3-NETS REALIZING A DIASSOCIATIVE LOOP IN A PROJECTIVE PLANE

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Abstract. A 3-net of order $n$ is a finite incidence structure consisting of points and three pairwise disjoint classes of lines, each of size $n$, such that every point incident with two lines from distinct classes is incident with exactly one line from each of the three classes. The current interest around 3-nets (embedded) in a projective plane $\text{PG}(2, K)$, defined over a field $K$ of characteristic $p$, arose from algebraic geometry; see [5, 12, 14, 17, 18]. It is not difficult to find 3-nets in $\text{PG}(2, K)$ as far as $0 < p \leq n$. However, only a few infinite families of 3-nets in $\text{PG}(2, K)$ are known to exist whenever $p = 0$, or $p > n$. Under this condition, the known families are characterized as the only 3-nets in $\text{PG}(2, K)$ which can be coordinatized by a group; see [10]. In this paper we deal with 3-nets in $\text{PG}(2, K)$ which can be coordinatized by a diassociative loop $G$ but not by a group. We prove two structural theorems on $G$. As a corollary, if $G$ is commutative then every non-trivial element of $G$ has the same order, and $G$ has exponent 2 or 3. We also discuss the existence problem for such 3-nets.

Keywords 3-net - projective plane - diassociative loop - Latin square - transversal design

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1. Introduction

The concept of a 3-net comes from classical differential geometry via the combinatorial abstraction of the concept of a 3-web. Formally, a 3-net of order $n$ is a finite incidence structure consisting of points and three pairwise disjoint classes of lines, each of size $n$, such that every point incident with two lines from distinct classes is incident with exactly one line from each of the three classes. It is well known that every 3-net can be coordinatized by a loop. The set $Q$ endowed with a binary operation “$\cdot$” is a quasigroup, if for any $a, b \in Q$, the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions in $Q$. A quasigroup with a multiplicative unit element is called a loop. For a general reference on nets, loops and quasigroups see for instance [1, 4].

In this paper we deal with 3-nets (embedded) in $\text{PG}(2, K)$, the projective plane over a field $K$ of characteristic $p \geq 0$. Such a 3-net, with line classes $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and coordinatizing loop $G = (G, \cdot)$, is equivalently defined by a triple of bijective maps from $G$ to $(\mathcal{A}, \mathcal{B}, \mathcal{C})$, say

$$
\alpha : G \rightarrow \mathcal{A}, \quad \beta : G \rightarrow \mathcal{B}, \quad \gamma : G \rightarrow \mathcal{C}
$$

such that $a \cdot b = c$ if and only if $\alpha(a), \beta(b), \gamma(c)$ are three concurrent lines in $\text{PG}(2, K)$, for any $a, b, c \in G$. If this is the case, the 3-net in $\text{PG}(2, K)$ is said to realize the loop $G$. 

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For the purpose of investigating 3-nets in $\text{PG}(2, K)$, the groundfield $K$ may be assumed to be algebraically closed. In order to present the key examples of embedded 3-nets, it is convenient to work with the dual concept. Formally, a dual 3-net of order $n$ in $\text{PG}(2, K)$ consists of a triple $(\Lambda_1, \Lambda_2, \Lambda_3)$ with $\Lambda_1, \Lambda_2, \Lambda_3$ pairwise disjoint point-sets of size $n$, called components, such that every line meeting two distinct components meets each component in precisely one point. We notice that finite dual 3-nets are also called transversal designs.

The following concepts and results have a detailed exposition in [10]. We say that an embedded dual 3-net is algebraic, if its point set $\Lambda_1 \cup \Lambda_2 \cup \Lambda_3$ is contained in a cubic curve $F$. If $F$ is reducible then we speak of pencil type, triangular type or conic-line type dual 3-net. Except for the pencil type, all algebraic (dual) 3-nets are coordinatized by either a cyclic group or by a direct product of two cyclic groups. Finite dihedral groups can be realized by dual 3-nets of tetrahedron type; in this case the point set is contained in six lines joining four independent points. Finally, we mention that the quaternion group $Q_8$ has an exceptional realization, cf. [16].

In recent years, finite 3-nets realizing a group have been investigated also in connection with complex line arrangements and resonance theory; see [2, 3, 5, 10, 11, 12, 14, 17, 18]. The following almost complete classification of such 3-nets is proven in [10].

**Theorem 1.1.** In the projective plane $\text{PG}(2, K)$ defined over an algebraically closed field $K$ of characteristic $p \geq 0$, let $(\Lambda_1, \Lambda_2, \Lambda_3)$ be a dual 3-net of order $n \geq 4$ which realizes a group $G$. If either $p = 0$ or $p > n$ then one of the following holds.

(I) $G$ is either cyclic or the direct product of two cyclic groups, and $(\Lambda_1, \Lambda_2, \Lambda_3)$ is algebraic.

(II) $G$ is dihedral and $(\Lambda_1, \Lambda_2, \Lambda_3)$ is of tetrahedron type.

(III) $G$ is the quaternion group of order 8.

(IV) $G$ has order 12 and is isomorphic to $\text{Alt}_4$.

(V) $G$ has order 24 and is isomorphic to $\text{Sym}_4$.

(VI) $G$ has order 60 and is isomorphic to $\text{Alt}_5$.

A computer aided exhaustive search shows that if $p = 0$ then (IV) (and hence (V), (VI)) does not occur; see [13]. It has been conjectured that this holds true in any characteristic.

In this paper we focus on 3-nets in $\text{PG}(2, K)$ which can be coordinatized by a diassociative loop $G$ different from a group. Recall that a loop $G$ is diassociative if any subloop generated by two elements is a group. There are two important classes of diassociative loops: Moufang loops and Steiner loops. Moufang loops are loops satisfying one (hence all) of the following identities.

$$z(x(yz)) = ((zx)z)y, \quad x(z(yz)) = ((xz)y)z, \quad (zx)(yz) = (z(xy))z.$$ 

In general, Moufang loops have a rich algebraic structure. This is not the case for Steiner loops. Steiner loops are diassociative loops of exponent two. Finite Steiner loops are in one-to-one connection with Steiner triple systems. For other classes of diassociative loops we refer to [9].

Our results consist of three structural theorems on $G$.

**Theorem 1.2.** In the projective plane $\text{PG}(2, K)$ defined over an algebraically closed field $K$ of characteristic $p \geq 0$, let $(\Lambda_1, \Lambda_2, \Lambda_3)$ be a dual 3-net of order $n \geq 4$ which realizes a diassociative loop $G$ different from a group. Let $d$ be the maximum of the
orders of the elements in $G$, and suppose that $d \geq 4$. If either $p = 0$ or $p > n$ then one of the following holds.

(a) $G$ has a unique subgroup $H$ of order $d$. Moreover, each element not in $H$ is an involution, and two such involutions either commute or their product is in $H$.

(b) $d = 4$, and $G$ has a subgroup isomorphic to one of the groups $Q_8$, $Alt_4$.

**Theorem 1.3.** With the same hypotheses as in Theorem 1.2 assume further that $G$ contains a subgroup isomorphic to $Q_8$ but no subgroup isomorphic to $Alt_4$. Then $G$ has a unique involution and the subgroup generated by any two non-commuting elements is isomorphic to $Q_8$.

It may be observed that a loop $G$ as in Theorem 1.3 defines a Steiner triple system in a natural way, namely the points are subgroups of $G$ of order 4 and the blocks are the subgroups isomorphic to $Q_8$, the point-block incidence being the set theoretic inclusion.

For a commutative loop $G$, neither (a) nor (b) of Theorem 1.2 can occur, and hence $d \leq 3$. More precisely, the following result holds.

**Corollary 1.4.** With the same hypotheses as in Theorem 1.2 assume further that $G$ is commutative. Then every non-trivial element in $G$ has the same order, and $G$ has exponent 2 or 3.

The quaternion group $Q_8$ has a counterpart in the class of Moufang loops. Let $\mathcal{O}$ be the division ring of real octonions and let $1, e_1, \ldots, e_7$ be an orthonormal basis. The set

$$
\mathcal{O}_{16} = \{ \pm 1, \pm e_1, \ldots, \pm e_7 \}
$$

forms a Moufang loop with a unique involution $-1$ and 14 elements of order 4. ($\mathcal{O}_{16}$ is also called the Cayley loop of order 16.)

**Theorem 1.5.** With the same hypotheses as in Theorem 1.2 assume further that $G$ is a Moufang loop. Then $G$ contains either the octonion loop $\mathcal{O}_{16}$, or it has a subgroup isomorphic to $Alt_4$.

An interesting issue which appears to be rather difficult is the existence and construction of 3-nets in the classical projective plane $PG(2, \mathcal{K})$ realizing a loop different from a group. All such examples available in the literature are 3-nets of order $n = 5, 6$, obtained by computer aided searches; see [16].

2. **Proof of Theorem 1.2**

Let $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$ be a 3-net of order $n$ coordinatized by a diassociative loop $G$ but not by a group. Let $g \in G$ be an element whose order is $d$, and let $\Psi = (\Psi_1, \Psi_2, \Psi_3)$ be the 3-net (subnet of $\Lambda$) coordinatized by $\langle g \rangle$. Take any element $h \in G$ not lying in the cyclic group generated by $g$, and consider the subgroup $H$ generated by $g$ and $h$. Obviously, $H$ is not a cyclic group. Since $H$ is a subloop of $G$, $H$ also realizes a 3-net $\Delta = (\Delta_1, \Delta_2, \Delta_3)$ in $PG(2, \mathcal{K})$. The classification [10] Theorem 1.1] applies to $H$ and yields one of the cases below, apart from the sporadic cases. Therefore, dismissing (b), we have that either

(i) $H$ is the direct product of two cyclic subgroups, and $\Delta_1 \cup \Delta_2 \cup \Delta_3$ lies on a plane non-singular cubic curve $\mathcal{F}_3$, or

(ii) $H$ is a dihedral group, and $\Delta$ is of tetrahedron type.
We first investigate case (i). As $G$ is not a group, it must contain an element $u \notin H$. Replacing $h, H$ by $u, U = \langle g, u \rangle$ in the above argument shows that $U$ realizes a 3-net $\Phi = (\Phi_1 \cup \Phi_2 \cup \Phi_3)$ in $PG(2, \mathbb{K})$, and that $U$ is either the direct product of two cyclic groups, or it is a dihedral group. The latter case here cannot actually occur. To show this, observe that if $U$ is dihedral then the maximality of $d$ implies that $u$ is an (involutory) element lying in some coset of $\langle g \rangle$. Since $\Phi$ is of tetrahedron type in this case, we have that $\Psi$ is a triangular 3-net in $U^1$. Therefore, $H \cap \Phi_3$ consists of $d > 3$ collinear points. Therefore, $U$ is a direct product of two cyclic groups and $\Phi_1 \cup \Phi_2 \cup \Phi_3$ lies on a non-singular plane cubic curve $F_1$. The intersection $F_3 \cap F_1$ contains all points of $\Psi$. Since $d > 3$, this yields $F_3 = F_1$.

Choose two elements $g_1, g_2 \in G$ which generate a subgroup $H$ isomorphic to $Q_8$. We remark that case (a) in Theorem 1.2 cannot occur since both $g$ and $h$ have order 4. Therefore, $d = 4$. Set $g_3 = g_1g_2$: then $\langle g_1 \rangle, \langle g_2 \rangle, \langle g_3 \rangle$ are the three cyclic subgroups of order 4 in $H$. Take an element $u \in G$ not lying in $H$.

Assume that $u$ is an involution. For $i = 1, 2, 3$, the group $U_i = \langle u, g_i \rangle$ contains at least two distinct involutions, and hence it is not isomorphic to $Q_8$. From Theorem 1.1 applied to $U_i$, we deduce that either $U_i$ is dihedral, or abelian.

We first investigate the case when all $U_i$'s are abelian. Clearly, $U_i = \langle g_i \rangle \times \langle u \rangle \cong C_4 \times C_2$, and case (a) of Theorem 1.2 holds for the sub 3-net $(\Delta^1, \Delta^2, \Delta^3)$ realizing $U_i$. Let $F_i$ be the cubic curve containing $\Delta^1 \cup \Delta^2 \cup \Delta^3$. Since $U_i$ is not cyclic, $F_i$ is nonsingular. These three sub 3-nets of order 8 share a sub 3-net of order 4, say $(\Omega_1, \Omega_2, \Omega_3)$, realizing the group $T = \langle u, g_1^2 = g_2^2 \rangle$. Since $|F_1 \cap F_2 \cap F_3| \geq 12 > 9$, this yields that $F_1 = F_2 = F_3$. But then the sub 3-net realizing $H$ lies on $F_1$ a contradiction since $Q_8$ is not abelian.

Assume now that $U_1, U_2$ are abelian and $U_3$ dihedral. With the same argument, the dual sub 3-nets realizing $U_1, U_2$ are contained in the nonsingular cubic curve $F$. The dual sub 3-net realizing $U_3$ is of tetrahedron type, which means that the four points of $\alpha(\langle g_3 \rangle)$ are collinear. The triple $\langle \alpha(\langle g_3 \rangle), \beta(\langle H \setminus \langle g_3 \rangle)), \gamma(\langle H \setminus \langle g_3 \rangle)) \rangle$ is a dual 3-net realizing $\langle g_3 \rangle$. On the one hand, the 8 points of $\beta(\langle H \setminus \langle g_3 \rangle)) \cup \gamma(\langle H \setminus \langle g_3 \rangle)$ are contained in a (possibly degenerate) conic $C$, see [2, Theorem 5.1]. On the other hand, $H \setminus \langle g_3 \rangle \subset \langle g_1 \rangle \cup \langle g_2 \rangle \subset U_1 \cup U_2$. This implies $\beta(\langle H \setminus \langle g_3 \rangle) \cup \gamma(\langle H \setminus \langle g_3 \rangle) \subset \mathcal{F}$ and $|\mathcal{F} \cap C| \geq 8$, a contradiction.

Assume that $U_1, U_2$ are dihedral. Hence the dual sub 3-nets realizing $U_1, U_2$ are of tetrahedron type yielding that the four points of $\alpha(\langle g_1 \rangle)$ and the four points of $\alpha(\langle g_2 \rangle)$ are contained in the lines $\ell_1, \ell_2$, respectively. However, $\alpha(1), \alpha(g_1^2 = g_2^2) \in \ell_1 \cap \ell_2$, thus, $\ell_1 = \ell_2$. Similarly, the six points of $\beta(\langle g_1 \rangle \cup \langle g_2 \rangle)$ and the six points of

3. Proof of Theorem 1.3

From the definition of a dual 3-net, there is a triple of bijective maps from $G$ to $(\Lambda_1, \Lambda_2, \Lambda_3)$, say $\alpha, \beta, \gamma$ respectively such that $a \cdot b = c$ in $G$ if and only if $\alpha(a), \beta(b), \gamma(c)$ are three collinear points in $PG(2, \mathbb{K})$, for any $a, b, c \in G$.

We first investigate the case when all $U_i$'s are abelian. Clearly, $U_i = \langle g_i \rangle \times \langle u \rangle \cong C_4 \times C_2$, and case (a) of Theorem 1.2 holds for the sub 3-net $(\Delta^1, \Delta^2, \Delta^3)$ realizing $U_i$. Let $F_i$ be the cubic curve containing $\Delta^1 \cup \Delta^2 \cup \Delta^3$. Since $U_i$ is not cyclic, $F_i$ is nonsingular. These three sub 3-nets of order 8 share a sub 3-net of order 4, say $(\Omega_1, \Omega_2, \Omega_3)$, realizing the group $T = \langle u, g_1^2 = g_2^2 \rangle$. Since $|F_1 \cap F_2 \cap F_3| \geq 12 > 9$, this yields that $F_1 = F_2 = F_3$. But then the sub 3-net realizing $H$ lies on $F_1$ a contradiction since $Q_8$ is not abelian.

Assume now that $U_1, U_2$ are abelian and $U_3$ dihedral. With the same argument, the dual sub 3-nets realizing $U_1, U_2$ are contained in the nonsingular cubic curve $F$. The dual sub 3-net realizing $U_3$ is of tetrahedron type, which means that the four points of $\alpha(\langle g_3 \rangle)$ are collinear. The triple $\langle \alpha(\langle g_3 \rangle), \beta(\langle H \setminus \langle g_3 \rangle)), \gamma(\langle H \setminus \langle g_3 \rangle)) \rangle$ is a dual 3-net realizing $\langle g_3 \rangle$. On the one hand, the 8 points of $\beta(\langle H \setminus \langle g_3 \rangle)) \cup \gamma(\langle H \setminus \langle g_3 \rangle)$ are contained in a (possibly degenerate) conic $C$, see [2, Theorem 5.1]. On the other hand, $H \setminus \langle g_3 \rangle \subset \langle g_1 \rangle \cup \langle g_2 \rangle \subset U_1 \cup U_2$. This implies $\beta(\langle H \setminus \langle g_3 \rangle) \cup \gamma(\langle H \setminus \langle g_3 \rangle) \subset \mathcal{F}$ and $|\mathcal{F} \cap C| \geq 8$, a contradiction.

Assume that $U_1, U_2$ are dihedral. Hence the dual sub 3-nets realizing $U_1, U_2$ are of tetrahedron type yielding that the four points of $\alpha(\langle g_1 \rangle)$ and the four points of $\alpha(\langle g_2 \rangle)$ are contained in the lines $\ell_1, \ell_2$, respectively. However, $\alpha(1), \alpha(g_1^2 = g_2^2) \in \ell_1 \cap \ell_2$, thus, $\ell_1 = \ell_2$. Similarly, the six points of $\beta(\langle g_1 \rangle \cup \langle g_2 \rangle)$ and the six points of
γ((g_1) ∪ (g_2)) are contained in the lines m, m', respectively. If U_3 is dihedral, then the dual sub 3-net realizing H is contained in ℓ_1 ∪ m ∪ m', which is impossible since H is not cyclic. If U_3 is abelian, then the sub 3-net realizing it is contained in the nonsingular cubic curve F. The second component of its sub 3-net (α((g_3)), β(H \ ⟨g_3⟩), γ(H \ ⟨g_3⟩)) is contained in m, hence |m ∩ F| ≥ 4, a contradiction. Assume that u has order 3. Then U is neither a dihedral group nor isomorphic to Q_8. From Theorem 1.2 U = ⟨g⟩ × ⟨u⟩ and hence U is a cyclic group of order 12 contradicting the remark at the beginning about case (a) in Theorem 1.2.

Therefore, G contains just one involution v, and if u ≠ v then u has order 4. Let u_1, u_2 ∈ G be any two distinct elements other than v. Since U contains no element of order 3, U = ⟨u_1, u_2⟩ is a 2-group of exponent 4 containing a unique involution. Since U has order bigger than 4, the only possibility is U ∼= Q_8.

**Remark 3.1.** Let S be the Steiner loop of order 10 corresponding to the Steiner triple system AG(2, 3). S has a central extension Q of order 20 all proper subloops are isomorphic to C_2, C_4, C_2 × C_4, or Q_8. In particular, Q is diassociative. By Theorem 1.3 Q has no projective realization despite all its subloops have.

4. **Proof of Theorem 1.5**

We start with three important facts on Moufang loops of small exponent. First, as diassociative loops of exponent 2 are commutative, Moufang loops of exponent 2 are elementary abelian groups. Second, by [6] Corollary 1 finite Moufang loops of exponent 3 are nilpotent. This implies that any proper finite Moufang loop of exponent 3 contains a subloop of order 27. The classification of small Moufang loops shows that Moufang loops of order 27 are groups. Thus, if G is a Moufang loop of exponent 3 then it contains a subgroup H of order 27. Since no such group H has a realization in PG(2, K) by Theorem 1.1 we have a contradiction.

Let us assume that G has an element g of order d > 4. Put U = ⟨g⟩. By Theorem 1.2 any h ∈ G \ U has order 2 and ⟨U, h⟩ is a dihedral group of order 2d. In particular, on the one hand, hU = Uh, and on the other hand, the involutions generate G. [7] Theorem 1 implies that U is a normal subloop of G. For any subset X of G, let Λ_i(X) denote the points of Λ_i, indexed by the elements of X. [10] Proposition 22 implies that any of the sets Λ_i(U), Λ_i(Uh) is contained in a line.

Choose an element h ∈ G \ U. Then we have four points P, Q, R, S such that the point sets Λ_1(U), Λ_2(U), Λ_3(U), Λ_1(Uh), Λ_2(Uh), Λ_3(Uh) are contained in the lines QR, RS, PR, SP, SQ, PQ. In fact, the points P, Q, R, S are the vertices of the tetrahedron type dual 3-net, corresponding to the dihedral group ⟨U, h⟩. Only the vertex S depends on the choice of h; S = S_h.

Choose elements h_1, h_2 ∈ G \ U such that ⟨U, h_1, h_2⟩ is a non-associative subloop of G. Let P, Q, R, S_{h_1}, S_{h_2}, S_{h_1h_2} be the vertices of the tetrahedron type dual nets of ⟨U, h_1⟩, ⟨U, h_2⟩ and ⟨U, h_1h_2⟩. The sets

Λ_1(Uh_1), Λ_2(Uh_2), Λ_3(Uh_1h_2)

of points form a triangular dual 3-net. [10] Proposition 10 implies S_{h_1} = S_{h_2} = S_{h_1h_2}, a contradiction.

Finally, assume that G has no subgroup isomorphic to Alt_4. By Theorem 1.3 G has a unique (central) involution u. As two non-commuting elements generate a subloop isomorphic to Q_8, the factor G/⟨u⟩ is an elementary abelian 2-group.
Thus, $G$ contains a non-associative subloop $S$ of order 16 with a unique involution. Using the classification of small Moufang loops in $S$, $S \cong O_{16}$ follows.

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