GEODESICS OF RANDOM RIEMANNIAN METRICS:
SUPPLEMENTARY MATERIAL

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Abstract. This is supplementary material for the main Geodesics article by the authors. In Appendix A we present some general results on the construction of Gaussian random fields. In Appendix B we restate our Shape Theorem from [10], specialized to the setting of this article. In Appendix C we state some straightforward consequences on the geometry of geodesics for a random metric. In Appendix D we provide a rapid introduction to Riemannian geometry for the unfamiliar reader. In Appendix E we present some analytic estimates which we use in the article. In Appendix F we present the construction of the conditional mean operator for Gaussian measures. In Appendix G we describe Fermi normal coordinates, which we use in our construction of the bump metric.

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Part IV. Supplemental Material (on arXiv)

Appendix A. Construction of Gaussian Random Fields

In this appendix, we construct Gaussian tensor fields on $\mathbb{R}^d$, which we use to generate random Riemannian metrics. A Gaussian 2-tensor field $\xi_{ij}(x)$ is entirely defined by its mean 2-tensor $m_{ij}(x) = \mathbb{E}\xi_{ij}(x)$ and its covariance 4-tensor $c_{ijkl}(x, y) = \mathbb{E}\xi_{ij}(x)\xi_{kl}(y)$. Throughout we assume the mean tensor is zero, and that the covariance tensor is symmetric, stationary and isotropic, and compactly supported. Such a Gaussian tensor field is the “source of randomness” for our random Riemannian metric, which we define pointwise by $g(x) = \varphi(\xi(x))$, where $\varphi$ is a function which sends a symmetric 2-tensor to a positive-definite one, and acts spectrally.

It $c(x)$ is a symmetric, stationary and isotropic, and compactly supported covariance function, we may generate a covariance tensor by setting $c_{ijkl}(x, y) := c(|x - y|) \cdot (\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk})$. It follows that $c_{iisi}(x, y) = 2c(|x - y|)$ and $c_{ijij}(x, y) = c(|x - y|)$ if $i \neq j$, with all other components equal to zero. It is not trivial that there exist Gaussian covariance functions $c$ satisfying these conditions. We present a family of examples $c_d$ due to Gneiting [Gne02 Equation (17)] which are compactly supported and 6-times differentiable at the origin.

Example A.1 (Gneiting’s covariance function). Let $d \geq 1$, and choose any integers $\kappa \geq 3$ and $\nu \geq \frac{d+1}{2} + \kappa$. Let $B$ denote the Beta function, and define

$$c_d(r) := c_{\nu,\kappa}(r) := \frac{1}{B(2\kappa, \nu + 1)} \int_r^1 u(u^2 - r^2)^{\kappa-1}(1 - u)^\nu \, du$$

when $0 \leq r \leq 1$, and set $c_{\nu,\kappa}(r) = 0$ for $r \geq 1$. Gneiting’s function $c_d(r)$ is non-trivial, compactly supported, and 6-times differentiable.\footnote{We remark that if $\nu < \frac{d+1}{2} + \kappa$, then Gneiting’s function $c_{\nu,\kappa}(r)$ is not a Gaussian covariance function.}

For a single example that works in low dimensions ($2 \leq d \leq 9$), we set $\kappa = 3$ and $\nu = 8$. In this case, formula (A.1) takes the explicit form $c_d(r) = -\frac{1}{5}(r - 1)^3(5 + 55r + 239r^2 + 429r^3)$. The covariance functions for larger values of $d$ are of a similar polynomial character.

Henceforth, let $\xi_{ij}(x)$ be a real-valued, stationary and isotropic Gaussian random field on $\mathbb{R}^d$ with mean zero and covariance tensor $c_{ijkl}(x, y) = \mathbb{E}\xi_{ij}(x)\xi_{kl}(y)$. Let $Q$ be the law of the random field $\xi$ on $\Omega = C^2(\mathbb{R}^d, \text{Sym})$.

Appendix B. The Shape Theorem

If $g$ is a random Riemannian metric, then $d_g$ is a random distance function, so $(\mathbb{R}^d, d_g)$ is a random metric space. Let $B_g(t) = \{ x : d_g(0, x) \leq t \}$ denote the ball of radius $t$ (with respect to this distance) centered at the origin in $\mathbb{R}^d$.

As a distance function, $d_g$ satisfies the triangle inequality on $\mathbb{R}^d$; Using Liggett’s version [Dur96] of Kingman’s subadditive ergodic theorem [Kin08], one easily sees that there exists some non-random constant $\mu \geq 0$ such that for each $v \in S^{d-1}$, we have that $\frac{1}{r}d_g(0, rv) \to \mu$ almost surely and in $L^1$. Due to rotation-invariance of our model, the constant $\mu$ does not depend on the direction $v$.

A priori, there is no guarantee that $\mu > 0$ or that this convergence is uniform; however, both of these statements are ensured by the Shape Theorem [LW10]. Heuristically, if $|x - y| \gg 1$, then $d_g(x, y) \sim \mu|x - y|$.

Let $B(r) = \{ x : |x| \leq r \}$ denote the Euclidean ball of radius $r$ centered at the origin. The Shape Theorem states that random Riemannian balls grow asymptotically like Euclidean balls: for large $t$, $B_g(t) \sim B(t/\mu)$.
almost surely. We will formally define the measure $\mathbb{P}$ in Section 2. The following theorem is an important consequence.

**Theorem B.1** (Shape Theorem). Let $\mathbb{P}$ be the measure on $(\Omega, \mathcal{F})$ introduced in Section 1.1. This measure satisfies $\mathbb{P}(\Omega_+) = 1$; let $g$ denote a random Riemannian metric with respect to $\mathbb{P}$. The following statements hold:

a) There exists a non-random constant $\mu > 0$ such that, with probability one, $\frac{1}{r}d_g(0, rv) \to \mu$ as $r \to \infty$, uniformly in the direction variable $v \in S^{d-1}$. The convergence also occurs in $L^2$, uniformly in $v$.

b) For all $\epsilon > 0$, with probability one, there exists a random constant $R_{shape} = R_{shape}(g)$ such that if $r \geq R_{shape}$, then $B(r - \epsilon r) \subseteq B_g(\mu r) \subseteq B(r + \epsilon r)$.

Equivalently, with probability one, the rescaled ball $\frac{1}{t}B_g(t)$ converges to the Euclidean ball $B(1/\mu)$ (in the Hausdorff topology on compact sets). The Euclidean ball $B(1/\mu)$ is called the limiting shape of the model.

c) With probability one, the Riemannian metric $g$ is geodesically complete. Consequently, with probability one, for all $x$ and $y$ in $\mathbb{R}^d$, there is a finite, minimizing geodesic $\gamma$ connecting $x$ to $y$ such that $d_g(x, y) = L_g(\gamma)$. With probability one, the topology on $\mathbb{R}^d$ generated by the metric $d_g(x, y)$ is complete.

**Proof.** The constant $\mu$ is independent of the direction $v$ since the measure $\mathbb{P}$ is rotationally-invariant. Part (a) is Proposition 3.3 of [LW10]. Part (b) is Theorem 3.1 of [LW10]. Part (c) is Corollary 3.5 of [LW10]. The equivalence of geodesic completeness and topological completeness is a consequence of the Hopf-Rinow theorem [Lee97]. \hfill \square

In [LW10], we called our model Riemannian first-passage percolation in analogy with lattice models of first-passage percolation. In such models, one assigns a random passage-time distribution $\tau$ to the bonds of the lattice $\mathbb{Z}^d$, and this generates a random metric $d_\tau$ on $\mathbb{Z}^d$ analogous to our random Riemannian metric $d_g$. In fact, Kingman [Kim08] developed his subadditive ergodic theory to analyze the distance function of lattice FPP. One can think of lattice first-passage percolation as a random perturbation of the flat Euclidean geometry of $\mathbb{Z}^d$. For comprehensive recent surveys of FPP, we suggest the surveys by Howard [How04] and Blair-Stahn [BS10]; the surveys by Kesten [Kes84, Kes87] are older, but contain many technical details.

To prove the Shape Theorem in our context, one tessellates $\mathbb{R}^d$ by unit cubes, and considers a dependent FPP model on the lattice formed by the centers of those cubes. For each $z \in \mathbb{Z}^d$, we define $\Lambda_z = \sup |g|$, the maximum eigenvalue of the random Riemannian metric $g$ on the cube centered at $z$. The random field $\Lambda_z$ induces a model of dependent FPP, and we use estimates from this setting to prove the Shape Theorem in the continuum. We revisit these techniques in Section 13.

In [LW10], our proof was based on the robust energy-entropy method of mathematical physics, where one shows that an event occurs with extremely low probability over one particular lattice path ("high energy"), but sums this over all possible lattice paths starting at the origin ("high entropy"). One proves convergence by showing that the "energy" beats the "entropy", which allows to apply the Borel-Cantelli lemma. A large-deviation estimate is essential to this method: the number of lattice paths at the origin grows exponentially in $n$, the length of each path, so the probabilities must decay exponentially in $n$ for the arguments to work. By adapting more carefully the proof of Cox and Durrett, one can weaken the assumptions to a finite moment bound. For example, the assumption $\mathbb{E} \min\{\Lambda_1, \cdots, \Lambda_{3^d}\}^{2^{3^d+1}} < \infty$ is sufficient for the Shape Theorem to hold.
It is a classical result that geodesics can be dually interpreted as a Hamiltonian flow on the cotangent bundle $T^*\mathbb{R}^d$, with Hamiltonian $H = g^{ij}(x)p_ip_j$. Armstrong and Souganidis [ASI2, ASU1] have proved a general result on stochastic homogenization of Hamilton-Jacobi equations in stationary, ergodic environments, and our shape theorem is a special case of their theorem.

**Appendix C. The Geometry of Geodesics**

We continue with the assumption that $d = 2$ in order to use Theorem 3.3 to say something about the plane geometry of geodetic curves. Let $\beta$ be a probability measure on the tangent bundle $T\mathbb{R}^2$ which is absolutely continuous with respect to the Lebesque measure. Let $(X,V)$ be chosen according to $\beta$, independently of the random metric $g$, and consider the geodesic $\gamma := \gamma_{X,V}$ with these random initial conditions. The curve $\gamma$ is a random plane curve, and Proposition C.1 demonstrates one consequence of this randomness.

We say that a plane curve $\gamma$ contains a straight line segment if the Euclidean acceleration $\ddot{\gamma}/|\dot{\gamma}|^2$ is constant on some interval. More generally, we say that the curve $\gamma$ contains a circular arc if the Euclidean normal acceleration $\dddot{\gamma}(t) = \frac{1}{|\dot{\gamma}|^2} \langle \ddot{\gamma}, \gamma \rangle$ is constant on some interval.

**Proposition C.1.** Suppose that $d = 2$. With probability one, the geodesic $\gamma$ contains neither straight line segments nor circular arcs.

**Proof.** For any $s$, let $U_s \in \mathcal{F}$ denote the event that the turning $w$ is constant on an interval containing $s$:

$$U_s = \{ \exists \, \delta > 0 \text{ s.t. } w \text{ is constant on the interval } (s - \delta, s + \delta) \},$$

and note that $U_s = \sigma_{-s}U_0$. We must show that $\mathbb{P}(\cup_s U_s) = 0$. We first prove that $\mathbb{P}(U_0) = 0$, then use a simple approximation argument and Theorem 3.3 to prove the lemma.

The event $U_0$ implies that

$$0 = \dot{w}(0) = \frac{1}{|\gamma(0)|^2} \left( \langle \dot{\gamma}, \dot{\gamma} \rangle + \langle \dot{\gamma}, \dddot{\gamma} \rangle - 3 \langle \dot{\gamma}, \dot{\gamma} \rangle \cdot \langle \gamma, \dddot{\gamma} \rangle \right),$$

which we can compute using the geodesic equation $\dddot{\gamma}^h = -\Gamma_{ij}^k \dot{\gamma}^i \dot{\gamma}^j$.

Since $\dot{w}(0)$ depends measurably on the random metric $g$, it is a random variable. In particular, the elements of the terms of the right side of (C.2) are rational functions of $|\dot{\gamma}(0)| = |\dot{\gamma}(0)| = 1/\sqrt{g_{11}(0)}$, the Christoffel symbols $\Gamma_{ij}^k(0)$ and their derivatives $\Gamma_{ij,l}^k(0)$. These are all real-valued random variables with continuous distributions, hence the random variable $\dot{w}(0)$ also has a continuous distribution. In particular, $\dot{w}(0) \neq 0$ almost surely. Consequently,

$$\mathbb{P}(U_0) \leq \mathbb{P}(\dot{w}(0) = 0) = 0.$$

Now, let $s_n$ be a (non-random) countable dense sequence in $\mathbb{R}$. If the event $U_s$ occurs for some (random) $s$, then it also must occur for some nearby $s_n$ (random $n$). Thus

$$\mathbb{P}(\cup_s U_s) = \mathbb{P}(\cup_n U_{s_n}) \leq \sum_{n=0}^{\infty} \mathbb{P}(\sigma_{s_n}^{-1}U_0) = 0,$$

since the measures $\mathbb{P} \circ \sigma_{s_n}^{-1}$ are all absolutely continuous with respect to the measure $\mathbb{P}$ by Theorem 3.3 and $\mathbb{P}(U_0) = 0$. □

**Appendix D. Riemannian Geometry Background**

We present a very rapid introduction to Riemannian geometry, working in coordinates on the fixed manifold $\mathbb{R}^d$. For a more detailed introduction, see the book by Lee [Lee97].
Let SPD be the set of symmetric, positive-definite matrices, and let \( \Omega_+ = C^2(\mathbb{R}^d, \text{SPD}) \) be the space of \( C^2 \)-smooth symmetric, positive-definite quadratic forms on \( \mathbb{R}^d \). Every element \( g \in \Omega_+ \) induces a Riemannian metric on \( \mathbb{R}^d \), that is, a smoothly varying inner product on the tangent bundle \( T\mathbb{R}^d \).

The flat Euclidean metric \( \delta \in \Omega_+ \) is the 2-tensor field which is everywhere equal to the identity matrix: \( \delta_{ij}(x) = \delta_{ij} \), where the symbol on the right-side is the Kronecker \( \delta \). Let \( e_i \) be the standard basis vectors in \( \mathbb{R}^d \). We write each Riemannian metric \( g \in \Omega_+ \) in coordinates by \( g_{ij}(x) = \langle e_i, g(x)e_j \rangle \), where the brackets denote the standard Euclidean inner product.

If \( v = v^i e_i \) and \( w = w^i e_i \) are tangent vectors to \( \mathbb{R}^d \) at some point \( x \) (following the Einstein convention of summing over repeated upper and lower indices), the inner product of \( w \) and \( w \) with respect to \( g \in \Omega_+ \) is \( \langle v, g(x)w \rangle = g_{ij}(x)v^i w^j \).

Fix some \( g \in \Omega_+ \). For a single tangent vector \( v \in T_x \mathbb{R}^d \), we denote by \( \|v\|_g = \sqrt{\langle v, g(x)v \rangle} = \sqrt{g_{ij}(x)v^i v^j} \) and \( |v| = \sqrt{\langle v, v \rangle} = \sqrt{\delta_{ij}v^i v^j} \) the Riemannian and Euclidean lengths of \( v \), respectively. For a \( C^1 \)-curve \( \gamma : [a, b] \to \mathbb{R}^d \), we define the Riemannian arc length of \( \gamma \) by \( L_g(\gamma) = \int_a^b \|\dot{\gamma}(t)\|_g \, dt \). We say that a curve is finite if it has finite Euclidean arc length; for our model, Theorem B.1 implies that finite curves have finite Riemannian length (for almost every \( g \)). The Riemannian distance between two points \( x \) and \( y \) is defined by \( d_g(x, y) = \inf_{\gamma} L_g(\gamma) \), where the infimum is over all \( C^1 \)-curves \( \gamma \) connecting \( x \) to \( y \).

We follow the Riemannian geometry convention and write the inverse of \( g \) as \( g^{ij}(x) = (g^{-1})_{ij}(x) \), so that \( g^{ij} g_{jk} \) is the identity matrix \( \delta_k^i \) (again, following the Einstein convention). We denote derivatives of \( g_{ij} \) by indices following a comma, so that \( g_{ij,k} := \frac{\partial}{\partial x^k} g_{ij} \) and \( g_{ij,kl} := \frac{\partial}{\partial x^k} \frac{\partial}{\partial x^l} g_{ij} \).

The Riemannian metric \( g \) induces a canonical covariant derivative, by way of the Levi-Civita connection. The expression \( \nabla_V W \) denotes the covariant derivative of a vector field \( W \) along the vector field \( V \). Let \( e_i \) denote the standard basis vectors in \( \mathbb{R}^d \); we use the same notation to denote the vector fields \( e_i(x) = e_i \) for all \( x \in \mathbb{R}^d \). The Christoffel symbols are defined by writing the covariant derivative in coordinates: \( \nabla_{e_i} e_j = \Gamma^k_{ij} e_k \).

The Riemann curvature tensor \( R = R_{ijkl}(g, x) \) quantifies how much the covariant derivatives fail to commute: \( R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z \). In this article, the only place we use the general Riemann curvature is Appendix [C] where we describe Fermi normal coordinates for general Riemannian manifolds. We will mainly use the scalar curvature \( K \). In the two-dimensional case, the Riemann curvature tensor is determined by the scalar curvature (this is formula [C.12]).

We can express the Christoffel symbols \( \Gamma^k_{ij}(g, x) \) and the scalar curvature \( K \) as polynomials in the metric and its first two derivatives:

\[
\Gamma^k_{ij}(g, x) = \frac{1}{2} g^{km} \left( g_{im,j} + g_{mj,i} - g_{ij,k} \right) \quad \text{and} \quad K(g, x) = g^{ij} \left( \Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^k_{ij} \Gamma^{kl}_{kl} - \Gamma^k_{il} \Gamma^{kl}_{kj} \right), \tag{D.1}
\]

where we evaluate the terms on the right side at the point \( x \).

A \( C^2 \)-curve \( \gamma \) is called a geodesic for the metric \( g \) if its (covariant) acceleration is zero: \( \nabla_{\dot{\gamma}} \dot{\gamma} = 0 \). In coordinates, geodesics are solutions to the geodesic equation

\[
\ddot{\gamma}^k(g, t) = -\Gamma^k_{ij}(g, \gamma(g, t)) \dot{\gamma}^i(g, t) \dot{\gamma}^j(g, t). \tag{D.2}
\]

We will often simplify our notation by writing \( \ddot{\gamma}^k = -\Gamma^k_{ij} \dot{\gamma}^i \dot{\gamma}^j \). The geodesic equation is the Euler-Lagrange equation for the Riemannian energy functional \( L_2^2(\gamma) := \frac{1}{2} \int \|\dot{\gamma}\|^2_g \).
Appendix E. Analytic Estimates

Throughout this article, it will be convenient to put a metric topology on the space of compact subsets of $\mathbb{R}^d$, in order to quantify what it means for two compact sets to be close to each other. For any compact set $D \subseteq \mathbb{R}^d$, let $D^\epsilon = \{ x : d_{\text{Euc}}(x, D) \leq \epsilon \}$ denote the $\epsilon$-neighborhood of $D$, i.e. the set of all points which are Euclidean distance at most $\epsilon$ from $D$. We define the Hausdorff metric on compact sets by

$$d_{\text{Haus}}(D, D') = \inf\{ \epsilon : D^\epsilon \subseteq D', D \subseteq (D')^\epsilon \}. \quad (E.1)$$

That is, $d_{\text{Haus}}(D, D') \leq \epsilon$ if and only if $D^\epsilon \subseteq D'$ and $D \subseteq (D')^\epsilon$. Let $\mathcal{C} = C(\mathbb{R}^d)$ be the space of compact sets equipped with the Hausdorff metric. This is a complete metric space, and balls under this metric are compact.

Next, we introduce the usual seminorms on differentiable tensor fields. Let $\alpha = 0, 1$ or 2. For any compact $D \subseteq \mathbb{R}^d$, we define the usual $C^\alpha(D)$-seminorm of a symmetric quadratic form $\xi \in \Omega$ by

$$\|\xi\|_{C^\alpha(D)} = \sup_{x \in D} \max_{i,j} |\xi_{ij}(x)|, \quad \|\xi\|_{C^{1}(D)} = \sup_{x \in D} \max_{i,j,k} \{|\xi_{ij}(x)|, |\xi_{ij,k}(x)|\}, \quad \text{etc.}$$

We equip the space $\Omega$ with the topology generated by the $C(D)$-seminorms. Consequently, the linear space $\Omega$ is a Fréchet space. The space $\Omega_+$ of $C^2$-smooth Riemannian metrics is dense in the open cone $C(\mathbb{R}^d, \text{SPD}) \subseteq \Omega$.

For any function $f$ and compact $D \subseteq \mathbb{R}^d$, let $\text{Lip}_D(f) = \limsup_{x \rightarrow 0} \sup_{D^x} |f(x) - f(y)|/|x - y|$ denote the Lipschitz constant of $f$ over an infinitesimal neighborhood of $D$. We define the $C^{\alpha,1}(D)$-seminorms of a tensor field $\xi$ as the maximum of its $C^{\alpha}(D)$-seminorm and the $\text{Lip}_D$-constant of the $\alpha$th derivatives of $\xi$:

$$\|\xi\|_{C^{\alpha,1}(D)} = \max_{i,j} \{ \|\xi\|_{C^\alpha(D)}, \text{Lip}_D(\xi_{ij}) \}, \quad \|\xi\|_{C^1(D)} = \max_{i,j,k} \{ \|\xi\|_{C^{1}(D)}, \text{Lip}_D(\xi_{ij,k}) \}, \quad \text{etc.}$$

We use the same notation to denote the corresponding seminorms for scalar functions (and higher dimensional tensor fields).

Let $\Omega^*$ denote the space of continuous linear functionals on $\Omega$. Since each $f \in \Omega^*$ is a measurable function on the probability space $\Omega$, it is a random variable. Of particular interest are the evaluation functionals $\delta^k_x \in \Omega^*$, defined by $\delta^k_x(\xi) = \xi_{ij}(x)$ for any $\xi \in \Omega$. For any compact subset $D \subseteq \mathbb{R}^d$, let $F^{(0)}_D$ denote the $\sigma$-algebra

$$F^{(0)}_D = \bigcap_{c > 0} \sigma(\delta^k_x : x \in D^c) \quad (E.2)$$

generated by the information of a random tensor field on an infinitesimal neighborhood of the set $D$. Let $\mathcal{F}_D$ denote the completion of the $\sigma$-algebra $F^{(0)}_D$.

Let $\|\cdot\|_D$ denote any of the $C^{\alpha}(D)$ or $C^{\alpha,1}(D)$-seminorms defined above for quadratic forms $\xi \in \Omega$. Note that this definition also uses an infinitesimal neighborhood of $D$. By definition, the seminorms $\|\cdot\|_D$ depend only on a tensor field on an infinitesimal neighborhood of the compact set $D$. Consequently,

the seminorms $\|\cdot\|_D$ are $\mathcal{F}_D$-measurable. \quad (E.3)

Recall that $Z_D(g) = \max\{ \|g\|_{C^{2,1}(D)}, \|g^{-1}\|_{C^{1,1}(D)} \}$ is the fluctuation functional for a random metric $g$ over the set $D$, defined in \ref{E.1}.

**Lemma E.1.** The map $(D, g) \mapsto Z_D(g)$ is jointly continuous in $D$ and $g$. The random variable $g \mapsto Z_D(g)$ is $\mathcal{F}_D$-measurable.
Proof. The proof of the lemma follows easily from the statement,
\[ \text{the map } (D, \xi) \mapsto \|\xi\|_D \text{ is jointly continuous.} \] (E.4)

To prove (E.4), we first fix a compact set \( D_0 \subset C \) and a tensor field \( \xi_0 \in \Omega \). By construction, the map \( D \mapsto \|\xi_0\|_D \) is continuous in the Hausdorff topology. Let \( \epsilon > 0 \), and suppose that \( D \subseteq D_0 \), that \( \|\xi\|_D - \|\xi_0\|_{D_0} \leq \epsilon \), and that \( \|\xi - \xi_0\|_D \leq \epsilon \). Then \( \|\xi\|_D - \|\xi_0\|_{D_0} \leq \|\xi - \xi_0\|_D + \|\xi_0\|_D - \|\xi_0\|_{D_0} \leq \|\xi - \xi_0\|_{D_0} + \epsilon \leq 2\epsilon \).

The \( F_D \)-measurability of \( Z_D \) follows trivially from the \( F_D \)-measurability of the seminorms \( \cdot \|_D \), i.e., statement (E.3). \( \square \)

The next lemma states that all the important geometric quantities are locally Lipschitz in the metric. That is, the functions \( \Gamma^{k}_{ij}(g, x) \), \( \Gamma^{k}_{ij,l}(g, x) \) and \( K(g, x) \) are locally Lipschitz at each point \((g, x) \in \Omega \times \mathbb{R}^d\).

Lemma E.2 (Uniform local Lipschitz estimate). Fix a compact set \( D \subset \mathbb{R}^d \) and a metric \( g \in \Omega_\pm \). There exists constants \( \epsilon \) and \( L \) (both varying continuously in \( D \) and \( g \)) such that if \( \|g' - g\|_{C^{2,1}(D)} < \epsilon \), then
\[ \|\Gamma(g', \cdot) - \Gamma(g, \cdot)\|_{C^{1,1}(D)} \leq L\|g' - g\|_{C^{2,1}(D)} \] (E.5)
and
\[ \|K(g', \cdot) - K(g, \cdot)\|_{C^{0,1}(D)} \leq L\|g' - g\|_{C^{2,1}(D)}. \] (E.6)

Proof. Consider the space \( \Omega_\pm \times \mathbb{R}^d \) equipped with the product pseudometric \( \|g' - g\|_{C^{2,1}(D)} \cdot |x' - x| \). The terms \((g, x) \mapsto (b_{ij}(x), b_{ij,k}(x), b_{ij,l}(x), g^{ij}(x), g^{ij,k}(x))\) are locally Lipschitz in \( \Omega_\pm \times \mathbb{R}^d \). Since the Christoffel symbols, their derivatives and the scalar curvature \( K \) are all polynomials in these terms, they are also locally Lipschitz. Let \( L \) be the largest such local Lipschitz constant, taken over all points in the compact set \( D \). Formulas (E.5) and (E.6) follow. \( \square \)

Recall that geodesics are solutions to the geodesic equation: \( \ddot{z}^k = -\Gamma^k_{ij} \dot{z}^i \dot{z}^j \). The geodesic equation is a second-order differential equation with locally-Lipschitz coefficients, so it is easily seen that geodesics exist, and are continuous in their initial conditions. Let \( \gamma_{x,v} \) be the unique geodesic with initial position \( x \) and velocity \( v \).

Lemma E.3. The map \((x, v, g, t) \mapsto \gamma_{x,v}(g, t)\) is jointly continuous.

Proof. Fix some metric \( g \in \Omega_+ \), and initial conditions \((x, v) \in (\mathbb{R}^d)^2\). Let \( T_g \) be the blow-up time of \( \gamma_g := \gamma_{x,v}(g, \cdot) \), and suppose that \( t < T_g \). Let \( D \subset \mathbb{R}^d \) be a compact set which contains the geodesic segment \( \gamma_g \big|_{[0, t+\epsilon]} \) in its interior.

The geodesic equation is a second-order ODE, with coefficients the Christoffel symbols. Lemma E.2 states that the Christoffel symbols \( \Gamma^k_{ij} \) are locally Lipschitz functions of the metric and position. Consequently, a theorem of smoothness of solutions of ODE\textsuperscript{41} implies that
\[ \sup_{s \in [0, t+\epsilon]} \left| \gamma_{x', v'}(g', s) - \gamma_{x, v}(g, s) \right| \leq C \left( \|\Gamma(g', \cdot) - \Gamma(g, \cdot)\|_{C^{0,1}(D)} \cdot |x - x'| \cdot |v - v'| \right), \] (E.7)
for some constant \( C \) depending on \( x, v \) and \( g \). By Lemma E.2, we have that \( \|\Gamma(g, \cdot) - \Gamma(g', \cdot)\|_{C^{0,1}(D)} \leq L\|g - g'\|_{C^{2,1}(D)} \), which implies the result. \( \square \)

\textsuperscript{41}A simple generalization of Theorem 31.8 of Arno\'ld [Arn98].
APPENDIX F. THE CONDITIONAL MEAN OPERATOR $m_D$

For a compact set $D \subseteq \mathbb{R}^d$, let $D^\epsilon$ denote the $\epsilon$-neighborhood of $D$. Recall that $C$ is the space of compact sets of $\mathbb{R}^d$, equipped with the Hausdorff metric. This means that $\epsilon = d_{\text{Haus}}(D, D_0)$ is the smallest value of $\epsilon$ for which $D \subseteq D_0^\epsilon$ and $D_0^\epsilon \subseteq D$.

If $A$ is any subset of $\Omega$, let $A^\perp \subseteq \Omega^*$ denote the annihilator of $A$:

$$A^\perp = \{ f \in \Omega^* : f(\xi) = 0 \text{ for all } \xi \in A \}.$$ 

**Lemma F.1.** Fix a compact set $D \subseteq \mathbb{R}^d$. On the subspace $K\Omega^*$ of $\Omega$, the restriction map $\eta_D$ has kernel $K(\eta^*_D X^*_D)^\perp$.

**Proof.** Let $f \in \Omega^*$. For all $\epsilon \in \eta^*_D X^*_D$, the symmetry of $K$ implies that $e(\eta_D K f) = f(\eta_D^* e)$. Thus $f \in (\eta^*_D X^*_D)^\perp$ if and only if $K f \in \ker \eta_D$. □

**Lemma F.1** implies that $\eta_D$ is injective on $K \eta^*_D X^*_D$, so the inverse map $\eta_D^{-1}$ is well-defined on $\eta_D K \eta^*_D X^*_D$.

**Lemma F.2.** For each compact $D \subseteq \mathbb{R}^d$, the linear map $\eta_D^{-1} : \eta_D K \eta^*_D X^*_D \to \Omega$ has operator norm 1.

**Proof.** The evaluation functionals are dense in the space $X^*_D$, so the operator norm of $\eta_D^{-1}$ is given by

$$\sup_{e \in X^*_D} \frac{\|K \eta^*_D e\|_{C(\mathbb{R}^d)}}{\|\eta_D e\|_{C(D)}} = \sup_{y \in D} \frac{\|c_y\|_{C(\mathbb{R}^d)}}{\|e_y\|_{C(D)}}.$$

(F.1)

The covariance defines an inner product, hence it satisfies the Cauchy-Schwarz inequality. For the stationary covariance function $c$, this implies that $c_y(x) = c(y, x) \leq \sqrt{c(y, y)c(x, x)} = c_y(y)$. Consequently, the numerator of (F.1) equals the denominator, and the ratio is constant and equal to 1. □

**Proof of Lemma 12.2**

**Proof of part (a).** By construction, $\eta_D \hat{m}_D = \eta_D \eta_D^{-1}$ is the identity operator on $\eta_D K \eta^*_D X^*_D$. This space is dense in $X_D$ and the operator $\eta_D \hat{m}_D$ is continuous, so the identity property follows.

**Proof of part (b).** By the finite-range dependence assumption on $c$, $c(y, x) = 0$ for any $y \in D$ and $|x - y| \geq 1$. Since $K$ is the integral operator with kernel $c$, we have $K \eta^*_D e(x) = \int_D c(x, y) d\mu_c(y) = 0$ for any $x \notin D^1$, so

$$K \eta^*_D X^*_D \subseteq \{ \xi \in \Omega : \xi(x) = 0 \text{ if } x \notin D^1 \}.$$ 

Clearly, $K \eta^*_D X^*_D = \hat{m}_D (\eta_D K \eta^*_D X^*_D)$. The space $\eta_D K \eta^*_D X^*_D$ is dense in $X_D$ and $\hat{m}_D$ is continuous, so this completes the proof.

**Proof of part (c).** If $\xi(y) = \xi'(y)$ for all $y \in D$, then $\eta_D(\xi) = \eta_D(\xi')$ hence $\hat{m}_D \eta_D \xi = \hat{m}_D \eta_D \xi'$. This proves that $\xi \mapsto \hat{m}_D \eta_D \xi$ is $F_D$-measurable.

**Proof of part (d).** Fix a compact set $D_0 \subseteq \mathbb{R}^d$ and a function $\xi_0 \in \Omega$. Write $\| \cdot \|$ for the seminorm $\| \cdot \|_{D^\epsilon}$. Let $(D_\alpha, \xi_\alpha) \to (D, \xi)$, and choose $\epsilon \in (0, 1)$ so that $D_\alpha \subset D^\epsilon$ and $\| \xi_\alpha - \xi \| \leq \epsilon$. In light of property (d), it suffices to prove that $\| m_{D_\alpha} \xi_\alpha - m_D \xi \| \to 0$.

Since the operator $K$ has dense image in $\Omega$ by Lemma 12.1, we may choose $f \in K X^*$ so that

$$\| K f - \xi \| < \epsilon,$$

(F.2)
Using the triangle inequality, we calculate
\[
\|m_{D_n} \xi_n - m_D \xi\| \leq \|m_{D_n} \xi_n - m_{D_n} \xi\| + \|m_{D_n} \xi - m_D K\| + \|m_D K - m_D \xi\| \\
\leq 3\epsilon + \|m_{D_n} Kf - m_D Kf\| 
\]
since the operators \(m_{D_n}\) and \(m_D\) all have operator norm 1 by Lemma [E.2].

Using the identity (12.6), we are able to transfer the problem into the setting of Hilbert spaces (as in Lemma [12.2]). Combining the factorization \(K = \iota^*\), the identity (12.6), and the fact that \(\iota\) is a unitary map from \(H\) to \(X\), we have
\[
\|m_{D_n} Kf - m_D Kf\| = \|m_{D_n} \iota^* f - m_D \iota^* f\| = \|\iota \pi_{D_n} \iota^* f - \iota \pi_D \iota^* f\|_H, 
\]
where we denote the Hilbert-space norm by \(\| \cdot \|_H\).

We now introduce the idea of finite-dimensional projections. For any \(\delta > 0\) and any compact \(D \subseteq \mathbb{R}^d\), define the \(\delta\)-mesh of \(D\) by \(M_D^\delta := D \cap \delta \mathbb{Z}\). Since the set \(D\) is compact, the set \(M_D^\delta\) is finite. Define the finite-dimensional subspaces
\[
H_D^\delta = \text{span}\{\iota^* \delta_x : x \in M_D^\delta\} \subseteq H,
\]
and let \(\pi_D^\delta\) denote the projection onto \(H_D^\delta\) in \(H\).

It is clear that \(\cup H_D^\delta = H_D\). Consequently, for each fixed \(f\), \(\lim_{\delta \to 0} \pi_D^\delta \iota^* f = \pi_D \iota^* f\) in \(H\). Since \(D_n \subseteq D\), we can choose \(\delta\) uniformly: there exists \(\delta > 0\) such that
\[
\text{for all } n, \quad \|\pi_D^\delta \iota^* f - \pi_D \iota^* f\|_H < \epsilon \quad \text{and} \quad \|\pi_D^\delta \iota^* f - \pi_D \iota^* f\|_H < \epsilon. 
\]

Since the sets \(D_n\) converge to \(D\) in the Hausdorff distance, for all sufficiently large \(n\), the \(\delta\)-meshes \(M_{D_n}^\delta\) and \(M_D^\delta\) are equal. Consequently, for all large \(n\),
\[
\pi_D^\delta = \pi_D. 
\]

We are ready to complete the proof. By applying the triangle inequality to (F.3), and using the estimates (F.4) and (F.6), we have for all large \(n\),
\[
\|\pi_{D_n} \iota^* f - \pi_D \iota^* f\|_H \leq \|\pi_{D_n} \iota^* f - \pi_D \iota^* f\|_H + \|\pi_D \iota^* f - \pi_D \iota^* f\|_H \\
\leq \epsilon + 0 + 2\epsilon. 
\]

By combining (F.3), (F.4) and (F.7), we have proved that \(\|m_{D_n} \xi_n - m_D \xi\| \leq 5\epsilon\). This completes the proof of Lemma [12.2].

\[\square\]

Appendix G. Proof of Theorem 14.1 Existence of Fermi Normal Coordinates

In Section 1.11 of [Poi04], Poisson derives the Fermi normal coordinates for the case of a pseudo-Riemannian metric in 4-dimensional spacetime. The same analysis also works for Riemannian metrics in arbitrary dimension. We focus on the general \(d\)-dimensional case here, then specialize to \(d = 2\) at the end of the proof to recover (G.2).

Let \(\gamma(t)\) denote a geodesic along an arbitrary Riemannian manifold \((M, g)\). Let \((\gamma(t), n_2(t), \ldots, n_d(t))\) be an orthonormal frame along \(\gamma\). Using the exponential map, define
\[
\Phi_g(t, x^2, \ldots, x^d) = \exp_{\gamma(t)}(x^i n_i(t)). 
\]
The coordinates \((t, x^2, \ldots, x^d)\) are called Fermi normal coordinates. It is clear that in these coordinates, the geodesic is along the \(t\)-axis, and the Christoffel symbols vanish. In the next lemma, we calculate the metric and its derivatives along the \(t\)-axis.

For notational convenience, we write symbols with more space, as with \(\Gamma_{ij}^k\) instead of \(\Gamma_{ij}^k\). We also write subscripts with commas to denote partial derivatives, as with \(\Gamma_{ij,l}^k\) for \(i, j, k\) and \(l\) not equal to 1.

**Lemma G.1.**

\[
g_{11}(t, x) = 1 - R_{1kl}(t)x^kx^l + O(x^3)
g_{ij}(t, x) = -\frac{2}{3}R_{klj}(t)x^kx^l + O(x^3)
g_{ij}(t, x) = \delta_{ij} - \frac{1}{3}R_{klj}(t)x^kx^l + O(x^3),
\]

for \(i, j, k\) and \(l\) not equal to 1.

**Proof.** It follows easily from the definition of the Christoffel symbols that

\[
g_{ij,k} = g_{im}\Gamma_{mj}^k + g_{mj}\Gamma_{ik}^m.
\]

The vanishing of the Christoffel symbols on the geodesic \(\gamma\) implies that \(g_{ij,k} \equiv 0\) along \(\gamma\). To compute the second derivatives of \(g_{ij}\), we will use the Riemann curvature tensor \(R_{ijkl}\), defined by

\[
R_{ijkl} = \Gamma_{ijkl}^k - \Gamma_{iklj}^k + \Gamma_{iml}^k\Gamma_{mj}^i - \Gamma_{mj}^k\Gamma_{iml}^i,
\]

following the physics convention of ordering the indices.

Since \(\Gamma_{ij}^k \equiv 0\) along the geodesic,

\[
\Gamma_{ij,1} = 0,
\]

for any \(i, j\) and \(k\). Plugging this into the definition \([G.4]\) of the Riemann curvature tensor gives

\[
\Gamma_{i1,l}^k = R_{i1l}^k,
\]

for any \(i, k\) and \(l\). The argument on page 23 of \([Po04]\) implies that

\[
\Gamma_{ij,l}^k = -\frac{1}{3}(R_{ijl}^k + R_{jil}^k),
\]

for any \(k\), and for \(i, j\) and \(l\) not equal to 1.

Since the metric is constant along \(\gamma\), \(g_{ij,1k} = 0\) for any \(i, j\) and \(k\). Thus it suffices to calculate \(g_{11,kl}, g_{1j,kl}\) and \(g_{ij,kl}\) for \(j, k\) and \(l\) not equal to 1.

Differentiating \([G.3]\) and noting that the terms with Christoffel symbols vanish, we have

\[
g_{ij,kl} = g_{im}\Gamma_{jk}^m + g_{mj}\Gamma_{ik}^m,
\]

along the geodesic. To calculate \(g_{11,kl}\), we plug in the formula \([G.6]\) for the first derivative of the Christoffel symbols to get

\[
g_{11,kl} = 2g_{1m}\Gamma_{k1,l}^m = 2g_{1m}R_{ml}^{k1,l} = 2R_{1kl}^1 = -2R_{1kl}^1,
\]

where the last line follows from the symmetry \(R_{klm} = R_{mlk}\) of the Riemann tensor. To calculate \(g_{1j,kl}\), we apply both expressions \([G.6]\) and \([G.7]\) for the Christoffel symbols to \([G.8]\) to get

\[
g_{1j,kl} = g_{1m}\Gamma_{jk}^m + g_{mj}\Gamma_{1k}^m = -\frac{1}{3}(R_{1jkl} + R_{1jkl}) + R_{jkl}^1
= -\frac{1}{3}R_{1jkl} + \frac{1}{3}(R_{1ljk} + R_{1klj}) - R_{1ljk}
= -\frac{2}{3}(R_{1jkl} + R_{1ljk})
\]

\([G.10]\).
where we use the symmetry $R_{jkl1} = -R_{1lkj}$, the Bianchi identity $R_{1jkl} = -R_{1ljk} - R_{1klj}$, and the symmetry $R_{1klj} = -R_{1kjl}$.

By a similar argument,

$$g_{ij,kl} = -\frac{1}{3} (R_{ikjl} + R_{ijkl} + R_{jikl}) = -\frac{1}{3} (R_{ikjl} + R_{iljk}),$$

where the middle two terms cancel by the symmetry $R_{ijkl} = -R_{jikl}$, and the last terms are equal by the symmetry $R_{jkil} = R_{iljk}$.

We now expand the metric $g(t, x)$ in a Taylor series around the point $(t, 0)$, noting that $g_{ij}(t, 0) = \delta_{ij}$, $g_{ij,k}(t, 0) = 0$, and using the values (G.9), (G.10) and (G.11) for the second derivative $g_{ij,kl}(t, 0)$ of the metric. Formula (G.2) follows. \(\square\)

In the case $d = 2$, formula (G.2) takes a particularly simple form, since the Riemann curvature tensor is determined by the scalar curvature $K(t)$ via the following identity:

$$R_{1212} = \frac{1}{2} K(t) \det g = \frac{1}{2} K(t)(g_{11} g_{22} - g_{12}^2).$$

Applying this, we have $R_{1212} = \frac{1}{2} K(t)$, and the terms with $R_{1222}$ and $R_{2222}$ vanish by the symmetries of the curvature tensor, so

$$g_{11}(t, x) = 1 - \frac{1}{2} K(t)x^2 + O(x^3), \quad g_{12}(t, x) = O(x^3), \quad \text{and} \quad g_{22}(t, x) = 1 + O(x^3).$$

(G.13)
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