Time crystal based on a levitated charged nanoparticle

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We propose a scheme to realize a time crystal via a levitated charged nanoparticle in a strong uniform magnetic field, where two torsional modes and one rotational mode are coupled. In the limit of strong magnetic field or large charge-to-mass ratio, two torsional modes can be eliminated, and the effective theory shows the remaining rotational mode behaves as a classical time crystal at finite temperature. Through canonical quantization, we solve the corresponding Hamiltonian. A quantum time crystal appears at the ground state when there is a magnetic flux. In this way, our model presents both classical and quantum time crystals.

Time crystal is a phase which spontaneously breaks time translational symmetry in the ground state \([1, 2]\). In 2012, Wilczek et al. proposed two models for time crystals. One is quantum [3] while the other is classical [4]. Later, Li et al. proposed that both the quantum space-time crystal and time quasicrystal can be realized experimentally using trapped ions [5]. Quantum time crystals have been discussed a lot in the following years [6–12]. A no-go theorem was proved that, for many-body systems with short range coupling and finite volume, the quantum time crystal does not exist in thermal equilibrium [11].

The no-go theorem can be bypassed if one considers systems in non-equilibrium. In this way, the discrete time crystal was theoretically proposed [13–16] and experimentally verified [17, 18]. Later, both the discrete space-time crystal and the discrete time quasicrystal were realized [19–22]. The discrete time crystal has also been discussed in topological quantum computation [23], cold atom [24, 25], etc. Recently, using long range coupling Hamiltonian or interacting gauge filed, the no-go theorem can also be bypassed, and the existence of quantum time crystals in the ground state has been proposed [26, 27]. However, it is practically challenging to realize these models in laboratory.

On the other hand, attentions on the classical time crystal are relatively low [28–33]. The original classical time crystal model contains singular solution points [4], which are difficult to be tested in experiments. The correspondence with the original classical time crystal is mostly found in cosmology [28–32]. Recently, Shapere and Wilczek showed that the “Sisyphus dynamics” could arise in the effective motion of a planar charged particle subjected to the magnetic field, and the classical time crystal Lagrangians emerges in the effective theory of their systems [34]. However, the amplitude of the Sisyphus dynamics depends on the external perturbations, and disappears in the ground state. Besides, asymmetric mass parameters are required in this model, which are difficult to realize.

In this letter, we propose a scheme to realize a time crystal based on a levitated charged nanoparticle placed in an uniform magnetic field, where two of the rotational modes of the nanoparticle are trapped, while the third one rotates freely. By eliminating the two trapped rotational (torsional) modes, we show nonzero angular velocity in the effective theory for the third rotational mode, for both the classical model in thermal equilibrium, and the quantum model in the ground state. This nonzero velocity is the hallmark of the continuous time translational symmetry breaking. Since the charges fixed inside the nanoparticle are interacting with each other through a long range all-to-all coupling (Coulomb interaction), the no-go theorem for quantum time crystal with short range coupling is not applied here. Furthermore, we find that the parameters required for the time crystal phase in our model are experimentally realizable.

Before we present our theoretical model, let us introduce the possible experimental setup. As shown in Figure 1, we consider a levitated charged nanoparticle in an optical tweezer or an ion trap [35–38]. Then the nanoparticle experiences both the force and torque in the trap until the mechanical equilibrium is reached where only one rotational mode is free. The three center of mass (c.m.) and other two rotational (torsional) modes are trapped [39–43]. In the following we ignore the coupling between c.m. and rotations, and only consider rotations and their own coupling [44]. Next, the nanoparticle is pierced through by a strong uniform magnetic field.
Figure 1. Schematic drawing of the experimental setup. A charged nanoparticle is trapped and levitated by external potential. All three translational degrees of freedom are fixed. The nanoparticle can freely rotate along the $Z''$ axis, but the other two rotational modes are doing small vibration. A strong uniform magnetic field $B_Y$ is applied along the $Y$ direction. Another weak magnetic field $B_Z \ll B_Y$ along the $Z$ direction can be used for achieving quantum time crystals.

Since the nanoparticle is charged, its rotation generates a magnetic moment coupling to the external magnetic field, which mixes the two torsional modes and the remaining free rotation.

Now let us examine the classical model. The Lagrangian for the rotation is given by

$$L = \frac{1}{2} m_{ij} \omega_i \omega_j + \frac{1}{2} \epsilon_{ij} B_i \omega_j - U(\bar{r}), \quad (1)$$

where $U(\bar{r})$ is the trapping potential, $\omega_i$ is the angular velocity, $B_i$ is the magnetic field, $m_{ij}$ and $\epsilon_{ij}$ are tensors defined as $m_{ij} = \int d^3r \rho_m(r)(r^2 - r_i r_j)$, and $\epsilon_{ij} = \int d^3r \rho_e(r)(r^2 - r_i r_j)$ with mass and charge density $\rho_m$ and $\rho_e$, and $i, j = 1, 2, 3$. We make several assumptions to simplify our model. First, consider the nanoparticle as a symmetric top such that $m_{ij}$ and $\epsilon_{ij}$ can be diagonalized simultaneously in the body frame, with principal moments $(m_1 = m_2, m_3)$, and $\{e_1 = e_2, e_3\}$ respectively. Second, we use the $ZY'Z''$ Euler angles (see Figure 1), and consider the potential $U(\alpha, \beta, \gamma)$ with equilibrium position $\alpha = \beta = 0$ but no constraint on $\gamma$, such that $U \approx u_1 \beta^2 / 2 + u_2 \alpha^2 / 2$. Third, we set the magnetic field as $B = (0, B_Y, B_Z)$, and $B_Z \ll B_Y$. Eventually, the Lagrangian and Hamiltonian in the limit of $\alpha, \beta \ll 1$ are given by (see Supplementary Materials I)

$$L = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} \left(m_3 + b^2 \dot{x}^2 / u_2\right) \dot{z}^2 + \phi \ddot{z} - \frac{1}{2} x_1 \dot{x}^2, \quad (2)$$

$$H = \frac{p_z^2}{2m_1} + \frac{(p_x - \phi)^2}{2(m_3 + b^2 \dot{x}^2 / u_2)} + \frac{1}{2} x_1 \dot{x}^2, \quad (3)$$

where $x = \beta$, $z = \alpha + \gamma$, $b = B_Y e_3 / 2$, and $\phi = B_Z e_3 / 2$.

Notice the total time derivative term $\phi \dot{z}$ has no effect in classical mechanics, so we ignore this term for the moment. However, in quantum mechanics, the gauge term $\phi \dot{z}$ accumulates a geometric phase in the wavefunction, since $z$ can rotate a full cycle. This phase changes the quantum spectrum and leads to nonzero angular velocity in the ground state, i.e. a quantum analogy of time crystal, as shown latter in this letter.

Figure 2. (a) Schematic picture of effective potential energy $V(x)$. If $|l| < l_c$, $V(x)$ behaves as the blue curve; if $|l| > l_c$, $V(x)$ behaves as the orange curve. The latter has a Mexican hat shape, which implies spontaneously symmetry breaking. (b) Schematic phase diagram of the classical time crystal. Time crystal phase exists when $q > 1$ and $T < u_2$. The higher the quality $q$, the larger the temperature range of the time crystal phase. As $q \to \infty$, any arbitrarily small thermal fluctuation kicks the system into the time crystal phase.

Let’s return to the classical model. Since $z$ is cyclic, $p_z = l$ is a constant. For a given $l$, the effective potential energy reads

$$V(x) = \frac{l^2}{2(m_3 + b^2 x^2 / u_2)} + \frac{1}{2} x_1 x^2, \quad (4)$$

where the first term is similar to the centrifugal potential except that $V(x = 0)$ is finite, while the conventional centrifugal barrier diverges as $x^{-2}$. There exists a critical angular momentum $l_c = m_3 \sqrt{u_1 u_2} / b$. If $0 < |l| < l_c$, there is one single equilibrium point $x = 0$ which is stable. The nontrivial case is $|l| > l_c$, in which we have three equilibrium points: $x_{\pm} = \pm \sqrt{l_c(l - l_c) / m_3 u_1}$. $x_0 = 0$. $x_{\pm}$ are stable with corresponding minima $V(x_{\pm}) = \left(|l|^2 - (l - l_c)^2\right) / 2m_3$, while $x_0$ is unstable with local maximum $V(0) = l^2 / 2m_3$. This creates a potential barrier $\Delta = |l - l_c|^2 / 2m_3$ at the center. The different potential profiles are shown in Figure 2 (a). If we choose $x(t = 0) = x_\pm$, then $\dot{z} = \sqrt{u_1 u_2} / b$ is a constant independent on $l$. Numerical simulations show that even if $x$ is doing small oscillation around $x_\pm$, the angular velocity averaged over a period is a constant $\bar{z} = \text{sign}(l) \sqrt{u_1 u_2} / b$ (see SM III). Since this property of $\bar{z}$ is quite robust when $|l| > l_c$, in the following we consider $\bar{z}$ as an order parameter. If $\bar{z} = \text{sign}(l) \sqrt{u_1 u_2} / b$, we say that the system is in the (classical) time crystal phase.

For convenience, we define the quality of the time crystal phase as $q \equiv b / \sqrt{m_3 u_2} = b / m_3 \omega_2$, where $\omega_{1,2} = \sqrt{u_{1,2} / m_3}$. In order to get some intuitions of how the trajectory $x(t)$ looks like in the time crystal phase, one may consider the oscillation around $x_\pm$ and take limit $m_3 u_2 / b^2 \ll \dot{x}^2$, or equivalently $q^2 x^2 \gg 1$, then the effective potential is replaced by $V(x) = l^2 u_2 / 2b^2 x^2 + x_1 x^2 / 2$. This procedure is justified if the oscillation amplitude is small and $q^2 x^2_\pm \gg 1$. Notice that $q^2 x^2_\pm = |l| / l_c$, this limit $|l| / l_c \gg 1$ is easily fulfilled within the time crys-
tal phase. The trajectory $x(t)$ can be solved analytically as
$$x(t) = \frac{1}{\sqrt{v_{12}}} \left[ E + \sqrt{E^2 - V^2(x_{\pm})} \sin(2\omega_1(t - t_0)) \right]^{1/2},$$
where $V(x_{\pm}) = |l|/\sqrt{v_{12}}$ are minimum of the double wells, $E > V(x_{\pm})$ is the total energy of the system, $t_0$ is a constant depends on the initial value $x(0)$. Thus $x(t)$ oscillates around $x_{\pm}$ with frequency $2\omega_1$, and in SM II we show the average angular velocity in one period is the same constant $\bar{\omega} = \text{sign}(l)/\sqrt{v_{12}}/b$.

The connection between our model to a classical time crystal can be seen as follows. First, in the limit of $m_{1,3} \to 0$ (or $q \gg 1$), in SM I we show that Eq. (2) reduces to exactly the same form as the classical time crystal Lagrangian in Ref. 4. Second, any finite but small $m_{1,3}$ will regulate the singularity of the classical time crystal Lagrangian, but the feature of a time crystal remains. Although the true ground state of this system is at rest, small perturbations can drive the system into the time crystal phase. Namely, if the initial condition satisfies $|l| \gg l_c$, which can be fulfilled by arbitrarily small perturbation in the limit $q \to \infty$, then $x = 0$ is no longer stable and the system will approach to $x_{\pm} \neq 0$, and as a result $\dot{z} = \text{sign}(l)/\sqrt{v_{12}}/b$. Numerical simulations show that even if the initial perturbation is big, a small damping will force the system stay close to $x_{\pm} \neq 0$, with nontrivial angular velocity $\dot{z} = \text{sign}(l)/\sqrt{v_{12}}/b$, until the energy is dissipated completely and the motion ceased (see SM III). The lifetime of this rotation is inversely proportional to the vacuum pressure, and could be larger than $6 \times 10^4$ s in experiments [45].

Next we want to estimate the temperature range such that the system is in the time crystal phase. The thermal fluctuations perturb the system with energy $E \sim T$ (here $k_B = 1$). To observe a time crystal phase, $T$ cannot be too big, otherwise our small angle approximation fails and the Lagrangian Eq. (2) is invalid. On the other hand, if $T$ is too small, there is no enough energy to reach the threshold angular momentum $l_c$. (The effective potential is not double-well when $|l| \leq l_c$ and $\dot{z} \neq \text{sign}(l)/\sqrt{v_{12}}/b$). Therefore, the temperature range for the time crystal phase is $l_c^2/2m_3 < T < v_{1,2}/q^2 < T < u_2$, which exists only when $q > 1$. The higher the $q$, the larger the range of temperature in the time crystal phase, and hence the higher the quality of the time crystal. In other words, the time crystal phase is only sensitive to a single dimensionless parameter, the quality $q$. A schematic picture of the phase diagram is shown as Figure 2 (b).

In thermal equilibrium, the statistical average of the angular velocity $\langle \dot{z} \rangle$ can be calculated, where $\dot{z} = p_z/\hbar b^2 x^2$, $\rho = e^{-H/T}/Z$ is the density matrix, $Z = \text{tr} e^{-H/T}$ is the partition function, and the Hamiltonian $H$ is defined in Eq. (3). Since the system has time reversal symmetry, $\langle \dot{z} \rangle = 0$ is guaranteed. Instead, one should look at the speed $\langle |\dot{z}| | \rangle$ and its relative fluctuation $\Delta |\dot{z}| | / \langle |\dot{z}| | \rangle$. In SM V we find that the relative fluctuation is smaller than 1 if $q^2T < 5.3u_2$. Therefore the system would rotate with a specific speed but change its rotational direction randomly by thermal fluctuations. In this way, the thermal equilibrium state breaks the time-translational symmetry, and is doubly degenerate. Thus, we call this state a classical time “polycrystal”.

Let us estimate the experimental feasibility of our classical time crystal model. Considering a hollow nanoparticle made by the hexagonal Boron nitride (h-BN) with radius $\sim 1\mu$m, thickness $\sim 10$nm, mass density $\sim 2g/cm^3$, and surface charge density $0.025e/\text{nm}^2$ trapped with tor- sional frequency $\sim 100Hz$ [46, 47], we can make the quality $q \sim 10$ by applying magnetic field $\sim 5T$. In order to observe the classical time crystal, the temperature should be $10mK \lesssim T \lesssim 50mK$ (See SM IV), which is reachable by feedback cooling [48].

Next, we study whether the time crystal phase exists in the quantum analog of our model. Before we proceed to the quantum model, it is heuristic and pedagogical to consider the semi-classical approach by applying Bohr-Sommerfeld quantization condition $\oint p_x dx = (n + 1/2)\hbar$, where $n \in \mathbb{Z}$ and $p_x = \sqrt{2m(E - V(x))}$ is the momentum. Again we take limit $m_3 \to 0$ to get analytical results. The spectrum is $E_n = V(x_{\pm}) + \omega_0^2(2n + 1)$, which looks like a simple harmonic oscillator (SHO) with frequency $2\omega_1$, and is consistent with the frequency of the classical trajectory.

Now we are well-prepared to solve the quantum model. The Schrödinger equation $H\Psi = E\Psi$ can be solved by separation of variables $\Psi(x, z) = \chi(x)\zeta(z)$, which leads to two ordinary differential equations

$$-\frac{h^2\chi''}{2m_1} + \left( \frac{\hbar^2\sigma(l)}{2m_1(x^2 + \xi^2)} + \frac{1}{2m_1\omega_1^2x^2} \right) \chi = E\chi, \quad (5)$$

$$h^2\zeta'' = -l^2\zeta, \quad (6)$$

where $\sigma = \frac{p_x^2 m_3}{\hbar^2 h^2}$, $\xi^2 = m_3 u_2/\hbar^2$. The boundary conditions are $\chi(\pm \infty) = 0$, $\zeta(z + 2\pi) = \zeta(z)$ respectively. Eq. (6) is trivial with solution $\zeta(z) = e^{ilz/\hbar}$, where $l/\hbar \in \mathbb{Z}$. To solve Eq. (5) analytically, we assume $\xi^2 \ll x^2$ which will be justified below. Following this assumption, the energy spectrum is determined by two

Figure 3. Spectrum $E_n$ and velocity $\langle \dot{z} \rangle$, presented in units such that $\hbar = \omega_1 = 1$, and $q = 10$. Numbers mark the curves with different quantum number $n$. Solid lines correspond to even $n$ (symmetric states) while dashed lines correspond to odd $n$ (antisymmetric states). The black curves marked by “A” correspond to Eqs. (7, 8). (a) Spectrum $E_n$ as a function of angular momentum $l$. (b) $\langle \dot{z} \rangle$ as a function of $l$. 

For symmetric states only for \( l \) simulations where \( q \) is actually periodic in \( t \), with frequency \( \dot{z} \). However, it is possible to get both \( \langle \dot{z} \rangle \neq 0 \) and periodic \( \langle z(0)z(t) \rangle \) even for the ground state. This indicates the existence of quantum time crystal [11]. To present the quantum time crystal, we consider the consequence of \( \phi \neq 0 \). It is well-known that this magnetic flux changes the spectrum, such that \( \sigma \propto (l - \phi)^2 \) in Eq. (7). Hence, as long as \( \phi/\hbar \notin \mathbb{Z} \), \( \sigma \) is non-vanishing even for the ground state, leading to nonzero \( \langle \dot{z} \rangle \) given by Eq. (8). As a result, \( \langle \dot{z} \rangle \) becomes a periodic function in \( \phi \), as shown in Figure 4. Distinguished from the classical model, here the time reversal symmetry is firstly broken by the magnetic flux, leading to \( \langle \dot{z} \rangle \neq 0 \), and then the time translational symmetry is broken simultaneously. In order to observe the quantum ground-state behavior experimentally, we require the temperature to be smaller than the excitation gap \( \sim \hbar\omega_1 \simeq 1\text{K} \), which could be realizably by sympathetic cooling with BEC [51].

In conclusion, we have proposed a scheme to realize the time crystal in a levitated charged nanoparticle. In our model, the time crystal appears in both classical thermal equilibrium and the quantum ground state. The conditions required for the time crystal phase in our model are feasible in laboratory. In future, it would be interesting to study how to realize the space-time crystal and time quasicrystal in classical systems, e.g., trapped ions crystals [52–54].

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\[ E_{nl} = \left( 2n + 1 + \sqrt{\sigma + 1/4} \right) \hbar \omega_1. \] (7)

We are interested in the expectation value of the angular velocity \( \dot{z} \)

\[ \langle \dot{z} \rangle = \frac{i}{\hbar} \langle [H, z] \rangle = \text{sign}(l) \frac{\sqrt{u_1 u_2/b}}{\sqrt{1 + (4\sigma)^{-1}}}. \] (8)

which is true for all eigenstates \(|nl\rangle\). Notice that the mechanical momentum \( \dot{z} \) is not the same as the canonical momentum \( p_z = l \). However, it is \( \dot{z} \) that determines how fast the system rotates. If \( \sigma \gg 1 \), then \( \langle \dot{z} \rangle \rightarrow \text{sign}(l) \sqrt{u_1 u_2/b} \), which recovers the classical result; if \( \sigma \ll 1 \), then \( \dot{z} \rightarrow 2l u_2 \sqrt{m_1 u_1/b^2} \hbar \), which is linear in \( l \). Furthermore, Eq. (8) allows us to evaluate the time correlation function \( \langle z(0)z(t) \rangle = \langle z(0)^2 \rangle + \langle z(0) \rangle \langle \dot{z}(0) \rangle t \), which linearly depends on time, and is true for any eigenstate. Since \( z \in \{0, 2\pi\} \) is an angular variable, \( \langle z(0)z(t) \rangle \) is actually periodic in \( t \), with frequency \( \propto \langle \dot{z}(0) \rangle \).

In Figure 3, we compare our analytical results Eq. (7, 8) (shown as black curves marked by “A”) with numerical simulations where \( l \geq 0 \) and \( q = 10 = \{50\} \). The most important feature resides in the parity of wavefunctions. For antisymmetric states with odd \( n \), numerical results are practically the same as Eqs. (7, 8); while for symmetric states with even \( n \), numeric curves coincide with Eqs. (7, 8) only when \( l \gg q \) (in units shown in Figure 3). This justifies that our assumption \( \xi^2 \ll x^2 \) is true for antisymmetric states independent on \( l \), but true for symmetric states only for \( l \gg q \).

Let us look at the energy spectrum shown in Figure 3 (a). For small \( l \), the spectrum is the same as the SHO \( E_n = n + 1/2 \); while for large \( l \) the spectrum approaches

the SHO plus an inverse square potential, and each energy level is doubly degenerate. It is even more interesting to look at \( \langle \dot{z} \rangle \) as shown in Figure 3 (b). For symmetric states there is a hump at \( l \sim q/2 \), while the hump disappears for antisymmetric states. Why is there a hump? Recall that the operator \( \dot{z} \) has the form \( p_z/(1 + q^2x^2) \) maximized at \( x = 0 \). Since the operator \( \dot{z} \propto p_z \), we expect \( \langle \dot{z} \rangle \) grows linearly in \( l \) at small \( l \ll q \). The antisymmetric states \( \psi_A \) must vanish at \( x = 0 \), and thus reduce \( \langle \psi_A | \dot{z} | \psi_A \rangle \); while symmetric states \( \psi_S \) can be nonzero at \( x = 0 \), and \( \langle \psi_S | \dot{z} | \psi_S \rangle \) gets enhanced. This explains why the states with even \( n \) rotates faster than the states with odd \( n \). However, the potential barrier starts to manifest itself and suppress \( \psi_S(x = 0) \) at \( l \gtrsim q \), so the results for states with different parity converge at large \( l \). This qualitatively explains the hump appeared with odd \( n \). Numerical calculations show that the larger the \( q \), the higher the hump, which demonstrates a sharp transition of \( \dot{z} \) at \( l \sim q/2 \).

For this system, the angular velocity vanishes for the ground state \( n = 0 \). However, it is possible to get both \( \langle \dot{z} \rangle \neq 0 \) and periodic \( \langle z(0)z(t) \rangle \) even for the ground state. This indicates the existence of quantum time crystal [11].

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SUPPLEMENTARY MATERIALS

I. DERIVATION OF THE LAGRANGIAN AND HAMILTONIAN

Here we derive Eq. (1-3) in the main text. Let us start with the Lagrangian of a particle with charge $q$ inside the electromagnetic field

$$ L = \frac{1}{2}mv^2 + q\mathbf{v} \cdot \mathbf{A} - q\Phi, \quad (S9) $$

where $\mathbf{A}$ is the vector potential and $\Phi$ is the electric potential. If the magnetic field is uniform, using the symmetric gauge $\mathbf{A} = \mathbf{B} \times \mathbf{r}/2$, we arrive at the Lagrangian for a charged rigid body around its center of mass by integrating over the body volume

$$ L = \int d^3r \left[ \frac{1}{2}\rho_m(r)(\mathbf{v} \times \mathbf{r})^2 + \frac{1}{2}\rho_e(r)(\mathbf{v} \times \mathbf{r}) \cdot (\mathbf{B} \times \mathbf{r}) - \rho_e(r)\Phi(r) \right]. \quad (S10) $$

A simple algebra leads to Eq. (1) in the main text. Using the $ZY'Z''$ Euler angles $(\alpha, \beta, \gamma)$, the Lagrangian can be written as

$$ L = \frac{1}{2}m_1(\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2) + \frac{1}{2}m_3(\dot{\alpha} \cos \beta + \dot{\gamma})^2 + \frac{B_x}{4}[(e_3 - e_1)\dot{\alpha} \cos \alpha \sin(2\beta) - 2e_1\dot{\beta} \sin \alpha + 2e_3\dot{\gamma} \cos \alpha \sin \beta] + \frac{B_y}{4}[(e_3 - e_1)\dot{\alpha} \sin \alpha \sin(2\beta) + 2e_1\dot{\beta} \cos \alpha + 2e_3\dot{\gamma} \sin \alpha \sin \beta] + \frac{B_z}{2}[(e_1 \sin^2 \beta + e_3 \cos^2 \beta)\dot{\alpha} + e_3\dot{\gamma} \cos \beta] - U(\alpha, \beta, \gamma). \quad (S11) $$

The energy of the system imposed by $m_{1,3} \to 0$ are given by

$$ E = \frac{1}{2}x^2(u_1 + b^2\dot{z}^2/u_2). \quad (S19) $$

Therefore, the total energy can be arbitrarily close to the ground state energy by choosing the initial condition $x(t = 0)$ small enough. Forced by Eq. (S17) and (S18), the system rotates with a constant angular speed $|\dot{z}| = \sqrt{u_1u_2}/b$, as long as $x \neq 0$. This effective dynamics reminds us the classical time crystal proposed by Shapere and Wilczek [4, 34]. One can also substitute Eq. (S14) into Eq. (S13) to arrive at

$$ L = \frac{1}{2}b^4y^2\left(\dot{z}^2 - \frac{u_1u_2}{b^2}\right), \quad (S20) $$

which is exactly the same as the classical time crystal Lagrangian [4, 34]. However, the small moment of inertia $m_{1,3}$ regulates the pathological property of the classical time crystal Lagrangian, so we expect to see that the system rotates in a constant velocity near the ground state with an excitation energy proportional to $m_{1,3}$. As $m_{1,3} \to 0$, the system behaves as a classical time crystal.

The correspondence between Eq. (2) within limits $m_{1,3} \to 0$ and the classical time crystal encourages us to study the dynamics of Eq. (2) when $m_{1,3}$ is small but finite. Notice that the phase space of the system defined by Eq. (2) has four instead of six DOFs. Treating Eq. (S15) as a constraint and eliminating $y$ from Eq. (2), we arrive at the Lagrangian Eq. (2) and the Hamiltonian Eq. (3) (by doing Legendre transformation) in the main text.
II. AVERAGED ANGULAR VELOCITY

In this subsection, we calculate the averaged angular velocity in both the classical and quantum models.

A. Classical model

Using the trajectory $x(t)$ given in the main text, we can calculate the the angular velocity averaged in a period

$$\bar{\omega} = \frac{\omega_1}{\pi} \int_0^{\pi/\omega_1} dt \frac{u_1 u_2}{b^2} \frac{d}{2 \omega_1 V(x_\pm)} = \frac{\omega_1 u_1 u_2}{\pi} \frac{1}{b^2} \frac{2\pi}{2 \omega_1 V(x_\pm)} = \text{sign}(l) \frac{\sqrt{u_1 u_2}}{b}. \quad (S21)$$

B. Quantum model

Here we prove Eq. (8) in the main text. The angular velocity is given by

$$\langle \dot{\gamma} \rangle = \frac{i}{\hbar} \langle n| [H, z]|n \rangle = \frac{u_2 l}{b^2} \int_0^{\infty} dx \frac{\langle n| x \rangle \langle x | n \rangle}{x^2}, \quad (S22)$$

The wave functions are

$$\langle x | n \rangle = \frac{(2n)!^{1/2} \rho^{n-\rho/2}}{\Gamma(n + 2s + 1/2)^{1/2}} L_n^{2s-1/2}(\rho), \quad (S23)$$

where $\rho = x^2 \omega_1 \hbar$ and $s = (1 + \sqrt{1 + 4\sigma})/4$. We will show that

$$\int_0^{\infty} dx \frac{d}{d x} \frac{\langle x | n \rangle \langle x | n \rangle}{x^2} = \frac{m_1 \omega_1}{\hbar} \frac{2}{\sqrt{1 + 4\sigma}}. \quad (S24)$$

This is equivalent to prove the following mathematical identity

$$\frac{n!}{\Gamma(n + \alpha + 1)} \int_0^{\infty} dx x^{\alpha-1} e^{-x} L_n^\alpha(x)^2 = 1/\alpha. \quad (S25)$$

Proof.

$$\int_0^{\infty} dx x^{\alpha-1} e^{-x} L_n^\alpha(x)^2 = \frac{1}{\alpha} \left[ x^{\alpha} e^{-x} L_n^\alpha(x)^2 \right]_0^\infty - \frac{1}{\alpha} \int x^{\alpha} e^{-x} L_n^\alpha(x)^2 dx$$

$$= \frac{1}{\alpha} \int_0^{\infty} dx x^{\alpha} e^{-x} L_n^\alpha(x)^2 \quad - \frac{2}{\alpha} \int_0^{\infty} dx x^{\alpha} e^{-x} L_n^\alpha(x) dL_n^\alpha/dx.$$  

Notice that

$$\frac{dL_n^\alpha}{dx} = -L_n^{\alpha+1} = - \sum_{m=0}^{n-1} L_m^\alpha,$$  

and the orthogonality of associated Laguerre polynomials

$$\int_0^{\infty} dx x^\alpha e^{-x} L_n^\alpha L_m^\alpha = \frac{\Gamma(n + \alpha + 1)}{n!} \delta_{mn}, \quad (S27)$$

we arrive at

$$\int_0^{\infty} dx x^\alpha e^{-x} L_n^\alpha dL_n^\alpha/dx = 0, \quad (S28)$$

and

$$\int_0^{\infty} dx x^\alpha e^{-x} L_n^\alpha(x)^2 = \frac{\Gamma(n + \alpha + 1)}{n!} \frac{1}{\alpha}. \quad (S29)$$

III. NUMERICAL SIMULATIONS

Here we provide the form of dimensionless Lagrangian and Hamiltonian and the corresponding results of numerical simulations. Generally speaking, each pair $m_1, m_2$ and $u_{1,2}$ are different only by some geometrical factors $\mu = m_1/m_2$, $\nu = c_1/c_2$, and $\eta \equiv \sqrt{u_2/u_1}$ depending on the shape of the nanoparticle. For a nano-ellipsoid or nano-dumbbell or other shapes of nanoparticles commonly used in levitated experiments, $\mu, \nu, \eta \sim 1$. In order to proceed numerical simulations, we choose the units of energy and time as $E_0 = u_2 = m_2 \omega_2^2$ and $\tau_0 = 1/\omega_2$ respectively. Without lost of generality, we use $B_X = 0$ and the trapping potential

$$U(\alpha, \beta) = -u_1 \cos \beta - u_2 \cos \alpha, \quad (S30)$$

such that Eq. (S11) reduces to the following

$$L = \frac{1}{2} \mu (\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2) + \frac{1}{2} (\dot{\alpha} \cos \beta + \dot{\gamma})^2$$

$$+ \frac{\eta}{2} (1 - \nu) \dot{\alpha} \sin \alpha \sin (2\beta) + 2 \nu \beta \cos \alpha + 2 \dot{\gamma} \sin \alpha \sin \beta$$

$$+ a q \left[ \nu \sin^2 \beta + \cos^2 \beta \right] \dot{\gamma} + \gamma \cos \beta + \eta^2 \cos \beta + \cos \alpha, \quad (S31)$$

where $a = B_X/B_Y$. Similarly, devided Eq. (2) by $E_0$, we have

$$L = \frac{1}{2} \mu \dot{x}^2 + \frac{1}{2} \dot{z}^2 + q x y \dot{z} + a \dot{z}^2 - \frac{1}{2} \gamma y^2 - \frac{1}{2} y^2, \quad (S32)$$

and Eqs. (2, 3) in the main text reduce as

$$L = \frac{1}{2} \mu \dot{x}^2 + \frac{1}{2} (1 + q^2 x^2) \dot{z}^2 + a \dot{z}^2 - \frac{1}{2} \gamma y^2, \quad (S33)$$

$$H = \frac{\dot{p}_x^2}{2} + \frac{(p_z - a q)^2}{2(1 + q^2 x^2)} + \frac{1}{2} \gamma y^2. \quad (S34)$$

Since generally $\mu, \nu, \eta \sim 1$, it is clear that the system is sensitive to only one dimensionless parameter, i.e. the quality of time crystal phase $q$. In order to get time crystal behavior in numerical simulations, we choose $q \gg 1$. 
will play a role in quantum mechanics. The value of \( a \) shows that \( \dot{z} \) stabilizes under damping is that the system slowly falls into one of the minimum of the double-well potential shown in Figure 2(a) in the main text. This state at the minimum of the double well is exactly in the classical time crystal phase.

The numerical simulation of quantum model is well discussed in the main text. Here we include Figure 7 the ground state energy as a function of magnetic flux. The true \( E_{0l}(\phi) \) traces on the bottom of different curves in Figure 7. Compare with \( \dot{z}(\phi) \) shown as Figure 4 in the main text, we find \( E_{0l}(\phi) \) vanishes at \( \phi \in \mathbb{Z} \). On the other hand, when \( \phi \) is half integers, both \( E_{0l}(\phi) \) and \( |\dot{z}(\phi)| \) get maximized while \( \dot{z}(\phi) \) jumps between its minimum and maximum values.

IV. EXPERIMENTAL APPLICABILITY

In this subsection, we want to estimate how large the quality \( q \) could be in experiments, and the temperature range to observe the time crystal phase. First we introduce the intrinsic frequency of the system \( \omega_{0} \equiv b/m_{3} = B_{Y} e_{3}/2m_{3} \), such that \( q = \omega_{0}/\omega_{2} \). Since \( e_{3}/m_{3} \) is essentially the charge-to-mass ratio of the particle, the larger the ratio, the higher the quality of the time crystal for a fixed magnetic field. In order to increase the ratio of charge-to-mass, consider the hollow nanoparticles, such that its mass is proportional to the surface area. On the other hand, the number of charges is also proportional to the surface area, so the charge-to-mass ratio doesn’t depends of the radius of a hollow nanoparticle as long as its thickness is small compared to the radius.

Consider a nanoparticle made by the hexagonal Boron nitride (h-BN) with thickness \( \sim 10 \text{nm} \), mass density...
\[ \sim 2g/cm^3, \text{ and surface charge density } \sim 0.025e/nm^2, \]

such that its charge-to-mass ratio is \( \sim 200C/kg \). If the magnetic field is \( \sim 5T \), then the intrinsic frequency is \( \sim 1kHz \). Thus \( \omega_2 \) must be much smaller than 1kHz in order to have high quality of the time crystal. For example, to make the quality \( q \sim 10 \), we require the oscillation frequency \( \omega_2 \sim 100Hz \). However, since \( u_2 = m_3\omega_2^2 \) and the time crystal phase can only be observed for \( u_2/q^2 < T < u_2 \), the smaller the \( \omega_2 \), the more difficult to cool down the nanoparticle to the time crystal phase. To compensate the effect of small \( \omega_2 \), we want \( m_3 \) big enough such that \( u_2 \) is realizable in experiments. Notice that the moment of inertia \( m \) is essentially the product of mass and radius square, we can control \( m_3 \) by increasing the radius of the nanoparticle and meanwhile fix its thickness. For the nanoparticle we consider above, if we require \( u_2 = 1K \), then the radius \( \sim 1\mu m \).

As a result, remarkably, if we put a hollow particle with radius \( \sim 1\mu m \) and thickness \( \sim 10nm \) inside the optical or ion trap such that the twisting frequency is \( \sim 100Hz \), then a classical time crystal phase appears within the temperature \( 10mK < T \lesssim 1K \) when we charge it and turn on a uniform magnetic field. In the discussion above we ignore the thermal fluctuation of \( |\dot{z}| \). Below we will show that the requirement of small fluctuation of \( |\dot{z}| \) in the time crystal phase restricts the temperature range to \( 10mK \lesssim T \lesssim 50mK \).

\[ \text{V. STATISTICAL AVERAGE OF ANGULAR SPEED} \]

Here we derive the upper bound of the temperature in the time crystal phase. First we study the classical statistical average of angular speed \( |\dot{z}| \) from Eq. (S34) with \( \mu = \eta = 1 \). The partition function reads

\[ Z = \text{tr} e^{-H/T} = 8\pi^{5/2}qT^2U(-1/2, 0, 2u), \quad (S36) \]

where we define a scaling variable \( u = (4q^2T)^{-1} \) for convenience. As \( u \to 0 \), \( Z \to 8\pi^2qT^2 \); while as \( u \to \infty \), \( Z \to (2\pi)^{5/2}T^{3/2} \). The absolute value of \( \dot{z} \) reads

\[ \langle |\dot{z}| \rangle = \frac{2q\sqrt{\pi}U(-1/2, 0, 2u)^{-1}}{(S37)} \]

where \( U \) is the confluent hypergeometric function, and \( \langle |\dot{z}| \rangle \to 1/q \) as \( u \to 0 \); while \( \langle |\dot{z}| \rangle \to \sqrt{2T}/\pi \) as \( u \to \infty \).

Next we want to calculate the fluctuation of \( |\dot{z}| \)

\[ \langle \Delta |\dot{z}| \rangle^2 = \frac{\langle |\dot{z}|^2 \rangle - \langle |\dot{z}| \rangle^2}{\langle |\dot{z}| \rangle^2}. \quad (S38) \]

We start with the second moment

\[ \langle |\dot{z}|^2 \rangle = \frac{e^{2K_0(u)}}{2q^2\sqrt{\pi}U(-1/2, 0, 2u)}. \quad (S39) \]

where \( K_0 \) is the zeroth-order second-typed Bessel function. Therefore we have

\[ \langle \Delta |\dot{z}| \rangle^2 / \langle |\dot{z}| \rangle^2 = \frac{\sqrt{\pi}}{2}e^{2K_0(u)}U(-1/2, 0, 2u) - 1. \quad (S40) \]

which only depends on the scaling variable \( u \). For \( u > 0.047 \), we get that \( \Delta |\dot{z}| < \langle |\dot{z}| \rangle \), and the lower bound of the relative fluctuation is \( \pi/2 - 1 \) as shown in Figure 8. The condition \( u > 0.047 \) or equivalently \( q^2T < 5.3u_2 \) corresponds to \( T < 53mK \) if \( q = 10 \) and \( u_2 = 1K \). This justifies the upper bound of temperature range in the time crystal phase discussed in the last section and the main text.

\[ \text{Figure 8. Relative fluctuation as a function of } u. \]