Brane induced cosmological acceleration and crossing of $w_{\text{eff}} = -1$

Nobuyuki MOTOYUI

Department of Physics, Faculty of Sciences, Ibaraki University, Bunkyo 2-1-1, Mito, 310-8512, Japan

Abstract

The cosmological observation indicates that the effective equation of state parameter $w_{\text{eff}}$ varies with $z$: It changes from $w_{\text{eff}} < -1$ to $w_{\text{eff}} > -1$ at $z \sim 0.2$. We investigate under which condition it exhibits such behaviors based on the five-dimensional braneworld scenario. It is possible in the model with or without an energy exchange between the four dimensional universe and the fifth dimension. However the curves of $w_{\text{eff}}$ are quite different between the two cases.

Keywords: Brane universe; dark energy; effective equation of state parameter.
PACS numbers: 95.36.+x, 11.25.Mj.
1 Introduction

The cosmological observations indicate that our universe have not only experienced the large amount of accelerating expansion in its infancy but it is still undergoing a small rate of accelerating expansion [1,2]. The present cosmological data also indicates that more than 70% of the total energy density is attributed to a component of dark energy density [3,4]. The source of driving force to the present accelerating expansion is thought to be dark energy. Although the physical origin of dark energy is still unknown, many cosmological candidates such as the cosmological constant or several kinds exotic matter like phantom fields [5,6], quintessence [7–9], or modifications of gravitational theory have been proposed. If we define the ratio of the pressure of the universe $P$ to its energy density $\rho$ as $w \equiv P/\rho$, then the cosmological constant is characterized by $w = -1$ and the phantom field is characterized by $w < -1$. If the cosmological constant is identified as dark energy, $w$ is a constant. On the other hand, $w$ varies with time if we adopt the assumptions of exotic matter. Although the simplest candidate of dark energy is the cosmological constant $\Lambda$, there is a well known 'fine tuning problem' between the energy density of cosmological constant $\rho_\Lambda$ and the radiation density $\rho_r$: The ratio of $\rho_\Lambda$ and $\rho_r$ depends on the energy scale and $\rho_\Lambda/\rho_r \sim 10^{-123}$ at the Planck scale, on the other hand $\rho_\Lambda/\rho_r \sim 10^{-54}$ at the electroweak scale because $\rho_\Lambda \simeq 10^{-47} GeV^4$ is a constant.

However the effective equation of state parameter is not necessarily a constant. Alternative ways for dark energy model in which either dark energy or its effective equation of state parameter is a function of time. The cosmological data indicates that the time varying dark energy model is a better fitting than a cosmological constant: the effective equation of state parameter of dark energy $w_{\text{eff}} \sim -1.21$ at $z = 0$ and it changes from $w_{\text{eff}} < -1$ to $w_{\text{eff}} > -1$ at $z \sim 0.2$ [10]. It indicates that $w_{\text{eff}}$ behaves as phantom-like at lower redshifts $z \lesssim 0.2$ and dust-like at higher redshifts. Since a phantom component with $w_{\text{eff}} < -1$ violates energy conditions, one may hope to make a dark energy model with the above nature in the framework of effective features, not caused by a phantom field. Then we may hope to do this behavior in the framework of braneworld scenario.

In the braneworld scenario, our universe is realized as a three dimensional hypersurface (brane) which is embedded in a higher dimensional spacetime (bulk). Gravity can propagate to the entire spacetime while the ordinary matter is confined on the brane. It received much attentions due to the possibilities that the compactification with the large extra dimensions [11], the solution to the large disparity between the electroweak scale and the Planck scale [12]. It is an important question that whether the standard four dimensional gravity is reproduced in these scenario. It is showed that the massless gravitons are trapped on the brane and the four dimensional gravity is reproduced [13].

It is an interesting question that how the present accelerating expansion of our universe can be treated in the framework of the braneworld scenario. The evolution of our universe is described as an effective Friedmann equation on a brane. It is modified from the usual Friedmann equation because of the presence of the fifth dimension [14,15]. The presence of the fifth dimension allows a five-dimensional bulk matter which propagates in the fifth dimension. It may interact with the ordinary matter on the brane and it could be able to lead the behavior that resembles to dark energy. In this picture, the bulk pressure and
the off-diagonal terms in the energy-momentum tensor affect the cosmological evolution of the brane. It has been shown that $w_{\text{eff}}$ crosses $w_{\text{eff}} = -1$ at lower redshift in the model which allows the energy exchange between the four dimensional universe and the fifth dimension [16–18].

This paper is organized as follows. In section 2, we present the general framework of the five-dimensional theory. Then we present the effective Friedmann equation and the conservation of energy-momentum based on [14]. In section 3, we derive the Hubble equation on the 3-brane. It is quite different from that of a standard four dimensional Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology in several points. Then we derive the effective equation of state parameter $w_{\text{eff}}$ of dark energy. In section 4, we examine the evolutions of $w_{\text{eff}}$ and investigate under which condition it exhibits crossing of $w_{\text{eff}} = -1$ at $z \sim 0$ and $w_{\text{eff}} \sim -1.21$ at $z = 0$ in two cases: (i) there is energy exchange between the four dimensional universe and the fifth dimension, and (ii) there is no energy exchange between the four dimensional universe and the fifth dimension. We also examine the evolutions of deceleration parameter and energy density components. Section 5 is devoted to summary.

2 General Framework and the effective Friedmann equation

We consider the five-dimensional action of the following form,

$$S = \int d^5 x \sqrt{-g(5)} \left[ \frac{1}{2 \kappa^2(5)} (R(5) - 2\Lambda(5)) + \mathcal{L}_B^{(m)} \right] + \int d^5 x \sqrt{-g_5} \left( -T_b + \mathcal{L}_b^{(m)} \right) \delta(y). \tag{1}$$

We denote the coordinate of fifth dimension by $y$ and it takes the values $-\infty < y < \infty$. We consider a 3-brane is located at $y = 0$ and assume a $Z_2$ symmetry for $y$ around $y = 0$. In the above expression, $R(5)$ is the five-dimensional Ricci scalar, $\Lambda(5)$ is the five-dimensional cosmological constant and $T_b$ is the tension of the 3-brane. We can include matter content in the bulk $\mathcal{L}_B^{(m)}$ or on the brane $\mathcal{L}_b^{(m)}$. We denote the five-dimensional metric as $g_{(5)AB}$, the four dimensional metric as $g_{\mu\nu}$ and the four dimensional metric on the brane as $g_b$. We define the signature of $g_{(5)AB}$ as $(-,+,+,+,+)$ and that of $g_{\mu\nu}$ as $(-,+,+,+)$. The line element is described as

$$ds^2 = g_{(5)AB} dx^A dx^B = g_{\mu\nu} dx^\mu dx^\nu + b^2(t,y) dy^2. \tag{2}$$

The capital Latin indices indicate $(0, \cdots, 4)$ and the Greek indices indicate $(0, \cdots, 3)$. The constant $\kappa(5)$ is related to the five-dimensional Newton constant $G(5)$ and the fifth-dimensional Planck mass $M(5)$ by the relation $\kappa^2(5) = 8\pi G(5) = M^{-3}(5)$. We assume that the five-dimensional metric is described as follows,

$$ds^2 = -n^2(t,y) dt^2 + a^2(t,y) \gamma_{ij} dx^i dx^j + b^2(t,y) dy^2, \tag{3}$$

where $\gamma_{ij}$ is a maximally symmetric FLRW metric. Its spatial curvature is parametrized by $K$ which takes the values $K = -1, 0, 1$. 

3
The energy-momentum tensor $T_{AB}$ is given as

$$T_{AB} = T^{(B)}_{AB} + T^{(b)}_{AB} - T_b \sqrt{-g_b} \, g_{\mu
u} \delta^\mu_A \delta^\nu_B \delta(y),$$

(4)

where $T^{(B)}_{AB}$ is the component which results from $L^{(m)}_b$ and $T^{(b)}_{AB}$ is the component which results from $L^{(m)}_b$. We assume the bulk energy-momentum tensor $T^{(B)}_{AB}$ as

$$T^{(B)}_{AB} = \begin{pmatrix} -\rho_B & 0 & Q \\ 0 & P_B \delta^i_j & 0 \\ -\frac{n^2}{b^2} Q & 0 & P_T \end{pmatrix}. \quad (5)$$

The brane energy-momentum tensor $T^{(b)}_{AB}$ is generally expressed as

$$T^{(b)}_{AB}|_{\text{brane}} = \frac{\delta(y)}{b} \begin{pmatrix} -\rho_b & 0 & 0 \\ 0 & P_b \delta^i_j & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (6)$$

In the above expression, $Q$ is responsible for the energy exchange between the four dimensional spacetime and the extra dimension. We generally allow the anisotropic choice of the bulk pressure: $P_B \neq P_T$. The dynamics of the five-dimensional universe is governed by the five-dimensional Einstein equations. They take the usual form,

$$G^{(5)}_{AB} \equiv R^{(5)}_{AB} - \frac{1}{2} g^{(5)}_{AB} R^{(5)} = -\Lambda^{(5)} g_{AB} + \kappa^2 T_{AB}. \quad (7)$$

Substituting (3), (5) and (6) into (7), we obtain the following equations,

$$3 \left\{ \frac{\dot{a}}{a} \left( \frac{\dot{a}}{a} + \frac{\dot{b}}{b} \right) - \frac{n^2}{b^2} \left( \frac{a''}{a} + \frac{a'}{a} \left( \frac{a'}{a} - \frac{b'}{b} \right) \right) + K \right\} = n^2 \left( \Lambda^{(5)} + \kappa^2 \rho_B + \frac{\kappa^2}{b} (\rho_b + T_b) \delta(y) \right), \quad (8)$$

$$\frac{a^2}{b^2} \left\{ \frac{a'}{a} \left( \frac{a'}{a} + 2 \frac{n'}{n} \right) - \frac{b'}{b} \left( \frac{n'}{n} + 2 \frac{a'}{a} \right) + 2 \frac{a''}{a} + \frac{n''}{n} \right\} + \frac{a^2}{n^2} \left\{ \frac{\dot{a}}{a} \left( -\frac{\dot{a}}{a} + 2 \frac{\dot{n}}{n} \right) - \frac{\dot{a}}{a} + 2 \frac{\dot{b}}{b} \left( -\frac{\dot{a}}{a} + \frac{\dot{n}}{n} \right) - \frac{\dot{b}}{b} \right\} + K = a^2 \left( -\Lambda^{(5)} + \kappa^2 P_B + \frac{\kappa^2}{b} (P_b - T_b) \delta(y) \right), \quad (9)$$

$$3 \left\{ \frac{n' \dot{a}}{n a} + \frac{a' \dot{b}}{a b} - \frac{\dot{a}'}{a} \right\} = -n^2 \kappa^2 Q, \quad (10)$$

$$3 \left\{ \frac{a'}{a} \left( \frac{a'}{a} + \frac{n'}{n} \right) - \frac{b^2}{n^2} \left( \frac{\dot{a}}{a} \left( \frac{\dot{a}}{a} - \frac{\dot{n}}{n} \right) + \frac{\dot{a}}{a} \right) - K \frac{b^2}{a^2} \right\} = b^2 \left( -\Lambda^{(5)} + \kappa^2 P_T \right), \quad (11)$$

where dots stand for differentiations with respect to $t$ and primes stand for differentiations with respect to $y$.

We make two assumptions to simplify these equations:
(i) the scale factor of extra dimension is normalized to $b(t, y) = 1$, 
(ii) the scale factor of temporal dimension at $y = 0$ is normalized to $n(t, 0) = 1$.

After these simplifications, Einstein equations on the brane are expressed as follows,

$$3 \left\{ \left( \frac{\dot{a}_b}{a_b} \right)^2 - \frac{a''_b}{a_b} - \left( \frac{a'_b}{a_b} \right)^2 + K \frac{1}{a_b^2} \right\} = \Lambda_{(5)} + \kappa_{(5)}^2 \rho_B + \kappa_{(5)}^2 (\rho_b + T_b) \delta(y),$$  \hspace{1cm} (12)

$$3 \left\{ \frac{a'_b}{a_b} \left( \frac{a'_b}{a_b} + 2n'_b \right) + 2 \frac{a''_b}{a_b} + n''_b \right\} - \left( \frac{\dot{a}_b}{a_b} \right)^2 - 2 \frac{\ddot{a}_b}{a_b} - K \frac{1}{a_b^2}$$

$$= - \Lambda_{(5)} + \kappa_{(5)}^2 (P_b + \kappa_{(5)}^2 (P_b - T_b) \delta(y),$$ \hspace{1cm} (13)

$$3 \left( \frac{n'_b}{a_b} - \frac{\dot{a}'_b}{a_b} \right) = - \kappa_{(5)}^2 Q,$$ \hspace{1cm} (14)

$$3 \left\{ \frac{a'_b}{a_b} \left( \frac{a'_b}{a_b} + n'_b \right) - \left( \frac{\dot{a}_b}{a_b} \right)^2 - \frac{\ddot{a}_b}{a_b} - K \frac{1}{a_b^2} \right\} = - \Lambda_{(5)} + \kappa_{(5)}^2 P_T,$$ \hspace{1cm} (15)

where the subscripts $b$ denotes that these functions are evaluated at $y = 0$.

We have to take into account of the junction conditions \[14\] when we solve the Einstein equations on the brane. Because the first derivatives of the metric with respect to $y$ can be discontinuous at $y = 0$ while the metric is required to be continuous across the brane, the delta functions appear in the second derivatives of the metric. We define the jump factor $\sharp f \sharp$ and the mean value $\sharp f \sharp$ of a function $f$ by the following equations,

$$\sharp f \sharp \equiv f(t, 0+) - f(t, 0-),$$ \hspace{1cm} (16)

$$\sharp f \sharp = \frac{f(t, 0+) + f(t, 0-)}{2}.$$ \hspace{1cm} (17)

We obtain the junction conditions as follows,

$$\sharp a' \sharp \equiv \frac{\dot{a}_b}{a_b} \bigg|_{t}$$

$$\sharp n' \sharp \equiv \frac{n'_b}{a_b} \bigg|_{t}$$

Taking into account the $Z_2$ symmetry around $y = 0$, we obtain

$$a'(t, +0) = \frac{-\kappa_{(5)}^2}{6} (\rho_b + T_b),$$ \hspace{1cm} (20)

$$n'(t, +0) = \frac{\kappa_{(5)}^2}{6} (2\rho_b + 3P_b - T_b).$$ \hspace{1cm} (21)

Taking the jump of (15), we also obtain a relation between mean values $\sharp a' \sharp$ and $\sharp n' \sharp$,

$$\frac{\sharp a' \sharp}{a_b} (P_b - T_b) = \frac{1}{3} (\rho_b + T_b) \frac{\sharp n' \sharp}{n_b}.$$ \hspace{1cm} (22)
Substituting (20) and (21) in (15), we obtain the effective Friedman equation in the limit $y \to +0$,

$$\frac{\ddot{a}_b}{a_b} + \left(\frac{\dot{a}_b}{a_b}\right)^2 + \frac{K}{a_b^2} = -\frac{\kappa_{(5)}^4}{36} (1 + 3w_b) \rho_b^2 + \frac{\kappa_{(5)}^4}{36} (1 - 3w_b) T_b \rho_b + \frac{1}{3} \left( \Lambda_{(5)} + \frac{\kappa_{(5)}^4}{6} T_b^2 \right) - \frac{\kappa_{(5)}^2}{3} P_T. \quad (23)$$

In the above expression, we used the equation of state for the brane: $P_b = w_b \rho_b$. Note that the 55-component of the bulk energy-momentum tensor $P_T$ and the quadratic term of the brane energy density $\rho_b$ appear on the right hand side of this equation. They will affect the cosmological evolution of the 3-brane.

The equations of Energy-momentum conservation are,

$$\nabla A T^A_B = \partial_A T^A_B + \Gamma^A_{DA} T^D_B - \Gamma^D_{BA} T^A_D = 0. \quad (24)$$

The 0-component and the 5-component of the above equations are

$$\dot{\rho}_B + \frac{n^2}{b^2} Q' + 3(P_B + \rho_B) \frac{\dot{a}}{a} + 3Q \frac{n^2}{b^2} \left( \frac{n'}{n} + \frac{a'}{a} \right) - Q \frac{n^2 b'}{b^3}$$

$$+ (P_T + \rho_B) \frac{\dot{b}}{b} + \delta(y) \left\{ \dot{\rho}_b + 3(P_b + \rho_b) \frac{\dot{a}}{a} + \rho_b \frac{\dot{b}}{b} \right\} = 0, \quad (25)$$

$$\dot{Q} + P_T' + Q \left( \frac{\dot{n}}{n} + 3 \frac{\dot{a}}{a} + \frac{\dot{b}}{b} \right) + (P_T + \rho_B) \frac{n'}{n} + 3(P_T - P_B) \frac{a'}{a} = 0. \quad (26)$$

Integrating (25) around $y = 0$ and using the $Z_2$ symmetry, we obtain the conservation of energy-momentum on the brane,

$$\dot{\rho}_b + 3(1 + w_b) \rho_b \frac{\dot{a}_b}{a_b} + 2Q(t) = 0, \quad (27)$$

where we used two assumptions $b(t, y) = 1$ and $n(t, 0) = 1$. Using (20) and (21), we obtain the conservation of energy-momentum in the limit $y \to +0$,

$$\dot{\rho}_B + Q' + 3(P_B + \rho_B) \frac{\dot{a}_b}{a_b} + \frac{\kappa_{(5)}^2}{2} Q \left\{ (1 + 3w_b) \rho_b - 2T_b \right\} = 0, \quad (28)$$

$$\dot{Q} + P_T' + 3Q \frac{\dot{a}_b}{a_b} + \frac{\kappa_{(5)}^2}{6} (P_T + \rho_B) \left\{ (2 + 3w_b) \rho_b - T_b \right\} - \frac{\kappa_{(5)}^2}{2} (P_T - P_B) (\rho_b + T_b) = 0, \quad (29)$$

where we used the two assumptions for the scale factors of extra dimension and the temporal dimension. We obtained an effective Friedmann equation on the 3-brane, the evolution equations for $\rho_b$, $\rho_B$ and $Q$.

The time evolution of the bulk energy density $\rho_B$ and the flow of energy from/to the extra dimension $Q$ are governed by (28) and (29). The bulk pressure $P_T$ contributes
to the effective Friedmann equation on a brane through (23). However we cannot fully determine \( Q \) and \( P_T \) since there are unknown functions \( Q' \) and \( P'_T \) in (28) and (29). We put two ansätze for \( Q \) and \( P_T \) as follows,

\[
Q = F \left( \frac{\dot{a}_b(t)}{a_b(t)} \right) a_b(t)^\mu, \quad P_T = Da_b(t)^\nu,
\]

where \( D, F, \mu \) and \( \nu \) are some constants. A justification of these ansätze is found in [17].

3 Hubble equation and effective equation of state

To transform the second order equation (23) to the first order equation, we introduce a new variable \( \chi(t) \) which is called dark energy variable [17],

\[
\chi \equiv \left( \frac{\dot{a}_b}{a_b} \right)^2 + \frac{K}{a_b^2} - 2\gamma \rho_b - \beta \rho_b^2 - \frac{\lambda}{2} + \frac{\kappa^2}{6} P_T,
\]

where we redefined some constants

\[
\beta \equiv \frac{\kappa^4}{36}, \quad \gamma \equiv \frac{\kappa^4}{36} T_b, \quad \lambda \equiv \frac{1}{3} \left( \Lambda_T + \frac{\kappa^4}{6} T_b^2 \right).
\]

Then (23) is rewritten as a pair of the first order equations,

\[
\left( \frac{\dot{a}_b}{a_b} \right)^2 + \frac{K}{a_b^2} = 2\gamma \rho_b + \beta \rho_b^2 + \chi + \frac{\lambda}{2} - \frac{\kappa^2}{6} P_T,
\]

\[
\dot{\chi} = -4 \frac{\dot{a}_b}{a_b} \chi + 4Q(\chi + \beta \rho_b) + \frac{\kappa^2}{6} \dot{P}_T.
\]

The equation (33) is analogous to the Hubble equation of standard four dimensional cosmology. However the quadratic term of brane energy density \( \rho_b \), the bulk pressure and the brane tension appear in the right hand side. The dark energy variable \( \chi(t) \) accounts for the non-standard contributions to the Friedmann equation (33). The evolution of \( \rho_b \) is determined by (27) and it is solved as

\[
\rho_b = - \left( \frac{a_b(0)}{a_b(t)} \right)^{3(1+w_b)} \int 2Q(t) \left( \frac{a_b(t)}{a_b(0)} \right)^{3(1+w_b)} dt.
\]

We treat \( w_b \) as a constant in the above expressions. Now we perform the integration of (34). With the ansätze (30), \( \rho_b \) is expressed as,

\[
\rho_b = \tilde{C}a_b(t)^{-3(1+w_b)} - \frac{2F}{3(1+w_b) + \mu} a_b(t)^\mu,
\]

7
where $\tilde{C}$ is some integration constant. With the ansätze (30), we can perform the integration of (34) and we obtain

$$\chi = 4F\gamma a_b^\mu - \frac{4F\beta}{\mu - 3w_b + 1}a_b^{\mu-3(1+w_b)} - \frac{4F^2\beta}{(3(1+w_b) + \mu)(2 + \mu)}a_b^{2\mu}$$

$$+ \frac{\kappa^2(5)D}{6(4 + \nu)}a_b^\nu + \frac{\mathcal{C}}{a_b^2},$$  

(37)

where $\mathcal{C}$ is some integration constant. Substituting (36) and (37) into (33), we obtain the Hubble equation on the brane,

$$\left(\frac{\dot{a}_b}{a_b}\right)^2 = \frac{\lambda}{2} - \frac{K}{a_b^2} + \frac{\mathcal{C}}{a_b^2} - \frac{2\kappa^2(5)D}{3(4 + \nu)}a_b^\nu - \frac{2\gamma\tilde{C}}{a_b^{3(1+w_b)}} + \frac{\beta\tilde{C}^2}{a_b^{6(1+w_b)}} - \frac{4F\gamma(1-3w_b)}{(4+\mu)(3(1+w_b) + \mu)}a_b^\mu - \frac{4F^2\beta(1+3w_b)}{(3(1+w_b) + \mu)^2(2 + \mu)}a_b^{2\mu}$$

$$+ \frac{8F\beta\tilde{C}(1+3w_b)}{(1-3w_b + \mu)(3(1+w_b) + \mu)}a_b^{\mu-3(1+3w_b)}.$$  

(38)

This equation is quite different from the Hubble equation of a standard four dimensional FLRW cosmology:

- The brane energy density appears in a linear and quadratic form, whereas it appears in a linear form in the standard four dimensional FLRW cosmology.

- There is a bulk pressure term $Da_b^\nu$ and an energy exchange term $Fa_b^\mu$ between the four dimensional spacetime and the extra dimension.

- There is a bulk radiation term $\mathcal{C}/a_b^4$.

We can write the Hubble equation (38) in the conventional form,

$$\left(\frac{\dot{a}_b}{a_b}\right)^2 = -\frac{K}{a_b^2} + \frac{\Lambda_{(4)}}{3} + \frac{8\pi G_{(4)}}{3}\rho_{eff},$$

(39)

$$\rho_{eff} \equiv \rho_b + \frac{\beta}{2\gamma}\rho_b^2 + \frac{X}{2\gamma} - \frac{\kappa^2(5)T_b}{12\gamma}P_T,$$

(40)

where $\rho_{eff}$ is the effective energy density. The four dimensional Newton constant $G_{(4)}$ and the four dimensional cosmological constant $\Lambda_{(4)}$ are defined as

$$G_{(4)} \equiv \frac{3\gamma}{4\pi} = \frac{4\pi G_{(5)}^2 T_b}{3},$$

(41)

$$\Lambda_{(4)} \equiv \frac{3}{2}\lambda = \frac{1}{2}\left(\Lambda_{(5)} + \frac{\kappa^4(5)T_b^2}{6}\right).$$

(42)

Deceleration parameter $q$ is defined by

$$q \equiv -\left(\frac{a_b}{\dot{a}_b}\right)^2 \left(\frac{\ddot{a}_b}{\dot{a}_b}\right),$$

(43)

8
where the acceleration behavior is described by

\[
\frac{\ddot{a}_b}{a_b} = \frac{\lambda}{2} - \frac{C}{a_b^4} + \frac{(2 + \nu)\kappa_5^2 D}{3(4 + \nu)} \dot{a}_b^\nu - \frac{(1 + 3w_b)\gamma \tilde{C}}{a_b^{3(1+w_b)}} - \frac{(2 + 3w_b)\beta \tilde{C}^2}{a_b^{6(1+w_b)}}
+ \frac{2F\gamma(-2 - \mu + 6w_b + 3w_b\mu)}{(4 + \mu)(3(1 + w_b) + \mu)} a_b^\mu - \frac{4F^2\beta(1 + 3w_b)(1 + \mu)}{(3(1 + w_b) + \mu)^2(2 + \mu)} a_b^{2\mu}
+ \frac{4F\tilde{C}\beta(1 + 3w_b)(1 + 3w_b - \mu)}{(3(1 + w_b) + \mu)(1 - 3w_b + \mu)} a_b^{\mu - 3(1 + w_b)}. \tag{44}
\]

We assume that \( K = 0, \Lambda (4) = 0 \) and the matter on the brane is all ordinary non-relativistic matter which is taken to be \( w_b = 0 \), and we obtain

\[
H(t)^2 = \left( \frac{\dot{a}_b}{a_b} \right)^2 = \frac{C}{a_b^4} - \frac{2\kappa_5^2 D}{3(4 + \nu)} a_b^\nu + \frac{2\gamma \tilde{C}}{a_b^3} + \frac{2\beta \tilde{C}^2}{a_b^6} - \frac{4F\gamma}{(4 + \mu)(3 + \mu)} a_b^\mu
- \frac{4F^2\beta}{(3 + \mu)^2(2 + \mu)} a_b^{2\mu} + \frac{8F\beta \tilde{C}}{(1 + \mu)(3 + \mu)} a_b^{\mu - 3}. \tag{45}
\]

In the above equation, \( H(t) \) gives the expansion rate of the four dimensional universe. Within the flat universe, in the presence of dark energy, the expansion rate is given as

\[
\frac{H^2(t)}{H_0^2} = \frac{\Omega_m}{a_b^3} + \frac{1 - \Omega_m}{a_b^{3(1+w)}}. \tag{46}
\]

where \( H_0^2 \equiv H^2(0), \Omega_m \) is the dimensionless matter density and \( w \) is the parameter of equation of state of the dark energy. The dimensionless dark energy density is given by \( 1 - \Omega_m \). The second term in the right hand side of (46) describes the contribution of the dark energy to the expansion rate of our universe. Following Linder et al. [19], we modify this equation as

\[
\frac{\delta H^2}{H_0^2} \equiv \frac{H^2(t)}{H_0^2} - \frac{\Omega_m}{a_b^3}, \tag{47}
\]

\[
w_{eff} \equiv -1 - \frac{1}{3} \frac{d \ln(\delta H^2)}{d \ln a_b}, \tag{48}
\]

where \( \delta H^2/H_0^2 \) accounts for any modification to the usual Hubble equation in the four dimensional spacetime and \( w_{eff} \) is the effective equation of state parameter. In our case, \( w_{eff} \) is given as

\[
w_{eff}(z) = -1 - \frac{1}{3} \left( -4C(z + 1)^4 - \frac{2\nu_5^2 D}{3(4 + \nu)}(z + 1)^{-\nu} - 6\beta \tilde{C}^2(z + 1)^6 \right.
- \frac{4\gamma \mu F}{(4 + \mu)(3 + \mu)}(z + 1)^{-\mu} - \frac{8\beta \mu F^2}{(3 + \mu)^2(2 + \mu)}(z + 1)^{-2\mu}
+ \left. \frac{8\beta(\mu - 3)\tilde{C} F}{(1 + \mu)(3 + \mu)}(z + 1)^{3-\mu} \right) \]

\]
\[
\left( C(z+1)^4 - \frac{2\kappa^2_{(5)}D}{3(4+\nu)}(z+1)^{-\nu} + \beta \tilde{C}^2(z+1)^6 
\right) \\
- \frac{4\gamma F}{(4+\mu)(3+\mu)}(z+1)^{-\mu} - \frac{4\beta F^2}{(3+\mu)^2(2+\mu)}(z+1)^{-2\mu} \\
+ \frac{8\beta \tilde{F}}{(1+\mu)(3+\mu)}(z+1)^{3-\mu},
\]

where we use the redshift parameter \( z \) defined as \( z(t) + 1 \equiv a(t_0)/a(t) \). Deceleration parameter is given by

\[
q(z) = -\left( -C(z+1)^4 + \frac{(2+\nu)\kappa^2_{(5)}D}{3(4+\nu)}(z+1)^{-\nu} - \gamma \tilde{C}(z+1)^3 - 2\beta \tilde{C}^2(z+1)^6 
\right) \\
- \frac{2F\gamma(2+\mu)}{(4+\mu)(3+\mu)}(z+1)^{-\mu} - \frac{4F^2\beta(1+\mu)}{(3+\mu)^2(2+\mu)}(z+1)^{-2\mu} \\
- \frac{4F\tilde{C}(1-\mu)}{(3+\mu)(1+\mu)}(z+1)^{3-\mu}. 
\]

\[
\left( C(z+1)^4 - \frac{2\kappa^2_{(5)}D}{3(4+\nu)}(z+1)^{-\nu} - \gamma \tilde{C}(z+1)^3 - 2\beta \tilde{C}^2(z+1)^6 
\right) \\
- \frac{4F\gamma}{(4+\mu)(3+\mu)}(z+1)^{-\mu} - \frac{4F^2\beta}{(3+\mu)^2(2+\mu)}(z+1)^{-2\mu} \\
+ \frac{8F\tilde{C}}{(1+\mu)(3+\mu)}(z+1)^{3-\mu}. 
\]

\section{Evolution of \( w_{\text{eff}} \) and the crossing \( w_{\text{eff}} = -1 \)}

We examine the evolution of the effective equation of state parameter \( w_{\text{eff}} \) and investigate under which condition it exhibits \( w_{\text{eff}} < -1 \). The cosmological data indicates that \( w_{\text{eff}} \sim -1.21 \) at \( z = 0 \) and it changes from \( w_{\text{eff}} < -1 \) to \( w_{\text{eff}} > -1 \) at \( z \sim 0.2 \) \cite{10}. We look for the set of parameters which exhibit such behavior with the assumptions that \( K = 0 \) and \( \lambda = 0 \). Each parameters must satisfy the following constraint which is obtained from (45),

\[
\frac{C}{H_0^2} - \frac{2\kappa^2_{(5)}D}{3(4+\nu)H_0^2} - \frac{4F\gamma}{(4+\mu)(3+\mu)H_0^2} \\
- \frac{4F^2\beta}{(3+\mu)^2(2+\mu)H_0^2} + \frac{8F\tilde{C}}{(1+\mu)(3+\mu)H_0^2} + \frac{2\gamma \tilde{C}}{H_0^2} + \frac{\beta \tilde{C}^2}{H_0^2} = 1. 
\]

In the above expression, \( 2\gamma \tilde{C}/H_0^2 \) corresponds to the dimensionless matter density and we assume that \( 2\gamma \tilde{C}/H_0^2 < 1 \). We consider two cases:
(i) there is energy exchange between the four dimensional universe and the fifth dimension,
(ii) there is no energy exchange between the four dimensional universe and the fifth dimension.

4.1 No quadratic brane energy density and no bulk radiation

We consider that there is energy exchange between the four dimensional universe and the fifth dimension. We assume that the brane energy density is much smaller than the brane tension: \( \rho_b \ll T_b \). In this case we can neglect the quadratic term in \( \rho_b \). We also assume that the bulk radiation is negligible: \( C = 0 \). We still assume that the brane matter is all ordinary matter \( (w_b = 0) \) and we use the ansatz \( (30) \). With these assumptions, we obtain the Hubble equation \( (38) \) and the acceleration behavior \( (44) \) as follows,

\[
\left( \frac{\dot{a}_b}{a_b} \right)^2 = -Aa_b^\nu - Ba_b^\mu + \frac{C}{a_b^3},
\]

\[
\frac{\ddot{a}_b}{a_b} = \frac{(2 + \nu)A}{2}a_b^\nu - \frac{(2 + \mu)B}{2}a_b^\mu - \frac{C}{2a_b^3},
\]

where we used the notations

\[
A \equiv \frac{2\kappa^2_5 w_B C_B}{3(4 + \nu)} , \quad B \equiv \frac{4F_1}{(4 + \nu)(3 + \mu)} , \quad C \equiv 2\gamma \tilde{C}.
\]

The parameter \( A \) corresponds to the contribution to Hubble parameter from the bulk matter, \( B \) corresponds to the contribution from the the energy exchange between the extra dimension and \( C \) corresponds to the contribution from the ordinary matter on the brane. The Hubble equation \( (52) \) is expressed as,

\[
-A - B + C = H_0^2.
\]

The first two terms correspond to effective dark energy density.

The deceleration parameter \( q \), the effective equation of state parameter \( w_{eff} \) and its present value are given by

\[
q = \frac{(2 + \nu)A}{(z + 1)^\nu} - \frac{(2 + \mu)B}{(z + 1)^\mu} - \frac{C}{2} (z + 1)^3.
\]

\[
w_{eff} = -1 - \frac{1}{3} \left( \frac{\nu A}{(z + 1)^\nu} + \frac{\mu B}{(z + 1)^\mu} \right),
\]

\[
w_{eff}(0) = -1 - \frac{1}{3} \left( \frac{\nu A + \mu B}{A + B} \right).
\]
We look for the set of parameters which satisfy $w_{eff} < 0$ for $0 < z < 0.2$ and $w_{eff} > 0$ for $z < 0.2$. The denominators of (57) and (58) correspond to the opposite sign of effective dark energy density and it must be negative. We can achieve $w_{eff} < -1$ when their numerators are negative. Then we obtain two conditions,

\[ A\nu(z+1)^{-\nu} + B\mu(z+1)^{-\mu} < 0, \]

\[ A(z+1)^{-\nu} + B(z+1)^{-\mu} < 0. \]

(59) (60)

for $z < 0.2$. We can achieve $w_{eff}(0) = -1.21$ when

\[ A(\nu - 0.63) + B(\mu - 0.63) = 0. \]

(61)

There is another constraint which comes from $w_{eff}(0.2) = -1$,

\[ \frac{\nu}{\mu} = -\frac{B}{A(1.2)^{\nu-\mu}}. \]

(62)

This equation determines the relative sign of $\mu$ and $\nu$. The allowed choices of parameters are found in [18] and they are:

(i) $A, B < 0$ when $\mu > 0, \nu < 0$,

(ii) $A, B < 0$ when $\mu < 0, \nu > 0$.

We look for the set of parameters which satisfy (61) and (62) under the constraint of (55). When we assume $\nu = -2$ and $C = 0.04$, we obtain $A = -0.30$, $B = -0.66$ and $\mu = 1.82$. Figure 1 show the behaviors of $w_{eff}(z)$ and $q(z)$ in this case. When we assume $\nu = -1$ and $C = 0.04$, we obtain $A = -0.57$, $B = -0.39$ and $\mu = 2.98$. Figure 2 show the behaviors of $w_{eff}(z)$ and $q(z)$ in this case. In each cases, $w_{eff}(z)$ and $q(z)$ increase with $z$ and $q(z)$ become positive $z \simeq 1.4$ in figure 1 and $z \simeq 2.8$ in figure 2. The behavior of $w_{eff}(z)$ and $q(z)$ with the parameters $(A, B, C, \mu, \nu) = (-1, -2, -2, 2, -2)$ is found in [18].

We consider the behavior of energy density components. We write the constraint (52) as

\[ \Omega_A(z) + \Omega_B(z) + \Omega_C(z) = 1, \]

(63)

where

\[ \Omega_A(z) \equiv -\frac{A}{(z+1)^{\nu}H^2(z)}, \quad \Omega_B(z) \equiv -\frac{B}{(z+1)^{\mu}H^2(z)}, \quad \Omega_C(z) \equiv \frac{C(z+1)^3}{H^2(z)}. \]

(64)

$\Omega_A$ corresponds to the contribution from the bulk matter, $\Omega_B$ corresponds to the contribution from the energy exchange between the extra dimension and $\Omega_C$ corresponds to the contribution from the brane matter. The behavior of each energy density components are shown in figures 3 and 4. In figure 3, $\Omega_B$ is dominant in small $z$ and $\Omega_A$ is dominant in large $z$. In figure 4, $\Omega_A$ is dominant even in small $z$. The linear contributions from the brane matter $\Omega_C$ are smaller than the other part, but they increase with $z$. $\Omega_C$ become subdominant at $z \sim 0.8$ in figure 3 and $z \sim 0.4$ in figure 4.
4.2 No energy flow to/from the extra dimension

We assume that there is no energy flow to/from the extra dimension: $T^{0}_{5} = T^{5}_{0} = 0$. It corresponds to $F = 0$. We put $F = 0$ and $P_T = D\alpha_b^\nu$ in (36) and (37), we obtain

$$\rho_b = \frac{\tilde{C}}{a_b^{3(1+w_b)}},$$  

(65)

$$\chi = \frac{\mathcal{C}}{a_b^4} + \frac{\kappa_{(5)}^2}{6(4+\nu)} D\nu a_b^\nu,$$  

(66)

where $\mathcal{C}$ and $\tilde{C}$ are some constants. The equation for the bulk matter (28) is expressed as,

$$\dot{\rho}_B = -3(P_B + \rho_B) \frac{\dot{a}_b}{a_b}$$  

(67)
where we assumed \( P_B = w_B \rho_B \) for the bulk matter. It is solved as
\[
\rho_B = C_B a_0^3 (1 + w_B)^b,
\]
\[ (68) \]
where \( C_B \) is some integration constant. We obtain the relation
\[
P_T = P_B = \frac{w_B C_B a_0^3 (1 + w_B)^b}{a_0^3 (1 + w_B)^b},
\]
\[ (69) \]
when we assume that the pressure of bulk matter is isotropic. We read \( \nu = -3(1 + w_B) \) and \( D = w_B C_B \) from \( (30) \). Assuming \( w_B = 0 \), the Hubble equation and the acceleration behavior is expressed as follows,
\[
\left( \frac{\dot{a}_b}{a_b} \right)^2 = \frac{A}{a_b^4} - \frac{B}{a_b^3 (1 + w_B)} + \frac{C}{a_b^3} + \frac{D}{a_b^6},
\]
\[ (70) \]
\[
\frac{\dot{a}_b}{a_b} = -\frac{A}{a_b^4} - \frac{B(1 + 3w_B)}{2a_b^3 (1 + w_B)} - \frac{C}{2a_b^3} - \frac{2D}{a_b^6},
\]
\[ (71) \]
where we used the notation
\[
A \equiv C, \quad B \equiv \frac{2\kappa^2 w_B C_B}{3(1 - 3w_B)}, \quad C \equiv 2\gamma \tilde{C}, \quad D \equiv \beta \tilde{C}^2.
\]
\[ (72) \]
Each parameters must satisfy the constraint from \( (70) \),
\[
A - B + C + D = H_0^2
\]
\[ (73) \]
where \( C \) corresponds to the energy density of ordinary matter. The deceleration parameter \( q \), the effective equation of state parameter \( w_{\text{eff}} \) and its present value are given by
\[
q = \left( A(1 + z)^4 + \frac{(1 + 3w_B)B}{2}(1 + z)^3(1 + w_B) + \frac{C}{2}(1 + z)^3 + 2D(1 + z)^6 \right)
\]
\[
/ \left( A(1 + z)^4 - B(1 + z)^3(1 + w_B) + C(1 + z)^3 + D(1 + z)^6 \right),
\]
\[ (74) \]
\[
w_{\text{eff}} = -1 + \frac{1}{3} \left\{ 4A(z + 1)^4 - 3B(1 + w_B)(z + 1)^3(1 + w_B) + 6D(z + 1)^6 \right\}
\]
\[
/ \left\{ A(z + 1)^4 - B(z + 1)^3(1 + w_B) + D(z + 1)^6 \right\},
\]
\[ (75) \]
\[
w_{\text{eff}}(0) = -1 + \frac{1}{3} \left( \frac{4A - 3B(1 + w_B) + 6D}{A - B + D} \right).
\]
\[ (76) \]
We look for the set of parameters which satisfy \( w_{\text{eff}} < 0 \) for \( 0 < z < 0.2 \) and \( w_{\text{eff}} > 0 \) for \( z < 0.2 \). The denominators of \( (75) \) and \( (76) \) correspond to the effective dark energy
density and it must be positive. We can achieve \( w_{\text{eff}} < -1 \) when their numerators are negative. Then we obtain two conditions for \( z < 0.2 \),

\[
4A(z + 1)^4 - 3B(1 + w_B)(z + 1)^3(1 + w_B) + 6D(z + 1)^6 < 0, \quad (77)
\]

\[
A(z + 1)^4 - B(z + 1)^3(1 + w_B) + D(z + 1)^6 > 0. \quad (78)
\]

They gives the upper and the lower bound to achieve \( w_{\text{eff}}(0) = -1.21 \). We can achieve \( w_{\text{eff}}(0) = -1.21 \) when

\[
4.63A - 3B(1.21 + w_B) + 6.63D = 0. \quad (79)
\]

There is another constraint which comes from \( w_{\text{eff}}(0.2) = -1 \),

\[
4A(1.2)^4 - 3B(1 + w_B)(1.2)^3(1 + w_B) + 6D(1.2)^6 = 0. \quad (80)
\]

We look for the set of parameters which satisfy (79) and (80) under the condition of (73). When we assume \( w_B = -1/3 \) and \( C = 0.04 \), we obtain \( A = -3.19 \), \( B = -3.19 \) and \( D = 0.96 \). Figure 5 show the behaviors of \( w_{\text{eff}}(z) \) and \( q(z) \) in this case. When we assume \( w_B = -2/3 \) and \( C = 0.04 \), we obtain \( A = -1.39 \), \( B = -1.83 \) and \( D = 0.52 \). Figure 6 show the behaviors of \( w_{\text{eff}}(z) \) and \( q(z) \) in this case. In each cases, \( q(z) \) become positive \( z \approx 0.4 \) in figure 5, \( z \approx 0.2 \) in figure 6 and \( z \approx 0.02 \) in figure 7.

We consider the behavior of energy density components. We write the constraint (70) as

\[
\Omega_A(z) + \Omega_B(z) + \Omega_C(z) + \Omega_D(z) = 1, \quad (81)
\]

where

\[
\Omega_A(z) \equiv \frac{C(z + 1)^4}{H^2(z)}, \quad \Omega_B(z) \equiv -\frac{2\kappa_5^2 w_B C_B(z + 1)^3(1 + w_B)}{3(1 - 3w_B)H^2(z)},
\]

\[
\Omega_C(z) \equiv \frac{2\gamma \tilde{C}(1 + z)^3(1 + w_b)}{H^2(z)}, \quad \Omega_D(z) \equiv \frac{\beta \tilde{C}^2(1 + z)^6(1 + w_b)}{H^2(z)}. \quad (82)
\]

\( \Omega_A \) corresponds to the contribution from the bulk radiation, \( \Omega_B \) corresponds to the contribution from the bulk matter, \( \Omega_C \) corresponds to the linear contribution from the brane matter and \( \Omega_D \) corresponds to the quadratic contribution from the brane matter. The behavior of each energy density components are shown in figures 8-10.

There are negative contributions from the bulk radiation to the total energy density in each cases. The contributions from the bulk matter are dominant in late time and the quadratic contributions from the brane matter are dominant in early time. The linear contributions from the brane matter are much smaller than the other contributions through the time.
Figure 5: Graph of $w_{\text{eff}}(z)$ and $q(z)$ with parameters $w_B = -1/3$, $A = -3.19$, $B = -3.19$, $C = 0.04$, $D = 0.96$.

Figure 6: Graph of $w_{\text{eff}}(z)$ and $q(z)$ with parameters $w_B = -2/3$, $A = -1.39$, $B = -1.83$, $C = 0.04$, $D = 0.52$.

Figure 7: Graph of $w_{\text{eff}}(z)$ and $q(z)$ with parameters $w_B = -1$, $A = -0.49$, $B = -1.22$, $C = 0.04$, $D = 0.23$.

5 Summary

The cosmological observation indicates that the effective equation of state parameter $w_{\text{eff}}$ varies with $z$: $w_{\text{eff}} \sim -1.21$ at $z = 0$ and it crosses $w_{\text{eff}} = -1$ at $z \sim 0.2$. We investigated that under which condition this behavior occurs based on the five-dimensional braneworld scenario. The Hubble equation on the 3-brane is quite different from that of a standard four dimensional FLRW cosmology: (i) The brane energy density appears in a linear and quadratic form. (ii) A bulk pressure term, a bulk radiation term, and an energy exchange term between the four dimensional spacetime and the extra dimension appear in the Hubble equation. They contribute to the effective equation of state parameter.

We considered two cases: (i) There is energy exchange between the four dimensional universe and the fifth dimension. (ii) There is no energy exchange between the four dimensional universe and the fifth dimension. In both cases, we obtained that the crossing of $w_{\text{eff}} = -1$ line and the universe changes from deceleration to acceleration at lower
redshift. However, the curves of $w_{\text{eff}}$ are different between two cases. Although it remains negative value in the case (i), it reaches $w_{\text{eff}} > 1$ in the case (ii). The curves of $q$ and the energy density components are also different between two cases. Especially, there are negative contributions from the bulk radiation to the total energy density in the case (ii). In the case (i), the linear contributions from the brane matter is smaller than the other part, but they increase with $z$. In the case (ii), they remain much smaller than the other contributions, however the sum of the linear and quadratic contribution is dominant.

References

[1] A. G. Riess et al., Astron. J. 116 (1998) 1009, arXiv:astro-ph/9805201.

[2] S. Perlmutter et al., Astrophys. J. 517 (1999) 565, arXiv:astro-ph/9812133.
[3] G. F. Hinshaw et al., Astro. Phys. J. Suppl. 208 (2013) 19, arXiv:1212.5226 [astro-ph.CO].

[4] C. L. Bennett et al., Astro. Phys. J. Suppl. 208 (2013) 20, arXiv:1212.5225 [astro-ph.CO].

[5] R.R. Caldwell, Phys. Lett. B545 (2002) 23, arXiv:astro-ph/9908168.

[6] R. R. Caldwell, M. Kamionkowski, N. N. Weinberg, Phys. Rev. Lett. 91 (2003) 071301, arXiv:astro-ph/0302506.

[7] R.R. Caldwell, R. Dave and P. J. Steinhardt, Phys. Rev. Lett. 80 (1998) 1582, arXiv:astro-ph/9708069.

[8] I. Zlatev, L. Wang and P. Steinhardt, Phys. Rev. Lett. 82 (1999) 896, arXiv:astro-ph/9807002.

[9] P. J. Steinhardt, L. Wang, I. Zlatev, Phys. Rev. D59 (1999) 123504, arXiv:astro-ph/9812313.

[10] U. Alam, V. Sahni, T. Saini and A. Starobinsky, Mon. Not. Roy. Astron. Soc. 354 (2004) 275, arXiv:astro-ph/0311364.

[11] N. Arkani-Hamid, S. Dimopoulos and G. R. Dvali, Phys. Lett. B429 (1998) 263, arXiv:hep-ph/9803315.

[12] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 3370, arXiv:hep-ph/9905221.

[13] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 4690, arXiv:hep-th/9906064.

[14] P. Binétruy, C. Deffayet and D. Langolis, Nucl. Phys. B565 (2000) 269, arXiv:hep-th/9905012.

[15] P. Binétruy, C. Deffayet, U. Ellwanger and D. Langolis, Phys. Lett. B477 (2000) 285, arXiv:hep-th/9910219.

[16] R. G. Cai, Y. Gong and B. Wang, JCAP 0603 (2006) 006, arXiv:hep-th/0511301.

[17] C. Bogdanos and K. Tamvakis, Phys. Lett. B646 (2007) 39, arXiv:hep-th/0609100.

[18] C. Bogdanos, A. Dimitriadis and K. Tamvakis, Phys. Rev. D75 (2007) 087303, arXiv:hep-th/0611094.

[19] E. V. Linder and A. Jenkins, Mon. Not. Roy. Astron. Soc. 346 (2003) 573, arXiv:astro-ph/0305286.