Emergence of isotropy and dynamic self-similarity in the birth of two-dimensional wave turbulence

Maciej Galka,1,* Panagiotis Christodoulou,1 Martin Gazo,1 Andrey Karailiev,1 Nishant Dogra,1 Julian Schmitt,1,2 and Zoran Hadzibabic1

1Cavendish Laboratory, University of Cambridge, J. J. Thomson Avenue, Cambridge CB3 0HE, United Kingdom
2Institut für Angewandte Physik, Universität Bonn, Wegelerstraße 8, 53115 Bonn, Germany

(Dated: March 18, 2022)

Turbulence, characterised by cascades of excitations across length scales, is ubiquitous in nature. Many of its statistical steady-state properties are universal, but little is established experimentally about how they emerge, starting from an equilibrium system. Here, we initiate a wave cascade in a two-dimensional Bose gas, using a large-scale oscillating force, and probe it on all length scales from the forcing one to the smallest one where energy is eventually dissipated. We observe two key phenomena conceptually associated with the development of turbulence – the emergence of statistical isotropy under anisotropic forcing, and the spatiotemporal scaling of the spectrum of excitations at times before any energy is dissipated. This new microscopic view on turbulence could enable further studies relevant for fields ranging from mesoscopic physics to cosmology.

Turbulence is a multi-scale phenomenon that is still not understood on a microscopic level, but is believed to generically feature cascades of excitations across different length scales [1–3], in contexts as diverse as ocean waves [4], nonlinear optics [5], supernovae [6], and financial markets [7]. In a direct cascade, conceptualised by Richardson a century ago [1], energy injected into a system at a large length scale flows without loss through momentum space until it is dissipated at some small length scale. More generally, turbulent systems can feature momentum-space fluxes of different conserved quantities, and also inverse cascades, from small to large lengthscales [8].

Steady-state cascades form under continuous injection and matching dissipation, and are reflected in stationary power-law spectra of the relevant quantities [2, 9–11]. While this hallmark of the fully developed, steady-state turbulence has been observed in a variety of contexts, comparatively little is established experimentally about how such steady states develop, starting from an equilibrium system.

Here, we observe this ‘birth of turbulence’ in a homogeneous two-dimensional (2D) atomic Bose gas [12, 13], which can support both direct [14] and inverse [15, 16] energy cascades. We force the gas on a large, system-size length scale, and realise a direct wave cascade. We study its birth by probing the system on all length scales, from the energy-injection one to the dissipation one, and observe the key theoretically expected phenomena outlined in Fig. 1A. First, according to this picture, even though the continuous energy injection at a large length scale (small wavenumber \( k = |k| \)) is anisotropic, beyond a sufficiently large \( k \) the cascade is statistically isotropic; such isotropy is believed to generically emerge in systems that, like our trapped gas, carry no net momentum [17] (see [11, 18] for a more general discussion). The second key phenomenon is dynamic scaling – the isotropic cascade front \( k_{\text{cf}} \) evolves algebraically in time \( t \), as \( k_{\text{cf}} \propto t^{-\beta} \) (with \( \beta < 0 \)), until it reaches the dissipation scale \( k_D \) and a steady state is established. In its wake, \( k_{\text{cf}} \) leaves an isotropic power-law momentum distribution, \( n_k(k) \propto k^{-\gamma} \) [11, 14], so the pre-steady-state \( n_k \) (at large \( k \)) follows the general form of self-similar spatiotemporal (dynamic) scaling:

\[
(t/t_0)^{-\alpha}n_k(k,t) = n_k((t/t_0)^\beta k,t_0),
\]

where \( t_0 \) is a reference time and in our case \( \alpha = \gamma \beta \) [11, 14]. Such scaling, known from classical surface growth [19, 20], and also seen in the relaxation dynamics of quantum gases [21–25], is hypothesised to be generic to far-from-equilibrium many-body quantum systems [26, 27] and proposed as a way to classify them into universality classes.

As outlined in Fig. 1B, we prepare a quasi-pure 2D superfluid of \(^{39}\)K atoms in a square optical box trap of size \( L \), and drive it anisotropically with a time-periodic force \( F \) created by a magnetic field gradient along one of the box axes, denoted \( x \) [13]. The gas is confined to the \( x,y \) plane by a harmonic potential with angular trap frequency \( \omega_z \), and has chemical potential \( \mu = N \hbar^2 g/(mL^2) \), where \( N \) is the atom number, \( \hbar = h/(2\pi) \) with \( h \) being the Planck constant, \( m \) is the atom mass, and \( g = \sqrt{8\pi \mu \omega_z}/\hbar a \) [28, 29], where \( a \) is the 3D s-wave scattering length, which we tune using a Feshbach resonance. The spatially-uniform force, \( F = F_0 \sin(\omega t) \), with \( \omega = \sqrt{\mu/m} \pi/L \), resonantly injects energy into a longest-wavelength phonon mode, with wavevector \( k_F = (\pi/L, 0) \). Our energy-injection scale is thus set by the system size, \( k_F = \pi/L \lesssim 0.1 \ \mu \text{m}^{-1} \), while the dissipation scale \( k_D = \sqrt{2\mu U_D/h^2} \approx 5 \ \mu \text{m}^{-1} \) is set by the trap depth \( U_D \) [30]. Between \( k_F \) and \( k_D \), the nature of excitations changes from phonons to particle-like matter waves at \( k = 1/\xi \), where \( \xi = L/\sqrt{\gamma N} \sim 1 \ \mu \text{m} \) is the 2D healing length [29].

We first summarise the universal features of our energy injection (Fig. 1C) and the resulting steady-state turbulence (Fig. 1D, E), and then study how such steady state gets established (Fig. 2, Fig. 3). To probe the gas on all length scales from \( k_F \) to \( k_D \), we use three complementary tools – using principal component analysis (PCA) [31, 32] we directly visualise the dynamics of the low-lying \( (k \sim k_F) \) discrete quantum states, with time-of-flight (TOF) expansion we study the emergent statistical behaviour at large \( k \), and using Bragg spectroscopy [33, 34] we bridge the \( k \)-space gap between these measurements.
To measure the energy flux $\epsilon$ injected at $k_F$ (averaged over half a drive period), we monitor the periodic displacement of the cloud’s centre of mass (COM), which is proportional to the density modulation due to the phonon excitation at $k_F$ (see Fig. 1B) [13]. Specifically, $\epsilon = NF_0v_0/2$, where $v_0$ is the amplitude of the COM speed. Identifying the natural scales for $F_0$ and $v_0$ as, respectively, $\mu/L$ [35] and the speed of sound $\sqrt{\mu/m} = \omega_L/\pi$, we plot the dimensionless $\epsilon/(N\mu\omega_F)$ versus the dimensionless drive strength $p = F_0L/\mu$, and find that it follows a universal curve, $\propto p^{-1.31(3)}$ (solid line). This scaling is in contrast with linear response (where $v_0 \propto F_0$, so $\epsilon \propto p^2$) and agrees with $\epsilon \propto p^{5/3}$ for a nonlinear transfer of energy to higher-lying excitations, as previously observed in 3D for a single interaction strength [36].

For sufficiently strong drives, $p \gtrsim 0.5$, and at sufficiently long times, in TOF we observe steady-state power-law distributions such as shown in Fig. 1D for $p = 0.85$ and $t = 5 \times 2\pi/\omega_L$. The line-integrated distributions parallel and perpendicular to the drive, $\vec{n}_{k,x}(k_x) = f dk_y n_k(k_x,k_y)$, are essentially identical, implying an isotropic $n_k$. Note, however, that due to finite-size effects these measurements are not accurate for $k \lesssim 0.6 \mu m^{-1}$ [37]. We also show the (azimuthally averaged) radial distribution $n_k(k)$, from which we extract $\gamma \approx 2.9$ (fitting in the range $1.5 - 3 \mu m^{-1}$). As shown in Fig. 1E, $\gamma$ is robust under changes of system parameters, including the box size, and drive strength; for different measurements we get a combined estimate $\gamma = 2.90(5)$.

To trace how such a steady state gets established, we start with the onset of the cascade at low $k$, by studying in-situ the spatiotemporal modulations of the gas density, $n$ (Fig. 2A); here we use our larger, 53-µm box, with parameters as for the red diamonds in Fig. 1C and $p = 0.6$, while below for Bragg and TOF measurements we use a 31-µm box with all parameters as in Fig. 1D [37]. Using PCA, we decompose $n(x,y,t)$ in an unbiased way as $\sqrt{\lambda} f(x,y) + \sum_{j=1}^{J-1} \sqrt{\lambda_j} f_j(x,y) b_j(t)$, with orthonormal $\{f_j(x,y)\}$ and $\{b_j(t)\}$, and $J$ equal to the number of different times for which we measure $n$. Here $f$ is the normalized time-averaged density profile and $f_j$ are the principal components of the modulations $\Delta n(x,y,t)$, with eigenvalues $\lambda_j$ decreasing with increasing $j$, and $\sum_{j=1}^{J-1} \lambda_j/\lambda = \langle (\Delta n)^2 \rangle/\langle n \rangle^2$, where $\langle ... \rangle$ denotes an average over both space and time. For weak modulations, $f_j$ directly visualise the wavefunctions of the underlying excitations through interference with the quasi-uniform condensate.

We find that the first four $f_j$ (see Fig. 2A), with $f_1$ showing the resonantly excited phonon, all closely resemble phonon wavefunctions with $k = jk_F$; here $J = 81$, but the first four modes account for 75% of the total (normalised) density variance $\sum_{j=1}^{80} \lambda_j/\lambda = 0.08$, and we do not identify any clear structures in the remaining ones. The directly driven $b_1$ oscillation at $\omega_F$ quickly reaches a steady state, while $b_2$ oscillates...
predominantly at $2\omega_F$ and with a discernible delay; $b_3$ and $b_4$ show more complex behaviour, but for $t > 2\pi/\omega_F$, all four $b_j$ are fitted well by $\sum_{i=1}^4 B_{j,i} \cos(\omega_i t + \phi_{j,i})$, which gives their harmonic cascade weights $\Lambda_{j,\ell} = B_{j,\ell}^2/(\sum_{i=1}^4 B_{j,i}^2)$. The non-linear cascade naturally results in the appearance of the diagonal terms $\Lambda_{j,j}$, corresponding to $j\omega_F$ phonons being created and revealed through interference with the condensate. The prominent off-diagonal ones, $\Lambda_{3,1}$ and $\Lambda_{4,2}$, can be partially explained by noting that two-phonon interference also contributes to $\Delta n$ (e.g., $B_{4,2}$ arising from interference of $k_F$ and $3k_F$ phonons); another contribution to $\Lambda_{3,1}$ arises from weak off-resonant direct driving of the $3k_F$ phonon.

Crucially, up to $4k_F$ all the dynamics are essentially one-dimensional. This can be qualitatively understood since in the presence of the condensate the dominant microscopic nonlinear process is a three-wave interaction, for which the requirement of momentum conservation supresses cross-directional coupling for $k \ll 1/\xi$, and here $4k_F \approx 0.15/\xi$ [38].

To follow the fate of the anisotropy at higher $k$, we use Bragg spectroscopy, which gives the line-integrated distributions $\tilde{n}_{k,x}$ and $\tilde{n}_{k,y}$ without any finite-size artefacts. Normalising $\tilde{n}_{k,x}(k)$ and $\tilde{n}_{k,y}(k)$ to unity for $k = 0$ [37], in Fig. 2B we show them for $t = 2\pi/\omega_F$. By this time the excitations already cascade to $k > 1/\xi$ and, while at low $k$ their distribution is clearly anisotropic, at $k \gtrsim 1/\xi = 1.0 \mu m^{-1}$ it is isotropic. Note that we have done additional measurements that show that at $k = 0$ the two axial distributions are distinct (with normalised $\tilde{n}_{k,x}(0) = 0.40(2)$ and $\tilde{n}_{k,y}(0) = 0.50(3)$) even for $t = 5 \times 2\pi/\omega_F$, when the steady state up to $k_D$ is already established.

Now turning to TOF to study the isotropic $n_k(k,t)$ for $k \gtrsim 1/\xi$ and $t \geq 2\pi/\omega_F$, in Fig. 3A we show the evolution of the compensated spectrum, $n_k k^3$, which highlights the propagation of its leading edge. From $n_k$, measured at half-periods of the drive, we extract the total kinetic energy in the isotropic cascade ($k > 1/\xi$), $E_c(t) = \int dk 2\pi k \varv k$, where $\varv_k = n_k k^2/(2m)$ [37] (Fig. 3B), and the cascade front $\kappa_{ct}(t)$ (Fig. 3C).

Beyond some time, $t^* \approx 3.5 \times 2\pi/\omega_F$, both $E_c$ and $\kappa_{ct}$ saturate, as expected for a steady state with matching energy injection and dissipation. Prior to that, the growth of $E_c$ is consistent with the independently measured $\varv$ injected at $k_F$ (solid line); note that the systematic error in $\varv$ is $\approx 20\%$, dominated by the errors in the calibration of $N$ and $L$ [37]. For $k_{ct} > 1/\xi$, associating a constant increase of $E_c$ with the growth of $k_{ct}$ leads to the scaling prediction [11, 30] $k_{ct} \propto t^{-\beta}$ (for $t < t^*$), with

$$\beta = -1/(d - 2 - \gamma),$$

where $d$ is the system dimensionality. The analytical theory of weak-wave turbulence [11, 14] predicts $\gamma = d = 2$ and $\beta = -1/2$ for $\ln(k_B/k_F) \gg 1$. This condition is not realised in our experiments, and our $\gamma$ is different (note that in a 3D gas $\gamma \approx 3.5$ was observed [35]). However, the relationship between $\beta$ and $\gamma$ in Eq. (2), which embodies the concept of dynamic scaling, should hold more generally [30], as its derivation is valid for any $\gamma < d + 2$ [11, 14]. Taking our experimental $\gamma = 2.90(5)$, we predict $\beta = -0.91(4)$, and in Fig. 3C show that $k_{ct}(t)$ agrees with this prediction (solid line). Alternatively, fitting $k_{ct} \propto t^{-\beta}$ for $t < 3 \times 2\pi/\omega_F$ gives a consistent $\beta = -0.85(7)$ (not shown), with the error dominated by the systematic uncertainty in $k_{ct}(t)$. 

---

**Fig. 2.** From low-$k$ anisotropy to high-$k$ isotropy. (A) PCA decomposition of the in-situ density modulations, for $p = 0.6$. The spatial structures, $f_j(x,y)$, of the first four modes show only excitations along the drive direction $x$. The temporal fits, $b_j(t) = \sum_{i=1}^4 B_{j,i} \cos(\omega_i t + \phi_{j,i})$ for $t > 2\pi/\omega_F$ (solid lines, with the dashed ones showing extrapolations to shorter $t$), give the harmonic weights $\Lambda_{j,\ell} = B_{j,\ell}^2/(\sum_{i=1}^4 B_{j,i}^2)$. The normalised PCA eigenvalues, $\lambda_j/\lambda$, are also shown for the next five modes, which do not show any clear structures. (B) Emergence of isotropy seen in Bragg spectroscopy, for $p = 0.85$ and $t = 2\pi/\omega_F$. Here $\tilde{n}_{k,x}(k)$ and $\tilde{n}_{k,y}(k)$ are normalised by their common value, $\tilde{n}^{\text{iso}}_k$, measured for $k = t = 0$. The emergence of isotropy is seen in the convergence of the two curves for $k \gtrsim 1/\xi = 1.0 \mu m^{-1}$; in the inset the dashed line indicates distributions measured with similar error bars for $p = 0$. All error bars (seen when larger than symbol sizes) correspond to standard statistical errors.
Fig. 3. Dynamic scaling in the pre-steady state. (A) Compensated spectra, \( n_k k^\gamma \), with \( \gamma = 2.9 \); note that \( 1/\xi = 1.0 \, \mu m^{-1} \). The error bars show standard statistical errors. (B) Total kinetic energy in the isotropic cascade, \( E_c(t) \), obtained by integrating over the spectra in (A). The slope of the solid line is equal to the independently measured flux \( \epsilon \) injected at \( k_0 \), and the horizontal dotted line is a guide to the eye. For \( t \gtrsim 3.5 \times 2\pi/\omega_k \) the system is in the steady state. (C) The cascade front, \( k_\text{cf}(t) \), defined by the intersections of the data in (A) with the horizontal dashed line; error bars show systematic uncertainties defined by the intersections with the two dotted lines. The solid line shows the theoretical prediction \( k_\text{cf} \propto t^{-\beta} \), with \( \beta = -0.91 \) based on Eq. (2). A fit for \( t < 3 \times 2\pi/\omega_k \) (not shown) gives a consistent \( \beta = 0.85(7) \). The shaded region corresponds to the independently estimated \( k_0 \propto \sqrt{\nu_0} \), including its uncertainty. (D) Compensated spectra from (A) rescaled according to Eq. (1), with \( \beta = -0.85 \), \( \alpha = \gamma\beta = -2.47 \), and the arbitrary \( t_0 \) set to \( 2.5 \times 2\pi/\omega_k \). Note that \( (t/t_0)^{\gamma\beta} (k/k_0)^\alpha n_k = n_k k^\gamma \), so when rescaling compensated spectra the y axis remains the same. We see both the dynamic scaling in the pre-steady state and its breakdown as the system reaches its non-thermal steady state.

In Fig. 3D we show the data from Fig. 3A rescaled according to Eq. (1). The collapse of the curves for \( t < t^* \) confirms the dynamic scaling in the pre-steady state, and we also show its breakdown at longer times. In the dynamics of closed quantum systems, such breakdown is expected when a system approaches equilibrium \([23, 26, 27]\); here it occurs when our driven gas reaches a non-thermal steady state.

Our experiments provide a complete, all-scales picture of the birth of 2D wave turbulence, and our microscopic view on the far-from-equilibrium dynamics could allow many further studies. It would be interesting to vary the energy-injection scale, explore excitations above an established turbulent steady state, and study decaying turbulence \([14]\). In a broader context, such studies could also allow quantum simulation of the post-inflationary cosmological reheating \([39]\). One could also search for scenarios in which the emergence of isotropy breaks down, for example by forcing the gas through a channel between two reservoirs \([40]\), so that turbulence forms in a moving frame.

Acknowledgments We thank Lena Dogra, Christoph Eigen, Nir Navon, and Giacomo Roati for discussions and comments on the manuscript. This work was supported by EPSRC [Grants No. EP/N011759/1 and No. EP/P009565/1], ERC (QBox and UniFlat), STFC [Grant No. ST/T006056/1] and QuantERA (NAQUAS, EPSRC Grant No. EP/R043396/1). J. S. acknowledges support by the DFG [Grants No. 277625399 and EXC 2004/1—390534769] and from Churchill College (Cambridge). Z. H. acknowledges support from the Royal Society Wolfson Fellowship.

* Corresponding author: mg850@cam.ac.uk
[1] L. F. Richardson, Weather prediction by numerical process (Cambridge University Press, 1922).
[2] A. N. Kolmogorov, The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers, Dokl. Akad. Nauk SSSR 30, 299 (1941).
[3] A. Obukhov, On the distribution of energy in the spectrum of turbulent flow, Dokl. Akad. Nauk SSSR 32, 22 (1941).
[4] P. A. Hwang, D. W. Wang, E. J. Walsh, W. B. Krabill, and R. N. Swift, Airborne measurements of the wavenumber spectra of ocean surface waves, Part I: Spectral slope and dimensionless spectral coefficient, J. Phys. Oceanogr. 30, 2753 (2000).
[5] U. Bortolozzo, J. Laurie, S. Nazarenko, and S. Residori, Optical wave turbulence and the condensation of light, J. Opt. Soc. Am. B 26, 2280 (2009).
[6] P. Mösta, C. D. Ott, D. Radice, L. F. Roberts, E. Schnetter, and R. Haas, A large-scale dynamo and magnetoturbulence in rapidly rotating core-collapse supernovae, Nature 528, 376 (2015).
[7] S. Ghashghaie, W. Breymann, J. Peinke, P. Talkner, and Y. Dodge, Turbulent cascades in foreign exchange markets, Nature 381, 767 (1996).
[8] R. H. Kraichnan, Inertial ranges in two-dimensional turbulence, The Physics of Fluids 10, 1417 (1967).
[9] H. L. Grant, R. W. Stewart, and A. Moilliet, Turbulence spectra from a tidal channel, J. Fluid Mech. 12, 241 (1962).
[10] M. A. Rutgers, Forced 2d turbulence: Experimental evidence of simultaneous inverse energy and forward enstrophy cascades, Phys. Rev. Lett. 81, 2244 (1998).
[11] V. E. Zakharov, V. S. L'Vov, and G. Falkovich, Kolmogorov spectra of turbulence (Springer Berlin, 1992).
[12] L. Chomaz, L. Corman, T. Bienaimé, R. Desbuquois, C. Weitemberg, S. Nascimbène, J. Beugnon, and J. Dalibard, Emergence of coherence via transverse condensation in a uniform quasi-two-dimensional Bose gas, Nat. Commun. 6, 6162 (2015).
[13] P. Christodoulou, M. Galka, N. Dogra, R. Lopes, J. Schmitt, and Z. Hadzibabic, Observation of first and second sound in a bkt superfluid, Nature 594, 191 (2021).
[14] S. Nazarenko, Wave turbulence (Springer, 2011).
[15] S. P. Johnstone, A. J. Groszek, P. T. Starkey, C. J. Billington, T. P. Simula, and K. Helmerson, Evolution of large-scale flow from turbulence in a two-dimensional superfluid, Science 364, 1267 (2019).
[16] G. Gauthier, M. T. Reeves, X. Yu, A. S. Bradley, M. A. Baker, T. A. Bell, H. Rubinsztein-Dunlop, M. J. Davis, and T. W. Neely, Giant vortex clusters in a two-dimensional quantum fluid, Science 364, 1264 (2019).
[17] G. E. Falkovich, Revised universality concept in the theory of turbulence, in Nonlinear Waves and Weak Turbulence: with Applications in Oceanography and Condensed Matter Physics, edited by N. Fitzmaurice, D. Gurarie, F. McCaughan, and W. A. Woyczynski (Birkhäuser Boston, Boston, MA, 1993) pp. 19–44.
[18] A. Monin and A. Yaglom, Statistical Fluid Mechanics, Volume II: Mechanics of Turbulence, Dover Books on Physics (Dover Publications, 2013).
[19] F. Family and T. Vicsek, Scaling of the active zone in the eden process on percolation networks and the ballistic deposition model, Journal of Physics A: Mathematical and General 18, L75 (1985).
[20] M. Kardar, G. Parisi, and Y.-C. Zhang, Dynamic scaling of growing interfaces, Phys. Rev. Lett. 56, 889 (1986).
[21] M. Prüfer, P. Kunkel, H. Strobel, S. Lannig, D. Linnemann, C.-M. Schmied, J. Berges, T. Gasenzer, and M. K. Oberthaler, Observation of universal dynamics in a spinor Bose gas far from equilibrium, Nature 563, 217 (2018).
[22] S. Erne, R. Bücker, T. Gasenzer, J. Berges, and J. Schmiedmayer, Universal dynamics in an isolated one-dimensional Bose gas far from equilibrium, Nature 563, 225 (2018).
[23] J. A. P. Glidden, C. Eigen, L. H. Dogra, T. A. Hilker, R. P. Smith, and Z. Hadzibabic, Bidirectional dynamic scaling in an isolated Bose gas far from equilibrium, Nature Physics 17, 457 (2021).
[24] A. D. García-Orozco, L. Madeira, M. A. Moreno-Armijos, A. R. Fritsch, P. E. S. Tavares, P. C. M. Castilho, A. Cidrim, G. Roati, and V. S. Bagnato, Universal dynamics of a turbulent superfluid Bose gas, arXiv:2107.07421.
[25] D. Wei, A. Rubio-Abadal, B. Ye, F. Machado, J. Kemp, K. Sraa, W. Hollerith, J. Rui, S. Gopalakrishnan, N. Y. Yao, I. Bloch, and J. Zeiher, Quantum gas microscopy of Kardar-Parisi-Zhang superdiffusion, arXiv:2107.00038.
[26] R. Micha and I. I. Tkachev, Relativistic Turbulence: A Long Way From Preheating to Equilibrium, Phys. Rev. Lett. 90, 121301 (2003).
[27] J. Berges, A. Rothkopf, and J. Schimdt, Nonthermal Fixed Points: Effective Weak Coupling for Strongly Correlated Systems Far from Equilibrium, Phys. Rev. Lett. 101, 041603 (2008).
[28] D. S. Petrov, M. Holzmann, and G. V. Shlyapnikov, Bose-Einstein condensation in quasi-2d trapped gases, Phys. Rev. Lett. 84, 2551 (2000).
[29] Z. Hadzibabic and J. Dalibard, Two-dimensional Bose fluids: An atomic physics perspective, Riv. Nuovo Cimento 34, 389 (2011).
[30] N. Navon, C. Eigen, J. Zhang, R. Lopes, A. L. Gaunt, K. Fujimoto, M. Tsubota, R. P. Smith, and Z. Hadzibabic, Synthetic dissipation and cascade fluxes in a turbulent quantum gas, Science 366, 382 (2019).
[31] S. R. Segal, Q. Diot, E. A. Cornell, A. A. Zozulya, and D. Z. Anderson, Revealing buried information: Statistical processing techniques for ultracold-gas image analysis, Phys. Rev. A 81, 053601 (2010).
[32] R. Dubessy, C. De Rossi, T. Badr, L. Longchambon, and H. Perrin, Imaging the collective excitations of an ultracold gas using statistical correlations, New Journal of Physics 16 (2014).
[33] M. Kozuma, L. Deng, E. W. Hagley, J. Wen, R. Lukwak, K. Helmerson, S. L. Rolston, and W. D. Phillips, Coherent splitting of Bose–Einstein condensates with optically induced Bragg diffraction, Phys. Rev. Lett. 82, 871 (1999).
[34] J. Stenger, S. Inouye, A. P. Chikkatur, D. M. Stamper-Kurn, D. E. Pritchard, and W. Ketterle, Bragg spectroscopy of a Bose-Einstein condensate, Phys. Rev. Lett. 82, 4569 (1999).
[35] N. Navon, A. L. Gaunt, R. P. Smith, and Z. Hadzibabic, Emergence of a turbulent cascade in a quantum gas, Nature 539, 72 (2016).
[36] J. Zhang, C. Eigen, W. Zheng, J. A. P. Glidden, T. A. Hilker, S. J. Garratt, R. Lopes, N. R. Cooper, Z. Hadzibabic, and N. Navon, Many-body decay of the gapped lowest excitation of a bose-einstein condensate, Phys. Rev. Lett. 126, 060402 (2021).
[37] See supplementary materials.
[38] S. Dyachenko, A. Newell, A. Pushkarev, and V. Zakharov, Optical turbulence: weak turbulence, condensates and collapsing filaments in the nonlinear schrödinger equation, Physica D 57, 96 (1992).
[39] A. Chatrchyan, K. T. Geier, M. K. Oberthaler, J. Berges, and P. Hauke, Analog cosmological reheating in an ultracold Bose gas, Phys. Rev. A 104, 023302 (2021).
[40] J.-P. Brantut, J. Meineke, D. Stadler, S. Krinner, and T. Esslinger, Conduction of ultracold fermions through a mesoscopic channel, Science 337, 1069 (2012).
[41] N. Tammuz, R. P. Smith, R. L. D. Campbell, S. Beattie, S. Moulder, J. Dalibard, and Z. Hadzibabic, Can a Bose gas be saturated?, Phys. Rev. Lett. 106, 230401 (2011).
[42] R. J. Fletcher, R. Lopes, J. Man, N. Navon, R. P. Smith, M. W. Zwierlein, and Z. Hadzibabic, Two- and three-body contacts in the unitary Bose gas, Science 355, 377 (2017).
[43] C. Pethick and H. Smith, Bose–Einstein Condensation in Dilute Gases (Cambridge University Press, 2002).
[44] N. Prokof’ev, O. Ruebenacker, and B. Svistunov, Critical point of a weakly interacting two-dimensional Bose gas, Phys. Rev. Lett. 87, 270402 (2001).
SUPPLEMENTARY MATERIALS

Experimental system

Our uniform two-dimensional gas of $^{39}$K atoms in the $|F, m_F⟩ = |1, 1⟩$ hyperfine ground state is optically confined in a potential sculpted using two Digital Micromirror Devices and calibrated as in [13]. Our atom number $N$ is calibrated with a systematic uncertainty of 15% using measurements of the critical temperature for Bose–Einstein condensation in a 3D harmonic trap [41], and our box size $L$ has a systematic uncertainty of $1.5 \mu m$. The systematic uncertainties in $ω_z$ and $g$ are $≤ 4%$. The trap depth $U_D$ is set by the in-plane potential and experimentally estimated from the measured light intensity. The driving force $F$ is created by a magnetic field gradient that can be applied along either axis of our square box, and we calibrated its magnitude, within 5%, by letting the gas expand under its influence and measuring the center-of-mass acceleration of the cloud. Changing the direction of the force is equivalent to changing the detection direction, and we use this symmetry to measure the momentum distribution both parallel and perpendicular to the forcing axis. We tune the scattering length $a$ exploiting a magnetic Feshbach resonance centred at 402.7 G [42]. For all our measurements $a$ is in the range $(20 - 160) a_0$, where $a_0$ is the Bohr radius.

Time-Of-Flight (TOF) measurements

We take absorption images of our gas after a variable time of ballistic expansion, $t_{TOF}$. Just before the expansion, we release the interaction energy by rapidly (< 0.1 ms) decompressing the out-of-plane confinement. TOF spectra are naturally convolved with the in-trap spatial distribution and are thus not accurate for $k ≲ m L/(ℏt_{TOF})$. We combine measurements with different $t_{TOF}$, between 8 and 35 ms, to extend the range of $k$ values we can reliably probe: the longest $t_{TOF}$ minimises finite-size effects at low $k$, while the shortest $t_{TOF}$ gives better signal-to-noise ratio at large $k$. For our main TOF measurements (Fig. 1D, Fig. 3), we use our smaller, 31-μm box to limit the low-$k$ range affected by finite-size effects. We repeated all measurements about 10 times under the same experimental conditions.

Principal Component Analysis (PCA)

PCA extracts orthonormal modes within a data set based on the spectral decomposition of its covariance matrix [31, 32]. Our data set (for Fig. 2A) is a series of in-situ density profiles obtained by absorption imaging at $J = 81$ different times, $t ∈ [0, 90] ms$, after the initiation of the driving. For each $t$ we repeated the experiment about 5 times and used the 81 averaged profiles to get the 80 PCA modes. In the $b_j(t)$ plots (which have 81 time steps), we filtered-out high-frequency noise using a 3-point moving average.

Bragg spectroscopy

The two far-off-resonant laser beams used for Bragg spectroscopy [33, 34] are detuned from each other by a frequency $Δν$ and have in-plane wavevectors $k_1$ and $k_2$ such that the recoil momentum $ℏk_r = ℏ(k_1 - k_2)$ is aligned with one of the box axes, and $k_r ≈ 15 \mu m^{-1}$ is larger than $k_D$, so the diffracted atoms leave the trap. Measuring the number of diffracted atoms, $N_{diff}$, as a function of $Δν$ gives the line-integrated distribution parallel to $k_r$. The duration of our Bragg pulse is $τ ≈ 1.5 ms$, which gives $k$-space resolution of $Δk = 2πm/(τℏk_r) ≈ 0.17 \mu m^{-1}$. The absolute values of $N_{diff}$, also depend on the intensity of the Bragg beams, so the measured $n_{k,x(y)}$ are not automatically normalised, and in the main text we normalise them to their common value measured for $k = t = 0$. All measurements were repeated about 20 times under the same experimental conditions.

Energy in the isotropic cascade

To calculate $E_c(t)$ (Fig. 3B) we integrate $ε_k$ for $k > 1/ξ$. Theoretically, the integral could be extended to infinity, but experimentally, integrating to unnecessarily large $k$ (where there is no real atomic population) just adds noise; instead, we cut off the integral at $6 \mu m^{-1}$, which is larger than both the $k_D = 5.1(6) \mu m^{-1}$ estimated from $U_D$ and the $k_0$ observed in the saturation of $k_T$ (Fig. 3C). Note that we consider only the increase of the kinetic energy at high $k$ and neglect any changes in the interaction energy. This is a good approximation since for $k > 1/ξ$ the excitation energy is mostly kinetic and since in a box trap changing $n_k$ does not change the gas density, so small changes in the total interaction energy arise only due to modification of the zero-distance second-order correlation function [43]. Calculating the size of this effect is non-trivial for a far-from-equilibrium gas, but based on an equilibrium calculation we could be underestimating the rate of growth of the total energy by up to 15%, which is less than the 20% systematic uncertainty in $ε$ injected at $k_F$. Finally, note that in Fig. 3 at $t ≈ t^*$ the fraction of all particles that is at $k > 1/ξ$ is about 30%, and that for our density and interaction strength (here $g = 0.018$) the Berezinskii-Kosterlitz-Thouless critical temperature for transition to superfluidity is $410$ nK [29, 44], corresponding to 8.5 kHz, so even the fully developed turbulent state has a very low energy per particle.