RATIONAL POINTS OF UNIVERSAL CURVES IN POSITIVE CHARACTERISTICS

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Abstract. For the moduli stack $\mathcal{M}_{g,n}/\mathbb{F}_p$ of curves over $\text{Spec}\mathbb{F}_p$ with the function field $K$, we show that if $g \geq 3$, then the only $K$-rational points of the generic curve over $K$ are its $n$ tautological points. Furthermore, we show that if $g \geq 4$ and $n = 0$, then Grothendieck’s Section Conjecture holds for the generic curve over $K$. This is an extension of Hain’s work in characteristic 0 to positive characteristics.

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1. Introduction

Suppose that \( C \) is a geometrically irreducible smooth projective curve over a field \( k \). Let \( G_k \) be the absolute Galois group of \( k \). Associated to the curve \( C \), there is a short exact sequence of algebraic fundamental groups:

\[
1 \to \pi_1(C, \bar{x}) \to \pi_1(C, \bar{x}) \to G_k \to 1,
\]

where \( \bar{k} \) is the separable closure of \( k \) and \( C_{\bar{k}} = C \otimes_k \bar{k} \). Each \( k \)-rational point \( x \) of \( C \) induces a section \( s_x \) of \( \pi_1(C, \bar{x}) \to G_k \), which is unique up to conjugation by elements of the geometric fundamental group \( \pi_1(C, \bar{x}) \). Grothendieck’s section conjecture states that when \( C \) is hyperbolic and \( k \) is a finitely generated infinite field, there is a bijection between the set of \( k \)-rational points and the set of conjugacy classes of sections of \( \pi_1(C, \bar{x}) \to G_k \) via the association \( x \mapsto [s_x] \). Hain proved in [14] that the sections conjecture holds for the restriction of the universal curve \( \mathcal{C} \to \mathcal{M}_g \) to its generic point \( \text{Spec} k(\mathcal{M}_g) \) with \( g \geq 5 \) and \( \text{char } k = 0 \). In this paper, we will extend his results to positive characteristics. In order to make this paper self-contained, the majority of results needed are cited from Hain’s original papers [11] and [14].

Before stating our main results, we need to introduce notations. A curve \( C/T \) of type \((g, n)\) is a proper smooth family \( C \to T \) of geometrically connected curves of genus \( g \) with distinct \( n \) sections \( s_i : T \to C \). Suppose that \( 2g - 2 + n > 0 \).

Let \( k \) be a field. Denote the moduli stack of curves of type \((g, n)\) over \( \text{Spec}(k) \) by \( \mathcal{M}_{g,n/k} \) and the universal curve over it by \( \mathcal{C}_{g,n/k} \). Let \( K \) be the function field of \( \mathcal{M}_{g,n/k} \). The generic curve of type \((g, n)\) over \( K \) with \( g \geq 3 \) is the pullback of the universal curve \( \mathcal{C}_{g,n/k} \) to the function field \( K \). The key ingredient that allows us to use Hain’s methods in positive characteristics is the comparison of algebraic fundamental groups of a certain finite étale cover of \( \mathcal{M}_{g,n} \). For a prime number \( \ell \), there is a finite étale Galois cover \( \mathcal{M}_{g,n}^\lambda \) of \( \mathcal{M}_{g,n/\mathbb{Z}[1/\ell]} := \mathcal{M}_{g,n/\mathbb{Z}} \otimes \text{Spec}(\mathbb{Z}[1/\ell]) \) that is representable by a scheme and has a smooth compactification over \( \mathbb{Z}[1/\ell] \) whose boundary is a relative normal crossing divisor over \( \mathbb{Z}[1/\ell] \). Such covers were explicitly constructed by Boggi, de Jong, and Pikaart in [4], [20], and [30].

Denote the moduli stack of curves of type \((g, n)\) over \( \text{Spec}(k) \) with an abelian level \( r \) by \( \mathcal{M}_{g,n/k}[r] \). When the ground field \( k \) contains an \( r \)th root of unity \( \mu_r(k) \), we always assume that \( \mathcal{M}_{g,n/k}[r] \) is a geometrically connected, smooth stack over \( \text{Spec}(k) \).

Suppose that \( p \) is a prime number, \( \ell \) is a prime number distinct from \( p \), and \( m \) is a nonnegative integer. Let \( \mathcal{C}_{g,n/\mathbb{F}_p}[\ell^m] \to \mathcal{M}_{g,n/\mathbb{F}_p}[\ell^m] \) be the universal curve over the stack \( \mathcal{M}_{g,n/\mathbb{F}_p}[\ell^m] \).

**Theorem 1.** Let \( K \) be the function field of \( \mathcal{M}_{g,n/\mathbb{F}_p}[\ell^m] \). If \( g \geq 4 \) or if \( g = 3 \), \( p \geq 3 \), and \( \ell = 2 \), then the only \( K \)-rational points of \( \mathcal{C}_{g,n/\mathbb{F}_p}[\ell^m] \) are its \( n \) tautological points.

The corresponding result in characteristic 0 follows from results in Teichmüller theory [19, 7] due to Hubbard, Earle and Kra. Our approach is to apply Hain’s algebraic methods in positive characteristics.

Let \( \mathbb{F}_q = \mathbb{F}_p(\zeta_{\ell^m}) \), where \( \zeta_{\ell^m} \) is a primitive \( \ell^m \)th root of unity.
Theorem 2. Let $C/L$ be the restriction of the universal curve $\mathcal{C}_{g/R_{s}}[\ell^m] \to \mathcal{M}_{g/R_{s}}[\ell^m]$ to the generic point $\text{Spec } L$ of $\mathcal{M}_{g/R_{s}}[\ell^m]$. Let $\bar{L}$ be the separable closure of $L$, and let $\bar{x}$ be a geometric point of $C_{\bar{L}}$. If $g \geq 4$, then the sequence

$$1 \to \pi_1(C_{\bar{L}}, \bar{x}) \to \pi_1(C, \bar{x}) \to G_L \to 1$$

does not split.

Corollary 3. The section conjecture holds for the generic curve $C/L$.

The first key tool used in this paper is the theory of specialization homomorphism from $[9, \text{SGA } 1, \S X, \text{XIII}].$ This allows us to compare the maximal pro-$\ell$ quotient of the fundamental groups of $\mathcal{M}^{\lambda}_{g,n/\mathbb{Q}_{\ell}}$ and $\mathcal{M}^{\lambda}_{g,n/\mathbb{F}_{p}}$ when $\ell \neq p$. The essential tools used in Hain’s original paper $[14]$ and this paper are weighted completion and relative completion of profinite groups. The theory of weighted completion was developed by Hain and Matsumoto in $[18]$. For a curve $C/T$, let $\text{GSp}(H_{\alpha}) := \text{GSp}(H_{\alpha}(\mathbb{C}_{T}, \mathbb{Q}_{\ell}(1)))$ with $\ell$ a prime not in the residue characteristic $\text{char}(T)$ of $T$. There are natural monodromy actions of $\pi_1(C, \bar{\eta})$ and $\pi_1(T, \bar{\eta})$ into $\text{GSp}(H_{\alpha})$ with the Zariski closure $R$ of their common images. One can take the weighted completion of $\pi_1(C, \bar{\eta}C)$ and $\pi_1(T, \bar{\eta}T)$ with respect to $R$ to obtain $\mathbb{Q}_{\ell}$-proalgebraic groups $\mathcal{G}_{C}$ and $\mathcal{G}_{T}$. These are extensions of $R$ by a prounipotent $\mathbb{Q}_{\ell}$-group. In this paper, $R$ is equal to the whole group $\text{GSp}(H_{\alpha})$. For the universal curve $\mathcal{C}_{g,n/k}[\ell^m] \to \mathcal{M}_{g,n/k}[\ell^m]$, the Zariski closure $\mathcal{G}^{\text{geom}}_{\mathcal{M}_{g,n/k}[\ell^m]}$ of the image in $\mathcal{G}^{\text{geom}}_{\mathcal{M}_{g,n/k}[\ell^m]}(\mathbb{Q}_{\ell})$ of the composite $\pi_1(\mathcal{M}_{g,n/k}[\ell^m], \bar{\eta}) \to \pi_1(\mathcal{M}_{g,n/k}[\ell^m], \bar{\eta}) \to \mathcal{G}^{\text{geom}}_{\mathcal{M}_{g,n/k}[\ell^m]}(\mathbb{Q}_{\ell})$ is an extension of the reductive group $\text{Sp}(H_{\alpha})$ by a prounipotent $\mathbb{Q}_{\ell}$-group and its Lie algebra $\mathfrak{g}^{\text{geom}}_{g,n}$ is a pro-object of the category of the $\mathcal{G}^{\text{geom}}_{\mathcal{M}_{g,n/k}[\ell^m]}$-modules. Each finite-dimensional $\mathcal{G}^{\text{geom}}_{\mathcal{M}_{g,n/k}[\ell^m]}$-module $V$ admits a natural weight filtration:

$$V = W_{0}V \supset W_{1}V \supset \cdots \supset W_{r}V$$

such that each weight graded quotient $\text{Gr}^{W}_{r}V$ is a $\text{GSp}(H_{\alpha})$-module of weight $r$. Each natural weight filtration induced on $\mathfrak{g}^{\text{geom}}_{g,n}$ satisfies the property that $\mathfrak{g}^{\text{geom}}_{g,n} = W_{0}\mathfrak{g}^{\text{geom}}_{g,n}$ and its prounipotent radical $\mathfrak{u}^{\text{geom}}_{g,n}$ is negatively weighted: $\mathfrak{u}^{\text{geom}}_{g,n} = W_{-1}\mathfrak{u}^{\text{geom}}_{g,n}$. Theorem 1 and 2 are proved by using the structure of the truncated Lie algebra $\text{Gr}^{W}_{\bullet}(\mathfrak{u}^{\text{geom}}_{g,n}/W_{-3})$, which is defined in section 13.

2. FUNDAMENTAL GROUPS

For a connected scheme $X$ and a choice of a geometric point $\bar{\eta} : \text{Spec } \Omega \to X$, we have the étale fundamental group of $X$ denoted by $\pi_1(X, \bar{\eta})$, which is defined as the automorphism group of the fibre functor. More generally, for a Galois category $\mathcal{C}$ with a fundamental functor $F$, we have the fundamental group $\pi_1(\mathcal{C}, F)$ such that $F$ is an equivalence of the category $\mathcal{C}$ and the category of finite sets on which $\pi_1(\mathcal{C}, F)$ acts continuously. When $\mathcal{C}$ is the category of finite étale covers $E$ of $X$ and $F = F_{\eta} : E \mapsto E_{\eta} := E \times_{X} \text{Spec } \Omega$, we have $\pi_1(\mathcal{C}, F) = \pi_1(X, \bar{\eta})$. When $X$ is a field $k$ and $\bar{k}$ is an algebraic closure of $k$, we have $\pi_1(\text{Spec } k, \text{Spec } \bar{k}) = \text{Gal}(k_{\text{sep}}/k)$, where $k_{\text{sep}}$ is the separable closure of $k$ in $\bar{k}$. In this paper, we will need the extension of this theory to the Deligne-Mumford stacks, which are constructed in $[27]$. 
2.1. Comparison theorem. Suppose that $k$ is a subfield of $\mathbb{C}$. Let $\bar{k}$ be the algebraic closure of $k$ in $\mathbb{C}$. For a geometrically connected scheme $X$ of finite type over $k$ and a geometric point $\bar{\eta} : \text{Spec} \mathbb{C} \to X$, there is a canonical isomorphism

$$\pi_1^{\text{top}}(X^{\text{an}}, \bar{\eta})^\wedge \cong \pi_1(X \otimes_k \bar{k}, \bar{\eta}),$$

where $X^{\text{an}}$ denotes the complex analytic variety associated to $X$ and $\pi_1^{\text{top}}(X^{\text{an}}, \bar{\eta})^\wedge$ denotes the profinite completion of the topological fundamental group of $X^{\text{an}}$ with the image of $\bar{\eta}$ as a base point. Furthermore, for a DM stack $\mathcal{X}$ over $k$, the corresponding analytical space denoted by $\mathcal{X}^{\text{an}}$ is an isomorphism and thus there is a projective system of profinite groups:

$$\pi_1^{\text{orb}}(\mathcal{X}^{\text{an}}, x)^\wedge \cong \pi_1(\mathcal{X} \otimes_k \bar{k}, x),$$

where $x : \text{Spec} \mathbb{C} \to \mathcal{X}$ is a geometric point of $\mathcal{X}$.

2.2. Fundamental groups of curves. Let $C$ be a smooth curve of genus $g$ over an algebraically closed field $k$ such that $C$ is a complement of $n \geq 0$ closed points of its smooth compactification. Fix a geometric point $\bar{\eta}$ of $C$. The fundamental group of a smooth curve does not change under extensions of algebraically closed fields of characteristic zero [32, 5.6.7], and thus we may assume that $k$ is a subfield of $C$. Then by the comparison theorem the fundamental group $\pi_1(C, \bar{\eta})$ of $C$ with base point $\bar{\eta}$ is isomorphic to the profinite completion of the group

$$\Pi_{g,n} := \langle a_1, b_1, \ldots, a_g, b_g, \gamma_1, \ldots, \gamma_n \mid [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] \gamma_1 \cdots \gamma_n = 1 \rangle.$$

When $\text{char} \ k = p > 0$, Grothendieck proved in [9] that the maximal prime-to-$p$ quotient $\text{of } \pi_1(C, \bar{\eta})$, denoted by $\pi_1(C, \bar{\eta})^{(p)}$, is isomorphic to the maximal prime-to-$p$ completion of the group $\Pi_{g,n}$.

2.3. Fundamental group of the generic point of a variety. Suppose that $X$ is a smooth variety over a field $k$. Let $K = k(X)$ be the function field of $X$ and $\bar{\eta}$ be a geometric point lying over the generic point of $X$. We may take this geometric point $\bar{\eta}$ as a base point for any open subvariety of $X$. For divisors $D \subset E$ of $X$ defined over $k$, there is a canonical surjection

$$\pi_1(X - E, \bar{\eta}) \to \pi_1(X - D, \bar{\eta})$$

and thus there is a projective system of profinite groups:

$$\{\pi_1(X - D, \bar{\eta})\}_D,$$

where $D$ is taken over the divisors of $X$ defined over $k$. Fix an algebraic closure $\bar{K}$ of $K$. Let $K_{\text{sep}}$ be the separable closure of $K$ in $\bar{K}$. Then the Zariski-Nagata [9, Theorem 3.1] implies

**Proposition 2.1.** The canonical surjection

$$\text{Gal}(K_{\text{sep}}/K) \to \varprojlim_D \pi_1(X - D, \bar{\eta})$$

is an isomorphism.

---

1Here the maximal prime-to-$p$ quotient $G^{(p)}$ of a profinite group $G$ is the projective limit of its finite continuous quotients of order prime to $p$. 

3. Representations of $\text{Sp}(H)$ and $\text{GSp}(H)$

3.1. The groups $\text{Sp}(H)$ and $\text{GSp}(H)$. Suppose $g \geq 1$. Let $A$ be a commutative ring and $H$ be a free $A$-module of rank $2g$. Fix a nondegenerate, skew symmetric bilinear form $q : H \otimes H \rightarrow A$. For an $A$-algebra $S$, denote $H \otimes_A S$ by $H_S$. The general symplectic group $\text{GSp}(H_S)$ is defined by

$$
\text{GSp}(H_S) = \{ \phi \in \text{GL}(H_S) | \phi^* q = \tau(\phi) q \ \text{for some} \ \tau(\phi) \in S^\times \}.
$$

Associating $\tau(\phi)$ to $\phi$ is a surjective homomorphism $\text{GSp}(H_S) \rightarrow \mathbb{G}_{m/S}$ and its kernel is the symplectic group $\text{Sp}(H_S)$.

We regard $\text{Sp}(H)$ and $\text{GSp}(H)$ as group schemes defined over $\mathbb{Z}$ and their group of $S$-rational points are identified with the groups $\text{Sp}(H_S)$ and $\text{GSp}(H_S)$, respectively. There is an exact sequence of group schemes over $\mathbb{Z}$:

$$
1 \rightarrow \text{Sp}(H) \rightarrow \text{GSp}(H) \xrightarrow{\iota} \mathbb{G}_m \rightarrow 1.
$$

A $\text{GSp}(H_A)$-module $A(n)$ is a free $A$-module of rank 1 with action of $\text{GSp}(H_A)$ via the $n$th power of the homomorphism $\tau$. Fixing an isomorphism $t : A \rightarrow A(1)$ mapping $1 \mapsto a_0$, we define a bilinear form $\theta := t \circ q$, which is a $\text{GSp}(H_A)$-equivariant, nondegenerate, skew symmetric bilinear form. For a $\text{GSp}(H_A)$-module $V$, we define $V(n)$ to be $V \otimes_A A(n)$. The dual pairing denoted by $\hat{\theta}$ is the map

$$
\hat{\theta} : A(1) \rightarrow \Lambda^2 H_A,
$$

which we view as an element of $\Lambda^2 H_A(-1)$ as well.

3.2. Key $\text{Sp}(H)$ and $\text{GSp}(H)$-representations. It follows from the fact that the irreducible representations of $\text{Sp}(H)_{\mathbb{Q}}$ and $\text{GSp}(H)_{\mathbb{Q}}$ are absolutely irreducible that for an extension $F \supset \mathbb{Q}$ of fields, the representations of $\text{Sp}(H_F)$ and $\text{GSp}(H_F)$ are obtained by extension of scalars from those of $\text{Sp}(H)$ and $\text{GSp}(H)$, respectively. Here we assume $H = H_{\mathbb{Q}_F}$. Define the central cocharacter

$$
\omega : \mathbb{G}_m \rightarrow \text{GSp}(H)
$$

by mapping $z \mapsto z^{-1}$ id, which we call the standard cocharacter. Each irreducible $\text{GSp}(H)$-representation admits weight $\omega(V)$ as a $\mathbb{G}_m(\mathbb{Q})$-representation. In particular, $H$ has weight $-1$. The composite

$$
\mathbb{G}_m \overset{\iota}{\rightarrow} \text{GSp}(H) \xrightarrow{\iota} \mathbb{G}_m
$$

is given by $z \mapsto z^{-2}$, and thus the representation $\mathbb{Q}(r)$ has weight $-2r$. If an irreducible representation $V$ has weight $\omega$, then the irreducible representation $V(r)$ has weight $\omega - 2r$. As mentioned in the introduction, the proof of the main results uses a truncated graded Lie algebra whose graded quotients are $\text{GSp}(H)$-representations. We will introduce $\text{GSp}(H)$-representations appearing in the graded quotients. For a partition $\lambda$ of a nonnegative integer $n$ into $s \leq g$ nonnegative integers: $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s \geq 0)$, denote the corresponding irreducible $\text{GSp}(H)$-representation $H_{[\lambda]}$ by $H_{[\lambda_1, \lambda_2, \ldots, \lambda_s]}$. The weight of $H_{[\lambda]}$ is given by $-(\lambda_1 + \cdots + \lambda_s)$. The following result can be easily obtained from the basic results in [8].

**Theorem 3.1.** Every irreducible representation of $\text{GSp}(H)$ is isomorphic to the representation of the form $H_{[\lambda]}(r)$, where $r \in \mathbb{Z}$ and $\lambda$ is a partition of an integer.
\( n \geq 0 \) into \( g \) parts. Moreover, each \( H_{|\lambda|}(r) \) restricts to an irreducible \( \text{Sp}(H) \)-representation and every isomorphism class of irreducible \( \text{Sp}(H) \)-representations occurs in this way. Finally, \( H_{|\lambda|}(r) = H_{|\lambda'|}(r') \) as \( \text{Sp}(H) \)-representations if and only if \( \lambda = \lambda' \).

The representations used in the proof of the main theorems are
\[
H_{|0|}(1) = \mathbb{Q}_\ell(1), \quad H_{|1|} = H_{\mathbb{Q}_\ell}, \quad H_{|12|}, \quad \text{and} \quad H_{|13|}(-1).
\]

We consider \( H_{|12|} \) and \( H_{|13|}(-1) \) as the quotient of \( \Lambda^2 H \) and \( \Lambda^3 H(-1) \), respectively. More explicitly, there are split exact sequences of \( \text{GSp}(H) \)-representations:
\[
0 \to H_{|12|} \to \Lambda^2 H \overset{\phi}{\to} \mathbb{Q}_\ell(1) \to 0,
\]
and
\[
0 \to H_{|13|}(-1) \to \Lambda^3 H(-1) \overset{\phi}{\to} H \to 0,
\]
where \( \phi \) is the twist of the map defined by
\[
\phi : x \wedge y \wedge z \mapsto \theta(x, y)z + \theta(y, z)x + \theta(z, x)y.
\]

It is easy to see that \( \bar{\theta}/g : \mathbb{Q}_\ell(1) \to \Lambda^2 H \) and \( \bar{\theta}/(g - 1) : u \mapsto \frac{u \wedge \bar{\theta}}{g - 1} \) are sections of \( \theta \) and \( \phi \), respectively. For the rest of this paper, we denote \( H_{|12|} \) by \( \Lambda_0^2 H \) and \( H_{|13|}(-1) \) by \( \Lambda_0^3 H \). Also, using Hain’s notation in [14], we denote \( H_{|22|} \) by \( H_{\mathbb{H}} \).

The following computations are made by using computer program LiE and used in section 10. Kabanov’s stability result [22] implies that the following decompositions are independent of \( g \) when \( g \geq 6 \).

**Proposition 3.2** ([11, 10.2]). If \( g \geq 3 \), we have:

(i)
\[
\Lambda^2 H_{|13|} = \begin{cases} 
H_{|16|} + H_{|14|} + H_{|12|} + H_{|2,12|} + H_{|22|} + H_{|0|} & : g \geq 6 \\
H_{|14|} + H_{|12|} + H_{|2,12|} + H_{|22|} + H_{|0|} & : g = 5 \\
H_{|12|} + H_{|2,12|} + H_{|22|} + H_{|0|} & : g = 4 \\
H_{|22|} + H_{|0|} & : g = 3
\end{cases}
\]

(ii)
\[
H_{|1|} \otimes H_{|13|} = \begin{cases} 
H_{|14|} + H_{|2,12|} + H_{|12|} & : g \geq 4 \\
H_{|2,12|} + H_{|12|} & : g = 3
\end{cases}
\]

4. **Monodromy Representation**

4.1. **Monodromy action in characteristic 0**. Suppose that \( T \) is a smooth geometrically connected variety over a field \( k \) of characteristic 0 and that \( f : C \to T \) is a curve of type \((g, n)\). Fix a geometric point \( \bar{\eta} : \text{Spec} \Omega \to T \) of \( T \) and denote the fiber of \( C \) over \( \bar{\eta} \) by \( C_{\bar{\eta}} \). For a prime number \( \ell \), denote \( H_{\ell}^{\lambda}(C_{\bar{\eta}}, \mathbb{Z}_\ell(1)) \) by \( H_{2\ell} \). This is equipped with the cup product pairing \( \theta : \Lambda^2 H_{2\ell} \to \mathbb{Z}_\ell(1) \), which is skew symmetric and nondegenerate. The choice of a symplectic basis of \( H_{2\ell} \) gives an isomorphism \( \text{GSp}(H_{2\ell}) \cong \text{GSp}_g(\mathbb{Z}_\ell) \). Let \( \bar{x} \) be a closed point of \( C_{\bar{\eta}} \) that lies over \( \bar{\eta} \).

**Lemma 4.1**. [14, Lemma 4.1] If \( g \geq 2 \), then the homomorphism \( \pi_1(C_{\bar{\eta}}, \bar{x}) \to \pi_1(C, \bar{x}) \) induced by \( i : C_{\bar{\eta}} \to C \) is injective.
The curve \( C \) is the pullback of the universal curve \( \mathcal{M}_{g,1/k} \) along the morphism \( \phi_f : T \to \mathcal{M}_{g/k} \) and we have the commutative diagram

\[
\begin{array}{ccc}
\pi_1(C, \bar{x}) & \xrightarrow{\phi_f^*} & \pi_1(T, \bar{\eta}) \\
\downarrow & & \downarrow \\
\pi_1(C, \bar{x}) & \xrightarrow{\pi_1(T, \bar{\eta})} & 1 \\
\end{array}
\]

whose rows are exact. Therefore, the homomorphism \( \pi_1(C, \bar{x}) \to \pi_1(C, \bar{x}) \) is injective. \( \square \)

Lemma 4.1 gives the exact sequence of algebraic fundamental groups

\[ 1 \to \pi_1(C, \bar{x}) \to \pi_1(C, \bar{x}) \to \pi_1(T, \bar{\eta}) \to 1. \]

Thus the conjugation action of \( \pi_1(C, \bar{x}) \) on \( \pi_1(C, \bar{x}) \) induces a natural monodromy representation

\[ \rho_{\bar{\eta}} : \pi_1(T, \bar{\eta}) \to \text{GSp}(H_\mathbb{Z}_\ell) \]

such that the diagram

\[
\begin{array}{ccc}
\pi_1(T, \bar{\eta}) & \xrightarrow{\rho_{\bar{\eta}}} & \text{GSp}(H_\mathbb{Z}_\ell) \\
\downarrow & & \downarrow \\
G_k & \xrightarrow{\chi_\ell} & \mathbb{Z}_\ell^\times
\end{array}
\]

commutes, where the left-hand vertical map is the canonical projection, the right-hand vertical map \( \tau \) is the natural surjection, and where \( \chi_\ell \) is the \( \ell \)-adic cyclotomic character.

**Remark 4.2.** Denote the smooth \( \mathbb{Z}_\ell \)-sheaf \( R^1 f_\ast \mathbb{Z}_\ell(1) \) over \( T \) by \( \mathbb{H}_{\mathbb{Z}_\ell} \). For a geometric point \( \bar{\eta} \) of \( T \), the monodromy action of \( \pi_1(T, \bar{\eta}) \) on the stalk \( H^1_{\mathbb{Z}_\ell}(C_{\bar{\eta}}, \mathbb{Z}_\ell(1)) \) of \( \mathbb{H}_{\mathbb{Z}_\ell} \) at \( \bar{\eta} \) coincides with \( \rho_{\bar{\eta}} \).

### 4.2. Monodromy action in characteristic \( p \)

Suppose that \( S \) is a connected scheme, and that \( f : X \to S \) is a proper smooth morphism of schemes whose fibers are geometrically connected. Let \( \bar{s} : \text{Spec} \Omega \to S \) be a geometric point of \( S \) and \( \bar{x} \) be a geometric point of the fiber \( X_{\bar{s}} \) of \( X \) with a value in \( \Omega \). Let \( \text{char} (S) \) be the set of residue characteristics of \( S \) and let \( L \) be the set of prime numbers not in \( \text{char}(S) \). The following results are from [9, SGA 1, Exposé XIII, 4.3, 4.4]. Let \( K \) be the kernel of the canonical homomorphism \( \pi_1(X, \bar{x}) \to \pi_1(S, \bar{s}) \) and \( N \) be the kernel of the projection \( K \to K^\mathbb{Z} \) where \( K^\mathbb{Z} \) is the maximal pro-\( L \) quotient of \( K \). Then \( N \) is a distinguished subgroup of \( \pi_1(X, \bar{x}) \) and we denote by \( \pi'_1(X, \bar{x}) \) the quotient of \( \pi_1(X, \bar{x}) \) by \( N \). Also we denote by \( \pi'_1(X_{\bar{s}}, \bar{x}) \) the maximal pro-\( L \) quotient of \( \pi_1(X_{\bar{s}}, \bar{x}) \). In general, the sequence

\[ \pi'_1(X_{\bar{s}}, \bar{x}) \to \pi'_1(X, \bar{x}) \to \pi_1(S, \bar{s}) \to 1 \]

is exact, but if the morphism \( f : X \to S \) admits a section, it becomes also left exact:

\[ 1 \to \pi'_1(X_{\bar{s}}, \bar{x}) \to \pi'_1(X, \bar{x}) \to \pi_1(S, \bar{s}) \to 1. \]
In this case, we obtain a monodromy action
\[ \rho_s : \pi_1(S, \bar{s}) \to \text{Out}(\pi_1^L(X, \bar{x})). \]

For the case where \( f : X \to S \) has no sections, we have the following result provided that \( S \) is locally noetherian.

**Proposition 4.3.** Suppose that \( S \) is a locally noetherian connected scheme, and that \( f : X \to S \) is a proper smooth morphism with geometrically connected fibers. If \( \bar{s} : \text{Spec} \, \Omega \to S \) is a geometric point of \( S \), and \( \bar{x} \) a geometric point of the geometric fiber \( X_{\bar{s}} \), then the sequence
\[ 1 \to \pi_1^L(X_{\bar{s}}, \bar{x}) \to \pi_1^L(X, \bar{x}) \to \pi_1(S, \bar{s}) \to 1 \]
is exact.

**Proof.** First we note that the sequence
\[ \pi_1(X_{\bar{s}}, \bar{x}) \to \pi_1(X, \bar{x}) \to \pi_1(S, \bar{s}) \to 1 \]
is exact [9, SGA 1, Exposé X], so that \( \pi_1(X_{\bar{s}}, \bar{x}) \) maps onto the kernel \( K \) of the canonical projection \( \pi_1(X, \bar{x}) \to \pi_1(S, \bar{s}) \). There is a commutative diagram
\[
\begin{array}{c}
1 \\
\downarrow \phi' \\
\downarrow \phi \\
1
\end{array}
\]
\[
\begin{array}{c}
\pi_1(X_{\bar{s}}, \bar{x}) \\
\downarrow \phi' \\
\downarrow \phi \\
\pi_1^L(X, \bar{x})
\end{array}
\]
\[
\begin{array}{c}
1 \\
\downarrow \phi'' \\
\downarrow \phi'' \\
1
\end{array}
\]
where the middle and right vertical maps are surjective, and \( N' \) is the kernel of the projection \( \pi_1(X_{\bar{s}}, \bar{x}) \to \pi_1^L(X_{\bar{s}}, \bar{x}) \). Since the middle map \( \phi \) is surjective, we see that \( \text{Ker}(\phi'') \) maps onto \( \text{Coker}(\phi') \). Consequently, \( \text{Coker}(\phi') \) is a pro-\( L \)-group. Thus, if \( \text{Coker}(\phi') \) is nontrivial, then \( N \) will admit a nontrivial finite \( L \)-quotient, contradicting the maximality of \( K^L \). Hence \( \phi' \) is surjective.

We claim now that the restriction to \( \text{Ker}(\phi) \) of the projection map \( \pi_1(X_{\bar{s}}, \bar{x}) \to \pi_1^L(X_{\bar{s}}, \bar{x}) \) is trivial. Consider the fiber product diagram
\[
\begin{array}{c}
X_{\bar{s}} \times_{\Omega} X_{\bar{s}} \\
\downarrow p_2 \\
X_{\bar{s}}
\end{array}
\]
\[
\begin{array}{c}
X_{\bar{s}} \\
\downarrow p_1 \\
X
\end{array}
\]
\[
\begin{array}{c}
\text{Spec} \, \Omega \\
\downarrow s \\
S
\end{array}
\]
where \( p_1, p_2 \) denote the 1st and 2nd projections, respectively, and \( s \) is the diagonal section. This diagram induces the commutative diagram of profinite groups
\[
\begin{array}{c}
1 \\
\downarrow \\
1 \\
\downarrow \\
1
\end{array}
\]
\[
\begin{array}{c}
\pi_1^L(X_{\bar{s}}, \bar{x}) \\
\downarrow \pi_1^L(X_{\bar{s}} \times_{\Omega} X_{\bar{s}}, s(\bar{x})) \\
\downarrow \pi_1(X_{\bar{s}} \times_{\Omega} X_{\bar{s}}, s(\bar{x})) \\
\downarrow \\
1
\end{array}
\]
\[
\begin{array}{c}
\pi_1^L(X_{\bar{s}}, \bar{x}) \\
\downarrow \pi_1(X \times_{\Omega} X, s(\bar{x})) \\
\downarrow \\
\pi_1(X, \bar{x}) \\
\downarrow \\
1
\end{array}
\]
where \( H \) is the kernel of the canonical map \( \pi_1(X_{\bar{s}}, \bar{x}) \to \pi_1(X, \bar{x}) \) and the first row is obtained by pulling back the middle exact sequence along the inclusion.
$H \to \pi_1(X_{\bar{s}}, \bar{x})$. The bottom two rows are exact by [9, SGA 1, Exposé XIII, 4.3, 4.4], and hence the right two squares are pullback squares. Note that $H$ is equal to $\text{Ker}(\phi)$. Denote also by $s_\ast$ the map $H \to \pi^1_1(X_{\bar{s}}, \bar{x}) \times H$ induced by the section $s$. By the commutativity of the diagram, $s_\ast(h) = (1, h)$ for all $h \in H$. Thus the composition

$$H \to \pi_1(X_{\bar{s}}, \bar{x}) \xrightarrow{\rho_1} \pi^1_1(X_{\bar{s}} \times_{\Omega} X_{\bar{s}}, s(\bar{x})) \xrightarrow{p_2} \pi^1_1(X_{\bar{s}}, \bar{x})$$

is trivial. Since $p_1 \circ s_\ast : \pi_1(X_{\bar{s}}, \bar{x}) \to \pi^1_1(X_{\bar{s}}, \bar{x})$ is equal to the canonical projection $\pi_1(X_{\bar{s}}, \bar{x}) \to \pi^1_1(X_{\bar{s}}, \bar{x})$, our claim holds. Therefore, $\text{Ker}(\phi''')$ is trivial by the Snake Lemma, and hence $\phi'''$ is an isomorphism.

\[\square\]

Suppose that $T$ is a locally noetherian connected scheme, and that $C \to T$ is a curve. Fix a prime number $\ell$ different from $\text{char}(T)$. Denote the maximal pro-$\ell$ quotient of $\pi_1(C_{\bar{n}}, \bar{x})$ by $\pi^1_1(C_{\bar{n}}, \bar{x})^{(1)}$. Then we have the exact sequence

$$1 \to \pi_1(C_{\bar{n}}, \bar{x})^{(1)} \to \pi^1_1(C, \bar{x}) \to \pi_1(T, \bar{n}) \to 1,$$

from which we obtain a natural monodromy action of $\pi_1(T, \bar{n})$ on $\text{Hom}(\pi_1(C_{\bar{n}}, \bar{x})^{(1)}, \mathbb{Z}_\ell(1)) \cong H^1_{et}(C_{\bar{n}}, \mathbb{Z}_\ell(1))$. Denote $H^1_{et}(C_{\bar{n}}, \mathbb{Z}_\ell(1))$ by $H_{Z\ell}$. The action of $\pi_1(T, \bar{n})$ respects the Weil pairing $\theta : \Lambda^2 H_{Z\ell} \to \mathbb{Z}_\ell(1)$. Hence we obtain a representation

$$\rho_{\bar{n}} : \pi_1(T, \bar{n}) \to \text{GSp}(H_{Z\ell}).$$

In particular, when $T$ is defined over a field $k$, we have the commutative diagram

$$\begin{array}{ccc}
\pi_1(T, \bar{n}) & \xrightarrow{\rho_{\bar{n}}} & \text{GSp}(H_{Z\ell}) \\
\downarrow & & \downarrow \\
G_k & \xrightarrow{\chi_{\ell}} & \mathbb{G}_m(\mathbb{Z}_\ell)
\end{array}$$

where the left-hand vertical map is the canonical projection, the right-hand vertical map $\tau$ is the natural surjection, and where $\chi_{\ell}$ is the $\ell$-adic cyclotomic character.

5. MODULI OF CURVES WITH A TEICHMÜLLER LEVEL STRUCTURE

Suppose that $C/T$ is a curve of type $(g, n)$. Let $L$ be the set of prime numbers distinct from $\text{char}(T)$. Associated to the curve $C/T$, there exists a pro-object $\pi^1_1(C'/T)$ of the category of locally constant sheaves of finite groups of order divisible by primes in $L$, where $C'/T$ is the curve obtained by removing the sections $s_1, \ldots, s_n$, see [6, §5]. This pro-object $\pi^1_1(C'/T)$ is a locally constant étale sheaf over $T$ such that each stalk $\pi^1_1(C'/T)_{\bar{n}}$ is isomorphic to the maximal pro-$L$ quotient of the fundamental group of the curve $C_{\bar{n}} - \{s_1(\bar{n}), \ldots, s_n(\bar{n})\}$. For a group $G$ whose order is divisible by primes in $L$, the sheaf of exterior homomorphisms

$$\mathcal{H}om^{ext}(\pi^1_1(C'/T), G)$$

is defined to be the quotient of the locally constant sheaf

$$\mathcal{H}om(\pi^1_1(C'/T), G)$$
by conjugation action of the sheaf $\pi_1^L(C'/T)$ on it. Then [6, 5.6] a Teichmüller structure $\alpha$ of level $G$ on the curve $C/T$ is a surjective exterior homomorphism

$$\alpha \in \Gamma(T, \mathcal{H}om(\pi_1^L(C'/T), G)).$$

5.1. Moduli stacks of curves with a non-abelian level structure. Suppose that $2g - 2 + n > 0$. Denote the Deligne-Mumford compactification [6] of $\mathcal{M}_{g,n}/\mathbb{Z}$ by $\overline{\mathcal{M}}_{g,n}/\mathbb{Z}$. Fix a prime number $\ell$. Finite étale coverings of $\mathcal{M}_{g,n}$ that are representable by a scheme and have a compactification that is smooth over $\text{Spec } \mathbb{Z}[1/\ell]$ are essential to our comparison between characteristic zero and positive characteristic. The existence of such coverings was established by

(i) de Jong and Pikaart for $n = 0$ and all $\ell$ in [20],
(ii) Boggi and Pikaart for $n > 0$ and odd $\ell$ in [4], and
(iii) Pikaart for $n > 0$ and $\ell = 2$ in [30].

Their results needed in this paper are summarized in the following statement:

**Proposition 5.1.** For all prime numbers $\ell$ and all $(g,n)$ satisfying $2g - 2 + n > 0$, there is a finite étale Galois covering $M \to \mathcal{M}_{g,n}[1/\ell] := \mathcal{M}_{g,n}/\mathbb{Z} \otimes \mathbb{Z}[1/\ell]$ over $\mathbb{Z}[1/\ell]$ that satisfies:

(i) $M$ is a separated scheme of finite type over $\mathbb{Z}[1/\ell]$;
(ii) the normalization $\overline{M}$ of $\mathcal{M}_{g,n}[1/\ell]$ with respect to $M$ is proper and smooth over $\mathbb{Z}[1/\ell]$;
(iii) the boundary $\overline{M} \setminus M$ is a relative normal crossing divisor over $\mathbb{Z}[1/\ell]$.

In fact, $M$ was taken to be the DM stack $G_\lambda \mathcal{M}_{g,n}/\mathbb{Z}[1/\ell]$ of curves of type $(g,n)$ with a Teichmüller structure of level $G$, where $G$ was specifically taken to be:

(i) the quotient of $\Pi_{g,0}$ by the normal subgroup generated by the third term of its lower central subgroup and all $\ell^k$th powers when $\ell$ is odd and $n = 0$;
(ii) the quotient of $\Pi_{g,0}$ by the normal subgroup generated by the fourth term of its lower central subgroup and all fourth powers when $\ell = 2$ and $n = 0$;
(iii) the quotient $\Pi_{g,n}/W^3\Pi_{g,n} \cdot \Pi_{g,n}^m$, where $W^3$ denotes the third term of the weight filtration of $\Pi_{g,n}$ defined in [4] when $\ell$ is odd and $n > 0$;
(iv) the quotient $\Pi_{g,n}/W^4\Pi_{g,n} \cdot \Pi_{g,n}^4$, where $W^4$ denotes the third term of the weight filtration of $\Pi_{g,n}$ defined in [4] when $\ell = 2$ and $n > 0$,

where $\Pi_{g,n}^k$ is the subgroup of $\Pi_{g,n}$ generated by all $k$th powers. In [6], $G$ is a finite quotient of $\Pi_{g,n}$ by a characteristic subgroup, but the same construction can be done when $G$ is a finite quotient of $\Pi_{g,n}$ by an invariant subgroup, see §5.4. For $n \geq 2$, the subgroups $W^k\Pi_{g,n} \cdot \Pi_{g,n}^k$ are not characteristic, but are invariant. For fixed prime numbers $p$ and $\ell \neq p$, denote by $M^\lambda_{g,n}$ or simply $M^\lambda$ the finite étale cover $M$ of $\mathcal{M}_{g,n}[1/\ell]$ given by the above proposition.

5.2. Moduli stacks of curves with an abelian level. When $G$ is a finite quotient by the subgroup $W^2\Pi_{g,n} \cdot \Pi_{g,n}^m$, we have $G \cong H_1(\Sigma_g, \mathbb{Z}/m\mathbb{Z})$, where $\Sigma_g$ is a closed oriented genus $g$ surface. In this case, we denote the moduli stack of $n$-pointed
smooth projective curves with the Teichmüller structure of level \( H_1(\Sigma_g, \mathbb{Z}/m/\mathbb{Z}) \) by \( \mathcal{M}_{g,n}[m] \). The stack \( \mathcal{M}_{g,n}[m] \) is representable by a scheme for \( m \geq 3 \) (See [2, Chapter XVI, Theorem 2.11]. It is well known that the Deligne-Mumford compactification \( \overline{\mathcal{M}_{g,n}[m]} \) is never smooth if \( g > 2 \).

5.3. Relative Pro-\( \ell \) Completion. The pro-\( \ell \) completion of a group \( \Gamma \) with \( H_1(\Gamma) \otimes \mathbb{F}_\ell = 0 \) is trivial. Thus the pro-\( \ell \) completions of the mapping class groups in genus at least 3 are trivial. On the other hand, their relative pro-\( \ell \) completions are large enough to give us the information of their structure. Here we recall from [17] the definition of and some basic facts about relative pro-\( \ell \) completion of a group. Suppose that:

(i) \( \Gamma \) is a discrete group or profinite group;
(ii) \( P \) is a profinite group;
(iii) \( \rho: \Gamma \to P \) is a continuous dense homomorphism.

**Definition 5.2.** The relative pro-\( \ell \) completion of \( \Gamma \) with respect to \( \rho \) consists of a profinite group \( \Gamma_{\text{rel}(\ell),\rho} \) and the natural homomorphisms \( \Gamma \to \Gamma_{\text{rel}(\ell),\rho} \) and \( \Gamma_{\text{rel}(\ell),\rho} \to P \) that make the diagram commute. It is characterized by the following universal mapping property: If \( G \) is a profinite group, \( \psi: G \to P \) a continuous homomorphism with pro-\( \ell \) kernel, and if \( \phi: \Gamma \to G \) is a continuous homomorphism whose composition with \( \psi \) is \( \rho \), then there is a unique continuous homomorphism \( \Gamma_{\text{rel}(\ell),\rho} \to G \) that makes the following diagram commute:

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{\rho} & P \\
\downarrow \ & & \downarrow \\
\Gamma_{\text{rel}(\ell),\rho} & \xrightarrow{\rho} & P \\
\end{array}
\]

When the context is clear, we will omit \( \rho \) from the notation and denote \( \Gamma_{\text{rel}(\ell),\rho} \) by \( \Gamma_{\text{rel}(\ell)} \).

The following propositions are the basic properties that are used in this paper.

**Proposition 5.3.** [17, Proposition 2.3](Naturality) Notations as in Definition 5.2. Suppose that \( \rho_j: \Gamma_j \to P_j \) for \( j = 1, 2 \) are continuous dense homomorphisms. If the diagram

\[
\begin{array}{ccc}
\Gamma_1 & \xrightarrow{\rho_1} & P_1 \\
\phi_\Gamma & \downarrow & \downarrow \phi_P \\
\Gamma_2 & \xrightarrow{\rho_2} & P_2 \\
\end{array}
\]

where \( \phi_\Gamma \) and \( \phi_P \) are continuous homomorphisms, commutes, then there is a unique continuous homomorphism \( \phi_{\text{rel}(\ell)} : \Gamma_{\text{rel}(\ell),\rho_1} \rightarrow \Gamma_{\text{rel}(\ell),\rho_2} \) that makes the diagram commute.

**Proof.** This follows from the universal mapping property. \( \Box \)

**Proposition 5.4.** [17, Proposition 2.1] A homomorphism \( \rho : \Gamma \rightarrow P \) from a discrete group to a profinite group induces a homomorphism \( \bar{\rho} : \Gamma^\wedge \rightarrow P \) from the profinite completion of \( \Gamma \) to \( P \). The natural homomorphism \( \Gamma \rightarrow \Gamma^\wedge \) induces a natural isomorphism \( \Gamma_{\text{rel}(\ell),\rho} \cong (\Gamma^\wedge)_{\text{rel}(\ell),\bar{\rho}} \).

**Proof.** By naturality, there is a homomorphism \( \phi : \Gamma_{\text{rel}(\ell),\rho} \rightarrow (\Gamma^\wedge)_{\text{rel}(\ell),\bar{\rho}} \). Since \( \Gamma_{\text{rel}(\ell),\rho} \) is a profinite group and the natural homomorphism \( \Gamma \rightarrow \Gamma_{\text{rel}(\ell),\rho} \) is continuous, it factors through \( \Gamma^\wedge \), and hence there is a homomorphism \( \psi : (\Gamma^\wedge)_{\text{rel}(\ell),\bar{\rho}} \rightarrow \Gamma_{\text{rel}(\ell),\rho} \). The universal mapping property implies that \( \phi \) and \( \psi \) are inverse to each other. \( \Box \)

**Proposition 5.5.** [17, Proposition 2.4] (Right exactness) Suppose that \( \rho_j : \Gamma_j \rightarrow P_j \) for \( j = 1, 2 \) and \( 3 \) are continuous dense homomorphisms as in the above definition. Suppose furthermore that the \( \Gamma_j \) are all discrete or all profinite groups. If the diagram

\[
1 \rightarrow \Gamma_1 \rightarrow \Gamma_2 \rightarrow \Gamma_3 \rightarrow 1,
\]

\[
1 \rightarrow P_1 \rightarrow P_2 \rightarrow P_3 \rightarrow 1
\]

where all the arrows are continuous and rows are exact, then the sequence

\( \Gamma_{\text{rel}(\ell),\rho_1} \rightarrow \Gamma_{\text{rel}(\ell),\rho_2} \rightarrow \Gamma_{\text{rel}(\ell),\rho_3} \rightarrow 1 \)

is exact.

**Proof.** See [17]. \( \Box \)

**Proposition 5.6.** [17, Lemma 2.6] Suppose that

\[
1 \rightarrow K \rightarrow P \xrightarrow{\rho} \mathcal{P} \rightarrow 1
\]

is a short exact sequence of profinite groups. Suppose that \( \rho : \Gamma \rightarrow P \) is a continuous dense homomorphism as in the above definition. Denote \( \psi \circ \rho \) by \( \bar{\rho} \). If \( K \) is a pro-\( \ell \)-group, then the natural homomorphism \( \Gamma_{\text{rel}(\ell),\rho} \rightarrow \Gamma_{\text{rel}(\ell),\bar{\rho}} \) is an isomorphism.
Proof: Since $K$ is a pro-$\ell$ group and the kernel of the natural homomorphism $\Gamma^{rel}(\ell,\rho) \rightarrow P$ is pro-$\ell$, the preimage $N$ of $K$ under the homomorphism $\Gamma^{rel}(\ell,\rho) \rightarrow P$ is also a pro-$\ell$ group, and hence, by the universal mapping property, there is a natural homomorphism $\Gamma^{rel}(\ell,\rho) \rightarrow \Gamma^{rel}(\ell,\tilde{\rho})$, which is an inverse of the natural homomorphism $\Gamma^{rel}(\ell,\rho) \rightarrow \Gamma^{rel}(\ell,\rho)$. □

Example 5.7. Let $\Gamma$ be a finite index normal subgroup of the mapping class group $\Gamma_{g,n}$. Denote by $\Gamma^\wedge_{g,n}$ and $\Gamma^\wedge$ the profinite completion of $\Gamma_{g,n}$ and $\Gamma$, respectively. Let $\rho : \Gamma^\wedge \rightarrow \text{Sp}(\mathbb{Z}_\ell)$ be the homomorphism obtained by composing with the standard representation $\Gamma_{g,n} \rightarrow \text{Sp}(\mathbb{Z}/\ell\mathbb{Z})$. Suppose that $\psi : \text{Sp}(\mathbb{Z}_\ell) \rightarrow \text{Sp}(\mathbb{Z}/\ell\mathbb{Z})$ is reduction mod $\ell$. Suppose that $\tilde{\rho}$ is trivial. Since $\ker \psi$ is a pro-$\ell$ group, there are natural isomorphisms $\Gamma^{rel}(\ell,\rho) \cong \Gamma^{rel}(\ell,\tilde{\rho})$.

5.4. Fundamental Groups of Finite Étale Covers of Moduli Stacks of Curves. Suppose that $g$ and $n$ are non-negative integers satisfying $2g - 2 + n > 0$. Fix a closed oriented genus $g$ surface $\Sigma_g$ and a finite subset $P = \{p_1, p_2, \ldots, p_n\}$ of $n$ distinct points in $\Sigma_g$. Denote the mapping class group of $(\Sigma_g, P)$ by $\Gamma_{\Sigma_g, P}$. This is defined to be the group of isotopy classes of orientation preserving homeomorphisms which fix $P$ pointwise. By the classification of surfaces, the homeomorphism class of $(\Sigma_g, P)$ depends only on $(g, n)$. Therefore, the group $\Gamma_{\Sigma_g, P}$ depends only on the pair $(g, n)$, and thus it is denoted by $\Gamma_{g,n}$. Denote the complement $\Sigma_g - P$ of $P$ in $\Sigma_g$ by $\Sigma_{g,n}$. Denote the topological fundamental group $\pi_1^{top}(\Sigma_{g,n}, *)$ of $\Sigma_{g,n}$ by $\Pi_{g,n}$. The standard presentation of $\Pi_{g,n}$ is

$$\Pi_{g,n} = \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \gamma_1, \ldots, \gamma_n | [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] \gamma_1 \cdots \gamma_n = 1 \rangle.$$ 

Note that $\Pi_{g,0} = \Pi_{g,n}/\langle \gamma_1, \ldots, \gamma_n \rangle$. The geometric automorphisms of $\Pi_{g,n}$ are defined to be the ones that fix the conjugacy class of every $\gamma_i$ and induce the identity on $H_2(\Pi_{g,0}, \mathbb{Z})$. Denote the group of geometric automorphisms of $\Pi_{g,n}$ by $\text{Aut}_{g,n}$ and the group of the inner automorphisms of $\Pi_{g,n}$ by $I_{g,n}$. $I_{g,n}$ is clearly a normal subgroup of $\text{Aut}_{g,n}$. It is well known that there is a canonical isomorphism

$$\Gamma_{g,n} \cong \text{Aut}_{g,n}/I_{g,n}.$$ 

(See [33, Theorem V.9]). The invariant subgroups of $\Pi_{g,n}$ are defined to be the ones that are stable under the action of $\text{Aut}_{g,n}$. For an invariant subgroup $K$ of $\Pi_{g,n}$, there is a natural representation

$$\Gamma_{g,n} \rightarrow \text{Out}(\Pi_{g,n}/K).$$

This representation is the key for the construction of $M^\wedge$

Let $k$ be a field of characteristic 0. For simplicity, assume that $k$ is contained in $\mathbb{C}$ and denote the algebraic closure of $k$ in $\mathbb{C}$ by $\bar{k}$. The moduli stack $\mathcal{M}_{g,n/\mathbb{C}}$ can be viewed as a complex analytic orbifold denoted by $\mathcal{M}_{g,n/\mathbb{C}}^{an}$. Denote the orbifold fundamental group of $\mathcal{M}_{g,n/\mathbb{C}}^{an}$ by $\pi_1^{orb}(\mathcal{M}_{g,n/\mathbb{C}}^{an}, \bar{\eta})$ with base point $\bar{\eta} \in \mathcal{M}_{g,n}(\mathbb{C})$. There is a natural isomorphism

$$\pi_1^{orb}(\mathcal{M}_{g,n/\mathbb{C}}, \bar{\eta}) \cong \Gamma_{g,n}.$$ 

Therefore, for each geometric point $\bar{\eta}$ of $\mathcal{M}_{g,n/k}$, there is an isomorphism

$$\pi_1(\mathcal{M}_{g,n/k}, \bar{\eta}) \cong \Gamma^\wedge_{g,n}.$$
which is uniquely determined up to inner automorphisms, and there is an exact sequence
\[ 1 \to \Gamma_{g,n} \to \pi_1(M_{g,n/L}, \bar{\eta}) \to \text{Gal}(\bar{k}/k) \to 1. \]

Let \( k \) be an algebraically closed field of characteristic \( p > 0 \). Denote the ring of \( p \)-adic Witt vectors over \( k \) by \( W(k) \). When \( k \) is clear from context, we denote \( W(k) \) by \( W \). It is a characteristic zero complete discrete valuation ring with the residue field \( k \). Fix an algebraic closure \( L \) of the fraction field of \( W(k) \). There is an isomorphism \( \Gamma_{g,n} \cong \pi_1(M_{g,n/L}, \bar{\eta}) \) of the geometric fundamental group of \( M_{g,n/L} \) with the profinite completion of the mapping class group \( \Gamma_{g,n} \). Fix a prime number \( \ell \neq p \). Let \( G = \Pi_{g,n}/W^3\Pi_{g,n} \cdot \Pi_{g,n}^{(\infty)} \) for odd \( \ell \) and \( G = \Pi_{g,n}/W^4\Pi_{g,n} \cdot \Pi_{g,n}^1 \) for \( \ell = 2 \), where the filtration \( W^* \) is defined in §5.1. Let \( M^\lambda \) be a finite étale cover of \( M_{g,n}[1/\ell] \) as in Proposition 5.1. Denote the kernel of the natural representation \( \Gamma_{g,n} \to \text{Out}(G) \) by \( \Gamma_{g,n}^\lambda \). Denote the Teichmüller space of the reference surface \( \Sigma_{g,n} \) by \( \mathcal{T}_{g,n} \). By construction, each connected component of the complex variety \( M^\lambda \otimes \mathbb{C} \) is isomorphic to the analytic space \( \mathcal{T}_{g,n}/\Gamma_{g,n}^\lambda \). Since \( \Gamma_{g,n}^\lambda \) acts on \( \mathcal{T}_{g,n} \) freely, we see that there is a natural conjugacy class of isomorphisms
\[ \pi_1(M^\lambda_{C}) \cong (\Gamma_{g,n}^\lambda)^\wedge, \]
where \( M^\lambda_{C} \) is a connected component of \( M^\lambda \otimes \mathbb{C} \). Since \( \ell \) is a unit in \( W \), there is a natural morphism \( \text{Spec} W \to \text{Spec} \mathbb{Z}[1/\ell] \). Choose a connected component of \( M^\lambda \otimes \mathbb{Z}[1/\ell] \) \( W \) and denote it by \( M^\lambda_{W} \). Denote its base changes to \( L \) and \( k \) by \( M^\lambda_{L} \) and \( M^\lambda_{k} \), respectively. Let \( \bar{\eta} \) and \( \bar{\xi} \) be a geometric point of \( M^\lambda_{L} \) and \( M^\lambda_{k} \), respectively. The scheme \( M^\lambda_{L} \) is a connected finite étale cover of \( M_{g,n/L} \) and there is an isomorphism \( \pi_1(M^\lambda_{L}, \bar{\eta}) \cong (\Gamma_{g,n}^\lambda)^\wedge \). Since the boundary of \( M^\lambda_{C} \) is a relative normal crossing divisor over \( \mathbb{Z}[1/\ell] \), the boundary of the Zariski closure of \( M^\lambda_{W} \) in \( M^\lambda \otimes \mathbb{C} \) is also a relative normal crossing divisor over \( W \). This allows us to define a specialization homomorphism of tame fundamental groups [9, Exposé XIII]
\[ sp : \pi_1(M^\lambda_{L}, \bar{\eta}) \to \pi_1(M^\lambda_{W}, \bar{\eta}) \cong \pi_1(M^\lambda_{W}, \bar{\xi}) \cong \pi_1(M^\lambda_{k}, \bar{\xi}), \]
where the left-hand map is induced by base change to \( L \), the map at middle is an isomorphism obtained by change of base points, and the right-hand map is the isomorphism induced by base change to \( k \).

**Theorem 5.8.** With notations as above, there is an isomorphism
\[ (\Gamma_{g,n}^\lambda)^{(\ell)} \cong \pi_1(M^\lambda_{k}, \bar{\xi})^{(\ell)}, \]
which is uniquely determined up to inner automorphisms.

**Proof.** The smoothness of \( M^\lambda_{W} \) over \( W \) implies that the specialization morphism \( sp \) is surjective. This surjective homomorphism induces an isomorphism
\[ sp^{(\ell)} : \pi_1(M^\lambda_{L}, \bar{\eta})^{(\ell)} \cong \pi_1(M^\lambda_{k}, \bar{\xi})^{(\ell)} \]
only upon taking maximal prime-to-\( p \) quotient. Hence we have an isomorphism
\[ sp^{(\ell)} : \pi_1(M^\lambda_{L}, \bar{\eta})^{(\ell)} \cong \pi_1(M^\lambda_{k}, \bar{\xi})^{(\ell)} \]
by taking maximal pro-\( \ell \) quotient.

\[ \square \]
Corollary 5.9. With notations as above, there are natural conjugacy classes of isomorphisms
\[
(\Gamma_{g,n}(\ell^n))(t) \cong \pi_1(\mathcal{M}_{g,n/k}[\ell^n])^{(t)}
\]
and
\[
\Gamma_{g,n}^{rel(t)} \cong \pi_1(\mathcal{M}_{g,n/k})^{rel(t)}.
\]

Proof. For \( A = L, W, \) and \( k, \) denote \( \mathcal{M}_{g,n/A} \) and \( \mathcal{M}_{g,n/A}[\ell^n] \) by \( \mathcal{M}_A \) and \( \mathcal{M}_A[\ell^n] \), respectively. Let \( \bar{\eta} \) and \( \bar{\xi} \) be geometric points of \( \mathcal{M}_k^\lambda \) and \( \mathcal{M}_k^\lambda \), respectively. Denote the images of \( \bar{\eta} \) and \( \bar{\xi} \) under morphisms by \( \bar{\eta} \) and \( \bar{\xi} \) also. The monodromy action \( \pi_1(\mathcal{M}_A)^{rel(t)} \to \text{Sp}(\mathbb{Z}/\ell\mathbb{Z}) \) factors through the finite group \( \Gamma_{g,n}/\Gamma_{g,n}^\lambda \), which is the automorphism group of \( \mathcal{M}_A^\lambda \) over \( \mathcal{M}_A \). Denote this finite group by \( G \). This implies that for \( A = W \) and \( A = k, \) there is an exact sequence
\[
1 \to \pi_1(M_A^\lambda, \bar{\xi})^{(t)} \to \pi_1(M_A^\lambda, \bar{\xi})^{rel(t)} \to G \to 1.
\]
Similarly, for \( A = L \) and \( A = W, \) there is an exact sequence
\[
1 \to \pi_1(M_A^\lambda, \bar{\eta})^{(t)} \to \pi_1(M_A^\lambda, \bar{\eta})^{rel(t)} \to G \to 1.
\]

Fix an isomorphism \( \pi_1(M_W^\lambda, \bar{\xi}) \cong \pi_1(M_W^\lambda, \bar{\eta}) \). These exact sequences fit into the commutative diagram
\[
\begin{array}{ccccccccc}
1 & \to & \pi_1(M_k^\lambda, \bar{\xi})^{(t)} & \to & \pi_1(M_k^\lambda, \bar{\xi})^{rel(t)} & \to & G & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \pi_1(M_L^\lambda, \bar{\xi})^{(t)} & \to & \pi_1(M_L^\lambda, \bar{\xi})^{rel(t)} & \to & G & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \pi_1(M_W^\lambda, \bar{\eta})^{(t)} & \to & \pi_1(M_W^\lambda, \bar{\eta})^{rel(t)} & \to & G & \to & 1 \\
\end{array}
\]
where the left-hand vertical maps are all isomorphisms and the map \( G \to G \) is an isomorphism induced by the fixed isomorphism \( \pi_1(M_W^\lambda, \bar{\xi}) \cong \pi_1(M_W^\lambda, \bar{\eta}) \). Therefore, the middle vertical maps are all isomorphisms and thus there are isomorphisms
\[
\pi_1(M_k^\lambda, \bar{\xi})^{rel(t)} \cong \pi_1(M_L^\lambda, \bar{\eta})^{rel(t)} \cong \Gamma_{g,n}^{rel(t)},
\]
which are unique up to conjugation by elements of \( \pi_1(M_k^\lambda, \bar{\xi})^{rel(t)} \). Similarly, let \( G' \) be the quotient of \( \pi_1(M_A[\ell^n]) \) by the finite index subgroup \( \pi_1(M_A^\lambda) \). It is a finite \( \ell \)-group. Using the exact sequences
\[
1 \to \pi_1(M_A^\lambda, \bar{\xi})^{(t)} \to \pi_1(M_A[\ell^n], \bar{\xi})^{(t)} \to G' \to 1,
\]
where \( A = W \) and \( A = k \), and
\[
1 \to \pi_1(M_A^\lambda, \bar{\eta})^{(t)} \to \pi_1(M_A[\ell^n], \bar{\eta})^{(t)} \to G' \to 1,
\]
where \( A = L \) and \( W \), we also have isomorphisms
\[
\pi_1(M_A[\ell^n], \bar{\xi})^{(t)} \cong \pi_1(M_L[\ell^n], \bar{\eta})^{(t)} \cong \Gamma_{g,n}[\ell^n]^{rel(t)},
\]
which are unique up to conjugation by elements of \( \pi_1(M_k[\ell^n], \bar{\xi})^{(t)} \).

\[\square\]
6. Review of Relative Completion

6.1. Relative completion of a discrete group. We recall the definition of relative completion. Suppose that:

(i) $\Gamma$ is a discrete group;
(ii) $R$ is a reductive algebraic group defined over $F$, where $F$ is a field of characteristic zero;
(iii) $\rho : \Gamma \to R(F)$ is a homomorphism with Zariski dense image.

**Definition 6.1.** The relative completion of $\Gamma$ with respect to $\rho$ consists of a proalgebraic $F$-group $G$, that is an extension $1 \to U \to G \to R \to 1$ where $U$ is a prounipotent $F$-group and a Zariski dense homomorphism $\tilde{\rho} : \Gamma \to G(F)$ whose composition with $G(F) \to R(F)$ is $\rho$. It is characterized by the following universal mapping property: If $G$ is an affine (pro)algebraic $F$-group that is an extension $1 \to U \to G \to R \to 1$ of $R$ by a (pro)unipotent group $U$, and if $\phi : \Gamma \to G(F)$ is a homomorphism whose composition with $G(F) \to R(F)$ is $\rho$, then there is a unique homomorphism of proalgebraic $F$-groups $\Phi : G \to G$ that commutes with the projections to $R$ and such that $\phi = \Phi(F) \circ \tilde{\rho}$:

$$
\begin{array}{ccc}
\Gamma & \xrightarrow{\tilde{\rho}} & G(F) \\
\phi \downarrow & & \Phi \downarrow \\
G(F) & \xrightarrow{\Phi} & R(F)
\end{array}
$$

For the construction of relative completion, see [15, §2].

An important property of relative completion is that it behaves well under base change. Suppose that $E$ is an extension of $F$. By extending scalars to $E$, every proalgebraic group $G$ over $F$ yields a proalgebraic group $G \otimes_F E$. Assume that the image of $\Gamma \to R(E) = (R \otimes_F E)(E)$ is Zariski dense in $R \otimes_F E$. By the universal mapping property of the relative completion $G_E$ of $\Gamma$ with respect to $\rho : \Gamma \to R(E)$, we obtain a natural homomorphism $G_E \to G \otimes_F E$.

**Theorem 6.2.** The natural homomorphism $G_E \to G \otimes_F E$ is an isomorphism.

When $R$ is the trivial group, one has $G = U$, which is a prounipotent group, and the pair $(U, \Gamma \to U)$ is called the unipotent completion of $\Gamma$ over $F$. It will be denoted by $\Gamma^un_{/F}$.

6.2. Relative completion of $\Gamma_{g,n}^\lambda$. Suppose $2g-2+n > 0$. Let $H_A = H_1(\Sigma_g, A)$, $\Gamma = \Gamma_{g,n}^\lambda$, and $R = \text{Sp}(H_3)$, where $H_A = H_1(\Sigma_g, A)$ is the first homology group of the compact reference surface $\Sigma_g$. Let $\rho : \Gamma_{g,n} \to \text{Sp}(H_3)$ be the representation of the mapping class group on the first homology of the surface. Since the image of $\rho$ is $\text{Sp}(H_2)$, $\rho$ is a Zariski dense representation. Denote by $G_{g,n}^{\text{geom}}$ the relative...
completion of $\Gamma_{g,n}$ with respect to $\rho$ and by $U_{g,n}^{geom}$ its prounipotent radical. Note that the base change theorem above implies that the relative completion of $\Gamma_{g,n}$ with respect to $\rho : \Gamma_{g,n} \to \text{Sp}(H_{Q_{\ell}})$ is $G_{g,n}^{geom} \otimes_{Q} Q_{\ell}$.

Let $\ell \geq 3$ be an odd prime number. Recall that $\Gamma_{g,n}^\lambda$ is the kernel of the natural representation $\Gamma_{g,n} \to \text{Out}(G)$, where $G = \Pi_{g,n}/W^3 \Pi_{g,n} \cdot \Pi_{g,n}^\ell$. The following theorem follows from [12, Cor. 6.7].

**Theorem 6.3.** Suppose that $g \geq 3$ and $n \geq 0$. The completion of $\Gamma_{g,n}^\lambda$ relative to the restriction of the standard representation $\rho : \Gamma_{g,n} \to \text{Sp}(H_{Q_{\ell}})$ is isomorphic to $G_{g,n}^{geom}$.

### 6.3. Continuous relative completion of a profinite group.

We will need the profinite analogue of relative completion, since our main objects are profinite groups. Here we take the coefficient field $F$ to be the field $Q_{\ell}$ for some prime number $\ell$.

**Definition 6.4.** Suppose that:

(i) $\Gamma$ is a profinite group;
(ii) $R$ is a reductive algebraic group defined over $Q_{\ell}$;
(iii) $\rho : \Gamma \to R(Q_{\ell})$ is a continuous homomorphism with Zariski-dense image.

The continuous relative completion of $\Gamma$ with respect to $\rho$ is a proalgebraic $Q_{\ell}$-group $\hat{\mathcal{G}}$ that is an extension

$$1 \to \mathcal{U} \to \mathcal{G} \to R \to 1$$

of $R$ by a prounipotent $Q_{\ell}$-group $\mathcal{U}$ and a continuous Zariski dense homomorphism $\tilde{\rho} : \Gamma \to \mathcal{G}(Q_{\ell})$ which lifts $\rho$ to $\mathcal{G}(Q_{\ell})$.

Like the relative completion of a discrete group, the continuous completion of a profinite group is also characterized by a universal mapping property that is the same as one for the discrete case except that all homomorphisms are required to be continuous in the $\ell$-adic profinite case.

Denote the profinite completion of a discrete group $\Gamma$ by $\Gamma^\wedge$. Consider $\Gamma$ as a topological group whose neighbourhoods of the identity are defined to be the finite index normal subgroups. We have the following theorem:

**Theorem 6.5 ([14, Thm. 6.3]).** Suppose that $\Gamma$ is a discrete group, $R$ is a reductive $Q_{\ell}$-group, and that $\rho : \Gamma \to R(Q_{\ell})$ is a continuous, Zariski dense representation. Let $\rho_{\ell} : \Gamma^\wedge \to R(Q_{\ell})$ be the continuous extension of $\rho$ to $\Gamma^\wedge$. If $\mathcal{G}$ and $\tilde{\rho} : \Gamma \to \mathcal{G}(Q_{\ell})$ is the completion of $\Gamma$ with respect to $\rho$, then:

(i) $\tilde{\rho}$ is continuous and thus induces a continuous homomorphism $\tilde{\rho}_{\ell} : \Gamma^\wedge \to \mathcal{G}(Q_{\ell})$;
(ii) $\mathcal{G}$ and $\tilde{\rho}_{\ell}$ is the continuous relative completion of $\Gamma^\wedge$ with respect to $\rho_{\ell}$.

Suppose that $\Gamma$ is a profinite group and that $\rho : \Gamma \to R(Z_{\ell})$ is a continuous homomorphism such that the composition with the inclusion $R(Z_{\ell}) \to R(Q_{\ell})$ has Zariski dense image. Recall that $\rho_{\ell}^{rel(\ell)} : \Gamma^{rel(\ell)\rho} \to R(Z_{\ell})$ is the relative pro-$\ell$ completion of $\Gamma$ with respect to $\rho$. Since $\Gamma \to \Gamma^{rel(\ell)\rho}$ is surjective, $\rho_{\ell}^{rel(\ell)} : \Gamma^{rel(\ell)\rho} \to R(Q_{\ell})$ has Zariski dense image.
Proposition 6.6. The continuous relative completion of $\Gamma^{rel(\ell),\rho}$ with respect to the homomorphism $\rho^{rel(\ell)} : \Gamma^{rel(\ell),\rho} \to R(\mathbb{Q}_\ell)$ is isomorphic to the continuous relative completion $\mathcal{G}$ of $\Gamma$ with respect to $\rho$.

Proof. Denote $\Gamma^{rel(\ell),\rho}$ by $\Gamma^{rel(\ell)}$. Denote the relative completion of $\Gamma^{rel(\ell)}$ with respect to the homomorphism $\rho^{rel(\ell)} : \Gamma^{rel(\ell)} \to R(\mathbb{Q}_\ell)$ by $G^{rel(\ell)}$. By functoriality, there exists a map $\phi : \Gamma^{rel(\ell)} \to G^{rel(\ell)}$. Denote the image of $\Gamma$ in $G^{(\mathbb{Q}_\ell)}$ by $N$. The kernel $K$ of $N \to R(\mathbb{Z}_\ell)$ lies in $U(\mathbb{Q}_\ell)$ and is pro-$\ell$. Thus, by the universal mapping property of $\Gamma^{rel(\ell)}$, there exists a unique continuous homomorphism $\Gamma^{rel(\ell)} \to N$ such that the diagram

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{\rho} & R(\mathbb{Q}_\ell) \\
\downarrow & & \downarrow \\
\Gamma^{rel(\ell)} & \xrightarrow{\rho^{rel(\ell)}} & R(\mathbb{Z}_\ell) \\
\end{array}
\]

commutes. Now, the universal mapping property of relative completion gives a map $\psi : G^{rel(\ell)} \to \mathcal{G}$, and it ensures that the maps $\phi$ and $\psi$ are inverse to each other. □

7. Review of Weighted Completion

7.1. Negatively weighted extensions. Suppose that $F$ is a field of characteristic 0, that $R$ is a reductive algebraic group defined over $F$, and that $w : \mathbb{G}_m \to R$ is a central cocharacter. Denote $\mathbb{G}_m/F$ by $\mathbb{G}_m$. Suppose that

\[
1 \to U \to G \to R \to 1
\]

is an extension of $R$ by a unipotent group $U$ in the category of algebraic $F$-groups. The abelianization $H_1(U)$ is an $R$-module, and therefore a $\mathbb{G}_m$-module via $w$. Thus we have the decomposition

\[
H_1(U) = \bigoplus_{r \in \mathbb{Z}} H_1(U)_r,
\]

where $\mathbb{G}_m$ acts on $H_1(U)_r$ via the $r$th power of its defining representation. We will say that this extension is negatively weighted with respect to $w$ if $H_1(U)_r = 0$ for all $r \geq 0$ and that a proalgebraic group $\mathcal{G}$ which is an extension of $R$ by a pronipotent group $U$ is negatively weighted if it is an inverse limit of negatively weighted extensions of $R$ by unipotent groups.

By the Levi decomposition, the extension

\[
1 \to U \to G \to R \to 1
\]

splits and any two splittings differ by conjugation by an element of $U$. Therefore there is a lift of the homomorphism $\mathbb{G}_m \to R$ to a homomorphism $\tilde{\omega} : \mathbb{G}_m \to \mathcal{G}$, and any two liftings are conjugate by an element of $U$. Fixing a lift $\tilde{\omega}$ of $\omega$, we can regard each finite-dimensional $\mathcal{G}$-module $V$ as a $\mathbb{G}_m$-module and have a decomposition

\[
V = \bigoplus_{n \in \mathbb{Z}} V_n.
\]
Define the weight filtration of $V$ by

$$W_n V := \bigoplus_{m \leq n} V_m.$$  

Although, by definition, this weight filtration depends on the choice of the lift $\tilde{\omega}$, it does not, and each $W_n V$ is a $G$-module [18, 3.8]. Main properties of negatively weighted extension used in this paper are summarized as the following.

**Proposition 7.1.** [18, Thms. 3.9 & 3.12] Suppose that $R$ is a reductive $F$-group and that $w : G_m \to R$ is a central cocharacter. If $G$ is a proalgebraic group that is a negatively weighted extension of $R$ with respect to $w$ by a prounipotent group, then every finite dimensional $G$-module $V$ has a natural weight filtration $W_\bullet$:

$$0 = W_n V \subset \cdots \subset W_{r-1} V \subset W_r V \subset \cdots \subset W_m V = V.$$  

It is characterized by the property that the action of $G$ on the $r$th weight graded quotient

$$\operatorname{Gr}^W_r V := W_r V / W_{r-1} V$$  

factors through $G \to R$ and is an $R$-module of weight $r$. The weight filtration is preserved by $G$-module homomorphisms and the functor $\operatorname{Gr}^W_\bullet$ on the category of finite-dimensional $G$-modules is exact.

Denote the Lie algebra of $G$, $R$, and $U$ by $\mathfrak{g}$, $\mathfrak{r}$, and $u$, respectively.

**Proposition 7.2.** [18, Prop. 4.5] The Lie algebras $\mathfrak{g}$, $\mathfrak{r}$, and $u$ admit natural weight filtrations $W_\bullet$ that satisfy

$$\mathfrak{g} = W_0 \mathfrak{g}, \quad u = W_{-1} \mathfrak{g}, \quad \mathfrak{r} = \operatorname{Gr}^W_0 \mathfrak{g}.$$  

7.2. **Weighted completion of a profinite group.** Weighted completion of a profinite group $\Gamma$ is similar to continuous relative completion. It plays an essential role in [14]. A key property of weighted completion is that it induces weight filtrations with strong exactness properties on the $\Gamma$-representations that factor through its weighted completion. Here we take $F$ to be $\mathbb{Q}_\ell$, where $\ell$ is a prime number. Denote $G_m / \mathbb{Q}_\ell$ by $G_m$. Suppose that:

(i) $\Gamma$ is a profinite group;
(ii) $R$ is a reductive algebraic group defined over $\mathbb{Q}_\ell$;
(iii) $w : G_m \to R$ is a central cocharacter;
(iv) $\rho : \Gamma \to R(\mathbb{Q}_\ell)$ is a continuous homomorphism with Zariski dense image.

**Definition 7.3** ([18, §4]). The weighted completion of $\Gamma$ with respect to $\rho$ and $w$ consists of a proalgebraic $\mathbb{Q}_\ell$-group $\mathcal{G}$, that is a negatively weighted extension

$$1 \to U \to \mathcal{G} \to R \to 1$$  

where $U$ is a prounipotent $\mathbb{Q}_\ell$-group and a continuous Zariski dense homomorphism $\bar{\rho} : \Gamma \to G(\mathbb{Q}_\ell)$ whose composition with $G(\mathbb{Q}_\ell) \to R(\mathbb{Q}_\ell)$ is $\rho$. It is characterized by the following universal mapping property: If $G$ is an affine (pro)algebraic $\mathbb{Q}_\ell$-group that is a negatively weighted extension

$$1 \to U \to G \to R \to 1$$  

of $R$ (with respect to $w$) by a (pro)unipotent group $U$, and if $\phi : \Gamma \to G(\mathbb{Q}_\ell)$ is a continuous homomorphism whose composition with $G(\mathbb{Q}_\ell) \to R(\mathbb{Q}_\ell)$ is $\rho$, then there
is a unique homomorphism of proalgebraic $\mathbb{Q}_\ell$-groups $\Phi : \mathcal{G} \rightarrow G$ that commutes with the projections to $R$ and such that $\phi = \Phi \circ \tilde{\rho}$:

$$
\begin{array}{c}
\Gamma \\
\downarrow \tilde{\rho} \\
G
\end{array}
\xrightarrow{\phi}
\begin{array}{c}
\phi \\
\downarrow \\
G \rightarrow R
\end{array}
$$

Suppose that $V$ is a finite-dimensional $R$-representation. $V$ can be decomposed as $V = \bigoplus_{n \in \mathbb{Z}} V_n$ under the $\mathbb{G}_m$-action through $\omega$. We say that $V$ is pure of weight $n$ if $V = V_n$, and that $V$ is negatively weighted if $V_n = 0$ for all $n \geq 0$. $V$ can be considered as a continuous $\Gamma$-module via the homomorphism $\rho : \Gamma \rightarrow R(\mathbb{Q}_\ell)$. Denote by $H^*_{cts}(\Gamma, V)$ the continuous cohomology of $\Gamma$ with coefficients in $V$.

**Proposition 7.4** ([18, Thms. 4.6 & 4.9]). For all finite-dimensional irreducible $R$-representations $V$ of weight $r$, there are natural isomorphisms

$$
\text{Hom}_R(H^1_{cts}(u), V) \cong \text{Hom}_R(G^W_1, H^1_{cts}(u), V) \cong \begin{cases} H^1_{cts}(\Gamma, V) & r < 0 \\ 0 & r \geq 0 \end{cases}
$$

and a natural injection $\text{Hom}_R(H^2_{cts}(u), V) \hookrightarrow H^2_{cts}(\Gamma, V)$ for $r \leq -2$, and $\text{Hom}_R(H^r_{cts}(u), V) = 0$ for $r > -2$.

### 7.3. An exactness criterion.
Relative and weighted completion are, in general, only right exact functors. Suppose that

$$
1 \rightarrow \pi \rightarrow \Gamma_2 \xrightarrow{\phi} \Gamma_1 \rightarrow 1
$$

is an exact sequence of profinite groups, that $R$ is a reductive $\mathbb{Q}_\ell$-group with a central cocharacter $\omega : \mathbb{G}_m \rightarrow R$, and that $\rho : \Gamma_1 \rightarrow R(\mathbb{Q}_\ell)$ is a continuous Zariski dense representation. Denote by $\mathcal{G}_1$ and $\mathcal{G}_2$ the weighted completions of $\Gamma_1$ and $\Gamma_2$ with respect to $\rho$ and $\rho \circ \phi$, respectively. Denote by $\mathcal{P}$ the continuous $\ell$-adic unipotent completion of $\pi$, see [18, A. 2].

**Proposition 7.5.** [14, 6.11] Suppose that $\mathcal{H}_1(\mathcal{P})$ is finite-dimensional, that the action of $\Gamma_1$ on $\mathcal{H}_1(\mathcal{P})$ induces a $\mathcal{G}_1$-action on $\mathcal{H}_1(\mathcal{P})$, and that the weight filtration induced on $\mathcal{H}_1(\mathcal{P})$ has finite dimensional graded quotients which vanish in weights $r \geq 0$. If $\mathcal{P}$ has trivial center, then the sequence of completions

$$
1 \rightarrow \mathcal{P} \rightarrow \mathcal{G}_2 \rightarrow \mathcal{G}_1 \rightarrow 1
$$

is exact.

### 8. Weighted Completion and Families of Curves

Suppose that $k$ is a field, that $T$ is a locally noetherian geometrically connected scheme over $k$, and that $C \rightarrow T$ is a curve of genus $g \geq 2$. Fix an algebraic closure $\overline{k}$ of $k$. Denote the base change to $\overline{k}$ of $C$ and $T$ by $C \otimes_k \overline{k}$ and $T \otimes_k \overline{k}$, respectively. Let $\overline{\eta} : \text{Spec} \Omega \rightarrow T \otimes_k \overline{k}$ be a geometric point of $T \otimes_k \overline{k}$. By abuse of notation, $\overline{\eta}$ also denotes the image of $\overline{\eta}$ in $T$. Denote the geometric fiber of $C \otimes_k \overline{k}$ over $\overline{\eta}$ by $C_{\overline{\eta}}$. Let $\overline{x}$ be a geometric point of the fiber $C_{\overline{\eta}}$. The images of $\overline{x}$ in $C \otimes_k \overline{k}$ and $C$ are also denoted by $\overline{x}$. Fix a prime number $\ell$ that is different from $\text{char}(k)$. In this

section, \(H_{Z_t} = H^1_{Z_t}(C, Z_t(1))\) and \(H_{Q_1} = H_{Z_t} \otimes Q_1\). Let \(R\) be the Zariski closure of the image of the natural monodromy representation

\[ \rho_{T, \eta} : \pi_1(T, \eta) \to \text{GSp}(H_{Q_1}). \]

Assuming that \(R\) contains the homotheties, we have the central cocharacter defined by

\[ \omega : \mathbb{G}_m \to R \quad z \mapsto z^{-1}\text{id}_H, \]

which we call the standard cocharacter.

**Lemma 8.1.** The monodromy representation \(\pi_1(C, \bar{x}) \to \text{GSp}(H_{Q_1})\) factors through \(\pi_1(T, \eta)\).

**Proof.** This follows immediately from the existence of the commutative diagram

\[
\begin{array}{ccc}
\pi_1(C, \bar{x}) & \to & \pi_1(T, \eta) \\
\downarrow & & \downarrow \\
1 & \to & \text{Inn}(\Pi^{(\ell)}) \\
\end{array}
\]

where \(\Pi^{(\ell)}\) denotes the maximal pro-\(\ell\) quotient \(\pi_1(C, \bar{x})\) of \(\pi_1(C, \bar{x})\) and rows are exact.

Since the canonical map \(\pi_1(C, \bar{x}) \to \pi_1(T, \eta)\) is surjective, it follows that the monodromy representation \(\pi_1(C, \bar{x}) \to R(Q_1)\) is also Zariski dense. Denote by \(\mathcal{G}_C\) and \(\mathcal{G}_T\) the weighted completions of \(\pi_1(C, \bar{x})\) and \(\pi_1(T, \eta)\) with respect to \(\omega\) and their monodromy representations to \(R\), respectively, and denote their prounipotent radicals by \(\mathcal{U}_C\) and \(\mathcal{U}_T\). Since the canonical map \(\pi_1(C \otimes_k \bar{k}, \bar{x}) \to \pi_1(T \otimes_k \bar{k}, \bar{\eta})\) is surjective, their images in \(R(Q_1)\) are equal. Denote their common Zariski closure by \(\text{G}_C^{\text{geom}}\), which is a reductive subgroup of \(R\). Denote by \(\mathcal{G}_C^{\text{geom}}\) and \(\mathcal{G}_T^{\text{geom}}\) the continuous relative completion of \(\pi_1(C, \bar{x})\) and \(\pi_1(T, \eta)\) with respect to their monodromy representations to \(\text{R}_C^{\text{geom}}(Q_1)\), respectively, and denote their prounipotent radicals by \(\mathcal{U}_C^{\text{geom}}\) and \(\mathcal{U}_T^{\text{geom}}\).

By pushing out the exact sequence

\[ \pi_1(C, \bar{x}) \to \pi_1(T, \eta) \to 1 \]

along the surjection \(\pi_1(C, \bar{x}) \to \pi_1(C, \bar{x})^{(\ell)}\), we obtain the exact sequence

\[ 1 \to \pi_1(C, \bar{x})^{(\ell)} \to \pi_1(T, \eta) \to 1 \]

that fits in the commutative diagram

\[
\begin{array}{ccc}
\pi_1(C, \bar{x}) & \to & \pi_1(T, \eta) \\
\downarrow & & \downarrow \\
1 & \to & \text{Inn}(\Pi^{(\ell)}) \\
\end{array}
\]

Denote by \(\mathcal{G}_C\) the weighted completion of \(\pi_1'(C, \bar{x})\) with respect to \(\omega\) and its monodromy representation \(\pi_1'(C, \bar{x}) \to R(Q_1)\).
Lemma 8.2. With the notations above, there is a canonical isomorphism
\[ G_C \cong G'_C. \]
Similarly, there is a canonical isomorphism
\[ G_C^{\text{geom}} \cong G'_C^{\text{geom}}. \]

Proof. By the functoriality of weighted completion, there is a unique map \( \phi : G_C \to G'_C \). Denote the kernel of \( \pi_1(C, \bar{x}) \to \pi'_1(C, \bar{x}) \) by \( N \). Recall that \( N \) is the kernel of the maximal pro-\( \ell \) quotient \( K \to K^{(\ell)} \), where \( K \) is the kernel of the canonical projection \( \pi_1(C, \bar{x}) \to \pi_1(T, \bar{\eta}) \). We have the commutative diagram:
\[
\begin{array}{ccc}
1 & \longrightarrow & N \\
\downarrow & & \downarrow \\
1 & \longrightarrow & G_C(Q_{\ell}) \\
\end{array}
\]
Since compact subgroups of \( U(Q_{\ell}) \) are pro-\( \ell \) groups, the left vertical map must be trivial. Hence the canonical map \( \pi_1(C, \bar{x}) \to G(Q_{\ell}) \) factors through \( \pi'_1(C, \bar{x}) \). By the universal property of weighted completion, there exists a unique map \( \psi : G'_C \to G_C \). It is easy to see that \( \phi \) and \( \psi \) are inverse to each other. \( \Box \)

Denote the continuous \( \ell \)-adic unipotent completion of \( \pi_1(C_{\bar{\eta}}, \bar{x}) \) by \( \mathcal{P} \). It is a prounipotent \( Q_{\ell} \)-group. Since compact subgroups of \( Q_{\ell} \)-points of a prounipotent group is pro-\( \ell \), the canonical map \( \pi_1(C_{\bar{\eta}}, \bar{x}) \to \mathcal{P} \) factors through \( \pi_1(C_{\bar{\eta}}, \bar{x})^{(\ell)} \), and furthermore there is a unique isomorphism \( \mathcal{P} \cong \pi_1(C_{\bar{\eta}}, \bar{x})^{(\ell), \text{un}} \) of \( \mathcal{P} \) and the unipotent completion of the maximal pro-\( \ell \) quotient of \( \pi_1(C_{\bar{\eta}}, \bar{x}) \), since both completions admit the same universal property.

Proposition 8.3. With the notation as above:

(i) There are exact sequences
\[
1 \to \mathcal{P} \to G_C \to G_T \to 1
\]
and
\[
1 \to \mathcal{P} \to G_C^{\text{geom}} \to G_T^{\text{geom}} \to 1
\]
of proalgebraic \( Q_{\ell} \)-groups such that the diagram
\[
\begin{array}{ccc}
1 & \longrightarrow & \pi_1(C_{\bar{\eta}}^{(\ell)}) \\
\downarrow & & \downarrow \\
1 & \longrightarrow & \pi_1(C_{\bar{\eta}}) \\
\downarrow & & \downarrow \\
1 & \longrightarrow & \mathcal{P}(Q_{\ell}) \\
\downarrow & & \downarrow \\
1 & \longrightarrow & G_C^{\text{geom}}(Q_{\ell}) \\
\downarrow & & \downarrow \\
1 & \longrightarrow & G_T^{\text{geom}}(Q_{\ell}) \\
\end{array}
\]
commutes.

(ii) Every section \( s \) of \( \pi_1(C, \bar{x}) \to \pi_1(T, \bar{\eta}) \) induces sections \( s^{(\ell)} \) and \( \bar{s}^{(\ell)} \) of \( \pi'_1(C, \bar{x}) \to \pi'_1(T, \bar{\eta}) \) and \( \pi'_1(C \otimes_k \bar{k}, \bar{x}) \to \pi'_1(T \otimes_k \bar{k}, \bar{\eta}) \), respectively, and
sections $\sigma$ and $\sigma_{\text{geom}}$ of $G_C \to G_T$ and $G_C^{\text{geom}} \to G_T^{\text{geom}}$, respectively, such that the diagram

\[
\begin{array}{ccc}
\pi'_1(C \otimes_k \bar{k}, \bar{x}) & \xrightarrow{s^{(\ell)}} & \pi_1(T \otimes_k \bar{k}, \bar{\eta}) \\
\downarrow & & \downarrow \\
\pi'_1(C, \bar{x}) & \xrightarrow{s^{(\ell)}} & \pi_1(T, \bar{\eta}) \\
\end{array}
\]

commutes.

Proof. The first part of the proposition follows from Proposition 7.5 with the right exactness of relative and weighted completion and the fact [25] that $P$ has trivial center and $H_1(P)$ is pure of weight $-1$. For the second part of the proposition, the sections $s^{(\ell)}$ and $\bar{s}^{(\ell)}$ are induced by base change to $\bar{k}$ and by pushout. Since the diagram

\[
\begin{array}{ccc}
\pi_1(T, \bar{\eta}) & \xrightarrow{\rho_T} & \pi'_1(C, \bar{x}) \\
\downarrow & & \downarrow \\
G_C^{\text{geom}}(\mathbb{Q}_\ell) & \xrightarrow{\sigma_{\text{geom}}} & G_T^{\text{geom}}(\mathbb{Q}_\ell) \\
\end{array}
\]

commutes and since $G_C$ is a negatively weighted extension of $R$, it follows from the universal mapping property of $G_T$ that there exists a unique map $\sigma : G_T \to G_C$, which is a section of $G_C \to G_T$. A similar argument applies for the section $\sigma_{\text{geom}}$. \hfill \Box

Denote the Lie algebras of $R$, $G_C$, $G_T$, $U_C$, $U_T$, $P$ by $\mathfrak{r}$, $\mathfrak{g}_C$, $\mathfrak{g}_T$, $\mathfrak{u}_C$, $\mathfrak{u}_T$, $\mathfrak{p}$, respectively. These admit natural weight filtrations as objects of the category of $G_C$-modules. By Proposition 7.1, their $r$th graded quotient is an $R$-module of weight $r$. Since $H_1(P) = H_1(p)$ is pure of weight $-1$, it follows that $p \cong W_{-1}p$, and by Proposition 7.2, we have

$$\mathfrak{g}_A = W_0\mathfrak{g}_A, \quad W_{-1}\mathfrak{g}_A = \mathfrak{u}_A, \quad \text{and} \quad \text{Gr}_W^0 \mathfrak{g}_A \cong \mathfrak{r},$$

where $A = C$ and $A = T$.

The following corollary follows immediately from the fact that the functor $\text{Gr}_W^*$ is exact on the category of $G_C$-modules.

**Corollary 8.4.** With the notation above:

There is an exact sequence

$$0 \to \text{Gr}_W^* p \to \text{Gr}_W^* \mathfrak{g}_C \to \text{Gr}_W^* \mathfrak{g}_T \to 0$$

of graded Lie algebras in the category of $R$-modules.
9. Weighted Completion of Arithmetic Mapping Class Groups

In this section, we summarize and extend the results of [14, §8]. Suppose that \( g \) and \( n \) are integers satisfying \( 2g - 2 + n > 0 \). Fix prime numbers \( p \) and \( \ell \neq p \).

Denote the finite Galois cover of the moduli stack \( M_{g,n} \) by \( M_{g,n}^\group \), the statement then will follow, if the composition \( R \) over the image of the monodromy representation \( \rho \) is of finite index in \( \Sp(\mathbb{Z}_p^{ur}) \), and denote it by \( M_{g,n}^{\group} \). Since \( \Sp(\mathbb{Z}_p^{ur}) \) is Zariski dense in \( \Sp(\mathbb{Z}_p) \), it can be defined over some finite unramified extension \( S \) of \( \mathbb{Z}_p \). Denote the fraction field and residue field of \( S \) by \( L \) and \( k \), respectively. Denote the absolute Galois group of \( L \) and \( k \) by \( G_L \) and \( G_k \), respectively.

Fix a geometric point \( \tilde{\eta} \) of \( M_{g,n}^\group \) and \( \tilde{\xi} \) of \( M_{g,n}^L \). Let \( C_{\bar{y}} \) be the fiber of the universal curve over \( \bar{y} \), where \( \bar{y} = \tilde{\eta} \) and \( \bar{y} = \tilde{\xi} \). Recall that for a \( \mathbb{Z}_\ell \)-module \( A \),

\[
H_A := H^1_{\acute{e}t}(C_{\bar{y}}, A(1)).
\]

Since the image of the \( \ell \)-adic cyclotomic character \( \chi_\ell : G_L \to \mathbb{Z}_\ell^\times \) is infinite, the image of \( \chi_\ell : G_L \to \mathbb{G}_m(\mathbb{Q}_\ell) \) is Zariski dense. The image of the monodromy representation

\[
\rho_{\eta,\mathbb{Q}_p,\mathbb{Z}} : \pi_1(M_{g,n}^\group, \tilde{\eta}) \to \Sp(H_{\mathbb{Z}_\ell})
\]

is of finite index in \( \Sp(H_{\mathbb{Z}_\ell}) \), and hence it is Zariski dense in \( \Sp(H_{\mathbb{Q}_\ell}) \). The commutative diagram

\[
\begin{array}{c}
1 \quad \pi_1(M_{g,n}^\group, \tilde{\eta}) \quad \pi_1(M_{g,n}^L, \tilde{\eta}) \quad G_L \quad 1 \\
\downarrow \rho_{\eta,\mathbb{Q}_p,\mathbb{Z}} \quad \downarrow \rho_L \quad \downarrow \chi_\ell \\
1 \quad \Sp(H_{\mathbb{Q}_\ell}) \quad \GSp(H_{\mathbb{Q}_\ell}) \quad \mathbb{G}_m(\mathbb{Q}_\ell) \quad 1
\end{array}
\]

implies that the image of the monodromy representation

\[
\rho_{L,\eta} : \pi_1(M_{g,n}^L, \tilde{\eta}) \to \GSp(H_{\mathbb{Q}_\ell})
\]

is also Zariski dense. Denote the weighted completion of \( \pi_1(M_{g,n}^L, \tilde{\eta}) \) with respect to \( \rho_{L,\eta} \) and the standard cocharacter \( \omega \) by

\[
\hat{\mathcal{G}}_{M_{g,n}^L} \text{ and } \hat{\rho}_{L,\eta} : \pi_1(M_{g,n}^L, \tilde{\eta}) \to \hat{\mathcal{G}}_{M_{g,n}^L}(\mathbb{Q}_\ell).
\]

Denote the pullback to \( M_{g,n}^{\group} \) of the universal curve \( C_{g,n} \to M_{g,n} \) by \( f : C_{g,n}^{\group} \to M_{g,n}^{\group} \). Let \( \pi : M_{g,n}^{\group} \to \mathbb{Z}_p^{ur} \) be the structure morphism of \( M_{g,n}^{\group} \) over \( \mathbb{Z}_p^{ur} \).

**Proposition 9.1.** The image of the monodromy representation

\[
\rho_{\eta,\mathbb{Q}_p,\mathbb{Z}}^\geom : \pi_1(M_{g,n}^{\group}, \tilde{\eta}) \to \Sp(H_{\mathbb{Z}_\ell})
\]

is pro-\( \ell \).

**Proof.** Since the kernel of the reduction map \( \Sp(H_{\mathbb{Z}_\ell}) \to \Sp(H_{\mathbb{Z}/(\ell^n)\mathbb{Z}}) \) is a pro-\( \ell \) group, the statement then will follow, if the composition

\[
\rho_{\eta,\mathbb{Q}_p,\mathbb{Z}}^\geom : \pi_1(M_{g,n}^{\group}, \tilde{\eta}) \to \Sp(H_{\mathbb{Z}_\ell}) \to \Sp(H_{\mathbb{Z}/(\ell^n)\mathbb{Z}})
\]

is pro-\( \ell \).
is trivial. By the proper-smooth base change theorem [23, Ch.6 §4], the sheaf $R^1 f_* \mu_\ell$ is a constructible locally constant étale sheaf on $M_{\mathbb{Z}_\ell}^\lambda$. Its fiber over a geometric point $\bar{y}$ is isomorphic to $H^1_{\text{ét}}(C_y, \mu_\ell) = (\mathbb{Z}/\ell^m\mathbb{Z})^{2g}$. Denote $R^1 f_* \mu_\ell$ by $\mathcal{F}$. Let $\bar{s}_1$ be a geometric point lying over the generic point and $\bar{s}_2$ be the closed point of $\mathbb{Z}_\ell^n$. By a generalization of the proper-smooth base change theorem [9, SGA 1 Exposé XIII, 2.9], the specialization morphism, induced by the specialization $\bar{s}_1 \to \bar{s}_2$,

$$H^0_{\text{ét}}(M_{\overline{\mathbb{Q}}_p}^\lambda, \mathcal{F}) = (\pi_* \mathcal{F})_{\bar{s}_2} \to (\pi_* \mathcal{F})_{\bar{s}_1} = H^0_{\text{ét}}(\overline{M}_{\overline{\mathbb{Q}}_p}^\lambda, \mathcal{F})$$

is an isomorphism. Note that $H^0_{\text{ét}}(\overline{M}_{\overline{\mathbb{Q}}_p}^\lambda, \mathcal{F}) = (\mathcal{F}_{\bar{\eta}})^{\pi_1(M_{\overline{\mathbb{Q}}_p}^\lambda, \bar{\eta})}$. Since the standard representation $\Gamma_{g,n}^\lambda \to \text{Sp}(H_{\mathbb{Z}_\ell})$ factors through the level $\ell^m$ subgroup $\Gamma_{g,n}[\ell^m]$, the composition with the reduction mod-$\ell^m$ map

$$\Gamma_{g,n}^\lambda \to \text{Sp}(H_{\mathbb{Z}/\ell^m\mathbb{Z}})$$

is trivial, and so is the monodromy representation

$$\pi_1(M_{\overline{\mathbb{Q}}_p}^\lambda, \bar{\eta}) \to \text{Sp}(H_{\mathbb{Z}/\ell^m\mathbb{Z}}).$$

Thus we have

$$(\mathcal{F}_{\bar{\eta}})^{\pi_1(M_{\overline{\mathbb{Q}}_p}^\lambda, \bar{\eta})} = (\mathbb{Z}/\ell^m\mathbb{Z})^{2g},$$

which implies that

$$(\mathcal{F}_{\bar{\xi}})^{\pi_1(M_{\overline{\mathbb{Q}}_p}^\lambda, \bar{\xi})} = (\mathbb{Z}/\ell^m\mathbb{Z})^{2g}.$$  

Therefore, the monodromy $\rho_{\text{geom}}^\lambda : \pi_1(M_{\overline{\mathbb{Q}}_p}^\lambda, \bar{\xi}) \to \text{Sp}(H_{\mathbb{Z}/\ell^m\mathbb{Z}})$ is trivial. \hfill \Box

**Corollary 9.2.** The image of the monodromy representation

$$\rho_{\overline{\mathbb{Q}}_p, \xi}^\text{geom} : \pi_1(M_{g,n, \overline{\mathbb{Q}}_p}[\ell^m], \bar{\xi}) \to \text{Sp}(H_{\mathbb{Z}_\ell})$$

is pro-$\ell$.

**Proof.** We use the same notation as in the proof of the above proposition. Denote the automorphism group of the étale cover $M_{\overline{\mathbb{Q}}_p}^\lambda \to M_{g,n, \overline{\mathbb{Q}}_p}[\ell^m]$ by $G$. Note that $H^0_{\text{ét}}(M_{\overline{\mathbb{Q}}_p}^\lambda, \mathcal{F})^G = H^0_{\text{ét}}(M_{g,n, \overline{\mathbb{Q}}_p}[\ell^m], \mathcal{F})$ and that $G$ acts trivially on $H^0_{\text{ét}}(M_{\overline{\mathbb{Q}}_p}^\lambda, \mathcal{F})$ as it acts trivially on $H^0_{\text{ét}}(M_{\overline{\mathbb{Q}}_p}^\lambda, \mathcal{F})$. Thus it follows that

$$H^0_{\text{ét}}(M_{g,n, \overline{\mathbb{Q}}_p}[\ell^m], \mathcal{F}) = (\mathbb{Z}/\ell^m\mathbb{Z})^{2g},$$

which implies that the monodromy $\pi_1(M_{g,n, \overline{\mathbb{Q}}_p}[\ell^m], \bar{\xi}) \to \text{Sp}(H_{\mathbb{Z}_\ell})$ has a pro-$\ell$ image. \hfill \Box

**Proposition 9.3.** The monodromy representation

$$\rho_{\overline{\mathbb{Q}}_p, \xi}^\text{geom} : \pi_1(M_{\overline{\mathbb{Q}}_p}^\lambda, \bar{\xi}) \to \text{Sp}(H_{\mathbb{Z}_\ell})$$

has a finite index image in $\text{Sp}(H_{\mathbb{Z}_\ell})$, and so does the monodromy representation

$$\rho_{\overline{\mathbb{Q}}_p, \xi}^\text{geom} : \pi_1(M_{g,n, \overline{\mathbb{Q}}_p}[\ell^m], \bar{\xi}) \to \text{Sp}(H_{\mathbb{Z}_\ell}).$$
Proof. Consider the diagram

\[
\begin{array}{cccccc}
1 & \rightarrow & \pi_1(C, \xi', \ell) & \rightarrow & \pi_1(C_{\mathbb{Z}_p}, \xi') & \rightarrow & \pi_1(M_{\mathbb{Z}_p}, \xi) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & \pi_1(C, \xi', \ell) & \rightarrow & \pi_1(C_{\mathbb{Z}_p}, \xi') & \rightarrow & \pi_1(M_{\mathbb{Z}_p}, \xi) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & \pi_1(C_{\mathbb{Z}_p}, \xi) & \rightarrow & \pi_1(C_{\mathbb{Z}_p}, \bar{x}) & \rightarrow & \pi_1(M_{\mathbb{Z}_p}, \bar{\eta}) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & \pi_1(C_{\mathbb{Z}_p}, \xi) & \rightarrow & \pi_1(C_{\mathbb{Q}_p}, \bar{x}) & \rightarrow & \pi_1(M_{\mathbb{Q}_p}, \bar{\eta}) & \rightarrow & 1,
\end{array}
\]

whose rows are exact and the vertical maps between the second and third rows are isomorphisms. This diagram commutes once we fix an isomorphism $\phi : \pi_1(C_{\mathbb{Z}_p}, \bar{x}) \cong \pi_1(C_{\mathbb{Z}_p}, \bar{x})$, which determines an isomorphism $\phi' : \pi_1(M_{\mathbb{Z}_p}, \xi) \cong \pi_1(M_{\mathbb{Z}_p}, \bar{\eta})$. Fix such an isomorphism. The proof of Proposition 9.1 also shows that the monodromy representation $\rho_{\mathbb{Z}_p, \xi} : \pi_1(C_{\mathbb{Z}_p}, \bar{x}) \rightarrow \text{Sp}(H_{\mathbb{Z}_p})$ also has a pro-$\ell$ image, since $H_1^{\mathbb{Z}_p}(M_{\mathbb{Z}_p}, F) \cong H_1^{\mathbb{Z}_p}(M_{\mathbb{Z}_p}, F)$ by the generalization of proper-smooth base change theorem. Thus it follows that the image of the monodromy representation $\pi_1(C_{\mathbb{Z}_p}, \bar{x}) \rightarrow \text{Sp}(H_{\mathbb{Z}_p})$ is also pro-$\ell$. This implies that the image of $\pi_1(C_{\mathbb{Z}_p}, \bar{x}')$ in $\text{Aut}(\pi_1(C_{\mathbb{Z}_p}, \bar{x})'(t))$ under its conjugation action on $\pi_1(C_{\mathbb{Z}_p}, \bar{x})'(t)$ is also pro-$\ell$, and hence this conjugation action factors through $\pi_1(C_{\mathbb{Z}_p}, \bar{x})'(t)$. Since the center of $\pi_1(C_{\mathbb{Z}_p}, \bar{x})'(t)$ is trivial, it follows that the composition

$\pi_1(C_{\mathbb{Z}_p}, \bar{x})'(t) \rightarrow \pi_1(C_{\mathbb{Z}_p}, \bar{x}')'(t) \rightarrow \text{Aut}(\pi_1(C_{\mathbb{Z}_p}, \bar{x})'(t))$

is injective. Thus by taking maximal pro-$\ell$ quotients of the above diagram, we obtain the commutative diagram

\[
\begin{array}{cccccc}
1 & \rightarrow & \pi_1(C_{\mathbb{Z}_p}, \bar{x}')'(t) & \rightarrow & \pi_1(C_{\mathbb{Z}_p}, \bar{x}')'(t) & \rightarrow & \pi_1(M_{\mathbb{Z}_p}, \xi) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & \pi_1(C_{\mathbb{Z}_p}, \bar{x}')'(t) & \rightarrow & \pi_1(C_{\mathbb{Z}_p}, \bar{x}')'(t) & \rightarrow & \pi_1(M_{\mathbb{Z}_p}, \xi) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & \pi_1(C_{\mathbb{Z}_p}, \bar{x})'(t) & \rightarrow & \pi_1(C_{\mathbb{Z}_p}, \bar{x})'(t) & \rightarrow & \pi_1(M_{\mathbb{Z}_p}, \bar{\eta}) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & \pi_1(C_{\mathbb{Z}_p}, \bar{x})'(t) & \rightarrow & \pi_1(C_{\mathbb{Q}_p}, \bar{x})'(t) & \rightarrow & \pi_1(M_{\mathbb{Q}_p}, \bar{\eta}) & \rightarrow & 1,
\end{array}
\]

whose rows are exact and vertical maps are all isomorphisms. From this diagram, we see that the diagram

\[
\begin{array}{ccc}
\pi_1(M_{\mathbb{Z}_p}, \bar{\eta}) & \rightarrow & \pi_1(M_{\mathbb{Z}_p}, \xi) \\
\downarrow & & \downarrow \\
\pi_1(M_{\mathbb{Z}_p}, \bar{\eta})'(t) & \xrightarrow{=} & \pi_1(M_{\mathbb{Z}_p}, \xi)'(t) \\
\downarrow & & \downarrow \\
\text{Sp}(H_{\mathbb{Z}_p}) & \xrightarrow{=} & \text{Sp}(H_{\mathbb{Z}_p})
\end{array}
\]

commutes, where the bottom isomorphism is induced by $\phi$. Since the composition of the two left-hand vertical maps is the standard representation $(\xi_{g,n})^\lambda \rightarrow$
Sp(H_{Z_{\ell}}), it has finite-index image, and so does the monodromy representation \( \rho^\text{geom}_{\bar{\ell}p,\xi} : \pi_1(M^\lambda_{\bar{\ell}p}, \bar{\xi}) \to \text{Sp}(H_{Z_{\ell}}) \). The density of the monodromy \( \pi_1(M_{g,n/\bar{\ell}p}[\ell^m], \bar{\xi}) \to \text{Sp}(H_{Z_{\ell}}) \) follows since its image in \( \text{Sp}(H_{Z_{\ell}}) \) contains the image of \( \pi_1(M^\lambda_{\bar{\ell}p}, \bar{\xi}) \to \text{Sp}(H_{Z_{\ell}}) \).

By Proposition 9.3, the image of \( \rho^\text{geom}_{\bar{\ell}p,\xi} : \pi_1(M^\lambda_{\bar{\ell}p}, \bar{\xi}) \to \text{Sp}(H_{Q_{\ell}}) \) is Zariski dense. Since the image of the \( \ell \)-adic cyclotomic character \( \chi : G_k \to \mathbb{Z}_{\ell}^\times \) is infinity, the image of \( \chi_{\ell} : G_k \to \mathbb{G}_m(Q_{\ell}) \) is Zariski dense. The commutative diagram

\[
\begin{array}{ccc}
1 & \to & \pi_1(M^\lambda_{\bar{\ell}p}, \bar{\xi}) \\
\downarrow{\rho^\text{geom}_{\bar{\ell}p}} & & \downarrow{\rho_k} \\
1 & \to & \text{Sp}(H_{Q_{\ell}}) \to \text{GSp}(H_{Q_{\ell}}) \to \mathbb{G}_m(Q_{\ell}) \to 1.
\end{array}
\]

implies that the monodromy representation

\[
\rho_{k,\xi} : \pi_1(M^\lambda_{k}, \bar{\xi}) \to \text{GSp}(H_{Q_{\ell}})
\]

has Zariski dense image. Denote the weighted completion of \( \pi_1(M^\lambda_{k}, \bar{\xi}) \) with respect to \( \rho_{k,\xi} \) and the standard cocharacter \( \omega \) by

\[
\mathcal{G}^\text{geom}_{M^\lambda_{k}} \text{ and } \check{\rho}_{k,\xi} : \pi_1(M^\lambda_{k}, \bar{\eta}) \to \mathcal{G}^\text{geom}_{M^\lambda_{k}}(Q_{\ell}).
\]

Let \( \bar{\gamma} \) denote \( \bar{\eta} \) and \( \bar{\xi} \). Similarly, we have the weighted completion of \( \pi_1(M^\lambda_{\bar{\ell}p}, \bar{\gamma}) \) with respect to \( \rho_{S,\bar{\gamma}} : \pi_1(M^\lambda_{\bar{\ell}p}, \bar{\gamma}) \to \text{GSp}(H_{Q_{\ell}}) \) and the cocharacter \( \omega \), denoted by

\[
\mathcal{G}^\text{geom}_{M^\lambda_{\bar{\ell}p}} \text{ and } \rho_{S,\bar{\gamma}} : \pi_1(M^\lambda_{\bar{\ell}p}, \bar{\gamma}) \to \mathcal{G}^\text{geom}_{M^\lambda_{\bar{\ell}p}}(Q_{\ell}).
\]

Recall that \( \mathcal{G}_{\text{geom}}^{g,n/\ell_{p}} \) and \( (\Gamma_{g,n})^\wedge \to \mathcal{G}_{\text{geom}}^{g,n/\ell_{p}}(Q_{\ell}) \) is the relative completion of \( (\Gamma_{g,n})^\wedge \) with respect to the standard representation \( (\Gamma_{g,n})^\wedge \to \text{Sp}(H_{Q_{\ell}}) \). For \( g \geq 3 \), the continuous relative completion of \( \pi_1(M^\lambda_{Q_{\ell}}, \bar{\eta}) \) with respect to its standard representation \( \rho_{Q_{\ell},\overline{\eta}}^\text{geom} \) is isomorphic to \( \mathcal{G}_{\text{geom}}^{g,n/\ell_{p}}(Q_{\ell}) \) by Theorem 6.3 and 6.5. Similarly, for \( g \geq 3 \), the continuous relative completion of \( \pi_1(M^\lambda_{g,n/\ell_{p}}[\ell^m], \overline{\eta}) \) with respect to its standard representation to \( \text{Sp}(H_{Q_{\ell}}) \) is isomorphic to \( \mathcal{G}_{\text{geom}}^{g,n/\ell_{p}} \) [11, Prop. 3.3]. When the field \( F \) is clear from context, we will denote \( \mathcal{G}_{\text{geom}}^{g,n/F} \) by \( \mathcal{G}_{\text{geom}}^{g,n} \).

**Proposition 9.4.** The continuous relative completion of \( \pi_1(M^\lambda_{\bar{\ell}p}, \bar{\xi}) \) with respect to \( \rho_{\bar{\ell}p,\xi}^\text{geom} \) is isomorphic to \( \mathcal{G}_{\text{geom}}^{g,n} \). Similarly, the continuous relative completion of \( \pi_1(M_{g,n/\bar{\ell}p}[\ell^m], \bar{\xi}) \) with respect to \( \rho_{\bar{\ell}p,\xi}^\text{geom} \) is isomorphic to \( \mathcal{G}_{\text{geom}}^{g,n} \).
Proof. Fix an isomorphism $\phi : \pi_1(M_\ell^{\text{geo}}_{\bar{\mathbb{Q}}_p}, \bar{\eta}) \cong \pi_1(M_\ell^{\text{geo}}_{\bar{\mathbb{Q}}_p}, \bar{\xi})$. We have the following commutative diagram

\[
\begin{array}{cccccc}
\pi_1(M_\ell^{\text{geo}}_{\bar{\mathbb{Q}}_p}, \bar{\eta}) & \to & \pi_1(M_\ell^{\text{geo}}_{\bar{\mathbb{Q}}_p}, \bar{\eta}) \\
\downarrow & & \downarrow \\
\pi_1(M_\ell^{\text{geo}}_{\bar{\mathbb{Q}}_p}, \bar{\xi}) & \to & \pi_1(M_\ell^{\text{geo}}_{\bar{\mathbb{Q}}_p}, \bar{\xi})
\end{array}
\]

where the isomorphism $\text{Sp}(H_{\mathbb{Q}_\ell}) \cong \text{Sp}(H_{\mathbb{Q}_\ell})$ is induced by the isomorphism $\phi$ and the isomorphisms on the second row are ones in the proof of Theorem 5.8. By taking the relative completion of each of the profinite groups with respect to its corresponding monodromy representation, we obtain the commutative diagram of proalgebraic $\mathbb{Q}_\ell$-groups

\[
\begin{array}{cccccc}
G_{M_\ell^{\text{geo}}_{\bar{\mathbb{Q}}_p}}^{\text{geo}} & \to & G_{M_\ell^{\text{geo}}_{\bar{\mathbb{Q}}_p}}^{\text{geo}} \\
\downarrow & & \downarrow \\
G_{M_\ell^{\text{geo}}_{\bar{\mathbb{Q}}_p}}^{\text{geo},(\ell)} & \to & G_{M_\ell^{\text{geo}}_{\bar{\mathbb{Q}}_p}}^{\text{geo},(\ell)}
\end{array}
\]

Since the vertical maps between the first and second rows are isomorphism by Proposition 6.6, it follows that

$G_{M_\ell^{\text{geo}}_{\bar{\mathbb{Q}}_p}}^{\text{geo}} \cong G_{M_\ell^{\text{geo}}_{\bar{\mathbb{Q}}_p}}^{\text{geo},(\ell)} \cong G_{M_\ell^{\text{geo}}_{\bar{\mathbb{Q}}_p}}^{\text{geo}}$.

A similar argument applies to the relative completion of $\pi_1(M_{g,n/\bar{\mathbb{F}}_p}[\ell^m], \bar{\xi})$.

\[\square\]

For a field $F$ whose $\ell$-adic cyclotomic character has an infinite image, denote the weighted completion of $G_F$ with respect to the $\ell$-adic cyclotomic character $\chi_\ell : G_F \to \mathbb{G}_m(\mathbb{Q}_\ell)$ and $\omega : z \mapsto z^{-2}$ by $A_F$.

Throughout the rest of this section, for a prime $\ell$, let $M$ denote the étale covers $M_{g,n}^{\text{geo}}$ and $M_{g,n}[\ell^m]$ of $M_{g,n/\mathbb{Z}[1/\ell]}$. As in above, we fix a connected component of the base change to $S$ of $M$ and denote it by $M_S$, where $S$ is some finite unramified extension of $\mathbb{Z}_p$ over which $M$ decomposes as a finite disjoint union of geometrically connected components. Recall that $L$ and $k$ are the fraction field and the residue field of $S$, respectively.

**Proposition 9.5** ([14, 8.1]). Applying weighted completion to the two right-hand columns and relative completion to the left-hand column of diagram

\[
\begin{array}{cccccc}
1 & \to & \pi_1(M_{\mathbb{Q}_\ell}, \bar{\eta}) & \to & \pi_1(M_L, \bar{\eta}) & \to & G_L & \to & 1 \\
\downarrow & & \uparrow^{\chi_\ell} & & \downarrow^{ho L} & & \downarrow & & \downarrow \\
1 & \to & \text{Sp}(H_{\mathbb{Q}_\ell}) & \to & G\text{Sp}(H_{\mathbb{Q}_\ell}) & \to & \mathbb{G}_m(\mathbb{Q}_\ell) & \to & 1
\end{array}
\]
gives a commutative diagram

\[
\begin{array}{ccc}
G_{g,n}^{geom} & \longrightarrow & G_{M_L} \\
\downarrow & & \downarrow \\
1 & \longrightarrow & Sp(H_{Q_L}) \\
\end{array}
\]

whose rows are exact. Similar results hold if we replace the sequence

1 \to \pi_1(M_{\bar{q},\bar{y}}) \to \pi_1(M_L,\bar{y}) \to G_L \to 1

with the exact sequence

1 \to \pi_1(M_{\bar{p},\bar{y}}) \to \pi_1(M_k,\bar{\xi}) \to G_k \to 1

and

1 \to \pi_1(M_{\bar{p},\bar{y}}) \to \pi_1(M_S,\bar{y}) \to \pi_1(S,\bar{y}) \to 1,

where \( \bar{y} = \bar{\eta} \) and \( \bar{y} = \bar{\xi} \).

Denote the pronipotent radicals of \( G_{M_{q,p}}^{geom}, G_{M_{q,p}}^{geom}, \) and \( G_{M_{q,p}}^{geom} \) by \( U_{M_{q,p}}^{geom}, U_{M_{q,p}}^{geom}, \) and \( U_{M_{q,p}}^{geom} \) respectively. Denote the Lie algebras of \( G_{M_{q,p}}^{geom}, G_{M_{q,p}}^{geom}, G_{M_{q,p}}^{geom}, G_{M_{q,p}}^{geom}, U_{M_{q,p}}^{geom}, U_{M_{q,p}}^{geom}, U_{M_{q,p}}^{geom}, \) and \( U_{M_{q,p}}^{geom} \) by \( \mathfrak{g}_{M_{q,p}}^{geom}, \mathfrak{g}_{M_{q,p}}^{geom}, \mathfrak{g}_{M_{q,p}}^{geom}, \mathfrak{u}_{M_{q,p}}^{geom}, \mathfrak{u}_{M_{q,p}}^{geom}, \mathfrak{u}_{M_{q,p}}^{geom}, \mathfrak{u}_{M_{q,p}}^{geom}, \) and \( \mathfrak{u}_{M_{q,p}}^{geom} \) respectively.

**Proposition 9.6.** Let \( F = L \) and \( k \) and \( \bar{y} = \bar{\eta} \) and \( \bar{\xi} \), respectively. If \( 2g - 2 + n > 0 \) and \( g \geq 3 \), then the natural action of \( \pi_1(M_{F,\bar{y}}) \) on \( \pi_1(M_{F,\bar{y}}) \) induces an action of \( G_{M_{F}}^{geom} \) on \( \mathfrak{g}_{M_{F}}^{geom} \). Therefore, \( \mathfrak{g}_{M_{F}}^{geom} \) and \( \mathfrak{u}_{M_{F}}^{geom} \) are pro-objects of the category of \( G_{M_{F}}^{geom} \)-modules, and thus admit natural weight filtrations.

**Proof.** The conjugation action of \( \pi_1(M_{F,\bar{y}}) \) on \( \pi_1(M_{F,\bar{y}}) \) induces a homomorphism

\[
\pi_1(M_{F,\bar{y}}) \to \text{Aut}(G_{M_{F}}^{geom})
\]

and thus a homomorphism

\[
\psi : \pi_1(M_{F,\bar{y}}) \to \text{Aut}(\mathfrak{g}_{M_{F}}^{geom}).
\]

Define the filtration \( D_{\bullet}\mathfrak{g}_{M_{F}}^{geom} \) by

\[
D_0\mathfrak{g}_{M_{F}}^{geom} = \mathfrak{g}_{M_{F}}^{geom}, \quad D_{-i}\mathfrak{g}_{M_{F}}^{geom} = L_i\mathfrak{u}_{M_{F}}^{geom}, \quad \text{for } i \geq 1
\]

where \( L_i\mathfrak{u}_{M_{F}}^{geom} \) is the \( i \)th term of the lower central series of \( \mathfrak{u}_{M_{F}}^{geom} \) with \( L_1\mathfrak{u}_{M_{F}}^{geom} = \mathfrak{u}_{M_{F}}^{geom} \). Proposition 9.4 implies that we have an isomorphism \( \mathfrak{u}_{g,n}^{geom} \cong \mathfrak{u}_{M_{F}}^{geom} \) and it is shown in [11] that each graded quotient of \( \mathfrak{u}_{g,n}^{geom} \) associated to the lower central series of \( \mathfrak{u}_{g,n}^{geom} \) is finite dimensional. Since every automorphism of \( \mathfrak{g}_{M_{F}}^{geom} \) preserves the filtration, we have \( \text{Aut}(\mathfrak{g}_{M_{F}}^{geom}) \cong \lim_{\to} \text{Aut}(\mathfrak{g}_{M_{F}}^{geom}/D_{-1}) \), and hence \( \text{Aut}(\mathfrak{g}_{M_{F}}^{geom}) \) is proalgebraic. Since \( p \) is a \( G_{M_{F}}^{geom} \)-module, it has a natural weight filtration, and the exactness of the functor \( G_{M_{F}}^{W} \) and the fact that \( H_1(p) \) has weight \(-1\) imply that the natural weight filtration on \( p \) agrees with its lower central filtration. Similarly, \( \text{Der} p \) has a natural weight filtration as a \( G_{M_{F}}^{geom} \)-module. Johnson’s computation of the abelianization of the Torelli group [21] implies that \( H_1(u_{g,n}^{geom}) \to \text{Gr}_{-1}W \) \( \text{Der} p \) is an isomorphism. Thus \( H_1(u_{g,1}^{geom}) \) is pure of weight \(-1\). For \( n > 1 \), the morphism \( M_{g,n,F} \to (M_{g,1,F})^n \) induces an injective map \( H_1(u_{g,n}^{geom}) \to H_1(u_{g,1}^{geom} \oplus n) \), and the morphism \( M_{g,1,F} \to M_{g,F} \) induces a surjective map \( H_1(u_{g,1}^{geom}) \to H_1(u_{g}^{geom}) \).
Therefore, $H_1(u_{g,n}^{\text{geom}})$ is pure of weight $-1$ for all $n \geq 0$ and so is $H_1(u_{g,n}^{\text{geom}})$ for all $n \geq 0$. This implies that each graded quotient $\text{Gr}_i^D u_{M_F}^{\text{geom}}$ is pure of weight $-i$ as a $G_{\Sp(H_{Q_{\ell}})}$-module. Now, the monodromy representation $\rho_{F,\bar{g}} : \pi_1(M_{\bar{F}}^{\bar{g}}, \bar{y}) \to G_{\Sp(H_{Q_{\ell}})}$ factors through $\text{Aut}(g_{M_F}^{\text{geom}})(\bar{Q}_{\ell})$. Since the image of $\rho_{F,\bar{g}}$ is Zariski dense in $G_{\Sp(H_{Q_{\ell}})}$, the Zariski closure of the image of $\pi_1(M_{F_{\bar{F}}}, \bar{y})$ in $\text{Aut}(g_{M_F}^{\text{geom}})(\bar{Q}_{\ell})$ maps onto $G_{\Sp(H_{Q_{\ell}})}$ with prounipotent kernel $K$. Since $K$ is negatively weighted, the universal mapping property of $G_{M_F}$ gives a map $\mathcal{G}_{M_F} \to \text{Aut}(g_{M_F}^{\text{geom}})$, which makes $g_{M_F}^{\text{geom}}$ and so $u_{M_F}^{\text{geom}}$ as $G_{M_F}$-modules.

**Remark 9.7.** The induced natural weight filtration on $g_{M_F}^{\text{geom}}$ indeed agrees with the filtration defined in this proof. The weight filtration clearly satisfies
\[
g_{M_F}^{\text{geom}} = W_0 g_{M_F}^{\text{geom}} \quad \text{and} \quad W_{-1} g_{M_F}^{\text{geom}} = u_{M_F}^{\text{geom}}.
\]
Again, the exactness of the functor $\text{Gr}_W^*$ and the fact that $H_1(u_{M_F}^{\text{geom}})$ has weight $-1$ imply that $W_{-r} u_{M_F}^{\text{geom}}$ is the $r$th term of the lower central series of $u_{M_F}^{\text{geom}}$. This coincidence allows us to apply the results of [11] in this paper.

**Proposition 9.8.** The isomorphisms
\[
g_{g,n}^{\text{geom}} \cong g_{M_{\bar{p}}}^{\text{geom}} \cong g_{M_{F}}^{\text{geom}}
\]
are morphisms in the category of $G_{M_{\ell}}$-modules.

**Proof.** First consider the diagram
\[
1 \longrightarrow \pi_1(M_{\bar{Q}_{\bar{p}}}, \bar{\eta}) \longrightarrow \pi_1(M_{\bar{F}}, \bar{\eta}) \longrightarrow G_{\bar{L}} \longrightarrow 1
\]
\[
1 \longrightarrow \pi_1(M_{g,n, Q_{\bar{p}}}, \bar{\eta}) \longrightarrow \pi_1(M_{g,n, \bar{L}}, \bar{\eta}) \longrightarrow G_{\bar{L}} \longrightarrow 1,
\]
whose rows are exact. $\pi_1(M_{\bar{L}}, \bar{\eta})$ acts on $\pi_1(M_{g,n, Q_{\bar{p}}}, \bar{\eta})$ by conjugation via the homomorphism $\pi_1(M_{\bar{L}}, \bar{\eta}) \to \pi_1(M_{g,n, \bar{L}}, \bar{\eta})$. This conjugation action induces an action of $G_{M_{\bar{L}}}$ on $g_{g,n}^{\text{geom}}$ and hence the isomorphism $g_{M_{\bar{p}}}^{\text{geom}} \to g_{g,n}^{\text{geom}}$ is a $G_{M_{\ell}}$-module homomorphism. Secondly, consider the diagram
\[
1 \longrightarrow \pi_1(M_{Z_{\bar{p}}}, \bar{\eta}) \longrightarrow \pi_1(M_{\bar{F}}, \bar{\eta}) \longrightarrow G_{\bar{L}} \longrightarrow 1
\]
\[
1 \longrightarrow \pi_1(M_{Z_{\bar{p}}}, \bar{\eta}) \longrightarrow \pi_1(M_{\bar{F}}, \bar{\eta}) \longrightarrow \pi_1(S, \bar{\eta}) \longrightarrow 1
\]
\[
1 \longrightarrow \pi_1(M_{Z_{\bar{p}}}, \bar{\xi}) \longrightarrow \pi_1(M_{\bar{F}}, \bar{\xi}) \longrightarrow \pi_1(S, \bar{\xi}) \longrightarrow 1
\]
\[
1 \longrightarrow \pi_1(M_{Z_{\bar{p}}}, \bar{\xi}) \longrightarrow \pi_1(M_{\bar{F}}, \bar{\eta}) \longrightarrow G_{\bar{L}} \longrightarrow 1.
\]
A choice of an isomorphism $\phi : \pi_1(M_{Z_{\bar{p}}}, \bar{\eta}) \cong \pi_1(M_{Z_{\bar{p}}}, \bar{\xi})$ determines isomorphisms $\pi_1(S, \bar{\eta}) \cong \pi_1(S, \bar{\xi})$ and $\pi_1(M_{\bar{F}}, \bar{\eta}) \cong \pi_1(M_{\bar{F}}, \bar{\xi})$, which makes the above diagram commute. Pushing out this diagram along the surjection $\pi_1(M_{Z_{\bar{p}}}, \bar{\xi}) \to$
\[ \pi_1(M_{Zur}, \bar{\xi})^{(\ell)} \text{ induces the commutative diagram} \]

\[
\begin{array}{cccccc}
1 & \rightarrow & \pi_1(M_{Qp}, \bar{\eta})^{(\ell)} & \rightarrow & \pi_1'(M_L, \bar{\eta}) & \rightarrow & G_L & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & \pi_1(M_{Zur}, \bar{\eta})^{(\ell)} & \rightarrow & \pi_1'(M_S, \bar{\eta}) & \rightarrow & \pi_1(S, \bar{\eta}) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & \pi_1(M_{Zur}, \bar{\xi})^{(\ell)} & \rightarrow & \pi_1'(M_S, \bar{\xi}) & \rightarrow & \pi_1(S, \bar{\xi}) & \rightarrow & 1 \\
\downarrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
1 & \rightarrow & \pi_1(M_{Fp}, \bar{\xi})^{(\ell)} & \rightarrow & \pi_1'(M_k, \bar{\xi}) & \rightarrow & G_k & \rightarrow & 1,
\end{array}
\]

where rows are exact and all the left-hand vertical maps and the vertical maps between the third and fourth rows are isomorphisms. Thus \( \pi_1(M_L, \bar{\eta}) \) acts on \( \pi_1(M_{Zur}, \bar{\xi})^{(\ell)} \) through the conjugation action of \( \pi_1'(M_k, \bar{\xi}) \) on \( \pi_1(M_{Fp}, \bar{\xi})^{(\ell)} \). Hence the induced isomorphism \( g_{geom}^{M_{Qp}} \cong g_{geom}^{M_{Fp}} \) is a \( G_{M_L} \)-module homomorphism. \( \square \)

Recall that for a prime number \( \ell \), the corresponding finite étale cover \( M_{g,n}^\lambda \) of \( M_{g,n} \) is defined over \( \mathbb{Z}[1/\ell] \). Suppose that \( F \) is a field of characteristic zero such that the image of the \( \ell \)-adic cyclotomic character \( \chi_{\ell} : G_F \rightarrow \mathbb{G}_m(\mathbb{Z}_\ell) \) is infinity and such that a connected component \( M_{\lambda}^\lambda \) of the base change to \( F \) of \( M_{g,n}^\lambda \) is geometrically connected.

**Proposition 9.9** ([14, 8.2]). If \( g \geq 3 \), then for all \( m \geq 1 \) the natural homomorphism

\[ G_{M_{g,n}/F}[m] \rightarrow G_{M_{g,n}/F} \]

is an isomorphism, and furthermore for all prime numbers \( \ell \geq 3 \) the natural homomorphisms

\[ G_{M_{Fp}}^\lambda \rightarrow G_{M_{g,n}/F}[\ell^m] \rightarrow G_{M_{g,n}/F} \]

are isomorphisms.

From this point, we will denote the weighted completions \( G_{M_{Qp}}^\lambda \), \( G_{M_{g,n}/F}[m] \), and \( G_{M_{g,n}} \) by simply \( G_{g,n}/F \) and omit \( F \) when \( F \) is clear from the context. Similarly, we will denote the Lie algebras \( g_{geom}^{M_{Qp}} \) and \( g_{geom}^{M_{Fp}} \) by \( g_{g,n}^{geom} \). They are pro-objects in the category of \( G_{g,n} \)-modules.

9.1. **Variants.** The comparison of the relative completions \( g_{geom}^{M_{Qp}} \) and \( g_{geom}^{M_{Fp}} \) can be extended to the relative completion of the universal curve over \( M \). Denote the pullback to \( M_{Zur} \) of the universal curve \( C_{g,n} \) by \( C_{Zur}^{(\ell)} \). The diagram of profinite
groups

\[
1 \longrightarrow \pi_1(C, \xi, \bar{x}')(\ell) \longrightarrow \pi_1(C_{p,\xi}, \bar{x}') \longrightarrow \pi_1(M_{p,\xi}, \bar{\xi}) \longrightarrow 1
\]

\[
1 \longrightarrow \pi_1(C, \xi, \bar{x}')(\ell) \longrightarrow \pi_1(C_{p,\xi}, \bar{x}') \longrightarrow \pi_1(M_{p,\xi}, \bar{\xi}) \longrightarrow 1
\]

\[
1 \longrightarrow \pi_1(C, \xi, \bar{x})(\ell) \longrightarrow \pi_1(C_{p,\xi}, \bar{x}) \longrightarrow \pi_1(M_{p,\xi}, \bar{\eta}) \longrightarrow 1
\]

\[
1 \longrightarrow \pi_1(C, \xi, \bar{x})(\ell) \longrightarrow \pi_1(C_{p,\xi}, \bar{x}) \longrightarrow \pi_1(M_{p,\xi}, \bar{\eta}) \longrightarrow 1
\]

commutes, where rows are exact, after fixing an isomorphism \(\pi_1(C_{2p,\xi}', \bar{x}') \cong \pi_1(C_{2p,\xi}, \bar{x})\), which determines isomorphisms \(\pi_1(C, \xi, \bar{x}') \cong \pi_1(C, \eta, \bar{x})\) and \(\pi_1(M_{2p,\xi}, \bar{\xi}) \cong \pi_1(M_{2p,\xi}, \bar{\eta})\). Applying continuous relative completion to this diagram with respect to their natural monodromy representation to \(Sp(H_{Qp})\) and taking Lie algebras, we obtain the commutative diagram

\[
\begin{array}{c}
1 \longrightarrow p \longrightarrow \mathfrak{g}_{C_{p,\xi}}^{\text{geom}} \longrightarrow \mathfrak{g}_{g,n}^{\text{geom}} \longrightarrow 1 \\
1 \longrightarrow p \longrightarrow \mathfrak{g}_{C_{p,\xi}}^{\text{geom}} \longrightarrow \mathfrak{g}_{g,n}^{\text{geom}} \longrightarrow 1 \\
1 \longrightarrow p \longrightarrow \mathfrak{g}_{C_{p,\xi}}^{\text{geom}} \longrightarrow \mathfrak{g}_{g,n}^{\text{geom}} \longrightarrow 1 \\
1 \longrightarrow p \longrightarrow \mathfrak{g}_{C_{p,\xi}}^{\text{geom}} \longrightarrow \mathfrak{g}_{g,n}^{\text{geom}} \longrightarrow 1 \\
\end{array}
\]

where rows are exact and all the left and right-hand vertical maps are isomorphisms. Proposition 8.3 implies that the map \(p \rightarrow \mathfrak{g}_{g,n}^{\text{geom}}\) is injective, since the composition \(p \rightarrow \mathfrak{g}_{g,n}^{\text{geom}} \ightarrow \mathfrak{g}\) is injective. Thus there is an isomorphism

\[
\mathfrak{g}_{C_{p,\xi}}^{\text{geom}} \cong \mathfrak{g}_{C_{p,\xi}}^{\text{geom}}.
\]

As there is an isomorphism \(\mathfrak{g}_{M_{p,\xi}}^{\text{geom}} \cong \mathfrak{g}_{g,n}^{\text{geom}}\), there is an isomorphism \(\mathfrak{g}_{C_{p,\xi}}^{\text{geom}} \cong \mathfrak{g}_{C_{p,\xi}}^{\text{geom}}\), and these isomorphisms are morphisms in the category of \(\mathcal{G}_{g,n}\)-modules. Hence we will denote the Lie algebras \(\mathfrak{g}_{C_{p,\xi}}^{\text{geom}}\) and \(\mathfrak{g}_{C_{p,\xi}}^{\text{geom}}\) by \(\mathfrak{g}_{C_{p,\xi}}^{\text{geom}}\). The canonical morphism \(\mathcal{G}_{g,n} \rightarrow \mathcal{G}_{g,n}\) makes \(\mathfrak{g}_{g,n}^{\text{geom}}\) a \(\mathcal{G}_{g,n}\)-module.

**Proposition 9.10.** Each section \(x\) of the universal curve \(f : \mathcal{C}_k \rightarrow M_k\) induces a well-defined \(\text{GSp}(H_{Qp})\)-equivariant section of \(\text{Gr}^W f_x : \text{Gr}^W \mathfrak{g}_{C_{p,\xi}}^{\text{geom}} \rightarrow \text{Gr}^W \mathfrak{g}_{g,n}^{\text{geom}}\).

**Proof.** By Proposition 8.3, each section \(x\) induces a section \(\sigma^{\text{geom}}\) of \(f_x : \mathfrak{g}_{C_{p,\xi}}^{\text{geom}} \rightarrow \mathfrak{g}_{M_{p,\xi}}^{\text{geom}}\), which is well defined up to conjugation by an element of \(P\). Thus the induced section \(d\sigma_x \circ df_x : \mathfrak{g}_{C_{p,\xi}}^{\text{geom}} \rightarrow \mathfrak{g}_{g,n}^{\text{geom}}\) is a morphism of \(\mathcal{G}_{g,n}\)-modules and is well defined up to addition of a section of the form \(\text{ad}(u) \circ d\sigma^{\text{geom}}\) with \(u\) an element of \(p\). Since \(\text{ad}(u) \in W_{-1} \cdot \text{Der} \mathfrak{g}_{C_{p,\xi}}^{\text{geom}}\), the sections \(d\sigma^{\text{geom}}\) and \(d\sigma^{\text{geom}} + \text{ad}(u) \circ d\sigma^{\text{geom}}\) induce the same section of \(\text{Gr}^W \mathfrak{g}_{C_{p,\xi}}^{\text{geom}} \rightarrow \text{Gr}^W \mathfrak{g}_{g,n}^{\text{geom}}\). Denote this section by \(\text{Gr}^W d\sigma^{\text{geom}}\). Since the action of \(\mathcal{U}_{C_{g,n}}\) on \(\mathfrak{g}_{C_{p,\xi}}^{\text{geom}}\) and \(\mathfrak{g}_{g,n}^{\text{geom}}\) is negatively
weighted, the graded Lie algebras \( \Gr^W_{\text{geom}} \) and \( \Gr^W_{\text{geom}} \) are \( \text{GSp}(H_{\mathbb{Q}}) \)-modules and \( \Gr^W_{\text{geom}} \) is \( \text{GSp}(H_{\mathbb{Q}_\ell}) \)-equivariant.

10. Generators and Relations

In [14], Hain notices that the structure of \( \Gr^W_{\text{geom}} / W_{-3} \) as a graded Lie algebra in the category of \( S_n \times \text{GSp}(H_{\mathbb{Q}}) \)-modules is an essential factor to understanding the rational points of the universal curve. In order to study the structure of \( \Gr^W_{\text{geom}} / W_{-3} \), Hain uses the computations in [11] where a weight filtration is constructed by using Hodge theory. In [14] and this paper, the weight filtration on \( \Gr^W_{\text{geom}} \) is constructed by using weighted completion. However, the weight filtration given by weighted completion agrees with the filtration produced by the lower central series of \( \Gr^W_{\text{geom}} \), and so does the Hodge theoretical weight filtration on \( \Gr^W_{\text{geom}} \). The agreement of the two construction follows from that in both construction.

In this section, we summarize Hain’s computation of the presentations of the Lie algebras \( \Gr^W_{\text{geom}} / W_{-3} \) and computations in [11]. The details of this section can be found in [14, §8,9] and [11].

10.1. \( S_n \) action on \( \Gr^W_{\text{geom}} \). We observe that \( \Gr^W_{\text{geom}} \) has an \( S_n \)-module structure as follows. Let \( F \) be a field of characteristic zero such that the \( \ell \)-adic cyclotomic character \( \chi : G_F \to \mathbb{Z}_\ell(1) \) has Zariski dense image. Fix an algebraic closure \( \bar{F} \) of \( F \). The projection morphisms \( \mathcal{M}_{g,n,F} \to C^n_{g,F}, (C, x_1, \ldots, x_n) \to (C, x_j) \) for \( j = 1, \ldots, n \) induce an inclusion \( \mathcal{M}_{g,n,F} \to C^n_{g,F} \), where \( C^n_{g,F} \) denotes the \( n \)th power of the universal curve over \( M_{g,F} \). There is a natural monodromy representation \( \rho : \pi_1(C^n_{g,F}, \bar{\eta}) \to \text{GSp}(H_{\mathbb{Q}_\ell}) \), which has Zariski dense image. Denote the weighted completion of \( \pi_1(C^n_{g,F}, \bar{\eta}) \) with respect to \( \rho \) and the standard cocharacter \( \omega \) by \( \hat{\mathcal{G}}_{g,n} \). Denote the continuous relative completion of \( \pi_1(C^n_{g,F}, \bar{\eta}) \) with respect to \( \rho \) by \( \hat{G}_{g,n} \). Denote the pro-exponential radical of \( \hat{\mathcal{G}}_{g,n} \) and \( \hat{G}_{g,n} \) by \( \hat{U}_{g,n} \) and \( \hat{U}_{g,n} \) respectively, and denote the Lie algebras of \( \hat{G}_{g,n} \), \( \hat{G}_{g,n} \), \( \hat{G}_{g,n} \), \( \hat{U}_{g,n} \), and \( \hat{U}_{g,n} \) respectively. The inclusion \( \mathcal{M}_{g,n,F} \to C^n_{g,F} \) induces an \( \hat{G}_{g,n,F} \)-module homomorphism \( \hat{G}_{g,n} \to \hat{G}_{g,n} \) and thus an \( S_n \times \text{GSp}(H) \)-module homomorphism \( \Gr^W_{\text{geom}} \to \Gr^W_{\text{geom}} \), where the action of \( S_n \) is induced by the action of \( S_n \) on \( C^n_{g,F} \) by permuting the \( n \) marked points.

**Proposition 10.1.** [14, 8.7] If \( g \geq 3 \), then
\[
\Gr^W_{\text{geom}} / W_{-3} \to \Gr^W_{\text{geom}}
\]
is an isomorphism for \( j = -1 \) and a surjective for \( j = -2 \) with kernel isomorphic to the \( S_n \times \text{GSp}(H_{\mathbb{Q}_\ell}) \)-module \( \bigoplus_{\ell<j} \mathbb{Q}_\ell(1) \), where \( S_n \) acts by permuting the factors.

10.2. Presentation of \( \Gr^W_{\text{geom}} / W_{-3} \). The action of \( \pi_1(M_{g,1,F}, \bar{\eta}) \) on \( p \) induces an action of \( \mathcal{G}_{g,1,F} \) on \( p \). Thus \( p \) has a natural weight filtration. This \( \mathcal{G}_{g,1,F} \)-action on \( p \) induces a Lie algebra homomorphism \( \mathcal{G}_{g,1} \to \text{Der} p \). Composing with the Lie algebra homomorphism \( \mathcal{G}_{g,1} \to \mathcal{G}_{g,1} \) induced by the natural map
\[\pi_1(M_{g,1}/F, \overline{\eta}) \to \pi_1(M_{g,1}/F, \overline{\eta}),\] we obtain a \(G_{g,1}\)-equivariant Lie algebra homomorphism \(\mathfrak{g}_{g,1}^{\text{geom}} \to \text{Der} \mathfrak{p}\) and hence a weight graded Lie algebra homomorphism \(\text{Gr}_g^{W} \mathfrak{g}_{g,1}^{\text{geom}} \to \text{Gr}_g^{W} \text{Der} \mathfrak{p} \cong \text{Der} \text{Gr}_g^{W} \mathfrak{p}\).

**Proposition 10.2** ([11, §9,10]). If \(g \geq 3\), then the homomorphism
\[\text{Gr}_j^{W} \mathfrak{g}_{g,1}^{\text{geom}} \to \text{Gr}_j^{W} \text{Der} \mathfrak{p}\]
is an isomorphism for \(j = -1\) and an injection for \(j = -2\).

Therefore, in order to compute the presentation of the Lie algebra \(\text{Gr}_W^{W} \mathfrak{g}_{g,1}^{\text{geom}} / W_{-3}\), we consider the action of \(\text{Gr}_W^{W} \mathfrak{g}_{g,1}^{\text{geom}} / W_{-3}\) on \(\text{Gr}_W^{W} \mathfrak{p}\). We have seen that the natural weight filtration on \(\mathfrak{p}\) induced by the \(G_{g,1}\) action agrees with the lower central series of \(\mathfrak{p}\). Recall that \(\breve{\theta}\) is a map \(\mathbb{Q}(1) \to \Lambda^2 H\), viewed as a map \(\breve{\theta} : \mathbb{Q}(1) \to L^2(H)\) via the canonical isomorphism \(\Lambda^2 H \cong L^2(H)\).

**Proposition 10.3** ([11, Thm. 5.8]). There is a natural \(G_{Sp}(H)\)-equivariant Lie algebra isomorphism
\[\text{Gr}_W^{W} \mathfrak{p} \cong L(H)/(\text{im} \breve{\theta})\].

**Remark 10.4.** Since inner automorphisms map to the identity element in \(G_{Sp}(H)\), they act trivially on this isomorphism and consequently this isomorphism does not depend on the choice of a base point.

The free Lie algebra \(L(V)\) generated by a \(F\)-vector space \(V\) is graded by bracket length: there is an isomorphism
\[L(V) \cong \bigoplus_{n \geq 1} L_n(V)\]

Now, since a derivation on \(L(H)\) is determined by its effect on \(H\), we see that \(\text{Der} L(H) \cong \text{Hom}_F(H, L(H))\) and that the derivation Lie algebra \(\text{Der} L(H)\) is graded:
\[\text{Der} L(H) \cong \bigoplus_{n \geq 1} \text{Der}_n L(H),\]
where \(\text{Der}_n L(H) := \text{Hom}_F(H, L_{n+1}(H))\).

We recall the following well known fact.

**Proposition 10.5.** If \(g \geq 2\), then there is a natural \(G_{Sp}(H)\)-equivariant graded Lie algebra homomorphism
\[\delta : L((\Lambda^3 H)(-1)) \to \text{Der} L(H)\]
satisfying the properties:

(i) \(\delta(u)\) kills the image of \(\breve{\theta}\) for all \(u \in L((\Lambda^3 H)(-1))\).
(ii) \(\delta(x \wedge \breve{\theta}) = \text{ad}(x) - \theta(x, )\breve{\theta}\) for all \(x \in H\).
(iii) \(\text{im} \{L_2((\Lambda^3 H)(-1)) \to \text{Der}_2 L(H)\} \cong H_{\mathfrak{m}} \oplus \Lambda^2_0 H \oplus F(1)\)

In fact, the homomorphism \(\delta\) is obtained by twisting the \(G_{Sp}(H)\)-equivariant linear mapping
\[\Lambda^3 H \to [\text{Der}_1 L(H)](1)\]
defined by
\[ x \wedge y \wedge z \mapsto (u \mapsto \theta(u, x)[y, z] + \theta(u, y)[z, x] + \theta(u, z)[x, y]).\]

By the property (i), \( \delta \) induces a \( \text{GSp}(H) \)-equivariant graded Lie algebra homomorphism
\[ L((\Lambda^3 H)(-1)) \to \text{Der} \, \text{Gr}^W_{\bullet} p, \]
which we denote by \( \delta \) also.

**Theorem 10.6** ([11, Cor. 5.7 and §9,11]). If \( g \geq 3 \), then there is a Lie algebra surjection
\[ q : L((\Lambda^3 H)(-1)) \to \text{Gr}^W_{\bullet} u_{g,1}^{\text{geom}} \]
that makes the diagram
\[
\begin{array}{c}
L((\Lambda^3 H)(-1)) \xrightarrow{q} \text{Gr}^W_{\bullet} u_{g,1}^{\text{geom}} \\
\downarrow \delta \downarrow \Rightarrow \\
\text{Gr}^W_{\bullet} \text{Der} p
\end{array}
\]
commute. Consequently, there are isomorphisms
\[ \text{Gr}^W_j u_{g,1}^{\text{geom}} \cong \text{Gr}^W_{j+1} \text{Der} p \cong \begin{cases} (\Lambda^3 H)(-1) = \Lambda^3_0 H + H : j = -1, \\ H_{\text{tr}} \oplus \Lambda^2_0 H : j = -2. \end{cases} \]

The exact sequence
\[ 0 \to p \to u_{g,1}^{\text{geom}} \to u_{g}^{\text{geom}} \to 0 \]
induces the exact sequence of weight graded Lie algebras
\[ 0 \to \text{Gr}^W_{\bullet} p \to \text{Gr}^W_{\bullet} u_{g,1}^{\text{geom}} \to \text{Gr}^W_{\bullet} u_{g}^{\text{geom}} \to 0. \]
The copy of \( H \) in \( \text{Gr}^W_{\bullet-1} u_{g,1}^{\text{geom}} \) can be identified as the image of the composition map
\[ H \xrightarrow{i} (\Lambda^3 H(-1)) \xrightarrow{q} \text{Gr}^W_{\bullet} u_{g,1}^{\text{geom}}, \]
where the inclusion \( i : H \to (\Lambda^3 H(-1)) \) is defined by \( x \mapsto x \wedge \tilde{\theta} \). This copy of \( H \) in \( \text{Gr}^W_{\bullet-1} u_{g,1}^{\text{geom}} \) corresponds to the inner derivations \( \text{ad}(x) \) in \( \text{Gr}^W_{\bullet-1} \text{Der} p \). Similarly, identify the copy of \( \Lambda^2_0 H \) in \( \text{Gr}^W_{\bullet-2} u_{g,1}^{\text{geom}} \) as the image of the composition map
\[ \Lambda^2 H \xrightarrow{\lambda^2} \Lambda^2((\Lambda^3 H)(-1)) \xrightarrow{q} \text{Gr}^W_{\bullet} u_{g,1}^{\text{geom}}. \]
In order to determine the presentation of \( \text{Gr}^W_{\bullet-1} u_{g,1}^{\text{geom}} / W^{-3} \), we need to determine the bracket
\[ \Lambda^2 \text{Gr}^W_{\bullet-1} u_{g,1}^{\text{geom}} = \Lambda^2((\Lambda^3 H)(-1)) \to \text{Gr}^W_{\bullet-2} u_{g,1}^{\text{geom}}. \]
Using the decomposition \( (\Lambda^3 H)(-1) = \Lambda^3_0 H \oplus H \), this bracket is decomposed into three \( \text{GSp}(H) \)-equivariant mappings:
\[ \Lambda^2 H \to \text{Gr}^W_{\bullet-2} u_{g,1}^{\text{geom}}, \quad H \oplus \Lambda^3_0 H \to \text{Gr}^W_{\bullet-2} u_{g,1}^{\text{geom}}, \quad \Lambda^2(\Lambda^3_0 H) \to \text{Gr}^W_{\bullet-2} u_{g,1}^{\text{geom}}. \]
These brackets can be computed in \( \text{Gr}^W_{\bullet} \text{Der} p \).
Proposition 10.7 ([11, §12]). If \( g \geq 3 \), then the three brackets are determined as:

\[
\begin{align*}
\text{im} \{ \Lambda^2 H \to \text{Gr}_2^W \text{Der} p \} &= \Lambda^3_0 H, \\
\text{im} \{ H \otimes \Lambda^3_0 H \to \text{Gr}_2^W \text{Der} p \} &= \Lambda^3_0 H, \\
\text{im} \{ \Lambda^2(\Lambda^3_0 H) \to \text{Gr}_2^W \text{Der} p \} &= \begin{cases} \\
H \oplus \Lambda^3_0 H & g = 3, \\
H \oplus \Lambda^3_0 H & g \geq 4. 
\end{cases}
\end{align*}
\]

10.3. Presentations of \( \text{Gr}_n^W u_{g,n}^{\text{geom}}/W_{-3} \). Hain introduces an \( S_n \times \text{GSp}(H) \)-module denoted by \( \Lambda^3_n H \). For \( u \in \Lambda^3 H(-1) \), denote the image of \( u \) in \( \Lambda^3_n H \) by \( \overline{u} \). For each positive integer \( n \), we define

\[
\Lambda^3_n H := \{(u_1, \ldots, u_n) \in (\Lambda^3 H(-1))^n | \overline{u}_1 = \cdots = \overline{u}_n \}.
\]

This \( S_n \times \text{GSp}(H) \) submodule of \( (\Lambda^3 H(-1))^n \) is isomorphic to \( \Lambda^3_n H \oplus H_{\oplus n} \) as an \( S_n \times \text{GSp}(H) \)-module.

Theorem 10.8 ([14, 9.11]). If \( g \geq 3 \) and \( n \geq 0 \), then there are natural \( S_n \times \text{GSp}(H) \)-equivariant isomorphisms

\[
H_1(\overline{u}^{\text{geom}}_{g,n}) \cong H_1(u^{\text{geom}}_{g,n}) \cong \text{Gr}_n^W u_n^{\text{geom}} \cong \Lambda^3_n H.
\]

There is an exact sequence

\[
0 \to \mathbb{Q}_\ell(1)_{n}^{(2)} \to \text{Gr}_n^W u_{g,n}^{\text{geom}} \to \text{Gr}_n^W u_{g,n}^{\text{geom}} \to 0
\]

of \( S_n \times \text{GSp}(H) \)-modules and an \( S_n \times \text{GSp}(H) \)-equivariant isomorphism

\[
\text{Gr}_n^W u_{g,n}^{\text{geom}} \cong H_1 \oplus (\Lambda^2_0 H)^n \oplus \mathbb{Q}_\ell(1)_{n}^{(2)}.
\]

In order to determine the presentations of \( \text{Gr}_n^W u_{g,n}^{\text{geom}}/W_{-3} \), we need to determined the bracket \( [\ , \ ] : \Lambda^2 \text{Gr}_n^W u_{g,n}^{\text{geom}} \to \text{Gr}_n^W u_{g,n}^{\text{geom}} \). For this purpose, we write

\[
\Lambda^3_n H = \Lambda^3_0 H \oplus H_1 \oplus H_2 \oplus \cdots \oplus H_n,
\]

where \( H_j \) corresponds to the \( j \)th marked point. By Theorem 10.8, \( \Lambda^2 \text{Gr}_n^W u_{g,n}^{\text{geom}} \) can be decomposed as

\[
\Lambda^2 \text{Gr}_n^W u_{g,n}^{\text{geom}} \cong \Lambda^2 \Lambda^3_0 H \oplus \bigoplus_{j=1}^{n} (H_j \otimes \Lambda^3_0 H) \oplus \bigoplus_{j=1}^{n} \Lambda^2 H_j \oplus \bigoplus_{i<j} H_i \otimes H_j,
\]

and we also have

\[
\text{Gr}_n^W u_{g,n}^{\text{geom}} \cong \bigoplus_{j=1}^{n} \Lambda^2_0 H_j \oplus \bigoplus_{i<j} \mathbb{Q}_\ell(1)_{ij} \oplus H_{\oplus n}.
\]

By [11, §13], we may choose this decomposition so that the bracket

\[
[\ , \ ] : \Lambda^2 H_j \to \text{Gr}_n^W u_{g,n}^{\text{proj}} \Rightarrow \Lambda^2_0 H_j
\]

is the quotient map and so that the bracket

\[
[\ , \ ] : H \otimes H \cong H_i \otimes H_j \to \text{Gr}_n^W u_{g,n}^{\text{proj}} \Rightarrow \mathbb{Q}_\ell(1)_{ij}
\]
is the cup product pairing $\theta$. Fix $\text{GSp}(H)$-equivariant projections

$$
c : \Lambda^2 \Lambda_0^3 H \to \Lambda_0^3 H \quad d : H \otimes \Lambda_0^3 H \to \Lambda_0^3 H$$

$$
e : \Lambda^2 H \to \Lambda_0^3 H \quad \psi : \Lambda^2 \Lambda_0^3 H \to \mathbb{Q}_l(1).$$

By Proposition 3.2, each of these projections are unique up to a scalar multiplication. Denote the $\text{GSp}(H)$-equivariant projections

$$
\Lambda^2 \text{Gr}_{-1}^W u_{g,n}^{\text{geom}} \to \Lambda^2 \Lambda_0^3 H \quad \theta \to \Lambda_0^3 H
$$

$$
\Lambda^2 \text{Gr}_{-1}^W u_{g,n}^{\text{geom}} \to H_i \otimes \Lambda_0^3 H \quad d_i \to \Lambda_0^3 H
$$

$$
\Lambda^2 \text{Gr}_{-1}^W u_{g,n}^{\text{geom}} \to \Lambda^2 H_j \quad \psi, \theta \to \Lambda_0^3 H
$$

$$
\Lambda^2 \text{Gr}_{-1}^W u_{g,n}^{\text{geom}} \to H_i \otimes H_j \quad \theta \to \mathbb{Q}_l(1)
$$

$$
\Lambda^2 \text{Gr}_{-1}^W u_{g,n}^{\text{geom}} \to \Lambda^2 \Lambda_0^3 H \quad \psi \to \mathbb{Q}_l(1)
$$

by $c$, $d_j$, $e_j$, $e_{ij}$, $\theta_i$, $\theta_{ij}$, and $\psi$, respectively. By Proposition 3.2 and the above decomposition of $\Lambda^2 \text{Gr}_{-1}^W u_{g,n}^{\text{geom}}$, we have

**Proposition 10.9** ([14, 9.12]). If $g \geq 3$ and $n \geq 0$, then

$$
\text{Hom}_{\text{GSp}(H)}(\Lambda^2 \text{Gr}_{-1}^W u_{g,n}^{\text{geom}}, \Lambda_0^3 H)
$$

has a basis

$$
\{d_1, \ldots, d_n, e_1, \ldots, e_n, e_{ij} : 1 \leq i < j \leq n\} : g = 3
$$

$$
\{c, d_1, \ldots, d_n, e_1, \ldots, e_n, e_{ij} : 1 \leq i < j \leq n\} : g \geq 4
$$

and

$$
\text{Hom}_{\text{GSp}(H)}(\Lambda^2 \text{Gr}_{-1}^W u_{g,n}^{\text{geom}}, \mathbb{Q}_l(1))
$$

has a basis

$$
\{\psi, \theta_1, \ldots, \theta_n, \theta_{ij} : 1 \leq i < j \leq n\}
$$

for all $g \geq 3$.

Denote the projections

$$
\text{Gr}_{-2}^W u_{g,n}^{\text{geom}} \to \Lambda_0^3 H_j
$$

$$
\text{Gr}_{-2}^W u_{g,n}^{\text{geom}} \to \mathbb{Q}_l(1)_{ij}
$$

$$
\text{Gr}_{-2}^W u_{g,n}^{\text{geom}} \to H_{ij}
$$

by $p_j$, $q_{ij}$, and $p_{ij}$, respectively.

**Proposition 10.10** ([14, 9.13],[11, §12]). If $g \geq 3$, then after rescaling $c$, $d$, and $\psi$ by nonzero constants if necessary,

$$
p_j \circ [ , ] = d_j + e_j \quad : g = 3
$$

$$
p_j \circ [ , ] = c + d_j + e_j \quad : g \geq 4
$$

$$
q_{ij} \circ [ , ] = \psi + \theta_{ij} - 1/g(\theta_i + \theta_j) \quad : g \geq 3
$$

Furthermore, $p_{ij}$ is non-zero and restricts to zero on each $H_j \otimes \Lambda_0^3 H$ and each $H_i \otimes H_j$ for $i \leq j$. 
11. The Lie Algebras $\mathfrak{d}_{g,n}$

In this section, we associate a graded two-step Lie algebra $\mathfrak{d}_{g,n}$ to the graded Lie algebra $\text{Gr}^W_{3} u_{g,n}^{\text{geom}}$. When $g \geq 4$, the $n$ tautological sections of the universal curve determine the sections of the projection $\mathfrak{d}_{g,n+1} \rightarrow \mathfrak{d}_{g,n}$.

For $g \geq 3$ and $n \geq 0$, define

$$
\mathfrak{d}_{g,n} = \left( \text{Gr}^W_{3} u_{g,n}^{\text{geom}} / W_{-3} \right) / (\Lambda^3_{n} H)^{\perp},
$$

where $(\Lambda^3_{n} H)^{\perp}$ denotes the $\text{GSp}(H)$-invariant complement of its $\Lambda^3_{n} H$-isotypical component. This is a graded Lie algebra in the category of $S_n \times \text{GSp}(H)$-modules and each $j$th graded quotient is given by

$$
(\mathfrak{d}_{g,n})_j = \begin{cases} 
\Lambda^3_{n} H & : j = -1 \\
(\Lambda^3_{n} H)^n & : j = -2 \\
0 & : j \leq -3 
\end{cases}
$$

Let $F$ be a number field, a finite extension of $\mathbb{Q}_p$, or a finite field of characteristic $p$. Recall that $M$ denotes the étale covers $M_{g,n}^{\lambda}$ and $\mathcal{M}_{g,n}[\ell]$. Assume that $M \otimes F$ is decomposed into finitely many geometrically connected components. Fix a component of $M \otimes F$ and denote it by $M_F$.

**Proposition 11.1** ([14, 10.3]). Each section $x$ of the universal curve $f : C_F \to M_F$ induces a well-defined $\text{GSp}(H_{\mathbb{Q}_p})$-equivariant section of $\mathfrak{d}_{g,n+1} \rightarrow \mathfrak{d}_{g,n}$.

**Proof.** By Proposition 9.10, each section $x$ of the family $C_F \to M_F$ induces a well-defined $\text{GSp}(H)$-equivariant section $\text{Gr}^W \mathfrak{d}_{\sigma_{x}^{\text{geom}}}$ of the projection $\text{Gr}^W_{3} f_{x} : \text{Gr}^W_{3} u_{g,n}^{\text{geom}} \to \text{Gr}^W_{3} u_{g,n}^{\text{geom}}$. The open immersion $\mathcal{M}_{g,n+1} / \mathbb{Q}_p \to \mathcal{C}_{g,n} / \mathbb{Q}_p$ induces a surjection $\pi_1(\mathcal{M}_{g,n+1} / \mathbb{Q}_p, \mathfrak{m}) \to \pi_1(\mathcal{C}_{g,n} / \mathbb{Q}_p, \mathfrak{m})$, which induces a Lie algebra surjection $\mathfrak{g}_{g,n+1}^{\text{geom}} \to \mathfrak{g}_{g,n}^{\text{geom}}$. This Lie algebra surjection is a $\mathcal{G}_{g,n+1}$-module homomorphism and hence it preserves the natural weight filtrations on $\mathfrak{g}_{g,n+1}^{\text{geom}}$ and $\mathfrak{g}_{g,n}^{\text{geom}}$. Moreover its kernel is isomorphic to $Q_{d}(1)^n$. Thus it follows that

$$
\left( \text{Gr}^W_{3} u_{g,n}^{\text{geom}} / W_{-3} \right) / (\Lambda^3_{n} H)^{\perp} \cong \left( \text{Gr}^W_{3} u_{g,n+1}^{\text{geom}} / W_{-3} \right) / (\Lambda^3_{n} H)^{\perp} = \mathfrak{d}_{g,n+1},
$$

and hence that the section $\text{Gr}^W_{3} \mathfrak{d}_{\sigma_{x}^{\text{geom}}}$ induces a section $s_x$ of $\mathfrak{d}_{g,n+1} \rightarrow \mathfrak{d}_{g,n}$. □

Using the decomposition $\Lambda^3_{n} H \cong \Lambda^3_{n} H \oplus H_1 \oplus H_2 \oplus \cdots \oplus H_n$, we denote the elements of $\Lambda^3_{n} H$ by $(v; u_1, \ldots, u_n)$, where $v$ is an element of $\Lambda^3_{n} H$ and $u_i$ is an element of $H_j$ for each $j = 1, \ldots, n$. The surjectivity of the map $H \otimes \Lambda^3_{n} H \rightarrow \Lambda^3_{n} H$ implies that the linear projection

$$
\Lambda^3_{n+1} H \rightarrow \Lambda^3_{n} H, \quad (v; u_0, u_1, \ldots, u_n) \mapsto (v; u_1, \ldots, u_n)
$$

induces a $\text{GSp}(H)$-equivariant Lie algebra projection

$$
\epsilon_n : \mathfrak{d}_{g,n+1} \rightarrow \mathfrak{d}_{g,n}.
$$

The following result is a key for understanding the rational points of the universal curve and follows from Schur’s lemma.
Proposition 11.2 ([14, 10.4]). If \( g \geq 4 \), then there are exactly \( n \) \( \text{GSp}(H) \)-equivariant sections of \( \epsilon_n \):

\[
s_j : (v; u_1, \ldots, u_n) \mapsto (v; u_j, u_1, \ldots, u_n)
\]

for each \( j = 1, \ldots, n \).

For \( g = 3 \), the sections of \( \epsilon_n \) are \( s_1, \ldots, s_n \) and the section

\[
\zeta_n : (v; u_1, \ldots, u_n) \mapsto (v; 0, u_1, \ldots, u_n).
\]

For \( n = 1 \), the section \( s_1 \) is induced by the tautological section of the universal curve \( \mathcal{C}_{g,1} \rightarrow \mathcal{M}_{g,1} \). For \( n > 1 \), considering the \( j \)th projection \( \mathcal{M}_{g,n} \rightarrow \mathcal{M}_{g,1} \), we can see that the \( j \)th tautological section induces the section \( s_j \). In deed, there is the commutative diagram

\[
\begin{array}{c}
\mathbf{d}_{g,n+1} \rightarrow \mathbf{d}_{g,2} \\
\downarrow s'_j \quad \downarrow \quad \downarrow s_j \\
\mathbf{d}_{g,n} \rightarrow \mathbf{d}_{g,1}
\end{array}
\]

where \( pr_j \) is the map induced by the \( j \)th projection \( \mathcal{M}_{g,n} \rightarrow \mathcal{M}_{g,1} \) and \( s'_j \) is the map induced by the \( j \)th tautological section. From this diagram, it is easy to see that the section \( s'_j \) agrees with the section \( s_j \).

When \( g = 3 \), there exists an extra section \( \zeta_n \) of \( \epsilon_n \). This is due to the fact that \( \Lambda^2 \Lambda^3 H \) does not contain \( \Lambda^3 H \). We will explain here that the section \( \zeta_n \) cannot be induced by a rational point. The point is that the section induced by a rational point respects a natural integral structure on \( \text{Gr}_{W_1}^{H} u_{g,n}^{\text{geom}} \) that comes from the image of the Torelli group in \( U_{g,n}^{\text{geom}} \), see [11]. Suppose that \( A \) is an integral domain with fraction field \( F \). For a positive integer \( n \), define a lattice in \( \Lambda^3 H_F \) by

\[
\Lambda^3 H_A = \{ (u_1, \ldots, u_n) \in (\Lambda^3 H_A)^n : \bar{u}_1 = \cdots = \bar{u}_n \} (-1).
\]

Proposition 11.3. [14, 10.7] Suppose that \( g \geq 3 \) and \( n \geq 0 \). If \( A \) is an integral domain and \( g-1 \not\in A^\times \), then \( \zeta_n \) does not restrict to a section of the projection

\[
\Lambda^3_{n+1} H_A \rightarrow \Lambda^3_n H_A, \quad (u_0, u_1, \ldots, u_n) \mapsto (u_1, \ldots, u_n)
\].

As a summary of this section, we have

Corollary 11.4 ([14, 10.8]). Suppose that \( n \geq 1 \). If \( g \geq 4 \) and any prime number \( \ell \), or if \( g = 3 \) and \( \ell = 2 \), then the only \( \text{GSp}(H_{Q_\ell}) \)-equivariant sections of \( \epsilon_n : \mathbf{d}_{g,n+1} \rightarrow \mathbf{d}_{g,n} \) that respect the integral structure on \( \text{Gr}_{W_1}^{H} \) are the ones induced by the tautological sections of the universal curve. In particular, there are no \( \text{GSp}(H) \)-equivariant sections of \( \mathbf{d}_{g,1} \rightarrow \mathbf{d}_{g,0} \).

12. The Characteristic Class of A Rational Point

In [14], Hain defined a characteristic class \( \kappa_x \) for a \( T \)-rational point \( x \) of the curve \( C \rightarrow T \), where \( T \) is a smooth variety over a field \( k \) with \( \text{char} (k) = 0 \). For our comparison purpose, we need to redefine this characteristic class for curves \( C \rightarrow T \), where \( T \) is defined over a more general base ring, e.g., \( \mathbb{Z}_p \). In this section, we will explain how this can be done and extend the results used in [14] to positive
characteristics. Let $B$ be a connected scheme. Suppose that $T$ is a geometrically connected smooth scheme over $B$ and that $f : C \to T$ is a curve of genus $g$. In this section, we associate a cohomology class $\kappa_x$ in $H^1_{\text{ét}}(T, R^1 f_* \mathbb{Q}(1))$ to a rational point $x \in C(T)$.

Denote the relative Jacobian of $f : C \to T$ by $\pi : J_{C/T} \to T$. $J_{C/T}$ is a family of Jacobians and is an abelian scheme over $T$. Note that $J_{C/T}$ has a zero section $s_0 : T \to J_{C/T}$. Let $\overline{\eta} : \text{Spec } \Omega \to T$ be a geometric point of $T$. Denote the fiber of $f$ over $\overline{\eta}$ by $C_{\overline{\eta}}$ and the fiber of $J_{C/T} \to T$ over $\overline{\eta}$ by $(J_{C/T})_{\overline{\eta}}$. Let $\overline{x}$ be a geometric point of $C_{\overline{\eta}}$. Note that $(J_{C/T})_{\overline{\eta}}$ is the jacobian variety of the curve $C_{\overline{\eta}}$. When $\ell$ is not in $\text{char}(T)$, there are natural isomorphisms

$$\pi_1((J_{C/T})_{\overline{\eta}}, \overline{x})^{(\ell)} \cong \pi_1(C_{\overline{\eta}}, \overline{x})^{(\ell), \text{ab}} \cong H^1_{\text{ét}}(C_{\overline{\eta}}, \mathbb{Z}_\ell(1)),$$

where $\text{ab}$ denotes maximal abelian quotient. Denote the lisse sheaf $R^1 f_* A(1)$ over $T$ by $\mathbb{H}_A$, where $A = \mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$. Then we have

$$H_A := H^1_{\text{ét}}(C_{\overline{\eta}}, A(1)) = (\mathbb{H}_A)_{\overline{\eta}}.$$

By Proposition 4.3, there is an exact sequence

$$1 \to \pi_1((J_{C/T})_{\overline{\eta}}, \overline{x})^{(\ell)} \to \pi_1'(J_{C/T}, \overline{x}) \to \pi_1(T, \overline{\eta}) \to 1.$$

Thus the zero section $s_0$ determines a splitting

$$\pi_1'(J_{C/T}, \overline{x}) \cong \pi_1((J_{C/T})_{\overline{\eta}}, \overline{x})^{(\ell)} \times \pi_1(T, \overline{\eta}) \cong H^1_{\text{ét}}(C_{\overline{\eta}}, \mathbb{Z}_\ell) \times \pi_1(T, \overline{\eta}),$$

which is well-defined up to conjugation action of $H^1_{\text{ét}}(C_{\overline{\eta}}, \mathbb{Z}_\ell)$. To each rational point $x \in C(T)$, we associate the divisor $D_x := (2g - 2)x - \omega_{C/T}$, where $\omega_{C/T}$ is the relative canonical divisor of the family $C \to T$. The divisor $D_x$ is homologically trivial on each geometric fiber, and hence gives a section of $J_{C/T} \to T$, which determines a $\kappa_x$ in

$$H^1_{\text{ét}}(\pi_1(T, \overline{\eta}), H^1_{\text{ét}}(C_{\overline{\eta}}, \mathbb{Z}_\ell)).$$

Tensoring with $\mathbb{Q}_\ell$, we obtain a class in $H^1_{\text{ét}}(T, \mathbb{H}_\ell)$, which we denote also by $\kappa_x$.

**Remark 12.1.** This class behaves well under base change.

### 12.1. Classes of the universal curve over $\mathcal{M}_{g,n}$.

Let $F$ be a field of characteristic zero. Suppose that $T$ is a noetherian geometrically connected scheme over $F$. Denote the class in $H^1_{\text{ét}}(\mathcal{M}_{g,1/F}, \mathbb{H}_\ell)$ of the tautological section of the universal curve $C_{g,1/F} \to \mathcal{M}_{g,1/F}$ by $\kappa$. This class is universal in the sense that for each rational point $x \in C(T)$, the class $\kappa_x \in H^1_{\text{ét}}(T, \mathbb{H}_\ell)$ is the pullback of $\kappa$, i.e., $\kappa_x = \phi^* \kappa$, where $\phi : T \to \mathcal{M}_{g,1/F}$ is the morphism induced by $x$. Denote the class of the $j$th tautological section of the universal curve $C_{g,n/F} \to \mathcal{M}_{g,n/F}$ by $\kappa_j$.

**Proposition 12.2 (14, 12.1).** If $g \geq 3$, $n \geq 0$, and $m \geq 1$, then for all fields $F$ of characteristic zero,

$$H^1_{\text{ét}}(\mathcal{M}_{g,n/F}[m], \mathbb{H}_\ell) = \mathbb{Q}_\ell \kappa_1 \oplus \mathbb{Q}_\ell \kappa_2 \oplus \cdots \oplus \mathbb{Q}_\ell \kappa_n.$$

\hfill $\square$

Suppose that $p$ is a prime number, and that $\ell$ is a prime number distinct from $p$ and $m$ is a positive integer such that $\ell^m \geq 3$. Denote a connected component of the base change to $\mathbb{Z}_p^{ur}$ of $\mathcal{M}_{g,n}[\ell^m]$ by $\mathcal{M}_{g,n}^{ur}[\ell^m]$. Denote the universal curve over
that makes the diagram of the exact sequence \( \pi C \) a geometric point of the fiber \( J \)
determines an isomorphism \( M \) over \( \bar{M} \)
made by \( \bar{M} \) and \( M \), respectively. Let \( \bar{\xi} \) and \( \bar{\eta} \) be geometric points of \( M_{\bar{p}}[\ell^m] \) and \( \bar{M}_{\bar{p}}[\ell^m] \), respectively. We consider \( \bar{\xi} \) and \( \bar{\eta} \) as geometric points of \( M_{g,n} / Z \) via canonical morphisms induced by base change. Denote the fiber over \( \bar{\xi} \) and \( \bar{\eta} \) of \( C_{Z_p} [\ell^m] \rightarrow \mathcal{M}_{Z_p} [\ell^m] \) by \( C_{\bar{\xi}} \) and \( C_{\bar{\eta}} \). Let \( \bar{x} \) and \( \bar{x}' \) be a geometric point of \( C_{\bar{\xi}} \) and \( C_{\bar{\eta}} \), respectively. By Proposition 4.3, we have the diagram (**) 
\[
\begin{array}{cccccccc}
1 & \rightarrow & \pi_1(C_{\bar{\xi}}, \bar{x}') & \rightarrow & \pi_1(J_{\bar{\xi}}[\ell^m], \bar{x}') & \rightarrow & \pi_1(\mathcal{M}_{\bar{p}}[\ell^m], \bar{\xi}) & \rightarrow & 1 \\
1 & \rightarrow & \pi_1(C_{\bar{\xi}}, \bar{x}) & \rightarrow & \pi_1(J_{\bar{\xi}}[\ell^m], \bar{x}) & \rightarrow & \pi_1(\mathcal{M}_{\bar{p}}[\ell^m], \bar{\xi}) & \rightarrow & 1 \\
1 & \rightarrow & \pi_1(C_{\bar{\eta}}, \bar{x}) & \rightarrow & \pi_1(J_{\bar{\eta}}[\ell^m], \bar{x}) & \rightarrow & \pi_1(\mathcal{M}_{\bar{p}}[\ell^m], \bar{\eta}) & \rightarrow & 1 \\
1 & \rightarrow & \pi_1(C_{\bar{\eta}}, \bar{x}') & \rightarrow & \pi_1(J_{\bar{\eta}}[\ell^m], \bar{x}') & \rightarrow & \pi_1(\mathcal{M}_{\bar{p}}[\ell^m], \bar{\eta}) & \rightarrow & 1 
\end{array}
\]
that commutes after fixing an isomorphism \( \pi_1(J_{\bar{\xi}}[\ell^m], \bar{x}') \cong \pi_1(J_{\bar{\xi}}[\ell^m], \bar{x}) \), which determines an isomorphism \( \pi_1(\mathcal{M}_{\bar{p}}[\ell^m], \bar{\xi}) \cong \pi_1(\mathcal{M}_{\bar{p}}[\ell^m], \bar{\eta}) \). The rows of the diagram are exact and the vertical maps between the second and third row are isomorphisms.

**Lemma 12.3.** Suppose that \( n \geq 1 \). If \( \bar{x} \) is a geometric point of \( \mathcal{M}_{\bar{p}}[\ell^m] \) and \( \bar{y} \) is a geometric point of the fiber \( C_{\bar{x}} \), then the sequence of the maximal pro-\( \ell \) quotients 
\[
1 \rightarrow \pi_1(C_{\bar{x}}, \bar{y})^{(\ell), ab} \rightarrow \pi_1(J_{\bar{p}}[\ell^m], \bar{y})^{(\ell)} \rightarrow \pi_1(\mathcal{M}_{\bar{p}}[\ell^m], \bar{y})^{(\ell)} \rightarrow 1 
\]
of the exact sequence 
\[
1 \rightarrow \pi_1(C_{\bar{x}}, \bar{y})^{(\ell), ab} \rightarrow \pi_1(J_{\bar{p}}[\ell^m], \bar{y}) \rightarrow \pi_1(\mathcal{M}_{\bar{p}}[\ell^m], \bar{y}) \rightarrow 1 
\]
is exact.

**Proof.** A tautological section induces the closed immersion \( \psi : C_{\bar{p}}[\ell^m] \rightarrow J_{\bar{p}}[\ell^m] \) that makes the diagram 
\[
(\ast) \quad 1 \rightarrow \pi_1(C_{\bar{x}}, \bar{y})^{(\ell)} \rightarrow \pi_1(C_{\bar{p}}[\ell^m], \bar{y}) \rightarrow \pi_1(\mathcal{M}_{\bar{p}}[\ell^m], \bar{y}) \rightarrow 1 
\]
\[
1 \rightarrow \pi_1(C_{\bar{x}}, \bar{y})^{(\ell), ab} \rightarrow \pi_1(J_{\bar{p}}[\ell^m], \bar{y}) \rightarrow \pi_1(\mathcal{M}_{\bar{p}}[\ell^m], \bar{y}) \rightarrow 1 
\]
commute, where the left-hand vertical map is the canonical projection. Denote the kernel of the projection \( \pi_1(C_{\bar{x}}, \bar{y})^{(\ell)} \rightarrow \pi_1(C_{\bar{x}}, \bar{y})^{(\ell), ab} \) by \( N \). Then \( \psi_* \) induces an isomorphism 
\[
\pi_1(C_{\bar{p}}[\ell^m], \bar{y}) / N \cong \pi_1(J_{\bar{p}}[\ell^m], \bar{y}).
\]
Since \( \pi_1(C_{\bar{x}}, \bar{y})^{(\ell)} \rightarrow \pi_1(C_{\bar{p}}[\ell^m], \bar{y})^{(\ell)} \) is injective, there is an isomorphism 
\[
\pi_1(C_{\bar{p}}[\ell^m], \bar{y})^{(\ell)} / N \cong \left( \pi_1(C_{\bar{p}}[\ell^m], \bar{y}) / N \right)^{(\ell)}. 
\]

Taking maximal pro-$\ell$ quotient of the diagram (**) and pushing out along the surjection $\pi_1(C_z, \tilde{y})(t) \to \pi_1(C_z, \tilde{y})(t)_{ab}$, we obtain the commutative diagram

\[
\begin{array}{c}
1 \to \pi_1(C_z, \tilde{y})(t)_{ab} \\
\downarrow \\
\pi_1(C_z, \tilde{y})(t)_{ab} \to \pi_1(J_{\mathcal{Z}_p}[m], \tilde{y})(t) \to \pi_1(\mathcal{M}_{\mathcal{Z}_p}[m], \tilde{y})(t) \to 1
\end{array}
\]

where the middle vertical map is an isomorphism. Thus it follows that the map $\pi_1(C_z, \tilde{y})(t)_{ab} \to \pi_1(J_{\mathcal{Z}_p}[m], \tilde{y})(t)$ is injective.

**Proposition 12.4.** Assume the notations above. If $g \geq 3$ and $n \geq 1$, then

\[
H^1_{\text{cts}}(\mathcal{M}_{g,n}/\mathbb{F}_p[m], \mathbb{H}_{\kappa_t}) = \mathbb{Q}_l\kappa_1 \oplus \mathbb{Q}_l\kappa_2 \oplus \cdots \oplus \mathbb{Q}_l\kappa_n.
\]

Moreover, we have

\[
H^2_{\text{cts}}(\mathcal{M}_{g,n}/\mathbb{F}_p[m], \mathbb{H}_{\kappa_t}) = \mathbb{Q}_l\kappa_1 \oplus \mathbb{Q}_l\kappa_2 \oplus \cdots \oplus \mathbb{Q}_l\kappa_n,
\]

where $\mathbb{F}_q = \mathbb{F}_p[\zeta_{m}]$ and $\zeta_{m}$ is a primitive $m$th root of unity.

**Proof.** By Lemma 12.3, taking pro-$\ell$ completion of the diagram (**) gives the commutative diagram

\[
\begin{array}{c}
1 \to \pi_1(C_t, \tilde{x})(t)_{ab} \\
\downarrow \\
\pi_1(C_t, \tilde{x})(t)_{ab} \to \pi_1(J_{\mathcal{Z}_p}[m], \tilde{x})(t) \to \pi_1(\mathcal{M}_{\mathcal{Z}_p}[m], \tilde{x})(t) \to 1
\end{array}
\]

whose rows are exact and the vertical maps between the second and third row are isomorphisms induced by change of base points. Furthermore, the maps

\[
\pi_1(\mathcal{M}_{\mathcal{Z}_p}[m], \tilde{y})(t) \to \pi_1(\mathcal{M}_{\mathcal{Z}_p}[m], \tilde{y})(t) \to \pi_1(\mathcal{M}_{\mathcal{Z}_p}[m], \tilde{y})(t) \to \pi_1(\mathcal{M}_{\mathcal{Z}_p}[m], \tilde{y})(t)
\]

are isomorphisms, and hence by exactness all the vertical maps are isomorphisms. This implies that there is an isomorphism

\[
H^1_{\text{cts}}(\pi_1(\mathcal{M}_{g,n}/\mathbb{F}_p[m], \mathbb{H}_{\kappa_t}))(\mathbb{H}_{\kappa_t}) \cong H^1_{\text{cts}}(\pi_1(\mathcal{M}_{g,n}/\mathbb{F}_p[m], \mathbb{H}_{\kappa_t}))(\mathbb{H}_{\kappa_t}).
\]

For $A = \mathbb{Q}_p$, $\mathbb{F}_p$, $\gamma = \tilde{y}$, $\tilde{x}$, and $\tilde{y} = \tilde{x}$, $\tilde{x}'$, respectively, the diagram

\[
\begin{array}{c}
1 \to \pi_1(C_t, \tilde{y})(t)_{ab} \\
\downarrow \\
\pi_1(C_t, \tilde{y})(t)_{ab} \to \pi_1(J_{\mathcal{A}}[m], \tilde{y}) \to \pi_1(\mathcal{A}_{\mathcal{A}}[m], \tilde{y}) \to 1
\end{array}
\]

is the pullback diagram along the surjection $\pi_1(\mathcal{A}_{\mathcal{A}}[m], \tilde{y}) \to \pi_1(\mathcal{A}_{\mathcal{A}}[m], \tilde{y})$. Thus there is a canonical isomorphism

\[
H^1_{\text{cts}}(\pi_1(\mathcal{M}_{g,n}/\mathbb{A}[m], \mathbb{H}_{\kappa_t}))(\mathbb{H}_{\kappa_t}) \cong H^1_{\text{cts}}(\pi_1(\mathcal{M}_{g,n}/\mathbb{A}[m], \mathbb{H}_{\kappa_t}))(\mathbb{H}_{\kappa_t}).
\]
Therefore, we have isomorphisms
\[
H^1_{\text{et}}(\mathcal{M}_{g,n}/\mathbb{Q}_q[\ell^m], \mathbb{H}_{\mathbb{Z}_\ell}) \cong H^1_{\text{cts}}(\pi_1(\mathcal{M}_{g,n}/\mathbb{Q}_q[\ell^m], \eta), (\mathbb{H}_{\mathbb{Z}_\ell})_\eta)
\]
\[
\cong H^1_{\text{cts}}(\pi_1(\mathcal{M}_{g,n}/\mathbb{Q}_q[\ell^m], \gamma), (\mathbb{H}_{\mathbb{Z}_\ell})_\xi)
\]
\[
\cong H^1_{\text{et}}(\mathcal{M}_{g,n}/\mathbb{Q}_q[\ell^m], \mathbb{H}_{\mathbb{Z}_\ell}).
\]
Under this isomorphism, the classes $\kappa_j$ of the $j$th tautological section correspond in $H^1_{\text{et}}(\mathcal{M}_{g,n}/\mathbb{Q}_q[\ell^m], \mathbb{H}_{\mathbb{Z}_\ell})$ and $H^1_{\text{et}}(\mathcal{M}_{g,n}/\mathbb{Q}_q[\ell^m], \mathbb{H}_{\mathbb{Z}_\ell})$. Hence our claim follows from Proposition 12.2. As to the second claim, the spectral sequence
\[
H^*(G_{\mathbb{Q}_q}, H^1_{et}(\mathcal{M}_{\mathbb{Q}_q}, \mathbb{H}_{\mathbb{Z}_\ell})) \Rightarrow H^{*+1}(\mathcal{M}_{\mathbb{Q}_q}, \mathbb{H}_{\mathbb{Z}_\ell})
\]
and the fact that $H^0_{et}(\mathcal{M}_{\mathbb{Q}_q}, \mathbb{H}_{\mathbb{Z}_\ell}) = 0$ imply that we have
\[
H^1_{et}(\mathcal{M}_{\mathbb{Q}_q}[\ell^m], \mathbb{H}_{\mathbb{Z}_\ell}) = H^1_{et}(\mathcal{M}_{\mathbb{Q}_q}[\ell^m], \mathbb{H}_{\mathbb{Z}_\ell})^{G_{\mathbb{Q}_q}} \subset H^1_{et}(\mathcal{M}_{\mathbb{Q}_q}[\ell^m], \mathbb{H}_{\mathbb{Z}_\ell}).
\]
Since the tautological sections are defined over $\mathbb{Z}$ and hence defined over $\mathbb{F}_q$ by base change, the corresponding classes $\kappa_j$'s lie in $H^1_{et}(\mathcal{M}_{\mathbb{Q}_q}[\ell^m], \mathbb{H}_{\mathbb{Z}_\ell})^{G_{\mathbb{Q}_q}}$. Tensoring with $\mathbb{Q}_\ell$, we have
\[
H^1_{et}(\mathcal{M}_{\mathbb{Q}_q}[\ell^m], \mathbb{Q}_\ell) = H^1_{et}(\mathcal{M}_{\mathbb{Q}_q}[\ell^m], \mathbb{Q}_\ell) = \mathbb{Q}_\ell\kappa_1 \oplus \mathbb{Q}_\ell\kappa_2 \oplus \cdots \oplus \mathbb{Q}_\ell\kappa_n.
\]
\[\square\]

12.2. The $\ell$-adic Abel-Jacobi map. Suppose that $\pi : A \to T$ is an abelian scheme over a smooth scheme over a field $F$ whose fibers are polarized abelian varieties. For a prime number $\ell$ not equal to char($F$), the $\ell$-adic Abel-Jacobi map agrees with the association
\[
A(T) \to H^1_{et}(T, R^1\pi_*\mathbb{Z}/\ell(1)), \quad x \mapsto \kappa_x.
\]

Lemma 12.5 ([14, 12.2]). If $\pi : A \to T$ is a family of polarized abelian varieties over a noetherian scheme $T$, then the kernel of the $\ell$-adic Abel-Jacobi map
\[
A(T) \to H^1_{et}(T, \mathbb{H}_{\mathbb{Z}_\ell})
\]
is the subgroup $\bigcap_{n} \ell^n A(T)$ of $\ell^n$-divisible points, where $\ell$ is not in char ($T$).

Corollary 12.6 ([14, 12.3]). With notations as above, if the group $A(T)$ of sections of $\pi : A \to T$ is finitely generated, then the kernel of
\[
A(T) \to H^1_{et}(T, \mathbb{H}_{\mathbb{Q}_\ell})
\]
is finite.

Remark 12.7. By a generalization of the Mordell-Weil Theorem [26] by Néron, when $T$ is a geometrically connected smooth variety over a field that is finitely generated over its prime subfield, $A(T)$ is finitely generated. This is the case, for example, for the universal curve $C_{g,n/\mathbb{Q}_q}[\ell^m] \to \mathcal{M}_{g,n/\mathbb{Q}_q}[\ell^m]$.

Applying this result to the relative Jacobian $\pi : J_{C/T} \to T$ associated to the family of curves $f : C \to T$, where $T$ is a geometrically connected smooth variety over a field $F$.

Corollary 12.8 ([14, 12.4]). Assume that the group of sections $J_{C/T}(T)$ of $\pi : J_{C/T} \to T$ is finitely generated. If $x$ and $y$ are sections of $f : C \to T$ and $\kappa_x = \kappa_y$, then $x - y$ is torsion in $J_{C/T}(T)$. 
Proof. Recall that the classes $\kappa_x$ and $\kappa_y$ are the images of $(2g - 2)x - \omega_{C/T}$ and $(2g - 2)y - \omega_{C/T}$, respectively. That $\kappa_x = \kappa_y$ implies that $(2g - 2)(x - y)$ lies in the kernel of the map $J_{C/T}(T) \to H_{\text{et}}^1(T, \mathbb{H}_Q)$. By Corollary 12.6, the divisor $(2g - 2)(x - y)$ is torsion. \hfill \square

12.3. The image of $\kappa_j$ in $\text{Hom}_{\text{GSp}(H)}(H_1(u_{g,n}^{\text{geom}}), H)$. Proposition 7.4 implies that there is a natural isomorphism

$$H_{\text{et}}^1(M_{g,n/\mathbb{Q}_p}[\ell^m], \mathbb{H}_Q) \cong \text{Hom}_{\text{GSp}(H)}(H_1(u_{g,n}^{\text{geom}}), H_{\mathbb{Q}_r}).$$

We can explicitly describe the image of the class $\kappa_j$ in $\text{Hom}_{\text{GSp}(H)}(H_1(u_{g,n}^{\text{geom}}), H_{Q_r})$.

Denote the $\text{GSp}(H)$-equivariant projection $\Lambda^3_{2}H \to H$ by $h$. This projection is induced by twisting the projection $\Lambda^3_{1}H \to H(1)$:

$$x \wedge y \wedge z \mapsto \theta(x, y)z + \theta(y, z)x + \theta(z, x)y.$$

Denote the $\text{GSp}(H)$-equivariant homomorphism $\Lambda^3_{n}H \to H$ by $h_j$.

**Proposition 12.9** ([14, 12.5 & 12.6],[16, 6.5]). If $g \geq 3$ and $n \geq 1$, for each $j = 1, \ldots, n$, the $\text{GSp}(H)$-equivariant homomorphism

$$H_1(u_{g,n}^{\text{geom}}) \cong \text{Gr}_{W_1}^W u_{g,n}^{\text{geom}} \cong \Lambda^3_{n}H \xrightarrow{h_j} H$$

corresponds to the class $\kappa_j$ under the isomorphism

$$H_{\text{et}}^1(M_{g,n/\mathbb{Q}_p}[\ell^m], \mathbb{H}_Q) \cong \text{Hom}_{\text{GSp}(H)}(H_1(u_{g,n}^{\text{geom}}), H).$$

Fixing an isomorphism $\pi_1(M_{g,n/\mathbb{Q}_p}[\ell^m], \bar{\eta}) \cong \pi_1(M_{g,n/\mathbb{Z}_p}[\ell^m], \bar{\xi})$ determines the isomorphisms $(\mathbb{H}_Q)_{\pi} \cong (\mathbb{H}_Q)_{\bar{\eta}}$ and $u_{g,n}^{\text{geom}} \cong u_{M_{g,n/\mathbb{Q}_p}[\ell^m]}^{\text{geom}}$ that make the diagram

$$H_{\text{et}}^1(M_{g,n/\mathbb{Q}_p}[\ell^m], \mathbb{H}_Q) \xrightarrow{\pi_1} \text{Hom}_{\text{GSp}(H)}(H_1(u_{g,n}^{\text{geom}}), (\mathbb{H}_Q)_{\bar{\eta}}) \xrightarrow{\pi_1} \text{Hom}_{\text{GSp}(H)}(H_1(u_{g,n}^{\text{geom}}), (\mathbb{H}_Q)_{\bar{\xi}})$$

commute. Hence we have

**Corollary 12.10.** If $g \geq 3$ and $n \geq 1$, for each $j = 1, \ldots, n$, the $\text{GSp}(H)$-equivariant homomorphism $2h_j$ corresponds to the class $\kappa_j$ under the isomorphism

$$H_{\text{et}}^1(M_{g,n/\mathbb{Z}_p}[\ell^m], \mathbb{H}_Q) \cong \text{Hom}_{\text{GSp}(H)}(H_1(u_{g,n}^{\text{geom}}), H).$$

**Remark 12.11.** The $\text{GSp}(H)$-equivariant projection

$$\Lambda^3_{n}H = \Lambda^3_{3}H \oplus H_1 \oplus \cdots \oplus H_n \to H_j$$

onto the $j$th copy of $H$ is equal to $h_j/(g - 1)$ and corresponds to the class $\kappa_j/(2g - 2)$ under this isomorphism.
13. Generic Sections of Fundamental Groups

The content of this section should be well known to experts. However, because of its key role in the proof of Theorem 2, we will give a brief introduction of the results needed in the proof.

Suppose that \( S \) is the spectrum of an excellent henselian discrete valuation ring \( R \) whose residue field \( k \) is a perfect field of characteristic \( p \geq 0 \). Denote the fraction field of \( R \) by \( K \). Fix an algebraic closure \( \bar{K} \) of \( K \). Suppose that \( \pi : X \to S \) is a proper smooth morphism with geometrically connected fibers. Let \( \bar{x} \) and \( \bar{x}' \) be geometric points of the fibers \( X_{\bar{k}} \) and \( X_{k} \), respectively. We also consider \( x \) and \( x' \) as geometric points of \( X \) via the morphisms \( j : X_{\bar{k}} \to X \) and \( i : X_{k} \to X \) induced by base change. Fixing an isomorphism \( \pi_1(X, \bar{x}) \cong \pi_1(X, \bar{x}') \) gives the commutative diagram (s)

\[
\begin{array}{ccc}
1 & \rightarrow & \pi_1(X_{\bar{k}}, \bar{x}) \\
\downarrow^{\text{sp}} & & \downarrow^{\text{sp}} \\
1 & \rightarrow & \pi_1(X_{k}, \bar{x}')
\end{array}
\]

whose rows are exact and vertical maps are surjective. The surjective maps

\[
\pi_1(X_{\bar{k}}, \bar{x'}) \rightarrow \pi_1(X_{k}, \bar{x'}), \quad \pi_1(X_{\bar{k}}, \bar{x}) \rightarrow \pi_1(X_{k}, \bar{x'})
\]

in the diagram are the specialization homomorphism defined in [9, SGA 1, X].

Denote the kernel of the natural map \( G_{\bar{k}} \rightarrow G_k \) by \( I_k \). It is the Galois group of the maximal unramified subextension \( K^\text{nr} \) in \( \bar{K} \) of \( K \). For a section \( s \) of \( \pi_1(X_{\bar{k}}, \bar{x}) \rightarrow G_{\bar{k}} \), we define the ramification of \( s \) to be the map

\[
\text{ram}_s = \text{sp} \circ s|_{I_k} : I_k \rightarrow \pi_1(X_{k}, \bar{x}').
\]

This sits in the commutative diagram

\[
\begin{array}{ccc}
1 & \rightarrow & I_k \\
\downarrow^\text{ram}_s & & \downarrow^{\text{spos}} \\
1 & \rightarrow & \pi_1(X_{k}, \bar{x'})
\end{array}
\]

\[
\begin{array}{ccc}
1 & \rightarrow & G_{\bar{k}} \\
\downarrow & & \downarrow \circlearrowleft \\
1 & \rightarrow & G_k
\end{array}
\]

\[
\begin{array}{ccc}
1 & \rightarrow & G_{\bar{k}} \\
\downarrow & & \downarrow \circlearrowleft \\
1 & \rightarrow & G_k
\end{array}
\]

\[
\begin{array}{ccc}
1 & \rightarrow & G_k
\end{array}
\]

From this, we see that \( \text{ram}_s^\text{ab} : I_k^\text{ab} \rightarrow \pi_1(X_{k}, \bar{x'})^\text{ab} \) is a \( G_k \)-equivariant map and that when \( \text{ram}_s \) is trivial, the section \( s \) induces a section \( s_0 \) of \( \pi_1(X_{k}, \bar{x'}) \rightarrow G_k \).

A section \( s \) with trivial \( \text{ram}_s \) is called unramified. A section of \( \pi_1(X_{\bar{k}}) \rightarrow G_{\bar{k}} \) induced by a rational point in \( X_K(K) \) is unramified.

Now, suppose that \( \ell \) is a prime number distinct from \( \text{char}(k) = p \). Pushing out the diagram (s) along the surjection \( \pi_1(X_{\bar{k}}) \rightarrow \pi_1(X_{k})^{(\ell)} \), we obtain the commutative diagram

\[
\begin{array}{ccc}
1 & \rightarrow & \pi_1(X_{\bar{k}}, \bar{x})^{(\ell)} \\
\downarrow^{\text{sp}^{(\ell)}} & & \downarrow^{\text{sp}^{(\ell)}} \\
1 & \rightarrow & \pi_1(X_{k}, \bar{x}')^{(\ell)}
\end{array}
\]

\[
\begin{array}{ccc}
1 & \rightarrow & \pi_1(X_{\bar{k}}, \bar{x})^{(\ell)} \\
\downarrow & & \downarrow \circlearrowleft \\
1 & \rightarrow & G_{k}^{(\ell)}
\end{array}
\]

\[
\begin{array}{ccc}
1 & \rightarrow & \pi_1(X_{\bar{k}}, \bar{x})^{(\ell)} \\
\downarrow & & \downarrow \circlearrowleft \\
1 & \rightarrow & G_{k}^{(\ell)}
\end{array}
\]

The restriction of the composite \( \text{sp}' \circ s' \) to \( I_k \) induces the map

\[
\text{ram}_s^{(\ell)} : \mathbb{Z}_\ell(1) \rightarrow \pi_1(X_{k})^{(\ell)}.
\]
Proposition 13.1 ([31, Prop. 91]). With the same notation as in above, suppose that the fibers of $\pi : X \to S$ are curves and that the residue field $k$ of $S$ is finitely generated over its prime subfield. Then $\text{ram}_s(I_k)$ is a free pro-$p$ group. In particular, $\text{ram}_s^{(1)}$ is trivial and each section of $\pi_1(X_k) \to G_K$ induces a section of $\pi_1^s(X_k) \to G_k$.

Proof. If $\text{ram}_s(I_k)$ is nontrivial, then we may find an open subgroup $N$ of $\pi_1(X_k)$ such that

$$\text{ram}_s(I_k) \subset N, \quad \text{and} \quad \text{ram}_s(I_k) \not\subset [N, N],$$

where $[N, N]$ denotes the closure of $[N, N]$ in $\pi_1(X_k)$. The subgroup $N$ corresponds to a finite étale cover of $X_k$. Since $N$ is the intersection of an open subgroup of $\pi_1(X)$ with $\pi_1(X_k)$, we may replace $X_k$ and $X$ with appropriate finite étale covers to reduce to the case where $N = \pi_1(X_k)$. Suppose $\text{ram}_s(I_k)$ has a finite quotient that is an $\ell$ group. Then $\text{ram}_s$ induces a nontrivial $G_k$-equivariant map

$$\text{ram}_s^{(1),ab} : \mathbb{Z}_\ell(1) \to \pi_1(X_k)^{(1),ab}.$$ 

This contradicts the Frobenius weights in étale cohomology. Therefore, if $p = 0$, $\text{ram}_s$ is unramified, and otherwise $\text{ram}_s(I_k)$ is a pro-$p$ group. Furthermore, the $p$-cohomological dimension of $\pi_1(X_k)$ is $\leq 1$, which implies that $\text{cd}_p(\text{ram}_s(I_k)) \leq 1$ and hence $\text{ram}_s(I_k)$ is a free pro-$p$ group. □

Suppose that $f : C \to T$ is a family of curves over an irreducible regular scheme $T$ of finite type over a field $F$. Let $L$ be the function field of $T$ and $\ell$ a prime number distinct from $\text{char}(F)$. Let $\tilde{\eta}$ be a geometric generic point of $C$. The image of $\tilde{\eta}$ in $T$ is a geometric generic point of $T$. In the following, fundamental groups are defined by using this choice of base points.

Proposition 13.2. Each section of $\pi_1(C_L) \to G_L$ induces a pro-$\ell$ section of $\pi_1(C) \to \pi_1(T)$. Consequently, there is a bijection between the set of conjugacy classes of pro-$\ell$ sections of $\pi_1(C_L) \to G_L$ and that of $\pi_1(C) \to \pi_1(T)$.

Proof. Each section $s$ of $\pi_1(C_L) \to G_L$ comes from the projective system of sections $\varprojlim U \to \pi_1(C_{U_i})$, where each $U_i$ is a complement of finitely many prime divisors of $T$. Thus it will be enough to show that for each open subscheme $U$ of $T$ that is a complement of a prime divisor, each section $s$ of $\pi_1(C_U) \to \pi_1(U)$ induces a pro-$\ell$ section of $\pi_1(C) \to \pi_1(T)$. Let $Y$ be a prime divisor of $T$ and $U$ its complement in $T$. Let $R$ be the henselization of the local ring of $T$ at $Y$. Then $R$ is an excellent henselian discrete valuation ring. Denote the fraction field of $R$ by $K$ and the residue field by $k$. We see that the fiber product $\text{Spec} R \times_T U$ is isomorphic to $\text{Spec} K$. We claim that the image of the inertia group $I_k$ in $\pi_1(U)$ is equal to the kernel of the canonical surjection $\pi_1(U) \to \pi_1(T)$. Clearly, the image of $I_k$ is contained in the kernel. Let $H$ be an open subgroup of $\pi_1(U)$ containing the image of $I_k$. Then the preimage $N$ in $G_K$ of $H$ under the homomorphism $G_K \to \pi_1(U)$ is an open subgroup of $G_K$ containing $I_k$. Since $I_k$ is the kernel of the canonical surjection $G_K \to \pi_1(\text{Spec} R)$, the image $N'$ of $N$ in $\pi_1(\text{Spec} R)$ is an open subgroup of $\pi_1(\text{Spec} R)$ whose preimage in $G_K$ is $N$. Its corresponding finite étale cover $\text{Spec} R$ pulls back to the finite étale cover of $K$ that corresponds to the subgroup $N$. Let $X$ be the finite étale cover of $U$ corresponding to $H$. Let $T'$ be the normalization
of $T$ with respect to $X$. Then $\nu : T' \to T$ is finite and the pullback of $\nu$ to $U$ is just $X \to U$. Pulling back $\nu$ along the canonical morphism $\text{Spec} \, R \to T$, we obtain a finite morphism $\nu' : T'' \to \text{Spec} \, R$, where $T''$ is a normal scheme. We may choose the component $Q$ of $T''$ whose fundamental group has the image in $\pi_1(\text{Spec} \, R)$ equal to $N'$. Let $W$ be the finite étale cover of $\text{Spec} \, R$ corresponding to $N'$. Then $Q$ and $W$ are isomorphic to each other over the generic point of $\text{Spec} \, R$. Pulling back $W$ along the morphism $Q \to \text{Spec} \, R$, we obtain a finite étale cover $Q'$ of $Q$ that admits a section $j$. The composite

$$Q \xrightarrow{j} Q' \to W$$

is a finite birational morphism of integral normal schemes and thus is an isomorphism. Therefore, $Q$ is unramified over the closed point of $\text{Spec} \, R$, which implies that $T'$ is unramified over the generic point of $Y$. By Zariski-Nagata [9, SGA 1 Thm. 3.1], $T'$ is unramified over whole $T$. Since $T'$ pulls back to $X$, it follows that $H$ contains the kernel of $\pi_1(U) \to \pi_1(T)$, whence our claim holds.

Now, each section $s$ of $\pi_1(C_U) \to \pi(U)$ induces a section $s_k$ of $\pi_1(C_K) \to K$. The above claim and the fact that $\text{ram}^{(i)}_k$ is unramified imply that the section $s$ descends to a pro-$\ell$ section of $\pi_1(C) \to \pi_1(T)$.

\[
\square
\]

14. The Proof of Theorem 1 and 2

Lemma 14.1. If $g \geq 3$ and $n \geq 1$, there is no $\text{GSp}(H)$-equivariant homomorphism

$$\Gr^W_{g,m} / W_{-3} \to \Gr^W_{g,2} / W_{-3}$$

that induces the map $(v; u_1, \ldots, u_n) \mapsto (v; u_1, u_1)$ on $\Gr^W_{g,1}$.

Proof. If $u \in H$, denote by $u^{(j)}$ the corresponding element in the $j$th copy of $H$ in

$$\Gr^W_{g,1} u_{g,m} = \Lambda^2_{g} H \oplus H_1 \oplus \cdots \oplus H_n.$$ 

Fix a symplectic basis $a_1, b_1, \ldots, a_g, b_g$ for $H$ and $(u \cdot v)$ denotes the intersection number of $u$ and $v$. Suppose that such a homomorphism $\phi$ exists. Recall that, for a positive integer $m$, we have

$$\Gr^W_{g,m} u_{g,m} = H_{g,m} \oplus \bigoplus_{1 \leq i < j \leq m} \mathbb{Q}_\ell(1)_{ij} \oplus \bigoplus_{j=1}^m \Lambda^2_{g} H_j,$$

where $\mathbb{Q}_\ell(1)_{ij}$ is spanned by $\sum_{k=1}^g [a_k^{(i)}, b_k^{(j)}]$. Denote the element $\sum_{k=1}^g [a_k^{(i)}, b_k^{(j)}]$ in $\Gr^W_{g,m} u_{g,m}$ by $\Theta_{ij}$. We claim first that $\phi$ vanishes on the $\mathbb{Q}_\ell(1)$ component. For any $i < j$ and $u, v \in H$, the bracket $[u^{(i)}, v^{(j)}]$ is computed in [11, §12] and is given by

$$[u^{(i)}, v^{(j)}] = \frac{(u \cdot v)}{g} \sum_{k=1}^g [a_k^{(i)}, b_k^{(j)}] \text{ in } \Gr^W_{g,m} u_{g,m}.$$ 

For $1 < j$, we have $\phi(v^{(j)}) = (0, 0)$ in $\Gr^W_{g,2} u_{g,2}$, and hence $[\phi(u^{(i)}), \phi(v^{(j)})] = 0$. Since $\phi$ is a homomorphism, it follows that $\phi(\Theta_{ij}) = 0$, and therefore $\phi$ vanishes on $\mathbb{Q}_\ell(1)_{ij}$ for all $1 \leq i < j \leq n$. 

\[
\square
\]
Next, we will compute $\sum_{k=1}^{g} \phi(a_k^{(1)}), \phi(b_k^{(1)})$ in $\text{Gr}^{W}_{-2} u^{\text{geom}}_{g,m}$. Denote the element $\sum_{k=1}^{g} [a_k^{(i)}, b_k^{(i)}]$ in $\text{Gr}^{W}_{-2} u^{\text{geom}}_{g,m}$ by $\Theta_i$. Theorem 12.6 in [11] implies that we have

$$
\phi(\Theta_1) = \sum_{k=1}^{g} \phi(a_k^{(1)}), \phi(b_k^{(1)}) = \sum_{k=1}^{g} [a_k^{(1)}, b_k^{(1)}] + \Theta_2 + 2\Theta_{12} = (\Theta_1 + \Theta_2 + 2\Theta_{12}) \in \text{Gr}^{W}_{-2} u^{\text{geom}}_{g,m}.
$$

But one has the relation [11, Thm. 12.6]

$$
\Theta_i + \frac{1}{g} \sum_{j \neq i} \Theta_{ij} = 0, \quad \text{for } 1 \leq i \leq m,
$$

in $\text{Gr}^{W}_{-2} u^{\text{geom}}_{g,m}$, so in $\text{Gr}^{W}_{-2} u^{\text{geom}}_{g,2}$

$$
\phi(\Theta_1) = \Theta_1 + \Theta_2 + 2\Theta_{12} = \frac{-1}{g} \Theta_{12} + \frac{-1}{g} \Theta_{12} + 2\Theta_{12} = \frac{2g - 2}{g} \Theta_{12} \neq 0.
$$

Therefore we have reach a contradiction.

Recall that $p$ is a prime number, $\ell$ is a prime number distinct from $p$, and $m$ is a nonnegative integer.

**Proposition 14.2.** Suppose that $g \geq 3$, $n \geq 1$, and $\ell^m \geq 3$. If $x$ is a section of the universal curve $C_{g,n/\bar{F}_p}[\ell^m] \rightarrow M_{g,n/\bar{F}_p}[\ell^m]$ and $\kappa_x = \kappa_j$, then $x$ is the $j$th tautological point $x_j$.

**Proof.** Without loss of generality, we may assume that $j = 1$. The section $x$ is defined over some finite extension $F_q$ of $\bar{F}_p$, which we may assume to contain a $\ell^m$th root of unity $\mu_{\ell^m}(\bar{F}_p)$. Thus we consider $x$ as a section of $C_{g,n/\bar{F}_q}[\ell^m] \rightarrow M_{g,n/\bar{F}_q}[\ell^m]$. Denote the relative Jacobian of $C_{g,n/\bar{F}_q}[\ell^m] \rightarrow M_{g,n/\bar{F}_q}[\ell^m]$ by $J$. By Corollary 12.8, $[x - x_1]$ is a torsion in $J(M_{g,n/\bar{F}_q}[\ell^m])$. Denote this torsion by $t$. If $t = 0$, then, since $g \geq 3$, $x = x_1$. If $t \neq 0$, then the sections $x$ and $x_1$ are disjoint, and hence they induce the morphism

$$
M_{g,n/\bar{F}_q}[\ell^m] \rightarrow M_{g,2/\bar{F}_q}[\ell^m] \quad y \mapsto (C_y; x_1(y), x(y)),
$$

where $C_y$ is the fiber at $y$ of $C_{g,n/\bar{F}_q}[\ell^m] \rightarrow M_{g,n/\bar{F}_q}[\ell^m]$. By Corollary 12.10, $\kappa_x = \kappa_1$ implies that the induced $\text{GSp}(H)$-equivariant homomorphism

$$
\phi : \text{Gr}^{W}_{1} u^{\text{geom}}_{g,n} = \Lambda_1^3 H \oplus H_1 \oplus \cdots \oplus H_n \rightarrow \text{Gr}^{W}_{1} u^{\text{geom}}_{g,2} = \Lambda_1^3 H \oplus H_1 \oplus H_2
$$

is given by

$$
(v; u_1, \ldots, u_n) \mapsto (v; u_1, u_1).
$$

This is impossible by Lemma 14.1. Thus $t = 0$, and we are done. \qed
Proof of Theorem 1. It is enough to show for the case $\ell^m \geq 3$. The valuative criterion of properness and the normality of $M_{g,n}/\mathcal{F}_p[\ell^m]$ implies that each $K$-rational point of $C_{g,n}/\mathcal{F}_p[\ell^m]$ gives a unique section of the universal curve. Hence we have $C_{g,n}/\mathcal{F}_p[\ell^m](K) = C_{g,n}/\mathcal{F}_p[\ell^m](M_{g,n}/\mathcal{F}_p[\ell^m])$. Let $x$ be a section of $C_{g,n}/\mathcal{F}_p[\ell^m] \to M_{g,n}/\mathcal{F}_p[\ell^m]$. The section $x$ induces a section $s_x$ of $\epsilon_n : \mathcal{O}_{g,n+1} \to \mathcal{O}_{g,n}$. By Corollary 11.4, we have $s_x = s_j$ for some $j \in \{1, \ldots, n\}$. Recall that $s_j$ is the section of $\epsilon_n$ induced by the $j$th tautological point. Corollary 12.10 implies that $\kappa_x = \kappa_j$ and thus we have $x = x_j$ by Proposition 14.2. 

Proof of Theorem 2. Suppose that there is a section $s$ of $\pi_1(C, \bar{x}) \to G_L$. By Corollary 13.2, the section $s$ induces a pro-$\ell$ section $s^{(\ell)}$ of $1 \to \pi_1(C, \bar{x})^{(\ell)} \to \pi_1(C, \bar{x}) \to \pi_1(M_{g,q}[\ell^m], \eta) \to 1$, which induces a $\text{GSp}(H)$-equivariant section of $\epsilon_0 : \mathcal{O}_{g,1} \to \mathcal{O}_{g,0}$. By Proposition 11.2, there is no $\text{GSp}(H)$-equivariant section of $\epsilon_0$. Therefore, there is no section of $\pi_1(C, \bar{x}) \to G_L$. 

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