Modelling of torsional wave propagation in a two-layer anisotropic inhomogeneous media

Tapas Ranjan Panigrahi¹, Sumit Kumar Vishwakarma¹, Dinesh Kumar Majhi²

¹Department of Mathematics, BITS-Pilani, Hyderabad Campus, Hyderabad-500078, India.
²Department of Mathematics, Patna College, Patna University, Patna-800005, India.
E-mail: tapas.infinity@gmail.com

Abstract. The current article reflects the behavior of torsional waves in a two-layer inhomogeneous model, where rigidities and density vary along with the depth parameter z. The variation for the upper layer and lower half-space has been taken as the product of two exponential functions and the product of one linear and exponential function, respectively. A mathematical model is developed by following the concept of elastic waves. The separation of the variable method has been imposed on both layers in order to calculate the displacement equations. Intrinsic boundary conditions are imposed to derive the frequency equation in closed form. It has been observed a substantial effect of rigidity and density on the phase velocity of the torsional wave. Graphical presentations have been performed in order to support the different findings of the dispersion equation.

Keywords: Torsional wave, phase velocity, half-space, inhomogeneous, dispersion equation.

1. Introduction

The research on the propagation of the torsional surface wave in different elastic media with layered media has been started over the past few years. Many seismologists have been carried out research on the propagation of torsional wave by following the theory of Love [1], whose comprehensive investigations on seismic wave propagation in non-homogeneous anisotropic layers created a large gap in the research of seismic wave propagation. Also, his involvement in the research described the unidentical characteristics in the isotropic media. It is known to all that the earth is consists of various heterogeneous geological parameters, and Bullen [2] suggested the density law inside the earth’s surface; the law states that the density varies at different rates within the earth for different layers. According to his research, density varies quadratically in depth from 413 km to 984 km and varies linearly on more than 984 km.

The amplitude of the torsional wave decays exponentially, and also this wave is known as a surface type wave. When the torsional wave propagates through a body, it gives a twist and is horizontally polarized. Vardoulakis [3] discovered the torsional wave propagation in the lower half-space, in which variation of shear moduli linearly with depth is considered without changing the density. Many seismologists followed the theory of Biot [4] and studied the torsional wave propagation on the layered medium lying over the half-space. Dey and Dutta [5] considered the initially stressed cylinder and explained the torsional wave propagation, whereas Pujol [6]...
and Chapman [7] focused on the elastic surface wave propagation in seismology. References can be given to Meissner [8], Ewing and Jaretzky [9], Bhattacharya [10], Vishwakarma et al. [11], Gupta et al. [12], Vishwakarma [13], and Vishwakarma et al. [14] for their recommended work on the surface type wave propagation and torsional surface wave propagation in a different heterogeneous medium.

The current paper has described the study of the torsional surface wave on a heterogeneous anisotropic medium overlying on an inhomogeneous half-space. Here, the heterogeneity for the upper layer is considered by varying the directional rigidities \( (N, L) \) and density \((\rho)\) as a product of two different types of exponential functions, whereas the heterogeneity in the half-space is formed by the product of one linear and one exponential function. The remarkable influence of inhomogeneity on the phase velocity has been described graphically, and the computation is performed numerically. The current research may be helpful for exploration purposes and the formation of seismic waves due to the artificial explosion.

2. Geometry of the problem

To examine the current problem’s behavior geometrically, here the coordinate system (cylindrical) has been considered, and the origin \( o \) has been kept at the common interface of the two-layered medium. In this model, \( r \)-axis has been considered in the direction of wave propagation, and \( z \)-axis has been kept in the direction of depth, as shown in Fig. 1. The thickness of the upper layer has been taken as \( H \). The inhomogeneity has been defined as follows:

**Upper medium:** \[ N = N_0 \beta_1 e^{\alpha z}, \quad L = L_0 \beta_1 e^{\alpha z}, \quad \rho = \rho_0 \beta_1 e^{\alpha z}, \]

where \( \alpha \) and \( \beta_1 \) are real numbers.

**Lower half-space:** \[ N = N_1 (1 - az) e^{\gamma z}, \quad L = L_1 (1 - az) e^{\gamma z}, \quad \rho = \rho_1 (1 - az) e^{\gamma z}, \]

where \( a \) and \( \gamma \) are real numbers.

![Figure 1. Geometry of the problem.](image-url)
3. Formulation of the problem

The equation of the motion is given by (Biot [4])

$$\frac{\partial S_{r\theta}}{\partial r} + \frac{\partial S_{\theta\theta}}{\partial z} + 2\frac{r}{s}S_{r\theta} = \rho \frac{\partial^2 v'}{\partial t^2},$$

(1)

where \( S_{ij}, i, j = r, \theta, z \) are the incremental stress components, \( v' \) is known as the displacement, \( t \) is the time, and \( \rho \) as density. Since we have interested in studying the torsional wave propagation, so the cylindrical coordinate system \((r, \theta, z)\) has been taken such that \( r \) is the radial distance, \( \theta \) is the azimuth, and \( z \) is the height.

The stress-strain relationships for the medium are

$$S_{r\theta} = 2Ne'_{r\theta}, \quad S_{z\theta} = 2Le'_{z\theta},$$

(2)

where \( N \) and \( L \) are known as the directional rigidities. \( e'_{r\theta} \), and \( e'_{z\theta} \) are the strain components. Also we can write,

$$e'_{r\theta} = \frac{1}{2} \left( \frac{\partial v'}{\partial r} - \frac{v'}{r} \right), \quad e'_{z\theta} = \frac{1}{2} \frac{\partial v'}{\partial z}.$$

Applying the Eq. (2) in Eq. (1), we get

$$N \left( \frac{\partial^2 v'}{\partial r^2} - \frac{v'}{r^2} + \frac{1}{r} \frac{\partial v'}{\partial r} \right) + \frac{\partial}{\partial z} \left( \frac{L}{\partial z} \frac{\partial v'}{\partial z} \right) = \rho \frac{\partial^2 v'}{\partial t^2}.$$  

(3)

Solution for the Eq. (3) may be assumed as:

$$v'(r, z, t) = v_j(r, z, t) = \psi_j(z)J_1(kr)e^{iwt}, \quad j = 1, 2,$$

(4)

where \( j = 1 \) is used in the solution process of upper layer and \( j = 2 \), is taken for half-space. \( J_1(kr) \) is the first kind of the Bessel function. \( w = kc \) such that \( k, \) and \( c \) are wave number and wave velocity, respectively and \( i = \sqrt{-1}. \)

Again using the Eq. (4) in Eq. (3), the equation can be written as

$$\frac{d^2 \psi_j}{dz^2} + \frac{1}{L} \frac{dL}{dz} d\psi_j - \frac{k^2 N}{L} \left( 1 - \frac{c^2 \rho}{N} \right) \psi_j = 0, \quad j = 1, 2.$$  

(5)

3.1. Solution for the upper layer

On using the variations \( N = N_0 \beta_1^2 e^{\alpha z}, L = L_0 \beta_1^2 e^{\alpha z}, \rho = \rho_0 \beta_1^2 e^{\alpha z} \), and \( j = 1 \) on the above Eq. (5), reduces to

$$\frac{d^2 \psi_1}{dz^2} + (\alpha + \text{Log}(\beta_1)) \frac{d\psi_1}{dz} - A_1 \psi_1 = 0,$$  

(6)

where \( A_1 = \frac{k^2 N_0}{4\alpha} \left( 1 - \frac{c^2}{\alpha} \right) \) and \( c_0^2 = \frac{N_0}{\rho_0} \). The solution for the Eq. (6) is

$$\psi_1(z) = C_1 e^{-\frac{1}{2}z \left( \alpha + \text{Log}(\beta_1) + \sqrt{4A_1 + (\alpha + \beta_1)^2} \right)} + C_2 e^{\frac{1}{2}z \left( \alpha + \text{Log}(\beta_1) - \sqrt{4A_1 + (\alpha + \beta_1)^2} \right)}.$$  

(7)

Hence, the displacement for the upper layer can be derived using the Eq. (4) and Eq. (7) i.e.

$$v_1 = \left( C_1 e^{-\frac{1}{2}z \left( \alpha + \text{Log}(\beta_1) + \sqrt{4A_1 + (\alpha + \beta_1)^2} \right)} + C_2 e^{\frac{1}{2}z \left( \alpha + \text{Log}(\beta_1) - \sqrt{4A_1 + (\alpha + \beta_1)^2} \right)} \right) J_1(kr)e^{iwt}.$$  

(8)
3.2. Solution for the lower half-space

The inhomogeneity parameters for the lower half-space are

\[ N = N_1 (1 - az) e^{\gamma z}, \quad L = L_1 (1 - az) e^{\gamma z}, \quad \text{and} \quad \rho = \rho_1 (1 - az) e^{\gamma z}, \]

where \( a \) and \( \gamma \) are real numbers, and \( az \neq 1 \). Using the above inhomogeneity conditions and \( j = 2 \) in Eq. (5), we get

\[ \frac{d^2 \psi_2}{dz^2} + \left( \gamma - \frac{a}{1 - az} \right) \frac{d\psi_2}{dz} - A_2 \psi_2 = 0, \tag{9} \]

\[ A_2 = \frac{k_2 N_1}{L_1} \left( 1 - \frac{c^2}{c_1^2} \right) \] and \( c_1^2 = \frac{N_1}{\rho_1} \). The solution for the Eq. (9) is

\[ \psi_2(z) = e^{-\frac{1}{2}z(\gamma + B_1)} \left( C_3 H_U \left[ \frac{\gamma + B_1}{2B_1}, 1, \frac{-B_1}{a} + B_1 \right] + C_4 L_L \left[ \frac{\gamma + B_1}{2B_1}, \frac{-B_1}{a} + B_1 \right] \right), \tag{10} \]

where \( H_U \) and \( L_L \) are the HypergeometricU and LaguerreL functions, respectively. \( C_3 \) and \( C_4 \) are arbitrary constants. \( a \neq 0 \), and \( B_1 = \sqrt{4A_2 + \gamma^2} \neq 0 \). Since the lower half-space has spread up to infinity i.e. the depth parameter \( z \to \infty \), so the solution for the lower half-space goes to zero (\( \psi_2 \to 0 \)). Hence, the displacement for the lower half-space can be determined by applying the reduced form (which is formed due to the above concept of half-space) of Eq. (10) in Eq. (4) i.e.

\[ v_2 = C_3 e^{-\frac{1}{2}z(\gamma + B_1)} H_U \left[ \frac{\gamma + B_1}{2B_1}, 1, \frac{-B_1}{a} + B_1 \right] J_1(kr)e^{iwt}. \tag{11} \]

4. Boundary conditions

(i). The upper layer is consist of without stress i.e.

\[ L_0 \frac{\partial v_1}{\partial z} = 0, \quad \text{at} \quad z = -H. \tag{12} \]

(ii). The continuity for displacement follows in the interface i.e.

\[ v_1 = v_2, \quad \text{at} \quad z = 0. \tag{13} \]

(iii). In the interface, stresses are also equal i.e.

\[ L_0 \frac{\partial v_1}{\partial z} = L_1 \frac{\partial v_2}{\partial z}, \quad \text{at} \quad z = 0. \tag{14} \]

5. Dispersion equation

Using the above boundary conditions (Eqs. (12-14)) in the displacement equations (Eq. 8 and Eq. 11 ) of the two medium we have

\[ \left( \alpha + \log(\beta) + \sqrt{4A_1 + (\alpha + \log(\beta))^2} \right) e^{\frac{1}{2}H \left( \alpha + \log(\beta) + \sqrt{4A_1 + (\alpha + \log(\beta))^2} \right)} C_1 + \]

\[ \left( \alpha + \log(\beta) - \sqrt{4A_1 + (\alpha + \log(\beta))^2} \right) e^{\frac{1}{2}H \left( \alpha + \log(\beta) - \sqrt{4A_1 + (\alpha + \log(\beta))^2} \right)} C_2 = 0 \tag{15} \]

\[ C_1 + C_2 - H_U \left[ \frac{\gamma + B_1}{2B_1}, 1, \frac{-B_1}{a} \right] C_3 = 0 \tag{16} \]

\[ - L_0 \left( \alpha + \log(\beta) + \sqrt{4A_1 + (\alpha + \log(\beta))^2} \right) C_1 - \]

\[ L_0 \left( \alpha + \log(\beta) - \sqrt{4A_1 + (\alpha + \log(\beta))^2} \right) C_2 + L_1 (\gamma + B_1) \left( H_U \left[ \frac{\gamma + B_1}{2B_1}, 1, \frac{-B_1}{a} \right] + H_U \left[ 1 + \frac{\gamma + B_1}{2B_1}, 2, \frac{-B_1}{a} \right] \right) C_3. \tag{17} \]
Considering the coefficients $C_1$, $C_2$, and $C_3$ of the above Eqs. (15-17), we will get the non trivial solution i.e.

$$\text{Re} \left( \begin{bmatrix} d'_{11} & d'_{12} & d'_{13} \\ d'_{21} & d'_{22} & d'_{23} \\ d'_{31} & d'_{32} & d'_{33} \end{bmatrix} \right) = 0.$$  \hspace{1cm} (18)

Where,

$$d'_{11} = \left( \alpha + \text{Log}(\beta) + \sqrt{4A_1 + (\alpha + \text{Log}(\beta))^2} \right) e^{\frac{1}{2}H(\alpha + \text{Log}(\beta) + \sqrt{4A_1 + (\alpha + \text{Log}(\beta))^2})}$$

$$d'_{12} = \left( \alpha + \text{Log}(\beta) - \sqrt{4A_1 + (\alpha + \text{Log}(\beta))^2} \right) e^{\frac{1}{2}H(\alpha + \text{Log}(\beta) - \sqrt{4A_1 + (\alpha + \text{Log}(\beta))^2})}$$

$$d'_{13} = 0$$

$$d'_{21} = d'_{22} = 1$$

$$d'_{23} = -H_U \left[ \frac{\gamma + B_1}{2B_1}, 1, -\frac{B_1}{a} \right]$$

$$d'_{31} = -L_0 \left( \alpha + \text{Log}(\beta) + \sqrt{4A_1 + (\alpha + \text{Log}(\beta))^2} \right)$$

$$d'_{32} = -L_0 \left( \alpha + \text{Log}(\beta) - \sqrt{4A_1 + (\alpha + \text{Log}(\beta))^2} \right)$$

$$d'_{33} = L_1 \left( \gamma + B_1 \right) \left( H_U \left[ \frac{\gamma + B_1}{2B_1}, 1, -\frac{B_1}{a} \right] + H_U \left[ 1 + \frac{\gamma + B_1}{2B_1}, 2, -\frac{B_1}{a} \right] \right)$$

Eq. (18) is known as the dispersion/frequency equation of the torsional wave propagation in the two layered model.

![Figure 2. Influence of dimensionless phase velocity against dimensionless wave number for $\alpha H = 1.1, \beta_1 H = 1.3, \text{and} \gamma H = 0.5.$](image)

6. Numerical computation and graphs

In this section graphical presentations are performed with the help of frequency Eq. (18) to know the impact of inhomogeneity parameters of both the layers. Various graphs have been plotted with the help of MATHEMATICA to study the significant effect of the non-homogeneity parameters ($a$, $\alpha$, $\beta_1$, and $\gamma$) on the dimensionless phase velocity $c'^2/c_0^2$ against the dimensionless wave number $kH$. The following explanations can be useful for understanding the problem very deeply.
Upper layer: (Gubbins [15])
\[ N_0 = 6.54 \times 10^{10} \text{ N/m}^2, \quad L_0 = 7.41 \times 10^{10} \text{ N/m}^2, \quad \rho_0 = 3430 \text{ kg/m}^2 \]

Half-space: (Gubbins [15])
\[ N_1 = 7.34 \times 10^{10} \text{ N/m}^2, \quad L_1 = 5.98 \times 10^{10} \text{ N/m}^2, \quad \rho_1 = 3195 \text{ kg/m}^2 \]

Figure 2 concludes the influence of the heterogeneity parameter \( a \) on the phase velocity of the torsional wave propagation. The curves 1, 2, and 3 are obtained against the values 0.1, 0.2, and 0.3, respectively for the inhomogeneity parameter \( a \). At a particular wave number, it can be clearly visible from the figure that the pattern of the curves are both increasing and decreasing. In the study of \( a \), one can find two wave front at a particular wave number. The curves are divided into two sets i.e. for the range 0 to 1 and 1 to 2. Considering a higher wave number, the phase velocity may go to 1. Here, the study shows that the lower half-space has a significant effect on the torsional wave propagation.

![Figure 3](image-url)

**Figure 3.** Influence of dimensionless phase velocity against dimensionless wave number for \( aH = 0.2, \beta_1 H = 1.3, \) and \( \gamma H = 0.5. \)

Figure 3 have been plotted to describe the impact of the inhomogeneity parameter \( \alpha \) on the phase velocity of torsional surface wave propagation. The curves are plotted against the values of \( \alpha \) as 1.1, 1.3, and 1.5. On comparing the figure 2 and figure 3, one can easily understand that the pattern of the curves in both the figures are nearly same. But the only difference one can visualize in figure 3 is that the pattern of the curves in both the groups i.e. ranges from 0-1, and 1-2 are decreasing in nature. On observing this figure minutely, one can depict the clear idea that the impact of the upper layer (figure 3) has more significant impact than the lower half-space (figure 2).

Figure 4 and figure 5 reflects the idea of calculating phase velocity of torsional wave propagation for the inhomogeneity parameters \( \beta_1 \) and \( \gamma \) associated with upper layer and lower half-space, respectively. The values for \( \beta_1 \) varies as 1.1, 1.3, and 1.5, whereas for \( \gamma \) it is 0.1, 0.5, and 0.9. One can visualize that the pattern of the curves for both the figures (i.e. figure 4, and figure 5) are converse to each other. On observing these two figures carefully, the observer can understand that the impact of the inhomogeneity parameters (\( \beta_1 \) and \( \gamma \)) have more effective than the previous two figures. It seems from both the figures that the curves are converges nearly equal to 1 for a large value of wave number.
7. Conclusion
The present article shows the influence of inhomogeneity parameters associated with the upper layer and lower half-space on the phase velocity of the torsional surface wave propagation in a two-layer inhomogeneous media. The closed-form of the frequency equation has been derived with the help of displacement equations and boundary conditions. The influences of the inhomogeneity parameters $a$, $\alpha$, $\beta_1$, and $\gamma$ on the phase velocity are noteworthy. The following conclusions can be drawn from the above figures:

- Two wavefronts can be visualized in all the above figures in which one set of curves ranges from 0 to 1 move from down to up and the second set of curves ranges from 1 to 2 moves from up to down.
- The impact of inhomogeneity parameters $\beta_1$ and $\gamma$ on the phase velocity have been more

Figure 4. Influence of dimensionless phase velocity against dimensionless wave number for $aH = 0.2$, $\alpha H = 1.3$, and $\gamma H = 0.5$.

Figure 5. Influence of dimensionless phase velocity against dimensionless wave number for $aH = 0.2$, $\alpha H = 1.3$, and $\beta_1 H = 1.1$. 
than the inhomogeneity parameters $a$ and $\alpha$.

- On enlarging the domain, one can depict the clear idea that the curves converge to 1.0 (approximately) irrespective of its pattern for all these above figures.

The study may be useful in signal processing, data analysis, and computing torsional wave propagation near the earth’s surface.

Acknowledgment
Authors extend their sincere thanks to SERB-DST, New Delhi, for providing financial support under Early Career Research Award with Ref. No. ECR/2017/001185. Authors are also thankful to DST New Delhi, for providing DST-FIST grant with Ref. no. 337 to Department of Mathematics, BITS-Pilani, Hyderabad Campus.

References
[1] Love A E H 1911 Some Problems of Geo-dynamics (London: Cambridge University Press)
[2] Bullen K E 1965 The problem of the earth’s density variation. B. Seismol. Soc. Am. 30(3) 235–250
[3] Vardoulakis I 1984 Torsional surface wave in inhomogeneous elastic media. Int. J. Numer. Anal. Methods Geomech. 8 287–296
[4] Biot M A 1965 Mechanics of Incremental Deformation (New York: John Willey and Sons)
[5] Dey S and Dutta D 1992 Torsional wave propagation in an initially stressed cylinder. Proc. Indian National Sci. Acad. 58(5) 425–429
[6] Pujol J 2003 Elastic Wave Propagation and Generation in Seismology (Cambridge: Cambridge University Press)
[7] Chapman C 2004 Fundamentals of Seismic Wave Propagation (Cambridge: Cambridge University Press)
[8] Meissner E 1921 Elastic oberflachenwellen mit dispersion in einem inhomogeneous medium. Vierteljahrschr. Naturf. Ges. 66 181–195
[9] Ewing W M and Jaretzky W S 1957 Elastic Waves in Layered Media (New York: McGraw Hill Book Company)
[10] Bhattacharya R C 1975 On the torsional wave propagation in a two layered circular cylinder with imperfect bonding. Proc. Indian National Sci. Acad. 41(6) 613–619
[11] Vishwakarma S K, Panigrahi T R and Kaur R 2019 SH-wave propagation in linearly varying fiber-reinforced viscoelastic composite structure uninitial stress. Arab. J. Geosci. 12 59
[12] Gupta S, Chattopadhyay A, Kundu S and Gupta A K 2010 Propagation of torsional surface wave in gravitating anisotropic porous half-space with rigid boundary. Int. J. Appl. Math. Mech. 6(11) 17–25
[13] Vishwakarma S K 2014 Torsional wave propagation in a self reinforced medium sandwiched between a rigid layer and viscoelastic half-space under gravity. Appl. Math. Comput. 242(1) 1–9
[14] Vishwakarma S K, Kaur R and Panigrahi T R 2019 Torsional wave frequency in heterogeneous earth crust lying over dry sandy semi-infinite substratum. Appl. Math. Mech. 40(10) 1399–1412
[15] Gubbins D 1990 Seismology and Plate Tectonics (Cambridge: Cambridge University Press)