Intertwining operators and Hirota bilinear equations

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Abstract

We give an interpretation of Hirota relations for τ-functions of hierarchies of integrable equations in terms of intertwining operators. This gives possibility to generalize the relations to the case of finite-dimensional Lie algebras and quantized universal enveloping algebras. An example of $U_q(sl_2)$ is presented.

1 Motivations. τ-function and matrix elements.

One of the most popular languages for algebraic study of nonlinear integrable equations is the language of τ-function. Usually τ-function is connected with initial variable $u$ via some logarithmic derivatives and can be treated as a point of infinite-dimensional grassmanian or of other homogeneous space for certain infinite-dimensional group $G$. For instance, for KP hierarchy $u = \frac{\partial^2 \log \tau}{\partial x^2}$ and $\tau$ is a point of $GL_\infty$-orbit $X$ of highest weight vector $v_0$ in basic representation $\Lambda_0$ of $GL_\infty$ in semiinfinite forms. The coordinates on $X$ are given by boson-fermion correspondence and the equations of motion in Hirota bilinear form could be obtained by Kac–Wakimoto construction [Kac], [KW] from quadratic Plucker-type equations of the orbit $X$. These equations are

$$\Omega \tau \otimes \tau = c \cdot \tau \otimes \tau$$

where $\Omega$ is an operator commuting with $g \otimes g$, $g \in G$, $c$ is a constant which one can compute acting by $\Omega$ on tensor product of highest weight vectors:

$$\Omega v_0 \otimes v_0 = c \cdot v_0 \otimes v_0$$

For KP hierarchy $\Omega = \sum_{j \in \mathbb{Z}} \psi_j \otimes \psi_j^*$, where $\psi_j$ and $\psi_j^*$ are free fermions. For hierarchies arising from level one irreducible representations of a Kac–Moody algebra $\widehat{g}$ an element $\Omega$ is divided Casimir operator,

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representing Killing form for \( \hat{\mathfrak{g}} \):

\[
\Omega = \sum e_i \otimes e^i
\]

Equivalently, \( \tau \)-function for KP hierarchy can be defined as a matrix element \[DJ1\]

\[
\tau^g(x_1, x_2, \ldots) = < v_0 \mid e^{H(\tau)} g \mid v_0 >
\]  

(1.1)

where \( g \in GL_\infty \) and

\[
e^{H(\tau)} = \exp(\sum_{i=1}^{\infty} x_i a_k)
\]

is a flow defined by commutative subalgebra \( \{ a_k, k \geq 1 \} \) of Lie algebra \( gl_\infty \):

\[
a_k = \sum_{j \in \mathbb{Z}} \psi_j \psi^*_j + k
\]

or by a commutative part of appropriate Heisenberg subalgebra for a Kac–Moody algebra \( \hat{\mathfrak{g}} \).

In these notations the basic relation which produces Hirota equations after the expansion over the diagonal is again the main property of \( \Omega \):

\[
\Omega g \otimes g = g \otimes g \Omega
\]

(1.2)

For instance, in KP case we have due to (1.2):

\[
\sum_{j \in \mathbb{Z}} < v_1 \mid e^{H(\tau)} \psi_j g \mid v_0 > < v_{-1} \mid e^{H(\tau)} \psi^*_j g \mid v_0 > =
\]

\[
= \sum_{j \in \mathbb{Z}} < v_1 \mid e^{H(\tau)} g \psi_j \mid v_0 > < v_{-1} \mid e^{H(\tau)} g \psi^*_j \mid v_0 > = 0
\]

(1.3)

where \( < v_k \mid \) is a left vacuum of charge \( k \):

\[
< v_k \mid = < v_0 \mid \psi^*_0 \cdots \psi^*_k, \quad k \geq 0,
\]

\[
< v_k \mid = < v_0 \mid \psi_{-1} \cdots \psi_{-k}, \quad k < 0,
\]

and, moving fermions \( \psi_j \) and \( \psi^*_j \) to the left in LHS of (1.3) we get desired functional relation in Hirota form

\[
\sum_{j \in \mathbb{Z}} S_j(\tau \tau) S_{j+1}(-\tilde{\partial}_y) \tau^{\beta}(\tau + \tau) \tau^{\beta}(\tau - \tau) = 0
\]

(1.4)

where \( \tilde{\partial}_y = \partial_{y_1}, \frac{1}{2} \partial_{y_2}, \ldots \) and \( S_j \) are Schur polynomials.

2 Hirota relations and Intertwining operators

We can see that the procedure of getting the bilinear equations for \( \tau \)-function could be broken into two steps:

(i) To find some (commutative) algebra which bosonize the representation;

(ii) To find bilinear relations for matrix elements of representation.

We could also have \( \tau \)-functions for different representations of the same \( G \). If we want to have corresponding \( \tau \)-functions being connected by a system of equations we should impose that all the representations involved could be bosonized by the same commutative algebra.

There is a general simple procedure of getting bilinear relations on matrix elements. Let us denote the matrix element \( < u g v > \) of \( g \) in representation \( V \) by \( \tau^g_{u,v} \):

\[
\tau^g_{u,v} = < u g v >, \quad v \in V, \quad u \in V^*, \quad g \in G
\]

(2.1)
Let now $V_1, V_2, V_3, V_4$ be representations of $G$, $v_i \in V_i$, $u_i \in V_i^*$ be some fixed vectors and $\Gamma = \sum_i \Gamma_i^1 \otimes \Gamma_i^2$ be an intertwining operator

$$\Gamma : V_1 \otimes V_2 \to V_3 \otimes V_4$$

(2.2)

that is,

$$\Gamma \Delta(x) = \Delta(x) \Gamma$$

(2.3)

where $\Delta$ is comultiplication in $G$. Then we have, since $\Delta(g) = g \otimes g$ for $g \in G$ and by [2,3], the following

Proposition 2.1

$$\sum_i \tau^{g}_{u_3 \Gamma_i^1 v_1 \tau^{g}_{u_4 \Gamma_i^2 v_2}} = \sum_i \tau^{g}_{u_3 \Gamma_i^1 v_1 \tau^{g}_{u_4 \Gamma_i^2 v_2}}$$

(2.4)

We should also keep bosonization picture for a tensor $\Gamma$ if it is constructed from some "local" data. There is a description of such a construction.

Let us fix some "vector" representation $W$ of Lie algebra $g$. It is the first fundamental representation of $g$ for finite-dimensional simple $g$; vector representation for $gl_\infty$ and evaluation representation for affine algebra $\hat{g}$, constructed by affinization of the first fundamental representation of $g$.

Let

$$\Phi_W : V_\lambda \otimes W \to V_\mu \quad \text{and} \quad \Psi_W : V_\lambda' \to W \otimes V_{\mu'}$$

be intertwining operators (that is $\Psi_W x = \Delta(x) \Psi_W$, $x \Phi_W = \Phi_W \Delta(x)$). We call them, as in [DJ2], vertex operators.

Then their composition

$$\Gamma = (\Phi_W \otimes id)(id \otimes \Psi_W) : V_\lambda \otimes V_{\lambda'} \to V_\mu \otimes V_{\mu'}$$

(2.5)

is an intertwining operator. Here $V_\lambda, V_{\lambda'}, V_\mu, V_{\mu'}$ are representations of $g$ with highest weights $\lambda, \lambda', \mu, \mu'$.

We can introduce the components of intertwining operators $\Phi_{W,i}$ and $\Psi_{W,i}$, where $w_i$ is a basis of $W$ as follows:

$$\Phi_{W,i}(v) = \Phi_W(v \otimes w_i),$$

$$\Psi_W(v) = \sum_i w_i \otimes \Psi_{W,i}(v).$$

Then $\Gamma = \sum_i \Phi_{W,i} \otimes \Psi_{W,i}$.

Example 1. Let $W$ be a vector representation of $gl_\infty$, $e_{i,j}(w_j) = w_i$ ($W$ could be realized in a space of free fermions $\psi_j$), $\Lambda_k$ be a representation of $gl_\infty$ in semiinfinite forms of charge $k$. Let

$$\Phi_W : \Lambda_k \otimes W \to \Lambda_{k+1}, \quad \Psi_W : \Lambda_k \to W \otimes \Lambda_{k-1}$$

be corresponding vertex operators. Their components $\Phi_{W,i}$ and $\Psi_{W,i}$ are known as operations of insertion and of removing a fermion $\psi_i$. If we apply tensor $\Gamma$, constructed by the relation (2.5) to a tensor product $\Lambda_0 \otimes \Lambda_0$:

$$\Gamma : \Lambda_0 \otimes \Lambda_0 \to \Lambda_1 \otimes \Lambda_{-1}$$

then the equations (2.4) turn to be Hirota relations [14] for KP hierarchy if we put, according to [LM],

$$u_3 = <v_1 | e^{H(\bar{\tau})}, \quad u_4 = <v_1 | e^{H(\bar{\tau})}, \quad v_1 = v_2 = |v_0>.$$  Moreover, following [UT], [DJ1], we can input an evolution of $\tau$ by "negative" times $y_j$ and define $\tau^a_n(\bar{\tau}, \bar{\varpi})$ as

$$\tau^a_n(\bar{\tau}, \bar{\varpi}) = <v_n | e^{H(\bar{\tau})}g e^{H(\bar{\varpi})} | v_n>$$

where $H'(\bar{\tau}) = \sum_{k \geq 1} u_k a_{-k}, a_{-k} = \sum_{j \in \mathbb{Z}} \psi_j \psi_j^{*},$ and $v_n$ is a vacuum of charge $n$.

Then the equation (1.4) for

$$\Gamma : \Lambda_n \otimes \Lambda_m \to \Lambda_{n+1} \otimes \Lambda_{m-1}$$

turns to Hirota relations

$$\sum_{j \in \mathbb{Z}} S_j(2\bar{\psi}) S_{j+n-m+1}(-\bar{\partial}_y) \tau^a_n(\bar{\tau} + \bar{\psi}, \bar{\varpi} + \bar{\psi}) \tau^a_m(\bar{\tau} - \bar{\psi}, \bar{\varpi} - \bar{\psi}) =$$

3
\[
\sum_{j \in \mathbb{Z}} S_{j+n-m+1}(-2\tau_j)S_j(\partial_{\bar{u}})r_{n+1}^g(\bar{\tau}, \bar{\gamma}_n, \bar{\gamma}_m, \bar{\gamma})r_{m-1}^g(\bar{\tau}, \bar{\gamma}, \bar{\gamma}_n, \bar{\gamma}) \tag{2.6}
\]

Technically for deriving (2.6) one needs only the commutation relations of vertex operators \( \Phi^W \) and \( \Psi^W \) with evolution operators \( e^{H(\bar{\tau})} \) and \( e^{H'(\bar{\tau})} \) and the action of \( \Phi_{W;i} \) and \( \Psi_{W;i} \) on highest weight vectors. If we put

\[ \Psi(z) = \sum_{i \in \mathbb{Z}} \Phi_{W,i} z^i, \quad \Psi^*(z) = \sum_{i \in \mathbb{Z}} \Psi_{W,i} z^{-i} \]

then these relations are standard ones [DJ1]:

\[ e^{H(\bar{\tau})} \Psi(z) = e^{\sum_{n \geq 1} n z^n} \Psi(z) e^{H(\bar{\tau})}, \quad e^{H'(\bar{\tau})} \Psi(z) = e^{\sum_{n \geq 1} n z^{-n}} \Psi(z) e^{H'(\bar{\tau})}, \]

\[ \Psi^*(z) e^{H(\bar{\tau})} = e^{\sum_{n \geq 1} n z^n} e^{H(\bar{\tau})} \Psi^*(z), \quad \Psi^*(z) e^{H'(\bar{\tau})} = e^{\sum_{n \geq 1} n z^{-n}} e^{H'(\bar{\tau})} \Psi^*(z), \]

where

\[ j(z) = \sum_{n \geq 1} a_n \frac{z^{-n}}{n}, \quad j'(z) = \sum_{n \geq 1} a_n \frac{z^n}{n} \]

For KdV and NLS hierarchies one should use level one irreducible representations \( \Lambda_0 \) and \( \Lambda_1 \) of \( \widehat{sl}_2 \) bosonized via correspondingly principal and homogenous Heisenberg subalgebras, and with ”vector” representation \( W_z \) being two-dimensional evaluation representation (in principal or homogeneous gradation),

\[ \Gamma = \oint \frac{dz}{z} \Phi_{W,a}(z) \otimes \Psi_{W,a}(z). \tag{2.7} \]

where \( a = \pm \) is an index of two-dimensional representation \( W_z \) of \( \widehat{sl}_2 \).

We should like to point out that in the case of affine algebra \( \widehat{g} \) vertex operators \( \Phi^W \) and \( \Psi^W \) are actually intertwining operators for algebra \( \widehat{g} \), where the full algebra \( \widehat{g} \) is obtained by adding grading element \( d \) to \( \widehat{g}' \). The integral in (2.7) arise since \( \Gamma \) should be intertwining operator for the full algebra \( \widehat{g} \).

**Example 2.** Using the technique of vertex operators we can describe Hirota presentation of finite-dimensional Toda lattice.

Let, for instance, \( g = A_1; \omega_1, \ldots, \omega_n \) be fundamental representations of \( A_n \), and

\[ \tau_n^g(\bar{\tau}, \bar{\gamma}) = \langle v_n | e^{H(\bar{\tau})} g e^{H'(\bar{\tau})} | v_n \rangle \]

where \( v_n \) is a highest weight vector of \( \omega_n \),

\[ H(x) = \sum_{k=1}^n x_k I_k, \quad H'(u) = \sum_{k=1}^n u_k I_{-k}, \]

\[ I_k = \sum_{i=1}^{n+1-k} e_{i,i+k}, \quad I_{-k} = \sum_{i=1}^{n+1-k} e_{i+k,i}, \]

\( e_{i,k} \in sl(n+1) \) are matrix units, \( W = \omega_1 \). If we use vertex operators

\[ \Phi_{\omega_1} : \omega_k \otimes \omega_1 \rightarrow \omega_{k+1}, \quad \Psi_{\omega_1} : \omega_k \rightarrow \omega_1 \otimes \omega_{k-1} \]

and, just as in example 1, introduce \( \Gamma : \omega_k \otimes \omega_m \rightarrow \omega_{k+1} \otimes \omega_{m-1} \) then we get the relations for \( \tau_n, \tau_m, \tau_{n+1} \) and \( \tau_{m-1} \) analogous to (2.4) (actually, higher times in (2.4) should be frozeed in appropriate way).
3 Noncommutative variant

The basic relation \((2.4)\) on matrix elements of representations could be rewritten as well for matrix elements of quantum deformation \(U_q(\mathfrak{g})\) of enveloping algebra \(U(\mathfrak{g})\), or, more generally, for matrix elements of representations of a Hopf algebra \(\mathcal{A}\). For this we should treat matrix elements as elements of dual Hopf algebra \(\mathcal{A}^*\) \((\mathcal{A}^* = \text{Fun}_q(\mathcal{A}))\).

Let us introduce some notations.

Let \(\mathcal{A}\) be a Hopf algebra, \(\mathcal{A}^*\) be a dual Hopf algebra and \(\{,\} : \mathcal{A}^* \otimes \mathcal{A} \to \mathbb{C}\) be canonical Hopf pairing. Let \(V\) be a representation of algebra \(\mathcal{A}\), \(v \in V, u \in V^*\). Define an element \(\tau_{u,v}\) by the relation

\[
\{\tau_{u,v}, x\} = <u | x | v>, \quad x \in \mathcal{A}
\]  

(3.1)

Let now \(V_1, V_2, V_3, V_4\) be representations of \(\mathcal{A}\), \(u_i \in V_i\), \(u_i \in V_i^*\) be some fixed vectors and \(\Gamma = \sum_i \Gamma_1^i \otimes \Gamma_2^i\) be an intertwining operator

\[
\Gamma : V_1 \otimes V_2 \to V_3 \otimes V_4
\]  

(3.2)

that is,

\[
\Gamma \Delta(x) = \Delta(x) \Gamma
\]  

(3.3)

**Proposition 3.1**

\[
\sum_i \tau_{u_3 \Gamma_1^i v_1} \tau_{u_4 \Gamma_2^i v_2} = \sum_i \tau_{u_3 \Gamma_1^i v_1} \tau_{u_4 \Gamma_2^i v_2}
\]  

(3.4)

**Proof.** We should prove that \(\{\text{LHS of (3.4)}, x\} = \{\text{RHS of (3.4)}, x\}\) for any \(x \in \mathcal{A}\). Let us remind that for a Hopf pairing \(\{,\}\) we have, in particular \(\{ab, c\} = \{a \otimes b, \Delta(c)\}\), or using the notation \(\Delta(c) = c^{(1)} \otimes c^{(2)}\), \(\{ab, c\} = \{a, c^{(1)}\}\{b, c^{(2)}\}\).

We have

\[
\sum_i \{\tau_{u_3 \Gamma_1^i v_1} \tau_{u_4 \Gamma_2^i v_2}, x\} = \sum_i \{\tau_{u_3 \Gamma_1^i v_1} \otimes \tau_{u_4 \Gamma_2^i v_2}, \Delta(x)\} =
\]

\[
= \sum_i < u_3 \Gamma_1^i x^{(1)} | v_1 > < u_4 \Gamma_2^i x^{(2)} | v_2 > = < u_3 \otimes u_4 \Gamma | \Delta(x) | v_1 \otimes v_2 > =
\]

\[
= < u_3 \otimes u_4 | \Delta(x) \Gamma | v_1 \otimes v_2 > = \sum_i < u_3 x^{(1)} | \Gamma_1^i v_1 > < u_4 x^{(2)} | \Gamma_2^i v_2 > =
\]

\[
= \sum_i \{\tau_{u_3, \Gamma_1^i v_1} \otimes \tau_{u_4, \Gamma_2^i v_2}, \Delta(x)\} = \sum_i \{\tau_{u_3, \Gamma_1^i v_1} \tau_{u_4, \Gamma_2^i v_2}, x\}.
\]

Let now \(g : \mathcal{A}^* \to \text{End} Y\) be a representation of an algebra \(\mathcal{A}^*\). Let us denote by \(\tau_{g, u,v}\) an operator \(g \tau_{u,v} \in \text{End} Y\). We have immediately the following generalization of Proposition 2.1

**Corollary 3.1** In the setting of Theorem 7.4 we have an equality

\[
\sum_i \tau_{g u \Gamma_1^i v_1} \tau_{g u \Gamma_2^i v_2} = \sum_i \tau_{g u \Gamma_1^i v_1} \tau_{g u \Gamma_2^i v_2}
\]  

(3.5)

In classical limit \(q = 1\) the Hopf pairing \(\{,\} : \text{Fun}(G) \otimes U(\mathfrak{g}) \to \mathbb{C}\) is given by the relation

\[
\{f, x\} = L_x f(e) \quad \text{for} \quad f(a) \in \text{Fun}(G), \quad a \in G.
\]  

(3.6)

where \(L_x\) is rightinvariant differential operator on \(G\) corresponding to element \(x \in U(\mathfrak{g})\). Comparing (3.6) with (3.1) we see that \(\tau_{u,v}\) can be identified with the following function of \(a \in G\): \(\tau_{u,v} = < u | a | v >\). A representation \(g\) of algebra \(\text{Fun}(G)\) is an evaluation of a function in a point \(g \in G\), that is

\[
\tau_{u,v}^g = \tau_{u,v}(g) = < u | g | v >.
\]
S is an antipode in of intertwining operator, Then \( W \) that it is important now that \( \Psi \) tensor \( \Gamma \). Let us look to the basic equation (3.4) in this particular case. Let presentations of vertex operators and of tensor \( \Gamma \) are actually equivalent to replacement of it is again an intertwining operator constructed from type I and type II intertwining operators. The two 

\[
\Gamma = (\Phi_W \otimes id)(id \otimes \Psi_W) : V_{\lambda} \otimes V_{\lambda'} \to V_{\mu} \otimes V_{\mu'}.
\]

Here \( V_{\lambda}, V_{\lambda'}, V_{\mu}, V_{\mu'} \) are representations of \( \mathcal{A} \); for \( \mathcal{A} = U_q(\mathfrak{g}) \) they are highest weight representations. Note that it is important now that \( \Psi_W \) creates \( W \) to the left and \( \Phi_W \) annihilate \( W \) from the right, that is, \( \Phi_W \) is type I vertex operator and \( \Psi_W \) is type II vertex operator in terminology of [DJ2], [IM].

Alternatively, we can use dual vertex operators

\[
\Phi_W : V_{\lambda} \to V_{\mu} \otimes W \quad \text{and} \quad \Psi_W : W \otimes V_{\lambda'} \to V_{\mu'}
\]

Then

\[
\Gamma = (id \otimes \Psi_W)(\Phi_W \otimes id) : V_{\lambda} \otimes V_{\lambda'} \to V_{\mu} \otimes V_{\mu'}
\]

is again an intertwining operator constructed from type I and type II intertwining operators. The two presentations of vertex operators and of tensor \( \Gamma \) are actually equivalent to replacement of \( W \) by \( W^*S \) where \( S \) is an antipode in \( \mathcal{A} \) due to canonical isomorphisms of the components of vertex operators [IM]:

\[
\Phi_{W, i} = \Phi_{W, s', i} : V_{\lambda} \to V_{\mu}, \quad \Psi_{W, i} = \Psi_{W, s', i} : V_{\lambda'} \to V_{\mu'}
\]

Here the dual bases \( \{w_i\} \) and \( \{w^{*i}\} \) of \( W \) and \( W^* \) are used; \( S' \) is an inverse to \( S \).

The defining commutation relations

\[
\Phi_W x = \Delta(x) \Phi_W, \quad x \Phi_W = \Phi_W \Delta(x), \quad (3.7)
\]

\[
\Psi_W x = \Delta(x) \Psi_W, \quad x \Psi_W = \Psi_W \Delta(x), \quad (3.8)
\]

could be rewritten also in terms of the components of the vertex operators (compare with [FR]). We summarize them in the following proposition.

**Proposition 3.2** The components of vertex operators satisfy the relations

\[
S(x^{(1)}) \Phi_{W, i} x^{(2)} = \sum_j \rho_i^j(x) \Phi_{W, j}, \quad S'(x^{(2)}) \Psi_{W, i} x^{(1)} = \sum_j \rho_i^j(x) \Psi_{W, j},
\]

\[
x^{(2)} \Phi_{W, i} S'(x^{(1)}) = \sum_j \rho_i^j(x) \Phi_{W, j}, \quad x^{(1)} \Psi_{W, i} S(x^{(2)}) = \sum_j \rho_i^j(x) \Psi_{W, j}.
\]

Here \( \rho_i^j \) is a matrix for the action of \( x \in \mathcal{A} \) in \( W \):

\[
x w_i = \sum_j \rho_i^j(x) w_j.
\]

If the Hopf algebra \( \mathcal{A} \) is quasitriangular, then the components of the vertex operators satisfy also the commutation relations of Zamolodchikov–Faddev algebra [ZF], [F], [IM]. We do not use them here.

Let now \( \mathcal{A} \) be a quantized universal algebra \( U_q(\mathfrak{g}) \). In this case an \( R \)-matrix gives one more example of tensor \( \Gamma \). Let us look to the basic equation (3.4) in this particular case. Let \( v_i \) be a basis of a representation \( V \) of \( U_q(\mathfrak{g}) \), \( u_i \) be a basis of a representation \( U \), \( < v_i, v_j >=< u_i, u_j >= \delta_{i, j} \), and \( \hat{R} : V \otimes U \to U \otimes V \) be an intertwining operator,

\[
\hat{R}(v_i \otimes u_j) = \sum_{a, b} R_{i, j}^{a, b} u_a \otimes v_b
\]
Denote \( T_i^j = \tau_{wi,ui} \), \( T_i^{ij} = \tau_{wi,ui} \), \( T = \{ T_i^j \} \), \( T' = \{ T'^{ij} \} \). Then the equation (3.4) means that

\[
R_{i,j}^{a,b} R_{b}^{k,l} T_i^{jl} = R_{c,d}^{k,l} T'_i T_j
\]

which is a traditional equation [FR1] for matrix elements

\[
RT_1 T'_2 = T'_2 T_1 R, \quad R = P\tilde{R}.
\]

Unfortunately, an \( R \)-matrix does not fit for deducing quantum Hirota-type equations; it is not built from local blocks and we cannot present in general a bosonized form for \( R \).

We would like to mention that the equations (3.4) with \( \Gamma \) being constructed from vertex operators for a fixed vector representation \( W \) give an alternative to (3.9) description of an algebra \( Fun_q(G) \) of matrix elements: there are no \( RTT \)-relations and we never permute matrices \( T \) and \( T' \), but, unlike to (3.9), where two representations of \( U_q(g) \) \( V \) and \( U \) are involved, we have four representations \( V_i \) which participate an equation (3.4). One may ask whether these equations describe completely an algebra \( Fun_q(G) \). Moreover, there are two variants of the question.

First, we can consider matrix elements of all integrable representations of \( U_q(g) \) with all the equations (3.4) built from vertex operators \( \Phi^W, \Phi_W, \Psi^W \) and \( \Psi_W \) for a given \( W \). On the other hand, we may restrict ourselves by matrix elements of fundamental representations and by vertex operators whose components act between them. In the case of \( g = A_n \) one can show that both algebras are isomorphic to \( Fun_q(G) \). It will be interesting to find a proof of analogous statement in general case.

4 An example of \( U_q(sl_2) \): \( q \)-difference Liouville equation

Here we present a toy example of \( U_q(sl_2) \) [KLMM]. An evolution in a representation of arbitrary spin \( j \) in this case can be defined via \( q \)-exponents of generators of \( U_q(sl_2) \). As a consequence of basic equations (3.4) we get noncommutative \( q \)-difference analog of Liouville equation and its generalization to arbitrary spin \( j \). Another approach is devided in [GS].

Let us fix the notations. We use the following description of \( U_q(sl_2) \):

\[
k_\alpha e_{\pm \alpha} = q^{\pm 1} e_{\pm \alpha} k_\alpha, \quad [e_\alpha, e_{-\alpha}] = \frac{k_\alpha - k_\alpha^{-1}}{q - q^{-1}}, \quad k_\alpha = q^{h_\alpha},
\]

\[
\Delta(e_\alpha) = e_\alpha \otimes 1 + k_\alpha^{-1} \otimes e_\alpha, \quad \Delta(e_{-\alpha}) = 1 \otimes e_{-\alpha} + e_{-\alpha} \otimes k_\alpha,
\]

\[
\exp_q(x) = 1 + x + \ldots + \frac{x^n}{(n)_q} + \ldots
\]

where

\[
(n)_q = \frac{q^n - 1}{q - 1} \quad \text{and} \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.
\]

For a representation \( V_j \) of spin \( j \) we define \( \tau_j(t,s) \in Fun_q(G) \) as

\[
\tau_j(t,s) = \tau_{<j|\exp_q^2(te_{\alpha})|\exp_{-2}(se_{-\alpha})|j>}.
\]

Equivalently, the Hopf pairing of \( \tau_j(t,s) \) with \( x \in U_q(g) \) is given by the relation

\[
\{ \tau_j(t,s), x \} = < j | \exp_{q^2}(te_{\alpha}) | x | \exp_{-2}(se_{-\alpha}) | j >\] 

where \( | j > \) is a highest weight vector, \( k_\alpha | j > = q^{j^2} | j > \). We denote by \( w_+ \) and \( w_- \) the standard basis of two-dimensional representation \( W \) of \( U_q(sl_2) \): \( e_\alpha w_- = w_+, e_{-\alpha} w_+ = w_- \) and by \( \Phi_\pm \) and \( \Psi_\pm \) the corresponding components of vertex operators \( \Phi_W \) and \( \Psi_W \):

\[
\Phi_\pm : V_j \rightarrow V_j', \quad \Psi_\pm : V_j \rightarrow V_j''\]

We can substitute \( \exp_{q^2}(te_{\alpha}) \) or \( \exp_{-2}(se_{-\alpha}) \) instead of \( x \) into defining relations

\[
x\Phi_W = \Phi_W \Delta(x), \quad \Psi_W x = \Delta(x)\Psi_W.
\]
The elements \( \exp_q(t e_\alpha) \) and \( \exp_{q^{-2}}(s e_\alpha) \) satisfy the following factorization properties with respect to comultiplication

\[
\Delta(\exp_q(t e_\alpha)) = \exp_q(k^{-1}_\alpha \otimes e_\alpha) \exp_q(t e_\alpha \otimes 1),
\]

\[
\Delta(\exp_{q^{-2}}(s e_\alpha)) = \exp_{q^{-2}}(1 \otimes s e_\alpha) \exp_{q^{-2}}(s e_\alpha \otimes k_\alpha).
\]

The relations (4.3)–(4.10) can be also deduced from Proposition 3.2.

The relations (4.1) and (4.2) follow from the addition theorem for \( \pm \) and \( \mp \).

As a result we get the following commutation relations of \( \Phi_\pm \) and \( \Psi_\pm \) with \( \exp_q(t e_\alpha) \) and \( \exp_{q^{-2}}(s e_\alpha) \):

\[
\exp_q(t e_\alpha) \Phi_+ = \Phi_+ \exp_q(t e_\alpha),
\]

\[
\exp_q(t e_\alpha) \Phi_- = (\Phi_+ k^{-1}_\alpha + \Phi_-) \exp_q(t e_\alpha),
\]

\[
\exp_{q^{-2}}(s e_\alpha) \Psi_+ = \Psi_+ \exp_{q^{-2}}(q s e_\alpha) - q t \Psi_- \exp_{q^{-2}}(q^{-1} s e_\alpha),
\]

\[
\exp_{q^{-2}}(s e_\alpha) \Psi_- = \Psi_- \exp_{q^{-2}}(q^{-1} s e_\alpha),
\]

\[
\Phi_+ \exp_{q^{-2}}(s e_\alpha) = \exp_{q^{-2}}(q^{-1} s e_\alpha) \Phi_+ - \exp_{q^{-2}}(q s e_\alpha) q^{-1} s \Phi_-.
\]

The relations (4.3)–(4.10) can be also deduced from Proposition 3.2.

We can use for instance \( \Phi_\pm : V_{j-1/2} \rightarrow V_j \) and \( \Psi_\pm : V_{j-1/2} \rightarrow V_j \) and substitute (4.3)–(4.10) into basic equation (3.4) for

\[
\Gamma = \Phi_+ \otimes \Psi_+ + \Phi_- \otimes \Psi_- : V_{j-1/2} \otimes V_{j-1/2} \rightarrow V_j \otimes V_j.
\]

Taking into account that

\[
\Phi_+ \mid j - 1/2 \geq j, \quad \Phi_- \mid j - 1/2 \geq q^{-2j+1} [2j]_q e_\alpha \mid j >,
\]

\[
\Psi_+ \mid j - 1/2 \geq -1, \quad \Psi_- \mid j - 1/2 \geq q^{-2j+1} [2j]_q e_\alpha \mid j >,
\]

\[
< j \mid \Phi_+ =< j - 1/2, \quad < j \mid \Phi_- =< j \mid \Psi_+ = 0, \quad < j \mid \Psi_- =< j - 1/2|
\]

and

\[
\exp_{q^{-2}}(s e_\alpha) e_\alpha = \partial_x(q^{-2}) \exp_{q^{-2}}(s e_\alpha),
\]

\[
c_\alpha \exp_q(t e_\alpha) = \partial_x(q^2) \exp_q(t e_\alpha)
\]

we get from (3.4) the following Hirota identities:

\[
\left( \partial_y(q^{y-2}) - q^{-2j} \partial_x(q^{x-2}) [2j]_q \right) + (q^{2j-2j-1} - q^{j-1} x) \partial_x(q^{y-2}) = \left( q^{2j-2j-1} - q^{j-1} x \right) \partial_x(q^{y-2})
\]

\[
\tau_j(u, x) \tau_j(v, y) = (v - q^{-2j} x) \tau_{j-1/2}(u, q^{-1} x) \tau_{j-1/2}(q^{-1} v, y)
\]

We can use \( q \)-difference analog of Taylor formula

\[
f(x) = f(a) + (x - a) \frac{\partial f}{\partial x} f(a) + \frac{(x - a)(x - qa)}{(2)_q!} \frac{\partial^2 f}{\partial x^2} f(a) + \ldots +
\]
and the equation (4.19) restricts a solution \( \tau \) which is Liouville equation in variable \( u,x \) asymmetry in \( \tau \). A general solution of equations (4.12), (4.15), (4.17)–(4.19).

\[
\frac{(x-a) \cdots (x-q^{n-1}a) \partial^q}{(n)_q!} \partial_x^n f(a) + \ldots +
\]

(4.13)

and expand both sides of equation (4.12) in a series

\[
\sum_{k,l \geq 0} P_{k,l}(x,u)(y-x')(y-q^{-2}x') \cdots (y-q^{-2(k-1)}x')(v-u')(v-q^{-2}u') \cdots (v-q^{-2(l-1)}u')
\]

(4.14)

where \( x' = xq^\alpha \), \( u' = uq^\beta \) for certain constants \( \alpha \) and \( \beta \) predicted by the form of the equation and get as a result a hierarchy of bilinear \( q \)-difference equations

\[
P_{k,l}(x,u) = 0
\]

(4.15)

For instance, in a particular case \( j = j' = \frac{1}{2} \) an equation (4.12) looks like

\[
\left(q\partial_y^{q^{-2}} - \partial_x^{q^{-2}} + (y-qx)\partial_y^{q^{-2}}\partial_x^{q^{-2}}\right)\tau_{1/2}(u,x)\tau_{1/2}(v,y) = qv - u
\]

(4.16)

Here we fix the normalization \( \tau_0 = 1 \).

By definition, an equation (4.15) has a solution

\[
\tau_{1/2}(u,x) = a + bu + cx + du
\]

where \( a = \tau_{<+,|+>} \), \( b = \tau_{<+,|->} \), \( c = \tau_{<-,|+>} \), \( d = \tau_{<-,|->} \). According to (3.9) the matrix

\[
T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

generates an algebra \( Fun_q(SL_2) \) and satisfy the relations [FRT]

\[
ab = q^{-1}ba, \quad ac = q^{-1}ca, \quad bd = q^{-1}db, \quad cd = q^{-1}dc, \quad bc = cb,
\]

\[
ad - q^{-1}bc = da - qbc = 1
\]

(4.17)

Note that the relation (4.17) gives only half of the relations (4.16). The rest of them one can deduce from the equation (3.4) for vertex operators \( \Phi_W^\pm : V_j \rightarrow V_{j-1/2}, \Phi_W^{1/2} : V_j \rightarrow V_{j-1/2} \), \( j = j' = 1/2 \).

An expansion (4.14) with \( \alpha = 1, \beta = -1 \) gives a hierarchy of equations with the first three listed below (for simplicity we write in these equations \( \tau \) instead of \( \tau_{1/2} \)).

\[
\tau(u,x) \cdot q^{q^{-2}} \tau(q^{-1}u,qx) = q^{q^{-2}} \tau(u,x) \cdot \tau(q^{-1}u,qx),
\]

(4.18)

\[
\tau(u,x) \cdot q^{q^{-2}} \tau(q^{-1}u,qx) - \partial_x^{q^{-2}} \tau(u,x) \cdot q^{q^{-2}} \tau(q^{-1}u,qx) = 1
\]

(4.19)

The first equation explains the commutation rule of \( \tau \) and its derivative, the second equation is a \( q \)-analog of Liouville equation and the third one should restrict a solution to a simple bilinear form on \( x \). Differentiation of the first equation enables one to rewrite (4.18) and (4.19) in different forms; so we see that there is no asymmetry in \( u, x, \tau \) and \( \partial^2 \tau \) in equations (4.18) and (4.19).

In the classical limit \( q = 1 \) \( \tau_{u,x} \) is commutative and the equation (4.17) is trivial; the equation (4.18) is

\[
\tau \partial_{ux}^2 \tau - \partial_u \tau \partial_x \tau = 1
\]

which is Liouville equation in variable \( \phi, \tau = e^\phi \):

\[
\phi_{xu} = e^\phi
\]

and the equation (4.19) restricts a solution \( \tau \) to a linear form of \( x \).

**Remarks.**

1. Let us note once more that we treat \( \tau \)-function as an element of algebra \( Fun_q(G) \). In representation \( q \) of \( Fun_q(G) \) \( \tau_{u,x} \) turns to be an operator-valued function being a solution of corresponding equations (4.12), (4.15), (4.17), (4.19).

2. A general solution \( \tau(u,x) \) of the equations (4.12) is a generating function of matrix coefficients of matrix elements of a representation of \( U_q(sl_2) \) of spin \( j \). It could be obtained directly from the universal \( T \)-matrix (see the next section).
5 The universal $T$-matrix and group-like elements

We can see that the ideology of intertwining operators allows one to write down a hierarchy of differential (or $q$-difference) equations provided there is known common bosonization of fundamental representations and of vertex operators $\Phi_i^W$, $\Phi_i^K$, $\Psi_i^W$ and $\Psi_i^K$, acting between them. The construction of such a bosonization is the main problem for the case of $U_q(\mathfrak{g})$.

However we can treat $\tau$-function as certain generating function of matrix elements and work directly with matrix elements. We have then the basic relation \([3.4]\) defining the connection between matrix elements of different representations. There is also a canonical way for producing matrix elements provided the structure of representation is known. Let us restrict ourselves to a case of $\mathcal{A} = U_q(\mathfrak{g})$ where $\mathfrak{g}$ is simple finite-dimensional Lie algebra.

Let $H = \mathcal{A} \otimes \mathcal{A}^* = U_q(\mathfrak{g}) \otimes \text{Fun}_q(G)$ and $T$ is a canonical tensor of a Hopf pairing of $\mathcal{A}$ and $\mathcal{A}^*$:

$$T = \sum e_i \otimes e^i, \quad \{e_i, e^j\} = \delta_{i,j}.$$  

$T$ is usually called as universal $T$-matrix \([\text{FRT}], \text{FG}\). The properties of Hopf pairing in terms of $T$ mean that

\[
\begin{align*}
(\Delta \otimes \text{id}) T &= T^{13}T^{23}, \\
(\text{id} \otimes \Delta) T &= T^{12}T^{13}, \\
(S \otimes \text{id}) T &= T^{-1}, \\
(\text{id} \otimes S) T &= T^{-1}.
\end{align*}
\]

Let $\rho : \mathcal{A} \to \text{End} V$ be a representation of $\mathcal{A} = U_q(\mathfrak{g})$ in a space $V$ with basis $\{v_i\}$. Then matrix elements $<v^j | (\rho \otimes \text{id}) T | v_i>$ are, by definition of $T$, the elements $\tau_{v^j, v_i}$ of an algebra $\mathcal{A}^* = \text{Fun}_q(G)$:

$$\tau_{v^j, v_i} = <v^j | (\rho \otimes \text{id}) T | v_i>. $$

Further, for a representation $g: \mathcal{A}^* = \text{Fun}_q(G) \to \text{End U}$ an operator $\tau^g_{v^j, v_i}$ can be also expressed in terms of $T$:

$$\tau^g_{v^j, v_i} = <v^j | (\rho \otimes g) T | v_i>. $$

The universal $T$-matrix can be written in factorized form (see \([\text{FG}]\) for $U_q(\mathfrak{gl}_n)$ case):

$$T = \prod_{\gamma \in \Delta_+} \exp_{q^{-\gamma, \gamma}} \gamma e_\gamma \otimes s_\gamma \cdot q^{d_{i,j} h_i \otimes h_j} \cdot \prod_{\gamma \in \Delta_+} \exp_{q^{\gamma, \gamma}} -a(\gamma) e_{-\gamma} \otimes s_{-\gamma}.$$  

(5.5)

Here $e_{\pm\gamma} \in U_q(\mathfrak{b}_{\pm})$ are Cartan–Weyl generators of $U_q(\mathfrak{g})$ constructed for a fixed reduced decomposition of the longest element $w_0$ of $q$-Weyl group or, equivalently, for a fixed normal ordering of a system of positive roots $\Delta_+$; $a(\gamma)$ are the constant coefficients normalizing the relation

$$[e_\gamma, e_{-\gamma}] = \frac{k_\gamma - k_\gamma^{-1}}{a(\gamma)}$$

and $d_{i,j}$ is an inverse matrix to symmetrized Cartan matrix of $\mathfrak{g}$. $U_q(\mathfrak{b}_{\pm})$ are Borel subalgebras of $U_q(\mathfrak{g})$.

The elements $s_\gamma, \gamma \in \Delta_+$ satisfy the same commutation relations as $e_{-\gamma} \in U_q(\mathfrak{b}_{-})$, the elements $s_{-\gamma}, \gamma \in \Delta_+$ satisfy the same commutation relations as $e_\gamma \in U_q(\mathfrak{b}_{+})$, $q^{h_i s_\gamma} = q^{(\alpha_i, \gamma)} s_\gamma q^{h_i}$, $q^{h_i s_{-\gamma}} = q^{(\alpha_i, -\gamma)} s_{-\gamma} q^{h_i}$ and $[s_\gamma, s_{-\gamma}] = 0$ for any $\gamma, \delta \in \Delta_+$.

An expression \([5.5]\) can be deduced from analogous expression for the universal $R$-matrix for $U_q(\mathfrak{g})$ \([\text{KR}], \text{KI}], \text{ES}\).

To show this we consider the double $\mathcal{D} = D(U_q(\mathfrak{g}))$ of $U_q(\mathfrak{g})$. The universal $R$-matrix for $\mathcal{D}$ belongs to $(U_q(\mathfrak{g}) \otimes 1) \otimes (1 \otimes \text{Fun}_q(G))$ where $0$ denotes an opposite comultiplication and enjoys the properties:

$$\Delta'(x) R = R \Delta(x)$$  

(5.6)

$$\Delta \otimes \text{id} R = R^{13} R^{23},$$  

(5.7)
(id \otimes \Delta)R = R^{13} R^{12}, \quad (5.8)
(S \otimes id)R = R^{-1}, \quad (5.9)
(id \otimes S')R = R^{-1}. \quad (5.10)

We are not interested here in the property \([5.6]\) and so have no need in commutation relations between \(U_q(\mathfrak{g})\) and \(Fun_q^0(G)\) inside \(D(U_q(\mathfrak{g}))\). In this setting we can identify \(U_q(\mathfrak{g})\) with \((U_q \mathfrak{g} \otimes 1)\) and \(Fun_q(G)\) with \((1 \otimes Fun_q^0(G))\) returning to original comultiplication. Under these identifications the universal \(R\)-matrix for \(D(U_q(\mathfrak{g}))\) coincides with universal \(T\)-matrix \(T\), the properties \([5.7]\)–\([5.10]\) correspond to \([5.1]\)–\([5.4]\).

We reconstruct the universal \(R\)-matrix for \(D(U_q(\mathfrak{g}))\) from the universal \(R\)-matrix for \(U_q(\mathfrak{g})\) using the isomorphisms of algebras

\[ Fun_q(G) \simeq U_q(b_+) \otimes U(h) \ U_q(b_-) \quad (5.11) \]

where \(U_q(b_\pm)\) are Borel subalgebras of \(U_q(\mathfrak{g})\) and \(U(h)\) is Cartan subalgebra.

Let \(R_U\) be the universal \(R\)-matrix for \(U_q(\mathfrak{g})\). In the same notations it has a factorized form \([KR], [LS], [KT]\).

\[ R_U = R \cdot K, \]

where

\[ R = \prod_{\gamma \in \Delta_+} \exp_q(-\gamma, \gamma) a(\gamma) e_\gamma \otimes e_{-\gamma}, \quad K = q^{d_{i,j} h_i \otimes h_j}; \]

(5.12)

The tensor \(R_U\) defines a pairing of \(U_q(b_+)\) with \(U_q(b_-)\).

We can rewrite its component \(R\) as

\[ R = \prod_{\gamma \in \Delta_+} \exp_q(-\gamma, \gamma) a(\gamma) e_\gamma \otimes s_\gamma \]

(5.13)

where \(s_\gamma = e_{-\gamma}\) is dual to \(a(\gamma)e_\gamma\) in \(U_q(b_-)\). Analogously, \((R_U^{21})^{-1}\) is again the universal \(R\)-matrix for \(U_q(\mathfrak{g})\).

It looks like

\[ (R_U^{21})^{-1} = K^{-1} \overline{R} \]

where

\[ \overline{R} = \prod_{\gamma \in \Delta_+} \exp_q(\gamma, \gamma) - a(\gamma)e_{-\gamma} \otimes s_{-\gamma} \]

with \(s_{-\gamma} = e_\gamma\) being dual to \(-a(\gamma)e_{-\gamma}\) in \(U_q(b_+)\).

The decomposition \([5.11]\) and the fact, that \(U_q(b_\pm)\) are Hopf subalgebras of \(U_q(\mathfrak{g})\) mean that we can compose the pairing of \(U_q(\mathfrak{g})\) with \(Fun_q(G)\) from \(R_U\) and \((R_U^{21})^{-1}\) after proper factorization over \(U(h)\), in other words,

\[ T = RK \overline{R}, \]

and we get the expression\([5.3]\).

Since \(s_{\pm,\gamma}\) and \(h_j\) are supposed to commute with \(U_q(\mathfrak{g})\) we can treat \(T\) as a product of \(q\)-exponents of \(e_{\pm,\gamma}\) with parameters \(s_{\pm,\gamma}\) and \(h_j\):

\[ T(s_{\pm,\gamma}, h_j) = \prod_{\gamma \in \Delta_+} \exp_q(-\gamma, \gamma) e_{\gamma} s_{\gamma} \cdot q^{d_{i,j} h_i \otimes h_j} \cdot \prod_{\gamma \in \Delta_+} \exp_q(\gamma, \gamma) e_{-\gamma} s_{-\gamma} \quad (5.14) \]

The property \([5.3]\) applied to a representation \(\rho \otimes \rho'\) of \(U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})\) means that

\[ \rho \otimes \rho' \Delta T(s_{\pm,\gamma}, h_j) = \rho(T(s_{\pm,\gamma}, h_j)) \otimes \rho'T(s_{\pm,\gamma}, h_j). \]

It could be treated as ”group-like” property of \(T\). The property \([5.2]\) means that

\[ T(s_{\pm,\gamma}^1, h_j^1)T(s_{\pm,\gamma}^2, h_j^2) = T(s_{\pm,\gamma}^3, h_j^3) \]

\[ T = R, \quad N \]

\[ N = \prod_{\gamma \in \Delta_+} \exp_q(-\gamma, \gamma) a(\gamma) e_\gamma \otimes e_{-\gamma}, \quad (5.12) \]

\[ N = \prod_{\gamma \in \Delta_+} \exp_q(\gamma, \gamma) a(\gamma) e_{-\gamma} \otimes e_{\gamma}, \quad (5.13) \]

\[ N = \prod_{\gamma \in \Delta_+} \exp_q(\gamma, \gamma) a(\gamma) e_\gamma \otimes e_{-\gamma}, \quad (5.14) \]

\[ N = \prod_{\gamma \in \Delta_+} \exp_q(-\gamma, \gamma) a(\gamma) e_{-\gamma} \otimes e_{\gamma}, \quad (5.15) \]
for mutually commuting parameters \{ s_{1,\gamma}, \hat{h}_{1}\}, \{ s_{2,\gamma}, \hat{h}_{2}\}; where parameters \{s_{3,\gamma}, \hat{h}_{3}\} satisfy the same commutation relations as \{s_{1,\gamma}, \hat{h}_{1}\} and \{s_{2,\gamma}, \hat{h}_{2}\} by themselves. Analogously, the last property \( [5.4] \) means that \( T^{-1}(s_{\pm,\gamma}, \hat{h}_{j}) \) is group-like element for an opposite comultiplication \( \Delta' \). This reformulation gives a hint to treat \( T \) in \( [5.14] \) as a \( q \)-analogue of a parametrization of Chevalley group \( S \). It is possible to try to follow the games which usually people play with Chevalley groups: One can define this “group” as a group generated by \( q \)-exponents of simple root vectors. Such a parametrization was presented in \( [MV] \) in a pretty nice form. It is also possible to write down the defining relations for the generators of this “group”. The relations turn to be \( q \)-analog of classical Chevalley relations. First they appear in \( [KT] \), eq. (5.10)-(5.12) and then in \( [MV] \) in the context of universal \( T \)-matrix.

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