I. Introduction

Quantum systems that are spatially separated can share information that cannot be accounted for by the relativistic laws of classical physics. This fundamental property of quantum mechanics, which plays a crucial role in quantum information, is known as entanglement and its measurement is still largely an open problem. There is not a unique way of quantifying entanglement; however, in bipartite systems one of the most popular and successful measure of entanglement is the von Neumann entropy \[ S(\rho) = -\text{Tr}(\rho \log \rho) \].

Suppose that the system is in a pure state $|\psi\rangle$. The density matrix is simply the projection operator $\rho_{PQ} = |\psi\rangle\langle\psi|$, where $P$ and $Q$ refer to the two parts and the Hilbert space $\mathcal{H} = \mathcal{H}_P \otimes \mathcal{H}_Q$. The von Neumann entropy is defined as

$$ S(\rho_P) = S(\rho_Q) = -\text{Tr}(\rho_P \log \rho_P) = -\text{Tr}(\rho_Q \log \rho_Q), $$

where

$$ \rho_P = \text{Tr}_Q \rho_{PQ}, \quad \rho_Q = \text{Tr}_P \rho_{PQ}. $$

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and $\text{Tr}_P$ and $\text{Tr}_Q$ denote the partial traces over the degrees of freedom of $P$ and $Q$, respectively.

One of the most interesting models where the entanglement of bipartite systems have been studied are one-dimensional quantum critical systems, in particular quantum spin chains. More precisely, in the simplest setting we consider a spin chain with $N$ spins. At zero temperature the Hamiltonian is in the ground state and in the thermodynamic limit $N \to \infty$ it undergoes a phase transition for some critical value of a parameter, e.g. the magnetic field. This quantum phase transition is characterized by an infinite spin-spin correlation length. We want to know what is the entanglement between the first $L$ spins and the rest of the chain. Several papers have addressed this problem in various contexts \cite{32, 31, 1, 35, 22, 24, 27, 25, 18, 20}. It is well known that the entanglement entropy grows as

\begin{equation}
S(\rho_P) \propto \log L, \quad L \to \infty.
\end{equation}

In this paper we study the entropy of a two block subsystem in the XX spin chain. More precisely, we consider the XX chain with zero magnetic field,

\begin{equation}
H_{XX} = - \sum_{l=1}^{N} \sigma_l^x \sigma_{l+1}^x + \sigma_l^y \sigma_{l+1}^y.
\end{equation}

The starting point for this analysis is an integral representation for the von Neumann entropy of the subsystem $P$ of spins on lattice sites

\begin{equation}
P = \{1, 2, \ldots, m\} \cup \{2m + 1, 2m + 2, \ldots, 3m\}
\end{equation}
in the XX spin chain. This was obtained by B.-Q. Jin and the fourth co-author in \cite{23}, and followed on from the success of this approach to calculating the entropy of a single block of spins \cite{22}. One of the main features of this representation is that the computation of the entanglement reduces to an integral involving the determinant of a block-matrix, whose two block-diagonal entries are Toeplitz determinants, see formulae (41), (48)–(51) in \cite{23}, or (2.1)–(2.4) below. This calculation would be the ultimate goal, but at the moment it is out of our reach. In this paper, instead, we consider a simplified example of a subsystem consisting of two blocks of spins separated by just one spin. The asymptotic analysis of this model is already much more difficult than that of a single block Hamiltonian. Indeed, we not only have to evaluate the asymptotics of the Toeplitz determinant itself, but we also need to extract detailed information on the asymptotic behaviour of the inverse Toeplitz matrix. It should also be noticed that besides its intrinsic interest as a physical a problem, the study of the asymptotics of Toeplitz determinants has a long history going back to Szegö \cite{33, 34} as such matrices are ubiquitous in mathematics and physics.

Recently, there has been considerable interest in computing $S(\rho_P)$ when $P$ is made of disjoint regions of space. Up to now this problem has received attention within the framework Conformal Field Theory (CFT) \cite{9, 10, 11, 2, 15, 16}. One-dimensional quantum critical systems can be described in terms of a massless CFT. More general holographic descriptions are given in \cite{36} and \cite{30}. When $P$ is one interval, then the coefficient of the logarithm in \eqref{1.3} is proportional to the central charge $c$, which is a characteristic of the theory \cite{8}. If the theory is bosonic, i.e. if $c$ is an integer, then in the two-interval case the von Neumann entropy depends on the compactification radius of the bosonic field \cite{10}. In the papers \cite{9, 10} the moments of the density matrix were obtained for two-intervals as ratios of Jacobi theta-functions. Unfortunately, they could not compute the analytic continuation of their formulae in terms of the exponent of the moments, which would have led them to an expression for the von Neumann entropy, except in the asymptotic limit of small intervals \cite{10}.

In this paper, for the two block/interval subsystem in the XX chain separated by one spin, we compute the mutual information between two intervals explicitly. Our approach is based on the Riemann-Hilbert method, which has the additional advantage of being mathematically rigorous.

\section{The main result}

Let $C$ denote the unit circle on the complex plane and

\begin{equation}
g: C \to \mathbb{C}, \quad g(z) = \begin{cases} 
1 & \Re z > 0, \\
-1 & \Re z < 0.
\end{cases}
\end{equation}
The Fourier coefficients of $g$ are
\[
g_l := \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{-il\theta} g(e^{i\theta}) d\theta = \oint_C z^{-\ell} g(z) \frac{dz}{2\pi i z} = \frac{2}{l\pi} \sin \frac{l\pi}{2} = \begin{cases} 0 & l \text{ is even,} \\ (-1)^{\frac{l-1}{2}} & l \text{ is odd.} \end{cases}
\]

In general, the $m \times m$ Toeplitz matrix and determinant with symbol $\phi \in L^\infty(C)$ will be denoted by $T_m[\phi]$ and $D_m[\phi]$, respectively. As it is well-known, the spectral norm (or operator norm) of $T_m[\phi]$ satisfies $\|T_m[\phi]\| \leq \|\phi\|_{\infty}$. In particular, as $T_m[g]$ is a self-adjoint matrix, we obtain the relation $\sigma(T_m[g]) \subseteq [-1, 1]$ for its spectrum.

Let $k, m, n \in \mathbb{N}$. We introduce the following matrix and determinant
\[
(2.1) \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathbb{C}^{(m+n)\times(m+n)}, \quad D(\lambda) = \det(\lambda I - A) \quad (\lambda \in \mathbb{C})
\]
where
\[
(2.2) \quad A_{11} = -T_m[g] \in \mathbb{C}^{m\times m}, \quad A_{22} = -T_n[g] \in \mathbb{C}^{n\times n}, \quad A_{12} = A_{21}^T = (A_{ij})_{i=1,\ldots,m;j=1,\ldots,n} \in \mathbb{C}^{m\times n},
\]
and
\[
(2.3) \quad A_{ij} = \begin{vmatrix} g_{i-j-m-k} & g_{i-m-1} & g_{i-m-2} & \cdots & g_{i-m-k} \\ g_{i-j-k} & g_0 & g_{-1} & \cdots & g_{-k} \\ g_{i-j-k} & g_1 & g_0 & \cdots & g_{-k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_{-j} & g_{k-1} & g_{k-2} & \cdots & g_0 \end{vmatrix},
\]
which is the determinant of a $(k + 1) \times (k + 1)$ matrix, $k \in \mathbb{N}$.

Define the quantity
\[
(2.4) \quad S(\rho_P) = \lim_{\varepsilon \searrow 0} \frac{1}{2\pi i} \oint_{\Gamma_\varepsilon} e(1 + \varepsilon, \lambda) \frac{d}{d\lambda} \ln D(\lambda) \ d\lambda,
\]
where
\[
e(x, v) := -\frac{x + v}{2} \ln \frac{x + v}{2} - \frac{x - v}{2} \ln \frac{x - v}{2}.
\]
The contour $\Gamma_\varepsilon$ goes around the $[-1, 1]$ interval once in the positive direction avoiding the cuts $(-\infty, -1 - \varepsilon] \cup [1 + \varepsilon, \infty)$ of $e(1 + \varepsilon, \cdot)$, see Figure 1. For instance $\Gamma_\varepsilon$ can be the circle $(1 + \frac{1}{2} \varepsilon)C$.

For a general $k, m, n$ we interpret the quantity in (2.4) as a measure of entanglement (kind of an entropy) between the subsystem
\[
(2.5) \quad P = \{1, 2, \ldots, m\} \cup \{m + k + 1, m + k + 2, \ldots, m + k + n\}.
\]
and the rest of the chain. The above subsystem consists of two blocks/intervals of $m$ and $n$ spins separated by a gap of length $k$. It was shown in [23] that in the special case when $k = m = n$, then $S(\rho_P)$ is the von Neumann entropy of (2.5).

![Figure 1. The cuts and the contour in (2.4).](image)
Our ultimate interest is to analyse $S(\rho_P)$ as $k, m, n \to \infty$, however, at this point the general problem seems to be far too complicated to attack directly. Therefore we decided to start with the easier case when the gap between the two intervals is fixed to be $k = 1$, that is, when (2.5) becomes

\[ P = \{1, 2, \ldots, m\} \cup \{m + 2, m + 3, \ldots, m + n + 1\}. \]

In this case the entries of $A_{12}$ in (2.3) become

\[ A_{ij} = \begin{vmatrix} g_{i-j-m-1} & g_{i-m-1} \\ g_{j} & g_{0} \end{vmatrix} = -g_{i-m-1} \cdot g_{-j}. \]

As we shall see, this simplest case already leads to a mathematically very challenging problem.

The asymptotic behaviour of the von Neumann entropy $S(\rho_P^{(m)})$ of the interval $\{1, 2, \ldots, n\}$ was calculated in [22]. In particular, it was shown there that

\[ \lambda \]

\[ \text{Therefore we obtain} \]

\[ \lim_{n \to \infty} \frac{1}{2\pi i} \oint_{\Gamma} e(1 + \varepsilon, \lambda) d\lambda \]

\[ \text{where } \phi(z) = g(z) + \lambda \text{ (} z \in C \text{). Therefore the problem of calculating the limiting behaviour of the entropy of (2.6) reduces to the calculation of the mutual information between the two intervals:} \]

\[ S(\rho_P^{(m)}) + S(\rho_P^{(n)}) - S(\rho_P) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \oint_{\Gamma} e(1 + \varepsilon, \lambda) d\lambda \]

\[ \text{To analyse the asymptotic behaviour of this quantity as } m, n \to \infty \text{ is still mathematically very complicated. However, as we expect this quantity to converge to a finite number, it makes sense to consider the following limit instead, where the } \varepsilon \text{ and } m, n \text{ limits are interchanged:} \]

\[ \lim_{\varepsilon \to 0} \lim_{m, n \to \infty} \frac{1}{2\pi i} \oint_{\Gamma} e(1 + \varepsilon, \lambda) d\lambda \]

\[ \text{We point out that a similar interchanged limit was considered in [19] [22] for the interval case. The value of the limit (2.8) is what we shall calculate and interpret as the mutual information between the two intervals. It will turn out that indeed this is a finite number, which is stated in our main theorem.} \]

**Theorem 2.1.** Let $\tilde{D}(\lambda) = \frac{D(\lambda)}{D_m(\phi) \cdot D_n(\phi)}$ ($\lambda \in C \setminus [-1, 1]$). The limiting mutual information between the two intervals of the subsystem $P$ from (2.6) is

\[ \lim_{\varepsilon \to 0} \lim_{m, n \to \infty} -\frac{1}{2\pi i} \oint_{\Gamma} e(1 + \varepsilon, \lambda) d\lambda \ln \tilde{D}(\lambda) d\lambda = 2 \ln 2 - 1 \approx 0.386294. \]

The main tool in the proof of the above theorem will be an asymptotic analysis of an inner product involving the inverse Toeplitz matrix $T_m(\phi)^{-1}$. We phrase the related statement in the next section as Lemma 3.2.

3. SOME PRELIMINARY CALCULATIONS

We introduce the notations

\[ \tilde{g}_1 = (g_{-m}, g_{-m+1}, \ldots, g_{-1})^T \in C^m, \quad \tilde{g}_2 = (g_{-1}, g_{-2}, \ldots, g_{-n})^T \in C^n, \]

\[ \tilde{G}_1 = T_m(\phi)^{-1} \tilde{g}_1 \in C^m, \quad \tilde{G}_2 = T_n(\phi)^{-1} \tilde{g}_2 \in C^n. \]

Notice that for all $\lambda \in C \setminus [-1, 1]$ we have

\[ \lambda I - A = \begin{pmatrix} T_m(\phi) & -\tilde{g}_1 \tilde{g}_2^T \\ -\tilde{g}_2 \tilde{g}_1^T & T_n(\phi) \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & T_n(\phi) \end{pmatrix} \cdot \begin{pmatrix} T_m(\phi) \tilde{G}_1 \tilde{G}_2^T \tilde{g}_2 \tilde{g}_1^T \\ 0 \end{pmatrix}. \]

Therefore we obtain

\[ D(\lambda) = D_m(\phi) \cdot D_n(\phi) \cdot \det \left( I - \tilde{G}_1 \tilde{g}_2^T \tilde{G}_2 \tilde{g}_1^T \right) = D_m(\phi) \cdot D_n(\phi) \cdot \left( 1 - \langle \tilde{G}_1, \tilde{g}_1 \rangle \langle \tilde{G}_2, \tilde{g}_2 \rangle \right) \]
where we used standard facts about rank-one matrices and the following identity for block-matrices:

$$I - \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} = \left( I - BC \begin{pmatrix} 0 \\ -B \end{pmatrix} \right) \begin{pmatrix} I & 0 \\ -C & I \end{pmatrix}.$$ 

In particular, we infer

$$\tilde{D}(\lambda) = 1 - \left< \tilde{\mathcal{E}}_1, \tilde{g}_1 \right> \left< \tilde{\mathcal{E}}_2, \tilde{g}_2 \right>. $$

Thus in order to compute the mutual information, we need to deal with the inner products $\left< \tilde{\mathcal{E}}_j, \tilde{g}_j \right>$. It turns out that it is sufficient to handle the case $j = 1$.

**Proposition 3.1.** Let us use the notations $\tilde{\mathcal{E}}_1^{(m)} = \tilde{g}_1$, $\tilde{\mathcal{E}}_1^{(n)} = \tilde{g}_1$, $\tilde{\mathcal{E}}_2^{(n)} = \tilde{g}_2$, $\tilde{\mathcal{E}}_2^{(n)} = \tilde{g}_2$, which indicates the $m$- or $n$-dependence of the vectors. Then, we have

$$\left< \tilde{\mathcal{E}}_2^{(n)}, \tilde{g}_2^{(n)} \right> = \left< \tilde{\mathcal{E}}_1^{(n)}, \tilde{g}_1^{(n)} \right>. $$

**Proof.** Consider the $n \times n$ matrix $J = (\delta_{i+j-n-1})_{i,j=0}^{n-1}$ where $\delta$ denotes the Kronecker delta symbol. Since we have $JT_n[g] = T_n[g]$, we obtain $JT_n[\phi]^{-1}J = T_n[\phi]^{-1}$, and thus

$$\left< \tilde{\mathcal{E}}_2^{(n)}, \tilde{g}_2^{(n)} \right> = \left< JT_n[\phi]^{-1}J \tilde{\mathcal{E}}_2^{(n)}, \tilde{g}_2^{(n)} \right> = \left< T_n[\phi]^{-1} \tilde{\mathcal{E}}_1^{(n)}, \tilde{g}_1^{(n)} \right>. $$

\[ \square \]

From now on, until the end of Section 8 our goal is to prove the following lemma, which then we shall apply in Section 9 to prove Theorem 2.1

**Lemma 3.2.** Define $\beta = \frac{1}{2\pi i} \ln \frac{\lambda + 1}{\lambda - 1}$. As $m \to \infty$ we have

$$\left< \tilde{\mathcal{E}}_1, \tilde{g}_1 \right> = \left< T_n[\phi]^{-1} \tilde{g}_1, \tilde{g}_1 \right> = i \tan \left( \frac{\pi}{2} \beta \right) + \mathcal{O}(m^{-\frac{1}{2}}),$$

where the error term is uniform in $\lambda$ on compact subsets of $|\lambda| > 1$.

In order to analyse $\left< \tilde{\mathcal{E}}_1, \tilde{g}_1 \right>$, we shall express it in terms of a Riemann–Hilbert problem (RHP) that arises in the theory of integrable operators, see [4, Section 5.6] or [12]. Define the kernel

$$K(z, s) = \frac{1 - \phi(s) z^m s^{-m} - 1}{2\pi i} \frac{\tilde{f}(z) \tilde{h}(s)}{z - s} = \frac{\tilde{f}(z)^T \tilde{h}(s)}{z - s} \quad (z, s \in \mathbb{C}),$$

where

$$\tilde{f}(z) = \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix} = \begin{pmatrix} z^m \\ 1 \end{pmatrix}, \quad \tilde{h}(s) = \begin{pmatrix} h_1(s) \\ h_2(s) \end{pmatrix} = \frac{1 - \phi(s)}{2\pi i} \begin{pmatrix} s^{-m} \\ -1 \end{pmatrix},$$

which also satisfy $\left< \tilde{f}(z), \tilde{h}(z) \right> = 0$ ($z \in \mathbb{C}$), where $\langle a, b \rangle = \sum_j a_j b_j$. This kernel defines a very special type of bounded singular integral operators on $L^2(\mathbb{C})$, namely a so-called (completely) integrable operator in the following way:

$$K[u](z) = \int_{\mathbb{C}} K(z, s) u(s) ds \quad (u \in L^2(\mathbb{C}), z \in \mathbb{C}),$$

where the integral is meant in the principal value sense, and we put the function in between $[ ]$.

By well-known properties of this operator, for all $\lambda \in \mathbb{C} \setminus [-1, 1]$ we have $0 \neq D_m[\phi] = \det(1 - K)$ and

$$T_m[\phi]^{-1} = \left( \left( (1 - K)^{-1}[z^j, z^k] \right)_{j,k=0}^{m-1} \right),$$

where $(\cdot, \cdot)$ denotes the complex inner product on $L^2(\mathbb{C})$. In particular, the connection between the two determinants can be shown by repeating the argument of [4, page 123]. In order to obtain (3.2) we observe that (by (5.157)-(5.158) in [4, page 123]) $1 - K$ has the block-matrix form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & T_m[\phi] & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

hence the $(2,2)$ block of $(1 - K)^{-1}$ is $T_m[\phi]^{-1}$. Furthermore, by [4, Theorem 5.21] or [12]

$$\tilde{F}(z) = \begin{pmatrix} F_1(z) \\ F_2(z) \end{pmatrix} := (1 - K)^{-1} \tilde{f}(z) = Y_{K(-)} \tilde{f}(z) = \begin{pmatrix} Y_{K,-11}(z) z^m + Y_{K,-12}(z) \\ Y_{K,-21}(z) z^m + Y_{K,-22}(z) \end{pmatrix},$$

\[ (3.3) \]
where $Y_K$ is the unique solution of the following RHP.

**Y$_K$–Riemann–Hilbert problem**

(3.4) \[ Y_K: \mathbb{C} \setminus C \to \mathbb{C}^{2 \times 2} \text{ is analytic,} \]

(3.5) \[ Y_{K+}(z) = Y_{K-}(z) \cdot \begin{pmatrix} \phi(z) & -(\phi(z) - 1)z^{-m} \\ -(\phi(z) - 1)z^{-m} & 2 - \phi(z) \end{pmatrix} \quad \text{(a.e. } z \in C) \]

(3.6) \[ Y_K(z) = I + O(z^{-1}) \text{ as } z \to \infty. \]

The unit circle is oriented in the usual positive direction, and the jump condition (3.5) is meant in the $L^2$ sense, see [4, Definition 5.16].

In the next section we will connect the $Y_K$–RHP with another RHP, but for the rest of this section our aim is to express the inner product $\langle \vec{G}_1, \vec{g}_1 \rangle$ in terms of $Y_K$.

**Proposition 3.3.** We have

\[ \langle \vec{G}_1, \vec{g}_1 \rangle = -\frac{2}{\pi} \sum_{\ell=1}^{\infty} \sum_{m=1}^{\infty} \sin \frac{\ell \pi}{2} M_{\ell,11}, \]

where

\[ Y_K(z) = I + \sum_{\ell=1}^{\infty} M_{\ell} z^{-\ell} \quad \text{as } z \to \infty. \]

**Proof.** First, by (3.2) we have

\[ \langle \vec{G}_1, \vec{g}_1 \rangle = \vec{g}_1^T T_m[\phi]^{-1} \vec{g}_1 = \sum_{j=0}^{m-1} g_{j-m} g_k m \begin{pmatrix} (1 - K)^{-1}[z^j], z^k \end{pmatrix} \]

\[ = \begin{pmatrix} (1 - K)^{-1} \left[ \sum_{j=0}^{m-1} g_{j-m} z^j, \sum_{k=0}^{m-1} g_k z^k \right] \end{pmatrix}. \]

Next, since $g(z) = (\phi(z) - 1) + (1 - \lambda)$, we calculate

\[ \sum_{j=0}^{m-1} g_{j-m} z^j = \sum_{j=0}^{m-1} \oint_C g(s) s^{m-j} z^j ds = \sum_{j=0}^{m-1} \oint_C (\phi(s) - 1) s^{m-j} z^j ds = -\oint_C K(z,s) s^m ds = -K[z^m](z). \]

Therefore, by (3.3) and (3.6)

\[ \langle \vec{G}_1, \vec{g}_1 \rangle = -\begin{pmatrix} (1 - K)^{-1} K[z^m], m \sum_{k=0}^{m-1} g_k z^k \end{pmatrix} = -\begin{pmatrix} (1 - K)^{-1}[z^m], z^m \sum_{k=0}^{m-1} g_k z^k \end{pmatrix} \]

\[ = -\begin{pmatrix} F_1, m \sum_{k=0}^{m-1} g_k z^k \end{pmatrix} = -\sum_{k=0}^{m-1} g_k \int_C F_1(z) z^{-k} \frac{dz}{2\pi iz} \]

\[ = -\sum_{k=0}^{m-1} g_k \oint_C Y_{K-,11}(z) z^{-k} + Y_{K-,12}(z) z^{-k} \frac{dz}{2\pi iz} \]

\[ = -\sum_{k=0}^{m-1} g_k \oint_C Y_{K-,11}(z) z^{-k} \frac{dz}{2\pi iz} = -\sum_{k=0}^{m-1} g_k M_{m-k,11}, \]

from which we conclude (3.7). \[ \square \]
4. Expressing the inner product in terms of the $R$–Riemann–Hilbert problem

Note that for all $\lambda \in \mathbb{C} \setminus [-1, 1]$ the function $\phi$ possesses Fisher–Hartwig singularities at $z_1 = i = e^{i \frac{\pi}{2}}$ and $z_2 = -i = e^{i \frac{3\pi}{2}}$; thus, we can apply the results in [13]. To be more precise, using the notation of (1.2) in [14], we can write $\phi$ in the following form:

$$\phi(z) = e^{V_0} g_1(z) g_2(z) z_1^{-\beta_1} z_2^{-\beta_2}$$

with $\alpha_1 = \alpha_2 = 0$, $\theta_1 = \frac{\pi}{2}$, $\theta_2 = \frac{3\pi}{2}$,

(4.1) \hspace{1cm} \beta = \beta(\lambda) := \beta_1 = -\beta_2 = \frac{1}{2\pi i} \ln \frac{\lambda + 1}{\lambda - 1} = \frac{1}{2\pi i} [\ln(\lambda + 1) - \ln(\lambda - 1)],

(4.2) \hspace{1cm} V(z) = V_0 = \frac{1}{2} [\ln(\lambda - 1) + \ln(\lambda + 1)],

(4.3) \hspace{1cm} g_1(z) g_2(z) = \begin{cases} 1 & \Re z > 0 \\ e^{-2i\pi\beta} & \Re z < 0 \end{cases} = \begin{cases} 1 & \Re z > 0 \\ \frac{1}{\lambda + 1} & \Re z < 0 \end{cases}

(4.4) \hspace{1cm} z_1^{-\beta_1} z_2^{-\beta_2} = e^{\frac{1}{2}[\ln(\lambda + 1) - \ln(\lambda - 1)]}.

Note that throughout this paper, $\ln z$ denotes the principal branch of the logarithm, that is, $-\pi < \arg z < \pi$. Since $\frac{\lambda + 1}{\lambda - 1}$ is a fractional linear map, we can easily examine the real- and imaginary parts of $\beta$. We have

(4.5) \hspace{1cm} \Re \beta = \frac{1}{2\pi} \arg \frac{\lambda + 1}{\lambda - 1}, \quad \Im \beta = \frac{1}{2\pi} \ln \left| \frac{\lambda + 1}{\lambda - 1} \right| = \frac{1}{2\pi} \ln \left| \frac{\lambda - 1}{\lambda + 1} \right|,

therefore we see that $\Im \beta$ stays bounded on compact subsets of $\mathbb{C} \setminus [-1, 1]$. In addition, $|\Re \beta| < \frac{1}{2}$ for all $\lambda \in \mathbb{C} \setminus [-1, 1]$. However, notice that a simple calculation gives that

$$|\Re \beta| < \frac{1}{4} \iff |\lambda| > 1,$$

which is the reason why we shall take $\Gamma_\varepsilon = (1 + \frac{1}{2} \varepsilon) C$ in (2.9) in our calculations. Let us also note that $\beta$ does not vanish on $\mathbb{C}$.

Next, we shall connect the $Y_K$–RHP with the $Y$–RHP, see e.g. [13] or [3] for details.

$Y$–Riemann–Hilbert problem for orthogonal polynomials on the circle

(4.6) \hspace{1cm} Y: \mathbb{C} \setminus C \rightarrow \mathbb{C}^{2 \times 2} \text{ is analytic,}

(4.7) \hspace{1cm} Y_+(z) = Y_-(z) \cdot \begin{pmatrix} 1 & \phi(z) z^{-m} \\ 0 & 1 \end{pmatrix} \quad (z \in C \setminus \{i, -i\}),

(4.8) \hspace{1cm} Y(z) = (I + O(z^{-1})) \begin{pmatrix} z^m & 0 \\ 0 & z^{-m} \end{pmatrix} \text{ as } z \rightarrow \infty,

(4.9) \hspace{1cm} Y(z) = \begin{pmatrix} O(1) & O(\ln |z + i|) \\ O(1) & O(\ln |z - i|) \end{pmatrix} \text{ as } z \rightarrow \pm i.

The jump condition (4.7) is meant in the sense that $Y$ is continuous up to $C$ from both sides, except at the points $\pm i$.

It is well-known that this RHP has a unique solution which can be given in terms of orthogonal polynomials. An easy calculation shows the following connection between the unique solutions $Y_K$ and $Y$:

(4.10) \hspace{1cm} Y_K(z) = \begin{cases} \sigma_3 Y(z) \sigma_3 \begin{pmatrix} z^m \\ 1 \end{pmatrix}^{-1}, & |z| < 1 \\ \sigma_3 Y(z) \sigma_3 \begin{pmatrix} z^m \\ 1 \end{pmatrix}^{-1}, & |z| > 1 \end{cases}

where $\sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the Pauli matrix. We point out that a similar connection was observed in [3]. Note that even though the jump conditions (3.5) and (4.7) are meant in different ways, one verifies easily that indeed the above $Y_K$ solves the $Y_K$–RHP in the $L^2$ sense. The advantage of
involving $Y$ in our analysis is that we can use the powerful results of [14], in particular, we can express our inner product in terms of the $R$–RHP which can be estimated effectively. Let us recall the $R$–RHP next, whose associated contour $\Gamma_R$ is shown in Figure 2. Notice that the circles $\partial U_1$ and $\partial U_2$ around $\pm i$ are oriented in the negative direction.

\begin{align}
R(\zeta) &= I + O(\zeta^{-1}) \quad \text{as } \zeta \to \infty.
\end{align}

The jump conditions (4.12)–(4.14) are meant in the sense that $R$ is continuous up to $\Gamma_R$ from each side. The functions $N$ and $P_j$ denote the global and local parametrices, respectively, see [14, Subsections 4.1–4.2]. Namely,

\begin{align}
N(z) &= \left\{ \begin{array}{ll}
\mathcal{D}(z)^{\sigma_3} & |z| > 1 \\
\mathcal{D}(z)^{\sigma_3} \left( \begin{array}{cc}
0 & 1 \\
-1 & 0 
\end{array} \right) & |z| < 1
\end{array} \right. 
\end{align}

where $\mathcal{D}(z) = \exp \left( \frac{1}{2\pi i} \int_{C} \frac{\ln \phi(s)}{s-z} ds \right)$ stands for the Szegő function. The local parametrices will be discussed in detail in Section 7.

From (4.10) we calculate

\begin{align}
Y_{K,11}(z) &= Y_{11}(z)z^{-m} + Y_{12}(z) \quad (|z| \geq 2),
\end{align}

If we trace back the transformations $Y \to T \to S \to R$ performed in [14], we obtain

\begin{align}
Y(z) &= R(z)N(z)z^{m\sigma_3} = R(z)\mathcal{D}(z)^{\sigma_3}z^{m\sigma_3} \quad (|z| \geq 2).
\end{align}

In particular,

\begin{align}
Y_{11}(z)z^{-m} &= R_{11}(z)\mathcal{D}(z), \quad Y_{12}(z) = R_{12}(z)\mathcal{D}(z)^{-1}z^{-m} \quad (|z| \geq 2).
\end{align}
Notice that \( Y_{12}(z) = O(z^{-m-1}) \) as \( z \to \infty \), hence by (3.7) it does not contribute to our inner product. Therefore we have

\[
M_{\ell,11} = d_{\ell} + \int_{|z|=1} (R_{11}(z) - 1) D(z) z^{\ell} \frac{dz}{2\pi iz} \quad (\ell = 1, \ldots, m),
\]

where \( D(z) = 1 + \sum_{j=1}^{\infty} d_j z^{-j} \) (\(|z| > 1\)). Thus, from (3.7) we obtain

\[
\langle \tilde{g}_1, \tilde{g}_1 \rangle = -\frac{2}{\pi} \sum_{\ell=1}^{m} \frac{\sin \frac{\ell \pi}{2}}{\ell} d_{\ell} - \frac{2}{\pi} \int_{|z|=2} (R_{11}(z) - 1) D(z) f_{m}(z) \frac{dz}{2\pi iz},
\]

where

\[
f_{m}(z) = \sum_{\ell=1}^{m} \frac{\sin \frac{\ell \pi}{2}}{\ell} z^{\ell}.
\]

Now, set \( M = \left[ \frac{m-1}{2} \right] \) and notice that for all \( z \in \mathbb{C}, \Re z \neq 0 \), we have

\[
f_{m}(z) = \sum_{k=0}^{M} \frac{(-1)^k}{2k+1} z^{2k+1} = \frac{1}{i} \sum_{k=0}^{M} \frac{(iz)^{2k+1}}{2k+1} = \frac{1}{i} \sum_{k=0}^{M} \int_{0}^{i\pi} s^{2k} ds = \frac{1}{i} \int_{0}^{i\pi} \frac{1 - s^{2M+2}}{1 - s^2} ds
\]

\[
= \int_{0}^{\pi} \frac{1 + (-1)^M y^{2M+2}}{1 + y^2} dy = \arctan z + (-1)^M \int_{0}^{\pi} \frac{y^{2M+2}}{1 + y^2} dy
\]

\[
= \frac{1}{2i} \ln \frac{z - i}{z + i} + \frac{\pi}{2} \text{sgn} \Re z + (-1)^M \int_{0}^{\pi} \frac{y^{2M+2}}{1 + y^2} dy
\]

\[
= \frac{1}{2i} \ln \frac{z - i}{z + i} + f_{m}(z) = \frac{1}{2i} \ln \frac{z - i}{z + i} + \frac{\pi}{2} \text{sgn} \Re z + f_{m,1}(z),
\]

where the integration is meant along a line segment and \( f_{m}, f_{m,1} \) are implicitly defined in the above equation-chain. Note that \( \frac{1}{2} \ln \frac{z - i}{z + i} \) and \( f_{m} \) are analytic in \( \mathbb{C} \setminus [-i, i] \), and that the integral expression \( f_{m,1} \) is analytic in \( \mathbb{C} \setminus ((i, i\infty) \cup [-i, -i\infty)) \). Since \( (R_{11}(z) - 1) D(z) = O(1/z) \) and \( \frac{1}{2} \ln \frac{z - i}{z + i} = O(1/z) \) as \( z \to \infty \), we easily obtain that

\[
\langle \tilde{g}_1, \tilde{g}_1 \rangle = \frac{2}{\pi} \sum_{\ell=1}^{m} \frac{\sin \frac{\ell \pi}{2}}{\ell} d_{\ell} - \frac{2}{\pi} \int_{|z|=2} (R_{11}(z) - 1) D(z) f_{m,1}(z) \frac{dz}{2\pi iz},
\]

To summarise, we have two kinds of contributions to the inner product, one which comes from the Szegö function and another coming from \( R - I \). Next, we compute the contribution coming from \( D(z) \).

5. The contribution from the Szegö function

Here we calculate the asymptotic behaviour of \( -\frac{2}{\pi} \sum_{\ell=1}^{m} \frac{\sin \frac{\ell \pi}{2}}{\ell} d_{\ell} = -\sum_{\ell=1}^{m} g_{\ell} d_{\ell} \) as \( m \to \infty \). For that, we need a formula for the Szegö function. By (4.8) (or (4.10)) in [14], a short calculation gives

\[
D(z) = \exp \left( \beta \cdot \ln \frac{z - i}{z + i} \right) = \left( \frac{z - i}{z + i} \right)^\beta \quad (|z| > 1),
\]

where the right-hand side is analytic outside \([-i, i]\), and

\[
D(z) = \left( \frac{z - i}{z + i} \right)^\beta \phi(z) \quad (|z| < 1).
\]

A simple calculation gives

\[
D(e^{i\theta}) = \left( \frac{e^{i\theta} - e^{i\frac{\pi}{4}}}{e^{i\theta} + e^{i\frac{\pi}{4}}} \right)^\beta = \left( i \frac{\sin \left( \theta - \frac{\pi}{4} \right)}{\cos \left( \theta - \frac{\pi}{4} \right)} \right)^\beta = \left( i \tan \left( \frac{\theta}{2} - \frac{\pi}{4} \right) \right)^\beta \quad (\theta \in \mathbb{R}).
\]
Therefore, since $D(1/z) = 1 + \sum_{j=1}^{\infty} d_j z^j$ belongs to the Hardy class $H^2$, we obtain the following expression for the limit of the sum:

$$
-\sum_{l=1}^{\infty} g_l d_l = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| e^{\pi i/2} \tan \left( \frac{\theta}{2} - \frac{\pi}{4} \right) \right|^2 d\theta.
$$

By the Cauchy–Schwarz inequality we get

$$
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| e^{\pi i/2} \tan \left( \frac{\theta}{2} - \frac{\pi}{4} \right) \right|^2 d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} D(e^{i\theta}) d\theta + \frac{1}{\pi} \int_{\pi/2}^{3\pi/2} D(e^{i\theta}) d\theta
$$

where we substituted $u = \tan \frac{\theta}{2}$, and similarly

$$
= -1 + \frac{1}{\pi} \int_{\pi/2}^{3\pi/2} \tan \left( \frac{\theta}{2} - \frac{\pi}{4} \right) d\theta = -1 + \frac{2 \cdot i \beta}{\pi} \int_{0}^{\infty} \frac{\beta}{1 + u^2} du = i \tan \left( \frac{\pi}{2} \beta \right),
$$

where we substituted $u = \tan \vartheta$ and used standard residue calculus.

We estimate the speed of convergence below.

**Proposition 5.1.** We have

$$
-\frac{2}{\pi} \sum_{m=1}^{\infty} \sin \frac{\ell \pi}{\ell} d_\ell = i \tan \left( \frac{\pi}{2} \beta \right) + O(m^{-\frac{1}{2}})
$$

as $m \to \infty$, where the error is uniform in $\lambda$ on compact subsets of $\mathbb{C} \setminus [-1, 1]$.

**Proof.** Note that

$$
|D(e^{i\theta})| = \left| e^{\pi i/2} \tan \left( \frac{\theta}{2} - \frac{\pi}{4} \right) \right|^2 = e^{-\frac{\beta}{2}} \left| e^{\frac{3\pi i}{2}} \tan \left( \frac{\theta}{2} - \frac{\pi}{4} \right) \right|^2 |\theta - \frac{\pi}{2}|^{\beta} |\theta - 3\pi/2|^{-\beta} |\beta|
$$

and similarly

$$
|D(e^{i\theta})| = \left| e^{-\pi i/2} \tan \left( \frac{\theta}{2} - \frac{\pi}{4} \right) \right|^2 \leq e^{-\frac{\beta}{2}} \left| e^{\frac{3\pi i}{2}} \tan \left( \frac{\theta}{2} - \frac{\pi}{4} \right) \right|^2 |\theta - \frac{\pi}{2}|^{\beta} |\theta + \frac{\pi}{2}|^{-\beta} |\beta|
$$

Hence the squared $L^2$-norm of the Szegő function can be estimated as follows:

$$
\frac{1}{2\pi} \int_{-\pi}^{\pi} |D(e^{i\theta})|^2 d\theta \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| e^{-\frac{\beta}{2}} \left| e^{\frac{3\pi i}{2}} \tan \left( \frac{\theta}{2} - \frac{\pi}{4} \right) \right|^2 |\theta - \frac{\pi}{2}|^{2\beta} |\theta + \frac{\pi}{2}|^{-2\beta} \right|^2 d\theta
$$

$$
= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \left| e^{-\frac{\beta}{2}} \left| e^{\frac{3\pi i}{2}} \tan \left( \frac{\theta}{2} - \frac{\pi}{4} \right) \right|^2 |\theta + \frac{\pi}{2}|^{-2\beta} \right|^2 \right|^2 d\theta \leq \frac{3}{1 - 2|\beta|}.
$$

Therefore by the Cauchy–Schwarz inequality we get

$$
\left| \sum_{l=m+1}^{\infty} g_l d_l \right| \leq \frac{2}{\pi} \sum_{l=m+1}^{\infty} \frac{1}{l} |d_l| \leq \frac{2}{\pi} \sqrt{\sum_{l=m+1}^{\infty} \frac{1}{l^2} \int_{-\pi}^{\pi} |D(e^{i\theta})|^2 d\theta}
$$

$$
\leq \frac{2}{\pi} \sqrt{\int_{-\pi}^{\pi} \frac{1}{x^2} dx} \sqrt{\int_{-\pi}^{\pi} |D(e^{i\theta})|^2 d\theta} \leq \frac{2}{\pi} m^{-1/2} e^{\frac{3\pi i}{2}} \sqrt{3} \sqrt{\frac{1}{1 - 2|\beta|}} = O(m^{-1/2})
$$

as $m \to \infty$, uniformly in $\lambda$ on compact subsets of $\mathbb{C} \setminus [-1, 1]$. 

Before we proceed with computing the contribution coming from $R - I$, we need some auxiliary calculations about the integral $\int_{m,1}$ defined in (4.19), and the local parametrices appearing in the analysis of $R$ in [14].
6. Estimation of $\tilde{f}_{m,1}$

We start with the following proposition.

**Proposition 6.1.** We have

$$e^{-mu} = (1 - u)^m + O\left(\frac{1}{m}\right)$$

as $m \to \infty$, uniformly in $u \in [0, 1]$.

**Proof.** As $\ln(1 - u) < -u$ ($0 < u < 1$), we have $e^{-mu} > (1 - u)^m$ ($0 < u \leq 1$). Note that

$$\frac{d}{du}(e^{-mu} - (1 - u)^m) = m((1 - u)^{m-1} - e^{-mu}) = 0 \iff -mu = (m-1)\ln(1-u) \quad (0 \leq u \leq 1).$$

Since $\ln(1 - u)$ is concave, we have at most two stationary points, and clearly one of them is $u = 0$. Simple calculations show that for $u = \frac{1}{m}$ the derivative is positive, and that for $u = \frac{\delta}{m}$ it is negative, therefore there is a second stationary point $\frac{1}{m} < \tilde{u}_m < \frac{\delta}{m}$ ($m \in \mathbb{N}, m > 5$). It is then obvious that for all $u \in [0, 1]$ we have

$$0 \leq e^{-mu} - (1 - u)^m \leq e^{-m\tilde{u}_m} - (1 - \tilde{u}_m)^m = e^{-m\tilde{u}_m}(1 - e^{m(\ln(1-\tilde{u}_m) + \tilde{u}_m)}) = O(1)(1 - e^{O(m\tilde{u}_m^2)}) = O(1)(1 - e^{O(1/m)}) = O\left(\frac{1}{m}\right)$$

as $m \to \infty$.

We proceed with the estimation of $\tilde{f}_{m,1}(z) = (-1)^M \int_0^z \frac{y^{2M+2}}{1+y^2} \, dy$ when $z$ is close to the cut $[-i, i]$.

**Lemma 6.2.** As $m \to \infty$, we have the following estimates which are uniform in $z$ and $t$:

(i)

$$\tilde{f}_{m,1}(z) = O\left(2^{-m}\right) \quad (|z| \leq \frac{1}{2}),$$

(ii)

$$\tilde{f}_{m,1}(it) = O\left(e^{-\frac{1}{2}\sqrt{m}}\frac{1}{\sqrt{m}}\right) \quad (-1 + \frac{1}{\sqrt{m}} \leq t \leq 1 - \frac{1}{\sqrt{m}})$$

(iii)

$$\tilde{f}_{m,1}(it) = \frac{1}{2i} \int_{m(1-t)}^\infty \frac{e^{-\zeta}}{\zeta} \, d\xi + O\left(\frac{1}{m \ln m}\right) = O(1) \quad (1 - \frac{1}{\sqrt{m}} \leq t \leq 1 - \frac{1}{m})$$

and $\tilde{f}_{m,1}(-it) = O(1)$.

**Proof.** (i) is obvious. Note that $\tilde{f}_{m,1}$ is an odd function, therefore it is enough to prove (ii)–(iii) for $t \geq 0$. For $0 \leq t \leq 1 - \frac{1}{m}$ we have

$$\tilde{f}_{m,1}(it) = \frac{(-1)^M}{2i} \int_0^t \frac{y^{2M+2}}{y - i} \, dy - \frac{(-1)^M}{2i} \int_0^t \frac{y^{2M+2}}{y + i} \, dy = \frac{(-1)^M}{2i} I_1(t) - \frac{(-1)^M}{2i} I_2(t),$$

where $I_1(t)$ and $I_2(t)$ are the first and second integrals, respectively. By substituting $y = ix$ and keeping in mind that $m \in \{2M + 1, 2M + 2\}$, we get for all $0 \leq t \leq 1 - \frac{1}{m}$ that

$$|I_2(t)| = \int_0^t \frac{x^{2M+2}}{x + 1} \, dx \leq \int_0^t x^{2M+2} \, dx = \frac{t^{2M+3}}{2M + 3} \leq \frac{(1 - \frac{1}{\sqrt{m}})^{2M+3}}{2M + 3} = \frac{(1 - \frac{1}{\sqrt{m}})^m}{m} = O\left(\frac{1}{m}\right)$$

as $m \to \infty$. Also we obtain that if $0 \leq t \leq 1 - \frac{1}{\sqrt{m}}$, then

$$|I_2(t)| \leq |I_1(t)| = \int_0^t \frac{x^{2M+2}}{1 - x} \, dx \leq \sqrt{m} \int_0^{1-\frac{1}{\sqrt{m}}} x^{2M+2} \, dx \leq \frac{1}{\sqrt{m}} \left(1 - \frac{1}{\sqrt{m}}\right)^m = O\left(\frac{1}{\sqrt{m}} e^{-\frac{1}{2}\sqrt{m}}\right)$$

as $m \to \infty$, which proves (ii).
Now, let \( 1 - \frac{1}{\sqrt{m}} \leq t \leq 1 - \frac{1}{m} \), then
\[
I_1(t) = (-1)^M \int_0^t \frac{x^{2M+2}}{1-x} \, dx = (-1)^M \int_0^{1-\frac{1}{\sqrt{m}}} \frac{x^{2M+2}}{1-x} \, dx + O \left( \frac{1}{\sqrt{m}} e^{-\frac{1}{2}\sqrt{m}} \right)
\]
\[
= (-1)^M \int_{\frac{1}{\sqrt{m}}}^1 \frac{(1-u)^{2M+2}}{u} \, du + O \left( \frac{1}{\sqrt{m}} e^{-\frac{1}{2}\sqrt{m}} \right)
\]
\[
= (-1)^M \int_{\frac{1}{\sqrt{m}}}^1 \frac{e^{-(2M+2)u}}{u} \, du + O \left( \frac{1}{\sqrt{m}} e^{-\frac{1}{2}\sqrt{m}} \right)
\]
\[
= (-1)^M \int_{\frac{1}{\sqrt{m}}}^1 \frac{e^{-(2M+2)u}}{u} \, du + O \left( \frac{1}{m} \ln m \right) = (-1)^M \int_{(2M+2)(1-t)}^{\frac{2M+2}{\sqrt{m}}} \frac{e^{-\zeta}}{\zeta} \, d\zeta + O \left( \frac{1}{m} \ln m \right)
\]
as \( m \to \infty \), uniformly in \( t \). What remains to prove is that the latter integral is \( \int_{\frac{2M+2}{\sqrt{m}}}^{\infty} \frac{e^{-\zeta}}{\zeta} \, d\zeta + O \left( \frac{1}{m} \ln m \right) \) as \( m \to \infty \), uniformly in \( t \), which follows from the following calculations:
\[
\int_{\frac{2M+2}{\sqrt{m}}}^{\infty} \frac{e^{-\zeta}}{\zeta} \, d\zeta \leq \int_{\frac{2M+2}{\sqrt{m}}}^{\infty} e^{-\frac{2M+2}{\sqrt{m}}} \, d\zeta \leq e^{-\sqrt{m}}
\]
and
\[
\int_{(2M+2)(1-t)}^{\frac{2M+2}{\sqrt{m}}} \frac{e^{-\zeta}}{\zeta} \, d\zeta = (2M+2)(1-t) \frac{e^{-m(1-t)}}{m(1-t)} \leq \frac{e^{-1}}{m}.
\]
\( \square \)

Now, we estimate near \( i \).

**Lemma 6.3.** Let \( 0 < c < 1 < C \), then the following holds as \( m \to \infty \), uniformly in \( \frac{c}{m} \leq |z-i| \leq \frac{C}{m} \), \( z \notin [i, i \infty) \):
\[
(6.1) \quad \tilde{f}_{m,z}(z) = \frac{1}{2i} \int_{1}^{\infty} \frac{e^{-\zeta}}{\zeta} \, d\zeta - \frac{1}{2i} \int_{1}^{im(z-i)} \frac{e^{-\zeta}}{\zeta} \, d\zeta + O \left( \frac{1}{m} \ln m \right) = O(1),
\]
where the path for the second integral lies in \( \zeta \in \mathbb{C} \setminus (-\infty, 0] \), \( c \leq |\zeta| \leq C \), as shown in Figure 3.
Proof. Notice that for all \( c < |u| < C \) we have
\[
(1 - \frac{u}{m})^{2M+2} - e^{-u} = e^{(2M+2)\ln(1-u/m)} - e^{-u} = e^{(2M+2)(-u/m + O(m^{-2}))} - e^{-u} = e^{-u} \left( e^{-(2M+2)u/m + (2M+2)O(m^{-2})} - 1 \right) = e^{-u} \left( e^{O(1/m)} - 1 \right) = O \left( \frac{1}{m} \right).
\]
Therefore, using the substitution \( y = i(1 - \frac{u}{m}) \), \( u = im(y - i) \), we obtain
\[
\widetilde{f}_{m,1}(z) = \widetilde{f}_{m,1}(i(1 - \frac{1}{m})) + (-1)^{M} \int_{i(1 - \frac{1}{m})}^{z} \frac{y^{2M+2}}{1 + y^{2}^{2}} dy
\]
\[
= \frac{1}{2i} \int_{1}^{\infty} e^{-\frac{\zeta}{\ln m}} d\zeta + O \left( \frac{1}{m} \ln m \right) + \frac{(-1)^{M}}{2i} \int_{i(1 - \frac{1}{m})}^{z} \frac{y^{2M+2}}{y - i} + O(1) dy
\]
\[
= \frac{1}{2i} \int_{1}^{\infty} e^{-\frac{\zeta}{\ln m}} \frac{1}{\zeta} d\zeta - \frac{1}{2i} \int_{1}^{im(z-i)\ln m} \frac{1}{u} du + O \left( \frac{1}{m} \ln m \right)
\]
\[
= \frac{1}{2i} \int_{1}^{\infty} e^{-\frac{\zeta}{\ln m}} \frac{1}{\zeta} d\zeta - \frac{1}{2i} \int_{1}^{im(z-i)} \frac{e^{-u}}{u} + O \left( \frac{1}{m} \ln m \right)
\]
\[
= \frac{1}{2i} \int_{1}^{\infty} e^{-\frac{\zeta}{\ln m}} \frac{1}{\zeta} d\zeta - \frac{1}{2i} \int_{1}^{im(z-i)} e^{-\frac{\zeta}{\ln m}} d\zeta + O \left( \frac{1}{m} \ln m \right).
\]
\( \square \)

We note that one can similarly estimate near \(-i\).

7. The local parametrices

In this section we shall compute how the local parametrices look like, with paying special attention to those parts that depend on \( m \). As the two cases are very similar, we shall only examine the parametrix \( P_{1} \) around \( i \) in detail. As in (4.12) and (4.23)–(4.24) in [14] we have
\[
\zeta = m \ln \frac{z}{i} \quad (z \in U_{1})
\]
and
\[
P_{1}(z) = E(z)\Psi_{1}(\zeta)F_{1}(z)^{-\sigma_{3}z^{2}m\sigma_{3}/2} \quad (z \in U_{1}),
\]
where \( \pm = + \) when \( |z| < 1 \), and \( \pm = - \) when \( |z| > 1 \). By equations (4.18)–(4.22) in [14], one easily sees that the auxiliary function \( F_{1}(z) \) is constant in \( U_{1} \), and its value is
\[
F_{1}(z) = F_{1} := \sqrt{(\lambda - 1)e^{i\pi\beta}} = \sqrt{(\lambda + 1)e^{-i\pi\beta}} \quad (z \in U_{1}).
\]
The function \( E(z) \) is analytic in a neighbourhood of \( U_{1} \) and is defined in (4.47)–(4.50) in [14]. What is important for our considerations is that
\[
E(z) = \begin{pmatrix} 0 & E_{12}(z) \\ E_{21}(z) & 0 \end{pmatrix} = m^{-\beta\sigma_{3}i\frac{2}{m}\sigma_{3}} \tilde{E}(z) = m^{-\beta\sigma_{3}i\frac{2}{m}\sigma_{3}} \begin{pmatrix} 0 & \tilde{E}_{12}(z) \\ \tilde{E}_{21}(z) & 0 \end{pmatrix},
\]
where \( \tilde{E}(z) \) is independent of \( m \) and analytic in a neighbourhood of \( U_{1} \). Furthermore,
\[
E_{12}(z) = i\frac{m}{2} D(z) \zeta^{-\beta} F_{1}^{-1} e^{i\pi\beta}, \quad \tilde{E}_{12}(z) = D(z) \left( \ln \frac{z}{i} \right)^{-\beta} F_{1}^{-1} e^{i\pi\beta} \quad (|z| < 1),
\]
and
\[
E_{21}(z) = -i\frac{m}{2} D(z)^{-1} \zeta^{\beta} F_{1} e^{-2i\pi\beta}, \quad \tilde{E}_{21}(z) = -D(z)^{-1} \left( \ln \frac{z}{i} \right)^{\beta} F_{1} e^{-2i\pi\beta} \quad (|z| < 1).
\]
The function \( \Psi_{1}(\zeta) \) is an auxiliary function which is the main ingredient in constructing the local parametrix in [14], and which is given explicitly in terms of the confluent hypergeometric function \( \psi(a, c, z) \). We recall the details now. Let the contours \( \Gamma_{1}, \ldots, \Gamma_{8} \) be defined as in Figure 4. In particular, each of them is a half line starting or ending at \( 0 \). Furthermore, \( \Gamma_{k} \cup \Gamma_{k+4} \) is a line \( (k = 1, 2, 3, 4) \), and when \( k = 1, 3 \), these unions are the imaginary and real axes, respectively.
These contours divide the complex plane into 8 sectors, denoted by I, II, ..., VIII as shown in Figure 4. The function $\Psi_1(\zeta)$ is analytic in $\mathbb{C} \setminus \bigcup_{k=1}^{8} \Gamma_k$, and is uniquely defined by

$$\Psi_1(\zeta) = \Psi_1^{(I)}(\zeta) = \begin{pmatrix} \psi(\beta, 1; \zeta) e^{i\pi\beta} e^{-\zeta/2} & -\psi(1 - \beta, 1, e^{-i\pi\zeta}) e^{i\pi\beta} e^{\zeta/2} \\ -\psi(1 + \beta, 1; \zeta) e^{i\pi\beta} e^{-\zeta/2} \Gamma(1 + \beta) \Gamma(-\beta) & \psi(-\beta, 1, e^{-i\pi\zeta}) e^{\zeta/2} \Gamma(1 - \beta) \Gamma(-\beta) \end{pmatrix} \quad (\zeta \in I),$$

and the following jump condition

$$\Psi_{1,+}(\zeta) = \Psi_{1,-}(\zeta) J_k(\zeta) \quad (\zeta \in \Gamma_k),$$

where the jump matrices $J_k$ are constant and are given by

$$J_1 = \begin{pmatrix} 0 & e^{-i\pi\beta} \\ -e^{i\pi\beta} & 0 \end{pmatrix}, \quad J_2 = J_5 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J_3 = J_7 = \begin{pmatrix} 0 & e^{i\pi\beta} \\ e^{-i\pi\beta} & 0 \end{pmatrix}, \quad J_4 = J_6 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

see (4.25)–(4.29) and (4.32) in [14]. Note that the functions $\psi(a, c, \zeta)$ and $\psi(a, c, e^{-i\pi\zeta})$ are defined on the universal covering of the punctured plane $\zeta \in \mathbb{C} \setminus \{0\}$, and that $\Psi_1^{(I)}(\zeta)$ is the analytic continuation of $\Psi_1|_I$ to $0 < \arg \zeta < 2\pi$.

8. The contribution from $R - I$

Recall that the integration in (2.9) will be taken over the circle $(1 + \varepsilon/2)C$, hence from now on we only consider the case when $|\lambda| > 1$, which implies $|\Re \beta| < 1/4$. In this section our aim is to show that for $|\lambda| > 1$ the integral

$$\int_{|z|=2} (R_{11}(z) - 1) \left( \frac{z - i}{z + i} \right)^{\beta} \tilde{f}_m(z) \frac{dz}{z}$$

introduced in (4.20) converges to 0, and thus the contribution to our inner product coming from $R - I$ is, roughly speaking, negligible. Of course, the contour of integration can be deformed to the outer boundary of the unbounded component of $\mathbb{C} \setminus \Gamma_R$. Since the integrand is analytic outside $\Gamma_R \cup [-i, i]$, the integrals over the other contours shown on Figure 5 vanish. Therefore, by a straightforward calculation we obtain the following expression for (8.1) where $\Delta(z) + I$ is the
jump in the \( R \)-RHP:

\[
\begin{align*}
\Gamma_R R^+ (z) - R^-(z) & = \int \left( \frac{z - i}{z + i} \right) \beta \frac{\tilde{f}_m(z)}{z} dz + \int \left( \frac{z - i}{z + i} \right) \beta \frac{\tilde{f}_m(z)}{z} dz \\
\Gamma_R R^- (z) \Delta(z) & = - \int \left( \frac{z - i}{z + i} \right) \beta \frac{\tilde{f}_m(z)}{z} dz + \int \left( \frac{z - i}{z + i} \right) \beta \frac{\tilde{f}_m(z)}{z} dz,
\end{align*}
\]

where \( \gamma_m \) is the union of two circles of radius \( 1/m \), four line segments and two half-circles of radius \( 1/2 \). More precisely, \( \gamma_m = C^+_m \cup [(1 - 1/m)i, i/2]_+ \cup [-i/2, -(1 - 1/m)i]_+ \cup C^-_m \cup [-(1 - 1/m)i, i/2]_- \cup [i/2, (1 - 1/m)i]_- \cup \{ z : |z| = 1/2 \} \) oriented in the positive direction where \( C^+_m \) is the circle around \( \pm i \) with radius \( 1/m \), and the line segment \( [-i, i] \) is oriented upwards, hence its \(-/+\) side is its right/left side.

\( \Gamma_R \) is the contour deformation of the dashed circle into the outer red contour. The contour integrals along the other red contours vanish.

We shall examine the two integrals in (8.3) separately, starting with the second one.

8.1. The integral over \( \gamma_m \).

**Proposition 8.1.** We have

\[
\int_{\gamma_m} (R_{11}(z) - 1) \left( \frac{z - i}{z + i} \right) \beta \frac{\tilde{f}_m(z)}{z} dz = O \left( m^{-1/4} \right) \quad \text{as } m \to \infty,
\]

which is uniform in \( \lambda \) on compact subsets of \( |\lambda| > 1 \).

**Proof.** We know from the standard analysis in the steepest descent method that \( \Delta(z) = O(m^{2|\Re(\beta)| - 1}) \) and hence \( R(z) - I = \frac{1}{2\pi i} \int_R R^{-1}(z) \Delta(z) ds = O(m^{2|\Re(\beta)| - 1}) \) as \( m \to \infty \), which is uniform in \( \lambda \) on compact subsets of \( |\lambda| > 1 \), and in \( z \) on \( \mathbb{C} \setminus \Gamma_R \) (see e.g. [14]), and note that there is some flexibility in choosing the parameters for \( \Gamma_R \), hence the Cauchy integral does not blow up as \( z \) gets closer to
\( \Gamma_R \). Also, elementary observations show that \( \left( \frac{z-i}{z+i} \right)^\beta = O(m^{[R\beta]} \) as \( m \to \infty \), which is uniform in \( \lambda \) on compact subsets of \(|\lambda| > 1\). Therefore, combining the above estimates with Lemmas 6.2 and 6.3 we conclude

\[
\int_{\gamma_m} (R_{11}(z) - 1) \left( \frac{z - i}{z + i} \right)^\beta \frac{f_m(z)}{z} \, dz = \int_{\gamma_m} O \left( m^{2[R\beta]-1} \right) O \left( m^{R\beta} \right) O \left( 1 \right) \, dz = O \left( m^{3[R\beta]-1} \right) = O \left( m^{-1/4} \right) \quad \text{as } m \to \infty. 
\]

\( \square \)

From now on, we estimate the integral over \( \Gamma_R \) from (8.3), which we split into two parts.

### 8.2. The integrals over \( \Sigma_j^{\prime\prime} \) and \( \Sigma_j^{\prime} \). First we deal with the integrals over the lenses.

**Proposition 8.2.** We have

\[
\int_{\Sigma_j^{\prime\prime}} (R_{-}(z)\Delta(z))_{11} \left( \frac{z - i}{z + i} \right)^\beta \frac{f_m(z)}{z} \, dz = O \left( m^{-1/2} \right),
\]

which is uniform in \( \lambda \) on compact subsets of \(|\lambda| > 1\).

**Proof.** Notice that for \( z \in \Sigma_j^{\prime\prime} \) we have

\[
\Delta(z) = D(z)^{\sigma_3} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \left( \phi(z)^{-1}z^m \right) \begin{pmatrix} 0 & 1 \end{pmatrix} D(z)^{-\sigma_3} = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}.
\]

Thus \( (R_{-}(z)\Delta(z))_{11} = 0 \), and we conclude that

\[
\int_{\Sigma_j^{\prime\prime}} (R_{-}(z)\Delta(z))_{11} \left( \frac{z - i}{z + i} \right)^\beta \frac{f_m(z)}{z} \, dz = 0 \quad (j = 1, 2).
\]

Next, for \( z \in \Sigma_j^{\prime} \) we have

\[
\Delta(z) = D(z)^{\sigma_3} \left( \phi(z)^{-1}z^{-m} \right) \begin{pmatrix} 0 & 1 \end{pmatrix} D(z)^{-\sigma_3} = \phi(z)^{-1}z^{-m} \left( \frac{z - i}{z + i} \right)^{-2\beta} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

Note that

\[
\left| \frac{f_m(z)}{z^m+1} \right| \leq \left| \frac{f_m(z)}{z^m+1} \right| + \frac{1}{2} \left| \ln \frac{z-i}{z+i} \right| \leq \sum_{\ell=1}^{m} |z|^{-\ell-1} + O(1) = \frac{1-|z|^{-m}}{|z|^{-1}} + O(1) = O(1)
\]

as \( m \to \infty \), uniformly in \( z \) on \( \Sigma_1^{\prime} \cup \Sigma_2^{\prime} \). Therefore,

\[
\int_{\Sigma_j^{\prime}} (R_{-}(z)\Delta(z))_{11} \left( \frac{z - i}{z + i} \right)^\beta \frac{f_m(z)}{z} \, dz = \int_{\Sigma_j^{\prime}} \phi(z)^{-1} \left( \frac{z - i}{z + i} \right)^{-\beta} R_{12,-}(z) \frac{f_m(z)}{z^m+1} \, dz
\]

and

\[
\int_{\Sigma_j^{\prime}} O(1)O \left( m^{2[R\beta]-1} \right) O(1) \, dz = O \left( m^{-1/2} \right)
\]

as \( m \to \infty \) \( (j = 1, 2) \), which is uniform in \( \lambda \) on compact subsets of \(|\lambda| > 1\). \( \square \)

Finally, we estimate the integrals over the circles.

### 8.3. The integrals over the circles \( \partial U_1 \) and \( \partial U_2 \). We shall only examine the integral over \( \partial U_1 \) and note that the case of \( \partial U_2 \) is very similar. We will deform the contour \( \partial U_1 \) inside the disk \( U_1 \). Recall the jump condition (4.14), and that the jump there \( P_1(z)N(z)^{-1} = \Delta(z) + I \) is analytic only in a neighbourhood of \( U_1 \setminus \left( \Sigma_1 \cup \Sigma_2 \cup \Sigma_1^{\prime} \cup \Sigma_2^{\prime} \right) \). The disk \( U_1 \) is cut into five components by \([-i, i] \cup \Sigma_1 \cup \Sigma_2 \cup \Sigma_1^{\prime} \cup \Sigma_2^{\prime} \), on all of which the integrand is analytic and continuous up to the
boundaries, except maybe at \( i \). Therefore, we can deform the five arcs in the way shown in Figure 6 and obtain the following:

\[
\int_{\partial U_1} (R_-(z) \Delta(z))_{11} \left( \frac{z - i}{z + i} \right)^{\beta} \frac{f_m(z)}{z} \, dz = - \int_{\gamma_m \cap U_1} (R(z) \Delta(z))_{11} \left( \frac{z - i}{z + i} \right)^{\beta} \frac{f_m(z)}{z} \, dz
\]

(8.7)

\[
+ \int_{\Sigma_{j}^{i,m} \cup \Sigma_{j}^{i,m} \cup \Sigma_{j}^{i,m} \cup \Sigma_{j}^{i,m}} (R(z) [\Delta_+(z) - \Delta_-(z)])_{11} \left( \frac{z - i}{z + i} \right)^{\beta} \frac{f_m(z)}{z} \, dz,
\]

where \( \gamma_m \) was defined just after (8.3), and \( \Sigma_{j}^{i,m} = \Sigma_j \cap U_1 \cap \{ z : |z - i| > 1/m \} \), \( \Sigma_{j}^{i,m} = \Sigma_j \cap U_1 \cap \{ z : |z - i| > 1/m \} \).

**Figure 6.** The contour deformation of \( \partial U_1 \).

First, we handle the integral over \( \gamma_m \cap U_1 \).

**Proposition 8.3.** We have

\[
\int_{\gamma_m \cap U_1} (R(z) \Delta(z))_{11} \left( \frac{z - i}{z + i} \right)^{\beta} \frac{f_m(z)}{z} \, dz = O \left( m^{-1/4} \right) \quad \text{as } m \to \infty,
\]

uniformly in \( \lambda \) on compact subsets of \( |\lambda| > 1 \). Moreover, the same estimation holds for \( U_2 \).

**Proof.** We have

\[
N(z)^{-1} = \left\{ \begin{array}{ll}
0 & |z| > 1, \\
-1 & |z| < 1,
\end{array} \right.
\]

hence we obtain \( N(z)^{-1} = O(|\lambda|_{m^{1/\beta}}) \) as \( m \to \infty \), uniformly in \( z \) on \( C_i^m \) and in \( \lambda \) on compact subsets of \( |\lambda| > 1 \). Using (7.1)–(7.4) we also obtain the following:

\[
P_1(z) = m^{-\beta} \sigma_3 O(1) \Psi_1 \left( m \ln \frac{z}{i} \right) O(1) = m^{-\beta} \sigma_3 O(1) = O(m^{1/\beta}) \quad \left( |z - i| = \frac{1}{m} \right)
\]
as $m \to \infty$, uniformly in $z$ and in $\lambda$ on compact subsets of $|\lambda| > 1$. Hence $\Delta(z) = P_1(z)N(z)^{-1} - I = O(m^{2|\Re \beta|})$, and therefore
\[
\int_{C_m^{\infty}} (R(z)\Delta(z))_{11} \left( \frac{z - i}{z + i} \right)^\beta \frac{f_m(z)}{z} \, dz = \int_{C_m^{\infty}} O(m^{2|\Re \beta|})O(m|\Re \beta|) \, dz = O\left(m^{-1/4}\right)
\]
as $m \to \infty$, uniformly in $\lambda$ on compact subsets of $|\lambda| > 1$.

Next we show that, by the formulae in Section [7], we have the following for $z \in II \cup III$:
\[
P_1(z)N(z)^{-1} - I = E(z)\Psi_1(\xi)F_1^{-\sigma_3}z^{\sigma_3} \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix} D(z)\sigma_3 - I
\]
\[
= F_1 z^{-\sigma_3/2} \begin{pmatrix} E_{12}(z) & \Psi_{1,22}(\xi) & 0 \\ E_{21}(z) & \Psi_{1,12}(\xi) & 1 \end{pmatrix} D(z)\sigma_3 - I
\]
\[
= F_1 z^{-\sigma_3/2} D(z)^{-1} \begin{pmatrix} E_{12}(z) & \Psi_{1,22}(\xi) & 0 \\ E_{21}(z) & \Psi_{1,12}(\xi) & 1 \end{pmatrix} D(z)\sigma_3 - I
\]
\[
= F_1 z^{-\sigma_3/2} D(z)^{-1} \begin{pmatrix} i\pi z D(z)\xi - \beta F_1 e^{-i\pi \beta} \Psi_{1,22}(\xi) & 0 \\ -i\pi z D(z)\xi - \beta F_1 e^{-i\pi \beta} \Psi_{1,12}(\xi) & 1 \end{pmatrix} D(z)\sigma_3 - I
\]
\[
= \begin{pmatrix} e^{i\pi \beta} e^{-i/2} z - \beta \Psi_{1,22}(\xi) & 0 \\ -i\pi z D(z)^{-1} F_1 e^{-i\pi \beta} \Psi_{1,12}(\xi) & 1 \end{pmatrix} D(z)\sigma_3 - I
\]
\[
= \begin{pmatrix} i\pi z D(z)^{-1} (e^{-i\pi \xi})^2 e^{-i\pi \beta} \psi(1 - \beta, 1, e^{-i\pi \xi}) \Gamma(1 - \beta) F_1^2 & 0 \\ -i\pi z D(z)^{-1} \xi - \beta F_1 e^{-i\pi \beta} \Psi_{1,12}(\xi) & 1 \end{pmatrix} D(z)\sigma_3 - I
\]
In the above we have concentrated on the first column only. Denote the radius of $U_1$ by $\varepsilon$. Then,
\[
(8.9) \quad \int_{(\gamma_m \cap U_1) \setminus C_m^{\infty}} (R(z)\Delta(z))_{11} \left( \frac{z - i}{z + i} \right)^\beta \frac{f_m(z)}{z} \, dz
\]
\[
= \int_{(\gamma_m \cap U_1) \setminus C_m^{\infty}} (R(z)\Delta(z))_{11} \left( \frac{z - i}{z + i} \right)^\beta \frac{f_m(z)}{z} \, dz = \int_{(\gamma_m \cap U_1) \setminus C_m^{\infty}} O(m|\Re \beta|) \frac{dz}{z}
\]
\[
= \int_{(\gamma_m \cap U_1) \setminus C_m^{\infty}} R_{11}(z) \left( (e^{-i\pi \xi})^2 \psi(-\beta, 1, e^{-i\pi \xi}) - 1 \right) O(m|\Re \beta|) \frac{dz}{z}
\]
\[
= \int_{(\gamma_m \cap U_1) \setminus C_m^{\infty}} R_{12}(z) D(z)^{-2} (e^{-i\pi \xi})^2 \psi(1 - \beta, 1, e^{-i\pi \xi}) O(m|\Re \beta|) \frac{dz}{z}
\]
Finally, by substituting $\xi = e^{-i\pi \xi}$, $d\xi = -\frac{d\xi}{m}$, and using the large $\xi$ asymptotics of the confluent hypergeometric function and the usual estimates for $R$ and $D$, we obtain
\[
= \int_{-\ln(1 - e)}^{\ln(1 - e)} \xi^{-\beta} \psi(-\beta, 1, \xi) - 1 \right) O(m|\Re \beta|) \frac{d\xi}{m} + \int_{-\ln(1 - e)}^{\ln(1 - e)} \xi^{-\beta} \psi(1 - \beta, 1, \xi) O(m|\Re \beta| - 1) \frac{d\xi}{m}
\]
\[
= O\left(m|\Re \beta| - 1\right) \int_{1}^{\ln(1 - e)} \xi^{-1/2} d\xi + O\left(m^5|\Re \beta| - 2\right) \int_{1}^{\ln(1 - e)} \xi^{-1/2} d\xi
\]
\[
= O\left(m|\Re \beta| - 1\right) \ln m + O\left(m^5|\Re \beta| - 2\right) \int_{1}^{\ln(1 - e)} \xi^{-1/2} d\xi = O\left(m^{-1/4}\right) \quad \text{as} \quad m \to \infty.
\]
\[\square\]

As last step, we consider the integrals over $\Sigma_j^{i,m}$ and $\Sigma_j^{\prime i,m}$ ($j = 1, 2$).

**Proposition 8.4.** We have
\[
(8.10) \quad \int_{\Sigma_1^{i,m} \cup \Sigma_2^{i,m} \cup \Sigma_1^{\prime i,m} \cup \Sigma_2^{\prime i,m}} (R(z)\Delta(z) - \Delta_- (z))_{11} \left( \frac{z - i}{z + i} \right)^\beta \frac{f_m(z)}{z} \, dz = O(m^{-1/4}).
\]
as $m \to \infty$, uniformly in $\lambda$ on compact subsets of $|\lambda| > 1$.

**Proof.** First, we notice that the integral in (8.7) over $\Sigma^{i,j}_{m}(j = 1, 2)$ vanishes. Indeed, we have $\Delta(z) - \Delta(-z) = (P_{1,+}(z) - P_{1,-}(z)) N(z)^{-1}$ and the jump of $P_{1}$ is exactly the same as that of $S$. Therefore, similarly as in (8.4) we obtain that $(R(z)[\Delta_{+}(z) - \Delta_{-}(z)])_{11} = 0$.

In the rest of the proof, we shall only deal with the integral over $\Sigma^{i,m}_{1}$, and note that the other integral over $\Sigma^{i,m}_{2}$ can be handled very similarly, since the local parametrix does not jump along $\Gamma_{7}$. Note that for $z \in \Sigma^{i,m}_{1}$ we have

$$
\Delta_{+}(z) - \Delta_{-}(z) = (P_{1,+}(z) - P_{1,-}(z)) N(z)^{-1} = P_{1,-}(z) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} D(z)^{-\sigma_{3}}
$$

$$
= m^{-\beta \sigma_{3} i \frac{\pi}{2}} \begin{pmatrix} 0 & \tilde{E}_{21}(z) \\ \tilde{E}_{21}(z) & 0 \end{pmatrix} \begin{pmatrix} \psi_{1,-}(\zeta) F_{1,-}\sigma_{3} z^{-m/2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \tilde{E}_{21}(z) & 0 \end{pmatrix} \begin{pmatrix} \psi_{1,-}(\zeta) F_{1,-}\sigma_{3} z^{-m/2} & 0 \\ 0 & 0 \end{pmatrix} D(z)^{-\sigma_{3}}
$$

Thus, the integral over $\Sigma^{i,m}_{1}$ has the following form:

$$
\int_{\Sigma^{i,m}_{1}} R_{11}(z)m^{-\beta i \frac{\pi}{2}} \tilde{E}_{21}(z) \psi_{1,-}(\zeta) \phi(z)^{-1} z^{-m/2} F_{1,1} \tilde{f}_{m}(z) \frac{dz}{z}
$$

$$
+ \int_{\Sigma^{i,m}_{1}} R_{12}(z)m^{\beta i \frac{\pi}{2}} \tilde{E}_{21}(z) \psi_{1,-}(\zeta) \phi(z)^{-1} z^{-m/2} F_{1,1} \tilde{f}_{m}(z) \frac{dz}{z}.
$$

Note that we have

$$
\Psi_{1}(\zeta) = \Psi_{1}(\zeta) J_{1}^{-1} J_{8}^{-1} = \Psi_{1}(\zeta) \begin{pmatrix} 1 & -e^{-i\beta} \\ e^{i\beta} & 0 \end{pmatrix} = \begin{pmatrix} -e^{-i\beta} \Psi_{1,11}(\zeta) \\ -e^{-i\beta} \Psi_{1,12}(\zeta) \end{pmatrix} \quad (\zeta \in \Pi).
$$

Therefore, if we use the estimation $|f_{m}(z)| \leq \sum_{l=1}^{m} \frac{|z|^{l-m}}{l} \leq \sum_{l=1}^{m} \frac{1}{l} (|z| > 1)$, the integral (8.11) becomes

$$
\int_{\Sigma^{i,m}_{1}} O \left( m^{-\beta} \right) \Psi_{1,-22}(\zeta) z^{-m/2} \tilde{f}_{m}(z) \frac{dz}{z} = \int_{\Sigma^{i,m}_{1}} O \left( m^{-\beta} \right) \begin{pmatrix} \Psi_{1,11}(\zeta) z^{-m/2} \\ \Psi_{1,12}(\zeta) z^{-m/2} \end{pmatrix} \frac{dz}{z}
$$

$$
= \int_{\Sigma^{i,m}_{1}} \left( m^{-\beta} \right) \left( F_{1,m}(z) - \frac{1}{2} \ln \frac{z-i}{z+i} \right) \frac{dz}{z}
$$

$$
= \int_{\Gamma_{0}\cap \{1+O(1/m) \leq |z| \leq Cm\}} \left( m^{-\beta} \ln m \right) \zeta^{-1-\beta} \frac{dz}{z} = O \left( m^{-\beta} m^{-1} (m^{1/4} + 1) \ln m \right) = O(m^{-1/2}).
$$
where $C > 0$ is a constant, $\frac{dz}{z} = \frac{dm}{m}$, and we used the large $\zeta$ asymptotics of $\psi(1 + \beta, 1; \zeta)$. The other integral (8.12) can be estimated somewhat similarly as follows:

$$\int_{\gamma_1} O \left( m^{-1+2|R\beta|} \right) m^\beta \Psi_{1,1}(\zeta) z^{-m/2} \tilde{f}_m(z) \frac{dz}{z} = \int_{\gamma_1} O \left( m^{-1+2|R\beta|} \right) m^\beta \Psi_{1,1}(\zeta) z^{-m/2} \tilde{f}_m(z) \frac{dz}{z}$$

$$= \int_{\gamma_1} O \left( m^{-1+2|R\beta|} \right) m^\beta \Psi(\beta, 1; \zeta) \tilde{f}_m(z) \frac{dz}{z}$$

$$= \int_{\Gamma \cap \{ 1 + O(1/m) \leq |\zeta| \leq Cm \}} O \left( m^{-1+2|R\beta|} \ln m \right) \left( \frac{\zeta}{m} \right)^{-\beta} \frac{d\zeta}{m}$$

$$= O \left( m^{-1+2|R\beta|} \ln m \right) \int_0^C t^{-R\beta} dt = O \left( m^{-1+2|R\beta|} \ln m \right) = O \left( m^{-1/2} \right).$$

With the above proof we have finished proving Lemma 3.2 which we use in the next section.

9. Calculating the Mutual Information

In this section we prove Theorem 2.1, that is, we calculate (2.9). We start with a lemma.

Lemma 9.1. We have

$$|\tan \left( \frac{\pi}{2} \beta \right)| < 1 \quad (\lambda \in \mathbb{C} \setminus [-1, 1]).$$

Proof. This is a simple geometric observation. Consider the parallelogram on the complex plane with vertices $0, e^{i\pi/2}, e^{-i\pi/2}$, and $e^{i\pi/2} + e^{-i\pi/2}$. Notice that $|\arg(e^{i\pi/2} + e^{-i\pi/2})| < \pi/4$, hence the angle in the parallelogram at $0$ is less than $\pi/2$. Therefore $|e^{i\pi/2} - e^{-i\pi/2}| < |e^{i\pi/2} + e^{-i\pi/2}|$, which completes the proof.  

Integration by parts gives the following for all $\varepsilon > 0$ and $m, n \in \mathbb{N}$:

$$\int_{\Gamma} e^{(1 + \varepsilon, \lambda)} \frac{d}{d\lambda} \ln \tilde{D}(\lambda) d\lambda = \frac{1}{2\pi i} \int_{\Gamma} \ln \tilde{D}(\lambda) \frac{d}{d\lambda} e^{(1 + \varepsilon, \lambda)} d\lambda$$

$$= \frac{1}{2\pi i} \int_{\Gamma} \ln \left( 1 + \left( \frac{\pi}{2} \beta \right)^2 + O(m^{-1/4}) \right) \frac{d}{d\lambda} e^{(1 + \varepsilon, \lambda)} d\lambda$$

Therefore, equation (2.9) becomes

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\Gamma} \ln \left( 1 + \tan^2 \left( \frac{\pi}{2} \beta \right) \right) \frac{d}{d\lambda} e^{(1 + \varepsilon, \lambda)} d\lambda$$

(9.1)

$$= \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\Gamma} e^{(1 + \varepsilon, \lambda)} \frac{d}{d\lambda} \ln \left( 1 + \tan^2 \left( \frac{\pi}{2} \beta \right) \right) d\lambda,$$

where we used integration by parts. Note that

$$\frac{d}{d\lambda} \ln \left( 1 + \tan^2 \left( \frac{\pi}{2} \beta \right) \right) = \frac{d}{d\beta} \ln \left( 1 + \tan^2 \left( \frac{\pi}{2} \beta \right) \right) \frac{d\beta}{d\lambda} = -i \tan \left( \frac{\pi}{2} \beta \right) \frac{1}{1 - \lambda^2}.$$

Hence, equation (9.1) becomes

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\Gamma} e^{(1 + \varepsilon, \lambda)} \tan \left( \frac{\pi}{2} \beta \right) \frac{1}{1 - \lambda^2} d\lambda.$$
Note that by Cauchy’s theorem there is a flexibility in choosing \( \Gamma_\varepsilon \). We observe that for \( |\lambda - 1| = \frac{\varepsilon}{2} \) we have \( \varepsilon (1 + \varepsilon, \lambda) = -\frac{1+\varepsilon+\lambda}{2} \ln \frac{1+\varepsilon+\lambda}{2} - \frac{1+\varepsilon-\lambda}{2} \ln \frac{1+\varepsilon-\lambda}{2} = O(\varepsilon) + O(\varepsilon \ln \varepsilon) = O(\varepsilon \ln \varepsilon) \) as \( \varepsilon \to 0 \). Therefore, if \( C_1^\varepsilon \) denotes the circle around 1 with radius \( \frac{\varepsilon}{2} \), then

\[
\lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{C_1^\varepsilon} e(1 + \varepsilon, \lambda) \tan \left( \frac{\pi}{2} \beta \right) \frac{1}{1 - \lambda^2} d\lambda = \int_{C_1^\varepsilon} O(\ln \varepsilon) d\lambda = 0.
\]

We similarly get that the integral over \( C_{-1}^\varepsilon \) converges to 0. Therefore, by the Lebesgue dominant convergence theorem, we conclude that (9.1) is equal to the following:

\[
\lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{-1+\frac{\varepsilon}{2}}^{1-\frac{\varepsilon}{2}} e(1 + \varepsilon, \lambda) \left[ \tan \left( \frac{\pi}{2} \beta_+ \right) - \tan \left( \frac{\pi}{2} \beta_- \right) \right] \frac{1}{1 - \lambda^2} d\lambda = \frac{1}{2\pi i} \int_{-1}^{1} e(1, \lambda) \left[ \tan \left( \frac{\pi}{2} \beta_+ \right) - \tan \left( \frac{\pi}{2} \beta_- \right) \right] \frac{1}{1 - \lambda^2} d\lambda
\]

(9.2)

where the interval \([-1, 1]\) is oriented from the left to the right, hence its +/- sides are its below/upper sides. Note that \( \lambda \to \beta \) transforms \( \mathbb{C} \setminus [-1, 1] \) onto \( \mathbb{C} \setminus (-\infty, 0] \). Also, if \((-\infty, 0]\) is oriented from the left to right, then the + side of \([-1, 1]\) is mapped onto the + side of \((-\infty, 0]\). In particular,

\[
\beta_\pm = \frac{1}{2\pi i} \left( \ln \left| \frac{\lambda+1}{\lambda-1} \right| \pm \pi i \right) = \frac{1}{2\pi i} \left( \ln \left( \frac{1+\lambda}{1-\lambda} \right) \pm \pi i \right) \quad (\lambda \in (-1, 1)),
\]

and hence a straightforward calculation shows that

\[
\tan \left( \frac{\pi}{2} \beta_+ \right) - \tan \left( \frac{\pi}{2} \beta_- \right) = 2\sqrt{1 - \lambda^2} \quad (\lambda \in (-1, 1)).
\]

Now, plugging in the above into (9.2) we obtain

\[
-\frac{1}{\pi} \int_{-1}^{1} \left( \frac{1+\lambda}{2} \ln \frac{1+\lambda}{2} + \frac{1-\lambda}{2} \ln \frac{1-\lambda}{2} \right) \frac{1}{\sqrt{1-\lambda^2}} d\lambda = -\frac{2}{\pi} \int_{0}^{1} \frac{t}{1-t} \ln t \, dt = -\frac{8}{\pi} \int_{0}^{1} \frac{v^2}{\sqrt{1-v^2}} \ln v \, dv
\]

\[
= \frac{8}{\pi} \left( \int_{0}^{1} \frac{v^2}{\sqrt{1-v^2}} \ln v \, dv - \int_{0}^{1} \frac{\ln v}{\sqrt{1-v^2}} \, dv \right) = 2 \ln 2 - 1.
\]

Above we performed two substitutions \( 2t = 1 + \lambda \) and \( t = v^2 \). Since the last two special integrals are well known to be equal to \(-\frac{\pi}{8} - \frac{\pi}{4} \ln 2 \) and \(-\frac{\pi}{2} \ln 2 \), respectively, the proof of Theorem 2.1 is complete.

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