Domain wall fermions in a waveguide:
the phase diagram at large
Yukawa coupling

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ABSTRACT: In this paper we return to a model with domain wall fermions in a waveguide. This model contains a Yukawa coupling $y$ which is needed for gauge invariance. A previous paper left the analysis for large values of this coupling incomplete. We fill the gap by developing a systematic expansion suitable for large $y$, and using this, we gain an analytic understanding of the phase diagram and fermion spectrum. We find that in a sense all the species doublers come back for large $y$. The conclusion that no lattice chiral gauge theory can be obtained from this approach therefore remains valid.

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1. Introduction

A recent proposal to obtain $d = 2n$ dimensional lattice chiral fermions from a domain wall in $d + 1$ dimensions [1] has generated a lot of interest. The basic observation is that if one introduces a domain wall in an odd dimensional theory of a Dirac fermion by introducing a mass parameter which changes sign across a hyperplane of codimension one (the “domain wall”), massless states occur that are bound to this domain wall. These states therefore can be interpreted as to live in one dimension less, if one takes the size of the mass parameter to be of the order of the cutoff (the spread of the bound state wavefunctions is of order of the inverse mass in the direction orthogonal to the hyperplane). Moreover, these states carry a definite chirality, which depends on the overall sign of the mass parameter.

Kaplan realized that the fact that the odd dimensional Dirac theory is entirely vector-like in nature, without any chiral symmetry appearing at the level of the action, can be used to implement this idea on the lattice. Because the action is vectorlike, a Wilson mass term can be introduced without breaking any symmetries of the action. This Wilson term eliminates the well known species doublers that are generated on the lattice. If we denote the coordinates of the $d$ dimensional space by $x$, and that of the extra $d + 1$-st coordinate by $s$, a local (i.e. single site) mass term is added which is positive for $s > 0$ and negative for $s < 0$. Kaplan showed that in this way the above described phenomenon carries over to the lattice, and that a chiral fermion emerges bound to the domain wall defect located at $s = 0$.

In a sense the theory of a massless chiral fermion constructed in this way is not entirely free of species doublers. If we consider the theory in a finite but large volume with (anti)periodic boundary conditions, an antidomain wall defect has to occur somewhere, and a chiral fermion with chirality opposite to that of the massless fermion bound to the domain wall occurs at the antidomain wall. In a finite volume the wavefunctions of these
two modes have a small overlap (approximately exponentially suppressed as $\exp(-mL)$, where $m$ is the absolute value of the mass parameter, and $L$ is the distance between the domain wall and the antidomain wall), and a tiny Dirac mass couples the two modes, which however become massless and decoupled in the infinite volume limit. This phenomenon was shown to be quite general (i.e. independent of the precise choice of boundary conditions) by Narayanan and Neuberger [2]. An alternate, equivalent construction employing free boundary conditions on a odd dimensional half space was proposed and discussed by one of us [3].

This theory was first studied in the presence of background gauge fields in order to assess the role of the anomaly. In the original paper [1], it was argued that a Goldstone–Wilczek current flowing off the domain wall should exist to reconcile the apparent conflict between the anomalous structure of a single even dimensional chiral fermion and the exact gauge invariance of the odd dimensional theory, much like the mechanism discussed by Callan and Harvey in the continuum case [4]. This was indeed confirmed in a series of papers [5,6,7]. In particular, it was shown in ref. [6] that a Goldstone–Wilczek current is generated in the region where the combined mass matrix (consisting of the local domain wall mass and the Wilson mass terms) is not positive definite, carrying anomalous charge between the domain wall and the antidomain wall in a way consistent with the chiral zeromode spectrum at those defects [8].

The situation becomes more complicated when one considers dynamical gauge fields. Obviously, one would like to introduce gauge fields that do not couple both to the zeromodes living at the antidomain wall and also to those at the domain wall, since this would render the resulting gauge theory vectorlike. The task is therefore to couple gauge fields in such a way that they do not couple to the fermion modes living at the antidomain wall. A concrete implementation has been proposed in which the gauge fields are confined to a “waveguide” around the domain wall, excluding a region around the antidomain wall
Within this waveguide, the gauge fields are taken to be $d$ dimensional, i.e. the gauge fields are independent of the direction orthogonal to the waveguide (the $s$ coordinate), and all link variables in this extra direction are equal to one [2,9]. (For a different approach, where the extent of space in the $s$-direction is kept strictly infinite, see refs. [11]. See also ref. [12].)

By introducing a waveguide, new defects are necessarily introduced into the theory, consisting of the boundaries of the waveguide. These are $d$ dimensional hyperplanes situated somewhere “halfway” between the domain wall and the antdomain wall. With the specific choice of gauge fields as described above, gauge invariance is explicitly broken at these boundaries. This is remedied by introducing a $d$ dimensional scalar field that lives only at the two boundaries, restoring gauge invariance in a way similar to the way fermion mass terms are made gauge invariant in the electroweak standard model. The Yukawa coupling between this scalar field and the fermion field is a free parameter of the model. It is important to point out that within the context of the waveguide model, such scalar fields are unavoidable. They are essentially the gauge degrees of freedom, which, due to the explicit breaking of gauge invariance, couple to the fermions. Moreover, their dynamics is necessarily nonperturbative, since their fluctuations are not controlled by a small parameter (such as the gauge coupling, which only controls the transverse degrees of freedom) [13,14].

The key question is now of course whether these new defects will lead to new fermionic zeromodes. Any such new zeromodes bound to the inside of a waveguide boundary will couple to the gauge field, and presumably destroy the chiral nature of the theory. In fact, by confining the gauge field to the waveguide, no Goldstone–Wilczek current can reach the antdomain wall anymore, and one may legitimately worry whether new chiral zeromodes are needed at the waveguide boundaries in order to absorb the anomalous charge carried by the Goldstone–Wilczek current between the domain wall and the boundary in the region
with nonpositive definite fermion mass matrix. However, it was argued in ref. [10] (which from now on we will refer to as $I$), that this is not necessarily the case, and that the existence or nonexistence of new fermionic zeromodes at the waveguide boundaries is a dynamical question.

In order to answer this question one needs to map out the phase diagram and spectrum of the theory. This was attempted in $I$, in which the waveguide model without gauge fields, but with the scalar field, was studied analytically and numerically as a function of the Yukawa coupling $y$ and the scalar hopping parameter $\kappa$ (the scalar field was taken to be radially frozen). It was found that always new zeromodes appear on the waveguide boundary in the region of nonpositive definite mass matrix, providing “mirror” modes to the chiral modes living on the domain wall in a one-to-one correspondence.

However, as described in $I$, the numerical simulations were notoriously difficult for large values of the Yukawa coupling, due to a bad signal to noise ratio for the fermion propagators in that region, and it was hard to arrive at any definite conclusion for large $y$. In the present paper we return to the question of the fermionic zeromode spectrum at the waveguide boundaries for large Yukawa coupling. It is well known that in general new phases occur for large values of the Yukawa coupling of lattice scalar-fermion models. Those phases can be distinguished from those at small values of the Yukawa coupling by the fermionic zeromode spectrum. In particular, it is in principle possible that no zeromodes occur at all at large $y$. This would be an important ingredient for constructing a genuinely chiral lattice gauge theory from domain wall fermions.

Here, we develop an expansion suitable for investigating the large $y$ region of the phase diagram. Again, we set the gauge field link variables equal to one, but keep the scalar field at the waveguide boundary. This allows us to find the zeromode spectrum for large $y$. We do find a new symmetric phase which is not connected to the symmetric phase at small values of $y$. However, the zeromode spectrum in fact turns out to be very rich, and
quite similar to that of the Smit-Swift [13,15,16] model for weak Yukawa coupling: all the doublers of the zeromode at the domain wall reappear at the waveguide boundaries!

In section 2 we define the waveguide model restricting ourselves to the case $d = 4$, and recast it in a way amenable to an expansion in $1/y$. In the next section we find the zeromode spectrum for $y = \infty$, and in section 4 we estimate the location of the phase transition between the symmetric and broken phases in the $\kappa-y$ plane for large $y$. We then briefly consider the zeromode spectrum at large but finite $y$. In the last section we present our conclusions.

2. The model

The waveguide implementation of the domain wall fermion model is defined by the action

$$S_{\text{domainwall}} = \sum_{s \in WG} \Psi_s (\bar{\Psi} (U) - W(U) + m_s) \Psi_s + \sum_{s \notin WG} \Psi_s (\bar{\Psi} - w + m_s) \Psi_s$$

$$+ \sum_{s \neq s'_0 - 1, s_0} \left[ \bar{\Psi}_s P_R \Psi_{s+1} + \bar{\Psi}_{s+1} P_L \Psi_s \right] - \sum_s \bar{\Psi}_s \Psi_s$$

$$+ y(\bar{\Psi}_{s'_0 - 1} V^\dagger P_R \Psi_{s'_0} + \bar{\Psi}_{s'_0} V P_L \Psi_{s'_0 - 1}) + y(\bar{\Psi}_{s_0} V P_R \Psi_{s_0 + 1} + \bar{\Psi}_{s_0 + 1} V^\dagger P_L \Psi_{s_0}).$$

(1)

In this equation we have suppressed the four dimensional coordinates, and only indicated the fifth coordinate explicitly. $\partial_{\mu}$ is the four dimensional symmetric nearest neighbor difference operator, and $w$ denotes the four dimensional Wilson mass term. $D_{\mu}(U)$ and $W(U)$ are gauge covariant versions of these. $m_s$ is the domain wall mass, which is taken to be positive for $s > 0$ and negative for $s < 0$. (We can take $m(0) = 0$ for instance.) $V$ is a radially frozen scalar field which takes its values in the gauge group $G$. The waveguide $WG$ consists of all lattice points with $s'_0 \leq s \leq s_0$, and the domain wall at $s = 0$ is inside the waveguide (i.e. $s'_0 < 0$ and $s_0 > 0$). With periodic boundary conditions, there will be an antidomain wall outside the waveguide, where $m_s$ again changes sign. We have chosen the
Wilson parameter $r = -1$, and the projection operators $P_R(L)$ are $P_R(L) = \frac{1}{2}(1 + (-)\gamma_5)$. For full details, see I. (Here we have chosen some conventions different from those in I, where for instance we chose $r = 1$.) This model contains a righthanded massless fermion bound to the domain wall and a lefthanded massless fermion bound to the antidomain wall, irrespective of the value of $y$ (cf. I).

In the present paper we are interested in the fermionic zeromode spectrum at the waveguide boundaries, and we will restrict ourselves to a simpler model in which only the waveguide boundary between $s = s_0$ and $s = s_0 + 1$ is present, with no other defects. We imagine all other defects to be very far away. Also, as in I, we will choose $U = 1$, i.e. we will turn off the gauge fields. Including a hopping term for the scalar field, the action for this situation is

$$S = S_F + S_B,$$

with

$$S_F = \sum_{x,y,s} \left( \overline{\Psi}_{x,s} \partial_{xy} \Psi_{y,s} + \frac{1}{2} \overline{\Psi}_{x,s} \Box_{xy} \Psi_{y,s} + m \overline{\Psi}_{x,s} \Psi_{x,s} \right) + \sum_{x,s \neq s_0} \left( \overline{\Psi}_{x,s} P_R \Psi_{x,s+1} + \overline{\Psi}_{x,s+1} P_L \Psi_{x,s} \right) - \sum_{x,s} \overline{\Psi}_{x,s} \Psi_{x,s} + y \sum_{x} \left( \overline{\Psi}_{x,s_0} V P_R \Psi_{x,s_0+1} + \overline{\Psi}_{x,s_0+1} V^\dagger P_L \Psi_{x,s_0} \right)$$

and

$$S_B = -\kappa \sum_{x,\mu} \text{tr}(V_{x} V^\dagger_{x+\mu} + h.c.),$$

where

$$\partial_{xy} = \frac{1}{2} \sum_{\mu} \gamma_\mu (\delta_{x+\mu,y} - \delta_{x-\mu,y}),$$

$$\Box_{xy} = \sum_{\mu} (\delta_{x+\mu,y} + \delta_{x-\mu,y} - 2 \delta_{x,y}).$$

We will always choose $0 < m < 1$. 

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This action is invariant under the transformations
\[
\Psi_{x,s} \rightarrow g \Psi_{x,s}, \quad s \leq s_0,
\]
\[
\Psi_{x,s} \rightarrow h \Psi_{x,s}, \quad s \geq s_0 + 1,
\]
\[
V \rightarrow gVh^\dagger,
\]
where \(g\) and \(h\) are elements of \(G\).

For \(y\) small and in the symmetric phase (i.e. \(\kappa < \kappa_c\)) one massless lefthanded fermion exists just inside the boundary, and a righthanded one just outside, with wavefunctions which fall off exponentially away from the boundary. In the broken phase, these two chiral modes pair up into a massive Dirac fermion with mass proportional to \(yv\) with \(v\) the vacuum expectation value of the field \(V\). Combined with the zeromodes at the domain wall and the antidomain wall, and turning on the gauge fields, this model results in a vectorlike gauge theory for small \(y\) (cf. I).

We would now like to investigate the situation for large \(y\). An expansion in \(1/y\) can be developed by a slight rewriting of the action, eq. (3). First, we will relabel the fermion variables
\[
\chi_s = \Psi_s, \quad s < s_0,
\]
\[
\chi_{R,s_0} = \Psi_{R,s_0}, \quad \chi_{L,s_0} = \Psi_{L,s_0 + 1},
\]
\[
\chi_s = \Psi_{s + 1}, \quad s > s_0,
\]
\[
\psi_R = \Psi_{R,s_0 + 1}, \quad \psi_L = \Psi_{L,s_0},
\]
where \(\Psi_{R,L} = P_{R,L} \Psi\), etc. Note that \(\psi\) is a four dimensional fermion field, i.e. independent of \(s\). Rescaling
\[
\psi \rightarrow \frac{1}{\sqrt{y}}\psi,
\]
the fermionic action becomes
\[
S_F = \sum_s (\bar{\psi}_s \gamma_5 \chi_s + \bar{\chi}_s a(s) (\Box + m - 1) \chi_s)
\]
\begin{align*}
+ \sum_s \left( \chi_{Ls} \chi_{Rs+1} + \chi_{Rs+1} \chi_{Ls} \right) + \bar{\psi}_L V \psi_R + \bar{\psi}_R V^\dagger \psi_L \\
+ \sqrt{\alpha} \left( \bar{\psi} (\Box + m - 1) \chi_{s_0} + \chi_{s_0} (\Box + m - 1) \psi \right) + \alpha \bar{\psi} \phi \psi,
\end{align*}

with

\begin{equation}
\alpha = \frac{1}{y}
\end{equation}

and

\begin{equation}
a(s) = 1 - \delta_{s,s_0}.
\end{equation}

In eq. (10) we again suppressed all \( x \) dependence.

3. The fermion spectrum on the boundary for \( y \to \infty \)

For \( y \to \infty \), \( \psi \) decouples, and the Dirac equation for \( \chi_s(p) = \sum_x e^{-ipx} \chi_{x,s} \) is

\begin{equation}
\begin{aligned}
&i \hat{s}(p) \chi_s(p) + a(s)(m - 1 - F(p)) \chi_s(p) + P_R \chi_{s+1}(p) + P_L \chi_{s-1}(p) = 0,
\end{aligned}
\end{equation}

with \( \hat{s}(p) = \sum_\mu \gamma_\mu \sin p_\mu \) and \( F(p) = \sum_\mu (1 - \cos p_\mu) \). We wish to look for solutions of this equation (in Minkowski space) which also satisfy the four dimensional massless Dirac equation, and which have a definite chirality. For such solutions

\begin{align*}
\chi_{Rs+1}(p) &= a(s)(1 - m + F(p)) \chi_{Rs}(p), \\
\chi_{Ls-1}(p) &= a(s)(1 - m + F(p)) \chi_{Ls}(p).
\end{align*}

These equations have normalizable solutions bound to the waveguide boundary due to the fact that \( a(s_0) = 0 \). The first equation has solutions inside the waveguide (\( i.e. \chi_{Rs} = 0 \) for \( s > s_0 \)) for \( F(p) > m \). Likewise, the equation for \( \chi_L \) has solutions outside (\( i.e. \chi_{Ls} = 0 \) for \( s < s_0 \)) for \( F(p) > m \).

This leads to a rich spectrum of chiral zeromodes at the waveguide boundary. Let

\( \pi_A \in \mathcal{E} \cup \mathcal{O}, \)
\[ \mathcal{E} = \{(0,0,0,0), (\pi,0,0,0), \ldots, (\pi,\pi,\pi,\pi)\} \]

\[ \mathcal{O} = \{(0,0,0,0), (0,\pi,0,0), \ldots, (0,\pi,\pi,\pi)\}. \]

The four dimensional Dirac equation has relativistic (continuum) solutions for \( p = \pi A + \tilde{p} \) with \( \tilde{p} \to 0 \). The chirality gets flipped if \( \pi A \in \mathcal{O} \), i.e. \( \chi_R(p) \) is lefthanded when \( p \approx \pi A \in \mathcal{O} \) [17]. The condition \( F(p) > m \) excludes the modes around \( p = 0 \), and results in a spectrum of 7 righthanded and 8 lefthanded zeromodes inside the waveguide boundary, and the mirror reflection of this outside the boundary. With the single righthanded zeromode on the domain wall and the lefthanded one on the antdomain wall the zeromode spectrum is nonchiral both inside and outside the waveguide. The fact that the spectrum outside is a mirror copy of the spectrum inside is in accordance with a symmetry of the action:

\[
\begin{align*}
\chi_{x,s} &\to \gamma_4 \chi_{x,2s_0-s}, \\
\psi_x &\to \gamma_4 \psi_{x}, \\
V_x &\to V^\dagger_{P_x},
\end{align*}
\]

where \( P_x = (-x_1, -x_2, -x_3, x_4) \).

The other boundary of the waveguide can be treated in exactly the same way, with now however \( m \) replaced by \( -m \), since this boundary is located on the other side of the domain wall. The condition for the existence of zeromodes becomes \( F(p) > -m \) on both sides of this boundary, which is satisfied for all \( p \). It follows that there are 8 righthanded and 8 lefthanded zeromodes both inside and outside this boundary for \( y \to \infty \).

We will later need the \( \chi \)-fermion propagator for \( y \to \infty \), which can be found as the inverse of the Dirac operator

\[
([i \not{s} (p) + a(s)(m - 1 - F(p))]\delta_{s,s''} + P_R \delta_{s+1,s''} + P_L \delta_{s-1,s''}) S_{s',s'}^{(0)}(p) = \delta_{s',s''}. \quad (17)
\]

We find, for \( s' = s_0 \)

\[
S_{s,s_0}^{(0)}(p) = \frac{-i \not{s} (p) + z(p) - b(p)}{s^2(p) + \mathcal{M}^2(p)} z^{s-s_0}(p) P_L, \quad s < s_0,
\]
\begin{equation}
S_{s,s_0}^{(0)}(p) = \frac{-i\hbar(p) + z(p) - b(p)}{s^2(p) + \mathcal{M}^2(p)} z^{s_0-s}(p) P_R, \quad s > s_0,
\end{equation}

\begin{equation}
S_{s_0,s_0}^{(0)}(p) = \frac{-i\hbar(p)}{s^2(p) + \mathcal{M}^2(p)},
\end{equation}

where

\begin{align*}
b(p) &= 1 - m + F(p) > 0, \quad \forall p, \\
z(p) &= \frac{1 + s^2(p) + b^2(p) + \sqrt{(1 + s^2(p) + b^2(p))^2 - 4b^2(p)}}{2b(p)} > 1, \quad \forall p, \\
\mathcal{M}^2(p) &= 1 - b(p)/z(p) \geq 0, \quad \forall p.
\end{align*}

Let us check that this propagator gives rise to the same zeromode spectrum as we found from the Dirac equation. For \( p = \pi_A \) we have \( b(p) = 2n + 1 - m \), with \( n \) equal to the number of components of \( \pi_A \) that are equal to \( \pi \). This leads to \( z = 1/(1 - m) \) for \( n = 0 \) and \( z = 2n + 1 - m \) for \( n \in \{1, 2, 3, 4\} \), and hence the propagator has a pole at \( p = \pi_A \) when \( \pi_A \neq 0 \), but not for \( p = 0 \).

For the other waveguide boundary \( b(p) = 1 + m + F(p) \), and in this case \( z = 2n + 1 + m \) for all \( n \), so that the propagator has massless poles at all \( \pi_A \), in accordance with what we found earlier.

4. The phase diagram for large \( y \)

In the limit \( y \to \infty \) the fields \( \chi, \psi \) and \( V \) decouple from each other, and the phase diagram of the model is dictated by \( S_B \), in particular by the bosonic hopping parameter \( \kappa \). For \( G = U(1) \) or \( SU(2) \) there will be a second order phase transition at \( \kappa = \kappa_c \). (The fermionic part of the action depends on \( V \) even for \( y \to \infty \), but the fermion determinant is independent of \( V \) in this limit.) The theory consists of a scalar field in the broken or symmetric phase, and free fermionic modes, described by the field \( \chi \).

We would now like to see how this changes if we turn on the coupling \( \alpha \). The effective action for \( V \) can be obtained by integrating out the fields \( \psi \) and \( \chi \) in an expansion in \( \alpha \).
Let us first integrate out $\chi$, which can be done exactly, resulting in an effective action for $\psi$:

$$S_{\psi} = \sum_x \left( \overline{\psi}_x V_x \psi_{Rx} + \overline{\psi}_{Rx} V_x^\dagger \psi_{Lx} \right)$$

\begin{equation}
= \sum_x \left( \overline{\psi}_x \partial_{xy} \psi_y - \left[ \overline{\psi}(\Box + m - 1) \right]_x S_{(x,s_0),(y,s_0)}[(\Box + m - 1)\psi]_y \right),
\end{equation}

where

$$S_{(x,s_0),(y,s_0)} = \int_p e^{ip(x-y)} S_{(s_0,s_0)}(p),$$

and

$$\int_p = \int \frac{d^4 p}{(2\pi)^4}.$$  \hspace{1cm} (22)

To order $\alpha^2$ the effective action for $V$ obtained by integrating out all fermion fields then becomes

$$S_{eff}(V) \equiv S_B(V)$$

\begin{equation}
= - \log \det \left[ \delta_{x,y} + \alpha (V_x^\dagger P_R + V_y P_L) \left( \partial_{x,y} - [(\Box + m - 1) S_{(s_0,s_0)}[(\Box + m - 1)\psi]]_x, y \right) \right]
\end{equation}

\begin{equation}
= - \left( \kappa + \frac{1}{2} \alpha^2 \left( 1 - \int_p \frac{s^2(p)b^2(p)}{s^2(p) + M^2(p)} \right) \right) \sum_{x,\mu} \text{tr}(V_x V_{x+\mu}^\dagger + h.c.)
\end{equation}

\begin{equation}
- 2\alpha^2 \sum_{xy} \text{tr}(V_x^\dagger V_y) \sum_{\mu} \int_{pq} e^{i(p-q)(x-y)} s_\mu(p)s_\mu(q) \frac{b^2(p)}{s^2(p) + M^2(p)} \frac{b^2(q)}{s^2(q) + M^2(q)}
\end{equation}

\begin{equation}
+ O(\alpha^4)
\end{equation}

(recall $s_\mu(p) = \sin p_\mu$).

There will be a similar contribution from the waveguide boundary on the other side of the domain wall, with the only difference that $m$ be replaced everywhere by $-m$.

This effective action contains long range self-interactions for the field $V$, due to the fact that the integrand of the last term has massless poles at $p = \pi_A \neq 0$ (and for all $\pi_A$ for the terms with $m \to -m$). It is well known that a meanfield approach gives the correct qualitative phase diagram for $\alpha = 0$. While this is in general not true for more complicated...
interactions, we expect it to hold for $\alpha$ small enough. If we apply mean-field, we obtain an estimate for the location of the critical line in the $\kappa$–$\alpha$ plane (valid for small $\alpha$):

$$\kappa + \frac{1}{2} \alpha^2 [I(m) + I(-m)] + O(\alpha^4) = \kappa_c,$$

with

$$I(m) = 1 - \int \frac{s^2(p)b^2(p)}{s^2(p) + M^2(p)} + \frac{1}{2} \int \frac{s^2(p)b^4(p)}{[s^2(p) + M^2(p)]^2},$$

which is a function of $m$ through $b$ and $M$. As an example, the value of $I$ is 187.4(3) for $m = 0.5$ and 371.3(6) for $m = -0.5$. The large values of $I(m)$ reflect the enhanced coupling of the fermions to the $V$ field (cf. the factors $b(p)$).

5. Fermion spectrum for large but finite $y$

We can integrate out the field $\psi$ in an expansion in $\alpha = 1/y$ in order to obtain an effective action for $\chi$ and $V$:

$$S_\chi = \sum_s (\overline{\chi}_s \gamma \chi_s + \overline{\chi}_s a(s)(\Box + m - 1)\chi_s) + \sum_s (\overline{\chi}_s P_R \chi_{s+1} + \overline{\chi}_{s+1} P_L \chi_s)$$

$$+ \alpha \sum_{s,s'} \overline{\chi}_{s0} \gamma_{s0} (\Box + m - 1) \gamma_{s0} (\Box + m - 1) \chi_{s0}$$

$$= \sum_{s,s'} \overline{\chi}_{s0} S_{s,s'}^{-1} \chi_{s0} - \alpha \sum_x [\overline{\chi}_{s0} (\Box + m - 1)]_x (V_x \chi_{s0} + V_x \chi_{s0}) [(\Box + m - 1) \chi_{s0}]_x$$

$$+ \alpha^2 \sum_{x,y} [\overline{\chi}_{s0} (\Box + m - 1)]_x (V_x \chi_{s0} + V_x \chi_{s0}) [\gamma x y (V_y \chi_{s0} + V_y \chi_{s0}) [(\Box + m - 1) \chi_{s0}]_y$$

$$+ O(\alpha^3).$$

To order $\alpha$, replacing $V$ by its expectation value $v_1$, the Dirac equation reads (in momentum space)

$$i \gamma (p) \chi_s + a(s)(m - 1 - F(p)) \chi_s + P_R \chi_{s+1} + P_L \chi_{s-1} - \alpha v (m - 1 - F(p))^2 \delta_{s_{s0} \chi_{s0}} = 0. \quad (27)$$

Effectively, $a(s)$ of eq. (12) no longer vanishes at $s = s_0$. Instead,

$$a(s_0) = \alpha v (1 - m + F(p)),$$

$$F(p) = \frac{1}{\alpha} \int \frac{s^2(p)b^2(p)}{s^2(p) + M^2(p)} + \frac{1}{2} \int \frac{s^2(p)b^4(p)}{[s^2(p) + M^2(p)]^2},$$

for $m = 0.5$ and $371.3(6)$ for $m = -0.5$. The large values of $I(m)$ reflect the enhanced coupling of the fermions to the $V$ field (cf. the factors $b(p)$).
and there are no zeromode solutions for $v \neq 0$. In the broken phase, the zeromodes pick up a mass of order $v/y$.

We see that in the broken phase, we effectively obtain a nonzero value for $a(s_0)$ (eq. (28)). This is only possible in the broken phase. The symmetry eq. (7) acts on $\chi_{s_0}$ as

$$
\chi_{Rs_0} \to g\chi_{Rs_0}, \quad \chi_{Ls_0} \to h\chi_{Ls_0},
$$

and this symmetry has to be broken in order to have a nonzero value for $a(s_0)$. Thus, a righthanded mode inside the waveguide can mix with a lefthanded mode outside the waveguide only if the symmetry eq. (7) is violated, i.e. only in the broken phase. Moreover, no righthanded and lefthanded modes which are both inside the waveguide can mix, because no lefthanded mode inside the waveguide carries the same lattice momentum as any righthanded mode inside the waveguide, and lattice momentum is exactly conserved.

(See the last section for further discussion of this point).

We will illustrate this by showing that the massless poles which exist at $y \to \infty$ (cf. eq. (18)) remain massless to second order in $\alpha = 1/y$, and that therefore the massless spectrum of the theory remains the same for large but finite $y$. For $G = U(1)$, to order $\alpha^2$, the inverse propagator in the symmetric phase becomes

$$
S^{-1}_{s,s'}(p) = S^{(0)}_{s,s'}^{-1}(p) - \Sigma_{s,s'}(p),
$$

$$
\Sigma_{s,s'}(p) = -i\alpha^2 \frac{\hat{s}(p)\hat{\delta}_{s,s_0}\delta_{s_0,s'}b^2(p)(L + K(p))}{\hat{s}^2(p) + M^2(p)},
$$

(29)

with

$$
L = \langle V_x V^\dagger_{x+\mu} \rangle
$$

(30)

and

$$
\int_l D(l-p) \frac{\hat{s}(l)b^2(l)}{s^2(l) + M^2(l)} \equiv \hat{s}(p)K(p).
$$

(31)

$D(p)$ is the scalar propagator. The $L$ term comes from a diagram with one order $\alpha^2$ vertex (cf. eq. (26)), and the $K(p)$ term comes from a diagram with two order $\alpha$ vertices. Since
\[ \Sigma(p) \text{ vanishes for all } p = \pi_A \text{ (eq. (15))}, \] the massless pole structure of the propagator is not changed by the \( \alpha^2 \) correction. (Note that the function \( K \) goes to a constant for \( p \to \pi_A \) in the symmetric phase.) We believe this result to be valid to all orders in \( \alpha \).

Let us conclude this section by considering the effective action for \( \psi \), eq. (20). This effective action is exact, and therefore should describe all zeromodes at the waveguide boundary for all values of the Yukawa coupling \( y \), as long as \( \psi \) couples to the field \( \chi \), \( i.e. \) for \( \alpha \neq 0 \). The term linear in \( \alpha \) contains the \( y \to \infty \) propagator for the field \( \chi \), and therefore the zeromodes at the waveguide boundary at large \( y \) are present in the effective action \( S_\psi \). For small \( y \), it was shown in \( I \) that there is only a zeromode at the waveguide boundary for \( \pi_A = 0 \), and this should also follow from \( S_\psi \). This is indeed the case: by rescaling the field \( \psi \) in a way appropriate for small values of \( y \) \( (i.e. \) undoing the transformation eq. (9)\), and choosing \( y = 0 \), we see that the \( \psi \) propagator for small \( y \) is defined by the order \( \alpha \) term in eq. (20) to lowest order in \( y \). In momentum space, this propagator reads

\[
S^{(0)}_\psi(p) = \frac{-i\delta(p)}{s^2(p)} \frac{s^2(p) + M^2(p)}{s^2(p) + M^2(p) + b^2(p)}.
\]

This expression only has a pole at \( p = 0 \), and none at \( p = \pi_A \) with \( \pi_A \neq 0 \) \( (cf. \) end of section 3\), and describes one massless Dirac fermion at the waveguide boundary, in accordance with the results of \( I \) at small \( y \).

### 6. Conclusion

In this paper we continued the investigation of the waveguide implementation of the domain wall fermion model \([9,10]\). In \( I \) we argued that the question as to whether this model can yield a chiral gauge theory is a dynamical question which cannot be decided on the basis of simple anomaly arguments. Therefore, a complete investigation of the phase diagram is needed in order to see what the massless fermion spectrum of the model is, as
a function of the parameters of the model. However, in I this analysis was left incomplete, because of difficulties with the numerical exploration of the phase diagram for large values of the Yukawa coupling.

Here we showed that the action for this model can be reformulated in such a way that a systematic strong coupling expansion in the Yukawa coupling can be performed. This allowed us to examine the strong Yukawa coupling region of the phase diagram analytically.

In the limiting case $\alpha = 0$ ($y \to \infty$) the fermions decouple from the scalar field, and we obtained a free fermion theory which can be solved exactly. We found a very rich zeromode spectrum. While the zeromode spectrum at the domain wall and antdomain wall is unchanged, in a sense all their doubler modes show up localized at the waveguide boundaries. The complete massless fermion spectrum inside as well as outside the waveguide is vectorlike. The gauge fields, which live only inside the waveguide, will couple to all the massless fermions inside, and the result is a vectorlike gauge theory. The situation is in fact worse than at weak Yukawa coupling, where inside the waveguide only one massless mirror mode emerged due to the interaction of the fermions with the scalar field.

The phase diagram consists of symmetric phases at weak and strong Yukawa coupling (for small enough values of the scalar hopping parameter), separated by a phase in which the gauge symmetry is spontaneously broken. We verified by an explicit calculation that the massless spectrum remains unchanged to first order in $\alpha^2$ in the symmetric strong coupling phase. While we did not provide a rigorous proof, there can be little doubt that this result extends to all orders in $\alpha^2$. There is an interesting relation between the present model and the Smit-Swift model [18]. The massless fermion spectrum at strong Yukawa coupling is similar to that of the Smit-Swift model at weak Yukawa coupling, and vice versa.

The question arises whether a more general framework could lead to qualitatively different results. For example, by a suitable modification of the scalar action, it might
be possible to arrange for the existence of an antiferromagnetic condensate which is $g$- and $h$-invariant. In that case, lattice momentum would be conserved only modulo $\pi$ (and not modulo $2\pi$) and mixing between righthanded and lefthanded modes from different corners of the Brillouin zone would be possible inside the waveguide boundary. Without fine tuning, we would expect that only a single lefthanded mode would survive inside the boundary. But since lefthanded and righthanded modes would pair to form massive states, the full massless spectrum inside the waveguide would remain vectorlike!

There is a good reason to believe that the phenomenon described in this example is completely general. The point is that we start (at $\alpha = 0$) with a well-defined theory that has a relativistic, albeit vectorlike, low energy spectrum. The question is whether one can achieve a chiral spectrum by making a continuous change of parameters which leaves the ($g$ and $h$) symmetries unbroken. Assuming the model maintains a consistent continuum limit throughout this process (which should be the case if we add only finite range operators to the lattice action), we expect that the low energy spectrum can be correctly described by a corresponding change of parameters in a relativistic effective lagrangian. But then the only way to decouple massless fermions is to pair them into massive Dirac fermions. Thus, since we started with a vectorlike spectrum, we will end up with a vectorlike spectrum.

We conclude that, although the phase diagram is richer than suspected in ref. I, the waveguide implementation of the domain wall fermion model does not lead to a lattice chiral gauge theory.

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