ON THE COMPLEXITY OF NONLINEAR MIXED-INTEGER OPTIMIZATION

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Abstract. This is a survey on the computational complexity of nonlinear mixed-integer optimization. It highlights a selection of important topics, ranging from incomputability results that arise from number theory and logic, to recently obtained fully polynomial time approximation schemes in fixed dimension, and to strongly polynomial-time algorithms for special cases.

1. Introduction. In this survey we study the computational complexity of nonlinear mixed-integer optimization problems, i.e., models of the form

\[
\begin{align*}
\max/\min & \quad f(x_1, \ldots, x_n) \\
\text{s.t.} & \quad g_1(x_1, \ldots, x_n) \leq 0 \\
& \quad \vdots \\
& \quad g_m(x_1, \ldots, x_n) \leq 0 \\
& \quad x \in \mathbb{R}^{n_1} \times \mathbb{Z}^{n_2},
\end{align*}
\]

where \(n_1 + n_2 = n\) and \(f, g_1, \ldots, g_m : \mathbb{R}^n \to \mathbb{R}\) are arbitrary nonlinear functions.

This is a very rich topic. From the very beginning, questions such as how to present the problem to an algorithm, and, in view of possible irrational outcomes, what it actually means to solve the problem need to be answered. Fundamental intractability results from number theory and logic on the one hand and from continuous optimization on the other hand come into play. The spectrum of theorems that we present ranges from incomputability results, to hardness and inapproximability theorems, to classes that have efficient approximation schemes, or even polynomial-time or strongly polynomial-time algorithms.

We restrict our attention to deterministic algorithms in the usual bit complexity (Turing) model of computation. Some of the material in the present survey also appears in [31]. For an excellent recent survey focusing on other aspects of the complexity of nonlinear optimization, including the performance of oracle-based models and combinatorial settings such as nonlinear network flows, we refer to Hochbaum [34]. We also do not cover the recent developments by Onn et al. [11, 13, 21, 23, 32, 44, 45] in the context of discrete convex optimization, for which we refer to the monograph [53]. Other excellent sources are [16] and [55].

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2. Preliminaries.

2.1. Presentation of the problem. We restrict ourselves to a model
where the problem is presented explicitly. In most of this survey, the func-
tions $f$ and $g_i$ will be polynomial functions presented in a sparse encoding,
where all coefficients are rational (or integer) and encoded in the binary
scheme. It is useful to assume that the exponents of monomials are given
in the unary encoding scheme; otherwise already in very simple cases the
results of function evaluations will have an encoding length that is expo-

dential in the input size.

In an alternative model, the functions are presented by oracles, such as
comparison oracles or evaluation oracles. This model permits to handle
more general functions (not just polynomials), and on the other hand it is
very useful to obtain hardness results.

2.2. Encoding issues for solutions. When we want to study the
computational complexity of these optimization problems, we first need to
discuss how to encode the input (the data of the optimization problem) and
the output (an optimal solution if it exists). In the context of linear mixed-
integer optimization, this is straightforward: Seldom are we concerned with
irrational objective functions or constraints; when we restrict the input to be
rational as is usual, then also optimal solutions will be rational.

This is no longer true even in the easiest cases of nonlinear optimiza-
tion, as can be seen on the following quadratically constrained problem in
one continuous variable:
\[
\max f(x) = x^4 \quad \text{s.t.} \quad x^2 \leq 2.
\]
Here the unique optimal solution is irrational ($x^* = \sqrt{2}$, with $f(x^*) = 4$),
and so it does not have a finite binary encoding. We ignore here the
possibilities of using a model of computation and complexity over the real
numbers, such as the celebrated Blum–Shub–Smale model \[14\]. In the
familiar Turing model of computation, we need to resort to approximations.

In the example above it is clear that for every $\epsilon > 0$, there exists a ra-
tional $x$ that is a feasible solution for the problem and satisfies $|x - x^*| < \epsilon$
(proximity to the optimal solution) or $|f(x) - f(x^*)| < \epsilon$ (proximity to the
optimal value). However, in general we cannot expect to find approxima-
tions by feasible solutions, as the following example shows.
\[
\max f(x) = x \quad \text{s.t.} \quad x^3 - 2x = 0.
\]
(Again, the optimal solution is $x = \sqrt{2}$, but the closest rational feasible
solution is $x = 0$.) Thus, in the general situation, we will have to use the
following notion of approximation:

**Definition 2.1.** An algorithm $\mathcal{A}$ is said to efficiently approximate
an optimization problem if, for every value of the input parameter $\epsilon > 0$,
it returns a rational vector $\mathbf{x}$ (not necessarily feasible) with $||\mathbf{x} - x^*|| \leq \epsilon$,
where $x^*$ is an optimal solution, and the running time of $A$ is polynomial in the input encoding of the instance and in $\log 1/\epsilon$.

### 2.3. Approximation algorithms and schemes.

The polynomial dependence of the running time in $\log 1/\epsilon$, as defined above, is a very strong requirement. For many problems, efficient approximation algorithms of this type do not exist, unless P = NP. The following, weaker notions of approximation are useful; here it is common to ask for the approximations to be feasible solutions, though.

**Definition 2.2.**

(a) An algorithm $A$ is an $\epsilon$-approximation algorithm for a maximization problem with optimal cost $f_{\text{max}}$, if for each instance of the problem of encoding length $n$, $A$ runs in polynomial time in $n$ and returns a feasible solution with cost $f_A$, such that

$$f_A \geq (1 - \epsilon) \cdot f_{\text{max}}. \quad (2.1)$$

(b) A family of algorithms $A_\epsilon$ is a polynomial time approximation scheme (PTAS) if for every error parameter $\epsilon > 0$, $A_\epsilon$ is an $\epsilon$-approximation algorithm and its running time is polynomial in the size of the instance for every fixed $\epsilon$.

(c) A family $\{A_\epsilon\}_\epsilon$ of $\epsilon$-approximation algorithms is a fully polynomial time approximation scheme (FPTAS) if the running time of $A_\epsilon$ is polynomial in the encoding size of the instance and $1/\epsilon$.

These notions of approximation are the usual ones in the domain of combinatorial optimization. It is clear that they are only useful when the function $f$ (or at least the maximal value $f_{\text{max}}$) are non-negative. For polynomial or general nonlinear optimization problems, various authors [9, 17, 65] have proposed to use a different notion of approximation, where we compare the approximation error to the range of the objective function on the feasible region,

$$|f_A - f_{\text{max}}| \leq \epsilon |f_{\text{max}} - f_{\text{min}}|. \quad (2.2)$$

(Here $f_{\text{min}}$ denotes the minimal value of the function on the feasible region.) It enables us to study objective functions that are not restricted to be non-negative on the feasible region. In addition, this notion of approximation is invariant under shifting of the objective function by a constant, and under exchanging minimization and maximization. On the other hand, it is not useful for optimization problems that have an infinite range. We remark that, when the objective function can take negative values on the feasible region, (2.2) is weaker than (2.1). We will call approximation algorithms and schemes with respect to this notion of approximation weak. This terminology, however, is not consistent in the literature; [16], for instance, uses the notion (2.2) without an additional attribute and instead reserves the word weak for approximation algorithms and schemes that give...
a guarantee on the absolute error:

\[ |f_A - f_{\text{max}}| \leq \epsilon. \] (2.3)

3. Incomputability. Before we can even discuss the computational complexity of nonlinear mixed-integer optimization, we need to be aware of fundamental incomputability results that preclude the existence of any algorithm to solve general integer polynomial optimization problems.

Hilbert’s tenth problem asked for an algorithm to decide whether a given multivariate polynomial \( p(x_1, \ldots, x_n) \) has an integer root, i.e., whether the Diophantine equation

\[ p(x_1, \ldots, x_n) = 0, \quad x_1, \ldots, x_n \in \mathbb{Z} \] (3.1)

is solvable. It was answered in the negative by Matiyasevich [48], based on earlier work by Davis, Putnam, and Robinson; see also [49]. A short self-contained proof, using register machines, is presented in [39].

**Theorem 3.1.**

(i) There does not exist an algorithm that, given polynomials \( p_1, \ldots, p_m \), decides whether the system \( p_i(x_1, \ldots, x_n) = 0, i = 1, \ldots, m \), has a solution over the integers.

(ii) There does not exist an algorithm that, given a polynomial \( p \), decides whether \( p(x_1, \ldots, x_n) = 0 \) has a solution over the integers.

(iii) There does not exist an algorithm that, given a polynomial \( p \), decides whether \( p(x_1, \ldots, x_n) = 0 \) has a solution over the non-negative integers \( \mathbb{Z}_+ = \{0, 1, 2, \ldots\} \).

(iv) There does not exist an algorithm that, given a polynomial \( p \), decides whether \( p(x_1, \ldots, x_n) = 0 \) has a solution over the natural numbers \( \mathbb{N} = \{1, 2, \ldots\} \).

These three variants of the statement are easily seen to be equivalent.

The solvability of the system \( p_i(x_1, \ldots, x_n) = 0, i = 1, \ldots, m \), is equivalent to the solvability of \( \sum_{i=1}^m p_i^2(x_1, \ldots, x_n) = 0 \). Also, if \((x_1, \ldots, x_n) \in \mathbb{Z}^n\) is a solution of \( p(x_1, \ldots, x_n) = 0 \) over the integers, then by splitting variables into their positive and negative parts, \( y_i = \max\{0, x_i\} \) and \( z_i = \max\{0, -x_i\} \), clearly \((y_1, z_1; \ldots; y_n, z_n)\) is a non-negative integer solution of the polynomial equation \( q(y_1, z_1; \ldots; y_n, z_n) := p(y_1 - z_1, \ldots, y_n - z_n) = 0 \).

(A construction with only one extra variable is also possible: Use the non-negative variables \( w = \max\{|x_i| : x_i < 0\} \) and \( y_i := x_i + w \).) In the other direction, using Lagrange’s four-square theorem, any non-negative integer \( x \) can be represented as the sum \( a^2 + b^2 + c^2 + d^2 \) with integers \( a, b, c, d \). Thus, if \((x_1, \ldots, x_n) \in \mathbb{Z}_+^n\) is a solution over the non-negative integers, then there exists a solution \((a_1, b_1, c_1, d_1; \ldots; a_n, b_n, c_n, d_n)\) of the polynomial equation \( r(a_1, b_1, c_1, d_1; \ldots; a_n, b_n, c_n, d_n) := p(a_1^2 + b_1^2 + c_1^2 + d_1^2, \ldots, a_n^2 + b_n^2 + c_n^2 + d_n^2) \).

The equivalence of the statement with non-negative integers and the one with the natural numbers follows from a simple change of variables.
Sharper statements of the above incomputability result can be found in [38]. All incomputability statements appeal to the classic result by Turing [64] on the existence of recursively enumerable (or listable) sets of natural numbers that are not recursive, such as the halting problem of universal Turing machines.

**Theorem 3.2.** For the following universal pairs $(\nu, \delta)$

$$(58, 4), \ldots, (38, 8), \ldots, (21, 96), \ldots, (14, 2.0 \times 10^5), \ldots, (9, 1.638 \times 10^{45}),$$

there exists a universal polynomial $U(x; z, u, y; a_1, \ldots, a_\nu)$ of degree $\delta$ in $4 + \nu$ variables, i.e., for every recursively enumerable (listable) set $X$ there exist natural numbers $z, u, y$, such that

$$x \in X \iff \exists a_1, \ldots, a_\nu \in \mathbb{N} : U(x; z, u, y; a_1, \ldots, a_\nu) = 0.$$

Jones explicitly constructs these universal polynomials, using and extending techniques by Matiyasevich. Jones also constructs an explicit system of quadratic equations in $4 + 58$ variables that is universal in the same sense. The reduction of the degree, down to 2, works at the expense of introducing additional variables; this technique goes back to Skolem [62].

In the following, we highlight some of the consequences of these results. Let $U$ be a universal polynomial corresponding to a universal pair $(\nu, \delta)$, and let $X$ be a recursively enumerable set that is not recursive, i.e., there does not exist any algorithm (Turing machine) to decide whether a given $x$ is in $X$. By the above theorem, there exist natural numbers $z, u, y$ such that $x \in X$ holds if and only if the polynomial equation $U(x; z, u, y; a_1, \ldots, a_\nu) = 0$ has a solution in natural numbers $a_1, \ldots, a_\nu$ (note that $x$ and $z, u, y$ are fixed parameters here). This implies:

**Theorem 3.3.**

(i) Let $(\nu, \delta)$ be any of the universal pairs listed above. Then there does not exist any algorithm that, given a polynomial $p$ of degree at most $\delta$ in $\nu$ variables, decides whether $p(x_1, \ldots, x_n) = 0$ has a solution over the non-negative integers.

(ii) In particular, there does not exist any algorithm that, given a polynomial $p$ in at most 9 variables, decides whether $p(x_1, \ldots, x_n) = 0$ has a solution over the non-negative integers.

(iii) There also does not exist any algorithm that, given a polynomial $p$ in at most 36 variables, decides whether $p(x_1, \ldots, x_n) = 0$ has a solution over the integers.

(iv) There does not exist any algorithm that, given a polynomial $p$ of degree at most 4, decides whether $p(x_1, \ldots, x_n) = 0$ has a solution over the non-negative integers (or over the integers).

(v) There does not exist any algorithm that, given a system of quadratic equations in at most 58 variables, decides whether it has a solution of the non-negative integers.
(vi) **There does not exist any algorithm that, given a system of quadratic equations in at most 232 variables, decides whether it has a solution of the integers.**

We remark that the bounds of $4 \times 9 = 36$ and $4 \times 58 = 232$ are most probably not sharp; they are obtained by a straightforward application of the reduction using Lagrange’s theorem.

For integer polynomial optimization, this has the following fundamental consequences. First of all, Theorem 3.3 can be understood as a statement on the feasibility problem of an integer polynomial optimization problem. Thus, the feasibility of an integer polynomial optimization problem with a single polynomial constraint in 9 non-negative integer variables or 36 free integer variables is undecidable, etc.

If we wish to restrict our attention to feasible optimization problems, we can consider the problem of minimizing $p^2(x_1, \ldots, x_n)$ over the integers or non-negative integers and conclude that unconstrained polynomial optimization in 9 non-negative integer or 36 free integer variables is undecidable. We can also follow Jeroslow [37] and associate with an arbitrary polynomial $p$ in $n$ variables the optimization problem

$$\min \ u$$
$$\text{s.t. } (1 - u) \cdot p(x_1, \ldots, x_n) = 0,$$
$$u \in \mathbb{Z}_+, \quad x \in \mathbb{Z}_n^+.$$  

This optimization problem is always feasible and has the optimal solution value 0 if and only if $p(x_1, \ldots, x_n) = 0$ is solvable, and 1 otherwise. Thus, optimizing linear forms over one polynomial constraint in 10 non-negative integer variables is incomputable, and similar statements can be derived from the other universal pairs above. Jeroslow [37] used the above program and a degree reduction (by introducing additional variables) to prove the following.

**Theorem 3.4.** The problem of minimizing a linear form over quadratic inequality constraints in integer variables is not computable; this still holds true for the subclass of problems that are feasible, and where the minimum value is either 0 or 1.

This statement can be strengthened by giving a bound on the number of integer variables.

4. **Hardness and inapproximability.** All incomputability results, of course, no longer apply when finite bounds for all variables are known; in this case, a trivial enumeration approach gives a finite algorithm. This is immediately the case when finite bounds for all variables are given in the problem formulation, such as for 0-1 integer problems.

For other problem classes, even though finite bounds are not given, it is possible to compute such bounds that either hold for all feasible solutions or for an optimal solution (if it exists). This is well-known for the case of linear
constraints, where the usual encoding length estimates of basic solutions are available. As we explain in section 6.2 below, such finite bounds can also be computed for convex and quasi-convex integer optimization problems.

In other cases, algorithms to decide feasibility exist even though no finite bounds for the variables are known. An example is the case of single Diophantine equations of degree 2, which are decidable using an algorithm by Siegel. We discuss the complexity of this case below.

Within any such computable subclass, we can ask the question of the complexity. Below we discuss hardness results that come from the number theoretic side of the problem (section 4.1) and those that come from the continuous optimization side (section 4.2).

4.1. Hardness results from quadratic Diophantine equations in fixed dimension. The computational complexity of single quadratic Diophantine equations in 2 variables is already very interesting and rich in structure; we refer to to the excellent paper by Lagarias. Below we discuss some of these aspects and their implications on optimization.

Testing primality of a number $N$ is equivalent to deciding feasibility of the equation

$$ (x + 2)(y + 2) = N $$

over the non-negative integers. Recently, Agrawal, Kayal, and Saxena showed that primality can be tested in polynomial time. However, the complexity status of finding factors of a composite number, i.e., finding a solution $(x, y)$ of (4.1), is still unclear.

The class also contains subclasses of NP-complete feasibility problems, such as the problem of deciding for given $\alpha, \beta, \gamma \in \mathbb{N}$ whether there exist $x_1, x_2 \in \mathbb{Z}_+$ with $\alpha x_1^2 + \beta x_2 = \gamma$. On the other hand, the problem of deciding for given $a, c \in \mathbb{N}$ whether there exist $x_1, x_2 \in \mathbb{Z}$ with $ax_1 x_2 + x_2 = c$, lies in $\text{NP} \backslash \text{coNP}$ unless $\text{NP} = \text{coNP}$.

The feasibility problem of the general class of quadratic Diophantine equations in two (non-negative) variables was shown by Lagarias to be in $\text{NP}$. This is not straightforward because minimal solutions can have an encoding size that is exponential in the input size. This can be seen in the case of the so-called anti-Pellian equation $x^2 - dy^2 = -1$. Here Lagarias proved that for all $d = 5^{2n+1}$, there exists a solution, and the solution with minimal binary encoding length has an encoding length of $\Omega(5^n)$ (while the input is of encoding length $\Theta(n)$). (We remark that the special case of the anti-Pellian equation is in $\text{coNP}$, as well.)

Related hardness results include the problem of quadratic congruences with a bound, i.e., deciding for given $a, b, c \in \mathbb{N}$ whether there exists a positive integer $x < c$ with $x^2 \equiv a \pmod{b}$; this is the NP-complete problem AN1 in [25].
From these results, we immediately get the following consequences on optimization.

**Theorem 4.1.**

(i) The feasibility problem of quadratically constrained problems in \( n = 2 \) integer variables is NP-complete.

(ii) The problems of computing a feasible (or optimal) solution of quadratically constrained problems in \( n = 2 \) integer variables is not polynomial-time solvable (because the output may require exponential space).

(iii) The feasibility problem of quadratically constrained problems in \( n > 2 \) integer variables is NP-hard (but it is unknown whether it is in NP).

(iv) The problem of minimizing a degree-4 polynomial over the lattice points of a convex polygon (dimension \( n = 2 \)) is NP-hard.

(v) The problem of finding the minimal value of a degree-4 polynomial over \( \mathbb{Z}_2^d \) is NP-hard; writing down an optimal solution cannot be done in polynomial time.

However, the complexity of minimizing a quadratic form over the integer points in polyhedra of fixed dimension is unclear, even in dimension \( n = 2 \). Consider the integer convex minimization problem

\[
\min \alpha x_1^2 + \beta x_2,
\]

s.t. \( x_1, x_2 \in \mathbb{Z}_+ \)

for \( \alpha, \beta \in \mathbb{N} \). Here an optimal solution can be obtained efficiently, as we explain in section 6.2 in fact, clearly \( x_1 = x_2 = 0 \) is the unique optimal solution. On the other hand, the problem whether there exists a point \((x_1, x_2)\) of a prescribed objective value \( \gamma = \alpha x_1^2 + \beta x_2 \) is NP-complete (see above). For indefinite quadratic forms, even in dimension 2, nothing seems to be known.

In varying dimension, the convex quadratic maximization case, i.e., maximizing positive definite quadratic forms is an NP-hard problem. This is even true in very restricted settings such as the problem to maximize \( \sum_i (w_i^T x)^2 \) over \( x \in \{0, 1\}^n \).

### 4.2. Inapproximability of nonlinear optimization in varying dimension

Even in the pure continuous case, nonlinear optimization is known to be hard. Bellare and Rogaway \[9, 10\] proved the following inapproximability results using the theory of interactive proof systems.

**Theorem 4.2.** Assume that \( \mathbb{P} \neq \mathbb{NP} \).

(i) For any \( \epsilon < \frac{1}{4} \), there does not exist a polynomial-time weak \( \epsilon \)-approximation algorithm for the problem of (continuous) quadratic programming over polytopes.

(ii) There exists a constant \( \delta > 0 \) such that the problem of polynomial programming over polytopes does not have a polynomial-time weak \( (1 - n^{-\delta}) \)-approximation algorithm.

Here the number \( 1 - n^{-\delta} \) becomes arbitrarily close to 0 for growing \( n \); note that a weak 0-approximation algorithm is one that gives no guarantee.
other than returning a feasible solution.

Inapproximability still holds for the special case of minimizing a quadratic form over the cube $[-1, 1]^n$ or over the standard simplex. In the case of the cube, inapproximability of the max-cut problem is used. In the case of the standard simplex, it follows via the celebrated Motzkin–Strasz theorem [51] from the inapproximability of the maximum stable set problem. These are results by Hästad [28]; see also [16].

5. Approximation schemes. For important classes of optimization problems, while exact optimization is hard, good approximations can still be obtained efficiently.

Many such examples are known in combinatorial settings. As an example in continuous optimization, we refer to the problem of maximizing homogeneous polynomial functions of fixed degree over simplices. Here de Klerk et al. [17] proved a weak PTAS.

Below we present a general result for mixed-integer polynomial optimization over polytopes.

5.1. Mixed-integer polynomial optimization in fixed dimension over linear constraints: FPTAS and weak FPTAS. Here we consider the problem

$$\max/\min \ f(x_1, \ldots, x_n)$$
subject to \[Ax \leq b\]
\[x \in \mathbb{R}_+ \times \mathbb{Z}^n,\]

where $A$ is a rational matrix and $b$ is a rational vector. As we pointed out above (Theorem [14]), optimizing degree-4 polynomials over problems with two integer variables ($n_1 = 0$, $n_2 = 2$) is already a hard problem. Thus, even when we fix the dimension, we cannot get a polynomial-time algorithm for solving the optimization problem. The best we can hope for, even when the number of both the continuous and the integer variables is fixed, is an approximation result.

We present here the FPTAS obtained by De Loera et al. [18–20], which uses the “summation method” and the theory of short rational generating functions pioneered by Barvinok [6, 7]. We review the methods below; the FPTAS itself appears in Section 5.1.3. An open question is briefly discussed at the end of this section.

5.1.1. The summation method. The summation method for optimization is the idea to use of elementary relation

$$\max\{s_1, \ldots, s_N\} = \lim_{k \to \infty} \sqrt[k]{s_1^k + \cdots + s_N^k},$$

which holds for any finite set $S = \{s_1, \ldots, s_N\}$ of non-negative real numbers. This relation can be viewed as an approximation result for $\ell_k$-norms.
Now if $P$ is a polytope and $f$ is an objective function non-negative on \( P \cap \mathbb{Z}^d \), let \( x^1, \ldots, x^N \) denote all the feasible integer solutions in \( P \cap \mathbb{Z}^d \) and collect their objective function values \( s_i = f(x^i) \) in a vector \( s \in \mathbb{Q}^N \). Then, comparing the unit balls of the \( \ell_k \)-norm and the \( \ell_\infty \)-norm (Figure 1), we get the relation

\[
L_k := N^{-1/k} \|s\|_k \leq \|s\|_\infty \leq \|s\|_k =: U_k.
\]

These estimates are independent of the function \( f \). (Different estimates that make use of the properties of \( f \), and that are suitable also for the continuous case, can be obtained from the Hölder inequality; see for instance [3].)

Thus, for obtaining a good approximation of the maximum, it suffices to solve a summation problem of the polynomial function \( h = f^k \) on \( P \cap \mathbb{Z}^d \) for a value of \( k \) that is large enough. Indeed, for \( k = \lceil (1 + 1/\epsilon) \log N \rceil \), we obtain \( U_k - L_k \leq \epsilon f(x^{\max}) \). On the other hand, this choice of \( k \) is polynomial in the input size (because \( 1/\epsilon \) is encoded in unary in the input, and \( \log N \) is bounded by a polynomial in the binary encoding size of the polytope \( P \)). Hence, when the dimension \( d \) is fixed, we can expand the polynomial function \( f^k \) as a list of monomials in polynomial time.

5.1.2. Rational generating functions. To solve the summation problem, one uses the technology of short rational generating functions. We explain the theory on a simple, one-dimensional example. Let us consider the set \( S \) of integers in the interval \( P = [0, \ldots, n] \). We associate with \( S \) the polynomial \( g(P; z) = z^0 + z^1 + \cdots + z^{n-1} + z^n \); i.e., every integer \( \alpha \in S \) corresponds to a monomial \( z^\alpha \) with coefficient 1 in the polynomial \( g(P; z) \). This polynomial is called the generating function of \( S \) (or of \( P \)). From the viewpoint of computational complexity, this generating function is of exponential size (in the encoding length of \( n \)), just as an explicit list of all the integers \( 0, 1, \ldots, n - 1, n \) would be. However, we can observe that \( g(P; z) \) is a finite geometric series, so there exists a simple summation formula that expresses it in a much more compact way:

\[
g(P; z) = z^0 + z^1 + \cdots + z^{n-1} + z^n = \frac{1 - z^{n+1}}{1 - z}. \tag{5.3}
\]

The “long” polynomial has a “short” representation as a rational function. The encoding length of this new formula is linear in the encoding length.
of \( n \). On the basis of this idea, we can solve the summation problem. Consider the generating function of the interval \( P = [0, 4] \),
\[
g(P; z) = z^0 + z^1 + z^2 + z^3 + z^4 = \frac{1}{1 - z} - \frac{z^5}{1 - z}.
\]
We now apply the differential operator \( \frac{d}{dz} \) and obtain
\[
\left( \frac{d}{dz} \right) g(P; z) = 1z^1 + 2z^2 + 3z^3 + 4z^4 = \frac{1}{(1 - z)^2} - \frac{-4z^5 + 5z^4}{(1 - z)^2}
\]
Applying the same differential operator again, we obtain
\[
\left( \frac{d}{dz} \right) \left( \frac{d}{dz} \right) g(P; z) = 1z^1 + 4z^2 + 9z^3 + 16z^4 = \frac{z + z^2}{(1 - z)^3} - \frac{25z^5 - 39z^6 + 16z^7}{(1 - z)^3}
\]
We have thus evaluated the monomial function \( h(\alpha) = \alpha^2 \) for \( \alpha = 0, \ldots, 4 \); the results appear as the coefficients of the respective monomials. Substituting \( z = 1 \) yields the desired sum
\[
\left. \left( \frac{d}{dz} \right) \left( \frac{d}{dz} \right) g(P; z) \right|_{z=1} = 1 + 4 + 9 + 16 = 30
\]
The idea now is to evaluate this sum instead by computing the limit of the rational function for \( z \to 1 \),
\[
\sum_{\alpha=0}^{4} \alpha^2 = \lim_{z \to 1} \left[ \frac{z + z^2}{(1 - z)^3} - \frac{25z^5 - 39z^6 + 16z^7}{(1 - z)^3} \right];
\]
this can be done using residue techniques.

We now present the general definitions and results. Let \( P \subseteq \mathbb{R}^d \) be a rational polyhedron. We first define its generating function as the formal Laurent series \( \tilde{g}(P; z) = \sum_{\alpha \in \mathbb{Z}^d \cap P} z^\alpha \in \mathbb{Z}[[z_1, \ldots, z_d, z_1^{-1}, \ldots, z_d^{-1}]] \), i.e., without any consideration of convergence properties. By convergence, one moves to a rational generating function \( g(P; z) \in \mathbb{Q}(z_1, \ldots, z_d) \).

The following breakthrough result was obtained by Barvinok in 1994.

**Theorem 5.1** (Barvinok [6]). Let \( d \) be fixed. There exists a polynomial-time algorithm for computing the generating function \( g(P; z) \) of a polyhedron \( P \subseteq \mathbb{R}^d \) given by rational inequalities in the form of a rational function
\[
g(P; z) = \sum_{i \in I} \epsilon_i \frac{z^{b_i}}{\prod_{j=1}^{d} (1 - z^{b_{ij}})} \quad \text{with } \epsilon_i \in \{\pm 1\}, \ a_i \in \mathbb{Z}^d, \ \text{and } b_{ij} \in \mathbb{Z}^d.
\]
5.1.3. Efficient summation using rational generating functions.

Below we describe the theorems on the summation method based on short rational generating functions, which appeared in [18, 20]. Let \( g(P; z) \) be the rational generating function of \( P \cap \mathbb{Z}^d \), computed using Barvinok’s algorithm. By symbolically applying differential operators to \( g \), we can compute a short rational function representation of the Laurent polynomial \( g(P, h; z) = \sum_{\alpha \in P \cap \mathbb{Z}^d} h(\alpha) z^\alpha \), where each monomial \( z^\alpha \) corresponding to an integer point \( \alpha \in P \cap \mathbb{Z}^d \) has a coefficient that is the value \( h(\alpha) \). As in the one-dimensional example above, we use the partial differential operators \( z_i \frac{\partial}{\partial z_i} \) for \( i = 1, \ldots, d \) on the short rational generating function. In fixed dimension, the size of the rational function expressions occurring in the symbolic calculation can be bounded polynomially. Thus one obtains the following result.

**Theorem 5.2** ( [19], Lemma 3.1).

(a) Let \( h(x_1, \ldots, x_d) = \sum_{\beta} c_{\beta} x^\beta \in \mathbb{Q}[x_1, \ldots, x_d] \) be a polynomial. Define the differential operator 

\[
D_h = h \left( z_1 \frac{\partial}{\partial z_1}, \ldots, z_d \frac{\partial}{\partial z_d} \right) = \sum_{\beta} c_{\beta} \left( z_1 \frac{\partial}{\partial z_1} \right)^{\beta_1} \cdots \left( z_d \frac{\partial}{\partial z_d} \right)^{\beta_d}.
\]

Then \( D_h \) maps the generating function \( g(P; z) = \sum_{\alpha \in P \cap \mathbb{Z}^d} z^\alpha \) to the weighted generating function \( (D_h g)(z) = g(P; h; z) = \sum_{\alpha \in P \cap \mathbb{Z}^d} h(\alpha) z^\alpha \).

(b) Let the dimension \( d \) be fixed. Let \( g(P; z) \) be the Barvinok representation of the generating function \( \sum_{\alpha \in P \cap \mathbb{Z}^d} z^\alpha \) of \( P \cap \mathbb{Z}^d \). Let \( h \in \mathbb{Q}[x_1, \ldots, x_d] \) be a polynomial, given as a list of monomials with rational coefficients \( c_{\beta} \) encoded in binary and exponents \( \beta \) encoded in unary. We can compute in polynomial time a Barvinok representation \( g(P, h; z) \) for the weighted generating function \( \sum_{\alpha \in P \cap \mathbb{Z}^d} h(\alpha) z^\alpha \).

Thus, we can implement the following algorithm in polynomial time (in fixed dimension).

**Algorithm 1** (Computation of bounds for the optimal value).

Input: A rational convex polytope \( P \subseteq \mathbb{R}^d \); a polynomial objective function \( f \in \mathbb{Q}[x_1, \ldots, x_d] \) that is non-negative over \( P \cap \mathbb{Z}^d \), given as a list of monomials with rational coefficients \( c_{\beta} \) encoded in binary and exponents \( \beta \) encoded in unary; an index \( k \), encoded in unary.

Output: A lower bound \( L_k \) and an upper bound \( U_k \) for the maximal function value \( f^* \) of \( f \) over \( P \cap \mathbb{Z}^d \). The bounds \( L_k \) form a nondecreasing, the bounds \( U_k \) a nonincreasing sequence of bounds that both reach \( f^* \) in a finite number of steps.

1. Compute a short rational function expression for the generating function \( g(P, z) = \sum_{\alpha \in P \cap \mathbb{Z}^d} z^\alpha \). Using residue techniques, compute \( |P \cap \mathbb{Z}^d| = g(P; 1) \) from \( g(P; z) \).

2. Compute the polynomial \( f^k \) from \( f \).

3. From the rational function \( g(P; z) \) compute the rational function representation of \( g(P, f^k; z) \) of \( \sum_{\alpha \in P \cap \mathbb{Z}^d} f^k(\alpha) z^\alpha \) by Theorem 5.2. Using
residue techniques, compute
\[ L_k := \left\lceil \sqrt[k]{g(P, f^k; 1)} / g(P; 1) \right\rceil \quad \text{and} \quad U_k := \left\lfloor \sqrt[k]{g(P, f^k; 1)} \right\rfloor. \]

From the discussion of the convergence of the bounds, one then obtains the following result.

**Theorem 5.3 (Fully polynomial-time approximation scheme).** Let the dimension \( d \) be fixed. Let \( P \subset \mathbb{R}^d \) be a rational convex polytope. Let \( f \) be a polynomial with rational coefficients that is non-negative on \( P \cap \mathbb{Z}^d \), given as a list of monomials with rational coefficients \( c_\beta \) encoded in binary and exponents \( \beta \) encoded in unary.

(i) Algorithm 1 computes the bounds \( L_k, U_k \) in time polynomial in \( k \), the input size of \( P \) and \( f \), and the total degree \( D \). The bounds satisfy
\[ U_k - L_k \leq f^* \cdot \left( \sqrt[k]{|P \cap \mathbb{Z}^d|} - 1 \right). \]

(ii) For \( k = (1 + 1/\epsilon) \log(|P \cap \mathbb{Z}^d|) \) (a number bounded by a polynomial in the input size), \( L_k \) is a \((1 - \epsilon)\)-approximation to the optimal value \( f^* \) and it can be computed in time polynomial in the input size, the total degree \( D \), and \( 1/\epsilon \). Similarly, \( U_k \) gives a \((1 + \epsilon)\)-approximation to \( f^* \).

(iii) With the same complexity, by iterated bisection of \( P \), we can also find a feasible solution \( x_\epsilon \in P \cap \mathbb{Z}^d \) with \(|f(x_\epsilon) - f^*| \leq \epsilon f^* \).

**5.1.4. Extension to the mixed-integer case by discretization.**

The mixed-integer case can be handled by discretization of the continuous variables. We illustrate on an example that one needs to be careful to pick a sequence of discretizations that actually converges. Consider the mixed-integer linear optimization problem depicted in Figure 2, whose feasible region consists of the point \((\frac{1}{2}, 1)\) and the segment \( \{(x, 0) : x \in [0, 1]\} \).

The unique optimal solution is \( x = \frac{1}{2}, z = 1 \). Now consider the sequence of grid approximations where \( x \in \frac{1}{m} \mathbb{Z}_{\geq 0} \). For even \( m \), the unique optimal solution to the grid approximation is \( x = \frac{1}{2}, z = 1 \). However, for odd \( m \), the unique optimal solution is \( x = 0, z = 0 \). Thus the full sequence of the optimal solutions to the grid approximations does not converge because it has two limit points; see Figure 2.

To handle polynomial objective functions that take arbitrary (positive and negative) values, one can shift the objective function by a large constant. Then, to obtain a strong approximation result, one iteratively reduces the constant by a factor. Altogether we have the following result.

**Theorem 5.4 (Fully polynomial-time approximation schemes).** Let the dimension \( n = n_1 + n_2 \) be fixed. Let an optimization problem \( (5.1) \) of a polynomial function \( f \) over the mixed-integer points of a polytope \( P \) and an error bound \( \epsilon \) be given, where

(I1) \( f \) is given as a list of monomials with rational coefficients \( c_\beta \) encoded in binary and exponents \( \beta \) encoded in unary,

(I2) \( P \) is given by rational inequalities in binary encoding,

(I3) the rational number \( \frac{1}{2} \) is given in unary encoding.
Fig. 2. A mixed-integer linear optimization problem and a sequence of optimal solutions to grid problems with two limit points, for even $m$ and for odd $m$.

(a) There exists a fully polynomial time approximation scheme (FPTAS) for the maximization problem for all polynomial functions $f(x, z)$ that are non-negative on the feasible region. That is, there exists a polynomial-time algorithm that, given the above data, computes a feasible solution $(x_\epsilon, z_\epsilon) \in P \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2})$ with

$$|f(x_\epsilon, z_\epsilon) - f(x_{\text{max}}, z_{\text{max}})| \leq \epsilon f(x_{\text{max}}, z_{\text{max}}).$$

(b) There exists a polynomial-time algorithm that, given the above data, computes a feasible solution $(x_\epsilon, z_\epsilon) \in P \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2})$ with

$$|f(x_\epsilon, z_\epsilon) - f(x_{\text{max}}, z_{\text{max}})| \leq \epsilon |f(x_{\text{max}}, z_{\text{max}}) - f(x_{\text{min}}, z_{\text{min}})|.$$

In other words, this is a weak FPTAS.

5.1.5. Open question. Consider the problem \ref{eq:general} for a fixed number $n_2$ of integer variables and a varying number $n_1$ of continuous variables. Of course, even with no integer variables present ($n_2 = 0$), this is NP-hard and inapproximable. On the other hand, if the objective function $f$ is linear, the problem can be solved in polynomial time using Lenstra’s algorithm. Thus it is interesting to consider the problem for an objective function of restricted nonlinearity, such as

$$f(x, z) = g(z) + c^\top x,$$

with an arbitrary polynomial function $g$ in the integer variables and a linear form in the continuous variables. The complexity (in particular the existence of approximation algorithms) of this problem is an open question.

6. Polynomial-time algorithms. Here we study important special cases where polynomial-time algorithms can be obtained. We also include cases here where the algorithms efficiently approximate the optimal solution to arbitrary precision, as discussed in section \ref{sec:approximation}.
6.1. Fixed dimension: Continuous polynomial optimization.

Here we consider pure continuous polynomial optimization problems of the form

\[
\begin{align*}
\min & \quad f(x_1, \ldots, x_n) \\
\text{s.t.} & \quad g_1(x_1, \ldots, x_n) \leq 0 \\
& \quad \vdots \\
& \quad g_m(x_1, \ldots, x_n) \leq 0 \\
& \quad x \in \mathbb{R}^n.
\end{align*}
\]  

When the dimension is fixed, this problem can be solved in polynomial time, in the sense that there exists an algorithm that efficiently computes an approximation to an optimal solution. This follows from a much more general theory on the computational complexity of approximating the solutions to general algebraic and semialgebraic formulae over the real numbers by Renegar [60], which we review in the following. The bulk of this theory was developed in [57–59]. Similar results appeared in [29]; see also [8, Chapter 14]). One considers problems associated with logic formulas of the form

\[
Q_1 x^1 \in \mathbb{R}^{n_1} : \ldots : Q_\omega x^\omega \in \mathbb{R}^{n_\omega} : P(y, x^1, \ldots, x^\omega)
\]  

with quantifiers \(Q_i \in \{\exists, \forall\}\), where \(P\) is a Boolean combination of polynomial inequalities such as

\[
g_i(y, x^1, \ldots, x^\omega) \leq 0, \quad i = 1, \ldots, m,
\]

or using \(\geq, <, >, =\) as the relation. Here \(y \in \mathbb{R}^{n_0}\) is a free (i.e., not quantified) variable. Let \(d \geq 2\) be an upper bound on the degrees of the polynomials \(g_i\). A vector \(\bar{y} \in \mathbb{R}^{n_0}\) is called a solution of this formula if the formula \((6.2)\) becomes a true logic sentence if we set \(y = \bar{y}\). Let \(Y\) denote the set of all solutions. An \(\epsilon\)-approximate solution is a vector \(y_\epsilon\) with \(\|y_\epsilon - \bar{y}\| < \epsilon\) for some solution \(\bar{y} \in Y\).

The following bound can be proved. When the number \(\omega\) of “blocks” of quantifiers (i.e., the number of alternations of the quantifiers \(\exists\) and \(\forall\)) is fixed, then the bound is singly exponential in the dimension.

**Theorem 6.1.** If the formula \((6.2)\) has only integer coefficients of binary encoding size at most \(\ell\), then every connected component of \(Y\) intersects with the ball \(\{\|y\| \leq r\}\), where

\[
\log r = \ell (md)^{2^{O(\omega)} n_0 n_1 \cdots n_k}.
\]

This bound is used in the following fundamental result, which gives a general algorithm to compute \(\epsilon\)-approximate solutions to the formula \((6.2)\).

**Theorem 6.2.** There exists an algorithm that, given numbers \(0 < \epsilon < r\) that are integral powers of 2 and a formula \((6.2)\), computes a set \(\{y_i\}\) of
\((md)^{O(\omega)n_{01}\ldots nk}\) distinct \(\epsilon\)-approximate solutions of the formula with the property that for each connected components of \(Y \cap \{\|y\| \leq r\}\) one of the \(y_i\) is within distance \(\epsilon\). The algorithm runs in time

\[(md)^{O(\omega)n_{01}\ldots nk}(\ell + md + \log \frac{1}{\epsilon} + \log r)^{O(1)}.\]

This can be applied to polynomial optimization problems as follows. Consider the formula

\[\forall x \in \mathbb{R}^n_1 : g_1(y) \leq 0 \wedge \cdots \wedge g_m(y) \leq 0 \wedge [g_1(x) > 0 \vee \cdots \vee g_m(x) > 0 \vee f(y) - f(x) < 0],\]

(6.3)

this describes that \(y\) is an optimal solution (all other solutions \(x\) are either infeasible or have a higher objective value). Thus optimal solutions can be efficiently approximated using the algorithm of Theorem 6.2.

6.2. Fixed dimension: Convex and quasi-convex integer polynomial minimization. In this section we consider the case of the minimization of convex and quasi-convex polynomials \(f\) over the mixed-integer points in convex regions given by convex and quasi-convex polynomial functions \(g_1, \ldots, g_m\):

\[
\begin{align*}
\min & \quad f(x_1, \ldots, x_n) \\
\text{s.t.} & \quad g_1(x_1, \ldots, x_n) \leq 0 \\
& \quad \vdots \\
& \quad g_m(x_1, \ldots, x_n) \leq 0 \\
& \quad x \in \mathbb{R}^n_1 \times \mathbb{Z}^{n_2},
\end{align*}
\]

(6.4)

Here a function \(g: \mathbb{R}^n \to \mathbb{R}^1\) is called \(\text{quasi-convex}\) if every lower level set \(L_\lambda = \{x \in \mathbb{R}^n: g(x) \leq \lambda\}\) is a convex set.

The complexity in this setting is fundamentally different from the general (non-convex) case. One important aspect is that bounding results for the coordinates of optimal integer solutions exists, which are similar to the ones for continuous solutions in Theorem 6.1 above. For the case of convex functions, these bounding results were obtained by \cite{40,63}. An improved bound was obtained by \cite{4,5}, which also handles the more general case of quasi-convex polynomials. This bound follows from the efficient theory of quantifier elimination over the real numbers that we referred to in section 6.1.

**Theorem 6.3.** Let \(f, g_1, \ldots, g_m \in \mathbb{Z}[x_1, \ldots, x_n]\) be quasi-convex polynomials of degree at most \(d \geq 2\), whose coefficients have a binary encoding length of at most \(\ell\). Let

\[
F = \{x \in \mathbb{R}^n : g_i(x) \leq 0 \text{ for } i = 1, \ldots, m\}
\]
be the (continuous) feasible region. If the integer minimization problem
\[ \min \{ f(x) : x \in F \cap \mathbb{Z}^n \} \]
is bounded, there exists a radius \( R \in \mathbb{Z}_+ \) of binary encoding length at most \( (md)^{O(n)} \ell \) such that
\[ \min \{ f(x) : x \in F \cap \mathbb{Z}^n \} = \min \{ f(x) : x \in F \cap \mathbb{Z}^n, \|x\| \leq R \}. \]

Using this finite bound, a trivial enumeration algorithm can find an optimal solution (but not in polynomial time, not even in fixed dimension). Thus the incomputability result for integer polynomial optimization (Theorem 3.3) does not apply to this case.

The unbounded case can be efficiently detected in the case of quasi-convex polynomials; see [5] and [52], the latter of which also handles the case of “faithfully convex” functions that are not polynomials.

In fixed dimension, the problem of convex integer minimization can be solved efficiently using variants of Lenstra’s algorithm [46] for integer programming. Lenstra-type algorithms are algorithms for solving feasibility problems. We consider a family of feasibility problems associated with the optimization problem,

\[ \exists x \in F_\alpha \cap \mathbb{Z}^n \text{ where } F_\alpha = \{ x \in F : f(x) \leq \alpha \} \text{ for } \alpha \in \mathbb{Z}. \] (6.5)

If bounds for \( f(x) \) on the feasible regions of polynomial binary encoding size are known, a polynomial time algorithm for this feasibility problem can be used in a binary search to solve the optimization problem in polynomial time. Indeed, when the dimension \( n \) is fixed, the bound \( R \) given by Theorem 6.3 has a binary encoding size that is bounded polynomially by the input data.

A Lenstra-type algorithm uses branching on hyperplanes (Figure 3) to obtain polynomial time complexity in fixed dimension. Note that only the
binary encoding size of the bound $R$, but not $R$ itself, is bounded polyno-
mically. Thus, multiway branching on the values of a single variable $x_i$ will
create an exponentially large number of subproblems. Instead, a Lenstra-
type algorithm computes a primitive lattice vector $w \in \mathbb{Z}^n$ such that there
are only few lattice hyperplanes $w^\top x = \gamma$ (with $\gamma \in \mathbb{Z}$) that can have a
nonempty intersection with $F_\alpha$. The width of $F_\alpha$ in the direction $w$, defined
as
\[
\max \{ w^\top x : x \in F_\alpha \} - \min \{ w^\top x : x \in F_\alpha \}
\] (6.6)

essentially gives the number of these lattice points. A lattice width direction
is a minimizer of the width among all directions $w \in \mathbb{Z}^n$, the lattice width
the corresponding width. Any polynomial bound on the width will yield a
polynomial-time algorithm in fixed dimension.

Exact and approximate lattice width directions $w$ can be constructed
using geometry of numbers techniques. We refer to the excellent tutorial
[24] and the classic references cited therein. The key to dealing with
the feasible region $F_\alpha$ is to apply ellipsoidal rounding. By applying the
shallow-cut ellipsoid method (which we describe in more detail below), one
finds concentric proportional inscribed and circumscribed ellipsoids that
differ by some factor $\beta$ that only depends on the dimension $n$. Then
any $\eta$-approximate lattice width direction for the ellipsoids gives a $\beta\eta$-
approximate lattice width direction for $F_\alpha$. Lenstra’s original algorithm
now uses an LLL-reduced basis of the lattice $\mathbb{Z}^n$ with respect to a norm
associated with the ellipsoid; the last basis vector then serves as an approx-
imate lattice width direction.

The first algorithm of this kind for convex integer minimization was
announced by Khachiyan [40]. In the following we present the variant
of Lenstra’s algorithm due to Heinz [30], which seems to yield the best
complexity bound for the problem published so far. The complexity result
is the following.

**Theorem 6.4.** Let $f, g_1, \ldots, g_m \in \mathbb{Z}[x_1, \ldots, x_n]$ be quasi-convex poly-
nomials of degree at most $d \geq 2$, whose coefficients have a binary en-
coding length of at most $\ell$. There exists an algorithm running in time $m\ell^{O(1)}d^{O(n)}\ell^{O(n^3)}$ that computes a minimizer $x^* \in \mathbb{Z}^n$ of the problem (6.4)
or reports that no minimizer exists. If the algorithm outputs a mini-
mizer $x^*$, its binary encoding size is $\ell d^{O(n)}$.

We remark that the complexity guarantees can be improved dramat i-
cally by combining Heinz’ technique with more recent variants of Lens tra’s
algorithm that rely on the fast computation of shortest vectors [33].

A complexity result of greater generality was presented by Khachiyan
and Porkolab [41]. It covers the case of minimization of convex polynomials
over the integer points in convex semialgebraic sets given by arbitrary (not
necessarily quasi-convex) polynomials.
Theorem 6.5. Let \( Y \subseteq \mathbb{R}^{n_0} \) be a convex set given by
\[
Y = \{ y \in \mathbb{R}^{n_0} : Q_1 x^1 \in \mathbb{R}^{n_1} : \cdots : Q_\omega x^\omega \in \mathbb{R}^{n_\omega} : P(y, x^1, \ldots, x^\omega) \}
\]
with quantifiers \( Q_i \in \{ \exists, \forall \} \), where \( P \) is a Boolean combination of polynomial inequalities
\[
g_i(y, x^1, \ldots, x^\omega) \leq 0, \quad i = 1, \ldots, m
\]
with degrees at most \( d \geq 2 \) and coefficients of binary encoding size at most \( \ell \).

There exists an algorithm for solving the problem
\[
\min \{ y^{n_0} : y \in Y \cap \mathbb{Z}^{n_0} \}
\]
in time
\[
\ell^{O(1)} (md)^{O(n_0^4)} \prod_{i=1}^{\omega} O(n_i)
\]

When the dimension \( n_0 + n_1 + \cdots + n_\omega \) is fixed, the algorithm runs in polynomial time. For the case of convex minimization where the feasible region is described by convex polynomials, the complexity bound of Theorem 6.5 however, translates to
\[
\ell^{O(1)} m^{O(n^2)} d^{O(n^4)},
\]
which is worse than the bound of Theorem 6.4 [30].

In the remainder of this subsection, we describe the ingredients of the variant of Lenstra’s algorithm due to Heinz. The algorithm starts out by “rounding” the feasible region, by applying the shallow-cut ellipsoid method to find proportional inscribed and circumscribed ellipsoids. It is well-known [26] that the shallow-cut ellipsoid method only needs an initial circumscribed ellipsoid that is “small enough” (of polynomial binary encoding size – this follows from Theorem 6.3) and an implementation of a shallow separation oracle, which we describe below.

For a positive-definite matrix \( A \) we denote by \( \mathcal{E}(A, \hat{x}) \) the ellipsoid
\[
\{ x \in \mathbb{R}^n : (x - \hat{x})^\top A (x - \hat{x}) \leq 1 \}.
\]

Lemma 6.1 (Shallow separation oracle). Let \( g_0, \ldots, g_{m+1} \in \mathbb{Z}[x] \) be quasi-convex polynomials of degree at most \( d \), the binary encoding sizes of whose coefficients are at most \( r \). Let the (continuous) feasible region \( F = \{ x \in \mathbb{R}^n : g_i(x) < 0 \} \) be contained in the ellipsoid \( \mathcal{E}(A, \hat{x}) \), where \( A \) and \( \hat{x} \) have binary encoding size at most \( \ell \). There exists an algorithm with running time \( m(\ln r)^{O(1)} d^{O(n)} \) that outputs
(a) “true” if
\[
\mathcal{E}((n + 1)^{-3} A, \hat{x}) \subseteq F \subseteq \mathcal{E}(A, \hat{x});
\]
(b) otherwise, a vector \( c \in \mathbb{Q}^n \setminus \{0\} \) of binary encoding length \( (l+r)(dn)^{O(1)} \)
with
\[
F \subseteq \mathcal{E}(A, \hat{x}) \cap \{ x \in \mathbb{R}^n : c^\top (x - \hat{x}) \leq \frac{1}{n+1} (c^\top A c)^{1/2} \}.
\]

Proof. We give a simplified sketch of the proof, without hard complexity estimates. By applying an affine transformation to \( F \subseteq \mathcal{E}(A, \hat{x}) \), we can assume that \( F \) is contained in the unit ball \( \mathcal{E}(I, 0) \). Let us denote as
usual by \(e_1, \ldots, e_n\) the unit vectors and by \(e_{n+1}, \ldots, e_{2n}\) their negatives. The algorithm first constructs numbers \(\lambda_{i1}, \ldots, \lambda_{id} > 0\) with
\[
\frac{1}{n + \frac{3}{2}} < \lambda_{i1} < \cdots < \lambda_{id} < \frac{1}{n + 1}
\]
and the corresponding point sets \(B_i = \{x_{ij} := \lambda_{ij}e_i : j = 1, \ldots, d\}\); see Figure 4(a). The choice of the bounds (6.9) for \(\lambda_{ij}\) will ensure that we either find a large enough inscribed ball for (a) or a deep enough cut for (b). Then the algorithm determines the (continuous) feasibility of the center \(0\) and the \(2n\) innermost points \(x_{i1}\).

**Case I.** If \(x_{i1} \in F\) for \(i = 1, \ldots, 2n\), then the cross-polytope \(\text{conv}\{x_{i1} : i = 1, \ldots, 2n\}\) is contained in \(F\); see Figure 4(b). An easy calculation shows that the ball \(\mathcal{E}((n+1)^{-3}, 0)\) is contained in the cross-polytope and thus in \(F\); see Figure 4. Hence the condition in (a) is satisfied and the algorithm outputs “true”.

**Case II.** We now discuss the case when the center \(0\) violates a polynomial inequality \(g_0(x) < 0\) (say). Let \(F_0 = \{x \in \mathbb{R}^n : g_0(x) < 0\} \supseteq F\). Due to convexity of \(F_0\), for all \(i = 1, \ldots, n\), one set of each pair \(B_i \cup B_{n+i}\) must be empty; see Figure 5(a). Without loss of generality, let us assume \(B_{n+i} \cap F_0 = \emptyset\) for all \(i\). We can determine whether a \(n\)-variate polynomial function of known maximum degree \(d\) is constant by evaluating it on \((d + 1)^n\) suitable points (this is a consequence of the Fundamental Theorem of Algebra). For our case of quasi-convex polynomials, this can be improved; indeed, it suffices to test whether the gradient \(\nabla g_0\) vanishes on the \(nd\) points in the set \(B_1 \cup \cdots \cup B_n\). If it does, we know that \(g_0\) is constant, thus \(F = \emptyset\), and so we can return an arbitrary vector \(c\). Otherwise, there is a point \(x_{ij} \in B_i\) with \(c := \nabla g_0(x_{ij}) \neq 0\); we return this vector as the desired normal vector of a shallow cut. Due to the choice of \(\lambda_{ij}\) as a number smaller than \(\frac{1}{n+1}\), the cut is deep enough into the ellipsoid \(\mathcal{E}(A, \hat{x})\), so that (6.8) holds.

**Case III.** The remaining case to discuss is when \(0 \in F\) but there exists a \(k \in \{1, \ldots, 2n\}\) with \(x_{k1} \notin F\). Without loss of generality, let \(k = 1\), and
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Fig. 5. The implementation of the shallow separation oracle. (a) Case II: The center 0 violates a polynomial inequality \( g_0(x) < 0 \) (say). Due to convexity, for all \( i = 1, \ldots, n \), one set of each pair \( B_i \cap F \) and \( B_{n+i} \cap F \) must be empty. (b) Case III: A test point \( x_{k1} \) is infeasible, as it violates an inequality \( g_0(x) < 0 \) (say). However, the center 0 is feasible at least for this inequality.

let \( x_{1,1} \) violate the polynomial inequality \( g_0(x) < 0 \), i.e., \( g_0(x_{1,1}) \geq 0 \); see Figure 5 (b). We consider the univariate polynomial \( \phi(\lambda) = g_0(\lambda e_i) \). We have \( \phi(0) = g_0(0) < 0 \) and \( \phi(\lambda_{1,1}) \geq 0 \), so \( \phi \) is not constant. Because \( \phi \) has degree at most \( d \), its derivative \( \phi' \) has degree at most \( d - 1 \), so \( \phi' \) has at most \( d - 1 \) roots. Thus, for at least one of the \( d \) different values \( \lambda_{1,1}, \ldots, \lambda_{1,d} \), say \( \lambda_{1,j} \), we must have \( \phi'(\lambda_{1,j}) \neq 0 \). This implies that \( c := \nabla g_0(x_{1,j}) \neq 0 \).

By convexity, we have \( x_{1,j} \notin F \), so we can use \( c \) as the normal vector of a shallow cut.

By using this oracle in the shallow-cut ellipsoid method, one obtains the following result.

**Corollary 6.1.** Let \( g_0, \ldots, g_m \in \mathbb{Z}[x] \) be quasi-convex polynomials of degree at most \( d \geq 2 \). Let the (continuous) feasible region \( F = \{ x \in \mathbb{R}^n : g_i(x) \leq 0 \} \) be contained in the ellipsoid \( E(A_0, 0) \), given by the positive-definite matrix \( A_0 \in \mathbb{Q}^{n \times n} \). Let \( \epsilon \in \mathbb{Q}^+ \) be given. Let the entries of \( A_0 \) and the coefficients of all monomials of \( g_0, \ldots, g_m \) have binary encoding size at most \( \ell \).

There exists an algorithm with running time \( m(\ell n)^{O(1)} d^{O(n)} \) that computes a positive-definite matrix \( A \in \mathbb{Q}^{n \times n} \) and a point \( \tilde{x} \in \mathbb{Q}^n \) with

(a) either \( E((n + 1)^{-3} A, \tilde{x}) \subseteq F \subseteq E(A, \tilde{x}) \)
(b) or \( F \subseteq E(A, \tilde{x}) \) and \( \text{vol} E(A, \tilde{x}) < \epsilon \).

Finally, there is a lower bound for the volume of a continuous feasible region \( F \) that can contain an integer point.

**Lemma 6.2.** Under the assumptions of Corollary 6.1, if \( F \cap \mathbb{Z}^n \neq \emptyset \), there exists an \( \epsilon \in \mathbb{Q}^+ \) of binary encoding size \( \ell(\text{dn})^{O(1)} \) with \( \text{vol} F > \epsilon \).

On the basis of these results, one obtains a Lenstra-type algorithm for the decision version of the convex integer minimization problem with the desired complexity. By applying binary search, the optimization problem can be solved, which provides a proof of Theorem 6.4.

**6.3. Fixed dimension: Convex integer maximization.** Maximizing a convex function over the integer points in a polytope in fixed dimension can be done in polynomial time. To see this, note that the op-
timal value is taken on at a vertex of the convex hull of all feasible integer points. But when the dimension is fixed, there is only a polynomial number of vertices, as Cook et al. [15] showed.

**Theorem 6.6.** Let \( P = \{ x \in \mathbb{R}^n : Ax \leq b \} \) be a rational polyhedron with \( A \in \mathbb{Q}^{m \times n} \) and let \( \phi \) be the largest binary encoding size of any of the rows of the system \( Ax \leq b \). Let \( P^I = \text{conv}(P \cap \mathbb{Z}^n) \) be the integer hull of \( P \). Then the number of vertices of \( P^I \) is at most \( 2^m n \phi (6n^2 \phi)^{n-1} \). Moreover, Hartmann [27] gave an algorithm for enumerating all the vertices, which runs in polynomial time in fixed dimension.

By using Hartmann’s algorithm, we can therefore compute all the vertices of the integer hull \( P^I \), evaluate the convex objective function on each of them and pick the best. This simple method already provides a polynomial-time algorithm.

7. **Strongly polynomial-time algorithms: Submodular function minimization.** In important specially structured cases, even strongly polynomial-time algorithms are available. The probably most well-known case is that of submodular function minimization. We briefly present the most recent developments below.

Here we consider the important problem of submodular function minimization. This class of problems consists of unconstrained 0/1 programming problems

\[
\min f(x) : x \in \{0, 1\}^n,
\]

where the function \( f \) is submodular, i.e.,

\[
f(x) + f(y) \geq f(\max\{x, y\}) + f(\min\{x, y\}).
\]

Here \( \max \) and \( \min \) denote the componentwise maximum and minimum of the vectors, respectively.

The fastest algorithm known for submodular function minimization seems to be by Orlin [54], who gave a strongly polynomial-time algorithm of running time \( O(n^5 T_{\text{eval}} + n^6) \), where \( T_{\text{eval}} \) denotes the running time of the evaluation oracle. The algorithm is “combinatorial”, i.e., it does not use the ellipsoid method. This complexity bound simultaneously improved that of the fastest strongly polynomial-time algorithm using the ellipsoid method, of running time \( O(n^5 T_{\text{eval}} + n^7) \) (see [50]) and the fastest “combinatorial” strongly polynomial-time algorithm by Iwata [35], of running time \( O((n^6 T_{\text{eval}} + n^7) \log n) \). We remark that the fastest polynomial-time algorithm, by Iwata [35], runs in \( O((n^4 T_{\text{eval}} + n^5) \log M) \), where \( M \) is the largest function value. We refer to the recent survey by Iwata [36], who reports on the developments that preceded Orlin’s algorithm [54].

For the special case of symmetric submodular function minimization, i.e., \( f(x) = f(1 - x) \), Queyranne [50] presented an algorithm of running time \( O(n^3 T_{\text{eval}}) \).
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REFERENCES

[1] L. Adleman and K. Manders, Reducibility, randomness and intractability, in Proc. 9th Annual ACM Symposium on Theory of Computing, 1977, pp. 151–163.
[2] M. Agrawal, N. Kayal, and N. Saxena, PRIMES is in P, Annals of Math., 160 (2004), pp. 781–793.
[3] V. Baldoni, N. Berline, J. A. De Loera, M. Köppe, and M. Vergne, How to integrate a polynomial over a simplex. To appear in Mathematics of Computation, eprint arXiv:0809.2083 [math.MG], 2008.
[4] B. Bank, J. Heintz, T. Krick, R. Mandel, and P. Solernò, Une borne optimale pour la programmation entière quasi-convexe, Bull. Soc. math. France, 121 (1993), pp. 299–314.
[5] B. Bank, T. Krick, R. Mandel, and P. Solernò, A geometrical bound for integer programming with polynomial constraints, in Fundamentals of Computation Theory, vol. 529 of Lecture Notes In Computer Science, Springer-Verlag, 1991, pp. 121–125.
[6] A. I. Barvinok, A polynomial time algorithm for counting integral points in polyhedra when the dimension is fixed, Mathematics of Operations Research, 19 (1994), pp. 769–779.
[7] A. I. Barvinok and J. E. Pommersheim, An algorithmic theory of lattice points in polyhedra, in New Perspectives in Algebraic Combinatorics, L. J. Billera, A. Björner, C. Greene, R. E. Simion, and R. P. Stanley, eds., vol. 38 of Math. Sci. Res. Inst. Publ., Cambridge Univ. Press, Cambridge, 1999, pp. 91–147.
[8] S. Basu, R. Pollack, and M.-F. Roy, Algorithms in Real Algebraic Geometry, Springer-Verlag, second ed., 2006.
[9] M. Bellare and P. Rogaway, The complexity of approximating a nonlinear program, in Pardalos [55].
[10] M. Bellare and P. Rogaway, The complexity of approximating a nonlinear program, Mathematical Programming, 69 (1995), pp. 429–441.
[11] Y. Berstein, J. Lee, H. Maruri-Aguilar, S. Onn, E. Riccomagno, R. Weismantel, and H. Wynn, Nonlinear matroid optimization and experimental design, SIAM Journal on Discrete Mathematics, 22 (2008), pp. 901–919.
[12] Y. Berstein, J. Lee, S. Onn, and R. Weismantel, Nonlinear optimization for matroid intersection and extensions, IBM Research Report RC24610, (2008).
[13] Y. Berstein and S. Onn, Nonlinear bipartite matching, Discrete Optimization, 5 (2008), pp. 53–65.
[14] L. Blum, M. Shub, and S. Smale, On a theory of computation and complexity over the real numbers: NP-completeness, recursive functions and universal machines, Bull. Am. Math. Soc., 21 (1989), pp. 1–46.
[15] W. J. Cook, M. E. Hartmann, R. Kannan, and C. McDiarmid, On integer points in polyhedra, Combinatorica, 12 (1992), pp. 27–37.
[16] E. de Klerk, The complexity of optimizing over a simplex, hypercube or sphere: a short survey, Central European Journal of Operations Research, 16 (2008), pp. 111–125.
[17] E. de Klerk, M. Laurent, and P. A. Parrilo, A PTAS for the minimization of polynomials of fixed degree over the simplex, Theoretical Computer Science, 361 (2006), pp. 210–225.
[18] J. A. De Loera, R. Hemmecke, M. Köppe, and R. Weismantel, FPTAS for mixed-integer polynomial optimization with a fixed number of variables, in 17th ACM-SIAM Symposium on Discrete Algorithms, 2006, pp. 743–748.
[19] ——, Integer polynomial optimization in fixed dimension, Mathematics of Operations Research, 31 (2006), pp. 147–153.
[20] ——, FPTAS for optimizing polynomials over the mixed-integer points of polytopes in fixed dimension, Mathematical Programming, Series A, 118 (2008), pp. 273–290.
[21] J. A. De Loera, R. Hemmecke, S. Onn, and R. Weismantel, N-fold integer programming, Disc. Optim., to appear, (2008).
[22] J. A. De Loera and S. Onn, All linear and integer programs are slim 3-way transportation programs, SIAM Journal of Optimization, 17 (2006), pp. 806–821.
[23] ———, Markov bases of three-way tables are arbitrarily complicated, Journal of Symbolic Computation, 41 (2006), pp. 173–181.
[24] F. Eisenbrand, Integer programming and algorithmic geometry of numbers, in 50 Years of Integer Programming 1958–2008, M. Jünger, T. Liebling, D. Naddef, W. Pulleyblank, G. Reinelt, G. Rinaldi, and L. Wolsey, eds., Springer-Verlag, 2010.
[25] M. R. Garey and D. S. Johnson, Computers and Intractability: A Guide to the Theory of NP-completeness, W. H. Freeman and Company, New York, NY, 1979.
[26] M. Grötschel, L. Lovász, and A. Schrijver, Geometric Algorithms and Combinatorial Optimization, Springer, Berlin, Germany, 1988.
[27] M. E. Hartmann, Cutting Planes and the Complexity of the Integer Hull, phd thesis, Cornell University, Department of Operations Research and Industrial Engineering, Ithaca, NY, 1989.
[28] J. Håstad, Some optimal inapproximability results, in Proceedings of the 29th Symposium on the Theory of Computing (STOC), ACM, 1997, pp. 1–10.
[29] J. Heintz, M. Roy, and P. Solernó, Sur la complexité du principe de Tarski–Seidenberg, Bull. Soc. Math. France, 118 (1990), pp. 101–126.
[30] S. Heinz, Complexity of integer quasiconvex polynomial optimization, Journal of Complexity, 21 (2005), pp. 543–556.
[31] R. Hemmecke, M. Köppe, J. Lee, and R. Weismantel, Nonlinear integer programming, in 50 Years of Integer Programming 1958–2008, M. Jünger, T. Liebling, D. Naddef, W. Pulleyblank, G. Reinelt, G. Rinaldi, and L. Wolsey, eds., Springer-Verlag, 2010.
[32] R. Hemmecke, S. Onn, and R. Weismantel, A polynomial oracle-time algorithm for convex integer minimization, Manuscript, (2008).
[33] R. Hildebrand and M. Köppe, A faster algorithm for quasi-convex integer polynomial optimization, eprint arXiv:1006.4661 [math.OC].
[34] D. Hochbaum, Complexity and algorithms for nonlinear optimization problems, Annals of Operations Research, 153 (2007), pp. 257–296.
[35] S. Iwata, A faster scaling algorithm for minimizing submodular functions, SIAM Journal on Computing, 32 (2003), pp. 833–840.
[36] ———, Submodular function minimization, Mathematical Programming, 112 (2008), pp. 45–64.
[37] R. G. Jeroslow, There cannot be any algorithm for integer programming with quadratic constraints, Operations Research, 21 (1973), pp. 221–224.
[38] J. P. Jones, Universal diophantine equation, Journal of Symbolic Logic, 47 (1982), pp. 403–410.
[39] J. P. Jones and Yu. V. Matiyasevich, Proof of recursive unsolvability of Hilbert’s tenth problem, The American Mathematical Monthly, 98 (1991), pp. 689–709.
[40] L. G. Khachiyan, Convexity and complexity in polynimial programming, in Proceedings of the International Congress of Mathematicians, August 16–24, 1983, Warszawa, Z. Ciesielski and C. Olech, eds., New York, 1984, North-Holland, pp. 1569–1577.
[41] L. G. Khachiyan and L. Porkolab, Integer optimization on convex semialgebraic sets., Discrete and Computational Geometry, 23 (2000), pp. 207–224.
[42] J. C. Lagarias, On the computational complexity of determining the solvability or unsolvability of the equation $x^2 - dy^2 = -1$, Transactions of the American Mathematical Society, 260 (1980), pp. 485–508.
[43] ———, Succinct certificates for the solvability of binary quadratic diophantine equations. e-print arXiv:math/0611209v1, 2006. Extended and updated ver-
sion of a 1979 FOCS paper.

[44] J. Lee, S. Onn, and R. Weismantel, *Nonlinear optimization over a weighted independence system*, IBM Research Report RC24513, (2008).

[45] __________, *On test sets for nonlinear integer maximization*, Operations Research Letters, 36 (2008), pp. 439–443.

[46] H. W. Lenstra, *Integer programming with a fixed number of variables*, Mathematics of Operations Research, 8 (1983), pp. 538–548.

[47] K. Manders and L. Adleman, *NP-complete decision problems for binary quadratic*, J. Comp. Sys. Sci., 16 (1978), pp. 168–184.

[48] Yu. V. Matiyasevich, *Enumerable sets are diophantine*, Doklady Akademii Nauk SSSR, 191 (1970), pp. 279–282, (Russian); English translation, Soviet Mathematics Doklady, vol. 11 (1970), pp. 354–357.

[49] __________, *Hilbert’s tenth problem*, The MIT Press, Cambridge, MA, USA, 1993.

[50] S. T. McCormick, *Submodular function minimization*, in Discrete Optimization, K. Aardal, G. Nemhauser, and R. Weismantel, eds., vol. 12 of Handbooks in Operations Research and Management Science, Elsevier, 2005.

[51] T. S. Motzkin and E. G. Straus, *Maxima for graphs and a new proof of a theorem of Turán*, Canadian Journal of Mathematics, 17 (1965), pp. 533–540.

[52] W. T. Ochichowska, *On boundedness of (quasi-)convex integer optimization problems*, Math. Meth. Oper. Res., 68 (2008).

[53] S. Onn, *Convex discrete optimization*, eprint arXiv:math/0703575, 2007.

[54] J. B. Orlin, *A faster strongly polynomial time algorithm for submodular function minimization*, Math. Program., Ser. A, 118 (2009), pp. 237–251.

[55] P. M. Pardalos, ed., *Complexity in Numerical Optimization*, World Scientific, 1993.

[56] M. Queyranne, *Minimizing symmetric submodular functions*, Mathematical Programming, 82 (1998), pp. 3–12.

[57] J. Renegar, *On the computational complexity and geometry of the first-order theory of the reals, part I: Introduction. Preliminaries. The geometry of semi-algebraic sets. The decision problem for the existential theory of the reals*, Journal of Symbolic Computation, 13 (1992), pp. 255–300.

[58] __________, *On the computational complexity and geometry of the first-order theory of the reals, part II: The general decision problem. Preliminaries for quantifier elimination*, Journal of Symbolic Computation, 13 (1992), pp. 301–328.

[59] __________, *On the computational complexity and geometry of the first-order theory of the reals. part III: Quantifier elimination*, Journal of Symbolic Computation, 13 (1992), pp. 329–352.

[60] __________, *On the computational complexity of approximating solutions for real algebraic formulae*, SIAM Journal on Computing, 21 (1992), pp. 1008–1025.

[61] C. L. Siegel, *Zur Theorie der quadratischen Formen*, Nachrichten der Akademie der Wissenschaften in Göttingen, II, Mathematisch-Physikalische Klasse, 3 (1972), pp. 21–46.

[62] T. Skolem, *Diophantische Gleichungen*, vol. 5 of Ergebnisse der Mathematik und ihrer Grenzgebiete, 1938.

[63] S. P. Tarasov and L. G. Khachiyan, *Bounds of solutions and algorithmic complexity of systems of convex diophantine inequalities*, Soviet Math. Doklady, 22 (1980), pp. 700–704.

[64] A. M. Turing, *On computable numbers, with an application to the Entscheidungsproblem*, Proceedings of the London Mathematical Society, Series 2, 42 (1936), pp. 230–265. Errata in ibidem, 43 (1937):544–546.

[65] S. A. Vavasis, *Polynomial time weak approximation algorithms for quadratic programming*, in Pardalos [55].