On the Lefschetz Standard Conjecture

José J. Ramón Marí

Abstract

The subject of the present paper is Grothendieck’s Lefschetz standard conjecture \( B(X) \). Our main result is that, if \( X \) is a projective smooth variety of dimension \( n \) and the conjecture \( B(Y) \) holds for the generic fibre \( Y \) (of dimension \( n - 1 \) over the field \( k(t) \)) of a suitable Lefschetz fibration of \( X \), then the operator \( \Lambda_X - p_X^{n+1} \) is algebraic. If in addition \( p_X^{n+1} \) is algebraic, then \( B(X) \) is settled. Along the way we establish the algebraicity of the Künneth projectors \( \pi_i X \) for \( i \neq n-1, n, n+1 \) under the above hypotheses.

1 Introduction

All varieties involved are assumed to be smooth and projective, unless otherwise stated. The notations on correspondences that we adopt are those of Kleiman [18], Jannsen [15], Scholl [21]. We fix a prime \( \ell \neq \text{char} k \), and make the harmless assumption that our field \( k \), of arbitrary characteristic, contains the \( \ell^N \)-th roots of unity for all \( N \); then we go on with the ‘heresy’ [13] \( \mathbb{Z}_\ell(1) \approx \mathbb{Z}_\ell \).

Let \( X \) be a smooth projective variety of dimension \( n \) over a field \( k \); we now fix a very ample line bundle \( L \), giving an immersion in \( \mathbb{P}^N \), which we replace if necessary by a tensor power \( L \otimes m \) to obtain condition (A) of Section 4. Let \( Y \) be a smooth hyperplane section; we write \( \xi_X := [Y] \in H^2(X)(1) \). Let \( A^*(X) \) will denote the graded ring of algebraic cycles modulo homological equivalence with coefficients over \( \mathbb{Q} \), and \( A^{n+r}_{n+r}(X \times X) \) will denote the ring of homological correspondences (with coefficients over \( \mathbb{Q} \)), \( o \) being the product considered. Note that the degree of a correspondence \( u \in CH^{\dim X+r}(X \times X') \) is \( r \) as usual [10], and the cohomological degree of \( u \), i.e. the degree of \( u \) as an operator in cohomology \( H^*(X) \rightarrow H^*(Y) \) will be \( 2r \). Given a subspace \( V \) of \( H^*(X) \), we denote by \( \psi_V \) the orthogonal projection onto \( V \). Following Kleiman [18], we denote the trace (or orientation) map by \( \langle \rangle : H^*(X) \rightarrow \mathbb{Q}_\ell \) and the Poincaré duality pairing by \( \langle , \rangle : H^i(X) \otimes H^{2n-i}(X) \rightarrow \mathbb{Q}_\ell \).

The Hard Lefschetz Theorem [7] states that the maps

\[ L^{n-i} : H^i(X) \rightarrow H^{2n-i}(X) \]

are isomorphisms (henceforth called Lefschetz isomorphisms). One can then define the primitive subspaces \( P^i(X) = \text{Ker} L^{n-i+1} \cap H^i(X) \), and one has a Lefschetz
decomposition of $H^*(X)$: $H^i(X) = \oplus L^j P^{i-2j}(X)$. Let $x = \sum L^j x_{i-2j}$ be the Lefschetz decomposition of $x \in H^i(X)$. Denote $i_1 = \max\{i-n,1\}$. We define the following operators of degree $-2$:

$$\Lambda x = \sum_{j \geq i_1} L^{j-1} x_{i-2j},$$
$$c\Lambda x = \sum_{j \geq i_1} j(n-i+j+1)L^{j-1}x_{i-2j}.$$  

We denote the Künneth projectors $H^*(X) \rightarrow H^i(X) \hookrightarrow H^*(X)$ by $\pi^i_X = \pi^i$. We define the operator of degree $0$

$$H = H_X = \sum_{i=0}^{2n} (n-i)\pi^i_X.$$  

The following operators are also essential: for $x = \sum L^j x_{i-2j} \in H^i(X)$, $p^j x = \delta_{i,k} x_k$, when $i \leq n$, and $p^j x = \delta_{i,k} x_{2n-k}$ for $k > n$; it is clear that $p^j$ is a projector for $i \leq n$. Whenever we have polarised varieties $X_i$, we will consider the induced polarisation on $X_1 \times X_2$, and so $L_{X_1 \times X_2} = L_{X_1} \otimes 1 + 1 \otimes L_{X_2}$. We will do likewise when we have an inclusion; for instance, let $i : Y \subset X$ denote an inclusion of a smooth hyperplane section. Then $\xi_Y = \iota^*\xi_X$ and $L_X = \iota_*\iota^* L_Y = \iota^*\iota_*$. We denote the space of vanishing cycles by $V(Y) = \text{Ker} \iota_*|H^{n-1}(Y) \subset H^{n-1}(Y)$, with $Y$ as above.

We recall the following result:

**Proposition 1.1** (Kleiman [18] 1.4.6, [1]) The operators $c\Lambda, L, H$ are an $sl_2$-triple; in other words, the following identities hold:

$$[c\Lambda, L] = H, [H, L] = -2L, [H, c\Lambda] = 2c\Lambda.$$  

The following conjecture was stated by Grothendieck, and is one of his standard conjectures [13]:

**B(X):** The operator $\Lambda$ is induced by an algebraic cycle; equivalently ([18] Prop. 2.3), all the operators in the $sl_2$-triple ($c\Lambda, L, H$) are algebraic.

The conjecture $B(X)$ is known for curves, surfaces, generalised flag varieties, abelian varieties and is stable under products and smooth hyperplane sections [18]. We will therefore assume that $n \geq 3$. For a discussion on this form of the conjecture–regarding the field of definition–see [7.3]. Another standard conjecture of Grothendieck, weaker than $B(X)$, (Kleiman [18] 2.4, [13]) regards the algebraicity of the Künneth projectors (again, we refer to [7.3]):

**C(X):** The Künneth projectors $\pi^i$ are algebraic for all $i = 0, \ldots, 2n$.

The main result of this paper states as follows.

**Main Theorem.** Let $X$ be smooth projective of dimension $n \geq 3$. Assume the conjecture $B(Y)$ for the general fibre $Y$ of a Lefschetz pencil of $X$ satisfying condition (A) (see Section 4). Then the operator $\Lambda X - p^{n+1}_X$ is algebraic.
The following result on $C(X)$ is proven in Proposition 7.1, although it may be viewed as a corollary of the Main Theorem (see subsection 7.2):

**Partial result on $C(X)$.** Assume $B(Y)$ for $Y$ as above. Then the Künneth projectors $\pi^i_X$ are algebraic for all $i \neq n - 1, n, n + 1$.

The departure point of our proof is essentially the algebraic cycle $\Lambda_Y$ on the generic fibre $Y/k(t)$ of a Lefschetz fibration of $X$ satisfying condition (A) of Section 4 (Katz [5] XVIII.5.3), $\rho: \tilde{X} \to \mathbb{P}^1$. In our proof, we pay special attention to the correspondences supported on $\tilde{X} \times_{\mathbb{P}^1} \tilde{X}$ (see 4.1), which turn out to preserve the Leray filtration of $\rho$, as will be seen in Proposition 4.10. Whenever we have an algebraic class $u$ in $A^{n-1+r}(Y \times Y)$, a lifting (or extension) of $u$ will denote a class supported on $\tilde{X} \times_{\mathbb{P}^1} \tilde{X}$ which yields $u$ after restriction to the generic fibre of $\mathbb{P}^1$. Our proof requires that the assumptions be over $k(t)$.

The consequences of establishing the full conjecture $B(X)$ for general $X$ would be remarkable. Not only would this yield a satisfactory category of pure motives in characteristic zero, but as shown by Y. André [1] it would imply the Variational Hodge Conjecture, hence the Hodge conjecture for arbitrary products of the form $A \times X_1 \times \cdots \times X_m$, where $A$ is an abelian variety and $X_i$ are K3 surfaces.

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## 2 General results

The results in this section need no more background than Kleiman [15]. For the sake of completeness, we include the proof of the following lemma.

**Lemma 2.1** Let $u \in A^{n+r}(X \times X)$ be a correspondence of degree $r$ on $X$. The following identity holds:

$$[H,u] = -2r \cdot u.$$  

**Proof:** One has $u\pi^i = \pi^{i+2r}u$, hence

$$uH = \sum (n-i)u\pi^i = \sum (n-i)\pi^{i+2r}u = Hu + 2r \cdot u.$$  

Isolating yields $[H,u] = -2r \cdot u$ as desired. □

**Lemma 2.2** Let $f: X' \to X$ be a generically finite, surjective morphism of smooth projective varieties. Assume that $C(X')$ holds; then $C(X)$ holds.

The lemma follows readily from the identity $\pi^i_X = \frac{1}{\deg(f)}f_* \pi^i_{X'}f^*$.

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Lemma 2.3 If \( j_1 + 2i_1 + j_2 + 2i_2 = 2n \), then the pieces \( L^{i_1}P^{j_1}(X) \) and \( L^{i_2}P^{j_2}(X) \) are orthogonal for \( j_1 \neq j_2 \). Let \( 0 \leq i \leq n \); then the operator \( p^i \) is a projector, and \( p^{2n-i} \) is a symmetric operator characterised by \( p^{2n-i} : L^{n-i}P^i(X) \to P^i(X) \) is given by the inverse of the Lefschetz isomorphism \( L^{n-i} \), and \( p^{2n-i}L^jP^k(X) = 0 \) if \( (j,k) \neq (n-i,i) \). The following identities hold:

\[
p^{2n-i}L^{n-i} = p^i, \quad L^{n-i}p^{2n-i} = t^ip^i.
\]

Proof: The first assertion implies the rest of the lemma. Suppose \( j_2 > j_1 \); then

\[
L^{i_2}P^{j_2}(X) \wedge L^{i_1}P^{j_1}(X) = L^{i_1+i_2}P^{j_1}(X) \wedge P^{j_2}(X) = 0;
\]

indeed, consider a smooth linear section \( \kappa : W \hookrightarrow X \) of codimension \( (i_1 + i_2) \). Then \( \kappa_*\kappa^*(P^{j_1}(X) \wedge P^{j_2}(X)) = 0 \), since \( j_2 > \frac{j_1+i_2}{2} = \dim W \), hence \( \kappa^*P^{j_2}(X) = 0 \) by \([18] 1.4.7, [5] \) Exp. XVIII (5.2.4). This proves the assertion. ■

Lemma 2.4 The operators \( L, \Lambda \) and \( \kappa \Lambda \) are symmetric.

Proof: The first two are well-known \([18] \). We prove \( \kappa \Lambda = t(\kappa \Lambda) \). By Lemma 2.3 one need only check the following. Let \( x = L^jx_{i-2j} \in L^jP^{n-2j}(X), y = L^{n-i+j+1}y_{i-2j} \in L^{n-i+j+1}P^{n-2j}(X) \); the following equality holds:

\[
\langle \kappa \Lambda L^jx_{i-2j}, L^{n-i+j+1}y_{i-2j} \rangle = \langle L^jx_{i-2j}, \kappa \Lambda L^{n-i+j+1}y_{i-2j} \rangle.
\]

Indeed, let \( c = \langle L^{n-i}x_{i-2j}, y_{i-2j} \rangle \). Then l.h.s. = \( j(n-i+j+1)c \) and r.h.s. = \( (n-i+j+1)jc \), since \( j = n-(2n-i+2) + (n-i+j+1)+1 \). The Lemma is thus established. ■

Proposition 2.5 1. The following non-commutative rings of operators are equal:

\[
\mathbb{Q}\langle L, \Lambda \rangle = \mathbb{Q}\langle L, \kappa \Lambda \rangle = \mathbb{Q}\langle L, p^n, \cdots, p^{2n} \rangle.
\]

2. \( B(X) \) holds if and only if, for all \( i < n \), the inverse

\[
\theta^i : H^{2n-i}(X) \to H^i(X)
\]

to the Lefschetz isomorphism \( L^{n-i} : H^i(X) \cong H^{2n-i}(X) \) is induced by an algebraic correspondence for \( i < n \).

The first assertion follows from Kleiman \([18] \) Prop. 1.4.4. The second is proved in op. cit., 2.3. ■

The morphisms \( \iota^*, \iota_* \) are well-behaved with respect to the Lefschetz decompositions of \( X, Y \) (see Kleiman \([18] \) Prop. 1.4.7). The following two lemmas relate the operators \( \Lambda_X, \Lambda_Y \).

Lemma 2.6 The following identity holds:

\[
(1) \quad \iota^*\Lambda_X = \Lambda_Y \iota^* + \sum_{j=n+1}^{2n-2} \iota^*L^{j-n-1}p^j_X.
\]
denoted as follows: 

Lefschetz fibration induces a fibration (henceforth called a)

We denote by $X$ the blowing-up of $Y$ by abus de langage $t_1 = \Delta$, and for any two $X \subset P$.

Lemma 2.8 The conjecture $C(X)$ holds if and only if the semisimple operator $H$ is algebraic.

Proof: The result follows readily from the identities

$$[\Delta_X] = id_{H^*} = \sum \pi^i$$ and $H^r = \sum (n - i) \pi^i$ for $r \in \mathbb{N}$. ■

3 The cohomology of Lefschetz pencils

For the basic results and the tone of this section we follow Katz [5] Exp. XVIII; we assume $k$ to be algebraically closed. For $X$ an $n$-dimensional variety and $L \hookrightarrow \mathbb{P}^N$ a suitable embedding, there exists a line $L \subset (\mathbb{P}^N)^\vee$ cutting the dual variety $X^\vee$ of $X \subset \mathbb{P}^N$ transversally; $L$ is then called a Lefschetz pencil. A basic property of $L$ is that, for every hyperplane $t \in L, X_t = X \cap H_t$ is either smooth or has a unique singular point which is an ordinary double point. The base locus of $L$ in $X$ will be denoted by $\Delta$, and for any two $t_1 \neq t_2 \in L$ one has a transversal intersection $X_{t_1} \cap X_{t_2} = \Delta$. Thus $\Delta$ is smooth of dimension $n - 2$: for any smooth member $Y = X_t$ as above, we will denote the canonical inclusion by $h : \Delta \hookrightarrow Y$. If $\tilde{X}$ denotes the blowing-up of $X$ centred at $\Delta$, projection induces a map $X - \Delta \to \mathbb{P}^1 \cong L$ which induces a fibration (henceforth called a Lefschetz fibration, or Lefschetz pencil by abus de langage):

$$\rho : \tilde{X} \to \mathbb{P}^1.$$

We denote by $f$ the blowing-up map $f : \tilde{X} \to X$. The full blow-up diagram will be denoted as follows:

$$\begin{align*}
\Delta & \xrightarrow{i} \tilde{X} \\
\downarrow g & \downarrow f \\
\Delta & \xrightarrow{j} X,
\end{align*}$$
where \( j \) is the canonical inclusion \( \Delta \subset X \). \( \tilde{\Delta} \) is then the exceptional divisor and, \( \Delta \) being a complete intersection, \( g \) is a trivial projective bundle; \( j \) denotes the inclusion \( \tilde{\Delta} \subset \tilde{X} \).

We describe the cohomology of \( \tilde{X} \) in the following proposition.

**Proposition 3.1** (Katz [5] Exp. XVIII Prop. 4.2) Notations and assumptions being as above. Then:

(i) the following homomorphisms are mutual inverses:

\[
H^\bullet(\tilde{X}) \xrightarrow{f_\ast \oplus g_\ast i^\ast} H^\bullet(X) \oplus H^{\bullet-2}(\Delta)(-1)
\]

and

\[
H^\bullet(X) \oplus H^{\bullet-2}(\Delta)(-1) \xrightarrow{f^\ast + i^\ast g^\ast} H^\bullet(\tilde{X}).
\]

(ii) Transport of structure via the above isomorphisms endows \( H^\bullet(X) \oplus H^{\bullet-2}(\Delta)(-1) \) with a structure of algebra, which expresses cup-product on \( \tilde{X} \) as follows. For \( a, b \in H^\bullet(X), x, y \in H^{\bullet-2}(\Delta)(-1) \) one has:

\[
\begin{align*}
(0 \oplus x) \wedge (0 \oplus y) &= -j_\ast(xy) \oplus 2L_\Delta xy, \\
(a \oplus 0) \wedge (b \oplus 0) &= ab \oplus 0, \\
(a \oplus 0) \wedge (0 \oplus y) &= 0 \oplus j^\ast(a)y, \\
(0 \oplus x) \wedge (b \oplus 0) &= 0 \oplus xj^\ast(b).
\end{align*}
\]

The Poincaré duality pairing is expressed as follows in terms of the above decomposition. If \( x \oplus y \in H^i(\tilde{X}), x' \oplus y' \in H^{2n-i}(\tilde{X}) \), then

\[
\langle x \oplus y, x' \oplus y' \rangle_{\tilde{X}} = \langle x, x' \rangle_X - \langle y, y' \rangle_\Delta.
\]

Let \( \iota : Y \hookrightarrow X \) denote the canonical inclusion of a smooth hyperplane section \( Y \) in \( X \). If \( Y = X_t \) is a smooth fibre \( \rho^{-1}(t) \) of \( \rho \), let \( k : Y \hookrightarrow \tilde{X} \) denote the canonical inclusion. The following result expresses the cohomology of \( k_* \) and \( k^\ast \) in terms of Proposition 3.1.

**Proposition 3.2** (Katz [5] Exp. XVIII 5.1.1) Notations and assumptions as above; the restriction homomorphism is expressed by

\[
k^\ast = \iota^\ast + h^\ast : H^\bullet(X) \oplus H^{\bullet-2}(\Delta)(-1) \to H^\bullet(Y)
\]

and the Gysin homomorphism has the expression

\[
k_* = \iota_* - h^* : H^{\bullet-2}(Y)(-1) \to H^\bullet(X) \oplus H^{\bullet-2}(\Delta)(-1).
\]

Since we will deal with the Lefschetz theory of both \( X \) and \( \tilde{X} \), the following discussion will help prove our Main Theorem.

**Choice of \( L_{\tilde{X}} \):** We know by Hartshorne [14] II.7.10, II.7.11 that the line bundle \( L_N = f^\ast L^N_X \otimes O_{\tilde{X}}(-\tilde{\Delta}) \) is very ample on \( \tilde{X} \) for \( N \geq N_0 \). Consider \( N = m + 1 \) such that \( m \geq N_0 \), and choose \( L_{\tilde{X}} := L_{m+1} \).
Proposition 3.3  Consider the polarisation on $\tilde{X}$ given by the divisor class $\xi_{\tilde{X}} = c_1(L_{m+1}) = m \cdot f^*\xi_X + \rho^*([t])$ for $t \in \mathbb{P}^1$ a regular value of $\rho$ (not necessary). Let $L_{\tilde{X}}$ be the Lefschetz operator of this polarisation. One also has $f^*\xi_X = \xi_X \oplus 0$ and $f^*(\xi_X) \wedge (x \oplus y) = L_X x \oplus L_\Delta y$. In terms of the decomposition of Proposition 3.1, $L^r_{\tilde{X}}$ is expressed as follows:

$$L^r_{\tilde{X}} (x \oplus 0) = m^{r-1}(m + r)L^r x \oplus -r \cdot m^{r-1}L^r-1\Delta^r x,$$

and

$$L_{\tilde{X}} (0 \oplus y) = r \cdot m^{r-1}L^{r-1}\Delta y \oplus m^{r-1}(m - r)L^r_\Delta y.$$

Proof: Using Propositions 3.1, 3.2, the obtain the following:

$$\xi_{\tilde{X}} = f^*\xi_X = [Y] \oplus 0, \quad [\Delta] = 0 \oplus 1\Delta, \quad [\rho^*(t)] = [Y] \oplus -1\Delta,$$

and $c_1(L_0) = (m + 1) \cdot [Y] \oplus -1\Delta$. Using the easy fact $\rho^*(t)^2 = 0$, we obtain

$$\xi^r_{\tilde{X}} = m^r f^r\xi_X + r \cdot m^{r-1}f^r(\xi_X)^{-1}\cdot \rho^*(t) = (m + r)m^{r-1}\xi_X \oplus -r \cdot m^{r-1}\xi^r_\Delta.$$

The Proposition now follows from Proposition 3.1(ii). ■

4  The Leray filtration of a Lefschetz pencil

Assume $k = \overline{k}$ as in the previous section. Choose a Lefschetz pencil on $X$, denoted by $\rho : \tilde{X} \to \mathbb{P}^1$.

Condition (A) of Katz [5] Exp. XVIII, 5.3 will be important for our purposes; we include it below.

**Condition (A):** Let $\nu : \mathcal{U} \subset \mathbb{P}^1$ be contained within the smooth locus of $\rho$. The adjunction morphisms

$$R^i \rho_* \mathbb{Q}_\ell \to \nu_* \nu^* R^i \rho_* \mathbb{Q}_\ell$$

are isomorphisms for all $0 \leq i \leq 2n - 2$ (independent of $\mathcal{U}$).

An immediate application of the weak Lefschetz theorem yields the first assertion of the following Lemma.

**Lemma 4.1** (Exp. XVIII Lemma 5.4, Th. 6.3, Cor. 6.4; [7]) If the Lefschetz pencil $\rho$ satisfies condition (A), then the sheaves $R^i \rho_* \mathbb{Q}_\ell$ are constant for $i \neq n - 1$. A suitable multiple of a given polarisation contains one such Lefschetz pencil.

**Theorem 4.2** ([5] 5.6, 5.6.8; [6] Sec. 2; [7]) For a Lefschetz pencil $\rho : \tilde{X} \to \mathbb{P}^1$, the Leray spectral sequence

$$E^{i, j}_2 = H^i(\mathbb{P}^1, R^j \rho_* \mathbb{Q}_\ell) \Rightarrow H^{i + j}(\tilde{X})$$

degenerates at $E_2$. For $k : Y = X_t \hookrightarrow \tilde{X}$ the inclusion map of a smooth fibre, the Leray filtration of $\rho$ can be interpreted as follows:
1. \( F^1_\rho H^*(\tilde{X}) = \text{Ker } k^* \), and \( Gr^0_\rho H^i(\tilde{X}) = H^0(\mathbb{P}^1, R^i\rho_*\mathbb{Q}_\ell); \)

2. \( F^2_\rho H^*(\tilde{X}) = \text{Im } k_* \) image of the Gysin map \( k_* \); one has an isomorphism \( F^2_\rho H^i(\tilde{X}) = H^2(\mathbb{P}^1, R^{i-2}\rho_*\mathbb{Q}_\ell) \simeq k_*H^{i-2}(Y). \)

The piece \( F^2_\rho H^*(\tilde{X}) \) coincides with the image of the Gysin homomorphism

\[ k_* : H^{i-2}(Y) \to H^*(\tilde{X}), \]

where \( Y \) is a smooth hyperplane section (as above); the piece \( F^1_\rho \) coincides with the kernel of \( k^* : H^*(\tilde{X}) \to H^*(Y) \). Furthermore, one has \( F^2_\rho H^i(\tilde{X}) = F^1_\rho H^{2n-i}(\tilde{X}). \)

The whole Theorem holds in general. One can prove the last assertion under condition (A) via the inclusion \( F^1_\rho H^{2n-i}(\tilde{X}) \subset F^2_\rho H^i(\tilde{X}) \) and a dimension count. For the general case see [9] Sec. 2, and [5], [11], [12] I.5; see also Looijenga [21] Sec. 5.

For the rest of this paper, we fix a Lefschetz pencil on \( X \) satisfying condition (A) above. We will denote by \( L \) the operator in \( H^*(\tilde{X}) \) given by \( L \bullet = f^*\xi_X \wedge \bullet \) (see Proposition 3.3), which has the following expression in terms of \( 3.1 \)

\[ L(x \oplus y) = L_X x \oplus L_\Delta y. \]

**Remark** Condition (A) induces a Lefschetz theory on the sheaves \( R^i\rho_*\mathbb{Q}_\ell \). One has Lefschetz isomorphisms

\[ L^{n-1-i} : R^i\rho_*\mathbb{Q}_\ell \simeq \nu_*\nu^*R^i\rho_*\mathbb{Q}_\ell \to \nu_*\nu^*R^{2n-2-i}\rho_*\mathbb{Q}_\ell \simeq R^{2n-2-i}\rho_*\mathbb{Q}_\ell, \]

where \( \nu : \mathcal{U} \subset \mathbb{P}^1 \) is such that \( \rho \) is smooth on \( \mathcal{U} \). We denote by \( \mathcal{P}^i_\rho = \ker L^{n-i} \) the primitive cohomology sheaves, and occasionally denote \( R^i\rho_*\mathbb{Q}_\ell \) by \( R^i \).

**Corollary 4.3**

1. The following isomorphisms hold:

\[ L^{n-1-i} : R^i\rho_*\mathbb{Q}_\ell \to R^{2n-2-i}\rho_*\mathbb{Q}_\ell. \]

2. Let \( \mathcal{P}^i_\rho = \ker L^{n-i} \subset R^i\rho_*\mathbb{Q}_\ell \). Then \( \mathcal{P}^i_\rho \) is constant of fibre \( P^i_\rho(X) \) if \( i \leq n-2 \) and \( \mathcal{P}^{n-1}_\rho = \mathcal{E}^{n-1}_\rho \oplus P^{n-1}(X)_{\mathbb{P}^1} \), where \( \mathcal{E}^{n-1}_\rho = \nu_*\nu^*\mathcal{E}^{n-1}_\rho \) for all \( \nu : \mathcal{U} \subset \mathbb{P}^1 \) within the smooth locus of \( \rho \); moreover \( \mathcal{E}^{n-1}_\rho = V(X_t) \) for all geometric points \( \mathcal{T} \to t \in \mathcal{U} \).

3. Let \( 0 \leq \epsilon \leq 2 \). The pairings

\[ R^i\rho_*\mathbb{Q}_\ell \times R^{2n-2-i}\rho_*\mathbb{Q}_\ell \to R^{2n-2}\rho_*\mathbb{Q}_\ell \simeq \mathbb{Q}_\ell \]

and

\[ L^{n-1-i} \bullet \cup \bullet : \mathcal{P}^i_\rho \times \mathcal{P}^i_\rho \to \mathbb{Q}_\ell \]

induce perfect pairings

\[ H^\epsilon(R^i\rho_*\mathbb{Q}_\ell) \otimes H^{2-\epsilon}(R^{2n-2-i}\rho_*\mathbb{Q}_\ell) \to H^2(\mathbb{P}^1, \mathbb{Q}_\ell) \]
and

\[ H^r(\mathcal{P}^i_\rho) \otimes H^{2-i}(\mathcal{P}^i_\rho) \rightarrow \mathbb{Q}_\ell \]

which agree with the ones resulting from Theorem 4.2, for instance, the pairing given by \( a \otimes b \mapsto (L^{n-i}a, b)_{\tilde{X}} \) in \( \text{Gr}^*_{\rho} H^i(\tilde{X}) \) equals the one in (5).

One has \( \dim H^0(\mathcal{P}^i_\rho) = \dim H^2(\mathcal{P}^i_\rho) \) for \( 0 \leq i \leq n-1 \).

4. The Lefschetz isomorphisms on sheaves translate also into their cohomology groups; in particular,

\[ H^r(\mathcal{P}^i_\rho) = \ker (L^{n-i} : H^r(\mathcal{R}^i) \rightarrow H^r(\mathcal{R}^{2n-i})). \]

5. \( \dim H^0(\mathcal{R}^i) = \dim H^2(\mathcal{R}^i) = b_i(X) \) for all \( i \). As a result, \( \dim H^0(\mathcal{P}^i_\rho) = \dim H^2(\mathcal{P}^i_\rho) = \dim P^i(X) \).

**Proof:** The result follows from Lemma 4.1, Theorem 4.2, Deligne 6.2.8 and 2.12, and Katz 5 XVIII Lemmas 5.4, 5.5, 5.6.9 and proof of Th. 5.6.8.

Let us check the last assertion for \( i = n-1 \); the morphism \( k_* = \iota_* \oplus -h^* : H^{n-1}(Y) \) has kernel \( V(Y) = \ker \iota_* \). Therefore \( \dim H^2(\mathcal{R}^{n-1}) = b_{n-1}(X) = b_{n-1}(Y) - \dim V(Y) \). The equality \( \mathcal{R}^i = \mathcal{P}^i_\rho \oplus LR^{i-2} \) yields \( \dim H^0(\mathcal{P}^i_\rho) = \dim P^i(X) \).

(Alternatively, use 5 XVIII Th. 5.6.)

We now view the computations of Proposition 3.3 in a different fashion.

**Lemma 4.4** Let \( x \oplus y \in H^i(\tilde{X}) \), and let \( r \in \mathbb{N} \). Then \( \xi_{\tilde{X}} - m \cdot f^*(\xi_X) = \rho^*([t]) = k_*(1_{H^r(Y)}) \in F^2_{\rho} \). Thus the expression

\[ (7) \quad (L^r_X - m^rL^r)(x \oplus y) = L^r_X(x \oplus y) - m^r(L^r x \oplus L^r\Delta y) = r \cdot m^{r-1}k_*L^{r-1}_Y(t^*x + h_*y) \]

belongs to \( F^2_{\rho} \).

**Proof:** By \( 4 \) we have \( L^s(x \oplus y) = f^*\xi_X^s \wedge (x \oplus y) = L^s x \oplus L^s\Delta y \). On the other hand, if \( Y = X_t \) is a smooth geometric fibre, then \( \rho^*(\text{pt.}) = k_*(1_{Y}) = \xi_{\tilde{X}} \oplus -1_{\Delta} \in H^2(\tilde{X}) \).

We have \( \rho^*(t) \wedge (x \oplus 0) = Lx \oplus j^*x = k_*(t^*x) \) and \( \rho^*(t) \wedge (0 \oplus y) = j_*y \oplus -L\Delta y = k_*(h_*y) \), whence \( \rho^*(t) \wedge (x \oplus y) = k_*(t^*x + h_*y) = Lx + j_*y - (j^*x + L\Delta y) \). Finally

\[
    r \cdot m^{r-1}L^r x + L^r - j_*y \oplus -L^{r-1}_\Delta (j^*x + L\Delta y) = L^r_X(x \oplus y) - m^r(L^r x \oplus L^r_\Delta y) =
    r \cdot m^{r-1}k_*L^{r-1}_Y(t^*x + h_*y)
\]

as desired. \( \blacksquare \)

**Corollary 4.5** Notations and assumptions being as above,

\[ L^{n-i}_X(P^i(X) \oplus 0) = L^{n-i}P^i(X) \oplus 0 = k_*L^{n-i}_Y \cdot i^*P^i(X) \subset F^2_{\rho} \]

and \( P^i(\tilde{X}) \supset P^i(X) \oplus 0 \). One has \( L^r_X k_*(y) = m^r k_*L^r_Y y = m^r L^r k_* y \) for all \( r \geq 0 \).
Proof: By [18] 1.4.7, $h^*L_Y^{n-1-i}P^i(Y) = 0$ and
\[ k_* : L_Y^{n-1-i}i^*P^i(X) \rightarrow L^{n-i}P^i(X) \oplus 0 \]
is an isomorphism. Let us prove the inclusion $P^i(X) \oplus 0 \subset P^i(\bar{X})$. By formula (7), it suffices to check that
\[ (n-i)m^{n-i}k_*L_Y^{n-1-i}P^i(Y) \subset L^{n-i}P^i(X) \oplus 0, \]

but this inclusion is clear. We have seen that the image of $P^i(X) \oplus 0$ via the Lefschetz isomorphism is precisely $L^{n-i}P^i(X) \oplus 0$, thus establishing the result. ■

Remark By Lemma [4.4], the operator $L \chi - m \cdot L$ vanishes on $Gr^*_p H^*(\bar{X})$. The same thing happens on the sheaves $\mathcal{R}^i$.

Corollary 4.6 The map $L^{n-i}$ and the Lefschetz isomorphism $L_X^{n-i}$ yield isomorphisms
\[ (P^i(X) \oplus 0) \oplus k_*H^{i-2}(Y) \sim k_*H^{2n-2-i}(Y). \]
The subspace $L_X^jP^i(X)\oplus0$ is linearly disjoint with $F^j$ for $j < n-i$, and $L^{n-i}P^i(X)\oplus0 \subset F^2$.

The first assertion follows from Corollary [4.5] and Corollary [4.3](5). The second assertion follows from the first. ■

Corollary 4.7 The natural map $P^i(X) \oplus 0 \rightarrow H^0(\mathcal{R}^i)$ of Theorem [4.2] induces an isomorphism
\[ P^i(X) \oplus 0 \cong H^0(\mathcal{P}^i) \]
for $0 \leq i \leq n-1$. The map $\rho^*(t) \wedge \bullet$ yields an isomorphism between $H^0(\mathcal{P}^i)$ and $H^2(\mathcal{P}^i)$. As a result, $H^2(\mathcal{P}^i) = LP^i(X) \oplus P^{i-2}(\Delta) \cap F^2H^{i+2}(\bar{X}) = k_*P^i(Y)$ for $i \leq n-1$.

Proof:
1. The dimensions are equal, and $L^{n-i}(P^i(X) \oplus 0) \subset F^2$, hence the map
\[ P^i(X) \oplus 0 \rightarrow H^0(\mathcal{R}^i) \]
induces an isomorphism onto $H^0(\mathcal{P}^i)$.
2. The class $[\rho^*(t)] \in \rho^*H^2(\mathbb{P}^1)$, hence $\rho^*(t) \wedge \bullet$ induces a map $H^0(\mathcal{P}^i) \rightarrow H^2(\mathcal{P}^i)$, which reads as follows:
\[ \rho^*(t) \wedge (x \oplus 0) = k_*k^*(x \oplus 0) = k_*t^*x. \]
Therefore its image is $k_*t^*P^i(X)$, whose dimension agrees with $\dim H^2(\mathcal{P}^i)$. The assertion is thus proven. ■
4.1 Absolute and relative correspondences

Let \( p : M \to B \) be a smooth projective morphism onto a smooth algebraic variety \( B \); denote the dimension of \( M \) by \( n \). In this section we will establish the usual properties of the composition law of correspondences on \( M \times_B M \); if \( u \) is a codimension-\((r - \dim B)\) cycle on \( M \times_B M \), the degree of \( u \) as a relative correspondence is defined to be \( r \), i.e. the same as that of \( u \) as a correspondence of \( M \); for instance, this definition makes the cycle \( \Delta_M \) into a relative correspondence of degree 0. See Fulton [10] Ch. 10, 16 for an introduction. The heart of this section is Lemma 4.8, where every identity holds modulo rational equivalence. If \( u, v \in CH^{n-1+\ast}(M \times_B M) \), then the composition of \( u, v \) relative to \( B \) is defined to be

\[
\circ_B u := p_{13*}(p_{12*}(u) \bullet p_{23*}(v)),
\]

where \( p_{ij}^B : M \times_B M \to M \times_B M \) are the canonical projections. \( \circ_B \) endows \( CH^{n-1+\ast}(M \times_B M) \) with a ring structure, and the usual properties hold. The upshot is Proposition 4.10.

**Lemma 4.8** Notations and assumptions as above: suppose that \( u, v \in CH^r(M \times_B M) \) are relative correspondences of degrees \( r, s \) respectively. If \( t \in B \) is a closed point, let \( \lambda_t : M_t \times M_t \hookrightarrow M \times_B M \) be the canonical inclusion, and let \( u_t = \lambda_t^*(u) \). Then:

1. The correspondence \( v \circ_B u \) is of degree \( r + s \);
2. for any \( t \in B \) one has: \( (v \circ_B u)_t = v_t \circ u_t \).
3. one can compare the composition laws \( \circ \) and \( \circ_B \) as follows. Let

\[
j : M \times_B M \hookrightarrow M \times M
\]

denote the natural inclusion; then \( j_* (v \circ_B u) = j_* v \circ j_* u \).

**Proof:** The first and second assertions are clear. For the third assertion, we need some notation: Let \( p^B_{ab} \) denote the \((a, b)\)-projections \( M \times_B M \to M \times_B M \) and \( p_{ab} \) denote the corresponding projections \( M^3 \to M^2 \). Let

\[
inc : M \times_B M \times_B M \hookrightarrow M \times M \times M
\]

be the natural inclusion. For each pair \( a \neq b \) in \( \{1, 2, 3\} \) we have a fibre product

\[
N_{ab} : \overset{j_{ab}}{\longrightarrow} M^3
\]

\[
\overset{j}{\longrightarrow} M^2.
\]

Denote by \( k_{ab} : M \times_B M \times_B M \hookrightarrow N_{ab} \) the natural inclusion. Then \( p^B_{ab} k_{ab} = p_{ab} \).

We wish to prove the identity

\[
j_{13*}^B (U \times_B M \bullet M \times_B V) = p_{13*} (U \times M \bullet M \times V);
\]

Proof...
by the above one has \( j_* p_{13}^B = p_{13} inc_* \). Thus it suffices to prove the following:

\[
inc_*(U \times_B M \times_B V) = U \times M \times V.
\]

In other words, one must check, for \( u, v \in CH^*(M \times_B M) \):

\[
(9) \quad p_{12}^*(j_* u) \cdot p_{23}^*(j_* v) = inc_*(p_{12}^B \circ (u) \cdot p_{23}^B \circ (v)).
\]

Consider now the following cartesian diagram of embeddings:

\[
\begin{array}{ccc}
M \times_B M \times_B M & \xrightarrow{inc} & M^3 \\
(k_{12}, k_{23}) \downarrow & & \downarrow \Delta_{M^3} \\
N_{12} \times N_{23} & \xrightarrow{j_{12}' \times j_{23}'} & M^3 \times M^3. \\
\end{array}
\]

Note that

\[
(11) \quad (k_{12}, k_{23}) = (k_{12} \times k_{23}) \Delta_{M \times_B M \times_B M}.
\]

Formula (9) is equivalent to the following:

\[
(12) \quad \Delta_{M^3}^*(p_{12} \times p_{23})^*(j_* \times j_*) = inc_* \Delta_{M \times_B M \times_B M}^*(p_{12}^B \times p_{23}^B)^*.
\]

Let us develop the l.h.s. Using Fulton [10] Prop. 1.7, one has \( p_{ab}^* j_* = j_{ab}' \circ p_{ab}' \) from the fibre product (8), hence

\[
\text{l.h.s.} = \Delta_{M^3}^*(j_{13}' \times j_{23}')^*(p_{12}' \times p_{23}')^*.
\]

Since all varieties involved are smooth and quasiprojective, (10) yields

\[
(13) \quad \Delta_{M^3}^*(j_{13}' \times j_{23}')^* = inc_*(k_{12}, k_{23})^*:
\]

Indeed, by the moving lemma [22] one can check the above for an algebraic cycle \( \zeta \) in \( N_{12} \times N_{23} \) that intersects properly with the image of \( (k_{12}, k_{23}) \). The identity (13) is easily established once for such \( \zeta \), and so for each Chow class in \( N_{12} \times N_{23} \). Using (11) and (13) yields

\[
(14) \quad \text{l.h.s.} = inc_* \Delta_{M^3}^*(k_{12} \times k_{23})^*(p_{12}' \times p_{23}')^*,
\]

which in turn equals

\[
inc_* \Delta_{M \times_B M \times_B M}^*(p_{12}^B \times p_{23}^B)^* = \text{r.h.s};
\]

the Lemma is thus established. ■

Remark Lemma 4.8 generalises accordingly when source and target of the relative correspondences are different; so do Lemma 4.9, Proposition 4.10.

Lemma 4.9 With the notations and assumptions of Lemma 4.8, we denote by \( t = t_* : M_t \to M \) the canonical inclusion. The following identity holds:

\[
(15) \quad j_* u \circ t_* = t_* \circ u_t.
\]
Proof of Lemma $[4.9]$ Consider the following cartesian diagram of embeddings:

\[
\begin{array}{ccc}
M_t \times M_t & \xrightarrow{1 \times \iota} & M_t \times M \\
\lambda_t & \downarrow & \downarrow 1 \times 1 \\
M \times B M & \xrightarrow{j} & M \times M.
\end{array}
\]

Analogously as shown in Lemma $[4.8]$ with (10), the following formula holds:

(15) \[(t \times 1)^* j_* \rho = (1 \times \iota)^* \lambda_t^* \] \[ j_*(u) \circ \iota_* = \iota_* \circ u_t. \]

Using Fulton [10] 16.1.1.(c) (see also Scholl [23] 1.10) we derive

\[ j_*(u) \circ \iota_* \rho = (1 \times \iota)^* \lambda_t^* \]

The proof is now complete. ■

Proposition 4.10 The correspondences supported on $D = \tilde{X} \times_{\mathbb{P}^1} \bar{X}$ preserve the Leray filtration. More precisely, if $u \in CH_*(\tilde{X} \times_{\mathbb{P}^1} \bar{X})$ and $j' : \tilde{X} \times_{\mathbb{P}^1} \bar{X} \hookrightarrow \tilde{X} \times \bar{X}$, then $[j'_*(u)]F^1_\rho \subseteq F^1_\rho$ for $i = 0, 1, 2$. In addition, if $u$ is supported on a finite set of fibres of the structure morphism $\tilde{X} \times_{\mathbb{P}^1} \bar{X} \to \mathbb{P}^1$, then $[j'_* u]F^2_\rho = 0, [j'_* u]H^*(\tilde{X}) \subseteq F^1_\rho$.

Proof of Proposition [4.10]

Choose a smooth fibre $Y = \tilde{X}_t$ of $\rho$. We will establish an identity identical to that of Lemma $[4.9]$ proved with the due care since $D$ is not smooth in general. Let $B \subseteq \mathbb{P}^1$ denote the smooth locus of $\rho$ and let $D_B := \rho^{-1}(B) \times_B \rho^{-1}(B)$ be the smooth locus of $D/\mathbb{P}^1$. Now $\lambda_t$ factors as

\[ X_t \times X_t \xrightarrow{\lambda_t'} D_B \xrightarrow{\nu'} D, \]

where $\nu'$ is an open immersion and $D_B$ is smooth. We may now define $\lambda_t^* = \lambda_t'^* \nu'^*$. The identity $(t \times 1)^* j'_* z = (1 \times \iota)^* \lambda_t^* z$ holds for any algebraic cycle on $D$: if $z$ is supported on $X_t \times X_t$, then both sides are 0 by Fulton [10] 10.1 (use op.cit. Cor. 6.3). If $z$ has no component contained in $X_t \times X_t$, then $z$ intersects properly with $X_t \times X_t$ and $j_* z$ intersects properly with $X_t \times \bar{X}$; again, it is easy to check that both sides agree (as algebraic cycles, no equivalence relation established). Just as in Lemma $[4.9]$ then for every correspondence $j'_* u$ on $\bar{X}$ supported in $D$ we have

(16) \[ j'_* u \circ \iota_* = \iota_* \circ u_t. \]

Now we check that any correspondence supported on $D = \tilde{X} \times_{\mathbb{P}^1} \bar{X}$ sends $F^2$ into $F^2$: indeed, (10) directly implies $j'_*(u)F^2_\rho \subseteq F^2_\rho$. Likewise, $j'_*(u)$ sends $F^1_\rho$ into $F^1_\rho$; suppose $x \in H^*(\bar{X})$ is such that $\nu'^* x = 0$. We have $\iota^* \circ j'_*(u) = u_t \iota^*$, hence $j'_*(u)F^1_\rho \subseteq F^1_\rho$. In the case when $u$ is supported on a finite union of subschemes $X_{s_1} \times X_{s_2}$, taking $t \neq s_1, \ldots, s_e$ in the above argument yields $u_t = 0$, thus proving the second assertion. The proof is now complete. ■
4.2 More on supports

We now return to the setting of Theorem 4.12 and assume the notations of Theorem 4.2 and Proposition 4.10.

Let $u, v$ be algebraic cycles supported on $D$. If $p_{12}^*(u), p_{23}^*(v)$ intersect properly, then the correspondence $v \circ u$ is easily seen to be supported on $D$; a similar argument works on Chow classes if an embedded desingularisation to $\tilde{X} \times \mathbb{P}^1 \tilde{X} \subset \tilde{X}^2$ exists – in the proof of Lemma 4.8 formula (14) required smoothness of the integral scheme $M \times_B M$; In a similar vein to 4.1 one derives the following statement.

Proposition 4.11 Notations and assumptions being as above, let $u = j'_*u_0, v = j'_*v_0$ be two correspondences supported on $D$. Let $j''_*: M \times_B M \subset M \times M$ denote the canonical inclusion.

The following statements hold (modulo rational equivalence).

1. Let $r, r_0$ denote the inclusions $r : M \times M \subset \tilde{X} \times \tilde{X}, r_0 : M \times_B M \subset D$. Then $r^*v \circ r^*u = j''_*(r^*_0v_0 \circ_B r^*_0u_0)$.

2. Let $\phi : Z \rightarrow D \times \tilde{X}$ be a De Jong alteration \[4\]. Let $z$ be such that $\phi_*(z) = p_{12}^*(u)$. Then $p_{12}^*(u)p_{13}^*(v)$ is supported on $\tilde{X} \times_{\mathbb{P}^1} \tilde{X} \times_{\mathbb{P}^1} \tilde{X}$; as a result, $v \circ u$ is supported on $D$.

Proof: The first statement follows from Lemma 4.8. Let us prove the second statement: for $u, z, \phi$ as above, one has $\phi_*(z') \bullet \phi_*(p_{23}^*(v)) = \phi_*(z' \bullet \phi_*(p_{23}^*(v)))$, where $z' \sim_{\text{rat}} z$ is such that $z', \phi_*(p_{23}^*(v))$ intersect properly. It is now apparent that the Chow class

$$\phi_*(z') \bullet \phi_*(p_{23}^*(v)) = \phi_*(z') \bullet p_{23}^*(v) = p_{12}^*(u) \bullet p_{23}^*(v)$$

is supported on $\tilde{X} \times_{\mathbb{P}^1} \tilde{X} \times_{\mathbb{P}^1} \tilde{X}$. Applying $p_{13*}$ yields the Chow class $v \circ u$, which is thererefore supported on $D$, thus completing the proof. 

4.3 Action on the Leray spectral sequence

(Again we assume $k$ algebraically closed.) Let $u$ be a correspondence of degree $r$ supported on $\tilde{X} \times_{\mathbb{P}^1} \tilde{X}$. Then $u$ induces a correspondence $u_\ell$ of degree $r$ on $X_\ell$ for each $t \in B(\overline{k})$, where $\nu : B \rightarrow \mathbb{P}^1$ is the smooth locus of $\rho$ as above. $u$ thus defines a homomorphism of $\ell$-adic sheaves for $0 \leq j \leq n - 1$:

$$u : \nu_*\nu^*R^j\rho_*\mathbb{Q}_\ell = R^j\rho_*\mathbb{Q}_\ell \rightarrow \nu_*\nu^*R^{j+2r}\rho_*\mathbb{Q}_\ell = R^{j+2r}\rho_*\mathbb{Q}_\ell$$

(using (A)), which in turn yields $\mathbb{Q}_\ell$-linear maps

$$H^i(R^j\rho_*\mathbb{Q}_\ell) \rightarrow H^i(R^{j+2r}\rho_*\mathbb{Q}_\ell).$$

These maps clearly agree with those induced on $Gr^r_{F_\nu}$ by $j_*u$ in Proposition 4.10 and so do the respective composition laws.

Remark 4.12 Morphisms induced in (17) and (18) depend only on the class of $u$ in $H^*(Y \times Y)$. Indeed, denote the generic point of $\mathbb{P}^1$ by $\eta$, and the image of $u$ in
\[ A^{n-1+r}(Y \times Y) \] by \( u_\eta \) or \([u]_Y\); suppose that \( u'_j - u''_j \) \( H^j(Y) = 0 \) for all \( j \). Then for a sufficiently small neighbourhood \( \nu_\eta : U_\eta \subset \mathbb{P}^1 \) of \( \eta \) one has \( u' - u'' = 0 \) \( R^j \rho_* \mathbb{Q}_\ell = R^j \rho_* \mathbb{Q}_\ell \) for all \( j \), hence \( u' - u'' \) induces 0 on \( Gr^\nu_{R^j_*} H^*(\tilde{X}) \). Here we used \[ 0 \] I.12.10, I.12.13 (see also \[ 13 \]) and the base change theorems in étale cohomology \[ 11 \]; \[ 9 \] I.6, I.7.

**Definition.** Let \( A \subset A^{n+r}(\tilde{X} \times \tilde{X}) \) denote the subring (see Proposition \[ 4.11 \] of homological correspondences supported on \( \tilde{X} \times \mathbb{P}^1 \tilde{X} \). Let \( J \) be the ideal of \( A^{n+r}(\tilde{X} \times \tilde{X}) \) consisting of the elements \( u \) such that \( uF^i_\rho \subset F^i_\rho \) for \( i = 0, 1, 2 \) (i.e. those inducing 0 on \( Gr^\nu_{R^i_*} H^*(\tilde{X}) \)). Let \( I \) be the ideal (see Lemma \[ 4.16 \]) of \( A \) consisting of the \( u \in A \) such that \( u = [j'_v] \) with \( v \) an algebraic cycle on \( \tilde{X} \times \mathbb{P}^1 \tilde{X} \) (with \( \mathbb{Q} \)-coefficients) inducing an homologically trivial class on \( H^*(Y \times Y) \). One has \( I \subset J \) by \[ 17 \]. We denote by \( K \subset A \) the subspace of all classes \( w = [j'_s w_0] \in A \) satisfying \([w_0]_Y H^1(Y) \subset V(Y) \).

The following Proposition sharpens Remark \[ 4.12 \] above.

**Proposition 4.13** Then the ideals \( I, J, K \) of \( A \) satisfy \( I \subset J, I^{\mathfrak{O}} = J^{\mathfrak{O}} = 0 \). Let \( w_0 \) be an algebraic cycle supported on \( D \), representing the correspondence \( w \in A^{n+r}(\tilde{X} \times \tilde{X}) \) of degree \( r \). Suppose that \( r \neq 0 \); then \( w_0 \in I \) if and only if \( w \in K \). In general, the following statements hold.

(i) \( J \subset K \). \( K \) is an ideal of \( A \).

(ii) Suppose that \( H^1(R^{n-1} \rho_* \mathbb{Q}_\ell) = 0 \). Then \( J = K \).

(iii) Assume that \( n \) is even or char \( k \neq 2 \). If \( H^1(R^{n-1} \rho_* \mathbb{Q}_\ell) \neq 0 \), then \( I = J \subset K \).

(iv) If \( n \) is even or char \( k \neq 2 \), then there exists \( d_0 \in \mathbb{N} \) such that, for every \( d \geq d_0 \), every Lefschetz fibration of degree-\( d \) hypersurfaces satisfies (iii).

(v) If \( n \) is even or char \( k \neq 2 \), and \( H^1(R^{n-1} \rho_* \mathbb{Q}_\ell) \neq 0 \), then any Chow class \( u \) supported on \( D \) such that \([j'_s u] = 0 \) satisfies \([u]_Y = 0 \).

**Proof:** The nilpotence assertion for \( I, J \) is clear. Now, let \( w_0 \) be as above with \( r \neq 0 \); we argue as in Remark \[ 4.12 \]. If \( R^i \) is constant (\( i \neq n-1 \)) then \([w_0]_Y H^i(Y) = 0 \) if and only if \( w_0 : R^i \to R^{i+2r} \) is 0. The same holds whenever \( R^{i+2r} \) is constant. Thus \( w \in I \iff w \in J \) whenever \( r \neq 0 \).

Assume \( r = 0 \). Then \( R^{n-1} = LR^{n-3} \oplus P^{n-1} \rho, \) where \( P^{n-1} = P^{n-1}(X)_{\mathbb{P}^1} \oplus \mathcal{E}^{n-1} \), and \( V(Y) = \mathcal{E}_1^{n-1} \) is the geometric generic fibre of \( \mathcal{E}^{n-1} \) (here \( \mathcal{E} \) is a geometric generic point of \( \mathbb{P}^1 \).) Hence \( w \in J \iff w_0 R^\bullet = w_0 \mathcal{E}^{n-1} \subset \mathcal{E}^{n-1} \). The \( \mathbb{Q}_\ell \)-adic sheaf \( w_0 \mathcal{E}^{n-1} \) has \([w_0]_Y V(Y) \) as its geometric generic fibre; all the pieces of \( H^*(Y) \) are monodromy invariant, except \( V(Y) \) which has no invariants (by \( (A) \)). This settles (i), (ii).

To prove (iii), recall that if \( n \) is even or char \( k \neq 2 \), then all the singularities of \( \rho \) are non-degenerate quadratic singularities of fibres, and the monodromy representation of \( \pi_1^{alg}(B, \eta) \) (\( B \) being the smooth locus of \( \rho \)) on \( V(Y) \) is absolutely irreducible (\[ 3 \], esp. XVIII Cor. 6.7). As a result, the \( \pi_1(B) \)-submodule \([w_0]_Y V(Y) \) is either 0 or \( V(Y) \), and so there are two possibilities for the inclusion of \( \mathbb{Q}_\ell \)-sheaves
$w_0\xi E^{n-1} \hookrightarrow E^{n-1}$: either the image or the cokernel of this inclusion are skyscraper sheaves. Since $H^1(\mathcal{R}^{n-1}) = H^1(E^{n-1}) \neq 0$, we have

$$wH^1(\mathcal{R}^{n-1}) = 0 \iff [w_0]_V(Y) = 0,$$

thus settling $(ii),(iii)$. Let us prove $(iv)$; by Lemma 6.2 (see [5] XVIII Th. 5.7), $H^1(\mathcal{R}^{n-1}) \simeq P^n(X) \oplus V(\Delta)$. If $P^n(X) \neq 0$ there is nothing to prove; if $P^n(X) = 0$ the assertion follows from the next elementary lemma.

**Lemma 4.14** (compare [5] XVIII Lemme 6.4.2) With the notations and hypotheses of this section (assuming $n \leq 3$), let $d \in \mathbb{N}$. Let $\Delta(d)$ denote degree-$d$ hypersurface sections intersecting transversally, and let $\Delta(d) = Y(d) \cap Y'(d)$. Then $b_{n-2}(\Delta(d))$ is a polynomial of degree $n$ in $d$.

**Proof of Lemma 4.14:** Let $c(X), c(\Delta)$ be the total Chern classes of $X, \Delta$ and let $j : \Delta \subset X$ denote the canonical inclusion. Let $\int_X$ denote the trace map on $X$, and $H = c_1(\mathcal{O}_X(1))$ with the present polarisation. Then, using $j^*j^*\alpha = d^2H^2 \cdot \alpha$, we obtain:

$$\chi(\Delta(d)) = \int_{\Delta(d)} c(\Delta(d)) = \int_{\Delta(d)} j^* \frac{c(X)}{(1 + d \cdot H)^2} = \int_X \frac{d^2H^2c(X)}{(1 + d \cdot H)^2},$$

which is a polynomial in $d$ of degree $n$ with lead term $(-1)^n\deg X \cdot d^n$. Isolating yields $b_{n-2}(\Delta(d)) = (-1)^n - \chi(\Delta(d)) + 2 \sum_{i \geq 1} (-1)^i b_{n-2-i}(X)$, thus completing the proof. $\blacksquare$

Taking $d \gg 0$, the $d$-uple embedding of $X \subset \mathbb{P}$ satisfies the hypotheses of $(iii)$, thus establishing $(iv)$.

It remains to prove $(v)$. By Corollary [4.13] $H^1(E^{n-1}) = H^1(\mathcal{R}^{n-1})$.

If $[j^*(u)] = 0 \in J$, then $u$ induces 0 on $H^\epsilon(\mathcal{R}^i)$ for all $\epsilon, i$ and from $(iii)$ we derive $[u]_V = 0$.

Proposition 4.13 is thus established. $\blacksquare$

**Corollary 4.15** If $n$ is even or char $k \neq 2$ and $H^1(\mathcal{R}^{n-1}) \neq 0$, then the restriction map

$$CH_{n-1-*}(D) \to A^{n-1+*}(Y \times Y)$$

factors through a ring homomorphism

$$\text{res}_Y : A \to A^{n-1+*}(Y \times Y)$$

whose kernel is $\text{res}_Y = \mathcal{I}$.

**Proof:** The Corollary follows from Proposition 4.13$(v)$. $\blacksquare$

The next Lemma is only necessary if $n$ is odd and char $k = 2$.

**Lemma 4.16** Notations and assumptions as above. The subspace $\mathcal{I} \subset A$ is an ideal.
Proof: Let $B \subset \mathbb{P}^1$ be the smooth locus of $\rho$, and $M = \rho^{-1}(B)$. We define the map $[j'_*]: CH_{n-1-\bullet}(D) \to A^{n+*}(\tilde{X} \times \tilde{X})$ (here $\bullet$ is the cycle map, and $D = \tilde{X} \times_{\rho_1} \tilde{X}$) whose image is precisely $\mathcal{A}$. If $r, r_0$ are as in Proposition 4.11 and $j'': M \times_B M \leftrightarrow M \times M$, we have a commutative diagram

$$
\begin{array}{ccc}
CH_{n-1-\bullet}(D) & \xrightarrow{[j'_*]} & A^{n+*}(\tilde{X} \times \tilde{X}) \\
[r_0^n] & \downarrow & \downarrow r^* \\
A^{n-1+*}(M \times_B M) & \xrightarrow{j''} & A^{n+*}(M \times M).
\end{array}
$$

We have $r^* \mathcal{A} = j''^* A^{n-1+*}(M \times_B M)$, and $r^*, j''_*$ are ring homomorphisms by Proposition 4.11. It is clear that $r^* \mathcal{I}$ is an ideal of $r^* \mathcal{A}$, which coincides with the image of $\text{Ker}(A^{n-1+*}(M \times_B M) \to A^{n-1+*}(\mathcal{Y} \times \mathcal{Y}))$ via $j''_*$. Now the kernel of $\mathcal{I} \to r^* \mathcal{I}$ consists of the correspondences supported on $\bigcup_{s \in \mathbb{P}^1} \text{singular } X_s \times X_s$. It is now clear that $\mathcal{I} \subset \mathcal{A}$ is an ideal. ■

Remark 4.17 By Proposition 4.13 above, there is a ring epimorphism

$$
\varphi = \varphi_{\mathcal{Y}} : A^{n-1+*}(\mathcal{Y} \times \mathcal{Y}) \to \mathcal{A}/\mathcal{I},
$$

which is an isomorphism if $n$ is even or char $k \neq 2$ by Corollary 4.15. It is not clear whether $\varphi_{\mathcal{Y}}$ is an isomorphism if $n$ is odd and char $k = 2$.

The next Corollary circumvents the possible non-isomorphy of $\varphi$ for the purposes of this paper. Its proof is straightforward.

Corollary 4.18 Let $a \subset A^{n-1+*}(\mathcal{Y} \times \mathcal{Y})$ be the ideal of correspondences $u$ such that $uH^*(\mathcal{Y}) \subset V(\mathcal{Y})$. One has a ring isomorphism induced by $\varphi$ above:

$$
A^{n-1+*}(\mathcal{Y} \times \mathcal{Y})/a \cong \mathcal{A}/\mathcal{K}.
$$

Consider a graded unital subalgebra $\mathcal{B} \subset A^{n-1+*}(\mathcal{Y} \times \mathcal{Y})$ such that $\mathcal{B} \cap a = 0$. Then $\varphi$ yields an isomorphism $\mathcal{B} \cong \varphi(\mathcal{B}) \subset \mathcal{A}/\mathcal{I}$, which maps isomorphically after composing with the quotient map $\mathcal{A}/\mathcal{I} \to \mathcal{A}/\mathcal{K}$.

■

5 The relative projectors

We have seen in Lemma 2.11 that, if $C(X)$ holds, then the ring of correspondences of $X$, $A^{\dim X+\bullet}(X \times X)$ decomposes through the adjoint action of $H_X$, $u \mapsto [H, u]$; the degree-0 correspondences are exactly those commuting with $H_X$, or equivalently, with the Künneth projectors $\pi_i^X$ for all $i$. We wish to translate this situation into the relative context presented in Section 4. Our first goal is to create natural relative analogues $\pi_{\rho}^i H_{\rho}$ of $\pi^i$ and of $H$, supported on $\tilde{X} \times_{\rho_1} \tilde{X}$. We will thereby create a splitting of the Leray filtration, and if $n$ is even or char $k \neq 2$ a section of the ring epimorphism resy.
Lemma 5.1 Assume $C(Y)$. Let $\pi^i \in A^n(\tilde{X} \times \tilde{X})$ be liftings of $\pi_Y^i$. Then $\pi^i$ are such that

$$\pi^i|Gr^*_F H^i(\tilde{X}) = \delta_{i,j - \epsilon}$$

for all $0 \leq i, j \leq 2n - 2$ and $\epsilon = 0, 1, 2$. The restriction of $\pi^i$ to $F^2H^*_Y(\tilde{X})$ is a projector which yields $0$ on $F^2H^i(\tilde{X})$ if $j \neq i + 2$ and the identity if $j = i + 2$. Thus the restriction to $F^2$ is clearly independent of the lifting chosen.

**Proof:** The proof is laid out in [4.3]. If $\pi^i \in A$ restricts to $\pi_Y^i$, then $\pi^i|\rho^k|\bar{Q}_{i,\epsilon} = \delta_{i,k}$ for all $i, k$. Applying $H^r(\mathbb{P}^1, \bullet)$ the Lemma follows.\[\square\]

Whatever the choice of liftings $\pi^i$, these operators commute with the Künneth projectors of $\tilde{X}$ by Lemma 2.1, this justifies the following definitions, which make sense under condition (A) of Section 4. We define $\pi^{i,0}$ after the relation $\pi_Y^i = t \pi^{2n - 2 - i}$.

**Notation-Definition.** Let $\pi^{i,2}$ denote the orthogonal projection onto $F^2H^{i+2}(\tilde{X})$, and $\pi^{i,0}$ denote the transpose $t \pi^{2n - 2 - i,2}$. We define $\pi^{n-1,1} := \pi^{n}_{\tilde{X}} - \pi^{n,0} - \pi^{n-2,2}$ and $\pi^{i,1} = 0$ otherwise. Then the $\pi^i, \epsilon$ form a complete orthogonal system of projectors, and provide a splitting for the Leray filtration $F^*_{\mu}$ of $H^*(\tilde{X})$.

The following Proposition is the relative equivalent to Lemma 2.8.

**Proposition 5.2** Let $H_{\rho} = \sum (n - 1 - i)\pi^i_{\rho}$. Then $H_{\rho}$ is (characterised as) the only semisimple (algebraic) skew-symmetric operator supported on $\tilde{X} \times_{\mathbb{P}^1} \tilde{X}$ mapping to $H_Y$ under the specialisation map. The complete orthogonal system of projectors $\{\pi^i_{\tilde{X}}\}$ yields a splitting of the Leray filtration $F^*_{\rho}$ of $H^*(\tilde{X})$.

**Proof:** The proof is elementary. Let us prove existence first. The correspondence $H_Y = \sum (n - 1 - i)\pi^i_Y$ lifts to a non-unique correspondence $\tilde{H}$ on $\tilde{X} \times \tilde{X}$ supported on $\tilde{X} \times_{\mathbb{P}^1} \tilde{X}$, which we may assume skew-symmetric. Indeed, $\tilde{1}H_Y = -H_Y$ implies that $\tilde{1}\tilde{H} + \tilde{H} \in J$ is nilpotent of order 3 by Proposition 4.13. Now, the minimal polynomial of $\tilde{H}$ divides $R(x) = P(x)^3$, where $P(x) = x \prod_{i=1}^{n-1}(x^2 - i^2)$. The fact that $R(x)$ is odd implies the following.

**Claim.** The semisimple part of the Jordan decomposition of $\tilde{H}$ is skew-symmetric.

**Proof of the Claim:** Write

$$R_i(x) = \prod_{k \in [-n+1,n-1], k \neq i} (x - k)^3.$$

Write

$$1 = \sum_{-n+1}^{n-1} R_i(x)a_i(x),$$

with $a_i(x)$ quadratic polynomials. It is clear that $R_i(x) = R_{-i}(x)$ if $i > 0$. Multiplying the above by $x$, one has $x = \sum_{-n+1}^{n-1} iR_i(x)a_i(x) + \sum R_i(x)a_i(x)(x - i)$, which yields the Jordan decomposition of $\tilde{H}$. Substituting $\tilde{H}$ into $x$, it follows that the first sum is skew-symmetric and semisimple (and, of course, algebraic).
Let $H'_\rho$ be a semisimple, algebraic, skew-symmetric lifting of $H_\rho$. By Proposition \ref{prop:lifting}, $H'_\rho$ agrees with $H_\rho$ on $F^2H^*(X) = \text{Im } \sum \pi^{i,2}$. Transposing yields

$$(H'_\rho - H_\rho)|\text{Im } \sum \pi^{i,0} = 0.$$  

It remains only to check equality on $\text{Im } \pi^{n-1,1}$: now $H'_\rho, H_\rho$ are nilpotent on $\text{Im } \pi^{n-1,1}$, hence 0 by semisimplicity, thus completing the proof. 

\begin{corollary}
With notations and assumptions of Proposition \ref{prop:lifting}, the relative projectors $\pi^i_\rho$ are the projections onto the primary components of the operator $H_\rho$, and $\pi^i_\rho = \pi^{2n-2-i}_\rho$. Moreover,

$$\pi^i_X = \pi^{i,0} + \pi^{i-1,1} + \pi^{i-2,2}$$

and

$$\pi^i = \pi^{i,0} + \pi^{i,1} + \pi^{i,2},$$

where $\pi^{i-1,1} = 0$ for $i \neq n$.

\begin{proof}
The first assertion follows from Proposition \ref{prop:lifting} and $\pi^i_\rho$ are thus polynomials in $H_\rho$. The second assertion follows from the fact that $H_\rho$ is semisimple skew-symmetric. The rest follows from the Leray spectral sequence of $\rho$, condition (A) and Lemma \ref{lem:primary}. 

\end{proof}

\end{corollary}

\begin{observation-definition}
Let $\tilde{u}$ be a correspondence of degree $r$ of $\mathcal{Y}$. Then

$$u = \sum \pi^{i+2r}_\rho \tilde{u} \pi^i_\mathcal{Y}.$$  

This goes along with (and in fact implies) the commutation relation in Lemma \ref{lem:primary}

We define for each $u \in \mathcal{A}$ the following element of $\mathcal{A}$:

$$u_\rho := \sum \pi^{i+2r}_\rho u \pi^i_\rho.$$  

It is clear by construction that $u_\rho - u \in \mathcal{J}$. If $u$ is a correspondence of degree $r$ on $\mathcal{Y}$, we will define $u_\rho$ to be $u'_\rho$ for $u'$ a lifting of $u$ in $\mathcal{A}$. Later we will see that this definition is consistent.

\begin{lemma}
The map $\psi : \mathcal{A} \to \mathcal{A}$ defined by $u \mapsto u_\rho$ satisfies $\mathcal{J} = \text{Ker } \psi$; in other words, $u_\rho = v_\rho$ if and only if $u - v$ induces 0 on $\text{Gr}_F^r$. The image $\psi(\mathcal{A})$ in degree $r$ consists of the $w \in \mathcal{A}$ such that $\pi^{i+2r}_\rho w = w \pi^i_\rho$ for all $i$. The map $\psi$ is a linear projector which induces a section $\sigma$ of the natural quotient map $\mathcal{A} \to \mathcal{A}/\mathcal{J}$ à la Wedderburn-Malcev, and commutes with transposition. As a result we have a well-defined ring homomorphism

$$A^{n-1+x}(\mathcal{Y} \times \mathcal{Y}) \to \psi(\mathcal{A})$$

defined by $u \mapsto u_\rho$, which agrees with the homomorphism $\psi \circ \text{proj}_\mathcal{J} \circ \varphi$.

\begin{proof}
It is clear that $\mathcal{J} = \text{Ker } \psi$. The image of $\psi$ is easily characterised as the subspace of $u$ such that $\psi(u) = u$ (easily seen to agree with the description $u \pi^i_\rho = \pi^{i+2r}_\rho u$ if $u$ is of degree $r$), whence $\psi^2 = \psi$. By Corollary \ref{cor:lifting}, $\psi(t u) = t \psi(u)$. Finally, the terms $v_\rho \circ u_\rho$ and $(v \circ u)_\rho$ differ by an element of $\mathcal{J} \cap \text{Im } \sigma_\mathcal{Y} = (0)$, thus proving that $\psi$ is a ring homomorphism. $\psi$ clearly induces a section $\mathcal{A}/\mathcal{J} \to \mathcal{A}$ of the quotient map, which gives rise to the map $u \mapsto u_\rho$ with target $A^{n-1+x}(\mathcal{Y} \times \mathcal{Y})$.

\end{proof}

\end{lemma}
6 A relative $\mathfrak{sl}_2$-triple

We have obtained a set of relative Künneth projectors under the hypothesis $C(\mathcal{V})$. In this section we assume $B(\mathcal{V})$ and we construct relative operators $c\Lambda_p, \Lambda_p$ lifting $c\Lambda_Y, \Lambda_Y$; this will give rise to a relative $\mathfrak{sl}_2$-triple $c\Lambda_p, L_p, H_p$ whose action on $H^*(X)$ will be exploited later.

**Proposition 6.1** The following assertions hold.

1. For any lifting $c\Lambda'$ of $c\Lambda_Y$, the correspondence $c\Lambda_p = \sum \pi_{i-2} \Lambda' \pi_i$ is symmetric and independent of the lifting $c\Lambda'$ chosen.

2. The operator $c\Lambda_p$ satisfies $c\Lambda_p \pi_{i,2} \subset \text{Im} \pi_{i-2,2}$ and $c\Lambda_p \pi_{i,0} \subset \text{Im} \pi_{i-2,0}$. In fact $c\Lambda_p \pi_{i,0} = \pi_{i-2,0} c\Lambda_p$ and $c\Lambda_p \pi_{i,2} = \pi_{i-2,2} c\Lambda_p$.

**Proof:** (1) follows from Lemma 6.1. (2) follows directly from Proposition 4.10 and Corollary 5.4. □

**Lemma 6.2** $\text{Im} \pi_{n-1,1} = P^n(X) \oplus V(\Delta)(-1)$ (compare Katz [5] Exp. XVIII Th. 5.7) and $\text{Im} \pi_{n,0}$ is the image of $\Delta(H^{n-2}(Y))$ via the inclusion

$$\iota_\ast \oplus h^\ast : H^{n-2}(X)(-1) \oplus H^{n-2}(Y)(-1) \rightarrow H^n(X) \oplus H^{n-2}(\Delta)$$

given by the decomposition of Proposition 6.1. On the other hand,

$$H^n(X) \cap F^1 \ast H^*(\tilde{X}) = P^n(X) \oplus 0.$$

**Proof:** The sought-for image of $\pi_{n-1,1}$ coincides with the orthogonal in $F^1 \ast H^n(\tilde{X})$ of $k_s H^{n-2}(Y) = \text{Im} \pi_{n-2,2}$.

Note the orthogonal decomposition

$$(21) \quad H^n(\tilde{X}) = (P^n(X) \oplus V(\Delta)) \oplus (\iota_\ast H^{n-2}(Y) \oplus h^\ast H^{n-2}(Y)(-1)).$$

The piece $P^n(X) \oplus V(\Delta)$ is clearly within $F^1 \ast$ and orthogonal to $F^2 \ast H^n(\tilde{X})$; by a dimension count (see Corollary 6.3) we have $(P^n(X) \oplus V(\Delta)) \oplus (\iota_\ast H^{n-2}(\tilde{X}))$.$ F^2 \ast H^n(\tilde{X}) = F^1 \ast H^n(\tilde{X})$; the equality $H^n(X) \cap F^1 \ast = P^n(X) \oplus 0$ is thus established.

We define $W = H^{n-2}(Y)(-1) \oplus$ and view it as a quadratic subspace of $H^n(\tilde{X})$ via \[
\begin{pmatrix}
\iota_* & 0 \\
0 & h^* 
\end{pmatrix}. \]

Write $W = W_1 \oplus W_2$ (not orthogonal), where $W_1 = \text{Im} \left( \begin{array}{c} 1 \\ -1 \end{array} \right)$ represents $F^2 \ast H^n(\tilde{X})$ and $W_2 = \text{Im} \left( \begin{array}{c} 1 \\ 1 \end{array} \right)$ — note that both $W_i$ are self-orthogonal.

We will show that $\pi_{n,0}$ is given by the projection onto $W_2$.

Let $w = (w_1, w_2), w' = (w'_1, w'_2) \in W = W_1 \oplus W_2$. Then $\langle w_1, w'_2 \rangle = 0$, and

$$\langle w_1, w'_1 + w'_2 \rangle = \langle w_1 + w_2, w'_2 \rangle,$$

which shows $\pi_{n,0} = \pi_{n-2,2}$. The Lemma is thus established. □

**Lemma 6.3** Notations and assumptions as above, let $i \leq n - 1$. Then

$$\text{Im} \pi_{i,0} = (P^i(X) \oplus 0) \oplus (\iota_\ast \oplus h^*) \pi_{i-2}.$$

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Proof: The first assertion is similar to Lemma 6.2. The piece $P^i(X) \oplus 0 \subset \text{Im } \pi^{i,0}$, since the image of the projector $f^* \rho_X f_*$ is contained in $\text{Im } \pi^{2n-i-2}$ by the equality $k_s L^{n-1} P^i(Y) = L^{n-i} P^i(X) \oplus 0$; the piece $(\iota_* \otimes h^*) \Delta(H^{i-2}(Y)) \subset \text{Im } \pi^{i,0}$ by a similar argument to Lemma 6.2. To prove the second assertion we note the following: if $x \in P^i(X)$ for $i \leq n - 2$, then $L^{n-1} P^i(X) \oplus 0 \cap k_s H^{2n-i-1}(Y) = 0$, since $j^*: P^i(X) \rightarrow P^i(\Delta)$ is injective; here we have used Lemma 4.4. The case $i = n - 1$ is obvious. ■

Finally we obtain the desired $sl_2$-triple.

**Proposition 6.4** We have a relative $sl_2$-triple $\Lambda, L, H$. A relative Lefschetz isomorphism holds:

$$L^i_p: \text{Im } \pi^{n-1-i}_p \rightarrow \text{Im } \pi^{n+1-i}_p$$

for $1 \leq i \leq n - 1$. The projectors $p^i_p$ are algebraic for $i \leq n - 1$, and we have symmetric operators $p^{n-j}_p$ derived from $p^{n+1-j}_p$ for $0 \leq j \leq n - 1$. The map $u \mapsto u_p$ yields an isomorphism of rings $\mathbb{Q}(\Lambda, \Lambda) \cong \mathbb{Q}(\Lambda, \Lambda)$ which preserves transposition.

**Proof:** The $sl_2$-identities $[H_p, \Lambda] = 2 \Lambda$, $[H_p, L_p] = -2L_p$ and $[\Lambda, L_p] = H_p$, and the isomorphism between $\mathbb{Q}(\Lambda, \Lambda)$ and $\mathbb{Q}(\Lambda, \Lambda)$ follow from Lemma 4.4 and Corollary 4.18. The operators $\frac{1}{m} L^i_p$, $L^i_p$ induce the same map on $\text{Gr}^q_{F^*_p}$ by Proposition 4.2 and Lemma 4.3. The ‘relative Lefschetz isomorphism’ can be checked by passing to $\text{Gr}^q_{F^*_p}$, or simply by using the identities $(\Lambda^i_p L^i_p - 1) \pi^{n-1-i}_p = 0$ for $i < n - 1$.

We now can view the Lefschetz theory of $\mathcal{Y}$ within $H^*(\tilde{X})$.

**Proposition 6.5** The relative primitive projectors $p^i_p$ for $i \leq n - 2$ are described as follows. $p^i_p[\Lambda(I) = f_\iota p_X f_\ast$, $p^i_p H^{i+1}(\tilde{X}) = 0$ and $\text{Im } p^i_p H^{i+2}(\tilde{X}) = k_s P^i(Y) = LP^n(X) \oplus P^{n-2}(\Delta) \cap F^2$.

**Proof:** By Lemma 6.3 and Corollary 4.7, $P^i(X) \oplus 0 \subset \text{Im } \pi^{i,0}$ is isomorphic to $H^0(\mathcal{P}) \subset H^0(\mathcal{R})$ via the obvious map. Since $\mathcal{P}$ is constant, the image of $p^i_p$ in degree $i + 1$ is zero; the last assertion follows from Corollary 4.4.

**Proposition 6.6** The projector $p^{n-1}_p$ satisfies the following properties:

1. $\pi^{n-1}_p = p^{n-1}_p$ and $p^{n-1}_p = t p^{n-1}_p$;
2. $\pi^{n-1,1}_p = \pi^{n-1,1}_p$, i.e. the orthogonal projection onto $P^n(X) \oplus V(\Delta) \subset P^n(\tilde{X})$;
3. $\pi^{n-2}_p p^{n-1}_p$ is the orthogonal projection onto $LP^{n-1}(X) \oplus 0$, and $\pi^{n-1,0}_p p^{n-1}_p$ is the projection onto $P^{n-1}(X) \oplus 0$.

In all, the projector $p^{n-1}_p$ can be expressed as

$$p^{n-1}_p = f^* p^{n-1}_X f_\ast + \pi^{n-1,1} + f^* t p^{n-1}_X f_\ast,$$

and $f_\ast p^{n-1}_p f^* = p^{n-1}_X + p^n_X + t p^{n-1}_X$.
Proof of Proposition [6.6]

1. is a straightforward consequence of Proposition [6.4]

2. Since \( R^{n-3} \rho_* Q_\ell \) is constant, one has \( H^1(R^{n-1} \rho_* Q_\ell) = H^1(\mathcal{P}^{n-1}) \); the rest follows from Lemma [6.2]

3. The computation \( H^2(\mathcal{P}^{n-1}) = L\mathcal{P}^{n-1}(X) \oplus 0 \) follows from Corollaries [4.5] [4.7]
Thus \( \text{Im} \pi^{n-1,2} p^{n-1} = L\mathcal{P}^{n-1}(X) \oplus 0 \). Using the Poincaré duality pairing yields \( \text{Im} \pi^{n-1,0} p^{n-1} = P^{n-1}(X) \oplus 0 \), by Lemma [2.3] and Corollary [4.5].

7 The Main Theorem

This section is devoted to proving the following result.

**Main Theorem.** Let \( X \) be a smooth, projective variety of dimension \( n \). Assume the Lefschetz standard conjecture for the generic fibre \( Y/k(t) \) of a Lefschetz pencil satisfying (A). Then \( \Lambda - p^{n+1} \) is algebraic.

We will prove this result in a series of steps, obtaining the algebraicity of the Künneth projectors \( \pi^i_X \) for \( i \neq n - 1, n, n - 1 \) in the course of our proof.

7.1 The algebraicity of some projectors

We start by proving the following.

**Proposition 7.1** \( B(Y) \), with \( Y \) as above, implies the algebraicity of the Künneth projectors \( \pi^i_X \) for \( i \leq n - 2 \) (hence that of \( \pi^i_X \) for \( i \geq n + 2 \)) and that of the primitive projectors \( p^0, \ldots, p^{n-2} \).

A couple of lemmas will be required to establish this Proposition.

**Lemma 7.2** The following statements hold.

(i) The identity \( \iota_* H^{i-2}(Y) = LH^{i-2}(X) \) holds for all \( 0 \leq i \leq 2n \), and \( \iota_* : H^{i-2}(Y) \to H^i(X) \) is injective for \( i \leq n \). For all \( i \leq n \),
\[
\text{Im}(\pi^i_X - p^i_X) = LH^{i-2}(X).
\]

(ii) For all \( i > n \), \( \text{Im}(\pi^i_X - t p^{2n-i}_X) = L^{i-n+1} H^{2n-i-2}(X) \).

(iii) Suppose \( B(Y) \) holds for \( Y \) a smooth hyperplane section of \( X \). Then for \( 0 \leq i \leq n \),
\[
\pi^i_X - p^i_X = \iota_* \Lambda Y \pi^i_Y \iota^* \text{ is algebraic. Thus the transposed operators } \pi^{2n-i} - L^{n-i} p^{2n-i} \text{ are algebraic for } 0 \leq i \leq n.
\]

(iv) The hypothesis \( B(Y) \) of the Main Theorem implies \( B(Y) \) for a suitable hyperplane section.
Proof: Statement (iv) follows by Proposition [2.5](2) and specialisation. The rest is straightforward. ■

We consider a suitable Lefschetz pencil for $X$, and prove the algebraicity of $\pi^i_X$ for $i \leq n - 2$.

It suffices to prove that the operators $\pi^i_X$ are algebraic for $i = 0, \cdots, n - 2$, since $\pi^{2n-i} = t_i \pi^i$ (Kleiman [18] showed already that $\pi^0, \pi^1$ are algebraic in general).

Lemma 7.3 The projectors $\pi^{i-2,2}$ are algebraic for $i \leq n$, and so are $\pi^{i-2,0}$.

Proof of Lemma 7.3:

- We know that $k_s = \iota_s \oplus -h^*$. Let $i \leq n$. Let us prove that

$$\pi^{i-2,2} = \pi^{i-2,2} f^s(\pi^i_X - p^i_X)f_s\pi^{i-2,2}.$$  

Indeed, by Lemma [7.2] the image of $k_s(y) = \iota_s(y) \oplus -h^*(y)$ by $(\pi^i_X - p^i_X)\pi^{i-2,2}$ is $\iota_s(y) \oplus 0$ if $y \in H^{i-2}(Y)$ and 0 otherwise. Applying $\pi^{i-2,2}$ to $\iota_s(y) \oplus 0$ yields $\iota_s(y) \oplus -h^*(y)$ by Lemma [7.2](i).

- Rewriting the previous step we get

$$\pi^{i-2,2} = \pi^{i-2} f^s(\pi^i_X - p^i_X)f_s\pi^{i-2}.$$  

- By the above, $\pi^{i-2,2}$ is algebraic for $i \leq n - 2$, and the operator $\pi^{i-2} - \pi^{i-2,2} = \pi^{i-2,0}$ is algebraic for $i \leq n - 2$. The Lemma is thus settled. ■

Lemma 7.4 The projectors $\pi^i_X$ are algebraic for $i \neq n - 1, n, n + 1$.

Proof: The proof is immediate, since for $i \leq n - 2$ the operator $\pi^i_X = \pi^{i,0} + \pi^{i-2,2}$ is algebraic by Lemma [7.3] ■

Proof of Proposition 7.1: It remains only to check that $p^i_X = \pi^i_X - (\pi^i_X - p^i_X)$ is algebraic for $i \leq n - 2$; this holds by Lemma [7.2](iii). Proposition 7.1 is thus established. ■

7.2 Proof of the Main Theorem

We finally prove the Main Theorem.

Assume that $B(\mathcal{Y})$ holds for $\mathcal{Y}$ the generic fibre of a Lefschetz fibration $\rho$ of $X$ satisfying condition (A).

By Lemma [2.6] we have the following identity:

$$t^*(\Lambda_X - p^{n+1}_X) - \Lambda_Y = \sum_{j=n+2}^{2n-2} t^*L^{j-n-1}p^j_X.$$  

Aside (not necessary): Assuming $B(Y)$, the l.h.s. of is algebraic if and only if $\Lambda_X - p^{n+1}_X$ is. This follows from the identity $t^*[t^*(\Lambda_X - p^{n+1}_X)] = \Lambda_X - p^{n+1}_X$.

The next step is to prove that the r.h.s. of (23) is algebraic. This will follow from the next Lemma ($j = 2n - i$).
Lemma 7.5 Assume $B(Y)$. For $0 \leq i \leq n-2$, the operator

$$L^{n-i-1}p_{X}^{2n-i} = \Lambda X^{i}p_{X}^{i} = f_{\ast}\Lambda p_{\ast}^{*}t_{\ast}^{i}$$

is algebraic.

Proof of Lemma 7.5 Let $i \leq n-2$; then $p_{X}^{i}$ is algebraic by Proposition 7.1. Consider the subspace $W = (L^{n-i-1}P(X) \oplus L^{n-i-2}P(\Delta)) \cap F_{\rho}^{2}$, which agrees with the image of

$$k_{x} = t_{x} \oplus (-h^{*}) : L^{n-2-i}P(Y) \rightarrow L^{n-i}P(X) \oplus L^{n-2-i}P(\Delta).$$

The first component is an isomorphism, and the second is injective, being bijective if $i < n-2$. On applying $L$, which coincides with $L_{\rho}$ on $F_{\rho}^{2}$, we have an isomorphism

$$L : W \xrightarrow{\sim} L^{n-i}P(X) \oplus 0 \subset F_{\rho}^{2};$$

the piece $L^{n-i}P(X) \oplus 0 = L^{n-i}P(\tilde{X})$ equals $k_{\ast}L_{\rho}^{n-i}P(\tilde{X})$ – see Corollary 4.5. $L$ is thus an isomorphism between $W'$ and $L^{n-i}P(X) \oplus 0$ (by Corollary 4.5, $m \cdot L$ and $L_{\tilde{X}}$ agree on $F_{\rho}^{2}$). The identity

$$L_{\rho}Y = 1_{Y} - \sum_{i=0}^{n-1} p_{\rho}^{i},$$

( [18] p. 372) translates by Proposition 6.4 into

$$L_{\rho}\Lambda_{\rho} = 1_{X} - \sum_{i=0}^{n-1} p_{\rho}^{i},$$

thus showing that $\Lambda_{\rho}$ defines the inverse isomorphism to $L : W \rightarrow L^{n-i}P(X) \oplus 0$. Taking the $X$-component yields the inverse

$$L^{n-i}P(X) \oplus 0 \rightarrow W \rightarrow L^{n-1-i}P(X)$$

of $L$, which coincides with $\Lambda_{X} | L^{n-i}P(X)$ – here we have used that $L$ agrees with $L_{\rho}$ on $F_{\rho}^{2}$, and that $p_{\rho}^{i}$ acts as $0$ on $H^{j}(\tilde{X})$ for $j \geq n+2$. We have thus proven that $\Lambda_{X}^{i}p_{X}^{n-i} = L^{n-i}p^{n+i} = f_{\ast}\Lambda_{\rho}p^{*}t_{\ast}^{i}p_{\rho}^{n-i}$ is algebraic by Propositions 6.4 and 7.1. ■

Lemma 7.6 Assuming the hypotheses of the Main Theorem, the operator

$$\Lambda_{X}\pi_{X}^{i} = \pi_{X}^{i-2}\Lambda_{X} = \Lambda_{X}(\pi_{X}^{i} - p_{X}^{i})$$

is algebraic for $i \leq n$. The operator $(\pi_{X}^{n-1} - p_{X}^{n-1})\Lambda_{X} = (\Lambda_{X}-p_{X}^{n+1})\pi_{X}^{n+1}$ is algebraic.

Proof: The identities are clear; let us prove algebraicity of the above operators. By Lemmas 2.6 and 7.2(iii) we have

$$\Lambda_{X}(\pi_{X}^{i} - p_{X}^{i}) = \Lambda_{X}\iota_{\ast}\Lambda_{Y} \pi_{Y}^{i}t^{*} = \iota_{\ast}\Lambda_{Y}^{2} \pi_{Y}^{i}t^{*} + \left( p_{X}^{n+1} + \sum_{j=n+2}^{2n-2} p_{X}^{j}L^{j-n+1} \right) \iota_{\ast}\Lambda_{Y} \pi_{Y}^{i}t^{*} =$$

$$= \iota_{\ast}\Lambda_{Y}^{2} \pi_{Y}^{i}t^{*} + \left( \sum_{j=n+2}^{2n-2} p_{X}^{j}L^{j-n+1} \right) \iota_{\ast}\Lambda_{Y} \pi_{Y}^{i}t^{*},$$

which is algebraic for $i \leq n + 1$ by Lemma 7.5, thereby establishing the Lemma. ■
Proof of Main Theorem: Under the hypotheses of this Section, the operator \( A_X - p_X^{n+1} \) is algebraic.

Indeed, we have proven in Lemma 7.6 that \( \pi_X^{n-2} A_X = \Lambda_X \pi_X^n \) and \( \pi_X^{n-3} A_X = \Lambda_X \pi_X^{n-1} \) are algebraic, as well as the algebraicity of \( (\Lambda_X - p_X^{n+1}) \pi_X^{n+1} \). It now remains to establish the algebraicity of \( \Lambda_X \sum_{i=n+2}^{2n} \pi_X^i \). Again, Lemma 7.6 shows that, for \( r \geq 2 \), the operator \( t (\Lambda_X \pi_X^{n+r}) = \pi_X^{n-r} \Lambda_X \) is algebraic. On the other hand, \( (\Lambda_X - p_X^{n+1}) \pi_X^{n+1} = \Lambda (\pi_X^{n+1} - t p_X^{n-1}) \). Altogether this shows that the operator

\[
A_X - p_X^{n+1} = \Lambda_X \left( \sum_{k=0}^{n} \pi_X^k + (\pi_X^{n+1} - t p_X^{n-1}) + \sum_{k=n+2}^{2n} \pi_X^k \right)
\]

is algebraic. The Main Theorem is thus established.

7.3 Final comments

On the field of definition of the correspondences \( c \Lambda, \Lambda, \pi \) we would like to say the following. The correspondence \( L \) is \( k \)-defined. Now assume that \( k \) is perfect: the operator \( H \) and all Künneth projectors are Galois invariants; if they are algebraic, an elementary argument furnishes \( k \)-defined algebraic representatives for \( H, \pi \). The operator \( c \Lambda \) is uniquely determined by its \( sl_2 \)-partners \( H, L \) (see e.g. [2] 5.2.2), hence Galois invariant, and again is represented by a \( k \)-defined algebraic class if it is algebraic. If \( k \) is arbitrary, one needs to descend from a purely inseparable finite extension \( k' \) to \( k \); which is standard, since the natural map \( X \times_k k' \to X \) is a homeomorphism. Thus our form of the conjectures is in fact equivalent to that of [13] [18], where it was assumed that \( k = \overline{k} \). This formulation was required in order to obtain algebraic cycles supported on \( \tilde{X} \times_{\tilde{p}} \tilde{X} \).

Restatement of the Main Theorem: In the language of Proposition 2.5 our Main Theorem shows precisely that \( \theta^i \) is induced by an algebraic cycle (to wit \( (\Lambda - p^{n+1})^{n-i} \)) for \( i \leq n - 2 \). By the proof of [18] Lemma 2.4, one derives that \( \pi_X^i \) is algebraic for \( i \neq n - 1, n, n + 1 \).

On the results needed in the Proof: The path we have travelled in order to settle our Main Theorem is not the shortest possible. Lemma 7.5 does not need \( \Lambda_\rho \) or \( \pi_\rho^i \), but merely arbitrary liftings of \( \Lambda_Y, \pi_Y^i \) to \( \tilde{X} \times_{\tilde{p}} \tilde{X} \), since the proof of Lemma 7.5 requires only working on \( F_\rho^2 \). Lemma 7.6 relies solely on Lemma 7.5 and material from Section 2 and the Main Theorem rests on Lemma 7.6. The use of Lemma 7.6 allows for a direct proof of the algebraicity of \( \Lambda - p^{n+1} \) without Proposition 7.1, but then one should use \( \theta^0, \cdots, \theta^{n-2} \) as in the proof of [18] Lemma 2.4, to derive that \( \pi_X^i \) is algebraic for \( i \neq n - 1, n, n + 1 \).

On the support of \( \pi_X^i \): We have proven that \( \pi^{i-2,2} \) is algebraic for \( i \leq n \). However, our proof does not necessarily imply that \( \pi^{i-2,2} \) is supported on \( D = \tilde{X} \times_{\tilde{p}} \tilde{X} \). In fact, it follows from Remark 4.12 and the proof of Proposition 4.13 that \( \pi^{i-2,2} \) is not supported on \( D \) unless it is 0 (i.e. when \( H^{i-2}(X) = 0 \)).
7.4 The operator \( p_X^{n+1} \)

This section is a complement to the Main Theorem. The operator \( p_X^{n+1} \) (and so \( \iota^* p_X^{n+1} \)) is of central importance in the Lefschetz theory of \( X \).

Lemma 7.7 Assume \( B(Y) \) for \( Y \) the general fibre of a Lefschetz pencil of \( X \) satisfying (A). The algebraicity of \( p_X^{n-1} \) implies that of \( p_X^n \) and the conjecture \( C(X) \).

The lemma follows from Proposition 6.6.

Lemma 7.8 Let \( X \) be a projective smooth, \( n \)-dimensional variety. The operator \( \iota^* p_X^{n+1} \) is algebraic if \( p_X^{n+1} \) is.

Proof: The result stems from the following identity:

\[
(t_*(\iota^* p_X^{n+1}) \iota^* p_X^n) = p_X^n L p_X^{n+1} = p_X^{n+1}. \]

Enclosed in \( p_X^{n+1} \) is information about the space of vanishing cycles, and also \( p_X^{n-1} \). For instance, as we shall see below, the algebraicity of \( p_X^{n+1} \) allows us to speak of the motive of vanishing cycles.

Proposition 7.9 With the above notations, the following statements hold.

1. The algebraicity of \( p_X^{n+1} \) implies that of \( p_X^{n-1} \).

2. Let \( V(Y) \) be the space of vanishing cycles of a smooth hyperplane section \( Y \), and \( e_{V(Y)} \) be the orthogonal projection \( H^*(Y) \rightarrow V(Y) \hookrightarrow H^*(Y) \). Then

\[
p_Y^{n-1} - e_{V(Y)} = \iota^* p_X^{n+1} \iota_*.
\]

3. If \( B(Y) \) holds and \( p_X^{n+1} \) is algebraic, then so is \( e_{V(Y)} \).

Proposition 7.10 The operator \( p_X^{n+1} \) cannot be obtained as \( f_*u f^* \), with \( u \) an algebraic cycle supported on \( D = \tilde{X} \times_{\mathbb{P}^1} \tilde{X} \).

Proof: It follows from Proposition 6.6 that \( L p_X^{n-1} \oplus 0 \subset F_\rho^2 \) and

\[
P^{n-1}(X) \oplus H^{n-3}(\Delta) \cap F_\rho^1 = 0;
\]

since every correspondence supported on \( D \) preserves the Leray filtration by Proposition 4.10, the assertion is clear.

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Email address: jjramon@maths.ucd.ie