Bayesian Learning of Occupancy Grids

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Abstract—Occupancy grids encode for hot spots on a map that is represented by a two dimensional grid of disjoint cells. The problem is to recursively update the probability that each cell in the grid is occupied, based on a sequence of sensor measurements. In this paper, we provide a new Bayesian framework for generating these probabilities that does not assume statistical independence between the occupancy state of grid cells. This approach is made analytically tractable through the use of binary asymmetric channel models that capture the errors associated with observing the occupancy state of a grid cell. Binary valued measurement vectors are the output of a physical layer detector in an imaging, radar, sonar, or other sensory system. We compare the performance of the proposed framework to that of classical formulations. The results show that the proposed framework identifies occupancy grids with lower false alarm and miss rates, and requires fewer observations of the surrounding area to generate an accurate estimate of occupancy probabilities when compared to classical formulations.

Index Terms—Occupancy grids, Bayesian estimation, sonar, robotic mapping

I. INTRODUCTION

THE problem of identifying hot spots on a map from a set of sequential observations has many applications, ranging from target localization to obstacle avoidance. There is a vast literature on various methods to solve such a problem. Occupancy grid estimation, the process of estimating the map given a set of observations, was introduced by Elfes and Moravec in the mid 1980s [1]. Several subsequent papers explored alternative methods for performing sensor fusion [2], for distributing sensor measurements over the occupancy grids, and for combining multiple occupancy grids [3], [4]. These methods make the assumption that the occupancy states of the grid cells are statistically independent by modeling the problem as a Markov Random Field (MRF) of order 0. This allows for the factorization of the joint occupancy distribution on the map into the product of occupancy distributions of individual grid cells.

Thrun [5] provided a new occupancy grid formulation, using forward sensor models, that accounts for statistical dependence between grid cells. His method assumes that the sensor provides range measurements from within a cone, and that the single range measurement comes from only a single source within the cone. This measurement is either a detection, a false alarm, or a measurement corresponding to the maximum range possible (no detection event). Each of these possible events is modeled with a distribution from the exponential family, and the process of identifying the occupancy grid becomes a most-likely-model selection process through the use of an expectation-maximization (EM) algorithm.

The recent work in [6], [7] use real antenna radiation patterns to better inform occupancy grid estimation and hence provide more realistic maps. In [8], [9], the authors employed Gaussian processes to model the dependence between grid cells. The use of Bayesian Occupancy Filters (BOF) [10] for generating occupancy grids has been studied in [11]–[13] though, unlike our method, their formulation assumes statistical independence between the grid cells.

Our formulation was developed with autonomous hunting for undersea mines in mind, but in no way depends on the use of a particular sensor system. We make use of a conic sensor radiation pattern as it provides an easy way to calculate what grid cells are being observed at any time, but any radiation pattern can be used without modification to our formulation.

In the context of a radar or sonar system, our methods can be thought of as a way to combine thresholded pulse-compressed returns for detection tasks, thus providing global location estimates of scatterers. The assumption of statistically independent grid cell occupancy states is not a realistic assumption for our autonomous navigation and sensing problems, where targets often occupy multiple grid cells.

The contributions of this paper are as follows. Using binary asymmetric channels (BACs), we build a measurement model for the interaction between the occupancy state of each grid cell and each sensor measurement. This measurement model allows us to consider the statistical dependence between grid cell occupancy states. Additionally, the use of BACs provides a method of modeling any physical layer detector by considering its statistical performance (probability of false alarm and probability of missed detection), and does not rely on a presumed distribution for sensor measurements. The proposed method is able to take as input either a full vector of observations, e.g., an entire ping from a sonar system, or a single ranging measurement. We also show that, when using our measurement model, the original formulation of Elfes [1], [3], which assumed statistical independence between each grid cell occupancy state, can be written as a special case of the methods proposed in this paper. The experimental results reveal that the proposed methods allow for the identification of small gaps between obstacles when measurements are taken from an area that includes objects on both sides of the gap, and in general produces more accurate maps when measurements

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are inconsistent.

The remainder of this paper is organized as follows. In Section II, we first define the occupancy grid estimation process along with the notation that will be used throughout this paper. Section III develops the Bayesian update equation for computing the posterior probability that a particular cell is occupied given the sequence of measurements. This computation is made analytically tractable by modeling the effect of each cell on each measurement as a network of BACs that randomly map occupancies into the so-called virtual occupancies. These virtual occupancies are then used to model the detection measurements. Experimental results for each case are presented in Section IV, for three distinct experiments. The first is a toy problem in which three different BAC networks, noted as the general case, and special cases one and two, are considered and compared. The second and third use synthesized sonar data, with the second experiment utilizing both special cases and the third experiment utilizing only one of the special cases. Concluding remarks are advanced in Section V.

II. OCCUPANCY GRID-DEFINITIONS & NOTATIONS

Let the environment under consideration be gridded into cells $c_i$ at coordinates $(x_i, y_i), i = 1, 2, \ldots, B$. To each cell is attached an indicator variable $b_i \in \{0, 1\}$, with $b_i = 1$ indicating that a cell $c_i$ is occupied by a scatterer of radiation, and $b_i = 0$ indicating that $c_i$ is empty. A cell with value 1 may be called a hot spot. These indicators may be organized in any convenient manner. Here, we choose to arrange them into a vector $b = [b_1, b_2, \ldots, b_B] \in \{0, 1\}^B$. It is common to call the set of indicators, organized in any manner, a map.

Given any map $\beta \in \{0, 1\}^B$, the problem is then to estimate the probability that $b = \beta$, given a sequence of measurements $J_S = \{j_1, j_2, \ldots, j_S\}$, where each of the $j_s, s \in \{0, 1\}^K$ is a binary vector of thresholded detection statistics. We denote this probability mass function as $p(b_j|J_S)$ and return our estimates as the marginal posterior probabilities $p(b_r = 1|J_S)$, where $b_r$ is the $r$th indicator random variable in $b$. Importantly, it is the set of marginal posterior probabilities $\{p(b_r = 1|J_S)\}_{r=1}^B$ that is returned by our algorithm. This set of probabilities can be organized in any order, and we choose to arrange them into a vector $p = [p(b_1 = 1|J_S), p(b_2 = 1|J_S), \ldots, p(b_B = 1|J_S)] \in [0, 1]^B$ such that the indices are consistent with $b$.

There is some ambiguity in the occupancy grid literature as to the precise definition of an occupancy grid or occupancy grid map [3]–[5], [11]–[15], and whether it is a map, as defined previously, or the set of marginal posterior probabilities. For this reason, we will refer to maps of the form taken by $b$ and $\beta$ as cellular occupancies, and the set of posterior probabilities, organized in any manner, as cellular posterior probabilities. It is common to illustrate, or plot, both cellular occupancies and cellular posterior probabilities to provide a visual representation for comparison. In this paper we reserve the occupancy grid name for both cellular occupancies and cellular posterior probabilities when they are illustrated, or plotted, in some way.

Before we present the mathematical machinery of occupancy grid estimation, we explain the approach with a simple figure. Consider Figure 1(a). The large rectangular region is the environment to be interrogated. This environment has been discretized into disjoint rectangular grid cells. Each element of $b$ represents the state of occupancy (occupied or empty) in one of these grid cells. There are two cylindrical objects, in grey, in the environment depicted in Figure 1(b), and each of the objects occupies multiple grid cells. Every entry of $b$ that corresponds to a grid cell containing a portion of a grey cylinder is equal to 1. All the other entries are equal to 0.

(a) Observation cone at time $s$, showing range locations of detections in with updated occupancy grid from green dashed lines and non-detections observations at time $s$. (b) Observation cone at time $s + 1$, in red dashed lines. Note the two separate detections above the object relate to false alarm events.

At each point in time, the sensor is effectively in a fixed position in its path. From this location, the sensor has a conic field of view that covers a subset of the grid cells of the occupancy map. In Figure 1(a), the sensor cone originates from the bottom of the figure, with an axis perpendicular to the sensor path, and contains one of the two cylinders. So in our example, the vertical axis captures range and the horizontal axis captures cross-range. The cone is divided into arc segments, shown in dashed lines, each intersecting multiple grid cells. Each of these arc segments corresponds to a range interval, for which we compute detection statistics.

The thresholded detection statistic for each range interval produces a binary decision (random variable) $j_{s,k}$ for that range interval, where subscript $s$ denotes the time at which the decision was made and subscript $k$ is the range index. In the figure, the red-dashed arcs depict the range intervals for which the detector has produced a zero ($j_{s,k} = 0$) and the green dashed arcs depict the range intervals for which the detector has produced a one ($j_{s,k} = 1$). The presence of a green dashed arc in an area that does not overlap with the cylindrical object in the sonar cone indicates a false positive (false alarm) and the presence of a red dash arc in a region that does overlap with the cylindrical object indicates a false negative (missed detection). An example of a false positive can be seen in the two isolated green-dashed arcs in top of the sensor cone in Figure 1(a), while an example of a false negative can be seen in the two isolated red-dashed arcs in the top of the sensor cone in Figure 1(b).
detection). As the sensor moves along its path, a sequence of such vectors is collected. We denote by $J_S = [j_1, \ldots, j_S]$ the matrix formed by arranging the collection of $S$ measurement vectors in a column-wise fashion.

We wish to develop a sequential Bayesian framework for updating the posterior probability of the occupancy of each grid cell $c_i$ with associated indicator random variable $b_i$, given the sequence of measurements in $J_S$, viewing $s$ as a running time index. That is, we wish to update the cellular posterior probabilities after each time step. The shade of pink in each grid cell in Figure 1(a) represents our current estimate for the occupancy of that grid cell, given all the measurements up to that point. This is the general idea of occupancy grid estimation. The following is a summary of the notation used in the development of our framework for occupancy grid estimation.

- $B$: The number of grid cells in a map, elements in cellular occupancies, elements in cellular posterior probabilities, and grid cells in an occupancy grid.
- $b$: A random vector taking values in $\{0, 1\}^B$ representing cellular occupancies.
- $B \in \{0, 1\}^B$: The virtual cellular occupancies after the occupancy state of each grid cell has been passed through a binary asymmetric channel (BAC).
- $p = [p(b_1 = 1|J_S), \ldots, p(b_B = 1|J_S)] \in [0, 1]^B$: The cellular posterior probabilities.
- $\mathcal{B}$: Set of all possible cellular occupancies $b$, $|\mathcal{B}| = 2^B$.
- $\mathcal{B}(r, \beta) = \{b|b_r = \beta\}$: Set of all cellular occupancies with $b_r = \beta$, $\beta \in \{0, 1\}$.
- $s$: Time index.
- $j_s \in \{0, 1\}^K$: Binary-valued measurement vector taken from the output of a detector at time $s$. $K$ is the number of samples taken in a single measurement vector at time $s$.
- $J_S = [j_1, \ldots, j_S]$: Collection of binary valued measurement vectors up to time $S$.

### III. Occupancy Grid Estimation Formulation

The occupancy grid estimation process produces the cellular posterior probabilities $p$. In much of the occupancy grid estimation literature [1], [3], [4], [6], [11], the joint distribution of the cellular occupancies is considered to be factored as $p(b) = \prod_i p(b_i)$. Similarly, when conditioned on the collection of measurements $J_S$, the joint distribution is considered to be factored as $p(b|J_S) = \prod_i p(b_i|J_S)$. These factorizations assume that the elements in the cellular occupancies are statistically independent while solving for the cellular posterior probabilities.

#### A. Model for Grid Cell and Measurement Interactions

The state $b_i$ of each grid cell influences each measurement $j_{s,k} \in J_s$. This can be represented by a directed acyclic graph (DAG) as in Figure 2, where the flow feeds forward from each $b_i$ to each $j_{s,k}$. The influence that each grid cell has on each measurement can be modeled by transition probabilities of a binary asymmetric channel [16]. The influence can be quantified by a function of distance between the grid cell and location of the measurement, grazing angle of the sonar platform in relation to the sea floor in sonar applications, or some other metric left to be decided by the designer of the system. Recalling that $j_{s,k}$, $k = 1, \ldots, K$ are binary decisions from the detector at time $s$, one can view a BAC as the device that models bit flips associated with false negatives and false positives for each range interval.

![Fig. 2. Causal chain of interaction between cell occupancies $b_r$ and measurements $j_{s,k}$.](image)

The binary occupancy information transmitted from each $b_i$ through a BAC and received at each $j_{s,k}$ node are logical OR’d together to produce the measurement $j_{s,k}$ for all $k$, with different BAC transition probabilities for each $(i, k)$ pair. Let $\hat{b}_i$ be the output of the BAC for each $c_i$. Then $j_{s,k} = \sum_{i=1}^{B} \hat{b}_i$ where the sum is Boolean. Here, $\hat{b}_i$ represents the estimated version of $b_i$ once passed through the channel. Consider the graphical representation of this model in Figure 3, and note that indicator $b_i$ is pinned at $b_r = 1$. This graphical model assigns $b_r = 1$ before passing it through the channel, whereas the remaining $b_i \neq b_r$ may be either a 1 or 0 before passing through the BAC.

The justification for the choice of the OR gate model is that an occupied cell reflects a transmitted signal back to a detector. If any of the signals received at the sensor has enough energy then it will be identified by the detector. This behavior is analogous to many signals being transmitted over electrical wires and connected to the circuitry that composes an OR gate.

#### B. Sequential Bayes’ Updating

Given the collection of $S$ sequential observations, $J_S = [j_1, \ldots, j_S]$, with $j_s \in \{0, 1\}^K$, we would like to estimate the
occupancy of each grid cell. Using Bayes’ rule, the update rule for the marginal probability of grid cell \( c_i \) being occupied or not, i.e. \( b_r = \beta \), given the entire set of measurements, \( J_S \), is given as

\[
p(b_r = 1 | J_S) = \frac{p(b_r = 1, J_S)}{p(j_s, J_{S-1})} = \frac{\sum_{b \in B(r,1)} p(b, j_S, J_{S-1})}{p(j_S, J_{S-1})}
\]

where \( \eta \) is a normalization term such that \( p(b_r = 1 | J_S) + p(b_r = 0 | J_S) = 1 \). The second to last line comes from the conditional independence of measurements. The second term on the right hand side of the last line, \( p(b_j | J_{S-1}) \), is the posterior probability from the previous time step \( S - 1 \). Likewise, the posterior probability vector \( p(b_j | J_S) \) can be written using the same measurement model as (1):

\[
p(b_j | J_S) = \frac{p(j_S, b_j | J_{S-1})}{p(j_S, J_{S-1})} = \mu p(j_S | b_j) p(b_j | J_{S-1}),
\]

where \( \mu \) is a normalization term such that \( \sum_{b \in B} p(b = \beta | J_S) = 1 \).

The vector \( j_s \) contains \( K \) measurements \( \{j_{s,k}\}_{k=1}^{K} \). Each of these measurements is conditionally independent given the map \( b \). Thus, we can write

\[
p(j_s | b) = \prod_k p(j_{s,k} | b).
\]

To better describe the model \( p(j_s, k | b) \) in (3), and specifically \( p(j_{s,k} = 0 | b) \), let \( \tilde{b} = \{b_i\}_{i=1}^{B} \) be the collection of BAC outputs described in III-A. These are merely latent variables that model the influence of each cell \( c_i \) on each measurement \( j_{s,k} \). Each \( j_{s,k} \) is a Boolean function of virtual occupancies; specifically \( j_{s,k} = \sum_{i=1}^{B} \tilde{b}_i \):

\[
p(j_{s,k} = 0 | b) = p(\tilde{b_i} = 0 \forall i | b) : OR \ gate \ assumption
\]

\[
= \prod_i p(\tilde{b}_i = 0 | b_i)
\]

\[
= \prod_i p_{ki}^{00}(1 - b_i) + p_{ki}^{01}b_i,
\]

where \( p(\tilde{b}_i = 0 | b_i) \) is implicitly parameterized by \( k \). For each pair \( (k, i) \) of measurement index \( k \) and cell \( c_i \), let \( p_{ki}^{00} \) and \( p_{ki}^{01} \) be the probability of detection and false alarm, respectively. Then, we have

\[
p_{ki}^{00} = p(\tilde{b}_i = 0 | b_i = 0) = 1 - p_{ki}^{01},
\]

\[
p_{ki}^{01} = p(\tilde{b}_i = 0 | b_i = 1) = 1 - p_{ki}^{01},
\]

where \( p_{ki}^{00} \) models the probability that \( b_i = 0 \) is correctly transmitted through the BAC and received as \( \tilde{b}_i = 0 \); while \( p_{ki}^{01} \) models the probability that \( b_i = 0 \) is incorrectly transmitted through the BAC and received as \( \tilde{b}_i = 1 \). Similarly, we can define \( p_{ki}^{11} = p_{ki}^{10} \) and \( p_{ki}^{11} = p_{ki}^{10} \). Note that the result in (4) can also be formulated through the idea of multiplying the probabilities of the min-terms that define the Boolean function \( f(\tilde{b} = \{b_i | i = 1, \ldots, B\}) \) as presented in [17].

Plugging all these into (1) and (2) gives the following closed-form expressions that can be used for computation of the Bayes updates:

\[
p(b_r = 1 | J_S) = \eta \sum_{b \in B(r,1)} p(j_S | b) p(b | J_{S-1})
\]

\[
= \eta \sum_{b \in B(r,1)} \prod_k (p(j_{s,k} = 0 | b)(1 - j_{s,k})
\]

\[
+ (1 - p(j_{s,k} = 0 | b)) j_{s,k}) p(b | J_{S-1})
\]

\[
\propto \sum_{b \in B(r,1)} \prod_k \prod_i \left[(p_{ki}^{00}(1 - b_i) + p_{ki}^{01}b_i)(1 - j_{s,k})
\right.
\]

\[
\left. + (1 - (p_{ki}^{00}(1 - b_i) + p_{ki}^{01}b_i)) j_{s,k}\right] p(b | J_{S-1}),
\]

and

\[
p(b | J_S) = \mu \prod_k \prod_i \left[(p_{ki}^{00}(1 - b_i) + p_{ki}^{01}b_i)(1 - j_{s,k})
\right.
\]

\[
\left. + (1 - (p_{ki}^{00}(1 - b_i) + p_{ki}^{01}b_i)) j_{s,k}\right] p(b | J_{S-1}).
\]

### C. Choices of Transition Probabilities

Methods for choosing the correct transition probabilities for each BAC in this framework can be divided into two different categories. The first category of methods involves a heuristic approach, where the designer imparts a belief into the choice of the transition probabilities. Potential heuristics would most commonly be distance-based, but could be any sort of scheme that is plausible in the setting of the problem. There have been research groups [6], [7] that implement a similar idea in their formulations by using the receiver antenna gain pattern as a
The second category of methods uses statistical estimation techniques to learn the transition probabilities given a collection of training data. These methods may have relative success with transition probabilities converging, with results depending on the environment. If, for example, the same sensor trajectory and measurement locations are used experiment to experiment, then convergence of the transition probabilities may be expected. Conversely, if the sensor takes measurements at different locations for each experiment, then the training data may not contain sufficient information for proper transition probability estimation.

**D. Special Cases**

Owing to computational issues, specifically the exponential scaling of the cardinality of \( B(r, 1) \) as the number of grid cells increases, there are two special cases that we consider to allow for computational feasibility by reducing the cardinality of \( B(r, 1) \) to \( 2^{B-1} \). For each special case, there is a tradeoff between the modeling of independence between grid cells and the size of problem that can be solved in reasonable time. Graphical representations of each special case can be seen in Figure 4.

The first special case (SC1) treats the grid cells outside of the observation cone at time \( s \) as having no influence on the measurement vector \( j_s \). This provides faster computation time while maintaining the statistical dependence between all of the grid cells within the sensor cone, but has the side effect of ignoring information imparted by neighboring occupied cells if an obstacle falls both within and outside of the sensor cone.

The second special case (SC2) accounts for grid cells outside of the cone and the grid cells within the sensor cone but outside a particular range gate as having no influence on the portion of the measurement vector \( j_s \) that falls inside the range gate. This further reduces the computation time, but only involves statistical dependence between neighboring grid cells within a small neighborhood surrounding each cell. Similar to the first special case, this can end up ignoring information imparted by neighboring occupied cells if an obstacle falls both within and outside of a particular range gate. A potential way to help mitigate this is to allow for overlapping range gates.

**Special Case 1 (SC1) - No influence between unobserved grid cells and measurements at time \( s \):** Here, we assume that the interaction between the measurement vector \( j_s \) and the grid cells that lie outside the observation cone at time \( s \) is negligible, and hence both the probability of detection \( p_{ki}^{11} \) and false alarm \( p_{ki}^{10} \) are 0.

Let us partition the grid indices \( \{1, 2, \cdots, B\} \) into two parts, \( I \) and \( O \), such that \( I \cup O = \{1, 2, \cdots, B\} \) and \( I \cap O = \emptyset \). We shall use the index notation \( i \in I \) and \( o \in O \).

The grid cells in a particular map \( b \) can then be partitioned into two disjoint sets, the set \( b_I = \{b_i | i \in I\} \) of grid cells inside the sensor cone and the set \( b_O = \{b_o | o \in O\} \) of grid cells outside such that \( b = b_I \cup b_O \). Evaluating (4) for this case gives

\[
p(j_{s,k} = 0 | b) = \prod_{o} p(\tilde{b}_o = 0 | b_o) \prod_{i} p(\tilde{b}_i = 0 | b_i)
\]

\[
= (1) \prod_{i} p(\tilde{b}_i = 0 | b_i)
\]

\[
= p(j_{s,k} = 0 | b_I),
\]

because the conditional probability of receiving a 0 through the BAC for each \( \tilde{b}_I \) is one. By making this assumption, the cardinality of \( B(r, 1) \) is reduced to \( |B(r, 1)| = 2^{B-1-|I|} \). For this special case, let \( b_I \) be the set of grid cells within the observation cone in the range gate shown in Figure 4(b). Although (7) remains consistent for this case, a restriction is put on \( j_s \) such that the delay at which the \( k \)th measurement is taken must coincide with the selected ranges. In this way, we calculate the posterior distributions of grid cell occupancies in each range gate separately. This further reduces the cardinality of \( B(r, 1) \) when compared to the first special case.

**Remarks:** One can imagine schemes to further reduce computational complexity by applying the first and second special cases in stages. For instance, suppose there were a very fine spatial grid defining the grid cells. Applying the second special case might still result in a cardinality of \( B(r, 1) \) that is too large for feasible computation. One might instead segment the observation cone into many smaller disjoint cones and attempting to apply SC1, then applying SC2 if the computation were still practically infeasible.

It is not hard to imagine that splitting the cone into smaller subregions and applying SC2 would eventually lead to treating the grid cells as being independent from one another. In other words, there would only be a single grid cell being updated at any one time, which leads to the following consequences. The set \( B(r, 1) \) in equation (5) contains only a single 1-dimensional element causing only a single term in the sum over \( B(r, 1) \), and the product over \( i \) reduces to a single term for the same reason. Additionally, the number of measurement samples \( 0 < K' \leq K \) taken in the range gate containing cell \( c_i \) is small (possibly only one sample), and the product over \( k \) would...
be restricted to the indices \( k \in \kappa = \{k, k + 1, \ldots, k + K'\} \), \( \kappa \geq 0 \) and \( k + K' \leq K \), that coincide with the range for cell \( c_r \). These simplifications reduce the generalized formulation into the same Bayesian update rule that one would derive by assuming independence between the occupancy state of grid cells, and independence between measurements:

\[
p(b_r = 1|J_S) = \eta \prod_{k \in \kappa} \left( p_{ki}^{01}(1 - j_{S,k}) + (1 - p_{ki}^{01})j_{S,k} \right) \times p(b_r = 1|J_{S-1}). \quad (8)
\]

In this case, the computational complexity would no longer scale exponentially in \( B \), but at the possible expense of degraded performance.

### IV. Experimental Results

In this section, we present experimental results for the general form defined in (5), as well as the two special cases. Three experiments were conducted, with an increasing number of grid cells per experiment. This was done because the general form and SCI are limited in the problems that they can tractably be applied to owing to the exponential scaling with the cardinality of \( B(r, 1) \).

All three experiments are designed for situations where classical occupancy grid estimation performs poorly—small gaps between objects that tend to be missed when a beam overlaps both objects and the gap at the same time [5].

We show through the series of the three experiments that the general formulation provides the best performance, and that although each special case gives up a bit of statistical dependence between the grid cells they each approximate the true occupancy significantly better than when assuming independence between the grid cell occupancy states.

In all experiments we assume a stationary map and an array of sensors taking observations of the map. The sensor array produces measurements that are thresholded detection in its formulation. Therefore, we benchmark the performance of the proposed methods with that of a Bayesian update [18] for each grid cell which uses our measurement model, while assuming independence between the grid cells, and independence between measurements. This is similar to the classical approaches, and will henceforth be referred to as the independence method (IM). The method for performing the Bayesian update on a 2-dimensional map, using IM, for experiments that use a cone shaped observation model is summarized as follows.

The algorithm for estimating occupancy grids using IM consists of the following steps:

1. Find the global position of each grid cell observed within the observation cone at time \( s \).
2. Determine the center line of the cone, and project the position of each observed grid cell onto the center line.
3. Associate each measurement \( j_{s,k} \) with its distance along the center line. Identify the set of all projected grid cells where projections equal this center line distance.
4. Update the probability of each grid cell \( c_r \), at location \((x_r, y_r)\), being occupied given the set of measurements on that grid cell, \( \{j_{s,k}\}_{k=\kappa}^{\kappa+K'} \), using (8).

### B. Metrics Used For Performance Comparison

To evaluate the performance of our methods, two different metrics are used. For both metrics, we consider the true cellular occupancy and the cellular posterior probabilities to be occupancy grids to facilitate direct comparison.

a) Sum of the Jensen-Shannon distance (SISD) \( D_{JS}(b_i||p_i) \) [19] over all \( i \) grid cells in an occupancy grid:

\[
\text{SISD} = \sum_i D_{JS}(b_i||p_i)
\]

\[
= \sum_i \frac{1}{2} \times D_{KL}(b_i||M_i) + \frac{1}{2} \times D_{KL}(p_i||M_i)
\]

\[
= -\sum_i \left[ \frac{1}{2} \times \sum_{x \in \mathcal{X}} b_i(x) \log \left( \frac{M_i(x)}{b_i(x)} \right) + \frac{1}{2} \times \sum_{x \in \mathcal{X}} p_i(x) \log \left( \frac{M_i(x)}{p_i(x)} \right) \right],
\]

where \( p_i \) is the probability mass function for the \( i \)th element from the cellular posterior probability occupancy grid \( p \), \( b_i \) is the probability mass function for the \( i \)th element from the ground truth occupancy grid \( b \), \( M_i(x) = \frac{1}{2} \times \left( p_i(x) + b_i(x) \right) \), and \( D_{KL}(\cdot, \cdot) \) is the Kullback-Leibler (KL) divergence [19]. The Jensen-Shannon distance is used in favor of the KL divergence as it is symmetric, positive, and always finite.

b) Similarity between the cellular posterior probability occupancy grid \( p \) to that of the true occupancy grid \( b \):

\[
\rho = \frac{\langle b, p \rangle_F}{\|b\|_F \|p\|_F},
\]
where \( \| \cdot \|_F \) is the Frobenius norm, and \( \langle \cdot , \cdot \rangle_F \) is the Frobenius inner product.

The occupancy grid was initialized with each grid cell having an occupancy probability of one half, providing a maximum-entropy prior.

C. Experiments with Toy Problem

The first set of experiments involves a toy problem that is sufficiently small such that all three methods can be compared. The toy environment used a 2-dimensional grid \( b \) comprised of 16 grid cells with equal distance of 0.5 units between the centers of individual grid cells. Each grid cell had a total of 9 different measurements sampled uniformly throughout the area it covers. The measurement vector \( j_s \) was comprised of 144 measurements \( j_s = [j_{s,1}, \cdots, j_{s,144}] \), 9 samples for each of the 16 cells, taken at equal distance covering the same overall area as the grid cells.

In this experiment, we compared the ability of each method to produce cellular posterior probability occupancy grids that accurately match the true underlying occupancy grid. All grid cells were observed at all times \( s \). SC1 is implemented as a neighborhood around grid cell locations instead of an observation cone. SC2 takes the same neighborhood used for SC1 and splits the neighborhood into two disjoint sections, updating each section separately.

The data used for the toy problem was synthesized by first choosing a \( b \) and finding the ideal \( j_{\text{ideal}} \) measurement vector by letting \( j_{\text{ideal},k} = 1 \) if it maps to an occupied grid cell, and \( j_{\text{ideal},k} = 0 \) otherwise. A series of 15 observations \( J = [j_{1}, \cdots, j_{15}] \) were then generated by passing each element of \( j_{\text{ideal}} \) through a BAC with \( p_d = 0.80 \) and \( p_a = 0.08 \). The values for \( p_d \) and \( p_a \) were chosen as to match the performance of the detector used in the sonar experiments in the next experiments.

As mentioned earlier, the BAC transition probabilities \( p_{00}^{b,k} \) and \( p_{01}^{b,k} \) were functions of the distance between the grid cell \( c_k \) and the measurement location of \( j_{s,k} \) with \( \alpha = 5 \). This choice of \( \alpha \) provided the best overall results for this toy problem given that there is not a physical interpretation of distance.

It is worth noting that the values of \( p_{00}^{b,k} \) and \( p_{01}^{b,k} \) can be learned through a training process as mentioned in Section III-C. In fact, it is a point of interest in this paper that the proposed methods outperform IM given a potential model mismatch for the chosen \( p_{00}^{b,k} \) and \( p_{01}^{b,k} \) versus the optimal values that best model the system.

Three different examples were run for the toy problem with the results shown in Figures 5, 6, and 7, respectively. Each set of figures shows the cellular occupancies \( b \), the cellular posterior probabilities \( p \) using the general formulation, SC1, SC2, and IM. In each of the figures, the probability of occupancy of a grid cell is represented by the gray level of the cell with a higher probability being associated with a darker cell.

An example of a single measurement \( j_s \) is shown in Figures 5(b), 6(b), and 7(b). Each of the measurement vectors \( j_s \) were sampled randomly according to the probability law associated with the true occupancy of that grid cell. If the grid cell ground truth is occupied, then the samples associated with that grid cell are chosen such that they are a 1 with probability \( p_d \) and a 0 with probability \( 1 - p_d \). Similarly, if the grid cell ground truth is empty, then the samples associated with that grid cell are chosen such that they are a 1 with probability \( p_a \) and a 0 with probability \( 1 - p_a \).

Figures 5(a) and (b) show an example of a checkerboard pattern ground truth and one possible realization of \( j_s \) measurements. As shown in Figures 5(c)-(f), the general formulation along with the SC1 generated occupancy grids that exactly represent the true occupancy of the map \( b \). SC2, however, incorrectly placed an occupied cell in the top-left corner of the map, likely because the neighborhood containing that cell only contained its two closest occupied neighbors, causing a bias towards occupancy in that cell. IM, on the other hand, misrepresents almost half of the map and provides an uncertain cell in the bottom right of the map.

Figures 6 and 7 show the same sets of results for two different maps. The significance of the map in Figure 6 is that it has two isolated occupied grid cells. All three of the proposed methods correctly represented the true occupancy, while the independent method missed the isolated cell on the left of the map. IM also estimated a high probability of occupancy in a small gap. This behavior of missing gaps is a common occurrence for IM [5] and one of the limitations that the proposed methods overcome.
Finally, Figure 7 shows a map that looks like the letters “J i”. Both the general formulation and SC1 correctly identify this map, while SC2 added an additional occupied cell in the bottom-left corner of the map on the hook of the “J”. Similar to the first example, this is likely due to the neighborhood of that cell containing more occupied neighbors so that false alarms show up in the measurements associated with that cell. The independent method missed the hook on the “J” while filling in the space between the dot and the body of the “i”.

The visual results discussed above are quantified in Table I, with the general formulation outperforming all other methods in essentially all metrics, and both special cases outperforming the independent method. The performance of the general method and SC1 are essentially identical. This is likely due to the neighborhoods of measurements, used in SC1, being sufficiently large and including enough measurements on surrounding grid cells to accurately estimate underlying occupancy states. SC2 gave up a little performance compared to the general method and SC1, due to the statistical dependence that it trades off for a reduction in computation complexity. It is interesting to note that the cardinality of $B(r, 1)$ was reduced from $2^{16}$ to $2^{8}$ for the SC1 and $2^{4}$ for SC2.

### Table I

| Example | Metric | GF   | SC1   | SC2   | IM    |
|---------|--------|------|-------|-------|-------|
| Ex°     | SJSD   | 1.801e-3 | 1.705e-2 | 6.939e-1 | 5.1252 |
| Ex°     | $\rho$ | .9998 | .9428 | .57783 |       |
| Ex°     | SJSD   | 6.671e-11 | 5.745e-11 | 1.168e-8 | 2.101 |
| Ex°     | $\rho$ | .67065 | .9428 | .875 |       |
| Ex°     | SJSD   | 2.2393e-5 | 2.6958e-5 | 6.936e-1 | 1.3863 |
| Ex°     | $\rho$ | 1 | 1 | 0.9428 | 0.875 |

### D. Experiments with Simulated Sonar Data

The experiments in this section used simulated sonar data, generated by the Personal Computer Shallow Water Acoustic Toolset (PC SWAT) simulation tool developed by NSWC Panama City, FL [20]. The purpose of these experiments is to mimic the behavior of an unmanned underwater vehicle (UUV) that is searching littoral zones for mine-like objects. The experiments used a side-looking sonar (SLS) system that directs acoustic radiation to the starboard side of the UUV.

The first experiment uses a short range, narrow beamwidth and coarse spatial gridding, which allows SC1 and SC2 to be used while the next experiment uses a longer range, wider beamwidth and a finer spatial gridding, which allows only SC2
to be used. Details about each experiment will be presented in their respective sections.

The stave data generated by PC SWAT was fed through an adaptive coherence estimator (ACE) detector [21]–[23] and then thresholded at a predetermined value that provides a desired $p_D$ and $p_H$.

1) Experiment 1 - Short, skinny beam: In this experiment the UUV is at 10 meters above the seafloor with two cylindrical objects located 5 meters above the seafloor in the sonar interrogation area. The targets were both 2 meters along the major axis with a radius of 0.25 meters. The UUV uses a single sonar projector and an 11-element uniform linear hydrophone array with 3° horizontal beamwidth. The transmit waveform was a linearly frequency modulated (LFM) chirp with center frequency $f_c = 80 \text{ kHz}$, bandwidth $BW = 20 \text{ kHz}$, and sampling frequency $f_s = 60 \text{ kHz}$. Hydrophone elements were separated by a half-wavelength of the carrier frequency, with the geometry required to achieve the desired beamwidth being designed by PC SWAT. A total of 200 pings were collected along a curved path, spaced at 0.01 meters apart. The plot of the magnitude of the raw stave data in dB for one hydrophone can be seen in Figure 8(a) for all pings. In the figure, the ping number is descending along the vertical axis and the sample number is ascending along the horizontal axis. Each ping received at time $s$ was passed through the ACE detector to generate the thresholded ACE test statistic which was then used as the input measurement vectors $j_s$. The output for all 200 pings forms $J = \{j_1, \cdots, j_{200}\}$. Only SC1, SC2 and IM were used for this experiment owing to the very large size of $B(r,1)$ when using a 2-D map.

The BAC transition probabilities $p_{ki}^{00}$ and $p_{ki}^{01}$ used for SC1 were chosen according to the model given earlier with $\alpha = 2$. However, for SC2 these probabilities were $p_{ki}^{00} = (1-p_d)/(1+0.96)^2$ and $p_{ki}^{01} = (1-p_h)/(1+0.96)^2$ i.e. did not change as a function of distance. This constant scaling across all cells in a range gate provided a better estimate for SC2, and makes sense as all measurements and grid cells within a range gate are sufficiently close. The choice of transition probabilities plays a large role in the performance of each method. Using the constant scaling for SC1 or the distance scaling for SC2 provided less accurate estimates.

The path of the UUV, placement of the targets, and their associated rotations are illustrated in Figure 8(c). The targets are marked by transparent green rectangles, showing the size and orientation of each target. In some of the images, the black grid cells occlude the green rectangles (e.g. in the ground truth images). The true occupancy grid in Figure 8(c) was generated by setting $b_i = 1$ for any grid cell that contained any significant part of a target, and a 0 otherwise. The path of the UUV follows the green-black gradient line. The last position of the UUV is marked in a black triangle, with the UUV traveling from the dark end of the line to the triangle. After each ping was received, an update to the cellular posterior probabilities was made. The occupancy grids generated by the SC1, SC2, and IM are illustrated in Figures 8(d), 8(e), and 8(f), respectively. Six range gates were used for SC2. We can see that SC1 and SC2 visually perform in a similar manner, surrounding the true occupied grid cells with areas of high probability of occupancy while providing an area of low probability of occupancy between the two targets. The IM visually missed the gap that exists between the two targets. When compared to SC1, it had more grid cells that are in an uncertain state with their posterior occupancy probabilities remaining within a small range around 0.5, instead of being close to 0 or 1. The performance for each of the methods was evaluated using SJSĐ and $\rho$ with the results presented in Table II.

![Fig. 8. Experiment 1 - (a) The stave data used as input to the ACE detector which generates (b) the series of measurement vectors $j_s$, $s = 1, \cdots, 200$. (c) The true occupancy grid. (d) (e) (f) The occupancy grids generated by special case one (SC1), special case two (SC2), and the independent method (IM), all with 0.25 x 0.25 meter grid cells. Green rectangles represent the true size, position, and rotation of the targets. Grey-scale value of pixels represent the computed posterior occupancy probability given the measurements.](image)

**TABLE II**

| Metric | SC1 | SC2 | IM |
|--------|-----|-----|----|
| SJSĐ   | 23.8306 | 20.2548 | 43.2365 |
| $\rho$  | 0.9768  | 0.9805  | 0.9673  |

As can be seen from Table II, SC2 slightly outperformed SC1. This is, in part, due to a smaller number of hot spots on the upper target in Figure 8(d). This is likely due to the larger mismatch in the collection of transition probabilities for SC1, which modeled the statistical dependence between grid cell
occupancy states poorly. Both special cases outperformed the independent grid cell method as it produced many false alarms, even though IM had a greater number of correct detection on each target.

Consider the situation where an occupancy grid is formed, but a process further down the line (human operator or autonomous navigation, for instance) requires a binary detection map instead of a list of probabilities. It is possible to generate an estimate of the underlying occupancy by applying a threshold $0 \leq \gamma \leq 1$ to the cellular posterior probabilities. Given the cellular posterior probabilities, which contains values between 0 and 1, what would be the appropriate value for the threshold $\gamma$, and does one method always outperform all others for all $\gamma$s? One can argue that the optimal $\gamma$ would be one that minimizes the probability of error. Figure 9 shows the probability of error as a function of the threshold for SC1, SC2, and the IM. The results presented in Table II are echoed in Figure 9, as SC2 provides a lower probability of error for all thresholds, followed by SC1, and finally IM.

This result, along with the results from the previous toy experiment, suggest that using SC2 provides performance similar to both the general form and SC1 while outperforming the independent method. Moreover, SC2 involves less computations than SC1. Comparing to IM, SC2 incurs only a minor increase in computation time while providing considerably better occupancy grid estimation results.

2) Experiment 2 - Long, wider beam: In many ways, this experiment is similar to Experiment 1. In this experiment, however, the UUV resides 10 meters above the seafloor with four cylindrical objects that are partially buried and proud of the seafloor in the sonar interrogation area. As before, the cylindrical targets were all 2 meters along the major axis with a radius of 0.25 meters. The horizontal beamwidth in this case is 10$^\circ$. A total of 300 pings were collected along a curved path, spaced at 0.1 meters apart. The plot of the magnitude of the raw stave data in dB for one hydrophones can be seen in Figure 10(a) for all pings with the ping number descending in the triangle. After each ping was received, an update to the true occupancy represented by transparent green rectangles, showing the size, position, and orientation of each target. The independent method (IM), all with $0.2 \times 0.2$ meter grid cells. Green rectangles represent the true size, position, and rotation of the targets. Grey-scale value of pixels represent the computed posterior occupancy probability given the measurements.

From Figure 10, it is seen that SC2 clearly identified the four separate targets by correctly separating them while generating hot spots over the majority of each target. The true occupancy represented in Figure 10(c) was generated by setting $b_r = 1$ for any grid cell that had any significant part of a target, and a 0 otherwise. The path of the UUV follows the green-to-black gradient line. The last position of the UUV is marked in a black triangle, with the UUV traveling from the dark end of the line to the triangle. After each ping was received, an update to the cellular posterior probabilities was made. The occupancy grids generated by the SC2, and IM are illustrated in Figures 10(d) and 10(e), respectively. One hundred and thirty two range gates were used for SC2. The BAC transition probabilities used for SC2 are the same as those in Experiment 1.

From Figure 10, it is seen that SC2 clearly identified the four separate targets by correctly separating them while generating hot spots over the majority of each target. It does this without producing too many false alarms. The IM also identifies the four targets, but produced many false alarms that join the four targets together. It produced the correct target shapes, but offset the hot spots slightly when compared to the actual target locations. The values for SJSD and $\rho$ are recorded in Table III, where SC2 can be seen to outperform IM. The value for $\rho$ is...
similar between the two methods, which is to be expected as both of the occupancy grids are similar to the truth. If only a small localized region around the four targets is considered, from -5 to 5 meters in the horizontal axis and -2.5 to 12.5 meters in the vertical axis, the value of $\rho$ goes down 0.855 and 0.671 for SC2 and IM, respectively. The large decrease for IM is due to the false alarm rate, specifically the offset in the hot spots when compared to the targets. The ability to separate the four targets is reflected through the SJSD as well.

As can be seen, the probability of error for SC2 is lower than that of IM for every threshold. This indicates that the addition of some complexity to the joint distribution model in the form of inter-cell statistical dependence, albeit small in the case of SC2, gives considerable improvements over IM.

![Fig. 11. Experiment 2 - Probability of error as a function of threshold $\gamma$ for SC2 and the IM.](image)

### V. Concluding Remarks

In this paper we presented a formulation for occupancy grid estimation that accounts for the statistical dependence between grid cell occupancy states and allows for vector-valued measurements. This contrasts with the classical methods in [1], [3] that consider the occupancy states of grid cells to be statistically independent, as well as the work by Thrun [5] that considers statistical dependence but only allows for scalar measurements through the use of correspondence variables. We showed that the classical independence method can be viewed as a special case of our formulation. Experimental results reveal that our formulation outperforms the classical independence method. More specifically, the proposed method identifies small gaps between closely positioned obstacles whereas the independent method groups them together. We also showed that although it is intractable to use the general formulation for real-world applications, modeling some correlation between the occupancy state of grid cells, as seen in SC1 and SC2, provides a good approximation to the general form while offering significantly reduced computational time.

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