A local hidden variable model of quantum correlation exploiting
the detection loophole

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Abstract
A local hidden variable model exploiting the detection loophole to reproduce exactly the
quantum correlation of the singlet state is presented. The model is shown to be compatible
with both the CHSH and the CH Bell inequalities. Moreover, it bears the same rotational
symmetry as spins. The reason why the model can reproduce the quantum correlation
without violating the Bell theorem is that in the model the efficiency of the detectors depends
on the local hidden variable. On average the detector efficiency is limited to 75%.

1 Introduction
Quantum theory is nonlocal in the sense that it predicts that distant systems can produce
outcomes whose correlation can not be explained by any model based only on local variables (i.e.
in which results ”here” happen independently of choices made at a distance), as demonstrated
by Bell [1]. The state of such ”quantum correlated” systems is said to be ”entangled”. Because
these outcomes are stochastic, there is no way to use them for signaling. Hence, despite this
nonlocality, there is no direct conflict with relativity [2].

All experiments to date are in perfect agreement with quantum theory. There is thus strong
evidence for the view that Nature is nonlocal. However, considering the importance of such
a conclusion, the experimental evidence should be analyzed very carefully and any additional
experimental tests of quantum entanglement are most welcome.

There is one logical loophole affecting all the experiments carried out so far that deserves
special attention. This is the ”detection loophole” [3, 4], which is the central issue of this letter.
The detection loophole is based on the following fact: in real experiments the efficiencies of the
detectors are such that the set of detected events is significantly smaller than the set of tested
quantum systems. One has thus to assume that the sample over which the statistics is measured
is a fair sample. From the point of view of quantum mechanics this assumption is almost trivial.
However, from the local hidden variable (lhv) point of view the opposite assumption is equally
almost trivial: if there are additional variables, unknown today, it is very plausible that the
actual value of these variables affects the probability to trigger a detector.

Admittedly, invoking the detection loophole to explain all the experimental tests of Bell
inequality is somewhat artificial. However, it is annoying that this loophole resists after almost 30
years [3] of research and progress! In this letter we present a simple lhv model which reproduces
analytically the quantum correlations corresponding to the singlet state of two spin $\frac{1}{2}$:

$$E(\vec{a}, \vec{b}) = -\vec{a}\vec{b}$$  (1)
where \( \vec{a} \) and \( \vec{b} \) represent the bases of the two distant measurements. The quantum correlation of any other maximally entangled states can also be reproduced by our model by applying to the lhv (which are vectors) the same local rotations that transform any maximally entangled state into the singlet. However, the model is compatible with all Bell inequalities. In our model the detector efficiency is of 75%. This is much larger than the efficiency used in all Bell tests so far, hence our model reproduces all existing experimental data of Bell tests.

In the next section we present our lhv model. Then, in section 3 we establish the connection with the Clauser-Horne (CH) Bell inequality \([6]\). In the conclusion we comment on the connection between our lhv model and the paper and the paper by Steiner \([7]\) in which the problem of minimal classical communication for simulating quantum correlation is considered and which inspired our work very much.

2 A local hidden variable model

In this section we present our lhv model, inspired by Steiner’s contribution \([6]\). We first present it in its simplest form, which is asymmetric between Alice and Bob, the two observers. Next we show how to make it symmetric. In this model, each spin \( \frac{1}{2} \) is characterized, in addition to its quantum state \( \rho \), by a normalized classical arrow \( \vec{\lambda} \) with uniform a priori probability distribution. If the spin is measured along a direction \( \vec{a} \), the outcome \( \pm 1 \) is determined by the sign of the scalar product \( \langle \vec{\sigma} \rangle_{\rho} = (\vec{\sigma} \cdot \vec{\lambda}) \cdot \vec{a} \), where \( \langle \vec{\sigma} \rangle_{\rho} = Tr(\vec{\sigma} \rho) \) denotes the expectation values of the Pauli matrices. For a single spin \( \frac{1}{2} \), the probability of an outcome +1 is then in accordance with the quantum prediction for a state \( \rho \):

\[
P_{+}(\vec{a}) = \frac{1}{2}(1 + \langle \vec{\sigma} \rangle_{\rho} \cdot \vec{a})
\]

In case of two spin \( \frac{1}{2} \) of total spin zero (i.e. singlet state), the vectors characterizing each spin are opposite: \( \vec{\lambda}_B = -\vec{\lambda}_A \). With these rules the correlation is linear (see Fig. 1) \([8]\):

\[
E_{lhv}(\vec{a},\vec{b}) = -1 + \frac{2\theta_{ab}}{\pi}
\]

where \( \theta_{ab} \in [0..\pi] \) is the angle between \( \vec{a} \) and \( \vec{b} \). They do not violate the CHSH-Bell inequality. So far measurements always produce outcomes, i.e. we did not exploit the detection loophole. Now, let’s assume that whenever a measurement in direction \( \vec{a} \) is performed, an outcome on Alice detector is produced only with probability \( |\vec{\lambda}_A \cdot \vec{a}| \). That is, on Alice side no outcome at all is produced in a ratio \( 1 - |\vec{\lambda}_A \cdot \vec{a}| \) of cases, while Bob always produces an outcome. This affects the correlation; indeed, when \( \vec{a} \) happens to be close to \( \vec{\lambda}_A \) then the probability that an outcome is produced is larger than when \( \vec{a} \) happens to be nearly orthogonal to \( \vec{\lambda}_A \). To compute the correlation function \( E(\vec{a},\vec{b}) \), we first need the mean probability \( p \) that an outcome is produced:

\[
p = \int_{S^2} \frac{d\vec{\lambda}}{4\pi} |\vec{a} \cdot \vec{\lambda}| = \frac{1}{2}
\]

Next, we need the conditional density probability distribution of the \( \vec{\lambda}_A \) given that an outcome is produced:

\[
\rho(\vec{\lambda} \mid \text{outcome produced}) = \frac{\rho(\vec{\lambda} \text{ and outcome produced})}{\text{Prob(outcome produced)}}
\]

\[
= \frac{1}{4\pi |\vec{a} \cdot \vec{\lambda}|}
\]

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= \frac{p}{2\pi |\vec{a} \cdot \vec{\lambda}|}
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\]
Now, the correlation can be computed (note that it is convenient to choose a reference frame such that \( \vec{b} = (0, 0, 1), \vec{a} = (\sin(\alpha), 0, \cos(\alpha)) \) and \( \vec{\lambda} = (\sqrt{1 - \eta^2 \cos(\phi)}, \sqrt{1 - \eta^2 \sin(\phi)}, \eta), d\vec{\lambda} = d\eta d\phi \):

\[
E(\vec{a}, \vec{b}) = \int_{S^2} d\vec{\lambda} \rho(\vec{\lambda} | \text{outcome produced}) \text{sign}(\vec{a} \cdot \vec{\lambda}) \text{sign}(\vec{b} \cdot \vec{\lambda})
\]

(7)

\[
= -\int_{S^2} d\vec{\lambda} \frac{\vec{a} \cdot \vec{\lambda}}{2\pi} \text{sign}(\vec{b} \cdot \vec{\lambda})
\]

(8)

\[
= -\int_{-1}^{1} d\eta \int_{0}^{2\pi} \frac{d\phi}{2\pi} \left( \eta \cos(\alpha) + \sqrt{1 - \eta^2} \cos(\phi) \sin(\alpha) \right) \text{sign}(\eta)
\]

(9)

\[
= -\cos(\alpha) = -\vec{a} \cdot \vec{b}
\]

(10)

Witch agrees exactly with the quantum correlation (1).

In summary, if Alice’s detector is allowed to fire only half the time, with a probability to fire that depends on the lhv \( \lambda_A \), while Bob’s detector always fires, then the quantum correlation can be recovered exactly! The asymmetric protocol presented above can clearly be made symmetric: it suffices to add an additional binary local hidden variable which determines when Alice and Bob exchange their roles. If this binary variable is randomly distributed, then both Alice’s and Bob’s detector will fires in 75% of cases, that is both detectors show an effective efficiency of 75%, much above the efficiencies of the detectors used in all actual experimental Bell tests. Note that in the model presented so far there is always at least one detector that fires. This may be considered as bizarre and corrected for by adding to the model yet another binary local hidden variable which programs both spins to produce no outcome at all. If this ”no outcome at all” happens with probability 1/9, then both detectors appear to be independent, each with an efficiency of \( \frac{8}{9} \times \frac{2}{3} = \frac{2}{3} \): probability of 2 outcomes = \( \left(\frac{2}{3}\right)^2 \), of 1 outcome = \( \frac{2}{3} \times \frac{1}{3} \) and of 0 outcome = \( \left(\frac{1}{3}\right)^2 \).

A simple software demo that simulates Alice and Bob “experiments” is available on our WEB side [5]. It allows two independent computers to produce data which violate the Bell inequality.

3 Connection to the Clauser-Horne inequality

The most wellknown Bell inequality is the CHSH one [9]:

\[
E(\vec{a}, \vec{b}) + E(\vec{a}, \vec{b}') + E(\vec{a}', \vec{b}) - E(\vec{a}', \vec{b}') \leq 2
\]

(11)

where \( E(\vec{a}, \vec{b}) = P_{++}(\vec{a}, \vec{b}) + P_{--}(\vec{a}, \vec{b}) - P_{+-}(\vec{a}, \vec{b}) - P_{-+}(\vec{a}, \vec{b}) \) is the correlation function with \( P_{ij}(\vec{a}, \vec{b}) \) the probabilities of outcomes \( ij = \pm \pm \) when the measurement bases are defined by the directions \( \vec{a} \) and \( \vec{b} \). In experiments one does not measure probabilities, but coincidence rate (i.e. coincidence counts per time unit) \( N_{ij}(\vec{a}, \vec{b}) \). The correlation function \( E(\vec{a}, \vec{b}) \) is thus evaluated experimentally as a ”renormalized correlation”:

\[
E(\vec{a}, \vec{b}) = \frac{N_{++} + N_{--} - N_{+-} - N_{-+}}{N_{++} + N_{--} + N_{+-} + N_{-+}}
\]

(12)

where for compactness we have dropped the indication of the bases. This renormalization, although natural in the frame of quantum mechanics, is questionable. Indeed, with this renormalization the lhv model presented in the previous section, although purely local, does violate the CHSH inequality [11]. Hence, in order to definitively prove all lhv models wrong, the experimental data should violate an inequality involving no ”renormalization”, ideally involving only count rates.
The CHSH inequality (11) can be deduced from the following trivial one:
\[ ab + ab' + a'b - a'b' = a(b + b') + a'(b - b') \leq 2 \] (13)
where \( a, a', b, b' \in \{-1, +1\} \). In order to obtain an inequality between count rates, we introduce numbers \( x = \frac{1}{2}(1 + a) \) and \( y = \frac{1}{2}(1 + b) \), \( x, y \in \{0, 1\} \) such that:
\[ ab + ab' + a'b - a'b' - 2 = 4(x + x'y + x'y' - (x + y)) \leq 0 \] (14)
From the inequality (14), one deduces the CH inequality (1):
\[ N_{++}(\vec{a}, \vec{b}) + N_{++}(\vec{a}', \vec{b}') - N_{++}(\vec{a}', \vec{b}) - N_{++}(\vec{a}, \vec{b}') \leq N_{+}(\vec{a}) + N_{+}(\vec{b}) \] (15)
where \( N_{+}(\vec{a}) \) and \( N_{+}(\vec{b}) \) denote the single counts on the first and second system, respectively.
Quantum mechanics predicts:
\[ N_{+}(\vec{a}) = P_{+}(\vec{a})\eta_A N \quad N_{+}(\vec{b}) = P_{+}(\vec{b})\eta_B N \]
where \( N \) is the total number of quantum systems tested per unit time and \( \eta_A \) and \( \eta_B \) denote the efficiencies of the first and second detector, respectively. Accordingly, assuming for simplicity \( \eta_A = \eta_B \equiv \eta \), quantum mechanics predicts a violation of the CH inequality (13) if and only if:
\[ \eta > \frac{P_{+}(\vec{a}) + P_{+}(\vec{b})}{P_{+}(\vec{a}, \vec{b}) + P_{+}(\vec{a}', \vec{b}') + P_{+}(\vec{a}', \vec{b}) - P_{+}(\vec{a}, \vec{b}') \}} \] (18)
The optimal is obtained for the same setting \( \vec{a}, \vec{a}', \vec{b}, \vec{b}' \) that maximize the violation of the CHSH inequality (11). For these one obtains \( \eta_{\text{threshold}} = \frac{2}{1 + \sqrt{2}} \approx 0.828 \). Hence, quantum mechanics predicts that the CH inequality (13) can be violate, and thus lhv models definitively ruled out, only with detector’s efficiencies higher than 82.8%! This is in agreement with our model in which the "efficiency" is of "only" 75%. Note however that in our model the detection probabilities on both sides are not independent: \( \eta_A \eta_B \neq \eta_A \cdot \eta_B \). Hence, the "relevant efficiency" is even smaller, in full agreement with the model with independent detectors discussed at the end of the previous section:
\[ \frac{\eta_A \eta_B}{\overline{\eta}_{\text{A}}} = \frac{p}{2(p + 1)} = \frac{2}{3} < \eta_{\text{threshold}} \] (19)
We conclude this section with some comments. First, it could be that some other inequality involving only count rates could be violated with detector efficiencies bellow 82.8%. However, our model proves that no such inequality could achieve a threshold bellow 75% (actually, no such inequality is known to us). Next, in our model there is always at least one detector that fires. This may seem unphysical. But this is a point that has never been ruled out experimentally. Moreover if one allows for lower detector efficiencies one could easily generalize our model and introduce cases where no detector at all fire. Finally, the optimal directions \( \vec{a}, \vec{a}', \vec{b}, \vec{b}' \) lie in a plane, hence a model reproducing the quantum correlation for settings restricted to a plane is also of interest. Such a model follows straightforwardly from Steiner contribution (7), with a detection efficiency of \( \frac{1}{2}(1 + \frac{2}{\pi}) \approx 81.8% \) and a "relevant efficiency" (taking into account the correlation between the detection probabilities) of 77.8%.
4 Conclusion

The detection loophole remains open after almost 30 years of research and progress, despite its importance for the understanding of physics and for applications in quantum communication. The model presented in section 2 underlines how simple and natural a local model can be, while reproducing exactly the quantum correlation function thanks to this loophole. This model also proves that there is no hope to close the detection loophole with detectors’ efficiencies below 75%.

Our lhv model was inspired by the CH model and by Steiner’s contribution. In the latter an apparently different problem was considered, namely that of the classical communication cost of simulating quantum correlation. The idea is that Alice and Bob have common classical information and exchange as little classical information as possible to reproduce the correlation, thus violating Bell inequality. Clearly, any lhv model provides a classical communication simulation model; instead of producing no outcome, Alice informs Bob not to use their common information and to go to the next one. The detection efficiency translates then into the communication cost. Conversely, a classical communication simulation model provides a lhv model. But in this direction the connection is not straightforward. Indeed, Bob’s action could depend on the information he receives from Alice in a more complicated way than merely going to the next common information. Nevertheless, a lhv obtains if Alice and Bob decide in advance to bet on the action Bob would have to take on each instance and that Alice merely produces no outcome whenever the actual action Bob should carry out differs from the bet. This can be very inefficient, but provides a lhv model.

The local symmetry of our model is the same rotational symmetry that characterizes spins. However, our model does not reproduce "coherent measurements", like the famous Bell measurements so powerful in quantum information processing, in particular for entanglement swapping. Nor does it account for partially entangled states (which have surprising characteristics, see [12]), nor for generalized (POVM) measurements. But, to date, neither Bell measurements nor tests of Bell inequality using partial entanglement have been performed. This work might provide additional motivation for such experiments.

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[5] http://www.gap-optique.unige.ch/News/BellSoft.asp. One program simulates the production of Alice data, while a similar program produces Bob's data, possibly on a different computer. A third program analyses then these data, reproducing the quantum correlation with the standard statistical
noise. The only common information to Alice’s and Bob’s programs are the lhv $\vec{\lambda}_A = -\vec{\lambda}_B$, encoded by a common pseudo-random number generator and a common seed (chosen by the operator).

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**Figure caption**

Correlation functions, with $\theta_{ab}$ the angle between the measurement directions on Alice and Bob sides. The dotted line corresponds to the local hidden variable model with 100% detection efficiency [4], the full line to our model with 75% detection efficiency and to quantum mechanics [4].
