A spacetime dual to the NUT spacetime

Mohammad Nouri-Zonoz\textsuperscript{1} *, Naresh Dadhich\textsuperscript{2} & D. Lynden-Bell\textsuperscript{1}
\textsuperscript{1}Institute of Astronomy, Madingley Road, Cambridge. CB3 0HA
\textsuperscript{2}IUCAA, Post Bag 4, Pune University Campus, Pune, 411 007, India.

March 24, 2022

Abstract

By decomposing the Riemann curvature into electric and magnetic parts, a duality transformation, which involves interchange of active and passive electric parts, has recently been proposed. It was shown that the Schwarzschild solution is dual to the one that describes the Schwarzschild particle with cloud of string dust or a global monopole. Following the same procedure we obtain the solution dual to the NUT spacetime.

PACS numbers: 0420J, 9880C

1 Introduction

In analogy with the electromagnetic field as well as the resolution of the Weyl curvature, we decompose the entire Riemann curvature components relative to a timelike unit vector into electric and magnetic parts (Dadhich 1997). The projection of Riemann tensor and its double dual are respectively identified as the active and passive electric parts, while the projection of the single dual is the magnetic part. The electromagnetic parts are the second rank 3-tensors orthogonal to the timelike resolving vector, the electric parts

*Email: mnzonoz@ast.cam.ac.uk
are symmetric while magnetic part is trace-free and consists of the symmetric Weyl magnetic part and an antisymmetric part representing energy flux. We can now write the vacuum Einstein equations entirely in terms of electromagnetic parts. It is in general symmetric in active and passive electric parts. The duality transformation we wish to consider is the interchange of active and passive parts, which would in general imply interchange of the Ricci and the Einstein tensors. This is because the former results from contraction of the Riemann while the latter from its double dual. The vacuum equation is obviously duality invariant. However it turns out that for the interesting vacuum solutions it is possible to break the symmetry and yet obtain the characteristic vacuum solution. This is so for all the black hole solutions (Dadhich 1997) and as we now show for the NUT solution. What really happens is that in obtaining the vacuum solution there always remains one extra equation corresponding to the Laplace equation. That it is implied by the others. Now throwing in some appropriate distribution on the right of it will not disturb the vacuum solution but it will now make the equation duality non-invariant. The solution of the dual equation gives a metric which can be interpreted as the vacuum spacetime with a global monopole. Barriola and Vilenkin (1989) have already obtained what can be called the spacetime of the Schwarzschild black hole with a global monopole (well outside the core of the monopole). It is shown that this is dual to the Schwarzschild solution (Dadhich 1997). Similarly the solution dual to the Kerr solution has also been constructed (Dadhich & Patel 1998a).

In the case of non-empty space, the duality transformation would imply $T^i_k \to T^i_k - (1/2)T g^i_k$, and in particular a fluid solution maps into a fluid solution with $\rho \to (\rho + 3p)/2$ and $p \to (\rho - p)/2$, and heat flux and pressure anisotropy remaining unaltered (Dadhich, Patel & Tikekar 1998c). It then follows that the stiff fluid is dual to dust, the radiation and the de Sitter models are self-dual, and perfect fluid with the equation of state $\rho + 3p = 0$ is dual to flat spacetime.

Note this duality transformation is different from Ehlers’ duality transformation (Ehlers 1962, Geroch 1971). The former interchanges the active and passive electric parts of the Riemann tensor and generates a global monopole. On the other hand the latter amounts to generating a symmetric magnetic part in the otherwise gravomagnetic part-free Schwarzschild solution to give the NUT space, which has a gravomagnetic monopole (Geroch 1971, Lynden-Bell & Nouri-Zonoz 1998).

In the next section we formulate the alternate vacuum equation in terms of
the electromagnetic parts of the field and obtain the solution dual to NUT. This will be followed in section 3 by studying geodesics of the dual solution to bring out the effect of its global monopole charge, and we finally conclude with a discussion.

2 Dual-NUT solution

We resolve the Riemann curvature relative to a timelike unit vector as follows:

\[ E_{ac} = R_{abcd} u^b u^d , \quad \tilde{E}_{ac} = *R *_{abcd} u^b u^d \]  

(1)

\[ H_{ac} = *R_{abcd} u^b u^d = H_{(ac)} + H_{[ac]} \]  

(2)

where

\[ H_{(ac)} = *C_{abcd} u^b u^d \]  

(3)

\[ H_{[ac]} = \frac{1}{2} \eta_{abcd} R_{ad} u^b u^d. \]  

(4)

Here \( C_{abcd} \) is the Weyl conformal curvature, \( \eta_{abcd} \) is the 4-dimensional volume element. The following relations are also satisfied,

\[ E_{ab} = E_{ba} , \quad \tilde{E}_{ab} = \tilde{E}_{ba} , \quad (E_{ab}, \tilde{E}_{ab}, H_{ab}) u^b = 0 \]

and

\[ H = H^a_a = 0, u^a u_a = 1. \]

In (Dadhich 1997), \( E_{ab} \) and \( \tilde{E}_{ab} \) are respectively termed as active and passive parts. The former refers to \( R_{0000} \) and the latter to \( R_{\alpha\alpha\beta\beta}, \alpha, \beta = 1, 2, 3 \) components of curvature. It can be argued (Dadhich 1997b) that distribution of matter-energy acts as gravitational charge for \( E_{ab} \) while gravitational field energy produces space curvature and acts as charge for \( \tilde{E}_{ab} \). In the context of the Schwarzschild field, it can be shown that the active part is derived from matter-energy and the passive part from the gravitational field energy. Using the above decomposition one can write the Ricci tensor in the following form

\[ R^b_a = E^b_a + \tilde{E}^b_a + (E + \tilde{E}) u_a u^b - \tilde{E} g^b_a + \frac{1}{2} (\eta_{amnc} H_{mn} u^b u^c + \eta_{bmn} H_{mn} u_a u_c) \]  

(5)
where $E = \tilde{E}^a_a$ and $\tilde{E} = \tilde{E}^a_a$. We define $E = \tilde{E} - \frac{T}{2}$ to be the gravitational charge density while $\tilde{E} = T_{ab} u^a u^b$ defines the energy density relative to the unit timelike vector $u^a$. The vacuum equation, $R_{ab} = 0$ is in general equivalent to

$$E \text{ or } \tilde{E} = 0 \quad , \quad H_{[ab]} = 0 = E^{ab} + \tilde{E}^{ab}. \quad (6)$$

This set is symmetric in $E_{ab}$ and $\tilde{E}_{ab}$.

Consider the set

$$H_{[ab]} = 0 = \tilde{E} \quad , \quad E_{ab} + \tilde{E}_{ab} = \lambda w_a w_b \quad (7)$$

where $w_a$ is the unit spacelike vector orthogonal to $u_a$ and $\lambda$ is a scalar function to be determined from the solution. Clearly this set is not symmetric in $E$ and $\tilde{E}$, but it will still yield Schwarzschild as the general solution for spherical symmetry (Dadhich, 1997). Not only that, it yields the Kerr solution as well, and we now show that it gives the NUT solution. Writing the metric in the following form (Newman et al 1963),

$$d s^2 = A (d t - 2 l \cos \theta d \phi)^2 - B d r^2 - (r^2 + l^2) (d \theta^2 + \sin^2 \theta d \phi^2), \quad (8)$$

where $A$ and $B$ are functions of $r$ and $t$ only. We wish to find the solution of equation (7) for the above metric. Now the equation $H_{[ab]} = 0$ will lead to the time independence of the solution, while $(E + \tilde{E})^2 = (E + \tilde{E})^3 = 0$ gives

$$AB = 1. \quad (9)$$

Substituting this in $\tilde{E} = 0$ leads to the NUT solution,

$$A = B^{-1} = 1 - \frac{2(mr + l^2)}{(r^2 + l^2)}. \quad (10)$$

Note that we did not use the extra equation $(E + \tilde{E})^1 = -\lambda$ which in this case gives $\lambda = 0$. This is why the introduction of $\lambda$ term on the right of eqn.(7) did not affect the vacuum solution and this happens for all stationary solutions including electrovac ones.

Now we define the duality transformation by

$$E_{ab} \leftrightarrow \tilde{E}_{ab} \quad , \quad H_{ab} \leftrightarrow -H_{ab} \quad (11)$$
which implies $R_{ab} \leftrightarrow G_{ab}$, because contraction of Riemann tensor is Ricci tensor while that of its double dual is Einstein tensor (Misner et al. 1973). Equation (7) then transforms into

$$H_{[ab]} = 0 = E_{ab} + \tilde{E}_{ab} = \lambda w_a w_b. \quad (12)$$

The general solution of this set of equations turns out to be

$$A = B^{-1} = C \left[ D - 2l^2/(r^2 + l^2) \right] + \frac{F}{\sqrt{r^2 + l^2}} \left[ D - l^2/(r^2 + l^2) \right]^{1/2} \quad (13a)$$

where $C$, $D$ and $F$ are constants. One can put $D = 1$ without loss of generality because it corresponds to the rescaling of the time coordinate and the NUT parameter $l$. To find the constants $F$ and $C$ we use the general solution of equation (12) for a spherically symmetric spacetime which has the following form (Dadhich 1997)

$$g_{00} = -g_{rr}^{-1} = 1 - 2k - 2m/r \quad (13b)$$

where $k$ is a constant. There are two different interpretations for the above metric. In the first one it has been interpreted as the spacetime associated with a particle of mass $m$ centered at the origin of the system of coordinates surrounded by a spherical cloud of strings of gauge invariant density $\frac{2k}{r}$ (Letelier 1979). In another interpretation it is shown to be an approximate solution of the Einstein equations for the spacetime outside a global monopole which has been formed as a result of a global $O(3)$ symmetry breaking into $U(1)$ (Barriola & Vilenkin, 1989). In the rest of this paper we will stick to the interpretation in terms of the global monopole and elaborate more on it. Now going back to the determination of the constants $F$ and $C$ we note that one expects (13a) to reduce to (13b) when $l = 0$ and as a result $C = 1 - 2k$ and $F = -2m$. So we have

$$A = B^{-1} = \left[ 1 - 2k - 2 \frac{mr + l^2(1 - 2k)}{(r^2 + l^2)} \right]. \quad (13c)$$

Again we have $AB = 1$, because the equation that yielded this remains unaltered, while $E = 0$ then gives the equation

$$A'' + 2rA'/(r^2 + l^2) + 4l^2 A/(r^2 + l^2)^2 = 0 \quad (14)$$
whose general solution is (13a). Equation (13c) is the spacetime dual to the NUT spacetime. Putting \( k = 0 \) one obtains NUT itself. The energy momentum distribution implied by solution (13c) is

\[
G_0^0 = G_1^1 = \frac{2k}{(r^2 + l^2)}
\]

which reduces to that of the Schwarzschild global monopole when \( l = 0 \). Global monopoles are supposed to be created as a result of global symmetry breaking in phase transitions in the early Universe (Vilenkin & Shellard 1994). The simplest radially symmetric global monopole is modelled by a triplet scalar,

\[
\phi^a = \eta f(r)x^a/r
\]

where \( x^a x^a = r^2 \) with the Lagrangian,

\[
L = \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a - \frac{1}{4} \lambda (\phi^a \phi^a - \eta^2)^2
\]

This model has a global \( O(3) \) symmetry which is spontaneously broken to \( U(1) \). At large \( r \) outside the monopole core, where \( f = 1 \), it would generate the stresses as given by (15) with \( l = 0 \) i.e., outside the monopole core, the energy-momentum tensor can be approximated as

\[
T_{00} \approx T_{11} \approx \eta^2/r^2 \quad T_2^2 = T_3^3 \approx 0
\]

The general solution of the Einstein equations with this energy-momentum tensor is given by (13b) where \( 2k = 8\pi G\eta^2 \) (Barriola & Vilenkin, 1989).

We now show that in the NUT case the scalar field configuration corresponding to (16) is

\[
\phi^a = \eta f(r)\frac{x^a}{\sqrt{l^2 + r^2}}
\]

where \( x^a x^a = r^2 + l^2 \) and

\[
x^1 = \sqrt{l^2 + r^2} \sin\theta \cos\phi
\]

\[
x^2 = \sqrt{l^2 + r^2} \sin\theta \sin\phi
\]

\[
x^3 = \sqrt{l^2 + r^2} \cos\theta.
\]
Now using metric (8) we can write the Lagrangian (17) for the scalar field $\phi$ (18) in the following form

$$L = \eta^2 f'^2 \frac{2B}{2} + \frac{\eta^2 f'^2}{(r^2 + l^2)} - \frac{\lambda}{4} \eta^4 (f^2 - 1)^2$$

(20)

where $f' = \partial_r f$. The equation of motion for the field $\phi$ will be given by

$$B^{-1} [f'' + f'(\frac{A'}{2A} - \frac{B'}{2B} + \frac{2}{r})] - \frac{2f}{r^2} - \lambda \eta^2 f (f^2 - 1) = 0.$$ 

(21)

This admits an approximate solution $f = 1$ for large $r$ when $O(r^{-2})$ is ignorable.

Then $T_{\mu\nu} = 2 \frac{\partial L}{\partial g_{\mu\nu}} - L g_{\mu\nu}$ leads to

$$T_0^0 = \frac{\eta^2 f'^2}{2B} + \frac{\eta^2 f'^2}{(r^2 + l^2)} - \frac{\lambda}{4} \eta^4 (f^2 - 1)^2$$

$$T_1^1 = -\frac{\eta^2 f'^2}{2B} + \frac{\eta^2 f'^2}{(r^2 + l^2)} + \frac{\lambda}{4} \eta^4 (f^2 - 1)^2$$

(22)

$$T_2^2 = T_3^3 = \frac{\eta^2 f'^2}{2B} + \frac{\lambda}{4} \eta^4 (f^2 - 1)^2.$$

Now for $r \to \infty$ and $f = 1$ we get

$$T_0^0 = T_1^1 = \frac{\eta^2}{r^2 + l^2}.$$ 

(23)

Comparing this with (15) we will see that

$$8\pi \eta^2 = 2k$$

(24)

which gives the constant $k$ in terms of the vacuum value of the scalar field.

It may also be noted that neither dual-Schwarzschild (13b) nor dual-NUT (13c) are asymptotically flat.
3 Geodesics

The metric of a NUT space with a global monopole has almost the same form as the NUT metric without a global monopole. Indeed one can write (13b) in the form

\[ A = B^{-1} = (1 - 2k) \left[ 1 - \frac{2(Mr + l^2)}{(r^2 + l^2)} \right] \]

where now \( M = \frac{m}{(1-2k)}. \) All the geodesics of NUT space, including the null ones, lie on spatial cones (Lynden-Bell and Nouri-Zonoz 1998). Following the same approach one can show that all the geodesics of NUT space with global monopole charge also lie on spatial cones with the semi-angle of the cone given by

\[ \tan \chi = \frac{2l\varepsilon}{L} \]

where

\[ \varepsilon = A(\dot{t} - 2l \cos \theta \dot{\phi}) \]

and “...” represents the differentiation with respect to an affine parameter. Thus the global monopole only contributes through \( A \) in the semi-angle of the cone. Its effect can be seen more explicitly by neglecting the mass and NUT parameters in the metric. We can then write

\[ ds^2 = (1 - 2k)dt^2 - \frac{dr^2}{(1 - 2k)} - r^2(d\theta^2 + \sin^2 d\phi^2) \]

which upon rescaling of \( t \) and \( r \) reads

\[ ds^2 = dT^2 - dR^2 - \left[ R^2(1 - 2k) \right] (d\theta^2 + \sin^2 d\phi^2). \]

This metric is dual to flat spacetime which can also be looked upon as a spacetime of uniform gravitational potential, \( g_{00} = g_{11}^{-1} = 1 + 2\phi \) (Dadhich 1997, 1997b). It describes a space with a deficit solid angle in which the area of a sphere of radius \( R \) is not \( 4\pi R^2 \), but \( 4\pi(1 - 2k)R^2 \). The surface \( \theta = \pi/2 \) has the geometry of a cone with the deficit angle \( \Delta \phi = 2\pi k \) (\( k \ll 1 \)). When a global monopole is added to a Schwarzschild black hole both the particle orbits and the Hawking radiation are merely rescaled (Dadhich et al 1998d).
4 Discussion

Using the decomposition of Riemann curvature with respect to a unit time like vector into electric and magnetic components we have defined a duality transformation between passive and active electric parts of the field. Though the vacuum equation was in general invariant under the duality transformation, yet it is possible to construct solutions dual to the well-known stationary solutions. This is because there was a free equation in the vacuum set which did not participate in determining the solution and hence could be tampered with suitably to break the duality symmetry (11) of the vacuum equation. The tampering would not however affect the vacuum solutions and would lead to distinct dual solutions which remarkably imbibe a global monopole. This is an interesting property of the field which is uncovered by the appropriate modification of the vacuum equation (Dadhich 1997).

Like the massless global monopole with $m = l = 0$ in the solution (13), non-static global texture spacetime with the equation of state $\rho + 3p = 0$ can be shown to be dual-flat (Dadhich 1997). Global monopoles and global textures which result from global symmetry breaking are stable topological defects. The most intriguing feature of the duality transformation is that it generates these topological defects in the original Einstein solution. In spherical symmetry it is possible to give a general prescription (Dadhich & Patel 1998b) for writing solution dual to any solution. Applications of the topological defects in cosmology have been considered extensively (Vilenkin and Shellard, 1994). Does the association of topological defects with the duality transformation indicate a manifestation of something deeper?

Acknowledgements

We thank one of the referees for introducing Letelier’s interpretation of metric (13b) to us. D.L-B and M.N-Z thank the members of IUCAA, Pune, for their warm hospitality during their stay there. M.N-Z acknowledges the support of the Ministry of Culture and Higher Education of Iran. D.L-B is a PPARC Senior Fellow.

References

[1] Barriola, M. and Vilenkin, A. (1989), Phys.Rev.Lett., 63, 341.
[2] Dadhich, N. (1997), On Electrogravity duality, gr-qc/9712021 submitted to GRG.

[3] Dadhich, N. (1997b), On the Schwarzschild field, gr-qc/9704068.

[4] Dadhich, N. & Patel, L. K. (1998a), Dual to Kerr solution, to be submitted.

[5] Dadhich, N. & Patel, L. K. (1998b), On spacetimes dual to spherically symmetric spacetimes, submitted.

[6] Dadhich, N., Patel, L. K. & Tikekar, R. (1998c) Class. Quantum Grav. 15, L27.

[7] Dadhich, N., Narayan, K. & Yajnik, U. (1998d), Pramana, 50, 307, gr-qc/9703034.

[8] Demiansky, M., Newman E. T., Bulletin De l’academie Polonaise des Sciences, Serie des Sciences math., astr. et phys.-Vol. XIV, No.11, 1966.

[9] Ehlers, J., Colloques internationaux C.N.R.S. No. 91 (Les theories relativistes de la gravitation), 275, 1962.

[10] Geroch, R., J.Math.Phys., 12, 918, 1971.

[11] Letelier, P. S., Phys.Rev.D, 6, 1294 (1979).

[12] Lynden-Bell, D., Nouri-Zonoz, M., Rev. Mod. Phys., Vol. 70, No. 2, April 1998.

[13] Misner, C.W., Thorne, K. S. and Wheeler, J. A., Gravitation, Freeman, 1973.

[14] Vilenkin, A. and Shellard, E. P. S., Cosmic strings and other topological defects, CUP, 1994.