Objective priors for divergence-based robust estimation

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Abstract

Objective priors for outlier-robust Bayesian estimation based on divergences are considered. It is known that the $\gamma$-divergence (or type 0 divergence) has attractive properties for robust parameter estimation (Jones et al. (2001), Fujisawa and Eguchi (2008)). This paper puts its focus on the reference and moment matching priors under quasi-posterior distribution based on the $\gamma$-divergence. In general, since such objective priors depend on unknown data generating mechanism, we cannot directly use them in the presence of outliers. Under Huber’s $\varepsilon$-contamination model, we show that the proposed priors are approximately robust under the condition on the tail of the contamination distribution without assuming any conditions on the contamination ratio. Some simulation studies are also illustrated.

Keywords: Divergence; Huber’s $\varepsilon$-contamination model; Moment matching prior; Reference prior; Robust estimation

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1 Introduction

The problem of robust parameter estimation against outliers have a long history. The excellent review for the classical robust estimation theory is given by Huber and Ronchetti (2009), for example. It is well-known that the maximum likelihood estimator (MLE) is not robust against outliers because the MLE is obtained by minimizing the Kullback Leibler (KL) divergence between the true and empirical distributions. To overcome this problem, we can use other (robust) divergences instead of the KL divergence. The robust parameter estimation based on divergences is one of central topics in modern robust statistics (e.g. Basu et al. (2011)). Such estimator was firstly proposed by Basu et al. (1998), and they called it the minimum density power divergence estimator. Jones et al. (2001) also proposed the type 0 divergence which is modified version of the density power divergence, and Fujisawa and Eguchi (2008) showed that the type 0 divergence has good robustness properties. The type 0 divergence is also called the $\gamma$-divergence, and there are various applications of the $\gamma$-divergence by many authors (e.g. Hirose et al. (2016), Kawashima and Fujisawa (2017) and so on).

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In Bayesian statistics, the robustness against outliers is also important issue, and divergence-based Bayesian methods have been proposed in recent years. Such methods are called quasi-Bayes or generalized Bayes method in some papers, and the corresponding posterior distribution is called quasi-posterior or general posterior. The quasi-posterior is the posterior distribution which is based on a general loss function instead of usual log likelihood function in order to overcome the model misspecification problem (see Bissiri et al. (2016)). In general, such general loss functions may not depend on an assumed statistical model. But, in this paper, we use loss functions which depend on the assumed model, that is, in particular, we use divergences or scoring rules as a loss function for quasi-posterior. Such quasi-posteriors have been proposed by many researches. For example, [Hooker and Vidyashankar (2014)] used the Hellinger divergence, [Ghosh and Basu (2016)] used the density power divergence, and [Nakagawa and Hashimoto (2020)] used the $\gamma$-divergence (see also [Jewson et al. (2018), Giummolè et al. (2019)]). The quasi-posterior based on the $\gamma$-divergence is called the $\gamma$-posterior in [Nakagawa and Hashimoto (2020)], and they showed that the $\gamma$-posterior is good robustness properties to overcome problems in [Ghosh and Basu (2016)].

In this paper, we consider objective priors for divergence-based quasi-posterior distributions. Although the selection of priors is important issue in Bayesian statistics, we often have no prior information in some practical situations. In such cases, we may use priors called objective or default priors, and we should select an appropriate objective prior in a given context. In particular, we consider reference and moment matching priors in this paper. The reference prior was firstly proposed by [Bernardo (1979)] and moment matching prior was proposed by [Ghosh and Liu (2011)]. However, such objective priors depend on an unknown data generating mechanism. For example, if we assume the Huber’s $\varepsilon$-contamination model (see e.g. [Huber and Ronchetti (2009)]) as a data generating distribution, many objective priors depend on unknown contamination ratio and unknown contamination distribution because these objective priors involve the expectations under the data generating distribution. Although [Giummolè et al. (2019)] derived some kinds of reference priors under quasi-posteriors based on some kinds of scoring rules, they did not discuss the robustness of such reference priors in their paper. Furthermore, their simulation studies largely depend on the assumption for contamination ratio, that is, they indirectly assume that the contamination ratio is very small. In this paper, we derive the moment matching priors under quasi-posterior in a similar way to [Ghosh and Liu (2011)], and show that reference and moment matching priors based on the $\gamma$-divergence does not approximately depend on such unknown quantities under a certain assumption for contamination distribution even if the contamination ratio is not small.

This paper is organized as follows: In Section 2 we review robust Bayesian estimation based on divergences referring to some previous studies. In Section 3 we derive moment matching priors based on quasi-posterior distribution by using the asymptotic expansion of quasi-posterior distribution given by [Giummolè et al. (2019)]. Furthermore, we show that reference and moment matching priors based on the $\gamma$-posterior are robust against unknown quantities in a data generating distribution. In Section 4 we compare empirical bias and mean squared error of estimators through some simulation studies. Furthermore, we discuss the selection of a tuning parameter in divergences by using the asymptotic relative efficiency.
2 Robust Bayesian estimation using divergences

In this section, we review $R^{(\alpha)}$- and $\gamma$-posterior distributions proposed by Ghosh and Basu (2016) and Nakagawa and Hashimoto (2020), respectively. Let $X_1, \ldots, X_n$ be independent and identically distributed (i.i.d.) random variables according to a contaminated probability density function on $\Omega$,

$$g(x) = (1 - \varepsilon)f(x) + \varepsilon\delta(x),$$

where $\varepsilon \in (0, 1)$ is a contamination ratio (see e.g. Huber and Ronchetti (2009)). Such data generating model is also called Huber’s $\varepsilon$-contamination model, and is widely used in modeling for robust estimation problem. Theoretical properties for estimators under Huber’s $\varepsilon$-contamination model are given by Chen et al. (2018), for example.

We now consider a parametric model $f_\theta(x) = f(x|\theta)$, indexed by an unknown parameter vector $\theta = (\theta_1, \ldots, \theta_p)^\top \in \Theta \subset \mathbb{R}^p$, where $\Theta$ is a parameter space of $\theta$. We also assume that the target density $f(x)$ is included in the parametric family of model $P = \{f_\theta(x) \mid \theta \in \Theta\}$, that is, $f(x) \equiv f_{\theta^*}(x)$ for some $\theta^* \in \Theta$. Hereafter, we will often omit arguments of functions for simplicity.

Over the past few years, the robust Bayesian estimators based on divergences have been developed by many researches (e.g. Hooker and Vidyashankar (2014), Ghosh and Basu (2016), Nakagawa and Hashimoto (2020)). Basu et al. (1998), Jones et al. (2001) and Fujisawa and Eguchi (2008) proposed robust estimation based on divergences in frequentist perspective. Ghosh and Basu (2016) proposed the $R^{(\alpha)}$-posterior defined by

$$\pi^{(\alpha)}(\theta|X_n) \propto \exp\left\{-nd_{\alpha}(\bar{g}, f_\theta)\right\} \pi(\theta) = \exp\left\{\sum_{i=1}^{n} q^{(\alpha)}(X_i; \theta)\right\} \pi(\theta),$$

where $d_{\alpha}(. , .)$ is the cross entropy of the density power divergence proposed by Basu et al. (1998), $\bar{g}$ is the empirical distribution of $X_n = (X_1, \ldots, X_n)^\top$, and

$$q^{(\alpha)}(X_i; \theta) := -\frac{1}{\alpha}f_\theta(X_i)^\alpha + \frac{1}{1 + \alpha} \int_{\Omega} f_\theta(x)^{1+\alpha} dx.$$ 

It is known that the posterior mean based on $R^{(\alpha)}$-posterior works well for the estimation of a location parameter in the presence of outliers. However, such estimator is known to be unstable in the case of the estimation for a scale parameter (see Nakagawa and Hashimoto (2020)). On the other hand, Nakagawa and Hashimoto (2020) proposed the $\gamma$-posterior defined by

$$\pi^{(\gamma)}(\theta|X_n) \propto \exp\left\{-\tilde{d}_{\gamma}(\bar{g}, f_\theta)\right\} \pi(\theta) = \exp\left\{\sum_{i=1}^{n} q^{(\gamma)}(X_i; \theta)\right\} \pi(\theta),$$

where $\tilde{d}_{\gamma}(. , .)$ is defined by

$$\tilde{d}_{\gamma}(g, f) = -\frac{1}{\gamma} \left\{\exp(-\gamma d_{\gamma}(g, f)) - 1\right\}$$

$$= -\frac{1}{\gamma} \left[\frac{\int_{\Omega} g(x)f(x)^{\gamma} dx}{\left(\int_{\Omega} f(x)^{(1+\gamma)} dx\right)^{(1+\gamma)/\gamma}} + \frac{1}{\gamma}\right]$$
which is the monotone transformation of the cross entropy for the \( \gamma \)-divergence (\( \gamma \)-cross entropy), and

\[
q^{(\gamma)}(X_i; \theta) := -\frac{1}{\gamma} f_{\theta}(X_i)^{\gamma} \left\{ \int_{\Omega} f_{\theta}(x)^{1+\gamma} dx \right\}^{-\gamma/(1+\gamma)}.
\]

We note that cross entropies \( d_{\alpha}(\cdot, \cdot) \) and \( \tilde{d}_{\gamma}(\cdot, \cdot) \) converge to the log likelihood function as \( \alpha \to 0 \) and \( \gamma \to 0 \), respectively. So, we can find that they are some kind of generalization of the log likelihood function. The monotone transformation of the \( \gamma \)-cross entropy is necessary to have the coherent Bayesian updating property (Bissiri et al. (2016)). Nakagawa and Hashimoto (2020) showed that the \( \gamma \)-posterior is robust against outliers for both location and scale parameters through their simulation studies.

Let \( d(\cdot, \cdot) \) be a cross entropy induced by a divergence which satisfies the identity

\[
\exp \left\{ -nd(\bar{g}, f_{\theta}) \right\} = \exp \left\{ \sum_{i=1}^{n} q^{(d)}(X_i; \theta) \right\}.
\]

For example, \( d_{\alpha}(\cdot, \cdot) \) and \( \tilde{d}_{\gamma}(\cdot, \cdot) \) satisfy the above equation, and the corresponding \( q^{(d)}(\cdot) \)s are \( q^{(\alpha)}(\cdot) \) and \( q^{(\gamma)}(\cdot) \), respectively. Furthermore, we assume the following regularity conditions on the density function \( f_{\theta}(x) = f(x; \theta) \) (\( \theta \in \Theta \subset \mathbb{R}^p \)). Let \( \theta_{\bar{g}} := \arg \min_{\theta \in \Theta} d(g, f_{\theta}) \).

(A1) The support of the density function does not depend on unknown parameter \( \theta \) and \( f_{\theta} \) is third order differentiable with respect to \( \theta \) in neighborhood \( U \) of \( \theta_{\bar{g}} \).

(A2) Interchange of the order of integration with respect to \( x \) and differentiation as \( \theta_{\bar{g}} \) is justified. The expectations \( \mathbb{E}_{g}[\partial_{i} q^{(d)}(X_1; \theta_{g})] \) and \( \mathbb{E}_{g}[\partial_{i} \partial_{j} q^{(d)}(X_1; \theta_{g})] \) are all finite and there exists \( M_{ijk}(x) \) such that

\[
\sup_{\theta \in U} \left| \partial_{i} \partial_{j} \partial_{k} q^{(d)}(x; \theta) \right| \leq M_{ijk}(x) \quad \text{and} \quad \mathbb{E}_{g}[M_{ijk}(X_1)] < \infty
\]

for all \( i, j, k = 1, \ldots, p \), where \( \partial_i = \partial/\partial \theta_i \) and \( \partial = \partial/\partial \theta \), and \( \mathbb{E}_{g}(\cdot) \) is expectation of \( X \) with respect to a probability density function \( g \).

(A3) For any \( \delta > 0 \), with probability one

\[
\sup_{\|\theta - \theta_{g}\| > \delta} \left\{ d(\bar{g}, f_{\theta}) - d(\bar{g}, f_{\theta_{g}}) \right\} < \varepsilon
\]

for some \( \varepsilon > 0 \) and for all sufficiently large \( n \).

The matrices \( I^{(d)}(\theta) \) and \( J^{(d)}(\theta) \) are defined by

\[
I^{(d)}(\theta) = \mathbb{E}_{g} \left[ \partial_{i} q^{(d)}(X_1; \theta) \partial_{i}^{\top} q^{(d)}(X_1; \theta) \right], \quad J^{(d)}(\theta) = -\mathbb{E}_{g} \left[ \partial \partial^{\top} q^{(d)}(X_1; \theta) \right],
\]

respectively. We also assume that \( I^{(d)}(\theta) \) and \( J^{(d)}(\theta) \) are positive definite matrices. Ghosh and Basu (2016) and Nakagawa and Hashimoto (2020) discussed several asymptotic properties of quasi-posterior distributions and the corresponding posterior means.
Theorem 1 (Ghosh and Basu (2016), Nakagawa and Hashimoto (2020)). Under the conditions \(\{A1\} \{A3\}\) we assume that \(\hat{\theta}^{(d)}_{n}\) is a consistent solution of \(\partial d(\hat{g}, f_{\theta}) = 0\) and \(\hat{\theta}^{(d)}_{n} \xrightarrow{p} \theta_{g}\) as \(n \to \infty\). Then for any prior density function \(\pi(\theta)\) which is continuous and positive at \(\theta_{g}\), it holds that

\[
\int \pi^{(d)}(t_{n}|X) = (2\pi)^{-p/2} \left| J^{(d)}(\theta_{g}) \right|^{1/2} \exp \left( -\frac{1}{2} t_{n}^\top J^{(d)}(\theta_{g}) t_{n} \right) \, dt_{n} \stackrel{p}{\to} 0 \tag{1}
\]

as \(n \to \infty\), where \(\pi^{(d)}(t_{n}|X_{n})\) is the quasi-posterior density function of the normalized random variable \(t_{n} = (t_{1}, \ldots, t_{p})^\top = \sqrt{n}(\theta - \hat{\theta}^{(d)}_{n})\) given \(X\).

Theorem 2 (Ghosh and Basu (2016), Nakagawa and Hashimoto (2020)). In addition to assumptions of Theorem 1, we assume that the prior density function \(\pi(\theta)\) has a finite expectation. Then it holds \(\sqrt{n}(\hat{\theta}^{(d)}_{n} - \hat{\theta}^{(d)}_{n}) \xrightarrow{p} 0\) as \(n \to \infty\), where \(\hat{\theta}^{(d)}_{n}\) is the quasi-posterior mean, that is, \(\hat{\theta}^{(d)}_{n} = \mathbb{E}[\theta|X_{n}] = \int_{\Theta} \theta \pi(\theta|X_{n}) d\theta\).

Therefore, the posterior mean \(\hat{\theta}^{(d)}_{n}\) is (first-order) asymptotically equivalent to the frequentist minimum divergence estimator \(\hat{\theta}^{(d)}_{n}\). Furthermore, we assume the additional condition as follows:

\(\{A4\}\) \(f_{\theta}\) is fifth order differentiable with respect to \(\theta\) in \(U\). Furthermore,

\[
\mathbb{E}_{q}[\partial_{i} \partial_{j} \partial_{k} q^{(d)}(x_{1}; \theta_{g})] \quad \text{and} \quad \mathbb{E}_{q}[\partial_{i} \partial_{j} \partial_{h} \partial_{k} q^{(d)}(x_{1}; \theta_{g})]
\]

are all finite and there exists \(M_{ijkl}s(x)\) such that

\[
\sup_{\theta \in U} \left| \partial_{i} \partial_{j} \partial_{k} \partial_{s} q^{(d)}(x; \theta) \right| \leq M_{ijkl}s(x) \quad \text{and} \quad \mathbb{E}_{q}[M_{ijkl}s(X_{1})] < \infty
\]

for all \(i, j, k, \ell, s = 1, \ldots, p\).

In terms of the higher-order asymptotic theory, Giummolè et al. (2019) derived the asymptotic expansion of such quasi-posterior distributions. We now introduce the notation which will be used in the rest of the paper. We use indices to denote derivatives of \(\hat{D}(\theta) = d(\hat{g}, f_{\theta})\) with respect to the components of the parameter \(\theta\). For example, \(\hat{D}_{ijk}(\theta) = \partial_{i} \partial_{j} \partial_{k} \hat{D}(\theta)\) and \(\hat{D}_{ijkl}(\theta) = \partial_{i} \partial_{j} \partial_{k} \partial_{l} \hat{D}(\theta)\) for \(i, j, k, \ell = 1, \ldots, p\). Then Giummolè et al. (2019) showed the following theorem.

Theorem 3 (Giummolè et al. (2019)). Under the conditions \(\{A1\} \{A4\}\) we assume that \(\hat{\theta}^{(d)}_{n}\) is a consistent solution of \(\partial d(\hat{g}, f_{\theta}) = 0\) and \(\hat{\theta}^{(d)}_{n} \xrightarrow{p} \theta_{g}\) as \(n \to \infty\). Then for any prior density function \(\pi(\theta)\) which is third order differentiable and positive at \(\theta_{g}\), it holds that

\[
\pi^{(d)}(t_{n}|X_{n}) = \phi \left( t_{n}; \hat{H}^{-1} \right) \left( 1 + n^{-1/2} A_{1}(t_{n}) + n^{-1} A_{2}(t_{n}) \right) + O_{p}(n^{-3/2}) \tag{2}
\]

where \(\phi(\cdot; A)\) is the density function of a \(p\)-variate normal distribution with zero mean vector
and covariance matrix $A$, and $\tilde{H} = J^{(d)}(\hat{\theta}_n^{(d)}), \tilde{H}^{-1} = (\tilde{h}^{ij})$, and

$$A_1(t_n) = \frac{\partial_i\pi(\hat{\theta}_n^{(d)})}{\pi(\hat{\theta}_n^{(d)})}t_i + \frac{1}{6}\sum_{i,j,k}D_{ijk}(\hat{\theta}_n^{(d)})t_it_jt_k,$$

$$A_2(t_n) = \sum_{i,j} \frac{1}{2}\frac{\partial_i\partial_j\pi(\hat{\theta}_n^{(d)})}{\pi(\hat{\theta}_n^{(d)})}(t_it_j - \tilde{h}^{ij}) - \sum_{i,j,k,\ell} \frac{1}{6}\frac{\partial_i\pi(\hat{\theta}_n^{(d)})}{\pi(\hat{\theta}_n^{(d)})}D_{ijk\ell}(\hat{\theta}_n^{(d)}) \left( t_it_jt_kt_\ell - 3\tilde{h}^{ij}\tilde{h}^{k\ell} \right)$$

$$- \sum_{i,j,k,\ell} \frac{1}{24}D_{ijk\ell}(\hat{\theta}_n^{(d)}) \left( t_it_jt_kt_\ell - 3\tilde{h}^{ij}\tilde{h}^{k\ell} \right)$$

$$+ \sum_{i,j,k,h,g,f} \frac{1}{12}D_{ijk\ell}D_{hgf}(2t_itjt_khgtf - 15\tilde{h}^{ij}\tilde{h}^{kh}\tilde{h}^{gf}).$$

**Proof.** The proof is given in appendix of Giummolè et al. (2019). \qed

As we mentioned before, quasi-posterior distributions depend on a cross entropy induced by a divergence and a prior distribution. If we have some information about unknown parameters $\theta$, we can use a prior distribution which takes in account such prior information. However, in the absence of prior information, we often use prior distributions called objective or default priors. Giummolè et al. (2019) proposed the reference prior for quasi-posterior distributions which is one of objective priors (see Bernardo (1979)). The reference prior $\pi_R$ is obtained by asymptotically maximizing the expected Kullback-Leibler divergence between prior and posterior distributions. As a generalization of the reference prior, Ghosh et al. (2011) discussed reference priors under a general divergence measure known as the $\alpha$-divergence (see also Liu et al. (2014)). The reference prior under the $\alpha$-divergence is given by asymptotically maximizing the expected $\alpha$-divergence

$$H(\pi) = \mathbb{E}[D^{(\alpha)}(\pi^{(d)}(\theta|X_n), \pi(\theta))],$$

where $D^{(\alpha)}$ is the $\alpha$-divergence defined as

$$D^{(\alpha)}(\pi^{(d)}(\theta|X_n), \pi(\theta)) = \frac{1}{\alpha(1-\alpha)} \int_{\Theta} \left\{ 1 - \left( \frac{\pi(\theta)}{\pi^{(d)}(\theta|X_n)} \right)^\alpha \right\} \pi^{(d)}(\theta|X_n)d\theta,$$

which corresponds to the KL divergence as $\alpha \to 0$, the Hellinger divergence for $\alpha = 1/2$, and the $\chi^2$-divergence for $\alpha = -1$. Giummolè et al. (2019) derived reference priors with the $\alpha$-divergence under quasi-posterior based on some kinds of proper scoring rules such as the Tsallis scoring rule and Hyvärinen scoring rule. We note that the Tsallis scoring rule is the same as the density power score of Basu et al. (1998) with minor notational modifications. However, they did not discuss the pseudo-spherical scoring rule which corresponds to the $\gamma$-divergence or the $\gamma$-cross entropy considered in this paper.

**Theorem 4.** (Giummolè et al. (2019)). When $|\alpha| < 1$, the reference prior which asymptotically maximizes the expected $\alpha$-divergence between quasi-posterior and prior distributions is given by

$$\pi_R(\theta) = \det(J^{(d)}(\theta))^{1/2}.$$

The result of Theorem 4 is similar to that of Ghosh et al. (2011) and Liu et al. (2014).
Objective priors such as the above theorem are useful because they can be determined by the data generating model. However, such priors do not have a statistical guarantee when the model is misspecified such as Huber’s $\varepsilon$-contamination model. In other words, the reference prior in Theorem 4 depends on data generating distribution $g$ because of

$$
J^{(d)}(\theta) = -E_g \left[ \partial \partial^\top g^{(d)}(X; \theta) \right],
$$

where $g(x) = (1 - \varepsilon)f_{\theta}(x) + \varepsilon\delta(x)$. We now consider some objective priors under the $\gamma$-posterior which is robust against such unknown quantities in next section.

## 3 Main results

In this section, we derive moment matching priors for quasi-posterior distributions, and show that the robustness of reference and moment matching priors under the $\gamma$-posterior.

### 3.1 Moment matching priors

The moment matching priors proposed by [Ghosh and Liu (2011)] are priors which matches the posterior mean and MLE up to the higher order (see also [Hashimoto (2019)]). In this section, we try to extend the results of [Ghosh and Liu (2011)] to the context of quasi posterior distributions. Our goal in this section is to find a prior such that the difference between the quasi-posterior mean $\tilde{\theta}^{(d)}_n$ and frequentist minimum divergence estimator $\hat{\theta}^{(d)}_n$ converges to zero up to the order of $o(n^{-1})$. From Theorem 3, we have the following theorem.

**Theorem 5.** Let $\tilde{\theta}^{(d)}_n = (\tilde{\theta}_1^{(d)}, \ldots, \tilde{\theta}_p^{(d)})$ and $\hat{\theta}^{(d)}_n = (\hat{\theta}_1^{(d)}, \ldots, \hat{\theta}_p^{(d)})$. Under the same assumptions as Theorem 3, it holds that

$$
n \left( \tilde{\theta}^{(d)}_\ell - \hat{\theta}^{(d)}_\ell \right) \xrightarrow{p} \sum_{i=1}^{p} \frac{\partial_i \pi(\theta)}{\pi(g)} h_{i\ell} + \frac{1}{6} \sum_{i,j,k} g_{ijk}^{(d)}(\theta_g) \left( h_{ij} h_{k\ell} + h_{ik} h_{j\ell} + h_{i\ell} h_{jk} \right)
$$

as $n \to \infty$, where $H = J^{(d)}(\theta_g)$, $H^{-1} = (h_{ij})$ and $g_{ijk}^{(d)}(\theta) = E_g \left[ \partial_i \partial_j \partial_k q^{(d)}(X_1; \theta) \right]$. Furthermore, if we set a prior which satisfies

$$
\frac{\partial_\ell \pi(\theta)}{\pi(\theta)} + \frac{1}{2} \sum_{i,j} g_{ij\ell}(\theta) h_{ij}(\theta) = 0
$$

for all $\ell = 1, \ldots, p$, then it holds that

$$
n \left( \tilde{\theta}^{(d)}_\ell - \hat{\theta}^{(d)}_\ell \right) \xrightarrow{p} 0
$$

for $\ell = 1, \ldots, p$ as $n \to \infty$, where $\{J^{(d)}(\theta)\}^{-1} = (h_{ij}(\theta))$.

Hereafter, the prior which satisfies the equation (4) for all $\ell = 1, \ldots, p$ is called moment matching prior and we denote it by $\pi_M$. 

7
Proof. From (2), we have

\[
\hat{\theta}_\ell^{(d)} = \int_\Theta \theta \pi^{(d)}(\theta | X_n) \, d\theta \\
= \hat{\theta}_\ell^{(d)} + \frac{1}{\sqrt{n}} \int_T t_\ell \pi^{\ast(d)}(t_n | X_n) \, dt_n \\
= \hat{\theta}_\ell^{(d)} + \frac{1}{n} \int_T t_\ell \phi \left(t_n; \hat{H}^{-1}\right) A_1(t_n) \, dt_n + O_p(n^{-3/2})
\]

for \( \ell = 1, \ldots, p \), where \( T = \{ t_n \in \mathbb{R}^p \mid \theta + n^{-1/2}t_n, \theta \in \Theta \} \). The integral in the above equation is calculated by

\[
\int_T t_\ell A_1(t_n) \phi \left(t_n; \hat{H}^{-1}\right) \, dt_n = \sum_{i=1}^p \partial_i \pi(\hat{\theta}_n^{(d)}) \int_T t_i t_\ell \phi \left(t_n; \hat{H}^{-1}\right) \, dt_n \\
+ \frac{1}{6} \sum_{i,j,k} \delta_{ijk}(\hat{\theta}_n^{(d)}) \int_T t_i t_j t_k \phi \left(t_n; \hat{H}^{-1}\right) \, dt_n \\
= \sum_{i=1}^p \partial_i \pi(\hat{\theta}_n^{(d)}) \tilde{h}_i^{\ell} \\
+ \frac{1}{6} \sum_{i,j,k} \delta_{ijk}(\hat{\theta}_n^{(d)}) \left( h_{ij} h_{jk}^{k} + h_{ik} h_{jk}^{j} + h_{ik} h_{jk}^{i} \right) + o_p(1).
\]

Therefore, we have

\[
n \left( \hat{\theta}_\ell^{(d)} - \tilde{\theta}_\ell^{(d)} \right) \overset{p}{\to} \sum_{i=1}^p \frac{\partial_i \pi(\theta)}{\pi(\theta)} h_i^{\ell} + \frac{1}{6} \sum_{i,j,k} \delta_{ijk}(\theta) \left( h_{ij} h_{jk}^{k} + h_{ik} h_{jk}^{j} + h_{ik} h_{jk}^{i} \right)
\]

as \( n \to \infty \).

In the case of \( p = 1 \), the moment matching prior is given by

\[
\pi_M(\theta) = C \exp \left\{ -\int_0^\theta \frac{g_3^{(d)}(t)}{2 J^{(d)}(t)} \, dt \right\}
\]

for a constant \( C \). This prior is very similar to that of Ghosh and Liu (2011), but the quantities \( g_3^{(d)}(t) \) and \( J^{(d)}(t) \) are different from it. Furthermore, we note that their quantities depend on the data generating distribution \( g \) expressed by \( g(x) = (1 - \varepsilon) f_\theta(x) + \varepsilon \delta(x) \) with unknown \( \varepsilon \) and \( \delta(x) \).

### 3.2 Robustness of objective priors

In general, reference and moment matching priors depend on the contamination ratio and distribution. Therefore, we cannot directly use such objective priors for the quasi-posterior distributions because the contamination ratio and distribution are unknown. In this section, we propose the priors which are robust against these unknown quantities. First of all, we
assume the following condition of the contamination distribution:

\[ \nu_{\theta} = \left\{ \int_{\Omega} \delta(x)f_{\theta}(x)^{\gamma_0}dx \right\}^{1/\gamma_0} \approx 0 \]  

(5)

for all \( \theta \in \Theta \) and an appropriately large constant \( \gamma_0 > 0 \) (see also Fujisawa and Eguchi (2008)). We note that we do not assume that the contamination ratio \( \varepsilon \) is sufficiently small.

This condition means that the contamination distribution \( \delta(x) \) mostly lies on the tail of any models \( f_{\theta}(x) \).

In other words, for an outlier \( x_0 \), it holds that \( f_{\theta}(x_0) \approx 0 \) for all \( \theta \in \Theta \).

We note that he assumption 5 is also a basis to show the robustness against outliers for the minimum \( \gamma \)-divergence estimator in Fujisawa and Eguchi (2008). Then we have the following theorem.

**Theorem 6.** Assume the condition (5), and suppose that there exist constant numbers \( M_s(s = 1, \ldots, 6) \) such that

\[
\sup_{\theta \in \Theta} \mathbb{E}_\delta \left[ \left| \partial_i \log f_{\theta}(X_1)^{1+\gamma} \right| \right] < M_1, \\
\sup_{\theta \in \Theta} \mathbb{E}_\delta \left[ \left| \partial_i \partial_j \log f_{\theta}(X_1)^{1+\gamma} \right| \right] < M_2, \\
\sup_{\theta \in \Theta} \mathbb{E}_\delta \left[ \left| \partial_i \log f_{\theta}(X_1) \partial_j \log f_{\theta}(X_1)^{1+\gamma} \right| \right] < M_3, \\
\sup_{\theta \in \Theta} \mathbb{E}_\delta \left[ \left| \partial_i \log f_{\theta}(X_1) \partial_j \partial_k \log f_{\theta}(X_1)^{1+\gamma} \right| \right] < M_4, \\
\sup_{\theta \in \Theta} \mathbb{E}_\delta \left[ \left| \partial_i \partial_j \partial_k \log f_{\theta}(X_1)^{1+\gamma} \right| \right] < M_5, \\
\sup_{\theta \in \Theta} \mathbb{E}_\delta \left[ \left| \partial_i \partial_j \partial_k \log f_{\theta}(X_1)^{1+\gamma} \right| \right] < M_6
\]

for \( i, j, k = 1, \ldots, p \). For \( \gamma + 1 \leq \gamma_0 \), it holds that

\[
J_{ij}^{(\gamma)}(\theta) = -\mathbb{E}_g \left[ \partial_i \partial_j q^{(\gamma)}(X_1; \theta) \right] = -(1-\varepsilon)\mathbb{E}_{f_{\theta}} \left[ \partial_i \partial_j q^{(\gamma)}(X_1; \theta) \right] + O(\varepsilon \nu^\gamma),
\]

\[
g_{ijk}^{(\gamma)}(\theta) = \mathbb{E}_g \left[ \partial_i \partial_j \partial_k q^{(\gamma)}(X_1; \theta) \right] = (1-\varepsilon)\mathbb{E}_{f_{\theta}} \left[ \partial_i \partial_j \partial_k q^{(\gamma)}(X_1; \theta) \right] + O(\varepsilon \nu^\gamma),
\]

where \( \nu := \sup_{\theta \in \Theta} \nu_{\theta} \).

The proof is given in appendix. Theorem 6 shows that expectations in the right-hand side of \( J_{ij}^{(\gamma)}(\theta) \) and \( g_{ijk}^{(\gamma)}(\theta) \) only depend on the model \( f_{\theta} \), but not depend on the contamination ratio and the contamination distribution. Hence, reference and moment matching priors for the \( \gamma \)-posterior can be obtained by only the parametric model \( f_{\theta} \). For example, for a normal distribution \( N(\mu, \sigma^2) \), reference and moment matching priors are given by

\[
\pi_R^{(\gamma)}(\mu, \sigma) = \sigma^{-3+1/(1+\gamma)} + O(\varepsilon \nu^\gamma), \quad \pi_M^{(\gamma)}(\mu, \sigma) = \sigma^{-(\gamma+7)/(2(1+\gamma))} + O(\varepsilon \nu^\gamma).
\]

(6)

However, reference and moment matching priors under \( R^{(\alpha)} \)-posterior depend on unknown quantities in the data generating distribution unless \( \varepsilon \approx 0 \), since \( J_{ij}^{(\alpha)}(\theta) \) and \( g_{ijk}^{(\alpha)}(\theta) \) have the
following forms:

\[
\mathbb{E}_g \left[ \partial_i \partial_j q^{(\alpha)}(X_1; \theta) \right] = (1 - \varepsilon)\mathbb{E}_{\theta} \left[ \partial_i \partial_j q^{(\alpha)}(X_1; \theta) \right] + \frac{\varepsilon}{1 + \alpha} \int_{\Omega} \partial_i \partial_j f_\theta(x)^{1+\alpha} dx + O(\varepsilon \nu^\alpha),
\]

\[
\mathbb{E}_g \left[ \partial_i \partial_j \partial_k q^{(\alpha)}(X_1; \theta) \right] = (1 - \varepsilon)\mathbb{E}_{\theta} \left[ \partial_i \partial_j \partial_k q^{(\alpha)}(X_1; \theta) \right] + \frac{\varepsilon}{1 + \alpha} \int_{\Omega} \partial_i \partial_j \partial_k f_\theta(x)^{1+\alpha} dx + O(\varepsilon \nu^\alpha).
\]

The priors given by (6) can be practically used under the condition (5) even if the contamination ratio \(\varepsilon\) is not small.

4 Simulation studies

We show performance of posterior means under reference and moment matching priors through some simulation studies. In this section, we suppose that the parametric model is the normal distribution with mean \(\mu\) and variance \(\sigma^2\), and consider the joint estimation problem for \(\mu\) and \(\sigma^2\). We also assume that the contamination distribution is the normal distribution with mean \(\nu\) and variance 1. We compare performances of estimators in terms of empirical bias and mean squared error (MSE) among three methods which include the ordinary KL-divergence based posterior, \(R^{(\alpha)}\)-posterior and \(\gamma\)-posterior (our proposal). We also adapt three joint prior distributions for \(\mu\) and \(\sigma^2\): (i) uniform prior, (ii) reference prior, (iii) moment matching prior.

Since exact calculations of posterior means are not easy, we use the importance sampling Monte Carlo algorithm using a proposal distribution \(N(\bar{x}, s^2)\) for \(\mu\) and IG(6, 5s) (the inverse gamma distribution) for \(\sigma\), where \(\bar{x} = n^{-1} \sum_{i=1}^{n} x_i\) and \(s^2 = (n - 1)^{-1} \sum_{i=1}^{n} (x_i - \bar{x})^2\) (for the details, see Robert and Casella (2004)). We carry out the importance sampling with 10,000 steps and we compute empirical bias and MSE for posterior means \((\hat{\mu}, \hat{\sigma})\) of \((\mu, \sigma)\) by 10,000 iterations. The simulation results are shown in Tables 1 to 4. Reference and the moment matching priors for the \(\gamma\)-posterior are given by (6), and those of \(R^{(\alpha)}\)-posterior are “formally” given as follows:

\[
\pi^{(\alpha)}_M(\mu, \sigma) \propto \sigma^{-2-\alpha}, \quad \pi^{(\alpha)}_M(\mu, \sigma) \propto \sigma^{-C_M/2},
\]

where \(C_M\) is a constant given by

\[
C_M = \frac{2 + \alpha^2}{1 + \alpha} + \frac{\alpha(1 + \alpha)^3(2 + \alpha) + (10 - \alpha^2(2 + \alpha(5 + \alpha(3 + \alpha))))\pi^{\alpha/2}}{(1 + \alpha)(-\alpha(1 + \alpha)^2 + (-2 + \alpha + \alpha^2 + \alpha^3)\pi^{\alpha/2})}.
\]

The word “formally” means that since reference and the moment matching priors for \(R^{(\alpha)}\)-posterior strictly depend on unknown contamination ratio and contamination distribution, we set \(\varepsilon = 0\) in these priors. On the other hand, our proposed objective priors does not need such assumption, but we assume only the condition (5). We note that Giannot et al. (2019) also use the same formal reference prior in their simulation studies.

Remark 1. Since the prior distributions in (6) and (7) are improper in general, we should show the posterior propriety under these priors. However, this is not easy because quasi-posterior distributions have very complex forms. To show the posterior propriety remains as
when sample size is small. On the other hand, from Tables 1 to 4, these posterior means under 
\( R \) and \( \mu \) have smaller biases and MSEs than those of ordinary posterior means 
(denoted by Bayes in tables) in the presence of outliers for large sample size. From Tables 1 to 4, 
\( \gamma \)-posterior and \( \alpha \)-posterior means have smaller biases and MSEs than 
ordinary posterior means for scale parameter \( \sigma \) under the uniform, reference and moment matching priors when 
\( \nu = 6 \) and the contamination ratio is \( \varepsilon = 0.00, 0.05 \) or 0.20. Tables 1 and 2 show that empirical biases 
of \( R^{(\alpha)} \)-posterior and \( \gamma \)-posterior means are smaller than those of ordinary posterior means 
in many cases. In particular, it is shown that biases and MSEs of \( \gamma \)-posterior means for scale parameter \( \sigma \) 
are drastically smaller than those of \( R^{(\alpha)} \)-posterior from Table 2 and 4.

Table 1: Empirical biases of posterior means for \( \mu \).

| \( \varepsilon \) | \( n \) | Bayesian & \( \alpha, \gamma \) & \( \mu \)-posterior | Uniform prior & \( \alpha \) & \( 0.3 \), \( 0.5 \), \( 0.7 \), \( 1.0 \) & \( \gamma \)-posterior | \( \gamma \) & \( 0.3 \), \( 0.5 \), \( 0.7 \), \( 1.0 \) |
|---|---|---|---|---|---|---|---|---|
| 0.00 | 20 | \(-0.002 \) | \(-0.003 \) | \(-0.002 \) | \( 0.001 \) | \( 0.005 \) | \(-0.003 \) | \(-0.003 \) | \(-0.002 \) | \(-0.001 \) |
| 0.00 | 50 | \(-0.002 \) | \(-0.001 \) | \(-0.001 \) | \( 0.000 \) | \( 0.005 \) | \(-0.001 \) | \(-0.001 \) | \( 0.000 \) | \( 0.000 \) |
| 0.00 | 100 | \( 0.000 \) | \( 0.000 \) | \( 0.000 \) | \( 0.001 \) | \( 0.001 \) | \( 0.000 \) | \( 0.000 \) | \( 0.000 \) | \( 0.000 \) |
| 0.05 | 20 | \( 0.298 \) | \( 0.075 \) | \( 0.098 \) | \( 0.172 \) | \( 0.239 \) | \( 0.064 \) | \( 0.046 \) | \( 0.060 \) | \( 0.100 \) |
| 0.05 | 50 | \( 0.301 \) | \( 0.020 \) | \( 0.009 \) | \( 0.016 \) | \( 0.134 \) | \( 0.017 \) | \( 0.004 \) | \( 0.002 \) | \( 0.002 \) |
| 0.05 | 100 | \( 0.301 \) | \( 0.012 \) | \( 0.004 \) | \( 0.002 \) | \( 0.013 \) | \( 0.011 \) | \( 0.003 \) | \( 0.001 \) | \( 0.001 \) |
| 0.20 | 20 | \( 1.192 \) | \( 0.800 \) | \( 0.815 \) | \( 0.973 \) | \( 1.092 \) | \( 0.755 \) | \( 0.596 \) | \( 0.615 \) | \( 0.752 \) |
| 0.20 | 50 | \( 1.198 \) | \( 0.638 \) | \( 0.362 \) | \( 0.478 \) | \( 0.964 \) | \( 0.600 \) | \( 0.215 \) | \( 0.112 \) | \( 0.103 \) |
| 0.20 | 100 | \( 1.201 \) | \( 0.578 \) | \( 0.158 \) | \( 0.108 \) | \( 0.445 \) | \( 0.537 \) | \( 0.065 \) | \( 0.015 \) | \( 0.007 \) |

Figure 1 shows results of empirical biases and MSEs of posterior means of \( \mu \) and \( \sigma \) under 
the uniform, reference and moment matching priors when \( \nu = 6 \) and varying the contamination.
Table 2: Empirical biases of posterior means for $\sigma$.

| $\epsilon$ | $n$ | $\beta$, $\gamma$ | $\alpha$, $\gamma$ | $\beta$-posterior | $\gamma$-posterior |
|------------|-----|-------------------|-------------------|------------------|------------------|
|            |     | $\alpha$, $\gamma$ | $\alpha$, $\gamma$ | $\beta$          | $\gamma$         |
|            |     | 0.0               | 0.3               | 0.5             | 0.7             | 1.0             | 0.3 | 0.5 | 0.7 | 1.0 |
| Uniform prior | 0.00 | 20               | 0.058             | 0.225           | 0.733           | 2.089           | 3.483 | 0.184 | 0.330 | 0.620 | 1.334 |
|            | 0.00 | 50               | 0.022             | 0.067           | 0.122           | 0.263           | 1.647 | 0.058 | 0.085 | 0.116 | 0.177 |
|            | 0.00 | 100              | 0.011             | 0.031           | 0.053           | 0.088           | 0.253 | 0.028 | 0.039 | 0.051 | 0.070 |
|            | 0.05 | 20               | 0.669             | 0.476           | 1.620           | 4.335           | 6.341 | 0.370 | 0.540 | 1.109 | 2.594 |
|            | 0.05 | 50               | 0.660             | 0.144           | 0.188           | 0.475           | 3.928 | 0.116 | 0.110 | 0.139 | 0.212 |
|            | 0.05 | 100              | 0.652             | 0.078           | 0.087           | 0.135           | 0.540 | 0.061 | 0.049 | 0.058 | 0.078 |
|            | 0.20 | 20               | 1.732             | 2.086           | 5.500           | 9.627           | 11.523 | 1.727 | 2.207 | 3.833 | 6.725 |
|            | 0.20 | 50               | 1.653             | 1.304           | 1.098           | 3.158           | 10.002 | 1.182 | 0.573 | 0.454 | 0.647 |
|            | 0.20 | 100              | 1.626             | 1.151           | 0.506           | 0.563           | 4.431 | 1.042 | 0.198 | 0.113 | 0.132 |
| Reference prior | 0.00 | 20               | -0.001            | 0.006           | -0.007          | -0.013          | -0.031 | -0.001 | -0.041 | -0.117 | -0.250 |
|            | 0.00 | 50               | 0.000             | 0.001           | -0.002          | -0.006          | -0.008 | 0.000 | -0.005 | -0.016 | -0.041 |
|            | 0.00 | 100              | 0.000             | 0.001           | -0.002          | -0.006          | -0.008 | 0.000 | -0.005 | -0.016 | -0.041 |
|            | 0.05 | 20               | 0.576             | 0.093           | 0.066           | 0.097           | 0.144 | 0.069 | 0.000 | -0.051 | -0.116 |
|            | 0.05 | 50               | 0.625             | 0.058           | 0.028           | 0.029           | 0.066 | 0.035 | -0.003 | -0.030 | -0.080 |
|            | 0.05 | 100              | 0.635             | 0.039           | 0.024           | 0.026           | 0.036 | 0.026 | 0.000 | -0.014 | -0.041 |
|            | 0.20 | 20               | 1.580             | 0.954           | 0.659           | 0.697           | 0.695 | 0.877 | 0.427 | 0.303 | 0.285 |
|            | 0.20 | 50               | 1.598             | 0.917           | 0.375           | 0.324           | 0.493 | 0.832 | 0.181 | 0.071 | 0.048 |
|            | 0.20 | 100              | 1.599             | 0.937           | 0.241           | 0.196           | 0.271 | 0.839 | 0.068 | 0.014 | 0.000 |
| Moment Matching prior | 0.00 | 20               | -0.039            | -0.044          | -0.083          | -0.186          | -0.529 | -0.039 | -0.061 | -0.090 | -0.102 |
|            | 0.00 | 50               | -0.015            | -0.016          | -0.029          | -0.067          | -0.275 | -0.014 | -0.019 | -0.027 | -0.048 |
|            | 0.00 | 100              | -0.007            | -0.014          | -0.032          | -0.147          | 0.006 | -0.008 | -0.012 | -0.019 |
|            | 0.05 | 20               | 0.516             | 0.021           | -0.029          | -0.113          | -0.415 | 0.016 | -0.021 | -0.023 | 0.042 |
|            | 0.05 | 50               | 0.601             | 0.026           | -0.002          | -0.037          | -0.243 | 0.017 | -0.011 | -0.021 | -0.031 |
|            | 0.05 | 100              | 0.623             | 0.027           | 0.010           | -0.005          | -0.121 | 0.017 | -0.003 | -0.010 | -0.018 |
|            | 0.20 | 20               | 1.481             | 0.736           | 0.395           | 0.225           | -0.157 | 0.717 | 0.373 | 0.361 | 0.606 |
|            | 0.20 | 50               | 1.559             | 0.808           | 0.276           | 0.165           | -0.070 | 0.748 | 0.162 | 0.084 | 0.104 |
|            | 0.20 | 100              | 1.579             | 0.872           | 0.197           | 0.135           | 0.001 | 0.787 | 0.061 | 0.019 | 0.022 |
Table 3: Empirical MSEs of posterior means for $\mu$.

| $\varepsilon$ | $n$  | $\alpha, \gamma$ | $R(\alpha)$-posterior | $\gamma$-posterior |
|---------------|------|------------------|------------------------|-------------------|
|               |      | 0.0              | 0.3 0.5 0.7 1.0        | 0.3 0.5 0.7 1.0   |
| Uniform prior |      |                  |                        |                   |
| 0.00          | 20   | 0.050            | 0.053 0.090 0.282 0.612| 0.053 0.057 0.078 0.152|
| 0.00          | 50   | 0.020            | 0.022 0.023 0.027 0.173| 0.022 0.023 0.025 0.028|
| 0.00          | 100  | 0.010            | 0.011 0.012 0.013 0.015| 0.011 0.012 0.013 0.015|
| 0.05          | 20   | 0.223            | 0.081 0.280 1.081 2.142| 0.075 0.076 0.159 0.471|
| 0.05          | 50   | 0.144            | 0.025 0.025 0.039 0.887| 0.025 0.025 0.027 0.030|
| 0.05          | 100  | 0.118            | 0.012 0.013 0.013 0.042| 0.012 0.013 0.014 0.016|
| 0.20          | 20   | 1.761            | 1.127 2.296 4.781 7.088| 1.031 0.906 1.402 2.708|
| 0.20          | 50   | 1.571            | 0.647 0.311 0.879 4.895| 0.613 0.188 0.088 0.078|
| 0.20          | 100  | 1.509            | 0.494 0.095 0.052 1.345| 0.463 0.046 0.019 0.019|
| Reference prior |      |                  |                        |                   |
| 0.00          | 20   | 0.050            | 0.054 0.062 0.077 0.124| 0.054 0.063 0.076 0.100|
| 0.00          | 50   | 0.020            | 0.022 0.024 0.027 0.033| 0.022 0.024 0.028 0.035|
| 0.00          | 100  | 0.010            | 0.011 0.012 0.013 0.016| 0.011 0.012 0.014 0.016|
| 0.05          | 20   | 0.223            | 0.065 0.067 0.086 0.372| 0.064 0.066 0.077 0.114|
| 0.05          | 50   | 0.144            | 0.024 0.026 0.028 0.042| 0.024 0.026 0.030 0.037|
| 0.05          | 100  | 0.118            | 0.012 0.013 0.014 0.016| 0.012 0.013 0.015 0.018|
| 0.20          | 20   | 1.761            | 0.744 0.385 0.564 2.433| 0.727 0.304 0.280 0.472|
| 0.20          | 50   | 1.571            | 0.497 0.111 0.057 0.164| 0.477 0.082 0.042 0.047|
| 0.20          | 100  | 1.509            | 0.410 0.041 0.020 0.021| 0.385 0.026 0.019 0.022|
| Moment Matching prior |      |                  |                        |                   |
| 0.00          | 20   | 0.050            | 0.055 0.064 0.080 0.124| 0.055 0.063 0.074 0.092|
| 0.00          | 50   | 0.020            | 0.022 0.025 0.028 0.041| 0.022 0.025 0.028 0.033|
| 0.00          | 100  | 0.010            | 0.011 0.012 0.014 0.018| 0.011 0.012 0.014 0.016|
| 0.05          | 20   | 0.223            | 0.063 0.067 0.085 0.241| 0.063 0.067 0.076 0.106|
| 0.05          | 50   | 0.144            | 0.024 0.026 0.030 0.043| 0.024 0.026 0.029 0.035|
| 0.05          | 100  | 0.118            | 0.012 0.013 0.014 0.019| 0.012 0.013 0.015 0.017|
| 0.20          | 20   | 1.761            | 0.648 0.295 0.394 1.884| 0.655 0.286 0.290 0.511|
| 0.20          | 50   | 1.571            | 0.453 0.088 0.044 0.061| 0.443 0.078 0.043 0.045|
| 0.20          | 100  | 1.509            | 0.385 0.034 0.018 0.022| 0.365 0.025 0.018 0.021|
Table 4: Empirical MSEs of posterior means for $\sigma$.

| $\varepsilon$ | $n$ | Bayes $\alpha, \gamma$ | $R^{(\alpha)}$-posterior $\alpha$ | $\gamma$-posterior $\gamma$ |
|---------------|-----|----------------------|-------------------------------|-------------------------|
|               |     | 0.0  | 0.3  | 0.5  | 0.7  | 1.0  | 0.3  | 0.5  | 0.7  | 1.0  |
| Uniform prior | 0.00 | 0.033 | 0.104 | 1.110 | 8.455 | 18.771 | 0.080 | 0.195 | 0.747 | 3.654 |
|               | 0.05 | 0.5  | 0.011 | 0.019 | 0.034 | 0.161 | 6.617 | 0.017 | 0.025 | 0.036 | 0.065 |
|               | 0.00 | 0.100 | 0.005 | 0.007 | 0.011 | 0.018 | 0.220 | 0.007 | 0.009 | 0.012 | 0.018 |
|               | 0.05 | 20   | 0.761 | 0.471 | 7.528 | 37.358 | 64.201 | 0.309 | 0.673 | 3.370 | 16.375 |
|               | 0.05 | 50   | 0.553 | 0.051 | 0.066 | 0.950 | 32.281 | 0.040 | 0.035 | 0.047 | 0.086 |
|               | 0.00 | 0.100 | 0.482 | 0.017 | 0.018 | 0.031 | 2.360 | 0.014 | 0.012 | 0.014 | 0.021 |
|               | 0.20 | 20   | 3.262 | 5.830 | 55.081 | 138.264 | 182.525 | 4.117 | 8.080 | 29.181 | 79.061 |
|               | 0.20 | 50   | 2.816 | 2.229 | 1.895 | 27.513 | 148.156 | 1.962 | 0.741 | 0.454 | 0.947 |
|               | 0.20 | 100  | 2.682 | 1.704 | 0.483 | 0.506 | 48.228 | 1.526 | 0.146 | 0.038 | 0.040 |
| Reference prior | 0.00 | 0.027 | 0.033 | 0.040 | 0.059 | 0.101 | 0.032 | 0.041 | 0.058 | 0.101 |
|               | 0.05 | 0.5  | 0.010 | 0.012 | 0.015 | 0.017 | 0.023 | 0.012 | 0.015 | 0.020 | 0.034 |
|               | 0.00 | 100  | 0.005 | 0.006 | 0.007 | 0.008 | 0.009 | 0.006 | 0.007 | 0.009 | 0.014 |
|               | 0.05 | 20   | 0.611 | 0.083 | 0.068 | 0.101 | 0.159 | 0.073 | 0.050 | 0.054 | 0.072 |
|               | 0.05 | 50   | 0.504 | 0.023 | 0.019 | 0.021 | 0.039 | 0.021 | 0.017 | 0.021 | 0.029 |
|               | 0.00 | 0.100 | 0.459 | 0.011 | 0.009 | 0.010 | 0.012 | 0.010 | 0.008 | 0.010 | 0.015 |
|               | 0.20 | 20   | 2.731 | 1.624 | 0.941 | 0.982 | 0.856 | 1.482 | 0.548 | 0.304 | 0.261 |
|               | 0.20 | 50   | 2.633 | 1.330 | 0.341 | 0.215 | 0.463 | 1.218 | 0.171 | 0.048 | 0.033 |
|               | 0.20 | 100  | 2.595 | 1.268 | 0.144 | 0.070 | 0.112 | 1.144 | 0.046 | 0.014 | 0.015 |
| Moment Matching prior | 0.00 | 0.026 | 0.031 | 0.040 | 0.063 | 0.286 | 0.032 | 0.042 | 0.054 | 0.079 |
|               | 0.05 | 0.5  | 0.010 | 0.012 | 0.015 | 0.019 | 0.083 | 0.012 | 0.015 | 0.020 | 0.028 |
|               | 0.00 | 100  | 0.005 | 0.006 | 0.007 | 0.009 | 0.028 | 0.006 | 0.007 | 0.009 | 0.013 |
|               | 0.05 | 20   | 0.525 | 0.058 | 0.046 | 0.052 | 0.192 | 0.057 | 0.048 | 0.056 | 0.101 |
|               | 0.05 | 50   | 0.470 | 0.020 | 0.017 | 0.018 | 0.068 | 0.019 | 0.017 | 0.021 | 0.026 |
|               | 0.00 | 0.100 | 0.443 | 0.010 | 0.008 | 0.009 | 0.021 | 0.009 | 0.008 | 0.010 | 0.014 |
|               | 0.20 | 20   | 2.411 | 1.132 | 0.441 | 0.186 | 0.057 | 1.137 | 0.461 | 0.385 | 0.867 |
|               | 0.20 | 50   | 2.507 | 1.120 | 0.222 | 0.082 | 0.023 | 1.065 | 0.153 | 0.054 | 0.051 |
|               | 0.20 | 100  | 2.532 | 1.148 | 0.106 | 0.040 | 0.012 | 1.054 | 0.043 | 0.015 | 0.016 |
Objective priors for divergence-based robust estimation
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Figure 1: The horizontal axis is the contamination ratio \( \varepsilon \). The red lines indicate the empirical biases and MSEs of \( \gamma \)-posterior means under the three priors when \( n = 100 \) and \( \varepsilon = 0.20 \). Similarly, the blue and green lines indicate that of \( R^{(\alpha)} \)-posterior and ordinary posterior means, respectively. “Uni”, “Ref” and “MM” indicate Uniform, Reference and Moment Matching priors, respectively.

Figure 2 also shows results of empirical biases and MSEs of posterior means of \( \mu \) and \( \sigma \) under the three types of priors when the contamination is \( \varepsilon = 0.20 \) and varying \( \nu \) from 0.0 to 10.0. From Figures 1 and 2 the performance of the \( \gamma \)-posterior means is reasonable for varying of \( \nu \) and \( \varepsilon \). In particular, the \( \gamma \)-posterior means of \( \sigma \) under reference and moment matching priors are better than other estimators. On the other hand, the performance of \( R^{(\alpha)} \)-posterior means also seems to be reasonable for the estimation of location parameter \( \mu \), but they have larger bias and MSE than that of \( \gamma \)-posterior means for the estimation of scale parameter \( \sigma \). Moreover, \( R^{(\alpha)} \)-posterior mean under the uniform prior seems to be unstable in many cases.

Remark 2. The selection of a tuning parameter \( \gamma \) (or \( \alpha \)) is very challenging and there is no optimal choice of \( \gamma \) to the best of our knowledge. The tuning parameter \( \gamma \) controls the degree of robustness, that is, if we set large \( \gamma \), then we obtain the higher robustness. However, there is a trade-off between robustness and efficiency of estimators. One of solutions for this problem is to use the asymptotic relative efficiency (see e.g. Ghosh and Basu (2016)). Note that Ghosh and Basu (2016) only deals with one-parameter case. In general, the asymptotic relative efficiency of the robust posterior mean \( \hat{\theta}^{(\gamma)} \) of \( p \)-dimensional parameter \( \theta \) relative to the usual posterior mean \( \hat{\theta} \) is defined by

\[
\text{ARE}(\hat{\theta}^{(\gamma)}, \hat{\theta}) := \left( \frac{\det(V(\theta))}{\det(V^{(\gamma)}(\theta))} \right)^{1/p}
\]
Figure 2: The horizontal axis is the location parameter $\nu$ of contamination distribution. The red lines indicate the empirical biases and MSEs of $\gamma$-posterior means under the three priors when $n = 100$ and $\varepsilon = 0.20$. Similarly, the blue and green lines indicate that of $R^{(\alpha)}$-posterior and ordinary posterior means, respectively. “Uni”, “Ref” and “MM” indicate Uniform, Reference and Moment Matching priors, respectively.

(see e.g. [Serfling (1980)]). This is the ratio of determinants of the covariance matrices, raised to the power of $1/p$, where $p$ is the dimension of the parameter $\theta$. We now calculate the $\text{ARE}(\hat{\theta}^{(\gamma)}, \hat{\theta})$ in our simulation setting. After some calculations, the asymptotic relative efficiency is given by

$$\text{ARE}(\hat{\theta}^{(\gamma)}, \hat{\theta}) = \left(\frac{2}{(1 + \gamma)^6(1 + 2\gamma)(2 + 4\gamma + 3\gamma^2)}\right)^{1/2} =: h(\gamma)$$

for $\gamma > 0$. We note that it holds $h(\gamma) \to 1$ as $\gamma \to 0$. Hence, we may be able to choose $\gamma$ to allow the small inflation of the efficiency. For example, if we require the value of the asymptotic relative efficiency $\text{ARE} = 0.95$, then we may choose the value of $\gamma$ as the solution of the equation $h(\gamma) = 0.95$ (see Table 5). The curve of the function $h(\gamma)$ is also given in Figure 3. A similar discussion is given by [Sugasawa (2020)] in terms of the MSE for small area estimation.

Table 5: The value of $\gamma$ and the corresponding asymptotic relative efficiency

| $\gamma$ | 0.01  | 0.1   | 0.3   | 0.5   |
|----------|-------|-------|-------|-------|
| ARE      | 0.951489 | 0.6222189 | 0.2731871 | 0.1359501 |
5 Concluding remarks

We considered objective priors for divergence-based robust Bayesian estimation. In particular, we showed that reference and moment matching priors under quasi-posterior based on the $\gamma$-divergence are robust against unknown quantities in a data generating distribution. The performance of the corresponding posterior means is illustrated through some simulation studies. However, the proposed objective priors are often improper, and to show the posterior propriety for them remains as the future plan. Our result should be extended to other settings. For example, Kanamori and Fujisawa (2015) proposed the estimation of the contamination ratio by using unnormalized model. To study such problem in Bayesian perspective is also challenging because it has a problem how to set a prior distribution for unknown contamination ratio. Furthermore, it would be also interesting to consider an optimal data-dependent choice of tuning parameter $\gamma$.

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Appendix

A.1 Some derivative functions

We now put
\[ \ell(x) = \log f_\theta(x), \quad \ell_i(x) = \partial_i \log f_\theta(x), \quad \ell_{ij}(x) = \partial_i \partial_j \log f_\theta(x) \quad \text{and} \quad \ell_{ijk}(x) = \partial_i \partial_j \partial_k \log f_\theta(x), \]
and let a norm \( \| \cdot \|_p : L_p(\Omega) \to \mathbb{R} \) be defined by
\[ \| h \|_p = \left( \int_{\Omega} |h(x)|^p dx \right)^{1/p}. \]

Then we obtain derivative functions of \( q^{(\alpha)}(x_j; \theta) \) with respect to \( \theta \) as follows:

\[
\begin{align*}
\partial_i q^{(\alpha)}(x; \theta) &= -f_\theta(x)^\alpha \ell_i(x) + \int_{\Omega} f_\theta(y)^{\alpha+1} \ell_i(y) dy \\
\partial_i \partial_j q^{(\alpha)}(x; \theta) &= -f_\theta(x)^\alpha \{ \alpha \ell_i(x) \ell_j(x) + \ell_{ij}(x) \} \\
&\quad + \int_{\Omega} f_\theta(y)^{\alpha+1} \{ (\alpha + 1) \ell_i(y) \ell_j(y) + \ell_{ij}(y) \} dy, \\
\partial_i \partial_j \partial_k q^{(\alpha)}(x; \theta) &= -f_\theta(x)^\alpha \{ \alpha^2 \ell_i(x) \ell_j(x) \ell_k(x) + \ell_{ijk}(x) \\
&\quad + \alpha (\ell_k(x) \ell_{ij}(x) + \ell_i(x) \ell_{jk}(x) + \ell_j(x) \ell_{ik}(x)) \} \\
&\quad + \int_{\Omega} f_\theta(y)^{\alpha+1} \{ (\alpha + 1)^2 \ell_i(y) \ell_j(y) \ell_k(y) + \ell_{ijk}(y) \\
&\quad + (\alpha + 1) \{ \ell_k(y) \ell_{ij}(y) + \ell_i(x) \ell_{jk}(y) + \ell_j(x) \ell_{ik}(y) \} \} dy, \\
\partial_i q^{(\gamma)}(x; \theta) &= -\frac{f_\theta(x)^\gamma}{\| f_\theta \|_{\gamma+1}} \ell_i(x) + \frac{f_\theta(x)^\gamma}{\| f_\theta \|_{\gamma+1}^{\gamma+1}} \int_{\Omega} f_\theta(y)^{\gamma+1} \ell_i(y) dy.
\end{align*}
\]
Similarly, we obtain derivative functions of $q^{(\gamma)}(x_j; \theta)$ as follows:

\[
\partial_j \partial_k q^{(\gamma)}(x_j; \theta) = -\frac{f_\theta(x)^\gamma}{\|f_\theta\|_{\gamma+1}^{\gamma+1}} (\gamma \ell_i(x) \ell_j(x) + \ell_{ij}(x))
\]

\[
+ \gamma \frac{f_\theta(x)^\gamma}{\|f_\theta\|_{\gamma+1}^{\gamma+1}} \left( \ell_j(x) \int_\Omega f_\theta(y)^{\gamma+1} \ell_i(x, y) dy + \ell_i(x) \int_\Omega f_\theta(y)^{\gamma+1} \ell_j(y) dy \right)
\]

\[
- (1 + 2\gamma) \frac{f_\theta(x)^\gamma}{\|f_\theta\|_{\gamma+1}^{\gamma+1}} \int_\Omega f_\theta(y)^{\gamma+1} \ell_i(y) dy \int_\Omega f_\theta(y)^{\gamma+1} \ell_j(y) dy
\]

\[
+ \frac{f_\theta(x)^\gamma}{\|f_\theta\|_{\gamma+1}^{\gamma+1}} \left( \int_\Omega f_\theta(y)^{\gamma+1} \left( (\gamma + 1) \ell_i(y) \ell_j(y) + \ell_{ij}(y) \right) dy \right).
\]

\[
\partial_k \partial_j q^{(\gamma)}(x; \theta) = -\frac{f_\theta(x)^\gamma}{\|f_\theta\|_{\gamma+1}^{\gamma+1}} (\gamma^2 \ell_i(x) \ell_j(x) \ell_k(x) + \ell_{ijk}(x))
\]

\[
- \gamma \frac{f_\theta(x)^\gamma}{\|f_\theta\|_{\gamma+1}^{\gamma+1}} (\ell_k(x) \ell_j(x) + \ell_j(x) \ell_{ik}(x) + \ell_i(x) \ell_{jk}(x))
\]

\[
+ \frac{f_\theta(x)^\gamma}{\|f_\theta\|_{\gamma+1}^{\gamma+1}} (\gamma^2 \ell_j(x) \ell_k(x) + \gamma \ell_{jk}(x)) \int_\Omega f_\theta(y)^{\gamma+1} \ell_i(y) dy
\]

\[
+ \frac{f_\theta(x)^\gamma}{\|f_\theta\|_{\gamma+1}^{\gamma+1}} (\gamma^2 \ell_i(x) \ell_k(x) + \gamma \ell_{ik}(x)) \int_\Omega f_\theta(y)^{\gamma+1} \ell_j(y) dy
\]

\[
+ \frac{f_\theta(x)^\gamma}{\|f_\theta\|_{\gamma+1}^{\gamma+1}} (\gamma^2 \ell_i(x) \ell_j(x) + \gamma \ell_{ij}(x)) \int_\Omega f_\theta(y)^{\gamma+1} \ell_k(y) dy
\]

\[
- (1 + 2\gamma) \frac{f_\theta(x)^\gamma}{\|f_\theta\|_{\gamma+1}^{\gamma+1}} \int_\Omega f_\theta(y)^{\gamma+1} \ell_i(y) dy \int_\Omega f_\theta(y)^{\gamma+1} \ell_j(y) dy
\]

\[
- (1 + 2\gamma) \frac{f_\theta(x)^\gamma}{\|f_\theta\|_{\gamma+1}^{\gamma+1}} \int_\Omega f_\theta(y)^{\gamma+1} \ell_j(y) dy \int_\Omega f_\theta(y)^{\gamma+1} \ell_k(y) dy
\]

\[
- (1 + 2\gamma) \frac{f_\theta(x)^\gamma}{\|f_\theta\|_{\gamma+1}^{\gamma+1}} \int_\Omega f_\theta(y)^{\gamma+1} \ell_i(y) dy \int_\Omega f_\theta(y)^{\gamma+1} \ell_k(y) dy
\]

\[
+ \frac{f_\theta(x)^\gamma}{\|f_\theta\|_{\gamma+1}^{\gamma+1}} \int_\Omega f_\theta(y)^{\gamma+1} \left( (\gamma + 1) \ell_i(y) \ell_j(y) \ell_k(y) + \ell_{ijk}(y) \right) dy
\]

\[
+ \frac{f_\theta(x)^\gamma}{\|f_\theta\|_{\gamma+1}^{\gamma+1}} \int_\Omega f_\theta(y)^{\gamma+1} \left( (\gamma + 1) \ell_i(y) \ell_k(y) + \ell_j(y) \ell_{ik}(y) + \ell_{ij}(y) \ell_{jk}(y) \right) dy.
\]
A.2 Proof of Theorem 6

Proof. From Hölder’s inequality and Lyapunov’s inequality, it holds that

\[ \mathbb{E}_\delta [ |f_{\theta}(X_1)^\gamma \ell_i(X_1)| ] \leq \nu^\gamma \left( \int_\Omega | \ell_i(x) |^{1+\gamma} \delta(x) dx \right)^{1/(1+\gamma)}, \]

\[ \mathbb{E}_\delta [ |f_{\theta}(X_1)^\gamma \ell_i(X_1) \ell_j(X_1)| ] \leq \nu^\gamma \left( \int_\Omega | \ell_i(x) \ell_j(x) |^{1+\gamma} \delta(x) dx \right)^{1/(1+\gamma)}, \]

\[ \mathbb{E}_\delta [ |f_{\theta}(X_1)^\gamma \ell_{ij}(X_1)| ] \leq \nu^\gamma \left( \int_\Omega | \ell_{ij}(x) |^{1+\gamma} \delta(x) dx \right)^{1/(1+\gamma)}, \]

\[ \mathbb{E}_\delta [ |f_{\theta}(X_1)^\gamma \ell_{ijk}(X_1)| ] \leq \nu^\gamma \left( \int_\Omega | \ell_{ijk}(x) |^{1+\gamma} \delta(x) dx \right)^{1/(1+\gamma)}, \]

\[ \mathbb{E}_\delta [ |f_{\theta}(X_1)^\gamma \ell_{ij}(X_1) \ell_{jk}(X_1)| ] \leq \nu^\gamma \left( \int_\Omega | \ell_{ij}(x) \ell_{jk}(x) |^{1+\gamma} \delta(x) dx \right)^{1/(1+\gamma)}, \]

\[ \mathbb{E}_\delta [ |f_{\theta}(X_1)^\gamma \ell_i(X_1) \ell_j(X_1) \ell_k(X_1)| ] \leq \nu^\gamma \left( \int_\Omega | \ell_i(x) \ell_j(x) \ell_k(x) |^{1+\gamma} \delta(x) dx \right)^{1/(1+\gamma)} \]  \hspace{1cm} (A.3)

for \( i, j, k = 1, \ldots, p \). By using (A.2) and (A.3), we have

\[ \mathbb{E}_\delta \left[ \partial_i \partial_j q^{(\gamma)}(X_1; \theta) \right] = O(\nu^\gamma), \]

\[ \mathbb{E}_\delta \left[ \partial_i \partial_j \partial_k q^{(\gamma)}(X_1; \theta) \right] = O(\nu^\gamma) \]

for \( i, j, k = 1, \ldots, p \). Therefore the proof is complete. \( \square \)