Effect of a contact line dynamics on oscillations of oblate bubble in a non-uniform electric field

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Abstract. The behaviour of a gas cylindrical bubble in a non-uniform alternating electric field is investigated. The bubble is surrounded by an incompressible fluid and is bounded in the axial direction by two parallel solid surfaces. It is assumed that the velocity of the contact line is proportional to the sum of the deviation of the contact angle and the velocity of fast relaxation processes, whose frequencies are proportional to twice the frequency of the electric field.

1. Introduction

The dynamics of fluid droplets (or gas bubble) in the electrical field, in particular, the dynamics, principles and basic mechanisms of bubble-based electrowetting (EW) and electrowetting-on-dielectric (EWOD) is still the focus of considerable researchers’ interest [1-5]. EWOD has proved to be an efficient tool in digital microfluidics that employs discrete droplets and various applications that use the principles of EWOD, such as bioanalysis or variable-focus liquid lenses.

The Young–Lippmann equation is the most frequently used fundamental equation for a theoretical description of the variable contact angle of the droplet under the action of direct current voltage [1-5]. It is possible to obtain zero contact angle (complete wetting and the contact angle does not change) by continuous increase of the applied voltage according to the Young–Lippmann equation, i.e. this equation applies to the case when voltage \( V \) which is smaller than the critical voltage \( V_\ast \) [1-4]. At \( V \geq V_\ast \) the electrowetting contact angle does not increase with increase in the applied voltage – the effect, which is generally known as contact angle saturation. The mechanism of contact angle saturation is still not fully understood and is the subject of broad discussion [1].

Another effective boundary condition was proposed on the basis of Hocking’s equation [6] for cylindrical drop in [7]:

\[
\frac{\partial \zeta^-}{\partial t} = \pm \Lambda^\ast \left( \frac{\partial \zeta^-}{\partial z} + A^\ast \cos(2\omega^\ast t) \right),
\]

where \( \zeta^- \) is the deviation of the drop surface from its equilibrium position, \( z^- \) is the axial coordinate, \( \Lambda^\ast \) is a phenomenological constant (the so-called wetting parameter or Hocking parameter), having the dimension of velocity, \( A^\ast \) is the effective amplitude, \( \omega^\ast \) is the frequency of electric current.

The Hocking equation [6] was used in the studies of oscillations of a capillary bridge [8], sessile drop and bubble [9-11], cylindrical drop and bubble [12-14], film interface [15] etc. The Hocking
parameter \( \Lambda \) was constant and real in all papers [6-15]. In paper [16], it was suggested that this parameter is a complex quantity. The substrate surfaces were considered as inhomogeneous in [7], i.e. the Hocking parameter was represented as a function of coordinates.

In the present paper, we consider the forced oscillations of cylindrical gas bubble under non-uniform AC. In order to describe the motion of the contact line the modified boundary condition (1) is used: the velocity of the contact line is proportional to the deviation of the contact angle and the speed of the fast relaxation processes, whose frequency is proportional to twice the AC frequency. The free oscillations of this bubble were studied in [13, 14].

2. Problem formulation

The problem formulation is largely consistent with the formulation developed in papers [7, 13, 14]. A gas bubble is surrounded by an incompressible liquid with a free external surface (fig. 1). The system is bounded by two parallel solid surfaces located at a distance \( h \) from each other. Suppose that in the equilibrium, the bubble and liquid volume are the circular cylinders, with the equilibrium radii \( r_0 \) and \( R_0 \) i.e. each equilibrium contact angle is \( 90^\circ \). The external non-uniform alternating electric field acts as an external force that causes the contact line motion.

![Figure 1. Problem geometry. 1 – electrode, 2 – dielectric layer.](image)

We assume that, on the one hand, the AC frequency \( \omega \) is large enough for the liquid viscosity to be ignored \(- \delta^+ = \nu (\omega^+)^{-1} \ll r_0^2 \), and, on the other hand, the oscillation frequency is sufficiently small, so that we can use the incompressibility condition for the external liquid \(- \omega^+ r_0^2 \ll c \) (\( \delta \) is the boundary-layer thickness, \( c \) is the sound velocity). The characteristic amplitude of oscillations of the drop \( A^+ \) is small compared to the equilibrium radius \( r_0^+ \): \( \varepsilon = A^+ (r_0^+)^{-1} \ll 1 \).

Since the problem is symmetric, it is convenient to introduce the cylindrical coordinates \((r^+, \alpha, z^+)\). The azimuthal angle \( \alpha \) is reckoned from the \( x \)-axis. Let the lateral surface of the bubble and the external surface of the liquid layer be described by the following equations, respectively:

\[
\begin{align*}
 r^+ &= r_0^+ + \xi^+ (\alpha, z^+, t^+), \\
 r^+ &= R_0^+ + \xi^+ (\alpha, z^+, t^+).
\end{align*}
\]

Following [13, 14], we use \( \sqrt{\rho \sigma / \sigma^+} \), \( r_0 \), \( h^+ \), \( A^+ \), \( A^+ \sigma^+ / r_0^2 \), \( A^+ \sqrt{\sigma^+ / \rho \sigma} r_0^+ \) as the scales for time, length, height, deviation of the bubble surface and the free surface from its equilibrium position, pressure, and velocity potential, respectively. Here, \( \sigma^+ \) is the surface tension and \( \rho^+ \) is the fluid density. Thus, the dimensionless boundary value problem is determined by (intermediate steps can be found in [7, 12-14])

\[
p_n = -\varphi^+ - \Delta \varphi = 0, \quad p_1 = -2n_\mu P_0^+ r_0^+ / \sigma^+ (\xi^+) \equiv -P_0 (\xi^+),
\]

(2)
\[\Delta = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \alpha^2} + b^2 \frac{\partial^2}{\partial z^2},\]

\[r = 1: \zeta_t = \varphi_r, \quad [P] = \zeta + \zeta_{ae} + b^2 \zeta_{ae}, \quad (3)\]

\[z = \pm 0.5: \varphi_z = 0, \quad (4)\]

\[r = 1, \quad z = \pm 0.5: \zeta_t = \mp \lambda (\zeta_x + \alpha f(\alpha) \cos(2\omega t)), \quad (5)\]

where \(p_c\) is the liquid pressure, \(\varphi\) is the potential of liquid velocity, \(p_g\) is the gas pressure in the bubble, \(n_p\) is the polytropic (e.g., adiabatic) exponent, \(P^*_g\) is the dimensional gas pressure in the bubble, \(P_0\) is the dimensionless equilibrium pressure inside the bubble. The boundary-value problem (2)–(5) involves six parameters (including \(P_0\)): the aspect ratio, the radius of the free surface, the wetting parameter, the AC frequency and AC amplitude

\[b = \frac{r_0^*}{h^*}, \quad R_0 = \frac{R_0^*}{r_0^*}, \quad \lambda = \Lambda \frac{\rho_0^* r_0^*}{\sigma^*}, \quad \omega = \omega^* \sqrt{\frac{\rho_0^* r_0^*}{\sigma^*}}, \quad \alpha = 0.5 A C \left(\sigma^2 \frac{\rho_0^* R_0^*}{h^*}\right)^{1/2},\]

where \(C = \varepsilon_0 \varepsilon_d d^{-1}\) is capacitance per unit area, \(\varepsilon_0\) and \(\varepsilon_d\) are the vacuum and the dielectric layer permittivity, respectively, \(d\) is the thickness of the dielectric film.

Note that the AC frequency is assumed to be comparable with the frequencies of both the shape and volume oscillations. It should be stressed that spatially uniform oscillations are of crucial importance for considering the bubble behavior in contrast to the incompressible drop problems [7,12], where the system is dominantly governed by the pressure difference in the liquid. In our system, to define the uniform part of the pressure field, it is necessary to specify an additional condition at a distance from the bubble. From the experimental viewpoint, the most convenient way to overcome this difficulty is to use a liquid layer with a free surface. For a spherical and hemispherical bubble, a similar way of treating this peculiarity of the pressure field has been used in a number of previous studies [9,11,17,18]. Also, for the cylindrical coordinates both the uniform pressure field and the velocity potential are proportional to \(\ln(r)\) [13, 14]. Consequently, the presence of a free surface is a prerequisite for limiting pressure at a distance from the bubble. This is the main difference between our problem and the problem of a spherical bubble and an incompressible drop, for which the uniform pressure decreases or is zero in the distance.

The limiting case of high pressure, \(P_0 \to \infty\), is associated with consideration of incompressible gas. In this situation, problem (2)–(5) is transformed to the problem of natural oscillations of an incompressible bubble immersed in a liquid [7,12-14]. Vise versa, for small values of \(P_0\), one would expect the collapse of the bubble. Note that hereinafter we assume that pressure in the ambient fluid is positive and the cavitation effects are ignored.

3. Uniform Electric Field

In order to look into the problem in detail, it is convenient to begin with a consideration of the uniform electric field, i.e. \(f(\alpha) = 1\) in (5). Since our concern is with the axisymmetric oscillation modes, the system of equations and boundary conditions (2)-(5) are assumed to be axially symmetric at \(f(\alpha) = 1\).

We use the well-behaved method of representing the solution in the form of a Fourier series. This method of solution was also used in [13,14]. In view of axial symmetry, the solution of the Laplace equation (2) with boundary condition (4) can be written as

\[\varphi(r, z, t) = \text{Re} \left( 2i \omega \sum_{k=0}^{\infty} \left( a_k R_k^c(r) + b_k R_k^c(r) \right) \cos(2k \pi z) e^{2i \omega t} \right), \quad (6)\]
where $R_0'(r) = \text{const}$, $R_0''(r) = \ln(r)$, $R_1'(r) = I_1(2\pi k b r)$, $R_1''(r) = \ln(r)$, $R_e'(r) = K_0(2\pi k b r)$, $I_0$ and $K_0$ are the modified Bessel functions. The kinematic condition on the free surface (the second condition in (3)) gives the expression for the surface deviation

$$\zeta(z,t) = \Re \left( \sum_{k=0}^{n} c_k \cos(2\pi k z) + d_k \cos \left( \frac{z}{b} \right) e^{2\omega t} \right),$$

(7)

**Figure 2.** Deviation of the contact line at the upper plate (a), the drop surface in central layer (b) and the contact angle (c) at the upper plate vs frequency ($\rho_i = 0.7$, $R_0 = 5$, $P_0 = 5$, $b = 1$, $a = 2.5$), $\lambda = 0.01$ – solid line, $\lambda = 1$ – dash, $\lambda = 100$ – dot.

The dependence of the surface oscillation amplitude at the layer center $\zeta_0$ and at the upper plate $\zeta_a$ on the forced frequency $\omega$, and the evolution of the bubble shape and the contact angle deviation at the upper plate are given in figures 2, 3 for different values of the Hocking parameter $\lambda$. From the graphs presented above it follows that the maximum amplitude is observed at the frequency of volume oscillations. The resonance amplitude decreases with increasing number of modes. The resonance amplitude of the bubble surface is maximal at those frequencies, at which the contact angle varies inessentially. An opposite phenomenon is observed for the amplitude of the contact line. The dynamics of the compressible bubble is consistent with the dynamics of the incompressible drop at the frequencies of surface oscillation modes.

**Figure 3.** Evolution of the drop surface (a,b) and the contact angle (c) at the upper plate ($\rho_i = 0.7$, $R_0 = 5$, $P_0 = 5$, $b = 1$, $a = 2.5$, $\lambda = 1$). $T = \pi \omega^{-1}$ is the oscillation period,

(a) – $\omega = 0.79$, (b) – $\omega = 8.07$.

(a)-(b) $t = 0$ – solid line, $t = 0.125T$ – dash, $t = 0.25T$ – dot, $t = 0.375T$ – dash-dot.

The Hocking parameter characterizes not only the chemical properties of the liquid-substrate pair but also the degree of surface treatment. At small values of the Hocking parameter the resonant amplitude is small (the interaction of the contact line with the plate is sufficiently large) as well the
energy dissipation (the displacement distance is negligible). But at large values of the parameter, the dissipation is also small, but the contact line weakly interacts with the substrate. As a result, the values of the resonant amplitudes are large.

With increasing gas pressure compression of the bubble decreases, the radial pulsations become small and the bubble behaviour agrees with the dynamics of an incompressible drop. The resonant amplitude of the oscillations decreases with increasing pressure (fig. 4). We note again that with increasing frequency, the compressibility of gas (value of $P_0$) is neglected.

![Figure 4](image)

**Figure 4.** Deviation of the contact line at the upper plate (a), the drop surface in the central layer (b) and the contact angle (c) at the upper plate vs frequency ($\rho_l = 0.7$, $R_0 = 5$, $\lambda = 1$, $b = 1$, $\alpha = 2.5$), $P_0 = 5$ – black line, $P_0 = 50$ – blue, $P_0 = 100$ – red.

As mentioned above, the distance from the external free surface affects the volume oscillations of the bubble. Since the natural logarithmic function $\ln(r)$ changes more slowly than the linear function $r$, the layer can be considered as infinite at $R_0 \gg 1$. In this case, solution (7) assumes finite values (with the exception of resonant cases). In figure 5, the surface oscillation amplitude is shown at different values of the radii $R_0$ and frequencies $\omega$.

![Figure 5](image)

**Figure 5.** Deviation of the contact line at the upper plate (a,b) and the drop surface in the central layer (c) at the upper plate ($\rho_l = 0.7$, $P_0 = 5$, $\lambda = 1$, $b = 1$, $\alpha = 2.5$),

(a) $\omega = 0.1$ – solid line, $\omega = 0.25$ – dash, $\omega = 0.5$ – dot, $\omega = 1$ – dash-dot,

(b,c) $R_0 = 2$ – solid line, $R_0 = 5$ – dash, $R_0 = 10^2$ – dot, $R_0 = 10^4$ – dash-dot.

### 4. Non-uniform electric field

The function $f(\alpha)$ in (5) is expanded into the Fourier series with respect to eigen functions of the Laplace operator. Let us considered a particular case of the non-uniform electric field:
\[ f(\alpha) = \left| \cos(\kappa \alpha) \right| = \left| \cos(\kappa \cos(\alpha)) \right| , \]
where \( \kappa \) is the wave number. The non-uniform AC field excites the azimuthal modes of forced oscillations.

The dependence of the surface oscillation amplitudes, the shape of the bubble, the contact line and the contact angle deviation on the forced frequency \( \omega \) is given in figures 6, 7 for \( \kappa = 1 \) and different values of the Hocking parameter \( \lambda \). The analysis of the forced oscillations has shown that the wavenumber \( \kappa \) is an effective amplitude. Also these values are larger in comparison with the values obtained in the uniform field (see also [7]). Most of the energy is transported to the main oscillation mode, with small peaks being associated with the frequencies of other modes. In our case, the contact line extends along the axis of the non-uniform AC field.

\[ \frac{\rho}{\rho_i} = \cos(\kappa \alpha) \]

![Figure 6. Deviation of the contact line at the upper plate (a), the drop surface in the central layer (b) and the contact angle (c) at the upper plate vs frequency (\( \rho_i = 0.7, R_0 = 5, P_0 = 5, b = 1, a = 2.5, \kappa = 1 \), \( \lambda = 0.01 \) – solid line, \( \lambda = 1 \) dash, \( \lambda = 100 \) dot.)](image)

\[ \frac{1}{T} \pi \omega = \]

![Figure 7. Evolution of the drop surface (a), the contact line (b) and the contact angle (c) at the upper plate (\( b = 1, \rho_i = 0.7, R_0 = 5, P_0 = 5, \omega = 0.79 \)). \( T = \pi \omega^{-1} \) is the oscillation period, (a)-(b) \( t = 0 \) solid line, \( t = 0.125T \) dash, \( t = 0.25T \) dot, \( t = 0.375T \) dash-dot.](image)

5. Conclusions

The model describing the behaviour of a cylindrical bubble between two parallel surfaces in an non-uniform alternating electric field was developed. The dynamics of the contact line was taken into account due to the application of an effective boundary condition.

It was shown that in the case of the uniform alternating electric field, the main contribution is made by radial oscillations. The investigation made in the framework of the proposed model allowed us to construct the amplitude-frequency characteristics for various parameters of the problem.
In the considered inhomogeneous field, the azimuth modes are excited, which increases the number of resonant frequencies. The basic modes of azimuth modes for finite values of the Hocking parameter can be suppressed by increasing the value of the geometric parameter.

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