Renormalized oscillation theory for symplectic eigenvalue problems with nonlinear dependence on the spectral parameter

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ABSTRACT

In this paper, we establish new renormalized oscillation theorems for discrete symplectic eigenvalue problems with Dirichlet boundary conditions. These theorems present the number of finite eigenvalues of the problem in the arbitrary interval $(a, b]$ using the number of focal points of a transformed conjoined basis associated with Wronskian of two principal solutions of the symplectic system evaluated at the endpoints $a$ and $b$. We suppose that the symplectic coefficient matrix of the system depends nonlinearly on the spectral parameter and that it satisfies certain natural monotonicity assumptions. In our treatment, we admit possible oscillations in the coefficients of the symplectic system by incorporating their non-constant rank with respect to the spectral parameter.

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1. Introduction

In this paper, we consider the discrete symplectic system

$$y_{k+1}(\lambda) = S_k(\lambda) y_k(\lambda), \quad k \in [0, N]_Z, \quad \lambda \in \mathbb{R},$$

(S$_\lambda$)

with the Dirichlet boundary conditions

$$x_0(\lambda) = 0 = x_{N+1}(\lambda), \quad (E_0)$$

where we use the notation $[M, N]_Z := [M, N] \cap \mathbb{Z}$ for the discrete interval with endpoints $M, N \in \mathbb{Z}$. The coefficient matrix $S_k(\lambda) \in \mathbb{R}^{2n \times 2n}$ of system (S$_\lambda$) with $n \times n$ blocks $A(\lambda), B(\lambda), C(\lambda), D(\lambda)$ depending nonlinearly on the spectral parameter $\lambda \in \mathbb{R}$ is assumed to be symplectic, i.e. for all $k \in [0, N]_Z$ and $\lambda \in \mathbb{R}$ we have

$$S_k^T(\lambda) J S_k(\lambda) = J, \quad S_k(\lambda) := \begin{pmatrix} A_k(\lambda) & B_k(\lambda) \\ C_k(\lambda) & D_k(\lambda) \end{pmatrix}, \quad J := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \quad (1)$$

In addition, we assume that the matrix $S_k(\lambda)$ piecewise continuously differentiable in $\lambda \in \mathbb{R}$, i.e. it is continuous on $\mathbb{R}$ and the derivative $\dot{S}_k(\lambda) := (d/d\lambda)S_k(\lambda)$ is piecewise...
continuous in the parameter $\lambda \in \mathbb{R}$ for all $k \in [0, N]_\mathbb{Z}$. Given the above symplectic matrix $S_k(\lambda)$, we consider the monotonicity assumption

$$
\Psi(S_k(\lambda)) = \Psi_k(\lambda) := \mathcal{J}^T S_k(\lambda) S_k^{-1}(\lambda) \geq 0, \quad k \in [0, N]_\mathbb{Z}, \quad \lambda \in \mathbb{R}
$$

(2)

for the $2n \times 2n$ matrix $\Psi_k(\lambda)$, which is symmetric for any $k \in [0, N]_\mathbb{Z}$ and $\lambda \in \mathbb{R}$ according to [32]. The notation $A \geq 0$ means that the matrix $A$ is symmetric and non-negative definite. Symplectic difference systems $(S_\lambda)$ cover as special cases many important difference equations, such as the second-order (or even order) Sturm–Liouville difference equations, symmetric three-term recurrence equations, and linear Hamiltonian difference systems, see [1,4,5,29,32]. A complete review of the history and development of qualitative theory of $(S_\lambda)$ is given in the new monograph [10] (see also the references therein).

Classical oscillation theorems connect the oscillation and spectral theories of $(S_\lambda)$. Assume that we need to know how many eigenvalues of $(S_\lambda), (E_0)$ are located in the given interval $(a, b) \subseteq \mathbb{R}$. Then, according to the global oscillations theorems in [6,11] the difference $l_d(Y^{[0]}(b), 0, N + 1) - l_d(Y^{[0]}(a), 0, N + 1)$ of the numbers of focal points calculated for the principal solutions $Y^{[0]}(b), Y^{[0]}(a)$ of $(S_\lambda)$ evaluated at the endpoints $\lambda = a$ and $\lambda = b$ presents the number of eigenvalues $\#\{\nu \in \sigma | a < \nu \leq b\}$ of $(S_\lambda), (E_0)$ in $(a, b)$. This result was proved in [6,11] for the coefficient matrix $S_k(\lambda)$ and $\Psi_k(\lambda)$ in the form

$$
S_k(\lambda) = \begin{pmatrix} I & 0 \\ -\lambda & I \end{pmatrix} \mu_k \quad \Psi_k(\lambda) = \Psi_k := \begin{pmatrix} \mu_k & 0 \\ 0 & 0 \end{pmatrix} \geq 0, \quad k \in [0, N]_\mathbb{Z}, \quad \lambda \in \mathbb{R},
$$

(3)

where $S_k$ is a constant symplectic matrix. The same result for $(S_\lambda)$ with the general non-linear dependence on $\lambda$ was originally proved in [32] for $B_k(\lambda) = \text{const}$ (here $B_k(\lambda)$ is the block of $S_k(\lambda)$ given by (1)) and then generalized in [29] to the case

$$
\text{rank} B_k(\lambda) = \text{const}, \quad \lambda \in \mathbb{R}, \quad k \in [0, N]_\mathbb{Z}.
$$

(4)

Then it was shown in [18] that assumption (4) plays a crucial role in the oscillation theory, in particular, if (4) is violated the number of focal points of the principal solution of $(S_\lambda)$ loses the monotonicity with respect to $\lambda$ and then the difference $l_d(Y^{[0]}(b), 0, N + 1) - l_d(Y^{[0]}(a), 0, N + 1)$ can be negative. In this case, it is necessary to incorporate oscillations of the block $B_k(\lambda)$ to present a proper generalization of the results in [6,11,29,32]. Moreover, it was proven in [18, Corollary 2.5] that condition

$$
\text{rank} B_k(\lambda) = \text{rank} B_k(\lambda^-) \quad \text{for all } \lambda \leq \lambda_0, \quad k \in [0, N]_\mathbb{Z},
$$

(5)

holds for some $\lambda_0 \in \mathbb{R}$ if and only if the real spectrum of $(S_\lambda), (E_0)$ is bounded from below. Observe that for a fixed $k \in [0, N]_\mathbb{Z}$ the symplectic matrix $S_k(\lambda)$ can be viewed as the fundamental matrix of the linear Hamiltonian differential system (with respect to $\lambda$)

$$
\dot{S}_k(\lambda) = \mathcal{J} \Psi_k(\lambda) S_k(\lambda), \quad \lambda \in \mathbb{R},
$$

(6)

with the symmetric Hamiltonian $\Psi_k(\lambda) \geq 0$ given by (2). In this context one can introduce the numbers

$$
\vartheta(S_k(\lambda_0)) = \vartheta_k(\lambda_0) := \text{rank} B_k(\lambda_0^-) - \text{rank} B_k(\lambda_0), \quad k \in [0, N]_\mathbb{Z}
$$

(7)

describing the multiplicities of proper focal points (see [28]) of $S_k(\lambda)(0 I)^T$ as a conjoined basis of (6) and then condition (5) means that system (6) is non-oscillatory for $\lambda$ near
In the recent paper [22], we generalized the results in [18] to the case of symplectic eigenvalue problems with general self-adjoint boundary conditions admitting possible oscillations of their coefficients with respect to \( \lambda \in \mathbb{R} \).

The renormalized and the more general relative oscillation theory makes it possible to replace the difference \( l_d(Y(b),0,N+1) - l_d(Y(a),0,N+1) \) of the numbers of focal points calculated for \( \lambda = a \) and \( \lambda = b \) by the number of focal points of only one transformed conjoined basis \( \tilde{Y}_k(a,b) \) associated with the Wronskian \( \tilde{Y}_k^T(b) \tilde{Y}_k(a) \) of \( Y_k(a) \) and \( \tilde{Y}_k(b) \). Remark that we refer to the renormalized oscillation theory of \( (S_\lambda) \) when the consideration concerns oscillations of the Wronskian of two conjoined bases of \( (S_\lambda) \) considered for different values of \( \lambda \). The relative oscillation theory investigates the oscillatory behaviour of Wronskians of conjoined bases of two symplectic systems with different coefficient matrices \( S_k(\lambda) \) and \( \hat{S}_k(\lambda) \), then all results of the renormalized theory follow from the relative oscillation theorems for the case \( S_k(\lambda) = \hat{S}_k(\lambda), \lambda \in \mathbb{R}, k \in [0,N]_\mathbb{Z} \).

The relative oscillation theory was developed for eigenvalue problems for the second-order Sturm–Liouville difference and differential equations (with a linear dependence on \( \lambda \) in \([2,3,23,26,33]\). In the recent papers \([24,25]\), the renormalized oscillation theory in \([23]\) is extended to the case of general linear Hamiltonian systems with block matrix coefficients, which are continuous counterparts of \( (S_\lambda) \).

The relative oscillation theory for two symplectic problems with Dirichlet boundary conditions under restriction (3) is presented in \([14,15]\), in \([16, \text{Theorem 5}]\) the renormalized oscillation theory for \( (S_\lambda) \), (3) is extended to the case of general self-adjoint boundary conditions. For the case of general nonlinear dependence on \( \lambda \) the first results of the relative oscillation theory for two matrix Sturm–Liouville equations were proved in \([17]\). In \([10, \text{Section 6.1}]\), we presented the relative oscillation theory for two symplectic eigenvalue problems with nonlinear dependence on \( \lambda \) and with general self-adjoint boundary conditions. All these results are derived under restriction (4) for \( S_k(\lambda) \).

The main results of this paper are devoted to the renormalized oscillation theory for \( (S_\lambda), (E_0) \) without condition (4). In this situation, the classical oscillation theorem (see \([19]\)) presents the number of finite eigenvalues \( \# \{ \nu \in \sigma \{ a < \nu \leq b \} \} \) of problem \( (S_\lambda), (E_0) \) in \( (a,b] \) incorporating oscillations of \( B_k(\lambda) \) in terms of numbers (7) (see Theorem 2.4 in Section 2)

\[
l_d(Y^{[0]}(b),0,N+1) - l_d(Y^{[0]}(a),0,N+1) + \sum_{a < \nu \leq b} \sum_{k=0}^{N} \tilde{\nu}_k(\nu) = \# \{ \nu \in \sigma \{ a < \nu \leq b \} \}.
\]

(8)

A similar formula can be proved for the so-called backward focal points \( l^*_d(Y^{[N+1]}(\lambda),0,N+1) \) of the principal solutions at \( N+1 \) (see Theorem 2.5 in Section 2). The main results of the paper (see Theorems 3.8, 3.12) present renormalized versions of Theorems 2.4 and 2.5, respectively. In more details, introducing a fundamental matrix \( \tilde{Z}_k^{[N+1]}(\lambda) \) of \( (S_\lambda) \) with the initial condition \( \tilde{Z}_k^{[N+1]}(\lambda) = I \) we have instead of (8) the following renormalized formula

\[
l_d((Z^{[N+1]}(a))^{-1}Y^{[0]}(b),0,N+1) + \sum_{a < \nu \leq b} \sum_{k=0}^{N} \tilde{\nu}_k(\nu) = \# \{ \nu \in \sigma \{ a < \nu \leq b \} \},
\]

(9)
where the numbers \( \tilde{\vartheta}_k(v) \) are associated with the transformed coefficient matrix 
\( \tilde{S}_k(\lambda) = (Z_{k+1}^{[N+1]}(a))^{-1}S_k(\lambda)Z_k^{[N+1]}(a) \) by analogy with (7). In (9), we have the number 
\( l_d((Z_k^{[N+1]}(a))^{-1}Y_k^{[0]}(b), 0, N + 1) \) which describes oscillations of the transformed conjoined basis 
\( (Z_k^{[N+1]}(a))^{-1}Y_k^{[0]}(b) \) associated with the Wronskian \( Y_k^{[N+1]}(a)T(\lambda)Y_k^{[0]}(b) \) of the principal solutions 
\( Y_k^{[N+1]}(a), Y_k^{[0]}(b) \) of (S_\lambda). The major advantage of using (9) instead of (8) is the calculation of only one number 
\( l_d((Z_k^{[N+1]}(a))^{-1}Y_k^{[0]}(b), 0, N + 1) \) instead of \( l_d(Y_k^{[0]}(b), 0, N + 1) \) especially in case of highly oscillatory principal solutions \( Y_k^{[0]}(b), Y_k^{[0]}(a) \). The price of this advantage is the necessity to evaluate the second addend in (9) which depends on the fundamental matrix \( Z_k^{[N+1]}(\lambda) \) of system \( (S_\lambda) \) evaluated for \( \lambda = a \). In Section 4 of the paper, we decide this problem presenting (9) in an invariant form incorporating oscillations of \( S_k(a) - S_k(\lambda), \lambda \in (a, b) \) instead of oscillations of blocks of \( \tilde{S}_k(\lambda) \). We proved in Section 4 (see Theorem 4.5) that (9) is equivalent to

\[
L_d((Z_k^{[N+1]}(a))^{-1}Y_k^{[0]}(b), 0, N + 1) + \sum_{a < \nu \leq b} \sum_{k=0}^{N} \rho_k(\nu) = \#\{\nu \in \sigma | a < \nu \leq b\},
\]

(10)

\[
\rho_k(\lambda) = \text{rank } (S_k(a) - S_k(\lambda)) - \text{rank } (S_k(a) - S_k(\lambda)) \geq 0,
\]

where \( L_d((Z_k^{[N+1]}(a))^{-1}Y_k^{[0]}(b), 0, N + 1) \) is the number of (forward) focal points of a 
\( 4n \times 2n \) conjoined basis associated with \( (Z_k^{[N+1]}(a))^{-1}Y_k^{[0]}(b) \) (see Remark 4.6). We call representation (10) invariant because after the replacement of the matrices \( S_k(\lambda), k \in [0, N]_\mathbb{Z} \) by \( R_k^{-1}S_k(\lambda)R_k \) formula (10) stays the same. Here \( R_k, k \in [0, N + 1]_\mathbb{Z} \) is an arbitrary sequence of symplectic transformation matrices which do not depend on \( \lambda \). In the last part of Section 4, we investigate the renormalized oscillation theory for systems (S_\lambda) under the assumption \( \rho_k(\lambda) = 0, k \in [0, N]_\mathbb{Z}, \lambda \in [a, b] \) which is necessary and sufficient for the equality \( L_d((Z_k^{[N+1]}(a))^{-1}Y_k^{[0]}(b), 0, N + 1) = \#\{\nu \in \sigma | a < \nu \leq b\} \). In particular, we show that this equality holds for any Hamiltonian difference system (see Corollary 4.12) under the monotonicity assumption \( \mathcal{H}_k(\lambda) \geq 0, k \in [0, N]_\mathbb{Z}, \lambda \in [a, b] \) for the discrete Hamiltonian \( \mathcal{H}_k(\lambda) \).

For the proof of the renormalized theorems, we involve new results of the oscillation theory for continuous case – for the differential Hamiltonian systems in form (6). Indeed, the theory presented in the paper is now a combination of two oscillation theories – for the discrete and for the continuous case. Using comparison theorems for the differential case (see [19, Theorem 2.2]), we present a new interpretation of the results of the discrete spectral theory in [6,11,12,18,22,29,32] which helps to provide the proof of Theorems 3.8, 3.12 in a compact form. In more details, consider a symplectic fundamental matrix \( Z_k(\lambda) \) of (S_\lambda) associated with the conjoined basis \( Y_k(\lambda) = Z_k(\lambda)(0, I)^T \) under the monotonicity assumption \( \Psi(Z_k(\lambda)) \geq 0 \) with respect to \( \lambda \). Then for arbitrary sequence of symplectic matrices \( R_k, k \in [0, N + 1]_\mathbb{Z} \) the matrix \( R_k^{-1}Y_k(\lambda) \) can be considered as a function of \( \lambda \in [a, b] \) with the (continuous) number of proper focal points \( l_k(R_k^{-1}Y_k, a, b) \) for any fixed index \( k \) and similarly, as a function of the discrete variable \( k \) with the (discrete) number of focal points \( l_d(R_k^{-1}Y_k(\lambda), 0, N + 1) \) for any fixed \( \lambda = \lambda_0 \). Introduce the following closed path \( \lambda = a, k \in [0, N + 1]_\mathbb{Z}, \lambda \in [a, b], k = N + 1; \lambda = b, k \in [0, N + 1]_\mathbb{Z}, \lambda \in [a, b], k = 0 \) in the plane \( (\lambda, k) \). Then, according to Theorem 3.5 proved in Section 3, we have the following representation for the sum of all focal points (in the continuous and in the discrete
settings) along this path
\[ l_d(R^{-1}Y(a),0,N+1) + l_c(R_{N+1}^{-1}Y_{N+1},a,b) - l_d(R^{-1}Y(b),0,N+1) - l_c(R_0^{-1}Y_0,a,b) \]
\[ = \sum_{k=0}^{N} l_c(\tilde{S}_k(0)I^T,a,b) \tag{11} \]
where \( l_c(\tilde{S}_k(0)I^T,a,b) \) is the number of focal points of \( \tilde{S}_k(\lambda)(0)I^T \) for \( \lambda \in (a,b) \) and \( \tilde{S}_k(\lambda) = R_k^{-1}S_k(\lambda)R_k \). For the case, \( R_k := I, k \in [0,N] \), and \( Z_0(\lambda) := I \) formula (11) turns into (8) because \( Y_k(\lambda) := Y_k^{[0]}(\lambda), l_c(Y_0,a,b) = 0, \) and the quantity \( l_c(Y_{N+1},a,b) \) presents the number of finite eigenvalues of \( (S_{\lambda}), (E_0) \). Observe that \( l_c(Y_{N+1},a,b) \) stays the same for the case of the non-constant matrices \( R_k \) under the condition \( R_{N+1} = I \). Then one can derive from (11) the representation of the number \( \#\{v \in \sigma | a < v \leq b \} \) using the transformed principal solutions \( R_k^{-1}Y_k^{[0]}(\lambda) \) (see Theorem 3.7).

The renormalized oscillation theory is connected with further simplifications of (11) when \( R_k \) is chosen in such a way that \( R_k^{-1}Y_k^{[0]}(\lambda) \) does not depend on \( k = 0,1,\ldots,N+1 \) for \( \lambda = b \) or for \( \lambda = a \). In particular, formula (9) is derived for the case \( R_k := Z_k^{[N+1]}(a), \) when \( l_d(R^{-1}Y^{[0]}(a),0,N+1) = 0. \) In the proof of the equivalence of (9) and (10), we also use separation results for the differential case (see [19, Theorem 2.3] and [30, Theorem 4.1]) to connect (in an explicit form) the numbers \( \rho_k(\lambda) \) in (10) with \( \tilde{\vartheta}_k(v) \) and \( \vartheta_k(v) \) in (9) and (8) (see Lemma 4.3).

The paper is organized as follows. In the next section, we recall the main notions of the discrete spectral and oscillation theory. We recall the classical global oscillation theorem from [18] (see Theorem 2.4) and present the version of this theorem in terms of backward focal points of the principal solution at \( N+1 \) (see Theorem 2.5). In Section 3, we recall some basic notions of the oscillation theory of linear differential Hamiltonian systems [7,28,31] including the recent results from [19,30] and prove Theorem 3.8 and Theorem 3.12 which are the renormalized versions of Theorems 2.4, 2.5. In Section 4, we derive Theorem 4.5 which generalizes [10, Theorem 6.4] to the case when assumption (4) is omitted. As an application of the results of Section 4, we present the renormalized oscillation theorem for the discrete linear Hamiltonian systems (see Example 4.11).

2. Classical oscillation and spectral theory for symplectic systems with the Dirichlet boundary conditions

We will use the following notation. For a matrix \( A \), we denote by \( A^T, A^{-1}, A^\dagger, \text{rank} A, \text{Ker} A, \text{Im} A, \text{ind} A, A \geq 0, A \leq 0, \) respectively, its transpose, inverse, Moore–Penrose pseudoinverse, rank (i.e. the dimension of its image), kernel, image, index (i.e. the number of its negative eigenvalues), positive semidefiniteness, negative semidefiniteness. If \( A(t) \) is a matrix-valued function, then by \( \text{rank} A(t_0^-), \text{ind} A(t_0^-) \) (rank \( A(t_0^+), \text{ind} A(t_0^+) \)) we mean the left-hand (the right-hand) limits of these quantities at \( t_0 \), provided these limits exist. We use the notation \( A(\lambda)|_a^b \) for the substitution \( A(b) - A(a) \) for functions of continuous argument \( \lambda \), and the similar notation \( A_k|_M^N = A_N - A_M \); for functions of discrete argument \( k \). We denote by \( \Delta A_k = A_{k+1} - A_k \) the forward difference operator. We say that \( A(\lambda) \in C^1_p \) if \( A(\lambda) \) is continuous with a piecewise continuous derivative (with respect to \( \lambda \)), and use the
notation \( Sp(2n) \) for the real matrix symplectic group in dimension \( 2n \), i.e. for real matrices \( S \) with the condition \( S^T J S = J \).

In this section, we recall some important notions and results of the discrete oscillation and spectral theory (see [10, Chapters 4,5]).

Recall that \( 2n \times n \) matrix solution \( Y(\lambda) = \begin{pmatrix} X(\lambda) \\ U(\lambda) \end{pmatrix} \) of \( (S_\lambda) \) with \( n \times n \) matrices \( X(\lambda), U(\lambda) \) is said to be a \textit{conjoined basis} if

\[
\operatorname{rank} \begin{pmatrix} X_k(\lambda) \\ U_k(\lambda) \end{pmatrix} = n \quad \text{and} \quad X_k(\lambda)^T U_k(\lambda) = U_k(\lambda)^T X_k(\lambda).
\]

(12)

Recall the definition of forward focal points and their multiplicities for conjoined bases of \( (S_\lambda) \), see [27, Definition 1]. We say that a conjoined basis \( Y(\lambda) \) of \( (S_\lambda) \) has a \textit{forward focal point} in the real interval \((k, k + 1)\) provided \( m_d(Y_k(\lambda)) := \operatorname{rank} M_k(\lambda) + \operatorname{ind} P_k(\lambda) \geq 1 \) and then the number \( m_d(Y_k(\lambda)) \) is its \textit{multiplicity}, where

\[
M_k(\lambda) := (I - X_{k+1}(\lambda) X_{k+1}^+(\lambda)) B_k(\lambda), \quad T_k(\lambda) := I - M_k^+(\lambda) M_k(\lambda),
\]

\[
P_k(\lambda) := T_k(\lambda) X_k(\lambda) X_{k+1}^+(\lambda) B_k(\lambda) T_k(\lambda),
\]

and the matrix \( P_k(\lambda) \) is symmetric. By a similar way (see [8,13]) one can introduce the multiplicities of backward focal points of a conjoined basis \( (S_\lambda) \) in the real interval \([k, k + 1)\). We define \( m^*_d(Y_k(\lambda)) := \operatorname{rank} \tilde{M}_k(\lambda) + \operatorname{ind} \tilde{P}_k(\lambda) \), where

\[
\tilde{M}_k(\lambda) := (I - X_k(\lambda) X_k^+(\lambda)) \tilde{B}_k^T(\lambda), \quad \tilde{T}_k(\lambda) := I - \tilde{M}_k(\lambda) \tilde{M}_k(\lambda),
\]

\[
\tilde{P}_k(\lambda) := \tilde{T}_k^T(\lambda) X_{k+1}(\lambda) X_k^+(\lambda) \tilde{B}_k^T(\lambda) \tilde{T}_k(\lambda).
\]

For the numbers \( m_d(Y_k(\lambda)), m_d^*(Y_k(\lambda)) \), we have the estimates

\[
0 \leq m_d(Y_k(\lambda)) \leq \operatorname{rank} B_k(\lambda) \leq n, \quad 0 \leq m_d^*(Y_k(\lambda)) \leq \operatorname{rank} B_k(\lambda) \leq n.
\]

(13)

Introduce the notation

\[
l_d(Y(\lambda), 0, N + 1) := \sum_{k=0}^{N} m_d(Y_k(\lambda)), \quad l_d^*(Y(\lambda), 0, N + 1) := \sum_{k=0}^{N} m_d^*(Y_k(\lambda))
\]

(14)

for the number of forward (resp., backward) focal points of the conjoined basis \( Y(\lambda) \) in the intervals \((0, N + 1)\) (resp., \([0, N + 1)\)) including their multiplicities.

Recall also that the conjoined basis \( Y_0^{[l]}(\lambda) \) with the initial condition \( Y_0^{[l]}(\lambda) = (0 \ I)^T \) for \( k = l \) is called the principal solution of \( (S_\lambda) \) at \( l \) (for \( l = 0, \ldots, N + 1 \)). The following important connection for the number of focal points of the principal solutions \( Y^{[0]}(\lambda), Y^{[N+1]}(\lambda) \) is proven in [13, Lemma 3.3]

\[
l_d(Y^{[0]}(\lambda), 0, N + 1) = l_d^*(Y^{[N+1]}(\lambda), 0, N + 1).
\]

(15)

To formulate the main results of this paper, we recall the notion of the comparative index introduced and collaborated in [13,14] (see also [10, Chapter 3]). According to [13], for
two real $2n \times n$ matrices $Y$ and $\hat{Y}$ satisfying condition (12), we define their comparative index $\mu(Y, \hat{Y})$ and the dual comparative index $\mu^*(Y, \hat{Y})$ by

$$
\mu(Y, \hat{Y}) := \text{rank } \mathcal{M} + \text{ind } D, \quad \mu^*(Y, \hat{Y}) := \text{rank } \mathcal{M} + \text{ind } (-D),
$$

where the $n \times n$ matrices $\mathcal{M}$ and $D$ are defined for $X := (I_0) Y$, $\hat{X} := (I_0) \hat{Y}$ as

$$
\mathcal{M} := (I - XX^\dagger) \hat{X}, \quad D := T w^T(Y, \hat{Y}) X^\dagger \hat{X} T, \quad T := I - \mathcal{M}^\dagger \mathcal{M},
$$

and where $w(Y, \hat{Y})$ is the Wronskian of $Y$ and $\hat{Y}$ given by $w(Y, \hat{Y}) = Y^T \mathcal{J} \hat{Y}$.

Note that the dual index can be presented in the form

$$
\mu^*(Y, \hat{Y}) = \mu(P_3 Y, P_3 \hat{Y}), \quad P_3 = \text{diag}(-I, I),
$$

where we use the notation from the book [10]. Moreover, one can verify that for arbitrary symplectic matrix $W$ we have that $P_3 WP_3$ is symplectic as well. The matrix $P_3$ together with $P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $P_2 = -P_3$ plays an important role in the duality principle in the comparative index theory for $(S_\lambda)$ (see [10, Chapter 3]). We have the following estimate for $\mu(Y, \hat{Y})$ and $\mu^*(Y, \hat{Y})$:

$$
\max\{\mu(Y, \hat{Y}), \mu^*(Y, \hat{Y})\} \leq \min \{\text{rank } \hat{X}, \text{rank } w(Y, \hat{Y})\} \leq n. \tag{17}
$$

We also need the following representation of $m_d(Y_k(\lambda))$ and $m_d^*(Y_k(\lambda))$ in terms of the comparative index (see [13, Lemmas 3.1,3.2]). Let $Y_k(\lambda)$ be a conjoined basis of $(S_\lambda)$ associated with a symplectic fundamental matrix $Z_k(\lambda)$ of this system, such that $Y_k(\lambda) = Z_k(\lambda)(0 I)^T$, then the multiplicities of forward and backward focal points are given by

$$
m_d(Y_k(\lambda)) = \mu(Y_{k+1}(\lambda), S_k(\lambda)(0 I)^T) = \mu^*(Z_{k+1}^{-1}(\lambda)(0 I)^T, Z_k^{-1}(\lambda)(0 I)^T), \tag{18}
$$

$$
m_d^*(Y_k(\lambda)) = \mu^*(Y_k(\lambda), S_k^{-1}(\lambda)(0 I)^T) = \mu(Z_k^{-1}(\lambda)(0 I)^T, Z_{k+1}^{-1}(\lambda)(0 I)^T). \tag{19}
$$

Remark that assumption (2) on the coefficient matrix $S_k(\lambda)$ plays a crucial role in the results of this paper. In the following proposition, we recall some properties of $\Psi(\cdot)$ from [22, Propositions 2.3, 3.1] and add several new ones needed for the subsequent proofs.

**Proposition 2.1:** Assume that $S_k(\lambda) \in C^1_p$, $\lambda \in \mathbb{R}$, $k \in [0, N]_z$ and (1), (2) hold. Then the following assertions are true.

(i) Condition (2) is equivalent to $\Psi(S_k^{-1}(\lambda)) \leq 0$ or

$$
\Psi(P_3 S_k^{-1}(\lambda) P_3) \geq 0, \quad P_3 = \text{diag}(-I, I). \tag{20}
$$

(ii) If $Z_k(\lambda)$ for $k \in [0, N + 1]_z$ is a symplectic fundamental matrix of system $(S_\lambda)$ such that $Z_0(\lambda) \in C^1_p$, $\lambda \in \mathbb{R}$ and

$$
\Psi(Z_k(\lambda)) \geq 0, \quad \lambda \in \mathbb{R}, \tag{21}
$$

for $k = 0$, then $Z_k(\lambda) \in C^1_p$, $\lambda \in \mathbb{R}$ for all $k \in [0, N + 1]_z$ and condition (21) holds for every $k \in [0, N + 1]_z$. 
(iii) If instead of (21), we have for $k = N + 1$

$$
\Psi(Z_k(\lambda)) \leq 0, \quad \lambda \in \mathbb{R},
$$

where a symplectic fundamental matrix $Z_k(\lambda)$ such that $Z_{N+1}(\lambda) \in C^1, \lambda \in \mathbb{R}$, then condition (22) holds for all $k \in [0, N + 1]$. Moreover, (22) is equivalent to

$$
\Psi(P_3 Z_k(\lambda) P_3) \geq 0, \quad \lambda \in \mathbb{R},
$$

where $P_3$ is defined by (20).

(iv) For arbitrary symplectic matrices $R$ and $P$ condition (2) is equivalent to

$$
\Psi(R^{-1} S_k(\lambda) P) \geq 0, \quad \lambda \in \mathbb{R}, \; k \in [0, N]_\mathbb{Z}.
$$

**Proof:** The equivalence of (2) and $\Psi(S_k^{-1}(\lambda)) \leq 0$ is proved in [22, Proposition 2.3 (iv)] while (20) is equivalent to $\Psi(S_k^{-1}(\lambda)) \leq 0$ because of the relation $\Psi(P_3 S_k^{-1}(\lambda) P_3) = -P_3 \Psi(S_k^{-1}(\lambda)) P_3 \geq 0$ which can be easily verified by direct computations.

The main statement in (ii) is proved in [22, Proposition 3.1]. The equivalence of (22) and (23) can be proved similarly to (i).

The proof of (22) for all $k \leq N + 1$ provided condition (22) holds for $k = N + 1$ follows from the relation (see [22, Proposition 2.3 (i)])

$$
\Psi(Z_k(\lambda)) = \Psi(S_k^{-1}(\lambda) Z_{k+1}(\lambda)) = \Psi(S_k^{-1}(\lambda)) + S_k^T(\lambda) \Psi(Z_{k+1}(\lambda)) S_k(\lambda)
$$

by induction using Proposition 2.1(i).

The equivalence of (2) and (24) was proved in [22, Proposition 2.3(iv)]. The proof is completed.

**Remark 2.2:** (i) In particular, condition (21) holds for the symplectic fundamental matrix $Z_k^{[0]}(\lambda)$ with the initial condition $Z_0^{[0]}(\lambda) = I$. Recall that in this case $Z_k^{[0]}(\lambda) (0 \; I)^T = Y_k^{[0]}(\lambda)$, where $Y_k^{[0]}(\lambda)$ is the principal solution of $(S_\lambda)$ at $k = 0$. By a similar way, we will use condition (23) for the symplectic fundamental matrix $Z_k^{[N+1]}(\lambda), \; Z_{N+1}^{[N+1]}(\lambda) = I$ associated with the principal solution $Y_k^{[N+1]}(\lambda)$ at $N + 1$.

(ii) It was proved in [18], [22, Theorem 2.4], [10, Theorem 5.1] that the monotonicity assumption (2) implies that for the block $B_k(\lambda)$ of $S_k(\lambda)$ the sets Ker$B_k(\lambda)$, Im$B_k(\lambda)$ are piecewise constant with respect to $\lambda \in \mathbb{R}$. By Proposition 2.1(ii), (iii) (see also [29], [22, Corollary 3.2], and [10, Theorem 5.3]), the same property also holds for the blocks $X_k^{[l]}(\lambda)$ of the principal solutions $Y_k^{[l]}(\lambda), \; l \in \{0, N + 1\}$ of $(S_\lambda)$, i.e. Ker$X_k^{[l]}(\lambda)$, Im$X_k^{[l]}(\lambda)$ are piecewise constant with respect to $\lambda \in \mathbb{R}$. Moreover, by Proposition 2.1(iv), a similar property also holds for blocks of the transformed matrices $R^{-1} S_k(\lambda) P$ and $R^{-1} Z_k^{[l]}(\lambda) P, \; l \in \{0, N + 1\}$ where $R$ and $P$ are arbitrary symplectic matrices.

(iii) The renormalized oscillation theory is closely related to the transformation theory [5,20], [10, Section 4.4] of $(S_\lambda)$. Instead of $(S_\lambda)$ consider the transformed symplectic system

$$
\tilde{y}_{k+1}(\lambda) = \tilde{S}_k(\lambda) \tilde{y}_k(\lambda) \quad k \in [0, N]_\mathbb{Z}, \quad \tilde{S}_k(\lambda) = \begin{pmatrix} \tilde{A}_k(\lambda) & \tilde{B}_k(\lambda) \\ \tilde{C}_k(\lambda) & \tilde{D}_k(\lambda) \end{pmatrix} \quad (TS_\lambda)
$$

derived from $(S_\lambda)$ using the transformation $\tilde{y}_k(\lambda) = R_k^{-1} y_k(\lambda)$. Here, $R_k \in Sp(2n)$ for $k \in [0, N + 1]_\mathbb{Z}$ and does not depend on $\lambda$. Then, the coefficient matrix $\tilde{S}_k(\lambda)$ of system $(TS_\lambda)$
is symplectic and according to Proposition 2.1(iv) obeys the monotonicity condition

\[
\Psi(\hat{S}_k(\lambda)) \geq 0, \quad \hat{S}_k(\lambda) = R_{k+1}^{-1} S_k(\lambda) R_k, \quad k \in [0, N]_Z, \quad \lambda \in \mathbb{R}.
\]  

(25)

By Proposition 2.1(ii)–(iv), we also have the monotonicity conditions for the transformed fundamental matrices \( R^{-1}_k Z^{[0]}_k(\lambda) \) and \( R^{-1}_k Z^{[N+1]}_k(\lambda) \), i.e.

\[
\Psi(R^{-1}_k Z^{[0]}_k(\lambda)) \geq 0, \quad \Psi(P_3 R^{-1}_k Z^{[N+1]}_k(\lambda) P_3) \geq 0, \quad k \in [0, N+1]_Z, \quad \lambda \in \mathbb{R}.
\]

In particular, taking the transformation matrix \( R_k \) in form \( R_k : Z^{[0]}_k(\beta) \) or \( R_k : Z^{[N+1]}_k(\beta) \) for \( \beta \in \{a, b\} \), where \( a, b \in \mathbb{R} \) are fixed values of \( \lambda \) we investigate oscillations of the Wronskians

\[
w(Y^{[N+1]}_k(\beta), Y^{[0]}_k(\lambda)) = Y^{[N+1]}_k T(\beta) J Y^{[0]}_k(\lambda) = -(I_0) (Z^{[N+1]}_k(\beta))^{-1} Y^{[0]}_k(\lambda),
\]

\[
w(Y^{[0]}_k(\beta), Y^{[N+1]}_k(\lambda)) = Y^{[0]}_k T(\beta) J Y^{[N+1]}_k(\lambda) = -(I_0) (Z^{[0]}_k(\beta))^{-1} Y^{[N+1]}_k(\lambda)
\]

associated with the upper blocks of \((Z^{[N+1]}_k(\beta))^{-1} Y^{[0]}_k(\lambda)\) or \((Z^{[0]}_k(\beta))^{-1} Y^{[N+1]}_k(\lambda)\).

Based on Remark 2.2(i),(ii) one can define the notion of a finite eigenvalue of problem \((S_\lambda), (E_0)\), see [32, Definition 4.4].

**Definition 2.3:** Assume (1),(2) and consider the principal solution \( Y^{[0]}_k(\lambda) \) of \((S_\lambda)\) at \( k = 0 \). Then, a number \( \lambda_0 \in \mathbb{R} \) is a finite eigenvalue of problem \((S_\lambda), (E_0)\) if

\[
\theta(\lambda_0) := \text{rank} X^{[0]}_{N+1}(\lambda_0^-) - \text{rank} X^{[0]}_{N+1}(\lambda_0) \geq 1.
\]

In this case, the number \( \theta(\lambda_0) \) is called the algebraic multiplicity of \( \lambda_0 \).

It follows from Definition 2.3 that under (1) and (2) the finite eigenvalues of \((S_\lambda), (E_0)\) are isolated.

By a similar way, Remark 2.2(ii) implies that the quantities \( \text{rank} B_k(\lambda) \) are constant on some left and right neighbourhoods of \( \lambda_0 \). Here, \( B_k(\lambda) \) is the block of \( S_k(\lambda) \) given by (1). Then, we introduce the notation (7) (see Section 1) to describe jumps of \( \text{rank} B_k(\lambda) \) in the left neighbourhood of \( \lambda_0 \).

Using these notions, we recall the global oscillation theorem for problem \((S_\lambda), (E_0)\) for the case when the block \( B_k(\lambda) \) of the matrix \( S_k(\lambda) \) has non-constant rank with respect to \( \lambda \) (see [18, formula (2.14) and Theorem 2.7]).

**Theorem 2.4:** For problem \((S_\lambda), (E_0)\) under assumptions (1) and (2) we have formula (8) presenting the number \( \#\{\nu \in \sigma | a < \nu \leq b \} \) of finite eigenvalues in \((a, b)\).
Moreover, under the assumption (5) the spectrum $\sigma$ of problem $(S_\lambda)$, $(E_0)$ is bounded from below, i.e. there exists $\lambda_{00} \in \mathbb{R}$ such that

$$\operatorname{rank} X_{N+1}^{[0]}(\lambda) = \operatorname{rank} X_{N+1}^{[0]}(\lambda^{-}) \text{ for all } \lambda < \lambda_{00}$$

(26)

and

$$l_d(Y^{[0]}(b), 0, N + 1) - m + \sum_{\nu \leq b} \sum_{k=0}^{N} \vartheta_k(\nu) = \#\{\nu \in \sigma | \nu \leq b\},$$

(27)

where the sums $\sum_{\nu \leq b} \sum_{k=0}^{N} \vartheta_k(\nu)$ and $\#\{\nu \in \sigma | \nu \leq b\} := \sum_{\nu \leq b} \theta(\nu)$ are finite and the constant $m$ is given by

$$m = l_d(Y^{[0]}(\lambda), 0, N + 1), \lambda < \min\{\lambda_0, \lambda_{00}\}. \quad (28)$$

Theorem 2.4 can be rewritten in terms of the principal solution $Y_{N+1}^{[N+1]}(\lambda)$ at $k = N + 1$.

**Theorem 2.5:** For problem $(S_\lambda)$, $(E_0)$ under assumptions (1) and (2), we have the following formula:

$$l^*_d(Y^{[N+1]}(b), 0, N + 1) - l^*_d(Y^{[N+1]}(a), 0, N + 1)$$

$$+ \sum_{a < \nu \leq b} \sum_{k=0}^{N} \vartheta_k(\nu) = \#\{\nu \in \sigma | a < \nu \leq b\},$$

(29)

connecting the number $\#\{\nu \in \sigma | a < \nu \leq b\}$ of finite eigenvalues in $(a, b]$ with the number $l^*_d(Y^{[N+1]}(\lambda), 0, N)$ of backward focal points of the principal solution of $(S_\lambda)$ at $k = N + 1$.

Moreover, under the assumption (5) we have

$$l^*_d(Y^{[N+1]}(b), 0, N + 1) - m^* + \sum_{\nu \leq b} \sum_{k=0}^{N} \vartheta_k(\nu) = \#\{\nu \in \sigma | \nu \leq b\},$$

(30)

where the constant $m^*$ is given by

$$m^* = l^*_d(Y^{[N+1]}(\lambda), 0, N + 1) = m = l_d(Y^{[0]}(\lambda), 0, N + 1), \lambda < \min\{\lambda_0, \lambda_{00}\}. \quad (31)$$

**Proof:** Indeed, using relation (15) and replacing $l(Y^{[0]}(\lambda), 0, N + 1)$ by $l^*(Y^{[N+1]}(\lambda), 0, N + 1)$ in (8), (27), and (28) we derive (29), (30), and (31).

The main purpose of this paper is to present renormalized versions of Theorems 2.4 and 2.5.
3. Oscillation theory for differential Hamiltonian systems and renormalized oscillation theorems

Recall now some basic concepts of the oscillation theory of the differential Hamiltonian systems

\[ y'(t) = J\mathcal{H}(t)y(t), \quad \mathcal{H}(t) = \begin{pmatrix} -C(t) & A^T(t) \\ A(t) & B(t) \end{pmatrix}, \quad \mathcal{H}(t) = \mathcal{H}^T(t), \]  

(32)

where we assume that \( \mathcal{H}(t) \) is piecewise continuous with respect to \( t \in [a, b] \) and the following Legendre condition

\[ B(t) \geq 0, \quad t \in [a, b]. \]  

(33)

Remark that the definition of a conjoined basis and the principal solution of (32) can be introduced by analogy with the discrete case (see Section 2).

**Definition 3.1:** Assume (33). We say (see [28]) that a point \( t_0 \in (a, b] \) is a (left) proper focal point of a conjoined basis \( Y(t) = (X(t), U(t)) \) of (32), provided

\[ m_c(Y(t_0)) = \text{def}X(t_0) - \text{def}X(t_0^-) = \text{rank}X(t_0^-) - \text{rank}X(t_0) \geq 1 \]  

(34)

and \( m_c(Y(t_0)) \) is its multiplicity.

Introduce the notation

\[ l_c(Y, a, b) = \sum_{\tau \in (a, b]} m_c(Y(\tau)) \]  

(35)

for the total number of proper focal points (including the multiplicities (34)) in \((a, b]\). Recall that by the Legendre condition \( B(t) \geq 0 \) all proper focal points are isolated (see [28, Theorem 3]) then the sum (35) is well-defined. Moreover, system (32) is called non-oscillatory on \((-\infty, b]\) if for every conjoined basis \( Y(t) \) of (32) the number of focal points on \((-\infty, b]\) is finite (see [31, Definition 2.3. and Theorem 2.2], where we replace \( \infty \) by \(-\infty\)).

Based on Definition 3.1 and Remark 2.2(ii) one can consider the numbers (7) as the multiplicities of left proper focal points of the conjoined basis \( S_k(\lambda)(0 I)^T = (\mathcal{B}_k(\lambda), \mathcal{D}_k(\lambda)) \) of the Hamiltonian differential system (6), i.e. for any \( a < b, \quad a, b \in \mathbb{R} \)

\[ \vartheta_k(\lambda) = m_c(S_k(\lambda)(0 I)^T), \quad \sum_{a \leq v \leq b} \vartheta_k(v) = l_c(S_k(0 I)^T, a, b). \]  

(36)

Moreover, under assumption (5) we have that system (6) is non-oscillatory as \( \lambda \to -\infty \).

By a similar way (see Remark 2.2) one can connect the notion of the algebraic multiplicity of finite eigenvalues of \( (S_{\lambda}), (E_0) \) with the notion of the multiplicity of proper focal points of the principal solution \( Y_{N+1}^{[0]}(\lambda) \) of \( (S_{\lambda}) \) considered as a function of \( \lambda \), i.e. we have
under (1) and (2) that
\[
\theta(\lambda) = m_c(Y_{N+1}^{[0]}(\lambda), \#\{\nu \in \sigma | a < \nu \leq b\} = l_c(Y_{N+1}^{[0]}, a, b).
\] (37)

The main result in [19, Theorem 2.2] connects the multiplicities of proper focal points of conjoined bases of two Hamiltonian systems under a majorant condition for their Hamiltonians and under the Legendre condition assumed for one of these systems. Here, we reformulate this result in a new notation convenient for the results of this paper.

**Theorem 3.2 (Comparison theorem for the continuous case):** Assume that
\[
Y(\lambda) := Z(\lambda)(0 I)^T, \quad \hat{Y}(\lambda) = \hat{Z}(\lambda)(0 I)^T,
\] (38)
where symplectic matrices \(Z(\lambda), \hat{Z}(\lambda) \in C_p^1\) obey the conditions
\[
\Psi(\hat{Z}^{-1}(\lambda)Z(\lambda)) = \hat{Z}^T(\lambda)(\Psi(Z(\lambda)) - \Psi(\hat{Z}(\lambda)))\hat{Z}(\lambda) \geq 0,
\]
\[
\phi(\hat{Z}(\lambda)) := (0 I)^T \Psi(\hat{Z}(\lambda)) (0 I)^T \geq 0, \lambda \in [a, b].
\] (39)

Then, for \(\hat{Y}(\lambda) := \hat{Z}^{-1}(\lambda)Y(\lambda)\), we have
\[
l_c(\hat{Y}, a, b) - l_c(Y, a, b) + l_c(\hat{Y}, a, b) = \mu(Y(a), \hat{Y}(a)) - \mu(Y(b), \hat{Y}(b)).
\] (40)

**Proof:** Here, we rewrite the main result in [19, Theorem 2.2] replacing the variable \(t\) by \(\lambda\) and using that \(Y(\lambda)\) and \(\hat{Y}(\lambda)\) are conjoined bases of systems in form (32) with the Hamiltonians \(\mathcal{H}(\lambda) = \Psi(Z(\lambda)), \hat{\mathcal{H}}(\lambda) = \Psi(\hat{Z}(\lambda))\), respectively. In this notation, the proof follows from the proof of Theorem 2.2 in [19].

As a corollary of Theorem 3.2 one can derive the following separation theorem (see [19, Theorem 2.3] and [30, Theorem 4.1]) which we reformulate in the notation introduced above.

**Theorem 3.3 (Separation theorem for the continuous case):** Under the notation and the assumptions of Theorem 3.2 suppose additionally that the matrix \(\hat{Z}^{-1}(\lambda)Y(\lambda)\) does not depend on \(\lambda \in \mathbb{R}\), then
\[
l_c(\hat{Y}, a, b) - l_c(Y, a, b) = \mu(Y(\lambda), \hat{Y}(\lambda))_{\lambda=b}^{\lambda=a}.
\] (41)

In the subsequent results, we will use the following corollary to Theorem 3.3. Remark that this property is well-known, see, for example, [7, Lemma 3.6].

**Corollary 3.4:** Assume that the symmetric matrix \(Q(\lambda) \in C_p^1\) is non-decreasing, i.e. \(\dot{Q}(\lambda) \geq 0, \lambda \in [a, b]\). Then rank \(Q(\lambda)\) is piecewise constant and
\[
l_c((Q I)^T, a, b) = \sum_{a < \lambda \leq b} (\text{rank } Q(\lambda^-) - \text{rank } Q(\lambda)) = \text{ind}Q(a) - \text{ind}Q(b).
\] (42)

**Proof:** Applying Theorem 3.3 to the case \(\hat{Z}(\lambda) := \begin{pmatrix} I & Q(\lambda) \\ 0 & I \end{pmatrix}\), \(Z(\lambda) := \hat{Z}(\lambda)J\), we see that \(\Psi(Z(\lambda)) = \Psi(\hat{Z}(\lambda)) = \text{diag}[0, \dot{Q}(\lambda)] \geq 0\). By (41) for \(\hat{Y}(\lambda) := \hat{Z}(\lambda)(0 I)^T\) and \(Y(\lambda) := Z(\lambda)(0 I)^T\), we derive \(l_c((Q I)^T, a, b) = \mu((I 0)^T, (Q(\lambda) I)^T)_{b}^{a} = \text{ind}Q(\lambda)_{b}^{a}\) or (42).
Another important corollary to Theorem 3.2 is the following theorem.

**Theorem 3.5:** Consider the discrete symplectic system

\[ y_{k+1}(\lambda) = W_k(\lambda) y_k(\lambda), \quad W_k^T(\lambda) J W_k(\lambda) = J, \quad k \in [0, N]_\mathbb{Z}, \]  

(43)

such that the symplectic matrix \( W_k(\lambda) \in C_p^1 \) obeys the condition

\[ \Psi(W_k(\lambda)) \geq 0, \quad k \in [0, N]_\mathbb{Z}, \quad \lambda \in \mathbb{R}, \]  

(44)

where the symmetric operator \( \Psi(\cdot) \) is defined by (2). Assume that \( Z_k(\lambda) \) is a symplectic fundamental matrix of (43) with \( Z_0(\lambda) \in C_p^1 \) and condition (21) holds for \( k = 0 \). Then for a conjoined basis \( \mathcal{Y}_k(\lambda) \) of (43) such that \( \mathcal{Y}_0(\lambda) = Z_0(\lambda)(0 \ 1)^T \) we have

\[ -\Delta l_c(\mathcal{Y}_k, a, b) + l_c(W_k(0 \ 1)^T, a, b) = \mu(\mathcal{Y}_{k+1}(\lambda), W_k(\lambda)(0 \ 1)^T)|_a^b = m_d(\mathcal{Y}_k(a)) - m_d(\mathcal{Y}_k(b)). \]

Moreover,

\[ l_c(\mathcal{Y}_{N+1}, a, b) - l_c(\mathcal{Y}_0, a, b) - \sum_{k=0}^N l_c(W_k(0 \ 1)^T, a, b) = l_d(\mathcal{Y}(b), 0, N + 1) - l_d(\mathcal{Y}(a), 0, N + 1). \]  

(46)

**Proof:** The condition \( \Psi(Z_0(\lambda)) \geq 0, \lambda \in \mathbb{R} \) and (44) imply (21) for \( k = 1, \ldots, N + 1 \) according Proposition 2.1(ii). Then, to prove (45) one can apply Theorem 3.2 for the case \( \mathcal{Z}(\lambda) := W_k(\lambda), Z(\lambda) := Z_{k+1}(\lambda) \), and then \( \mathcal{Z}(\lambda) := \mathcal{Z}^{-1}(\lambda)Z(\lambda) = \mathcal{Z}_k(\lambda) \) for \( k \in [0, N]_\mathbb{Z} \). It is clear that under this settings assumptions (39) of Theorem 3.2 are satisfied. Applying (40) and using (18) for the comparative indices in the right-hand side we derive identity (45). Summing (45) from \( k = 0 \) to \( k = N \) we derive (46). The proof is completed. \[ \blacksquare \]

**Remark 3.6:** (i) For the special case \( \mathcal{Y}_k(\lambda) := Y_k^{[0]}(\lambda) \) putting \( W_k(\lambda) \equiv S_k(\lambda), k \in [0, N]_\mathbb{Z} \), we have in (46) that \( l_c(Y_0^{[0]}, a, b) = 0 \) while according to Definition 2.3 the quantity \( l_c(Y_{N+1}^{[0]}, a, b) \) is equal to the number \( \# \{ v \in \sigma | a < v \leq b \} = \sum_{a < v \leq b} \theta(v) \) of finite eigenvalues of \( (S_z) \), \( (E_0) \) in \( (a, b) \).

(ii) Moreover, using the finiteness of the sum \( \sum_{a < v \leq b} \vartheta_k(v) \) we have \( \sum_{a < v \leq b} \sum_{k=0}^N \vartheta_k(v) = \sum_{k=0}^N \sum_{a < v \leq b} \vartheta_k(v) \) and under assumption (5) we have a similar property as \( a \to -\infty \), i.e. \( \sum_{v \leq b} \sum_{k=0}^N \vartheta_k(v) = \sum_{k=0}^N \sum_{v \leq b} \vartheta_k(v) \). So we see that Theorem 2.4 follows from Theorem 3.5.

Applying Theorem 3.5, we also show that for the calculation of the number \( \# \{ v \in \sigma | a < v \leq b \} \) of finite eigenvalues of \( (S_z) \), \( (E_0) \) it is possible to use the transformation \( \tilde{Y}_k(\lambda) = R_k^{-1} Y_k^{[0]}(\lambda) \) of the principal solution \( Y_k^{[0]}(\lambda) \) of \( (S_z) \) with the symplectic transformation...
matrix $R_k$ which does not depend on $\lambda$ and obeys the condition
\[ R_{N+1} = I. \]  
(47)

Then, by Remark 2.2(iii) $\tilde{Y}_k(\lambda)$ is a conjoined of the transformed symplectic system $(TS_\lambda)$ with condition (25). Moreover, according to Remark 2.2(ii), for the block $\tilde{B}_k(\lambda)$ of $\tilde{S}_k(\lambda)$ in $(TS_\lambda)$ there exists the finite limit
\[ \vartheta(\tilde{S}_k(\lambda_0)) = \tilde{\vartheta}_k(\lambda_0) := \text{rank } \tilde{B}_k(\lambda_0^-) - \text{rank } \tilde{B}_k(\lambda_0), \quad k \in [0,N_z], \lambda_0 \in \mathbb{R}. \]  
(48)

Introduce the notation
\[ n_R(\lambda_0) := \sum_{k=0}^{N} \tilde{\vartheta}_k(\lambda_0), \]  
(49)

where the index $R$ denotes the transformation matrix $R_k$ in the definition of $\tilde{S}_k(\lambda)$. We have the following result.

**Theorem 3.7:** Assume (1) and (2), then for the conjoined basis $\tilde{Y}_k(\lambda) = R_k^{-1} Y_k^{[0]}(\lambda)$ of $(TS_\lambda)$, where the symplectic matrix $R_k$ obeys condition (47) we have the following formula:
\[ l_d(R^{-1} Y^{[0]}(b),0,N+1) - l_d(R^{-1} Y^{[0]}(a),0,N+1) \]
\[ + \sum_{a < v \leq b} n_R(v) = \# \{ v \in \sigma \mid a < v \leq b \} \]  
(50)

connecting the number of finite eigenvalues of $(S_\lambda)$, $(E_0)$ in $(a,b]$ with the number of forward focal points of the conjoined basis $\tilde{Y}_k(\lambda) = R_k^{-1} Y_k^{[0]}(\lambda)$ and the numbers given by (48) and (49).

Moreover, under assumption (5) the sum $\sum_{a < v \leq b} n_R(v)$ is finite as $a \to -\infty$, then for any $a < \lambda_1 = \min \sigma$ the left-hand side of (50) presents the number $\# \{ v \in \sigma \mid v \leq b \}$ of finite eigenvalues of $(S_\lambda)$, $(E_0)$ less than or equal to $\lambda := b$.

**Proof:** Applying (46) to the case $\mathcal{Y}_k(\lambda) := \tilde{Y}_k(\lambda) = R_k^{-1} Y_k^{[0]}(\lambda)$ and $W_k(\lambda) := \tilde{S}_k(\lambda)$ we have
\[ l_c(\tilde{Y}_{N+1},a,b) - l_c(\tilde{Y}_0,a,b) - \sum_{k=0}^{N} l_c(\tilde{S}_k(0)^T,a,b) \]
\[ = l_d(\tilde{Y}(b),0,N+1) - l_d(\tilde{Y}(a),0,N+1). \]  
(51)

We see that $\tilde{Y}_0 = R_0^{-1} Y_0^{[0]}(\lambda) = R_0^{-1}(0 I)^T$ does not depend on $\lambda$, then $l_c(\tilde{Y}_0,a,b) = 0$, while $\tilde{Y}_{N+1}(\lambda) = R_{N+1}^{-1} Y_{N+1}^{[0]}(\lambda) = Y_{N+1}^{[0]}(\lambda)$ due to condition (47). Moreover, according to (36) (see also Remark 3.6(ii)), we have $\sum_{k=0}^{N} l_c(\tilde{S}_k(0)^T,a,b) = \sum_{a < v \leq b} n_R(v)$. Similarly, $l_c(Y_{N+1}^{[0]},a,b) = \# \{ v \in \sigma \mid a < v \leq b \}$ by (37). Substituting these equalities into (51), we complete the proof of (50).
As in the proof of [18, Corollary 2.5], we have from (50)
\[
\left| \sum_{a < v \leq b} n_R(v) - \#\{v \in \sigma | a < v \leq b\} \right| = \left| l_d(R^{-1}Y^{[0]}(a), 0, N + 1) - l_d(R^{-1}Y^{[0]}(b), 0, N + 1) \right| \leq n(N + 1), \tag{52}
\]
where we have used estimate (13). So we see from (52) that the sum \( \sum_{a < v \leq b} n_R(v) \) is finite as \( a \to -\infty \) if and only if the spectrum \( \sigma \) of \((S_\lambda), (E_0)\) is bounded from below. But condition (5) is equivalent to (26), see [18, Corollary 2.5], then \( \#\{v \in \sigma | a < v \leq b\} \) is bounded for any fixed \( b \in \mathbb{R} \) as \( a \to -\infty \). Finally (50) for \( a < \lambda_1 = \min \sigma \) calculates the number of finite eigenvalues of \((S_\lambda), (E_0)\) less than or equal to \( b \).

For the special choice of the transformation matrix \( R_k \) in form \( R_k := Z_k^{[N+1]}(a) \) or \( R_k := Z_k^{[N+1]}(b) \) in the formulation of Theorem 3.7, we derive the renormalized oscillation theorem for forward focal points.

**Theorem 3.8:** Let \( Z_k^{[N+1]}(\lambda) \) be symplectic fundamental matrix of \((S_\lambda)\) such that \( Z_{N+1}^{[N+1]}(\lambda) = I, \lambda \in \mathbb{R} \). Then, under the assumptions of Theorem 3.7 for \( R_k := Z_k^{[N+1]}(a) \) formula (50) takes the form
\[
l_d((Z^{[N+1]}(a))^{-1}Y^{[0]}(b), 0, N + 1) + \sum_{a < v \leq b} n_{Z^{[N+1]}(a)}(v) = \#\{v \in \sigma | a < v \leq b\}. \tag{53}
\]
Similarly, if \( R_k := Z_k^{[N+1]}(b) \), then instead of (50) we have
\[
-l_d((Z^{[N+1]}(b))^{-1}Y^{[0]}(a), 0, N + 1) + \sum_{a < v \leq b} n_{Z^{[N+1]}(b)}(v) = \#\{v \in \sigma | a < v \leq b\}. \tag{54}
\]
If, additionally, (5) holds, then for any \( a < \lambda_1 = \min \sigma \) formulae (53) and (54) present the number \( \#\{v \in \sigma | v \leq b\} \) of finite eigenvalues of \((S_\lambda), (E_0)\) less than or equal to \( \lambda := b \).

**Proof:** For the special choice \( R_k := Z_k^{[N+1]}(\beta), \beta \in (a, b) \) in (50) we have \( l_d((Z^{[N+1]}(\beta))^{-1}Y^{[0]}(\beta), 0, N + 1) = 0 \) because \( Z_k^{[N+1]}(\beta) \) and \( Z_k^{[0]}(\beta) \) solve the same symplectic system \((S_\lambda)\) for \( \lambda := \beta, \beta \in (a, b) \). Then, all assertions of Theorem 3.8 follow from Theorem 3.7.

**Example 3.9:** Consider an example given in [18, Example 2.9] which illustrated applications of formula (8) in Theorem 2.4. Here, we use this example to illustrate applications of (53) in Theorem 3.8. Consider problem \((S_\lambda), (E_0)\) for the trigonometric difference system with
\[
S_k(\lambda) = \begin{pmatrix} \cos(\lambda) & \sin(\lambda) \\ -\sin(\lambda) & \cos(\lambda) \end{pmatrix}, \quad k \in [0, N]_\mathbb{Z}. \tag{55}
\]
We have \( \Psi(S_k(\lambda)) = I \geq 0 \), then the monotonicity condition (2) holds for \( \lambda \in \mathbb{R} \). The principal solution of \((S_\lambda), (E_0)\) with (55) has the form \( Y_k^{[0]}(\lambda) = [\sin(k\lambda) \cos(k\lambda)]^T \), then
the finite eigenvalues of this problem \( \lambda_p = \pi p/(N + 1), p \in \mathbb{Z} \). Point out that the spectrum \( \sigma \) of this problem is unbounded. However, one can use Theorems 2.4 and 3.8 to calculate the number of finite eigenvalues in any half-open interval \((a, b]\). The multiplicities of focal points of \( Y_k^{[0]}(\lambda) \) in \((k, k + 1]\) are given by

\[
m_d(Y_k^{[0]}(\lambda)) = \begin{cases} 
1, & \lambda = \pi p/(k + 1), \lambda \neq \pi l, p, l \in \mathbb{Z}; \\
1, & \sin(\lambda) \sin((k + 1)\lambda) < 0; \\
0, & \text{otherwise.}
\end{cases}
\] (56)

So we see that \( m_d(Y_k^{[0]}(\lambda)) = m_d(Y_k^{[0]}(\lambda + \pi l)), l \in \mathbb{Z} \) and then \( l_d(Y^{[0]}(\lambda), 0, N + 1) \) is periodic with the minimal period \( T = \pi \) and non-decreasing in any interval \([a, b] \subseteq [\pi l, \pi(l + 1)], l \in \mathbb{Z} \). We have

\[
l_d(Y^{[0]}(\lambda), 0, N + 1) = \begin{cases} 
0, & \lambda \in [0, \pi/(N + 1)]; \\
1, & \lambda \in [\pi/(N + 1), 2\pi/(N + 1)); \\
& \vdots \\
N, & \lambda \in [N\pi/(N + 1), \pi]; \\
l_d(Y^{[0]}(\lambda), 0, N + 1) = l_d(Y^{[0]}(\lambda + \pi l), 0, N + 1), l \in \mathbb{Z}.
\end{cases}
\]

Put

\[
a = q\pi/(N + 1), \quad b = r\pi/(N + 1), \quad r > q, r, q \in \mathbb{Z}.
\] (57)

Then there are \( r - q \) finite eigenvalues of \((S_\lambda), (E_0)\) located in \((a, b]\). For example, if \( r = N, q = N - 1, N > 0 \), then in (8)

\[
\sum_{a<v\leq b} \sum_{k=0}^{N} \vartheta_k(v) = 0, \quad l_d(Y^{[0]}(b), 0, N + 1) = N, \quad l_d(Y^{[0]}(a), 0, N + 1) = N - 1,
\]

and the number \#\{\( v \in \sigma | a < v \leq b \)\} of finite eigenvalues in \((a, b]\) is equal to 1. Note that to apply (8), we have to calculate \#\{\( v \in \sigma | a < v \leq b \)\} using the pair \( Y^{[0]}(a), Y^{[0]}(b) \) of highly oscillatory (for \( N \gg 1 \)) conjoined bases of \((S_\lambda)\). In contrast, applying (53) in Theorem 3.8 we deal with only one slowly oscillatory conjoined basis \( \tilde{Y}_k(b) \) of \((TS_\lambda)\) with \( R_k := Z_k^{[N+1]}(a) \). Indeed, the matrix \( Z_k^{[N+1]}(\lambda), k = 0, \ldots, N + 1 \) has the form

\[
Z_k^{[N+1]}(\lambda) = \begin{pmatrix} 
\cos((N - k + 1)\lambda) & -\sin((N - k + 1)\lambda) \\
\sin((N - k + 1)\lambda) & \cos((N - k + 1)\lambda)
\end{pmatrix},
\]

then

\[
\tilde{Y}_k(b) = (Z_k^{[N+1]}(a))^{-1} Y^{[0]}(b) = \begin{pmatrix} 
\sin(\beta_k) \\
\cos(\beta_k)
\end{pmatrix}, \quad \beta_k = (b - a)k + (N + 1)a
\] (58)

and by analogy with (56) we have

\[
m_d(\tilde{Y}_k(b)) = \begin{cases} 
1, & \beta_{k+1} = \pi p, b - a \neq \pi l, p, l \in \mathbb{Z}; \\
1, & \sin(b - a) \sin(\beta_k) \sin(\beta_{k+1}) < 0; \quad k \in [0, N]_{\mathbb{Z}}, \\
0, & \text{otherwise},
\end{cases}
\] (59)
where $\beta_k$ is given by (58). The matrix $\tilde{S}_k(\lambda)$ associated with $R_k := Z_k^{[N+1]}(a)$ has the form $\tilde{S}_k(\lambda) = S_k^{-1}(a)S_k(\lambda)$, then $\tilde{B}_k(\lambda) = \sin(\lambda - a)$. We see that for the case $a = \pi(N - 1)/(N + 1)$, $b = \pi N/(N + 1)$ the matrix $\tilde{B}_k(\lambda)$ has constant rank for $\lambda \in (a, b)$, i.e. $\sum_{a < \nu < b} n_z^{(1)}(a)(\nu) = 0$, while according to (59) $m_y(\tilde{Y}_N(b)) = 1$, $m_y(\tilde{Y}_k(b)) = 0$, $k \neq N$. Finally, in this case $l_d(\tilde{Y}(b), 0, N + 1) = 1$ calculates the number of finite eigenvalues in $(a, b]$.

Consider another situation when the block of $B_k(\lambda)$ of the matrix $S_k(\lambda)$ has non-constant rank in $(a, b)$. Assume that $a$, $b$ are given by (57), where $q = N$ and $r = N + 2$, then rank $B_k(\lambda) = 0$ for $\lambda = \pi \in (a, b)$ and for the given case $\lambda = N + 1$, $l_d(Y^{[0]}(b), 0, N + 1) = 1$, $l_d(Y^{[0]}(a), 0, N + 1) = N$, then, according to (8) we have the correct result $1 - N + N + 1 = 2$ for the number of finite eigenvalues of $(S_k(\lambda))$, (E0).

Applying Theorem 3.8 instead of Theorem 2.4, we again have that $\tilde{B}_k(\lambda) = \sin(\lambda - a)$ has constant rank in $(a, b)$, i.e. $\sum_{a < \nu < b} n_z^{(1)}(a)(\nu) = 0$, while according to (59) $m_y(\tilde{Y}_N(b)) = 1$, and $m_y(\tilde{Y}_k(b)) = 1$ for $l = \lfloor N/2 \rfloor$. Finally, as before $l_d(\tilde{Y}(b), 0, N + 1) = 2$ calculates the number of finite eigenvalues in $(a, b)$. So we see that in this situation we deal with the slowly oscillatory conjoined basis $\tilde{Y}_k(b)$ and the non-oscillatory coefficient matrix $\tilde{S}_k(\lambda)$.

By a similar way one can prove an analogue of Theorem 3.5 for the time-reversed system (43).

**Theorem 3.10:** Consider symplectic system (43) under assumption (44). Suppose that for $k = N + 1$ a symplectic fundamental matrix $Z_k(\lambda)$ of (43) is piecewise continuously differentiable and obeys condition (22). Then for a conjoined basis of (43), such that $Y_k(\lambda) := Z_k(\lambda)(0 I)^T$ we have

$$
\Delta l_c(Y_k(a), b) + l_c(W_k(0 I)^T, a, b) = \mu^*(Y_k(\lambda), W_k^{-1}(\lambda)(0 I)^T)_{(b)}^a
$$

$$
= m_y^*(Y_k(a)) - m_y^*(Y_k(b)).
$$

Moreover,

$$
l_c(Y_N+1(a), b) - l_c(Y_0(a), b) + \sum_{k=0}^N l_c(W_k(0 I)^T, a, b)
$$

$$
= l_d^*(Y(a), 0, N + 1) - l_d^*(Y(b), 0, N + 1).
$$

**Proof:** By Proposition 2.1(iii), we have that conditions (22), (23) hold for $Z_k(\lambda)$ with $k \in [0, N + 1]$. Then, we prove (60) applying Theorem 3.2 to the case $\tilde{Z}(\lambda) := P_3W_k^{-1}(\lambda)P_3$, $Z(\lambda) := P_3Z_k(\lambda)P_3$, and then $\tilde{Z}(\lambda) := \tilde{Z}^{-1}(\lambda)Z(\lambda) = P_3Z_k^{-1}(\lambda)P_3$ for $k = 0, \ldots, N$, where the matrix $P_3$ is given by (16). It is clear that under this settings assumptions (39) of Theorem 3.2 are satisfied. Note also that by Definition 3.1 and the relation $W_k^{-1}(\lambda) = -JW_k^T(\lambda)J$ we have $l_c(P_3W_k^{-1}(\lambda), a, b) = l_c(W_k(0 I)^T, a, b)$. Using (16) and (19) for the comparative indices $\mu(P_3Y_k(\lambda), P_3W_k^{-1}(\lambda)P_3(0 I)^T) = \mu^*(Y_k(\lambda), W_k^{-1}(\lambda)(0 I)^T) = m_y^*(Y_k(\lambda))$ for $\lambda = a$ and $\lambda = b$ in the right-hand side of (40) we derive identity (60).
Summing (60) from \( k = 0 \) to \( k = N \) we derive (61). The proof is completed. \( \blacksquare \)

Consider the transformation \( \tilde{Y}_k(\lambda) = R_k^{-1}Y_k^{[N+1]}(\lambda) \) of the principal solution \( Y_k^{[N+1]}(\lambda) \) of \((S_\lambda)\) at \( k = N + 1 \) with the symplectic transformation matrix \( R_k \) which does not depend on \( \lambda \) and obeys the condition

\[
R_0 = I.
\]

Using the notation (48) and (49) associated with the transformation matrix \( R_k \) with property (62) one can formulate the following analogue of Theorem 3.7 for backward focal points.

**Theorem 3.11:** Assume (1) and (2), then for the conjoined basis \( \tilde{Y}_k(\lambda) = R_k^{-1}Y_k^{[N+1]}(\lambda) \) of \((S_\lambda)\), where the symplectic matrix \( R_k \) obeys condition (62) we have the following formula

\[
\text{l}^a_d(R^{-1}Y^{[N+1]}(b), 0, N + 1) - \text{l}^a_d(R^{-1}Y^{[N+1]}(a), 0, N + 1) + \sum_{a < v \leq b} n_R(v) = \#\{v \in \sigma \mid a < v \leq b\}
\]

connecting the number of finite eigenvalues of \((S_\lambda)\), \((E_0)\) in \((a, b)\) with the number of backward focal points of the conjoined basis \( \tilde{Y}_k(\lambda) \) and the numbers given by (48) and (49).

Moreover, under assumption (5) the sum \( \sum_{a < v \leq b} n_R(v) \) is finite as \( a \to -\infty \), then for any \( a < \lambda_1 = \min \sigma \) the left-hand side of (63) presents the number \( \#\{v \in \sigma \mid v \leq b\} \) of finite eigenvalues of \((S_\lambda)\), \((E_0)\) less than or equal to \( \lambda := b \).

**Proof:** Applying (61) to the case \( \mathcal{Y}_k(\lambda) := \tilde{Y}_k(\lambda) = R_k^{-1}Y_k^{[N+1]}(\lambda) \) and \( W_k(\lambda) := \tilde{S}_k(\lambda) \), we have

\[
\text{l}_c(\tilde{Y}_{N+1}, a, b) - \text{l}_c(\tilde{Y}_0, a, b) + \sum_{k=0}^{N} \text{l}_c(\tilde{S}_k(0 \, I)^T, a, b) = \text{l}_d^a(\tilde{Y}(a), 0, N + 1) - \text{l}_d^b(\tilde{Y}(b), 0, N + 1).
\]

We see that \( \tilde{Y}_{N+1} = R_{N+1}^{-1}Y_{N+1}^{[N+1]}(\lambda) = R_{N+1}^{-1}(0 \, I)^T \) does not depend on \( \lambda \), then \( \text{l}_c(\tilde{Y}_{N+1}, a, b) = 0 \), while \( \tilde{Y}_0(\lambda) = R_0^{-1}Y_0^{[N+1]}(\lambda) = Y_0^{[N+1]}(\lambda) \) due to condition (62). According to (36), we have \( \sum_{k=0}^{N} \text{l}_c(\tilde{S}_k(0 \, I)^T, a, b) = \sum_{a < v \leq b} n_R(v) \). Similarly, \( \text{l}_c(Y_0^{[N+1]}, a, b) = \text{l}_c(Y_0^{[N+1]}, a, b) = \#\{v \in \sigma \mid a < v \leq b\} \) because of (37) and the property \( -X_0^{[N+1]}(\lambda) = X_{N+1}^{[0]}(\lambda), \, \lambda \in \mathbb{R} \) which follows from the Wronskian identity

\[
w(Y_k^{[0]}(\lambda), Y_k^{[N+1]}(\lambda)) = Y_k^{[0]}(\lambda) \mathcal{J} Y_k^{[N+1]}(\lambda) = \text{const}
\]

for the principal solutions \( Y_k^{[0]}(\lambda), Y_k^{[N+1]}(\lambda) \) of \((S_\lambda)\).

Substituting these equalities into (64), we complete the proof of (63).
As in the proof of Theorem 3.7, we have from (63)
\[
\left| \sum_{a < \nu \leq b} n_R(\nu) - \#\{\nu \in \sigma \mid a < \nu \leq b\} \right|
= \left| l^*_d(R^{-1}Y^{[0]}(a), 0, N + 1) - l^*_d(R^{-1}Y^{[0]}(b), 0, N + 1) \right| \leq n(N + 1),
\]
where we have used estimate (13). Repeating the same arguments as in the proof of Theorem 3.7, we complete the proof of Theorem 3.11.

For the special choice of the transformation matrix \( R_k := Z_k^{[0]}(a) \) or \( R_k := Z_k^{[0]}(b) \) in the formulation of Theorem 3.11, we derive the renormalized oscillation theorem for backward focal points.

**Theorem 3.12:** Under the assumptions of Theorem 3.11 consider the transformation matrix \( R_k := Z_k^{[0]}(a) \). Then formula (63) takes the form
\[
l^*_d((Z^{[0]}(a))^{-1}Y^{[N+1]}(b), 0, N + 1) + \sum_{a < \nu \leq b} n_{Z^{[0]}(a)}(\nu) = \#\{\nu \in \sigma \mid a < \nu \leq b\}, \quad (65)
\]
while for the choice of \( R_k \) in form \( R_k := Z_k^{[0]}(b) \) we have instead of (63)
\[
-l^*_d((Z^{[0]}(b))^{-1}Y^{[N+1]}(a), 0, N + 1) + \sum_{a < \nu \leq b} n_{Z^{[0]}(b)}(\nu) = \#\{\nu \in \sigma \mid a < \nu \leq b\}. \quad (66)
\]
Moreover, under assumption (5) formulae (65) and (66) with \( a < \lambda_1 = \min \sigma \) present the number \( \#\{\nu \in \sigma \mid \nu \leq b\} \) of finite eigenvalues of \((S_{\lambda}), (E_0)\) less than or equal to \( \lambda := b\).

**Proof:** The proof follows from Theorem 3.11 using that \( l^*_d(Z^{[0]}^{-1}(\beta)Y^{[N+1]}(\beta), 0, N + 1) = 0, \beta \in \{a, b\}\) (see the proof of Theorem 3.8).

### 4. Relative oscillation numbers and renormalized oscillation theorems

In this section, we present another approach to the results of the renormalized and relative oscillation theory based on discrete comparison theorems. This approach is presented in [10, Section 6.1] under restriction (4). Here, we generalize the results from [10, Section 6.1] referred to the renormalized theory deleting condition (4).

For the first step in this direction, we recall the discrete comparison theorem, see [14, Theorem 2.1], [9, Theorem 3.3] and a notion of the relative oscillation numbers (see [17], [9, Definition 3.2], [10, Section 4.2]). Introduce the notation
\[
\langle S \rangle = \begin{pmatrix} \mathcal{X} \\ \mathcal{U} \end{pmatrix}, \quad \mathcal{X} = \begin{pmatrix} I & 0 \\ A & B \end{pmatrix}, \quad \mathcal{U} = \begin{pmatrix} 0 & -I \\ C & D \end{pmatrix}, \quad S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (67)
\]
for \( S \in Sp(2n) \) separated into \( n \times n \) blocks \( A, B, C, D \). In [14, Lemma 2.3], we proved that \( 4n \times 2n \) matrices \( \langle S \rangle, \langle \hat{S} \rangle \) associated with \( S, \hat{S} \in Sp(2n) \) obey (12) (with \( n \) replaced by \( 2n \)) and then the comparative index for the pair \( \langle S \rangle, \langle \hat{S} \rangle \) is well defined. The main
properties of the comparative index $\mu(\langle S \rangle, \langle \hat{S} \rangle)$ are proved in [14, Lemma 2.3] (see also [10, Section 3.3]).

Recall the notion of the relative oscillation numbers for two symplectic difference systems.

**Definition 4.1:** Let $Y_k$, $\hat{Y}_k$ be conjoined bases of the symplectic systems

$$Y_{k+1} = S_k Y_k, \quad \hat{Y}_{k+1} = \hat{S}_k \hat{Y}_k$$

(68)

associated with symplectic fundamental matrices $Z_k$, $\hat{Z}_k$ such that (38) hold.

Then the relative oscillation number is defined as

$$
\#_k(Z, Z) = \mu((\hat{Z}_k^{-1} Z_k), (\hat{Z}_k^{-1} Z_{k+1})) - \mu((S_k), (\hat{S}_k))
$$

$$
= \mu((S_k), (\hat{S}_k)) - \mu((\hat{Z}_k^{-1} Z_k), (\hat{Z}_k^{-1} Z_{k+1})).
$$

(69)

The following comparison result was proved in [14, Theorem 2.1].

**Theorem 4.2:** Let $Y_k$, $\hat{Y}_k$ be conjoined bases of (68) associated with symplectic fundamental matrices $Z_k$, $\hat{Z}_k$ such that (38) holds, then

$$
l_d(\hat{Y}, 0, N + 1) - l_d(Y, 0, N + 1) + \mu(Y_k, \hat{Y}_k)_{0}^{N+1} = \#(\hat{Z}, Z, 0, N),
$$

(70)

where

$$
\#(\hat{Z}, Z, 0, N) = \sum_{k=0}^{N} \#_k(\hat{Z}, Z)
$$

(71)

and $l_d(\hat{Y}, 0, N + 1)$, $l_d(Y, 0, N + 1)$ are the numbers of forward focal points in $(0, N + 1]$ defined by (14).

Applying Theorem 4.2 to $(S_k)$ for the case $\hat{S}_k := S_k(b)$, $S_k := S_k(a)$ and $\hat{Z}_k := Z_k^{[N+1]}(b)$, $Z_k := Z_k^{[0]}(a)$, we see that formula (70) can be rewritten in the form (see [14, Corollary 2.4])

$$
l_d(Y^{[0]}(b), 0, N + 1) - l_d(Y^{[0]}(a), 0, N + 1) = \#(Z^{[N+1]}(b), Z^{[0]}(a), 0, N).
$$

(72)

Substituting (72) into the left-hand side of (8) we derive

$$
\#(Z^{[N+1]}(b), Z^{[0]}(a), 0, N) + \sum_{a < \mu \leq b} \sum_{k=0}^{N} \vartheta_k(\mu) = \#\{v \in \sigma | a < v \leq b\},
$$

(73)

where the relative oscillation numbers $\#(Z^{[N+1]}(b), Z^{[0]}(a), 0, N)$ are given by (69) and (71).

Main questions which we answer in this section are connected with simplifications of the sum in the left-hand side of (73) which leads to the proof of formula (10) and the central
result of this section, Theorem 4.5. In particular, we investigate sufficient conditions for the majorant condition (see \cite[Section 2]{9})

\[
\mu((S_k(b), S_k(a))) = 0
\]  

which makes the relative oscillation numbers non-negative

\[
\#(Z^{[N+1]}(b), Z^{[0]}(a), 0, N) \geq 0.
\]  

We begin with the following interesting oscillation result which is derived as a corollary to Theorem 3.3.

**Lemma 4.3:** For a symplectic matrix \( W(\lambda) \in C^1_p \) assume that

\[
\Psi(W(\lambda)) \geq 0, \lambda \in \mathbb{R}.
\]  

Then for any \( \lambda \in \mathbb{R} \) and for arbitrary constant matrix \( \hat{W} \in Sp(2n) \) there exists the limit rank \( \hat{W} - W(\lambda^-) \). For the non-negative numbers

\[
\rho_{\hat{W}}(W(\lambda)) := \text{rank} (\hat{W} - W(\lambda^-)) - \text{rank} (\hat{W} - W(\lambda))
\]  

and

\[
\vartheta(W(\lambda)) := \text{rank} B(\lambda^-) - \text{rank} B(\lambda), \quad W(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix},
\]  

we have the connection

\[
\sum_{a < \nu \leq b} \rho_{\hat{W}}(W(\nu)) - \sum_{a < \nu \leq b} \vartheta(W(\nu)) = \mu((W(\lambda)), \langle \hat{W} \rangle)_{a}^{b},
\]  

where the matrix \( \langle W(\lambda) \rangle \) is defined by (67).

The proof of Lemma 4.3 is postponed to Appendix 1. For the subsequent proofs, we need the following properties of the numbers \( \rho_{\hat{W}}(W(\lambda)) \) given by (77).

**Proposition 4.4:** Under the assumptions and the notation of Lemma 4.3 the following properties of numbers (77) hold.

(i) Let \( R, P \) be arbitrary symplectic matrices which do not depend on \( \lambda \), then

\[
\rho_{R^{-1}\hat{W}(P)}(R^{-1}W(\lambda)P) = \rho_{\hat{W}}(W(\lambda)), \quad \lambda \in \mathbb{R}.
\]  

In particular, for the coefficient matrices \( S_k(\lambda), \tilde{S}_k(\lambda) \) of systems \((S_\lambda)\) and \((TS_\lambda)\) we have for any fixed \( \beta \in \mathbb{R} \)

\[
\rho_{S_k(\beta)}(\tilde{S}_k(\lambda)) = \rho_{S_k(\beta)}(S_k(\lambda)), \quad \lambda \in \mathbb{R}.
\]  

(ii) For any \( a < b, a, b \in \mathbb{R} \) there exist the limits \( \text{rank} (W(a) - W(\lambda^-)), \text{rank} (W(b) - W(\lambda^-)) \) and for the numbers

\[
\rho_{W(a)}(W(\lambda)) := \text{rank} (W(a) - W(\nu))_{a}^{\lambda}, \quad \rho_{W(b)}(W(\lambda)) := \text{rank} (W(b) - W(\nu))_{a}^{\lambda}.
\]  


and \( \vartheta(W(\lambda)) \) defined by (78) we have

\[
\sum_{a < v \leq b} \rho_{W(a)}(W(v)) - \sum_{a < v \leq b} \vartheta(W(v)) = -\mu((W(b), (W(a))), (83)
\]

\[
\sum_{a < v \leq b} \rho_{W(b)}(W(v)) - \sum_{a < v \leq b} \vartheta(W(v)) = \mu((W(a)), (W(b))). \tag{84}
\]

(iii) For the numbers (82), we have the connection

\[
\sum_{a < v \leq b} \rho_{W(b)}(W(v)) - \sum_{a < v \leq b} \rho_{W(a)}(W(v)) = \text{rank } (W(b) - W(a)).
\]

**Proof:** The proof of (i) follows from the definition of numbers (77). Indeed, using the non-singularity of the symplectic matrices \( R, P \), we have

\[
\rho_{R^{-1}\hat{W}P}(R^{-1}W(\lambda)P) := \text{rank } (R^{-1}(\hat{W} - W(\lambda))P) - \text{rank } (R^{-1}(\hat{W} - W(\lambda))P)
\]

\[
= \text{rank } (\hat{W} - W(\lambda)) - \text{rank } (\hat{W} - W(\lambda)) := \rho_{\hat{W}}(W(\lambda)).
\]

In particular, putting \( R := R_{k+1}, P := R_k, W(\lambda) := \hat{S}_k(\lambda), \hat{W} := \hat{S}_k(\beta) \) we derive (81).

(ii) For the proof of (83) or (84), we put in (79) \( \hat{W} := W(a) \) or \( \hat{W} := W(b) \) and use that \( \mu((W(\lambda)), (W(\lambda))) = 0 \) for \( \lambda = a \) or \( \lambda = b \).

(iii) Subtracting (83) from (84), we derive

\[
\sum_{a < v \leq b} \rho_{W(b)}(W(v)) - \sum_{a < v \leq b} \rho_{W(a)}(W(v)) = \mu((W(a)), (W(b))) + \mu((W(b)), (W(a))).
\]

Then we apply the property of the comparative index \( \mu(Y, \hat{Y}) + \mu(\hat{Y}, Y) = \text{rank } w(Y, \hat{Y}) \) (see [13, p.448]), where the rank of the Wronskian of \( \langle \hat{S}, S \rangle \) is equal to \( \text{rank } (\hat{S} - S) \) according to [14, Lemma 2.3 (i)]. The prove of property (iii) is completed.

Introduce the following notation:

\[
L_d(\hat{Z}^{-1}Y, 0, N + 1) := \sum_{k=0}^{N} \mu((\hat{Z}_{k+1}^{-1}Z_{k+1}), (\hat{Z}_{k}^{-1}Z_{k})), \tag{85}
\]

\[
L_d^*(Z^{-1}\hat{Y}, 0, N + 1) := \sum_{k=0}^{N} \mu((\hat{Z}_{k}^{-1}Z_{k}), (\hat{Z}_{k+1}^{-1}Z_{k+1})). \tag{86}
\]

for the sums of the terms \( \mu((\hat{Z}_{k+1}^{-1}Z_{k+1}), (\hat{Z}_{k}^{-1}Z_{k})) \) and \( \mu((\hat{Z}_{k}^{-1}Z_{k}), (\hat{Z}_{k+1}^{-1}Z_{k+1})) \) in (69).

By analogy with the proof of Proposition 4.4(iii), we derive the connection

\[
L_d(\hat{Z}^{-1}Y, 0, N + 1) + L_d^*(Z^{-1}\hat{Y}, 0, N + 1) = \sum_{k=0}^{N} \text{rank } (\hat{S}_k - S_k), \tag{87}
\]
where we have used that
\[
\mu((\hat{Z}_k^{-1}Z_k), (\hat{Z}_{k+1}^{-1}Z_{k+1})) + \mu((\hat{Z}_{k+1}^{-1}Z_{k+1}), (\hat{Z}_k^{-1}Z_k)) = \text{rank } (\Delta(\hat{Z}_k^{-1}Z_k)) = \text{rank } (\hat{S}_k - S_k).
\]

The main result of this section is the following theorem which generalizes [10, Theorem 6.4] to the case when (4) does not hold.

**Theorem 4.5:** Assume (1) and (2), then for any \(a \in \mathbb{R}\) there exists the limit \(\text{rank } (S_k(a) - S_k(\lambda^-))\) and for the numbers \(\rho_{S_k(a)}(S_k(\lambda))\) defined by
\[
\rho_k(\lambda) := \rho_{S_k(a)}(S_k(\lambda)) = \text{rank } (S_k(a) - S_k(\lambda^-)) - \text{rank } (S_k(a) - S_k(\lambda)),
\]
we have
\[
L^*_d((Z^{[0]}(a))^{-1}Y^{[N+1]}(b), 0, N + 1) + \sum_{a < v \leq b} \sum_{k=0}^{N} \rho_k(v) = \#\{v \in \sigma | a < v \leq b\}, \tag{89}
\]
where
\[
L^*_d((Z^{[0]}(a))^{-1}Y^{[N+1]}(b), 0, N + 1) = L^*_d((Z^{[N+1]}(a))^{-1}Y^{[0]}(b), 0, N + 1) \tag{90}
\]
and \(L^*_d(\cdot), L^*_{d}(\cdot)\) are given by (85), (86).

Moreover, if (5) holds, then the sum \(\sum_{a < v \leq b} \sum_{k=0}^{N} \rho_k(v)\) is finite as \(a \to -\infty\) and for \(a < \lambda_1 < \min \sigma\) formula (89) presents the number \(\#\{v \in \sigma | v \leq b\}\) of finite eigenvalues of \((S_z), (E_0)\) less than or equal to \(b\).

**Proof:** Putting \(W(\lambda) := S_k(\lambda)\) in (83) and using the notation \(\vartheta(S_k(\lambda)) = \vartheta_k(\lambda)\) given by (7) we see that
\[
\sum_{a < v \leq b} \rho_{S_k(a)}(S_k(v)) + \mu((S_k(b), S_k(a))) = \sum_{a < v \leq b} \vartheta_k(v), \tag{91}
\]
where \(\rho_{S_k(a)}(S_k(\lambda))\) is given by (88). Summing (91) for \(k = 0, 1, \ldots, N\) and then substituting the representation for \(\sum_{k=0}^{N} \sum_{a < v \leq b} \vartheta_k(v) = \sum_{a < v \leq b} \sum_{k=0}^{N} \vartheta_k(v)\) into the left-hand side of (73) we cancel the same addends \(\mu((S_k(b), S_k(a)))\) in the first representation (69) of the relative oscillation numbers and in (91). Incorporating notation (86), we derive identity (89).

For the proof of (90), we replace the roles of \(a\) and \(b\) in (72) and derive
\[
l_d(Y^{[0]}(a), 0, N + 1) - l_d(Y^{[0]}(b), 0, N + 1) = \#(Z^{[N+1]}(a), Z^{[0]}(b), 0, N).
\]
So we see that \(-\#(Z^{[N+1]}(a), Z^{[0]}(b), 0, N) = \#(Z^{[N+1]}(b), Z^{[0]}(a), 0, N)\), then using for the relative oscillation numbers in the previous identity the representations associated with \(\mu((S_k(b), S_k(a))), k = 0, 1, \ldots, N\) (see (69)) we cancel these terms and derive (90).

Using estimate (17) for the comparative index \(\mu((S_k(b), S_k(a))) \leq \text{rank } (S_k(b) - S_k(a)) \leq 2n\) in (91), we see that the sum \(\sum_{a < v \leq b} \rho_{S_k(a)}(S_k(v))\) is finite if and only if \(\sum_{a < v \leq b} \vartheta_k(v)\) is finite as \(a \to -\infty\). Then (5) is sufficient for the finiteness of
The proof is completed.

Remark 4.6: (i) One can verify directly that (65), (53) in Theorems 3.12 and 3.8 are equivalent to (89), (90) using the representations for (86) and (85) in form

\[ L_d^*(Z^{-1} \hat{Y}, 0, N + 1) = l_d^*(Z^{-1} \hat{Y}, 0, N + 1) + \sum_{k=0}^{N} \mu(\langle \hat{S}_k, \langle l \rangle \rangle), \quad \hat{S}_k = Z_{k+1}^{-1} \hat{S}_k Z_k, \]  

(92)

\[ L_d(\hat{Z}^{-1} Y, 0, N + 1) = l_d(\hat{Z}^{-1} Y, 0, N + 1) + \sum_{k=0}^{N} \mu(\langle \hat{S}_k, \langle l \rangle \rangle), \quad \hat{S}_k = \hat{Z}_{k+1}^{-1} S_k \hat{Z}_k \]  

(93)

which is based on [14, Lemma 2.3(v)] and (19), (18) (see also [10, Remark 4.46]). Next one can apply (83) putting \( W(\lambda) := \hat{S}_k(\lambda) \) to see that (89) is equivalent to (65) and that (89) with \( L_d^*(Z^{[0]}(a))^{-1} Y^{[N+1]}(b), 0, N + 1 \) replaced by \( L_d((Z^{[N+1]}(a))^{-1} Y^{[0]}(b), 0, N + 1) \) is equivalent to (53).

(ii) Using the relation \( \sum_{a<v \leq b} (\rho_{S_k(b)}(S_k(v)) - \rho_{S_k(a)}(S_k(v))) = \text{rank} (S_k(b) - S_k(a)) \) according to Proposition 4.4(iii) and incorporating connection (87) one can derive the equivalent form of (54), (66) in Theorems 3.8 and 3.12

\[-L_d((Z^{[N+1]}(b))^{-1} Y^{[0]}(a), 0, N + 1) + \sum_{a<v \leq b} \sum_{k=0}^{N} \rho_{S_k(b)}(S_k(v)) = \# \{ v \in \sigma \mid a < v \leq b \}, \]

where

\[ L_d((Z^{[N+1]}(b))^{-1} Y^{[0]}(a), 0, N + 1) = L_d^*((Z^{[0]}(b))^{-1} Y^{[N+1]}(a), 0, N + 1). \]  

(94)

(iii) The main advantage of Theorem 4.5 is the invariant form of the sum \( \sum_{a<v \leq b} \sum_{k=0}^{N} \rho_k(v) \) which does not depend on the (unknown) transformation matrices \( Z_k(\beta), l \in \{0, N+1\}, \beta \in \{a, b\} \) as it takes place in Theorems 3.8 and 3.12. The price of this advantage is the necessity to calculate \( L_d^*(Z^{-1} \hat{Y}, 0, N + 1) \) or \( l_d(\hat{Z}^{-1} Y, 0, N + 1) \) instead of \( l_d^*(Z^{-1} \hat{Y}, 0, N + 1) \) or \( l_d(\hat{Z}^{-1} Y, 0, N + 1) \) according to connections (92), (93). Remark also that \( L_d(Z^{-1} \hat{Y}, 0, N + 1) \) and \( l_d(\hat{Z}^{-1} Y, 0, N + 1) \) have the same meaning as \( l_d^*(Z^{-1} \hat{Y}, 0, N + 1) \) and \( l_d(\hat{Z}^{-1} Y, 0, N + 1) \), i.e. present the number of backward and forward focal points of conjoined bases of some augmented systems associated with \( (T_{S_k}) \) (see [10, Remark 4.46] for more details).

As a corollary to Theorems 3.8, 3.12 and 4.5 consider the important special case associated with the condition

\[ \rho_k(\lambda) := \rho_{S_k(a)}(S_k(\lambda)) = \text{rank} (S_k(a) - S_k(\lambda^-)) - \text{rank} (S_k(a) - S_k(\lambda)) = 0, \]

\[ \lambda \in (a, b], \quad k \in [0, N]. \]  

(95)
Theorem 4.7: Assume (1), (2). Then, for any \(a < b\) condition (95) is necessary and sufficient for the representation

\[
L_d^*([Z]_0(a))^{-1} Y_{N+1}^N(b), 0, N+1 = L_d([Z]_N^{-1}(a))^{-1} Y_0^N(b), 0, N+1
\]

\[
= \#\{\nu \in \sigma | a < \nu \leq b\}. \tag{96}
\]

Similarly, condition (95) is necessary and sufficient for the representations of the sums

\[
\sum_{a < \nu \leq b} n_{Z_0}(\nu) = \sum_{k=0}^N \mu(\langle \tilde{S}_k(b) \rangle, \langle I \rangle) \leq n(N+1), l \in \{0,N+1\} \tag{97}
\]

in (53) and (65), where \(\tilde{S}_k(b)\) is the coefficient matrix of \((TS_\lambda)\) associated with \(R_k := Z[l]_0(a), l \in \{0,N+1\}\).

**Proof:** By (89) and (90), we see that (95) is equivalent to (96), where we use that numbers (88) are non-negative.

Putting in (83) of Proposition 4.4(ii) \(W(\lambda) := \tilde{S}_k(\lambda)\), where \(\tilde{S}_k(\lambda) = (Z[l]_k^{-1}(a))^{-1} S_k(\lambda) Z[l]_k^{-1}(a), l \in \{0,N+1\}\) and incorporating that \(\vartheta(\tilde{S}_k(\lambda)) = \tilde{\vartheta}_k(\lambda)\) for \(\tilde{\vartheta}_k(\lambda)\) given by (48) we have

\[
\sum_{a < \nu \leq b} \rho_{\tilde{S}_k(a)}(\tilde{S}_k(\nu)) - \sum_{a < \nu \leq b} \tilde{\vartheta}(\nu) = -\mu(\langle \tilde{S}_k(b) \rangle, \langle \tilde{S}_k(a) \rangle).
\]

Observe also that \(\tilde{S}_k(a) = (Z[l]_{k+1}(a))^{-1} S_k(a) Z[l]_k^{-1}(a) = I, l \in \{0,N+1\}\) then summing the above identity from \(k = 0\) to \(k = N\) and incorporating property (81) we derive for \(l \in \{0,N+1\}\)

\[
\sum_{a < \nu \leq b} n_{Z[l]_0}(\nu) = \sum_{a < \nu \leq b} \sum_{k=0}^N \rho_{\tilde{S}_k(a)}(S_k(\nu)) + \sum_{k=0}^N \mu(\langle \tilde{S}_k(b) \rangle, \langle I \rangle). \tag{98}
\]

By (98), we see that (95) is equivalent to (97), where we estimate the comparative index \(\mu(\langle \tilde{S}_k(b) \rangle, \langle I \rangle) \leq n\) using (17). The proof is completed.

Next we formulate the simplest sufficient criteria for (95).

**Corollary 4.8:** Assume (1), (2) and

\[
\text{rank } \mathcal{B}_k(\lambda) = \text{rank } \mathcal{B}_k(\lambda), \lambda \in (a,b), k \in [0,N]_\mathbb{Z} \tag{99}
\]

for the block \(\mathcal{B}_k(\lambda)\) of \(S_k(\lambda)\) given by (1). Then condition (95) and the majorant condition (74) hold. Moreover, according to Theorem 4.7 we also have (96) and (97).

**Proof:** For the proof, we use (91). Applying (99), we see that \(\sum_{a < \nu \leq b} \vartheta_k(\nu) = 0\). Remark that both addends in the left-hand side of (91) are non-negative, then it follows that

\[
\mu(\langle S_k(b) \rangle, \langle S_k(a) \rangle) = 0, \quad \rho_{S_k(a)}(S_k(\lambda)) = 0, k \in [0,N], \lambda \in (a,b) \tag{100}
\]

Then we have proved (95) and (74). Moreover, by (95) we have that (96) and (97) hold.
Remark 4.9: (i) Consider problem \((S_\lambda), (E_0)\) under (2) and the additional assumption for the matrix \(\tilde{S}_k(\lambda)\)

\[
\tilde{A}_k(\lambda) \equiv A_k, \quad \tilde{B}_k(\lambda) \equiv B_k, \quad k \in [0, N]_\mathbb{Z}, \quad \lambda \in \mathbb{R}
\]

which covers the special case (3). We see that condition (99) is satisfied for all \(a < b\), then by Corollary 4.8 all identities in Theorem 4.7 hold. Moreover, one can verify directly (see also [10, Subsection 6.1.3]) that the block \(\tilde{B}_k(\lambda)\) of the matrix \(\tilde{S}_k(\lambda)\) associated with \(R_k = Z_k^{[l]}(0), l \in [0, N + 1]\) is symmetric, \(\tilde{B}_k(a) = 0\), and by (2) we have \(\tilde{B}_k(\lambda) \geq 0\). Finally, applying Corollary 3.4 we derive for the left-hand side of (97) \(\sum_{a < \nu \leq b} n_{Z_k^{[l]}(a)}(\nu) = 0, l \in [0, N + 1]\), then, instead of (96) we have by Theorems 3.8, 3.12 (see also (92), (93))

\[
l_d^*((Z^{[0]}(a))^{-1}Y^{[N+1]}(b), 0, N + 1)
\]

\[
= l_d((Z^{[N+1]}(a))^{-1}Y^{[0]}(b), 0, N + 1) = \#\{\nu \in \sigma | a < \nu \leq b\},
\]

i.e. the number of finite eigenvalues of \((S_\lambda), (E_0)\) in \((a, b]\) can be calculated using the number of backward focal points in \([0, N + 1]\) of the conjoined basis \((Z^{[0]}(a))^{-1}Y^{[N+1]}(b)\) which equals the number of forward focal points in \((0, N + 1]\) of the conjoined basis \((Z^{[N+1]}(a))^{-1}Y^{[0]}(b)\). This result was proved in [10, Theorem 6.9 and Remark 6.10(i)], where we used a different proof.

(ii) The most important special case of \((S_\lambda)\) for which condition (99) is satisfied is discrete matrix Sturm–Liouville eigenvalue problems with nonlinear dependence on \(\lambda \in \mathbb{R}\). Renormalized and more general relative oscillation theorems for these problems are presented in [17] (see also [10, Section 6.1]).

(iii) Observe that under the assumptions of Corollary 4.8 formula (96) is equivalent to

\[
\#(Z^{[N+1]}(b), Z^{[0]}(a), 0, N) = \#\{\nu \in \sigma | a < \nu \leq b\},
\]

where for the relative oscillation numbers \(\#(Z^{[N+1]}(b), Z^{[0]}(a), 0, N)\) given by (71) with \(\tilde{Z} := Z^{N+1}(b), Z := Z^{[0]}(a)\) we have majorant condition (74) and then

\[
\#_k(Z^{[N+1]}(b), Z^{[0]}(a)) = \mu((Z_k^{[N+1]}(b))^{-1}Z_k^{[0]}(a)), (Z_{k+1}^{[N+1]}(b))^{-1}Z_{k+1}^{[0]}(a)) \geq 0
\]

i.e. (75) holds. This result was derived in [10, Theorem 6.4 and Remark 6.5(i)], where we used a different proof.

Combining the proof of Corollary 4.8 and Proposition 4.4(i) one can generalize Corollary 4.8 as follows.

**Corollary 4.10:** Assume (1) and (2) and suppose that there exist symplectic matrices \(R\) and \(P\) such that for the matrix

\[
\tilde{S}_k(\lambda) = R^{-1}\tilde{S}_k(\lambda)P = \left(\tilde{A}_k(\lambda) \quad \tilde{B}_k(\lambda)\right)
\]

condition (99) is satisfied, i.e.

\[
\text{rank} \tilde{B}_k(\lambda^-) = \text{rank} \tilde{B}_k(\lambda), \quad \lambda \in (a, b], \quad k \in [0, N]_\mathbb{Z}.
\]

Then, we have (95) and according to Theorem 4.7 identities (96) and (97) hold.
Proof: As it was proved in Corollary 4.8 condition (103) is sufficient for \( \rho_{S_k(\lambda)}(\tilde{S}_k(\lambda)) = 0 \), then by Proposition 4.4(i) we prove that \( \rho_{S_k(\lambda)}(S_k(\lambda)) = 0 \). □

In particular, criterion (103) is satisfied if one of the blocks \( A_k(\lambda), C_k(\lambda), D_k(\lambda) \) in representation (1) of \( S_k(\lambda) \) is non-singular in \((a, b]\). This assumption is true for the block \( A_k(\lambda) \) of \( S_k(\lambda) \) associated with the most important special case of \((S_\lambda)\), with the discrete Hamiltonian systems [1]. Remark that condition (99) is not assumed.

Example 4.11: Consider the discrete Hamiltonian eigenvalue problem
\[
\Delta x_{k+1} = A_k(\lambda)x_{k+1} + B_k(\lambda)u_k, \quad \Delta u_k = C_k(\lambda)x_{k+1} - A_k^T(\lambda)u_k,
\]
\[\det(I - A_k(\lambda)) \neq 0, \quad k = 0, \ldots, N,\] (104)
\[x_0(\lambda) = x_{N+1}(\lambda) = 0,\] (105)
with the Hamiltonian
\[
\mathcal{H}_k(\lambda) = \mathcal{H}_k^T(\lambda), \quad \mathcal{H}_k(\lambda) = \begin{pmatrix} -C_k(\lambda) & A_k^T(\lambda) \\ A_k(\lambda) & B_k(\lambda) \end{pmatrix},
\]
which obeys the monotonicity condition (see [32, Example 7.9])
\[
\dot{\mathcal{H}}_k(\lambda) \geq 0, \quad \lambda \in \mathbb{R}.\] (106)
For the Hamiltonian system (104) rewritten in form \((S_\lambda)\) the matrix \( S_k(\lambda) \) is given by
\[
S_k(\lambda) = \begin{pmatrix} (I - A_k(\lambda))^{-1} & (I - A_k(\lambda))^{-1}B_k(\lambda) \\ C_k(\lambda)(I - A_k(\lambda))^{-1} & C_k(\lambda)(I - A_k(\lambda))^{-1}B_k(\lambda) + I - A_k^T(\lambda) \end{pmatrix},\] (107)
The block \( A_k(\lambda) = (I - A_k(\lambda))^{-1} \) of \( S_k(\lambda) \) is non-singular, then there exists the matrices \( R = I \) and \( P = J \) such that \( \tilde{S}_k(\lambda) := S_k(\lambda)J \) has the non-singular block \( \tilde{B}_k(\lambda) \), i.e. condition (103) is satisfied. Applying Corollary 4.10, we derive that \( S_k(\lambda) \) given by (107) obeys condition (95).

Corollary 4.12: Consider problem (104), (105) under assumption (106), then for any \( a < b \) we have condition (95) for matrix (107) and according to Theorem 4.7 identity (96) holds for the number \( \#\{\nu \in \sigma | a < \nu \leq b\} \) of finite eigenvalues of (104), (105).

Recall that condition (99) for \( S_k(\lambda) \) is not assumed, moreover, applying Corollary 3.4 for \( Q(\lambda) := B_k(\lambda) = A_k^{-1}(\lambda)B_k(\lambda) \) we see that
\[
\sum_{a < \nu \leq b} \vartheta_k(\nu) = \text{ind}B_k(a) - \text{ind}B_k(b) = \mu(\langle S_k(b) \rangle, \langle S_k(a) \rangle),\] (108)
where we have used that \( \text{rank} \tilde{B}_k(\lambda) = \text{rank} B_k(\lambda) \), (42) and (91).

Remark 4.13: The results of this paper can be further developed to the case of two symplectic eigenvalue problems in form of \((S_\lambda)\) with different coefficient matrices \( S_k(\lambda), \tilde{S}_k(\lambda) \), which are the subject of the relative oscillation theory. This theory is developed in [10, Chapter 6] for \( S_k(\lambda), \tilde{S}_k(\lambda) \) under restriction (4). Using new comparison results in [21] we are going to generalise the relative oscillation theory to the case when this condition is omitted.
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Appendix. Proof of Lemma 4.3

Introduce the $4n \times 4n$ matrices

\[ \tilde{Z}(\lambda) = R^{-1}(I, W(\lambda) \tilde{W}^{-1})R, \quad Z(\lambda) = R^{-1}(I, W(\lambda)), \]

\[ \{I, W(\lambda)\} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & A(\lambda) & 0 & B(\lambda) \\ 0 & 0 & I & 0 \\ 0 & C(\lambda) & 0 & D(\lambda) \end{pmatrix}, \quad R = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -I & I & 0 \\ 0 & I & I & 0 \\ -I & 0 & 0 & -I \\ -I & 0 & 0 & I \end{pmatrix}. \]
Then, it is easy to verify that $R$ is symplectic (and orthogonal) and $\{I, W(\lambda)\}$, $Z(\lambda), \hat{Z}(\lambda) \in \mathbb{R}^{4n \times 4n}$ are symplectic provided $W(\lambda), \hat{W} \in Sp(2n)$, moreover
\[
\Psi(Z(\lambda)) = R^T \Psi([I, W(\lambda)]) R = R^T [0, \Psi(W(\lambda))] R \geq 0
\]
and
\[
\Psi(\hat{Z}(\lambda)) = R^T \Psi([I, W(\lambda) \hat{W}^{-1}]) R = R^T [0, \Psi(W(\lambda) \hat{W}^{-1})] R = R^T [0, \Psi(W(\lambda))] R \geq 0
\]
provided (76) holds. Observe that the assumptions of Theorem 3.3 are satisfied because
\[
\hat{Z}^{-1}(\lambda) Z(\lambda) = R^{-1}[I, \hat{W}] \text{ does not depend on } \lambda.
\]
Applying (41), we have
\[
l_c(\hat{Z}(\lambda)(0 I)^T, a, b) - l_c(Z(\lambda)(0 I)^T, a, b) = \mu(Z(\lambda)(0 I)^T, \hat{Z}(\lambda)(0 I)^T) = \mu^*(Z^{-1}(\lambda)(0 I)^T, \hat{Z}(\lambda)(0 I)^T)
\]
Moreover, the upper blocks of the matrices $\hat{Z}(\lambda)(0 I)^T, Z(\lambda)(0 I)^T$ have the form
\[
(I 0) \hat{Z}(\lambda)(0 I)^T = \frac{1}{2} J(I - W(\lambda) \hat{W}^{-1}), (I 0) Z(\lambda)(0 I)^T = \frac{1}{\sqrt{2}} \begin{pmatrix} -I & -A^T(\lambda) \\ 0 & -B^T(\lambda) \end{pmatrix},
\]
where the rank of the second matrix in the above formula is equal to $n + \text{rank } B(\lambda)$. Then
\[
l_c(\hat{Z}(\lambda)(0 I)^T, a, b) = \sum_{a < \nu \leq b} \rho_{\hat{Z}}(W(\nu)), l_c(Z(\lambda)(0 I)^T, a, b) = \sum_{a < \nu \leq b} \vartheta(W(\nu)).
\]
Substituting the representations derived above into (A2), we prove formula (79).