IMPROVED WELL-POSEDNESS FOR THE QUADRATIC DERIVATIVE NONLINEAR WAVE EQUATION IN 2D

VIKTOR GRIGORYAN\textsuperscript{1} AND ALLISON TANGUAY\textsuperscript{2}

Abstract. In this paper we consider the Cauchy problem for the nonlinear wave equation (NLW) with quadratic derivative nonlinearities in two space dimensions. Following Grünrock’s result in 3D, we take the data in the Fourier-Lebesgue spaces $\hat{H}^r_s$, which coincide with the Sobolev spaces of the same regularity for $r = 2$, but scale like lower regularity Sobolev spaces for $1 < r < 2$. We show local well-posedness (LWP) for the range of exponents $s > 1 + \frac{3}{2r}, \frac{3}{2} < r \leq 2$. On one end this recovers the best known result on the Sobolev scale, $H^{\frac{7}{4}}$, while on the other end establishes the $\hat{H}^{\frac{3}{2}}_{s_2}$ result, which scales like the Sobolev $H^{\frac{5}{3}}$, thus, corresponding to a $\frac{1}{12}$ derivative improvement on the Sobolev scale.

1. Introduction

Consider the quadratic derivative nonlinear wave equation (NLW),
\begin{equation}
\Box u = (\partial_t u)^2 \quad \text{in } \mathbb{R}^{1+n},
\end{equation}
where $\partial_t u$ is the space-time gradient of $u$, and no special structure is assumed in the nonlinearity. We study the local well-posedness (LWP) question for equation (1) with initial data taken in Fourier-Lebesgue spaces $\hat{H}^r_s$. Thus, we consider the Cauchy problem for (1) with initial conditions
\begin{equation}
(u, \partial_t u)|_{t=0} = (f, g) \in \hat{H}^r_s \times \hat{H}^r_{s-1}.
\end{equation}
Our goal is to establish local well-posedness for a range of the exponents $(r, s)$, which improves on previously known Sobolev space results. The Fourier-Lebesgue spaces $\hat{H}^r_s$ have previously appeared in literature in the context of various equations, and were used to achieve improved regularity results. See for example [10], [23], [6], [7], [2], [8], [9].

Equation (1) is invariant under the scaling
\begin{equation}
(t, x) \mapsto (\lambda t, \lambda x).
\end{equation}
That is, if $u$ is a solution to (1) in $\mathbb{R}^{1+n}$, then so is the scaled function $u_\lambda(t, x) = u(\lambda t, \lambda x)$. The homogeneous Sobolev norm of the initial data then scales as
\begin{equation}
\|u_\lambda(0, \cdot)\|_{H^s(\mathbb{R}^n)} = \lambda^{s - \frac{n}{2}} \|u(0, \cdot)\|_{H^s(\mathbb{R}^n)},
\end{equation}
and $s_c = \frac{n}{2}$ is called the critical exponent for equation (1). As the $\hat{H}^{s_c}$ norm of the initial data is preserved under its scaling. From general scaling considerations, it is expected that local well-posedness holds for data in the Sobolev space $H^s$ for $s > s_c$ (subcritical regime), global existence holds for small data in $\hat{H}^{s_c}$ (critical regime), and some form of ill-posedness holds for data in $H^s$ for $s < s_c$ (supercritical regime).

Consequently, local well-posedness for (1) is expected to hold for data in the Sobolev space $H^s$, $s > \frac{n}{2}$. This can be proved in dimensions $n \geq 4$ with a Strichartz estimate approach; however, it is known to be false in dimensions $n = 2, 3$. 

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In dimension $n = 3$, Ponce and Sideris [18] proved LWP for $s > 2$. This is sharp in light of the counterexamples of Linblad [15], [16], [17]. If the nonlinearity has a null-form structure, Klainerman and Machedon [12] improved the range for LWP to $s > \frac{3}{4}$, which is almost critical in 3D.

In dimension $n = 2$, the best known result for (1) is for $s > \frac{7}{4}$, which can be shown using the Strichartz estimates approach. As we were unable to locate a reference for this result, we sketch its proof in Appendix A. If the nonlinearity has a null-form structure, the LWP results can again be improved. For the $Q_0$ null-form, almost critical LWP was established by Klainerman and Selberg [14] in the context of wave maps. For the other null forms, Zhou [24] established LWP for data in $\hat{H}^s$ with $s > \frac{5}{4}$, and also showed that this is sharp.

Recently, Grünrock showed in [9] that one can eliminate the gap to the almost criticality for the equation (1) in dimension $n = 3$, by taking the initial data in the Fourier-Lebesgue spaces $\hat{H}^r_s$. These spaces are defined as the closure of the set of Schwartz functions under the norm,

$$\|f\|_{\hat{H}^r_s} = \|\langle \xi \rangle^s \hat{f}\|_{L^2_\xi^r}, \quad \frac{1}{r} + \frac{1}{r'} = 1,$$

where $\hat{f}$ denotes the (spatial) Fourier transform of $f$, and $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$. The norm of the corresponding homogeneous space is $\|f\|_{\hat{H}^r_s} = \|\langle \xi \rangle^s \hat{f}\|_{L^2_\xi^r}$. The norms of the initial data in the homogeneous Fourier-Lebesgue spaces scale as

$$\|u_\lambda(0, \cdot)\|_{\hat{H}^r_s(R^n)} = \lambda^{s - \frac{n}{2}} \|u(0, \cdot)\|_{\hat{H}^r_s(R^n)},$$

so the critical exponent for these spaces is $s_c^r = \frac{5}{4}$. In terms of scaling of the norms, we therefore observe the following correspondence between the homogeneous Sobolev and Fourier-Lebesgue spaces,

$$(4) \quad \hat{H}^r_s \sim \hat{H}^\sigma, \quad \text{if} \quad \sigma = s + n \left(\frac{1}{2} - \frac{1}{r}\right).$$

Grünrock’s result established LWP for data in the space $\hat{H}^r_s$ for $s > \frac{3}{4} + 1$, $1 < r \leq 2$. This range of exponents almost reaches the critical pair at the endpoint $r = 1$. He relies on free wave interaction estimates of Foschi-Klainerman [4], which have a factor of $\|\tau| - |\xi|\|^{-\frac{1}{2}}$. This factor becomes unbounded near the null cone $|\tau| = |\xi|$ in dimension $n = 2$. Thus, Grünrock’s result does not directly generalize to dimension $n = 2$. However, if this factor can be offset by cancellations in the nonlinearity along the null-cone, then his arguments can be salvaged. This approach was explored by the first author and A. Nahmod in [3], who achieve almost critical LWP for the null-form problem in 2D.

The main result of this paper is the following.

**Theorem 1.1.** Let $\frac{3}{2} < r \leq 2$, $s > \frac{3}{2} + 1$, then the Cauchy problem (1)-(2) is locally well-posed for data in the space $\hat{H}^r_s \times \hat{H}^r_{s-1}$.

**Remark 1.2.** Well-posedness here is understood in the sense of Theorem [3,1] with the solution found in the space $Z^r_{s,b}$, which will be defined in the next section.

**Remark 1.3.** At one extreme, $(s,r) = (\frac{7}{4}, 2)$, our result coincides with the local well-posedness for data in $H^{\frac{7}{4}+}$, while at the other extreme, $(s,r) = (2, \frac{3}{2})$, we obtain local well-posedness in the space $\hat{H}^{\frac{3}{2}+}_{2^+}$. By the scaling correspondence (4), this last space scales like the Sobolev space $H^{\frac{7}{4}+}$, which gives a $\frac{1}{12}$ derivative improvement on the $H^{\frac{7}{4}+}$ Sobolev data result.

\footnote{Alternatively, one can use $\langle \xi \rangle = 1 + |\xi| \approx \sqrt{1 + |\xi|^2}$.}
Remark 1.4. An analogous result to Theorem 1.1 can be proved for the equation
\( \Box u = \partial (u^2) \).
This equation is invariant under the scaling \( u \mapsto u_\lambda \), where \( u_\lambda(t, x) = \lambda u(\lambda t, \lambda x) \). And the critical exponent on the Sobolev scale is \( s_c = \frac{n}{2} - 1 \), while on the Fourier-Lesbegue scale it is \( s'_c = \frac{n}{r} - 1 \). The best known result for (5) in dimension \( n = 2 \) for data in \( H^s \) is for \( s > \frac{3}{4} \), which, again, can be shown using Strichartz estimates. A similar argument to the one we use for equation (1) will show that the Cauchy problem for (5) with data in \( \hat{H}^s_r \) is locally well-posed for \( s > \frac{3}{2r}, \frac{3}{2} < r \leq 2 \).

The region for the parameters \( (s, \frac{1}{r}) \), for which Theorem 1.1 holds is shaded in Figure 1. Notice that the bottom and right edges of the region are not included. If our estimates did not break down for \( r \leq \frac{3}{2} \), the \( s \)-line \( s = \frac{3}{2r} + 1 \) would have picked up the point \( (s, r) = (\frac{5}{7}, 1) \), which corresponds to the Sobolev space \( H^\frac{3}{2} \). This is still above the critical regularity, as can be seen from the picture, where the solid line represents the critical relation \( s'_c = \frac{2}{r} \).

![Figure 1](image-url)

**Figure 1.** The shaded region represents the range of indices for which LWP for data in space \( \hat{H}^r_s \times \hat{H}^r_{s-1} \) holds.

The outline of the rest of the paper is: in Section 2 we introduce the solution spaces, and reduce Theorem 1.1 to a bilinear estimate. This reduction is achieved by utilizing the general LWP theorem stated in Appendix B. In Section 3 we establish bilinear Fourier restriction estimates in \( \hat{L}^r \) spaces. These estimates are used in Section 4 to establish the main bilinear estimate (8).

As was mentioned earlier, we also sketch the proof of the LWP for equation (1) with data in the Sobolev space \( H^\frac{7}{4} + \times H^\frac{3}{4} + \varepsilon \) by the Strichartz estimates in Appendix A.

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The first author would also like to thank Magdalena Czubak for showing him the proof of the \( H^\frac{7}{4} \) LWP via the Strichartz estimates, which is sketched in Appendix A.
2. The $X^{r}_{s,b}$ space, reduction to a bilinear estimate

We will obtain the local in time solution via a contraction principle in the time restricted $X^{r}_{s,b}$ space. This space is a Fourier-$L^{r'}$ analogue of the wave-Sobolev space $X^{s,b}$. It is defined by its norm

$$
\|u\|_{X^{r}_{s,b}} = \|\langle \xi \rangle^{-s} (\tau - |\xi|^{b}) \hat{u}\|_{L^{r'}},
$$

where $\hat{u}$ stands for the time-space Fourier transform of $u$. The time restricted space is defined as

$$
X^{r}_{s,b;T} = \{ u | U|_{-T,T} \times \mathbb{R}^{n} : U \in X^{r}_{s,b} \},
$$

with its norm given by

$$
\|u\|_{X^{r}_{s,b;T}} = \inf \left\{ \| U\|_{X^{r}_{s,b}} : U|_{[-T,T] \times \mathbb{R}^{n}} = u \right\}.
$$

If $b > \frac{1}{2}$, the so-called transfer principle allows one to transfer multilinear estimates for free solutions of the homogeneous equation $\Box u = 0$ with data in $\hat{H}^{r}_{s}$ spaces to general elements of $X^{r}_{s,b}$ spaces. A trivial consequence of the transfer principle is the following crucial embedding

$$
X^{r}_{s,b} \subset C(\mathbb{R}, \hat{H}^{r}_{s}),
$$

thus, also $X^{r}_{s,b;T} \subset C([-T,T], \hat{H}^{r}_{s})$, which guarantees that the solutions found in the $X^{r}_{s,b}$ spaces are the proper continuations of the initial data in $\hat{H}^{r}_{s}$ spaces. For details on the transfer principle see [19] for $L^{2}$ based spaces, [11] for the linear case on the $X^{r}_{s,b}$ spaces, and [14] for the $X^{r}_{s,b}$ multilinear case.

We need to separately estimate the time derivative of the solution, since the wave operator is of second order in time. For this, we define our solution space, $Z^{r}_{s,b}$, by its norm

$$
\|u\|_{Z^{r}_{s,b}} = \|u\|_{X^{r}_{s,b}} + \|\partial_{t} u\|_{X^{r-1}_{s,b}}.
$$

The time restricted space $Z^{r}_{s,b;T}$ and its norm are defined as before.

We will also use the simplified notation $\hat{L}^{r}_{s,x} = X^{0}_{s,0}$, and $\hat{L}^{r} = \hat{H}^{r}$, with the last norm taken either with respect to the time, or space variables.

Relying on the general local well-posedness Theorem [13,1] we will prove Theorem [11] by establishing the following two estimates

\[ (6) \parallel \partial_{u}\partial v \parallel_{X^{r-1}_{s-1,b+1}} \lesssim \parallel u \parallel_{Z^{r}_{s,b}} \parallel v \parallel_{Z^{r}_{s,b}}, \]

and

\[ (7) \parallel (\partial u)^{2} - (\partial v)^{2} \parallel_{X^{r-1}_{s-1,b+1}} \lesssim \left( \parallel u \parallel_{Z^{r}_{s,b}} + \parallel v \parallel_{Z^{r}_{s,b}} \right) \parallel u - v \parallel_{Z^{r}_{s,b}}. \]

The second estimate immediately follows from the first one due to the quadratic nature of the nonlinearity. Thus, we only need to prove estimate (6).

If $b + \epsilon - 1 < 0$, it is easy to see that estimate (6) will follow from the estimate

\[ (8) \parallel uv \parallel_{X^{r}_{s,0}} \lesssim \parallel u \parallel_{X^{r}_{s,b}} \parallel v \parallel_{X^{r}_{s,b}}, \]

where $\sigma = s - 1$, and $u, v$ are now general elements of $X^{\sigma}_{s,b}$. The rest of this paper is dedicated to proving (8) for $\sigma > \frac{1}{2}$, $\frac{1}{2} < r \leq 2$, and some $b, \epsilon$, with $\frac{1}{2} < b < 1$, $\epsilon \in (0,1-b)$.

Remark 2.1. In [9] Grünrock relies on the general local well-posedness theorem for the first order equation $\partial_{t} u - \partial^{2} (D) u = N(u)$, which he proved in the earlier paper [6]. For this, he reformulates equation (11) as a system of first order equations for $u_{\pm} = u \pm i(1-\Delta_{x})^{-1} \partial_{t} u$. If we were to use this approach in our context, and the nonlinearity contains time derivatives
of $u$, we would have to place the left hand sides of estimates (6), (7) in the space $Z_{s-1,b}^r$. These estimates in addition to (8) would also require the estimate
\begin{equation}
\|uv\|_{X_{s-1,b}^{r+1}} \lesssim \|u\|_{X_{s,b}} \|v\|_{X_{s,b}}.
\end{equation}
This is due to the fact that the absolute value of the symbol of the first order operator, $|\tau - \phi(\xi)|$, does not control $|\tau|$. However, Grünrock uses the transfer principle and does his estimates only on free solutions, which allows an easy control of the time derivatives by the spatial derivatives, and essentially produces the three dimensional analogue of (9) for free from (8). As we mentioned before, this approach fails in two dimensions. Nevertheless, one can indeed show (9) with the arguments we employ here and a hyperbolic Leibniz type rule. See [22] for the details, where also more general estimates of the form (6) are proved.

3. Bilinear Fourier restriction estimates

By duality, we can reformulate the multiplicative estimate (8) as the trilinear integral estimate
\begin{equation}
I \lesssim \|F_0\|_{L^p_{\xi,k}} \|F_1\|_{L^p_{\tau,\xi}} \|F_2\|_{L^p_{\tau,\xi}},
\end{equation}
where
\[
I = \iint_{\mathbb{R}^3} F_0(X_0) F_1(X_1) F_2(X_2) \delta(X_0 + X_1 + X_2) dX_0 dX_1 dX_2 \frac{\langle \xi_0 \rangle^{-\sigma} \langle \xi_1 \rangle^{\sigma} \langle \xi_2 \rangle^{\sigma} \langle |\tau_1| - |\xi_1| \rangle^b \langle |\tau_2| - |\xi_2| \rangle^b}{\langle \tau \rangle^b},
\]
$X_j = (\tau_j, \xi_j)$ for $j = 0, 1, 2$, and $F_1, F_2$ stand for
\[
F_1(\tau_1, \xi_1) = \langle \xi_1 \rangle^{\sigma} \langle |\tau_1| - |\xi_1| \rangle^b \tilde{u}(\tau_1, \xi_1),
\]
\[
F_2(\tau_2, \xi_2) = \langle \xi_2 \rangle^{\sigma} \langle |\tau_2| - |\xi_2| \rangle^b \tilde{v}(\tau_2, \xi_2),
\]
and to simplify notation, $p = r'$.

Since $\xi_0 + \xi_1 + \xi_2 = 0$ in the above integral, by triangle inequality, $|\xi_j| \leq |\xi_k| + |\xi_l|$ for all permutations $(j, k, l)$ of $(0, 1, 2)$. Hence, also, $\langle \xi_j \rangle \lesssim \langle \xi_k \rangle + \langle \xi_l \rangle$. This implies that only low-high-high (LHH), high-low-high (HLH), and high-high-low (HHL) interactions are permitted. We then split $I = I_{LHH} + I_{HLH} + I_{HHL}$, with the following correspondence:

- $\text{LHH} \leftrightarrow \langle \xi_0 \rangle \lesssim \langle \xi_1 \rangle \sim \langle \xi_2 \rangle$
- $\text{HLH} \leftrightarrow \langle \xi_1 \rangle \lesssim \langle \xi_0 \rangle \sim \langle \xi_2 \rangle$
- $\text{HHL} \leftrightarrow \langle \xi_2 \rangle \lesssim \langle \xi_0 \rangle \sim \langle \xi_1 \rangle$.

The letters $M, N, L$ and their indexed counterparts will denote dyadic numbers of the form $2^j, j \in \{0, 1, 2, \ldots \}$, and we will use the following notation for Fourier restrictions.

\[
F^N(X) = \chi_{\langle \xi \rangle \sim N} F(X),
\]
\[
F^{N,L}(X) = \chi_{\langle |\tau| - |\xi| \rangle \sim L} F^N(X),
\]
\[
F^{N,L,\pm}(X) = \chi_{|\tau - \xi| \geq 0} F^{N,L}(X).
\]

One immediately has the following dyadic summation estimates:
\[
\sum_{N} \|F^N\|^p_{L^p} \sim \|F\|^p_{L^p},
\]
\[
\sum_{L} \|F_{N,L}\|^p_{L^p} \sim \|F^N\|^p_{L^p},
\]
\[
\sum_{L} \|F_{N,L,\pm}\|^p_{L^p} \lesssim \|F^N\|^p_{L^p}.
\]
Using the trilinear convolution form

\[
J(F_0, F_1, F_2) = \iiint F_0(X_0) F_1(X_1) F_2(X_2) \delta(X_0 + X_1 + X_2) \, dX_0 \, dX_1 \, dX_2,
\]

we have

\[
I \lesssim \sum_{N,L} J(F_0^{N_0}, F_1^{N_1,L_1}, F_2^{N_2=L_2}) \frac{N_0^{-\sigma} N_1^\sigma N_2^\sigma L_1^1 L_2^2}{N_0^0 N_1^{N/2} N_2^0 L_1^1 L_2^2},
\]

where \( N = (N_0, N_1, N_2), \ L = (L_1, L_2). \)

The proof of estimate (12) relies on bilinear Fourier restriction estimates on \( \mathbb{R}^{1+2} \) of the form

\[
\|P_{A_0}(P_{A_1} u_1 : P_{A_2} u_2)\|_{L^r} \leq C \|P_{A_1} u_1\| \|P_{A_2} u_2\|_{L^r},
\]

where \( A_0, A_1, A_2 \subset \mathbb{R}^{1+2} \) are measurable sets, and the projection \( P_A \) is defined by \( \tilde{P}_A u = \chi_A u \). We are interested in these restriction estimates with \( A_0, A_1 \) and \( A_2 \) being thickened subsets of the null cone \( K = \{ (\tau, \xi) \in \mathbb{R}^{1+2} : |\tau| = |\xi| \} \). Following Selberg [20, 21], we introduce the following notation for thickened upper \( (\tau \geq 0) \) and lower \( (\tau \leq 0) \) cones, truncated in the spatial frequency \( \xi \) by balls, annuli and angular sectors:

\[
K_{N,L}^\pm = \{ (\tau, \xi) \in \mathbb{R}^{1+2} : |\xi| \lesssim N, \tau = \pm |\xi| + O(L) \},
\]

\[
\hat{K}_{N,L}^\pm(\omega) = \{ (\tau, \xi) \in \hat{K}_{N,L}^\pm : \theta(\pm \xi, \omega) \leq \gamma \},
\]

where \( N, L, \gamma > 0; \omega \in S^1; \theta(a,b) \) denotes the angle between nonzero \( a, b \in \mathbb{R}^2 \).

By duality, (12) is equivalent to

\[
J(F_0^{-A_0}, F_1^{A_1}, F_2^{A_2}) \leq C \|F_0^{-A_0}\|_{L^r} \|F_1^{A_1}\|_{L^p} \|F_2^{A_2}\|_{L^p},
\]

where \( F^{A}(X) = \chi_A(X) F(X); F_j = \tilde{u}_j, \ j = 1, 2; \) and \( J \) is given by (11).

We start with the following elementary lemma, which is a direct extension of [21 Lemma 1.1] to \( \tilde{L}^r \) (see also [20 Lemma 3.1]). We use \( |E| \) for the volume of the set \( E \subset \mathbb{R}^{1+2} \).

**Lemma 3.1.** The estimate (12) holds with \( C \) equal to an absolute constant times

\[
\min \left( \sup_{X \in A_0} |A_1 \cap (X - A_2)|^{\frac{1}{r}}, \sup_{X \in A_1} |A_0 \cap (X + A_2)|^{\frac{1}{p}} |A_2|^{\frac{1}{2} - \frac{1}{r}}, \right.
\]

\[
\left. \sup_{X \in A_2} |A_0 \cap (X + A_1)|^{\frac{1}{p}} |A_1|^{\frac{1}{2} - \frac{1}{r}} \right),
\]

provided this quantity is finite.

**Proof.** For the first bound, we use the dual formulation, and rewrite

\[
J(F_0^{-A_0}, F_1^{A_1}, F_2^{A_2}) = \int_{X_0} \int_{X_1} \chi_{X_0 \cap (X_0 - A_2)} \chi_{X_0 \cap A_0} F_1^{A_1}(X_1) F_2^{A_2}(X_0 - X_1) F_0^{-A_0}(-X_0) \, dX_1 \, dX_0
\]

\[
\leq \int_{X_0} \left[ \sup_{X_0 \in A_0} |A_1 \cap (X_0 - A_2)|^{\frac{1}{r}} \right] \left[ \int_{X_1} \left( F_1^{A_1}(X_1) F_2^{A_2}(X_0 - X_1) \right)^p \, dX_1 \right]^{\frac{1}{p}} \times F_0^{-A_0}(-X_0) \, dX_0,
\]
where we used Hölder’s inequality in the $X_1$ variable. Now applying Hölder’s inequality in the $X_0$ variable and using Fubini’s theorem, we bound the above by

$$\lesssim \sup_{X_0 \in A_0} |A_1 \cap (X_0 - A_2)|^{\frac{2}{p}}$$

$$\times \left[ \int_{X_0} \left( F_1^{A_1}(X_1) \right)^p \left( F_2^{A_2}(X_0 - X_1) \right)^p dX_1 dX_0 \right]^{\frac{1}{p}} \left[ \int_{X_0} \left( F_0^{-A_0}(-X_0) \right)^r dX_0 \right]^{\frac{1}{r}}$$

$$\lesssim \sup_{X_0 \in A_0} |A_1 \cap (X_0 - A_2)|^{\frac{2}{p}}$$

$$\times \left[ \int_{X_1} \left( F_1^{A_1}(X_1) \right)^p \int_{X_0} \left( F_2^{A_2}(X_0 - X_1) \right)^p dX_0 dX_1 \right]^{\frac{1}{p}} \|F_0^{-A_0}\|_{L^r}$$

$$\lesssim \sup_{X_0 \in A_0} |A_1 \cap (X_0 - A_2)|^{\frac{2}{p}} \|F_0^{-A_0}\|_{L^r} \|F_1^{A_1}\|_{L^p} \|F_2^{A_2}\|_{L^p}.$$

The proofs of the second and third bounds are similar, so we prove only the second bound. Using the dual formulation we now rewrite

$$J(F_0^{-A_0}, F_1^{A_1}, F_2^{A_2})$$

$$= \int_{X_1} \int_{X_0} \chi_{X_0 \in A_0 \cap (X_1 + A_2)} \chi_{X_1 \in A_1} F_1^{A_1}(X_1) F_2^{A_2}(X_0 - X_1) F_0^{-A_0}(-X_0) dX_1 dX_0$$

$$\lesssim \int_{X_1} \left[ \sup_{X_1 \in A_1} |A_0 \cap (X_1 + A_2)|^{\frac{2}{p}} \right] \left[ \int_{X_0} \left( F_0^{-A_0}(-X_0) F_2^{A_2}(X_0 - X_1) \right)^r dX_0 \right]^{\frac{1}{r}}$$

$$\times F_1^{A_1}(X_1) dX_1$$

$$\lesssim \sup_{X_1 \in A_1} |A_0 \cap (X_1 + A_2)|^{\frac{2}{p}}$$

$$\times \left[ \int_{X_1} \left( F_0^{-A_0}(-X_0) \right)^r \left( F_2^{A_2}(X_0 - X_1) \right)^r dX_0 dX_1 \right]^{\frac{1}{r}} \|F_1^{A_1}\|_{L^p}$$

$$\lesssim \sup_{X_1 \in A_1} |A_0 \cap (X_1 + A_2)|^{\frac{2}{p}} \|F_0^{-A_0}\|_{L^r} \|F_2^{A_2}\|_{L^r} \|F_1^{A_1}\|_{L^p}$$

$$\lesssim \sup_{X_1 \in A_1} |A_0 \cap (X_1 + A_2)|^{\frac{2}{p}} \|A_2\|^{\frac{1}{r}} \|F_0^{-A_0}\|_{L^r} \|F_1^{A_1}\|_{L^p} \|F_2^{A_2}\|_{L^p}.$$

Using the above lemma, we can now prove the necessary bilinear restriction estimates in $\hat{L}^r$, analogous to those established by Selberg in [21, Theorem 2.1].

**Proposition 3.1.** Let $\frac{3}{2} < r \leq 2$, then the estimate

$$\left\| P_{K_{N_0}^{\pm_0}} \left( P_{K_{N_1,1}^{\pm_1}} u_1 \cdot P_{K_{N_2,2}^{\pm_2}} u_2 \right) \right\|_{\hat{L}^r} \leq C \|u_1\|_{\hat{L}^r} \|u_2\|_{\hat{L}^r}$$

holds with

$$C \sim (N_{\min}^{12})^{\frac{3}{2}} (N_{\min}^{12})^{\frac{1}{2}} (L_{\min}^{12})^{\frac{1}{2}},$$

$$C \sim (N_{\min}^{12})^{\frac{1}{2}} (N_{\min}^{12})^{\frac{1}{2}} (L_{\max}^{12})^{\frac{1}{2}},$$

regardless of the choice of signs $\pm_j$.

**Remark 3.2.** The restriction $\frac{3}{2} < r$ is only needed for (16). Furthermore, the bound with (15) can be trivially generalized to any dimension, since Lemma 3.1 which is the only
ingredient in its proof, holds irrespective of the dimension of the space. In $n$ dimensions, \([15]\) looks like
\[
C \sim (N^{012})_\min \frac{2}{p} (N^{12})_\min \frac{2}{q} (L^{12})_\min \frac{1}{r}.
\]

**Proof of** \([15]\). The bound with \([15]\) immediately follows from Lemma \([3.1]\). To see this, we split into the cases $N_0 \ll N_1 \sim N_2$ (LHH), $N_1 \lesssim N_0 \sim N_2$ (HLH), and $N_2 \lesssim N_0 \sim N_1$ (HHL), and by symmetry consider only LHH and HLH cases.

In the HLH case, $N_1 \lesssim N_2 \sim N_0$, we estimate the volume of the set
\[
E = K_{N_1,L_1}^{\pm 1} \cap (X_0 - K_{N_2,L_2}^{\pm 2}) \subset \{ \xi_1 \leq N_1; \tau_1 = \pm 1 |\xi_1| + O(L_1), \tau_0 - \tau_1 = \pm 2 |\xi_0 - \xi_1| + O(L_2) \},
\]
for any $X_0 \in K_{N_0,L_0}^{\pm 0}$. Integrating first in $\tau_1$ and then in $\xi_1$, we obtain $|E| \lesssim N_1^2 L_1^{12}$. Raising to the power $1/r$ and using the first bound from Lemma \([3.1]\), gives the desired result.

For the LHH regime, i.e. when $N_0 \ll N_1 \sim N_2$, due to symmetry we can assume $L_1 \leq L_2$, and estimate the volume of the set
\[
E = K_{N_0,L_0}^{\pm 0} \cap (X_2 + K_{N_1,L_1}^{\pm 1}) \subset \{ \xi_0 \leq N_0; \tau_0 - \tau_2 = \pm 1 |\xi_0 - \xi_2| + O(L_1) \},
\]
for any $X_2 \in K_{N_2,L_2}^{\pm 2}$. Again, integrating first in $\tau_0$ and then in $\xi_0$, we have $|E| \leq N_0^2 L_1$, which coupled with $|K_{N_1,L_1}^{\pm 1} | \leq N_1^2 L_1$ and the third bound in Lemma \([3.1]\) gives the desired result.

**Proof of** \([16]\). We may assume $L^{12}_{\max} \ll N^{012}_{\min}$, since otherwise \([13]\) is already better than \([16]\). Additionally, we may replace $K_{N_j,L_j}^{\pm j}$ by $K_{N_j,L_j}^{\pm j}$ due to the presence of the factor $(N^{012})_\min \frac{1}{p} (N^{12})_\min \frac{1}{q} (L^{12})_\min$ in the bound. Indeed, using the dual formulation and decomposing the balls $|\xi_j| \lesssim N_j$ into annuli $|\xi_j| \sim M_j$ for dyadic $0 < M_j \leq N_j$, we can sum over $M^{12}_{\max} \leq N^{012}_{\min}$ using the factor $(M^{12}_{\min})_p^{-1}$ and the fact that the two largest of the $M_j$’s are comparable. For the rest of the sum we use the duality of $l^p$ and $l^r$ in the HLH and HHL cases, or the factor $(N^{12}_{\min})_\min \frac{1}{q} \frac{1}{r}$ in the LHH case. We now proceed with the proof separately in the HLH and LHH cases. In what follows, we use the calculations and ideas of \([21]\) Section 3] in a substantial way.

- **The HLH case:** $N_1 \lesssim N_0 \sim N_2$. By Lemma \([3.1]\), we reduce to estimating the volume of the set
\[
E = K_{N_0,L_0}^{\pm 0} \cap (X_0 - K_{N_1,L_1}^{\pm 1})
\]
uniformly in $X_0 \in K_{N_0,L_0}^{\pm 0}$. But by \([21]\) Subsection 3.1, $|E| \lesssim N^\frac{3}{2} L^{12}_{\min} (L^{12}_{\max})^\frac{1}{2}$. We then have
\[
|E|^{\frac{1}{r}} \lesssim \left( N^\frac{3}{2} L^{12}_{\min} (L^{12}_{\max})^\frac{1}{2} \right)^\frac{1}{r} \lesssim N^\frac{3}{2} (L^{12}_{\min})^\frac{1}{q} (L^{12}_{\max})^\frac{1}{r}.
\]

- **The LHH case:** $N_0 \ll N_1 \sim N_2$. We may assume $L_1 \leq L_2$ by symmetry, and need to prove that \([3.1]\) holds with
\[
C \sim N^\frac{1}{p} N^\frac{1}{q} L^\frac{1}{r} L^\frac{1}{p}.
\]

Recall that \([14]\) is equivalent to the dual estimate \([13]\). By a simple change of variables in the integral in the trilinear form on the left, we can also see that \([13]\) is equivalent to the product estimate
\[
\left\| P_{K_{N_0,L_0}^{\pm 0}} u_0 \left( P_{K_{N_1,L_1}^{\pm 1}} u_1 \cdot P_{K_{N_0,L_0}^{\pm 2}} u_0 \right) \right\|_{L^p} \leq C \| u_1 \|_{L^r} \| u_0 \|_{L^p},
\]
where $u_0$ is the inverse Fourier transform of $F_0$. So it suffices to prove \([19]\) for $C$ given by \([18]\).
Denoting \( \theta_{01} = \theta(\pm_1 \xi_1, \pm_0 \xi_0) \), we split into the cases \( \theta_{01} \lesssim \gamma_0 \), and \( \theta_{12} \gg \gamma_0 \), where
\[
\gamma_0 = \left( \frac{L_2}{N_1} \right)^{\frac{1}{q}}.
\]

Assuming without loss of generality that \( \tilde{u}_1, \tilde{u}_0 \geq 0 \), we have by \cite{21} Lemma 2.4, 2.5,
\[
\| \mathcal{P}_{N_2}(u_1 u_0) \|_{L^p} \lesssim \sum_{\omega, \omega_1 \in \Omega(\gamma_0)} \chi_{\theta(\omega, \omega_1) \leq \gamma_0} \| \mathcal{F} \mathcal{P}_{N_2}(u_1^{\gamma_0} \omega_1 u_0^{\gamma_0} \omega) \|_{L^r} + \sum_{\gamma \leq \gamma_0} \sum_{\omega, \omega_1 \in \Omega(\gamma)} \chi_{\theta(\omega, \omega_1) \leq \gamma_0} \| \mathcal{F} \mathcal{P}_{N_2}(u_1^{\gamma_0} \omega_1 u_0^{\gamma_0}) \|_{L^r} = \Sigma_1 + \Sigma_2,
\]
where \( \mathcal{F} \) denotes the time-space Fourier transform; \( \Omega(\gamma) \) is a maximal \( \gamma \)-separated subset of \( S^1 \), and \( u_0^{\gamma} \omega = \chi_{\theta(\xi, \omega) < \gamma} \).

For \( \Sigma_1 \), we estimate the volume of the set \( E = \hat{K}_{N_0}^{\pm_0} (\omega_0) \cap (X_2 + \hat{K}_{N_1}^{\pm_1} (\omega_1)) \), where \( X_2 \) is any point in the support of \( \tilde{u}_2^0 \). Notice that
\[
E \subset \{ (\tau_0, \xi_0) : |\xi_0| \leq N_0, \theta(\xi_0, \omega_0) < \gamma_0, \tau_0 = \pm_1 (\xi_0 - \xi_2) + r_2 + O(L_1) \},
\]
and hence \( |E| \lesssim N_0 N_1 \gamma_0 L_1 \leq N_0 N_1 \gamma_0 L_1 \). Further observing that \( |A_1| \approx N_1 \cdot N_1 \gamma_0 L_1 \), we have by Lemma \( 3.1 \) that
\[
\| \mathcal{F} \mathcal{P}_{N_2}(u_1^{\gamma_0} \omega_1 u_0^{\gamma_0}) \|_{L^r} \lesssim (N_0 N_1 \gamma_0 L_1) \frac{1}{2} (N_2^2 \gamma_0 L_1)^{\frac{1}{2} - \frac{1}{p}} \| u_1^{\gamma_0} \|_{L^p} \| u_0^{\gamma_0} \|_{L^r},
\]
and hence,
\[
\Sigma_1 \lesssim \frac{N_0^\frac{1}{p} N_1^{2 - \frac{1}{p}} \gamma_0^\frac{1}{2} L_1^\frac{1}{2}}{p} \sum_{\omega, \omega_1 \in \Omega(\gamma_0)} \chi_{\theta(\omega, \omega_1) \leq \gamma_0} \| u_1^{\gamma_0} \|_{L^p} \| u_0^{\gamma_0} \|_{L^r},
\]
where in the last step we used Hölder’s inequality, relying on the almost orthogonality condition (see \cite{21} (10)).

Finally, using \( \gamma_0 = (L_2/N_1)^{\frac{1}{q}} \) gives the desired estimate.

For \( \Sigma_2 \) we again estimate the volume of the set \( E = \hat{K}_{N_0} \cap (X_2 + A_1) \), using now the fact that \( 3 \gamma \leq \theta(\omega_1, \omega_0) \leq 12 \gamma \). First observe that
\[
E = \{ (\tau_0, \xi_0) : \xi_0 \in R, \tau_0 = \tau_2 \pm_1 |\xi_0 - \xi_2| + O(L_1) \},
\]
where
\[
R = \{ \xi_0 \in \mathbb{R}^2 : |\xi_0| \lesssim N_0, \xi_0 \in \Gamma(\omega_0), \pm_1 (\xi_0 - \xi_2) \in \Gamma(\omega_1) \}.
\]
Then for a fixed \( \tau_0 \),
\[
f(\xi_0) := \pm_1 |\xi_0 - \xi_2| = \tau_0 - \tau_2 + O(L_1),
\]
and we define
\[
e = \nabla f(\xi_0) = \pm_1 \frac{\xi_0 - \xi_2}{|\xi_0 - \xi_2|}.
\]
Choosing coordinates \( (\xi_0^1, \xi_0^2) \), so that \( \frac{\xi_0^1 - \xi_0^2}{|\xi_0^1 - \xi_0^2|} = (1, 0) \), we have for all \( \xi_0 \in E \),
\[
\partial_1 f(\xi_0) = e \cdot \frac{\xi_0^1 - \xi_0^2}{|\xi_0^1 - \xi_0^2|} = \frac{\cos \theta(e, \omega_1) - \cos \theta(e, \omega_0)}{|\omega_1 - \omega_0|}.
\]
Noticing that \( \theta(e, \omega_1) \leq \gamma \), the fact that \( \theta(\omega_1, \omega_0) \geq 3\gamma \) implies that \( \theta(e, \omega_0) \geq 2\gamma \). Then from the monotonic decreasing property of the cosine function for small angles,

\[
\partial_1 f(\xi_0) \geq \frac{\cos \gamma - \cos 2\gamma}{|\omega_1 - \omega_0|} \geq \frac{\gamma^2}{|\omega_1 - \omega_0|} \sim \gamma,
\]

where we used the expansion of \( \cos \gamma \) around \( \gamma = 0 \) and that \( |\omega_1 - \omega_0| \sim \gamma \). Then integrating first in the \( \xi_0^\eta \) direction, we obtain

\[
|E| \lesssim |\{\xi_0 \in R : \tau_0 \in T, f(\xi_0) = \tau_0 - \tau_2 + O(L_1)\}| |T|
\]

\[
\lesssim \frac{L_1}{\gamma} |\{\xi_0^\eta : |\xi_0^\eta - \xi_0^\gamma| = \pm 1(\tau_0 - \tau_2) + O(L_1)\}| |T| \lesssim \frac{L_2}{\gamma} L_1 N_0,
\]

where \( T = \{\tau_0 : \tau_0 = \tau_2 \pm 1 |\xi_0^\gamma| + O(N_0) + O(L_1)\} \).

Using this estimate for the volume of \( E \), and that \( |A_1| \lesssim N_0^2 \gamma L_1 \), as before, we have from Lemma 3.1

\[
\|\mathcal{F}P_{N_0}(u_1^{\gamma, 0} u_2^{\omega, 0})\|_{L^r} \lesssim \left( \frac{N_0 L_1 L_2}{\gamma} \right)^{\frac{1}{p}} (N_0^2 \gamma L_1)^{\frac{1}{p} - \frac{1}{r}} \|u_1^{\gamma, 0}\|_{L^r} \|\tilde{u}_0^{\omega, 0}\|_{L^r}.
\]

Hence,

\[
\Sigma_2 \lesssim N_0^\frac{1}{p} L_1^\frac{1}{p} N_1^{\frac{2}{p} - \frac{1}{p}} L_2^{\frac{1}{p}} \sum_{\gamma_0 < \gamma < 1} \sum_{\omega_1, \omega_2 \in \Omega(\gamma)} \chi_{3\gamma \leq \theta(\omega_1, \omega_0) \leq 12\gamma} \|u_1^{\gamma, 0}\|_{L^r} \|\tilde{u}_0^{\omega, 0}\|_{L^r}
\]

\[
\lesssim N_0^\frac{1}{p} L_1^\frac{1}{p} N_1^{\frac{2}{p} - \frac{1}{p}} L_2^{\frac{1}{p}} \gamma_0^{\frac{1}{p} - \frac{2}{p}} \|u_1\|_{L^r} \|\tilde{u}_0\|_{L^r},
\]

where we estimated the sum in \( \omega_1, \omega_2 \) as in the case of \( \Sigma_1 \), and summed in \( \gamma \) using the restriction \( r > \frac{2}{\gamma} \), which guarantees that the exponent is negative. But \( \gamma_0 = (L_2/N_1)^{\frac{1}{2}} \), so \( \gamma_0^{\frac{1}{2} - \frac{2}{p}} = L_2^{\frac{1}{2} - \frac{2}{p}} N_1^{\frac{1}{2} - \frac{2}{p}} \), and, thus,

\[
\Sigma_2 \lesssim N_0^\frac{1}{p} N_1^{\frac{2}{p} - \frac{1}{p}} L_1^{\frac{1}{p}} L_2^{\frac{1}{p}} \|u_1\|_{L^r} \|\tilde{u}_0\|_{L^r}.
\]

\[\square\]

4. The proof of estimate \( \square \)

We start by establishing some simple summation lemmas.

**Lemma 4.1.** Let \( A, B \in \mathbb{R} \), then

\[
\sum_{N_0 \lesssim N_1} N_0^A \lesssim N_1^B,
\]

provided (i) \( B \geq A \), (ii) \( B \geq 0 \) and (iii) \( A = B = 0 \) is excluded.

**Proof.** The statement of the lemma follows immediately from the trivial dyadic summation rule

\[
\sum_{N_0} \chi_{N_0 \leq N_1} N_0^A \sim \begin{cases} N_1^A & \text{if } A > 0, \\ \log(N_1) & \text{if } A = 0, \\ 1 & \text{if } A < 0. \end{cases}
\]

\[\square\]

The next two dyadic summation lemmas will be used for estimating HLH (and by symmetry, HHL) and LHH interactions respectively.
Lemma 4.2. Let \( A, B \in \mathbb{R} \). The estimate
\[
\sum_{N} \chi_{N_1 \leq N_0 \sim N_2} \frac{N_A}{N_B} \| F_0^{N_0} \|_{L^r} \| F_1^{N_1} \|_{L^p} \| F_2^{N_2} \|_{L^p} \lesssim \| F_0 \|_{L^r} \| F_1 \|_{L^p} \| F_2 \|_{L^p}
\]
holds, provided that (i) \( B \geq A \), (ii) \( B \geq 0 \) and (iii) \( A = B = 0 \) is excluded.

Proof. The proof of this lemma is contained in the proof of \([1, \text{Lemma 2.1}]\), where, after applying Lemma 4.1, one simply needs to use the duality of \( L^p \) and \( L^r \) (Hölder’s inequality for \( L^p \)), instead of the Cauchy-Schwartz inequality.

\[\square\]

Lemma 4.3. Let \( A, B \in \mathbb{R} \). The estimate
\[
\sum_{N} \chi_{N_0 \leq N_1 \sim N_2} \frac{N_A}{N_B} \| F_0^{N_0} \|_{L^r} \| F_1^{N_1} \|_{L^p} \| F_2^{N_2} \|_{L^p} \lesssim \| F_0 \|_{L^r} \| F_1 \|_{L^p} \| F_2 \|_{L^p}
\]
holds, provided that (i) \( B > A \), (ii) \( B > 0 \).

Proof. The proof is similar to the proof of the previous lemma, with the only difference in that one cannot sum in \( N_1 \sim N_2 \) using Hölder’s inequality, since \( L^p \) is not self-dual. But due to the strict inequalities in the hypothesis, we have by Lemma 4.1
\[
\sum_{N} \chi_{N_0 \leq N_1 \sim N_2} \frac{N_A}{N_B} \| F_0^{N_0} \|_{L^r} \| F_1^{N_1} \|_{L^p} \| F_2^{N_2} \|_{L^p} \lesssim \| F_0 \|_{L^r} \| F_1 \|_{L^p} \sum_{N_1 \sim N_2} N_1^{-r} \| F_1^{N_1} \|_{L^p} \| F_2^{N_2} \|_{L^p}
\]
\[
\lesssim \| F_0 \|_{L^r} \| F_1 \|_{L^p} \left( \frac{N_1}{L_1} \right) \left( \frac{N_2}{L_2} \right) \| F_2^{N_2} \|_{L^p}
\]
\[
\lesssim \| F_0 \|_{L^r} \| F_1 \|_{L^p} \| F_2 \|_{L^p}
\]
\[\square\]

We are now ready to prove the key estimate (8).

Proof of estimate (8). It is enough to prove the dual estimate (10). We will use dyadic decompositions, and by symmetry can assume \( L_1 \leq L_2 \). Estimate (10) then reduces to showing
\[
\sum_{N} \frac{S_N}{N_0^{-r} N_1^r N_2^s} \lesssim \| F_0 \|_{L^r} \| F_1 \|_{L^p} \| F_2 \|_{L^p},
\]
where
\[
S_N = \sum_{L} \chi_{L_1 \leq L_2} \frac{J(F_0^{N_0}, F_1^{N_1, L_1}, F_2^{N_2, L_2})}{L_1^b L_2^b},
\]
with \( J \) again defined as in (11). By symmetry, we only need to consider the HLH and LHH cases.

- The HLH case: \( N_1 \lesssim N_0 \sim N_2 \). From (15), (16) and the dual formulation we know that the estimate
\[
J(F_0^{N_0}, F_1^{N_1, L_1}, F_2^{N_2, L_2}) \leq C \| F_0^{N_0} \|_{L^r} \| F_1^{N_1, L_1} \|_{L^p} \| F_2^{N_2, L_2} \|_{L^p}
\]
holds with
\[
C \sim N_1^{\frac{3}{4}} L_1^{\frac{3}{4}} \left[ \min(N_1, L_2) \right]^{\frac{1}{4}}.
\]
Hence, we split into cases \( L_2 \leq N_1 \) and \( L_2 > N_1 \). In the first case (16) will be used, while the second case requires (15).
• The subcase $L_2 \leq N_1$. In this case using (23) gives
\[
S_N \lesssim N_1^{\frac{3}{2}} \sum_L L_1^{\frac{1}{r}-b} L_2^{\frac{1}{r}-b} \|F_0\|_{L^r} \|F_1^{N_1}\|_{L^p} \|F_2^{N_2}\|_{L^p}
\lesssim N_1^{\frac{3}{2\sigma}} \|F_0\|_{L^r} \|F_1^{N_1}\|_{L^p} \|F_2^{N_2}\|_{L^p},
\]
as $b > \frac{1}{r}$. But then by Lemma 4.2
\[
\sum_N \frac{S_N}{N_0^{-\sigma} N_1^{\frac{3}{2\sigma}} N_2^\sigma} \lesssim \sum_N \chi_{N_1 \leq N_0 \approx N_2} \frac{N_1^{\frac{3}{2}-\sigma}}{N_0^{1-\sigma}} \|F_0\|_{L^r} \|F_1^{N_1}\|_{L^p} \|F_2^{N_2}\|_{L^p}
\lesssim \|F_0\|_{L^r} \|F_1\|_{L^p} \|F_2\|_{L^p},
\]
as $\sigma > \frac{3}{2r}$.

• The subcase $L_2 > N_1$. In this case (23) reads $C \sim N_1^{\frac{3}{2}} L_1^\frac{1}{r}$, hence,
\[
S_N \lesssim N_1^{\frac{3}{2}} \sum_L \chi_{L_2 > N_1} L_1^{\frac{1}{r}-b} L_2^{\frac{1}{r}-b} \|F_0\|_{L^r} \|F_1^{N_1}\|_{L^p} \|F_2^{N_2}\|_{L^p}
\lesssim N_1^{\frac{3}{2}-b} \|F_0\|_{L^r} \|F_1^{N_1}\|_{L^p} \|F_2^{N_2}\|_{L^p}.
\]

Then by Lemma 4.2 gives,
\[
\sum_N \frac{S_N}{N_0^{-\sigma} N_1^{\frac{3}{2\sigma}} N_2^\sigma} \lesssim \sum_N \chi_{N_1 \leq N_0 \approx N_2} N_1^{\frac{3}{2}-b-\sigma} \|F_0\|_{L^r} \|F_1^{N_1}\|_{L^p} \|F_2^{N_2}\|_{L^p}
\lesssim \|F_0\|_{L^r} \|F_1\|_{L^p} \|F_2\|_{L^p},
\]
as $\sigma > \frac{3}{2r}$ and $b > \frac{1}{r}$.

Notice that this subcase shows that estimate (15) will only establish (8) for $\sigma > \frac{2}{r}$, which is above the best known result, and, in fact, coincides with what one can prove with classical energy estimates.

• The LHH case: $N_0 \lesssim N_1 \sim N_2$. From (15), (16) we know that (22) holds with
\[
(24) \quad C \sim N_0^{\frac{1}{r}} N_1^{\frac{3}{2r}-\frac{1}{p}} L_1^{\frac{1}{r}} \min(N_0^{\frac{1}{r}} N_1^{\frac{3}{2r}-\frac{1}{p}}, L_2^{\frac{1}{r}}).
\]

Hence, we split into cases $L_2 \leq N_0^{\frac{2}{r}}/N_1^{\frac{2}{r}-1}$ and $L_2 > N_0^{\frac{2}{r}}/N_1^{\frac{2}{r}-1}$. Again, we will use (16) in the first case, while the second case requires (15).

• The subcase $L_2 \leq N_0^{\frac{2}{r}}/N_1^{\frac{2}{r}-1}$. In this case (24) becomes $C \sim N_0^{\frac{1}{r}} N_1^{\frac{3}{2r}-\frac{1}{p}} L_1^{\frac{1}{r}} L_2^{\frac{1}{r}}$, and we have
\[
S_N \lesssim N_0^{\frac{3}{2}} N_1^{\frac{3}{2}-b} \sum_L L_1^{\frac{1}{r}-b} L_2^{\frac{1}{r}-b} \|F_0\|_{L^r} \|F_1^{N_1}\|_{L^p} \|F_2^{N_2}\|_{L^p}
\lesssim N_0^{\frac{3}{2}} N_1^{\frac{3}{2}-b} \|F_0\|_{L^r} \|F_1^{N_1}\|_{L^p} \|F_2^{N_2}\|_{L^p}.
\]

Lemma 4.3 then gives
\[
\sum_N \frac{S_N}{N_0^{-\sigma} N_1^{\frac{3}{2\sigma}} N_2^\sigma} \lesssim \sum_N \chi_{N_1 \leq N_0 \approx N_2} \frac{N_0^{\frac{1}{r}+\sigma}}{N_1^{2\sigma\frac{1}{2r}+(\frac{1}{p})}} \|F_0\|_{L^r} \|F_1^{N_1}\|_{L^p} \|F_2^{N_2}\|_{L^p}
\lesssim \|F_0\|_{L^r} \|F_1\|_{L^p} \|F_2\|_{L^p},
\]
since $\sigma > \frac{3}{2r}$ implies that $2\sigma > \frac{3}{2r} - \frac{1}{p}$, and $2\sigma - \frac{3}{2r} + \frac{1}{p} > \sigma + \frac{1}{p}$. 
• The subcase $L_2 > N_0^{2r}/N_1$. Now (24) becomes $C \sim N_0^{\frac{2}{r}} N_1^{\frac{2}{p} - \frac{2}{r}} L_1^b$, and

$$S_N \lesssim N_0^{\frac{2}{p}} N_1^{\frac{2}{p} - \frac{2}{r}} \sum L \chi_{L_2 > N_0^{\frac{2}{r}}/N_1^r} \frac{2r}{p - (2r - 1)b} \| L \|_1 L_2^{1-b} \| L \|_1 \| L \|_1 \| L \|_p \| F_0 \|_1 \| L \|_1 \| L \|_1 \| L \|_p$$

$$\lesssim N_0^{\frac{2}{p}} N_1^{\frac{2}{p} - \frac{2}{r} + (\frac{2r}{p} - 1)b} \| F_0 \|_1 \| L \|_1 \| L \|_1 \| L \|_p \| F_2 \|_1 \| L \|_p \| F_2 \|_1 \| L \|_p.$$

Lemma 4.3 then gives

$$\sum \frac{S_N}{N_0^{\sigma} N_1^{\sigma} N_2^{\sigma}} \lesssim \sum \chi_{N_0 \leq N_1} \frac{N_0^{\sigma} N_1^{\sigma - 2r + \frac{2r}{p}}} {\| F_0 \|_1 \| L \|_1 \| L \|_1 \| L \|_p} \| F_1 \|_1 \| L \|_1 \| L \|_1 \| L \|_p \| F_2 \|_1 \| L \|_p,$$

as

$$2\sigma - \frac{2}{r} + \frac{2}{p} - \left(\frac{2p}{p} - 1\right) b = 2\sigma - \frac{2}{r} + 2 \left(1 - \frac{1}{p}\right) - 2r \left(1 - \frac{1}{p}\right) b + b$$

$$= 2\sigma - \frac{4}{p} + 2 - 2rb + 3b$$

$$\geq 2\sigma - \frac{4}{p} + 2 - 2rb > 0,$$

since $\sigma > \frac{3}{p}$, and we can chose $b$ to be as close to $\frac{1}{p}$ as we want, and the difference of the exponents of $N_1$ and $N_0$ is

$$\left(2\sigma - \frac{2}{p} + \frac{2}{p} - \left(\frac{2r}{p} - 1\right) b\right) - \left(\sigma - \frac{2}{p} - \frac{2r}{p} \right) = \sigma - \frac{2}{r} + b > 0.$$

□

**APPENDIX A. LOCAL WELL-POSEDNESS FOR SoboI LE data in $H^{\frac{7}{2}}$**

For simplicity, we will only sketch the proof for the equation

$$(25)$$

$\Box u = (Du)^2,$

where $D$ stands for the spatial gradient. To treat the general equation (1), which may contain time derivatives in the nonlinearity as well, one can apply the arguments below to the equation $\Box(\partial_t u) = \partial_t (Du)^2$ with appropriate initial data, for which the estimates can be closed at one derivative less regularity. This, along with the presented argument, will allow one to close the estimates for $\partial u$ in $H^{\frac{7}{2}}$.

We start by stating the Strichartz estimates for the wave equation. For proofs of these estimates see (11) or (5).

A pair of indices $(p, q)$ is called wave-admissible in dimension $n$, if it belongs to the set

$$A = \left\{ (p, q) : 2 \leq p \leq \infty, 2 \leq q \leq \infty, \frac{2}{p} + \frac{n-1}{q} \leq \frac{n-1}{2} \right\}.$$

We will use the notation $p'$ for the Lebesgue conjugate of the index $p$, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

**Theorem A.1** (Strichartz estimates). If $(q, r)$ and $(\tilde{q}, \tilde{r})$ are wave-admissible, then

$$\| D^s u \|_{L_t^p L_x^q} \lesssim \| (u_0, u_1) \|_{H^{2} \times H^{1}} + \| D^{\tilde{q}} \Box u \|_{L_t^{\tilde{p}} L_x^{\tilde{q}}},$$

provided the following scaling condition holds

$$\frac{1}{q} + \frac{n}{r} - s = \frac{n}{2} - \gamma = \frac{1}{q'} + \frac{n}{r'} - 2 - \tilde{s}.$$
The pair \((p, q) = (4, \infty)\) barely fails to be wave-admissible in dimension \(n = 2\). However, to keep the notation simple, we will assume that it is wave-admissible, and proceed with the estimates using this pair. For the proper proof one can take for example \((p, q) = \left( \frac{12}{11}, \frac{8}{3} \right)\) for a small \(\epsilon > 0\), which will give \(s = \frac{1}{4} + \epsilon\), and the estimates will close in \(H^{\frac{7}{4} + \epsilon}\) instead of \(H^{\frac{5}{4}}\).

For the following choice of indices in dimension \(n = 2\),
\[
p = 4, q = \infty, s = \frac{1}{4}, \quad \gamma = 1; \quad \vec{p}' = 1, \vec{q}' = 2, \vec{s} = 0,
\]
for which the scaling condition is satisfied, one gets the estimate
\[
||D^\frac{1}{4} u||_{L^4_t L^\infty_x} \lesssim ||(u_0, u_1)||_{H^\frac{7}{4} \times L^2} + ||\Box u||_{L^1_t L^2_x}.
\]
Applying this estimate to \(D^\frac{3}{4} u\) leads to (26)
\[
||Du||_{L^4_t L^\infty_x} \lesssim ||(u_0, u_1)||_{H^{\frac{7}{4}} \times H^{\frac{3}{4}}} + ||\Box u||_{L^1_t H^{\frac{3}{4}}_x}.
\]
On the other hand, by the classical energy inequality for \((25)\) in \(S_T = [0, T) \times \mathbb{R}^2\),
\[
||Du(t, \cdot)||_{H^{\frac{3}{4}}} \lesssim ||u(0, \cdot)||_{H^{\frac{7}{4}}} + ||\partial_t u(0, \cdot)||_{H^{\frac{7}{4}}} + \int_0^T ||(DuDu)(t', \cdot)||_{H^{\frac{3}{4}}} dt'.
\]
But, by Hölder’s inequality, the fractional Leibniz rule, and estimate (26),
\[
\int_0^T ||(DuDu)(t', \cdot)||_{H^{\frac{3}{4}}} dt' \leq ||u||_{L^1_t H^{\frac{3}{4}}_x} \lesssim T^{\frac{3}{4}} ||Du||_{L^4_t L^\infty_x} ||Du||_{L^\infty_t H^{\frac{3}{4}}_x} \lesssim T^{\frac{3}{4}} ||(u_0, u_1)||_{H^{\frac{7}{4}} \times H^{\frac{3}{4}}} + ||\Box u||_{L^1_t H^{\frac{3}{4}}_x} ||Du||_{L^\infty_t H^{\frac{3}{4}}_x}.
\]
Taking \(T\) small, the last estimate combined with the previous energy inequality is enough to show the LWP for \((25)\) with data in \(H^{\frac{7}{4}} \times H^{\frac{3}{4}}\) using the Pickard’s iteration method. For the proper proof for a wave admissible pair \((p, q)\), one would additionally need to use the Sobolev’s inequality for the spatial norm in the previous to last estimate in order to place \(Du\) in \(L^2_t\).

**Appendix B. The General Well-Posedness Theorem**

Below we state the general local well-posedness theorem for nonlinear wave equations with data in the Fourier-Lebesgue space \(H^s \times \hat{H}^s_{-1}\). We only include the main idea of the proof here, as, with minor differences, it closely follows the proof of the analogous theorem for \(L^2\) based spaces. For the proof in the \(L^2\) case we refer the reader to [19, Theorem 4.1] (see also [13, Theorem 5.3]).

Let us consider the Cauchy problem
\[
(27) \quad \Box u = F(u, \partial u), \quad (t, x) \in \mathbb{R}^{1+n},
\]
\[
(28) \quad (u, \partial_t u)|_{t=0} = (f, g) \in \hat{H}^s \times \hat{H}^s_{-1},
\]
where \(\partial u\) stands for the space-time gradient of \(u\), and \(F\) is a smooth function with \(F(0) = 0\).

**Theorem B.1** (c.f Theorem 14 in [19]). Assume \(s \in \mathbb{R}, \frac{1}{r} < b < 1, \epsilon \in (0, 1 - b)\). If
\[
||F(u, \partial u)||_{X^s_{-1, b; \epsilon, -1}} \leq A_\sigma(||u||_{Z^s_{1, b}}) ||u||_{Z^s_{1, b}}
\]
for all $\sigma \geq s$, and
\[
\|F(u, \partial_1u) - F(v, \partial_1v)\|\leq A_s(\|u\|_s + \|v\|_s)\|u - v\|_s,
\]
for all $u, v \in Z_{s,b}^r$, where $A_s : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing and locally Lipschitz for every $\sigma \geq s$, then

- (existence) There exists $u \in Z_{s,b}^r$, which solves \( (\ref{eq:27}) \) \( \cdot \) \( \ref{eq:28} \) on \([0,T] \times \mathbb{R}^n\), where $T = T(\|f\|_{\dot{H}^s_x} + \|g\|_{\dot{H}^s_x}) > 0$ depends continuously on $\|f\|_{\dot{H}^s_x} + \|g\|_{\dot{H}^s_x}$.
- (uniqueness) The solution is unique in the class $Z_{s,b}^r$, i.e., if $u, v \in Z_{s,b}^r$ both solve \( (\ref{eq:27}) \) \( \cdot \) \( \ref{eq:28} \) on \([0,T] \times \mathbb{R}^n\) for some $T > 0$, then
\[
u(t) = v(t) \quad \text{for } t \in [0,T],
\]
- (Lipschitz) The solution map
\[
(f, g) \mapsto u, \quad \dot{H}^r_s \times \dot{H}^r_{s-1} \to Z_{s,b}^r
\]
is locally Lipschitz.
- (higher regularity) If the data has higher regularity
\[
f \in \dot{H}^r_s, g \in \dot{H}^r_{s-1} \quad \text{for } \sigma > s,
\]
then $u \in C([0,T], \dot{H}_{s,b}^r) \cap C^1([0,T], \dot{H}_{s-1,b}^r)$ for any $T > 0$ for which $u$ solves \( (\ref{eq:27}) \) \( \cdot \) \( \ref{eq:28} \).

In particular, if $(f, g) \in S$, then $u \in C^\infty([0,T] \times \mathbb{R}^n)$.

The proof of the general LWP theorem relies on a fixed point argument by showing that the solution $\Phi$, given by
\[
\Phi u(t) = \cos(tD) \cdot f + D^{-1} \sin(tD) g + \int_0^t D^{-1} \sin((t - t')D) \cdot F(u, \partial_1u) dt',
\]
is a contraction of some closed ball in $Z_{s,b,T}^r$ for appropriately chosen $T$. This is achieved through the following estimate for the solution of the inhomogeneous equation $\Box u = F(t,x)$ with initial data $(f, g)$,
\[
\|\Phi u\|_{Z_{s,b}^r} \leq C \left( \|f\|_{\dot{H}^r_x} + \|g\|_{\dot{H}^r_{s-1}} + T^{s/2}\|F\|_{X_{s-1,b+1}^{r}} \right).
\]
The above estimate exhibits a time factor in front of the inhomogeneity, if one is allowed to place $F$ in $X_{s-1,b+1}^{r}$, instead of the natural $X_{s,b-1}^{r}$. Exploiting the time factor in this estimate by choosing $T$ small, and relying on the estimates in the theorem, one can show that $\Phi$ is a contraction of some closed ball in the space $Z_{s,b,T}^r$, and hence has a fixed point.

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1 DEPARTMENT OF MATHEMATICS, OCCIDENTAL COLLEGE, 1600 CAMPUS ROAD, LOS ANGELES, CALIFORNIA 90041

E-mail address: vgrigoryan@oxy.edu

2 MATHEMATISCHES INSTITUT, UNIVERSITÄT TÜBINGEN, AUF DER MORGENSTELLE 1072076 TÜBINGEN

E-mail address: ajtanguay@gmail.com