Quantum total detection probability from repeated measurements I.  
The bright and dark states

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We investigate a form of quantum search, where a detector repeatedly probes some quantum particle with fixed rate $1/\tau$ until it is first successful. This is a quantum version of the first-passage problem. We focus on the total probability, $P_{\text{det}}$, that the particle is eventually detected in some state, for example on a node $r_d$ on a graph, after an arbitrary number of detection attempts. For finite graphs, and more generally for systems with a discrete spectrum, we provide an explicit formula for $P_{\text{det}}$ in terms of the energy eigenstates. We provide two derivations of this formula, one proceeding directly from the renewal equation for the generating function of the detection amplitude and a second based on a study of the so-called “dark states.” The dark states are those that are never detected, and constitute a subspace of the Hilbert space. All states orthogonal to the dark space in finite systems are in fact “bright”, with $P_{\text{det}} = 1$. Dark states can arise either from degeneracies of the energy spectrum, or from energy levels that have no projection on the detection state. We demonstrate how breaking the degeneracy infinitesimally can restore $P_{\text{det}} = 1$. For finite systems, it is found that $P_{\text{det}}$ is independent of the measurement frequency $1/\tau$, except for special resonant values. Our formula for $P_{\text{det}}$ fails for infinite systems, in which case the result is only an upper bound of $P_{\text{det}}$. We show how this breakdown occurs in the case of the infinite line. In two follow-ons to this paper, we unravel the relation between $P_{\text{det}}$ and the system’s symmetry.

I. INTRODUCTION

Recently the first detection problem for quantum dynamics has attracted increasing interest [1–20], due in part to its potential relevance for the readout of certain quantum computations. More fundamentally, it sheds light on hitting time processes and measurement theory in quantum theory [21–25]. The classical counterpart of this topic is the first passage time problem, which has a vast number of applications in many fields of science [26–31]. In its simplest guise, a classical random walker initially located on a particular vertex of a graph is considered, and the question of interest is: When will the particle arrive at a target state, for instance another vertex of the graph, for the first time? For the quantum system, we investigate unitary evolution on a graph, with a particle in an initial state $|\psi_{\text{in}}\rangle$, which could be, e.g., a localized state on a vertex $|r_{\text{in}}\rangle$. This evolution is repeatedly perturbed by detection attempts for another state $|\psi_d\rangle$ called the detection state (for example, another localized vertex state of the system $|r_d\rangle$, see below). In this situation, the concept of first arrival is not meaningful, but we can register the first detected arrival time. The protocol of measurement (i.e., the epochs of the detection attempts) crucially determines this first detection time [18]. We consider a stroboscopic protocol, i.e., a sequence of identical measurements with fixed inter-attempt time $\tau$, continued until the first successful detection (see definitions below). One of the general aims in this direction of research is to gain information on the statistics of this event, which is inherently random by the basic laws of quantum mechanics. Our approach explicitly incorporates repeated, strong measurements into the definition of the first detected arrival, and therein differs from other quantum search setups [1, 32–49] and from the time-of-arrival problem [50–58].

In some cases, quantum search is by far more efficient than its classical counterpart. In particular for the hypercube and for certain trees it was shown that quantum search can be exponentially faster than possible classically [2, 59–61]. Indeed, while the classical random walker repeatedly re-samples its trajectory, a quantum walker may benefit from the constructive interference of its wave function. This mechanism enables a quantum walker to achieve much faster detection times than his classical counterpart. In the same way, however, certain initial conditions suffer from destructive interference, such that the desired state is never detected and yields a diverging mean detection time [2–4, 6, 18]. We call such initial conditions dark states. The classical walk, if the process is ergodic, does not possess dark states and hence in this sense performs “better”, since detection on an ergodic finite graph is guaranteed. In the quantum problem, the presence of the detector splits up the total Hilbert space into a dark space [2, 7, 62] and its complement. These play the roles of the ergodic components in a classical random walk. In contrast to the classical situation, they are not generated by a separation of state space (alone), but rather by destructive interference.

The main focus of this series of papers is the total detection probability $P_{\text{det}}$. This is the probability to detect the particle eventually, irrespective of the number of attempts. In a finite system, if the initial and detection states coincide, $P_{\text{det}}$ is always unity [9]. However, when the initial state differs from the detection state, the initial state can have an overlap with some dark states, which are undetectable, and the overlap of the initial state with the dark space gives no contribution.

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to the $P_{\text{det}}$. We derive an explicit formula for $P_{\text{det}}$ in terms of the eigenstates $|E_{l,m}\rangle$ of the unitary propagator $\hat{U}(\tau) := e^{-i\tau\hat{H}/\hbar}$ (or, the Hamiltonian $\hat{H}$), the $n$th state with quasienergy $\lambda_l = E_l/\hbar$ mod $2\pi$, such that 

$$\hat{U}(\tau)|E_{l,m}\rangle = e^{-i\lambda_l\tau}|E_{l,m}\rangle; \quad P_{\text{det}}(\psi_{\text{in}}) = \sum_t |\sum_{m=1}^{g_l} \langle \psi_{\text{d}}|E_{l,m}\rangle \langle E_{l,m}|\psi_{\text{in}}\rangle|^2 / \sum_{m=1}^{g_l} |\langle E_{l,m}|\psi_{\text{d}}\rangle|^2. \quad (1)$$

For most values of $\tau$, there is a one-to-one correspondence between the set of values of $E_l$ and those of $\lambda_l$. Only, if there are one or more pairs of energies $\{E_k, E_l\}$, such that the resonance condition

$$(E_k - E_l)\tau/\hbar = 0 \mod 2\pi \quad (2)$$

is satisfied are there fewer quasienergies than energies. The primed sum runs over those distinct quasienergy sectors that have non-zero overlap with the detected state, (i.e., excluding any completely dark sectors) with the inner sum running over the $g_l$ degenerate states of quasienergy level $\lambda_l$. Thus, except for the zero-measure set of resonant values of $\tau$, $P_{\text{det}}$ does not depend on the detection period at all. However, at these resonant $\tau$s, $P_{\text{det}}$ changes dramatically [63]. This formula is obviously invariant under a change of basis within any quasienergy sector. We give two different derivations of this formula, one based on our general renewal formula [17] for the probability of detection after $n$ measurements, and a more elementary proof based on an explicit formula for the bright and dark states in terms of the propagator’s eigenstates. This proof requires showing that, on a finite graph, all states that are orthogonal to the bright space are bright, and not changed by the unitary propagator. We give two different derivations of this formula, one based on our general renewal formula [17] for the probability of detection after $n$ measurements, and a more elementary proof based on an explicit formula for the bright and dark states in terms of the propagator’s eigenstates. This proof requires that, on a finite graph, all states that are orthogonal to the bright space are bright, and not changed by the unitary propagator.

The rest of this paper is organized as follows: In Sec. II we will introduce our model. There we derive our main result, first using the renewal formula in Sec. III. We then discuss the splitting of the Hilbert space into bright and dark part in Sec. IV using this in Sec. V to re-derive our main formula. Examples are discussed in Sec. VI. The case of the infinite line is analyzed in Sec. VII. We discuss some implications of our results in Sec. VIII and provide some technical details in the Appendices.

II. STROBOSCOPIC DETECTION PROTOCOL

One does not simply observe the first arrival of a quantum particle in some target state $|\psi_{\text{d}}\rangle$, because it does not have a trajectory in the classical sense. The measurement, the detection, must be explicitly incorporated into the dynamics. Following Refs. [9, 10, 12, 13, 16–20] this can be done by adhering to the stroboscopic detection protocol, where detection in state $|\psi_{\text{d}}\rangle$ is attempted at the times $\tau, 2\tau, 3\tau, \ldots$ and so on. The detection period $\tau$ between the detection attempts is a parameter of the experimentalist’s choice. The experiment we have in mind follows this protocol:

1. Prepare the system in state $|\psi_{\text{in}}\rangle$ at time $t = 0$ and set $n = 0$.

2. The system evolves unitarily for time $\tau$ with the evolution operator $\hat{U} (\tau) := e^{-i\tau\hat{H}/\hbar}$; the wave function is then $|\psi(n\tau + \tau^-)\rangle = \hat{U}(\tau)|\psi(n\tau^-)\rangle$. [The $- (+)$ superscript denotes the limit from the below (above)]. Increase $n$ by one, $n = n + 1$.

3. Attempt to detect the system in the state $|\psi_{\text{d}}\rangle$ with a strong, collapsing measurement.

   a. With conditional probability 

   $$\|\hat{D}|\psi(n\tau^-)\rangle\|^2 = \langle \psi_{\text{d}}|\psi(n\tau^-)\rangle^2, \quad \text{the system is successfully detected. Here,}$$

   $$\hat{D} = |\psi_{\text{d}}\rangle\langle\psi_{\text{d}}| \quad (3)$$

   is the projector onto the detection state. The detection time is $t = n\tau$ and the experiment ends.

   b. Otherwise, the measurement failed to detect the system in the target state. The wave function is instantaneously projected to the state that has no overlap with the detection state $|\psi_{\text{d}}\rangle$. This is the collapse postulate [65]. Mathematically the wave function directly after the unsuccessful measurement is equal to $|\psi(n\tau^-)\rangle = N_n (\mathbb{I} - \hat{D})|\psi(n\tau^-)\rangle$, where $N_n$ is a normalization constant, and $\mathbb{I}$ is the identity operator. After constructing the new wave function, jump back to step two. This loop is repeated until the system is finally detected in step 3a.

After following this procedure many times, one may construct a histogram for the first successful detection number $n$.

As shown by Dhar, et al. [12], the overall probability of detection at measurement $n$ is

$$F_n = \|\hat{D}\hat{U}(\tau) [\mathbb{I} - \hat{D}] \hat{U}(\tau)]^{n-1} |\psi_{\text{in}}\rangle\|^2 \quad (4)$$
and the probability of no detection in the first n measurements is
\[
S_n(\psi_{in}) = 1 - \sum_{m=1}^{n} F_m = \|[(1 - \hat{D})\hat{U}(\tau)]^n|\psi_{in}\rangle\|^2 = \|\hat{S}^n|\psi_{in}\rangle\|^2, \tag{5}
\]
where \(\hat{S} := (1 - \hat{D})\hat{U}(\tau)\) is the survival operator. The dependence on the initial state is specified in the notation. The dependence on the detection state, however, will be suppressed throughout the article. Clearly, \(S_n\) involves \(n\) compound steps of unitary evolution followed by unsuccessful detection. The main focus of this paper is the total detection probability, the probability to be eventually detected, i.e., the probability to “not survive”:
\[
P_{\text{det}}(\psi_{in}) = \sum_{n=1}^{\infty} F_n = 1 - \lim_{n \to \infty} S_n(\psi_{in}). \tag{6}
\]
An initial state \(|\psi_{in}\rangle\) that is never detected is called a dark state with respect to the detection state \(|\psi_{d}\rangle\); for these states \(P_{\text{det}}(\psi_{in}) = 0\) and \(S_n(\psi_{in}) = 1\) for all \(n\). Similarly, a bright state is detected with probability one; i.e., \(P_{\text{det}}(\psi_{in}) = 1\), and \(S_n(\psi_{in}) \to 0\). Of course we may also have states that are neither dark nor bright.

Our theory is developed in generality, valid for any finite dimensional Hamiltonian \(\hat{H}\). Besides the initial and detection state, \(\hat{H}\) and the detection period \(\tau\) are the missing ingredients that enter the stroboscopic detection protocol via the evolution operator \(\hat{U}(\tau) := e^{-i\tau\hat{H}/\hbar}\). We assume that a diagonalization of the latter is available:
\[
\hat{U}(\tau) = \sum_{l} e^{-i\lambda_l} \hat{P}_l, \quad \hat{P}_l := \sum_{m=1}^{g_l} |E_{l,m}\rangle\langle E_{l,m}|. \tag{7}
\]
Here, \(|E_{l,m}\rangle\) are the eigenstates and \(\hat{P}_l\) are the eigenspace projectors, that gather all eigenstates of the \(g_l\)-fold degenerate quasienergy level \(\lambda_l = \tau E_l/\hbar \mod 2\pi\), so that all \(\lambda_l\) in Eq. (7) are distinct. This form is easily obtained from a similar diagonalization of the Hamiltonian \(\hat{H} = \sum_{l} E_l \sum_{m=1}^{g_l} |E_{l,m}\rangle\langle E_{l,m}|\). For convenience, we label the states by their energy \(E_l\), rather than by their quasienergy \(\lambda_l\), since we focus on non-resonant values of \(\tau\).

Although the only effect of \(\tau\) on \(P_{\text{det}}\) is discontinuous, \(\tau\) has a more profound effect on other quantities like the mean detection time. It is an important parameter of the stroboscopic detection protocol. In the limit \(\tau \to 0\), when the system is observed almost continuously, the Zeno effect freezes the dynamics [66–69] and detection can become impossible.

### III. FROM THE RENEWAL EQUATION TO THE TOTAL DETECTION PROBABILITY

One proof of Eq. (1) starts from the first detection amplitudes \(\varphi_n\) [17], which in turn yields the first detection probabilities \(F_n = |\varphi_n|^2\), and so \(P_{\text{det}} = \sum_{n=1}^{\infty} F_n\).

The generating function for these amplitudes \(\varphi(z) = \sum_{n=1}^{\infty} z^n \varphi_n\) can be expressed in terms of \(\hat{U}(\tau)\) (for details, see Refs. [17, 18]):
\[
\varphi(z) = z \langle \psi_{d} | \hat{U}(\tau) | \psi_{in} \rangle / \langle \psi_{d} | 1 - \hat{U}(\tau) | \psi_{d} \rangle. \tag{8}
\]
As shown in [17], \(P_{\text{det}}\) can be obtained from \(\varphi(z)\):
\[
P_{\text{det}} = \sum_{m,n=1}^{\infty} \delta_{m,n} \varphi_n \varphi_n = \frac{1}{2\pi} \int_0^{2\pi} d\theta \sum_{m,n=1}^{\infty} \varphi^*_n \varphi_m e^{i\theta(n-m)}
= \frac{1}{2\pi} \int_0^{2\pi} d\theta |\varphi(e^{i\theta})|^2, \tag{9}
\]
i.e., as the integral of the generating function’s modulus on the unit circle.

Using the quasienergy representation of the evolution operator, (7), the generating function (8) is expressed as the fraction of two expressions:
\[
\varphi(z) = \sum_{l} \langle \psi_{d} | \hat{P}_l | \psi_{in} \rangle \frac{e^{-i\lambda_l}}{1 - ze^{-i\lambda_l}} = \frac{(K\mu e^{-i\theta})(z)}{(K\mu)(z)}. \tag{10}
\]
\((K\mu)(z)\) is the so-called Cauchy transform of the function \(\mu(\theta), 0 < \theta < 2\pi\), defined by [70]:
\[
(K\mu)(z) = \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{\mu(\theta)}{1 - ze^{-i\theta}}. \tag{11}
\]
In Ref. [20], we showed that the denominator of \(\varphi(z)\) is a Cauchy transform of the so-called wrapped measurement spectral density of states:
\[
\mu(\theta) := 2\pi \langle \psi_{d} | \delta(e^{-i\theta} - \hat{U}(\tau)) | \psi_{d} \rangle
= 2\pi \sum_{l} \langle \psi_{d} | \hat{P}_l | \psi_{d} \rangle \delta(\theta - \lambda_l), \tag{12}
\]
where \(\lambda_l\) is the phase corresponding to the \(l\)-th quasienergy level: \(\lambda_l := \tau E_l/\hbar \mod 2\pi\). All \(\lambda_l\) are distinct.

The numerator is a Cauchy transform as well, but of a product of functions \(\mu(\theta)\nu(\theta)\nu(\theta)\):}
\[
\mu(\theta)\nu(\theta) e^{-i\theta} := \sum_{l} \langle \psi_{d} | \hat{P}_l | \psi_{d} \rangle \delta(\theta - \lambda_l) \frac{\langle \psi_{d} | \hat{P}_l | \psi_{in} \rangle e^{-i\lambda_l}}{\langle \psi_{d} | \hat{P}_l | \psi_{d} \rangle).
\tag{13}
\]
The primed sum excludes all completely dark energy levels, i.e., those for which \( \hat{P}_1 |\psi_d\rangle = 0 \), so that \( \langle \psi_d | \hat{P}_1 |\psi_d\rangle \) does not vanish in the denominator of \( \nu(\theta) \). Putting the numerator and denominator together, we find that \( \varphi(z) = \frac{z(K\mu e^{-i\theta})/(K\mu(z)) = z(\nu \mu e^{-i\theta})(z) \text{ is a so-called normalized Cauchy transform} \) [70]. For objects like this, Aleksandrov's theorem allows one to compute the total detection probability [70, Prop. 10.2.3]:

\[
P_{\text{det}} = \frac{1}{2\pi} \int_0^{2\pi} d\lambda \left| \langle \psi_d | \hat{P}_1 |\psi_d\rangle \right|^2 = \frac{1}{2\pi} \int_0^{2\pi} d\lambda \mu(\theta)|\nu(\lambda)|^2, \tag{14}
\]

where we used Eq. (9) for the first equality. An explicit example of how Aleksandrov's theorem works in practice for a two-level system is given in App. B.

Since \( \mu(\theta) \) is just a sum of delta functions at the quasienergies \( \lambda_l \), and \( \nu(\lambda_l) \) is known, the required integral is easily determined:

\[
P_{\text{det}} = \frac{1}{2\pi} \int_0^{2\pi} d\lambda \mu(\lambda)|\nu(\lambda)|^2 = \sum_l \langle \psi_d | \hat{P}_1 |\psi_d\rangle \left| \nu(\lambda_l) \right|^2 = \sum_l \frac{\nu(\lambda_l)^2}{\sum_{m=1}^{g_l} \langle E_{l,m} | \psi_d \rangle^2} \tag{15}
\]

We have thus obtained Eq. (1).

It is important to note that Aleksandrov's theorem in this form only applies to operators with a point spectrum. If the spectrum has a continuous component, then Aleksandrov's theorem turns into an inequality [70, Prop. 10.2.3], and we have

\[
P_{\text{det}} \leq \frac{1}{2\pi} \int_0^{2\pi} d\lambda \mu(\lambda)|\nu(\lambda)|^2. \tag{16}
\]

Ref. [20] discusses \( \mu(\lambda) \) in depth for systems with a continuous spectrum. (Note that this reference defines \( \nu(\lambda) \) in a slightly different way.) This proof does not make any explicit reference to dark or bright states (except for the exclusion of dark levels) and so is not physically transparent. We therefore present an alternate proof in the next section.

\IV. A PARTITION OF THE HILBERT SPACE

\A. Dark States

In this section we discuss the partition of the Hilbert space into a bright and dark part. In Sec. II, we defined dark states as those with \( F_n = 0 \), \( S_n = 1 \) for all \( n \). In particular, we focus on stationary dark states, which are invariant under unitary evolution as well as under the detection attempts. During the course of the detection protocol only their phase is affected. Hence they remain dark for all times. In view of the diagonalization (7), there are two ways dark states can arise.

\a. Completely dark quasienergy levels

Consider an quasienergy level \( E_l \) that has no overlap in the detection state. That means none of the level's eigenstates overlaps with \( |\psi_d\rangle \), i.e. \( \hat{P}_1 |\psi_d\rangle = 0 \) or \( \langle E_{l,m} |\psi_d\rangle = 0 \) for all \( m = 1, \ldots, g_l \). Alternatively, one can write \( D |E_{l,m}\rangle = 0 \). We also denote these states as \( |\delta_{l,m}\rangle = |E_{l,m}\rangle \). If we take one of these as an initial state, we have:

\[
\hat{S} |\delta_{l,m}\rangle = (1 - \hat{D})\hat{U}(\tau) |\delta_{l,m}\rangle = (1 - \hat{D})e^{-i\tau E_l/\hbar} |\delta_{l,m}\rangle = e^{-i\tau E_l/\hbar} |\delta_{l,m}\rangle.
\]

Thus, \( |\delta_{l,m}\rangle \) is an eigenstate of the survival operator with an eigenvalue on the unit circle, and so \( S_n(\delta_{l,m}) = ||\hat{S}^n |\delta_{l,m}\rangle|| = 1 \). Thus, all the \( |\delta_{l,m}\rangle \) are dark states. We call quasienergy levels that do not appear in a decomposition of the detection state completely dark quasienergy levels. All \( g_l \) associated eigenstates are dark.

\b. Degenerate energy levels

Consider now an quasienergy level \( E_l \) that does overlap with the detection state, but which is degenerate, such that \( g_l > 1 \). Construct the projection of the detection state on this sector,

\[
|\beta_l\rangle = \frac{\hat{P}_1 |\psi_d\rangle}{\sqrt{\langle \psi_d | \hat{P}_1 |\psi_d\rangle}} \tag{17}
\]

It will turn out that \( |\beta_l\rangle \) is a bright state. All states within this sector which are orthogonal to \( |\beta_l\rangle \) are orthogonal to the detector, since if \( |\beta_l\rangle \) is such a state,

\[
\langle \psi_d | \delta_{l,j} \rangle = \langle \psi_d | \hat{D} |\delta_{l,j}\rangle \propto \langle \beta_l |\delta_{l,j}\rangle = 0 \tag{18}
\]

and so are dark. These states constitute a \( g_l - 1 \) dimension subspace of this quasienergy sector, any state lying within which is dark. It can be convenient to have an explicit basis for this subspace, which can be generated by a determinantal formula similar to that used in the Gram-Schmidt procedure. We define

\[
|\delta_{l,j}\rangle = N_j \begin{bmatrix} |E_1\rangle & |E_2\rangle & \cdots & |E_{g_l}\rangle \\ \langle \psi_d | E_1 \rangle & \langle \psi_d | E_2 \rangle & \cdots & \langle \psi_d | E_{g_l} \rangle \\ \langle \delta_1 | E_1 \rangle & \langle \delta_1 | E_2 \rangle & \cdots & \langle \delta_1 | E_{g_l} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \delta_{j-1} | E_1 \rangle & \langle \delta_{j-1} | E_2 \rangle & \cdots & \langle \delta_{j-1} | E_{g_l} \rangle \\ \end{bmatrix} \tag{19}
\]

where \( N_j \) is a normalization factor and the subscript \( l \) was omitted on the right-hand side. This determinant has the obvious properties that (i) it yields an eigenstate of \( \hat{U}(\tau) \); (ii) it is orthogonal to \( |\psi_d\rangle \), since \( \langle \psi_d | \delta_{l,j}\rangle \) is a determinant with two identical rows; and (iii) that it is orthogonal to all \( |\delta_{l,m}\rangle \) with \( m < j \). For the same reason, \( \langle \delta_{l,m} |\delta_{l,j}\rangle \) is a determinant with two identical rows.

This formula gives

\[
|\delta_{l,1}\rangle = N_1 \left( \langle \psi_d | E_{l,2} \rangle |E_{l,1}\rangle - \langle \psi_d | E_{l,1} \rangle |E_{l,2}\rangle \right) \tag{20}
\]
and then recursively computes $|\delta_{l,m}\rangle$ from all previous $|\delta_{l,j}\rangle$. The result is a set of $g_l - 1$ normalized and mutually orthogonal stationary dark states. As $|\delta_{l,m}\rangle$ is constructed only from the eigenstates $|E_{l,1}\rangle, |E_{l,2}\rangle, \ldots, |E_{l,m+1}\rangle$, we find that many of the $|\langle E_{l,m}| E_{l,j}\rangle|$ in Eq. (19) vanish, namely all those with $j > m + 1$. Consequently the matrix in Eq. (19) is lower triangular except for the first rows. This enables us to compute the determinant explicitly:

$$
|\delta_{l,m}\rangle = \sum_{j=1}^{m} \frac{[\alpha_{l,j}^2 |E_{l,m+1} - \alpha_{l,m+1}^* \alpha_{l,j} |E_{l,j}\rangle]}{\sqrt{\sum_{j'=1}^{m+1} \sum_{j''=1}^{m+2} |\alpha_{l,j}^2| |\alpha_{l,j'}^2|^2}}. 
$$

(21)

Here we abbreviated $\alpha_{l,m} = \langle E_{l,m}| \psi_d \rangle$. Eq. (21) contains Eq. (20) as a special case.

B. Bright States

We have seen above how dark states arise, and constructed a set of stationary dark states, which are not only energy eigenstates, but (unit modulus eigenvalue) eigenstates of the survival operator. We have also seen that each degenerate level that was not completely dark yielded an energy eigenstate $|\beta\rangle$ which we claimed was not only not dark, but is bright. Similarly, every non-degenerate level that is not completely dark also turns out to be bright, so that $|E_l\rangle = |\beta\rangle$. These bright states, while energy eigenstates, are not in general eigenstates of the survival operator. Together, the $|\beta\rangle$ states, one arising from each not-totally-dark level, span the orthogonal complement of the dark space.

We now demonstrate that they are indeed bright. It was already mentioned that the stationary dark state $|\delta_{l,m}\rangle$ is a (right-)eigenstate of $\tilde{\hat{S}}$ with eigenvalue $e^{-i\tau E_l}/\hbar$, lying on the unit circle. It is easy to see that these are the only eigenvalues on the unit circle, since $\tilde{\hat{D}}$ must annihilate it. Since $\tilde{\hat{S}}$ is the product of a projector with a unitary matrix, it can have no eigenvalues outside the unit disk, and so all other eigenvalues $\zeta$ must lie inside the unit disk, i.e. $|\zeta| < 1$.

In fact, these eigenvalues are directly related to the poles of the generating function $\varphi(z)$ discussed above. We may now use the definition $\tilde{\hat{S}} := (1 - \tilde{\hat{D}})\hat{U}(\tau)$ of the survival operator and rewrite its characteristic polynomial as $\det[(1 - \hat{U}(\tau) + |\psi_d\rangle\langle\psi_d| \hat{U}(\tau)]$. An application of the matrix determinant lemma yields:

$$
\det[(1 - \tilde{\hat{S}}) = \det(\zeta I - \hat{U}(\tau))] \langle \psi_d | (\zeta I - \hat{U}(\tau))^{-1} | \psi_d \rangle .
$$

(22)

The last term can be identified with the denominator of $\varphi(z)$ of Eq. (8). Inverting the equation yields:

$$
\langle \psi_d | \frac{1}{1 - \zeta \hat{U}(\tau)} | \psi_d \rangle = \frac{\det(\zeta I - \tilde{\hat{S}})}{\det(\zeta I - \hat{U}(\tau))}.
$$

(23)

This relation shows that $z_p$ is a (finite) pole of $\varphi(z)$ if and only if $z_p = 1/|\beta\rangle$ is an eigenvalue of $\hat{S}$, but not of $\hat{U}(\tau)$. These are exactly the eigenvalues of $\hat{S}$ that lie inside the unit disk. $\varphi(z)$ is by construction analytic in the unit disk. Therefore all of its poles must lie outside the unit disk and have $|z_p| > 1$. Consequently all of these non-trivial eigenvalues of $\hat{S}$ - which are the poles’ reciprocals - must lie inside the unit circle, $|\zeta| < 1$. The corresponding eigenvectors belong to the complement of the dark space, i.e., the bright space. These vectors span the bright space, and it is clear that any superposition of bright eigenstates $|\beta\rangle = \sum_l b_l |\beta_l\rangle$ will yield an exponentially decaying survival probability. So $S_\infty(\beta) = 0$ and the state is bright, as claimed. The decay rate is determined by the eigenvalue $\zeta_{\text{max}}$ closest to the unit circle.

There are two subtleties in this argument. One must allow for the possibility that there are degenerate eigenvalues of $\tilde{\hat{S}}$, and the operator is not diagonalizable. Showing that $|\beta\rangle$ is bright in this case is a bit technical, and we present the argument in Appendix A. The other point to be raised is that if the system is infinite, the eigenvalues might approach arbitrarily closely to the unit circle, and there might not be an exponential decay of the survival probability. This indeed happens, and in such systems, states in the complement to the dark space are not necessarily bright. This would be discussed in more detail in Sec. VII.

The just described transition from $\{ |E_{l,m}\rangle \}_{m=1}^{g_l}$ to $|\beta\rangle$ and $\{ |\delta_{l,m}\rangle \}_{m=1}^{g_l-1}$ is a change from one orthonormal basis to another. Still, all involved states are eigenstates, and are thus invariant under $\hat{U}(\tau)$. The special feature of the new representation is that each individual stationary dark state and all bright eigenstates together are additionally invariant under the detection process. In the language of Refs. [7, 62], $\mathcal{H}_B$ and $\mathcal{H}_D$ are so-called invariant subspaces. The action of the survival operator $\hat{S}$ may change a particular superposition of bright eigenstates $|\beta\rangle = \sum_l b_l |\beta_l\rangle$ into some other superposition of $|\beta_l\rangle$, but it can never generate a dark state.

V. $P_{\text{det}}$ Calculated from the Bright Space

We can see from the above discussion that, as noted by Krovi and Brun [2], the limit $\lim_{n\to\infty} S_n(\psi_{\text{in}})$ of an arbitrary initial state’s survival probability is equal to its overlap with the dark space: $\| \hat{P}_{\mathcal{H}_D} |\psi_{\text{in}}\rangle \|^2$, at least for a finite system. Complementarily, the total detection probability must be equal to the overlap with the bright space.

The projector onto the bright space has the form:

$$
\hat{P}_{\mathcal{H}_B} = \sum_l |\beta_l\rangle\langle\beta_l|, \text{ where the sum excludes all completely dark energy levels, so that they do not possess a bright state, and } |\beta_l\rangle \text{ are the bright eigenstates of Eq. (17).}
$$
Identifying $P_{\text{det}}(\psi_{\text{in}}) = \langle \psi_{\text{in}} | \hat{P} \hat{n} | \psi_{\text{in}} \rangle$, we find:

$$P_{\text{det}}(\psi_{\text{in}}) = \sum_{l} \left| \langle \psi_{l} | \hat{P} | \psi_{\text{in}} \rangle \right|^{2}.$$  \hfill (24)

Plugging the definition of the eigenspace projectors $\hat{P}$ that is found in Eq. (7) into Eq. (24), yields Eq. (1). Note that the sum only runs over those quasienergy levels $E_{l}$ which are not completely dark, so that the denominator in Eq. (1) can never vanish.

We can now make the following observations:

(a) When the initial state coincides with the detection state, we get $P_{\text{det}} = 1$ in agreement with Ref. [9]. This means that the detection state $| \psi_{d} \rangle$ itself is always bright.

(b) The total detection probability does not depend on the sampling rate. $P_{\text{det}}$ is $\tau$-independent. This is true except for the resonant detection periods $\tau_{c}$, defined in Eq. (2). At these points the number of bright states suddenly changes and $P_{\text{det}}$ will exhibit a discontinuous drop.

(c) In systems with only non-degenerate quasienergy levels that all overlap with the detection state, i.e. $\langle E_{l} | \psi_{d} \rangle \neq 0$, any initial state is detected with probability one. In this case there are no dark states, all eigenstates are bright. Such a behavior is classical, in the sense that a classical ergodic random walk also finds its target with probability one. It implies that some disorder in the system, which removes all the degeneracy, may increase the detection probability provided that none of the eigenstates is orthogonal to the detection state. In systems with degenerate quasienergy levels, one may still be able to find initial states which are detected with probability one, but this will not be true generically for all initial states.

VI. EXAMPLES

Let us now consider some examples that demonstrate the relation of $P_{\text{det}}$ to the dark and bright states. The examples all consist of graphs of nodes connected by links, which represent hopping between the connected nodes, and we consider only initial and detection states which are localized on individual nodes. In all cases but the ring with a magnetic field, we take the hopping strengths to be uniform, with value $\gamma$, and zero on-site energies. We also assume that $\tau$ does the resonant condition (2).

A. A ring

Let us now consider a ring with $L$ (even) sides. Its Hamiltonian takes the form:

$$\hat{H} = -\gamma \sum_{r=1}^{L} \left[ |r\rangle \langle r+1| + |r\rangle \langle r-1| \right].$$  \hfill (25)

We employ periodic boundary conditions and identify $|r + L\rangle = |r\rangle$.

The free wave states diagonalize the Hamiltonian, such that the energy levels are given by:

$$E_{l} = -2\gamma \cos \frac{2\pi l}{L},$$  \hfill (26)

with $l = 0, 1, \ldots, L/2$. Except for $E_{0}$ and $E_{L/2}$, which are non-degenerate, all energy levels possess two eigenstates:

$$\begin{align*}
|E_{0}\rangle &= \sum_{r=1}^{L} \frac{|r\rangle}{\sqrt{L}}, \\
|E_{L/2}\rangle &= \sum_{r=1}^{L} \frac{(-1)^{r} |r\rangle}{\sqrt{L}}, \\
|E_{l,1}\rangle &= \sum_{r=1}^{L} e^{i \frac{2\pi l r}{L}} |r\rangle, \\
|E_{l,2}\rangle &= \sum_{r=1}^{L} e^{-i \frac{2\pi l r}{L}} |r\rangle.
\end{align*}$$  \hfill (27)

Picking the localized states $|r_{in}\rangle$ and $|r_{d}\rangle = |L\rangle$ as initial and detection state, we find, using Eq. (1)

$$P_{\text{det}}(r_{in}) = \frac{2}{L} + \frac{2}{L} \sum_{r=1}^{L/2-1} \cos 2 \frac{2\pi l r_{in}}{L} = \begin{cases} 1, & r_{in} = \frac{L}{2}, \frac{L}{2}, \\
\frac{1}{2}, & \text{else} \end{cases}.$$  \hfill (28)

For almost all sites, we find $P_{\text{det}} = 1/2$ except when $r_{in}$ and $r_{d}$ coincide or are on opposing sides of the ring. This is in accordance to our results from Ref. [18]. The same result appears for rings of odd sizes, albeit it is not possible to place initial and detection site on opposite sites of the ring.

The energy basis bright states are given by the cosine waves, whereas the stationary dark states are given by sine waves:

$$|\beta_{l}\rangle = N_{l} \sum_{r=1}^{L} \cos 2 \frac{2\pi l r}{L} |r\rangle, \quad |\delta_{l}\rangle = \sqrt{\frac{2}{L}} \sum_{r=1}^{L} \sin 2 \frac{2\pi l r}{L} |r\rangle,$$

where $N_{l} = 1/\sqrt{L}$ for $l = 0, L/2$ and $N_{l} = \sqrt{2/\pi}$ otherwise. $|\beta_{0}\rangle$ and $|\delta_{L/2}\rangle$ are not defined. Thus, in this case, the deviation of $P_{\text{det}}$ from unity is solely due to the degeneracies of the energy spectrum, which of course arise from the parity symmetry of the system.

B. A Ring with a magnetic field

In the ring, as we saw, the degeneracy of an energy level gives rise to dark states, which in turn lead to a deficit in the total detection probability. What happens if this degeneracy is lifted? To explore this, we add to the ring model of Eq. (25) a magnetic field:

$$\hat{H} = -\gamma \sum_{r=1}^{L} \left( e^{i \alpha} |r\rangle \langle r+1| + e^{-i \alpha} |r\rangle \langle r-1| \right).$$  \hfill (30)

The magnetic field strength is proportional to $\alpha$ and its vector is normal to the plane in which the ring lies.
FIG. 1. Total detection probability for a ring with magnetic field. Initial and detection state are: \( r_d = 0 \) and \( r_{in} = 1 \). Bottom: The energy levels of Eq. (31) as a function of \( \alpha \). Degeneracy occurs when they cross, this occurs for special values of \( \alpha \). Top: The total detection probability as a function of \( \tau \) and \( \alpha \). We usually find \( P_{\text{det}} = 1 \), but a deficit occurs when there is a degeneracy in the quasienergy levels which appears as dark lines in the figure. This happens for certain values of \( \alpha \) (vertical lines, coinciding with the intersection of two energy levels in the bottom) and for special combinations of \( \tau \) and \( \alpha \). Here \( P_{\text{det}} \) was approximated by \( P_{\text{det}} = 1 - S_{\alpha} \), with \( N = 50 \). As \( N \) is taken to infinity the dark lines will become infinitely thin.

This field splits up the two-fold degeneracy of the energy levels. The eigenstates are still the free wave states of Eq. (27), although the index \( l \) now runs from zero to \( L = 1 \). The new levels are

\[
E_l = -2\gamma \cos \left( \frac{2\pi l}{L} + \alpha \right). \tag{31}
\]

Except for special values of \( \alpha \), namely when \( \alpha \) is an integer multiple of \( 2\pi/L \), these are all distinct. Similarly, a degenerate quasienergy level can only appear when condition (2) for resonant \( \tau \) is fulfilled. Since a localized detection state \( |r_d\rangle \) has overlap with all eigenstates and since all energy levels are non-degenerate, we find

\[
P_{\text{det}}(r_{in}) = 1 \tag{32}
\]

with the exception of special combinations of \( \tau \) and \( \alpha \). This is nicely demonstrated in Fig. 1, \( P_{\text{det}} \) as a function of \( \tau \) and \( \alpha \), as well as the energy levels as a function of \( \alpha \) for a ring with \( L = 6 \). We find that the total detection probability is almost everywhere unity except along some lines in \( (\alpha, \tau) \)-space which parametrize the resonant combinations at which degeneracy occurs. What is presented is \( 1 - S_{50} \), rather than \( 1 - S_{\infty} \). Thus, the lines have finite width. Nevertheless, the extreme sensitivity to the presence of any symmetry breaking is apparent.

This can be quantified by considering the value of \( n_{1/2} \) for which \( S_n \) falls below its \( \alpha = 0 \) limit of \( 1/2 \) (for the detection site not identical or diametrically opposite the initial site). This is plotted in Fig. 2, where even for tiny value of \( \alpha = 10^{-6} \), the presence of a magnetic field can be unequivocally detected after a series of runs of 50 measurement attempts. We see that \( n_{1/2} \) grows only logarithmically with diminishing \( \alpha \), so that detection of very small magnetic fields is in principle possible. This can be understood by looking at the behavior of \( F_n \), as depicted in Fig. 3. We see that, for the small values of \( \alpha \) depicted, \( F_n \) is essentially independent of \( \alpha \) for not too large \( n \). For small, finite \( \alpha \) and large \( n \), \( S_n \) decays very slowly, as a result of the presence of slow modes of \( \mathcal{S} \). These modes, which for \( \alpha = 0 \) were dark states with unit modulus eigenvalue, now are at a distance of order \( \alpha^2 \) from the unit circle, leading to a decay rate of order \( \alpha^2 \). These modes have to contribute a total of \( 1/2 \) to \( P_{\text{det}} \). Given their slow \( O(\alpha^2) \) exponential decay, once this slow decay sets in, (at \( n_c \), say), \( F_n \) must behave as \( F_n \sim F_{n_c} \exp(-\alpha d_0(n - n_c)) \), for some \( \alpha \)-independent \( d_0 \), with \( F_{n_c} \sim \alpha^{-2} \), to give an \( \alpha \)-independent sum. Thus, in Fig. 2 we see that \( F_{n_c} \) for \( \alpha = 10^{-4} \) is roughly a factor of 100 smaller than for \( \alpha^{-3} \). Since, for \( n < n_c \), the decay rate of \( F_n \) is set by the slowest non-dark state (with an \( \alpha \)-independent decay rate, \( d_0 \)), we get that \( F_{n_c} \sim \exp(-d_0 n) \), so that

\[
n_c \sim -\ln \alpha \tag{33}
\]

Since, for \( \alpha = 0 \), \( P_{\text{det}} = 1/2 \), \( 1 - S_n \) will first exceed this value at an \( n_{1/2} \) of order \( n_c \).

C. Square with center node

We next consider a square with a single node in the middle connected to the all others. The states localized on the corners of the square are denoted by \(|1\rangle, |2\rangle, |3\rangle \) and \(|4\rangle \), and the additional node in the center of the
diagonals is state $|0\rangle$. The Hamiltonian in matrix form is

$$
\hat{H} = -\gamma \begin{pmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0
\end{pmatrix}. 
$$

(34)

The energy levels are $E_1 = -(1 + \sqrt{5})\gamma$, $E_2 = (\sqrt{5} - 1)\gamma$, $E_3 = 2\gamma$, and $E_4 = 0$, and $E_4$ is two-fold degenerate. To find $P_{\text{det}}$ we obtain the eigenvectors:

$$
\begin{align*}
|E_1\rangle &= (\sqrt{5} - 1, 1, 1, 1, 1)^T / \sqrt{10 - 2\sqrt{5}}, \\
|E_2\rangle &= (-1 - \sqrt{5}, 1, 1, 1, 1)^T / \sqrt{10 + 2\sqrt{5}}, \\
|E_3\rangle &= (0, -1, -1, 1, 1)^T / 2, \\
|E_{4,1}\rangle &= (0, 1, -1, -1, 1)^T / 2, \\
|E_{4,2}\rangle &= (0, 1, 1, -1, -1)^T / 2.
\end{align*}
$$

(35)

Notice that, as opposed to the ring, where there were no completely dark levels, here there are two energy levels, $E_3$ and $E_4$, which are completely dark with respect to a measurement in the center. These sectors are dark because these states are not invariant under rotation by $\pi/2$ (where $|E_3\rangle$ transforms to $-|E_3\rangle$, and $|E_{4,1}\rangle$ transforms to $-|E_{4,2}\rangle$), whereas the center is, and so there can be no overlap with the center. As per our general discussion, this gives rise to a deviation of $P_{\text{det}}$ from unity when measuring at exactly this node, i.e., $|r_4\rangle = |0\rangle$. The bright eigenstates are given by the projections of the detection state into the energy subspaces, hence $|\beta_1\rangle = |E_1\rangle$ and $|\beta_2\rangle = |E_2\rangle$. As mentioned the energy levels $E_3$ and $E_4$ are completely dark and will be excluded in the sum of Eq. (1) that yields:

$$
P_{\text{det}}(r_{in}) = \begin{cases} 1, & r_{in} = 0 \\
\frac{1}{\gamma}, & r_{in} \neq 0.
\end{cases}
$$

(36)

For other detection states, say $r_d = 1$, as other initial states are the same up to a rotation, there are no dark energy levels, but $E_4$ possesses one dark state due to degeneracy: $\delta_4 := (|E_{4,1}\rangle - |E_{4,2}\rangle)/\sqrt{2}$. All energy levels participate in Eq. (1):

$$
P_{\text{det}}(r_{in}) = \begin{cases} 1, & r_{in} = 0, 1, 3 \\
\frac{1}{\gamma}, & r_{in} = 2, 4.
\end{cases}
$$

(37)

D. Additional examples

We have worked out a number of additional examples where $P_{\text{det}} < 1$ and the associated dark states for other simple geometries of graphs. These include a complete graph, a star graph, a hypercube and a tree graph. The results are presented in the appendix, and summarized together with the previous examples in Fig. 4. The detection node is represented by an open circle, All other (filled) circles are possible initial states and the numbers next to them represent the corresponding total detection probability. It is interesting to note that in all these cases, $P_{\text{det}}$ is rational, and except for the tree, is simply 1 over an integer. We will expand upon this observation in the second paper of this series.

VII. THE INFINITE LINE

We have seen above that finite systems are very different than infinite systems, at least as far as the properties of $P_{\text{det}}$ are concerned. Whereas Grünbaum, et al. [9] showed that, when the detection and initial states coincide, $P_{\text{det}}$ is unity for every finite system, for an infinite system such as the infinite line, it is always less than unity [17]. We wish here to amplify on this point by comparing the finite ring of size $L$ with its infinite counterpart. Above we exhibited the dark states for the finite ring. It is clear that the dark states remain dark even in the infinite-$L$ limit. However, the bright states of the finite ring, while remaining orthogonal to every dark state, are not detected with unit probability in the infinite-$L$ limit. Instead, they are dim, and $P_{\text{det}}$ for these states depends essentially on $\gamma$.

To get an insight into this phenomenon, we first consider the space-time picture of the undetected probability density (i.e., the position distribution at time $n\tau$, normalized to $S_n$). We start with the particle at $x = L/2$ on a ring of large length $L$, measuring at the same point. As time progresses, the density spreads out ballistically from the initial location in both directions. All long as not enough time has passed to allow the two “wings”
FIG. 4. Total detection probability for graphs. A quantum particle is put on a graph, whose node's describe the particle’s possible position states. The links describe allowed transitions each with equal strength. The particle starts localized on some node of the graph (full circles) and we attempt to detect it on the node with the open circle. The numbers denote \( P_{\text{det}} \) for this initial state. (A) Square with a center node, (B) ring of \( L \) sites, (C) complete graph with \( L \) sites, (D) star graph with \( L \) sites in the periphery (E) hypercube (here of dimension \( d = 3 \) (F) binary tree with 2 generations.

of the distribution to meet at the opposite end of the ring \( (x = 0 = L) \), a time of order \( L \), the density has an overall amplitude proportional to \( 1/t \), so that, given the extension over a distance proportional to \( t \), the total undetected density is of order unity. It should be noted that this is the same exact scaling behavior of the measurement-free density. This situation is presented in Fig. 5, where the scaled density is presented after 238 and 476 steps. The total undetected probability, \( S_n \), is 0.6006 at both times, the difference being only of order \( 4 \cdot 10^{-6} \), while the average distance to the origin of the surviving particle has doubled from 386 to 770. Thus, in the limit of large \( L \), the total undetected probability remains of order unity, spreading out to infinity, for all times less than the huge system traversal time of order \( L \).

An alternative picture arises from considering the stationary bright states which are the eigenvectors of the survival operator with eigenvalues lying inside the unit circle. For large finite \( L \), these eigenvalues span the range in magnitude from 0 to very near unity. For times \( N\tau \) shorter than the time to propagate across the system, (i.e., for the wings of the distribution to meet, as described above) the relatively quickly decaying stationary bright states with eigenvalues with small magnitude contribute to \( \sum_{n \leq N} F_n = 1 - S_N \), while the states with eigenvalues near the unit circle do not. These slow states, then, are essentially dark over this time-scale. We find that for \( L = 1000 \), there are 267 fast eigenvalues (out of 501) further than a distance .002 from the unit circle, and for \( L = 2000 \), there are 531 fast eigenvalues (out of 1001) further than a distance .001 from the unit circle (for \( \gamma \tau = \hbar \)); i.e., twice as many. Examining the eigenvectors, they extend over the entire system, so that an initial state localized near the origin will have a squared overlap with each bright mode proportional to \( 1/L \). Thus, the number of effectively dark states, those that do not contribute in time \( N\tau \) to \( P_{\text{det}} \), scales with \( L \). Hence, the undetected probability is of order 1, since it arises from order \( L \) modes times a number of order \( 1/L \) overlap for each mode.

FIG. 5. The surviving (i.e., as yet undetected) particle probability density \( P_n(x) \) after \( n = 238 \) and 576 measurements, in a ring of length \( L = 2000 \) and a detection interval of \( \gamma \tau / \hbar = 1 \). For clarity, only every 5th point is plotted.

VIII. DISCUSSION

In a classical random walk in a finite state space, the “bright space” would correspond to the ergodic component, let’s call it \( B \), that the target site \( r_d \) resides in. That means \( B \) consists of all states that have access to \( r_d \). The total arrival/detection probability would then be \( P_{\text{det}} = \sum_{r \in B} p_n(r) \). In particular, all the example systems studied herein are classically ergodic, with \( P_{\text{det}} = 1 \). A deficit in this probability is a signal for ergodicity breaking. In a quantum setup, the dark space is due to destructive interference.

These interference effects are a double-edged sword. Constructive interference may render a quantum walk more efficient than a classical one, in the sense that the average first detection time \( \langle n\tau \rangle \) is smaller for the quantum search. But in the same way, it weakens the process’s reliability, in that carelessly chosen initial states...
yield $P_{\text{det}} < 1$. Ref. [9] considered the return problem, $|\psi_{\text{in}}\rangle = |\psi_{\text{d}}\rangle$ and showed that this mean is equal to the number $\omega$ of different bright states in the system. An in-depth analysis for the transition problem ($|\psi_{\text{in}}\rangle \neq |\psi_{\text{d}}\rangle$) or for unbounded systems is still lacking. Finding the optimal detection period $\tau^*$ which minimizes the mean first detection time constitutes an important follow-up project to this paper.

Introduction of disorder breaks up the degeneracy of the energy levels and thus enhances $P_{\text{det}}$. This was nicely demonstrated in Refs. [7, 71] where on-site energy disorder was used to disrupt the dark space in a non-Hermitean system. However, all this should be taken with a grain of salt, since disorder may as well lead to a very large average detection time. A similar deceleration occurs in the Zeno limit. As $\tau \to 0$, the mean detection time $\langle T \rangle$ diverges when $|\psi_{\text{in}}\rangle \neq |\psi_{\text{d}}\rangle$, see Ref. [18]. All this implies that minimizing the average detection time, while maximizing the detection probability by modifying either $\tau$ or $\hat{H}$ is not a simple task.

Another possible way to increase the $P_{\text{det}}$ could be to attempt detection on more than one state, see Refs. [4, 10]. Then each quasienergy level would yield more than one bright state. For a completely reliable quantum search (in the sense that $P_{\text{det}}(\psi_{\text{in}}) = 1$ for all $|\psi_{\text{in}}\rangle$) it is necessary that there are more detection states than the largest degeneracy $g_i$. This was already pointed out in Refs. [2–4, 6].

The detection period $\tau$ only enters Eq. (1) in the degeneracy $g_i$ of the quasienergy levels. Thus $P_{\text{det}}$ is almost independent of $\tau$ apart from the resonant values defined by Eq. (2). The non-intuitive effects of these resonances on the first detection statistics have already been briefly discussed in our prior publications [17–20], and in more detail on Ref. [63].

The $\tau$-independence of the total detection probability will survive when irregularities are introduced in the sampling times. The lack of the total control over the detector can be modeled by a random sequence of detection times $\{\tau_1, \tau_2, \ldots\}$, as has been done, e.g. in Ref. [6], where the sampling times were given by a Poisson process. A strong finding of ours is that $P_{\text{det}}$ does actually not depend on $\tau$ at all for systems with a discrete spectrum. As the set of resonant detection periods [defined by Eq. (2)] has zero measure, and thus all of our results are expected to hold for non-stroboscopic sampling as well.

The terminology dark states is borrowed from atomic physics and quantum optics where it describes forbidden transitions or non-emissive states [72]. In blinking quantum dots, light emission can be quenched through non-radiative channels [73]. In these cases the coupling of the material system to an electromagnetic field and the radiative lifetime of an excited state play an important role that is here taken by the detection period $\tau$.

After infinitely many unsuccessful detection attempts only the dark component of the initial wave function survives. The dark space is invariant under unitary evolution and strong detection [62, 74]. Similar behavior can be found in the long-time behavior of general open quantum dynamics [75]. An interesting question would be which of the here described features will carry over when one employs weak measurements or open quantum dynamics [21, 24, 76, 77].

We have thus seen how in an infinite dimensional system, it can be that $P_{\text{det}}(\beta) < 1$. Our Eq. (1) still constitutes an upper bound for the true value of $P_{\text{det}}$, which may become $\tau$-dependent. As shown in Ref. [9, 17, 19], $P_{\text{det}}(\tau)$ is highly non-trivial in this situation. More powerful tools must be invented to compute the total detection probability in such systems.

**IX. SUMMARY**

We have investigated herein the total probability of detection, $P_{\text{det}}$, in a quantum system that is stroboscopically probed in its detection state. An explicit formula for this in terms of the energy eigenstates was produced via the renewal equation previously derived for the generating function for the detection amplitude, with the help of the Aleksandrov theorem for Cauchy transforms. An alternate derivation for the formula was also given via an analysis of the dark and bright subspaces that comprise the total Hilbert space. The dark states are those energy eigenstates that have no overlap with the detector and thus are never detected. There were found to be two classes of dark states: those that belong to completely dark energy levels, which have no overlap whatsoever with the detection state, and those which perfur appear in degenerate energy levels which are not completely dark. The bright states are those eigenstates which are detected with probability unity, were shown to constitute, in a finite system, the orthogonal complement to the dark subspace. They were shown to belong to the spectrum of the survival operator $\hat{S}$ inside the unit disk. An explicit set of basis states for the subspaces was constructed. From this, $P_{\text{det}}$ was calculated as the overlap of the initial state with the bright space, reproducing the original result. We considered several examples, showing in particular how lifting the degeneracy in the energy spectrum discontinuously changes $P_{\text{det}}$. Finally, the breakdown of our formula for $P_{\text{det}}$ in an infinite system was discussed in the context of the infinite line.

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Appendix A: Additional properties of the Survival operator

Here, we will give a more detailed discussion of $\hat{S}$'s properties. In the main text we already mentioned that the stationary dark states are eigenstates of $\hat{S}$ with eigenvalue on the unit circle and that the stationary bright states belong to the spectrum inside the unit disk. In view of the fact that $\hat{S}$ is not a diagonalizable matrix in general, these statements need to be refined.

First, we note that the stationary dark states $|\delta_{l,m}\rangle$ of Eq. (21) are in general the right-eigenstates of $\hat{S}$ that correspond to eigenvalues $e^{-i\tau E_{l}/\hbar}$ on the unit circle. In particular, these eigenvalues coincide with eigenvalues of $\hat{U}(\tau)$. Next, Eq. (22) that relates the poles of the generating function $\varphi(z)$ and the spectrum of $\hat{S}$ is generally true, as well. Its algebraic eigenvalues $\zeta$ are given by the zeros of the characteristic polynomial $\det[\mathbb{1} - \hat{S}]$.

It remains to be shown that the survival probability $S_{n}(\bar{\lambda}) = \|\hat{S}^{n}|\bar{\lambda}\rangle\|$ of some bright state decays to zero as $n \to \infty$. To see this, consider the restriction of $\hat{S}$ to the bright space $\hat{S}_{B} = \hat{P}_{B} \hat{S} \hat{P}_{B}$. This removes all trivial eigenvalues that lie on the unit circle from $\hat{S}$. This operator is defective in general and can not be brought into a diagonal form. It can be brought into a Jordan normal form, though:

$$\hat{S}_{B} = \sum_{l} [\zeta_{l} \hat{M}_{l} + \hat{N}_{l}]. \quad (A1)$$

Here $\zeta_{l}$ are the algebraic eigenvalues of $\hat{S}_{B}$ that we determined from $\varphi(z)$, and the matrices $\hat{M}_{l}$ and $\hat{N}_{l}$ act on the (generalized) eigenspaces of $\zeta_{l}$. $\hat{N}_{l}$ is a nilpotent matrix (i.e. $\hat{N}_{l}^{p_{l}} = 0$, for some integer $p_{l} > 0$) and $\hat{M}_{l}$ is the projector to the corresponding eigenspace. In particular, $\hat{M}_{l} \hat{M}_{l} = \hat{M}_{l}$. The rank of $\hat{M}_{l}$ is equal to the algebraic multiplicity of $\zeta_{l}$, $\hat{M}_{l}$ and $\hat{N}_{l}$ commute, and $\hat{M}_{l} \hat{N}_{l} = \hat{N}_{l} \hat{M}_{l} = \delta_{l,m} \hat{N}_{l}$. Using these properties, one finds for $n > p_{l}$

$$(\zeta_{l} \hat{M}_{l} + \hat{N}_{l})^{n} = \zeta_{l}^{l} \hat{M}_{l} + \sum_{m=1}^{p_{l}} \binom{n}{m} \zeta_{l}^{n-m} \hat{N}_{l}^{m}, \quad (A2)$$

and therefore for any superposition $|\bar{\lambda}\rangle$ of bright states:

$$S_{n}(\bar{\lambda}) = \|\hat{S}_{B}^{n}|\bar{\lambda}\rangle\|^{2} = \sum_{l} |\zeta_{l}|^{2n} C_{l,n}(\bar{\lambda}). \quad (A3)$$

The coefficients are given by:

$$C_{l,n}(\bar{\lambda}) = \|\hat{M}_{l} + \zeta_{l}^{-1} \hat{N}_{l}|\bar{\lambda}\rangle\|^{2}. \quad (A4)$$

Using Eq. (A2), the triangle inequality, as well as $|\zeta_{l}|^{-m} < |\zeta_{l}|^{-p_{l}}$ and $\binom{n}{m} \leq \binom{p_{l}}{m} \leq n^{m}/m!$, for $n > 2p_{l}$ and $m < p_{l}$, one can bound these coefficients with:

$$C_{l,n}(\bar{\lambda}) \leq \|\hat{M}_{l} + \zeta_{l}^{-1} \hat{N}_{l}|\bar{\lambda}\rangle\|^{2} + \sum_{m=1}^{p_{l}-1} \binom{n}{m} |\zeta_{l}|^{-2m} \|\hat{N}_{l}^{m}|\bar{\lambda}\rangle\|^{2}$$

$$\leq \|\hat{M}_{l}|\bar{\lambda}\rangle\|^{2} + \binom{n}{p_{l}} |\zeta_{l}|^{-2p_{l}} \sum_{m=1}^{p_{l}-1} \|\hat{N}_{l}^{m}|\bar{\lambda}\rangle\|^{2}. \quad (A5)$$

Hence $C_{l,n}(\bar{\lambda}) \leq \alpha n^{2p_{l}}$. Considering the largest of all $p_{l}$ denoted by $p_{*}$ and the eigenvalue $\zeta_{*}$ closest to the unit circle, we find that the survival probability will eventually decay exponentially

$$S_{n}(\bar{\lambda}) \leq O(n^{2p_{*}}|\zeta_{*}^{2n}|). \quad (A6)$$

From writing $[1 - \hat{D}]$ in the energy basis and inspecting the definition of $\hat{S}_{B}$, it is evident that each $\zeta_{l}$ has to be a convex sum of the phase factors $e^{i\tau E_{l}/\hbar}$. In fact, Ref. [9] shows that the non-trivial eigenvalues $\zeta_{l}$ can be obtained from the stationary points of a certain 2d-Coulomb force fields and must lie inside the unit disk. This approach is investigated in detail in Ref. [63]. Therefore we find another reason why the non-trivial eigenvalues obey $|\zeta_{l}| < 1$.

This is easily seen for a system with only two bright states $|\beta_{1}\rangle$ and $|\beta_{2}\rangle$ corresponding to two energies $E_{1}$ and $E_{2}$. The detection state is decomposed into the bright states via $|\psi_{a}\rangle = a|\beta_{1}\rangle + b|\beta_{2}\rangle$, where $|a|^{2} + |b|^{2} = 1$. In this case, $\hat{S}_{B}$ reads in a matrix notation:

$$\hat{S}_{B} = \begin{pmatrix} 1 - |a|^{2} & ab^{*} & e^{-i\tau E_{1}} & 0 \\ ba^{*} & 1 - |b|^{2} & e^{-i\tau E_{2}} & 0 \\ e^{i\tau E_{1}} & e^{i\tau E_{2}} & |a|^{2} & e^{-i\tau E_{2}} \\ 0 & 0 & e^{-i\tau E_{1}} & |b|^{2} \end{pmatrix}. \quad (A7)$$

This matrix has one vanishing eigenvalue $\zeta_{1} = 0$. The other one is equal to $\zeta_{2} = |a|^{2} e^{-i\tau E_{2}/\hbar} + |b|^{2} e^{-i\tau E_{1}/\hbar}$, i.e. it is a convex sum of the phase factors, which lies inside the unit circle. Similar reasoning applies to systems with more than two bright states.

In systems with finite dimensional Hilbert space we are finished, because there is some minimum decay rate $\lambda^{*} = \min\{-2 \ln |\zeta_{l}| \} > 0$. Infinity dimensional systems may behave more subtly. In the thermodynamic limit the eigenvalues $\zeta_{l}$ can get infinitely close to the unit circle. In that case unity is an accumulation point of the sequence $\{\zeta_{l}\}$ and $\Lambda^{*} = \inf\{-2 \ln |\zeta_{l}| \} = 0$. In the view of the spectral theorem of Ref. [9], there are two options: Either, one is lucky and $S_{n}$ decays to zero, albeit slower than exponentially fast, or not. $S_{n} \to 0$ will still hold in the first case. The spectral theorem of Ref. [9] says, that this is the case when the spectrum of $\hat{U}(\tau)$ has no absolutely continuous part. If such a part of the spectrum is present, on the other hand, then $S_{n}$ will converge to some positive value, and we find $P_{\text{det}}(\beta) < 1$. This is
obviously only possible in infinite dimensional systems. Some more mathematical details about $\mathcal{S}_B$ can be found in Appendix A of Ref. [10].

**Appendix B: Aleksandrov’s theorem in a two-level system**

A proof of the powerful Eq. (14) lies far outside the scope of this paper, the interested reader is referred to [70]. In order to nevertheless help to understand the origin of the equality, we consider the case when there are only two different energy levels, $E_1$ and $E_2$. The Hamiltonian of such a system would read $H = P_1 E_1 + P_2 E_2$. It is not necessary to specify the eigenstates and degeneracies, as long as the overlap of each eigenspace with the detection state is given. For simplicity, we assume that $\langle \psi_d | P_1 | \psi_d \rangle = \langle \psi_d | P_2 | \psi_d \rangle = 1/2$. (In fact, the Aleksandrov ansatz does not “see” individual eigenstates $|E_{l,m}\rangle$, but just the overlap with the eigenspace projectors $P_l$.) We can replace $\langle \psi_d | P_1 | \psi_{in} \rangle = \nu(\lambda_1) \langle \psi_d | P_1 | \psi_d \rangle = \nu(\lambda_1)/2$, and similarly with $\nu(\lambda_2)$. $\varphi(z)$ reads:

$$\varphi(z) = \frac{ze^{-i\lambda_1}\nu(\lambda_1)}{1 - ze^{-i\lambda_1}} + \frac{ze^{-i\lambda_2}\nu(\lambda_2)}{1 - ze^{-i\lambda_2}}.$$  

(B1)

Its absolute value on the unit circle equals:

$$|\varphi(e^{i\theta})|^2 = \frac{\nu(\lambda_1)e^{i\lambda_1}(e^{i\lambda_2} - e^{i\theta}) + \nu(\lambda_2)e^{i\lambda_2}(e^{i\lambda_1} - e^{i\theta})}{e^{i\lambda_1} + e^{i\lambda_2} - 2e^{i\theta}} \times \frac{\nu(\lambda_1)e^{-i\lambda_1}(1 - e^{i\theta-\lambda_2}) + \nu(\lambda_2)e^{-i\lambda_2}(1 - e^{i\theta-\lambda_1})}{2 - e^{i\theta}(e^{-i\lambda_1} + e^{-i\lambda_2})}.$$  

(B2)

When this expression is plugged into Eq. (14), the left-hand side integral can be treated by a variable change $z = e^{i\theta}$, and $d\theta = dz/(iz)$. The result is a complex contour integral which is solved by residue inspection. The integrand has only two simple poles inside the unit circle. One lies at the origin $z = 0$ and the other is defined by the linear term in the denominator of the first line in Eq. (B2). After some lengthy algebra, we find that the prefactors of the cross-terms $\nu(\lambda_1)^*\nu(\lambda_2)$ and $\nu(\lambda_2)^*\nu(\lambda_1)$ vanish. Collecting the residues of the diagonal terms, however, we find one half:

$$\oint_{|z|=1} \frac{dz}{2\pi i z} \frac{e^{i\lambda_2}(1 - ze^{-i\lambda_2})}{e^{i\lambda_1} + e^{i\lambda_2} - 2z - z(e^{-i\lambda_1} + e^{-i\lambda_2})} = \frac{1}{2}.$$  

(B3)

We obtain the same result for the integral proportional to $|\nu(\lambda_2)|^2$. Hence:

$$P_{\text{det}} = \frac{1}{2} \left[ |\nu(\lambda_1)|^2 + |\nu(\lambda_2)|^2 \right],$$  

(B4)

which is the result of Eq. (14) and Eq. (1) for the chosen example system. A demonstration with more than two energy levels using the same method first becomes tedious and soon unfeasible. In a system with $w$ bright states there are $w$ different phases $\lambda_i$ in $\mu(\theta)$ and $w$ poles in the integrand of Eq. (9) after switching again to $z = e^{i\theta}$. Aleksandrov’s theorem “magically” ensures that the residues of these poles are exactly given by $|\nu(\lambda_i)|^2 \langle \psi_d | P_l | \psi_d \rangle$ thus giving Eq. (1).

**Appendix C: Additional Example Graphs and their $P_{\text{det}}$ and dark states**

In this appendix, we present the calculation for the spectrum, eigenstates, $P_{\text{det}}$ and dark states for the collection of graphs illustrated in Fig. 4(C-F), (A) and (B) having been presented in the main text.

1. **Complete graph**

We now consider a graph with $L$ nodes, in which each node is connected to each other node, see C1C. Its Hamiltonian reads:

$$\hat{H} = -\gamma \sum_{r,r'=1}^L (1 - \delta_{r,r'}) |r\rangle\langle r'|$$  

(C1)

We pick one node as the detection node, say $r_d$. The system has only two energy levels, namely $E_1 = -\gamma(L-1)$ and $E_2 = \gamma$, the latter being $(L-1)$-fold degenerate. The eigenstates are:

$$|E_1\rangle = |j\rangle := \sum_{r=1}^L \frac{|r\rangle}{\sqrt{L}}, \quad |E_{2,m}\rangle = \sum_{r=1}^L \frac{e^{i2\pi mc/L} |r\rangle}{\sqrt{L}}.$$  

(C2)

where $m = 1, 2, \ldots, L-1$ and $|j\rangle$ is the uniform state over all nodes. Therefore, none of these energy levels is completely dark with respect to a localized detection state. The two bright eigenstates are:

$$|\beta_1\rangle = |j\rangle = \frac{1}{\sqrt{L}} \sum_{r=1}^L |r\rangle, \quad |\beta_2\rangle = \frac{\sqrt{L} |r_d\rangle - |j\rangle}{\sqrt{L-1}}.$$  

(C3)

Computing the overlap with the bright space, we find:

$$P_{\text{det}}(r_{in}) = \begin{cases} 1, & r_{in} = r_d \\ \frac{1}{L-1}, & r_{in} \neq r_d \end{cases}.$$  

(C4)

2. **Star**

The next example is a star graph with a center node $|0\rangle$ and $L$ nodes in the periphery, see Fig. C1D. The Hamiltonian reads:

$$\hat{H} = -\gamma \sum_{r=1}^L [ |0\rangle\langle r| + |r\rangle\langle 0|].$$  

(C5)
The system has three energy levels \( E_1 = -\gamma \sqrt{L}, \) \( E_2 = 0 \) and \( E_3 = \gamma \sqrt{L}, \) of which \( E_2 \) is \((L - 1)\)-fold degenerate. The eigenstates are:

\[
\begin{cases}
|E_1\rangle = \frac{|0\rangle + |j\rangle}{\sqrt{2}}, & |E_3\rangle = \frac{|0\rangle - |j\rangle}{\sqrt{2}} \\
|E_{2,m}\rangle = \frac{1}{\sqrt{L}} \sum_{r=1}^{L} e^{i \frac{2 \pi}{L} m r} |r\rangle,
\end{cases}
\]  

where \( m = 1, 2, \ldots, L - 1. \) \(|j\rangle := \sum_{r=1}^{L} |r\rangle / \sqrt{L}\) is the uniform state over the periphery.

If the detection takes place on the center node, the energy level \( E_2 \) is completely dark. Would detection take place in the periphery, there would be no completely dark energy levels. \( P_{\text{det}} \) is computed from Eq. (1). The calculations are similar to the ones of the complete graph:

\[
P_{\text{det}}(r_{in}) = \begin{cases} 
1, & r_d = r_{in} \\
\frac{1}{L}, & r_d = 0, r_{in} \neq 0 \\
1, & r_d \neq 0, r_{in} = 0 \\
\frac{1}{L}, & r_d \neq 0, r_{in} \neq 0
\end{cases}
\]  

3. Hypercube

The next example is the \( d \)-dimensional hypercube. An example graph for \( d = 3 \) is depicted in Fig. C1E. In this system there are \( 2^d \) different states which can be represented by a \( d \)-length string consisting only of zeros and ones, i.e. a string of \( d \) bits, \(|\psi\rangle = \otimes_{j=1}^{d} |b_j\rangle\), where \( b_j = 0_j, 1_j \) is the \( j \)-th bit. The allowed transitions correspond to bit flips, and the Hamiltonian can be written as a sum of Pauli \( \hat{\sigma}_x \)-matrices:

\[
\hat{H} = -\gamma \sum_{j=1}^{d} \hat{\sigma}_x^{(j)},
\]  

where \( \hat{\sigma}_x^{(j)} \) only acts on \(|b_j\rangle\), and \( \hat{\sigma}_x^{(j)} |0_j\rangle = |1_j\rangle, |1_j\rangle = |0_j\rangle. \) (As an example: \( \hat{\sigma}_x^{(1)} |110\rangle = |010\rangle, \hat{\sigma}_x^{(2)} |110\rangle = |100\rangle, \) and \( \hat{\sigma}_x^{(3)} |110\rangle = |111\rangle. \) The relation to a word of \( d \) bits with bit-flip transitions make this model particularly relevant for the quantum computation field [2, 61]. The energy levels are \( E_l = -\gamma (2l - d), \) \( l = 0, 1, \ldots, d \), each is \( \left(\begin{array}{c} d \\ l \end{array}\right) \)-fold degenerate. Let \( x_j = \pm 1 \) and define \(|x_j\rangle := (|0_j\rangle + x_j |1_j\rangle) / \sqrt{2}\), then \( \hat{\sigma}_x^{(j)} |x_j\rangle = x_j |x_j\rangle \). Hence the energy eigenstates have the form:

\[
|E_{l,m}\rangle = \otimes_{j=1}^{d} |x_j\rangle,
\]  

such that \( \sum_{j=1}^{d} x_j = 2l - d \) and \( m \) enumerates all \( \left(\begin{array}{c} d \\ l \end{array}\right) \) combinations. In each bit, we have the important relation:

\[
\langle b_j | x_j \rangle = \frac{1}{\sqrt{2}} (-1)^{b_j} \sqrt{1 - x_j},
\]  

which has a negative sign when \( b_j = 1 \) and \( x_j = -1 \) and a positive sign otherwise.

We try to detect the system in the node with all bits equal to zero, i.e. \(|r_d\rangle = \otimes_{j=1}^{d} |0_j\rangle\). From Eq. (C10) we find that all energy eigenstates have the same overlap with the detection state: \( \langle r_d | E_{l,m} \rangle = 2^{d/2} \). Therefore one finds the following bright eigenstates:

\[
|\beta_l\rangle = \frac{1}{\sqrt{\left(\begin{array}{c} d \\ l \end{array}\right)}} \sum_{\{x_j\}} \otimes_{j=1}^{d} |x_j\rangle,
\]  

where the sum runs over all combinations \( \{x_j\} \), where \(|\{x_j\}\rangle = \sum_{j=1}^{d} x_j = 2l - d \).

We now pick any localized initial state \(|r_{in}\rangle = \otimes_{j=1}^{d} |b_{in,j}\rangle\), which differs from \(|r_d\rangle\) in exactly \( \xi \) bits. (\( \xi \) is the so-called Hamming-distance between both states.) \( P_{\text{det}} \) only depends on \( \xi \) and can be obtained from Eq. (1), and Eq. (C11) and some complicated combinatoric computation. We do not present this here, and instead refer to Ref. [42], where \( P_{\text{det}} \) was computed in a different way:

\[
P_{\text{det}}(r_{in}) = \frac{1}{\left(\begin{array}{c} d \\ \xi \end{array}\right)}.
\]

This shows that only two transitions in a hypercube are actually reliable, in the sense that \( P_{\text{det}} = 1 \) the return to the initial node and the traversal to the opposing node, see Fig. 4E.

4. A tree graph

Similar to the hypercube, trees are important examples from quantum computation, in particular because
they feature an exponential speedup [59, 60] relative to classical algorithms.

We will consider here only a small binary tree with two generations, see Fig. C1F. The top of the tree is called the root, the bottom nodes are called leaves of the tree. Its Hamiltonian reads:

\[ \hat{H} = -\gamma \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \]  

The corresponding energy levels are: \( E_1 = -2\gamma \), \( E_2 = -\sqrt{2}\gamma \), \( E_3 = 0 \), being thrice degenerate, as well as, \( E_4 = \sqrt{2}\gamma \) and \( E_5 = 2\gamma \). The energy eigenstates are:

\[
\begin{align*}
|E_1\rangle &= (2, 2, 2, 1, 1, 1, 1)^T/4 \\
|E_2\rangle &= (0, -\sqrt{2}, \sqrt{2}, -1, -1, 1, 1)^T/\sqrt{8} \\
|E_{3,1}\rangle &= (0, 0, 0, 1, -1, 0, 0)^T/\sqrt{2} \\
|E_{3,2}\rangle &= (0, 0, 0, 0, 0, 1, -1)^T/\sqrt{2} \\
|E_{3,3}\rangle &= (0, 0, 0, 1, 1, 1, 1)^T/\sqrt{8} \\
|E_4\rangle &= (0, \sqrt{2}, -\sqrt{2}, -1, -1, 1, 1)^T/\sqrt{8} \\
|E_5\rangle &= (2, -2, -2, 1, 1, 1, 1)^T/4
\end{align*}
\]

We see that some of the energy levels may be completely dark depending on the detection state.

a. Detection on the root. The choice \( |r_d\rangle = |0\rangle \) renders \( E_2 \) and \( E_4 \) completely dark. The third energy level yields the bright eigenstate \( |\beta_3\rangle = |E_{3,3}\rangle \). Doing the calculations with Eq. (1), we find the total detection probability equal to 1/2 for nodes \( |1\rangle \) and \( |2\rangle \) and equal to 1/4 for the leaves, i.e.

\[
P_{\text{det}}(r_{in}) = \begin{cases} 
1, & r_{in} = 0 \\
\frac{1}{2}, & r_{in} = 1, 2 \\
\frac{1}{4}, & \text{otherwise}
\end{cases}
\]  

b. Detection in the middle. Now, we choose \( |r_d\rangle = |1\rangle \). Then \( E_3 \) is completely dark. All remaining energy levels are non-degenerate, hence there are no other dark states. We find from Eq. (1):

\[
P_{\text{det}}(r_{in}) = \begin{cases} 
\frac{1}{2}, & r_{in} = 0 \\
1, & r_{in} = 1, 2 \\
\frac{3}{8}, & \text{otherwise}
\end{cases}
\]  

c. Detection in the leaves. We choose \( |r_d\rangle = |3\rangle \), then there are two dark state in the energy level \( E_3 \). The bright state in this level is

\[
|\beta_3\rangle = \frac{2}{\sqrt{40}} (|0\rangle - 5|3\rangle + 4|4\rangle - 5|5\rangle - 6|6\rangle).
\]

The remaining bright states are equal to the eigenstates. This results in:

\[
P_{\text{det}}(r_{in}) = \begin{cases} 
\frac{3}{5}, & r_{in} = 0, 4 \\
1, & r_{in} = 1, 2, 3 \\
\frac{2}{5}, & r_{in} = 5, 6
\end{cases}
\]

All these results are summarized in Fig. 4.

[1] E. Bach, S. Coppersmith, M. P. Goldschen, R. Joynt, and J. Watrous, Journal of Computer and System Sciences 69, 562 (2004).
[2] H. Krovi and T. A. Brun, Physical Review A 73, 032341 (2006).
[3] H. Krovi and T. A. Brun, Physical Review A 74, 042334 (2006).
[4] H. Krovi and T. A. Brun, Physical Review A 75, 062332 (2007).
[5] M. Stefaňák, I. Jex, and T. Kiss, Physical Review Letters 100, 020501 (2008).
[6] M. Varbanov, H. Krovi, and T. A. Brun, Physical Review A 78, 022324 (2008).
[7] F. Caruso, A. W. Chin, A. Datta, S. F. Huelga, and M. B. Plenio, The Journal of Chemical Physics 131, 105106 (2009), https://aip.scitation.org/doi/pdf/10.1063/1.3223548.
[8] E. Agliari, O. Müllken, and A. Blumen, International Journal of Bifurcation and Chaos 20, 271 (2010).
[9] F. A. Grünbaum, L. Velázquez, A. H. Werner, and R. F. Werner, Communications in Mathematical Physics 320, 543 (2013).
[10] J. Bourgain, F. A. Grünbaum, L. Velázquez, and J. Wilkening, Communications in Mathematical Physics 329, 1031 (2014).
[11] P. L. Krapivsky, J. M. Luck, and K. Mallick, Journal of Statistical Physics 154, 1430 (2014).
[12] S. Dhar, S. Dasgupta, and A. Dhar, Journal of Physics A: Mathematical and Theoretical 48, 115304 (2015).
[13] S. Dhar, S. Dasgupta, A. Dhar, and D. Sen, Physical Review A 91, 062115 (2015).
[14] P. Sinkovicz, Z. Kurucz, T. Kiss, and J. K. Asbóth, Physical Review A 91, 042108 (2015).
[15] P. Sinkovicz, T. Kiss, and J. K. Asbóth, Physical Review A 93, 050101 (2016).
[16] S. Lahiri and A. Dhar, Physical Review A 99, 012101 (2019).
[17] H. Friedman, D. A. Kessler, and E. Barkai, Journal of Physics A: Mathematical and Theoretical 50, 04LT01 (2017).
[18] H. Friedman, D. A. Kessler, and E. Barkai, Physical Review E 95, 032141 (2017).
[19] F. Thiel, E. Barkai, and D. A. Kessler, Physical Review Letters 120, 040502 (2018).
[20] F. Thiel, D. A. Kessler, and E. Barkai, Physical Review A 97, 062105 (2018).
