Liouville quantum gravity and the Brownian map II: geodesics and continuity of the embedding

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Abstract

We endow the $\sqrt{8/3}$-Liouville quantum gravity sphere with a metric space structure and show that the resulting metric measure space agrees in law with the Brownian map. Recall that a Liouville quantum gravity sphere is a priori naturally parameterized by the Euclidean sphere $S^2$. Previous work in this series used quantum Loewner evolution (QLE) to construct a metric $d_Q$ on a countable dense subset of $S^2$. Here we show that $d_Q$ a.s. extends uniquely and continuously to a metric $d_Q$ on all of $S^2$. Letting $d$ denote the Euclidean metric on $S^2$, we show that the identity map between $(S^2, d)$ and $(S^2, d_Q)$ is a.s. Hölder continuous in both directions. We establish several other properties of $(S^2, d_Q)$, culminating in the fact that (as a random metric measure space) it agrees in law with the Brownian map. We establish analogous results for the Brownian disk and plane.

Our proofs involve new estimates on the size and shape of QLE balls and related quantum surfaces, as well as a careful analysis of $(S^2, d_Q)$ geodesics.
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1 Introduction

1.1 Overview

This article is the second in a three part series that proves the equivalence of two fundamental and well studied objects: the $\sqrt{8/3}$-Liouville quantum gravity (LQG) sphere and the Brownian map (TBM). Both of these objects can be understood as random measure-endowed surfaces. However, an instance $S$ of the $\sqrt{8/3}$-LQG sphere comes with a conformal structure, which means that it can be parameterized by the Euclidean sphere $S^2$ in a canonical way (up to Möbius transformation), and an instance of TBM comes with a metric space structure. The problem is to endow each object with the other’s structure in a natural way, and to show that once this is accomplished the two objects agree in law. To briefly summarize the current series of articles:

1. The first article [MS15b] used a “quantum natural time” form of the so-called quantum Loewner evolution (QLE), as introduced in [MS13b], to define a distance $d_Q$ on a countable, dense collection of points $(x_n)$ chosen as i.i.d. samples from the area measure that lives on the instance $S$ of a $\sqrt{8/3}$-LQG sphere. Moreover, it was shown that for any $x$ and $y$ sampled from the area measure on $S$, the value $d_Q(x, y)$ is a.s. determined by $S$, $x$, and $y$. This implies in particular that the distance function $d_Q$, as defined on $(x_n)$, is a.s. determined by $S$ and the sequence $(x_n)$, so that there is no additional randomness required to define $d_Q$.

2. The current article shows that there is a.s. a unique continuous extension $\overline{d}_Q$ of $d_Q$ to all of $S$, and that the pair $(S, \overline{d}_Q)$, interpreted as a random metric measure space, agrees in law with TBM. Moreover, $\overline{d}_Q$ is a.s. determined by $S$. Thus a $\sqrt{8/3}$-LQG sphere has a canonical metric space structure.
3. The third article [MS16] will show that it is a.s. possible to recover $S$ when one is given just the metric measure space structure of the corresponding instance of TBM. In other words, the map (established in the current article) from $\sqrt{8/3}$-LQG sphere instances to instances of TBM is a.e. invertible — which means that an instance of TBM can a.s. be embedded in the sphere in a canonical way (up to Möbius transformation) — i.e., an instance of TBM has a canonical conformal structure. In particular, this allows us to define Brownian motion on Brownian map surfaces, as well as various forms of SLE and CLE.

Thanks to the results in these three papers, every theorem about TBM can be understood as a theorem about $\sqrt{8/3}$-LQG, and vice versa.

But let us focus on the matter at hand. Assume that we are given an instance $S$ of the $\sqrt{8/3}$-LQG sphere, endowed with the metric $d_Q$ on a countable dense set $(x_n)$. How shall we go about extending $d_Q$ to $d_Q^*$?

By way of analogy, let us recall that in an introductory probability class one often constructs Brownian motion by first defining its restriction to the dyadic rationals, and second showing (via the so-called Kolmogorov-Čentsov theorem [KS91, RY99]) that this restriction is a.s. a Hölder continuous function on the dyadic rationals, and hence a.s. extends uniquely to a Hölder continuous function on all of $\mathbb{R}_+$. The work in [MS15b] is analogous to the first step in that construction (it constructs $d_Q$ on a countable dense set), and Sections 3, 4, and 5 of the current article are analogous to the second step. These sections derive Hölder continuity estimates that in particular imply that $d_Q$ can a.s. be continuously extended to all of $S^2$.

Precisely, these sections will show that if $(x_n)$ are interpreted as points in $S^2$ (which parameterizes $S$), then for some fixed $\alpha, \beta > 0$ it is a.s. the case that, for some (possibly random) $C_1, C_2 > 0$,

$$C_1 d(x_i, x_j)^\alpha \leq d_Q(x_i, x_j) \leq C_2 d(x_i, x_j)^\beta \quad (1.1)$$

where $d$ is the Euclidean metric on $S^2$. This will immediately imply that $d_Q$ can be uniquely extended to a continuous function $\overline{d}_Q : S^2 \times S^2 \to \mathbb{R}$ that satisfies the same bounds, i.e.,

$$C_1 d(x_i, x_j)^\alpha \leq \overline{d}_Q(x_i, x_j) \leq C_2 d(x_i, x_j)^\beta, \quad (1.2)$$

and is also a metric on $S^2$. Another way to express the existence of $C_1$ and $C_2$ for which (1.2) holds is to say that the identity map between $(S^2, d)$ and $(S^2, \overline{d}_Q)$ is a.s. Hölder continuous (with some deterministic exponent) in both directions. We will also show that the metric $\overline{d}_Q$ is a.s. geodesic, i.e. that it is a.s. the case that every pair of points $x, y$ can be connected by a path whose length with respect to $\overline{d}_Q$ is equal to $\overline{d}_Q(x, y)$.

Once we have established this, Sections 6, 7, and 8 will show that this geodesic metric space agrees in law with TBM. The proof makes use of several basic results about
LQG spheres derived in [MS15c], and along with several properties that follow from the manner in which $d_Q$ was constructed in [MS15b]. A fundamental part of the argument is to show that certain paths that seem like they should be geodesics on the LQG-sphere side actually are geodesics w.r.t. $d_Q$, which will be done by studying a few approximations to these geodesics. We will ultimately conclude that, as a random metric measure space, $(S, d_Q)$ satisfies the properties that were shown in [MS15a] to uniquely characterize TBM.

We remark that the results of the current series of articles build on a large volume of prior work by the authors and others on imaginary geometry [MS12a, MS12b, MS12c, MS13a], conformal welding [She15], conformal loop ensembles [She09, SW12], and the mating of trees in infinite and finite volume settings [DMS14, MS15c], as well as the above mentioned works on quantum Loewner evolution [MS13b] and TBM [MS15a]. We also cite foundational works by many other authors on Liouville quantum gravity, Schramm-Loewner evolution, Lévy trees, TBM, continuous state branching processes, and other subjects. There has been a steady accumulation of theory in this field over the past few decades, and we hope that the proof of the equivalence of TBM and $\sqrt{8/3}$-LQG will be seen as a significant milestone on this continuing journey.

1.2 Main results

In this subsection, we state the results summarized in Section 1.1 more formally as a series of theorems. In [MS15b], it was shown that if $S$ is a unit area $\sqrt{8/3}$-LQG sphere $\dQ$ and $(x_n)$ is an i.i.d. sequence chosen from the quantum measure on $S$ then a variant of the QLE($8/3, 0$) processes introduced in [MS13b] induces a metric space structure $d_Q$ on $(x_n)$ which is almost surely determined by $S$. Our first main result is that the map $(x_i, x_j) \mapsto d_Q(x_i, x_j)$ almost surely extends to a function $\overline{d}_Q$ on all of $S^2 \times S^2$ such that $(x, y) \mapsto \overline{d}_Q(x, y)$ is Hölder continuous on $S^2 \times S^2$.

**Theorem 1.1.** Suppose that $S = (S^2, h)$ is a unit area $\sqrt{8/3}$-LQG sphere, $(x_n)$ is an i.i.d. sequence chosen from the quantum measure on $S$, and $d_Q$ is the associated QLE($8/3, 0$) metric on $(x_n)$. Then $(x_i, x_j) \mapsto d_Q(x_i, x_j)$ is almost surely Hölder continuous with respect to the Euclidean metric $d$ on $S^2$. In particular, $d_Q$ uniquely extends to a Hölder continuous function $\overline{d}_Q: S^2 \times S^2 \to R_+$. Finally, $\overline{d}_Q$ is almost surely determined by $S$.

Our next main result states that $\overline{d}_Q$ induces a metric on $S^2$ which is isometric to the metric space completion of $d_Q$, and provides some relevant Hölder continuity.

**Theorem 1.2.** Suppose that $S = (S^2, h)$ is a unit area $\sqrt{8/3}$-LQG sphere and that $\overline{d}_Q$ is as in Theorem 1.1. Then $\overline{d}_Q$ defines a metric on $S^2$ which is almost surely isometric to the metric space completion of $d_Q$. Moreover, the identity map from $(S^2, d)$ to $(S^2, \overline{d}_Q)$ is almost surely Hölder continuous in both directions where $d$ denotes the Euclidean metric on $S^2$. 

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Recall that a metric space \((M, d)\) is said to be \emph{geodesic} if for all \(x, y \in M\) there exists a path \(\gamma_{x,y}\) whose length is equal to \(d(x,y)\). Our next main result is that the metric space \(d_\mathcal{Q}\) is almost surely geodesic.

**Theorem 1.3.** Suppose that \(\mathcal{S} = (S^2, h)\) is a unit area \(\sqrt{8/3}\)-LQG sphere and that \(d_\mathcal{Q}\) is as in Theorem 1.1. The metric space \(d_\mathcal{Q}\) is almost surely geodesic. Moreover, it is a.s. the case that for all \(x, y \in S^2\), each geodesic path \(\gamma_{x,y}\), viewed as a map from a real time interval to \((S^2, d)\), is Hölder continuous.

Combining Theorem 1.2 and Theorem 1.3 with the axiomatic characterization for TBM given in [MS15a] and the results in the first paper of this series [MS15b], as well as some additional work carried out in the present article, we will find that the law of the metric space with metric \(d_\mathcal{Q}\) is indeed equivalent to the law of TBM.

**Theorem 1.4.** Suppose that \(\mathcal{S} = (S^2, h)\) is a unit area \(\sqrt{8/3}\)-LQG sphere and that \(d_\mathcal{Q}\) is as in Theorem 1.1. Then the law of the metric measure space \((S^2, d_\mathcal{Q}, \mu_h)\) is the same as that of the unit area Brownian map. Moreover, \(d_\mathcal{Q}\) is almost surely determined by \(\mathcal{S}\).

Theorem 1.4 implies that there exists a coupling of the law of a \(\sqrt{8/3}\)-LQG unit area sphere \(\mathcal{S}\) and an instance \((M, d, \nu)\) of TBM such that the metric measure space \((S^2, d_\mathcal{Q}, \mu_h)\) associated with \(\mathcal{S}\) is almost surely isometric to \((M, d, \nu)\). Moreover, by the construction of \(d_\mathcal{Q}\) given in [MS15b] we have that \(d_\mathcal{Q}\) and hence \((M, d, \nu)\) is almost surely determined by \(\mathcal{S}\). That is, the structure \((M, d, \nu)\) of TBM is a measurable function of \(\mathcal{S}\). The converse is the main result of the subsequent work in this series [MS16]. In other words, it will be shown in [MS16] that TBM almost surely determines its embedding into \(\sqrt{8/3}\)-LQG via QLE\((8/3, 0)\).

We can extract from Theorem 1.4 the equivalence of the QLE\((8/3, 0)\) metric on a unit boundary length \(\sqrt{8/3}\)-quantum disk [DMS14] and the random metric disk with boundary called the Brownian disk. The Brownian disk is defined in different ways in [BM15] and [MS15a] and is further explored in [AL15]. The equivalence of the Brownian disk definitions in [BM15] and [MS15a] will be established in the forthcoming work [JM]. We can similarly extract from Theorem 1.4 the equivalence of the QLE\((8/3, 0)\) metric on a \(\sqrt{8/3}\)-quantum cone [She15, DMS14] and the Brownian plane [CL12]. We state this result as the following corollary.

**Corollary 1.5.** (i) Suppose that \(\mathcal{D} = (D, h)\) is a unit boundary length \(\sqrt{8/3}\)-LQG disk. Then the law of \((D, d_\mathcal{Q}, \mu_h)\) is the same as that of the unit boundary length Brownian disk. Moreover, the identity map from \((D, d)\) to \((D, d_\mathcal{Q})\) is almost surely Hölder continuous in both directions where \(d\) denotes the Euclidean metric on \(D\).

(ii) Suppose that \(\mathcal{C} = (C, h, 0, \infty)\) is a \(\sqrt{8/3}\)-quantum cone. Then the law of \((C, d_\mathcal{Q}, \mu_h)\) is the same as that of the Brownian plane. Moreover, the identity map from \((C, d)\)
to \((\mathbb{C}, \overline{d}_Q)\) is almost surely locally Hölder continuous (i.e., Hölder continuous on compact sets) in both directions where \(d\) denotes the Euclidean metric on \(\mathbb{C}\).

In both cases, \(\overline{d}_Q\) is almost surely determined by the underlying quantum surface.

Part (i) of Corollary 1.5 follows from Theorem 1.4 because both a unit boundary length quantum disk and the Brownian disk can be realized as the complement of the filled metric ball. That is, if \((\mathcal{S}, x, y)\) denotes a doubly-marked instance of TBM (resp. \(\sqrt{8/3}\)-LQG surface) then for each \(r > 0\) the law of the \(y\)-containing component of the complement of the ball centered at \(x\) of radius \(r\) conditioned on its boundary length is that of a Brownian disk (resp. quantum disk), weighted by its area. Indeed, this follows in the case of the Brownian disk from its construction given in [MS15a] and this follows in the case of a \(\sqrt{8/3}\)-quantum cone from the basic properties of QLE(\(8/3\), 0) established in [MS15b].

Part (ii) of Corollary 1.5 follows from Theorem 1.4 because a \(\sqrt{8/3}\)-quantum cone is given by the local limit of a \(\sqrt{8/3}\)-LQG sphere near a quantum typical point and likewise the Brownian plane is given by the local limit of TBM near a typical point sampled from TBM’s intrinsic area measure [CL12, Theorem 1].

It will also be shown in [MS16] that the unit boundary length Brownian disk (resp. Brownian plane) almost surely determines its embedding into the corresponding \(\sqrt{8/3}\)-LQG surface via QLE(\(8/3\), 0).

The proofs of Theorems 1.1–1.4 and Corollary 1.5 require us to develop a number of estimates for the Euclidean size and shape of the regions explored by QLE(\(8/3\), 0). While we do not believe that our estimates are in general optimal, we are able to obtain the precise first order behavior for the Euclidean size of a metric ball in \(\overline{d}_Q\) centered around a quantum typical point. We record this result as our final main theorem.

Throughout this work, we will make use of the following notation. We will write \(B(z, \epsilon)\) for the open Euclidean ball centered at \(z\) of radius \(\epsilon\) and write \(B_Q(z, \epsilon)\) for the ball with respect to \(\overline{d}_Q\).

**Theorem 1.6.** Suppose that \(\mathcal{S} = (\mathbb{S}^2, h)\) is a unit area \(\sqrt{8/3}\)-LQG sphere and that \(z\) is picked uniformly from the quantum measure on \(\mathcal{S}\). Then we have (in probability) that

\[
\frac{\log \text{diam } B_Q(z, \epsilon)}{\log \epsilon} \to 6 \quad \text{as} \quad \epsilon \to 0.
\]

That is, the typical Euclidean diameter of \(B_Q(z, \epsilon)\) for quantum typical \(z\) is \(\epsilon^{6(1+o(1))}\) as \(\epsilon \to 0\). The same also holds if we replace \(\mathcal{S}\) with the unit boundary length \(\sqrt{8/3}\)-LQG disk or a finite mass open subset of a \(\sqrt{8/3}\)-quantum cone.

To put this result in context, recall that a typical radius \(\epsilon\) ball in TBM has Brownian map volume \(\epsilon^4\) and that we expect that TBM can be covered by \(\epsilon^{-4}\) such balls. If
the overall Brownian map has unit area, then among these $\epsilon^{-4}$ balls the *average* ball has to have Euclidean volume of order at least $\epsilon^4$. But “average” and “typical” can be quite different. Theorem 1.6 states that in some sense a *typical* Brownian map ball has Euclidean diameter of order $\epsilon^6$ and hence Euclidean volume of order at most $\epsilon^{12}$, much smaller than this average. Based on this fact it is natural to conjecture that when a random triangulation with $n^4 = N$ triangles is conformally mapped to $S^2$ (with three randomly chosen vertices mapping to three fixed points on $S^2$, say) *most* of the triangles end up with Euclidean volume of order $n^{-12} = N^{-3}$, even though the *average* triangle has Euclidean volume of order $n^{-4} = N^{-1}$.

We remark that there are approximate variants of Theorem 1.6 that could have been formulated without the metric construction of this paper. This is because even before one constructs a metric on $\sqrt{8/3}$-LQG, it is possible to construct a set one would expect to “approximate” a radius $\epsilon$ ball in the random metric: one does this by considering a typical point $x$ and taking the *Euclidean* ball centered at $x$ with radius chosen so that its LQG volume is exactly $\epsilon^4$. Scaling results involving these “approximate metric balls” are derived e.g. in [DS11]. Once Theorems 1.1 and 1.2 are established, Theorem 1.6 is deduced by bounding the extent to which the “approximate metric balls” differ from the actual radius $\epsilon$ balls in the random metric.

1.3 Outline

As partially explained above, the remaining sections of the paper can be divided into three main parts (not counting the open problem list in Section 9):

1. Section 2 provides background definitions and results.

2. Sections 3, 4, and 5 establish the fact that $d_\mathcal{Q}$ a.s. extends uniquely to $d_\mathcal{Q}$ (Theorem 1.1), along with the Hölder continuity of the identity map between $(S^2, d)$ and $(S^2, d_\mathcal{Q})$ (Theorem 1.2), that $d_\mathcal{Q}$ is geodesic (Theorem 1.3), and the scaling exponent describing the Euclidean size of typical small metric balls (Theorem 1.6). These results are proved in Section 5 using estimates derived in Sections 3 and 4.

3. Sections 6, 7, and 8 establish the fact that, when viewed as a random metric measure space, $(S^2, d_\mathcal{Q})$ has the law of TBM (Theorem 1.4). This is proved in Section 8 using estimates derived in Sections 6 and 7.

The reader who mainly wants to know how to interpret an instance of the $\sqrt{8/3}$-LQG sphere as a random metric measure space homeomorphic to the sphere can stop reading after the first two parts. Theorems 1.1, 1.3, and 1.6 provide a way to endow an instance of the $\sqrt{8/3}$-LQG sphere with a metric $d_\mathcal{Q}$ and answer some of the most basic questions about the relationship between $(S^2, d)$ and $(S^2, d_\mathcal{Q})$. These four theorems are already significant and it remains very much an open problem to establish analogs of these
theorems for the \( \gamma \)-LQG sphere when \( \gamma \neq \sqrt{8/3} \). On the other hand, the third part
may be the most interesting for many readers, as this is where the long-conjectured
relationship between TBM and LQG is finally proved.

We conclude this introduction below with Section 1.4, which gives a brief synopsis of
the proof strategies employed in the later parts of the paper, along with summaries of
some of the lemmas and propositions obtained along the way. Section 1.4 is meant as
a sort of road map of the paper, to help the reader keep track of the overall picture
without getting lost, and to provide motivation and context for the many estimates we
require.

1.4 Strategy

1.4.1 Remark on scaling exponents

Throughout this paper, for the sake of intuition, the reader should keep in mind the
“1-2-3-4 rule” of scaling exponents for TBM and for corresponding discrete random
surfaces. Without being too precise, we will try to briefly summarize this rule here,
first in a discrete context. Consider a uniform infinite planar triangulation centered
at a triangle \( y \) and let \( \partial B(y, r) \) denote the outer boundary of the set of triangles in the
dual-graph ball \( B(y, r) \). The rule states that the length of a geodesic from \( y \) to \( \partial B(y, r) \)
is \( r \), the outer boundary length \( |\partial B(y, r)| \) is of order \( r^2 \), the sum \( \sum_{i=0}^{r} |\partial B(y, r)| \) is of
order \( r^3 \), and the volume of \( B(y, r) \) (as well as the volume of the whole region cut off
from \( \infty \) by \( \partial B(y, r) \)) is of order \( r^4 \).

The \( r^3 \) exponent corresponds to the number of triangles explored by the first \( r \) layers
of the peeling process, as presented e.g. in [Ang03]. Also, as explained e.g. in [MS13b,
Section 2], if the vertices of the planar triangulation are colored with i.i.d. coin tosses,
one can define an “outward-reflecting” percolation interface starting at \( y \) and (by com-
parison with the peeling procedure) show that the length of a percolation interface (run
until \( r^4 \) triangles have been cut off from \( \infty \)) is also of order \( r^3 \), while the outer boundary
of the set of triangles in that interface should have length of order \( r^2 \).

The continuum analog of this story is that the Hausdorff dimension \( d_H \) of a set \( S \) on
TBM (defined using the intrinsic metric on TBM) should be

- \( d_H = 1 \) if \( S \) is a geodesic,
- \( d_H = 2 \) if \( S \) is the outer boundary of a metric ball, or the outer boundary of an
  (appropriately defined) SLE\(_6\) curve, or an (appropriately defined) SLE\(_{8/3}\) curve,
- \( d_H = 3 \) if \( S \) is an (appropriately defined) SLE\(_6\) curve itself, or if \( S \) is the union of
  the outer boundaries of balls of radius \( r \) (as \( r \) ranges over an interval of values), and
• $d_H = 4$ if $S$ is an open subset of the entire Brownian map.

Similarly, on an instance of the $\sqrt{8/3}$-LQG sphere, the number of Euclidean balls of quantum area $\delta$ required to cover a geodesic, a metric ball boundary (or SLE$_{8/3}$ curve), an SLE$_6$ curve, and the entire sphere should be respectively of order $\delta^{-1/4}$, $\delta^{-1/2}$, $\delta^{-3/4}$ and $\delta^{-1}$.

We will not prove these precise statements in this paper (though in the case of SLE$_6$ or the entire sphere the scaling dimension follows from the KPZ theorem as stated e.g. in [DS11]). On the other hand, in the coming sections we will endow all of these sets with fractal measures that scale in the appropriate manner: i.e., if one adds a constant to $h$ so that overall volume is multiplied by $C^4$, then geodesic lengths are multiplied by $C$, metric ball boundary lengths are multiplied by $C^2$, and QLE trace measures and SLE$_6$ quantum natural times are both multiplied by $C^3$.

As discussed later in Section 9 (see Problem 9.7) very little is known about how to construct distance functions on $\gamma$-LQG surfaces when $\gamma \neq \sqrt{8/3}$. In particular, the analog of the “1-2-3-4” rule for $\gamma$-LQG surfaces with $\gamma \neq \sqrt{8/3}$ has never been completely worked out. The “1” should presumably remain unchanged (a geodesic always has dimension one) but the “4” should presumably be replaced by the fractal dimension of the surface, which is expected to increase from 2 to 4 continuously as $\gamma$ increases from 0 to $\sqrt{8/3}$ (see [MS13b, Section 3] for further discussion of this point, including a controversial conjectural formula due to Watabiki that applies to all $\gamma \in [0, 2]$). The “3” should be replaced by two possibly distinct values (the quantum dimensions of the QLE($\gamma^2, 0$) trace and of SLE$_{\kappa'}$, both drawn on a $\gamma$-LQG surface, where $\kappa' = 16/\gamma^2$), while the “2” should also be replaced by two possibly distinct values (the quantum dimensions of the outer boundaries of the QLE($\gamma^2, 0$) trace and of SLE$_{\kappa'}$, when each is generated up to a stopping time).

1.4.2 Remark on variants of measures on unit area surfaces

The unit area Brownian map, or unit area $\sqrt{8/3}$-LQG sphere, is not always the easiest or most natural object to work with directly. If one considers a doubly marked unit area surface, together with an SLE$_6$ curve from one endpoint to the other, then the disks cut out by the SLE$_6$ cannot be completely conditionally independent of one another (given their boundary length) because we know that the total sum of their areas has to be 1. To produce a setting where this type of conditional independence does hold exactly, we will often be led to consider either

1. probability measures on the space of infinite volume surfaces, such as the Brownian plane and the (to be shown to be equivalent) $\sqrt{8/3}$-LQG cone with a $\sqrt{8/3}$-log singularity, or
2. *infinite* measures on the space of *finite* volume surfaces, where the law of the total area $A \in (0, \infty)$ is (up to multiplicative constant) an infinite measure given by $A^\alpha dA$ for some $\alpha$, and where once one conditions on a fixed value of $A$, the conditional law of the surface is a rescaled unit area Brownian map or the (to be shown to be equivalent) unit area $\sqrt{8/3}$-LQG sphere.

In order to simplify proofs, we will prove some of our results *first* in the setting where they are easiest and cleanest, and only later transfer them to the other settings. We will do a fair amount of work in the quantum cone setting in Sections 3, 4, and 5, a fair amount of work in the (closely related) quantum wedge setting in Section 6, and a fair amount of work in the “infinite measure on space of finite volume surfaces” setting in Sections 7 and 8.

1.4.3 Strategy for background

This is a long and somewhat technical paper, but many of the estimates we require in later sections can be expressed as straightforward facts about classical objects like the Gaussian free field, Poisson point processes, stable Lévy process, and continuous state branching processes (which can be understood as time-changed stable Lévy processes). In Section 2 we enumerate some of the background results and definitions necessary for the current paper and suggest references in which these topics are treated in more detail.

We begin Section 2 by recalling the definitions of quantum disks, spheres, cones, and wedges, as well as the construction of quantum Loewner evolution given in [MS13b]. We next make an elementary observation: that the proof of the standard Kolmogorov-Čentsov theorem — which states that a.s. $\gamma$-Hölder continuity of a random field $X_u$, indexed by $u \in [0,1]^d$, can be deduced from estimates on moments of $|X_u - X_v|$ — can be adapted to bound the law of the corresponding $\gamma$-Hölder norm. We then proceed to give some bounds on the probability that maximal GFF circle averages are very large. We finally define continuous state branching processes and present a few facts about them to be used later, along with some basic observations about stable Lévy processes and Poisson point processes.

1.4.4 Strategy for constructing metric and proving Hölder continuity

We will consider a QLE$(8/3, 0)$ process $\Gamma_r$ (a random increasing family of closed sets indexed by $r$) defined on a certain infinite volume quantum surface called a quantum cone. To establish the desired Hölder continuity, we will need to control the law of the amount of time it takes a QLE growth started at a generic point $x_i$ to reach a generic point $x_j$, and to show that, in some appropriate local sense, these random quantities
can a.s. be uniformly bounded above and below by random constants times appropriate powers of $|x_i - x_j|$. 

To this end, we begin by establishing some control on how the Euclidean diameter of $\Gamma_r$ (started at zero) changes as a function of $r$. We do not a priori have a very simple way to describe the growth of the Euclidean diameter of $\Gamma_r$ as a function of $r$. On the other hand, based on the results in [MS15b, MS13b], we do have a simple way to describe the evolution of the boundary length of $\Gamma_r$, which we denote by $B_r$, and the evolution of the area cut off from $\infty$ by $\Gamma_r$, which we denote by $A_r$. These processes can be described using the continuous state branching processes discussed in Section 2.

Sections 3, 4, and 5 are a sort of a dance in which one first controls the most accessible relationships (between $r$, $A_r$ and $B_r$) and other reasonably accessible relationships (between Euclidean and quantum areas of Euclidean disks, or between Euclidean and quantum lengths of boundary intervals — here uniform estimates are obtained from basic information about the GFF) and then combines them to address the a priori much less accessible relationship between $r$ and the Euclidean diameter of $\Gamma_r$, and then uses this to address the general relationship between $|x_i - x_j|$ and the amount of time it takes for a branching QLE exploration to get from $x_i$ to $x_j$.

As explained in Section 1.4.1 one would expect $B_r$ to be of order $r^2$, and it is natural to expect

$$\sup_{0 \leq s \leq r} B_s$$

(1.3)

to also be of order $r^2$. Similarly, as explained in Section 1.4.1 we expect $A_r$ to be of order $r^4$. In Section 3 we obtain three important results:

1. Lemma 3.1 uses standard facts about continuous state branching processes to bound the probability that (1.3) is much larger or smaller than $r^2$.

2. Lemma 3.2 uses standard facts about CSBPs to bound the probability that $A_r$ is much smaller than $r^4$.

3. Proposition 3.4 uses simple Gaussian free field estimates to put a lower bound on the probability of the event that (within a certain region of an appropriately embedded quantum cone) the quantum mass of every Euclidean ball is at most some universal constant times a power of that ball’s radius. In what follows, it will frequently be useful to truncate on this event — i.e., to prove bounds conditioned on this event occurring.

Section 4 uses the estimates from Section 3 to begin to relate $r$ and the Euclidean diameter of $\Gamma_r$. There are a number of incremental lemmas and propositions used internally in Section 4 but the results cited in later sections are these:
1. Propositions 4.1 and 4.2 begin the game of relating \( r \) and the Euclidean diameter of \( \Gamma_r \). Proposition 4.1 states that \textit{on the event} described in Proposition 3.4, the Euclidean diameter of \( \Gamma_r \) is very unlikely to be less than some power of \( r \), and Proposition 4.2 states that (without any truncation) the Euclidean diameter of \( \Gamma_r \) is very unlikely to more than some other power of \( r \). (In fact, under a certain truncation, a bound on the fourth moment of diam(\( \Gamma_r \)) is given.) To show that \( \Gamma_r \) is unlikely to have small Euclidean diameter, one applies the bounds from Section 3 in a straightforward way. (If \( \Gamma_r \) had small Euclidean diameter, then either \( A_r \) would be unusually small or a small Euclidean-diameter region would have an unusually large amount of quantum mass, both scenarios that were shown in Sections 3 to be improbable.) To show that \( \Gamma_r \) is unlikely to have large Euclidean diameter, the hard part is to rule out the possibility that \( \Gamma_r \) has large diameter despite having only a moderate amount of quantum area — perhaps because it has lots of long and skinny tentacles. On the other hand, we understand the law of the quantum surface that forms the complement of \( \Gamma_r \) (it is independent of the surface cut off by \( \Gamma_r \) itself, given the boundary length) and can use this to show (after some work) that these kinds of long and skinny tentacles do not occur.

2. Corollary 4.3 (which follows from Propositions 4.1 and 4.2) implies that the total quantum area cut off by \( \Gamma_r \) has a certain power law decay on the special event from Proposition 3.4. (The power law exponent one obtains after truncating on this event is better than the one that can be derived using the direct relationship between \( r \) and \( A_r \) without this truncation.)

3. Proposition 4.4 shows that when \( h \) is an appropriately normalized GFF with free boundary conditions, the boundary length measure is \textit{very} unlikely to be much smaller than one would expect it to be.

4. Lemma 4.6 (used in the proof of Proposition 4.4, as well as later on) is an elementary but useful tail bound on the maximum (over a compact set \( K \)) of the projection of the Gaussian free field onto the space of functions harmonic on some \( U \supseteq K \).

In Section 5 we use the estimates from Section 4 to show that the QLE(8/3, 0) metric extends to a function which is Hölder continuous with respect to the Euclidean metric. This will allow us to prove Theorem 1.1 and Theorem 1.2. We will also give the proof of Theorem 1.3 and Theorem 1.6 in Section 5.

### 1.4.5 Strategy for proving metric measure space has law of TBM

Sections 6, 7, and 8 will show that the law of \( (\mathbb{S}^2, \mathcal{d}_Q) \) is the law of TBM. They will do this by making use of the axiomatic characterization of TBM given in [MS15a]. Let us recall some notation and results from [MS15a].
A triple \((S,d,\nu)\) is called a **metric measure space** (or **mm-space**) if \((S,d)\) is a complete separable metric space and \(\nu\) is a measure on the Borel \(\sigma\)-algebra generated by the topology generated by \(d\), with \(\nu(S) \in (0, \infty)\). We remark that one can represent the same space by the quadruple \((S,d,\tilde{\nu},m)\), where \(m = \nu(S)\) and \(\tilde{\nu} = m^{-1}\nu\) is a probability measure. This remark is important mainly because some of the literature on metric measure spaces requires \(\nu\) to be a probability measure. Relaxing this requirement amounts to adding an additional parameter \(m \in (0, \infty)\).

Two metric measure spaces are considered equivalent if there is a measure-preserving isometry from a full measure subset of one to a full measure subset of the other. Let \(\mathcal{M}\) be the space of equivalence classes of this form. Note that when we are given an element \((S,d,\nu)\) of \(\mathcal{M}\), we have no information about the behavior of \(S\) away from the support of \(\nu\).

Next, recall that a measure on the Borel \(\sigma\)-algebra of a topological space is called **good** if it has no atoms and it assigns positive measure to every open set. Let \(\mathcal{M}_{\text{SPH}}\) be the space of geodesic metric measure spaces that can be represented by a triple \((S,d,\nu)\) where \((S,d)\) is a geodesic metric space homeomorphic to the sphere and \(\nu\) is a good measure on \(S\).

Note that if \((S_1,d_1,\nu_1)\) and \((S_2,d_2,\nu_2)\) are two such representatives, then the a.e. defined measure-preserving isometry \(\phi: S_1 \to S_2\) is necessarily defined on a dense set, and hence can be extended to the completion of its support in a unique way so as to yield a continuous function defined on all of \(S_1\) (similarly for \(\phi^{-1}\)). Thus \(\phi\) can be uniquely extended to an everywhere defined measure-preserving isometry. In other words, the metric space corresponding to an element of \(\mathcal{M}_{\text{SPH}}\) is uniquely defined, up to measure-preserving isometry.

As we are ultimately interested in probability measures on \(\mathcal{M}\), we will need to describe a \(\sigma\)-algebra \(\mathcal{F}\) on \(\mathcal{M}\), and more generally a \(\sigma\)-algebra \(\mathcal{F}_k\) on elements of \(\mathcal{M}\) with \(k\) marked points. We will also need that \(\mathcal{M}_{\text{SPH}}\) belongs to that \(\sigma\)-algebra, so that in particular it makes sense to talk about measures on \(\mathcal{M}\) that are supported on \(\mathcal{M}_{\text{SPH}}\). We would like to have a \(\sigma\)-algebra that can be generated by a complete separable metric, since this would allow us to define regular conditional probabilities for all subsets. Such a \(\sigma\)-algebra is introduced in [MS15a].

Let \(\mathcal{M}_{\text{SPH}}^2\) denote the space of sphere-homeomorphic metric measure spaces with two distinct marked points \(x\) and \(y\). Given an element of this space, one can consider the union of the boundaries \(\partial B^*(x,r)\) taken over all \(r \in [0,d(x,y)]\), where \(B^*(x,r)\) is the set of all points cut off from \(y\) by the closed metric ball \(\overline{B(x,r)}\). (That is, \(B^*(x,r)\) is the complement of the component of \(\overline{B(x,r)}\) containing \(y\).) This union is called the **metric net** from \(x\) to \(y\) and it comes equipped with certain structure (e.g., there is a distinguished leftmost geodesic from any point on the net back to \(x\)). When \(\mathcal{M}_{\text{SPH}}^2\) is an instance of the doubly marked Brownian map, the metric net is the so-called \(\alpha\)-stable Lévy net, as defined in [MS15a Section 3.3], with \(\alpha = 3/2\).
In fact multiple equivalent constructions of the \( \alpha \)-stable Lévy net appear in [MS15a, Section 3]. (See Figure 1.1 for an informal description of the Lévy net.) We now cite the following from [MS15a, Theorem 4.6].

![Diagram of a doubly-marked sphere](image)

Figure 1.1: Shown is a doubly-marked sphere \((S, x, y)\) equipped with a metric \(d\). We assume that, for each \(r \in (0, d(x, y))\), \(\partial B^\bullet(x, d(x, y) - r)\) comes equipped with a boundary length measure. For a fixed value of \(r \in (0, d(x, y))\), the points \(x_1, x_2\) shown in the illustration are assumed to be sampled from the boundary measure on \(\partial B^\bullet(x, d(x, y) - r)\) and the red paths are leftmost geodesics from \(x_1, x_2\) back to \(x\).

Roughly, the metric net of \((S, x, y)\) from \(x\) to \(y\) has the law of the \(3/2\)-Lévy net if it is the case that boundary lengths of the clockwise and counterclockwise segments of \(\partial B^\bullet(x, d(x, y) - (r + s))\) between the leftmost geodesics from \(x_1, x_2\) back to \(x\) evolve as independent \(3/2\)-stable CSBPs as \(s\) varies in \([0, d(x, y) - r]\). The main focus of Section 8 is to show that the metric net associated with a \(\sqrt{8/3}\)-LQG sphere has the law of a \(3/2\)-Lévy net.

**Theorem 1.7.** The doubly marked Brownian map measure \(\mu_{\text{SPH}}^2\) is the unique (infinite) measure on \((\mathcal{M}_{\text{SPH}}^2, \mathcal{F})\) which satisfies the following properties, where an instance is denoted by \((S, d, \nu, x, y)\).

1. Given \((S, d, \nu)\), the conditional law of \(x\) and \(y\) is that of two i.i.d. samples from \(\nu\). In other words, the law of the doubly marked surface is invariant under the Markov step in which one “forgets” \(x\) (or \(y\)) and then resamples it from the given measure.

2. The law of the metric net from \(x\) to \(y\) (an infinite measure) agrees with the law of a \(3/2\)-Lévy net. More precisely: the metric net of \((S, d, x, y)\) can be coupled with the \(3/2\)-Lévy net in such a way that there is a.s. a unique homeomorphism between the two doubly marked topological spaces such that under this homeomorphism all of the distinguished left and right geodesics in the Lévy net map to actual geodesics (of the same length) in \((S, d)\).
3. Fix $r > 0$ and consider the circle that forms the boundary $\partial B^*(x, r)$ (an object that is well-defined a.s. on the finite-measure event that the distance from $x$ to $y$ is at least $r$). Then the inside and outside of $B^*(x, r)$ (each viewed as a marked metric measure space) are conditionally independent, given the boundary length of $\partial B^*(x, r)$ (as defined from the Lévy net structure).

The ultimate goal of Sections 6, 7, and 8 is to show that the metric measure space we construct using QLE satisfies the conditions of Theorem 1.7.

- The fact that our metric space is topologically a sphere and that the map is a Hölder continuous homeomorphism is proved in Sections 3–5.
- It is obvious from our construction of the doubly marked $\sqrt{8/3}$-LQG sphere that its law is preserved by the operation of forgetting the points $x$ and $y$ and resampling them independently from the underlying measure.
- The independence of the inside and outside of the filled metric ball follows from the construction of QLE($8/3, 0$) given in [MS15b], but care is needed to deal with a distinction between forward and reverse explorations, see Section 7.
- The fact that the metric net has the law of a $3/2$-Lévy net is proved in Section 8.

In order to do this, we will recall that some hints of the relationship with TBM, and more specifically with the $3/2$-Lévy net, were already present in [MS15b]. One can define the “outer boundary length” process for growing QLE clusters and for growing Brownian map metric balls, and it was already shown in [MS15b] that both of these processes can be understood as continuous state branching process excursions, and that their laws agree. In both cases, the “jumps” correspond to times at which disks of positive area are “swallowed” by the growing process; these disks are removed from the “unexplored region” at these jump times. In both cases, it is possible to reverse the “unexplored region” process so that disks of positive area are “glued on” (at single “pinch points”) at these jump times, and in both cases one can show that the location of the pinch point is uniformly random, conditioned on all that has happened before. One can use this to generate a coupling between the Lévy net and QLE. However, it is not obvious that the geodesic paths of the Lévy net actually correspond to geodesics of $\mathcal{d}_Q$. This is the part that takes a fair amount of work and requires the analysis of a sequence of geodesic approximations.

In Section 6, we will prove moment bounds for the quantum distance between the initial point and tip of an SLE$_6$ on a $\sqrt{8/3}$-quantum wedge as well as between two boundary points on a $\sqrt{8/3}$-quantum wedge separated by a given amount of quantum length. These bounds will be used later to control the law of the length of certain geodesic approximations.
In Section 7 we will describe the time-reversal of the SLE$_6$ and QLE($8/3, 0$) unexplored-domain processes and deal with some technicalities regarding time reversal definitions. The QLE definition on an LQG sphere involves “reshuffling” every $\delta$ units of time during a certain time interval $[0, T]$ parameterizing a Lévy process excursion; but technically speaking if $T$ is random and not necessarily a multiple of $\delta$, it makes a difference whether one marks the increments starting from 0 (so their endpoints are $\delta, 2\delta, \ldots$) or starting from $T$ (so that endpoints are $T - \delta, T - 2\delta, \ldots$). Part of the purpose of Section 7 is to show that (unsurprisingly) this subtle distinction does not matter in the limit.

Finally Section 8 will use the results of Sections 6 and 7 to control various geodesic approximations and ultimately show that the geodesics of $d_Q$ correspond to the Lévy net in the expected way. This will enable us to complete the proofs of Theorem 1.4 and Corollary 1.5.

2 Preliminaries

The purpose of this section is to review some background and to establish a number of preliminary estimates that will be used to prove our main theorems. We begin in Section 2.1 by reminding the reader of the construction of quantum disks, spheres, cones, and wedges. We will then construct QLE($8/3, 0$) on a $\sqrt{8/3}$-quantum cone in Section 2.2. This process is analogous to the QLE($8/3, 0$) process constructed in [MS15b] on a $\sqrt{8/3}$-LQG sphere. Next, we will establish a quantitative version of the Kolmogorov-Čentsov theorem in Section 2.3. Then, in Section 2.4, we will use the results of Section 2.3 to bound the extremes of the GFF. Finally, we record a few basic facts about continuous state branching processes in Section 2.5, an estimate of the tail of the supremum of an $\alpha$-stable process in Section 2.6.1, and an estimate of the tail of the Poisson distribution in Section 2.6.2.

2.1 Quantum disks, spheres, cones, and wedges

The purpose this section is to give a brief overview of the construction of quantum disks, spheres, cones, and wedges. We refer the reader to [DMS14, Section 4] for a much more in depth discussion of these objects. See also the discussion in [She15, MS15c].

Suppose that $h$ is an instance of the Gaussian free field (GFF) on a planar domain $D$ and $\gamma \in (0, 2)$. The $\gamma$-LQG measure associated with $h$ is formally given by $e^{\gamma h(z)} dz$ where $dz$ denotes Lebesgue measure on $D$. Since $h$ does not take values at points, it is necessary to use a regularization procedure in order to make sense of this expression rigorously. This has been accomplished in [DS11], for example, by considering the approximation $e^{\gamma^2/2}e^{h_{\epsilon}(z)} dz$ where $h_{\epsilon}(z)$ denotes the average of $h$ on $\partial B(z, \epsilon)$ and $e^{\gamma^2/2}$
is the normalization factor which is necessary for the limit to be non-trivial. A marked quantum surface is an equivalence class of triples consisting of a domain $D$, a vector of points $z \in D$, and a distribution $h$ on $D$ where two triples $(D, h, z)$ and $(\tilde{D}, \tilde{h}, \tilde{z})$ are said to be equivalent if there exists a conformal transformation $\varphi: D \to \tilde{D}$ which takes each element of $z$ to the corresponding element of $\tilde{z}$ and such that $h = \tilde{h} \circ \varphi + Q \log |\varphi'|$ where $Q = \frac{2}{\gamma} + \frac{\gamma^2}{2}$. We will refer to a particular choice of representative of a marked quantum surface as its embedding. In order to specify the law of a marked quantum surface, we only have to specify the law of $h$ with one particular choice of embedding.

Throughout, we consider the infinite strip $S = \mathbb{R} \times [0, \pi]$ and the infinite cylinder $Q = \mathbb{R} \times [0, 2\pi]$ (with the top and the bottom identified). We denote by $Q_{\pm} = \mathbb{R}_{\pm} \times [0, 2\pi]$ (with the top and bottom identified) the positive and negative half-infinite cylinders. For $X \in \{S, Q, Q_{\pm}, C, H\}$, we let $H(X)$ be the closure of $C^\infty_0(X)$ with respect to the Dirichlet inner product

$$((f, g)_\nabla = \frac{1}{2\pi} \int \nabla f(x) \cdot \nabla g(x) dx.$$ \hfill (2.1)

For $X \in \{S, Q, Q_{\pm}\}$, we note that $H(X)$ admits the orthogonal decomposition $H_1(X) \oplus H_2(X)$ where $H_1(X)$ (resp. $H_2(X)$) consists of those functions on $X$ which are constant (resp. have mean zero) on vertical lines; see, e.g. [DMS14, Lemma 4.2]. For $X = C$, we have that $H(C)$ admits the orthogonal decomposition $H_1(C) \oplus H_2(C)$ where $H_1(C)$ (resp. $H_2(C)$) consists of those functions on $C$ which are radially symmetric about 0 (resp. have mean zero on circles centered at 0). The same is likewise true for $H(H)$ except with circles centered at 0 replaced by semicircles centered at 0.

The starting point for the construction of the unit boundary length quantum disk as well as the unit area quantum sphere is the infinite excursion measure $\nu_B^{\text{BES}}$ associated with the excursions that a Bessel process of dimension $\delta$ ($\text{BES}^{\delta}$) makes from 0 for $\delta \in (0, 2)$. This measure can be explicitly constructed as follows.

- Sample a lifetime $t$ from the infinite measure $c_\delta t^{\delta/2-2} dt$ where $dt$ denotes Lebesgue measure on $\mathbb{R}_+$ and $c_\delta > 0$ is a constant.

- Given $t$, sample a $\text{BES}^{4-\delta}$ bridge from 0 to 0 of length $t$.

The law of a $\text{BES}^{\delta}$ process with $\delta \in (0, 2)$ can then be sampled from by first picking a Poisson point process (p.p.p.) $\Lambda$ with intensity measure $dud\nu_\delta$ where $du$ denotes Lebesgue measure on $\mathbb{R}_+$ and then concatenating together the elements $(u, e) \in \Lambda$ ordered by $u$. It is still possible to sample a p.p.p. $\Lambda$ as above when $\delta \leq 0$, however it is not possible to concatenate together the elements of $\Lambda$ in chronological order to form a continuous process because there are too many short excursions. (See [PY82] as well as the text just after [PY96, Theorem 1].)
2.1.1 Quantum disks

As explained in [DMS14, Definition 4.21], one can use $\nu_\delta^{\text{BES}}$ to define an infinite measure $\mathcal{M}$ on quantum surfaces $(\mathcal{S}, h)$ as follows.

- Take the projection of $h$ onto $\mathcal{H}_1(\mathcal{S})$ to be given by $2\gamma^{-1}\log Z$ where $Z$ is sampled from $\nu_\delta^{\text{BES}}$ with $\delta = 3 - \frac{4}{\gamma}$, reparameterized (by all of $\mathbb{R}$) to have quadratic variation $2du$.

- Take the projection of $h$ onto $\mathcal{H}_2(\mathcal{S})$ to be given by the corresponding projection of a free boundary GFF on $\mathcal{S}$ sampled independently of $Z$.

The above construction defines a doubly marked quantum surface parameterized by the infinite cylinder; however it only determines $h$ up to a free parameter corresponding to “horizontal translation.” We will choose this horizontal translation depending on the context.

If we condition $\mathcal{M}$ on the quantum boundary length being equal to 1, then we obtain the law of the unit boundary length quantum disk. More generally, we can sample from the law of $\mathcal{M}$ conditioned on having quantum boundary length equal to $L$ by first sampling from the law of the unit boundary length quantum disk and then adding $\frac{2}{\gamma} \log L$ to the field. We will denote this law by $M_{\text{DISK}}^L$. The points which correspond to $\pm \infty$ are independently and uniformly distributed according to the quantum boundary length measure conditional on $\mathcal{S}$ [DMS14, Proposition 5.11]. The law $M_{\text{DISK}}^L$ is obtained by weighting $M_{\text{DISK}}^L$ by its quantum area. This corresponds to adding an extra marked point which is uniformly distributed from the quantum measure.

2.1.2 Quantum spheres

As is also explained in [DMS14, Definition 4.21], one can use $\nu_\delta^{\text{BES}}$ to define an infinite measure $\mathcal{M}_{\text{BES}}$ on doubly-marked quantum surfaces $(\mathcal{Q}, h, -\infty, +\infty)$ as follows.

- Take the projection of $h$ onto $\mathcal{H}_1(\mathcal{Q})$ to be given by $2\gamma^{-1}\log Z$ where $Z$ is sampled from $\nu_\delta^{\text{BES}}$ with $\delta = 4 - \frac{8}{\gamma^2}$, reparameterized to have quadratic variation $du$.

- Take the projection of $h$ onto $\mathcal{H}_2(\mathcal{Q})$ to be given by the corresponding projection of a whole-plane GFF on $\mathcal{Q}$ sampled independently of $Z$.

As in the case of quantum disks, we have not yet fully specified $h$ as a distribution on the infinite cylinder because there is still one free parameter which corresponds to the “horizontal translation.” We will choose this horizontal translation depending on the context.
If we condition on the quantum area associated with $M_{\text{BES}}$ to be equal to 1, then we obtain the law of the unit area quantum sphere. Given $S$, the points which correspond to $\pm \infty$ are uniformly and independently distributed according to the quantum measure [DMS14, Proposition 5.15].

As explained in [MS15c], in the special case that $\gamma = \sqrt{8/3}$ the measure $M_{\text{BES}}$ admits another description in terms of the infinite excursion measure for a $3/2$-stable Lévy process with only upward jumps from its running infimum; see [Ber96] for more details on this measure. In this construction, one uses that if we start off with a quantum sphere sampled from $M_{\text{BES}}$ and then draw an independent whole-plane SLE$_6$ process $\eta'$ from $-\infty$ to $+\infty$, then the law of ordered, oriented (by whether $\eta'$ traverses the boundary points in clockwise or counterclockwise order — i.e., whether the loop is on the left or right side of $\eta'$), and marked (last point on the disk boundary visited by $\eta'$) disks cut out by $\eta'$ can be sampled from as follows:

- Sample an excursion $e$ from the infinite excursion measure for $3/2$-stable Lévy processes with only upward jumps from its running infimum. (The time-reversal $e(T - \cdot)$ of $e: [0, T] \rightarrow \mathbb{R}_+$ at time $t$ is equal to the quantum boundary length of the component of $S \setminus \eta'([0, t])$ which contains $y$.)
- For each jump of $e$, sample a conditionally independent quantum disk whose boundary length is equal to the size of the jump.
- Orient the boundary of each quantum disk either to be clockwise or counterclockwise with the toss of a fair coin flip and mark the boundary of each with a uniformly chosen point from the quantum measure.

Moreover, it is shown in [MS15c] that the information contained in the doubly-marked sphere and $\eta'$ can be uniquely recovered from the ordered collection of marked and oriented disks.

A quantum sphere produced from $M_{\text{BES}}$ is doubly marked. If we parameterize the surface by $Q$ as described above, the marked points are located at $\pm \infty$. In general, we will indicate such a doubly marked quantum sphere with the notation $(S, x, y)$ where $S$ denotes the quantum surface and $x, y$ are the marked points and we will indicate the corresponding measure by $M_{\text{SPH}}^2$.

### 2.1.3 Quantum cones

Fix $\alpha < Q$. An $\alpha$-quantum cone [DMS14, Section 4.3] is a doubly marked quantum surface which is homeomorphic to $C$. The two marked points are referred to as the “origin” and “infinity.” Bounded neighborhoods of the former all almost surely contain a finite amount of mass and neighborhoods of the latter contain an infinite amount of mass. It is convenient to parameterize a quantum cone by either $Q$ or $C$, depending on
the context. In the former case, we will indicate the quantum cone with the notation \((Q, h, -\infty, +\infty)\) (meaning that \(-\infty\) is the origin and \(+\infty\) is infinity) and the law of \(h\) can be sampled from by:

- Taking the projection of \(h\) onto \(\mathcal{H}_1(Q)\) to be given by \(2\gamma^{-1}\log Z\) where \(Z\) is a BES\(\delta\) with \(\delta = 2 + \frac{1}{\gamma}(Q - \alpha)\), reparameterized to have quadratic variation \(du\).

- Taking the projection of \(h\) onto \(\mathcal{H}_2(Q)\) to be given by the corresponding projection of a whole-plane GFF on \(Q\).

It is often convenient in the case of quantum cones to take the horizontal translation so that the projection of \(h\) onto \(\mathcal{H}_1(Q)\), which can be understood as a function of one real variable (since it is constant on vertical line segments), last hits 0 on the line \(\text{Re}(z) = 0\).

When \(h\) is an instance of the GFF, the projection of \(h\) onto \(\mathcal{H}_1(Q)\) is (as a function of the horizontal coordinate) a Brownian motion with drift. In order to construct an \(h\) that corresponds to an instance of the quantum cone, we can take the projection onto \(\mathcal{H}_1(Q)\) to be as follows:

- For \(u < 0\), it is equal to \(B_{-u} + (Q - \alpha)u\) where \(B\) is a standard Brownian motion with \(B_0 = 0\).
- For \(u \geq 0\), it is equal to \(\tilde{B}_u + (Q - \alpha)u\) where \(\tilde{B}\) is a standard Brownian motion independent of \(B\) conditioned so that \(\tilde{B}_u + (Q - \alpha)u \geq 0\) for all \(u \geq 0\).

The definition of \(\tilde{B}\) involves conditioning on an event with probability zero, but it is explained in [DMS14, Remark 4.3], for example, how to make sense of this conditioning rigorously.

If we parameterize by \(C\) instead of \(Q\), we first sample the process \(A_u\) by:

- For \(u > 0\) taking it to be \(B_u + \alpha u\) where \(B\) is a standard Brownian motion with \(B_0 = 0\).
- For \(u \leq 0\) taking it to be \(\tilde{B}_{u} + \alpha u\) where \(\tilde{B}\) is a standard Brownian motion with \(\tilde{B}_0 = 0\) conditioned so that \(\tilde{B}_u + (Q - \alpha)u > 0\) for all \(u \geq 0\).

Then we take the projection of \(h\) onto \(\mathcal{H}_1(C)\) to be equal to \(A_{e^{-u}}\) and the projection of \(h\) onto \(\mathcal{H}_2(C)\) to be the corresponding projection for a whole-plane GFF. We will use the notation \((C, h, 0, \infty)\) for a quantum cone parameterized by \(C\) where 0 (resp. \(\infty\)) is the origin (resp. infinity).

We will refer to the particular embedding of a quantum cone into \(C\) described just above as the \textit{circle average embedding}.
As explained in [DMS14, Theorem 1.18], it is natural to explore a $\sqrt{8/3}$-quantum cone (parameterized by $C$) with an independent whole-plane SLE$_6$ process $\eta'$ from 0 to $\infty$. If one parameterizes $\eta'$ by quantum natural time [DMS14], then the quantum boundary length of the unbounded component of $C \setminus \eta'([0, t])$ evolves in $t$ as a $3/2$-stable Lévy process with only downward jumps conditioned to be non-negative [DMS14, Corollary 12.2]. (See [Ber96, Chapter VII, Section 3] for more details on the construction of a Lévy process with only downward jumps conditioned to be non-negative. In particular, [Ber96, Chapter VII, Proposition 14] gives the existence of the process started from 0.) Moreover, the surface parameterized by the unbounded component of $C \setminus \eta'([0, t])$ given its quantum boundary length is conditionally independent of the surfaces cut off by $\eta'|_{[0,t]}$ from $\infty$. If the quantum boundary length is equal to $u$, then we will write this law as $m^u$. By scaling, we can sample from the law of $m^u$ by first sampling from the law $m^1$ and then adding the constant $2\gamma^{-1}\log u$, $\gamma = \sqrt{8/3}$, to the field. (One can think of a sample produced from $m^u$ as corresponding to a quantum disk with boundary length equal to $u$ and conditioned on having infinite quantum area.)

It is also shown in [DMS14] that it is natural to explore a $\gamma$-quantum cone $(C, h, 0, \infty)$ with a space-filling SLE$_{\kappa'}$ process $\eta'$ [MS13a] from $\infty$ to $\infty$ which is sampled independently of the quantum cone and then reparameterized by quantum area, i.e., so that $\mu_h(\eta'(s,t)) = t-s$ for all $s < t$ and normalized so that $\eta'(0) = 0$. It is in particular shown in [DMS14, Theorem 1.13] that the joint law of $h$ and $\eta'$ is invariant under the operation of translating so that $\eta'(t)$ is taken to 0. That is, as doubly-marked path-decorated quantum surfaces we have that

$$(h, \eta') \overset{d}{=} (h(\cdot - \eta'(t)), \eta'(\cdot - t) - \eta'(t))$$

This fact will be important for us in several places in this article.

### 2.1.4 Quantum wedges

Fix $\alpha < Q$. An $\alpha$-quantum wedge [DMS14, Section 4.2] (see also [She15]) is a doubly-marked surface which is homeomorphic to $H$. As in the case of a quantum cone, the two marked points are the origin and infinity. It is natural to parameterize a quantum wedge either by $S$ or by $H$. In the former case, we can sample from the law of the field $h$ by:

- Taking its projection onto $\mathcal{H}_1(S)$ to be given by $2\gamma^{-1}\log Z$ where $Z$ is a BES$^\delta$ with $\delta = 2 + \frac{2}{3}(Q - \alpha)$ reparameterized to have quadratic variation $2du$.

- Taking its projection onto $\mathcal{H}_2(S)$ to be given by the corresponding projection of a GFF on $S$ with free boundary conditions.

As in the case of an $\alpha$-quantum cone, we can also describe the projection of $h$ onto $\mathcal{H}_1(S)$ in terms of Brownian motion [DMS14, Remark 4.5]. In fact, the definition is
the same as for an $\alpha$-quantum cone except with $B_u, \~B_u$ replaced by $B_{2u}, \~B_{2u}$. (The variance is twice as much because the strip is half as wide as the cylinder.)

If we parameterize the surface with $H$, then we can sample from the law of the field $h$ by (see [DMS14, Definition 4.4]):

- Taking its projection onto $H_1(H)$ to be given by $A e^{-u}$ where $A$ is as in the definition of an $\alpha$-quantum cone parameterized by $C$ except with $B_u, \~B_u$ replaced by $B_{2u}, \~B_{2u}$.
- Taking its projection onto $H_2(H)$ to be given by the corresponding projection of a GFF on $H$ with free boundary conditions.

2.2 QLE(8/3, 0) on a $\sqrt{8/3}$-quantum cone

In [MS15b, Section 6], we constructed a “quantum natural time” [DMS14] variant of the QLE(8/3, 0) process from [MS13b] on a $\sqrt{8/3}$-LQG sphere and showed that this process defines a metric on a countable, dense set of points chosen i.i.d. from the quantum area measure on the sphere. In many places in this article, it will be convenient to work on a $\sqrt{8/3}$-quantum cone instead of a $\sqrt{8/3}$-LQG sphere. We will therefore review the construction and the basic properties of the process in this context. We will not give detailed proofs here since they are the same as in the case of the $\sqrt{8/3}$-LQG sphere. We refer the reader to [MS15b, Section 6] for additional detail.

We suppose that $(C, h, 0, \infty)$ is a $\sqrt{8/3}$-quantum cone and that $\eta'$ is a whole-plane SLE$_6$ from 0 to $\infty$ sampled independently of $h$ and then reparameterized by quantum natural time. Fix $\delta > 0$. We define the $\delta$-approximation of QLE(8/3, 0) starting from 0 as follows. First, we take $\Gamma^\delta_t$ to be the complement of the unbounded component of $C \setminus \eta'(\{0, t\})$ for each $t \in [0, \delta]$. We also let $g_t^\delta : C \setminus \Gamma^\delta_t \to C \setminus D$ be the unique conformal map with $|g_t^\delta(z) - z| \to 0$ as $z \to \infty$. Fix $j \in \mathbb{N}$ and suppose that we have defined paths $\eta'_1, \ldots, \eta'_j$ and a growing family of hulls $\Gamma^\delta_t$ with associated uniformizing conformal maps $(g_t^\delta)$ for $t \in [0, j\delta]$ such that the following hold:

- The conditional law of the surface parameterized by the complement of $\Gamma^\delta_{j\delta}$ given its quantum boundary length $\ell$ is the same as in the setting of exploring a $\sqrt{8/3}$-quantum cone with an independent whole-plane SLE$_6$. That is, it is given by $m^\ell$.
- $\eta'_j/j\delta$ is distributed uniformly according to the quantum boundary measure on $\partial \Gamma^\delta_{j\delta}$ conditional on $\Gamma^\delta_{j\delta}$ (as a path decorated quantum surface).
- The joint law of the components (viewed as quantum surfaces) separated from $\infty$ by time $j\delta$, given their quantum boundary lengths, is the same as in the case of whole-plane SLE$_6$. That is, they are given by conditionally independent quantum disks given their boundary lengths and their boundary lengths correspond
Figure 2.1: **Left:** Independent whole-plane SLE$_6$ from 0 to $\infty$ drawn on top of a $\sqrt{8/3}$-quantum cone. **Middle:** We can represent the path-decorated surface as a collection of $\delta$-quantum natural time length necklaces which serve to encode the bubbles cut off by the SLE$_6$ in each of the $\delta$-length intervals of time. Each necklace has an inner and an outer boundary, is doubly marked by the initial and terminal points of the SLE$_6$, the necklaces are conditionally independent given their inner and outer boundary lengths, and each necklace is almost surely determined by the collection of marked and oriented bubbles cut off by the SLE$_6$ in the corresponding time interval. The length of the outer boundary of each necklace is equal to the length of the inner boundary of the next necklace. If we glue together the necklaces as shown, then we recover the $\sqrt{8/3}$-quantum cone decorated by the independent SLE$_6$. **Right:** If we “rotate” each of the necklaces by a uniformly random amount and then glue together as shown, the underlying surface is a $\sqrt{8/3}$-quantum cone which is decorated with the $\delta$-approximation to QLE($8/3, 0$). The left and right pictures are naturally coupled together so that the bubbles cut out by the SLE$_6$ and QLE($8/3, 0$) are the same as quantum surfaces and the evolution of the boundary length of both is the same, up to a time-change.

We then let $\eta'_{j+1}$ be an independent radial SLE$_6$ starting from a point on $\partial \Gamma^\delta_j$ which is chosen uniformly from the quantum boundary measure conditionally independently of everything else (i.e., we resample the location of the tip $\eta'_j(j\delta)$ of $\eta'_j$). For each $t \in [j\delta, (j + 1)\delta]$, we also let $\Gamma^t$ be the complement of the unbounded component of $C \setminus (\Gamma^\delta_{j\delta} \cup \eta'_{j+1}([0, t]))$. Then by the construction, all three properties described above
are satisfied by the process up to time \((j + 1)\delta\).

By repeating the compactness argument given in [MS15b, Section 6], we see that there exists a sequence \((\delta_k)\) which tends to 0 as \(k \to \infty\) along which the \(\delta\)-approximations converge and the limiting process satisfies properties which are analogous to the three properties described above.

We note that it is shown in [MS15b] that if \((x_n)\) is a sequence of points chosen i.i.d. from the quantum measure on a \(\sqrt{8/3}\)-LQG sphere, then the joint law of the hitting times of the \((x_n)\) by the subsequentially limiting QLE(8/3, 0) does not depend on the choice of sequence \((\delta_k)\). Since the law of a \(\sqrt{8/3}\)-quantum cone in a fixed neighborhood of 0 is mutually absolutely continuous with respect to that of a \(\sqrt{8/3}\)-LQG sphere (provided we choose the same embedding for both), it follows that the same is also true for QLE(8/3, 0) on a \(\sqrt{8/3}\)-quantum cone. This alone does not imply that the \(\delta\)-approximations to QLE(8/3, 0) converge as \(\delta \to 0\) (in other words, it is not necessary to pass along a sequence of positive numbers \((\delta_k)\) which tend to 0 as \(k \to \infty\)) because these hitting times may not determine the law of the process itself. This, however, will be a consequence of the continuity results established in the present article.

In the case of a whole-plane SLE\(_6\) exploration of a \(\sqrt{8/3}\)-quantum cone, we know from [DMS14, Corollary 12.2] that the boundary length of the outer boundary evolves as a 3/2-stable Lévy process with only downward jumps conditioned to be non-negative. The compactness argument of [MS15b, Section 6] also implies that the subsequentially limiting QLE(8/3, 0) with the quantum natural time parameterization has the same property.

Recall from [MS15b] that we change time from the quantum natural time to the quantum distance time parameterization using the time-change

\[
\int_0^t \frac{1}{X_s} ds
\]  

(2.2)

where \(X_s\) is the quantum boundary length of the outer boundary of the process at quantum natural time \(s\). (The intuition for using this particular time change is that in the Eden growth model, the rate at which new edges are added to the outer boundary of the cluster is proportional to the boundary length of the cluster.) If we perform this time-change, then the outer boundary length of the QLE(8/3, 0) evolves as the time-reversal of a 3/2-stable continuous state branching process (CSBP; we will give a review of CSBPs in Section 2.5 below).

**Lemma 2.1.** Suppose that \((D, h)\) has the law of a quantum disk with boundary length \(L > 0\) and that \(z \in D\) is distributed uniformly according to the quantum area measure. Then the QLE(8/3, 0) starting from \(z\) hits \(\partial D\) first at a point chosen uniformly from the quantum boundary measure. Moreover, the QLE(8/3, 0) stopped upon first hitting \(\partial D\) intersects \(\partial D\) at a unique point almost surely. Finally, if \(D_L\) has the law of the amount of quantum distance time required by the QLE(8/3, 0) to hit \(\partial D\) then \(D_L \overset{d}{=} L^{1/2} D_1\).
Proof. The first assertion of the lemma follows from the construction and the proof of the metric property given in [MS15b] (see [MS15b Lemma 7.7]). The second assertion is also established in [MS15b Lemma 7.6].

We will deduce the final assertion of the lemma using the following scaling calculation. Recall that if we add the constant $C$ to the field then quantum boundary length is scaled by the factor $e^{\gamma C/2}$ and that quantum natural time is scaled by the factor $e^{3\gamma C/4}$ (see [MS15c Section 6.2]). Equivalently, if we start off with a unit boundary length quantum disk, $L > 0$, and we scale the field so that the boundary length is equal to $L$ then quantum natural time is scaled by the factor $L^{3/2}$. Recall also that if $X_t$ denotes the quantum boundary length of the outer boundary of the QLE($8/3,0$) growth at quantum natural time $t$, then the quantum distance time elapsed by quantum natural time $T$ is equal to

$$
\int_0^T \frac{1}{X_s} ds. \tag{2.3}
$$

Combining (2.3) with the scaling given for boundary length and quantum natural time given above, we see that if we start out with a unit boundary length quantum disk and then scale the field so that the boundary length is $L$, then the amount of quantum distance time elapsed by the resulting QLE($8/3,0$) is given by

$$
\int_0^{L^{3/2}T} \frac{1}{LX_{L^{-3/2}s}} ds. \tag{2.4}
$$

Making the substitution $t = L^{-3/2}s$ in (2.4), we see that (2.4) is equal to

$$
L^{1/2} \int_0^T \frac{1}{X_t} dt. \tag{2.5}
$$

The final claim follows from (2.5).

Using the same scaling argument used to establish Lemma 2.1, we can also determine how quantum distances scale when we add a constant $C$ to the field.

**Lemma 2.2.** Suppose that $(D,h)$ is a $\sqrt{8/3}$-LQG surface and let $d_Q$ be the distance function associated with the QLE($8/3,0$) metric. Fix $C \in \mathbb{R}$. Then the distance function associated with the field $h + C$ is given by $e^{\gamma C/4}d_Q$ with $\gamma = \sqrt{8/3}$.

We note that $d_Q$ is a priori only defined on a countable dense subset of $D$ chosen i.i.d. from the quantum area measure. However, upon completing the proof of Theorem 1.1 and Theorem 1.2 the same scaling result immediately extends to $d_Q$ by continuity.

**Proof of Lemma 2.2.** This follows from the same argument used to establish (2.3), (2.4), and (2.5).
2.3 Quantitative Kolmogorov-Čentsov

The purpose of this section is to establish a quantitative version of the Kolmorogov-Čentsov continuity criterion [KS91, RY99]. We will momentarily apply this result to the case of the circle average process for the GFF, which will be used later to establish the continuity results for QLE(8/3, 0).

**Proposition 2.3** (Kolmogorov-Čentsov continuity criterion). Suppose that \((X_u)\) is a random field indexed by \(u \in [0, 1]^d\). Assume that there exist constants \(\alpha, \beta, c_0 > 0\) such that for all \(u, v \in [0, 1]^d\) we have that

\[
E[|X_u - X_v|^{\alpha}] \leq c_0 |u - v|^{d + \beta}. \tag{2.6}
\]

Then there exists a modification of \(X\) (which we shall write as \(X\)) such that for each \(\gamma \in (0, \alpha/\beta)\) there exists \(M > 0\) such that

\[
|X_u - X_v| \leq M|u - v|^\gamma \quad \text{for all } u, v \in [0, 1]^d. \tag{2.7}
\]

Moreover, if we define \(M\) to be \(\sup_{u, v} |X_u - X_v|/|u - v|^\gamma\), then there exists \(c_1 > 0\) depending on \(\alpha, \beta, \gamma, c_0\) such that

\[
P[M \geq t] \leq c_1 t^{-\alpha} \quad \text{for all } t \geq 1. \tag{2.8}
\]

The first statement of the proposition is just the usual Kolmogorov-Čentsov continuity criterion. One sees that (2.8) holds by carefully following the proof. For completeness, we will work out the details here.

**Proof of Proposition 2.3**. Applying Chebyshev’s inequality, we have from (2.6) that

\[
P[|X_u - X_v| \geq \delta] \leq c_0 \delta^{-\alpha} |u - v|^{d + \beta} \quad \text{for all } u, v \in [0, 1]^d. \tag{2.9}
\]

For each \(k\), let \(D_k\) consist of those \(x \in [0, 1]^d\) with dyadic rational coordinates that are integer multiples of \(2^{-k}\). Let \(\tilde{D}_k\) consist of those pairs \(\{u, v\}\) in \(D_k\) which are adjacent, i.e., differ in only one coordinate and have \(|u - v| = 2^{-k}\). By (2.9), we have that

\[
P\left[|X_u - X_v| \geq t2^{-\gamma k}\right] \leq c_0 t^{-\alpha} 2^{-k(d + \beta - \alpha \gamma)} \quad \text{for all } u, v \in \tilde{D}_k. \tag{2.10}
\]

Noting that \(|\tilde{D}_k| = O(2^{dk})\), by applying a union bound and using (2.10) we have for some constant \(c_1 > 0\) that

\[
P\left[\max_{\{u, v\} \in D_k} |X_u - X_v| \geq t2^{-\gamma k}\right] \leq c_1 t^{-\alpha} 2^{-k(\beta - \alpha \gamma)}. \tag{2.11}
\]

Thus, by a further union bound and using (2.11), we have for some constant \(c_2 > 0\) that

\[
P\left[\sup_{k \in \mathbb{N}} \max_{\{u, v\} \in \tilde{D}_k} 2^{\gamma k}|X_u - X_v| \geq t\right] \leq c_2 t^{-\alpha}. \tag{2.12}
\]

It is not difficult to see that there exists some constant \(c_3 > 0\) such that on the event \(\sup_{k \in \mathbb{N}} \max_{\{u, v\} \in \tilde{D}_k} 2^{\gamma k}|X_u - X_v| \leq t\) considered in (2.12) we have that \(|X_u - X_v| \leq c_3 t|u - v|^\gamma\) for all \(u, v \in \cup_k D_k\). This, in turn, implies the result. \(\square\)
2.4 GFF extremes

In this section, we will establish a result regarding the tails of the maximum of the circle average process associated with a whole-plane GFF. We refer the reader to [DS11, Section 3] for more on the construction of the circle average process. We also refer the reader to [She15, Section 3.2] for more on the whole-plane GFF.

**Proposition 2.4.** Suppose that \( h \) is a whole-plane GFF. For each \( r > 0 \) and \( z \in \mathbb{C} \) we let \( h_r(z) \) be the average of \( h \) on \( \partial B(z,r) \). We assume that the additive constant for \( h \) has been fixed so that \( h_1(0) = 0 \). For each \( \xi \in (0,1) \) there exists a constant \( c_0 > 0 \) such that for each fixed \( r \in (0,1/2) \) and all \( \delta > 0 \) we have that

\[
P\left[ \sup_{z \in B(0,1/2)} |h_r(z)| \geq (2 + \delta) \log r^{-1} \right] \leq c_0 r^{2\delta(1-\xi)}. \tag{2.13}
\]

Before giving the proof of Proposition 2.4, we are first going to deduce from it a result which bounds the behavior of \( h_r(z) \) for all \( z \in \mathbb{C} \).

**Corollary 2.5.** Suppose that we have the same setup as described in Proposition 2.4. There exists a constant \( c_0 > 0 \) such that for each \( k \in \mathbb{N} \) and all \( r \in (0,e^{k/2}) \) we have that

\[
P\left[ \sup_{z \in B(0,e^{k/2})} |h_r(z) - h_{e^k}(0)| \geq (2 + 2\delta)(\log r^{-1} + 2k) \right] \leq c_0 \left( \frac{r}{e^k} \right)^\delta e^{-(1+\delta)k/2}. \tag{2.14}
\]

The same likewise holds if \( \alpha < Q \) and \( h = h_1 + \alpha \log |\cdot| \) where \((\mathbb{C},h_1,0,\infty)\) is an \( \alpha \)-quantum cone with the circle average embedding.

Before establishing Corollary 2.5, we first record the following Gaussian tail bound, which is easy to derive directly from the standard Gaussian density function.

**Lemma 2.6.** Suppose that \( Z \sim N(0,1) \). Then we have that

\[
P[Z \geq \lambda] \leq \sqrt{\frac{2}{\pi}} \lambda^{-1} \exp\left(-\frac{\lambda^2}{2}\right) \quad \text{as} \quad \lambda \to \infty.
\]

**Proof of Corollary 2.5.** We are first going to deduce the result in the case of a whole-plane GFF from Proposition 2.4 and a union bound.

Note that \( h - h_{e^k}(0) \) has the law of the whole-plane GFF with the additive constant fixed so that \( h_{e^k}(0) = 0 \). By the scale invariance of the whole-plane GFF, we have for each \( k \in \mathbb{N} \) and \( r \in (0,e^{k/2}) \) that

\[
P\left[ \sup_{z \in B(0,e^{k/2})} (h_r(z) - h_{e^k}(0)) \geq (2 + \delta)(\log r^{-1} + 3k/2) \right]
\]
\[ \mathbb{P} \left[ \sup_{z \in B(0,1/2)} h_{r e^{-k}}(z) \geq (2 + \delta)(\log r^{-1} + 3k/2) \right] \]
\[ \mathbb{P} \left[ \sup_{z \in B(0,1/2)} h_s(z) \geq (2 + \delta)(\log s^{-1} + k/2) \right] \quad \text{(with } s = re^{-k}) \]
\[ \leq c_0 s^\delta e^{-(1+\delta/2)k} = c_0 \left( \frac{r}{e^k} \right)^\delta e^{-(1+\delta/2)k} \quad \text{(by Proposition 2.4 with } \xi = 1/2). \quad (2.15) \]

Since \( h_{e^{-k}}(0) \) is a Gaussian random variable with mean 0 and variance \( k \), it follows from Lemma 2.6 that
\[ \mathbb{P} \left[ h_{e^{-k}}(0) \geq (1 + 2\delta)k + \delta \log r^{-1} \right] \lesssim \exp \left( -\frac{(1 + 2\delta)k + \delta \log r^{-1})^2}{2k} \right) \]
\[ \lesssim \left( \frac{r}{e^k} \right)^\delta e^{-(1/2+\delta)k}. \quad (2.16) \]

Combining (2.15), (2.16) gives the result in the setting of a whole-plane GFF.

We will now extract the corresponding result for an \( \alpha \)-quantum cone. Suppose that
\( h = h_1 + \alpha \log |\cdot| \) where \( h_1 \) is an \( \alpha \)-quantum cone with \( \alpha < Q \) and the embedding as in the statement of the corollary. In this setting, \( h|_D \) has the same law as a whole-plane GFF with the additive constant fixed so that its average on \( \partial D \) is equal to 0. For each \( z \in \mathbb{C} \) and \( r > 0 \) we let \( h_{1,r}(z) \) be the average of \( h_1 \) on \( \partial B(z,r) \). Then we have that \( h_{1,r}(0) \) for \( r \geq 0 \) evolves as \( B_r - ar \) where \( B \) is a standard Brownian motion conditioned so that \( B + (Q - \alpha)r \geq 0 \) for all \( r \geq 0 \). Therefore \( h_r(0) \) evolves as a standard Brownian motion \( B \) conditioned so that \( B + (Q - \alpha)r \geq 0 \) for all \( r \geq 0 \). From this, the result in the case of an \( \alpha \)-quantum cone follows. \( \Box \)

**Lemma 2.7.** Suppose that we have the same setup as in Proposition 2.4. For each \( \alpha > 0 \) there exists a constant \( c_0 > 0 \) such that the following is true. For all \( z, w \in B(0,1/2) \) and \( r, s \in (0,1/2) \) we have that
\[ \mathbb{E} \left[ |h_r(z) - h_s(w)|^\alpha \right] \leq c_0 \left( \frac{|(z,r) - (w,s)|}{r \wedge s} \right)^{\alpha/2}. \]

**Proof.** This is the content of [HMP10, Proposition 2.1] in the case of a GFF on a bounded domain \( D \subseteq \mathbb{C} \) with Dirichlet boundary conditions. The proof in the case of a whole-plane GFF is the same. \( \Box \)

**Proof of Proposition 2.4** By combining Lemma 2.7 (with a sufficiently large value of \( \alpha \)) with Proposition 2.3 we have that the following is true. For each \( \zeta > 0 \), there exists \( M > 0 \) (random) such that for all \( z, w \in B(0,1/2) \) and \( r \in (0,1/2) \) we have that
\[ |h_r(z) - h_r(w)| \leq Mr^{-1/2+\zeta}|z - w|^{1/2-\zeta}. \quad (2.17) \]
Moreover, Lemma 2.7 and Proposition 2.3 imply that, for each \( \alpha > 0 \), there exists a constant \( c_0 > 0 \) depending only on \( \alpha \) such that:

\[
P[M \geq t] \leq c_0 t^{-\alpha} \quad \text{for all} \quad t \geq 1. \tag{2.18}
\]

Fix \( a_0 \in (0, 1) \), \( j \in \mathbb{N} \), and let \( E_{j,a_0} = \{ M \geq e^{a_0 j/4} \} \). On \( E^c_{j,a_0} \), (2.17) implies that

\[
|h_e^{-j}(z) - h_e^{-j}(w)| \leq Me^{j(1/2-\varsigma)}|z - w|^{1/2-\varsigma} \\
\leq e^{-a_0(1/4-\varsigma)j} \quad \text{for all} \quad |z - w| \leq e^{-(1+a_0)j}. \tag{2.19}
\]

Combining Lemma 2.6 with the explicit form of the variance of \( h_e \) [DST11, Proposition 3.2], we have that there exists a constant \( c_1 > 0 \) such that for each \( \alpha, \delta > 0 \) that

\[
P[h_e(z) \geq (\alpha + \delta) \log \epsilon^{-1}] \leq c_1 \exp \left( -\frac{(\alpha + \delta)^2 (\log \epsilon^{-1})^2}{2 \log \epsilon^{-1}} \right) \leq c_1 e^{2j(1/2+\alpha\delta)} \tag{2.20}
\]

We are now going to use (2.20) to perform a union bound over a grid of points with spacing \( e^{-(1+a_0)j} \). The result will then follow by combining this with (2.18) and (2.19).

Let \( C_{j,a_0} = \{ z \in e^{-j(1+a_0)}Z^2 : z \in B(0, 1/2) \} \). Note that \( |C_{j,a_0}| \approx e^{2j(1+a_0)} \). By (2.20), we have that

\[
P[h_e^{-j}(z) \geq (2 + \delta)j] \leq c_1 e^{2j(1+\delta)j}. \tag{2.21}
\]

Consequently, by a union bound and (2.21), there exists a constant \( c_2 > 0 \) such that with

\[
F_{j,a_0} = \left\{ \max_{z \in C_{j,a_0}} h_e^{-j}(z) \leq (2 + \delta)j \right\} \quad \text{we have} \quad P[F^c_{j,a_0}] \leq c_2 e^{2j(a_0-\delta)}. \tag{2.22}
\]

Suppose that \( u \in B(0, 1/2) \) is arbitrary. Then there exists \( z \in C_{j,a_0} \) such that \( |u - z| \leq \sqrt{2} \cdot e^{-j(1+a_0)} \). On \( E^c_{j,a_0} \), by (2.19) we have for a constant \( c_3 > 0 \) that

\[
|h_e^{-j}(z) - h_e^{-j}(u)| \leq c_3 e^{-a_0(1/4-\varsigma)j}. \tag{2.23}
\]

Thus, on \( E^c_{a_0} \cap F_{j,a_0} \), we have that

\[
h_e^{-j}(u) \leq c_3 e^{-a_0(1/4-\varsigma)j} + h_e^{-j}(z) \leq c_3 e^{-a_0(1/4-\varsigma)j} + (2 + \delta)j.
\]

That is,

\[
\sup_{u \in B(0, 1/2)} h_e^{-j}(u) \leq c_3 e^{-a_0(1/4-\varsigma)j} + (2 + \delta)j.
\]

Choose \( \alpha > 0 \) sufficiently large so that, applying (2.18) with this value of \( \alpha \), we have that

\[
P[E_{j,a_0}] \leq c_0 e^{2j(a_0-\delta)}. \tag{2.24}
\]

By (2.20) and (2.23), we have that

\[
P[E^c_{j,a_0} \cap F_{j,a_0}] \geq 1 - (c_0 + c_2)e^{2j(a_0-\delta)} = 1 - c_4 e^{2j(a_0-\delta)}
\]

where \( c_4 = c_0 + c_2 \). This proves the result for \( r = e^{-j} \). The result for general \( r \in (0, 1/2) \) is proved similarly. \( \square \)
2.5 Continuous state branching processes

The purpose of this section is to record a few elementary properties of continuous state branching processes (CSBPs); see [LG99, Kyp06] for an introduction.

Suppose that \( Y \) is a CSBP with branching mechanism \( \psi \). Recall that this means that \( Y \) is the Markov process on \( \mathbb{R}_+ \) whose transition kernels are characterized by the property that

\[
\mathbb{E}[\exp(-\lambda Y_t) \mid Y_s] = \exp(-Y_s u_t(\lambda)) \quad \text{for all} \quad t > s \geq 0
\]  

(2.24)

where \( u_t(\lambda), t \geq 0, \) is the non-negative solution to the differential equation

\[
\frac{\partial u_t}{\partial t}(\lambda) = -\psi(u_t(\lambda)) \quad \text{for} \quad u_0(\lambda) = \lambda.
\]  

(2.25)

Let

\[
\Phi(q) = \sup\{\theta \geq 0 : \psi(\theta) = q\}
\]  

(2.26)

and let

\[
\zeta = \inf\{t \geq 0 : Y_t = 0\}
\]  

(2.27)

be the extinction time for \( Y \). Then we have that [Kyp06, Corollary 10.9]

\[
\mathbb{E}\left[e^{-q \int_0^\zeta Y_s ds}\right] = e^{-\Phi(q)Y_0}.
\]  

(2.28)

A \( \psi \)-CSBP can be constructed from a Lévy process with only positive jumps and vice-versa [Lam67] (see also [Kyp06, Theorem 10.2]). Namely, suppose that \( X \) is a Lévy process with Laplace exponent \( \psi \). That is,

\[
\mathbb{E}[\exp(-\lambda X_t) = \exp(\psi(\lambda) t)
\]

Let

\[
s(t) = \int_0^t \frac{1}{X_u} du \quad \text{and} \quad s^*(t) = \inf\{r > 0 : s(r) > t\}.
\]  

(2.29)

Then the time-changed process \( Y_t = X_{s^*(t)} \) is a \( \psi \)-CSBP. That is, \( Y_{s(t)} = X_t \). Conversely, if \( Y \) is a \( \psi \)-CSBP and we let

\[
t(s) = \int_0^s Y_u du \quad \text{and} \quad \tau(s) = \inf\{r > 0 : t(r) > s\}
\]  

(2.30)

then \( X_s = Y_{\tau(s)} \) is a Lévy process with Laplace exponent \( \psi \). That is, \( X_{t(s)} = Y_s \).

We will be interested in the particular case that \( \psi(u) = u^\alpha \) for \( \alpha \in (1, 2) \). For this choice, we note that

\[
u_t(\lambda) = (\lambda^{1-\alpha} + (\alpha - 1)t)^{1/(1-\alpha)}.
\]  

(2.31)

Combining (2.24) and (2.31) implies that \( u^\alpha \)-CSBPs (which we will also later refer to as \( \alpha \)-stable CSBPs) satisfy a certain scaling property. Namely, if \( Y \) is a \( u^\alpha \)-CSBP starting from \( Y_0 \) then \( \tilde{Y}_t = \beta^{1/(1-\alpha)}Y_{\beta^t} \) is a \( u^\alpha \)-CSBP starting from \( \tilde{Y}_0 = \beta^{1/(1-\alpha)}Y_0 \). In particular, if \( Y \) is a \( u^{3/2} \)-CSBP starting from \( Y_0 \) then \( \tilde{Y}_t = \beta^{-2}Y_{\beta^t} \) is a \( u^{3/2} \)-stable CSBP starting from \( \tilde{Y}_0 = \beta^{-2}Y_0 \).
2.6 Tail bounds for stable processes and the Poisson law

2.6.1 Supremum of an $\alpha$-stable process

Lemma 2.8. Suppose that $X$ is an $\alpha$-stable process with $X_0 = 0$ and without positive jumps. For each $t \geq 0$, let $S_t = \sup_{s \in [0,t]} X_s$. There exist constants $c_0, c_1 > 0$ such that

$$
P[S_t \geq u] \leq c_0 \exp(-c_1 t^{-1/\alpha} u).$$  \hspace{1cm} (2.32)

Proof. For each $t \geq 0$, we let $S_t = \sup_{s \in [0,t]} X_s$. Fix $q > 0$ and let $\tau(q)$ be an exponential random variable with parameter $q$ which is sampled independently of $X$. Let $\Phi(\lambda) = a_0^{-1/\alpha} \lambda^{1/\alpha}$ be the inverse of the Laplace exponent $\psi(\lambda) = a_0 \lambda^\alpha$ of $X$. By [Ber96, Chapter VII, Corollary 2], we have that $S_{\tau(q)}$ has the exponential distribution with parameter $\Phi(q)$. In particular, we have that

$$
P[S_{\tau(q)} \geq u] = \exp(-\Phi(q)u).$$

Therefore we have that

$$
P[S_{q^{-1}} \geq u] \leq P[S_{\tau(q)} \geq u \mid \tau(q) \geq q^{-1}]$$

$$
\leq c_0 P[S_{\tau(q)} \geq u]$$

$$
\leq c_0 \exp(-\Phi(q)u)
$$

where $c_0 = 1/P[\tau(q) \geq q^{-1}] = e$. \hfill \Box

2.6.2 Poisson deviations

Lemma 2.9. If $Z$ is a Poisson random variable with mean $\lambda$ then for each $\alpha \in (0,1)$ we have that

$$
P[Z \leq \alpha \lambda] \leq \exp\left(\lambda(\alpha - \alpha \log \alpha - 1)\right).$$  \hspace{1cm} (2.33)

Similarly, for each $\alpha > 1$ we have that

$$
P[Z \geq \alpha \lambda] \leq \exp\left(\lambda(\alpha - \alpha \log \alpha - 1)\right).$$  \hspace{1cm} (2.34)

Proof. Recall that the moment generating function for a Poisson random variable with mean $\lambda$ is given by $\exp(\lambda(e^t - 1))$. Therefore the probability that a Poisson random variable $Z$ of mean $\lambda$ is smaller than a constant $c$ satisfies for each $\beta > 0$ the inequality

$$
P[Z \leq c] = P[e^{-\beta Z} \geq e^{-\beta c}] \leq e^{\beta c} \mathbb{E}[e^{-\beta Z}] = \exp(\beta c + \lambda(e^{-\beta} - 1)).$$

If we take $c = \alpha \lambda$, the above becomes

$$
P[Z \leq \alpha \lambda] \leq \exp(\lambda(\alpha \beta + e^{-\beta} - 1)).$$

Note that $\beta \mapsto \alpha \beta + e^{-\beta} - 1$ is minimized with $\beta = -\log \alpha$ and taking $\beta$ to be this value implies the lower bound. The upper bound is proved similarly. \hfill \Box
3 Quantum boundary length and area bounds

The purpose of this section is to derive tail bounds for the quantum boundary length of the outer boundary of a QLE(8/3, 0) (Section 3.1), for the quantum area surrounded by a QLE(8/3, 0) (Section 3.2), and also to establish the regularity of the quantum area measure on a γ-quantum cone (Section 3.3). The estimates established in this section will then feed into the Euclidean size bounds for QLE(8/3, 0) derived in Section 4.

3.1 Quantum boundary length of QLE(8/3, 0) hull

Lemma 3.1. Suppose that \((C, h, 0, \infty)\) is a \(\sqrt{8/3}\)-quantum cone, let \((\Gamma_r)\) be the QLE(8/3, 0) starting from 0 with the quantum distance parameterization, and for each \(r > 0\) let \(B_r\) be the quantum boundary length of the outer boundary of \(\Gamma_r\). There exist constants \(c_0, \ldots, c_3 > 0\) such that for each \(r > 0\) and \(t > 1\) we have both

\[
P\left[ \sup_{0 \leq s \leq r} B_s \leq r^2/t \right] \leq c_0 e^{-c_1 t^{1/2}} \quad \text{and} \quad P\left[ \sup_{0 \leq s \leq r} B_s \geq r^2 t \right] \leq c_2 e^{-c_3 t}. \tag{3.1}
\]

Recall from the construction of QLE(8/3, 0) on a \(\sqrt{8/3}\)-quantum cone given in Section 2.2 that \(B\) evolves as the time-reversal of a 3/2-stable CSBP starting from 0 conditioned to be non-negative. Consequently, Lemma 3.1 is in fact a statement about 3/2-stable CSBPs. In order to prove Lemma 3.1, we will make use of the scaling property for 3/2-stable CSBPs explained at the end of Section 2.5. Namely, if \(Y\) is a 3/2-stable CSBP starting from \(Y_0 = x\) and \(\alpha > 0\) then \(\alpha^{-2}Y_{\alpha^{-2}}\) is a 3/2-stable CSBP starting from \(x\). We will also make use of the relationship between the time-reversal of a 3/2-stable CSBP conditioned to be non-negative and the law of a 3/2-stable CSBP run until the first time that it hits 0. Results of this type in the context of Lévy processes with only downward jumps are explained in [Ber96, Chapter VII, Section 4]. In particular, combining the Lamperti transform (2.29) and [Ber96, Chapter VII, Theorem 18] implies that the following is true. If \(Y\) is a 3/2-stable CSBP starting from \(Y_0 > 0\) and \(\zeta = \inf\{t \geq 0 : Y_t = 0\}\), then \(Y_{\zeta-t}\) for \(t \in [0, \zeta]\) evolves as a 3/2-stable CSBP conditioned to be non-negative stopped at the last time that it hits \(x\).

Proof of Lemma 3.1. Let \(Y\) be a 3/2-stable CSBP and let \(\zeta = \inf\{t > 0 : Y_t = 0\}\) starting from \(Y_0\). For each \(x \geq 0\), we let \(\mathbb{P}^x[\cdot]\) be the law under which \(Y_0 = x\).

Using the time-reversal result for CSBPs mentioned just above, in order to prove the first inequality of (3.1) it suffices to show that the following is true. There exist constants \(c_0, c_1 > 0\) such that the probability that there is an interval of length at least \(r\) during which \(Y\) is contained in \([0, r^2/t]\) is at most \(c_0 e^{-c_1 t^{1/2}}\) under the law \(\mathbb{P}^x\) with \(x \geq r^2/t\). By applying scaling as described at the end of Section 2.5, it in turn suffices to show that the probability of the event \(E\) that there is an interval of length at least 33
$t^{1/2}$ during which $Y$ is contained in $[0,1]$ is at most $c_0 e^{-ct^{1/2}}$ under the law $P^x$ with $x \geq 1$.

To see that this is the case, we define stopping times inductively as follows. Let $\tau_0 = \inf\{t \geq 0 : Y_t \leq 1\}$ and $\sigma_0 = \zeta \wedge \inf\{t \geq \tau_0 : Y_t \geq 2\}$. Assuming that we have defined stopping times $\tau_0, \ldots, \tau_k$ and $\sigma_0, \ldots, \sigma_k$ for some $k \in \mathbb{N}$, we let $\tau_{k+1} = \inf\{t \geq \sigma_k : Y_t \leq 1\}$ and $\sigma_{k+1} = \zeta \wedge \inf\{t \geq \tau_{k+1} : Y_t \geq 2\}$. Let $N = \min\{k : Y_{\sigma_k} = 0\}$. Then $N$ has the geometric distribution. Note that there exist constants $c_0, c_1$ such that for each $k$, we have that $P[\sigma_k - \tau_k \geq t^{1/2} | N \geq k] \leq c_0 e^{-ct^{1/2}}$ because in each round of length 1, $Y$ has a uniformly positive chance of exiting $(0,2)$. Observe that

$$P[E] \leq \sum_k P[\sigma_k - \tau_k \geq t^{1/2}, N \geq k]$$

$$= \sum_k P[\sigma_k - \tau_k \geq t^{1/2} | N \geq k] P[N \geq k]$$

$$\leq c_0 e^{-ct^{1/2}} \sum_k P[N \geq k] = E[N] c_0 e^{-ct^{1/2}}. \quad (3.2)$$

The first inequality of (3.1) thus follows by possibly increasing the value of $c_0$.

We will prove the second inequality of (3.1) using again the aforementioned time-reversal result for CSBPs. Namely, it suffices to show that there exist constants $c_2, c_3 > 0$ such that the probability that there is an interval of length at most $r$ in which $Y$ starts at $r^2 t$ and then exits at 0 is at most $c_2 e^{-ct}$. By scaling, it suffices to show that there exist constants $c_2, c_3 > 0$ such that the probability of the event $E$ that there is an interval of length at most $t^{-1/2}$ in which $Y$ starts at 1 and then exits at 0 is at most $c_2 e^{-ct}$. To show that this is the case we assume that we have defined stopping times $\sigma_k, \tau_k$ and $N$ as in our proof of the first inequality of (3.1). Note that (recall (2.24) and (2.31))

$$P^x[\zeta \leq v] = \lim_{\lambda \to \infty} E^x[\exp(-\lambda Y_v)] = \lim_{\lambda \to \infty} \exp(-x u_v(\lambda)) = \exp(-4x/v^2). \quad (3.3)$$

Evaluating (3.3) at $x = 1$ and $v = t^{-1/2}$ implies that there exist constants $c_2, c_3 > 0$ such that $P[\sigma_k - \tau_k \leq t^{-1/2} | N \geq k] \leq c_2 e^{-ct}$. Thus the second inequality in (3.1) follows the calculation in (3.2) used to complete the proof of the first inequality of (3.1). \quad \Box

### 3.2 Quantum area of QLE$(8/3,0)$ hull

**Lemma 3.2.** Let $(C, h, 0, \infty)$ be a $\sqrt{8/3}$-quantum cone, let $(\Gamma_r)$ be the QLE$(8/3,0)$ growing from 0 with the quantum distance parameterization, and for each $r > 0$ let $A_r$ be the quantum area of $\Gamma_r$. There exist constants $a_0, c_0, c_1 > 0$ such that

$$P[A_r \leq r^4/t] \leq c_0 \exp(-c_1 t^{a_0}) \quad \text{for all} \quad r > 0, \: t \geq 1. \quad (3.4)$$
Before we give the proof of Lemma 3.2, we first need to record the following fact.

**Lemma 3.3.** There exists a constant $c_0 > 0$ such that the following is true. Suppose that $(S, h)$ has the law of a quantum disk with quantum boundary length $\ell$. Then

$$E[\mu_h(S)] = c_0 \ell^2. \quad (3.5)$$

**Proof.** Recall that the law of a quantum disk with boundary length $\ell$ can be sampled from by first picking $(S, h)$ from the law of the unit boundary length quantum disk and then taking the field $h + 2\gamma^{-1} \log \ell$. Note that adding $2\gamma^{-1} \log \ell$ to the field has the effect of multiplying quantum boundary lengths (resp. areas) by $\ell$ (resp. $\ell^2$). [MS15c, Proposition 6.5] implies that the law of a quantum disk with given boundary length weighted by its quantum area makes sense as a probability measure which is equivalent to the quantum area having finite expectation. Combining this with the aforementioned scaling implies the result.

**Proof of Lemma 3.2.** For each $r > 0$, we let $B_r$ be the quantum length of the outer boundary of $\Gamma_r$. Fix $r > 0$. Then we know from Lemma 3.1 that there exist constants $c_0, c_1 > 0$ such that

$$P\left[ \sup_{0 \leq s \leq r} B_s \leq r^2/t \right] \leq c_0 \exp(-c_1 t^{1/2}) \quad \text{for each} \quad t \geq 1. \quad (3.6)$$

Suppose that $X$ is a $3/2$-stable Lévy process with only downward jumps and let $P^x[\cdot]$ be the law under which $X_0 = x$. Let $W$ have law $P[\cdot | X \geq 0]$ and write $P^w[\cdot]$ for the law of $W$ under which $W_0 = w$. Then we know that the law of $B$ is equal to the law of $W$ under $P^0$ after performing the time change as in (2.30) (recall the importance of this time-change in the context of QLE$(8/3, 0)$, as discussed around (2.2)). Fix $t \geq 1$. It then follows from (3.6) that the probability that $W$ hits $r^2/t$ before the time which corresponds to when $\Gamma$ has quantum radius $r$ is at least $1 - c_0 \exp(-c_1 t^{1/2})$.

We are now going to argue that, by possibly adjusting the values of $c_0, c_1 > 0$, we have that the probability that $W$ takes less than $r^3/t^3$ units of time to hit $r^2/t$ is at most $c_0 \exp(-c_1 t)$. To see this, we let $\tau$ be the first time that $W$ hits $r^2/(2t)$. Then it suffices to show that the probability that $W$ starting from $r^2/(2t)$ takes less than $r^3/t^3$ time to hit $r^2/t$ is at most $c_0 \exp(-c_1 t)$. Since the probability that a $3/2$-stable Lévy process with only downward jumps starting from $r^2/(2t)$ to hit $r^2/t$ before hitting 0 is uniformly positive in $r > 0$ and $t \geq 1$ (by scaling), it suffices to show that the probability that $X$ starting from $r^2/(2t)$ hits $r^2/t$ in less than $r^3/t^3$ time is at most $c_0 \exp(-c_1 t)$. This, in turn, follows from Lemma 2.8.

Suppose that $0 < a < b < \infty$. The number of downward jumps made by $X$ in time $r^3/t^3$ of size between $a$ and $b$ is distributed as a Poisson random variable with mean given by a constant times

$$\frac{r^3}{t^3} \int_a^b s^{-5/2} ds = \frac{2}{3} \cdot \frac{r^3}{t^3} (a^{3/2} - b^{3/2}). \quad (3.7)$$
In particular, the number of jumps made by $X$ in time $r^3/t^3$ of size between $\frac{1}{2}r^2t^{-8/3}$ and $r^2t^{-8/3}$ is Poisson with mean proportional to $t$. Therefore it follows from Lemma 2.9 that there exist constants $c_2, c_3 > 0$ such that the probability of the event that the number of such jumps is fewer than $1/2$ its mean is at most $c_2 \exp(-c_3 t)$. It follows from the argument of the previous paragraph that the same holds for $W$. We note that each of the jumps of $W$ corresponds to a quantum disk cut out by $\Gamma|_{[0,r]}$ and the size of the jump corresponds to the quantum boundary length of the disk. Moreover, the probability that fewer than $1/2$ of these disks have quantum area which is larger than $1/2$ of the conditional expectation of the quantum area given its quantum boundary length is at most $c_4 \exp(-c_5 t)$ where $c_4, c_5 > 0$ are constants. By Lemma 3.3, the conditional mean of the quantum area of such a quantum disk given its quantum boundary length is proportional to $r^4 t^{-16/3}$. Combining all of our estimates implies (3.4).

### 3.3 Regularity of the quantum area measure on a $\gamma$-quantum cone

The purpose of this section is to record an upper bound for the quantum area measure associated with a $\gamma$-quantum cone.

**Proposition 3.4.** Fix $\gamma \in (0, 2)$ and let

$$\alpha = \frac{(\gamma^2 - 4)^2}{4(4 + \gamma^2)}. \quad (3.8)$$

Suppose that $(C, h, 0, \infty)$ is a $\gamma$-quantum cone with the circle average embedding. Fix $\zeta \in (0, \alpha)$ and let $H_{R, \zeta}$ be the event that for every $z \in C$ and $s \in (0, R)$ such that $B(z, s) \subseteq D$ we have that $\mu_{h}(B(z, s)) \leq s^{\alpha-\zeta}$. Then $P[H_{R, \zeta}] \to 1$ as $R \to 0$ with $\zeta > 0$ fixed.

**Proof.** We first suppose that $h$ is a whole-plane GFF on $C$ with the additive constant fixed so that $h_1(0) = 0$ and let $\mu_{h}$ be the associated quantum area measure. Fix $q \in (0, 4/\gamma^2)$. Then [RV10, Proposition 3.7] implies that there exists a constant $c_q > 0$ such that with

$$\xi(q) = \left(2 + \frac{\gamma^2}{2}\right) q - \frac{\gamma^2}{2} q^2$$

we have that

$$E[\mu_{h}(B(z, s))^q] \leq c_q s^{\xi(q)}. \quad (3.9)$$

Let $\alpha$ be as in (3.8) and fix $\zeta \in (0, \alpha)$. It therefore follows from (3.9) and Markov’s inequality that

$$P[\mu_{h}(B(z, s)) \geq s^{\alpha-\zeta}] \leq c_q s^{\xi(q)-(\alpha-\zeta)q}. \quad (3.10)$$
Let
\[ q^* = \frac{4 + \gamma^2}{2\gamma^2} \in \left(0, \frac{4}{\gamma^2}\right) \]
be the value of \( q \) that maximizes \( \xi(q) \). Note that
\[ \alpha = \frac{\xi(q^*) - 2}{q^*} \]
so that the exponent on the right side of (3.10) with \( q = q^* \) is strictly larger than 2. Therefore applying the Borel-Cantelli lemma along with (3.10) on a dyadic partition of \( \mathbf{D} \) implies the result in the case of the whole-plane GFF.

We are now going to deduce the result in the case of a \( \gamma \)-quantum cone from the result in the case of the whole-plane GFF using absolute continuity. We suppose now that \((\mathbf{C}, \tilde{h}, 0, \infty)\) is a \( \gamma \)-quantum cone with the circle average embedding. If \( B \subseteq \mathbf{D} \) is any box with positive distance to 0, we have that the law of \( h|_B \) is mutually absolutely continuous with respect to the law of \( \tilde{h}|_B \). In particular, if we define \( \tilde{H}^B_{R, \zeta} \) in the same manner as \( H_{R, \zeta} \) except with \( \tilde{\mu} \) restricted to \( B \) in place of \( \mu \) then we have that
\[ \mathbf{P} \left[ \tilde{H}^B_{R, \zeta} \right] \to 1 \quad \text{as} \quad R \to 0 \quad \text{with} \quad \zeta \in (0, \alpha) \text{ fixed}. \]

Let \( \tilde{\eta}' \) be a space-filling \( \text{SLE}_{\kappa'} \) from \( \infty \) to \( \infty \) sampled independently of \( \tilde{h} \) and then reparameterized by quantum area as assigned by \( \tilde{h} \). That is, we have that \( \tilde{\mu}(\tilde{\eta}'([s, t])) = t - s \) for all \( s < t \). We normalize time so that \( \tilde{\eta}'(0) = 0 \). Then we know from [DMS14, Theorem 1.13] that the joint law of \((\tilde{h}, \tilde{\eta}')\) is the same as the joint law of the field which arises by taking \((\tilde{h}(\cdot - \tilde{\eta}'(t)), \tilde{\eta}'(\cdot + t) - \tilde{\eta}'(t))\) and then rescaling so that the new field has the circle average embedding.

Note that for \( t > 0 \) small we have that \( \tilde{\eta}'(t) \) has probability arbitrarily close to 1 of being in a box \( B \) as above with rational coordinates. The result therefore follows by scaling.

\[ \square \]

4 \hspace{1em} \textbf{Euclidean size bounds for QLE(8/3, 0)}

The purpose of this section is to establish bounds for the Euclidean size of a QLE(8/3, 0) process growing on \( \sqrt{8/3} \)-quantum cone. The lower bound is obtained in Section 4.1 by combining Proposition 3.4 established just above with the lower bound on the quantum area cut off from \( \infty \) by a QLE(8/3, 0) established in Lemma 3.2. In Section 4.2 we will first give an upper bound on the Euclidean diameter of a QLE(8/3, 0) and then combine this with the results of Section 3.3 to obtain an upper bound on the quantum area of the hull of a QLE(8/3, 0).
4.1 Diameter lower bound

**Proposition 4.1.** Suppose that $(C, h, 0, \infty)$ is a $\sqrt{8/3}$-quantum cone with the circle average embedding. Let $H_{R, \zeta}$ be the event from Proposition 3.4. There exist constants $c_0, \ldots, c_3 > 0$ depending only on $R, \zeta$ such that the following is true. Let $(\Gamma_r)$ be the hull of a QLE$(8/3, 0)$ process starting from 0 parameterized by quantum distance. For each $r \in (0, R)$ we have that

$$\Pr[\text{diam}(\Gamma_r) \leq r c_0, H_{R, \zeta}] \leq c_1 \exp(-c_2 r^{-c_3}). \quad (4.1)$$

**Proof.** This follows by combining (3.4) of Lemma 3.2 with the definition of $H_{R, \zeta}$. \qed

4.2 Diameter upper bound

**Proposition 4.2.** Suppose that $(C, h, 0, \infty)$ is a $\sqrt{8/3}$-quantum cone with the circle average embedding. Let $(\Gamma_r)$ be a QLE$(8/3, 0)$ process starting from 0 with the quantum distance parameterization. There exist constants $c_4 > 0$ and $a_1 > 4$ such that

$$\Pr[\text{diam}(\Gamma_r) \geq r^{a_1}] \leq c_3 r^\delta \text{ for all } r > 0. \quad (4.2)$$

Moreover, there exist constants $c_4 > 0$ and $a_1 > 4$ such that

$$\mathbb{E}[\text{diam}(\Gamma_r)^4 \mathbf{1}_{\{\text{diam}(\Gamma_r) \leq 1\}}] \leq c_4 r^{a_1} \text{ for all } r > 0. \quad (4.3)$$

The part of Proposition 4.2 asserted in (4.2) will be used in the proof of Theorem 1.1 and Theorem 1.2. The part which is asserted in (4.3) will be used in the proof of the main result of [MS16].

We will divide the proof of Proposition 4.2 into three steps. The first step, carried out in Section 4.2.1, is to give a tail bound for the quantum boundary length of $\partial \mathbb{Q}_+$ assigned by a free boundary GFF on $\mathbb{Q}_+$ with the additive constant fixed so that the average on $\partial \mathbb{Q}_+$ is equal to 0. Using the resampling characterization of the unexplored region of a $\sqrt{8/3}$-quantum cone established in [MS15c], we will then deduce from this in Section 4.2.2 that it is very unlikely for the harmonic extension of the values of the field from $\partial \mathbb{Q}_+$ to $\mathbb{Q}_+$ restricted to $\mathbb{Q}_+ + r$ to be large where $r > 0$ is fixed. We will then use this result to complete the proof of Proposition 4.2 in Section 4.2.3.

Before we proceed to the proof, we will first deduce an upper bound on the quantum area in the hull of a QLE$(8/3, 0)$.

**Corollary 4.3.** Let $H_{R, \zeta}$ be as in Proposition 3.4. For every $\beta > 0$ there exists $r_0, \alpha \in (0, 1)$ such that the following is true. Let $(C, h, 0, \infty)$ be a $\sqrt{8/3}$-quantum cone with the circle average embedding and let $(\Gamma_r)$ be a QLE$(8/3, 0)$ starting from 0 with the
\[ \psi(\partial B(0,2)) = h \circ \psi + Q \log |\psi'| \]

Figure 4.1: Illustration of the argument used to prove Proposition 4.2. Shown on the left is a QLE(8/3, 0) process \( \Gamma \) on a \( \sqrt{8/3} \)-quantum cone \( (C, h, 0, \infty) \) starting from 0 run up to quantum distance time \( \epsilon > 0 \). If \( \ell \) denotes the quantum boundary length of \( \Gamma_\epsilon \), then the conditional law of the surface parameterized by \( C \setminus \Gamma_\epsilon \) is given by \( \mathbf{m}^\ell \). The map \( \varphi \) takes \( C \setminus \Gamma_\epsilon \) to \( C \setminus D \) with \( \varphi(z) - z \to 0 \) as \( z \to \infty \). To bound \( \text{diam}(\Gamma_\epsilon) \), it suffices to bound the Euclidean length of \( \psi(\partial B(0,2)) \) where \( \psi = \varphi^{-1} \). By solving for \( \log |\psi'| \) in the change of coordinates formula \( \tilde{h} = h \circ \psi + Q \log |\psi'| \) for quantum surfaces and using that \( \log |\psi'| \) is harmonic, it in turn suffices to bound the extremes of the harmonic extensions of \( h \) and \( \tilde{h} \) from \( \partial \Gamma_\epsilon \) to \( C \setminus \Gamma_\epsilon \) and from \( \partial D \) to \( C \setminus D \).

4.2.1 Quantum boundary length tail bounds for the free boundary GFF

We turn to establish a tail bound for the quantum boundary length assigned by a free boundary GFF on \( Q_+ \) to \( \partial Q_+ \) where the additive constant is set so that its average on \( \partial Q_+ \) is equal to 0. This result is analogous to [DS11, Lemma 4.5] and we will make use of a similar strategy for the proof.

**Proposition 4.4.** Suppose that \( h \) is a free boundary GFF on \( Q_+ \) with additive constant fixed so that its average on \( \partial Q_+ \) is equal to 0. There exist constants \( c_0, c_1 > 0 \) such that

\[
P[A_r \geq r^\alpha, H_{R, \zeta}] \leq c_0 r^\beta \quad \text{for all} \quad r \in (0, r_0).
\]
that the following is true. Let $B$ be the quantum boundary length of $\partial Q_+$ and $\widetilde{B} = 2\gamma^{-1}\log B$. Then

$$P[\widetilde{B} \leq \eta] \leq c_0 e^{-c_1 \eta^2} \quad \text{for all } \eta \in \mathbb{R}_-. \quad (4.4)$$

We need three preparatory lemmas in order to establish Proposition 4.4.

**Lemma 4.5.** Suppose that $f : \mathbb{R}_- \rightarrow [0, 1]$ is an increasing function such that there exist constants $c_0, c_1 > 0$, $\alpha \in (1/\sqrt{2}, 1)$, and $\eta_0 \in \mathbb{R}_-$ such that

$$f(\eta) \leq e^{-c_0 \eta^2} + (f(\alpha \eta + c_1))^2 \quad \text{for all } \eta \leq \eta_0. \quad (4.5)$$

Then there exists a constant $c_2 > 0$ and $\eta_1 \in \mathbb{R}_-$ such that

$$f(\eta) \leq e^{-c_2 \eta^2} \quad \text{for all } \eta \leq \eta_1. \quad (4.6)$$

**Proof.** We set $a_0 = \eta$ and then inductively set $a_{k-1} = \alpha a_k + c_1$ for $k \geq 1$. We note that if $\eta$ is sufficiently small (depending only on $\alpha$ and $c_1$), then we have that $a_k \leq \eta$ for all $k$. Let

$$p_k = \frac{f(a_k)}{e^{-c_0 a_k^2}} \quad \text{for each } k \in \mathbb{N}.$$ 

Then for $\eta \leq \eta_0$ we have that

$$p_k \leq 1 + p_{k-1}^2 e^{-c_0(2\alpha^2 - a_k^2)} \quad \text{(by (4.5))}$$

$$\leq 1 + p_{k-1}^2 e^{-c_2 a_k^2}$$

$$\leq 1 + p_{k-1}^2 e^{-c_2 \eta^2} \quad \text{(since $a_k \leq \eta$ for all $k$)}$$

where $c_2 > 0$ is a constant. Hence $p_k$ is dominated by the sequence $q_k$ defined by

$$q_k = 1 + q_{k-1}^2 e^{-c_2 \eta^2} \quad \text{with } q_0 = 1.$$ 

By choosing $\eta$ sufficiently negative so that $1 + 4e^{-c_2 \eta^2} \leq 2$, we see that $q_k \leq 2$ for all $k$ hence $p_k \leq 2$ for all $k$. This implies that $f(a_k) \leq 2e^{-c_0 a_k^2}$ for all $k$. The result then follows from the monotonicity of $f$. \qed

**Lemma 4.6.** Suppose that $h$ is a GFF with zero boundary conditions on a bounded domain $D$, $U \subseteq D$ is open with $\text{dist}(\partial U, \partial D) > 0$, and $K \subseteq U$ is compact. Let $\tilde{h}$ be the projection of $h$ onto the subspace of functions in $H(D)$ which are harmonic on $U$. There exist constants $c_0, c_1 > 0$ depending only on $U$, $K$, and $D$ such that

$$P\left[\sup_{z \in K} |\tilde{h}(z)| \geq \eta\right] \leq c_0 e^{-c_1 \eta^2} \quad \text{for all } \eta \geq 0. \quad (4.7)$$

The same is also true if $h$ is a whole-plane GFF with the additive constant fixed so that its average on $\partial D$ is equal to 0, $U \subseteq D$ is open with $\text{dist}(U, \partial D) > 0$, and $K \subseteq U$ is compact.
Proof. We will give the proof in the case that $h$ is a GFF on a bounded domain $D$ with zero-boundary conditions. The proof in the case of the whole-plane GFF is analogous.

Fix $r_0 > 0$ such that $z \in K$ implies that $B(z, r_0)$ has distance at least $r_0$ to $\partial U$ and let $r_1 = \frac{1}{2}r_0$ and $r_2 = \frac{1}{2}r_1$. Fix $z \in K$ and, for $w \in B(z, r_1)$, let $\mu_{z,w}$ denote harmonic measure in $B(z, r_1)$ as seen from $w$. Then we can write

$$\tilde{h}(w) = \int \tilde{h}(u) d\mu_{z,w}(u)$$

and therefore

$$|\tilde{h}(w)| \leq \int |\tilde{h}(u)| d\mu_{z,w}(u).$$

Note that there exists a constant $c_0 > 0$ such that

$$\sup_{w \in B(z, r_2)} |\tilde{h}(w)| \leq c_0 \int |\tilde{h}(u)| d\mu_{z,z}(u).$$

By the compactness of $K$, it suffices to show that there exist constants $c_1, c_2 > 0$ such that

$$\mathbb{P}\left[\int |\tilde{h}(u)| d\mu_{z,z}(u) \geq \eta\right] \leq c_1 e^{-c_2 \eta^2} \text{ for all } \eta \geq 0.$$

Fix $\alpha > 0$. By two applications of Jensen’s inequality, we have that

$$\mathbb{E}\left[\exp\left(\alpha \int |\tilde{h}(u)| d\mu_{z,z}(u)\right)^2\right] \leq \int \mathbb{E}\left[e^{\alpha |\tilde{h}(u)|^2}\right] d\mu_{z,z}(u). \quad (4.8)$$

The right hand side of (4.8) is finite for $\alpha > 0$ small enough uniformly in $z \in K$ since $\tilde{h}(u)$ is a Gaussian with variance which is uniformly bounded over $u \in \partial B(z, r_1)$ for $z \in K$. This, in turn, implies the result. \qed

Lemma 4.7. Suppose that $h$ is a GFF on $D$ with zero boundary conditions. Let $B$ be the quantum boundary length of $[-1/2, 1/2]$ measured using the field $\sqrt{2} h$ and let $\tilde{B} = 2\gamma^{-1} \log B$. There exist constants $c_0, c_1 > 0$ such that

$$\mathbb{P}\left[\tilde{B} < \eta\right] \leq c_0 e^{-c_1 \eta^2} \text{ for all } \eta \in \mathbb{R}_-.$$

By the odd/even decomposition \cite[Section 3.2]{She15}, it follows that the law of the restriction of $\sqrt{2} h$ as in the statement of Lemma 4.7 is mutually absolutely continuous with respect to the law of the corresponding restriction of a free boundary GFF on $H$. Consequently, the quantum boundary length of $[-1/2, 1/2]$ assigned by $\sqrt{2} h$ is well-defined.
Proof of Lemma 4.7. Let $\tilde{h}$ be the projection of $h$ onto the subspace of functions which are harmonic in $C_- = B(\frac{1}{4}, \frac{1}{4})$ and $C_+ = B(\frac{3}{4}, \frac{1}{4})$. Then we have that $\hat{h} = h - \tilde{h}$ is given by a pair of independent zero-boundary GFFs in $C_-, C_+$. Let $B_-$ (resp. $B_+$) be the quantum boundary length of $[-3/4, -1/4]$ (resp. $[1/4, 3/4]$) computed using the GFF $\sqrt{2} h$. Let $\bar{h}$ be the infimum of $\tilde{h}$ on $[-3/4, -1/4] \cup [1/4, 3/4]$ and let $\bar{B}_\pm = 2\gamma^{-1} \log B_\pm$. Then $\bar{B}_-, \bar{B}_+$ are independent, $\bar{B}_-, \bar{B}_+ \overset{d}{=} \tilde{B} - Q \log 4$, and

$$\bar{B} \geq \max(\bar{B}_-, \bar{B}_+) + \sqrt{2} h.$$  (4.9)

For each $\eta \leq 0$, we let $f(\eta) = P[\tilde{B} < \eta]$. Fix $\alpha > 0$. Then we have that

$$f(\eta) \leq P[h \leq \alpha\eta] + P[\tilde{B} < \eta, h > \alpha\eta]$$

$$\leq c_0 e^{-c_1 \alpha^2 \eta^2} + P[\tilde{B}_- + \sqrt{2}\alpha \eta < \eta, \tilde{B}_+ + \sqrt{2}\alpha \eta < \eta]$$  (Lemma 4.6 and (4.9))

$$\leq c_0 e^{-c_1 \alpha^2 \eta^2} + (f(\bar{\alpha}\eta - Q \log 4))^2$$  (with $\bar{\alpha} = 1 - \sqrt{2}\alpha$).

Assume that $\alpha > 0$ is chosen sufficiently small so that $\bar{\alpha} \in (1/\sqrt{2}, 1)$. Then Lemma 4.5 implies that there exist constants $c_2, c_3 > 0$ such that $f(\eta) < c_2 e^{-c_3 \eta^2}$, which gives the result. \hfill \box

Proof of Proposition 4.4. Lemma 4.7 implies the result when we work in the modified setting that $h$ is a GFF on $D$ with zero boundary conditions and $B$ is the quantum boundary length of $[-1/2, 1/2]$ measured using $\sqrt{2} h$. We will deduce the result from this and conformal mapping. We begin by letting $\varphi$ be a Möbius transformation which sends $[-1/2, 1/2]$ to $X = \{ \frac{1}{2} e^{\theta} : \theta \in [0, \pi] \}$, i.e. the semi-circle of radius $1/2$ in $\overline{D}$ centered at the origin, and let $\hat{h} = h \circ \varphi^{-1} + Q \log |(\varphi^{-1})'|$. Let $\hat{B}$ be the quantum boundary length assigned to $X$ by $\sqrt{2} \hat{h}$. Since $(\varphi^{-1})'$ is bounded from above and below on $X$, it follows that there exist constants $c_0, c_1 > 0$ such that

$$P[2\gamma^{-1} \log \hat{B} < \eta] \leq c_0 e^{-c_1 \eta^2} \text{ for all } \eta \in \mathbb{R}.$$  

Two applications of Lemma 4.6 and the Markov property imply that the same is true for the quantum length $\hat{B}$ assigned to $X$ by $\sqrt{2} \hat{h}$ where $\hat{h}$ is a zero-boundary GFF on $D$ and therefore by a union bound the same is true for the quantum length assigned to $\frac{1}{2} \partial D$ by $\sqrt{2} \hat{h}$. The result for the whole-plane GFF then follows by applying the Markov property and Lemma 4.6 again. Finally, the result for the GFF on $\mathbb{Q}$ with free boundary conditions follows by using the odd/even decomposition [She15, Section 3.2] of the free boundary GFF on $\mathbb{Q}_+$ in terms of the whole-plane GFF on $\mathbb{Q}$. \hfill \box
4.2.2 Harmonic tail bound for the unexplored region of a quantum cone

We are now going to use Proposition 4.4 to show that the harmonic extension of the boundary values of \( h \) sampled from \( m^1 \) (recall the definition from Section 2.1.3) is unlikely to be large when restricted to \( Q_+ + r \) for any fixed \( r > 0 \).

**Proposition 4.8.** For each \( r > 0 \) there exist constants \( c_0, c_1 > 0 \) such that the following is true. Suppose that \( (Q_+, h) \) has the law \( m^1 \). Let \( h \) be the harmonic extension of the values of \( h \) from \( \partial Q_+ \) to \( Q_+ \). Then we have that

\[
P\left[ \sup_{z \in Q_+ + r} h(z) \geq \eta \right] \leq c_0 e^{-c_1 \eta^2} \quad \text{for all} \quad \eta \in \mathbb{R}_.
\]

We will need to collect two preliminary lemmas before we give the proof of Proposition 4.8. The first result gives that Proposition 4.4 holds when we choose the additive constant for \( h \) in a slightly different way.

**Lemma 4.9.** Fix \( r > 0 \). Suppose that we have the same setup as in Proposition 4.4, let \( h \) be the function which is harmonic in \( Q_+ \) with boundary values given by those of \( h \) on \( \partial Q_+ \), and that we have taken the additive constant for \( h \) so that \( \sup_{z \in \partial Q_+ + r} h(z) \) is equal to 0. Then (4.4) still holds.

**Proof.** This follows by a union bound using Proposition 4.4 with Lemma 4.6. \( \square \)

**Lemma 4.10.** For each \( r > 0 \), consider the law \( P_r \) on random fields \( h_r \) defined as follows.

1. Sample \( h \) from \( m^1 \)
2. Take \( h_r \) to be equal to \( h \) in \( Q_+ + r \) and then sample \( h_r \) in the annulus \([0, r] \times [0, 2\pi] \) in \( Q_+ \) as a GFF with Dirichlet boundary conditions on \( \partial Q_+ + r \) given by those of \( h \) and free boundary conditions on \( \partial Q_+ \).

Let \( \mathfrak{h} \) denote the harmonic extension of the values of \( h_r \) from \( \partial Q_+ \) to \( Q_+ \) and let \( A = \sup_{z \in Q_+ + 1} \mathfrak{h}(z) \). Let \( B \) denote the quantum boundary length of \( \partial Q_+ \) and let \( \bar{B} = 2\gamma^{-1} \log B \). Fix \( x,y \in \mathbb{R} \) and let \( I_{u,\epsilon} = [u, u + \epsilon] \) for \( u \in \{x, y\} \). There exists a constant \( c_0 > 0 \) such that

\[
\limsup_{\epsilon \to 0} \limsup_{r \to \infty} \frac{P_r[A \in I_{x,\epsilon}]}{P_r[B \in I_{y,\epsilon}]} \leq c_0 e^{(Q-\gamma)(x-y)/2}.
\]

**Proof.** For each \( r > 0 \) we let \( W_r \) be the average of \( h \) on \( \partial Q_+ + r \). The resampling properties for \( m^1 \) (see, e.g., [MS15c, Proposition 6.5]) imply that

\[
W_r = (Q - \gamma)r + U_r + X \quad (4.10)
\]
where $U_r$ is a standard Brownian motion with $U_0 = 0$ and $X$ is almost surely finite. Under $P_r$, the conditional law of the average of the field on $\partial Q_+$ given $W_r$ is that of a Gaussian random variable with mean $W_r$ and variance $2r$. It therefore follows that there exists a constant $c_0 > 0$ such that

$$P_r[A \in I_{x,\epsilon} | W_r] \leq \frac{c_0 \epsilon}{\sqrt{r}} e^{-(W_r-x)^2/4r}.$$  \hfill (4.11)

We similarly have that there exists a constant $c_1 > 0$ such that

$$P_r[\tilde{B} \in I_{y,\epsilon} | W_r] \geq \frac{c_1 \epsilon}{\sqrt{r}} e^{-(W_r-y)^2/4r}.$$  \hfill (4.12)

The result follows by combining (4.11) and (4.12) and using (4.10).

**Proof of Proposition 4.8** Let $P_r$, $A$, $B$, $\tilde{B}$, $I_{x,\epsilon}$, and $I_{y,\epsilon}$ be as in Lemma 4.10. By Bayes’ rule we have that

$$P_r[A \in I_{x,\epsilon} | \tilde{B} \in I_{y,\epsilon}] = \frac{P_r[A \in I_{x,\epsilon}] P_r[\tilde{B} \in I_{y,\epsilon} | A \in I_{x,\epsilon}]}{P_r[\tilde{B} \in I_{y,\epsilon}]}.$$  \hfill (4.13)

Let $f_y(x)$ be the conditional density of $A$ at $x$ given $\tilde{B} = y$ and let $g_x(y)$ be the conditional density of $\tilde{B}$ at $y$ given $A = x$. By Lemma 4.10 and (4.13), there exists a constant $c_0 > 0$ such that

$$f_y(x) = \lim_{\epsilon \to 0} \lim_{r \to \infty} \frac{1}{\epsilon} P_r[A \in I_{x,\epsilon} | \tilde{B} \in I_{y,\epsilon}] \leq c_0 e^{(Q-\gamma)(x-y)/2} g_x(y).$$  \hfill (4.14)

Proposition 4.4 implies that there exist constants $c_1, c_2 > 0$ such that

$$\int_{y-1}^{y} g_x(s) ds \leq c_1 e^{-c_2(x-y)^2} \text{ for all } y < x.$$  \hfill (4.15)

Combining (4.14) and (4.15) implies that there exist constants $c_3, c_4 > 0$ such that, under $m^1$, we have that the probability that the supremum of the harmonic extension of the field from $\partial Q_+$ to $Q_+$ restricted to $\partial Q_+ + 1$ is at least $\eta$ is at most $c_3 e^{-c_4 \eta^2}$. \hfill $\square$

**4.2.3 Proof of Proposition 4.2**

Suppose that $(C, h, 0, \infty)$ is a $\sqrt{8/3}$-quantum cone with the circle average embedding as in the statement of the proposition and let $(\Gamma_r)$ be the QLE($8/3, 0$) growing from 0 to $\infty$.

Throughout, we let $\gamma = \sqrt{8/3}$. Fix $\epsilon > 0$ and let $\ell_\epsilon$ be the quantum boundary length of the outer boundary of $\Gamma_\epsilon$. Let $\varphi : C \setminus \Gamma_\epsilon \to C \setminus D$ be the unique conformal
transformation with \( \varphi(\infty) = \infty \) and \( \varphi'(\infty) > 0 \) and let \( \psi = \varphi^{-1} \). We then let \( h_1 = h \circ \psi + Q \log |\psi'| - 2\gamma^{-1} \log \ell \) so that \((C \setminus D, h_1)\) has the law \( \mathfrak{m}^1 \).

Let \( R^* = 4\pi \sup_{z \in \partial B(0,2)} |\psi'(z)| \) and note that

\[
\text{diam}(\Gamma_c) \leq \int_{\partial B(0,2)} |\psi'(z)| dz \leq R^* \tag{4.16}
\]

where \( dz \) denotes Lebesgue measure on \( \partial B(0,2) \). It therefore suffices to show that there exist \( a_0, c_0 > 0 \) such that

\[
\mathbf{P}[R^* \geq \epsilon^{a_0}] \leq c_0 \epsilon^{\delta}. \tag{4.17}
\]

Fix \( \zeta > 0 \) and let \( E_1 = \{ \ell \leq \epsilon^{2-\zeta} \} \). By Lemma 3.1 we have for constants \( c_1, c_2 > 0 \) that \( \mathbf{P}[E_1^c] \leq c_1 \exp(-c_2 \epsilon^{-4}) \). It therefore suffices to work on \( E_1 \).

Write \( h_2 = h + \gamma \log |\cdot| \). By the change of coordinates formula for quantum surfaces, we have on the event \( E_1 \) that

\[
Q \log |\psi'| = \frac{2}{\gamma} \log \ell + \gamma \log |\psi(\cdot)| + h_1 - h_2 \circ \psi
\]

\[
\leq \frac{4 - 2\zeta}{\gamma} \log \epsilon + \gamma \log |\psi(\cdot)| + h_1 - h_2 \circ \psi. \tag{4.18}
\]

Let \( h_1 \) (resp. \( h_2 \)) be the function which is harmonic in \( C \setminus D \) (resp. \( C \setminus \Gamma_c \)) with boundary values given by those of \( h_1 \) (resp. \( h_2 \)) on \( \partial D \) (resp. \( \partial \Gamma_c \)). Proposition 4.8 implies that there exist constants \( c_3, c_4 > 0 \) such that with

\[
E_2 = \left\{ \sup_{z \in \partial B(0,2)} h_1(z) \leq \frac{\zeta}{\gamma} \log \epsilon^{-1} \right\} \quad \text{we have} \quad \mathbf{P}[E_2^c] \leq c_3 \exp(-c_4 \zeta^2 (\log \epsilon^{-1})^2).
\]

Therefore it suffices to work on \( E_2 \).

Since the left side of (4.18) is harmonic in \( C \setminus D \) it follows that (4.18) holds with \( h_1, h_2 \) in place of \( h_1, h_2 \) so that on \( E_1 \cap E_2 \) we have for \( z \in \partial B(0,2) \) that

\[
Q \log |\psi'(z)| \leq \frac{4 - 2\zeta}{\gamma} \log \epsilon + \gamma \log |\psi(z)| + h_1(z) - h_2(\psi(z))
\]

\[
\leq \frac{4 - 3\zeta}{\gamma} \log \epsilon + \gamma \log |\psi(z)| - h_2(\psi(z)). \tag{4.19}
\]

For each \( r > 0 \) and \( z \in C \), we let \( h_{2,r}(z) \) denote the average of \( h_2 \) on \( \partial B(z, r) \). For each \( \delta > 0 \) we let

\[
k_\delta = \inf \left\{ k \in \mathbb{N} : \inf_{z \in B(0, \epsilon/2) \setminus \Gamma_c} (h_2(z) + (2 + 2\delta)(\log (\text{dist}(z, \Gamma_c))^{-1} + 2j)) \geq 0 \right\}
\]

and
\[ \tilde{k}_\delta = \inf \left\{ k \in \mathbb{N} : \inf_{j \geq k, z \in B(0,e^j/2)} \left( h_{2,r}(z) + (2 + 2\delta) \left( \log r^{-1} + 2j \right) \right) \geq 0 \right\}. \]

Corollary 2.5 together with a union bound implies that there exist constants \(c_5, c_6 > 0\) such that
\[
P\left[ \tilde{k}_\delta \geq \frac{1}{2\gamma} \log \epsilon^{-1} \right] \leq c_5 \epsilon^{c_6(1+\delta)}. \tag{4.20}
\]

We claim that (4.20) holds with \(k_{3}\) in place of \(\tilde{k}_\delta\). Indeed, suppose that \(w \in C \setminus \Gamma_{\epsilon}\) and \(r = \text{dist}(w, \Gamma_{\epsilon})\). Then we can write
\[
h_{2,r}(w) = h_2(w) + Z_{w,r} \tag{4.21}
\]
where \(Z_{w,r}\) is a mean-zero Gaussian random variable independent of \(h_2\) with variance which is uniformly bounded in \(w\). This implies the claim.

For \(z \in B(0, 2)\) we note that
\[
|\psi(z)| \leq \text{diam}(\psi(B(0, 2))) \leq R^*. \tag{4.22}
\]

Suppose we are working on the event that \(k_{\delta} < (2\gamma)^{-1} \log \epsilon^{-1}\). Then we have that
\[
- \inf_{z \in \partial B(0, 2)} h_2(\psi(z)) \leq \frac{2 + 2\delta}{\gamma} \log \epsilon^{-1} - (2 + 2\delta) \log R^*. \tag{4.23}
\]

Thus using (4.22) and (4.23) we have from (4.19) the upper bound
\[
Q \log R^* \leq \frac{2 - 2\delta - 3\zeta}{\gamma} \log \epsilon + \gamma \log R^* - (2 + 2\delta) \log R^* + c_7 \tag{4.24}
\]
where \(c_7 > 0\) is a constant. Rearranging (4.24) gives for a constant \(c_8 > 0\) that
\[
\log R^* \leq \frac{2 - 2\delta - 3\zeta}{\gamma(2 + 2\delta)} \log \epsilon + c_8. \tag{4.25}
\]

By making very \(\delta > 0\) very large, we see from (4.25) that \(P[R^* \geq \epsilon^{-1/\gamma}]\) decays to 0 faster than any power of \(\epsilon\). Combining this with the scaling properties of quantum cones (i.e., the law of a quantum cone with the circle average embedding is invariant under the operation of adding a constant to the field and then applying a change of coordinates so that the field has the circle average embedding) and the quantum natural time (Lemma 2.2) implies (4.2).

On the event that \(\text{diam}(\Gamma_{\epsilon}) \leq 1\), we can use Proposition 2.4 in place of Corollary 2.5. In this case, we get for constants \(c_9, c_{10} > 0\) that
\[
\log R^* \leq \frac{4 - 3\zeta}{\gamma(Q - \gamma + 2 + \delta)} \log \epsilon + c_9. \tag{4.26}
\]
off an event which occurs with probability at most $c_10\varepsilon^{2\delta(1-\zeta)}$. This implies (4.3) because we uniformly have that

$$4 \left( \frac{4-3\zeta}{\gamma(Q-\gamma+2+\delta)} \right) + 2\delta(1-\zeta) > 4 \quad \text{for all} \quad \delta > 0$$

provided we fix $\zeta > 0$ small enough.

\[ \square \]

5 H"older continuity of the QLE(8/3, 0) metric

We will prove Theorems 1.1–1.3 and Theorem 1.6 in this section. We will prove the first two results in the setting of a $\sqrt{8/3}$-quantum cone. As we will see, this setting simplifies some aspects of the proofs because a quantum cone is invariant under the operation of multiplying its area by a constant.

To prove Theorem 1.1, we suppose that $\mathcal{C} = (\mathcal{C}, h, 0, \infty)$ is a $\sqrt{8/3}$-quantum cone. We want to get an upper bound on the amount of quantum distance time that it takes for the QLE$(8/3, 0)$ process $(\Gamma_r)$ starting from 0 to hit a point $w \in \mathcal{C}$ with $|w|$ small. There are two possibilities if $(\Gamma_r)$ does not hit $w$ in a given amount of quantum distance time $r$. First, it could be that $w$ is contained in the hull of $\Gamma_r$ in which case we can use the bound established in Section 5.1 just below for the quantum diameter of the hull of $\Gamma_r$ to get that the quantum distance of 0 and $w$ is not too large. The second possibility is that $w$ is not contained in the hull of $\Gamma_r$ in which case due to our lower bound on the Euclidean hull diameter established in Section 4.1, we would get that the distance of $w$ to the hull of $\Gamma_r$ is much smaller than the Euclidean diameter of $\Gamma_r$. This implies that if we apply the unique conformal map which takes the unbounded component of the complement of $\Gamma_r$ to $\mathcal{C} \setminus \mathcal{D}$ which fixes and has positive derivative at $\infty$ then the image of $w$ will have modulus which is very close to 1. Therefore we need to get an upper bound on the quantum distance of those points in a surface sampled from $m^1$ parameterized by $\mathcal{C} \setminus \mathcal{D}$ which are close to $\partial \mathcal{D}$. We accomplish this in Section 5.2.

To prove Theorem 1.2 we suppose that $\mathcal{C} = (\mathcal{C}, h, 0, \infty)$ is a $\sqrt{8/3}$-quantum cone. We want to get an upper bound on the amount of quantum distance time that it takes for the QLE$(8/3, 0)$ process $(\Gamma_r)$ starting from 0 to hit a point $w \in \mathcal{C}$ with $|w|$ small. There are two possibilities if $(\Gamma_r)$ does not hit $w$ in a given amount of quantum distance time $r$. First, it could be that $w$ is contained in the hull of $\Gamma_r$ in which case we can use the bound established in Section 5.1 just below for the quantum diameter of the hull of $\Gamma_r$ to get that the quantum distance of 0 and $w$ is not too large. The second possibility is that $w$ is not contained in the hull of $\Gamma_r$ in which case due to our lower bound on the Euclidean hull diameter established in Section 4.1, we would get that the distance of $w$ to the hull of $\Gamma_r$ is much smaller than the Euclidean diameter of $\Gamma_r$. This implies that if we apply the unique conformal map which takes the unbounded component of the complement of $\Gamma_r$ to $\mathcal{C} \setminus \mathcal{D}$ which fixes and has positive derivative at $\infty$ then the image of $w$ will have modulus which is very close to 1. Therefore we need to get an upper bound on the quantum distance of those points in a surface sampled from $m^1$ parameterized by $\mathcal{C} \setminus \mathcal{D}$ which are close to $\partial \mathcal{D}$. We accomplish this in Section 5.2.

We put all of our estimates together to complete the proof of Theorem 1.1 in Section 5.3.1 using a Kolmogorov-Çentsov type argument, except we subdivide our space using a sequence of i.i.d. points chosen from the quantum measure rather than the usual dyadic subdivision.

We will prove Theorem 1.2 using a similar argument in Section 5.3.2 using the upper bound on the Euclidean diameter of a QLE$(8/3, 0)$ hull established in Section 4.2.

The estimates used to prove Theorem 1.1 and Theorem 1.2 will easily lead to the proofs of Theorem 1.3 in Section 5.3.3 and Theorem 1.6 in Section 5.3.5.
5.1 Quantum diameter of QLE(8/3, 0) hull

We are now going to give an upper bound on the tail of the quantum diameter of a QLE(8/3, 0) on a $\sqrt{8/3}$-quantum cone. In other words, we will bound the tail of the amount of additional time it requires a QLE(8/3, 0) on a $\sqrt{8/3}$-quantum cone run for a given amount of time to fill all of the components that it has separated from $\infty$. In what follows, it will be necessary to truncate on the event \( H_{R,\zeta} \) from Proposition 3.4 in order to ensure that the tail decays to 0 sufficiently quickly.

Lemma 5.1. Suppose that \((\Gamma_r)\) is a QLE(8/3, 0) process on a $\sqrt{8/3}$-quantum cone \((C, h, 0, \infty)\) starting from 0 with the quantum distance parameterization. Let \( H_{R,\zeta} \) be the event as in Proposition 3.4. Fix \( \epsilon > 0 \) and let \( d^*_\epsilon \) be the supremum of the amount of time that it takes \((\Gamma_r)\) to fill all of the quantum disks which have been separated from \( \infty \) by quantum distance time \( \epsilon \). For every \( \beta > 0 \) there exists \( \alpha \in (0, 1) \) and \( c_0 > 0 \) such that

\[
\mathbb{P}[d^*_\epsilon \geq \epsilon^\alpha, H_{R,\zeta}] \leq c_0 \epsilon^\beta. \tag{5.1}
\]

We note that on the event in (5.1) the quantum diameter of the hull of \( \Gamma_\epsilon \) is at most \( 2(\epsilon + \epsilon^\alpha) \).

The main input into the proof of Lemma 5.1 is the following lemma which gives the tail for the amount of time that it takes a QLE(8/3, 0) growth starting from the boundary of a quantum disk to hit every point in the disk.

Lemma 5.2. Fix \( 0 < a < \infty \) and suppose that \((D, h)\) is a unit boundary length quantum disk conditioned to have quantum area at most \( a \). Let \( d^* \) be the amount of time that it takes the QLE(8/3, 0) exploration starting from \( \partial D \) to hit every point in \( D \). There exists a constant \( c_0 > 0 \) depending only on \( a \) such that

\[
\mathbb{P}[d^* \geq r] \leq c_0 \exp(-\frac{3}{2}(1 + o(1))r^{4/3})
\]

where the \( o(1) \) term tends to 0 as \( r \to \infty \).

The QLE(8/3, 0) exploration from \( \partial D \) is defined because the conditional law of the components cut off from \( \infty \) by the QLE exploration are given by conditionally independent quantum disks given their boundary length.

Proof of Lemma 5.2. This is a consequence of [MS15a, Proposition 4.18] and the branching structure of QLE(8/3, 0). \( \square \)

Proof of Lemma 5.1. Suppose that \( \alpha, \alpha' \in (0, 1) \). We will adjust their values in the proof. For each \( \epsilon > 0 \), we let \( \tau_\epsilon \) (resp. \( \sigma_\epsilon \)) be the first \( r > 0 \) such that \((\Gamma_r)\) cuts off a bubble with quantum diameter (resp. area) at least \( \epsilon^\alpha \) (resp. \( \epsilon^{\alpha'} \)). Fix \( \beta > 0 \). We have that

\[
\mathbb{P}[d^* \geq \epsilon^\alpha, H_{R,\zeta}] = \mathbb{P}[\tau_\epsilon \leq \epsilon, H_{R,\zeta}]
\]
\[
\leq P[\sigma_\epsilon \leq \tau_\epsilon \leq \epsilon, \ H_{R,\zeta}] + P[\tau_\epsilon \leq \sigma_\epsilon, \ H_{R,\zeta}]
\leq P[\sigma_\epsilon \leq \epsilon, \ H_{R,\zeta}] + P[\tau_\epsilon \leq \sigma_\epsilon].
\]

If we let \( A_\epsilon \) be the quantum area separated by \( \Gamma_\epsilon \) from \( \infty \), then the first term above is bounded from above by \( P[A_\epsilon \geq e^{\alpha'}, \ H_{R,\zeta}] \). Corollary 4.3 implies that we can make \( \alpha' \in (0, 1) \) small enough so that this probability is at most \( c_0 e^{\beta} \). The lemma thus follows because Lemma 5.2 implies that by making \( \alpha \in (0, 1) \) small enough, there exists a constant \( c_1 > 0 \) such that \( P[\tau_\epsilon \leq \sigma_\epsilon] \leq c_1 e^{\beta} \).

5.2 Euclidean disks are filled by QLE(8/3, 0) growth

Figure 5.1: Illustration of the setup and the argument of Proposition 5.3, which shows that Euclidean disks are filled by the QLE(8/3, 0) growth. Left: a QLE(8/3, 0) process \( \Gamma_r \) starting from the origin of a \( \sqrt{8/3} \)-quantum cone run up to a given radius \( r > 0 \). The dashed curve indicates the range of \( \Gamma \) at time \( r + \epsilon^\zeta \) for \( \zeta > 0 \) very small. Middle: The map \( \psi \) is the unique conformal map from \( \mathbb{C} \setminus \Gamma_r \) to \( Q_+ \) with \( \infty \) sent to \( +\infty \) and with positive derivative at \( \infty \). The region bounded by the dashed curve is the image under \( \psi \) of the corresponding region on the left. Right: The map \( \varphi \) is the unique conformal map from the unbounded complementary component of the dashed region to \( Q_+ \) with \( \varphi(z) - z \to 0 \) as \( z \to +\infty \). To prove the result (see Figure 5.2 for an illustration), we show that by making \( \zeta > 0 \) sufficiently small the event that for every \( z \) with \( \text{Re}(z) \in [\epsilon/2, \epsilon] \) we have that \( \text{Re}(\varphi(z)) < \epsilon/4 \) occurs with overwhelming probability. Iterating this implies there exists \( \beta > 0 \) such that, with overwhelming probability, the QLE(8/3, 0) growing from \( Q_+ \) absorbs all such \( z \) in time \( e^{\beta} \).

We will now give an upper bound on the amount of quantum distance time that it takes for the QLE(8/3, 0) hull growing in \( Q_+ \) from \( \partial Q_+ \) to fill a neighborhood of \( \partial Q_+ \) where the quantum surface has law \( m_1 \). Similar to the setting of Lemma 5.2 considered above, it makes sense to talk about the QLE(8/3, 0) hull growing from \( \partial Q_+ \) because \( m_1 \)
Figure 5.2: (Continuation of Figure 5.1) Shown on the left is a copy of the middle part of Figure 5.1 scaled so that the law of the surface is given by $m^1$. Suppose that $\text{Re}(z) \in [\epsilon/2, \epsilon]$ and that $\varphi$ is as in Figure 5.1. In order to show that $\text{Re}(\varphi(z)) \leq \epsilon/4$ with overwhelming probability, we place semi-disks of radius $\epsilon/(\log \epsilon - 1)^2$ with equal spacing $\epsilon/(\log \epsilon - 1)$ along $\partial Q_+$. Shown on the right is an enlargement of one of the semi-disks. The restriction of the field to each semi-disk is mutually absolutely continuous with respect to the law of a quantum disk. By making such a comparison, we see that if we pick a uniformly random point inside of the semi-disk and then grow the QLE($8/3, 0$) starting from that point until it hits the boundary, then there is a positive chance that the QLE($8/3, 0$) first exits in $\partial Q_+$ and does so in time at most $\epsilon^\sigma$. By the metric property, the range of this QLE($8/3, 0$) is then contained in the QLE($8/3, 0$) growing from $\partial Q_+$ for time $\epsilon^\sigma$. Since the behavior of the field in each of the semi-disks is approximately independent, with overwhelming probability, there cannot be a collection of consecutive semi-disks so that the event does not occur for any of them. In particular, there must exist a semi-disk which close enough to $z$ to show that $\text{Re}(\varphi(z) - z)$ decreases by a definite amount. Iterating this yields the desired bound.

gives the conditional law of the unbounded component when performing a QLE($8/3, 0$) exploration of a $\sqrt{8/3}$-quantum cone, after rescaling so that the boundary length is equal to 1. The main result is the following proposition.

**Proposition 5.3.** Suppose that $(Q_+, h)$ has law $m^1$. For each $\beta > 0$ there exist constants $c_0, \alpha, \zeta > 0$ such that the following is true. Let $E_{\alpha, \zeta}$ be the event that every $z \in Q_+$ with $\text{Re}(z) < \epsilon^\alpha$ is contained in the QLE($8/3, 0$) hull of radius $\epsilon^\xi$ growing from $\partial Q_+$. Then $P[E_{\alpha, \zeta}] \leq c_0 e^{\beta}$. Moreover, if we fix $\sigma > 0$ and let $A_{\alpha, \sigma, \epsilon}$ be the event that the quantum area of $\{z \in Q_+ : \text{Re}(z) < \epsilon^\alpha\}$ is at most $\epsilon^\sigma$, then (with $\alpha$ fixed) for each $\beta > 0$ there exists $\zeta \in (0, 1)$ such that $P[E_{\alpha, \zeta}, A_{\alpha, \sigma, \epsilon}] \leq c_0 e^{\beta}$.

We begin by recording an elementary lemma which gives the Radon-Nikodym derivative of the GFF with mixed boundary conditions when we change the boundary conditions on the Dirichlet part.

**Lemma 5.4.** Suppose that $D \subseteq \mathbb{C}$ is a bounded Jordan domain and $\partial D = \partial^F \cup \partial^D$ where $\partial^F, \partial^D$ are non-empty, disjoint intervals. Let $h_1, h_2$ be GFFs on $D$ with free
(resp. Dirichlet) boundary conditions on $\partial^F$ (resp. $\partial^D$). Let $U \subseteq D$ be open with positive distance from $\partial^D$ and let $g$ be the function which is harmonic in $D$ with Neumann (resp. Dirichlet) boundary conditions $\partial^F$ (resp. $\partial^D$) where the Dirichlet boundary conditions are given by the difference of those of $h_1$ and $h_2$. The Radon-Nikodym derivative $Z$ of the law of $h_1|_U$ with respect to the law of $h_2|_U$ is given by

$$Z = \exp \left( \langle h_2|_U, g|_U \rangle - \| g|_U \|_{\mathcal{F}}^2 / 2 \right). \quad (5.2)$$

Proof. We first recall that if $h$ is a GFF on a domain $D \subseteq \mathbb{C}$ and $f \in H(D)$ then the Radon-Nikodym derivative of the law of $h + f$ with respect to the law of $h$ is given by $\exp(\langle h, f \rangle - \| f \|_{\mathcal{F}}^2 / 2)$. (This is proved by using that the Radon-Nikodym derivative of the law of a $N(\mu, 1)$ random variable with respect to the law of a $N(0, 1)$ random variable is given by $e^{\mu - \mu^2 / 2}$.) We can extract from this the result as follows. We let $\tilde{g} = g\phi$ where $\phi \in C_0^\infty(D)$ with $\phi|_U \equiv 1$. Then we have that $\langle h_2 + \tilde{g} \rangle|_U$ has the law of $h_1|_U$. Moreover, with $W = D \setminus U$ we have that the Radon-Nikodym derivative of the law of $h_2 + \tilde{g}$ with respect to the law of $h_2$ is given by a normalizing constant times

$$\exp \left( \langle (h_2|_U, \tilde{g}|_U \rangle - \| h_2|_W, \tilde{g}|_W \|_{\mathcal{F}} \right) = \exp \left( \langle h_2|_U, g|_U \rangle + \langle h_2|_W, \tilde{g}|_W \|_{\mathcal{F}} \right).$$

Since $U, W$ are disjoint, we have that the random variables $\langle h_2|_U, g|_U \rangle$, $\langle h_2|_W, \tilde{g}|_W \|_{\mathcal{F}}$ are independent. The result thus follows by integrating over the randomness of $(h_2|_W, \tilde{g}|_W)$. $\square$

Suppose that we are in the setting of Lemma 5.4 and that there exists a constant $M > 0$ such that

$$\sup_{z, w \in U} |g(z) - g(w)| \leq M.$$ 

Then elementary regularity estimates for harmonic functions imply that there exists a constant $c_0 > 0$ such that

$$\| g|_U \|_{\mathcal{F}}^2 \leq c_0 M^2 |U| \quad (5.3)$$

where $|U|$ denotes the Lebesgue measure of $U$. Inserting the bound (5.3) into (5.2) and using, for example, the Cauchy-Schwarz inequality gives us a uniform lower bound on the probability of an event which depends on $h_2|_U$ in terms of the probability of the corresponding event computed using the law of $h_1|_U$. We will make use of this fact shortly.

**Lemma 5.5.** Suppose that $D \subseteq \mathbb{C}$ is a bounded Jordan domain and $\partial D = \partial^F \cup \partial^D$ where $\partial^F, \partial^D$ are non-empty, disjoint intervals. There exists $U \subseteq D$ open with positive distance to $\partial^D$, $p > 0$, and $K < \infty$ such that the following is true. Suppose that $h$ is a GFF on $D$ with free (resp. Dirichlet) boundary conditions on $\partial^F$ (resp. $\partial^D$) where the Dirichlet part differs from a given constant $A$ by at most $K$. Pick $z \in D$ uniformly from the quantum measure. Then the QLE$(8/3, 0)$ starting from $z$ has chance at least $p$ of hitting $\partial U$ first in $\partial^F$ before reaching quantum distance time $e^{\gamma/4(A+K)}$, $\gamma = \sqrt{8/3}$. 

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Proof. This follows by combining Lemma 5.4 and Lemma 2.1.

We will now argue that if we place small neighborhoods at evenly spaced points on $\partial \mathcal{Q}_+$ then the law of the field sampled from $m^1$ restricted to each such neighborhood is approximately independent of the field restricted to the other neighborhoods, up to an additive constant.

Lemma 5.6. Suppose that $h$ has the law of a GFF on the annulus $D = [0, 2\pi]^2 \subseteq \mathcal{Q}_+$ (so that the top and bottom of $[0, 2\pi]^2$ are identified) with free (resp. Dirichlet) boundary conditions on the left (resp. right) side of $\partial D = \partial F \cup \partial D$. Fix $\epsilon > 0$ very small and let $x_1, \ldots, x_n$ be equally spaced points on $\partial F$ with spacing $\epsilon (\log \epsilon - 1)^{-1}$. Let $r = r_\epsilon = \epsilon (\log \epsilon - 1)^{-2}$. For each $k$, let $U_k = B(x_k, r) \cap \mathcal{Q}_+$ and let $h_k$ be the function which is harmonic in $\mathcal{Q}_+ \setminus \bigcup_{j \neq k} U_j$ with boundary conditions given by those of $h$ on $\bigcup_{j \neq k} \partial U_j$ and Neumann boundary conditions on $\partial \mathcal{Q}_+ \setminus \bigcup_{j \neq k} U_j$. Let

$$\Delta_k = \sup_{z, w \in U_k} |h_k(z) - h_k(w)|.$$

For each $M > 0$ there exist constants $K, c_0 > 0$ such that if $E = \{ \max_k \Delta_k \leq K \}$ then

$$P[E] \leq c_0 \epsilon M.$$

Proof. By the odd/even decomposition of the GFF with mixed boundary conditions (see [DS11, Section 6.2] or [She15, Section 3.2]), we can represent $h$ as the even part of a GFF $h^\dagger$ on the annulus $D^\dagger = [-2\pi, 2\pi] \times [0, 2\pi] \subseteq \mathcal{Q}$ (so that the top and the bottom are identified) with Dirichlet boundary conditions. Fix a value of $1 \leq k \leq n$. The conditional law of $h^\dagger$ in $B(x_k, r)$ given its values on $B(x_j, r)$ for $j \neq k$ is given by that of the sum of a GFF on $D^\dagger \setminus \bigcup_{j \neq k} B(x_j, r)$ with zero boundary conditions and a harmonic function $h^\dagger_k$. By the odd/even decomposition, we note that

$$h_k(z) = \frac{1}{\sqrt{2}} (h^\dagger_k(z) + h^\dagger_k(z^*))$$

where $z^*$ is the reflection of $z$ about the vertical axis through 0. Proposition 2.4 implies that there exists a constant $c_0 > 0$ such that the probability of the event that $|h_k(z)| \leq M \log \epsilon^{-1}$ for all $z \in B(x_k, r \log \epsilon^{-1})$ is at least $1 - c_0 \epsilon^{2M}$. Elementary regularity results for harmonic functions then tell us that there exists a constant $c_1 > 0$ such that, on this event, we have

$$\sup_{z, w \in B(x_k, r)} |h_k(z) - h_k(w)| \leq \frac{c_1 M \log \epsilon^{-1}}{r \log \epsilon^{-1}} \times r = c_1 M.$$

Applying a union bound over $1 \leq k \leq n$ implies the result.

We now combine Lemma 5.5 and Lemma 5.6 to argue that with high probability the QLE(8/3, 0) hull grows from a “dense” set of points near $\partial \mathcal{Q}_+$.
Lemma 5.7. Suppose that $\gamma = \sqrt{8/3}$, $\alpha \in (0, Q - 2)$, and let $\beta = \gamma(Q - 2 - \alpha)/4$. Suppose that we have the same setup as in Lemma 5.6 and fix $1 \leq k \leq n$. There exist $p > 0$ and $M < \infty$ such that the following is true. Assume that $w$ is picked uniformly from the quantum area measure in $U_k$. Given $\Delta_k \leq K$, $h_k(x_k) \leq (2 + \alpha) \log \epsilon^{-1}$, and $h_k$, the conditional probability that the QLE($8/3, 0$) starting from $w$ exits $U_k$ in $\partial Q_+$ in at most $\epsilon^\beta$ quantum distance time is at least $p$.

Proof. We will deduce the result from Lemma 5.5. We note that if we perform a change of coordinates from $U_k$ to $\{ z \in D : \text{Re} (z) \geq 0 \}$ via the map $z \mapsto \epsilon^{-(\log \epsilon - 1)/4} (z - x_k)$ then the correction to the field which comes from the change of coordinates is $Q \log \epsilon - 2 \log \log \epsilon^{-1}$. By the definition of the event that we assume to be working on, we have that

$$\sup_{z \in U_k} h_k(z) \leq (2 + \alpha) \log \epsilon^{-1} + K.$$ 

The result thus follows as

$$\frac{\gamma}{4} ((2 + \alpha) \log \epsilon^{-1} + Q \log \epsilon) = \beta \log \epsilon.$$

We now prove a result which, when combined with Lemma 5.7, will give a lower bound on the rate at which the distance of the metric ball growth from a given point decreases.

Lemma 5.8. There exists a constant $c_0 > 0$ such that the following is true. Fix $\epsilon > 0$ and suppose that $K \subseteq \overline{Q_+}$ is compact such that:

- $Q_+ \setminus K$ is simply connected and
- For every $z \in Q_+$ with $\text{Re}(z) \in [\epsilon/2, \epsilon]$ there exists $w \in K$ with $\text{Re}(w) \geq \epsilon(\log \epsilon^{-1})^{-2}/2$ and $|z - w| \leq \epsilon$.

Let $\phi_K : Q_+ \setminus K \to Q_+$ be the unique conformal map which fixes $+\infty$ and has positive derivative at $+\infty$. For all $z \in Q_+ \setminus K$ with $\text{Re}(z) \in [\epsilon/2, \epsilon]$ we have that

$$\text{Re}(z) - \text{Re}(\phi_K(z)) \geq \frac{c_0 \epsilon}{(\log \epsilon^{-1})^4}. \quad (5.4)$$

Proof. Let $E^w$ denote the expectation under the law where $B$ is a standard Brownian motion starting from $w \in Q_+$ and let $\sigma$ be the first time that $B$ leaves $Q_+ \setminus K$. As $\text{Re}(w) - \text{Re}(\phi_K(w))$ is harmonic in $Q_+ \setminus K$ and $\text{Re}(\phi_K(z)) \to 0$ as $z \in Q_+ \setminus K$ tends to $K \cup \partial Q_+$, we therefore have that $\text{Re}(z) - \text{Re}(\phi_K(z)) = E^w[\text{Re}(B_\sigma)]$. From the assumptions, we thus see that the probability of the event that $\text{Re}(B_\sigma) \geq \epsilon(\log \epsilon^{-1})^{-2}/4$ is at least a constant times $(\log \epsilon^{-1})^{-2}$ when $B_0 = z$. Combining implies the result. \(\square\)
**Proof of Proposition 5.3** Fix \( \epsilon > 0 \). Let \( U_1, \ldots, U_n \) be as in Lemma 5.6 and Lemma 5.7. Assume that \( \gamma = \sqrt{8/3} \) and let \( \alpha \in (0, Q - 2) \), \( \beta = \gamma(Q - 2 - \alpha)/4 \), \( p > 0 \), and \( M < \infty \) be as in Lemma 5.7. Let \( K \) be the hull of the QLE(8/3, 0) grown from \( \partial Q_+ \) for quantum distance time \( \epsilon^\beta \). Lemma 5.6 and Lemma 5.7 together imply that with probability at least \( 1 - \epsilon^{2\alpha} \) for every \( z \in Q_+ \) with \( \text{Re}(z) \in [\epsilon/2, \epsilon] \) there exists \( w \in K \) such that \( \text{Re}(w) \geq \epsilon/(\log \epsilon - 1)^2 \) and \( |z - w| \leq \epsilon \). Let \( \phi_K \) be as in Lemma 5.8. Then we have that \( \text{Re}(z) - \text{Re}(\phi_K(z)) \geq c_0 \epsilon/(\log \epsilon - 1)^4 \).

If we iterate this procedure a constant times \((\log \epsilon - 1)^4 \) times then we see that the following is true. Suppose that \( K \) denotes the hull of the QLE(8/3, 0) grown from \( \partial Q_+ \) for quantum distance time given by a constant times \((\log \epsilon - 1)^4 \) \( \epsilon^\beta \) and let \( \phi_K \) be as above. Then on an event which occurs with probability at least \( 1 - c_1(\log \epsilon - 1)^4 \epsilon^{2\alpha} \) for a constant \( c_1 > 0 \) we have for all \( z \in Q_+ \) with \( \text{Re}(z) \in \left[ \frac{\epsilon}{2}, \epsilon \right] \) that \( \text{Re}(z) - \text{Re}(\phi_K(z)) \geq \frac{\epsilon}{4} \).

We now turn to prove the second assertion of the proposition (namely when we truncate on the amount of quantum area which is close to \( \partial Q_+ \)). The reason that we had the exponent of \( \alpha \) in the above is that we needed the field to have average at most \((2 + \alpha) \log \epsilon - 1\) in each of the \( B(x_j, r) \). Thus, we just need to argue that if we truncate on the amount of quantum area close to \( \partial Q_+ \) being at most \( \epsilon^\sigma \), then with very high probability the field averages are not larger than \((2 + \alpha) \log \epsilon - 1\) for some fixed value \( \alpha \in (0, Q - 2) \). This, in turn, follows from [DSTI Lemma 4.6]. Indeed, [DSTI Lemma 4.6] tells us that inside such a ball it is very unlikely for the field to assign mass smaller than

\[ \epsilon^{\gamma Q} \times \epsilon^{-\gamma(2 + \alpha)} = \epsilon^{\gamma(Q - 2 - \alpha)} \]

and it is easy to see that we can make this exponent larger than \( \sigma > 0 \) provided we make \( \alpha \) sufficiently close to \( Q - 2 \).

\[ \square \]

### 5.3 Proof of Hölder continuity

#### 5.3.1 Proof of Theorem 1.1

The first step (Proposition 5.9) in the proof of Theorem 1.1 is to combine the estimates of the previous sections to bound the moments of the Euclidean diameter of a QLE(8/3, 0) starting from 0 on a \( \sqrt{8/3} \)-quantum cone. The purpose of the subsequent lemmas is to transfer this estimate to the setting in which the QLE(8/3, 0) is starting from another point.

**Proposition 5.9.** Suppose that \((C, h, 0, \infty)\) is a \( \sqrt{8/3} \)-quantum cone with the circle average embedding. For every \( \beta > 0 \) there exist constants \( c_0, \alpha > 0 \) such that the following is true. With \( Y_\epsilon = \sup_{z \in B(0, \epsilon)} d_Q(0, z) \) and \( H_{R, \zeta} \) as in Proposition 3.4 we have that

\[ \mathbb{P}[Y_\epsilon \geq \epsilon^\alpha, H_{R, \zeta}] \leq c_0 \epsilon^\beta \quad \text{for all } \epsilon \in (0, 1). \]
Proof. Fix $\beta > 0$. Let $\alpha, \delta, \xi > 0$ be parameters. We will adjust their values in the proof. Let $\Gamma$ be the hull of the QLE$(8/3, 0)$ exploration starting from $0$ and stopped at the first time that it reaches quantum radius $\epsilon^\beta$. Let $E$ be the event that the quantum diameter of the hull of $\Gamma$ is smaller than $\epsilon^\alpha$ and let $F$ be the event that $B(0, \epsilon) \subseteq \Gamma$. Note that $E \cap F$ implies $Y_\epsilon < \epsilon^\alpha$. Thus we have that

$$
P[Y_\epsilon \geq \epsilon^\alpha, H_{R, \xi}] \leq \P[E^c \cap H_{R, \xi}] + \P[F^c \cap H_{R, \xi}].$$

By Lemma 5.1 (and the comment just after the statement), we know that by making $\alpha/\delta > 0$ small enough we have that $\P[E^c \cap H_{R, \xi}] \leq c_0 \epsilon^\beta$ for a constant $c_0 > 0$.

Thus, we are left to bound $\P[F^c \cap H_{R, \xi}]$. Let $\tilde{\Gamma}$ be the hull of the QLE$(8/3, 0)$ process grown for quantum distance time $\epsilon^\delta/2$. We let $G$ be the event that the Euclidean diameter of $\tilde{\Gamma}$ is at least $\epsilon^\xi$. Then we have that

$$
P[F^c \cap H_{R, \xi}] \leq \P[F^c \cap G \cap H_{R, \xi}] + \P[G^c \cap H_{R, \xi}]. \tag{5.5}$$

By adjusting the value of $c_0 > 0$ if necessary and making $\delta > 0$ small enough, Proposition 4.1 implies that the second term in (5.5) is bounded by $c_0 \epsilon^\delta$. To handle the first term, we let $\varphi : C \setminus \tilde{\Gamma} \to Q_+$ be the unique conformal map with $\varphi(\infty) = +\infty$ and $\varphi'(\infty) > 0$. Since the diameter of $\tilde{\Gamma}$ on $G$ is at least $\epsilon^\xi$, it follows from the Beurling estimate that there exists a constant $c_1 > 0$ such that on $G$ we have $\sup_{y \in B(0, \epsilon)} \Re(\varphi(y)) \leq c_1 \epsilon^{(1-\xi)/2}$. Thus by possibly decreasing the value of $\xi > 0$ and increasing the value of $c_0 > 0$, Proposition 5.3 implies that $\P[F^c \cap G \cap H_{R, \xi}] \leq c_0 \epsilon^\beta$. \qed

We next show that QLE$(8/3, 0)$ defines a metric on a countable, dense subset of a $\sqrt{8/3}$-quantum cone.

Lemma 5.10. Let $\eta'$ be a space-filling SLE on a $\sqrt{8/3}$-quantum cone $(C, h, 0, \infty)$ from $\infty$ to $\infty$ parameterized by quantum area. Then QLE$(8/3, 0)$ defines a metric $d_Q$ on $\{\eta'(t) : t \in Q\}$.

Proof. Throughout the proof, we will write $d_Q(s, t)$ for $d_Q(\eta'(s), \eta'(t))$. First, we note that Proposition 5.9 implies that $d_Q(s, t) < \infty$ almost surely for any fixed $s, t \in R$. Fix $s \in R$ and suppose that we have recentered the quantum cone so that $\eta'(t) = 0$ and then we rescale so that we have the circle average embedding. By [DMST14] Theorem 1.13, the resulting field has the same law as $h$. Then Proposition 4.2 implies that the diameter of the QLE$(8/3, 0)$ running from $\eta'(t) = 0$ stopped at the first time that it hits $\eta'(s)$ is finite almost surely and that the same is true when we swap the roles of $\eta'(s)$ and $\eta'(t)$. Fix $R > 0$. As the restriction of $h$ to $B(0, R)$ is mutually absolutely continuous with respect to the corresponding restriction of a quantum sphere with large area, by fixing $R > 0$ sufficiently large it follows from the main result of [MS15b] that $d_Q(s, t) = d_Q(t, s)$. Applying the same argument but with three points implies that the triangle inequality is satisfied. \qed
Lemma 5.11. For each $p \in (0, 1)$ there exists $s_1, s_2 \in \mathbb{R}$ with $s_1 < s_2$, $\eta \in \mathbb{D}$ \{0\}, and $r_1 > 0$ with $B(z_0, r_1) \subseteq \mathbb{D}$ \{0\} such that the following is true. Suppose that $(C, h, 0, \infty)$ is a $\sqrt{8/3}$-quantum cone with the circle average embedding and let $\eta'$ be a space-filling SLE$_6$ process from $\infty$ to $\infty$ sampled independently of $h$ and then reparameterized according to $\sqrt{8/3}$-LQG area. Then with

$$E(z_0, r_1, s_1, s_2) = \{B(z_0, r_1) \subseteq \eta'([s_1, s_2]) \subseteq \mathbb{D}\} \quad (5.6)$$

we have that $P[E(z_0, r_1, s_1, s_2)] \geq p$.

Proof. First, we consider the ball $B(\frac{1}{2}, \frac{1}{4})$. Fix $p > 0$. Then we know that there exists $R_0 > 0$ such that for all $R \geq R_0$ we have that $\eta'([-R^2, R^2])$ (with the Lebesgue measure parameterization) contains $B(\frac{1}{2}, \frac{1}{4})$ with probability at least $p$. Fix $\epsilon > 0$. By rescaling space by the factor $\epsilon/R$, we have that the probability that $\eta'([\epsilon^2, \epsilon^2])$ (with the Lebesgue measure parameterization) contains $B(\frac{1}{2}, \frac{1}{4})$ is at least $p$. The result follows because by making $\epsilon > 0$ sufficiently small, we can find $\delta > 0$ such that $\eta'([-\delta, \delta])$ (with the quantum area parameterization) is contained in $\mathbb{D}$ and contains $\eta'([\epsilon^2, \epsilon^2])$ (with the Lebesgue measure parameterization) with probability at least $p$. \hfill \Box

Lemma 5.12. Suppose that $(C, h, 0, \infty)$ is a $\sqrt{8/3}$-quantum cone with the circle average embedding. Let $\eta'$ be a space-filling SLE$_6$ process from $\infty$ to $\infty$ sampled independently of $h$ and then reparameterized by $\sqrt{8/3}$-LQG area. For each $t \in \mathbb{R}$ and $r > 0$, let $\tilde{h}_t^{\epsilon, r}$ be the field which is obtained by translating so that $\eta'(t)$ is sent to the origin and then rescaling by the factor $r$. Fix $0 < s_1 < s_2$. For each $t \in [s_1, s_2]$, let $R(t)$ be such that $\tilde{h}_t^{\epsilon, R(t)}(0)$ has the circle average embedding. Fix $r_1 > 0$ and $z_0 \in \mathbb{D}$ so that $B(z_0, r_1) \subseteq \mathbb{D}$ \{0\} and let $E(z_0, r_1, s_1, s_2)$ be as in (5.6). There exist constants $c_0, c_1 > 0$ such that for each $d \in (0, 1)$ with

$$F(z_0, r_1, s_1, s_2, d) = \{\forall t \in [s_1, s_2] : \eta'(t) \in B(z_0, r_1), R(t) \in [d, d^{-1}]\} \quad (5.7)$$

we have

$$P[F(z_0, r_1, s_1, s_2, d)^c \cap E(z_0, r_1, s_1, s_2)] \leq c_0 d^{c_1}. \quad (5.8)$$

Proof. Proposition 2.4 implies that there exist constants $c_0, c_1 > 0$ such that

$$P \left[ \sup_{z \in B(z_0, r_1)} h_r(z) + Q \log r > 0 \right] \leq c_0 r^{c_1} \quad (5.9)$$

for all $r \in (0, \text{dist}(B(z_0, r_1), \partial \mathbb{D}))$. (Note that we can apply Proposition 2.4 here because, by our normalization, the law of $h$ restricted to $\mathbb{D}$ is equal to that of a whole-plane GFF plus $-\gamma \log |z|$ normalized to have average equal to 0 on $\partial \mathbb{D}$.) This implies the desired upper bound for the probability that $R(t) > d^{-1}$ for some $t \in [s_1, s_2]$ with $\eta'(t) \in B(z_0, r_1)$. The desired upper bound for the probability that $R(t) < d$ for some
\[ t \in [s_1, s_2] \text{ with } \eta'(t) \in B(z_0, r_1) \text{ follows because Corollary 2.5 implies that there exists } c_2, c_3 > 0 \text{ such that for each } R > 2 \text{ we have that} \]
\[
P \left[ \inf_{r \geq R} \left( \inf_{z \in B(z_0, r_1)} h_r(z) + Q \log r \right) \leq 0 \right] \leq c_2 R^{-c_3}. \]

Lemma 5.13. Suppose that \((C, h, 0, \infty)\) is a \(\sqrt{8/3}\)-quantum cone with the circle average embedding. Let \(E(z_0, r_1, s_1, s_2)\) be as in (5.6). Let \((w_j)\) be an i.i.d. sequence of points picked from \(\mu = \mu_h\) restricted to \(\eta([s_1, s_2])\) and let \(N = \delta (\log \epsilon^{-1} e^{-\gamma Q - (2 + \delta) \gamma})\) where \(\gamma = \sqrt{8/3}\). Let \(G\) be the event that \(\{w_1, \ldots, w_N\} \cap B(z_0, r_1)\) forms an \(\epsilon\)-net of \(B(z_0, r_1)\). There exists a constant \(c_0 > 0\) which depends on \(z_0, r_1\) such that
\[
P[G^c \cap E(z_0, r_1, s_1, s_2)] \leq c_0 \epsilon^\delta. \]

Proof. Let \(z_1, \ldots, z_k\) be the elements of \(\xi \mathbb{Z}^2\) which are contained in \(B(z_0, r_1)\). Proposition 2.4 implies that for each \(\xi \in (0, 1)\) there exists a constant \(c_0 > 0\) such that
\[
P \left[ \min_{1 \leq j \leq k} h_\epsilon(z_j) \leq (2 + \delta) \log \epsilon \right] \leq c_0 \epsilon^{2(1-\xi)}. \quad (5.10) \]

Combining [DSTT] Lemma 4.6] with (5.10), we have by possibly adjusting the values of \(c_0 > 0\) and \(\xi\) that
\[
P \left[ \min_{1 \leq j \leq k} \mu_h(B(z_j, \epsilon)) \leq \epsilon^{\gamma Q + (2 + \delta) \gamma} \right] \leq c_0 \epsilon^{2(1-\xi)}. \quad (5.11) \]

On the complement of the event in (5.11), the probability that none of \(w_1, \ldots, w_N\) are contained in \(B(z_j, \epsilon)\) is at most
\[
(1 - \epsilon^{\gamma Q + (2 + \delta) \gamma})^N \leq \exp(-Ne^{\gamma Q + (2 + \delta) \gamma}) \leq \epsilon^\delta. \]

Combining this with (5.11) implies the result. \[ \square \]

We will now use Proposition 5.9 and Lemmas 5.11–5.13 to prove that \(d_Q\) is Hölder continuous with positive probability on \(B(z_0, r_1)\). We will afterwards explain how to deduce from this the almost sure local Hölder continuity of \(d_Q\) on all of \(C\), thus finishing the proof of Theorem 1.1.

Lemma 5.14. Suppose that \((C, h, 0, \infty)\) is a \(\sqrt{8/3}\)-quantum cone with the circle average embedding. On the events \(E(z_0, r_1, s_1, s_2), F(z_0, r_1, s_1, s_2, d)\) from (5.6), (5.7), we have that the quantum distance \(d_Q\) restricted to pairs of points in \(B(z_0, r_1)\) is almost surely Hölder continuous.
Proof. Throughout, we shall assume that we are working on the event $H_{R,\zeta}$ of Proposition \ref{prop:holder} and we will prove the almost sure Hölder continuity of this event. We note that it suffices to do so since Proposition \ref{prop:holder} implies that $P[H_{R,\zeta}] \to 1$ as $R \to 0$ with $\zeta$ fixed.

For each $j$, we let $N_j = e^{\beta j}$ (note that $9 > (Q + 3)\gamma$ for $\gamma = \sqrt{8/3}$) and we pick $\mathcal{U}_j = \{w^j_1, \ldots, w^j_{N_j}\}$ i.i.d. from the $\sqrt{8/3}$-LQG area measure restricted to $\eta'(\{s_1, s_2\})$. Equivalently, we can first pick $t^j_1, \ldots, t^j_{N_j}$ i.i.d. from $[s_1, s_2]$ uniformly using Lebesgue measure and then take $w^j_i = \eta'(t^j_i)$. We assume that the $\mathcal{U}_j$ are also independent as $j$ varies. By Lemma \ref{lem:quant diam} the probability of the event $E_j$ that $\mathcal{U}_j$ forms an $e^{-j}$-net of $B(z_0, r_1)$ is at least $1 - c_0 e^{-j}$ where $c_0 > 0$ is a constant.

By \cite[Theorem 1.13]{DMS14}, we have that the joint law of $(C_h, h, 0, \infty)$ and $\eta'$ is invariant under the operation of translating so that $\eta'(t^j_i)$ is sent to the origin and then rescaling so that the resulting surface has the circle average embedding. Let $h^{i,j}$ be the resulting field. Proposition \ref{prop:holder} implies that for each $\beta > 0$ we can find $\alpha > 0$ such that for each $i, j$, the probability that the quantum diameter of $B(0, e^{-j})$ as measured using the field $h^{i,j}$ is larger than $e^{-\alpha j}$ is at most $c_1 e^{-\beta j}$ where $c_1 > 0$ is a constant. Thus on the event $E_j$, this implies that the probability that the quantum diameter of $B(w^j_i, e^{-j})$ is larger than $e^{-\alpha j}$ is at most $c_2 e^{-\beta j}$ for a constant $c_2 > 0$. Therefore by a union bound, the probability that the quantum diameter of any of the $B(w^j_i, e^{-j})$ is larger than $e^{-\alpha j}$ is at most $c_1 e^{(\theta - \beta)j}$. Assume that $\beta > 9$. Thus by the Borel-Cantelli lemma, it follows that there almost surely exists $J_0 < \infty$ (random) such that $j \geq J_0$ implies that $\mathcal{U}_j$ is an $e^{-j}$-net of $B(z_0, r_1)$ and the maximal quantum diameter of $B(w^j_i, e^{-j})$ is $e^{-\alpha j}$ for all $1 \leq i \leq N_j$.

Let $\mathcal{U} = \bigcup_j \mathcal{U}_j$. We will now extract the Hölder continuity of $(z, w) \mapsto d_{Q}(z, w)$ for $z, w \in \tilde{\mathcal{U}} = \mathcal{U} \cap B(z_0, r_1)$. Assume that $z, w, z', w' \in \tilde{\mathcal{U}}$ and assume that $d_{Q}(z, w) \geq d_{Q}(z', w')$. By repeated applications of the triangle inequality, we have that

$$d_{Q}(z, w) - d_{Q}(z', w') \leq d_{Q}(z, z') + d_{Q}(z', w) - d_{Q}(z', w')$$

$$\leq d_{Q}(z, z') + d_{Q}(z', w') + d_{Q}(w', w) - d_{Q}(z', w')$$

$$= d_{Q}(z, z') + d_{Q}(w, w').$$  \hspace{1cm} (5.12)

Consequently, it suffices to show that there exist constants $M, a > 0$ such that $d_{Q}(z, w) \leq M|z - w|^a$ for all $z, w \in \tilde{\mathcal{U}}$. It in fact suffices to show that this is the case for all $z, w \in \tilde{\mathcal{U}}$ with $|z - w| \leq e^{-J_0}$ and $J_0$ as above. Indeed, if this is the case and $z, w \in \tilde{\mathcal{U}}$ are such that $|z - w| > e^{-J_0}$ then we can find $z_0 = z, z_1, \ldots, z_{n-1}, z_n = w \in \tilde{\mathcal{U}}$ with $n \leq e^{J_0}$ and then we can use the triangle inequality to get that

$$d_{Q}(z, w) \leq \sum_{j=1}^{n} d_{Q}(z_{j-1}, z_j) \leq M ne^{-a \cdot J_0} \leq M e^{J_0} |z - w|^a.$$  \hspace{1cm} (5.13)

Fix $z, w \in \tilde{\mathcal{U}}$ with $|z - w| \leq e^{-J_0}$ and take $j_0 \in \mathbb{N}$ so that $e^{-j_0 - 1} \leq |z - w| \leq e^{-j_0}$. Then we can find $u_0 = v_0$ in $\mathcal{U}_{j_0}$ such that $|u_0 - w| \leq e^{-j_0}$ and $|v_0 - z| \leq e^{-j_0}$. This
implies that \( d_Q(u_0, w) \leq e^{-\alpha_0 j} \) and \( d_Q(v_0, z) \leq e^{-\alpha_0 j} \). For each \( j \geq j_0 + 1 \), we can inductively find \( u_{j-j_0}, v_{j-j_0} \in U_j \) with \( |u_{j-j_0} - u_{j-j_0-1}| \leq e^{-\alpha j}, |v_{j-j_0} - v_{j-j_0-1}| \leq e^{-\alpha j} \), hence \( d_Q(u_{j-j_0}, u_{j-j_0-1}) \leq e^{-\alpha j} \) and \( d_Q(v_{j-j_0}, v_{j-j_0-1}) \leq e^{-\alpha j} \). Therefore we have that

\[
d_Q(z, w) \leq d_Q(u_0, w) + d_Q(v_0, z) + \sum_{i=1}^{\infty} d_Q(u_i, u_{i-1}) + \sum_{i=1}^{\infty} d_Q(v_i, v_{i-1})
\]

\[
\leq M_0 \sum_{i=0}^{\infty} e^{-\alpha (i+j_0)} \leq M_1 e^{-\alpha j_0} \leq M_2 |z - w|^\alpha
\]

where \( M_0, M_1, M_2 > 0 \) are constants. Therefore \( (z, w) \mapsto d_Q(z, w) \) is almost surely Hölder continuous on \( \tilde{U} \). Since the set \( \tilde{U} \) is almost surely dense in \( B(z_0, r_1) \), it follows that \( d_Q \) almost surely extends to be Hölder continuous in \( (z, w) \) for \( z, w \in B(z_0, r_1) \).

**Proof of Theorem 1.1.** By Lemma 5.14 we know that on the events \( E(z_0, r_1, s_1, s_2) \) and \( F(z_0, r_1, s_1, s_2, d) \) we have the almost sure Hölder continuity of \( d_Q \) restricted to pairs \( z, w \in B(z_0, r_1) \). Fix \( t \in (s_1, s_2) \). Since translating by \( -\eta'(t) \) and then rescaling so that the surface has the circle average embedding is a measure preserving transformation, it follows that the probability that \( d_Q \) is Hölder continuous in a neighborhood of the origin is at least the probability \( p \) of \( E(z_0, r_1, s_1, s_2) \) and \( F(z_0, r_1, s_1, s_2, d) \). Since the law of \((C, h, 0, \infty)\) is invariant under the operation of multiplying its area by a constant, we have that the probability that \( d_Q \) is Hölder continuous in any compact subset of \( C \) is also at least \( p \). The almost sure continuity of the QLE(8/3,0) metric on a \( \sqrt{8/3} \)-quantum cone restricted to compact subsets of \( C \) follows because by Lemma 5.11 and Lemma 5.12 we can make \( p \) as close to 1 as we want. The result in the case of a \( \sqrt{8/3} \)-quantum sphere follows by absolute continuity.

**5.3.2 Proof of Theorem 1.2**

We let \( E(z_0, r_1, s_1, s_2) \) be as in [5.6] and let \( F(z_0, r_1, s_1, s_2, d) \) be as in [5.7]. As in the proof of Lemma 5.14, we shall assume throughout that we are working on the event \( H_{R,\xi} \) of Proposition 3.4. For each \( j \), we let \( N_j = e^{\beta j} \) and we pick \( \mathcal{U}_j = \{w^j_1, \ldots, w^j_{N_j}\} \) i.i.d. from the \( \sqrt{8/3} \)-LQG measure restricted to \( \eta'([s_1, s_2]) \). Equivalently, we can first pick \( t^j_1, \ldots, t^j_{N_j} \) i.i.d. from \([s_1, s_2]\) uniformly and then take \( w^j_i = \eta'(t^j_i) \). We assume that the \( \mathcal{U}_j \) are also independent as \( j \) varies. By the proof of Lemma 5.14 we know that the probability that every point in \( B(z_0, r_1) \) is within quantum distance at most \( e^{-\alpha j} \) of some point in \( \mathcal{U}_j \) is at least \( 1 - c_0 e^{-\beta j} \) for constants \( \alpha, \beta, c_0 > 0 \). Moreover, by we can make \( \beta > 0 \) as large as we want by possibly decreasing the value of \( \alpha > 0 \). It follows from Proposition 4.2 that the probability that the Euclidean diameter \( B_Q(w^j_i, e^{-\alpha j}) \) is larger than \( e^{-\bar{\alpha} j} \) is at most \( c_1 e^{-\bar{\beta} j} \) for constants \( \bar{\alpha}, \bar{\beta}, c_1 > 0 \). Moreover, by adjusting
the values of $\tilde{\alpha}, c_1$ we can make $\tilde{\beta}$ as large as we want. In particular, by taking $\beta, \tilde{\beta} > 9$
so that
\[
\sum_j N_j \cdot e^{-\beta j} < \infty \quad \text{and} \quad \sum_j N_j \cdot e^{-\tilde{\beta} j} < \infty,
\]
it follows from the Borel-Cantelli lemma that there almost surely exists $J_0 < \infty$ (random) such that $j \geq J_0$ implies that every point in $B(z_0, r_1)$ is contained in $B_Q(u^j, e^{-\alpha j})$ for some $j$ and the Euclidean diameter of $B_Q(u^j, e^{-\alpha j})$ is smaller than $e^{-\tilde{\alpha} j}$.

As explained in (5.12) in the proof of Lemma 5.14, it suffices to show that there exist constants $M, \alpha > 0$ such that $|z - w| \leq M(d_Q(z, w))^\alpha$ for all $z, w \in B(z_0, r_1)$. In fact, it suffices to show that this is the case for all $z, w \in B(z_0, r_1)$ such that $d_Q(z, w) \leq e^{-\tilde{\alpha}_0}$. So, suppose that $z, w \in B(z_0, r_1)$ are such that $d_Q(z, w) \leq e^{-\tilde{\alpha}_0}$ and let $j_0 \in \mathbb{N}$ be such that $e^{-\alpha(j_0 + 1)} \leq d_Q(z, w) \leq e^{-\alpha j_0}$. Then we can find $w_0 = v_0 \in U_{j_0}$ such that $d_Q(u_0, w) \leq e^{-\alpha j_0}$ and $d_Q(v_0, z) \leq e^{-\alpha_0 j_0}$. This implies that $|u_0 - w_0| \leq e^{-\tilde{\alpha}_0}$ and $|v_0 - z| \leq e^{-\tilde{\alpha}_0 j_0}$. For each $j \geq j_0 + 1$, we can inductively find $u_{j - j_0}, v_{j - j_0} \in U_j$ with $d_Q(u_{j - j_0}, u_{j - j_0 - 1}) \leq e^{-\alpha j}, d_Q(v_{j - j_0}, v_{j - j_0 - 1}) \leq e^{-\alpha j}$ hence $|u_{j - j_0} - u_{j - j_0 - 1}| \leq e^{-\tilde{\alpha} j}$ and $|v_{j - j_0} - v_{j - j_0 - 1}| \leq e^{-\tilde{\alpha} j}$. We therefore have that
\[
|z - w| \leq |z - v_0| + |w - w_0| + \sum_{i=1}^{\infty} |v_i - v_{i-1}| + \sum_{i=1}^{\infty} |w_i - w_{i-1}|
\leq M_0 \sum_{i=1}^{\infty} e^{-(i + j_0)\tilde{\alpha}} \leq M_1 e^{-\tilde{\alpha}_0 j_0} \leq M_2 d_Q(z, w)^{\alpha/\tilde{\alpha}}
\]
where $M_0, M_1, M_2$ are constants. Therefore $(z, w) \mapsto |z - w|$ is Hölder continuous with respect to the metric defined by $d_Q$ on $B(z_0, r_1)$ on the event $F(z_0, r_1, s_1, s_2, d)$. The argument explained in the proof of Theorem 1.1 then implies that $(z, w) \mapsto |z - w|$ is almost surely Hölder continuous with respect to $d_Q$ when restricted to a compact subset of $\mathbb{C}$. The continuity in the case of a $\sqrt{8/3}$-quantum sphere follows by absolute continuity.

\[\square\]

5.3.3 Existence and continuity of geodesics: proof of Theorem 1.3

We now show that the metric space that we have constructed is almost surely geodesic. We will then show (Proposition 5.18) that geodesics between quantum typical points are almost surely unique and (Proposition 5.19) that there is almost surely a unique geodesic from a typical point on the boundary of a metric ball to its center (assuming this center is also a “typical” point).

**Proposition 5.15.** Suppose that $S$ is a $\sqrt{8/3}$-LQG sphere and let $d_Q$ be the corresponding QLE($8/3, 0$) metric. Then $(S, d_Q)$ is almost surely geodesic.
Proposition 5.15 is a consequence of the following two general observations about compact metric spaces and the proof of the metric property given in [MS15b]. In this section, we will use the notation \((X, d)\) for a metric space, \((X, d, \mu)\) for a metric measure space, and let \(B(x, r) = \{y \in X : d(x, y) < r\}\) be the open ball of radius \(r\) centered at \(x\) in \(X\).

**Lemma 5.16.** Suppose that \((X, d)\) is a compact metric space. Then \((X, d)\) is geodesic if and only if for all \(x, y \in X\) there exists \(z \in X\) such that \(d(x, z) = d(z, y) = d(x, y)/2\).

**Proof.** Suppose that \((X, d)\) is geodesic and \(x, y \in X\). Then there exists a geodesic \(\eta: [0, d(x, y)] \to X\) connecting \(x\) and \(y\) and \(z = \eta(d(x, y)/2)\) satisfies \(d(x, z) = d(y, z) = d(x, y)/2\).

Conversely, we suppose that for all \(x, y \in X\) there exists \(z \in X\) with \(d(x, z) = d(y, z) = d(x, y)/2\). Fix \(x, y \in X\). We iteratively define a function \(\eta\) on the dyadic rationals in \([0, 1]\) as follows. We first pick \(z\) so that \(d(x, z) = d(z, y) = d(x, y)/2\) and set \(\eta(1/2) = z\). By iterating this construction in the obvious way, we have that \(\eta\) satisfies

\[\left|\eta(rd(x, y)) - \eta(sd(x, y))\right| = d(x, y)|r - s|.\]

Hence it is easy to see that \(\eta\) extends to a continuous map \([0, 1] \to X\) which (after reparameterizing time by the constant factor \(d(x, y)\)) is a geodesic connecting \(x\) and \(y\).

**Lemma 5.17.** Suppose that \((X, d, \mu)\) is a good-measure endowed compact metric space (recall the definition from Section 1.4.3). Then \((X, d)\) is geodesic if and only if the following property is true. Suppose that \(x, y\) are chosen from \(\mu\) and \(U \in [0, 1]\) uniformly with \(x, y, U\) independent. With \(r = Ud(x, y)\) and \(\overline{r} = d(x, y) - r\) there almost surely exists \(z \in \partial B(x, r) \cap \partial B(y, \overline{r})\) that \(d(x, z) + d(z, y) = d(x, y)\).

**Proof.** It is clear that if \(X\) is geodesic then the property in the lemma statement holds because we can take \(z\) to be a point along a geodesic from \(x\) to \(y\) in \(\partial B(x, r) \cap \partial B(y, \overline{r})\). Suppose that the property in the lemma statement holds. We will show that \((X, d)\) is geodesic by verifying the condition from Lemma 5.16. Suppose that \((x_n), (y_n)\) are independent i.i.d. sequences chosen from \(\mu\), that \((U_n)\) is an i.i.d. sequence of uniform random variables in \([0, 1]\) which are independent of \((x_n), (y_n)\), and \(r_n = d(x_n, y_n)\) and \(\overline{r}_n = d(x_n, y_n) - \overline{r}_n\). The following is then almost surely true for all \(x, y \in X\) distinct and \(k \in \mathbb{N}\). Then there exists \(n_k\) such that \(d(x_{n_k}, x) < 1/k\), \(d(y_{n_k}, y) < 1/k\), and \(|U_{n_k} - 1/2| < 1/k\). Let \(z_{n_k} \in \partial B(x_{n_k}, r) \cap \partial B(y_{n_k}, r)\) be such that \(d(x_{n_k}, z_{n_k}) + d(z_{n_k}, y_{n_k}) = d(x_{n_k}, y_{n_k})\). Let \((\overline{z}_m)\) be a convergent subsequence of \(z_{n_k}\) and let \(z = \lim_m \overline{z}_m\). By the continuity of \(d\), we have that \(d(x, z) + d(z, y) = d(x, y)\) and \(d(x, z) = d(y, z) = d(x, z)/2\).

**Proof of Proposition 5.15.** This follows by combining Lemma 5.16 and Lemma 5.17 and the construction of \(d_Q\) given in [MS15b].
Proof of Theorem 1.3. The first part of the theorem follows from Proposition 5.15. The second part of the theorem follows because a geodesic on a $\sqrt{8/3}$-LQG sphere is 1-Lipschitz with respect to the QLE(8/3,0) metric, so the result follows by combining with Theorem 1.1 and Theorem 1.2. □

Proposition 5.18. Suppose that $S$ is a $\sqrt{8/3}$-LQG sphere and that $\bar{d}_Q$ is the associated QLE(8/3,0) metric. Assume that $x,y \in S$ are picked uniformly from the quantum measure. Then there almost surely exists a unique geodesic connecting $x$ and $y$.

Proof. Proposition 5.15 implies that there exists at least one geodesic $\eta$ connecting $x$ and $y$. Suppose that $\bar{\eta}$ is another geodesic. By [MS15b, Lemma 7.6], if we let $r = Ud(x,y)$ where $U$ is uniform in $[0,1]$ independently of everything else and $\bar{r} = d(x,y) - r$ then $\partial B(x,r) \cap \partial B(y,\bar{r})$ almost surely intersect at a unique point. This implies that $\eta(Ud(x,y)) = \bar{\eta}(Ud(x,y))$ which, in turn, implies that on a set of full Lebesgue measure in $[0,d(x,y)]$ we have that $\eta(t) = \bar{\eta}(t)$. Therefore $\gamma = \bar{\gamma}$ by the continuity of the paths. □

Proposition 5.19. Suppose that $S$ is a $\sqrt{8/3}$-LQG sphere, let $\bar{d}_Q$ be the associated QLE(8/3,0) metric, and assume that $x,y \in S$ are chosen independently from the quantum measure. Fix $r > 0$ and assume that we are working on the event that $d_Q(x,y) > r$. Suppose that $z$ is chosen uniformly from the quantum boundary measure on the boundary of the filled metric ball centered at $x$ of radius $r$. Then there is almost surely a unique geodesic from $z$ to $x$. The same holds if we replace $r$ with $d_Q(x,y) - r$.

Proof. This result is a consequence of Proposition 5.18 because we can sample from the law of $z$ by growing a metric ball from $y$ and taking $z$ to be the unique intersection point of this ball with the filled metric ball starting from $x$. □

5.3.4 The internal metric

We next turn to construct the internal metric $d^U_Q$ associated with the restriction of $\bar{d}_Q$ to a domain $U$. The almost sure finiteness of $d^U_Q$ will rely on Theorems 1.1,1.3

Proposition 5.20. Suppose that $(C, h, 0, \infty)$ is a $\sqrt{8/3}$-quantum cone and that $U \subseteq C$ is a domain. For each $x,y \in U$ we let $U_{x,y}$ be the set of paths in $U$ which connect $x,y$ and, for $\eta \in U_{x,y}$, we let $\ell_Q(\eta)$ be the $\bar{d}_Q$-length of $\eta$. We let

$$d^U_Q(x,y) = \inf_{\eta \in U_{x,y}} \ell_Q(\eta) \quad \text{for} \quad x,y \in U.$$ 

Then $d^U_Q$ defines a metric on $U$. The same holds with a $\sqrt{8/3}$-LQG sphere in place of the $\sqrt{8/3}$-quantum cone.
Suppose that \( x, y \in U \) there exists \( \eta \in U_{x,y} \) with \( \ell_Q(\eta) < \infty \). We may assume without loss of generality that \( U \) is bounded. We fix \( c, \beta, r_0 > 0 \) so that \( B_Q(x, r) \) contains \( B(x, cr^\beta) \) for all \( x \in U \) and \( r \in (0, r_0) \). Such \( c, \beta, r_0 \) exist by Theorem \[1.2\].

We then pick points \( x_0 = x, x_1, \ldots, x_k = y \) and radii \( r_0, \ldots, r_k \) such that \( B_Q(x, r_j) \subseteq U \) for all \( 0 \leq j \leq k \) and \( B(x_j, cr_j^\beta) \cap B(x_{j+1}, cr_{j+1}^\beta) \neq \emptyset \) for all \( 0 \leq j \leq k - 1 \). For each \( 0 \leq j \leq k - 1 \), we let \( y_j \in B(x_j, cr_j^\beta) \cap B(x_{j+1}, cr_{j+1}^\beta) \).

Then there exists \( d_Q \)-geodesics connecting \( x_j \) to \( y_j \) and \( y_j \) to \( x_{j+1} \), which are respectively contained in \( B_Q(x_j, r_j) \) and \( B_Q(x_{j+1}, r_{j+1}) \) hence also in \( U \). Concatenating these paths yields an element of \( U_{x,y} \) with \( d_Q \)-length at most

\[
r_0 + 2(r_1 + \cdots + r_{k-1}) + r_k < \infty,
\]
as desired.

5.3.5 Proof of Theorem \[1.6\]

We assume for simplicity that \((C, h, 0, \infty)\) is a \( \sqrt{8/3} \)-quantum cone with the circle average embedding. Theorem \[1.1\] and Theorem \[1.2\] together imply that \( B_Q(0, 1) \) has Euclidean diameter which is finite and positive, so that the quantum area of the smallest Euclidean ball \( B \) containing \( B_Q(0, 1) \) is finite and positive. If we add \( C = 4\gamma^{-1}\log \epsilon \) to \( h, \gamma = \sqrt{8/3} \), then the radius of our quantum ball becomes \( \epsilon \) and the quantum area of the smallest Euclidean ball centered at 0 which contains it is of order \( \epsilon^4 \). To finish the proof, we just need to calculate the typical radius of the Euclidean ball centered at 0 with quantum area of order \( \epsilon^4 \). By [DS11, Lemma 4.6], we have that the conditional expectation of the amount of quantum mass in \( B(0, r) \) given \( h_r(0) \) is equal to \( e^{\gamma h_r(0)/(2+\gamma^2/2)\log r} \). Since \( h_{c,t}(0) \) evolves as a standard Brownian motion plus the linear drift \( \gamma t \) in \( t \), the typical value of \( h_r(0) \) is of order \( \sqrt{\log r^{-1} + \gamma \log r^{-1}} \). Therefore the dominant term in the exponent of the aforementioned conditional expectation is equal to \((2 - \gamma^2/2)\log r = (2/3)\log r \). Taking \( r = \epsilon^\alpha \) for some \( \alpha > 0 \), we see that the typical amount of quantum mass in \( B(0, \epsilon^\alpha) \) is given by \( e^{(2\alpha/3)(1+o(1))} \) as \( \epsilon \to 0 \).

Equating \( 2\alpha/3 \) to 4 yields that \( \alpha = 6 \). The result follows because the local behavior of quantum sphere, disk, or quantum cone in a bounded open set near a quantum typical point looks like the behavior of \( h \) near 0.

6 Distance to tip of SLE\(_6\) on a \( \sqrt{8/3} \)-quantum wedge

The main purpose of this section is to prove the following two propositions, which will be used in Section 8 to show that the metric net between two quantum typical points in a \( \sqrt{8/3} \)-LQG sphere has the law of the 3/2-Lévy net. The first (Proposition 6.1) bounds the moments of the distance in a \( \sqrt{8/3} \)-quantum wedge between the origin and
the tip of an independent SLE$_{6}$ run for $\delta$ units of quantum natural time and the second (Proposition 6.2) bounds the moments of the distance between the origin and a point which is $\delta$ units of quantum length along the boundary from the origin in a $\sqrt{8/3}$-quantum wedge. Throughout, we will use the spaces $\mathcal{H}_1(\mathcal{X})$ and $\mathcal{H}_2(\mathcal{X})$ introduced in Section 2.1.

**Proposition 6.1.** Suppose that $(\mathcal{S}, h, -\infty, +\infty)$ is a $\sqrt{8/3}$-quantum wedge and let $\eta'$ be an independent SLE$_{6}$ process from $-\infty$ to $+\infty$ with the quantum natural time parameterization. There exists $p_0 > 1$ such that for all $p \in (0, p_0)$ there exists a constant $c_p > 0$ such that the following is true. Let $D_\delta$ be the quantum distance between $-\infty$ and $\eta'(\delta)$ (with respect to the internal QLE($8/3$, 0) metric associated with $h$). Then we have that

$$E[D_\delta^p] = c_p \delta^{p/3}.$$  \hspace{1cm} (6.1)

For each $\alpha > 0$, let $u_{\alpha,\delta}$ be where the projection of $h$ onto $\mathcal{H}_1(\mathcal{S})$ first hits $\alpha \log \delta$, let $F_{\alpha,\delta} = \{\sup_{t \in [0, \delta]} \text{Re}(\eta'(t)) \leq u_{\alpha,\delta} - 1\}$ and let $D_{\alpha,\delta}$ be the quantum distance between $-\infty$ and $\eta'(\delta)$ with respect to the internal QLE($8/3$, 0) metric associated with $\mathcal{S}_- + u_{\alpha,\delta}$. For each $p \in (0, p_0)$ there exists $\alpha > 0$ and a constant $c_p > 0$ such that

$$E[D_{\alpha,\delta}^p 1_{F_{\alpha,\delta}}] \leq c_p \delta^{p/3}$$  \hspace{1cm} (6.2)

Finally, there exists $\alpha_0 > 0$ such that for all $\alpha \in (0, \alpha_0)$ and each $k > 0$ there exists a constant $c_k > 0$ such that

$$P[F_{\alpha,\delta}^c] \leq c_k \delta^k.$$  \hspace{1cm} (6.3)
As we will see in the proof of Proposition 6.1, the exponent \( p/3 \) in (6.1) arises because adding a constant \( C \) to \( h \) has the effect of scaling the amount of quantum natural time elapsed by \( \eta' \) by the factor \( e^{3\gamma C/4} \), \( \gamma = \sqrt{8/3} \) [MS15c, Section 6.2] and the quantum distance by the factor \( e^{\gamma C/4} \) (Lemma 2.2). In particular, the quantum distance behaves like the quantum natural time to the power 1/3.

**Proposition 6.2.** Suppose that \( (\mathcal{S}, h, -\infty, +\infty) \) is a \( \sqrt{8/3} \)-quantum wedge. There exists \( p_0 > 1 \) such that for all \( p \in (0, p_0) \) there exists a constant \( c_p > 0 \) such that the following is true. For each \( \delta > 0 \) we let \( x_{\delta} = \inf\{x \in \mathbb{R} : \nu_h([0, x]) \geq \delta\} \) and let \( D_\delta \) be the quantum distance between \(-\infty\) and \( x_\delta \) (with respect to the internal \( \text{QLE}(8/3, 0) \) metric associated with \( h \)). Then we have that

\[
E[D_\delta^p] = c_\delta \delta^{p/2}. \tag{6.4}
\]

For each \( \alpha > 0 \), let \( u_{\alpha, \delta} \) be where the projection of \( h \) onto \( \mathcal{H}_1(\mathcal{S}) \) first hits \( \alpha \log \delta \), let \( F_{\alpha, \delta} = \{x_\delta \leq u_{\alpha, \delta} - 1\} \), and let \( D_{\alpha, \delta} \) be the quantum distance between \(-\infty\) and \( x_\delta \) with respect to the internal metric associated with \( \mathcal{S}_- + u_{\alpha, \delta} \). For each \( p \in (0, p_0) \) there exists \( \alpha > 0 \) and a constant \( c_{\alpha, p} > 0 \) such that

\[
E[D_{\alpha, \delta}^p] \leq c_{\alpha, p} \delta^{p/2}. \tag{6.5}
\]

Finally, there exists \( \alpha_0 > 0 \) such that for all \( \alpha \in (0, \alpha_0) \) and each \( k > 0 \) there exists a constant \( c_k > 0 \) such that

\[
P[F_{\alpha, \delta}] \leq c_k \delta^k. \tag{6.6}
\]

As we will see in the proof of Proposition 6.2, the exponent \( p/2 \) in (6.4) arises because adding a constant \( C \) to \( h \) has the effect of scaling quantum length by the factor \( e^{\gamma C/2} \), \( \gamma = \sqrt{8/3} \), and the quantum distance by the factor \( e^{\gamma C/4} \) (Lemma 2.2). In particular, the quantum distance behaves like the quantum length to the power 1/2.

We note that (6.2), (6.3) of Proposition 6.1 and (6.5), (6.6) of Proposition 6.2 also hold in the setting of a quantum disk \( (\mathcal{S}, h) \) sampled from \( \mathcal{M}^\text{BES} \) provided we condition on the event that the projection of \( h \) onto \( \mathcal{H}_1(\mathcal{S}) \) exceeds \( \alpha \log \delta \). Indeed, this follows because in this case the law of \( h \) restricted to the part of \( \mathcal{S} \) up until where the projection of \( h \) onto \( \mathcal{H}_1(\mathcal{S}) \) first hits \( \alpha \log \delta \) is the same as the corresponding restriction of a \( \sqrt{8/3} \)-quantum wedge. In fact, we will be applying these results in the setting of a quantum surface whose law is closely related to that of a quantum disk below. As we will see, however, it will be more convenient to establish the above estimates in the setting of a quantum wedge because the of the exact scaling properties that a quantum wedge possesses.

We will break the proof of Proposition 6.1 into two steps. The first step (carried out in Section 6.1) is to establish (6.3). The second step (carried out in Section 6.2) is to establish a moment bound between deterministic points in \( \mathcal{S}_- \). As we will see upon completing the proof of Proposition 6.1, the proof of Proposition 6.2 will follow from the same set of estimates used to prove Proposition 6.1 (though in this case the argument turns out to be simpler).
6.1 Size of path with quantum natural time parameterization

The purpose of this section is to bound the size of an SLE_6 path drawn on top of an independent $\sqrt{8/3}$-quantum wedge equipped with the quantum natural time parameterization.

**Proposition 6.3.** Suppose that $(\mathcal{S}, h, -\infty, +\infty)$ has the law of a $\sqrt{8/3}$-quantum wedge. Let $\eta'$ be an SLE_6 from $-\infty$ to $+\infty$ which is sampled independently of $h$ and then parameterized according to quantum natural time. Let $u_{\alpha, \delta}$ be where the projection of $h$ onto $\mathcal{H}_1(\mathcal{S})$ first hits $\alpha \log \delta$. There exists $\alpha_0 > 0$ such that for all $\alpha \in (0, \alpha_0)$ and each $k > 0$ there exists a constant $c_k > 0$ such that

$$
\mathbb{P}\left[ \sup_{t \in [0, \delta]} \text{Re}(\eta'(t)) \geq u_{\alpha, \delta} \right] \leq c_k \delta^k.
$$

That is, (6.3) from Proposition 6.1 holds.

We will prove Proposition 6.3 by first bounding in Lemma 6.4 the number of quantum disks cut out by $\eta'|_{[0, \delta]}$ with large quantum area (a small, positive power of delta) and then argue in Lemma 6.5 and Lemma 6.6 that if we run $\eta'$ until it first hits the line $\text{Re}(z) = u_{\alpha, \delta}$ for $\alpha > 0$ small then it is very likely to cut out a large number of quantum disks with large quantum area.

**Lemma 6.4.** Suppose that we have the same setup as in Proposition 6.3. There exist $\alpha, \beta > 0$ such that for each $n \in \mathbb{N}$ there exists a constant $c_n > 0$ such that the following is true. The probability that $\eta'|_{[0, \delta]}$ separates from $+\infty$ at least $n$ quantum disks with quantum area at least $\delta^\alpha$ is at most $c_n \delta^{3n}$.

**Proof.** Fix $k \in \mathbb{N}$. We will prove the result by giving an upper bound on the probability that $\eta'|_{[0, \delta]}$ cuts out at least $k$ quantum disks with boundary length in three different regimes and then we will sum over all possibilities so that at least $n$ quantum disks with quantum area at least $\delta^\alpha$ are cut out by $\eta'|_{[0, \delta]}$.

Recall that for each $j \in \mathbb{Z}$ the number $N_j$ of quantum disks cut out by $\eta'|_{[0, \delta]}$ with quantum boundary length in $(e^{-j-1}, e^{-j}]$ is distributed as a Poisson random variable with mean $\lambda_j$ which is given by a constant times $\delta e^{3/2j}$ and that the $N_j$ are independent. Moreover, recall from Lemma 3.3 that the expected quantum area in such a disk is given by a constant times $e^{-2j}$ (i.e., a constant times the square of its boundary length). Therefore the probability that a given such disk has quantum area at least $\delta^\alpha$ is, by Markov’s inequality, at most a constant times $e^{-2j} \delta^{-\alpha}$. By Lemma 2.9 there exists a constant $c_0 > 0$ such that

$$
\mathbb{P}[N_j \geq 2\lambda_j] \leq \exp(-c_0 \delta e^{3/2j}).
$$

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The upper bound in (6.7) is negligible compared to any power of \( \delta \) as \( \delta \to 0 \) provided we have for some \( \epsilon > 0 \) fixed that \( j \geq \ell_0 = \frac{2}{3}(1+\epsilon) \log \delta^{-1} \). Let \( E_{j,k} \) be the event that \( \eta'_{[0,\delta]} \) cuts out at least \( k \) quantum disks with quantum area at least \( \delta^\alpha \) and quantum boundary length in \((e^{-j-1}, e^{-j}]\). It thus follows that there exists a constant \( c_1 > 0 \) such that

\[
P[E_{j,k}] \leq c_1(\delta e^{3/2j})^k \times \left(e^{-2j} \delta^{-\alpha}\right)^k = c_1 \delta^{(1-\alpha)k} e^{-jk/2} \quad \text{for all} \quad j \geq \frac{2}{3}(1+\epsilon) \log \delta^{-1}. \tag{6.8}
\]

The number \( N \) of quantum disks separated by \( \eta'_{[0,\delta]} \) from \( +\infty \) with quantum boundary length between \( \delta^{2/3(1+\epsilon)} \) and \( \delta^{1/2} \) is Poisson with mean \( \lambda \) proportional to \( \delta^{-\epsilon} \). By Lemma 2.9 we have for a constant \( c_2 > 0 \) that

\[
P[N \geq 2\lambda] \leq \exp\left(-c_2 \delta^{-\epsilon}\right), \tag{6.9}
\]

hence decays to 0 faster than any power of \( \delta \) as \( \delta \to 0 \). By Lemma 3.3, the expected quantum area in such a quantum disk is at most a constant times \( \delta \) so that, as before, the probability that any such disk has quantum area at least \( \delta^\alpha \) is, by Markov’s inequality, at most a constant times \( \delta^{1-\epsilon} \). Let \( F_k \) be the event that there are at least \( k \) such disks. Combining, it follows that there exists a constant \( c_3 > 0 \) such that

\[
P[F_k] \leq c_3 \delta^{-ek} \times \delta^{(1-\alpha)k} = c_3 \delta^{(1-\alpha-\epsilon)k}. \tag{6.10}
\]

Finally, the number of quantum disks separated by \( \eta'_{[0,\delta]} \) from \( +\infty \) with boundary length larger than \( \delta^{1/2} \) is Poisson with mean proportional to \( \delta \times \delta^{-3/4} = \delta^{1/4} \). Thus if we let \( G_k \) be the event that there are at least \( k \) such quantum disks cut out by \( \eta'_{[0,\delta]} \) then it follows that there exists a constant \( c_4 > 0 \) such that

\[
P[G_k] \leq c_4 \delta^{k/4}. \tag{6.11}
\]

We can deduce the result from (6.8), (6.10), and (6.11) as follows. For each sequence \( a = (i,j,k_{\ell_0}, k_{\ell_0+1}, \ldots) \) of non-negative integers with

\[
i + j + \sum_{\ell \geq \ell_0} k_\ell = n
\]

we let \( F_{\mathbf{a}} \) be the event that \( \eta'_{[0,\delta]} \) separates from \( +\infty \) at least \( i \) (resp. \( j \)) quantum disks of quantum area at least \( \delta^\alpha \) and quantum boundary length in \([\delta^{2/3(1+\epsilon)}, \delta^{1/2}]\) (resp. larger than \( \delta^{1/2} \)) and at least \( k_\ell \) quantum disks of quantum area at least \( \delta^\alpha \) and quantum boundary length in \( (e^{-\ell-1}, e^{-\ell}] \) for each \( \ell \geq \ell_0 \). Assume that we have chosen \( \alpha, \epsilon \) such that \( 1-\alpha - \epsilon \in (0,1/4) \). Then (6.8), (6.10), and (6.11) together imply that there exists a constant \( c_5 > 0 \) such that

\[
P[F_{\mathbf{a}}] \leq c_5 \delta^{(1-\alpha-\epsilon)n} \prod_{\ell \geq \ell_0} e^{-\ell k_\ell/2}.
\]

The result follows by summing over all such multi-indices \( \mathbf{a} \). \( \Box \)
Lemma 6.5. Suppose that we have the same setup as in Proposition 6.3. For each $k \in \mathbb{Z}$, we let $\tau_k = \inf\{t \geq 0 : \text{Re}(\eta(t)) \geq k\}$. For each $\rho \in (0, 1)$ there exists $r > 0$ such that the conditional probability given $\eta|_{[0, \tau_k]}$ that there exists $z \in [k + r, k + 1 - r] \times [0, \pi]$ such that $\eta|_{[\tau_k, \tau_{k+1}]}$ separates $B(z, r)$ from $+\infty$ is at least $\rho$.

**Proof.** This follows from the conformal Markov property for SLE$_6$. \hfill \Box

Lemma 6.6. Suppose that we have the same setup as in Proposition 6.3 and let $\tau_k$ be as in Lemma 6.5. For each $\beta > 0$ there exists $\alpha > 0$ such that the following is true. Fix $\delta > 0$, let $k = u_{\alpha, \delta}$, and let $m \in \mathbb{N}$. For each $p > 0$ there exists a constant $c_p > 0$ such that the probability that $\eta|_{[0, \tau_k]}$ separates from $+\infty$ fewer than $m$ quantum disks of quantum area at least $\delta^\beta$ is at most $c_p \delta^p$.

**Proof.** Fix $\rho \in (0, 1)$ and let $r > 0$ be as in Lemma 6.5 for this value of $\rho$. Fix $\sigma > 0$ small and let $n = \sigma \log \delta^{-1}$. Lemma 6.5 implies that the number of components separated by $\eta|_{[\tau_k - n, \tau_k]}$ from $+\infty$ which contain a Euclidean disk of radius at least $r$ is stochastically dominated from below by a Binomial random variable with parameters $(n, \rho)$. In particular, by choosing $\rho$ sufficiently close to 1 we have that the probability that the number of components separated by $\eta|_{[\tau_k - n, \tau_k]}$ from $+\infty$ which contain a Euclidean disk of radius at least $r$, all contained in $[k - n, k] \times [0, \pi]$, is smaller than $n/2$ is at most a constant times $\delta^p$.

Assume that we are working on the complementary event that $\eta|_{[\tau_k - n, \tau_k]}$ separates at least $n/2$ such components $U_1, \ldots, U_{n/2}$ and, for each $j$, we let $z_j$ be such that $B(z_j, r) \subseteq U_j$. For each $1 \leq j \leq n/2$, we let $h_j$ be the harmonic extension of the values of $h$ from $\partial B(z_j, r)$ to $B(z_j, r)$. By the Markov property for the GFF with free boundary conditions, we know that the conditional law of the restriction of $h$ to $B(z_j, r)$ given its values outside of $B(z_j, r)$ is that of a GFF in $B(z_j, r)$ with zero boundary conditions plus $h_j$ conditioned so that $u_{\alpha, \delta}$ is the first time that $\alpha \log \delta$ is hit by the projection of $h$ onto $\mathcal{H}_1(S)$. For $\gamma = \sqrt{8/3}$, let

$$X_j^* = \sup_{w \in B(z_j, r/2)} |h(w) - (Q - \gamma)(\text{Re}(z_j) - u_{\alpha, \delta}) - \alpha \log \delta|.$$  

The argument used to prove Lemma 6.6 implies that there exist constants $c_0, c_1 > 0$ such that

$$\mathbb{P}[X_j^* \geq \eta] \leq c_0 \exp \left(- \frac{c_1 \eta^2}{\sigma \log \delta^{-1}} \right) \quad \text{for all} \quad \eta \geq 0. \quad (6.12)$$

Fix $\epsilon > 0$. It follows from (6.12) that there exist constants $c_2, c_3 > 0$ such that

$$G = \left\{ \max_{1 \leq j \leq n/2} X_j^* \geq \epsilon \log \delta^{-1} \right\} \quad \text{we have} \quad \mathbb{P}[G] \leq c_2 \exp \left(- \frac{c_3 \epsilon^2}{\sigma \log \delta^{-1}} \right). \quad (6.13)$$

In particular, by making $\sigma > 0$ small enough we can make it so that $\mathbb{P}[G]$ decays to 0 faster than any fixed positive power of $\delta$. Conditional on $G^c$, for each $j$ we have that
the probability that the quantum area associated with $B(z_j, r)$ is at least a constant times $a\delta^{\gamma(\epsilon+\alpha)} \times \delta^{2\alpha/3}$ is at least $\tilde{\rho}$ where $\tilde{\rho} \to 1$ as $a \to 0$. We note that these events are conditionally independent given the values of $h$ on the complement of $\cup_j B(z_j, r)$ and on the event $G^c$. Therefore by choosing $a > 0$ small enough, the probability that we have fewer than $n/4$ disks with quantum area at least $a\delta^{\gamma(\epsilon+\alpha)}$ tends to 0 faster than any power of $\delta$. Combining implies the result.

Proof of Proposition 6.3. The result follows by combining Lemma 6.4 with Lemma 6.6.

6.2 Quantum distance bounds

Proposition 6.7. Suppose that $(S, h, -\infty, +\infty)$ has the law of a $\sqrt{8/3}$-quantum wedge with the embedding into $S$ so that the projection of $h$ onto $H_1(S)$ first hits 0 at $u = 0$. There exist $p > 1$ and constants $c_0, c_1 > 0$ such that the following is true. For each $k \in \mathbb{Z}$ with $k < 0$ we let $D_k$ be the quantum distance from $k + i\pi/2$ to $k + 1 + i\pi/2$ (i.e., the midpoint of the line segment in $S$ with Re($z$) = $k$ to the midpoint of the line segment with Re($z$) = $k + 1$) with respect to the QLE($8/3, 0$) internal metric in $[k-1, k+2] \times [\pi/4, 3\pi/4]$. Then we have that

$$E[D_k^p] \leq c_0 e^{c_1 k}. \quad (6.14)$$

Let $\tilde{D}_k$ denote the quantum distance between the points $k + 3\pi/4i$ and $k + i\pi/4$ with respect to the QLE($8/3, 0$) internal metric in $[k-1, k+1] \times [0, \pi]$. We also have (for the same values of $p, c_0, c_1$) that

$$E[\tilde{D}_k^p] \leq c_0 e^{c_1 k}. \quad (6.15)$$

We need to collect several lemmas before giving the proof of Proposition 6.7.

Lemma 6.8. Suppose that $(S, h)$ is a $\sqrt{8/3}$-quantum wedge with the embedding into $S$ as in the statement of Proposition 6.7. Fix $k \in \mathbb{Z}$ with $k < 0$ and suppose that $B(z, r) \subseteq [k, k+1] \times [\pi/4, 3\pi/4]$. With $\gamma = \sqrt{8/3}$, let

$$\xi(q) = \left(2 + \frac{\gamma^2}{2}\right) q - \frac{\gamma^2}{2} q^2 = \frac{10}{3} q - \frac{4}{3} q^2.$$ 

For each $M \in \mathbb{R}$, we let $A_M$ be the event that the value of the projection of $h$ onto $H_1(S)$ at $k+1$ is in $[M, M+1]$. For each $q \in (0, 4/\gamma^2) = (0, 3/2)$ there exists a constant $c_q > 0$ such that

$$E[\mu_h(B(z, r))^q | A_M] \leq c_q e^{\gamma M, \xi(q)}.$$ 

Proof. This follows from the standard multifractal spectrum bound for the moments of the quantum measure; recall Proposition 3.4.
Fix $k \in \mathbb{Z}$ with $k < 0$. For $\alpha, \beta > 0$ and $j \in \mathbb{N}$, we say that a point $z \in S$ with $\Re(z) \in [k, k+1]$ is $(\alpha, \beta, j)$-good if:

1. $B(z, e^{-j}) \subseteq B_\varrho(z, e^{-\alpha j})$ and
2. $B_\varrho(z, e^{-\alpha j}) \subseteq B(z, e^{-\beta j})$.

**Lemma 6.9.** Suppose that $(S, h)$ is a $\sqrt{8/3}$-quantum wedge with the embedding into $S$ as in the statement of Proposition 6.7. Suppose that $z \in [-2, -1] \times [\pi/4, 3\pi/4]$. For each $\epsilon > 0$ there exists $\alpha, \beta, c_0 > 0$ such that the probability that $z$ is $(\alpha, \beta, j)$-good is at least $1 - c_0 e^{-(25/12 - \epsilon)j}$ for each $j \in \mathbb{N}$.

**Proof.** Fix $\epsilon > 0$. By Lemma 6.8 and Markov’s inequality with $q = 5/4$ so that $\xi(q) = 25/12$, we can find $\alpha > 0$ and a constant $c_0 > 0$ such that with

$$E = \{\mu_h(B(z, e^{-j})) \leq e^{-\alpha j}\} \quad \text{we have} \quad P[E^c] \leq c_0 e^{-(25/12 - \epsilon)j}.$$ 

Let $Z$ be the Radon-Nikodym derivative between the law of the restriction of $h$ to $B(z, 1/4)$ and the law of a whole-plane GFF on $\mathcal{C}$ restricted to $B(z, 1/2)$ with the additive constant fixed so that its average on $\partial B(z, 1)$ is equal to 0. Then Lemma 4.6 and Lemma 5.4 together imply that for each $p > 0$ there exists a constant $c_p < \infty$ such that

$$E[Z^p] \leq c_p. \quad (6.16)$$

For each $\delta > 0$ we let $\phi_\delta$ be a $C^\infty_0$ function which agrees with $w \mapsto \log |w - z|$ in $B(z, 1) \setminus B(z, \delta)$. It is not hard to see that there is a constant $c_1 > 0$ so that we can find such a function $\phi_\delta$ so that $\|\phi_\delta\|^2 \leq c_1 \log \delta^{-1}$. Combining this with (6.16) implies that the Radon-Nikodym derivative $Z_\delta$ between the law of the restriction of $h$ to $B(z, 1/4) \setminus B(z, \delta)$ and the law of a $\sqrt{8/3}$-quantum cone $(\mathcal{C}, \tilde{h}, z, \infty)$ restricted to $B(z, 1/4) \setminus B(z, \delta)$ with the circle average embedding satisfies the property that for each $p > 0$ there is a constant $\tilde{c}_p > 0$ such that

$$E[Z_\delta^p] \leq \tilde{c}_p \delta^{p^2/2}. \quad (6.17)$$

Proposition 5.3 implies that there exists $c_2, c_3 > 0$ such that, conditional on $E$, the probability that the QLE$(8/3, 0)$ distance using the quantum cone between every point in $B(z, e^{-k}) \setminus B(z, e^{-k-1})$ and $\partial B(z, e^{-k-1})$ is at least $e^{-c_2 k}$ with probability at least $1 - e^{-c_3 k}$. Combining this with (6.17) and the Cauchy-Schwarz inequality, we see that the same is true under $h$ (though with possibly different constants $c_2, c_3$). Iterating this and summing over $k$ implies that there exists $\alpha > 0$ so that the probability of the first part of being $(\alpha, \beta, j)$-good is at least $1 - c_0 e^{-(25/12 - \epsilon)j}$.

The same change of measures argument but using (4.2) of Proposition 4.2 in place of Proposition 5.3 yields a similar lower bound of the probability of the second $(\alpha, \beta, j)$-good condition.
Proof of Proposition 6.7. We will first prove the result for \( k = -2 \) and then explain how to extract the result for other values of \( k \) from this case. Let \( a \) (resp. \( b \)) be the midpoint of the line \( \text{Re}(z) = -2 \) (resp. \( \text{Re}(z) = -1 \)) in \( S \). Let \( j \in \mathbb{N} \) and for each \( 0 \leq \ell \leq e^j \) we let
\[
z_\ell = -2 + \frac{\ell}{e^j} + i \frac{\pi}{2}
\]
be the midpoint of the line \( \text{Re}(z) = -2 + \ell/e^j \) in \( S \). Let \( G_j \) be the event that \( a = z_0, \ldots, z_{e^j} = b \) are all \( (\alpha, \beta, j) \)-good. On \( G_j \), we have that
\[
\mathbb{E}[d_Q^p(a, b)] \leq c_0 \sum_{j} e^{(1-\alpha)pj} e^{-(13/12-\epsilon)j} < \infty.
\]
Fix \( \epsilon > 0 \) small so that \( 13/12 - \epsilon > 1 \). By Lemma 6.9, we know for a constant \( c_0 > 0 \) that
\[
\mathbb{P}[G_j^c] \leq c_0 e^j \times e^{-(25/12-\epsilon)j} = c_0 e^{-(13/12-\epsilon)j}.
\]
Fix \( p > 1 \) so that \( (1-\alpha)p - (13/12-\epsilon) < 0 \). Combining (6.18) and (6.19), we see that
\[
\mathbb{E}[d_Q^p(a, b)] \leq c_0 \sum_{j} e^{(1-\alpha)pj} e^{-(13/12-\epsilon)j} \leq c_0 e^{c_1 k}.
\]
This completes the proof of the result for \( k = -2 \).

We will now generalize the result to all \( k \in \mathbb{Z} \) with \( k \leq -2 \). Lemma 2.2 implies that adding a constant \( C \) to the field scales distances by the factor \( e^{\gamma C/4} \) for \( \gamma = \sqrt{8/3} \). Let \( X \) be the projection of \( h \) onto \( H_1(S) \). Then we can write \( X_u = B_{-2u} + (Q - \gamma)u \) for \( u \leq 0 \) where \( B \) is a standard Brownian motion with \( B_0 = 0 \) conditioned so that \( X_u \leq 0 \) for all \( u \leq 0 \). If we let \( a \) (resp. \( b \)) be the midpoint of the line \( \text{Re}(z) = k \) (resp. \( \text{Re}(z) = k + 1 \)) in \( S \) then it follows that for a constant \( c_1 < \infty \) we have that (with \( \gamma = \sqrt{8/3} \))
\[
\mathbb{E}[d_Q^p(a, b) \mid B_{-2(k+2)}^*(k+2)] \leq c_1 \exp \left( \frac{p\gamma}{4} (B_{-2(k+2)}^* + (Q - \gamma)(k + 2)) \right)
\]
where \( B_{-2(k+2)}^* = \sup_{t \in [0,2]} B_{-2(k+2-t)} \). Using that \( X_u \) conditioned to be negative for all \( u \leq 0 \) is stochastically dominated from above by \( X_u \) conditioned to be negative only for \( u = k + 2 \), it is easy to see that there exists \( p_0 > 1 \) and constants \( c_0, c_1 > 0 \) such that for all \( p \in (0, p_0) \) we have that
\[
\mathbb{E}\left[ \exp \left( \frac{p\gamma}{4} (B_{-2(k+2)}^* + (Q - \gamma)(k + 2)) \right) \right] \leq c_0 e^{c_1 k}.
\]
Combining (6.20) and (6.21) implies the result. \( \square \)
6.3 Proof of moment bounds

We now have the necessary estimates to complete the proofs of Proposition 6.1 and Proposition 6.2.

Proof of Proposition 6.1. Fix $\alpha > 0$ small and let $\beta = \alpha/2$. Recall from the proposition statement that $F_{\alpha,\delta}$ is the event that $\eta'[0,\delta]$ is contained in $S_\delta + u_{\alpha,\delta} - 1$. We let $\gamma_1$ be the shortest $\partial_Q$-length path from $-\infty$ to the midpoint of the vertical line through $\text{Re}(z) = u_{\beta,\delta}$ contained in $[\pi/4, 3\pi/4] \times \mathbb{R}$. We then let $\gamma_2$ be the shortest $\partial_Q$-path from $a = u_{\beta,\delta} + i\pi/4 - 2$ to $b = u_{\beta,\delta} + i3\pi/4 - 2$ contained in $[u_{\beta,\delta} - 5, u_{\beta,\delta}] \times [0, \pi]$.

Let $f_\delta$ be the unique conformal map from the component of $S \setminus \eta'(0, \delta)$ with $+\infty$ on its boundary to $S$ with $|f_\delta(z) - z| \rightarrow 0$ as $z \rightarrow +\infty$. We also let $\tilde{\gamma}_3$ be the shortest path with respect to the internal QLE$(8/3, 0)$ metric associated with $h \circ f_\delta^{-1} + Q \log |(f_\delta^{-1})'|$ from $-\infty$ to the midpoint of the line with $\text{Re}(z) = u_{\beta,\delta}$ contained in $[\pi/4, 3\pi/4] \times \mathbb{R}$ and let $\gamma_3 = f_\delta^{-1}(\tilde{\gamma}_3)$. Note that $\gamma_1$ crosses $\gamma_2$. Moreover, standard distortion estimates for conformal maps imply that on $F_{\alpha,\delta}$ we have that $\gamma_3$ also crosses $\gamma_2$ for all $\delta > 0$ small enough. It therefore follows that, on $F_{\alpha,\delta}$, the distance between $-\infty$ and $\eta'(\delta)$ is bounded from above by the sum of the $\partial_Q$ lengths of $\gamma_1, \gamma_2, \gamma_3$. Thus our first goal will be to show that the lengths of these three paths have a finite $p$th moment for some $p > 1$. We will then deduce the result from this using a scaling argument.

We begin by bounding the length of $\gamma_1$. Fix $p > 1$ and $\epsilon > 0$. Let $D_k$ be as in the statement of Proposition 6.7. Let $n = [u_{\beta,\delta}]$ and let $d_1 = \sum_{k=-\infty}^{n} D_k$. Then the length of $\gamma_1$ is bounded by $d_1$. Suppose that $p > 1$. By Jensen’s inequality, with $c_\epsilon = \sum_{k=0}^{\infty} \epsilon^{-ck}$, we have that

$$d_1^p = \left( \sum_{k=-\infty}^{n} D_k \right)^p = \left( \sum_{k=-\infty}^{n} D_k e^{-\epsilon(n-k)} e^{\epsilon(n-k)} \right)^p \leq c_\epsilon^{p-1} \sum_{k=-\infty}^{n} D_k^p e^{-\epsilon(n-k)p}. \tag{6.22}$$

Thus to bound $\mathbb{E}[d_1^p]$ it suffices to bound the expectation of the right side of (6.22). Proposition 6.7 and scaling together imply that there exists $p > 1$ and constants $c_0, c_1 > 0$ such that $\mathbb{E}[D_k^p] \leq c_0 e^{c_1(k-n)} \delta^{c_1 \alpha}$ for each $k$. Thus by choosing $\epsilon > 0$ sufficiently small, by inserting this into (6.22) we see that (possibly adjusting $c_0, c_1$)

$$\mathbb{E}[d_1^p] \leq c_0 \delta^{c_1 \alpha}. \tag{6.23}$$

The same argument also implies that the $p$th moments of the lengths of $\gamma_2$ and $\gamma_3$ are both at most $c_0 \delta^{c_1 \alpha}$ (possibly adjusting $c_0, c_1$).

Combining everything implies that there exists $p > 1$ such that (possibly adjusting $c_0, c_1$)

$$\mathbb{E}[D_\delta^p 1_{F_{\alpha,\delta}}] \leq c_0 \delta^{c_1 \alpha}. \tag{6.24}$$

Fix $\alpha' > 0$ which is much smaller than $\alpha$. Then we have that

$$\mathbb{E}[D_\delta^p 1_{F_{\alpha',\delta}}] \leq c_0 \delta^{c_1 \alpha} + \mathbb{E}[D_\delta^p 1_{F_{\alpha',\delta} \cap F_{\delta,\delta}}]$$
\[ \leq c_0 \delta^{c_1 \alpha} + E[D_{\delta}^{p'} 1_{F_{\alpha', \delta}}]^{1/p'} P[F_{\alpha, \delta}]^{1/q'} \]

for conjugate exponents \( p', q' \). By Proposition 6.3, the second term decays to 0 faster than any polynomial of \( \delta \), hence for each \( \alpha' > 0 \) small enough (possibly adjusting \( c_0, c_1 \) depending on \( \alpha' \))

\[ E[D_{\delta}^{p'} 1_{F_{\alpha', \delta}}] \leq c_0 \delta^{c_1 \alpha}. \quad (6.25) \]

We will now use a scaling calculation to remove the truncation and obtain the correct exponent in (6.25). Namely, we know that if we add \( C \) to the field then quantum natural time gets scaled by \( e^{3\gamma C/4} \), for \( \gamma = \sqrt{8/3} \) (see [MS15c, Section 6.2]) and quantum distance gets scaled by \( e^{\gamma C/4} \) (Lemma 2.2). Applying this in the setting of (6.24) with \( C = \alpha' \log \delta^{-1} \), we see that (possibly adjusting \( c_0, c_1 \))

\[ E[D_{\delta}^{p'} 1_{F_{\alpha, \delta}}] \leq c_0 \delta^{c_1 \alpha}. \]

By writing \( \delta \) in place of \( \delta^{1-3\alpha'/4} \), we thus see that (possibly adjusting \( c_0, c_1 \))

\[ E[D_{\delta}^{p'} 1_{F_{\alpha, \delta}}] \leq c_0 \delta^{c_1} \text{ for all } \delta \in (0, 1). \]

By iterating this, it is not difficult to see that \( E[D_{\delta}^{p}] < \infty \). Combining this with scaling again (i.e., quantum natural time scales as the third power of quantum distance) implies (6.1).

The final assertions of the proposition are immediate from the first and Proposition 6.3. \( \Box \)

**Proof of Proposition 6.2.** This follows from the same argument used to prove Proposition 6.1, except we have to explain why the analog of Proposition 6.3 holds in this setting. This, in turn, is a consequence of Proposition 4.4. \( \Box \)

### 7 Reverse explorations of \( \sqrt{8/3} \)-LQG spheres

In [MS15b, MS15c] we constructed forward explorations of doubly-marked \( \sqrt{8/3} \)-LQG spheres sampled from the infinite measure \( M_{\text{SPH}}^2 \) by QLE\((8/3, 0)\) and SLE\(_6\), respectively.

The purpose of this section is to describe the time-reversals of the unexplored-domain processes which correspond to these explorations.

#### 7.1 Time-reversal of SLE\(_6\) unexplored-domain process

Suppose that \((S, x, y)\) has the law of a \( \sqrt{8/3} \)-LQG sphere decorated with a whole-plane SLE\(_6\) process \( \eta' \) from \( x \) to \( y \) sampled from \( M_{\text{SPH}}^2 \). We assume that \((S, x, y)\) is embedded into \( \mathcal{Q} \) with \( x \) taken to \(-\infty\), \( y \) to \(+\infty\), and we assume that \( \eta' \) has the quantum natural time parameterization. For each \( t \), we let \( U_t \) be the component of \( \mathcal{Q} \setminus \eta'([0, t]) \) containing \(+\infty\). We recall from [MS15c] that:

- The quantum boundary length of \( U_t \) evolves as the time-reversal of a 3/2-stable Lévy excursion \( e: [0, T] \to \mathbb{R}_+ \) with only upward jumps [MS15c, Theorem 1.2]. (Recall that the Lévy excursion measure is an infinite measure.)
Figure 7.1: **Left:** Part of the time-reversal of the unexplored-domain process associated with a whole-plane SLE$_6$ process $\eta'$ on a doubly-marked quantum sphere $(\mathcal{S}, x, y)$ from $x$ to $y$. If $T$ denotes the (random) amount of quantum natural time required by $\eta'$ to go from $x$ to $y$ and $\delta > 0$, then the green region corresponds to the component of $\mathcal{S} \setminus \eta'([0, T - \delta])$ which contains $y$. This surface is doubly marked by the interior point $y$ and the boundary point $\eta'(T - \delta)$. The union of the blue and green regions corresponds to the component of $\mathcal{S} \setminus \eta'([0, T - 2\delta])$ containing $y$. This surface is also doubly marked, with the interior point being equal to $y$ and the boundary point equal to $\eta'(T - 2\delta)$. The red region is defined similarly. Each of the green, blue, and red regions may individually be viewed as a doubly-marked surface, where in this case the surface is marked by the first and last point visited by $\eta'$. **Right:** We separate the three surfaces on the left hand side into three “necklaces.” As on the left, each necklace has two marked points. Each necklace also has two marked boundary segments, which we call the “top” and “bottom” of the necklace. The top corresponds to the boundary segment which is not part of the circular arc (hence filled by $\eta'$) and the bottom the part of the circular arc which is bold. If we glue together the necklaces as shown (with the tip of one necklace identified with the initial point of the next), then we can recover the left hand picture.

- Given its quantum boundary length, the conditional law of the quantum surface $(U_t, h)$ is that of a quantum disk weighted by its quantum area [MS15c, Proposition 6.4] and, given $(U_t, h)$ as a quantum surface, $+\infty$ is uniformly distributed from the quantum measure.

- Given the quantum surface $(U_t, h)$, the point $\eta'(t) \in \partial U_t$ is uniformly distributed according to the quantum length measure [MS15c, Proposition 6.4].
We recall also from [MS15b] that these three properties were crucial in the proof that QLE$(8/3, 0)$ defines a metric on a quantum sphere.

In Section 7.2 we will describe the time-reversal of the unexplored-domain process for QLE$(8/3, 0)$, which will be useful for using the characterization given in Theorem 1.7 of TBM to complete the proof of Theorem 1.4. The following analog of the third property mentioned just above in the case of the forward exploration of a quantum sphere will be essential for that construction.

**Lemma 7.1.** Using the notation introduced just above, we have that

(i) The quantum boundary length of $U_{(T-t)_+}$ evolves in $t$ as a $3/2$-stable Lévy excursion with only upward jumps from 0 to 0 of length $T$. (We emphasize that $T$ is the length of the Lévy excursion and is random and that $t$ can be bigger than $T$.)

(ii) For each $t \geq 0$, on the event that $T > t$ we have that $\eta'(T - t)$ is uniformly distributed according to the quantum boundary length measure on $\partial U_{T-t}$ given the quantum surface $(U_{T-t}, h)$.

**Proof.** The first statement is immediate from the definition of $M_{\text{SPH}}^2[t]$; see also [MS15c, Theorem 1.2] as mentioned above.

For the second statement, we assume that we are working on the event that $T > t$. Then we know that the conditional law of $(U_{T-t}, h)$ given its quantum boundary length can be sampled from as follows. We sample from the law of a quantum disk weighted by its quantum area with the given boundary length and decorated by an independent SLE$_6$ process starting from a uniformly random marked boundary point and targeted at a uniformly random marked interior point *conditioned* on taking quantum natural time exactly equal to $t$ to reach its target point. The second statement immediately follows from this description and the analogous property in the case of the forward exploration of a $\sqrt{8/3}$-LQG sphere by SLE$_6$.

Throughout, we let $M_{\text{SPH,R}}^{2,t}$ denote infinite measure on doubly marked surfaces $(U_{(T-t)_+}, h)$ decorated by a path $\eta'$ as considered in Lemma 7.1 (The subscript “R” is to indicate that this law corresponds to a time-reversal.) We emphasize again that $T > 0$ is random. On the event that $t < T$, the quantum surface $(U_{(T-t)_+}, h) = (U_{T-t}, h)$ has the topology of a disk. In this case, one marked point is on the disk boundary and the other marked point is in the interior. The marked points respectively correspond to the starting and ending points of the restriction of $\eta'$ to $U_{(T-t)_+}$. On the event that $t \geq T$, the quantum surface $(U_{(T-t)_+}, h) = (U_0, h)$ has the topology of a sphere. In this case, both of the marked points are both contained in the interior of the surface and they correspond to the starting and ending points of $\eta'$.

For $s, t > 0$ we note that there is a natural coupling of $M_{\text{SPH,R}}^{2,s}$ and $M_{\text{SPH,R}}^{2,t+s}$ because we can produce both laws from $M_{\text{SPH}}^2$, as described just above. The proof of part (ii) of
Lemma 7.1 implies that the path decorated quantum surface which is parameterized by $U_{(T - s - t)} \setminus U_{(T - t)}$ is conditionally independent of the path-decorated quantum surface parameterized by $U_{(T - t)}$ given the quantum boundary length of $U_{(T - t)}$. We note that both quantum surfaces are doubly marked: $U_{(T - t)}$ is marked by the initial and target points of $\eta'$ and $U_{(T - s - t)} \setminus U_{(T - t)}$ is also marked by the initial and target points of the SLE$_6$. We will prove below in Proposition 7.2 that both $\partial U_{(T - t)}$ and $\partial (U_{(T - s - t)} \setminus U_{(T - t)})$ are conformally removable, hence the usual removability arguments imply that the two path decorated surfaces together with their marked points almost surely determine the path decorated surface parameterized by $U_{(T - s - t)}$. In particular, this gives us a way of describing a two-step sampling procedure for producing a sample from $\mathcal{M}^{2, t}_{\text{SPH}, R}$. Namely, we:

- Produce a sample from $\mathcal{M}^{2, t}_{\text{SPH}, R}$, and then,
- Given the boundary length, we can glue on a conditionally independent surface which corresponds to another $s$ units of quantum natural time and obtain a sample from $\mathcal{M}^{2, t+s}_{\text{SPH}, R}$. We will refer to this operation either as “zipping in $s$ units of quantum natural time of SLE$_6$” or “gluing in an SLE$_6$ necklace with quantum natural time length $s$.” We note that this operation involves adding at most $s$ units of quantum natural time, however it may involve adding less in the case that all of the boundary length is exhausted in fewer than $s$ units. We note that if we iterate this procedure for long enough, then we will eventually be left with a sample from $\mathcal{M}^{2}_{\text{SPH}}$ (i.e., $\mathcal{M}^{2}_{\text{SPH}}$ is the limit of $\mathcal{M}^{2, t}_{\text{SPH}, R}$ as $t \to \infty$).

We recall that a quantum disk weighted by its quantum area with given boundary length $L$ can be encoded in terms of the time-reversal of a 3/2-stable Lévy excursion with only upward jumps starting from 0 and stopped the last time that it hits $L$. Consider, on the other hand, the surface that arises in the time-reversal of the SLE$_6$ unexplored-domain process on a quantum disk/sphere with a given amount $t$ of quantum natural time. Note that such a surface can be encoded using the time-reversal of a 3/2-stable Lévy excursion with only upward jumps and conditioned to have length $t$ stopped at time $t$. Moreover, the first law conditioned on having quantum natural time exactly equal to $t$ is equal to the second law conditioned on having quantum boundary length exactly equal to $L$.

We now turn to show that $\partial U_{(T - t)}$ is almost surely conformally removable for each $t$.

**Proposition 7.2.** Fix $0 < t < T$ and suppose that $U_{(T - t)}$ is as above. Then $\partial U_{(T - t)}$ is almost surely conformally removable. In particular, for any $s > 0$, the overall surface is uniquely determined by the length-$s$ SLE$_6$ necklaces which arise in the time-reversal of the unexplored-domain process.

We will prove the result by relating the law of the quantum surface $(U_{(T - t)}, h)$ to that of a quantum disk. This will allow us to make use of the removability results established in [MS13b] and [MS15b]. This type of comparison will also be useful for us later on.
Figure 7.2: Shown on the left is sample produced from $M_{t}^{2,\mathcal{SPH},R}$ parameterized by $\mathbf{D}$ and conditioned on having quantum boundary length $L > 0$. We assume that we have fixed $\zeta \in (0, t)$ and the green region is the sample produced from $M_{\mathcal{SPH},R}^{2,\mathcal{SPH},R}$ which is naturally coupled with the big surface. As explained in (7.1), the law of the time-reversal of the boundary length of the reverse domain exploration process can be sampled from by weighting the law of the time-reversal of a $3/2$-stable Lévy excursion in a certain way. As the latter describes the evolution of the boundary length of the unexplored-domain process associated with an instance of $M_{1,\text{DISK}}^{L}$, if we weight an instance of the latter by (7.1), then we can couple the two processes to agree up to a certain time. As explained in Lemma 7.3, this implies that we can couple the surfaces up to a certain time to be the same. Therefore we may view the first law as arising from the second by “cutting out” the red surface and then gluing in the green surface. Thus if we have a time at which the boundary of the reverse exploration disjoint from the inner part, as shown, it is going to be conformally removable because this implies that it is absolutely continuous with respect to the boundary when we do the forward exploration on a quantum disk.

For $s, L > 0$, we let $\rho(s, L)$ be the density at $s$ with respect to Lebesgue measure on $\mathbb{R}_{+}$ of the law of the amount of quantum natural time required by an SLE$_6$ to reach its target point where the underlying quantum surface is a quantum disk with boundary length $L$ and weighted by its quantum area. Equivalently, $\rho(s, L)$ is the density at $s$ with respect to Lebesgue measure on $\mathbb{R}_{+}$ of the law of the length of the time-reversal of a $3/2$-stable Lévy excursion starting from the last time it hits $L$ to first reach $0$. Fix $t, \zeta > 0$. By a Bayes’ rule calculation, it is easy to see that the Radon-Nikodym derivative of

- The law of the part of the surface cut out by an SLE$_6$ in the first $t - \zeta$ units of quantum natural time of a quantum disk weighted by its quantum area conditioned to have quantum natural time equal to $t$ with respect to

- The law of the part of the surface cut out by an SLE$_6$ in the first $t - \zeta$ units of
quantum natural time of a quantum disk weighted by its quantum area conditioned to have quantum natural time at least $t$

is given by

$$\frac{\rho(\zeta, X_{t-\zeta})}{\rho(t, X_0)}.$$  \hspace{1cm} (7.1)

For $t$ and $\zeta$ fixed and $X_0$ constrained to lie in $[C^{-1}, C]$ where $C > 1$ is a fixed constant, it is easy to see that this Radon-Nikodym derivative is a bounded, continuous function of the value of $X_{t-\zeta}$ (and the bound only depends on $t$, $C$, and $\zeta$).

We consider three laws on surface/path pairs:

Law 1: a quantum disk weighted by its quantum area with quantum boundary length equal to $L$ (i.e., $M^L_{\text{DISK}}$) decorated by an independent SLE$_6$ conditioned to have quantum natural time at least $t$.

Law 2: a quantum disk weighted by its quantum area with quantum boundary length equal to $L$ (i.e., $M^L_{\text{DISK}}$) decorated by an independent SLE$_6$ conditioned to have quantum natural time at least $t$ weighted by the Radon-Nikodym derivative in (7.1) (with the value of $\zeta$ fixed).

Law 3: The path-decorated quantum surface which arises by running the time-reversal of the SLE$_6$ unexplored-domain process for $t$ units of quantum natural time, i.e. sampled from $M^{2t}_{\text{SPH,R}}$.

Then we know that we can transform from Law 2 to Law 1 by unweighting by the Radon-Nikodym derivative in (7.1) and then weighting by the Radon-Nikodym derivative for the boundary length. We now record the fact that we can transform from Law 3 to Law 2 by cutting out the last $\zeta$ units of quantum natural time of the SLE$_6$ and then gluing in the path decorated surface associated with a time-reversed 3/2-stable Lévy excursion starting from $X_{t-\zeta}$ and stopped upon hitting 0.

**Lemma 7.3.** Suppose that $(U, h)$ is a quantum surface decorated by an SLE$_6$ process $\eta'$ sampled from $M^L_{\text{DISK}}$ weighted by the Radon-Nikodym derivative as in (7.1). Suppose also that $(U_t, h)$ is the SLE$_6$ reverse unexplored-domain process associated with a sample from $M^2_{\text{SPH}}$ conditioned on requiring at least $t$ units of quantum natural time for the SLE$_6$ to reach its target point. Then the law of the quantum surface $(U_t \setminus U_\zeta, h)$ is equivalent to the corresponding part of the quantum surface produced from Law 2 described just above.

**Proof of Proposition 7.2.** Since we can choose $\zeta > 0$ small enough so that the last $\zeta$ units of quantum natural time of the path is disjoint from the outer boundary of the first $t-\zeta$ when we produce the path-decorated surface from $M^{2t}_{\text{SPH,R}}$, it follows that all almost sure properties of the boundary of a quantum disk also hold for the boundary of the surface produced from $M^{2t}_{\text{SPH,R}}$. This implies the assertion of the proposition. \(\Box\)
7.2 Reverse QLE$(8/3, 0)$ metric exploration

Figure 7.3: **Left:** Shown is the same collection of necklaces as in Figure 7.1 except each of the necklaces has been “rotated” by a uniformly random amount. **Right:** If we glue together the picture this way as shown, then the law of the overall quantum surface will be the same as in Figure 7.1, but the natural growth process which is associated with the construction is the $\delta$-approximation to the time-reversal of the unexplored-domain for QLE$(8/3, 0)$. The law of this process is not the same as the time-reversal for the $\delta$-approximation to QLE$(8/3, 0)$ because in this process, the necklace which reaches the terminal point has quantum length $\delta$ while the necklace which reaches the initial point has quantum length in $[0, \delta]$. The purpose of Section 7.2 is to show that this asymmetry disappears in the $\delta \to 0$ limit. As in the case of the $\delta$-approximation to QLE$(8/3, 0)$, the process on the left is naturally coupled with an SLE$_6$ on a $\sqrt{8/3}$-LQG sphere so that the bubbles formed have the same law.

We will now describe the time-reversal of the process which corresponds to the unexplored-domain for QLE$(8/3, 0)$. In contrast to the setting of SLE$_6$, this process does not have a marked point along the domain boundary (but does have a marked point which is in the interior).

We first suppose that we have fixed $\delta > 0$ and that $(D, h)$ has the law $M^{2,\delta}_{SPHL}$ described just above. We assume that the surface has been embedded so that the target point of the path is 0. Recall from Part (ii) of Lemma 7.1 that the seed of the path is distributed uniformly at random from the quantum boundary measure on $\partial D$.

We construct the rest of the doubly-marked quantum sphere inductively using the following procedure. We suppose that we have constructed doubly-marked quantum
surfaces \((D, h^\delta_1, x^\delta_1),..., (D, h^\delta_j, x^\delta_j)\) where \((D, h^\delta)\) is equal to the initial surface \((D, h)\) described above and \(x^\delta_1 \in \partial D\) is the starting point of the path. We take the embedding of each surface into \(D\) so that the second marked point (i.e., the target point of the path) is equal to 0. As a quantum surface, each \((D, h^\delta_j)\) will have the law of the corresponding marginal under \(M^2_{\text{SPH,R}}\). To construct \((D, h^\delta_j+1, x^\delta_{j+1})\) from \((D, h^\delta_j, x^\delta_j)\), we

1. Sample \(x \in \partial D\) uniformly at random from the quantum boundary length measure on \(\partial D\) associated with \(h^\delta_j\). We note that the law of the doubly marked quantum surface \((D, h^\delta_j, x)\) is the same as the law of the doubly marked quantum surface \((D, h^\delta, x^\delta)\) by Part (ii) of Lemma 7.1.

2. Evolve the SLE\(_6\) unexplored-domain process for \(\delta\) units of quantum natural time with the marked boundary point given by \(x\) (as explained in Section 7.1) to obtain the doubly marked surface \((D, h^\delta_{j+1}, x^\delta_{j+1})\) where \(x^\delta_{j+1} \in \partial D\) is the marked boundary point and the embedding into \(D\) is taken so that the second marked point (i.e., the target point of the path) is equal to 0.

This procedure terminates once the boundary length of the surface has reached 0 so that, as explained at the end of Section 7.1, we have constructed a doubly-marked quantum sphere, i.e., a sample produced from \(M^2_{\text{SPH}}\). We note that the final segment of SLE\(_6\) exploration will almost surely not have length \(\delta\) since it is a zero-measure event that the quantum natural time associated with the entire surface is an integer multiple of \(\delta\) (as this would correspond to having a Lévy excursion whose length is an integer multiple of \(\delta\)). We let \((S^\delta, x^\delta, y^\delta)\) be the resulting doubly marked quantum sphere (which has law \(M^2_{\text{SPH}}\)) and we let \((\bar{\Gamma}_r^\delta)\) be the (reverse) growth process on \((S^\delta, x^\delta, y^\delta)\) which is given by following the SLE\(_6\) necklaces from \(y^\delta\) to \(x^\delta\) as in the construction given just above. We take the time-parameterization of \((\bar{\Gamma}_r^\delta)\) to be given by quantum distance time. We call \((\bar{\Gamma}_r^\delta)\) the \(\delta\)-approximation to the reverse metric exploration.

The time-reversal of \((\bar{\Gamma}_r^\delta)\) does not have the same law as the unexplored region associated with the \(\delta\)-approximation to (forward) QLE\((8/3,0)\). Indeed, note that the first (resp. last) SLE\(_6\) in the construction of \((\bar{\Gamma}_r^\delta)\) has (resp. almost surely does not have) quantum natural time \(\delta\). The same is also true for the \(\delta\)-approximation to (forward) QLE\((8/3,0)\). Consequently, after time-reversing, there is an asymmetry in the amount of quantum natural time associated with the first and last SLE\(_6\) necklaces. We will need to spend a few pages dealing with what might seem like a trivial point, which is that this asymmetry disappears in the \(\delta \to 0\) limit. This, in turn, will imply that in the limit as \(\delta \to 0\), the joint law of the growth processes \((\bar{\Gamma}_r^\delta)\) and \((S^\delta, x^\delta, y^\delta)\) converge weakly to a doubly-marked quantum sphere \((S, x, y)\) and the time-reversal of the unexplored-domain associated with the filled QLE\((8/3,0)\) metric ball starting from \(x\) and targeted at \(y\).

**Proposition 7.4.** In the limit as \(\delta \to 0\), the pair \((S^\delta, x^\delta, y^\delta)\) and \((\bar{\Gamma}_r^\delta)\) converges to a doubly-marked quantum sphere \((S, x, y)\) and the time-reversal \((\bar{\Gamma}_r)\) of the unexplored
region associated with the filled metric ball starting from $x$ and targeted at $y$. Moreover, for each $r > 0$, conditional on the quantum boundary length of $\tilde{\Gamma}_r$, we have that the quantum surface $S \setminus \tilde{\Gamma}_r$ is independent of the surface corresponding to $\tilde{\Gamma}_r$. Finally, the quantum boundary length of $(\tilde{\Gamma}_r)$ evolves as a $3/2$-stable CSBP in $r$.

Before we give the proof of Proposition 7.4, we first record the following convergence result for the forward $\delta$-approximations to QLE($8/3, 0$) (which is a consequence of \[MS15b\] and the earlier results of this article).

**Lemma 7.5.** Suppose that $(S^\delta, x^\delta, y^\delta)$ is a doubly-marked quantum sphere together with the forward $\delta$-approximation $(\Gamma^\delta_r)$ to QLE($8/3, 0$) from $x^\delta$ to $y^\delta$. Then we have that, as $\delta \to 0$, the joint law of $(S^\delta, x^\delta, y^\delta)$ and $(\Gamma^\delta_r)$ converges weakly to that of a doubly marked quantum sphere together with the QLE($8/3, 0$) filled metric ball starting from $x$ and targeted at $y$.

**Proof.** The results of \[MS15b\] imply that if we take any sequence $(\delta_k)$ of positive numbers tending to 0 as $k \to \infty$ then there exists a subsequence $(\delta_{j_k})$ of $(\delta_k)$ along which we have the weak convergence of the joint law of $(S^\delta, x^\delta, y^\delta)$ and $(\Gamma^\delta_r)$ to the joint law of a doubly marked quantum sphere $(S, x, y)$ and a growth process $(\Gamma_r)$ from $x$ to $y$. It was not proved in \[MS15b\] that $(\Gamma_r)$ does not depend on the choice of subsequence $(\delta_{j_k})$. However, it was shown in \[MS15b\] that if $(x_n)$ is any i.i.d. sequence in $S$ chosen from the quantum measure on $S$ then the joint law of the hitting times of the $(x_n)$ by $(\Gamma_r)$ does not depend on the choice of subsequence and, moreover, the hitting times are almost surely determined by $S$. For a general growth process, these hitting times are not enough to determine the growth process itself (for example, by taking for a given value $r$ the closure of those $(x_n)$ which are hit at or before time $r$). The continuity results that we have established in the present article (Theorem 1.1 and Theorem 1.2), however, imply that this is the case for QLE($8/3, 0$). This, in particular, implies that every subsequential limit of the $(\Gamma^\delta_r)$ is the same, which implies the existence of the limit and that the limit is almost surely determined by the limiting surface $S$, as desired. \(\square\)

We note that Proposition 7.4 does not immediately follow from Lemma 7.5 because, as mentioned just above, the unexplored region associated with the $\delta$-approximation to (forward) QLE($8/3, 0$) does not have the same law as the time-reversal of $(\tilde{\Gamma}_r^\delta)$.

The two processes do, however, have the same law if we condition on the amount of quantum natural time required by each to be a fixed multiple of $\delta$. We will use this exact symmetry to complete the proof of Proposition 7.4. In order to do so, we need to strengthen Lemma 7.5 to get that the joint law of $(\Gamma^\delta_r)$ and $(S^\delta, x^\delta, y^\delta)$ converges weakly as $\delta \to 0$ to $(S, x, y)$ and $(\Gamma_r)$ when we have fixed the amount of quantum natural time required by the QLE($8/3, 0$) to go from $x$ to $y$.

**Lemma 7.6.** Fix $T > 0$. The statement of Lemma 7.5 holds even if we condition on the amount of quantum natural time required by $(\Gamma^\delta_r)$ to reach $y^\delta$ to be equal to $T$. 

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That is, the joint law of \((S^\delta, x^\delta, y^\delta)\) and \((\Gamma_r^\delta)\) conditioned on \((\Gamma_r^\delta)\) requiring \(T\) units of quantum natural time to reach \(y^\delta\) converges weakly as \(\delta \to 0\) to the joint law of \((S, x, y)\) and \((\Gamma_r)\) conditioned on \((\Gamma_r)\) requiring \(T\) units of quantum natural time to reach \(y\).

**Proof.** Fix \(T > 0\). To prove this strengthened result, we will make a comparison between the law of the surface conditioned so that \((\Gamma_r^\delta)\) requires exactly \(T\) units of quantum natural time to reach \(y^\delta\) with the law of the unconditioned surface. This will allow us to deduce the result from Lemma 7.5.

It will be convenient in the proof to assume that \((\Gamma_r^\delta)\) has the quantum natural time parameterization (rather than the quantum distance parameterization). Once we have established the result in this setting, the result in the case of the quantum distance parameterization follows by applying a time change.

For each \(s, L > 0\), we let \(\rho(s, L)\) be the density at \(s\) with respect to Lebesgue measure on \(\mathbb{R}_+\) of the law of the amount of quantum natural time required by the \(\delta\)-approximation to QLE(8/3, 0) to reach its target point where the underlying quantum surface is a quantum disk weighted by its quantum area with quantum boundary length \(L\). We note that this is the same as the density at \(s\) with respect to Lebesgue measure on \(\mathbb{R}_+\) as that of the amount of time that it takes for the time-reversal of a 3/2-stable Lévy excursion with only upward jumps starting from the last time that it hits \(L\) to hit 0 at time \(s\). In particular, this density does not depend on \(\delta\). We also let \(\rho(s)\) be the density at \(s\) with respect to Lebesgue measure on \(\mathbb{R}_+\) of the law of the amount of quantum natural time required by \((\Gamma_r^\delta)\) to reach its target point. This density also does not depend on \(\delta\). A simple Bayes’ rule calculation implies that the Radon-Nikodym derivative of the law of \((\Gamma_r^\delta)\) up to quantum natural time \(S\) conditioned on requiring quantum natural time exactly equal to \(T > S\) to reach \(y^\delta\) with respect to the law under which we have not conditioned on the total quantum natural time is given by:

\[
Z_{S,T,X} = \frac{\rho(T - S, X)}{\rho(T)}
\]

where \(X\) is the quantum boundary length of \((\Gamma_r^\delta)\) at time \(S\). We note that \(Z_{S,T,X}\) is a bounded, continuous function alone of the value of the time-reversal of the Lévy excursion associated with the doubly marked surface at time \(S\). Thus since we have the weak convergence of the process up to time \(S\) before weighting the law by (7.2), we also have the weak convergence of the process up to time \(S\) after weighting the law by (7.2). That is, we get the weak convergence up to time \(S < T\) when we have fixed the amount of quantum natural time to be equal to \(T\). The result thus follows since we have the convergence of the process and quantum surface thus formed up to time \(S\) for all \(S < T\).

**Proof of Proposition 7.4** We are going to prove the first assertion of the proposition by constructing a coupling of forward/reverse QLE(8/3, 0) and then make use of
Lemma 7.6. We first suppose that we have a doubly-marked quantum sphere \((S, x, y)\) sampled from \(M_{2^{\text{SPH}}}\) conditioned so that the amount of quantum natural time required by the path to go from \(x\) to \(y\) being exactly equal to 1. We take \(\delta = 1/k\) with \(k \in \mathbb{N}\) so that \(1/\delta \in \mathbb{N}\). We then take this surface and “reshuffle” the seed of the path by resampling it uniformly from the quantum boundary measure at each integer multiple of \(\delta\) time increment. We then obtain a doubly-marked quantum sphere \((S_\delta, x_\delta, y_\delta)\) decorated with a forward \(\delta\)-approximation to QLE\((8/3, 0)\) conditioned on taking exactly 1 unit of quantum natural time to go from \(x_\delta\) to \(y_\delta\). Lemma 7.1 implies that time-reversing the unexplored region of forward QLE\((8/3, 0)\) has the law of the \(\delta\)-approximation to the time-reversal of the QLE\((8/3, 0)\) unexplored-domain process. Lemma 7.6 implies that, as \(\delta \to 0\), we have that the joint law of \((S^\delta, x^\delta, y^\delta)\) and \((\Gamma^\delta_r)\) converge weakly to the joint law of a doubly marked quantum sphere \((S, x, y)\) and the filled metric ball \((\Gamma_r)\) from \(x\) to \(y\) conditioned on the amount of quantum natural time required \((\Gamma^\delta_r)\) to reach \(y\) being exactly equal to 1. Therefore, by the construction of our coupling, the joint law of \((S^\delta, x^\delta, y^\delta)\) and the \(\delta\)-approximation \((\tilde{\Gamma}^\delta_r)\) to the time-reversal of the unexplored-domain for QLE\((8/3, 0)\) also converge weakly as \(\delta \to 0\) and the two limits are the same.

To finish the proof, we need to generalize to the setting in which:

- The amount of quantum natural time is not fixed and
- The value of \(\delta\) that we use in the \(\delta\)-approximations to the time-reversal of the QLE\((8/3, 0)\) unexplored-domain process does not depend on the total amount of quantum natural time required for the process to go from \(y\) to \(x\).

We can reduce the setting of general quantum natural times to the setting in which we have fixed the quantum natural time by weighting by a Radon-Nikodym derivative which is analogous to that in (7.2). Namely, we let \(\sigma(s, X)\) be the density at \(s\) with respect to Lebesgue measure on \(\mathbb{R}_+\) of the amount of quantum natural time required by \((\tilde{\Gamma}^\delta_r)\) to hit \(x^\delta\) given that its boundary length is equal to \(X\) and we let \(\sigma(s)\) be the density at \(s\) with respect to Lebesgue measure on \(\mathbb{R}_+\) of the amount of quantum natural time required by \((\tilde{\Gamma}^\delta_r)\) to go from \(y^\delta\) to \(x^\delta\). Then the Radon-Nikodym derivative between the law of \((\tilde{\Gamma}^\delta_r)\) requiring quantum natural time exactly equal to \(T\) with respect to its unconditioned law, both up to quantum natural time \(S\), is equal to:

\[
Y_{S,T,X} = \frac{\sigma(T - S, X)}{\sigma(S)}.
\]

We note that the definition of \(Y_{S,T,X}\) in (7.3) takes the same form as in (7.2) except it is defined in terms of the reverse rather than the forward exploration. Moreover, as in (7.2), since this is a function of the boundary length process, it does not depend on the value of \(\delta\).
Fix a sequence \((\delta_k)\) of positive numbers with \(\delta_k \to 0\) as \(k \to \infty\). For each \(k\), we let \(T_k = \delta_k \lceil \delta_k^{-1} \rceil\). Note that \(T_k/\delta_k\) is an integer for every \(k\) and that \(T_k \to 1\) as \(k \to \infty\). In particular, if we condition on the surface having exactly \(T_k\) units of quantum natural time then the \(\delta_k\)-approximations to forward and reverse QLE agree. Since \((7.3)\) is also continuous in \(T_k\), we can use the value of \(T_k\) associated with the given value of \(\delta_k\) and the result follows.

The final assertions of the proposition are obvious from the construction.

\[\square\]

7.3 Filled-metric ball complements and metric bands

![Figure 7.4: Left: An instance \((\mathcal{S}, x, y)\) of a doubly-marked \(\sqrt{8/3}\)-LQG sphere decomposed into four metric bands. Note that a metric band can have the topology of either a disk or an annulus. Right: If we mark the inside and outside of each metric band, then we can uniquely reconstruct \((\mathcal{S}, x, y)\) by gluing the bands together according to boundary length, with the marked point on each band identified with the corresponding marked point on the next band.]

Suppose that we have a doubly-marked quantum sphere \((\mathcal{S}, x, y)\) and that we run the time-reversal of the QLE\((8/3, 0)\) unexplored-domain process for \(r\) units time. We let \(\mathcal{C}_r\) be the law of the surface which is parameterized by the process up to time \(r\), conditionally on the event that \(d_Q(x, y) \geq r\). Proposition \(7.4\) implies that the conditional law of the remaining surface depends only on the quantum boundary length \(\ell\) of the outer boundary of what we have explored so far. Moreover, by scaling, this law can be sampled from by first sampling from the law in the case that the boundary length is equal to 1 and then scaling so that the quantum boundary length is equal to \(\ell\). Recall that this has the effect of scaling quantum distances by \(\ell^{1/2}\) (Lemma \(2.2\)) and quantum areas by \(\ell^2\). If we start off with such a surface of quantum boundary length \(\ell\), and then we explore the metric ball in reverse for \(s\) units of distance, then we refer to this
surface as a (reverse) metric band of inner boundary length $\ell$ and width $s$. We call $B_{\ell,s}$ the law on such surfaces. We make the following observations about $B_{\ell,s}$:

**Proposition 7.7.** Fix $\ell, s > 0$ and suppose that $B$ has the law $B_{\ell,s}$. Then $B$ is topologically either an annulus or a disk (it is a disk in the case that the target point has distance less than $s$ from the boundary of the band). Moreover, if we fix a sequence $(r_k)$ of positive numbers with $\sum_k r_k = \infty$ and we decompose $(\mathcal{S}, x, y)$ into its successive bands $B_k$ of width $r_k$, then the $B_k$ are conditionally independent given the quantum length of their inner and outer boundaries.

We note that in the statement of Proposition 7.7, there exists $k_0$ (random) such that $B_k = \emptyset$ for all $k \geq k_0$.

**Proof of Proposition 7.7.** The first assertion follows from the continuity results established earlier (Theorem 1.1 and Theorem 1.2).

The second assertion is immediate from the construction. \qed

We next turn to establish a result regarding the regularity of the boundary of a sample produced from either $C_r$ or $B_{\ell,s}$ and regarding the regularity of the quantum boundary measure for a sample produced from $C_r$. As in the case of the reverse exploration of a doubly marked sphere $(\mathcal{S}, x, y)$ produced from $M^2_{SPH}$, we will prove these results by making a comparison between $C_r$ and a quantum disk.

By the construction of reverse QLE$(8/3, 0)$ with the quantum distance time parameterization, we know that the conditional law of the region cut off by the reverse QLE$(8/3, 0)$ growth up exploring for $r$ units of quantum distance time given its quantum boundary length is given by that of a quantum disk weighted by its quantum area decorated with a marked point $y$ chosen uniformly from the quantum measure conditioned on the event that $y$ has distance $r$ to the boundary. We note that the Radon-Nikodym derivative of:

- The conditional law of the unexplored region when performing a reverse QLE$(8/3, 0)$ exploration for $r$ units of quantum natural time given the quantum boundary length is equal to $\ell$ with respect to

- A quantum disk with quantum boundary length $\ell$ weighted by its quantum area marked by a uniformly random point $y$ conditioned on the distance of $y$ to the boundary being at least $r$

is given by (via a Bayes’ rule calculation)

$$\frac{\rho(r, \ell)}{\rho(r)}$$

(7.4)

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where $\rho(r, \ell)$ is equal to the density at $r$ of the amount of time required by the time-reversal of a $3/2$-stable CSBP excursion starting from the last time it hits $\ell$ to require $r$ units of time to hit 0 and $\rho(r)$ is the density at $r$ for a $3/2$-stable CSBP excursion to take time $r$. Thus by a cutting/gluing argument analogous to what we used just above, we have that the law of the surface near the boundary is absolutely continuous with respect to that of a quantum disk with the same boundary length.

**Proposition 7.8.** Fix $r > 0$ suppose that $(\mathcal{S}, x, y)$ is a sampled from $M_{2_{\text{SPH}}}$ conditioned on $d_{Q}(x, y) \geq r$, and let $U$ be the region separated from $x$ by the reverse QLE($8/3, 0$) exploration by quantum distance time $r$. In any embedding of $(\mathcal{S}, x, y)$, we have that $\partial U$ is almost surely a Hölder domain. That is, if $\varphi : D \to U$ is any conformal transformation, then $\varphi$ is Hölder continuous on $\partial D$. The same holds in the setting of a metric band. In particular, $\partial U$ is almost surely conformally removable and the same likewise holds for a metric band embedded in $(\mathcal{S}, x, y)$.

*Proof.* It is explained in [MS15b, Proposition 5.2] that it follows from [MS13b, Theorem 8.1] that if $(D, h)$ is any quantum surface whose law is absolutely continuous with respect to that of a quantum disk and $\varphi : D \to \mathcal{S}$ is any embedding of $(D, h)$ into a bigger surface whose law is somehow related to the GFF, then $\varphi$ is almost surely Hölder continuous. The result thus follows in the present setting because, as explained just above, by applying a cutting/gluing operation and weighting by a Radon-Nikodym derivative, we can transform from the law in the present setting to the law of a quantum disk.

Suppose that we are in the setting of Proposition 7.7 and we mark the outer boundary of each $B_k$ with a point chosen uniformly at random from the quantum measure, so that each metric band is doubly marked (one point on the inside boundary and one point on the outside boundary). Then the removability of each $\partial B_k$ established in Proposition 7.8 (combined with the usual removability arguments, e.g., [She15]) implies that there is almost surely a unique way to glue these doubly marked metric bands together to reconstruct the original doubly-marked quantum sphere $(\mathcal{S}, x, y)$. That is, the doubly marked bands almost surely determine the entire doubly-marked quantum surface.

This decomposition will be important for us in Section 8.3 in which we show that the quantum boundary lengths between geodesics along the boundary of a filled metric ball evolve as independent $3/2$-stable CSBPs.

We finish this section by describing the limiting behavior of a metric band near a typical point in terms of a $\sqrt{8/3}$-quantum wedge.

**Proposition 7.9.** Suppose that we have a sample $B$ from $B_{L,r}$ and that $z$ is chosen uniformly from the quantum measure on the outer boundary of $B$. Suppose that $(\mathcal{H}, h, 0, \infty)$ has the law of a $\sqrt{8/3}$ quantum wedge. Let $\mathcal{H}$ be the surface which consists of those points which have distance at most 1 to $\partial \mathcal{H}$ or which are cut off from $\infty$ by those points.
with distance at most 1 from \( H \). Fix \( R > 0 \) and let \( \mathcal{S} \) (resp. \( \tilde{\mathcal{S}} \)) be the quantum surface which consists of those points in \( B \) (resp. \( \tilde{B} \)) whose distance to \( z \) (resp. 0) is at most \( R \) or are cut off from \( \partial B \) (resp. \( \partial \tilde{B} \)) by those points with distance at most \( R \) to \( z \) (resp. 0). Then the law of the marked quantum surface \((\mathcal{S}, z)\) converges weakly as \( L \to \infty \) to the law of the marked quantum surface \((\tilde{\mathcal{S}}, z)\).

**Proof.** The Radon-Nikodym derivative arguments used to prove Proposition 7.4 imply that if one picks a quantum typical point on the boundary of a reverse metric exploration and zooms in then one obtains in the limit a \( \sqrt{8/3} \)-quantum wedge. Indeed, this follows because the corresponding fact for quantum disks is immediate from the definition and the aforementioned Radon-Nikodym derivative arguments imply that the boundary behavior of a reverse metric exploration is the same as the boundary behavior of a quantum disk.

\[ \square \]

## 8 Emergence of the 3/2-Lévy net

In this section we will see the 3/2-Lévy net structure \([MS15a]\) appear in the \( \sqrt{8/3} \)-LQG sphere. We will establish this by successively considering three different approximations to geodesics, all of which connect a quantum typical point on the boundary of the reverse metric exploration back to the root of the metric ball.

We will describe the first approximation in Section 8.1. It is based on the \( \delta \)-approximation to the reverse metric exploration process (Section 7.2). Using Proposition 6.1 and Proposition 6.2 established in Section 6, we will then show in Section 8.2 that these first approximations to geodesics converge (at least along a subsequence) to limiting continuous paths. These subsequential limits serve as our second approximation to geodesics.

Although it may not be obvious from the construction that the first and second approximations to geodesics are related to actual geodesics, these approximations will be useful to analyze. This is because, as we will show in Section 8.2.6, it will follow from the construction that if one considers two such second approximations to geodesics and then performs a reverse metric exploration, then the quantum lengths of the two segments of the boundary of the reverse metric exploration between the two paths evolve as independent 3/2-stable CSBPs. In fact, we will show that this holds more generally for any finite collection of such paths. At this point, we will start to see some of the (breadth first) 3/2-Lévy net structure from \([MS15a]\) to emerge.

We will see in the proofs that these second approximations to geodesics are finite length paths but we will not rule out in the construction that they can be strictly longer than an actual geodesic. This will lead us to our third approximation to geodesics, which will be paths whose expected length is at most \((1+\epsilon)\) times the length of an actual geodesic with the additional property that the quantum boundary lengths between such paths along
the boundary of a reverse metric exploration evolve approximately like independent $3/2$-stable CSBPs. We will then use that quantum boundary lengths and quantum distances have different scaling exponents to deduce that the quantum boundary lengths between any finite collection of actual geodesics also evolve as independent $3/2$-stable CSBPs.

Once we have finished all of this, it will not require much additional work in Section 8.4 to combine the results of this article with Theorem 1.7 to complete the proof of Theorem 1.4.

Throughout this section, for a doubly-marked surface $(\mathcal{S}, x, y)$ and $r > 0$, we will write $\overline{B}_Q(x, r)$ for the hull of the closure of $B_Q(x, r)$ relative to $y$. That is, $\overline{B}_Q(x, r)$ is equal to the hull of the $\text{QLE}(8/3, 0)$ growth from $x$ to $y$ with the quantum distance parameterization stopped at time $r$.

### 8.1 First approximations to geodesics

Fix $r, \delta > 0$. Suppose that we have a doubly-marked quantum sphere $(\mathcal{S}, x, y)$ sampled from $\mathcal{M}^2_{\text{SPH}}$ which is decorated by the growth process which arises by first performing the reverse metric exploration from $y$ to $x$ up to quantum distance time $r$, and then, in the unexplored region (i.e., $\overline{B}_Q(x, d_Q(x, y) - r)$), running the $\delta$-approximation to the reverse metric exploration as described in Section 7.2. Let $\overline{\Gamma}^{r, \delta}$ be the corresponding growth process with the quantum natural time parameterization, where we take time 0 to correspond to where the reverse metric exploration has first reached $r$ units of quantum distance time. That is, $\overline{\Gamma}^{r, \delta}_0 = \mathcal{S} \setminus \overline{B}_Q(x, d_Q(x, y) - r)$. For each $u \geq 0$, we let $X^{r, \delta}_u$ be the quantum boundary length of $\partial \overline{\Gamma}^{r, \delta}_u$. Then $X^{r, \delta}$ evolves as a $3/2$-stable Lévy process with only upward jumps.

We augment the construction of $\overline{\Gamma}^{r, \delta}$ by simultaneously building what we will call a first approximation to a geodesic as follows. For each $j$, we let $N_j^{r, \delta}$ be the $j$th SLE$_6$ necklace in the construction of $\overline{\Gamma}^{r, \delta}$. We note that $N_j^{r, \delta}$ is encoded by $X^{r, \delta}|_{[(j-1)\delta, j\delta]}$ and the corresponding collection of quantum disks. We can divide the outer boundary of $N_j^{r, \delta}$ into two parts: the bottom and the top (see Figure 8.1). The second part is what gets glued to $\overline{\Gamma}^{r, \delta}_{(j-1)\delta}$ and it is marked by the tip of the SLE$_6$ segment and the bottom is marked by the initial point of the path. Let $T_j^{r, \delta}$ (resp. $B_j^{r, \delta}$) denote the quantum length of the top (resp. bottom) of $N_j^{r, \delta}$. Then we note that $B_j^{r, \delta} - T_j^{r, \delta} = X_j^{r, \delta} - X_{(j-1)\delta}^{r, \delta}$.

For $u \in [0, \delta]$ we let $X^{r, \delta, L}_u$ (resp. $X^{r, \delta, R}_u$) denote the change in the left (resp. right) boundary length of the SLE$_6$ which forms $N_j^{r, \delta}$ as it is being zipped in so that

$$X^{r, \delta}_{(j-1)\delta + u} - X^{r, \delta}_{(j-1)\delta} = X^{r, \delta, L}_u + X^{r, \delta, R}_u.$$ 

Then we have that

$$T_j^{r, \delta} = X^{r, \delta}_{(j-1)\delta} - \left( \inf_{u \in [0, \delta]} X^{r, \delta, L}_u + \inf_{u \in [0, \delta]} X^{r, \delta, R}_u \right).$$ (8.1)
Figure 8.1: Shown is an SLE$_6$ necklace $\mathcal{N}$ of length $\delta$. When referring to the boundary of $\mathcal{N}$, we mean the boundary of the region which is cut off from the target point by the corresponding SLE$_6$. (We will only show this part of the necklace in illustrations in subsequent figures.) We can divide the boundary of $\mathcal{N}$ into two parts: the top (heavy red) and the bottom (blue), as shown. The top is marked by the terminal point of the SLE$_6$ and the bottom is marked by the initial point. If $T$ (resp. $B$) denotes the quantum length of the top (resp. bottom) of the necklace and $X$ is the 3/2-stable Lévy process with only upward jumps which encodes the change in the boundary length of the time-reversal of the unexplored domain process as one glues in $\mathcal{N}$ then we have that $B - T = X_\delta - X_0$.

and

$$B_{j}^{r,\delta} = X_{j}^{r,\delta} - \left( \inf_{u \in [0,\delta]} X_{u}^{j,r,\delta,L} + \inf_{u \in [0,\delta]} X_{u}^{j,r,\delta,R} \right).$$ (8.2)

As $X_{j}^{r,\delta,L}$ and $X_{j}^{r,\delta,R}$ evolve as 3/2-stable Lévy processes with only upward jumps, it follows that $T_{j}^{r,\delta}$ has an exponential moment (recall Lemma 2.8). This will be important for us in our later arguments.

We suppose that $w_{0}^{r,\delta}$ is picked uniformly from $\partial \hat{\Gamma}_{0}^{r,\delta}$ using the quantum boundary measure. Assume that we have defined $w_{0}^{r,\delta}, \ldots, w_{j-1}^{r,\delta}$. Then we inductively define $w_{j}^{r,\delta}$ as follows. If $w_{j-1}^{r,\delta}$ is contained in the interval of $\partial \hat{\Gamma}_{(j-1)\delta}$ to which the top of $\mathcal{N}_{j}^{r,\delta}$ is glued, then we take $w_{j}^{r,\delta}$ to be equal to the marked point on the bottom of $\mathcal{N}_{j}^{r,\delta}$ (see Figure 8.2). Otherwise, we take $w_{j}^{r,\delta}$ to be equal to $w_{j-1}^{r,\delta}$ (see Figure 8.3).

We then form a path $\eta_{r,\delta}$, our first approximation to a geodesic, by connecting the points $w_{0}^{r,\delta}, \ldots, w_{n}^{r,\delta}$ with paths $\eta_{j}^{r,\delta}$ where we take $\eta_{j}^{r,\delta}$ to be the shortest path in the
Figure 8.2: Illustration of one step in the construction of the first approximations to geodesics. **Left:** The disk represents the surface parameterized by $\Gamma_{r,\delta}^{(j-1)\delta}$. Shown is the event $A_{r,\delta}^{j\delta}$ that the top of the SLE$_6$ necklace $N_{r,\delta}^{j\delta}$ is glued to a boundary segment which contains the marked boundary point $w_{r,\delta}^{j\delta}$ at step $j$. **Right:** The disk represents the surface parameterized by $\Gamma_{r,\delta}^{j\delta}$, which is formed by gluing $N_{r,\delta}^{j\delta}$ to $\Gamma_{r,\delta}^{(j-1)\delta}$. The path $\eta_{r,\delta}^{j\delta}$, indicated in green, is a shortest path in the internal metric associated with $\Gamma_{r,\delta}^{j\delta}$ which connects the marked boundary point $w_{r,\delta}^{j\delta}$ at step $j$ to the marked boundary point $w_{r,\delta}^{j-1\delta}$ from step $j-1$.

The internal metric of the surface which has been explored by time $j$ (i.e., the surface parameterized by $\Gamma_{r,\delta}^{j\delta}$) between $w_{r,\delta}^{j\delta}$ and $w_{r,\delta}^{j-1\delta}$. We note that $\eta_{r,\delta}^{j\delta}$ for a given value $j$ is typically constant because $T_{r,\delta}^{j\delta}$ is typically of order $\delta^{2/3}$ while the quantum length of $\partial \Gamma_{r,\delta}^{j\delta}$ is typically of order 1. Thus, the probability that $w_{r,\delta}^{j\delta} \neq w_{r,\delta}^{j-1\delta}$ is of order $\delta^{2/3}$. In particular, the number of $j$ such that $w_{r,\delta}^{j\delta} \neq w_{r,\delta}^{j-1\delta}$ is of order $\delta^{-1/3}$.

We have so far defined a single path $\eta_{r,\delta}^{j\delta}$. By repeating this construction with independently chosen initial points on $\partial \Gamma_{r,\delta}^{0\delta}$, we can construct many such paths.

8.2 Second approximations to geodesics

Fix $r > 0$. We will now show that the joint law of $(S, x, y)$ and the paths $\eta_{r,\delta}^{j\delta}$ constructed in Section 8.1 (first approximations to geodesics) just above converges weakly, at least along a subsequence $(\delta_k)$, to a limiting doubly-marked quantum sphere $(S, x, y)$ with law $M_{SPH}^{2}$ decorated by a path $\eta^r$ which connects a uniformly random point on the boundary of the reverse metric exploration at time $r$ to $x$.
Figure 8.3: (Continuation of Figure 8.2) Illustration of one step in the construction of an approximate geodesic. Left: Shown is the case that \((A^r_\delta)_j\) occurs, i.e., the top of the SLE\(_6\) necklace \(N^r_\delta\) is glued to a boundary segment which does not contain the marked point \(w^{r_\delta}_{j-1}\) from step \(j-1\). Right: Shown is the surface parameterized by \(\tilde{\Gamma}^{r_\delta}_{j}\). In this case, \(\eta^{r_\delta}_j\) is the constant path which is equal to the point \(w^{r_\delta}_j = w^{r_\delta}_{j-1}\).

The exact topology that we use here is not important, but to be concrete we will make the following choice. By applying a conformal transformation, we can parameterize \((S, x, y)\) using \(S^2\) with \(x\) (resp. \(y\)) taken to the south (resp. north) pole and the starting point of \(\eta^{r_\delta}\) taken to a fixed point on the equator. We use the uniform topology on paths on \(S^2\) and the weak topology on measures on \(S^2\) for the area measure which encodes the quantum surface.

We will refer to the path \(\eta^r\) as our second approximation to a geodesic because it is a finite length path between a point on \(\partial B \begin{scriptsize}(x, d_Q(x, y) - r)\end{scriptsize}\) and \(x\). In the process of proving the existence of \(\eta^r\), we will also show that it has certain properties that will be useful for us in the next section. We will later show that the quantum boundary length of the two segments along the boundary of a metric ball between two such paths started at uniformly random points evolve as independent 3/2-stable CSBPs and, more generally, that the same is true for any finite number of paths.

**Proposition 8.1.** Fix \(r > 0\). There exists a sequence \((\delta_k)\) of positive numbers with \(\delta_k \to 0\) as \(k \to \infty\) such that the following is true. The joint law of the doubly marked quantum surfaces \((S, x, y)\) and paths \(\eta^{r_\delta_k}\) converges weakly (using the topology described just above) to that of a limiting doubly marked quantum surface/path pair \((S, x, y), \eta^r\) where the marginal of \((S, x, y)\) is given by \(M^2_{\text{SPH}}\) and the following hold.

(i) Almost surely, \(\eta(t) \in \partial B^\bullet_Q(x, d_Q(x, y) - (r + t))\) for all \(t \in [0, d_Q(x, y) - r]\).
(ii) For each $t \geq 0$, given the quantum boundary length of $\partial B_Q^\bullet(x, \bar{d}_Q(x, y) - (r + t))$, the quantum surface parameterized by $B_Q^\bullet(x, \bar{d}_Q(x, y) - (r + t))$ and marked by the pair $(\eta(t), x)$ is independent of the quantum surface parameterized by $S \setminus B_Q^\bullet(x, \bar{d}_Q(x, y) - (r + t))$ and marked by the pair $(\eta(t), y)$.

Fix $T > 0$, $C > 1$, and let $E_{C,T}^r$ be the event that the quantum boundary length of $\partial B_Q^\bullet(x, \bar{d}_Q(x, y) - (r + t))$ stays in $[C^{-1}, C]$ for $t \in [0, T]$ and let $\ell_T^r$ be the arc length of $\eta^r|_{[0,T]}$. Then there exists a constant $K > 0$ depending only on $C, T$ such that $\mathbb{E}[\ell_T^r 1_{E_{C,T}^r}] \leq K$. In particular, for every $\epsilon > 0$ there exists $\delta > 0$ such that if we have an event $Q$ which occurs with probability at most $\delta$ then $\mathbb{E}[\ell_T^r 1_{E_{C,T}^r \cap Q}] \leq \epsilon$.

Finally, by passing to a further subsequence if necessary, we can construct a coupling of a countable collection of paths which each satisfy (i) and (ii), which start at a countable dense set of points chosen i.i.d. from the quantum boundary measure on $\partial B_Q^\bullet(x, \bar{d}_Q(x, y) - r)$, and do not cross.

We will break the proof of Proposition 8.1 into several steps which are carried out in Sections 8.2.1–8.2.5. The part of the proof contained in Section 8.2.1 is instructive to read on a first reading because it provides some intuition as to why the second approximations should be related to geodesics. The estimates from Sections 8.2.2–8.2.5 may be skipped on a first reading, since the material here is mainly technical and is focused on transferring the moment bounds from Section 6 to the present setting.

We will establish the statement regarding the evolution of the quantum boundary lengths between a finite number of paths as in Proposition 8.1 in Section 8.2.6.

8.2.1 Step count distance passes to limit

We begin by establishing a lemma which we will later argue implies part (i) of Proposition 8.1. This will be important because it will imply that along any subsequence which $\eta^{r,\delta}$ converges we have that the limiting path $\eta^r$ does not trace along $\partial B_Q^\bullet(x, \bar{d}_Q(x, y) - (r + t))$ for any value of $t$. Equivalently, this will imply that $\eta^r$ is a continuous path if we parameterize it according to its distance from $x$ and the proof will show that this is in fact the natural parameterization to use for $\eta^r$.

Lemma 8.2. There exists a constant $c > 0$ such that the following is true. For each $j$, we let $A_j^{r,\delta}$ be the event that $w_j^{r,\delta} \neq w_{j-1}^{r,\delta}$ and let $I_j^{r,\delta} = 1_{A_j^{r,\delta}}$. Fix any value of $t > 0$ and let

$$N = \min \left\{ m \geq 1 : c^{-1} \delta^{1/3} \sum_{j=1}^{m} I_j^{r,\delta} \geq t \right\}.$$ 

On the event that $\bar{d}_Q(x, y) > r + t$, we have that $\bar{d}_Q(w_N^{r,\delta}, x)$ converges in probability as $\delta \to 0$ to $\bar{d}_Q(x, y) - (r + t)$.
Proof. Note that $\sum_{j=1}^{m} I_{j}^{\tau}$ counts the number of times that the marked point moves in $m\delta$ units of quantum natural time. That is, $\sum_{j=1}^{m} I_{j}^{\tau}$ is the “step count distance” of $w_{i}^{\tau}$ to $\partial B_{Q}^\ast(x, d_{Q}(x, y) - r)$ because it counts the number of steps that the marked point has taken after $m$ SLE$_{6}$ necklaces have been added in the reverse exploration.

For each $s$, we let $\mathfrak{s} = [\delta^{-1}s]\delta$. Assume that $u > 0$ is fixed, let $\epsilon > 0$, and $\tau^{j,\delta} = u \wedge \inf\{s \geq 0 : X_{s}^{\tau} = \epsilon\}$. As $X_{s}^{\tau}$ is a non-negative càdlàg process with only upward jumps and whose law does not depend on $\delta$, it is easy to see that (in probability)

$$\int_{0}^{\tau^{j,\delta}} \frac{1}{X_{s}^{\tau}} ds \to \int_{0}^{\tau^{j,\delta}} \frac{1}{X_{s}^{\tau}} ds \quad \text{as} \quad \delta \to 0.$$

Let $F_{s}^{\tau}$ be the filtration generated by $X_{s}^{\tau}$ and recall from (8.1) that $T_{j}^{\tau,\delta}$ is the quantum boundary length of the top of $\mathcal{N}_{j}^{\tau,\delta}$. We assume that $\delta > 0$ is sufficiently small so that $\delta^{2/3} \leq \epsilon$. Let $Q_{j}^{\tau,\delta} = \{T_{j}^{\tau,\delta} \leq X_{(j-1)\delta}^{\tau,\delta}\}$. On the event that $j\delta \leq \tau^{j,\delta}$ so that $X_{j\delta}^{\tau,\delta} \geq \epsilon$, using that $1_{Q_{j+1}^{\tau,\delta}} = 1 - 1_{(Q_{j+1}^{\tau,\delta})^c}$, we have for a constant $c > 0$ that

$$P[A_{j+1}^{\tau,\delta}, Q_{j+1}^{\tau,\delta} \mid F_{j}^{\tau,\delta}] = E \left[ \frac{T_{j+1}^{\tau,\delta}}{X_{j\delta}^{\tau,\delta}} 1_{Q_{j+1}^{\tau,\delta}} \mid F_{j}^{\tau,\delta} \right] = c\delta^{2/3} - P[(Q_{j+1}^{\tau,\delta})^c \mid F_{j}^{\tau,\delta}], \quad (8.3)$$

(The constant $c$ appearing in (8.3) is the value of $c$ that we take in the statement of the lemma.) Let $n = [\delta^{-1}\tau^{\epsilon,\delta} / c]$ and let $G_{\epsilon}^{\tau,\delta}$ be the event that $T_{j}^{\tau,\delta} \leq X_{j\delta}^{\tau,\delta}$ for all $j$ such that $\tau_{\epsilon}^{\tau,\delta} \geq j\delta$. By Lemma 2.8, we have that $P[G_{\epsilon}^{\tau,\delta}] \to 1$ as $\delta \to 0$ with $r, \epsilon$ fixed. Consequently, it follows that

$$\delta^{1/3} \sum_{j=1}^{n} I_{j}^{\tau,\delta} 1_{(Q_{j}^{\tau,\delta})^c} \to 0$$

in probability as $\delta \to 0$ with $r, \delta$ fixed. Using that $P[(Q_{j+1}^{\tau,\delta})^c \mid F_{j}^{\tau,\delta}] \to 0$ as $\delta \to 0$ faster than any power of $\delta$ (Lemma 2.8) on the event that $j\delta \leq \tau^{\epsilon,\delta}$, and using the notation $o(1)$ to indicate terms which tend to 0 as $\delta \to 0$ with $r, \epsilon$ fixed, we have that

$$E \left[ \left( c^{-1}\delta^{1/3} \sum_{j=1}^{n} I_{j}^{\tau,\delta} 1_{Q_{j}^{\tau,\delta}} - \int_{0}^{\tau^{j,\delta}} \frac{1}{X_{s}^{\tau}} ds \right)^2 \right]$$

$$= \delta^{2/3} E \left[ \sum_{j,k=1}^{n} (c^{-1}I_{j}^{\tau,\delta} 1_{Q_{j}^{\tau,\delta}} - \delta^{2/3}(X_{(j-1)\delta}^{\tau,\delta})^{-1})(c^{-1}I_{k}^{\tau,\delta} 1_{Q_{k}^{\tau,\delta}} - \delta^{2/3}(X_{(k-1)\delta}^{\tau,\delta})^{-1}) \right]$$

$$= \delta^{2/3} E \left[ \sum_{j=1}^{n} (c^{-1}I_{j}^{\tau,\delta} 1_{Q_{j}^{\tau,\delta}} - \delta^{2/3}(X_{(j-1)\delta}^{\tau,\delta})^{-1})^2 \right] + o(1) \quad \text{(by (8.3))}$$

$$= \delta^{2/3} E \left[ \sum_{j=1}^{n} \left( c^{-2}I_{j}^{\tau,\delta} 1_{Q_{j}^{\tau,\delta}} + \delta^{4/3}(X_{(j-1)\delta}^{\tau,\delta})^{-2} - 2c^{-1}\delta^{2/3}I_{j}^{\tau,\delta}(X_{(j-1)\delta}^{\tau,\delta})^{-1}1_{Q_{j}^{\tau,\delta}} \right) \right] + o(1)$$

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For the first summand in (8.4) we have that
\[ c^{-2} \delta^{2/3} \mathbb{E} \left[ \sum_{j=1}^{n} I_{j}^{r,\delta} Q_{j}^{r,\delta} \right] = c^{-1} \delta^{4/3} \mathbb{E} \left[ \sum_{j=1}^{n} (X_{(j-1)\delta}^{r,\delta})^{-1} \right] + o(1) \quad \text{(by (8.3))} \]
\[ \leq c^{-1} u e^{-1} \delta^{1/3} + o(1) \to 0 \quad \text{as} \quad \delta \to 0. \] (8.5)

To bound the second summand in (8.4), we can use the deterministic bound
\[ \delta^{2} \sum_{j=1}^{n} (X_{j\delta}^{r,\delta})^{-2} \leq u e^{-2} \delta \to 0 \quad \text{as} \quad \delta \to 0. \] (8.6)

Combining (8.5) and (8.6) implies that (8.4) tends to 0 as \( \delta \to 0 \). This completes the proof because the boundary of the time-reversal of the \( \delta \)-approximation to the reverse metric exploration at quantum distance time \( r + \int_{0}^{u} (X_{s}^{r,\delta})^{-1} ds \) converges as \( \delta \to 0 \) to the boundary of the radius \( d_{Q}(x,y) - (r + \int_{0}^{u} (X_{s}^{r,\delta})^{-1} ds) \) ball.

\[ \square \]

### 8.2.2 Conditional law of necklace given top glued to marked point

Conditioning on the event \( A_{j}^{r,\delta} \) that \( w_{j}^{r,\delta} \neq w_{j-1}^{r,\delta} \) introduces a bias into the law of \( \mathcal{N}_{j}^{r,\delta} \) because necklaces with longer top boundary lengths are more likely to be glued to a given marked boundary point. As we will see in the following lemma, this bias corresponds to weighting the law of \( \mathcal{N}_{j}^{r,\delta} \) by the quantum boundary length \( T_{j}^{r,\delta} \) of its top.

**Lemma 8.3.** We have that:

(i) The conditional law of \( \mathcal{N}_{j}^{r,\delta} \) given \( A_{j}^{r,\delta} \) and \( X_{(j-1)\delta}^{r,\delta} \) on \( \{T_{j}^{r,\delta} \leq X_{(j-1)\delta}^{r,\delta}\} \) is that of an \( \text{SLE}_{6} \) necklace weighted by \( T_{j}^{r,\delta} \).

(ii) Given \( A_{j}^{r,\delta} \), \( \{T_{j}^{r,\delta} \leq X_{(j-1)\delta}^{r,\delta}\} \), and \( X_{(j-1)\delta}^{r,\delta} \) we have that \( w_{j-1}^{r,\delta} \) is distributed uniformly from the quantum boundary measure on the boundary of the top of \( \mathcal{N}_{j}^{r,\delta} \).

**Proof.** The first assertion of the lemma is a standard sort of Bayes’ rule style calculation. Fix an event \( \mathcal{A} \) such that \( \mathbb{P}[\mathcal{N}_{j}^{r,\delta} \in \mathcal{A} | X_{(j-1)\delta}^{r,\delta}] > 0 \) and \( \mathcal{A} \subseteq \{T_{j}^{r,\delta} \leq X_{(j-1)\delta}^{r,\delta}\} \). We have that
\[ \mathbb{P}[\mathcal{N}_{j}^{r,\delta} \in \mathcal{A} | A_{j}^{r,\delta}, X_{(j-1)\delta}^{r,\delta}] = \frac{\mathbb{P}[A_{j}^{r,\delta}, \mathcal{N}_{j}^{r,\delta} \in \mathcal{A}, X_{(j-1)\delta}^{r,\delta}]}{\mathbb{P}[A_{j}^{r,\delta} | X_{(j-1)\delta}^{r,\delta}]} \mathbb{P}[\mathcal{N}_{j}^{r,\delta} \in \mathcal{A} | X_{(j-1)\delta}^{r,\delta}]. \] (8.7)
We can read off from (8.7) the Radon-Nikodym derivative of the law of $N_r,\delta_j$ given $A_r,\delta_j, X_r,\delta_j (j-1)\delta$ on the event that $\{T_r,\delta_j \leq X_r,\delta_j (j-1)\delta\}$ with respect to the unconditioned law of $N_r,\delta_j$. Fix $\epsilon, a, b > 0$. Assume that on $A$ we have that $T_r,\delta_j \in [a, a + \epsilon]$ where $a + \epsilon \leq X_r,\delta_j (j-1)\delta$. Then we have that

$$\frac{a}{X_r,\delta_j} \leq \mathbb{P}[A_r,\delta_j \mid N_r,\delta_j \in A, X_r,\delta_j (j-1)\delta] \leq \frac{a + \epsilon}{X_r,\delta_j (j-1)\delta}. \quad (8.8)$$

The first assertion follows by combining (8.7) and (8.8) and sending $\epsilon \to 0$.

The second assertion of the lemma is obvious from the construction. \hfill \Box

### 8.2.3 Comparison of explored surface to a quantum disk

In order to make use of Proposition 6.1 and Proposition 6.2 in the proof of Proposition 8.1 given just below we will need to make a comparison between:

- the quantum surface which arises when running the $\delta$-approximation of the reverse metric exploration in the setting of a $\sqrt{8/3}$-quantum sphere and
- a $\sqrt{8/3}$-quantum wedge.

We will accomplish this with a cutting/gluing argument which is analogous to that given in Section 7.2 and Section 7.3, but will require some additional steps.

We note that the law of the surface parameterized by $\tilde{\Gamma}_{r,\delta_j}$ can be sampled from as follows. Given that its boundary length is equal to $L$, we first produce a sample from the law of the time-reversal of a $3/2$-stable Lévy excursion conditioned to have maximum at least $L$ run until the last time that it hits $L$ conditioned to hit 0 precisely after being run for $j\delta$ units of Lévy process time, then $r$ units of CSBP time (i.e., after performing a time-change as in (2.29)). The surface is then constructed by associating with each jump a conditionally independent quantum disk whose boundary length is equal to the size of the jump. In the first $j\delta$ units of quantum natural time, each chunk of surface which corresponds to $\delta$-units of quantum natural time corresponds to an $\text{SLE}_6$ necklace and the necklaces are glued together by gluing the tip of one necklace onto the previous necklaces at a uniformly random point chosen from the quantum boundary measure. In the last $r$ units of quantum distance time, the surface is given by a reverse metric exploration.

We can make a comparison between the law of the surface parameterized by $\tilde{\Gamma}_{r,\delta_j}$ and that of a quantum disk weighted by its quantum area with quantum boundary length equal to $L$ as follows. First, we recall that this latter law can be encoded using the time-reversal of a $3/2$-stable Lévy excursion with maximum at least $L$ starting from where it last hits $L$ and then run until it first hits 0.
Let $\rho(s, L)$ be the density with respect to Lebesgue measure on $\mathbb{R}_+$ for hitting 0 at time $s$ for the time-reversal of a $3/2$-stable CSBP excursion starting from when it last hits $L$ and let $\rho_{j, \delta}(s, L)$ be the density at $s$ for the time-reversal of a $3/2$-stable CSBP excursion starting from when it last hits $L$ to hit 0 after $j\delta$ units of Lévy process time and then $s$ units of CSBP time. Let $X$ (resp. $L$) be the boundary length of the surface at time $r - \zeta$ (resp. the initial surface). Then the Radon-Nikodym derivative between the law of the process which encodes the first type of surface described above up until $r - \zeta$ units of quantum distance time after $j\delta$ units of quantum natural time with respect to the law of the second type of surface described above up until the same time, by a Bayes’ rule calculation, is equal to

$$\frac{\rho(\zeta, X)}{\rho_{j, \delta}(r, L)}.$$  

(8.9)

For $r$, $\zeta$, and $L$ fixed, it is easy to see that this Radon-Nikodym derivative is a bounded, continuous function of $X$ (and the bound only depends on $r$, $\zeta$, $L$). Moreover, if $C > 1$ and $r$, $\zeta$ are fixed, the bound is also uniform in $L \in [C^{-1}, C]$.

We consider three laws on disk-homeomorphic growth-process-decorated quantum surfaces with fixed quantum boundary length $L$:

**Law 1:** A quantum disk weighted by its quantum area with quantum boundary length equal to $L$ (i.e., $M_{1L}^{\text{DISK}}$) decorated by the growth process which evolves as the $\delta$-approximation to QLE($8/3, 0$) for $j\delta$ units of quantum natural time and then as a QLE($8/3, 0$) for $r$ units of quantum distance time conditioned not to hit the uniformly random marked point.

**Law 2:** A quantum disk weighted by its quantum area with quantum boundary length equal to $L$ (i.e., $M_{1L}^{\text{DISK}}$) decorated by the growth process which evolves as the $\delta$-approximation to QLE($8/3, 0$) for $j\delta$ units of quantum natural time and then as a QLE($8/3, 0$) for $r$ units of quantum distance time weighted by the Radon-Nikodym derivative in (8.9) (with the value of $\zeta$ fixed).

**Law 3:** The growth-process-decorated quantum surface which arises by running the reverse metric exploration for $r$ units of quantum distance time and then the $\delta$-approximation to the reverse metric exploration for $j\delta$ units of quantum natural time, conditioned on having quantum boundary length $L$ at the terminal time.

Then we know that:

- We can transform from Law 3 to Law 2 by cutting out the last $\zeta$ units of quantum distance time of the QLE($8/3, 0$) and then gluing in a quantum disk weighted by quantum area decorated by a uniformly random marked point conditioned on the metric exploration from the boundary of the disk taking time at least $\zeta$ to
hit the marked point. The continuation of the growth process is given by the metric exploration from the boundary of the disk which has been glued in. (This is analogous to the argument illustrated in Figure 7.2.)

- We can transform from Law 2 to Law 1 by unweighting by the Radon-Nikodym derivative in (8.9).

As we will see momentarily, Law 1 is the one which is easiest to make the comparison with a $\sqrt{8/3}$-quantum wedge (hence apply Proposition 6.1 and Proposition 6.2) while to prove Proposition 8.1 we will need to work with Law 3.

### 8.2.4 Comparison of explored surface near $w_{j_0}^{r,\delta}$ to a $\sqrt{8/3}$-quantum wedge

We are now going to introduce events on which we will truncate when making the comparison to a $\sqrt{8/3}$-quantum wedge. In what follows, we will indicate a quantity associated with Law 1 (resp. Law 2) using the notation $\dot{a}$ (resp. $\ddot{a}$). In other words, one (resp. two) dots indicates Law 1 (resp. Law 2). We will indicate quantities associated with Law 3 in a manner which is consistent with the notation from the preceding text.

Suppose that $(D, h)$ is a quantum surface decorated with the growth process $\Gamma^r$ with Law 3 described just above in the case that $j = 0$ (i.e., the surface generated by running the reverse metric exploration for exactly $r$ units of quantum distance time). We assume that we have taken the embedding of the surface so that the marked point is equal to 0. That is, we assume that $\Gamma^r$ is targeted at 0. Fix a function $\phi \in C^\infty_0(S)$. For each $r > 0$, $M, C > 1$, and $\zeta \in (0, r)$ we let $\Psi^\phi_M$ be the set of conformal transformations $\psi : D \to D$ where $D \subseteq S$ contains supp$(\phi)$ with $|\psi'(z)| \in [M^{-1}, M]$ for all $z \in \psi^{-1}(\text{supp}(\phi))$ and let $G^{r, \delta}_{\zeta, M, C}$ be the event that:

$$\Gamma^r([r - \zeta, r]) \subset B(0, 1/100) \quad \text{and} \quad \inf\{(h + Q \log |\psi'|, |\psi'|^2 \phi \circ \psi) : \psi \in \Psi^\phi_M\} \geq -C.$$

**Lemma 8.4.** For $r, M$ fixed, the measure under the law considered just above of the set of surface/growth process pairs with quantum distance time equal to $r$ for which $G^{r, \delta}_{\zeta, M, C}$ occurs tends to 1 as $\zeta \to 0$ and $C \to \infty$ uniformly in $\delta$.

**Proof.** This follows from the argument given in [DMS14 Proposition 10.18 and Proposition 10.19] as in [DMS14 Section 10].

Let $(S, h_j^{r,\delta}, \Gamma^{r,\delta,j})$ be the growth-process decorated surface with Law 3 which arises after performing $j$ steps of the time-reversal of the SLE$_6$ unexplored-domain process after $r$ units of quantum distance time exploration with $(D, h)$ as the initial surface (i.e., with $j = 0$) and we let $(S, \tilde{h}_j^{r,\delta}, \tilde{\Gamma}^{r,\delta,j})$ and $(S, \ddot{h}_j^{r,\delta}, \ddot{\Gamma}^{r,\delta,j})$ be growth-process decorated surfaces with Law 1 and Law 2, respectively. We assume that $(S, h_j^{r,\delta}, \Gamma^{r,\delta,j})$ and $(S, \tilde{h}_j^{r,\delta}, \tilde{\Gamma}^{r,\delta,j})$ have been coupled together so that the surfaces parameterized by $\Gamma^{r,\delta,j}$ and $\tilde{\Gamma}^{r,\delta,j}$ agree
except for the last $\zeta$ units of quantum distance time. In other words, it is possible to transform from the former to the latter using the cutting/gluing operation described just above. We take the embedding for $(S, \tilde{h}^r, \delta)$ into $S$ by taking the tip of the SLE$_6$ necklace just glued in (i.e., at time $j$) to be $-\infty$ and we then pick another point uniformly from the quantum boundary measure in the complement of the interval with quantum length $2C^{-1}$ centered at the tip and send this point to $+\infty$. We take the horizontal translation so that the target point $\tilde{z}^r_j$ of $\tilde{\Gamma}^r_{j,\delta}$ has real part equal to 0.

For each $j$, we let $\tilde{w}^r_j$ (resp. $\tilde{N}^r_j$) be the point on $\partial \tilde{\Gamma}^r_{j,\delta}$ (resp. SLE$_6$ necklace) which corresponds to $w^r_j$ (resp. $N^r_j$). Under the coupling that we have constructed, we have that $\tilde{N}^r_j$ is equal to $N^r_j$ (as path decorated quantum surfaces).

We also let $F^r_{j,M,C}$ be the event that:

1. The quantum boundary length of $(S, h^r_j, \delta)$ is in $[C^{-1}, C]$.
2. The quantum area of $(S, h^r_j, \delta)$ is in $[C^{-1}, C]$.
3. The Euclidean distance between $\partial S \cup \Gamma^r_{j,\delta}$ and the support of $\phi$ is at least $M^{-1}$.

Note that the second condition of the definition of $F^r_{j,M,C}$ implies that the following is true. Let $\psi$ be the unique conformal transformation from $S \setminus \Gamma^r_j \to D$ with $\psi(\tilde{z}^r_j) = 0$ and $\psi'(\tilde{z}^r_j) > 0$. Then, by the distortion theorem, there exists a constant $c_0 > 0$ such that $|\psi'(z)| \leq c_0M$ for all $z$ in the support of $\phi$. Thus if we assume that we are working on the event $G^r_{j,M,C} = \psi \in \Psi^0_M$, by the change of coordinates formula for quantum surfaces we have that $(h^r_j, \phi) = (h + Q \log |\psi'|, |\psi'|^2 \phi \circ \psi) \geq -C$.

**Lemma 8.5.** For each $C > 1$ and $\zeta > 0$ there exists a constant $K > 0$ such that on the event that the quantum boundary length of $(S, h^r_j, \delta)$ is in $[C^{-1}, C]$ we have that the Radon-Nikodym derivative between the law of $(S, h^r_j, \Gamma^r_{j,\delta})$ and $(S, h^r_j, \tilde{\Gamma}^r_{j,\delta})$ is at most $K$.

**Proof.** This follows by combining the observations made just after (8.9).

**Lemma 8.6.** Suppose that $(S, \hat{h})$ has the Bessel quantum disk law conditioned on the event that $\sup_{r \in \mathbb{R}} (\hat{h}, \phi(\cdot + r)) \geq 0$ and let $r^*$ be the value of $r \in \mathbb{R}$ at which the supremum is achieved. Let $Y$ be equal to the value of the projection of $\hat{h}$ onto $\mathcal{H}_1(S)$ at $r^*$. There exist constants $c_0, c_1 > 0$ such that

$$\mathbb{P}[Y^* - (\hat{h}, \phi(\cdot + r^*)) \geq u] \leq c_0e^{-c_1u^2} \quad \text{for all } u \geq 0.$$

**Proof.** This is an immediate from the construction.
Lemma 8.7. We assume that we are working on $G_{\zeta,M,C}^{r,\delta} \cap F_{j,M,C}^{r,\delta}$. There exist constants $c_0, c_1 > 0$ depending only on $C, M, \zeta$ such that the following is true. The probability that the supremum of the projection of $\tilde{h}_j^{r,\delta}$ onto $\mathcal{H}_1(\mathcal{S})$ is smaller than $u$ is at most $c_0 e^{-c_1 u^2}$ for all $u \in \mathbb{R}_-$. 

Proof. Let $\psi: \mathcal{S} \setminus \Gamma_j \to \mathcal{D}$ be the unique conformal map with $\psi(\tilde{z}_j^{r,\delta}) = 0$ and $\psi'(\tilde{z}_j^{r,\delta}) > 0$. As explained above, it follows from the definition of the event $G_{\zeta,M,C}^{r,\delta} \cap F_{j,M,C}^{r,\delta}$ that $(h + Q \log |\psi'|, |\psi'|^2 \phi \circ \psi) \geq -C$. Applying the change of coordinates rule for quantum surfaces, this implies that $(\tilde{h}_j^{r,\delta}, \phi) \geq -C$. Note that on $G_{\zeta,M,C}^{r,\delta} \cap F_{j,M,C}^{r,\delta}$, the law of $\tilde{h}_j^{r,\delta}$ (modulo horizontal translation) is absolutely continuous with bounded Radon-Nikodym derivative with respect to the law on distributions which comes from the Bessel law conditioned so that $\sup_{r \in \mathbb{R}}(\tilde{h}, \phi(\cdot + r)) \geq -C$. Consequently, the result follows by applying Lemma 8.6.

Assuming that $\zeta, M, C$ are fixed, we can choose $c$ sufficiently large so that with $c_0, c_1$ as in the statement of Lemma 8.7 we have with

$$u_0 = -c\sqrt{\log \delta^{-1}}$$

that $c_0 e^{-c_1 u_0^2} \leq \delta^2$. We let $\tilde{u}_{j,0,\alpha} \in \mathbb{R}$ be where the projection of $\tilde{h}_j^{r,\delta}$ onto $\mathcal{H}_1(\mathcal{S})$ first hits $\alpha \log \delta$; we take $\tilde{u}_{j,0,\alpha} = +\infty$ if the supremum of this projection is smaller than $\alpha \log \delta$. For each $j \in \mathbb{N}$ and $\alpha > 0$, we let $T_{j,\alpha}$ be the event that

1. The supremum of the projection of $\tilde{h}_j$ onto $\mathcal{H}_1(\mathcal{S})$ is at least $u_0$.
2. $T_{j,\alpha} \leq \delta^{2/3 - \alpha}$.
3. $\tilde{N}_j^{r,\delta}$ is contained in $\mathcal{S}_- + \tilde{u}_{j,0,\alpha}$.
4. Let $(\mathcal{S}, \tilde{h}_{j-1})$ be the quantum surface which is given by re-embedding the quantum surface $(\mathcal{S}, \tilde{h}_j)$ so that the point on $\partial \mathcal{S}$ where the tip of $\tilde{N}_j^{r,\delta}$ is glued to form the quantum surface $(\mathcal{S}, \tilde{h}_j^{r,\delta})$ is sent to $-\infty$ (with $+\infty$ fixed and the horizontal translation left unspecified). Let $\tilde{u}_{j-1,0,\alpha}$ be where the projection of $\tilde{h}_{j-1}$ onto $\mathcal{H}_1(\mathcal{S})$ first hits $\alpha \log \delta$. Then the interval of the boundary of $(\mathcal{S}, \tilde{h}_{j-1})$ to where $\tilde{N}_j^{r,\delta}$ gets glued to is contained in $\partial \mathcal{S}_- + \tilde{u}_{j-1,0,\alpha}$.

We then let $E_{j,M,C,0}^{r,\delta} = G_{\zeta,M,C}^{r,\delta} \cap F_{j,M,C}^{r,\delta} \cap H_{j,\alpha}^{r,\delta}$ and $E_{j,M,C,0}^{n,\delta} = \cap_{j=1}^n E_{j,M,C,0}^{r,\delta}$.

We will now combine the estimates established earlier to get that it is possible to adjust the parameters in the definition of $E_{\zeta,M,C,0}^{n,\delta}$ so that it occurs with probability as close to 1 as we like.

Lemma 8.8. For every $\epsilon, a_0 > 0$ there exists $M, C > 1, \alpha, \zeta, \delta_0 > 0$, and $\phi \in C_0^\infty(\mathcal{S})$ such that $\delta \in (0, \delta_0)$ implies that $\mathbb{P}(E_{\zeta,M,C,0}^{n,\delta}) \leq \epsilon$ where $n = [a_0 \delta^{-1}]$. 

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Proof. We explained just after the definition of $G_{r,\varepsilon,M,C}$ why there exists $M, C > 1$ and $\varepsilon > 0$ such that $P[\overline{G_{r,\varepsilon,M,C}]}] \leq \varepsilon$. Therefore it is left to explain why we have the corresponding property for $\bigcap_{j=1}^n F_{j,M,C}$ and $\bigcap_{j=1}^n H_{j,\alpha}$.

From the definition of $F_{j,M,C}$, elementary distortion estimates for conformal maps, and elementary estimates for Lévy processes, it is easy to see that by choosing $M, C > 1$ sufficiently large and by making the support of $\phi$ sufficiently small, we have that $P[\overline{\bigcap_{j=1}^n F_{j,M,C}}] \leq \varepsilon$.

It is left to explain why $P[\overline{\bigcap_{j=1}^n H_{j,\alpha}}] \leq \varepsilon$. The first two parts of the definition follow from Lemma 2.8 and Lemma 8.7. The second two parts of the definition respectively follow from (6.3) of Proposition 6.1 and (6.6) of Proposition 6.2.

8.2.5 Moment bounds

For each $j$, we let $D_{r,\delta}^j$ (resp. $\bar{D}_{r,\delta}^j$) denote the quantum distance between $w_{r,\delta}^j$ (resp. $\bar{w}_{r,\delta}^j$) and $w_{r,\delta}^{j+1}$ (resp. $\bar{w}_{r,\delta}^{j+1}$) with respect to the internal metric of $(S, h_{r,\delta}^j)$ (resp. $(S, \bar{h}_{r,\delta}^j)$).

We let $S_{r,\delta}^j$ be the event that the shortest path from $w_{r,\delta}^j$ to $w_{r,\delta}^{j+1}$ does not hit the part of the surface that we cut out in order to transform from $(S, h_{r,\delta}^j)$ to $(S, \bar{h}_{r,\delta}^j)$. On $S_{r,\delta}^j$, we have that $D_{r,\delta}^j = \bar{D}_{r,\delta}^j$. We note that this is the case for all $1 \leq j \leq n$ on $E_{r,\delta}^n$.

Fix $a_0 > 0$ and let $n = \lfloor a_0 \delta^{-1} \rfloor$ as in the statement of Lemma 8.8. Suppose that $Q$ is any event. Using that $D_{r,\delta}^j = 0$ on $(A_{r,\delta}^j)^c$ in the last step, we have that

$$E\left[\sum_{j=1}^n D_{r,\delta}^j 1_{E_{r,\delta}^n} Q\right] = E\left[\sum_{j=1}^n \bar{D}_{r,\delta}^j 1_{E_{r,\delta}^n} Q\right]$$

$$\leq \sum_{j=1}^n E[\bar{D}_{r,\delta}^j 1_{F_{r,\delta}^j \cap H_{r,\delta}^j} Q]$$

$$= \sum_{j=1}^n E[\bar{D}_{r,\delta}^j 1_{F_{r,\delta}^j \cap H_{r,\delta}^j} Q | A_{r,\delta}^j] P[A_{r,\delta}^j]. \quad (8.11)$$

We next aim to bound the right hand side of (8.11).

**Lemma 8.9.** There exist constants $c_0, \sigma > 0$ such that

$$E[\bar{D}_{r,\delta}^j 1_{F_{r,\delta}^j \cap H_{r,\delta}^j} Q | A_{r,\delta}^j] \leq c_0 \sigma^{1/3} P[Q]. \quad (8.12)$$

**Proof.** We let $\tilde{D}_{r,\delta}^j$ be the quantum distance (with respect to the internal metric of $(S, \bar{h}_{r,\delta}^j)$) between the tip of $\tilde{N}_{r,\delta}^j$ and a point which is chosen uniformly at random from the quantum measure on the top of $\tilde{N}_{r,\delta}^j$. Conditionally on $A_{r,\delta}^j$, we have by Lemma 8.3
that $\dot{D}_{j} \overset{d}{=} \ddot{D}_{j}$. Let $p > 1$ be such that both Proposition 6.1 and Proposition 6.2 apply and let $q \in (1, \infty)$ be conjugate to $p$, i.e., $p^{-1} + q^{-1} = 1$. We begin by noting that:

\[
E[\dot{D}_{j}^{r,\delta} 1_{F_{j,M,C}^{r,\delta} \cap H_{j,\alpha}^{r,\delta}} | A_{j}^{r,\delta}] = E[\dot{D}_{j}^{r,\delta} 1_{F_{j,M,C}^{r,\delta} \cap H_{j,\alpha}^{r,\delta}} ] \\
\leq c_0 E[\dot{D}_{j}^{r,\delta} T_{j}^{r,\delta} \left( 1_{F_{j,M,C}^{r,\delta} \cap H_{j,\alpha}^{r,\delta}} \right)] \quad \text{(by Lemma 8.3)} \\
\leq c_0 E[(\dot{D}_{j}^{r,\delta})^p 1_{F_{j,M,C}^{r,\delta} \cap H_{j,\alpha}^{r,\delta}}]^{1/p} E[(T_{j}^{r,\delta})^q]^{1/q} \quad \text{(Hölder’s inequality)} \\
\leq c_1 E[(\dot{D}_{j}^{r,\delta})^p 1_{F_{j,M,C}^{r,\delta} \cap H_{j,\alpha}^{r,\delta}}]^{1/p} \quad \text{(by Lemma 2.8)}.
\]

We note that the constant $c_1$ depends on $q$. By Lemma 8.5 we know that there exists a constant $K > 0$ such that

\[
E[(\dot{D}_{j}^{r,\delta})^p 1_{F_{j,M,C}^{r,\delta} \cap H_{j,\alpha}^{r,\delta}}]^{1/p} \leq KE[(\dot{D}_{j}^{r,\delta})^p 1_{H_{j,\alpha}^{r,\delta}}]^{1/p}
\]

where $\hat{E}$ denotes the expectation under the law $(S, \hat{h}_{j,\alpha}^{r,\delta}, \hat{\Gamma}_{j,\alpha}^{r,\delta})$. We let $\hat{D}_{j}^{1,r,\delta}$ denote the quantum distance between the base and the tip of $\hat{N}_{j}^{r,\delta}$ and we let $\hat{D}_{j}^{2,r,\delta}$ denote the quantum distance between the tip of $\hat{N}_{j}^{r,\delta}$ and the uniformly random point $\hat{w}_{j}^{r,\delta}$ on the top of $\hat{N}_{j}^{r,\delta}$ in the surface which arises after cutting out $\hat{N}_{j}^{r,\delta}$. We will establish (8.12) by bounding the $p$th moments of $\hat{D}_{j}^{1,r,\delta}$ and $\hat{D}_{j}^{2,r,\delta}$.

By the definition of the event $H_{j,\alpha}^{r,\delta}$, we have that $\Re(\hat{w}_{j}^{r,\delta}) \leq \hat{w}_{j,\alpha}^{r,\delta}$. Note that the law of the field $\hat{h}_{j}^{r,\delta}$ in $S_{\pm} + \hat{w}_{j,\alpha}^{r,\delta}$ is absolutely continuous with bounded Radon-Nikodym derivative to the law of a $\sqrt{8/3}$-quantum wedge with the usual embedding into $S$ restricted to the part of $S$ up to where the projection of the field onto $H_{1}(S)$ first hits $\alpha \log \delta$. Consequently, it follows from Proposition 6.2 that for a constant $c_2 > 0$ we have that

\[
\hat{E}[(\hat{D}_{j}^{2,r,\delta})^p 1_{H_{j,\alpha}^{r,\delta}}]^{1/p} \leq c_2 \delta^{1/3}.
\]  

(8.13)

It similarly follows from Proposition 6.1 that, by possibly increasing the value of $c_2 > 0$, we have that

\[
\hat{E}[(\hat{D}_{j}^{1,r,\delta})^p 1_{H_{j,\alpha}^{r,\delta}}]^{1/p} \leq c_2 \delta^{1/3}.
\]  

(8.14)

Combining (8.13) and (8.14) implies the result.  

Proof of Proposition 8.1. We take the path that we have constructed and parameterize it according to arc length with respect to the quantum distance. Using (8.11) and the fact that the conditional probability of $A_{j}^{r,\delta}$ given that the boundary length is not too short is of order $\delta^{2/3}$ (since $T_{j}^{r,\delta}$ is of order $\delta^{2/3}$ and has exponential moments), it follows from Lemma 8.8 and Lemma 8.9 that the length of $\eta^{r,\delta}|_{[0,T]}$ is tight as $\delta \to 0$. Since the length of $\eta^{r,\delta}|_{[0,T]}$ is equal to its Lipschitz constant (as we assume $\eta^{r,\delta}$ to be parameterized according to arc length on the interval $[0,1]$), it follows that the law of
\(\eta^r|_{[0,T]}\) is in fact tight as \(\delta \to 0\). This completes the proof of the tightness of the law of \(\eta^r|_{[0,T]}\) for each \(T > 0\).

Let \(\eta^r\) be any subsequential limit. Lemma 8.2 implies that for any fixed \(t > 0\) we have that \(\eta^r(t) \in \partial B^\bullet_\delta(x, \widehat{d}_\delta(x, y) - (t+r))\) almost surely. Therefore this holds almost surely for all \(t \in Q_+\) simultaneously and, combining with the continuity of \(\eta^r\), we have that \(\eta^r(t) \in \partial B^\bullet_\delta(x, \widehat{d}_\delta(x, y) - (t+r))\) for all \(t > 0\) almost surely.

The conditional independence statement in the limit is immediate since it holds for the approximations.

The final assertion of the proposition is immediate from the argument given above. \(\square\)

### 8.2.6 Boundary lengths between second approximations of geodesics

**Proposition 8.10.** Fix \(r > 0\) and suppose that \((S, x, y)\) has the law \(M_{\text{SPH}}^2\) conditioned so that \(\widehat{d}_\delta(x, y) > r\). Suppose that \(x_1, \ldots, x_k\) are picked independently from the quantum boundary measure on \(\partial B^\bullet_\delta(x, \widehat{d}_\delta(x, y) - r)\) and then reordered to be counterclockwise. We let \(\eta^r_j, \ldots, \eta^r_k\) be second approximations to geodesics starting from \(x_1, \ldots, x_k\) as constructed in Proposition 8.1. For each \(1 \leq j \leq k\) and each \(t \in [0, \widehat{d}_\delta(x, y) - r]\) we let \(X^r_{t,j}\) be the quantum boundary length of the counterclockwise segment of \(\partial B^\bullet_\delta(x, \widehat{d}_\delta(x, y) - (r + t))\) between \(\eta^r_j(t)\) and \(\eta^r_{j+1}(t)\) (with the convention that \(\eta^r_{k+1} = \eta^r_1\)). Given \(X^r_{0,1}, \ldots, X^r_{0,k}\), the processes \(X^r_{t,j}\) evolve as independent \(3/2\)-stable CSBPs with initial values \(X^r_{0,j}\).

**Proof.** We will prove the result in the case that \(k = 2\) for simplicity; the proof for general values of \(k \in \mathbb{N}\) with \(k \geq 3\) follows from the same argument. See Figure 8.4 for an illustration. We will prove the result by showing that \(X^r_{t,1}, X^r_{t,2}\) have the property that if we reparameterize the time for each using the time change \(\int_0^t X^r_{s,j} ds\), then the resulting processes evolve as independent \(3/2\)-stable Lévy processes. This suffices because if we invert the time change, then the Lamperti transform (recall (2.30)) implies that the resulting processes (i.e., we recover \(X^{r,1}, X^{r,2}\)) are independent \(3/2\)-stable CSBPs.

We fix \(\delta > 0\) and consider the boundary lengths between two points as in the construction of the first approximations to geodesics. As earlier, for each \(j\) we let \(w^{r,1,\delta}_j, w^{r,2,\delta}_j\) be the locations of the two marked points when we have glued on the \(j\)th SLE\(_6\) necklace. We let \(X^r_{t,1,\delta}, X^r_{t,2,\delta}\) be the quantum boundary lengths between the points and assume that we have the quantum distance parameterization for the overall boundary length process \(X^r_{t,\delta}\) in the \(\delta\)-approximation to the reverse metric exploration. We let \(\sigma^{r,\delta}_j\) be the \(j\)th time that the top of a necklace gets glued to one of the marked points and we let \(\tau^{r,\delta}_j\) be the end time of that necklace (\(\tau^{r,\delta}_j = \delta\) units of quantum natural time after \(\sigma^{r,\delta}_j\)). For each \(t > 0\) we let \(I^r_{t,\delta} = 1\) (resp. \(I^r_{t,\delta} = 0\)) if the starting point of the necklace being glued in at time \(t\) is in the counterclockwise (resp. clockwise) segment between \(w^{r,1,\delta}_j\) and \(w^{r,2,\delta-k,\delta}_j\). Let

\[
A^r_{t,\delta} = \int_0^t I^r_{s,\delta} X^r_{s,\delta} ds.
\]
Figure 8.4: Illustration of the argument to prove Proposition 8.10, which states that the boundary lengths between second approximations to geodesics evolve as independent $3/2$-stable CSBPs. The green region parameterizes $\tilde{\Gamma}_r^r\delta$, which we recall is equal to the reverse metric exploration at time $r$ and the disk parameterizes $\tilde{\Gamma}_r^r\delta_{(j-1)\delta}$. The orange paths are first approximations to geodesics starting from points $w_r^r,1,\delta_0$, $w_r^r,2,\delta_0$, which are independently chosen from the quantum boundary measure on $\partial \tilde{\Gamma}_r^r\delta_0$. Shown is the SLE$_6$ necklace $N_r^r\delta_j$ which is about to be glued to the surface parameterized by $\tilde{\Gamma}_r^r\delta_{(j-1)\delta}$. In the case that the top of $N_r^r\delta_j$ is contained in the counterclockwise (resp. clockwise) segment from $w_r^r,1,\delta_{j-1}$ to $w_r^r,2,\delta_{j-1}$, the boundary length of the corresponding segment gets an increment of $\delta$ units of Lévy process time. In the case that the top of $N_r^r\delta_j$ is glued to an interval which contains either $w_r^r,1,\delta_{j-1}$ or $w_r^r,2,\delta_{j-1}$, then the boundary lengths of both segments are changed. Since the probability that this happens is of order $\delta^2/3$ (i.e., proportional to the quantum length of the top of $N_r^r\delta_j$) and there are of order $\delta^{-1}$ necklaces overall, the number of such necklaces will be of order $\delta^{-1/3}$. Since the change to the boundary lengths which result from such a necklace is of order $\delta^{2/3}$, the overall change to the boundary lengths which results from such necklaces will be of order $\delta^{1/3}$, hence tend to 0 in the $\delta \to 0$ limit.

We will first argue that $A_{r,j,\delta}^{r,j,\delta} - \int_0^t X_{s,j,\delta}^{r,j,\delta} ds \to 0$ in $L^1$ as $\delta \to 0$.

For each $s \geq 0$ we let $\tilde{s} = \lceil \delta^{-1}s \rceil \delta$. We have that

$$\mathbb{E} \left| \int_0^t I_{s,j,\delta} X_{s,j,\delta}^{r,j,\delta} ds - \int_0^t X_{s,j,\delta}^{r,j,\delta} ds \right| \leq \mathbb{E} \left| \int_0^t I_{s,j,\delta} X_{s,j,\delta}^{r,j,\delta} ds - \int_0^t X_{\tilde{s},j,\delta}^{r,j,\delta} ds \right| + \int_0^t \mathbb{E} |X_{s,j,\delta}^{r,j,\delta} - X_{\tilde{s},j,\delta}^{r,j,\delta}| ds \quad (8.15)$$

We note that the second term on the right hand side of (8.15) tends to 0 as $\delta \to 0$. 

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because we have for each fixed \( s \in [0, t] \) that \( |X_{s}^{r,j,\delta} - X_{\tau}^{r,j,\delta}| \to 0 \) in probability as \( \delta \to 0 \) and there exists a constant \( c > 0 \) and \( p > 1 \) so that \( \mathbb{E}|X_{s}^{r,j,\delta} - X_{\tau}^{r,j,\delta}|^{p} \leq c \) for all \( s \in [0, t] \). The third term on the right hand side of (8.15) tends to 0 as \( \delta \to 0 \) for the same reason.

We will now argue that the first term on the right hand side of (8.15) tends to 0 if we take a limit as \( \delta \to 0 \). To see this, we assume that \( t \) is an integer multiple of \( \delta \), we let \( L = t/\delta \). Since the values of \( I_{r,j,\delta}^{s}, X_{\tau}^{r,\delta}, X_{\tau}^{r,j,\delta} \) do not change in an interval of the form \((k\delta, (k+1)\delta] \), we can write the first term on the right hand side of (8.15) as the expectation of the absolute value of

\[
\int_{0}^{t} \left( I_{s}^{r,j,\delta} - \frac{X_{s}^{r,j,\delta}}{X_{\tau}^{r,\delta}} \right) X_{\tau}^{r,\delta} \, ds = \delta \sum_{k=0}^{L} \Delta_{k}^{r,\delta} X_{k\delta}^{r,\delta} \quad \text{where}
\]

\[
\Delta_{k}^{r,\delta} = \left( I_{k\delta}^{r,j,\delta} - \frac{X_{k\delta}^{r,j,\delta}}{X_{k\delta}^{r,\delta}} \right).
\]

Note that the \( |\Delta_{k}^{r,\delta}| \leq 1 \) for each \( k \) and

\[
\mathbb{E}[I_{k\delta}^{r,j,\delta} \mid X_{k\delta}^{r,j,\delta}, X_{k\delta}^{r,\delta}] = \frac{X_{k\delta}^{r,j,\delta}}{X_{k\delta}^{r,\delta}}.
\]

Consequently, \( \sum_{k=0}^{n} \Delta_{k}^{r,\delta} X_{k\delta}^{r,\delta} \) is a martingale whose increments have a uniformly bounded \( p \)th moment for some \( p > 1 \) (as \( X_{k\delta}^{r,\delta} \) has a uniformly bounded \( p \)th moment for some \( p > 1 \)). It therefore follows that the first term in the right hand side of (8.15) tends to 0 as \( \delta \to 0 \).

We let \( B_{t}^{r,j,\delta} \) be the right-continuous inverse of \( A_{t}^{r,j,\delta} \). For a given value of \( t > 0 \) and each \( k \), we also let \( \tau_{k}^{r,j,\delta} = t \wedge A_{k\delta}^{r,j,\delta} \) and \( \sigma_{k}^{r,j,\delta} = t \wedge A_{k\delta}^{r,\delta} \). Then we note that we can write

\[
X_{B_{t}^{r,j,\delta}}^{r,j,\delta} = \sum_{k} \left( X_{B_{t}^{r,j,\delta}}^{r,j,\delta} - X_{\tau_{k}^{r,j,\delta}}^{r,j,\delta} \right) + \sum_{k} \left( X_{B_{t}^{r,j,\delta}}^{r,j,\delta} - X_{\sigma_{k}^{r,j,\delta}}^{r,j,\delta} \right).
\]

To finish the proof, we need to show that in the limit as \( \delta \to 0 \) we have that \( X_{B_{t}^{r,j,\delta}}^{r,j,\delta} \) evolves as a 3/2-stable Lévy process. We will establish this by showing that the first term in (8.17) in the \( \delta \to 0 \) limit evolves as 3/2-stable Lévy process and the second term in (8.17) tends to 0 as \( \delta \to 0 \).

We begin with the second term in the right hand side of (8.17). We note that the probability that a necklace hits one of the marked points is proportional to the quantum length of the top of the necklace. By Lemma 2.8 we know that it is exponentially unlikely for this length to be larger than a constant times \( \delta^{2/3} \). Since the total number of necklaces is of order \( \delta^{-1} \), we see that there will be with high probability \( \delta^{-1/3} \) necklaces whose top is glued to one of the marked points. The change in the boundary
length for the left (resp. right) side of (8.17) evolves like a 3/2-stable Lévy process and these Lévy processes are independent across necklaces. So the overall magnitude of the error which comes from necklaces of this type is dominated by the supremum of the absolute value of a 3/2-stable Lévy process run for time of order \( \delta \times \delta^{-1/3} = \delta^{2/3} \). We conclude that the amount of change which comes from these time intervals tends to 0 as \( \delta \to 0 \).

We now turn to the first term in the right hand side of (8.17). In each of the other intervals we know that the boundary length evolves as a 3/2-stable Lévy process. The total amount of Lévy process time for each of the two sides is equal to \( t \) minus the time which corresponds to those necklaces whose top was glued to a marked point. As we have just mentioned above, this corresponds to time of order \( \delta^{2/3} \) and therefore makes a negligible contribution as \( \delta \to 0 \).

8.3 Third approximations to geodesics and the 3/2-Lévy net

We will now show that the statement of Proposition 8.10 holds in the setting of geodesics starting from the boundary of a filled metric ball.

**Proposition 8.11.** Fix \( r > 0 \) and suppose that \((\mathcal{S}, x, y)\) has the law \( \mathcal{M}_{SPH}^0 \) conditioned so that \( d_Q(x, y) > r \). Suppose that \( x_1, \ldots, x_k \) are picked independently from the quantum boundary measure on \( \partial B_{\infty}^3(x, d_Q(x, y) - r) \) and then reordered to be counterclockwise. We let \( \eta_1^r, \ldots, \eta_k^r \) be the almost surely unique (recall Proposition 5.19) geodesics from \( x_1, \ldots, x_k \) to \( x \). For each \( 1 \leq j \leq k \) and \( t \in [0, d_Q(x, y) - r] \) we let \( X_{i,j}^r \) be the quantum boundary length of the counterclockwise segment of \( \partial B_{\infty}^3(x, d_Q(x, y) - (r + t)) \) between \( \eta_j^r(t) \) and \( \eta_{j+1}^r(t) \) (with the convention that \( \eta_{k+1}^r = \eta_1^r \)). Given \( X_{0,1}^{r,1}, \ldots, X_{0,k}^{r,k} \), the processes \( X_{i,j}^{r,j} \) evolve as independent 3/2-stable CSBPs with initial values \( X_{0,j}^{r,j} \).

In order to prove Proposition 8.11 we will need to construct our third approximations to geodesics. We will carry this out in Section 8.3.1. We will then compare these third approximations with the second approximations in Section 8.3.2. This comparison together with a scaling argument will lead to Proposition 8.11.

8.3.1 Construction of third approximations to geodesics

**Lemma 8.12.** For each \( \epsilon > 0 \) and \( C > 1 \) there exists \( L_0, M_0 > 0 \) such that for all \( L \geq L_0 \) and \( M \geq M_0 \) the following is true. Suppose that \( \mathcal{B} \) has the law \( \mathcal{B}_{L,1} \) (i.e., is a metric band with inner boundary length \( L \) and width 1) and that \( w \) is chosen uniformly at random from the quantum measure restricted to the inner boundary of the band. Let \( \eta \) be the path from \( w \) to the outer boundary of the band as constructed in Proposition 8.1 (i.e., a second approximation to a geodesic), let \( z \) be the point on the outside of the band where this path terminates, and let \( I_M \) be the interval of quantum
Figure 8.5: Illustration of the statement of Lemma 8.12. Shown is a metric band $B$ with inner boundary length $L$ and width 1 together with a second approximation of a geodesic $\eta$ starting from a uniformly random point $w$ chosen on the inner boundary of $B$ and the length $M$ interval $I_M$ starting from the point $z$ where $\eta$ terminates on the outer boundary of $B$. Lemma 8.12 implies that, if $M$ is large enough, then the expected distance from $w$ to $I_M$ is at most $1 + \epsilon$.

...length $M$ on the outside of the band which is centered at $z$. Let $E_C$ be the event that the quantum boundary length of the outer boundary of the reverse metric exploration starting from the inner boundary of $B$ and terminating at the outer boundary of $B$ stays in $[C^{-1}L, CL]$. Conditionally on $E_C$, we have that the expected distance inside of $B$ starting from $z$ to a point on $I_M$ is at most $1 + \epsilon$.

See Figure 8.5 for an illustration of the statement of Lemma 8.12. We call a shortest length path in a metric band as in the statement of Lemma 8.12 which connects $z$ to the closest point to $z$ along $I_M$ a third approximation to a geodesic. We first record the following before giving the proof of Lemma 8.12.

**Lemma 8.13.** For each $\epsilon, D > 0$ there exists $L_0, M_0 > 0$ such that for all $L \geq L_0$ and $M \geq M_0$ the following is true. Suppose that $B$ has the law $B_{L,1}$ (i.e., is a metric band with inner boundary length $L$ and width 1) and that $w$ is chosen uniformly at random from the quantum measure restricted to the inner boundary of the band. Let $\eta$ be the path from $w$ to the outer boundary of the band as constructed in Proposition 8.1 (i.e., a second approximation to a geodesic) let $z$ be the point on the outside of the band where this path terminates, and let $I_M$ be the interval of quantum length $M$ on the outside of the band which is centered at $z$. The probability that the quantum distance between the complement of $I_M$ in the outer boundary to $w$ is at least $D$ is at least $1 - \epsilon$.

**Proof.** This is a consequence of Proposition 7.9 combined with Proposition 8.1. \hfill $\square$
Proof of Lemma 8.12. For each \(j\), we let \(I_j\) be the interval of quantum length \(j\) centered at \(z\) as in the statement of the proposition and let \(X_j\) be the distance from \(w\) to \(I_j\) inside of \(B\). Then we have that \(X_{j+1} \leq X_j\) for every \(j\). We also have that \(X_0\) is at most the length of \(\eta\). We also know that \(X_j = 1\) on the event \(F_j\) that the geodesic terminates in \(I_j\) since \(B\) has width 1. Then we have that

\[
E[X_j \mid E_C] = E[X_j(1_{F_j} + 1_{F^c_j}) \mid E_C] \leq 1 + E[X_01_{F_j} \mid E_C].
\]

Lemma 8.13 implies that by adjusting the parameters, we can make \(P[F^c_j \mid E_C]\) as small as we like. Therefore the result follows from the uniform integrability of the length of \(\eta\) on \(E_C\) established in Proposition 8.1.

8.3.2 Subsequential limits of rescalings of concatenations of third approximations of geodesics

We need to establish one more lemma before completing the proof of Proposition 8.11.

Lemma 8.14. Fix \(\alpha \in (1, 2)\) and for \(y_0 \geq 0\) let \(Y_t\) be an \(\alpha\)-stable CSBP with \(Y_0 = y_0\). There exists a constant \(c_0 > 0\) depending only on \(\alpha\) such that

\[
\sup_{y_0 \geq 0} E[|Y_1 - y_0|] \leq c_0.
\]

Proof. Since an \(\alpha\)-stable CSBP is both non-negative and a martingale, we have that

\[
E[|Y_1 - y_0|] \leq E[Y_1] + y_0 = 2y_0. \tag{8.18}
\]

It follows from (8.18) that the claim clearly holds for all \(y_0 \leq 1\) with \(c_0 = 2\). We now suppose that \(y_0 \geq 1\) and let \(\tau = \inf\{t \geq 0 : Y_t \leq 1\}\). We then have that

\[
E[|Y_1 - y_0|] = E[|Y_1 - y_0|1_{\{\tau \leq 1\}}] + E[|Y_1 - y_0|1_{\{\tau > 1\}}]. \tag{8.19}
\]

For the first term, we note that

\[
E[|Y_1 - y_0|1_{\{\tau \leq 1\}}] \leq E[Y_11_{\{\tau \leq 1\}}] + y_0 P[\tau \leq 1] = (1 + y_0)P[\tau \leq 1], \tag{8.20}
\]

where we used the strong Markov property for \(Y\) at the stopping time \(\tau\) in the final equality. It is easy to see that there exists \(p_\alpha > 0\) which does not depend on the value of \(y_0\) such that

\[
P[Y_2 = 0 \mid \tau \leq 1] \geq p_\alpha.
\]

By rearranging the lower bound for this conditional probability in the first step below and in the second step using the explicit form of the Laplace transform of the law of \(Y_2\) (recall (2.24) and (2.31)), we have for a constant \(c_\alpha > 0\) depending only on \(\alpha\) that

\[
P[\tau \leq 1] \leq \frac{1}{p_\alpha} P[Y_2 = 0] \leq \frac{1}{p_\alpha} \lim_{\lambda \to \infty} E[e^{-\lambda Y_2}] = \frac{1}{p_\alpha} e^{-c_\alpha y_0}. \tag{8.21}
\]
Figure 8.6: Illustration of the comparison argument used to prove Proposition 8.11, which implies that the boundary lengths between geodesics from the boundary of the reverse metric exploration up to time $r$ back to the root evolve as independent $3/2$-stable CSBPs. Each of the $k$ layers shown represents a metric band of a fixed width (where each band is as illustrated in Figure 7.4). Each of the blue paths represents a second approximation to a geodesic and each of the red paths represents a third approximation of a geodesic. Note that the terminal point of each of the red paths is contained in a green interval centered at the corresponding second approximation to a geodesic. These green intervals each have quantum length equal to a fixed constant $M$. The evolution of the boundary lengths between the blue paths is given by independent $3/2$-stable CSBPs. By construction, the evolution of the boundary lengths between the red paths is then close to being that of independent $3/2$-stable CSBPs. Due to the way that boundary lengths and quantum distances scale, this error can be made to be arbitrarily small by first taking $k$ to be large and then rescaling. We will then argue that we can make the red paths as close to geodesics as we like by taking $\epsilon > 0$ very small, which in turn implies that boundary lengths between geodesics evolve as independent $3/2$-stable CSBPs.

Inserting the upper bound in (8.21) into (8.20) implies that the first term in the right side of (8.19) is clearly bounded in $y_0 \geq 1$. By the Lamperti transform (2.29), we note that we can represent $Y$ as the time change of a $3/2$-stable Lévy process $X$. On the event $\{\tau > 1\}$, the amount of Lévy process time which has passed in one unit of time for $Y$ is at most 1. Consequently, if we let $S_t$ (resp. $I_t$) denote the running infimum (resp. supremum) of $X$ then we have that the second term in (8.19) is bounded from above by

$$
E[|Y_1 - Y_0| 1_{\{\tau > 1\}}] \leq E[S_1 - X_0] + E[X_0 - I_1].
$$

(8.22)

This completes the proof as both expectations on the right hand side of (8.22) are finite.
We can now complete the proof of Proposition 8.11. See Figure 8.6 for an illustration of the argument.

Proof of Proposition 8.11. Fix $\alpha \in (0, 1)$. Fix $\epsilon > 0$, $C > 1$, and let $L_0, M_0$ be as in Lemma 8.12 for these values of $\epsilon, C$. Fix $r, s > 0$ and assume that $(S, x, y)$ is sampled from $M_{SPH}^2$ conditioned on both:

- $d_Q(x, y) > r\sqrt{L}$ and
- the quantum boundary length of the boundary of the reverse metric exploration at quantum distance time $r\sqrt{L}$ being in $[L, CL]$.

That is, the quantum boundary length of $\partial B_Q^\bullet(x, d_Q(x, y) - r\sqrt{L})$ is contained in $[L, CL]$. Let $(B_j)$ be the sequence of width-1 metric bands in the reverse exploration from $S$ starting from $\partial B_Q^\bullet(x, d_Q(x, y) - r\sqrt{L})$ and targeted towards $x$. We let $E$ be the event that the boundary length of the reverse metric exploration starting from quantum distance time $r\sqrt{L}$ stays in $[C^{-1}L, CL]$ for quantum distance time $s\sqrt{L}$. By scaling quantum lengths by $L^{-1}$ so that quantum distances scale by $L^{-1/2}$ (recall also the scaling rules for $3/2$-stable CSBPs from Section 2.5), we observe that the conditional probability of $E$ assigned by $M_{SPH}^2$ conditioned as described just above is bounded from below by a positive constant which depends only on $C$ and $s$.

Assume that we have chosen $L \geq L_0$, $M \geq M_0$, and that we have picked $w_1^{1,r}$ from the quantum measure on the inner boundary of $B_1$. Let $w_1^{1,r}$ be the point where the second approximation to a geodesic starting from $w_0^{1,r}$ hits the outside of $B_1$. Lemma 8.12 implies that, conditionally on $E$, the expected length of the shortest path starting from $w_0^{1,r}$ and terminating in the quantum length $M$ interval centered at $w_1^{1,r}$ is at most $1 + \epsilon$. Let $z_1^{1,r}$ be the terminal point of this path. Assuming that $w_1^{1,r}, \ldots, w_k^{1,r}$ and $z_1^{1,r}, \ldots, z_k^{1,r}$ have been defined, we let $w_{k+1}^{1,r}$ be the terminal point of the second approximation of a geodesic starting from $z_k^{1,r}$ across the band $B_{k+1}$ and let $z_{k+1}^{1,r}$ be the terminal point of the shortest path starting from $z_k^{1,r}$ and terminating in the boundary length $M$ interval centered at $w_{k+1}^{1,r}$. Lemma 8.12 implies that, conditionally on $E$, the expected length of the this path is at most $1 + \epsilon$.

Let $u_0^{1,r}, u_1^{1,r}, \ldots$ be the points on the inner boundaries of the successive metric bands visited by a second approximation to a geodesic starting from $u_0^{1,r} = w_0^{1,r}$.

For each $k$, we let $S_k^{1,r}$ be the quantum length of the shorter boundary segment between $z_k^{1,r}$ and $u_k^{1,r}$. We know from the construction that

$$S_k^{1,r} \leq \Upsilon_k^{1,r} + \Delta_k^{1,r} + S_{k-1}^{1,r}$$

where:

- $|\Upsilon_k^{1,r}| \leq M$ and

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• $\Delta_k = Y_k - Y_0$ and $Y_k$ has the law of a $3/2$-stable CSBP starting from $Y_0 = S_{k-1}^1$.

Lemma 8.14 implies that there exists a constant $c_1 > 0$ such that $E[|\Delta_k^1|] \leq c_1$ uniformly in $k$. Consequently, the expectation of the absolute value of the overall contribution to $S_k^L$ coming from this term is at most $c_1 k$.

Combining, we thus see that

$$E[S_k^L] \leq (c_0 M + c_1)k.$$  \hfill (8.23)

Recall that if we rescale so that distances are multiplied by $L^{-1/2}$, then quantum lengths are rescaled by the factor $L^{-1}$. Therefore if we rescale so that distances are rescaled by $L^{-1/2}$, we have a pair of paths $\gamma^r$ and $\tilde{\gamma}^r$ which connect $\partial B$ on $\partial B$ to that of a pair of independent $\frac{3}{2}$-stable CSBPs when performing a reverse metric exploration. Indeed, Proposition 8.10 implies that the boundary lengths of the two boundary segments between $\gamma^r$ and $\tilde{\gamma}^r$ evolve as independent $3/2$-stable CSBPs when performing a reverse metric exploration. Indeed, Proposition 8.10 implies that this is the case for second approximations to geodesics and, as this property is scale invariant, it also holds for rescalings of second approximations to geodesics.

We can take $M$ to be of constant order as $L \to \infty$, it follows from (8.24) that by taking $L$ to be very large we can arrange so that the distance between the tips of $\gamma^r$ and $\tilde{\gamma}^r$ is arbitrarily small. In particular, as $L \to \infty$, we find that the evolution of the boundary lengths of the two segments of $\partial B^*_{\infty}(x, d(x, y) - (r + t))$ converges to that of a pair of independent $3/2$-stable CSBPs. As $\epsilon \to 0$, the length of $\tilde{\gamma}^r$ and $\tilde{\gamma}^2$ converges to $s$. Thus Proposition 5.19 implies that $\tilde{\gamma}^r \to 0$ to the almost surely unique geodesic which connects their starting points back to $x$. This statement holds uniformly in $L$. Thus the result follows by taking limits first as $L \to \infty$, then as $\epsilon \to 0$, and then finally as $C \to \infty$.

8.3.3 $\sqrt{8/3}$-LQG metric net is the $3/2$-stable Lévy net

We will now combine Proposition 7.4 with Proposition 8.15 to show that the law of the metric net from $x$ to $y$ in a sample $(\mathcal{S}, x, y)$ from $\mathcal{M}_{\text{SPH}}^2$ is the same as in the $3/2$-stable Lévy net of [MST15a].

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**Proposition 8.15.** Suppose that \((S, x, y)\) has law \(M_{SPH}^2\). Then the law of the metric net from \(x\) to \(y\) associated with \(S\) is equal to that of the \(3/2\)-stable Lévy net.

*Proof.* We recall that there are several equivalent definitions of the \(3/2\)-stable Lévy net which are given in [MS15a]. We will make the comparison here between the construction of the \(3/2\)-stable Lévy net given in [MS15a, Section 3.6], which is based on a breadth-first approach.

Since we have shown in Proposition 7.4 that the overall boundary length of a filled metric ball in a \(\sqrt{8/3}\)-LQG sphere evolves as a \(3/2\)-stable CSBP excursion when one explores the ball in the reverse direction, it follows that we can couple a \(\sqrt{8/3}\)-LQG sphere with a \(3/2\)-stable Lévy net excursion so that these boundary lengths agree.

We recall from the construction given in [MS15a, Section 3.6] that the boundary lengths between geodesics starting from equally spaced points on the boundary of a metric ball of radius \(d - r\) where \(r > 0\) and \(d\) is the length of the \(3/2\)-stable CSBP excursion used to generate the \(3/2\)-stable Lévy net evolve as independent \(3/2\)-stable CSBPs as the radius of the ball varies between \(d - r\) and 0. The same is also true if the geodesics start from randomly chosen points on the boundary of the ball and then ordered to be counterclockwise.

For any fixed value of \(r\), Proposition 8.11 implies that the same is true for the boundary lengths between the geodesics in a reverse metric exploration of \((S, x, y)\) which start from points chosen i.i.d. from the quantum boundary measure. Therefore, for a fixed value of \(r\), we can couple these boundary lengths to be the same as in the \(3/2\)-stable Lévy net. By sending the number of geodesics considered to \(\infty\), we can couple so that the boundary lengths of all of the (leftmost) geodesics starting from a fixed radius back to \(x\) agree with the corresponding boundary lengths in the \(3/2\)-stable Lévy net instance.

Now, using the conditional independence for the reverse exploration established in Proposition 7.4, we can arrange so that the same is true for any finite number of \(r\) values. Sending the number of \(r\) values to \(\infty\), we obtain a coupling in which the whole metric net agrees, from which the result follows. \(\square\)

### 8.4 Proof of Theorem 1.4 and Corollary 1.5

*Proof of Theorem 1.4.* Proposition 8.15 implies that, in a \(\sqrt{8/3}\)-LQG sphere sampled from \(M_{SPH}^2\), we have that the metric net from \(x\) to \(y\) has the same law as in the \(3/2\)-stable Lévy net. [DMS14, Proposition 5.11] implies that the construction is invariant under the operation of resampling \(x\) and \(y\) independently from the quantum measure area. We also have the conditional independence of the unexplored region going in the forward direction from the construction of QLE(8/3, 0) given in [MS15b]. Therefore all of the hypotheses of Theorem 1.7 are satisfied, hence our metric measure space
is almost surely isometric to TBM. If we condition on the total mass of the surface being equal to 1 then the resulting metric measure space is isometric to the standard unit-area Brownian map measure.

Proof of Corollary 1.3. As explained just after the statement of Corollary 1.3 this immediately follows from Theorem 1.4.

9 Open problems

We now state a number of open problems which are related to the present work.

Problem 9.1. Compute the Hausdorff dimension of the outer boundary of a QLE(8/3, 0) process, stopped at a deterministic time \( r \). In other words, consider the outer boundary of a \( \mathbb{Q}_Q \) metric ball of radius \( r \), interpret this as a random closed subset of the Euclidean sphere or plane, and compute its (Euclidean) Hausdorff dimension.

To begin to think about this problem, suppose that \( z \) is chosen from the boundary measure on a filled metric ball boundary. What does that surface look like locally near \( z \)? We understand that the “outside” of the filled metric ball near \( z \) should look locally like a weight 2 quantum wedge, and that the inside should be an independent random surface—somewhat analogous to a quantum wedge—that corresponds to the local behavior of a filled metric ball at a typical boundary point. If we had some basic results about the interplay between metric, measure, and conformal structure near \( z \), such as what sort of (presumably logarithmic) singularity the GFF might have near \( z \), this could help us understand the number and size of the Euclidean balls required to cover the boundary.

Problem 9.2. Compute the Hausdorff dimension of a \( \sqrt{8/3} \)-LQG geodesic (interpreted as a random closed subset of the Euclidean sphere or plane).

As in the case of a metric ball, we can also consider the local structure near a point \( z \) chosen at random from the length measure of a geodesic between some distinct points \( a \) and \( b \). This \( z \) lies on the boundary of a metric ball (of appropriate radius) centered at \( a \), and also on the boundary of a metric ball centered at \( z \). These two ball boundaries divide the local picture near \( z \) into four pieces, two of which look like independent weight 2 wedges, and the other two of which look like the surfaces one gets by zooming in near metric ball boundaries. As before, if we knew what type of GFF thick point \( z \) corresponded to, this could enable to extract the dimension.

We emphasize that the KPZ formula cannot be applied in the case of either Problem 9.1 or Problem 9.2 because in both cases the corresponding random fractal is almost surely determined by the underlying quantum surface. To the best of our knowledge, there are no existing physics predictions for the answers to Problem 9.1 and Problem 9.2.
**Problem 9.3.** Show that a geodesic between two quantum typical points on a $\sqrt{8/3}$-LQG sphere is almost surely conformally removable.

A solution of Problem 9.3 would imply the independence of geodesic slices from the boundary of a filled metric ball back to its root. This, in turn, would allow us to use [MS15a, Theorem 1.1] in place of Theorem 1.7 (i.e., [MS15a, Theorem 4.6]) to check that the metric on $\sqrt{8/3}$-LQG induced by QLE($8/3,0$) is isometric to TBM. We note that the coordinate change trick used to prove the removability of the outer boundary of QLE given in [MS13b] does not apply in this particular setting because we do not have an explicit description of the field which describes the quantum surface in a geodesic slice. Related removability questions include establishing the removability of SLE$_{\kappa}$ for $\kappa \in [4,8)$ as well as the entire QLE($8/3,0$) trace (as opposed to just its outer boundary).

In [MS16], we will show that the embedding of TBM into $\sqrt{8/3}$-LQG constructed in this article is almost surely determined by the instance of TBM, up to Möbius transformation. This implies that TBM comes equipped with a unique conformal structure, which in turn implies that we can define Brownian motion on TBM, up to time-change, by taking the inverse image of a Brownian motion on the corresponding $\sqrt{8/3}$-LQG instance under the embedding map. The existence of the process with the correct time change was constructed in [Ber13, GRV13] and some rough estimates of its associated heat kernel have been obtained in [MRV14, AK14]. Following the standard intuition from heat kernel theory, one might guess that the probability that a Brownian motion gets from $x$ to $y$ in some very small $\epsilon$ amount of time should scale with $\epsilon$ in a way that depends on the metric distance between $x$ and $y$ (since any path that gets from $x$ to $y$ in a very short time would probably take roughly the shortest possible path). This leads to the following question (left deliberately vague for now), which could in principle be addressed using the techniques of this paper independently of [MS16].

**Problem 9.4.** Relate the heat kernel for Liouville Brownian motion in the case that $\gamma = \sqrt{8/3}$ to the QLE($8/3,0$) metric.

It has been conjectured that the heat kernel $p_t(x,y)$ should satisfy (for some constants $c_0, c_1 > 0$) the bound

$$\frac{c_0}{t} \exp\left(-\frac{\bar{d}_Q(x,y)^{4/3}}{c_0 t^{1/3}}\right) \leq p_t(x,y) \leq \frac{c_1}{t} \exp\left(-\frac{\bar{d}_Q(x,y)^{4/3}}{c_1 t^{1/3}}\right).$$

(9.1)

See, for example, the discussion in [DB09].

A number of versions of the KPZ relation [KPZ88] have been made sense of rigorously in the context of LQG [BS09, BGRV14, BJRV13, DMS14, DRS14, DS11, RV11, GHM15, Aru15]. One of the differences between these formulations is how the “quantum dimension” of the fractal set is computed.
Problem 9.5. Does the KPZ formula hold when one computes Hausdorff dimensions using $QLE(8/3, 0)$ metric balls?

In this article, we have constructed the metric space structure for $\sqrt{8/3}$-LQG and have shown that in the case of a quantum sphere, quantum disk, and quantum cone the corresponding metric measure space has the same law as in the case of TBM, the Brownian disk, and the Brownian plane, respectively. The construction of the metric is a local property of the surface, so we also obtain the metric for any other $\sqrt{8/3}$-LQG surface. One particular example is the torus. The natural law on $\sqrt{8/3}$-torii is described in [DRV15] and the Brownian torus, the scaling limit of certain types of random planar maps, will be constructed in [BM].

Problem 9.6. Show that the $\sqrt{8/3}$-LQG torus of [DRV15], endowed with the metric defined by $QLE(8/3, 0)$ using the methods of this paper, agrees in law (as a random metric measure space) with the Brownian torus of [BM].

Finally, a major open problem is to rigorously describe an analog of TBM that corresponds to $\gamma$-LQG with $\gamma \neq \sqrt{8/3}$, to extend the results of this paper to that setting. A partial step in this direction appears in [GHS16], which shows the existence of a certain distance scaling exponent (but does not compute it explicitly).

Problem 9.7. Construct a metric on $\gamma$-LQG when $\gamma \neq \sqrt{8/3}$. Work out the appropriate dimension and scaling relations (as discussed in Section 1.4.1).

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