Free vibration analysis of shield cutter head using element-free method

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Abstract: Free vibration of cutter head is very important to the shield tunneling machine in design, manufacture and construction. It is a free vibration problem of simply-connected circular plates with clamped supported boundary. In this paper, a element-free key to solve the problems of cutter head vibration by the Radial Basis Function is put forward. In order to save the computational time, the real-part MFS is used to carry out the mathematical analysis which is different from the previous complex value calculation. For the cutter head, the eigenfunctions were obtained by analytic deduction using the discrete element-free model. It is found that the spurious eigenfunction relies on the equations employed, but the true eigenfunction is related to the boundary conditions. By using Burton & Miller methods, the true eigenfunction was extracted. A numerical case of clamped cutter head was adopted to valid the effectiveness of the present method.

1. Introduction
Cutter head is one of the core components of the shield machine (W.C., et al, 2017). In the complex geological construction, the cutter head of the shield tunneling machine will produce high impact and vibration due to the uneven cutting tool (G.J., et al, 2017; S.H., et al, 2017). Therefore, it is very important for the design, manufacture and construction of the shield tunneling machine to carry out the research on the vibration of the cutter head. The analysis for free vibration of cutter head has been of great concern to many investigators during these years (Q., et al, 2015; K., et al, 2017). In essence, this is a free vibration problem of simply-connected circular plates with clamped supported boundary. For this problem, numerical simulation is often used to study, such as finite difference method (FDM), finite element method (FEM) and boundary element method (BEM). Nevertheless, there are also some shortcomings in these methods (J., et al, 2015; M.N., et al, 2017; M.M., et al, 2017). Recently, more and more scholars have concerned element-free method because of the advantages, i.e., element free, no singular and hypersingular integrals (H., et al, 2017). In this paper, mesh less methods for solving the problems of cutter head vibration using the radial basis function (RBF) are proposed. In addition, a clamped cutter head case was demonstrated analytically and numerically to see the validity of the present method.
2. Analysis of free vibration of simply-connected plate using element-free method

The governing equation for a free flexural vibration of a plate as shown in Fig. 1 is written as follows:

\[ \nabla^4 u(x) = \lambda^2 u(x), \quad x \in \Omega \]  \hspace{1cm} (1)

Where \( u \) is the lateral displacement, \( \lambda = \frac{\omega \rho h}{D} \), \( \lambda \) is the frequency parameter, \( \omega \) is the circular frequency, \( \rho_0 \) is the surface density, \( D \) is the flexural rigidity expressed as \( D = \frac{E h^3}{12(1-v^2)} \) in terms of Young’s modulus \( E \), the Poisson ratio \( v \) and the cutterhead thickness \( h \), and \( \Omega \) is the domain of the cutterhead.

![Fig. 1 Figure sketch for the plate problem](image)

The kernel function \( U_f \) is the fundamental solution which satisfies

\[ \nabla^4 U_f (s, x) - \lambda^2 U_f (s, x) = -\delta(x - s) \]  \hspace{1cm} (2)

Where \( \delta(x - s) \) is the Dirac-Delta function, and \( s \) and \( x \) are the source and field points, respectively. We have

\[ U_f (s, x) = \frac{i}{8 \alpha^2} \left[ H_0^1(\lambda \gamma) + H_1^1(\lambda \gamma) \right] \]

\[ = \frac{1}{8 \alpha^2} \left[ Y_0(\lambda \gamma) - iJ_0(\lambda \gamma) + \frac{2}{\pi} K_0(\lambda \gamma) \right] \]  \hspace{1cm} (3)

Where \( \gamma \equiv |s - x|^2 = -1 \), \( H_0^1(\lambda \gamma) \) is the first kind zeroth-order Hankel function, \( J_0(\lambda \gamma) \) and \( Y_0(\lambda \gamma) \) are the first kind and second kind zeroth-order Bessel functions, and \( K_0(\lambda \gamma) \) is the second kind zeroth-order modified Bessel function. Since \( I_0(\lambda \gamma) \) is the homogeneous solution of the biharmonic operator, we can add it to the fundamental solution for satisfying the Hilbert transform of causal constraint. Then, the complete kernel function \( U_f (s, x) \) is obtained as

\[ U_f (s, x) = \frac{1}{8 \alpha^2} \left[ Y_0(\lambda \gamma) - iJ_0(\lambda \gamma) + \frac{2}{\pi} K_0(\lambda \gamma) \right] \]  \hspace{1cm} (4)

We also can express the displacement field of cutterhead vibration in other form through the indirect equations, that is

\[ u(x) = \sum_{j=1}^{2N} P(s_j, x) \phi_j + \sum_{j=1}^{2N} Q(s_j, x) \phi_j \]  \hspace{1cm} (5)

In which there are \( 2N \) boundary node number. \( s_j \) and \( x \) are on behalf of the source points and field points, respectively. The two kernels \( P \) and \( Q \) can be get from the following three kernels,

\[ \Theta(s, x) = K_{\phi, \phi}(U(s, x)) \]  \hspace{1cm} (6)

\[ M(s, x) = K_{\phi, \omega}(U(s, x)) \]  \hspace{1cm} (7)

\[ P(s, x) = K_{\omega, \omega}(U(s, x)) \]  \hspace{1cm} (8)
where the subscript "s" denotes the source point, $K_s(t)$, $K_{m,s}(t)$, and $K_{v,s}(t)$ mean the operators for the source point

$$K_{s,t}(t) = \frac{\partial t}{\partial t}$$  \hspace{1cm} (9)

$$K_{m,s}(t) = v \nabla^2(t) + (1-v) \frac{\partial^2 t}{\partial n^2}$$  \hspace{1cm} (10)

$$K_{v,s}(t) = \frac{\partial \nabla^2(t)}{\partial n} + (1-v) \frac{\partial}{\partial t} \left( \frac{\partial^2 t}{\partial n^2} \right)$$  \hspace{1cm} (11)

Where $n_s$ and $t_s$ are normal vectors and tangent vectors of source points, respectively. In order to avoid the singularity, the collocation points were arranged on the real boundary which has the radius of the source points were located on the fictitious-boundary radius ($a'$). It can be seen in Fig. 2.

Similarly, $\Theta$, $M$ and $V$ kernels can be generated sixteen kernels using the above operators, and the results can be seen in Fig. 3.

Fig. 2 Figure sketch for node distribution

Fig. 3 The construction of sixteen kernels

The Eq. (5) is calculated by the three operators including Eqs. (9), (10) and (11), and we have

$$\theta(x) = K_{\theta,s}(u(x))$$  \hspace{1cm} (12)

$$m(x) = K_{m,s}(u(x))$$  \hspace{1cm} (13)

$$v(x) = K_{v,s}(u(x))$$  \hspace{1cm} (14)

Where $\theta$ is the gradient, $m$ is the moment, and $v$ is the shear force. Equation (5) and Eqs. (12)-(14) can be expressed by the following forms.

$$\{ u \} = [P]\{ \varphi \} + [Q]\{ \psi \}$$  \hspace{1cm} (15)

$$\{ \theta \} = [P_\theta]\{ \varphi \} + [Q_\theta]\{ \psi \}$$  \hspace{1cm} (16)

$$\{ m \} = [P_m]\{ \varphi \} + [Q_m]\{ \psi \}$$  \hspace{1cm} (17)

$$\{ v \} = [P_v]\{ \varphi \} + [Q_v]\{ \psi \}$$  \hspace{1cm} (18)
Where $\{\phi\}$ and $\{\psi\}$ are the vectors of pending modulus. When the boundary conditions of the cutter head is clamped, we have

$$\{0\} = [U]\{\phi\} + [\Theta]\{\psi\}$$  \tag{19}$$

$$\{0\} = [U_0]\{\phi\} + [\Theta_0]\{\psi\}$$  \tag{20}$$

By assembling Eqs. (19) and (20) together, we have

$$[SM_1^c][\nu] = \{0\}$$  \tag{21}$$

Where the superscript “$C$” denotes the clamped case, the subscript “1” means using the $U - \Theta$ formulation and

$$[SM_1^c] = \begin{bmatrix} U & \Theta \\ U_0 & \Theta_0 \end{bmatrix}_{4N \times 4N}$$  \tag{22}$$

According to the nontrivial solution condition of matrix, i.e.

$$\text{det}\[SM_1^c\] = 0$$  \tag{23}$$

By solving the above equations, eigenvalues can be obtained.

3. Analytical derivations for the eigenvalue of cutter head using the MFS

For easy analysis, $x = (\rho, \phi)$ and $s = (R, \Theta)$ uses polar coordinate representation. The $U$ kernel function is simplified. It is shown as below:

$$U(R, \Theta; \rho, \phi) = \frac{1}{8\lambda^2} \sum_{j=1}^\infty \left\{ J_i(\lambda \rho) \left[ Y_i(\lambda R) - i J_i(\lambda R) \right] ight\}$$

$$+ \frac{2}{\pi} (-1)^j \left[ (-1)^j K_i(\lambda R) - i J_i(\lambda R) \right] \cos(1(\theta - \varphi)), R > \rho$$  \tag{24}$$

$$U(R, \Theta; \rho, \phi) = \frac{1}{8\lambda^2} \sum_{j=1}^\infty \left\{ J_i(\lambda R) \left[ Y_i(\lambda \rho) - i J_i(\lambda \rho) \right] ight\}$$

$$+ \frac{2}{\pi} (-1)^j \left[ (-1)^j K_i(\lambda \rho) - i J_i(\lambda \rho) \right] \cos(1(\theta - \varphi)), R < \rho$$  \tag{25}$$

Where the high and low corner mark ",i" and ",e" stand for the inner area $(R > \rho)$ and outer area $(R < \rho)$ respectively.

Because circular boundary has the character of rotation symmetry, the sixteen influence matrices are denoted by the elements as shown in Fig. 3.

$$K_{ij} = K\left(R, \Theta; \rho, \phi_i\right)$$  \tag{26}$$

Where we can take any of the sixteen kernels in Fig.3 as the kernel $K$. $\phi_i$ and $\Theta_j$ denote the viewing angle and boundary point, respectively. Applying the superposition technology to the 2N concentration intensity along the boundary, the influence matrix is obtained,

$$[K] = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{2N-2} & a_{2N-1} \\
 a_{2N-1} & a_0 & a_1 & \cdots & a_{2N-3} & a_{2N-2} \\
 a_{2N-2} & a_{2N-1} & a_0 & \cdots & a_{2N-4} & a_{2N-3} \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 a_1 & a_2 & a_3 & \cdots & a_{2N-1} & a_0 \end{bmatrix}$$  \tag{27}$$

Where the elements of the first row can be obtained by

$$a_{i,j} = K(s_i, x_j)$$  \tag{28}$$
Due to the symmetry character of influence coefficient, the matrix $[K]$ in Eq. (27) becomes a symmetric loop. The eigenvalues of the sixteen matrices in Fig. 3 are get with the aid of degenerate kernels and orthogonality. Here we only list the eigenvalues of kernel function $U$ and $\Theta$ for the latter deduction. Since the matrix $[U]$ is a symmetric circulant, the $[U]$ matrix can be decomposed into

$$[U] = \varphi \sum_{\nu} \varphi^{-1}$$

Where

$$\sum_{\nu} = \text{diag}(U_{1}^{U}, U_{-1}^{U}, \ldots, U_{-N}^{U}, U_{N}^{U})$$

Similarly, the other matrices in Fig. 3 can be decomposed after altering their diagonal elements. So we can rewrite Equation (22) as

$$[SM_{i}^{C}] = \left[ \Phi \sum_{\nu} \Phi^{-1} \Phi \sum_{\nu} \Phi^{-1} \right]_{4N \times 4N}$$

4. Mathematical analysis using the real-part MFS
To the real-part MFS, the kernel function $U^{\Re}(s,x)$ is the real-part of the fundamental solution superimposing $2N$ concentration intensity along the boundary, the four influence matrices can be determined easily. Based on the circulant property and orthogonal conditions, the eigenvalues for the influence matrices, can be obtained by using real-part of the eigenvalues in Eqs. (29) - (32). Based on the conventional MFS, source points must be distributed outside the domain with radius $a'$ as shown in Fig. 2. The radii, $\rho = a$ and $R = a'$, are utilized. For the clamped cutter head by using the $U - \Theta$ formulation, Eq. (38) reduces to

$$\det [SM_{R}^{C}] = \prod_{(x,y) \in S} \frac{\pi^{2}}{a'^{2}}$$

$$\left\{ J_{\nu} (\lambda \alpha) I_{\nu} (\lambda \alpha) + J_{\nu} (\lambda \alpha) I_{\nu} (\lambda \alpha) \right\}$$

The former term may be the eigenequation, i.e.

$$\left\{ J_{\nu+1} (\lambda \alpha') I_{\nu} (\lambda \alpha') + J_{\nu} (\lambda \alpha') I_{\nu+1} (\lambda \alpha') \right\} = 0$$

Or the latter term

$$\left\{ K_{\nu+1} (\lambda \alpha') Y_{\nu} (\lambda \alpha') + K_{\nu} (\lambda \alpha') Y_{\nu+1} (\lambda \alpha') \right\} = 0$$

5. A numerical analysis example for verification
A cutter head subjected to clamped boundary conditions is considered. The radius of the cutter head is 6 meter, and the Poisson ratio is 0.33. For the real-part MFS, forty-six field points are arranged on the real boundary and forty-six source points are arranged on the fictitious boundary with radius $a' = 6.2$ m. The $U - \Theta$ formulation is selected. Fig. 4 shows the solving process between determinants versus frequency of cutter head by the real-part MFS. Fig. 5 shows the solving process between determinant and frequency of cutter head by applying Burton & Miller method on the real-part MFS. In Fig. 5, we find that true eigenvalues are obtained only and spurious eigenvalues are eliminated. The true eigenvalues are extracted by Burton & Miller method and agree well with the analytical prediction.
6. Conclusions
The vibration analysis of cutterhead is very important in shield construction. An element-free method to solve the vibration Eigen frequency of cutter head with a fixed boundary is put forward. The fundamental equation of the method is derived in detail. The discrete model of element-free is solved to obtained the Eigen equations. It is deduced that, the true Eigen frequency can be get always , but the spurious Eigen frequency is closely related the solving way. In aid of Burtoon and Miler way, we can extract the true Eigen frequency easily. Through the numerical analysis of shield cutterhead, the validity of this method is verified, and the degree of compliance is very high.

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