FUNCTIONAL EQUATIONS RELATED TO THE DIRICHLET LAMBDA AND BETA FUNCTIONS

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Abstract. We give closed-form expressions for the Dirichlet beta function at even positive integers and for the Dirichlet lambda function at odd positive integers, based on the function $J(s)$ defined via convergent integral. We also show fundamental relations between Dirichlet lambda and beta functions and the function $J(s)$.

1. Introduction

We will use the definitions involving the Dirichlet lambda function and the Dirichlet beta function. The Dirichlet lambda and beta function are defined as

$$\lambda(s) = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} = \left(1 - \frac{1}{2^s}\right) \zeta(s) \quad \Re(s) > 1$$

$$\beta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^s} \quad \Re(s) > 0$$

where $\zeta(s)$ is the Riemann zeta function. The values of the Dirichlet lambda function at even positive integers and Dirichlet beta function at odd positive integers are given as

$$\lambda(2m) = (2^{2m} - 1) \frac{(-1)^{m-1} \pi^{2m}}{2(2m)!} B_{2m} \quad m \in \mathbb{N}$$

$$\beta(2m - 1) = \frac{(-1)^{m-1} E_{2m-2}}{2(2m-2)!} \left(\frac{\pi}{2}\right)^{2m-1} \quad m \in \mathbb{N}$$

where $B_{2m}$ is Bernoulli number and $E_{2m}$ is Euler number.

In this paper, we define the integral function $J(s)$ which can be written for all $\Re(s) > 0$

$$J(s) = \frac{1}{\Gamma(s+1)} \frac{2}{\pi} \int_{0}^{\pi} x^s \sin(x) \,dx$$

where $\Gamma$ denotes the Gamma function.

The function $J(s)$ gives closed-form expressions for the Dirichlet lambda function at odd positive integers and for the Dirichlet beta function at even positive integers.

**Theorem 1.** The values of the Dirichlet lambda function at odd positive integers are denoted by $J(s)$ as follows:

$$\lambda(2m + 1) = \sum_{k=1}^{m} [(-1)^{k-1} \lambda(2m - 2k + 2)] + (-1)^m \beta(1) J(2m)$$

where for all $m \in \mathbb{N}$

**Theorem 2.** The values of the Dirichlet beta function at even positive integers are denoted by $J(s)$ as follows:

$$\beta(2m) = \sum_{k=1}^{m} (-1)^{k-1} \beta(2m - 2k + 1) J(2k - 1)$$

where for all $m \in \mathbb{N}$
For example,

\[
\begin{align*}
\beta(2) &= \beta(1)J(1) \\
\beta(4) &= \beta(3)J(1) - \beta(1)J(3) \\
\beta(6) &= \beta(5)J(1) - \beta(3)J(3) + \beta(1)J(5) \\
\beta(8) &= \beta(7)J(1) - \beta(5)J(3) + \beta(3)J(5) - \beta(1)J(7)
\end{align*}
\]

\[\vdots\]

and

\[
\begin{align*}
\lambda(3) &= \lambda(2)J(1) - \beta(1)J(2) \\
\lambda(5) &= \lambda(4)J(1) - \lambda(2)J(3) + \beta(1)J(4) \\
\lambda(7) &= \lambda(6)J(1) - \lambda(4)J(3) + \lambda(2)J(5) - \beta(1)J(6) \\
\lambda(9) &= \lambda(8)J(1) - \lambda(6)J(3) + \lambda(4)J(5) - \lambda(2)J(7) + \beta(1)J(8)
\end{align*}
\]

\[\vdots\]

2. Preliminary Lemmas

In this section, we start with several Lemmas used in proving Theorems 1 and 2.

**Lemma 1.** If \(n\) is a positive integer, then

\[
\begin{align*}
\sum_{k=1}^{n} \cos((2k-1)x) &= \frac{1}{2} \csc(x) \sin(2nx) \quad (8) \\
\sum_{k=1}^{n} \sin((2k-1)x) &= \csc(x) \sin^2(nx) \quad (9)
\end{align*}
\]

**Proof.** Consider the following sum,

\[
S = \sum_{k=1}^{n} \cos((2k-1)x) + i \sum_{k=1}^{n} \sin((2k-1)x) = \sum_{k=1}^{n} e^{i(2k-1)x}
\]

Since \(S\) is a geometric series with common ratio \(e^{2ix}\)

\[
S = \frac{e^{ix}(1 - e^{2nix})}{1 - e^{2ix}} = \frac{(e^{-nix} - e^{nix})e^{nix}}{e^{-ix} - e^{ix}} = \frac{-2i \sin(nx)(\cos(nx) + i \sin(nx))}{2i \sin(x)} = \frac{1}{2} \csc(x) \sin(2nx) + i \csc(x) \sin^2(nx)
\]

Therefore,

\[
\begin{align*}
\sum_{k=1}^{n} \cos((2k-1)x) &= \frac{1}{2} \csc(x) \sin(2nx) \\
\sum_{k=1}^{n} \sin((2k-1)x) &= \csc(x) \sin^2(nx)
\end{align*}
\]

\[\square\]

**Lemma 2.** If \(n\) is a positive integer, then

\[
\sum_{k=1}^{n} (-1)^{k-1} \cos((2k-1)x) = \sec(x) \sin^2 \left( \frac{n(\pi - 2x)}{2} \right) \quad (10)
\]
Proof. Consider the following sum,

\[
S = \sum_{k=1}^{n} (-1)^{k-1} \cos((2k-1)x) + i \sum_{k=1}^{n} (-1)^{k-1} \sin((2k-1)x) = \sum_{k=1}^{n} (-1)^{k-1} e^{i(2k-1)x} = e^{ix} \frac{1 - (-1)^n e^{2nx}}{1 + e^{2ix}} = \frac{1 - (-1)^n \cos(2nx) - (-1)^n i \sin(2nx)}{2 \cos(x)}
\]

Taking the real part,

\[
\Re(S) = \frac{1 - (-1)^n \cos(2nx)}{2 \cos(x)} = \frac{1 - \cos(n\pi) \cos(2nx)}{2 \cos(x)} = \frac{1 - \cos(n\pi - 2nx)}{2 \cos(x)} = \frac{\sin^2(n(\pi - 2x)/2)}{\cos(x)}
\]

Therefore,

\[
\sum_{k=1}^{n} (-1)^{k-1} \cos((2k-1)x) = \sec(x) \sin^2 \left( \frac{n(\pi - 2x)}{2} \right)
\]

\[\square\]

**Lemma 3.** Let \( A \) be a \( n \times n \) matrix, and \( A_{ij} = \sin \left( \frac{(2i - 1)(2j - 1)\pi}{4n} \right) \), then \( A^{-1} = \frac{2}{n} A \)

Proof. Note that \((i, j)\)th element of the matrix \( A^2 \). The \( A^2 \) is the \( n \times n \) matrix whose \((i, j)\)th entry is given by

\[
A_{ij}^2 = \sum_{m=1}^{n} \left[ \sin \left( \frac{(2i - 1)(2m - 1)\pi}{4n} \right) \sin \left( \frac{(2j - 1)(2m - 1)\pi}{4n} \right) \right]
\]

If \( i = j \), we have

\[
A_{ij}^2 = \sum_{m=1}^{n} \sin^2 \left( \frac{(2i - 1)(2m - 1)\pi}{4n} \right)
\]

By using the identity \( \sin^2(x) = \frac{1}{2}(1 - \cos(2x)) \) and Lemma III

\[
A_{ij}^2 = \sum_{m=1}^{n} \left[ \frac{1}{2} - \frac{1}{2} \cos \left( \frac{(2i - 1)(2m - 1)\pi}{2n} \right) \right] = \frac{n}{2} - \frac{1}{2} \sin((2i - 1)\pi) \csc \left( \frac{(2i - 1)\pi}{2n} \right)
\]

If \( i \neq j \), we have

\[
A_{ij}^2 = \frac{1}{2} \sum_{m=1}^{n} \left[ \cos \left( \frac{(2i - 2j)(2m - 1)\pi}{4n} \right) - \cos \left( \frac{(2i + 2j - 2)(2m - 1)\pi}{4n} \right) \right]
\]

\[
= \frac{1}{4} \sin((i - j)\pi) \csc \left( \frac{(i - j)\pi}{2n} \right) - \frac{1}{4} \sin((i + j - 1)\pi) \csc \left( \frac{(i + j - 1)\pi}{2n} \right)
\]

Thus, if \( i = j \), the expression evaluates to \( n/2 \) and if \( i \neq j \), the this expression evaluates to 0. By the two cases above,

\[
A^2 = \frac{n}{2} I_n
\]

where \( I_n \) is \( n \times n \) identity matrix. Therefore \( A \) is non-singular and

\[
A^{-1} = \frac{2}{n} A
\]

\[\square\]

**Lemma 4.** Let \( B \) be a \( n \times n \) matrix, and \( B_{ij} = \cos \left( \frac{(2i - 1)(2j - 1)\pi}{4n} \right) \), then \( B^{-1} = \frac{2}{n} B \)
Proof. Note that \((i, j)\)th element of the matrix \(B^2\). The \(B^2\) is the \(n \times n\) matrix whose \((i, j)\)th entry is given by

\[
B_{ij}^2 = \sum_{m=1}^{n} \left[ \cos \left( \frac{(2i - 1)(2m - 1)\pi}{4n} \right) \cos \left( \frac{(2j - 1)(2m - 1)\pi}{4n} \right) \right]
\]

If \(i = j\), we have

\[
B_{ij}^2 = \sum_{m=1}^{n} \left[ \frac{1}{2} + \frac{1}{2} \cos \left( \frac{(2i - 1)(2m - 1)\pi}{2n} \right) \right] = \frac{n}{2} + \frac{1}{2} \sin((2i - 1)\pi) \csc \left( \frac{(2i - 1)\pi}{2n} \right)
\]

By using the identity \(\cos^2(x) = \frac{1}{2}(1 + \cos(2x))\) and Lemma 1,

\[
B_{ij}^2 = \sum_{m=1}^{n} \left[ \frac{1}{2} + \frac{1}{2} \cos \left( \frac{(2i - 1)(2m - 1)\pi}{2n} \right) \right] = \frac{n}{2} + \frac{1}{2} \sin((2i - 1)\pi) \csc \left( \frac{(2i - 1)\pi}{2n} \right) = \frac{n}{2}
\]

If \(i \neq j\), we have

\[
B_{ij}^2 = \frac{1}{2} \sum_{m=1}^{n} \left[ \cos \left( \frac{(2i - 2j)(2m - 1)\pi}{4n} \right) + \cos \left( \frac{(2i + 2j - 2)(2m - 1)\pi}{4n} \right) \right]
\]

\[
= \frac{1}{4} \sin((i - j)\pi) \csc \left( \frac{(j - i)\pi}{2n} \right) + \frac{1}{4} \sin((i + j - 1)\pi) \csc \left( \frac{(i + j - 1)\pi}{2n} \right) = 0
\]

Finally, the expression for \(i = j\) evaluates to \(n/2\), and the expression for \(i \neq j\) evaluates to 0. By the two cases above,

\[
B^2 = \frac{n}{2} I_n
\]

where \(I_n\) is \(n \times n\) identity matrix. Therefore \(B\) is non-singular and

\[
B^{-1} = \frac{2}{n} B
\]

\[\square\]

**Lemma 5.** Let \(f(s)\) be an infinite series defined by

\[
f(s) = \frac{1}{\Gamma(s + 1)} \lim_{n \to \infty} \frac{1}{n} \sum_{p=1}^{n} \left( \frac{(2p - 1)\pi}{4n} \right)^{s} \sin \left( \frac{(2p - 1)\pi}{4n} \right)
\]

where \(\Re(s) > 0\), then \(f(s) = J(s)\).

Proof. \(f(s)\) is represented by difference of two infinite series as follows:

\[
f(s) = \frac{1}{\Gamma(s + 1)} \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left( \frac{\pi}{2} \frac{k}{2n} \right)^{s} \sin \left( \frac{\pi}{2} \frac{k}{2n} \right) - \frac{1}{\Gamma(s + 1)} \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left( \frac{\pi}{2} \frac{2k}{2n} \right)^{s} \sin \left( \frac{\pi}{2} \frac{2k}{2n} \right)
\]

By substituting \(2m = n\),

\[
f(s) = \frac{1}{\Gamma(s + 1)} \lim_{n \to \infty} \frac{2}{n} \sum_{k=1}^{n} \left( \frac{\pi}{2} \frac{k}{n} \right)^{s} \sin \left( \frac{\pi}{2} \frac{k}{n} \right) - \frac{1}{\Gamma(s + 1)} \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left( \frac{\pi}{2} \frac{2k}{2n} \right)^{s} \sin \left( \frac{\pi}{2} \frac{2k}{2n} \right)
\]

\[
= \frac{1}{\Gamma(s + 1)} \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left( \frac{\pi}{2} \frac{k}{n} \right)^{s} \sin \left( \frac{\pi}{2} \frac{k}{n} \right)
\]
Let $\Delta x = (\frac{\pi}{2}) \frac{1}{n}$, $x_k = (\frac{\pi}{2}) \frac{k}{n}$, $f(x) = \frac{x^s}{\sin(x)}$, then

$$f(s) = \frac{1}{\Gamma(s+1)} \lim_{n \to \infty} \frac{2}{n} \sum_{k=1}^{n} f(x_k) \Delta x = \frac{1}{\Gamma(s+1)} \frac{2}{\pi} \int_{0}^{\frac{\pi}{2}} f(x) dx$$

$$= \frac{1}{\Gamma(s+1)} \frac{2}{\pi} \int_{0}^{\frac{\pi}{2}} \frac{x^s}{\sin(x)} dx = J(s)$$

Lemma 6. Let $W(s)$ be a divergent function defined by

$$W(s) = \frac{1}{\Gamma(s+1)} \lim_{n \to \infty} \frac{1}{n} \sum_{p=1}^{n} \left( \frac{(2p-1)\pi}{4n} \right)^s \cos \left( \frac{(2p-1)\pi}{4n} \right)$$

then $W(s)$ where $m \in \mathbb{N}$ is denoted by $J(s)$ as follows:

$$W(m) = \sum_{k=0}^{m} \frac{(-1)^k}{(m-k)!} \left( \frac{\pi}{2} \right)^{m-k} J(k)$$

3. Proof of the Theorems

The expression $x(\pi - x)$ where $0 \leq x \leq \pi$ can be expanded to a Fourier sine series as follows:

$$x(\pi - x) = \frac{8}{\pi} \left\{ \frac{\sin(x)}{1^3} + \frac{\sin(3x)}{3^3} + \frac{\sin(5x)}{5^3} + \cdots \right\}$$

Using the Dirichlet lambda and beta function values, we have

$$\sum_{k=1}^{\infty} \frac{\sin((2k-1)x)}{(2k-1)^3} = \lambda(2) x - \beta(1) \frac{x^2}{2!}$$
Let \( f_n(x) = \sum_{k=1}^{\infty} \frac{\sin((2k-1)x)}{(2k-1)^n} \) and \( g_n(x) = \sum_{k=1}^{\infty} \frac{\cos((2k-1)x)}{(2k-1)^n} \), then the multiple integrals on both sides of Eq. (15) with respect to \( x \) from 0 to \( x \) are given by the functional equations.

\[
f_{2m+1}(x) = \sum_{k=1}^{\infty} \frac{\sin((2k-1)x)}{(2k-1)^{2m+1}} = \sum_{k=1}^{m} \left\{ \lambda(2m - 2k + 2) \frac{(-1)^{k-1} x^{2k-1}}{(2k-1)!} \right\} + (-1)^m \beta(1) \frac{x^{2m}}{(2m)!} \tag{16}
\]

\[
g_{2m}(x) = \sum_{k=1}^{\infty} \frac{\cos((2k-1)x)}{(2k-1)^{2m}} = \sum_{k=1}^{m} \left\{ \lambda(2m - 2k + 2) \frac{(-1)^{k-1} x^{2k-2}}{(2k-2)!} \right\} + (-1)^m \beta(1) \frac{x^{2m-1}}{(2m-1)!} \tag{17}
\]

where \( m \in \mathbb{N} \). The constant of integration is determined by boundary conditions at \( f_n(0) = 0 \) and \( g_n(0) = \lambda(n) \).

If \( a_k = \sin \left( \frac{(2k-1)(2p-1)\pi}{4n} \right) \) and \( b_k = \cos \left( \frac{(2k-1)(2p-1)\pi}{4n} \right) \) where \( p = 1, 2, 3, \ldots, n \), periodic sequences \( a_k \) and \( b_k \) satisfy as follow:

\[
a_k = (-1)^{m+1} a_{2mn-(k-1)} = (-1)^m a_{2mn+k} \tag{18}
\]

\[
b_k = (-1)^{m} b_{2mn-(k-1)} = (-1)^m a_{2mn+k} \tag{19}
\]

where \((1 \leq k \leq n)\) and \( m \in \mathbb{N} \). For example, if \( n = 10 \) and \( k = 6 \), then \( a_6 = a_{15} = -a_{26} = -a_{35} = a_{46} = \ldots \) and \( b_6 = -b_{15} = -b_{26} = b_{35} = b_{46} = \ldots \).

Thus, \( f_{2m+1} \left( \frac{(2p-1)\pi}{4n} \right) \) and \( g_{2m} \left( \frac{(2p-1)\pi}{4n} \right) \) are given by the functional equations.

\[
f_{2m+1} \left( \frac{(2p-1)\pi}{4n} \right) = \sum_{q=1}^{n} \left[ \sin \left( \frac{(2q-1)(2p-1)\pi}{4n} \right) \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)(2n-(2n-(2q-1)))^{2m+1}} \right] \tag{20}
\]

\[
g_{2m} \left( \frac{(2p-1)\pi}{4n} \right) = \sum_{q=1}^{n} \left[ \cos \left( \frac{(2q-1)(2p-1)\pi}{4n} \right) \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)(2n-(2n-(2q-1)))^{2m}} \right] \tag{21}
\]

When \( p \) has the values \( 1, 2, \ldots, n \), we get \( n \) functional equations which can be written as

\[
F = AX \tag{22}
\]

\[
G = BY \tag{23}
\]

where

\[
A = \begin{pmatrix}
\sin \left( \frac{\pi}{4n} \right) & \sin \left( \frac{3\pi}{4n} \right) & \ldots & \sin \left( \frac{(2n-1)\pi}{4n} \right) \\
\sin \left( \frac{3\pi}{4n} \right) & \sin \left( \frac{\pi}{4n} \right) & \ldots & \sin \left( \frac{3(2n-1)\pi}{4n} \right) \\
\vdots & \vdots & \ddots & \vdots \\
\sin \left( \frac{(2n-1)\pi}{4n} \right) & \sin \left( \frac{3(2n-1)\pi}{4n} \right) & \ldots & \sin \left( \frac{(2n-1)^2\pi}{4n} \right)
\end{pmatrix} \tag{24}
\]
To calculate \( X \) and \( Y \), we need to use Lemma 2 and Lemma 3. By Lemma 2 and 3, the \( X = A^{-1}F = \frac{2}{n}AF \) and \( Y = B^{-1}G = \frac{2}{n}BG \) as follows:

\[
X = \begin{pmatrix}
\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)2n-(2n-1)}^{2m+1} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)2n+(2n-1)}^{2m+1} \\
\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)2n-(2n-3)}^{2m+1} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)2n+(2n-3)}^{2m+1} \\
\vdots \\
\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)2n-(2n-1)}^{2m+1} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)2n+(2n-1)}^{2m+1}
\end{pmatrix}
\]

(25)

(26)

(27)

(28)
\[ Y = \left( \begin{array}{cccc}
\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)2n - (2n - 1)} & \ldots & \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)2n + (2n - 1)} \\
\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)2n - (2n - 3)} & \ldots & \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)2n + (2n - 3)} \\
\vdots & \ddots & \vdots \\
\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)2n - (2n - (2n - 1))} & \ldots & \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)2n + (2n - (2n - 1))} \\
\end{array} \right) \]

\[ X = \left( \begin{array}{cccc}
\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)2n - (2n - 1)} & \ldots & \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)2n + (2n - 1)} \\
\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)2n - (2n - 3)} & \ldots & \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)2n + (2n - 3)} \\
\vdots & \ddots & \vdots \\
\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)2n - (2n - (2n - 1))} & \ldots & \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)2n + (2n - (2n - 1))} \\
\end{array} \right) \]

\[ \lambda(2m + 1) = \lim_{n \to \infty} \sum_{k=1}^{n} X_{k,1} \]

Thus, sum of all elements in a matrix \( X \) is represented as

\[ \lambda(2m + 1) = \lim_{n \to \infty} \frac{2}{n} \sum_{p=1}^{n} \left[ f_{2m+1}(\frac{p-1}{4n}) \sum_{q=1}^{n} \sin\left(\frac{(2p-1)(2q-1)\pi}{4n}\right) \right] \]
In order to obtain the expression $\beta$, using the Eq. (16), we have

$$\lambda(2m + 1) = \lim_{n \to \infty} \frac{2}{n} \sum_{p=1}^{n} f_{2m+1} \left(\frac{(2p-1)\pi}{4n}\right) \frac{\sin^2 \left(\frac{(2p-1)\pi}{4n}\right)}{\sin \left(\frac{(2p-1)\pi}{4n}\right)}$$

$$= \lim_{n \to \infty} \frac{1}{n} \sum_{p=1}^{n} f_{2m+1} \left(\frac{(2p-1)\pi}{4n}\right) \frac{1}{\sin \left(\frac{(2p-1)\pi}{4n}\right)}$$

Using the Eq. (16), we have

$$\lambda(2m + 1) = \lim_{n \to \infty} \frac{1}{n} \sum_{p=1}^{n} \left[ \sum_{k=1}^{m} (-1)^{k-1} \lambda(2m - 2k + 2) \frac{(2p-1)\pi}{(2k-1)!} \frac{1}{\sin \left(\frac{(2p-1)\pi}{4n}\right)} \right]^{2k-1}

+ \lim_{n \to \infty} \frac{1}{n} \sum_{p=1}^{n} \left[ (-1)^{m} \beta(1) \frac{(2p-1)\pi}{(2m)!} \frac{1}{\sin \left(\frac{(2p-1)\pi}{4n}\right)} \right]^{2m}$$

Using the Lemma [5] we have

$$\lambda(2m + 1) = \sum_{k=1}^{m} \left[ (-1)^{k-1} \lambda(2m - 2k + 2)J(2k - 1) \right] + (-1)^{m} \beta(1)J(2m)$$

The proof of Theorem 1 was completed.

### 3.2. Dirichlet Beta Function at Even Positive Integers.

In Eq. (30), we know that sum of all elements in a matrix $Y$ is equal to $\lambda(2m)$

$$\lambda(2m) = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^{2m}} = \lim_{n \to \infty} \sum_{k=1}^{n} Y_{k,1}$$

where

$$Y = \begin{pmatrix}
\cos \left(\frac{\pi}{4n}\right) & \cos \left(\frac{3\pi}{4n}\right) & \ldots & \cos \left(\frac{(2n-1)\pi}{4n}\right) \\
\cos \left(\frac{3\pi}{4n}\right) & \cos \left(\frac{9\pi}{4n}\right) & \ldots & \cos \left(\frac{3(2n-1)\pi}{4n}\right) \\
\vdots & \vdots & \ddots & \vdots \\
\cos \left(\frac{(2n-1)\pi}{4n}\right) & \cos \left(\frac{3(2n-1)\pi}{4n}\right) & \ldots & \cos \left(\frac{(2n-1)^2\pi}{4n}\right)
\end{pmatrix}$$

$$g_{2m}(\pi) \begin{pmatrix}
\frac{\pi}{4n} \\
\frac{3\pi}{4n} \\
\vdots \\
\frac{(2n-1)\pi}{4n}
\end{pmatrix}$$

In order to obtain the expression $\beta(2m)$, we define the matrix $Z$ as follows:

$$Z = \begin{pmatrix}
\begin{pmatrix}
\cos \left(\frac{\pi}{4n}\right) \\
-\cos \left(\frac{3\pi}{4n}\right) \\
\vdots \\
(-1)^{n-1} \cos \left(\frac{(2n-1)\pi}{4n}\right)
\end{pmatrix} & g_{2m}(\pi) & \ldots & g_{2m}(\frac{(2n-1)\pi}{4n}) \\
\end{pmatrix}$$

$$+ \begin{pmatrix}
\begin{pmatrix}
\cos \left(\frac{(2n-1)\pi}{4n}\right) \\
-\cos \left(\frac{3(2n-1)\pi}{4n}\right) \\
\vdots \\
(-1)^{n-1} \cos \left(\frac{(2n-1)^2\pi}{4n}\right)
\end{pmatrix} & g_{2m}(\frac{(2n-1)\pi}{4n}) & \ldots & g_{2m}(\frac{(2n-1)^2\pi}{4n})
\end{pmatrix}$$
Then sum of all elements in a matrix $Z$ is equal to $\beta(2m)$.

$$
\beta(2m) = \lim_{n \to \infty} \sum_{p=1}^{n} g_{2m} \left( \frac{(2p-1)\pi}{4n} \right) \sum_{q=1}^{n} (-1)^{q-1} \cos \left( \frac{(2p-1)(2q-1)\pi}{4n} \right)
$$

By using the Lemma 6, we have

$$
\beta(2m) = \lim_{n \to \infty} \sum_{p=1}^{n} g_{2m} \left( \frac{(2p-1)\pi}{4n} \right) \sin^2 \left( \frac{n}{2} \left( \pi - \frac{(2p-1)\pi}{2n} \right) \right)
$$

$$
= \lim_{n \to \infty} \frac{1}{n} \sum_{p=1}^{n} g_{2m} \left( \frac{(2p-1)\pi}{4n} \right) \sin^2 \left( \frac{(2p-1)\pi}{4n} \right)
$$

Using the Eq. (17), we have

$$
\beta(2m) = \lim_{n \to \infty} \frac{1}{n} \sum_{p=1}^{n} \left[ \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\lambda(2m-2k+2)}{2k-1)!} \left( \frac{(2p-1)\pi}{4n} \right)^{2k-2} \frac{(2m-1)!}{\cos \left( \frac{(2p-1)\pi}{4n} \right)} \right]
$$

$$
+ \lim_{n \to \infty} \frac{1}{n} \sum_{p=1}^{n} \left[ (-1)^m \beta(1) \left( \frac{(2p-1)\pi}{4n} \right) \frac{(2m-1)!}{\cos \left( \frac{(2p-1)\pi}{4n} \right)} \right]
$$

Using the Lemma 6, we have

$$
\beta(2m) = \sum_{k=1}^{m} \left[ (-1)^{k-1}\lambda(2m-2k+2)W(2k-2) \right] + (-1)^m \beta(1)W(2m-1)
$$

(31)

$$
= \sum_{k=1}^{m} \left[ (-1)^{k-1}\lambda(2m-2k+2) \sum_{q=0}^{2k-2} \left\{ (-1)^q \frac{1}{{(2k-2)-q)!} \left( \frac{\pi}{2} \right)^{(2k-2)-q} J(q) \right\} \right]
$$

$$
+ (-1)^m \beta(1) \sum_{q=0}^{2m-1} \left\{ (-1)^q \frac{1}{(2m-1)-q)!} \left( \frac{\pi}{2} \right)^{(2m-1)-q} J(q) \right\}
$$

The index of summation $q$ takes on integer values from 0 to $2k-2$.

Now, expand the inner summation(which involves $q$).

$$
\beta(2m) = \sum_{k=1}^{m} \left[ (-1)^{k-1}\lambda(2m-2k+2) \left( \frac{\pi}{2} \right)^{2k-2} J(+0) \right] + (-1)^m \beta(1) \left( \frac{\pi}{2} \right)^{2m-1} J(+0)
$$

$$
+ \sum_{k=2}^{m} \left[ (-1)^{k+1}\lambda(2m-2k+2) \left( \frac{\pi}{2} \right)^{(2k-2)-1} J(1) \right]
$$

$$
+ \sum_{k=2}^{m} \left[ (-1)^{k+1}\lambda(2m-2k+2) \left( \frac{\pi}{2} \right)^{(2k-2)-2} J(2) \right]
$$

$$
+ \sum_{k=2}^{m} \left[ (-1)^{k+1}\lambda(2m-2k+2) \left( \frac{\pi}{2} \right)^{(2k-2)-(2m-2)} J(2k-2) \right]
$$

$$
+ \sum_{k=m} \left[ (-1)^{3m-1}\beta(1) \left( \frac{\pi}{2} \right)^{(2m-1)-(2m-1)} J(2m-1) \right]
$$
Change the index of summation $k$ so that it would start from 1.

\[
\beta(2m) = \sum_{k=1}^{m} \left\{ (-1)^{k-1} \frac{\lambda(2m - 2k + 2)}{(2k - 2)!} \left( \frac{\pi}{2} \right)^{2k-2} \right\} + (-1)^m \beta(1) \left( \frac{\pi}{2} \right)^{2m-1} J(0) \\
+ \sum_{k=1}^{m-1} \left\{ (-1)^{k-1} \frac{\lambda(2m - 2k)}{(2k - 1)!} \left( \frac{\pi}{2} \right)^{2k-1} \right\} + (-1)^m \beta(1) \left( \frac{\pi}{2} \right)^{2m-2} J(1) \\
+ \sum_{k=1}^{m-1} \left\{ (-1)^k \frac{\lambda(2m - 2k)}{(2k - 2)!} \left( \frac{\pi}{2} \right)^{2k-2} \right\} + (-1)^m \beta(1) \left( \frac{\pi}{2} \right)^{2m-3} J(2) + \cdots \\
+ \left[ (-1)^m \lambda(2) \beta(2m - 2) \right] \right\} + (-1)^m \beta(1) \beta(2m - 1)
\]

Using the Eq. (16) and Eq. (17)

\[
\beta(2m) = \sum_{k=1}^{m} \left\{ (-1)^{k-1} g_{2m - 2k + 2} \left( \frac{\pi}{2} \right) J(2k - 2) + (-1)^k f_{2m - 2k + 1} \left( \frac{\pi}{2} \right) J(2k - 1) \right\}
\]

Since $g_{2m} \left( \frac{\pi}{2} \right) = 0$ and $f_{2m+1} \left( \frac{\pi}{2} \right) = \beta(2m - 1)$ (See Eq. (16) and Eq. (17))

\[
\beta(2m) = \sum_{k=1}^{m} \left\{ (-1)^{k-1} \beta(2m - 2k + 1) \beta(2k - 1) \right\}
\]

The proof of Theorem 2 was completed.

4. THE INTEGRAL FUNCTION $J(n)$

**Lemma 7.** The function $\frac{1}{2} \ln \left( \tan \frac{x}{2} \right)$ can be expanded as an infinite series,

\[
\sum_{k=1}^{\infty} \frac{\cos((2k - 1)x)}{2k - 1} = -\frac{1}{2} \ln \left( \tan \frac{x}{2} \right)
\]

(32)

where $x \in \mathbb{R}$

**Proof.** Let $f(x) = \sum_{k=1}^{\infty} e^{i(2k-1)x}$, then we have

\[
f(x) = \sum_{k=1}^{\infty} e^{i(2k-1)x} = \frac{e^{ix}}{1 - e^{2ix}} = \frac{1}{2i} \frac{e^{ix}}{e^{-ix} - e^{ix}} = \frac{1}{2i} \frac{e^{2ix} + 1}{e^{ix} + 1} = \frac{1}{2} \csc(x)
\]

By integrating the $f(x)$, we have

\[
\frac{1}{i} e^{ix} + \frac{1}{3i} e^{3ix} + \frac{1}{5i} e^{5ix} + \cdots = \frac{1}{2} \ln \left( \tan \frac{x}{2} \right) + C
\]

\[
e^{ix} + \frac{1}{3} e^{3ix} + \frac{1}{5} e^{5ix} + \cdots = -\frac{1}{2} \ln \left( \tan \frac{x}{2} \right) + Ci
\]

where $C$ is constant of integration.

Taking the real part,

\[
\cos(x) + \frac{1}{3} \cos(3x) + \frac{1}{5} \cos(5x) + \cdots = -\frac{1}{2} \ln \left( \tan \frac{x}{2} \right)
\]

\[
\square
\]

**Lemma 8.** The Euler number $E_n$ is represented as

\[
\frac{d^{2n}}{dx^{2n}} \csc \left( \frac{\pi}{2} \right) = (-1)^n E_{2n}
\]

(33)

where $n \in \{\mathbb{N}, 0\}$
Proof. The expression for $\csc(x)$ can be expanded to a Taylor series at $x = \pi/2$ as follows:

$$
csc(x) = 1 + \frac{1}{2!} (x - \frac{\pi}{2})^2 + \frac{5}{4!} (x - \frac{\pi}{2})^4 + \frac{61}{6!} (x - \frac{\pi}{2})^6 + \cdots = \sum_{m=0}^{\infty} \frac{E_m}{m!} (x - \frac{\pi}{2})^m
$$

The definition of the $m$-th term of a Taylor series at $x = \pi/2$ is

$$
\left\{ \frac{d^m}{dx^m} f \left( \frac{\pi}{2} \right) \right\} \frac{1}{m!} (x - \frac{\pi}{2})^m
$$

If $m = 2n$, then $E_m = E_{2n}$ and if $m = 2n + 1$, then $E_{2n} = 0$.

Therefore,

$$
\frac{d^{2n}}{dx^{2n}} \csc \left( \frac{\pi}{2} \right) = (-1)^n E_{2n}
$$

Theorem 3. The function $J(n)$ where $n \in \mathbb{N}$ can be expressed as an infinite series,

$$
J(n) = \sum_{k=0}^{\infty} \frac{(-1)^k E_{2k}}{(n+2k+1)!} \left( \frac{\pi}{2} \right)^{n+2k}
$$

where $E_k$ is Euler number.

Proof. The function $J(n)$ where $n \in \mathbb{N}$ is defined as

$$
J(n) = \frac{1}{n! \pi} \int_0^{\pi/2} x^n \sin(x) dx
$$

Integrating by parts,

$$
J(n) = \frac{1}{n! \pi} \left[ x^{n+1} \{ \csc(x) \} - \frac{x^{n+2} \left\{ \frac{d}{dx} \csc(x) \right\}}{(n+1)(n+2)} + \frac{x^{n+3} \left\{ \frac{d^2}{dx^2} \csc(x) \right\}}{(n+1)(n+2)(n+3)} - \cdots \right]_{0}^{\pi/2}
$$

By using Lemma 8 we have

$$
J(n) = \frac{2}{\pi} \left[ \frac{x^{n+1}}{(n+1)!} \{ \csc(x) \} - \frac{x^{n+2}}{(n+2)!} \left\{ \frac{d}{dx} \csc(x) \right\} + \frac{x^{n+3}}{(n+3)!} \left\{ \frac{d^2}{dx^2} \csc(x) \right\} - \cdots \right]_{0}^{\pi/2}

\begin{align*}
J(n) &= \frac{2}{\pi} \left[ \frac{E_0}{(n+1)!} \left( \frac{\pi}{2} \right)^{n+1} - \frac{E_2}{(n+3)!} \left( \frac{\pi}{2} \right)^{n+3} + \frac{E_4}{(n+5)!} \left( \frac{\pi}{2} \right)^{n+5} - \cdots \right] \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k E_{2k}}{(n+2k+1)!} \left( \frac{\pi}{2} \right)^{n+2k}
\end{align*}
$$

The proof of Theorem 3 was completed.

Theorem 4. The function $J(2n-1)$ and $J(2n)$ where $n \in \mathbb{N}$ can be calculated directly in special forms as

$$
\frac{\pi}{4} J(2n-1) = (-1)^n \sum_{k=0}^{n-1} \left\{ (-1)^k \beta(2n-2k) \frac{1}{(2k)!} \left( \frac{\pi}{2} \right)^{2k} \right\}
$$

$$
\frac{\pi}{4} J(2n) = (-1)^n \left[ \lambda(2n+1) - \sum_{k=0}^{n-1} \left\{ (-1)^k \beta(2n-2k) \frac{1}{(2k+1)!} \left( \frac{\pi}{2} \right)^{2k+1} \right\} \right]
$$

where $E_k$ is Euler number.
Proof. The expression for \( \csc(x) \) can be expanded to a Taylor series at \( x = \pi/2 \) as follows:

\[
\csc(x) = 1 + \frac{1}{2!} \left( x - \frac{\pi}{2} \right)^2 + \frac{5}{4!} \left( x - \frac{\pi}{2} \right)^4 + \frac{61}{6!} \left( x - \frac{\pi}{2} \right)^6 + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k E_{2k}}{(2k)!} \left( x - \frac{\pi}{2} \right)^{2k}
\]

Integrating both sides of the formula with respect to \( x \), we have

\[
\ln \left( \tan \frac{x}{2} \right) = \left( x - \frac{\pi}{2} \right) + \frac{1}{3!} \left( x - \frac{\pi}{2} \right)^3 + \frac{5}{5!} \left( x - \frac{\pi}{2} \right)^5 + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k E_{2k}}{(2k+1)!} \left( x - \frac{\pi}{2} \right)^{2k+1}
\]

The constant of integration is determined by \( x = \pi/2 \). By using Lemma [7] we have

\[
\sum_{k=0}^{\infty} \frac{(-1)^k E_{2k}}{(2k+1)!} \left( x - \frac{\pi}{2} \right)^{2k+1} = 2 \sum_{k=0}^{\infty} \frac{-\cos((2k+1)x)}{(2k+1)}
\]

The multiple integral on both sides with respect to \( x \) is given by the functional equations. The constant of integration is determined by \( x = \pi/2 \).

\[
\sum_{k=0}^{\infty} \frac{(-1)^k E_{2k}}{(2k+2n)!} \left( x - \frac{\pi}{2} \right)^{2k+2n} = 2(-1)^n \sum_{k=0}^{\infty} \frac{\cos((2k+1)x)}{(2k+1)^{2n+1}} + 2(-1)^{n-1} \sum_{k=0}^{n-1} \frac{(-1)^k \beta(2n-2k)}{(2k)!} \left( x - \frac{\pi}{2} \right)^{2k}
\]

By substituting \( x = \pi \), we have

\[
\sum_{k=0}^{\infty} \frac{(-1)^k E_{2k}}{(2k+2n+1)!} \left( \frac{\pi}{2} \right)^{2k+2n+1} = 2(-1)^n \left[ \sum_{k=0}^{n-1} \frac{\lambda(2n+1) - \sum_{k=0}^{n-1} (-1)^k \beta(2n-2k)}{(2k+1)!} \left( \frac{\pi}{2} \right)^{2k+1} \right]
\]

Using the Theorem [3] we have

\[
\frac{\pi}{4} J(2n-1) = (-1)^n \sum_{k=0}^{n-1} \left[ (-1)^k \beta(2n-2k) \frac{\lambda(2n+1) - \sum_{k=0}^{n-1} (-1)^k \beta(2n-2k)}{(2k)!} \left( \frac{\pi}{2} \right)^{2k+1} \right]
\]

The proof of Theorem 4 was completed.

\[\square\]

Remark 1. Similarly to the Theorem [3] the expressions \( \frac{1}{(2m-1)!} \left( \frac{\pi}{2} \right)^{2m-1} \) and \( \frac{1}{(2m)!} \left( \frac{\pi}{2} \right)^{2m} \), where \( m \in \mathbb{N} \) can be calculated directly special forms as

\[
\frac{\pi}{4} \left\{ \frac{1}{(2m-1)!} \left( \frac{\pi}{2} \right)^{2m-1} \right\} = (-1)^{m-1} \sum_{k=0}^{m-1} \left[ (-1)^k \lambda(2m-2k) \frac{1}{(2k)!} \left( \frac{\pi}{2} \right)^{2k} \right]
\]

\[\text{(37)}\]

\[
\frac{\pi}{4} \left\{ \frac{1}{(2m)!} \left( \frac{\pi}{2} \right)^{2m} \right\} = (-1)^m \left[ \beta(2m+1) - \sum_{k=0}^{m-1} \left[ (-1)^k \lambda(2m-2k) \frac{1}{(2k+1)!} \left( \frac{\pi}{2} \right)^{2k+1} \right] \right]
\]

\[\text{(38)}\]
Proof. By substituting $x = \pi/2$ in Eq. (16) and Eq. (17), we have

$$\beta(2m + 1) = m \sum_{k=1}^{m} \left\{ \lambda(2m - 2k + 2) \frac{(-1)^{k-1} (\pi/2)^{2k-1}}{(2k-1)!} \right\} + (-1)^m \beta(1) \frac{(\pi/2)^{2m}}{(2m)!},$$

$$0 = \sum_{k=1}^{m} \left\{ \lambda(2m - 2k + 2) \frac{(-1)^{k-1} (\pi/2)^{2k-2}}{(2k-2)!} \right\} + (-1)^m \beta(1) \frac{(\pi/2)^{2m-1}}{(2m-1)!}.$$

Change the index of summation $k$ so that it would start from 0.

$$\beta(2m + 1) = \sum_{k=0}^{m-1} \left\{ \lambda(2m - 2k) \frac{(-1)^{k} (\pi/2)^{2k+1}}{(2k+1)!} \right\} + (-1)^m \beta(1) \frac{(\pi/2)^{2m}}{(2m)!},$$

$$0 = \sum_{k=0}^{m-1} \left\{ \lambda(2m - 2k) \frac{(-1)^{k-1} (\pi/2)^{2k}}{(2k)!} \right\} + (-1)^m \beta(1) \frac{(\pi/2)^{2m-1}}{(2m-1)!}.$$

Therefore,

$$\frac{\pi}{4} \left\{ \frac{1}{(2m-1)!} \left( \frac{\pi}{2} \right)^{2m-1} \right\} = (-1)^m \sum_{k=0}^{m-1} \left\{ (-1)^k \lambda(2m - 2k) \frac{1}{(2k)!} \left( \frac{\pi}{2} \right)^{2k} \right\}$$

$$\frac{\pi}{4} \left\{ \frac{1}{(2m)!} \left( \frac{\pi}{2} \right)^{2m} \right\} = (-1)^m \beta(2m + 1) - \sum_{k=0}^{m-1} \left\{ (-1)^k \lambda(2m - 2k) \frac{1}{(2k+1)!} \left( \frac{\pi}{2} \right)^{2k+1} \right\}.$$

□

References

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