Spectral data and solvent theory
for regular matrix polynomials

Nir Cohen, Edgar Pereira

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Abstract: This paper contains a re-evaluation of the spectral approach and
factorizability for regular matrix polynomials. In addition, solvent theory is
extended from the monic and comonic cases to the regular case. The clas-
sification of extended solvents (bisolvents) is shown to be equivalent to the
classification of all the regular first order right factors for a general matrix
polynomial.

I Introduction

It is of interest to extend the fundamental theorem of algebra (roots and factorization)
from complex polynomials to complex matrix polynomials, i.e. expressions of the form

\[ P(\lambda) = \sum_{i=0}^{k} A_i \lambda^i \quad A_i \in M_{n,m}(\mathbb{C}), \quad \lambda \in \mathbb{C}. \]

(I) KW theory. This 19th century approach, which is limited to matrix pencils,
extends the usual notion of complex eigenvalues, including roots ”at infinity”. Initially,
Weierstrass considered only regular pencils, and provided a complete analysis of roots
(also called eigenvalues) via strict equivalence and the Weierstrass (or W) canonical
form. Kronecker extended Weierstrass’ work to singular pencils in terms of additional
structure: right and left null indices ([8], p. 25), later described in detail by Forney

1Department Mathematics, UFRN - Universidade Federal de Rio Grande do Norte, Natal, Brazil (nir@ccet.ufrn.br, edgar@ccet.ufrn.br).
These considerations have led to the Kronecker-Weierstrass (KW) form under strict equivalence (8, p. 24).

**II) Standard pair theory.** Although complex roots can be defined for higher order matrix polynomials, the associated analysis of "eigenvectors" in terms of state space models became possible only after the emergence of the state space approach in engineering in 1960, due to Kalman [12]. A series of papers in the 1970-80s provided a complete state space analysis for regular matrix polynomials in terms of strong equivalence and reduction to a companion pencil. The eigenvector data and the eigenvalue data are stored in two separate matrices ([X, Y], T ⊕ Z) called standard pair [4] or decomposable pair [16]. Notably, these matrices satisfy the following algebraic relations:

\[
\sum_{i=0}^{k} A_i X T^i = 0, \quad \sum_{i=0}^{k} A_i Y Z^{k-i} = 0
\]

plus a certain controllability condition. The matrix polynomial can be reconstructed in an essentially unique way from this data. See [17], [18], [19], [4], [16], [14], [15].

**III) Factorization.** The description of factorizations of the type \( P(\lambda) = Q(\lambda) R(\lambda) \) are far more difficult, mainly due to the commutativity issue. Only the monic \( 2 \times 2 \) case is fully analyzed [5]. Some results are known for finding a first order right factor, specially in the case of quadratic matrix polynomials, see [9], [16].

**IV) Solvent theory.** Solvent theory has a different, truly non-commutative starting point, where roots are not complex numbers but \( m \times m \) matrices. Formally, considering \( P(\cdot) \) as a left multiplication operator on \( M_m(\mathbb{C}) \), solvents are defined as solutions \( S \in M_m(\mathbb{C}) \) of the matrix equation

\[
\sum_{i=0}^{k} A_i S^i = 0.
\]

The scope of this theory is, to date, quite limited: the classification of solvents is complete only in the essentially monic case (\( n = m \) and \( \det A_k \neq 0 \) [21]. Nevertheless, solvent theory shows close ties with both the standard pair approach and factorization, especially first order right factors.

A similar classification of "cosolvents" is available in the comonic case, based on the "reverse equation"

\[
\sum_{i=0}^{k} A_i S^{k-i} = 0,
\]
leading to a mirror image of cosolvent theory.

The objective of this paper is to integrate the solvent and cosolvent approaches in a non-trivial way to a unique theory which is able to describe all the regular first order right factors of an arbitrary matrix polynomial. We use the concept of bisolvent, related with the algebraic relation

\[ \sum_{i=0}^{k} A_i S_1^i S_2^{k-i} = 0. \]

Clearly, commutativity is a subtle issue here, and so we find it necessary to restrict our scope to commuting pairs \((S_1, S_2)\); as we show below, however, there is absolutely no need to consider non-commuting bisolvents!

In fact, we focus in this paper a special type of commuting bisolvents, which we call "separable". These are, formally, triples \((S_1, S_2, \Pi)\) with \(\Pi \in M_m(\mathbb{C})\) idempotent, which also satisfy

\[
S_1 = \Pi S_1 \Pi + (I - \Pi), \quad S_2 = \Pi + (I - \Pi) S_2 (I - \Pi). \quad (1)
\]

The text is self sufficient, but the first few preliminary sections are quite brief as they mostly summarize existing results.

\section{Strict equivalence and KW theory}

The following basic definitions will be used throughout.

\textbf{Definition 1} \begin{enumerate}
\item[(i)] \(M_{nm}, M_n\) and \(GL_n\) will denote the sets of \(n \times m, n \times n\) and non-singular \(n \times n\) (complex) matrices.
\item[(ii)] A (complex) \(n, m, k\)-polynomial is an expression of the form \(P(\lambda) = \sum_{i=1}^{k} A_i \lambda^i\) with \(A_i \in M_{nm}\) and \(A_0, A_k \neq 0\).
\item[(iii)] \(P(\lambda)\) is called monic if \(A_k = I_n\), comonic if \(A_0 = I_n\), regular if \(n = m\) and \(\det P(\lambda) \neq 0\) (singular otherwise), unimodular if \(\det P(\lambda)\) is a non-zero constant, i.e. having a polynomial inverse.
\item[(iv)] The reverse \(n, m, k\) polynomial is \(\tilde{P}(\lambda) = \lambda^k P(1/\lambda) = \sum_{i=0}^{k} A_i \lambda^{k-i}\).
\end{enumerate}

Although in our discussion we include considerations on singular matrix polynomials the scope of the presented theory here in general is limited to regular matrix polynomials.
A central question is the classification of matrix pencils up to strict equivalence.

**Definition 2**  
(i) Two $n, m, k$ polynomials $P(\lambda)$ and $P'(\lambda)$ are called strictly equivalent if $QP(\lambda) = P'(\lambda)R$ with $Q \in GL_n$ and $R \in GL_m$.  
(ii) An $n, m, 1$ polynomial is called an $n, m$ pencil.  
(iii) A Weierstrass pencil is a direct sum of monic and comonic Jordan pencils.  
(iv) Denote by $R_\kappa(\lambda)$ the matrix pencil formed from the first $\kappa$ rows of the pencil $\lambda I + J$, where $J$ is a single nilpotent Jordan block of size $(\kappa + 1)$. We call $R_\kappa$ (resp. $R_\kappa^T(\lambda)$) a right null block (resp. left null block) of index $\kappa$.  
(v) A KW pencil is a direct sum of the form

$$T(\lambda) = \begin{pmatrix} T'(\lambda) & 0 \\ 0 & 0 \end{pmatrix}, \quad T'(\lambda) = \oplus T_i(\lambda)$$

consisting of four types of units $T_i(\lambda)$: monic and comonic Jordan pencils, right and left null blocks.

The following is the main result of the KW theory.

**Theorem 3**  
(i) Every matrix pencil $C(\lambda)$ is strictly equivalent to a KW pencil $T(\lambda)$.  
(ii) $C(\lambda)$ is regular iff $T(\lambda)$ is Weierstrass. In addition, $P(\lambda)$ is essentially monic, i.e. $\det A_k \neq 0$ (resp. essentially comonic) iff $T(\lambda)$ is monic (resp. comonic).  
(iii) The KW form is unique up to two trivial operations: (i) reordering the blocks; (ii) replacing a monic Jordan block of eigenvalue $\lambda \neq 0$ by a comonic Jordan block of the same size for $1/\lambda$, and vice versa.

To avoid the last source of non-uniqueness, it is commonplace to require that all comonic Jordan blocks be nilpotent; namely, the monic part includes all the finite eigenvalues.

The KW form defines the full set of strict equivalence invariants: the eigenvalues (eigenvalues of monic blocks plus inverse eigenvalues of comonic blocks), their partial multiplicities, right and left null indices.
III  Strong equivalence and linearization

We now discuss the extension of the KW theory to higher order polynomials.

**Definition 4**  
(i) We say that $P(\lambda)$ and $P'(\lambda)$ (of possibly different sizes $n \times m$ and $n' \times m'$) are equivalent if there exist unimodular matrix polynomials $Q(\lambda), R(\lambda)$ and integers $p, q$ so that

$$[P(\lambda) \oplus I_p]Q(\lambda) = R(\lambda)[C(\lambda) \oplus I_q].$$

Clearly, $n - m = n' - m'$.

(ii) $P(\lambda)$ and $P'(\lambda)$ are said to be strongly equivalent ($P \sim P'$) if both the direct pair $P, P'$ and reverse pair $\tilde{P}, \tilde{P}'$ are equivalent.

(iii) $C(\lambda)$ is a linearization for $P(\lambda)$ if $C(\lambda)$ is a matrix pencil and $C \sim P$.

**Theorem 5**  
(i) Strict equivalence implies strong equivalence.

(ii) Two matrix pencils are strongly equivalent iff they have the same eigenvalues and partial multiplicities and the same number (but not necessarily values) of left and right null indices.

(iii) So, for regular pencils strong equivalence implies strict equivalence.

Theorem 5(ii) does not extend to singular pencils. In fact, strongly equivalent singular pencils may have different null indices.

Order reduction and companion forms are used in extending Theorem 5 to higher order. Coefficient-based order reduction is a powerful method in ODEs and engineering applications. For monic matrix polynomials, four monic companion forms are widely used in the engineering literature, which may be described as "up, down, right" and "left" \[11, 2\]; throughout this work we restrict attention to the down and right companion pencils defined by

$$C_d(\lambda; P) = \begin{pmatrix} I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \\ 0 & 0 & \cdots & 0 & A_m \end{pmatrix} \lambda - \begin{pmatrix} 0 & I & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & I \\ -A_0 & -A_1 & \cdots & -A_{m-2} & -A_{m-1} \end{pmatrix},$$

$$C_r(\lambda; P) := C_d(\lambda; P^T)^T.$$
in which the coefficients $A_i$ occupy, respectively, the lowest block row and the rightmost block column. In general the sizes of these two pencils are different: resp, $mk \times [m(k - 1) + n] \times mk$ and $[n(k - 1) + m] \times nk$. Other companion forms, besides these four, are discussed in [6] and [1]. A pencil is considered as its own linearization, i.e. $C_d(\lambda; P) = C_r(\lambda; P) = P(\lambda)$.

**Theorem 6**  
(i) Any matrix polynomial $P(\lambda)$ admits linearizations; for example, all the companion forms including $C_d(\lambda; P)$ and $C_r(\lambda; P)$.

(ii) If the $n, n, k$ polynomial $P(\lambda)$ is regular then all its regular linearizations are of size $nk$, regular, and pairwise strictly equivalent.

(iii) If $P_i(\lambda)$ have equal degree and $C_i(\lambda)$ are linearizations for $P_i(\lambda)$ then $\oplus C_i$ is a linearization for $\oplus P_i$.

Item (iii) may fail for polynomials of different degrees, due to discrepancies at the infinite eigenvalue. For example, a direct sum of monic polynomials of different degrees is not monic, and so does not have a monic linearization.

Item (ii) fails in the singular case, as $P(\lambda)$ admits linearizations of various sizes.

**IV The Jordan chain approach**

In order to encode spectral information efficiently we need to translate the data in matricial rather than modular terms. Namely, using Taylor expansion of root vectors at a neighborhood of a root, we obtain certain chains of ”eigenvectors” and ”generalized eigenvectors”. With this, a direct generalization of the usual concept of Jordan chains of a matrix is obtained. The problem is that the matricial representation is adapted to encode eigenvector data but not null vector data.

**Definition 7**  
(i) A Jordan $\kappa$-chain for $P(\lambda)$ at $a \in \mathbb{C}$ is a sequence $v_1, \ldots, v_\mu$ with $v_1 \neq 0$ so that

$$
\frac{1}{j!} P^{(j-1)}(a)v_1 + \cdots + P(a)v_j = 0 \quad (0 \leq j \leq \kappa - 1).
$$

(ii) A Jordan $\kappa$-cochain at $\lambda = \infty$ for $P(\lambda)$ is a Jordan $\kappa$-chain for $\tilde{P}(\lambda)$ at $\lambda = 0$. 

A Jordan system is a finite set of chains and cochains associated with complex points in which the eigenvectors (initial vectors) in each chain associated with $\lambda = a$ are LI.

In the regular case, this procedure makes sense and it is clear that a Jordan system is limited in size. Indeed, the number of chains at each eigenvalue is limited by the linear independence requirement, and the degree of each chain is limited due to size of respective Jordan block.

**Lemma 8** Let $P(\lambda)$ be regular.

(i) The chain lengths due to a maximal Jordan system coincide with the partial multiplicities at $\lambda$ in the Kronecker form.

(ii) The chain lengths of a non-maximal Jordan system for $P(\lambda)$ are majorized by these.

This will be clarified in the next section.

**V Matricial spectral analysis**

In the regular case both the companion and KW linearizations are of the same size, hence they are strictly equivalent. Thus, a maximal system of Jordan chains can be calculated from the strict equivalence between them. This procedure, developed in the 1970-80s (see [17], [18], [4], [16]) partly goes back to work of Keldyš on ODEs in the 1950s [13], although we find those Taylor’s expansions, ”Jordan chains” (see Roth [23]), in a previous work with solutions for ”unilateral equation in matrices”, that is $\sum_{i=0}^{k} A_i X^i = 0$, for a rectangular matrix $X$. The following analysis using the term ”standard pair” is due to [4], see also [16] where the term ”decomposable pair” is used instead.

When Jordan chains are encoded matricially, some algebraic relations become evident.

**Definition 9** An standard pair for $P(\lambda)$ is a pair $([X, Y], T \oplus Z)$ where

(i) $X \in M_{m,p}$, $T \in M_{p,p}$ satisfy the algebraic relation $\sum_{i=0}^{k} A_i X^i T^i = 0$.

(ii) $Y \in M_{m,q}$, $Z \in M_{q,q}$ satisfy the algebraic relation $\sum_{i=0}^{k} A_i Y Z^{k-i} = 0$. 

(iii) the matrix

\[ Q_k = \begin{pmatrix}
  x & yZ^{k-1} \\
  x^T & yZ^{k-2} \\
  \vdots & \vdots \\
  x^{T_{k-2}} & yZ \\
  x^{T_{k-1}} & y
\end{pmatrix} \in M_{nk,p+q} \]

has full row rank (= p + q).

(iv) We call the pair Jordan if \( T \oplus Z \) is in Jordan form.

The matrix \( Q_k \) is sometimes called controllable matrix, this is due to the fact that a pair satisfy item (iii) above if and only the matrix \( Q_k \) constructed with any \( j > 1 \) instead of \( k \), is of full rank, such pair is called controllable.

We observe that if the dimension of \( Z \) is zero \( (q = 0) \), then \( (X, T) \) is a standard pair for \( P(\lambda) \), conversely if the dimension of \( T \) is zero \( (p = 0) \), then \( (Y, Z) \) is a standard pair for \( P(\lambda) \). Furthermore if \( (X, T) \) is a standard pair for \( P(\lambda) \) with \( T \) being nonsingular then \( (X, T^{-1}) \) is a standard pair for \( \tilde{P}(\lambda) \), equivalent conclusions are obtained if \( Z \) or both \( T \) and \( Z \) are nonsingular. In the case that the pair \( ([X, Y], T \oplus Z) \) is Jordan, with \( T \oplus Z \) being nonsingular and \( (T \oplus Z)^{-1} = R(\tilde{T} \oplus \tilde{Z})R^{-1} \), where \( \tilde{T} \oplus \tilde{Z} \) is also in Jordan form, then \( ([\tilde{X}, \tilde{Y}], (\tilde{T} \oplus \tilde{Z}) \) is a standard pair for \( \tilde{P}(\lambda) \), where \( \tilde{X} = XR \) and \( \tilde{Y} = YR \).

**Example 1.** The special case of a first order monic polynomial is suggestive. An standard pair \( (v, a) \) for an \( n \times n \) matrix \( A \) induces a Jordan pair \( (X = v, J = a) \) for \( P(\lambda) = \lambda I - A \). The algebraic relation in question reduces to \( Av - Iv\lambda = 0 \).

Similarity for standard pairs is defined, quite naturally, in terms of the gauge transformation

\[
([X_1, X_2], T_1 \oplus T_2) \rightarrow ([X_1 G, X_2 H], G^{-1}T_1G \oplus H^{-1}T_2H) \quad (G \in GL_p, H \in GL_q),
\]

and it is clear that the entire similarity orbit of an standard pair for \( P(\lambda) \) consists of standard pairs. However, there exists an additional degree of freedom not present in the monic or comonic cases: spectral inversion.

**Definition 10** Spectral inversion of a standard pair \( ([X_1, X_2], T_1 \oplus T_2) \) is defined as follows: assume that \( X_1 = [X_1', X_2'] \) and \( T_1 = T_1' \oplus T_2' \) with \( T_2' \) non-singular. Then spectral inversion replaces \( ([X_1, X_2], T_1 \oplus T_2) \) by \( ([X_1', [X_2', X_2]], T_1' \oplus (T_2'^{-1} \oplus T_2)) \). Namely, \( p \) is decreased and \( q \) is increased, and spectral content of \( (X_1, T_1) \) is transferred to \( (X_2, T_2) \).
The inverse operation considered before is also a spectral inversion. The zero eigenvalues of $T$ and $Z$ do not suffer spectral inversion.

Large standard pairs can be constructed from smaller ones by merging:

$$X = [X_1, \ldots, X_l], \quad T = \oplus_{i=1}^l T_i, \quad Y = [Y_1, \ldots, Y_l], \quad Z = \oplus_{i=1}^l Z_i.$$ 

These maximal standard pairs correspond to the maximal Jordan systems discussed in the last section.

If we require that $Z$ be nilpotent, this means that $T$ will have all the information about the finite spectra and $Z$ all the information of the infinite spectra of $P(\lambda)$.

**Theorem 11**

(i) Every regular $n, k P(\lambda)$ admits a standard pair of size $nk$.

(ii) $[X, Y], T \oplus Z$ defines $P(\lambda)$ via an equivalence relation with its companion pencil $C_d(\lambda; P)$. In particular, the finite eigenvalues, and respective partial multiplicities, of $P(\lambda)$ match with the union of eigenvalues of $T$ and inverse eigenvalues of $Z$, and the infinite eigenvalue, with respective partial multiplicities, of $P(\lambda)$ matches with zero eigenvalue of $Z$.

(iii) $P(\lambda)$ determines uniquely $([X, Y], T \oplus Z)$ up to similarity and spectral inversion (in particular, assuming $Z$ nilpotent, up to similarity).

(iv) $([X, Y], T \oplus Z)$ is a Jordan pair (i.e. $T \oplus Z$ is Jordan) iff $[X, Y]$ defines a complete system of Jordan chains for $P(\lambda)$.

Indeed, consider the strict equivalence $C_d(\lambda; P)Q = RT(\lambda)$ between the companion $C_d(\lambda; P)$ and any separable linearization $T(\lambda) = (\lambda I - T) \oplus (\lambda Z - I)$ (e.g. the Weierstrass form). By direct calculation $R, Q$ must have the form

$$Q = \begin{pmatrix} x & yZ^{m-1} \\ xT & yZ^{m-2} \\ \vdots & \vdots \\ xT^{m-2} & yZ^{m-1} \\ xT^{m-1} & y \end{pmatrix}, \quad R = \begin{pmatrix} x & yZ^{m-2} \\ xS & yZ^{m-3} \\ \vdots & \vdots \\ xS^{m-2} & y \\ A_mXS^{m-1} & -S^{m-1}A_iYZ^{m-1-i} \end{pmatrix} \quad (3)$$

for some data

$$X \in M_{m,p}, \quad Y \in M_{m,q}, \quad T \in M_p, \quad Z \in M_q \quad (p, q \in \mathbb{N}).$$

In particular, if $T, Z$ are Jordan matrices, columns of $X, Y$ form a complete set of Jordan chains for $P(\lambda)$. \hfill \square
Let $\gamma$ be a simple closed Jordan curve system in $\mathbb{C}$. If we seek only standard pairs separated by $\gamma$, i.e. eigenvalues of $T$ are in $int(\gamma)$ and inverse eigenvalues of $Z$ are in $ext(\gamma)$, then spectral inversion is not an option and the uniqueness is up to similarity only. For example, standard pairs with $Z$ nilpotent.

Clearly, in Theorem 11 $P(\lambda)$ is essentially monic (resp. essentially comonic) iff (up to spectral inversion) $q = 0$ (resp. $p = 0$).

**Example 2.** Consider the matrix $A$ in Example 1. If $J$ is the Jordan form of $A$, $P(\lambda) = \lambda I - A$ has the monic Weierstrass pencil $\lambda I - J$ and the monic companion $\lambda I - A$.

Let $X$ be the similarity matrix between the two pencils, $\lambda(I - A)X = X(\lambda I - J)$. As is well known, the columns of $X$ provide a complete set of Jordan chains for $A$. Thus, $(X, J)$ is a standard pair. Here $X = Q_1$ serves as a controllability matrix.

### VI Solvent theory

An $m \times m$ matrix is a solvent (resp. cosolvent) for $P(\lambda) = \sum_{i=0}^{k} A_i \lambda^i$ if it satisfies $\sum_{i=0}^{k} A_i S^i = 0$ (resp. $\sum_{i=0}^{k} A_i S^{k-i} = 0$). Solvents and cosolvents help describe right first order factors of $P(\lambda)$ which are, respectively, monic and comonic.

The solvents can be obtained from a standard pair $([X, Y], T \oplus Z)$ when $Z$ is nilpotent, that is, $T$ is similar to the finite part and $Z$ is similar to the infinite part of the canonical (Weierstrass) form of $P(\lambda)$, in such case for the sake of simplicity we will refer $T$ as being the finite part and $Z$ as being the infinite part of the standard pair.

**Theorem 12** (i) All the solvents $S$ of $P(\lambda)$ can be constructed from a maximal controllable standard pair $([X, Y], T \oplus Z)$, with $Z$ being nilpotent, by a restriction to an $m$-dimensional $T$-invariant subspace. Namely, $S = \tilde{X} \tilde{T} \tilde{X}^{-1}$ where $(\tilde{X}, \tilde{T})$ are the restrictions of $(X, T)$ to the subspace.

(ii) In particular, the spectrum of $S$ is subordinate to that of $T$.

(iii) $\lambda I - S$ is a right monic factor iff $S$ is a solvent.

The conditions above can be stated in a generalization of eigenpairs (see [21]), where the necessary and sufficient conditions for the existence of a solvent are in terms of the Jordan chains of $P(\lambda)$ and the partial multiplicities of the respective eigenvalues.
The number of solvents will depend on the different pairs \((\tilde{X}, \tilde{T})\) that can be obtained in the theorem above from \((X, T)\) or equivalent with the different nonsingular matrices constructed with the leading vectors of the Jordan chains associated with the finite eigenvalues. This number can be zero, a finite number or infinitely many.

**Example 3.** Consider the following polynomial \(P(\lambda) = A_1\lambda + A_0\), where \(A_1\) is singular and \(A_0\) is nonsingular. It is immediate that \(P(\lambda)\) has no solvents. On the other hand, it is more complicated to construct a polynomial with no solvents for the monic case (see [21]).

We recall that if the \(\text{deg}(\det(P(\lambda))) = M\), then \(M\) is the number of finite eigenvalues, including multiplicities. Furthermore, if \(M = mn\), \(P(\lambda)\) has only finite eigenvalues, otherwise if \(M < mn\), \(P(\lambda)\) has infinite eigenvalues, including the multiplicities.

Thus, the maximum finite number of solvents in the case of \(\text{deg}(\det(P(\lambda))) = M\) will be \({M\choose n}\), and it occurs when \(P(\lambda)\) has \(M\) eigenvectors that are linearly independent \(n\) by \(n\).

Infinitely many solvents occur when there exists a solvent \(X_1\) with an eigenvalue having the geometric multiplicity less in \(X_1\) than in \(P(\lambda)\).

A similar complete classification of "cosolvents" is based on the "reverse equation"

\[
\sum_{i=0}^{k} A_i S^{k-i},
\]

leading to a mirror image of cosolvent theory:

**Theorem 13**  
(i) All the cosolvents \(S\) of \(P(\lambda)\) can be constructed from a maximal controllable standard pair \([X, Y], T \oplus Z\), with \(T\) being nilpotent, by restriction to an \(m\)-dimensional \(Z\)-invariant subspace, in the form \(S = \tilde{Y} \tilde{Z} \tilde{Y}^{-1}\).

(ii) In particular, the spectrum of \(S\) is subordinate to that of \(Z\).

(iii) \(I - \lambda S\) is a right comonic factor iff \(S\) is a cosolvent.

The relation between solvents and cosolvents is a direct consequence from their constructions. So, we have that if \(S\) is a nonsingular solvent of \(P(\lambda)\), then \(S^{-1}\) is a cosolvent.
of $P(\lambda)$. Moreover, with the same arguments used before, we can estimate the number of cosolvents and also of "infinite solvents", that is nilpotent solvents of $\tilde{P}(\lambda)$.

**Example 4.** Consider a quadratic ($k = 2$) matrix polynomial $P(\lambda) = A_2\lambda^2 + A_1\lambda + A_0$, where

$$
A_2 = \begin{bmatrix} 0 & 0 \\ -2 & 2 \end{bmatrix}, A_1 = \begin{bmatrix} 1/7 & -1/7 \\ 27/7 & -41/7 \end{bmatrix}, A_0 = \begin{bmatrix} 2/7 & -2/7 \\ 174/7 & -146/7 \end{bmatrix}
$$

are matrices of order 2 ($n = 2$).

We have that $([X,Y], T \oplus Z)$, with

$$
X = \begin{bmatrix} 1 & 1 \\ 1 & 8 \end{bmatrix}, T = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}, Y = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, Z = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
$$

is a maximal standard pair for $P(\lambda)$, with $Z$ nilpotent. $P(\lambda)$ has only one solvent

$$
S_1 = XTX^{-1} = \begin{bmatrix} 18/7 & -4/7 \\ 32/7 & -18/7 \end{bmatrix}.
$$

We can compute the cosolvents performing an spectral inversion, $(Y', Z')$, with

$$
Y' = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 8 & 1 & 2 \end{bmatrix}, Z' = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

Therefore, with the leading vectors of the Jordan chains in $Y'$ we can construct 3 cosolvents

$$
\tilde{S}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 9/14 & -1/7 \\ 8/7 & -9/14 \end{bmatrix},
$$

$$
\tilde{S}_2 = \begin{bmatrix} 1 & 1 \\ 8 & 1 \end{bmatrix} \begin{bmatrix} -1/2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/14 & -1/14 \\ 4/7 & -4/7 \end{bmatrix}
$$

and
\[ \tilde{S}_3 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \]

the cosolvent \( \tilde{S}_3 \) is nilpotent and so it is an "infinite solvent" of \( P(\lambda) \).

VII Bisolvents and regular right factors

Observe that a regular matrix polynomial may have regular first order factors without admitting any solvent or cosolvent. We now extend solvent theory so as to yield regular right factors in some of such cases.

**Definition 14**  
(i) A bisolvent for the \( m \times m \) matrix \( P(\lambda) \) is a pair \( (S_1, S_2) \) of commuting \( m \times m \) matrices which satisfy

\[ (ii) \sum_{i=0}^{k} A_i S_1 S_2^{k-i} = 0. \]  

\[ (6) \]

(iii) We call the bisolvent separable if for some \( m \times m \) idempotent \( \Pi \) we have

\[ S_1 = \Pi S_1 \Pi + (I - \Pi), \quad S_2 = \Pi + (I - \Pi) S_2 (I - \Pi). \]  

\[ (7) \]

\((S_1, S_2, \Pi)\) is a solvent if \( \Pi = I \) and cosolvent if \( \Pi = 0 \).

Observe that, any pair \((S_1, S_2)\) such that \( S_1 S_2 = 0 \) and both \( S_1 \) and \( S_2 \) are nilpotent with an index \((k - 1)\) satisfies item (ii) of this definition, in special \((S_1 = 0, S_2 = 0)\).

We are interested in bisolvents that fulfill the item (iii), we will see that in this case \( S(\lambda) = \lambda S_2 - S_1 \) has spectral information of \( P(\lambda) \) and are right divisors.

Observe that, in a separable bisolvent, \( S_1 \) and \( S_2 \) normally fail to be a solvent or a cosolvent. Nevertheless, solvents and cosolvents can be viewed as special cases: \( \Pi = I \) and \( S_2 = I \) for solvents, \( \Pi = 0 \) and \( S_1 = I \) for cosolvents.

As we shall see, the commutativity assumption does not limit the scope of bisolvent theory. The separability hypothesis implies that the pencil \( \lambda S_1 + S_2 \) is commutative and regular.
The special case $P(\lambda) = \lambda I_n - A$ is instructive. Here, the roots of $P(\lambda)$ coincide with the eigenvalues of $A$, and the corresponding Jordan basis of "eigenvectors" for $P(\lambda)$ are the actual (nominal or generalized) eigenvectors of $A$. Let $J$ be the Jordan form for $A$ and let $X$ be the similarity matrix between them: $AX = XJ$. Then $(X, J)$ is a standard pair for $P(\lambda)$. Another one (the companion standard pair) is $(I, A)$. The various standard pairs form a single similarity orbit through the gauge transformation $(X, T) \rightarrow (XG, G^{-1}TG)$.

So, in this special case, $P(\lambda)$ admits many standard pairs but only a single solvent, viz. $A$. The proceeding in Theorem 12 (i) for producing the solvent from a standard pair reduces to the equation $A = XTX^{-1}$.

The following procedure defines potential separable bisolvents for $P(\lambda)$, and at the same time, potential right divisors.

Let $([X, Y], T \oplus Z)$ be any sufficiently large standard pair for $P(\lambda)$. Choose an $m$-dimensional $T \oplus Z$-invariant subspace $V_1 \oplus V_2$ and let $(\tilde{X}, \tilde{Y}, \tilde{T}, \tilde{Z})$ be the corresponding restriction:

$$
\tilde{X} = X|_{V_1}, \quad \tilde{Y} = Y|_{V_2}, \quad \tilde{T} = P_{V_1}T|_{V_1}, \quad \tilde{Z} = P_{V_2}Z|_{V_2}.
$$

By specifying bases in $V_1$ and $V_2$ we may consider both $Q_0 = [\tilde{X}, \tilde{Y}]$ and $\tilde{T} \oplus \tilde{Z}$ as $m \times m$ matrices. In the same basis, $I \oplus 0$ represents an idempotent, i.e. the projection onto $V_1$ and along $V_2$. Now, assuming $Q_0$ non-singular, define

$$
S_1 = \tilde{X}(\tilde{T}_1 \oplus I)\tilde{X}^{-1}, \quad S_2 = \tilde{X}(I \oplus \tilde{T}_2)\tilde{X}^{-1}, \quad \Pi = \tilde{X}(I \oplus 0)\tilde{X}^{-1}.
$$

This shows that indeed we have a separable bisolvent, and a right factor, which can be enunciate as follows.

**Lemma 15** As long as $Q_0$ is invertible, $(S_1, S_2)$ is a separable bisolvent and $S(\lambda) = S_1\lambda - S_2$ is a regular first order right factor.

**Proof.**

$$
\sum_{i=0}^{k} A_iS_1^iS_2^{k-i} = [\sum_{i=0}^{k} A_i\tilde{X}T^i] [\sum_{i=0}^{k} A_i\tilde{Y}Z^{k-i}]Q_0^{-1} = [\sum_{i=0}^{k} A_iXT^i|_{V_1}] [\sum_{i=0}^{k} A_iYZ^{k-i}|_{V_2}]Q_0^{-1},
$$

and the RHS vanishes since $([X, Y], T \oplus Z)$ is a standard pair. Thus, $(S_1, S_2)$ is a bisolvent, and clearly with $\Pi = I_p \oplus 0_{m-p}$, for a suitable $p$, it is separable.
The following results show that separable bisolvent theory is a full-fledged generalization of solvent/cosolvent theories, and an adequate tool in handling the regular case:

**Theorem 16**  
(i) All the separable bisolvents $S$ of $P(\lambda)$ can be constructed from a "maximal controllable bistandard pair" $([X,Y], T \oplus Z)$ by restriction to an $m$-dimensional $T \oplus Z$-invariant subspace. Namely,

$$S_1\lambda - S_2 = [\tilde{X}, \tilde{Y}][(I\lambda - \tilde{T}) \oplus (I - \lambda\tilde{Z})][\tilde{X}, \tilde{Y}]^{-1}$$

where $([\tilde{X}, \tilde{Y}], \tilde{T} \oplus \tilde{Z})$ are the restrictions of $([X,Y], T \oplus Z)$ to the subspace.

(ii) In particular, the KW spectrum of $S_1\lambda - S_2$ is subordinate to that of $(I\lambda - T) \oplus (I - Z\lambda)$.

(iii) $\lambda A + B$ is a right regular factor iff $(S_1, S_2) = (XA, XB)$ is a separable bisolvent for some $X \in GL_m$. In other words, separable bisolvents describe all the right factors up to left equivalence!

Item (iii) is central in establishing the natural status of bisolvent theory in general factorization procedures. It also implies a simplification in the calculation of the left factor $Q(\lambda) = P(\lambda)(S_1\lambda - S_2)^{-1}$ for a given right factor.

Important conclusions can be derived from previous results, suppose that $P(\lambda)$ is a regular matrix polynomial, let $B, BS$ and $F$ be the respective sets of bisolvents, separable bisolvents and the regular first order right factors, so $BS \subset B, BS \subset F$ and moreover $BS = B \cap F$.

**Example 5.** We demonstrate a subtlety which arises due to Theorem 16 (ii) (spectral inversion). Consider the matrix polynomial

$$P(\lambda) = \lambda \oplus (\lambda - 1) \oplus 1 = \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and with it two non-similar standard pairs:

$$X = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad T = 0 \oplus 1, \quad Z = 0$$

$$\hat{X} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \hat{Y} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \hat{T} = 0, \quad \hat{Z} = 1 \oplus 0$$

related via a simple spectral inversion of the eigenvalue $\lambda = 1$. $P(\lambda)$ admits the bisolvent, $S_1 = 0 \oplus 1 \oplus 1, S_2 = 1 \oplus 1 \oplus 0$, which is separable w.r.t. two distinct idempotents,
\[ \Pi = 1 \oplus 0 \oplus 0 \] and \[ \hat{\Pi} = 1 \oplus 1 \oplus 0 \]. It can be easily verified that \((S_1, S_2, \Pi)\) (resp. \((S_1, S_2, \hat{\Pi})\)) is a restriction of \((X, Y, T, Z)\) (resp. \((\hat{X}, \hat{Y}, \hat{T}, \hat{Z})\)). In other words, spectral inversion does not affect the bisolvent but may affect its separating idempotent. \(\square\)

The existence of a non-nilpotent solvent \(S\) always leads a separable bisolvent with a spectral inversion, with \(\lambda S_1 - S_2 = (\lambda I - S)Q\), with an appropriated \(Q\), the same can be concluded with a cosolvent. Furthermore we have.

**Theorem 17**  
(i) Let \((S_1, S_2)\) be a separable bisolvent if \(S_1\) is nonsingular, then \(S = S_1^{-1} S_2\) is a solvent and if \(S_2\) is nonsingular, then \(\tilde{S} = S_2^{-1} S_1\) is a cosolvent.

(ii) In particular, if \((P(\lambda))\) has no zero nor infinite eigenvalues, then \((S_1, S_2)\) is a separable bisolvent if and only if \(S = S_1^{-1} S_2\) is a solvent and if and only if \(\tilde{S} = S_2^{-1} S_1\) is a cosolvent.

In one word, in Item (ii) if \(P(\lambda)\) has no solvents it will have no cosolvents nor separable bisolvents, anyway this condition is very restrictive, so we can conclude that the importance of bisolvents appears when a zero eigenvalue or (and) an infinite eigenvalue exist. We can suppose without loss of generality that in Theorem \(\text{[10]}\) \(Z\) is nilpotent, then it will be clear that we can have separable bisolvents and therefore right regular first order factors, with no solvents or (and) cosolvents at all.

**Example 6.** Consider a quadratic \((k = 2)\) matrix polynomial \(P(\lambda) = A_2 \lambda^2 + A_1 \lambda + A_0\), where

\[
A_2 = \begin{bmatrix}
-33/10 & 3/2 & 8/5 \\
-16/5 & 1 & 7/5 \\
19/10 & -1/2 & -4/5
\end{bmatrix}, \quad
A_1 = \begin{bmatrix}
-788/215 & 11/43 & 281/215 \\
931/215 & -22/43 & -347/215 \\
1097/215 & -134/43 & -589/215
\end{bmatrix}, \quad (11)
\]

\[
A_0 = \begin{bmatrix}
-60/43 & 76/43 & 74/43 \\
30/43 & -81/43 & 6/43 \\
-36/43 & 80/43 & 10/43
\end{bmatrix}, \quad (12)
\]

are matrices of order 2 \((n = 3)\).

We have that \(([X, Y], T \oplus Z)\), with

\[
X = \begin{bmatrix}
2 & 2 & 5 \\
4 & 4 & 2 \\
1 & 1 & 2
\end{bmatrix}, \quad
T = \begin{bmatrix}
3 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad
Y = \begin{bmatrix}
1 & 1 & 3 \\
-1 & -1 & 1 \\
3 & 3 & 4
\end{bmatrix}, \quad
Z = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]
is a maximal standard pair for $P(\lambda)$, with $Z$ nilpotent. We can not construct any nonsingular matrix of size $n$ with the leading vectors of $X$ nor $Z$, therefore $P(\lambda)$ has no solvents nor cosolvents. Besides that, with the standard pair above we get two separable bisolvents: $(S_1, S_2)$ and $(S'_1, S_2)$, where $S_1 = U_1J_1U_1^{-1}$, $S_2 = U_1J_2U_1^{-1}$ and $S'_1 = U_1J_3U_1^{-1}$, in which

\[
U_1 = \begin{bmatrix}
2 & 5 & 1 \\
4 & 2 & -1 \\
1 & 2 & 3
\end{bmatrix},
J_1 = \begin{bmatrix}
3 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix},
J_2 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix},
J_3 = \begin{bmatrix}
2 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

VIII Bisolvents: the additive formalism

Separable bisolvents can also be described using an equivalent additive formalism.

**Definition 18** An additive (separable) bisolvent is a triple of $m \times m$ matrices $(P_1, P_2, \Pi)$ with $\Pi$ idempotent, which satisfies

\[
(i) \quad \sum_{i=0}^{k} A_i (P_1^i + P_2^{k-i}) = 0 \quad (13)
\]

\[
(ii) \quad P_1 = \Pi T_1 \Pi, \quad P_2 = (I - \Pi)T_2(I - \Pi).
\]

The two separable theories, multiplicative and additive, are completely equivalent. Indeed, given $P_1, P_2, \Pi$ we define $S_1, S_2$ via

\[
S_1 = P_1 + (I - \Pi), \quad S_2 = P_2 + \Pi.
\]

Conversely, given $S_1, S_2, \Pi$ we define

\[
P_1 = \Pi S_1 \Pi, \quad P_2 = (I - \Pi)S_2(I - \Pi).
\]

In both cases we have $P_1^i + P_2^j = S_1^i S_2^j$ holds for all $i, j \geq 0$.

Both bisolvent pairs are commutative. In the additive formalism, in fact, $P_1 P_2 = P_2 P_1 = 0$.

IX Conclusions

The new bisolvent formulation can be an important contribution to the factorization theory on regular matrix polynomials permitting the description, up to equivalence, of all right regular first order factors in contrast with solvents/cosolvents which restrict it to monic/comonic factors.
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