UNIMODALITY, LOG-CONCAVITY, REAL-ROOTEDNESS
AND BEYOND

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1. Introduction

Many important sequences in combinatorics are known to be log–concave or unimodal, but many are only conjectured to be so although several techniques using methods from combinatorics, algebra, geometry and analysis are now available. Stanley [90] and Brenti [25] have written extensive surveys of various techniques that can be used to prove real–rootedness, log–concavity or unimodality. After a brief introduction and a short section on probabilistic consequences of real–rootedness, we will complement [25, 90] with a survey over new techniques that have been developed, and problems and conjectures that have been solved. I stress that this is not a comprehensive account of all work that has been done in the area since op. cit.. The selection is certainly colored by my taste and knowledge.

If \( A = \{a_k\}_{k=0}^n \) is a finite sequence of real numbers, then

- \( A \) is **unimodal** if there is an index \( 0 \leq j \leq n \) such that \( a_0 \leq \cdots \leq a_{j-1} \leq a_j \geq a_{j+1} \geq \cdots \geq a_n \).

- \( A \) is **log–concave** if

\[
 a_j^2 \geq a_{j-1}a_{j+1}, \quad \text{for all } 1 \leq j < n.
\]

- the generating polynomial, \( p_A(x) := a_0 + a_1x + \cdots + a_nx^n \), is called **real–rooted** if all its zeros are real. By convention we also consider constant polynomials to be real–rooted.

We say that the polynomial \( p_A(x) = \sum_{k=0}^n a_k x^k \) has a certain property if \( A = \{a_k\}_{k=0}^n \) does. The most fundamental sequence satisfying all of the properties above is the \( n \)th row of Pascal’s triangle \( \{\binom{n}{k}\}_{k=0}^n \). Log–concavity follows easily from the explicit formula \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \):

\[
 \binom{n}{k}^2 \frac{\binom{n}{k+1}}{\binom{n}{k-1}} = \frac{(k+1)(n-k+1)}{k(n-k)} > 1.
\]

The following lemma relates the three properties above.

**Lemma 1.1.** Let \( A = \{a_k\}_{k=0}^n \) be a finite sequence of nonnegative numbers.

- If \( p_A(x) \) is real–rooted, then the sequence \( A' = \{a_k/\binom{n}{k}\}_{k=0}^n \) is log–concave.

- If \( A' \) is log–concave, then so is \( A \).

- If \( A \) is log–concave and positive, then \( A \) is unimodal.

**Proof.** Suppose \( p_A(x) \) is real–rooted. Let \( a_k = \binom{n}{k}b_k \), for \( 1 \leq k \leq n \). By the Gauss–Lucas theorem below, the polynomial

\[
 \frac{1}{n} p_A'(x) = \sum_{k=0}^n \frac{k}{n} \binom{n}{k} b_k x^{k-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} b_{k+1} x^k
\]

is real–rooted. The operation

\[
 x^n p_A(1/x) = \sum_{k=0}^n \binom{n}{k} b_{n-k} x^k,
\]

preserves real–rootedness. Let \( 1 \leq j \leq n-1 \). Applying the operations (1.1) and (1.2) appropriately to \( p_A(x) \), we end up with the real–rooted polynomial

\[
 b_{j-1} + 2b_j x + b_{j+1}x^2,
\]

and thus \( b_j^2 \geq b_{j-1}b_{j+1} \). This proves the first statement.
The term-wise (Hadamard) product of a positive and log–concave sequence and a log–concave sequence is again log–concave. Since \( \binom{n}{k} \) is positive and log–concave, the second statement follows.

The third statement follows directly from the definitions. \( \square \)

**Example 1.1.** Natural examples of log–concave polynomials which are not real–rooted are the \( q \)-factorial polynomials,

\[
[n]_q! = [n]_q \cdot [n-1]_q \cdots [2]_q \cdot [1]_q,
\]

where \([k]_q = 1 + q + \cdots + q^{k-1}\). The polynomial \([n]_q!\) is the generating polynomial for the number of inversions over the symmetric group \( S_n \):

\[
[n]_q! = \sum_{\pi \in S_n} q^{\text{inv}(\pi)},
\]

where

\[
\text{inv}(\pi) = |\{1 \leq i < j \leq n : \pi(i) > \pi(j)\}|,
\]

see [94]. The easiest way to see that \([n]_q!\) is log–concave is to observe that \([k]_q\) is log–concave. Log–concavity of \([n]_q!\) then follows from the fact that if \(A(x)\) and \(B(x)\) are generating polynomials of positive log–concave sequences, then so is \(A(x)B(x)\), see [60].

**Example 1.2.** Examples of unimodal sequences that are not log–concave are the \( q \)-binomial coefficients

\[
\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.
\]

These are polynomials with nonnegative coefficients

\[
\left[ \begin{array}{c} n \\ k \end{array} \right]_q = a_0(n,k) + a_1(n,k)q + \cdots + a_{k(n-k)}(n,k)q^{k(n-k)}, \quad (1.3)
\]

which are unimodal and symmetric. There are several proofs of this fact, see [90]. For example the Cayley–Sylvester theorem, first stated by Cayley in the 1850’s and proved by Sylvester in 1878, implies unimodality of (1.3), see [90]. However \([4]_2 = 1 + q + 2q^2 + q^3 + q^4\), which is not log–concave.

For a proof of the following fundamental theorem we refer to [82].

**Theorem 1.2** (The Gauss–Lucas theorem). Let \( f(x) \in \mathbb{C}[x] \) be a polynomial of degree at least one. All zeros of \( f'(x) \) lie in the convex hull of the zeros of \( f(x) \).

**Example 1.3.** Let \( \{S(n,k)\}_{k=0}^n \) be the Stirling numbers of the second kind, see [94]. Then \( \tilde{S}(n,k) := k!S(n,k) \) counts the number of surjections from \([n] := \{1, 2, \ldots, n\}\) to \([k]\). For a surjection \( f : [n+1] \to [k] \), let \( j = f(n+1) \). Conditioning on whether \( |f^{-1}(\{j\})| = 1 \) or \( |f^{-1}(\{j\})| > 1 \), one sees that

\[
\tilde{S}(n+1,k) = k\tilde{S}(n,k-1) + k\tilde{S}(n,k), \quad \text{for all } 1 \leq k \leq n+1. \quad (1.4)
\]

Let \( E_n(x) = \sum_{k=1}^n \tilde{S}(n,k)x^k \). Then (1.4) translates as

\[
E_{n+1}(x) = xE_n(x) + x(x+1)E'_n(x) = x\frac{d}{dx}\left((x+1)E_n(x)\right).
\]

By induction and the Gauss–Lucas theorem, we see that \( E_n(x) \) is real–rooted, and that all its zeros lie in the interval \([-1, 0]\) for all \( n \geq 1 \). Later, in Example 7.1 we will see that the operation of dividing the \( k \)th coefficient by \( k! \), for each \( k \),
preserves real–rootedness. Hence also the polynomials \( \sum_{k=1}^{n} S(n,k)x^k, n \geq 1, \) are real–rooted.

A generalization of finite nonnegative sequences with real–rooted generating polynomials is that of Pólya frequency sequences. A sequence \( \{a_k\}_{k=0}^{\infty} \subseteq \mathbb{R} \) is a Pólya frequency sequence (PF for short) if all minors of the infinite Toeplitz matrix \((a_{i-j})_{i,j=0}^{\infty}\) are nonnegative. In particular, PF sequences are log–concave. PF sequences are characterized by the following theorem of Edrei [42], first conjectured by Schoenberg.

**Theorem 1.3.** A sequence \( \{a_k\}_{k=0}^{\infty} \subseteq \mathbb{R} \) of real numbers is PF if and only if its generating function may be expressed as

\[
\sum_{k=0}^{\infty} a_k x^k = C x^m e^{ax} \prod_{k=0}^{\infty} (1 + \alpha_k x) \prod_{k=0}^{\infty} (1 - \beta_k x),
\]

where \( C, a \geq 0, m \in \mathbb{N}, \alpha_k, \beta_k \geq 0 \) for all \( k \in \mathbb{N}, \) and \( \sum_{k=0}^{\infty} (\alpha_k + \beta_k) < \infty. \)

Hence a finite nonnegative sequence is PF if and only if its generating polynomial is real–rooted. This was first proved by Aissen, Schoenberg and Whitney [1]. Theorem 1.3 provides — at least in theory — a method of proving combinatorially that a combinatorial polynomial with nonnegative coefficients is real–rooted. Namely to find a combinatorial interpretation of the minors of \((a_{i-j})_{i,j=0}^{\infty}.\) This method was used by e.g. Gasharov [52] to prove that the independence polynomial of a \((3+1)\)-free graph is real–rooted. For more on PF sequences in combinatorics, see [24].

### 2. Probabilistic consequences of real–rootedness

Below we will explain two useful probabilistic consequences of real–rootedness. For further consequences, see Pitman’s survey [79]. If \( X \) is a random variable taking values in \( \{0, \ldots, n\}, \) let \( a_k = P[X = k] \) for \( 0 \leq k \leq n, \) and let

\[
p_X(t) = a_0 + a_1 t + \cdots + a_n t^n,
\]

be the partition function of \( X. \) Then \( X \) has mean

\[
\mu = \mathbb{E}[X] = \sum_{k=0}^{n} kP[X = k] = p'_X(1),
\]

and variance

\[
\text{Var}(X) = \mathbb{E}[X^2] - \mu^2 = p''_X(1) + p'_X(1) - p'_X(1)^2.
\]

The following theorem of Bender [4] has been used on numerous occasions to prove asymptotic normality of combinatorial sequences, see e.g. [4, 5, 8].

**Theorem 2.1.** Let \( \{X_n\}_{n=1}^{\infty} \) be a sequence of random variables taking values in \( \{0, 1, \ldots, n\} \) such that

1. \( p_{X_n}(t) \) is real–rooted for all \( n, \)
2. \( \text{Var}(X_n) \to \infty. \)

Then the distribution of the random variable

\[
\frac{X_n - \mathbb{E}[X_n]}{\sqrt{\text{Var}(X_n)}}
\]

converges to the standard normal distribution \( N(0, 1) \) as \( n \to \infty. \)
Example 2.1. Let $X_n$ be the random variable on the symmetric group $\mathfrak{S}_n$ counting the number of cycles in a uniform random permutation. Since the number of permutations in $\mathfrak{S}_n$ with exactly $k$ cycles is the signless Stirling number of the first kind $c(n,k)$ (see [64]),

$$p_{X_n}(t) = \frac{1}{n!} x(x+1) \cdots (x+n-1).$$

Thus $X_n$ has mean $H_n = 1 + 1/2 + \cdots + 1/n$ and variance

$$\sigma_n^2 = H_n - \sum_{k=1}^{n} k^{-2}.$$

Hence the distribution of the random variable $\frac{X_n - H_n}{\sigma_n}$ converges to the standard normal distribution $N(0,1)$ as $n \to \infty$.

For more examples using Theorem 2.1, see [4], and for recent examples, see [5,8].

A simple consequence of Lemma 1.1 is that if a polynomial $a$ has only real and nonpositive zeros, then there is either a unique index $m$ such that $a_m = \max_k a_k$, or two consecutive indices $m \pm 1/2$ (whence $m$ is a half-integer) such that $a_m \pm 1/2 = \max_k a_k$. The number $m = m(\{a_k\}_{k=0}^{n})$ is called the mode of $\{a_k\}_{k=0}^{n}$. A theorem of Darroch [40] enables us to easily compute the mode.

Theorem 2.2. Suppose $\{a_k\}_{k=0}^{n}$ is a sequence of nonnegative numbers such that the polynomial $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ is real-rooted. If $m$ is the mode of $\{a_k\}_{k=0}^{n}$, and $\mu := p'(1)/p(1)$ its mean, then

$$[\mu] \leq m \leq \lceil \mu \rceil.$$

Applying Theorem 2.2 to the signless Stirling numbers of the first kind $\{c(n,k)\}_{k=1}^{n}$ (Example 2.1), we see that

$$[H_n] \leq m(\{c(n,k)\}_{k=1}^{n}) \leq \lceil H_n \rceil.$$

3. Unimodality and $\gamma$-nonnegativity

We say that the sequence $\{h_k\}_{k=0}^{d}$ is symmetric with center of symmetry $d/2$ if $h_k = h_{d-k}$ for all $0 \leq k \leq d$. A property called $\gamma$-nonnegativity, which implies symmetry and unimodality, has recently been considered in topological, algebraic and enumerative combinatorics.

The linear space of polynomials $h(x) = \sum_{k=0}^{d} h_k x^k \in \mathbb{R}[x]$ which are symmetric with center of symmetry $d/2$ has a basis

$$B_d := \{x^k (1+x)^{d/2-k} \}_{k=0}^{d/2}.$$  

If $h(x) = \sum_{k=0}^{d/2} \gamma_k x^k (1+x)^{d/2-k}$, we call $\{\gamma_k\}_{k=0}^{d/2}$ the $\gamma$-vector of $h$. Since the binomial numbers are unimodal, having a nonnegative $\gamma$-vector implies unimodality of $\{h_k\}_{k=0}^{n}$. If the $\gamma$-vector of $h$ is nonnegative, then we say that $h$ is $\gamma$-nonnegative. Let $\Gamma_+^d$ be the convex cone of polynomials that have nonnegative coefficients when expanded in $B_d$. Clearly

$$\Gamma_+^m \cdot \Gamma_+^n := \{fg : f \in \Gamma_+^m \text{ and } g \in \Gamma_+^n\} \subseteq \Gamma_+^{m+n}. \quad (3.1)$$
3. Properties of the modified Foata–Strehl action as the modified Foata–Strehl action, or the MFS-action for short. Hence the group

\[ \text{Orb} \]

acts on \( S_n \) via the functions

\[ \pi(\theta) = \hat{\pi} = \pi \circ \theta \]

where

\[ \pi \text{-nonnegativity. Let } \pi \in S_n \text{ be a permutation written as a word } (\pi(i) = a_i), \text{ and set } a_0 = a_{n+1} = n + 1. \]

If \( k \in [n] \), then \( a_k \) is a

- **valley** if \( a_{k-1} > a_k < a_{k+1} \),
- **peak** if \( a_{k-1} < a_k > a_{k+1} \),
- **double ascent** if \( a_{k-1} < a_k < a_{k+1} \), and
- **double descent** if \( a_{k-1} > a_k > a_{k+1} \).

Define functions \( \varphi_x : S_n \to S_n, x \in [n] \), as follows:

- If \( x \) is a double descent, then \( \varphi_x(\pi) \) is obtained by moving \( x \) into the slot between the first pair of letters \( a_i, a_{i+1} \) to the right of \( x \) such that \( a_i < x < a_{i+1} \);
- If \( x \) is a double ascent, then \( \varphi_x(\pi) \) is obtained by moving \( x \) to the slot between the first pair of letters \( a_i, a_{i+1} \) to the left of \( x \) such that \( a_i > x > a_{i+1} \);
- If \( x \) is a valley or a peak, then \( \varphi_x(\pi) = \pi \).

There is a geometric interpretation of the functions \( \varphi_x, x \in [n] \), first considered in [87]. Let \( \pi = a_1a_2 \cdots a_n \in S_n \) and imagine marbles at the points \( (i, a_i) \in \mathbb{N} \times \mathbb{N} \), for \( i = 0, 1, \ldots, n + 1 \). For \( i = 0, 1, \ldots, n \) connect \((i, a_i)\) and \((i + 1, a_{i+1})\) with a wire. Suppose gravity acts on the marbles, and that \( x \) is not at an equilibrium. If \( x \) is released it will slide and stop when it has reached the same height again. The resulting permutation is \( \varphi_x(\pi) \), see Fig. [1].
The functions \( \varphi_x \) are commuting involutions. Hence for any subset \( S \subseteq [n] \), we may define the function \( \varphi_S : \mathfrak{S}_n \to \mathfrak{S}_n \) by
\[
\varphi_S(\pi) = \prod_{x \in S} \varphi_x(\pi).
\]
Hence the group \( \mathbb{Z}_2^n \) acts on \( \mathfrak{S}_n \) via the functions \( \varphi_S, S \subseteq [n] \). For example
\[
\varphi_{\{2,3,7,8\}}(573148926) = 857134926.
\]

For \( \pi \in \mathfrak{S}_n \), let \( \text{Orb}(\pi) = \{g(\pi) : g \in \mathbb{Z}_2^n\} \) be the orbit of \( \pi \) under the action. There is a unique element in \( \text{Orb}(\pi) \) which has no double descents and which we denote by \( \hat{\pi} \).

**Theorem 3.2.** Let \( \pi = a_1a_2 \cdots a_n \in \mathfrak{S}_n \). Then
\[
\sum_{\sigma \in \text{Orb}(\pi)} x^{\text{des}(\sigma)} = x^{\text{des}(\hat{\pi})}(1 + x)^{n-1-2\text{des}(\hat{\pi})} = x^{\text{peak}(\pi)}(1 + x)^{n-1-2\text{peak}(\pi)},
\]
where \( \text{des}(\pi) = |\{i \in [n] : a_i > a_{i+1}\}| \) and \( \text{peak}(\pi) = |\{i \in [n] : a_{i-1} < a_i > a_{i+1}\}| \).

**Proof.** If \( x \) is a double ascent in \( \pi \) then \( \text{des}(\varphi_x(\pi)) = \text{des}(\pi) + 1 \). It follows that
\[
\sum_{\sigma \in \text{Orb}(\pi)} x^{\text{des}(\sigma)} = x^{\text{des}(\hat{\pi})}(1 + x)^a,
\]
where \( a \) is the number of double ascents in \( \hat{\pi} \). If we delete all double ascents from \( \hat{\pi} \) we get an alternating permutation
\[
n + 1 > b_1 < b_2 > b_3 < \cdots > b_{n-a} < n + 1,
\]
with the same number of descents. Hence \( n - a = 2\text{des}(\hat{\pi}) + 1 \). Clearly \( \text{des}(\hat{\pi}) = \text{peak}(\pi) \) and the theorem follows.

For a subset \( T \) of \( \mathfrak{S}_n \) let
\[
A(T; x) := \sum_{\pi \in T} x^{\text{des}(\pi)}.
\]

**Corollary 3.3.** If \( T \subseteq \mathfrak{S}_n \) is invariant under the \( \mathbb{Z}_2^n \)-action, then
\[
A(T; x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i(T)x^i(1 + x)^{n-1-2i},
\]
where
\[
\gamma_i(T) = 2^{-n+1+2i}|\{\pi \in T : \text{peak}(\pi) = i\}|.
\]
In particular \( A(T, x) \) is \( \gamma \)-nonnegative.

**Proof.** It is enough to prove the theorem for an orbit of a permutation \( \pi \in \mathfrak{S}_n \). Since the number of peaks is constant on \( \text{Orb}(\pi) \) the equality follows from Theorem 3.2.

**Example 3.1.** Recall that the **Eulerian polynomials** are defined by
\[
A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{des}(\pi)+1}, \tag{3.2}
\]
see [94]. By Corollary 3.3
\[
A_n(x)/x = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i x^i(1 + x)^{n-1-2i},
\]

\[
A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{des}(\pi)+1}.
\]
where
\[ \gamma_n = 2^{n+1+2i} \{ \pi \in \mathcal{S}_n : \text{peak}(\pi) = i \} \].

**Example 3.2.** This example is taken from [18]. The stack-sorting operator \( S \) may be defined recursively on permutations of finite subsets of \( \{1, 2, \ldots\} \) as follows. If \( w \) is empty, then \( S(w) := w \). If \( w \) is nonempty, write \( w \) as the concatenation \( w = LmR \) where \( m \) is the greatest element of \( w \), and \( L \) and \( R \) are the subwords to the left and right of \( m \), respectively. Then \( S(w) := S(L)S(R)m \).

If \( \sigma, \tau \in \mathcal{S}_n \) are in the same orbit under the \( \mathbb{Z}^n \)-action, then it is not hard to prove that \( S(\sigma) = S(\tau) \), see [18]. Let \( r \in \mathbb{N} \). A permutation \( \pi \in \mathcal{S}_n \) is said to be \( r \)-stack sortable if \( S^r(\pi) = 12 \cdots n \). Denote by \( \mathcal{S}_n^r \) the set of \( r \)-stack sortable permutations in \( \mathcal{S}_n \). Hence \( \mathcal{S}_n^r \) is invariant under the \( \mathbb{Z}^n_r \)-action for all \( n, r \in \mathbb{N} \), so Corollary 3.3 applies to prove that for all \( n, r \in \mathbb{N} \)

\[ A(\mathcal{S}_n^r; x) = \sum_{i=0}^{[n/2]} \gamma_i(\mathcal{S}_n^r)x^i(1+x)^{n-1-2i}, \]

where
\[ \gamma_i(\mathcal{S}_n^r) = 2^{-n+1+2i} \{ \pi \in \mathcal{S}_n^r : \text{peak}(\pi) = i \} \].

Unimodality and symmetry of \( A(\mathcal{S}_n^r; x) \) was first proved by Bona [7]. Bona conjectured that \( A(\mathcal{S}_n^r; x) \) is real-rooted for all \( n, r \in \mathbb{N} \). This conjecture remains open for all \( 3 \leq r \leq n-3 \), see [18].

More generally, if \( A \subseteq \mathcal{S}_n \), then the polynomial
\[ \sum_{\pi \in \mathcal{S}_n} x^{\text{des}(\pi)} \]

is \( \gamma \)-nonnegative.

Postnikov, Reiner and Williams [81] modified the \( \mathbb{Z}^n \)-action to prove Gal’s conjecture (see Conjecture 3.6) for so called chordal nestohedra.

In [88], Shareshian and Wachs proved refinements of the \( \gamma \)-positivity of Eulerian polynomials. Let

\[ A_n(q, p, s, t) = \sum_{k=0}^{n} A_{n,k}(q, p, t)s^k = \sum_{\sigma \in \mathcal{S}_n} q^{\text{maj}(\sigma)}p^{\text{des}(\sigma)}t^{\text{exc}(\sigma)}s^{\text{fix}(\sigma)}, \]

where
\[ \text{exc}(\sigma) = |\{ i : \sigma(i) > i \}|, \]
\[ \text{fix}(\sigma) = |\{ i : \sigma(i) = i \}|, \]
\[ \text{maj}(\sigma) = \sum_{i : \sigma(i) > \sigma(i+1)} i. \]

**Theorem 3.4.** Let \( B_d = \{ t^k(1+t)^{d-2k} \}_{k=0}^{[d/2]} \).

1. The polynomial \( A_{n,0}(q, p, q^{-1}t) \) has coefficients in \( \mathbb{N}[q, p] \) when expanded in \( B_n \).
2. If \( 1 \leq k \leq n \), then \( A_{n,k}(q, 1, q^{-1}t) \) has coefficients in \( \mathbb{N}[q] \) when expanded in \( B_{n-k} \).
3. The polynomial \( A_n(q, 1, 1, q^{-1}t) \) has coefficients in \( \mathbb{N}[q] \) when expanded in \( B_{n-1} \).
Gessel [53] has conjectured a fascinating property which resembles $\gamma$-nonnegativity for the joint distribution of descents and inverse descents:

**Conjecture 3.5** (Gessel, [18, 53, 78]). If $n$ is a positive integer, then there are nonnegative numbers $c_n(k, j)$ for all $k, j \in \mathbb{N}$ such that

$$\sum_{\pi \in S_n} x^{\text{des}(\pi)} y^{\text{des}(\pi^{-1})} = \sum_{k, j \in \mathbb{N}, k + 2j \leq n-1} c_n(k, j)(1 + xy)^n - k - 1 - 2j. \quad (3.3)$$

The existence of integers $c_n(k, j)$ satisfying (3.3) follows from symmetry properties, see [78]. The open problem is nonnegativity.

### 3.2. $\gamma$-nonnegativity of $h$-polynomials.

In topological combinatorics the $\gamma$-vectors were introduced in the context of face numbers of simplicial complexes [15, 51]. The $f$-polynomial of a $(d-1)$-dimensional simplicial complex $\Delta$ is

$$f_\Delta(x) = \sum_{k=0}^{d} f_{k-1}(\Delta)x^k,$$

where $f_k(\Delta)$ is the number of $k$-dimensional faces in $\Delta$, and $f_{-1}(\Delta) := 1$. The $h$-polynomial is defined by

$$h_\Delta(x) = \sum_{k=0}^{d} h_k(\Delta)x^k = (1 - x)^d f_\Delta(x/(1 - x)), \quad \text{or equivalently,} \quad (3.4)$$

$$f_\Delta(x) = (1 + x)^d h_\Delta(x/(1 + x)).$$

Hence $f_\Delta(x)$ and $h_\Delta(x)$ contain the same information. If $\Delta$ is a $(d-1)$-dimensional homology sphere, then the Dehn–Sommerville relations (see [91]) tell us that $h_\Delta(x)$ is symmetric, so we may expand it in the basis $B_d$. Recall that a simplicial complex $\Delta$ is flag if all minimal non-faces of $\Delta$ have cardinality two. Motivated by the Charney–Davis conjecture below, Gal made the following intriguing conjecture:

**Conjecture 3.6** (Gal, [51]). If $\Delta$ is a flag homology sphere, then $h_\Delta(x)$ is $\gamma$-nonnegative.

Gal’s conjecture is true for dimensions less than five, see [51]. If $h_\Delta(x)$ is symmetric with center of symmetry $d/2$, then $h_\Delta(-1) = 0$ if $d$ is odd, and $h_\Delta(-1) = (-1)^{d/2}\gamma_{d/2}(\Delta)$ if $d$ is even. Hence Gal’s conjecture implies the Charney–Davis conjecture:

**Conjecture 3.7** (Charney–Davis, [30]). If $\Delta$ is a flag $(d-1)$-dimensional homology sphere, where $d$ is even, then $(-1)^{d/2}h_\Delta(-1)$ is nonnegative.

Postnikov, Reiner and Williams [81] proposed a natural extension of Conjecture 3.6.

**Conjecture 3.8.** If $\Delta$ and $\Delta'$ are flag homology spheres such that $\Delta'$ geometrically subdivides $\Delta$, then the $\gamma$-vector of $\Delta'$ is entry-wise larger or equal to the $\gamma$-vector of $\Delta$.

Conjecture 3.8 was proved for dimensions $\leq 4$ in a slightly stronger form by Athanasiadis [3]. In [3], Athanasiadis also proposes an analog of Gal’s conjecture for local $h$-polynomials.
3.3. **Barycentric subdivisions.** The collection of faces of a regular cell complex $\Delta$ are naturally partially ordered by inclusion; if $F$ and $G$ are open cells in $\Delta$, then $F \leq G$ if $F$ is contained in the closure of $G$, where we assume that the empty face is contained in every other face. A **Boolean cell complex** is a regular cell complex such that each interval $[\emptyset, F] = \{ G \in \Delta : G \leq F \}$ is isomorphic to a Boolean lattice. Hence simplicial complexes are Boolean. The **barycentric subdivision**, $\text{sd}(\Delta)$, of a Boolean cell complex, $\Delta$, is the simplicial complex whose $(k-1)$-dimensional faces are strictly increasing flags $F_1 < F_2 < \cdots < F_k$,

where $F_j$ is a nonempty face of $\Delta$ for each $1 \leq j \leq k$. The $f$-polynomials and $h$-polynomials for cell complexes are defined just as for simplicial complexes.

Brenti and Welker [27] investigated positivity properties, such as real-rootedness and $\gamma$-positivity, of the $h$-polynomials of complexes under taking barycentric subdivisions. This was done by using analytic properties — obtained in [16,27] — of the linear operator that takes the $f$-polynomial of a Boolean complex to the $f$-polynomial of its barycentric subdivision. These analytic properties will be discussed in Section 7.1. In this section we describe the topological consequences of the analytic properties.

Let $E : \mathbb{R}[x] \to \mathbb{R}[x]$ be the linear operator defined by its image on the binomial basis:

$$E \left( \binom{x}{k} \right) = x^k, \quad \text{for all } k \in \mathbb{N},$$

where $\binom{x}{k} = \frac{x(x-1) \cdots (x-k+1)}{k!}$. The operator $E$ appears in several combinatorial settings. Using the binomial theorem one sees

$$E(f)(x) = \sum_{n=0}^{\infty} f(n) \frac{x^n}{(1+x)^{n+1}}.$$

It follows e.g. from the theory of $P$–partitions (or from [1,4] and induction) that

$$E(x^n) = E_n(x) = \sum_{k=1}^{n} k! S(n,k) x^k, \quad \text{for all } n \geq 1,$$

where $\{S(n,k)\}_{k=0}^{n}$ are the Stirling numbers of the second kind, see [94,102].

The following lemma was proved by Brenti and Welker [27].

**Lemma 3.9.** For any Boolean cell complex $\Delta$,

$$f_{\text{sd}(\Delta)} = E(f_{\Delta}).$$

**Proof.** By definition

$$f_{\text{sd}(\Delta)}(x) = \sum_{F \in \Delta} W_F(x),$$

where $W_\emptyset = 1$ and

$$W_F(x) = \sum_{k=1}^{\dim F+1} x^k \left[ \emptyset < F_1 < \cdots < F_k = F \right],$$

if $F \neq \emptyset$. Since $\Delta$ is Boolean, there is a one–to–one correspondence between flags $\emptyset < F_1 < \cdots < F_k = F$, $1 \leq k \leq \dim F + 1$, and ordered set-partitions of $[n]$, where $n = \dim F + 1$. Hence $W_F(x) = E_n(x) = E(x^n)$, and the lemma follows. \qed
Lemma 3.10. Let $\Delta$ be a $(d-1)$-dimensional Boolean cell complex. If $h_\Delta(x)$ is symmetric, then so is $h_{\text{sd}(\Delta)}(x)$.

**Proof.** By (3.4), $h_\Delta(x)$ is symmetric if and only if $(-1)^d f_\Delta(-1 - x) = f_\Delta(x)$. Let $I : \mathbb{R}[x] \to \mathbb{R}[x]$ be the algebra automorphism defined by $I(x) = -1 - x$. It was observed in [16, Lemma 4.3] that

$$I \circ E = E \circ I,$$

from which the lemma follows. □

Corollary 3.11 ([27]). Let $\Delta$ be a Boolean cell complex. If the $h$-polynomial of $\Delta$ has nonnegative coefficients, then all zeros of $h_{\text{sd}(\Delta)}(x)$ are nonpositive and simple.

If $h_\Delta(x)$ is also symmetric, then $h_{\text{sd}(\Delta)}(x)$ is $\gamma$-nonnegative.

**Proof.** The first conclusion follows immediately from Theorem 7.7, Lemma 3.9 and (3.4). The second conclusion follows from Remark 3.1 and Lemma 3.10. □

The second conclusion of Corollary 3.11 was strengthened in [71], where it was shown that with the same hypothesis, the $\gamma$-vector of $\text{sd}(\Delta)$ is the $f$-vector of a balanced simplicial complex.

If $\Delta$ is a Boolean cell complex and $k$ is a positive integer, let $\text{sd}^k(\Delta)$ be the simplicial complex obtained by a $k$-fold application of the subdivision operator $\text{sd}$. Most of the following corollary appears in [27].

Corollary 3.12. Let $\Delta$ be a $(d-1)$-dimensional Boolean cell complex with reduced Euler characteristic $\bar{\chi}(\Delta)$, where $d \geq 2$. There exists a number $N(\Delta)$ such that

1. all zeros of $h_{\text{sd}^n(\Delta)}(x)$ are real and simple for all $n \geq N(\Delta)$,
2. if $(-1)^d \bar{\chi}(\Delta) \geq 0$, then all zeros of $h_{\text{sd}^n(\Delta)}(x)$ are nonpositive and simple for all $n \geq N(\Delta)$,
3. if $(-1)^d \bar{\chi}(\Delta) < 0$, then all zeros of $h_{\text{sd}^n(\Delta)}(x)$ except one are nonpositive and simple for all $n \geq N(\Delta)$.

Moreover,

$$\lim_{n \to \infty} \frac{1}{d^n} f_{\text{sd}^n(\Delta)}(x) = f_{d-1}(\Delta) p_d(x),$$

where $p_d(x)$ is the unique monic degree $d$ eigenpolynomial of $E$ (see Theorem 7.8).

**Proof.** The identity (3.6) follows from the proof of Theorem 7.8 by choosing $f = f_\Delta(x)/f_{d-1}(\Delta)$. By Theorem 7.8, all zeros of $p_d(x)$ are real, simple and lie in the interval $[-1, 0]$. In view of (3.6), all zeros of $f_{\text{sd}^n(\Delta)}(x)$ will be real and simple for $n$ sufficiently large. The same holds for $h_{\text{sd}^n(\Delta)}(x)$ by (3.4).

Assume $(-1)^d \bar{\chi}(\Delta) \geq 0$. By Theorem 7.8, $p_d(0) = p_d(-1) = 0$. Since $f_{\text{sd}^n(\Delta)}(-1) = f_\Delta(-1) = \bar{\chi}(\Delta)$, we see by (3.6) that for all $n$ sufficiently large all zeros of $f_{\text{sd}^n(\Delta)}(x)$ are simple and lie in $[-1, 0]$ (since $f_{\text{sd}^n(\Delta)}(x)$ has the correct sign to the left of $-1$). By (3.4) this is equivalent to (2). Statement (3) follows similarly. □

Corollary 3.13. Let $\Delta$ be a $(d-1)$-dimensional Boolean cell complex such that $h_\Delta(x)$ is symmetric and $(-1)^d \bar{\chi}(\Delta) \geq 0$. Then there is a number $N(\Delta)$ such that $h_{\text{sd}^n(\Delta)}(x)$ is $\gamma$-nonnegative whenever $n \geq N(\Delta)$.

**Proof.** Combine Remark 3.1, Lemma 3.10 and Corollary 3.12. □
3.4. Unimodality of \( h^*\)-polynomials. Let \( P \subset \mathbb{R}^n \) be an \( m \)-dimensional integral polytope, i.e., all vertices have integer coordinates. Ehrhart \,[43,44]\ proved that the function
\[
i(P,r) = |rP \cap \mathbb{Z}^n|,
\]
which counts the number of integer points in the \( r \)-fold dilate of \( P \), is a polynomial in \( r \) of degree \( m \). It follows that we may write
\[
\sum_{r=0}^{\infty} i(P,r)x^r = \frac{h_0^*(P) + h_1^*(P)x + \cdots + h_m^*(P)x^m}{(1-x)^{m+1}},
\]
(3.7)
Stanley \,[89]\ proved that the coefficients of the polynomial, \( h_p^*(x) \), in the numerator of (3.7) are nonnegative, and Hibi \,[59]\ conjectured that \( h_p^*(x) \) is unimodal whenever it is symmetric. Hibi \,[59]\ proved the conjecture for \( n \leq 5 \). However Payne and Mustaţă \,[69,74]\ found counterexamples to Hibi’s conjecture for each \( n \geq 6 \). Let us mention a weaker conjecture that is still open. An integral polytope \( P \) is Gorenstein if \( h_p^*(x) \) is symmetric, and \( P \) is integrally closed if each integer point in \( rP \) may be written as a sum of \( r \) integer points in \( P \), for all \( r \geq 1 \).

**Conjecture 3.14** (Ohsugi–Hibi, \,[73]\). If \( P \) is a Gorenstein and integrally closed integral polytope, then \( h_p^*(x) \) is unimodal.

Inspired by work of Reiner and Welker \,[83]\, Athanasiadis \,[2]\ provided conditions on an integral polytope \( P \) which imply that \( h_p^*(x) \) is the \( h \)-polynomial of the boundary complex of a simplicial polytope. Hence, by the \( g \)-theorem (see \,[91]\), \( h_p^*(x) \) is unimodal. Athanasiadis used this result to prove the following conjecture of Stanley. An **integer stochastic matrix** is a square matrix with nonnegative integer entries having all row- and column sums equal to each other. Let \( H_n(r) \) be the number of \( n \times n \) integer stochastic matrices with row- and column sums equal to \( r \). The function \( r \mapsto H_n(r) \) is the Ehrhart polynomial of the integral polytope \( P_n \) of real doubly stochastic matrices. Stanley \,[91]\ conjectured that \( h_p^*(x) \) is unimodal for all positive integers \( n \), and Athanasiadis’ proof of Stanley’s conjecture was the main application of the techniques developed in \,[2]\. Subsequently Bruns and Römer \,[28]\ generalized Athanasiadis results to the following general theorem.

**Theorem 3.15.** Let \( P \) be a Gorenstein integral polytope such that \( P \) has a regular unimodular triangulation. Then \( h_p^*(x) \) is the \( h \)-polynomial of the boundary complex of a simplicial polytope. In particular, \( h_p^*(x) \) is unimodal.

4. **Log–concavity and matroids**

Several important sequences associated to matroids have been conjectured to be log–concave. Progress on these conjectures have been very limited until the recent breakthrough of Huh and Huh–Katz \,[61,62]\. Recall that the **characteristic polynomial** of a matroid \( M \) is defined as
\[
\chi_M(x) = \sum_{F \in L_M} \mu(\emptyset,F)x^{r(M) - r(F)} = \sum_{k=0}^{r} (-1)^k w_k(M)x^{r(M) - k},
\]
where \( L_M \) is the lattice of flats, \( \mu \) its Möbius function, \( r \) is the rank function of \( M \) and \( \{(-1)^k w_k(M)\}_{k=0}^{r(M)} \) are the **Whitney numbers of the first kind**. The sequence \( \{w_k(M)\}_{k=0}^{r(M)} \) is nonnegative, and it was conjectured by Rota and Heron to be...
unimodal. Welsh later conjectured that \( \{ w_k(M) \}_{k=0}^{r(M)} \) is log–concave. It is known that \( \chi_M(1) = 0 \). Define the reduced characteristic polynomial by

\[
\bar{\chi}_M(x) = \chi_M(x)/(x - 1) =: \sum_{k=0}^{r-1} (-1)^k v_k(M)x^{r(M)-1-k}.
\]

Note that if \( \{ v_k(M) \}_{k=0}^{r(M)-1} \) is log–concave, then so is \( \{ w_k(M) \}_{k=0}^{r(M)} \), see [90].

**Theorem 4.1** (Huh–Katz, [62]). If \( M \) is representable over some field, then the sequence \( \{ v_k(M) \}_{k=0}^{r(M)-1} \) is log–concave.

Since the chromatic polynomial of a graph is the characteristic polynomial of a representable matroid we have the following corollary:

**Corollary 4.2** (Huh, [61]). Chromatic polynomials of graphs are log–concave.

Let

\[
f_M(x) = \sum_{k=0}^{r(M)} (-1)^k f_k(M)x^{r(M)-k},
\]

where \( f_k(M) \) is the number of independent sets of \( M \) of cardinality \( k \). Hence \( f_M(x) \) is the (signed) \( f \)-polynomial of the independence complex of \( M \). Now, \( f_M(x) = \bar{\chi}_M(x) \), where \( M \times e \) is the free coextension of \( M \), see [29, 65]. Also if \( M \) is representable over some field, then so is \( M \times e \). Hence the following corollary is a consequence of Theorem 4.1

**Corollary 4.3.** If \( M \) is representable over some field, then \( \{ f_k(M) \}_{k=0}^{r(M)} \) is log–concave.

This corollary, first noted by Lenz [65], verifies the weakest version of Mason’s conjecture below for the class of representable matroids.

**Conjecture 4.4** (Mason). Let \( M \) be a matroid and \( n = f_1(M) \). The following sequences are log–concave:

\[
\{ f_k(M) \}_{k=0}^{r(M)} , \quad \{ k! f_k(M) \}_{k=0}^{r(M)} , \quad \text{and} \quad \left\{ f_k(M)/(\binom{n}{k}) \right\}_{k=0}^{r(M)}.
\]

The proofs in [61, 62] use involved algebraic machinery which falls beyond the scope of this survey. It is unclear if the method can be extended to the case of non–representable matroids.

### 5. Infinite Log–Concavity

Consider the operator \( \mathcal{L} \) on sequences \( \mathcal{A} = \{ a_k \}_{k=0}^{\infty} \subset \mathbb{R} \) defined by \( \mathcal{L}(\mathcal{A}) = \{ b_k \}_{k=0}^{\infty} \), where

\[
b_0 = a_0^2 \quad \text{and} \quad b_k = a_k^2 - a_{k-1}a_{k+1}, \quad \text{for} \; k \geq 1.
\]

This definition makes sense for finite sequences by regarding these as infinite sequences with finitely many nonzero entries. Hence a sequence \( \mathcal{A} \) is log–concave if and only if \( \mathcal{L}(\mathcal{A}) \) is a nonnegative sequence. A sequence is \( k \)-fold log–concave if \( \mathcal{L}^j(\mathcal{A}) \) is a nonnegative sequence for all \( 0 \leq j \leq k \). A sequence is infinitely log–concave if it is \( k \)-fold log–concave for all \( k \geq 1 \). Although similar notions were studied by Craven and Csordas [37, 38], the following questions asked by Boros and Moll [13] spurred the interest in infinite log-concavity in the combinatorics community:
(A) For $m \in \mathbb{N}$, let $\{d_\ell(m)\}_{\ell=0}^m$ be defined by
\[
d_\ell(m) = 4^{-m} \sum_{k=\ell}^m 2^k \binom{2m-2k}{m-k} \binom{m+k}{\ell} k^k.
\]
Is the sequence $\{d_\ell(m)\}_{\ell=0}^m$ infinitely log-concave?

(B) For $n \in \mathbb{N}$, is the sequence $\{\binom{n}{k}\}_{n,k=0}^\infty$ infinitely log-concave?

Question (A) is still open. However, Chen et al. [31] proved 3-fold log-concavity of $\{d_\ell(m)\}_{\ell=0}^m$ by proving a related conjecture of the author which implies 3-fold log-concavity of $\{d_\ell(m)\}_{\ell=0}^m$, for each $m \in \mathbb{N}$, by the work of Craven and Csordas [38].

In connection to (B), Fisk [47], McNamara–Sagan [68], and Stanley [95] independently conjectured the next theorem from which (B) easily follows. We may consider $\mathcal{L}$ to be an operator on the generating function of the sequence, i.e.,
\[
\mathcal{L} \left( \sum_{k=0}^\infty a_k x^k \right) = \sum_{k=0}^\infty (a_{k+1}^2 - a_k a_k - 1) x^k.
\]

**Theorem 5.1** ([19]). If $f(x) = \sum_{k=0}^\infty a_k x^k$ is a polynomial with real and nonpositive zeros only, then so is $\mathcal{L}(f)$. In particular, the sequence $\{a_k\}_{k=0}^\infty$ is infinitely log-concave.

The proof of Theorem 5.1 uses multivariate techniques, and will be given in Section 9.4.

There is a simple criterion on a nonnegative sequence $A = \{a_k\}_{k=0}^\infty$ that guarantees infinite log-concavity [38,68]. Namely
\[
a_k^2 \geq r a_{k-1} a_{k+1}, \quad \text{for all } k \geq 1,
\]
where $r \geq (3 + \sqrt{5})/2$.

McNamara and Sagan [68] conjectured that the operator $\mathcal{L}$ preserves the class of PF sequences. In particular they conjectured that the columns of Pascal’s triangle $\{\binom{n+k}{k}\}_{n,k=0}^\infty$, where $k \in \mathbb{N}$, are infinitely log–concave. In [20], Chasse and the author found counterexamples to the first mentioned conjecture and proved the second. They considered PF sequences that are interpolated by polynomials, i.e., PF sequences $\{p(k)\}_{k=0}^\infty$ where $p$ is a polynomial, and asked when classes of such sequences are preserved by $\mathcal{L}$.

Let $\mathcal{P}$ be the following class of PF sequences which are interpolated by polynomials
\[
\{\{p(k)\}_{k=0}^\infty \in \text{PF} : p(x) \in \mathbb{R}[x] \text{ and } p(-j) = p(-j+1) = 0 \text{ for some } j \in \{0,1,2\} \}.
\]

**Theorem 5.2** ([20]). The operator $\mathcal{L}$ preserves the class $\mathcal{P}$. In particular each sequence in $\mathcal{P}$ is infinitely log–concave.

Note that for each $k \in \mathbb{N}$, $\{\binom{n+k}{k}\}_{n=0}^\infty \in \mathcal{P}$. The following corollary solves the above mentioned conjecture of McNamara and Sagan.

**Corollary 5.3.** The columns of Pascal’s triangle are infinitely log–concave, i.e., for each $k \in \mathbb{N}$, the sequence $\{\binom{n+k}{k}\}_{n=0}^\infty$ is infinitely log–concave.

Let us end this section with an interesting open problem posed by Fisk [47].
are roughly 4 there are no counterexamples to the Neggers conjecture on so the 10-vertex counterexamples are minimal among all posets, but there could be other computer search that there are no counterexamples to the Stanley conjecture with smallest counterexamples among all posets. In this direction, we have confirmed by com-
or derivations.

In a second search, we discovered that the smallest narrow counterexamples for Stanley’s conjecture were first found by the author in [14], and shortly thereafter naturally labeled counterexamples were found by Stembridge in [97], see Fig. 2.

Problem 1. Suppose all zeros of \( \sum_{k=0}^{n} a_k x^k \) are nonpositive. If \( d \in \mathbb{N} \), are all zeros of

\[
\sum_{k=0}^{n} \det(a_{k+i-j})_{i,j=0}^d \cdot x^k,
\]

where \( a_i = 0 \) if \( i \notin \{0, \ldots, n\} \), nonpositive?

Hence the case \( d = 1 \) of Problem 1 is Theorem 5.1

6. The Neggers–Stanley conjecture

It is natural to ask if the real–rootedness of the Eulerian polynomials may be extended to generating polynomials of linear extensions of any poset. Define a labeled poset to be a poset of the form \( P = ([n], \leq_P) \), where \( n \) is a positive integer. The Jordan–Hölder set of \( P \),

\[
\mathcal{L}(P) = \{ \sigma \in \mathfrak{S}_n : i < j \text{ whenever } \sigma(i) <_P \sigma(j) \},
\]

is the set of all linear extensions of \( P \). Here \( < \) denotes the usual order on the integers. The \( P \)-Eulerian polynomial is defined by

\[
W_P(x) = \sum_{\sigma \in \mathcal{L}(P)} x^{\text{des}(\sigma)+1}.
\]

Recall that \( P \) is naturally labeled if \( i < j \) whenever \( i <_P j \). Neggers [70] conjectured in 1978 that \( W_P(x) \) is real–rooted for any naturally labeled poset \( P \), and Stanley extended the conjecture to all labeled posets in 1986, see [24, 25, 102]. Counterexamples to Stanley’s conjecture were first found by the author in [14], and shortly thereafter naturally labeled counterexamples were found by Stembridge in [97], see Fig. 2.

However, this does not seem to be the end of the story. Recall that a poset \( P \) is graded if all maximal chains in \( P \) have the same size.

Theorem 6.1 (Reiner and Welker, [84]). If \( P \) is a graded and naturally labeled poset, then \( W_P(x) \) is unimodal.
Reiner and Welker proved Theorem 6.1 by associating to $P$ a simplicial polytope whose $h$–polynomial is equal to $W_P(x)$, and then invoking the $g$–theorem for simplicial polytopes.

Theorem 6.1 was refined in [15, 18] to establish $\gamma$–nonnegativity for the $P$–Eulerian polynomials of a class of labeled posets which contain the graded and naturally labeled posets. Let $E(P) = \{(i, j) : j \text{ covers } i\}$ be the Hasse diagram of a labeled poset $P$. Define a function $\epsilon : E(P) \to \{-1, 1\}$, by

$$
\epsilon(i, j) = \begin{cases} 
1 & \text{if } i < j, \\
-1 & \text{if } j < i.
\end{cases}
$$

A labeled poset $P$ is sign–graded if for all maximal chains $x_0 <_P x_1 <_P \cdots <_P x_k$ in $P$, the quantity

$$r = \sum_{i=1}^k \epsilon(x_{i-1}, x_i)$$

is the same, see Fig 3. Note that a naturally labeled poset is sign–graded if and only if it is graded.

**Figure 3.** A sign–graded poset of rank 1.

**Theorem 6.2.** If $P$ is sign–graded, then $W_P(x)$ is $\gamma$–nonnegative.

Two proofs are known for Theorem 6.2. The first proof [15] uses a partitioning of $\mathcal{L}(P)$ into Jordan–Hölder sets of refinements of $P$ for which $\gamma$–positivity is easy to prove. The second proof [18] uses an extension to $\mathcal{L}(P)$ of the $\mathbb{Z}_2^n$–action described in Section 3.1.

Here are two questions left open regarding the Neggers–Stanley conjecture.

**Question 1.** Are the coefficients of $P$–Eulerian polynomials log–concave or unimodal?

**Question 2.** Are $P$–Eulerian polynomials of graded (or sign–graded) posets real–rooted?

The work in [15] was generalized by Stembridge [98] to certain Coxeter cones. Let $\Phi$ be a finite root system in a real Euclidian space $V$ with inner product $\langle \cdot, \cdot \rangle$. A Coxeter cone is a closed convex cone of the form

$$\Delta(\Psi) = \{\mu \in V : \langle \mu, \beta \rangle \geq 0 \text{ for all } \beta \in \Psi\},$$

where $\Psi \subseteq \Phi$. This cone is a closed union of cells of the Coxeter complex defined by $\Phi$, so it forms a simplicial complex which we identify with $\Delta(\Psi)$. A labeled Coxeter cone is a cone of the form

$$\Delta(\Psi, \lambda) = \{\mu \in \Delta(\Psi) : \langle \mu, \beta \rangle > 0 \text{ for all } \beta \in \Psi \text{ with } \langle \lambda, \beta \rangle < 0\},$$
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where $\Delta(\Psi)$ is a Coxeter cone and $\lambda \in V$. Hence $\Delta(\Psi, \lambda)$ may be identified with a relative complex inside $\Delta(\Psi)$. When $\Phi$ is crystallographic, Stembridge defines what it means for a (labeled) Coxeter cone to be graded. In type $A$, graded labeled Coxeter cones correspond to sign–graded posets.

**Theorem 6.3** (Stembridge, [98]). The $h$-vectors of graded labeled Coxeter cones are $\gamma$-nonnegative.

7. Preserving real–rootedness

If a sequence of polynomials satisfies a linear recursion, then to prove that the polynomials are real–rooted it is sufficient to prove that the defining recursion “preserves” real–rootedness. Hence it is natural, from a combinatorial point of view, to ask which linear operators on polynomials preserve real–rootedness. This question has a rich history that goes back to the work of Jensen, Laguerre and Pólya, see the survey [39]. In his thesis, Brenti [24] studied this question focusing on operators occurring naturally in combinatorics.

Let us recall Pólya and Schur’s [80] celebrated characterization of diagonal operators preserving real–rootedness. A sequence $\Lambda = \{\lambda_k\}_{k=0}^{\infty}$ of real numbers is called a multiplier sequence (of the first kind), if the linear operator $T_{\Lambda} : \mathbb{R}[x] \to \mathbb{R}[x]$ defined by

$$T_{\Lambda}(x^k) = \lambda_k x^k, \quad k \in \mathbb{N},$$

preserves real–rootedness.

**Theorem 7.1** (Pólya and Schur, [80]). Let $\Lambda = \{\lambda_k\}_{k=0}^{\infty}$ be a sequence of real numbers, and let

$$G_{\Lambda}(x) = \sum_{k=0}^{\infty} \frac{\lambda_k}{k!} x^k,$$

be its exponential generating function. The following assertions are equivalent:

1. $\Lambda$ is a multiplier sequence.
2. For all nonnegative integers $n$, the polynomial

$$T((x + 1)^n) = \sum_{k=0}^{n} \binom{n}{k} \lambda_k x^k,$$

is real–rooted, and all its zeros have the same sign.
3. Either $G_{\Lambda}(x)$ or $G_{\Lambda}(-x)$ defines an entire function that can be written as

$$G_{\Lambda}(\pm x) = C x^m e^{ax} \prod_{k=1}^{\infty} (1 + \alpha_k x),$$

where $m \in \mathbb{N}$, $C \in \mathbb{R}$, $a \geq 0$, $\alpha_k \geq 0$ for all $k \in \mathbb{N}$, and $\sum_{k=1}^{\infty} \alpha_k < \infty$.
4. $G_{\Lambda}(x)$ defines an entire function which is the limit, uniform on compact subsets of $\mathbb{C}$, of real–rooted polynomials whose zeros all have the same sign.

**Example 7.1.** Let $\Lambda = \{1/k!\}_{k=0}^{\infty}$. Then $T_{\Lambda}((x + 1)^n) = L_n(-x)$, where $L_n(x)$ is the $n$th Laguerre polynomial. Since orthogonal polynomials are real–rooted (REF) we see that (2) of Theorem 7.1 is satisfied, and thus $\Gamma$ is a multiplier sequence.

Only recently a complete characterization of linear operators preserving real–rootedness was obtained by Borcea and the author in [10]. This characterization is
in terms of a natural extension of real–rootedness to several variables. A polynomial $P(x_1, \ldots, x_m) \in \mathbb{C}[x_1, \ldots, x_m]$ is called stable if
\[
\text{Im}(x_1) > 0, \ldots, \text{Im}(x_m) > 0 \text{ implies } P(x_1, \ldots, x_n) \neq 0.
\]
By convention we also consider the identically zero polynomial to be stable. Hence a univariate real polynomial is stable if and only if it is real–rooted. Let $\alpha_1 \leq \cdots \leq \alpha_n$ and $\beta_1 \leq \cdots \leq \beta_m$ be the zeros of two real–rooted polynomials. We say that these zeros interlace if
\[
\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \beta_2 \leq \cdots \text{ or } \beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \cdots.
\]
By convention, the “zeros” of any two polynomials of degree 0 or 1 interlace. Interlacing zeros is characterized by a linear condition as the following theorem which is often attributed to Obreschkoff describes:

**Theorem 7.2** (Satz 5.2 in [72]). Let $f, g \in \mathbb{R}[x] \setminus \{0\}$. Then the zeros of $f$ and $g$ interlace if and only if all polynomials in the linear space
\[
\{\alpha f + \beta g : \alpha, \beta \in \mathbb{R}\}
\]
are real–rooted.

Let $\mathbb{R}_n[x] = \{p \in \mathbb{R}[x] : \deg p \leq n\}$. The symbol of a linear operator $T : \mathbb{R}_n[x] \to \mathbb{R}[x]$ is the bivariate polynomial
\[
G_T(x, y) = T((x + y)^n) := \sum_{k=0}^{n} \binom{n}{k} T(x^k)y^{n-k} \in \mathbb{R}[x, y].
\]

**Theorem 7.3** ([10]). Let $T : \mathbb{R}_n[x] \to \mathbb{R}[x]$ be a linear operator. Then $T$ preserves real–rootedness if and only if (1), (2) or (3) below is satisfied.

1. $T$ has rank at most two and is of the form
   \[
   T(p) = \alpha(p)f + \beta(p)g,
   \]
   where $\alpha, \beta : \mathbb{R}[x] \to \mathbb{R}$ are linear functionals and $f, g$ are real–rooted polynomials whose zeros interlace.
2. $G_T(x, y)$ is stable.
3. $G_T(x, -y)$ is stable.

**Example 7.2.** The operators of type (1) are the ones achieved by Theorem 7.2. An example of an operator of type (2) is $T = d/dx$, because then $G_T(x, y) = n(x + y)^{n-1}$. An example of an operator of type (3) is the algebra automorphism, $S : \mathbb{R}[x] \to \mathbb{R}[x]$, defined by $S(x) = -x$. Indeed $T$ is of type (2) if and only if $T \circ S$ is of type (3).

To illustrate how Theorem 7.3 may be used let us give a simple example from combinatorics.

**Example 7.3.** The Eulerian polynomials satisfy the recursion $A_{n+1}(x) = T_n(A_n(x))$, where
\[
T_n = x(1-x) \frac{d}{dx} + (n+1)x,
\]
see [94]. The symbol of $T_n : \mathbb{R}_n[x] \to \mathbb{R}[x]$ is
\[
T_n((x + y)^n) = x(x + y)^{n-1}(x + (n+1)y + n),
\]
which is trivially stable. Hence $A_n(x)$ is real–rooted for all $n \in \mathbb{N}$ by Theorem 7.3. This was first proved by Frobenius [50].
A characterization of stable polynomials in two variables — and hence of the symbols of preservers of real-rootedness — follows from Helton and Vinnikov’s characterization of real-zero polynomials in [58], see [11].

**Theorem 7.4.** Let \( P(x, y) \in \mathbb{R}[x, y] \) be a polynomial of degree \( d \). Then \( P \) is stable if and only if there exists three real symmetric \( d \times d \) matrices \( A, B \) and \( C \) and a real number \( r \) such that
\[
P(x, y) = r \cdot \det(xA + yB + C),
\]
where \( A \) and \( B \) are positive semidefinite and \( A + B \) is the identity matrix.

For the unbounded degree analog of Theorem 7.3 we define the symbol of a linear operator \( T : \mathbb{R}[x] \to \mathbb{R}[x] \) to be the formal powers series
\[
\tilde{G}_T(x, y) = T(e^{-xy}) := \sum_{n=0}^{\infty} (-1)^n \frac{T(x^n)}{n!} y^n \in \mathbb{R}[x][[y]].
\]
The Laguerre–Pólya class, \( \mathcal{L}P_n \), is defined to be the class of real entire functions in \( n \) variables which are the uniform limits on compact subsets of \( \mathbb{C} \) of real stable polynomials. For example \( \exp(-x_1x_2 - x_3x_4 + 2x_5) \in \mathcal{L}P_5 \) since it is the limit of the stable polynomials
\[
\left(1 - \frac{x_1x_2}{n}\right)^n \left(1 - \frac{x_3x_4}{n}\right)^n \left(1 + \frac{x_5}{n}\right)^n.
\]

**Theorem 7.5 ([10]).** Let \( T : \mathbb{R}[x] \to \mathbb{R}[x] \) be a linear operator. Then \( T \) preserves real-rootedness if and only if (1), (2) or (3) below is satisfied.

(1) \( T \) has rank at most two and is of the form
\[
T(p) = \alpha(p)f + \beta(p)g,
\]
where \( \alpha, \beta : \mathbb{R}[x] \to \mathbb{R} \) are linear functionals and \( f, g \) are real-rooted polynomials whose zeros interlace.

(2) \( \tilde{G}_T(x, y) \in \mathcal{L}P_2 \).

(3) \( \tilde{G}_T(x, -y) \in \mathcal{L}P_2 \).

There are, as of yet, no analogs of Theorems 7.3 and 7.5 for linear operators that preserve the property of having all zeros in a prescribed interval (other than \( \mathbb{R} \) itself).

**Problem 2.** Let \( I \subset \mathbb{R} \) be an interval. Characterize all linear operators on polynomials that preserve the property of having all zeros in \( I \).

For polynomials appearing in combinatorics the case when \( I = (-\infty, 0] \) is the most important.

### 7.1. The subdivision operator

An example of an operator of the kind appearing in Problem 2 is the “subdivision” operator \( E : \mathbb{R}[x] \to \mathbb{R}[x] \) in Section 3.2. The following theorem by Wagner proved the Neggers–Stanley conjecture for series–parallel posets, see [102,103].

**Theorem 7.6 ([103]).** If all zeros of \( E(f) \) and \( E(g) \) lie in the interval \([-1, 0]\), then so does the zeros of \( E(fg) \).

As we have seen in Section 3.2, the next theorem has consequences in topological combinatorics.
Theorem 7.7. If
\[ f(x) = \sum_{k=0}^{d} h_k x^k (1 + x)^{d-k}, \]
where \( h_k \geq 0 \) for all \( 0 \leq k \leq d \), then all zeros of \( \mathcal{E}(f) \) are real, simple and located in \([-1,0]\).

The main part of the next theorem was proved by Brenti and Welker in \([27]\), while (2) was proved in \([41]\). We take the opportunity to give alternative simple proofs below.

Theorem 7.8. For each integer \( n \geq 2 \), \( \mathcal{E} \) has a unique monic eigen-polynomial, \( p_n(x) \), of degree \( n \).
Moreover,
1. all zeros of \( p_n(x) \) are real, simple and lie in the interval \([-1,0]\);
2. \( p_n(x) \) is symmetric around \(-\frac{1}{2}\), i.e.,
\[ (-1)^n p_n(-1 - x) = p_n(x). \]

Proof. Let \( n \geq 2 \). Consider the map \( \phi: [-1,0]^n \to [-1,0]^n \) defined as follows. Let \( \theta = (\theta_1, \ldots, \theta_n) \in [-1,0]^n \). Since \( \mathcal{E} \) preserves the property of having all zeros in \([-1,0]\) (Theorems 7.6 and 7.7), we may order the zeros of \( \mathcal{E}(x - \theta_1) \cdots (x - \theta_n) \) as \(-1 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq 0 \). Let \( \phi(\theta) := (\alpha_1, \ldots, \alpha_n) \). By Hurwitz’ theorem on the continuity of zeros \([82]\), \( \phi \) is continuous. Hence by Brouwer’s fixed point theorem \( \phi \) has a fixed point, which then corresponds to a degree \( n \) eigen-polynomial, \( p_n \), of \( \mathcal{E} \).
It follows by examining the leading coefficients that the corresponding eigenvalue is \( n! \).

Set \( p_0 := 1 \) and \( p_1 := x + \frac{1}{2} \). Let \( f \) be an arbitrary monic polynomial of degree \( n \geq 2 \), and let \( T = n!^{-1} \mathcal{E} \). Now by expanding \( f \) as a linear combination of \( \{p_k\}_{k=0}^{n} \),
\[ f = \sum_{i=0}^{n} a_i p_i, \]
we see that
\[ \lim_{k \to \infty} T^k(f) = \lim_{k \to \infty} \sum_{i=0}^{n} \left( \frac{n!}{n!} \right)^k a_i p_i = p_n, \]
since \( a_n = 1 \). Hence \( p_n \) is unique. By choosing \( f \) to be \([-1,0]-\)rooted, we see that \( T^k(f) \) is also \([-1,0]-\)rooted for all \( k \). By Hurwitz’ theorem, so is \( p_n \). Since \( p_n \) is \([-1,0]-\)rooted, it is certainly of the form displayed in Theorem 7.7. By Theorem 7.7 again, the zeros of \( p_n = n!^{-1} \mathcal{E}(p_n) \) are distinct.

Property (2) follows immediately from (3.5).

It is easy to see that the coefficients of \( p_n(x) \) are rational numbers for each \( n \geq 2 \).

Question 3. Is there a closed formula for \( p_n(x) \)? What are the generating functions
\[ A(x,y) = \sum_{n=0}^{\infty} p_n(x)y^n \quad \text{and} \quad B(x,y) = \sum_{n=0}^{\infty} \frac{p_n(x)}{n!}y^n. \]

Note that \( \mathcal{E}(B) = A \).
8. Common interleavers

A powerful technique for proving that families of polynomials are real–rooted is that of compatible polynomials. This was employed by Chudnovsky and Seymour [35] to prove a conjecture of Hamidoune and Stanley on the zeros of independence polynomials of clawfree graphs. Subsequently an elegant alternative proof was given by Lass [63], by proving a Mehler formula for independence polynomials of clawfree graphs. An independent set in a finite and simple graph $G = (V,E)$ is a set of pairwise non-adjacent vertices. The independence polynomial of a claw is $1 + 4x + 3x^2 + x^3$, which has two non–real zeros. A graph is clawfree if no induced subgraph is a claw. The next theorem was posed as a question by Hamidoune [57] and later as a conjecture by Stanley [92].

Theorem 8.1 ([35,63]). If $G$ is a clawfree graph, then all zeros of $I(G,x)$ are real.

Let $f,g \in \mathbb{R}[x]$ be two real–rooted polynomials with positive leading coefficients. We say that $f$ is an interleaver of $g$ (written $f \ll g$) if

$$\cdots \leq \alpha_2 \leq \beta_2 \leq \alpha_1 \leq \beta_1,$$

where $\{\alpha_i\}_{i=1}^n$ and $\{\beta_i\}_{i=1}^m$ are the zeros of $f$ and $g$, respectively. By convention we also write $0 \ll 0$, $0 \ll h$ and $h \ll 0$, where $h$ is any real–rooted polynomial with positive leading coefficient. If $f \ll g$ and $f \neq 0$, we say that $f$ is a proper interleaver of $g$. The polynomials $f_1(x), \ldots, f_m(x)$ are $k$–compatible, where $1 \leq k \leq m$, if

$$\sum_{j \in S} \lambda_j f_j(x)$$

is real–rooted whenever $S \subseteq [m]$, $|S| = k$ and $\lambda_j \geq 0$ for all $j \in S$. The following theorem was used in Chudnovsky and Seymour’s proof of Theorem 8.1.

Theorem 8.2 (Chudnovsky–Seymour, [35]). Suppose that the leading coefficients of $f_1(x), \ldots, f_m(x) \in \mathbb{R}[x]$ are positive. The following are equivalent.

1. $f_1(x), \ldots, f_m(x)$ are 2-compatible;
2. For all $1 \leq i < j \leq m$, $f_i(x)$ and $f_j(x)$ have a proper common interleaver;
3. $f_1(x), \ldots, f_m(x)$ have a proper common interleaver;
4. $f_1(x), \ldots, f_m(x)$ are $m$-compatible.

Theorem 8.2 is useful in situations when the polynomials of interest may be expressed as a nonnegative sums of similar polynomials. In order to prove that the polynomials of interest are real–rooted it then suffices to prove that the similar polynomials are 2–compatible.

A sequence $F_n = (f_i)_{i=1}^n$ of real–rooted polynomials is called interlacing if $f_i \ll f_j$ for all $1 \leq i < j \leq n$. Let $\mathcal{F}_n$ be the family of all interlacing sequences $(f_i)_{i=1}^n$ of polynomials, and let $\mathcal{F}_n^+$ be the family of $(f_i)_{i=1}^n \in \mathcal{F}_n$ such that $f_i$ has nonnegative coefficients for all $1 \leq i \leq n$. We are are interested in when an $m \times n$ matrix
$G = (G_{ij}(x))$ of polynomials maps $\mathcal{F}_n$ to $\mathcal{F}_m$ (or $\mathcal{F}^+_n$ to $\mathcal{F}^+_m$) by the action

$$G \cdot F_n = (g_1, \ldots, g_m)^T,$$

where $g_k = \sum_{i=0}^{n} G_{ki}f_i$ for all $1 \leq k \leq m$.

This problem was considered by Fisk [46, Chapter 3], who proved some preliminary results. Since this approach has been proved successful in combinatorial situations, see [86] where it was used to prove e.g. that the type $D$ Eulerian polynomials are real–rooted, we take the opportunity to give a complete characterization for the case of nonnegative polynomials.

**Lemma 8.3.** If $(f_i)_{i=1}^n$ and $(g_i)_{i=1}^n$ are two interlacing sequences of polynomials, then the polynomial

$$f_1g_n + f_2g_{n-1} + \cdots + f_ng_1$$

is real–rooted.

**Proof.** By Theorem 8.2 it suffices to prove that the sequence $(f_ig_{n+1-i})_{i=1}^n$ is 2-compatible. If $i < j$, then $f_ig_{n+1-j}$ is a common interleaver of $f_ig_{n+1-i}$ and $f_jg_{n+1-j}$. Hence the lemma follows from Theorem 8.2. \hfill \square

See [86] for a proof of the following lemma.

**Lemma 8.4.** Let $f$ and $g$ be two polynomials with nonnegative coefficients. Then $f \ll g$ if and only if for all $\lambda, \mu > 0$, the polynomial

$$(\lambda x + \mu)f + g$$

is real–rooted.

**Theorem 8.5.** Let $G = (G_{ij}(x))$ be an $m \times n$ matrix of polynomials. Then $G : \mathcal{F}^+_n \to \mathcal{F}^+_m$ if and only if

1. $G_{ij}$ has nonnegative coefficients for all $i \in [m]$ and $j \in [n]$, and
2. for all $\lambda, \mu > 0$, $1 \leq i < j \leq n$ and $1 \leq k < \ell \leq m$

$$(\lambda x + \mu)G_{kj}(x) + G_{\ell j}(x) \ll (\lambda x + \mu)G_{ki}(x) + G_{\ell i}(x). \tag{8.1}$$

**Proof.** Let

$$g_k = \sum_{i=0}^{n} G_{ki}f_i.$$

By Lemma 8.4 $G : \mathcal{F}^+_n \to \mathcal{F}^+_m$ if and only if for all $k < \ell$ and $\lambda, \mu > 0$

$$(\lambda x + \mu)g_k + g_\ell = \sum_{i=0}^{n} ((\lambda x + \mu)G_{ki} + G_{\ell i})f_i =: \sum_{i=0}^{n} h_{n+1-i}f_i$$

is real–rooted and has nonnegative coefficients. The sufficiency follows from Lemma 8.3 since if (8.1) holds, then the sequence $(h_i)_{i=1}^n$ is interlacing. To prove the necessity, let $i < j$ and $(f_r)_{r=1}^n$ be the interlacing sequence defined by

$$f_r(x) = \begin{cases} 1 & \text{if } r = i, \\ \alpha x + \beta & \text{if } r = j, \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha, \beta > 0$. Hence if $G : \mathcal{F}^+_n \to \mathcal{F}^+_m$, then $h_{n+1-i} + (\alpha x + \beta)h_{n+1-j}$ is real–rooted for all $\alpha, \beta > 0$. Thus $h_{n+1-j} \ll h_{n+1-i}$, by Lemma 8.4 which is (8.1). \hfill \square
Corollary 8.6. Let $G = (G_{ij})$ be an $m \times n$ matrix over $\mathbb{R}$. Then $G : F_n^+ \to F_m^+$ if and only if $G$ is TP$_2$, i.e., all minors of $G$ of size less than three are nonnegative.

Proof. By Theorem 8.5 we may assume that all entries of $G$ are nonnegative. Now

$$(\lambda x + \mu)G_{kj} + G_{\ell j} \ll (\lambda x + \mu)G_{ki} + G_{\ell i}$$

for all $\lambda, \mu > 0$ if and only if

$$xG_{kj} + G_{\ell j} \ll xG_{ki} + G_{\ell i},$$

which is seen to hold if and only if $G_{ki}G_{\ell j} \geq G_{\ell i}G_{kj}$.

If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ are integers such that $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq n$, let $G_\lambda = (g_\lambda^{ij}(x))$ be the $m \times n$ matrix with entries

$$g_\lambda^{ij}(x) = \begin{cases} x & \text{if } 1 \leq j \leq \lambda_i \\
1 & \text{otherwise.} \end{cases}$$

The following corollary was first proved in [86].

Corollary 8.7. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ are integers such that $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq n$, then $G_\lambda : F_n^+ \to F_m^+$.

Proof. The possible $2 \times 2$ sub-matrices of $G_\lambda$ are

$$\begin{pmatrix} x & x \\ x & x \end{pmatrix}, \begin{pmatrix} x & 1 \\ x & x \end{pmatrix}, \begin{pmatrix} x & 1 \\ x & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ x & x \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ x & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

By Theorem 8.5 we need to check (8.1) for these matrices. For example for the second matrix from the right we need to check

$$(\lambda + 1)x + \mu \ll x(\lambda x + \mu + 1),$$

for all $\lambda, \mu > 0$, which is equivalent to

$$-\frac{\mu + 1}{\lambda} \leq -\frac{\mu}{\lambda + 1},$$

which is certainly true. The other cases follows similarly.

Example 8.1. Let $n$ be a positive integer and define polynomials $A_{n,i}(x), i \in [n]$, by

$$A_{n,i}(x) = \sum_{\sigma \in S_n} x^{\text{des}(\sigma)}.$$

By conditioning on $\sigma(2) = k$, where $\sigma \in S_n$ and $\sigma(1) = i$, we see that

$$A_{n+1,i}(x) = \sum_{k \leq i} xA_{n,k}(x) + \sum_{k \geq i} A_{n,k}(x), \quad 1 \leq i \leq n + 1.$$

Hence if $A_n = (A_{n,i}(x))_{i=1}^n$, then

$$A_{n+1} = G_{(1,2,\ldots,n)} : A_n.$$

Since $A_2 = (1, x)$, we have by induction and Corollary 8.7 that $A_n$ is an interlacing sequence of polynomials for all $n \geq 2$. 
8.1. s-Eulerian polynomials. Corollary 8.7 was used by Savage and Visontai [86] to prove real-rootedness of a large family of $h^*$-polynomials. Let $s = \{s_i\}_{i=1}^n$ be a sequence of positive integers. Define an integral polytope $P_s$ by

$$P_s = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : 0 \leq \frac{x_1}{s_1} \leq \frac{x_2}{s_2} \leq \cdots \leq \frac{x_n}{s_n} \leq 1 \right\}.$$  

The s–Eulerian polynomial may be defined as the $h^*$-polynomial of $P_s$:

$$\sum_{k=0}^{\infty} s(P_s, k) x^k = \frac{E_s(x)}{(1-x)^{n+1}}.$$ 

Savage and Schuster [85] provided a combinatorial description of s-Eulerian polynomials. The s-inversion sequences are defined by

$$I_s = \{ \mathcal{E} = (e_1, \ldots, e_n) \in \mathbb{N}^n : e_i/s_i < 1 \text{ for all } 1 \leq i \leq n \}.$$ 

The ascent statistic on $I_s$ is defined as

$$\text{asc}(\mathcal{E}) = |\{i \in [n] : e_i/s_i < e_{i-1}/s_{i-1} \}|,$$

where $\mathcal{E} = (e_1, \ldots, e_n)$, $e_0 = 0$ and $s_0 = 1$.

**Theorem 8.8** ([85]).

$$E_s(x) = \sum_{\mathcal{E} \in I_s} x^{\text{asc}(\mathcal{E})}.$$ 

It turns out that several much studied families of polynomials in combinatorics are s–Eulerian polynomials for various $s$. For example the $n$th ordinary Eulerian polynomial corresponds to $s = (1, 2, \ldots, n)$, while the $n$th Eulerian polynomial of type $B$ corresponds to $s = (2, 4, \ldots, 2n)$. If $s = (s_1, \ldots, s_n)$, let

$$E_{s,i}(x) = \sum_{\mathcal{E} \in I_s, e_n = i} x^{\text{asc}(\mathcal{E})}.$$ 

It is not hard to see that the polynomials $E_{s,i}(x)$ satisfy the following recurrences which make them ideal for an application of Corollary 8.7.

**Lemma 8.9** ([86]). If $s = (s_1, \ldots, s_n)$, $n > 1$, is a sequence of positive integers and $0 \leq i < n$, then

$$E_{s,i}(x) = \sum_{j=0}^{t_i-1} x E_{s',j}(x) + \sum_{j=t_i}^{s_n-1} E_{s',j}(x),$$

where $s' = (s_1, \ldots, s_{n-1})$ and $t_i = \lceil is_{n-1}/s_n \rceil$.

An application of Corollary 8.7 proves the following theorem.

**Theorem 8.10** ([86]). If $s = (s_1, \ldots, s_n)$ is a sequence of positive integers, then $E_s(x)$ is real-rooted. Moreover if $n > 1$, then the sequence $\{E_{s,i}(x)\}_{i=0}^{s_n-1}$ is interlacing.
8.2. Eulerian polynomials for finite Coxeter groups. For undefined terminology on Coxeter groups we refer to [6]. Let \((W,S)\) be a Coxeter system. The \emph{length} of an element \(w \in W\) is the smallest number \(k\) such that
\[
w = s_1 s_2 \cdots s_k, \quad \text{where } s_i \in S \text{ for all } 1 \leq i \leq n.
\]
Let \(\ell_W(w)\) denote the length of \(w\). The \emph{(right) descent set} of \(w\) is
\[
D_W(w) = \{s \in S : \ell_W(ws) < \ell_W(w)\},
\]
and the \emph{descent number} is \(\text{des}_W(w) = |D_W(w)|\). The \emph{W–Eulerian polynomial} of a finite Coxeter group \(W\) is the polynomial
\[
\sum_{w \in W} x^{\text{des}_W(w)}
\]
which is known to be the \(h\)-polynomial of the Coxeter complex associated to \(W\), see [26]. The type \(A\) Eulerian polynomials are the common Eulerian polynomials.

In [26], Brenti conjectured that the Eulerian polynomial of any finite Coxeter group is real–rooted. Brenti’s conjecture is true for type \(A\) and \(B\) Coxeter groups [26,50], and one may check with the aid of the computer that the conjecture holds for the exceptional groups \(H_3, H_4, F_4, E_6, E_7,\) and \(E_8\). Moreover, the Eulerian polynomial of the direct product of two finite Coxeter groups is the product of the Eulerian polynomials of the two groups. Hence it remains to prove Brenti’s conjecture for type \(D\) Coxeter groups. The type \(D\) case resisted many attempts, and it was not until very recently that the first sound proof was given by Savage and Visontai [86]. Their proof used compatibility arguments and ascent sequences. We will give a similar proof below that avoids the detour via ascent sequences.

Recall that a combinatorial description of a rank \(n\) Coxeter group of type \(B\) is the group \(B_n\) of signed permutations \(\sigma : \{\pm n\} \to \{\pm n\}\), where \(\{\pm n\} = \{\pm 1, \ldots, \pm n\}\), such that \(\sigma(-i) = -\sigma(i)\) for all \(i \in \{\pm n\}\). An element \(\sigma \in B_n\) is conveniently encoded by the \emph{window notation} as a word \(\sigma_1 \cdots \sigma_n\), where \(\sigma_i = \sigma(i)\). The type \(B\) \emph{descent number} of \(\sigma\) is then
\[
\text{des}_B(\sigma) = |\{i \in [n] : \sigma_{i-1} > \sigma_i\}|,
\]
where \(\sigma_0 := 0\), see [26]. The \(n\)th \emph{type \(B\) Eulerian polynomial} is thus
\[
B_n(x) = \sum_{\sigma \in B_n} x^{\text{des}_B(\sigma)}.
\]

A combinatorial description of a rank \(n\) Coxeter group of type \(D\) is the group \(D_n\) consisting of all elements of \(B_n\) with an even number of negative entries in their window notation. The type \(D\) \emph{descent number} of \(\sigma \in D_n\) is then
\[
\text{des}_D(\sigma) = |\{i \in [n] : \sigma_{i-1} > \sigma_i\}|,
\]
where \(\sigma_0 := -\sigma_2\), see [26]. The \(n\)th \emph{type \(D\) Eulerian polynomial} is
\[
D_n(x) = \sum_{\sigma \in D_n} x^{\text{des}_D(\sigma)}.
\]
For \(n \geq 2\) and \(k \in [\pm n]\), let
\[
D_{n,k}(x) = \sum_{\substack{\sigma \in D_n \\ \sigma_n = -k}} x^{\text{des}_D(\sigma)}.
\]
If \( k \notin [\pm n] \), we set \( D_{n,k}(x) := 0 \). The following table is conveniently generated by the recursion in Lemma 8.12 below.

| \( k \) | \( D_{2,k} \) | \( D_{3,k} \) | \( D_{4,k} \) |
|-------|-------------|-------------|-------------|
| -4    | 0           | 0           | \( (x + 1)(x^2 + 10x + 1) \) |
| -3    | 0           | \( (x + 1)^2 \) | \( 2x(x + 1)(x + 5) \) |
| -2    | \( x(3 + x) \) | \( x(3x^2 + 14x + 7) \) | |
| -1    | \( x(2x(x + 1)) \) | \( x(5x^2 + 14x + 5) \) | (8.2) |
| 1     | \( x(2x(x + 1)) \) | \( x(5x^2 + 14x + 5) \) | |
| 2     | \( x(3x + 1) \) | \( x(7x^2 + 14x + 3) \) | |
| 3     | 0           | \( x(x + 1)^2 \) | \( 2x(x + 1)(5x + 1) \) |
| 4     | 0           | 0           | \( x(x + 1)(x^2 + 10x + 1) \) |

Note that the type \( D \) descents make sense for any element of \( B_n \), where \( n \geq 2 \).

**Lemma 8.11.** If \( n \geq 2 \), then

\[
D_{n,k}(x) = \frac{1}{2} \sum_{\sigma \in B_n \atop \sigma_n = -k} x^{\text{des}(\sigma)}. \tag{8.3}
\]

**Proof.** For \( k \in [n] \), let \( \phi_k : B_n \to B_n \) be the involution that swaps the letters \( k \) and \( -k \) in the window notation of the permutation. Clearly \( \phi_1 \) is a bijection between \( D_n \) and \( B_n \setminus D_n \) which preserves the type \( D \) descents for all \( n \geq 2 \). This proves (8.3) for \( k \notin \{1, -1\} \).

For \( k \in [\pm n] \), let \( B_n[k] \) be the set of \( \sigma \in B_n \) with \( \sigma_n = k \). Then \( \phi_1 \) is a bijection between \( B_n[1] \) and \( B_n[-1] \) which preserves the type \( D \) descents for all \( n \geq 2 \). Similarly let \( D_n[k] \) be the set of \( \sigma \in D_n \) with \( \sigma_n = k \). Now \( B_n[1] = D_n[1] \cup \phi_1(D_n[-1]) \) and \( B_n[-1] = D_n[-1] \cup \phi_1(D_n[1]) \), where the unions are disjoint. Hence to prove (8.3) for \( k = \pm 1 \), it remains to prove \( D_{n,1}(x) = D_{n,-1}(x) \). We prove this by induction on \( n \geq 2 \), where the case \( n = 2 \) is easily checked.

Consider the involution \( \phi_2 \phi_1 : B_n[1] \to B_n[-1] \), where \( n \geq 3 \). Then \( \phi_2 \phi_1 \) preserves type \( D \) descents on \( \sigma \) unless \( \sigma_{n-1} = \pm 2 \). Hence it remains to prove that the type \( D \) descent generating polynomials of \( D_n[2,1] \cup D_n[-2,1] \) and \( D_n[2,-1] \cup D_n[-2,-1] \) agree, where \( D_n[k, \ell] \) is the set of \( \sigma \in D_n \) such that \( \sigma_{n-1} = k \) and \( \sigma_n = \ell \). By induction we have

| Set       | Generating polynomial of set |
|-----------|-----------------------------|
| \( D_n(2,1) \) | \( xD_{n-1,1}(x) \) |
| \( D_n(-2,1) \) | \( xD_{n-1,1}(x) \) |
| \( D_n(2,-1) \) | \( xD_{n-1,1}(x) \) |
| \( D_n(-2,-1) \) | \( xD_{n-1,1}(x) \) |

and the lemma follows. \( \square \)

**Lemma 8.12.** If \( n \geq 2 \) and \( i \in [\pm n] \), then

\[
D_{n+1,i}(x) = \sum_{k \leq i} xD_{n,k}(x) + \sum_{k > i} D_{n,k}(x), \quad \text{if } i < 0 \text{ and}
\]

\[
D_{n+1,i}(x) = \sum_{k \leq i} xD_{n,k}(x) + \sum_{k > i} D_{n,k}(x), \quad \text{if } i > 0.
\]

**Proof.** The lemma follows easily by using the alternative description (8.3) of \( D_{n,i}(x) \), and keeping track of \( \sigma_n \), where \( \sigma \in D_{n+1}[-i] \). We leave the details to the reader. \( \square \)
Theorem 8.13. Let \( n \geq 2 \). The type \( D \) Eulerian polynomial \( D_n(x) \) is real-rooted. Moreover for each \( k \in [\pm n] \), the polynomial \( D_{n,k}(x) \) is real-rooted, and if \( n \geq 4 \), then the sequence \( D_n := (D_{n,k}(x))_{k \in [\pm n]} \) is interlacing.

Proof. One may easily check that \( D_n(x) \) and \( D_{n,k}(x) \) are real-rooted whenever \( 2 \leq n \leq 4 \) and \( k \in [\pm n] \), see (8.2). The sequence \( D_4 \) is interlacing, see (8.2). By Lemma 8.12 up to a relabeling of \( [\pm n] \),
\[
D_{n+1} = G_n^k D_n,
\]
where \( G_n \) is a weakly increasing sequence. The matrix \( G_n \) is of the type appearing in Corollary 8.7. Hence the theorem follows from Corollary 8.7.

Theorem 8.14 (Frobenius [50], Brenti [26], Savage–Visontai [86]). The Eulerian polynomial of any finite Coxeter group is real-rooted.

Remark 8.15. For \( n \geq 1 \) and \( i \in [\pm n] \), define
\[
B_{n,i}(x) = \sum_{\sigma \in B_n \atop \sigma(i) = -i} x^\text{des}(\sigma).
\]

Then \( B_{n,i} \) satisfies the same recursion as in Lemma 8.12 because the proof is ignorant to what happens in the far left in the window notation of an element of \( B_n \). Moreover this recursion is valid for all \( n \geq 1 \). Since \((B_{1,-1}(x), B_{1,1}(x)) = (1, x)\) is interlacing, induction and Corollary 8.7 implies that the sequence \((B_{n,i}(x))_{i \in [\pm n]}\) is an interlacing sequence of polynomials for all \( n \geq 1 \).

9. Multivariate techniques

To prove that a family of univariate polynomials are real-rooted it is sometimes easier to work with multivariate analogs of the polynomials. As alluded to in Section 7, a fruitful generalization of real-rootedness for multivariate polynomials is that of (real-) stable polynomials. There are several benefits in a multivariate approach; the proofs sometimes become more transparent, several powerful inequalities are available for multivariate stable polynomials, it may give you a better understanding for the combinatorial problem at hand. An important class of stable polynomials are determinantal polynomials.

Proposition 9.1. Let \( A_1, \ldots, A_n \) be positive semidefinite hermitian matrices, and \( A_0 \) a hermitian matrix. Then the polynomial
\[
P(x_1, \ldots, x_n) = \det(A_0 + x_1 A_1 + \cdots + x_n A_n)
\]
is either stable or identically zero.

Proof. By Hurwitz' theorem [33, Footnote 3, p. 96] and a standard approximation argument we may assume that \( A_1 \) is positive definite. Let \( x = (a_1 + ib_1, \ldots, a_n + ib_n) \in \mathbb{C}^n \) be such that \( a_j \in \mathbb{R} \) and \( b_j > 0 \) for all \( 1 \leq j \leq n \). We need to prove that \( P(x) \neq 0 \). Now \( P(x) = \det(iB - A) \), where \( B = b_1 A_1 + \cdots + b_n A_n \) is positive definite and \( A = -A_0 - a_1 A_1 - \cdots - a_n A_n \) is hermitian. Hence \( B \) has a square root and thus \( P(x) = \det(B) \det(iI - B^{-1/2}AB^{-1/2}) \neq 0 \), where \( I \) is the identity matrix, since \( B^{-1/2}AB^{-1/2} \) is hermitian and has real eigenvalues only.

For \( n = 2 \) a converse of Proposition 9.1 holds, see Theorem 7.4. The analog of Theorem 7.4 for \( n \geq 3 \) fails to be true by a simple count of parameters. For possible partial converses of Proposition 9.1, see the survey [100].
Recently attempts have been made to find appropriate multivariate analogs of frequently studied real–rooted univariate polynomials in combinatorics. Let us illustrate by describing a multivariate Eulerian polynomial. For $\sigma \in S_n$ let

\[ DB(\sigma) = \{ \sigma(i) : \sigma(i - 1) > \sigma(i) \}, \quad \text{and} \]

\[ AB(\sigma) = \{ \sigma(i) : \sigma(i - 1) < \sigma(i) \}, \]

where $\sigma(0) = \sigma(n+1) = \infty$, be the set of descent bottoms and ascent bottoms of $\sigma$, respectively. Let $A_n(x, y)$ be the polynomial in $\mathbb{R}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ defined by

\[ A_n(x, y) = \sum_{\sigma \in S_n} w(\sigma), \quad \text{where} \quad w(\sigma) = \prod_{i \in DB(\sigma)} x_i \prod_{j \in AB(\sigma)} y_j. \]

For example, $w(573148926) = x_5x_4x_1x_2y_5y_1y_4y_6y_2y_6$. Generate a permutation $\sigma'$ in $S_n$ by inserting the letter 1 in a slot between two adjacent letters in a permutation $\sigma_0\sigma_1 \cdots \sigma_n$ of $\{2, 3, \ldots, n\}$ (where $\sigma_0 = \sigma_n = \infty$). Note that there is an obvious one–to–one correspondence between the slots and the variables appearing in $w(\sigma')$.

Thus if we insert 1 in the slot corresponding to the variable $z$, then

\[ w(\sigma) = x_1y_1 \frac{\partial}{\partial z} w(\sigma') \]

since the descent/ascent bottom corresponding to $z$ in $\sigma'$ will be destroyed, and 1 becomes an ascent- and descent bottom. We have proved

\[ A_n(x, y) = x_1y_1 \left( \sum_{j=2}^{n} \frac{\partial}{\partial x_j} + \frac{\partial}{\partial y_j} \right) A_{n-1}(x^*, y^*), \]

where $x^* = (x_2, \ldots, x_n)$ and $y^* = (y_2, \ldots, y_n)$. To prove that $A_n(x, y)$ is stable for all $n$ it remains to prove that the operators of the form $\sum_{j=2}^{n} \frac{\partial}{\partial x_j}$ preserve stability. Stability preservers were recently characterized in [9]. The following theorem is the algebraic characterization. For $\kappa \in \mathbb{N}$, let $C_\kappa[x_1, \ldots, x_n]$ be the linear space of all polynomials that have degree at most $\kappa_i$ in $x_i$ for each $1 \leq i \leq n$. The symbol of a linear operator $T : C_\kappa[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_m]$ is the polynomial

\[ G_T(x_1, \ldots, x_m, y_1, \ldots, y_n) = T((x_1 + y_1)^{\kappa_1} \cdots (x_n + y_n)^{\kappa_n}), \]

where $T$ acts on the $y$-variables as if they were constants.

**Theorem 9.2 ([9]).** Let $T : C_\kappa[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_m]$ be a linear operator of rank greater than one. Then $T$ preserves stability if and only if $G_T$ is stable.

The symbol of the operator $T = \sum_{j=1}^{n} \partial/\partial x_j$ is

\[ G_T = (x_1 + y_1)^{\kappa_1} \cdots (x_n + y_n)^{\kappa_n} \sum_{j=1}^{n} \frac{\kappa_j}{x_j + y_j}. \]

Hence if $\text{Im}(x_j) > 0$ and $\text{Im}(y_j) > 0$ for all $1 \leq j \leq n$, then $\text{Im}(x_j + y_j)^{-1} < 0$, and hence the symbol is non-zero. Thus $G_T$ is stable and by induction and Theorem 9.2 $A_n(x, y)$ is stable for all $n \geq 1$.

The multivariate Eulerian polynomials above and more general Eulerian-like polynomials were introduced in [22] and used to prove the Monotone Column Permanent Conjecture of Haglund, Ono and Wagner [55]. Suppose $A = (a_{ij})_{i,j=1}^{n}$ is a real matrix which is weakly increasing down columns. Then the Monotone Column
Permanent Conjecture stated that the permanent of the matrix \((a_{ij} + x)^n\) for all \(i,j=1\), where \(x\) is a variable, is real-rooted. Subsequently multivariate Eulerian polynomials for colored permutations and various other models have been studied [23,32,56,101].

9.1. **Stable polynomials and matroids.** Let \(E\) be a finite set and let \(x = (x_e)_{e \in E}\) be independent variables. The *support* of a multiaffine polynomial

\[
P(x) = \sum_{S \subseteq E} a(S) \prod_{e \in S} x_e,
\]

is the set system \(\text{Supp}(P) = \{S \subseteq E : a(S) \neq 0\}\). Choe, Oxley, Sokal and Wagner [33] proved the following striking relationship between stable polynomials and matroids.

**Theorem 9.3.** The support of a homogeneous, multiaffine and stable polynomial is the set of bases of a matroid.

Hence Theorem 9.3 suggests an alternative way of representing matroids. A matroid \(M\), with set of bases \(B\), has the *half-plane property* (HPP) if its bases generating polynomial

\[
P_M(x) = \sum_{B \in B} \prod_{e \in B} x_e
\]

is stable, and \(M\) has the *weak half-plane property* (WHPP) if there are positive numbers \(a(B), B \in B\), such that

\[
\sum_{B \in B} a(B) \prod_{e \in B} x_e
\]

is stable. For example, the Fano matroid \(F_7\) is not WHPP, see [17]. The fact that graphic matroids are HPP is a consequence of the *Matrix–tree theorem* and Proposition 9.1. Suppose \(V = [n]\), and let \(\{\delta_e\}_{e=1}^n\) be the standard basis of \(\mathbb{R}^n\). The *weighted Laplacian* of a connected graph \(G = (V,E)\) is defined as

\[
L_G(x) = \sum_{e \in E} x_e(\delta_{e_1} - \delta_{e_2})(\delta_{e_1} - \delta_{e_2})^T,
\]

where \(e_1\) and \(e_2\) are the vertices incident to \(e \in E\). We refer to [99, Theorem VI.29] for a proof of the next classical theorem that goes back to Kirchhoff and Maxwell. Let \(T_G(x)\) be the *spanning tree polynomial* of \(G\), i.e., the bases generating polynomial of the graphical matroid associated to \(G\).

**Theorem 9.4** (Matrix–tree theorem). For \(i \in V\), let \(L_G(x)_{ii}\) be the matrix obtained by deleting the column and row indexed by \(i\) in \(L_G(x)\). Then

\[
T_G(x) = \det(L_G(x)_{ii}).
\]

Clearly the matrices in the pencil \(L_G(x)_{ii}\) are positive semidefinite. Hence that graphic matroids are HPP follows from Theorem 9.4 and Proposition 9.1. A similar reasoning proves that all regular matroids are HPP, and that all matroids representable over \(\mathbb{C}\) are WHPP, see [17,33]. On the other hand, the Vamos cube \(V_8\) is not representable over any field, and still \(V_8\) is HPP [105]. For further results on the relationship between stable polynomials and matroids we refer to [17,21,33,105].
9.2. Strong Rayleigh measures. Stability implies several strong inequalities among the coefficients. Note that the multivariate Eulerian polynomial above is multiaffine, i.e., it is of degree at most one in each variable. We may view multiaffine polynomials with nonnegative coefficients as discrete probability measures. If $E$ is a finite set, $x = (x_e)_{e \in E}$ are independent variables, and

$$P(x) = \sum_{S \subseteq E} a(S) \prod_{e \in S} x_e,$$

is a multiaffine polynomial with nonnegative coefficients normalized so that $P(1, \ldots, 1) = 1$, we may define a discrete probability measure $\mu$ on $2^E$ by setting $\mu(S) = a(S)$ for each $S \in 2^E$. Then $P_{\mu} := P$ is the multivariate partition function of $\mu$. A discrete probability measure $\mu$ is called strong Rayleigh if $P_{\mu}$ is stable. Hence the measure $\mu_n$ on $2^{[2n]}$, defined by

$$\mu_n(S) = \frac{1}{n!} |\{\sigma \in S_n : DB(\sigma) \cup \{i + n : i \in AB(\sigma)\} = S\}|$$

is strong Rayleigh. A fundamental strong Rayleigh measure is the uniform spanning tree measure, $\mu_G$, associated to a connected graph $G = (V, E)$. This is the measure on $2^E$ defined by

$$\mu_G(S) = \frac{1}{t} \begin{cases} 1 & \text{if } S \text{ is a spanning tree,} \\ 0 & \text{otherwise} \end{cases}$$

where $t$ is the number of spanning trees of $G$. The uniform spanning tree measures — and more generally the uniform measure on the set of bases of any HPP matroid — is strong Rayleigh by the discussion in Section 9.1.

A general class of strong Rayleigh measures containing the uniform spanning tree measures is the class of determinantal measures, see [66]. Let $C$ be a hermitian $n \times n$ contraction matrix, i.e., a positive semidefinite matrix with all its eigenvalues located in the interval $[0, 1]$. Define a probability measure on $2^{[n]}$ by

$$\mu_C(\{T : T \supseteq S\}) = \det C(S), \quad \text{for all } S \subseteq [n],$$

where $C(S)$ is the submatrix of $C$ with rows and columns indexed by $S$. Using Proposition 9.1 it is not hard to prove that $\mu_C$ is strong Rayleigh, see [12].

Negative dependence is an important notion in probability theory, statistics and statistical mechanics, see the survey [75]. In [12] several strong negative dependence properties of strong Rayleigh measures were established. Identify $2^E$ with $\{0, 1\}^E$. A probability measure $\mu$ on $\{0, 1\}^n$ is negatively associated if

$$\int fg d\mu \leq \int f d\mu \int g d\mu,$$

whenever $f, g : \{0, 1\}^n \to \mathbb{R}$ are increasing functions depending on disjoint sets of variables, i.e., $f(\eta)$ only depends on the variables $\eta_i, i \in A$, and $g(\eta)$ only depends on the variables $\eta_i, i \in B$, where $A \cap B = \emptyset$. In particular setting $f(\eta) = \eta_i$ and $g(\eta) = \eta_j$, where $i \neq j$, we see that $\mu$ is pairwise negatively correlated i.e.,

$$\mu(\eta : \eta_i = 1) \leq \mu(\eta : \eta_i = 1) \mu(\eta : \eta_j = 1).$$

Example 9.1. For $n = 2$, a discrete probability measure $\mu$ defined by $\mu(\emptyset) = a, \mu(\{1\}) = b, \mu(\{2\}) = c, \mu(\{1\}) = d$, with $a + b + c + d = 1$ is pairwise negatively correlated if and only if $d(a + b + c + d) \leq (b + d)(c + d)$, i.e., if and only if $ad \leq bc$.

Also, it is easy to see that a real polynomial $a + bx_1 + cx_2 + dx_1x_2$ is stable if
An example of an exclusion process for \( n = 4 \) and only if \( ad \leq bc \). By the next theorem the notions strong Rayleigh, negative association and pairwise negative correlation agree for \( n = 2 \).

**Theorem 9.5** ([12]). If \( \mu \) is a discrete probability measure which is strong Rayleigh, then it is negatively associated.

Recently Pemantle and Peres [77] proved general concentration inequalities for strong Rayleigh measures. A function \( f : \{0, 1\}^n \to \mathbb{R} \) is *Lipschitz-1* if

\[
|f(\eta) - f(\xi)| \leq d(\eta, \xi), \quad \text{for all } \eta, \xi \in \{0, 1\}^n,
\]

where \( d \) is the *Hamming distance*, \( d(\eta, \xi) = |\{i \in [n] : \eta_i \neq \xi_i\}| \).

**Theorem 9.6** (Pemantle and Peres [77]). Suppose \( \mu \) is a probability measure on \( \{0, 1\}^n \) whose partition function is stable and has mean \( m = \mathbb{E}(\sum_{i=1}^n \eta_i) \). If \( f \) is any Lipschitz-1 function on \( \{0, 1\}^n \), then

\[
\mu(\eta : |f(\eta) - \mathbb{E}f| > a) \leq 5 \exp\left(-\frac{a^2}{16(m + 2m)}\right).
\]

### 9.3. The symmetric exclusion process.

The symmetric exclusion process (with creation and annihilation) is a Markov process that models particles jumping on a countable set of sites. Here we will just consider the case when we have a finite set of sites \([n]\). Given a *symmetric* matrix \( Q = (q_{ij})_{i,j=1}^n \) of nonnegative numbers and vectors \( b = (b_i)_{i=1}^n \) and \( d = (d_i)_{i=1}^n \) of nonnegative numbers, define a continuous time Markov process on \( \{0, 1\}^n \) as follows. Let \( \eta \in \{0, 1\}^n \) represent the configuration of the particles, with \( \eta_i = 1 \) meaning that site \( i \) is occupied, and \( \eta_i = 0 \) that site \( i \) is vacant. Particles at occupied sites jump to vacant sites at specified rates. More precisely, these are the transitions in the Markov process, which we denote by \( \text{SEP}(Q, b, d) \), see Fig. 4.

- (J) A particle jumps from site \( i \) to site \( j \) at rate \( q_{ij} \): The configuration \( \eta \) is unchanged unless \( \eta_i = 1 \) and \( \eta_j = 0 \), and then \( \eta_i \) and \( \eta_j \) are exchanged in \( \eta \).
- (B) A particle at site \( i \) is created (is born) at rate \( b_i \): The configuration \( \eta \) is unchanged unless \( \eta_i = 0 \), and then \( \eta_i \) is changed from a zero to a one in \( \eta \).
- (D) A particle at site \( i \) is annihilated (dies) at rate \( d_i \): The configuration \( \eta \) is unchanged unless \( \eta_i = 1 \), and then \( \eta_i \) is changed from a one to a zero in \( \eta \).

It was proved in [12, 104] that \( \text{SEP}(Q, b, d) \) preserves the family of strong Rayleigh measures.
Theorem 9.7. If the initial distribution of a symmetric exclusion process SEP\((Q, b, d)\) is strong Rayleigh, then the distribution is strong Rayleigh for all positive times.

An immediate consequence of Theorem 9.7 is that the stationary distribution (if unique) of the symmetric exclusion process is strong Rayleigh.

Corollary 9.8. If a symmetric exclusion process SEP\((Q, b, d)\) is irreducible and positive recurrent, then the unique stationary distribution is strong Rayleigh.

Proof. Choose an initial distribution which is strong Rayleigh. Then the partition function, \(P_t(x)\), of the distribution at time \(t > 0\), by Theorem 9.7, is stable for all \(t > 0\). The partition function of the stationary distribution is given by \(\lim_{t \to \infty} P_t(x)\). By Hurwitz’ theorem [33, Footnote 3, p. 96] the partition function of the stationary distribution is stable, i.e., the stationary distribution is strong Rayleigh.  

In view of Corollary 9.8, it would be interesting to find the stationary distributions of SEP\((Q, b, d)\) for specific parameters \(Q, b,\) and \(d\). This was achieved by Corteel and Williams [36] for the parameters

\[
q_{ij} = \begin{cases} 
1 & \text{if } |j - i| = 1 \text{ and } |j| > |i|, \\
0 & \text{if } |j - i| > 1. 
\end{cases},
\]

\[b = (\alpha, 0, \ldots, 0, \delta), \]

\[d = (\gamma, 0, \ldots, 0, \beta). \tag{9.1}\]

Hence the particles jump on a line, where particles are only allowed to jump to neighboring sites, and be created and annihilated at the endpoints. The description of the stationary distribution is in terms of combinatorial objects called staircase tableaux. The special case when \(\delta = \gamma = 0\) is related to multivariate Eulerian polynomials. The excedence set, \(X(\sigma) \subseteq [n]\), of a signed permutation \(\sigma \in B_n\) was defined by Steingrımsson [96] as

\[i \in X(\sigma) \text{ if and only if } \begin{cases} 
|\sigma(i)| > i, \text{ or; } \\
\sigma(i) = -i.
\end{cases}\]

If \(\sigma \in B_n\), let \(|\sigma| \in S_n\) be the permutation where \(i \mapsto |\sigma(i)|\) for all \(1 \leq i \leq n\). A cycle \(c \) of \(|\sigma|\) is called a negative cycle if \(\sigma(j) < 0\), where \(|\sigma(j)|\) is the maximal element of \(c\). Otherwise \(c\) is called a positive cycle of \(\sigma\). Let \(c_-(\sigma)\) and \(c_+ (\sigma)\) be the number of negative- and positive cycles of \(\sigma\), respectively.

Theorem 9.9 ([23]). The multivariate partition function of the symmetric exclusion process on \(2^n\) with parameters as in (9.1), with \(\delta = \gamma = 0\), is a constant multiple of

\[
\sum_{\sigma \in B_n} \left(\frac{2}{\alpha}\right)^{c_-(\sigma)} \left(\frac{2}{\beta}\right)^{c_+(\sigma)} \prod_{i \in X(\sigma)} x_i. \tag{9.2}
\]

Note that by Corollary 9.8 the polynomial (9.2) is stable.

Problem 3. Find the stationary distribution of SEP\((Q, b, d)\) for parameters other than (9.1).
9.4. The Grace–Walsh–Szegö theorem, and the proof of Theorem 5.1

The proof of Theorem 5.1 is an excellent example of how multivariate techniques may be used to prove statements about the zeros of univariate polynomials. The proof uses a combinatorial symmetric function identity and the Grace–Walsh–Szegö theorem, which is undoubtedly one of the most useful theorems governing the location of zeros of polynomials, see [82].

A circular region is a proper subset of the complex plane that is bounded by either a circle or a straight line, and is either open or closed.

Theorem 9.10 (Grace–Walsh–Szegö). Let \( f \in \mathbb{C}[z_1, \ldots, z_n] \) be a multiaffine and symmetric polynomial, and let \( K \) be a circular region. Assume that either \( K \) is convex or that the degree of \( f \) is \( n \). For any \( \zeta_1, \ldots, \zeta_n \in K \) there is a number \( \zeta \in K \) such that \( f(\zeta_1, \ldots, \zeta_n) = f(\zeta, \ldots, \zeta) \).

The second ingredient in the proof of Theorem 5.1 is the following symmetric function identity. Let \( e_k(x) \) be the \( k \)th elementary symmetric function in the variables \( x = (x_1, \ldots, x_n) \).

Lemma 9.11. For nonnegative integers \( n \),

\[
\sum_{k=0}^{n} (e_k(x)^2 - e_{k-1}(x)e_{k+1}(x)) = e_n(x) \sum_{k=0}^{\lfloor n/2 \rfloor} C_k e_{n-2k} \left( x + \frac{1}{x} \right), \tag{9.3}
\]

\( x + 1/x = (x_1 + 1/x_1, \ldots, x_n + 1/x_n) \) and \( C_k = (2k \choose k)/(k+1) \), \( k \in \mathbb{N} \), are the Catalan numbers.

Proof. For undefined symmetric function terminology, we refer to [93, Chapter 7]. The polynomial \( e_k(x)^2 - e_{k-1}(x)e_{k+1}(x) \) is the Schur–function \( s_{2k}(x) \), where \( 2^k = (2, 2, \ldots, 2) \). We may rewrite (9.3) as

\[
\sum_{k=0}^{n} s_{2k}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} C_k \sum_{|S|=2k} \prod_{i \in S} x_i \prod_{j \notin S} (1 + x_j^2). \tag{9.4}
\]

By the combinatorial definition of the Schur–function, the left hand side of (9.4) is the generating polynomial of all semi–standard Young tableaux with entries in \( \{1, \ldots, n\} \), that are of shape \( 2^k \) for some \( k \in \mathbb{N} \). Call this set \( A_n \). Given \( T \in A_n \), let \( S \) be the set of entries which occur only ones in \( T \). By deleting the remaining entries we obtain a standard Young tableau of shape \( 2^k \), where \( 2^k = |S| \). There are exactly \( C_k \) standard Young tableaux of shape \( 2^k \) with set of entries \( S \), see e.g. [93] Exercise 6.19.ww. It is not hard to see that the original semi–standard Young tableau is then determined by the set of duplicates. This explains the right hand side of (9.4).

Proof of Theorem 5.1. Let \( P(x) = \sum_{k=0}^{n} a_k x^k = \prod_{k=0}^{n} (1 + \rho_k x) \), where \( \rho_k > 0 \) for all \( 1 \leq k \leq n \), and let

\[
Q(x) = \sum_{k=0}^{n} (a_k^2 - a_{k-1}a_{k+1}) x^k.
\]
Suppose there is a number \( \zeta \in \mathbb{C} \), with \( \zeta \notin \{ x \in \mathbb{R} : x \leq 0 \} \), for which \( Q(\zeta) = 0 \). We may write \( \zeta \) as \( \zeta = \xi^2 \), where \( \text{Re}(\xi) > 0 \). By (9.3),

\[
0 = Q(\zeta) = a_n \xi^n \sum_{k=0}^{\lfloor n/2 \rfloor} C_k e_{n-2k} \left( \rho_1 \xi + \frac{1}{\rho_1 \xi}, \ldots, \rho_n \xi + \frac{1}{\rho_n \xi} \right).
\]

Since \( \text{Re}(\rho_j \xi + 1/(\rho_j \xi)) > 0 \) for all \( 1 \leq j \leq n \), the Grace–Walsh–Szegö Theorem provides a number \( \eta \in \mathbb{C} \), with \( \text{Re}(\eta) > 0 \), such that

\[
0 = \sum_{k=0}^{\lfloor n/2 \rfloor} C_k e_{n-2k} (\eta, \ldots, \eta) = \sum_{k=0}^{\lfloor n/2 \rfloor} C_k \left( \frac{n}{2k} \right) \eta^{n-2k} =: \eta^n q_n \left( \frac{1}{\eta^2} \right).
\]

Since \( \text{Re}(\eta) > 0 \), we have \( 1/\eta^2 \in \mathbb{C} \setminus \{ x \in \mathbb{R} : x \leq 0 \} \). Hence, the desired contradiction follows if we can prove that all the zeros of the polynomial \( q_n(x) \) are real and negative. This follows from the identity

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} C_k \left( \frac{n}{2k} \right) x^k (1 + x)^{n-2k} = \sum_{k=0}^{n} \frac{1}{n+1} \binom{n+1}{k+1} x^k = \frac{1}{n+1} (1-x)^n P^{(1,1)}_n \left( \frac{1+x}{1-x} \right),
\]

where \( \{ P^{(1,1)}_n(x) \} \) are Jacobi polynomials, see [83, p. 254]. The zeros of the Jacobi polynomials \( \{ P^{(1,1)}_n(x) \} \) are located in the interval \((-1, 1)\). Note that the first identity in the equation above follows immediately from (9.3). \( \square \)

10. Historical notes

Here are some complementary historical notes about the origin of some of the central notions of this chapter.

Although some combinatorial polynomials such as the Eulerian polynomials have been known to be \( \gamma \)-positive for at least 45 years [48], Gal [51] and the author [15] realized the relevance of \( \gamma \)-positivity to topological combinatorics and in particular to the Charney–Davis conjecture.

Multivariate stable polynomials and similar classes of polynomials have been studied in many different areas. For their importance in control theory, see [45] and the references therein. In statistical mechanics they play an important role in Lee and Yang’s approach to the study of phase transitions [64, 106]. In PDE theory so called hyperbolic polynomials play an important part in the existence of a fundamental solution to a linear PDE with constant coefficients, see [60]. The importance of stable polynomials in matroid theory was first realized in [33]. An important application of stable polynomials to a problem in combinatorics is Gurvits proof of a vast generalization of the Van der Waerden conjecture, [54]. A recent application is the spectacular solution to the Kadison–Singer problem by Marcus et al. [67]. See the surveys [76, 104] for further applications of stable polynomials.

The notion of HPP and WHPP matroids were introduced in Choe et al. [33]. The strong Rayleigh property was introduced for matroids by Choe and Wagner [34], and extended to discrete probability measures and studied extensively in [12].
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