\( \mathcal{N} = 2 \) gauge theories and
degenerate fields of Toda theory

**Abstract:** We discuss the correspondence between degenerate fields of the \( \mathcal{W}_N \) algebra and punctures of Gaiotto’s description of the Seiberg-Witten curve of \( \mathcal{N} = 2 \) superconformal gauge theories. Namely, we find that the type of degenerate fields of the \( \mathcal{W}_N \) algebra, with null states at level one, is classified by Young diagrams with \( N \) boxes, and that the singular behavior of the Seiberg-Witten curve near the puncture agrees with that of \( \mathcal{W}_N \) generators. We also find how to translate mass parameters of the gauge theory to the momenta of the Toda theory.

**Keywords:** Toda field theory, Seiberg-Witten theory
1. Introduction

Recently, Gaiotto [1] showed that the S-duality group of a large class of four-dimensional $\mathcal{N} = 2$ superconformal field theories can be understood by realizing them by compactifying the six-dimensional $\mathcal{N} = (2, 0)$ theory, which describes the low-energy dynamics of $N$ coincident M5-branes, on a Riemann surface $C$ with punctures$^1$. Each puncture was shown to be labeled by a Young diagram with $N$ boxes, by analyzing the linear

$^1$For an extensive review, see §3 of [2]
quiver gauge theories which fall within this class of theories. The Seiberg-Witten curve \( \Sigma \) of the theory is then given by an \( N \)-folded cover of the base \( C \):

\[
x^N + \phi^{(2)}(z)x^{N-2} + \cdots + \phi^{(N)}(z) = 0.
\]

(1.1)

Here, \( z \) is a local coordinate of \( C \) and \( \phi^{(k)}(z) \) is a degree-\( k \) differential on \( C \). The Young diagram labeling the puncture specifies the poles of \( \phi^{(k)} \), as will be reviewed later.

Soon, it was demonstrated in \cite{3} by Alday, Gaiotto and one of the authors that Nekrasov’s partition function of this class of theories with \( N = 2 \) gives the correlation function of the two-dimensional Liouville theory, which lives on the Riemann surface \( C \) on which the M5-branes are compactified. There, a puncture on the Riemann surface was identified with an insertion of the exponential of the Liouville boson. It was also noted that the “semiclassical” limit of the vacuum expectation value of the energy-momentum tensor \( T(z) \) of the 2D theory gave \( \phi^{(2)}(z) \) appearing in Eq. (1.1):

\[
\langle T(z) \rangle \longrightarrow \phi^{(2)}(z).
\]

(1.2)

This discussion was then generalized by Wyllard \cite{4} to \( N > 2 \). There, the corresponding two-dimensional theory is the \( A_{N-1} \) conformal Toda field theory, which is a natural extension of the Liouville theory which is equivalent to the \( A_1 \) Toda field theory. The \( A_{N-1} \) Toda theory has the \( W_N \) algebra as its symmetry, which includes not only the energy-momentum tensor \( T(z) = W^{(2)}(z) \) but also chiral primary fields \( W^{(3)}(z), \ldots, W^{(N)}(z) \) of dimension 3, 4, \ldots, \( N \). For \( N = 3 \), Mironov and Morozov \cite{5} checked this proposal up to instanton number 2. There have been works to better understand why this relation holds \cite{6, 7, 8, 9}.

When setting up the correspondence for general \( N \), it was found by \cite{4} that different types of punctures gave different types of states of the Toda theory. Namely, the “full puncture”, labeled by the Young diagram \( [1^N] \), corresponds to general Toda momenta, whereas the “simple puncture”, whose diagram is \( [N-1, 1] \), corresponds to a restricted set of Toda momenta and gives a degenerate state with null states in the level-1 descendants. However, it has not been understood what are the corresponding states of the Toda theory for punctures labeled by other types of Young diagrams. Our objective is to answer this question.

The \( A_{N-1} \) Toda theory describes, in the minisuperspace approximation, a quantum-mechanical wave in a real space of dimension \( N - 1 \), scattering off \( N - 1 \) exponential walls. It is convenient to parameterize the space as\(^2\) \( \vec{\varphi} = (\varphi_1, \ldots, \varphi_N) \) constrained by

\(^2\)The authors apologize for the simultaneous usage of \( \phi^{(k)} \) as the differentials in the Seiberg-Witten curve, and \( \varphi_k \) as the bosons of the Toda field theory. They hope it will not cause deep confusion. \( i \) in this paper stands for the imaginary unit.
\[ \sum \varphi_k = 0. \] Then the potential is given by \[ \mu \sum_{k=1}^{N-1} e^{\varphi_k - \varphi_{k+1}}. \] The states correspond to the vertex operator \( e^{i \vec{\beta} \cdot \vec{\phi}} \). We will see that one of the level-1 descendants becomes null when \( \beta_k = \beta_{k+1} \) for some \( k \), i.e., the wave is parallel to the \( k \)-th wall. We will then see that the Young diagram \([l_1, \ldots, l_s]\) where \( \sum l_k = N \), specifies the subspace of the momentum space given by the form

\[
\vec{\beta} = (\beta_{(1)}, \ldots, \beta_{(1)}, \beta_{(2)}, \ldots, \beta_{(2)}, \ldots, \beta_{(s)}, \ldots, \beta_{(s)}).
\]

(1.3)

It has been long known that the standard physical states of the Toda theory have the momenta of the form

\[
\vec{\beta} = \vec{p} - iQ \vec{\rho}
\]

(1.4)

where \( Q \) determines the background charge, \( \vec{\rho} \) is the Weyl vector, and \( \vec{p} \) is a real vector specifying the direction of the propagation. This never satisfies the condition \( \beta_k = \beta_{k+1} \) because of the imaginary part. Instead, we will find a strong indication that the physical state of the class specified by a Young diagram \( Y \) has the momenta

\[
\vec{\beta} = \vec{p}_Y - iQ \vec{\rho}_Y
\]

(1.5)

where \( \vec{p}_Y \) is a general real vector of the form (1.3) and \( \vec{\rho}_Y \) a certain real vector determined later in the paper. Therefore, the physical states for a generic diagram \( Y \) are not a special case of the physical states for the diagram \([1^N]\) given by (1.4).

We will then study the behavior of the differentials \( \phi^{(k)}(z) \) appearing in the Seiberg-Witten curve Eq. (1.1) close to a puncture labeled by \( Y \) in the massive gauge theory. We will see that their behavior agrees with the semiclassical limit of the behavior of the generators of the \( W_N \) algebra, close to the insertion of the degenerate field of the corresponding type. In other words, we show that \( \phi^{(k)}(z) \) behaves as the semiclassical limit of \( \langle W^{(k)}(z) \rangle \), at least close to the punctures:

\[
\langle W^{(k)}(z) \rangle \rightarrow \phi^{(k)}(z).
\]

(1.6)

The rest of the paper is organized as follows. In §2, we review the Seiberg-Witten curve of the linear quiver of SU gauge groups to remind ourselves how the Young diagram arises in this setup. In §3 we give a brief review the \( A_{N-1} \) Toda field theory and the \( W_N \)-algebra, including its free-boson realization. In §4, we construct the null states at level 1, and show that their structure is characterized by a Young diagram. We then study how to identify physical momenta for a given diagram, and also analyze the semiclassical limit of these states. In §5, we study the behavior of the Seiberg-Witten curve near the punctures. We will see that the null state condition in the semiclassical limit is exactly reproduced. We conclude in §6 with a discussion on the future directions.
2. Review: Young diagrams from Seiberg-Witten curve

Let us briefly recall how the punctures are labeled by Young diagrams, following the discussion of [1, 2]; readers familiar with this point can skip this section. Consider a four-dimensional $\mathcal{N} = 2$ linear quiver gauge theory with a chain of $n$ SU groups

$$\text{SU}(d_1) \times \text{SU}(d_2) \times \cdots \times \text{SU}(d_{n-1}) \times \text{SU}(d_n),$$

(2.1)

with a bifundamental hypermultiplet between each adjoining gauge group and $k_a$ fundamental hypermultiplets for $\text{SU}(d_a)$. To make every gauge coupling constant marginal, the number of the fundamental hypermultiplets must satisfy

$$k_a = 2d_a - d_{a+1} - d_{a-1} = (d_a - d_{a+1}) - (d_{a-1} - d_a),$$

(2.2)

where we define $d_0 = d_{n+1} = 0$. Since $k_a$ is non-negative, we have

$$d_1 < d_2 < \cdots < d_{l-1} < d_l = \cdots = d_r > d_{r+1} > \cdots > d_{n-1} > d_n.$$  

(2.3)

In the following, we denote $d_l = \cdots = d_r =: N$ and $w_a := d_{a+1} - d_a$. Since $\sum_{a=0}^{l-1} w_a = d_l = N$, the left tail is characterized by a Young diagram that has a row of width $w_a$ for each $a \leq l$. The diagram for the right tail can be assigned in the same way. We will denote a Young diagram by listing inside $[\ldots]$ the height of the columns in the decreasing order; $k$ columns with the same height $h$ are sometimes abbreviated as $h^k$ inside $[\ldots]$. Then, the tail $4 > 3 > 2$ corresponds to the diagram $[3, 1]$, while the tail $12 > 8 > 4$ is $[3^4]$. The puncture $[1^N]$ is called the full puncture, corresponding to the tail consisting of just one SU($N$) gauge group, and $[N - 1, 1]$ is called the simple puncture, corresponding to the tail $N > N - 1 > \cdots > 3 > 2$. Now it is easy to see that there is no distinction between the simple and the full punctures when $N = 2$, because both are punctures of type $[1^2]$.

Let us consider the flavor symmetry of the theory. A bifundamental hypermultiplet carries U(1) flavor symmetry. It can be associated with a simple puncture. The flavor symmetry which acts on fundamental hypermultiplets in a tail can be read off from the Young diagram, say $[l_1, l_2, \ldots, l_s]$. Each column corresponds to a fundamental hypermultiplet and its height stands for the gauge group it couples to. So, if there are $N_h$ columns whose height is $h$, we have $U(N_h)$ flavor symmetry. Overall U(1) symmetry is carried by a simple puncture, so the flavor symmetry associated with the diagram is

$$S\left(\prod_{h>0} U(N_h)\right).$$  

(2.4)
The Seiberg-Witten curve of these linear quiver gauge theories was originally found in [10]. The curve is given by a polynomial of two complex variables \((v, z)\) of degree \(N\) in \(v\) and of degree \(n + 3\) in \(z\). It is sufficient for our further analysis to keep only the left tail in the general form. Therefore, for simplicity, we set \(l = r = n\) so that we have a full puncture on the right hand side of the quiver. The curve is then given by

\[
\prod_{i=1}^{N} (v - \tilde{m}_i) z^{n+1} + c_n (v^N - M_n v^{N-1} - u_n^{(2)} v^{N-2} - u_n^{(3)} v^{N-3} - \ldots - u_n^{(N-1)} v - u_n^{(N)}) z^n
\]

\[
+ \cdots + c_j \prod_{k \text{ s.t. } l_k > j} (v - m_k)^{l_k - j} (v^{d_j} - M_j v^{d_j-1} - u_j^{(2)} v^{d_j-2} - \ldots - u_j^{(d_j-1)} v - u_j^{(d_j)}) z^j
\]

\[
+ c_0 \prod_{k=1}^{s} (v - m_k)^{l_k} = 0.
\]

(2.5)

Here, \(c_j\) are the gauge coupling parameters and \(u_j^{(a)}\) parameterize the Coulomb branch. \(\tilde{m}_1, \ldots, \tilde{m}_N\) are the mass parameters associated to the SU\((N)\) flavor symmetry of the hypermultiplets coupled to SU\((d_n)\), \(m_k\) controls the mass of the extra fundamental hypermultiplets coupled to SU\((d_a)\) with \(a < n\), and \(M_a\) controls the mass of the bifundamental hypermultiplets.

It is convenient to rewrite the curve in the following way [1]. First, we collect the terms of (2.5) with respect to \(v\) and write it as

\[
\Delta(z) v^N - M(z) v^{N-1} + \sum_{k>1} \psi^{(k)}(z) v^{N-k} = 0.
\]

(2.6)

Then, we perform the shift

\[
v \to v - \frac{M(z)}{N \Delta(z)}
\]

(2.7)

to eliminate the \(v^{N-1}\) term, and finally we change the coordinates from \((v, z)\) to \((x = v/z, z)\). The final outcome is

\[
x^N + \sum_{k=2}^{N} \phi^{(k)}(z) x^{N-k} = 0.
\]

(2.8)

We regard \(z\) as a local coordinate of a sphere. \(\phi^{(k)}(z)(dz)^k\) can then be naturally thought of as a degree-\(k\) differential.

There are \(n + 3\) punctures on the sphere. Two of them are at \(z = 0, \infty\), and are labeled by the Young diagrams associated with the tails. The other \(n + 1\) are at the zeroes of \(\Delta(z) = z^{n+1} + \sum_k c_k z^k = 0\). \(\phi^{(k)}(z)\) has a pole of order \(k\) at each of
the puncture; the Young diagram labeling a puncture is reflected by the polynomial relations among the residues of $\phi^{(k)}(z)$ at the puncture.

The Seiberg-Witten differential is now $\lambda = x dz / z$, and its residue at the puncture gives the mass parameters of the flavor symmetry associated with it. It can be read off from Eq. (2.3):

$$\left( m'_1, \ldots, m'_i, \ldots, m'_s, \ldots, m'_l \right)$$

(2.9)

where $m'_k = m_k - m$ with $m$ appropriately chosen to have $\sum_k l_k m'_k = 0$. This pattern agrees with the flavor symmetry (2.4).

In the M-theory point of view, this setup describes the low-energy effective theory of $N$ coincident M5-branes wrapping the sphere. At the punctures, there are codimension-two defects whose types are specified by the Young diagram.

Under the correspondence of the 4D gauge theory and the 2D Toda field theory, punctures on the Riemann surface translate into insertions of primary operators. As we reviewed in this section, punctures are classified by Young diagrams. So it is natural to conjecture that the corresponding primaries are also classified by the same diagram. We will see below that the structure of the level-1 null states in $\mathcal{W}_N$ algebra, which governs the symmetry of the Toda theory, is also labeled by the Young diagram with $N$ boxes, and that null state conditions are directly reflected by the behavior of the Seiberg-Witten curve near the puncture.

3. Review: $A_{N-1}$ Toda theory and $\mathcal{W}_N$ algebra

In this section, we summarize rudimentary materials concerning the $A_{N-1}$ Toda theory and the representation theory of $\mathcal{W}_N$ algebra. The standard references are [11, 12]; for the modern developments, see [13] and the references therein. Readers who are familiar with these structures should proceed to the next section.

3.1 $A_{N-1}$ Toda theory

The $A_{N-1}$ Toda field theory is given by the action

$$S = \int d^2 \sigma \sqrt{g} \left[ \frac{1}{8\pi} g^{xy} \partial_x \vec{\varphi} \cdot \partial_y \vec{\varphi} + \mu \sum_{k=1}^{N-1} e^{b\vec{e}_k \cdot \vec{\varphi}} + \frac{Q}{4\pi} R \vec{\rho} \cdot \vec{\varphi} \right].$$

(3.1)

Here $\vec{\varphi} = (\varphi_1, \varphi_2, \ldots, \varphi_N)$ with the condition $\sum \varphi_k = 0$ parameterize the Cartan subspace of the algebra $A_{N-1}$, $b$ is a real parameter, $Q = b + 1/b$, $e_k$ is the $k$-th simple root of $A_{N-1}$ given by $(0, \ldots, 1, -1, \ldots, 0)$, and $\vec{\rho}$ is the Weyl vector of $A_{N-1}$ given by
the condition $\vec{e}_k \cdot \vec{\rho} = 1$ for all $k$. Explicitly it is given as, $\vec{\rho} = (\frac{N-1}{2}, \frac{N-3}{2}, \cdots, -\frac{N-1}{2})$. This theory is conformal with the central charge

$$c = (N-1) + 12Q^2 \vec{\rho} \cdot \vec{\rho} = (N-1)(1 + N(N+1)Q^2).$$

(3.2)

In the minisuperspace approximation, it describes the propagation of a quantum-mechanical wave in the space of $\vec{\varphi}$, scattered by $N-1$ exponential potential walls perpendicular to $\vec{e}_k$. Therefore, an eigenstate of the Hamiltonian roughly corresponds to a linear superposition of the waves with different momenta, related by the Weyl reflections by the walls. In the conformal field theory language, the primary field $e^{i\vec{\beta} \cdot \vec{\varphi}}$ is known to correspond to the propagating mode with the momenta $\vec{\beta} + iQ\vec{\rho}$, with $L_0 = \vec{\beta} \cdot \vec{\beta} / 2 + iQ\vec{\rho} \cdot \vec{\beta}$. Two such operators $e^{i\vec{\beta} \cdot \vec{\varphi}}$ and $e^{i\vec{\beta'} \cdot \vec{\varphi}}$ are in fact proportional to each other if

$$\vec{\beta} + iQ\vec{\rho} = w(\vec{\beta'} + iQ\vec{\rho})$$

(3.3)

for an element $w$ of the Weyl group, i.e. the reordering of the components. The proportionality constant is called the reflection amplitude and was determined in [14]. The symmetry of the theory is described by the $W_N$ algebra, which will be discussed below.

For $N = 2$ with $\vec{\varphi} = (\varphi, -\varphi)$, this theory reduces to the standard Liouville theory. Let us write $\vec{\beta} = (\beta, -\beta)$. Then $e^{i\vec{\beta} \cdot \vec{\varphi}} = e^{2i\beta \varphi}$, and the Weyl reflection (3.3) becomes

$$\beta \rightarrow -iQ - \beta.$$  

(3.4)

### 3.2 $W_N$ algebra

$W_N$ algebra is generated by the energy-momentum tensor $T(z) = W^{(2)}(z)$ and $N-2$ chiral primary fields $W^{(r)}(z)$ with $r = 3, \cdots, N$, where $r$ also gives the dimension of $W^{(r)}$. The explicit form of the algebra for $N = 3$ was first determined in [15],

$$W^{(2)}(z)W^{(2)}(0) \sim \frac{c}{2z^4} + \frac{2}{z^2}W^{(2)}(0) + \frac{1}{z}\partial W^{(2)}(0) + \cdots,$$

(3.5)

$$W^{(2)}(z)W^{(3)}(0) \sim \frac{3}{z^2}W^{(3)}(0) + \frac{1}{z}\partial W^{(3)}(0) + \cdots,$$

(3.6)

$$W^{(3)}(z)W^{(3)}(0) \sim \frac{c}{3z^6} + \frac{2}{z^4}W^{(2)}(0) + \frac{1}{z^3}\partial W^{(2)}(0) + \frac{1}{z^2}\left(\frac{3}{10}\partial^2 W^{(2)}(0) + 2q^2\Lambda(0)\right)$$

$$+ \frac{1}{z}\left(\frac{1}{15}\partial^3 W^{(2)}(0) + q^2\partial\Lambda(0)\right) + \cdots,$$

(3.7)

where,

$$\Lambda := (W^{(2)})^2 : - \frac{3}{10}\partial^2 W^{(2)}, \quad q^2 = \frac{16}{22 + 5c}.$$  

(3.8)
The operator $\mathcal{W}^{(2)}$ generates Virasoro algebra and the operators $\mathcal{W}^{(3)}, \ldots, \mathcal{W}^{(n)}$ represent the extra symmetries. The appearance of the nonlinear term $\Lambda$ is a characteristic feature of a $\mathcal{W}$-algebra. As the dimension of the generators is getting higher, we have more complicated nonlinear terms in the algebra.

In order to manage the algebra, a representation in terms of free bosons was developed \[13, 11\]. Roughly speaking, this corresponds to taking $\mu = 0$ in the Lagrangian \(3.1\). The operators $W^{(r)}$ are systematically produced through an $N$-th order differential operator:

$$R_N = \prod_{m=1}^{N} \left( Q \frac{d}{dz} + \tilde{h}_m \partial_z \vec{\varphi} \right) = \sum_k W^{(k)}(z) \left( Q \frac{d}{dz} \right)^{N-k}. \quad (3.9)$$

The relation between $\vec{\varphi}$ and $W^{(k)}$ is called as quantum Miura transformation. In the following, we refer to $R_N$ as “Lax operator” for simplicity by using the terminology of solvable system. Here, $\vec{\varphi}(z) = (\varphi_1(z), \ldots, \varphi_N(z))$ are free bosons which satisfies the operator-product expansion: $\varphi_j(z)\varphi_k(0) \sim -\delta^{jk} \log(z)$. We write their components as

$$\varphi_j(z) = x_j + a_{j,0} \log z - \sum_{s \neq 0} \frac{a_{j,s}}{s z^s}. \quad (3.10)$$

$\tilde{h}_m$ are vectors in $\mathbb{R}^N$ and defined by $(h_j)_k = \delta_{jk} - \frac{1}{N}$. Since it satisfies $\sum_{m=1}^{N} \tilde{h}_m = 0$, one component $\sum_k \varphi_k$ of $\vec{\varphi}$ is decoupled. The definition \(3.9\) gives $W^{(0)}(z) = 1$ and $W^{(1)}(z) = 0$. The Virasoro generator is

$$W^{(2)}(z) = -\frac{1}{2} : (\partial_z \vec{\varphi})^2 : + Q \vec{\rho} \cdot \partial^2_z \vec{\varphi} \quad (3.11)$$

with the central charge \(3.2\). It is important to note that $W^{(k)}$ defined by the quantum Miura transformation is not primary in general. For example, a primary $W^{(3)}$ is given by \[12\]

$$\hat{W}^{(3)}(z) = W^{(3)}(z) - \left( \frac{N-2}{2} \right) Q \partial W^{(2)}(z). \quad (3.12)$$

$W^{(3)}(z)$ quoted in \(3.7\) is then $W^{(3)}(z) = i \sqrt{3} q \hat{W}^{(3)}(z)$.

The Lax operator \(3.9\) plays an essential role to derive the relation between the $\mathcal{W}_N$ algebra with the $A_{N-1}$ Toda equation \[16\]. It was shown that the Toda fields $\varphi_\rho$ may be rewritten as the Wronskians of the solutions to $R_N \psi = 0$. Geometrical aspects of the correspondence was given in \[17\]. Such correspondence will, however, not be explored further in this paper since we have to combine the left and right mover to give
the Toda fields. Instead, we will treat Toda fields \( \varphi \) as free fields and treat the Toda potential as chiral perturbation.

In this free-boson representation, the primary fields are defined by the vertex operators

\[
V_{\vec{\beta}}(z) := e^{i\vec{\beta} \cdot \vec{\varphi}(z)}. \tag{3.13}
\]

The corresponding state is defined as \( |\vec{\beta}\rangle := \lim_{z \to 0} V_{\vec{\beta}}(z)|0\rangle \), and satisfies

\[
a_{k,0}|\vec{\beta}\rangle = -i\beta_k|\vec{\beta}\rangle. \tag{3.14}
\]

It is a highest-weight state of the \( \mathcal{W}_N \) algebra:

\[
W^{(r)}_0|\vec{\beta}\rangle = \Delta^{(r)}(\vec{\beta})|\vec{\beta}\rangle, \quad W^{(r)}_s|\vec{\beta}\rangle = 0 \quad (s > 0) \tag{3.15}
\]

where \( W^{(r)}_s \) are the modes of the generators defined by

\[
W^{(r)}(z) = \sum_s W^{(r)}_sz^{-s-r} \tag{3.16}
\]

and the eigenvalues of the zero modes are given by

\[
\Delta^{(2)}(\vec{\beta}) = \Delta(\vec{\beta}) = \frac{1}{2}\vec{\beta} \cdot \vec{\beta} + iQ\vec{\rho} \cdot \vec{\beta}, \tag{3.17}
\]

\[
\Delta^{(k)}(\vec{\beta}) = (-1)^k \sum_{1 \leq j_1 \leq j_2 \leq \ldots \leq j_k \leq N} \prod_{m=1}^{k} (i\hbar_{j_m} \cdot \vec{\beta} + Q(k-m)) \tag{3.18}
\]

This condition is equivalent to the operator-product expansion

\[
W^{(k)}(z)\tilde{V}_{\vec{\beta}}(0) = \frac{\Delta^{(k)}(\vec{\beta})}{z^k}\tilde{V}_{\vec{\beta}}(0) + \mathcal{O}(z^{-k+1}). \tag{3.19}
\]

One important fact is that \( \Delta^{(k)}(\vec{\beta}) \) is invariant under the shifted action of the Weyl group given by (3.3). Therefore, \( \Delta^{(k)}(\vec{\beta}) \) is given by a symmetric polynomial of the components of \( \vec{\beta} + iQ\vec{\rho} \).

### 3.3 Screening charges and null states

To construct irreducible representations, it is essential to understand how null states are produced in the Verma module over the primary state \( |\vec{\beta}\rangle \). A null state satisfies the highest-weight condition (3.15) and has vanishing inner product with all the state in the module. Null states can be constructed by applying the so-called screening operators.
$S_j^{(\pm)}$ to the primary state with a particular value of the momenta $\vec{\beta}'$, which we will explain below.

The screening operator is the integral of a special type of the vertex operators and commutes with all the $W$ generators:

$$S_j^{(\pm)} = \int \frac{dz}{2\pi i} V_j^{(\pm)}(z) = \int \frac{dz}{2\pi i} : e^{i\alpha_{\pm} \vec{e}_j \cdot \vec{\phi}(z)} :,$$

$$[W_r^{(k)}, S_j^{(\pm)}] = 0.$$  \hspace{1cm} (3.20)

To achieve this, we need to impose $\Delta(\alpha_{\pm} \vec{e}_j) = 1$ in particular. This determines the parameters $\alpha_{\pm}$ to be

$$(\alpha_+, \alpha_-) = i(b, 1/b).$$  \hspace{1cm} (3.21)

Then it is easy to see that the state

$$(S_j^{(\pm)})^{\ell \pm} |\vec{\beta} - \ell_{\pm} \alpha_{\pm} \vec{e}_j\rangle$$

satisfies the highest-weight condition by using the property (3.21), thus giving a null state in the module over $|\vec{\beta}\rangle$ if it is nonzero. This state vanishes unless $\vec{\beta}$ satisfies

$$\vec{e}_j \cdot \vec{\beta} = (1 - \ell_j^+ \alpha_+ + (1 - \ell_j^-) \alpha_-), \quad (\ell_j^+, \ell_j^- = 1, 2, \ldots),$$

$$\beta_j = \beta_{j+1}.$$  \hspace{1cm} (3.22)

for some $j$, because of a nontrivial phase in the contour integration. If this condition is satisfied, there is a null state at level $\ell_j^+ \ell_j^-$ by applying the screening charges $S_j^{(\pm)}$.

It is possible to choose $\vec{\beta}$ such that $\vec{\beta}$ satisfies the condition (3.24) for all $j = 1, \ldots, N - 1$. It is also possible that the generated null states have null states in their own module. The celebrated minimal model of $\mathcal{W}_N$ algebra was constructed this way. On the other hand, the Toda theory which is relevant in our paper has central charge $c > N - 1$ and the primary fields in general are not completely degenerate. Therefore, we need to pay attention to less a restrictive set of null states, which will be discussed in the next section.

4. Level-1 null states and Young diagrams

4.1 General consideration

In the following, we will focus on the null states which appear at level 1. The existence of level-1 null states implies $\ell_j^+ = \ell_j^- = 1$ for some $j$. The condition (3.24) then becomes

$$\vec{e}_j \cdot \vec{\beta} = \beta_j = \beta_{j+1}.$$  \hspace{1cm} (4.1)
More generally, let $M_{kl}$ the matrix of the inner products of $W^{(k)}_0|\vec{\beta}\rangle$, $(k = 2, \ldots, N)$ and $\vec{e}_l \cdot \vec{a}_0|\vec{\beta}\rangle$ $(l = 1, \ldots, N - 1)$. Then it is known that

$$\det(M_{kl}) \propto \prod_{m<n} (\beta_n - \beta_m + i(n-m-1)Q) \quad (4.2)$$

The factors on the right hand side with $n > m + 1$ are the images of the conditions $(4.1)$ under the shifted Weyl action $(3.3)$.

We consider the following conditions labeled by the Young diagram $Y = [l_1, l_2, \cdots, l_s]$

$$\vec{\beta} = (\beta_1, \cdots, \beta_N) = (\underbrace{\beta(1)}_{l_1}, \cdots, \underbrace{\beta(1)}_{l_1}, \cdots, \underbrace{\beta(s)}_{l_s}, \cdots, \underbrace{\beta(s)}_{l_s}) \quad (4.3)$$

where

$$\sum_{k=1}^{N} \beta_k = \sum_{k=1}^{s} k\beta(k) = 0. \quad (4.4)$$

The number of null states in the level-1 descendants is $\sum_{k=1}^{s}(l_k - 1)$.

Let us make a few observations. First, the condition $(4.4)$ implies that the null state associated with Young diagram $[N]$ necessarily has $\vec{\beta} = 0$. This corresponds to the insertion of the vacuum, and agrees with the fact that the puncture of type $[N]$ does nothing on the gauge theory side. Second, the form $(4.3)$ is exactly the same as the form of the residue of the Seiberg-Witten differential associated to the same type of the Young diagram, $(2.9)$. Therefore, it is natural to map the mass terms and the Toda momenta by identifying $(2.9)$ and $(4.3)$. However, there is a slight problem here, because the Weyl group of the flavor symmetry acts linearly on the masses, but acts on the momenta by the shifted action $(3.3)$. We need to take care of this discrepancy, to which we come back in §4.3.

### 4.2 Explicit form of the null states

The simplest nontrivial level-1 null state is the one for $N = 3$ with the Young diagram $[2,1]$. This is the puncture relevant in the analysis in [4, 5] of the correspondence between the SU(3) gauge theory with six flavors and the Toda theory. Therefore, let us derive the explicit form of this null state.

For $N = 3$, we have only two states $W^{(2)}_{-1}|\vec{\beta}\rangle$, $W^{(3)}_{-1}|\vec{\beta}\rangle$ at level 1, and a linear combination of them should become null. To construct level-1 states, we need only to keep $\vec{a}_0$ and $\vec{a}_{-1}$ in the expression for $W^{(r)}$. The condition $\vec{e}_1 \cdot \vec{\beta} = 0$ can be solved by
writing $\vec{\beta} = (\beta, \beta, -2\beta)$. Then we find, after an explicit calculation,
\begin{align*}
W^{(2)}_0 |\vec{\beta}\rangle &= 3\beta(\beta - iQ)|\vec{\beta}\rangle, \\
W^{(2)}_{-1} |\vec{\beta}\rangle &= i\vec{\beta} \cdot \vec{a}_{-1} |\vec{\beta}\rangle, \\
W^{(3)}_0 |\vec{\beta}\rangle &= -2i\beta(\beta - iQ)(\beta - 2iQ)|\vec{\beta}\rangle, \\
W^{(3)}_{-1} |\vec{\beta}\rangle &= (\beta - iQ)\vec{\beta} \cdot \vec{a}_{-1} |\vec{\beta}\rangle.
\end{align*}
Both $W^{(2)}_{-1} |\vec{\beta}\rangle$ and $W^{(3)}_{-1} |\vec{\beta}\rangle$ are proportional to $\vec{\beta} \cdot \vec{a}_{-1} |\vec{\beta}\rangle$. Then a linear combination of them is zero. After the redefinition (3.12), the null state is given as
\begin{equation}
(2W^{(2)}_0 \hat{W}^{(3)}_{-1} - 3W^{(2)}_{-1} \hat{W}^{(3)}_0) |\vec{\beta}\rangle \approx 0
\end{equation}
in the Verma module. Indeed, this state vanishes in the free-boson representation. In this rather simple case, the null state condition can be found by directly studying the structure of the Verma module as in [5], but the free-boson representation is crucial to obtain the general structure.

4.3 Physical states

In the Liouville theory, the states whose momentum are of the form $\beta = p - iQ/2$ play a special role. They describe the propagation of the wave in the $\varphi$ space with momentum $p$, and they are the states to be inserted and integrated over in the intermediate channel when one calculates the four-point function by combining two three-point functions. On this class of states, the $L_0$ eigenvalue $p^2 + Q^2/4$ is real and bounded from below. Finally the reflection (3.4) acts on $p$ by just flipping the sign: $p \rightarrow -p$. These features motivated the authors of [3] to identify the SU(2) mass parameter $m$ with $p$ under the correspondence of the SU(2) gauge theory and the Liouville theory.

In the Toda theory, the states whose momenta are of the form
\begin{equation}
\vec{\beta} = \vec{p} - iQ\vec{\rho}
\end{equation}
have similar features. Namely, the $L_0$ eigenvalue is real and bounded from below; the shifted Weyl action on $\vec{\beta}$, (3.3), acts linearly on $\vec{p}$; and they appear in the intermediate channel when calculating four- and higher-point functions. These facts led Wyllard in [4] to identify $\vec{p}$ with the SU($N$) mass parameters of the gauge theory. Let us call this class of states the physical states.

What are the physical states for a general Young diagram $Y = [l_1, \ldots, l_s]$? The problem is that the conditions (4.10) and (4.3) are incompatible. Still, under the SU($N$) gauge theory–Toda theory correspondence, one expects to integrate over a real subspace of the momenta (4.3) to get the partition function of the gauge theory. We propose the solution to this question below.
Let us denote by $P_Y$ the set of real vectors of the form (4.3). Take the formula of $L_0$, (3.17). When $\vec{\beta} \in P_Y + iP_Y$, we have

$$L_0 = \frac{1}{2} \vec{\beta} \cdot \vec{\beta} + iQ\rho \cdot \vec{\beta} = \frac{1}{2} \vec{\beta} \cdot \vec{\beta} + iQ\rho_Y \cdot \vec{\beta}$$

(4.11)

where $\rho_Y$ is the projection of $\rho$ onto $P_Y$. Explicitly, $\rho_Y$ is given by

$$\rho - \rho_Y = \rho_{l_1} \oplus \rho_{l_2} \oplus \cdots \oplus \rho_{l_s}$$

(4.12)

where $\rho_k = (k - 1/2, k - 3/2, \ldots, 1 - k/2)$ (4.13)

is the Weyl vector of $A_{k-1}$, and $\oplus$ signifies that we concatenate the components of the vectors to form a vector with more components. It is straightforward to check that $\rho_Y \in P_Y$. Then, (4.11) becomes

$$L_0 = \frac{1}{2}(\vec{\beta} + iQ\rho_Y) \cdot (\vec{\beta} + iQ\rho_Y) + \frac{1}{2}Q^2 \rho_Y \cdot \rho_Y.$$  

(4.14)

Now it is easy to see that $L_0$ is positive definite when the momenta is of the form

$$\vec{\beta} = \vec{p} - iQ\rho_Y, \quad \vec{p} \in P_Y,$$

i.e. when $\vec{p}$ has the form

$$\vec{p} = (p(1), \ldots, p(1), \ldots, p(s), \ldots, p(s)).$$

(4.15)

(4.16)

Next, consider the shifted action of the Weyl group on $\vec{\beta}$. It acts linearly on $\vec{\beta} + iQ\rho$. Therefore, on the momenta (4.13), it acts linearly on $\vec{p} + iQ(\vec{p} - \rho_Y)$. The form of $\vec{p} - \rho_Y$, (4.12), then means that the Weyl group of the flavor symmetry (2.4), which acts on $\vec{\beta}$ via the shifted formula (3.3), acts linearly on $\vec{p}$ in (4.13).

Finally, consider the eigenvalues $\Delta^{(k)}$, (3.13), of the zero modes of the generators $W^{(k)}$. As we discussed there, they are linear combinations of the symmetric polynomials of the components of $\vec{\beta} + iQ\rho$. The symmetric polynomials of the components of $\vec{\beta} + iQ\rho$ are, in turn, polynomials with real coefficients of the power sums of the components of $\vec{\beta} + iQ\rho$. For the momenta of the form (4.13), these power sums are

$$\sum_{k=1}^{N} [\beta_k + iQ(p_k - \rho_{Y,k})]^{\ell} = \sum_{k=1}^{s} \sum_{m=1}^{l_s} \left[p_{(k)} + iQ\left(\frac{l_s - 1}{2} - m\right)\right]^{\ell},$$

(4.17)

which are all real. Therefore, if $\vec{\beta}$ is of the form (4.13), the elementary symmetric polynomials of the components of $\vec{\beta}$ are real, just as when $\vec{\beta}$ is of the form (4.10).
This property is a necessary condition to have a unitary representation of the $\mathcal{W}_N$ algebra, because the linear combinations of the zero modes $W_0^{(k)}$ corresponding to the elementary symmetric polynomials should be all Hermitian. This suggests that the state of the Toda theory with momenta of the form (4.13) is unitary.

Combining these observations, we propose that the momenta of the form (4.13) correspond to the mass parameters (2.9) of the gauge theory associated to the same diagram $Y$ under the identification $m'_k = p_{(k)}$. One corollary is that the massless puncture corresponds to the momenta $\beta = -iQ\rho_Y$.

4.4 Null states in the semiclassical limit

The correspondence of the level-1 null states and the punctures of the Seiberg-Witten curve becomes more illuminating if we take the limit where $Q$ is very small compared to any component of $\vec{\varphi}$. In this limit, the action of the derivative on $\vec{\varphi}$ in (3.9) can be neglected and we can replace the derivative by a parameter,

$$Q \frac{d}{dz} \to ix.$$  \hspace{1cm} (4.18)

We call this operation the semiclassical limit in the following, because it replaces the pair $(Qd/dz, z)$ which satisfies the Heisenberg commutation relation with the pair $(x, z)$ with the Poisson bracket $\{x, z\} = 1$. $Q$ serves the role of the Planck constant $\hbar$. Readers should bear in mind that we still treat $\vec{\varphi}$ as quantum operators acting on the quantum Hilbert space generated from $|\vec{\beta}\rangle$. We note that, in the context of integrable system, such a limit is called “dispersionless limit” [18, 19].

The Lax operator is then replaced by its semiclassical version:

$$r_N|\vec{\beta}\rangle = \prod_{m=1}^N \left( x - i\hbar m \cdot \partial \vec{\varphi}(z) \right)|\vec{\beta}\rangle = \sum_k w^{(k)}(z) x^{N-k} |\vec{\beta}\rangle.$$  \hspace{1cm} (4.19)

Here, $r_N$ and $w^{(r)}$ are the semiclassical limits of $R_N$ and $W^{(r)}$ up to powers of $i$. We define the modes of $w^{(k)}(z)$ as usual,

$$w^{(k)}(z) = w_0^{(k)} z^{-k} + w_1^{(k)} z^{-k+1} + \cdots.$$  \hspace{1cm} (4.20)

In the following, we use the abbreviation

$$A^m(z) = i\hbar m \cdot \partial \vec{\varphi}(z),$$  \hspace{1cm} (4.21)

to simplify the formulas; the relation $A^m(z)|\vec{\beta}\rangle \sim (\beta_m/z + O(1))|\vec{\beta}\rangle$ can be used to evaluate $w^{(r)}$ and $r_N$.\hfill\(-14\)-
As we already noted, the relevant part of $A^m(z)$ in the construction of the level-1 null state is $A^m(z) = A^m_0 z^{-1} + A^m_1 + \cdots$. To see the correspondence with the Seiberg-Witten curve, it is convenient to consider $r'_N = z^N r_N$ and to introduce $v = z x$. The Lax operator (4.19) becomes

$$r'_N |\vec{\beta}\rangle = \sum_{k=0}^{N} (w^{(k)}_0 + w^{(k)}_{-1} z + \cdots) v^{N-k} |\vec{\beta}\rangle$$

(4.22)

$$= (w_0(v) + w_{-1}(v) z + \cdots) |\vec{\beta}\rangle$$

(4.23)

where

$$w_s(v) := \sum_{k=1}^{N} w_s^{(k)} v^{N-k}.$$  

(4.24)

We note that $w^{(0)}(z) = 1$, $w^{(1)}(z) = 0$ because $\sum_m A^m_s = 0$. The explicit form of $w_0(v)$ and $w_{-1}(v)$ becomes, after dropping the irrelevant part,

$$w_0(v) |\vec{\beta}\rangle = \prod_{m=1}^{N} (v - \beta_m) |\vec{\beta}\rangle, \quad w_{-1}(v) |\vec{\beta}\rangle = w_0(v) \sum_{m=1}^{N} \frac{A^m_1}{v - \beta_m} |\vec{\beta}\rangle.$$  

(4.25)

Then, to find the null states in the semiclassical limit, we need to show that a linear combination of $w^{(k)}_{-1}$ becomes null whenever any neighboring pair of $\beta_m$ coincides.

This can be seen very easily. Suppose $\beta_1 = \beta_2 = \beta$. Then the function $w_0(v)$ has the second order zero at $v = \beta$:

$$w_0(v) \propto (v - \beta)^2.$$  

(4.26)

Eq. (4.25) then implies that $w_{-1}(v)$ is zero at $v = \beta$:

$$w_{-1}(\beta) |\vec{\beta}\rangle = \sum_{m=2}^{N} \beta^{N-m} w^{(m)}_{-1} |\vec{\beta}\rangle \approx 0.$$  

(4.27)

This gives the explicit form of the null state in terms of $w^{(m)}_{-1}$.

In a similar manner, we can write down the level-1 null states for the singularity associated with the Young diagram $[l_1, l_2, \cdots, l_s]$. Call the corresponding components of $A^m_0$ as $\beta_{(1)}, \beta_{(2)}, \ldots, \beta_{(s)}$. Then we have a set of null states:

$$w_{-1}(\beta_{(k)}) |\vec{\beta}\rangle \approx w'_{-1}(\beta_{(k)}) |\vec{\beta}\rangle \approx \cdots \approx \frac{d^{l_k-2}}{dv^{l_k-2}} w_{-1}(\beta_{(k)}) |\vec{\beta}\rangle \approx 0$$  

(4.28)

for each $k$ with $l_k > 1$. It gives rise to the desired $\sum_{k=1}^{s} (l_k - 1)$ null states at level 1.
At this point, it is illuminating to point out a similarity between the behavior of Lax operator near the singularity with the mass-deformed version of the Seiberg-Witten curve (2.5). The null state conditions (4.28) implies that the behavior of Lax operator $r_N'$ in the vicinity of $V_{\vec{\beta}}(0)$ is,

$$\langle \cdots r_N'(z) V_{\vec{\beta}}(0) \rangle = c_0 \prod_k (v - \beta(k))^l_k + c_1 z \prod_k (v - \beta(k))^{l_k-1} (v^{d_1} + \cdots) + \cdots. \quad (4.29)$$

The structure of the first two terms is exactly the same as the last two terms in (2.5) if we replace $m_k \to \beta(k)$. It implies that the behavior of $r_N'$ at $z = 0$ coincides with the Seiberg-Witten curve near the corresponding puncture.

The null states constructed above do not look like the quantum null state for $N = 3$ (4.9). This type of null state can be constructed when the Young diagram is given in the form $[r, N - r]$. Let us take $\vec{\beta} = (\beta(1), \cdots, \beta(1), \beta(2), \cdots, \beta(2))$ with $r\beta_1 + (N - r)\beta_2 = 0$. Then $w_{0,1}(v)$ is given by

$$w_0(v)|\vec{\beta}\rangle = (v - \beta(1))^r(v - \beta(2))^{N-r}|\vec{\beta}\rangle,$$

$$w_{-1}(v)|\vec{\beta}\rangle = (\beta(1) - \beta(2)) (v - \beta(1))^{l-1}(v - \beta(2))^{N-r-1} A_{-1}|\vec{\beta}\rangle \quad (4.30)$$

where $A_{-1} = \sum_{m=1}^r A_{-1}$. Expanding in terms of $v$, one finds

$$w_{-1}^{(m)}|\vec{\beta}\rangle = \frac{\beta(1) - \beta(2)}{N\beta(1)\beta(2)} mw_0^{(m)} A_{-1}|\vec{\beta}\rangle. \quad (4.31)$$

Therefore, we arrive at identities which are the direct analog of the null state (4.9),

$$(mw_0^{(m)} w_{-1}^{(n)} - nw_0^{(n)} w_{-1}^{(m)})|\vec{\beta}\rangle \approx 0, \quad (m, n = 2, \cdots, N). \quad (4.32)$$

We will see in the next section that these properties also arise naturally from the analysis of the Seiberg-Witten curve.

5. Behavior of Seiberg-Witten curve at punctures

In this section, we study the behavior of the differentials $\phi^{(k)}$ in the Seiberg-Witten curve close to the puncture. We will explicitly see that the null state condition of $\mathcal{W}$-algebra reproduces this behavior, under the identification $\langle W^{(k)}(z) \rangle = \phi^{(k)}(z)$ in the semiclassical limit.
5.1 SU(3) with 6 flavors

In §4, we explicitly obtained the level-1 null state condition (4.9) of $W_3$ algebra. To see the correspondence to the Seiberg-Witten curve, we should consider the mass-deformed Seiberg-Witten curve (2.5) of the SU(3) gauge theory with six flavors. This is given by

$$(v - \tilde{m}_1)(v - \tilde{m}_2)(v - \tilde{m}_3)z^2 + c_1(v^3 - u(2)v - u_3)z + c_0(v - m_1)(v - m_2)(v - m_3) = 0.$$  \hfill (5.1)

As we reviewed in §2, we need to shift $v$ to eliminate $v^2$ terms to make it of the form (2.8). Then we obtain the equation

$$x^3 + \phi^{(2)}(z)x + \phi^{(3)}(z) = 0$$ \hfill (5.2)

where $x = v/z$. Four punctures exist on the sphere at $z = 0, \infty, z_0, z_1$, where $z_{0,1}$ are two solutions of $z^2 + c_1z + c_0 = 0$. The punctures at $z = 0, \infty$ are “full” punctures, while the punctures at $z = z_0, z_1$ are “simple” punctures.

Let us study the behavior of the curve near the simple puncture at $z = z_0$. For that purpose, we take the Laurent expansion of $\phi^{(2)}(z)$ and $\phi^{(3)}(z)$ at $z = z_0$:

$$\phi^{(2)}(z) = \frac{\phi^{(2)}_0}{(z - z_0)^2} + \frac{\phi^{(2)}_1}{(z - z_0)^1} + \mathcal{O}((z - z_0)^0),$$

$$\phi^{(3)}(z) = \frac{\phi^{(3)}_0}{(z - z_0)^3} + \frac{\phi^{(3)}_1}{(z - z_0)^2} + \mathcal{O}((z - z_0)^{-1}), \hfill (5.3)$$

whose explicit form can be found by specializing the general result below in §5.3. We find the relation

$$2\phi^{(2)}_0 \phi^{(3)}_{-1} - 3\phi^{(3)}_0 \phi^{(2)}_{-1} = 0,$$ \hfill (5.4)

which is the same as the form of the null state condition of $W_3$ algebra (4.9). For the full punctures, no such relations exist.

5.2 SU(3) × SU(2)

In the previous section, we considered the simple puncture of the SU(3) theory with six flavors; this simple puncture carried the U(1) symmetry of the bifundamental hypermultiplets. The simple puncture also arises as the puncture coming from the superconformal case, at $z = 0$ or $z = \infty$. Let us check that the differentials $\phi^{(k)}(z)$ in this case behave in the same manner.

Consider the quiver with the gauge group SU(3) × SU(2), for which the puncture at $z = 0$ is simple. Again, we start from the mass-deformed Seiberg-Witten curve (2.5).
We shift \( v \) to eliminate \( v^2 \) terms, and set \( x = v/z \), to make the curve into the form (2.8). To study the puncture at \( z = 0 \), we perform the Laurent expansion at \( z = 0 \) as in (5.3), and find

\[
\begin{align*}
\phi_0^{(2)} &= -\frac{(m_1 - m_2)^3}{3}, \\
\phi^{(2)}_{-1} &= \frac{c_1}{3c_0}(m_1^2 + 2m_2^2 - (m_1 + 2m_2)M_1 - 3u_2^{(2)}), \\
\phi_0^{(3)} &= \frac{2(m_1 - m_2)^3}{27}, \\
\phi^{(3)}_{-1} &= -\frac{c_1}{9c_0}(m_1 - m_2)(m_1^2 + 2m_2^2 - (m_1 + 2m_2)M_1 - 3u_2^{(2)}).
\end{align*}
\] (5.5)

From this, we indeed find the same relation as Eq. (5.4),

\[
2\phi_0^{(2)}\phi^{(3)}_{-1} - 3\phi_0^{(3)}\phi_{-1}^{(2)} = 0. \quad (5.6)
\]

This shows that the Seiberg-Witten curve near all the simple punctures satisfies the same relation, independently of its position.

### 5.3 SU(N) with 2N flavors

As a generalization, let us study the simple puncture of the SU(N) theory with 2N flavors. The mass-deformed Seiberg-Witten curve in this case is given by

\[
(v - \tilde{m}_1)(v - \tilde{m}_2)\cdots(v - \tilde{m}_N)z^2 + c_1(v^N - u^{(2)}v^{N-2} - u^{(3)}v^{N-3} - \cdots - u^{(N)})z \\
+ c_0(v - m_1)(v - m_2)\cdots(v - m_N) = 0. \quad (5.7)
\]

Let us denote by \( z_{0,1} \) two solutions of \( \Delta(z) = z^2 + c_1z + c_0 = 0 \) as before.

The full punctures are at \( z = 0, \infty \) and the simple punctures are at \( z = z_0, z_1 \), just as in the SU(3) case. We take the Laurent expansion of \( \phi^{(k)}(z) \) at the puncture \( z = z_0 \):

\[
\phi^{(k)}(z) = \frac{\phi_0^{(k)}}{(z - z_0)^k} + \frac{\phi_{-1}^{(k)}}{(z - z_0)^{k-1}} + \mathcal{O}((z - z_0)^{k-2}). \quad (5.8)
\]

Explicit calculation leads to the result

\[
\frac{\phi_{-1}^{(k)}}{k\phi_0^{(k)}} = \frac{2a_1}{a_1z_0 + b_1z_1} - \frac{2z_0 - z_1}{z_0(z_0 - z_1)} \\
- \frac{N}{N - 1} \frac{z_0 - z_1}{z_0(a_1z_0 + b_1z_1)^2} \left(a_2z_0 + u^{(2)}(z_0 + z_1) + b_2z_1\right), \quad (5.9)
\]

\[\text{--- 18 --} \]
where
\[ a_1 = \sum_k \tilde{m}_k, \quad a_2 = \sum_{k<l} \tilde{m}_k \tilde{m}_l, \quad b_1 = \sum_k m_k, \quad b_2 = \sum_{k<l} m_k m_l. \] (5.10)

Therefore, \( \phi_{-1}^{(k)}/k\phi_0^{(k)} \) is independent of \( k \). This means that
\[ k\phi_0^{(k)} \phi_{-1}^{(j)} - j\phi_0^{(j)} \phi_{-1}^{(k)} = 0 \] (5.11)
is satisfied for all \( j, k = 2, \cdots, N \). This is exactly the form we found in §4.4 as the null state condition of \( \mathcal{W}_N \) algebra (4.32) in the semiclassical limit.

5.4 Puncture of type \([r, N-r] \)

As a final generalization, we consider a puncture with the Young diagram \([r, N-r] \), to compare what we considered in §4.4 in the Toda theory side. The quiver which realizes this puncture is the one with the gauge group

\[ \text{SU}(N) \times \text{SU}(N-1) \times \cdots \times \text{SU}(2(N-r)) \times \text{SU}(2(N-r-1)) \times \cdots \times \text{SU}(2). \] (5.12)

The mass-deformed Seiberg-Witten curve in this case is given in (2.5), where \( s = 2, l_1 = r \) and \( l_2 = N-r \). Besides the puncture at \( z = 0 \), a full puncture is at \( z = \infty \) and there are simple punctures at the solutions of \( \Delta(z) = 0 \).

We perform the Laurent expansion of \( \phi^{(k)}(z) \) at the puncture at \( z = 0 \) of type \([r, N-r] \) as follows:
\[ \phi^{(k)}(z) = \frac{\phi_0^{(k)}}{z^k} + \frac{\phi_{-1}^{(k)}}{z^{k-1}} + O(z^{-k+2}) \] (5.13)
and we find that
\[ \frac{\phi_{-1}^{(k)}}{k\phi_0^{(k)}} = \frac{1}{r(N-r)} \frac{c_1 r(M_1 - m_1)m_1 + (N-r)(M_1 - m_2)m_2 + Nu_1^{(2)}}{c_0 (m_1 - m_2)^2} \] (5.14)
is independent of \( k \). Therefore,
\[ k\phi_0^{(k)} \phi_{-1}^{(j)} - j\phi_0^{(j)} \phi_{-1}^{(k)} = 0 \] (5.15)
is satisfied for all \( j, k = 2, \cdots, N \). Therefore, as we mentioned in §4.4, we can surely show that Eq. (4.32) is satisfied in this general case.
6. Conclusions

In this paper, we studied the structure of the null states in the level-1 descendants of the $\mathcal{W}_N$ algebra, and found it to be labeled by a Young diagram with $N$ boxes. This Young diagram controls the Toda momenta of the corresponding primary operator insertion, and also controls the behavior of the generators $W^{(k)}(z)$ of the $\mathcal{W}_N$ algebra close to the insertion.

Under the correspondence between $SU(N)$ quiver gauge theories and the $A_{N-1}$ Toda field theory, the Young diagram labeling the primary operator maps to the Young diagram labeling the puncture of the Riemann surface on which $N$ M5-branes are wrapped. It was known that the Young diagram controls the behavior of the differentials $\phi^{(k)}(z)$ appearing in the Seiberg-Witten curve close to the puncture; we found that the semiclassical limit of $\langle W^{(k)}(z) \rangle$ behaves exactly as $\phi^{(k)}(z)$ does, thus giving another indication that the Toda theory gives the “quantization” of the Seiberg-Witten curve.

We also discussed the real subspace of the allowed momenta for a given Young diagram. In this real subspace, the zero modes of the generators of the $\mathcal{W}_N$ algebra, $W^{(k)}_0$, have real eigenvalues for all $k$. Our proposal is that these special momenta describe the propagation of the waves along the intersection of a subset of the exponential walls of the Toda theory, and the subset is controlled by the Young diagram.

In this paper, we discussed the correspondence between the 4D gauge theory and the Toda field theory only at the vicinity of a puncture. Therefore, the obvious next step is to study the correspondence on a whole Riemann surface with a number of punctures, each labeled by a Young diagram.

It should be straightforward to carry out an analysis directly analogous to the ones performed in [3, 4, 5], for general linear quiver gauge theories with $SU$ gauge groups. Namely, we predict that Nekrasov’s partition function and the partition function on $S^4$ of these gauge theories correspond to the conformal block of the $\mathcal{W}_N$ algebra and the correlation function of the Toda field theory, respectively, under the mapping we proposed in this paper. To calculate the partition function, we will need to integrate over the real subspace which we discussed in §4.3. The fusion rule among the primary operators labeled by different Young diagrams would also be important in the matching.

It would also be interesting to understand the relation of the degenerate states of the Toda theory and the puncture of the Hitchin system, which underlies Gaiotto’s construction of the punctures [2, 9]. Another direction of the study would be to connect our analysis with the analysis of the matrix model [11, 20] which also gave the interpretation that the Toda field theory is the quantization of the Seiberg-Witten curve.
Acknowledgments

Y. M. and Y. T. would like to thank the hospitality of the Yukawa Institute for Theoretical Physics where this work was initiated during the workshop “Branes, Strings and Black Holes.” Y. T. is supported in part by the NSF grant NO. PHY-0503584, and by the Marvin L. Goldberger membership at the Institute for Advanced Study. Y. M. is partially supported by KAKENHI (#20540253) from MEXT, Japan. S. S. is partially supported by Global COE Program “the Physical Sciences Frontier,” MEXT, Japan.

References

[1] D. Gaiotto, “\(\mathcal{N} = 2\) Dualities,” arXiv:0904.2715 [hep-th].

[2] D. Gaiotto, G. W. Moore, and A. Neitzke, “Wall-Crossing, Hitchin Systems, and the Wkb Approximation,” arXiv:0907.3987 [hep-th].

[3] L. F. Alday, D. Gaiotto, and Y. Tachikawa, “Liouville Correlation Functions from Four-dimensional Gauge Theories,” Lett. Math. Phys. 91 (2010) 167–197, arXiv:0906.3219 [hep-th].

[4] N. Wyllard, “\(A_{N-1}\) Conformal Toda Field Theory Correlation Functions from Conformal \(\mathcal{N}=2\) \(SU(N)\) Quiver Gauge Theories,” JHEP 11 (2009) 002, arXiv:0907.2189 [hep-th].

[5] A. Mironov and A. Morozov, “On Agt Relation in the Case of \(U(3)\),” Nucl. Phys. B825 (2010) 1–37, arXiv:0908.2569 [hep-th].

[6] R. Dijkgraaf and C. Vafa, “Toda Theories, Matrix Models, Topological Strings, and \(\mathcal{N}= 2\) Gauge Systems,” arXiv:0909.2453 [hep-th].

[7] G. Bonelli and A. Tanzini, “Hitchin Systems, \(\mathcal{N}= 2\) Gauge Theories and W-Gravity,” arXiv:0909.4031 [hep-th].

[8] L. F. Alday, F. Benini, and Y. Tachikawa, “Liouville/Toda Central Charges from M5-Branes,” arXiv:0909.4776 [hep-th].

[9] D. Nanopoulos and D. Xie, “Hitchin Equation, Singularity, and \(N=2\) Superconformal Field Theories,” JHEP 03 (2010) 043, arXiv:0911.1990 [hep-th].

[10] E. Witten, “Solutions of Four-Dimensional Field Theories via M-Theory,” Nucl. Phys. B500 (1997) 3–42, arXiv:hep-th/9703166.

[11] V. A. Fateev and S. L. Lukyanov, “The Models of Two-Dimensional Conformal Quantum Field Theory with \(Z_N\) Symmetry,” Int. J. Mod. Phys. A3 (1988) 507.
[12] P. Bouwknegt and K. Schoutens, “W Symmetry in Conformal Field Theory,” Phys. Rept. 223 (1993) 183–276, arXiv:hep-th/9210010.

[13] V. A. Fateev and A. V. Litvinov, “Correlation Functions in Conformal Toda Field Theory I,” JHEP 11 (2007) 002, arXiv:0709.3806 [hep-th].

[14] C.-R. Ahn, V. A. Fateev, C.-J. Kim, C. Rim, and B. Yang, “Reflection Amplitudes of ADE Toda Theories and Thermodynamic Bethe Ansatz,” Nucl. Phys. B565 (2000) 611–628, arXiv:hep-th/9907072.

[15] V. A. Fateev and A. B. Zamolodchikov, “Conformal Quantum Field Theory Models in Two-Dimensions Having $\mathbb{Z}_3$ Symmetry,” Nucl. Phys. B280 (1987) 644–660.

[16] A. Bilal and J.-L. Gervais, “Systematic Approach to Conformal Systems with Extended Virasoro Symmetries,” Phys. Lett. B206 (1988) 412.

[17] J.-L. Gervais and Y. Matsuo, “Classical $A_N$ W Geometry,” Commun. Math. Phys. 152 (1993) 317–368, arXiv:hep-th/9201026.

[18] I. Krichever, “The Dispersionless Lax equations and topological minimal models,” Commun. Math. Phys. 143 (1992) 415–429.

[19] K. Takasaki and T. Takebe, “Integrable hierarchies and dispersionless limit,” Rev. Math. Phys. 7 (1995) 743–808, arXiv:hep-th/9405096.

[20] H. Itoyama, K. Maruyoshi, and T. Oota, “Notes on the Quiver Matrix Model and 2D-4D Conformal Connection,” arXiv:0911.4244 [hep-th].