A Fast, Principled Working Set Algorithm for Exploiting Piecewise Linear Structure in Convex Problems

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Abstract
By reducing optimization to a sequence of smaller subproblems, working set algorithms achieve fast convergence times for many machine learning problems. Despite such performance, working set implementations often resort to heuristics to determine subproblem size, makeup, and stopping criteria. We propose BlitzWS, a working set algorithm with useful theoretical guarantees. Our theory relates subproblem size and stopping criteria to the amount of progress during each iteration. This result motivates strategies for optimizing algorithmic parameters and discarding irrelevant components as BlitzWS progresses toward a solution. BlitzWS applies to many convex problems, including training $\ell_1$-regularized models and support vector machines. We showcase this versatility with empirical comparisons, which demonstrate BlitzWS is indeed a fast algorithm.

1. Introduction
Many optimization problems in machine learning have useful structure at their solutions. For sparse regression, the optimal model makes predictions using a fraction of available features. For support vector machines, easy-to-classify examples have no influence on the optimal model. In this work, we exploit such structure to train these models efficiently.

Working set algorithms exploit structure by reducing optimization to a sequence of simpler subproblems. Each subproblem considers only a priority subset of the problem’s components—the features likely to have nonzero weight in sparse regression, for example, or training examples near the margin in SVMs. Likely the most prominent working set algorithms for machine learning are those of the LIBLINEAR library (Fan et al., 2008), an efficient software package for training linear models. By using working set and related “shrinking” (Joachims, 1999) heuristics, LIBLINEAR converges very quickly. Other successful applications of working sets include algorithms proposed by Osuna et al. (1997), Zanghirati and Zanni (2003), Tsochantaridis et al. (2005), Kim and Park (2008), Roth and Fischer (2008), Obozinski et al. (2009) and Friedman et al. (2010).
Despite the usefulness of working set algorithms, there is limited theoretical understanding of these methods. For LIBLINEAR, except for guaranteed convergence to a solution, there are no guarantees with regard to working sets and shrinking. As a result, critical aspects of working set algorithms typically rely on heuristics rather than principled understanding.

We propose BlitzWS, a working set algorithm accompanied by useful theoretical analysis. Our theory explains how to prioritize components of the problem in order to guarantee a specified amount of progress during each iteration. To our knowledge, BlitzWS is the first working set algorithm with this type of guarantee. This result motivates a theoretically justified way to select each subproblem, making BlitzWS’s choice of subproblem size, components, and stopping criteria more principled and robust than those of prior approaches.

BlitzWS solves instances of a novel problem formulation, which formalizes our notion of “exploiting structure” in problems such as sparse regression. Specifically, we define the objective function as a sum of many piecewise terms. Each piecewise function is comprised of simpler subfunctions, some of which we assume to be linear. Exploiting structure amounts to selectively replacing piecewise terms in the objective with linear subfunctions. This results in a modified objective that can be much simpler to minimize. By solving a sequence of such subproblems, BlitzWS rapidly converges to the original problem’s solution.

In addition to BlitzWS, we propose a closely related safe screening test called BlitzScreen. First proposed by El Ghaoui et al. (2012), safe screening identifies problem components that are guaranteed to be irrelevant to the solution. Compared to prior screening tests, BlitzScreen (i) applies to a larger class of problems, and (ii) simplifies the objective function by a greater amount.

We include empirical evaluations to showcase the usefulness of BlitzWS and BlitzScreen. We find BlitzWS significantly outperforms LIBLINEAR in many cases, especially for sparse logistic regression problems. Perhaps surprisingly, although our screening test improves on many prior tests, we find that screening often has negligible effect on overall convergence times. In contrast, BlitzWS improves convergence times significantly in nearly all cases.

This work builds upon two previous conference papers (Johnson and Guestrin, 2015, 2016). New contributions include refinements to the proposed algorithm, improved theoretical results, and additional empirical results. An open-source implementation of BlitzWS is available at the web address http://github.com/tbjohns/blitzml.

We organize the remainder of this paper as follows. In §2, we introduce BlitzWS for a simple constrained problem, emphasizing BlitzWS’s main concepts. In §3 we introduce a piecewise problem formulation, which encompasses a larger set of problems than we consider in §2. In §4, we define BlitzWS for the general piecewise problem. This section contains more detail compared to §2, including analysis of approximate subproblem solutions and a method for selecting algorithmic parameters. In §5 we introduce BlitzScreen and explain its relation to BlitzWS. In §6, we demonstrate the usefulness of BlitzWS and BlitzScreen in practice. We discuss conclusions in §7.

2. BlitzWS for a simple constrained problem

In this section, we introduce BlitzWS for computing the minimum norm vector in a polytope. Given vectors $a_i \in \mathbb{R}^n$ and scalars $b_i \in \mathbb{R}$ for $i = 1, 2, \ldots, m$, we solve

$$\minimize_{x \in \mathbb{R}^n} \psi_{\text{MN}}(x) := \frac{1}{2} \|x\|^2 \quad \text{s.t.} \quad \langle a_i, x \rangle \leq b_i \quad i = 1, \ldots, m.$$

(PMN)
We refer to the special case of BlitzWS that solves this min-norm problem as “BlitzMN.” For now, we consider only this simple problem to emphasize the algorithm’s concepts rather than its capability of solving a variety of problems.

(PMN) has important applications to machine learning. Most notably, $\ell_1$-regularized least squares problems—the “lasso” (Tibshirani 1996)—can be transformed into an instance of (PMN) using duality. We discuss using BlitzWS to solve the lasso more in §3.2.2.

2.1 Overview of BlitzMN

During each iteration $t$, BlitzMN selects a working set of constraints, $W_t$. BlitzMN then computes the minimizer of $\psi_{MN}$ subject only to constraints in $W_t$, storing the result as $x_t$:

$$x_t \leftarrow \arg\min \left\{ \frac{1}{2} \|x\|^2 \mid \langle a_i, x \rangle \leq b_i \text{ for all } i \in W_t \right\}.$$

We refer to the task of computing $x_t$ as “subproblem $t$.”

Typical working set algorithms select each $W_t$ using heuristics. BlitzMN improves upon this with two main novelties. First, in addition to $x_t$, BlitzMN introduces a second iterate, $y_t$. This iterate is feasible (satisfies all constraints in (PMN)) for all $t$.

The iterate $y_t$ is necessary for BlitzMN’s second novelty, which is the principled choice of each working set. BlitzMN selects $W_t$ in a way that guarantees quantified progress during iteration $t$. This amount of progress is determined by a progress parameter, $\xi_t \in (0, 1]$. For now we assume $\xi_t$ is given, but in §4.9 we discuss a way to automatically select $\xi_t$.

If $\xi_t = 1$, then BlitzMN is guaranteed to return (PMN)’s solution upon completion of iteration $t$. As $\xi_t$ decreases toward zero, BlitzMN guarantees less progress (which we quantify more precisely in §2.4). At a high level, BlitzMN combines two ideas to ensure this progress:

- **Including in the working set constraints that are active at the previous subproblem solution**: BlitzMN includes in $W_t$ all constraints for which $\langle a_i, x_{t-1} \rangle = b_i$. This ensures that $\psi_{MN}(x_t) \geq \psi_{MN}(x_{t-1})$.

- **Enforcing an equivalence region**: For subproblem $t$, BlitzMN defines an “equivalence region,” $S_\xi$, which is a subset of $\mathbb{R}^n$. BlitzMN selects $W_t$ in a way that ensures subproblem $t$ and (PMN) are identical within $S_\xi$ (i.e., within $S_\xi$, the feasible region is preserved).

Establishing this equivalence region has two major implications. First, if $x_t \in S_\xi$, then $x_t$ solves not only subproblem $t$ but also (PMN). This is because subgradient values are preserved within the equivalence region. Second, if $x_t$ does not equal the solution, then it must be the case that $x_t \notin S_\xi$. We design $S_\xi$ to ensure “$\xi_t$ progress” in this case.

2.2 Making working sets tractable with suboptimality gaps

We measure BlitzMN’s progress during each iteration in terms of a suboptimality gap. Since $x_t$ minimizes $\psi_{MN}$ subject to a subset of constraints, it follows that $\psi_{MN}(x_t) \leq \psi_{MN}(x^*)$, where $x^*$ solves (PMN). Thus, given the feasible point $y_t$, we can define the suboptimality gap

$$\Delta_t = \psi_{MN}(y_t) - \psi_{MN}(x_t) \geq \psi_{MN}(y_t) - \psi_{MN}(x^*).$$

Later, we analyze the improvement in $\Delta_t$ between iterations $t - 1$ and $t$. Maximizing this improvement leads to a principled method for selecting each working set.
2.3 Converging from two directions: iterate \( y_t \) and line search

BlitzMN initializes \( y_0 \) as a feasible point. After the algorithm computes \( x_t \) during iteration \( t \), BlitzMN performs a line search update to \( y_t \). Specifically, \( y_t \) is the point on the segment \([y_{t-1}, x_t]\) that is closest to \( x_t \) while remaining feasible. Put differently, assuming that \( x_t \) violates at least one constraint, BlitzMN updates \( y_t \) so that (i) \( y_t \) lies on the segment \([y_{t-1}, x_t]\), (ii) \( y_t \) satisfies all \( m \) constraints, and (iii) unless \( y_t = x_t \), there exists a “limiting constraint” \( i_{\text{limit}} \) for which \( \langle a_{i_{\text{limit}}}, x_t \rangle > b_{i_{\text{limit}}} \) and \( \langle a_{i_{\text{limit}}}, y_t \rangle = b_{i_{\text{limit}}} \).

To perform this line search update, BlitzMN computes

\[
\alpha_t = \min_{i : \langle a_i, x_t \rangle > b_i} \frac{b_i - \langle a_i, y_{t-1} \rangle}{\langle a_i, x_t \rangle - \langle a_i, y_{t-1} \rangle}
\]

and subsequently defines \( y_t = \alpha_t x_t + (1 - \alpha_t) y_{t-1} \). In the special case that \( \langle a_i, x_t \rangle \leq b_i \) for all \( i \), we define \( \alpha_t = 1 \) (and hence \( y_t = x_t \)). BlitzMN has converged in this case, since \( \Delta_t = 0 \).

Because \( x_t \) minimizes \( \psi_{\text{MN}} \) subject to a subset of constraints, \( \psi_{\text{MN}}(x_t) \leq \psi_{\text{MN}}(y_{t-1}) \). By convexity of \( \psi_{\text{MN}} \), this implies \( \psi_{\text{MN}}(y_t) \leq \psi_{\text{MN}}(y_{t-1}) \). Recall also from §2.1 that \( \psi_{\text{MN}}(x_t) \) is nondecreasing with \( t \). Therefore, \( \Delta_t \) is nonincreasing with \( t \).

We have not yet quantified how much \( \Delta_t \) decreases between iterations. We next derive a rule for selecting \( W_t \) that guarantees this suboptimality gap decreases by a specified amount.

2.4 Quantifying suboptimality gap progress during iteration \( t \)

In §2.1, we established that BlitzMN includes in \( W_t \) all constraints that are active at \( x_{t-1} \). We now add additional constraints to the working set in order to guarantee a specified amount of progress toward convergence. In particular, given a progress coefficient \( \xi_t \in (0, 1] \), we design \( W_t \) such that

\[
\Delta_t \leq (1 - \xi_t) \Delta_{t-1} .
\]

Applying properties of convexity and the definition of \( y_t \), we derive the following bound:

**Lemma 2.1.** Assume \( \alpha_t > 0 \), and define \( \beta_t = \alpha_t (1+\alpha_t)^{-1} \). Assume that \( W_t \) includes all constraints that are active at \( x_{t-1} \). Then we have

\[
\Delta_t \leq \frac{1 - 2\beta_t}{1 - \beta_t} \left[ \Delta_{t-1} - \beta_t \frac{1}{2} \| y_t - \beta_t x_{t-1} - (1 - \beta_t) y_{t-1} \|^2 - \beta_t \frac{1}{2} \| x_{t-1} - y_{t-1} \|^2 \right] .
\]

We prove this result in Appendix A. Since the special case that \( \alpha_t = 0 \) is unnecessary for understanding the algorithm’s concepts, we ignore this case except in later proofs.

The working set affects only two variables in (3): \( \beta_t \) and \( y_t \). The other variables—\( x_{t-1}, y_{t-1}, \) and \( \Delta_{t-1} \)—are given when selecting \( W_t \). By understanding how \( W_t \) affects \( \beta_t \) and \( y_t \), we can choose \( W_t \) so that the right side of (3) is upper bounded by \( (1 - \xi_t) \Delta_{t-1} \).

Assume for a moment that \( \beta_t \) is given when BlitzMN defines \( W_t \). In this scenario, it is straightforward to select \( W_t \) in a way that guarantees (2). For the special case that \( \beta_t = 1/2 \), (3) simplifies to \( \Delta_t \leq 0 \) (2) holds, regardless of \( W_t \). In the case that \( \beta_t < 1/2 \), we have \( \alpha_t < 1 \). Applying the definition of \( \alpha_t \) in (1), there exists a limiting constraint \( i_{\text{limit}} \) such that

\[
\langle a_{i_{\text{limit}}}, x_t \rangle > b_{i_{\text{limit}}} \quad \text{and} \quad \langle a_{i_{\text{limit}}}, y_t \rangle = b_{i_{\text{limit}}} .
\]

1. While computing a feasible \( y_0 \) could be difficult in general, this is not an issue for the applications we consider. When solving the lasso’s dual (§3.2.2), for example, BlitzWS defines \( y_0 = 0 \), which is feasible in this case.
Since $x_t$ violates constraint $i_{\text{limit}}$, we have $i_{\text{limit}} \not\in \mathcal{W}_t$. To apply this fact, note $y_t$ appears in Lemma 2.1 through the quantity $\|y_t - \beta_t x_{t-1} - (1 - \beta_t) y_{t-1}\|$. BlitzMN chooses $\mathcal{W}_t$ to ensure this norm equals a threshold $\tau_\xi(\beta_t)$ at minimum. To achieve this, we define the “equivalence ball”

$$B_\xi(\beta_t) = \{ x \mid \| x - \beta_t x_{t-1} - (1 - \beta_t) y_{t-1} \| < \tau_\xi(\beta_t) \}. \quad (5)$$

BlitzMN includes $i$ in $\mathcal{W}_t$ if there exists an $x \in B_\xi(\beta_t)$ such that $\langle a_i, x \rangle \geq b_i$. This preserves (PMN)'s feasible region within $B_\xi(\beta_t)$. Since $i_{\text{limit}} \not\in \mathcal{W}_t$, this implies that no point on the boundary of constraint $i_{\text{limit}} = y_t$ included, due to (4)—lies within $B_\xi(\beta_t)$. By our definition of $B_\xi(\beta_t)$ in (5), this guarantees that $\|y_t - \beta_t x_{t-1} - (1 - \beta_t) y_{t-1}\| \geq \tau_\xi(\beta_t)$. We illustrate this concept in Figure 1.

Having linked $\tau_\xi(\beta_t)$ to Lemma 2.1, we can define $\tau_\xi(\beta_t)$ to produce our desired bound, (2):

$$\tau_\xi(\beta_t) = \beta_t \sqrt{2 \Delta_{t-1}} \left[ 1 + \frac{\beta_t}{1-\beta_t} \left( 1 - \frac{\|x_{t-1} - y_{t-1}\|^2}{2 \Delta_{t-1}} \right) - \frac{1-\xi_t}{1-2\beta_t} \right]^{1/2}. \quad (6)$$

This leads to the following result:

**Lemma 2.2.** Assume $\beta_t$ is known when selecting $\mathcal{W}_t$, and assume $\beta_t > 0$. For all $i \in [m]$, let $\mathcal{W}_t$ include constraint $i$ if either $\{ x \mid \langle a_i, x \rangle \geq b_i \} \cap B_\xi(\beta_t) \neq \emptyset$ or $\langle a_i, x_{t-1} \rangle = b_i$. Then

$$\Delta_t \leq (1 - \xi_t) \Delta_{t-1}.$$ 

We prove Lemma 2.2 in Appendix B. On its own, this lemma is impractical, as it assumes knowledge of $\beta_t$ when choosing $\mathcal{W}_t$. Since $\beta_t$ is unknown BlitzMN considers all possible $\beta$ when selecting $\mathcal{W}_t$. To do so, we define the equivalence region

$$\mathcal{S}_\xi = \bigcup_{\beta \in (0,1/2)} B_\xi(\beta).$$

This new equivalence region leads to the following result, which we prove in Appendix C:

**Theorem 2.3** (Guaranteed progress during iteration $t$ of BlitzMN). During iteration $t$ of BlitzMN, consider any progress coefficient $\xi_t \in (0, 1)$. For all $i \in [m]$, let $\mathcal{W}_t$ include constraint $i$ if either $\mathcal{S}_\xi \cap \{ x \mid \langle a_i, x \rangle \geq b_i \} \neq \emptyset$ or $\langle a_i, x_{t-1} \rangle = b_i$. Then

$$\Delta_t \leq (1 - \xi_t) \Delta_{t-1}.$$
increases with $\xi$ (set of points within a fixed distance from a line segment) for which testing if $S$ is a “teardrop” shape. When $\xi = 1$, $S$ is a ball with center $\frac{1}{2}(x_{t-1} + y_{t-1})$. To generate the figure, let $\|x_{t-1} - y_{t-1}\|^2 / \Delta_{t-1} = 1$.

**Figure 2: Geometry of equivalence regions.** As $\xi_t$ increases, the size of $S$ increases, the number of constraints in $W_t$ increases, and the amount of guaranteed progress increases. For small $\xi_t$, $S$ has a “teardrop” shape. When $\xi_t = 1$, $S$ is a ball with center $\frac{1}{2}(x_{t-1} + y_{t-1})$. To generate the figure, let $\|x_{t-1} - y_{t-1}\|^2 / \Delta_{t-1} = 1$.

### 2.5 Computing $W_t$ efficiently with capsule approximations

Figure 2 contains renderings of $S$. Note how $S$ grows in size as $\xi_t$ increases, since $\tau_\xi(\beta)$ also increases with $\xi_t$ due to (6). Unfortunately, using $S$ to select $W_t$ is problematic in practice, since testing if $S \cap \{x \mid (a_i, x) \geq b_i\} \neq \emptyset$ is not simple. To reduce computation, BlitzMN constructs $W_t$ using a relaxed equivalence region, $S_c^{\text{cap}}$. This set is the convex hull of two balls:

$$S_c^{\text{cap}} = \text{conv}(B_1^{\text{cap}} \cup B_2^{\text{cap}}),$$

where

$$B_1^{\text{cap}} = \{ x \mid \| x - c_1^{\text{cap}} \| < r^{\text{cap}} \}, \quad B_2^{\text{cap}} = \{ x \mid \| x - c_2^{\text{cap}} \| < r^{\text{cap}} \},$$

$$c_1^{\text{cap}} = y_{t-1} + \frac{x_{t-1} - y_{t-1}}{\|x_{t-1} - y_{t-1}\|} (d_{\text{min}}^{\text{cap}} + r^{\text{cap}}), \quad c_2^{\text{cap}} = y_{t-1} + \frac{x_{t-1} - y_{t-1}}{\|x_{t-1} - y_{t-1}\|} (d_{\text{max}}^{\text{cap}} - r^{\text{cap}}),$$

$$d_{\text{min}}^{\text{cap}} = \inf_{\beta : \tau_\xi(\beta) > 0} \beta \| x_{t-1} - y_{t-1} \| - \tau_\xi(\beta), \quad d_{\text{max}}^{\text{cap}} = \sup_{\beta : \tau_\xi(\beta) > 0} \beta \| x_{t-1} - y_{t-1} \| + \tau_\xi(\beta),$$

and $r^{\text{cap}} = \sup_{\beta : \tau_\xi(\beta) > 0} \tau_\xi(\beta)$.

In Appendix I.1, we prove that $S \subseteq S_c^{\text{cap}}$. Illustrated in Figure 3, $S_c^{\text{cap}}$ is the smallest “capsule” (set of points within a fixed distance from a line segment) for which $S \subseteq S_c^{\text{cap}}$. We define the capsule using three scalars: $d_{\text{min}}^{\text{cap}}$, $d_{\text{max}}^{\text{cap}}$, and $r^{\text{cap}}$ (in addition to the points $x_{t-1}$ and $y_{t-1}$). The radius of the capsule is $r^{\text{cap}}$, while $d_{\text{min}}^{\text{cap}}$ and $d_{\text{max}}^{\text{cap}}$ parameterize the capsule’s endpoints. Determining these three parameters requires solving three 1-D optimization problems, which we can solve efficiently:

**Theorem 2.4** (Computing capsule parameters is quasiconcave). For each $s \in \{-1, 0, +1\}$, the function $q_s(\beta) = s\beta \| x_{t-1} - y_{t-1} \| + \tau_\xi(\beta)$ is quasiconcave over $\{ \beta \mid q_s(\beta) > 0 \}$. Thus, $d_{\text{min}}^{\text{cap}}$, $d_{\text{max}}^{\text{cap}}$, and $r^{\text{cap}}$ are suprema of 1-D quasiconcave functions.
Figure 3: Capsule approximation. To simplify computation, BlitzMN constructs \( W_t \) using \( S_{\xi}^{\text{cap}} \), which is a relaxation of \( S_{\xi} \). \( S_{\xi} \) is shaped like a teardrop when \( \xi_t \) is small, while \( S_{\xi}^{\text{cap}} \) is the smallest capsule (convex hull of two balls with equal radius) that contains \( S_{\xi} \). For the figure, \( \xi_t = 0.4 \), and \( \| x_{t-1} - y_{t-1} \|_2^2 / \Delta_{t-1} = 1 \).

We prove Theorem 2.4 in Appendix D. Making use of this result, BlitzMN computes \( d_{\text{cap}}^{\text{min}}, d_{\text{cap}}^{\text{max}} \), and \( r_{\text{cap}} \) using the bisection method. Empirically we find the computational cost of computing \( S_{\xi}^{\text{cap}} \) is negligible compared to the cost of solving each subproblem (unless (P) is very small).

After computing \( S_{\xi}^{\text{cap}} \), it is simple to test whether \( S_{\xi}^{\text{cap}} \cap \{ x \mid \langle a_i, x \rangle \geq b_i \} \neq \emptyset \). The condition is true iff \( B_1^{\text{cap}} \) or \( B_2^{\text{cap}} \) intersect \( \{ x \mid \langle a_i, x \rangle \geq b_i \} \). It follows that \( S_{\xi}^{\text{cap}} \cap \{ x \mid \langle a_i, x \rangle \geq b_i \} \neq \emptyset \) iff

\[
 b_i - \max \left\{ \langle a_i, c_1^{\text{cap}} \rangle, \langle a_i, c_2^{\text{cap}} \rangle \right\} < \| a_i \| r_{\text{cap}} .
\]

BlitzMN includes in the working set any \( i \) for which either the above inequality is satisfied or \( \langle a_i, x_{t-1} \rangle = b_i \). Since \( S_{\xi} \subseteq S_{\xi}^{\text{cap}} \), this \( W_t \) satisfies the conditions for Theorem 2.3 ensuring the suboptimality gap decreases by at least a \( 1 - \xi_t \) factor during iteration \( t \).

2.6 BlitzMN definition and convergence guarantee

We formally define BlitzMN in Algorithm 1. BlitzMN assumes an initial feasible point \( y_0 \) and initializes \( x_0 \) as the zero vector (“subproblem 0” is implicitly defined as minimizing \( \psi_{MN} \) subject to no constraints). The suboptimality gap decreases with the following guarantee:

**Theorem 2.5** (Convergence bound for BlitzMN). For any iteration \( T \) of Algorithm 1, define the suboptimality gap \( \Delta_T = \psi_{MN}(y_T) - \psi_{MN}(x_T) \). For all \( T > 0 \), we have

\[
\Delta_T \leq \Delta_0 \prod_{t=1}^{T} (1 - \xi_t) .
\]

We have yet to address several practical considerations for BlitzMN. This includes analysis of approximate subproblem solutions and a procedure for selecting \( \xi_t \) during each iteration. We address these details in §4 in the context of our more general working set algorithm, BlitzWS. Before that, we define a more general problem formulation.

3. Exploiting piecewise linear structure in convex problems

Rather than exploiting irrelevant constraints, BlitzWS exploits piecewise linear structure. In this section, we reformulate the objective function to accommodate this more general concept.
Algorithm 1 BlitzMN for solving (PMN)

input feasible point \( y_0 \) and method for choosing \( \xi_t \) for all \( t \)
initialize \( x_0 \leftarrow 0 \)
for \( t = 1, \ldots, T \) until \( x_T = y_T \) do

Choose progress coefficient \( \xi_t \in (0, 1] \)
\( S^\text{cap}_\xi \leftarrow \text{compute_capsule_region}(\xi_t, x_{t-1}, y_{t-1}) \) \# see §2.5
\( W_t \leftarrow \{ i \in [m] \mid \text{include_in_working_set}(i, S^\text{cap}_\xi, x_{t-1}) \} \)
\( x_t \leftarrow \text{argmin} \left\{ \frac{1}{2} \| x \|^2 \mid \langle a_i, x \rangle \leq b_i \text{ for all } i \in W_t \right\} \)
\( y_t \leftarrow \text{compute_extreme_feasible_point}(x_t, y_{t-1}) \)
return \( y_T \)

function \( \text{include_in_working_set}(i, S^\text{cap}_\xi, x_{t-1}) \)
\( c^\text{cap}_1, c^\text{cap}_2, r^\text{cap} \leftarrow \text{get_capsule_centers_and_radius}(S^\text{cap}_\xi) \) \# see §2.5
if \( b_i - \max \{ \langle a_i, c^\text{cap}_1 \rangle, \langle a_i, c^\text{cap}_2 \rangle \} < \| a_i \| r^\text{cap} \) or \( \langle a_i, x_{t-1} \rangle = b_i \) then
return true
return false

function \( \text{compute_extreme_feasible_point}(x_t, y_{t-1}) \)
\( \alpha_t \leftarrow 1 \)
for \( i \in [m] \) do
if \( \langle a_i, x_t \rangle > b_i \) then
\( \alpha_t \leftarrow \min \left\{ \alpha_t, \frac{b_i - \langle a_i, y_{t-1} \rangle}{\langle a_i, x_t \rangle - \langle a_i, y_{t-1} \rangle} \right\} \)
return \( \alpha_t x_t + (1 - \alpha_t) y_{t-1} \)

3.1 Piecewise problem formulation

For the remainder of this work, we consider convex optimization problems of the form

\[
\text{minimize } f(x) := \psi(x) + \sum_{i=1}^{m} \phi_i(x). \tag{P}
\]

We assume each \( \phi_i \) is piecewise. That is, for each \( \phi_i \), we assume a domain-partitioning function \( \pi_i : \mathbb{R}^n \rightarrow \{1, 2, \ldots, p_i\} \) and corresponding subfunctions \( \phi^{(1)}_i, \phi^{(2)}_i, \ldots, \phi^{(p_i)}_i \) such that

\[
\phi_i(x) = \begin{cases} 
\phi^{(1)}_i(x) & \text{if } \pi_i(x) = 1, \\
\vdots & \\
\phi^{(p_i)}_i(x) & \text{if } \pi_i(x) = p_i.
\end{cases}
\]

We assume that \( \psi, \phi_i, \) and \( \phi^{(k)}_i \) for all \( i \) and \( k \) are convex lower semicontinuous functions. We also assume that \( \psi \) is 1-strongly convex. (We can adapt this formulation to the more general case that \( \psi \) is \( \gamma \)-strongly convex for some \( \gamma > 0 \) by scaling \( f \) by \( \gamma^{-1} \).)

Let \( \mathcal{X}^{(k)}_i \) denote the \( k \)th subdomain of \( \phi_i \): \( \mathcal{X}^{(k)}_i = \{ x \mid \pi_i(x) = k \} \). Denoting (P)’s solution by \( x^* \), we use \( \mathcal{X}^{(k)}_i \) to denote the subdomain of \( \phi_i \) that contains \( x^* \).
BlitzWS efficiently solves (P) by exploiting \( f \)'s piecewise structure. We focus on instances of (P) for which the piecewise functions are the primary obstacle to efficient optimization (generally problems for which \( m \) is large). We also focus on instances of (P) for which many \( \phi_i^{(k)} \) subfunctions are linear. We base our methods on the following proposition:

**Proposition 3.1** (Exploiting piecewise structure at \( x^* \)). For each \( i \in [m] \), assume knowledge of \( \pi_i(x^*) \) and whether \( x^* \in \text{bd}(\mathcal{X}^{(i)}_i) \). Define \( \phi_i^* = \phi_i \) if \( x^* \in \text{bd}(\mathcal{X}^{(i)}_i) \) and \( \phi_i^* = \phi_i(\pi_i(x^*)) \) otherwise. Then \( x^* \) is also the solution to

\[
\min_{x \in \mathbb{R}^n} f^*(x) := \psi(x) + \sum_{i=1}^m \phi_i^*(x).
\]

(P*)

Proposition 3.1 states that if \( f \)'s minimizer does not lie on the boundary of \( \mathcal{X}^{(i)}_i \), then replacing \( \phi_i \) with the subfunction \( \phi_i(\pi_i(x^*)) \) in \( f \) does not change the objective's minimizer. We can verify this by observing that \( f^* \) preserves the subgradient of \( f \) at \( x^* \), which implies that \( 0 \in \partial f^*(x^*) \).

Despite matching solutions, solving (P*) can require much less computation than solving (P). This is especially true when many \( \phi_i^* \) are linear subfunctions. In this case, the linear subfunctions collapse into a single linear term, making \( f^* \) simpler to minimize than \( f \). We next provide some examples to illustrate this idea.

### 3.2 Piecewise linear structure in machine learning

We now describe several instances of (P) that are importance to machine learning.

#### 3.2.1 Inactive constraints in constrained optimization

We first consider constrained optimization (for which (PMN) is a special case):

\[
\min_{x \in \mathbb{R}^n} \psi(x) \\
\text{s.t.} \quad \sigma_i(x) \leq 0 \quad i = 1, \ldots, m.
\]

(\text{PC})

If each \( \sigma_i \) is convex and \( \psi \) is 1-strongly convex, this problem can be transformed into an instance of (P) using implicit constraints. For each \( i \in [m] \), define \( \phi_i \) as

\[
\phi_i(x) = \begin{cases} 
0 & \text{if } \sigma_i(x) \leq 0, \\
+\infty & \text{otherwise}.
\end{cases}
\]

There are two subdomains for each \( \phi_i \)—the constraint's feasible and infeasible regions. Since \( x^* \) must satisfy all constraints, note \( \mathcal{X}^{(i)}_i \) represents constraint \( i \)'s feasible region.

Let us consider Proposition 3.1 in the context of (PC). Define \( \mathcal{W}^* = \{ i \mid \sigma_i(x^*) = 0 \} \), the set of constraints that are active at (PC)'s solution. The condition \( x^* \in \text{bd}(\mathcal{X}^{(i)}_i) \) implies \( \sigma_i(x^*) = 0 \) and \( i \in \mathcal{W}^* \). To define each \( \phi_i^* \), we let \( \phi_i^* = \phi_i \) for all \( i \in \mathcal{W}^* \) and \( \phi_i^* = 0 \) otherwise. Applying Proposition 3.1 we see that \( x^* \) also solves

\[
\min_{x \in \mathbb{R}^n} \psi(x) \\
\text{s.t.} \quad \sigma_i(x) \leq 0 \quad \text{for } i \in \mathcal{W}^*.
\]

(\text{PC}*)

That is, we have reduced (PC) to a problem with only \( |\mathcal{W}^*| \) constraints. Since often \( |\mathcal{W}^*| \ll m \), solving (PC*) can be significantly simpler than solving the original problem.
Table 1: Smooth loss examples. The dual of \( \ell_1 \)-regularized smooth loss minimization is a strongly convex constrained problem. Each feature corresponds to a dual constraint. We can use working sets to make convergence times less dependent on the number of features. In the table, \( L_j \) is the loss for training example \( j \), \( (a_j, b_j) \) is the \( j \)th example, and \( s \) is a design parameter.

### 3.2.2 Zero-valued weights in sparse optimization

Optimization with sparsity-inducing penalties is popular in machine learning—see Bach et al. (2012) for a survey. Here we consider learning \( \ell_1 \)-regularized linear models.

Let \( (a_j, b_j) \) be a collection of \( n \) training examples where \( a_j \in \mathbb{R}^m \) is a feature vector and \( b_j \in \mathcal{B} \) is a corresponding label. Typically \( \mathcal{B} = \{ -1, +1 \} \) for classification problems, while \( \mathcal{B} = \mathbb{R} \) for regression. We can fit parameters of a linear model to this data by solving

\[
\text{minimize}_{\omega \in \mathbb{R}^m} \sum_{j=1}^{n} L_j((a_j, \omega)) + \lambda \| \omega \|_1. \tag{PL1}
\]

Above \( \lambda > 0 \) is a tuning parameter, and \( L_j \) is a loss function (parameterized by \( b_j \)). When \( \lambda \) is sufficiently large, a solution \( \omega^* \) is sparse, meaning most entries of \( \omega^* \) equal 0.

(PL1) is not directly an instance of (P), since this problem is not 1-strongly convex in general. Assuming each \( L_j \) is 1-smooth, however, we can transform (PL1) into an instance of (PC) by considering the problem’s dual (Borwein and Zhu 2005, Chapter 4):

\[
\text{minimize}_{x \in \mathbb{R}^n} \sum_{j=1}^{n} L_j^*(x_j) \quad \text{s.t.} \quad |\langle A_i, x \rangle| \leq \lambda \quad i = 1, \ldots, m. \tag{PL1D}
\]

By solving (PL1), we can efficiently recover (PL1D)’s solution and vice versa. In the dual problem, \( A_i \in \mathbb{R}^n \) refers to the \( i \)th column (feature) of the \( n \times m \) design matrix \([a_1, \ldots, a_n]^T\). \( L_j^* \) denotes the convex conjugate of \( L_j \). Since each \( L_j \) is 1-smooth, the \( L_j^* \) terms are 1-strongly convex (Rockafellar and Wets, 1997 Chapter 12). We include several examples of smooth loss functions and their convex conjugates in Table 1.

This dual transformation allows BlitzWS to exploit sparsity to solve (PL1) efficiently. Due to the correspondence between features and dual constraints, zero entries in \( \omega^* \) correspond to constraints that are unnecessary for computing (PL1D)’s solution. That is, if we define \( W^* = \{ i \mid \omega_i^* \neq 0 \} \), then we can also compute \( x^* \) by minimizing the dual objective subject only to constraints in \( W^* \). Since \( \omega^* \) is sparse, solving the problem with only \( |W^*| \) constraints typically requires much less computation than solving (PL1D) directly.
Table 2: Losses with piecewise linear subfunctions. For \( \ell_2 \)-regularized learning, we can leverage piecewise losses to reduce the training time’s dependence on the number of observations. Above, \( L_i \) is the loss for example \( i \), \((a_i, b_i)\) is the \( i \)th training example, and \( s \) is a design parameter.

3.2.3 Training examples in support vector machines

Our final example considers support vector machines and, more generally, \( \ell_2 \)-regularized loss minimization problems with piecewise loss functions. Given training examples \((a_i, b_i)\) and a tuning parameter \( C > 0 \), we can learn a linear model by solving

\[
\text{minimize}_{x \in \mathbb{R}^n} {\frac{1}{2}} \|x\|^2 + C \sum_{i=1}^{m} L_i(x). \tag{PL2}
\]

Each \( L_i \) is a loss function (parameterized by \( a_i \) and \( b_i \)). Often each \( L_i \) has piecewise linear components; we include some examples of such losses in Table 2.

When each \( L_i \) has piecewise linear components, we can solve the problem quickly by exploiting piecewise structure. Consider (PL2) instantiated with hinge loss. Given knowledge of \( \pi_i(x^*) \) and whether \( x^* \in \text{bd}(X_i(x^*)) \) for each \( i \) (for this problem, this implies knowledge of the sign of \( 1 - b_i \langle a_i, x^* \rangle \)), we can define

\[
L_i^*(x) = \begin{cases} 
0 & \text{if } 1 - b_i \langle a_i, x^* \rangle < 0, \\
1 - b_i \langle a_i, x \rangle & \text{if } 1 - b_i \langle a_i, x^* \rangle > 0, \\
L_i(x) & \text{if } 1 - b_i \langle a_i, x^* \rangle = 0.
\end{cases}
\]

Applying Proposition 3.1, we can compute \( x^* \) by solving

\[
\text{minimize}_{x \in \mathbb{R}^n} {\frac{1}{2}} \|x\|^2 + C \sum_{i=1}^{n} L_i^*(x). \tag{PSVM*}
\]

If we define \( W^* = \{i \mid 1 - b_i \langle a_i, x^* \rangle \neq 0\} \), the benefit becomes clear. (PSVM*) has the same solution as

\[
\text{minimize}_{x \in \mathbb{R}^n} {\frac{1}{2}} \|x\|^2 + \langle a^*, x \rangle + C \sum_{i \in W^*} L_i(x), \tag{PSVM**}
\]

where the vector \( a^* \) is easily computable. We have reduced (PL2) from a problem with \( m \) training examples to a problem with \( |W^*| \) training examples. Since often \( |W^*| \ll m \), we can solve (PSVM*) much more efficiently.
4. BlitzWS working set algorithm

BlitzWS extends BlitzMN to problems of the form (P). In this section, we adapt main concepts from §2 to this piecewise formulation. We also address some practical considerations for BlitzWS.

4.1 Working set algorithms for piecewise objectives

To generalize working set algorithms to our piecewise problem formulation, we generalize the form of each subproblem. During each iteration $t$, BlitzWS minimizes a “relaxed objective,”

$$f_t(x) = \psi(x) + \sum_{i=1}^{m} \phi_{i,t}(x),$$

where for $i \in [m]$, we have $\phi_{i,t} \in \{\phi_i\} \cup \{\phi_i^{(1)}, \phi_i^{(2)}, \ldots, \phi_i^{(p_i)}\}$. That is, BlitzWS defines each $\phi_{i,t}$ as either (i) the original piecewise function, $\phi_i$, or (ii) one of $\phi_i$’s simpler subfunctions. The “working set” is the set of indices corresponding to piecewise functions in $f_t$.

4.2 Line search and $y_t$ in BlitzWS

Like BlitzMN, BlitzWS maintains two iterates, $x_t$ and $y_t$. In BlitzMN, $y_t$ is the feasible point on the segment $[y_{t-1}, x_t]$ with smallest objective value.

Extending this concept to the piecewise problem, BlitzWS initializes $y_0$ such that $f(y_0)$ is finite. After computing $x_t$, BlitzWS updates $y_t$ using the rule

$$y_t \leftarrow \text{argmin}\{f(x) \mid x \in [y_{t-1}, x_t]\}.$$

BlitzWS can compute this update using the bisection method; we elaborate on this in §4.8.3.

4.3 Equivalence regions in BlitzWS

BlitzWS’s choice of each $\phi_{i,t}$ is a choice of where in $\phi_i$’s domain the algorithm ensures that $\phi_{i,t}$ and $\phi_i$ are equivalent. If $\phi_{i,t} = \phi_i^{(1)}$, then $\phi_{i,t}(x) = \phi_i(x)$ is only guaranteed for $x \in \mathcal{X}_i^{(1)}$.

Like BlitzMN, BlitzWS uses equivalence regions to ensure quantifiable progress during each iteration. Given a progress parameter $\xi_t \in (0, 1]$, BlitzWS defines $\mathcal{S}_{\xi_t}^{\text{cap}}$ exactly as in §2.5.

$$\mathcal{S}_{\xi_t}^{\text{cap}} = \text{conv}(\mathcal{B}_1^{\text{cap}} \cup \mathcal{B}_2^{\text{cap}}).$$

Recall that $\mathcal{B}_1^{\text{cap}}$ and $\mathcal{B}_2^{\text{cap}}$ are balls with centers $c_1^{\text{cap}}$ and $c_2^{\text{cap}}$, respectively.

BlitzWS selects each $\phi_{i,t}$ so that for all $x \in \mathcal{S}_{\xi_t}^{\text{cap}}$, we have $f_t(x) = f(x)$. To establish this equivalence region, BlitzWS defines each $\phi_{i,t}$ so that $\phi_{i,t}(x) = \phi_i(x)$ for all $x \in \mathcal{S}_{\xi_t}^{\text{cap}}, i \in [m]$. This property results from BlitzWS’s first sufficient condition for including a particular $i \in [m]$ in the working set (there are three conditions total). For each $i \in [m], k = \pi_i(c_i^{\text{cap}})$, BlitzWS includes $i$ in the working set—that is, BlitzWS defines $\phi_{i,t} = \phi_i$—if the following:

(C1) There exists an $x \in \mathcal{S}_{\xi_t}^{\text{cap}}$ such that $x \notin \mathcal{X}_i^{(k)}$, meaning equivalence between $\phi_i$ and $\phi_i^{(k)}$ within $\mathcal{S}_{\xi_t}^{\text{cap}}$ is not guaranteed.

We will define conditions (C2) and (C3) soon. If any of these sufficient conditions are satisfied, BlitzWS includes $i$ in the working set. Otherwise, BlitzWS defines $\phi_{i,t} = \phi_i^{(k)}$, where $k = \pi_i(c_i^{\text{cap}})$. 

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4.4 Generalizing suboptimality gaps for BlitzWS

Condition (C2) for constructing the working set allows us to generalize the suboptimality gap to our piecewise formulation. Specifically, for each \( i \in [m] \), \( k = \pi_i (c_i^{cap}) \), BlitzWS defines \( \phi_{i,t} = \phi_i \) if:

(C2) There exists an \( x \in \mathbb{R}^n \) such that \( \phi^{(k)}_i (x) > \phi_i (x) \)—that is, \( \phi^{(k)}_i \) does not lower bound \( \phi_i \).

From (C2) and (7), it follows that \( f_i (x) \leq f(x) \) for all \( x \). Importantly, we can use this property to define a suboptimality gap. Since \( x_t \) minimizes \( f_t \), we have \( f_t (x_t) \leq f_t (x^*) \leq f(x^*) \). Thus, given any \( y_t \) such that \( f(y_t) \) is finite, we have the suboptimality gap

\[
\Delta_t = f(y_t) - f_t(x_t) \geq f(y_t) - f(x^*) .
\]

We note (C2) does not generally affect the computational cost of minimizing \( f_t \). The advantage of minimizing \( f_t \) instead of \( f \) results from the fact that linear \( \phi_{i,t} \) functions collapse into a single linear term. In the case that \( \phi^{(k)}_i \) is linear, then \( \phi^{(k)}_i \) lower bounds \( \phi_i \) as a result of convexity (assuming \( x_i^{(k)} \) has non-empty interior).

4.5 Ensuring \( f_t (x_t) \) is nondecreasing with \( t \)

BlitzMN defines each working set to ensure \( \psi_{MN}(x_t) \) is nondecreasing with \( t \). Generalizing this idea, BlitzWS ensures \( f_t (x_t) \geq f_{t-1} (x_{t-1}) \) for all \( t \). This property follows from the third condition for including \( i \) in the working set. For each \( i \in [m] \), \( k = \pi_i (c_i^{cap}) \), BlitzWS defines \( \phi_{i,t} = \phi_i \) if:

(C3) \( \phi^{(k)}_i \) does not upper bound \( \phi_{i,t-1} \) in a neighborhood of \( x_{t-1} \)—there exists a \( g_i \in \partial\phi_{i,t-1} (x_{t-1}) \) and \( x \in \mathbb{R}^n \) such that

\[
\phi^{(k)}_i (x) < \phi_{i,t-1} (x_{t-1}) + \langle x - x_{t-1}, g_i \rangle .
\]

Because of (C3), we have \( f_t (x) \geq f_{t-1} (x_{t-1}) + \langle x - x_{t-1}, g_{t-1} \rangle \) for all \( x \) and \( g_{t-1} \in \partial f_{t-1} (x_{t-1}) \). Since \( x_{t-1} \) minimizes \( f_{t-1} \), we have \( 0 \in \partial f_{t-1} (x_{t-1}) \). It follows that \( f_t (x) \geq f_{t-1} (x_{t-1}) \) for all \( x \), which implies that \( f_t (x_t) \geq f_{t-1} (x_{t-1}) \).

4.6 BlitzWS definition and convergence guarantee

We define BlitzWS in Algorithm 2. BlitzWS initializes \( y_0 \) such that \( f(y_0) \) is finite, while \( x_0 \) is the minimizer of a function \( f_0 \). BlitzWS defines \( f_0 \) so that \( x_0 \) is easy to compute. For example, for (PC), we can define \( \phi_{i,0} (x) = 0 \) for all \( i \), making \( x_0 \) the unconstrained minimizer of \( \psi \).

During each iteration \( t \), BlitzWS chooses a progress coefficient \( \xi_t \), which parameterizes the equivalence region \( S_k^{cap} \). BlitzWS defines \( f_t \) according to §4.3, §4.4, and §4.5. Given \( f_t \), BlitzWS computes \( x_t \leftarrow \arg\min f_t (x) \). At the end of iteration \( t \), the algorithm updates \( y_t \) via line search.

Together, conditions (C1), (C2), and (C3) guarantee quantified progress toward convergence during iteration \( t \). In particular, we have the following convergence result for BlitzWS:

**Theorem 4.1** (Convergence bound for BlitzWS). For any iteration \( T \) of Algorithm 2, define the suboptimality gap \( \Delta_T = f(y_T) - f_T(x_T) \). For all \( T > 0 \), we have

\[
\Delta_T \leq \Delta_0 \prod_{t=1}^{T} (1 - \xi_t) .
\]
Algorithm 2 BlitzWS for solving (P)

**input** initial \( y_0 \) such that \( f(y_0) < +\infty \), linear functions \( (\phi_{i,0})_{i=1}^m \) for which \( \phi_{i,0}(x) \leq \phi_i(x) \) \( \forall x \), and method for choosing \( \xi_t \)

\[ x_0 \leftarrow \arg\min f_0(x) := \psi(x) + \sum_{i=1}^m \phi_{i,0}(x) \]

**for** \( t = 1, \ldots, T \) **until** \( f_T(x_T) = f(y_T) \)**

**# Form subproblem:**

Choose progress coefficient \( \xi_t \in (0, 1] \)

\[ S_{\xi}^{\text{cap}} \leftarrow \text{compute}_\text{capsule}_\text{region}(\xi_t, x_{t-1}, y_{t-1}) \quad \# \text{see §2.5} \]

**for** \( i = 1, \ldots, m \) **do**

**if** (C1) or (C2) or (C3) **then**

# Include \( i \) in working set:

\[ \phi_{i,t} \leftarrow \phi_i \]

**else**

\[ \phi_{i,t} \leftarrow \phi_i^{(k)} \] where \( k \) is the subdomain for which \( S_{\xi}^{\text{cap}} \subseteq X_i^{(k)} \)

**# Solve subproblem:**

\[ x_t \leftarrow \arg\min f_t(x) := \psi(x) + \sum_{i=1}^m \phi_{i,t}(x) \]

**# Perform line search update:**

\[ y_t \leftarrow \arg\min_{x \in [y_{t-1}, x_t]} f(x) \]

**return** \( y_T \)

### 4.7 Accommodating approximate subproblem solutions

BlitzWS can minimize \( f_t \) using any subproblem solver. Since solvers are usually iterative, it is important to only compute \( x_t \) approximately. Computing \( x_t \) to high precision would require time that BlitzWS could instead use to solve subproblem \( t+1 \).

To accommodate approximate solutions, we make several adjustments to BlitzWS. Most significantly, the subproblem solver returns three objects: (i) an approximate subproblem solution, \( z_t \), where \( f_t(z_t) \) is finite, (ii) a function \( f_t^{\text{LB}} \) that lower bounds \( f_t \), and (iii) \( x_t = \arg\min f_t^{\text{LB}}(x) \), which is a “dual” approximate minimizer of \( f_t \). We assume \( f_t^{\text{LB}} \) takes the form

\[
f_t^{\text{LB}}(x) = \left[ \psi(z_t) + \langle g_{\psi}^{\text{LB}}, x - z_t \rangle + \frac{1}{2} \|x - z_t\|^2 \right] + \sum_{i=1}^m \left[ \phi_{i,t}(z_t) + \langle g_i^{\text{LB}}, x - z_t \rangle \right],
\]

where \( g_{\psi}^{\text{LB}} \in \partial\psi(z_t) \) and \( g_i^{\text{LB}} \in \partial\phi_i(z_t) \) for each \( i \). Since \( \psi \) is 1-strongly convex and each \( \phi_i \) is convex, we have \( f_t^{\text{LB}}(x) \leq f_t(x) \) for all \( x \). Also, because \( f_t^{\text{LB}} \) is a simple quadratic function, it is straightforward to compute \( x_t \).

Together, \( z_t, x_t, \) and \( f_t^{\text{LB}} \) allow us to quantify the precision of the approximate subproblem solutions in terms of suboptimality gap. Since \( x_t \) minimizes \( f_t^{\text{LB}} \), it follows that

\[
f_t(z_t) - f_t^{\text{LB}}(x_t) \geq f_t(z_t) - \min_x f_t(x),
\]
We note that when subproblem \( t \) is solved exactly, we can define \( f_t^{\text{LB}} \) such that this “subproblem suboptimality gap” is zero. To do so, we define \( g_t^{\text{LB}} \) and each \( g_i^{\text{LB}} \) such that \( g_0^{\text{LB}} + \sum_{i=1}^{m} g_i^{\text{LB}} = 0 \). In this case, \( z_t \) also minimizes \( f_t^{\text{LB}} \), which implies that \( x_t = z_t \) and \( f_t(z_t) - f_t^{\text{LB}}(x_t) = 0 \).

To bound the effect of approximate subproblem solutions, BlitzWS chooses a subproblem termination threshold \( \epsilon_t \in [0, 1) \). We require that \( z_t, x_t, \) and \( f_t^{\text{LB}} \) satisfy two conditions:

\[
\begin{align*}
&f_t(z_t) - f_t^{\text{LB}}(x_t) \leq \epsilon_t \Delta_{t-1}, \\
\text{and} &f_t^{\text{LB}}(x_t) - f_{t-1}^{\text{LB}}(x_{t-1}) \geq (1 - \epsilon_t)^{1/2} \| z_t - x_{t-1} \|^2 .
\end{align*}
\]

The first condition bounds the subproblem suboptimality gap. The second condition lower bounds the algorithm’s dual progress. The parameter \( \epsilon_t \) trades off the precision of subproblem \( t \)’s solution with the amount of time used to solve this subproblem—smaller \( \epsilon_t \) values imply more precise subproblem solutions. While not obvious in this context, we note the second condition is always satisfied once the subproblem solution is sufficiently precise. We prove this fact in Appendix I.2.

In addition to the changes already discussed, we make three final modifications to BlitzWS to accommodate approximate subproblem solutions. First, we redefine BlitzWS’s suboptimality gap to ensure \( \Delta_t \geq f(y_t) - f(x^*) \). Specifically, we define \( \Delta_t = f(y_t) - f_t^{\text{LB}}(x_t) \).

The second final change is that instead of searching along the segment \([x_t, y_{t-1}]\), BlitzWS updates \( y_t \) by performing line search along \([z_t, y_{t-1}]\):

\[
y_t \leftarrow \arg\min_{x \in [z_t, y_{t-1}]} f(x) .
\]

The last change to BlitzWS adjusts condition (C3) from §4.5 Specifically, for each \( i \in [m] \) and \( k = \pi_i(e^{\text{cap}}) \), BlitzWS defines \( \phi_{i,t} = \phi_i \) if:

\[
\text{(C3)} \quad \phi_i^{(k)} \text{ does not upper bound } \phi_{i,t-1}^{\text{LB}} \quad \text{for some } x \in \mathbb{R}^n, \text{ we have } \phi_i^{(k)}(x) < \phi_{i,t-1}^{\text{LB}}(x) .
\]

This change guarantees that \( f_t \) upper bounds \( f_{t-1}^{\text{LB}} \). Compared to the original (C3) condition from §4.5, our new (C3) guarantees that \( \phi_{i,t} \) upper bounds \( \phi_{i,t-1} \) in a neighborhood of \( z_t \) as opposed to a neighborhood of \( x_t \).

Taking these changes into account, we have the following convergence result for BlitzWS with approximate subproblem solutions (proven in Appendix F):

**Theorem 4.2** (Convergence bound for BlitzWS with approximate subproblem solutions). Consider BlitzWS with approximate subproblem solutions. For any iteration \( T \), define the suboptimality gap \( \Delta_T = f(y_T) - f_T^{\text{LB}}(x_T) \). For all \( T > 0 \), we have

\[
\Delta_T \leq \Delta_0 \prod_{t=1}^{T} (1 - (1 - \epsilon_t) \xi_t) .
\]

This result clearly describes the effect of approximate subproblem solutions. By solving subproblem \( t \) with tolerance \( \epsilon_t \in [0, 1) \), it is guaranteed that BlitzWS makes \( (1 - \epsilon_t) \xi_t \) progress during iteration \( t \). When \( \epsilon_t = 0 \), we recover our original convergence bound, Theorem 4.1.

The parameters \( \xi_t \) and \( \epsilon_t \) allow BlitzWS to trade off between subproblem size, time spent solving subproblems, and progress toward convergence. We next explore these trade-offs in more detail.
4.8 Bottlenecks of BlitzWS

Each iteration $t$ of BlitzWS has three stages: select subproblem $t$, solve subproblem $t$, and update $y_t$. Here we discuss the amount of computation that each stage requires.

4.8.1 Time Required to Form Each Subproblem

The time-consuming step for forming subproblem $t$ is testing condition (C1). This step requires checking if $S_{\xi}^{\text{cap}} \subseteq \mathcal{X}_i^{(k)}$ for all $i \in [m]$.

Recall $S_{\xi}^{\text{cap}} = \text{conv} (B_1^{\text{cap}} \cup B_2^{\text{cap}})$, where $B_1^{\text{cap}}$ and $B_2^{\text{cap}}$ are balls with centers $c_1^{\text{cap}}$ and $c_2^{\text{cap}}$ and radius $r^{\text{cap}}$. If $\mathcal{X}_i^{(k)}$ is convex, then $S_{\xi}^{\text{cap}} \subseteq \mathcal{X}_i^{(k)}$ iff $B_1^{\text{cap}} \subseteq B_2^{\text{cap}} \subseteq \mathcal{X}_i^{(k)}$. Unfortunately, when $\mathcal{X}_i^{(k)}$ is convex, testing whether $B_1^{\text{cap}} \subseteq \mathcal{X}_i^{(k)}$ is not convex in general. Even so, in the common scenarios that $\mathcal{X}_i^{(k)}$ is a half-space or ball, we can check if $B_1^{\text{cap}} \subseteq \mathcal{X}_i^{(k)}$ in $O(n)$ time. In other cases, we can often approximately check this condition efficiently.

Let us first consider the case that $\mathcal{X}_i^{(k)}$ is a half-space. For some $\mathbf{a}_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$, we can write $\mathcal{X}_i^{(k)} = \{ \mathbf{x} \mid \langle \mathbf{a}_i, \mathbf{x} \rangle \leq b_i \}$. Then $B_1^{\text{cap}} \subseteq \mathcal{X}_i^{(k)}$ iff

$$\langle \mathbf{a}_i, \mathbf{c}_1^{\text{cap}} \rangle - b_i < \| \mathbf{a}_i \| r^{\text{cap}}.$$  

Alternatively, suppose that $\mathcal{X}_i^{(k)}$ is a ball: $\mathcal{X}_i^{(k)} = \{ \mathbf{x} \mid \| \mathbf{x} - \mathbf{a}_i \| \leq b_i \}$. Then $B_1^{\text{cap}} \subseteq \mathcal{X}_i^{(k)}$ iff

$$\| \mathbf{a}_i - \mathbf{c}_1^{\text{cap}} \| + r^{\text{cap}} < b_i.$$  

When $\mathcal{X}_i^{(k)}$ is neither a half-space nor a ball, one option may be to approximate condition (C1) by defining a ball $\mathcal{X}_i^{(k)}$ such that $\mathcal{X}_i^{(k)} \subseteq \mathcal{X}_i^{(k)}$. If $B_1^{\text{cap}} \subseteq \mathcal{X}_i^{(k)}$, then $B_1^{\text{cap}} \subseteq \mathcal{X}_i^{(k)}$. If $B_1^{\text{cap}} \subseteq \mathcal{X}_i^{(k)}$ and $B_2^{\text{cap}} \subseteq \mathcal{X}_i^{(k)}$, then we must have $S_{\xi}^{\text{cap}} \subseteq \mathcal{X}_i^{(k)}$. By approximating condition (C1) in this manner, $f_t$ and $f$ remain equivalent within $S_{\xi}$, meaning Theorem 4.1 and Theorem 4.2 still apply.

4.8.2 Time Required to Solve Subproblem $t$

The time required to solve subproblem $t$ depends mainly on three factors: the progress coefficient, the subproblem termination threshold, and the subproblem solver. Larger values of $\xi_t$ result in a larger equivalence region, increasing the size of the working set due to (C1).

BlitzWS can use any solver to minimize $f_t$, but to be effective, the time required to solve each subproblem must increase with the working set size. This is usually the case but not always. For example, in the distributed setting, communication bottlenecks can affect convergence times greatly. Depending on the algorithm and implementation, some distributed solvers require $O(n)$ communication per iteration, while the communication for other solvers may scale with the working set size. The $O(n)$ case is not desirable for BlitzWS, since the amount of time needed to solve each subproblem depends less directly on the working set size.

4.8.3 Time Required to Update $y_t$

Updating $y_t$ requires minimizing $f$ along the segment $[x_t, y_{t-1}]$. BlitzWS can perform this update using the bisection method, which requires evaluating $f$ a logarithmic number of times. In this case, it is not necessary to compute $y_t$ exactly. Our analysis requires only that $f(y_t) \leq f(y'_t)$, where $y'_t$ is the point on the segment $[x_t, y_{t-1}]$ that is closest to $x_t$ while remaining in the closure of $S_{\xi}$.  

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In many cases it is also straightforward to compute $y_t$ exactly. For constrained problems like (PMN), $y_t$ is the extreme feasible point on the segment $[y_{t-1}, x_t]$. If the constraints are linear or quadratic, BlitzWS can compute $y_t$ in closed form.

### 4.9 Choosing algorithmic parameters in BlitzWS

Each BlitzWS iteration uses a progress parameter, $\xi_t \in (0, 1]$, and termination threshold, $\epsilon_t \in [0, 1)$. We could assign $\xi_t$ and $\epsilon_t$ constant values for all $t$. As we will see in §6, however, values of $\xi_t$ and $\epsilon_t$ that work well for one problem may result in slow convergence times for other problems. For this reason, it is beneficial to choose these parameters in an adaptive manner.

To adapt the parameter choices to each problem, we model as functions of $\xi_t$ and $\epsilon_t$ both (i) the time required to complete iteration $t$ and (ii) BlitzWS’s progress during this iteration. Using these models, BlitzWS selects $\xi_t$ and $\epsilon_t$ by approximately optimizing the trade-offs between subproblem size, iteration duration, and convergence progress.

To model the time required for BlitzWS to complete iteration $t$, we define the function

$$
\hat{T}_t(\xi, \epsilon) = \hat{C}_{t,\text{setup}} + \hat{C}_{t,\text{solve}} \cdot \text{ProblemSize}(\xi) \epsilon^{-1}.
$$

Above, ProblemSize($\xi$) measures the size of subproblem $t$ as a function of $\xi$. For (PMN), we define

$$
\text{ProblemSize}(\xi) = \sum_{i \in W_t(\xi)} \text{NNZ}(a_i),
$$

where $W_t(\xi)$ denotes the working set when $\xi_t = \xi$, and $\text{NNZ}(a_i) = \|a_i\|_0$.

BlitzWS adapts the scalars $\hat{C}_{t,\text{setup}}$ and $\hat{C}_{t,\text{solve}}$ from iteration to iteration. During each iteration $t$, BlitzWS measures the time required to solve subproblem $t$, denoted $T_{t,\text{solve}}$, as well as the time taken for all other steps of iteration $t$, denoted $T_{t,\text{setup}}$. Upon completion of iteration $t$, BlitzWS estimates $\hat{C}_{t,\text{setup}}$ and $\hat{C}_{t,\text{solve}}$ by solving for the appropriate value in the model:

$$
\hat{C}_{t,\text{setup}} = T_{t,\text{setup}}, \quad \text{and} \quad \hat{C}_{t,\text{solve}} = \frac{T_{t,\text{solve}}}{\text{ProblemSize}(\xi_t)}.
$$

When selecting subproblem $t$, BlitzWS defines $\hat{C}_{t,\text{setup}}$ and $\hat{C}_{t,\text{solve}}$ by taking the median of the five most recent estimates for these parameters. For example,

$$
\hat{C}_{t,\text{setup}} = \text{median}(\hat{C}_{t-1,\text{setup}}, \hat{C}_{t-2,\text{setup}}, \ldots, \hat{C}_{t-5,\text{setup}}).
$$

If $t \leq 5$, then the algorithm takes the median of only the past $t - 1$ parameter estimates. Since this is not possible when $t = 1$ ($\hat{C}_{0,\text{setup}}$ does not exist), BlitzWS does not model the time required for iteration 1. Instead, during iteration 1, we define $\xi_1$ as the smallest value in $(0, 1]$ such that $f_t = f$, but we solve the subproblem crudely by terminating the subproblem solver after one iteration.

In addition to modeling the time for iteration $t$, BlitzWS applies Theorem 4.2 to model the suboptimality gap upon completion of this iteration:

$$
\Delta_t(\xi, \epsilon) = \max \{(1 - (1 - \epsilon)\xi C_t(\text{progress})), \epsilon \Delta_{t-1}\}.
$$

The parameter $C_t(\text{progress}) \geq 1$ accounts for looseness in the theorem’s bound, which guarantees that $\Delta_t \leq (1 - (1 - \epsilon_t)\xi_t)\Delta_{t-1}$. The $\max \{\cdot\}$ in (9) results from the fact that we should not expect that...
\( \Delta_t \leq \epsilon_t \Delta_{t-1} \), regardless of looseness in our bound. This is because as a termination condition for subproblem \( t \), we only assume that the subproblem suboptimality gap does not exceed \( \epsilon_t \Delta_{t-1} \).

BlitzWS estimates \( C_t^{\text{progress}} \) in the same way that the algorithm estimates \( C_t^{\text{solve}} \) and \( C_t^{\text{setup}} \)—by solving for the appropriate parameter after iteration \( t \) and taking the median over past estimates:

\[
\hat{C}_t^{\text{progress}} = \frac{1}{(1 - \epsilon_t) \xi_t} \left[ 1 - \frac{\Delta_t}{\Delta_{t-1}} \right], \quad \hat{\epsilon}_t = \frac{f_t(x_t) - f_t^{\text{LB}}(x_t)}{\Delta_{t-1}},
\]

and

\[
C_t^{\text{progress}} = \max \left\{ 1, \text{median}(\hat{C}_{t-1}^{\text{progress}}, \hat{C}_{t-2}^{\text{progress}}) \right\}.
\]

Here the \( \max \{1, \cdot \} \) guarantees that \( C_t^{\text{progress}} \geq 1 \)—this ensures the value of \( \hat{\Delta}(\xi, \epsilon) \) is at most the bound predicted by Theorem 4.2. In this case, we take the median of only the past two estimates for \( C_t^{\text{progress}} \), allowing \( \xi_t \) to change significantly between iterations if necessary (unlike \( C_t^{\text{setup}} \), for example, it is unclear whether we should expect \( C_t^{\text{progress}} \) to be approximately constant for all \( t \)).

Having modeled both the time for iteration \( t \) and the progress during iteration \( t \), BlitzWS combines \( T_t \) and \( \Delta_t \) to approximately optimize \( \xi_t \) and \( \epsilon_t \). Specifically, BlitzWS defines

\[
\xi_t, \epsilon_t = \arg \max_{\xi, \epsilon} -\log(\hat{\Delta}_t(\xi, \epsilon)/\Delta_{t-1})/T_t(\xi, \epsilon).
\]

With (10), BlitzWS values time as if the algorithm converges linearly. That is, a subproblem that requires an additional second to solve should result in a \( \Delta_t \) that is smaller by a multiplicative factor.

BlitzWS solves (10) approximately with grid search, considering 125 candidates for \( \xi_t \) and 10 candidates for \( \epsilon_t \). The candidates for \( \xi_t \) span between \( 10^{-6} \) and 1, while the candidates for \( \epsilon_t \) span between 0.01 and 0.7. Later in §6, we examine some of these parameter values empirically.

We also enforce a time limit when solving each subproblem. In addition to the termination conditions described in §4.7, we also terminate subproblem \( t \) if the threshold \( \epsilon_t \) is not reached before a specified amount of time elapses. We define the time limit as \( C_t^{\text{solve}} \text{ProblemSize}(\xi_t) \epsilon_t^{-1} \), which is the estimated time for solving the subproblem in (8).

### 4.10 Relation to prior algorithms

Many prior algorithms also exploit piecewise structure in convex problems. The classic simplex algorithm (Dantzig, 1965), for example, exploits redundant constraints in linear programs.

In the late 1990s and early 2000s, working set algorithms became important to machine learning for training support vector machines. Using working sets, Osuna et al. (1997) prioritized computation on training examples with suboptimal dual value. Joachims (1999) improved the choice of working sets based on a first-order “steepest feasible direction” strategy—an idea that Zoutendijk (1970) originally proposed for constrained optimization. To further reduce computation, Joachims developed a “shrinking” heuristic, which freezes values of specific dual variables that satisfy a condition during several consecutive iterations.

Later works refined and extended these working set ideas. Fan et al. (2005) as well as Glasmachers and Igel (2006) used second-order information to improve working sets for kernelized SVMs. Unlike BlitzWS, these approaches apply only to working sets of size two, which is limiting but nevertheless practical for kernelized SVMs. Zanghirati and Zanni (2003) and Zanni et al. (2006) considered larger working sets and parallel algorithms. Tschantaridis et al. (2005) extended working set ideas to structured prediction problems. Hsieh et al. (2008) combined shrinking with
dual coordinate ascent to train linear SVMs. This resulted in a very fast algorithm, and the popular LibLINEAR library (Fan et al., 2008) uses this approach to train linear SVMs today.

Similar coordinate descent strategies work well for training ℓ₁-regularized models. Friedman et al. (2010) proposed a fast algorithm that combines working sets with coordinate descent and a proximal Newton strategy. Similarly, Yuan et al. (2010) found that combining CD with shrinking heuristics leads to a fast algorithm for sparse logistic regression. Today LibLINEAR uses a refined version of this approach to train such models (Yuan et al., 2012), which applies working sets and shrinking in a two-layer prioritization scheme.

These are just two of many algorithms that incorporate working sets to speed up sparse optimization. For lasso-type problems, many additional studies combine working set (Scheinberg and Tang, 2016; Massias et al., 2017) or active set (Wen et al., 2012; Solntsev et al., 2015; Keskar et al.) strategies with standard algorithms. Researchers have also applied working sets to many other sparse problems—see e.g. (Lee et al., 2007; Bach, 2008; Kim and Park, 2008; Roth and Fischer, 2008; Obozinski et al., 2009; Friedman et al., 2010; Schmidt and Murphy, 2010).

More generally, prioritizing components of the objective continues to be an important idea for scaling model training. Many works consider importance sampling to speed up stochastic optimization (Needell et al., 2014; Zhao and Zhang, 2015; Csiba et al., 2015; Vainsencher et al., 2015; Perekrestenko et al., 2017a). Harikandeh et al. (2015), Stich et al. (2017b), and Johnson and Guestrin (2017) use alternative strategies to improve first-order algorithms.

To our knowledge, BlitzWS is the first working set algorithm that selects each working set in order to guarantee an arbitrarily large amount of progress during each iteration. Prior algorithms choose working sets in intuitive ways, but there is little understanding of the resulting progress. In contrast, our theory for BlitzWS provides justification for the algorithm, avoids possible pathological scenarios, and inspires new ideas, such as our approach to tuning algorithmic parameters.

We note that because BlitzWS can use any subproblem solver, BlitzWS could also use importance sampling, shrinking, or another strategy when solving each subproblem.

5. BlitzScreen safe screening test

In this section, we introduce BlitzScreen, a safe screening test that relates closely to BlitzWS. Like BlitzWS, BlitzScreen involves minimizing a relaxed objective instead of the original objective, f. Unlike BlitzWS, BlitzScreen guarantees that the relaxed objective and f have the same minimizer.

5.1 BlitzScreen definition

BlitzScreen requires three ingredients:

1. An approximate minimizer of f, denoted y₀, for which f(y₀) is finite.
2. A 1-strongly convex function, denoted f₀, that satisfies f₀(x) ≤ f(x) for all x.
3. The minimizer of f₀, denoted x₀.

One way to construct such a f₀ uses a subgradient g₀ ∈ ∂f(y₀). Given such a point, we can define

\[ f₀(x) = f(y₀) + ⟨g₀, x − y₀⟩ + \frac{1}{2} ∥x − y₀∥². \]

We can easily compute x₀ = argmin f₀(x), and since f is 1-strongly convex, f₀ lower bounds f.

Regardless of how we define f₀, we have the following screening result.
Letting $S_1 = \{ x \mid \| x - \frac{1}{2}(x_0 + y_0) \| < \sqrt{\Delta_0 - \frac{1}{4} \| x_0 - y_0 \|^2} \}$. For any $i \in [m]$, define $k$ such that the subdomain $X_i^{(k)}$ contains $\frac{1}{2}(x_0 + y_0)$. Then if $S_1 \subseteq X_i^{(k)}$, we can safely replace $\phi_i$ with $\phi_i^{(k)}$ in $f$. That is, for all $i \in [m]$, if we let

$$\phi_{i,S} = \begin{cases} 
\phi_i^{(k)} & \text{if } S_1 \subseteq X_i^{(k)}, \\
\phi_i & \text{otherwise},
\end{cases}$$

then the “screened objective” $f_S(x) := \psi(x) + \sum_{i=1}^{m} \phi_{i,S}(x)$ has the same minimizer as $f$.

We prove Theorem 5.1 in Appendix G. The proof relies on the equivalence of $f_S$ and $f$ within $S_1$. As long as $f_S(x) = f(x)$ for all $x \in S_1$, these objectives have the same minimizer. Note the safe region size greatly depends on the approximate solutions. When $\Delta_0$ is large, $S_1$ is large and $\phi_{i,S} = \phi_i$ for many $i$. If $\Delta_0$ is small, minimizing $f_S$ can be significantly simpler than minimizing $f$.

Applying BlitzScreen requires checking whether $S_1 \subseteq X_i^{(k)}$ for each $i$. This condition is closely related to (C1) in BlitzWS, and our remarks in §4.8.1 about testing (C1) also apply to screening. In many scenarios, we can evaluate whether $S_1 \subseteq X_i^{(k)}$ in $O(n)$ time.

### 5.2 Example: BlitzScreen for $\ell_1$-regularized learning

As an example, we apply BlitzScreen to $\ell_1$-regularized loss minimization:

$$\min_{\omega \in \mathbb{R}^n} g_{L1}(\omega) := \sum_{j=1}^{n} L_j((a_j, \omega)) + \lambda \| \omega \|_1.$$  \hspace{1cm} (PL1)

If each $L_j$ is 1-smooth, we can transform the problem into its 1-strongly convex dual:

$$\min_{x \in \mathbb{R}^n} f_{L1D}(x) := \sum_{j=1}^{n} L_j^*(x_j) + \sum_{i=1}^{m} \phi_i(x).$$  \hspace{1cm} (PL1D)

Above, each implicit constraint defines $\phi_i(x) = 0$ if $|\langle A_i, x \rangle| \leq \lambda$ and $\phi_i(x) = +\infty$ otherwise. Successfully screening a constraint in (PL1D) corresponds to eliminating a feature from (PL1).

To apply BlitzScreen, we assume an approximate solution to (PL1), which we denote by $\omega_0$. Letting $L_j^*(\cdot)$ represent the derivative of $L_j(\cdot)$, we define

$$x_0 = [L_1^*((a_1, \omega_0)), \ldots, L_n^*((a_n, \omega_0))]^T, \quad \text{and} \quad f_0(x) = \frac{1}{2} \| x - x_0 \|^2 - g_{L1}(\omega_0).$$  \hspace{1cm} (11)

Using properties of duality, we show in Appendix I.3 that $f_0$ indeed lower bounds $f_{L1D}$.

BlitzScreen also requires a $y_0 \in \mathbb{R}^n$ such that $f_{L1D}(y_0)$ is finite, meaning $y_0$ must satisfy all constraints. We define $y_0$ by scaling $x_0$ toward 0 until this requirement is satisfied:

$$y_0 = \max_{i \in [m]} \frac{\lambda}{|\langle A_i, x_0 \rangle|} x_0.$$  \hspace{1cm} (12)

We note there exist more advanced strategies for defining $y_0$ (Massias et al., 2018), but we do not consider such ideas in this work. With (12), we have the following screening test for (PL1).
Corollary 5.2 (BlitzScreen for (PL1)). Given any $\omega_0$ that does not solve (PL1), define $f_0$, $x_0$, and $y_0$ as in (11) and (12). Define $\Delta_0 = f_{L1D}(y_0) + g_{L1}(\omega_0)$. For any $i \in [m]$, if

$$\lambda - \frac{1}{2} \langle A_i, \frac{1}{2}(x_0 + y_0) \rangle \geq \|A_i\|_1 \sqrt{\Delta_0} - \frac{1}{2} \|x_0 - y_0\|^2,$$

we can safely remove $\phi_i$ from (PL1D), which implies that $\omega^*_i = 0$ for all $\omega^*$ that solve (PL1).

5.3 Relation to prior screening tests

BlitzScreen improves upon prior screening tests in a few ways, which we summarize as follows:

- **More broadly applicable:** Prior works have derived separate screening tests for different objectives, including sparse regression (El Ghaoui et al., 2012; Xiang and Ramadge, 2012; Tibshirani et al., 2012; Liu et al., 2014; Wang et al., 2015), sparse group lasso (Wang and Ye, 2014), as well as SVM and least absolute deviation problems (Wang et al., 2014). Extending screening to each new objective requires substantial new derivations. In contrast, BlitzScreen applies in a unified way to all instances of our piecewise problem formulation.

  Recently Raj et al. (2016) proposed a general recipe for deriving screening tests for different problems. Unlike this approach, BlitzScreen is an explicit screening test.

- **Adaptive:** Before recently, most safe screening tests relied on knowledge of an exact solution to a related problem. For example, El Ghaoui et al. (2012)’s test requires the solution to an identical problem but with greater regularization. This is disadvantageous for a few reasons, one of which is that screening only applies as a preprocessing step prior to optimization.

  Recent works have proposed adaptive (also called “dynamic”) safe screening tests (Bonnefoy et al., 2014, 2015; Fercoq et al., 2015; Johnson and Guestrin, 2015; Ndiaye et al., 2015; Zimmert et al., 2015; Shibagaki et al., 2016; Raj et al., 2016; Ndiaye et al., 2016, 2017). Adaptive screening tests increasingly simplify the objective as the quality of the approximate solution improves. BlitzScreen is an adaptive screening test.

- **More effective:** Prior to BlitzScreen, the “gap safe sphere” tests proposed by Fercoq et al. (2015) were state-of-the-art adaptive screening tests, as were the closely related tests proposed by Johnson and Guestrin (2015), Zimmert et al. (2015), Shibagaki et al. (2016), Raj et al. (2016) and Ndiaye et al. (2017). Each of these tests applies to a different class of objectives, but they relate to BlitzScreen in the same way. With the exception of Zimmert et al.’s result (which is a special case of BlitzScreen for SVM problems), we can recover these prior screening tests as special cases of BlitzScreen but only by replacing $S_1$ with a larger set. Specifically, if we replace $S_1$ in Theorem 5.1 with the larger ball

$$S_{\text{Gap}} = \left\{ x \mid \|x - y_0\| \leq \sqrt{2\Delta_0} \right\},$$

then the resulting theorem is a more general version of these existing tests. The main difference is that $S_{\text{Gap}}$ is at least a factor $\sqrt{2}$ larger than the radius of $S_1$. As a result, BlitzScreen is more effective at simplifying the objective.

5.4 Relation to BlitzWS

We can view safe screening as a working set algorithm that converges in one iteration. To solve “subproblem 1,” we minimize the screened objective, $f_S$. The subproblem solution also solves (P).
Our next theorem shows that in the case of BlitzScreen and BlitzWS, this relation goes further:

**Theorem 5.3** (Relation between equivalence regions in BlitzScreen and BlitzWS). Given points \(x_0\) and \(y_0\), function \(f_0\), and suboptimality gap \(\Delta_0\) that satisfy the requirements for Theorem 5.1, define the ball \(S_1\) as in Theorem 5.1. In addition, consider the equivalence region \(S_\xi\) from §2 with parameter choices \(\xi_t = 1, x_t-1 = x_0, y_{t-1} = y_0,\) and \(\Delta_{t-1} = \Delta_0\). Then

\[
S_1 = S_\xi.
\]

We prove Theorem 5.3 in Appendix H. When \(\xi_1 = 1\), using BlitzWS is nearly equivalent to applying BlitzScreen. The only minor difference is that BlitzWS may not simplify the objective as much as BlitzScreen, since BlitzScreen does not consider conditions analogous to (C2) and (C3).

Importantly, it is usually not desirable for a working set algorithm to converge in one iteration. Since screening tests only make “safe” simplifications to the objective, screening tests often simplify the problem only a modest amount. In fact, unless a good approximate solution is already known, screening can fail to simplify the objective at all. We find it is usually better to simplify the objective aggressively, correcting erroneous choices later as needed. This is precisely the working set approach. As part of the next section, we support this observation with empirical results.

### 6. Empirical evaluation

This section demonstrates the performance of BlitzWS and BlitzScreen in practice.

#### 6.1 Comparing the scalability of BlitzWS and BlitzScreen

We first consider a group lasso task and a linear SVM task. In each case, we examine how BlitzWS and BlitzScreen affect convergence times as the problem grows larger. To our knowledge, such scalability tests are a novel contribution to research on safe screening.

**6.1.1 Scalability tests for group lasso application**

For our first experiment, we consider the group lasso objective (Yuan and Lin, 2006):

\[
g_{GL}(\omega) := \frac{1}{2} \|A\omega - b\|^2 + \lambda \sum_{i=1}^{m} \|\omega_{G_i}\|.
\]

Here \(G_1, \ldots, G_m\) are disjoint sets of feature indices such that \(\bigcup_{i=1}^{m} G_i = [q]\). Let \(\omega^* \in \mathbb{R}^q\) denote a minimizer of \(g_{GL}\). If \(\lambda > 0\) is sufficiently large, then \(\omega^*_{G_i} = 0\) for many \(i\).

We transform this problem into an instance of (P) by considering the dual problem:

\[
\min_{\omega \in \mathbb{R}^n} \quad f_{GL}(\omega) := \frac{1}{2} \|x + b\|^2 - \frac{1}{2} \|b\|^2
\]

\[
\text{s.t.} \quad \|A_{G_i}^T x\| \leq \lambda \quad i = 1, \ldots, m.
\]

(PGD)

Each feature group corresponds to a constraint in the dual problem. Constraints that do not determine the dual solution correspond to zero-valued groups in the primal solution.

We apply group lasso to perform feature selection for a loan default prediction task. Using data available from Lending Club, we train a boosted decision tree model to predict whether a loan will

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2. **URL:** https://www.kaggle.com/wendykan/lending-club-loan-data.
Following Qin et al. (2013), our implementation computes an optimal update to
During an iteration, BCD updates weights in one group, keeping the remaining weights unchanged.

et al., 2010). Our implementation uses the block coordinate descent approach of Qin et al. (2013).

There exist many algorithms for minimizing \( g_{\text{GL}} \) (Yuan and Lin, 2006; Liu et al., 2009; Kim
et al., 2010). Our implementation uses the block coordinate descent approach of Qin et al. (2013).

During an iteration, BCD updates weights in one group, keeping the remaining weights unchanged.
Following Qin et al. (2013), our implementation computes an optimal update to \( \omega_G \), for roughly the cost of multiplying a dual vector \( x \in \mathbb{R}^n \) by \( A_G \). Each update requires solving a 1-D optimization problem, which we solve with the bisection method.

We implement BCD in C++. Using the same code base, we also implement the following:
- **BlitzWS**: To solve each subproblem, we use BCD.
- **BlitzWS + BlitzScreen**: After solving each BlitzWS subproblem, we apply BlitzScreen.
- **BCD + BlitzScreen**: After every five epochs, we apply BlitzScreen.
- **BCD + gap safe screening**: After every five passes over the groups, we apply gap safe screening (Ndiaye et al., 2015). This implementation is identical to BCD + BlitzScreen except we replace \( S_1 \) in BlitzScreen with the \( S_{\text{Gap}} \) region defined in (13).

BlitzWS and the screening tests require checking if a region \( S = \{ x \mid \| x - c \| \leq r \} \) is a subset of a set \( X_{i} = \{ x \mid \| A_{G_i}^T x \| \leq \lambda \} \). Computing this is nontrivial, so we apply relaxation ideas from §4.8.1 and §5.1. We define a set \( X_{i} = \{ x \mid \| A_{G_i}^T x \| \leq \lambda \} \), and the algorithms test if \( S \subseteq X_{i} \). For each \( i \in [m] \), we let \( L_i = \max_{k \in G_i} \| A_k \| \) and define \( X_{i} = \{ x \mid \| A_{G_i}^T c \| + L_i \| x - c \| \leq \lambda \} \).

We perform data preprocessing to standardize groups in \( A \). For each \( i \), we scale \( A_{G_i} \), so the variances of each column sum to one. Our implementations include an unregularized bias variable. We can easily accommodate this bias term by adding the constraint \( (x, 1) = 0 \) to (PGD).

To test the scalability of BlitzWS and BlitzScreen, we create nine smaller problems from the original group lasso problem. We consider problems with \( m = 990 \), 330, and 110 groups by subsampling groups uniformly without replacement. We consider problems with \( n = 480k \), 160k, and 53.3k training examples by subsampling examples. For each problem, we define \( \lambda \) so that exactly 10% of the groups have nonzero weight in the optimal model.

We evaluate performance using the relative suboptimality metric:

\[
\text{Relative suboptimality} = \frac{g_{\text{GL}}(\omega_T) - g_{\text{GL}}(\omega^*)}{g_{\text{GL}}(\omega^*)}.
\]

Here \( \omega_T \) is the weight vector at time \( T \). We take the optimal solution to be BlitzWS’s solution after optimizing for twice the amount of time as displayed in each figure.

Figure 4 shows the results of these scalability tests. Our first takeaway is that for this problem, the number of training examples does not greatly affect the impact of BlitzWS and screening; as \( n \) increases, the relative performance of each algorithm is remarkably consistent. As the number of groups increases, we observe a different trend. When \( m = 110 \), the screening tests provide some speed-up compared to BCD without screening, particularly once the relative suboptimality reaches \( 6 \times 10^{-4} \). When \( m = 990 \), however, screening provides much less benefit. In this case, despite
being state-of-the-art for safe screening, BlitzScreen has no impact on convergence progress until relative suboptimality reaches $10^{-5}$.

In contrast to safe screening, we find BlitzWS achieves significant speed-ups compared to BCD, regardless of $m$. We also note BlitzScreen provides no benefit when combined with BlitzWS. This is because BlitzWS already effectively prioritizes BCD updates.

### 6.1.2 Scalability Tests for Linear SVM Application

We perform similar scalability tests for a linear SVM problem ([PL2] with hinge loss). We consider a physics prediction task involving the Higgs boson (Adam-Bourdarios et al., 2014). We perform feature engineering using XGBoost (Chen and Guestrin, 2016), which achieves good accuracy for this problem (Chen and He, 2014). Using each leaf in the ensemble as a feature, the data set contains $n = 8010$ features and $m = 10^7$ examples.
There exist many algorithms for solving this problem (Zhang, 2004; Joachims, 2006; Shalev-Shwartz et al., 2007; Teo et al., 2010). We use dual coordinate ascent, which is simple and fast (Hsieh et al., 2008). We implement DCA in C++. Like the group lasso comparisons, we implement BlitzWS using the same code base. For each algorithm, we also implement BlitzScreen.

By subsampling training instances without replacement, we test the scalability of BlitzWS and BlitzScreen using \( m = 10^7 \), \( 3.2 \times 10^5 \), and \( 10^4 \) training instances. For each problem and each algorithm, we plot relative suboptimality vs. time—here we measure relative suboptimality using the dual objective. We choose \( C \) using five-fold cross validation.

We also test the performance of BlitzScreen using a range of \( C \) parameters. We show these results using heatmaps, where the \( y \)-axis indicates epochs completed by DCA, and the \( x \)-axis indicates \( C \). The shading of the heat map depicts the fraction of training instances that BlitzScreen screens successfully at each point in the algorithm.

Figure 5 includes results from these comparisons. Similar to the group lasso case, we see BlitzScreen provides some speed-up when \( m \) is small. But as \( m \) increases, BlitzScreen has no impact on convergence times until the relative suboptimality is much smaller. In contrast, BlitzWS provides improvements that, relative to the DCA solver, do not degrade as \( m \) grows larger.

**6.2 Comparing BlitzWS to LIBLINEAR**

LIBLINEAR is one of the most popular and, to our knowledge, one of the fastest solvers for sparse logistic regression and linear SVM problems. Here we test how BlitzWS compares.
For sparse logistic regression, **LIBLINEAR** uses working sets and shrinking to prioritize computation (Yuan et al., 2012). For linear SVM problems, **LIBLINEAR** applies only shrinking (Joachims, 1999). We can view shrinking as a working set algorithm that initializes the working set with all components (i.e., \( \mathcal{W}_t = [m] \) and \( f_t = f \)); then while solving the subproblem, shrinking progressively removes elements from \( \mathcal{W}_t \) using a heuristic.

### 6.2.1 Sparse Logistic Regression Comparisons

Our **LIBLINEAR** comparisons first consider sparse logistic regression ((PL1) with logistic loss). There are many efficient algorithms for solving this problem (Shalev-Shwartz and Tewari, 2009; Xiao, 2010; Bradley et al., 2011; Defazio et al., 2014; Xiao and Zhang, 2014; Fercoq and Richtárík, 2015). To solve each subproblem, our BlitzWS implementation uses an inexact proximal Newton algorithm (“ProxNewton”). We use coordinate descent to compute each proximal Newton step. **LIBLINEAR** uses the same ProxNewton strategy (Yuan et al., 2012).

We compare BlitzWS with **LIBLINEAR** version 2.11. We compile BlitzWS and **LIBLINEAR** with GCC 4.8.4 and the -O3 optimization flag. We compare with two baselines: our ProxNewton subproblem solver (no working sets) and ProxNewton combined with BlitzScreen. We perform screening as described in §5.2 after each ProxNewton iteration.

We compare the algorithms using data from the LIBSVM data repository. Tasks include spam detection (Webb et al., 2006), malicious URL identification (Ma et al., 2009), text classification (Lewis et al., 2004), and educational performance prediction (Yu et al., 2010).

We perform conventional preprocessing on each data set. We standardize all features to have unit variance. We remove features with fewer than ten nonzero entries. We include an unregularized bias term in the model. To accommodate this term, we add the constraint \((1, x) = 0\) to (PL1D). Since **LIBLINEAR** implements an \( \ell_1 \)-regularized bias term, we slightly modify **LIBLINEAR** to (i) use regularization 0 for the bias variable, and (ii) always include the bias term in the working set.

We solve each problem using three \( \lambda \) values: \( 0.2 \lambda_{\text{max}}, 0.02 \lambda_{\text{max}}, \) and \( 0.002 \lambda_{\text{max}} \). Here \( \lambda_{\text{max}} \) is the smallest regularization value for which the problem’s solution, \( \omega^* \), equals 0. For each problem, we report the fraction of nonzero entries in \( \omega^* \), which we denote by \( s^* \). We also report a weighted version of this quantity, which we define as \( s^*_W = \frac{1}{\text{NNZ}(A)} \sum_{i: \omega^*_i \neq 0} \text{NNZ}(A_i) \). Here \( \text{NNZ}(A_i) \) denotes the number of nonzero entries in column \( i \) of the design matrix.

With the exception of the spam detection problem, we solve each problem using a m4.2xlarge Amazon EC2 instance with 2.3 GHz Intel Xeon E5-2686 processors, 46 MB cache, and 32 GB memory. Due to memory requirements, we use a r3.2xlarge instance with 61 GB memory and Intel Xeon E5-2670 processors for the spam detection problem.

Figure 6 contains the results of these comparisons. In many cases, we see that BlitzWS converges in much less time than **LIBLINEAR**. Considering that **LIBLINEAR** is an efficient, established library, these results show that BlitzWS is indeed a fast algorithm.

We also note that BlitzWS provides significant speed-ups compared to the non-working set approach. The amount of speed-up depends on the solution’s sparsity, which is not surprising since we designed BlitzWS to exploit the solution’s sparsity.

---

3. URL: https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/.
We compare BlitzWS to its subproblem solver and LIBLINEAR. BlitzWS provides consistent optimization speed-ups.
BlitzWS progress parameters for sparse logistic regression. Plots show $\xi_t$ values that BlitzWS uses to produce the results in Figure 6. As regularization decreases, BlitzWS adapts by decreasing $\xi_t$.

Impact of BlitzWS’s capsule approximation. We plot the working set size vs. possible choices of the $\xi_t$ progress parameter (dashed curves). Each of BlitzWS’s first seven iterations corresponds to a different colored curve. We also plot the working set size when using the teardrop region, $S_t$, to select each working set (solid curves). The close alignment of curves indicate the capsule approximation performs well.

6.2.2 Adaptation to regularization strength

We find BlitzWS outperforms BlitzScreen because BlitzWS adapts its $\xi_t$ progress parameter to each problem. In contrast, ProxNewton + BlitzScreen is approximately equivalent to using BlitzWS with $\xi_t = 1$ for all iterations (as discussed in §5.4). Figure 7 contains plots of BlitzWS’s chosen $\xi_t$ parameters for each logistic regression problem. When $\lambda = 0.2\lambda_{max}$, BlitzWS uses large $\xi_t$ values, and screening (i.e., $\xi_t = 1$) also tends to perform well. As $\lambda$ decreases, screening becomes ineffective, while BlitzWS adapts by choosing smaller values of $\xi_t$. 
6.2.3 Impact of Capsule Approximation

For the sparse logistic regression problems, we examine how BlitzWS’s capsule approximation affects each working set. We log BlitzWS’s state—$x_{t-1}$, $y_{t-1}$, and $\Delta_{t-1}$—prior to selecting each working set. Then offline, we compute working sets for many values of $\xi_t$.

We record working set sizes for each problem, iteration, and $\xi_t$ value. In each case, we construct one working set using $S_\xi$ and a second working set using the capsule approximation. To calculate the working set using $S_\xi$, we discretize the definition of this set using 200 values of $\beta$.

We measure the capsule approximation’s impact by comparing the sizes of $W_\xi$ and $W_{\xi}^{cap}$, where the teardrop determines $W_\xi$ and the capsule determines $W_{\xi}^{cap}$. If $|W_{\xi}^{cap}| \approx |W_\xi|$, then BlitzWS’s capsule approximation has little impact on the makeup of each working set.

Figure 8 contains the results of this experiment. Observe that in all cases, $|W_{\xi}^{cap}| \approx |W_\xi|$. This suggests that $S_{\xi}^{cap}$ is a very good approximation of $S_\xi$.

6.2.4 Linear SVM Comparisons

We also compare BlitzWS with LIBLINEAR for training linear SVMs. The BlitzWS implementation is the same as described in §6.1.2. LIBLINEAR also uses a DCA-based algorithm.

For these comparisons, we use the same data sets, compilation settings, and hardware as we used in §6.2.1. For each data set, we compute a practical value of $C$ using five-fold cross validation, which we denote by $C_{cv}$. We compare using three values of $C$: $0.1C_{cv}$, $C_{cv}$, and $10C_{cv}$. We also report the solution’s “sparsity,” denoted $s^*$, which we define as the fraction of training examples that are unbounded support vectors at the solution.

Figure 9 includes results from these comparisons. BlitzWS consistently provides speed-up compared to LIBLINEAR, often during early iterations.

7. Discussion

We proposed BlitzWS, a principled yet practical working set algorithm. Unlike prior algorithms, BlitzWS selects subproblems in a way that maximizes guaranteed progress. We also analyzed the consequences of solving BlitzWS’s subproblems approximately, and we applied this understanding to adapt algorithmic parameters as iterations progress.

In practice, BlitzWS is indeed a fast algorithm. Compared to the popular LIBLINEAR library, BlitzWS achieves very competitive convergence times. Another appealing quality of BlitzWS is its capability of solving a variety of problems. This includes constrained problems, sparse problems, and piecewise loss problems. This flexibility results from §3’s novel piecewise problem formulation. We find this formulation is a useful way of thinking about sparsity and related structure in optimization.

We also proposed a state-of-the-art safe screening test called BlitzScreen. Unlike prior screening tests, BlitzScreen applies to a large class of problems. Because of its relatively small safe region, BlitzScreen also simplifies the objective by a greater amount. Unfortunately, we found that in many practical scenarios, BlitzScreen had little impact on the algorithm’s progress. While disappointing, we think this observation is an important contribution.

Exploiting piecewise structure can lead to large optimization speed-ups. Our analysis of BlitzWS and BlitzScreen provides a foundation for exploiting this structure in a principled way. We hope these contributions may serve as a starting point for future approaches to scalable optimization.
BlitzWS Dual Coordinate Ascent DCA + BlitzScreen LibLinear

\[ \text{web} \text{spam} \quad \begin{cases} \quad m = 2.8 \times 10^5, \ n \approx 4.4 \times 10^5, \ \text{NNZ} \approx 1.0 \times 10^9 \\ \quad C = 0.1C_{cv}, \ s^* \approx 0.06 \\ \quad C = C_{cv}, \ s^* \approx 0.06 \\ \quad C = 10C_{cv}, \ s^* \approx 0.05 \end{cases} \]

\[ \text{url} \quad \begin{cases} \quad m \approx 1.9 \times 10^6, \ n \approx 1.5 \times 10^5, \ \text{NNZ} \approx 2.1 \times 10^8 \\ \quad C = 0.1C_{cv}, \ s^* \approx 0.02 \\ \quad C = C_{cv}, \ s^* \approx 0.02 \\ \quad C = 10C_{cv}, \ s^* \approx 0.02 \end{cases} \]

\[ \text{kdda} \quad \begin{cases} \quad m \approx 6.7 \times 10^6, \ n \approx 2.2 \times 10^6, \ \text{NNZ} \approx 2.2 \times 10^8 \\ \quad C = 0.1C_{cv}, \ s^* \approx 0.1 \\ \quad C = C_{cv}, \ s^* \approx 0.1 \\ \quad C = 10C_{cv}, \ s^* \approx 0.1 \end{cases} \]

\[ \text{rcv1}\_\text{test} \quad \begin{cases} \quad m \approx 5.4 \times 10^7, \ n \approx 3.3 \times 10^4, \ \text{NNZ} \approx 4.0 \times 10^7 \\ \quad C = 0.1C_{cv}, \ s^* \approx 0.03 \\ \quad C = C_{cv}, \ s^* \approx 0.04 \\ \quad C = 10C_{cv}, \ s^* \approx 0.05 \end{cases} \]

Figure 9: Convergence comparisons for linear SVMs. BlitzWS also leads to convergence time improvements when training linear SVMs. For more difficult problems, plot markers represent to multiple iterations.
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Appendix A. Proof of Lemma 2.1

Proof. We start with the definition of $y_t$:

$$
\Delta_t = \psi_{MN}(y_t) - \psi_{MN}(x_t)
$$

$$
= \psi_{MN}(\alpha_t x_t + (1 - \alpha_t)y_{t-1}) - \psi_{MN}(x_t)
$$

$$
= (1 - \alpha_t) \left[ \Delta_{t-1} - \frac{1}{2} \alpha_t \|x_t - y_{t-1}\|^2 - [\psi_{MN}(x_t) - \psi_{MN}(x_{t-1})] \right]. \quad (14)
$$

Since $\psi_{MN}$ is 1-strongly convex,

$$
\psi_{MN}(x_t) \geq \psi_{MN}(x_{t-1}) + \langle \nabla \psi_{MN}(x_{t-1}), x_t - x_{t-1} \rangle + \frac{1}{2} \|x_t - x_{t-1}\|^2
$$

$$
\Rightarrow \psi_{MN}(x_t) - \psi_{MN}(x_{t-1}) \geq \frac{1}{2} \|x_t - x_{t-1}\|^2. \quad (15)
$$

Above, we have used the fact that $\langle \nabla \psi_{MN}(x_{t-1}), x_t - x_{t-1} \rangle \geq 0$, which must be true because $\psi_{MN}(x_{t-1}) \leq \psi_{MN}(x_t)$. Combining (15) with (14), we have

$$
\Delta_t \leq (1 - \alpha_t) \left[ \Delta_{t-1} - \frac{1}{2} \alpha_t \|x_t - y_{t-1}\|^2 - \frac{1}{2} \|x_t - x_{t-1}\|^2 \right]. \quad (16)
$$

Next, we use the algebraic fact

$$
\alpha_t \|x_t - y_{t-1}\|^2 + \|x_t - x_{t-1}\|^2 = (1 + \alpha_t) \left\| x_t - \frac{x_{t-1} + \alpha_t y_{t-1}}{1 + \alpha_t} \right\|^2 + \frac{\alpha_t}{1 + \alpha_t} \|x_t - x_{t-1}\|^2. \quad (17)
$$

To simplify notation, we define $d_{t-1} = \|x_t - x_{t-1}\|$. Applying the assumption that $\alpha_t > 0$, we can write $x_t = \frac{y_t - (1 - \alpha_t)y_{t-1}}{\alpha_t}$. Substituting this equality into (17), we have

$$
\alpha_t \|x_t - y_{t-1}\|^2 + \|x_t - x_{t-1}\|^2 = (1 + \alpha_t) \left\| y_t - \frac{(\alpha_t y_{t-1} - x_{t-1} + \alpha_t y_{t-1})}{1 + \alpha_t} \right\|^2 + \frac{\alpha_t}{1 + \alpha_t} d_{t-1}^2
$$

$$
= \frac{1 + \alpha_t}{\alpha_t^2} \left\| y_t - \frac{\alpha_t (x_{t-1} + y_{t-1})}{1 + \alpha_t} \right\|^2 + \frac{\alpha_t}{1 + \alpha_t} d_{t-1}^2. \quad (18)
$$

Inserting (18) into (16), we see that

$$
\Delta_t \leq (1 - \alpha_t) \left[ \Delta_{t-1} - \frac{1 + \alpha_t}{\alpha_t^2} \frac{1}{2} \left\| y_t - \frac{\alpha_t (x_{t-1} + y_{t-1})}{1 + \alpha_t} \right\|^2 - \frac{\alpha_t}{1 + \alpha_t} d_{t-1}^2 \right].
$$

Using the definition $\beta_t = \alpha_t (1 + \alpha_t)^{-1}$, we can plug in $\alpha_t = \beta_t (1 - \beta_t)^{-1}$ to complete the proof. \hfill \Box

Appendix B. Proof of Lemma 2.2

Proof. If $\beta_t = \frac{1}{2}$, we have $\Delta_t = 0$ by Lemma 2.1. The bound holds in this case because $\Delta_{t-1} (1 - \xi_t) \geq 0$. For the remainder of the proof, we assume that $\beta_t < \frac{1}{2}$, which implies that $\alpha_t < 1$.

Since $\alpha_t < 1$, there exists a constraint $i_{\text{limit}} \notin W_t$ for which $\langle a_{i_{\text{limit}}}, y_t \rangle = b_i$. Since $i_{\text{limit}} \notin W_t$, we must have $B_\xi(\beta_t) \cap \{x : \langle a_{i_{\text{limit}}}, x \rangle \geq b_i \} = \emptyset$. Thus, $y_t \notin B_\xi(\beta_t)$. Since $B_\xi(\beta_t)$ is a ball with center $\beta_t x_{t-1} + (1 - \beta_t)y_{t-1}$ and radius $\tau_\xi(\beta_t)$, this implies that

$$
\|y_t - \beta_t x_{t-1} - (1 - \beta_t)y_{t-1}\| \geq \tau_\xi(\beta_t).
$$
To simplify notation, we define \( d_{t-1} = \|x_{t-1} - y_{t-1}\| \). Combining with Lemma 2.1 and plugging in the definition of \( \tau_\ell(\beta_t) \), we have

\[
\Delta_t \leq \frac{1 - 2\beta_t}{1 - \beta_t} \left[ \Delta_{t-1} - \frac{1 - \beta_t}{\beta_t^2} \frac{1}{2} \tau_\ell(\beta_t)^2 - \beta_t \frac{1}{2} d_{t-1}^2 \right] \\
= \frac{1 - 2\beta_t}{1 - \beta_t} \left[ \Delta_{t-1} - (1 - \beta_t) \Delta_{t-1} \left[ 1 + \frac{\beta_t}{1 - \beta_t} \left( 1 - \frac{d_{t-1}^2}{2\Delta_{t-1}} \right) - \frac{1 - \xi_t}{1 - 2\beta_t} \right] + \beta_t \frac{1}{2} d_{t-1}^2 \right] \\
\leq \frac{1 - 2\beta_t}{1 - \beta_t} \left[ \Delta_{t-1} \left( \frac{1 - \xi_t}{1 - \beta_t} \right) \right] \\
= (1 - \xi_t) \Delta_{t-1} .
\]

Appendix C. Proof of Theorem 2.3

The proof can be divided into three cases: \( \beta_t = 1/2, \beta_t \in (0, 1/2) \), and \( \beta_t = 0 \). Here we present the proof of Theorem 2.3 for only the main case that \( \beta_t \in (0, 1/2) \), and we rely on the proof in Appendix F (a more general proof) for the edge cases.

Partial proof. Assuming \( \beta_t < 1/2 \) implies \( \alpha_t < 1 \). This implies there exists a \( i_{\text{limit}} \in \mathcal{W}_t \) such that \( (a_{i_{\text{limit}}}, y_t) = b_t \). Since \( i_{\text{limit}} \notin \mathcal{W}_t \), we must also have \( S_t \cap \{ x : (a_{i_{\text{limit}}}, x) \geq b_t \} = \emptyset \), which implies \( y_t \notin S_t \). Since \( y_t \notin S_t \), then for all \( \beta \in (0, 1/2) \), we have \( y_t \notin B_\ell(\beta) \). Applying the definition of \( B_\ell(\beta) \), we have \( \|y_t - \beta x_{t-1} - (1 - \beta_t)y_{t-1}\| \geq \tau_\ell(\beta) \) for all \( \beta \in (0, 1/2) \). Thus,

\[
\|y_t - \beta x_{t-1} - (1 - \beta_t)y_{t-1}\| \geq \tau_\ell(\beta_t) .
\]

At this point, we can combine (20) with Lemma 2.1 to achieve the desired bound. The result follows from the same steps as Lemma 2.2’s proof, starting at (19).

Appendix D. Proof of Theorem 2.4

Proof. Plugging in definitions of \( q_\ell \) and \( \tau_\beta \), we have

\[
q_\ell(\beta) = s\beta \|x_{t-1} - y_{t-1}\| + \sqrt{2\Delta_{t-1}} \left[ 1 + \frac{\beta}{1 - \beta} \left( 1 - \frac{\|x_{t-1} - y_{t-1}\|^2}{2\Delta_{t-1}} \right) \right]^{1/2} .
\]

To simplify notation, define \( d_{t-1} = \|x_{t-1} - y_{t-1}\| \). Taking the log and substituting \( \beta = \theta(1+\theta)^{-1} \), we have

\[
\tilde{q}_\ell(\theta) = \log \left( \frac{\theta}{1 + \theta} \right) + \log \left( s d_{t-1} + \sqrt{2\Delta_{t-1}} \left[ 1 + \theta \left( 1 - \frac{d_{t-1}^2}{2\Delta_{t-1}} \right) \right]^{1/2} \right) .
\]

Since \( \frac{\theta}{1 + \theta} \) is concave, \( \theta \) is concave, \( -\frac{1 + \theta}{1 - \theta} \) is concave and nondecreasing, and \( \log(\cdot) \) is concave and nondecreasing, we see that \( \tilde{q}_\ell(\theta) \) is log-concave on \( \{ \theta : \tilde{q}_\ell(\theta) > 0 \} \). (Here we have also used the facts that \( 1 - \frac{d_{t-1}^2}{2\Delta_{t-1}} \geq 0 \) and \( 1 - \xi \geq 0 \).) Since all log-concave functions are quasiconcave and quasiconcavity is preserved under composition with the increasing function \( \theta = \beta(1-\beta)^{-1} \) (on the domain \( 0 < \beta \leq 1/2 \)), it must be the case that \( q_\ell(\beta) \) is quasiconcave.

\[ \square \]
Appendix E. Proof of Theorem 2.5

Proof. In §I.1, we prove that $S_{\xi} \subseteq S_{\xi}^{\text{cap}}$. Since $S_{\xi} \subseteq S_{\xi}^{\text{cap}}$, we have

$$S_{\xi} \cap \{x : |\langle A_i, x \rangle| \geq \lambda\} \neq \emptyset \Rightarrow S_{\xi}^{\text{cap}} \cap \{x : |\langle A_i, x \rangle| \geq \lambda\} \neq \emptyset.$$ 

Thus, during iteration $t$ of Algorithm 1, condition (i) for Theorem 2.3 is satisfied. That is, for any $i \in [m]$, if $S_{\xi} \cap \{x : |\langle A_i, x \rangle| \geq \lambda\} \neq \emptyset$, then $i \in W_t$. Since condition (ii) of the theorem is satisfied by our definition of Algorithm 1, we have by Theorem 2.3, $\Delta_t \leq (1 - \xi_t)\Delta_{t-1}$ for all $t \geq 1$. The theorem then follows from induction. \hfill \Box

Appendix F. Proof of Theorem 4.1 and Theorem 4.2

Since Theorem 4.1 is a special case of Theorem 4.2, we prove both theorems by proving Theorem 4.2. To recover Theorem 4.1, we define $\epsilon_t = 0, f_t^{LB} = f_t$, and $x_t = z_t = \arg\min_x f_t(x)$.

Proof. We will prove that for all $t > 0$, we have

$$\Delta_t \leq (1 - (1 - \epsilon_t)\xi_t)\Delta_{t-1}. \tag{21}$$

To prove (21) for any $t > 0$, let us define the scalar

$$\theta_t = \max \{\theta \in [0, 1] : \theta z_t + (1 - \theta) y_{t-1} \in \text{cl}(S_{\xi}^{\text{cap}})\}$$

and point $y'_t = \theta_t z_t + (1 - \theta_t) y_{t-1}$. Above, cl(·) denotes the closure of a set. Note $y_{t-1} \in \text{cl}(S_{\xi}^{\text{cap}})$. Since $y_t$ minimizes $f$ along $[y_{t-1}, z_t]$, it follows that $f(y_t) \leq f(y'_t)$. Due to (C1), we have that $f_t(x) = f(x)$ for all $x \in S_{\xi}^{\text{cap}}$. Since $y'_t \in \text{cl}(S_{\xi}^{\text{cap}})$ and $f$ is convex lower semicontinuous, it follows that $f_t(y'_t) = f(y'_t)$. Beginning with the definition of $\Delta_t$, we can write

$$\Delta_t = f(y_t) - f_t^{LB}(x_t) \leq f(y'_t) - f_t^{LB}(x_t) = f_t(y'_t) - f_t^{LB}(x_t).$$

We divide the remainder of the proof into three cases.

Case 1: $\theta_t = 1$ In this case, $y'_t = z_t$, and it follows that

$$\Delta_t \leq f_t(y'_t) - f_t^{LB}(x_t) = f_t(z_t) - f_t^{LB}(x_t) \leq \epsilon_t \Delta_{t-1} \leq (1 - (1 - \epsilon_t)\xi_t)\Delta_{t-1}.$$ 

Above, the second-to-last step results from termination conditions for subproblem $t$, while the final step is true because $\xi_t \in (0, 1]$.

Case 2: $\theta_t \in (0, 1)$ Applying the definition of $y'_t$, the fact that $f_t$ is 1-strongly convex, the fact that $f_t(x) \leq f(x)$ for all $x$, and the definition of $\Delta_{t-1}$, we have

$$\Delta_t \leq f_t(y'_t) - f_t^{LB}(x_t)$$

$$= f_t(\theta_t z_t + (1 - \theta_t) y_{t-1}) - f_t^{LB}(x_t)$$

$$\leq \theta_t f_t(z_t) + (1 - \theta_t) f_t(y_{t-1}) - \frac{1}{2} (1 - \theta_t) \theta_t \|z_t - y_{t-1}\|^2 - f_t^{LB}(x_t)$$

$$\leq \theta_t f_t(z_t) + (1 - \theta_t) f_t(y_{t-1}) - \frac{1}{2} (1 - \theta_t) \theta_t \|z_t - y_{t-1}\|^2 - f_t^{LB}(x_t)$$

$$= (1 - \theta_t) \Delta_{t-1} - (1 - \theta_t) [f_t^{LB}(x_t) - f_t^{LB}(x_{t-1})] +$$

$$\theta_t [f_t(z_t) - f_t^{LB}(x_t)] - \frac{1}{2} (1 - \theta_t) \theta_t \|z_t - y_{t-1}\|^2.$$
From termination conditions for subproblem $t$, we have that $f_t(z_t) - f_t^L(x_t) \leq \epsilon_t \Delta t - 1$ and also that $f_t^L(x_t) - f_t^L(x_{t-1}) \geq (1 - \epsilon_t) \frac{1}{2} \|z_t - x_{t-1}\|^2$. Thus,

$$
\Delta t \leq (1 - \theta_t)\Delta t - 1 - (1 - \theta_t)(1 - \epsilon_t) \frac{1}{2} \|z_t - x_{t-1}\|^2 + \theta_t \epsilon_t \Delta t - 1 - \frac{1}{2}(1 - \theta_t) \theta_t \|z_t - y_{t-1}\|^2
$$

$$
\leq \Delta t - 1 - (1 - \epsilon_t) \left[ \theta_t \Delta t - 1 + \frac{1}{2}(1 - \theta_t) \left( \theta_t \|z_t - y_{t-1}\|^2 + \|z_t - x_{t-1}\|^2 \right) \right].
$$

We next use the fact

$$
\theta_t \|z_t - y_{t-1}\|^2 + \|z_t - x_{t-1}\|^2 = (1 + \theta_t) \left[ \|z_t - \frac{x_{t-1} + \theta_t y_{t-1}}{1 + \theta_t}\|^2 + \frac{\theta_t}{1 + \theta_t} \|x_{t-1} - y_{t-1}\|^2 \right].
$$

To simplify notation slightly, we define $d_{t-1} = \|x_{t-1} - y_{t-1}\|$. Applying the assumption that $\theta_t > 0$, we can write $z_t = \frac{y_{t-1} - (1 - \theta_t) y_{t-1}}{\theta_t}$. Substituting this equality into (23), we have

$$
\theta_t \|z_t - y_{t-1}\|^2 + \|z_t - x_{t-1}\|^2 = (1 + \theta_t) \left[ \|y'_{t-1} - \frac{x_{t-1} + \theta_t y_{t-1}}{1 + \theta_t}\|^2 + \frac{\theta_t}{1 + \theta_t} d_{t-1} \right].
$$

Inserting (24) into (22), it follows that

$$
\Delta t \leq \Delta t - 1 - (1 - \epsilon_t) \left[ \theta_t \Delta t - 1 + \frac{1 - \theta_t^2}{\beta_t^2} \left( \|y'_{t-1} - \frac{x_{t-1} + \theta_t y_{t-1}}{1 + \theta_t}\|^2 + \frac{\theta_t}{1 + \theta_t} d_{t-1} \right) \right].
$$

Let us denote the quantity within the brackets above by $P$ (for “progress” toward convergence). Also, let us define $\beta_t = \theta_t(1 + \theta_t)^{-1}$, which implies $\theta_t = \beta_t(1 - \beta_t)^{-1}$. We see that

$$
P = \theta_t \Delta t - 1 + \frac{1 - \theta_t^2}{\beta_t^2} \left( \|y'_{t-1} - \frac{x_{t-1} + \theta_t y_{t-1}}{1 + \theta_t}\|^2 + \frac{\theta_t}{1 + \theta_t} d_{t-1} \right)
$$

$$
= \frac{\beta_t}{1 - \beta_t} \Delta t - 1 + \frac{1 - 2\beta_t}{\beta_t^2} \left( \|y'_{t-1} - \beta_t x_{t-1} - (1 - \beta_t) y_{t-1}\|^2 + \frac{\beta_t(1 - 2\beta_t)}{1 - \beta_t} d_{t-1} \right)
$$

$$
= \frac{1 - 2\beta_t}{1 - \beta_t} \left[ \frac{\beta_t}{1 - \beta_t} \Delta t - 1 + \frac{1 - \beta_t}{\beta_t^2} \left( \|y'_{t-1} - \beta_t x_{t-1} - (1 - \beta_t) y_{t-1}\|^2 + \beta_t \frac{1}{2} d_{t-1} \right) \right].
$$

Since $\theta_t < 1$, by definition of $\theta_t$ and $y'_{t-1}$, we must have $y'_{t-1} \in \text{bd}(S_{\xi}^c)$. Since $S_{\xi}^c$ is an open set and $y'_{t-1} \not\in \text{bd}(S_{\xi}^c)$, we have $y'_{t-1} \not\in S_{\xi}^c$. Furthermore, since $S_{\xi}^c \supseteq S_{\xi} \supseteq B_\xi(\beta_t)$, it follows that $y'_{t-1} \not\in B_\xi(\beta_t)$. By definition of $B_\xi(\beta_t)$, we have

$$\|y'_{t-1} - \beta_t x_{t-1} - (1 - \beta_t) y_{t-1}\| \geq \tau_\xi(\beta_t).$$

Plugging in the definition of $\tau_\xi(\beta_t)$, it follows that

$$
\frac{1 - \beta_t}{\beta_t^2} \left( \|y'_{t-1} - \beta_t x_{t-1} - (1 - \beta_t) y_{t-1}\|^2 \right) \geq (1 - \beta_t) \Delta t - 1 \left[ 1 + \frac{\beta_t}{1 - \beta_t} \left( 1 - \frac{d_{t-1}^2}{\Delta t - 1} \right) - \frac{\xi_t}{1 - \beta_t} \right] +
$$

$$
= \Delta t - 1 - \beta_t \frac{1}{2} d_{t-1}^2 - \frac{1 - \beta_t}{1 - \beta_t} \xi_t \Delta t - 1 + \beta_t \frac{1}{2} d_{t-1}^2
$$

$$
= \left[ \frac{1 - \beta_t}{1 - \beta_t} \xi_t \Delta t - 1 - \beta_t \frac{1}{2} d_{t-1}^2 \right].
$$
Plugging this result into (26), we have
\[
P \geq \frac{1-2\beta t}{1-\beta t} \left[ \frac{\beta_t}{1-2\beta_t} \Delta_{t-1} + \left[ \frac{1-\beta_t}{1-2\beta_t} \xi_t \Delta_{t-1} - \frac{\beta_t}{2} \frac{d_{t-1}^2}{\Delta_{t-1}} \right] + \frac{\beta_t}{2} d_{t-1}^2 \right] \geq \frac{1-2\beta t}{1-\beta t} \left[ \frac{1-\beta_t}{1-2\beta_t} \xi_t \Delta_{t-1} \right] = \xi_t \Delta_{t-1}.
\]
By combining (27) with (25), we obtain the desired bound.

**Case 3:** $\theta_t = 0$ Using the definition of $y_t'$, we have
\[
\Delta_t \leq f_t(y_t') - f_t^L(x_t)
= f_t(y_{t-1}) - f_t^L(x_t)
\leq f(y_{t-1}) - f_t^L(x_t)
= \Delta_{t-1} - \left[ f_t^L(x_t) - f_t^L(x_{t-1}) \right]
\leq \Delta_{t-1} - (1 - \epsilon_t) \frac{1}{2} \|z_t - x_{t-1}\|^2.
\]
Above, the last step follows from termination conditions for subproblem $t$.

Since $\theta_t = 0$, it follows from the definition of $\theta_t$ that $\beta_t z_t + (1 - \beta_t) y_{t-1} \notin \text{cl}(S_{\xi}^{\text{cap}})$ for all $\beta \in (0, 1/2)$. Since $S_{\xi}^{\text{cap}} \supseteq S_{\xi} \supseteq B_{\xi}(\beta)$ for all $\beta \in (0, 1/2)$, then $\beta z_t + (1 - \beta) y_{t-1} \notin \text{cl}(B_{\xi}(\beta))$ for all $\beta \in (0, 1/2)$. By definition of $B_{\xi}(\beta)$, this means that
\[
\|\beta z_t + (1 - \beta) y_{t-1} - \beta x_{t-1} - (1 - \beta) y_{t-1}\| > \tau_{\xi}(\beta)
\Rightarrow \|z_t - x_{t-1}\| > \frac{\tau_{\xi}(\beta)}{\beta}.
\]
This implies that
\[
\|z_t - x_{t-1}\| \geq \lim_{\beta \to 0^+} \frac{\tau_{\xi}(\beta)}{\beta} = \frac{d}{\delta^2} \tau_{\xi}(\beta) \bigg|_{\beta = 0} = \sqrt{2 \Delta_{t-1} \xi_t}.
\]
By combining (29) with (28), we obtain the result.

\[ \square \]

**Appendix G. Proof of Theorem 5.1**

Proof. We need to show that $x^* = \text{argmin}_x f(x) = \text{argmin}_x f_0(x) = x_S$. First note that since $f_0$ is a $1$-strongly convex lower bound on $f$, and $x_0$ minimizes $f_0$, it follows that
\[
f(x^*) \geq f_0(x_0) + \frac{1}{2} \|x^* - x_0\|^2.
\]
Since $f$ is $1$-strongly convex, and $x^*$ minimizes $f$, we have
\[
f(y_0) \geq f(x^*) + \frac{1}{2} \|y_0 - x^*\|^2.
\]
Combining (31) with (30), we have
\[ f(x^*) + f(y_0) \geq f_0(x_0) + \frac{1}{2} \|x^* - x_0\|^2 + f(x^*) + \frac{1}{2} \|y_0 - x^*\|^2 \]
\[ \Rightarrow \Delta_0 \geq \|x^* - \frac{1}{2}(x_0 + y_0)\|^2 + \frac{1}{4} \|x_0 - y_0\|^2 \]
\[ \Rightarrow x^* \in \text{cl}(S_1). \]

By construction, \( f_S(x) = f(x) \) for all \( x \in S_1 \). Since \( S_1 \) is an open set, if \( x^* \in S_1 \), then
\[ \partial f_S(x^*) = \partial f(x^*) \Rightarrow 0 \in \partial f_S(x^*) \Rightarrow x^* = x_S. \]

For the remainder of the proof, we consider the case that \( x^* \in \text{bd}(S_1) \). In this case, (31) holds with equality (since (32) holds with equality), meaning
\[ f(y_0) = f(x^*) + \frac{1}{2} \|y_0 - x^*\|^2. \]  

(33)

Define \( z = \frac{1}{2}(y_0 + x^*) \). Note \( z \in S_1 \), since \( x^* \in \text{bd}(S_1) \), \( y_0 \in \text{cl}(S_1) \), \( x^* \neq y_0 \), and \( S_1 \) is an open ball. Also, since \( z \) lies on the segment \([x^*, y_0]\), and \( f \) is 1-strongly convex, (33) implies that
\[ f(z) = f(x^*) + \frac{1}{2} \|z - x^*\|^2. \]

This implies that \( z - x^* \in \partial f(z) \), since \( f(x^*) + \frac{1}{2} \|x - x^*\|^2 \leq f(x) \) for all \( x \). Because \( z \in S_1 \), it follows that \( z - x^* \in \partial f_S(z) \). Since \( f_S \) is 1-strongly convex, then for all \( x \), we have
\[ f_S(x) \geq f(z) + \langle z - x^*, x - z \rangle + \frac{1}{2} \|x - z\|^2 \]
\[ = f(x^*) + \frac{1}{2} \|z - x^*\|^2 + \langle z - x^*, x - z \rangle + \frac{1}{2} \|x - z\|^2 \]
\[ \geq f(x^*). \]

(34)

At the same time, since \( f_S(x) = f(x) \) for all \( x \in S_1 \), and \( f_S \) is lower semicontinuous, we must have \( f_S(x^*) = f(x^*) \). Combined with (34), it follows that \( x^* \) minimizes \( f_S \).

\[ \square \]

**Appendix H. Proof of Theorem 5.3**

**Proof.** Consider any \( x_{S_1} \in S_1 \). For some \( \delta > 0 \), we have
\[ \|x_{S_1} - \frac{1}{2}(x_0 + y_0)\| = \sqrt{\Delta_0 - \frac{1}{4} \|x_0 - y_0\|^2} - \delta \]

From the definition of \( S_\xi \), we know \( S_\xi \supseteq B_\xi(\beta) \) for all \( \beta \in (0, 1/2) \). Recall that \( B_\xi(\beta) \) is a ball with center \( \beta x_0 + (1 - \beta)y_0 \) and radius \( \tau_\xi(\beta) \). We have
\[ \lim_{\beta \to 1/2^-} [\tau_\xi(\beta) - \|x_{S_1} - \beta x_0 - (1 - \beta)y_0\|] = \sqrt{\Delta_0 - \frac{1}{4} \|x_0 - y_0\|^2 - \|x_{S_1} - \frac{1}{2}(x_0 + y_0)\|} \]
\[ = \delta > 0. \]

Thus, for some \( \beta \in (0, 1/2) \), we have \( x_{S_1} \in B_\xi(\beta) \), implying \( x_{S_1} \in S_\xi \). We have shown \( S_1 \subseteq S_\xi \).

To show that \( S_\xi \subseteq S_1 \), consider any \( x_{S_\xi} \in S_\xi \). Since \( \xi_t = 1 \), for all \( \beta \in (0, 1/2) \) we have
\[ \tau_\xi(\beta) = \beta \sqrt{2\Delta_0 \left[ 1 + \frac{\beta}{1 - \beta} \left( 1 - \frac{\|x_0 - y_0\|^2}{2\Delta_0} \right) \right]} = 2\beta \sqrt{\Delta_0 - \frac{1}{4} \|x_0 - y_0\|^2}. \]
Thus, we again used the fact $\frac{\|x_0 - y_0\|^2}{\Delta_0} \leq 1$, which follows from $f(y_0) \geq f(x_0) + \frac{1}{2} \|y_0 - x_0\|^2$.

From the definition of $\bar{S}_\xi$, there exists a $\beta \in (0, 1/2)$ such that $x_{\bar{S}_\xi} \in B_\xi(\beta)$. From this, we see

$\|x_{\bar{S}_\xi} - \beta x_0 - (1 - \beta)y_0\| < \tau_\xi(\beta)$

$\Rightarrow \|x_{\bar{S}_\xi} - \beta x_0 - (1 - \beta)y_0\| < 2\beta \sqrt{\Delta_0 - \frac{1}{4} \|x_0 - y_0\|^2}$

$\Rightarrow \|x_{\bar{S}_\xi} - \frac{1}{2}(x_0 + y_0) + \frac{1}{2} (1 - \beta)(x_0 - y_0)\| < 2\beta \sqrt{\Delta_0 - \frac{1}{4} \|x_0 - y_0\|^2}$

$\Rightarrow \|x_{\bar{S}_\xi} - \frac{1}{2}(x_0 + y_0)\| < (1 - 2\beta)\frac{1}{2} \|x_0 - y_0\| + 2\beta \sqrt{\Delta_0 - \frac{1}{4} \|x_0 - y_0\|^2}$

$\Rightarrow \|x_{\bar{S}_\xi} - \frac{1}{2}(x_0 + y_0)\| < \sqrt{\Delta_0 - \frac{1}{4} \|x_0 - y_0\|^2}$

$\Rightarrow x_{\bar{S}_\xi} \in S_1$.

Note we again used the fact $\frac{\|x_0 - y_0\|^2}{\Delta_0} \leq 1$, which implies $\frac{1}{2} \|x_0 - y_0\| \leq \sqrt{\Delta_0 - \frac{1}{4} \|x_0 - y_0\|^2}$.

**Appendix I. Miscellaneous proofs**

**I.1 Proof that teardrop equivalence region is a subset of capsule equivalence region**

**Theorem I.1.** Define $S_\xi$ and $S_\xi^{\text{cap}}$ as in §2.4 and §2.5. Then $S_\xi \subseteq S_\xi^{\text{cap}}$.

**Proof.** Recall $S_\xi^{\text{cap}}$ is the set of points within a distance $r_{\text{cap}}$ from the segment $[c_1^{\text{cap}}, c_2^{\text{cap}}]$, where

$c_1^{\text{cap}} = \beta_1 x_{t-1} + (1 - \beta_1)y_{t-1}, \quad c_2^{\text{cap}} = \beta_2 x_{t-1} + (1 - \beta_2)y_{t-1},$

$\beta_1 = \frac{\Delta_{\text{cap}} + r_{\text{cap}}}{\|x_{t-1} - y_{t-1}\|}, \quad \beta_2 = \frac{\Delta_{\text{min}} - r_{\text{cap}}}{\|x_{t-1} - y_{t-1}\|}.$

Consider any $x' \in S_\xi$. By definition of $S_\xi$, there exists a scalar $\beta' \in (0, 1/2)$ such that

$\|x' - (\beta' x_{t-1} + (1 - \beta')y_{t-1})\| < \tau_\xi(\beta').$

In the case that $\beta_1 \leq \beta' \leq \beta_2$, then $\beta' x_{t-1} + (1 - \beta')y_{t-1}$ falls on the segment $[c_1^{\text{cap}}, c_2^{\text{cap}}]$. This implies that $x' \in S_\xi^{\text{cap}}(S_\xi)$, since

$\|x' - (\beta' x_{t-1} + (1 - \beta')y_{t-1})\| < \tau_\xi(\beta') \leq r_{\text{cap}}.$

In the case that $\beta' \leq \beta_1$, we have

$\|x' - c_1^{\text{cap}}\| \leq \|x' - (\beta' x_{t-1} + (1 - \beta')y_{t-1})\| + \|\beta' x_{t-1} + (1 - \beta')y_{t-1} - c_1^{\text{cap}}\|

< \tau_\xi(\beta') + (\beta_1 - \beta') \|x_{t-1} - y_{t-1}\|

= \left[\tau_\xi(\beta') - \beta' \|x_{t-1} - y_{t-1}\|\right] + \beta_1 \|x_{t-1} - y_{t-1}\|

\leq -c_{\text{min}} + \left[d_{\text{min}} + r_{\text{cap}}\right]

= r_{\text{cap}}.$

Thus, $x' \in S_\xi^{\text{cap}}$ if $\beta' \leq \beta_1$. A similar argument implies $x' \in S_\xi^{\text{cap}}$ when $\beta' \geq \beta_2$.

Thus, for all $\beta'$, we have $x' \in S_\xi$, which implies $S_\xi \subseteq S_\xi^{\text{cap}}$. \qed
I.2 Proof that dual progress termination condition can be satisfied

Theorem I.2. For the BlitzWS algorithm with approximate subproblem solutions described in §4.7, if subproblem \( t \) is solved exactly, then it is always the case that

\[
 f^\text{LB}_t(x_t) - f^\text{LB}_{t-1}(x_{t-1}) \geq (1 - \epsilon_t) \frac{1}{2} \| z_t - x_{t-1} \|^2 .
\]

Proof. If subproblem \( t \) is solved exactly, then \( f_t(z_t) = f^\text{LB}_t(x_t) \), since \( x_t = z_t \). Due to condition (C3) in §4.7, we have \( f_t(x) \geq f^\text{LB}_t(x) \) for all \( x \). Thus,

\[
 f_t(x) \geq f^\text{LB}_t(x_t) + \frac{1}{2} \| x - x_{t-1} \|^2 \\
 \Rightarrow f_t(z_t) \geq f^\text{LB}_t(x_t) + \frac{1}{2} \| z_t - x_{t-1} \|^2 \\
 \Rightarrow f^\text{LB}_t(x_t) - f^\text{LB}_t(x_{t-1}) \geq \frac{1}{2} \| z_t - x_{t-1} \|^2 \\
 \Rightarrow f^\text{LB}_t(x_t) - f^\text{LB}_t(x_{t-1}) \geq (1 - \epsilon_t) \frac{1}{2} \| z_t - x_{t-1} \|^2 .
\]

\( \square \)

I.3 Proof that \( f_0 \) lower bounds \( f_{L1D} \) in §5.2

Theorem I.3. For any \( \omega_0 \in \mathbb{R}^m \), define \( f_0, f_{L1D} \), and \( x_0 \) as in §5.2. Then \( f_0(x) \leq f_{L1D}(x) \forall x \).

Proof. Let \([x_0]_j\) denote the \( j \)th entry of \( x_0 \). For all \( x_j \), the Fenchel-Young inequality implies

\[
 L^*_j(x_j) - x_j \langle a_j, \omega_0 \rangle \geq -L_j(\langle a_j, \omega_0 \rangle) .
\]

When \( x_j = L'_j(\langle a_j, \omega_0 \rangle) \), this inequality holds with equality, implying \( L^*_j(x_j) - x_j \langle a_j, \omega_0 \rangle \) is minimized when \( x_j = [x_0]_j \). By assuming that \( L_j \) is 1-smooth, \( L^*_j \) is 1-strongly convex. Thus,

\[
 \sum_{j=1}^n \left[ L^*_j(x_j) - x_j \langle a_j, \omega_0 \rangle \right] \geq \frac{1}{2} \| x - x_0 \|^2 + \sum_{j=1}^n \left[ L^*_j([x_0]_j) - [x_0]_j \langle a_j, \omega_0 \rangle \right] \\
 = \frac{1}{2} \| x - x_0 \|^2 - \sum_{j=1}^n L_j(\langle a_j, \omega_0 \rangle) .
\]

Applying this result, we have

\[
 f_0(x) \leq f_{L1D}(x) \iff \frac{1}{2} \| x - x_0 \|^2 - g_{L1}(\omega_0) \leq \sum_{j=1}^n L^*_j(x_j) + \sum_{i=1}^m \phi_i(x) \\
 \iff -\sum_{j=1}^n x_j \langle a_j, \omega_0 \rangle - \lambda \| \omega_0 \|_1 \leq \sum_{i=1}^m \phi_i(x) \\
 \iff -\langle A\omega_0, x \rangle - \lambda \| \omega_0 \|_1 \leq \sum_{i=1}^m \phi_i(x) . \tag{35}
\]

Thus, it remains to prove (35). For each \( i \), note \( \phi_i(x) = +\infty \) if \( |\langle A_i, x \rangle| > \lambda \). Thus, we must only consider the case \(|\langle A_i, x \rangle| \leq \lambda \), which implies \( \phi_i(x) = 0 \). Assuming \(|\langle A_i, x \rangle| \leq \lambda \), we have

\[
 -[\omega_0]_i \langle A_i, x \rangle - \lambda \| \omega_0 \|_1 \leq \| \omega_0 \|_1 |\langle A_i, x \rangle| - \lambda \| \omega_0 \|_1 \\
 = \| \omega_0 \|_1 (|\langle A_i, x \rangle| - \lambda) \\
 \leq 0 \\
 = \phi_i(x) .
\]

Summing over \( i \in [m] \) proves (35). \( \square \)