Abstract: Gauge fields have a natural metric interpretation in terms of horizontal distance. The latest, also called Carnot-Carathéodory or subriemannian distance, is by definition the length of the shortest horizontal path between points, that is to say the shortest path whose tangent vector is everywhere horizontal with respect to the gauge connection. In noncommutative geometry all the metric information is encoded within the Dirac operator $D$. In the classical case, i.e. commutative, Connes’s distance formula allows to extract from $D$ the geodesic distance on a riemannian spin manifold. In the case of a gauge theory with a gauge field $A$, the geometry of the associated $U(n)$-vector bundle is described by the covariant Dirac operator $D + A$. What is the distance encoded within this operator? It was expected that the noncommutative geometry distance $d$ defined by a covariant Dirac operator is intimately linked to the Carnot-Carathéodory distance $d_H$ defined by $A$. In this paper we make precise this link, showing that the equality of $d$ and $d_H$ strongly depends on the holonomy of the connection. Quite interestingly we exhibit an elementary example, based on a 2 torus, in which the noncommutative distance has a very simple expression and simultaneously avoids the main drawbacks of the riemannian metric (no discontinuity of the derivative of the distance function at the cut-locus) and of the subriemannian one (memory of the structure of the fiber).

1. Introduction

Noncommutative geometry [2] enlarges differential geometry beyond the scope of riemannian spin manifold and gives access, among various examples, to spaces obtained as the product of the continuum by the discrete. It allows to describe in a single and coherent geometrical object the space-time of the standard model of elementary particles\(^1\) coupled with euclidean general relativity [1]. Specifically the diffeomorphism group of general relativity appears as the automorphism group of $C^\infty (M)$, the algebra

\(^{1}\) with massless neutrinos. Massive Dirac neutrinos are easily incorporated in the model [15] as long as one of them remain massless. Otherwise more substantial changes might be required.
of smooth functions over a compact riemannian spin manifold $M$, while the gauge group of the strong and electroweak interactions emerges as the group $U(A_I)$ of unitary elements of a finite dimensional algebra $A_I$ (modulo a lift to the spinor [11]). Remarkably unitaries not only act as gauge transformations but also acquire a metric signification via the so called fluctuations of the metric. This paper aims at studying in detail the analogy introduced in [3] between a simple kind of fluctuations of the metric, those governed by a connection 1-form on a principal bundle, and the associated Carnot-Carathéodory metric.

A noncommutative geometry consists in a spectral triple

$$\mathcal{A}, \mathcal{H}, D,$$

where $\mathcal{A}$ is an involutive algebra commutative or not, $\mathcal{H}$ an Hilbert space carrying a representation $\Pi$ of $\mathcal{A}$, and $D$ a selfadjoint operator on $\mathcal{H}$. Together with a chirality $\Gamma$ and a real structure $J$ both acting on $\mathcal{H}$, they satisfy a set of properties [3] providing the necessary and sufficient conditions for 1) an axiomatic definition of riemannian spin geometry in terms of commutative algebra 2) its natural extension to the noncommutative framework. Points are recovered as pure states $\mathcal{P}(\mathcal{A})$ of $\mathcal{A}$, in analogy with the commutative case where

$$\mathcal{P}(C^\infty (M)) \simeq M, \quad (1)$$

$$\omega_x (f) = f(x) \quad (2)$$

for any pure state $\omega_x$ and smooth function $f$. A distance $d$ between states $\omega, \omega'$ of $\mathcal{A}$ is defined by

$$d(\omega, \omega') \doteq \sup_{a \in \mathcal{A}} \{|\omega(a) - \omega'(a)|; \| [D, \Pi(a)] \| \leq 1 \}, \quad (3)$$

where the norm is the operator norm on $\mathcal{H}$. In the commutative case,

$$\mathcal{A}_E = C^\infty (M), \quad \mathcal{H}_E = L_2(M, S), \quad D_E = -i \gamma^\mu \partial_\mu \quad (4)$$

with $\mathcal{H}_E$ the space of square integrable spinors and $D_E$ the ordinary Dirac operator of quantum field theory, $d$ coincides with the geodesic distance defined by the riemannian structure of $M$. Thus (3) is a natural extension of the classical distance formula, all the more as it does not involve any notion ill-defined in a quantum framework such as trajectory between points.

Carnot-Carathéodory metrics (or sub-riemannian metrics) [14] are defined on manifolds $P$ equipped with an horizontal distribution, that is to say a (smooth) specification at any point $p \in P$ of a subspace $H_p P$ of the tangent space $T_p P$. The Carnot-Carathéodory distance $d_H$ between $p$ and $q$ is the length of the shortest path $c$ joining $p$ and $q$ whose tangent vector is everywhere horizontal,

$$d_H(p, q) = \inf_{\dot{c}(t) \in H_c P} \int_0^1 \| \dot{c}(t) \| dt. \quad (5)$$

If there is no horizontal path from $p$ to $q$ then $d_H(p, q)$ is infinite. Any point at finite distance from $p$ is said to be accessible

$$\text{Acc}(p) \doteq \{ q \in P; \ d_H(p, q) < +\infty \}. \quad (6)$$