On the possibility of Dark Energy from corrections to the Wheeler-DeWitt equation

William Nelson and Mairi Sakellariadou
King’s College London, Department of Physics, Strand WC2R 2LS, London, U.K.

We present a method for approximating the effective consequence of generic quantum gravity corrections to the Wheeler-DeWitt equation. We show that in many cases these corrections can produce departures from classical physics at large scales and that this behaviour can be interpreted as additional matter components. This opens up the possibility that dark energy (and possible dark matter) could be large scale manifestations of quantum gravity corrections to classical general relativity. As a specific example we examine the first order corrections to the Wheeler-DeWitt equation arising from loop quantum cosmology in the absence of lattice refinement and show how the ultimate breakdown in large scale physics occurs.

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Quantum cosmology can be studied within the context of mini-superspace models, reducing the full quantum field theory to a quantum mechanical system of finite degrees of freedom. Applying this to the evolution of a three metric results in the Wheeler-DeWitt equation (WDW) equation, which breaks down near the classical big-bang singularity. Loop Quantum Gravity (LQG), another approach to canonical quantisation, uses triads and connections rather than metrics and extrinsic curvatures as the basic variables, with more success, whilst string and brane theories remove the singular behaviour by the extra dynamical states that become available at small scales. Irrespective of the full underlying theory, the WDW equation must be recovered as a semi-classical approximation to the dynamical equations of the theory, if classical general relativity is to be produced at large scales. We can then ask whether Quantum Gravity (QG) corrections can have any effects on large scale physics.

At first sight it would appear unlikely that this is possible. However, the wave-function solutions to the WDW equation oscillate and so their derivatives can become significant at large scales. If the QG corrections depend on the derivatives of the wave-function, then it is possible that they become significant at macroscopic scales. This is the case for Loop Quantum Cosmology (LQC), which predicts that the evolution of the universe is dictated by a difference, rather than a differential equation. If the lattice size of this difference equation is fixed, as was assumed until recently, not all the oscillations of the wave-function can be supported, leading to a breakdown in large scale classical physics.

We investigate the effects of a general class of corrections to the WDW equation, considering all possible derivative terms and show that, at least initially, these corrections mimic the behaviour of additional matter components. This raises the exciting possibility that QG corrections could, at late times, produce a cosmological constant like effect. It is also possible that such corrections may produce additional matter components such as dark matter, however this would only be true for a limited class of corrections. What is generally true is that QG correction terms that dominate at small scale can also produce significant large scale effects, which can be well approximated by the classical behaviour of additional matter sources.

By considering the mini-superspace model of an homogeneous and isotropic universe, the WDW equation reads

\[ \frac{1}{a} \frac{d}{da} \left[ \frac{1}{a} \frac{d}{da} (a \psi (a)) \right] - \frac{9k}{16\pi^2 l_P^2} a \psi (a) + \frac{3}{2\pi^2 l_P^2} H \phi \psi (a) = 0 \]

where \( a \) is the scale factor, \( k = 0, \pm 1 \) is the curvature, \( H \) is the matter Hamiltonian; in our units \( \hbar = c = 1 \).

In general, \( H \) will have several terms, with different dependence on \( a \) and \( \phi \). Being only interested in the large scale (\( a \gg l_P \)) behaviour of \( H \), we expect one term to dominate and we approximate it by \( [3/(2\pi l_P^2)] H = \epsilon (\phi) a^\delta \). With this approximation the WDW equation can be written in as

\[ \frac{1}{a} \frac{d}{da} \left[ \frac{1}{a} \frac{d}{da} (a \psi (a)) \right] - \frac{9k}{16\pi^2 l_P^2} a \psi (a) + \epsilon (\phi) a^\delta \psi (a) = 0 \]

We only consider the flat case, so that solutions to Eq. (2) are analytically tractable. If this equation breaks down at small scales, due to some QG effects, extra terms should become significant on these scales. These new QG correction terms will be a function of \( \psi (a) \) and its derivatives, so the full underlying equation can be written as

\[ \frac{1}{a} \frac{d}{da} \left[ \frac{1}{a} \frac{d}{da} (a \psi (a)) \right] + \epsilon (\phi) a^\delta \psi (a) + a_0 f (\psi (a), \partial_a \psi, \partial_a^2 \psi, \cdots ) = 0 \]

where \( a_0 \) is the scale at which the new terms become important. At large scales, \( a \gg a_0 \), one might naively assume that the QG corrections can simply be ignored, however this assumes that \( \psi (a) \) and all its derivatives remain small at all scales, which is easily shown not to always be consistent. This has been explored within the context of LQC, in which the exact form of the \( a_0 \) corrections are known.

Assuming \( a_0 \ll \epsilon (\phi) \), whilst \( f (\psi (a), \partial_a \psi, \cdots ) \) is
small, we can find approximate solutions to Eq. (3) as

\[ 
\psi(a) \approx C_1 \left[ \frac{\sqrt{\epsilon} a^{(3+\delta)/2}}{3+\delta} \right]^{-1} 
+ C_2 \left[ \frac{\sqrt{\epsilon} a^{(3+\delta)/2}}{3+\delta} \right]^{-1} \psi(a), 
\]

where \( J \) and \( Y \) are Bessel functions of the first kind and second kind, respectively, \( C_1 \) and \( C_2 \) are integration constants; for clarity the \( \phi \) dependence has been suppressed. We can thus evaluate approximate solutions to the derivatives:

\[ 
\partial_a \psi(a) \approx \left[ 1 - \sqrt{\epsilon} Z_1(a) a^{(3+\delta)/2} \right] a^{-1} \psi(a), 
\]

\[ 
\partial_a^2 \psi(a) \approx \left[ Z_1(a) a^{(3+\delta)/2} - \sqrt{\epsilon} a^{3+\delta} \right] \sqrt{\epsilon} a^{-2} \psi(a), 
\]

\[ 
\partial_a^3 \psi(a) \approx \left[ \epsilon Z_1 (9+3\delta)/2 - 3Z_1 a^{(3+\delta)/2} \right] \sqrt{\epsilon} a^{-3} \psi(a) 
- \sqrt{\epsilon} (1+\delta) a^{3+\delta} \sqrt{\epsilon} a^{-3} \psi(a), 
\]

etc.

where

\[ 
Z_1(a) \equiv C_1 \int \frac{d\alpha}{\alpha} \left[ \frac{2\sqrt{\epsilon} a^{(3+\delta)/2}}{3+\delta} \right] \psi(a)^{-1}. 
\]

Taking the large argument expansion of the Bessel functions, which is the large \( a \) limit for \( \delta > -3 \), one obtains

\[ 
\lim_{a \to \infty} Z_1(a) = \frac{C_1 \sin(A(a)) - C_2 \cos(A(a))}{C_1 \cos(A(a)) + C_2 \sin(A(a))}, 
\]

\[ 
= \lim_{a \to \infty} \frac{\psi^*(a)}{\psi(a)}, 
\]

where \( \psi^*(a) \) is the solution corresponding to choosing the integration factors \( C_1 \to -C_2 \) and \( C_2 \to C_1 \) compared to the solution given in Eq. (1). This approximation is valid only when the correction terms are still small compared to the matter component, however this method will give the correct effective initial behaviour of QG corrections.

In particular, the corrections become non-negligible for scales at which the corrections are of the order of \( a_{0}^{-1} \) and the approximation breaks down when they are of the order of \( \epsilon(\phi) a_{0}^{4} \). It is clear that for \( \delta > -1 \) the derivative terms grow, in agreement with Ref. [7]. In addition, for \( \delta > -1 \), the terms containing the highest powers of \( \epsilon \) grow fastest and hence are always dominant, implying

\[ 
\partial_a \psi(a) \approx -\sqrt{\epsilon} \lim_{a \to \infty} \frac{\psi^*(a)}{\psi(a)} a^{(1+\delta)/2} \psi(a), 
\]

\[ 
\partial_a^2 \psi(a) \approx -\epsilon a^{1+\delta} \psi(a), 
\]

\[ 
\partial_a^3 \psi(a) \approx \epsilon^{3/2} \lim_{a \to \infty} \frac{\psi^*(a)}{\psi(a)} a^{3(1+\delta)/2} \psi(a), 
\]

\[ 
\partial_a^4 \psi(a) \approx \epsilon a^{2(1+\delta)} \psi(a), 
\]

etc.

This allows us to approximate the initial effect of large scale breakdown of classical physics (breakdown in preclassicality [8]) as a second matter component. In general the \( \beta^{th} \) derivative is proportional to \( a^{\beta(1+\delta)/2} \), thus correction terms like \( a^{\alpha} \partial_a^\beta \psi(a) \) can be approximated by additional terms in Eq. (2), that scale like \( a^{\alpha+\beta(1+\delta)/2} \).

It is then possible to consider these correction terms as being produced from an effective Hamiltonian. A useful pedagogical example is to consider classical matter terms (although the method is, of course, general), where the effective Hamiltonian that mimics the correction terms is

\[ 
H_{\text{cor}} \propto a^{\alpha+\beta(1+\delta)/2}. 
\]

An example of how these approximations compare to the exact case is given in Fig. (1).

We have shown how hypothetical correction terms to the WDW equation can mimic the behaviour of additional matter sources. In particular, such corrections can be used to produce an effective cosmological constant. For example, consider \( H_{\phi} \propto a^{0} \), i.e., a matter dominated universe (see Table I); \( \delta = 0 \). If we want the quantum corrections to mimic a vacuum energy then we need \( H_{\text{cor}} \propto a^{3} \), implying \( \alpha + \beta/2 = 3 \). Thus, correction terms like \( a_{0} a^{3} \), \( a_{0} a^{2} \psi^2/da^2 \), \( a_{0} a \psi(a)/da \), \( a_{0} a^2 \psi(a)/da^3 \), etc., with \( a_{0} < 0 \), all resemble a vacuum energy in the presence of a matter dominated universe (see Fig. (1)). In the presence of other types of matter \( \delta \) changes and so the form of the correction terms that give vacuum energy like behaviour are different. In particular, for a universe dominated by radiation, we have \( \delta = -1 \) and the only correction term that can mimic the behaviour of dark energy is \( a_{0} a^3 \). Considering a universe dominated by a field that scales like \( H_{\phi} \propto a^{2} \), i.e., \( \delta = -2 \), then a correction term like \( a_{0} a^{-3} \psi^2(a)/da^3 \) mimics a (negative) cosmological constant like term. This is precisely the dominant correction from LQC.

Up to now we have only discussed how the correction terms scale with \( a \), however there is clearly also a dependence on the constant part of the matter Hamiltonian, \( \rho_{0} \) (through \( \epsilon(\phi) \)). In principle this differentiates these correction terms from other models of dark energy (cosmological constant, quintessence, etc.), as does the fact that as these correction terms begin to dominate the WDW equation this approximation will break down and the be-

| Term          | \( \rho \) | \( H_{\phi} \) | \( \delta \) |
|---------------|-----------|-------------|---------|
| Matter        | \( a^{-3} \) | \( a^0 \) | 0       |
| Radiation     | \( a^{-4} \) | \( a^{-1} \) | -1      |
| Vacuum energy | \( a^0 \) | \( a^3 \) | 3       |
bearsome of the wave-function will be drastically altered.

In addition to corrections that mimic the behaviour of dark energy, it is also possible to find correction terms that have a dark matter like form, for which Eq. \(10\) is

\[
\mathcal{H}_{\text{cor}} \propto a^{\alpha + \beta (1 + \delta)/2} \propto a^0 .
\]

For example, in a matter dominated universe (i.e., \(\delta = 0\)),

\[
a_0 a^{-1/2} \frac{d \psi}{da}, \quad a_0 a^{-1} \frac{d^2 \psi}{da^2}, \quad a_0 a^{-3/2} \frac{d^3 \psi}{da^3}, \quad \text{etc.}
\]

all produce additional matter like terms. Notice that, unlike the dark energy case, these corrections do not scale faster than original matter component and so will never dominate Eq. \(3\). In addition, these correction matter terms will be closely related to the physical matter Hamiltonian degrees of freedom. For example,

\[
a_0 a^{-2} \frac{d^3 \psi}{da^4} \approx a_0 \varepsilon(\phi)^2 \psi(a),
\]

which amount to replacing \(\varepsilon(\phi) \rightarrow \tilde{\varepsilon}(\phi) = \varepsilon(\phi) + a_0 \varepsilon(\phi)^2\) in Eq. \(3\). Within the mini-superspace model used here it is only possible to explore homogeneous, isotropic solutions and so we cannot say that these dark matter like correction terms would produce the necessary behaviour to explain structure formation, galaxy and cluster dynamics, etc. Another difficulty with this type of correction comes from the fact that the energy density of dark matter is approximately five times that of standard matter. If correction terms were to produce dark matter in the presence of a matter dominated universe, \(a_0\) would have to be fine tuned to ensure that \(a_0 \ll \varepsilon(\phi)\) (the approximation used here) and \(a_0 \varepsilon(\phi)^2/2 - 1 \approx 5\). Similar dark matter like correction terms can be produced for a radiation, or vacuum energy, dominated universe; if these corrections arose due to the the presence of a vacuum energy, one may explain the coincidence problem, i.e., that the dark matter degrees of freedom would be dictated by those of the dark energy, however again significant tuning would probably be required.

LQG is a background independent, non-perturbative method of quantising gravity. Reducing the symmetries of the theory to a specific cosmological model makes the theory tractable and ensures a large scale continuum limit. The theory is based on holonomies of the standard Ashtekar variables \(2\), namely the triad and connections. In an isotropic model these can be parameterised by single variables, \(\tilde{p}\) and \(\tilde{\varepsilon}\) respectively, which in terms of standard cosmological variables are \(|\psi| = a^2\), \(\tilde{c} = k + \gamma \dot{a}\), \((\gamma\) is an ambiguity parameter known as the Barbero-Immirzi parameter). Define \(p = \tilde{p} V_0^0/3\) and \(c = eV_0^0/3\), where \(V_0\) is the volume of a fiducial cell \(\mathcal{R}\) related via the classical identity, \(\{c, p\} = 8\pi l_P^2 \tilde{\gamma}/3\). By analogy with the full LQG theory, \(c\) is quantised via its holonomies, \(h = \exp(i\mu \tilde{c}/2\), where \(\mu\) is an arbitrary real number. Then \(\mathcal{R}, \tilde{p}|\mu\rangle = |\mu|\langle\mu| \equiv 4\pi l_P^2 \tilde{\gamma}/3|\mu\rangle\), and \(e^{i\mu \tilde{c}/2}|\mu\rangle = e^{\mu \tilde{c}/2}|\mu\rangle |\mu + \mu_0\rangle\).

The action of the gravitational part of the Hamiltonian constraint on the basis states \(|\mu\rangle\) reads \(4\)

\[
\hat{H}_g|\mu\rangle = \frac{3}{256\pi^2 l_P^4 \tilde{\gamma}^3/3 l_0^3} \left\{ \left[ S(\mu) + S(\mu + 4\mu_0) \right]|\mu + 4\mu_0\rangle \\
- 4S(\mu)|\mu\rangle + \left[ S(\mu) + S(\mu - 4\mu_0) \right]|\mu - 4\mu_0\rangle \right\} (14)
\]

where

\[
S(\mu) = (4\pi l_P^2 \tilde{\gamma}/3)^{3/2} |\mu + \mu_0|^{3/2} - |\mu - \mu_0|^{3/2} .
\]

The Hamiltonian constraint is \(\left\{ \hat{H}_g, \hat{H}_\phi \right\} |\psi\rangle = 0\), where \(|\psi\rangle = \sum_\mu \psi_\mu |\mu\rangle\). Taking the continuum limit \(\psi_\mu \rightarrow \psi(\mu)\) and using the expansion \(4\)

\[
|\mu + \alpha \mu_0|^{3/2} - |\mu + \beta \mu_0|^{3/2} = \mu^{3/2} \left[ \frac{3\mu^2}{2\mu} (\alpha - \beta) + \frac{3\mu_0^2}{2\mu^2} (\alpha^2 - \beta^2) - \frac{3\mu_0^3}{16\mu^3} (\alpha^3 - \beta^3) + \cdots \right],
\]

FIG. 1: Solutions to the WDW equation, for \(\delta = 0\) (i.e.,
matter dominated universe), \(\varepsilon(\phi) = 5 \times 10^{-3}\), \(a_0 = 9 \times 10^{-6}\) (with \(l_P = 1\)). The QG corrections are \(a_0 a^2 \frac{d^3 \psi}{da^3}\). The solutions are calculated numerically: the bare solution (solid) excludes QG corrections entirely whilst the approx. solution (dotted) approximates them as discussed in the text. For small \(a\) the QG corrections do not play a role, however as \(a\) becomes larger the effects of the QG corrections become significant.
we can expand Eq. (14).
Changing variables, using $a^2 = 4\pi l_p^2 \gamma |\mu|/3$, we obtain
\[
\mathcal{H}_a \psi(a) = \frac{2\pi l_p^2}{3} \left( \frac{1}{a} \frac{d}{da} \left[ \frac{1}{a} \frac{d}{da} [a \psi(a)] \right] + \frac{d}{da} \left[ \frac{1}{a} \frac{d}{da} \right] \psi(a) \right) + \frac{2\pi l_p^2 a_0}{9} \left( \frac{1}{a^3} \frac{d^4}{da^4} [a \psi(a)] - 4 \frac{d^3}{da^3} \psi(a) + \frac{47}{8a^5} \frac{d^2}{da^2} \psi(a) \right) + \frac{1}{2a^6} \frac{d}{da} \psi(a) - \frac{135}{16a^2} \psi(a) ,
\]
where $a_0 = 16\pi^2 l_p^4 \gamma^2 \mu_0^2/9$. The full Hamiltonian constraint is easily written in the same form as Eq. (3). As we have shown, for $\delta > -1$ the higher derivative terms scale faster. Since in this case the lower derivative terms are further suppressed by powers of $a$, to a good approximation the effective large scale equation to solve is
\[
\frac{1}{2} \left( \frac{1}{a} \frac{d}{da} \left[ \frac{1}{a} \frac{d}{da} [a \psi(a)] \right] + \frac{d}{da} \left[ \frac{1}{a} \frac{d}{da} \right] \psi(a) \right) + \frac{a_0}{3} (\phi)^2 a^{2\delta-1} \psi(a) + \epsilon (\phi) a^\delta \psi(a) \approx 0 ,
\]
where $3\mathcal{H}_a/(2\pi l_p^2) = \epsilon (\phi) a^\delta$ and we used Eq. (14) to approximate the fourth order derivative term in Eq. (17) by term that resembles an additional matter component.

LQC is a concrete example of the general methods for approximating the QG corrections. For a universe dominated by a (classical) matter content with $\delta = 0$, the leading correction term acts as an effective radiation field, $\mathcal{H}_{\text{cor}} = a_0 \epsilon (\phi)^2 a^{-1}/3$, i.e., the correction term mimics radiation. For a radiation dominated universe with $\delta = -1$, the correction term acts as $\mathcal{H}_{\text{cor}} = a_0 \epsilon (\phi)^2 a^{-3}$. For $\delta = 3$, i.e., a universe dominated by a vacuum energy, such as during inflation, the correction term acts like $\mathcal{H}_{\text{cor}} = a_0 \epsilon (\phi)^2 a^5$. Notice that in this case the corrections scale faster than the existing matter component, which motivated the modelling of lattice refinement ($\mu_0 \rightarrow \tilde{\mu}(\mu)$ is no longer a constant) in loop quantum cosmology [3, 6, 7]. For a universe dominated by a source with $\delta = 2$, the dominant correction term acts as $\mathcal{H}_{\text{cor}} = a_0 \epsilon (\phi)^2 a^3$, i.e. it mimics a negative cosmological constant (i.e. an accelerated contraction), whilst $\delta = 1/2$ leads to the correction term mimicking dark matter.

We have shown how generic QG corrections to the WDW equation mimic the behaviour of additional matter sources, at least whilst they are smaller than the existing matter components. This simple procedure can be used to examine, to first order, the effect of any QG correction under consideration. Here, as a specific example, we used the first order corrections from loop quantum cosmology to examine what the breakdown of large scale classical physics looks like in the absence of lattice refinement. The fact that quantum corrections introduce extra matter components opens up the possibility that they may be able to explain the current cosmological acceleration and possibly dark matter. That these correction components become significant only at specific scales is encouraging, as this may provide an explanation why classical general relativity is valid on sub-galactic scales, but requires the input of additional matter on super-galactic scales. However it remains to be seen if it is possible for such corrections to meet observational constraints.

This work can be used in two complementary ways: top-down and bottom-up. The former is to apply this method to any fundamental theory that can produce a WDW like equation on large scales and so characterise the new, phenomenological effects of the theory. The latter would estimate the types of QG corrections that would be necessary to produce certain desirable effects (e.g., dark energy) at particular scales. The case of loop quantum cosmology illustrates how both approaches can be used to mutual benefit. The underlying theory was used to calculate the form of the QG corrections to the WDW equation. The phenomenological consequence (eventual domination of the correction terms and hence a break down in large scale classical physics) highlighted the importance of modelling lattice refinement within cosmological models.

Considering the phenomenological consequences of a theory is the first step to testing it against experimental and observational data. We have shown here that for QG this principle can be invaluable in guiding our search for the full theory.

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[11] Note that the factor ordering is not unique. We will be interested only in the scaling behaviour of $\psi(a)$ at large scales where the factor ordering is of no consequence.