Critical Insight into the Cosmological Sector of Loop Quantum Gravity

Jakub Bilski\(^1\) and Antonino Marcianò\(^2\)

\(^1\)Institute for Theoretical Physics and Cosmology, Zhejiang University of Technology, 310023 Hangzhou, China

\(^2\)Center for Field Theory and Particle Physics & Department of Physics, Fudan University, 200433 Shanghai, China

We critically analyze Quantum Reduced Loop Gravity, an attempt to extract the cosmological sector of Loop Quantum Gravity. We reconsider the reduction procedure applied to the states of the kinematical Hilbert space of the full theory, developing a comparative analysis with previous efforts in the literature. We show that the constraints of the model were formerly instantiated in an inconsistent fashion, leading to an overconstrained dynamics. We then scrutinize alternative implementations of symmetry-reduction. While remaining unaffected by the shortcomings encountered in Quantum Reduced Loop Gravity, these latter procedures may bridge the gap between the full theory and former endeavors in Loop Quantum Cosmology.

I. INTRODUCTION

Despite over the last three decades optimistic acclamation was often bestowed upon Loop Quantum Gravity (LQG) \(^1\), no clear resolution of the quantization of the Hamiltonian constraint problem was sorted out. We therefore still ignore the physical Hilbert space of the theory, and consequently its ground state. The very same structure of the vacuum, which unfortunately is still very far from being tackled within the theory, is suggested to be non-trivial by considerations based on non-abelian gauge theories\(^i\). Already two decades ago, the overall situation of incompleteness eventually urged to grasp more insights by a development of a parallel simpler theory, Loop Quantum Cosmology (LQC) \(^3, 4\), established to deal with the quantization of the symmetry-reduced phase space of the full theory, LQG. However, the link between LQG and LQC is neither obvious nor obviously able to shed light on the quantization issues, as the longstanding lack of solutions to the latter, despite the development of LQC, proves. Other approaches then naturally followed, often motivated with the purpose of linking LQC to the full theory of LQG, since quantization and symmetry-reduction need not, \textit{a priori}, commute. Several possibilities were investigated within the literature \(^5\), trying to unravel how the quantum configuration spaces of LQC can be embedded into the full theory. Lights on the use of spinfoam techniques was sought in \(^6\), while coherent state techniques were proposed within the Group Field Theory approach in \(^7, 8\).

Quantum reduced loop gravity (QRLG) is chronologically one of the latest attempts, developed in \(^9\) — for a review see \(^13\). It relies on imposing weak gauge-fixing conditions to the states of the kinematical Hilbert space of the full theory, LQG. This peculiarity was argued to allow recovering the cosmological sector directly from LQG. Classically, the gravitational systems considered are those ones described by dreibein fields gauge-fixed to a diagonal form. The gauge-fixing conditions are then applied weakly on the kinematical Hilbert space of the full theory. As a result, Bianchi I models were thought to be successfully recovered in the framework — see \(^14\) for a description of the Bianchi I extension to LQC. Furthermore, there are studies seeking to show that within the semiclassical limit, QRLG reproduces the effective Hamiltonian of LQC \(^11, 15\), in the \(\mu_0\) regularization scheme\(^ii\). It has been also claimed that the effective improved dynamics proposed in \(^17\), can be inferred in this framework by averaging over the ensemble of the classically equivalent states \(^18\).

For the aforementioned reasons, QRLG was conjectured to provide a novel derivation of earlier results of LQC, including the realization of the singularity-resolution scenario. But disregarding the emanation of LQG, the full theory will remain unsolved until the Hamiltonian constraint problem will be solved, and matter fields will be taken into account. Within the framework of QRLG, since this introduces a graph structure underlying the description of the continuous universe at the classical level, and since the origin of the discretization must be recovered at the quantum level, the quantization of the matter fields shall be achieved via the same tools of LQG \(^21, 22\). This was providing encouragement that QRLG might have offered a framework to test the implications of the loop quantization of matter fields, as first suggested by the analysis of a scalar matter field in \(^23\), and then by the implementation of gauge vector fields in \(^24, 25\) and related applications.

---

\(^i\) The well known non-trivial structure of the vacuum that is present in gauge theories that can be easily extended to quantum theories of gravity, by inspections of the instantonic solutions, as argued in Ref. \(^2\).

\(^ii\) The Dapor-Liegener model called Cosmological Complexifier-Coherent Loop Quantum Gravity, shortly Cosmological Coherent Quantum Gravity (CCQG) has been first proposed in \(^19\). Its detailed constructional description has been presented in \(^20\).
QRLG initially appeared as a promising way to reconcile a cosmological toy model, LQC, while a full theory remained under construction. But this did not come without its flaws. Actually, as we will question throughout the paper, the shortcomings of the model are enough to verify its internal inconsistency. Specifically, we will argue about the fate of the theory, re-examining the reduction procedure applied to the states of the kinematical Hilbert space of LQG, and developing a comparative analysis with previous attempts formulated in the literature of QRLG, while seeking to unravel the cosmological sector of the full theory. We show how the eigenvalues of the geometrical operators on cuboidal lattices coincide with analog expressions defined in LQG. In particular, we recover exactly the minimal eigenvalue of the area operator, which is used as a regulator in LQC. We then compute the action of the Hamiltonian Constraint Operator on generic lattices characterized by hexavalent nodes. We show that constraints are not unaffected by the flaws of QRLG, this new procedure may eventually represent a novel bridge between the full theory and the other attempts belonging to the LQC scenario. In particular, the paper is organized as follows. In Sec. II we introduce lattice regularization in LQG, and provide a regularized expression for the geometric operators of the theory. In Sec. III we reconsider the quantum reduction map derived in QRLG, and shed light on the inconsistencies previously arisen. We also propose alternative reduction patterns, to avoid shortcomings. In Sec. IV we comment on the kinematical properties of a suitable simplified version of QRLG. Finally, in Sec. V we present conclusions and outlooks on future investigations to be carried out.

Through the paper, the metric signature is specified by $(-, +, +, +)$. The fundamental constant of LQG (representing the quantum of action analogously to $\hbar$ in quantum mechanics) reads $k = \frac{1}{2}\gamma \hbar c = 8\pi \gamma l_p^3$, where $\gamma$ and $l_p$ are the Immirzi parameter and the Planck length respectively and for simplicity we set the speed of light to $c = 1$. The metric tensor, $g_{\mu\nu} = e_{\mu}^a e_{\nu}^b \eta_{ab}$, can be cast in terms of $e_{\mu}^a$, the co-vierbein fields, and the flat Minkowski metric $\eta_{ab}$. The spatial metric tensor is denoted with $q_{ab} = e_{a}^i e_{b}^j \delta_{ij}$, where $e_{a}^i$ and $e_{b}^i = e_{b}^j q^{ab} \delta_{ij}$ denote co-dreibeins and dreibeins, respectively. A regularization via a cuboidal graph structure with the three directions of the orientation of links can be chosen, and adapted to the fiducial metric $\eta_{ab}$. Analogously, the constant orthonormal triad $e_i^a$ and co-triad $e_i^a$ are defined. Lowercase Latin indices $a, b, ... = 1, 2, 3$ label coordinate on each Cauchy hypersurface constructed by ADM decomposition \cite{26}, while $i, j, ... = 1, 2, 3$ are $su(2)$ internal indices and $\delta_{ij}$ stands for the Kronecker delta. Generators of $su(2)$ are defined as $\tau^i = -\frac{1}{2}\sigma^i$, where $\sigma^i$ are Pauli matrices (see Appendix A). Indices written in the bracket ( ) are not summed, while for every other repeated pair the Einstein convention is applied.

II. REGULARIZATION AND OPERATORS

In this section we introduce lattice regularization in LQG, and provide a regularized expression for the geometric operators of the theory. We start from classical general relativity, minimally coupled to the Standard Model of particle physics, and cast the theory within the Hamiltonian ADM formalism \cite{26}, in terms of the real Ashtekar-Barbero gravitational variables \cite{27}. Thus we consider the decomposition of the line element

$$ds^2 := g_{\mu\nu} dx^\mu dx^\nu = (N^a N_a - N^2) dt^2 + 2 N_a dtdx^a + q_{ab} dx^a dx^b,$$

where $N$ is the lapse function and $N^a$ the shift vector. We then look into the kinematical state space of the theory, discussing some subtleties of the reduction procedure.

II.1. Classical theory

We define the Einstein-Hilbert action with the cosmological constant term, minimally coupled to the free fields of the Standard Model,

$$S := S(g) + S^{(A)} + \int_M dt^4 x \sqrt{-g} \mathcal{L}^{\text{matter}}, \quad (2)$$

1 The meaning of this structure should be clear. When we define the link between the volume of the fiducial cell $V_0 := l_p^3 l_p^2 l_p^2$ and the physical volume $V$ via the scale factor, we find that $a^3 = V/V_0$. It is worth mentioning that considering the Bianchi I universe, the scale factor is defined as $a := (a_1 a_2 a_3)^{1/3}$. 
where $L^{(\text{matter})}$ encodes the Yang-Mills field, the scalar field, and the Dirac field.

In this section, we focus on the first two terms of $S$. The starting point in the construction of the Hamiltonian constraint operator (HCO) in LQG would be the Einstein-Hilbert action, which reproduces the classical equation of motion,

$$S^{(g)} + S^{(A)} := \frac{1}{\kappa} \int d^4x \sqrt{-g}R - \frac{2\Lambda}{\kappa} \int d^4x \sqrt{-g},$$

(3)

where $R$ is the Ricci scalar. For completeness, we kept the cosmological constant $\Lambda$ in the action. Here $g$ stands for the determinant of the metric tensor $g_{\mu\nu}$, and the gravitational coupling constant reads $\kappa = 16\pi G$.

The canonical quantization procedure in LQG is applied to the Hamiltonian obtained from action $S^{(g)}$, which is derived in the ADM formalism while using the Ashtekar variables. The latter are the Ashtekar-Barbero connection $A_{a}^{i} = \Gamma_{a}^{i} + \gamma K_{a}^{i}$ and the densitized dreibein $E_{i}^{a} = \sqrt{q}\epsilon_{i}^{a}$. Here, $\Gamma_{a}^{i} := \frac{1}{2}\epsilon^{ijk}\Gamma_{jka} = -\frac{1}{2}\epsilon^{ijk}e_{k}^{b}(\partial_{a}e_{b}^{j} - \Gamma_{ab}^{ij})$ is the spin-connection and $\gamma K_{a}^{i} = \gamma \delta_{a}^{i}$ is the extrinsic curvature. These form a canonically conjugate pair of variables, with a Poisson structure given by

$$\{A_{a}^{i}(t, x), E_{j}^{b}(t, y)\} = \frac{\gamma^{2}}{2}\delta_{a}^{i}\delta_{j}^{b}\delta^{(3)}(x - y).$$

(4)

An important remark is that the Ashtekar variables are introduced by a canonical point transformation on the gravitational phase-space from the ADM canonical variables when the latter are written in the first-order form as $(K_{a}^{i}, E_{i}^{a})$. From now on, for consistency of notation, we will use superscript $(A)$ rather than $(g)$, in order to denote objects describing gravitational degrees of freedom. Since we foliate spacetime and restrict our analysis to three-dimensional spatial hypersurfaces with metric tensor $g_{ab}$ on it, we reserve the term ‘metric’ only to this object.

The Hamiltonian, which is obtained by the Legendre transform of (3), reads

$$H^{(A)} + H^{(A)} = \int_{\Sigma_{t}} d^{3}x \left( A_{a}^{i}G_{a}^{i}(A) + N^{a}V_{a}(A) + N(H^{(A)} + H^{(A)}) \right),$$

(5)

where the three elements

$$G^{(A)} := \frac{1}{\gamma\kappa} \int_{\Sigma_{t}} d^{3}x A_{a}^{i}D_{a}E_{i}^{a},$$

(6)

$$V^{(A)} := \frac{1}{\gamma\kappa} \int_{\Sigma_{t}} d^{3}x N^{a}F_{a}^{b}(A),$$

(7)

and

$$H^{(A)} = \frac{1}{\kappa} \int_{\Sigma_{t}} d^{3}x N\left( \frac{1}{\sqrt{q}}(F_{ab}^{i} - (\gamma^{2} + 1)\epsilon_{ilm}K_{a}^{i}K_{m}^{l})\epsilon^{ijk}E_{k}^{b} + 2\Lambda\sqrt{q} \right),$$

(8)

are called respectively the Gauss, the diffeomorphism (or vector) and the Hamiltonian (or scalar) constraints. These constraints impose respectively an internal SU(2), a spatial diffeomorphism and a time reparametrization invariance. Hence the Hamiltonian constraint describes dynamics on the SU(2) and spatial diffeomorphisms (or in short, diffeomorphisms) invariant subspace. Objects $A_{a}^{i}$, $N^{a}$ and $N$ are Lagrange multipliers. The quantity $F_{a}^{b}$ denotes the curvature of the Ashtekar connection, while $D_{a}$ is a metric and dreibein compatible covariant derivative.

II.2. Lattice regularization

Lattice regularization in LQG is performed in two steps. In the first step, we begin from the imposition of the so-called ‘Thiemann trick’, which goes as

$$\frac{1}{E_{i}^{a}}(\sqrt{|E|})^{n} = \frac{2}{n}\frac{\delta V^{n}}{\delta E_{i}^{a}} = \frac{4}{n\gamma\kappa} \left\{ A_{a}^{i}, V^{n} \right\},$$

(9)

$$K_{a}^{i} = \frac{\delta K}{\delta E_{i}^{a}} = \frac{2}{\gamma\kappa} \left\{ A_{a}^{i}, K \right\},$$

(10)

where $|E| = q$ is the absolute value of the determinant of $E_{i}^{a}$ and $K = \int d^{3}x K_{a}^{i}E_{i}^{a}$. 

The second step is to regularize the spatial hypersurfaces \( \text{via} \) a virtual granulation. It is realized by a construction of small solid objects — grains, which fill all of the spacelike Cauchy hypersurface and intersect each other only in lower-dimensional submanifolds. This granulation of space is controlled by the parameter \( \varepsilon \), where the limit \( \varepsilon \to 0 \) corresponds to the granulating object of a trivial volume or, in other words, corresponds to taking the regulator to zero. This is done in a way similar to taking the decoupling limit in effective field theories — decreasing the volume of the grains while at the same time increasing their number, in a way such that they always fill out the entire space. The standard choice for the shape of the solids is a tetrahedron. Consequently, the procedure is called a triangulation. The detailed description of this method can be found in [1, 2, 9]. An alternative, much simpler choice for the shape of the solids is a cuboid or even a cube — albeit resulting in fixing some of the gauge freedom of the theory. This is the case of the ‘cubulation’ procedure used in QRLG [9].

As a consequence of the granulation of the space, a regularization of the dynamical variables is introduced. After quantization, the effect of the regularization is to remove both the gravitational singularities (the initial singularity in a classical cosmology and the black hole singularity) and the UV-singularities of quantum matter fields [11]. Finally, there is an identification between the space and a graph \( \Gamma \) that is created as a consequence of the granulation. This identification is realized by a duality: \( \Gamma \) consists of links and nodes, hence in the dual graph we get respectively faces and volumes of the grains of \( \Gamma^* \).

At the level of the canonical variables, the regularization is realized as follows. The Ashtekar connection \( A^i_a \) is recovered from the holonomy \( h_a(v) := h_{iv}(v) \) (being a solution to the equation of a parallel transport of the connection) along the \( l^a(v) \) link emanated from the \( v \) node,

\[
h_a(v) := \mathcal{P} \exp \left( \int_{l^a} ds A^i_a(l(s)) \tau^i l^b(b) \right).
\]

Consequently, the curvature of the connection \( F_{ab}^i(v) \) is turned into the holonomy around the loop \( a \bowtie b \) that starts from the initial point of link \( l^a \), goes along this link and through the shortest polygon chain, it returns along link \( l^b \) to the initial point. It is worth noting that the a \( \bowtie b \) loop is constructed, connecting paths along links \( l^a \) and \( (l^b)^{-1} \) intersected at the node \( v \), with a path set by the arch \( \alpha_{ab} \). Moreover, we assume to fix the orientation of this path according to the orientation of the given loop of links.

This regularization is realized via the relations

\[
h_p^{-1}(h_p, V) = \varepsilon (A_a, V) P_p^a + O(\varepsilon^2),
\]

\[
2\varepsilon^{pq} h_{q\bowtie r} = \varepsilon^{pqr} (h_{q\bowtie r} - h_q^{-1}) = \epsilon^{ab(c)} \varepsilon^2 F_{ab} P_c^p + O(\varepsilon^4),
\]

where \( F_{ab} = F_{ab}^i \tau_i \), \( A_a = A_a^i \tau_i \) and \( p, q, r \ldots \) label directions of links of \( \Gamma \), while \( P_p^a \) is the projector onto these directions. As a result, we obtain all the gravitational dynamical variables written in terms of \( h_p, h_{q\bowtie r}, V \) and (in the case of the gravitational contribution to the Hamiltonian constraint) \( K \). Namely, the scalar constraint density reads,

\[
H^{(A)} = \frac{1}{\kappa} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^3} \int d^3 x \ N \varepsilon^{pqr} \left( \frac{2^3}{\gamma^3} \varepsilon \left( h_p h_q h_r^{-1} \{V, h_r \} \right) \right)
\]

\[
- \frac{2^5(\gamma^2 + 1)}{\gamma^3 \kappa^3} \varepsilon \left( h_p^{-1} \{K, h_p \} h_q^{-1} \{K, h_q \} h_r^{-1} \{V, h_r \} \right).
\]

Having derived the lattice-regularized scalar constraint, the quantization method is straightforward and can be implemented \( \text{via} \) the Dirac procedure [29]. We turn Poisson brackets into commutators multiplied by \( 1/i\hbar \) and change the dynamical variables into operators. The latter ones are the holonomy and flux of the densitized dreibein operator, as well as the geometrical operators, \( i.e. \) the volume, the area and the length operators. It is worth mentioning that the holonomy and flux are the canonical pair, while all the geometrical operators are constructed as smeared appropriate combinations of the densitized dreibein operators.

The kinematical Hilbert space of LQG (or its reduced equivalents, like CCQG or QRLG) is a direct sum of cylindrical functions of (possibly reduced, \( i.e. \) diagonal) connections along the links of the graph \( \Gamma \) (the general graph or cuboidal one \( ^{1/2} \Gamma \) in the reduced case). In the case of LQG, the kinematical Hilbert space is equipped with an inner product defined as an integral over cylindrical functions with an SU(2)-invariant Haar measure.

II.3. DeWitt coordinate representation

The Alesci-Cianfrani model is based on an appropriate projection of the SU(2) group elements to the three U(1) subgroups defined along the directions of the \( t^p := \vec{t^p} \cdot \vec{t} \) basis vectors, constructed as rotations of \( t^3 = u^3 \) (being
one of the Lie algebra generators) into the unit vector $\tilde{p}^i$ that projects $t^i$ onto $t^3$ and $\tau^i$ onto $\tau^3$, respectively — see Appendices A and B. This projection is based on the Livine-Speziale SU(2) coherent states [30], which are defined along $t^p = P^i t^i$ and minimize uncertainty of the gravitational momentum operator (defined in (19)).

QRLG has been constructed as a cosmological model with the general relativistic diffeomorphism invariance broken down to the Bianchi I symmetry, supposedly in a more rigorously defined way than in the case of LQC. The latter one, already before the quantization, replaces the Ashtekar connection in the definition of holonomy (11) with a diagonal variable defined as $A_a^{(1)}(\mathcal{L}QC) := \bar{c}_{(i)} e_a^{(i)} / l_0^{(i)}$ [3], with $l_0^{(i)}$ being the length of the $\delta_{e_a^{(i)}}$ side of the fiducial elementary cell, while $\bar{c}_{(i)}$ is a constant. As a result, the real SU(2) holonomy becomes replaced with a complex one, invariant under U(1) transformations. Considering a link $l^{(i)}$ of a length $\varepsilon l_0^{(i)}$, one finds the explicit form of the LQC holonomy,

$$h_{(i)}|_{\text{LQC}} = \left( \frac{1}{2} - i \varepsilon \bar{c}_{(i)} \right) \exp \left( \frac{i}{2} \varepsilon \bar{c}_{(i)} \right) + \left( \frac{1}{2} + i \varepsilon \bar{c}_{(i)} \right) \exp \left( - \frac{i}{2} \varepsilon \bar{c}_{(i)} \right) = \exp \left( \varepsilon \bar{c}_{(i)} \tau^{(i)} \right),$$

where the object $\exp \left( \pm \frac{i}{2} \varepsilon \bar{c}_{(i)} \right)$ is the complex U(1) holonomy. It is worth mentioning that an explicit form of the factor $\varepsilon$ distinguishes between so called $\mu$-scheme [3, 4] or $\bar{\mu}$-scheme [16, 17].

Notice that to recover a similar appearance of the holonomy as in LQC, Alesci and Cianfrani redefined the real LQG holonomy (11) to be the imaginary one $h_l = \exp(\pm i \int A^l_{ij} s^i \bar{s}^j)$ [3]. Then to preserve the structure of the theory, i.e. $h_l \in SU(2)$, they replaced the generators of the $su(2)$ representation with the self-adjoint operators $s^i := \sigma^i / 2$ (c see Appendix A). To avoid confusion, we are going to keep the standard notation, and show that there is no difference between $\exp(\pm \frac{i}{2} \varepsilon c_{(i)})$ and the reduced real LQG holonomy in (11). In both cases, the action of the densitized dreibein operator leads to a real eigenvalue.

In LQG, QRLG and LQC we use the DeWitt-like representation [31] of the Ashtekar variables,

$$A_a^i \rightarrow \tilde{A}_a^i(\ldots) = A_a^i(\ldots),$$

$$E^a_i \rightarrow \tilde{E}^a_i(\ldots) = -i k \delta_{\delta A_a^i} |\ldots\rangle.$$  

Here, $|\ldots\rangle$ denotes a standard basis vector in LQG or QRLG, which is defined in Sec. II.4. Notice that the operators in [16] and [17] do not correspond geometrically to their classical equivalents, since the Ashtekar connection has dimension of a length $^{-1}$, while the eigenvalue of the densitized dreibein operator has dimension of a $k \times$ length, and thus length$^3$.

The proper rescaling has been suggested by LQC [3], and later adapted to QRLG extending Bianchi I metric to an inhomogeneous model [3]. It defines a pair of canonical variables $(c_{(i)}, p^{(j)})$, which in the case of LQC are spatially constant, $c_{(i)} \rightarrow \bar{c}_{(i)}$, $p^{(j)} \rightarrow \tilde{p}^{(j)}$. The map from the Ashtekar variables to the reduced ones reads

$$R A_a^i(t, x) := \frac{1}{l_0^{(i)}} \bar{c}_{(i)}(t, x^{(i)}) \theta_a^i,$$

$$R E^a_i(t, x) := \frac{l_0^{(i)}}{\sqrt{V}_0} p^{(i)}(t, x^{(i)}) \sqrt{\theta_q^a} \theta_q^i,$$

where $V_0 := l_0^{(1)} l_0^{(2)} l_0^{(3)}$. The canonical Poisson relation for the reduced variables is summarized in Appendix C.

Now we are ready to prove that the eigenvalue of the $\tilde{p}^i$ operator acting on the state based on a reduced holonomy is real. For simplicity, we assume that the holonomy is oriented along the third internal direction, i.e. along the link $l^3$. Following Alesci and Cianfrani [3, 13], we impose the DeWitt representation (17) — this is not rigorous and completely correct (see Sec. IV), but is sufficient for our purposes — getting

$$\tilde{p}^i e^{c_{(i)} \tau^3} = -i k \delta_{\delta c_{(i)}} e^{c_{(i)} \tau^3} = -i k \varepsilon \bar{c}_{(i)} \bar{c}_{(i)} \tau^3 e^{c_{(i)} \tau^3} = -m k \bar{c}_{(i)} \bar{c}_{(i)} \tau^3 e^{c_{(i)} \tau^3}.$$

Notice also that in the last step we used the $-im$ eigenvalue of the $\tau^3$ generator in the spherical basis (see [A14]). The result in (20) coincides (up to the sign, being dependent on the orientation of links, and a result of a convention in a definition of the Lie algebra generators) with all the articles considering QRLG.

II.4. State space: SU(2)-coherent spin network

Another feature of the Alesci-Cianfrani construction, apart from the rotation onto the $t^p$ directions, is the modification of intertwiners. They are the components of the kinematical Gauss-invariant Hilbert space of LQG, introduced as a
result of the implementation of the Gauss constraint \([0]\) at the quantum level. The intertwiners, in the definition of
spin-network states (the basis states of the theory), connect SU(2) irreducible representations attached to the links of
the \(\Gamma\) graph. Consequently, intertwiners are thought to be attached to the nodes of \(\Gamma\), and act as projectors enforcing
the SU(2) gauge invariance via the group averaging. Within the case of the spin-network states rotated onto three
orthogonal directions, unique intertwiners are represented by the Clebsch-Gordan coefficients, or equivalently by the
Wigner 3-j symbols. It is worth noting that intertwiners do not appear in LQC. Therefore QRLG, as long as it is
formulated by the projection on the Gauss-invariant Hilbert space of LQG, retains a more adherent structure to the
original theory than LQC.

The spin-network states, defined as \(\langle h| \Gamma ; j^l, i_v \rangle\), are supported on the graphs \(\Gamma\), labelled by the spins \(j^l\) (representing
\(su(2)\) irreducible representations of the holonomies attached to each links \(l\) of \(\Gamma\)) and the intertwiners \(i_v\) (implementing
SU(2) invariance at each node \(v\) of \(\Gamma\)). In QRLG the basis-like states are \(\langle h| \Gamma ; j^p, i_v \rangle_R\) and involve Wigner matrices,
which are rotated to the \(t^p\) directions and projected on the coherent Livine-Speziale states \([30]\), with maximal spin number,
\[
j^p \to \bar{j}^p := j_{\text{max}}.
\]
Moreover, the secondary spin quantum number (called also the magnetic number), taking values \(m^p = -j^p, -j^p + 1, \ldots, j^p\) and being the eigenvalue of the \(s^3\) internal angular momentum generator (as in particle physics) in the spherical
basis (this notation is explained in Appendix A), is fixed to
\[
m^p \to \bar{m}^p := \pm \bar{j}^p.
\]

The reduced spin-network space is constructed as a space of solutions of the constraints \([6]\) and \([7]\) and will be
denoted as \(R\mathcal{H}^{(gr)}_{\Gamma} \). In the case of the Alesci-Cianfrani model, the SU(2) invariance is replaced with the three
\(U(1)\) symmetries along the directions of the links of \(\Gamma\), while the diffeomorphism constraint is restricted to the
implementation of an invariance under spatial diffeomorphisms, which do not generate any off-diagonal components.
The former restriction is an internal gauge fixing realized by the projection on the coherent states. The latter one
can be interpreted as an external gauge fixing of the geometry, which restricts a generic \(\Gamma\) graph the cuboidal one,
\(\hat{R}\Gamma\). A precise construction of the Hilbert space of the full theory, LQG, \(\mathcal{H}^{(gr)}_{\hat{R}\Gamma}\), can be found e.g. in [1], while that for
QRLG is given in [9, 10]. Details of the mechanism and consequences of rotational transformations imposed on the
LQG spin-network are discussed in the Sec. [II.4].

Finally, the problem of solving the Hamiltonian constraint at the quantum level recasts as the problem of finding
solutions of the action \(\hat{H} | \Gamma; J, I \rangle\) — in the reduced case, the action \(\hat{R}\hat{H} | \Gamma; J, I \rangle_R\). Here, \(J \ni m^l\) and \(I \ni i_v\) are the
set of spin numbers and the set of intertwiners, attached respectively to all links and all nodes of a given graph. For
simplicity, we omitted labeling with \(^{\hat{R}\Gamma}\) the quantities \(J\) and \(I\) inside the \(\ldots\rangle_R\) ‘kets’ describing reduced states.

As we already mentioned, the projection of the Wigner matrices on the coherent states, simultaneously projects
SU(2) intertwiners. As a consequence, states become decomposed as follows,
\[
\langle h| \Gamma; J, I \rangle_R := \sum_{v \in \Gamma} \langle j^l, i_v | \bar{m}^p, t^p \rangle \prod_{l \in \Gamma} \mathcal{D}_{\bar{m}^p, m^p}(\bar{h}_p),
\]
where \(\langle j^l, i_v | \bar{m}^p, t^p \rangle\) are the reduced \(U(1)\) intertwiners, while \(\mathcal{D}_{\bar{m}^p, m^p}(\bar{h}_p)\) is the Wigner D-matrix, with a fixed irreducible
representation \(j^p \to \bar{j}^p = |\bar{m}|^p\) attached to the \(l^i\) link.

It is important to notice that the basis-like states are not orthonormal within the scalar product given by the expression
\[
\langle \Gamma; m^p, i_v | M, m'^p, i'_v \rangle_R = \delta_{\Gamma, M} \prod_{l \in \Gamma} \prod_{v \in \Gamma} \delta_{j^l, j'^l} \delta_{i^l, i'^l} \langle \bar{m}^p, t^p | j^p, i_v \rangle \langle j'^p, i'_v | m'^p, t^p \rangle.
\]
The term \(\langle \bar{m}^p, t^p | j^p, i_v \rangle \langle j'^p, i'_v | m'^p, t^p \rangle\) represents a product of \(U(1)\) phases. It is also worth mentioning that by
definition any Hilbert space is complete, i.e. it has an orthonormal basis. Therefore we impose the normalization
\[
\langle \Gamma; J \rangle_R := \prod_{v \in \Gamma} \prod_{i_v \in I} \left( \langle j^l, i_v | \bar{m}^p, t^p \rangle \right)^{-1} \langle \Gamma; J \rangle_R,
\]
to drop the phase dependence. As a result, the state space of QRLG, namely \(R\mathcal{H}^{(gr)}_{\Gamma} \), gets rid of its dependence on
intertwiners placed at the nodes \(v\). Therefore this can be understood as a Hilbert space, becoming \(R\mathcal{H}^{(gr)}_{\Gamma} = \otimes_{l \in \Gamma} \mathcal{H}_p\),
with \(\mathcal{H}_p\) denoting the \(U(1)\) Hilbert space associated to each orthogonal direction. In other words, we simplify the
structure of the sum over intertwiners into a contraction of Kronecker delta functions oriented along the link-directions.
Let us present one more argument why the state provided by expression (25) can be considered as the spin-network of QRLG. From the point of view of the Dirac program of canonical quantization of constrained systems \[29\], one should impose constraints one by one, to recover the physical phase-space. Notice that for the $U^3(1)$ symmetry, the vector constraint vanishes identically\([1]\). The Gauss constraint becomes reduced to abelian transformations of Lie group along three orthogonal directions, $\tilde{h}_i \rightarrow \tilde{h}_i' = g_{i(i)} \tilde{h}_i g_{i(i)}^{-1} = \tilde{h}_i$, where $\tilde{h}_i, g_{i(i)} \in U(1)$. Hence there is no reason to introduce the construction of intertwiners (which is necessary in LQG formulated in terms of $SU(2)$ group elements). Then from the geometrical perspective of the $SU(2)$ to $U(1)$ reduced theory, these transformations are simply the phase transformations, with generators being C numbers. This allows us to perform the normalization as defined in (25).

Another way to reproduce this result is moving the reduced intertwiner $\langle j^i, i_c | \tilde{m} e^{\tau(t)} \rangle$ in (23) to the right hand-side of the expression, and then rescaling the $U(1)$ holonomy. For consistency, let us assume to move all the intertwiners in the unnormalized space towards positive orientation. Then, since for a given node-link pair, the intertwiners are fixed spin $\tilde{m}_i$-dependent functions, we simply rescale appropriately $\varepsilon$ in $|e^{\varepsilon \tilde{e}_i e^{\tau(t)}}\rangle$.

Finally, let us discuss why the reduced intertwiners, which appear to be only a redundant complication, are still present after the $SU(2)$ to $U(1)$ reduction in the original formulation of QRLG \[9\,13\]. These are a consequence of the reduction of a partially constrained kinematical Hilbert space of LQG. Notice that all the constraints, including the Gauss \[4\], the diffeomorphism \[7\] and the Hamiltonian one \[8\], are first class secondary constraints. They are independent, therefore after the quantization they should be imposed on the spin-network in any order, but necessarily during the same step, without any manipulations on the structure of the Hilbert space after implementation of only one of the constraints. As a result we would obtain a physical Hilbert space. Only by convenience — to simplify calculations — we first impose the Gauss constraint, then the diffeomorphism one and finally the Hamiltonian constraint. Hence for a consistency, the reduction procedure should be performed either on the kinematical, or on the physical Hilbert space and not on the gauge invariant kinematical Hilbert space (after imposition of only the Gauss constraint). The former choice does not generate the reduced intertwiners, because the kinematical Hilbert space is the space of cylindrical functions over the $\Gamma$ graph, equipped with the Ashtekar-Lewandowski measure \[32\,33\], without yet introduced the $SU(2)$ intertwiners. The latter choice, i.e. the reduction of the physical Hilbert space — up to the present stage of the development of LQG — is impossible to achieve. The Hamiltonian of LQG is so complicated that the full structure of the physical Hilbert space remains unknown. This argumentation is developed in further analyses, contained in Sec. III.2, where we present the proper order of implementation of the constraints leading to a well defined and simplified theory.

II.5. Gravitational field operators in LQG

Let us now discuss the generic model of LQG. While taking into account the cosmological constant’s sector, the whole difficulty in finding a solution to the equation $H^{(A)}|\Gamma; j^i, i_c \rangle$ becomes the derivation of the action of the volume operator,

$$\hat{\mathcal{V}}|\Gamma; J, I \rangle.$$ (26)

The gravitational Hamiltonian $H^{(A)}$ produces two classes of equations for the eigenvalues of the $su(2)$ traces of the operators in (14). As usual in the standard literature, we are going to call the first one the Euclidean term,

$$\text{tr} \left( \hat{h}_{p \gamma q} \hat{h}_{r}^{-1} \hat{\mathcal{V}} \hat{h}_{r} \right) |\Gamma; J, I \rangle.$$ (27)

The second, being the most complicated object, has been named the Lorentzian term and it is given by the formula

$$\text{tr} \left( \hat{h}_{p}^{-1} \hat{K}_{v} \hat{h}_{p} \hat{h}_{q}^{-1} \hat{K}_{v} \hat{h}_{q} \hat{h}_{r}^{-1} \hat{\mathcal{V}} \hat{h}_{r} \right) |\Gamma; J, I \rangle.$$ (28)

It is worth noting that equation (26) is solvable for simple configurations of states. However a problem, which arises is the fact that there is an ambiguity in the choice of the definition of the volume operator \[34\,35\]. Besides that, as we will see in the next sections, in order to derive actions of the complete set of all the Standard Model matter fields, we need rather some powers of $\hat{\mathcal{V}}_v$. Hence, instead of focusing only on the cosmological constant sector described by formula (26), we need to solve the following action,

$$\left( \hat{\mathcal{V}}_v \right)^n |\Gamma; J, I \rangle,$$ (29)

---

\[1\] Precisely speaking, reduced diffeomorphisms map directions of links into themselves. Hence for a diagonal form of the dreibein, the directions restrict the lattice to be cuboidal. Therefore fixing holonomies to the ones of the diagonal connections, which are attached to the cuboidal lattice, we neglect the vector constraint.
\(n\) being a positive rational number.

In the case of equation (27), the solutions for standard LQG have been found only for single-node states of a particular valency \(v = 1\), and for coherent complexifier states \(|\Phi\rangle\) [19, 20]. However, in the case of the reduced graph, a general solution exists \([11]\). This latter takes a simpler form upon inclusion of the corrections from the reduction procedure discussed in section (III.2) — see Sec. [IV].

Derivation of the Lorentzian term in equation (28) is even more demanding. As in the case of the Euclidean contribution to the Hamiltonian constraint, the result for a general case with a big number of nodes of different valency is rather impossible to be achieved. In the reduced model at the classical level this term does not appear any more. This is a consequence of the diagonalization of the spatial metric tensor (which is a correct assumption if one considers only a leading order term in the semiclassical analysis of QRLG). However, a precise approach to quantization of the Hamiltonian constraint has to be applied to the Lorentzian term as well. We expect that the next to the leading order corrections to the matrix element, expanded around the classical configuration, are of the same order of significance as the corrections from the expansion of the Euclidean term. Moreover, the corrections from the expansion of the Lorentzian term in the framework of QRLG could be different with respect to the ones obtained by Alesci and Cianfrani in \([11]\), as it happens in the case of different approaches to LQC.\(^{[12]}\)

### III. QUANTUM REDUCTION OF SPIN NETWORK

QRLG has been constructed as an alternative to LQC. It was thought to retain a definite advantage with respect to the latter theory, since it was believed to provide a precisely defined reduction procedure of LQG at the quantum level. It comes together with significant simplifications with respect to the full theory. As was already mentioned in the latter theory, since it was believed to provide a precisely defined reduction procedure of LQG at the quantum level. QRLG has been constructed as an alternative to LQC. It was thought to retain a definite advantage with respect to LQC.\(^{[39]}\)

Let us begin with the volume operator. The regularized action of this operator \([29]\) in full LQG has complicated structure \([11, 39]\). Neglecting ineffectiveness (from the point of view of applications) of a direct regularization of fluxes, we can write the action of \(\mathcal{V}_v\) defined around a neighborhood centered at the \(v\) node as

\[
(\mathcal{V}_v)^n|\Gamma; j^i, i_v\rangle = \left[ \int d^3x \left( \frac{1}{3!} \epsilon^{ijk} \epsilon_{pqr} \hat{E}_i(S^p) \hat{E}_j(S^q) \hat{E}_k(S^r) \right) \right]^{\frac{1}{2}} (\Gamma; j^i, i_v)_R^n, (30)
\]

where it has been assumed that the operator of a volume to a given power equals that power of the volume operator.\(^{[12]}\) The irregularity and complication of a structure of the general graph directly prevents from getting solution to equation (30). Since the same operator appears in other equations such as (27) and (28), and in the gravitational contributions to the HCO’s of matter fields, it follows that these cannot be solved either. The situation is much simpler in the Alesci-Cianfrani model with the regular cuboidal, self-dual graph.

The self-duality of \(\mathcal{R}\Gamma^*\) should be understood in a geometrical way. The faces dual to the links of \(\mathcal{R}\Gamma\) and the polyhedra dual to the nodes of this graph are respectively rectangles and cuboids. They are elements of the dual space. Then, the dual graph \(\mathcal{R}\Gamma^*\) is constructed from the edges and vertices of the cuboids. The result is the \(\mathcal{R}\Gamma^*\) graph, congruent to \(\mathcal{R}\Gamma\). Moreover, identifying the edges and vertices with some lattice’s links and nodes, respectively, leads to an analogous structure to \(\mathcal{R}\mathcal{H}_1^{(gr)}\). Then one can choose some averaging procedure that translates \(|\tilde{m}\rangle\) spin numbers attached to \(\mathcal{R}\Gamma\) onto the ones along the links of \(\mathcal{R}\Gamma^*\), emanated from the nodes shifted by a half link distance. As a result, one obtains a Hilbert space \(\mathcal{R}\mathcal{H}_1^{(gr)} \cong \mathcal{R}\mathcal{H}_1^{(gr)}\), where \(\mathcal{R}\mathcal{H}_1^{(gr)}\) is the normalized Hilbert space of QRLG. It is worth noting that a similar identification for state space of QRLG including intertwiners, \(\mathcal{R}\mathcal{H}_{1,v}^{(gr)}\), is generally not true — except for the homogeneous case, in which this identification is natural. This identification is not correct due to the presence of intertwiners placed at nodes of \(\mathcal{R}\Gamma^*\), which should not be related to the ones in \(\mathcal{R}\Gamma\), but should provide a gauge invariance in \(\mathcal{R}\mathcal{H}_1^{(gr)}\).

Another relevant feature of \(\mathcal{R}\Gamma^*\) is that it is a fixed graph, conversely to the graph structure, which supports LQG. The latter one is the uncountable (almost direct) sum of disjoint graphs, hence it is non-separable. Thus it can represent continuous geometries, being embedded in a differential manifold. The former one decomposes into a direct product of three fixed graphs. Each one supports a family of states, which corresponds to a fixed one-dimensional

---

\(^{[1]}\) See e.g. \([36]\) (for trivalent nodes) or \([37]\) (for tetravalent nodes).

\(^{[2]}\) The next to the leading order corrections arising from the Euclidean and Lorentzian terms are different with respect to each other both in the case of LQC \([38]\) and of the cosmological sector of LQG \([19]\). In the latter example we refer to the expectation value of HCO with respect to coherent states peaked on the phase-space variables assuming the Friedmann-Lemaître-Robertson-Walker symmetry.

\(^{[3]}\) This assumption is better legitimated in QRLG, where the volume operator is an eigenoperator of the reduced spin network (see Sec. \([IV]\)).
geometry. Moreover, the reduction procedure restricts both the canonical pair as well as all the geometrical operators to the ones that preserve the structure of $R\Gamma$. Therefore each space of cylindrical functions over a one-dimensional lattice is a superselection sector with corresponding graph-preserving (also called non-graph-changing) operators. Expectation values of these operators are Dirac observables. Each of these sectors is equipped with the U(1) Haar measure on the Bohr compactifications of the real line. It is also worth mentioning that in the homogeneous limit, this polymer-like structure simplifies into a collection of lines (via the well-known map — the Euler’s formula), equipped with the Lebesgue measures.

Notice that the kinematical Hilbert space of LQC is not separable by an analogous argument. The main difference is that the action of HCO connects different superselection sectors [16]. Then HCO is modified to preserve these sectors, while the physical Hilbert space is constructed from the states on which this HCO acts.

### III.1. Reduction procedure I: projected space approach

The first complete description of the reduction of LQG to QRLG was shown in [9]. Here we review the procedure, pointing out all the assumptions, which we can classify into two categories: i) an external additional modification not being a standard method of a field theory or LQG; ii) an internal gauge fixing introduced as an additional constraint at the quantum level. We also emphasize which steps in this method we consider to be incorrect. Finally, we distinguish a reduction of states that we label by sub-point a) and a reduction of operators labeled by b).

#### I)

The first assumption in this method is solving the Gauss constraint in (14) quantized and imposed on $\mathcal{H}_{kin}^{(gr)}$. As a result, we obtain the Gauss-invariant Hilbert space of LQG, already discussed in Sec. II.4. This is a standard procedure in the theory, but we placed it here, since it is a modification of the unconstrained Hilbert space before the next steps of reduction take place — these have to be done before solving HCO. It is worth mentioning that the order of resolution of the vector constraint operator is not influential into this analysis. However, all three first class constraints as the elements of the standard Dirac’s method of quantization [29], should be solved one by one without any intermediate modifications. Therefore we already found this first step to be extremely problematic. Moreover, later in this procedure, in Va), another first class constraint will be introduced, acting on the already reduced (Gauss-invariant) spin network.

#### II)

The next additional modification of LQG is a restriction on $\Gamma$ to be cuboidal,

$$\Gamma \rightarrow R\Gamma$$  \hspace{1cm} (31)

This is a cosmologically motivated simplification, replacing rotational invariance of directions at the quantum level with the translations along rigid Cartesian directions, $x^a \rightarrow x^p = x^a 0 e^p_a$. Notice that it has no influence on the canonical operators, but it restricts loop holonomies $h_{\eta \subset \Gamma}$ to rectangular paths.

#### III)

The third externally introduced assumption is a freezing of the internal symmetry. The SU(2) generators become fixed along directions of $R\Gamma$ lattice, $\tau^p = P^p_i \tau^i$. Notice that the internal rotation operator $P^p_i$, has to be fixed with respect to the orthogonal Cartesian frame spanned by $t^i$. Therefore, the translational invariance of the Ashtekar variables in [18] and [19] with the fixed spatial directions, $x^p$ has to be fixed as well. A change in the scaling of spatial direction, would affect the structure of the $P^p_i$ operator (and the choice of the $\rho$ vector — compare with Appendix B).

a) The basis states of the reduced spin-network are then chosen to be the U(1)$_p$-invariant, diagonalizing holonomies in [IIIb]).

b) The reduced holonomies take the following general form, $h_p = e^{i \alpha(p) \tau^p}$. Acting on states in [IIa)], they result in eigenvalue $e^{-i \alpha(p) m^{(p)}}$, where $\alpha(p)$ is a function of the Ashtekar connection and of a link, while $m^{(p)}$ is the magnetic quantum number corresponding to the spin attached to the $l^p$ link — see Appendix A.

#### IV)

Another modification of the theory is an introduction of the SU(2) projected spin-network by lifting up the reduced U(1)$_p$ state space (for a general idea of projected spin-networks see [31]).

a) The projected states are constructed by an extension of the U(1)$_p$ ones to the subsector of SU(2) restricted to generators $\tau^p$. This can be done by convolutions of $\bigotimes^3_p U(1)_p$ and SU(2) characters.

---

1. Another possible solution that tames this problem is called the integral Hilbert space method. In this model a separable Hilbert space for LQC is constructed in terms of an integral of superselection sectors equipped with a Lebesgue measure (see [10]).
b) Notice that, according to the construction of the projected spin-networks in [11], ‘gluing’ subspaces in order to lift them into the ones associated to a more general symmetry, should not affect the matrix elements of the operators. Therefore, at this step, we keep unchanged the reduced form of holonomies given in Appendix B.

V) a) Finally, considering now the lifted SU(2) state space, one can solve the following gauge fixing constraint,

$$E_j^a 0 c^k (E_j^b 0 c^k - E_k^b 0 c^j) = 0 .$$

(32)

Solving it at the quantum level, i.e. replacing densitized dreibeins with flux operators

$$\hat{E}^p(S) := \int_S dx^a dx^b \epsilon_{abc} \hat{E}^c_p ,$$

(33)

and identifying the internal directions with the lattice ones, we ensure that the constraint \( \epsilon^{ijk} E_a^j 0 c^k = 0 \) is implemented weakly. Notice that in this way we break the SU(2) group of rotations into U(1) in a controlled way, introducing another first class constraint in the form of (32). Solutions to this equation at the quantum level (see [9] for the details) are realized by the states corresponding to the quantum numbers satisfying the relations

$$j^p \to \infty$$

(34)

and

$$m^p \approx \bar{m}^p .$$

(35)

Therefore, when defining the reduced spin-network, the limit in (34) is approximated by (21), while the approximation in (35) becomes simplified by the sharp equality, as in (22). It is worth noting that although this reduction step is implemented only with respect to the states, it was introduced as a new first class constraint, thus it should impose a restriction on the state space, with no influence on the operators. This interpretation prevents from the argument of breaking a consistency between states and operators, while implementing the modification.

VI) b) The last assumption is to replace the rotated holonomies obtained in IIIb) with general ones, restricted only by the rotation of generators into \( \tau_p \). In this way the mechanism for lifting up the U(1)_p states into SU(2) ones in [IVa) is extended into the operators. Moreover, releasing the internal fixing of generators, one can restore the translational invariance along \( x^p = x^a 0 c^a_p \) of the Ashtekar variables in (18) and (19). Notice that without any changes, this step can be also implemented before point [Va]. However, this additional modification is not a consequence of the projected spin-network technique, and it results in a much wider spectrum of operators than in the case we discussed in [IVa]. Moreover, since this assumption is introduced to change only the operators, we consider it as being inconsistent, breaking the balance in a simultaneous reduction of states and operators.

Summarizing, the procedure introduced in [9] appears to be complicated, and in several steps also problematic. However, in [10] a simpler method, which we discuss in the next subsection, has been proposed.

III.2. Reduction procedure II: external gauge approach

The reduction scheme described in the previous subsection can be simplified and performed in a more controlled way. In the original construction, four additional modifications and one extra first class constraint are needed. Here, referring to the method described in [10], we obtain the same results using two constraints and two additional modifications.

1) We begin from the same problematic assumption as in [1], first solving the Gauss constraint.

2) a) Next, we introduce a new gauge fixing equation that classically describes the vanishing of the off-diagonal entries of the dreibein (and densitized dreibein) matrices,

$$E^a_i E^b_i |_{a \neq b} = 0 .$$

(36)

Notice that this constraint is introduced as a global condition, breaking the SO(3) invariance of the \( q_{ab} \) spatial metric, and leads to Ashtekar variables that in the most general form can be represented by expressions (18) and (19). At the quantum level, this leads to the large-\( j \) limit solution in (34) and the cuboidal graph structure (31). It is worth mentioning that the implementation of the new constraint in (36) at the quantum level influences only states, not modifying operators.
Then, we fix holonomies and states to the ones with frozen generators \( \tau^p \), and the ones resulting in the diagonal eigenvalues of the holonomies, respectively. This is the same step discussed in III.

4) a) The fourth condition is again a gauge fixing constraint, the same as in [Va]. The only difference with respect to the method in subsection III.1 concerns the implementation of the gauge fixing to \( \text{U}(1) \), not on the \( \text{SU}(2) \)-invariant space, but on the \( \bigotimes_p \text{U}(1)_p \) state space.

b) Finally, in this simplified construction of QRLG, the additional modification, already described in [VIIb], is introduced. Hence again holonomies are promoted into general \( \text{SU}(2) \) operators, while states are not changed in an analogous way. As noticed previously, this appears to be an inconsistent step.

Concluding the enhanced reduction procedure, the number of external modifications to the theory has been decreased, but the same problematic steps as in Sec. III.1 still appear in Sec. III.2.

### III.3. Reduction procedure III: perspicuous assumptions approach

Finally, let us propose another model for the reduction procedure, which is a direct analog of the QRLG method, but at the same time remains free from the problematic assumptions.

A. a) First, we introduce the gauge fixing equation [36], leading to the classical condition selecting only diagonal entries of the densitized dreibein matrices. In the same way as in (2a), it corresponds to the states associated with large-\( j \) spin numbers and the cuboidal graphs. Notice that even if we did not impose previously the Gauss constraint, this condition acts exactly like in the standard QRLG, which we described in subsection III.2.

b) As a second step, we fix the holonomies to be the ones with rotated generators \( \tau^p \) and we fix states to the ones resulting in the diagonal eigenvalues of the holonomies. Performing the same step as in III and 3, we simplify the spin-network’s structure to the three copies of the \( \text{U}(1)_p \) state spaces, on which the reduced holonomies, \( h_p = e^{\alpha_p(\tau^p)} \) act.

c) Finally, we gauge fix the system again, in the same way as specified in the point 4a) and analogously to [Va]. This results into a \( \text{U}(1) \)-invariant Hilbert space and three different copies of the \( \text{U}(1) \) operators, which is the last step in the procedure. Notice that we obtained an abelian model classically corresponding to the homogeneous variables,

\[
R\tilde{A}^i_a(t) := \frac{1}{\sqrt{\lambda(t)}} \tilde{c}^{(1)}_a(t) \tilde{q}^i_a,
\]

\[
R\tilde{E}^a_i(t) := \frac{1}{\sqrt{\lambda(t)}} \tilde{p}^{(1)}_a(t) \sqrt{q^j} \tilde{c}^j_i.
\]

As a result, imposing two first class constraints and one external assumption, we end up with a simplified version of QRLG. However, the most significant benefits of this procedure are the omissions of steps I) and 1), as well as VIb). The former ones not only affect the balance between the reduced states and the operators, while states are not changed (2), but also result in the appearance of the reduced intertwiners in (23) that break the orthogonality of the state space in the scalar product’s definition entering (24). The latter ones affect the balance between the reduced states and the operators. Moreover, they generalize the homogeneous variables entering in (37) and (38) into the inhomogeneous form appearing in (18) and (19), respectively. This leads to a restoration of the reduced Gauss constraint,

\[
RG^{(A)} = \frac{1}{\kappa} \int_{\Sigma_t} d^3 x \ A^i_a \partial_a R\tilde{E}^a_i
\]

and the reduced diffeomorphism constraint,

\[
RV^{(A)} = \frac{1}{\kappa} \int_{\Sigma_t} d^3 x \ N^a \left( \partial_a R\tilde{A}^i_b - \partial_b R\tilde{A}^i_a \right) R\tilde{E}^b_i.
\]

The former expression corresponds to \( \propto \partial_p p^{(p)}(t, x^{(p)}) \), and generates the \( \text{U}(1)_p \)-invariance. The constraint in (40) simplifies to \( \propto \partial_p c^{(p)}(t, x^{(p)}) \) — the terms \( \propto \partial_p c^{(p)} |_{p \neq q} \) vanish. Imposing the Gauss constraint, \( \partial_p p^{(p)} = 0 \), and neglecting the boundary term, the reduced diffeomorphism constraint can be recast as an expression proportional to
Introducing the simplified formulation with a correctly defined dynamics, QRLG became a simple U(1) theory. Now the states are defined as follows,

\[
\langle \bar{h}|\Gamma; J \rangle := \prod_{\nu \in \Gamma} p_{\nu} \bar{D}_{\mu \nu} \bar{m}_{\mu \nu}(\bar{h}_{\nu}),
\]

where \( p_{\nu} \bar{D}_{\mu \nu} \bar{m}_{\mu \nu}(\bar{h}_{\nu}) \) are the same Wigner \( D \)-matrices as in the definition in [23]. These states span a well-defined Hilbert space, with an orthogonality condition given by the scalar product

\[
\langle \Gamma; \bar{m}^p | \Gamma', \bar{m}'^q \rangle = \delta_{\Gamma, \Gamma'} \prod_{l \in \Gamma} \delta_{j^p, j'^q}.
\]

Rewriting the total Hamiltonian in [5] in terms of the homogeneous variables defined in (37) and (38), considerably simplifies dynamics. Indeed, this takes the form

\[
H_{T}^{(\Lambda)} + H_{T}^{(A)} = H^{(\Lambda)} + H^{(A)},
\]

while the constraints in (6) and (7) vanish identically.

All the results of original QRLG, concerning flux-dependent operators remain unchanged. For instance the eigenequation of the reduced flux operator acting at a uniquely defined point \( l^{(q)} \cap S \), reads

\[
\hat{E}^{p}(S)|l^{(q)}, \bar{m}^{(q)}\rangle = -i k \int_{S} dx^{a} dx^{t} \epsilon_{rst} \frac{\delta}{\delta A_{r}^{p}(x^{t})} |l^{(q)}(y^{(q)}), \bar{m}^{(q)}\rangle
\]

\[
= -i k \delta^{p}_{q} \langle \tau^{q} |l^{(q)}, \bar{m}^{(q)}\rangle = -k \bar{m}^{(q)} \delta^{q}_{p} |l^{(q)}, \bar{m}^{(q)}\rangle.
\]

It is worth mentioning that the difference in the sign with respect to the Alesci-Cianfrani result in [9,13] comes from the initial choice of the \( \tau^{q} \) generators and the real holonomies — we already discussed that in Sec. II.3.

Notice also that since a holonomy operator acts on states as a multiplication, the aforementioned eigenequation is the only expression that imposes significant constraints on the structure of the states. Moreover, it is easy to see that formula (44) has the form of a simple differential equation, with the solution

\[
|l^{(q)}, \bar{m}^{(q)}\rangle = \exp \left( -i \bar{m}^{(q)} \int dx^{a} \bar{A}_{a}^{(q)} \right).
\]

Thus, without any loss of generality, we may consider this expression as a definition of a state.

Let us now derive the action of a square of the volume operator defined on an open neighborhood \( B_{l_{0}(v)} \),

\[
(\hat{V}_{v, l_{0}})^{2}|\Gamma; J \rangle = k^{3} \prod_{p} \left| \bar{m}_{l(p)}(v) + \bar{m}_{l(p)}(v - l_{0}^{(p)}) \right| \Gamma; J \rangle.
\]

Here, \( v \in \Gamma \) is a hexavalent node, i.e. there are six links emanated from \( v \), while \( v - l_{0}^{(p)} \) denotes a nearest neighbor node in a negatively oriented \( p \) direction in a distance \( l_{0}^{(p)} \). Operators \( \hat{E}_{i}(S^{p})|_{i=p} \) from expression (30) act at \( l^{(p)}(v - l_{0}^{(p)}) \cap S \).
and at \( I^{(p)}(v) \cap S_+ \) — the vertices in brackets represent initial points of the positively oriented collinear links connected at \( v \). Notice also that to derive the eigenvalue of \( \hat{V}_{v,l_0} \), we used the eigenvalue of the flux operator in (44), assuming an unidirectional orientation of a collinear link frame \( y^{(q)} \) and a coordinate frame \( x^{(r)} \).

It is worth mentioning that the derivation of the square root of the result in (46) would be problematic. However, the square root of an analogous matrix element is well defined,

\[
\langle \hat{V}_{v,l_0} \rangle := \sqrt{\langle \Gamma; J | (\hat{V}_{v,l_0})^2 | \Gamma; J \rangle}.
\]

(47)

It also coincides with the analogous expression in the original formulation of QRLG [9].

Finally, we can find the eigenvalue of the \( U(1) \) holonomy operators, derived on the basis states in (45),

\[
\hat{h}^{(p)} = \exp \left( i \hat{c}^{(p)} - i \hat{\bar{m}}^{(p)} \int \bar{A}(r) \right) \hat{\bar{V}}_{v,l_0} \hat{\bar{h}}_{r}.
\]

(48)

Notice that the reduction defined in Sec. III.3 is performed at the quantum level, hence we cannot \( \textit{a priori} \) replace the \( \hat{h}^{(p)} \) operator with its eigenvalue \( \hat{h}^{(p)}(v) \) in the Hamiltonian constraint. This operation can be done after we act HCO on (45).

Then, in order to derive the action of HCO in our simplified approach, we find that the eigenvector of expression (27) reads

\[
\sum_{\nu \in \Gamma} \text{tr}\left( \hat{h}^{-1} \hat{V}_{v,l_0} \hat{h} \right) | \Gamma; J \rangle = -\frac{i}{2} \epsilon^{pqr} \sum_{\nu \in \Gamma} \sin(\epsilon \hat{c}_r) \sin(\epsilon \hat{\bar{m}}_q) \prod_{s=1}^3 \left| \bar{m}(s)(v) + \bar{m}(s)(v - l_0^{(s)}) \right|^{1/2} \times \left( \left| \bar{m}(r)(v) + \bar{m}(r)(v - l_0^{(r)}) + \epsilon \right| \right) - \left| \bar{m}(r)(v) + \bar{m}(r)(v - l_0^{(r)}) - \epsilon \right| | \Gamma; J \rangle.
\]

(49)

Here, by analogy with (47), we assumed that the eigenvalue of the square root of the volume operator square equals to the square root of the eigenvalue of \( (\hat{V}_{v,l_0})^2 \). Expanding the result in (49) arbitrarily around \( \bar{m}(v) \approx \bar{m}(v - l_0^{(s)}) \approx \infty \) or around \( \epsilon \approx 0 \), leads to the well known eigenvalue of the Euclidean contribution to HCO in LQC [3, 4]. In order to derive the action of the operator in (28), and then the full Lorentzian term, one should use the quantum relation introduced in [25], namely

\[
\hat{K}_v = -\frac{2}{\kappa^2} \epsilon^{pqr} \left[ \text{tr}\left( \hat{h}^{-1} \hat{V}_{v,l_0} \hat{h} \right), \hat{V}_{v,l_0} \right].
\]

(50)

Details of the derivation of HCO for the simplified (and improved) version of QRLG are in preparation. However, we can already recognize that the form of the operator realizes the Bianchi I extension of the modified LQC introduced in [38], with the inverse volume corrections in the form of

\[
1 + \frac{\epsilon^2}{8 \left( \bar{m}(r)(v) + \bar{m}(r)(v - l_0^{(r)}) \right)^2} + \mathcal{O} \left( \frac{\epsilon^4}{\bar{m}(r)} \right).
\]

(51)

arising from each trace analogous to the expression (49).

V. CONCLUSIONS

We have argued that shortcomings arise within the initial model of Quantum Reduced Loop Gravity proposed by Alesci and Cianfrani, proving its \textit{internal inconsistency}. We arrived to these conclusions re-examining the reduction procedure applied to the states of the kinematical Hilbert space of Loop Quantum Gravity, and developing a comparative analysis with previous attempts formulated in the literature of QRLG. Constraints were formerly inconsistently implemented within the framework of the reduced model, which was leading to an overconstrained dynamics.

This analysis motivated us to develop the alternative implementations of the symmetry-reduction procedure, which we have discussed here in detail. In particular, we remark that the \( U(1) \) version we alternatively proposed here,
They form observable in quantum theories. They are also unitary, hence they preserve norm and thereby probability amplitude.

The Pauli matrices

Applying SU

so far could not help in solving the Hamiltonian constraint in the full theory. Instead, our proposal reckons on the program of bridging the gap between the full theory and former endeavors in LQC.

We have been then shifting away from the paradigm of quantizing a symmetry reduced space, a procedure that so far could not help in solving the Hamiltonian constraint in the full theory. Instead, our proposal reckons on the program of bridging the gap between the full theory and former endeavors in LQC.

**Appendix A: su(2) representations and spin representations**

The Pauli matrices \( \sigma^i \) are a set of three \( 2 \times 2 \) complex matrices. They are Hermitian (self-adjoint), thus they represent observables in quantum theories. They are also unitary, hence they preserve norm and thereby probability amplitude. They satisfy normal commutation relations,

\[
[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k. \tag{A1}
\]

The \( su(2) \) Lie algebra generators \( t^i := -i\sigma^i \) of the SU(2) group are antihermitian (skew-Hermitian) and unitary. They form \( su(2) \) basis and satisfy the following commutation relations,

\[
[t^i, t^j] = 2i\epsilon^{ijk}t^k. \tag{A2}
\]

The \( so(3) \cong su(2) \) Lie algebra generators \( \{ u^1 := \frac{1}{\sqrt{2}}(\sigma^1+i\sigma^2), u^2 := -\frac{1}{\sqrt{2}}(\sigma^1-i\sigma^2), u^3 := -i\sigma^3 \} \) of the rotation group are antihermitian and unitary. They form spherical basis and satisfy the following commutation relations,

\[
[u^1, u^2] = 2iu^3, \ [u^1, u^3] = 2iu^1, \ [u^2, u^3] = -2iu^2. \tag{A3}
\]

The Hermitian generators of the spin representation in particle physics are defined as \( s^i := \frac{1}{2}\sigma^i \).

The antihertian generators of \( su(2) \) that form the spin representation in LQG are defined as \( \tau^i := \frac{1}{2}t^i \). They are the preferable choice for keeping both holonomies and Ashtekar connections real.

Finally, we should define the antihertian generators of the \( so(3) \) spherical representation of spin in LQG, \( v^i := \frac{1}{2}u^i \).

Knowing the standard spin basis (equivalently the angular momentum basis) defined by the \( i\sigma^j \) operators (commonly used in the particle physics), we find

\[
v^1|j, m\rangle = -i\sqrt{j(j+1) - m(m+1)}|j, m+1\rangle, \\
v^2|j, m\rangle = i\sqrt{j(j+1) - m(m-1)}|j, m-1\rangle, \\
v^3|j, m\rangle = -im|j, m\rangle, \tag{A4}
\]

\[
v^i v^j |j, m\rangle = -j(j+1)|j, m\rangle, \tag{A5}
\]

where \( j = 0, 1/2, 1, \ldots \) and \( m = -j, -j+1, \ldots, j \).

It is worth noting that all the representations discussed above are proper. Furthermore, the sign in front of each triple of generators is conventional. Reversing the sign, we impose anomalous commutation relations instead of the normal ones in (A1) and (A2).

**Appendix B: Wigner D-matrices of diagonal holonomies**

Let us define a unitary rotation matrix \( \bar{\rho} \) as follows,

\[
\bar{\rho}^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \bar{\rho}^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \quad \bar{\rho}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{B1}
\]

Applying SU(2) covariance of generators \( \tau^i \), we can consider the rotation of a basis frame as a matrix operation,

\[
e^{\bar{\sigma} \cdot \bar{\tau} \cdot \bar{\epsilon}} = \bar{\rho}^1 e^{\tau^3 \bar{\epsilon}} \rho^1. \tag{B2}
\]
Then the following relation holds,

\[ D^{(j)}_{mn}(e^{\tau^3 c}) = D^{(j)}_{mm'}(\rho^1) D^{(j)}_{m'n}(e^{\tau^3 c}) D^{(j)}_{n'n}(\rho) = e^{-imc}, \]

where we used the fact that for a diagonal SU(2) element, the Wigner $D$-matrix takes a particularly simple form,

\[ D^{(j)}_{mn}(e^{\tau^3 c}) = e^{-imc}\delta_{mn}. \]

**Appendix C: Symplectic structure and Poisson brackets**

The symplectic structure of LQG, $\Omega_{LQG}(\delta_1, \delta_2)$, corresponding to (4) reads

\[ \Omega_{LQG} = 2\gamma\kappa\int d^3 x \left( \delta_1 A^a_i(x) \delta_2 E^a_i(x) - \delta_2 A^a_i(x) \delta_1 E^a_i(x) \right). \]

Thus the analogous structure for QRLG becomes

\[ \Omega_{QRLG} = 2\gamma\kappa \sum_i \int d^3 x \left( \delta_1 c^{(i)}(t, x^{(i)}) \delta_2 p^{(i)}(t, x^{(i)}) - \delta_2 c^{(i)}(t, x^{(i)}) \delta_1 p^{(i)}(t, x^{(i)}) \right), \]

while considering the LQC limit, one finds

\[ \Omega_{LQC} = 2\gamma\kappa \sum_i \int d\bar{c}^{(i)}(t) \wedge d\bar{p}^{(i)}(t) = \frac{2}{\gamma\kappa} \bar{c}^{(i)}(t) \wedge d\bar{p}^{(i)}(t). \]

Therefore the Poisson bracket (4) for the reduced variables reads

\[ \{c_i(t, x^{(i)}), p^j(t, y^{(j)})\} = \frac{\kappa\gamma}{2} \delta^j_i \delta(x^{(i)} - y^{(j)}) = \frac{\kappa}{\hbar} \delta^j_i \delta(x^{(i)} - y^{(j)}). \]

Analogously, in the homogeneous limit one finds

\[ \{\bar{c}_i(t), \bar{p}^j(t)\} = \frac{\kappa\gamma}{2} \delta^j_i. \]

**Acknowledgments**

A.M. acknowledges support by the NSFC, through the grant No. 11875113, the Shanghai Municipality, through the grant No. KBH1512299, and by Fudan University, through the grant No. JJH1512105. J.B. is supported in part by the NSFC, through the grants No. 11375153 and 11675145.
