LOWER BOUNDS OF POTENTIAL BLOW-UP SOLUTIONS OF
THE THREE-DIMENSIONAL NAVIER-STOKES EQUATIONS IN
$\dot{H}^{\frac{7}{2}}$

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ABSTRACT. We improve previous known lower bounds for Sobolev norms of
potential blow-up solutions to the three-dimensional Navier-Stokes equations
in $\dot{H}^{\frac{7}{2}}$. We also present an alternate proof for the lower bound for the $\dot{H}^{\frac{7}{2}}$
blow-up.

1. INTRODUCTION

We consider the three-dimensional incompressible Navier-Stokes equations
\begin{align*}
\frac{\partial u}{\partial t} + (u \cdot \nabla) u &= -\nabla p + \nu \Delta u, \\
\nabla \cdot u &= 0,
\end{align*}
(1.1)
$u(x,0) = u_0(x),$
where the velocity $u(x,t)$ and the pressure $p(x,t)$ are unknowns, $\nu > 0$ is the kinematic viscosity coefficient, the initial data $u_0(x) \in L^2(\Omega)$, and the spatial domain $\Omega$
may have periodic boundary conditions or $\Omega = \mathbb{R}^3$. The question of the regularity
of solutions to (1.1) remains open and is one of the Clay Mathematics Institute
Millennium Prize problems.

In 1934, Leray published his formative work [8] on the fluid equations. Before
we discuss his seminal work further, we present relevant definitions.

Definition 1.1. For all smooth, divergence free test functions $\phi(x)$, and times
$0 \leq t_0 < t$, a weak solution of (1.1) with divergence-free, finite energy initial data
$u_0$ is an $L^2$-valued function $u(x,t)$ that satisfies
\begin{align*}
\int_\Omega u(x,t) \cdot \phi(x) \, dx - \int_\Omega u(x,t_0) \cdot \phi(x) \, dx \\
= \nu \int_{t_0}^t \int_\Omega u(x,\tau) \cdot \Delta \phi(x) \, dx \, d\tau + \int_{t_0}^t \int_\Omega (u(x,\tau) \cdot \nabla \phi(x)) u(x,\tau) \, dx \, d\tau.
\end{align*}

Definition 1.2. A Leray-Hopf weak solution is a weak solution of (1.1) that sat-
sifies the energy inequality
\begin{align*}
\frac{1}{2} \|u(t)\|_{L^2}^2 + \nu \int_{t_0}^t \|\nabla u(\tau)\|_{L^2}^2 \, d\tau \leq \frac{1}{2} \|u(0)\|_{L^2}^2.
\end{align*}
for $0 \leq t_0 < t$ and $t_0$ almost everywhere.

Definition 1.3. A Leray-Hopf weak solution of (1.1) is regular on time interval $I$
if the Sobolev norm $H^s$ of the solution is continuous for $s > \frac{1}{2}$.
Leray proved the existence of global weak solutions to (1.1) and proved that regular solutions are unique in the class of Leray-Hopf solutions. He also showed that if $\|u\|_{H^1}$ is continuous on $[0, T^*)$ and blows up at time $T^*$, then

$$\|u(t)\|_{H^1(\mathbb{R}^3)} \geq \frac{c}{(T^* - t)^{\frac{s}{2}}}.$$  

Moreover, the bound for $L^p$ norms for $3 < p < \infty$,

$$\|u(t)\|_{L^p(\mathbb{R}^3)} \geq \frac{c_p}{(T^* - t)^{\frac{s}{2p}}},$$

have been known for a long time (see [8] and [6]). The Sobolev embedding $\dot{H}^s(\mathbb{R}^3) \subset L^{\frac{6}{3-2s}}(\mathbb{R}^3)$ and (1.2) yield that

$$\|u(t)\|_{\dot{H}^s(\Omega)} \geq \frac{c}{(T^* - t)^{\frac{s}{2}}},$$

for $\frac{1}{2} < s < \frac{3}{2}$ and $\Omega = \mathbb{R}^3$. Robinson, Sadowski, and Silva extended (1.4) in [9] for $\frac{3}{2} < s < \frac{5}{2}$ for the whole space and in the presence of periodic boundary conditions. This bound is considered optimal for those values of $s$.

When $s > \frac{5}{2}$, Benameur [1] showed

$$\|u(t)\|_{\dot{H}^s(\mathbb{R}^3)} \geq \frac{c(s)\|u(t)\|_{L^2(\mathbb{R}^3)^\frac{5-2s}{s}}}{(T^* - t)^{\frac{s}{2}}},$$

which was improved upon by Robinson, Sadowski, and Silva in [9] to

$$\|u(t)\|_{\dot{H}^s(\Omega)} \geq \frac{c(s)\|u_0\|_{L^2(\Omega)^\frac{5-2s}{s}}}{(T^* - t)^{\frac{s}{4}}},$$

when $\Omega = T^3$ or $\Omega = \mathbb{R}^3$.

The border cases $s = \frac{3}{2}$ and $s = \frac{5}{2}$ required separate treatment. Robinson, Sadowski, and Silva were able to show (1.4) for $s = \frac{3}{2}$ on $\mathbb{R}^3$, but required an epsilon correction for the case with periodic boundary conditions. In [5], Cortissoz, Montero, and Pinilla improved the bound for $s = \frac{3}{2}$ in the presence of periodic boundary conditions. They found

$$\|u(t)\|_{\dot{H}^\frac{3}{2}(T^3)} \geq \frac{c}{\sqrt{(T^* - t)|\log(T^* - t)|}},$$

They also addressed the boundary case $s = \frac{5}{2}$ and found

$$\|u(t)\|_{\dot{H}^\frac{5}{2}(\Omega)} \geq \frac{c}{(T^* - t)|\log(T^* - t)|},$$

when $\Omega = T^3$ or $\Omega = \mathbb{R}^3$.

In this paper, we improve the bound for the $\dot{H}^\frac{3}{2}(\Omega)$-norm to the optimal bound (1.4) in the case with periodic boundary conditions. Our method is not contingent on rescaling arguments and thus works when $\Omega = \mathbb{R}^3$ or when $\Omega$ has periodic boundary conditions. We stress the importance of the $\dot{H}^\frac{3}{2}$ norm, which scales to the $L^\infty$ norm and corresponds to the uncovered limit of (1.4). We also give an alternative proof to the bound for the $\dot{H}^\frac{3}{2}(\Omega)$-norm. Our methods differ from previous works as we utilize Littlewood-Paley decomposition of solutions $u$ of (1.1).
for much of the paper. We denote wave numbers as \( \lambda_q = 2^q \) (in some wave units). For \( \psi \in C^\infty(\Omega) \), define
\[
\psi(\xi) = \begin{cases} 
1 & : |\xi| \leq \frac{1}{2} \\
0 & : |\xi| > 1.
\end{cases}
\]
Next define \( \phi(\xi) = \psi(\xi/\lambda) - \psi(\xi) \) and \( \phi_q(\xi) = \phi(\xi/\lambda_q) \). Then
\[
u = \sum_{q=-\infty}^{\infty} u_q,
\]
in the sense of distributions, where the \( u_q \) is the \( q^{th} \) Littlewood-Paley piece of \( u \).

On \( \mathbb{R}^3 \), the Littlewood-Paley pieces are defined as
\[
u_q(x) = \int_{\mathbb{R}^3} u(x-y) \mathcal{F}^{-1}(\phi_q)(y) \, dy,
\]
where \( \mathcal{F} \) is the Fourier transform. In the periodic case, the Littlewood-Paley pieces are given by
\[
u_q(x) = \sum_{k \in \mathbb{Z}^3} \hat{u}(k) \phi_q(k) e^{ik \cdot x},
\]
where \( (1.8) \) holds provided \( u \) has zero-mean. Moreover, \( u_q = 0 \) in the periodic case when \( q < 0 \). We will use the notation
\[
u_{\leq Q} = \sum_{q \leq Q} u_q, \quad \nu_{\geq Q} = \sum_{q \geq Q} u_q.
\]
We define the homogeneous Sobolev norm of \( u \) as
\[
\|u\|_{\dot{H}^s} = \left( \sum_{q=-\infty}^{\infty} \lambda_q^{2s} \|u_q\|_{L^2}^2 \right)^{\frac{1}{2}}.
\]
Note that it corresponds to the nonhomogeneous Sobolev norm \( H^s \) in the periodic case.

We also integrate some other notions into this discussion, such as that of the dissipation wave number in connection to Kolmogorov’s ideas regarding the inertial range of turbulent flows. We define it as follows:

**Definition 1.4.** The dissipation wave number \( \Lambda(t) \) is defined by
\[
\Lambda(t) := \min_{p>q} \{ \lambda_p : \lambda_p \|u_p\|_{L^2} < c \}.
\]

The time-dependent function \( \Lambda(t) \) separates the low frequency inertial range, where the nonlinear term dominates, from the high frequency dissipative range, where the viscous forces take over in determining the dynamics of the equation. In \( [3] \), Cheskidov and Shvydkoy showed if \( \Lambda(t) \in L^\infty(0,T) \), then \( u(t) \) is regular up to time \( T \). This was improved in \( [4] \), where they showed that \( \Lambda(t) \in L^\frac{5}{2} \) implies regularity and that \( \Lambda(t) \in L^1 \) for all Leray-Hopf solutions.

We suppress \( L^p \) norm notation as \( \| \cdot \|_p := \| \cdot \|_{L_p} \). We will also suppress the notation for domains for integrals and functional spaces, i.e. \( \int := \int_{\Omega} \). All \( L^p \) and Sobolev spaces are over \( \Omega \), where \( \Omega \) either has periodic boundary conditions or is the whole space \( \mathbb{R}^3 \), as described in the introduction (unless explicitly otherwise stated). The methods of proof apply to either domain. Sobolev spaces are denoted by \( H^s \) and homogeneous Sobolev spaces by \( \dot{H}^s \). We will use the symbol \( \lesssim \) (or \( \gtrsim \)) to denote that an inequality holds up to an absolute constant.
2. Bounding Blow-Up for $s = 3/2$

We begin by testing the weak formulation of the Navier-Stokes equation with $\lambda_q^{2s}(u_q)_q$ to obtain

$$
\frac{d}{dt} \left( \lambda_q^{2s} \| u_q \|_2^2 \right) = -\nu \lambda_q^{2s+2} \| u_q \|_2^2 + 2 \lambda_q^{2s} \int [(u \otimes u)_q \cdot \nabla u_q] \, dx.
$$

In the typical fashion, we write

$$
(u \otimes u)_q = u_q \otimes u + u \otimes u_q + r_q(u, u),
$$

for $q > -1$, where the remainder function is given by

$$
r_q(u, u)(x) = \int F^{-1}(\phi_q)(y)(u(x-y) - u(x)) \otimes (u(x-y) - u(x)) \, dy.
$$

Thus we rewrite the nonlinear term as

$$
\int \text{Tr}[\{(u \otimes u)_q \cdot \nabla u_q\}] \, dx = \int r_q(u, u) \cdot \nabla u_q \, dx - \int u_q \cdot \nabla u_{\leq q+1} \cdot u_q \, dx.
$$

We refer the reader to [2] for more details on the method used above and in the following Lemma.

**Lemma 2.1.** For $\frac{1}{\sigma} + \frac{1}{\sigma'} = 1$, the nonlinear term in (2.1) can be bound above by

$$
\int \text{Tr}[\{(u \otimes u)_q \cdot \nabla u_q\}] \, dx \leq \lambda_q^{-1} \| u_q \|_{\sigma'} \sum_{p=-\infty}^{q} \lambda_p^2 \| u_p \|_{2\sigma'}^2 + \lambda_q \| u_q \|_{\sigma'} \sum_{p=q+1}^{\infty} \| u_p \|_{2\sigma} + \sum_{p=-\infty}^{q+1} \lambda_p^2 \| u_p \|_{2\sigma},
$$

**Proof.** We examine the two terms on the right-hand side of (2.4) separately. For the remainder term,

$$
\int r_q(u, u) \cdot \nabla u_q \, dx \leq \| r_q(u, u) \|_{\sigma} \lambda_q \| u_q \|_{\sigma'} \leq \lambda_q \| u_q \|_{\sigma'} \left( \lambda_q^{-2} \sum_{p=-\infty}^{q} \lambda_p^2 \| u_p \|_{2\sigma'}^2 + \sum_{p=q+1}^{\infty} \| u_p \|_{2\sigma} \right),
$$

where $\frac{1}{\sigma} + \frac{1}{\sigma'} = 1$. Later we will also require that $\sigma' \geq 2$. For the second term of (2.4),

$$
\int u_q \cdot \nabla u_{\leq q+1} \cdot u_q \, dx \leq \| u_q \|_{2}^2 \sum_{p=-\infty}^{q+1} \lambda_p \| u_p \|_{\infty} \leq \| u_q \|_{2}^2 \sum_{p=-\infty}^{q+1} \lambda_p^2 \| u_p \|_{2}.
$$

Thus

$$
\frac{d}{dt} \sum_{q=-\infty}^{\infty} \left( \lambda_q^{2s} \| u_q \|_2^2 \right) \leq - \sum_{q=-\infty}^{\infty} \left( \nu \lambda_q^{2s+2} \| u_q \|_2^2 \right) + 2(A + B + C),
$$

(2.8)
where

\[(2.9) \quad A = \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^{q} \lambda_q^{2s-1} \|u_q\|_{\sigma'} \lambda_p^2 \|u_p\|_{2\sigma}^2, \]

\[(2.10) \quad B = \sum_{q=-\infty}^{\infty} \sum_{p=q+1}^{\infty} \lambda_q^{2s+1} \|u_q\|_{\sigma'} \|u_p\|_{2\sigma}^2, \]

\[(2.11) \quad C = \sum_{q=-\infty}^{q+1} \sum_{p=-\infty}^{\infty} \lambda_q^{2s} \|u_q\|_{\frac{q}{2}} \lambda_p^5 \|u_p\|_2. \]

**Lemma 2.2.** Optimal kernels for (2.9) and (2.10) are achieved when \(\sigma = \sigma' = 2\).

**Proof.** We choose \(\sigma\) and \(\sigma'\) by optimizing the kernel that occurs in the estimates of the nonlinear terms (2.9) and (2.10). First we rewrite (2.9):

\[ A = \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^{q} K_A \left( \lambda_p^{(2s-\frac{3}{2}) \frac{2s+1}{4}} \|u_p\|_{2\sigma}^{\frac{3}{2} - \frac{3}{4}} \right) \]

\[ \cdot \left( \lambda_q^{s - \frac{1}{2}} \|u_q\|_{\sigma'} \lambda_p^{\frac{3}{2} - (2s-\frac{3}{2}) \frac{2s+1}{4} + s - \frac{3}{4}} \|u_p\|_{2\sigma}^{\frac{3}{2} - s} \right), \]

where

\[(2.12) \quad K_A = \lambda_p^{\frac{3}{2} - \frac{3}{4} - s}. \]

We require \(\frac{3}{4} + \frac{3}{2} - s > 0\) since \(p \leq q\), thus we must maximize \(f(\sigma') = \frac{3}{2} - \frac{3}{4} + \frac{3}{2}\) for \(\sigma' \in [2, \infty]\) to optimize regularity. Hence

\[(2.13) \quad \sigma' = 2, \]

and thus

\[(2.14) \quad \sigma = 2. \]

We check for consistency with (2.10). We rewrite it as

\[ B = \sum_{q=-\infty}^{\infty} \sum_{p=q+1}^{\infty} K_B \left( \lambda_p^{(2s-\frac{3}{2}) \frac{2s+1}{4}} \|u_p\|_{2\sigma}^{\frac{3}{2} - \frac{3}{4}} \right) \]

\[ \cdot \left( \lambda_q^{s - \frac{1}{2}} \|u_q\|_{\sigma'} \lambda_p^{\frac{3}{2} - (2s-\frac{3}{2}) \frac{2s+1}{4} + s - \frac{3}{4}} \|u_p\|_{2\sigma}^{\frac{3}{2} - s} \right), \]

where

\[(2.15) \quad K_B = \lambda_p^{\frac{3}{2} - \frac{3}{4} - s}. \]

We require \(\frac{3}{2} - \frac{3}{4} - s < 0\) since \(p > q\). If \(\sigma' = 2\), we have \(s > 0\), as needed. \(\square\)

Then (2.9)-(2.11) become

\[(2.16) \quad A = \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^{q} \lambda_q^{2s-1} \|u_q\|_2 \lambda_p^2 \|u_p\|_4^2, \]

\[(2.17) \quad B = \sum_{q=-\infty}^{\infty} \sum_{p=q+1}^{\infty} \lambda_q^{2s+1} \|u_q\|_2 \|u_p\|_4^2. \]
\( C = \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^{q+1} \lambda_q^{2s} \|u_q\|_2^2 \lambda_p^{\frac{3}{2}} \|u_p\|_2. \)

**Theorem 2.3.** Let \( u \) be a solution to the (1.1) with finite energy initial data and let \( s = \frac{3}{2} \). Then the solution \( u \) satisfies the Riccati-type differential inequality

\[
\frac{d}{dt} \sum_{q=-\infty}^{\infty} \left( \lambda_q^{2s} \|u_q\|_2^2 \right) \lesssim \sum_{q=-\infty}^{\infty} \left( \lambda_q^{2s} \|u_q\|_2^2 \right)^{\frac{2s+1}{2s+2}}.
\]

**Proof.** We bound the nonlinear terms. First, we estimate (2.16) for \( s = \frac{3}{2} \). We apply Bernstein’s inequality in three-dimensions and we rewrite the sum

\[
A = \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^{q} \lambda_q^{\frac{3}{2}} \|u_q\|_2 \lambda_p^{\frac{3}{2}} \|u_p\|_2^2
\]

Next, we sum in \( q \), to yield

\[
A \lesssim \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^{q} \lambda_q^{\frac{3}{2}} \left( \frac{\nu}{3} \lambda_p^5 \|u_q\|_2^2 \right) + \lambda_q^{\frac{3}{2}} \left( \nu^{-1} \lambda_p^3 \|u_p\|_2^2 \right)^2,
\]

which indeed amounts to the particular case of

\[
A \lesssim \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^{q} \lambda_q^{\frac{3}{2}} \left( \frac{\nu}{3} \lambda_p^5 \|u_q\|_2^2 \right) + \lambda_q^{\frac{3}{2}} \left( \nu^{-1} \lambda_p^3 \|u_p\|_2^2 \right)^{\frac{2s+1}{2s+2}}.
\]

Next we sum in \( p \) for the first term and exchange the order of summation and sum in \( q \) for the second term of (2.20):

\[
A \lesssim \sum_{q=-\infty}^{\infty} \left( \nu^{-1} \lambda_q^{2s} \|u_q\|_2^2 \right)^{\frac{2s+1}{2s+2}} + \frac{\nu}{3} \sum_{q=-\infty}^{\infty} \left( \lambda_q^{2s+2} \|u_q\|_2^2 \right).
\]

Next, we estimate (2.17). By Bernstein’s inequality for three-dimensions, we have

\[
B = \sum_{q=-\infty}^{\infty} \sum_{p=q+1}^{\infty} \lambda_q^{2s+1} \|u_q\|_2 \|u_p\|_2^2
\]

\[
\lesssim \sum_{q=-\infty}^{\infty} \sum_{p=q+1}^{\infty} \lambda_q^{2s+1} \|u_q\|_2 \lambda_p^{\frac{3}{2}} \|u_p\|_2^2.
\]

We rewrite the sum to look like

\[
B \lesssim \sum_{q=-\infty}^{\infty} \sum_{p=q+1}^{\infty} \lambda_p^{-(s+1)} \left( \lambda_q^{s} \|u_q\|_2 \right) \left( \lambda_p^{s} \|u_p\|_2 \right)^{\epsilon} \left( \lambda_p^{\frac{3}{2} - s} \epsilon + \|u_p\|_2^{2-\epsilon} \right).
\]
where
\[ \epsilon = s - \frac{1}{2}. \]
We require \( \epsilon \) to be nonnegative, thus we need
\[ s \geq \frac{1}{2}. \]  
We apply Young’s inequality with the exponents
\[ \theta_1 = \frac{2(2s + 1)}{2s - 1}, \quad \theta_2 = \frac{2(2s + 1)}{(s - \frac{1}{2})(2s - 1)}, \quad \theta_3 = \frac{2}{2 - s}, \]
where indeed \( \frac{1}{\theta_1} + \frac{1}{\theta_2} + \frac{1}{\theta_3} = 1 \). We require \( \theta_1, \theta_2, \) and \( \theta_3 \) to be positive as well, which adds the restrictions
\[ \frac{1}{2} < s < \frac{5}{2}. \]
Thus by Young’s inequality,
\[ B \lesssim \sum_{q=-\infty}^{\infty} \sum_{p=q+1}^{\infty} \lambda_{p-q}^{-\frac{(s+1)/3}{2}} \left( \nu^{-1} \lambda_q^{2s} \|u_q\|_2^2 \right)^{\frac{2s+1}{2s}} + \sum_{q=-\infty}^{\infty} \sum_{p=q+1}^{\infty} \left[ \lambda_{p-q}^{-\frac{(s+1)/3}{2}} \left( \nu^{-1} \lambda_q^{2s} \|u_q\|_2^2 \right)^{\frac{2s+1}{2s}} + \lambda_{p-q}^{-\frac{(s+1)/3}{2}} \left( \nu^{-1} \lambda_{p}^{2s+2} \|u_p\|_2^2 \right) \right]. \]
Next we sum in \( p \) for the first term and exchange the order of summation and sum in \( q \) for the other two terms of (2.24):
\[ B \lesssim \sum_{q=-\infty}^{\infty} \left( \nu^{-1} \lambda_q^{2s} \|u_q\|_2^2 \right)^{\frac{2s+1}{2s}} + \sum_{p=-\infty}^{\infty} \left[ \left( \nu^{-1} \lambda_q^{2s} \|u_q\|_2^2 \right)^{\frac{2s+1}{2s}} + \left( \nu^{-1} \lambda_p^{2s+2} \|u_p\|_2^2 \right) \right]. \]
Note the summation in \( q \) converges since \( s > -1 \) by (2.24). Then for \( \frac{1}{2} < s < \frac{5}{2} \), in particular when \( s = \frac{3}{2} \), we have
\[ B \lesssim \sum_{q=-\infty}^{\infty} \left( \nu^{-1} \lambda_q^{2s} \|u_q\|_2^2 \right)^{\frac{2s+1}{2s}} + \nu \sum_{q=-\infty}^{\infty} \left( \lambda_q^{2s+2} \|u_q\|_2^2 \right). \]
Finally, we estimate (2.18) for \( s = \frac{3}{2} \). We rewrite the sum
\[ C = \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^{q+1} \lambda_q^{3-\delta} \|u_q\|_2^2 \lambda_p^{\delta} \|u_p\|_2^2 \]
\[ = \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^{q+1} \lambda_{q-p}^{3-\delta} \left( \lambda_q^{\frac{2}{2}} \|u_q\|_2^2 \right)^{2-\delta} \left( \lambda_p^{\frac{2}{2}} \|u_p\|_2^2 \right)^{\delta} \left( \lambda_{q-p}^{\frac{2}{2}} \|u_p\|_2 \right)^{1-\delta}, \]
where \( \delta \) is a small positive number we choose. We apply Young’s inequality with
\[ \theta_1 = \frac{4}{2-\delta}, \quad \theta_2 = \frac{2}{\delta}, \quad \theta_3 = \frac{4}{\delta}, \quad \theta_4 = \frac{2}{1-\delta}, \]
where we require $\delta < 1$ to ensure the exponents are all positive and indeed $\frac{1}{b_2} + \frac{1}{b_3} + \frac{1}{d_4} = 1$. Then we have

$$C \lesssim \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^{q+1} \lambda_{q-p}^{-\delta/4} \left( \nu^{-1} \lambda_{q}^{2s} \|u_{q}\|_{2}^{2} \right)^{\frac{2s+1}{3}} + \frac{\nu}{3} \lambda_{q}^{2s+2} \|u_{q}\|_{2}^{2} \right),
$$

which is a particular case of

$$C \lesssim \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^{q+1} \lambda_{q-p}^{-\delta/4} \left( \nu^{-1} \lambda_{q}^{2s} \|u_{q}\|_{2}^{2} \right)^{\frac{2s+1}{3}} + \frac{\nu}{3} \sum_{q=-\infty}^{\infty} \lambda_{q}^{2s+2} \|u_{q}\|_{2}^{2} \right),
$$

when $s = \frac{4}{3}$. For the first two terms of (2.27), we sum in $p$. For the third and fourth terms, we exchange the order of summation and sum in $q$ to arrive at

$$C \lesssim \sum_{q=-\infty}^{\infty} \left( \nu^{-1} \lambda_{q}^{2s} \|u_{q}\|_{2}^{2} \right)^{\frac{2s+1}{3}} + \frac{\nu}{3} \sum_{q=-\infty}^{\infty} \lambda_{q}^{2s+2} \|u_{q}\|_{2}^{2} \right).
$$

Note $\delta$ positive ensures the summation in $q$ converges. Rewriting the above inequality yields

$$C \lesssim \sum_{q=-\infty}^{\infty} \left( \nu^{-1} \lambda_{q}^{2s} \|u_{q}\|_{2}^{2} \right)^{\frac{2s+1}{3}} + \frac{\nu}{3} \lambda_{q}^{2s+2} \|u_{q}\|_{2}^{2} \right).
$$

We use the estimates (2.21), (2.26), and (2.28) in (2.8) to get the Ricatti-type differential inequality

$$\frac{d}{dt} \sum_{q=-\infty}^{\infty} \left( \lambda_{q}^{2s} \|u_{q}\|_{2}^{2} \right) \lesssim \sum_{q=-\infty}^{\infty} \left( \nu^{-1} \lambda_{q}^{2s} \|u_{q}\|_{2}^{2} \right)^{\frac{2s+1}{3}},$$

for $s = \frac{4}{3}$.

**Theorem 2.4.** Let $u$ be so solution to (1.1) with finite energy initial data such that $u$ loses regularity at time $T^*$. Then for $s = \frac{3}{2}$ the homogeneous Sobolev norm $\|u\|_{H^s}$ is bound below by

$$\frac{1}{\sqrt{T^* - t}} \lesssim \|u\|_{H^{\frac{3}{2}}(\Omega)},$$

for $\Omega = T^{3} \text{ or } \Omega = \mathbb{R}^{3}$.

**Proof.** Let $y = \|u\|_{H^s}^{2}$. By Theorem 2.2, $y$ satisfies the differential inequality

$$\frac{d}{dt} y(t) \lesssim y(t)^{\frac{2s+1}{\frac{3}{2} - s}},$$

where $s = \frac{3}{2}$. The proof proceeds as above with different estimates for the bounds.
for $s = \frac{3}{2}$. Rearranging the inequality and integrating from time $t$ to blow-up time $T^*$ yields

$$
\int_{y(t)}^{\infty} \frac{dw}{w^{s+1}} \lesssim \int_{t}^{T^*} d\tau,
$$

which becomes

$$
\frac{1}{y(t)^{\frac{s}{s-1}}} \lesssim T^* - t.
$$

Then, as desired

$$
\frac{1}{\sqrt{T^* - t}} \lesssim \|u\|_{H^s},
$$

for $s = \frac{3}{2}$.

\[ \Box \]

Remark 2.5. Arguments similar to those for Theorem 2.3 can be employed to show (2.29) for $\frac{1}{2} < s < \frac{3}{2}$ on $\Omega$, whether $\Omega$ is $\mathbb{R}^3$ or if it has periodic boundary conditions. For $\frac{3}{2} < s < \frac{5}{2}$, a far less technical method (more like that employed in the following section) can be used to bound the nonlinear term and then achieve (1.4) on $\Omega$.

3. Bounding Blow-Up for $s = \frac{5}{2}$

In this section, we offer an alternate proof of Theorem 1.3 of [5] for the other border case left $s = \frac{5}{2}$ left open in [4].

**Theorem 3.1.** Let $u$ be so solution to (1.1) with finite energy initial data such that $u$ loses regularity at time $T^*$. Then for $s = \frac{5}{2}$ the homogeneous Sobolev norm $\|u\|_{H^s}$ is bound below by

$$
\frac{1}{T^* - t} \lesssim \|u\|_{H^s(\Omega)} \left( 2 + \log \|u\|_{H^s(\Omega)} \right)
$$

when $\|u\|_{H^s} \geq 1$, and

$$
\frac{1}{T^* - t} \lesssim \|u\|_{H^s(\Omega)}
$$

when $0 < \|u\|_{H^s} < 1$, for $\Omega = \mathbb{T}^3$ or $\Omega = \mathbb{R}^3$.

**Proof.** We follow a similar process as in Section 3 of [4]. We test (1.1) with $\partial^{\alpha} u$ on the time interval $[0,T)$ for $|\alpha| \leq s$, a multiindex. This yields

$$
\frac{1}{2} \frac{d}{dt} \sum_{|\alpha| \leq s} \|\partial^{\alpha} u\|^2 \leq -\nu \|u\|_{H^{s+1}}^2 + \sum_{1 \leq i \leq s} \int \partial^{\alpha - i} u \cdot \nabla \partial^i u \cdot \partial^\alpha u.
$$

Incompressibility, as noted in [4], reduces the above to

$$
\frac{1}{2} \frac{d}{dt} \|u\|_{H^s}^2 \leq -\nu \|u\|_{H^{s+1}}^2 + \sum_{1 \leq i \leq s} \int \partial^{\alpha - i} u \cdot \nabla \partial^i u \cdot \partial^\alpha u.
$$

(3.3)
We use the Besov-type Logarithmic Sobolev inequality in Lemma 3.2 of [4] on the right-hand side of (3.3) to find
\[- \nu \|u\|_{H^{s+1}}^2 + \sum_{1 \leq \alpha \leq s \atop i < \alpha} \left| \int \partial^{\alpha-i} u \cdot \nabla \partial^i u \cdot \partial^\alpha u \right| \lesssim \|u\|_{L^\infty} (1 + \log \|u\|_{H^s}) \|u\|_{H^s}^2,
\]
thus
\[- \nu \|u\|_{H^{s+1}}^2 + \sum_{1 \leq \alpha \leq s \atop i < \alpha} \left| \int \partial^{\alpha-i} u \cdot \nabla \partial^i u \cdot \partial^\alpha u \right| \lesssim \sup_{q \leq Q} \|\nabla u_q\|_\infty (1 + \log \|u\|_{H^s}) \|u\|_{H^s}^2,
\]
for \(s > \frac{5}{2}\). By Bernstein’s inequality in three-dimensions, we have
\[(3.4) \quad - \nu \|u\|_{H^{s+1}}^2 + \sum_{1 \leq \alpha \leq s \atop i < \alpha} \left| \int \partial^{\alpha-i} u \cdot \nabla \partial^i u \cdot \partial^\alpha u \right| \lesssim \sup_{q \leq Q} \lambda_{q, s}^\gamma \|u_q\|_2 (1 + \log \|u\|_{H^s}) \|u\|_{H^s}^2.
\]
Denote \(y = \|u\|_{H^s}^2\), then (3.3) and (3.4) yield the differential inequality
\[(3.5) \quad \frac{d}{dt} y \lesssim \sup_{q \leq Q} \lambda_{q, s}^\gamma \|u_q\|_2 (1 + \frac{1}{2} \log y) y.
\]
For \(\gamma \in [0, \frac{5}{2}), \frac{5}{2s} - \frac{5}{s} \leq 1\), and by Definition 1.12 then
\[
\frac{d}{dt} y \lesssim \Lambda^\gamma \sup_{q \leq Q} \lambda_{q, s}^\gamma \|u_q\|_2 (1 + \frac{1}{2} \log y) y
\]
\[
\lesssim \Lambda^\gamma \sup_{q \leq Q} \left( \lambda_{q, s}^{\frac{5}{2}} \|u_q\|_2^{\frac{5}{2s}} \right) \|u_q\|_2^{1 - \frac{5}{2s} + \frac{5}{s}} (1 + \frac{1}{2} \log y) y
\]
\[
= \Lambda^\gamma \|u_q(0)\|_2^{1 - \frac{5}{2s} + \frac{5}{s}} y^\beta (1 + \frac{1}{2} \log y),
\]
where
\[(3.6) \quad \beta = 1 + \frac{5}{4s} - \frac{\gamma}{2s}
\]
and we require
\[(3.7) \quad s \geq \frac{5}{2} - \gamma.
\]
We rearrange the above differential inequality and integrate from time \(t\) to blowup time \(T^*\) to find
\[(3.8) \quad \int_{y(t)}^{\infty} \frac{d\omega}{\omega^\beta (1 + \frac{1}{\omega} \log \omega)} \lesssim \|u_q(0)\|_2^{1 - \frac{5}{2s} + \frac{5}{s}} \int_t^{T^*} \Lambda(\tau)^\gamma d\tau.
\]
We examine the integral on the left-hand side of (3.8) in two cases: first when \(y \geq 1\), then for \(0 < y < 1\).
For $y \geq 1$, we apply the change of variables
\[ v = 2(\beta - 1)(1 + \frac{1}{2}\log w) \]
to yield
\[
\int_0^\infty \frac{dw}{w^\beta (1 + \frac{1}{2}\log w)} = 2e^{2(\beta - 1)} \int_0^\infty \frac{e^{-v}}{v} dv = 2e^{2(\beta - 1)} \Gamma((\beta - 1)(2 + \log y)),
\]
Note for $y \geq 1$, we may replace $\log(\cdot)$ with $\log(\cdot)$. Here $\Gamma(0, x)$ is the upper incomplete Gamma function. By Jensen’s inequality, $\Gamma(0, x)$ enjoys the lower bound
\[
\Gamma(0, x) \geq e^{-x} + x.
\]
Hence, applying (3.10) to the right-hand side of (3.9) yields
\[
\int_0^\infty \frac{dw}{w^\beta (1 + \frac{1}{2}\log w)} \geq 2e^{2(\beta - 1)} e^{-((\beta - 1)(2 + \log y))} = 2y^{1-\beta} \Gamma(0, (\beta - 1)(2 + \log y)).
\]
Then (3.8) becomes
\[
2y^{1-\beta} \Gamma(0, (\beta - 1)(2 + \log y)) \approx 2 \|u_q(0)\|^{1-\frac{\beta}{2} + \frac{\gamma}{2}} \int_t^{T^*} \Lambda(\tau) \gamma d\tau,
\]
which yields
\[
\frac{2\|u_q(0)\|^{1-\frac{\beta}{2} + \frac{\gamma}{2}}}{T^* - t} \approx y^{\beta-1} (1 + (\beta - 1)(2 + \log y)) = y^{\frac{\gamma}{4s}} \left( 1 + \frac{5 - 2\gamma}{4s} (2 + \log y) \right).
\]
The best bound for this method is achieved when $\gamma = 0$, so (3.12) becomes
\[
\frac{2\|u_q(0)\|^{1-\frac{\beta}{2} + \frac{\gamma}{2}}}{T^*$ - $t} \approx y^{\frac{\gamma}{4s}} \left( 1 + \frac{5}{4s} (2 + \log y) \right).
\]
Thus
\[
\frac{2\|u_q(0)\|^{1-\frac{\beta}{2} + \frac{\gamma}{2}}}{(T^*$ - $t)^{\frac{\gamma}{4s}}} \approx \|u\|^{\frac{\gamma}{4s}} \left( 1 + \frac{5}{2s} + \frac{5}{4s} \log \|u\|^{\frac{\gamma}{2s}} \right),
\]
which is valid for $s \geq \frac{5}{2}$. Specifically, however, when $s = \frac{5}{2}$, we have
\[
\frac{1}{T^*$ - $t} \approx \|u\|^{\frac{\gamma}{2s}} \left( 2 + \log \|u\|^{\frac{\gamma}{2s}} \right),
\]
which implies
\[
\frac{1}{(T^*$ - $t) \log(T^*$ - $t)} \approx \|u\|^{\frac{\gamma}{4s}}.
\]
When $0 < y < 1$, then (3.3) simplifies to
\[
\int_0^\infty \frac{dw}{w^\beta} \approx \|u_q(0)\|^{1-\frac{\beta}{2} + \frac{\gamma}{2}} \int_t^{T^*} \Lambda(\tau) \gamma d\tau,
\]
and yields the lower bound
\begin{align}
\|u_q(0)\|_{2}^{-\frac{(1-\frac{2}{p}+\frac{3}{2})}{2}} \int_{t}^{T^*} \Lambda^\gamma \, d\tau \lesssim y^{\frac{s-2s}{2}}.
\end{align}

When \( \gamma = 0 \), then
\begin{align}
\|u_q(0)\|_{2}^{\frac{1}{2}-1} \frac{1}{T^* - t} \lesssim y^{\frac{s}{2}},
\end{align}

so
\begin{align}
\|u_q(0)\|_{2}^{\frac{1}{2}-\frac{2s}{5}} \frac{1}{(T^* - t)^{\frac{2s}{5}}} \lesssim \|u\|_{H^s},
\end{align}

for \( s \geq \frac{5}{2} \). In particular when \( s = \frac{5}{2} \), we have the bound
\begin{align}
\frac{1}{T^* - t} \lesssim \|u\|_{H^s},
\end{align}

provided \( \|u\|_{H^s} < 1 \).

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