We analyze the role of boundaries in the infrared behavior of quantum field theories. By means of a novel method we calculate the vacuum energy for a massless scalar field confined between two homogeneous parallel plates with the most general type of boundary properties. This allows the discrimination between boundary conditions which generate attractive or repulsive Casimir forces between the plates. In the interface between both regimes we find a very interesting family of boundary conditions which do not induce any type of Casimir force. We analyze the effect of the renormalization group flow on these boundary conditions. Even if the Casimirless conformal invariant conditions are physically unstable under renormalization group flow they emerge as a new set of conformally invariant boundary conditions which are anomaly free.

Keywords: Vacuum Energy, Casimir Effect and Renormalization Group flow

1. Introduction

The role of boundaries of quantum systems in new physical phenomena has been a focus of increasing activity in different areas of physics. In general, the presence of boundaries enhances quantum aspects of the system. Boundary properties play a relevant role in the double slit Young experiment, Aharonov-Bohm effect, Casimir effect [1], appearance of edge states and quantization of conductivity in the quantum Hall effect, and graphene physics. On the other hand, the physics of boundary conditions also reaches the very foundations of fundamental physics: topological fluctuations in quantum gravity, black hole quantum physics, quantum holography, string theories and D-branes, AdS/CFT correspondence and M-theory. There are also implications in cosmology, where the recently observed suppression and alignment of quadrupolar and octupolar terms, might be related to boundary conditions or non-trivial topology of the universe.

Boundary phenomena determine the structure of the vacuum and the low energy
behavior of the quantum field theories. In massless theories these effects are amplified because the existence of long distance correlations allows boundary effects to percolate the whole interior region. In that case the vacuum energy is highly dependent on the geometry of the bodies and the physical properties of the boundaries [4,5,6,7,8,9,10].

In this paper we shall focus on the dependence of vacuum energy on boundary conditions in a massless field theory on a domain bounded by two homogeneous parallel plates. The behavior of this energy with the distance between the plates is the basis for the Casimir effect. The energy due to vacuum fluctuations of the quantum fields induces a force between the plates and this force is always attractive for two identical plates because of the Kenneth-Klich theorem [11]. This theorem shows from very general principles that the force induced by the quantum vacuum fluctuations between two identical bodies is always attractive. However, there is an enormous interest in getting physical configurations where the Casimir force is repulsive instead of attractive. One reason for this interest are the technical applications to micro-mechanical devices (MEMS). Another reason is that the existence of repulsive or null Casimir forces allows a more accurate analysis of micro-gravity effects, because of the tight competition between gravity and Casimir force at short distances. In particular, they are useful to verify the recently formulated conjectures about the violation of Newton gravitational law at sub-millimeter scales [12].

All methods used to achieve a repulsive Casimir effect are based on plates with different properties. In fact, new repulsive regimes of the Casimir effect have been found between plates with different dielectric properties [13], non-local boundary conditions [14] and between a metallic plate with a hole and a needle pointing to the hole center. In this paper we consider the most general boundary conditions for two plates to analyze in great detail the transition from the attractive Casimir regime to the repulsive Casimir regime [15,16,17,18]. We also analyze the boundary renormalization group flow of boundary conditions and the stability properties of the Casimirless boundary conditions [18]. Although in practice, only some of these boundary conditions can be physically implemented, the advances on nano-science allows to construct new materials (metamaterials) with very special characteristics, which can allow in near future the implementation of new types of boundary conditions.

2. Quantum Fields in Bounded Domains

Let us consider a free complex scalar field $\phi$ confined in a domain between two parallel plates $\Omega = \{x \in \mathbb{R}^3; 0 < x_3 < L\}$. The quantum Hamiltonian given by

$$H = \frac{1}{2} \int_{\Omega} d^3x \left( |\pi(x)|^2 + \phi^*(x)(-\Delta + m^2) \phi(x) \right)$$

(1)

corresponds to an infinite number of decoupled harmonic oscillators associated to the Fourier modes of the operator $-\Delta + m^2$ acting on static fields confined in $\Omega$. The stability conditions require that the corresponding oscillating frequencies have
to be real and positive. This condition can be fulfilled for any value of the mass only if all eigenvalues of the Laplacian operator $-\Delta$ are real and nonnegative, i.e. $-\Delta$ is a self-adjoint non-negative operator. The positivity condition can be relaxed for a fixed geometry and given mass, but the independence on the size of $\Omega$, e.g. in the large volume limit, becomes equivalent to mass independence. In all cases the consistency is guaranteed by the positivity of the self-adjoint extension of $-\Delta$.

Because of the homogeneity of the plates the boundary conditions are translation invariant along the plates. Thus, the most general homogeneous boundary condition which satisfies the selfadjointness condition of $-\Delta$ is given by \[ \varphi - iL \dot{\varphi} = U(\varphi + iL \dot{\varphi}) , \] (2)

where $U$ is any $2 \times 2$ unitary matrix, $\varphi$ is the boundary value of $\phi$ and $\dot{\varphi}$ the normal derivative of $\phi$ at the boundary $\partial \Omega$. The positivity condition of $-\Delta$ imposes further restrictions in $U$. In consequence, the set of all boundary conditions which preserve unitarity are given by (2) with unitary matrices $U$ whose eigenvalues, $\lambda = e^{i\alpha}$ satisfy that $0 \leq \alpha \leq \pi$. In summary, $U$ has to be of the form \[ U(\alpha, \beta, \theta, \varphi) = e^{i\alpha} (\cos(\beta) I + i \sin(\beta) \mathbf{n} \cdot \mathbf{\sigma}) \] (3) with $0 \leq \alpha \leq 2\pi$, $-\pi/2 \leq \beta \leq \pi/2$ and $0 \leq \alpha \pm \beta \leq \pi$, where $\mathbf{\sigma}$ are the Pauli matrices and $\mathbf{n}$ an unitary vector of the $S^2$ sphere.

Some symmetries of the classical theory can be broken upon quantization by quantum interactions. In the case of fields confined in bounded domains only the symmetries which leave the boundary invariant can be preserved for some boundary conditions. However, in the case of scale invariance ($x \rightarrow x/\Lambda$) the presence of the boundaries does not automatically imply the breaking of the symmetry at the quantum level because the rescaling involved in the Wilson renormalization group transformation restores the system back to the same boundary domain $\Omega$. Now, scale invariance in the massless quantum field theory can still be broken because not all boundary conditions preserve this symmetry. In fact, it has been shown in Refs. [22,23] that the renormalization group acts on the space of boundary conditions according to the flow \[ \Lambda U_\Lambda \partial_\Lambda U_\Lambda = U_\Lambda \partial_t U_t = \frac{1}{2} \left( U_\Lambda^\dagger - U_\Lambda \right), \] (4) where $\Lambda = \Lambda_0 e^t$.

The only boundary conditions which preserve scale invariance are the fixed points of the renormalization group flow (4), i.e. boundary conditions whose unitary operators $U$ are Hermitian unitary matrices $U^\dagger = U = U^{-1}$. \[ \alpha'(\Lambda) + \frac{1}{\Lambda} \sin(\alpha) \cos(\beta) = 0; \ \beta'(\Lambda) + \frac{1}{\Lambda} \cos(\alpha) \sin(\beta) = 0; \ \mathbf{n}' = 0 , \] (5)

which defines a vector field that can be extended to the whole group $U(2)$. 


All fixed points are located at the corners of the rhombus in figure 1. The upper and lower corners correspond to Dirichlet and Neumann \((U = \mp 1)\) boundary conditions. The other fixed points are located at the other two corners and correspond to a \(S^2\) manifold given by \(U = n \cdot \sigma\), \(n\) being an arbitrary unit vector of \(\mathbb{R}^3\), which includes pseudo-periodic and quasiperiodic boundary conditions.

For mixed boundary conditions \(U = e^{2i \arctan e^{-t}}\), the RG flows from Dirichlet boundary conditions (ultraviolet fixed point) toward Neumann boundary conditions (infrared fixed point) \([22,23]\).

![Fig. 1. Renormalization group flow in the space of consistent boundary conditions. The box represents the projection on the plane \((\beta, \alpha)\) of the space of selfadjoint extensions of \(-\Delta\), and the grey rhombus the subset of non-negative selfadjoint extensions. Notice that fixed points are located at the corners of the rhombus. Neumann boundary conditions are at the lowest corner which is the only stable fixed point. This point is the final attractor of the whole renormalization group flow.](image)

3. Vacuum energy.

The Casimir effect in massless quantum theories is a consequence of the scale symmetry anomaly which arises in the form of finite size corrections to the vacuum energy. Within the global framework of boundary conditions formulated above it is possible to analyze with complete generality the characterization of attractive and repulsive regimes generated by this anomaly.

The vacuum energy given by the sum of the eigenvalues of \(\frac{1}{2} \sqrt{- \Delta_U}\) is ultraviolet divergent, but the Casimir effect is associated to some finite volume corrections of the vacuum energy which are UV finite and universal. In a heat kernel regularization of UV divergences

\[
E_U^{(L,\epsilon)} = \frac{1}{2} \text{tr} \sqrt{- \Delta_U} e^{\epsilon \Delta_U},
\]

the Casimir energy can be obtained from the asymptotic expansion in powers of
\[ \frac{\sqrt{\epsilon}}{L} \] of the vacuum energy per unit plate area \( A \) \[ (2) \]

The eigenvalues \( \lambda_n = (k^1)^2 + (k^2)^2 + k_n^2 \) of the Laplacian operator \( -\Delta_U \) are given in terms of the zeros \( k_n \) of the spectral function \[ (3) \]

and two arbitrary real parallel components \( k^1, k^2 \). The spectral function \( h_U(k) \) is obtained from the determinant of the coefficients of the eigenvalue equation of \( -\Delta_U \) for plane waves with momenta \((0,0,k)\). The vacuum energy can be formally given in terms of the spectral function \( h_U \) \[ (4) \]

Using the heat kernel regularization \[ (1) \] and the asymptotic expansion \[ (2) \] it is possible to compute in a very efficient way from expression \[ (4) \] the Casimir energy for arbitrary boundary conditions.

In some cases the Casimir energy can be computed analytically \[ (16,15,23,28,25,17,18) \]. The results summarized in table 1 show that many of the conditions (e.g. Dirichlet, Neumann, periodic) give rise to attractive forces between the plates, others (e.g. antiperiodic, Zaremba) induce repulsive forces, and between these two types of boundary conditions there exist a family of boundary conditions.

| Condition | Casimir Energy |
|-----------|----------------|
| \( U_n = -U_d = \mathbb{I} \) | \( c_n = c_d = -\frac{\pi^2}{1440} \) |
| \( U_{ap} = -\sigma_1 \) | \( c_{ap} = \frac{7\pi^2}{120} \) |
| \( U_p = \sigma_1 \) | \( c_p = -\frac{\pi^2}{90} \) |
| \( U_z = \pm \sigma_3 \) | \( c_z = \frac{7\pi^2}{11520} \) |
| \( U_{qp} = \cos \theta \sigma_3 + \sin \theta \sigma_1 \) | \( c_{qp} = \frac{127\pi^2}{11520} - \frac{3\pi^2}{32} - \frac{116^2}{96} - \frac{40^3 + 2\theta^3}{96\pi} + \frac{\theta^4}{48\pi^2}; \theta \in [0, \pi] \) |
| \( U_{pp} = \cos \xi \sigma_1 - \sin \xi \sigma_2 \) | \( c_{pp}(\xi) = -\frac{\pi^2}{90} + \frac{\xi^2}{12} - \frac{\xi^4}{12\pi^2} + \frac{\xi^4}{48\pi^2}; \xi \in [0, 2\pi] \) |

Table 1. Casimir energies for different boundary conditions obtained by different analytical methods: Dirichlet (d), Neumann (n), antiperiodic (ap), periodic (p), Zaremba (z), quasiperiodic (qp) and pseudoperiodic (pp).
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with no Casimir force [15]. In the case of quasi-periodic boundary conditions for \( \theta_{qp}^\pm = \pi \left\{ \frac{1}{2} \pm (1 - (1 - 2\sqrt{2/15})^\pm) \right\} \), the Casimir energy vanishes which signals the transition from attractive to repulsive regimes of the Casimir effect. Indeed, something similar occurs in the case of pseudo-periodic boundary conditions where there are two values of \( \xi \) with vanishing Casimir energy, and therefore there is no force between plates \( \xi_{pp}^\pm = \pi \left\{ 1 \pm (1 - 2\sqrt{2/15})^\pm \right\} \).

Another particular case of interest is the case of fixed points of the renormalization group which are saddle points and are located at left and right corners of the rhombus of Fig. 1, i.e. boundary conditions corresponding to points on the unit sphere \( S^2 \) for values \( \alpha = \pm \beta = \frac{\pi}{2} \). These include periodic, anti-periodic, quasi-periodic, and pseudo-periodic boundary conditions. The Casimir energy given by

\[
E(n_1) = \frac{1}{L^2} \left( \frac{\pi^2}{90} + \frac{(\arccos n_1)^2}{12} - \frac{(\arccos n_1)^3}{12\pi} + \frac{(\arccos n_1)^4}{48\pi^2} \right), \tag{5}
\]

with \( \arccos n_1 \in [0, 2\pi] \) has two attractive and repulsive regimes separated by a one dimensional circle of Casimirless boundary conditions given by \( \alpha = \beta = \frac{\pi}{2}; n_1 = \cos \pi \left\{ 1 \pm (1 - 2\sqrt{2/15})^\pm \right\} \).

The subspace of Casimirless boundary conditions is unstable under the renormalization group flow, because it only intersects the manifold of fixed points at the \( S^2 \) sphere of saddle fixed points. Obviously, Dirichlet and Neumann boundary conditions have always a non-vanishing attractive Casimir energy.

For more general boundary conditions it is possible to numerically evaluate the Casimir energy. In this way one can find the complete set of boundary conditions which give rise to attractive Casimir forces and those which give rise to repulsive forces [17,18]. On the other hand, boundary conditions for identical plates correspond to \( \beta = 0 \) and from the numerical calculations it is shown that all these boundary conditions are always in the attractive regime, which is in agreement with the Kenneth-Klich theorem [11].

In summary, the global analysis of the dependence of infrared properties of field theories on the nature of boundary conditions unveils many interesting physical effects. However, the characteristics of boundary conditions which encode the attractive or repulsive nature of the Casimir energy are still unknown, although the algorithm found in the previous section provides the simplest mechanism to determine such a character. The powerful method based on the use of the spectral function for the calculation of the Casimir effect permits to analyze from a global perspective the properties of the Casimir energy as a function over the space of consistent boundary conditions. On the other hand it will be very interesting to understand the special role of the Casimirless boundary conditions which are also fixed points of the boundary renormalization group. Even if these Casimirless conformal invariant conditions are physically unstable under renormalization group flow they provide a new set of conformally invariant boundary conditions which are anomaly free. The existence of similar conditions in 1+1 dimensions opens a new approach for the study of string theory in non-critical dimensions. The role of such conformally invariant boundary conditions in the corresponding string theory
deserves further study.

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