Learning Strategies for Radar Clutter Classification

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Abstract—In this paper, we address the problem of classifying clutter returns in order to partition them into statistically homogeneous subsets. The classification procedure relies on a model for the observables including latent variables that is solved by the expectation-maximization algorithm. The derivations are carried out by accounting for three different cases for the structure of the clutter covariance matrix. A preliminary performance analysis highlights that the proposed technique is a viable means to cluster clutter returns over the range.

Index Terms—Clutter, Diagonal Loading, Expectation-Maximization, Heterogeneous Environment, Interference Classification, Radar.

I. INTRODUCTION

In the past ten years, improvements in digital architectures and miniaturization technologies have yielded a significant impact in the evolution of radar systems which, consequently, are being equipped with more and more reliable and sophisticated functions [1], [2]. This increase in computational resources has led the radar community to devise detection/estimation algorithms capable of facing with challenging scenarios and, more importantly, of capitalizing on specific a priori knowledge about either the system or the environment or both. In this context, a few examples related to the structural information about the interference covariance matrix are provided by [3]–[9], where, at the design stage, it is assumed that the system illuminates the surveillance area through a symmetrically spaced linear array of sensors. This assumption lends both the interference covariance matrix and the steering vector a special structure which yields interesting processing gains at the price of an additional computational load [10], [11].

Other approaches relying on a priori information exploit the possible symmetries in the interference spectral properties [5], [12], [13]. As a matter of fact, ground clutter returns collected by a monostatic steady radar experience a symmetric power spectral density centered around zero-Doppler frequency [14], [15]. Remarkably, such property allows to double data used to estimate the clutter covariance matrix. Therefore, the above knowledge-based strategies represent an effective means to deal with situations where the amount of training data, used for the estimation of the interference covariance matrix, is limited (sample-starved condition) otherwise leading to low-quality estimates and, consequently, to a detection performance degradation. Besides the mentioned approaches, other widely used techniques to come up with suitable estimates of the interference covariance matrix consist in the regularization (or shrinkage) of the sample covariance matrix towards a given matrix [16]–[18].

However, in practice, it is not seldom to meet situations where the presence of inhomogeneities makes the interference properties estimation an even more difficult task due to the fact that such outliers should be censored as proposed in [19]–[22]. In these contributions, suitable techniques to detect and suppress the outliers are devised in order to make the training set homogeneous. In fact, the homogeneity assumption for secondary data is a very common in detector design [23]–[26] and references therein] and when it is no longer valid the performance degradation might become severe [27]. A more complete approach to the problem of generating homogeneous training sets would envisage an additional architectural layout capable of integrating and fusing information coming from potential heterogeneous sources to depict a clear picture of the clutter properties. These sources can be internal or external to the system and comprise mapping data, communication links, tracker feedback, or other inputs [28]–[31].

Now, note that environment maps might be useful to identify clutter edges and to cluster data into homogeneous subsets, whose cardinality can be increased by exploiting a priori information about the clutter properties as described before. Thus, classifying (or, otherwise stated, clustering) clutter returns would represent a desirable feature for modern radar systems. Examples of clutter classifiers are provided by [32], [33], where the authors build up a neural network or process suitable features to distinguish between echoes from weather, birds, and aircrafts. Other classifiers are aimed at identifying the distribution for clutter data [34]–[36], the specific structure of the clutter covariance matrix [37], or the variability of clutter power over the range bins [38].

In this paper, we focus on the problem of partitioning training data into homogeneous subsets and we assume that only partial information about the environment is available at the radar receiver, namely that a given number of clutter boundaries is present. Then, we design a classification procedure capable of partitioning the secondary data set into subsets containing statistically homogeneous data. To this end, we jointly exploit the expectation-maximization (EM) algorithm [39] and the latent variable model [40]. The latter tool allows us to introduce hidden random variables which represent the classes, namely, uniform clutter regions, to which each range cell belongs. Thus, at the end of the procedure, the clustering is accomplished by estimating the a posteriori probability that
a range bin belongs to a specific class. More importantly, we consider three different models for the covariance matrix of the disturbance and more precisely the following:

- the disturbance of each class is characterized by its own Hermitian covariance matrix;
- different classes share a common structure of the covariance matrix, but they have different power values (clutter-dominated environment);
- noise returns consist of a thermal noise component (whose power is independent of the class) plus a clutter component; as in the previous case clutter returns share the same structure of the clutter covariance matrix, but each class is characterized by its own clutter power.

The preliminary performance analysis deals for the moment with the second model and shows the effectiveness of the proposed method in clustering data.

The remainder of the paper is organized as follows. The next section contains the problem formulation, whereas Section III is devoted to the design of the classification architectures. Illustrative examples and discussion about the classification performance are provided in Section IV. Finally, in Section V, we draw the conclusions and lay down possible future research lines. Derivations are confined to the Appendices.

A. Notation

In the sequel, vectors and matrices are denoted by boldface lower-case and upper-case letters, respectively. The $i$th entry of a vector $a$ is represented by $a(i)$ whereas symbols $\det(\cdot)$, $\text{Tr}(\cdot)$, $(\cdot)^T$, and $(\cdot)^\dagger$ denote the determinant, trace, transpose, and conjugate transpose, respectively. As to numerical sets, $\mathbb{N}$ is the set of natural numbers, $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^{N \times M}$ is the Euclidean space of $(N \times M)$-dimensional real matrices (or vectors if $M = 1$), $\mathbb{C}$ is the set of complex numbers, and $\mathbb{C}^{N \times M}$ is the Euclidean space of $(N \times M)$-dimensional complex matrices (or vectors if $M = 1$). $\mathbf{I}$ and $\mathbf{0}$ stand for the identity matrix and the null vector or matrix of proper size. Given $a_1, \ldots, a_N \in \mathbb{C}^{N \times 1}$, diag$(a_1, \ldots, a_N)$ $\in \mathbb{C}^{N \times N}$ indicates the diagonal matrix whose $i$th diagonal element is $a_i$. The acronym pdf and pmf stand for probability density function and probability mass function, respectively, whereas the conditional pdf of a random variable $x$ given another random variable $y$ is denoted by $f(x|y)$. Finally, we write $x \sim \mathcal{CN}(m, M)$ if $x$ is a complex circular $N$-dimensional normal vector with mean $m$ and positive definite covariance matrix $M$ while given a matrix $X = [x_1 \cdots x_M] \in \mathbb{C}^{N \times M}$, writing $X \sim \mathcal{CN}(m, M, I)$ means that $x_i \sim \mathcal{CN}(m, M)$, $i = 1, \ldots, M$, and the $x_i$s are statistically independent.

II. PROBLEM FORMULATION AND PRELIMINARY DEFINITIONS

Consider a radar system equipped with $N \geq 2$ space, time, or space-time channels which illuminates the operating area consisting of $K$ range bins. The signals backscattered by these range cells are suitably conditioned and sampled by the signal-processing unit to form $N$-dimensional complex vectors denoted by $z_1, \ldots, z_K$. Now, let us assume that, from a statistical point of view, the observed environment is temporally stationary, whereas its statistical properties may change over the range due, for instance, to the presence of clutter boundaries [11]. Otherwise stated, we assume that the set of vectors can be partitioned into $L$ subsets of statistically homogeneous data; the $l$th subset is denoted by

$$\Omega_l = \{z_{i_1}, \ldots, z_{i_{K_l}}\}$$

where $K_l$, $l = 1, \ldots, L$, denotes its cardinality. Thus, the elements of $\Omega_l$ share the same distributional parameters which are generally different from those associated to the distribution of $\Omega_m$, $m \neq l$. Specifically, we assume that

$$[z_{i_1}, \ldots, z_{i_{K_l}}] \sim \mathcal{CN}_N(\mathbf{0}, M_l, I), \quad l = 1, \ldots, L,$$

where $M_l$ is unknown.

Summarizing, we are interested in estimating the subsets $\Omega_l$ along with the associated unknown parameter $M_l$, $l = 1, \ldots, L$. To this end, in the next section we devise a classification procedure relying on the joint exploitation of the expectation maximization (EM) algorithm [39] and the latent variable model [40]. Moreover, besides the most general structure for the clutter covariance matrix, we consider two additional models which account for possible clutter power variations and diagonal loading due to thermal noise (a point better explained in the next section).

III. CLASSIFICATION ARCHITECTURE DESIGNS

Data classification task is accomplished by introducing $K$ independent and identically distributed discrete random variables, $c_k$s say, which take on values in $\{1, \ldots, L\}$ with unknown pmf

$$P(c_k = l) = p_l, \quad k = 1, \ldots, K,$$

and such that when $c_k = l$, then $z_{i_k} \sim \mathcal{CN}(\mathbf{0}, M_l)$. Under this assumption, it naturally follows that the pdf of $z_k$ can be written as

$$f(z_k; \theta) = \frac{1}{L} \sum_{l=1}^{L} p_l f(z_k|c_k = l; M_l) = E_{c_k} \left[ f(z_k|c_k; \theta) \right],$$

where $E_{c_k} [\cdot]$ denotes the statistical expectation with respect to $c_k$.

$$\theta = [\mathbf{p}^T, \mathbf{\sigma}^T]^T, \quad \mathbf{p} = [p_1 \cdots p_L]^T, \quad \mathbf{\sigma} = [\nu^T(M_1) \cdots \nu^T(M_L)]^T, \quad \nu(\cdot)$$

a vector-valued function selecting the generally distinct entries of the matrix argument, and

$$f(z_k|c_k = l; M_l) = \frac{1}{\pi^N \det(M_l)} \exp \left\{ -\text{Tr}[M_l^{-1} z_k z_k^\dagger] \right\}.$$  \footnote{Recall that $\sum_{l=1}^{L} p_l = 1.$}
since it is a simple iterative algorithm that provides closed-form updates for the parameter estimates at each step and reaches at least a local stationary point. To this end, let us write the joint log-likelihood of \( Z = [z_1 \cdots z_K] \) as follows
\[
\mathcal{L}(Z; \theta) = \sum_{k=1}^{K} \log \sum_{c_k=1}^{L} f(z_k, c_k; \theta) = \sum_{k=1}^{K} \log \left( \sum_{l=1}^{L} p_l f(z_k | c_k = l; M_i) \right). \tag{7}
\]
As observed before, the EM algorithm is a recursive approach to the estimation of the parameter \( \theta \); its \( h \)th iteration is aimed at computing \( \hat{\theta}^{(h)} \) starting from the estimate at the previous iteration, \( \hat{\theta}^{(h-1)} \), say, to form a nondecreasing sequence of log-likelihood values, namely
\[
\mathcal{L}(Z; \theta^{(h)}) \geq \mathcal{L}(Z; \hat{\theta}^{(h-1)}). \tag{8}
\]
Obviously, an initial estimate of \( \theta \), say \( \hat{\theta}^{(0)} \), is necessary to initialize the algorithm as well as a reasonable stopping criterion as, for instance, a maximum number of iterations, say \( h_{\text{max}} \). The EM consists of two steps referred to as the E-step and the M-step, respectively. The E-step leads to the computation of the following quantity
\[
g_k^{(h-1)}(l) = p(c_k = l | z_k; \hat{\theta}^{(h-1)}) = \frac{f(z_k | c_k = l; \hat{M}_1 | \hat{\theta}^{(h-1)}) \hat{p}_k^{(h-1)}}{f(z_k; \hat{\theta}^{(h-1)})} = \frac{f(z_k | c_k = l; \hat{M}_1 | \hat{\theta}^{(h-1)}) \hat{p}_k^{(h-1)}}{\sum_{l'=1}^{L} f(z_k | c_k = l'; \hat{M}_{l'}^{(h-1)}) \hat{p}_l^{(h-1)}}, \tag{9}
\]
whereas the M-step requires to solve the following problem
\[
\hat{\theta}^{(h)} = \arg \max_{\theta} \sum_{k=1}^{K} \sum_{l=1}^{L} g_k^{(h-1)}(l) \log f(z_k | c_k = l; M_i) p_l = \arg \max_{\theta} \left\{ \sum_{k=1}^{K} \sum_{l=1}^{L} g_k^{(h-1)}(l) \log f(z_k | c_k = l; M_i) p_l \right\}. \tag{10}
\]
Note that the maximization with respect to \( p_l \), \( l = 1, \ldots, L \), is independent of that over \( M_i \), \( i = 1, \ldots, L \), and, hence, we can proceed by separately addressing these two problems. Starting from the optimization over \( p \), observe that it can be solved by using the method of Lagrange multipliers, to take into account the constraint
\[
\sum_{l=1}^{L} p_l = 1. \tag{11}
\]
Therefore, it is not difficult to show that
\[
\hat{p}_k^{(h)} = \frac{1}{K} \sum_{k=1}^{K} g_k^{(h-1)}(l). \tag{12}
\]
Finally, in order to come up with the estimates of \( M_1, \ldots, M_L \), we solve the following problem
\[
\hat{\sigma}^{(h)} = \arg \max_{\sigma} \sum_{k=1}^{K} \sum_{l=1}^{L} \hat{q}_k^{(h-1)}(l) \log f(z_k | c_k = l; M_i), \tag{13}
\]
where three different forms for the \( M_i, i = 1, \ldots, L \), are considered, namely
1) \( M_i \) is a positive definite Hermitian matrix;
2) \( M_i = \sigma_{c,i}^2 I \), where \( \sigma_{c,i}^2 > 0 \) represents the clutter power which might vary over the range profile when a clutter edge occurs, while \( I \) is the common structure shared by the interference of the \( K \) range bins;
3) \( M_i = \sigma_n^2 I + R_i \), where \( \sigma_n^2 > 0 \) is the unknown thermal noise power and \( R_i \in \mathbb{C}^{N \times N} \) denotes the clutter contribution to the interference of the \( l \)th range bin whose rank, \( r_l \), say, is assumed for the moment known.

Then, the estimates of the unknown parameters for the above cases are provided by the following propositions.

**Proposition 1.** Assume that \( K \geq N \), then an approximation to the relative maximum point of
\[
g_1(M_1, \ldots, M_L) = \sum_{k=1}^{K} \sum_{l=1}^{L} \hat{q}_k^{(h-1)}(l) \log f(z_k | c_k = l; M_i) \tag{14}
\]
has the following expression
\[
\hat{M}_l^{(h)} = \frac{\sum_{k=1}^{K} \hat{q}_k^{(h-1)}(l) z_k z_k^*}{\sum_{k=1}^{K} \hat{q}_k^{(h-1)}(l)}, \quad l = 1, \ldots, L. \tag{15}
\]

**Proof.** See Appendix A \( \square \)

**Proposition 2.** Given the function
\[
g_2(\sigma_{c,i}^2, M) = \sum_{k=1}^{K} \sum_{l=1}^{L} \hat{q}_k^{(h-1)}(l) \log f(z_k | c_k = l; \sigma_{c,i}^2 M) \tag{16}
\]
where \( \sigma_{c,i}^2 = [\sigma_{c,i}^2 \cdots \sigma_{c,L}^2]^T \) and \( K \geq N \), an approximation to the relative maximum point can be achieved by means of the following cyclic procedure with respect to the iteration index \( t \), \( t = 1, \ldots, t_{\text{max}} \) (with \( t_{\text{max}} \) a proper design parameter)
\[
(\hat{\sigma}_{c,i}^2)^{(1)}; \ldots; \hat{\sigma}_{c,L}^2) = \left\{ \sum_{k=1}^{K} \hat{q}_k^{(h-1)}(l) z_k^*(M^{(t_{\text{max}},(h-1)-1)} z_k) \right\}, \tag{17}
\]
\[
\hat{M}^{(t)}(h) = \frac{1}{K} \sum_{k=1}^{K} \sum_{l=1}^{L} \hat{q}_k^{(h-1)}(l) \frac{z_k z_k^*}{(\hat{\sigma}_{c,i}^2)^{(t)}}, \tag{18}
\]
\( t = 1, \ldots, t_{\text{max}} \), and
\[
(\hat{\sigma}_{c,i}^2)^{(t)}; \ldots; \hat{\sigma}_{c,L}^2) = \left\{ \sum_{k=1}^{K} \hat{q}_k^{(h-1)}(l) z_k^*(M^{(t_{-1},(h-1)-1)} z_k) \right\}, \tag{19}
\]
\( t = 2, \ldots, t_{\text{max}}, l = 1, \ldots, L. \)

**Proof.** See Appendix B \( \square \)
Proposition 3. Assume that $r_l$, $l = 1, \ldots, L$, is known, then an approximation to the relative maximum point of the function \[ g_{\alpha}(\sigma_n^2, R_1, \ldots, R_L) = \sum_{k=1}^{K} \sum_{l=1}^{L} q_k^{(h-1)}(l) \log f(z_k | c_k = l; \sigma_n^2 I + R_l), \] can be obtained as follows
\[
\hat{\sigma}_n^{(h)} = \frac{\sum_{l=1}^{K} \sum_{k=1}^{L} q_k^{(h-1)}(l) (N - r_l)}{\sum_{l=1}^{K} \sum_{k=1}^{L} q_k^{(h-1)}(l)}, \quad \hat{r}_{\hat{l}} = \max \left\{ \frac{\hat{\gamma}_{l,1}^{(h-1)}}{\sum_{k=1}^{L} q_k^{(h-1)}(l)}, 0 \right\}, \ldots, \max \left\{ \frac{\hat{\gamma}_{l,r_l}^{(h-1)}}{\sum_{k=1}^{L} q_k^{(h-1)}(l)}, 0 \right\}. \]
where $\hat{\gamma}_{l,n}^{(h)}$ is the unitary matrix whose columns are the eigenvectors corresponding to the eigenvalues $\gamma_{l,1}^{(h-1)} \geq \gamma_{l,2}^{(h-1)} \geq \ldots \geq \gamma_{l,N}^{(h-1)}$ of the matrix
\[
S_{l}^{(h-1)} = \sum_{k=1}^{K} q_k^{(h-1)}(l) z_k z_k^H.
\]

Proof. See Appendix C. \qed

Note that the last proposition supposes that $r_l$, $l = 1, \ldots, L$, is known. However, it is clear that such assumption does not exhibit a practical value; however, the results provided by Proposition 3 can suitably be exploited to construct an estimator for $r$ which, then, can be used for estimation/detection purposes.

\[ r = \arg \min_r \left\{ 2 \sum_{l=1}^{L} r_l \log \left( \frac{\gamma_{l,m}^{(h-1)}}{\sum_{k=1}^{K} q_k^{(h-1)}(l)} \right) \sum_{k=1}^{K} q_k^{(h-1)}(l) \right. \]
\[ + 2 \sum_{l=1}^{L} (N - r_l) \log \left( \frac{\hat{\gamma}_{l,r_l}^{(h-1)}}{\sum_{k=1}^{K} q_k^{(h-1)}(l)} \right) \sum_{k=1}^{K} q_k^{(h-1)}(l) \]
\[ + 2 \sum_{l=1}^{L} r_l \sum_{k=1}^{K} q_k^{(h-1)}(l) + \frac{2}{\sigma_n^2(h)} \sum_{l=1}^{L} \sum_{m=r_l+1}^{N} \gamma_{l,m}^{(h)} + \xi(r) \right\}, \]
where $\xi(r)$ is a penalty term related to the number of unknown parameters and has the following expression $\xi(r) = \sum_{l=1}^{L} [r_l(2N - r_l) + 1] k_p$ with
\[
k_p = \begin{cases} 
2, & \text{AIC,} \\
1 + a, & a \geq 1, \text{ BIC,} \\
\log(2KN), & \text{GIC.} 
\end{cases}
\]

Notice that we are neglecting some constants that do not depend on $r_l$ and, hence, do not enter the decision process.

| CNR [dB] | RMSE | RMSE/K | [%] |
|----------|------|--------|-----|
| case 1   | 25   | 27.5   | 30  |
| case 2   | 20   | 25     | 30  |
| case 3   | 15   | 25     | 30  |

\[
\forall k = 1, \ldots, K : z_k \sim \mathcal{CN}(0, \hat{M}_{ik})
\]

\[ i_k = \arg \max_{l=1, \ldots, L} q_k^{(h_{\text{max}})}(l). \]

IV. Numerical Examples

A. Classification Performance

In this section, we investigate the behavior of the proposed classification architecture through numerical examples on simulated data. All the numerical examples assume $L = 3$, $N = 16$, $K = 96$, $K_1 = 2N$, $K_2 = N + 2$ and $K_3 = K - (K_1 + K_2)$.

We generate $z_l$ as in (2) with $M_l = \sigma_{c,l}^2 M$ and $M = I + M_c$. The $(i,j)$th entry of $M_c$ is given by $M_c(i,j) = \rho^{|i-j|}$ where $\rho = 0.9$ is the one-lag correlation coefficient.

As for the estimation of the covariance structure, the procedure of Proposition 2 is applied. A maximum number of 10 iterations has been set for both the EM and the alternating maximization procedure (hmax = 10 and tmax = 10), representing a good compromise in terms of computational load and estimation accuracy for this preliminary analysis. As for the initialization, we set equiprobable priors and $L = 3$ covariance matrices with a common random Hermitian structure with three different clutter power levels.

Three cases for the clutter power levels are analyzed. Particularly, a power ratio between the maximum and the minimum clutter power is considered of 5, 10, and 15 dB, respectively (as indicated in Table I). The three classification examples are shown in Figure 1 and 3, respectively, where the estimated clutter classes are represented by "x" red stems, whereas the true ones by the "o" blue stems.

Finally, we resort to standard Monte Carlo counting techniques by evaluating the root mean square error (RMSE) of the classification error (defined as the RMS of the absolute difference between the estimated number of not correctly classified classes and the true one) over 1000 independent trials. Results are shown in Table II.

V. Conclusions

This paper proposes several algorithms to classify clutter radar echoes with the goal of partitioning the possibly heterogeneous training data set into homogeneous subsets, which, then, can be used for estimation/detection purposes. The algorithms have been designed using the EM algorithm in conjunction with the latent variable model. More precisely,
considering three different structures for the clutter covariance matrix (from the most general case of a Hermitian structure to the specific one where diagonal loading is accounted for) three different classification architectures have been introduced. Preliminary performance analysis has focused on the second (i.e., intermediate) case and has shown the capability of the proposed approach to solve the problem of clutter data clustering.

Future research tracks include the design of clustering algorithms in the presence of outliers, which can be discarded once identified. Another issue is related to further structures for the clutter covariance matrix that can improve the estimation quality and, hence, detection performance of those receivers relying on such estimates. Finally, the design architectures for the joint detection and classification of clutter edges represent an important extension of this work. These topics represent the current research activity.

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APPENDIX A

PROOF OF PROPOSITION 1

Let us consider the following problem

$$\tilde{\sigma}^{(h)} = \arg \max_{\sigma} g_1(M_1, \ldots, M_L),$$

which is tantamount to solving

$$\tilde{M}_l^{(h)} = \arg \max_{M_l} \left[ -\log \det(M_l) - z_k^\dagger M_l^{-1} z_k \right] d(M_l)$$

for each $l = 1, \ldots, L$. To this end, we set to zero the first derivative of $d(M_l)$ with respect to $M_l$ 43, namely

$$\frac{\partial}{\partial M_l} d(M_l) = -(M_l^T)^{-1} \sum_{k=1}^K q_k^{(h-1)}(l)$$

$$+ (M_l^T)^{-1} \left[ \sum_{k=1}^K q_k^{(h-1)}(l) z_k z_k^\dagger \right] (M_l^T)^{-1} = 0.$$ 

The solution of the above equation is given by

$$\tilde{M}_l^{(h)} = \frac{\sum_{k=1}^K q_k^{(h-1)}(l) z_k z_k^\dagger}{\sum_{k=1}^K q_k^{(h-1)}(l)},$$

and the proof is complete.

APPENDIX B

PROOF OF PROPOSITION 2

In order to come up with the estimates of the $\sigma_{c,l}^2$'s and $M$, we set to zero the first derivatives of $g_2(\sigma_{c,l}^2, M)$ with respect
to the $\sigma_{c,l}^2$'s and $M$, namely

$$\forall l = 1, \ldots, L : \frac{\partial g_3(\sigma_{c,l}^2, M)}{\partial \sigma_{c,l}^2} = - \sum_{k=1}^{K} q_k^{(h-1)}(l) \times \left( \frac{N}{\sigma_{c,l}^2} - \frac{1}{\sigma_{c,l}^2} z_k^T M^{-1} z_k \right) = 0 \quad (33)$$

and

$$\frac{\partial g_3(\sigma_{c,l}^2, M)}{\partial M} = - \sum_{k=1}^{K} \sum_{l=1}^{L} q_k^{(h-1)}(l) \times \left( M^{-1} - \frac{1}{\sigma_{c,l}^2} z_k z_k^T M^{-1} \right)^T = 0. \quad (34)$$

The equations can be re-written as

$$\sigma_{c,l}^2 = \frac{\sum_{k=1}^{K} q_k^{(h-1)}(l) z_k^T M^{-1} z_k}{N \sum_{k=1}^{K} q_k^{(h-1)}(l)}, \quad l = 1, \ldots, L, \quad (35)$$

and

$$M = \frac{1}{K} \sum_{k=1}^{K} \sum_{l=1}^{L} q_k^{(h-1)}(l) \frac{z_k z_k^T}{\sigma_{c,l}^2}, \quad (36)$$

respectively, where we have used the fact that

$$\sum_{k=1}^{K} \sum_{l=1}^{L} q_k^{(h-1)}(l) = K. \quad (37)$$

Since the equation system formed by (35) and (36) does not admit a closed-form solution, we propose to resort to alternating maximization; based on $\hat{\sigma}_{c,l}^2$ and $\hat{M}^{(h-1)}$ we first compute the $\hat{\sigma}_{c,l}^2$'s by plugging $\hat{M}^{(h-1)}$ into egs. (35); then, we compute $\hat{M}^{(h)}$ by plugging the $\hat{\sigma}_{c,l}^2$'s into eq. (36). This procedure can be iterated obtaining, after $t$ iterations, the $\hat{\sigma}_{c,l}^2$'s and $\hat{M}^{(h)}$. To conclude the proof we observe that both EM and alternating maximization lead to a non decreasing sequence of likelihood values [44].

**APPENDIX C**

**PROOF OF PROPOSITION 3**

First we re-write (20) as follows

$$g_3(\sigma_n^2, R_1, \ldots, R_L) = \sum_{k=1}^{K} \sum_{l=1}^{L} q_k^{(h-1)}(l) \left[ - \log \det (\sigma_n^2 \mathbf{I} + R_l) - N \log \pi - z_k^T (\sigma_n^2 + R_l)^{-1} z_k \right]$$

and also as

$$g_3'(\sigma_n^2, R_1, \ldots, R_L) = \sum_{k=1}^{K} \sum_{l=1}^{L} q_k^{(h-1)}(l) \left[ - \log \det (\sigma_n^2 \mathbf{I} + R_l) - \mathbb{E} \left[ (\sigma_n^2 + R_l)^{-1} S_k \right] \right] \quad (38)$$

where $S_k = z_k z_k^T$. Now, let us consider the eigendecomposition of $R_l$, namely

$$R_l = \mathbf{U}_l \Lambda_l \mathbf{U}_l^T,$$

where $\mathbf{U}_l \in \mathbb{C}^{N \times N}$ is a unitary matrix whose columns are the eigenvectors of $R_l$; $\Lambda_l$ is the corresponding diagonal matrix of the eigenvalues of $R_l$; $\Lambda_l$ can be represented as $\Lambda_l = \text{diag}(\lambda_{l,1}, \ldots, \lambda_{l,r_l}, 0, \ldots, 0) \in \mathbb{R}^{N \times N}$ with $\lambda_{l,1} \geq \ldots \geq \lambda_{l,r_l} > 0$. It follows that the objective function becomes

$$g_3'(\sigma_n^2, R_1, \ldots, R_L) = \sum_{l=1}^{L} \sum_{k=1}^{K} q_k^{(h-1)}(l) \left\{ - \log \det (\sigma_n^2 \mathbf{I} + \Lambda_l) - \mathbb{E} \left[ (\sigma_n^2 \mathbf{I} + \Lambda_l)^{-1} \mathbf{U}_l^T S_k \right] \right\}$$

and

$$= \sum_{l=1}^{L} \left\{ - \left( \sum_{k=1}^{K} q_k^{(h-1)}(l) \right) \log \left[ (\sigma_n^2)^{N-r_l} \prod_{m=1}^{r_l} (\sigma_n^2 + \lambda_{l,m}) \right] \right\} - \mathbb{E} \left[ (\sigma_n^2 \mathbf{I} + \Lambda_l)^{-1} \mathbf{U}_l^T S_k \right]$$

where

$$S_k^{(h)} = \sum_{k=1}^{K} q_k^{(h-1)}(l) S_k.$$

Replacing $S_k^{(h-1)}$ by its eigendecomposition, we also come up with

$$\sum_{l=1}^{L} \left\{ - \left( \sum_{k=1}^{K} q_k^{(h-1)}(l) \right) \log \left[ (\sigma_n^2)^{N-r_l} \prod_{m=1}^{r_l} (\sigma_n^2 + \lambda_{l,m}) \right] \right\} - \mathbb{E} \left[ (\sigma_n^2 \mathbf{I} + \Lambda_l)^{-1} \mathbf{U}_l^T S_k^{(h)} \right]$$

where $\Gamma_l^{(h)} = \text{diag}(\gamma_{l,1}^{(h)}, \ldots, \gamma_{l,N}^{(h)})$ with $\gamma_{l,1}^{(h)} \geq \ldots \geq \gamma_{l,N}^{(h)}$ being the eigenvalues of $S_k^{(h-1)}$ and $O_l^{(h)}$ the unitary matrix of the corresponding eigenvectors. As a consequence, the objective function (38) can also be recast as

$$g_3''(\sigma_n^2, \mathbf{V}_l, \Lambda_l, l = 1, \ldots, L) = \sum_{l=1}^{L} \left\{ - \left( \sum_{k=1}^{K} q_k^{(h-1)}(l) \right) \log \left[ (\sigma_n^2)^{N-r_l} \prod_{m=1}^{r_l} (\sigma_n^2 + \lambda_{l,m}) \right] \right\} - \mathbb{E} \left[ (\sigma_n^2 \mathbf{I} + \Lambda_l)^{-1} \mathbf{V}_l^{(h-1)} \right]$$

where

$$\arg \max_{\mathbf{V}_l} -\mathbb{E} \left[ (\sigma_n^2 \mathbf{I} + \Lambda_l)^{-1} \mathbf{V}_l^{(h-1)} \right] = \mathbf{I},$$

which implies that $U_l^{(h)} = O_l^{(h-1)}$. Then, we obtain

$$g_3''(\sigma_n^2, \Lambda_l, l = 1, \ldots, L) = \max_{\mathbf{V}_l} g_3''(\sigma_n^2, \mathbf{V}_l, \Lambda_l, l = 1, \ldots, L) \quad (39)$$

$$= \sum_{l=1}^{L} \left\{ - \left( \sum_{k=1}^{K} q_k^{(h-1)}(l) \right) \log \left[ (\sigma_n^2)^{N-r_l} \prod_{m=1}^{r_l} (\sigma_n^2 + \lambda_{l,m}) \right] \right\}$$

and

$$\sum_{l=1}^{L} \log (\sigma_n^2 + \lambda_{l,m}) - \sum_{m=1}^{N} \frac{\gamma_{l,m}^{(h)}}{\sigma_n^2} - \sum_{m=r_l+1}^{N} \frac{\gamma_{l,m}^{(h)}}{\sigma_n^2}.$$
As the next step towards the final result, we set to zero the first derivative of the above objective function with respect to $\lambda_{l,m}$, namely

$$\frac{\partial}{\partial \lambda_{l,m}} \left[ -q^{(h-1)}(l) \log(\sigma_n^2 + \lambda_{l,m}) - \frac{\gamma_{l,m}^{(h-1)}}{\sigma_n^2 + \lambda_{l,m}} \right] = 0$$

$$\Rightarrow -q^{(h-1)}(l) \frac{1}{\sigma_n^2 + \lambda_{l,m}} + \frac{\gamma_{l,m}^{(h-1)}}{(\sigma_n^2 + \lambda_{l,m})^2} = 0$$

$$\Rightarrow \lambda_{l,m} = \begin{cases} \frac{\gamma_{l,m}^{(h-1)}}{q^{(h-1)}(l) - \sigma_n^2}, & \sigma_n^2 < \frac{\gamma_{l,m}^{(h-1)}}{q^{(h-1)}(l)}; \\ 0, & \text{otherwise}. \end{cases} \quad (40)$$

After replacing $\lambda_{l,m}$ with $\hat{\lambda}_{l,m}$ in (39), the last optimization is

$$\max_{\sigma_n^2} \sum_{l=1}^L \left\{ -q^{(h-1)}(l)(N - r_l) \log(\sigma_n^2) - q^{(h-1)}(l) \sum_{m=1}^{r_l} \gamma_{l,m}^{(h-1)} \right\} \log \left( \frac{\gamma_{l,m}^{(h-1)}}{q^{(h-1)}(l)} \right) - r_l q^{(h-1)}(l) - \sum_{m=r_l+1}^N \gamma_{l,m}^{(h-1)} \right\},$$

which can be solved by finding the zeros of the following function

$$\frac{\partial}{\partial \sigma_n^2} \left[ \sum_{l=1}^L \left\{ -q^{(h-1)}(l)(N - r_l) \log(\sigma_n^2) - q^{(h-1)}(l) \sum_{m=1}^{r_l} \gamma_{l,m}^{(h-1)} \right\} \log \left( \frac{\gamma_{l,m}^{(h-1)}}{q^{(h-1)}(l)} \right) - r_l q^{(h-1)}(l) - \sum_{m=r_l+1}^N \gamma_{l,m}^{(h-1)} \right\} = -1 \sigma_n^2 q^{(h-1)}(l)(N - r_l) + \frac{1}{(\sigma_n^2)^2} \sum_{l=1}^L \sum_{m=r_l+1}^N \gamma_{l,m}^{(h-1)}.$$

The result is

$$\hat{\sigma}_n^{2(h)} = \sum_{l=1}^L \sum_{m=r_l+1}^N \gamma_{l,m}^{(h-1)} / \sum_{l=1}^L q^{(h-1)}(l)(N - r_l). \quad (41)$$

Finally, the estimate of $\hat{\lambda}_{l,m}$, $l = 1, \ldots, L, m = 1, \ldots, r_l$, is given by

$$\hat{\lambda}_{l,m}^{(h)} = \begin{cases} \frac{\gamma_{l,m}^{(h-1)}}{q^{(h-1)}(l)} - \hat{\sigma}_n^{2(h)}; & \hat{\sigma}_n^{2(h)} < \frac{\gamma_{l,m}^{(h-1)}}{q^{(h-1)}(l)}; \\ 0, & \text{otherwise}. \end{cases} \quad (42)$$

and the proof is complete.

REFERENCES

[1] W. L. Melvin and J. A. Scheer, Principles of Modern Radar: Advanced Techniques, S. Publishing, Ed., Edison, NJ, 2013.

[2] M. Richards, W. Melvin, J. Scheer, J. Scheer, and W. Holm, Principles of Modern Radar: Radar Applications, Volume 3, ser. Electromagnetics and Radar. Institution of Engineering and Technology, 2013.

[3] J. Liu, W. Liu, H. Liu, B. Chen, X. G. Xia, and F. Dai, “Average SINR Calculation of a Persymmetric Sample Matrix Inversion Beamformer,” IEEE Transactions on Signal Processing, vol. 64, no. 8, pp. 2135–2145, April 2016.

[4] J. Liu, S. Sun, and W. Liu, “One-step persymmetric GLRT for subspace signals,” IEEE Transaction on Signal Processing, vol. 14, no. 67, pp. 3639–3648, July 15 2019.

[5] G. Foglia, C. Hao, G. Giunta, and D. Orlando, “Knowledge-aided adaptive detection in partially homogeneous clutter: Joint exploitation of persymmetric and symmetric spectrum,” Digital Signal Processing, vol. 67, no. Supplement C, pp. 131 – 138, 2017.

[6] L. Cai and H. Wang, “A Persymmetric Multiband GLR Algorithm,” IEEE Transactions on Aerospace and Electronic Systems, vol. 28, no. 3, pp. 806–816, 1992.

[7] P. Wang, Z. Sahinoglu, M. Pun, and H. Li, “Persymmetric Parametric Adaptive Matched Filter for Multichannel Adaptive Signal Detection,” IEEE Transactions on Signal Processing, vol. 60, no. 6, pp. 3322–3328, 2012.
[31] A. Benavoli, L. Chisci, A. Farina, S. Immediata, L. Timmoneri, and G. Zappa, “Knowledge-based system for multi-target tracking in a littoral environment,” IEEE Transactions on Aerospace and Electronic Systems, vol. 42, no. 3, pp. 1100–1119, 2006.

[32] S. Haykin and C. Deng, “Classification of radar clutter using neural networks,” IEEE Transactions on Neural Networks, vol. 2, no. 6, pp. 589–600, 1991.

[33] S. Haykin, W. Stehwien, C. Deng, P. Weber, and R. Mann, “Classification of radar clutter in an air traffic control environment,” Proceedings of the IEEE, vol. 79, no. 6, pp. 742–772, 1991.

[34] V. Anastassopoulos and G. A. Lampropoulos, “High resolution radar clutter classification,” in Proceedings International Radar Conference, 1995, pp. 662–667.

[35] M. A. Darzikolaei, A. Ebrahimzade, and E. Gholami, “Classification of radar clutters with Artificial Neural Network,” in 2015 2nd International Conference on Knowledge-Based Engineering and Innovation (KBEI), 2015, pp. 577–581.

[36] P. Formont, F. Pascal, G. Vasile, J. Ovarlez, and L. Ferro-Famil, “Statistical Classification for Heterogeneous Polarimetric SAR Images,” IEEE Journal of Selected Topics in Signal Processing, vol. 5, no. 3, pp. 567–576, 2011.

[37] V. Carotenuto, A. De Maio, D. Orlando, and P. Stoica, “Model Order Selection Rules for Covariance Structure Classification in Radar,” IEEE Transactions on Signal Processing, vol. 65, no. 20, pp. 5305–5317, 2017.

[38] J. Liu, F. Biondi, D. Orlando, and A. Farina, “Training Data Classification Algorithms for Radar Applications,” IEEE Signal Processing Letters, vol. 26, no. 10, pp. 1446–1450, 2019.

[39] A. P. Dempster, N. M. Laird, and D. B. Rubin, “Maximum likelihood from incomplete data via the EM algorithm,” Journal of the Royal Statistical Society (Series B - Methodological), vol. 39, no. 1, pp. 1–38, 1977.

[40] K. Murphy, Machine Learning: A Probabilistic Perspective, ser. Adaptive Computation and Machine Learning series. MIT Press, 2012.

[41] M. A. Richards, J. A. Scheer, and W. A. Holm, Principles of Modern Radar: Basic Principles. Raleigh, NC: Scitech Publishing, 2010.

[42] L. Yan, P. Addabbo, C. Hao, D. Orlando, and A. Farina, “New ECCM Techniques Against Noise-like and/or Coherent Interferers,” IEEE Transactions on Aerospace and Electronic Systems, 2019.

[43] A. Hjørungnes, Complex-Valued Matrix Derivatives: With Applications in Signal Processing and Communications. Cambridge University Press, 2011.

[44] E. Conte, A. De Maio, and G. Ricci, “Recursive Estimation of the Covariance Matrix of a Compound-Gaussian Process and Its Application to Adaptive CFAR Detection,” IEEE Transactions on Signal Processing, vol. 50, no. 8, pp. 1908–1915, 2002.

[45] L. Mirsky, “On the trace of matrix products,” Mathematische Nachrichten, vol. 20, pp. 171–174, 1959.