ON $\mathbb{Q}$-CONIC BUNDLES, II

SHIGEFUMI MORI AND YURI PROKHOROV

Abstract. A $\mathbb{Q}$-conic bundle germ is a proper morphism from a threefold with only terminal singularities to the germ $(Z \ni o)$ of a normal surface such that fibers are connected and the anticanonical divisor is relatively ample. We obtain the complete classification of $\mathbb{Q}$-conic bundle germs when the base surface germ is singular. This is a generalization of [MP06], which further assumed that the fiber over $o$ is irreducible.

1. Introduction

This note is a continuation of our previous work [MP06] where we studied the local structure of $\mathbb{Q}$-conic bundles.

(1.1) Definition. A $\mathbb{Q}$-conic bundle is a projective morphism $f: X \to Z$ from a threefold with only terminal singularities to a surface such that

(i) $f_*\mathcal{O}_X = \mathcal{O}_Z$ and all fibers are one-dimensional,
(ii) $-K_X$ is $f$-ample.

For $f: X \to Z$ as above and for a point $o \in Z$, we call the analytic germ $(X, f^{-1}(o)_{\text{red}})$ a $\mathbb{Q}$-conic bundle germ.

In [MP06] we completely classified $\mathbb{Q}$-conic bundle germs over a singular base and such that the central fiber is irreducible. For convenience of quotations we reproduce briefly the classification. For more detailed explanations we refer to the original paper [MP06].

(1.2) Theorem. Let $f: (X,C) \to (Z,o)$ be a $\mathbb{Q}$-conic bundle germ, where $C$ is irreducible and $(Z,o)$ is singular. Then we are in one of the following cases:

---

The research of the first author was supported in part by JSPS Grant-in-Aid for Scientific Research (B)(2), No. 16340004. The second author was partially supported by grants CRDF-RUM, No. 1-2692-MO-05 and RFBR, No. 05-01-00353-a, 06-01-72017.
In this paper we consider the case where the base surface is singular and the central fiber is reducible. Our main result is the following.

**Theorem.** Let \( f : (X, C) \rightarrow (Z, o) \) be a \( \mathbb{Q} \)-conic bundle germ. Assume that \( C \) is reducible and the base surface \((Z,o)\) is singular. Then \((Z,o)\) is Du Val of type \( A_1 \) and \((X,C)\) is the \( \mu_2 \)-quotient of the index-two \( \mathbb{Q} \)-conic bundle \( f' : (X', C') \rightarrow (Z', o') \) over a smooth base, where \( \mu_2 \) acts on \((Z', o')\) freely in codimension one. Moreover, \( C' \) has four irreducible components, \( \mu_2 \) does not fix any of them and \( X \) has a unique non-Gorenstein point \( P \). Furthermore, \( X' \) is given by the following two equations in \( \mathbb{P}(1,1,1,2)_{y_1,...,y_4} \times \mathbb{C}_{u,v}^2 \)

\[
\begin{align*}
y_1^2 - y_2^2 &= \psi_1(y_1, \ldots, y_4; u, v), \\
y_2^2 - y_3^2 &= \psi_2(y_1, \ldots, y_4; u, v),
\end{align*}
\]

where \( \mu_2 \) acts as follows:

\[
(y_1, y_2, y_3, y_4; u, v) \mapsto (-y_1, -y_2, y_3, -y_4; -u, -v).
\]

Here \( \psi_i = \psi_i(y_1, \ldots, y_4; u, v) \) are weighted quadratic in \( y_1, \ldots, y_4 \) with respect to \( \text{wt}(y_1, \ldots, y_4) = (1,1,1,2) \) and \( \psi_i(y_1, \ldots, y_4; 0,0) = 0 \). The following are the only possibilities:

**1.3.1** \((X, P)\) is a cyclic quotient singularity of type \( \frac{1}{4}(1,1,1,-1) \) and for any component \( C_i \subset C \) germ \((X, C_i)\) is of type \((IA')\),

**1.3.2** \((X, P)\) is a singularity of type \( cAx/4 \) and for any component \( C_i \subset C \) germ \((X, C_i)\) is of type \((II')\).

Conversely, if the quotient \((X, C) = (X', C')/\mu_2\), where \((X', C')\) and the action of \( \mu_2 \) are as above, has only terminal singularities, then \((X, C)\) is a conic bundle germ over \( \mathbb{C}_{u,v}^2/\mu_2 \) with reducible central fiber \( C \).

Below are a series of explicit examples of \( \mathbb{Q} \)-conic bundles as in [1.3].

| Type       | No.       | singularities                                                                 | \((Z,o)\) |
|------------|-----------|-------------------------------------------------------------------------------|-----------|
| toroidal   | (1.2.1)   | \( \frac{1}{n}(1, a, -a) \) and \( \frac{1}{n}(-1, a, -a) \), \( \text{gcd}(n, a) = 1 \) | \( A_{n-1} \) |
| \((IA)+(IA)\) | (1.2.2)   | \( \frac{1}{n}(a, -1, 1) \) and \( \frac{1}{n}(a+1, 1, -1) \), \( n = 2a + 1 \) | \( A_{n-1} \) |
| \((IE')\)  | (1.2.3)   | \( \frac{1}{8}(5, 1, 3) \)                                                 | \( A_3 \)  |
| \((ID')\)  | (1.2.4)   | \( cA/2 \) or \( cAx/2 \)                                                 | \( A_1 \)  |
| \((IA')\)  | (1.2.5)   | \( \frac{1}{4}(1,1,3) \) \quad \((+\text{III})\)                           | \( A_1 \)  |
| \((II')\)  | (1.2.6)   | \( cAx/4 \) \quad \((+\text{III})\)                                       | \( A_1 \)  |
(1.3.3) Example. Consider the subvariety $X' \subset \mathbb{P}(1, 1, 1, 2) \times \mathbb{C}^2$ defined by the following two equations:

\[
\begin{align*}
    y_1^2 - y_3^2 + u^{2k+1}y_4 + v^2y_2^2 &= 0, \\
    y_2^2 - y_3^2 + vy_4 &= 0.
\end{align*}
\]

The projection $f': X' \to \mathbb{C}^2$ is a $\mathbb{Q}$-conic bundle of index 2 (cf. [MP06, 12.1.3]). Define the action of $\mu_2$ on $X'$ as follows

\[
(y_1, y_2, y_3, y_4; u, v) \mapsto (-y_1, -y_2, y_3, -y_4; -u, -v).
\]

Then $X'/\mu_2 \to \mathbb{C}^2/\mu_2$ is a $\mathbb{Q}$-conic bundle with a unique non-Gorenstein point $P$. The point $P$ is of type (1.3.1) if $k = 0$ and of type (1.3.2) if $k \geq 1$.

The basic idea of the proof is to reduce the problem of classifying $\mathbb{Q}$-conic bundles $(X, C)$ as in Theorem (1.3) to the case where the central fiber is irreducible by applying the MMP to a $\mathbb{Q}$-factorialization $(X^q, C^q)$. Then the resulting $\mathbb{Q}$-conic bundle $(\bar{X}, \bar{C})$ belongs to the list (1.2). We trace back from $(\bar{X}, \bar{C})$ to $(X, C)$. It turns out that in many cases the steps of the MMP do not affect the singularities of $(\bar{X}, \bar{C})$. Here we use some results about divisorial contractions and flips (see §2) based on [KM92] and [Kaw96]. Then the base change trick allows us to show that $(X, C)$ is a $\mu_2$-quotient of an index-two conic bundle, see §3.

Acknowledgments. The work was carried out at Research Institute for Mathematical Sciences (RIMS), Kyoto University. The second author would like to thank RIMS for invitations to work there in February 2007, for hospitality and wonderful conditions of work.

2. Preliminary results on extremal contractions

(2.1) Let $(E^\sharp, P^\sharp)$ be a Du Val singularity. (We assume that $(E^\sharp, P^\sharp)$ is singular.) Assume that $\mu_m$ acts on $E^\sharp$ freely outside $P^\sharp$ and the quotient $(E, P) = (E^\sharp, P^\sharp)/\mu_m$ is also Du Val. Then there is a $\mu_m$-equivariant embedding $(E^\sharp, P^\sharp) \subset (\mathbb{C}^3_{x,y,z}, 0)$ such that $x, y, z$ and the equation of $E^\sharp$ are semi-invariant. Let $F^\sharp \subset \mathbb{C}^3$ be the locus of points at which the action of $\mu_m$ is not free. By our assumption $F^\sharp$ is a curve. Define the invariant $\varsigma(E^\sharp, P^\sharp, \mu_m)$ as the local intersection number
According to [Rei87, 4.10] we have only the following cases:

| \(m\) | \(\left(E^\sharp, P^\sharp\right) \to \left(E, P\right)\) | \(\varsigma\left(E^\sharp, P^\sharp, \mu_m\right)\) |
|---|---|---|
| any | \(A_{r-1} \to A_{mr-1}\) | \(r\) |
| 4 | \(A_{2r-2} \to D_{2r+1}\) | \(2r - 1\) |
| 2 | \(A_{2r-1} \to D_{r+2}\) | \(2\) |
| 3 | \(D_4 \to E_6\) | \(2\) |
| 2 | \(D_{r+1} \to D_{2r}\) | \(r\) |
| 2 | \(E_6 \to E_7\) | \(3\) |

(2.1.2) Let \((W, P)\) be a three-dimensional terminal singularity of index \(m > 1\) and let \(E \in \left| -K_W \right|\) be a divisor having a Du Val singularity at \(P\). Assume that \((W, P)\) is not a cyclic quotient. Let \(\pi: \left(W^\sharp, P^\sharp\right) \to (W, P)\) be the index-one \(\mu_m\)-cover and let \((W^\sharp, P^\sharp) = \{\phi = 0\} \subset \mathbb{C}^4_{x_1, x_2, x_3, x_4}\) be a \(\mu_m\)-equivariant embedding. Let \(E^\sharp := \pi^{-1}(E)\) and \(F^\sharp \subset \mathbb{C}^3\) be the locus of points at which the action of \(\mu_m\) is not free. Since \(\pi\) is free in codimension two, \(F^\sharp\) is a curve. Recall that the local intersection number \((E^\sharp \cdot F^\sharp)_0\) is called the axial multiplicity of \((W, P)\) [Mor88, 1a.5]. We denote it by \(\text{am}(W, P)\). By the classification of terminal singularities we may assume that \(F^\sharp\) is the \(x_4\)-axis, and either \(\text{wt}(x_1, x_2, x_3, x_4, \phi) \equiv (1, -1, a, 0, 0) \mod m\), or \(m = 4\) and \(\text{wt}(x_1, x_2, x_3, x_4, \phi) \equiv (1, -1, a, 2, 2) \mod 4\), where \(\gcd(a, m) = 1\). Since \((E^\sharp, P^\sharp)\) is a Du Val singularity, its Zariski tangent space at the origin is three-dimensional. Hence there is a \(\mu_m\)-stable hypersurface \(H^\sharp \subset \mathbb{C}^4\) such that \(E^\sharp = H^\sharp \cap W^\sharp\) and \(H^\sharp\) is smooth.

(2.1.3) Claim. \(F^\sharp \subset H^\sharp\).

Proof. Let \(\psi\) be the \(\mu_m\)-semi-invariant equation of \(H^\sharp\). Then \(\text{wt} \psi \equiv a\). Hence \(\psi\) does not contain terms \(x_4^k\) and so it vanishes on \(F^\sharp\). □

(2.1.4) We define the invariant \(\varsigma(W, E, P)\) as the local intersection number \((E^\sharp \cdot F^\sharp)_0\) inside \(H^\sharp\). Clearly it coincides with \(\varsigma(E^\sharp, P^\sharp, \mu_m)\) defined above.

(2.1.5) Lemma. Assume that \((W, P)\) is not a cyclic quotient singularity. The invariant \(\varsigma(W, E, P)\) does not depend on the choice of \(E\) and \(\varsigma(W, E, P) = \text{am}(W, P)\).

Proof. Both sides of the equality coincide with the order of vanishing of \(\phi|_{F^\sharp}\). □
(2.1.6) Corollary. Let \((W,P)\) is a three-dimensional terminal singularity of index \(m > 1\) which is not a cyclic quotient and let \(E \in |-K_{(W,P)}|\) be a member having a Du Val singularity of \(A\)-type at \(P\). Then \(E\) is isomorphic to a general member \(E_{\text{gen}} \in |-K_{(W,P)}|\).

Proof. By the above lemma we have \(\varsigma(E^\sharp, P^\sharp, \mu_m) = \varsigma(E_{\text{gen}}^\sharp, P_{\text{gen}}^\sharp, \mu_m) = \text{am}(W,P)\). Then the statement follows by the first line in (2.1.1). □

(2.2) Proposition. Let \(\varphi: (V, \Gamma) \to (W,o)\) be the analytic germ of a divisorial extremal contraction of threefolds with terminal singularities (in particular, \(W\) is \(\mathbb{Q}\)-Gorenstein) such that the central fiber \(\Gamma := \varphi^{-1}(o)_{\text{red}}\) is one-dimensional and irreducible.

(i) The point \((W,o)\) cannot be of type \(cAx/4\).

(ii) If \((W,o)\) is of type \(cAx/2\), then \((V,\Gamma)\) has a unique non-Gorenstein point which is of type \((II'\lor)\).

(iii) If \((W,o)\) is analytically isomorphic to
\[
\{x_1x_2 + x_3^2 + x_4^{2k} = 0\}/\mu_2(1,1,0,1),
\]
then \((V,\Gamma)\) has a unique non-Gorenstein point \(P\) which is locally imprimitive of index 4 and splitting degree 2. Moreover, \(P \in (V,\Gamma)\) is either of type \((II')\) or \((IA')\) and in the second case \((X,P)\) is a cyclic quotient singularity.

Proof. For the proof we assume that \((W,o)\) is of type \(cAx/4, cAx/2\), or as in (2.2.1). We will use the classification [KM92, Th. 2.2]. Let \(m\) be the index of \((W,o)\). Then the the canonical class \(K_W\) is an \(m\)-torsion element in \(\text{Cl}_{\text{sc}}(W,o)\). Its pull-back \(\varphi^*K_W\) is a well-defined Cartier divisor on \(V \setminus \Gamma\) such that \(m(\varphi^*K_W) \sim 0\). Hence the group \(\text{Cl}_{\text{sc}}(V,\Gamma)\) contains an \(m\)-torsion element, say \(\xi\). By the classification [KM92, Th. 2.2] \(\text{Cl}_{\text{sc}}(V,\Gamma)\) can contain a torsion only when \((V,\Gamma)\) is of type 
\(\{x_1x_2 + x_3^2 + x_4^{2k} = 0\}/\mu_2(1,1,0,1),\)
then \((V,\Gamma)\) has a unique non-Gorenstein point \(P\) which is locally imprimitive of index 4 and splitting degree 2. Moreover, \(P \in (V,\Gamma)\) is either of type \((II')\) or \((IA')\) and in the second case \((X,P)\) is a cyclic quotient singularity.

Assume that \((V,\Gamma)\) is of type \((k1A)\) (with a point of type \((IA')\), \((II')\), or \((k2A)\)).

Assume that \((V,\Gamma)\) is of type \((k2A)\). Then by [KM92, Th. 2.2] a general member \(D \in |-K_V|\) and its image \(\varphi(D) \in |-K_W|\) have only Du Val singularities. Moreover, \((\varphi(D), o)\) is a singularity of type \(A\), and so \((W,o)\) is of type \(cA/\ast\). Clearly, the contraction \(\varphi|_D: D \to \varphi(D)\) is crepant. By our assumptions \((W,o)\) is a singularity given by (2.2.1). So, \(\text{am}(W,o) = 2\). By Corollary [2.1.6] the singularity \((\varphi(D), o)\) is of type \(A_3\). Since \(\varphi_D: D \to \varphi(D)\) is crepant and \(V\) has two singular points, the only possibility is that \(D\) has two singularities of type \(A_1\). But in this case \(V\) is of index two and then by [KM92, Th. 4.7] \(V\) has a unique non-Gorenstein point, a contradiction.

In the remaining cases \((II')\) and \((k1A)\), \(V\) has a unique non-Gorenstein point \(P\). Then \((V,\Gamma)\) is locally imprimitive at \(P\) and the
splitting degree equals $m$. In particular, the index of $P$ is $> m$ \cite[Cor. 1.16]{Mor88}. Thus if $(V, \Gamma)$ is of type $(\Pi')$, then we are in the case (ii) or (iii).

Assume that $(V, \Gamma)$ is of type $(k1A)$. Then by \cite[Th. 2.2]{KM92} a general member $D \in |-K_V|$ does not contain $\Gamma$, has only Du Val singularity at $P := \{D \cap \Gamma\}$, and $\varphi|_D : D \to \varphi(D)$ is an isomorphism. Hence $\varphi(D) \in |-K_W|$ has a Du Val singularity of type $A$ at $o$. In this case, $(W, o)$ cannot be of type $cAx/\ast$. Thus $(W, o)$ is given by (2.2.1). By Corollary (2.1.6) $D \simeq \varphi(D)$ is of type $A_3$. Since the index of $(V, P)$ is $> 2$, $(V, P)$ must be a cyclic quotient singularity $\frac{1}{4}(1, 1, -1)$. So we are in the case (iii). This proves the proposition.

\(\square\)

\textbf{(2.3) Proposition.} Let $\chi : (V, \Gamma) \longrightarrow (V^+, \Gamma^+)$ be a flip of threefolds with terminal singularities with irreducible flipping curve $\Gamma$. Then $(V^+, \Gamma^+)$ contains none of the following configurations of singularities:

(i) two cyclic quotient singularities $P_1^+$ and $P_2^+$ of indices $m_1$ and $m_2$ with $\gcd(m_1, m_2) > 1$ such that $(V^+, \Gamma^+)$ is locally primitive at $P_1^+$ and $P_2^+$;

(ii) an imprimitive point $P^+$ of splitting degree $s > 1$.

\textit{Proof.} By \cite[Cor. 13.4]{KM92} $\Gamma^+$ is irreducible. Assume that one of the cases (i)-(ii) holds. As in \cite[Cor. 1.12]{Mor88} there is a $d$-torsion element $\xi^+ \in \Cl^{\infty}V^+$ for some $d > 1$. Its proper transform $\xi$ on $V$ is a $d$-torsion element in $\Cl^{\infty}V$. In \cite[Th. 2.2]{KM92} flips are classified into 6 types $(k1A)$, $(k2A)$, $(cD/3)$, $(IIA)$, $(IC)$, $(kAD)$ according to a general member of the anticanonical linear system $|-K_V|$ \cite[Th. 2.2]{KM92}. The group $\Cl^{\infty}V$ can contain a torsion only in cases $(k1A)$ and $(k2A)$ (in all other cases the flipping variety is locally primitive and indices of non-Gorenstein points are coprime, cf. \cite[Cor. 1.12]{Mor88}). The torsion elements $\xi$ and $\xi^+$ induce the following cyclic $\mu_d$-coverings:

\begin{equation}
(V', \Gamma') \xrightarrow{\xi} (V^+, \Gamma^+)
\end{equation}

Consider the flipping diagram

\begin{equation}
(V, \Gamma) \xrightarrow{\chi} (V^+, \Gamma^+)
\end{equation}

\begin{equation}
(W, o) \xrightarrow{\varphi^+} (V^+, \Gamma^+)
\end{equation}

\begin{equation}
(W, o) \xrightarrow{\varphi} (V', \Gamma')
\end{equation}
By [Mor88] Th. 7.3, 9.10 and [KM92] Th. 2.2, a general member $D \in (-K_V)$ has only Du Val singularities. Since the restriction $\varphi_D : D \to \varphi(D)$ is crepant, the same holds for $\varphi(D) \in (-K_W)$. Further, if we put $D^+ = \chi(D)$, then $D^+ \in (-K_{V^+})$ and $D^+$ also has only Du Val singularities. Since $K_{V^+} \cdot \Gamma^+ > 0$, $D^+ \supseteq \Gamma^+$.

(2.3.2) First we consider the case where our flip is of type (k1A). Then $V$ has a unique non-Gorenstein point $P$ and $P$ is of type $cA/\ast$. In this case $D \cap \Gamma = \{P\}$ and $(\varphi(D), o) \simeq (D, P)$ is of type $A_s$. Since $	ext{Cl}^* V$ has a torsion, $(V, \Gamma)$ is locally imprimitive at $P$.

(2.3.3) Assume that we are in the case (i). We claim that $V^+$ has at least one Gorenstein singular point. Indeed, since the germ $(V, \Gamma)$ has only one non-Gorenstein point, it is locally imprimitive and in the diagram (2.3.1) $\pi$ is the splitting cover [Mor88] Cor. 1.12. Here $\Gamma'$ has exactly $d$ components and $V'^{+}$ is the relative canonical model of $V'$. Since $(V^{+}, \Gamma^{+})$ is locally primitive at $P_1^{+}$ and $P_2^{+}$, the curve $\Gamma'^{+}$ is irreducible. Now the map $\chi'$ can be decomposed as follows

$$\chi' : V' = V'_0 \to V'_1 \to \cdots \to V'_n \to V'^{+},$$

where every $V'_i \to V'_{i+1}$ is a flip along an irreducible curve and $V'_{n} \to V'^{+}$ is a crepant small contraction (cf. [KM92] Proof of 13.5). Every step $V'_i \to V'_{i+1}$ preserves the number of components of the central fiber. Hence the crepant contraction $V'_{n} \to V'^{+}$ is nontrivial and gives us a Gorenstein non-$Q$-factorial point $Q \in \Gamma^+ \subset V^+$. This proves our claim. Thus the divisor $D^+$ has at least three singular points: $P_1^+$, $P_2^+$, and $Q$. But then $\varphi^+_D : D^+ \to \varphi(D)$ contracts $\Gamma^+$ to a Du Val singularity of type $D_s$ or $E_s$, a contradiction.

(2.3.4) Now we assume that we are in the case (ii). We claim that the log divisor $K_{D^+} + \Gamma^+$ is not plt at $P^+$. Indeed, in the diagram (2.3.1) $\pi^+$ is the splitting cover (see [Mor88] Cor. 1.12.1). In particular, $\pi^+$ is étale outside $P^+$, $\pi^+-1(P^+)$ is one point, and $\Gamma'^+_{s}$ has $s > 1$ irreducible components, all of them pass through $\pi^{+\ast-1}(P^+)$. Let $D'^+ := \pi^+-1(D^+)$. Since $\Gamma'^+_{s}$ is singular at $\pi^{+\ast-1}(P^+)$, the log divisor $K_{D'^+} + \Gamma'^+$ is not plt at this point. This proves our claim because the restriction $\varphi^+_D : D'^+ \to D^+$ is étale in codimension one (see, e.g., [Kol92] Cor. 20.4)). Now since the contraction $D^+ \to \varphi(D)$ is crepant, $D^+$ is dominated by the minimal resolution $D_{\min}$ of $\varphi(D)$:

$$D_{\min} \to D^+ \to \varphi(D).$$

Since $K_{D^+} + \Gamma^+$ is not plt, the exceptional divisor of $D_{\min} \to \varphi(D)$ is not a chain of smooth rational curves. Hence $(\varphi(D), o)$ is not a singularity of type $A_s$, a contradiction.

(2.3.5) Finally, we consider the case where our flip is of type (k2A). These flips are described in [Mor02]. We will use notation of [Mor02].
By [Mor02, Th. 4.7] \((V^+, \Gamma^+)\) is locally primitive. Hence we have the case (i). Moreover, \(V^+\) has exactly two singular points and they are analytically isomorphic to germs of the following \(cA/m_i\) singularities:

\[
\{\xi_i \eta_i = G_{k-i}(\zeta_i^{m_i}, u^{e(k+2-i)})\}/\mu_{m_i} \subset \mathbb{C}^4_{\xi_i, \eta_i, \zeta_i, u}/\mu_{m_i}(1, -1, a_i, 0),
\]

where \(k, a_i\) are some positive numbers and \(e(j)\) is some function. Hence these points coinide with \(P_1^+\) and \(P_2^+\). Since \(P^+_i \in \Gamma^+ \subset V^+\) are cyclic quotient singularities, we have \(e(k) = e(k+1) = 1\) \((u\) needs to be eliminated). If we put \(\delta := a_1 m_2 + a_2 m_1 - m_1 m_2\), then \(\delta \geq d\) and by definition [Mor02, Def. 3.2] we have \(e(3) = 0, e(4) = \delta \alpha_1 \geq d > 1, e(5) = (\delta^2 \rho_2 - 1) \alpha_1 + \delta \alpha_2 \geq d > 1\) \(\text{see [Mor02, Rem. 3.6]}\). Thus, \(k \geq 6\). On the other hand, by [Mor02, Lemma 3.5, Cor. 3.7] we have \(k \leq 5\), a contradiction.

\(\square\)

**(2.4) Proposition.** Let \(\varphi: (V, \Gamma) \to (W, o)\) be the germ of a birational crepant contraction of threefolds with terminal singularities, where \(\Gamma\) is irreducible.

(i) \((V, \Gamma)\) contains at most two non-Gorenstein points.

(ii) If \((V, \Gamma)\) is imprimitive at some point \(P\), then \((W, o)\) cannot be a singularity of type \(cA/\ast\).

**Proof.** For the proof we assume that \(V\) is not Gorenstein. Since \(\varphi\) is crepant, the point \((W, o)\) is not Gorenstein. Let \(m\) be its index. Let \(D \in |-K_{(W,o)}|\) be a general member and let \(S := \varphi^{-1}(D)\). Then \(S \in |-K_{(V,\Gamma)}|\) and both \(S\) and \(D\) have only Du Val singularities. Moreover, the restriction map \(\varphi_S: S \to D\) is crepant. Hence \(S\) is dominated by the minimal resolution \(D^{\min}\) of \(D\) and obtained from \(D^{\min}\) by contracting all but one exceptional curves.

First assume that \((V, \Gamma)\) has at least three non-Gorenstein points, say \(P, Q,\) and \(R\). By the classification of Du Val singularities \((D, o)\) is a singularity of type \(D_4\) or \(E_8\) and \(S\) is obtained from \(D\) by blowing up the exceptional curve corresponding to the central vertex in the Dynkin diagram. In this case exceptional curves on \(D^{\min}\) over \((S, P), (S, Q)\) and \((S, R)\) form strings and the proper transform of \(\Gamma\) is adjacent to the ends of them. This means that the log divisor \(K_S + \Gamma\) is plt. The latter implies that the germ \((V, \Gamma)\) is locally primitive (cf. (2.3.4)). Now consider the index-one cover \(\pi: (W^\sharp, o^\sharp) \to (W, o)\). It
induces the following diagram

\[
\begin{array}{ccc}
(V^\sharp, \Gamma^\sharp) & \xrightarrow{\nu} & (V, \Gamma) \\
\downarrow \varphi^\sharp & & \downarrow \varphi \\
(W^\sharp, \omega^\sharp) & \xrightarrow{\pi} & (W, \omega)
\end{array}
\]

Since \((V, \Gamma)\) is locally primitive, \(\Gamma^\sharp = \pi^\sharp(\omega^\sharp)\) is irreducible. The group \(\mu_m\) naturally acts on \(\Gamma^\sharp \simeq \mathbb{P}^1\) and has exactly two fixed points. Thus we may assume that \(\nu^{-1}(R)\) contains no fixed points. But then \(\nu^{-1}(R)\) consists of \(m > 1\) non-Gorenstein points of the same index. By \cite[Cor. 1.12]{Mor88} there is a torsion element in \(\text{Cl}^{\text{sc}}(V^\sharp, \Gamma^\sharp) \simeq \text{Cl}^{\text{sc}}(W^\sharp, \omega^\sharp)\).

This contradicts the fact that \(W^\sharp \setminus \{\omega^\sharp\}\) is simply connected. Thus (i) is proved.

Now assume that \((V, \Gamma)\) contains an imprimitive point \(P\). By the proof of (i) \(S\) has at most two singular points and the log divisor \(K_S + \Gamma\) is not plt at \(P\). On the other hand, assume that \((D, o)\) is a point of type \(A_\ast\). Then the exceptional curves of the minimal resolution \(D^{\text{min}} \to S\) and \(\Gamma\) form a chain. Hence \(K_S + \Gamma\) is not plt, a contradiction. \(\square\)

**Proposition (cf. \cite[1.14]{Mor88}).** Let \(f: (X, C) \to (Z, o)\) be the germ of a contraction from a threefold with only terminal singularities to a surface such that

(i) \(-K_X\) is nef and big,
(ii) \(C := f^{-1}(o)_{\text{red}}\) is a curve having at least three components,
(iii) each \(K_X\)-trivial component \(C_j \subset C\) contains a non-Gorenstein point.

Then \(X\) has index \(> 1\) at all singular points of \(C\).

**Proof.** By the Kawamata-Viehweg vanishing theorem we have \(R^1 f_* \mathcal{O}_X = 0\). Hence \(C\) is a union of \(\mathbb{P}^1\)’s whose configuration is a tree. Let \(P \in C\) be a singular point and let \(C_i \subset C\) be a component passing through \(P\). We have \(\text{gr}^{0}_{C_i} \omega \simeq \mathcal{O}(-1)\). Indeed, take a positive integer \(m\) such that \(mK_X\) is Cartier. Then there is a natural embedding \((\text{gr}^{0}_{C_i} \omega)^{\otimes m} \to \mathcal{O}_{C_i}(mK_X)\). Since \(K_X \cdot C_i \leq 0\) we have \(\text{deg} \text{gr}^{0}_{C_i} \omega \leq 0\). Moreover, if \(K_X \cdot C_i < 0\), then \(\text{deg} \text{gr}^{0}_{C_i} \omega < 0\). Assume that \(K_X \cdot C_i = 0\). Since \(C_i\) contains a non-Gorenstein point, the above embedding is not an isomorphism and so again \(\text{deg} \text{gr}^{0}_{C_i} \omega < 0\). On the other hand, \(C_i\) is contractible over \(Z\). Hence, by the Grauert-Riemenschneider vanishing theorem we have \(H^1(\text{gr}^{0}_{C_i} \omega) = 0\). This shows \(\text{gr}^{0}_{C_i} \omega \simeq \mathcal{O}(-1)\).

Now let \(C_j\) be another component of \(C\) passing through \(P\). As above, \(\text{gr}^{0}_{C_j} \omega \simeq \mathcal{O}(-1)\). Consider the following exact sequence

\[
0 \to \text{gr}^{0}_{C_i \cup C_j} \omega \to \text{gr}^{0}_{C_i} \omega \oplus \text{gr}^{0}_{C_j} \omega \to \mathcal{F} \to 0,
\]
where Supp $\mathcal{F} = P$. Since $C_i \cup C_j \neq C$, $C_i \cup C_j$ is contractible over $Z$ and again by the Grauert-Riemenschneider vanishing $H^1(gr_{C_i \cup C_j}^0 \omega) = 0$. This implies $gr_{C_i \cup C_j}^0 \omega \simeq gr_{C_i}^0 \omega \oplus gr_{C_j}^0 \omega$. So $gr_{C_i \cup C_j}^0 \omega$ is not locally free at $P$ and this point cannot be Gorenstein. \(\square\)

3. The proof of the main theorem

(3.1) Notation. Let $f : (X, C) \to (Z, o)$ be a $\mathbb{Q}$-conic bundle germ with reducible central fiber $C$. Then $\rho(X/Z) > 1$. Recall that according to [MP06, Th. 1.2.7] $(Z, o)$ is either smooth or Du Val of type $A$. We assume that $(Z, o)$ is singular of type $A_{n-1}$, $n \geq 2$.

(3.1.1) Lemma. Notation as above.

(i) If $(X, C)$ has a point $P$ such that either
   (a) $P$ is of type $cAx/4$, or
   (b) for each component $C_i \subset C$ passing through $P$ the germ $(X, C_i)$ is locally imprimitive at $P$.
   Then $P$ is the only non-Gorenstein point on $X$.

(ii) Conversely, if $P$ is a unique non-Gorenstein point on $X$, then all the components $C_i \subset C$ pass through $P$ and the germ $(X, C_i)$ is locally imprimitive at $P$. If furthermore $(X, P)$ is of index 4, then $(X, C)$ is a quotient of an index two $\mathbb{Q}$-conic bundle germ $(X', C')$ over a smooth base by $\mu_2$, where the action is free in codimension one, $C'$ has four irreducible components and $\mu_2$ does not fix any of them.

Proof. Let $P \in X$ be a point as in (i). For each component $C_i \subset C$ passing through $P$ the germ $(X, C_i)$ is an extremal neighborhood and by [KM92, Th. 2.2] $(X, C_i)$ has no non-Gorenstein point other than $P$. Since each singular point of $C$ is not Gorenstein [Kol99, Prop. 4.2], [MP06, 4.4.2] and $C$ is connected, $P$ is the only non-Gorenstein point on the whole $X$.

Now assume that $P$ is the only non-Gorenstein point. Consider the base change [MP06, 2.4]: $(X', C') \to (X, C)$. Here $(X', C')$ is a conic bundle germ over a smooth base and $X' \to X$ is an étale outside $P$ $\mu_n$-cover. Thus $(X, C) = (X', C')/\mu_n$. If $\mu_n$ fixes a component $C'_i \subset C$, then there are two $\mu_n$-fixed points on $C_i$ and they give us two non-Gorenstein points on $X$, a contradiction. So the first assertion of (ii) is proved.

Finally assume that $(X, P)$ is of index 4. Since the index of $(X, P)$ is divisible by $n$, $n = 4$ or 2. If $n = 4$, then $X$ is Gorenstein. In this case, by [Pro97, Th. 2.4] $C$ is irreducible, a contradiction. Thus
$n = 2$ and $(X', C')$ is of index 2. By the above, $\mu_2$ does not fix any component of $C'$. On the other hand, $C'$ has at most four components [MP06, Th. 12.1]. Hence $C'$ has exactly four components. This proves the lemma. \hfill $\square$

(3.1.2) Let $q : X^a \to X$ be a $\mathbb{Q}$-factorialization. (It is possible that $q$ is the identity map.) Run the MMP over $Z : X^a = X_0 \to X_{N+1} = \tilde{X}$. Since $X/Z$ is a rational curve fibration, $X_{N+1}$ is not a minimal model over $Z$. Therefore, at the end we get an extremal contraction $\tilde{f} : \tilde{X} \to \tilde{Z}$ of Fano type over $Z$. Since the composition $f^a : X^a \to Z$ has only one-dimensional fibers, $Z = \tilde{Z}$ and $X^a \to \tilde{X}$ is a sequence of flips and extremal divisorial contractions that contract a divisor to a curve which is not contained in the fiber over $o \in Z$. Thus we have the following diagram:

$$
(X^a, C^a) \xrightarrow{g_0} (X_1, C_1) \to \cdots \to (X_N, C_N) \xrightarrow{g_N} (\tilde{X}, \tilde{C})
$$

Here each $X_k$ has a morphism $f_k : X_k \to Z$ with connected one-dimensional fibers and $C_k := f_k^{-1}(o)$ is the central fiber (with reduced structure). Since $\rho(\tilde{X}/Z) = 1$, $\tilde{f} : \tilde{X} \to \tilde{Z}$ is a $\mathbb{Q}$-conic bundle with irreducible central fiber $\tilde{C}$. Since the base $(Z, o)$ is singular, $\tilde{X}$ is not Gorenstein. So $\tilde{f}$ is classified in [MP06], see also (1.2).

(3.1.3) Note that each component of the central fiber $C_k$ is contractible and the resulting variety is again projective over $Z$ (because it has one-dimensional fibers over $Z$). Hence each component of $C_k$ generates an extremal ray (not necessarily $K$-negative). This implies that all our flipping curves are irreducible and all the divisorial contractions have irreducible fibers. Note also that all the varieties $X_k$ are analytically $\mathbb{Q}$-factorial at each point on $C_k$ (again because $X_k \to Z$ has one-dimensional fibers, cf. [Mor88, Proof of 1.7]).

The following is the key argument in the proof.

(3.2) Proposition. In the above notation one of the following holds.

(3.2.1) There is a component $C^a_0 \subset C^a$ containing two cyclic quotient singularities $P^a$ and $Q^a$ of index $n$. No other components of $C^a$ pass through $P^a$ and $Q^a$.  

11
(3.2.2) There is a point $P^k \in (X^q, C^q)$ of index $m > 1$ which is contained in only one component $C_0^q \subset C^q$ and such that $(X^q, C_0^q)$ is locally imprimitive at $P^q$. The following are the possibilities for $(n, m)$: $(4, 8)$, $(2, 4)$, and $(2, 2)$.

(3.2.3) There is a point $P^q \in (X^q, C^q)$ which is contained in exactly two components $C_0^q, C_1^q \subset C^q$ and such that both germs $(X^q, C_i^q)$ are locally imprimitive at $P^q$. The point $(X^q, P^q)$ is of type $cAx/4$ or $\frac{1}{4}(1, 1, -1)$. Here $n = 2$.

Moreover, there is an $n$-torsion element $\xi^q \in Cl^se(X^q, C^q)$ which is not Cartier at $P^q$ (and at $Q^q$ is the case (3.2.1)).

Proof. Since $(Z, o)$ is of type $A_{n-1}$, there is an $n$-torsion element $\eta \in Cl((Z, o))$. Put $\xi := f^*\eta$, $\xi_t := f_t^*\eta$, and $\xi^q := f^q*\eta$.

Assume that $(X, \mathcal{C})$ is either toroidal or of type (IA)+(IA). Let $\bar{P}, \bar{Q}$ be the singular points of $\bar{X}$. Then $\bar{\xi}$ is not Cartier at $\bar{P}$ and $\bar{Q}$. We claim that the map $\psi: \bar{X} \rightarrow X^q$ is an isomorphism near $\bar{P}$ and $\bar{Q}$. Indeed, by induction, since $\bar{P}, \bar{Q}$ are cyclic quotient singularities of index $n$, there is no divisorial contractions over these points by [Kaw96] and by Proposition (2.3) on each step the proper transform of $\bar{C}$ cannot be a flipped curve. So if we put $P^q := \psi(P), Q^q := \psi(Q)$, and $C_0^q := \psi(C)$, we get the case (3.2.1).

Now assume that $(\bar{X}, \bar{C})$ is of type (IE$^\vee$), (IA$^\vee$), or (II$^\vee$). Let $\bar{P}$ be a (unique) non-Gorenstein point. Then $(\bar{X}, \bar{P})$ is either a cyclic quotient singularity or of type $cAx/4$ and again $\bar{\xi}$ is not Cartier at $\bar{P}$. Moreover, $(\bar{X}, \bar{C})$ is locally imprimitive at $\bar{P}$. As above, there is no divisorial contractions over $\bar{P}$ by [Kaw96] and Proposition (2.2) and the proper transform of $\bar{C}$ cannot be a flipped curve by Proposition (2.3) Put $P^q := \psi(\bar{P})$ and $C_0^q := \psi(\bar{C})$. We get the case (3.2.2).

Finally consider the case where $(\bar{X}, \bar{C})$ is of type (ID$^\vee$). Then $n = 2$, i.e., $(Z, o)$ is of type $A_1$. Let $\bar{P}$ be a (unique) non-Gorenstein point. Then $(\bar{X}, \bar{C})$ is locally imprimitive at $\bar{P}$ and $(\bar{X}, \bar{P})$ is of type $cA/2$ or $cAx/2$. Moreover, in the first case, $(\bar{X}, \bar{P})$ is analytically isomorphic to a singularity given by (2.2.1). If there is no divisorial contractions over $\bar{P}$, we can argue as above and get the case (3.2.2). Otherwise on some step, the map $\psi_{k+1}: \bar{X} \rightarrow X_{k+1}$ is an isomorphism near $\bar{P}$ and there is a divisorial contraction $g_k: X_k \rightarrow X_{k+1}$ which blows up a curve passing through $P_k = \psi_{k+1}(\bar{P})$. Let $C_{k,0} = g_k^{-1}(P_{k+1})$ and let $C_{k,1}$ be the proper transform of $\bar{C}$ on $X_k$. By Proposition (2.2) $X_k$ has exactly one non-Gorenstein point $P_k$ on $C_{k,0}$. Moreover, $P_k$ is either a cyclic quotient singularity $\frac{1}{4}(1, 1, -1)$ or of type $cAx/4$ and $(X_k, C_{k,0})$ is locally imprimitive at $P_k$ of splitting degree $2$. Note that $\xi_k = g_k^\ast \xi_{k+1}$ is non-Cartier at all points of $C_{k,0}$. Since $P_k$ is the only
non-Gorenstein point on $C_{k,0}$, $\xi_k$ is not Cartier at $P_k$. Now if $C_{k,1}$ does not pass through $P_k$, then as above we get the case (3.2.2). Assume that $C_{k,0} \cap C_{k,1} = \{P_k\}$.

We claim that $(X_k, C_{k,1})$ is locally imprimitive at $P_k$. Indeed, $\xi_k$ defines the double cover $\pi_k: (X'_k, C'_k) \to (X_k, C_k)$ which is étale outside $\Sing X_k$. Since $\xi_k$ is not Cartier at $P_k$, $\pi_k$ does not split over $P_k$. Hence, $C'_{k,1} := \pi_k^{-1}(C_{k,1})$ is connected. On the other hand, since $(\bar{X}, \bar{C})$ is locally imprimitive at $\bar{P}$, the curve $C'_{k,1}$ is reducible. This means that $C_{k,1}$ is locally imprimitive at $P_k$. Finally as above the map $X_k \dashrightarrow X^q$ is an isomorphism near $P_k$. We get case (3.2.3). \hfill $\square$

(3.3) Proposition. Notation as in (3.1). Then $(X, C)$ contains only one non-Gorenstein point $P$. This point is either a cyclic quotient $\frac{1}{4}(1, 1, -1)$ or of type $cAx/4$. Moreover, for each component $C_i \subset C$ the germ $(X, C_i)$ is imprimitive at $P$ and $(Z, 0)$ is of type $A_1$.

Proof. By Proposition (3.2) there is a component $C^q_0 \subsetneq C^q$ as in (3.2.1), (3.2.2), or (3.2.3). First assume that $C^q_0$ is not contracted by $q: X^q \to X$. Put $C_0 := q(C^q_0)$. Then $(X, C_0)$ is an extremal neighborhood. In the case (3.2.1) it has two cyclic quotient singularities at $q(P^q)$ and $q(Q^q)$ and no other components of $C$ pass through $q(P^q)$ and $q(Q^q)$. On the other hand, $C \neq C_0$ and intersection points $C_0 \cap (C - C_0)$ are non-Gorenstein [Kol99, Prop. 4.2], [MP06, 4.4.2]. Thus the extremal neighborhood $(X, C_0)$ has at least three non-Gorenstein points. This contradicts Mor88, Th. 6.2). Similarly, in the case (3.2.2) $(X, C_0)$ is locally imprimitive at $q(P^q)$ and no other components of $C$ pass through $q(P^q)$. We get a contradiction by Lemma (3.1.1). Consider the case (3.2.3). If $C^q_1$ is not contracted by $q$, then we are done by Lemma (3.1.1). If $C^q_1$ is contracted by $q$, then $q(C_1)$ is a point of type $cAx/4$ by Proposition (2.4) and because $P^q$ is of index 4. Then again the assertion follows by Lemma (3.1.1).

From now on we assume that $q$ contracts $C^q_0$, i.e., $K_{X^q} \cdot C^q_0 = 0$. In the case (3.2.3) by symmetry and by the above arguments we may assume that $q$ contracts $C^q_1$. Consider the decomposition

$$q: X^q \xrightarrow{\delta} X^\delta \xrightarrow{\delta} X,$$

where $\delta$ contracts all the $K_{X^q}$-trivial components of $C^q$ except for $C^q_0$. Put $C^\delta := \delta(C^q)$ and $C^\delta_0 := \delta(C^q_0)$. Thus $-K_{X^\delta}$ is nef and big over $Z$ and $C^\delta_0$ is the only $K_{X^\delta}$-trivial curve on $X^\delta/Z$. Let $C^{\delta\delta} := C^\delta - C^\delta_0$. Then $C^{\delta\delta}$ has at least two components. Let $P := \delta(C^\delta_0)$ and $R^\delta := C^{\delta\delta} \cap C^\delta_0$. By Proposition (2.5) $R^\delta$ in not Gorenstein.
In the case $3.2.1$, $C^δ_0$ contains at least three non-Gorenstein points: $R^δ, P^δ := ϕ(P^q),$ and $Q^δ := ϕ(Q^q)$. This contradicts Proposition (2.4).

In the case $3.2.2$, $P^δ := ϕ(P^q)$ is a locally imprimitive point of $(X^δ, C^δ_0)$. By Proposition (2.4) the singularity $(X, P = δ(C^δ_0))$ is not of type $cA/*$. If the index of $(X, P)$ is $≥ 4$, then $(X, P)$ is of type $cAx/4$ and we can apply Lemma (3.1.1). Thus we assume that $(X, P)$ is of index 2 and $n = 2$. Let $C_i ⊂ C$ be a component passing through $P$. By [Mor88, Cor. 1.16] $(X, C_i)$ is primitive at $P$. Further, $ξ := f^*η = q^*ξ^q$ is a 2-torsion element of $Cl^{sc}(X, C)$ and is not Cartier at $P$. This defines a double étale in codimension one cover $(X', C'_i) → (X, C_i)$ which does not splits over $P$. Hence there is a point $Q ∈ (X, C_i)$ of even index. This contradicts the classification [KM92, Th. 2.2] (cf. [Mor07]).

Consider the case $3.2.3$. Then $P^δ := ϕ(P^q)$ is a point of index $≥ 4$ (because $ϕ$ is a crepant contraction). Recall that $ϕ$ contracts $C^q_1$ by our assumption. Then by Proposition (2.4) $(X^δ, P^δ)$ is a point of type $cAx/4$. As in the proof of Proposition (2.4), let $D ∈ |−K_{(X, δ(P^δ))}|$ be a general element and let $S := δ^{-1}(D)$. Then both $D$ and $S$ have only Du Val singularities and the contraction $δ_S: S → D$ is crepant. Since $(S, P^δ)$ is not of type $A_*$, the germ $(D, P)$ also cannot be of type $A_*$. Hence, $(X, P)$ is not of type $cA/*$ and so it is of type $cAx/4$ (because its index is $≥ 4$). Then the assertion follows by Lemma (3.1.1). □

(3.4) Explicit forms. By Proposition (3.3) and Lemma (3.1.1) $f: (X, C) → (Z, o)$ is a quotient of an index-two $Q$-conic bundle $f': (X', C') → (Z', o')$ over a smooth base by $μ_2$, where $μ_2$ acts on $X'$ and $Z'$ freely in codimension one. By [MP06] Prop. 12.1.10] there is a $μ_2$-equivariant diagram

$$
\begin{array}{c}
X' \leftarrow \mathbb{P}(1, 1, 1, 2) \times \mathbb{C}^2 \\
\downarrow f \\
\mathbb{C}^2
\end{array}
$$

where the actions of $μ_2$ on $(\mathbb{C}^2, 0) ≅ (Z', o')$ and $\mathbb{P}(1, 1, 1, 2)$ are linear. Further, we can make coordinates $y_1, y_2, y_3, u, v$ in $\mathbb{P}(1, 1, 1, 2)$ and $\mathbb{C}^2$ to be semi-invariant. By [MP06] Th. 12.1] $X'$ is given by two semi-invariant equations

$$
\begin{cases}
q_1(y_1, y_2, y_3) - \psi_1(y_1, \ldots, y_4; u, v) = 0, \\
q_2(y_1, y_2, y_3) - \psi_2(y_1, \ldots, y_4; u, v) = 0,
\end{cases}
$$

where $\psi_1$ and $\psi_2$ are given by

$$
\psi_1 = y_1 \psi_1^0, \quad \psi_2 = y_1 \psi_2^0,
$$

and $ψ_i^0$ are polynomials in $y_1, y_2, y_3$. The $q_i$ are polynomials in $y_1, y_2, y_3$ with coefficients in $\mathbb{C}[u, v]$.
where $\psi_1$ and $q_i$ are weighted quadratic in $y_1, \ldots, y_4$ with respect to $\omega((y_1, \ldots, y_4)) = (1, 1, 1, 2)$ and $\psi_1(y_1, \ldots, y_4; 0, 0) = 0$. Since the action of $\mu_2$ on $Z \cong \mathbb{C}^2$ is free outside 0, this action is given by $u \mapsto -u$, $v \mapsto -v$. Modulo multiplication on $\pm 1$ and permutations of $y_1, y_2, y_3$, we may assume also that $y_1 \mapsto -y_1$, $y_2 \mapsto -y_2$, $y_3 \mapsto y_3$. Otherwise all the points of $\{y_4 = 0\} \cap C'$ are fixed by $\mu_2$, while $P$ is the only non-Gorenstein on $X$.

The central fiber $C'$ is defined by $q_1 = q_2 = 0$. By Lemma 3.1.1, $C'$ has exactly four components and $\mu_2$ does not fix any of them. Thus we may assume that $C' = \cup C'_i$, $i = 1, 2, 3, 4$ and $\mu_2$ interchanges $C'_1$ and $C'_2$ (resp. $C'_3$ and $C'_4$). For any two components $C'_i \neq C'_j$ of $C'$, there is a linear form $l_{i,j}(y_1, \ldots, y_3)$ that vanishes along $C'_i \cup C'_j$. Then quadratic forms $l_{1,2}l_{3,4}$, $l_{1,3}l_{2,4}$, $l_{1,4}l_{2,3}$ vanish along $C'$. Hence they belong to the pencil $\lambda_1q_1 + \lambda_2 q_2$ and semi-invariant. This implies that the action of $\mu_2$ on the pencil is trivial. Moreover, we can put $q_1 = l_{1,2}l_{3,4}$ and $q_2 = l_{1,3}l_{2,4}$. In view of the $\mu_2$-action we may assume that $l_{1,3} = y_1 + y_3$, $l_{2,4} = y_1 - y_3$, $l_{1,4} = y_2 + y_3$, $l_{2,3} = y_2 - y_3$ after some linear coordinate change of $y_1, y_2, y_3$.

We claim that $y_4 \mapsto -y_4$. The arguments below are similar to those in the proof of [MP06, Lemma 12.1.12]. Assume to the contrary that $y_4 \mapsto y_4$. Let $U \subset \mathbb{P}(1, 1, 1, 2)$ be the chart $y_4 \neq 0$. Then $U \cong \mathbb{C}^3_{z_1, z_2, z_3}/\mu_2(1, 1, 1)$. Let $X'$ be the pull-back of $X \cap (U \times \mathbb{C}^2_{u,v})$ on $\mathbb{C}^3_{z_1, z_2, z_3} \times \mathbb{C}^2_{u,v}$ and let $P \subset X'$ be the preimage of $P$. Since the induced map $X' \to X$ is étale in codimension one, $(X', P') \to (X, P)$ is the index-one cover. Hence $(X', P') \to (X, P)/\mu_2$ is also the index-one cover of the terminal point $(X, P)/\mu_2$ of index 4 (the last is true because the action of $\mu_2$ is free in codimension one). Hence the morphism is a $\mu_4$-covering by the structure of terminal singularities. However $(X, P)/\mu_2$ is the quotient of $(X', P')$ by commuting $\mu_2$-actions:

$$(z_1, z_2, z_3, u, v) \mapsto (-z_1, -z_2, -z_3, u, v), (z_1, -z_2, z_3, -u, -v)$$

This is a contradiction, and we have $y_4 \mapsto -y_4$ as claimed. This finishes the proof of Theorem (1.3).

References

[Kaw96] Y. Kawamata, Divisorial contractions to 3-dimensional terminal quotient singularities. In *Higher-dimensional complex varieties* (Trento, 1994), pages 241–264. de Gruyter, Berlin, 1996.

[KM92] J. Kollár and S. Mori. Classification of three-dimensional flips. *J. Amer. Math. Soc.*, 5(3):533–703, 1992.

[Kol92] J. Kollár, editor. *Flips and abundance for algebraic threefolds*. Société Mathématique de France, Paris, 1992. Papers from the Second Summer
Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, Astérisque No. 211 (1992).

[Kol99] J. Kollár. Real algebraic threefolds. III. Conic bundles. *J. Math. Sci. (New York)*, 94(1):996–1020, 1999. Algebraic geometry, 9.

[Mor88] S. Mori. Flip theorem and the existence of minimal models for 3-folds. *J. Amer. Math. Soc.*, 1(1):117–253, 1988.

[Mor02] S. Mori. On semistable extremal neighborhoods. In *Higher dimensional birational geometry (Kyoto, 1997)*, volume 35 of *Adv. Stud. Pure Math.*, pages 157–184. Math. Soc. Japan, Tokyo, 2002.

[Mor07] S. Mori. Errata to [KM92]. *J. Amer. Math. Soc.*, 20(1):269–271, 2007.

[MP06] S. Mori and Yu. Prokhorov. On Q-conic bundles, 2006.

[Pro97] Yu. G. Prokhorov. On the complementability of the canonical divisor for Mori fibrations on conics. *Sbornik. Math.*, 188(11):1665–1685, 1997.

[Rei87] M. Reid. Young person’s guide to canonical singularities. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 345–414. Amer. Math. Soc., Providence, RI, 1987.

Shigefumi Mori: RIMS, KYOTO UNIVERSITY, OIWAKE-CHO, KITASHIRAKAWA, SAKYO-KU, KYOTO 606-8502, JAPAN

E-mail address: mori@kurims.kyoto-u.ac.jp

Yuri Prokhorov: DEPARTMENT OF ALGEBRA, FACULTY OF MATHEMATICS, MOSCOW STATE UNIVERSITY, MOSCOW 117234, RUSSIA

E-mail address: prokhor@mech.math.msu.su