Density of normal binary covering codes

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December 31, 2003

Abstract

A binary code with covering radius $R$ is a subset $C$ of the hypercube $Q_n = \{0, 1\}^n$ such that every $x \in Q_n$ is within Hamming distance $R$ of some codeword $c \in C$, where $R$ is as small as possible. For a fixed coordinate $i \in [n]$, define $C^{(i)}_b$, for $b \in \{0, 1\}$, to be the set of codewords with a $b$ in the $i$th position. Then $C$ is normal if there exists an $i \in [n]$ such that for any $v \in Q_n$, the sum of the Hamming distances from $v$ to $C^{(i)}_0$ and $C^{(i)}_1$ is at most $2R + 1$. We newly define what it means for an asymmetric covering code to be normal, and consider the worst case asymptotic densities $\nu^*(R)$ and $\nu^*_+(R)$ of constant radius $R$ symmetric and asymmetric normal covering codes, respectively. Using a probabilistic deletion method, and analysis adapted from previous work by Krivelevich, Sudakov, and Vu, we show that both are bounded above by $e(R \log R + \log R + \log \log R + 4)$, giving evidence that minimum size constant radius covering codes could still be normal.

1 Introduction

The problem of finding a small set of $n$-bit binary string codewords such that every $n$-bit binary string is within $R$ bit-flips of a codeword is the classical coding theory question of finding binary covering codes of length $n$ and radius $R$. Much effort has been made to determine the minimum or optimal size of the smallest binary covering codes for various values of $n$ and $R$, as well as for constant $R$ as $n$ tends to infinity (cf. Chapter 12 of [3]), with asymptotically tight bounds having been achieved only in the case of $R = 1$. One method by Graham and Sloane [6], which has produced best-known upper bounds on the optimal size of covering codes for many values of $n$ and $R$, involves considering a special class of so-called normal codes (cf. entries marked with “Q” in Table 6.1 of [3]). These codes admit to an efficient concatenation operation, called amalgamated direct sum (ADS), by which good longer codes are constructed from shorter codes. In this paper, we extend this concatenation operation to give an asymptotic upper bound on the optimal size of constant radius normal covering codes which nearly approaches the corresponding best-known bound for unrestricted codes. Our extension employs a probabilistic deletion method, and a recursive construction motivated by [4] and [7], from which several analytical techniques are also borrowed. This result provides positive evidence for an unsolved conjecture: for general $n$ and $R$, does there

*Research supported in part by NSF grant DMS-9977354.
exist an optimal code which is also normal? We also newly define normality for asymmetric codes, in which every \( n \)-bit string must be obtainable from a codeword by flipping at most \( R \) 1’s to 0’s, and we adapt the above-mentioned extended concatenation operation to give an asymptotic bound on the optimal size of normal asymmetric codes for constant \( R \).

2 Definitions and the ASDS construction

Let \( Q_n := \{ x = (x_1, x_2, \ldots, x_n) : x_i \in \{0, 1\} \} \) be the set of \( n \)-bit strings, or binary \( n \)-vectors, with algebraic structure inherited from the vector space \( \mathbb{F}_2^n \) and partial ordering inherited from the boolean lattice (i.e., \( x \preceq y \) provided \( x_i \leq y_i \) for all \( 1 \leq i \leq n \)). Define the weight, or level, of \( x \in Q_n \) to be \( w(x) := \sum_{i=1}^n x_i \), that is, the number of 1’s in \( x \). Define the Hamming distance between \( x \) and \( y \) to be \( d(x, y) := w(x - y) \); for a set \( Y \subseteq Q_n \), \( d(x, Y) := \min\{d(x, y) : y \in Y\} \), with \( d(x, Y) = \infty \) when \( Y = \emptyset \). The undirected ball in \( Q_n \) with center \( x \) and radius \( R \), denoted by \( B_n(x, R) \), is the set \( \{ y \in Q_n : d(x, y) \leq R \} \). We sometimes refer to such a ball as an \( R \)-ball. The size of \( B_n(x, R) \) is independent of \( x \) and is denoted by \( b_n(R) \). The covering radius of a set \( C \subseteq Q_n \) is the smallest integer \( R \geq 0 \) such that \( Q_n = \bigcup_{c \in C} B_n(c, R) \). The usual definition of a binary covering code, which for our purposes we refer to as a symmetric binary covering code of length \( n \) and radius \( R \), or more simply an \((n, R)\)-code, is a set of codewords \( C \subseteq Q_n \) with covering radius \( R \). We use \( K(n, R) \) to denote the minimum size of any \((n, R)\)-code. A lower bound for \( K(n, R) \) is obtained by considering that the minimum conceivable number of \( R \)-balls needed to cover \( Q_n \) is \( 2^n / b_n(R) \), which gives the (folkloric) sphere bound

\[
K(n, R) \geq \frac{2^n}{b_n(R)} = \frac{2^n}{\binom{n}{\leq R}},
\]

where we define \( \binom{n}{\leq R} := \sum_{i=0}^{\lfloor R \rfloor} \binom{n}{i} \). The sphere bound motivates the definition of the density of an \((n, R)\)-code \( C \), which is \( \frac{|C|}{2^n / \binom{n}{\leq R}} \). The optimal density of an \((n, R)\)-code is \( \mu(n, R) := \frac{K(n, R)}{2^n / \binom{n}{\leq R}} \), and the asymptotic worst-case density of an \((n, R)\)-code is

\[
\mu^*(R) := \limsup_{n \to \infty} \mu(n, R).
\]

It is known that \( \mu^*(1) = 1 \) by Theorem 12.4.11 of [3] due to Kabatyanskii and Panchenko; whether \( \mu^*(R) = 1 \) for constant \( R \neq 1 \) is a central conjecture in coding theory.

In order to define asymmetric covering codes, we first define upward and downward directed \( R \)-balls. An upward directed ball in \( Q_n \) with center \( x \) and radius \( R \) is defined as the set \( B^+_n(x, R) := B_n(x, R) \cap \{ y \in Q_n : x \leq y \} \), and the corresponding downward directed ball is \( B^-_n(x, R) := B_n(x, R) \cap \{ y \in Q_n : y \preceq x \} \). We write \( b^+_n(x, R) \) or \( b^-_n(x, R) \) for the sizes of the upward or downward directed \( R \)-balls centered at \( x \in Q_n \), respectively, and sometimes instead write \( b^+_n(l, R) \) or \( b^-_n(l, R) \), where \( l \) is the weight \( w(x) \) of \( x \), since directed ball size depends only on \( n \), \( R \), and the weight of the center \( x \). In particular,

\[
b^+_n(l, R) = b^-_n(n - l, R) = \binom{n - l}{\leq R}.
\]

The asymmetric distance \( d^+(x, Y) \) between a vector \( x \in Q_n \) and a set \( Y \subseteq Q_n \) is defined by \( d^+(x, Y) := \min\{d(x, y) : y \in Y \text{ and } x \preceq y\} \), to reflect the fact that \( x \) can be covered by
$B_n(y, R)$ provided that $d^+(x, y) \leq R$. A set $C \subseteq Q_n$ downward $R$-covers $Q_n$ provided that $Q_n = \bigcup_{c \in C} B_n(c, R)$, and the asymmetric covering radius of $C$ is the smallest $R$ for which $C$ downward $R$-covers $Q_n$. We say that such a set $C$ with asymmetric covering radius $R$ is an asymmetric binary covering code of length $n$ and radius $R$, or more simply, an $(n, R)^+$-code. Analogous to the notation for symmetric codes, we define $K^+(n, R)$ to be the minimum size of an $(n, R)^+$-code. Since the typical downward directed $R$-ball size in $Q_n$ is $\left(\frac{n/2}{\leq R}\right)$, following [7] we define the density of an $(n, R)^+$-code $C$ to be $\frac{|C|}{2^{n/\left(\frac{n/2}{\leq R}\right)}}$; an alternate definition for small values of $n$ and $R$ is given in Theorem 2 of [4]. The optimal density of an $(n, R)^+$-code is $\mu_+(n, R) := \frac{K^+(n, R)}{2^{n/\left(\frac{n/2}{\leq R}\right)}}$, and the asymptotic worst-case density of an $(n, R)^+$-code is

$$\mu^*_+(R) := \limsup_{n \to \infty} \mu_+(n, R).$$

For properties of $(n, R)^+$-codes, especially for constant $R$ or constant $n - R$, see [4].

The concatenation of two vectors $x \in Q_n$ and $y \in Q_{n'}$ is the vector $(x, y) \in Q_{n+n'}$ determined by $(x, y) := (x_1, \ldots, x_n, y_1, \ldots, y_{n'})$. The direct sum of two sets $X \subseteq Q_n$ and $Y \subseteq Q_{n'}$ is $X \oplus Y := \{(x, y) : x \in X, y \in Y\} \subseteq Q_{n+n'}$. The following proposition is straightforward and presented without proof, as it is well-known in the symmetric case.

**Proposition 1 (Direct sum of codes).** Let $C$ be an $(n, R)$-code ($(n, R)^+$-code), and let $C'$ be an $(n', R')$-code ($(n', R')^+$-code). Then $C \oplus C'$ is an $(n+n', R+R')$-code ($(n+n', R+R')^+$-code).

We have reminded the reader of the direct sum construction because it is the basis of the amalgamated direct sum and amalgamated semi-direct sum constructions to be defined.

### 2.1 Normal codes

We now present normal symmetric covering codes, introduced in [3]; our notation follows that of Chapter 4 in [3]. Let $[n] := \{1, \ldots, n\}$. For a fixed coordinate $i \in [n]$ and a set $X \subseteq Q_n$, define $X^{(i)}_0 := \{x \in X : x_i = 0\}$, and $X^{(i)}_1 := \{x \in X : x_i = 1\}$; thus $C^{(i)}_0$ and $C^{(i)}_1$ partition a code $C \subseteq Q_n$ based on the $i$th codeword coordinate. The norm of $C$ with respect to the $i$th coordinate is

$$N^{(i)} := \max_{x \in Q_n} \left\{d(x, C^{(i)}_0) + d(x, C^{(i)}_1)\right\}.$$

The **minimum norm** of a code $C$ with length $n$ is defined to be

$$N_{\min}(C) := \min_{i \in [n]} N^{(i)}.$$

A code $C$ has **norm** $N$ provided $N_{\min}(C) \leq N$. In other words, $C$ has norm $N$ provided there is a coordinate $i$ such that $d(x, C^{(i)}_0) + d(x, C^{(i)}_1) \leq N$ for all $x \in Q_n$. A code with covering radius $R$ is **normal** provided it has norm $N = 2R + 1$ and its minimum norm $N_{\min}$ is $2R + 1$ or $2R$, since if a code has norm $N$, its covering radius is $R \leq N/2$. If $N^{(i)} \leq 2R + 1$, then coordinate $i$ is **acceptable** with respect to $2R + 1$. We shall refer to such a code as a symmetric normal $(n, R)$-code, or equivalently a normal $(n, R)$-code. Define $K_\nu(n, R)$ to be the size of the smallest
normal \((n, R)\)-code, \(\nu(n, R) := \frac{K_+(n,R)}{2^n / \binom{2n}{n}}\) to be the optimal density of a normal \((n, R)\)-code, and 
\(\nu^*(R) := \limsup_{n \to \infty} \nu(n, R)\) to be the asymptotic worst-case density of a normal \((n, R)\)-code. By Theorem 4.4.2 of [3] due to Honkala and Hämäläinen, and independently van Wee [8], all optimal \((n, 1)\)-codes with length \(n \geq 3\) are normal. Therefore \(\nu^*(1) = \mu^*(1) = 1\), but it is unknown whether equality holds for \(R > 1\).

The asymmetric norm of a code is newly defined here and is similar to the (symmetric) norm above. Notation which coincides with that of the symmetric norm will be made clear from context. The asymmetric norm of a code \(\mathcal{C}\) of length \(n\) with respect to coordinate \(i\) is

\[
N^{(i)}(\mathcal{C}) := \max \left\{ \max_{x \in (Q_n)_0} \left\{ d^+(x, \mathcal{C}_0^{(i)}) + d^+(x, \mathcal{C}_1^{(i)}) \right\}, \max_{x \in (Q_n)_1} \left\{ 2 \cdot d^+(x, \mathcal{C}_1^{(i)}) + 1 \right\} \right\};
\]

The departure from the definition of the (symmetric) norm with respect to coordinate \(i\) is due to the fact that a vector \(x \in (Q_n)_1^{(i)}\) cannot be covered by any downward directed ball centered in \((Q_n)_0^{(i)}\). The minimum asymmetric norm \(N_{\text{min}}\) of \(\mathcal{C}\) is

\[
N_{\text{min}}(\mathcal{C}) := \min_{i \in [n]} N^{(i)}(\mathcal{C}).
\]

Therefore if a code \(\mathcal{C}\) has asymmetric norm \(N\), there is a coordinate \(i\) such that all words \(x\) with \(x_i = 0\) satisfy \(d^+(x, \mathcal{C}_0^{(i)}) + d^+(x, \mathcal{C}_1^{(i)}) \leq N\), and all words \(x\) with \(x_i = 1\), for which \(d^+(x, \mathcal{C}_0^{(i)}) = \infty\), satisfy \(d^+(x, \mathcal{C}_1^{(i)}) \leq (N - 1)/2\). An \((n, R)^+\)-code is asymmetric normal, or simply normal if the context is clear, provided it has asymmetric norm \(N = 2R + 1\) and its minimum asymmetric norm \(N_{\text{min}}\) is \(2R + 1\) or \(2R\). If \(N^{(i)} \leq 2R + 1\), then coordinate \(i\) is acceptable with respect to \(2R + 1\). Define \(K^+_+(n,R)\) to be the size of the smallest normal \((n, R)^+\)-code, \(\nu_+(n, R) := \frac{K^+_+(n,R)}{2^n / \binom{2n}{n}}\) to be the optimal density of a normal \((n, R)^+\)-code, and \(\nu^*_+(R) := \limsup_{n \to \infty} \nu_+(n, R)\) to be the asymptotic worst-case density of a normal \((n, R)^+\)-code.

### 2.2 Amalgamated direct sum (ADS) of normal codes

Two normal codes can be concatenated in a more efficient construction than the basic direct sum. The construction is the same regardless of whether considering symmetric or asymmetric codes, and so we present the two cases simultaneously in the following theorem, the symmetric case of which is due to Graham and Sloane [8]. The theorem in the symmetric case is often stated in terms of the covering radius, but in this paper the norm is more central to our purpose.

**Theorem 2 (ADS of normal codes).** Let \(A\) be a normal symmetric (asymmetric) code of length \(n_A\) and norm \(N_A\) with the last coordinate acceptable, and let \(B\) be a normal symmetric (asymmetric) code of length \(n_B\) and norm \(N_B\) with the first coordinate acceptable. Then their amalgamated direct sum (ADS)

\[
A \oplus B := \{(a, 0, b) : (a, 0) \in A, (0, b) \in B\} \cup \{(a, 1, b) : (a, 1) \in A, (1, b) \in B\}
\]

is a normal symmetric (asymmetric) code of length \(n_A + n_B - 1\) and norm \(N_A + N_B - 1\) with respect to coordinate \(n_A\).
Proof. The proof of the symmetric case essentially appears in the proof of Theorem 4.1.8 and the remarks following Theorem 4.1.14, both of §. We now adapt the same proof for the asymmetric case, from which the reader may easily reconstruct the symmetric case.

Let \( C = A \odot B \). Then \( C \) clearly has length \( n_A + n_B - 1 \), as it is constructed by overlapping a single coordinate of \( A \) and \( B \). Let \( z \in C \). First suppose \( z = (x, 0, y) \), where \((x, 0) \in Q_{n_A} \). Computing, we have

\[
d^+(z, C_0^{(n_A)}) + d^+(z, C_1^{(n_A)}) \leq d^+((x, 0), A_0^{(n_A)}) + d^+((0, y), B_0^{(1)}) + d^+((x, 0), A_1^{(n_A)}) + d^+((0, y), B_1^{(1)}) - 1 \leq N_A + N_B - 1.
\]

Now suppose \( z = (x, 1, y) \), where \((x, 1) \in Q_{n_A} \). Then we have

\[
2 \cdot d^+(z, C_1^{(n_A)}) + 1 \leq (2 \cdot d^+((x, 1), A_1^{(n_A)}) + 1) + (2 \cdot d^+((1, y), B_1^{(1)}) + 1) - 1 \leq N_A + N_B - 1.
\]

Therefore \( C \) has asymmetric norm \( N_A + N_B - 1 \) with respect to coordinate \( n_A \).

The size of \( A \odot B \) depends on the relative sizes of \( A_0^{(n_A)} \) versus \( A_1^{(n_A)} \) and of \( B_0^{(1)} \) versus \( B_1^{(1)} \). We define a code \( C \) to be balanced if \( |C_0^{(i)}| = |C_1^{(i)}| \), where \( i \) is the coordinate with respect to which the ADS is taken. The major consequence of Theorem 2 for code density is as follows. Two codes \( A \) and \( B \) of lengths \( n_A \) and \( n_B \) and covering radii \( R_A \) and \( R_B \), respectively, form a direct sum of size \( |A| \cdot |B| \), length \( n_A + n_B \), and covering radius \( R_A + R_B \). If in addition both codes are normal and at least one is balanced, their amalgamated direct sum is of size \( |A| \cdot |B|/2 \), length \( n_A + n_B - 1 \), and covering radius at most \( R_A + R_B \). Since

\[
|A| \cdot |B| \left( \frac{n}{2^n} \right) > \frac{|A| \cdot |B|}{2} \left( \frac{n-1}{2^{n-1}} \right) \quad \text{and} \quad |A| \cdot |B| \left( \frac{n/2}{2^n} \right) \geq \frac{|A| \cdot |B|}{2} \left( \frac{(n-1)/2}{2^{n-1}} \right),
\]

the density of the direct sum code is at least as large as that of the corresponding ADS code in both the symmetric and asymmetric case.

2.3 Amalgamated semi-direct sum (ASDS) of normal codes

We now define the central construction of this paper, the amalgamated semi-direct sum. The idea behind this construction is as follows. With length \( n \) fixed, and target norm \( N \) (and implicitly radius \( R \leq N/2 \)), we probabilistically choose a candidate code \( S \). Any strings \( x \in Q_n \) which violate the target norm \( N \) in coordinate \( n \) contribute to a “patch” \( T \). Together, this “patched” code \((S, T)\) can be incorporated into a modified amalgamated direct sum resulting in a longer code with some desired norm, which in turn bounds the covering radius of the resulting code.

More formally, for a fixed \( N > 0 \), a norm \( N \)-patched symmetric code of length \( n \) is a 2-tuple \((S, T)\), where \( S, T \subseteq Q_n \), such that there exists a coordinate \( i \in [n] \) so that for all \( x \in Q_n \) either

(I) \( d(x, S_0^{(i)}) + d(x, S_1^{(i)}) \leq N \), or

(II) \( \{x, x + e_i\} \subseteq T \), where \( x + e_i \) is \( x \) with the \( i \)th coordinate flipped.
When \( N \) and \( n \) are clear from context, the terminology \textit{norm-patched code} may also be used. Any coordinate \( i \) achieving these properties is called \textit{acceptable} for \((S, T)\) with respect to \( N \). If a vector \( v \in Q_n \) violates condition (I), we say it is \textit{missed} by \( S \) with respect to coordinate \( i \). Note that if \((S, T)\) is a norm \( N \)-patched code, then \( S \cup T \) is a normal \((n, R)\)-code with radius \( R \leq \lfloor N/2 \rfloor \).

A \textit{norm \( N \)-patched asymmetric code} of length \( n \) is defined similarly, except that \((S, T)\) must satisfy for some coordinate \( i \in [n] \) the following altered conditions: for all \( x \in (Q_n)_0^{(i)} \), either \( d^+(x, S_0^{(i)}) + d^+(x, S_1^{(i)}) \leq N \) or \( \{x, x+e_i\} \subseteq T \); and for all \( x \in (Q_n)_1^{(i)} \), either \( 2 \cdot d^+(x, S_1^{(i)}) + 1 \leq N \) or \( x \in T \). A vector \( x \in (Q_n)_0^{(i)} \) is \textit{missed} by \( S \) with respect to coordinate \( i \) provided \( d^+(x, S_0^{(i)}) + d^+(x, S_1^{(i)}) > N \), and a vector \( x \in (Q_n)_1^{(i)} \) is \textit{missed} by \( S \) w.r.t. \( i \) provided \( 2 \cdot d^+(x, S_1^{(i)}) + 1 > N \). With these definitions we have the following new theorem.

**Theorem 3 (ASDS of norm-patched and normal codes).** Suppose \((S, T)\) is a norm \( N \)-patched symmetric (asymmetric) code of length \( n \) with coordinate \( n \) acceptable, \( K_1 \) is a symmetric (asymmetric) code of length \( n' \) and norm \( N' \) with first coordinate acceptable, and \( K_2 \) is a symmetric (asymmetric) code of length \( n' \) and norm \( N + N' - 1 \) with first coordinate acceptable. Then the amalgamated semi-direct sum

\[
(S, T) \natural K_1 \natural K_2 := (S \natural K_1) \cup (T \natural K_2)
\]

is a symmetric (asymmetric) code of length \( n + n' - 1 \) and norm \( N + N' - 1 \) with coordinate \( n \) acceptable.

**Proof.** First, consider the symmetric case. Define \( C = (S, T) \natural K_1 \natural K_2 \) and let \( z \in Q_{n+n'-1} \). Suppose \( z = (x, 0, y) \) where \((x, 0) \in Q_n \). If \( d((x, 0), S_0^{(n)}) + d((x, 0), S_1^{(n)}) \leq N \), then we have

\[
d(z, C_0^{(n)}) + d(z, C_1^{(n)}) \leq d((x, 0), S_0^{(n)}) + d((0, y), (K_1)_0^{(1)}) + d((x, 0), S_1^{(n)}) + d((0, y), (K_1)_1^{(1)}) - 1 \\
\leq N + N' - 1.
\]

Otherwise we must have \( \{(x, 0), (x, 1)\} \subseteq T \), so that

\[
d(z, C_0^{(n)}) + d(z, C_1^{(n)}) \leq d((x, 0), T_0^{(n)}) + d((0, y), (K_2)_0^{(1)}) + d((x, 0), T_1^{(n)}) + d((0, y), (K_2)_1^{(1)}) - 1 \\
\leq 1 + (N + N' - 1) - 1 = N + N' - 1.
\]

That \( d(z, C_0^{(n)}) + d(z, C_1^{(n)}) \leq N + N' - 1 \) when \( z \) is of the form \((x, 1, y)\) follows by an analogous verification, proving the theorem in the symmetric case.

For the asymmetric case, the proof that any \( z \) of the form \((x, 0, y)\) for \((x, 0) \in Q_n \) satisfies \( d^+(z, C_0^{(n)}) + d^+(z, C_1^{(n)}) \leq N + N' - 1 \) is nearly identical to the symmetric case and is omitted. Now suppose \( z \) is of the form \((x, 1, y)\) where \((x, 1) \in Q_n \). If \( 2 \cdot d^+(x, 1, C_1^{(n)}) + 1 \leq N \), then

\[
2 \cdot d^+(z, C_1^{(n)}) + 1 \leq (2 \cdot d^+(x, 1, C_1^{(n)}) + 1) + (2 \cdot d^+(1, y, (K_1)_1^{(1)}) + 1) - 1 \\
\leq N + N' - 1.
\]
Otherwise we must have \((x, 1) \in T\), so that
\[
2 \cdot d^+(z, C_1^{(n)}) + 1 \leq 2 \cdot d^+((x, 1), T_1^{(n)}) + (2 \cdot d^+((1, y), (K_2)_1^{(1)}) + 1)
\leq N + N' - 1;
\]
therefore the theorem also holds in the asymmetric case.

Again, we chose to present the theorem in terms of norms of codes rather than radii to suit our purpose in developing the main density theorems of the next two sections. Additionally, it will be convenient to choose \(S\) and \(T\) to be balanced with respect to the acceptable coordinate, so that the size of the resulting ASDS can be readily determined.

### 3 Asymptotic density of normal symmetric codes

We now present the main theorem on the asymptotic worst-case density of constant radius normal symmetric codes. The framework and analysis of the theorem borrows from that of Theorem 1.2 (and Corollaries 1.3-1.4) of [7] in the following sense. We develop here a more careful probabilistic deletion method in Lemma 5 for selecting a norm-patched code \((S, T)\), which is tailored for normal codes and our ASDS construction. We must also compute a preliminary asymptotic bound on the sizes of \(|S|\) and \(|T|\) in Corollary 6 before employing a recursive ASDS construction. We then adapt Theorem 1.2 of [7] and its supporting analysis from the setting of unrestricted codes and the so-called semi-direct sum, to the case of normal codes and our ASDS construction, in order to obtain the main density theorem on \(\nu^*(R)\). The proof of Theorem 4 follows these supporting results.

**Theorem 4.** Let \(R \geq 2\). Then
\[
\nu^*(R) \leq e(R \log R + \log \log R + 4).
\]

**Lemma 5 (Selection of a norm-patched code).** For every positive constant \(x\) and positive integer \(N \leq n\), there exist (disjoint) sets \(S_0 \subseteq (Q_n)_0^{(n)}\) and \(S_1 \subseteq (Q_n)_1^{(n)}\) each of size at most
\[
x2^{n-1} \frac{b_{n-1}(\lfloor \frac{N-1}{2} \rfloor) + b_{n-1}(\lceil \frac{N-1}{2} \rceil - 1)}{b_{n-1}(\lfloor \frac{N-1}{2} \rfloor) + b_{n-1}(\lceil \frac{N-1}{2} \rceil - 1)}
\]
and a set \(T \subseteq Q_n\) of size at most \(\tau(n, N, x) :=
\[
2^{n+1} \sum_{i=0}^{N-1} \exp \left(-x \frac{b_{n-1}(i - 1) + b_{n-1}(N - i - 1)}{b_{n-1}(\lfloor \frac{N-1}{2} \rfloor) + b_{n-1}(\lceil \frac{N-1}{2} \rceil - 1)} + \frac{b_{n-1}(i - 1) + b_{n-1}(N - i - 1)}{2^{n-1}} \right)
+ 2^{n+1} \exp \left(-x \frac{b_{n-1}(N - 1)}{b_{n-1}(\lfloor \frac{N-1}{2} \rfloor) + b_{n-1}(\lceil \frac{N-1}{2} \rceil - 1)} + \frac{b_{n-1}(N - 1)}{2^{n-1}} \right),
\]
\(1\)
such that \((S_0 \cup S_1, T)\) is a balanced norm \(N\)-patched symmetric code.
Proof. Let
\[
k = \left\lceil \frac{x 2^{n-1}}{b_{n-1} \left\lceil \frac{n-1}{2} \right\rceil + b_{n-1} \left\lfloor \frac{n-1}{2} \right\rfloor - 1} \right\rceil,
\]
and choose uniformly at random subsets \( S_0 \subseteq (Q_n)_0^{(n)} \) and \( S_1 \subseteq (Q_n)_1^{(n)} \) each of size \( k \). A vector \( v \in Q_n \) is missed by \( S \) if \( d(v, S_0) + d(v, S_1) > N \); otherwise, there exists an \( i \in \{0, 1, \ldots, N\} \) such that \( d(v, S_0) = i \) and \( d(v, S_1) \leq N - i \). For \( b \in \{0, 1\} \) classify the missed vertices as follows:
\[
B_{b,1} := \{ u \in (Q_n)_b^{(n)} : 0 \leq d(u, S_b) < N \text{ and } d(u, S_{1-b}) > N - d(u, S_b) \}
\]
\[
B_{b,2} := \{ u \in (Q_n)_b^{(n)} : d(u, S_b) \geq N \}.
\]
Let the patch be the balanced set
\[
T = \bigcup_{b \in \{0, 1\}} (B_{b,1} \cup B_{b,2}) + \{0, e_n\},
\]
where addition is done by taking all possible combinations of one vector from each set and adding coordinate-wise mod 2. Thus \( T \) contains all missed vertices, and \( S \cup T \) is a norm \( N \)-patched code.

By linearity of expectation and symmetry with respect to the \( n \)th coordinate,
\[
\mathbb{E}(|T|) = \sum_{b \in \{0, 1\}} 2\mathbb{E}(|B_{b,1}|) + 2\mathbb{E}(|B_{b,2}|) = 4\mathbb{E}(|B_{0,1}|) + 4\mathbb{E}(|B_{0,2}|)
\]
\[
= 4 \cdot 2^{n-1} \sum_{i=0}^{N-1} \Pr[d(v, S_0) = i \mid v \in (Q_n)_0^{(n)}] \cdot \Pr[d(v, S_1) > N - i \mid v \in (Q_n)_0^{(n)}] + 4 \cdot 2^{n-1} \Pr[d(v, S_0) \geq N \mid v \in (Q_n)_0^{(n)}].
\]

For \( v \in (Q_n)_0^{(n)}, \Pr[d(v, S_0) = i] = \Pr[d(v, S_0) > i - 1] - \Pr[d(v, S_0) > i], \) and for fixed \( i, \Pr[d(v, S_0) > i - 1] \) dominates \( \Pr[d(v, S_0) > i] \) as \( n \to \infty \); therefore we estimate \( \mathbb{E}(|T|) \) by
\[
\mathbb{E}(|T|) \leq 2^{n+1} \sum_{i=0}^{N-1} \Pr[d(v, S_0) > i - 1 \mid v \in (Q_n)_0^{(n)}] \cdot \Pr[d(v, S_1) > N - i \mid v \in (Q_n)_0^{(n)}] + 2^{n+1} \Pr[d(v, S_0) \geq N \mid v \in (Q_n)_0^{(n)}]. \tag{2}
\]

Suppose \( 0 \leq j < N \). For \( v \in (Q_n)_0^{(n)}, \) if \( d(v, S_0) \) is to be more than \( j \), then \( S_0 \) must not contain any of the vertices in \( B_n(v, j) \cap (Q_n)_0^{(n)} \). This intersection can be reached from \( v \) by fixing the \( n \)th coordinate of \( v \) and changing at most \( j \) of the remaining \( n - 1 \) coordinates. In particular, \( |B_n(v, j) \cap (Q_n)_0^{(n)}| = b_{n-1}(j) \). Along with the corresponding computation for \( d(v, S_1) \), we have
\[
\Pr[d(v, S_0) > j \mid v \in (Q_n)_0^{(n)}] = \binom{2^{n-1} - b_{n-1}(j)}{k} \binom{2^{n-1}}{k}, \quad \text{and}
\]
\[
\Pr[d(v, S_1) > j \mid v \in (Q_n)_0^{(n)}] = \binom{2^{n-1} - b_{n-1}(j - 1)}{k} \binom{2^{n-1}}{k}.
\]

Now the bound on \( \mathbb{E}(|T|) \) in (2) becomes
\[
\mathbb{E}(|T|) \leq 2^{n+1} \frac{2^{n-1}}{k} \sum_{i=0}^{N-1} \frac{2^{n-1} - b_{n-1}(i - 1)}{k} \frac{2^{n-1} - b_{n-1}(N - i - 1)}{k}
\]
\[
= 2^{n+1} \frac{2^{n-1}}{k} \sum_{i=0}^{N-1} \frac{2^{n-1} - b_{n-1}(i - 1)}{k} \frac{2^{n-1} - b_{n-1}(N - i - 1)}{k}.
\]
\[ +2^{n+1} \binom{2^{n-1}}{k}^{-1} \left( \frac{2^{n-1} - b_{n-1}(N - 1)}{k} \right). \]  

Using the estimate

\[ \frac{(m-d)}{m} = \frac{(m-d) \cdots (m-d-k+1)}{m \cdots (m-k+1)} \leq \left( \frac{m-d}{m} \right)^k = \left( 1 - \frac{d}{m} \right)^k \leq e^{-k \frac{d}{m}} \]  

borrowed from the proof of Lemma 2.2 of [7], (3) becomes

\[ \mathbb{E}(|T|) \leq 2^{n+1} \sum_{i=0}^{N-1} \exp \left( -k \frac{b_{n-1}(i-1) + b_{n-1}(N-i-1)}{2^{n-1}} \right) \]

\[ + 2^{n+1} \exp \left( -k \frac{b_{n-1}(N-1)}{2^{n-1}} \right) \]

\[ \leq \tau(n,N,x). \]

Since there exists a $T$ of size at most $\mathbb{E}(|T|)$, the result follows. \[ \square \]

In practice, what is important is the expected size of the patch $T$ as $n \to \infty$. We have the following asymptotic upper bounds on $|S|$, and on $|T|$ via $\tau(n,N,x)$.

**Corollary 6.** Let $N \geq 2$ be fixed. Then the asymptotic size of $S := S_0 \cup S_1$ in Lemma 5 is

\[ |S| \sim \begin{cases} 
\frac{2n^{2n-1}}{b_{n-1}(N-1)/2}, & \text{if } N \text{ is odd} \\
\frac{2n^{2n-1}}{b_{n-1}(N/2-1)}, & \text{if } N \text{ is even}, 
\end{cases} \]

and the size of the patch $T$ is bounded above asymptotically by

\[ \tau(n,N,x) \sim \begin{cases} 
2^{n+2} e^{-x}, & \text{if } N \text{ is odd} \\
2^{n+1} e^{-x}, & \text{if } N \text{ is even}. 
\end{cases} \]

**Proof.** The calculation for $|S|$ is easily verified. For the size of $T$, note that for constant $R$ the asymptotic size of an $R$-ball in $Q_n$ is $b_n(R) \sim n^R/R!$. The proof proceeds by identifying which exponential terms $\exp(\cdot)$ in (11) are not swallowed in the limit. If $N$ is odd, then \( \left\lfloor \frac{N-1}{2} \right\rfloor = \frac{N-1}{2} \) and \( \left\lceil \frac{N-1}{2} \right\rceil - 1 = \frac{N-1}{2} - 1 \). The only terms which survive are the \( i = \left\lfloor \frac{N-1}{2} \right\rfloor, \left\lceil \frac{N-1}{2} \right\rceil + 1 \) terms of the summation in (11), which each converge to $\exp(-x)$. If $N$ is even, then \( \left\lfloor \frac{N-1}{2} \right\rfloor = \left\lceil \frac{N-1}{2} \right\rceil - 1 = \frac{N}{2} - 1 \), and the only exponential term of the summation in (11) which does not vanish corresponds to $i = \frac{N}{2}$, and also converges to $\exp(-x)$. For all other exponential terms in both cases, the numerator dominates since at least one of the two balls has radius larger than $\max\{ \left\lfloor \frac{N-1}{2} \right\rfloor, \left\lceil \frac{N-1}{2} \right\rceil - 1 \}$. \[ \square \]

The following technical lemma, due to Krivelevich, Sudakov, and Vu [7, Lemma 2.1], allows a tight analysis of the upper bound on $\nu^*(R)$ given by a recursive ASDS construction. We quote the lemma without proof and then continue to the proof of the main theorem in the symmetric case.

\[ 9 \]
Lemma 7 (Krivelevich, Sudakov, Vu). Let \((f_n), (a_n), (b_n)\) and \(s_n\) be sequences of positive numbers where

\[
\limsup_{n \to \infty} f_n \leq f, \quad \limsup_{n \to \infty} a_n \leq a, \quad \limsup_{n \to \infty} b_n \leq b < 1
\]

and

\[
s_n \leq a_n f_{[n/y]} + b_n s_{[n/y]}
\]

where \(y > 1\) is a constant. Then

\[
\limsup_{n \to \infty} s_n \leq \frac{af}{1 - b}
\]

Proof of Theorem 4. Let \(n\) be sufficiently large \((n \geq R\) suffices), and let \(n_1 = \lfloor n/R \rfloor\) and \(n_1' = n - n_1 + 1\). The selection of these particular parameters in the bounding of \(\mu^*(R)\) is due to \[7\], and we find them to be suitable for the ASDS construction as well. We use Lemma 5 to select a length \(n_1'\) balanced norm \((2R - 1)\)-patched code \((S, T)\), where \(|S|\) and \(|T|\) are bounded above as given in the lemma. Let \(K_1\) be an optimal normal \((n_1, 1)\)-code, and let \(K_2\) be an optimal normal \((n_1, R)\)-code. Now perform the ASDS of \((S, T)\) with \((K_1, K_2)\). By Theorem 3 the resulting code is length \(n\) and has norm \(2R + 1\), and so has covering radius at most \(R\). Therefore there exists a normal \((n, R)\)-code with size at most \(|(S, T)\cap (K_1, K_2)|\), and the optimal density of such a code is

\[
\nu(n, R) \leq \left( \frac{|S|K_1}{2} + \frac{|T||K_2|}{2} \right) \left( \frac{n}{2n_1} \right)
\]

\[
\leq \frac{1}{2} \frac{x^{2n_1'} - 1}{b_{n_1'-1}(R - 1) + b_{n_1'-1}(R - 2)} \nu(n_1, 1) \left( \frac{n}{2n_1} \right) + \frac{\tau(n_1', 2R - 1, x)}{2} \nu(n, R) \left( \frac{n}{2n_1} \right)
\]

Define \(s_n := \nu(n, R), f_n := \nu(n, 1),\)

\[
a_n := \frac{1}{2} \frac{x^{2n_1'} - 1}{b_{n_1'-1}(R - 1) + b_{n_1'-1}(R - 2)} \left( \frac{n}{2n_1} \right), \quad b_n := \frac{1}{2} \frac{x^{2n_1'} - 1}{b_{n_1'-1}(R - 1) + b_{n_1'-1}(R - 2)} \left( \frac{n}{2n_1} \right) \nu(n_1', 2R - 1, x);
\]

note that

\[
\limsup_{n \to \infty} a_n = x \left( \frac{R}{R - 1} \right)^{R-1} \leq ex, \quad \text{and} \quad \limsup_{n \to \infty} b_n = 4R e^{-x}, \quad (5)
\]

by Corollary 6. Therefore by Lemma 7 when \(4R e^{-x} < 1\), we have

\[
\nu^*(R) \leq \frac{ex}{1 - 4e^{-x}R^R} \nu^*(1).
\]

Setting \(f(x) = \frac{ex}{1 - 4e^{-x}R^R}\) and minimizing over \(x > 0\) such that \(4e^{-x}R^R < 1\), the derivative of \(f\) is

\[
f'(x) = \frac{e(1 - 4(1 + x)e^{-x}R^R)}{(1 - 4e^{-x}R^R)^2}.
\]

The numerator \((1 - 4(1 + x)e^{-x}R^R)\) has two roots, one positive and one negative, and \(f(x)\) reaches its minimum at the positive root. Let this root be \(x_0\), for which \(4e^{-x_0}R^R = \frac{1}{(1 + x_0)}\), and so

\[
\nu^*(R) \leq e(x_0 + 1) \nu^*(1).
\]
Since \((1 - 4(1 + x)e^{-x}R^R)\) is negative on \([0, x_0]\) and increasing at \(x_0\), we can bound \(x_0\) slightly above by choosing an approximation for \(x_0\) which yields a positive value in the numerator of \(f'(x)\). Choosing \(x_0 = (R \log R + \log R + \log \log R + 3)\) ensures for \(R \geq 2\) that \(e^{x_0} > 4(1 + x_0)R^R\). By Theorem 4.4.2 in [3], all optimal \((n, 1)\)-codes with length \(n \geq 3\) are normal; and by Theorem 12.4.11 in [3], \(\mu^*(1) = 1\); these results allow the replacement of \(\nu^*(1)\) with 1 to obtain the desired result. \(\square\)

4 Asymptotic density of normal asymmetric codes

We now present the asymmetric version of Theorem [3] that is, a bound on the asymptotic worst-case density of constant radius normal asymmetric codes. The proof proceeds along the lines of that of the symmetric case, with the most notable deviation occurring in the probabilistic selection of the norm-patched asymmetric code \((S, T)\) due to a more complicated definition of \(T\). However, we obtain a simplified asymptotic upper bound on \(|T|\) which allows us to employ the same analysis on the recursive ASDS construction as before. The proof of Theorem [3] follows that of Corollary [10].

Theorem 8. Let \(R \geq 2\). Then

\[
\nu^+_*(R) \leq e(R \log R + \log R + \log \log R + 4).
\]

Because of the asymmetry of the covering condition for \((n, R)^+\)-codes, we prefer to concentrate on the vast majority of vertices of \(Q_n\) which have weight close to \(n/2\). Define a vector \(u \in Q_n\) to be rare if \(|w(u) - n/2| > \sqrt{2(R + 1)n \ln n}\), and define

\[
hi(n, R) := \min \{n, \lceil (n + \sqrt{2(R + 1)n \ln n})/2 \rceil\}, \quad \text{and}\quad
lo(n, R) := \max \{0, \lfloor (n - \sqrt{2(R + 1)n \ln n})/2 \rfloor\}.
\]

Then the set of rare vectors of \(Q_n\) (with respect to asymmetric radius \(R\)) is

\[
Q_n^{\text{rare}} := \{ u \in Q_n : w(u) < lo(n, R) \text{ or } w(u) > hi(n, R) \}.
\]

The Chernoff bound states that the number of vertices \(u \in Q_n\) with \(w(u) > (n + \sqrt{j \cdot n \ln n})/2\) is at most \(2^n n^{-j^2/2}\) (cf. [13] Theorem A.1.1). Thus \(|Q_n^{\text{rare}}| < 2^{n+1} n^{-R-1} \in O(2^n n^{-R-1})\), which would have density \(O(1/n)\) as a \((n, R)^+\)-code, except that for all but finitely many \(n\), \(|Q_n^{\text{rare}}|\) doesn’t downward \(R\)-cover \(Q_n\).

Lemma 9 (Selection of a norm-patched asymmetric code). For every positive constant \(x\) and for positive integers \(N \leq n\), there exist (disjoint) sets \(S_0 \subseteq (Q_n)^{(n)}\) and \(S_1 \subseteq (Q_n)^{(n)}\) each of size at most

\[
x 2^{n-1} b_{n-1}^+ (hi(n, R), \lceil \frac{N-1}{2} \rceil) + b_{n-1}^+ (hi(n, R), \lfloor \frac{N-1}{2} \rfloor - 1),
\]

and a set \(T \subseteq Q_n\) of size at most \(\tau^+(n, N, x) := O(2^n n^{-R-1}) + 2^n \sum_{i=0}^{N-1} \exp \left( -x \frac{b_{n-1}^+ (hi(n, R), i - 1) + b_{n-1}^+ (hi(n, R), N - i - 1)}{b_{n-1}^+ (hi(n, R), \lceil \frac{N-1}{2} \rceil) + b_{n-1}^+ (hi(n, R), \lfloor \frac{N-1}{2} \rfloor - 1)} \right)\)
such that \((S_0 \cup S_1, T)\) is a balanced norm \(N\)-patched asymmetric code.

**Proof.** Let

\[
k = \left\lfloor \frac{x 2^{n-1}}{b_{n-1}^+(hi(n, R), \left\lfloor \frac{N}{2} \right\rfloor) + b_{n-1}^+(hi(n, R), \left\lceil \frac{N}{2} \right\rceil)} \right\rfloor,
\]

and choose uniformly at random subsets \(S_0 \subseteq (Q_n)_0^{(n)}\) and \(S_1 \subseteq (Q_n)_1^{(n)}\) each of size \(k\). A vector \(v \in (Q_n)_0^{(n)}\) is missed by \(S\) if \(d^+(v, S_0) + d^+(v, S_1) > N\); otherwise, there exists an \(i \in \{0, 1, \ldots, N\}\) such that \(d^+(v, S_0) = i\) and \(d^+(v, S_1) \leq N - i\). A vector \(v \in (Q_n)_1^{(n)}\) is missed by \(S\) provided \(2d^+(v, S_1) + 1 > N\). We classify the missed vertices as follows:

\[
B_{0,1}^+ := \{ u \in (Q_n)_0^{(n)} \setminus Q_n^{\text{rare}} : 0 \leq d^+(u, S_0) < N \text{ and } d^+(u, S_1) > N - d^+(u, S_0) \}
\]

\[
B_{0,2}^+ := \{ u \in (Q_n)_0^{(n)} \setminus Q_n^{\text{rare}} : d^+(u, S_0) \geq N \}
\]

\[
B_1^+ := \{ u \in (Q_n)_1^{(n)} \setminus Q_n^{\text{rare}} : 2d^+(u, S_1) + 1 > N \}.
\]

Let the patch be the balanced set

\[
T = Q_n^{\text{rare}} \cup \left( (B_{0,1}^+ \cup B_{0,2}^+ \cup B_1^+) + (e_n \cup 0) \right);
\]

then \(T\) contains all missed vectors, and \((S, T)\) is a balanced norm \(N\)-patched asymmetric code. By linearity of expectation,

\[
\mathbb{E}(|T|) = |Q_n^{\text{rare}}| + 2\mathbb{E}(|B_{0,1}^+|) + 2\mathbb{E}(|B_{0,2}^+|) + 2\mathbb{E}(|B_1^+|)
\]

\[
= O(2^n n^{-R-1}) + 2 \sum_{v \in (Q_n)_0^{(n)} \setminus Q_n^{\text{rare}}} N-1 \sum_{i=0}^N \Pr[d^+(v, S_0) = i] \cdot \Pr[d^+(v, S_1) > N - i]
\]

\[
+ 2 \sum_{v \in (Q_n)_0^{(n)} \setminus Q_n^{\text{rare}}} \Pr[d^+(v, S_0) \geq N] + 2 \sum_{v \in (Q_n)_1^{(n)} \setminus Q_n^{\text{rare}}} \Pr[2d^+(v, S_1) + 1 > N].
\]

Similar to the symmetric case, replacing \(\Pr[d^+(v, S_0) = i]\) above with \(\Pr[d^+(v, S_0) > i - 1]\) yields a good upper bound for \(\mathbb{E}(|T|)\). Using the definition of asymmetric distance, for any vector \(v \in (Q_n)_0^{(n)}\) of weight \(l\) and any \(i\),

\[
\Pr[d^+(v, S_0) > i] = \left(2^{n-1} - b_{n-1}^+(l, i)\right) / \left(\binom{2^{n-1}}{k}\right),
\]

and
Again using the estimate \( \binom{m-d}{k}/\binom{m}{k} \leq e^{-kd/m} \) in (4), this allows a regrouping of the expression for \( \mathbb{E}(|T|) \) by weight of \( v \). We have

\[
\mathbb{E}(|T|) \leq O(2^n n^{-R-1}) + 2^{hi(n,R)} \sum_{l=lo(n,R)}^{n-1} \frac{n-1}{l} \left( \sum_{i=0}^{N-1} \exp \left(-k \frac{b_{n-1}^+(l, i-1) - b_{n-1}^+(l, N-i-1)}{2^{n-1}} \right) \right)
\]

\[
+ \exp \left(-k \frac{b_{n-1}^+(l, N-1)}{2^{n-1}} \right) + 2^{hi(n,R)} \sum_{l=lo(n,R)}^{n-1} \frac{n-1}{l-1} \exp \left(-k \frac{b_{n-1}^+(l-1, \frac{N-l}{2})}{2^{n-1}} \right)
\]

\[
\leq O(2^n n^{-R-1}) + 2^n \left[ \sum_{i=0}^{N-1} \exp \left(-k \frac{b_{n-1}^+(hi(n,R), i-1) + b_{n-1}^+(hi(n,R), N-i-1)}{2^{n-1}} \right) \right]
\]

\[
+ \exp \left(-k \frac{b_{n-1}^+(hi(n,R), N-1)}{2^{n-1}} \right) + 2^n \exp \left(-k \frac{b_{n-1}^+(hi(n,R) - 1, \frac{N-1}{2})}{2^{n-1}} \right)
\]

\[
\leq \tau^+(n, N, x).
\]

Since there exists a \( T \) with size at most \( \mathbb{E}(|T|) \), the result follows. \( \square \)

Just as in the symmetric case, what is important about Lemma 9 is the asymptotic behavior of \( |T| \) as \( n \) tends to infinity. Accordingly, we have the following corollary.

**Corollary 10.** Let \( N \geq 2 \) be fixed. Then the asymptotic size of \( S := S_0 \cup S_1 \) in Lemma 9 is

\[
|S| \sim \begin{cases} 
\frac{2^n}{(n/2)^R}, & \text{if } N = 2R + 1 \text{ is odd} \\
\frac{2^n}{(n/2)^{R-1}/(R-1)!}, & \text{if } N = 2R \text{ is even},
\end{cases}
\]

and the size of the patch \( T \) is bounded above asymptotically by

\[
\tau^+(n, N, x) \sim \begin{cases} 
3 \cdot 2^n e^{-x}, & \text{if } N = 2R + 1 \text{ is odd} \\
2^n (e^{-x} + e^{-x/2}), & \text{if } N = 2R \text{ is even}.
\end{cases}
\]

**Proof.** The asymptotic size of an upward asymmetric \( R \)-ball \( B_n^+(v, R) \) for constant \( R \) where \( w(v) = hi(n, R) \) is

\[
b_n^+(hi(n, R), R) = \binom{lo(n,R)}{\leq R} \sim \frac{(n/2)^R}{R!}.
\]

The calculation for \( |S| \) is now easily verified. The proof of the bound for \( T \) proceeds, similarly to the proof of Cor. 9 by identifying what exponential terms \( \exp(\cdot) \) in (6) are not swallowed in the limit. \( \square \)
Proof of Theorem \[\text{IX}\] Let \( n \geq R \), \( n_1 = \lfloor n/R \rfloor \), and \( n'_1 = n - n_1 + 1 \) as in the proof of Theorem \[\text{IV}\]. We use Lemma \[\text{IX}\] to select a length \( n'_1 \) balanced norm \((2R - 1)\)-patched asymmetric code \((S,T)\), where \(|S|\) and \(|T|\) are bounded above as given in the lemma. Let \( K_1 \) be an optimal normal \((n_1,1)^+\)-code, and let \( K_2 \) be an optimal normal \((n_1,R)^+\)-code. By Theorem \[\text{IX}\] the ASDS of \((S,T)\) with \((K_1,K_2)\) has length \( n \) and norm \( 2R + 1 \). Therefore there exists a normal \((n,R)^+\)-code with size at most \(|(S,T)| \geq (K_1,K_2)|\), and so

\[
\nu_+(n,R) \leq \left( \frac{|S||K_1|}{2} + \frac{|T||K_2|}{2} \right) \left( \frac{|n/2|}{2R} \right)^{2n}
\]

\[
\leq \frac{1}{2} b_{n'_1-1}^+ (hi(n'_1,R - 1), R - 1) + b_{n'_1-1}^+ (hi(n'_1,R - 1), R - 2) \nu_+(n_1,1) \frac{2^{n_1} (\lfloor n/2 \rfloor)}{(\lfloor n/2 \rfloor)^2} 2^n.
\]

Define \( s_n := \nu_+(n,R) \), \( f_n := \nu_+(n,1) \),

\[
a_n := \frac{1}{2} b_{n'_1-1}^+ (hi(n'_1,R - 1), R - 1) + b_{n'_1-1}^+ (hi(n'_1,R - 1), R - 2) \frac{2^{n_1} (\lfloor n/2 \rfloor)}{(\lfloor n/2 \rfloor)^2} 2^n, \text{ and } b_n := \frac{1}{2} (\lfloor n/2 \rfloor)^2 \tau^+(n'_1, 2R - 1, x);
\]

note that

\[
\limsup_{n \to \infty} a_n = x \left( \frac{R}{R - 1} \right)^{R-1} \leq ex, \text{ and } \limsup_{n \to \infty} b_n = 3R^e e^{-x},
\]

by Corollary \[\text{X}\]. Therefore by Lemma \[\text{X}\] when \( 3R^e e^{-x} < 1 \), we have

\[
\nu_+(R) \leq \frac{ex}{1 - 3e^{-x}R^e} \nu_+(1).
\]

Similar to the proof of Theorem \[\text{IV}\] letting \( x_0 = R \log R + \log R + \log \log R + 3 \) ensures for \( R \geq 2 \) that the denominator of the right-hand side is positive, and gives the desired result. \( \Box \)

5 Open questions

The primary open question, in the author’s opinion, is the value of \( \mu_+^*(1) \), the asymptotic worst-case density of radius 1 asymmetric covering codes, for which we believe no respectable upper bound has been published. This question is likely to be quite hard, as it is related to the question of finding covering numbers, specifically, the smallest number of \( l \)-subsets of \([n]\) which contain all \((l-1)\)-subsets of \([n]\) (cf. \[\text{I}\]). A more routine open question is to determine for which values of \( n \) and \( R \) the ADS or ASDS constructions yield best known upper bounds on \( K^+(n,R) \). In general, the best known lower and upper bounds on \( K^+(n,R) \) (see \[\text{III} \] \[\text{II} \] \[\text{V}\]) are still open to significant improvement.
Acknowledgement

Thanks are due to Iiro Honkala and Simon Litsyn for assistance in identifying previous results.

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