Pebbling on Jahangir graphs
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Abstract: The pebbling number of a graph \( G \), \( f(G) \), is the least \( p \) such that, however \( p \) pebbles are placed on the vertices of \( G \), we can move a pebble to any vertex by a sequence of moves, each move taking two pebbles off one vertex and placing one on an adjacent vertex. In this paper, we will show the pebbling number of Jahangir graphs \( J_{n,m} \) with \( n \) even, \( m \geq 8 \).

Keywords: pebbling number, Jahangir graph, graph parameters.

1 Introduction

Pebbling in graphs was first introduced by Chung[2]. We only consider simple connected graphs. For a given graph \( G \), the pebbling distribution \( D \) of \( G \) is a projection from \( V(G) \) to \( N \), \( D : V(G) \rightarrow N \), where, \( D(v) \) is the number of pebbles on \( v \), the total number of pebbles on a subset \( A \) of \( V \) is given by \( |D(A)| = \sum_{v \in A} D(v) \), \( |D| = |D(V)| \) is the size of \( D \).

A pebbling move consists of the removal of two pebbles from a vertex and the placement of one pebble on an adjacent vertex. Let \( D \) and \( D' \) be two pebbling distribution of \( G \), we say that \( D \) contains \( D' \) if \( D(v) \geq D'(v) \) for all \( v \in V(G) \), we say that \( D' \) is reachable from \( D \) if there is some sequence (probably empty) of pebbling moves start from \( D \) and resulting in a distribution that contains \( D' \). For a graph \( G \), and a vertex \( v \), we call \( v \) a root if the goal is to place pebbles on \( v \); If \( t \) pebbles can be moved to \( v \) from \( D \) by a sequence of pebbling moves, then we say that \( D \) is \( t \)-fold \( v \)-solvable, and \( v \) is \( t \)-reachable from \( D \). If \( D \) is \( t \)-fold \( v \)-solvable for every vertex \( v \), we say that \( D \) is \( t \)-solvable.

Computing the pebbling number is difficult in general. The problem of deciding if a given distribution on a graph can reach a particular vertex was shown in [13] to be NP-complete, even for planar graphs[12]. The problem of deciding whether a graph \( G \) has pebbling number at most \( k \) was shown in [13] to be \( \Pi_2^P \)-complete.

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Given a root vertex $v$ of a tree $T$, then we can view $T$ be a directed graph $\overrightarrow{T}_v$ with each edge directed to $v$. A path partition is a set of nonoverlapping directed paths the union of which is $\overrightarrow{T}_v$. A path partition is said to majorize another if the nonincreasing sequence of the path size majorizes that of the other (that is $(a_1, a_2, \ldots, a_r) > (b_1, b_2, \ldots, b_l)$ if and only if $a_i > b_i$ where $i = \min\{j : a_j \neq b_j\}$). A path partition of a tree $T$ is said to be maximum if it majorizes all other path partitions.

**Theorem 1.1** [1, 2, 14] Let $T$ be a tree, $(a_1, \ldots, a_n)$ is the size of the maximum path partition of $\overrightarrow{T}_v$, then $$f(T,v) = \sum_{i=1}^{n} 2^{a_i} - n + 1.$$  

**Lemma 1.2** ([16]) The pebbling numbers of the cycles $C_{2n+1}$ and $C_{2n}$ are $$f(C_{2n+1}) = 2 \left\lfloor \frac{2^{n+1}}{3} \right\rfloor + 1, f(C_{2n}) = 2^n.$$  

The following lemma is important in the proof of our main result.

**Lemma 1.3** ([14], No-Cycle-Lemma) If we have a graph $G$ with a certain distribution of pebbles and a vertex $v$ of $G$ such that $m$ pebbles can be moved to $v$, then there always exists an acyclic orientation $H$ for $G$ such that $m$ pebbles can still be moved to $v$ in $H$.

In this paper, we will show the pebbling number of Jahangir graphs $J_{n,m}$ when $n$ is even and $m \geq 8$, which generalized the result of A. Lourdusamy etc.[13].

**2 Main Result**

**Definition 2.1** Jahangir Graph $J_{n,m}$ ($m \geq 3$) has $nm + 1$ vertices, that is, a graph consisting of a cycle $C_{nm}$ with one additional vertex which is adjacent to $m$ vertices of $C_{nm}$ at distance $n$ to each other on $C_{nm}$.
In this paper, we use the following notations: For a graph $G$, assume $v \in V(G)$, $A$ is a subset of $V(G)$, we use $p(v)$ and $p(A)$ to denote the number of pebbles on $v$ and $A$, respectively. We use $p_i$ to denote $p(v_i)$ for short.

Let $w$ be the root vertex of $G$. We say that a pebbling step from $u$ to $v$ is greedy if $\text{dist}(v, w) < \text{dist}(u, w)$, and that a graph $G$ is greedy if from any distribution with $f(G)$ pebbles on $G$, one pebble can be moved to any specified root vertex $w$ with greedy pebbling moves.

Clarke etc. asked the following question.

**Proposition 2.2** [13] Is every bipartite graph greedy?

Here we give a counterexample.

**Theorem 2.3** $J_{2,3}$ is not greedy.

**Proof.** Assume $J_{2,3}$ is shown as Figure 1. We know that $f(J_{2,3}) = 8$, if $p_2 = p_6 = 3, p_1 = p_7 = 1$, then we can not move a pebble on $v_4$ with greedy pebbling moves. 

By the following construction, we can show a set of bipartite graphs that are not greedy.

**Lemma 2.4** [11, 17] Given a graph $G$ and a vertex $v$ on $V(G)$, form $G'$ from $G$ by adding a new vertex, $u$, with $N(u) = N(v)$. If $G$ is Class 0, then $G'$ is also Class 0.

**Corollary 2.5** Assume $J_{2,3}$ is shown as Figure 1. Let $G_m$ be obtained from $J_{2,3}$ by adding $m$ new vertices, each of which is connected with all of $\{v_1, v_3, v_5\}$. Then $G_m$ is not greedy.

**Proof.** By Lemma 2.4, $G_m$ is Class 0, Let $p_2 = p_6 = 3, p_3 = p_4 = p_5 = 0, p(v) = 1$ otherwise, then we can not move a pebble on $v_4$ with greedy pebbling moves. So $G_m$ is not greedy.

A. Lourdusamy etc. gave the pebbling number of $J_{2,m}$.

**Theorem 2.6** [13] $f(J_{2,m}) = 2m + 10, m \geq 8$.

In this paper, we will show the pebbling number of $J_{n,m}$ ($n$ is even, $m \geq 8$).

Assume $J_{n,m}$ is obtained from $C_{nm} = v_0v_1 \cdots v_{nm-1}$ with an additional vertex $u$ which is adjacent to $v_{ni}$ $(0 \leq i \leq m-1)$. Let $P_i = v_{ni}v_{ni+1} \cdots v_{n(i+1)}$ for $i = 0, 1, \ldots, m-1$.

**Lemma 2.7** Let $L_n = v_0v_1 \cdots v_n$ be a path with length $n$, $D$ is a pebbling distribution on $L_n$, then

1) If neither of $v_0$ and $v_n$ are reachable, then $|D| \leq f(C_n) - 1$.
2) If only one of $v_0$ and $v_n$ is reachable, but not 2-reachable, then $|D| \leq f(C_{n+1}) - 1$.
3) If both of $v_0$ and $v_n$ are reachable, but not 2-reachable, then $|D| \leq f(C_{n+2}) - 1$.

**Proof.** 1) If neither of $v_0$ and $v_n$ are reachable, then we consider the same pebbling distribution $D$ on a new graph $C_{n-1} = v_0v_1 \cdots v_{n-1}v_0$, So $D$ is not $v_0$-solvable on $C_n$, thus $|D| \leq f(C_n) - 1$.
Lemma 2.8 \( f(C_{n-1}) + f(C_{n+1}) \geq 2f(C_n) \) for \( n \geq 2 \).

Lemma 2.9 Assume \( D \) is a pebbling distribution on \( J_{n,m} \) (\( m \geq 8 \)), if none of \( u, v_0 \) and \( v_n \) are reachable, and \( D(P_0) = 0 \), then we can get the following tight upper bounds.

\[
|D| \leq \alpha = \begin{cases} 
\frac{m}{2}(f(C_n) + f(C_{n+2}) - 2) - f(C_{n+2}) + 1, & \text{if } m \text{ is even}, \\
\frac{m-1}{2}(f(C_n) + f(C_{n+2}) - 2) + f(C_{n+1}) - f(C_{n+2}), & \text{if } m \text{ is odd}.
\end{cases}
\]

Proof. Note that \( J_{n,m} \) divide the cycle \( C_{nm} \) to paths \( P_i = v_{ni}v_{ni+1} \cdots v_{n(i+1)} \) for \( i = 0, 1, \ldots, m - 1 \) with length \( n \).

We only need to consider the contribution of the pebbling distribution on every path \( P_i \) to its endpoints.

If the pebbles on \( P_i \) can make both of its endpoints reachable, the number of pebbles is denoted by \( L \) (Large); If the pebbles on \( P_i \) can only make one of its endpoints reachable, the number of pebbles is denoted by \( M \) (Middle); If the pebbles on \( P_i \) can make neither of its endpoints reachable, the number of pebbles is denoted by \( S \) (Small). We also use \( L, M \) or \( S \) to name the path with \( L, M \) or \( S \) pebbles on it, respectively.

We will give a pebbling distribution with as many as possible pebbles.

Claim:

1. Any two paths with \( L \) pebbles cannot be adjacent, \( D(P_0) = 0 \), neither of \( P_1 \) and \( P_{m-1} \) is \( L \).

2. At least one \( S \) is between any two \( L \). Otherwise, there is a sequence \( L, M, M, \ldots, M, L \), then one endpoint is reachable from two paths respectively.

3. At least one \( S \) is between \( L \) and \( P_0 \). Otherwise, there is a sequence \( 0, M, M, \ldots, M, L \), then one endpoint is reachable from two paths or the endpoint of \( P_0 \) is reachable from \( L \) which is adjacent to \( P_0 \).

4. We may assume that there does not exist two \( M \) adjacent. By Lemma 2.8, if we replaced them by \( S, L \), we can get a new distribution with at least the same number of pebbles.

Assume there are \( k \) paths with \( L \) pebbles, from the Claim above, there are at least \( k + 1 \) paths with \( S \) pebbles, the number of pebbles on each of the left \( m - 2k - 2 \) paths is at most \( M \). Thus \( |D| \leq kL + (k + 1)S + (m - 2k - 2)M \).

If \( m \) is even, then at most \( \frac{m}{2} - 1 \) paths are \( L \). By Lemma 2.8 \( |D| \leq kL + (k + 1)S + (m - 2k - 2)M \leq kL + (k + 1)S + \frac{m-2k-2}{2}(L + S) \).

If \( m \) is odd, then at most \( \frac{m-3}{2} \) paths are \( L \). By Lemma 2.8 \( |D| \leq kL + (k + 1)S + (m - 2k - 2)M \leq kL + (k + 1)S + \frac{m-2k-3}{2}(L + S) + M \).
The upper bounds of $L, M, S$ are given by Lemma 2.7, and we are done.
Now we give the distribution $D^*$ to show that the bounds are tight:
For the path sequence $P_0, P_1, \ldots, P_{m-1}$,
If $m$ is even, the sequence of distribution is 0, $S, L, S, L, \ldots, S$;
If $m$ is odd, the sequence of distribution is 0, $S, L, \ldots, S, L, M, S$, where $M$ is $v_{n(m-1)}$-solvable.

Let $L, M$ and $S$ get the upper bounds in Lemma 2.7, we are done.

Let $\varsigma$ be a sequence of pebbling moves from the distribution $D_1$ to $D_2$ on the graph $G$, then we say that the number of the pebbles cost in $\varsigma$ is $|D_1| - |D_2|$. The pebbling move along one edge $\{w, v\}$ from $w$ to $v$ induce a directed edge $(w, v)$, similarly, we can define the directed graph $\vec{G}$ induced by $\varsigma$, in which we allow some edges have no direction. The source vertex of $\vec{G}$ is the vertex that its out-degree is greater than 0, and its in-degree is 0; The sink vertex of $\vec{G}$ is the vertex that its in-degree is greater than 0, and its out-degree is 0 (the vertex with no directed edges has the out-degree 0 and in-degree 0).

Assume $w$ and $v$ are adjacent, the sequence of pebbling moves $\eta$ remove $2\beta$ pebbles from $w$, and place $\beta$ pebbles on $v$, then we can say that $\eta$ moves $\beta$ pebbles from $w$ to $v$. Now we generalize it to two nonadjacent vertices. We paint the pebbles on $w$ red, paint the pebbles on other vertices black. If a pebbling move remove two pebbles from $v_1$, and place one pebble on $v_2$. We consider three cases before this pebbling move.

(1) There are at least two red pebbles on $v_1$, then we remove two red pebbles from $v_1$, and place one red pebble on $v_2$;
(2) There is only one red pebble on $v_1$, then we remove one red and one black pebble from $v_1$, and place one red pebble on $v_2$;
(3) There is no red pebble on $v_1$, then we remove two black pebbles from $v_1$, and place one black pebble on $v_2$.

If $\gamma$ pebbles on $v$ are red after a sequence of pebbling moves $\phi$, then we say that $\phi$ moves $\gamma$ pebbles from $w$ to $v$.

**Theorem 2.10** Let $n$ be an even integer, $m \geq 8$, $f(J_{n,m}) = f_{2n^2+1-1}(C_{n+2}) + \alpha + 1$, where $\alpha$ is shown in Lemma 2.7.

**Proof.** First we show that $f(J_{n,m}, v_2) = f_{2n^2+1-1}(C_{n+2}) + \alpha + 1$, assume the target vertex is $v_2$.

**Lower bound:** Let the distribution $D$ be obtained from the distribution $D^*$ given in the proof of Lemma 2.9 by adding $f_{2n^2+1-1}(C_{n+2})$ pebbles on $v_{4n+n/2}$.

We only need to show that $D$ is not $v_2$-solvable. We show it by contradiction. If $D$ was $v_2$-solvable, then there exist a sequence of pebbling moves so that one pebble can be moved from $D$ to $v_2$. Assume $\tau$ is such a sequence of pebbling moves with the number of the pebbles cost minimum.

According to the No-Cycle-Lemma, the directed graph $\vec{J}$ induced by $\tau$ has no directed cycle. For $\tau$ is the sequence of pebbling moves that cost the minimum number of pebbles, $\vec{J}$ has only one sink vertex $v_2$. If the out-degree of some vertex $v$ is not 0, then $\tau$ moves one pebble from $v$ to $v_2$.

Moreover, we claim the following:
• \((u, v_{ni}) \notin J\) for \(2 \leq i \leq m - 1\).

Assume that \((u, v_{nj}) \in J\) for some \(2 \leq j \leq m - 1\), we may assume that one pebble has been moved from \(u\) to \(v_{nj}\). We paint this pebble red, and we paint other pebbles black. Note that \(J\) has only one sink vertex \(v_S\), so the red pebble must be moved along \(P_{j-1}\) to \(v_{n(j-1)}\) (or along \(P_{j+1}\) to \(v_{n(j+1)}\)). Then we choose the pebbling move from \(u\) to \(v_{n(j-1)}\) (or \(v_{n(j+1)}\)) instead of this sequence of pebbling moves to get a new sequence of pebbling moves with less pebbles cost than \(\tau\), which is a contradiction to the assumption that \(\tau\) costs the least pebbles.

• \((v_{4n}, v_{4n-1}) \notin J\) and \((v_{5n}, v_{5n+1}) \notin J\).

Note that for the path sequence \(P_0, P_1, P_2, P_3, P_4, P_5\), the sequence of pebbles is \(0, S, L, S, L', S\), where \(L'\) is obtained from \(L\) by adding \(f_{22+1} J_{n+2}(C_{n+2})\) pebbles on \(v_{4n+n/2}\). We paint the pebbles on \(P_4\) red, and paint the other pebbles black. Assume \((v_{4n}, v_{4n-1}) \in J\), then one red pebble has been moved from \(v_{4n}\) to \(v_{4n-1}\), according to the No-Cycle-Lemma, and the assumption that \(\tau\) costs the least pebbles, we can get that this red pebble must be moved along \(P_3\) to \(v_{3n}\), now we can get \((v_{3n}, u) \notin J\), for if \((v_{3n}, u) \in J\), then the red pebble is moved to \(u\), so we choose the pebbling move from \(v_{4n}\) to \(u\) instead of such sequence of pebbling moves, and we get a new sequence of pebbling moves with less pebbles cost than \(\tau\), which is a contradiction. From Claim 1, we can get \((u, v_{3n}) \notin J\). We know that \(v_{3n}\) is not a sink vertex in \(J\), so we can get \((v_{5n}, v_{3n-1}) \in J\). By a similar argument, we can get that the red pebble must be moved along the path sequence \(P_3, P_2, P_1\) to \(v_n\). It costs us at least five red pebbles to move one red pebble to \(v_n\), so we can choose a new sequence of pebbling moves instead of this sequence: remove four red pebbles from \(v_{4n}\), and add two red pebbles on \(u\), and then remove these two red pebbles and add one red pebble to \(v_n\), which is a contradiction. Similarly, we can show that \((v_{5n}, v_{5n+1}) \notin J\).

According to the claim above, we can get that the pebbles must be moved from \(P_1\) to \(u\), then from \(u\) to \(P_0\), directly. At most \(2^{\frac{n}{2} + 1} - 1\) red pebbles can be moved to \(u\), so we can not move one red pebble to \(v_S\).

**Upper bound:** Assume \(f_{22+1} J_{n+2}(C_{n+2}) = \alpha + 1\) pebbles are placed on \(J_{n,m}\). Let \(C^i = P_i \cup u\) be the cycle induced by \(P_i\) and \(u\) in \(J_{n,m}\).

We first consider the case: \(p(P_0) = 0\).

We only need to show that with \(f_{t-1}(C_{n+2}) + \alpha + 1\) pebbles on \(J_{n,m}\), such that \(p(P_0) = 0\), one can move \(t\) pebbles to \(C^0\) without the pebbles on \(S\). So we may assume that \(p(u) = 0\). We use induction on \(t\). It holds for \(t = 1\) by Lemma 2.9. Assume it holds for \(t - 1\) \((t \geq 2)\). For the case \(t\), by Lemma 2.9, one of the following holds: 1) Two \(L\) are adjacent; 2) one \(L\) and one \(M\) are adjacent, so that two pebbles can be moved to the joint vertex, and so one pebble can be moved to \(u\); 3) The number of the pebbles on some path \(P_j\) is larger than \(f(C_{n+2})\). Case 1. Two \(L\) are adjacent. Assume that \(P_k\) and \(P_{k+1}\) both have large number of pebbles. We can move one pebble to \(v_{nk}\) from each path, and so one pebble can be moved to \(u\). Then we replace the distribution of pebbles left on \(P_k\) and \(P_{k+1}\) by the distribution with \(2^{n/2} - 1\) pebbles on \(P_k\) and \(P_{k+1}\) respectively, such that none
of its endpoints are reachable (just put $2^{n/2} - 1$ pebbles on $v_{nk+n/2}$ and $v_{n(k+1)+n/2}$, respectively). Then the total number of the new distribution of $J_{n,m} \setminus u$ is

$$f_{t-1}(C_{n+2}) + \alpha + 1 - |P_k| - |P_{k+1}| + 2(2^{n/2} - 1) \geq f_{t-1}(C_{n+2}) + \alpha + 1 - 2(2^{n/2+1} - 1) + 2(2^{n/2} - 1) = f_{t-2}(C_{n+2}) + \alpha + 1.$$ 

By induction, $t - 1$ pebbles can be moved to $u$ from the new distribution, and we do not use the pebbles on $P_k$ or $P_{k+1}$, that means we can move $t - 1$ pebbles from the original distribution, and we are done.

Case 2. The proof is similar to Case 1.

Case 3. The number of the pebbles on some path $P_j$ is larger than $f(C_{n+2})$, we can move one pebble to $u$ from $P_j$ with at most $2^{n/2+1} - p(P_0)$ pebbles cost (for the even cycle is greedy). Then we left at least $f_{t-2}(C_{n+2}) + \alpha + 1$ pebbles on $J_{n,m} \setminus u$, and we can move $t - 1$ pebbles to $u$ with the left pebbles by induction, and we are done.

Now if $p(P_0) > 0$, then we only need to move $2^{n/2+1} - p(P_0)$ pebbles from $J_{n,m} \setminus \{P_0 \cup u\}$ to $u$ (then we have $2^{n/2+1}$ pebbles on $P_0 \cup u$, so one pebble can be moved to $v_{n/2}$). The number of pebbles on $J_{n,m} \setminus \{P_0 \cup u\}$ is $f_{2^{n/2+1} - (C_{n+2}) + \alpha + 1 - p(P_0)}$, which is larger than $f_{2^{n/2+1} - (C_{n+2}) + \alpha + 1}$. So we are done from the argument above.

If the target vertex is not $v_{n/2}$, we only need to check the upper bound, we may assume that the target vertex belongs to $P_0$, then by a similar argument, one can show that $2^{n/2+1}$ pebbles can be moved to $P_0 \cup u$, and so one pebble can be moved to the target vertex with $f_{2^{n/2+1} - (C_{n+2}) + \alpha + 1}$ pebbles on $J_{n,m}$, and we are done.

3 Remark

Let $n = 2$, then we can get $f(J_{2,m}) = 2m + 10$ for $m \geq 8$ from Theorem [2.10], which is just Lemma [2.6]. For $m < 8$, there may exist a sequence of pebbling moves with least pebbles cost, which does not through the vertex $u$, so we cannot get the tight lower bound with the method in Theorem [2.10] but we can still get the upper bound by a similar argument.

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