Localization of hidden Chua attractors by the describing function method

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Abstract: In this paper the Chua circuit with five linear elements and saturation non-linearity is studied. Numerical localization of self-excited attractor in the Chua circuit model can be done by computation of trajectory with initial data in a vicinity of an unstable equilibrium. For a hidden attractor its basin of attraction does not overlap with a small vicinity of equilibria, so it is difficult to find the corresponding initial data for localization. This survey is devoted to the application of describing function method for localization of hidden periodic and chaotic attractors in the Chua model. We use a rigorous justification of the describing function method, based on the method of small parameter, to get the initial data for the visualization of the hidden attractors. A new configuration of hidden Chua attractors is presented.

Keywords: Chua circuit, hidden attractor, self-excited attractor, describing function method

1. INTRODUCTION

In the initial period of the development of the theory of nonlinear oscillations (first half of the XX century) main attention of researchers was paid to analysis and synthesis of oscillating systems for which the oscillation existence problem can be solved relatively easily. The structure of many applied systems considered was such that the existence of oscillations was "almost obvious" - the oscillation was excited from an unstable equilibrium (so called self-excited oscillation). From a computational point of view this allows one to use a standard computational procedure, in which after a transient process a trajectory, started from a point of unstable manifold in a neighborhood of equilibrium, reaches a state of oscillation, therefore one can easily identify it. The use of the term self-excited oscillation or self-oscillations can be traced back to the works of H.G. Barkhausen and A.A. Andronov, where it describes the generation and maintenance of a periodic motion in mechanical and electrical models by a source of power that lacks any corresponding periodicity (e.g., a stable limit cycle in the van der Pol oscillator) (Andronov et al., 1966; Jenkins, 2013).

Attractor is called a self-excited attractor if its basin of attraction intersects any arbitrarily small open neighborhood of an equilibrium, otherwise it is called a hidden attractor (Leonov et al., 2011, 2012; Leonov and Kuznetsov, 2013; Leonov et al., 2015; Kuznetsov, 2016). We use the notion “self-excited” for attractors of dynamical systems to describe the existence of transient process from a small vicinity of an unstable equilibrium to an attractor.

If there is no such a transient process for an attractor, it is called a hidden attractor. For example, hidden attractors are attractors in systems without equilibria or with only one stable equilibrium (a special case of multistability and coexistence of attractors). Some examples of hidden attractors can be found in Shahzad et al. (2015); Brezetskyi et al. (2015); Jafari et al. (2015); Zhusubaliyev et al. (2015); Saha et al. (2015); Semenov et al. (2015); Feng and Wei (2015); Li et al. (2015); Feng et al. (2015); Sprott (2015); Pham et al. (2015); Vaidyanathan et al. (2015); Danca (2016); Zelinka (2016); Dudkowski et al. (2016); Kuznetsov et al. (2017); Danca et al. (2017); Kiseleva et al. (2016).

The self-excited and hidden classification of attractors was introduced by Leonov and Kuznetsov in connection with the discovery of hidden chaotic attractor in the Chua system (Kuznetsov et al., 2010; Leonov et al., 2011; Kuznetsov et al., 2013):

\[
\begin{align*}
\dot{x} &= \alpha(y - x(m_1 + 1)) - \alpha \psi(x), \\
\dot{y} &= x - y + z, \\
\dot{z} &= -(\beta y + \gamma z), \\
\psi(x) &= (m_0 - m_1) \text{sat}(x) = \\
&= \frac{1}{2}(m_0 - m_1)(|x + 1| - |x - 1|),
\end{align*}
\]

(1)

where $\alpha$, $\beta$, $\gamma$, $m_0$, $m_1$ are parameters. This system provides a mathematical model, describing the behavior of the Chua circuit (Chua and Lin, 1990; Chua, 1992, 1995) with five linear elements and saturation non-linearity (see Fig. 1). Until this discovery only self-excited chaotic attractors had been found in Chua circuits (see Fig. 2...
and, e.g. works (Matsumoto, 1990; Lozi and Ushiki, 1993; Bilotta and Pantano, 2008)). Note that L. Chua himself, analyzing various cases of attractors existence in Chua circuit, does not admit the existence of hidden attractor in his circuits (Chua, 1992).

We consider only one type of the Chua circuits, while there are known various modifications of Chua circuit (see, e.g. (Banerjee, 2012; Semenov et al., 2015)) where hidden oscillations can also be localized (Chen et al., 2015b; Bao et al., 2015; Chen et al., 2015a; Menacer et al., 2016).

Fig. 1. Chua circuit with two resistors, one inductor, two capacitors (red) and one nonlinear resistor called “Chua diode” (green).

Fig. 2. Self-excited attractors in Chua system (1): (2a) – two symmetric spiral attractors, (2b) – two symmetric Rössler-like attractors.

2. HIDDEN ATTRAJECTORS LOCALIZATION VIA DESCRIBING FUNCTION METHOD

In this section an effective analytical-numerical approach for hidden oscillations localization, based on the describing function method (DFM), the method of small parameter and continuation method, is demonstrated.

2.1 Describing function method

The describing function method (DFM) is a searching method for oscillations which are close to the harmonic periodic oscillations of nonlinear systems of automatic control. This method is not strictly mathematically justified and is one of approximate methods of analysis of control systems (see, e.g. (Krylov and Bogolyubov, 1947; Khalil, 2002)). One of the first examples, where the describing function method gives untrue results, is due to Tsypkin (1984). Remark that well-known Aizerman’s and Kalman’s conjectures on the absolute stability of nonlinear control systems are valid from the standpoint of the describing function method (what explains why these conjectures were put forward). Nowadays various counterexamples to these conjectures (nonlinear systems where the only equilibrium, which is stable, coexists with a hidden periodic oscillation) are known (see, e.g. Pliss (1958); Fitts (1966); Barabanov (1988); Bernat and Llibre (1996); Leonov et al. (2010); Bragin et al. (2011); Leonov and Kuznetsov (2011, 2013); the corresponding discrete examples are considered in Alli-Oke et al. (2012); Heath et al. (2015)).

Let us recall a classical way of applying the DFM. Consider a system with one scalar non-linearity in the Lur’e form

\[
\frac{dx}{dt} = P_0 x + q^* x, \quad x \in \mathbb{R}^n
\]

where \( P \) is a constant \((n \times n)\)-matrix, \( q, r \) are constant \(n\)-dimensional vectors, * denotes transpose operation, \( q^* \) is a continuous piecewise-differentiable scalar function, and \( q(0) = 0 \).

In order to find a periodic oscillation, a certain coefficient of harmonic linearization \( k \) (assume that such \( k \) exists) is introduced in such a way that the matrix \( P_0 = P + kqr^* \) of the linear system

\[
\frac{dx}{dt} = P_0 x, \quad x \in \mathbb{R}^n
\]

has a pair of pure imaginary eigenvalues \( \pm i \omega_0 (\omega_0 > 0) \), and the rest eigenvalues have negative real parts.

Introduce a transfer function

\[
W(p) = r^* (P - pI)^{-1} q,
\]

where \( p \) is a complex variable, \( I \) is a unit matrix. Transfer function \( W(p) \) is applied to define the values of \( k \) and \( \omega_0 \). The number \( \omega_0 > 0 \) is defined from the equation

\[
\text{Im} W(i \omega_0) = 0
\]

and \( k \) is defined by the formula

\[
k = -\left( \text{Re} W(i \omega_0) \right)^{-1}.
\]

If such \( \omega_0 \) and \( k \) exist, then system (2) has a periodic solution \( x(t) \) for which

\[
\sigma(t) = r^* x(t) \approx a \cos \omega_0 t.
\]

Following the DFM, the amplitude \( a \) can be obtained from the equation

\[
\int_0^{2\pi/\omega_0} (\psi(a \cos \omega_0 t)a \cos \omega_0 t - k(a \cos \omega_0 t)^2) dt = 0.
\]

Rewrite system (2) as follows

\[
\frac{dx}{dt} = P_0 x + q^* x^0,
\]

where \( \varphi(\sigma) = \psi(\sigma) - k \sigma \). As it is mentioned above classical DFM is not strictly mathematically justified and can lead to untrue results, however for the systems with a small parameter it can be rigorously justified. For that let us change \( \varphi(\sigma) \) by \( \varepsilon \varphi(\sigma) \) and consider the existence of a periodic solution for system

\[
\frac{dx}{dt} = P_0 x + \varepsilon q^* x^0.
\]
To define the initial data $x^0(0)$ of the periodic solution, system (7) is transformed by a linear non-singular transformation $x = Sy$ to the form

$$
\begin{align*}
\dot{y}_1 &= -\omega_0 y_2 + \varepsilon b_1 \varphi(y_1 + c_1 y_3), \\
\dot{y}_2 &= -\omega_0 y_1 + \varepsilon b_2 \varphi(y_1 + c_1 y_3), \\
\dot{y}_3 &= A_3 y_3 + \varepsilon b_3 \varphi(y_1 + c_3 y_3),
\end{align*}
$$

(8)

where $y_1$, $y_2$ are scalars, $y_3$, $b_3$, and $c_3$ are $(n-2)$-dimensional vectors, $b_1$ and $b_2$ are real numbers; $A_3$ is a constant $(n-2) \times (n-2)$ matrix all eigenvalues of which have negative real parts. Without loss of generality, it can be assumed that for the matrix $A_3$ there exists a positive number $d > 0$, such that $y_3^* (A_3 + A_3^*) y_3 \leq -2d |y_3|^2$, $orall y_3 \in \mathbb{R}^{n-2}$.

Introduce the describing function

$$
\Phi(\alpha) = \int_0^{2\pi/\omega_0} \varphi(\cos(\omega_0 t) a) \cos(\omega_0 t) dt
$$

(9)

and assume the existence of its derivative.

**Theorem 1.** [Leonov and Kuznetsov (2013)] If there exists a positive number $a_0$ such that

$$
\Phi(a_0) = 0, \quad b_1 \frac{d\Phi(\alpha)}{d\alpha}\bigg|_{\alpha=a_0} < 0,
$$

(10)

then system (7) has a stable periodic solution with initial data

$$
x^0(0) = S \left( y_1(0), y_2(0), y_3(0) \right)^*,
$$

where $y_1(0) = a_0 + O(\varepsilon)$, $y_2(0) = 0$, $y_3 = O(\varepsilon)$ and with the period $T = \frac{2\pi}{\omega_0} + O(\varepsilon)$.

### 3. Hidden Attractors Localization in Chua Circuit via the Describing Function Method

In this section we apply the above approach for hidden attractors localization in Chua circuit. Let us write Chua system (1) in the Lur'e form (2) (see, e.g. Leonov et al., 2011) with

$$
\begin{align*}
P &= \begin{pmatrix}
-\alpha(m_1 + 1) & \alpha & 0 \\
1 & -1 & -1 \\
0 & 0 & -\beta - \gamma
\end{pmatrix}, & \quad Q &= \begin{pmatrix}
-\alpha \\
0 \\
0
\end{pmatrix}, \\
\mathbf{r} &= \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}, & \quad \psi(\sigma) &= (m_0 - m_1) \text{sat}(\sigma).
\end{align*}
$$

(11)

Introduce a coefficient $k$ and a small parameter $\varepsilon$, and represent (11) in the form (7) with

$$
P_0 = P + k \mathbf{r} \varphi^* = \begin{pmatrix}
-\alpha(m_1 + 1 + k) & \alpha & 0 \\
1 & -1 & -1 \\
0 & 0 & -\beta - \gamma
\end{pmatrix},
$$

(12)

$$
\varphi(\sigma) = \psi(\sigma) - k \sigma = (m_0 - m_1) \text{sat}(\sigma) - k \sigma,
$$

and $\lambda_{11}^{P_0} = -i \omega_0$, $\lambda_{12}^{P_0} = -d < 0$.

By the non-singular linear transformation $x = Sy$ system (12) is reduced to the form

$$
\frac{dy}{dt} = Ay + b \varepsilon \varphi(u^* y),
$$

(13)

where

$$
A = \begin{pmatrix}
0 & -\omega_0 & 0 \\
\omega_0 & 0 & 0 \\
0 & 0 & -d
\end{pmatrix}, \quad b = \begin{pmatrix}
b_1 \\
b_2 \\
1
\end{pmatrix}, \quad c = \begin{pmatrix}
1 \\
h
\end{pmatrix}.
$$

The transfer function $W_A(p)$ of system (13) can be represented as

$$
W_A(p) = \frac{-b_1 p + b_2 \omega_0}{p^2 + \omega_0^2} + \frac{h}{p + d}.
$$

Further, using the equality of transfer functions of systems (12) and (13) one can obtain

$$
W_A(p) = r^*(P_0 - p I)^{-1} q.
$$

This implies the following relations

$$
k = \frac{-\alpha (m_1 + m_1 \gamma + \gamma) + \omega_0^2 - \gamma - \beta}{\alpha (1 + \gamma)},
$$

$$
d = \frac{\alpha + \omega_0^2 - \beta + 1 + \gamma + \gamma^2}{1 + \gamma},
$$

$$
h = \frac{\alpha (\gamma + \beta - (1 + \gamma) d + d^2)}{\omega_0^2 + d^2},
$$

$$
b_1 = \frac{\alpha (\gamma + \beta - \omega_0^2 - (1 + \gamma) d)}{\omega_0^2 + d^2},
$$

$$
b_2 = \frac{\alpha (1 + \gamma - d) \omega_0^2 + (1 + \gamma) d)}{\omega_0 (\omega_0^2 + d^2)}.
$$

(14)

Since by the non-singular linear transformation $x = Sy$ system (12) can be reduced to the form (13) for the matrix $S$ the following relations

$$
A = S^{-1} P_0 S, \quad b = S^{-1} q, \quad c^* = r^* S.
$$

(15)

are valid. After solving these matrix equations, one can obtain the transformation matrix

$$
S = \begin{pmatrix}
s_{11} & s_{12} & s_{13} \\
s_{21} & s_{22} & s_{23} \\
s_{31} & s_{32} & s_{33}
\end{pmatrix},
$$

where

$$
s_{11} = 1, \quad s_{12} = 0, \quad s_{13} = -h,
$$

$$
s_{21} = m_1 + 1 + k, \quad s_{22} = \frac{-\omega_0}{\alpha},
$$

$$
s_{23} = -h \frac{(\alpha (m_1 + 1 + k) - d)}{\alpha},
$$

$$
s_{31} = \frac{\alpha (m_1 + 1 + k) - \omega_0^2}{\alpha},
$$

$$
s_{32} = -\alpha (\beta + \gamma) (m_1 + k) + \alpha \beta - \gamma \omega_0^2,
$$

$$
s_{33} = \frac{h \alpha (m_1 + k) (d - 1) + d (1 + \alpha d)}{\alpha}.
$$

(16)

Using Theorem 1 one obtains the initial data

$$
x(0) = Sy(0) = S \begin{pmatrix}
a_0 \\
0 \\
0
\end{pmatrix} = \begin{pmatrix}
a_0 s_{11} \\
a_0 s_{21} \\
a_0 s_{31}
\end{pmatrix}.
$$

(17)

Back to Chua system denotations, for the determination of the initial data of starting solution for multistage procedure, it can be obtained

$$
x(0) = a_0, \quad y(0) = a_0 (m_1 + 1 + k),
$$

$$
z(0) = a_0 \frac{\alpha (m_1 + k) - \omega_0^2}{\alpha}.
$$

\footnote{Such transformation exists for non-degenerate transfer functions.}
Consider system (1) with the parameters
\[ \alpha = 8.4562, \quad \beta = 12.0732, \quad \gamma = 0.0052, \quad m_0 = -0.1768, \quad m_1 = -1.1468. \]  
(18)
Note that for the considered values of parameters there are three equilibria in the system: the zero equilibrium \( F_0 = (0, 0, 0) \) is a stable focus-node and two symmetric equilibria
\[ S_{\pm} = \pm \frac{m_1 - m_0}{m_1 + \frac{\alpha}{\gamma + \beta}}, \quad (\gamma (m_1 - m_0) (\gamma + \beta) m_1 + \beta) \]
are saddle-foci with one-dimensional unstable manifolds.

Let us try to apply the DFM and define an initial data for periodic oscillation. Using (4) and (5) for parameters (18) one obtains following starting frequency and a coefficient of harmonic:
\[ \omega_0 = 2.0392, \quad k = 0.2098. \]  
(19)
Assuming \( a \geq 1 \), describing function (9) and its derivative for Chua system (1) can be rewritten as follows:
\[ \Phi(a) = 2(m_0 - m_1) \left[ \frac{\pi a}{2} - \frac{1}{a} \sqrt{1 - \frac{1}{a^2}} - a \arccos \frac{1}{a} \right] - \pi a k, \]
\[ \frac{d\Phi(a)}{da} = 2(m_0 - m_1) \left[ \frac{\pi}{2} - \frac{1}{a} \sqrt{1 - \frac{1}{a^2}} - \arccos \frac{1}{a} \right] - \pi k. \]

For parameters (18) and (19) one obtains initial amplitude \( a_0 = 5.8576 \) that satisfies the conditions of Theorem 1. Thus, by (17) initial data for the oscillation are as follows
\[ x(0) = 5.8576, \quad y(0) = 0.3694, \quad z(0) = -8.3686. \]  
(20)
In our numerical experiments we skip the multistep procedure based on the small parameter method and apply initial data (20) for hidden attractors localization in the initial system (i.e., system (1) in the form (6), \( \varepsilon = 1 \)). It turns out that in this case this is enough for localization of two symmetric hidden chaotic attractors \( A_{\text{hid}} \) in the Chua system (see Fig. 3). For attractor \( A_{\text{hid}} \) one should take symmetric initial data \( x(0) = -5.8576 \), \( y(0) = -0.3694 \), \( z(0) = 8.3686 \).

Consider system (1) with another values of the parameters
\[ \alpha = 8.4, \quad \beta = 12, \quad \gamma = -0.005, \quad m_0 = -1.2, \quad m_1 = -0.05. \]  
(21)
Note that for the considered values of parameters the zero equilibrium \( F_0 \) is a saddle-focus with one-dimensional unstable manifold and two symmetric equilibria \( S_{\pm} \) are stable focus-nodes. Again let us apply the DFM and define an initial data for periodic oscillation. Note that equation (4) for parameters (21) has two positive solutions and by (5) we obtain following starting frequencies and coefficients of harmonic:
\[ \omega_0 = 2.0260, \quad k = -0.8890 \]  
(22)
and
\[ \omega_0 = 3.2396, \quad k = -0.1244. \]  
(23)
For parameters (21) and (22) one obtains initial amplitude \( a_0 = 1.5187 \) that satisfies the conditions of Theorem 1. Thus, by (17) initial data for the oscillation are as follows
\[ x(0) = 1.5187, \quad y(0) = 0.0926, \quad z(0) = -2.1682. \]  
(24)
Using these initial data for original system (1) it is possible to localize two symmetric hidden chaotic attractors \( A_{\text{hid}} \) (see Fig 4).

CONCLUSIONS

In this paper we discuss the use of describing function method for searching periodic oscillations in its application to the famous Chua circuit. Despite the fact that DFM is an approximate analytical method (which does not guarantee the true results), the application of DFM to the Chua system allows us to localize hidden chaotic and periodic attractors. In particular, for certain values of parameters we obtain a new configuration of co-existing hidden attractors (two symmetric chaotic and stable limit cycle) in the Chua system.

4. ACKNOWLEDGMENTS

This work was supported by the Russian Science Foundation (14-21-00041).
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