QUASIPOSITIVE ANNULI (CONSTRUCTIONS OF QUASIPOSITIVE KNOTS AND LINKS, IV)

LEE RUDOLPH
Department of Mathematics and Computer Science
Clark University, Worcester, Massachusetts 01610

ABSTRACT

The modulus of quasipositivity $q(K)$ of a knot $K$ was introduced as a tool in the knot theory of complex plane curves, and can be applied to Legendrian knot theory in symplectic topology. It has also, however, a straightforward characterization in ordinary knot theory: $q(K)$ is the supremum of the integers $f$ such that the framed knot $(K, f)$ embeds non-trivially on a fiber surface of a positive torus link. Geometric constructions show that $-\infty < q(K)$, calculations with link polynomials that $q(K) < \infty$. The present paper aims to provide sharper lower bounds (by optimizing the geometry with positive plats) and more readily calculated upper bounds (by modifying known link polynomials), and so to compute $q(K)$ for various classes of knots, such as positive closed braids (for which $q(K) = \mu(K) - 1$) and most positive pretzels. As an aside, it is noted that a recent result of Kronheimer & Mrowka implies that $q(K) < 0$ if $K$ is slice.

Keywords: fence diagram, link polynomial, modulus of quasipositivity, positive plat

1. Review of Background Material

By default, manifolds are piecewise-smooth, compact, unbounded, and oriented; $-X$ denotes $X$ with orientation reversed.

1.1. Annular Seifert Surfaces and Framed Links

A surface is annular if each component is an annulus. A Seifert surface $S \subset S^3$ is a 2-submanifold-with-boundary such that each component of $S$ has non-empty boundary. Let $K \subset S^3$ be a knot, $f \in \mathbb{Z}$; by $A(K, f)$ denote any annulus in $S^3$ with $K \subset \partial A(K, f)$, $\text{link}(K, \partial A(K, f) \setminus K) = -f$. Up to ambient isotopy, $A(K, f)$ depends only on $K$ and $f$, and $A(K, f) = A(-K, f) = -A(K, f)$. Let $L \subset S^3$ be a link with components $K_i$, $f : L \to \mathbb{Z}$ a framing of $L$ (i.e., a continuous function); by $A(L, f)$ denote any union of pairwise disjoint annuli $A(K_i, f(K_i))$. Any annular Seifert surface has the form $A(L, f)$. A framed link $(L, f)$ is embedded on a Seifert surface $S$ if $L \subset S$ and the regular neighborhood $N_S(L)$ is $A(L, f)$.

Lemma 1. If $(K_k, f_k)$ is embedded on $S_k$, $k = 1, 2$, then the connected sum $(K_1 \# K_2, f_1 + f_2)$ embeds on a boundary-connected sum $S_1 \# S_2$ ($\varepsilon \in \{+,-\}$).

Proof. Fig. 1. \qed
Remark 1. If $S$ is a Seifert surface with connected boundary $K$, then $(K, 0)$ and $(-K, 0)$ both embed on $S$. If $K$ is non-invertible, that is, not isotopic to $-K$, then $(K \parallel -K, 0)$ embeds on $S \setminus -S$ but need not embed on $S \setminus S$.

1.2. Closed Braids, Plats, Band Representations, and Braided Surfaces

For present purposes, we may define the $n$-string braid group by its standard presentation

$$B_n := \text{gp} \left( \sigma_i, 1 \leq i \leq n-1 \mid [\sigma_i, \sigma_j] = \sigma_j^{-1} \sigma_i, |i - j| = 1 \right).$$

A braidword in $B_n$ is a $k$-tuple $b = (\sigma_{i(1)}^{\varepsilon(1)}, \ldots, \sigma_{i(k)}^{\varepsilon(k)}), \varepsilon(s) \in \{1, -1\}$; the braid of $b$ is $\beta(b) := \sigma_{i(1)}^{\varepsilon(1)} \cdots \sigma_{i(k)}^{\varepsilon(k)}$. It is usual to picture a braidword by a braidword diagram with $2n$ loose ends, $n$ at the top and $n$ at the bottom, cf. Fig. 2.

Let $\beta \in B_n$. Fig. 2(1) indicates the construction of a link $\hat{\beta} \subset S^3$, called the closed braid of $\beta$: let $b$ be any braidword with $\beta = \beta(b)$; then a link diagram for $\hat{\beta}(b) := \beta(b)$ consists of the braidword diagram of $b$, together with $n$ arcs joining the loose ends at the bottom to those at the top so as to create no new crossings. The closed braid $\hat{\beta}$ depends only on $\beta$ (not on $b$), is well-defined up to ambient isotopy, and has the canonical orientation indicated in Fig. 2(1).
Let $n$ be even. A *plat-plan* is a pair $\Pi = (\pi_{\cup}, \pi_{\cap})$ of permutations in $S_n$ each of which is the product of $n/2$ disjoint transpositions such that $s < t < \pi(s) \Rightarrow s < \pi(t) < \pi(s)$. Fig. 3(2) indicates how, given a plat-plan $\Pi$ and $\beta \in B_n$, to construct an unoriented link $\beta^\Pi \subset S^3$, called the *Pi-plat* of $\beta$: let $b$ be any braidword with $\beta = \beta(b)$; then a link diagram for $\beta^\Pi(b) := \beta(b)^\Pi$ consists of the braidword diagram of $b$, together with $n/2$ arcs joining the loose ends at the bottom according to $\pi_{\cap}$, and $n/2$ arcs joining the loose ends at the top according to $\pi_{\cup}$, all so as to create no new crossings. The Pi-plat $\beta^\Pi$ depends only on $\beta$ (not on $b$) and is well-defined up to ambient isotopy; it has no canonical orientation. Where possible, the particular plat-plan $\Pi$ is suppressed and a Pi-plat is simply called a *plat*.

**Remark 2.** The *writhe* (i.e., algebraic crossing number) of a knot diagram (as opposed to a multi-component link diagram) derived as above from a braidword diagram of $b$, whether for the closed braid or the Pi-plat of $\beta(b)$, depends only on $\beta(b)$; in the case of the closed braid, this writhe equals $e(\beta(b))$, the exponent sum of $\beta(b)$ (with respect to the standard generators $\sigma_i$ of $B_n$), but there seems to be no similarly neat expression for the writhe of a plat.

A positive embedded band in $B_n$ is one of the $\binom{n}{2}$ braids

$$\sigma_{i,j} := (\sigma_i \cdots \sigma_{j-2})\sigma_{j-1}(\sigma_i \cdots \sigma_{j-2})^{-1}, \quad 1 \leq i < j \leq n;$$

a negative embedded band is the inverse of a positive embedded band. An embedded band representation of length $k$ in $B_n$ is a $k$-tuple $b = (b(1), \cdots, b(k))$ of embedded
bands; as with braidwords (which are embedded band representations, since \( \sigma_i = \sigma_{i,i+1} \)), write \( \beta(b) := b(1) \cdots b(k), \hat{\beta}(b) := \hat{\beta}(\beta(b)), \beta^\Pi(b) := \beta(b)^\Pi \). There is a straightforward construction of a braided Seifert surface

\[
S(b) = \bigcup_{s=1}^n h_i^{(0)} \cup \bigcup_{t=1}^k h_j^{(1)}
\]

given as the union of \( n \) 0-handles and \( k \) 1-handles, with \( \partial S(b) = \hat{\beta}(b) \) (cf. Fig. 4).

![Fig. 4. A braided Seifert surface \( S(b) \), \( b = (\sigma_{1,2}, \sigma_{2,3}, \sigma_{1,3}) \).](image)

Call \( b \) weakly annular if \( S(b) \) is annular, and annular if also every 0-handle \( h_i^{(0)} \) of \( S(b) \) is attached to two 1-handles \( h_i^{(1)}, h_j^{(1)} \) (rather than to only a single 1-handle).

**1.3. Quasipositivity**

An embedded band representation \( b \) is quasipositive if each \( b(t) \) is positive. A Seifert surface \( S \) is quasipositive if, for some quasipositive \( b \), \( S \) is ambient isotopic to \( S(b) \). In this case, \( b \) may always be taken to be such that no 0-handle of \( S(b) \) is attached to precisely one 1-handle; in particular, if \( S \) is annular, then \( b \) may also be taken to be annular (rather than merely weakly annular).

**Example 1.** A braidword \( p \) is quasipositive if and only if it is positive (i.e., \( \varepsilon(t) = 1 \) for all \( t \)). A positive closed braid \( \hat{\beta}(p) \) has many special properties (cf. Fig. 4), among them that each component of \( S(p) \) is a fiber surface, so the split components of \( (S^3, \hat{\beta}(p)) \) are fibered links. In particular, for \( m, n > 0 \), the torus link \( O\{m, n\} \) is the non-split positive closed braid \( \hat{\beta}(o\{m, n\}) \), where \( o\{m, n\}(t) := \sigma_t \in B_m \) for \( t \equiv i \) (mod \( m-1 \)), \( 1 \leq t \leq n(m-1) \); \( O\{m, n\} \) is the link of the singularity at
the origin of the complex plane curve \((z, w) \in \mathbb{C}^2 : z^m + w^n = 0\), as well as the link at infinity of the same curve, so its fiber surface \(S(o \{m, n\})\) can be realized as \(\{(z, w) \in S^3 : z^m + w^n \geq 0\} \subset S^3 := \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}\).

Let \(S\) be a surface. A full subset \(X\) of \(S\) is one such that no component of \(S \setminus X\) is contractible (e.g., a collar of \(\partial S\) is full iff no component of \(S\) is \(D^2\); a simple closed curve on \(S\) is full iff it does not bound a disk on \(S\)).

**Theorem 1.** (iii) If \(S\) is a Seifert surface, then the following are equivalent.

(i) \(S\) is quasipositive.

(ii) \(S\) is a full subsurface of a fiber surface of a positive closed braid.

(iii) \(S\) is a full subsurface of a fiber surface of a positive torus link.

(iv) \(S\) is a full subsurface of a fiber surface of \(O\{n, n\}\) for some \(n > 0\).

**Corollary 1.** A full subsurface of a quasipositive Seifert surface is quasipositive.

The *modulus of quasipositivity* of a knot \(K\) is

\[
q(K) := \sup\{ f \in \mathbb{Z} : A(K, f) \text{ is quasipositive} \} = \sup\{ f : A(K, f) \text{ is full on some quasipositive surface} \} = \sup\{ f : \text{for some } n, A(K, f) \text{ is full on } S(o \{n, n\})\}
\]

(the equalities following from Corollary 1). For any \(K\), \(-\infty < q(K)\) by \(\mathbb{3}\) and \(q(K) < \infty\) by \(\mathbb{4}\).

**Proposition 1.** (\(\mathbb{4}\)) For any knot \(K\), if \(f \leq q(K)\) then \(A(K, f)\) is quasipositive.

**Proof.** For \(n \geq 1\), a collar of a component of the boundary of a fiber surface of \(O\{n + 1, n + 1\}\) is an annulus \(A(O, -n)\); the proposition follows from Lemma \(\mathbb{4}\) Corollary 1 and the observation that any boundary-connected sum of quasipositive Seifert surfaces is quasipositive (e.g., \(S(\sigma_1, \sigma_1, \sigma_2, \sigma_2, \sigma_2) \# S(\sigma_1, \sigma_1, \sigma_1)\)) is either \(S(\sigma_1, \sigma_1, \sigma_2, \sigma_2, \sigma_3, \sigma_4, \sigma_4, \sigma_4)\) or \(S(\sigma_1, \sigma_1, \sigma_2, \sigma_2, \sigma_2, \sigma_1, \sigma_4, \sigma_4, \sigma_4)\). \(\square\)

**Corollary 2.** For any knots \(K_k\), \(q(\#_k K_k) \geq \sum_k q(K_k)\).

**Proof.** By Lemma \(\mathbb{4}\) \((\#_k K_k, \sum_k f_k)\) embeds on \(\#_k A(K_k, f_k)\) (since \(A(K, f) = -A(K, f)\) for all \((K, f)\)). \(\square\)

In Proposition \(\mathbb{4}\) this is improved to \(q(\#_k K_k) + 1 \geq \sum_k (q(K_k) + 1)\).

### 1.4. *Link Polynomials*

If \(L_+\), \(L_0\), and \(L_-\) are links with diagrams which are identical except as indicated in Fig. \(\mathbb{4}\), and the visible crossing in \(L_+\) involves one component (resp., two components), then we say we are in case 1 (resp., case 2), we let \(p\) (resp., \(q\)) be the linking number of the right-hand visible component of \(L_0\) with the rest of \(L_0\) (resp., the linking number of the lower visible component of \(L_+\) with the rest of \(L_+\)), and we denote by \(L_\infty\) the link indicated in Fig. \(\mathbb{4}(1)\) (resp., Fig. \(\mathbb{4}(2)\)).
The oriented (or FLYPMOTH \cite{10, 11}) and semi-oriented (or Kauffman \cite{12}) polynomials $P_L(v, z) \in \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ and $F_L(a, x) \in \mathbb{Z}[a^{\pm 1}, x^{\pm 1}]$ of a link $L$ can be defined recursively as follows:

\[
P_{L^0}(v, z) = 1 = F_{L^0}(a, x) \text{ if } L = O \text{ is an unknot},
\]

\[
P_{L^+}(v, z) = vzP_{L^0}(v, z) + v^2P_{L^-}(v, z),
\]

\[
F_{L^+}(a, x) = a^{-1}xF_{L^0}(a, x) - a^{-2}F_{L^-}(a, x) + \begin{cases} a^{-4p-1}xF_{L_\infty}(a, x) & \text{in case 1,} \\ a^{-4q+1}xF_{L_\infty}(a, x) & \text{in case 2} \end{cases}
\]

For any ring $\mathfrak{R}$, for any Laurent polynomial $H(s) \in \mathfrak{R}[s^{\pm 1}]$, write $\text{ord}_s H(s) := \sup\{n \in \mathbb{Z} : s^{-n}H(s) \in \mathfrak{R}[s] \subset \mathfrak{R}[s^{\pm 1}]\}$, $\text{deg}_s H(s) := -\text{ord}_s H(s^{-1})$. Let $c(L)$ be the number of components of $L$. Easy inductions establish the following estimates.

**Lemma 2.** For every link $L$, $\text{ord}_z P_L \geq 1 - c(L)$ and $\text{ord}_x F_L \geq 1 - c(L)$.

**Corollary 3.** $\left((z^{c(L)-1}P_L(v, z))\right|_{z=0}$ and $\left((x^{c(L)-1}F_L(a, x))\right|_{x=0}$ are well-defined link invariants (in $\mathbb{Z}[v^{\pm 1}]$ and $\mathbb{Z}[a^{\pm 1}]$, respectively).

Let $R_L(v) := (z^{c(L)-1}P_L(v, z))|_{z=0}$.

**Lemma 3.** (cf. \cite{15}) $\left((-\sqrt{-1}x)^{c(L)-1}F_L(a, x))\right|_{x=0} = R_L(\sqrt{-1}a^{-1})$.

**Lemma 4.** $R_L(v)$ can be calculated recursively as follows:

\[
R_O(v) = 1,
\]

\[
R_{L^+}(v) = (2 - k)vR_{L^0}(v) + v^2R_{L^-}(v) \text{ in case } k (k = 1, 2).
\]

**Corollary 4.** If links $L_1$ and $L_2$ are disjoint (i.e., if $L_1 \cup L_2$ is a link), then

\[
R_{L_1 \cup L_2}(v) = (v^{-1} - v)v^{2\text{link}(L_1, L_2)} R_{L_1}(v)R_{L_2}(v).
\]
If \( L \) has components \( K_i \), then its total linking is \( \tau(L) := \sum_{i<j} \text{link}(K_i, K_j) \). If \( f \) is a framing of \( L \), then the total framing of the framed link \((L, f)\) is \( \varphi(L, f) := \sum_i f(K_i) \). Define the framed polynomial to be

\[
(L, f)(v, z) := (-1)^\varepsilon(L)(1 + (v^{-1} - v)z^{-1} \sum_{L'} (-1)^\varepsilon(L') P_{\partial A(L', f(L'))})
\]

where \( L' \) runs through the non-empty sublinks of \( L \).

**Proposition 2.** \( \{L, f\} = v^{-2\varphi(L, f)}\{L, 0\} \).

The framed polynomial provides a bridge between the oriented and semi-oriented polynomials, as the following result (proved in [1]) makes plain.

**Theorem 2.** \((1 + (v^{-2} + v^2)z^{-2})F_L(v^{-2}, z^2) \equiv v^{\tau(L)}\{L, 0\}(v, z) \mod 2 \).

Let \( F^*_L := (F_L \mod 2) \in (\mathbb{Z}/2\mathbb{Z})[a^{\pm 1}, x^{\pm 1}], G^*_L(a) := (x^{1-\varepsilon(L)}F^*_L(a, x)) \mod 2 \). Thus \( G^*_L(a) = R_L(a^{-1}) \mod 2 \), and can be calculated using the formulas in Lemma 4 reduced mod 2.

**Lemma 5.** The polynomial \( G^*_L \) can be calculated recursively as follows:

\[
G^1_{L+}(a) = 1, \\
G^1_{L-}(a) = a^{-4G^1_{L+}(a)} + a^{-1}G^1_{L0}(a) + \begin{cases} a^{-4p-1}G^1_{L+}(a) & \text{in case 1,} \\ a^{-4q+1}G^1_{L+}(a) & \text{in case 2.} \end{cases}
\]

2. Lower Bounds for the Modulus of Quasipositivity

2.1. **Summary of Results**

In this section we use fences to show that, in contrast to positive closed braids (some special properties of which were mentioned above in Example 3), positive plats are not at all special: every (unoriented) link is ambient isotopic to a plat of a positive braidword (Construction 3). We then express \( q(K) \) in terms of positive plat realizations of \( K \) (Corollary 3), thus bounding \( q(K) \) from below (Theorem 3).

2.2. **Fences**

A post is a vertical segment \( \{x\} \times [a, b] \subset \mathbb{R}^2, a < b \). A wire is a horizontal segment \( [c, d] \times \{y\} \subset \mathbb{R}^2, c < d \). A fence \( \Phi \subset \mathbb{R}^2 \) is the union of \( n \geq 1 \) posts with pairwise distinct abscissae and \( k \geq 0 \) wires with pairwise distinct ordinates, such that both endpoints of each wire lie on posts; cf. Fig. 3(1).

Let \( X(\Phi) \) denote the set of abscissae of posts of \( \Phi \), \( Y(\Phi) \) the set of ordinates of wires of \( \Phi \); let \( x_1 < \cdots < x_n \) be the elements of \( X(\Phi) \), \( y_1 < \cdots < y_k \) the elements of \( Y(\Phi) \). For \( 1 \leq t \leq k \), define \( i(t) \) and \( j(t) \) by the requirement that \([x_{i(t)}, x_{j(t)}]\) be a wire of \( \Phi \). A graph of \( \Phi \) (cf. Fig. 3(2)) is any 1-dimensional polyhedron \( \text{gr } \Phi \subset (\mathbb{R}^2 \times \{0\}) \cup (\mathbb{R} \times Y(\Phi) \times [0, \infty]) \subset \mathbb{R}^3 \) such that

(i) \( \text{gr}(\Phi) \cap (\mathbb{R}^2 \times \{0\}) \) is the union of the posts of \( \Phi \subset \mathbb{R}^2 = \mathbb{R}^2 \times \{0\} \),
Fig. 7. A fence $\Phi$ and a graph $\text{gr}(\Phi)$.

(ii) the restriction to the closure in $\mathbb{R}^3$ of $\text{gr}(\Phi) \cap (\mathbb{R} \times \text{Y}(\Phi) \times [0, \infty[)$ of the projection $\text{pr}_{1,2}$ is a homeomorphism onto the union of the wires of $\Phi$.

Any two graphs of $\Phi$ are vertically isotopic with $\Phi$ fixed.

A charge on $\Phi$ is a function $\varepsilon : \text{Y}(\Phi) \to \{1, -1\}$; $(\Phi, \varepsilon)$ is a charged fence.

2.3. Constructions with Fences

Construction 1. Given an embedded band representation $b$ of length $k$ in $B_n$, with $b(t) := \sigma_{i(t),j(t)}$ for $t = 1, \ldots, k$, construct a charged fence $(\Phi[b], \varepsilon[b])$ as follows: the posts of $\Phi[b]$ are $\{s\} \times [1, k]$ for $s = 1, \ldots, n$, its wires are $[i(t), j(t)] \times \{t\}$ for $t = 1, \ldots, k$, and the charge is $\varepsilon[b](t) := \varepsilon(t)$.

In particular, if $b$ is quasipositive, then $\varepsilon[b](t) = 1$ for all $t$; we will write + (rather than 1) for this charge.

Construction 2. Conversely, given a charged fence $(\Phi, \varepsilon)$, construct an embedded band representation $b[\Phi, \varepsilon]$ of length $k := \text{card}(\text{Y}(\Phi))$ in $B_{\text{card}(\text{X}(\Phi))}$, by setting $b[\Phi, \varepsilon](t) := \sigma_{i(t),j(t)}^{\varepsilon(t)}$.

Constructions 1 and 2 show that embedded band representations and charged fences are essentially the same: $b \mapsto (\Phi[b], \varepsilon[b])$ and $(\Phi, \varepsilon) \mapsto b[\Phi, \varepsilon]$ are mutual inverses up to an obvious equivalence relation on charged fences.

A fence $\Phi$ is annular if $\text{gr}(\Phi)$ is a link. Embellished with over-crossings as in Fig. 7(2), an annular fence becomes an unoriented link diagram for the link which is its graph; call such a diagram an annular fence diagram.

Lemma 6. ([2, 16]) All unoriented links have annular fence diagrams.
**Construction 3.** Given \( \Phi \subset \mathbb{R}^2 \), an annular fence with respect to coordinates \((x, y)\), put \( \xi := x - y \), \( \eta := x + y \), and consider the restriction \( \eta|\Phi \); this has an equal number, say \( m(\Phi) \), of local maxima (at “upper right” corners of \( \Phi \)) and local minima (at “lower left” corners). At each local extremum, apply one of the procedures illustrated in Fig. 8. This replaces \( \Phi \) by an annular fence \( \Phi' \) such that

\[
\begin{align*}
\text{(i) } & \text{gr}(\Phi') \text{ is isotopic to gr}(\Phi) \text{ (the link diagrams obtained from } \Phi \text{ and } \Phi' \text{ are regularly homotopic)}, \\
\text{(ii) } & \eta|\Phi' \text{ also has } m(\Phi) \text{ local maxima (and } m(\Phi) \text{ local minima), and} \\
\text{(iii) } & \eta|\Phi' \text{ has a single local maximum value and a single local minimum value.}
\end{align*}
\]

When we view the link diagram derived from \( \Phi' \) in \((\xi, \eta)\)-coordinates, we see the diagram of a positive braidword in \( B_{2m}(\Phi) \), say \( p[\Phi] \), with loose ends joined at the local extrema of \( Y|\Phi' \) according to a suitable plat-plan, say \( \Pi[\Phi] \), so that \( \beta^\Pi(p[\Phi]) \) and gr(\( \Phi \)) are ambient isotopic, cf. Fig. 9.

**Construction 4.** Conversely, given a positive braidword \( p \) in \( B_{2m} \) and a plat-plan \( \Pi \), construct a diagram for \( \beta^\Pi(p) \) such that each segment has slope 1 or -1,
cf. Fig. 10 viewed in \((\xi, \eta)\)-coordinates, the underlying graph of this diagram is an annular fence \(\Phi[p, \Pi]\).

![Fig. 10](image)

Constructions 3 and 4 show that annular fence diagrams and positive plats are essentially the same: \((p, \Pi) \mapsto \Phi[p, \Pi]\) and \(\Phi \mapsto (p[\Phi], \Pi[\Phi])\) are mutual inverses up to obvious equivalences (for annular fence diagrams, the equivalence is an adaptation of regular homotopy).

### 2.4. Fences and the Modulus of Quasipositivity

We have seen that an annular embedded band representation \(b\), an annular braided surface \(S(b)\), a charged annular fence, and an appropriately framed positive plat all convey the very same information. Specialize to the case that \(b\) is quasipositive, i.e., \(\varepsilon = +\). Nothing is lost by assuming that \(S(b)\) is a single annulus, say \(A(K, f)\). Let \(p := p[\Phi[b]], \Pi := \Pi[\Phi[b]]\), and \(m := m(\Phi[b])\), so that \(\beta^\Pi(p)\) is a positive plat on \(2m\) strings that realizes (the unoriented knot underlying) \(K\).

**Proposition 3.** In this case, the framing \(f\) is equal to the writhe of the annular fence diagram of \(\Phi[b]\) (equivalently, the positive plat diagram of \(\beta^\Pi(p)\)) diminished by \(m\).

**Proof.** It is clear that each crossing in the annular fence diagram makes the same contribution to \(f\) as to the writhe; Fig. 11 shows how each of the \(m\) “upper right” corners adds 1 to the linking of two components of the boundary of the annulus, and thus diminishes the framing by 1. □

**Corollary 5.** \(q(K)\) is the maximum, over all \(m\) and all realizations of \(K\) as a plat \(\beta^\Pi(p)\) of a positive braidword \(p\) in \(B_{2m}\), of the writhe of \(\beta^\Pi(p)\) diminished by \(m\).

**Theorem 3.** If \(\beta^\Pi(p)\) is any realization of \(K\) as a positive \(\Pi\)-plat on \(2m\) strings, then \(q(K)\) is greater than or equal to the writhe of \(\beta^\Pi(p)\) diminished by \(m\); equivalently, if \(gr(\Phi)\) is any realization of \(K\) as the graph of an annular fence, then \(q(K)\)
Fig. 11.

is greater than or equal to the writhe of the annular fence diagram of $\Phi$ diminished by $m(\Phi)$.

**Corollary 6.** ([3]) If $p$ is a positive braidword in $B_n$, then $q(\bar{\beta}(p)) \geq e(\beta(p)) - n$.

**Proof.** As is well-known, if $\beta \mapsto \beta^{(n)}$ denotes the injection $B_n \rightarrow B_{2n}$ which takes $\sigma_i \in B_n$ to $\sigma_i \in B_{2n}$, and $\pi_{i,\pi_j} = (1 2n)(2 2n - 1) \cdots (n n + 1)$, then for all $\beta \in B_n$, the closed braid $\hat{\beta}$ and the $\Pi$-plat $(\beta^{(n)})^{\Pi}$ are ambient isotopic. □

**Proposition 4.** For any knots $K_k$, $q(\hat{\bigcup_k} K_k) + 1 \geq \sum_k (q(K_k) + 1)$.

**Proof.** If $\Phi$ is any annular fence, then the top of the rightmost post of $\Phi$ is a local maximum of $\eta|\Phi$, and no point of the leftmost post of $\Phi$ is a local maximum of $\eta|\Phi$. Let $\Phi_k$ be an annular fence such that $gr(\Phi_k) = K_k$ and the writhe of the annular fence diagram of $\Phi_k$ is $q(K_k) + m(\Phi_k)$. By applying appropriate translations and homotheties, we may assume that the rightmost post of $\Phi_k$ is the leftmost post of $\Phi_{k+1}$ for $k = 1, \ldots, N - 1$. The fence obtained from $\bigcup_k \Phi_k$ by deleting all the interiors of the common posts (which may be suggestively denoted by $\hat{\bigcup_k} \Phi_k$) is annular, the writhe of its fence diagram is the sum of the writhes of the fence diagrams of the $\Phi_k$, $m(\hat{\bigcup_k} \Phi_k) - 1 = \sum_k (m(\Phi_k) - 1)$, and $gr(\hat{\bigcup_k} \Phi_k) = \hat{\bigcup_k} gr(\Phi_k)$; so the proposition follows from Theorem 3. □

**Remark 3.** I do not know if the inequality in Proposition 4 is ever strict.

**Historical remark.** Fences are my synthesis of (i) some diagrams that H. Morton used to describe certain Hopf-plumbed fiber surfaces in 1982 at Les-Plans-sur-Bex, Switzerland, and (ii) “square bridge projections” as described by H. Lyon in 1977 in Blacksburg, Virginia ([16]): an unoriented link is in “square bridge position” if and only if it is the graph of an annular fence (see below). Square bridge projections have frequently been rediscovered—for instance by Thurston, Erlandsson [17], and Kuhn [18], who (jointly and severally) call them “barber-pole projections”.

### 3. Upper Bounds for the Modulus of Quasipositivity

In this section we derive various upper bounds for $q(K)$ (Corollaries 8, 9, 10, and 11), all given in terms of link polynomials and based on a fundamental result of Morton and Franks & Williams.
Corollary 10. ([19], [20]) For any knot \( K \), for all \( \beta \in B_n \), we have \( \text{ord}_v P_{\beta} \geq \epsilon(\beta) - n + 1 \).

Corollary 7. ([4], [21]) If \( b \) is a quasipositive annular embedded band representation, then \( \text{ord}_v P_{\beta(b)} \geq 1 \). Equivalently, if \( A(L, f) \) is quasipositive, then \( \text{ord}_v P_{\partial A(L, f)} \geq 1 \).

Corollary 8. For any knot \( K \), \( q(K) \leq -1 + \text{ord}_v R_K \).

Proof. For all \( L \), \( \text{ord}_v P_L \leq \text{ord}_v R_L \). By Corollary 1\( \hat{\ } \), \( R_{\partial A(K, f)} = (v^{-1} - v)v^{-2f} R_K(v)^2 \) for all \((K, f)\). Let \( \partial A(K, f) \) be quasipositive; then \( 1 \leq \text{ord}_v P_{\partial A(K, f)} \leq \text{ord}_v R_{\partial A(K, f)} \leq -1 - 2f + 2 \text{ord}_v R_K \leq -1 + 2 \text{ord}_v R_K - 2q(K) \) by Corollary 4.

Theorem 4. ([14], [19]) For all \( (K, f) \), \( \text{ord}_v P_{\partial A(K, f)} \geq 1 \).

Proof. Let \( A \) be quasipositive; then \( 1 \geq \text{ord}_v \min\{\text{ord}_v A, \text{ord}_v A\} \) if \( \text{ord}_v A = \text{ord}_v A \).

Corollary 9. ([4], [21]) For any knot \( K \), \( q(K) \leq \frac{1}{2} \text{ord}_v \{K, 0\} \).

Theorem 6. If \( A(L, f) \) is quasipositive, then \( \text{deg}_{z^2} F_L^* \leq -1 - \varphi(L, f) - 2\tau(L) \).

Proof. If \( A(L, f) \) is quasipositive, then
\[
0 \leq \text{ord}_v \{L, f\} - (-1)^{\varphi(L)} \\
\leq \text{ord}_v \{ (1 + \{L, f\}) \mod 2 \} \\
\leq \text{ord}_v (1 + v^{-4\varphi(L) - 2\varphi(L)}(1 + (v^{-2} + v^2)z^2)^2)F_L^*(v^{-2}, z^2)).
\]

But if \( \text{deg}_a(F_L^*) \geq -1 - \varphi(L, f) - 2\tau(L) \), then
\[
0 > -2 - 2\varphi(L, f) - 4\tau(L) - 2\text{deg}_a(F_L^*) \\
= \text{ord}_v (v^{-4\varphi(L) - 2\varphi(L)}(1 + (v^{-2} + v^2)z^2)^2)F_L^*(v^{-2}, z^2)) \\
= \text{ord}_v (1 + v^{-4\tau(L) - 2\varphi(L)}(1 + (v^{-2} + v^2)z^2)^2)F_L^*(v^{-2}, z^2))
\]
since \( \text{ord}_v 1 = 0 \) and \( \text{ord}_v (A + B) = \min\{\text{ord}_v A, \text{ord}_v B\} \) if \( \text{ord}_v A \neq \text{ord}_v B \).

Corollary 10. ([4], [21]) For any knot \( K \), \( q(K) \leq -1 - \text{deg}_a F_K^* \).

Corollary 11. For any knot \( K \), for \( k = 0, 1 \), \( q(K) \leq -1 - \text{deg}_a G_K^k \).

Proof. This follows from Corollary 1\( \hat{\ } \) since \( \text{deg}_a F_K^* = \max\{\text{deg}_a G_K^0, \text{deg}_a G_K^1\} \).
(The case \( k = 0 \) also follows from Theorem 4 and Corollary 8.)

4. Some Calculations of the Modulus of Quasipositivity

4.1. Summary of results

To illustrate the usefulness of the upper and lower bounds derived in the preceding sections, we compute \( q(K) \) exactly for various infinite classes of knots. (The
Proof. For $k$ my guess is that it is at best an inequality, and involves not only $\mu$ ring; there may be some generalization of Corollary 12 to other fibered knots, but enhancement $\lambda$.

Combine Theorem 7 with Corollaries 6 and 8.

**Theorem 7.** If $p$ is a positive braidword in $B_n$, then $\text{ord}_e R_{\hat{\beta}(p)} = e(\beta(p)) - n + 1$, and the coefficient of $e^{(\beta(p)) - n + 1}$ in $R_{\hat{\beta}(p)}$ is a positive integer.

**Proof.** If $n = 1$ then $p$ is empty, $\hat{\beta}(p)$ is an unknot, and the conclusion holds for $p$. Let $n > 1$, and assume the conclusion for all positive braidwords on fewer than $n$ strings, and all positive braidwords on $n$ strings with exponent sum less than $e(\beta(p))$. If $\hat{\beta}(p)$ has more than one component, let $\emptyset \neq L' \subset \hat{\beta}(p)$, $\emptyset \neq L'' := \hat{\beta}(p) \setminus L'$; then there are positive braidwords $p', p''$ in $B_n$ and $B_n'$ (where $n' + n'' = n$), with $L' = \hat{\beta}(p')$, $L'' = \hat{\beta}(p'')$, $2\text{link}(L', L'') = e(\beta(p)) - e(\beta(p')) - e(\beta(p''))$; by the inductive hypothesis and Corollary 4, the conclusion holds for $p$. If $\hat{\beta}(p)$ has exactly one component, then one of the alternatives in Lemma 7 is the case. If no generator appears more than once, then each generator appears exactly once, so $\hat{\beta}(p) = O$ and the conclusion holds for $p$. If a generator appears more than once, then let $q$ be as in Lemma 7, $q := (q(2), \ldots, q(e(\beta(p))))$, $q'' := (q(3), \ldots, q(e(\beta(p))))$; by Lemma 7, $R_{\hat{\beta}(p)} = R_{\hat{\beta}(q)} = vR_{\hat{\beta}(q')} + v^2 R_{\hat{\beta}(q'')}$ so by the inductive hypothesis the conclusion holds for $p$. □

**Corollary 12.** Let $p$ be a positive braidword in $B_n$ such that $\hat{\beta}(p)$ is a knot. Then $q(\hat{\beta}(p)) = e(\beta(p)) - n$.

**Proof.** Combine Theorem 7 with Corollaries 6 and 8. □

**Remark 4.** In terms of the Milnor number of the fibered knot $\hat{\beta}(p)$, Corollary 12 says that $q(\hat{\beta}(p)) = \mu(\hat{\beta}(p)) - 1$. The appearance of $\mu$ is undoubtedly a red herring; there may be some generalization of Corollary 12 to other fibered knots, but my guess is that it is at best an inequality, and involves not only $\mu$ but also the enhancement $\lambda$ (which happens to equal 0 for a positive closed braid); $\lambda$ is sensitive to handedness, as $q$ seems to be and $\mu$ manifestly is not), cf. [20] and references cited therein.

**4.2. Positive Closed Braids**

**Lemma 7.** (1) Let $p$ be a positive braidword in $B_n$. Either no generator $\sigma_i$ of $B_n$ appears more than once in $p$, or there is a positive braidword $q$ in $B_n$ such that $q(1) = q(2)$, $e(\beta(q)) = e(\beta(p))$, and $\hat{\beta}(q)$ is isotopic to $\hat{\beta}(p)$.

**Theorem 7.** If $p$ is a positive braidword in $B_n$, then $\text{ord}_e R_{\hat{\beta}(p)} = e(\beta(p)) - n + 1$, and the coefficient of $e^{(\beta(p)) - n + 1}$ in $R_{\hat{\beta}(p)}$ is a positive integer.

**Proof.** If $n = 1$ then $p$ is empty, $\hat{\beta}(p)$ is an unknot, and the conclusion holds for $p$. Let $n > 1$, and assume the conclusion for all positive braidwords on fewer than $n$ strings, and all positive braidwords on $n$ strings with exponent sum less than $e(\beta(p))$. If $\hat{\beta}(p)$ has more than one component, let $\emptyset \neq L' \subset \hat{\beta}(p)$, $\emptyset \neq L'' := \hat{\beta}(p) \setminus L'$; then there are positive braidwords $p', p''$ in $B_n$ and $B_n'$ (where $n' + n'' = n$), with $L' = \hat{\beta}(p')$, $L'' = \hat{\beta}(p'')$, $2\text{link}(L', L'') = e(\beta(p)) - e(\beta(p')) - e(\beta(p''))$; by the inductive hypothesis and Corollary 4, the conclusion holds for $p$. If $\hat{\beta}(p)$ has exactly one component, then one of the alternatives in Lemma 7 is the case. If no generator appears more than once, then each generator appears exactly once, so $\hat{\beta}(p) = O$ and the conclusion holds for $p$. If a generator appears more than once, then let $q$ be as in Lemma 7, $q := (q(2), \ldots, q(e(\beta(p))))$, $q'' := (q(3), \ldots, q(e(\beta(p))))$; by Lemma 7, $R_{\hat{\beta}(p)} = R_{\hat{\beta}(q)} = vR_{\hat{\beta}(q')} + v^2 R_{\hat{\beta}(q''})$ so by the inductive hypothesis the conclusion holds for $p$. □

**Corollary 12.** Let $p$ be a positive braidword in $B_n$ such that $\hat{\beta}(p)$ is a knot. Then $q(\hat{\beta}(p)) = e(\beta(p)) - n$.

**Proof.** Combine Theorem 7 with Corollaries 6 and 8. □

**Remark 4.** In terms of the Milnor number of the fibered knot $\hat{\beta}(p)$, Corollary 12 says that $q(\hat{\beta}(p)) = \mu(\hat{\beta}(p)) - 1$. The appearance of $\mu$ is undoubtedly a red herring; there may be some generalization of Corollary 12 to other fibered knots, but my guess is that it is at best an inequality, and involves not only $\mu$ but also the enhancement $\lambda$ (which happens to equal 0 for a positive closed braid); $\lambda$ is sensitive to handedness, as $q$ seems to be and $\mu$ manifestly is not), cf. [20] and references cited therein.

**4.3. Two-strand Torus Knots**

**Theorem 8.** $q(O\{2, 2k + 1\})$ equals $2k - 1$ for $k \geq 0$, $-1$ for $k = -1$, and $4k + 2$ for $k \leq -2$.

**Proof.** The cases with $k \geq -1$ are already done: if $k \geq 0$, then represent $O\{2, 2k + 1\}$ as the closed positive 2-string braid $\sigma_1^{2k+1}$ and apply Corollary 12; if $k = -1$, then note that $O\{2, -1\} = O$.
Let $k \leq -2$. Put $r := -(2k + 1) \geq 3$ and represent $O\{2,2k+1\}$ as the positive II-plat $(\sigma_1 \sigma_3 \ldots \sigma_5)^{2k}$ on $2r$ strings, where $\pi_{\perp} = \pi_{\cap} = (1 2r)(2 3) \cdots (2r - 2 2r - 1)$. Then the writhe of the plat diagram is $-r$, so by Theorem 8, $q(O\{2,2k+1\}) \geq -r - \frac{1}{2}(2r) = 4k + 2$. To prove the opposite inequality and finish the proof, it suffices by Corollary 1 to show that $\deg_a G^{1}_{O(2,2k+1)} \geq -4k-3$ for $2k+1 \leq -3$. By Lemma 3 $G^{1}_{O(2,-3)} = a^2 + a^3 + a^5$, $G^{1}_{O(2,-4)} = a^3 + a^6 + a^7$, $G^{1}_{O(2,-5)} = a^5 + a^8 + a^9$, and $G^{1}_{O(2,m)} = a^3 G^{1}_{O(2,m+3)} + a^{-2m-3} + a^{-2m-1}$ for $m \leq -6$; by induction we have $\deg_a G^{1}_{O(2,m)} = -2m - 1$ for $m \leq -3$. \[\Box\]

4.4. Some Positive Pretzel Knots

Let $\pi_{\perp} = \pi_{\cap} = (1 6)(2 3)(4 5) \in S_6$. The unoriented link $P(r, s, t) := (\sigma_1^r \sigma_3^s \sigma_5^t)^{2k}$ is called a pretzel; it is positive iff $r, s,$ and $t$ are all non-negative. Note that $P(r, s, t)$ is a knot iff two or three of $r, s, t$ are odd, and that $P(r, s, t)$ is ambient isotopic to $P(s, t, r)$.

**Theorem 9.** If $r, s, t \geq 1$ are odd, then $q(P(r, s, t)) = -3 + r + s - t$.

**Proof.** In the braidword diagram of $(\sigma_1, \ldots, \sigma_1, \sigma_3, \ldots, \sigma_3, \sigma_5, \ldots, \sigma_5)$, in this case, the crossings that correspond to $\sigma_1$ and $\sigma_3$ are positive and those that correspond to $\sigma_5$ are negative, so the writhe of the associated knot diagram of $P(r, s, t)$ is $r + s - t$; by Theorem 8, $q(P(r, s, t)) \geq -3 + r + s - t$.

To establish the opposite inequality, it suffices, by Corollary 8, to show that $\ord_a R_{P(r, s, t)} = -2 + r + s - t$. If $t = 0$, then $P(r, s, 0) = O\{2, r\} \mp O\{2, s\}$, while if $t \geq 2$, then (by considering any one of the negative crossings) $R_{P(r, s, t)} = v^{-2} R_{P(r, s, t-2)} - v^{-1} R_{O(2, r+s)}$; easy inductions complete the proof. \[\Box\]

**Remark 5.** A similar calculation shows that $-3 - r - s - t \leq q(P(r, s, t)) \leq -2 - r - s - t$ when $r, s, t \geq 1$ are all odd.

5. The Modulus of Quasipositivity of a Slice Knot

Kronheimer & Mrowka [22] have recently announced a gauge-theoretic proof of the following long-conjectured result.

**Theorem 10.** Let $\Gamma \subset C^2$ be a smooth complex-algebraic curve. If $\Gamma$ intersects $S^3 := \{ (z, w) \in C^2 : |z|^2 + |w|^2 = 1 \}$ transversely, then no smooth orientable surface $S \subset D^4 := \{ (z, w) \in C^2 : |z|^2 + |w|^2 \leq 1 \}$ without closed components, such that $\partial S = \Gamma \cap S^3$, has larger Euler characteristic than $\Gamma \cap D^4$.

A knot $K \subset S^3$ is slice if $K = \partial D$ for some smooth 2-disk $D \subset D^4$.

**Proposition 5.** If $K$ is a slice knot, then $q(K) \leq 0$.

**Proof.** Let $K$ be a knot. If $K$ is slice, then $\partial A(K, 0)$ bounds a surface in $D^4$ of Euler characteristic two (namely, the union of two disjoint smooth 2-disks). On the other hand, if $q(K) \geq 0$, then (by Proposition 1) the annulus $A(K, 0)$ is quasipositive, and it follows from 1 (cf. also 21) that there exists a smooth complex-algebraic curve $\Gamma$ such that $\Gamma \cap S^3$ is a link of type $\partial A(K, 0)$ (the intersection being transverse), while $\Gamma \cap D^4$ is a surface of Euler characteristic zero (namely, a “push-in” of $A(K, 0)$). Now the proposition follows from Theorem 14. \[\Box\]
**Remark 6.** Another interesting consequence of Theorem 10 is that, for quasipositive knots, slice implies ribbon.

**Acknowledgments**

This research was partially supported by NSF grant DMS-8801959.

**Addendum (December 2001)**

Bennequin’s proof [23] that the maximal Thurston–Bennequin invariant $TB(K)$ of a knot in $S^3$ is an integer (rather than $\infty$) sparked considerable interest in finding ways to compute $TB(K)$, or at least bound it above (see, e.g., [24]).

In [22], it was shown that $TB(K)$ is identical to the modulus of quasipositivity $q(K)$; thus the various bounds on, and calculations of, $q$ derived above are equally bounds on, or calculations of, $TB$. Similar (sometimes sharper) results for $TB$ have been derived using a variety of different methods by a number of researchers, among them Fuchs & Tabachnikov [26], Tabachnikov [27], Epstein [28], Chmutov and Goryunov [29], Kanda [30], Tanaka [31], Goryunov and Hill [32], Etnyre and Honda [33], Ng [34], and Ferrand [35].

Torisu [36] and Etnyre & Honda [37] have announced that $TB + 1$ is additive; in light of [22], this settles the issue raised in Remark 3, by showing that $q + 1$ is additive (the inequality in Proposition 4 may be replaced by an equation).

**References**

[1] Lee Rudolph, *Braided surfaces and Seifert ribbons for closed braids*, Comment. Math. Helvetici **58** (1983) 1–37.

[2] Lee Rudolph, *Constructions of quasipositive knots and links*, I, in *Nœuds, Tresses, et Singularités* (L’Ens. Math. Mono. 31), ed. C. Weber, Kundig, Geneva (1983) 233–245.

[3] Lee Rudolph, *Constructions of quasipositive knots and links*, II, Contemp. Math. **35** (1984) 485–491.

[4] Lee Rudolph, *Special positions for surfaces bounded by closed braids*, Rev. Mat. Iberoamericana **1** (1985) 93–133.

[5] J. R. Stallings, *Constructions of fibred knots and links*, Proc. Symp. Pure Math. XXXII, Part 2 (1979) 55–60.

[6] J. Birman and R. F. Williams, *Knotted periodic orbits in dynamical systems, I: Lorenz’s equations*, Topology **22** (1983) 47–82.

[7] Lee Rudolph, *Nontrivial positive braids have positive signature*, Topology **21** (1983) 325–327.

[8] Lee Rudolph, *A characterization of quasipositive Seifert surfaces (Constructions of quasipositive knots and links, III)*, Topology **31** (1992) 231–237.

[9] Lee Rudolph, *A congruence between link polynomials*, Math. Proc. Camb. Phil. Soc. **107** (1990) 319–327.

[10] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett, A. Oceaneau, *A new polynomial invariant of knots and links*, Bull. Amer. Math. Soc. **12** (1985) 239–246.

[11] J. H. Przytycki and P. Traczyk, *Invariants of links of Conway type*, Kobe J. Math. **4** (1987) 115–139.

[12] Louis H. Kauffman, *On Knots*, Ann. Math. Studies **115**, Princeton University Press, Princeton, N. J. (1987).
13. W. B. R. Lickorish, *A relationship between link polynomials*, Math. Proc. Camb. Phil. Soc. **100** (1986) 109–112.
14. H. R. Morton, *Seifert circles and knot polynomials*, Math. Proc. Camb. Phil. Soc. **90** (1986) 107–110.
15. W. B. R. Lickorish and K. Millett, *The new polynomial invariants of knots and links*, Math. Mag. **61** (1988) 3–23.
16. Herbert Lyon, *Torus knots in the complements of links and surfaces*, Mich. Math. J. **27** (1980) 39–46.
17. T. Erlandsson, *Geometry of contact transformations in dimension three*, Ph.D. Thesis, Uppsala (1981).
18. Nathaniel Kuhn, *A conjectural inequality on the slice genus of links*, Ph.D. Thesis, Princeton University (1984).
19. J. Franks and R. F. Williams, *Braids and the Jones-Conway polynomial*, Trans. Amer. Math. Soc. **303** (1987) 97–108.
20. Lee Rudolph, *Quasipositivity and new knot invariants*, Rev. Math. Univ. Comp. Madrid **2** (1989) 85–109.
21. Lee Rudolph, *Some topologically locally-flat surfaces in the complex projective plane*, Comment. Math. Helvetici **59** (1984) 592–599.
22. P. Kronheimer and T. Mrowka, *Gauge theory for embedded surfaces*, I, preprint (1991); Topology **32** (1993) 773–826.

**Supplementary Bibliography (December 2001)**

23. Daniel Bennequin, *Entrelacements et équations de Pfaff*, Astérisque **107-108** (1983) 87–161.
24. Lee Rudolph, *An obstruction to sliceness via contact geometry and ‘classical’ gauge theory*, Invent. Math. **119** (1995) 155–163.
25. Yakov Eliashberg, *Legendrian and transversal knots in tight contact 3-manifolds*, in *Topological methods in modern mathematics* (Stony Brook, NY, 1991), ed. L. Goldberg and A. Phillips, Publish or Perish, Houston (1993) 171–193.
26. Dmitry Fuchs and Serge Tabachnikov, *Invariants of Legendrian and transverse knots in the standard contact space*, Topology **36** (1997) 1025–1053.
27. Sergei Tabachnikov, *Estimates for the Bennequin number of Legendrian links from state models for knot polynomials*, Math. Res. Lett. **4** (1997) 143–156.
28. J. Epstein, *On the Invariants and Isotopies of Legendrian and Transverse Knots*, Ph.D. thesis, U. C. Davis (1997).
29. S. Chmutov and V. Goryunov, *Polynomial invariants of Legendrian links and wave fronts*, in *Topics in singularity theory*, Amer. Math. Soc., Providence (1997) 25–43; [http://www.liv.ac.uk/~goryunov/papers/poly.ps](http://www.liv.ac.uk/~goryunov/papers/poly.ps).
30. Yutaka Kanda, *On the Thurston-Bennequin invariant of Legendrian knots and non-exactness of Bennequin’s inequality*, Invent. Math. **133** (1998) 227–242.
31. Toshifumi Tanaka, *Maximal Bennequin numbers and Kauffman polynomials of positive links*, Proc. Amer. Math. Soc. **127** (1999) 3427–3432; [http://www.ams.org/jourcgi/amsjournal?pii=piid=SO0029-9930(99)049831](http://www.ams.org/jourcgi/amsjournal?pii=piid=SO0029-9930(99)049831).
32. V. V. Goryunov and J. W. Hill, *A Bennequin number estimate for transverse knots*, in *Singularity theory (Liverpool, 1996)*, London Math. Soc. Lecture Note Ser. **263**, Cambridge Univ. Press, Cambridge (1999) 265–280; [http://www.liv.ac.uk/~goryunov/papers/trans.ps](http://www.liv.ac.uk/~goryunov/papers/trans.ps).
33. John Etnyre and Ko Honda, *Knots and contact geometry*, preprint (2000); [http://arxiv.org/abs/math.GT/0006112](http://arxiv.org/abs/math.GT/0006112).
[34] Lenhard L. Ng, Maximal Thurston-Bennequin number of two-bridge links, Algebr. Geom. Topol. 1 (2001) 427–434; http://www.maths.warwick.ac.uk/agt/AGTVol1/agt-1-21.abs.html.

[35] Emmanuel Ferrand, On Legendrian knots and polynomial invariants, to appear in Proc. Amer. Math. Soc. (2001); http://www.ams.org/joucgi/amsjournal?pg1=pii&sl=S0002993901061536.

[36] Ichiro Torisu, On the additivity of the Thurston-Bennequin invariant of Legendrian knots, preprint (2001); http://arxiv.org/abs/math.GT/0103023.

[37] Etnyre and Honda, unpublished (cited in [36]), 2001.