A short note on the multiplicative energy of the spectrum of a set

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Abstract

We obtain an upper bound for the multiplicative energy of the spectrum of an arbitrary set from $\mathbb{F}_p$, which is the best possible up to the results on exponential sums over subgroups.

1 Introduction

Let $p$ be a prime number and let $A$ be a subset of the prime field $\mathbb{F}_p$. Denote by $\hat{A}(r)$, $r \in \mathbb{F}_p$, the Fourier transform of the characteristic function of the set $A$, namely,

$$
\hat{A}(r) = \sum_{a \in A} e^{-2\pi i ar/p}.
$$

Given a real number $\varepsilon \in (0, 1]$, define

$$
\text{Spec}_\varepsilon(A) = \{ r \in \mathbb{F}_p : |\hat{A}(r)| \geq \varepsilon|A| \}. \tag{1}
$$

The set $\text{Spec}_\varepsilon(A)$ is called the spectrum or the set of large exponential sums of our set $A$. Such sets are studied in [18, Section 4.6], further, in [2—5], [11—14] and in many other papers. The spectrum appears naturally in any additive problem and, hence, it is important to know the structure of these sets. It is well–known that $\text{Spec}_\varepsilon(A)$ has strong additive properties, see, e.g., [2], [3], [13]. This fact was used in [15] to obtain a new property of the spectrum, namely, that $\text{Spec}_\varepsilon(A)$ has poor multiplicative structure. It coincides with the philosophy of the sum–product phenomenon, see, e.g., [18] that says that both additive and multiplicative structures do not exist simultaneously. Previously, we used the modern sum–product tools, see [9], [10] to demonstrate this poor multiplicative structure. Here we apply the main sum–product result of [9] directly and obtain

**Theorem 1** Let $A \subseteq \mathbb{F}_p$ be a set, $|A| = \delta p$ and $R \subseteq \text{Spec}_\varepsilon(A) \setminus \{0\}$ be any set. Suppose that $p \leq \varepsilon^2|A|^3$. Then

$$
|\{(x, y, z, w) \in R^4 : xy = zw\}| \ll \varepsilon^{-4}\delta^{-1}|R|^{3/2}. \tag{2}
$$

Estimate (2) is stronger than the results of [15, Section 4] and moreover one can show (see Remarks 6, 9) that the bound in Theorem 1 is sharp up to our current knowledge of some number–theoretical questions. Also, in this paper we study other multiplicative characteristics of the spectrum, see Theorem 5 and Theorem 7, formula (15). As a byproduct we obtain by the
same method a purely sum–product result, namely, a new lower bound on $AA + AA$ for sets with small subset.

All logarithms are to base 2. The signs $\ll$ and $\gg$ are the usual Vinogradov symbols. If we have a set $A$, then we will write $a \ll b$ or $b \gg a$ if $a = O(b \cdot \log^c |A|)$, $c > 0$.

2 Notation and preliminary results

In this paper $p$ is an odd prime number, $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ and $\mathbb{F}_p^* = \mathbb{F}_p \setminus \{0\}$. We denote the Fourier transform of a function $f : \mathbb{F}_p \to \mathbb{C}$ by $\hat{f}$,

$$\hat{f}(\xi) = \sum_{x \in \mathbb{F}_p} f(x)e(-\xi \cdot x),$$

where $e(x) = e^{2\pi i x/p}$. We rely on the following basic identities. The first one is called the Plancherel formula and its particular case $f = g$ is called the Parseval identity

$$\sum_{x \in \mathbb{F}_p} f(x)g(x) = \frac{1}{p} \sum_{\xi \in \mathbb{F}_p} \hat{f}(\xi)\overline{\hat{g}(\xi)}. \quad (4)$$

Another particular case of (4) is

$$\sum_{y \in \mathbb{F}_p} \left| \sum_{x \in \mathbb{F}_p} f(x)g(y - x) \right|^2 = \frac{1}{p} \sum_{\xi \in \mathbb{F}_p} |\hat{f}(\xi)|^2 |\hat{g}(\xi)|^2. \quad (5)$$

In this paper we use the same letter to denote a set $A \subseteq \mathbb{F}_p$ and its characteristic function $A : \mathbb{F}_p \to \{0, 1\}$. Also, we write $f_A(x)$ for the balanced function of a set $A \subseteq \mathbb{F}_p$, namely, $f_A(x) = A(x) - |A|/p$.

Let $A \subseteq \mathbb{F}_p$ be a set, and $\varepsilon \in (0, 1]$ be a real number. We have defined the set $\text{Spec}_\varepsilon(A)$ in (1) already. Clearly, $0 \in \text{Spec}_\varepsilon(A)$, and $\text{Spec}_0(A) = -\text{Spec}_\varepsilon(A)$. For further properties of $\text{Spec}_\varepsilon(A)$ see, e.g., [2], [3], [13], [15]. Usually, we denote by $\delta$ the density of our set $A$, that is, $\delta = |A|/p$. From Parseval identity (4), we have a simple upper bound for size of the spectrum, namely,

$$|\text{Spec}_\varepsilon(A)| \leq \frac{p}{|A|\varepsilon^2} = \frac{1}{\delta \varepsilon^2}. \quad (6)$$

Put $E^+(A, B)$ for the common additive energy of two sets $A, B \subseteq \mathbb{F}_p$ (see, e.g., [13]), that is,

$$E^+(A, B) = \left| \{(a_1, a_2, b_1, b_2) \in A \times A \times B \times B : a_1 + b_1 = a_2 + b_2 \} \right|.$$ If $A = B$, then we simply write $E^+(A)$ instead of $E^+(A, A)$ and the quantity $E^+(A)$ is called the additive energy in this case. One can consider $E^+(f)$ for any complex function $f$ as well. Sometimes we use representation function notations like $r_{AB}(x)$ or $r_{A+B}(x)$, which counts the number of ways $x \in \mathbb{F}_p$ can be expressed as a product $ab$ or a sum $a + b$ with $a \in A, b \in B$, respectively. Put $\sigma^+(A) = \sum_{x \in A} r_{A-A}(x)$. Further clearly

$$E^+(A, B) = \sum_x r^2_{A+B}(x) = \sum_x r^2_{A-B}(x) = \sum_x r_{A-A}(x)r_{B-B}(x).$$
and by (5),
\[ E^+(A, B) = \frac{1}{p} \sum_{\xi} |\hat{A}(\xi)|^2 |\hat{B}(\xi)|^2. \] (7)

Similarly, one can define \( E^x(A, B), E^x(A), E^x(f) \) and so on.

Now we recall some results from the incidence geometry, see, e.g., [18, Section 8]. First of all, we need a general design bound for the number of incidences, see [16, 19, 20]. Let \( \mathcal{P} \subseteq \mathbb{F}_q^3 \) be a set of points and \( \Pi \) be a collection of planes in \( \mathbb{F}_q^3 \). Having \( p \in \mathcal{P} \) and \( \pi \in \Pi \), we write
\[ I(p, \pi) = \begin{cases} 1 & \text{if } q \in \pi \\ 0 & \text{otherwise.} \end{cases} \]

Put \( I(\mathcal{P}, \Pi) = \sum_{p \in \mathcal{P}, \pi \in \Pi} I(p, \pi) \). We have (see [16])

**Lemma 2** For any functions \( \alpha : \mathcal{P} \to \mathbb{C}, \beta : \Pi \to \mathbb{C} \) one has
\[ |\sum_{p, \pi} I(p, \pi) \alpha(p) \beta(\pi)| \leq q \|\alpha\|_2 \|\beta\|_2, \] (8)
provided either \( \sum_{p \in \mathcal{P}} \alpha(p) = 0 \) or \( \sum_{\pi \in \Pi} \beta(\pi) = 0 \).

Of course, similar arguments work not just for points/plane incidences but, e.g., points/lines incidences and so on. A much more deep result on incidences is contained in [9] (or see [7, Theorem 8]). We formulate a combination of these results and Lemma 2, see [16].

**Theorem 3** Let \( p \) be an odd prime, \( \mathcal{P} \subseteq \mathbb{F}_q^3 \) be a set of points and \( \Pi \) be a collection of planes in \( \mathbb{F}_q^3 \). Suppose that \( |\mathcal{P}| \leq |\Pi| \) and that \( k \) is the maximum number of collinear points in \( \mathcal{P} \). Then the number of point–planes incidences satisfies
\[ I(\mathcal{P}, \Pi) - \frac{|\mathcal{P}||\Pi|}{p} \ll |\mathcal{P}|^{1/2}|\Pi| + k|\mathcal{P}|. \] (9)

Finally, we recall a simple asymptotic formula for the number of points/lines incidences in the case when the set of points forms a Cartesian product, see [17] and also [16].

**Theorem 4** Let \( A, B \subseteq \mathbb{F}_p \) be sets, \( |A| \leq |B| \), \( \mathcal{P} = A \times B \), and \( \mathcal{L} \) be a collection of lines in \( \mathbb{F}_p^2 \). Then
\[ I(\mathcal{P}, \mathcal{L}) - \frac{|A||B||\mathcal{L}|}{p} \ll |A|^{3/4}|B|^{1/2}|\mathcal{L}|^{3/4} + |\mathcal{L}| + |A||B|. \] (10)
3 The proof of the main results

Let \( A \subseteq \mathbb{F}_p \) be a set. We write

\[
E_k^\varepsilon(A) = \sum_x r_{A/A}^\varepsilon(x)
\]

for any \( k \geq 1 \). Our aim is to obtain an upper bound for \( E_2^\varepsilon \)-energy of the spectrum but before that we prove an optimal result for \( E_4^\varepsilon \) which is interesting in its own right. We use arguments similar to [8].

**Theorem 5** Let \( A \subseteq \mathbb{F}_p \) be a set, \(|A| = \delta p \) and \( R = \text{Spec}_\varepsilon(A) \setminus \{0\} \). Then

\[
E_4^\varepsilon(R) \leq \varepsilon^{-16} \delta^4 \left( \frac{E^\varepsilon(f_A)}{|A|^3} \right)^2.
\] (11)

**Proof.** Applying formula (7) and the definition of the spectrum (11), we notice that

\[
\frac{(\varepsilon|A|)^4}{p} \cdot r_{R/\lambda}(\lambda) \leq p^{-1} \sum_{x \in R, \lambda \in R} |\hat{A}(x)|^2 |\hat{A}(\lambda x)|^2 \leq p^{-1} \sum_x |\hat{f}_A(x)|^2 |\hat{f}_A(\lambda x)|^2 = E^\varepsilon(f_A, \lambda f_A).
\]

Hence

\[
E_4^\varepsilon(R) \leq (\varepsilon|A|)^{-16} p^4 \sum_{\lambda} E^\varepsilon(f_A, \lambda f_A)^4 = (\varepsilon|A|)^{-16} p^4 \sum_{\lambda} r_{(f_A-f_A)/(f_A-f_A)}^4(\lambda) = (\varepsilon|A|)^{-16} p^4 \cdot \sigma.
\]

By the Dirichlet principle there is \( \Delta > 0 \) and a set \( P \) such that \( \Delta < |r_{f_A-f_A}(\lambda)| \leq 2\Delta \) on \( P \) and

\[
\sigma \lesssim \Delta^4 \sum_{\lambda} r_{(f_A-f_A)/P(\lambda)}^4(\lambda) = \Delta^4 \sum_{\lambda} |\{\lambda p = a_1 - a_2 : p \in P\}|^4,
\]

where \( a_1, a_2 \) have weights \( f_A(a_1), f_A(a_2) \), correspondingly. Let \( \tau > 0 \) and \( S_\tau \) be the set of all \( \lambda \) such that \( |r_{(f_A-f_A)/P}(\lambda)| \geq \tau \). Since \( r_{(f_A-f_A)/P}(\lambda) = r_{(A-A)/P}(\lambda) + \delta^2 |P| \), it follows that

\[
\tau |S_\tau| \leq \sum_{\lambda \in S_\tau} |r_{(f_A-f_A)/P}(\lambda)| \leq |A|^2 |P| + \delta^2 |P|^2 = 2|A|^2 |P|.
\]

In particular, \( |S_\tau| \leq 2|A|^2 |P|/\tau \). The number of the solutions to the equation \( sp = a_1 - a_2 \) can be interpreted as the number of incidences between the set of lines \( \mathcal{L} = S_\tau \times A \), counting with the weight \( f_A(a_1) \) and the sets of points \( P = A \times P \), again counting with the weight \( f_A(a_2) \). Applying Theorem 4 we obtain

\[
\tau |S_\tau| = I(P, \mathcal{L}) \ll |A|^{3/2} |P|^{1/2} |S_\tau|^{3/4} + |S_\tau||A| + |A||P|.
\] (12)

If the first term dominates, then we have

\[
|S_\tau| \ll |A|^6 |P|^2/\tau^4.
\] (13)
In view of the inequality $|S_\tau| \leq 2|A|^2|P|/\tau$ one can suppose that $\tau^3 \gg |A|^4|P| \gg |A|^3$ because otherwise it is nothing to prove. It gives us that $\tau \gg |A|$ and hence the second term in (13) is negligible. We will consider the case when the third term in (12) dominates and we know that $\tau^3 \gg |A|^5|P|$. In other words, if we consider the ordering

$$|r_{(f_A-fA)}/P(s_1)| \geq |r_{(fA-fA)}/P(s_2)| \geq \ldots \geq |r_{(fA-fA)}/P(s_j)| \geq \ldots ,$$

then there is an effective bound $|r_{(fA-fA)/P(s_j)}| \leq |A||P|j^{-1}$ for $j \geq J := (|P|/|A|)^{2/3}$. Again, by summation we obtain

$$\sigma \ll \sum_{j \geq J} (|A||P|/j)^4 \ll |A|^4|P|^4J^{-3} \ll |P|^2|A|^6$$

and it gives the same bound for $E^+_4(R)$. This completes the proof. \hfill \Box

**Remark 6** Let $A$ be a multiplicative subgroup of order $p^{2/3}$. Then the best known bound for the Fourier coefficients of $A$ is $|\hat{A}(r)| < \sqrt{p}$, $\forall r \neq 0$, see, e.g., [8]. On the other hand, taking $R$ equals a coset of $A$ belonging to $\text{Spec}_{\varepsilon}(A) \setminus \{0\}$, we see that $E^+_4(R) \gg |R|^5 = |A|^5$. Applying formulae (4), (7), we get

$$E^+(f_A) < \left( \max_{r \neq 0} |\hat{A}(r)| \right)^2 |A|$$

and hence estimate (11) of Theorem 5 is tight (up to our current knowledge of the Fourier coefficients of multiplicative subgroups).

Unfortunately, the method of the proof of Theorem 5 works for $E^+_4(R)$ but not for $E^+_k(R)$ with $k < 4$. In this case we obtain

**Theorem 7** Let $A \subseteq \mathbb{F}_p$ be a set, $|A| = \delta p$ and $R \subseteq \text{Spec}_{\varepsilon}(A) \setminus \{0\}$ be any set. Suppose that $p \leq \varepsilon^2|A|^3$. Then

$$E^x(R) \ll \varepsilon^{-4} \delta^{-1} |R|^{3/2}. \quad (14)$$

Similarly,

$$\sigma^x(R) \lesssim \varepsilon^{-4} \delta^{-1} |R|^{3/4} \left( \frac{E^+(f_A)}{|A|^3} \right)^{1/2} + \varepsilon^{-4} \delta^{-1} \left( 1 + \frac{|R|}{|A|} \right). \quad (15)$$
Proof. Using the Fourier transform similar to the proof of Theorem 3 we have
\[
\left(\frac{\varepsilon |A|^4}{p}\right) E^\times (R) \leq p^{-1} \sum_{\lambda, \mu \in R} \sum_{x} |\hat{f}_{A}(\lambda x)|^2 |\hat{f}_{A}(\mu x)|^2 = \sum_{x} r_{(f_{A} - f_{A})R}(x).
\]
Clearly, the last quantity can be interpreted as points/planes incidences (with weights), see [1]. Here the number of the points and planes is at most \(O(|A|^2|R|)\). Finally, using our assumption, we get from (6)
\[
|R| \leq \frac{p}{\varepsilon^2 |A|} \leq |A|^2.
\]
Applying Theorem 3 we obtain
\[
\sum_{x} r_{(f_{A} - f_{A})R}(x) \ll |A|^3|R|^{3/2}.
\]
It follows that
\[
E^\times (R) \ll \varepsilon^{-4} \delta^{-1}|R|^{3/2}
\]
as required.

Similarly,
\[
\sigma^\times (R) \leq (\varepsilon |A|)^{-4} \sum_{\lambda \in R} \sum_{x} |\hat{f}_{A}(x)|^2 |\hat{f}_{A}(\lambda x)|^2 = \varepsilon^{-4} \delta^{-1}|A|^{-3} \sum_{\lambda \in R} r_{(f_{A} - f_{A})/(f_{A} - f_{A})}(\lambda).
\]
After that we can use the arguments and the notation from the proof of Theorem 5 (with \(S_r = R\) and derive that
\[
\sum_{\lambda \in R} r_{(f_{A} - f_{A})/(f_{A} - f_{A})}(\lambda) \lesssim \Delta |P|^{1/2} |R|^{3/4} |A|^{3/2} + \Delta |A|(|P| + |R|) \ll
\]
\[
\ll (E^+(f_{A}))^{1/2} |R|^{3/4} |A|^{3/2} + |A|^3 + |A|^2 |R|.
\]
Here we have used a trivial bound \(\Delta \leq 2|A|\). It gives us
\[
\sigma^\times (R) \lesssim \varepsilon^{-4} \delta^{-1}|R|^{3/4} (E^+(f_{A})/|A|^3)^{1/2} + \varepsilon^{-4} \delta^{-1} + \varepsilon^{-4} \delta^{-1}|R|/|A|
\]
and this coincides with (15). \(\square\)

Example 8 Let \(\varepsilon \gg 1\), \(R = \text{Spec}_\varepsilon (A) \setminus \{0\}\), and let size of \(R\) is comparable with the upper bound which is given by (6), namely, \(|R| \gg \delta^{-1} \varepsilon^{-2} \gg \delta^{-1}\). Then \(E^\times (R) \lesssim |R|^{5/2}\). It means that we have a non–trivial estimate for the multiplicative energy of the spectrum in this case. Similarly, we always have \(E^+(f_{A}) < |A|^3\), so \(\sigma^\times (R) \lesssim |R|^{7/4} + |R|^2/|A|\).

Remark 9 The same construction as in Remark 6 shows the tightness of bounds (14), (15), again up to our current knowledge of the Fourier coefficients of multiplicative subgroups.

In the same vein we obtain a result on the growth of \(AA + AA\), which improves [16, Theorem 32] for small \(E^\times (A)\).
Theorem 10 Let $A \subseteq \mathbb{F}_p$ be sets. Then
\[
\sum_x r_{AA}^2(x) - \frac{|A|^8}{p} \lesssim |A|^4 (E_4^\alpha(A))^{1/2} + E_4^\alpha(A)|A|^2.
\] (16)

Proof. Without losing of the generality, one can assume that $0 \notin A$. We need to estimate the number of the solutions to the equation
\[
a_1/a \cdot a_1'/a' + a_2/a \cdot a_2'/a' - a_3/a \cdot a_3'/a' = 1,
\]
where $a, a', a_j, a'_j \in A$, $j = 1, 2, 3$. Put
\[
C_4^\alpha(A)(\alpha, \beta, \gamma) := |A \cap \alpha A \cap \beta A \cap \gamma A|.
\]
One can check that
\[
\sum_{\alpha, \beta, \gamma} C_4^\alpha(A)(\alpha, \beta, \gamma) = |A|^4,
\]
and
\[
\sum_{\alpha, \beta, \gamma} C_4^\alpha(A)(\alpha, \beta, \gamma)^2 = E_4^\alpha(A).
\] (17)

In these terms, we want to bound the sum
\[
\sigma := \sum_{\alpha, \beta, \gamma} \sum_{\alpha', \beta', \gamma'} C_4^\alpha(A)(\alpha, \beta, \gamma) C_4^\alpha(A)(\alpha', \beta', \gamma') \delta(\alpha \alpha' + \beta \beta' - \gamma \gamma' = 1),
\]
where $\delta(x = 1)$ equals one iff $x = 1$. Using the Dirichlet principle as in the proof of Theorems 5, 7 we find two numbers $\Delta_1, \Delta_2 > 0$ and two corresponding sets of points and planes $\mathcal{P}, \Pi$ such that
\[
\sigma \lesssim \Delta_1 \Delta_2 \sum_{\alpha, \beta, \gamma} \sum_{\alpha', \beta', \gamma'} \mathcal{P}(\alpha, \beta, \gamma) \Pi(\alpha', \beta', \gamma') \delta(\alpha \alpha' + \beta \beta' - \gamma \gamma' = 1).
\]

Without losing of the generality, suppose that $|\mathcal{P}| \leq |\Pi|$. Also, notice that $|\mathcal{P}|, |\Pi| \leq |A|^4$. Applying Theorem 3 (previously inserting the balanced function $f_A(x) = A(x) - |A|/p$ as in the proofs of Theorems 5, 7) with the maximal number of collinear points $k \leq |A|^2$ and using formula (17), combining with Lemma 8 we get
\[
\sigma \lesssim \Delta_1 \Delta_2 |\mathcal{P}||\Pi|^{1/2} + \Delta_1 \Delta_2 k |\mathcal{P}| + |\mathcal{P}|^{1/2} E_4^\alpha(f_A) \leq
\]
\[
\leq (\Delta_2^2 |\Pi|)^{1/2} \Delta_1 |\mathcal{P}| + k(\Delta_1^2 |\mathcal{P}|)^{1/2} (\Delta_2^2 |\Pi|)^{1/2} + |A|^2 E_4^\alpha(f_A) \leq
\]
\[
\leq (E_4^\alpha(A))^{1/2} |A|^4 + E_4^\alpha(A)|A|^2.
\]

This completes the proof. \hfill \Box

Corollary 11 Let $A \subseteq \mathbb{F}_p$, $|A + A| = K|A|$ and $|A + A|^3|A| \leq p^3$. Then
\[
|AA + AA| \gtrsim \min\{p, \Omega_K(|A|^2)\}.
\]
Proof. Using [7, Lemma 18] (where, actually, a better dependence on $K$ is suggested) or just applying the arguments of the proof of Theorem 5, we get 

$$E^\chi_4(r_{B+C}) - \frac{|B|^8|C|^8}{p^3} \lesssim E^+(B, C)^2|B|^3|C|^3.$$ 

Putting $B = A + A$, $C = -A$ and noting that $|A|A(x) \leq r_{B+C}(x)$, we obtain 

$$E^\chi_4(A) - \frac{|A + A|^8}{p^3} \lesssim |A + A|^5|A|^{-1}.$$ 

(18) 

Obviously, by the Cauchy–Schwartz inequality, we have 

$$|A|^8 \leq |AA + AA| \cdot \sum_x r_{AA+AA}^2(x).$$ 

(19) 

By Theorem 10 we get 

$$\sum_x r_{AA+AA}^2(x) - \frac{|A|^8}{p} \lesssim |A|^4(E^\chi_4(A))^{1/2} + E^\chi_4(A)|A|^2.$$ 

If the term $|A|^8$ dominates in the last formula, we have from (19) that $|AA + AA| \gg p$. Otherwise in view of (18) and our condition $|A + A|^3|A| \leq p^3$, we see that 

$$|A|^8 \lesssim |AA + AA| \cdot \left( |A|^4(E^\chi_4(A))^{1/2} + E^\chi_4(A)|A|^2 \right) \ll |AA + AA| \cdot K^5|A|^6.$$ 

This completes the proof. 

Considering $A = \{1, 2, \ldots, n\}$, where $n$ is sufficiently small comparable to $p$, we see that Corollary 11 is the best possible up to logarithms. 

References

[1] E. Aksoy Yazici, B. Murphy, M. Rudnev, I. D. Shkredov, Growth Estimates in Positive Characteristic via Collisions, International Mathematics Research Notices, Volume 2017, Issue 23 (2017), 7148–7189, https://doi.org/10.1093/imrn/rnw206

[2] T. F. Bloom, A quantitative improvement for Roth’s theorem on arithmetic progressions, doi: 10.1112/jlms/jdw010.

[3] M.–C. Chang, A polynomial bound in Freiman’s theorem, Duke Math. J. 113 (2002) no. 3, 399–419.

[4] B. Green, Some constructions in the inverse spectral theory of cyclic groups, Comb. Prob. Comp. 12 (2003) no. 2, 127–138.

[5] B. Green, Spectral structure of sets of integers, Fourier analysis and convexity (survey article, Milan 2001), Appl. Numer. Harmon. Anal., Birkhauser Boston, Boston, MA (2004), 83–96.
[6] S. V. Konyagin, I. Shparlinski, *Character sums with exponential functions*, Cambridge University Press, Cambridge, 1999.

[7] B. Murphy, G. Petridis, Ol. Roche–Newton, M. Rudnev, I. D. Shkredov, *New results on sum–product type growth over fields*, arXiv:1702.01003v2 [math.CO] 8 Feb 2017, accepted.

[8] G. Petridis, *Products of Differences in Prime Order Finite Fields*, arXiv:1602.02142 [math.CO] 5 Feb 2016.

[9] M. Rudnev, *On the number of incidences between planes and points in three dimensions*, Combinatorica, 2017. First published online, doi:10.1007/s00493-016-3329-6.

[10] M. Rudnev, I. D. Shkredov, and S. Stevens, *On the energy variant of the sum–product conjecture*, arXiv:1607.05053.

[11] T. Sanders, *On certain other sets of integers*, Journal d’Analyse Mathmatique 116.1 (2012): 53–82.

[12] T. Sanders, *On Roth’s theorem on progressions*, Annals of Mathematics (2011): 619–636.

[13] I. D. Shkredov, *On Sets of Large Exponential Sums*, Izvestiya of Russian Academy of Sciences, 72:1 (2008), 161–182.

[14] I. D. Shkredov, *On sumsets of dissociated sets*, Online Journal of Analytic Combinatorics, 4 (2009), 1–26.

[15] I. D. Shkredov, *An application of the sum–product phenomenon to sets having no solutions to several linear equations*, Sbornik Math., 209:4 (2018), 117–142.

[16] I. D. Shkredov, *On asymptotic formulae in some sum–product questions*, arXiv:1802.09066v2 [math.NT] 2 Mar 2018.

[17] S. Stevens, F. de Zeeuw, *An improved point-line incidence bound*, arXiv: 1609.06284v2 [math.CO] 7 Oct 2016.

[18] T. Tao, V. Vu, *Additive combinatorics*, Cambridge University Press 2006.

[19] P. Thang, M. Tait, C. Timmons, *A Szemerédi–Trotter type theorem, sum–product estimates in finite quasifields, and related results*, Journal of Combinatorial Theory, Series A 147 (2017): 55–74.

[20] L. A. Vinh, *A Szemerédi–Trotter type theorem and sum–product estimate over finite fields*, Eur. J. Comb. 32:8 (2011), 1177–1181.

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