Professor Chen Ping Yang’s early significant contributions to mathematical physics

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In the 60’s Professor Chen Ping Yang with Professor Chen Ning Yang published several seminal papers on the study of Bethe’s hypothesis for various problems of physics. The works on the lattice gas model, critical behaviour in liquid-gas transition, the one-dimensional (1D) Heisenberg spin chain, and the thermodynamics of 1D delta-function interacting bosons are significantly important and influential in the fields of mathematical physics and statistical mechanics. In particular, the work on the 1D Heisenberg spin chain led to subsequent developments in many problems using Bethe’s hypothesis. The method which Yang and Yang proposed to treat the thermodynamics of the 1D system of bosons with a delta-function interaction leads to significant applications in a wide range of problems in quantum statistical mechanics. The Yang and Yang thermodynamics has found beautiful experimental verifications in recent years.

I. INTRODUCTION

It was a very sad news that Professor Chen Ping Yang passed away in this May. To our mind, he was a very humble physicist in our mathematical physics community. He went to Brown University to pursue his undergraduate studies in the summer of 1948. Later, he completed his Master Degree of Science at Harvard University in 1953, and his PhD at the Johns Hopkins University in 1960. He then taught physics at The Ohio State University until his retirement in 1998. Although he did not publish many scientific papers \cite{1–6}, his early contributions to mathematical physics made in the 60’s are seminal and influential. His works on physical problems using Bethe’s hypothesis \cite{7} opened various research areas of mathematical physics at that time.

We shall focus here primarily on two works, which Professor Chen Ping Yang did in collaboration with professor Chen Ning Yang, on the one-dimensional (1D) Heisenberg spin chain and the thermodynamics of 1D delta-function interacting Bose gas. The work on the Heisenberg spin chain consists of a series of papers published in the mid-60’s, in which Yang and Yang \cite{1,2} presented for the first time a rigorous analysis of the Bethe ansatz equations for the 1D Heisenberg spin chain throughout the full range of anisotropic parameter $\Delta$ and magnetic field $H$. Moreover, they obtained the ground state energy, the magnetization, the pressure-volume phase diagram and the critical behaviour of magnetization of the model, thereby adding to the results obtained earlier by Hulthén \cite{8}, Orbach \cite{9}, Griffith \cite{10}, Walker \cite{11} des Cloizeaux and Pearson \cite{12} and others. The key importance of their series of papers is the initiation of mathematical analysis in the study of the transcendental Bethe ansatz equations for physical problems in 1D. Their original inventions led to immediate applications in many research areas of mathematical physics.

The work on the thermodynamics of the Bose gas was the paper \cite{24} published in 1969, where Yang and Yang proposed the grand canonical ensemble to calculate the thermodynamics of the 1D Bose gas with delta-function interaction. This was the first exact thermodynamics of many-body interacting systems and led to a significant step to treat macroscopic properties of integrable systems. They showed that the thermodynamics can be determined from the minimization of the Gibbs free energy in terms of particle and hole densities. Such a minimization condition gives rise to the so called Yang-Yang thermodynamic Bethe ansatz equation that determines the dressed energy of the particles in terms of quasi-momenta, interaction strength, chemical potential, and temperature. This method has profound and influential impact on quantum statistical mechanics. The equation they obtained permanently bears the name Yang-Yang thermodynamic Bethe ansatz equation.

The 60’s were arguably be the most exciting time in the history of quantum integrable models. A number of notable Bethe ansatz integrable models in a variety of fields...
of physics were solved at that time, including the Lieb-Liniger Bose gas [12], the Yang-Gaudin model [14, 15], the Hubbard model [16], the SU(N) interacting Fermi gases [17], etc. Professor Chen Ning Yang discovered the necessary condition for the Bethe ansatz solvability, which is now known as the Yang-Baxter equation, i.e. the factorization condition—the scattering matrix of a quantum many-body system can be factorized into a product of many two-body scattering matrices. In the early 70's Professor Rodney Baxter [18] independently showed that such a factorization relation also occurred as the conditions for commuting transfer matrices in 2D lattice models in statistical mechanics. A short time later, the study of Yang-Baxter lattice gas model and found a very interesting feature of the specific heat near the phase transition, displaying a sharp peak. The aim of this short communication is to elaborate further on the two works introduced above. As a matter of fact, we have personally benefited a lot from those works of Professor Chen Ping Yang with Professor Chen Ning Yang. Here we wish to express our highest respect to the humble physicist, who made significant contributions to mathematical physics and statistical mechanics.

In particular, besides the above-mentioned two works, Professor Chen Ping Yang published other interesting papers [3-6]. Together with Professor Chen Ning Yang [4] he studied the lattice gas model and found a very interesting feature of the specific heat near the phase transition, displaying a sharp peak. The aim of this short communication is to elaborate further on the two works introduced above. As a matter of fact, we have personally benefited a lot from those works of Professor Chen Ping Yang with Professor Chen Ning Yang. Here we wish to express our highest respect to the humble physicist, who made significant contributions to mathematical physics and statistical mechanics in the 60's. See Fig. 1 for a memory of Professor Chen Ping Yang.

II. 1D HEISENBERG SPIN CHAIN

In a series of papers [1, 2], Professor Chen Ping Yang, in collaboration with Professor Chen Ning Yang, studied the solutions of the Bethe ansatz equations for the 1D anisotropic Heisenberg spin chain described by the Hamiltonian

$$\mathcal{H} = -\frac{1}{2} \sum \left\{ \sigma_x \sigma'_x + \sigma_y \sigma'_y + \Delta \sigma_z \sigma'_z \right\} - H \sum \sigma_z, \quad (1)$$

where $\sigma_{x,y,z}$ and $\sigma'_{x,y,z}$ are Pauli matrices of different projections on a particular site and on a neighboring site, respectively, $H$ is the external magnetic field, and $\Delta$ is a real anisotropic parameter. For different values of choices: $\Delta = \pm 1$ correspond to the isotropic ferromagnetic and antiferromagnetic Heisenberg chain, respectively; $\Delta > 1$ and $\Delta < -1$ lead to the gapped phases in which the energy spectrum has a gap; $|\Delta| < 1$ corresponds to an anisotropic Heisenberg spin chain in which the energy spectrum is gapless. Hulthén [8], des Cloizeaux and Pearson [12] studied the antiferromagnetic case with $\Delta = -1$. Walker [11] studied the particular case of $\Delta \leq -1$.

Since Bethe proposed a particular wave function to obtain the spectrum of the model (1) in 1931 [7], there were very few publications on the Bethe's method for about 30 years. In Yang and Yang's series of papers published in the mid-60's, they [2] carried out an analytical study of the Bethe ansatz equations for the Heisenberg spin chain throughout the full range of anisotropic parameter $\Delta$ in a presence of magnetic field. Yang and Yang proved that the ground state energy is an analytical function of $\Delta$ and magnetization $y$ denoted as $f(\Delta, y)$ for $|\Delta| < 1$, where $y = \frac{1}{2} \sum \sigma_z$, and $L$ is the total number of sites. In particular, they built up a rigorous analysis of the Bethe ansatz equations of the model so that the function of the ground state $f(\Delta, y)$ was given explicitly.

In this series of papers, they used the inverse tangent function to define the phase shift of the exchange of two spins, namely

$$\Theta(p, q) = 2 \tan^{-1} \left[ \frac{\Delta \sin \left( \frac{\pi p}{\Delta} \right)}{\cos \left( \frac{\pi p}{\Delta} \right) - \Delta \cos \left( \frac{\pi q}{\Delta} \right)} \right], \quad (2)$$

where $p$ and $q$ are the quasimomenta of two exchanged spins. Thus quasimomenta $p_i$'s are within the following range:

$$-\pi < p_j < \pi, \quad \text{for } \Delta \leq -1$$

$$-(\pi - \mu) < p_j < \pi - \mu, \quad \text{for } -1 < \Delta < 1,$$

$$0 \leq \mu < \pi, \quad \text{and } \cos \mu = -\Delta.$$

These regions uniquely define the values of the inverse tangent function [2] for different values of $\Delta$ and therefore one can conveniently represent the Bethe ansatz equations as an integral form in the thermodynamic limit, namely,

$$1 = 2 \pi \rho - \int_{-Q}^{Q} \frac{\partial \Theta(p, q)}{\partial p} \rho(q) dq. \quad (4)$$

This form of the Bethe ansatz equations can be systematically analyzed in a whole range of $H$ for $1 \leq \Delta \leq 1$. It turns out that Yang and Yang’s inverse tangent form [2] can be in general used for the study of other quantum integrable systems.

Moreover, taking the advantage of the Bethe ansatz equation [4], Yang and Yang analyzed the analyticity of the ground state energy function $f(\Delta, y)$ with respect to $\Delta$ and $y$. For a small value of $y$, they calculated the energy function using the Wiener-Hopf method. Furthermore, Yang and Yang presented a very insightful result of the magnetization vs magnetic field for different values of $\Delta$, see Fig. 2. This series of papers [2] naturally formed the fundamental basis for studying the 6-vertex model [3, 25, 27]. This work immediately opened wide applications in 1D many-body problems, see reviews [10, 23].
III. YANG-YANG THERMODYNAMICS

In 1969 Professor Chen Ping Yang in collaboration with Professor Chen Ning Yang published his seminal work on the thermodynamics of the Lieb-Liniger Bose gas. They proposed for the first time the quasimomenta subject to the Bethe ansatz equation of the model. They started their formalism from the distribution function of the quasi-Bethe ansatz equations of the model. They proposed for the first time the grand canonical ensemble description of the integrable model using the Bethe ansatz equation. Technically, the thermodynamics of the Lieb-Liniger Bose gas. They argued that the TBA equation provides a prototype of quantum statistical mechanics. It encodes not only the quantum statistical effect but also the rich quantum many-body dynamical interaction effect. As is mentioned in the commentary by Professor Chen Ning Yang, “It shows the subtlety in the definitions of the vacuum, the interaction, and the excitation spectrum”. Moreover, Professor Chen Ping Yang showed its connection to both bosonic and fermionic statistics using the TBA equation. The particle and hole densities in terms of the quasimomenta reveal such subtle changes under a change of the temperature and interaction strength.

Building on the Yang-Yang’s approach, Professor Minoru Takahashi has made further important contributions to the thermodynamics of 1D integrable models.

In view of the grand canonical ensemble, there exists a quantum phase transition at the chemical potential \( \mu_c = 0 \) at zero temperature. Yang and Yang’s TBA equation provides a precise understanding of the universal thermodynamics, the quantum criticality and the quantum liquid in 1D Lieb-Liniger gas, see a review. Ultracold bosonic atoms trapped in a quasi-1D geometry are ultimately related to the integrable models of quantum gases. Based on the TBA, particularly striking examples were the measurements of the thermodynamics and quantum fluctuations, the dynamic structure factor, the quantum criticality and the Tomonaga-Luttinger liquid (TLL).

In a recent paper, the density profiles of quasi-1D trapped ultracold \(^{87}\)Rb atoms were measured by in situ absorption imaging. The density scaling law and the equation of states were obtained by rescaling these

\[ s = \int_{-\infty}^{\infty} \left[ \rho \ln \left( 1 + \frac{\rho}{\rho_h} \right) - \rho \ln \left( \frac{\rho}{\rho_h} \right) \right] dk. \]

Here we would like to emphasize that such a subtle connection between the Bethe ansatz microscopic states and the macroscopic state of the system play a key role in the Yang-Yang method.

Maximizing the entropy is the next key step in the Yang-Yang approach. The Gibbs free energy per unit length is given by \( G/L = E/L - \mu n - Ts \) with the relation to the free energy \( F = G + \mu N \). Here \( \mu \) is the chemical potential, \( n \) is the linear density. It is important to note that the entropy \( s \), the energy \( E \), and the density \( n \) are functions of the particle and hole densities subject to the Bethe ansatz equation. The minimization condition \( \frac{\partial G}{\partial T} = 0 \) with respect to particle density \( p \) leads to the so-called Yang-Yang thermodynamic Bethe ansatz (TBA) equation

\[ \varepsilon(k) = k^2 - \mu - \frac{Tc}{\pi} \int_{-\infty}^{\infty} \frac{dq}{\varepsilon^2 + (k - q)^2} \ln \left( 1 + e^{\frac{-\varepsilon(q)}{T}} \right), \]

which determines the thermodynamics of the system in a whole temperature regime. Using the Bethe ansatz equation again, the Gibbs free energy per length \( p = - \left( \frac{\partial G}{\partial \mu} \right)_{\mu, c} \) gives

\[ p = \frac{T}{2\pi} \int dk \ln \left( 1 + e^{-\varepsilon(k)/T} \right). \]
measurements at different temperatures and chemical potentials. Based on the obtained equation of states, two crossover branches that distinguish the quantum critical regime from the classical gas and the TLL were observed through the double-peak structure of the specific heat, see Fig. 3. Furthermore, the measured propagations of density disturbances, the Luttinger parameters and also the power-law behavior in the momentum profiles confirm the existence of the TLL. The updated observations of such many-body phenomena have revealed the beauty of Yang and Yang’s grand canonical ensemble approach to interacting many-body systems.

In summary, we have presented two significant works which Professor Chen Ping Yang did in the 60’s. Those works leads to significant applications in a wide range of problems in quantum statistical mechanics and mathematical physics. His contributions are a remarkable legacy to physics.

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[1] C. N. Yang and C. P. Yang, Phys. Rev. 147, 303 (1966).
[2] C. N. Yang and C. P. Yang, Phys. Rev. 150, 321 (1966); C. N. Yang and C. P. Yang, Phys. Rev. 150, 327 (1966); C. N. Yang and C. P. Yang, Phys. Rev. 151, 258 (1966).
[3] S. Sutherland, C. N. Yang, and C. P. Yang, Phys. Rev. Lett. 19, 588 (1967).
[4] C. N. Yang and C. P. Yang, Phys. Rev. Lett. 13, 303 (1964).
[5] C. P. Yang, Phys. Rev. A 2, 154 (1970).
[6] C. N. Yang and C. P. Yang, Phys. Lett. 20, 9 (1966); C. P. Yang, Phys. Rev. Lett. 19, 586 (1967); Alfred C. T. Wu, C. P. Yang, Kurt Fuchel and Sidney Heller, Phys. Rev. Lett. 12, 57 (1964); Alfred C. T. Wu and C. P. Yang, Phys. Rev. D 1, 3180 (1970); B. D. Metcalf and C. P. Yang, Phys. Rev. B 18, 2304 (1978); C. N. Yang and C. P. Yang, Trans. NY Acad. Sci. 40, 267 (1980).
[7] H. A. Bethe, Z. Physik 71, 205 (1931).
[8] L. Hulthen, Arkiv. Mat. Mstron. Fysik 26 A, 11 (1938).
[9] R. Orbach, Phys. Rev. 112, 309 (1958).
[10] R. B. Griffiths, Phys. Rev. 113, A768 (1964).
[11] L. R. Walker, Phys. Rev. 116, 1089 (1959).
[12] J. des Cloizeaux and J. J. Pearson, Phys. Rev. 128, 2131 (1962).
[13] E. H. Lieb and W. Liniger, Phys. Rev. 130, 1605 (1963).
[14] C. N. Yang, Phys. Rev. Lett. 19, 1312 (1967).
[15] M. Gaudin, Phys. Lett. A 24, 55 (1967).
[16] E. H. Lieb and F. Y. Wu, Phys. Rev. Lett. 20, 1445 (1968).
[17] B. Sutherland, Phys. Rev. Lett. 20, 98 (1968).
[18] R. J. Baxter, Ann. Phys. (N. Y.) 70, 193 (1972); R. J. Baxter, Ann. Phys. (N. Y.) 70, 323 (1972).
[19] V. E. Korepin, A. G. Izergin, and N. M. Bogoliubov, Quantum Inverse Scattering Method and Correlation Functions (Cambridge: Cambridge University Press), 1993.
[20] B. Sutherland, Beautiful Models: 70 years of exactly solved quantum many-body problems (Singapore: World Scientific), 2004.
[21] M. Takahashi Thermodynamics of One-Dimensional Solvable Models (Cambridge: Cambridge University Press), 1999.
[22] M. A. Cazalilla, R. Citro, T. Giannarici, E. Orignac and M. Rigol, Rev. Mod. Phys. 83, 1405 (2011).
[23] X. W. Guan, M. T. Batchelor and C. Lee, Rev. Mod. Phys. 85, 1633 (2013).
[24] C. N. Yang and C. P. Yang, J. Math. Phys. 10, 1115 (1969).
[25] I. Nolden, J. Stat. Phys. 67, 155 (1992).
[26] J. D. Shore and D. J. Buckman, Phys. Rev. Lett. 72, 604 (1994); D. J. Buckman and J. D. Shore, J. Stat. Phys. 78, 1227 (1995).
[27] H. Y. Huang, F. Y. Wu, H. Kunz and D. Kim, Physica A 228, 1 (1996).
[28] C. N. Yang. Selected papers 1945-1980 W. H. Freeman and Company, 1983.
[29] M. Takahashi, Prog. Theor. Phys. 46, 401 (1971); M. Takahashi, Prog. Theor. Phys. 46, 1388 (1971).
[30] M. Takahashi, Prog. Theor. Phys. 47, 69 (1972); M. Takahashi, Prog. Theor. Phys. 50, 1519 (1973); M. Takahashi, Prog. Theor. Phys. 52, 103 (1973).
[31] Y.-Z. Jiang, Y.-Y. Chen and X.-W. Guan, Chinese Phys. B 24, 050311 (2015).
[32] A. H. van Amerongen, J. J. P. van Es, P. Wicke, K.V. Kheruntsyan, and N. J. van Druten, Phys. Rev. Lett. 100, 090402 (2008).
[33] J. Armijo, T. Jacqmin, K.V. Kheruntsyan, and I. Bouchoule, Phys. Rev. Lett. 105, 230402 (2010).
[34] J. Armijo, T. Jacqmin, K.V. Kheruntsyan, and I. Bouchoule, Phys. Rev. A 83, 021605(R) (2011).
[35] J. Armijo, Phys. Rev. Lett. 108, 225306 (2012).
[36] T. Jacqmin, J. Armijo, T. Berrada, K.V. Kheruntsyan, and I. Bouchoule, Phys. Rev. Lett. 106, 230405 (2011).
[37] H.-P. Stimming, N. J. Mauser, J. Schmiedmayer, and I. E. Mazets, Phys. Rev. Lett. 105, 015301 (2010).
[38] P. Krüger, P., S. Hofferberth, I. E. Mazets, I. Lesanovsky, and J. Schmiedmayer, Phys. Rev. Lett. 105, 265302 (2010).
[39] Y. Sagi, M. Brook, I. Almog, and N. Davidson, Phys. Rev. Lett. 108, 093002 (2012).
[40] A. Vogler, R. Labouvie, F. Stubenrauch, G. Barontini, V. Guerrera, and H. Ott. Phys. Rev. A 88, 031603 (2013).
[41] F. Meinert, M. Panfil, M. J. Mark, K. Lauber, J.-S. Caux, and H.-C. Nägerl, Phys. Rev. Lett. 115, 085301 (2015).
[42] B. Yang, Y.-Y. Chen, Y.-G. Zheng, H. Sun, H.-N. Dai, X.-W. Guan, Z.-S. Yuan, and J.-W. Pan, Phys. Rev. Lett. 119, 165701 (2017).