The double queen Dido’s problem
Lorenzo Cavallina, Antoine Henrot, Shigeru Sakaguchi

To cite this version:
Lorenzo Cavallina, Antoine Henrot, Shigeru Sakaguchi. The double queen Dido’s problem. The Journal of Geometric Analysis, In press, 10.1007/s12220-020-00549-1. hal-02501399

HAL Id: hal-02501399
https://hal.science/hal-02501399
Submitted on 6 Mar 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
The double queen Dido’s problem *

Lorenzo Cavallina†, Antoine Henrot‡, and Shigeru Sakaguchi†

Abstract

This paper deals with a variation of the classical isoperimetric problem in dimension $N \geq 2$ for a two-phase piecewise constant density whose discontinuity interface is a given hyperplane. We introduce a weighted perimeter functional with three different weights, one for the hyperplane and one for each of the two open half-spaces in which $\mathbb{R}^N$ gets partitioned. We then consider the problem of characterizing the sets $\Omega$ that minimize this weighted perimeter functional under the additional constraint that the volumes of the portions of $\Omega$ in the two half-spaces are given. It is shown that the problem admits two kinds of minimizers, which will be called type I and type II, respectively. These minimizers are made of the union of two spherical domes whose angle of incidence satisfies some kind of “Snell’s law”. Finally, we provide a complete classification of the minimizers depending on the various parameters of the problem.

Key words. Isoperimetric problem, Dido’s problem, constrained minimization problem, weighted manifold, two-phase.

AMS subject classifications. 49Q20.

1 Introduction and main result

The classical isoperimetric inequality has a long history even if one had to wait until the fifties for a rigorous proof for general sets by E. De Giorgi (see [11] for some history). There is also some important literature on isoperimetric problems with densities, see for

---

*This research was partially supported by the Grants-in-Aid for Scientific Research (B) No.18H01126 and JSPS Fellows No.18J11430 of Japan Society for the Promotion of Science. This work was also partially supported by the project ANR-18-CE40-0013 SHAPO financed by the French Agence Nationale de la Recherche (ANR).
†Research Center for Pure and Applied Mathematics, Graduate School of Information Sciences, Tohoku University, Sendai, 980-8579, Japan (cava@ims.is.tohoku.ac.jp, sigersak@tohoku.ac.jp).
‡Institut Elie Cartan de Lorraine, CNRS UMR 7502 and Université de Lorraine, BP 70239 54506 Vandoeuvre-les-Nancy, France (antoine.henrot@univ-lorraine.fr).
example [15, Chapter 18] for an introduction. Let a positive function \( f: \mathbb{R}^N \to \mathbb{R}_+ \) be given. For any sufficiently smooth set \( E \), we define the weighted volume and perimeter of \( E \) to be
\[
|E|_f = \int_E f \, dx, \quad P_f(E) = \int_{\partial E} f \, d\mathcal{H}^{N-1}.
\]
(1.1)

In probability theory, it is quite common to use the Gaussian density \( f(x) = \exp(-|x|^2) \) for which the isoperimetric sets are half-spaces, see [6] and [15, Chapter 18]. Another classical choice is radial functions like \( f(x) = |x|^q \) (see e.g. [3, 5, 9]), for which the isoperimetric sets are usually balls. A much less studied density is a piecewise constant density. In the paper [8], A. Cañete, M. Miranda and N. Vittone studied some particular cases related to the characteristic functions of half-planes, strips and balls. Our aim here is to consider a variant of this study by considering two-half spaces in \( \mathbb{R}^N \) with different constant densities together with a cost \( \gamma \geq 0 \) (possibly 0) on the hyperplane which is the interface. Concerning Dido’s problem in the half space (with a constant density), the solution is given by a half ball, see e.g. [10] for the proof of a more general result, namely that the half ball has the smallest possible (relative) perimeter than any other set of the same volume outside a convex domain. As an application, for our problem, if we do not put any cost on the interface (that is, \( \gamma = 0 \)), the problem decouples and the solution of the isoperimetric problem will be the union of two half balls.

Let us now fix the notations and set the problem in more detail. Let \( \mathbb{R}^N_\pm \) denote the following left and right open half-spaces of \( \mathbb{R}^N \):
\[
\mathbb{R}^N_- = \{(x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : x_1 < 0\}, \quad \mathbb{R}^N_+ = \{(x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : x_1 > 0\},
\]
and let \( \Sigma \) be the vertical hyperplane
\[
\Sigma = \{(x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : x_1 = 0\}.
\]
For a given set of finite perimeter \( \Omega \subset \mathbb{R}^N \), put
\[
\Omega_\pm = \Omega \cap \mathbb{R}^N_\pm, \quad \Gamma_\pm = \partial \Omega \cap \mathbb{R}^N_\pm, \quad \Gamma_0 = \partial \Omega \cap \Sigma.
\]
(1.2)

For given constants \( V_\pm, \rho_\pm > 0 \) and \( \gamma \geq 0 \), we consider the following constrained minimization problem:
\[
\min \left\{ \mathcal{F}(\Omega) : \Omega \subset \mathbb{R}^N \text{ is of finite perimeter and } \rho_\pm |\Omega_\pm| = V_\pm \right\},
\]
(1.3)
where
\[
\mathcal{F}(\Omega) = \rho_- P(\Omega, \mathbb{R}^N_-) + \rho_+ P(\Omega, \mathbb{R}^N_+) + \gamma \mathcal{H}^{N-1}(\Gamma_0).
\]
(1.4)
In the above, we used the notations $|\cdot|$, $P$ and $}\mathcal{H}^{N-1}$ for the Lebesgue measure, the relative perimeter in the sense of De Giorgi (see [13]) and the $(N-1)$-dimensional Hausdorff measure, respectively. We remark that, if the boundary $\partial \Omega$ coincides with the reduced boundary $\partial^* \Omega$, then

$$F(\Omega) = P_f(\Omega),$$

where $P_f$ is the weighted perimeter introduced in (1.1) and $f$ is the piecewise constant function defined as $f(x) = \rho_\pm$ for $x_1 \gtrless 0$ and $f(x) = \gamma$ if $x_1 = 0$.

The aim of this paper is to give a complete characterization of the minimizers of (1.3) for all values of the parameters $V_\pm$, $\rho_\pm$ and $\gamma$. This paper is organized as follows. In section 2 we show that, if $\gamma > 0$, any minimizer $\Omega$ of (1.3), if it exists, must be connected and both $\Gamma_\pm$ are spherical caps. This fact allows for only two types of minimizers of (1.3): one where the boundaries of the two spherical caps coincide (type I) and another one where $\mathcal{H}^{N-1}(\Gamma_0) > 0$ (type II). In section 3 we show the existence of minimizers for problem (1.3) by means of a standard compactness argument. In section 4 we derive some geometrical transmission conditions that describe the angle of incidence between $\Gamma_\pm$ and $\Sigma$. By means of these conditions, we are able to reduce the number of potential minimizers to just two: one for each type (up to translations). Finally, in section 5 we find a threshold $\gamma^* = \gamma^*(V_\pm, \rho_\pm)$ such that the minimizer of (1.3) is of type II for $0 < \gamma < \gamma^*$ and of type I for $\gamma \geq \gamma^*$.

2 Geometrical properties of minimizers

Here we will study the geometrical properties of a minimizer of (1.3) provided that at least one exists. The question of existence will be then addressed in the next section. For this purpose, let us first utilize the Schwarz symmetrization (see [14, p. 238] for its definition and properties). Let $\ell$ be a line orthogonal to $\Sigma$. For a set of finite perimeter $\Omega$ in $\mathbb{R}^N$ with $|\Omega| < +\infty$, let $\Omega^*$ denote the Schwarz symmetrization of $\Omega$ around $\ell$. Then we have

**Lemma 2.1.** If $\Omega$ is a set of finite perimeter in $\mathbb{R}^N$ with $|\Omega| < +\infty$, then $\Omega^*$ is also a set of finite perimeter in $\mathbb{R}^N$ and the following hold

$$|\Omega^*_\pm| = |\Omega_\pm|, \quad P(\Omega^*, \mathbb{R}^N) \leq P(\Omega, \mathbb{R}^N), \quad \mathcal{H}^{N-1}(\Gamma_0^*) \leq \mathcal{H}^{N-1}(\Gamma_0) \quad \text{and hence} \quad F(\Omega^*) \leq F(\Omega),$$

respectively, where $\Omega^*_\pm$ and $\Gamma_0^*$ follow notations (1.2).

**Proof.** By means of the Schwarz symmetrization, Fubini’s theorem yields that $|\Omega^*_\pm| = |\Omega_\pm|$, respectively. Since the regularity of $\partial \Omega$ on $\Sigma$ is not transparent, we employ the following
approximation argument. In view of [14, Exercise 15.13, p. 173], we may find a decreasing sequence of positive numbers \( \{\varepsilon_n\}_n \) with \( \lim_{n \to \infty} \varepsilon_n = 0 \) such that for every \( n \in \mathbb{N} \) each left half-space \( H_{-\varepsilon_n} = \{x \in \mathbb{R}^N : x_1 < -\varepsilon_n\} \) satisfies the following:

\[
\mu_{\Omega \cap H_{-\varepsilon_n}} = \mu_\Omega \upharpoonleft H_{-\varepsilon_n} + \varepsilon_1 \mathcal{H}^{N-1} \upharpoonleft (\Omega \cap \partial H_{-\varepsilon_n}),
\]

where \( \mu_{\Omega \cap H_{-\varepsilon_n}} \) and \( \mu_\Omega \) are the Gauss-Green measures of the two sets of finite perimeter \( \Omega \cap H_{-\varepsilon_n} \) and \( \Omega \), respectively, (see [14, Chapter 12] for the definition and some basic properties) and \( \varepsilon_1 = (1, 0, \ldots, 0) \in \mathbb{R}^N \). Hence it follows from (2.5) that for every \( n \in \mathbb{N} \)

\[
P(\Omega \cap H_{-\varepsilon_n}) = P(\Omega, H_{-\varepsilon_n}) + \mathcal{H}^{N-1}(\Omega \cap \partial H_{-\varepsilon_n}).
\]

By the same argument as above, we may also have that for every \( n \in \mathbb{N} \)

\[
P(\Omega^* \cap H_{-\varepsilon_n}) = P(\Omega^*, H_{-\varepsilon_n}) + \mathcal{H}^{N-1}(\Omega^* \cap \partial H_{-\varepsilon_n}),
\]

if we replace \( \Omega \) with its Schwarz symmetrization \( \Omega^* \). By means of the Schwarz symmetrization, we notice that for every \( n \in \mathbb{N} \)

\[
(\Omega \cap H_{-\varepsilon_n})^* = \Omega^* \cap H_{-\varepsilon_n} \quad \text{and} \quad \mathcal{H}^{N-1}(\Omega \cap \partial H_{-\varepsilon_n}) = \mathcal{H}^{N-1}(\Omega^* \cap \partial H_{-\varepsilon_n}).
\]

Then, the inequality [14, Theorem 19.11, p. 238] shows that

\[
P(\Omega^* \cap H_{-\varepsilon_n}) = P((\Omega \cap H_{-\varepsilon_n})^*) \leq P(\Omega \cap H_{-\varepsilon_n}).
\]

Therefore, by combining (2.6) and (2.7) with the second equality of (2.9), we conclude that for every \( n \in \mathbb{N} \)

\[
P(\Omega^*, H_{-\varepsilon_n}) \leq P(\Omega, H_{-\varepsilon_n}).
\]

Now, since the monotonically increasing sequence of sets \( \{H_{-\varepsilon_n}\}_n \) converges to \( \mathbb{R}^N_- \), letting \( n \to \infty \) gives

\[
P(\Omega^*, \mathbb{R}^N_-) \leq P(\Omega, \mathbb{R}^N_-).
\]

Similarly, using right half-space \( H_{\varepsilon_n} = \{x \in \mathbb{R}^N : x_1 > \varepsilon_n\} \) instead of \( H_{-\varepsilon_n} \) gives

\[
P(\Omega^*, \mathbb{R}^N_+) \leq P(\Omega, \mathbb{R}^N_+).
\]

Also, we introduce the set \( F_n = \{x \in \mathbb{R}^N : -\varepsilon_n < x_1 < \varepsilon_n\} \) instead of the two half-spaces, and similarly we obtain that for every \( n \in \mathbb{N} \)

\[
P(\Omega \cap F_n) = P(\Omega, F_n) + \mathcal{H}^{N-1}(\Omega \cap \partial H_{-\varepsilon_n}) + \mathcal{H}^{N-1}(\Omega \cap \partial H_{\varepsilon_n}),
\]

\[
P(\Omega^* \cap F_n) = P(\Omega^*, F_n) + \mathcal{H}^{N-1}(\Omega^* \cap \partial H_{-\varepsilon_n}) + \mathcal{H}^{N-1}(\Omega^* \cap \partial H_{\varepsilon_n}).
\]
Here the inequality \[14, \text{Theorem 19.11, p. 238}\] shows again that
\[ P(\Omega^{*} \cap F_{n}) \leq P(\Omega \cap F_{n}), \]
and hence similarly
\[ P(\Omega^{*}, F_{n}) \leq P(\Omega, F_{n}). \]
Now, since the monotonically decreasing sequence of sets \( \{F_{n}\}_{n} \) converges to \( \Sigma \), letting \( n \to \infty \) gives
\[ \mathcal{H}^{N-1}(\Gamma^{*}_{0}) \leq \mathcal{H}^{N-1}(\Gamma_{0}). \tag{2.12} \]
Finally, collecting \( (2.10), (2.11) \) and \( (2.12) \) yields that
\[ \mathcal{F}(\Omega^{*}) \leq \mathcal{F}(\Omega). \]

**Theorem 2.2.** If \( \Omega \) is a minimizer of \( (1.3) \), then each of \( \Gamma_{\pm} \) is either a spherical cap or a sphere.

**Proof.** Let \( \Omega \) be a minimizer of \( (1.3) \). In particular, both \( \Omega_{\pm} \) are isoperimetric sets on their own, that is, \( \Omega_{\pm} \) minimize perimeter in \( \mathbb{R}^{N}_{\pm} \), respectively, with a volume constraint in the sense of Gonzales, Massari and Tamanini \[12, \text{p. 27}\]. Therefore their regularity result \[12, \text{Theorem 2, p. 29}\] implies that \( \Gamma_{\pm} \) must be analytic surfaces up to a singular set of dimension at most \( N-8 \) (that is to say that the singular set is empty when \( N \leq 7 \)).

It follows from Lemma \( (2.1) \) that the Schwarz symmetrization \( \Omega^{*} \) of \( \Omega \) is also a minimizer of \( (1.3) \) with \( \mathcal{F}(\Omega^{*}) = \mathcal{F}(\Omega) \), since \( \Omega \) does. Hence the equalities also hold in \( (2.10) \) and \( (2.11) \). These two equalities together with \[14, \text{Theorem 19.11, p. 238}\] yield that for almost every \( t \in \mathbb{R} \), the set \( \Omega_{t} = \{ x \in \Omega : x_{1} = t \} \) is \( \mathcal{H}^{N-1} \)-equivalent to an \( (N-1) \)-dimensional ball, whose radius will be denoted by \( R(t) \geq 0 \). By combining this information with the regularity of \( \Gamma_{\pm} \) mentioned at the beginning of the proof, we infer that each \( \Omega_{\pm} \) needs to enjoy axial symmetry with respect to some straight line orthogonal to the hyperplane \( \Sigma \) and each \( \Gamma_{\pm} \) does not contain any flat parts. In particular, both \( \Gamma_{\pm} \) are analytic everywhere except at most at those points where \( \Gamma_{\pm} \) intersect their axis of symmetry. In other words, we know that \( \Gamma_{\pm} \) are analytic at every point \( x \in \Gamma_{\pm} \) whose first component \( x_{1} \) belongs to
\[ I = \{ t \in \mathbb{R} : R(t) > 0 \}. \]

We will now show that each \( \Gamma_{\pm} \) is a spherical cap or a sphere, as claimed. To fix ideas, let us consider the following rearrangement of \( \Omega_{\pm} \). Take some positive value \( t_{0} \in I \). By the
above, we know that $\Omega_+$ is an axially symmetric set and the intersection $D = \overline{\Omega_+} \cap \{x_1 = t_0\}$ is an $(N - 1)$-dimensional closed ball of positive radius $R(t_0) > 0$. Now, let $B$ denote the closed ball in $\mathbb{R}^N$ determined by

$$B \cap \{x_1 = t_0\} = D \quad \text{and} \quad |B \cap \{x_1 > t_0\}| = |\Omega_+ \cap \{x_1 > t_0\}|.$$  \hspace{1cm} (2.13)

Define

$$\widetilde{B} = (B \cap \{x_1 \leq t_0\}) \cup \left(\overline{\Omega_+} \cap \{x_1 > t_0\}\right).$$  \hspace{1cm} (2.14)

![Diagram](image.png)

Figure 1: The construction employed in the proof of Theorem 2.2.

Since $B$ and $\widetilde{B}$ have the same volume by construction, then, by the isoperimetric inequality, $\mathcal{H}^{N-1}(\partial B) \leq \mathcal{H}^{N-1}(\partial \widetilde{B})$, with equality holding true if and only if $B = \widetilde{B}$. By (2.13) and the minimality of $\Omega$, this implies that $\overline{\Omega_+} \cap \{x_1 > t_0\} = B \cap \{x_1 > t_0\}$ and, hence, $\Gamma_+ \cap \{x_1 > t_0\}$ must be a spherical cap. In particular, the set $I \cap [t_0, \infty)$ is connected. As a matter of fact, we claim that the set $I \cap (0, \infty)$ is connected. Indeed, let us assume, by contradiction, that there exists some value $t_1 \in I \cap (0, t_0)$ that belongs to a different connected component. Now, performing one more time the rearrangement with $t_1$ instead of $t_0$ yields that the set $I \cap [t_1, \infty)$ must be connected, which is a contradiction. Therefore, $I \cap (0, \infty)$ is connected and $\Gamma_+ \cap \{x_1 > t_0\}$ is a spherical cap. By analyticity, this implies that the whole $\Gamma_+$ must be either a spherical cap or a sphere, as claimed. The same conclusion holds true for $\Gamma_-$. \hfill \Box

Theorem 2.2 suggests to us that the following two possibilities for a minimizer will be taken into account.

**Definition 2.3.** Let $\Omega$ be a minimizer of (1.3). Then, $\Omega$ is connected and $\Gamma_\pm$ are spherical caps. This can only happen in one of the following two ways (see also Figure 2).
• **Type I minimizer.** The boundaries of the manifolds $\Gamma_\pm$ coincide and $\overline{\Omega} \cap \Sigma$ is an $(N-1)$-dimensional ball. In this case, $\mathcal{H}^{N-1}(\Gamma_0) = 0$.

• **Type II minimizer.** $\overline{\Omega} \cap \Sigma$ is an $(N-1)$-dimensional ball, but this time the boundaries of the manifolds $\Gamma_\pm$ are $(N-2)$-spheres whose radii have distinct lengths and centers do not necessarily coincide. In this case, $\mathcal{H}^{N-1}(\Gamma_0) > 0$.

![Diagram showing Type I and Type II minimizers](image)

Figure 2: The only two possible types of minimizers.

### 3 Existence of a minimizer and its regularity

Here, we will finally prove the existence of a minimizer for Problem (1.3). In order to do this, we will first need to generalize the rearrangement technique employed in the proof of Theorem 2.2 to a general set of finite perimeter. Indeed, the situation is quite standard in each half-space since we deal then with a classical isoperimetric problem, but the difficulty is to deal with the part of the boundary which may be on $\Sigma$ (since, a priori, we have little information about the regularity of the set there).

**Lemma 3.1.** Let $\Omega$ be a set of finite perimeter in $\mathbb{R}^N$ with $|\Omega| < +\infty$. Then, there exists a set of finite perimeter $\Omega^*$ in $\mathbb{R}^N$ which is axially symmetric with respect to $\ell$ such that each of $\Gamma_{\pm} = \partial \Omega^* \cap \mathbb{R}_\pm^N$ is either a spherical cap or a sphere and the following hold:

$$|\Omega^*_\pm| = |\Omega^*_\mp|, \quad P(\Omega^*, \mathbb{R}^N_{\pm}) \leq P(\Omega^*, \mathbb{R}_\pm^N), \quad \mathcal{H}^{N-1}(\Gamma^*_0) \leq \mathcal{H}^{N-1}(\Gamma^*_0) \quad \text{and} \quad F(\Omega^*) \leq F(\Omega^*),$$

respectively, where $\Omega_{\pm}$ and $\Gamma_0$ follow notations (1.2).
Proof. By Lemma 2.1 we know that the set $\Omega^*$ is well defined. Now, using the same notation as in Lemma 2.1 by (2.7) and (2.8) we have that, for every $n \in \mathbb{N}$

$$\mathcal{H}^{N-1}(\Omega^* \cap \partial H_{-\varepsilon_n}) \leq P(\Omega^*) \quad \text{and similarly} \quad \mathcal{H}^{N-1}(\Omega^* \cap \partial H_{\pm\varepsilon_n}) \leq P(\Omega^*).$$

Therefore, by the Bolzano-Weierstrass theorem, up to a subsequence, we may assume that, as $n \to \infty$, the sequences of the radii of the $(N-1)$-dimensional balls $\Omega^* \cap \partial H_{\pm\varepsilon_n}$ converge to some nonnegative numbers $r_{\pm}$, respectively. Let $D_{\pm}$ be the two $(N-1)$-dimensional closed balls in $\Sigma$ centered at $\ell \cap \Sigma$ with radii $r_{\pm}$, respectively. Let $B_{\pm}$ denote the two closed balls in $\mathbb{R}^N$ determined by

$$B_{\pm} \cap \Sigma = D_{\pm} \quad \text{and} \quad |B_{\pm} \cap \mathbb{R}^N| = |\Omega^*_\pm|,$$  \hspace{1cm} (3.15)

respectively.

Figure 3: The construction employed in the proof of Lemma 3.1. For ease of understanding, the boundary of the set $\Omega^*$ is denoted by a bold line, while the interior of $\Omega^*$ is shaded.

Then we set

$$\Omega^\sharp = (B_- \cap \mathbb{R}^N) \cup (B_+ \cap \mathbb{R}^N).$$  \hspace{1cm} (3.16)

Hence $|\Omega^\sharp_-| = |\Omega^\sharp_+|$, respectively. Notice that if $r_+$ (or $r_-$) equals 0, then $\Omega^\sharp_+$ (or $\Omega^\sharp_-$) equals the ball $B_+$ (or $B_-)$). Moreover, for every $n \in \mathbb{N}$, let $B^n_{\pm}$ denote the two closed balls in $\mathbb{R}^N$ determined by

$$B^n_{\pm} \cap \partial H_{\pm\varepsilon_n} = \Omega^\sharp \cap \partial H_{\pm\varepsilon_n} \quad \text{and} \quad |B^n_{\pm} \cap H_{\pm\varepsilon_n}| = |\Omega^\sharp \cap H_{\pm\varepsilon_n}|,$$  \hspace{1cm} (3.17)
respectively. Then, for every \( n \in \mathbb{N} \), we set

\[
\Omega^n = (B^n_+ \cap H_{-\varepsilon_n}) \cup (\Omega^* \cap F_n) \cup (B^n_- \cap H_{\varepsilon_n}).
\]  

(3.18)

Hence \(|\Omega^n_\pm| = |\Omega^*_\pm| \), respectively. Define

\[
\tilde{B}^n_\pm = (\Omega^* \cap H_{\pm\varepsilon_n}) \cup (B^n_\pm \setminus H_{\pm\varepsilon_n}),
\]

respectively. Since \( B^n_\pm \) and \( \tilde{B}^n_\pm \) have the same volume respectively, we have from the classical isoperimetric inequality that \( \mathcal{H}^{N-1}(\partial \tilde{B}^n_\pm) \leq P(\tilde{B}^n_\pm) \) respectively. Hence it follows that for every \( n \in \mathbb{N} \)

\[
\mathcal{H}^{N-1}(\partial B^n_\pm \cap \overline{H_{\pm\varepsilon_n}}) \leq P(\Omega^*, H_{\pm\varepsilon_n}),
\]

respectively. Then, we observe that for every \( n \in \mathbb{N} \)

\[
P(\Omega^n, \mathbb{R}^N_\pm) \leq P(\Omega^*, \mathbb{R}^N_\pm) \quad \text{and} \quad \mathcal{H}^{N-1}(\Gamma^n_0) = \mathcal{H}^{N-1}(\Gamma^*_0),
\]

respectively, where \( \Gamma^n_0 = \partial \Omega^n \cap \Sigma \). Another observation is that \( \{\Omega^n\}_n \) converges to \( \Omega^* \) in the sense of their characteristic functions as \( n \to \infty \). Hence the lower semicontinuity of the perimeter yields that

\[
P(\Omega^1, \mathbb{R}^N_\pm) \leq \liminf_{n \to \infty} P(\Omega^n, \mathbb{R}^N_\pm) \leq P(\Omega^*, \mathbb{R}^N_\pm) \quad \text{and} \quad P(\Omega^1, F_\varepsilon) \leq \liminf_{n \to \infty} P(\Omega^n, F_\varepsilon),
\]

(3.19)

where \( \varepsilon > 0 \) is an arbitrary number and \( F_\varepsilon = \{ x \in \mathbb{R}^N : -\varepsilon < x_1 < \varepsilon \} \). Since

\[
P(\Omega^n, F_\varepsilon) = \mathcal{H}^{N-1}(\partial \Omega^n \cap (F_\varepsilon \setminus F_n)) + P(\Omega^*, F_n),
\]

we see that

\[
\lim_{n \to \infty} P(\Omega^n, F_\varepsilon) = \lim_{n \to \infty} \mathcal{H}^{N-1}(\partial \Omega^n \cap (F_\varepsilon \setminus F_n)) + \mathcal{H}^{N-1}(\Gamma^*_0).
\]

By recalling that as \( n \to \infty \) the sequence of radii of the \((N-1)\)-dimensional balls \( \Omega^* \cap \partial H_{\pm\varepsilon_n} \) converges to the nonnegative numbers \( r_\pm \), respectively, we infer that

\[
\lim_{\varepsilon \to 0} \liminf_{n \to \infty} \mathcal{H}^{N-1}(\partial \Omega^n \cap (F_\varepsilon \setminus F_n)) = 0,
\]

and hence

\[
\lim_{\varepsilon \to 0} \liminf_{n \to \infty} P(\Omega^n, F_\varepsilon) = \mathcal{H}^{N-1}(\Gamma^*_0).
\]

Moreover,

\[
\lim_{\varepsilon \to 0} P(\Omega^1, F_\varepsilon) = |\mathcal{H}^{N-1}(D_+) - \mathcal{H}^{N-1}(D_-)| = \mathcal{H}^{N-1}(\Gamma^*_0).
\]

Therefore it follows from (3.19) that

\[
\mathcal{H}^{N-1}(\Gamma^*_0) \leq \mathcal{H}^{N-1}(\Gamma^*_0), \quad P(\Omega^1, \mathbb{R}^N_\pm) \leq P(\Omega^*, \mathbb{R}^N_\pm) \quad \text{and} \quad \mathcal{F}(\Omega^1) \leq \mathcal{F}(\Omega^*),
\]

which completes the proof. \( \square \)
Theorem 3.2. The problem (1.3) has a minimizer $\Omega$. Moreover it is of one of the two types described in Definition 2.3.

Proof. Let $\Omega^k$ be a minimizing sequence. Following (1.2) we introduce $\Omega^k_\pm$, $\Gamma^k_\pm$ and $\Gamma^k_0$. By Lemma 3.1 we may assume that $\Gamma^k_\pm$ are spherical caps. Moreover, let $D^k_\pm$ denote the two $(N-1)$-dimensional balls given by the intersections $\Omega^k_\pm \cap \Sigma$ and let $R^k_\pm$ be their radii. In particular, since $\Omega^k$ is a minimizing sequence for the functional $F$, we know that the sequence of radii $R^k_\pm$ must be bounded (as $H^{N-1}(\Gamma^k_\pm)$ would diverge to infinity otherwise).

This means that the whole sequence of sets $\Omega^k$ is contained in a large enough compact set. By compactness (compact embedding from the BV-space to $L^1_{\text{loc}}$, see [13, Chapter 2]), up to extracting a subsequence, we can assume that:

- $\Omega^k_\pm$ converges to some $\Omega_\pm$ (in the sense of characteristic functions),
- each $D^k_\pm$ converges to a ball $D_\pm$ (convergence of their radii).

Moreover, by property of this convergence, in particular the lower semicontinuity of the perimeter, we have

$$\rho_\pm |\Omega_\pm| = V_\pm, \quad P(\Omega, \mathbb{R}^N) \leq \liminf_{k \to \infty} P(\Omega^k, \mathbb{R}^N).$$

Together with the convergence of the balls $D^k_\pm$ and the fact that $H^{N-1}(\Gamma_0) = |H^{N-1}(D_-) - H^{N-1}(D_+)|$, this implies that $F(\Omega) \leq \liminf_{k \to \infty} F(\Omega^k)$ and thus $\Omega$ is a minimizer of problem (1.3). The second part of the statement directly follows from Theorem 2.2. In particular, to rule out the case of a sphere in one of the half-space, we can use Corollary 4.2 below. \qed

4 Complete characterization of the two types of minimizers

4.1 Some preliminary geometrical lemmas

By Theorem 3.2, we know that there are only two kinds of minimizers, which, without loss of generality, can be assumed to be symmetric with respect to rotations around the $x_1$ axis. Under this additional symmetry assumption, each candidate set becomes a (piecewise) hypersurface of revolution, whose generatrix can be uniquely described by four parameters (the radii $R_\pm$, and the angles of incidence $\alpha$ and $\beta$, or the radii $R_\pm$ and the pair of centers $a$ and $b$) as shown in Figure 4. We will refer to each candidate set by means of those parameters as $\Omega(\alpha, \beta, R_\pm)$. In what follows, we are going to give explicit formulas for
computing the Lebesgue measure of $\Omega_\pm(\alpha, \beta, R_\pm)$ and the $(N - 1)$-dimensional Hausdorff measure of $\Gamma_\pm(\alpha, \beta, R_\pm)$ and $\Gamma_0(\alpha, \beta, R_\pm)$. We will also employ the use of the following shorthand notation:

$$\begin{align*}
J_- &= J_-(\alpha) = \int_{\alpha}^{\pi} (\sin \theta)^{N-2} d\theta, \\
J_+ &= J_+(\beta) = \int_{0}^{\beta} (\sin \theta)^{N-2} d\theta, \\
I_- &= I_-(\alpha) = \int_{\alpha}^{\pi} (\sin \theta)^{N} d\theta, \\
I_+ &= I_+(\beta) = \int_{0}^{\beta} (\sin \theta)^{N} d\theta, \\
s_- &= s_-(\alpha) = \sin \alpha, \\
s_+ &= s_+(\beta) = \sin \beta, \\
c_- &= c_-(\alpha) = \cos \alpha, \\
c_+ &= c_+(\beta) = \cos \beta.
\end{align*}$$

Lemma 4.1. Let $\Omega = \Omega(\alpha, \beta, R_\pm)$ and let $\omega$ denote the $(N - 1)$-dimensional volume of the $(N - 1)$-dimensional unit ball. Then the following formulas hold true.

$$|\Omega_\pm| = \omega R_\pm^{N} I_\pm, \quad \mathcal{H}^{N-1}(\Gamma_\pm) = (N-1)\omega R_\pm^{N-1} J_\pm, \quad \mathcal{H}^{N-1}(\Gamma_0) = \omega \left| (R_- s_-)^{N-1} - (R_+ s_+)^{N-1} \right|.$$  

Proof. We will prove only the formulas corresponding to the subscript $-$, as the others are analogous. First, by Cavalieri’s principle as

$$|\Omega_-| = \omega \int_{-R_-}^{R_-} \left( \sqrt{R_-^2 - x^2} \right)^{N-1} dx.$$  

Now, the substitution $x = R_- \cos \theta$ yields

$$|\Omega_-| = \omega \int_{\alpha}^{\pi} R_-^N (\sin \theta)^{N} d\theta = \omega R_-^N I_-. $$  

The value of $\mathcal{H}^{N-1}(\Gamma_\pm)$ can be easily computed by considering $\Gamma_-$ as a hypersurface of revolution in $\mathbb{R}^N$. Indeed, if we set $y(x) = \sqrt{R_-^2 - x^2}$, by the area formula for surfaces of
revolution in general dimension given in $[1]$ we get:

$$\mathcal{H}^{N-1}(\Gamma_-) = (N-1)\omega \int_{-R_-}^{R_- \cos \alpha} y(x)^{N-2} \sqrt{1 + (y'(x))^2} \, dx.$$ 

Now, recalling that $\sqrt{1 + (y'(x))^2} = R_- / y(x)$ and performing the substitution $x = R_- \cos \theta$ yield

$$\mathcal{H}^{N-1}(\Gamma_-) = (N-1)\omega \int_0^\pi R_-^{N-1}(\sin \theta)^{N-2} \, d\theta = (N-1)\omega R_-^{N-1} J_-.$$ 

Finally, the value of $\mathcal{H}^{N-1}(\Gamma_0)$ is immediately computed by noticing that, in this case, $\Gamma_0$ is just an $(N-1)$-dimensional spherical shell whose outer and inner radii are given by $\max(R_-s_-, R_+s_+)$ and $\min(R_-s_-, R_+s_+)$ respectively.

**Corollary 4.2.** If $\Omega$ is an optimal set, then neither of $\Gamma_\pm$ can be a whole sphere.

**Proof.** Without loss of generality, assume that $\Omega$ is a type II minimizer and the part of boundary $\Gamma_- \pm$ is a whole sphere. This corresponds to $\alpha = 0$ in the previous notations.

To get the thesis, it suffices to prove that the total perimeter strictly decreases when we increase $\alpha$. Let us first assume that $\Gamma_\pm$ is not a whole sphere, then $\Gamma_0$ is not empty and $\mathcal{H}^{N-1}(\Gamma_0)$ will decrease if $\alpha$ increases. Let us now look at $\mathcal{H}^{N-1}(\Gamma_-)$. By the volume constraint and Lemma 4.1, the radius of any spherical cap (including the case of the sphere where $\alpha = 0$) is given by

$$R_-(\alpha) = \left( \frac{V_-}{\omega \rho_- I_-(\alpha)} \right)^{1/N}$$

while the perimeter is given in terms of $\alpha$ by

$$\mathcal{H}^{N-1}(\Gamma_-(\alpha)) = (N-1)\omega J_-(\alpha) \left( \frac{V_-}{\omega \rho_- I_-(\alpha)} \right)^{(N-1)/N}.$$ 

Therefore, we want to study the dependence on $\alpha$ (near $\alpha = 0$) of the function

$$g : \alpha \mapsto J_-(\alpha) \left( I_-(\alpha) \right)^{(1-N)/N}.$$ 

Its derivative is given by

$$g'(\alpha) = \sin^{N-2}(\alpha) I_-(\alpha) \frac{1}{N} \left( \left( 1 - \frac{1}{N} \right) J_-(\alpha) \sin^2 \alpha - I_-(\alpha) \right)$$

that shows that the derivative is negative when $\alpha$ goes to 0 proving the claim.

When both $\Gamma_-$ and $\Gamma_+$ are whole spheres, the proof works as well replacing each sphere by a spherical cap such that $\Gamma_0$ remains of zero $(N-1)$-measure (in other words, we can replace both $\Gamma_\pm$ with two “slightly perturbed” spherical caps in such a way that the two boundaries of the manifolds $\Gamma_\pm$ coincide).
Lemma 4.3. The following identity holds true for all $\alpha$ and $\beta$.

$$NI_\pm = (N - 1)J_\pm \mp s^{N-1}_\pm c_\pm.$$  

Proof. The identity corresponding to the subscript $-$ follows from integration by parts:

$$I_- = \int_\alpha^\pi (\sin \theta)^N d\theta = (\sin \alpha)^{N-1} \cos \alpha + (N - 1) \int_\alpha^\pi (\sin \theta)^{N-2} (\cos \theta)^2 d\theta.$$  

Now, since $(\cos \theta)^2 = 1 - (\sin \theta)^2$ for all $\theta$, we get

$$I_- = s^{N-1}_- c_- + (N - 1)J_- - (N - 1)I_-,$$

which is exactly what we wanted. The case corresponding to the subscript $+$ is analogous and will be therefore omitted. \qed

4.2 Characterization of type I minimizers

Theorem 4.4 (Snell’s law for type I minimizers). Let $\Omega = \Omega(\alpha, \beta, R_\pm)$ be a type I minimizer for (1.3). Then the following identity holds true.

$$\rho_- \cos \alpha = \rho_+ \cos \beta.$$  

Let $\Omega$ be a type I minimizer of (1.3) and let $\Omega_\pm$ and $\Gamma_\pm$ be defined according to (1.2). Let $U = \Omega \cap \{x_2 = 0\}$, $U_\pm = U \cap \mathbb{R}^N_\pm$, and $U_0 = U \cap \{x_1 = 0\}$. Now, the set $\partial \Omega \cap \{x_2 > 0\}$ can be expressed as the graph of a function $u : U \to (0, \infty)$. In particular, since we know that $\partial \Omega$ is the union of two spherical caps, we get

$$u(x) = \begin{cases} \sqrt{R^2_- - (x - ae_1)^2} & \text{for } x \in U_-, \\ \sqrt{R^2_+ - (x - be_1)^2} & \text{for } x \in U_+, \end{cases} \quad (4.21)$$

where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^N$, $ae_1$ and $be_1$ are the centers of the spheres that generate $\Gamma_\pm$ respectively, and $R_\pm$ are the corresponding radii. Notice that, since $\Omega$ is of type I by hypothesis, the function $u$ defined in (4.21) admits a continuous extension along the interface $U_0$.

Let $\rho = \rho_- \mathcal{X}_{U_-} + \rho_+ \mathcal{X}_{U_+}$, where $\mathcal{X}_A$ denotes the indicator function of the set $A$ (namely, $\mathcal{X}_A(x) = 1$ if $x \in A$ and $\mathcal{X}_A(x) = 0$ otherwise). For any function $w \in H^1_0(U)$, let

$$\mathcal{G}(w) = \rho \int_U \sqrt{1 + |\nabla w|^2} \, dx.$$
Notice that, by construction, \( \mathcal{F}(\Omega) = 2\mathcal{G}(u) \). In particular, since by definition, \( \Omega \) is a minimizer of (1.3), then \( u \) must be a critical point for the following Lagrangian
\[
\mathcal{L}(w) = \mathcal{G}(w) + \mu_- \int_{U_-} w \, dx + \mu_+ \int_{U_+} w \, dx,
\]
where \( \mu_\pm \) are the Lagrange multipliers associated to the volume constraints on \( \Omega_\pm \) respectively. Therefore, for all \( v \in H^1_0(U) \), we must have
\[
\left. \frac{d}{dt} \right|_{t=0} \mathcal{L}(u + tv) = 0.
\]
An explicit computation of the Gâteaux derivative above yields
\[
\int_U \rho \frac{\nabla u \cdot \nabla v}{\sqrt{1 + |\nabla u|^2}} \, dx + \int_U \mu v \, dx = 0 \quad \text{for all } v \in H^1_0(U),
\]
where we set \( \mu = \mu_-\mathcal{X}_{U_-} + \mu_+\mathcal{X}_{U_+} \). Equation (4.22) is nothing else than the weak form of
\[
-\text{div} \left( \rho \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = \mu \quad \text{in } U.
\]
By a standard result concerning elliptic PDE’s in divergence form with piecewise constant coefficients, we get that the quantity
\[
\rho \frac{\nabla u \cdot e_1}{\sqrt{1 + |\nabla u|^2}}
\]
has no jump along the interface \( U_0 \). An explicit computation with (4.21) at hand yields
\[
\rho_- \frac{(x - ae_1) \cdot e_1}{\sqrt{R^2_+ - (x - ae_1)^2}} = \rho_+ \frac{(x - be_1) \cdot e_1}{\sqrt{R^2_+ - (x - be_1)^2}} \quad \text{for } x \in U_0.
\]
Since, by construction, \( \sqrt{R^2_+ - (x - ae_1)^2} = \sqrt{R^2_+ - (x - be_1)^2} \) for \( x \in U_0 \), the equality in (4.23) simplifies to
\[
\frac{\rho_- a}{R_-} = \frac{\rho_+ b}{R_+}.
\]
Or, equivalently
\[
\rho_- \cos \alpha = \rho_+ \cos \beta.
\]
4.3 Characterization of type II minimizers

**Theorem 4.5 (Snell’s law for type II minimizers).** Let $\Omega = \Omega(\alpha, \beta, R_\pm)$ be a type II minimizer for (1.3). If $R_- \sin \alpha > R_+ \sin \beta$, then

$$\rho_+ \cos \alpha = \gamma = \rho_+ \cos \beta.$$  \hspace{1cm} (4.25)

On the other hand, if $R_- \sin \alpha < R_+ \sin \beta$, then

$$\rho_- \cos \alpha = -\gamma = \rho_+ \cos \beta.$$  \hspace{1cm} (4.26)

**Proof.** For $\alpha \in [0, \pi)$ and $\beta \in (0, \pi]$, let us define the following

$$R_- := R_-(\alpha) = \sqrt{\frac{V_-}{\rho_\omega I_-(\alpha)}}, \quad R_+ := R_+(\beta) = \sqrt{\frac{V_+}{\rho_\omega I_+(\beta)}}.$$  \hspace{1cm} (4.27)

This way, by the first formula of Lemma 4.1 we are sure that the volume constraints

$$\rho_\pm \mid_{\Omega_\pm(\alpha, \beta, R_-, R_+)} = V_\pm$$  \hspace{1cm} (4.28)

are satisfied. Notice that, differentiating (4.27) yields

$$\frac{d}{d\alpha} R_-(\alpha) = \frac{s_N R_-}{N L_-}, \quad \frac{d}{d\beta} R_+(\beta) = -\frac{s_N R_+}{N L_+}.$$  \hspace{1cm} (4.29)

Let now $\Omega(\alpha, \beta) = \Omega(\alpha, \beta, R_-(\alpha), R_+(\beta))$ be a type II minimizer of (1.3). In particular, the pair $(\alpha, \beta)$ is a minimizer of the functional $F(\alpha, \beta) = F(\Omega(\alpha, \beta)):

$$F(\alpha, \beta) = \rho_- \mathcal{H}^{N-1}(\Gamma_-(\alpha)) + \rho_+ \mathcal{H}^{N-1}(\Gamma_+(\beta)) + \gamma \omega \left| R_+^{N-1}(\alpha)s_2^{N-1} - R_-^{N-1}(\beta)s_2^{N-1} \right|$$

$$= (N-1) \rho_- \omega R_-^{N-1} J_- + (N-1) \rho_+ \omega R_+^{N-1} J_+ + \gamma \omega \left| R_+^{N-1}s_2^{N-1} - R_-^{N-1}s_2^{N-1} \right|.$$  \hspace{1cm} (4.30)

Moreover, since $(\alpha, \beta)$ is a minimizer of $F(\cdot, \cdot)$ by hypothesis, we have

$$\frac{d}{d\alpha} F(\alpha, \beta) = 0 = \frac{d}{d\beta} F(\alpha, \beta).$$

In what follows, we will compute the partial derivative of $F$ with respect to the first variable, $\alpha$, at $(\alpha, \beta)$. Differentiating (4.30) with respect to $\alpha$ with (4.29) at hand, under the assumption that $R_-(\alpha) \sin \alpha > R_+(\beta) \sin \beta$ yields

$$0 = \frac{d}{d\alpha} F(\alpha, \beta) = (N-1) \omega R_-^{N-1} \left( \rho_-(N-1) \frac{s_2^{N-1} J_-}{N L_-} - \rho_- s_2^{N-2} + \gamma s_2^{N-1} \frac{N L_-}{N L_+} + \gamma s_2^{N-2} c_- \right).$$
Now, by Lemma (4.3), we get
\[ 0 = \frac{d}{d\alpha} F(\alpha, \beta) = \frac{(N - 1)\omega R_{-1}^{N-1} s_{-}^{N-2}}{N I_{-}} \left( (N - 1)J_{-} c_{-} + s_{-}^{N-1} \right) (\gamma - \rho_{-} c_{-}). \] (4.31)

This implies that
\[ \rho_{-} \cos \alpha = \gamma \]
as wanted. The condition concerning the angle \( \beta \) is analogous. Finally, the optimality condition (4.26) follows from (4.25) by replacing \( \gamma \) by \( -\gamma \) in (4.30). \( \square \)

**Lemma 4.6.** Let
\[ L_1(\alpha) = \frac{\sin^N(\alpha)}{I_{-}(\alpha)} \quad \text{and} \quad L_2(\beta) = \frac{\sin^N(\beta)}{I_{+}(\beta)}. \]
Then \( L_1 \) is a strictly increasing function in the interval \((0, \pi)\), while \( L_2 \) is strictly decreasing in the same interval.

**Proof.** We will just show that the function \( L_1 \) is strictly increasing, since the proof for \( L_2 \) is analogous. First of all, we compute the derivative of \( L_1 \) with respect to \( \alpha \):
\[ \frac{d}{d\alpha} L_1(\alpha) = \frac{s_{-}^{N-1}}{T_{-}^{2}} \left( N c_{-} I_{-} + s_{-}^{N+1} \right). \]
We will show that \( f_1 = N c_{-} I_{-} + s_{-}^{N+1} \) is strictly positive for all \( \alpha \in [0, \pi) \). In particular, notice that \( f_1(\pi) = 0 \), hence it suffices to show that \( f_1 \) is strictly decreasing in the interval \((0, \pi)\). Another derivative with respect to \( \alpha \) yields
\[ \frac{d}{d\alpha} f_1(\alpha) = s_{-} \left(-N I_{-} + s_{-}^{N-1} c_{-}\right). \]
We claim that \( f_2 = -N I_{-} + s_{-}^{N-1} c_{-} \) is negative in the interval \((0, \pi)\). To this end, notice that \( f_2(\pi) = 0 \), hence we just need to show that \( f_2 \) is a strictly increasing function in the interval \((0, \pi)\). This is indeed true, as
\[ \frac{d}{d\alpha} f_2 = (N - 1)s_{-}^{N} + (N - 1)s_{-}^{N-2} c_{-}^{2} = (N - 1)s_{-}^{N} > 0. \]
Therefore \( L_1 \) is a strictly increasing function as claimed. \( \square \)

**Lemma 4.7.** Let \( \rho_{-} > 0, 0 \leq \gamma < \min\{\rho_{-}, \rho_{+}\} \) and \( V_{\pm} > 0 \) be given. Then the following three candidate minimizers \( \Omega_{\gamma}, \Omega_{-\gamma} \) and \( \Omega^{*} \) are well defined (and uniquely characterized by their defining properties).
(i) The set \( \Omega_\gamma \) is the unique candidate minimizer of type II of the form \( \Omega(\alpha, \beta, R_\pm) \) that satisfies both the volume constraints (4.28) and the Snell’s law (4.25).

(ii) The set \( \Omega_\gamma^{-} \) is the unique candidate minimizer of type II of the form \( \Omega(\alpha, \beta, R_\pm) \) that satisfies both the volume constraints (4.28) and the Snell’s law (4.26).

(iii) The set \( \Omega^* \) is the unique candidate minimizer of type I of the form \( \Omega(\alpha, \beta, R_\pm) \) that satisfies the volume constraints (4.28), the Snell’s law (4.24) and the equality \( R_\pm \sin \alpha = R_\pm \sin \beta \).

Proof. The points (i)-(ii) can be treated together, by considering the set \( \Omega_\gamma \) that satisfies conditions (4.28)–(4.25) for \( |\gamma| < \min(\rho_-, \rho_+) \). Notice that this amounts to solving a nonlinear system of 4 equations in 4 variables, which nicely decouples. We get

\[
\alpha = \arccos \left( \frac{\gamma}{\rho_-} \right), \quad \beta = \arccos \left( \frac{\gamma}{\rho_+} \right),
\]

\[
R_- = \sqrt{\frac{V_-}{\rho_- I_-(\alpha)}}, \quad R_+ = \sqrt{\frac{V_+}{\rho_+ I_+(\beta)}},
\]

(4.32)

A key observation to show (iii) relies on the fact that any set \( \Omega^* \) satisfying (4.28)–(4.24) is indeed a particular case of \( \Omega_\gamma \) satisfying (4.28)–(4.25) for some constant \( \gamma \in \mathbb{R} \) to be determined. We just need to determine the value of \( \gamma \) such that \( (\alpha, \beta, R_\pm) \), defined by (4.32), satisfy \( R_- \sin \alpha = R_+ \sin \beta \). To this end we will look for the zeros of the following function

\[
L(\gamma) = R_-^N \sin^N(\alpha) - R_+^N \sin^N(\beta) = \frac{V_-}{\omega \rho_-} L_1(\alpha) - \frac{V_+}{\omega \rho_+} L_2(\beta),
\]

where \( \alpha \) and \( \beta \) are considered to be functions of \( \gamma \) by the first two relations in (4.32) and the functions \( L_1, L_2 \) are defined in Lemma (4.6). Now, since \( \alpha \) and \( \beta \) are strictly decreasing functions of \( \gamma \), Lemma (4.6) implies that \( \gamma \mapsto L(\gamma) \) is strictly decreasing too. This ensures uniqueness for \( \Omega^* \). As far as existence is concerned, one could analyze the limit cases and conclude by the intermediate value theorem or simply notice that the existence of a type I candidate minimizer derives from the existence of a minimizer of (1.3) for \( \gamma > \min(\rho_-, \rho_+) \) (i.e. when no candidate minimizers of type II can be defined).

In what follows, let \( (\alpha^*, \beta^*) \in (0, \pi)^2 \) denote the unique pair such that

\[
\Omega^* = \Omega(\alpha^*, \beta^*).
\]

Furthermore, let

\[
\gamma^* = \rho_- \cos \alpha^* \left( = \rho_+ \cos \beta^* \right).
\]

(4.33)
Notice that the sign of $\gamma^*$ is determined by the given constants $V_\pm$ and $\rho_\pm$. Indeed if $V_-/\rho_- \geq V_+/\rho_+$, then $\gamma^* \geq 0$. This is a consequence of the fact that

$$L(0) = \frac{1}{\omega \int_{\pi/2}^{\pi} \sin^N(\theta) \, d\theta} \left( \frac{V_-}{\rho_-} - \frac{V_+}{\rho_+} \right)$$

and the map $\gamma \mapsto L(\gamma)$ is strictly decreasing, as stated in the proof of point (iii) of Lemma 4.7.

5 Proof of the main result

We are ready to prove the main result of this paper.

**Theorem 5.1.** Let $\gamma \geq 0$, $\rho_\pm > 0$ and $V_\pm > 0$ be given. Moreover, without loss of generality let $V_-/\rho_- > V_+/\rho_+$. Then, the minimizers of (1.3) can be characterized as follows.

(i) If $\gamma < \gamma^*$, $\Omega_\gamma$ is the only minimizer of (1.3) up to suitable translations.

(ii) If $\gamma = \gamma^*$, then $\Omega_{\gamma^*} = \Omega^*$ is the only minimizer of (1.3) up to suitable translations.

(iii) If $\gamma > \gamma^*$, then $\Omega^*$ is the only minimizer of (1.3) up to suitable translations.

**Proof.** By Lemma 4.7 we know that (up to suitable translations) there are only three candidates minimizers for (1.3), namely $\Omega_\gamma$, $\Omega_{-\gamma}$ and $\Omega^*$. First of all, we will show that the set of candidate minimizers can be reduced to just $\Omega_\gamma$ and $\Omega^*$. Indeed, if $\gamma = 0$, then $\Omega_\gamma = \Omega_{-\gamma}$. The case $\gamma > 0$ is a bit more complicated. First, notice that $L(0) > 0$ by (4.34). Moreover, since the map $\gamma \mapsto L(\gamma)$ is strictly decreasing, then $L(-\gamma) > 0$: this implies that $R_- \sin \alpha > R_+ \sin \beta$ and hence, by Theorem 4.5, $\Omega_{-\gamma}$ should satisfy (4.25) (instead of (4.26)) in order to be a minimizer.

We will now prove part (i) of the theorem by contradiction. Let $\gamma < \gamma^*$ and assume by contradiction that the candidate of type I, $\Omega^* = \Omega(\alpha^*, \beta^*)$, is a minimizer. In particular, this implies that $\alpha^*$ is a minimum point of the functional $f(\alpha) = F(\Omega(\alpha, \beta^*))$ which was previously explicitly computed in (4.30). Since $\Omega(\alpha^*, \beta^*)$ is of type I by construction, the term inside the absolute value bars in (4.30) vanishes at $(\alpha^*, \beta^*)$. In particular, by the first part of Lemma 4.6 we know that the term inside the absolute value bars in (4.30) is negative for $\beta = \beta^*$ and $\alpha < \alpha^*$, and positive for $\beta = \beta^*$ and $\alpha > \alpha^*$. Therefore, the same
computations that lead to (4.31) give us
\[
\frac{d}{d\alpha} f(\alpha) = \begin{cases} 
A(\alpha)(-\gamma - \rho \cos \alpha) & \text{for } 0 < \alpha < \alpha^*, \\
A(\alpha)(\gamma - \rho \cos \alpha) & \text{for } \alpha^* < \alpha < \pi,
\end{cases}
\]
where
\[
A(\alpha) = \frac{(N-1)\omega R_{N-1}^{N-1}s_{N-2}^{N-2}}{NJ_-} \left( (N-1)J_-c_- + s_{N-1}^{N-1} \right) > 0.
\]
In particular, since
\[
\gamma < \gamma^* = \rho \cos \alpha^*
\]
by assumption, both left and right derivatives of \( f \) at \( \alpha^* \) are negative, which violates the assumption made about \( \alpha^* \) being a minimum point of \( f \).

Part (ii) is obvious because, by definition, the characteristic property of \( \gamma^* \) is
\[
\Omega_{\gamma^*} = \Omega^*.
\]
Since we previously ruled out \( \Omega_{-\gamma} \) as a competitor, we obtain that \( \Omega_{\gamma^*} = \Omega^* \) is the unique minimizer up to suitable translations.

We will now take \( \gamma > \gamma^* \) and prove part (iii) of the theorem. It will suffice to show that \( \Omega_{\gamma} \) is not a minimizer. Indeed, since \( L(\gamma^*) = 0 \) by construction, and \( \gamma \mapsto L(\gamma) \) is strictly decreasing by Lemma 4.6, we get that \( L(\gamma) < 0 \). In other words, \( \Omega_{\gamma} \) satisfies \( R_- \sin \alpha < R_+ \sin \beta \). If \( \Omega_{\gamma} \) were indeed a minimizer, then by Theorem 4.5 it should satisfy the Snell’s law (4.26) (instead of (4.25)). Since \( \gamma \) is not 0 in this case, this is a contradiction.

References

[1] D. Aberra & K. Agrawal, Surfaces of revolution in \( n \) dimensions, Int. J. Math. Educ. Sci. Technol., 38 (2007), 843–852.

[2] A.D. Alexandrov, Uniqueness theorems for surfaces in the large V, Vestnik Leningrad Univ., 13 (1958), 5–8 (English translation: Trans. Amer. Math. Soc., 21 (1962), 412–415).

[3] A. Alvino, F. Brock, F. Chiacchio, A. Mercaldo & M.R. Posteraro, Some isoperimetric inequalities on \( \mathbb{R}^N \) with respect to weights \( |x|^\alpha \), J. Math. Anal. Appl., 451 (2017), 280–318.

[4] L. Ambrosio & E. Paolini, Partial regularity for quasi minimizers of perimeter, Papers in memory of Ennio De Giorgi (Italian), Ricerche Mat., 48 (1999), suppl., 167–186.
[5] M.F. Betta, F. Brock, A. Mercaldo & M.R. Posteraro, A weighted isoperimetric inequality and applications to symmetrization, J. Inequal. Appl., 4 (1999), 215–240.

[6] C. Borell The Brunn-Minkowski inequality in Gauss space, Invent. Math., 30 (1975), 207–216.

[7] Y.D Burago & V.A. Zalgaller, Geometric Inequalities. (Translated from the Russian by A.B. Sosinski.) Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] (285). Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1988.

[8] A. Cañete, M. Miranda & D. Vittone, Some Isoperimetric Problems in Planes with Density, J. Geom. Anal., 20 (2010), 243–290.

[9] C. Carroll, A. Jacob, C. Quinn & R. Walters, The isoperimetric problem on planes with density, Bull. Aust. Math. Soc., 78 (2008), 177–197.

[10] J. Choe, M. Ghomi & M. Ritoré, The relative isoperimetric inequality outside convex domains in \( \mathbb{R}^N \), Calc. Var. Partial Differential Equations, 29 (2007), 421–429.

[11] E. De Giorgi, Sulla proprietà isoperimetrica dell’ipersfera, nella classe degli insiami aventi frontiera orientata di misura finita, Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. Sez. 1, 8 (1958), 33–44.

[12] E. Gonzales, U. Massari & I. Tamanini, On the regularity of boundaries of sets minimizing perimeter with a volume constraint, Indiana Univ. Math. J., 32 (1983), 25–37.

[13] A. Henrot & M. Pierre, Shape Variation and Optimization: A Geometrical Analysis, English version of the French publication with additions and updates, EMS Tracts in Mathematics, 28, European Mathematical Society (EMS), Zürich, 2018.

[14] F. Maggi, Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory, Cambridge Studies in Advanced Mathematics, 135, Cambridge University Press, Cambridge, 2012.

[15] F. Morgan, Geometric Measure Theory: A Beginner’s Guide, Academic Press Inc., San Diego, 4th edition, 2008.

[16] I. Tamanini, Boundaries of Caccioppoli sets with Hölder continuous normal vector, J. Reine Angew. Math., 334 (1982), 27–39.