A RING THEORETIC APPROACH TO THE FINITE REPRESENTATION TYPE

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Abstract. An Artin algebra $\Lambda$ is said to be of finite Cohen-Macaulay type, CM-finite for short, if the full subcategory $\text{Gprj-} \Lambda$ of finitely generated Gorenstein projective $\Lambda$-modules is of finite representation type. If $\Lambda$ is a CM-finite algebra, then we denote by $\text{Aus}(\text{Gprj-} \Lambda)$ the stable Cohen-Macaulay Auslander algebra, i.e. $\text{End}_\Lambda(G)$, where $G$ is a basic representation generator of $\text{Gprj-} \Lambda$. In this paper, we will explain how by defining an equivalence relation on the elements of algebra $\text{Aus}(\text{Gprj-} \Lambda)$ can be used to give a characterization for $\text{Aus}(\text{Gprj-} \Lambda)$ to be of finite representation type, or equivalently, the CM-finiteness of the algebra of $2 \times 2$ lower triangular matrices over $\Lambda$, where $\Lambda$ is a CM-finite Artin algebra over an algebraic closed field. Then, by presenting some examples we will show how our results work.

1. Introduction

One of the main task in the representation theory of Artin algebras is to determining the (finite or infinite) representation type of a given algebra, and more ideal to give a complete classification of the indecomposable modules. Until now, different methods has been invented for this kind of representation theoretic problems. By help of them, some classes of algebras, for instance the hereditary or self-injective algebras of finite representation type, were completely understood. Since the representation theoretic problems are dealing with modules category, so it is more reasonable to going through the modules to solve our problems. But, in this case, we often will be involved in a creature, exactly the category of finitely generated modules over the given Artin algebra, which is often hard to be controlled. For some certain algebras, we will explain how the ground set of those algebras can be used to decide the set of isomorphism classes of indecomposable modules over the algebras to be finite or infinite, or the algebra to be of finite representation type or infinite representation type. The class of Artin algebras over algebraic closed field, which our results work for, are the stable Cohen-Macaulay Auslander algebras. The assumption of algebraic closed we need for the validity of the second Brauer-Thrall conjecture for the given Artin algebra, see the first of Section 3 for something more about this conjecture. This kind of algebras are produced by the (finitely generated) Gorenstein projective modules. Gorenstein projective modules were first introduced by Auslander and Bridger [AB] over commutative Noetherian rings (in which case they are called G-dimension zero modules) as a generalization of finitely generated projective modules, then extension of this notion to any (not necessarily finitely generated) module over any (not necessarily commutative Noetherian) ring by Enochs and Jenda [EJ] led to the definition of Gorenstein projective modules.

Artin algebra $\Lambda$ is said to be CM-finite, if there are only finitely many isomorphism classes of finitely generated Gorenstein projective modules. The stable Cohen-Macaulay Auslander

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algebra of a CM-finite algebra is the quotient algebra of the endomorphism algebra of the direct sum of all pairwise non-isomorphic indecomposable Gorenstein projective modules modulo the ideal consisting of those endomorphisms factor through a projective module. In our main result, Theorem 3.7, we shall show for such algebras not to be necessary to looking for within the modules category for determining their representation types. Only by defining an equivalence relation on the set of their element, we are able to determine its representation type just by computing the equivalence classes of the relation.

Whenever, $\Lambda$ is a self-injective of finite representation type, as discussed in 2.3, we can see the studying of representation type of the stable Cohen-Macaulay Auslander algebras is related to studying of the submodule categories. Investigation of the submodule category goes back to Garrott Birkhoff in 1934, recently for instance in [RS1], [RS2] and [XZZ], has been studied more by some modern methods, like the Auslander-Reiten theory and covering theory. In the last section, by presenting some examples we will show how our abstract results are applicable. A surprising event in the examples occur is that we use only some elementary instruments for determining their representation type.

2. Preliminaries

2.1. Functor category. Let $\mathcal{A}$ be an additive category and $\mathcal{C}$ a subcategory of $\mathcal{A}$. We denote by $\text{Hom}_{\mathcal{A}}(X,Y)$ the set of morphisms from $X$ to $Y$. we denote by $\text{ind-}\mathcal{A}$ the set of isomorphisms classes of indecomposable objects in $\mathcal{A}$. An $\mathcal{A}$-module is a contravariant additive functor from $\mathcal{A}$ to the category of abelian groups. We call an $\mathcal{A}$-module $F$ finitely presented if there exists an exact sequence $\text{Hom}_{\mathcal{A}}(-,X) \xrightarrow{f} \text{Hom}_{\mathcal{A}}(-,Y) \rightarrow F \rightarrow 0$. We denote by $\text{mod-}\mathcal{A}$ the category of finitely presented $\mathcal{A}$-modules. We call $\mathcal{C}$ contravariantly (resp. covariantly) finite in $\mathcal{A}$ if $\text{Hom}_{\mathcal{A}}(-,X)|_{\mathcal{C}}$ (resp. $\text{Hom}_{\mathcal{A}^{op}}(-,X)|_{\mathcal{C}}$) is a finitely generated $\mathcal{C}$-module for any $X$ in $\mathcal{A}$. We call $\mathcal{C}$ functorially finite if it is contravariantly and covariantly finite. It is known that if $\mathcal{C}$ is a contravariantly finite subcategory of abelian category $\mathcal{A}$, then $\text{mod-}\mathcal{C}$ is an abelian category, see [AHK, §2] for more details.

Let $\mathcal{A}$ be an abelian category with enough projectives and $\mathcal{X}$ consists of all projective objects of $\mathcal{A}$. We consider the stable category of $\mathcal{X}$, denoted by $\mathcal{X}^{\mathbb{L}}$. The objects of $\mathcal{X}^{\mathbb{L}}$ are the same as the objects of $\mathcal{X}$, which we usually denote by $\underline{X}$, and the morphisms are given by $\text{Hom}_{\mathcal{X}^{\mathbb{L}}}(\underline{X},\underline{Y}) = \text{Hom}_{\mathcal{X}}(X,Y)/\mathcal{P}(X,Y)$, where $\mathcal{P}(X,Y)$ is the subgroup of $\text{Hom}_{\mathcal{X}}(X,Y)$ consisting of those morphisms from $X$ to $Y$ which factor through a projective object in $\mathcal{A}$. We also denote by $\underline{f}$ the residue class of $f : X \rightarrow Y$ in $\text{Hom}_{\mathcal{X}}(\underline{X},\underline{Y})$. In order to simplify, we will use $(-,X)$, resp. $(-,\underline{X})$, for the representable functor $\text{Hom}_{\mathcal{X}}(-,X)$, resp. $\text{Hom}_{\mathcal{X}^{\mathbb{L}}}(\underline{-},\underline{X})$, in $\text{mod-}\mathcal{X}$, resp. $\text{mod-}\mathcal{X}^{\mathbb{L}}$. Moreover, in the case that the subcategory $\mathcal{X}$ is contravariantly finite in $\mathcal{A}$, then the category of finitely presented $\mathcal{X}^{\mathbb{L}}$-modules, $\text{mod-}\mathcal{X}^{\mathbb{L}}$, by the equivalence proved in [AHK, Proposition 4.1], can be identified with those of functors in $\text{mod-}\mathcal{X}$ such that vanish on all projective objects in $\mathcal{A}$. We use this identification completely free for some certain subcategories which we will be dealing with throughout the paper later.

We assume throughout this paper that $\Lambda$ is an Artin algebra over field $k$. A subcategory $\mathcal{X}$ of $\text{mod-}\Lambda$, the category of finitely generated right $\Lambda$-modules, is always a full subcategory of $\text{mod-}\Lambda$ closed under isomorphisms, finite direct sums and direct summands. The subcategory $\mathcal{X}$ is called of finite representation type if $\text{Ind-}\mathcal{X}$ is a finite set. An Artin algebra $\Lambda$ is called of finite representation type, or simply representation-finite, if $\text{mod-}\Lambda$ is of finite representation type. If $\mathcal{X}$ is of finite representation type, then it admits a representation generator, i.e., there exists $X \in \mathcal{X}$ such that $\mathcal{X} = \text{add-}X$, the subcategory of $\text{mod-}\Lambda$ consisting of all direct summands of
all finite direct sums of copies of $X$. It is known that $\text{add}-X$ is a functorially finite subcategory of $\text{mod}-\Lambda$. To avoid complicated notations, for $\text{prj}-\Lambda \subset \mathcal{X} \subset \text{mod}-\Lambda$, we show $\text{Hom}_{\mathcal{X}}(X,Y)$, resp. $\text{Hom}_{\mathcal{X}}(X,\Sigma Y)$, by $\text{Hom}_\Lambda(X,Y)$ resp. $\text{Hom}_\Lambda(X,\Sigma Y)$. Set $\text{Aus}(\mathcal{X}, X) = \text{End}_\Lambda(X)$, where $\mathcal{X}$ is a subcategory with representation generator $X$. Clearly $\text{Aus}(\mathcal{X}, X)$ is an Artin algebra. It is known that the evaluation functor $\zeta_X : \text{mod}-\mathcal{X} \to \text{mod}-\text{Aus}(\mathcal{X}, X)$ defined by $\zeta_X(F) = F(X)$, for $F \in \text{mod}-\mathcal{X}$, is an equivalence of categories. It also induces an equivalence of categories $\text{mod}-\mathcal{X} \simeq \text{mod}-\text{Aus}(\mathcal{X}, \underline{X})$, only by the restriction. Recall that $\text{Aus}(\mathcal{X}, X) = \text{End}_\Lambda(X)/\mathcal{P}$, where $\mathcal{P} = \mathcal{P}(X, X)$. The Artin algebra $\text{Aus}(\mathcal{X}, X)$, resp. $\text{Aus}(\underline{X}, \underline{X})$, is called the relative, resp. stable, Auslander algebra of $\Lambda$ with respect to the subcategory $\mathcal{X}$ and with the representation generator $X$. For the case $\mathcal{X} = \text{mod}-\Lambda$, we clearly delete "relative", and also when $X$ is basic we delete "$X$" in the notation and just call the relative (stable) Auslander of $\Lambda$. In fact, if $X'$ is another representation generator of $\mathcal{X}$, then $\text{Aus}(\mathcal{X}, X)$, resp. $\text{Aus}(\underline{X}, \underline{X})$, and $\text{Aus}(\mathcal{X}, X')$, resp. $\text{Aus}(\underline{X}, \underline{X}')$, are Morita equivalent. But if both are basic, in this case we have stronger situation, that is, $\text{Aus}(\mathcal{X}, X) \simeq \text{Aus}(\mathcal{X}, X')$, resp. $\text{Aus}(\underline{X}, \underline{X}) \simeq \text{Aus}(\underline{X}, \underline{X})$, as isomorphism of algebras.

The fact, which we need later, from [A, Chapter 2], is any simple functor $\mathcal{S}$ in $\text{mod}-\underline{\mathcal{X}}$ is isomorphic to $(-, \underline{X})/r(-, \underline{X})$, where $X$ is a non-projective indecomposable module in $\mathcal{X}$ and $r(-, \underline{X})$ is the radical functor of $(-, \underline{X})$. The notions of the radical of a functor and also simple functors are defined in analogy with modules over rings. Further, if $\mathcal{X}$ is functorially finite and closed under extensions, then $S$ has the following minimal projective resolution in $\text{mod}-\mathcal{X}$

$$0 \to (-, Z) \xrightarrow{(\zeta_X f)} (-, Y) \xrightarrow{(\zeta_X g)} (-, X) \to S \to 0,$$

where $0 \to Z \xrightarrow{f} Y \xrightarrow{g} X \to 0$ is an almost split sequence in $\mathcal{X}$, see [AS] for details and definition of almost split sequences for subcategories.

2.2. Gorenstein projective modules. A complex

$$P^{\bullet} : \cdots \to P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \to \cdots$$

of finitely generated projective $\Lambda$-modules is said to be totally acyclic provided it is acyclic and the Hom complex $\text{Hom}_\Lambda(P^{\bullet}, \Lambda)$ is also acyclic. A $\Lambda$-module $M$ is said to be (finitely generated) Gorenstein projective provided that there is a totally acyclic complex $P^{\bullet}$ of finitely generated projective $\Lambda$-modules such that $M \cong \text{Ker}(d^0)$ [E.J]. We denote by $\text{Gprj}-\Lambda$ the full subcategory of $\text{mod}-\Lambda$ consisting of all Gorenstein projective modules.

An Artin algebra $\Lambda$ is of finite Cohen-Macaulay type, or simply, CM-finite, if there are only finitely many isomorphism classes of indecomposable finitely generated Gorenstein projective $\Lambda$-modules. Clearly, $\Lambda$ is a CM-finite algebra if and only if there is a finitely generated module $E$ such that $\text{Gprj}-\Lambda = \text{add}-E$. In this case, $E$ is called to be a Gorenstein projective representation generator of $\text{Gprj}-\Lambda$. If $\text{gldim} \Lambda < \infty$, then $\text{Gprj}-\Lambda = \text{prj}-\Lambda$, so $\Lambda$ is CM-finite. If $\Lambda$ is self-injective, then $\text{Gprj}-\Lambda = \text{mod}-\Lambda$, so $\Lambda$ is CM-finite if and only if $\Lambda$ is representation-finite. If $E$ is a basic Gorenstein projective representation generator of $\text{Gprj}-\Lambda$, then the relative (stable) Auslander algebra $\text{Aus}(\text{Gprj-}\Lambda) = \text{End}_\Lambda(E)$, resp. $\text{Aus}(\underline{\text{Gprj-}}\Lambda) = \underline{\text{End}}_\Lambda(E)$, is called the (stable) Cohen-Macaulay Auslander algebra of $\Lambda$.

2.3. An equivalence. In this subsection, we bring some facts concerning an equivalence proved in [H, Theorem 3.3]. Let $H(\text{mod-}\Lambda)$ be the morphism category over $\text{mod-}\Lambda$. Indeed, the objects in $H(\text{mod-}\Lambda)$ are the maps in $\text{mod-}\Lambda$, and morphisms are given by commutative diagrams. We can consider the objects in $H(\text{mod-}\Lambda)$ as the representations over the quiver $\mathbb{A}_2 : v \to w$ by $\Lambda$-modules and $\Lambda$-morphisms in $\text{mod-}\Lambda$, usually denoted by $\text{rep}(\mathbb{A}_2, \Lambda)$. But, we know by a general
fact the category \( \text{rep}(A, \Lambda) \) is equivalent to the category of finitely generated module over the path algebra \( \Lambda A_2 \cong T_2(\Lambda) \), where \( T_2(\Lambda) \) is the algebra of \( 2 \times 2 \) lower triangular matrices with entries of \( \Lambda \).

For a given subcategory \( \mathcal{X} \) of \( \text{mod-}\Lambda \), we assign the subcategory \( S_\mathcal{X}(\Lambda) \) of \( H(\text{mod-}\Lambda) \) consisting of morphisms \( A \xrightarrow{f} B \) satisfying:

1. \( f \) is a monomorphism;
2. \( A, B \) and \( \text{Coker}(f) \) belong to \( \mathcal{X} \).

Whenever \( \mathcal{X} \) is the entire of \( \text{mod-}\Lambda \), we use \( S(\Lambda) \) for \( S_{\text{mod-}\Lambda}(\Lambda) \), and called the submodule category as in [RS2].

As introduced in [H, Construction 3.2], the functor \( \Psi : S_\mathcal{X}(\Lambda) \to \text{mod-}\mathcal{X} \) is defined by sending an object \( A \xrightarrow{f} B \) to the functor \( F \) in \( \text{mod-}\mathcal{X} \) induced by the short exact sequence \( 0 \to A \xrightarrow{f} B \to \text{Coker}(f) \to 0 \) in \( \text{mod-}\Lambda \), that \( F \) can be fitted in the following exact sequence

\[
0 \to (-, A) \xrightarrow{\cdot f} (-, B) \to (-, \text{Coker}(f)) \to F \to 0
\]

in \( \text{mod-}\mathcal{X} \).

For a nice subcategory \( \mathcal{X} \), the functor \( \Psi \) induces an equivalences of categories. More precisely:

**Theorem 2.1.** Let \( \mathcal{X} \) be a subcategory of \( \text{mod-}\Lambda \) including \( \text{proj-}\Lambda \), contravariantly finite and closed under kernels of epimorphisms. Consider the full subcategory \( \mathcal{V} \) of \( S_\mathcal{X}(\Lambda) \) formed by finite direct sums of objects in the form of \( (X \xrightarrow{1} X) \) or \( (0 \to X) \), that \( X \) runs through all of objects in \( \mathcal{X} \). Then the functor \( \Psi \), defined in the above, induces the following equivalence of categories

\[
S_\mathcal{X}(\Lambda)/\mathcal{V} \cong \text{mod-}\mathcal{X}.
\]

In fact, the above theorem is a relative version of the equivalences in [RZ] and [E].

Denote by \( \mathcal{C} \) the full additive subcategory of \( S_\mathcal{X}(\Lambda) \) consisting of all indecomposable objects in \( S_\mathcal{X}(\Lambda) \) not isomorphic to an object of the form either \( X \xrightarrow{1} X \) or \( 0 \to X \) with \( X \) indecomposable module in \( \mathcal{X} \). Assume that \( \mathcal{X} \) is of finite representation type. Then based on the above theorem in Theorem 2.1, one can see there is a bijection between the indecomposable modules in \( \mathcal{C} \) and the indecomposable modules in \( \text{mod-}\text{Aus}(\mathcal{X}) \). Hence, \( S_\mathcal{X}(\Lambda) \) is a subcategory of finite representation type of \( H(\text{mod-}\Lambda) \) if and only if the algebra \( \text{Aus}(\mathcal{X}) \) is representation-finite. This observation plays an important role in our main result.

Since the categories \( \text{rep}(A_2, \Lambda) \) and \( H(\text{mod-}\Lambda) \) are equivalent to \( \text{mod-}T_2(\Lambda) \), then by these equivalences we can naturally define the notion of Gorenstein projective representation (mor-}

phism) in \( \text{rep}(A_2, \Lambda) \) \( (H(\text{mod-}\Lambda)) \), coming from the concept of Gorenstein projective modules. There is the following local characterization of Gorenstein projective representations in \( \text{rep}(A_2, \Lambda) \):

**Lemma 2.2.** ([EHS, Theorem 3.5.1] or [LZ, Theorem 5.1]) Let \( X \xrightarrow{f} Y \) be a representation in \( \text{rep}(A_2, \Lambda) \). Then \( X \xrightarrow{f} Y \) is a Gorenstein projective representation if and only if (1) \( X, Y \) and \( \text{coker}(f) \) are in \( \text{Gprj-}\Lambda \), and (2) \( f \) is a monomorphism.

By using the above Lemma, we observe \( S_{\text{Gprj-}\Lambda}(\Lambda) \cong \text{Gprj-}T_2(\Lambda) \). Thus by the observation given in the above, we can say for CM-finite algebra \( \Lambda \): \( T_2(\Lambda) \) is CM-finite if and only if the stable Cohen-Macaulay Auslander algebra, \( \text{Aus}(\text{Gprj-}\Lambda) \), is representation-finite. In particular, if assume \( \Lambda \) is a self-injective of finite type, then \( T_2(\Lambda) \) is CM-finite if and only if the stable Auslander algebra, \( \text{Aus}(\text{mod-}\Lambda) \), is representation-finite.
3. The results

The second Brauer-Thrall conjecture, that is, an Artin algebra Λ is infinite representation type if and only if there are infinitely many positive integers \( n_1, n_2, \cdots \) such that for each \( i \in \mathbb{N} \) there are infinitely many non-isomorphic indecomposable modules of Jordan-Hölder length \( n_i \), has been proved for Artin algebras over an algebraically closed field. The original proof can be found in [NR], a sketch of the proof in English is given in [R]. The problem is still open for Artin algebras over arbitrary fields or commutative artinian rings.

From now on, we assume that \( \Lambda \) is a finite dimension algebra over an algebraic closed field \( k \).

Let \( \epsilon, \epsilon' \) be two short exact sequences

\[
\epsilon : \quad 0 \to N \to L_1 \to M \to 0
\]

and

\[
\epsilon' : \quad 0 \to N \to L_2 \to M \to 0.
\]

The short exact sequence \( \epsilon \) is EE-related to \( \epsilon' \), denoted by \( \epsilon \sim_{EE} \epsilon' \), if and only if there is the following commutative diagram

\[
\begin{array}{cccccc}
0 & \to & N & \to & L_1 & \to & M & \to & 0 \\
\downarrow{\sigma_1} & & \downarrow{\sigma_2} & & \downarrow{\sigma_3} & & \\
0 & \to & N & \to & L_2 & \to & M & \to & 0,
\end{array}
\]

with \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) isomorphisms. This relation on the class of the short exact sequences with beginning term \( N \) and ending term \( M \) makes an equivalence relation. Let \( \text{EExt}^1_\Lambda(M,N) \) denote the set of the equivalence classes of the short exact sequences

\[
0 \to N \to ? \to M \to 0
\]

respect to the equivalence relation \( \sim_{EE} \). In fact, the relation \( \sim_{EE} \) between the short exact sequences is nothing but the notion of isomorphisms of objects in the category of complexes, whenever we consider the short exact sequences as objects in it. Moreover, the so-called Baer Sum of the abelian group structure \( \text{Ext}^1_\Lambda(M,N) \) since acts compatibly with the relation \( \sim_{EE} \), thus \( \text{EExt}^1_\Lambda(M,N) \) also carries an abelian group structure. Note that definition of the Baer sum relies on pull-backs and push-outs, which both are categorical notions. Also an element \([\epsilon]_{EE}\) in the abelian group \( \text{EExt}^1_\Lambda(M,N) \) is zero, if \( \epsilon \) is EE-related to a split sequence with ending terms by \( N \) and \( M \). Recall that two short exact sequences \( \epsilon \) and \( \epsilon' \) are in the same equivalence class in the extension group \( \text{Ext}^1_\Lambda(M,N) \), say \( \epsilon \sim_E \epsilon' \), if there is the following commutative diagram

\[
\begin{array}{cccccc}
0 & \to & N & \to & L_1 & \to & M & \to & 0 \\
\downarrow{\sigma_2} & & & & & & \\
0 & \to & N & \to & L_2 & \to & M & \to & 0,
\end{array}
\]

Therefore, if two short exact sequences are in the same equivalence class respect to \( \sim_E \), then trivially also lie in the same equivalence relation respect to \( \sim_{EE} \).

In the sequel, we define another equivalence relation on the class of the short exact sequences. Let \( \epsilon : 0 \to N \oplus M \to H \to X \to 0 \), \( \epsilon' : 0 \to N \oplus M \to H' \to X \to 0 \) be two short exact sequences. The short exact sequence \( \epsilon \) is EE-related to \( \epsilon' \), denoted by \( \epsilon \sim_{EE} \epsilon' \), if and only if
there is a commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & N \oplus M \\
\sigma_1 & & \sigma_2 \\
0 & \longrightarrow & N \oplus M \\
\sigma_3 & & \sigma_3 \\
& H & \longrightarrow X & 0 \\
& & \downarrow H' & \\
& H' & \longrightarrow X & 0,
\end{array}
\]

with \(\sigma_1, \sigma_2, \sigma_3\) isomorphisms, and further \(\sigma_1(N) = N, \sigma_1(M) = M\). Similarly, the relation \(\sim_{\text{EE}}\) is an equivalence, and denoted by \(\text{Ext}_A^1(X, N \oplus M)\) the set of equivalence classes of such short exact sequences respect to the relation \(\sim_{\text{EE}}\). Again, we can enhance \(\text{Ext}_A^1(X, N \oplus M)\) by an abelian group structure, by using the Baer Sum. The new relation is stronger than \(\sim_{\text{EE}}\), but weaker than \(\sim_E\), i.e., \(E \sim E' \Rightarrow \sim_{\text{EE}} E' \Rightarrow \sim_E E'\). The connection among these three relations give us the epimorphisms of abelian groups

\[
\begin{align*}
\pi_0 : \text{Ext}_A^1(X, N \oplus M) & \longrightarrow \text{Ext}_A^1(X, N \oplus M), \quad [\epsilon]_E \mapsto [\epsilon]_{\text{EE}}, \\
\pi_1 : \text{Ext}_A^1(X, N \oplus M) & \longrightarrow \text{Ext}_A^1(X, N \oplus M), \quad [\epsilon]_{\text{EE}} \mapsto [\epsilon]_{\text{EE}}.
\end{align*}
\]

Dually, we have a similar relation \(\sim_{\text{EE}}\) for the class of the short exact sequences in the form \(0 \rightarrow X \rightarrow? \rightarrow M \oplus N \rightarrow 0\), and denoted by \(\text{Ext}_A^1(M \oplus N, X)\) the corresponding set of equivalence classes.

From [SY, Section 3 of Chapter III], we have an isomorphism of abelian groups \(h : \text{Ext}_A^1(X, N \oplus M) \rightarrow \text{Ext}_A^1(X, N) \times \text{Ext}_A^1(M, X)\), by sending \(\epsilon_E \mapsto ([\epsilon_1]_E, [\epsilon_2]_E)\), where \(\epsilon_1, \epsilon_2\), is the push-out \(\epsilon\) along the natural epimorphism \(N \oplus M \rightarrow N\), resp. \(N \oplus M \rightarrow M\). In a way analogous to \(h\), we can define map \(h' : \text{Ext}_A^1(X, N \oplus M) \rightarrow \text{Ext}_A^1(X, N) \times \text{Ext}_A^1(X, M)\), only by sending \(\epsilon_E \mapsto ([\epsilon_1]_{\text{EE}}, [\epsilon_2]_{\text{EE}})\). Here, note that the way of defining the relation \(\sim_{\text{EE}}\) helps us to prove easily \(h'\) is well-defined. The map \(h'\) satisfies the following commutative diagram

\[
\begin{array}{ccc}
\text{Ext}_A^1(X, N \oplus M) & \longrightarrow & \text{Ext}_A^1(X, N) \times \text{Ext}_A^1(M, X) \\
\downarrow \pi_0 & & \downarrow \delta \\
\text{Ext}_A^1(X, N \oplus M) & \longrightarrow & \text{Ext}_A^1(X, N) \times \text{Ext}_A^1(X, M),
\end{array}
\]

where \(\delta, \delta'\) are defined naturally as \(\pi_0\) or \(\pi_1\). Now since maps in the top row and columns are epimorphisms of abelian groups, then we see \(h'\) is also a homomorphism of abelian groups, even more an epimorphism, and using this fact directly to prove \(h'\) so is an isomorphism. In fact, let \([\epsilon]_{\text{EE}} \in \text{Ext}_A^1(X, N \oplus M)\) be in the kernel of \(h'\). Then \(h'([\epsilon]_{\text{EE}}) = ([\epsilon_1]_{\text{EE}}, [\epsilon_2]_{\text{EE}}) = 0\), so \(\epsilon_1\) and \(\epsilon_2\) both are isomorphic to split exact sequences. This implies that \([\epsilon]_{\text{EE}}\) is a split sequence since \(h\) is an isomorphism, and consequently \([\epsilon]_{\text{EE}}\). Hence \(h'\) is a monomorphism, and so an isomorphism.

**Lemma 3.1.** If \(\text{Ext}_A^1(X, X)\) is a finite set, then for each natural number \(n\), the set \(\text{Ext}_A^1(X^n, X^n)\) so is finite.

**Proof.** First, note that since \(\sim_{\text{EE}}\) and \(\sim_{\text{EE}}\) are stronger than \(\sim_{\text{EE}}\), then for every \(M, N\) and \(X\) in mod-\(\Lambda\), we have if \(\text{Ext}_A^1(X, M \oplus N)\), resp. \(\text{Ext}_A^1(M \oplus N, X)\) is a finite set, then \(\text{Ext}_A^1(M \oplus N, X)\), resp. \(\text{Ext}_A^1(M \oplus N, X)\), so is finite. By using isomorphism \(h\) and also its dual, we can deduce for every \(X, N\) and \(M\) in mod-\(\Lambda\), \(\text{Ext}_A^1(X, N \oplus M)\), resp. \(\text{Ext}_A^1(N \oplus M, X)\), is
a finite set if and only if $\Ext^1_{\Lambda}(X,N)$ and $\Ext^1_{\Lambda}(X,M)$, resp. $\Ext^1_{\Lambda}(N,X)$ and $\Ext^1_{\Lambda}(M,X)$ so are finite. Now by invoking several times these facts we obtain the desired result.

In the following result a new characterization of the relative Auslander algebras to be representation-finite is given in terms of $\Ext$ groups. Although this characterization seems to be also interesting in its own right. However, we do not have plan to study this abelian groups more in this paper. We only use this notion here as a tool in the proof of Theorem 3.7 to give a ring theoretic criteria for representation-finiteness which is the main purpose of the present paper.

**Proposition 3.2.** Let $\mathcal{X}$ be a subcategory of $\mod-\Lambda$ containing $\prj-\Lambda$ and of finite representation type. If $\mathcal{X}$ is closed under extensions and kernels of epimorphisms in $\mathcal{X}$, then the following assertions are equivalent.

1. The subcategory $\mathcal{S}_{\mathcal{X}}(\mod-\Lambda)$ of $\mod-T_2(\Lambda)$ is of finite representation type;
2. The relative stable Auslander algebra, $\text{Aus}(\mathcal{X})$, is representation-finite;
3. If $X$ is a representation generator of $\mathcal{X}$, then $\Ext^1_{\Lambda}(X,X)$ is a finite abelian group.

**Proof.** The equivalence (1) and (2) follows from Theorem 2.1. Assume (3) holds. We claim for each natural number $n > 0$, the number of pairwise non-isomorphism indecomposable modules $F$ of $\mod-\mathcal{X}$ with the composition length $l(F) = n$ is finite. Of course, for the case $n = 1$ is clear since we are dealing with the set of all pairwise non-isomorphic simple modules which is always finite over Artin algebras. So assume $n > 1$. Let $F$ be an indecomposable module in $\mod-\mathcal{X}$ with $l(F) = n$. Then there is a sequence of submodules

$$0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{n-1} \subseteq F_n = F$$

such that for each $i \in \{1, \cdots, n\}$ the module $F_i/F_{i-1}$ is simple. For each $i \in \{1, \cdots, n\}$, we have the short exact sequence

$$(i) \quad 0 \to F_{i-1} \to F_i \to F_i/F_{i-1} \to 0.$$  

For $i = 2$, we have $0 \to F_1 \to F_2 \to F_2/F_1 \to 0$ with simple ending terms. We observe from 2.1, $F_2/F_1 \simeq (-, X_2)/r(-, X_2)$ and $F_1 \simeq (-, X_1)/r(-, X_1)$ for some non-projective indecomposable modules $X_1$ and $X_2$ in $\mathcal{X}$, and also the minimal projective resolutions

$$0 \to (-, Z_1) \to (-, Y_1) \to (-, X_1) \to F_1 \simeq (-, X_1)/r(-, X_1) \to 0,$$

and

$$0 \to (-, Z_2) \to (-, Y_2) \to (-, X_2) \to F_2 \simeq (-, X_2)/r(-, X_2) \to 0.$$  

Now by using the horseshoe lemma for the short exact sequence (2), we find the following projective resolution

$$0 \to (-, Z_1 \oplus Z_2) \to (-, Y_1 \oplus Y_2) \to (-, X_1 \oplus X_2) \to F_2 \to 0$$

(not minimal in general) of $F_2$ in $\mod-\mathcal{X}$. If we continue this process for each short exact sequence (i) associated to $i \in \{2, \cdots, n\}$ by using the projective resolution of $F_{i-1}$, obtained by the previous steps, and the minimal projective resolution of simple module $F_i/F_{i-1}$, as we did for the case $i = 2$, then we get the following projective resolution

$$0 \to (-, \oplus_{i=1}^n Z_i) \to (-, \oplus_{i=1}^n Y_i) \to (-, \oplus_{i=1}^n X_i) \to F \to 0$$

in $\mod-\mathcal{X}$. Since $X$ is a representation generator of $\mathcal{X}$ then the $X_i$ are direct summands of $X$. By viewing the short exact sequence $\eta: 0 \to \oplus_{i=1}^n Z_i \to \oplus_{i=1}^n Y_i \to \oplus_{i=1}^n X_i \to 0$, obtained by the above exact sequence, and considering $\eta$ as an element in $\Ext^1(\mathcal{X}^n, \mathcal{X}^n)$, by adding the split exact sequences if needed, as we can correspond $F$ by the element $\eta$ of $\Ext^1(\mathcal{X}^n, \mathcal{X}^n)$. But
by (3), $\text{EExt}^1(X, X)$ is a finite set, so $\text{EExt}^1_\Lambda(X_n, X^n)$ a finite set by Lemma 3.1, now by using the correspondence we get our claim. If $\text{Aus}(X)$ is infinite representation type, then since the second Brauer-Thrall conjecture holds for $\text{Aus}(X)$, we can consider $\text{Aus}(X)$ as an Artin algebra over an algebraically closed field due to our assumption on $\Lambda$, it leads to a contradiction to our claim. So we proved (3) implies the equivalent conditions (2) and (1). It remains to show the converse. Assume (2) holds. Let $\epsilon$ be a short exact sequence with ending terms by $X$, say, $\epsilon : 0 \to X \to V \to X \to 0$. Viewing $\epsilon$ as an object in the category of complexes over $\Lambda$-modules, then it can be decomposed uniquely as $\epsilon \simeq \epsilon' \oplus \epsilon_1 \oplus \epsilon_2$, where $\epsilon' : 0 \to Y \xrightarrow{f} \Lambda \xrightarrow{g} Z \to 0$ with $f$ and $g$ radical morphisms, $\epsilon_1 : 0 \to 0 \to U \xrightarrow{1_U} U \to 0$ and $\epsilon_2 : 0 \to W \xrightarrow{1_W} W \to 0 \to 0$ for some $U, W$ in $X$. Indeed, $\epsilon'$ has no summand in the form $\epsilon_1$ and $\epsilon_2$. The short exact sequence $\epsilon'$, by applying the Yoneda functor, induces the following exact sequence

$$(\dagger) \quad 0 \to (-, Y) \to (-, V) \to (-, Z) \to F \to 0,$$

in $\text{mod}-X$. Since $f$ and $g$ are radical, then one can see the sequence $(\dagger)$ is a minimal projective resolution of $F$ in $\text{mod}-X$. Let $\{G_1, \cdots G_n\}$ be the set of all pairwise indecomposable modules in $\text{mod}-X$, that is finite by (2). By considering $F$ as an object in $\text{mod}-X$, we have the unique decomposition $F \simeq G^{d_1}_1 \oplus \cdots \oplus G^{d_n}_n$ into the indecomposable modules in $\text{mod}-X$. Take the following minimal projective resolution of each $G_i$

$$(\dagger\dagger) \quad 0 \to (-, A_i) \to (-, B_i) \to (-, C_i) \to G_i \to 0,$$

in $\text{mod}-X$. We know minimal projective resolutions are closed under getting finite sums, this fact gives us the following exact sequence

$$(\dagger\dagger\dagger) \quad 0 \to (-, A_{i_1}^{d_{i_1}} \oplus \cdots \oplus A_{i_n}^{d_{i_n}}) \to (-, B_{i_1}^{d_{i_1}} \oplus \cdots \oplus B_{i_n}^{d_{i_n}}) \to (-, C_{i_1}^{d_{i_1}} \oplus \cdots \oplus C_{i_n}^{d_{i_n}}) \to F \to 0,$$

which is again a minimal projective resolution of $F$. By comparing $(\dagger)$ and $(\dagger\dagger\dagger)$ as two minimal projective resolutions of $F$, we get $Z \simeq C^{d_1}_1 \oplus \cdots \oplus C^{d_n}_n$ and further $\epsilon' \sim_{EE} \eta^{d_1}_1 \oplus \cdots \oplus \eta^{d_n}_n$, here the short exact sequences $\eta_i : 0 \to A_i \to B_i \to C_i \to 0$ induced by the sequences $(\dagger\dagger)$, and $\eta^{d_i}_i$ means getting (component wise) sums of $\eta$ with itself $d_i$ times. Since $d_1 l(C_1) + \cdots + d_n l(C_n) = l(Z) \leq l(X)$ with $d_i \geq 0$, we have $l(C_i) \geq 0$, then there exist finitely many numbers $d_i$ satisfying the inequality. Consequently, $\epsilon'$ is $EE$-related to finitely many short exact sequences, up to isomorphism by considering them as objects in the category of complexes, with ending terms of direct summands of $X$. On the other hand, since by our assumption $X$ is of finite representation type, then there is a finite number of short exact sequences, up to isomorphisms of complexes, in the form $\epsilon_1$ and $\epsilon_2$. Finally, $\epsilon$ lies in one of the equivalence classes of finitely many short sequences with ending terms $X$, which are obtained as a direct sum of split exact sequences in the form $\epsilon_1, \epsilon_2$ and the short exact sequence induced by the minimal projective resolution of $G^{d_1}_1 \oplus \cdots \oplus G^{d_n}_n$, where there is a finite number of choices for $d_1, \cdots, d_n$, as discussed in the above. The proof is completed.

**Corollary 3.3.** Let $\Lambda$ be a representation-finite algebra. Then the following assertions are equivalent.

(1) The submodule category $S(\Lambda)$ is of finite representation type;

(2) The stable Auslander algebra of $\Lambda$, $\text{Aus}(\text{mod}-\Lambda)$, is representation-finite;

(3) If $M$ is a representation generator of $\Lambda$, then $\text{EExt}^1(\Lambda, M)$ is a finite abelian group.

**Proof.** It follows from Proposition 3.2 only by taking $X = \text{mod}-\Lambda$.

**Corollary 3.4.** Let $\Lambda$ be a CM-finite algebra. Then the following assertions are equivalent.
A RING THEORETIC APPROACH TO THE FINITE REPRESENTATION TYPE

1. $T_2(\Lambda)$ is CM-finite
2. The stable Cohen-Macaulay Auslander algebra, $\text{Aus}(\text{Gprj-}\Lambda)$, is representation-finite;
3. If $G$ is a representation generator of $\text{Gprj-}\Lambda$, then $\text{EExt}_\Lambda^1(G, G)$ is a finite abelian group.

Proof. It follows from Proposition 3.2 only by taking $\mathcal{X} = \text{Gprj-}\Lambda$, and of course using the local characterization given in Lemma 2.2. □

Let $\mathcal{C}$ be an additive category. Take $X$ and $Y$ of $\mathcal{C}$. We define a relation $\sim_H$ on $\text{Hom}_\mathcal{C}(X, Y)$ by setting $f \sim_H g$ for $f, g \in \text{Hom}_\mathcal{C}(X, Y)$ if there exists a commutative diagram

$$
\begin{array}{c}
X \\
v \\
\downarrow \downarrow \\
X \\
g \\
\downarrow \downarrow \\
Y \\
\end{array}
\begin{array}{c}
\begin{array}{c}
\rightarrow \\
f \\
\rightarrow \\
U \\
\rightarrow \\
G \\
\rightarrow \\
0 \\
\end{array}
\begin{array}{c}
\rightarrow \\
\downarrow \\
\rightarrow \\
\downarrow \\
\rightarrow \\
\downarrow \\
\rightarrow \\
0 \\
\end{array}
\end{array}
$$

with $u, v$ automorphisms in $\mathcal{C}$.

Denote by $\text{HHom}_\mathcal{C}(X, Y)$, resp. $\text{EEnd}_\mathcal{C}(X)$, the set of all equivalence classes of $\text{Hom}_\mathcal{C}(X, Y)$, resp. $\text{End}_\mathcal{C}(X)$, under the equivalence relation $\sim_H$. For the case $\mathcal{C} \subseteq \text{mod-}\Lambda$, we only write $\text{HHom}_\Lambda(G, G')$, resp. $\text{EEnd}_\Lambda(G)$, instead of $\text{HHom}_\mathcal{C}(G, G')$, resp. $\text{EEnd}_\mathcal{C}(G)$, for every $G, G'$ in $\mathcal{C}$.

In the following construction we provide naturally a map from $\text{HHom}_\Lambda(G, G')$ to $\text{EExt}_\Lambda^1(G, \Omega(G'))$ and vice versa, for Gorenstein projective modules $G$ and $G'$.

Construction 3.5. Suppose that $G$ and $G'$ are Gorenstein projective modules. Let a class $[f]_H$ in $\text{HHom}_\Lambda(G, G')$ be given. Define by $\theta([f]_H)$ the equivalence class in $\text{EExt}_\Lambda^1(G, \Omega(G'))$ of the short exact sequence in the top row of the following diagram

$$
\begin{array}{c}
0 \\
\rightarrow \\
\Omega(G') \\
\rightarrow \\
U \\
\rightarrow \\
G \\
\rightarrow \\
0 \\
\end{array}
\begin{array}{c}
\rightarrow \\
\downarrow \\
\rightarrow \\
\downarrow \\
\rightarrow \\
\downarrow \\
\rightarrow \\
0 \\
\end{array}
\begin{array}{c}
\rightarrow \\
\downarrow \\
\rightarrow \\
\downarrow \\
\rightarrow \\
\downarrow \\
\rightarrow \\
0 \\
\end{array}
\begin{array}{c}
0 \\
\rightarrow \\
\Omega(G') \\
\rightarrow \\
P \\
\rightarrow \\
G' \\
\rightarrow \\
0 \\
\end{array}
$$

obtained by taking the pull-back along $f$ of the short exact sequence in the bottom row. Further, for a class $[\epsilon : 0 \rightarrow \Omega(G') \rightarrow V \rightarrow G \rightarrow 0]_{EE}$ in $\text{EExt}_\Lambda^1(G, \Omega(G'))$, since $P$ is an injective object in the exact category $\text{Gprj-}\Lambda$ then there is $l : V \rightarrow P$ so that commutes the following diagram

$$
\begin{array}{c}
0 \\
\rightarrow \\
\Omega(G') \\
\rightarrow \\
V \\
\rightarrow \\
G \\
\rightarrow \\
0 \\
\end{array}
\begin{array}{c}
\rightarrow \\
\downarrow \\
\rightarrow \\
\downarrow \\
\rightarrow \\
\downarrow \\
\rightarrow \\
0 \\
\end{array}
\begin{array}{c}
\rightarrow \\
\downarrow \\
\rightarrow \\
\downarrow \\
\rightarrow \\
\downarrow \\
\rightarrow \\
0 \\
\end{array}
\begin{array}{c}
0 \\
\rightarrow \\
\Omega(G') \\
\rightarrow \\
P \\
\rightarrow \\
G' \\
\rightarrow \\
0 \\
\end{array}
$$

Now, define $\delta([\epsilon]_{EE}) = g$. Note that the morphism $g$ in $\text{Gprj-}\Lambda$ is independent of the choice $l$ in the above. The mappings $\theta$ and $\delta$ are well-defined and also mutually inverse isomorphisms.

In fact, the functions, introduced in the above construction, are inspired by the mutually inverse $k$-isomorphisms in $\text{Ext}_\Lambda^1(G, \Omega(G')) \cong \text{Hom}_\Lambda(G, G')$, which behave also well in our setting.

Let $\Gamma$ be an Artin algebra. We define a relation $\sim_c$ on the elements of $\Gamma$ by setting

$$a \sim_c b \iff au = vb \text{ for some unit elements } u, v \text{ of } \Gamma$$

for all $a, b \in \Gamma$. But $\sim_c$ is an equivalence relation and let $A(\Gamma)$ show the set of equivalence classes of $\Gamma$ under the relation.
Remark 3.6. Let $I$ be a two-sided ideal of $\Gamma$ and $\Gamma/I$ the quotient algebra of $\Gamma$ modulo $I$. Since the residue class of an unit element is again an unit element in $\Gamma/I$, so we can deduce that if $|A(\Gamma)| < \infty$, then $|A(\Gamma/I)| < \infty$.

Theorem 3.7. Let $\Lambda$ be a CM-finite algebra. Then the following assertions are equivalent.

1. $T_2(\Lambda)$ is CM-finite;
2. The stable Cohen-Macaulay Auslander algebra, $\text{Aus}(\text{Gprj-}\Lambda)$, is representation-finite;
3. If $G$ is a representation generator of $\text{Gprj-}\Lambda$, then $\text{EExt}^1_\Lambda(G, G)$ is a finite abelian group;
4. $|A(\text{Aus}(\text{Gprj-}\Lambda))| < \infty$.

Proof. According to Proposition 3.2, it suffices to prove only (3) $\iff$ (4). Assume (3) holds. If $G$ is a basic representation generator of $\text{Gprj-}\Lambda$, then, by (3), also $\text{EExt}^1_\Lambda(G, G)$ is a finite abelian group. We observe by a simple modification of Theorem 8.4 in page 397 of [ASS], where the result stated for the modules over a self-injective, but the proofs work for any Frobenius category, in particular $\text{Gprj-}\Lambda$, that is, the syzygy functor makes a bijective map from the set of all algebraic non-isomorphic non-projective indecomposable Gorenstein projective modules into itself. Thus, this observation implies $\text{O}(G) \simeq G$ in $\text{Gprj-}\Lambda$. Now Construction 3.5 in conjunction with (3) yields the cardinality of $\text{HHom}_\Gamma(G, G)$, or $\text{EEnd}_\Lambda(G)$, is finite. But $A(\text{Gprj-}\Lambda) = \text{End}_\Lambda(G)$, by the definition of the equivalence relation $\sim_H$ on $\text{End}_\Lambda(G)$, is the same as the equivalence relation $\sim_c$ on the algebra $\text{Aus}(\text{Gprj-}\Lambda)$. Hence, $|\text{EEnd}_\Lambda(G)| = |A(\text{Aus}(\text{Gprj-}\Lambda))|$, so we get (4). The proof of (4) $\Rightarrow$ (3) can be proved in a similar way for the case that $G$ is a basic representation generator of $\mathcal{X}$. If $G$ is an arbitrary representation generator of $\mathcal{X}$. Let $G_0$ be a basic representation generator of $\mathcal{X}$. Then there exists $n > 0$ such that $G$ is a direct summand of $G_0^n$. By Lemma 3.1, $\text{EExt}^1_\Lambda(G_0^n, G_0^n)$ is finite. Since we can correspond the elements in $\text{EExt}^1_\Lambda(G^n, G^n)$ by the ones in $\text{EExt}^1_\Lambda(G_0^n, G_0^n)$, by adding some split exact sequences in case of need, so $\text{EExt}^1_\Lambda(G^n, G^n)$ is finite, as required. □

Trivially, since the projective objects vanish in the stable category $\text{Gprj-}\Lambda$, then we can assume $G$ to be a direct sum of pairwise non-isomorphic non-projective indecomposable $\Lambda$-modules in the above theorem. As a direct consequence of the above we get the following theorem over the self-injective algebras.

Theorem 3.8. Let $\Lambda$ be a representation-finite self-injective algebra. Then the following assertions are equivalent.

1. The submodule category $\mathcal{S}(\Lambda)$ is of finite representation type;
2. The stable Auslander algebra of $\Lambda$, $\text{Aus}(\text{mod-}\Lambda)$, is representation-finite;
3. If $M$ is a representation generator of $\Lambda$, then $\text{EExt}^1_\Lambda(M, M)$ is a finite abelian group;
4. $|A(\text{Aus}(\text{mod-}\Lambda))| < \infty$.

4. Examples and Remarks

In this section we shall provide some examples to support our abstract results in the previous section. Throughout this section, we assume that $k$ is an algebraic closed filed.

Example 4.1. Let $A = k\mathbb{Q}/I$ be a quadratic monomial algebra, i.e. the ideal $I$ is generated by paths of length two. By [CSZ, Theorem 5.7], $\text{Gprj-}\Lambda \simeq T_1 \times \cdots \times T_n$ such that the underlying categories of triangulated categories $T_i$ are equivalent to semisimple abelian categories $\text{mod-k}^{d_i}$, for some natural numbers $d_i$. Hence $\text{mod-}\text{Gprj-}\Lambda$ is a semisimple abelian category and consequently $\text{Aus}(\text{Gprj-}\Lambda)$ a semisimple Artin algebra. Thus Theorem 3.7 implies $T_2(A)$ is CM-finite.
Also we can see the CM-finiteness of triangular matrix rings over quadratic monomial algebras by other equivalent conditions in Theorem 3.7. For this consider the the following quiver $Q$.

$$
\begin{array}{ccc}
1 & \overset{\alpha}{\rightarrow} & 2 \\
\beta & \searrow & \\
&& 3
\end{array}
$$

Let $I$ be the ideal generated by $\beta \alpha$ and $\alpha \beta$, and let $A = kQ/I$. We denote by $S_i$ the simple $A$-module corresponding to the vertex $i$ for $1 \leq i \leq 3$. Due to the classification of non-projective indecomposable Gorenstein projective modules by the perfect paths given in [CSZ, Theorem 4.4], as done in [CSZ, Example 4.4], we observe that there exist precisely two perfect paths $\alpha A \simeq S_2$ and $\beta A \simeq S_1$. Trivially, $\text{Hom}_A(S_i, S_j) = \delta_{ij}k$ then we have $\text{Aus}(\text{Gprj}-A) \simeq (\begin{smallmatrix} k & 0 \\ 0 & 0 \end{smallmatrix})$. But we have for any diagonal matrix $M = (\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix})$: (1) if $a$ and $b$ are non-zero, then $M \sim_c (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})$, (2) if $a \neq 0$ and $b = 0$, then $M \sim_c (\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix})$ and (3) if $b \neq 0$ and $a = 0$, then $M \sim_c (\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix})$. Therefore, the stable Cohen Macaulay algebra over $A$ has precisely 3 equivalence classes respect to $\sim_c$, as expected by Theorem 3.7 for the finiteness of the equivalence classes.

**Example 4.2.** In [RS1] was shown that for $n \leq 5$, $S(k[x]/(x^n))$ is of finite representation type, while for $n \geq 7$ it is a wild representation type where one can not expect for any classification. For $n \leq 5$ one may can directly check the finiteness of $S(k[x]/(x^n))$ by computing the equivalence classes of their stable Auslander algebras, as we want to do for the case $n = 3$. Let $A$ be the self-injective algebra $k[x]/(x^3)$. There are two pairwise non-isomorphic indecomposable non-projective modules $S = (x)/(x^3)$ and $T = (x^2)/(x^3)$ in mod-$A$. By a routine computation we obtain the following $k$-isomorphisms

$$
\text{Hom}_A(S, S) \simeq k, \quad \text{Hom}_A(S, T) \simeq k, \quad \text{Hom}_A(T, S) \simeq k, \quad \text{Hom}_A(T, T) \simeq k^2.
$$

By using these isomorphisms we get $\text{Aus}(\text{mod}-A) = \text{End}_A(S \oplus T)$ is isomorphic to

$$
\Gamma = \{ (\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}) \mid a, b, c, d, 1 \text{ and } 2 \in k \},
$$

where the multiplication in $\Gamma$ is given by

$$
(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}) (\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}) = (\begin{smallmatrix} ae & af + ch \\ be + d, g & dh + h, d + h, d + h, b, f \end{smallmatrix}).
$$

we observe an element $\left(\begin{smallmatrix} b & c \\ d & d \end{smallmatrix}\right)$ is an unit in $\Gamma$ if and only if $ad_1 \neq 0$. Fix a non-zero element $A = \left(\begin{smallmatrix} b & c \\ d & d \end{smallmatrix}\right)$ of $\Gamma$. If $ad_1 \neq 0$, then $A \sim_\subset 1$. Assume $ad_1 = 0$. Then two cases happen.

(i) $a = 0$ for this case the following situations occur:

1. If $c = d_1 = b = 0$, then $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and so $A \sim_\subset \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$;
2. If $c \neq 0$ and $b = d_1 = d_2 = 0$, then $\left(\begin{smallmatrix} c & 0 \\ 0 & 0 \end{smallmatrix}\right)A = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and so $A \sim_\subset \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$;
3. If $c \neq 0$, $d_2 \neq 0$ and $b = d_1 = 0$, then $\left(\begin{smallmatrix} c & 0 \\ 0 & d_2 \end{smallmatrix}\right)A = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and so $A \sim_\subset \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$;
4. If $c \neq 0$, $d_2 \neq 0$, $d_1 \neq 0$ and $b = 0$, then $\left(\begin{smallmatrix} c & 0 \\ 0 & d_2 \end{smallmatrix}\right)A = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and so $A \sim_\subset \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$;
5. If $c \neq 0$, $d_2 \neq 0$, $d_1 = 0$ and $b \neq 0$, then $\left(\begin{smallmatrix} c & 0 \\ 0 & d_2 \end{smallmatrix}\right)A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and so $A \sim_\subset \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$;
6. If $c \neq 0$, $d_1 \neq 0$ and $b = d_2 = 0$, then $\left(\begin{smallmatrix} c & 0 \\ 0 & d_1 \end{smallmatrix}\right)A = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and so $A \sim_\subset \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$;
(7) If $b \neq 0$, $d_1 \neq 0$ and $c = d_2 = 0$, then $A \left( \frac{1}{bd_1}, \frac{1}{d_2} \right) = \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$ and so $A \sim_c \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$;

(8) If $d_2 \neq 0$, $d_1 \neq 0$, $c \neq 0$ and $b = 0$, then $\left( \frac{1}{bd_1}, \frac{1}{d_2} \right) = \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$ and so $A \sim_c \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$;

(9) If $d_2 \neq 0$, $d_1 \neq 0$ and $b = c = 0$, then $A \left( \frac{1}{d_1} \right) = \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$ and so $A \sim_c \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$;

(10) If $d_2 \neq 0$, $d_1 \neq 0$, $b \neq 0$ and $c = 0$, then $A \left( \frac{1}{bd_1}, \frac{1}{d_2} \right) = \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$ and so $A \sim_c \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$;

(11) If $d_2 \neq 0$, $b \neq 0$ and $c = d_1 = 0$, then $A \left( \frac{1}{bd_1}, \frac{1}{d_2} \right) = \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$ and so $A \sim_c \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$;

(12) If $d_2 = d_1 = c = 0$ and $b \neq 0$, then $A \left( \frac{1}{d_1}, \frac{1}{d_2} \right) = \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$ and so $A \sim_c \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$;

(13) If $d_2 = b = c = 0$ and $d_1 \neq 0$, then $A \left( \frac{1}{d_1} \right) = \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$ and so $A \sim_c \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$;

(14) If $d_1 \neq 0$, $c \neq 0$, $b \neq 0$ and $d_2 = 0$, then $A \left( \frac{1}{bd_1}, \frac{1}{c} \right) = \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$ and now by using (6) we get $A \sim_c \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$;

(15) If $d_1 \neq 0$, $c \neq 0$, $b \neq 0$ and $d_2 \neq 0$, then $A \left( \frac{1}{bd_1}, \frac{1}{d_2} \right) = \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$ and now again by using (6) we get $A \sim_c \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$;

(ii) If $d_1 = 0$ and $a \neq 0$ we have the following cases:

(16) If $b = c = d_2 = 0$, then $A \left( \frac{1}{a} \right) = \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$ and so $A \sim_c \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$;

(17) If $b = d_2 = 0$, and $c \neq 0$ then $A \left( \frac{1}{a}, \frac{1}{c} \right) = \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$ and so $A \sim_c \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$;

(18) If $c \neq 0$, $d_2 \neq 0$, and $b = 0$ then $A \left( \frac{1}{a}, \frac{1}{d_2} \right) = \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$ and so $A \sim_c \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$;

(19) If $b \neq 0$, $d_2 \neq 0$, and $c = 0$ then $A \left( \frac{1}{a}, \frac{1}{d_2} \right) = \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$ and so $A \sim_c \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$;

(20) If $b = c = d_2$ and $d_2 \neq 0$, then $A \left( \frac{1}{a}, \frac{1}{d_2} \right) = \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$ and so $A \sim_c \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$;

(21) If $d_2 = c = 0$, and $b \neq 0$ then $A \left( \frac{1}{a}, \frac{1}{c} \right) = \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$ and so $A \sim_c \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$;

(22) If $b \neq 0$, $c \neq 0$, and $d_2 = 0$ then $A \left( \frac{1}{a}, \frac{1}{c} \right) = \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$ and by (18) we obtain $A \sim_c \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$;

(23) If $b \neq 0$, $c \neq 0$, and $d_2 \neq 0$ then $A \left( \frac{1}{a}, \frac{1}{c} \right) = \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$ and by (17), when $d_2 = ba^{-1}c$, or again by (18), when $d_2 \neq ba^{-1}c$, we obtain $A \sim_c \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$ or $A \sim_c \left( \begin{smallmatrix} 0 & 0 \\ 0 & (1,0) \end{smallmatrix} \right)$.

By the above computations we deduce the non-zero and non-unit elements of $\Gamma$ are partitioned to at most 11 equivalence classes. Now by considering the equivalence classes for $0$ and $1_{11}$, we have at most 13 equivalence classes for the relation $\sim_c$ over the set of elements of $\Gamma$. Thus by Theorem 3.7, $T_2(k[x]/(x^3))$ is CM-finite, of course as expected by [RS1], but the interesting point here is we did by doing only an elementary method. We can continue to determining CM-finiteness of $T_2(k[x]/(x^n))$ for higher $n$ with this elementary method although we may encounter with their own difficulties to compute the equivalence classes. Conversely, we can use the results of [RS1] to see that $A(\text{Aus}(\text{mod-k[x]}/(x^n)))$ is finite for $n \leq 5$ and infinite for $n > 6$. Let us point
out, as discussed in [RZ], Aus$(\text{mod-k}[x]/(x^n)) \simeq \Pi_{n-1}$, where $\Pi_{n-1}$ denotes the preprojective algebra of type $\mathbb{A}_n$.

**Example 4.3.** Let $Q$ be the quiver

$$
\begin{array}{c}
1 \\
\alpha \\
\beta
\end{array}
$$

Set $A = kQ/I$, where $kQ$ is the path algebra of the quiver $Q$ and with ideal $I$ generated by the paths $\alpha\beta\alpha$ and $\beta\alpha\beta$. In fact, $A$ is a self-injective Nakayama algebra with loewy length 3. By the classification of indecomposable modules over Nakayama algebras, see for example Theorem 3.5 of [ASS] in page 169, or the classification by the perfect paths given in [CSZ], any non-projective indecomposable $A$-module is isomorphic to one of the modules $S_i = P_i/\text{rad}P_i$, $T_i = P_i/\text{rad}^2P_i$, $i = 1, 2$ and the $P_i$ are the corresponding indecomposable projective modules to the vertex $i$.

These modules can be represented as the following

$S_1 : k \begin{array}{cr} 0 & 1 \\ 0 & 0 \end{array}$, $T_1 : k \begin{array}{cr} 0 & 1 \\ 0 & 0 \end{array}$, $S_2 : k \begin{array}{cr} 0 & 1 \\ 0 & 0 \end{array}$, $T_2 : k \begin{array}{cr} 0 & 1 \\ 0 & 0 \end{array}$.

By doing a simple computation we can get $\text{Aus}(\text{mod-A}) = \text{End}_k(S_1 \oplus T_1 \oplus S_2 \oplus T_2)$ is isomorphic to the algebra $\Gamma$ consisting of all the following $4 \times 4$ matrices

$$
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
$$

where the entries belong to $k$. The multiplication of the elements in $\Gamma$ is given by the following way

$$
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
= 
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
$$

Consider an element $M = \begin{pmatrix} a_1 & 0 & 0 & a_2 \\ a_3 & a_4 & a_5 & 0 \\ 0 & a_6 & a_7 & a_8 \\ 0 & a_9 & a_{10} & 0 \end{pmatrix}$ in $\Gamma$. By viewing this multiplication can be seen $M$ is an unit element in $\Gamma$ if and only if $a_1a_4a_7a_{10} \neq 0$ in $k$. By definition every multiplication

$$
\begin{pmatrix}
w_1 & 0 & 0 & w_2 \\
0 & w_4 & w_5 & 0 \\
0 & w_6 & w_7 & 0 \\
0 & w_8 & w_9 & w_{10} \\
\end{pmatrix}
$$

of the elements of $\Gamma$ in the following belongs to the corresponding equivalence class of $M$ respect to $\sim_c$

$$
\begin{pmatrix}
u_1 & 0 & 0 & v_2 \\
u_3 & 0 & 0 & v_4 \\
u_5 & 0 & 0 & v_6 \\
u_7 & 0 & 0 & v_8 \\
\end{pmatrix}
\begin{pmatrix}
u_1 & 0 & 0 & v_2 \\
u_3 & 0 & 0 & v_4 \\
u_5 & 0 & 0 & v_6 \\
u_7 & 0 & 0 & v_8 \\
\end{pmatrix}
= 
\begin{pmatrix}
u_1 & 0 & 0 & v_2 \\
u_3 & 0 & 0 & v_4 \\
u_5 & 0 & 0 & v_6 \\
u_7 & 0 & 0 & v_8 \\
\end{pmatrix}
$$

where $u_1v_4v_7v_{10} \neq 0$ and $v_1u_4v_7v_{10} \neq 0$. We claim that the $u_i$ and $v_i$ can be chosen in the way such that the entries $w_i$ in the multiplication will get either 0 or 1, and consequently there is a finite number of the equivalent classes of $\Gamma$ respect to $\sim_c$. Now, we get the following equations system:

$$
a_1u_1v_1 = w_1 \\
a_1u_1v_2 + a_2u_1v_{10} + a_{10}u_2v_{10} = w_2 \\
a_1u_3v_1 + a_3u_4v_1 + a_4u_4v_3 = w_3 \\
a_4u_4v_4 = w_4 \\
a_1u_3v_2 + a_3u_4v_2 + a_4u_4v_5 + a_2u_3v_{10} + a_5u_4v_{10} + a_{10}u_5v_{10} = w_5
$$
\[ a_4u_6v_4 + a_6u_7v_4 + a_7u_7v_6 = w_6 \]
\[ a_7u_7v_7 = w_7 \]
\[ a_4u_8v_4 + a_6u_9v_4 + a_8u_{10}v_4 + a_7u_9v_6 + a_9u_10v_6 + a_{10}u_{10}v_8 = w_8 \]
\[ a_7u_9v_7 + a_9u_{10}v_7 + a_{10}u_{10}v_9 = w_9 \]
\[ a_{10}u_{10}v_{10} = w_{10} \]

The left side of the above equations system is obtained by doing a simple multiplication in \( \Gamma \). Let us call the above equations system by \( \langle \rangle \). Since the situation for the case \( a_1a_2a_7a_{10} \neq 0 \) is clear. In fact, in this case \( M \sim 1_\Gamma \). So we can concentrate only on the case \( a_1a_2a_7a_{10} = 0 \). For this case, we have the following different situations for which of them we want to explain how the \( u_i \) and \( v_i \) satisfying \( u_1u_4u_7u_{10} \neq 0 \) and \( v_1v_4v_7v_{10} \neq 0 \) can be chosen as to force the \( w_i \) to be either zero or 1, by providing the simpler equations systems.

**Case 1** If \( a_1 = a_4 = a_7 = a_{10} = 0 \), then by taking \( u_2 = v_3 = u_4 = v_6 = 0 \), then we have more simpler equations: \( a_2u_1v_{10} = w_2 \), \( a_3u_4v_1 = w_3 \), \( a_5u_4v_{10} = w_5 \), \( a_6u_7v_4 = w_6 \), \( a_8u_{10}v_4 = w_8 \), \( a_9u_{10}v_7 = w_9 \). It is not really difficult to see that based on the rest \( a_i \), i.e. \( i \neq 1, 4, 7, 10 \), to be zero or non-zero, then we assign the corresponding \( w_i \) to be either zero or 1, respectively, and next one can easily choose the suitable \( u_i \) and \( v_i \) in the corresponding equations, as required.

More explanation, first let \( \eta : k \rightarrow k \) be a function by sending \( a \in k \) to 0 if \( a = 0 \) and to 1 if \( a \neq 0 \), then by the function we can give the values to the variables \( w_i \) and find the equations system: \( a_2u_1v_{10} = \eta(a_2), a_3u_4v_1 = \eta(a_3), a_5u_4v_{10} = \eta(a_5), a_6u_7v_4 = \eta(a_6), a_8u_{10}v_4 = \eta(a_8), a_9u_{10}v_7 = \eta(a_9) \). But the obtained equations system can be solved very easily, namely, take \( u_1 = 1 \) then by the equation \( a_2v_{10} = \eta(a_2) \), we choose \( v_{10} = a_2^{-1} \) if \( \eta(a_2) = 1 \) and arbitrary nonzero element of \( k \) if \( \eta(a_2) = 0 \), then we can accordingly continue by the other equations to obtain the other \( u_i \) and \( v_i \). For example for the case when all the rest \( a_i \) are zero it is enough to take \( u_4 = v_4 = 1, v_{10} = a_2^{-1}, u_1 = a_2a_7^{-1}, v_1 = a_3^{-1}, u_{10} = a_6^{-1}, v_7 = a_8a_9^{-1}, u_7 = a_9^{-1} \).

Moreover, all the \( w_i \) except \( i = 1, 4, 7, 10 \) will get 1. The other \( u_i \) and \( v_i \) which are not appeared in this way, exactly those \( u_i \) and \( v_i \) are disappeared by assuming \( a_1 = a_4 = a_7 = a_{10} = 0 \) in the equations system \( \langle \rangle \), are useless and we can take any value of \( k \) for them.

Let us remark here for all the next cases the absent \( u_i \) and \( v_i \) which are not appeared in the corresponding way are assumed automatically to be 0. In addition, for the next cases when we receive to a simple form of the equitations system as the Case 1 we stop and leave it to the reader to see that it has a solution as required.

**Case 2** If \( a_4 = a_7 = a_{10} = 0, \) and \( a_1 \neq 0, \) then the following situations happen:

(a) If \( a_5 = 0, \) then by taking \( v_2 = u_3 = u_9 = v_6 = 0, \) we have the equations system:
\[ a_1u_1v_1 = w_1, a_2u_1v_{10} = w_2, a_3u_4v_1 = w_3, a_6u_7v_4 = w_6, a_8u_{10}v_4 = w_8, a_9u_{10}v_7 = w_9. \]
But it is not difficult to see that this system has a solution in the way as we like.

(b) If \( a_5 \neq 0 \) and \( a_2 = 0 \) then by taking \( v_2 = u_9 = v_6 = 0, u_3 = v_{10} = 1, u_4 = a_5^{-1}, \) so we have the equations:
\[ a_1u_1v_1 = w_1, (a_1 + a_3a_5^{-1})v_1 = w_3, a_6u_7v_4 = w_6, a_8u_{10}v_4 = w_8, a_9u_{10}v_7 = w_9. \]
Hint: First finding \( v_1 \) from the equation \( (a_1 + a_3a_5^{-1})v_1 = \eta(a_1 + a_3a_5^{-1}) \), and then \( u_1 \) from the equation \( a_1u_1v_1 = w_1, \) and so forth.

(c) If \( a_5 \neq 0 \) and \( a_2 \neq 0 \), then by taking \( v_2 = u_9 = v_6 = 0, u_3 = v_{10} = 1, u_4 = a_2a_5^{-1}, \) then we have the equations:
\[ a_1u_1v_1 = w_1, a_2u_1v_{10} = w_2, (a_1 - a_3a_2a_5^{-1})v_1 = w_3, a_6u_7v_4 = w_6, a_8u_{10}v_4 = w_8, a_9u_{10}v_7 = w_9. \]
Similarly the system has a solution.

**Case 3** If \( a_1 = a_7 = a_{10} = 0, \) and \( a_4 \neq 0, \) then by taking \( v_2 = v_3 = v_5 = v_6 = u_3 = u_6 = u_8 = u_9 = 0, \) so we have the equations system:
\[ a_2u_1v_{10} = w_2, a_3u_4v_1 = w_3, a_4u_4v_4 = \]
$w_4, a_5 u_4 v_{10} = w_5, a_6 u_7 v_4 = w_6, a_8 u_{10} v_4 = w_8, a_9 u_{10} v_7 = w_9$. Similarly the system has a solution.

For deleting some cases to be checked we need to define an automorphism algebra over $\Gamma$ as follows. Let $\Psi: \Gamma \to \Gamma$ be defined in the following way

$$\begin{pmatrix} a & 0 & 0 & d \\ c & b & 0 & e \\ 0 & f & h & 0 \\ 0 & g & l & 1 \end{pmatrix} \mapsto \begin{pmatrix} h & 0 & 0 & f \\ l & i & 0 & g \\ 0 & d & a & 0 \\ 0 & c & e & b \end{pmatrix}.$$ 

It is clear $\Psi$ is an automorphism of algebras. Hence the equivalence relation $\sim_c$ is preserved by $\Psi$. We shall use this automorphism to simplify our computation in the below.

The two cases $a_1 = a_4 = a_7 = 0, a_{10} \neq 0$ and $a_1 = a_4 = a_{10} = 0, a_7 \neq 0$ no need to investigate in view of cases 2, 3 and the automorphism algebra $\Psi$. So until now we have considered all situations in which at most one of the diagonal entries of $M$ are nonzero. Let us continue the other cases.

**Case 4** If $a_1 = a_{10} = 0, a_4 \neq 0, a_7 \neq 0$, then

(a) If $a_9 = 0$, then by taking $v_2 = v_3 = v_5 = v_6 = u_3 = u_6 = u_8 = w_9 = 0$, then we obtain the equations system: $a_2 u_1 v_{10} = w_2, a_3 u_4 v_1 = w_3, a_4 u_4 v_4 = w_4, a_5 u_4 v_{10} = w_5, a_6 u_7 v_4 = w_6, a_7 u_7 v_7 = w_7, a_8 u_{10} v_4 = w_8$. But this equations system can be solved easily.

(b) If $a_9 \neq 0$, by taking $v_2 = v_3 = v_5 = v_6 = u_3 = u_6 = u_9 = 0, v_7 = 1, u_7 = a_{7}^{-1}, u_{10} = a_9^{-1}, u_8 = -a_8 a_{7}^{-1} a_{4}^{-1}$, we have the equation system: $a_2 u_1 v_{10} = w_2, a_3 u_4 v_1 = w_3, a_4 u_4 v_4 = w_4, a_5 u_4 v_{10} = w_5, a_6 u_7 v_4 = w_6, a_7 u_7 v_7 = w_7, a_8 u_{10} v_4 = w_8$. But no hard to solve this equations system as we want.

**Case 5** If $a_4 = a_7 = 0, a_1 \neq 0, a_{10} \neq 0$, this case no need to check because is the same as Case 4 by using the symmetry reason obtained by the $\Psi$.

**Case 6** If $a_1 = a_4 = 0, a_7 \neq 0, a_{10} \neq 0$, then

(a) If $a_8 = 0$, then by taking $u_2 = v_2 = v_6 = v_8 = v_9 = u_2 = u_3 = u_5 = u_9 = 0$ and $u_{10} = 1, u_{10} = a_{10}^{-1}$, then we have: $a_2 u_1 = w_2, a_3 u_4 v_1 = w_3, a_5 u_4 = w_5, a_6 u_7 v_4 = w_6, a_7 u_7 v_7 = w_7, a_9 a_{10} v_7 = w_9$.

(b) If $a_8 \neq 0, a_6 \neq 0$, then by taking $u_2 = v_3 = u_5 = u_9 = v_2 = v_6 = v_8 = v_9 = 0$ and $u_{10} = 1, a_{10} = a_{10}^{-1}$, then we have: $a_2 a_{10} u_1 = w_2, a_3 u_4 v_1 = w_3, a_5 a_{10} u_4 = w_5, a_6 u_7 v_4 = w_6, a_7 u_7 v_7 = w_7, a_9 a_{10} v_7 = w_9$.

(c) If $a_8 \neq 0, a_6 \neq 0$, then by taking $u_2 = v_3 = u_5 = v_2 = v_6 = v_8 = v_9 = 0, u_{10} = 1, v_{10} = a_{10}^{-1}, u_9 = -a_8 a_{6}^{-1}$, then we have: $a_2 a_{10} u_1 = w_2, a_3 u_4 v_1 = w_3, a_5 a_{10} u_4 = w_5, a_6 u_7 v_4 = w_6, a_7 u_7 v_7 = w_7, a_9 a_{6}^{-1} v_7 = w_9$.

No need to check the case $a_7 = a_{10} = 0, a_1 \neq 0, a_4 \neq 0$ since it follows from Case 6 by using the symmetry reason.

**Case 7** If $a_1 = a_7 = 0, a_4 \neq 0, a_{10} \neq 0$, then

(a) If $a_5 = 0$, then by taking $u_2 = v_3 = v_5 = u_3 = u_5 = u_6 = u_8 = u_9 = v_6 = v_8 = v_9 = 0$ and $u_{10} = 1, v_{10} = a_{10}^{-1}$, then we have: $a_2 a_{10} u_1 = w_2, a_3 u_4 v_1 = w_3, a_4 u_4 v_4 = w_4, a_6 u_7 v_4 = w_6, a_7 u_7 v_7 = w_7, a_9 a_{10} v_7 = w_9$.

(b) If $a_5 \neq 0, a_6 = 0$, then by taking $u_2 = v_3 = v_5 = u_3 = u_5 = u_6 = u_9 = v_6 = v_8 = v_9 = 0, u_4 = 1, v_{10} = a_{10}^{-1}, v_4 = a_{4}^{-1}, u_8 = -a_8 a_{5} a_{4}^{-1} a_{10}^{-1}, u_{10} = a_5 a_{10}^{-1}$, then we have: $a_2 a_{5} a_{10} u_1 = w_2, a_3 u_4 v_1 = w_3, a_9 a_{5} a_{10} v_7 = w_9$. 

(c) If \( a_5 \neq 0 \), \( a_6 \neq 0 \), then by taking \( u_2 = v_3 = v_2 = v_5 = u_3 = u_5 = u_6 = u_8 = v_8 = v_9 = v_3 = 1 \), \( u_4 = a_4^{-1} \), \( u_7 = a_6^{-1} \), \( v_{10} = a_5^{-1} a_4^{-1} \), \( u_9 = -a_9 a_9^{-1} a_{10}^{-1} a_6^{-1} \), \( u_{10} = a_5 a_4^{-1} a_{10}^{-1} \), we then have: \( a_2 a_4 a_5^{-1} u_1 = w_2 \), \( a_3 a_4^{-1} v_1 = w_3 \), \( a_9 a_5 a_4^{-1} a_{10}^{-1} v_7 = w_9 \).

**Case 8** If \( a_4 = a_{10} = 0 \), \( a_7 \neq 0 \), \( a_1 \neq 0 \), then

(a) If \( a_5 - a_3 a_2 a_1^{-1} = 0 \), then by taking \( u_3 = v_5 = 0 \), \( u_1 = v_{10} = u_{10} = 1 \), \( v_1 = a_1^{-1} \), \( v_2 = -a_2 a_1^{-1} \), \( u_9 = -a_9 a_7^{-1} \), then we have: \( a_3 a_1^{-1} u_4 = w_3 \), \( a_6 u_7 v_4 = w_6 \), \( a_7 u_7 v_7 = w_7 \), \((a_8 - a_9 a_6 a_7^{-1})v_4 = w_8\).

(b) If \( a_5 - a_3 a_2 a_1^{-1} \neq 0 \), then by taking \( u_1 = u_{10} = 1 \), \( u_3 = v_5 = 0 \), \( v_1 = a_1^{-1} \), \( u_9 = -a_9 a_7^{-1} \), then we have: \( a_3 a_1^{-1} u_4 = w_3 \), \((a_5 - a_3 a_2 a_1^{-1})u_4 v_{10} = 1 \), \( v_2 = -a_2 a_1^{-1} v_{10} \), \( a_6 u_7 v_4 = w_6 \), \( a_7 u_7 v_7 = w_7 \), \((a_8 - a_9 a_6 a_7^{-1})v_4 = w_8\).

**Case 9** If \( a_1 = 0 \), \( a_4 \neq 0 \), \( a_7 \neq 0 \), \( a_{10} \neq 0 \), then by taking \( v_2 = v_5 = u_3 = u_5 = u_9 = v_8 = 0 \), \( u_1 = v_1 = u_7 = 1 \), \( u_2 = u_{10}^{-1} \), \( v_3 = -a_3 a_4^{-1} \), \( u_6 = -a_6 a_4^{-1} \), \( v_7 = a_7^{-1} \), \( v_9 = -a_9 a_7^{-1} a_{10}^{-1} \), we then have: \( a_4 u_4 v_4 = w_4 \), \( a_5 u_4 v_{10} = w_5 \), \( u_8 = -a_8 a_4^{-1} u_{10} \), \( a_{10} u_{10} v_{10} = w_{10} \).

**Case 10** If \( a_4 = 0 \), \( a_1 \neq 0 \), \( a_7 \neq 0 \), \( a_{10} \neq 0 \), then by taking \( v_2 = u_9 = v_8 = 0 \), \( u_1 = v_7 = v_4 = u_4 = 1 \), \( v_1 = a_1^{-1} \), \( u_7 = a_7^{-1} \), \( v_2 = -a_2 a_1^{-1} \), \( u_3 = -a_3 a_1^{-1} \), \( u_5 = a_{10}^{-1} (a_2 a_3 a_1^{-1} - a_5) \), \( v_6 = -a_6 a_7^{-1} \), \( v_9 = -a_9 a_1^{-1} \), then we obtain: \((a_8 - a_9 a_7^{-1} a_6)u_{10} = w_8 \), \( a_{10} u_{10} v_{10} = w_{10} \).

In view of Theorem 3.7 and the above computations then \( T_2(A) \) is CM-finite, or equivalently the stable Auslander algebra of \( A \) is representation-finite.

**Remark 4.4.** As we have seen in the above example, sometimes representation theoretical problems can be translated to solving some equations systems, here there might be a hope to use some computational methods especially using some software in mathematics to solve the equations systems and then an application using it to determining the representation type of a subcategory or an algebra. Note that since we need in the above example the \( w_i \) to be zero or 1, and \( u_1 u_4 v_7 u_{10} \neq 0 \) and \( v_1 v_4 v_7 v_{10} \neq 0 \), by adding the equations \( w_i (w_i - 1) = 0 \), \( 1 \leq i \leq 10 \), and \( u_1 u_4 u_7 u_{10} y_1 = 1 \), \( v_1 v_4 v_7 v_{10} y_2 = 1 \) to 10 equations in (7) of the above example, then we get a equations system with 22 equations by 32 variable, including \( u_i, v_i, w_i \) and \( y_1, y_2 \), and 10 parameters \( a_i \). So, in this way we can translate the required conditions to equations.

Also, based on the above example and the other examples done by the author we conjecture: Let \( A \) be a CM-finite algebra. If for any pair of indecomposable Gorenstein projective modules \( G \) and \( G' \), \( \text{Hom}_A(G, G') \) is isomorphic to either \( k \) or 0, then \( T_2(A) \) is a CM-finite algebra.

It is not difficult to see that the self-injective Nakayama algebras with lowey length than the number of pairwise non-isomorphic simple modules satisfying the conditions of the above conjecture.

Finally, in Example 4.2 we can observe that \( A(\text{Aus}(\text{mod}-k[x]/(x^3))) \) has at most 13 elements. On the other hand, it was proved in [RS1] there are, up to isomorphism, \( 2 + 2(n - 1)/6 - n \) indecomposable modules in \( \text{Gprj}-T_2(k[x]/(x^3)) \), where \( n < 6 \), in particular for \( n = 3 \) we have 14 indecomposable modules in \( \text{Gprj}-T_2(k[x]/(x^3)) \). In general, it might be interesting to see that whether there is a relation between the cardinal number of indecomposable modules, up to isomorphism, in \( \text{Gprj}-T_2(A) \) for a given CM-finite algebra \( A \) and the cardinal number \(|A(\text{Aus}(\text{Gprj}-A))|\). From Theorem 3.7 just we know the finiteness these two cardinal numbers is the same.

**Remark 4.5.** Two algebras \( A \) and \( A' \) are said to be Gorenstein stably equivalent if the sable categories \( \text{Gprj}-A \) and \( \text{Gprj}-A' \) are equivalent. Hence if \( A \) and \( A' \) are CM-finite and Gorenstein
stably equivalent, then stable Cohen-Macaulay Auslander algebra $\text{Aus}(\text{Gprj-}\Lambda)$ is representation-finite if and only if $\text{Aus}(\text{Gprj-}\Lambda')$ so is. Recall the singular category $\mathcal{D}_{\text{sg}}(\Lambda)$ is defined to be the Verdier quotient of the bounded derived category $\mathcal{D}^b(\Lambda)$ of finitely generated modules over $\Lambda$ by the full subcategory $\mathcal{K}^b(\text{prj-}\Lambda)$ consisting of bounded complexes of finitely generated $\Lambda$-modules. Similarly, two algebras $\Lambda$ and $\Lambda'$ are said to be singularly equivalent if their singular categories $\mathcal{D}_{\text{sg}}(\Lambda)$ and $\mathcal{D}_{\text{sg}}(\Lambda')$ are equivalent as triangulated categories. The singular equivalences recently attained much attention, see for instance [ZZ]. By a known theorem due to Buchweitz or Happel, we know over Gorenstein algebras there exist the equivalence $\mathcal{D}_{\text{sg}}(\Lambda) \simeq \text{Gprj-}\Lambda$. Then when we are dealing with two Gorenstein algebras to be singularly equivalent so they are indeed Gorenstein stably equivalent. In addition, we have the notion of derived equivalence, i.e. $\Lambda$ and $\Lambda'$ are derived equivalent if $\mathcal{D}^b(\Lambda) \simeq \mathcal{D}^b(\Lambda')$ as triangulated categories. From [AHV, Theorem 4.1.2], we obtain if $\Lambda$ and $\Lambda'$ are derived equivalent, then $\text{Gprj-}\Lambda \simeq \text{Gprj-}\Lambda'$. Thus, if $\Lambda$ and $\Lambda'$ are derived equivalent, then they also are Gorenstein stably equivalent, and consequently by help of Theorem 3.7, if one of them to be CM-finite algebra, then the representation-finiteness, resp. CM-finiteness, of their corresponding Cohen-Macaulay Auslander algebras, resp. triangular matrix algebras, are same.

We conclude this paper by this remark that our results are true for any Artin algebra satisfying the second Brauer-Thrall conjecture.

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