Preorders on Subharmonic Functions and Measures
with Applications to the Distribution of Zeros
of Holomorphic Functions

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Abstract—Let $X$ be a class of extended numerical functions on a domain $D$ of $d$-dimensional Euclidean space $\mathbb{R}^d$, $H \subset X$. Given $u \in X$ and $M \in X$, we write $u \prec_H M$ if there is a function $h \in H$ such that $u + h \leq M$ on $D$. We consider this special preorder $\prec_H$ for a pair of subharmonic functions $u$ and $M$ on $D$ in cases when $H$ is the space of all harmonic functions on $D$ or $H$ is the convex cone of all subharmonic functions $h \neq -\infty$ on $D$. Main results are dual equivalent forms for this preorder $\prec_H$ in terms of balayage processes for Riesz measures of subharmonic functions $u$ and $M$, for Jensen and Arens–Singer (representing) measures, for potentials of these measures, and for special test functions generated by subharmonic functions on complements $D \setminus S$ of non-empty precompact subsets $S \subset D$. Applications to holomorphic functions $f$ on a domain $D \subset \mathbb{C}^n$ relate to the distribution of zero sets of functions $f$ under upper restrictions $|f| \leq \exp M$ on $D$. If a domain $D \subset \mathbb{C}$ is a finitely connected domain with non-empty exterior or a simply connected domain with two different points on the boundary of $D$, then our conditions for the distribution of zeros of $f \neq 0$ with $|f| \leq \exp M$ on $D$ are both necessary and sufficient.

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1. INTRODUCTION

Let $(X, +, 0, \leq)$ be a preordered monoid equipped with a binary operation–addition $+$ with a neutral element $0$ and a reflexive and transitive preorder $\leq$, and let $H \subset X$ be a preordered submonoid in $X$, $0 \in H$. We can define a special preorder $\prec_H$ on $X$:

$$x' \prec_H x \quad \text{if there exists } h \in H \text{ such that } x' + h \leq x.$$  \hfill (1.1)

This preorder $\prec_H$ is weaker than the original preorder $\leq$. In [10, Ch. VI, 8], [16, § 4.1], another specific order $\preceq_H$ of M. Brelot for some ordered groups $(X, +, 0, \leq)$ with

$$H \subset X^+ := \{x \in X : 0 \leq x\}$$

was considered: $x' \preceq x$ if there exists $h \in H$ such that $x' + h = x$. This specific order $\preceq_H$ is stronger than the original order $\leq$. Generally speaking, our special preorder $\prec_H$ is even strictly weaker than the original order $\leq$ and the specific Brelot order $\preceq_H$. In this article, we study only “subharmonic” and “harmonic” versions of preorder $\prec_H$.

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Throughout this article, $d \in \mathbb{N} := \{1, 2, \ldots \}$, $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$, $\mathbb{R}$ is the real line, $\mathbb{R}^d$ is the $d$-dimensional Euclidean space $\mathbb{R}^d$ with the Euclidean norm $|x| := \sqrt{x_1^2 + \cdots + x_d^2}$ of $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and the distance function $\text{dist}(\cdot, \cdot); O \subset \mathbb{R}^d$ is a non-empty open set, and a fixed point $o \in O$ is often considered as an origin point of $O$, unless they are a notation for big-$O(\cdot)$ or little-$o(\cdot)$.

$L_1 \subseteq L_1^1(\Omega) := \{u \in \text{sbh}(\Omega): u \neq -\infty\} \subset L_1^1(\Omega)$, where $\text{sbh}(\Omega)$ is the convex cone over $\mathbb{R}$ of all subharmonic (convex for $d = 1$) functions on $\Omega$ (the subharmonic version), or $H := \text{har}(\Omega) := \text{sbh}(\Omega) \cap (-\text{sbh}(\Omega))$ is the vector space over $\mathbb{R}$ of all harmonic (affine for $d = 1$) functions on $\Omega$ (the harmonic version). If $u \leq M$ on $\Omega$ for $u \in \text{sbh}(\Omega)$ and $M \in \text{sbh}(\Omega)$, then $u(x) \leq M(x)$ at each $x \in \Omega$.

Our main goal is to obtain dual equivalent conditions for the preorder (1.1) in the case of an arbitrary pair of subharmonic functions $u, M \in \text{sbh}(\Omega)$ in the role of $x', x \in X$ in terms of their Riesz measures. Our dual descriptions use both a linear balayage and an affine balayage for Riesz measures, for Jensen measures and for their potentials in the subharmonic version $H := \text{sbh}(\Omega)$ (see Theorem 1 in Sec. 2), and for Riesz measures, for Arens–Singer measures, and for their potentials in the harmonic version $H := \text{har}(\Omega)$ (see Theorem 2 in Sec. 3). Our interest in the preorder $\prec_H$ on subharmonic functions is largely motivated by its applications to the non-triviality of weighted spaces of holomorphic functions [29], [30], [32], [34], [41], to the description of zero sets for functions of this spaces [22]-[24], [50, 111], [26], [32], [88, 111], [34], [36], [39], [5], [41]-[44, 9.2, 9.3], [46], [52], [58], [59], to the approximation by exponential and similar systems in function spaces [25], [28], [39], [45], to the representation of meromorphic functions [26], [27], [33], [34], [36], [37], [38], [44, 9.4], [52], etc.

We denote by $Y^X$ the set of all functions $f : X \to Y$, where values $f(x)$ is not necessarily defined for all $x \in X$, and the domain of definition of function $f$ is a set

$$\text{dom } f := \{x \in X: \text{there is } y \in Y \text{ such that } f(x) = y\} = f^{-1}(Y) \subset X.$$  

**Definition 1** [see [44], [58], [59], [41]; cf. [6], [53], [56, Ch. 9], [16, Ch. 7], [10], [17], [21], [31, §4], [32, §7]]. Let $(\cdot, \cdot) \in \mathbb{R}^{X \times Y}$, i.e., \(\text{dom } (\cdot, \cdot) \ni (x, y) \mapsto (x, y) \in \mathbb{R}\).

Let $x', x \in X$, $x' \in Y \subset Y'$, and $(x', x) \times Y' \subset \text{dom } (\cdot, \cdot)$. We write $x' \preceq_r y'$ and say that $x$ is an affine $Y'$-balayage of $x'$, if there is a constant $c \in \mathbb{R}$ such that $(x', y) \leq (x, y) + c$ for each $y \in Y'$, but if we can choose this constant $c := 0$, then $x$ is a linear $Y'$-balayage of $x'$ and we write $x' \preceq_r y'$ in this case.

Let $y' \in Y$, $y \in Y$, $X' \subset X$, and $X' \times \{y', y\} \subset \text{dom } (\cdot, \cdot)$. We write $y' \preceq x'$ and say that $y$ is an affine $X'$-balayage of $y'$, if there is a constant $c \in \mathbb{R}$ such that $(x, y') \leq (x, y) + c$ for each $x \in X'$, but if we can choose this constant $c := 0$, then $y$ is a linear $X'$-balayage of $y'$ and we write $y' \preceq x'$ in this case.

Let $\mathbb{R}^d_\infty := \mathbb{R}^d \cup \{\infty\}$ is the Alexandroff one-point compactification of $\mathbb{R}^d$, and let $Q$ be a subset in $\mathbb{R}^d_\infty$. We denote by $B(Q)$ the class of all Borel subsets in $Q$. The closure $\text{clo} Q$, the interior $\text{int} Q$ and the boundary $\partial Q$ of $Q$ will always be taken relative to $\mathbb{R}^d_\infty$. For $Q' \subset Q \subset \mathbb{R}^d_\infty$ we write $Q' \subset Q$ if $\text{clo} Q' \subset \text{int} Q$.

In our article, we use the linear and affine balayage only in the following variants:

$[X] X$ is a subset of one of the classes $\text{Meas}^+(Q)$ of all (Radon positive) measures $\mu$ on $Q \in B(\mathbb{R}^d_\infty)$ [9], [62, Appendix A], or $\text{Meas}^+(Q) := \text{Meas}^+(Q) - \text{Meas}^+(Q)$ of charges [53] or $\text{Meas}^\text{cmp}(Q)$ of charges $\mu \in \text{Meas}(Q)$ with a compact support $\text{supp} \mu \in Q$ or $\text{Meas}^\text{cmp}(Q) := \text{Meas}^\text{cmp}(Q) \cap \text{Meas}^+(Q)$ or $\text{Meas}^\text{cmp}(Q)$ of probability measures or $\text{Meas}^\text{cmp}(Q) := \text{Meas}^\text{cmp}(Q) \cap \text{Meas}^+(Q)$ or Jensen or Arens–Singer (representing) measures.
[Y] Y ⊂ sbh(Q), where sbh(Q) consists of the restrictions to Q of subharmonic functions on open sets O ⊃ Q; har(Q) := sbh(Q) ∩ (−sbh(Q)).

[[r, s]] the function ⟨r, s⟩ ∈ ℝX×Y from Definition 1 is defined as the Lebesgue integral with respect to a measure or charge:

\[ \langle \mu, v \rangle := \int v \, d\mu, \quad (\mu, v) \in X \times Y. \]

It is important the following. If we explicitly indicate a subset Q′ ⊂ Q such that measures from X′ or functions from Y′ are defined only on Q′, then we pass from measures μ ∈ X and functions v ∈ Y to their restrictions μ |Q′ and v |Q′ to Q′. So, if u ∈ sbh(Q) and M ∈ sbh(Q), but Q′ ⊂ Q and M(Q′) ⊂ Meas(Q′), then the relation \( u \preceq_{M(Q')} M \) means that there is a constant \( C \in \mathbb{R} \) such that

\[ \int_{Q'} u \, d\mu \leq \int_{Q'} M \, d\mu + C \quad \text{for each } \mu \in M(Q'). \quad (1.2) \]

Similarly, if θ ∈ Meas(Q) and μ ∈ Meas(Q), but Q′ ⊂ Q and S(Q′) ⊂ sbh(Q′), then the relation \( \theta \preceq_{S(Q')} \mu \) means that there is a constant \( C \in \mathbb{R} \) such that

\[ \int_{Q'} v \, d\theta \leq \int_{Q'} v \, d\mu + C \quad \text{for each } v \in S(Q'). \quad (1.3) \]

**Example 1** [[18, 3], [49], [1], [50], [14], [15], [11], [19], [22], [23], [32], [35], [29], [41], [45], [5], [57]]. We denote by \( \delta_o \in \text{Meas}^{1+}(O) \) the Dirac measure with sup\( \delta_o = \{o\} \). If \( \mu \in \text{Meas}_o^{1+}(O) \) is a linear sbh(O)-balayage of \( \delta_o \), then \( \mu \) is called a Jensen measure for \( o \). The convex set of all these Jensen measures is denoted by \( J_o(O) \subset \text{Meas}^{1+}(O) \). If \( D \subset O \) is a domain with non-polar boundary \( \partial D \), \( o \in D \), and a function \( \omega_D : D \times B(\partial D) \to [0, 1] \) is the Chebyshev measure for \( D \) [62, 4.3], [20], then the harmonic measure \( \omega_D(o, \cdot) \) at \( o \) belongs to \( J_o(D) \cap \text{Meas}^{1+}(\partial D) \). If \( b_d \) is the measure of the unit ball \( B \subset \mathbb{R}^d \) and \( s_{d-1} \) is the area of the unit sphere \( \partial B \subset \mathbb{R}^d, s_0 := 2 \), then the restriction of \( \frac{1}{bd} \lambda_d \) to \( B \) and the measure \( \frac{1}{s_{d-1}} \sigma_{d-1} \), where \( \sigma_{d-1} \) is the usual measure of area on the unit sphere \( \partial B \), belong to \( J_0(rB) \) if \( r > 1 \).

**Example 2** [[18, 3], [65], [26], [35], [32], [37], [52]]. If \( \mu \in \text{Meas}_o^{1+}(O) \) is a linear sah(O)-balayage of \( \delta_o \), then \( \mu \) is called an Arens–Singer measure for \( o \in O \). The convex set of all these Arens–Singer measures is denoted by \( \text{AS}_o(O) \supset J_o(O) \). Arens–Singer measures are often referred to as representing measures.

**2. MAIN RESULT FOR THE SUBHARMONIC VERSION**

Given \( f \in F \subset \overline{\mathbb{R}}^X \), we set

\[ f^+ : x \mapsto \max\{0, f(x)\} \quad \text{for each } x \in \text{dom } f, \quad F^+ := \{ f \in F : f = f^+ \}; \]

\[ F^\updownarrow := \big\{ f \in \overline{\mathbb{R}}^X : \text{there is an increasing sequence } (f_j)_{j \in \mathbb{N}}, f_j \in F, \]

\[ \text{dom } f_j = X, \text{ such that } f(x) = \lim_{j \to \infty} f_j(x) \text{ for each } x \in X \quad \text{(we write } f_j \uparrow f \text{).} \]

**Proposition 1.** Let \( F \subset \overline{\mathbb{R}}^X \) be closed relative to the max-operation. If \( f_{kj} \in F, k \in \mathbb{N}, j \in \mathbb{N}, \text{ dom } f_{kj} = X, \text{ and } f_{kj} \uparrow f_k \uparrow f, \text{ then } F \ni \max_{k,j \in \mathbb{N}} f_{kj} \uparrow f, (F^\updownarrow)^\updownarrow = F^\updownarrow. \)

Everywhere below, if some proposition is not proved or not commented, then this proposition is obvious and formulated only for the convenience of references to it.

Throughout this article, \( D \) is a non-empty domain in \( \mathbb{R}^d \), and a point \( o \in D \) or, more generally, a non-empty subset \( S_o \subset D, S_o \in \mathcal{B}(D) \), will play a role of an origin for \( D \). The theory of distributions or generalized functions uses test finite positive functions to define the usual order relation on distributions or measures/charges. We consider various classes of test functions generated by subharmonic functions.
near the boundary $\partial D$ of $D$ or on $D \setminus S_o$ to study the order relation $\prec_H$ on subharmonic functions [45], [46], [47], [43, 21], [40], [58], [41, Theorem 1], [59, Definition 1].

So, we use the class $G_o(D \setminus S_o) \subset J_P(D)$ of all extended Green’s functions $g_D(\cdot, o)$ together with the class $\Omega_o(D \setminus S_o) \subset J_o(D)$ of all harmonic measures $\omega_D(o, \cdot)$, whose domains $D' \subset \mathbb{R}^d$ run through all regular (for the Dirichlet problem) domains such that $S_o \subset D' \subset D$. We also use various wider classes of test subharmonic functions on $D \setminus S_o$ and their extensions. We define subclasses $\text{sbh}_e(D \setminus S_o)$ of functions that \textit{vanish on the boundary} $\partial D \subset \mathbb{R}^d$:

$$\text{sbh}_0(D \setminus S_o) := \left\{ v \in \text{sbh}(D \setminus S_o) : \lim_{D \ni x' \to x} v(x') = 0 \text{ for each } x \in \partial D \right\},$$

(2.1)

and \textit{vanish near the boundary} $\partial D$:

$$\text{sbh}_{00}(D \setminus S_o) := \left\{ v \in \text{sbh}(D \setminus S_o) : v \equiv 0 \text{ on } D \setminus S(v) \text{ for some } S(v) \subset D \right\}.$$  \hspace{1cm} (2.2)

Under notations

$$\text{sbh}^+(Q) := (\text{sbh}(Q))^+, \quad Q \subset \mathbb{R}^d,$$

(2.3)

and using (2.1), (2.2), we define classes of \textit{test functions}

$$\text{sbh}_0(D \setminus S_o; \leq b_+) := \text{sbh}_0(D \setminus S_o) \bigcap \text{sbh}(D \setminus S_o; \leq b_+),$$

(2.4)

$$\text{sbh}_0^+(D \setminus S_o; \leq b_+) := \left( \text{sbh}_0(D \setminus S_o; \leq b_+) \right)^+, \quad \text{sbh}_{00}^+(D \setminus S_o; \leq b_+) := \left( \text{sbh}_{00}(D \setminus S_o; \leq b_+) \right)^+, \quad \text{sbh}_{00}(D \setminus S_o; \leq b_+) := \text{sbh}_{00}(D \setminus S_o) \bigcap \text{sbh}(D \setminus S_o; \leq b_+).$$

A single-point set $\{o\}$ is denoted as $o$. For example, $O \setminus o := O \setminus \{o\}, \mathbb{R} \setminus 0 := \mathbb{R} \setminus \{0\}, o \cup Q := \{o\} \cup Q$ for $Q \subset \mathbb{R}^d$, etc.

\textbf{Definition 2} ([62], [20], [53]). For $q \in \mathbb{R}$ and $d \in \mathbb{N}$, we set

$$k_q(t) := \begin{cases} \ln t & \text{if } q = 0, \\ -\frac{q}{|q|} t^{-q} & \text{if } q \in \mathbb{R} \setminus \{0\}, \quad t \in \mathbb{R}^+ \setminus 0, \end{cases} \hspace{1cm} (2.5k)$$

$$K_{d-2}(x, y) := \begin{cases} k_{d-2}(|x - y|) & \text{if } x \neq y, \\ -\infty & \text{if } x = y \text{ and } d \geq 2, \\ 0 & \text{if } x = y \text{ and } d = 1, \end{cases} \hspace{1cm} (2.5K)$$

\textbf{Example 3} ([18, 3], [65], [26], [32], [35], [37, Definition 6], [52, §4]). A function $V \in \text{sbh}_{00}(O \setminus o)$ is called an \textit{Arens–Singer potential on $O$ with pole at $o \in O$} if

$$V(y) \leq -K_{d-2}(o, y) + O(1) \quad \text{for } o \neq y \to o.$$ \hspace{1cm} (2.6)

The class of all Arens–Singer potentials on $O$ with pole at $o \in O$ denote by \textit{ASP$(O \setminus o)$}. Besides, we use a subclass

$$\text{ASP}^1(O \setminus o) := \left\{ V \in \text{ASP}(O \setminus o) : V(y) = -K_{d-2}(o, y) + O(1) \text{ for } o \neq y \to o \right\}.$$ \hspace{1cm} (2.7)

\textbf{Example 4} ([18, 3], [1], [35], [57], [37, Definition 8], [50, III.C.], [39], [45], [5]). The class $J_P(O \setminus o) := (\text{ASP}(O \setminus o))^+$ is the class of \textit{Jensen potentials on $O$ with pole at $o \in O$}. Besides, we use a subclass
$JP^1(O\backslash o) := (ASP^1(O\backslash o))^+$. For $D \in O$, the extended Green’s function $g_D(\cdot, o)$ with pole at $o \in D$ \cite[5.7.2-4]{20}, \cite[Ch. 5.2]{21},

$$g_D(x, o) = \begin{cases} 
g_D(x, o) \text{ if } x \in D \backslash o \\
0 \text{ if } x \in \mathbb{R}^d_\infty \backslash \text{clos } D \\
\limsup_{D \ni x' \to x} g_D(x', o) \text{ if } x \in \partial D 
\end{cases} \in \text{sbh}^+(\mathbb{R}^d_\infty \backslash o), \quad (2.8)$$

belongs to $JP^1(O\backslash o)$.

Let $Q \subset \mathbb{R}^d_\infty$. The class $C^\infty(Q)$ consists of the restrictions to $Q$ of all infinitely differentiable function on open sets $O \subset \mathbb{R}^d_\infty$ that include $Q$, but $C(Q)$ is the vector space over $\mathbb{R}$ of all continuous functions on $Q$. For $Q \in \mathcal{B}(\mathbb{R}^d_\infty)$, the class $C^\infty(Q) d\lambda_d$ consists of all charges $\mu \in \text{Meas}(Q)$ with an infinitely differentiable density, i.e., $d\mu = g d\lambda_d$, where $g \in C^\infty(Q)$. The Riesz measure of $u \in \text{sbh}_+(O)$ is a measure

$$\Delta_u := c_d \Delta u \in \text{Meas}^+(O), \quad c_d := \frac{1}{s_{d-1} \max\{1, d-2\}}, \quad (2.9)$$

where $\Delta$ is the Laplace operator acting in the sense of the distribution theory, or the theory of generalized functions. If $-\infty \in \text{sbh}(O)$ is the minus-infinity function on $O$, then $\Delta_{-\infty}(S) := +\infty$ for each $S \subset O$.

**Theorem 1** (subharmonic version). Let $M \in \text{sbh}(D) \cap C(D)$ be a function with Riesz measure $\Delta_M$, let $u \in \text{sbh}_+(D)$ be a function with Riesz measure $\Delta_u$, and let the boundary $\partial D$ be non-polar. Then the following nine statements are equivalent:

1. $u \preceq_{\text{sbh}_+(D)} M$, i.e., there is $h \in \text{sbh}_+(D)$ such that $u + h \leq M$ on $D$.
2. $u \preceq_{J_o(D)} M$, i.e., $M$ is an affine $J_o(D)$-balayage of $u$ (see (1.2)).
3. There is a non-empty subset $S_o \subset D$ such that the function $M$ is an affine $\left(J_o(D) \cap \text{Meas}^+_{\text{cmp}}(D \backslash S_o) \cap (C^\infty(D) d\lambda_d)\right)$-balayage of $u$.
4. There is a subset $S_o \subset D$ such that $u \preceq_{\text{Meas}^+(D \backslash S_o)} M$.
5. There is a subset $S_o \subset D$ such that $\Delta_u \preceq_{G_o(D \backslash S_o)} \Delta_M$ (see (1.3)).
6. $\Delta_u \preceq_{Jp(D \backslash o)} \Delta_M$, i.e., $\Delta_M$ is an affine $Jp(D \backslash o)$-balayage of $\Delta_u$.
7. For each $S_o$ satisfying

$$o \in \text{int } S_o \subset S_o \subset D \subset \mathbb{R}^d, \text{ and } b_+ \in \mathbb{R}^+ \backslash 0,$$

$\Delta_M$ is an affine $\text{sbh}_0^+(D \backslash S_o) \leq b_+$-balayage of $\Delta_u$.
8. There are a subset $S_o \subset D$ and a number $b_+ > 0$ as in (2.10) such that $\Delta_M$ is an affine $\left(\text{sbh}_0^+(D \backslash S_o) \leq b_+) \cap C^\infty(D \backslash S_o)\right)$-balayage of $\Delta_u$.
9. There is a non-empty subdomain $D_o \subset D$ containing $o \in D_o$ such that $\Delta_M$ is an affine $\left(Jp^1(D \backslash o) \cap \text{har}(D \backslash o) \cap C^\infty(D \backslash o)\right)$-balayage of $\Delta_u$.

**Comments** (to Theorem 1). The equivalence $\{s1\} \Leftrightarrow \{s2\}$ is proved in \cite[Theorem 7.2]{32}. The inclusion $\Omega_o(D \backslash S_o) \subset J_o(D)$ gives the implications $\{s2\} \Rightarrow \{s4\}$, and the inclusion

$$J_o(D) \cap \text{Meas}^+_{\text{cmp}}(D \backslash S_o) \cap (C^\infty(D) d\lambda_d) \subset J_o(D)$$

gives the implication $\{s2\} \Rightarrow \{s3\}$. The implication $\{s4\} \Rightarrow \{s1\}$ for $d = 2$ is proved partially in \cite[Main Theorem]{37} and more generally in \cite[Theorem 4]{41}. The combination of these proofs is transferred almost verbatim to the cases $d = 1$ and $d > 2$. We omit this transfer. The equivalence $\{s4\} \Leftrightarrow \{s5\}$ follows easily from the classical Poisson–Jensen formula \cite[4.5]{62}, \cite[3.7]{20}. The implication $\{s6\} \Rightarrow \{s5\}$ is obvious, since $\mathcal{G}_o(D \backslash S_o) \subset Jp(D \backslash o)$. The implication $\{s2\} \Rightarrow \{s6\}$ is a special case of \cite[Theorem 6]{37}. The implication $\{s7\} \Rightarrow \{s8\}$ follows from inclusions (3.4) of Proposition 3 below. Thus, if we prove the
implication [s6]⇒[s7] (see Sec. 8) and the chain of implications [s8]⇒[s9]⇒[s3]⇒[s1] (see Sec. 10–11), then Theorem 1 will be completely proved.

Remark 1. Only the proof of implication [s8]⇒[s9] uses that the boundary ∂D is non-polar. Thus, all other implications explicitly written above in the Comments to Theorem 1 are true for any domain D ⊂ ℝ^d.

3. MAIN RESULT FOR THE HARMONIC VERSION

“Subharmonic” Theorem 1 has a “harmonic” counterpart. We need some additional definitions and notations. Denote by B(x, r) := \{x' ∈ ℝ^d : |x' - x| < r\} ⊂ ℝ^d an open ball centered at x ∈ ℝ^d with radius r ∈ ℝ^+, B := B(0, 1);

Q^{ur} := \bigcup \{B(x, r) : x \in Q\}

is the outer r-parallel open set for Q ⊂ ℝ^d [64, Ch. 1, §4].

Proposition 2. Let a subset S ⊂ ℝ^d be connected, and r ∈ ℝ^+\{0\}. Then S^{ur} is a domain. If r' ∈ ℝ^+ and r' > r, then there is a regular (for the Dirichlet problem) domain D'_r ⊂ ℝ^d [62, 4.1], [20] such that

S^{ur} ⊂ D'_r ⊂ S^{ur'}.

(3.1)

We supplement the classes (2.3) of subharmonic functions from Sec. 2 with subclasses that are positive near the boundary ∂D:

\[ \text{sbh}_+(D \setminus S_0) := \{v ∈ \text{sbh}(D \setminus S_0) : 0 ≤ v \text{ on } D \setminus S(v) \text{ for some } S(v) ∈ D\}, \]

\[ \text{sbh}_+(D \setminus S_0) := \text{sbh}_0(D \setminus S_0) \bigcap \text{sbh}_+(D \setminus S_0) \subseteq \text{sbh}_0(D \setminus S_0), \]

\[ \text{sbh}_+(D \setminus S_0; ≤ b_+) := \text{sbh}_+(D \setminus S_0) \bigcap \text{sbh}(D \setminus S_0; ≤ b_+). \]

(3.2)

We define the average of \( v ∈ L^1(∂B(x, r)) \) on a sphere ∂B(x, r) as

\[ v^{or}(x) := \frac{1}{s_{d-1}} \int_{∂B} v(x + rs) \, dσ_{d-1}(s). \]

For given constants

\[ -∞ < b_- < 0 < b_+ < +∞, \quad 0 < 4r < \text{dist}(S_0, ∂D), \]

(3.3)

using (2.4), we define the following classes of test functions with some restrictions from above and from below:

\[ \text{sbh}_+(D \setminus S_0; r, b_- < b_+) := \{v ∈ \text{sbh}_+(D \setminus S_0; ≤ b_+) : b_- ≤ v \text{ on } S_0^{(4r)} \setminus S_0\}, \]

\[ \text{sbh}_+(D \setminus S_0; r, b_- < b_+) := \left(\text{sbh}_+(D \setminus S_0; r, b_- < b_+)\right)^\uparrow, \]

\[ \text{sbh}_0(D \setminus S_0; r, b_- < b_+) := \{v ∈ \text{sbh}_0(D \setminus S_0; ≤ b_+) : b_- ≤ v \text{ on } S_0^{(4r)} \setminus S_0\}, \]

\[ \text{sbh}_+(D \setminus S_0; ⋃ r, b_- < b_+) := \{v ∈ \text{sbh}_+(D \setminus S_0; ≤ b_+) : b_- ≤ v^{or} \text{ on } S_0^{(3r)} \setminus S_0^{(2r)}\}, \]

\[ \text{sbh}_+(D \setminus S_0; ⋃ r, b_- < b_+) := \left(\text{sbh}_+(D \setminus S_0; r, b_- < b_+)\right)^\uparrow. \]

(3.4)

Proposition 3. We have the following inclusions:

\[ \text{sbh}_0(D \setminus S_0, ≤ b_+) ⊂ \text{sbh}_0(D \setminus S_0; r, b_- < b_+) \]

\[ \cap \]

\[ \text{sbh}_0(D \setminus S_0, ≤ b_+) ⊂ \text{sbh}_+(D \setminus S_0; r, b_- < b_+) \]

(3.4)

\[ \cap \]

\[ \text{sbh}_0(D \setminus S_0, ≤ b_+) ⊂ \text{sbh}_+(D \setminus S_0; ⋃ r, b_- < b_+) \]
In (3.4), generally speaking, all inclusions are strict.

**Proof.** Inclusions (3.4) immediately follow from definitions. Examples from [49, XI B2], [58, Example] show that all “horizontal” inclusions are strict. The first line of “vertical” inclusions is strict in an obvious way. The second line of “vertical” inclusions is strict in the case when there are irregular points on the boundary $\partial D$ of the domain $D$ [20, Lemma 5.6], since the limit values of the Green’s function $g_D$ at such points are not zero, even if these limit values exist [20, Theorem 5.19].

**Theorem 2** (harmonic version). If the conditions of Theorem 1 are fulfilled, then the following eight statements are equivalent:

- $[h1]$ $u \prec_{\text{har}(D)} M$, i.e., there is $h \in \text{har}(D)$ such that $u + h \leq M$ on $D$.
- $[h2]$ $u \preceq_{AS_o(D)} M$, i.e., $M$ is an affine $AS_o(D)$-balayage of $u$ (see (1.2)).
- $[h3]$ There is a non-empty set $S_0 \subseteq D$ such that the function $M$ is an affine \((AS_o(D) \cap \text{Meas}_{\text{cmp}}^+ (D \setminus S_0) \cap (C^\infty(D) \, d\lambda_d))\)-balayage of $u$.
- $[h4]$ $\Delta_u \preceq_{\text{ASP}(D \setminus o)} \Delta_M$, i.e., $\Delta_M$ is an affine $\text{ASP}(D \setminus o)$-balayage of $\Delta_u$ (see (3.3)).
- $[h5]$ For each connected set $S_0$ from (2.10) and for any constants $r, b_\pm$ from (3.3) there is a constant $C \in \mathbb{R}$ such that
\[
\int_{D \setminus S_0} v \, d\Delta_u \leq \int_{D \setminus S_0(r)} v \, d\Delta_M + C \quad \text{for each} \quad v \in \text{sbh}^1_{+0}(D \setminus S_o; r, b_-, b_+). \tag{3.5}
\]
- $[h6]$ For each connected set $S_0$ from (2.10) and for any constants $r, b_\pm$ from (3.3), $\Delta_M$ is an affine $\text{sbh}^1_{+0}(D \setminus S_o; r, b_-, b_+)$-balayage of $\Delta_u$.
- $[h7]$ There are connected set $S_0$ as in (2.10) and constants as in (3.3) such that $\Delta_M$ is an affine \((\text{sbh}_{+0}^1(D \setminus S_o; r, b_- < b_+) \cap C^\infty(D \setminus S_o))\)-balayage of $\Delta_u$.
- $[h8]$ There is a non-empty subdomain $D_o \subseteq D$ containing $o \in D_o$ such that $\Delta_M$ is an affine \((\text{ASP}^1(D \setminus o) \cap \text{har}(D \setminus o) \cap C^\infty(D \setminus o))\)-balayage of $\Delta_u$.

**Comments** (to Theorem 2). The implication $[h1] \Rightarrow [h2]$ is a very special case of [37, Proposition 3.5] or [44, Corollary 8.1-1]. The implication $[h2] \Rightarrow [h3]$ is obvious, since
\[AS_o(D) \cap \text{Meas}_{\text{cmp}}^+ (D \setminus S_o) \cap (C^\infty(D) \, d\lambda_d) \subset AS_o(D).\]

The implication $[h2] \Rightarrow [h4]$ is noted in [37, Theorem 6, (3.20)\(\Rightarrow\)(3.22)]. The implication $[h6] \Rightarrow [h7]$ follows from $\text{sbh}_{+0}^1(D \setminus S_o; r, b_- < b_+) \cap C^\infty(D \setminus S_o) \subset \text{sbh}_{+0}^1(D \setminus S_o; r, b_- < b_+)$. Thus, if we prove the implications $[h4] \Rightarrow [h5] \Rightarrow [h6]$ (see Sec. 8) and $[h7] \Rightarrow [h8] \Rightarrow [h3] \Rightarrow [h1]$ (see Sec. 10-11), then Theorem 2 will be completely proved.

**Remark 2.** Only the proof of implication $[h7] \Rightarrow [h8]$ uses that the boundary $\partial D$ is non-polar. Thus, all other implications explicitly written above in the Comments to Theorem 2 are true for any domain $D$.

### 4. GLUING THEOREMS

**Gluing Theorem 1** ([62, Theorem 2.4.5]. [48, Corollary 2.4.4]). Let $\mathcal{O}$ be an open set in $\mathbb{R}^d$, and let $\mathcal{O}_0$ be a subset of $\mathcal{O}$. If $u, u_0 \in \text{sbh}(\mathcal{O})$, and
\[
\limsup_{\mathcal{O}_0 \ni x' \to x} u_0(x') \leq u(x) \quad \text{for each} \quad x \in \mathcal{O} \cap \partial \mathcal{O}_0, \tag{4.1}
\]
then the formula
\[
U := \begin{cases} 
\max\{u, u_0\} & \text{on } \mathcal{O}_0, \\
u & \text{on } \mathcal{O} \setminus \mathcal{O}_0
\end{cases} \tag{4.2}
\]
defines a subharmonic function on $\mathcal{O}$.
Gluing Theorem 2. Let \( O, O_0 \) be a pair of open subsets in \( \mathbb{R}^d \), and \( v \in \text{sbh}(O) \), \( v_0 \in \text{sbh}(O_0) \) be a pair of functions such that

\[
\limsup_{x' \to x} v(x') \leq v_0(x) \quad \text{for each } x \in O_0 \cap \partial O, \quad (4.3.0)
\]

\[
\limsup_{x' \to x} v_0(x') \leq v(x) \quad \text{for each } x \in O \cap \partial O_0. \quad (4.3.1)
\]

Then the function

\[
V := \begin{cases} 
    v_0 & \text{on } O_0 \setminus O, \\
    \max\{v_0, v\} & \text{on } O \cap \partial O, \\
    v & \text{on } O \setminus O_0 
\end{cases} \quad (4.4)
\]

is subharmonic on \( O_0 \cup O \).

Proof. It is enough to apply the Gluing Theorem 1 twice:

0) to one pair of functions

\[
u := v_0 \in \text{sbh}(O), \quad \mathcal{O} := O;
\]

\[
u_0 := v \big|_{O \cap O_0} \in \text{sbh}(O \cap O_0), \quad \mathcal{O}_0 := O \cap O_0 \subset O_0,
\]

under condition (4.3.0), which corresponds to condition (4.1);

1) to another pair of functions

\[
u := v \in \text{sbh}(O), \quad \mathcal{O} := O;
\]

\[
u_0 := v_0 \big|_{O_0 \cap O} \in \text{sbh}(O_0 \cap O), \quad \mathcal{O}_0 := O_0 \cap O \subset O,
\]

under condition (4.3.1), which corresponds to condition (4.1).

These two glued subharmonic functions coincide at the open intersection \( O \cap O_0 \), and we obtain subharmonic function \( V \) on \( O_0 \cup O \) defined by (4.4).

Gluing Theorem 3 (quantitative version) Let \( O \) and \( O_0 \) be a pair of open subsets in \( \mathbb{R}^d \), and \( v \in \text{sbh}(O) \) and \( g \in \text{sbh}(O_0) \) be a pair of functions such that

\[
-\infty < m_v \leq \inf_{x \in O \cap \partial O_0} v(x), \quad (4.7m)
\]

\[
\sup_{x \in O_0 \cap \partial O} \limsup_{x' \to x} v(x') \leq M_v < +\infty, \quad (4.7M)
\]

\[
-\infty < \sup_{x \in O \cap \partial O_0} \limsup_{x' \to x} g(x') \leq m_g < M_g \leq \inf_{x \in O_0 \cap \partial O} g(x) < +\infty. \quad (4.7g)
\]

If we choose

\[
v_0 := \frac{M_v^+ + m_v^-}{M_g - m_g} (2g - M_g - m_g) \in \text{sbh}(O_0) \quad (4.8)
\]

then the function \( V \) from (4.4) is subharmonic on \( O_0 \cup O \).

Proof. The function \( v_0 \) from definition (4.8) is subharmonic on \( O_0 \), since this function \( v_0 \) has a form \( \text{const}^+ g + \text{const} \) with constants \( \text{const}^+ \in \mathbb{R}^+, \text{const} \in \mathbb{R} \). In addition, by construction (4.8), for each \( x \in O_0 \cap \partial O \), we obtain

\[
\limsup_{y \to x} v(y) \leq \frac{M_v^+ + m_v^-}{M_g - m_g} (2g - M_g - m_g) \quad (4.7m) - (4.7M)
\]

\[
\leq \frac{M_v^+ + m_v^-}{M_g - m_g} \left( 2 \inf_{x \in O_0 \cap \partial O} g(x) - M_g - m_g \right) = \inf_{x \in O_0 \cap \partial O} \frac{M_v^+ + m_v^-}{M_g - m_g} (2g(x) - M_g - m_g) \quad (4.7g)
\]
Thus, we have (4.3.0). Besides, by construction (4.8), for each \( x \in \Omega \cap \partial \Omega_0 \), we obtain
\[
\limsup_{x' \to x} v_0(x') \overset{(4.8)}{=} \frac{M_v^+ + m_v^-}{M_g - m_g} \left( 2 \limsup_{x' \to x} g(x') - M_g - m_g \right)
\]
\[
\overset{(4.7g)}{\leq} \frac{M_v^+ + m_v^-}{M_g - m_g} (2m_g - M_g - m_g) = -(M_v^+ + m_v^-) \leq -m_v \leq m_v
\]
\[
\overset{(4.7m)}{\leq} \inf_{x \in \Omega \cap \partial \Omega_0} v(x) \leq v(x) \quad \text{for each } x \in \Omega \cap \partial \Omega_0.
\]
Thus, we have (4.3.1), and Gluing Theorem 3 follows from Gluing Theorem 2.

**Remark 4.** Theorems of this section can be easily transferred to the cone of plurisubharmonic functions [48, Corollary 2.9.15]. We sought to formulate our theorems and their proofs with the possibility of their fast transport to plurisubharmonic functions and to abstract potential theories with more general constructions based on the theories of harmonic spaces and sheaves [4], [16], [7], [8], [6], [55], [3], etc.

5. GLUING WITH GREEN’S FUNCTION

Recall that a set \( E \subset \mathbb{R}^d \) is called polar if there is \( u \in \text{shb}_u(\mathbb{R}^d) \) such that
\[
\left( E \subset (-\infty)_u := \{ x \in \mathbb{R}^d : u(x) = -\infty \} \right) \Longleftrightarrow \left( \text{Cap}^* E = 0 \right),
\]
where the set \((-\infty)_u\) is minus-infinity \( G_\delta \)-set for the function \( u \), and
\[
\text{Cap}^*(E) := \inf_{E \subset \Omega = \text{int} \Omega} \sup_{C \subset \partial \Omega \subset \Omega} \int_{\Omega} \mu(x) d\mu(y)
\]
is the outer capacity of \( E \subset \mathbb{R}^d \) [62], [20], [21], [17], [53].

Let \( \Omega \) be an open proper subset in \( \mathbb{R}^d_\infty \). Consider a point \( o \in \mathbb{R}^d \) and subsets \( S_o, S \subset \mathbb{R}^d_\infty \) such that
\[
\mathbb{R}^d \ni o \in \text{int} S_o \subset S \subset S \subset \text{int} \Omega = \Omega \subset \mathbb{R}^d_\infty \neq \emptyset.
\]
Let \( D \) be a domain in \( \mathbb{R}^d_\infty \) with non-polar boundary \( \partial D \) such that
\[
o \overset{(5.1)}{\in} \text{int} S_o \subset S \subset D \subset S \subset \Omega.
\]
Such domain \( D \) possesses the extended Green’s function \( g_D(\cdot, o) \) with pole at \( o \in D \) (see Example 4, (2.8)) with the following properties:
\[
g_D(\cdot, o) \in \text{shb}^+(\mathbb{R}^d_\infty \setminus o) \subset \text{shb}^+(\Omega \setminus o),
\]
\[
g_D(\cdot, o) = 0 \quad \text{on} \quad \mathbb{R}^d_\infty \setminus \text{cl} D \supset \Omega \setminus \text{cl} D \supset \Omega \setminus S,
\]
\[
g_D(\cdot, o) \in \text{har} \left( D \setminus o \right) \subset \text{har} \left( S_o \setminus o \right) \subset \text{har} \left( B(o, r_o) \setminus o \right)
\]
for a number \( r_o \in \mathbb{R}^+ \setminus 0 \), \( g_D(o, o) := +\infty \),
\[
g_D(x, o) \overset{(2.5K)}{=} -K_{d-2}(x, o) + O(1) \quad \text{as} \quad o \neq x \to o.
\]
Besides, the following strictly positive number
\[
0 < M_g := \inf_{x \in \partial S_o} g_D(x, o) = \text{const}^+_o, S_o, S,
\]

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depends only on \( o, S_o, S, D, d \), and, by the minimum principle, we have

\[
g_D(x, o) - M_g \geq 0 \quad \text{for each } x \in S_o \setminus o, \tag{5.3M+}\]

where by \( \text{const}_{a_1, a_2, \ldots} \in \mathbb{R} \) and \( \text{const}_{a_1, a_2, \ldots}^+ \in \mathbb{R}^+ \) we denote constants and constant functions, in general, depend on \( a_1, a_2, \ldots \) and, unless otherwise specified, only on them, but the dependence on dimension \( d \) of \( \mathbb{R}^d \), a domain \( D \), and open sets \( O \) or \( O \) will be not specified and not discussed. Properties (5.3) for the extended Green’s function \( g_D(\cdot, o) \) are well known under conditions (5.1)–(5.2) [62, 4.4], [20, 5.7].

**Gluing Theorem 4.** Under conditions (5.1), suppose that a function \( v \in \text{sbh}(O \setminus S_o) \) satisfies constraints from above and from below in the form

\[
-\infty < m_v \leq \inf_{S \setminus S_o} v \leq \sup_{S \setminus S_o} v \leq M_v < +\infty. \tag{5.4}\]

Every domain \( D \) with inclusions (5.2) possesses the extended Green’s function \( g_D(\cdot, o) \) with pole \( o \in \text{int} S_o \), properties (5.3) and the constant \( M_g \) of (5.3M) such that for the function

\[
v_o := \frac{M_v^+ + m_v^-}{M_g}(2g_D(\cdot, o) - M_g) \in \text{sbh}(\mathbb{R}^d \setminus o) \subset \text{sbh}(\text{int} S \setminus o), \tag{5.5V}\]

we can to define a subharmonic function

\[
V := \begin{cases} v_o & \text{on } S_o, \\ \sup\{v_o, v\} & \leq v_o^+ + v^+ & \text{on } S \setminus S_o & \in \text{sbh}(O \setminus o) \\ v & \text{on } O \setminus S & \end{cases} \tag{5.5V}\]

on \( O \setminus o \) satisfying the conditions

\[
V \in \text{har} (S_o \setminus o) \subset \text{har} (B(o, r_o) \setminus o) \quad \text{for a number } r_o \in \mathbb{R}^+ \setminus o, \tag{5.5h}\]

\[
v(x) \leq V(x) \leq M_v^+ + 2\frac{M_v^+ + m_v^-}{M_g}g_D(x, o) \quad \text{for each } x \in S \setminus S_o, \tag{5.5+}\]

\[
0 \leq V(x) \leq 2\frac{M_v^+ + m_v^-}{M_g}g_D(x, o) \quad \text{for each } x \in S_o \setminus o, \tag{5.5.0+}\]

\[
V(x) = -2\frac{M_v^+ + m_v^-}{M_g}K_{d-2}(x, o) + O(1) \quad \text{as } o \neq x \to o. \tag{5.5o}\]

**Proof.** It is enough to apply Gluing Theorem 3 with

\[
O := O \setminus \text{clos} S_o, \quad O_0 := \text{int} S \setminus o, \quad g := g_D(\cdot, o), \quad m_g := 0
\]

in accordance with the reference marks indicated over relationships in (5.4)–(5.5).

**Remark 5.** The choice of \( D \) and \( M_g \) in (5.3M) and (5.5o) is entirely determined by the mutual arrangement of \( S_o \in S \).

**Gluing Theorem 5.** Let \( O \subset \mathbb{R}^d \) be an open subset, and \( S_o \subset \mathbb{R}^d \) be a connected set such that there is a point

\[
o \in \text{int} S_o \subset S_o \subset O. \tag{5.6}\]

Let \( r \in \mathbb{R}^+ \) be a number such that

\[
0 < 4r < \text{dist}(S_o, \partial O), \tag{5.7}\]
$D_r := D'_r$ be a domain from Proposition 2 satisfying (3.1) with $2r$ instead of $r$ and $r' := 3r > 2r$, $v \in \text{sbh}_+(O \setminus S_o)$, and $M_v \in \mathbb{R}$ be a constant such that

$$v \leq M_v < +\infty \quad \text{on } S_o^{u(4r)} \setminus S_o,$$  

(5.8M)

$$m_v := \inf \{ u^r(x) : x \in S_o^{u(3r)} \setminus S_o^{u(2r)} \}.$$  

(5.8m)

$$M_g := \inf_{x \in \partial S_o^{u(2r)}} g_{D_r}(x, o) = \text{const}^+_{o,S_o,r,D_r}.$$  

(5.8g)

Then $M_g > 0$, $m_v > -\infty$, and there is a subharmonic function $V \in \text{sbh}_+(O \setminus o)$ satisfying conditions (5.5h)–(5.5o), i.e.,

$$0 < V \in \text{har}^+(S_o \setminus o) \quad \text{on } S_o \setminus o,$$  

(5.9h)

$$V = v \quad \text{on } O \setminus S_o^{u(4r)},$$  

(5.9=)

$$v(x) \leq V(x) \leq M_v^+ + \frac{2M_v^+ + m_v^-}{M_g} g_{D_r}(x, o) \quad \text{for each } x \in S_o^{u(4r)} \setminus S_o,$$  

(5.9+)

$$0 < V(x) \leq \frac{2M_v^+ + m_v^-}{M_g} g_{D_r}(x, o) \quad \text{for each } x \in S_o \setminus o,$$  

(5.9.0+)

$$V(x) \overset{(5.5o)}{=} -\frac{2M_v^+ + m_v^-}{M_g} K_{d-2}(x, o) + O(1) \quad \text{as } o \neq x \to o.$$  

(5.9o)

**Proof.** We have $M_g > 0$ for (5.8g) by (5.3M)–(5.3M+), and $m_v > -\infty$ since the function $v^r$ is continuous on $\text{clos} \left( S_o^{u(3r)} \setminus S_o^{u(2r)} \right)$ [21, Theorem 1.14]. Using the Perron–Wiener–Brelot method [62, Ch. 4], [20, 2.6], [21, Ch. 8], [17, Ch. VIII, 2], we construct the function

$$\tilde{v} := \sup \left\{ w \in \text{sbh}_+(O \setminus S_o) : w \leq v \text{ on } (O \setminus S_o) \setminus \left( S_o^{u(4r)} \setminus \text{clos } S_o^{u(3r)} \right) \right\}$$

and its upper regularization

$$\tilde{v}^*(x) := \limsup_{x' \to x} \tilde{v}(x'), \quad x \in \text{int } (O \setminus S_o).$$

By construction, the function $\tilde{v}^*$ is subharmonic on int $(O \setminus S_o)$, harmonic on $S_o^{u(4r)} \setminus \text{clos } S_o^{u(3r)}$ and $\tilde{v}^* = v$ on $(O \setminus S_o) \setminus (S_o^{u(4r)} \setminus \text{clos } S_o^{u(3r)})$. It follows from the principle of subordination (domination) for harmonic continuations and the maximum principle that

$$-\infty < m_v \leq \tilde{v}^* \quad \text{on } S_o^{u(3r)} \setminus S_o^{u(2r)}, \quad \tilde{v}^* \leq M_v \quad \text{on } S_o^{u(4r)} \setminus \text{clos } S_o.$$  

(5.10)

In Gluing Theorem 4, we choose the set $S_o^{u(2r)}$ as $S_o$, $S_o^{u(3r)}$ as $S$, and $\tilde{v}^*$ as $v$. We have (5.4) for $\tilde{v}^*$ in view of (5.10). Then, by construction (5.5v)–(5.5V) and conditions (5.5h)–(5.5o), we get series of conclusions (5.9) of Gluing Theorem 5 with $S_o^{u(2r)}$ instead of $S_o$. But we can replace $S_o^{u(2r)}$ with $S_o$ back in estimates (5.9+)–(5.9.0+) by virtue of condition (5.8M), as well as in (5.9h) since $S_o \subset S_o^{u(2r)}$. The possibility of replacing a constant $\text{const}^+_{o,S_o,r,D_r}$ with $\text{const}^+_{o,S_o,r}$ follows from Remark 4.

### 6. GLUING OF TEST FUNCTIONS

**Proposition 4** ([43, 3.2.1]). A function $v \in \text{sbh}_{+0}(D \setminus S_o)$ continues as subharmonic function on $\mathbb{R}_\infty^d \setminus S_o$ by the rule

$$v(x) := \begin{cases} v(x) & \text{at } x \in D \setminus S_o, \\ 0 & \text{at } x \in \mathbb{R}_\infty^d \setminus D \end{cases} \in \text{sbh}(\mathbb{R}_\infty^d \setminus S_o).$$  

(6.1)
Gluing Theorem 6. If \( D \) is a domain together with (2.10), and \( b_\pm, r \) are constants satisfying (3.3), then there is a constant

\[
B := 2 \frac{b_+ - b_-}{\text{const}_o} := \text{const}_o > 0
\]

such that for any test subharmonic function \( v \in \text{sbh}_+ (D \setminus S_o; \sigma) \) or \( b_- < b_+ \) we can construct a subharmonic function \( V \in \text{sbh}_+ (\mathbb{R}^d_\infty \setminus o) \) with properties

\[
0 < V \in \text{har}_+ (S_o \setminus o) \quad \text{on} \quad S_o \setminus o,
\]

\[
V (5.9h) = v \quad \text{on} \quad D \setminus S^{4(4r)},
\]

\[
v(x) \leq V (x) \leq b_+ + B g_D (x, o) \quad \text{for each} \quad x \in S^{4(4r)} \setminus S_o,
\]

\[
0 < V (x) \leq B g_D (x, o) \quad \text{for each} \quad x \in S_o \setminus o,
\]

\[
V (x) \in \text{har}_+ (S_o \setminus o) \quad \text{as} \quad o \neq x \to o
\]

\[
V \equiv 0 \quad \text{on} \quad \mathbb{R}^d_\infty \setminus D.
\]

Besides, for each \( v \in \text{sbh}_{++} (D \setminus S_o; \sigma) \) or \( b_- < b_+ \) we get a function \( V : \mathbb{R}^d_\infty \setminus o \to \mathbb{R} \) with the same properties (6.3h)–(6.3o) as the limit of an increasing sequence of functions satisfying the conditions (6.3h)–(6.3o), but with a weaker property instead of (6.3o), more precisely

\[
V \equiv 0 \quad \text{on} \quad \mathbb{R}^d_\infty \setminus \text{clos} \ D, \quad V \geq 0 \quad \text{on} \quad \partial D,
\]

and such function \( V \) is not necessarily upper semi-continuous on \( \text{clos} \ D \setminus S_o \).

This is also true for every positive function

\[
v \in \text{sbh}_0^+ (D \setminus S_o; \leq b_+) \quad \text{or} \quad v \in \text{sbh}_0^{++} (D \setminus S_o; \leq b_+),
\]

together with an additional property of the positivity of \( V \geq 0 \) on \( \mathbb{R}^d_\infty \setminus o).

Proof. By Proposition 4 we can consider the function \( v \in \text{sbh}_+ (D \setminus S_o; \sigma) \) as defined on \( \mathbb{R}^d_\infty \setminus S_o \) by (6.1), i.e., \( v \equiv 0 \) on \( \mathbb{R}^d_\infty \setminus D \), and \( v \in \text{sbh}_+ (\mathbb{R}^d_\infty \setminus S_o) \). By Gluing Theorem 5 with open set

\[
C := \mathbb{R}^d_\infty \setminus S_o \quad \text{with constants} \quad M_v := b_+, \quad m_v := b_- \quad \text{and} \quad M_o := \text{const}_o > 0,
\]

and a constant \( B \) from (6.2), we construct a function \( V \in \text{sbh}_0 (\mathbb{R}^d_\infty \setminus o), V (\infty) := 0 \), with properties (5.9) that go into properties (6.3h)–(6.3o) together with identity (6.3o). Note that we use the principle of domination in (6.3h)–(6.3o) for Green’s functions to replace \( D \) with \( D \), since a domain \( D \) from (5.9) is a subdomain of \( D \) provided (3.3).

7. APPROXIMATION OF TEST FUNCTIONS

Theorem 3. Let \( b_\pm, r \) are constants from (3.3), \( v \in \text{sbh}_+ (D \setminus S_o; \sigma) \) (respectively, \( v \in \text{sbh}_0^{++} (D \setminus S_o; \leq b_+) \)). Then there is a constant

\[
B := \frac{\text{const}_o}{\text{const}_o^{++}}, \quad B > 0,
\]

and an increasing sequence \( (V_n)_{n \in \mathbb{N}} \) of Arens–Singer (respectively, Jensen) potentials \( V_n \in \text{ASP}_1 (D \setminus o) \) (respectively, \( V_n \in \text{JP}_1 (D \setminus o) \)), \( n \in \mathbb{N} \), such that

\[
0 < V_n \in \text{har}_- (\text{int} S_o \setminus o) \quad \text{on} \quad S_o \setminus o
\]

\[
BV_n \nrightarrow V \quad \text{on} \quad D \setminus o.
\]
where \( V : \mathbb{R}^d_\infty \setminus o \to \mathbb{R} \) is a function with properties (6.3h)–(6.3o), (6.3.0'),
\[
V_n(x) = -K_{d-2}(x,o) + O(1) \quad \text{as } o \neq x \to o, \tag{7.2o}
\]
\[
BV_n \leq b_+ + B_g D(\cdot, o) \quad \text{on } S^{u(4r)}_o \setminus o. \tag{7.2+}
\]

**Proof.** The classes \( ASP(D \setminus o) \) and \( JP(D \setminus o) \) are closed relative to the max–operation. By Propositions 11 and 4, it suffices to prove Theorem 3 only for functions
\[
v \in \text{sbh}_+ (D \setminus S_o; \text{or}, b_- < b_+) \quad \text{(resp. } v \in \text{sbh}_0^+ (D \setminus S_o; \leq b_+)); v \equiv 0 \text{ on } \mathbb{R}^d_\infty \setminus D).
\]

Denote by \( \text{conn}(Q, x) \in \text{Conn } Q \) a connected component of \( Q \subset \mathbb{R}^d_\infty \) containing \( x \). For a function \( v \in \text{sbh}_+ (D \setminus S_o; \text{or}, b_- < b_+) \), we consider a function \( V \) from Gluing Theorem 6 with properties (6.3). For each number \( n \in \mathbb{N} \) we put in correspondence an open set \( O_n := \{ x \in \mathbb{R}^d_\infty \setminus o : V(x) < 1/n \} \supset O_{n+1}, \) and a function \( v_n \) such that
i) this function \( v_n \) vanishes on all connected components \( \text{conn}(O_n, x) \in \text{Conn } O_n \) that meet with complement \( \mathbb{R}^d_\infty \setminus D \) of \( D \), i.e., \( v_n \equiv 0 \) on every connected component \( \text{conn}(O_n, x) \) with \( x \in \mathbb{R}^d_\infty \setminus D,
\]
ii) \( v_n := 1/n \) on \( V \) on \( \mathbb{R}^d_\infty \setminus o. \)

By construction i)–ii), these functions \( v_n \) are subharmonic on \( \mathbb{R}^d_\infty \setminus 0 \) with \( \text{supp}\, v_n \in D \). Therefore, \( v_n \in \text{sbh}_0 (D \setminus o) \). Besides, these functions \( v_n \) form an increasing sequence \( v_n \nearrow V \) on \( \mathbb{R}^d_\infty \setminus o. \)

In view of (6.3o), there exists the limit \( v_n(x) = -K_{d-2}(x,o) + O(1) \) as \( o \neq x \to o. \) Thus, if we set \( V_n := \frac{1}{n} v_n \), then we have (2.6). Hence every function \( V_n \) belongs to \( ASP^1(D \setminus o) \) by Example 3, (2.7), and properties (7.2) are fulfilled.

In the case \( v \in \text{sbh}_0^+ (D \setminus S_o; \leq b_+) \), we consider the functions \( v_n^+ \) after i)–ii) instead of \( v_n \) and we obtain \( V_n \in JP(D \setminus o) \) with properties (7.2) by Example 4.

8. FROM ARENS–SINGER AND JENSEN POTENTIALS TO TEST FUNCTIONS

**Proof of implications** [s6]⇒[s7] and [h4]⇒[h5] of Theorems 1, 2. By Definition 1 in variant (1.3), the condition \( \Delta_n \triangleq_{J P(D \setminus o)} \Delta_M \) from [s6] (resp. \( \Delta_n \triangleq_{A S P(D \setminus o)} \Delta_M \) from [h4]) means that there is a constant \( C_1 \in \mathbb{R} \) such that
\[
\int_{D \setminus o} v \, d\Delta_n \leq \int_{D \setminus o} v \, d\Delta_M + C_1 \quad \text{for each } v \in J P(D \setminus o), \tag{8.1}
\]

or
\[
\int_{D \setminus o} v \, d\Delta_n \leq \int_{D \setminus o} v \, d\Delta_M + C_1 \quad \text{for each } v \in A S P(D \setminus o), \tag{8.1}
\]

Let \( v \in \text{sbh}_0^+ (D \setminus S_o; \leq b_+) \) in the case [s6] or \( v \in \text{sbh}_0^+ (D \setminus S_o; \leq b_+) \) in the case [h4], respectively. By Theorem 3, there are a constant from (7.1) and an increasing sequence of Jensen (respectively, Arens–Singer) potentials \( V_n \in J P(D \setminus o) \) (respectively, \( V_n \in A S P(D \setminus o) \)), \( n \in \mathbb{N} \), satisfying (7.2). Hence (8.1) entails
\[
\int_{D \setminus o} BV_n \, d\Delta_n \leq \int_{D \setminus o} BV_n \, d\Delta_M + BC_1 \leq \int_{D \setminus o} V \, d\Delta_M + BC_1, \tag{8.2}
\]
where the function \( V := \lim_{n \to \infty} BV_n \) on \( D \setminus o \) has all the properties (6.3h), (6.3o), (6.3.0'), and the constant \( BC_1 \in \mathbb{R} \) independent of \( V_n \), \( n \in \mathbb{N} \). Let’s represent the integral on the right-hand side of inequalities (8.2) as a sum of integrals:
\[
\int_{D \setminus o} V \, d\Delta_M = \left( \int_{D \setminus S_o^{u(4r)}} + \int_{S_o^{u(4r)} \setminus S_o} + \int_{S_o \setminus o} \right) V \, d\Delta_M \tag{8.3}
\]
\[
\leq \int_{D \setminus S_o^{u(4r)}} v \, d\Delta_M + b_+ \Delta_M (S_o^{u(4r)} \setminus S_o) + B \int_{S_o^{u(4r)} \setminus o} g_D (x, o) \, d\Delta_M
\]
\[
\leq \int_{D \setminus S_o^{u(4r)}} v \, d\Delta_M + C_2, \quad \text{where } C_2 = \text{const}_S^+ a, b_+, B, u, M
\]

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is a constant independent of $v$. In addition, in the case $v \in \text{sabh}_0^+(D \setminus S_0; b_+)$, the function $v$ is positive on $D \setminus S_0$, and we have
\begin{equation}
\int_{D \setminus S_0} V \, d\Delta_M \leq \int_{D \setminus S_0} v \, d\Delta_M + C_2 \quad \text{in the case } \text{s6}, \tag{8.4}
\end{equation}
If the integrals in the right-hand sides of (8.3) and (8.4) are equal to $+\infty$, then there is nothing to prove. Otherwise, by Beppo Levi’s monotone convergence theorem for Lebesgue integral, (8.2) and (7.2/) together with (8.3) and (8.4) entails
\begin{equation}
\int_{D \setminus S_0} V \, d\Delta_u \leq \int_{D \setminus S_0} v \, d\Delta_M + C_2, \quad \text{where } S_0^* \equiv \begin{cases} S_0 & \text{or} S_o^{U(4r)} \end{cases}, \text{ respectively.} \tag{8.5}
\end{equation}
According to (6.3h)–(6.3+), it is follows from (8.5) that
\begin{equation}
\int_{D \setminus S_o} v \, d\Delta_u \leq \int_{S_0^*} V \, d\Delta_u + \int_{D \setminus S_0} v \, d\Delta_u \tag{6.3h-)
\end{equation}
\begin{equation}
\leq \int_{S_0^*} V \, d\Delta_u + \int_{S_o^{U(4r)} \setminus S_0} V \, d\Delta_u + \int_{D \setminus S_o^{U(4r)}} v \, d\Delta_u \tag{6.3=)
\end{equation}
\begin{equation}
= \int_{D \setminus S_0} V \, d\Delta_u \leq \int_{D \setminus S_0} v \, d\mu_M + C, \tag{8.5}
\end{equation}
where the constant $C$ is independent of $v$, and $S_0^*$ is defined in (8.5).

Proof of implication [h5] $\Rightarrow$ [h6] of Theorem 2. If
\begin{equation}
v \in \text{sabh}_0^+(D \setminus S_o; r, b_- < b_+) \subset \text{sbh}_0^+(D \setminus S_o; r, b_- < b_+),
\end{equation}
then this function $v$ is bounded from below on $S_o^{U(4r)} \setminus S_o$ by the constant $b_- \in (-\mathbb{R}^+) \setminus 0$, and (3.5) implies
\begin{equation}
\int_{D \setminus S_0} v \, d\Delta_u \leq \int_{D \setminus S_0} v \, d\Delta_M - \int_{S_o^{U(4r)} \setminus S_o} v \, d\Delta_M + C
\end{equation}
\begin{equation}
\leq \int_{D \setminus S_0} v \, d\Delta_M - \Delta_M(S_o^{U(4r)} \setminus S_o) b_- + C \quad \text{for each } v \in \text{sbh}_0^+(D \setminus S_o; r, b_- < b_+),
\end{equation}
where the constant $(-\Delta_M(S_o^{U(4r)} \setminus S_o) b_- + C)$ is independent of $v$.

9. ARENS–SINGER AND JENSEN MEASURES AND THEIR POTENTIALS

**Definition 3** ([62], [35, Definition 2], [43, 3.1, 3.2], [13]). Let $\mu \in \text{Meas}_{\text{cmp}}(\mathbb{R}^d)$ be a charge with compact support. Its potential is a function
\begin{equation}
\text{pt}_\mu : \mathbb{R}^d \to \mathbb{R}, \quad \text{pt}_\mu(y) (2.5k) := \int K_{d-2}(x, y) \, d\mu(x),
\end{equation}
where the kernel $K_{d-2}$ is defined in Definition 2 by the function $k_0$ from (2.5k),
\begin{equation}
\text{dom } \text{pt}_\mu \equiv \left\{ y \in \mathbb{R}^d : \int_0^{\mu^-(y, t)} \frac{dt}{t^{d-1}} < +\infty \right\} \bigcup \left\{ y \in \mathbb{R}^d : \int_0^{\mu^+(y, t)} \frac{dt}{t^{d-1}} < +\infty \right\},
\end{equation}
and $\mathbb{R}^d \setminus \text{dom } \text{pt}_\mu$ is a bounded polar set in $\mathbb{R}^d$ with $\text{Cap}^*(\mathbb{R}^d \setminus \text{dom } \text{pt}_\mu) = 0$.

**Proposition 5.** If
\begin{equation}
\mu \in \text{Meas}_{\text{cmp}}^+(\mathbb{R}^d), \quad L \in \mathbb{R}^d, \quad L \setminus \emptyset \neq \emptyset,
\end{equation}
then
\begin{equation}
\inf_{x \in L} \text{pt}_\mu(x) \geq \mu(\mathbb{R}^d) k_{d-2}(\text{dist}(L, \text{supp}\mu)),
\end{equation}

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\[
\inf_{x \in L} \text{pt}_{\mu - \delta_o}(x) \geq \mu(\mathbb{R}^d)k_{d-2}(\text{dist}(L, \text{supp}\mu)) - k_{d-2}\left(\sup_{x \in L} |x| + |o|\right). \tag{9.30}
\]

**Proof.** The case \(d = 1\) is trivial. Let \(d \geq 2\). If \(\text{dist}(L, \text{supp}\mu) = 0\), then the right-hand sides in inequalities (9.3) are equal to \(-\infty\), and inequalities (9.3) are true. Otherwise, by Definition 3, we obtain
\[
\text{pt}_\mu(x) = \int k_{d-2}(|x - y|) \, d\mu(y) = \inf_{y \in \text{supp}\mu} k_{d-2}(|x - y|) \mu(\mathbb{R}^d)
\]
\[
\geq k_{d-2}\left(\inf_{y \in \text{supp}\mu} |x - y|\right) \mu(\mathbb{R}^d) = \mu(\mathbb{R}^d)k_{d-2}(\text{dist}(x, \text{supp}\mu)),
\]
since the function \(k_q\) from (2.5k) is increasing. We obtain inequality (9.3i) after applying the operation \(\inf_{x \in L}\) to both sides of inequality (9.4). Using (9.3i), we have
\[
\inf_{x \in L} \text{pt}_{\mu - \delta_o}(x) \geq \inf_{x \in L} \text{pt}_\mu(x) - \sup_{x \in L} k_{d-2}(|x - o|)
\]
\[
\geq \mu(\mathbb{R}^d)k_{d-2}(\text{dist}(L, \text{supp}\mu)) - k_{d-2}\left(\sup_{x \in L} |x| + |o|\right),
\]
and it is inequality (9.30).

**Duality Theorem** ([35, Proposition 1.4, Duality Theorem]). The mapping
\[
\mathcal{P}_o : \mu \mapsto \text{pt}_{\mu - \delta_o} \tag{9.5}
\]
is an affine bijection from \(\text{AS}_o(O)\) onto \(\text{PAS}(O \setminus o)\) (resp. \(\text{J}_o(O)\) onto \(\text{JP}(O \setminus o)\)) with inverse mapping
\[
\mathcal{P}_o^{-1} : V \xrightarrow{(2.9)} c_d\Delta V \big|_{\mathbb{R}^d \setminus o} + \left(1 - \limsup_{o \not\to o} \frac{V(y)}{-K_{d-2}(o, y)}\right) \delta_o. \tag{9.6}
\]

Let \(o \in \text{int} Q = Q \subseteq O\). The restriction of \(\mathcal{P}_o\) to the class
\[
\left\{ \mu \in \text{AS}_o(O) : \text{supp}\mu \cap Q = \emptyset \right\} \tag{9.7}
\]
\(
\left(\text{resp. } \left\{ \mu \in \text{J}_o(O) : \text{supp}\mu \cap Q = \emptyset \right\} \right)
\)
define an affine bijection from class (9.7) onto class (see (2.7), Examples 3, 4)
\[
\text{ASP}^1(O \setminus o) \bigcap \text{har}(Q \setminus o) \tag{9.8}
\]
\[
\left(\text{resp. } \text{JP}^1(O \setminus o) \bigcap \text{har}(Q \setminus o) \right).
\]
The restriction of \(\mathcal{P}_o\) to the class
\[
\left\{ \mu \in \text{AS}_o(O) : \text{supp}\mu \cap Q = \emptyset \right\} \bigcap \left(\mathcal{C}^\infty(O) \, d\lambda_d\right) \tag{9.9}
\]
\[
\left(\text{resp. } \left\{ \mu \in \text{J}_o(O) : \text{supp}\mu \cap Q = \emptyset \right\} \bigcap \left(\mathcal{C}^\infty(O) \, d\lambda_d\right) \right)
\]
define an affine bijection from class (9.9) onto class
\[
\text{ASP}^1(O \setminus o) \bigcap \text{har}(Q \setminus o) \bigcup \mathcal{C}^\infty(O \setminus o) \tag{9.10}
\]
\[
\left(\text{resp. } \text{JP}^1(O \setminus o) \bigcap \text{har}(Q \setminus o) \bigcup \mathcal{C}^\infty(O \setminus o) \right).
\]
This transition from the main bijection \(\mathcal{P}_o\) to the bijection from (9.7) onto (9.8) or from (9.9) onto (9.10) by restriction of \(\mathcal{P}_o\) to (9.7) or (9.9) is quite obvious.

**Poisson–Jensen formula** ([35, Proposition 1.2]). If \(u \in \text{sh}(D)\), \(u(o) \neq -\infty\), then
\[
u(o) = \int_D u \, d\mu - \int_{D \setminus o} \text{pt}_{\mu - \delta_o} d\Delta u \quad \text{for each } \mu \in \text{AS}_o(D).
\]
10. EMBEDDINGS OF ARENS–SINGER AND JENSEN POTENTIALS INTO CLASSES OF TEST FUNCTIONS

Throughout this Sec. 10, the boundary $\partial D$ of $D \ni o$ is non-polar, i.e., $\text{Cap}^*(\partial D) > 0$.

**Proposition 6** (a variant of Prágl–Lindelöf principle). If $v \in \text{sbl}(D\setminus o)$ satisfies the conditions

$$\limsup_{o \not \to x} \frac{v(x)}{-K_{D-2}(x, o)} \leq 0, \quad \limsup_{D \ni x \to \partial D} v(x) \leq 0,$$

then $v \leq 0$ on $D\setminus o$. In particular, if a function $V \in \text{sbl}(D\setminus o)$ satisfies the conditions

$$\limsup_{o \not \to x} \frac{V(x)}{-K_{D-2}(x, o)} \leq c \in \mathbb{R}^+, \quad \limsup_{D \ni x \to \partial D} V(x) \leq 0,$$

then $V \leq cg_D(\cdot, o)$ on $D\setminus o$.

**Proof.** By conditions (10.1), for any $a \in \mathbb{R}^+ \setminus 0$, we have

$$v(x) - ag_D(x, o) \leq O(1), \quad o \not \to x; \quad \limsup_{D \ni x \to \partial D} (v(x) - ag_D(x, o)) \leq 0.$$

Hence the function $v - ag_D \in \text{sbl}(D\setminus o)$ has the removable singularity at the point $o$ [20, Theorem 5.16], and the function

$$\begin{cases} v - ag_D & \text{on } D\setminus o, \\ \limsup_{o \not \to x} (v(x) - ag_D(x, o)) & \text{at } o \end{cases}$$

(10.3)

is subharmonic on $D$ and

$$\limsup_{D \ni x \to \partial D} (v(x) - ag_D(x, o)) \leq 0.$$

By the maximum principle, the function (10.3) is negative on $D$, and $v \leq ag_D(\cdot, o)$ on $D\setminus o$ for an arbitrary $a > 0$. Thus, $v \leq 0$ on $D$. In particular, for $v := V - cg_D(\cdot, o)$, under the conditions (10.2), we have (10.1) and obtain $V - cg_D(\cdot, o) \leq 0$ on $D$.

**Theorem 4** (on embedding). Let $S_o$ be a subset from (2.10), and let $r, b_\pm$ are constants from (3.3). For any domain $D_o$ satisfying

$$o \in \text{int} S_o \subseteq S_o^{\text{int}(4r)} \subseteq D_o \subseteq D,$$

we can find a constant $B = \text{const}^+_{o, S_o, r, D_o} \in \mathbb{R}^+ \setminus o$ such that

$$\text{ASP}^1(D\setminus o) \cap \text{har}(D_o\setminus o) \subset \text{sbl}_{\infty}(D\setminus S_o; r, -B < B),$$

(10.5A)

$$\text{JP}^1(D\setminus o) \cap \text{har}(D_o\setminus o) \subset \text{sbl}_{\infty}(D\setminus S_o; \leq B),$$

(10.5J)

where, in the case (10.5A), we assume that the subset $S_o$ is connected.

**Proof.** Let $V \in \text{ASP}^1(D\setminus o) \cap \text{har}(D_o\setminus o) \supset \text{JP}^1(D\setminus o) \cap \text{har}(D_o\setminus o)$.

**Lemma 1.** If $V \in \text{ASP}(D\setminus o)$, then $V \leq g_D(\cdot, o)$ on $D$.

**Proof of Lemma 1.** The Arens–Singer potentials from Example 3 satisfy conditions (10.2) of Proposition 6 with $c := 1$. Hence $V \leq g_D(\cdot, o)$ on $D$.

By Lemma 1 we have

$$\sup_{x \in S_o^{\text{int}(4r)} \setminus o} V(x) \leq \sup_{x \in S_o^{\text{int}(4r)} \setminus o} g_D(x, o) =: B' = \text{const}^+_{o, S_o, r} \in \mathbb{R}^+ \setminus o.$$

(10.6)

If $V \in \text{JP}^1(D\setminus o) \cap \text{har}(D_o\setminus o)$, then $V \geq 0$ on $\mathbb{R}^d \setminus o$, and we obtain (10.5J) with $B := B' = \text{const}^+_{o, S_o, r}$. Otherwise, we use

**Lemma 2.** Under the conditions (10.4), there is a constant $B'' \in -\mathbb{R}^+$ such that

$$\inf_{x \in S_o^{\text{int}(4r)} \setminus o} V(x) \geq B'' = \text{const}^+_{o, S_o, r, D_o} > -\infty,$$

(10.7)
For every $V \in \text{ASP}^1(D \setminus o) \cap \text{har}(D_o \setminus o)$.

**Proof of Lemma 2.** By Duality Theorem in version (9.7)-(9.8), the Riesz measure $\Delta_V = c_d \Delta V$ is a Arens–Singer probability measure, $\Delta_V \in \text{Meas}^1_{\text{comp}}(D \setminus D_o)$ and $V = \text{pt}_V - \delta_o$. By Proposition 5 with $\mu := \Delta_V$ and $L := S_{o}^{c, (4r)}$, we have

$$\inf_{x \in S_{o}^{c, (4r)} \setminus o} V(x) = \inf_{x \in S_{o}^{c, (4r)} \setminus o} \text{pt}_V - \delta_o(x)$$

$$\geq \Delta_V(\mathbb{R}^d)k_{d-2}(\text{dist}(S_{o}^{c, (4r)}, \text{supp}\Delta_V)) - k_{d-2} \left( \sup_{x \in S_{o}^{c, (4r)}} |x| + |o| \right)$$

$$\geq k_{d-2}(\text{dist}(S_{o}^{c, (4r)} \setminus o, \partial D_o)) + \text{const}_{o, s, r} \quad \text{in the case } [s8] \quad \text{in the case } [h7] \quad \text{in the case } [s8] \quad \text{in the case } [h7]$$

and we obtain (10.7).

If we set $B^{(10.6)} := \max\{B', (-B'')^+\}$, then (10.5A) follows from (10.6) and (10.7).

**Proof of implications [s8] $\Rightarrow$ [s9] and [h7] $\Rightarrow$ [h8].** There is a domain $D_o$ satisfying (10.4). According to (2.10), we can choose a point $o \in \text{int } S_o$ such that $u(o) \neq -\infty$. The latter means that

$$-\infty < \int_{S_o} k_{d-2}(|x - o|) \, d\Delta_u(x) \overset{\text{i.e.}}{=} \int_{S_o} g_D(x, o) \, d\Delta_u(x) < +\infty.$$ 

Thus, by Lemma 1, we obtain

$$\int_{S_o} V \, d\Delta_u \leq \int_{S_o} g_D(\cdot, o) \, d\Delta_u \quad \forall \, V \in \text{ASP}^1(D \setminus o).$$

(10.8)

Besides, by Lemma 2, we have

$$-\infty < \text{const}_{o, S, r, D, o, M} = B''\Delta_M(S_o) \leq \int_{S_o} V \, d\Delta_M \quad \forall \, V \in \text{ASP}^1(D \setminus o) \cap \text{har}(D_o).$$

(10.9)

The conditions [s8] or [h7] means that there is a constant $C_2 \in \mathbb{R}$ such that

$$\int_{D \setminus S_o} v \, d\Delta_u \leq \int_{D \setminus S_o} v \, d\Delta_M + C_2$$

for each $v \in \{\text{in the case } [s8], \quad \text{in the case } [h7]\}$

$$v^+_b := \text{sbh}_{00}(D \setminus S_o; l \leq b) \bigcap C^\infty(D \setminus S_o), \quad b := b_+,$$

$$v_b := \text{sbh}_{00}(D \setminus S_o; r, -b < b) \bigcap C^\infty(D \setminus S_o), \quad b := \min\{b_+, -b_-\}.$$ 

Let $B \in \mathbb{R}^+ \setminus 0$ be a constant from Theorem 4 on embedding with inclusions (10.5). We multiply both sides of inequality (10.5) by the number $B/b$ and obtain

$$\int_{D \setminus S_o} v \, d\Delta_u \leq \int_{D \setminus S_o} v \, d\Delta_M + BC_2$$

for each $v \in \{\text{in the case } [s8], \quad \text{in the case } [h7]\}$

Hence, by inclusions (10.5) from Theorem 4 on embedding, we have an inequality

$$\int_{D \setminus S_o} V \, d\Delta_u \leq \int_{D \setminus S_o} V \, d\Delta_M + BC_2$$

for each $V \in \{\text{in the case } [s8], \quad \text{in the case } [h7]\}$

which together with (10.8) (resp. (10.9)) gives an inequality

$$\int_{D \setminus o} V \, d\Delta_u \leq \int_{D \setminus o} V \, d\Delta_M + C$$

(10.11)
for each $V \in (10.5)$

$$\left\{ \begin{array}{l}
J P^1(D \setminus o) \cap \text{har} \ (D_o \setminus o) \cap C^\infty(D \setminus o) \text{ in the case } [s8], \\
A S P^1(D \setminus o) \cap \text{har} \ (D_o \setminus o) \cap C^\infty(D \setminus o) \text{ in the case } [h7],
\end{array} \right.$$ 

where $C := C_1 + B C_2 - B'' \Delta_M(S_o) = \text{const.}_o, S_o, r, D_o, u, M \in \mathbb{R}$ is a constant independent of $V$. By Definition 1 in the form (1.3), (10.11) means that the measure $\Delta_M$ is an affine

$$(J P^1(D \setminus o) \cap \text{har} \ (D_o \setminus o) \cap C^\infty(D \setminus o)) \text{-balayage of the measure } \Delta_u$$

in the case [s8] or an affine

$$(A S P^1(D \setminus o) \cap \text{har} \ (D_o \setminus o) \cap C^\infty(D \setminus o)) \text{-balayage of the measure } \Delta_u$$

in the case [h7], respectively. The implications [s8] $\Rightarrow$ [s9] and [h7] $\Rightarrow$ [h8] are proved.

**Proof of implications** [s9] $\Rightarrow$ [s9] and [h7] $\Rightarrow$ [h8]. Under the statement [s9] (resp. [h8]), there is a constant $C \in \mathbb{R}$ such that inequality (10.11) is fulfilled for each potential

$$V \in \left\{ \begin{array}{l}
A S P^1(D \setminus o) \cap \text{har} \ (D_o \setminus o) \cap C^\infty(D \setminus o), \\
J P^1(D \setminus o) \cap \text{har} \ (D_o \setminus o) \cap C^\infty(D \setminus o),
\end{array} \right. \quad \text{respectively.} \quad (10.12)$$

We choose $o \in S_o := D_o$ so that $u(o) \neq -\infty$. By the Duality Theorem and the generalized Poisson–Jensen formula from Sec. 9, we have two equalities

$$\int_{D \setminus o} V \ d\Delta_u = u(o) + \int_D u \ d\Delta_V, \quad \int_{D \setminus o} V \ d\Delta_M = M(o) + \int_D M \ d\Delta_V$$

for each $V \in A S P^1(D \setminus o) \cap \text{har} \ (D_o \setminus o) \cap C^\infty(D \setminus o)$, where, in view of affine bijection $P_o$ from (9.9) onto (9.10), the Riesz measures $\Delta_V^{(2,9)} := c_d \Delta V$ of potentials $V$ from (10.12) run through all Arens–Singer (resp. Jensen) measures in

$$\begin{align*}
\mathcal{M}_A^\infty(D \setminus D_o) &:= A S_o(D) \cap \text{Meas}^{+1}_{\text{cmp}}(D \setminus D_o) \cap (C^\infty(D) \ d\lambda_d), \\
\mathcal{M}_J^\infty(D \setminus D_o) &:= J_o(D) \cap \text{Meas}^{+1}_{\text{cmp}}(D \setminus D_o) \cap (C^\infty(D) \ d\lambda_d). 
\end{align*}$$

Using (10.11) with (10.12) and (10.13) with (10.14), we obtain

$$\int_D u \ d\mu \leq \int_D M \ d\mu + (C + M(o) - u(0))$$

for each $\mu \in (10.14)$

$$\begin{align*}
\mathcal{M}_A^\infty(D \setminus D_o) \text{ for the case } [h8], \\
\mathcal{M}_J^\infty(D \setminus D_o) \text{ for the case } [s9].
\end{align*}$$

where the constant $(C + M(o) - u(0))$ is independent of $\mu$. Thus, by Definition 1 in the form (1.2), we get $u \leq \mathcal{M}_A^\infty(D \setminus D_o) M$ (resp. $u \leq \mathcal{M}_J^\infty(D \setminus D_o) M$). This is exactly [h3] (resp. [s3]) for $S_o := D_o$.

11. FROM ARENS–SINGER AND JENSEN MEASURES TO SUBHARMONIC FUNCTIONS

Our final part of the proof of Theorems 1 and 2 uses the following corollary from the more general results of [44].

**Theorem 5** ([44, Ch. 2, 8.2, Corollary 8.1, II, 1, (i)-(ii)]). Let $D \subset \mathbb{R}^d$ be a domain, $H$ be a convex cone in $\text{shh}_o(D)$ containing constants, $D_o \subset D$ be a subdomain, $o \in D_o$. Suppose that one of the following two conditions is fulfilled:

(a) for any locally bounded from above sequence of functions $(h_k)_{k \in \mathbb{N}} \subset H$, the upper semicontinuous regularization of the upper limit $\limsup_{k \to \infty} h_k$ belongs to $H$ provided that $\limsup_{k \to \infty} h_k(x) \neq -\infty$ on $D$;
(b) $H$ is sequentially closed in $L^1_{loc}(D)$.

Let $u \in \text{sbh}_*(D)$, $M \in C(D)$, and $0 \neq \vartheta \in \text{Meas}^+(D)$, $\text{supp} \vartheta \subset D_0 \Subset D$. If there is a constant $C \in \mathbb{R}$ such that

$$
\int_D u \, d\mu \leq \int_D M \, d\mu + C \quad \text{for each } \mu \in \mathcal{M}^∞(D \backslash D_0) \text{ such that } \vartheta \preceq_H \mu,
$$

(11.1)

where $\mathcal{M}^∞(D \backslash D_0) := \text{Meas}^+(D \backslash D_0) \cap (C^∞(D) \, d\lambda)$,

then there is a function $h \in H$ such that $u + h \leq M$ on $D$.

Proof of implications $\{h3\} \Rightarrow \{h1\}$ and $\{s3\} \Rightarrow \{s1\}$. We choose the convex cone

$$
H := \begin{cases} 
\text{har}(D) \text{in the case } [h3] \text{satisfying the condition (b) of Theorem 5,} \\
\text{sbh}_*(D) \text{in the case } [s3] \text{satisfying the condition (a) of Theorem 5,}
\end{cases}
$$

respectively. The statement $[h3]$ (resp. $[s3]$) means that we have (11.1) for $\vartheta := \delta_o$ if we choose a domain $D_0$ so that $S_0 \Subset D_0 \Subset D$. By Theorem 5, there is a function $h \in \text{har}(D)$ (resp. $h \in \text{sbh}_*(D)$) such that $u + h \leq M$ on $D$. Thus, $u \prec_H M$, and Theorem 2 (resp. Theorem 1) is proved.

12. Applications to the Distribution of Zero Sets of Holomorphic Functions of One and Several Complex Variables

For $n \in \mathbb{N}$ we denote by $\mathbb{C}^n$ the $n$-dimensional complex space over $\mathbb{C}$ with the standard norm $|z| := \sqrt{|z_1|^2 + \cdots + |z_n|^2}$ for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and the distance function $\text{dist}(\cdot, \cdot)$. By $\mathbb{C}_\infty^n := \mathbb{C}^n \cup \{\infty\}$, and $\mathbb{C}_\infty := \mathbb{C}_\infty^n$ we denote the one-point Alexandroff compactifications of $\mathbb{C}^n$, and $\mathbb{C}; |\cdot| := +\infty$. If necessary, we identify $\mathbb{C}^n$ and $\mathbb{C}_\infty^n$ with $\mathbb{R}^{2n}$ and $\mathbb{R}_\infty^{2n}$ respectively (over $\mathbb{R}$), and the preceding terminology and concepts are naturally transferred from $\mathbb{R}^{2n}$ to $\mathbb{C}^n$ and from $\mathbb{R}_\infty^{2n}$ to $\mathbb{C}_\infty^n$.

We use the outer Hausdorff $p$-measure $\sigma_p$ with $p \in \mathbb{N}_0$ [12, A6]:

$$
\sigma_p(S) := b_p \lim_{0<r \to 0} \inf \left\{ \sum_{j \in \mathbb{N}} r_j^p : S \subset \bigcup_{j \in \mathbb{N}} B(x_j, r_j), 0 \leq r_j < r \right\}, \quad S \subset \mathbb{R}^d,
$$

(12.1H)

$$
b_p :\begin{cases} 
1 & \text{if } p = 0, \\
\frac{s_p - 1}{p} & \text{if } p \in \mathbb{N},
\end{cases}
$$

is the volume of the unit ball $\mathbb{B}$ in $\mathbb{R}^p$.

Thus, for $p = 0$, for any $Q \subset \mathbb{R}^d$, its Hausdorff 0-measure $\sigma_0(Q)$ is the cardinality $\#Q$ of $Q$, $\sigma_d = \lambda_d$ on $\mathcal{B}(\mathbb{R}^d)$, and $\sigma_{d-1} := \sigma_{d-1} |_{\partial \mathbb{B}}$ on $\partial(\mathbb{B})$.

For a subset $Q \subset \mathbb{C}^n$, the class $\text{Hol}_d(Q)$ consists of restrictions to $Q$ of holomorphic functions $f$ on an open set $\Omega_f \supset Q$; $\text{Hol}_d(Q) := \text{Hol}(Q) \backslash 0$.

12.1. Zeros of Holomorphic Functions of Several Variables

The counting function (or multiplicity function, or divisor) of zeros of function $f \in \text{Hol}_d(D)$ on a domain $D \subset \mathbb{C}^n$ is a function $\text{Zero}_f : D \to \mathbb{N}_0$ that can be defined as [12, 1.5, Proposition 2], [12, Ch. 1, 1.2], [63, §11], [39, Ch. 4].

$$
\text{Zero}_f(z) := \max \left\{ p \in \mathbb{N}_0 : \limsup_{D \ni z' \to z} \frac{|f(z')|}{|z' - z|^p} < +\infty \right\}, \quad z \in D,
$$

(12.2Z)

with the support set $\text{supp} \text{Zero}_f = \{ z \in D : f(z) = 0 \}$. For $f = 0 \in \text{Hol}(D)$, by definition, $\text{Zero}_0 \equiv +\infty$ on $D$. Each counting function of zeros $\text{Zero}_f$ is associated with a counting measure of zeros $n_{\text{Zero}_f} \in \text{Meas}^+(D)$ defined as a Radon measure:

$$
n_{\text{Zero}_f}(c) := \int_D c \, d\text{Zero}_f := \int_D c \text{Zero}_f \, d\text{Meas}^{-2}, \quad c \in C_0(D) := \{ c \in C(D) : \text{supp} c \Subset D \},
$$

(12.2R)
or, equivalently, as a Borel measure on $D$:

$$n_{\text{zero}}(Q) = \int_Q \text{Zero}_f \, d\sigma_2n-2 \quad \text{for each compact subset } Q \subset D.$$  \hspace{1cm} (12.2B)

**Poincaré–Lelong formula** ([54], [12]). If $f \in \text{Hol}_s(D)$, i.e., $\ln |f| \in \text{sbh}_s(D)$, then

$$\Delta \ln |f| = c_{2n} \Delta \ln |f|^{(2.9)} = \sum \frac{(n-1)!}{2^n \max\{1,2n-2\}} \Delta \ln |f| = n_{\text{zero}}.$$ \hspace{1cm} (12.3)

Let $Z: D \to \mathbb{R}^+$ be a function on $D$. We call this function $Z$ a *subdivisor of zeros* for function $f \in \text{Hol}(D)$ if $Z \leq \text{Zero}_f$ on $D$. Integrals with respect to a positive measure whose integrands contain a subdivisor are everywhere below treated as upper integrals $f^+$ [9], [16].

### 12.2. Zeros of Holomorphic Functions of one Variable

Let $D \subset \mathbb{C}$. The counting function of zeros of $f \in \text{Hol}_s(D)$ is the function [39, 0.1]

$$\text{Zero}_f(z) \overset{(12.2Z)}{=} \max \left\{ p \in \mathbb{N}_0 : \frac{f}{(\cdot - z)^p} \in \text{Hol}(D) \right\}, \quad z \in D,$$

and the counting measure of zeros for $f$ is defined as a Radon measure:

$$n_{\text{Zero}}(c) \overset{(12.2R)}{=} \sum_{z \in D} \text{Zero}_f(z) c(z) = \int_D c \text{Zero}_f \, d\sigma_0, \quad c \in C_0(D),$$

or as a Borel measure on $D$:

$$n_{\text{Zero}}(Q) \overset{(12.2B)}{=} \sum_{z \in S} \text{Zero}_f(z), \quad Q \subset D.$$ 

In this case, the support set $\text{suppZero}_f$ is a locally finite set of isolated points in $D$. An *indexed set* $Z := \{z_k\}_{k=1,2,\ldots}$ of points $z_k \in D$ is *locally finite in* this domain $D$ if $\#\{k : z_k \in Q\} < +\infty$ for each subset $Q \subset D$. The *counting measure* $n_Z \in \text{Meas}^+(D)$ of this indexed set $Z$ is defined as

$$n_Z := \sum_k \delta_{z_k}, \text{ or, equivalently, } n_Z(Q) := \sum_{z_k \in Q} 1 \text{ for each } Q \subset D.$$ 

Let $Z$ and $Z'$ be a pair of indexed locally finite sets in $D$. By definition, $Z = Z'$ if $n_Z = n_Z'$, and $Z' \subset Z$ if $n_{Z'} \leq n_Z$. An indexed set $Z$ is the *zero set* of $f \in \text{Hol}_s(D)$ if $n_Z = n_{\text{Zero}_f}$. A function $f \in \text{Hol}(D)$ *vanishes on* $Z$ if $Z \subset \text{Zero}_f$.

### 12.3. Zero Sets of Holomorphic Functions with Restrictions on their Growth

The following Theorem 6 develops results from [43, Main Theorem, Theorems 1-3], and from [58, Theorem 1]. Both integrals on the right-hand sides of inequalities (12.7) and (12.8), and a pair of integrals in inequality (12.9) below are, generally speaking, upper integrals in the sense N. Bourbaki [9], [16]. We denote by $\text{dsbh}(O) := \text{sbh}(O) - \text{sbh}(O)$ the class of all $\delta$-*subharmonic* functions on $O \subset \mathbb{R}^d$ [2], [43, 3.1].

**Theorem 6.** Let $D \neq \emptyset$ be a domain in $\mathbb{C}^n$, let

$$M_+ \in \text{sbh}_s(D) \cap C(D), \quad M_- \in \text{sbh}_s(D), \quad M := M_+ - M_- \in \text{dsbh}(D),$$  \hspace{1cm} (12.5)

are functions with Riesz measures $\Delta_{M_+}, \Delta_{M_-} \in \text{Meas}^+(D)$ and Riesz charge $\Delta_M = \Delta_{M_+} - \Delta_{M_-} \in \text{Meas}(D)$, respectively, and let $f \in \text{Hol}_s(D)$ be a function such that

$$|f| \leq \exp M \quad \text{on } D.$$ \hspace{1cm} (12.6)

**Then**
Remark 1. We get the statement
\[ \int_{D_0} v \text{Zero}_f \, d\mu_{2n-2} \leq \int_{D_0} v \, d\Delta_M + \int_{S_0^{(4r)} \setminus S_0} (-v) \, d\Delta_M + C \tag{12.7} \]
for each \( v \in \text{sbh}_+^+(D_0; r, b_- < b_+) \).

Remark 2. For any connected subset \( S_0 \subset D \) from (2.10) and for any numbers \( r, b_\pm \) from (3.3), i.e.,
\( 0 < 4r < \text{dist}(S_0, \partial D), -\infty < b_- < 0 < b_+ < +\infty \), there is a constant \( C \in \mathbb{R} \) such that
\[ \int_{D_0 \setminus S_0} v \text{Zero}_f \, d\mu_{2n-2} \leq \int_{D_0 \setminus S_0} v \, d\Delta_M + \int_{S_0^{(4r)} \setminus S_0} (-v) \, d\Delta_M + C \tag{12.8} \]
for each \( v \in \text{sbh}_+^+(D_0; r, b_- < b_+) \).

Remark 3. For any set \( S_0 \) from (2.10), constant \( b_+ \in \mathbb{R}^+ \), and subdivisor \( Z \leq \text{Zero}_f \), there is a constant \( C \in \mathbb{R} \) such that
\[ \int_{D_0 \setminus S_0} v \, d\mu_{2n-2} \leq \int_{D_0 \setminus S_0} v \, d\Delta_M + C \tag{12.9} \]
for each \( v \in \text{sbh}_+^+(D_0; \leq b_+) \).

Besides, the implication \([Z1] \Rightarrow [Z2]\) is true.

Proof. We can rewrite (12.6) as
\[ \text{sbh}_+(D) \ni u := \ln |f| + M_- \leq M_+ \quad \text{(12.5)} \]
and, by implication \([h1] \Rightarrow [h5]\) of Theorem 2 together with Remark 2, there is a constant \( C \in \mathbb{R} \) such that
\[ \int_{D_0 \setminus S_0} v \, d(\Delta_{\ln |f|} + \Delta_{M_-}) \leq \int_{D_0 \setminus S_0} v \, d\Delta_{M_+} + C \tag{3.5} \]
for each \( v \in \text{sbh}_+^+(D_0; \geq r, b_- < b_+) \). Hence, by Poincaré–Lelong formula (12.3),
\[ \int_{D_0 \setminus S_0} \text{Zero}_f \, d\mu_{2n-2} \leq \int_{D_0 \setminus S_0} v \, d(\Delta_{M_+} - \Delta_{M_-}) - \int_{S_0^{(4r)} \setminus S_0} v \, d\Delta_{M_-} \tag{12.8} \]
for each \( v \in \text{sbh}_+^+(D_0; \geq r, b_- < b_+) \), and we obtain the statement \([Z1]\) with (12.7). Similary, by Poincaré–Lelong formula (12.3) and by implication \([h1] \Rightarrow [h6]\) of Theorem 2 together with Remark 2, we obtain the statement \([Z2]\) with (12.8). Besides, the implication \([Z1] \Rightarrow [Z1]\) follows from the estimate
\[ \int_{S_0^{(4r)} \setminus S_0} |v| \, d|\Delta_M| \leq \max\{b_+, -b_-\} |M_+(S_0^{(4r)} \setminus S_0)| \quad \text{for each} \ v \in \text{sbh}_+^+(D_0; \geq r, b_- < b_+) \]
Finally, by the Poincaré–Lelong formula, and by implication \([s1] \Rightarrow [s7]\) of Theorem 1 together with Remark 1, we get the statement \([Z3]\) with (12.9), since
\[ \int_{D_0 \setminus S_0} v \, d\mu_{2n-2} \leq \int_{D_0 \setminus S_0} v \, d\mu_{2n-2} \tag{12.9} \]
for every positive function \( v \in \text{sbh}_+^+(D_0; \leq b) \).

Remark 5. If \( n > 1 \) and the function \( M \) from (12.6) is plurisubharmonic, then the scale of necessary conditions for the distribution of zeros of \( f \) can be much wider than in Theorem 6. It should include other characteristics related to the Hausdorff measure of smaller dimension than \( 2n - 2 \). We plan to explore this elsewhere. In particular, the analytical and polynomial disks should play a key role for this approach (see [51, Ch. 3], [11], [60], [61], [41, § 4], etc.).
12.4. The case of a finitely connected domain D ⊂ C

We denote by Conn Q the set of all connected components of Q ⊂ \mathbb{R}^d.

Everywhere below, the domain D ⊂ C is finitely connected in \mathbb{C}_\infty with number of component
\#\text{Conn} \partial D < +\infty, among which there is at least one component containing two different points.
In this case the boundary \partial D of D is non-polar.

Lemma 3 ([37, Lemma 2.1]). If h ∈ \text{har} (D) is harmonic, then there are c < \#Conn \partial D - 1 and a function g ∈ Hol (D) without zeros in D, i.e., with \text{infra}_g = 0, such that

\[\ln|g(z)| \leq h(z) + c^+ \ln(1 + |z|)\quad \text{for all } z \in D.\]  (12.10)

If \text{clos} D \neq \mathbb{C}_\infty, then we can choose c := 0.

Theorem 7. Let M ∈ \text{dsbh}(D) be a function from (12.5) with M_+ ∈ C(D), and Z := \{z_k\}_k be an indexed locally finite set in D of points z_k ∈ D. If there are a connected set S_o as in (2.10), numbers b_\pm, r as in (3.3), and C ∈ \mathbb{R} such that

\[\sum_{z_k \in D \setminus S_o} v(z_k) \leq \int_{D \setminus S_o} v \, d\Delta_M + C \quad \text{for each } v \in \text{sbh}_{b_o}(D \setminus S_o; r, b_- < b_+) \cap C^\infty(D \setminus S_o),\]  (11)

then there are a real number c < \#Conn \partial D - 1 and a function f ∈ Hol (D) with zero set Z such that

\[\ln|f(z)| \leq M(z) + c^+ \ln(1 + |z|)\quad \text{for each } z \in D,\]  (12.12)

where c := 0 if \text{clos} D \neq \mathbb{C}_\infty.

Proof. We can rewrite relation (12.11) as

\[\int_{D \setminus S_o} v \, d(n_Z + \Delta_{M_-}) \leq \int_{D \setminus S_o} v \, d\Delta_{M_+} + C,\]

where the constant C is independent of \text{v ∈ sbh}_{b,o}(D \setminus S_o; r, b_- < b_+) \cap C^\infty(D \setminus S_o). By Definition 1 in the form (1.3), the Riesz measure \Delta_{M_+} of function M_+ is an affine \text{sbh}_{b,o}(D \setminus S_o; r, b_- < b_+) \cap C^\infty(D \setminus S_o))-balayage of n_Z + \Delta_{M_-} ∈ \text{Meas}^+(D). There is a function u ∈ \text{sbh}_{b}(D) with the Riesz measure n_Z + \Delta_{M_-} [2, Theorem 1]. It follows from implication [h7]⇒[h1] of Theorem 2 that there exists a function h ∈ \text{har} (D) such that u + h ≤ M_+. According to one of Weierstrass theorems, there is a function f_z ∈ Hol (D) with the zero set \text{Zero}_{f_z} = Z. Hence, using Weyl’s lemma for Laplace’s equation, we have a representation u = \text{ln} |f_z| + M_- + H, where H ∈ \text{har} (D), and

\[\ln |f_z| + H + h \leq M_+ - M_- \quad \text{(5.3)}\quad \text{on } D.\]

By Lemma 3, there is a function g ∈ Hol (D) without zeros such that

\[\ln|g| \leq \ln |f_z| + H + h + c^+ \ln(1 + |z|)\quad \text{on } D,\]

where c is a constant from Lemma 3. Hence

\[\ln |f_z| + \ln |g| \leq M + c^+ \ln(1 + |z|)\quad \text{on } D.\]

If we set f := gf_z ∈ Hol (D), then Z = \text{Zero}_f, and we have (12.12).

The intersection of Theorem 7 with Theorem 6, [22], gives the following criterium.

Theorem 8. Under the conditions of Theorem 7, if the domain D is simply connected with \#\partial D > 1 or \text{clos} D ≠ \emptyset, then the following four assertion are equivalent:

[z1] There is a function f ∈ Hol (D) with \text{Zero}_f = Z such that |f| ≤ \text{exp} M on D.

[z2] For any connected set S_o from (2.10) and for any numbers r, b_\pm from (3.3), there is a constant C ∈ \mathbb{R} such that

\[\sum_{z_k \in D \setminus S_o} v(z_k) \leq \int_{D \setminus S_o} v \, d\Delta_M + \int_{S_o} (-v) \, d\Delta_{M_-} + C\]
for each test function \( v \in \text{sbh}_{10}^+(D \setminus S_0; r, b_-, b_+). \)

\[[z3]\] For any connected set \( S_0 \) from (2.10) and for any numbers \( r, b_\pm \) as in (3.3), there is a constant \( C \in \mathbb{R} \) such that
\[
\sum_{z_k \in D \setminus S_0} v(z_k) \leq \int_{D \setminus S_0} v \, d\Delta_M + C
\]
for each test function \( v \in \text{sbh}_{10}^+(D \setminus S_0; r, b_-, b_+). \)

\[[z4]\] There are connected set \( S_0 \) as in (2.10), numbers \( r, b_\pm \) as in (3.3), and a constant \( C \) such that we have (12.13) for each \( v \in \text{sbh}_{100}(D \setminus S_0; r, b_-, b_+) \cap C^\infty(D \setminus S_0). \)

**Proof.** The implications \([z1] \Rightarrow [z2] \Rightarrow [z3] \Rightarrow [z4]\) follows from Theorem 6 with implication \([Z1] \Rightarrow [Z2]\) and Proposition 3. The implications \([z4] \Rightarrow [z1]\) follow from Theorem 7, where \( c = 0 \) in (12.12) according to the properties of the domain \( D \).

**Remark.** A special case of Theorem 8 was announced in [58, Theorem 2], and equivalences \([z1] \Leftrightarrow [z3] \Leftrightarrow [z4]\) were proved in [59, Theorem 1] in a slightly different way. Besides, the works [43, Main Theorem, Theorems 1–3], [41, Theorems 2,4,5] contain a wide range of necessary or sufficient conditions under which there exists a function \( f \in \text{Hol}_+(D) \) that vanishes on \( Z \) and satisfies the inequality \(|f| \leq \exp M \) on \( D \). These results do not follow directly from Theorem 8, but a significant part of these conditions follows from Theorems 1 and 6 in stronger forms.

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**CONFLICT OF INTEREST**

The authors declare that they have no conflict of interest.

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