Group Velocity of Discrete-Time Quantum Walks

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I. INTRODUCTION

While classical random walks have found important applications in classical computing, quantum random walks are anticipated to lead to significant applications in quantum computing. We will here focus on discrete-time quantum walks. These were first introduced by Aharonov \textit{et al.}\textsuperscript{1} as an example of a quantum physical process that exhibit striking differences from classical processes. Such differences have motivated a search for quantum algorithms based on quantum walks that outperform their classical counterparts \textsuperscript{2,3,4}.

Aharonov \textit{et al.}\textsuperscript{5} developed the basic theory of quantum walks on graphs with a focus on mixing-time properties. They proved that quantum walks mix at most polynomially faster than classical random walks. Marquezino \textit{et al.}\textsuperscript{6} presented an analytical expression for the mixing time of discrete-time quantum walks on the hypercube.

Particularly important in this context is the concept of hitting time. For classical random walks, the hitting time is unambiguously defined as the average time the walker takes to hit the final vertex for the first time after departing from the initial vertex. The generalization for quantum walks is not straightforward, however, since measurements disturb the movement of the walker, see \textit{e.g.}\textsuperscript{7}. One possibility is to let the walk evolve unmeasured, \textit{i.e.}, unitarily, until the arrival probability at the final vertex is above some threshold. Another possibility is to perform a partial measurement at each step of the walk to check whether the walker has already reached the final vertex. Both definitions of arrival times have drawbacks. In the first case, the walker is not confirmed to have hit the final vertex, as there is only a probability of hitting it. In the second case, the quantum walk has been modified by the repeated measurements so that one is actually calculating the hitting time of a non-unitary walk that is effectively subject to a quantum Zeno effect. Nevertheless, at least for the symmetric walk on the hypercube\textsuperscript{8}, the two strategies for defining the hitting time yield similar results. We note that the hitting time can, in general, be infinite. Krovi and Brun\textsuperscript{8} analyzed the conditions for infiniteness of the hitting time for walks on Cayley graphs of Abelian groups. This includes the special cases of quantum walks on the line and on hypercubes.

We show that certain types of quantum walks can be modeled as waves that propagate in a medium with phase and group velocities that are explicitly calculable. Since the group and phase velocities indicate how fast wave packets can propagate causally, we propose the use of these wave velocities in a new definition for the hitting time of quantum walks. The new definition of hitting time has the advantage that it requires neither the specification of a walker’s initial condition nor of an arrival probability threshold. We give full details for the case of quantum walks on the Cayley graphs of Abelian groups. This includes the special cases of quantum walks on the line and on hypercubes.
standard definitions of hitting times for quantum walks on one-dimensional lattices and on the $n$-dimensional hypercube. For the lattice, the hitting times coincide approximately. For the hypercube, the group-velocity hitting time is $O(n \sqrt{N})$, which is greater than the result from the standard definitions that is essentially $O(n)$. We also analyze the group velocity of quantum walks on Cayley graphs of finite Abelian groups.

The paper is organized as follows. In Sec. II we review the theory of quantum walks on Cayley graphs of finite Abelian groups and present a general procedure for calculating the group velocity. In Sec. III we present finite Abelian groups and present a general procedure for calculating the group velocity. In Sec. IV we compare the group-velocity hitting time with the standard defining time. For the hypercube, the group-velocity hitting time is approximately $\frac{1}{2} \sqrt{N}$. For the hypercube, the group-velocity hitting time is approximately $\frac{1}{2} \sqrt{N}$. The Fourier transform on $G$ is given by the operator

$$F_{G} = \frac{1}{\sqrt{|G|}} \sum_{g, h \in G} \chi_{g}(h)|g|/|h|, \quad (3)$$

where $\chi_{g}$ is a character of $G$ given by

$$\chi_{g}(h) = \prod_{j=1}^{n} \omega_{N_{j}}^{g_{j} h_{j}}, \quad (4)$$

where $\omega_{N_{j}} = \exp(\frac{2\pi i}{N_{j}})$ is the $N_{j}$-primitive root of unity.

The Fourier basis is an orthonormal set of vectors defined by

$$|\tilde{x}_{h}\rangle = \frac{1}{\sqrt{|G|}} \sum_{g \in G} \chi_{g}(h^{-1})|g\rangle, \quad (5)$$

where $h \in G$. That basis is interesting because any vector $|h\rangle|\tilde{x}_{g}\rangle$ is an eigenvector of the shift operator, in fact

$$S|h\rangle|\tilde{x}_{g}\rangle = \chi_{h}(g)|h\rangle|\tilde{x}_{g}\rangle, \quad (6)$$

which can be proved by using Eq. (4).

II. QUANTUM WALKS ON CAYLEY GRAPHS

A Cayley graph encodes the structure of a discrete group. Let $G$ be a finite group and $S$ a generating set. The Cayley graph $\Gamma(G, S)$ is a directed graph such that the vertex set is identified with $G$ and the edge set consists of pairs of the form $(g, gh)$ for all $g \in G$ and $h \in S$. The Cayley graph depends in an essential way on the generating set. It is interesting to diminish that dependence by demanding that $S$ be a symmetric set, that is, if $h \in S$ then $h^{-1} \in S$, where $h^{-1}$ is the inverse of $h$.

In that case, the Cayley graph is a symmetric regular graph of degree $|S|$ with no loops, where $|S|$ is the cardinality of $S$. From now on we consider only symmetric generating sets.

Coined quantum walks can be defined on $\Gamma(G, S)$ in the following way. Let $H_{S}$ be the Hilbert space spanned by states $|h\rangle$ where $h \in S$. $H_{S}$ is the coin or internal space. Let $H_{G}$ be the Hilbert space spanned by states $|g\rangle$ where $g \in G$. $H_{G}$ is the physical stage where the walk takes place. The evolution operator for one step of the walk is $U = S \circ (C \otimes I)$ where

$$C = \sum_{h_{1}, h_{2} \in S} C_{h_{1}, h_{2}} |h_{1}\rangle \langle h_{2}| \quad (1)$$

is the coin operator, $I$ is the identity operator in $H_{G}$, and $S$ is the shift operator given by

$$S|h\rangle = |h\rangle |gh\rangle. \quad (2)$$

As we see from the last equation, if the walker is in vertex $g$ and the result of the coin toss is $h$, then the walker moves to its neighboring vertex $gh$.

The analysis of the evolution is simplified in the Fourier space. Let us suppose that $G$ is an Abelian group. In that case, $G$ is a direct sum of cyclic groups, that is, there are integers $N_{1}$ to $N_{n}$ such that $G$ is isomorphic to $\mathbb{Z}_{N_{1}} \times \cdots \times \mathbb{Z}_{N_{n}}$, where $\mathbb{Z}_{N}$ is the additive group modulo $N$. Any element $g \in G$ can be written as a $n$-tuple $(g_{1}, \cdots, g_{n})$. Such decomposition can be determined efficiently. The Fourier transform on $G$ is given by the operator

$$F_{G} = \frac{1}{\sqrt{|G|}} \sum_{g, h \in G} \chi_{g}(h)|g|/|h|, \quad (3)$$

where $\chi_{g}$ is a character of $G$ given by

$$\chi_{g}(h) = \prod_{j=1}^{n} \omega_{N_{j}}^{g_{j} h_{j}}, \quad (4)$$

where $\omega_{N_{j}} = \exp(\frac{2\pi i}{N_{j}})$ is the $N_{j}$-primitive root of unity.

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which can be proved by using Eq. (4).

III. HITTING TIMES

Using the group velocity, $v_{g}$, we can calculate the traveling time from vertex $g_{1}$ to $g_{2}$. Taking edges of length
one, the time is given by $d/v$, where $d$ is the length of the shortest path connecting the vertices. We define the group-velocity hitting time as the length of the shortest path divided by the maximal value of the group velocity.

It is interesting to compare that “physical” hitting time notion with the mathematical definition that generalizes the well-known classical hitting time notion. In the classical case, the evolution is governed by a stochastic matrix and the hitting time is the expected time the walker takes to hit vertex $g_2$ for the first time starting from vertex $g_1$. In the quantum case, there is more than one notion of quantum hitting time $[6, 10]$. Either one lets the walk evolve unitarily after leaving from vertex $g_1$ and checks when the probability at vertex $g_2$ is above some threshold, or one performs a partial measurement at each step to measure when the walker has reached vertex $g_2$. The first notion has the following definition.

**Definition III.1 (One-shot hitting time)** Given a threshold $0 < p < 1$ and an initial condition $|\phi_0\rangle$ for the coin state, the one-shot hitting time from vertex $g_1$ to $g_2$ of the discrete-time quantum walk $U$ is

$$\min\{T \mid |\phi_0\rangle \in H_S : \sum_{h \in S} |\langle h | (g_2 U^T | \phi_0 \rangle | g_1 \rangle|^2 \geq p\}.$$ 

The hitting time may be infinite if one chooses $p$ too high. On the other hand, it is advisable to take $p$ as high as possible to have a good chance to find the walker on vertex $g_2$.

The second notion has two definitions. Let us first define the concurrent hitting time $[7]$.

**Definition III.2 (Concurrent hitting time)** A discrete-time quantum walk $U$ has a $(T, p)$ concurrent hitting time from vertex $g_1$ to $g_2$, if the $|g_2\rangle$-measured walk from $U$ with the initial state $|\phi_0\rangle |g_1\rangle$ has a probability greater or equal to $p$ of stopping at a time $t \leq T$.

A walk is called $|g\rangle$-measured when we perform a measurement at each step of the evolution with the projectors $P = I \otimes |g\rangle \langle g|$ and $Q = I - P$. If $P$ is measured the process stops, otherwise the iteration is continued.

Krovi and Brun $[10]$ proposed an alternative definition which does not have a threshold $p$.

**Definition III.3 (Average hitting time)** A discrete-time quantum walk $U$ with initial state $\rho_1 = |\Psi\rangle \langle \Psi|$ where $|\Psi\rangle = |\phi_0\rangle |g_1\rangle$ has a $(g_1, g_2)$ average hitting time

$$\sum_{t=1}^{\infty} tp(t),$$

where

$$p(t) = \text{Tr}\{PU(QU)^t\rho_1(U^\dagger Q)^t U^\dagger P\},$$

$P = I \otimes |g_2\rangle \langle g_2|$ and $Q = I - P$.

Note that the wave function is not renormalized after the measurement at each step.

A drawback of both the one-shot and the concurrent hitting times is that they depend on a choice of threshold probability. Intuitively, the threshold should not be chosen too low, because else the hitting times would reflect the arrival of mere traces of probability. Exponentially suppressed traces of probability often arrive quickly but in practice cannot be considered a useful criterion for the arrival of the walker. The threshold probability also should not be set too high, as this could lead to an infinite hitting time. Apart from these arguments it appears difficult, however, to further constrain any choice of threshold probability.

Let us consider, therefore, that, intuitively, the walker carries information and that it is necessary to wait for this information to arrive at the final vertex. Information travels in a medium with what is called the signal velocity which, for normal dispersion, should be given by the maximum value of the group velocity. This is the case here, as the analysis for walks on the line and on the hypercube will show. The medium is passive, since the evolution is unitary, and the dispersion relations are well behaved with the group velocity staying below the phase velocity.

**IV. QUANTUM WALKS ON HYPERCUBES**

The $n$-dimensional hypercube is the Cayley graph of the group $\mathbb{Z}_2^n$. Let us represent the group elements by binary $n$-tuples $x = (x_{n-1}, \ldots, x_1, x_0)$ and the generating set by $\{e_j, 0 \leq j < n\}$, where $e_j$ has a single 1 entry in the $(n-j)^{th}$ component. In this case the vertices $(0, \cdots, 0)$ and $(1, \cdots, 1)$ are opposite corners. The shift operator $[2]$ reduces to

$$S|e_j\rangle |x\rangle = |e_j\rangle |x \oplus e_j\rangle,$$

where $\oplus$ is the $n$-tuple binary sum. The character is given by $\chi_x(e_j) = (-1)^x$ and the matrix elements of the reduced evolution operator are $\langle e_i | U_k | e_j \rangle = (-1)^k C_{ij}$. From this point on, let us particularize the analysis to the $n$-dimensional Grover coin, $C_{ij} = 2/n - \delta_{ij}$, which obeys the permutation symmetry of the hypercube. For this coin, we can calculate explicitly the eigenvalues of $U_k$. They are given in the following table $[6]$.

| Hamming weight $|k|$ | Eigenvalue $\lambda_k$ |
|------------------|-------------------|
| $|k| = 0$         | $-1$              |
| $1 \leq |k| \leq n - 1$ | $1$               |
| $|k| = n$         | $e^{i\omega_k}$   |
The quantity \( \omega_k \) is defined by
\[
\cos \omega_k \equiv 1 - \frac{2|k|}{n}, \tag{9}
\]
Notice that the eigenvalues depend only on \( n \) and on the Hamming weight of \( k \), defined as \( |k| \equiv \sum_{j=0}^{n-1} k_j \).

One may define the velocity of a classical walker as the derivative of the Hamming distance as function of time. We define accordingly the group velocity of the quantum walker as
\[
 v_g = \frac{d\omega}{d|k|}, \tag{10}
\]
where \( \omega \) is the angular frequency. By examining the eigenvalue table, we see that the group velocity is not zero only if \( 0 < |k| < n \), and it is given by
\[
 v_g = \frac{\pm 1}{\sqrt{|k|(n - |k|)}} \tag{11}
\]

Fig. 1 depicts \( v_g \) as function of the wave number when \( n = 100 \). The maximum velocity is \( 1/\sqrt{n-1} \) when \( |k| = 1 \) or \( |k| = n - 1 \) and the minimum is \( 2/n \) when \( |k| = n/2 \). Fig. 1 also depicts the phase velocity and the dispersion relation. For \( k < 85 \) the phase velocity is greater than the group velocity. For small \( k \) the dispersion relation has negative concavity and for \( 15 < k < 85 \) it is close to a straight line. Those facts indicate that the maximal group velocity which is achieved at \( k = 1 \) is the signal velocity. Then the time for the walker to go to the opposite corner of the hypercube is \( n/v_{g,\max} \approx n/\sqrt{n} \) when \( n \) is large. It is interesting to compare this time with the current definitions of hitting time.

![Figure 1: Group velocity, phase velocity and \( \omega_k \) of the quantum walk on the hypercube with \( n = 100 \) as function of \( |k| \). The axes are in log scale.](image)

Taking \( \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} |e_j\rangle \otimes |0, \cdots, 0\rangle \) as the initial condition, the one-shot hitting time from vertex \((0, \cdots, 0)\) to \((1, \cdots, 1)\) is either \( \lfloor \frac{2}{3} n \rfloor \) or \( \lceil \frac{2}{3} n \rceil \) for \( p = 1 - O(\log^3 n) \) with the condition that the hitting time and \( n \) have the same parity \([7]\). Note that, as discussed in \([7]\), one cannot increase the threshold probability beyond \( p = 1 - O(\log^3 n) \) without getting infinite one-shot hitting times, because the threshold probability is very close to the maximum value of the probability distribution at the final vertex.

Now when \( n \) is large, with high probability the walker hits the opposite corner at time \( O(n) \). This is faster than the \( O(n/\sqrt{n}) \) scaling of the group-velocity hitting time. We will discuss the interesting origin of the difference in the scaling behavior in the last section.

Also, using the same initial condition given above, one obtains that the walk has \( (\frac{2}{3} n, \Omega(\frac{1}{n \log n})) \) concurrent hitting time \([7]\). Note that in this case the probability of finding the walker at the final vertex is close to zero for large \( n \). This result is not a contradiction with the group-velocity hitting time for the unitary walk because the evolution in this case is non-unitary, i.e., the walk is a different physical process, due to the repeated measurements demanded by the definition of the concurrent hitting time.

In comparison, also the value of the average hitting time obtained in Ref. \([10]\) is smaller than the group-velocity hitting time. This again is not a contradiction, because the approach of \([10]\) does not also not describe the same physics as we do here, due to the non-unitary evolution caused by the repeated measurements assumed in the definition of the average hitting time \([10]\).

### V. QUANTUM WALKS ON A 1-D LATTICE

In this example, the group-velocity based hitting time and the one-shot hitting time are essentially in agreement for a suitable choice of the threshold probability.

An one-dimensional lattice is the Cayley graph of the additive group of integers \( \mathbb{Z} \) with \( S = \{1, -1\} \) as the generating set. Since \( \mathbb{Z} \) is infinite, the theory of Sec. \([11]\) does not apply straightforwardly in this case. We make the necessary modifications in this Section.

The shift operator \([12]\) reduces to
\[
 S|j\rangle|n\rangle = |j\rangle|n + j\rangle. \tag{12}
\]
Without loss of generality, we can use the Hadamard matrix as coin operator \([11]\), which is given by
\[
 C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{13}
\]

The generic state of the walk is given by
\[
 |\psi(t)\rangle = \sum_{j=\{1,-1\}} \sum_{n=-\infty}^{\infty} \psi_{j,n}(t)|j\rangle|n\rangle, \tag{14}
\]
and the probability distribution by
\[
 P_n(t) = \sum_{j=\{1,-1\}} |\psi_{j,n}(t)|^2. \tag{15}
\]
The transformed amplitudes are
\[
\tilde{\psi}_{j,k} = \sum_{n=-\infty}^{\infty} e^{ikn} \psi_{j,n},
\]
where \( k \in [-\pi, \pi] \). The reduced evolution operator \( U_k \), which acts on \(|\psi_k\rangle = \sum_j \tilde{\psi}_{j,k} |j\rangle\), is given by
\[
U_k = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-ik} & e^{-ik} \\ e^{ik} & -e^{ik} \end{pmatrix}.
\]

In the eigenbasis, \( U_k \) is given by
\[
U_k = \begin{pmatrix} \lambda_k^1 & 0 \\ 0 & \lambda_k^2 \end{pmatrix},
\]
where \( \lambda_k^1 = e^{-i\omega_k} \) and \( \lambda_k^2 = e^{i(\pi/2 + \omega_k)} \), and \( \omega_k \) is defined as the angle in \([-\pi/2, \pi/2] \) such that \( \sin(\omega_k) = \sin(k)/\sqrt{2} \).

The Hamiltonian associated with that evolution operator is
\[
H_k = \begin{pmatrix} \omega_k & 0 \\ 0 & -\pi - \omega_k \end{pmatrix}.
\]

We can now calculate the group velocity \( v_g = \frac{d\omega}{dk} \), where \( \omega \) is the angular frequency, obtaining
\[
v_g^+ = \frac{\pm \cos(k)}{\sqrt{1 + \cos^2(k)}}.
\]

The phase velocity are characterized by two values, which are
\[
v_{ph}^+ = \frac{1}{k} \arcsin\left( \frac{\sin k}{\sqrt{2}} \right)
\]
and \( v_{ph}^- = -v_{ph}^+ - \pi/k \).

Fig. 2 depicts \( v_g^+ \) as function of the wave number. The maximum velocity is \( 1/\sqrt{2} \) when \( k = 0 \) and the minimum velocity is the opposite value when \( k = \pm \pi \). The phase velocity \( (v_{ph}^+) \) is equal to the group velocity for \( k = 0 \) and is greater than the group velocity when \( k > 0 \). The dispersion relation has negative concavity for \( 0 < k < \pi \). Those facts indicate that the group velocity is in fact the signal velocity. For the second values of phase and group velocities, we have \( v_{ph} < v_g^- \) when \( 0 < k < \pi \). This anomalous case involves velocities that are smaller or equal to the maximum group velocity \( v_g^+ \) at \( k = 0 \).

It is interesting to relate the group velocity with the probability distribution. Fig. 3 depicts the probability distribution of the Hadamard walk at \( t = 100 \). Note that the distribution is clearly non-zero in the region \(-v_{g_{max}}^t < n < v_{g_{max}}^t \). One can verify that the probability distribution is not exactly zero for \(|n| > v_{g_{max}}^t \) although very small.

It is trivial to calculate the one-shot hitting time for the lattice case. One can read it directly from the probability distribution. For example, in Fig. 3 we see that the plot has a sharp peak at around \( n_{max} = t/\sqrt{2} \). If we take \( p \) as the value \( P_{n_{max}}(t) \) obtained from Eq. (14), which is the most natural one to take, the one-shot hitting time is \( \sqrt{2} n \). In this case, the one-shot and the group-velocity hitting time yield the same value approximately.

The calculation of the average hitting time is somewhat tricky, because it is defined in terms of an only slowly converging series. Our numerical results indicate that the average hitting time is again smaller than group-velocity hitting time.

Note that parameter \( n \) for the line has a different meaning when compared to the parameter \( n \) for the hypercube. In the line, \( n \) is a linear distance to the origin while in the hypercube \( n \) is a dimension.
VI. DISCUSSION AND CONCLUSIONS

Quantum walkers often behave similarly to wave packets in media. It is the case, in particular, when the walker’s lattice possesses an Abelian symmetry group which allows the use of a normal mode decomposition.

In this situation, the walker can be described as a wave packet which over time propagates and disperses. When the walk is unitary the walker’s wave packet effectively travels in a medium which is neither absorptive nor amplifying. For such passive media, the group velocity is known to be a good measure of the speed with which the wave packet can propagate information.

This motivated us to use the group velocity as the basis for a new definition of hitting time. The new hitting time is defined as the distance divided by the maximal group velocity for any wave number, i.e., also for any wave packet. Therefore, among all possible initial conditions for the quantum walker’s wave packet, the group-velocity based definition of hitting time yields the optimum. The group velocity based hitting time also does not depend on a choice of threshold probability. Instead, this hitting time depends only on the coin operator and on the symmetry group which defines the Cayley graph.

Within this approach, we calculated the hitting times for discrete-time quantum walks on Cayley graphs of general Abelian groups. For the special cases of the hypercubes and the one-dimensional lattice, we compared the group velocity based hitting times with hitting times obtained with respect to previous definitions of the hitting time. While we found general agreement in the case of the quantum walk on the line, we found for the hypercubes that the group velocity based hitting times are generally scaling slower with $n$ (the dimension) than the hitting times with respect to previous definitions.

To explain the apparent discrepancy, let us first consider the fact that the group velocity based hitting time is larger than the concurrent and average hitting times. That there is a discrepancy is not surprising, because the described physical processes are different. In the case of the group velocity based hitting time calculation, the quantum walk is unitary while in the other cases the quantum walk is non-unitary due to the performance of measurements.

More significant and interesting is the fact that, for the same unitary quantum walk on the hypercube, the group velocity based hitting time is larger than the one-shot hitting time.

The group velocity based hitting time is determined by how fast the fastest wave packet can travel. In comparison, the one-shot hitting time is based on the idea that the arrival of the walker can be recognized by the arrival of a certain threshold probability. In the case studied in the literature, where the walker is asked to reach the diagonally opposite vertex in the hypercube, the threshold probability was optimized and it is in fact close to one. It would appear, therefore, that the arrival of such a large threshold probability indicates the arrival of a wave packet. How, therefore, can the group velocity based hitting time scale slower than the one-shot hitting time?

To this end, let us consider the larger picture. In principle, in eventual practical applications in quantum computing, it is important how fast the walker can arrive at any vertex – not only the vertex opposite to the starting vertex. What, however, is the one-shot hitting time with respect to the walker’s arrival at a vertex other than the one diagonally opposite? The calculation of the one-shot hitting time for arrival at the diagonally opposite vertex gives a partial answer. It was shown there that the threshold for arrival can be chosen very high, in fact converging to 1. The walker is exceedingly likely to arrive there. This also shows that the threshold for the walker’s arrival at other vertices must be chosen small in order to obtain a finite value for the one-shot hitting time. In the larger picture, where we ask how fast a quantum walker can visit any vertex in the graph, this indicates that the one-shot hitting time is a difficult measure to use. This is because suitable threshold probabilities and therefore the one-shot hitting times can heavily depend on the end vertex considered. The reason for that is apparently the possibility of strong destructive or constructive interferences, that lead, in particular, to a very significant enhancement of the arrival probability at the diagonally opposite vertex.

The group velocity based hitting time, on the other hand, does not require the consideration of threshold probabilities. Nor does it seem to depend on whether or not the initial and final positions of a wave packet that represent the walker are in a highly symmetric relationship such as being diagonally opposed to another. In the larger picture, where we ask how fast a quantum walker can visit any arbitrary vertex of the graph, the group velocity based hitting time should therefore provide a more reliable measure of that speed.

Nevertheless, for completeness, let us address the remaining question regarding the special case of the walker arriving at the diagonally opposite vertex. How can it be that, as the one-shot hitting time calculation has shown, the walker can quite reliably arrive at the diagonally opposite vertex faster than the group velocity would indicate is possible?

To see this, let us recall that wave packets tend to possess leading small amplitude waves that travel faster than the group velocity. These are the Brillouin and Sommerfeld precursors, which can also be viewed as evanescent waves. In general, in passive media, in normal circumstances, precursors stay small during the propagation. In active media, precursors may be amplified, thereby leading to an apparent speed up of the wave packet. We conjecture that a similar situation prevails here, even though the medium is passive (since the evolution is unitary). Namely, it could be that the precursors of the quantum walker’s wave packet, on its way from the initial vertex to the diagonally opposite vertex, constructively interfere so as to lead to an effectively amplified precursor arriv-
ing at the diagonally opposite vertex, before the arrival of the main wave packet. This, and also the relationship between the group velocity and the notion of mixing time should be interesting to explore further. We remark that an earlier conjecture by one of us (AK) has been confirmed, see [12], that the precursor phenomenon occurs for the quantum walk on the line.

It should also be interesting to determine the group velocities for more general quantum walks. The method we have described here is applicable to Cayley graphs of Abelian groups. A suitable generalization may be applicable to non-Abelian Cayley graphs, or to even to more general graphs that allow quantum walks, as long as these possess some form of normal mode decomposition.

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