ON THE CONCENTRATION OF CERTAIN ADDITIVE FUNCTIONS

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Abstract. We study the concentration of the distribution of an additive function, when the sequence of prime values of $f$ decays fast and has good spacing properties. In particular, we prove a conjecture by Erdős and Kátai on the concentration of $f(n) = \sum_{p|n} (\log p)^{-c}$ for $c > 1$.

1. Introduction

An arithmetic function $f : \mathbb{N} \to \mathbb{R}$ is called additive if $f(mn) = f(m) + f(n)$ whenever $(m, n) = 1$. According to the Kubilius probabilistic model of the integers, statistical properties of additive functions can be modeled by statistical properties of sums of independent random variables. We describe this model in the case that $f$ is a strongly additive function, that is $f$ satisfies the relation $f(n) = \sum_{p|n} f(p)$; the general case is slightly more involved. Let $\mathbb{P}$ denote the set of prime numbers and consider a sequence of independent Bernoulli random variables $\{X_p : p \in \mathbb{P}\}$ such that

\[
\text{Prob}(X_p = 1) = \frac{1}{p} \quad \text{and} \quad \text{Prob}(X_p = 0) = 1 - \frac{1}{p}.
\]

The random variable $X_p$ can be thought as a model of the characteristic function of the event $\{n \in \mathbb{N} : p|n\}$. Then a probabilistic model for $f$ is given by the random variable $\sum_p f(p)X_p$.

The above model and well-known facts from probability theory lead to the prediction that the values of $f$ follow a certain distribution, possibly after rescaling them appropriately. In fact, the Erdős-Wintner theorem \cite{ErdosWintner} states that if the series

\[
\sum \frac{f(p)}{p}, \quad \sum \frac{f^2(p)}{p}, \quad \sum \frac{1}{p}
\]

converge, then $f$ has a limit distribution, in the sense that there is a distribution function $F : \mathbb{R} \to [0, 1]$ such that

\[
F_x(u) := \frac{1}{[x]} |\{n \leq x : f(n) \leq u\}| \to F(u) \quad \text{as} \quad x \to \infty
\]

for every $u \in \mathbb{R}$ that is a point of continuity of $F$; the characteristic function of $F$ is given by

\[
\hat{F}(\xi) = \prod_p \left\{ \left(1 - \frac{1}{p} \right) \sum_{k \geq 0} \frac{e^{i\xi f(p^k)}}{p^k} \right\}.
\]

Conversely, if $f$ possesses a limit distribution, then the three series in (1.1) converge.
One way to measure the regularity of the distribution of the set \( \{ f(n) : n \in \mathbb{N} \} \) is by its concentration. In general, given a distribution function \( G : \mathbb{R} \to [0,1] \), we define its concentration function to be

\[
Q_G(\epsilon) = \sup_{u \in \mathbb{R}} \{ G(u + \epsilon) - G(u) \}.
\]

We seek estimates for \( Q_{F_1}(\epsilon) \), or for \( Q_F(\epsilon) \) if \( f \) possesses a limit distribution. There are various such results in the literature, a historic account of which is given in [1]. The most general estimate on \( Q_{F_1}(\epsilon) \) is due to Ruzsa [12]. Improving upon bounds by Erdős [5] and Halász [9], he showed that

\[
Q_{F_1}(\epsilon) \leq \left( \min_{\lambda \in \mathbb{R}} \left\{ \lambda^2 + \sum_{p \leq x} \min\{1, (f(p) - \lambda \log p)^2/p\} \right\} \right)^{-1/2}.
\]

This result is best possible, as it can be seen by taking \( f(n) = c \log n \) or \( f(n) = \omega(n) = \sum_{p \mid n} 1 \). However, both of these functions satisfy \( f(p) \gg 1 \). So, a natural question is whether it is possible to improve upon (1.2) in the case that \( f(p) \) decays to zero. Erdős and Kátai [7], building on earlier work of Tjan [14] and Erdős [6], showed the following result:

**Theorem 1** (Erdős, Kátai [7]). Let \( f : \mathbb{N} \to \mathbb{R} \) be an additive function such that

\[
\sum_{p > t^A} \frac{|f(p)|}{p} \ll \frac{1}{t} \quad (t \geq 1), \quad |f(p_1) - f(p_2)| \gg \frac{1}{p_2^c} \quad (p_1, p_2 \in \mathbb{P}, p_1 < p_2),
\]

for some constants \( A \) and \( B \). Then

\[
Q_F(\epsilon) \asymp_{A,B} \frac{1}{\log(1/\epsilon)} \quad (0 < \epsilon \leq 1/2);
\]

except for \( A \) and \( B \), the implied constant depends on the implied constants in the assumptions of the theorem too.

On the other hand, when \( f(p) \ll 1/p^\delta, p \in \mathbb{P}, \) for some \( \delta > 0 \), then (1.2) applied to \( f/\epsilon \) yields an upper bound for \( Q_F(\epsilon) \) that is never better that \( 1/\sqrt{\log \log(1/\epsilon)} \).

Also, Erdős and Kátai studied \( Q_F(\epsilon) \) in the case that \( f(p) = (\log p)^{-c}, p \in \mathbb{P}, \) for some \( c \geq 1 \). They showed that

\[
\begin{align*}
\epsilon^{1/c} &\ll Q_F(\epsilon) \ll \epsilon^{1/c} (\log \log(1/\epsilon))^{-1} &\text{if } c > 1, \\
\epsilon &\ll Q_F(\epsilon) \ll \epsilon (\log(1/\epsilon)) (\log \log(1/\epsilon))^{-1} &\text{if } c = 1,
\end{align*}
\]

for \( 0 < \epsilon \leq 1/3 \). Furthermore, they conjectured that when \( c > 1 \), then

\[
Q_F(\epsilon) \asymp c^{1/c} \quad (0 < \epsilon \leq 1).
\]

The conjecture of Erdős and Kátai was proven for \( c \) large enough by La Bretèche and Tenenbaum in [1]:

**Theorem 2** (La Bretèche, Tenenbaum [1]). Let \( c \geq 1 \) and \( f : \mathbb{N} \to \mathbb{R} \) be an additive function such that \( |f(p)| \asymp (\log p)^{-c} \) for every \( p \in \mathbb{P} \) and

\[
|f(p_1) - f(p_2)| \gg \frac{p_2 - p_1}{p_2(\log p_2)^{c+1}} \quad (p_1, p_2 \in \mathbb{P}, p_1 < p_2).
\]
If \( c \) is large enough, then we have that

\[
Q_F(\epsilon) \approx \epsilon^{1/c} \quad (0 < \epsilon \leq 1);
\]

the implied constant depends at most on the implied constants in the assumptions of the theorem.

La Bretèche and Tenenbaum derived their theorem from a general upper bound on \( Q_F(\epsilon) \) that they showed when the sequence of prime values of \( f \) satisfies certain regularity assumptions. Their method uses a result from the theory of functions of Bounded Mean Oscillation, first introduced by Diamond and Rhoads [2] in this context to study the concentration of \( f(n) = \log(\phi(n)/n) \).

In the present paper we improve upon Theorems 1 and 2. In particular, we settle the Erdős-Kátaï conjecture for every \( c > 1 \). We phrase our results in terms of the distribution function

\[
\mathcal{F}_y(u) = \prod_{p \leq y} \left(1 - \frac{1}{p}\right) \sum_{\substack{p \mid n \Rightarrow p \leq y \\ f(n) \leq u}} \frac{1}{n} \quad (u \in \mathbb{R}),
\]

defined for every \( y \geq 1 \). From a technical point of view, this function is more natural to work with than \( F_x \). Indeed, a calculation of the characteristic function of \( \mathcal{F}_y \) immediately implies that \( \mathcal{F}_y \) converges to \( F \) weakly, provided that the latter is well defined. It is relatively easy to pass from estimates for \( Q_{\mathcal{F}_y}(\epsilon) \) to estimates for \( Q_{F_x}(\epsilon) \).

Our first result is Theorem 3 below, which will be proven in Section 3. For the upper bound in it we use a combination of ideas from [6, 7], whereas the lower bound is a straightforward application of Theorem 1.2 in [1]. Observe that by letting \( y \to \infty \) in Theorem 3, we deduce as special cases\(^1\) Theorems 1 and 2.

**Theorem 3.** Consider an additive function \( f : \mathbb{N} \to \mathbb{R} \) for which there is a set of primes \( \mathcal{P} \) and a constant \( c \in [1, 2] \) such that

\[
|f(p)| \ll \frac{1}{(\log p)^c} \quad (p \in \mathcal{P}), \quad \text{and} \quad \sum_{p \in \mathcal{P} \setminus \mathcal{P}} \frac{1}{p} \ll 1.
\]

For \( t \geq 2 \) set

\[
g(t) = \sup \left\{ \frac{|f(p)| (\log p)^c}{(\log t)^c} : p \geq t, p \in \mathcal{P} \right\}
\]

and assume that there is some \( A \geq 1 \) such that

\[
|f(p_2) - f(p_1)| \gg \min \left\{ \frac{g(p_2)(p_2 - p_1)}{p_2 \log p_2}, g(p_2^A) \right\} \quad (p_1, p_2 \in \mathcal{P}, p_1 < p_2).
\]

Then for \( 0 < \epsilon \leq 1/2 \) and \( y \geq K(\epsilon)^{1/A} \), where \( K(\epsilon) = \min\{n \in \mathbb{N} : n \geq 3, g(n) \leq \epsilon\} \), we have that

\[
\frac{1}{\log K(\epsilon)} \ll Q_{\mathcal{F}_y}(\epsilon) \ll_A \min \left\{ \frac{1}{\epsilon - 1}, \frac{\log \frac{1}{\epsilon}}{\log \epsilon} \right\};
\]

except for \( A \), the implied constants depend on the implied constants in the assumptions of the theorem too.

As an immediate corollary, we deduce the Erdős-Kataï conjecture:

\(^1\)To deduce Theorem 1 take \( \mathcal{P} = \{p \in \mathcal{P} : |f(p)| \leq p^{-1/2A}\} \); see also the proof of Theorem 4 in page 11.
Corollary 1. Let $c \geq 1$ and $f : \mathbb{N} \to \mathbb{R}$ be an additive function with $f(p) = (\log p)^{-c}$ for all $p \in \mathbb{P}$. For $0 < \epsilon \leq 1/2$ we have that

$$e^{1/c} \ll Q_F(\epsilon) \ll \min \left\{ \frac{1}{c-1}, \log \frac{1}{\epsilon} \right\} e^{1/c}.$$ 

Proof. This follows by applying Theorem 3 with $\min\{c, 2\}$ in place of $c$, $A = 1$, and $y \geq \exp\{10e^{-1/c}\}$, and then letting $y \to \infty$. \hfill \Box

Remark 1.1. When $0 < c < 1$, the behavior of $Q_F$ for $f$ as in Corollary 1 is different. As Gérard Tenenbaum has pointed out to us in a private communication, in this case we have that

$$(1.3) \quad Q_F(\epsilon) \approx e^{1/c} (0 < \epsilon \leq 1).$$

Corollary 1 and relation (1.3) give the concentration of an additive function $f$ with $f(p) = (\log p)^{-c}$, $p \in \mathbb{P}$, for all positive values of $c$ except for $c = 1$, which is the only case remaining open.

Remark 1.2. There is a heuristic argument which motivates Theorem 3. Assume for simplicity that the sequence $\{|f(p)|(\log p)^c : p \in \mathbb{P}\}$ is decreasing so that the assumptions of Theorem 3 are satisfied with $g(p) = |f(p)|$ for all $p \in \mathbb{P}$. For every integer $n$ we have that

$$\sum_{p | n, p \geq K(\epsilon)} |f(p)| \leq \epsilon \sum_{p | n, p \geq K(\epsilon)} \frac{(\log K(\epsilon))^c}{(\log p)^c}.$$ 

Since for a typical integer $n$ the sequence \{log log $p : p | n\}$ is distributed like an arithmetic progression of step 1 [10, Chapter 1], we find that

$$\sum_{p | n, p \geq K(\epsilon)} |f(p)| \approx \epsilon \sum_{j \geq \log \log K(\epsilon)} \frac{(\log K(\epsilon))^c}{e^{c^j}} \ll \epsilon.$$ 

So only the prime divisors of $n$ lying in $[1, K(\epsilon)]$ are important for the size of $Q_F(\epsilon)$. Note that for a prime number $p < K(\epsilon)$ we have that $|f(p)| > \epsilon$. Therefore if $a$ and $b$ are composed of primes from $[1, K(\epsilon)]$, then it is reasonable to expect that $|f(a) - f(b)|$ is big compared to $\epsilon$, unless $a$ and $b$ have a large common factor. This leads to the prediction that $Q_F(\epsilon) \approx 1/\log K(\epsilon)$, which is confirmed by Theorem 3 when $c > 1$. However, when $c < 1$ this heuristic fails, as [1,2] shows, and the underlying reason is combinatorial: The pigeonhole principle implies the lower bound $Q_F(\epsilon) \gg_F \epsilon$ for the concentration function of any distribution function $F$ (see also [7, Remark 1, p. 297]).

Finally, our methods yield the following strengthening of Theorem 3 which will be proven in Section 3.

Theorem 4. Let $f : \mathbb{N} \to \mathbb{R}$ be an additive function for which there exists positive constants $\eta, A$ and $B$ such that

$$\sum_{p > t^A} \frac{|f(p)|}{p} \ll \frac{1}{t} \quad \text{and} \quad \sum_{u < f(p) \leq u + 1/t^B} 1 \ll t^{1-\eta} \quad (t \geq 1, u \in \mathbb{R}).$$

\footnote{The symbol ‘\ll’ here is used in a non-rigorous fashion to denote ‘roughly less than’. Similarly, the symbol ‘\approx’ means ‘roughly equal to’.}
Then for \( y \geq \epsilon^{-1/B} \) we have
\[
Q_{F_y}(\epsilon) \asymp_{\eta,A,B} \frac{1}{\log(1/\epsilon)};
\]
except for \( \eta, A \) and \( B \), the implied constant depends on the implied constants in the assumptions of the theorem too.

**Notation.** For an integer \( n \) we denote with \( P^+(n) \) and \( P^-(n) \) its largest and smallest prime factors, respectively, with the notational convention that \( P^+(1) = 1 \) and \( P^-(1) = \infty \).

### 2. The main results

In this section we state two general results and derive Theorems 3 and 4 from them in Section 3. The first one of them, Theorem 5, is novel, whereas the second one, Theorem 6, is a corollary of Theorem 1.2 in [1].

**Theorem 5.** Let \( f : \mathbb{N} \to \mathbb{R} \) be an additive function for which there is a set of primes \( \mathcal{P} \) and a decreasing function \( P_f : (0, 1] \to [2, +\infty) \) such that
\[
(2.1) \quad \sum_{p \in \mathcal{P} \setminus \mathcal{P}} \frac{1}{p} \ll 1,
\]
\[
(2.2) \quad |f(p)| \leq \epsilon \quad (0 < \epsilon \leq 1, \ p \in \mathcal{P}, \ p > P_f(\epsilon)).
\]
Furthermore, assume that there is some \( \lambda \in (0, 1] \) and some \( \rho \geq 1 \) such that for all \( u \in \mathbb{R} \), \( 0 < \epsilon < \delta \leq 1 \) and \( 1 \leq z \leq w \leq \min\{P_f(\delta), P_f(\epsilon)^\lambda\} \) we have the estimate
\[
(2.3) \quad \sum_{p \in \mathcal{P}^+(z,w) \cup \{u < f(p) \leq u + \epsilon\}} \frac{1}{p} \ll \begin{cases} 
\epsilon/\delta & \text{if } z \geq (\delta/\epsilon)^\rho, \\
\epsilon/\delta + \log^2(z+1) & \text{else}.
\end{cases}
\]

Let \( 0 < \epsilon \leq \delta \leq 1 \) such that \( P_f(\delta) \leq P_f(\epsilon)^\lambda \), set \( q_j = P_f(2^j \delta) \) for \( j \geq 0 \), and consider \( J \in \{0\} \cup \{j \in \mathbb{N} : 2^j \leq 1/\delta \} \) and \( q_j \geq (2^j \delta/\epsilon)^\rho \). For \( y \geq q_0 \) we have that
\[
(2.4) \quad Q_{F_y}(\epsilon) \ll \frac{1}{\log q_0} + \frac{\epsilon}{\lambda \delta} \sum_{j=1}^{J} \frac{1}{2^j \log q_j} + \frac{\epsilon}{\delta} \sum_{j=0}^{J} \frac{\log q_j}{2^j \log q_j};
\]
the implied constant depends at most on the implied constants in (2.1) and (2.3).

**Remark 2.1.** The parameter \( \lambda \) and the set \( \mathcal{P} \) are introduced to make Theorem 5 more applicable. One can think of \( P_f \) defined by \( f(p) = \max\{p \in \mathbb{P} : |f(p)| > \epsilon\} \). Condition (2.3) can be motivated as follows. Assume that
\[
(2.5) \quad \sum_{p \geq P_f(\eta)} \frac{|f(p)|}{p} \ll \eta \quad (0 < \eta \leq 1).
\]
We have that \( |f(p)| \approx \delta \) for \( p \in (P_f(2\delta), P_f(\delta)) \). So if the sequence \( \{f(p) : p \in \mathbb{P}\} \) is ‘well-spaced’, then we expect that
\[
\sum_{P_f(2\delta) < p \leq P_f(\delta), 0 < u < f(p) \leq u + \epsilon} \frac{1}{p} \approx \frac{\epsilon}{\delta^2} \sum_{P_f(2\delta) < p \leq P_f(\delta)} \frac{1}{p} \ll \frac{\epsilon}{\delta};
\]
by (2.5).
Proof of Theorem 5. For \( w > z \) we denote with \( \mathcal{P}(z, w) \) the set of integers all of whose prime factors belong to \( \mathcal{P} \cap [z, w] \). For every \( u \in \mathbb{R} \) we have that

\[
\sum_{n \in \mathcal{P}(1, q_0) \setminus \mathcal{N}} \frac{1}{n} \leq \sum_{p > q_j} \sum_{P^+(n) \leq q_0} \frac{1}{n} \ll \log q_0 \sum_{p > q_j} \frac{1}{p^2} \ll \frac{\log q_0}{q_j \log q_j} \leq \begin{cases} 1 & \text{if } J = 0, \\ \frac{\epsilon \log q_j}{\delta 2^j \log q_j} & \text{if } J \geq 1, \end{cases}
\]

(2.6)

by (2.1).

Next, fix \( v \in \mathbb{R} \) and let

\[
\mathcal{N} = \{ n \in \mathcal{P}(1, q_0) : v < f(n) \leq v + \epsilon, p > q_j \Rightarrow p^2 \nmid n \}.
\]

Clearly,

\[
\sum_{n \in \mathcal{P}(1, q_0) \setminus \mathcal{N}} \frac{1}{n} \ll \sum_{p > q_j} \sum_{P^+(n) \leq q_0} \frac{1}{n} \ll \log q_0 \sum_{p > q_j} \frac{1}{p^2} \ll \frac{\log q_0}{q_j \log q_j} \leq \begin{cases} 1 & \text{if } J = 0, \\ \frac{\epsilon \log q_j}{\delta 2^j \log q_j} & \text{if } J \geq 1, \end{cases}
\]

(2.7)

which is admissible in both cases. So it suffices to bound \( \sum_{n \in \mathcal{N}} 1/n \) from above.

We partition \( \mathcal{N} \) into certain subsets and estimate the contribution of each one of them to \( \sum_{n \in \mathcal{N}} 1/n \) separately. Let \( \mathcal{N}_0 = \mathcal{P}(1, q_j) \). Next, given \( n \in \mathcal{N} \setminus \mathcal{N}_0 \) we write \( n = apb \), where \( p \) is prime and \( P^+(a) \leq q_j < p < P^+(b) \). For \( 1 \leq j \leq J \) let \( \mathcal{N}_j \) be the set of \( n \in \mathcal{N} \) with \( q_j < p \leq q_{j-1} \). First, we bound \( \sum_{n \in \mathcal{N}_0} 1/n \). Writing \( n = mp^+ \) we find that

\[
\sum_{n \in \mathcal{N}_0} \frac{1}{n} = O(1) + \sum_{P^+(m) \leq q_j} \frac{1}{m} \sum_{p' \in \mathcal{P} \cap [P^+(m), q_j]} \frac{1}{p'} \ll 1 + \sum_{P^+(m) \leq q_j} \frac{1}{m} \left( \frac{\epsilon}{2^j \delta} + \frac{1}{\log^2(1 + P^+(m))} \right) \ll 1 + \frac{\epsilon \log q_j}{2^j \delta},
\]

(2.8)

by (2.3) with \( v - f(m), 2^j \delta, P^+(m) \) and \( q_j \) in place of \( u, \delta, z \) and \( w \), respectively. Next, we bound \( \sum_{n \in \mathcal{N}_j} 1/n \) for \( j \in \{ 1, \ldots, J \} \). We have that

\[
\sum_{n \in \mathcal{N}_j} \frac{1}{n} \leq \sum_{b \in \mathcal{P}(q_j, q_{j-1})} \frac{1}{b} \sum_{a_1 \in \mathcal{P}(q_j, q_{j-1})} \frac{1}{a_1} \sum_{a_2 \in \mathcal{P}(1, q_j^2)} \frac{1}{a_2} \sum_{p \in \mathcal{P} \cap [q_j, q_{j-1}]} \frac{1}{p} \ll 1 + \frac{\epsilon \log q_j}{2^j \delta},
\]

(2.9)
Fix some \( w \in \mathbb{R} \) and consider \( a_2 \in \mathcal{P}(1,q_j) \) and \( p \in \mathcal{P} \cap (q_j, q_{j-1}] \) with \( w < f(a_2) + f(p) \leq w + \epsilon \), as above. Since \( |f(p)| \leq 2\delta \) for \( p \in \mathcal{P} \cap (q_j, +\infty) \), by (2.2), we must have that \( f(a_2) = w + O(2\delta) \).

So

\[
\sum_{a_2 \in \mathcal{P}(1,q_j)} \frac{1}{a_2} \sum_{\substack{p \in \mathcal{P} \cap (q_j, q_{j-1}] \atop w-f(a_2)<f(p)\leq w-f(a_2)+\epsilon}} \frac{1}{p} = \sum_{a_2 \in \mathcal{P}(1,q_j)} \frac{1}{a_2} \sum_{\substack{p \in \mathcal{P} \cap (q_j, q_{j-1}] \atop w-f(a_2)<f(p)\leq w-f(a_2)+\epsilon}} \frac{1}{p} \ll \frac{\epsilon}{2\delta} \sum_{a_2 \in \mathcal{P}(1,q_j)} \frac{1}{a_2},
\]

by the first part of (2.3) applied with \( w - f(a_2), 2^{j-1}\delta, q_j \) and \( q_{j-1} \) in place of \( u, \delta, z \) and \( w \), respectively, since \( q_j \geq q_{j-1} \geq (2^j\delta/\epsilon) \rho \geq (2^j\delta/\epsilon)^\rho \). Finally, writing \( a_2 = mP^+(a_2) = mp' \), we find that

\[
\sum_{a_2 \in \mathcal{P}(1,q_j)} \frac{1}{a_2} = O(1) + \sum_{m \in \mathcal{P}(1,q_j)} \frac{1}{m} \sum_{p' \in \mathcal{P}(1,m)q_j} \frac{1}{p'}.
\]

For every \( m \in \mathbb{N} \) we have that

\[
\sum_{p' \in \mathcal{P}(1,m)q_j} \frac{1}{p'} \ll \sup_{y \in \mathbb{R}} \left\{ \sum_{p' \in \mathcal{P}(1,m)q_j} \frac{1}{p'} \right\} \ll \frac{2^j\delta}{2^j\delta} + \frac{1}{\log^2(1+P^+(m))},
\]

by (2.3) with \( y, 2^j\delta, 2^{j-1}\delta, P^+(m) \) and \( q_j \) in place of \( u, \epsilon, \delta, z \) and \( w \), respectively. So we find that

\[
\sum_{a_2 \in \mathcal{P}(1,q_j)} \frac{1}{a_2} \ll 1 + \sum_{m \leq q_j} \frac{1}{m} \frac{2^j\delta}{2^j\delta} + \frac{1}{\log^2(1+P^+(m))} \ll 1 + \frac{\lambda \log q_j}{2^{j-1}}.
\]

Combining the above estimate with (2.9) and (2.10) implies that

\[
\sum_{n \in \mathcal{N}_j} \frac{1}{n} \ll \frac{\epsilon}{2^j\delta} \left( 1 + \frac{\lambda \log q_j}{2^{j-1}} \right) \sum_{b \in \mathcal{P}(q_j,q_0)} \frac{1}{b} \sum_{a_1 \in \mathcal{P}(q_j,q_j)} \frac{1}{a_1} \ll \frac{\epsilon}{2^j\delta} \left( 1 + \frac{\lambda \log q_j}{2^{j-1}} \right) \frac{\log q_0}{\lambda \log q_j},
\]

which together with relations (2.6), (2.7) and (2.8) completes the proof. \( \square \)

As we mentioned above, the next theorem is a corollary of Theorem 1.2 in [1]. Notice also its similarity with Theorem 2 in [7].

**Theorem 6.** Let \( f : \mathbb{N} \to \mathbb{R} \) be an additive function and \( 0 < \epsilon < 1 \). If there is a set of primes \( \mathcal{P} \) and some \( M \geq 2 \) such that

\[
\sum_{p \in \mathcal{P} \cap \mathcal{P}} \frac{1}{p} \ll 1 \quad \text{and} \quad \sum_{p \in \mathcal{P} \cap \mathcal{P}^+} \frac{|f(p)|}{p} \ll \epsilon,
\]

then for \( y \geq M \) we have that

\[
Q_{xy}(\epsilon) \gg \frac{1}{\log M};
\]
the implied constant depends at most on the implied constants implicit in the assumptions of
the theorem.

Proof. Without loss of generality, we may assume that $M$ is big enough. Let

$$C = \frac{1}{\epsilon} \sum_{p \in P, p > M} \frac{|f(p)|}{p} \ll 1.$$  

Define $g : \mathbb{N} \to \mathbb{R}$ by

$$g(n) = \begin{cases} f(n) & \text{if } p \in [1, y] \cap P \text{ for all primes } p | n, \\ 0 & \text{else}, \end{cases}$$

and call $G$ its distribution function. Then Theorem 1.2 in [1] yields that

$$Q_G(3C\epsilon) \geq \left(1 - 2 \cdot \frac{C\epsilon}{3C\epsilon} + o(1)\right) \prod_{p \leq M} \left(1 - \frac{1}{p}\right) \gg \frac{1}{\log M} \quad (M \to \infty).$$

So, by the pigeonhole principle, we deduce that

$$(2.11) \quad Q_G(\epsilon) \gg \frac{Q_G(3C\epsilon)}{C + 1} \gg c \frac{1}{\log M}.$$  

Finally, we have that

$$Q_G(\epsilon) = \sup_{u \in \mathbb{R}} \left\{ \prod_{p \in P \cap [1, y]} \left(1 - \frac{1}{p}\right) \sum_{p | n \Rightarrow p \in P \cap [1, y]} \frac{1}{n} \right\} \leq Q_{F_y}(\epsilon) \prod_{p \in \mathcal{P} \setminus \mathcal{P}} \left(1 - \frac{1}{p}\right)^{-1} \ll Q_{F_y}(\epsilon),$$

which together with (2.11) completes the proof of the theorem. □

3. PROOF OF THEOREMS 3 AND 4

We conclude the paper by showing how to deduce Theorems 3 and 4 by the results of Section 2.

Proof of Theorem 3. Before we delve into the details of the proof, we note some properties of the functions $g$ and $K$. The definition of $g$ implies that the function $t \to g(t)(\log t)^c$ is decreasing and, in particular, that $g$ is strictly decreasing. Also, for $p \in \mathcal{P}$ with $p \geq K(\epsilon)$ we have that

$$(3.1) \quad |f(p)| \leq g(p) \leq g(K(\epsilon))(\log(K(\epsilon)))^c \leq \epsilon(\log(K(\epsilon)))^c.$$  

Finally, we claim that

$$(3.2) \quad \log(K(\delta) - 1) \geq \frac{1}{2} \left(\frac{\epsilon}{\delta}\right)^{1/c} \log(K(\epsilon) - 1) \quad (0 < \epsilon \leq \delta \leq 1).$$

Indeed, if $K(\delta) = K(\epsilon)$, this holds trivially, and if $K(\delta) < K(\epsilon)$, then we have that

$$1 \geq \frac{g(K(\epsilon) - 1)(\log(K(\epsilon) - 1))^c}{g(K(\delta))(\log K(\delta))^c} \geq \frac{\epsilon(\log(K(\epsilon) - 1))^c}{\delta(\log K(\delta))^c} \geq \frac{\epsilon(\log(K(\epsilon) - 1))^c}{\delta(2 \log(K(\delta) - 1))^c},$$

since $K(\epsilon) \geq 3$ for all $\epsilon \in (0, 1]$. In any case, (3.2) holds.

Now we are ready to proceed to the proof of the theorem. First, note that the hypothesis of Theorem 3 is satisfied with $M = K(\epsilon)$ and $\mathcal{P}$, by (3.1), and the desired lower bound
follows. For the upper bound we show that we may apply Theorem 3 with $P_f = K - 1$, $\mathcal{P}$, $\lambda = 1/A$ and $\rho = 2$. Condition (2.1) holds by assumption and condition (2.2) follows immediately by (3.1). Lastly, we show (2.3) with $\lambda = 1/A$ and $\rho = 2$. First, we prove some intermediate estimates. Let $0 < \epsilon \leq \delta \leq 1$, $u \in \mathbb{R}$ and $3 \leq z \leq w \leq \min \{P_f(\delta), P_f(\epsilon)^{1/4}\}$ with $z \geq w^{1/4}$. By assumption, there is an absolute constant $C > 0$ such that

$$|f(p_1) - f(p_2)| \geq \frac{2}{C} \min \left\{ \frac{g(p_2)(p_2 - p_1)}{p_2 \log p_2}, g(p_2^A) \right\} \quad (p_1 < p_2, p_1, p_2 \in \mathbb{P}).$$

We claim that

$$\sum_{p \in \mathcal{P} \cap (z, w)} \frac{1}{p} \ll \begin{cases} \min \{\epsilon / \delta, 1 / \log w\} & \text{if } z \geq (\delta / \epsilon)^{4/3}, \\ \min \{\epsilon / \delta, 1 / \log w\} \log (2\delta / \epsilon) + 1 / z & \text{if } z < (\delta / \epsilon)^{4/3}, \\ \min \{\epsilon / \delta, 1 / \log w\} + \log^{-3}(z + 1) & \text{if } z < (\delta / \epsilon)^{4/3}. \end{cases}$$

Indeed, if $p_1 < p_2$ both belong to the set \{p \in \mathcal{P} \cap (z, w) : u < f(p) \leq u + \min \{\epsilon, \delta / \log w\}/C\}, then we have that

$$\frac{1}{C} \min \left\{ \epsilon, \frac{\delta}{\log w} \right\} > |f(p_1) - f(p_2)| \geq \frac{2}{C} \min \left\{ \frac{g(p_2)(p_2 - p_1)}{p_2 \log p_2}, g(p_2) \right\} \geq \frac{2}{C} \min \left\{ \frac{\delta(p_2 - p_1)}{p_2 \log p_2}, \epsilon \right\}$$

and, consequently,

$$0 < p_2 - p_1 \leq \frac{p_2 \log p_2}{2} \min \left\{ \frac{\epsilon}{\delta}, \frac{1}{\log w} \right\} \leq \frac{p_2}{2}.$$

Relation (3.3) then follows, by the Brun-Titchmarsh inequality [13, Theorem 9, p. 73] for the first and the second part, and by the Prime Number Theorem [13, Theorem 1, p. 167] for the third part. Next by breaking the interval $(u, u + \epsilon]$ into at most $1 + C \max \{1, \epsilon(\log w) / \delta\} \leq 1 + C \log w$ intervals of the form $(v, v + \epsilon)$, where $v \leq \max \{1, \epsilon(\log w) / \delta\}$, we deduce that

$$\sum_{p \in \mathcal{P} \cap (z, w)} \frac{1}{p} \ll \begin{cases} \epsilon / \delta & \text{if } z \geq (\delta / \epsilon)^{4/3}, \\ (\epsilon / \delta)(2\delta / \epsilon) + (\log z) / z & \text{if } z < (\delta / \epsilon)^{4/3}, \\ \epsilon / \delta + \log^{-2} z & \text{if } z < (\delta / \epsilon)^{4/3}. \end{cases}$$

We are now in position to show (2.3): consider $0 < \epsilon \leq \delta \leq 1$, $u \in \mathbb{R}$ and $3 \leq z \leq w \leq \min \{P_f(\delta), P_f(\epsilon)^{1/4}\}$. Note that for all $j \geq 0$ with $2^j \leq 1 / \delta$ we have that

$$\log P_f(2^j \delta) \geq 2^{-1 - j/\epsilon} \log P_f(\delta) \geq 2^{-1 - j/\epsilon} \log w \geq 4^{-j} \log w,$$

by (3.2). Consequently, relation (3.5) with $2^j \delta$, $\max \{z, w^{4^{-j-1}}\}$ and $w^{4^{-j}}$ in place of $\delta$, $z$ and $w$, respectively, implies that

$$\sum_{p \in \mathcal{P} \cap (z, w)} \frac{1}{p} \ll \begin{cases} \epsilon / 2^j \delta & \text{if } z \geq (\delta / \epsilon)^2 \text{ and } 2^j \leq \sqrt{\delta / \epsilon}, \\ 2^j \delta \log \left( \frac{2^j \delta}{\epsilon} \right) + 4^{-j} \log w_{\epsilon / \delta + \frac{16^j}{(\log w)^2}} & \text{if } z \geq (\delta / \epsilon)^2 \text{ and } 2^j > \sqrt{\delta / \epsilon}, \end{cases}$$

Summing the above inequality over $j \geq 0$ with $2^j \leq 1 / \delta$ and $w^{4^{-j}} \geq z$ completes the proof of (2.3). In conclusion, we may apply Theorem 5 with $P_f = K - 1$, $\rho = 2$ and $\lambda = 1/A$. 
Consider $0 < \epsilon \leq 1/2$ small enough so that $K(\epsilon) \geq 3^A$ and $\delta = g\left(\lceil K(\epsilon)^{1/A} \rceil \right)$ lies in $[\epsilon, 1/2]$. Since $g$ is strictly decreasing, the definition of $K$ implies that

\[(3.6) \quad K(\delta) = \lceil K(\epsilon)^{1/A} \rceil .\]

In particular, $P_f(\delta) \leq P_f(\epsilon)^{1/A}$. For $j \in \mathbb{Z}$ with $1 \leq 2^j \leq 1/\delta$ we set $q_j = P_f(2^j \delta) = K(2^j \delta) - 1$. Note that

\[(3.7) \quad \log q_j \geq 2^{-1-(j-i)/\epsilon} \log q_i \quad (0 \leq i < j, 2^j \leq 1/\delta),\]

by \((3.2)\). Set

\[J = \max \left(\{0\} \cup \{j \in \mathbb{N} : 2^j \leq 1/\delta \text{ and } q_j \geq (2^j \delta/\epsilon^2)\}\right).\]

Then Theorem 5 and relation \((3.7)\) imply that for $y \geq K(\epsilon)^{1/A} \geq q_0$

\begin{align*}
Q_{\tau_y}(\epsilon) & \ll_A \frac{1}{\log q_0} + \frac{\epsilon}{\delta} \sum_{j=1}^{J} \frac{1}{2^j \log q_j} + \frac{\epsilon}{\delta} \sum_{j=0}^{J} \frac{\log q_j}{2^j \log q_j} \\
& \ll \frac{1}{\log q_0} + \frac{\epsilon}{\delta} \sum_{j=1}^{J} \frac{1}{2^j \log q_0} \frac{\log q_j}{2^j} + \frac{\epsilon}{\delta} \sum_{j=0}^{J} \frac{\log q_j}{2^j \log q_0} \\
& \ll \min \left\{ \frac{1}{\epsilon \log q_0}, 1 + J\epsilon/\delta \right\} \frac{\log q_j}{\log q_0} + \frac{\epsilon}{\delta} \log q_j \\
& \ll \min \left\{ \frac{1}{\epsilon \log q_0}, 1 + J\epsilon/\delta \right\} \frac{\log q_j}{\log q_0} + \frac{\epsilon}{\delta} \max \{1, (J + 1)(c - 1)\} \frac{\log q_j}{\log q_0}.
\end{align*}

Finally, note that if $2^{J+1} \leq 1/\delta$, then the maximality of $J$ and \((3.7)\) imply that

\[\log q_J \leq 4 \log q_{J+1} \leq 8 \log(2^{J+1} \delta/\epsilon) \ll J + 1 + \log(\delta/\epsilon).\]

On the other hand, if $2^{J+1} > 1/\delta$, then $q_J \leq K(1/2) \ll 1$, since $g(t) \ll (\log t)^{-c}$. In any case, we find that $\log q_J \ll J + 1 + \log(\delta/\epsilon)$, which together with the inequality

\[\frac{J\epsilon}{\delta} \ll \frac{\epsilon \log(1/\delta)}{\delta} \leq \log \frac{1}{\epsilon},\]

and relations \((3.6)\) and \((3.8)\) completes the proof of the theorem. \(\square\)

**Proof of Theorem 4.** The lower bound is an immediate consequence of Theorem 6. For the upper bound, note that

\[\sum_{|f(p)| > p^{-1/2A}} \frac{1}{p} \leq \sum_p \frac{|f(p)|}{p^{1-1/2A}} \leq \sum_{k \geq 1} 2^{k/2A} \sum_{2^{k-1} \leq p \leq 2^k} \frac{|f(p)|}{p} \ll \sum_{k \geq 1} \frac{1}{2^{k/2A}} \ll_A 1.\]

So if we set $\mathcal{P} = \{p : |f(p)| \leq p^{-1/2A}\}$, $P_f(\epsilon) = \epsilon^{-2A}$, $\lambda = 1/(2AB)$, and $\rho = 1/\eta$, then the assumptions of Theorem 5 are satisfied. Applying this theorem with $\delta = \epsilon^{1/2AB}$ and $J = 0$, we deduce that

\[Q_{\tau_y}(\epsilon) \ll_{\eta, A, B} \frac{1}{\log(1/\delta)} + \frac{\epsilon}{\delta} \ll_{A, B} \frac{1}{\log(1/\epsilon)} \quad (y \geq \delta^{-2A}),\]

which completes the proof. \(\square\)
ACKNOWLEDGMENTS

I would like to thank Gérald Tenenbaum for pointing out relation (1.3) to me. I am also grateful to Régis de la Bretèche and Maksym Radziwill for some helpful comments. Finally, I would like to acknowledge the referee for exposing some inaccuracies in an earlier version of the paper. This manuscript was largely written while enjoying the hospitality of Université Paris-Diderot.

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