Energy-based Control and Observer Design
for higher-order infinite-dimensional
Port-Hamiltonian Systems

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Abstract: In this paper, we present a control-design method based on the energy-Casimir method for infinite-dimensional, boundary-actuated port-Hamiltonian systems with two-dimensional spatial domain and second-order Hamiltonian. The resulting control law depends on distributed system states that cannot be measured, and therefore, we additionally design an infinite-dimensional observer by exploiting the port-Hamiltonian system representation. A Kirchhoff-Love plate serves as an example in order to demonstrate the proposed approaches.

Keywords: infinite-dimensional systems, partial differential equations, boundary actuation, port-Hamiltonian systems, structural invariants, observer design

1. INTRODUCTION

The port-Hamiltonian (pH) system representation has turned out to be a powerful tool for the description of systems governed by ordinary differential equations (ODEs) as well as partial differential equations (PDEs). With respect to the infinite-dimensional case, especially the well-known Stokes-Dirac scenario, see, e.g., van der Schaft and Maschke (2002); Le Gorrec et al. (2005), is widely used, as the underlying structure — in particular so-called power ports — can be exploited for the controller design like in Macchelli et al. (2017) for instance. A key feature of this approach is the use of energy variables replacing spatial derivatives that occur in the Hamiltonian. This has the consequence that differential operators, which generate the mentioned power ports, appear in the interconnection mapping. Exemplarily, strain variables are used for a proper pH-description of mechanical systems, which implies that for spatially two-dimensional systems besides the PDEs also certain compatibility conditions have to be fulfilled, see, e.g., Brugnoli et al. (2019) for a pH-formulation of a Kirchhoff plate based on Stokes-Dirac structures.

However, from the author’s point of view in particular for mechanical systems allowing for a variational characterisation an approach based on jet-bundle structures, see, e.g., Einsbrenner and Schlacher (2005); Schöberl and Schlacher (2015), is quite suitable, as the deflection of the system under consideration appears as system state. This is especially beneficial for position control, see, e.g., Malzer et al. (2020) for the controller design for infinite-dimensional systems with two-dimensional spatial domain and in-domain actuation based on the well-known energy-Casimir method. This pH-system formulation heavily exploits so-called jet variables (or derivative coordinates), which are of particular importance with respect to the generation of power ports, and therefore for the design of boundary-control schemes, see Rams and Schöberl (2017) for the controller design based on the energy-Casimir method for spatially one-dimensional systems. In this paper, one of the intentions is to adapt the energy-Casimir method for boundary-actuated pH-systems with 2-dimensional spatial domain. However, we find that for this scenario the control law depends on system states that are distributed over a part of the boundary, which therefore cannot be measured. In light of this aspect, a further objective is to develop an infinite-dimensional observer by exploiting the pH-formulation.

Note that in Toledo et al. (2020) passive observers for distributed-parameter pH-systems are developed based on Stokes-Dirac structures. Moreover, in Malzer et al. (2021) an observer is derived within the jet-bundle framework for an in-domain actuated vibrating string, where the convergence of the observer error is verified by means of functional analytic methods. At this point, let us mention that in this contribution we focus on energy considerations and neglect detailed stability investigations. However, the mentioned approaches are restricted to systems with 1-dimensional spatial domain, whereas we present an observer-design method being able to cope with spatially 2-dimensional systems implying a rise of complexity. Here, the intention is to exploit boundary-power ports in order to impose a desired behaviour on the observer error.

Therefore, the main contributions of this paper are as follows: i) in Section 4, we extend the energy-Casimir method to boundary-controlled pH-systems with 2-dimensional spatial domain; ii) as for that scenario the control law depends on distributed system states, an observer-design method based on the pH-formulation is presented in Section 5. To demonstrate the capability of the presented approaches, we study a Kirchhoff-Love plate as running example.

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2. NOTATION AND PRELIMINARIES

Throughout this contribution, differential-geometric methods are exploited as underlying framework, where a notation similar to those of Saunders (1989) is used. Moreover, tensor notation and Einstein’s convention on sums are applied to keep formulas short and readable, where the range of the indices is omitted when it is clear from the context. The standard symbols ∧, d and ∇ denote the exterior wedge product, the exterior derivative and the Hook operator enabling the natural contraction between tensor fields, respectively. By $C^\infty(M)$ we denote the set of all smooth functions on a manifold $M$.

To properly describe distributed-parameter systems with 2-dimensional spatial domain in a differential-geometric setting, first we introduce a bundle $\pi: E \rightarrow B$, with $(z^i, x^\alpha)$ denoting the independent coordinates of the base manifold $B$ and $(z^i, x^\alpha, \dot{z}^i, \dot{x}^\alpha)$, $i = 1, 2$, $\alpha = 1, \ldots, n$, those of the total manifold $E$. Next, we consider the 1st-order jet manifold $J^1(E)$ equipped with the coordinates $(z^i, z^j, x^\alpha, x^{\alpha\beta}, \dot{x}^\alpha, \dot{x}^\alpha)$, where we already used ordered multi-indices $[10]$ and $[01]$ respectively. Thus, an ordered multi index $[\alpha_1, \alpha_2, \ldots, \alpha_r]$, where $\alpha_i \leq J_i$ and $0 \leq \# J < r$ denoting the corresponding order, also allows to introduce higher-order jet manifolds $J^r(E)$ possessing the coordinates $\{z^i, z^j, x^\alpha, x^{\alpha\beta}, \ldots, \dot{x}^\alpha, \dot{x}^\alpha\}$, where $x_{[00]}^\alpha = x^\alpha$.

A further important differential-geometric object is a tangent bundle $\tau^*_E: T^*(E) \rightarrow E$, where $T(C)$ possesses the coordinates $(z^i, x^\alpha, z^j, \dot{z}^j, \dot{x}^\alpha)$, which allows to introduce a vector field $v: E \rightarrow T(E)$ reading $v = \dot{z}^j \dot{\theta}_j$. In this paper, we are particularly interested in vertical vector fields $v: E \rightarrow V(E)$, where $v = \dot{v}^\alpha \partial_\alpha$, which can be defined by means of vertical tangent bundles $v_E: V(E) \rightarrow E$ endowed with $(z^i, x^\alpha, \dot{z}^j, \dot{x}^\alpha)$.

Moreover, the total derivative $d_{[i]} v = \partial_i \dot{z}^j \dot{\theta}_j + x_{[i+1]}^\alpha \dot{\theta}_\alpha$, where $\dot{\theta}_\alpha = \partial \dot{z}^\alpha$, and $[1]$ represents a multi index containing only zeros except the $i$th entry which is one, enables to prolong a vertical vector field to the $r$th-order jet manifold along with $J^r(E) = (z^i, x^\alpha, z^j, \dot{z}^j, \dot{x}^\alpha)$, where $d_{[i]} = (d_{[0]})^i + (d_{[1]})^i$ and $1 \leq \# J < r$.

Next, we consider a cotangent bundle $\tau^*_E: T^*(E) \rightarrow E$ equipped with the coordinates $(z^i, x^\alpha, \dot{z}^i, \dot{x}^\alpha)$ and the fibre basis $dz^i, dx^\alpha$, which allows to introduce a 1-form $\omega: E \rightarrow T^*(E)$ locally given as $\omega = \omega_i dz^i + \omega^\alpha dx^\alpha$. Moreover, in this paper we study Hamiltonian densities $H = H_\Omega$, which depend on 2nd-order jet variables, i.e. $H \in C^\infty(J^2(E))$, and can be constructed by means of special pullback umbilics omitted here for ease of presentation. Here, $\Omega = dz^i \wedge dx^\alpha$ denotes a volume form, whereas $\Omega_1 = \partial_1 \Omega$ corresponds to a boundary-volume form. Moreover, the formal change of a Hamiltonian functional $\mathcal{H} = J_\Omega \Omega$ along the solutions of an evolutionary vector field $v = \dot{v}^\alpha \partial_\alpha$, corresponding to a set of PDEs $\dot{x}^\alpha = v^\alpha$ with $v^\alpha \in C^\infty(J^2(E))$ and the time $t$ as evolution parameter of the solution, is of great significance, where we use the Lie-derivative reading $L_\xi(\omega)$ for a differential form $\omega$. Due to the fact that $H \in C^\infty(J^2(E))$, we are interested in $\mathcal{H} = J_\Omega L_\xi(\Omega)$. In fact, for the system configuration under investigation, i.e. 2nd-order Hamiltonian density and 2-dimensional spatial domain, the determination of $\mathcal{H}$ is a non-trivial task. However, in Schöberl and Schlacher (2018) an approach based on so-called Cartan forms is presented allowing to introduce the boundary operators

\begin{align}
\delta^{0,1} \dot{\eta} &= (\partial_\alpha \dot{\eta}_1 - d_{[0]}(\partial_{[0]} \dot{\eta}_1 - d_{[0]}(\partial_{[0]} \dot{\eta}_1)))d\omega^\alpha \wedge \Omega_2 \label{eq:1a}
\delta^{0,2} \dot{\eta} &= \partial_\alpha \dot{\eta}_2 d\omega^\alpha \wedge \Omega_2, \label{eq:1b}
\end{align}

where $\Omega_2$ denotes a boundary-volume form in coordinates adapted to the boundary. Moreover, for $H \in C^\infty(J^2(E))$ and $\dim(B) = 2$, the variational derivative reads

\begin{align}
\delta H = \delta_\alpha H \omega^\alpha &\wedge \Omega \label{eq:2}
\end{align}

with $\delta_\alpha = \partial_\alpha - d_{[0]}(\partial_{[0]} \dot{\eta}_1 - d_{[0]}(\partial_{[0]} \dot{\eta}_1)) + d_{[0]}(\partial_{[0]} \dot{\eta}_1) + d_{[0]}(\partial_{[0]} \dot{\eta}_1)$. Thus, based on Schöberl and Schlacher (2018), it is possible to introduce the following theorem.

**Theorem 1.** (Decomposition Theorem). (Rams, 2018, Theorem 3.2) Let $v$ be an evolutionary vector field and $H \in C^\infty(J^2(E))$ a second-order density. Then, by exploiting the domain operator (2) and the boundary operators (1), the integral $\int_B L_H \dot{\eta}_2(\Omega)\Omega$ can be decomposed into

\begin{align}
\mathcal{H} = \int_B v_\alpha \delta_\alpha \dot{\eta} + \int_{\partial B} v_\alpha \delta^{0,1}_\alpha \dot{\eta} + \int_{\partial B} v_\alpha \delta^{0,2}_\alpha \dot{\eta}. \label{eq:3}
\end{align}

Finally, let us mention that the bundle $\pi: E \rightarrow B$ enables the construction of further differential-geometric objects such as the tensor bundle $W_2^2(E) = T^*(E) \wedge \Omega^2 T^*(B)$, where $\omega = \omega_\alpha dx^\alpha \wedge \Omega$, with $\omega_\alpha \in C^\infty(J^2(E))$, denotes an element.

3. INFINITE-DIMENSIONAL PH-SYSTEMS

Next, we discuss the port-Hamiltonian framework for systems with 2nd-order Hamiltonian, see Rams and Schöberl (2017), where we focus on systems with 2-dimensional spatial domain. The presented approach is based on an underlying jet-bundle structure as well as on a certain power-balance relation, where we exploit Theorem 1.

Thus, we consider systems with 2nd-order Hamiltonian $H$, i.e. $H \in C^\infty(J^2(E))$, and a 2-dimensional, rectangular spatial domain $B = \{z^1, z^2 \mid z^1 \in [0, L_1], z^2 \in [0, L_2]\}$, where the boundary is divided into $\partial B_1 = \{z^1, z^2 \mid z^1 \in [0, L_1], z^2 = 0\}$, $\partial B_2 = \{z^1, z^2 \mid z^1 \in [0, L_1], z^2 = 0\}$, $\partial B_3 = \{z^1, z^2 \mid z^1 = L_1, z^2 \in [0, L_2]\}$ and $\partial B_4 = \{z^1, z^2 \mid z^1 \in [0, L_1], z^2 = L_2\}$. The boundary is divided into $\partial B_1 = \{z^1, z^2 \mid z^1 \in [0, L_1], z^2 = 0\}$, $\partial B_2 = \{z^1, z^2 \mid z^1 \in [0, L_1], z^2 = 0\}$, $\partial B_3 = \{z^1, z^2 \mid z^1 = L_1, z^2 \in [0, L_2]\}$ and $\partial B_4 = \{z^1, z^2 \mid z^1 \in [0, L_1], z^2 = L_2\}$, see Fig. 1. Then, a pH-formulation can be given as

\begin{align}
\dot{x}_\beta = (J_{\beta} - \mathcal{R}(\delta)) \delta \eta \label{eq:4}
\end{align}

together with appropriate boundary conditions. The objects $\mathcal{J}$ and $\mathcal{R}$ describe the internal power flow and dissipation effects, respectively, and can be interpreted as mappings of the form $\mathcal{J}, \mathcal{R}: W_2^2(E) \rightarrow \mathcal{V}(E)$. Next, the formal change of the Hamiltonian functional $\mathcal{H}$ along solutions of $\dot{\eta}$, which can be written as

\begin{align}
\mathcal{H} = \int_B \mathcal{R}(\delta \eta) \delta \eta + \int_{\partial B} (\dot{x}_\beta \delta \eta_\beta + \dot{x}_{[01]} \delta \eta_{[01]} \delta \eta) \label{eq:5}
\end{align}

by substituting $\dot{v} = \dot{x}$ with (3) in Theorem 1, is of particular interest and states a power-balance relation if $\mathcal{H}$ corresponds to the total energy of the system. Moreover, a local coordinate representation of (3) can be given by

\begin{align}
\dot{x}_\alpha = (J_{\alpha} - \mathcal{R}(\delta) \delta) \delta \eta_\alpha, \quad \alpha, \beta = 1, \ldots, n. \label{eq:6}
\end{align}
While the interconnection map $J$ is skew-symmetric, i.e., the coefficients fulfill $J^\alpha\beta = -J^\beta\alpha \in C^\infty(J^4(E))$, the dissipation map $R$ is symmetric and positive semi-definite, implying that the coefficients meet $R^{\alpha\beta} = R^{\beta\alpha} \in C^\infty(J^4(E))$ and $|R^{\alpha\beta}| \geq 0$ for the coefficient matrix. Thus, the power-balance relation

$$\mathcal{H} = -\int_{\partial B} \delta_\alpha(\mathcal{H})R^{\alpha\beta}(\partial_\beta(\mathcal{H}))\Omega + \int_{\partial B} (\delta_\alpha\delta^{\alpha\beta\gamma\delta}H + \delta^{\alpha\beta\gamma\delta}\mathcal{H})\Omega_2$$

allows to introduce boundary ports in a straightforward manner by exploiting the boundary operators (1). Now, we assume that the boundary $\partial B$ is divided into an acted boundary $\partial B_a = \partial B_1 \cup \partial B_2$ and an unactuated boundary $\partial B_a = \partial B_1 \cup \partial B_3$. This has the consequence that for the actuated boundary $\partial B_a$ we are able to set

$$(\delta^{\alpha\beta\gamma\delta}H)|_{\partial B_a} = u_{0,1}\delta^{\alpha\beta\gamma\delta}, \quad (\delta^{\alpha\beta\gamma\delta}\mathcal{H})|_{\partial B_a} = u_{0,2}\delta^{\alpha\beta\gamma\delta},$$

while no power flow takes place over the unactuated boundary $\partial B_a$, i.e., $(\delta^{\alpha\beta\gamma\delta}H)|_{\partial B_a} = 0$ and $(\delta^{\alpha\beta\gamma\delta}\mathcal{H})|_{\partial B_a} = 0$. It should be noted that the roles of inputs and outputs cannot be uniquely defined; however, for our purposes, where we intend to exploit mechanical quantities such as forces and bending moments as inputs and velocities and angular velocities, respectively, as collocated output quantities, we confine ourselves to the parameterization

$$\delta^{\alpha\beta\gamma\delta}H|_{\partial B_a} = B^{\alpha\beta\gamma\delta}_{0,1}u_{0,1}, \quad \delta^{\alpha\beta\gamma\delta}\mathcal{H}|_{\partial B_a} = B^{\alpha\beta\gamma\delta}_{0,2}u_{0,2}, \quad (6a)$$

for the boundary inputs as well as

$$B^{\alpha\beta\gamma\delta}_{0,1}\mathcal{H}|_{\partial B_a} = y_{\alpha\beta\gamma\delta}^{0,1}, \quad B^{\alpha\beta\gamma\delta}_{0,2}\mathcal{H}|_{\partial B_a} = y_{\alpha\beta\gamma\delta}^{0,2}, \quad (6b)$$

for the outputs, where $z_1 = 1, \ldots, l_1, \mu = 1, \ldots, l_2$.

**Example 2.** (Boundary-actuated Kirchhoff-Love plate). We consider a rectangular, boundary-actuated Kirchhoff-Love plate depicted in Fig. 1 and governed by

$$\rho Aw = -D_E(w_{[40]} + 2w_{[22]} + w_{[04]}), \quad (7)$$

see (Meirovitch, 1997, p. 448, Eq. (7.333)). The boundary conditions are discussed below, as they are of particular interest here. For the sake of simplicity we assume the material parameters $\rho, A, D_E > 0$ in (7) to be constant. If we introduce the generalised moment $p = \rho Aw$ as well as the Hamiltonian density $\mathcal{H} = T + V$, where $T = \frac{1}{2}p^2$, $V = \frac{1}{2}D_E((w_{[40]})^2 + (w_{[22]})^2 + 2\nu w_{[40]}w_{[22]} + 2(1-\nu)(w_{[11]})^2)$ with Poisson’s ratio $\nu$, we find an appropriate pH-system representation according to

$$\begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \delta_\mathcal{H} \\ \delta_\mathcal{H} \end{bmatrix} = \begin{bmatrix} -D_E(w_{[40]} + 2w_{[22]} + w_{[04]}) \end{bmatrix}.$$

Thus, the power-balance relation (4) reads

$$\begin{align} \mathcal{H} &= \int_0^{L_1} (Q_1\dot{w} + M_1\dot{w}_{[01]})|_{z_1 \in [0,L_2]} \, dz_1 \\
&\quad + \int_0^{L_2} (Q_2\dot{w} + M_2\dot{w}_{[10]})|_{z_1 \in [0,L_2]} \, dz_2, \quad (8) \end{align}$$

where the boundary relations $Q_1 = D_E(w_{[40]} + (2-\nu)w_{[21]}), \quad M_1 = -D_E(w_{[20]} + \nu w_{[02]}), \quad Q_2 = -D_E(w_{[20]} + (2-\nu)w_{[12]}), \quad M_2 = D_E(w_{[20]} + \nu w_{[02]})$ can be determined by evaluating the boundary operators (1) \(^1\).

With regard to control-engineering purposes, the boundary ports in (8) can be used to extract or deliver power. In fact, we assume that the plate is clamped at the boundary $\partial B_1$ and free at $\partial B_3$, i.e., we have the boundary conditions

$$w_{[10]} = 0 \quad \text{for} \quad \partial B_1, \quad M_2 = 0 \quad \text{for} \quad \partial B_3, \quad (9)$$

implying that the corresponding power ports vanish identically. However, the shear forces $Q_1|_{\partial B_2}$ and $Q_1|_{\partial B_3}$ at the actuated boundary shall be generated by piezo-like actuators, which are supposed to be perfectly attached at the boundaries $\partial B_2$ and $\partial B_1$. The forces supplied by these actuators, which are spatially distributed over (a part of) $\partial B_2$ and $\partial B_1$, see Fig. 1 and 2, can be described by $Q_1|_{\partial B_2} = A_{11}u_{m_1}^1$ and $Q_1|_{\partial B_3} = A_{22}u_{m_2}^2$, where the characteristic functions read

$$A_1 = -\Psi_1^{\alpha\beta\gamma\delta}(\tanh(\sigma z_2^1) - \tanh(\sigma(z_2^1 - \frac{L_2}{2})))|_{z_1 = 0}$$

$$A_2 = -\Psi_2^{\alpha\beta\gamma\delta}(\tanh(\sigma z_2^2) - \tanh(\sigma(z_2^2 - \frac{L_2}{2})))|_{z_1 = L_2},$$

with the material parameters hidden in $\Psi$ and $\sigma \in \mathbb{R}^+$. Thus, the voltages $u_{m_1}^1, u_{m_2}^2$ applied to the piezo-like actuators serve as manipulated variables. In light of this aspects, for (6) we set

$$Q_1|_{\partial B_2} = B_{11}^{\alpha\beta\gamma\delta}u_{m_1}^1, \quad B_{11}^{\alpha\beta\gamma\delta}\mathcal{H}|_{\partial B_a} = y_{11}^{\alpha\beta\gamma\delta}, \quad Q_1|_{\partial B_3} = B_{22}^{\alpha\beta\gamma\delta}u_{m_2}^2, \quad B_{22}^{\alpha\beta\gamma\delta}\mathcal{H}|_{\partial B_a} = y_{22}^{\alpha\beta\gamma\delta}, \quad (10)$$

with $B_{11}^{\alpha\beta\gamma\delta} = A_1, \quad B_{22}^{\alpha\beta\gamma\delta} = A_2$, and $u_{m_1}^1 = u_{m_1}^1, \quad u_{m_2}^2 = u_{m_2}^2$, to assign the roles of inputs and outputs. Note that since we have the shear forces $Q_1|_{\partial B_1}, Q_1|_{\partial B_3}$ as inputs solely, the relations $M_1|_{\partial B_2} = 0, \quad M_1|_{\partial B_3} = 0$ complete the boundary conditions (9). Hence, by virtue of the plate configuration and the assignment (10), the power-balance relation reads

$$\mathcal{H} = \int_{\partial B_1} u_{m_1}^1 y_{11}^{\alpha\beta\gamma\delta} \, dz_1 + \int_{\partial B_2} u_{m_2}^2 y_{22}^{\alpha\beta\gamma\delta} \, dz_2.$$

\(^1\) Note that for the boundaries $\partial B_2$ and $\partial B_3$ we have the adapted coordinates $z_1^1 = z_1$ and $z_1^2 = z_2$ const., i.e. the boundary operators indeed read (1); however for $\partial B_1$ and $\partial B_3$ the coordinates adapted to the boundary are $z_1^1 = z_1$ and $z_1^2 = z_1$ const. implying that the coordinates in (1) need to be swapped.
4. BOUNDARY CONTROL BASED ON STRUCTURAL INVARIANTS

Now, the objective is to adapt the energy-Casimir method to boundary-actuated pH-systems with 2nd-order Hamiltonian and 2-dimensional spatial domain, where the intention is to design a dynamic controller for the boundary-actuated Kirchhoff-Love plate discussed in Ex. 2 by exploiting a certain interconnection of plant and controller.

4.1 Control by Interconnection

Thus, we develop a dynamic controller based on structural invariants, which allows to shape the total energy of the closed loop and to inject additional damping in order to increase the dissipation rate. In light of the aspect that we only consider systems with lumped inputs, the interconnection of a plant (3) and a finite-dimensional controller, beneficially given in the pH-formulation,

\[ x_c^\alpha = (J_c^{\alpha\beta} - R_c^{\alpha\beta}) \delta_{\beta\alpha} H_c + G_c^{\alpha\alpha} u_c, \alpha, 1 + G_c^{\alpha\beta} u_c, \alpha, 2 \]

(11a)

\[ y_c^{\alpha, 1} = G_c^{\alpha\beta} \delta_{\alpha\beta} H_c, \quad y_c^{\alpha, 2} = G_c^{\alpha\beta} \delta_{\alpha\beta} H_c, \]

(11b)

with \( \alpha, \beta = 1, \ldots, n_c \), \( \alpha = 1, \ldots, l_1 \) and \( \mu = 1, \ldots, l_2 \), is motivated, where we splitted the controller inputs into two different parts to take into account the two different categories of boundary ports. The idea is to couple the finite-dimensional controller to the infinite-dimensional plant at the actuated boundary \( \partial B_a = \partial B_2 \cup \partial B_1 \) in a power-conserving manner fulfilling

\[ \int_{\partial B_a} (u_{\alpha, 1} | y_c^{\alpha, 1} + u_{\alpha, 2} | y_c^{\alpha, 2}) + u_{c, \alpha, 1} | y_c^{\alpha, 1} + u_{c, \alpha, 2} | y_c^{\alpha, 2} = 0. \]

(12)

Note that this is quite different compared to Rams and Schöberl (2017), where boundary-actuated systems with 1-dimensional spatial domain are considered, as well as to Malzer et al. (2020), where the energy-Casimir method is investigated for in-domain actuated systems with 2-dimensional spatial domain, as we have to integrate over the 1-dimensional boundary here. Hence, if we choose

\[ u_{c, \alpha, 1} = \int_{\partial B_a} K_{\alpha, 1} | y_c^{\alpha, 1}, \quad u_{\alpha, 1} = -K_{\alpha, 1} | y_c^{\alpha, 1}, \]

(13a)

\[ u_{c, \alpha, 2} = \int_{\partial B_a} K_{\alpha, 2} | y_c^{\alpha, 2}, \quad u_{\alpha, 2} = -K_{\alpha, 2} | y_c^{\alpha, 2}, \]

where \( K_{\alpha, 1} \) and \( K_{\alpha, 2} \) denote appropriate mappings – that can be interpreted as degrees of freedom but are set to the identity matrix for the most part – the relation (12) is satisfied. As a consequence, the closed-loop system can be formulated as a pH-system described by the Hamiltonian

\[ \mathcal{H}_c = \mathcal{H} + H_c. \]

Moreover, because of the power-conserving interconnection (12), the formal change of \( \mathcal{H}_c \) along solutions of the closed-loop system reads

\[ \mathcal{H}_c = -\int_B \delta_c(H) \mathcal{R}^{\alpha\beta} \delta_c(H) \Omega - \partial_{\alpha_c}(H_c) R_c^{\alpha\beta} \delta_{\alpha_c}(H_c), \]

highlighting that the controller allows to inject damping.

Remark 3. At this point let us stress that we assume that the closed-loop solutions exist. In principle, the well-posedness of the closed-loop system would need to be verified by means of functional analysis. However, in this paper the emphasis is on a formal, geometric approach focusing on energy considerations, where the relations \( \mathcal{H}_c > 0 \) and \( \mathcal{H}_c \leq 0 \) serve as necessary conditions for (possible) stability investigations in the sense of Liapunov.

4.2 Controller Design

Next, we are interested in certain functionals allowing for a relation between plant and controller states in order to shape the Hamiltonian of the closed loop. Thus, in accordance with Rams and Schöberl (2017), we consider

\[ \mathcal{F}^\lambda = x_c^\lambda + \int_B C^\lambda \Omega, \quad C^\lambda \in C^{\infty}(J^2(\mathcal{E})), \]

(14)

with \( \lambda = 1, \ldots, n_c \leq n_c \), where it should be mentioned that here we have \( \dim(B) = 2 \). Thus, the functionals (14) have to meet \( \mathcal{F}^\lambda = 0 \) independently of \( \mathcal{H} \) and \( H_c \) to qualify as structural invariants.

Theorem 4. (Structural Invariants). Consider the closed-loop system stemming from the interconnection of the plant (3) and the controller (11) by means of (13). Then, (14) are structural invariants if they meet the conditions

\[ (J_c^{\alpha\beta} - R_c^{\alpha\beta}) = 0 \quad (15a) \]

\[ \delta_c C^\lambda(J^\alpha^\beta - R^\alpha^\beta) = 0 \quad (15b) \]

\[ (G_c^{\alpha\alpha} K_{\alpha, 1} B_1^{\alpha\alpha} + \delta_c C^\lambda) |_{\partial B_a} = 0 \quad (15c) \]

\[ (G_c^{\alpha\beta} K_{\alpha, 1} B_2^{\alpha\beta} + \delta_c C^\lambda) |_{\partial B_a} = 0 \quad (15d) \]

\[ (\dot{x}_c^{\alpha\beta} - \lambda_c^{\alpha\beta} x_c^{\alpha\beta} + \dot{\lambda}_c^{\alpha\beta} \lambda_c^{\alpha\beta}) |_{\partial B_a} = 0. \]

(15e)

Proof. To prove the conditions (15), we substitute \( v \) and \( \delta \) by \( \dot{x} \) and \( \lambda_c \), respectively, in the decomposition Theorem 1. Moreover, by inserting the plant and controller dynamics described by (5) and (11a), respectively, as well as the coupling (13) together with the boundary-output assignments (6b), we are able to deduce

\[ \mathcal{F}^\lambda = (J_c^{\alpha\beta} - R_c^{\alpha\beta}) \delta_c H_c + \int_{\partial B_a} G_c^{\alpha\beta} K_{\alpha, 1} B_1^{\alpha\beta} \dot{x}_c^{\alpha\beta} \Omega_2 + \int_{\partial B_a} G_c^{\alpha\beta} K_{\alpha, 1} B_2^{\alpha\beta} \dot{x}_c^{\alpha\beta} \Omega_2 + \int_B \delta_c C^\lambda(J^\alpha^\beta - R^\alpha^\beta) \delta_c H \Omega + \int_B \dot{x}_c^{\alpha\beta} \delta_c C^\lambda \Omega_2 + \int_{\partial B_a} \dot{x}_c^{\alpha\beta} \lambda_c^{\alpha\beta} (\lambda_c^{\alpha\beta}) \Omega_2 = 0, \]

enabling to find the conditions (15).

Note that – in contrast to (Rams and Schöberl, 2017, Eq. (17)), where pH-systems with 1-dimensional spatial domain are considered – the conditions (15) basically hold for systems with 1- or 2-dimensional spatial domain; however, the differences are hidden in the geometric objects that of course strongly depend on the spatial dimension. Moreover, the conditions (15) clearly distinguish from those of Prop. 2 in Malzer et al. (2020), where Casimir conditions for in-domain actuated systems with 2-dimensional spatial domain are studied.

Remark 5. It should be stressed that the proposed control scheme can be exploited for nonlinear systems as well, see (Malzer et al., 2018, Sec. 4), where a Casimir-based controller for a nonlinear Euler-Bernoulli beam structure is presented. However, in this paper we intend to combine the proposed controller with an infinite-dimensional observer, where the design method is restricted to the linear scenario, and therefore, in the following we derive a controller for a linear Kirchhoff-Love plate.
Example 6. (Casimir-based Controller for Ex. 2). Next, we design a controller for the Kirchhoff-Love plate to stabilise the (approximated) configuration

\[
w^d = \begin{cases} \alpha(z)^2 & \text{for } 0 \leq z^1 < \frac{L_0}{4} \\ b(z^1 - \frac{L_0}{4}) + \frac{L_0^2}{4} & \text{for } \frac{L_0}{4} \leq z^1 \leq L_1 \end{cases}
\]

with \(a, b > 0\). In light of the fact that for the system under investigation we have two output densities given in (10), we intend to relate two controller states to the plant. To this end, we make the trivial choice \(K_{0,1} = I\), with \(I\) denoting the identity matrix, and \(K_{0,2} = 0\) (as we have no boundary-actuation corresponding to the category \(\partial, 2\)) regarding the design parameters, and thus, the interconnection of plant and controller reads

\[
u = \int_{\partial B_2} A_1 \frac{\partial}{\partial z^1} dz^1, \quad \nu_2 = -y_{c,1}
\]

\[
u_2 = \int_{\partial B_4} A_2 \frac{\partial}{\partial z^1} dz^1, \quad \nu_2 = -y_{c,2}
\]

Hence, if we set the parameters \(G_{c,1,1} = \frac{1}{2}\) and \(G_{c,2,1} = 0\), from (15c) – (15e) we find

\[
(\Lambda_1 + \delta_0 w^1 C^1) |_{\partial B_2} = 0, \quad (A_2 + \delta_0 w^2 C^2) |_{\partial B_4} = 0,
\]

where \(\delta_0 w^1 C^1|_{\partial B_2} = 0\) and \(\delta_0 w^2 C^2|_{\partial B_4} = 0\) have to be met since \(K_{0,2} = 0\), as well as for the unactuated boundary

\[
(\delta_0 w^1 C^1 + \rho \delta_0 w^2 C^2 + \dot{w}_{(0)})(\delta_0 w^2 C^2 + \rho \delta_0 w^2 C^2) |_{\partial B_2} = 0
\]

for \(\lambda = 1, 2\). Moreover, for the domain condition (15b) we have \(\delta_0 w C^1 = 0\) and \(\delta_0 w C^2 = 0\), which are trivially satisfied if \(C^1\) and \(C^2\) stem from total derivatives. In light of this aspects, we find that a possible choice for Casimir functions is given by \(C^1 = d_{(0)}(\frac{\sqrt{2}}{L_0} z^1 \Lambda_1 w)\) and \(C^2 = d_{(0)}(\frac{\sqrt{2}}{L_0} \Lambda_2 w)\), which enable to deduce the relations

\[
x^1 = \int_{\partial B_2} A_1 u_0 dz^1 + \kappa^1, \quad x^2 = \int_{\partial B_4} A_2 u_0 dz^1 + \kappa^2
\]

with \(\kappa^1, \kappa^2\) depending on the initial states of the plant and the controller. Compared to boundary controllers for 1-dimensional systems, like in Rams and Schöberl (2017) for instance, this is a major difference, as we have weighted system states integrated over \(\partial B_2\). However, the chosen Casimir functionals only assign a part of the controller dynamics. In fact, two further controller states shall be exploited to inject damping, and thus, the controller is described by

\[
J_c - R_c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -R_{c,3}^3 & R_{c,4}^3 - R_{c,4}^4 \end{bmatrix}
\]

\[
G_c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ G_{c,1,1} & G_{c,1,2} \\ G_{c,1,3} & G_{c,2,1} \\ G_{c,1,4} & G_{c,2,2} \\ G_{c,2,3} & G_{c,3,1} \\ G_{c,2,4} & G_{c,3,2} \\ G_{c,3,3} & G_{c,3,4} \end{bmatrix}
\]

Moreover, to properly shape the closed-loop Hamiltonian \(\mathcal{H}_{cl}\), we set the controller Hamiltonian to

\[
H_c = \frac{1}{2}(x_{c,1}^2 - x_{c,2}^2 - \frac{a + b}{c_0} x_{c,1}^2)
\]

\[
+ \frac{\gamma}{2}(x_{c,2}^2 - x_{c,3}^2 - \frac{a + b}{c_0} x_{c,2}^2) + \frac{1}{2} M_{c,\nu,\nu} x_{c,\nu}^2 x_{c,\nu}^2
\]

\[2\] Due to the special force characteristic, our actuators are able to generate equivalent bending moments.
5. OBSERVER DESIGN

In this section, we present an energy-based observer-design method, where the intention is to exploit the pH-system representation and to introduce error-injection terms based on available measurements that are located at the boundary. To properly determine these observer-correction terms, we apply a design approach based on energy balancing – also presented in (Malzer et al., 2018, Sec. 5) with regard to the controller design for spatially 1-dimensional systems – for the observer-error system, which relies on energy shaping and damping injection. For that purpose, it is assumed to have measurements $y_{m,1}^1$ and $y_{m,2}^2$ (corresponding to deflections and angular displacements) and their observer-correction terms $\dot{\hat{\alpha}}$ and $\ddot{\hat{\alpha}}_2$, respectively, which have to be determined depending on the observer inputs to (20a) and (20b), regarding the observer-error system the plant inputs are cancelled. The boundary-inputs read

$$\begin{align}
\dot{\hat{\alpha}}_1 = (J^\alpha \beta - R^\alpha \beta)\delta \hat{\beta} \hat{H}, \quad (19)
\end{align}$$

with $\hat{\alpha}, \hat{\beta} = 1, \ldots, n$, where the boundary conditions need to be determined and $\delta \hat{\beta} = \delta \hat{H}$ is the copy of the Hamiltonian density depending on the observer states $\hat{x}$. Due to the fact that we assume to have measurements available only at the boundary $\partial B_m = \partial B_1 \cup \partial B_2$, we intend to introduce no observer-correction terms at $\partial B_m = \partial B_1 \cup \partial B_2$. Furthermore, as we have no actuation at this part of the boundary, no power flow takes place at $\partial B_m$, i.e.,

$$\begin{align}
\int_{\partial B_m} \dot{\hat{x}} \delta \alpha \hat{\beta} \hat{H} = 0, \quad \int_{\partial B_m} \dot{\hat{x}}[01] \delta \beta \hat{H} = 0.
\end{align}$$

As a consequence, by exploiting the decomposition Theorem 1, the formal change of $\mathcal{H} = \int_B \hat{H} \Omega$ reads

$$\dot{\mathcal{H}} = -\int_B \delta \hat{\beta}(\hat{H}) R^{\hat{\beta}} \delta \hat{\beta}(\hat{H}) \Omega + \int_{\partial B_m} \left( \dot{\hat{\beta}} R^{\hat{\beta}} \hat{H} + \dot{\hat{x}}[01] \delta \beta \hat{H} \right) \Omega_2,$$

and allows to introduce observer-correction terms as

$$\begin{align}
\delta \alpha \hat{\beta}(\hat{H}) |_{\partial B_m} &= B^{\alpha_1}_{\beta_1} \delta \alpha \hat{\beta} + B^{\alpha_2}_{\beta_2} \delta \alpha \hat{\beta}, \\
\delta \beta \hat{\beta}(\hat{H}) |_{\partial B_m} &= B^{\beta_1}_{\beta_1} \delta \beta \hat{\beta} + B^{\beta_2}_{\beta_2} \delta \beta \hat{\beta},
\end{align}$$

with $\beta = 1, \ldots, \beta_0, \hat{\beta} = 1, \ldots, \hat{\beta}_2$ denoting the components of the mapping $B^{\alpha_1}_{\beta_1}$, $B^{\alpha_2}_{\beta_2}$, respectively, which have to be determined depending on the number of available measurements. The observer inputs (20a) and (20b) comprise the inputs of the plant (6a) as well as the observer-correction terms $\dot{\hat{\alpha}}_1$, $\dot{\hat{\alpha}}_2$, where $\dot{\hat{\alpha}}_1$, $\dot{\hat{\alpha}}_2$ denote the components of the mappings $B^{\hat{\alpha}_1}_{\hat{\beta}_1}$, $B^{\hat{\alpha}_2}_{\hat{\beta}_2}$, respectively, which have to be determined depending on the spatial position of the available measurements. In fact, $B^{\hat{\alpha}}_{\hat{\beta}}$ and $B^{\hat{\beta}}_{\hat{\beta}}$ shall be chosen such that for the resulting observer-error system the observer-correction terms $\dot{\hat{k}}$, $\ddot{\hat{k}}$ are collocated to the error terms including the available measurements. As a consequence, we are able to define

$$\begin{align}
\dot{\hat{\beta}}_1 &= (J^\alpha \beta - R^\alpha \beta) \delta \hat{\beta} \hat{H}, \\
\dot{\hat{\beta}}_2 &= (J^\alpha \beta - R^\alpha \beta) \delta \hat{\beta} \hat{H},
\end{align}$$

which correspond to the observer-equivalent of the measurements $y_{m,1}^1, y_{m,2}^2$. Next, we study the observer error $\hat{x} = \hat{x} - \hat{x}$, where the dynamics can be deduced by substituting (5) and (19) in $\hat{x} = \hat{x} - \hat{x}$, and can be formulated as

$$\begin{align}
\dot{\hat{x}} &= (J^\alpha \beta - R^\alpha \beta) \delta \hat{\beta} \hat{H},
\end{align}$$

since we confine ourselves to linear systems. To determine the boundary-port relations of the observer-error system, we study the formal change of $\hat{\mathcal{H}} = \int_B \hat{H} \Omega$, where $\hat{H}$ exhibits the same form as $H$ but depends on error coordinates $\hat{x}$, which follows to

$$\dot{\mathcal{H}} = -\int_B \delta \hat{\beta}(\hat{H}) R^{\hat{\beta}} \delta \hat{\beta}(\hat{H}) \Omega + \int_{\partial B_m} \left( \dot{\hat{\beta}} R^{\hat{\beta}} \hat{H} + \dot{\hat{x}}[01] \delta \beta \hat{H} \right) \Omega_2.$$

The restriction to linear systems implies $\delta \beta \hat{\beta} = \delta \beta \hat{\beta}$ and $\delta \beta \hat{\beta} = \delta \beta \hat{\beta}$ and thus, we set the observer inputs to (20a) and (20b), regarding the observer-error system plant inputs are cancelled. The boundary-inputs read

$$\begin{align}
\delta \beta \hat{\beta}(\hat{H}) |_{\partial B_m} &= B^{\beta_1}_{\beta_1} \delta \beta \hat{\beta}, \\
\delta \beta \hat{\beta}(\hat{H}) |_{\partial B_m} &= B^{\beta_2}_{\beta_2} \delta \beta \hat{\beta},
\end{align}$$

while the collocated boundary-outputs are given by

$$\begin{align}
\delta \beta \hat{\beta}(\hat{H}) |_{\partial B_m} &= \delta \beta \hat{\beta},
\end{align}$$

where we have $y_{m,1}^1 = y_{m,2}^1 y_{m,1}^2$ and $y_{m,2}^2 = y_{m,2}^2 y_{m,2}^2$ with the measurements $y_{m,1}^1, y_{m,2}^2$ as well as the corresponding observer quantities $\tilde{y}_{m,1}^1, \tilde{y}_{m,2}^2$ according to (20c). Let us stress again that the coefficients $B^{\hat{\alpha}}_{\hat{\beta}}$ take into account the spatial position of the available measurements.

Consequently, the dynamics of the observer error are reformulated as a pH-system, where the boundary-correction terms $\dot{\hat{k}}_1$ and $\dot{\hat{k}}_2$ shall be determined such that the observer-error system exhibits a desired behaviour. To this end, we apply the energy-balancing approach presented in (Malzer et al., 2018, Sec. 5). In particular, the observer-correction terms $\dot{\hat{k}}_1$ and $\dot{\hat{k}}_2$ shall be used to shape the error Hamiltonian $\hat{\mathcal{H}}$ and to inject damping into the observer-error system.

**Theorem 7.** (Observer Design). Consider the observer-error system (21) with the boundary-inputs and -outputs (23), where the observer-correction terms are splitted according to $\dot{\hat{k}}_1 = \dot{\hat{k}}_1 + \gamma \hat{\beta}$ and $\dot{\hat{k}}_2 = \dot{\hat{k}}_2 + \gamma \hat{\beta}$. Thus, if we find a $\hat{H}_a$ such that the matching conditions

$$\begin{align}
(J^\alpha \beta - R^\alpha \beta) \delta \hat{\beta} \hat{H}_a &= 0, \\
\delta \beta \hat{\beta}(\hat{H}_a) |_{\partial B_m} &= 0,
\end{align}$$

are fulfilled, the energy-shaping correction terms

$$\begin{align}
\int_{\partial B_m} B^{\beta_1}_{\beta_1} \delta \beta \hat{H}_a \Omega_2 &= -\int_{\partial B_m} \delta \beta \hat{H}_a \Omega_2, \\
\int_{\partial B_m} B^{\beta_2}_{\beta_2} \delta \beta \hat{H}_a \Omega_2 &= -\int_{\partial B_m} \delta \beta \hat{H}_a \Omega_2
\end{align}$$

map the observer-error system (21) into the target system.
\[ \dot{\varepsilon} = (J_{\delta\beta} - R_{\delta\beta})\delta_{\beta}\mathcal{H}_d, \]  
(26)

ensuring that \( \mathcal{H}_d = \int_{\partial B} \mathcal{H}_d \, d\Omega \), with \( \mathcal{H}_d = \mathcal{H} + \mathcal{H}_n \), exhibits a certain minimum. Thus, if we set the new inputs of the target system, which can be parameterized as

\[ (\delta_{\alpha}^{\mathcal{H}_d})|_{\partial B_m} = B^{\delta_{\alpha}^{\mathcal{H}_d} \hat{p}}_{\delta_{\beta}} \cdot \delta_{\beta}\mathcal{H}_d, \]  
(27)

and are referred to as damping-injection inputs, to

\[ \delta_{\beta}^{\alpha} = -K_{\delta_{\alpha}} \delta_{\beta}^{\alpha}, \]

with appropriate positive definite mappings \( K_{\delta_{\alpha}} \) and \( K_{\delta_{\beta}} \), the desired error system (26) is dissipative.

**Proof.** The matching condition (24a) follows immediately by substituting the ansatz \( \mathcal{H}_d = \mathcal{H} + \mathcal{H}_n \) in

\[ (J_{\delta\beta} - R_{\delta\beta})\delta_{\beta}\mathcal{H}_d = (J_{\delta\beta} - R_{\delta\beta})\delta_{\beta}\mathcal{H}_d, \]

where (24a) is trivially satisfied if \( \mathcal{H}_n \) stems from total derivatives. Next, we compare the power-balance relation of the error-Hamiltonian, which reads

\[ \dot{\mathcal{H}} = -\int_{\partial B} (\delta_{\beta}) \mathcal{H}_{\delta\beta} \delta_{\beta}\mathcal{H}_d + \int_{\partial B_m} \left( \delta_{\beta}^{\mathcal{H}_d} + \int_{\partial B_m} \mathcal{H}_{\delta\beta} \delta_{\beta}\mathcal{H}_d \right) \hat{p} \Omega_d = 0, \]

which allows to find the matching conditions (24b) and (24c). Thus, for the formal change of \( \mathcal{H}_d \) we have

\[ \dot{\mathcal{H}} = -\int_{\partial B} (\delta_{\beta}) \mathcal{H}_{\delta\beta} \delta_{\beta}\mathcal{H}_d + \int_{\partial B_m} \left( \delta_{\beta}^{\mathcal{H}_d} + \int_{\partial B_m} \mathcal{H}_{\delta\beta} \delta_{\beta}\mathcal{H}_d \right) \hat{p} \Omega_d, \]

enabling to introduce the inputs according to (27). Therefore, a comparison of the system targets with (23a), where we insert \( \delta_{\alpha}^{\mathcal{H}_d} = \beta_{\alpha}^{\mathcal{H}_d} + \gamma_{\alpha}^{\mathcal{H}_d} \) and \( \delta_{\beta}^{\mathcal{H}_d} = \beta_{\beta}^{\mathcal{H}_d} + \gamma_{\beta}^{\mathcal{H}_d} \), allowing to write

\[ \delta_{\alpha}^{\mathcal{H}_d} = \beta_{\alpha}^{\mathcal{H}_d} + \gamma_{\alpha}^{\mathcal{H}_d}, \quad \delta_{\beta}^{\mathcal{H}_d} = \beta_{\beta}^{\mathcal{H}_d} + \gamma_{\beta}^{\mathcal{H}_d}, \]

yields the energy-shaping correction terms \( \beta_{\alpha}^{\mathcal{H}_d} \gamma_{\alpha}^{\mathcal{H}_d} = -\left( \delta_{\alpha}^{\mathcal{H}_d} \mathcal{H}_d \right) \partial B_m \) and \( \beta_{\beta}^{\mathcal{H}_d} \gamma_{\beta}^{\mathcal{H}_d} = -\left( \delta_{\beta}^{\mathcal{H}_d} \mathcal{H}_d \right) \partial B_m \). Hence, if we exploit the boundary-input and -output parameterisation according to (27) and (23b), respectively, as well as the damping-injection laws (28), relation (29) reads

\[ \dot{\mathcal{H}} = -\int_{\partial B} (\delta_{\beta}) \mathcal{H}_{\delta\beta} \delta_{\beta}\mathcal{H}_d + \int_{\partial B_m} \left( \delta_{\beta}^{\mathcal{H}_d} + \int_{\partial B_m} \mathcal{H}_{\delta\beta} \delta_{\beta}\mathcal{H}_d \right) \hat{p} \Omega_d, \]

Example 8. (Observer Design for Ex. 2) Now, regarding the observer design we assume that the plate deflection \( w(q, t) = y_{m1} \), together with the corresponding velocity \( \dot{w}(q, t) = y_{m2} \) as well as \( w(\frac{3\pi}{4}, L_2, t) = y_{m3} \) and \( \dot{w}(\frac{3\pi}{4}, L_2, t) = y_{m4} \) are available as measurement quantities, where the positions are marked by \( x \) in Fig. 1. Following the presented approach, we introduce the dynamics of the ipH-observer as a copy of the plant

\[ \begin{bmatrix} \hat{\omega} \\ \hat{\rho} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \delta_{\beta}\mathcal{H} \\ \delta_{\beta}\mathcal{H} \end{bmatrix} = \left[-D_E(\hat{w}_{[40]} + \hat{w}_{[22]} + \hat{w}_{[04]}), \right], \]

with the observer density

\[ \mathcal{H} = \frac{1}{2pA}\mathbf{u}^2 + p \left( \frac{2}{w_{[20]}^2} + (\hat{w}_{[02]}^2) \right), \]

depending on the observer states \( \hat{w} \) and \( \hat{\rho} \). Note that in accordance with the plate configuration the observer dynamics are restricted to the boundary conditions \( \hat{w} = 0 \), \( \hat{w}_{[01]} = 0 \) for \( \partial B_1 \), \( \hat{M}_1 = 0 \) for \( \partial B_2 \), \( \hat{Q}_2 = 0 \), and \( \hat{M}_2 = 0 \) for \( \partial B_3 \), where the relations \( \hat{Q}_1 = D_E(\hat{w}_{[01]} + (\nu(\hat{w}_{[11]})) \) (30)

Thus, in light of (20) we write

\[ \hat{Q}_1 = B^{\hat{Q}_1}_{[01]} \hat{w}_{[01]} + B^{\hat{Q}_1}_{[11]} \hat{w}_{[11]}, \quad \hat{B}_1^{\hat{Q}_1}_{[01]} B^{\hat{Q}_1}_{[01]} = \hat{m}_1, \]

with the components \( B^{\hat{Q}_1}_{[01]} \) of the mapping \( B^{\hat{Q}_1} \) and the error-injection terms \( B^{\hat{Q}_1}_{[01]} \) to be determined. As we have measurements available at \( (z^1 = \frac{3\pi}{4}, z^2 = 0) \) and \( (z^1 = \frac{3\pi}{4}, z^2 = L_2) \), we set \( B^{\hat{Q}_1}_{[01]} = \delta(z^1 = \frac{3\pi}{4}) \) for \( z^2 = 0 \) and \( B^{\hat{Q}_1}_{[01]} = \delta(z^1 = \frac{3\pi}{4}) \) for \( z^2 = L_2 \) with \( \delta(z) \) denoting the Dirac delta function. To derive proper error-injection terms \( \delta_{\alpha}^{\mathcal{H}_d} \), we study the dynamics of the observer error, which can be introduced by means of \( \hat{w} = w - \hat{w} \), \( \hat{\rho} = p - \hat{p} \) together with the corresponding derivatives and can be written as

\[ \begin{bmatrix} \hat{w} \\ \hat{\rho} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \delta_{\beta}\mathcal{H} \\ \delta_{\beta}\mathcal{H} \end{bmatrix} = \left[-D_E(\hat{w}_{[40]} + \hat{w}_{[22]} + \hat{w}_{[04]}), \right] \]

Example 7. (Kirchhoff-Love Plate) In light of the available measurements \( w(\frac{3\pi}{4}, 0) \), and \( w(\frac{3\pi}{4}, L_2) \), for the energy-balancing scheme we choose

\[ \begin{align*}
\hat{H}_1 &= d_{[01]}(k_1 \delta(z^1 - \frac{3\pi}{4}) \hat{w}^2), \\
\hat{H}_2 &= d_{[01]}(k_2 \delta(z^1 - \frac{3\pi}{4}) \hat{w}^2),
\end{align*} \]
(31a)

\[ \begin{align*}
\hat{H}_3 &= d_{[01]}(k_3 \delta(z^1 - \frac{3\pi}{4}) \hat{w}^2), \\
\hat{H}_4 &= d_{[01]}(k_4 \delta(z^1 - \frac{3\pi}{4}) \hat{w}^2),
\end{align*} \]
(31b)

where we intentionally write \( \hat{H}_4 \), such that \( \int_{\partial B} \hat{H}_2 \frac{\partial s}{\partial z} d\Omega \) yields \( \hat{\omega}^2 \), \( k_2 \delta(z^1 - \frac{3\pi}{4}) \hat{w}^2 d\Omega \) by means of Stoke’s Theorem as \( d_{[01]} \Omega = -dz^1 \) for instance. Thus, the ansatz (31) fulfills the matching conditions (24) and yields \( \beta_{\alpha}^{\beta_{\alpha}} = -k_1 \hat{w}^2 \) and \( \beta_{\alpha}^{\beta_{\alpha}} = -k_2 \hat{w}^2 \), respectively, by evaluating (25). Moreover, if we use the damping-injection laws
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