Symmetry Dependence of Localization in Quasi-1-dimensional Disordered Wires

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The crossover in energy level statistics of a quasi-1-dimensional disordered wire as a function of its length $L$ is used, in order to derive its averaged localization length, without magnetic field, in a magnetic field and for moderate spin orbit scattering strength. An analytical function of the magnetic field for the local level spacing is obtained, and found to be in excellent agreement with the magnetic field dependent activation energy, recently measured in low-mobility quasi-one-dimensional wires [1]. This formula can be used to extract directly and accurately the localization length from magneto-resistance experiments. In general, the local level spacing is shown to be proportional to the excitation gap of a virtual particle, moving on a compact symmetric space. PACS-numbers: 72.15.Rn,73.20.Fz,02.20.Qs

In disordered wires quantum interference results in localization of all states for arbitrary disorder strength, if the wire is infinite [2,3]. It has been discovered that the localization length depends on the global symmetry of the wire [4]: $L_{c} = \beta \tau \hbar v S D_{0}$, where $\beta = 1, 2, 4$, corresponding to no magnetic field, finite magnetic field, and strong spin-orbit scattering or magnetic impurities, respectively. $\nu(E)$ is the electronic density of states in the wire. $D_{0}$ is the classical diffusion constant of the electrons in the wire, and $S$ its crosssection. This result was obtained by calculating the spatial decay of the density correlation function for wires whose thickness exceeds the mean free path $l$. Independently, it was obtained by calculating the transmission probability through thin, few channel wires [5]. Recently, the doubling of the localization length was observed in sub-micron thin wires of doped GaAs by Khavin, Gershenson and Bogdanov, who found a continuously decreasing activation energy when the magnetic field is increased, saturating indeed at one half of its field free value [6]. However, the crossover as function of moderate symmetry breaking fields defied any attempt to study it with nonperturbative methods. The only results were based on a heuristic approach by Bouchaud [7], a kind of semiclassical analysis by Imry and Lerner [8] and numerical studies [9]. Recently, however, Kolesnikov and Efetov [10] succeeded to tackle that complex problem and derived the density-density correlation function with the supersymmetry method in the crossover regime. The result does not scale with a single localization length.

Here, the problem of the crossover is addressed from a different perspective. The statistics of discrete energy levels of a finite coherent, disordered metal particle is an efficient way to characterize its properties [11]. This can be studied by calculating a disorder averaged autocorrelation function between two energies at a distance $\omega$ in the energy level spectrum. It is an oscillatory function whose amplitude decays with a power law when the energy levels in the vicinity of the central energy $E$ are extended. A Gaussian decaying function is a strong indication that all states are localized.

The autocorrelation function of spectral determinants (ASD) is defined by $C(\omega) = C(\omega)/C(0)$, $C(\omega) = \langle \det(E + \omega/2 - H)\det(E - \omega/2 - H) \rangle$. $E$ is a central energy. We consider the Hamiltonian of disordered electrons

$$H = [p - qA]^2/2m + V(x),$$

(1)

where $q$ is the electron charge, $c$ the velocity of light. $V(x)$ is taken to be a Gaussian distributed random function $\langle V(x) \rangle = 0$, and $\langle V(x)V(x') \rangle = (\Delta SL/2\pi)\delta(x - x')$, which models randomly distributed, uncorrelated impurities in the sample. $\Delta$ is the mean level spacing, $1/\tau$ the elastic scattering rate. The vector potential is used in the gauge $A = (-By,0,0)$, where $x$ is the coordinate along the wire of length $L$, $y$ the one in the direction perpendicular both to the wire and the magnetic field $B$, which is directed perpendicular to the wire. The angular brackets denote averaging over impurities.

The ASD appears as an intermediate step in the Grassman replica trick and as the compact sector of the supersymmetric field theory of disordered systems [12]. It is a non-self-averaging quantity. It has been shown recently, that the ASD can distinguish between an uncorrelated spectrum of localized states and a correlated spectrum of extended states in the vicinity of $E$. This way, the metal-insulator crossover as a function of the length of a disordered wire has been explored with the ASD. A localization length $\xi_c$ in a moderate magnetic field has been derived as the crossover length scale and shown to coincide with $L_c(\beta = 2)$ [11]. The ASD also yields qualitative information on the location of localized states in a quantum-Hall-system [12].

The impurity averaged ASD can be written as a partition function [12]

$$\bar{C}(\omega) = \text{Tr} \exp(-L\bar{H}|Q|),$$

(2)

where $\bar{H}$ is an effective Hamiltonian of matrices $Q$ on a compact manifold, determined by the symmetries of the Hamiltonian $H$ of disordered electrons. Thus, the problem reduces to the one of finding the spectrum of the effective Hamiltonian $\bar{H}$.

There is a finite gap $E_G$ between the ground state energy and the energy of the next excited state of $\bar{H}(\omega = 0)$.
For a long wire, $L_E G \gg 1$, the ASD becomes, according to Eq. 2, $C(\omega) = \exp(-\text{const.}L\omega^2/E_G)$. This is typical for a a spectrum of localized states [12]. In the other limit $L_E G \ll 1$, all modes of $\hat{H}$ do contribute to the trace in the partition function Eq. 2 with equal weight, yielding the correlation function of a spectrum of extended states [1]. Thus, the crossover length $\xi, \sim 1/E_G$ can be identified with an averaged localization length.

Being a product of two spectral determinants, $4\alpha$-component Grassman fields are needed to get the functional integral representation of the ASD. Here, $\alpha = 1$, when the Hamiltonian is independent of the spin of the electrons, and each level is doubly spin degenerate. There is one pair of Grassman fields for each determinant in the ASD and each pair is composed of a Grassman field and each pair is composed of a Grassman field and the corresponding time reversed one. One notes a global invariance under rotations of these vectors.

Averaging over impurities according to Eq. (1), one gets an interacting theory of Grassman fields with inter- action strength $1/\tau$. It can be decoupled with a Gaussian integral over $4\alpha \times 4\alpha$ matrices which preserve the global rotational invariance. The Grassman fields can now be integrated out exactly. Finding the saddle point of the integral over matrices and integrating out longitudinal (massive) modes, for $\omega < 1/\tau$ and $1/\tau \gg \Delta$, the ASD reduces to a functional integral over transverse modes $Q$, being elements of respective symmetric spaces. For disordered wires whose thickness $W$ is larger than the mean free path $l$ but smaller than the magnetic length $W < l_B$, the action reduces to the one of the one-dimensional compact nonlinear sigma model [13,14],

$$C(\omega) = \int \prod_x dQ(x) \exp(-F[Q]), \quad (3)$$

$$F[Q] = \alpha \frac{1}{16} L_{CU} \int_0^L dx \text{Tr} \left[ (\nabla^2 Q(x))^2 - \frac{q^2}{\hbar^2} B^2 [Q, \tau_3]^2 \right] + i \alpha \frac{\pi \omega}{4 \Delta} \int \frac{dx}{L} \text{Tr} \Lambda_3 Q(x). \quad (4)$$

where $L_{CU} = L_C (\beta = 2) = 2\pi \hbar \nu SD_0$ is the localization length in the wire in a moderately strong magnetic field [4]. The integral over the crosssection $S$ of the wire is done, giving $y^2$. Here, and in the following, $\Lambda_i$ are the Pauli matrices in the subsbasis of the left and the right spectral determinant, $\tau_i$ the ones in the subbasis spanned by time reversal and $\sigma_i$ the ones in the space spanned by the spinor, for $i = 1, 2, 3$. The fluctuations of the matrices $Q$ are transverse, $Q^2 = 1$, and restricted by global rotational invariance to $Q^+ = Q$. For $\alpha = 1$ in addition $Q^T C = C Q$, where $C = -i\sigma_2$. This constrains $Q$ to be an element of the group defined on the compact symplectic symmetric space $Sp(2)/(Sp(1) \times Sp(1))$. For a moderate magnetic field, $Q$ is reduced to a $2 \times 2$- matrix by the broken time reversal symmetry. This reduces the space of $Q$ to $U(2)/(U(1) \times U(1))$. For $\alpha = 2$ the matrix $C$ is, due to the time reversal of the spinor, substituted by $i\sigma_3 \tau_1$. Both magnetic impurities and spin-orbit scattering reduce the Q matrix to unity in spin space. Thus, $C$ has effectively the form $\tau_1$. The condition $Q^T C = C Q$ leads therefore to a new symmetry class, when the spin symmetry is broken but the time reversal symmetry remains intact. This is the case for moderately strong spin-orbit scattering. Then, $Q$ are $4 \times 4$- matrices on the orthogonal symmetric space $O(4)/(O(2) \times O(2))$ [15]. With magnetic impurities both the spin and time reversal symmetry is broken, and the $Q$- matrices are in the unitary symmetric space $U(2)/(U(1) \times U(1))$ as for a moderate magnetic field and spin degenerate levels. The difference in the prefactor $\alpha$ in the action, Eq. (3), remains. One can extend this approach to other compact symmetric spaces with physical realizations, see Ref. [16] for a complete classification. For $\omega/\Delta < L_{CU}/L$, the spatial variation of $Q$ can be neglected and one retains the same ASD as for random matrices of orthogonal, unitary and symplectic symmetry, characterizing the energy levels of an ergodic particle without magnetic field, with magnetic field, and with spin-orbit scattering, respectively [17]. The confusing fact, that random orthogonal matrices result for the ASD in a functional integral over compact symplectic $Q$- matrices and vice versa, results from the sign change of the Grassman variables under time reversal.

We can derive the corresponding Hamiltonian $\tilde{H}$ by means of the transfer matrix method, reducing the one-dimensional integral in Eq. (3) to a single functional integral. Thus, the ASD is obtained in the simple form of Eq. 2, with the effective Hamiltonian

$$\tilde{H}(\omega = 0) = \frac{1}{\alpha L_{CU}} (-4 \Delta Q - \frac{1}{16} X^2 Tr Q^2[Q^3], \quad (5)$$

$\Delta Q$ is that part of the Laplacian on the symmetric space, which couples to $Tr[\Lambda_3 Q]$. $X = 2\pi \alpha L_{CU} (y^2)^{1/2} B/\phi_0$, where $\phi_0 = q/\hbar$ is the flux quantum.

The problem is now equivalent to a particle with “mass” $(\alpha/8) L_{CU} (E)$ moving on the symmetric space of $Q$ in a harmonic potential with “frequency” $X/\alpha L_{CU}$, and in an external field $i\hbar (\pi/4) \omega / (L\Delta)$, in “time” $x$, the coordinate along the wire. To find the ASD as a function of $\omega$ and the length of the wire $L$, one can do a Fourier analysis in terms of the spectrum and eigenfunctions of the effective Hamiltonian at zero frequency, $\tilde{H}(\omega = 0)$ [18]. For $L \gg \xi$, only the gap between the ground and the first excited state of $\tilde{H}(\omega = 0)$, determines the ASD.

Without magnetic field, $B = 0$, the Laplacian is

$$\Delta Q = \partial_{\lambda_C} (1 - \lambda_C^2) \partial_{\lambda_C} + 2 \frac{1 - \lambda_C^2}{\lambda_C} \partial_{\lambda_C}$$
where \( \lambda_{C,D} \in [-1,1] \). Its ground state is 1 and the first excited state is \( \lambda_{C} \lambda_{D} \). Thus, the gap is
\[
E_{G}(B = 0) = 16/L_{CU}.
\]
For moderate magnetic field, with the condition \( L_{CU}(y^2)^{(1/2)}B \gg \phi_{0} = h/q \) all degrees of freedom arising from time reversal invariance are frozen out, due to the term \( Tr_{Q}[Q,\tau_{3}]^2 = 16(\lambda_{C}^2 - 1) \) which fixes \( \lambda_{C}^2 = 1 \). Then, the Laplacian reduces to
\[
\Delta_{Q} = \partial_{\lambda_{D}}(1 - \lambda_{D}^2)\partial_{\lambda_{D}}.
\]
Its eigenfunctions are the Legendre polynomials. There is a gap above the isotropic ground state of magnitude
\[
E_{G}(X \gg 1) = 8/L_{CU}.
\]
For moderate magnetic impurity scattering exceeding the local level spacing, \( 1/\tau_{3} > \Delta_{C} \), the Laplacian is given by Eq.
(8). Thus, due to \( \alpha = 2 \), the gap is reduced to \( E_{G}(1/\tau_{3} > \Delta_{C}) = 4/L_{CU} \). For moderately strong spin-orbit scattering \( 1/\tau_{SO} > \Delta_{C} \), the Laplace operator is
\[
\Delta_{Q} = \sum_{l=1,2} \partial_{\lambda_{l}}(1 - \lambda_{l}^2)\partial_{\lambda_{l}},
\]
where \( \lambda_{1,2} \in [-1,1] \). The ground state is \( \psi_{0} = 1 \), the first excited state is doubly degenerate, \( \psi_{11} = \lambda_{1}, \psi_{12} = \lambda_{2} \). Thus, the gap is the same as for magnetic impurities,
\[
E_{G}(1/\tau_{SO} > \Delta_{C}) = 4/L_{CU}.
\]
An external magnetic field lifts this degeneracy but does not change the gap.

Using the crossover in energy level statistics as the definition of a localization length as above, we get in a quasi-1-dim. wire,
\[
\xi_{x} = 1/E_{G}(\beta) = (1/16)^{\beta}L_{CU},
\]
where \( \beta = 1,2,4 \) corresponding to no magnetic field, finite magnetic field, and strong spin-orbit scattering or magnetic impurities, respectively. Comparing with the known equation for the localization length, \( L_{c} \), we find that the dependence of the ratios \( \beta \) on the symmetry are in perfect agreement with the result as obtained from the spatial decay of the density-density correlation function \( \langle \phi_{x} \phi_{x} \rangle \), while it defers by the overall constant 1/8.

This relation can be proven directly. The ASD at zero frequency \( C(0)_{L} \) of the wire of length \( L \), becomes, when the wire is divided into two parts, \( C(0)_{L/2}^2 \). For \( L \to \infty \), we find that the relative difference is:
\[
f(L) = \frac{C(0)_{L/2}^2}{C(0)_{L}} - 1 = 2\exp(-LE_{G}/2),
\]
expansively decaying with the length \( L \). \( f(L) \) can be estimated, following an argument by Mott [2]: When the two halves of the wire get connected, the eigenstates of the two separate halves become hybridized and the Eigenenergy of a state \( \psi_{n} \) is changed by \( \pm \lambda_{C} \exp(-2x_{n}/L_{C}) \). \( x_{n} \) is random, depending on the position of an eigenstate with closest energy in the other half of the wire. Thus, averaging over \( x_{n} \) gives:
\[
f(L) \sim \exp(-4L/L_{C}).
\]
Comparison with Eq. (13) yields indeed \( 1/L_{C} = 8E_{G} \).

The Crossover Behaviour of the Localization Length

This direct relation of the ASD to the spatial overlap between wavefunctions allows us to extend this approach to get an analytical solution for the crossover behaviour of the localization length and the local level spacing as a magnetic field is turned on, and without strong spin-orbit scattering. While a heuristic approach [3] and numerical studies [4] seemed to indicate a continuous increase of the localization length, the analytical result [10] does show that both limiting localization lengths \( L_{C}(\beta = 1) \) and \( L_{C}(\beta = 2) \) are present in the crossover regime and that there is no single parameter scaling. This is explained by arguing that the far tails of the wavefunctions do cover a large enough area to have fully broken time reversal symmetry, decaying with the length scale \( L_{c}(\beta = 2) \) even if the magnetic field is too weak to affect the properties of the bulk of the wavefunction, which does decay at smaller length scales with the shorter localization length \( L_{C}(\beta = 1) \), corresponding to the time reversal symmetric case. The quantity studied there is the impurity averaged correlation function of local wavefunction amplitudes and its momenta at a fixed energy \( c: Y(\epsilon) = \langle \sum_{a} | \psi_{a}(0) \rangle | \psi_{a}(x) \rangle |^{2} \delta(\epsilon - \epsilon_{a}) \rangle \). It is averaged over a distribution of eigenfunctions in different impurity representations. Thus, each eigenfunction could decay exponentially with a single localization length, but with a distribution which has two maxima, at \( L_{C}(\beta = 1) \) and \( L_{C}(\beta = 2) \), whose weight is a function of the magnetic field in the crossover regime. It is interesting to ask if this property of the eigenfunctions does have a consequence on the energy level statistics as well, or, if that is still governed by a single parameter as the magnetic field is varied.

The effective Hamiltonian for moderate magnetic fields is given, without spin dependent scattering, \( \alpha = 1 \), by:
\[
\hat{H} = \frac{1}{L_{CU}}(-4\Delta_{Q} + X^{2}(1 - \lambda_{C}^{2})),
\]
where the Laplacian is Eq. [1-3] and \( X = 2\pi\phi/\phi_{0} \), where \( \phi \) is the magnetic flux through an area \( A = L_{CU}(y^{2})^{1/2} \).

In the limit \( X \to 0 \) the ground state and first excited state approach 1, \( \lambda_{C}\lambda_{D} \), respectively. In the limit \( X \gg 1 \), \( \lambda_{C}^{2} \) becomes fixed to 1. Thus, the Ansatz \( \psi_{0}(\lambda_{C}) = \exp(A_{0}(1 - \lambda_{C}^{2})) \), and \( \psi_{1}(\lambda_{C},\lambda_{D}) = \lambda_{C}\lambda_{D} \exp(A_{1}(1 -
with the suggestion by Bouchaud [6].

This solution is valid in both the limits \( X \ll 1 \) and \( X \gg 1 \), interpolating the region \( X \approx 1 \). The resulting ratio of local energy level spacings \( \Delta C(B)/\Delta C(0) = E_C(B)/E_C(0) \), is shown in Fig. 5 to be in excellent qualitative agreement with experimental data for the magnetic field dependent activation energy, measured recently in transport experiments [1].

There is a quantitative discrepancy by a factor \( .6 \) between the best fit \( X = .36H/Oe \), and \( X = 2\pi\phi/\phi_0 \), \( \phi = \mu_0 H L_{CU} (y^2)^{(1/2)} \). With the experimental parameters \( \alpha = 1, L_{CO} = .61\mu m \), width \( W = 2\mu m \) of sample 5 in Ref. [5] and \( y^2 = W^2/12 \) for a 2-dimensional wire, it yields \( X = .021H/Oe \). We note that smooth confinement can give \( y^2 > W^2/12 \), thus explaining this discrepancy and the observed difference between \( W \) as obtained from the sample resistance and estimated from the analysis of the weak localization magnetoresistance, which also depends on \( y^2 \) [21]. When \( W^2B > \phi_0 \) or \( H > H^* = 2.1 \times 10^{-11} \text{Oe}[m^2/W^2] = 5250\text{Oe} \), the derivation has to be extended to the 2-dimensional nonlinear sigma-model, and Eq. [16] seizes to be valid. Then, \( L_C \) increases as function of magnetic field more strongly.

The asymptotic behaviour, of \( \delta L_C(B) \sim B^2 \) for small and \( \sim 1/B \) at large magnetic fields of Eq. [16] does agree with the suggestion by Bouchaud [6].

The two length scale physics of Ref. [10] has thus no consequence for the energy level statistics as studied with the ASD. Since we could also show a direct relation between this spectral statistics and the exponential decay of localized eigenfunctions, it is suggestive to be the non self averaging property of the ASD which washes out the two scale physics.

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![Graph](image.png)

FIG. 1. The activation gap ratio \( T_0(H)/T_0(0) \) as a function of the magnetic field \( H \) in Oe of sample 5 measured at temperature \( T = .3K \) as reported in Ref. 1, together with the theoretical curve.

[1] M. E. Gershenson, Yu. B. Khavin, A. G. Mikhailchuk, H.M. Bozler, A. L. Bogdanov, Phys. Rev. Lett. 79, 725 (1997); Yu. B. Khavin,, M. E. Gershenson, A. L. Bogdanov, Phys. Rev. B 58, 8009 (1998).
[2] B. Kramer, A. MacKinnon, Rep. Prog.Phys.56, 1469 (1993); C. W. J. Beenakker, Rev. Mod. Phys. 69, 731 (1997).
[3] K. B. Efetov, Adv. Phys. 32, 53 (1983); Supersymmetry in Disorder and Chaos Cambridge University Press, Cambridge (1997)
[4] K. B. Efetov, A. I. Larkin, Zh. Eksp Teor. Fiz. 85, 764 (1983) (Sov. Phys. JETP 58, 444 (1983)).
[5] N. Dorokhov, Sov. Phys. JETP Lett. 36, 318 (1982); Sov. Phys. JETP 58, 606 (1983).
[6] J. P. Bouchaud, J. Phys. 1 (France) 1, 985 (1991).
[7] J. V. Lerner, Y. Imry, Europhys. Lett. 29, 49 (1995).
[8] J.-L. Pichard, M. Sanquer, K. Slevin, and P. Debray, Phys. Rev. Lett. 65, 1812 (1990)
[9] M. Leadbeater, V. I. Falko, C. J. Lambert,Phys. Rev. Lett. 81, 1274 (1998), H. Schomerus, C. W. J. Beenakker, Phys. Rev. Lett. 84, 3927 (2000).
[10] A. V. Kolesnikov, K. B. Efetov,Phys. Rev. Lett. 83, 3689 (1999); cond-mat/0006337 (2000).
[11] S. Kettemann, Phys. Rev. B 59 4799 (1999).
[12] S. Kettemann, A. Tsvelik, Phys. Rev. Lett. 82, 3689 (1999).
[13] F. Wegner, Z. Physik B 36 (1979) 1209 ; B 38 (1979) 207 Nucl. Phys. B 316 (1989) 663.
[14] K. B. Efetov, A. I. Larkin, D. E. Khmel’nitskii, Zh. Eksp. Teor. Fiz. 79 (1980) 1120 (Sov. Phys. JETP 52 (1980) 568).
[15] S. Hikami, Prog. Theor. Phys. 64, 1466 (1980).
[16] M. R. Zirnbauer J. Math. Phys. 37, 4986 (1996).
[17] S. Kettemann, D. Klakow, U. Smilansky, J. Phys. A 30, 3643 (1997).
[18] M. R. Zirnbauer Phys. Rev. Lett. 69, 1584 (1992), A. D. Mirlin, A. Muller-Groeling, M. R. Zirnbauer, Ann. Phys. ( New York) 236, 325 (1994); P. W. Brouwer, K. Frahm, Phys. Rev. B 53, 1490 (1996).
[19] S. Helgason, Differential Geometry and Symmetric Spaces Academic Press, New York (1962); R. Gilmore, Lie Groups, Lie Algebras and Some of their Applications, Wiley, New York (1974).
[20] N. F. Mott, Conduction in Non-Crystalline Materials, Oxford (1987).
[21] Yu. B. Khavin., M. E. Gershenson, A. L. Bogdanov, Phys. Rev. Lett. 81, 1066 (1998).