The Cayley Trick and Triangulations of Products of Simplices

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Abstract. We use the Cayley Trick to study polyhedral subdivisions of the product of two simplices. For arbitrary (fixed) \( l \geq 2 \), we show that the numbers of regular and non-regular triangulations of \( \Delta^l \times \Delta^k \) grow, respectively, as \( k^{\Theta(k)} \) and \( 2^{\Omega(k^2)} \).

For the special case of \( \Delta^2 \times \Delta^k \), we relate triangulations to certain class of lozenge tilings. This allows us to compute the exact number of triangulations up to \( k = 15 \), show that the number grows as \( e^{\beta k^2/2 + o(k^2)} \) where \( \beta \simeq 0.32309594 \) and prove that the set of all triangulations is connected under geometric bistellar flips. The latter has as a corollary that the toric Hilbert scheme of the determinantal ideal of \( 2 \times 2 \) minors of a \( 3 \times k \) matrix is connected, for every \( k \).

We include “Cayley Trick pictures” of all the triangulations of \( \Delta^2 \times \Delta^2 \) and \( \Delta^2 \times \Delta^3 \), as well as one non-regular triangulation of \( \Delta^2 \times \Delta^5 \) and another of \( \Delta^3 \times \Delta^3 \).

Introduction

The polyhedral Cayley Trick gives a canonical bijection between mixed subdivisions of the Minkowski sum \( \sum_{i=1}^{k} P_i \) of several polytopes \( P_1, \ldots, P_k \in \mathbb{R}^d \) and all polyhedral subdivisions of a certain polytope \( \mathcal{C}(P_1, \ldots, P_k) \subset \mathbb{R}^d \times \mathbb{R}^{k-1} \) called the Cayley embedding of \( P_1, \ldots, P_k \). The correspondence was first developed by Sturmfels [19] for the case of coherent subdivisions, and then generalized to all subdivisions by Huber et al. [10].

Originally, the trick was devised as a way of understanding and computing fine (i.e., minimal with respect to refinement) mixed subdivisions, taking advantage of the much deeper knowledge and specific software that exists for triangulations. But the trick can be also used in reverse, to understand triangulations of the \( d + k - 1 \)-dimensional polytope \( \mathcal{C}(P_1, \ldots, P_k) \) in terms of a \( d \)-dimensional object. This is what we do here.

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Specially interesting is the case when all the $P_i$’s are copies of a simplex $\Delta^{l-1}$. Then, the Cayley Trick relates polyhedral subdivisions of $\Delta^{k-1} \times \Delta^{l-1}$ to mixed subdivisions of the dilation $k\Delta^{l-1}$. If, moreover, we fix $l = 3$, then the mixed subdivisions we have to study are essentially the same as lozenge tilings of a triangle of size $k$. Using this interpretation we prove:

**Theorem 1.**

1. The graph of flips between triangulations of $\Delta^2 \times \Delta^k$ is connected (Theorem 4.4) and it has diameter $\Theta(k^2)$ (Corollary 4.5).
2. The number of triangulations of $\Delta^2 \times \Delta^k$ grows as $e^{\frac{3}{2}L\left(\frac{\pi}{3}\right)k^2 + o(k^2)},$
   where $L(x)$ is the Lobachevsky function
   $$L(x) = -\int_0^x \log|2\sin t|dt.$$
3. The number of triangulations of $\Delta^2 \times \Delta^{k-1}$, for $k = 1, \ldots, 16$ is $k!$ times the number shown in Table 1.

| $k$ | lozenge tilings of $k\Delta^2$ | $k$ | lozenge tilings of $k\Delta^2$ |
|-----|-------------------------------|-----|-------------------------------|
| 1   | 1                             | 9   | 15952438877                   |
| 2   | 3                             | 10  | 198341709785                  |
| 3   | 18                            | 11  | 355891365876534               |
| 4   | 187                           | 12  | 91655826195854811             |
| 5   | 3135                          | 13  | 33726014269095727260          |
| 6   | 81462                         | 14  | 17665249123640876125464       |
| 7   | 3198404                       | 15  | 1313039906766418384962727     |
| 8   | 186498819                     | 16  | 1381341177861864458167635925  |

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| 15  | 1313039906766418384962727     |
| 16  | 1381341177861864458167635925  |

Table 1. Tilings of $k\Delta^2$ into $k$ triangles and $\binom{k}{2}$ lozenges. Times $k!$, this is the number of triangulations of $\Delta^2 \times \Delta^{k-1}$.

The number $\frac{3}{2}L\left(\frac{\pi}{3}\right) \approx 0.323$ that appears in part (2) of this statement is the maximum asymptotic normalized entropy of lozenge tilings of a planar region, as computed in [4]. The exact number of triangulations of $\Delta^2 \times \Delta^k$ that appears in part (3) had previously been computed only up to $k = 5$ [6, 15]. Part (1) of the statement is interesting for two reasons. On the one hand, there are not many examples where the graph of flips is known to be connected. Essentially, only the case of dimension at most 2 (classical), codimension at most 3 [1] and that of cyclic polytopes [14]. On the other hand, since all triangulations of products of simplices are unimodular, the graph of flips has a very direct interpretation in toric algebraic geometry; see Theorem 2 below.

Besides the results in Theorem 1, the Cayley Trick allows to picture triangulations of $\Delta^2 \times \Delta^k$ as 2-dimensional objects. We include pictures of all (non-isomorphic) triangulations of $\Delta^2 \times \Delta^2$ and $\Delta^2 \times \Delta^3$ (Figures 5 and 14). Also, of non-regular triangulations of $\Delta^2 \times \Delta^5$ and $\Delta^3 \times \Delta^3$ and of a non-regular coarse subdivision of $\Delta^2 \times \Delta^7$ (Figures 6 and 7). $\Delta^2 \times \Delta^5$ and $\Delta^3 \times \Delta^3$ are the minimal products of simplices that have non-regular triangulations. To the best of
our knowledge, our subdivision of $\Delta^2 \times \Delta^7$ is the first known non-regular coarse subdivision of a product of simplices.

Subdivisions and triangulations of $\Delta^{k-1} \times \Delta^{l-1}$ are interesting from several perspectives. They have been studied for their own sake in [3, 6] and [8] Sect. 7.3.D and have been used as building blocks to find efficient triangulations of high dimensional cubes [9, 12] or to find disconnected flip-graphs [16, 18]. Also, $\Delta^{k-1} \times \Delta^{l-1}$ is an example of a totally unimodular polytope; that is, a lattice polytope all the simplices of which have the same volume. This implies it is equidecomposable [2], i.e., all its triangulations have the same $f$-vector. From the $f$-vector of any of its triangulations one can recover, for example, its Erhart polynomial.

From a more algebraic point of view, the toric ideal associated to the vertex set of $\Delta^{k-1} \times \Delta^{l-1}$ is a fundamental object. It is the determinantal ideal generated by the $2 \times 2$-minors of a $k \times l$-matrix, and the variety associated to it is the Segre embedding of $\mathbb{CP}^{k-1} \times \mathbb{CP}^{l-1}$ in $\mathbb{CP}^{kl-1}$. The study of this ideal has connections to enumeration, sampling and optimization for contingency tables and transportation problems (cf. [20], Chapter 5).

In this context, part (1) of Theorem 1 has the following algebro-geometric application: Every lattice point set defines a toric (binomial) ideal and a toric Hilbert scheme. Contrary to standard Hilbert schemes, toric Hilbert schemes are sometimes non-connected [18]. In fact, for a totally unimodular polytope, connectivity of the corresponding toric Hilbert scheme is equivalent to connectivity of the graph of triangulations (cf. [14] and Theorem 10.13 in [20]). Hence:

**Theorem 2.** The toric Hilbert scheme of the determinantal ideal of $2 \times 2$ minors of a $3 \times k$ matrix is connected.

Another recent source of interest in triangulations of $\Delta^{k-1} \times \Delta^{l-1}$ comes from tropical geometry. Develin and Sturmfels [5] have proved that the combinatorial types of point configurations with $k$ points in the tropical space of dimension $l - 1$ are in bijective correspondence to the regular subdivisions of $\Delta^{k-1} \times \Delta^{l-1}$. Via this relation, we prove:

**Theorem 3 (Corollary 5.5).** For every fixed $l \geq 2$, the number of regular subdivisions of $\Delta^l \times \Delta^k$ grows as $2^{\Theta(k \log k)}$ while the number of non-regular subdivisions grows as $2^{\Omega(k^2)}$.

The structure of the paper is as follows: After a first section with preliminaries on the Cayley Trick, Section 2 shows how to study mixed subdivisions from a purely geometric point of view. The results hold for arbitrary mixed subdivisions, but have a specially simple form for dilations of a simplex. Sections 3 and 4 contain our main results, on triangulations of $\Delta^2 \times \Delta^{k-1}$. The first explores the relation between mixed subdivisions of $k \Delta^2$ and lozenge tilings, and the second uses it to prove Theorem 1. Finally, Section 5 describes the relation between regular subdivisions of $\Delta^l \times \Delta^k$ and tropical polytopes in $l$-space, and proves Theorem 3.

1. The Cayley Trick

Except for Theorem 1.5, the contents of this section are special cases of results from [10].
1.1. Polyhedral subdivisions. Let \( P \) be a polytope in \( \mathbb{R}^d \). A cell of \( P \) is any sub-polytope of the same dimension as \( P \) and whose vertices are a subset of those of \( P \). A polyhedral subdivision of \( P \) is any family of cells which cover \( P \) and intersect properly, meaning that \( B \cap B' \) is a common face of both \( B \) and \( B' \) for every pair of cells \( B \) and \( B' \) in the subdivision. This definition is a special case of the definition of polyhedral subdivisions of a point configuration \([7, 6, 8, 22]\).

Polyhedral subdivisions form a poset under the refinement relation
\[
S \leq S' \iff \forall B \in S \exists B' \in S' : B \subseteq B'.
\]
The minimal elements in this poset, i.e., those subdivisions whose cells are all simplices, are the triangulations of \( P \). The unique maximal element in the poset is the trivial subdivision, which has the whole polytope \( P \) as its only cell. The subdivisions which refine only the trivial one are called coarse.

A polyhedral subdivision is called regular (or, sometimes, coherent) if it can be obtained as the orthogonal projection of the lower facets of a \( d+1 \)-dimensional polytope. Equivalently, \( S \) is regular if there is a height function \( h : \text{vertices}(P) \to \mathbb{R} \) such that for every cell \( B \in S \) the points \( \{(v, h(v)) : v \in B\} \subset \mathbb{R}^{d+1} \) lie in a hyperplane that passes strictly below all other points \( \{(w, h(w)) : w \notin B\} \).

1.2. Mixed subdivisions. Let \( P_1, \ldots, P_k \subset \mathbb{R}^d \) be convex polytopes. We in principle do not need to assume them to be full-dimensional, but we require their Minkowski sum to be. This Minkowski sum is defined as:
\[
\{ x_1 + \cdots + x_k : x_i \in P_i \}.
\]

**Definition 1.1.** A Minkowski cell of the Minkowski sum \( \sum_{i=1}^k P_i \) is any full-dimensional polytope \( B = \sum_{i=1}^k B_i \), where each \( B_i \) is a (perhaps not full-dimensional) polytope with vertices among those of \( P_i \). A mixed subdivision of \( \sum_{i=1}^k P_i \) is any family \( S \) of Minkowski cells which cover \( \sum P_i \) and intersect properly as Minkowski sums, meaning that for any two cells \( B = \sum B_i \) and \( B' = \sum B'_i \) in \( S \) and for every \( i \in \{1, \ldots, k\} \), the polytopes \( B_i \) and \( B'_i \) intersect properly (their intersection is a face of both).

**Remark 1.2.** We have intentionally decided to slightly abuse notation in these definitions and in the rest of the paper, to simplify it and to make the geometry more apparent. Indeed, although we speak of “mixed subdivisions of \( P = \sum P_i \)”, the concept depends not only on the polytope \( P \) but also on the particular Minkowski decomposition of it that we are given. The same occurs in the definition of proper intersection of \( B \) and \( B' \). To resolve this ambiguity, every time we write a Minkowski sum, the expression \( \sum_{i=1}^k Q_i \) should be formally understood as an ordered \( k \)-tuple of polytopes \( (Q_1, \ldots, Q_k) \). Only in sentences like “a family of Minkowski cells covers \( \sum P_i \)” we are referring to the underlying polytope resulting from the sum.

In particular, if there are different ways of obtaining a certain subpolytope of \( P \) as a sum of subpolytopes from the \( P_i \)'s, we consider these as different Minkowski cells and we have to specify which of them we are using in a particular mixed subdivision. For example, part (a) of Figure 1 shows a mixed subdivision of the Minkowski sum of two equal squares. Below the figure, each of the six Minkowski cells is expressed as a sum of sub-polytopes of the squares \( a_1a_2a_3a_4 \) and \( b_1b_2b_3b_4 \). Clearly, the exchange of all the \( a \)'s and \( b \)'s in these expressions would still provide a
(different) mixed subdivision, with the same picture. On the other hand, considering $B_4 = a_1 + b_1 b_3 b_4$ and $B_5 = a_1 a_2 a_3 + b_4$ would not provide a mixed subdivision, since these two Minkowski cells do not intersect properly.

**Figure 1.** (a) A mixed subdivision of the Minkowski sum of two equal squares. (b) A subdivision which is not mixed

Part (b) of the figure shows another reason why the “labeling” of cells as Minkowski sums is important. There, the same Minkowski sum of two equal squares is decomposed into cells which intersect properly as polytopes and which individually can be considered Minkowski cells, but which cannot be labeled in a way that makes them intersect properly in the Minkowski sense. In other words, the picture is not compatible with any mixed subdivision.

**Remark 1.3.** Our treatment of mixed subdivision is formally different, but equivalent to, the one in [10]. There, in order to keep track of the Minkowski decompositions of cells, instead of speaking of the sum of polytopes $\sum P_i$ the authors speak of the sum of point sets $\sum A_i$, where each $A_i$ is the vertex set of $P_i$. $\sum A_i$ is considered as a multiset, where different “points” (different Minkowski sums) may have identical geometric coordinates.

For the reader familiar with the theory of fiber polytopes (cf. Chapter 10 in [22]), mixed subdivisions of $\sum P_i$ can also be described as the subdivisions which are $\Pi$-induced, where $\Pi$ is the natural projection

$$\Pi : \Delta^{\# \text{vertices}(P_i)-1} \times \cdots \times \Delta^{\# \text{vertices}(P_k)-1} \to P_1 + \cdots + P_k.$$ 

This is Lemma 2.4 in [10].

As in the case of subdivisions, mixed subdivisions form a poset. A Minkowski cell $B = \sum B_i$ is smaller than (or contained in) another one $B' = \sum B'_i$ if for every $i$ we have $B_i \subseteq B'_i$. We write $B \leq B'$ if this happens. This induces the following refinement relation among mixed subdivisions:

$$S \leq S' \iff \forall B \in S \exists B' \in S' : B \leq B'.$$

The unique maximal element in this poset is again the trivial subdivision with only one Minkowski cell $B = \sum P_i$. The minimal elements are called fine mixed
subdivisions. In Proposition 1.2, we will see that fine mixed subdivisions are characterized in terms of what Minkowski cells they use. This is analogous to the fact that "fine polyhedral subdivisions" (i.e., triangulations) of a polytope are the polyhedral subdivisions whose cells are simplices. As in the case of subdivisions, we call coarse mixed subdivisions those which only refine the trivial one. They do not have an easy intrinsic characterization, as far as we know.

A mixed subdivision of $P_1, \ldots, P_k$ is called coherent if it can be obtained as the orthogonal projection of the lower facets of a $d+1$-dimensional Minkowski sum $P_1 + \cdots + P_k \subseteq \mathbb{R}^{d+1}$, where each $P_i$ orthogonally projects to the corresponding $P_i$. Equivalently, $S$ is regular if there are height functions $h_i : \text{vertices}(P_i) \to \mathbb{R}$ such that for every cell $B = \sum B_i \in S$ the points
\[
\{(v_1, h_1(v_1)) + \cdots + (v_k, h_k(v_k)) : v_i \in B_i \text{ for all } i\} \subseteq \mathbb{R}^{d+1}
\]
lie in a hyperplane that passes strictly below all other points
\[
\{(w_1, h_1(w_1)) + \cdots + (w_k, h_k(w_k)) : w_i \notin B_i \text{ for some } i\}.
\]

1.3. The Cayley Trick. As before, let $P_1, \ldots, P_k$ be polytopes in $\mathbb{R}^d$. Let $e_1, \ldots, e_k$ be an affine basis in $\mathbb{R}^{k-1}$ and let $\mu_i : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^{k-1}$ be the inclusion $\mu_i(x) = (x, e_i)$. We define
\[
C(P_1, \ldots, P_k) := \text{conv}(\bigcup_{i=1}^k \mu_i(P_i)).
\]
We call $C(P_1, \ldots, P_k)$ the Cayley embedding of $P_1, \ldots, P_k$. We will primarily be interested in the Cayley embedding of $k$ copies of the same polytope $P$. Clearly, it equals $P \times \Delta^{k-1}$, where $\Delta^{k-1}$ is the simplex with vertices $e_1, \ldots, e_k$.

The Cayley Trick (see [10] for a full exposition) is a poset isomorphism between polyhedral subdivisions of the Cayley embedding and mixed subdivisions of the Minkowski sum. More precisely, observe that for any choice of affine coordinates $(\lambda_1, \ldots, \lambda_k)$ in the simplex $\Delta^{k-1}$, the intersection of $C(P_1, \ldots, P_k)$ with the affine subspace $\mathbb{R}^d \times \{\sum \lambda_i e_i\}$ equals the "weighted Minkowski sum" $\sum \lambda_i P_i$. Choosing $\lambda_1 = \cdots = \lambda_k = 1/k$ gives a (scaled) copy of the standard Minkowski sum. What the Cayley Trick says is that:

THEOREM 1.4 (Cayley Trick [10] Theorem 3.1).

1. For any polyhedral subdivision of $C(P_1, \ldots, P_k)$, intersecting its cells with $\mathbb{R}^d \times \{\sum \lambda_i e_i\}$ we get a mixed subdivision of $\frac{1}{k} \sum P_i$.

2. This correspondence is a poset isomorphism between polyhedral subdivisions of $C(P_1, \ldots, P_k)$ and mixed subdivisions of $\sum P_i$, and bijects regular subdivisions of $C(P_1, \ldots, P_k)$ to coherent mixed subdivisions of $\sum P_i$.

Part (1), and hence one direction of part (2), are straightforward: every full-dimensional cell in a subdivision of $C(P_1, \ldots, P_k)$ is itself a Cayley embedding $C(B_1, \ldots, B_k)$ of certain subpolytopes $B_i \subseteq P_i$, and hence its intersection with $\mathbb{R}^d \times \{\sum \lambda_i e_i\}$ is a Minkowski cell in $\frac{1}{k} \sum P_i$. The other direction, that every mixed subdivision arises in this way, can be easily proved using different values of $\lambda = (\lambda_1, \ldots, \lambda_k) \in \Delta^{k-1}$: The "Minkowski intersection property" of the cells in a mixed subdivision $S$ of $P_1 + \cdots + P_k$ guarantees that $S$ induces a mixed subdivision of $\sum \lambda_i P_i$ for every $\lambda$ (by just replacing each Minkowski cell $B = \sum B_i$ by its weighted version $\lambda B := \sum \lambda_i B_i$ and the cells $C(B_1, \ldots, B_k) = \cup_{\lambda \in \Delta^{k-1}} \lambda B$ for $B \in S$ form the desired polyhedral subdivision of $C(P_1, \ldots, P_k)$.
1.4. The case $P_1 = \cdots = P_k$. When all the polytopes $P_i$ are copies of a single polytope $P \subset \mathbb{R}^d$, then $\mathcal{C}(P, \ldots, P) = P \times \Delta^{k-1}$. Hence, the Cayley Trick is a bijection between subdivisions of the product $P \times \Delta^{k-1}$ and mixed subdivisions of the dilation $kP$ of $P$. The main topic of this paper is to take advantage of this fact in order to study triangulations of $P \times \Delta^{k-1}$, via fine mixed subdivisions of $P + \cdots + P$. Here comes a first result in this direction.

**Theorem 1.5.** In the action of the affine symmetry group of $\Delta^{k-1}$ on triangulations of $P \times \Delta^{k-1}$, every orbit has exactly $k!$ elements (that is, the action is free). In particular, the number of triangulations of $P \times \Delta^{k-1}$ is divisible by $k!$.

**Proof.** The action of the permutation group, when regarded in the corresponding mixed subdivision of $kP$, amounts to a mere permutation of the Minkowski summands of every Minkowski cell, without affecting the cell itself (as a polytope). In particular, if some permutation sends a mixed subdivision $S$ to itself, it will send every Minkowski cell of $S$ to itself. Hence, it suffices to prove that for every fine mixed subdivision $S$ and every permutation $\sigma$ there is a Minkowski cell $B = \sum B_i$ that is not invariant under reordering of the summands (that is to say, in which $B_i \neq B_{\sigma(i)}$ for some $i$).

Finding such cells is easy. Take any $i$ such that $i \neq \sigma(i)$ and let $B = \sum B_i$ be a cell in which $B_i$ is full-dimensional. These are easy to find in the Cayley embedding: they are the ones with a full-dimensional (in $P$) intersection with the face $P \times \{e_i\}$ of $P \times \Delta^{k-1}$, where $e_i$ is the $i$th vertex of $\Delta^{k-1}$. □

Theorem 1.5 can be proved directly (and easily) in the world of triangulations of $P \times \Delta^{k-1}$. What we want to emphasize is that the proof is much more transparent using the Cayley Trick. In particular, we have not found this result in the bibliography about triangulations of the product of two simplices [6, 3].

2. The labeling of a mixed subdivision

2.1. General case. We said earlier (and illustrated with Figure 1) that there may be different ways in which a given polyhedral subdivision of $P_1 + \cdots + P_k$ can be labeled as a mixed subdivision. In this section we address the problem of what additional information is needed to make the mixed subdivision unique. We start with a straightforward, but useful, observation:

**Lemma 2.1.** Let $I \subset \{1, \ldots, k\}$ be a subset of indices. Let $S$ be a mixed subdivision of $P_1 + \cdots + P_k$. Then, the following is a mixed subdivision of $\sum_{i \in I} P_i$:

$$S|_I := \left\{ \sum_{i \in I} B_i : \sum_{i \in \{1, \ldots, k\}} B_i \in S \right\}.$$  

**Proof.** To see that the cells in $S|_I$ cover $\sum_{i \in I} P_i$ one can use a limiting process: for each $\lambda = (\lambda_1, \ldots, \lambda_k) \in (\mathbb{R}_+)^k$ the cells $\{\lambda B : B \in S\}$ form a mixed subdivision of $\sum_{i \in \{1, \ldots, n\}} \lambda_i P_i$. If we fix $\lambda_i = 1$ for the indices in $S$ and make all other $\lambda$’s go to zero, the Minkowski sum tends to $\sum_{i \in I} P_i$. That the cells intersect properly in the Minkowski sense is straightforward. □

This result can be easily understood in the Cayley embedding: $\mathcal{C}(P_i : i \in I)$ is a face of $\mathcal{C}(P_1, \ldots, P_k)$ and, certainly, every polyhedral subdivision of $\mathcal{C}(P_1, \ldots, P_k)$ induces a subdivision of it.
Theorem 2.2. Let $S$ be a polyhedral subdivision of $P_1 + \cdots + P_k$ and suppose that a labeling of it as a mixed subdivision exists. Assume further that all the $P_i$ have the same dimension. If we know:

1. The subdivisions $S_i := S|_{\{i\}}$ induced by $S$ in each of the individual $P_i$’s, and
2. Which cell of $S$ collapses to each full-dimensional cell of each $S_i$, 
then the whole mixed subdivision can be recovered from that data (that is to say, $S$ is the only mixed subdivision compatible with that information and there is an algorithm to recover the whole labeling).

Proof. We work with one $i$ at a time. That is to say, we are going to fix $i$ and show how to recover the $i$-th summand $B_i$ of a certain cell $B \in S$. We argue by induction on the codimension of $B_i$. The hypotheses give us the case of codimension zero.

So, suppose that we already know the $i$-th summand of all cells for which that summand has dimension $d' + 1$. Let $B_i$ be a $d'$-dimensional cell in $S_i$. Our goal is to determine all maximal cells of $S$ whose $i$-th component is precisely $B_i$:

- First, if a cell $C$ of $S$ has been determined to have as $i$-th summand a cell $C_i \in S_i$ that contains $B_i$ (hence as a face), then the faces of $C$ in any of the directions defined by the (relatively open) normal cone) of $B_i$ in $C_i$ will have $B_i$ as their $i$-th component.

- Second, if a full-dimensional cell $C'$ has been determined by the previous rule to have a face with $i$-th summand equal to $B_i$ then its $i$-th summand contains $B_i$ as a face. In particular, it either has been determined already or equals $B_i$.

Our claim is that applying these two rules we can recover all the cells whose $i$-th summand is $B_i$. A way to see it is in the process that gives $S_i$ as the limit of mixed subdivisions of $\lambda (P_1 + \cdots + P_k) + (1 - \lambda)P_i$ then $\lambda$ goes to zero. When $\lambda$ is very close to zero, all the cells whose $i$-th summand is $B_i$ become very close to $B_i$ itself and certainly we can go from any of them to one that becomes close to the afore-mentioned $C_i$ traversing only cells whose $i$-th summand is $B_i$. $\square$

2.2. Fine and pure subdivisions. Lemma 2.1 makes it easy to show that fine mixed subdivisions can be characterized in terms of what Minkowski cells they are allowed to use:

Proposition 2.3. A mixed subdivision $S$ is fine if and only if for every Minkowski cell $B = \sum B_i$ in $S$, the $B_i$’s are all simplices and $\dim(B) = \sum \dim(B_i)$ (in other words, the $B_i$’s lie in independent affine subspaces).

Proof. The “if” direction is straightforward: If all the cells are as claimed then no mixed cell can be properly contained in one of $S$ and, hence, $S$ is minimal. For the “only if” direction, a direct proof can be given but it is simpler to use the Cayley Trick. The Cayley Trick implies that fine mixed subdivisions are those for which the associated polyhedral of $C(P_1, \ldots, P_k)$ is a triangulation. In other words, those that use only Minkowski cells $B = \sum B_i$ whose associated Cayley cells $C(B_1, \ldots, B_k)$ are simplices. This condition is easily seen to be equivalent to the one in the statement. $\square$

We will refer to the special mixed cells described in Proposition 2.3 as fine mixed cells.
Corollary 2.4. For a fine mixed subdivision, the information in (1) and (2) of Theorem 2.2 can be recovered if we know, for each $i$, what maximal cells of $S$ have a full-dimensional $i$-th summand.

Proof. In a fine mixed subdivision, if a cell has a full-dimensional summand then all the other summands are 0-dimensional (points). Hence, the information we are given is what maximal cells of $S$ form the subdivision $S_i$, for every $i$, except we are not told how to arrange them to subdivide $P_i$. If we show how to do that, then Theorem 2.2 gives the rest. It is actually enough to find out, for every cell $B$ of $S$, what is (up to translation) the $i$th summand $B_i$ of that cell. If we know this, we know how to write $B = B_i + C_i$ for each cell (where the $C_i$ is the sum of all the other summands) and we can scale down the $C_i$ components of all cells to recover (in the limit) the subdivision $P_i$.

Suppose then that we have identified (up to translation) all the $i$-th summands of dimension greater than a certain $d'$. Exactly as in Theorem 2.2 it is then possible to propagate along $S$ all the $d'$ dimensional faces of those summands, hence getting all the $i$-th summands of dimension $d'$.

Figure 2 illustrates this result. The figure shows all the triangulations of the 3-dimensional cube (the Cayley embedding of two equal squares), pictured as mixed subdivisions. Only one representative modulo the symmetries of the square and modulo the exchange of the labels 1 and 2 is shown. In each picture, the two triangles labeled 1 are the triangulation of the bottom square and the two triangles labeled 2 are the triangulation of the top square of the cube. Knowing that information is enough to recover the mixed subdivision labeling (and hence, the corresponding triangulation of the 3-cube).

Figure 2. The 74 triangulations of a regular cube

It may be of interest also to define pure mixed subdivisions as those for which every Minkowski cell $B = \sum B_i$ satisfies $\dim(B) = \sum \dim(B_i)$. If all the $P_i$’s are simplices then pure is equivalent to fine. In general, pure mixed subdivisions form a lower ideal in the poset of mixed subdivisions and contain all the fine ones. Their Minkowski cells are (combinatorially) products of their summands. Corollary 2.4 and Theorem 1.5 hold for pure mixed subdivisions, with the same proofs.
2.3. The product of two simplices. The product $\Delta^p \times \Delta^q$ of two simplices can be considered a Cayley embedding in two ways: $q + 1$ copies of $\Delta^p$ or $p + 1$ copies of $\Delta^q$. It is easy to conclude from the previous results that in this case the “Minkowski labeling” of a subdivision can totally be neglected.

**Lemma 2.5.** Let $B \subset \mathbb{R}^d$ be a polytope and $\Delta \subset \mathbb{R}^d$ be a $d$-simplex. If there is a way of writing $B$ as a Minkowski sum of (positive dimensional, not necessarily distinct) faces of $\Delta$, then this way is unique, modulo reordering.

**Proof.** We argue by induction on the dimension of $\Delta$. The case where $\Delta$ is a segment is trivial: $B$ is the sum of as many copies of $\Delta$ as indicated by its length.

For the inductive step, let $\Delta_0$ be a facet of $\Delta$, and let $p_0$ be the opposite vertex. For every polytope $P$ in $\mathbb{R}^d$ let $P_0$ denote the face of $P$ in the direction of $\Delta_0$ (that is, the face containing the outer normal vector to $\Delta_0$ in its relatively open outer normal cone) and let $P_1$ denote the face in the opposite direction (that is, the face whose outer normal cone contains the opposite vector).

Since $B$ can be written as a Minkowski sum of faces of $\Delta$, every face of it can. In particular, the faces $B_0$ and $B_1$. Moreover, any decomposition $B = F_1 + \cdots + F_k$ will restrict to decompositions $B_0 = F_{01} + \cdots + F_{0k}$ and $B_1 = F_{11} + \cdots + F_{1k}$.

By inductive hypothesis, the decompositions of $B_0$ and $B_1$ are unique, so we can assume we know them, except perhaps for the ordering and for the fact that some $F_i$ may be points and we cannot recover them.

For every $i$, $F_{ii}$ must be either equal to $F_0$ or to the point $p_0$. This allows us to assume that we have matched each $F_{ii}$ to its $F_{ii}$ and that we can recover $F_i$, with the only exception of the summands where both $F_0$ and $F_1$ are points (and hence $F_i$ is a segment containing $p_0$). But after we subtract from $B$ all the summands which are not segments, what remains is a parallelotope (Minkowski sum of linearly independent segments) whose Minkowski decomposition is straightforward, and unique.

**Theorem 2.6.** Let $\Delta$ be a simplex. Let $S$ be a polyhedral subdivision of $k\Delta$ and assume that every cell $B$ can be written as a Minkowski sum of faces of $\Delta$. Then:

1. $S$ can be labeled as a mixed subdivision.
2. The labeling is unique, modulo reordering of the $k$ summands.

**Proof.** Uniqueness follows immediately from Theorem 2.2 and Lemma 2.5. Indeed, the $S_i$’s are known because the simplex has only the trivial subdivision, and which cells collapse to full-dimensional in $S_{\{i\}}$ is also known (modulo reordering of the factors): those which have a full-dimensional summand in their unique decompositions as Minkowski sum of faces of $\Delta$.

To prove existence, let us show how to find the $i$th summand of all cells of $S$. First, identify the cells that have the full simplex $\Delta$ as one of the summands in their (unique, by Lemma 2.5) Minkowski decomposition. There will be exactly $k$ such cells, counted with multiplicity if some have $\Delta$ as a repeated summand. Assign the numbers $1$ to $k$ to them arbitrarily.

Once this is done, fix an index $i \in \{1, \ldots, k\}$. As in the previous results, starting from that cell we can conclude what the $i$th summand of every other cell should be, starting with those where this summand has codimension 1, then 2, etc. The only difficulty is to show that this assignment is globally consistent, meaning that the $i$th summand obtained for a given cell $B$ is independent of the path that led from the cell with a full-dimensional $i$th summand to $B$. 
To show consistency we use the following idea: assume without loss of generality that \( i = k \), and in each cell of \( S \) shrink by a factor of \( \lambda \) the \( k \)-th summand obtained by the previous method. Since being a polyhedral subdivision is a local property, this produces a polyhedral subdivision of \((k - 1)\Delta + \lambda \Delta\). In the limit where \( \Delta = 0 \) we get a polyhedral subdivision of \((k - 1)\Delta\) all of whose cells are Minkowski sums and, by induction on \( k \), the labelings are consistent (and unique). ✓

3. Mixed subdivisions of \( k\Delta^2 \) and lozenge tilings

The triangulations of \( \Delta^1 \times \Delta^{k-1} \) are well-understood. Their number is \( k! \) and they form one only orbit under the action of the affine symmetry group \( S_k \) of \( \Delta^{k-1} \) (see [8], Section 7.3.C]). All this can be easily derived from the Cayley Trick: The Minkowski sum of \( k \) copies of a segment is just a segment \( k \) times longer, and the fine mixed subdivisions of it are the \( k! \) ways of placing the segments one after another (in other words, the fine mixed subdivisions differ only by the “labeling”). This and the next section are devoted to the next case, \( \Delta^2 \times \Delta^{k-1} \).

By the Cayley Trick, subdivisions (resp., triangulations) of \( \Delta^2 \times \Delta^{k-1} \) are in bijection with the mixed subdivisions (resp., fine mixed subdivisions) of the Minkowski sum of \( k \) copies of the triangle \( \Delta^2 \). Let us denote \( T_k \) this Minkowski sum, which we think of as an equilateral triangle of size \( k \). Hence \( T_1 = \Delta^2 \) is a triangle of unit size, whose vertices we denote \( a, b \) and \( c \).

3.1. Fine mixed subdivisions of \( T_k \). Fine mixed cells must be sums of faces from the summand triangles and the sum of dimensions of the faces involved must be 2. This leaves two possibilities: either one of the \( k \) triangles plus vertices from the other ones or the sum of two non-parallel edges of two triangles plus vertices from the other ones. In other words, they are the upward triangles and lozenge tiles in the next definition:

DEFINITION 3.1. Let \( T_k \) be an equilateral triangle in the plane with side length \( k \). Consider it tiled into \( k^2 \) equilateral triangles of side length 1, \((k^2 + k)/2\) of them parallel to \( T_k \) (we call them upward triangles) and \((k^2 - k)/2\) of them opposite to \( T_k \) (downward triangles). See Figure 3 where \( k = 4 \).

A lozenge in \( T_k \) is the union of a pair of adjacent triangles in the tiling (one upward and one downward). A lozenge tiling (or rhombus tiling) of \( T_k \) is a decomposition of \( T_k \) into \((k^2 - k)/2\) lozenges and \( k \) upward triangles.

![Figure 3. (a) The triangular tiling of \( T_4 \). (b) The tiles allowed in a lozenge tiling of \( T_k \)](image)

The terms “lozenge” and “rhombus” tilings are synonyms in the literature but they are used normally in a sense different from ours; they refer to tilings by only lozenges, of a shape containing as many upward as downward triangles. For
example, a classical object of study is the set of lozenge tilings of the centrally symmetric hexagon with sides of lengths \(a, b, c, a, b\) and \(c\). They are in bijection with plane partitions that fit into an \(a \times b \times c\) box. A classical result of MacMahon gives the number of them. See, for instance, [13], for more information on this subject. Theorem 2.6 implies:

**Theorem 3.2.** Every lozenge tiling of \(T_k\) admits a labeling as a mixed subdivision of \(k\) copies of a triangle. Moreover, in any such a labeling,

1. Each of the \(k\) copies of \(\Delta^2\) appears exactly once as a summand in one of the upward triangles of the tiling.
2. Specifying an assignment of the \(k\) copies of \(\Delta^2\) to the \(k\) upward triangles uniquely determines the labeling of the mixed subdivision.

There is a more direct way of proving Theorem 3.2 that explicitly tells how to get the labelings. First, a simple counting argument shows that there are exactly \(k\) upward triangles in every lozenge tiling; in the triangular tiling of \(T_k\) there are \(k\) more upward than downward triangles. After assigning the numbers 1 to \(k\) to the \(k\) upward triangles arbitrarily, we can define the \(i\)th zone of the lozenge tiling as the union of the \(i\)th upward triangle plus the lozenges that are obtained from it by parallel sweep of its three edges along the tiling. Figure 4 shows the four zones in a certain lozenge tiling of \(T_4\). The name “zone” is borrowed from a somewhat similar concept in zonotopal tilings.

The complement of (the relative interior of) any of the zones consists of three regions, one containing each of the three vertices of \(T_k\). We label the three regions as \(a, b\) or \(c\) depending on the vertex of \(T_k\) they contain. Every lattice point of \(T_k\) lies in exactly one of the three regions. Then, the \(i\)th summand of a cell \(B\) in the tiling equals the convex hull of the vertices of \(T_1\) corresponding to the (closed) regions of the complement of the \(i\)th zone intersected by \(B\).

![Figure 4. The four zones in a certain lozenge tiling of \(T_4\)](image)

As an example, the central triangle in figure 4 gets labeled as

\[
\{b\} + \{a, b, c\} + \{a\} + \{c\}
\]

**Corollary 3.3.** There is a bijection between lozenge tilings of \(T_k\) and orbits of triangulations of \(\Delta^2 \times \Delta^{k-1}\) by the action of the symmetry group of \(\Delta^{k-1}\). In particular, the number of triangulations of \(\Delta^2 \times \Delta^{k-1}\) equals \(k!\) times the number of lozenge tilings of \(T_k\).

As a first application of this result, we can “draw” all the triangulations of \(\Delta^2 \times \Delta^2\), that is to say, all mixed subdivisions of \(T_3\). Modulo the symmetries of the triangle, there are 5 lozenge tilings of \(T_3\), displayed in Figure 5. Each represents as many \(S_3\)-orbits of triangulations as lozenge tilings in its orbit modulo...
the symmetries of the triangle\(^1\). Hence, the number of lozenge tilings of \(T_3\) is \(3 + 6 + 1 + 2 + 6 = 18\) and the number of triangulations of \(\Delta^2 \times \Delta^2\) is \(18 \cdot 6 = 108\). The reader can compare Figure 5 with Figure 39 in [8] (page 150), where a different representation of the triangulations of \(\Delta^2 \times \Delta^2\) is used. There, the vertices of \(\Delta^{k-1} \times \Delta^{l-1}\) are represented as a \(k \times l\) grid, and each simplex of a triangulation is represented by marking some squares in the grid. Incidentally, comparing the two figures the reader can easily detect an error in the adjacency graph of one of the triangulations of [8, Fig. 39].

![Figure 5. The 108 triangulations of \(\Delta^2 \times \Delta^2\)](image)

### 3.2. Non-fine mixed subdivisions of \(T_k\).

What should a non-fine mixed subdivision look like? In the first place, it must be a polyhedral subdivision in the usual sense, with vertex set contained in the \(\binom{k+2}{2}\) lattice points of \(T_k\). Second, it must possess fine mixed refinements, hence each cell must be a convex union of lozenge tiles and upward triangles. We consider any such union as a hexagon with three pairs of parallel edges, each pair parallel to one edge of \(T_k\). Hexagons may degenerate to have some edges of length zero.

**Lemma 3.4.** Let \(B\) be a convex union of triangles of the triangular tiling. Then, the following properties are equivalent:

1. \(B\) is a Minkowski sum of faces (perhaps with repetition) of the unit upward triangle.
2. \(B\) can be tiled by lozenges and upward triangles.
3. \(B\) contains at least as many upward triangles as downward triangles.
4. In each of the three pairs of parallel sides of \(B\) the one with the same outer normal as a side of the unit upward triangle is at least as long as the opposite one.

**Proof.** If \(B\) has a pair of opposite sides of positive length, then reducing both by one unit does not affect whether \(B\) satisfies any of the properties. Hence, there is no loss of generality in assuming that \(B\) has no pair of opposite sides, hence it is a triangle (or a point). If it is an upward triangle of any size (or a point) then the four conditions hold; if it is a downward triangle then none of them does. \(\blacksquare\)

As in the fine case, Theorem 2.6 has the following consequence. Observe, however, that the orbits of the permutation group may now have cardinality smaller than \(k!\) (an extreme example of this is the trivial subdivision).

**Theorem 3.5.** A polygonal subdivision of \(T_k\) can be labeled as a mixed subdivision if and only if all the cells are in the conditions of Lemma 3.4. The labeling is unique modulo the action of the permutation group.

---

\(^1\)Observe that in this sentence there are of two different \(S_3\)-actions: one on the labels, the other on \(T_3\). They correspond to the symmetries in the two factors of \(\Delta^2 \times \Delta^2\).
A direct way of getting the Minkowski labeling in this case is as follows: We define the excess of a cell as the difference between upward and downward triangles contained in it. Our assumption is that all cells have non-negative excess. Actually, the excess of a cell has an interpretation in any of the four settings of Lemma 3.4: it is the number of upward triangles in a Minkowski decomposition, the number of upward triangles in a lozenge tiling, and the difference in length between any of the three pairs of opposite edges.

The total excess in the tiling is clearly $k$. Let us distribute the numbers 1 through $k$ to the different cells, giving a cell as many numbers as its excess. In much the same way as we did for lozenge tilings, we can define the $k$ zones of the polyhedral subdivision: The $i$-th zone contains the cell to which we assigned the label $i$ and then three arms, obtained as the cells adjacent to it in the directions towards the three edges of $T_k$, and then the ones adjacent to these, and so on. The main difference with the zones in a lozenge tiling is that now, as we travel along an arm, the edges that we cross may increase in length from one cell to the next. Also, the $i$th and $j$th zones may coincide (if the $i$th and $j$th excess reside in the same cell) or an arm of one zone be contained in the other zone.

As in the case of lozenge tilings, the definition of $i$-th zone classifies cells into seven types: the one labeled $i$, the ones in the three arms and the ones in the three regions of the complement of the zone (some of the last six types may be empty). This classification says whether the $i$-th summand in the mixed cell expression of a given cell is going to be \{a, b, c\} (if the cell is labeled $i$), \{a, b\}, \{a, c\} or \{c, b\} (if the cell lies in one of the three arms, depending on the edge of $T_k$ they are heading to) or just \{a\}, \{b\}, or \{c\} (if the cells in the complement of the zone).

As an example, the right part of Figure 6 shows a valid polygonal subdivision of $T_8$. The cells labeled 1 to 8 have excess 1. The two unlabeled cells have excess zero. The dots, arrow and shading in the figure are there for later use. In order to illustrate the above concepts, Table 2 shows the mixed subdivision labeling (i.e., the Minkowski decomposition) of the ten cells. Cells 1 to 8 appear first in the list, then the shaded hexagon and finally the parallelogram. The eight columns of summands correspond to the eight zones.

4. Subdivisions of $\Delta^2 \times \Delta^{k-1}$

In this section we list several properties of triangulations and subdivisions of $\Delta^2 \times \Delta^{k-1}$ which can be derived from representing them as lozenge tilings.

4.1. Non-regular subdivisions of $\Delta^2 \times \Delta^{k-1}$. In order for a mixed subdivision to be coherent it has first to be regular as a subdivision in the standard sense (i.e., the projection of the lower hull of a polytope in one dimension more). Hence:

**Proposition 4.1.** The tilings of Figure 6 represent a non-regular triangulation of $\Delta^5 \times \Delta^2$ and a coarse non-regular subdivision of $\Delta^2 \times \Delta^7$, respectively.

**Proof.** The proof of non-regularity is sketched in the picture in both cases. In the left, if a lifting existed, there would be no loss of generality (by addition of an affine function to all the heights) in assuming that the three neighbors of the central point get height zero. The central point must then get a negative height that we denote $-a$. We let $b, c, d, e, f$ and $g$ denote the heights of certain boundary points, as shown in the figure. From these heights some others can be deduced, and
\{a, b, c\} + \{a, c\} + \{a, c\} + \{a\} + \{a\} + \{a, b\} + \{a\}.
\{b\} + \{a, b, c\} + \{a, c\} + \{a\} + \{a\} + \{a, b\} + \{a\}.
\{b\} + \{b\} + \{a, b, c\} + \{a\} + \{a, b\} + \{a\} + \{b\} + \{b\}.
\{b\} + \{b\} + \{b\} + \{a, b, c\} + \{a\} + \{a, b\} + \{a\} + \{b\} + \{b\}.
\{b\} + \{c\} + \{c\} + \{c\} + \{c\} + \{c\} + \{a, b, c\} + \{a\} + \{a, b\}.
\{c\} + \{c\} + \{c\} + \{c\} + \{c\} + \{c\} + \{a, c\} + \{a, c\} + \{a\} + \{a, b\}.
\{c\} + \{c\} + \{c\} + \{c\} + \{c\} + \{c\} + \{a, c\} + \{a, c\} + \{a\} + \{a, b\}.
\{b\} + \{b\} + \{c\} + \{c\} + \{c\} + \{c\} + \{a, b\} + \{b\} + \{b\}.

Table 2. Minkowski decomposition of the cells in the non-regular subdivision of $\Delta^2 \times \Delta^7$ of Figure 6 (right)

in particular the figure shows how to conclude that $b > d$. The same arguments applied cyclically show that $d > f$ and $f > b$, which is impossible.

For the picture on the right, there is no loss of generality in assuming height zero for all the vertices of the shaded cell. Then, the seven marked points can easily be proved to get all the same height, but this contradicts convexity at the edge between region 5 and its adjacent parallelogram. That the right picture represents a coarse subdivision follows from the fact that it is coarse as a subdivision of $T_6$ in the standard sense. □

Non-regular triangulations of the product of two simplices were first constructed by de Loera [6] for $\Delta^3 \times \Delta^3$. He also proved that all triangulations of $\Delta^2 \times \Delta^k$ are regular, up to $k = 4$. Later, Sturmfels [20], constructed a non-regular triangulation of $\Delta^2 \times \Delta^5$, hence concluding that $\Delta^k \times \Delta^l$ has non-regular triangulations if and only if $(k - 1)(l - 1) \geq 4$. In particular, the non-regular lozenge tiling of $T_6$
that we show is smallest possible. As for our second example, to the best of our knowledge it is the first known coarse non-regular subdivision of the product of two simplices. Observe that coarse subdivisions of polytopes in general, and of products of simplices in particular, are not well-understood objects.

The Cayley Trick can also be used to picture non-regular triangulations of $\Delta^3 \times \Delta^3$. Figure 7 is our attempt to do so. The picture shows (an explosion of) the coherent mixed subdivision of $4\Delta^3$ produced by the following lifting matrix. Each row represents the lifting of one of the four copies of $\Delta^3$:

\[
\begin{pmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0
\end{pmatrix}.
\]

This subdivides $4\Delta^3$ into 14 cells: four parallelepipeds in the four corners of $4\Delta^3$; four tetrahedra incident to the center of the four facets of $4\Delta^3$; and six “Minkowski sums of two triangles” along the six edges of $4\Delta^3$. Another way of describing this subdivision is that it is obtained by cutting $4\Delta^3$ with the four planes through its centroid and parallel to its facets.

This mixed subdivision is not fine, because the six special cells along edges of $4\Delta^3$ can be refined into two triangular prisms each, as Figure 8 shows. If the six cells are refined in the particular “skew” way sketched by dashed lines in Figure 7 then the fine mixed subdivision obtained is not coherent. (It is actually a non-regular subdivision; the proof is easy and left to the reader). Hence, it corresponds to a non-regular triangulation of $\Delta^3 \times \Delta^3$.

It is interesting to observe that $\Delta^{k-1} \times \Delta^{l-1}$ has non-regular triangulations if and only if there is a matroid on $k + l$ elements and of rank $k$ which is not representable over the reals. This was first noticed in [20], where the subdivisions in Figure 7 and the left part of Figure 6 were related, respectively, to the Vamos and the non-Pappus matroids.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure7.png}
\caption{A regular subdivision of $\Delta^3 \times \Delta^3$ and a non-regular refinement of it.}
\end{figure}
4.2. Lozenge flips versus bistellar flips. A basic concept to understand the set of all triangulations of a polytope is that of geometric bistellar flip. Roughly speaking, it is the minimum possible difference (the “elementary move”) between two triangulations. One simple definition, (see [17]) is that two triangulations differ by a bistellar flip if and only if they are the only two refinements of a certain polyhedral subdivision. A more explicit definition that says what the difference between the two triangulations has to be for this to happen is contained, for example, in [8, 7]. We do not need it here.

Definition 4.2. We say that two lozenge tilings of $T_k$ differ by a lozenge flip if one can be obtained from the other by one of the four substitutions of tiles shown in Figure 9. More precisely, the first three will be called trapezoid flips and the last one a hexagon flip.

Proposition 4.3. Let $L$ and $L'$ be two labeled lozenge tilings of $T_k$, corresponding to two triangulations $S$ and $S'$ of $\Delta^2 \times \Delta^{k-1}$. Then, $S$ and $S'$ differ by a bistellar flip if and only if the following three properties hold:

1. $L$ and $L'$ differ by a lozenge flip.
2. The labeling of upward triangles is the same in $L$ and $L'$ (except for the displacement of the triangle in case of a trapezoid flip).
3. For a trapezoid flip, the arm of the triangle affected by the flip in the direction of the big edge of the trapezoid does not change with the flip (it is only translated).

Proof. Let us look at what a polyhedral subdivision of $T_k$ has to look like in order to admit only two lozenge refinements. First, all cells must be convex and individually admit only two lozenge refinements. The possibilities are a hexagon as the one in a hexagon flip, a trapezoid, or the union of two parallel lozenges with a common edge. If a hexagon arises, then its refinements are independent of the refinement of any other cell, which means that no other refinable cell can be
present. Hence, the two tilings differ by a hexagon-flip. If a trapezoid or union of two lozenges arises, then the edge (or edges) of length two in that cell must be propagated up to the boundary of \( T_k \) on one side and to a trapezoid on the other side. The flip is a trapezoid flip and satisfies the arm condition in the statement. That the labels must be the same in the two lozenge tilings is trivial.

As an example, the two lozenge tilings on the right part of Figure 10 represent triangulations which differ by a bistellar flip, because they are the two refinements of the subdivision in the bottom-right. The two lozenge tilings on the left differ by a lozenge flip, but not by a bistellar flip.

![Figure 10. A trapezoid flip is not always a bistellar flip](image)

**Theorem 4.4.**

1. The set of all (labeled) lozenge tilings of \( T_k \) is connected under trapezoid flips.
2. The set of triangulations of \( \Delta^2 \times \Delta^{k-1} \) is connected under geometric bistellar flips.

**Proof.** Any triangle not on the bottom row of a lozenge tiling is adjacent to a lozenge below it. Performing the trapezoid flip there produces a triangle one level lower. With this idea, we can eventually arrive to a lozenge tiling with all its triangles on the bottom row, and there is only one such tiling. We have not taken care of labels, but once we have the tiling with all triangles in the bottom row there is a sequence of three trapezoid flips which exchanges two consecutive triangles. Hence, any permutation of the labels can be implemented as a sequence of trapezoid flips, too. This proves part (1).

For part (2), we proceed similarly. Not all trapezoid flips are bistellar flips, but we can prove that unless all the triangles are in the bottom row there must be some trapezoid flip which decreases the height of the triangle involved and which is a geometric bistellar flip. To see this, start a triangle \( i \) with maximum height in the tiling. This implies that all the rows above that triangle are tiled with vertical
lozenges, as in Figure 11. Let us consider the trapezoid flip that would decrease the height of \( i \) and, more specifically, at the downward looking triangle next to the lozenge below \( i \), in the direction of the big side of the trapezoid. There are two possibilities for the lozenge containing that downward triangle: if it is a vertical lozenge, then the trapezoid flip at \( i \) is a geometric bistellar flip, and we are done. If it is not, then above it there is another triangle at the same height as \( j \), and we get a trapezoid flip looking in the same direction and “closer to the boundary”. Repeating this process we must eventually arrive at a trapezoid flip which is itself a bistellar flip. \( \square \)

**Figure 11.** Every lozenge tiling has *some* trapezoid flip which is a bistellar flip.

Further analysis of the above proof gives bounds on the diameter:

**Corollary 4.5.** The graph of lozenge tilings of \( T_{k+1} \) and the graph of triangulations of \( \Delta^2 \times \Delta^k \) both have diameter in \( \Theta(k^2) \).

More precisely, the graph of unlabeled lozenge tilings has diameter at least \( \binom{k}{2} \) and the graph of triangulations of \( \Delta^2 \times \Delta^k \) has diameter at most \( 5\binom{k}{2} \).

**Proof.** For a quadratic lower bound, observe that at least \( \binom{k}{2} \) lozenge flips are needed to go from the lozenge tiling with all triangles on the bottom to the lozenge tiling with all triangles on one side. This is so because each lozenge flip changes the height of only one triangle and only by one unit.

For an upper bound, the process in the proof of Theorem 4.4 shows how to go from any lozenge tiling to the one with all triangles in the bottom by a sequence of \( \sum(\text{height}_{T}(i)) \leq \binom{k}{2} \) bistellar flips. As mentioned there, once we are in that lozenge tiling we can permute any two labelings with three times the number of pairs of indices which are ordered differently, that is, at most \( 3\binom{k}{2} \) bistellar flips. With another \( \binom{k}{2} \) flips we can go back to the second lozenge tiling. \( \square \)

The constants in the previous statement can surely be improved. For example, instead of going to the tiling with all triangles on the bottom, we can choose to go to the tiling with all triangles on one side, which provides a different definition of height. For each triangle, the sum of its three heights is clearly \( k - 1 \), so that with respect to one of the three sides we get

\[
\sum(\text{height}_{T}(i)) + \sum(\text{height}_{T'}(i)) \leq 2k(k - 1)/3
\]

instead of the \( 2\binom{k}{2} = k(k - 1) \) used in the proof.
QUESTION 4.6. Is there a triangulation of $\Delta^2 \times \Delta^{k-1}$ with less than $2k - 2$ bistellar flips? Observe that $2k - 2$ is the dimension of the corresponding secondary polytope, hence it is a lower bound for the number of flips of every regular triangulation. Also, it is easy to prove that every lozenge tiling has at least $2k - 2$ lozenge flips: The $k$ upward triangles have a total of $3k$ sides and at most $k + 3$ of them are in the boundary. This implies there are at least $2k - 3$ trapezoid flips and that there are exactly that number if and only if three triangles are at the corners and the others are on the boundary. In this case, the lozenges produce a lozenge tiling of a simply connected region in the standard sense. Since lozenge tilings of a simply connected region are connected by hexagon flips, there has to be at least one hexagon flip.

4.3. Counting lozenge tilings. The number of lozenge tilings of $T_k$ can be computed in the following way. Let $k$ be fixed, and let $S$ be a subset of $\{1, 2 \ldots, k\}$. We classify the lozenge tilings of $T_k$ according to what triangles they have in the bottom line. More precisely, let

- $f_k(S)$ denote the number of lozenge tilings of $T_k$ which have triangles exactly in the positions of the bottom line given by $S$.
- $g_k(S)$ denote the number of lozenge tilings of $T_k$ which have triangles at least in the positions of the bottom line given by $S$.

Clearly,

$$g_k(S) = \sum_{S' \subseteq S} f_k(S').$$

But, moreover,

**Proposition 4.7.** Let $S = \{s_1, \ldots, s_j\} \neq \emptyset$, where $1 \leq s_1 < \cdots < s_j \leq k$. If $j = 1$, then $f_k(S) = g_{k-1}(\emptyset)$. If $j > 1$, then:

$$f_k(S) = \sum_{s_1 \leq s'_1 < s_2} g_{k-1}({\{s'_1, \ldots, s'_{j-1}\}}) \quad \vdots \quad \sum_{s_{j-1} \leq s'_{j-1} < s_j}$$

**Proof.** Between every two triangles of the bottom row there must be one and only one vertical lozenge. Once we fixed the positions $s'_1, \ldots, s'_{j-1}$ of these vertical lozenges, the ways to complete the lozenge tiling are exactly the same as the lozenge tilings of $T_{k-1}$ containing triangles in (at least) the positions $s'_1, \ldots, s'_{j-1}$ of the bottom row. \qed

Table 3 shows all the values of $f_k(S)$ and $g_k(S)$ with $k = 1, 2, 3$, as well as the values of $f_4(S)$, computed using the recursive equations (1) and (2). Adding all the entries of $f_4(S)$ we get the number of lozenge tilings of $T_4$, which is $g_4(\emptyset) = 187$. Hence, the number of triangulations of $\Delta^2 \times \Delta^3$ is $187 \times 4! = 4488$.

The numbers shown in Table 1 in the introduction were computed with an implementation of these recursive formulas in Maple. The computation is clearly exponential in time, since we need to compute $2^k$ values of $f_k(S)$ and $g_k(S)$ for each $k$. In practice, the computation of each value took about five times the previous one: 21 seconds for $k = 10$ and 70 hours for $k = 16$. By Corollary 3.3, multiplying the $k$th number by $k!$ we get the number of triangulations of $\Delta^{k-1} \times \Delta^2$. A direct approach
allowed Jesús de Loera and Jörg Rambau [6, 15] to compute these numbers of triangulations only up to $k = 4$ and $k = 6$ respectively.

4.4. The asymptotic number of lozenge tilings. Let $l_k$ denote the number of lozenge tilings of $T_k$. It is easy to show that $l_k$ is in $e^{\Theta(k^2)}$:

- Since a lozenge tiling can be specified by which of the three upward neighbors of each of the $(k^2 - k)/2$ downward triangles forms a lozenge with it, $l_k \leq 3^{(k^2-k)/2} < 3^{k^2/2}$.
- Assume $k$ is a multiple of 3. $T_k$ can be tiled into $3 \left( \frac{k}{3} \right) = \frac{k^2-3k}{6}$ hexagons plus $k$ boundary trapezoids (see Figure 12), each of which can independently be refined in two ways. Hence, $l_k \geq 2^{(k^2+3k)/6}$.

In particular, it is asymptotically not relevant to distinguish between labeled and unlabeled lozenge tilings. We can as well think of $l_k$ as the number of triangulations of $\Delta^2 \times \Delta^{k-1}$.

The property that the logarithm of the number of tilings is proportional to the area for dilations of a given shape is well-known in the context of usual lozenge

| $S$ | $\emptyset$ | 1 | 2 | 1, 2 | 3 | 1, 3 | 2, 3 | 1, 2, 3 |
|-----|-----------|---|---|-------|---|------|-----|--------|
| $f_1(S)$ | 0 | 1 |   |   |   |   |   |  |
| $g_1(S)$ | 1 | 1 |   |   |   |   |   |  |
| $f_2(S)$ | 0 | 1 | 1 | 1 |   |   |   |  |
| $g_2(S)$ | 3 | 2 | 2 | 1 |   |   |   |  |
| $f_3(S)$ | 0 | 3 | 3 | 2 | 3 | $2 + 2 = 4$ | 2 | 1 |
| $g_3(S)$ | 18 | 10 | 8 | 3 | 10 | 5 | 3 | 1 |
| $f_4(S)$ | 0 | 18 | 18 | 10 | 18 | $10 + 8 = 18$ | 8 | 3 |
| $f_4(S \cup \{4\})$ | 18 | $10 + 8 + 10 = 28$ | 18 | $8 + 10 = 18$ | $3 + 5 = 8$ | 10 | $5 + 3 = 8$ | 3 | 1 |

Table 3. The number of triangulations of $\Delta^2 \times \Delta^3$ computed by hand. It is $4!$ times the sum of entries in the last two rows.

Figure 12. Proof of a quadratic lower bound for $\log(l_k)$ (left) and a lozenge tileable region of nearly constant boundary height (right)
tilings, as follows from the following result of Cohn, Kenyon and Propp [4] (they consider mostly the case of domino tilings, i.e., perfect matchings in a sub-region of the square grid, but the case of lozenge tilings arises as a particular case in which certain edges are forbidden in the matching). Let \( R^* \) be a simply-connected region in the plane, and let \( (R_n)_{n \in \mathbb{N}} \) be a sequence of lozenge-tileable (in the standard sense) simply-connected regions such that \( R_n/n \) converges to \( R^* \). Let \( f_n : \partial R_n \to \mathbb{R} \) be the boundary height function of the region \( R_n \) (defined below). Assume that, after scaling it down, \( f_n/n \) converges to a certain function \( f^* : \partial R^* \to \mathbb{R} \).

**Lemma 4.8** (Cohn et al. [4]). In the above conditions,

(1) The logarithm of the number of lozenge tilings of \( R_n \) divided by the area of \( R_n \) (measured in lozenge tiles) converges to a constant that depends only on \( R^* \) and \( f^* \).

(2) This constant is maximized if \( f^* = 0 \). In this case it equals

\[
\frac{3}{\pi} L \left( \frac{\pi}{3} \right) \approx 0.32306594.
\]

Here, \( L(x) \) is the Lobachevsky function, defined as

\[
L(x) = -\int_0^x \log |2 \sin t| \, dt.
\]

The constant in part (1) (and its specific instance in part (2)) is computed as an integral of the average (in a well-defined sense) extension of the boundary height function \( f^* \) to the interior of \( R^* \).

The boundary height function of a simply connected union \( R \) of triangles of the regular triangular tiling is defined as follows: choose alternating signs for the six directions of edges in the tiling. Starting at any particular boundary vertex, give height zero to that vertex and then propagate the height along the boundary cycle of \( R \), increasing or decreasing the height by one depending on the direction of the edge traversed. A simply connected region is lozenge-tileable if and only if the height becomes 0 again when you return to the starting point [21]. The right part of Figure 12 is an example of a tileable region with nearly constant boundary function.

The following statement says that the asymptotic entropy per unit tile is the same in our lozenge tilings of \( T_k \) as in classical lozenge tilings of a simply connected region with nearly constant boundary height function. The proof we give is essentially glued from personal communications to the author by J. Propp, H. Cohn and, specially, David Wilson:

**Theorem 4.9.**

\[
l_k = e^{\beta L_k} + o(k^2),
\]

where \( \beta = \frac{2}{3} L \left( \frac{\pi}{3} \right) \approx 0.32306594 \) is the asymptotic entropy per unit tile of regions with nearly constant boundary height function, as given by Lemma 4.8.

**Proof.** For the lower bound, apply part (2) of Lemma 4.8 to the region on the right of Figure 12.

For the upper bound, let \( f(k) \) be any function such that, asymptotically, \( 1 << f(k) << \sqrt{k} \) (for example, \( f(k) = k^{1/4} \)). Let \( S_k \) be a tiling of \( T_k \) into a triangular grid of about \( k^2/f(k)^2 \) triangles of size about \( f(k) \).

For each lozenge tiling of \( T_k \), we cut \( T_k \) without breaking tiles but otherwise as close as possible to the tiling \( S_k \). If a tile overlaps two cells of \( S_k \), we choose,
for instance, to give that tile to the bottom of the two. The total perimeter of
the cells in $S_k$ is clearly in $\Theta(k^2/f(k)) \subset o(k^2)$. Hence, the number of possible
ways of cutting $T_k$ produced in this way is in $2^{o(k^2)}$, and will not affect the final
asymptotics. Our task is to bound the number of lozenge tilings compatible with a
specific cutting.

For this, we consider independently the cells of $S_k$ that only contain lozenges
and those that contain at least a triangle in the lozenge tiling. Although this is not
relevant for the asymptotics, observe that which cells contain triangles (and how
many of them) is fixed by the cutting: for a specific cell, the number of triangles is
the difference between upward and downward triangles of the triangular unit grid
contained in it.

Since at most $k$ cells contain triangles, we do not need to care much about
their number of tilings. The easy argument that each downward triangle must be
matched to one of at least three upward triangles shows that the number of tilings
of each cell is at most $3^{p(k)/2}$. Hence, the number of tilings

\[ e^{\beta \frac{k^2}{2} + o(f(k)^2)} \]

and will not affect the final
asymptotics. Our task is to bound the number of lozenge tilings compatible with a
specific cutting.

For the cells that are tiled only with lozenges, we are in the situation of
Lemma 4.8: the number of lozenge tilings of each is at most

\[ e^{\beta f(k)^2} \]

and the combined number is at most

\[ (e^{\beta k^2/2} + o(f(k)^2))^{k^2/f(k)^2} = e^{\beta k^2/2 + o(k^2)}. \]

\[ \square \]

5. Tropical polytopes

Develin and Sturmfels have recently started developing the theory of polytopes
in tropical space \[5\]. We here give a brief account of their main results, specially
in their relations to subdivisions of the product of two simplices.

The tropical projective space of dimension $l-1$, denoted $\mathbb{T}P^{l-1}$, is the quotient
of $\mathbb{R}^l$ by the equivalence relation $v \sim v + (\lambda, \ldots, \lambda)$, for every $v \in \mathbb{R}^l$ and every
$\lambda \in \mathbb{R}$. By normalizing one of the coordinates (say the first one) to be equal to zero
we can identify $\mathbb{T}P^{l-1}$ to $\mathbb{R}^{l-1}$.

The tropical hyperplane defined by a vector $(a_1, \ldots, a_l) \in \mathbb{R}^l$ is the set of points
$v \in \mathbb{T}P^{l-1}$ such that the minimum of the numbers $v_i + a_i$ is achieved twice. Clearly,
the hyperplanes defined by $(a_1, \ldots, a_l)$ and by $(a_1 + \lambda, \ldots, a_l + \lambda)$ coincide, so
we may say that a hyperplane is defined by a point $a \in \mathbb{T}P^{l-1}$. The hyperplane
defined by $(0, \ldots, 0)$ is the set of points $v \in \mathbb{T}P^{l-1} = \mathbb{R}^{l-1}$ such that $v$ either lies
in the boundary of the positive orthant or has minimum coordinate negative and
repeated at least twice. Said in a more compact (and invariant) form, it equals the
$l-2$-skeleton of the normal fan of the simplex with vertices $O, e_1, \ldots, e_{l-1}$. The
translation of this hyperplane by the vector $-a \in \mathbb{R}^l$ gives the hyperplane defined by
$a$.

If $H$ is the hyperplane defined by a point $v \in \mathbb{T}P^{l-1}$, here we call $-H$ the antihyperplane
defined by $v$. For the purposes of this paper, the following consequence
of the results in \[5\] Section 3] can be taken as a definition of tropical convex hull:
Proposition 5.1. Let \( v_1, \ldots, v_k \) be a finite set of points in tropical \( l-1 \) space. Then, its tropical convex hull \( \text{tconv}(v_1, \ldots, v_n) \) equals the union of all bounded cells in the polyhedral arrangement of tropical anti-hyperplanes given by \( v_1, \ldots, v_k \).

The left part of Figure 13 shows an example of this. The tropical convex hull of the five dots equals the shaded region, including its boundary and the horizontal segment that reaches to point number 5. Develin and Sturmfels make no clear distinction between the tropical convex hull as a subset of \( \mathbb{T}^d \) and the polyhedral complex in the above statement, and use the term “tropical polytope” referring to both. Here we will use “tropical polytope” referring to the region and call “tropical order type” of the point set the polyhedral complex. Two point sets are combinatorially equivalent if they have the same bounded complex, in a labeled sense (with the label of each cell indicating its relative position in each of the anti-hyperplanes. This is essentially what Develin and Sturmfels call the “type” of a cell).

The connection to mixed subdivisions is given in the following statement, paraphrased from Section 4 of [5]:

Theorem 5.2. Let \( M \) and \( M' \) be two \( k \times l \) real matrices. Then, the columns of \( M \) and \( M' \) produce the same tropical order type if and only if they produce the same regular mixed subdivision of \( k\Delta^{l-1} \) (where the \( i \)th column specifies the heights to lift the vertices of the \( i \)th copy of \( \Delta^{l-1} \)).

Corollary 5.3. There is the same number of order types of \( k \) points in tropical \( l-1 \) space as coherent mixed subdivisions of \( k\Delta^{l-1} \); that is, regular subdivisions of \( \Delta^{k-1} \times \Delta^{l-1} \).

Figure 13 illustrates the correspondence between a tropical point configuration with 5 points in 2-space and a mixed subdivision of \( 5\Delta^2 \). As is easy to check, there is a 1-to-1 dimension (and order) reversing correspondence between the cells defined by the tropical point set and the cells in the mixed subdivision. Unbounded cells in the tropical point set correspond to boundary cells in the mixed subdivision. Corresponding cells are orthogonal to one another. Actually, the link of a cell in the tropical point set is the normal fan of the corresponding cell in the mixed subdivision.

As another example, Figure 14 shows all the fine mixed subdivisions of \( 3\Delta^2 \), placed and numbered to exactly match the 35 types of “tropical quadrangles” as shown in Figure 6 of [5]. (This list was originally computed by J. Rambau [15], and its order is the one given as output by TOPCOM, that performs a “breadth-first search” on the graph of flips).
We now use Corollary 5.3 to give a bound on the number of regular subdivisions of $\Delta^{k-1} \times \Delta^{l-1}$. For this, we extend the tropical arrangement of $k$ anti-hyperplanes that defines $t\text{conv}(v_1, \ldots, v_n)$ to a (usual) affine arrangement $\mathcal{H}$ of $k\binom{l}{2}$ hyperplanes. Indeed, the anti-hyperplane corresponding to a point $v_i = (v_{i,1}, \ldots, v_{i,l-1}) \in \mathbb{R}^{l-1} \cong \mathbb{P}^{l-1}$ is a $(l-2)$-dimensional polyhedral complex with $\binom{l}{2}$ maximal cells, lying respectively in the following hyperplanes:

\begin{align*}
H(i; j, l) := \{ x_j = v_{i,j} \}, & \quad \text{for } j = 1, \ldots, l - 1, \\
H(i; j, k) := \{ x_j - x_k = v_{i,j} - v_{i,k} \}, & \quad \text{for } 1 \leq j < k \leq l - 1.
\end{align*}

Figure 14. The 35 symmetry classes of triangulations of $\Delta^{2} \times \Delta^{3}$.
Clearly, point sets with different tropical order type produce different (labeled) hyperplane arrangements. Then:

**Theorem 5.4.** For any \( k \) and \( l \) the number of regular subdivisions of \( \Delta^{l-1} \times \Delta^{k-1} \) is bounded above by:

\[
\left( \frac{e}{2}kl \right)^{l(l-1)(k-1)}
\]

**Proof.** We can assume that \((k-1)(l-1) \geq 6\), because if this is not the case, the bound can be proved by direct inspection: if \( l = 2 \) then the number of triangulations is \( k! \) and if \( l = k = 3 \) then the number of triangulations is 108 by Table 1.

We need to bound the number of different arrangements \( \mathcal{H} \) that can be produced for varying \((v_1, \ldots, v_k) \in \mathbb{R}^{kl}\). Two arrangements are “equal” if they have the same chirotope, that is to say, if every determinant of \( l \) of the \( k(\frac{l}{2}) \) hyperplanes has the same sign in the two arrangements.

From the definition of the hyperplanes in equation (3) it is clear that each of the determinants that define the chirotope of \( \mathcal{H} \) is a linear functional on the \( kl \) variables \((v_{i,j})\). Since the tropical order type is invariant under addition of a constant to a row or column of the matrix \((v_{i,j})\), we can assume \( v_{0,j} = v_{i,0} = 0 \) for every \( i \) and \( j \), leaving only \((l-1)(k-1)\) variables.

Hence, the order type of \( \mathcal{H} \) appears represented as a cell in a huge linear hyperplane arrangement of \( (k(\frac{l}{2}))^l \) hyperplanes in \( \mathbb{R}^{(k-1)(l-1)} \). This gives the statement, since the number of cells in an arrangement of \( N \) hyperplanes in \( \mathbb{R}^D \) is maximal for simple arrangements, in which case it equals

\[
\sum_{i=0}^{D} 2^i \binom{N}{i} \leq (D+1)2^D \binom{N}{D} \leq \frac{(D+1)2^D}{D!} N^D \leq N^D.
\]

In the first inequality we assume that \( D \leq N/2 \), which always happens for \( N = \left( \frac{k\ell}{2} \right)^l \) and \( D = (k-1)(l-1) \); in the last inequality we have used our assumption that \( D = (k-1)(l-1) \geq 6 \). \( \square \)

Observe that our bound is quite rough not only because different arrangements \( \mathcal{H} \) may represent the same tropical order type, but also because the sign of many of the \( k(\frac{l}{2}) \) determinants considered in the proof is constant (independent of the \( v_{i,j} \)'s). We believe the actual number of regular subdivisions to be in \( (kl)^{O(kl)} \). Anyway, for fixed \( l \) our bound gives the exact asymptotic behavior of the number of regular subdivisions:

**Corollary 5.5.** For any fixed \( l \geq 2 \), the number of regular subdivisions of \( \Delta^{l-1} \times \Delta^{k-1} \) is in \( k^{O(k)} \). For \( l \geq 3 \) the number of all subdivisions is in \( 2^{O(k^2)} \).

**Proof.** For regular subdivisions, the upper bound follows from the previous theorem and the lower bound is trivial, since a single orbit of regular triangulations has already \( k! \in k^{O(k)} \) elements.

For all subdivisions, Theorem 5.4 gives the case \( l = 3 \) and the others follow immediately: any subdivision of a particular \( \Delta^2 \times \Delta^{k-1} \) face of \( \Delta^{l-1} \times \Delta^{k-1} \) can be extended to the whole polytope. \( \square \)
Bibliographic remark: The writing of this paper has spanned an unusually long period of time, the first drafts dating back to 1998. Previous versions of it have been cited as “in preparation” under the title Applications of the polyhedral Cayley Trick to triangulations of polytopes.

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