Gauge invariance and electron spectral functions in underdoped cuprates

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The single particle spectral function for the normal state of underdoped high \( T_c \) cuprates is studied within the slave particle framework. We find that the presence of a massless dynamical gauge field - a direct consequence of the quantum order - explains the broad, but not totally incoherent, line-shapes observed in experiments. The issue of the negative anomalous dimension of a recently proposed gauge invariant single particle amplitude is also considered. We show how the anomalous behavior of the single particle amplitude can be incorporated within the slave particle approach and, thus reinterpreted, lead to physical phenomenology.

I. INTRODUCTION

In recent years gauge theories have become a prominent tool in the description of strongly correlated electron systems \[1,2,3\]. The gauge field appears because we attempt to use fermion operators to describe excitations of a strongly correlated state. We know that the fermion operators are useful at low energies only when the ground state wave function can be described by a Slater determinant. If the ground state is a strongly correlated state, we cannot use the fermion/boson operators to write down a free low energy effective theory. However if we insist on writing everything in terms of fermion/boson operators, one finds that we can use fermion/boson operators to describe the low energy physics of a strongly correlated state provided that gauge fields are introduced. This is why low energy effective theories of strongly correlated states, such as the quantum Hall states and spin liquid states, are all gauge theories.

Realizing that the electron operator is not a good starting point to describe a strongly correlated state, we can rewrite the electron operator as a product of several other operators in order to study those states. These operators are called parton operators (the spinon and holon operators are examples of parton operators). We then construct mean-field states in the Hilbert space of partons. After identifying the gauge structure – the transformations between partons that leave the electron operator unchanged, we can project the mean-field state onto the physical Hilbert space (i.e. the gauge invariant space) and obtain a strongly correlated electron state. Although the parton mean-field state is a Slater determinant, the projected state is not and can be used to describe the correlations in certain strongly correlated states. This procedure, in its general form, is called projective construction, which is a generalization of the slave-boson approach\[4,5,6\]. The generic projective construction and the associated gauge structure was discussed in detail for quantum Hall states in \[7\]. The resulting effective theory arising from the projective construction naturally contains a gauge structure. In this paper, we will use the projective-construction/ slave-boson-approach to study the electron spectral function in underdoped high \( T_c \) superconductors.

One convenient way to describe dynamical gauge theories is via the path integral approach. Traditionally, however, the formalism which allows for most direct comparison with experimental results is based on Green’s functions and diagrammatic techniques. This turns out to be a kind of nuisance since the standard expression for the single particle Green’s function (the cornerstone of diagrammatic perturbation theory) is not gauge invariant. The lack of gauge invariance makes it hard to extract any meaningful quantitative results based on these Green’s functions.

In the first part of the paper we present a calculation of a particular gauge invariant Green’s function for 2+1 dimensional massless Dirac particles coupled to a dynamical gauge field. The motivation for this analysis arises from an effective continuum description - the Algebraic Spin Liquid (ASL)\[8,9\] which we believe underlies the strange phenomenology of the normal state of underdoped high temperature superconductors. Within this continuum model, which is rooted in the slave-boson approach to strongly correlated systems, the spin degrees of freedom are carried by massless Dirac particles called spinons. Our early attempt to interpret the resulting spinon propagator as describing the propagation of the physical particle is however thwarted by the appearance of a negative anomalous dimension for the particular gauge invariant spinon amplitude at the ASL fixed point. On reinterpreting the proposed amplitude as representing a two-particle spinon-holon propagator, we find that the main effect of the gauge fluctuation is to bind the two particles (via its effect on the vertex) to a more coherent entity. Thus we are led to the heuristic description of the physical electron propagator presented in the last section of the paper where in addition to the spinon contribution we include the bosonic holon contribution. Not surprisingly the resulting spectrum has no quasiparticle peak, however it is also not totally incoherent once the gauge fluctuations are included. The emerging picture for the electron propagation is thus one of spinons and holons (separate at the mean field level) whose attractive interaction mediated via the gauge field leads to
the formation of a more coherent structure on top of the
incoherent mean-field background \[10\]. In analyzing this
problem we have been guided by a first quantized path
integral approach to the physical Green’s function which
makes the gauge dependence of the amplitude particu-
larly transparent.

II. PATH INTEGRAL FORMULATION

Before plunging into the second quantized formalism
let us look at the problem in the light of single particle
quantum mechanics. Our starting point is the following
definition for a gauge-invariant Green’s function in first
quantized notation

\[
G(\vec{x}) = \int D\alpha D\beta e^{-\int_0^\beta d\tau \left(L[x(\tau)] + L[a]\right)} e^{-i \int_\alpha \alpha \mu d\tau \mu}
\]

where \( | \) is the straight line connecting 0 to \( \vec{x} \), \( \mu = 0..2 \) and the metric is Euclidean. The action for the gauge
field is given by

\[
S[\alpha] = \frac{1}{2} \int \frac{d^3 q}{(2\pi)^3} a_\mu(q) \frac{1}{4} \sqrt{q^2} (\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}) a_\nu(-q)
\]

which is obtained by integrating out massless Dirac
fermions (the spinons) \[See Eq. (8)\]. The action for the
gauge field is given by

\[
\frac{1}{2} \int D\alpha D\beta e^{-\int_0^\beta d\tau \left(L[x(\tau)] + L[a]\right)} e^{-i \int_\alpha \alpha \mu d\tau \mu}
\]

where \( \int_\alpha \) indicates a path from 0 to \( \vec{x} \) described by \( x(\tau) \).
\( L[x(\tau)] \) remains unspecified since we will be concerned
with different types of particles. Putting the above to-
gether we obtain the following expression for the Green’s
function

\[
G(\vec{x}) = \int D\alpha D\beta e^{-\int_0^\beta d\tau \left(L[x(\tau)] + L[a]\right)} e^{i \int_\alpha \alpha \mu \mu} d\tau \mu
\]

This expression exhibits the gauge dependence in a par-
ticularly appealing form where the line integrals close
up into a contour, of which one side is the propagat-
ing path of the particle and other side the straight-line
path needed to make the gauge invariance manifest. Via
Stokes theorem we see that the sum of the line integrals
is nothing but the flux going through the contour. So
far our expressions have full rotational symmetry in order
to perform the calculation we now choose to parameter-
ize the particle world-lines by their \( x_0 = \tau \) coordinate.
Hence \( \int a_\mu d\tau \mu \) goes over to \( \int (a(x) \frac{d\vec{x}}{d\tau} + a_0) d\tau \) where \( a, x \)
denote the 2 spatial components of the corresponding
3-vectors. In order to simplify our analysis further we
consider the propagation of the particle only along the
time direction i.e. \( G(0, \tau) \). Considering the propagation
between equal spatial coordinates we have no contribu-
tion from the 2-velocity component along the straight
line path which simplifies this part of the phase factor
to \( \int a_0 d\tau \). Next we want to perform the average over
the gauge fluctuations which is straightforward given the
Gaussian weight \( 2 \). Reading off the gauge propagator

\[
D \left( a_\mu(q) a_\nu(\vec{k}) \right) = \frac{1}{(2\pi)^3} \delta(q+\vec{k}) \left[ \frac{1}{4} \sqrt{q^2} (\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}) \right]^{-1}
\]

In order to invert the polarization we chose the \( a_0 = 0 \)
gauge which yields

\[
D \left( a_\mu(q) a_\nu(\vec{k}) \right) = (2\pi)^3 \delta(q+\vec{k}) \delta_{\mu,i} \delta_{\nu,j} \left( \delta_{i,j} + \frac{q_i q_j}{q^2} \right) \frac{4}{\sqrt{q^2}}
\]

where \( i, j = 1, 2 \) denote the spatial components of \( q \). Note
particularly that in choosing the \( a_0 = 0 \) gauge we got rid
of the remaining contribution of the straight line path to
the phase factor in the expression for the Green’s func-
tion. Now we perform the average over the gauge fluctua-
tions

\[
\langle \exp(-i \int_0^\tau a_\mu d\tau \mu) \rangle = \exp\left( -\frac{1}{2} \int_0^\tau \int_0^\tau \dot{x}(\tau) \dot{x}(\tau') d\tau d\tau' \sum_{\vec{q},\vec{k}} \exp\left( \frac{i q \vec{x} \cdot \vec{k'}}{\sqrt{1 + \frac{1}{\tau^2}} + \frac{1}{\tau^2}} \right) \langle a(\vec{q}) a(\vec{k}) \rangle \right)
\]

The calculation is straightforward albeit some care has
to be taken with the boundary conditions. The averaging
results in

\[
\exp\left( -\frac{1}{4\pi^2} \int_0^\tau \int_0^\tau \theta(\tau - \tau') d\tau d\tau' \times
\left[ \frac{\dot{x}(\tau) \dot{x}(\tau') + 1}{|\dot{x} - \tau|^2 + (\tau - \tau')^2} \right] - \frac{1}{\tau^2 + (\tau - \tau')^2} \right)
\]

The last three terms in the above expression \( 2 \) arise
from the boundary conditions and thus explicitly depend
on the direction of propagation. In order to extract the
dependence on \( \vec{x} - \vec{x}_i \) explicitly we can now exploit the
rotational invariance in the Euclidean formulation to re-
express the above in vector notation.

\[
\exp\left( -\frac{1}{4\pi^2} \int_{\vec{x}_i - \vec{n}}^{\vec{x}_f - \vec{n}} \int_{\vec{x}_i - \vec{n}}^{\vec{x}_f - \vec{n}} \theta(\vec{n} \cdot \vec{n} - \vec{x} \cdot \vec{n}) \times
\left[ \frac{d\vec{x} \cdot d\vec{x}}{(\vec{x} - \vec{x})^2} - \frac{d\vec{x} \cdot d\vec{n} d\vec{x} \cdot \vec{n}}{|\vec{x} - (\vec{x} \cdot \vec{n})\vec{n}|^2} \right] - \frac{d\vec{x} \cdot d\vec{n} d\vec{x} \cdot \vec{n}}{|\vec{x} - (\vec{x} \cdot \vec{n})\vec{n}|^2} \right)
\]

with \( \vec{n} = \frac{\vec{x}_f - \vec{x}_i}{|\vec{x}_f - \vec{x}_i|} \) the unit vector along the classical
straight line path. Expressions \( 2 \) and \( 3 \) exhibit nicely
the retarded nature of the effective interaction. In prin-
cept, to obtain the propagator \( G(\vec{x}) \), we would now have
to perform the average over the trajectories with the free

particle action appropriate for the particle in question. For a massless particle we can neglect the spatial part of the retardation since it is down by a factor of the speed of “light” (the spinon velocity) which we have set equal to one. Hence we can see that only the term proportional to $\dot{x}(t) \dot{x}'(t')$ gives a non-vanishing contribution whose effect is to renormalize the mass of the particle. For a massless relativistic particle however both spatial and temporal retardation are equally important and the argument above cannot be applied. We find that the integrals in (6) are all dimensionless. In the large $\tau$ limit, those integrals have the form $c_1 \tau + c_2 \ln \tau$, where $c_0$ is the short distance cut-off scale. The $c_1$ term corresponds to mass generation. Within second quantization, as we will discuss next, the mass term cannot be generated if we regularize our theory at short distances in a way consistent with the underlying lattice symmetries + gauge structure [12]. Thus we can set the regularization dependent term $c_1$ to zero. The true leading term has a form $c_2 \ln \tau$ ($c_2$ is regularization independent). Those contributions can only change the exponent of the free Green’s function: $1/\tau^2 \rightarrow 1/\tau^{2-2\alpha}$. The effect of the gauge interaction is to modify the exponent in the algebraic decay of the Green’s function, which means that the gauge interaction is a marginal perturbation.

### III. MASSLESS DIRAC SPINORS

Having used the first quantized version of the path integral to guide our intuition we will henceforth return to field theory and in particular to the problem of massless Dirac spinors coupled to a U(1) gauge field. The Euclidean model which we are concerned with is the following

$$Z = \int D\bar{\Psi}D\Psi e^{i\int d^3x \sum_{\sigma=1}^{N} \bar{\Psi}_{\sigma}(\partial_{\mu} - ia_{\mu})\gamma_{\mu}\Psi_{\sigma})}$$

(7)

with the dynamics for the gauge field given by

$$Z_a = \int D\bar{a}D\bar{\phi} \exp\left(-\frac{1}{2} \int d^3q \bar{a}_{\mu}(\bar{\phi})\Pi_{\mu\nu}a_{\nu}(-\bar{\phi})\right)$$

$$\Pi_{\mu\nu} = \frac{N}{8} \sqrt{2} \left(\delta_{\mu\nu} - \frac{q_{\mu}q_{\nu}}{q^2}\right)$$

(8)

The Fermi field $\Psi_{\sigma}$ is a $4 \times 1$ spinor $\psi_{\sigma} = (\psi_{\sigma,1}^+, \psi_{\sigma,2}^+)$ with $\psi_{\sigma,1} = (f_{\sigma,1}^+, f_{\sigma,1}^{-})$, $\psi_{\sigma,2} = (f_{\sigma,2}^+, f_{\sigma,2}^{-})$. The notation (1,2,e,o) arises naturally when the theory is derived as the low energy effective description of the so called $\pi$-flux phase of Hubbard type lattice models [12]. There 1,2 denotes two types of Fermi points and e,o denote even and odd lattice sites respectively. Notice that we have generalized to $N$ fermion species in order to have a more controlled perturbation theory ($N=2$ in the physical case). The $4 \times 4$ $\gamma_{\mu}$ matrices form a representation of the Dirac algebra

\[ \{\gamma_{\mu}, \gamma_{\nu}\} = 2\delta_{\mu\nu} \quad (\mu, \nu = 0, 1, 2) \] and are taken to be

\[ \gamma_0 = \begin{pmatrix} 0 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} \]

where $\sigma_\mu$ the Pauli matrices. Finally note that $\bar{\Psi}_\epsilon = \Psi_{\epsilon}^{\dagger}\gamma_0$.

From the path integral discussion above we have learned a way to calculate a particular gauge invariant Green’s function for the Fermi fields. Note that a certain choice for the gauge has to be made before we can invert the polarization operator to obtain the gauge propagator. In order to get rid of the contribution from the straight line path in $G$, we can fix the gauge in such a way as to make it vanish along that very line. In this particular gauge, $G$ has a form of a normal one-particle Green’s function and can be calculated using the usual method through the self energy:

\[ G(\vec{p}) = G_0(\vec{p}) + G_0(\vec{p}) (\Sigma(\vec{p})) G_0(\vec{p}) + \cdots \]

where $\Sigma$ to lowest order in $1/N$ is the exchange diagram depicted in Fig. 1. However, the above trick only works for one particular direction $\vec{x}$ in real space. For different directions, we need to choose different gauges. We can put this into a more precise mathematical form. By choosing a gauge $\vec{n} \cdot \vec{a} = 0$, we can calculate the self energy and the corresponding Green’s function (which is denoted as $G_{\vec{n}}(\vec{x})$) through diagrams. However, $G_{\vec{n}}(\vec{x})$ is not equal to the gauge invariant Green’s function $G(\vec{x})$. But the two Green’s functions are closely related:

\[ G(\vec{x}) = G_{\vec{n}}(\vec{x}) |_{\vec{n} \cdot \vec{x} = |\vec{x}|} \]

(10)

In the following, we will calculate $G$ through calculating $G_{\vec{n} = \vec{x}}$.

To leading order in the gauge field fluctuations the fermion self-energy is

\[ -\Sigma(\vec{p}) = \int \frac{d^3q}{(2\pi)^3} i\gamma^\mu G_0(\vec{p} - \vec{q}) i\gamma^\nu D_{\mu\nu}(\vec{q}) \]

where $G_0 = \frac{-ik\gamma^\nu}{k^2}$ is the free fermion Green’s function. The sign convention for the self energy has been chosen such that

\[ G^{-1}(k) = ik\gamma^\mu + \Sigma(k) \]
To invert the gauge polarization, we will choose the $a_0 = 0$ gauge. This implies that our calculation gives a correct result for the equal-space propagation only. Employing relativistic invariance however, we can immediately extend the result to all of real space. With the above gauge choice

$$D_{\mu\nu}(\vec{q}) = \frac{1}{N} \frac{1}{\sqrt{q^2}} \delta_{\mu\nu} \left( \delta_{ij} + \frac{q_i q_j}{q_0^2} \right)$$

and we arrive at the following expression for the self-energy at $T=0$

$$\frac{N}{8} \int \frac{d^3 \vec{p}}{(2\pi)^3} \left[ \left( \frac{q_0 - p_0}{\sqrt{\vec{p}^2}} \right)^2 + 2 \frac{(q_0 - p_0) q_0^2}{\sqrt{\vec{p}^2} q_0^2} \right]$$

Hence

$$\Sigma_i(p) = -i \gamma_i p_0 \frac{1}{N \left( \frac{q_0}{p_0} \right) \gamma_0} \left( \frac{\Lambda}{|\vec{p}|} \right)^2$$

If we could read off the anomalous dimension of the fermion field and found the Green’s function $G_\tau(x)$

$$G_\tau(x) \propto \frac{\tau^0}{x^{2+2\gamma}}$$

Since the above $G_\tau$ has no explicit $\tau$ dependence, we find

$$G(x) = G_\tau(\vec{x}) \propto \frac{x^0}{x^{2+2\gamma}}$$

However, the coefficients of the two terms in Eq. (12) are not equal and the above calculation of $G(\vec{x})$ is not valid. But noting that the coefficients in Eq. (12) have the same sign - corresponding to a negative $\gamma_\Psi$ - this allows us to conclude that $G_\tau(\tau)$ has the form

$$G_\tau(\tau) \propto \frac{\tau^0}{\tau^{2+2\gamma}}$$

We see that the effect of gauge interactions is to make the propagator decay slower. In the path integral picture, the gauge interactions bring the particle path closer to the straight-line path so the area enclosed by the two paths is smaller. Since the particle path is closer to the straight-line path in the $\tau$ direction, this leads us to believe that the second $p_1$ term in Eq. (12) is not important compared to the first $p_0$ term. Hence

$$\gamma_\Psi = -\frac{N}{N \left( \frac{\Lambda}{N} \right) \gamma_0} \left( \frac{\Lambda}{N} \right)^2$$

This gives us

$$G \propto \frac{x^0}{x^{2+2\gamma}}$$

We would like to mention that the renormalization program within the temporal axial gauge is not established (see Ref. [4] for a discussion on renormalization within non-covariant gauges). To check and to confirm the result Eq. (13), we follow a different route where we consider in addition to the massless fermions coupled to the gauge field a heavy boson field whose “current” will take care of the straight line path in the equal space correlater. This type of formulation is borrowed from heavy quark effective theory approaches to QCD [5]. The task at hand is thus a calculation of the wavefunction renormalization of the boson fermion bilinear $j = \bar{\Psi} b$ with the following Lagrangian

$$\mathcal{L} = \bar{\Psi} \left( \partial_\mu \gamma^\mu + ia_\nu \gamma^\nu \right) \Psi - b^* (\partial_\nu + ia_\nu) b + \mathcal{L}_a$$

In order to calculate the wavefunction renormalization of the above composite we need the field strength renormalizations $Z_\Psi, Z_b$ for both the fermion and the boson field as well as the vertex renormalizations $Z_\Gamma$. To obtain the boson wavefunction renormalization we calculate its self energy due to the gauge field fluctuations Fig. [2]. As can be seen from the above Lagrangian, the boson being static only couples to the Coulomb field. Thus we arrive at the following expression for the boson self energy.

$$-\Sigma_b(p_0) = \int \frac{d^3 \vec{q}}{(2\pi)^3} \frac{\gamma_0 D_{\mu\nu}(\vec{q}) \gamma_0}{i(p_0 - q_0)}$$

FIG. 2: Exchange diagram used in the calculation of the boson self-energy.
FIG. 3: Boson-Fermion vertex with the $p_0$ flowing along the static boson line

whose divergent part evaluates (see appendix) to

$$i\Sigma_b(p_0) = p_0 \frac{8}{\pi^2 N^3 - d}$$

where the sign convention for the self energy has been chosen in accord with the one for the fermion field mentioned above.

Thus we find for the wavefunction renormalization the minimal

$$G_b(p_0) = \frac{1}{ip_0 + \Sigma(p_0)} = \frac{Z_b}{ip_0} \quad Z_b = 1 + \frac{8}{\pi^2 N^3 - d}$$

This result has been obtained in the Landau gauge. The fermion wavefunction renormalization can be obtained similarly from the self energy and reads in the Landau gauge

$$Z_\Psi = 1 + \frac{8}{3\pi^2 N^3 - d}$$

Finally we need to obtain the vertex renormalization. From Fig. 3 we can read off the following expression for the vertex correction

$$\mathbb{I} \int \frac{d^3l}{(2\pi)^3} \frac{\gamma^\mu ( -i ) \gamma^\nu ( k + l ) \gamma^c \delta_{\mu\nu} i D_{\mu\nu}(l)}{(k + l)^2 (p_0 + l_0)}$$

where $\mathbb{I}$ is the fermion-boson vertex whose renormalization is sought. By simple powercounting we see that the loop integration is only logarithmically divergent and hence we can evaluate it by setting all external momenta equal to zero which leaves us with

$$\mathbb{I} \int \frac{d^3l}{(2\pi)^3} \left[ \frac{\gamma^0 l_0 - \gamma^\mu \gamma^\nu l_\mu l_\nu}{l^5} \right] = \mathbb{I} \int \frac{d^3l}{(2\pi)^3} \left[ \frac{1}{l^3} - \frac{l^2}{l^5} \right] = 0$$

where the first term was simplified by the fact that only the $\gamma^0 l_0$ term survives integration over spatial momenta. Thus we conclude that within the Landau gauge the vertex doesn’t renormalize to one loop order.

We now have all the ingredients to calculate the wave function renormalization of the fermion boson bilinear in the minimal form

$$Z_j = Z_{b_1/2} Z_{\Psi}^{1/2} Z_{\Gamma} = 1 + \frac{16}{N} \frac{1}{3\pi^2} \frac{1}{3 - d}$$

(22)

This result is gauge invariant. From the above discussion we can see that within the axial temporal gauge there is no Coulombic interaction and hence both the vertex correction and the boson wavefunction renormalization are absent which would suggest that

$$Z_\Psi(a_0 = 0) = Z_j^2 = 1 + \frac{16}{N} \frac{1}{3\pi^2} \frac{2}{3 - d}$$

which yields

$$\gamma_\Psi = -\frac{16}{N} \frac{1}{3\pi^2}$$

for the anomalous dimension of the fermion field.

An important point to note here is the fact that the gauge fluctuations do not generate a mass or a chemical-potential term perturbatively. In a diagram calculation, a regularization dependent mass/chemical-potential term actually appears in the self-energy. We have set such a regularization dependent mass/chemical-potential term equal to zero. In the next section we will see that our Dirac-$U(1)$ gauge system is the low energy effective description of the so called staggered flux (SF) phase (the mean-field phase corresponding to the pseudogap regime within the $SU(2)$ slave boson theory). The lattice SF phase provides a short distance regularization. Now the question is whether such a regularization generates a regularization dependent mass/chemical-potential term or not. One way to answer the above question is through a symmetry argument. In Ref. [11], it is shown that any change of the mean-field SF ansatz cannot generate a mass term or chemical potential term if the change does not break any symmetries and does not break the $U(1)$ gauge structure. This implies that if the regularization is consistent with the underlying lattice symmetries and does not break the $U(1)$ gauge structure, then the regularization cannot generate any mass/chemical-potential terms. This is the reason why we can set the regularization dependent mass/chemical-potential term equal to zero. We see that the massless Dirac fermion is protected by the lattice symmetries and by the $U(1)$ gauge structure (or more precisely, protected by the quantum order[11]). Thus what we really have learned from the diagram calculation is that the mass or chemical-potential is not generated spontaneously from the infrared divergences (at the perturbative level). Combining the symmetry consideration for the high energy cutoff and the diagram calculation for the infrared, we find that no mass or chemical-potential is generated from the self energy corrections to the fermion Green’s function. In contrast, the first quantized path integral analysis does not know about the underlying lattice symmetries and, as a result allows for corrections to the mass or chemical-potential...
due to dressing with the short-distance gauge fluctuations (see discussion in section 3). We would like to mention that the Dirac-$U(1)$ gauge system was studied in detail in Ref. [13] using a different gauge fixing. It was shown that the mass/chemical-potential term is not generated perturbatively to all orders in a $1/N$ expansion. Furthermore note that since the conserved current (that couples to $a_{\mu}$) cannot have any anomalous dimension, the gauge interaction is an exact marginal perturbation.

Hence we can take the anomalous dimension seriously and obtain the full Green’s function in the form

$$G(\vec{k}) = -iC \frac{\gamma^{\mu}}{k^{2} - 2\gamma^{\mu}} \gamma^{\nu} = \frac{16}{N} \frac{1}{3\pi^{2}} \quad (24)$$

where $C$ is determined by the energy range over which the effective theory is supposed to be valid. Comparing this dressed propagator with the free fermion Green’s function $G_{0} = \frac{\gamma^{\mu}}{k^{2}}$ we see that the inclusion of the gauge fluctuations has destroyed the coherent quasiparticle pole.

At this point a couple of comments are in order. So far we have not addressed the fact that the sign of the particle pole. In hindsight this can be understood as a sign of confinement - the effect of the gauge fluctuations is the tendency to form a bound state between the two particles. This picture of confinement is one of central importance to the question of spin charge separation.

In the following we will outline a calculation of the electron spectral function which is more directly rooted in the SU(2) approach of Wen and Lee [4]. Here we include the effect of the gauge fluctuations - the confining tendency, gleaned from the above analysis - more heuristically.

IV. CONFINEMENT BY HAND

It was mentioned above that the problem of massless Dirac particles coupled to a gauge field is intimately related to a possible model description of the copper oxide planes in the high $T_{c}$ superconductors [12]. To keep the discussion reasonably self-contained we will first relate some of the details of the mean field flux state as it appears in the SU(2) slave boson formulation [4].

The slave boson theories approach the copper oxide problem from the insulating side; restricting the single particle Hilbert space on each Copper site to three states with the doubly occupied one prohibited via the use of a Lagrangian multiplier field. Within a path integral implementation of the above mentioned constraint it is the Lagrange multiplier that gets promoted to the temporal component of a gauge field. The starting point for the investigation is the t-J Hamiltonian which is believed to contain the essential physics and exhibits nicely the competition of kinetic versus local spin fluctuations. Within the SU(2) formulation the physical electron operator is represented as follows:

$$c_{\uparrow i} = \frac{1}{\sqrt{2}} h_{1} \psi_{\uparrow i} = \frac{1}{\sqrt{2}} (b_{1i}^{\dagger} f_{\uparrow i} + b_{2i}^{\dagger} f_{\downarrow i})$$

$$c_{\downarrow i} = \frac{1}{\sqrt{2}} h_{1} \psi_{\downarrow i} = \frac{1}{\sqrt{2}} (b_{1i}^{\dagger} f_{\downarrow i} - b_{2i}^{\dagger} f_{\uparrow i})$$

where the following SU(2) doublets were introduced

$$\psi_{\uparrow i} = \left( f_{\uparrow i}^{\dagger}, f_{\downarrow i}^{\dagger} \right), \quad \psi_{\downarrow i} = \left( f_{\downarrow i}^{\dagger}, -f_{\uparrow i}^{\dagger} \right), \quad h_{i} = \left( b_{1i}, b_{2i} \right)$$

The $\psi_{\uparrow i}, \psi_{\downarrow i}$ are the two fermion fields representing the destruction of a spin up and spin down on site $i$ respectively and $h_{i}$ is the doublet of bosonic fields keeping track of the doped holes. The procedure is now to put this representation into the t-J Hamiltonian

$$H = P \sum_{\langle ij \rangle} \left[ J (\vec{S}_{i} \cdot \vec{S}_{j} - 1/4 n_{i} n_{j}) - t (c_{\sigma i}^{\dagger} c_{\sigma j} + h.c.) \right] P$$

which on performing a Hubbard-Stratonovich transformation to the appropriate bosonic bond variables yields the following partition function

$$Z = \int Dh_{i} D\psi_{i} D\tilde{a}_{0} DU e^{-L_{0}/L}$$

$$L = \frac{J}{2} \sum_{<ij>} Tr[U_{ij}^{\dagger} U_{ij}] + \frac{1}{2} \sum_{i,j,\sigma} \psi_{\sigma i}^{\dagger} (\partial_{\tau} \psi_{\sigma i} + JU_{ij}) \psi_{\sigma j}$$

$$+ \sum_{i,l} a_{il}^{\dagger} \left( \frac{1}{2} \psi_{\sigma i}^{\dagger} \tau^{\dagger} \psi_{\sigma i} + h_{l}^{\dagger} \tau^{\dagger} h_{l} \right)$$

$$+ \sum_{i,j} h_{l}^{\dagger} \left( (\partial_{\tau} - \mu) \delta_{ij} + \tilde{b} U_{ij} \right) h_{l}$$

The $\tilde{a}_{0}$ fluctuations incorporate the projection to the space of SU(2) singlets. Furthermore note that $J = 3J/8$, $\tilde{b} = t/2$ [13] and the matrix $U_{ij}$ in the form

$$U_{ij} = \begin{pmatrix} -\chi_{ij} & \Delta_{ij} \\ \Delta_{ij} & \chi_{ij} \end{pmatrix}$$
contains the Hubbard-Stratonovich fields which classify the part of the phase diagram we are looking at. The mean-field phase diagram (see Introduction) is found by minimizing the free energy for a given number of bosons with respect to the bond variables \( U_{ij} \). The phase which will be of interest to us is the so-called staggered flux (sF) phase which can be represented as

\[
U_{i,i+\hat{x}} = -\tau^3 \chi - i(-)^{i\hat{x}+\hat{y}} \Delta \\
U_{i,i+\hat{y}} = -\tau^3 \chi + i(-)^{i\hat{x}+\hat{y}} \Delta
\]

(28)

The slave boson representation results in the following expression for the physical electron Green's function

\[
G(\vec{r}, \tau) = -\langle T_\tau (c_\uparrow(\vec{r}, \tau)c_\downarrow(\vec{0}, 0)) \rangle \\
= \frac{1}{2} \langle T_\tau ((h^1(\vec{r}, \tau)\psi_\uparrow(\vec{r}, \tau)\psi_\downarrow(\vec{0}, 0)h(\vec{0}, 0)) \rangle
\]

In the sF state we don't have any anomalous fermion-fermion pairing (both \( \tau^3 \) and \( \mathbb{I} \) are diagonal in isospin space) and hence this simplifies to

\[
G(\vec{r}, \tau) = -\frac{1}{2} \left( b_1^\dagger(\vec{r}, \tau)b_1(\vec{0}, 0)f_\uparrow(\vec{r}, \tau)f_\downarrow(\vec{0}, 0) \right) \\
+ \langle b_2^\dagger(\vec{r}, \tau)b_2(\vec{0}, 0)f_\uparrow(\vec{r}, \tau)f_\downarrow(\vec{0}, 0) \rangle
\]

The angle brackets \( \langle . . . \rangle \) mean averaging with respect to the partition function \( \Omega \). The complication with this task however is the presence of light bosons whose tendency to condense prohibits sensible perturbation theory. Our strategy here will be to first calculate the following mean field decomposition

\[
G(\vec{r}, \tau)_{MF} = -\frac{1}{2} \left( b_1^\dagger(\vec{0}, 0)b_1(\vec{r}, \tau)f_\uparrow(\vec{r}, \tau)f_\downarrow(\vec{0}, 0) \right) \\
- \frac{1}{2} \langle b_2^\dagger(\vec{0}, 0)b_2(\vec{r}, \tau)f_\uparrow(\vec{r}, \tau)f_\downarrow(\vec{0}, 0) \rangle
\]

and then incorporate the confining effect of the gauge field by making the algebraic decay of the spinon Green's function in real space slower. The motivation for this is rooted in the result of the previous section (the negative anomalous dimension for the spinon amplitude at the ASL fixed point) and we will discuss the implied approximations down below. With the help of

we can express the Fourier transform of \( G(\vec{r}, \tau)_{MF} \) in the form

\[
G(\vec{k}, \omega_n) = \frac{-1}{2} \int \frac{d\vec{q} d\nu_n}{(2\pi)^3} G_{f\uparrow}(\vec{q}, \nu_n)G_{b_1}(\vec{k} - \vec{q}, \omega_n - \nu_n) \\
+ \frac{1}{2} \int \frac{d\vec{q} d\nu_n}{(2\pi)^3} G_{f\downarrow}(\vec{q}, \nu_n)G_{b_2}(\vec{k} - \vec{q}, \omega_n - \nu_n)
\]

To make contact with our previous discussions let us concentrate on the fermionic part of the theory for the moment. From the form \( (28) \) of the mean field gauge we see that translational invariance is explicitly broken. After Fourier transformation the fermionic Lagrangian is given in the \( f \) basis by

\[
L_{MF} = \sum_{\vec{q}, \omega_n, \sigma} \left( f_{\uparrow \sigma}^\dagger(\vec{q}, \omega_n) \right) \left( f_{\uparrow \sigma}(\vec{q}, \omega_n + \tilde{Q}) \right) \\
- \omega_n \mathbb{1} + \epsilon_f(\vec{q})\sigma_3 + \eta_f(\vec{q})\sigma_2 \left( f_{\sigma}(\vec{q}, \omega_n) \right)
\]

where \( \sum' \) denotes a summation over the magnetic Brillouin zone which is half the size of the original one and takes account of the fact that in the sF phase the real space unit cell has been doubled with principal axis along \( \pm \hat{x} \pm \hat{y} \). The \( \sigma \) matrices operate in \( \vec{q}, \vec{q} + \tilde{Q} \) space with \( \tilde{Q} = (\pi, \pi) \). Furthermore note that

\[
\epsilon_f(\vec{q}) = -2\tilde{J}\chi(\cos(q_x a) + \cos(q_y a)) \quad (29) \\
\eta_f(\vec{q}) = -2\tilde{J}\Delta(\cos(q_x a) - \cos(q_y a)) \quad (30)
\]

We can now simply obtain

\[
G_{f\uparrow}(\vec{k}, \nu_n) = \frac{i\nu_n + \epsilon_f - i\eta_f}{\nu_n^2 + \tilde{E}_f^2} \quad \tilde{E}_f^2 = \epsilon_f^2 + \eta_f^2 \quad (31)
\]

\[
G_{f\downarrow}(\vec{k}, \nu_n) = \frac{i\eta_f - i\nu_n + \epsilon_f}{\nu_n^2 + \tilde{E}_f^2} \quad (32)
\]

by inverting the matrix in the above Lagrangian \( L_{MF} \). Analogously we find for the boson Green’s functions

\[
G_{b_1}(\vec{k}, \omega_m) = \frac{i\omega_m - \mu + \epsilon_b - i\eta_b}{(i\omega_m)^2 - \tilde{E}_b^2} \quad \tilde{E}_b^2 = \epsilon_b^2 + \eta_b^2 \\
G_{b_2}(\vec{k}, \omega_m) = \frac{i\omega_m - \mu + \epsilon_b - i\eta_b}{(i\omega_m)^2 - \tilde{E}_b^2} \quad (33)
\]

where \( \mu \) is the chemical potential.

The standard way to proceed is now to calculate the convolution integrals in momentum space to obtain the mean field Green’s function. As outlined above we will however follow the different route of first calculating the Fourier transform of the above Green’s functions.

Let us first concentrate on the boson part which can be written as

\[
G_{b_1}(\vec{k}, \omega_m) = \frac{E_b + \epsilon_b + i\eta_b}{2E_b(i\omega_m - \mu + E_b)} + \frac{E_b - \epsilon_b - i\eta_b}{2E_b(i\omega_m - \mu - E_b)}
\]

(33)
Expanding for small $\tilde{k}$ we obtain
\[ E_0(\tilde{k}) \approx 4\tilde{\chi} - i\chi(ka)^2 \]
\[ \epsilon(\tilde{k}) \approx -4\tilde{\chi} + i\chi(ka)^2 \]
\[ \eta(\tilde{k}) \approx i\Delta a^2[k_x^2 - k_y^2] \]
Since $-\mu - E_b(0) > 0$ we can neglect the contribution from the first term and Fourier transform the second to obtain
\[ G_{b_1}(\vec{r}, \tau) = \frac{1}{2} \Theta(-\tau)e^{i(\mu + 4t\chi)|\tau|}| \begin{array}{c} \text{sign}(\tau) + \frac{\epsilon_f - i\eta_f}{E_f} \end{array} \]
\[ \int \frac{d^2k}{(2\pi)^2} e^{ik\cdot\vec{r}} \left[ 1 - \frac{\epsilon_b + i\eta_b}{E_b} \right] e^{-i\chi(ka)^2|\tau|} = \Theta(-\tau)e^{i(\mu + 4t\chi)|\tau|}| \begin{array}{c} \text{sign}(\tau) + \frac{\epsilon_f - i\eta_f}{E_f} \end{array} \] (34)
(35)
(36)
where $m_b = \frac{1}{2\sqrt{\pi\alpha}}$ is the band mass of the boson. (34) is nothing but the propagator for a particle of mass $m_b$ in 2d. Fourier transforming $G_{f_1}(\tilde{k}, \omega_n)$ with respect to frequency yields
\[ G_{f_1}(\tilde{k}, \tau) = \frac{1}{2} \Theta(-\tau)e^{i(\mu + 4t\chi)|\tau|}| \begin{array}{c} \text{sign}(\tau) + \frac{\epsilon_f - i\eta_f}{E_f} \end{array} \]
\[ \int \frac{d^2k}{(2\pi)^2} e^{ik\cdot\tilde{r}} \left[ 1 - \frac{\epsilon_b + i\eta_b}{E_b} \right] e^{-i\chi(ka)^2|\tau|} \]
\[ = \Theta(-\tau)e^{i(\mu + 4t\chi)|\tau|}| \begin{array}{c} \text{sign}(\tau) + \frac{\epsilon_f - i\eta_f}{E_f} \end{array} \] (37)
Next we expand the dispersion about the node $Q_1 = \left( \frac{x}{2a}, \frac{y}{2a} \right)$
\[ \epsilon_f(Q_1 + \tilde{p}) = 2\chi\tilde{J}a(\tilde{p}_x + \tilde{p}_y) \equiv v_f p_1 \]
\[ \eta_f(Q_1 + \tilde{p}) = -2\Delta\tilde{J}a(-\tilde{p}_x + \tilde{p}_y) \equiv v_p p_2 \]
where $v_f = 2\sqrt{2}\chi\tilde{J}a$ and $v_p = 2\sqrt{2}\Delta\tilde{J}a$ and we resort to the notation $\tilde{p} = \tilde{p} \times 2d$ vectors.
We now split up the momentum integral $\int d^2k/(2\pi)^2 e^{ik\cdot\tilde{r}}$ where $Q_i$ with $i = 1...4$ corresponds to $(\pi/2, \pi/2), (-\pi/2, -\pi/2), (\pi/2, -\pi/2), (-\pi/2, \pi/2)$ in this order and we have set a the lattice spacing equal to unity.
The momentum Fourier transform about $Q_1$ is then given by
\[ \int Q_1 \frac{d^2\tilde{p}}{(2\pi)^2} e^{i(Q_1 + \tilde{p})\cdot\tilde{r}} = \int \frac{d^2\tilde{p}}{(2\pi)^2} e^{i(Q_1 + \tilde{p})\cdot\tilde{r}} \]
\[ \left[ \text{sign}(\tau) + \frac{v_f p_1 + i v_p p_2}{\sqrt{(v_f p_1)^2 + (v_p p_2)^2}} \right] = \frac{e^{iQ_1\cdot\tilde{r}} [\tau + i\tilde{x}_1 - \tilde{y}_1]}{4\pi v_f v_2 \left( \tau^2 + \tilde{r}_1^2 \right)} \]
\[ \text{with } \tilde{x}_1 = \frac{x+y}{\sqrt{v_f v_p}} \quad \tilde{y}_1 = \frac{y-x}{\sqrt{v_f v_p}} \]
Analogously we can obtain the contributions about the other three nodes - hence
\[ G_{f_1}(\tilde{r}, \tau) = \int \frac{d^2k}{(2\pi)^2} e^{ik\cdot\tilde{r}} \frac{1}{2} e^{-E_k|\tau|} \left[ \text{sign}(\tau) + \frac{\epsilon_f - i\eta_f}{E_f} \right] \]
\[ = \frac{e^{iQ_1\cdot\tilde{r}} [\tau + i\tilde{x}_1 - \tilde{y}_1]}{4\pi v_f v_2 \left( \tau^2 + \tilde{r}_1^2 \right)} + \frac{e^{iQ_2\cdot\tilde{r}} [\tau - i\tilde{x}_2 + \tilde{y}_2]}{4\pi v_f v_2 \left( \tau^2 + \tilde{r}_2^2 \right)} \]
\[ + \frac{e^{iQ_3\cdot\tilde{r}} [\tau - i\tilde{y}_3 + \tilde{x}_3]}{4\pi v_f v_2 \left( \tau^2 + \tilde{r}_3^2 \right)} + \frac{e^{iQ_4\cdot\tilde{r}} [\tau + i\tilde{y}_4 - \tilde{x}_4]}{4\pi v_f v_2 \left( \tau^2 + \tilde{r}_4^2 \right)} \]
where $\tilde{x}_3 = \tilde{x}_4 = \frac{x+y}{2\sqrt{v_f v_p}} \quad \tilde{y}_3 = \tilde{y}_4 = \frac{y-x}{2\sqrt{v_f v_p}}$ and $\tilde{r}_1 = \tilde{r}_2$. Identical calculations as given above lead to the expressions for $G_{f_1}(\tilde{r}, \tau)$ and $G_{b_2}(\tilde{r}, \tau)$. Putting all of them together we finally obtain
\[ G^e(\tilde{r}, \tau) = \Theta(-\tau)e^{i(\mu + 4t\chi)|\tau|} \left( \frac{m_b}{2\pi} \right) e^{-\frac{m_b^2}{4\pi|\tau|}} \times \]
\[ \left\{ \begin{array}{l} e^{iQ_1\cdot\tilde{r}} [\tau + i\tilde{x}_1 - \tilde{y}_1] \frac{1}{4\pi v_f v_2 \left( \tau^2 + \tilde{r}_1^2 \right)} \\
+ e^{iQ_2\cdot\tilde{r}} [\tau - i\tilde{x}_2 + \tilde{y}_2] \frac{1}{4\pi v_f v_2 \left( \tau^2 + \tilde{r}_2^2 \right)} \\
+ e^{iQ_3\cdot\tilde{r}} [\tau - i\tilde{y}_3 + \tilde{x}_3] \frac{1}{4\pi v_f v_2 \left( \tau^2 + \tilde{r}_3^2 \right)} \\
+ e^{iQ_4\cdot\tilde{r}} [\tau + i\tilde{y}_4 - \tilde{x}_4] \frac{1}{4\pi v_f v_2 \left( \tau^2 + \tilde{r}_4^2 \right)} \end{array} \right\} \]
(38)
At this point let us make the following replacement in the spinon part of the physical hole propagator.
\[ \frac{1}{(\tau^2 + \tilde{r}_i^2)^{\frac{3}{2}}} \rightarrow \frac{\Gamma_0}{(\tau^2 + \tilde{r}_i^2)^{\frac{3}{2}}} \]
(39)
This procedure is motivated by revisiting the following first quantized path integral formulation for the electron Green's function
\[ G(\tilde{r}, \tau) = N^{-1} \int DaDx f D\epsilon(-\tau) dr(L_f + L_b + L_a) e^{i\int_{i\tau} a\cdot dx - i\int_{i\tau} a\cdot dx} \]
\[ e^{i\int_{i\tau} a\cdot dx - i\int_{i\tau} a\cdot dx} \]
where $\int_{i\tau}$ integrates along the boson path $x_b(\tau)$ and $\int_{\tau}$ integrates along the fermion path $x_f(\tau)$. $L_f, b, a$ are Lagrangians for the fermion, the boson, and the gauge fields. Let $| be the straight line connecting $(0, 0)$ and $(\tilde{r}, \tau)$. Since the fermion is relativistic, the area spanned by the fermion path $\gamma_f$ and $| is of order $L_f^2$, where $L_f^2 = r^2 + v_f^2 \tau^2$ and $v_f$ is the fermion velocity. For the non-relativistic boson, the area spanned by the boson path $\gamma_b$ and $| is of order $v_b r \sqrt{\tau^2/2m_b}$, with $m_b$ the boson mass. For $\tau > 1/m_b v_f^2$, the area between the loop spanned by $\gamma_f$ and $\gamma_b$ mainly comes from the area between $\gamma_f$ and $|$ and we can approximate $\int_{\gamma_b} by $\gamma_f$.
\[ G = N^{-1} \int DaDx f D\epsilon(-\tau) dr(L_f + L_b + L_a) e^{i\int_{i\tau} a\cdot dx - i\int_{i\tau} a\cdot dx} \]
\[ \gamma_f \]
(40)
\[ G_f = \]
\[ N^{-1} \int DaDx f D\epsilon(-\tau) dr(L_f + L_a) e^{i\int_{i\tau} a\cdot dx - i\int_{i\tau} a\cdot dx} \]
where $G_{b_0}$ is the free boson Green's function (not coupled to the gauge fields). Note that the fermion Green's function $G_f$ is just the gauge invariant Green's function discussed in the first part of the paper. Hence we see that by performing the above replacement (39) with $\alpha = 2\gamma_{\Phi} \sim 0.54$ we make the transition from the free spinon Green's function to the particular gauge invariant spinon amplitude discussed in section. Furthermore we note here that our previous attempt of identifying the
FIG. 4: The top figure shows the good agreement between a direct evaluation of the convolution integral in momentum space (dashed line) and the result based on equation (44) the Fourier Transform with $\alpha = 0$. The bottom figure depicts the evolution of the spectrum as we increase $\alpha$. The value $\alpha = 0.54$ is “motivated” by our earlier analysis. Note that $t/J = 3$ and we have used the implementation of the hypergeometric function given in “Numerical Recipes in C”, W.H. Press et al. [20].

physical hole Green’s function with the gauge invariant spinon amplitude corresponds to neglecting the $\frac{1}{m_b}$ term in (38) which arises from the fluctuations of the holon about the classical path (the static straight line in the $m_b \to \infty$ limit). In what follows we will analyze the full expression (38) under the replacement (41) where the fluctuations of the holons are assumed decoupled from the gauge field (hence the free boson $\frac{1}{m_b}$ contribution from the fluctuations around the classical path). This in particular implies that the area between the loop spanned by $f$ and $b$ is solely given by the relativistic spinon path combined with the straight line segment which from the previous discussion is only exact in the limit $m_b \to \infty$ - the static holon case. Thus the value $\alpha = 2\gamma \Phi \approx 0.54$ reflects the total gauge fluctuation effects strictly speaking only for the static holon case. Granted these limitations of the present analysis let us press on and analyze the effect of a finite positive $\alpha$. Note also that we introduced

$\Gamma$ with units of energy to balance dimensions whose value will be on the order of the lattice scale.

Thus our task now is to calculate

$$G^e(\vec{k}, \omega_n) = \int d^2 r d\tau e^{-i\vec{k} \cdot \vec{r} + i\omega_n \tau} G^e(\vec{r}, \tau) \quad (41)$$

To do this we first concentrate on $\vec{k} = Q_1 + \vec{p}$ and note that the other three points are related by symmetry. From expression (38) we see that only the first term in curly brackets contributes since all the other terms have large phases even in the infrared limit. We are thus left
with the evaluation of
\[
G^\sigma(Q_1 + \bar{p}, \omega_n) = \int d^2r \int_0^\infty d\tau e^{-i\vec{p} \cdot \vec{r}} e^{-[i\omega_n + |\mu| - 4\tau\chi] \tau} \times \frac{m_b L^{-\alpha}}{2\pi \tau} e^{-\frac{|\mu x^2|}{4\pi\tau}} \left[ \frac{\tau - i\frac{\sqrt{\tau^2 + r^2}}{2\pi}}{4\pi(\tau^2 + r^2)^{\frac{3}{2}}} \right]^{\frac{\nu}{2}}
\]  

(42)

where we have simplified to the case \( v_f = v_2 = v \) and have taken care of the fact that our analysis only allows us to calculate the hole spectral function which was indicated by the \( \Theta(-\tau) \) factor in the boson propagator.

Next let us analyze expression (42) in some limiting cases where the integrals can be performed exactly. The general result and some more technical points can be found in the appendix.

The simplest point to analyze is the expression right at the node \( \vec{p} = 0 \) where the terms proportional to \( x \times y \) average to zero on angular integration and we are left with

\[
G^\sigma(Q_1, \omega_n) = K_1 \int_0^\infty d\tau r \int_0^\infty d\tau e^{-[i\omega_n + |\mu| - 4\tau\chi] \tau} \times \frac{e^{-\frac{m_b x^2}{4\pi\tau}}}{(\tau^2 + r^2)^\frac{3}{2}}
\]

where \( K_1 = \frac{m_b L^{-\alpha}}{2\pi} L = v/\Gamma \). The \( r \) integral can be evaluated via

Gradshteyn Ryzhik 5 edition (GR5) 3.382 (4)

\[
\int_0^\infty dx (x + \beta)^\nu e^{-\mu x} dx = \frac{1}{\mu^{1+\nu}} e^{\beta \mu} \Gamma(\nu + 1, \beta \mu)
\]

[arg\beta] < \pi, \ Re \mu > 0

where \( G(x, y) \) is an incomplete Gamma function. The resulting \( \tau \) integral can again be performed with the help of (GR5) 6.455 (1)

\[
\int_0^\infty x^{\mu - 1} e^{-\beta x} \Gamma(\nu, \alpha x) dx = \frac{\alpha^\nu \Gamma(\mu + \nu)}{\mu(\alpha + \beta)^{\mu + \nu}} 2 F_1 \left( 1, \mu + \nu; \mu + 1; \frac{\beta}{\alpha + \beta} \right)
\]

\( \text{Re} (\alpha + \beta) > 0, \ Re \mu > 0, \ Re (\mu + \nu) > 0 \)

where \( 2 F_1 \) is a hypergeometric function and we have the following identifications \( \beta = [i\omega_n + |\mu| - 4\tau\chi] - \frac{m_b x^2}{2} \alpha = \frac{m_b x^2}{2} \mu = \frac{\alpha + 1}{2} \nu = \frac{\alpha - 1}{2} \). Putting this together we arrive at

\[
G^\sigma(Q_1, \omega_n) = \frac{m_b \Gamma(\nu + 1)}{4\pi(\alpha + 1)} e^{\beta \mu} \Gamma(\nu + 1, \beta \mu)
\]

(43)

\[
2 F_1 \left( 1, \alpha; \frac{\alpha + 3}{2}; 1 - \frac{m_b \nu^2}{2[i\omega_n + |\mu| - 4\tau\chi]} \right)
\]

We are now ready to perform the analytic continuation \( i\omega_n \rightarrow \omega - i0^+ \) to obtain the hole spectral function at the node \( Q_1 \)

\[
A_-(Q_1, \omega) = \frac{m_b \Gamma(\alpha)}{4\pi(\alpha + 1)} \Theta(-\omega) \left[ \sin(\pi \alpha) \pi [\omega - |\mu| + 4\tau\chi]^{-\alpha} \Theta(\omega - |\mu| + 4\tau\chi) \right.
\]

\[
\text{Re} 2 F_1 \left( 1, \alpha; \frac{\alpha + 3}{2}; 1 - \frac{m_b \nu^2}{2[i\omega_n + |\mu| - 4\tau\chi - i0^+]} \right)
\]

(44)

\[
+ \frac{\cos(\pi \alpha)}{\pi} [\omega - |\mu| + 4\tau\chi]^{-\alpha} \Theta(\omega - |\mu| + 4\tau\chi) \]

\[
\text{Im} 2 F_1 \left( 1, \alpha; \frac{\alpha + 3}{2}; 1 - \frac{m_b \nu^2}{2[i\omega_n + |\mu| - 4\tau\chi - i0^+]} \right)
\]

Note that in principal we also have a contribution from \( \Theta(|\mu| - 4\tau\chi + \omega) \) however since the chemical potential is exponentially close to the bottom of the band we will set \( |\mu| \sim 4\tau\chi \) for our calculation. It is nevertheless important in the derivation of the above that \( |\mu| - 4\tau\chi = 0^+ \) for the integrals to converge. In Fig. 10 we show the effect of a non-zero \( \alpha \) on the spectrum at the node. The top figure simply compares the two results for the spectrum based on a direct evaluation of the convolution between single

boson and spinon Green’s functions (dashed line) and

the evaluation of Eq. (44) for the case \( \alpha \sim 0 \) where they should agree. The bottom portion of Fig. 10 depicts the evolution of the spectrum as we increase \( \alpha \) away from zero. Though still rather incoherent a shift of spectral weight toward lower energies is apparent. This piling up of low energy spectrum is directly associated with a nonzero alpha which in turn is a consequence of the gauge interaction between the holon and the spinon responsible
for the incoherent mean-field spectrum. Thus we see that when our reference system is the incoherent spectrum associated with the independent propagation of spinon and holon the gauge fluctuations have the desirable and intuitive effect of binding the two degrees of freedom back into the physical hole. This should be compared with our initial attempt (in the first half of the paper) to reference the gauge fluctuation effects with respect to simply a free spinon plus a coherent boson. A nearly coherent boson (with a well defined phase) is however not consistent with a massless gauge field whose origin lies in the strong local phase fluctuations of both holon and spinon.

Fig. shows a similar comparison for momentum $p_x = p_y = -0.1$ along the diagonal. This result is based on the general equation derived in Appendix C. The evolution with $\alpha$ follows the expected trend. The onset of the spectrum centered at the boson energy.

We can now view this as a 3d integral over the positive hemisphere where $\bar{\tau} = R cos \theta \quad r = R sin \theta$ and after integrating out $\phi$ and $R$ we are left with

$$G^s(Q, \omega_n) = \frac{K_1}{v} \int_0^1 dx \frac{\Gamma(\alpha)}{[\phi (x, 0) - |\phi| - 4\bar{\tau} \chi] + \frac{m^2 v^2}{2x^2}(1 - x^2)\alpha}$$

After analytic continuation we arrive at the following expression for the hole spectral function

$$A_-(Q, \omega) = \Theta(-\omega) \frac{K_1}{v} \frac{sin \pi \alpha \Gamma(\alpha)}{\pi} \int_0^1 dx \Theta \left[ |\omega| - |\mu| + 4\bar{\tau} \chi + \frac{m_b v^2}{2} \frac{m_b v^2}{2x^2} \right]^{x^\alpha v^\alpha} [x^2 (|\omega| - |\mu| + 4\bar{\tau} \chi) - \frac{m_b v^2}{2x^2}(1 - x^2)]$$

which in the $\alpha \rightarrow 0^+$ limit reduces to

$$A_-(Q, \omega) = \Theta(-\omega) \frac{K_1}{v} \frac{1}{\sqrt{2 |\omega| - |\mu| + 4\bar{\tau} \chi + m_b v^2}}$$

We thus find that $\frac{1}{m_b}$ determines the slope of the spectrum at the onset. In Fig. we depict the effect of decreasing the mass of the exponential by a factor of 3 ($t = 0.7$) and 4 respectively for the case $\alpha = 0$. Note that in addition to becoming steeper in accord with the result just derived for the case $\bar{\tau} = 0$ the onset of the spectrum is pushed to higher frequencies initially simply following the increase in $\frac{1}{m_b}$. The curves seem to converge to a step function, however, the onset of the spectrum does not converge to $pv$ the energy of the spinon but rather $\frac{pv}{2}$.

In order to see this dependence most clearly let us re-derive an expression for the spectrum at the node. Introducing $\bar{\tau} = \tau v$ we would like to evaluate

$$G^s(Q, \omega_n) = \frac{m_b}{8\pi^2} e^{-\frac{m_b v^2}{\bar{\tau}^2 + \bar{r}^2} |\bar{\tau} - \bar{r}|^{\alpha - 3}}$$

We see that our cavalier treatment of boson-boson correlations which in the absence of a full theory for the spinon holon + gauge system we have totally neglected. Boson correlations will tend to reduce the density of low energy boson states which are responsible for the large distance behavior of the boson correlator. On the crudest level we might hope to capture this reduction by simply decreasing $m_b$ which determines the free boson density of states.

VI. $m_b = 0$

As was indicated in the previous section the free boson dispersion determines both the onset an the shape of the low energy spectrum. Correlations will however reduce the low energy modes whose effect we tried to emulate by reducing the boson density of states. Let us now set $m_b = 0$ from the start. The rational behind this section is the attempt to capture the correlations corresponding to collective bosons that are not condensed. For nearly condensed bosons the appropriate collective variables are the density and phase and we shall assume that we can neglect fluctuations in the density and only the correlations of the phase field determine the infrared physics.

$$\langle b^\dagger(\tau, \bar{r})b(0, 0) \rangle = \rho_0 e^{-i\theta(\tau, \bar{r})} e^{i\theta(0, 0)}$$

where the dynamics of $\theta$ is given by a 2d XY-model. The $\theta$ Green’s function can thus be determined and the resulting boson correlation function reads

$$\langle b^\dagger(\tau, \bar{r})b(0, 0) \rangle = \rho_0 e^{-\frac{1}{4\pi^2 m_b} \sqrt{\bar{r}^2 + \bar{r}^2}}$$
which in the infrared limit we can write suggestively as
\[ \langle b^\dagger(\tau, \vec{r})b(0, 0) \rangle \simeq \rho_0 e^{-\frac{1}{4\pi v_b\chi_b}} \left( 1 + \frac{1}{4\pi v_b\chi_b\sqrt{r^2 + v_b^2\tau^2}} \right) \]
\( (47) \)

We will take the second term in parenthesis as reflecting the uncondensed but correlated boson contribution to the long distance behavior of the boson propagator.

\[ G^\rho(Q_1 + \vec{p}, \omega_n) = L^{-\alpha} \int d^2 r \int_0^\infty dr e^{-i\vec{p}\cdot\vec{r}} e^{-[i(\omega_n + |\mu| - 4\bar{\chi})\tau]} \rho_0 e^{-\frac{1}{4\pi v_b\chi_b\sqrt{r^2 + v_b^2\tau^2}}} \frac{1}{4\pi v_b\chi_b\sqrt{r^2 + v_b^2\tau^2}} \left[ \tau - i \frac{r + s\bar{\chi}}{4\beta} \right] v \]

Let us first use the exponential parametrization
\[ \frac{1}{(r^2 + v^2\tau^2)^\mu} = \frac{1}{\Gamma(\mu)} \int_0^\infty ds s^{\mu-1} e^{-s(r^2 + v^2\tau^2)} \]
which allows us to perform both the \( r \) integration via (GR5) 6.631 (4)
\[ \int_0^\infty dr r^{\nu+1} J_\nu(\beta r) e^{-\alpha r^2} = \frac{\beta^\nu}{(2\alpha)^{\nu+1}} \Gamma(\nu+1) \exp\left( -\frac{\beta^2}{4\alpha} \right) \]
and the \( \tau \) integral via (GR5) 3.462 (1)
\[ \int_0^\infty x^{\nu-1} e^{-\beta x^2 - \gamma x} dx = (2\beta)^{-\nu/2} \Gamma(\nu) \exp\left( -\frac{\gamma^2}{8\beta} \right) \times D_{-\nu}\left( \frac{\gamma}{\sqrt{2\beta}} \right) \]
where \( D_{-\nu} \) is a parabolic cylinder function.

\( \chi_b \) denotes the boson compressibility \( v_b \) is the velocity of sound of the collective bosons and \( l \) is some lattice scale cut-off determining the wavefunction renormalization which will give rise to a reduction of the superfluid density \( \rho_0 \).

Taking the above correlation function as the holon part of the physical hole Green’s function we are faced with the evaluation of

Once the \( r \) and \( \tau \) integrals are done we are left with the integrals over the parameters \( s, u \). Concentrating for the moment on the term proportional to \( \tau \) in the numerator we obtain after \( r \) and \( \tau \) integration
\[ \frac{2\pi K}{\Gamma\left( \frac{1}{2} \right) \Gamma\left( \frac{1}{2} \right)} \int_0^\infty ds du \frac{s^{1/2} u^{1/2}}{4(s + u)(sv_b^2 + uv^2)} D_{-2}\left( \frac{i|\omega_n + |\mu| - 4\bar{\chi}|^2}{\sqrt{2(sv_b^2 + uv^2)}} \right) \]

Next we perform a change of variables
\[ v_b^2 s = t_1 s_1 \quad v^2 u = (1 - t_1) s_1 \]
which changes the parameter integration to
\[ \frac{2\pi K}{\Gamma\left( \frac{1}{2} \right) \Gamma\left( \frac{1}{2} \right)} \int_0^1 dt_1 \int_0^\infty ds_1 \frac{s_1}{v_b^2 v_b^2} \left( \frac{t_1 s_1}{v_b^2} \right)^{-1/2} \]
\[ (1 - t_1)s_1 \left( \frac{1}{v^2} - \frac{\nu^2}{8} \right) e^{-\frac{\nu^2}{4s_1} + \frac{\nu^2}{4s_1} + \frac{\nu^2}{4s_1}} \times \exp \left( \frac{i\omega_n + |\mu| - 4\bar{\chi}^2}{8s_1} \right) D_{-2} \left[ \frac{i\omega_n + |\mu| - 4\bar{\chi}^2}{\sqrt{2s_1}} \right] \]

Finally we can perform the integral over \( s_1 \) via (GR) 7.725 (6)

\[ \int_0^\infty e^{-z t + \frac{\beta}{2} D - \nu [2k(t^2)]} dt = \frac{2^{1 - \beta - \frac{\nu}{2}} \Gamma(\beta)(z + k)^{-\frac{\nu}{2}}}{\Gamma \left( \frac{\nu + \beta + 1}{2} \right)} F_2 \left( \frac{\nu, \beta + \frac{1}{2}}{2}; \frac{z + k}{2} \right) \]

with the identification \( \beta = \alpha, \nu = 2, k = \frac{[i\omega_n + |\mu| - 4\bar{\chi}^2]}{8} \). It turns out that for the numerical evaluation of the final parameter integral over \( t_1 \) it is easier if we use the following quadratic transformation formula for the hypergeometric function

\[ F(a, b; a - b + 1; z) = \frac{1}{(1 - z)^{a - b + 1}} \times \]

Putting everything together we obtain for the hole Green’s function

\[ G^w(Q_1 + \bar{p}, \omega_n) = K \left\{ \int_0^1 dt t^{-\frac{1}{2}}(1 - t)^{\frac{1}{2}} \left( \frac{t v^2 + (1 - t) u^2}{v^2 v_b^2} \right)^{\frac{1}{2} - \frac{1}{2} + \frac{1}{2}} \right. \]

with the case of finite boson mass the spectra for zero \( \alpha \) are broad without any special features at the onset of the spectrum. The confining tendency of the gauge fluctuations can again be nicely observed on increasing \( \alpha \).

**VII. CONCLUSION**

The topic of this paper - the single particle spectral function - was analyzed in a myriad of ways. After briefly addressing the problem of gauge invariance in the context of the single particle Green’s function we gave a first quantized analysis of a particle coupled to gauge fluctuations. The proposed gauge invariant quantity to reflect
In order to get reasonably smooth graphs we first calculated the spectra with a resolution $\Delta \omega = 0.001J$ and then averaged 5 points to plot the spectrum respectively.

The propagation of the particle was a loop, one segment of which was made up by the particle path propagating from $\vec{r}_i$ to $\vec{r}_f$. The loop was completed by a straight line segment connecting the two destinations. We pointed out that if a particular gauge (an axial gauge with $\vec{a} \cdot \vec{u} = 0$ with $\vec{u} \propto \vec{r}_f - \vec{r}_i$) was chosen for the gauge propagator the contribution of the straight line segment to the particle amplitude could be made to vanish. With this idea we then analyzed the amplitude in second quantized formulation and obtained the dressed Green’s function with a slower decay in space-time. However this approach has some spurious infrared problems related to the incomplete gauge fixing in the temporal gauge and required special (sometimes ad hoc) regularizations (Leibbrandt prescription). To confirm the result, we then resorted to discuss the above amplitude as a gauge invariant two-body propagator with one “body” infinitely heavy and hence giving rise to the straight line path introduced by hand earlier. Thus freed from choosing a particular gauge we performed a proper analysis of this propagator. The resulting anomalous dimension for the gauge invariant amplitude agrees with the previous calculation. It however could not and should not be interpreted as representing the propagation of a physical particle.

This forced on us a reinterpretation of the gauge field - not primarily as a scattering mechanism - but rather a confining field which wants to form a bound state between the two charged particles propagating under the mutual attraction mediated by it. This change in perspective led to the real space analysis of the second part where we incorporated the gauge fluctuation effect heuristically into a slower decay of the spinon contribution to the hole propagator. We saw how this change in the spinon Green’s function led to the piling up of low energy spectral weight. We associate this with the confining tendencies of the gauge field. Our treatment of the bosonic holons as free, however clearly overestimated the density of low energy holon states which led to the spectrum to be peaked about the holon energy $\frac{e^2}{2m_e}$. This is not seen in experiments (with the possible exception of the single hole spectrum which seems to have a parabolic band near $(\pi/2, \pi/2)$ compared with the Dirac spectrum observed at higher doping levels). In order to overcome this we argued for the holon Green’s function to be determined in the infrared limit by collective correlations rather than the free particle behavior. This remedied the problem of the edge of the spectrum tracing out $\frac{e^2}{2m_e}$ and led to the spectra depicted in Fig.[8] and Fig.[9]. It is however clear that at the level of our present study the real value of $\alpha$ is not known and hence we are forced to treat it as a phenomenological parameter. A proper microscopic analysis of the holon + spinon + gauge system is needed to put the heuristic results proposed in this paper on a firm foundation.

We would like to remark that the electron spectral function discussed here is the high energy spectral function, which is valid for high energies and/or high temperatures. This is because we have ignored the boson condensation and the instanton effect. For $T < 150K$, the boson-condensation/instanton-effect will be important which will lead to a spin-charge recombination due to the confinement of $U(1)$ gauge field. In this case a $\delta$-function peak will appear in the electron spectral function, and low energy excitations will be described by electron-like quasiparticles. However, the Fermi surfaces of those well defined quasiparticles are formed by four small segments.\[9\]

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VIII. APPENDIX A: FERMION SELF ENERGY IN THE $a_0 = 0$ GAUGE

In this appendix we present details of the evaluation of the self energy in the $a_0 = 0$ gauge at $T=0$

\[
\frac{N}{8} \Sigma(p) = \gamma^0 \int \frac{d^3 q}{(2\pi)^3} \left[ \frac{(q_0 - p_0)q^2 + 2(q_0 - p_0)q_0^2}{\sqrt{q^2(p^2 - q^2)^2}q_0^2} \right] + \gamma^\alpha \int \frac{d^3 q}{(2\pi)^3} \frac{(p_x - q_x)(q_x^2 - q_0^2)^2 + 2q_x q_0 (p_y - q_y)}{\sqrt{q^2(p^2 - q^2)^2}q_0^2} + \gamma^\beta \int \frac{d^3 q}{(2\pi)^3} \frac{(p_y - q_y)(q_y^2 - q_0^2)^2 + 2q_y q_0 (p_x - q_x)}{\sqrt{q^2(p^2 - q^2)^2}q_0^2}
\]

(49)

Let us take a closer look at the $\gamma^0$-term which has two contributions

\[
\int \frac{d^3 q}{(2\pi)^3} \left[ \frac{(q_0 - p_0)q^2}{\sqrt{q^2(p^2 - q^2)^2}q_0^2} + \frac{(q_0 - p_0)}{\sqrt{q^2(p^2 - q^2)^2}} \right]
\]

(50)

First look at term $\mathcal{B}$ which can be calculated in the standard way using dimensional regularization combined with the Feynman trick in the generalized form

\[
\int \frac{d^3 q}{(2\pi)^3} \frac{2(q_0 - p_0)}{\sqrt{q^2(p^2 - q^2)^2}} d_{d_0} = -\frac{p_0}{6\pi^2} \frac{1}{3 - d}
\]

With $A = (p^2 - q^2)^2$ $B = q^2$ we find for $\mathcal{B}$

\[
\text{Term } \mathcal{A} \text{ exhibits in addition to the UV divergence which presumably will be taken care of by some appropriate regularization procedure an infrared divergence due to the $q_0^2$ in the denominator. This divergence is an indication that our gauge fixing $a_0 = 0$ has a large residual gauge freedom in the form of time independent gauge transformations. This reflects the fact that in using the temporal gauge we have “lost” Gauss’ law which within the Lagrangian formulation is obtained as a constrained on the gauge fields by varying the action with respect to $a_0$. In order to deal with this problem we follow Leibbrandt and Staley [22] which have introduced a prescription for this unphysical pole in the form}

\[
\frac{1}{q_0} \rightarrow \lim_{\mu \rightarrow 0} \frac{q_0}{(q_0^2 + \mu^2)^2}
\]

where the limit $\mu \rightarrow 0$ is taken after the integrations over loop momenta and Feynman parameters is completed. This leaves us with the following expression for $\mathcal{A}$

\[
\mathcal{A} = \lim_{\mu \rightarrow 0} \int \frac{d^3 q}{(2\pi)^3} \frac{(q_0 - p_0)q^2 q_0^2}{\sqrt{q^2(p^2 - q^2)^2} (q_0^2 + \mu^2)^2}
\]

(51)

To compute this integral [23] we first combine the denominator using exponential parametrization

\[
\frac{1}{(p^2 - q^2)^2 \sqrt{q^2(q_0^2 + \mu^2)^2}} = \int_0^\infty dt_1 dt_2 dt_3 \frac{t_3}{\sqrt{\pi t_2}} \exp \left[ -t_1(p^2 - q^2)^2 - t_2 q^2 - t_3(q_0^2 + \mu^2)^2 \right]
\]

then we use dimensional regularization to compute

\[
\int \frac{d^d q}{(2\pi)^d} \frac{d}{d(t_1 + t_2) dt_3} e^{\left[-(t_1 + t_2)(p^2 - q^2)^2 - 2t_1 p^2 q - t_3 q_0^2\right]} = \left\{ \frac{1}{(\pi t_1 + t_2)^{d/2}} \frac{\sqrt{t_1 + t_2}}{(t_1 + t_2 + t_3)^{d/2}} e^{\left[\frac{t_1 p^2}{(t_1 + t_2)^2} - \frac{t_1^2 q_0^2}{(t_1 + t_2)^2} - \frac{t_2^2 q_0^2}{(t_1 + t_2)^2} - \frac{t_3^2 q_0^2}{(t_1 + t_2 + t_3)^2}\right]} \right\}
\]
In order to compute the divergent part of this integral we concentrate on the terms with no powers of $p$ in the numerator from the differentiation since it is not hard to convince oneself that all terms of higher order in $p$ converge in $d = 3$. Thus to extract the $1/3-d$ divergence we have

$$p_0 \int \frac{d^3q}{(2\pi)^3} \sqrt{q^2(\vec{p} - \vec{q})^2 q_0^2 + \mu^2} q^2 \sim$$

$$- \frac{p_0}{2\sqrt{\pi} (2\pi)^d} \int_0^\infty dt_1 dt_2 dt_3 \frac{t_3}{t_3} \left[ \frac{1}{3-d}(t_1 + t_2)^{-1} (1-d) \right] - \frac{3}{2} \left[ (1 + t_2)(1 - t) \right]$$

Expanding the term $\{\cdots\}^{(d-3)/2}$ for $d \to 3$ and integrating over the remaining two parameters we finally obtain

$$\int \frac{d^3q}{(2\pi)^3} \frac{p_0 q^2}{\sqrt{q^2(\vec{p} - \vec{q})^2 q_0^2}}|_{\text{div}} = - \frac{p_0}{2\pi^2 3 - d} 1/3 - d$$

In a similar fashion we arrive at

$$\int \frac{d^3q}{(2\pi)^3} \frac{q_0 q^2}{\sqrt{q^2(\vec{p} - \vec{q})^2 q_0^2}}|_{\text{div}} = \frac{p_0}{\pi^2 3 - d}$$

and thus finally

$$\frac{N}{8} \Sigma_0|_{\text{div}} = \gamma_0 \left[ \frac{3p_0}{2\pi^2 3 - d} - \frac{p_0}{6\pi^2 3 - d} \right]$$

$$= \frac{4}{3\pi^2} \gamma_0 \frac{p_0}{3 - d}$$

The spatial components of the self energy can be evaluated similarly to give

$$\frac{N}{8} \Sigma_x = \frac{2}{3\pi^2} \sum_{i=x,y} \gamma_i p_i \frac{1}{3 - d}$$

where the following results were used

$$\int \frac{d^3q}{(2\pi)^3} \frac{q_k q_j}{\sqrt{q^2(\vec{p} - \vec{q})^2 q_0^2}}|_{\text{div}} = - \frac{\delta_{jk}}{2\pi^2 3 - d} \frac{1}{3 - d}$$

$$\int \frac{d^3q}{(2\pi)^3} \frac{q_j q^2}{\sqrt{q^2(\vec{p} - \vec{q})^2 q_0^2}}|_{\text{div}} = - \frac{p_j}{3\pi^2 3 - d}$$

$$\int \frac{d^3q}{(2\pi)^3} \frac{q^2}{\sqrt{q^2(\vec{p} - \vec{q})^2 q_0^2}}|_{\text{div}} = - \frac{1}{2\pi^2 3 - d}$$

with $j, k = 1, 2$ the spatial components.
IX. APPENDIX B: STATIC BOSON SELF ENERGY

To calculate the static boson self energy we need to evaluate

\[- \Sigma_b(p_0) = \int \frac{d^3q}{(2\pi)^2} \frac{i\gamma_0 D_{00}(\vec{q})i\gamma_0}{i(p_0 - q_0)} \quad (54)\]

where \(D_{00} = \frac{8}{N} q^2 \) in the Landau gauge. To combine the denominators we use the following variant of Feynman’s trick

\[
\frac{1}{a^\alpha b^\beta} = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty \frac{y^{\beta-1}dy}{(a + yb)^{\alpha + \beta}}
\]

which in our case gives

\[
\frac{1}{|\vec{q}|^3(p_0 - q_0)} = \frac{3}{2} \int_0^\infty \frac{dy}{(q^2 + y(p_0 - q_0))^{5/2}}
\]

Combining the denominator \((\vec{q} - y/2\vec{v})^2 - y^2/4 + yp_0\) with \(\vec{v} = (1, 0, 0)\) and shifting the momentum to \(\vec{k} = \vec{q} - y/2\vec{v}\) we obtain

\[
i \frac{N}{8} \Sigma_b(p_0) = \frac{3}{2} \int_0^\infty dy \int \frac{d^4k}{(2\pi)^d} \frac{k^2}{|k^2 + \Delta|^5/2}
\]

which with

\[
\int \frac{d^4k}{(2\pi)^d} \frac{k_\mu k_\nu}{|k^2 + \Delta|^{5/2}} = \frac{\delta_{\mu\nu}}{2} \frac{\Gamma(\frac{3-d}{2})}{\Gamma(\frac{d}{2})} \Delta^{-\frac{d-1}{2}}
\]

\[
\Delta = yp_0 - \frac{y^2}{4}
\]

The final parameter integral is calculated using \[14\]

\[
\int_0^\infty y^\alpha (ay + b)^\beta dy = \frac{b^{\alpha+\beta+1}}{a^{\alpha+1}} \frac{\Gamma(-1 - \alpha - \beta)\Gamma(1 + \alpha)}{\Gamma(-\beta)}
\]

and yields

\[
\int_0^\infty dy (yp_0 - \frac{y^2}{4})^{\frac{d-1}{2}} = \frac{p_0 \Gamma(3 - d)}{\frac{d}{2} \Gamma(\frac{d-1}{2})}
\]

which finally gives

\[
\Sigma_b(p_0) = -\frac{8}{N} \frac{p_0}{\pi^2} \frac{1}{3 - d} \quad (55)
\]

X. APPENDIX C: FT OF THE HOLE GREEN’S FUNCTION

In this appendix we analyze

\[
G^x(Q_1 + \vec{p}, \omega_n) = \int d^2r \int_0^\infty d\tau e^{-i\vec{p} \cdot \vec{r}} e^{-[\omega_n + |\mu| - 4\chi] \tau} \frac{mbL^{-\alpha}}{2\pi\tau} e^{-\frac{m_v^2}{4\tau^2}} \frac{\tau - \frac{i \varepsilon + \mu}{\sqrt{4\varepsilon}}}{4\pi(\tau^2 + \varepsilon^2)^{3/2}}
\]

more fully than was done in the main body of this paper. Let us split the above integral into three parts

\[
I_1 = \int d^2r \int_0^\infty d\tau e^{-i\vec{p} \cdot \vec{r}} e^{-[\omega_n + |\mu| - 4\chi] \tau} \frac{mbL^{-\alpha}}{2\pi\tau} e^{-\frac{m_v^2}{4\tau^2}} \frac{\tau v}{4\pi(\tau^2 v^2 + \varepsilon^2)^{3/2}}
\]

\[
I_2 = \int d^2r \int_0^\infty d\tau e^{-i\vec{p} \cdot \vec{r}} e^{-[\omega_n + |\mu| - 4\chi] \tau} \frac{mbL^{-\alpha}}{2\pi\tau} e^{-\frac{m_v^2}{4\tau^2}} \frac{-i \frac{\mu}{\sqrt{2}}}{4\pi(\tau^2 v^2 + \varepsilon^2)^{3/2}}
\]

\[
I_3 = \mathbb{I}_2(x \rightarrow y)
\]
and first concentrate on $I_1$. After angular integration we arrive at

$$I_1 = K_1 \int_0^\infty d\tau \int_0^\infty dr e^{-[i\omega_n + |\mu| - 4\xi_\tau] \tau} J_0(pr) \frac{e^{-\frac{m_b v^2}{2\tau}}}{(\tau^2 v^2 + r^2)^{3/2}}$$

with $J_0$ the zeroth order Bessel function and $K_1 = \frac{m_b L^{-\alpha}}{4\pi}$. Next we utilize the series representation of $J_0$

$$J_\nu(x) = \left(\frac{1}{2}\right)^\nu \sum_{k=0}^\infty \frac{(-1)^k x^k}{k! \Gamma(\nu + k + 1)}$$

which converges absolutely for all $x$. This fact allows us to integrate the series term by term

$$I_1 = K_1 \sum_{k=0}^\infty \frac{(-1)^k}{k! \Gamma(\nu + k + 1)} \int_0^\infty dr e^{-[i\omega_n + |\mu| - 4\xi_\tau] \tau} \int_0^\infty dx x^k e^{-\frac{m_b v^2}{2\tau}} \left[1 + \frac{x}{\tau^2 v^2}\right]^{\frac{\alpha - 3}{2}}$$

Using GR5 (3.383 (5))

$$\int_0^\infty e^{-px} x^{\nu-1}(1 + ax)^{-\nu} dx = a^{-\nu}\Gamma(q)\Psi(q, q + 1 - \nu; p/a) \quad \text{Re} \nu > 0 \quad \text{Re} p > 0 \quad \text{Re} a > 0$$

where $\Psi(a, b; z)$ denotes a confluent Hypergeometric series. The $\tau$ integration can again be performed with the help of GR5 (7.621 (6))

$$\int_0^\infty t^{b-1}\Psi(a,c;t)e^{-st}\,dt = \frac{\Gamma(b)\Gamma(b - c + 1)}{\Gamma(a + b - c + 1)} s^{-b} \mathbf{2}F_1(a, b; a + b - c + 1; 1 - s^{-1})$$

with $\text{Re} b > 0 \quad \text{Re} c < \text{Re} b + 1$ and the identifications $b = 2k + \alpha \quad a = k + 1 \quad c = k + \frac{1+\alpha}{2} \quad s = 2[i\omega_n + |\mu| - 4\xi_\tau]/(m_b v^2)$ yielding to

$$I_1 = \frac{m_b \Gamma^\alpha}{8\pi} \sum_{k=0}^\infty \frac{(-\frac{\xi_\tau}{2})^k}{k!} [i\omega_n + |\mu| - 4\xi_\tau]^{-2k-\alpha} \frac{\Gamma(2k + \alpha)\Gamma(k + \frac{1+\alpha}{2})}{\Gamma(2k + 1 + \frac{\alpha}{2})} 2F_1\left(k + 1, 2k + \alpha; 2k + \frac{3 + \alpha}{2}; 1 - \frac{m_b v^2}{2[i\omega_n + |\mu| - 4\xi_\tau]}\right)$$

(57)

Notice that this result reduces to the one quoted in the main body of the thesis for the case $p = 0$ where only the $k = 0$ term contributes in the sum.

Next let’s look at $I_2$ which can be cast into

$$I_2 = \frac{m_b L^{-\alpha}}{4\sqrt{2\pi}} \int_0^\infty \frac{dt}{\tau} \frac{d}{dp} \int_0^\infty dx e^{-\frac{m_b \nu}{2\tau}(\tau^2 v^2 + r^2)^{-3/2}} J_0(p \sqrt{x})$$

With $\frac{d}{dp} J_0(p \sqrt{x}) = -J_1(p \sqrt{x}) p / p$ the above program can now be repeated step by step. Thus combining $I_1, I_2$ and $I_3$ we find for the physical hole Green’s function

$$G^\nu(Q_1 + \vec{p}, \omega_n) =$$

$$= \frac{m_b \Gamma^\alpha}{8\pi} \sum_{k=0}^\infty \frac{(-\frac{\nu}{2})^k}{k!} [i\omega_n + |\mu| - 4\xi_\nu]^{-2k-\alpha} \frac{\Gamma(2k + \alpha)\Gamma(k + \frac{1+\alpha}{2})}{\Gamma(2k + 1 + \frac{\alpha}{2})} 2F_1\left(k + 1, 2k + \alpha; 2k + \frac{3 + \alpha}{2}; 1 - \frac{m_b v^2}{2[i\omega_n + |\mu| - 4\xi_\nu]}\right)$$

(58)
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