As a typical characteristic of ergodicity, the equality of the time average and the space average is pointed out. However, there exist phenomena in which the time average is not equivalent to the space average in infinite ergodic systems. The Boole transformation is known as a one-dimensional map, which is proven that the transformation preserves the Lebesgue measure (infinite measure) and is ergodic. In this paper it is proven that countably infinite number of one-parameterized one dimensional maps which are generalized from the Boole transformation exactly preserve the Lebesgue measure and are ergodic at certain parameters. Additionally we show that in these maps the normalized Lyapunov exponent obeys the Mittag-Leffler distribution of order 1/2 by numerical simulation.

I. INTRODUCTION

Chaos theory has developed statistical physics through ergodic theory. In a chaotic dynamics, it is difficult to predict future orbital state from past information because the system is unstable, which is characterized by the sensitivity to initial conditions. However, from its mixing property, one can characterize the system statistically using the invariant density function. The relation between microscopic dynamics and density function is important when macroscopic properties are led from microscopic dynamics, and ergodicity plays a significant role in this derivation.

In the case of a dynamical system \((X, T, \mu)\) with a normalized ergodic invariant measure \(\mu\) where \(X\) and \(T\) represent the phase space and a map, respectively, for an observable \(f \in L^1(\mu)\), a time average \(\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f \circ T^i\) converges to the phase average \(\int_X f d\mu\) in almost all region. In systems with a normalized Lyapunov measure, we can characterize their stability by Lyapunov exponent \(\lambda\), which is defined as \(\lambda \defeq \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log |T'(x_i)|\) when \(|T'(x_i)|\) is a \(L^1\) class function for the measure \(\mu\). Usually, since it is difficult to obtain the information at infinite time, we use numerical simulations or apply the ergodicity to calculate the Lyapunov exponent as \(\lambda = \int_X \log |T'(x)| \, d\mu\). For example, for logistic map \(x_{n+1} = ax_n(1 - x_n)\), the Lyapunov exponent \(\lambda\) is obtained analytically as \(\lambda = \int_{-\infty}^{\infty} \left( -\frac{2}{x} \right) \frac{\sqrt{1 - \alpha}}{\pi x'^2} \, dx\) for \(0 < \alpha < 1\).

On the other hand, how about the case of infinite ergodic case? Consider the Boole transformation \(x_{n+1} = T(x_n) = x_n - 1/x_n^2\), which corresponds to \(\alpha = \beta = 1\) for the generalized Boole transformations where the dynamical system preserves the Lebesgue measure as an infinite ergodic system. That means it holds that \(\int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^{\infty} f \left( x - \frac{1}{x} \right) \, dx\) where \(f\) is a \(L^1\) function with respect to \(dx\). For an observable \(|T'|\), although the usual time average \(\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log |T'(x_i)|\) converges to zero, the phase average is as \(\int_{-\infty}^{\infty} \log \left( |1 + 1/x^2| \right) \, dx = 2\pi\), so that the time average does not consistent with the phase average.

In infinite ergodic systems, the Darling, Kac andAaronson theorem says that if the observable \(f\) is positive \(L^1\) function in terms of invariant measure \(\mu\), the time average using the return sequence \(a_n\) converges in distribution. In the case of the Boole transformation, by defining the return sequence \(a_n \defeq \sqrt{n}\), the distribution of \(\frac{1}{n} \sum_{i=0}^{n-1} \log |T'(x_i)|\) converges to the Mittag-Leffler distribution of order 1/2. That is, interesting phenomena are observed which are different from the usual ergodic theory and the standard statistical mechanics.
lowing $L^1$ class observables converge to the Mittag-Leffler distribution such as Lempel-Ziv complexity\textsuperscript{11}, the transformed observation function for the correlation function\textsuperscript{11}, normalized Lyapunov exponent\textsuperscript{11}, normalized diffusion coefficient\textsuperscript{11} and that non-$L^1$ class observables such as time average of position\textsuperscript{15} converges to generalized arc-sin distribution\textsuperscript{14,18} or other distribution\textsuperscript{37}.

Infinitesimal densities were observed in the context with the long time limit of solution of Fokker-Planck equation for Brownian motion\textsuperscript{38,39} and semiclassical Monte Carlo simulations of cold atoms\textsuperscript{20}.

In order to characterize the instability of systems with infinite measure, several quantities were invented such as Lyapunov pair\textsuperscript{27} and generalized Lyapunov exponent\textsuperscript{24,25}.

In relation to Lyapunov exponent, the change of stability of systems characterizes their dynamical properties and is important phenomenon. In particular, critical phenomena at which systems become unstable from stable called as routes to chaos has attracted a lot of interest to investigate \textquoteleft\textquoteleft ergodic\textquoteright\ which is invariant under Hamiltonian dynamical system\textsuperscript{20}.

In the following one can extend the Range A to the newly defined Range B such that \textquoteleft\textquoteleft $0 < |\alpha| < 1$ and $K = 2N'$ or $\frac{1}{K^2} < |\alpha| < 1$ and $K = 2N' + 1$\textquoteright\ where $N' \in \mathbb{N}$. At first, define a function $F_K : \mathbb{R} \setminus A \rightarrow \mathbb{R} \setminus A$ such as

\[ F_K(\cot \theta) \overset{\text{def}}{=} \cot(K\theta) \quad (1) \]

where $K \in \mathbb{N} \setminus \{1\}$ and $A$ represents a set of point $x \in \mathbb{R}$ such that for finite iteration $n \in \mathbb{Z}$, $F^0_K(x)$ reaches the singular point.

Let us define Range A as \textquoteleft\textquoteleft $0 < |\alpha| < 1$ and $K = 2N''$ or $\frac{1}{K} < |\alpha| < 1$ and $K = 2N'' + 1$\textquoteright\ where $N'' \in \mathbb{N}$. When $(K, \alpha)$ are in Range A, the super generalized Boole (SGB) transformations are \textit{exact} and when $\alpha > 1$, the any orbits diverge to the infinity and the SGB transformations do not preserve measure over real line\textsuperscript{37}.

In the following we consider the super generalized Boole (SGB) transformations\textsuperscript{37} at first, define a function $F_K : \mathbb{R} \setminus A \rightarrow \mathbb{R} \setminus A$ such as

\[ F_K(\cot \theta) \overset{\text{def}}{=} \cot(K\theta) \quad (1) \]

where $K \in \mathbb{N} \setminus \{1\}$ and $A$ represents a set of point $x \in \mathbb{R}$ such that for finite iteration $n \in \mathbb{Z}$, $F^0_K(x)$ reaches the singular point.

In the following one can extend the Range A to the newly defined Range B such that \textquoteleft\textquoteleft $0 < |\alpha| < 1$ and $K = 2N''$ or $\frac{1}{K} < |\alpha| < 1$ and $K = 2N'' + 1$\textquoteright\ where $N'' \in \mathbb{N}$. Let us define the Range A’ as

\[ \begin{cases} -1 < \alpha < 0 & \text{in the case of } K = 2N', \\ -1 < \alpha < -\frac{1}{K} & \text{in the case of } K = 2N' + 1, \end{cases} \]

where $N' \in \mathbb{N}$. In the following extension from $\alpha$ to $|\alpha|$ can be proven in the similar way as the reference 37. If the density function at the time $n$ ($f_n(x)$) is denoted as

\[ f_n(x) = \frac{1}{\pi} \frac{\gamma}{x^2 + \gamma^2} , \]

$f_{n+1}(x)$ is given by

\[ f_{n+1}(x) = \frac{1}{\pi} \frac{|\alpha|KG_K(\gamma)}{x^2 + (|\alpha|)^2K^2G^2_K(\gamma)} \]

according to the Perron-Frobenius equation where $G_K(\gamma)$ corresponds to the the K-angle formula of the cot function\textsuperscript{37}. Then the scale parameter $\gamma$ is changed in a single iteration as

\[ \gamma \mapsto |\alpha|KG_K(\gamma). \]
Then, by changing the parameter from \( \alpha \) to \( |\alpha| \), we can prove straightforwardly that the SGB transformations \( \{S_{K,\alpha}\} \) preserve the Cauchy distribution and the scale parameter can be chosen uniquely when the parameters \( (K, \alpha) \) are in Range B. In terms of exactness, it holds that \( S'_{K,\alpha}(\theta) < 0 \) when \( (K, \alpha) \) are in the Range A’. Then, \( S'_{K,\alpha}(\theta) \) is also the monotonic function. Thus, we can prove that the SGB transformations \( \{S_{K,\alpha}\} \) are exact when the parameters \( (K, \alpha) \) are in Range B by considering the intervals \( \{I_{j,n}\} \). In the case of \( \alpha < -1 \), one straightforwardly sees that orbits diverge to the infinity and the SGB transformations do not preserve measure over real line.

From above discussion, we know that the SGB transformations are exact when the parameters \( (K, \alpha) \) are in range B and the systems are dissipative when \( |\alpha| > 1 \). However, the ergodic property at the critical point \( \alpha = \pm 1 \) was unsettled.

Then, what happens at \( \alpha = \pm 1 \)? Since the statistical properties drastically change before and after the value of \( \alpha = \pm 1 \), the ergodic property of the critical SGB transformations at \( \alpha = \pm 1 \) is important. As we know, the Boole transformation which corresponds to the case of \( K = 2, \alpha = 1 \) preserves the Lebesgue measure and are ergodic. In the following section, we show that all the SGB transformations at \( \alpha = \pm 1 \) preserve the Lebesgue measure for any \( K \in \mathbb{N}\setminus\{1\} \). Table I shows the explicit form of \( S_{K,\pm 1} \) for \( K = 2, 3, 4, 5 \) and 6.

![FIG. 1: Return maps of \( S_{\pm 1,1}, S_{\pm 1,4} \) and \( S_{\pm 1,6} \). The function \( f(x) = x \) represents the set of fixed points.](image)

### III. INFINITE ERGODICITY FOR \( \alpha = 1, -1 \)

**Theorem III.1.** The SGB transformations at \( \alpha = \pm 1 \) preserve the Lebesgue measure.

**Proof.** The goal is to prove that
\[
|S_{K,\pm 1}^{-1}I| = |I|
\]

for any interval \( I \) where \( |\cdot| \) denotes the length of an interval. It is sufficient to verify this for intervals of \( I = (0, \eta), \eta > 0 \) and \( I = (\eta, 0), \eta < 0 \).

(I) In the case of \( \alpha = 1 \).

(II) In the following, we will prove Eq. \( 3 \) holds for \( \eta > 0 \); the proof for \( \eta < 0 \) is similar.) The map \( S_{K,1} \) increases monotonically. We have that
\[
x_{n+1} = S_{K,1}(x_n)
\]

and for \( x_{n+1} = 0, x_n = \cot \theta_n, \theta_n \in \arccot(\mathbb{R}\setminus B) \subset [0, \pi] \) satisfies the following relation:
\[
K\theta_n = \frac{\pi}{2} + m\pi, m \in \mathbb{Z}
\]
\[
\theta_n = \frac{2\pi}{K} + m\pi,
\]
\[
0 \leq \frac{2\pi}{K} + \frac{m\pi}{2} \leq \pi.
\]
The range of possible values for \( m \) is \( m = 0, 1, 2, \cdots, K-1 \). Then for \( x_n \) such that \( x_{n+1} = 0 \), it follows that
\[
x_n = \cot \left( \frac{3\pi}{2K} + \frac{m\pi}{2} \right), m = 0, 1, 2, \cdots, K-1.
\]

For \( x_n \) such that \( x_{n+1} = K \cot(\theta_n) = \eta \), it follows that
\[
K\theta_n = \cot^{-1} \left( \frac{\eta}{K} \right) + m\pi
\]
\[
\theta_n = \frac{1}{K} \cot^{-1} \left( \frac{\eta}{K} \right) + m\pi,
\]
\[
0 \leq \frac{1}{K} \cot^{-1} \left( \frac{\eta}{K} \right) + m\pi \leq \pi.
\]

Here, since
\[
0 < \cot^{-1} \left( \frac{\eta}{K} \right) < \frac{\pi}{2},
\]
the range of possible values for \( m \) is given by
\[
-\frac{1}{2} < -\frac{1}{\pi} \cot^{-1} \left( \frac{\eta}{K} \right) \leq m \leq K - \frac{1}{\pi} \cot^{-1} \left( \frac{\eta}{K} \right) < K,
\]
that is \( m = 0, 1, 2, \cdots, K-1 \). Then \( \theta_n \) and \( x_n \) are given by
\[
\theta_n = \frac{1}{K} \cot^{-1} \left( \frac{\eta}{K} \right) + \frac{m\pi}{K}, m = 0, 1, 2, \cdots, K-1,
\]
\[
x_n = \cot \left\{ \frac{1}{K} \cot^{-1} \left( \frac{\eta}{K} \right) + \frac{m\pi}{K} \right\}
\]
where \( \eta = K \cot(\theta_n) \). Because the \( S_{K,1} \) increases monotonically and the cot function decreases monotonically for \( \theta \in [0, \pi] \), the interval that is mapped from \( (0, \eta) \) by \( S_{K,1}^{-1} \) is
\[
\bigcup_{m=0}^{K-1} \left( \cot \left( \frac{\pi}{2K} + \frac{m\pi}{K} \right), \cot \left\{ \frac{1}{K} \cot^{-1} \left( \frac{\eta}{K} \right) + \frac{m\pi}{K} \right\} \right).
\]

Then we have that
\[
\left| S_{K,1}^{-1}(0, \eta) \right| = \sum_{m=0}^{K-1} \left| \cot \left\{ \frac{1}{K} \cot^{-1} \left( \frac{\eta}{K} \right) + \frac{m\pi}{K} \right\} - \cot \left( \frac{\pi}{2K} + \frac{m\pi}{K} \right) \right|.
\]
In the following discussion, we consider
\[ \sum_{m=0}^{K-1} \cot \left( \frac{\pi}{2K} + \frac{m}{K} \pi \right). \]

(i) Case \( K = 2N \). For \( \sum_{m=0}^{K-1} \cot \left( \frac{\pi}{2K} + \frac{m}{K} \pi \right) \), adding the terms corresponding to \( m = 0 \) and \( m = K-1 \), we obtain
\[
\frac{\cot \left( \frac{\pi}{2K} \right) + \cot \left( \frac{\pi}{2K} + \frac{K-1}{K} \pi \right)}{\frac{\cot \left( \frac{\pi}{2K} \right) + \cot (\pi - \frac{\pi}{2K})}{\pi}} = 0. \quad (13)
\]
Adding the terms corresponding to \( m = l \) and \( m = K-l \), we obtain
\[
\frac{\cot \left( \frac{\pi}{2K} \right) + \cot \left( \frac{\pi}{2K} + \frac{K-1}{K} \pi \right)}{\frac{\cot (\pi - \frac{\pi}{2K})}{\pi}} = 0.
\]

(ii) Case \( K = 2N + 1 \). We have
\[
\sum_{m=0}^{K-1} \cot \left( \frac{\pi}{2K} + \frac{m}{K} \pi \right) = \sum_{m=0}^{K-1} \cot \left( \frac{\pi}{2K} + \frac{m}{K} \pi \right) + \cot \left( \frac{K-1 + 1}{2K} \pi \right) + \sum_{m=0}^{K-1} \cot \left( \frac{\pi}{2K} + \frac{m}{K} \pi \right).
\]

Much as in (i), because the term corresponding to \( m = l \) negates the term corresponding to \( m = K-1-l \), \( l = 0, \ldots, \frac{K-3}{2} \), it follows that
\[
\sum_{m=0}^{K-1} \cot \left( \frac{\pi}{2K} + \frac{m}{K} \pi \right) = 0.
\]

Thus, we have that
\[
\left| S_{K,1}^{-1}(0, \eta) \right| = \sum_{m=0}^{K-1} \cot \left\{ \frac{1}{K} \cot^{-1} \left( \frac{\eta}{K} \right) + \frac{m}{K} \pi \right\}. \quad (18)
\]

In the following discussion, we calculate Eq. (18). Let the \( K \) roots \( x_n \) of the equation \( \eta = S_{K,1} x_n \) be denoted \( \xi_i, i = 0, \ldots, K-1 \). Because the map \( S_{K,1}(x) \) corresponds to the \( K \)-angle formula of the cot function, \( \eta \) is given by
\[
\eta = x_{n+1} = S_{K,1}(x_n) = \frac{x_n^K + (K-2) \text{ th and the smaller order terms}}{K x_n^{K-1} + (K-3) \text{ th and the smaller order terms}}. \quad (19)
\]
where \( x_{n+1} = \eta \). Then it follows that
\[
x_n^K - \eta x_n^{K-1} + (K-2 \text{ th and the smaller order terms}) = 0.
\]

Because by definition \( \xi_i \) is a root of the above \( K \)-th degree equation, it follows that \((x_n - \xi_0)(x_n - \xi_1) \cdots (x_n - \xi_{K-1}) = 0\). According to the relation between the roots and coefficients of a \( K \)-th degree equation, we have that
\[
\eta = \sum_{m=0}^{K-1} \xi_m = \sum_{m=0}^{K-1} \cot \left\{ \frac{1}{K} \cot^{-1} \left( \frac{\eta}{K} \right) + \frac{m}{K} \pi \right\}. \quad (20)
\]

Therefore, because
\[
\left| S_{K,1}^{-1}(0, \eta) \right| = \eta. \quad (21)
\]

Eq. (13) holds.

(II) In the case of \( \alpha = -1 \).

Consider the case \( \eta > 0 \) as in (I). Because the map \( S_{K,-1} \) decreases monotonically,
\[
\left| S_{K,-1}^{-1}(0, \eta) \right| = \sum_{m=0}^{K-1} \left| \cot \left( \frac{\pi}{2K} + \frac{m}{K} \pi \right) - \cot \left\{ \frac{1}{K} \cot^{-1} \left( \frac{\eta}{K} \right) + \frac{m}{K} \pi \right\} \right|
\]
\[
= - \sum_{m=0}^{K-1} \cot \left\{ \frac{1}{K} \cot^{-1} \left( \frac{\eta}{K} \right) + \frac{m}{K} \pi \right\}. \quad (23)
\]
For the map \( S_{K,-1} \), the following relation holds:

\[
\eta = -K \frac{K(x_n - \eta)^+ (K-2) \text{th and the smaller order terms}}{K^2(x_n - \eta)^+ (K-3) \text{th and the smaller order terms}} \]

\[
x_n^K + \eta^K x_n^{K-1} + (K-2) \text{th and the smaller order terms} = 0.
\]

(24)

According to the relation between the roots and coefficients of a \( K \)-th-degree equation, we have the relation:

\[
-\eta = \sum_{m=0}^{K-1} \xi_m = \sum_{m=0}^{K-1} \cot \left( \frac{1}{K} \cot^{-1} \left( \frac{-\eta}{K} \right) + \frac{m}{K} \pi \right)
\]

\[
\therefore -\sum_{m=0}^{K-1} \cot \left( \frac{1}{K} \cot^{-1} \left( \frac{-\eta}{K} \right) + \frac{m}{K} \pi \right) = \eta.
\]

(25)

Therefore, it follows that

\[
\left| S_{K,-1}^{-1}(0, \eta) \right| = -\sum_{m=0}^{K-1} \cot \left( \frac{1}{K} \cot^{-1} \left( \frac{-\eta}{K} \right) + \frac{m}{K} \pi \right) = \eta
\]

and Eq. (3) holds.

\[ \square \]

At \( \alpha = \pm 1 \), the SGB transformations preserve the Lebesgue measure for any \( K \geq 2 \). Thus, for the SGB transformations the measure for the whole set cannot be normalized to unity. Then we define the ergodicity for the system with the infinite measure as follows (according to Definition III.1):

**Definition III.1 (ergodicity).** Let \( (X, \mathcal{A}, \mu) \) be a measurable space. \( S : X \to X \) is a measurable transformation on the measure space \( (X, \mathcal{A}, \mu) \). The transformation \( S \) is called ergodic if every invariant set \( A \in \mathcal{A} \) is such that either \( \mu(A) = 0 \) or \( \mu(X \setminus A) = 0 \).

**Theorem III.2.** The SGB transformations at \( \alpha = \pm 1 \) are ergodic.

**Proof.** For the map \( S_{K,\pm 1} \) substituting \( \cot(\pi \theta_n) \) into \( x_n \in \mathbb{R \setminus B} \), one has the induced map \( \tilde{S}_{K,\pm 1} : X \overset{\text{def}}{=} \frac{1}{\pi} \text{arccot}(\mathbb{R \setminus B}) \to X \) such that

\[
\theta_{n+1} = \tilde{S}_{K,\pm 1}(\theta_n) = \frac{1}{\pi} \cot^{-1} \{ \pm K \cot(\pi K \theta_n) \}. \quad (27)
\]

The Figure 2 shows the relation between \( \mathbb{R \setminus B} \) and \( X \) in the range of \(-10 < x_n < 10\).

Since we eliminate countably infinite number of points whose measure is 0 from \((0, 1)\) to obtain the set \( X, X \subset (0, 1) \). The map \( \tilde{S}_{K,\pm 1} \) has topological conjugacy with the map \( S_{K,\pm 1} \), so that the ergodic properties of \( \tilde{S}_{K,\pm 1} \) are the same as those of \( S_{K,\pm 1} \). In terms of absolute value of the derivative of \( \tilde{S}_{K,\pm 1} \), it holds that

\[
\left| \tilde{S}_{K,\pm 1}'(\theta) \right| = \frac{K^2 \left\{ 1 + \cot^2(\pi K \theta) \right\}}{K^2 \cot^2(\pi K \theta) + 1} > 1, \forall \theta \in X. \quad (28)
\]

Take the contraposition for Definition III.1 and we will show that

\[
\text{for any } A \text{ s.t. } \mu(A) \neq 0, \text{ and } \mu(A^c) \neq 0 \Rightarrow A \text{ is not invariant}. \quad (29)
\]

Figure 2 illustrates the case of \( \{I_{j,1}\} \) for \( K = 3, 4, \text{ and } 5 \) at \( \alpha = 1 \).
FIG. 3: Solid lines correspond to the transformation $\bar{S}_K$, which has exact topological conjugacy with the super generalized Boole transformation $S_{K,1}$, where $K = 3, 4$ and 5. Dashed line corresponds to the line $\theta_{n+1} = \theta_n$.

Since the absolute value of the derivative $\bar{S}'_{K,1}$ on any $I_{j,n}$ is larger than ( $\theta | \cot^2(\pi K \theta) = \infty \in \mathbb{R}$ ) unity, the length of the interval $I_{j,n}$ becomes infinitesimal as $n \to \infty$. Then, for any set $A$ such that $\mu(A) \neq 0$, it follows that

$$\exists p, q \text{ s.t. } I_{p,q} \subset A.$$  \hfill (31)

From the definition of $I_{p,q}$, it follows that

$$\bar{S}_{K,1}^q I_{p,q} = X,$$
$$\therefore \bar{S}_{K,1}^q A = X.$$  \hfill (32)

Next, for any set $A$ such that $\mu(A) \neq 0$, it follows that

$$A \neq X.$$  \hfill (33)

Then, for any $A$ such that $\mu(A) \neq 0$ and $\mu(A^c) \neq 0$, it follows that

$$\exists q \in \mathbb{N} \text{ s.t. } \bar{S}_{K,1}^q A = X \text{ and } A \neq X.$$  \hfill (34)

This means that the set $A$ is not invariant. Therefore, Theorem III.2 holds.

IV. NORMALIZED LYAPUNOV EXPONENT

According to the Darling-Kac-Aaronson theorem\textsuperscript{5}, for infinite measure $m$, for a conservative, ergodic, measure preserving map $T$ and for a function $f$ such as $f \in L^1(m), f \geq 0, \int_X f dm > 0$ where $X$ is a set on which the map $T$ is defined, normalized time average of $f$ converges to the normalized Mittag-Leffler distribution such as\textsuperscript{1,12,41}.

$$\frac{1}{a_n} \sum_{k=0}^{n-1} f \circ T_k \to \left( \int_X f dm \right) Y_\gamma,$$  \hfill (35)

where $a_n$ is the return sequence and $Y_\gamma$ is a random variable which obeys the normalized Mittag-Leffler distribution of order $\gamma$. In the case of the Boole transformation, the return sequence is obtained as $a_n = \sqrt{2/\pi}$.

In the case of this SGB transformations at $\alpha = \pm 1$, consider $f$ as $\log |dS_{K,\pm 1}|$ and we clarify whether the normalized Lyapunov exponent converges to the normalized Mittag-Leffler distribution by numerical simulation.

We have that $\log |dS_{K,\pm 1}| \in L^1(\mu)$.

We calculate the normalized Lyapunov exponents such as

$$\lambda = \frac{c(K)}{\sqrt{n}} \sum_{i=0}^{n-1} \log |dS_{K,\pm 1}(x_i)|$$  \hfill (37)

where $c$ are the normalization constants to make the mean values equal to unity. Figures 4a, 4b, 4c, 5a, 5b.
and (c) show the density function of the normalized Lyapunov exponents for \((K, \alpha) = (3, 1), (4, 1), (5, 1), (3, -1), (4, -1)\) and \((5, -1)\), respectively, which confirms that their normalized Lyapunov exponents are distributed according to the normalized Mittag-Leffler distribution of order \(\frac{1}{2}\).

**FIG. 4:** Relation between the density functions of normalized Lyapunov exponent and normalized Lyapunov exponent in SGB transformation for \(K = 3, 4, 5\) \((\alpha = 1)\). The number of initial points is \(M = 10^5\) and the number of iteration is \(N = 10^5\). Initial points are distributed to obey the normal distribution whose mean and variance are 0 and 1, respectively. The bar graph represents the numerical simulation of the normalized Lyapunov exponents and the solid line represents the normalized Mittag-Leffler distributions of order \(\frac{1}{2}\).

**FIG. 5:** Relation between the density functions of normalized Lyapunov exponent and normalized Lyapunov exponent in SGB transformation for \(K = 3, 4, 5\) \((\alpha = -1)\). The number of initial points is \(M = 10^5\) and the number of iteration is \(N = 10^5\). Initial points are distributed to obey the normal distribution whose mean and variance are 0 and 1, respectively. The bar graph represents the numerical simulation of the normalized Lyapunov exponents and the solid line represents the normalized Mittag-Leffler distributions of order \(\frac{1}{2}\).

**Figure** shows the relation between normalization constant \(c(K)\) and \(K\) at \(\alpha = \pm 1\). We can see that \(c(K)\) tends to decrease as \(K\) increases. At \((K, \alpha) = (2, 1)\), we know that \(c(K) = \frac{1}{2\sqrt{2}} \approx 0.354\) from \(a_n = \frac{\sqrt{2\pi n}}{\Gamma(\frac{3}{2})}\). Figure \(\text{\textbullet}\) is consistent with this result and from the fact that the points at \((K, \alpha) = (2, -1), (3, 1)\) and \((3, -1)\) are on \(g(K)\)
and that \( \int \ln |S_{2, -1}(x)| dx = \int \ln |S_{3, \pm 1}(x)| dx = 2\pi \), we conjecture that for \( S_{2, -1} \), the return sequence \( a_n \) is given by \( a_n = \frac{\sqrt{2n}}{\pi} \) and that for \( S_{3, \pm 1} \), \( a_n = \frac{\sqrt{3n}}{\pi} \).

\[ \begin{align*}
\alpha = & 1 \quad + \\
\alpha = & -1 \quad o \\
g(K) & \\
\end{align*} \]

**FIG. 6:** The relation between normalization constant \( c(K) \) and parameter \( K \). The function \( g(K) \) is rewritten as \( g(K) = \frac{1}{2\sqrt{K}} \).

\[ a = 1 \quad + \\
\alpha = -1 \quad o \\
\begin{align*}
g(K) & \\
\end{align*} \]

\[ \begin{align*}
\alpha = & 1 \quad + \\
\alpha = & -1 \quad o \\
\begin{align*}
g(K) & \\
\end{align*} \]

**FIG. 6:** The relation between normalization constant \( c(K) \) and parameter \( K \). The function \( g(K) \) is rewritten as \( g(K) = \frac{1}{2\sqrt{K}} \).

\[ \begin{align*}
\alpha = & 1 \quad + \\
\alpha = & -1 \quad o \\
\begin{align*}
g(K) & \\
\end{align*} \]

\[ \begin{align*}
\alpha = & 1 \quad + \\
\alpha = & -1 \quad o \\
\begin{align*}
g(K) & \\
\end{align*} \]

**V. CONCLUSION**

In this paper, we showed the statistical ergodic property of one dimensional chaotic maps, the super generalized Boole (SGB) transformations \( S_{K, \alpha} \) at \( \alpha = \pm 1 \). That is, for infinite number of \( K \), we proved that the \( S_{K, \pm 1} \) preserve the Lebesgue measure and that the dynamical systems are \textit{ergodic} for \( K \geq 2 \). In the case of \( K = 2 \) (the Boole transformation), Adler and Wiss proved its ergodicity in unbounded region, but in our method, we proved the ergodicity by transforming the unbounded domain to the bounded domain using topological conjugacy. In the previous work, the authors proved that the SGB transformations are \textit{exact} for \( 0 < \alpha < 1 \) \( (K = 2N + 1) \), \( N \in \mathbb{N} \) and they are dissipative for \( \alpha > 1 \). The result of this paper connects these two regions in the same way of the generalized Boole transformations. Then, we demonstrated that the normalized Lyapunov exponents actually obey the Mittag-Leffler distribution of order \( \frac{1}{2} \) for \( (K, \alpha) = (3, 1), (4, 1), (5, 1), (3, -1), (4, -1) \) and \( (5, -1) \). In this numerical experiments, the form of Mittag-Leffler distribution does not depend on the value of \( K \) although there is a relation between \( c \) and \( K \). Owing to these results, we obtain a class of countably infinite number of critical maps in the sense of \textit{Type} 1 or \textit{Type} 3 intermittency which preserve the Lebesgue measure and are proven to be ergodic with respect to the Lebesgue measure.

In previous works various indicators were proposed to characterize the instability when the corresponding Lyapunov exponent is zero such as generalized Lyapunov exponent and Lyapunov pair. It is fully expected that these infinite critical SGB transformations will be used as represented indicator maps in order to detect chaotic criticality since the ergodic properties are exactly obtained.

**ACKNOWLEDGMENTS**

One of the author, Ken-ichi Okubo, thanks to Dr. Takuma Akimoto for his fruitful advice.

1. T. Akimoto and Y. Aizawa, Chaos \textbf{20}, 033110 (2010).
2. R. L. Adler and B. Weiss, Isr. J. Math. \textbf{16}, 263–278 (1973).
3. G. D. Birkhoff, Proc. Natl. Acad. Sci. USA \textbf{17}, 656–660 (1931).
4. M. V. Jakobson, Commun. Math. Phys. \textbf{81}, 39–88 (1981).
5. J. Aaronson, \textit{An introduction to infinite ergodic theory}, 50 (American Mathematical Soc., 1997).
6. K. Umeno and K. Okubo, Prog. Theor. Exp. Phys. \textbf{2016}, 021A01 (2016).
7. K. Umeno, Phys. Rev. E \textbf{58}, 2644 (1998).
8. K. Umeno, Nonlinear Theory and Its Applications, IEICE \textbf{7}, 14–20 (2016).
9. G. Boole, Philos. Trans. R. Soc. London, \textbf{745–803} (1857).
10. S. Shinkai and Y. Aizawa, Prog. Theor. Phys. \textbf{116}, 503–515 (2006).
11. T. Akimoto and Y. Aizawa, J. Korean Phys. Soc. \textbf{50}, 254 (2007).
12. T. Akimoto and T. Miyaguchi, Phys. Rev. E \textbf{82}, 030102 (2010).
13. N. Korabel and E. Barkai, Phys. Rev. Lett. \textbf{108}, 060604 (2012).
14. M. Thaler, Ergod. Theory Dyn. Syst. \textbf{22}, 1289–1312 (2002).
15. M. Thaler and R. Zweimüller, Probab. Theory Relat. Fields \textbf{135}, 15–52 (2006).
16. T. Akimoto, J. Stat. Phys. \textbf{132}, 171 (2008).
17. T. Akimoto, S. Shinkai, and Y. Aizawa, J. Stat. Phys. \textbf{158}, 476–493 (2015).
18. D. A. Kessler and E. Barkai, Phys. Rev. Lett. \textbf{105}, 120602 (2010).
19. A. Dechant, E. Lutz, E. Barkai, and D. Kessler, J. Stat. Phys. \textbf{145}, 1524–1545 (2011).
20. P. C. Holz, A. Dechant, and E. Lutz, Europhys. Lett. \textbf{109}, 23001 (2015).
21. N. Korabel and E. Barkai, Phys. Rev. Lett. \textbf{102}, 050601 (2009).
22. N. Korabel and E. Barkai, Phys. Rev. E \textbf{82}, 016209 (2010).
23. N. Korabel and E. Barkai, J. Stat. Mech.: Theory Exp. \textbf{2013}, P08010 (2013).
24. F. T. Hoe and Z. Deng, Phys. Rev. A \textbf{35}, 847 (1987).
25. P. Manneville and Y. Pomeau, Phys. Lett. A \textbf{75}, 1–2 (1979).
26. Y. Pomeau and P. Manneville, Commun. Math. Phys. \textbf{74}, 189–197 (1980).
27. E. Ott and J. C. Sommerer, Phys. Lett. A \textbf{188}, 39–47 (1994).
28. H. Lamba and C. Budd, Phys. Rev. E \textbf{50}, 84 (1994).
29. B. Huberman and J. Rudnick, Phys. Rev. Lett. \textbf{45}, 154 (1980).
30. M. S. Milosavljevic, J. N. Blakey, A. N. Beal, and N. J. Corron, Phys. Rev. E \textbf{95}, 062223 (2017).
31. Z. Liu, Y.-C. Lai, and M. A. Matías, Phys. Rev. E \textbf{67}, 052003 (2003).
32. J. Crutchfield, M. Nauenberg, and J. Rudnick, Phys. Rev. Lett. \textbf{46}, 933 (1981).
33. H. L. Swinney, Physica D \textbf{7}, 3–15 (1983).
34. K. He and A. C.-L. Chian, Phys. Rev. E \textbf{69}, 026207 (2004).
35. A. Collet and Y. K. Chembo, Chaos \textbf{24}, 013113 (2014).
36. L. Bakemeier, A. Alvermann, and H. Fehske, Phys. Rev. Lett. \textbf{114}, 013601 (2015).
37. K. Okubo and K. Umeno, Prog. Theor. Exp. Phys. \textbf{2018}, 103A01 (2018).
38. I. Landau and E. Lifshitz, \textit{Statistical Physics: Volume 5} (Pergamon Press, 1970).
39. G. Gallavotti, \textit{Nonequilibrium and irreversibility} (Springer, 2014).
[40] A. Lasota and M. C. Mackey, *Probabilistic properties of deterministic systems* (Cambridge university press, 2008).

[41] T. Akimoto, M. Nakagawa, S. Shinkai, and Y. Aizawa, Phys. Rev. E 91, 012926 (2015).