Rigorous results on Schrödinger operators with certain Gaussian random potentials in multi-dimensional continuous space*

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Schrödinger operators with certain Gaussian random potentials in multi-dimensional Euclidean space possess almost surely an absolutely continuous integrated density of states and no absolutely continuous spectrum at sufficiently low energies.

1 Introduction

Schrödinger operators with Gaussian random potentials in $d$-dimensional Euclidean space $\mathbb{R}^d$, $d \geq 1$, find wide-spread applications in physics. They are used, for example, to model aspects of disordered electronic systems such as heavily doped and highly compensated semiconductors [BEE+, SE]. Over several decades theoretical physicists have developed a good insight into the spectral characteristics of these operators by combining intuitive ideas with approximation techniques and numerical studies. On the other hand there are still only few rigorous results available [K, CL, PF]. Our goal here is to present two new ones, which in the physics literature are often taken for granted. More precisely, for $\mathbb{R}^d$-homogeneous Gaussian random potentials with certain covariance functions we are able to prove (i) the existence of the density of states, that is, the absolute continuity of the integrated density of states and (ii) the almost-sure absence of the absolutely continuous spectrum at sufficiently low energies.

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2 Results

Let \((\Omega, \mathcal{A}, P)\) be a complete probability space and let \(V : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}\), \((\omega, x) \mapsto V(\omega)(x)\) be a real-valued homogeneous Gaussian random field indexed by \(\mathbb{R}^d\) with zero mean, \(\int_\Omega dP(\omega) V(\omega)(0) = 0\), and covariance function \(C(x) := \int_\Omega dP(\omega) V(\omega)(x) V(\omega)(0)\) obeying \(C(0) > 0\) and the

Assumptions 2.1. The covariance function \(C\) admits the representation

\[
C(x) = \int_{\mathbb{R}^d} d^d y \ u(x + y) u(y) .
\]

Here \(d^d x\) denotes the Lebesgue measure on \(\mathbb{R}^d\) and \(u\) is some real-valued Borel-measurable function on \(\mathbb{R}^d\) having the properties

(i) Non-negativity: \(u(x) \geq 0\) for all \(x \in \mathbb{R}^d\).

(ii) Hölder continuity: there exist constants \(a > 0\) and \(\alpha \in ]0, 1]\) such that

\[
|u(x + y) - u(x)| \leq a |y|_\infty^\alpha
\]

for all \(x, y \in \mathbb{R}^d\), where \(|x|_\infty := \sup \{|x_k| : 1 \leq k \leq d\}\) is the usual supremum norm of \(x = (x_1, \ldots, x_d) \in \mathbb{R}^d\).

(iii) Sufficiently fast decay at infinity: there exist constants \(b > 0\) and \(\beta > (25d/2) - 1\) such that

\[
|u(x)| \leq \frac{b}{|x|_\infty^\beta}
\]

for all \(x \in \mathbb{R}^d\).

Remarks 2.2.

(i) The existence of the representation (2.1) with some square-integrable function \(u\) is equivalent to the fact that \(C\) is the Fourier transform of a finite measure which is absolutely continuous with respect to \(d^d x\).

(ii) Assumptions 2.1 imply that \(C\) is a non-negative Hölder continuous function on \(\mathbb{R}^d\) which tends sufficiently fast to zero at infinity. Due to the Hölder continuity of \(C\) it follows from [F, Thm. 3.2.2] that there is a version of \(V\) which is jointly measurable with respect to the sigma-algebra \(\mathcal{A}\) and the Borel sigma-algebra of \(\mathbb{R}^d\). Moreover, the realisations of this version are \(P\)-almost surely continuous [F, Thm. 4.1.1]. The vanishing of
Gaussian random potentials

C at infinity implies mixing and hence ergodicity of \( V \). Taken together, a compromise is required between local dependence and global independence of \( V \).

(iii) Since \( C(0) > 0 \) and \( C \) is continuous there is a length \( \ell > 0 \) such that

\[
\gamma := \inf_{|x|_{\infty} < \ell/2} \{ C(x)/C(0) \} > 0.
\]

(iv) An example of a covariance function satisfying Assumptions 2.1 is the Gaussian

\[
C(x) = \sigma^2 \exp\{-x^2/2\xi^2\}, \quad \sigma, \xi > 0.
\]

With the Gaussian random field \( V \) and a bounded open (hyper-) cube \( \Lambda \subset \mathbb{R}^d \) we associate the finite-volume random Schrödinger operator

\[
H^{(\omega)}_{\Lambda,X} := \left( -\frac{1}{2} \Delta + V^{(\omega)} \right)_{\Lambda,X}, \quad X = D \text{ or } X = N
\]

acting on the Hilbert space \( L^2(\Lambda) \) of complex-valued functions on \( \Lambda \) which are square-integrable over \( \Lambda \) with respect to \( d^d x \). According to the subscript \( X \) the functions in the domain of \( H^{(\omega)}_{\Lambda,X} \) obey either a Dirichlet or a Neumann condition on the boundary of \( \Lambda \). Here, as usual, \( \Delta := \sum_{k=1}^d \partial^2/\partial x_k^2 \) is the \( d \)-dimensional Laplacian and \( V \) appears as a random potential which acts as a multiplication operator. We also consider the infinite-volume random Schrödinger operator

\[
H^{(\omega)} := -\frac{1}{2} \Delta + V^{(\omega)}
\]

which acts on the Hilbert space \( L^2(\mathbb{R}^d) \).

Remarks 2.3. According to standard arguments it is known [K, CL, PF] that

(i) the finite-volume operator \( H^{(\omega)}_{\Lambda,X} \) is self-adjoint on the domain of the free operator \(-\frac{1}{2} \Delta_{\Lambda,X}\) for \( P \)-almost all \( \omega \). The infinite-volume operator \( H^{(\omega)} \) is essentially self-adjoint for \( P \)-almost all \( \omega \) on the dense subspace \( C^\infty_0(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \), consisting of arbitrarily often differentiable complex-valued functions with compact support in \( \mathbb{R}^d \).

(ii) the spectrum of \( H^{(\omega)}_{\Lambda,X} \) is discrete for \( P \)-almost all \( \omega \). Hence the finite-volume integrated density of states \( N^{(\omega)}_{\Lambda,X}(E) \), defined as the number of eigenvalues of \( H^{(\omega)}_{\Lambda,X} \) which are smaller than \( E \) and counted with their multiplicities, exists for \( P \)-almost all \( \omega \).
(iii) there is a non-random unbounded distribution function $N$ on $\mathbb{R}$, called the infinite-volume integrated density of states, and a set $\Omega_0 \in \mathcal{A}$ of maximal probability, $P(\Omega_0) = 1$, such that the macroscopic-limit relation

$$N(E) = \lim_{\Lambda \uparrow \mathbb{R}^d} \frac{N_{\Lambda,X}^{(\omega)}(E)}{|\Lambda|}$$

holds for both boundary conditions $X$, for all $\omega \in \Omega_0$ and for all those $E \in \mathbb{R}$ which are continuity points of $N$. Here $|\Lambda|$ denotes the volume of the cube $\Lambda \subset \mathbb{R}^d$ measured by $d^d x$. The infinite-volume integrated density of states may also be expressed as an average of the localised spectral projection of the infinite-volume operator according to

$$N(E) = \frac{1}{|\Lambda|} \int_{\Omega} dP(\omega) \text{Tr} \left( \chi_{\Lambda} \Theta(E - H^{(\omega)}) \chi_{\Lambda} \right).$$

(2.7)

Here $\Lambda \subset \mathbb{R}^d$ is an arbitrary bounded cube with $|\Lambda| > 0$. Its indicator function $\chi_{\Lambda}$ appears in (2.7) as a multiplication operator. Furthermore, $\Theta$ denotes Heaviside's unit-step function and $\text{Tr}$ stands for the trace over $L^2(\mathbb{R}^d)$. The leading asymptotic low- and high-energy behaviour of $N$ is given by

$$\lim_{E \to -\infty} \frac{\ln N(E)}{E^2} = -\frac{1}{2C(0)},$$

$$\lim_{E \to \infty} \frac{N(E)}{E^{d/2}} = \frac{1}{(2\pi)^{d/2} \Gamma(1 + d/2)},$$

(2.8) (2.9)

where $\Gamma$ denotes Euler's gamma function.

(iv) the topological support of the measure associated with $N$ equals the real line $\mathbb{R}$ and coincides with the spectrum $\sigma(H^{(\omega)})$ of the infinite-volume operator for $P$-almost all $\omega$. The spectral components, the absolutely continuous, the singular continuous and the pure point spectrum, arising in the Lebesgue decomposition $\sigma(H^{(\omega)}) = \sigma_{ac}(H^{(\omega)}) \cup \sigma_{sc}(H^{(\omega)}) \cup \sigma_{pp}(H^{(\omega)})$ are also non-random closed sets for $P$-almost all $\omega$.

As our first result we state a Wegner estimate for the finite-volume situation in

**Theorem 2.4.** Let the finite-volume operator $H_{\Lambda,X}^{(\omega)}$ be defined as in (2.4) with a homogeneous Gaussian random potential satisfying Assumptions 2.1.
Then for every energy $E \in \mathbb{R}$ there exists a constant $0 < W(E) < \infty$, which is independent of $\Lambda$ and $X$, such that the finite-volume integrated density of states $N^{(\omega)}_{\Lambda,X}$ obeys

$$\int_\Omega dP(\omega) \left| N^{(\omega)}_{\Lambda,X}(E_1) - N^{(\omega)}_{\Lambda,X}(E_2) \right| \leq |\Lambda| |E_1 - E_2| W(E) \quad (2.10)$$

for all $E_1, E_2 \leq E$ and all bounded open cubes $\Lambda \subset \mathbb{R}^d$ with $|\Lambda| \geq \ell^d$. (The length $\ell$ is defined in Remark 2.2(iii).)

The Lipschitz continuity (2.10) of the averaged finite-volume integrated density of states implies by the non-randomness of the infinite-volume integrated density of states and Fatou’s lemma the following

**Corollary 2.5.** Under the assumptions of Theorem 2.4 the infinite-volume integrated density of states $N$ is absolutely continuous on any bounded interval and its derivative, the density of states, is locally bounded in the sense that

$$0 \leq \frac{dN(E)}{dE} \leq W(E) \quad (2.11)$$

for Lebesgue-almost all $E \in \mathbb{R}$.

**Remarks 2.6.**

(i) Theorem 2.4 and Corollary 2.5 are proved in [FHLM] under weaker assumptions than the ones used here. Actually it is shown there that Theorem 2.4 is a consequence of a Wegner estimate which holds for all continuum Schrödinger operators whose random potential admits a certain one-parameter decomposition. For the present case the basic idea is to write

$$V^{(\omega)}(x) =: U^{(\omega)}(x) + V^{(\omega)}(0) \frac{C(x)}{C(0)}. \quad (2.12)$$

Since the Gaussian random variable $V(0)$ is stochastically independent of the (non-homogeneous) Gaussian random field $U$, spectral averaging with respect to $V(0)$ is easily performed and one may adapt the line of reasoning laid down in [CH] to prove Theorem 2.4.

(ii) It follows from the proof of Theorem 2.4 that the Wegner constant $W(E)$ may be taken as

$$W(E) = \exp\{tE + t^2C_E/2\} \left(2\ell^{-1}E + (2\pi t)^{-1/2}\right)^d, \quad (2.13)$$
where \( t > 0 \) is arbitrary and may be considered as a variational parameter. In (2.13) we are using the constants

\[
\ell_E := \inf\{|E|^{-1/2}, \ell\}, \quad (2.14)
\]

\[
b_E := \inf_{|x|_{\infty} < \ell_E/2} \{C(x)/C(0)\} \geq \gamma, \quad (2.15)
\]

\[
C_E := C(0)(2 - b_E^2). \quad (2.16)
\]

The choice \( t = (2C_E)^{-1}(-E + \sqrt{E^2 + 2C_E/\pi}) \) shows that \( W \) has the same low- and high-energy behaviour as \( N \) except that the constant on the right-hand side in (2.9) is to be replaced by \( 3d e^{1/2\pi} / \sqrt{2\pi C(0)} \).

As our second result we state for the infinite-volume situation the almost-sure absence of the absolutely continuous spectrum at sufficiently low energies in

**Theorem 2.7.** Let the infinite-volume operator \( H^{(\omega)} \) be defined as in (2.5) with a homogeneous Gaussian random potential satisfying Assumptions 2.1. Then there exists a finite energy \( E_0 < 0 \) such that

\[
\sigma_{ac}(H^{(\omega)}) \cap [-\infty, E_0] = \emptyset \quad \text{for } P\text{-almost all } \omega. \quad (2.17)
\]

**Remarks 2.8.**

(i) The heart of the proof of Theorem 2.7 is a multi-scale analysis in the spirit of the fundamental work [FS]. Its technical realisation is patterned after [DK] and [CH] in order to cope with a correlated random potential and a continuous space, respectively. A key role in the proof is played by the Wegner estimate Theorem 2.4 above. In case of a Gaussian random field with a sufficiently fast decaying strong-mixing coefficient a proof of the statement of Theorem 2.7 is outlined in [FLM1] and will be completed in [FLM2]. In case of the present assumptions on \( V \) a more refined strategy is required in that different Gaussian random fields \( V_{L_j} \) are used on the different length scales \( L_j \) of the multi-scale analysis. The correlation radius of \( V_{L_j} \) is of order \( L_{j-1} \) and in the macroscopic limit \( V_{L_j} \) tends sufficiently fast to the given random field \( V \) in \( P\)-mean square. This is why Assumption 2.1(iii) requires such a strong decay of \( u \). Again we defer the details of the proof to [FLM2].

(ii) So far we have not yet been able to rule out the singular continuous spectrum for sufficiently low energies. But we have some hope that the
decomposition \((2.12)\) of \(V\) may be used to proceed along the lines in [SW, H, CH].

(iii) Theorem \(2.7\) applies in particular to a Gaussian random field with the Gaussian covariance function \((2.3)\). For one space dimension this is even an example of a so-called deterministic stochastic process [IR, ch. IV, §9, Thm. 7].

(iv) The absence of the (absolutely) continuous spectrum is in general not sufficient to imply vanishing transport coefficients. Confer the recent critical discussions [S, RJLS, BCM, C] on the relation between quantum dynamics and spectral characteristics.

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