Probabilistic sensitivity of Nash equilibria in multi-agent games: a wait-and-judge approach

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Abstract—Motivated by electric vehicle charging control problems, we consider multi-agent noncooperative games where, following a data driven paradigm, unmodeled externalities acting on the players’ objective functions are represented by means of scenarios. Building upon recent developments in scenario-based optimization, based on the evaluation of the computed solution, we accompany the Nash equilibria of the uncertain game with an a posteriori probabilistic robustness certificate, providing confidence on the probability that the computed solution remains unaffected when a new uncertainty realisation is encountered. The latter constitutes, to the best of our knowledge, the first application of the so-called scenario approach to multi-agent Nash equilibrium problems. The efficacy of our approach is demonstrated in simulation for the charging coordination of an electric vehicle fleet.

I. INTRODUCTION

The role of game theory in the management and control of cyberphysical systems [1] is now consolidated, counting on a significant research effort in the last decades [2]. The significance of the concept of Nash equilibrium (NE) in this context comes from the ability to predict the resolution of conflicts between selfish, rational agents [3]. NE problem formulations have become popular candidates for distributed and decentralized control architectures, as they naturally lend themselves to price-based implementations, easily fitting in the operational requirements of liberalized resource markets [4]. Applications in smart grid, transportation, and IT are envisioned among the possible beneficiaries [5–9]. In connection with public infrastructures, a large number of studies followed the seminal work of Wardrop [10], focusing on the relation between the achieved equilibria and the social welfare; among recent results, we mention [7], [11]–[14].

According to the definition of NE [15], the aforementioned studies address deterministic settings where the systems conditions (e.g., prices in an incentive scheme) are assumed to be known in advance. Such an assumption is not reflected in realistic scenarios, where the relevant information may be subject to significant uncertainties: this concept becomes paramount in the framework of cyberphysical systems, where the heterogeneity and complexity of the interactions between the constituting parts heavily hinders the task of predicting the system behaviour [16]–[18]. Indeed, since the complete knowledge of the game was questioned by [19], uncertainty has been widely addressed in noncooperative games, by adopting stochastic or robust (worst-case) approaches. In the first case, both chance-constrained (risk-averse) [20], [21] or expected payoff criteria [22]–[26] have been considered. Due to their approach, these studies pivot around given hypotheses on the underlying probability distribution of the uncertainty. In the second case, results build upon robust control theory [3], [27], and as such they hold for given characterizations of the uncertainty set. We refer the reader to [10], [28] for an extended literature review.

We depart by previous robust game-theoretical literature by characterizing uncertainty on the objective function through a data-driven methodology. In particular, our results are founded on recent developments in the scenario optimization [29]. In this framework, uncertainty is characterized by a finite set of scenarios [30] obtainable, e.g., from historical data [31]. One essential feature of the proposed approach is to allow the definition of probabilistic performance certificates for the solution—without requiring any additional knowledge on the uncertainty except for the available set of scenarios used in the solution computation [32]. Notably, the results presented in [33] constitute a breakthrough in this sense: under mild convexity assumptions, these provide tight a priori bounds on the probability of constraint violation of a scenario program solution. However, in the present context, these results can be quite conservative in general, as they depend on the dimension of the decision space: in the considered setting, a considerable overhead is due to the size of the scenario set. Moreover, in multi-agent games, this translates into a number of decision variables increasing with the number of players. Nonetheless, the theory presented in [29] allows to circumvent this problem by means of an a posteriori analysis of the solution: in this case, the only determining factor in the robustness performance is the cardinality of the subset of samples constraining the solution.

In this paper, we contribute to the state-of-the-art of NE problem solution algorithms in the following aspects:

- We address the robustness of strategic equilibria in multi-agent settings, characterized as NEs of noncooperative minimax games. We propose a partially-cooperative approach to achieve robustness to uncertainty globally affecting the players’ objective functions. We note that although this approach shares theoretical grounds with [34], we depart from it by admitting the presence of multiple maximisers. Besides expanding the scope of the proposed game model, this is pivotal to the application of the adopted scenario-based framework.

- We propose a decentralized solution approach: relying on the action of a coordinator, no communication between

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the agents is contemplated, thus relaxing the communicational requirements of related results (e.g., [25]). To circumvent the nondifferentiability of the game, hindering the application of conventional decentralised techniques, we follow the methodology in [35] and resort to an augmented game. Major advantages of this approach are (i) the solution computation enjoys the same convergence properties of state-of-the-art decentralized algorithms for monotone games [3], without imposing strong monotonicity assumptions that may not be satisfied for games of practical interest, and (ii) the uncertain component of the objective function can fall within the broader class of weakly convex functions [35].

We note that, as an alternative, the minmax problem can be cast into the class of generalized Nash games by an epigraph reformulation: however, decentralized solution algorithms for this class of games impose generally restrictive assumptions, e.g., affine coupling constraints [7], [9], [11], which do not necessarily fit the format of the resulting epigraphic constraint.

- We adopt a data-driven paradigm and represent the unmodelled uncertainty by means of scenarios [30]. We then accompany the solution with prescribed robustness levels, while dropping the standard requirements on the knowledge of the uncertainty [29]. To the best of our knowledge, this work sets the first step in the formalization of scenario-based robustness in noncooperative games. In doing this, we overcome significant limitations of stochastic approaches, namely the computational cost of Monte Carlo simulations, and the absence of guarantees for the solution.

The rest of the paper is structured as follows. In Section II we provide a motivating example, and introduce a scenario-based noncooperative minmax problem. We present the main result of the paper in Section III. Section IV provides a decentralized construction of the equilibrium strategy, while Section V contains the proof of the main result. In Section VI we provide for a wide class of games a computationally efficient methodology to determine an upper bound of the number of uncertainty samples supporting the NE, a quantity that lies at the core of the main result. Section VII provides a numerical example on an electric-vehicle charging coordination problem, while Section VIII concludes the paper and provides some directions for future work.

II. PROBLEM STATEMENT

A. Motivating example: coordinated EV charging

Let the set \( \mathcal{N} = \{1, \ldots, N\} \) designate the finite population of EV agents. The demand profile of the EVs (henceforth the strategy) \( x_i \in \mathbb{R}^q \) of agent \( i \in \mathcal{N} \) must fulfill individual constraints described by the set \( \mathcal{X}_i \subset \mathbb{R}^q \). We denote by \( x = (x_i)_{i \in \mathcal{N}} \in \mathcal{X} \subset \mathbb{R}^N \), with \( n = qN \), the collection of strategies relative to all the agents, where \( \mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_N \).

Each agent determines its strategy as a response to a pricing signal received from a coordinator, and synthesized as a function of the global strategy \( x \); such a signal can represent, for instance, the variable unit cost of a consumed resource. The possible influence of uncertainty (e.g., externalities acting on the energy spot market) on the price is modelled by the parameter \( \theta \in \Theta \). In particular, we assume that a nominal and an uncertain component can be distinguished: thus, given \( x \) and \( \theta \), the utility of agent \( i \in \mathcal{N} \) (that is to be minimized) is given by \(- (x_i^T p_0(x) + x_i^T p_0(\theta))\), where \( p_0 \) and \( p_0 \) respectively represent the nominal and the uncertain component of the price.

B. Noncooperative minmax game

To cope with the uncertain component of the price, we consider the minimisation of its average impact on the value of \( U_i \), for all \( i \in \mathcal{N} \). In other words, each agent \( i \in \mathcal{N} \) computes

\[
x_i^* = \arg\min_{x_i \in \mathcal{X}_i} J_i(x_i, x_{-i})
\]

where \( J_i : \mathbb{R}^n \to \mathbb{R} \) is defined as

\[
J_i(x_i, x_{-i}) = x_i^T p_0(x_i, x_{-i}) + \max_{\theta \in \Theta} \frac{1}{N} \sum_{j \in \mathcal{N}} x_j^T p_0(x_i, x_{-i}, \theta).
\]

We can write (2) in a more general form as

\[
J_i(x_i, x_{-i}) = f_i(x_i, x_{-i}) + \max_{\theta \in \Theta} g(x_i, x_{-i}, \theta),
\]

where \( f_i(x) \) expresses local objectives as a function of local and (possibly) global strategies whereas, for some common objective \( g(x, \theta) \), \( \max_{\theta \in \Theta} g(x, \theta) \) evaluates the worst-case realization over the uncertainty set \( \Theta \). As a consequence, the solution of (1) entails a certain level of cooperation between the agents, which can be facilitated in settings where their interests are (at least to some degree) aligned (e.g., electric vehicles participating in the same aggregation plan, or belonging to a centrally managed fleet).

The setting described above is modelled by a noncooperative minmax game, defined by the tuple \( \mathcal{G} = \langle \mathcal{N}, (\mathcal{X}_i)_{i \in \mathcal{N}}, (J_i)_{i \in \mathcal{N}}, \Theta \rangle \), where \( \mathcal{N} \) is the set of players, \( \mathcal{X}_i \), \( J_i \) are respectively the strategy set and the cost function for each player \( i \in \mathcal{N} \), and \( \Theta \) is the uncertainty set.

We consider the following blanket assumptions:

**Assumption 1:** For every fixed \( x_{-i} \in \mathcal{X}_{-i} \) and \( \theta \in \Theta \), the function \( f_i(\cdot, x_{-i}) + g(\cdot, x_{-i}, \theta) \) is convex and continuously differentiable for all \( i \in \mathcal{N} \). Furthermore, the local constraint set \( \mathcal{X}_i \) is nonempty, compact and convex for all \( i \in \mathcal{N} \), and \( \Theta \subset \mathbb{R}^d \) is bounded.

**Assumption 2:** Let \( \mathcal{D}_x \subset \mathbb{R}^n \) be an open convex set such that \( \mathcal{X} \subset \mathcal{D}_x \). The function \( g \) is twice differentiable on \( \mathcal{D}_x \) and, for each \( i \in \mathcal{N} \), \( f_i \) is twice differentiable on \( \mathcal{D}_x \).

We note that for \( f_i(\cdot, x_{-i}) + g(\cdot, x_{-i}, \theta) \) to be convex it is sufficient that \( f_i(\cdot, x_{-i}) \) is weakly convex on \( \mathcal{X}_i \) and \( g(\cdot, \theta) \) is weakly convex on \( \mathcal{X} \), with constant \( \lambda^f \) and \( \lambda^g \) respectively, such that \( \lambda^f + \lambda^g \geq 0 \); for a definition of weak convexity see [16]–[17].

C. Scenario-based approach

The solution of the robust game \( \mathcal{G} \) requires each agent to solve a minmax optimisation problem. Even in cases where this is numerically tractable (for example when the uncertainty of the value of \( g(\cdot, \theta) \) is well-behaved), the attainment
of a robust solution still requires the full characterisation of the set $\Theta$. Here we suppose instead that, however large, the available set of data (e.g., historical data) only allows to achieve a partial characterisation of the uncertainty. Following a data-based approach, it is still possible to consider a “sampled” version of the problem, using a finite set of uncertainty realizations $\theta_1, \ldots, \theta_M \in \Theta$. This results in the finite minmax game $\tilde{G} = \langle \mathcal{N}, (\mathcal{X}_i)_{i \in \mathcal{N}}, (\tilde{f}_i)_{i \in \mathcal{N}}, \{\theta_j\}_{j=1}^M\rangle$, where

$$\tilde{f}_i(x_i, x_{-i}) = f_i(x_i, x_{-i}) + \max_{m \in \{1, \ldots, M\}} g(x_i, x_{-i}, \theta_m),$$

and $\max_{m \in \{1, \ldots, M\}} g(x, \theta_m)$ approximates $\max_{\theta \in \Theta} g(x, \theta)$ in $\tilde{G}$. We consider the following solution concept for $\tilde{G}$:

**Definition 3 (Nash equilibrium):** Let $\tilde{\Omega} \subseteq \tilde{X}$ denote the set of Nash equilibria of $\tilde{G}$, defined as

$$\tilde{\Omega} \triangleq \{ \tilde{x} = (\tilde{x}_i)_{i \in \mathcal{N}} \in \tilde{X} : \tilde{x}_i \in \arg \min_{\tilde{x}_i \in \mathcal{X}_i} \tilde{f}_i(x_i, \tilde{x}_{-i}), \forall i \in \mathcal{N} \}.$$  \hspace{1cm} (5)

We point out that $\tilde{w}$ is a random variable, subject to the particular extraction of $M$ elements from $\Theta$. As a consequence, the composition of the set $\Omega$ is subject to the given multieextraction $(\theta_1, \ldots, \theta_M) \in \Theta^M$. For notational simplicity, we will not make this dependence explicit in the rest of the document.

### III. A posteriori Robustness Certification

A question that naturally arises is how robust a solution $x^* \in \Omega$ is against unknown scenarios—i.e., scenarios not included in the available dataset. In the remainder of this section we show that a formal answer to this question can be provided through a sensitivity analysis of the solution over the set of samples $\{\theta_1, \ldots, \theta_M\}$ used for its derivation. Most importantly, this performance can be performed without requiring any further knowledge of the uncertainty except for the aforementioned $M$ samples. The main result—relying on the developments in scenario-based optimization recently presented in $[29]$—is a robustness certificate that quantifies the probability for a NE of the approximated game $\tilde{G}$ to remain an equilibrium when a new scenario of uncertainty is realized. The statement of this result relies on the definition of a $\sigma$-algebra $\mathcal{D}$ on the uncertainty set $\Theta$, with assigned probability $\mathbb{P}$. This confers necessary measurability properties of the probability space that enable the use of fundamental results from statistical learning theory; see, e.g., $[32]$ and references therein.

**A. Main results**

In order to proceed we provide some basic definitions. Let $\hat{\Phi} : \Theta^M \to \Omega$ be a single-valued mapping from the set of $M$-multisamples to the set of equilibria of $\tilde{G}$; for notational convenience, the dimension of the domain of $\hat{\Phi}$ is to be considered implicitly determined by the size of the multisample.

**Definition 4 (Support sample $[33]$):** Fix any i.i.d. multisample $(\theta_1, \ldots, \theta_M) \in \Theta^M$, and let $x^* = \hat{\Phi}(\theta_1, \ldots, \theta_M)$ be a solution of the finite minmax game $\tilde{G}$. Let $x^0 = \hat{\Phi}(\theta_1, \ldots, \theta_{k-1}, \theta_{k+1}, \ldots, \theta_M)$ be the solution obtained by discarding the sample $\theta_k$. We call the latter a support sample if $x^0 \neq x^*$.

The next definition is adapted from $[29]$:

**Definition 5 (Compression set):** Fix any i.i.d. multisample $(\theta_1, \ldots, \theta_M) \in \Theta^M$, and let $x^* = \hat{\Phi}(\theta_1, \ldots, \theta_M)$ be a solution of the finite minmax game $\tilde{G}$. Consider any subset $\tilde{C} \subseteq \{\theta_1, \ldots, \theta_M\}$, and let $x^0 = \hat{\Phi}(\tilde{C})$. We call $\tilde{C}$ a compression set if $x^0 = x^*$.

The notion of compression set has appeared in the literature under different names; its properties are studied in full detail in $[29]$, where it is designated as support subsample. Here we adopt the term compression set as in $[36], [37]$ to avoid confusion with Definition 4.

Let $\mathcal{C}(\theta_1, \ldots, \theta_M)$ be the collection of all compression sets associated with the $M$-multisample $\{\theta_1, \ldots, \theta_M\}$. We refer to compression cardinality $\tilde{n}$ as the size $|\tilde{C}|$ (where $|\cdot|$ returns the cardinality of its argument) of some compression set $\tilde{C} \subseteq \{\theta_1, \ldots, \theta_M\}$. Note that $\mathcal{C}$—hence in general also $\tilde{n}$—is itself a random variable as it depends on the multisample $\{\theta_1, \ldots, \theta_M\}$.

Given $\theta \in \Theta$, let $\tilde{\Omega}^\perp$ designate the NE set of the game $\tilde{G}^\perp = \langle \mathcal{N}, (\mathcal{X}_i)_{i \in \mathcal{N}}, (\tilde{J}_i)_{i \in \mathcal{N}}, \{\theta_j\}_{j=1}^M \cup \{\theta\} \rangle$, defined over the $M + 1$ scenarios $\{\theta_1, \ldots, \theta_M, \theta\}$. Then, for any $x^* \in \tilde{\Omega}$, let

$$V(x^*) \triangleq \mathbb{P}\{\theta \in \Theta : x^* \notin \tilde{\Omega}^\perp\}.$$  \hspace{1cm} (6)

Finally, following $[29]$, let $\varepsilon : \{0, \ldots, M\} \to [0, 1]$ be a function satisfying

$$\varepsilon(M) = 1,$$

$$\sum_{k=0}^{M-1} \binom{M}{k} (1 - \varepsilon(k))^{M-k} = \beta,$$  \hspace{1cm} (7)

for any fixed $\beta \in (0, 1)$. We can now state our main result:

**Theorem 6:** Fix $\beta \in (0, 1)$ and let $\varepsilon(\cdot)$ be defined as in (7).

Under Assumptions 1, 2 the following hold:

(i) There exists a single-valued decentralized mapping $\tilde{\Phi} : \Theta^M \to \tilde{\Omega}$;

(ii) Let $x^* = \hat{\Phi}(\theta_1, \ldots, \theta_M)$, with $\{\theta_m\}_{m=1}^M$ being independent random samples from $\Theta$. Then

$$\mathbb{P}^M\{\{\theta_1, \ldots, \theta_M\} \in \Theta^M : V(x^*) > \varepsilon(\tilde{n})\} \leq \beta,$$  \hspace{1cm} (8)

where $\tilde{n} \leq M$ is the cardinality of any given compression set of $\{\theta_1, \ldots, \theta_M\}$.

**Proof:** See Section V.

Theorem 6 (point (ii)) shows that a single-valued mapping $\tilde{\Phi} : \Theta^M \to \tilde{\Omega}$ from the set of $M$-multisamples to the set of NE of the game $\tilde{G}$ indeed exists, and provides a decentralized way to construct it without imposing (standard) strong requirements on the monotonicity of the game. Theorem 6 (point (i)), shows that any solution $x^* \in \tilde{\Omega}$ returned by this mapping can be endowed with probabilistic guarantees on its robustness against uncertainty. The level of guarantee is determined in a wait-and-judge fashion [38], as it depends on the observed compression cardinality $\tilde{n}$, which in turn depends on the samples $\{\theta_m\}_{m=1}^M$. 


B. Discussion

A fundamental interpretation of Theorem 6 is that it quantifies, with a given confidence level, the probability that the NE \( x^* \), computed on the basis of the randomly extracted samples \( \{\theta_1, \ldots, \theta_M\} \in \Theta^M \), remains a solution of the game \( \tilde{G} \) when a new sample \( \theta \in \Theta \) is produced.

As a byproduct of our analysis we also obtain an additional significant result. Let \( V_c(x^*) \triangleq \Pr\{\theta \in \Theta : g(x^*, \theta) \geq \max_{\nu \in \{1, \ldots, M\}} g(x^*, \theta_m)\} \). From arguments discussed in detail in Section V it follows that

\[
P^M\{\theta_1, \ldots, \theta_M\} \in \Theta^M : V_c(x^*) > \varepsilon(\bar{n})\} \leq \beta. \quad (9)
\]

As a consequence of (9), the probability that the (predicted) individual utility \( \tilde{J}_i(x^*) \) achieved by a given NE \( x^* \) of \( \tilde{G} \) is not met by effect of a new uncertainty realization is, with confidence at least \( 1 - \beta \), lower than \( \varepsilon(\bar{n}) \).

A tighter bound could be obtained by means of the results of [38]; however, this would require the imposition of a non-degeneracy assumption on the problem, which cannot generally be verified even in convex settings. The notion of degeneracy implies that solving a problem using only the support samples does not yield the same outcome had all the samples been employed. In other words, support samples as identified by Definition 4 form a strict subset of any compression set in \( \mathcal{C} \); for a deeper discussion on degeneracy and minimal support in scenario-based contexts, we refer the reader to [38], [39]. Moreover, here we only assume that for any \( \theta \in \Theta \), \( g(\cdot, \theta) \) is weakly convex, thus imposing a non-degeneracy assumption would be quite restrictive. In view of not restricting the class of problems considered, we limit attention to confidence levels in the form of (7).

For practical purposes, the compression cardinality \( \bar{n} \) is bounded by \( M \) by Definition 5. An a posteriori estimate of the compression cardinality can be obtained through different methodologies, whose design may be tuned on the specific case. It follows from (8) that the closer the estimate to the minimal cardinality of the compression sets in \( \mathcal{C} \), the stronger the probabilistic guarantees on the robustness performance of the solution. For completeness, Algorithm 1 reports from [29, §II] a greedy procedure of general application, which is a straightforward implementation of Definition 4. Algorithm 1 allows to estimate (an upper bound to the minimal compression cardinality. In presence of degenerate samples, the algorithm can be run repeatedly to achieve an irreducible compression set. However, Algorithm 1 presents two major drawbacks: (i) the computational cost is generally high, as the value of \( \Phi(\cdot) \)—i.e., a solution of the game \( \tilde{G} \)—is required at least \( M \) times, and this would in turn require to run an iterative algorithm that converges to \( x^* \) only asymptotically; (ii) in practice, limited numerical accuracy makes the evaluation of the condition at Step 3 a hard task.

To alleviate these, we show in Proposition 13 Section VI that for a wide class of games capturing also the electric vehicle coordination example of Section VII we can efficiently determine a compression set based on a direct observation of the solution \( x^* = \Phi(\theta_1, \ldots, \theta_M) \) returned by the proposed algorithm.

IV. DECENTRALIZED NE COMPUTATION

In this section we study in more detail the existence of a single-valued mapping \( \Phi \), necessary for the proof of Theorem 6. In particular, we consider the case in which the image of \( \Phi \) corresponds to the limit of a decentralized solution algorithm for the game \( \tilde{G} \). To address this, we characterize the NEs of \( \tilde{G} \) as solutions of variational inequalities (VI). Established results in this framework allow to define sufficient conditions for the existence of equilibria, and set the foundations for the design of decentralized solution procedures [6].

A. VI analysis

At the core of the use of the VI framework for the modelling of noncooperative games solutions is the correspondence between the so-called VI problem, which takes the form (for a given domain \( D \subseteq \mathbb{R}^d \))

\[
\begin{align}
\text{find } z^* \in D & \\
\text{s.t. } (z - z^*)^T F(z^*) \geq 0, \forall z \in D,
\end{align}
\]

and the first-order optimality conditions corresponding to a NE (see Definition 3) [40, §1.4.2]. This model naturally hinges on the differentiability of the problem at hand; however, we can observe that, due to the \( \max \) operator, the players’ objective functions defining \( \tilde{G} \) are in general nondifferentiable.

With this in mind, let us define the augmented game \( \tilde{G} \) between \( N + 1 \) players. In \( \tilde{G} \) each agent \( i \in \mathcal{N} \), given \( x_{-i} \) and \( y = (y_m)_{m=1} \) will compute

\[
x_i \in \arg \min_{\nu_i \in \mathcal{A}_i} f_i(\nu_i, x_{-i}) + \sum_{m=1}^M y_m g(\nu_i, x_{-i}, \theta_m), \quad (11)
\]

where \( g(x, y) \) follows from the equivalence

\[
\max_{m \in \{1, \ldots, M\}} g(x, \theta_m) = \max_{y \in \Delta} \sum_{m=1}^M y_m g(x, \theta_m), \quad (12)
\]

with \( \Delta = \{ y \in \mathbb{R}^M : y \geq 0, \sum_{m=1}^M y_m = 1 \} \) being the simplex in \( \mathbb{R}^M \) [41, Lemma 6.2.1]. The additional player (the coordinator), given \( x \), will act instead as a maximizing player for the uncertain component of \( \tilde{J}_i \), \( i \in \mathcal{N} \),

\[
y \in \arg \max_{\nu \in \Delta} \hat{g}(x, \nu). \quad (13)
\]
The following relation holds between \( \hat{G} \) and \( \tilde{G} \):

**Lemma 7 (Thm. 1 [33]):** Let the pair \( (x^*,y^*) \in X \times \Delta \) be a NE of the game \( \tilde{G} \). Then \( x^* \) is a NE of \( \hat{G} \).

Note that \( \hat{G} \) is differentiable under Assumption 1. Therefore this fundamental result (holding for all randomly extracted set of uncertainty realizations \( \{\theta_1,\ldots,\theta_M\} \in \Theta \)) enables the characterization of the NEs of \( \hat{G} \) by means of VIs.

We start by defining the mapping \( F(x,y) : X \times \Delta \to \mathbb{R}^{(n+M)} \) as the pseudo-gradient [40, Sec. 1.4.1]

\[
F(x,y) \triangleq \left[ \begin{array}{c}
(\nabla_x f_i(x) + \nabla_y g(x,y))_{i \in N} \\
-(\nabla_y g(x,y))_{m=1}
\end{array} \right]
\]

(14).

By letting \( z \triangleq (x,y) \) and \( D \triangleq X \times \Delta \) we see that (10b) represents the concatenation of the first-order optimality conditions for the \( N + 1 \) individual problems described by (11) and (13). In the following, we refer to the problem described by (10b) and (13) as VI(\( F,X,\Delta \)).

It turns out that under Assumption 1 any NE of \( \tilde{G} \) can be expressed as the solution of the VI(\( F,X,\Delta \)) [40, Prop. 1.4.2]. A link with the equilibria of \( \hat{G} \) is formalised next:

**Proposition 8:** Let Assumption 1 hold, and \( z^* = (x^*,y^*) \) be a solution of the VI(\( F,X,\Delta \)). Then

(i) \( x^* \) is a NE of \( \hat{G} \);

(ii) The set \( \Omega \) is nonempty.

**Proof:** (i) By [40, Prop. 1.4.2], \( z^* = (x^*,y^*) \) is a solution of \( \tilde{G} \) if and only if it solves the VI(\( F,X,\Delta \)); then the statement follows readily from Lemma 7 (i); (ii): Given Assumption 1 and the compactness of \( \Delta \), the VI(\( F,X,\Delta \)) has at least one solution [40, Cor. 2.2.5]. Nonemptiness of \( \Omega \) then follows from the previous point.

**B. Monotonicity of \( F \)**

The development of algorithms for the solution of VI problems relies upon the monotonicity of the mapping \( F \) in [10], which plays a role analogous to convexity in optimization [8].

**Definition 9 (Monotonicity):** A mapping \( F : D \to \mathbb{R}^d \), with \( D \subseteq \mathbb{R}^d \) and convex, is

- monotone on \( D \) if \( (z - w)^\top (F(z) - F(w)) \geq 0 \) for all \( z,w \in D \),
- strongly monotone on \( D \) if there exists \( c > 0 \) such that \( (z - w)^\top (F(z) - F(w)) \geq c \|z - w\|^2 \) for all \( z,w \in D \).

The following result is instrumental in our discussion:

**Lemma 10:** Let Assumptions 1 and 2 hold. Then

(i) \( F(x,y) \) in (14) is monotone on \( X \times \Delta \);

(ii) The game \( \hat{G} \) admits multiple NEs.

**Proof:** (i): First, note that by Assumption 2 \( F(x,y) \) is continuously differentiable on its domain. Let \( F^x \) and \( F^y \) respectively denote the first \( n \) and the last \( M \) rows of \( F \), i.e.,

\[
F^x(x,y) = (\nabla_x f_i(x) + \nabla_x g(x,y))_{i \in N}, \quad \text{and} \quad F^y(x,y) = (\nabla_y g(x,y))_{m=1}
\]

1. Compactness of \( X \) in Assumption 1 is only needed for \( \hat{G} \), closedness is sufficient for \( \tilde{G} \).

By definition of Jacobian we have

\[
JF(x,y) = \begin{bmatrix}
J_x f^x(x,y) & J_y f^x(x,y) \\
-J_x f^y(x,y) & -J_y f^y(x,y)
\end{bmatrix}
\]

(15).

where \( J_x f^x = (\nabla^2_{xx} f_i(x) + \nabla^2_{xt} g(x,y))_{i,j \in N} \) and \( J_y f^x = (\nabla^2_{xt} f_i(x) + \nabla^2_{yt} g(x,y))_{i \in N} \) (similar definitions apply to the remaining terms). Assumption 1 implies the existence of \( \chi^f \in \mathbb{R} \) and \( \chi^g \in \mathbb{R}^M \) such that, for all \( (x,y) \in X \times \Delta \),

\[
\nu^\top (\nabla^2_{xx} f_i(x))_{i,j} \nu \geq \chi^f \|\nu\|^2,
\]

(16)

\[
\nu^\top (\nabla^2_{xt} f_i(x))_{i,j} \nu \geq \chi^g \|\nu\|^2, \forall \nu \in \mathbb{R}^T
\]

(17)

with \( \chi^f + \chi^g \geq 0 \) for the convexity assumption to hold. Summing the above inequalities yields

\[
\nu^\top (\nabla^2_{xx} f_i(x) + \nabla^2_{xt} f_i(x))_{i,j} \nu \geq \chi^f + \chi^g \|\nu\|^2, \forall \nu \in \mathbb{R}^T
\]

(18)

which corresponds to \( J_x f^x \) \( \geq 0 \) in turn, from (15), implies \( \nu^\top JF(x,y) \nu \geq 0 \) for all \( \nu \in \mathbb{R}^{n+M} \). The statement then follows directly from [40, Prop. 2.3.2].

(ii): By [8, Thm. 41] the monotonicity of \( F \) implies the VI(\( F,X,\Delta \)) admits multiple solutions: this together with [40, Prop. 1.4.2]—stating the correspondence between the solutions of the VI(\( F,X,\Delta \)) and the NEs of \( \hat{G} \)—concludes the proof.

**C. Decentralized algorithm for monotone VI and equilibrium selection**

There are two main challenges: firstly, due to the possible presence of multiple equilibria, standard decentralized algorithms for VIs are not guaranteed to converge on monotone problems; a tighter condition, namely strong monotonicity, is required on the VI mapping \( F \). Secondly, given the previous point is addressed, a tie-break rule needs to be put in place to select a unique solution in the presence of multiple NEs and fulfill the single-valued character of \( \hat{G} \) as required by Theorem 6.

A solution to the first issue comes from [8, 42]: these results show that proximal algorithms can be employed to retrieve a solution of a monotone VI by solving a particular sequence of strongly monotone problems, derived by regularizing the original problem. However, despite the fact that a single solution is obtained through these techniques, using different initial conditions in the underlying algorithms a different equilibrium may be returned. To alleviate this and construct a single-valued mapping, we employ the results of [8]. In particular, [8, Algorithm 4] allows to solve, as a specific case,

\[
\text{find } z^* = \arg \min_{z} \frac{1}{2} \|z^*\|^2
\]

(19a)

s.t. \( (z - z^*)^\top F(z^*) \geq 0, \forall z \in D \),

(19b)

where \( F \) is monotone. By (19) the minimum-norm equilibrium of the game \( \hat{G} \) can be specified as the limit point of
the algorithm, thus recovering a suitable formulation for the single-valued mapping $\Phi$.

Therefore, we proceed by considering the regularized game \( \hat{G}^{\tau, \bar{z}} \) where \( \tau \) and \( \bar{z} = (\bar{x}, \bar{y}) \) are the designated step size and centre of regularization, respectively. Then, given the tuple \( \{x_i, y, \bar{x}_i\} \), each player \( i \in N \) solves the following problem

\[
x = \arg \min_{\nu_i \in \mathcal{X}} f_i(\nu_i, x_i) + \hat{g}(\nu_i, x_i, y_i, y) + \frac{\eta}{2} \| (\nu_i, x_i, y_i, y) \|^2 + \frac{\tau}{2} \| \nu_i - \bar{x}_i \|^2,
\]

while the coordinator (player \( N + 1 \)), given \( \{x, y, \bar{x}\} \), solves

\[
y = \arg \max_{\nu \in \Delta} \hat{g}(x, \nu) - \frac{\eta}{2} \| (x, \nu) \|^2 - \frac{\tau}{2} \| \nu - \bar{y} \|^2,
\]

with \( \eta, \tau \in \mathbb{R}_+ \). Note that Assumption 1 still holds for (20–21). By taking the pseudo-gradient of the above as in (14), we have from [40 Prop. 1.4.2] that \( z^* = (x^*, y^*) \) is a NE of \( \hat{G}^{\tau, \bar{z}} \) if and only if it satisfies the VI

\[
(z - \bar{z})^T (F(z^*) + \eta z^* + \tau (z^* - \bar{z})) \geq 0, \quad \forall z, \bar{z} \in \mathcal{X} \times \Delta.
\]

The next lemma is key to the use of decentralized algorithms for strongly monotone VIs on our case.

**Lemma 11:** Let Assumptions 1 and 2 hold; let \( F \) be defined as in (14), and \( \eta \geq 0 \) given. Then, for any \( \tau > 0 \) and \( \bar{z} \in \mathcal{X} \times \Delta \), the regularized game \( \hat{G}^{\tau, \bar{z}} \) defined by (20–21) has a unique NE.

**Proof:** Let \( F^{\tau, \bar{z}}(z) \) \( \hat{F}(z) + \eta z + \tau (z - \bar{z}) \). We note that \( JF^{\tau, \bar{z}} \geq \tau I \) since, for all \( \nu \neq 0 \),

\[
\nu^T JF^{\tau, \bar{z}} \nu = \nu^T (JF + (\eta + \tau) I) \nu \geq \nu^T JF + \nu^T \nu \geq \| \nu \|^2,
\]

where the last inequality follows from Lemma 10. By definition of convexity this implies \( (z - w)^T (F^{\tau, \bar{z}}(z) - F^{\tau, \bar{z}}(w)) \geq \tau \| z - w \|^2 \) for all \( z, w \in \mathcal{X} \times \Delta \), corresponding to the definition of strongly monotone mapping. The result then follows by [8, Thm. 41].

Now let \( S^* (\cdot) \) denote the solution of the VI\((F^{\tau, \bar{z}}, \mathcal{X} \times \Delta)\). Building on Lemma 11 the idea is to achieve a NE of \( \hat{G} \) by updating the centre of regularization of \( \hat{G}^{\tau, \bar{z}} \) on the basis of an iterative method in the form \( z^{k+1} = S^* (\hat{z}^k) \), until convergence to the fixed point \( z^* = S^* (z^*) \) corresponding to the NE of \( \hat{G}^{\tau, \bar{z}} \) satisfying (19). Algorithm 2 allows to establish such a connection, formalised in Lemma 12.

**Lemma 12** (Thm. 21 [8]): Consider the minmax game \( \hat{G} \) defined by (11–13) and the regularized augmented game \( \hat{G}^{\tau, \bar{z}} \) defined by (20–21), and let Assumptions 1 and 2 hold. Let \{\eta^k\}_{k=0}^{\infty} be any sequence satisfying \( \eta(k) \to 0 \) for all \( k \), \( \sum_k \eta(k) = \infty \), and \( \lim_{k \to \infty} \eta(k) = 0 \). Let \( \tau > 0 \) be such that with \( \tau = \tau \) steps 3-11 of Algorithm 2 constitute a block contraction [43], and let \( \hat{z}(k) \to \infty \) denote the sequence generated by the Algorithm. For any \( \tau \geq \bar{\tau} \), there exists \( \gamma_{\text{in}}, \gamma_{\text{out}} > 0 \) such that \( \hat{z}(k) \sim \bar{z} \) is bounded, and \( z^* = (x^*, y^*) \) solution of (19) such that \( \| \hat{z}(k) - z^* \| \to 0 \) for \( k \to \infty \). Moreover, \( z^* \in \Omega \).

**Proof:** By [8, Thm. 21], Algorithm 2 asymptotically converges to a solution of (19). By Proposition 8 (19b) is equivalent to the game \( \hat{G} \), whose solution set is nonempty and—by Lemma 7—contained in \( \Omega \).

The main implication of Lemma 12 is to provide a constructive, decentralized implementation of the single-valued mapping \( \Phi: \Theta^M \to \Omega \) thus formally proving point i) in Theorem 6.

**V. PROOF OF THEOREM 6**

i) A proof for the existence of a single-valued mapping \( \Phi \) is provided by the construction of Section IV-C. In this regard, we emphasize that Assumptions 1–2 are specifically tailored to our interest in a multi-agent setting: they may be relaxed in case \( \Phi \) corresponds to a centralized algorithm.

ii) Fix any \( (\theta_1, \ldots, \theta_M) \in \Theta^M \), and let \( x^* = \Phi(\theta_1, \ldots, \theta_M) \in \Omega \). Let \( \gamma^* = \max_{m \in \{1, \ldots, M\}} g(x^*, \theta_m) \) and for each \( i \in N \) notice that \( (x^*_i, \gamma^*) \) will belong to the set of minimizers of the following epigraphic reformulation of \( \hat{G} \):

\[
(x^*_i, \gamma^*) \in \arg \min \{ f_i(x_i, x^* - \cdot) + \gamma \}_{x_i, \gamma} \quad \text{s.t.} \ x_i \in \mathcal{X}_i, \quad \text{g}(x_i, x^* - \cdot, \theta_m) \leq \gamma, \quad \forall m \in \{1, \ldots, M\},
\]

For any \( \theta \in \Theta \), let \( \Gamma_{\theta} \triangleq \{ (x^*, \gamma^*) : \text{g}(x^*, \theta) \leq \gamma^* \} \). By (24c) it follows that

\[
(x^*, \gamma^*) \in \Gamma_{\theta_m}, \quad \forall m \in \{1, \ldots, M\}.
\]

By the last statement and point (i), [29 Assum. 1] is fulfilled. Therefore, by [29 Thm. 1] we have that

\[
\mathbb{P}^M \{ (\theta_1, \ldots, \theta_M) \in \Theta^M : \mathbb{P} \{ \theta \in \Theta : (x^*, \gamma^*) \in \Gamma_{\theta} \} \geq 1 - \varepsilon(n) \} \geq 1 - \beta,
\]

where \( \bar{n} \) is the cardinality of some compression set for the equilibrium \( x^* \) returned by \( \Phi \), and \( \varepsilon(\cdot) \) is defined by [7] for fixed \( \beta \in (0, 1) \). Note that (26) corresponds to (9).

We now proceed to demonstrate the claim in (8). Recall that, by (14), (19) and Proposition 8, we can obtain \( x^* \in \Omega \)
as solution of the following optimization program (note the slight abuse of notation as by \((x^*, y^*)\) we denote both the optimizer and the corresponding decision variables).

\[
\min_{(x^*, y^*) \in \mathcal{X} \times \Delta} \frac{1}{2} \| (x^*, y^*) \|^2 \tag{27a}
\]

s.t. \(\sum_{i \in \mathcal{N}} (x_i - x_i^*)^T \nabla x_i (f_i(x^*) + \hat{g}(x^*, y^*)) - M \sum_{m=1}^M (y_m - y_m^*) \nabla y_m \hat{g}(x^*, y^*) \geq 0, \forall x \in \mathcal{X}, y \in \Delta. \tag{27b}\)

where \((x^*, y^*)\) is a NE of \(\hat{G}\). By definition of \(\hat{g}\) in \((11)\), and recalling \(\hat{g}(\sum_{m=1}^M y_m^* g(x^*, \theta_m)) = g(x^*, \theta_m), \) \((27b)\) can be equivalently written as

\[
\sum_{i \in \mathcal{N}} (x_i - x_i^*)^T \nabla x_i (f_i(x^*) + \sum_{m=1}^M y_m^* g(x^*, \theta_m)) + M \sum_{m=1}^M y_m^* g(x^*, \theta_m) - \sum_{m=1}^M y_m g(x^*, \theta_m) \geq 0, \forall x \in \mathcal{X}, y \in \Delta. \tag{28}\)

As \((28)\) holds for all \(y \in \Delta\), we have that \((27b)\) is equivalent to the following inequality being satisfied for all \(x \in \mathcal{X}\),

\[
\sum_{i \in \mathcal{N}} (x_i - x_i^*)^T \nabla x_i (f_i(x^*) + \sum_{m=1}^M y_m^* g(x^*, \theta_m)) + M \sum_{m=1}^M y_m^* g(x^*, \theta_m) - \sum_{m=1}^M y_m g(x^*, \theta_m) \geq 0, \forall x \in \mathcal{X}, y \in \Delta. \tag{29}\)

For a given \(\theta \in \Theta\), recall from Section \(\text{III-A}\) the definitions of \(\hat{G}^+\)—the game associated with the samples \(\{\theta_1, \ldots, \theta_M\} \cup \{\theta\}\) — and its NE set \(\Omega^+\). Moreover, let \(\hat{G}^+\) denote the associated augmented game. Analogously to \((29)\), any solution \((x^+, y^+)\) in \(\mathcal{X} \times \Delta^+\) — where \(\Delta^+\) is the simplex in \(\mathbb{R}^{M+1}\) — of the augmented game \(\hat{G}^+\) will satisfy the following VI.

\[
\sum_{i \in \mathcal{N}} (x_i - x_i^*)^T \nabla x_i (f_i(x^*) + \sum_{m=1}^M y_m^* g(x^*, \theta_m)) + M \sum_{m=1}^M y_m^* g(x^*, \theta_m) - \sum_{m=1}^M y_m g(x^*, \theta_m) \geq 0. \tag{30}\]

Note the analogy between \((30)\) and \((29)\), with the additional terms corresponding to the new sample \(\theta \) \((\text{w}M+1\) is the additional decision variable corresponding to the new sample).

We are interested in quantifying the probability of \(x^+ \in \Omega^+\). To this end, notice that if \(g(x^*, \theta) \leq \gamma^*\), then \(x^+ = x^*\) and \(y^+ = (y^*\top, 0)^\top\) constitute a feasible pair for \((30)\). This is due to the fact that under this choice \(y_M^+ = 0\) and

\[
\max \left\{ \max_{m \in \{1, \ldots, M\}} g(x^*, \theta_m), g(x^*, \theta) \right\} = \max \left\{ \gamma^*, g(x^*, \theta) \right\} = \gamma^* = \max_{m \in \{1, \ldots, M\}} g(x^*, \theta_m),
\]

hence \((30)\) reduces to \((29)\). Applying Proposition \(8\) to \(\hat{G}^+\) and \(\hat{G}^+\), we have that if \((x^+, y^+)\) satisfies \((30)\) (i.e., it is an NE of the augmented game \(\hat{G}^+\)) then \(x^+ \in \Omega^+\). Therefore, \(x^+ \in \Omega^+\) whenever \(g(x^*, \theta) \leq \gamma^*\), or in other words

\[
P\{\theta \in \Theta: (x^*, \gamma^*) \in \Gamma_\theta\} = P\{\theta \in \Theta: g(x^*, \theta) \leq \gamma^*\} \leq P\{\theta \in \Theta: x^+ \in \Omega^+\}. \tag{31}\]

By \((26)\) and \((31)\), \((3)\) follows, thus concluding the proof. \(\blacksquare\)

VI. COMPUTATION OF THE COMPRESSION CARDINALITY

We provide in this section a computationally efficient way to determine a compression set—and hence the compression cardinality, used in Theorem \(6\)—as a direct byproduct of the algorithm developed to construct the mapping \(\hat{\Phi}\). To achieve this, we impose certain uniqueness requirements as detailed in the subsequent proposition. However, it should be noted that the additional structure of Part 2 in Proposition \(13\) below implies only uniqueness of the aggregate strategy, and multiple equilibria may exist.

Proposition 13: Let Assumptions \(1, 2\) hold. We further assume that for all \(M \in \mathbb{N}\), either

1) \(\hat{G}\) admits a unique NE;
2) or, \(g\) depends on the aggregate strategy \(\sigma(x): x \mapsto \sum_{i \in \mathcal{N}} x_i\), and \(\hat{G}\) admits a unique NE aggregate strategy \(\sigma(x)\).

Then, the set \(\mathcal{Y}^* \triangleq \{m \in \{1, \ldots, M\} : y_m^* > 0\}\) constitutes a compression set such that \(x^+ = \hat{\Phi}(\theta_1, \ldots, \theta_M) = \hat{\Phi}(\mathcal{Y}^*)\).

Proof: Part 1: Uniqueness of NE. Fix any \((\theta_1, \ldots, \theta_M) \in \Theta^M\) and notice that it forms a (trivial) compression set for \(x^*\). Let \((x^+, y^+)\) be a solution of the augmented game \(\hat{G}\), where \(y^* = (y_m^*)_{m=1}^M\). To prove that \(x^+ = \hat{\Phi}(\mathcal{Y}^*)\) it suffices to show that the solution of \(\hat{\Phi}\) remains unaltered after removing all samples from \(\{\theta_1, \ldots, \theta_M\}\) whose associated component of \(y^*\) is zero.

To this end, suppose that at least one such sample exists: without loss of generality, assume \(y_M^* = 0\) (i.e., that sample has index \(M\)). We will first show that \(\{\theta_1, \ldots, \theta_{M-1}\}\) is a compression set, i.e., \(x^+ = \hat{\Phi}(\theta_1, \ldots, \theta_{M-1})\). Let \(\hat{G}^- = \langle \mathcal{N}, \mathcal{X}_i \rangle_{i \in \mathcal{N}}; \{\tilde{J}_i\}_{i \in \mathcal{N}}; \{\theta_j\}_{j=1}^{M-1}\rangle\) be the game associated with the samples \(\{\theta_1, \ldots, \theta_{M-1}\}\). Moreover, let \(\hat{G}^-\) denote the associated augmented game, and \(\Delta^-\) the simplex in \(\mathbb{R}^{M-1}\). Since \((x^*, y^*)\) is an NE of \(\hat{G}\), it will satisfy the VI in

\[\sum_{i \in \mathcal{N}} (x_i - x_i^*)^T \nabla x_i (f_i(x^*) + \sum_{m=1}^M y_m^* g(x^*, \theta_m)) + M \sum_{m=1}^M y_m^* g(x^*, \theta_m) - \sum_{m=1}^M y_m g(x^*, \theta_m) \geq 0. \tag{30}\]

2With a slight abuse of notation, in the second part of the proposition it is to be understood that for all \(i \in \mathcal{N}\) and for any given \(x_{-i}\), Assumptions \(1, 2\) refer to the function \(f_i(\cdot, x_{-i}) + g(\sigma(\cdot, x_{-i}), \theta)\).
The derivation until the discussion right after (32) remains the proof of the first part. 
\[ \hat{x} \] for the sake of contradiction that (\( \hat{x} \)) showing that \( x \) can be verified that the decision variable \( \hat{x} \) satisfies the VI in (29) for the game with \( M \) and \( \max \) we have that \( \hat{x} \) is an NE for \( G \). However, due to the uniqueness assumption, \( \hat{x} \) has to be the only NE of \( G \), showing that \( x = \hat{\Phi}(\theta_1, \ldots, \theta_M) \).

Following the same procedure, removing one by one all samples for which the associated elements of \( y \) are zero, shows that \( x = \hat{\Phi}(\theta_1, \ldots, \theta_M) = \Phi(\gamma) \), thus concluding the proof of the first part.

Part 2: Uniqueness of NE aggregate. The proof follows the same arguments as in Part 1 with the following modifications. The derivation until the discussion right after (32) remains unaltered, showing that \( x = \Phi(\hat{\theta}_1, \ldots, \hat{\theta}_M) \) is a NE of \( G \). To prove that \( x = \Phi(\hat{\theta}_1, \ldots, \hat{\theta}_M) \) it suffices to show that \( x = \Phi(\hat{\theta}_1, \ldots, \hat{\theta}_M) \) is the minimum norm NE. We thus assume for the sake of contradiction that \( (\hat{x}, \hat{y}) \in X \times \Delta \) is the NE of \( G \) that achieves the minimum norm, i.e., \(|(\hat{x}, \hat{y})| < \|(x^*, (y^*)^{M-1})\|\). We distinguish two cases:

Case 1: \( g(\sigma(\hat{x}), \theta_M) \leq \max_{m \in \{1, \ldots, M-1\}} g(\sigma(\hat{x}), \theta_m) \). Under the condition of this case observe that \( (\hat{x}, (y^*)^M) \) satisfies the VI in (29) for the game with \( M \) samples. However, as \( (x^*, y^*) \) is the minimum norm equilibrium for that game, we have that \(|(x^*, y^*)| < \|(\hat{x}, (y^*)^0)\|\). Overall, recalling \( y^M = 0 \), \(|(x^*, (y^*)^M)| < \|(x^*, y^*)| \leq \|(\hat{x}, (y^*)^0)\| \) thus establishing a contradiction. We then can show that \( x = \hat{\Phi}(\theta_1, \ldots, \theta_M) = \Phi(\gamma) \) as in the last sentence of Part 1.

Case 2: \( g(\sigma(\hat{x}), \theta_M) > \max_{m \in \{1, \ldots, M-1\}} g(\sigma(\hat{x}), \theta_m) \). We will show that, under our assumptions, this case cannot occur. By the uniqueness assumption we have that \( \hat{\sigma}(\hat{x}) = \sigma(x^*) \) for any equilibrium \( \hat{x} \neq x^* \) (the NE is not necessarily unique, but all equilibria have the same aggregate). We then have
\[
g(\sigma(x^*), \theta_M) = g(\sigma(\hat{x}), \theta_M) > \max_{m \in \{1, \ldots, M-1\}} g(\sigma(\hat{x}), \theta_m) \geq g(\sigma(\hat{x}), \theta_m) = g(\sigma(x^*), \theta_m),
\]
for any \( m = 1, \ldots, M-1 \). Therefore
\[
g(\sigma(x^*), \theta_M) > \max_{m \in \{1, \ldots, M-1\}} g(\sigma(x^*), \theta_m).
\]
Consider now the epigraphic reformulation of \( G \) in (24). By direct computation of the KKT optimality conditions [41, §6.2.1] corresponding to (24) and to (13), respectively, it can be verified that the decision variable \( y \in \Delta \) introduced in the augmented game of (11)–(13) is a shadow price for the constraint (24c). By the complementary slackness condition it holds
\[
y_m^* (g(\sigma(x^*), \theta_m) - \gamma^*) = 0, \ \forall m \in \{1, \ldots, M\}.
\]
By observing that \( y_M^* = 0 \) implies \( g(\sigma(x^*), \theta_m) \leq \gamma^* \) we obtain
\[
\max_{m \in \{1, \ldots, M\}} g(\sigma(x^*), \theta_m) = \max_{m \in \{1, \ldots, M-1\}} g(\sigma(x^*), \theta_m).
\]
From (35) it follows \( \max_{m \in \{1, \ldots, M-1\}} g(\sigma(x^*), \theta_m) = \max_{m \in \{1, \ldots, M-1\}} g(\sigma(x^*), \theta_m) \geq g(\sigma(x^*), \theta_M) \). This last statement establishes a contradiction with (33), which concludes the proof of the second part.

Based on the shadow price interpretation of \( y \) (see proof of Proposition [13] if \( g(x^*, \theta_M) < \gamma^* \) (inactive constraint) then \( y_M^* = 0 \). Moreover, for nondegenerate problems it holds \( y_m^* > 0 \) if \( g(x^*, \theta_m) = \gamma^* \), and as a result, \( \gamma^* \) provides an exact enumeration of the active constraints (hence of a minimal compression set), which in this case would correspond to the support ones. In the degenerate case, however, the aforementioned relation holds only in one direction, i.e., \( y_m^* > 0 \Rightarrow g(x^*, \theta_m) = \gamma^* \) as support constraints do not coincide with the active ones. In this case, \( \gamma^* \) still identifies some compression set of the \( M \)-multisample used for the derivation of \( x^* \in \Omega \), albeit not necessarily of minimal cardinality. It is thus worth remarking that Proposition [13] does not provide guarantees that an irreducible compression set is determined; this can be obtained by Algorithm [1] as specified in Section III-B (see also [29]). However, the important implication of Proposition [13] is that the cardinality of a compression set is readily available by inspecting \( y^* \) and does not require an iterative procedure.

VII. Numerical Example: Coordinated EV Charging Problem Revisited

We revisit here the example of Section II-A. In the following, the strategy vector \( x_i \) of each electric vehicle (EV) describes the charging demand over \( q = T \) time slots. For simplicity we consider time slots of unit length (1 h). We suppose that the energy price in (2) is well approximated by an affine function of the aggregate strategy \( \sigma(x): x \mapsto \sum_{i \in N} x_i \). Accordingly, \( p_q(x) = A_0 \sigma(x) + b_0 \), and \( p_q(x, \theta_m) = A_m \sigma(x) + b_m \), where \( A_m, b_m \in \mathbb{R}^{T \times T} \). Moreover, we assume the charging constraints are subject to \( \theta_j \), \( \hat{\theta} \{ x_i \in \mathbb{R}^T : 1^T x_i \geq E_i, \ 0 \leq x_i \leq P_i, \ \forall j = \{1, \ldots, T\} \} \). However, it should be noted that even though this example fits in the class of aggregative games, it does not necessarily meet the uniqueness requirement of Part 2, Proposition [13]. However,
we have empirically observed that the main conclusion of the proposition still holds, namely, the minimum norm solution returned by Algorithm 2 remains unaltered when the algorithm is fed only with the samples with indices in $\mathcal{D}^\gamma$. Informally, this happens due to the fact that for any feasible problem instance $1^T x_i \geq E_i$ will always be binding at the optimum, and as result $1^T \sigma(x)$ will be constant for any NE $x$ (see also transparent plane in Figure 1); a detailed investigation of this issue is topic of current research.

We analyse the results of several randomly generated cases, differing in the parameters characterizing the EV constraints $\mathcal{X}_g$, selected from a uniform random distribution: specifically, $P_i \in [6, 15]$ kW, and $E_i$ is chosen to be feasible in the specified time interval ($\sim 0–35$ kWh per 12 h interval). Regarding the uncertainty samples, the pairs $\{A_m, b_m\}_{m=1}^M$ are i.i.d. extracted from (i) a lognormal distribution for the diagonal entries of $A_m$, and (ii) a positive-valued uniform distribution for the vectors $b_m$. The nominal electricity price, i.e., the diagonal entries $\{a_{ii}\}_{i=1}^T$ of the matrix $A_0$, have been derived by rescaling a winter weekday demand profile in the UK [44], whereas $b_0 = 0$. The robust charging schedules have been obtained by implementing Algorithm 2 with $\gamma_{\text{inn}} = 10^{-14}$, $\gamma_{\text{out}} = 10^{-2}$ and $\tau \in [4, 6]$; the setting considered in this example can be cast into a decentralized QP optimization, which has been efficiently solved on a dual-core 7th gen. Intel processor using Matlab.

Table I shows the mean robustness performance of several solutions (with $N = 20$, $T = 24$), obtained from different sets of $M = 500$ samples, grouped by the observed compression cardinality $\bar{n}$. The violation rate $V(x^\ast)$ of each solution is recorded over $10^6$ newly extracted samples (according to the same aforementioned distributions). Consistently with [29], we note that the observed value of $\bar{n}$ is indicative of the confidence level on the equilibrium robustness. The experimental results are compared with the theoretical bound provided by Theorem 3. In this case, the conservativeness of the latter can be addressed to the relatively small number of samples (500) employed for the computation of the NE. We observe that a much tighter bound on the violation probability ($\sim 2–3\%$) can be achieved in the same case by increasing the size of the sample to $M = 2000$; this naturally comes at the price of an increase of the computational requirements, which is nonetheless (practically) independent of the number of agents, due to the decentralized scheme employed.

A visual representation of the concept of support constraint is given in Fig. 1. The plot depicts the curves expressing the uncertain cost term $N^g(x^\ast, \theta_m)$ associated to a subset $m \in \{53, 72, 282, 566\}$ of the $M = 1000$ samples used for the derivation of the NE $x^\ast$. Values are plotted as a function of the aggregate demand $\sigma(x)|_{t=1}, \sigma(x)|_{t=2}$ on an interval around $\sigma(x^\ast)$. We have empirically observed that this setting reveals some structure with respect to the considered uncertainty. In particular, a specific relation between $\bar{n}$ and the dimension of the individual decision variable $T$ holds in place. As shown in Fig. 2, experimental results suggest that $\bar{n}$ is bounded by $T$. Interestingly, at the same time we observed no sensitivity to the number of agents $N$.

Finally, Figs. 3-4 show the convergence of Algorithm 2 in the computation of the NE for $N = 20$ EVs, $T = 24$ and $M = 500$. A dominant linear convergence rate can be observed, and less than 4000 outer loop iterations were needed to meet the desired exit accuracy $\gamma_{\text{out}}$. The inner loop enjoys similar convergence rate (not shown for space reasons), and less than 30 iterations (15 in average) are needed to achieve an error smaller than $\gamma_{\text{inn}} = 10^{-14}$.

![Fig. 1](image1.png)  
**Fig. 1.** Case with $N = 20$, $T = 2$: Uncertain cost component $N^g(x^\ast, \theta_m)$ for a subset $m \in \{53, 72, 282, 566\}$ of the $M = 1000$ samples used for the derivation of the NE $x^\ast$. Curves are plotted as a function of the aggregate demand $\sigma(x)|_{t=1}, \sigma(x)|_{t=2}$ on an interval around $\sigma(x^\ast)$, plotted in red.

![Fig. 2](image2.png)  
**Fig. 2.** Sensitivity of $\bar{n}$ to the dimension of the individual decision variable $T$. The plot shows the average compression cardinality $\bar{n}$ observed over 50 trials, corresponding to different randomly generated cases, for different values of $T$ and $M$. In all cases, $\bar{n}$ is bounded by $T$. Also, we observed no sensitivity to the variation of the number of agents $N$.  

| $\bar{n}$ [-] | 2   | 4   | 6   | 7   | 9   |
|---|---|---|---|---|---|
| Empirical rate [%] | 0.98 | 1.09 | 1.26 | 1.33 | 1.33 |
| Theorem 3 bound [%] | 8.06 | 9.76 | 10.55 | 12.06 | 12.06 |

**TABLE I**

| $\bar{n}$ [-] | 2   | 4   | 6   | 7   | 9   |
|---|---|---|---|---|---|
| Violation rate | 0.98 | 1.09 | 1.26 | 1.33 | 1.33 |
| Theorem 3 bound [%] | 8.06 | 9.76 | 10.55 | 12.06 | 12.06 |
VIII. CONCLUSION

We considered multi-agent games in the presence of uncertainty. We robustified agents’ strategies against unknown externalities acting on the objective function by defining a suitable noncooperative minmax game. In doing this, we adopted a data driven paradigm where uncertainty is represented by means of scenarios. Building on results from [29], we showed how the NEs of the minmax game can be accompanied with robust sensitivity certificate, providing confidence on the probability that the computed strategies remain unaffected when a new uncertainty realisation is encountered. Nonetheless, experimental results suggest that a priori bounds could be achieved, at least in the considered case. Our current research activities focus on this direction. Future work will also evaluate the possibility of using the readily available shadow prices associated to the support samples as a tuning parameter to achieve the desired tradeoff between robustness and performance [39], [45].

REFERENCES

[1] F. Lamahbi-Lagarigue, A. Annaswamy, S. Engell, A. Isaksson, P. Karponek, R. M. Murray, H. Nijmeijer, T. Samad, D. Tilbury, and P. V. den Hof, “Systems & control for the future of humanity, research agenda: Current and future roles, impact and grand challenges,” Annual Reviews in Control, vol. 43, pp. 1 – 64, 2017.
[2] T. Baar and G. Olsder, Dynamic Noncooperative Game Theory, 2nd Edition. Society for Industrial and Applied Mathematics, 1998.
[3] M. Aghassi and D. Bertsimas, “Robust game theory,” Mathematical Programming, vol. 107, no. 1, pp. 231–273, Jun 2006.
[4] F. Facchinei and C. Kanzow, “Generalized Nash equilibrium problems,” 4OR, vol. 5, no. 3, pp. 173–210, Sep 2007.
[5] A. Mohsenian-Rad, V. W. S. Wong, J. Jatskevich, R. Schober, and A. Leon-Garcia, “Autonomous demand-side management based on game-theoretic energy consumption scheduling for the future smart grid,” IEEE Transactions on Smart Grid, vol. 1, no. 3, pp. 320–331, Dec 2010.
[6] A. De Paola, F. Fele, D. Angeli, and G. Strbac, “Distributed coordination of price-responsive electric loads: A receding horizon approach,” in 2018 IEEE Conference on Decision and Control (CDC), Dec 2018, pp. 6033–6040.
[7] D. Paccagnan, B. Gentile, F. Parise, M. Kamgarpour, and J. Lygeros, “Nash and Wardrop equilibria in aggregative games with coupling constraints,” IEEE Transactions on Automatic Control, pp. 1–1, 2018.
[8] G. Scutari, F. Facchinei, J. S. Pang, and D. P. Palomar, “Real and complex monotone communication games,” IEEE Transactions on Information Theory, vol. 60, no. 7, pp. 4197–4231, July 2014.
[9] I. Atzeni, L. G. Orde, G. Scutari, D. P. Palomar, and J. R. Fonolosa, “Noncooperative day-ahead bidding strategies for demand-side expected cost minimization with real-time adjustments: A GNEP approach,” IEEE Transactions on Signal Processing, vol. 62, no. 9, pp. 2397–2412, May 2014.
[10] J. G. Wardrop, “Some theoretical aspects of road traffic research.” Proceedings of the Institution of Civil Engineers, vol. 1, no. 3, pp. 325–362, 1952.
[11] S. Grammatico, “Dynamic control of agents playing aggregative games with coupling constraints,” IEEE Transactions on Automatic Control, vol. 62, no. 9, pp. 4537–4548, Sep. 2017.
[12] L. Deori, K. Margellos, and M. Prandini, “Price of anarchy in electric vehicle charging control games: When Nash equilibria achieve social welfare,” Automatica, vol. 96, pp. 150 – 158, 2018.
[13] D. Paccagnan, F. Parise, and J. Lygeros, “On the efficiency of Nash equilibria in aggregative charging games,” IEEE Control Systems Letters, vol. 2, no. 4, pp. 629–634, Oct 2018.
[14] F. Fele, A. De Paola, D. Angeli, and G. Strbac, “A framework for receding-horizon control in infinite-horizon aggregative games,” Annual Reviews in Control, vol. 45, pp. 191 – 204, 2018.
[15] J. Nash, “Non-cooperative games,” Annals of Mathematics, vol. 54, no. 2, pp. 286–295, 1951.
[16] J.-S. Pang, S. Sen, and U. V. Shanbhag, “Two-stage non-cooperative games with risk-averse players,” Mathematical Programming, vol. 165, no. 1, pp. 235–290, Sep 2017.
[17] S. K. Khaitan and J. D. McCalley, “Design techniques and applications of cyberphysical systems: A survey,” IEEE Systems Journal, vol. 9, no. 2, pp. 350–365, Jun 2015.
[18] K. Kim and P. R. Kumar, “Cyberphysical systems: A perspective at the centennial,” Proceedings of the IEEE, vol. 100, no. Special Centennial Issue, pp. 1287–1308, May 2012.
[19] J. C. Harsanyi, “Games with incomplete information played by bayesian players, iiii part i. the basic model,” Management Science, vol. 14, no. 3, pp. 159–182, 1967.
[20] P. Couchman, B. Kouvaritakis, M. Cannon, and F. Prashad, “Gaming strategy for electric power with random demand,” IEEE Transactions on Power Systems, vol. 20, no. 3, pp. 1283–1292, 2005.
[21] V. V. Singh, O. Jouini, and A. Lissner, “Existence of Nash equilibrium for chance-constrained games,” Operations Research Letters, vol. 44, no. 5, pp. 640 – 644, 2016.
[22] U. Raval and U. Shanbhag, “On the characterization of solution sets of smooth and non-smooth convex stochastic Nash games,” SIAM Journal on Optimization, vol. 21, no. 3, pp. 1168–1199, 2011.
[23] J. Joshaal, A. Nedic, and U. V. Shanbhag, “Regularized iterative stochastic approximation methods for stochastic variationalinequality problems,” IEEE Transactions on Automatic Control, vol. 58, no. 3, pp. 594–609, March 2013.
[24] H. Xu and D. Zhang, “Stochastic Nash equilibrium problems: sample average approximation and applications,” Computational Optimization and Applications, vol. 55, no. 3, pp. 597–645, Jul 2013.
[25] C. Yu, M. van der Schaar, and A. H. Sayed, “Distributed learning for stochastic generalized Nash equilibrium problems,” IEEE Transactions on Signal Processing, vol. 65, no. 15, pp. 3893–3908, Aug 2017.
[26] J. Lei and U. V. Shanbhag, “A randomized inexact proximal best-response scheme for potential stochastic Nash games,” in 2017 IEEE
[27] R. Nishimura, S. Hayashi, and M. Fukushima, “Robust Nash equilibria in $n$-person non-cooperative games: Uniqueness and reformulation,” Pacific J. Optim., no. 5, 2009.

[28] M. Hu and M. Fukushima, “Existence, uniqueness, and computation of robust Nash equilibria in a class of multi-leader-follower games,” SIAM Journal on Optimization, vol. 23, no. 2, pp. 894–916, 2013.

[29] M. C. Campi, S. Garatti, and F. A. Ramponi, “A general scenario theory for non-convex optimization and decision making,” IEEE Transactions on Automatic Control, pp. 1–1, 2018.

[30] G. C. Calafiore and M. C. Campi, “The scenario approach to robust control design,” IEEE Transactions on Automatic Control, vol. 51, no. 5, pp. 742–753, May 2006.

[31] A. J. Conejo, M. Carrión, and J. M. Morales, Uncertainty Characterization via Scenarios. Boston, MA: Springer US, 2010, pp. 63–119.

[32] T. Alamo, R. Tempo, and E. F. Camacho, “Randomized strategies for probabilistic solutions of uncertain feasibility and optimization problems,” IEEE Transactions on Automatic Control, vol. 54, no. 11, pp. 2545–2559, Nov 2009.

[33] M. Campi and S. Garatti, “The exact feasibility of randomized solutions of uncertain convex programs,” SIAM Journal on Optimization, vol. 19, no. 3, pp. 1211–1230, 2008.

[34] J. Zazo, S. Zazo, and S. V. Macua, “Robust worst-case analysis of demand-side management in smart grids,” IEEE Transactions on Smart Grid, vol. 8, no. 2, pp. 662–673, March 2017.

[35] F. Facchinei, J.-S. Pang, and G. Scutari, “Non-cooperative games with minmax objectives,” Computational Optimization and Applications, vol. 59, no. 1, pp. 85–112, Oct 2014.

[36] S. Floyd and M. Warmuth, “Sample compression, learnability, and the Vapnik-Chervonenkis dimension,” Machine Learning, vol. 21, no. 3, pp. 269–304, Dec 1995.

[37] K. Margellos, M. Prandini, and J. Lygeros, “On the connection between compression learning and scenario based single-stage and cascading optimization problems,” IEEE Transactions on Automatic Control, vol. 60, no. 10, pp. 2716–2721, Oct 2015.

[38] M. C. Campi and S. Garatti, “Wait-and-judge scenario optimization,” Mathematical Programming, vol. 167, no. 1, pp. 155–189, Jan 2018.

[39] G. Calafiore, “Random convex programs,” SIAM Journal on Optimization, vol. 20, no. 6, pp. 3427–3464, 2010.

[40] F. Facchinei and J.-S. Pang, Finite-Dimensional Variational Inequalities and Complementarity Problems. Springer-Verlag New York, 2003.

[41] B. Rustem and M. Howe, Algorithms for Worst-Case Design and Applications to Risk Management. Princeton University Press, 2002.

[42] A. Kannan and U. Shanbhag, “Distributed computation of equilibria in monotone Nash games via iterative regularization techniques,” SIAM Journal on Optimization, vol. 22, no. 4, pp. 1177–1205, 2012.

[43] D. P. Bertsekas and J. N. Tsitsiklis, Parallel and Distributed Computation: Numerical Methods. Upper Saddle River, NJ, USA: Prentice-Hall, Inc., 1989.

[44] National Grid. (2019, Mar.) Historical demand data. [Online]. Available: [https://www.nationalgrideso.com/balancing-data/data-explorer](https://www.nationalgrideso.com/balancing-data/data-explorer)

[45] M. C. Campi and S. Garatti, “A sampling-and-discarding approach to chance-constrained optimization: Feasibility and optimality,” Journal of Optimization Theory and Applications, vol. 148, no. 2, pp. 257–280, Feb 2011.