MAPPINGS OF MULTIMODE BOSE ALGEBRA PRESERVING NUMBER OPERATORS

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Abstract

We define a class of deformed multimode oscillator algebras which possess number operators and can be mapped to multimode Bose algebra. We construct and discuss the states in the Fock space and the corresponding number operators.
Recently, much interest has been devoted to the study of quantum groups \[1\] and to generalizations (deformations) of oscillator algebras \[2\][3][4][5][6]. Any type of single-mode deformed oscillator can be related (mapped) to a single-mode Bose oscillator. Deformations of multimode oscillators have also been studied \[7\]. For Pusz-Woronowicz oscillators covariant under the $SU_q(n)(SU_q(n|m))$ quantum algebra (superalgebra) \[2][7\] and anyonic-type algebras \[8\] there exist mappings to multimode Bose algebra. However, there are deformed generalized quon algebras \[9\] for which such mappings do not exist. It is essential that all these algebras are associative and the R-matrix approach to them has been pursued \[10\]. Multimode deformed oscillator algebras are important in possible physical applications, for example in q-deformed field theory \[8\] and generalized statistics \[12\].

Our aim is to define and analyze a general class of deformed multimode oscillator algebras which possess well-defined number operators for each type of oscillator, and can be mapped to a multimode Bose algebra. These mappings can be viewed as generalized Jordan-Wigner/Klein transformations. We discuss the corresponding algebras. For special values of parameters, one recovers all known such algebras \[2\][3][5][6][7][8][10\]. Finally, we construct and discuss the states in the Fock space and the corresponding number operators for deformed oscillators.

Let us define the annihilation and creation operators $b_i, b_i^+$, ($i \in S$) satisfying the Bose algebra

\[
[b_i, b_j^+] = \delta_{ij}, \quad \forall i, j \in S \\
[b_i, b_j] = 0
\]

(1)

where $S = 1, 2, \ldots n$ or S is a set of sites on a lattice. The number operators $N_i$
satisfy
\[ [N_i, b_j] = -b_i \delta_{ij}, \quad \forall i, j \in S \]
\[ [N_i, b_j^+] = b_i^+ \delta_{ij} \]  \hspace{1cm} (2)
\[ N_i = b_i^+ b_i, \quad \forall i, j \in S. \]

Now we define generalized Jordan-Wigner transformations of the above Bose algebra, equations (1) and (2), as

\[ a_i = b_i e^{\sum c_{ij} N_j} \sqrt{\frac{\varphi_i(N_i)}{N_i}} \]  \hspace{1cm} (3)

where \( c_{ij} \) are complex numbers and \( \varphi_i(N_i) \) is an arbitrary (complex) function with \( \varphi_i(0) = 0, \lim_{\epsilon \to 0} \frac{\varphi_i(\epsilon)}{\epsilon} = 1, |\varphi_i(1)| = 1, \forall i \in S \). We assume that \( |\varphi_i(N)| \) are bijective, monotonically increasing functions or that \( \varphi_i(N) = \frac{1-(-1)^N}{2} \), implying \( a_i^2 = 0 \). The mappings (3) generalize the mappings considered in [7][10].

It is important to note that the number operators are preserved, i.e.

\[ N_i^{(a)} = N_i^{(b)} \equiv N_i, \quad \forall i \in S. \]  \hspace{1cm} (4)

Then it is easy to find the corresponding deformed algebra:

\[ a_i a_j = e^{c_{ji}-c_{ij}} a_j a_i, \quad i \neq j \]
\[ a_i a_j^+ = e^{c_{ji}+c_{ij}} a_j^+ a_i, \quad i \neq j \]
\[ a_i a_i^+ = |\varphi_i(N_i + 1)| e^{\sum (c_{ij}+c_{ji}) N_j} e^{(c_{ii}+c_{ii}^*)} \]
\[ a_i^+ a_i = |\varphi_i(N_i)| e^{\sum (c_{ij}+c_{ji}) N_j}. \]  \hspace{1cm} (5)

Note that \( a_i^2 \neq 0 \), unless \( b_i \varphi_i(N_i) b_i \varphi_i(N_i) = 0 \). For example, if \( \varphi_i(N_i) = \frac{1-(-1)^N}{2} \), then \( a_i^2 = 0 \), implying the hard-core condition for the \( i^{th} \) oscillator. Generally, there are other mappings of Bose algebra, equations (1) and (2), but, in general,
they do not have the number operators \(N_i^{(a)}\), and equation (4) does not hold for other mappings than those in equation (3).

We point out that the complete deformed algebra is associative owing to the mapping of Bose algebra. The Fock space for the deformed algebra is spanned by powers of the creation operators \(a_i^+, i \in S\), acting on the vacuum \(|0>^{(a)} = |0>^{(b)} \equiv |0>\). The states in the Fock space are specified by the eigenvalues of the number operators \(N_i\), namely \(|n_1, n_2, ..., n_i, ..., >^{(a)} = |n_1, n_2, ..., n_i, ..., >^{(b)}\). (If there exists a number \(n_i^{(0)} \in \mathbb{N}\), such that \(\varphi_i(n_i^{(0)}) = 0\), then \(N_i = 0, 1, ...(n_i^{(0)} - 1)\)).

States with unit norm are \(|n_1, ..., n_n> = \frac{(a_i^+)_{n_i}^{(a)इ} \cdots (a_i^+)_{n_n}^{(a)इ}}{\sqrt{[\varphi_1(n_1)]! \cdots [\varphi_n(n_n)]!}} e^{-\frac{1}{2} \sum_j \theta_{ji} (c_{ij} + c_{ij}^*) n_i n_j} |0, 0, ..., 0>\)

\[\varphi_i(n_i) = |\varphi_i(n_i)| e^{(c_{ii} + c_{ii}^*) n_i}.\]

where \(\theta_{ij}\) is the step function. (For anyons in \((2 + 1)\) dimension \[8\], \(\theta\) is the angle function.)

Furthermore, the matrix elements of the operators \(a_i, a_i^+, i \in S\), are

\[<... (n_i - 1) ... | a_i | ... n_i > = <... n_i ... | a_i^+ | ... (n_i - 1) ... >^* = \sqrt{\varphi_i(n_i)} e^{\frac{1}{2} \sum_j (c_{ij} + c_{ij}^*) n_j} \prod_{j \neq i} \delta_{n_j, n_j'}\]

We also find for any \(k = 0, 1, 2, ...\) that

\[ (a_i^+)^k(a_j)^k = \frac{[\varphi_j(N_j)]!}{[\varphi_j(N_j - k)]!} e^{k \sum_{(c_{ij} + c_{ij}^*) N_i}} \]

\[ (a_j)^k(a_j^+)^k = \frac{[\varphi_j(N_j + k)]!}{[\varphi_j(N_j)]!} e^{k \sum_{(c_{ij} + c_{ij}^*) N_i}} \]

The norms of arbitrary linear combinations of the states in equation (6) in the Fock space corresponding to deformed algebra, are positive definite owing to the mapping.
of Bose algebra, equations (1) and (2). Namely, $|n_1, n_2, \ldots n_i, \ldots \rangle^{(a)} = |n_1, n_2, \ldots n_i, \ldots \rangle^{(b)} \equiv |n_1, n_2, \ldots n_i, \ldots \rangle$.

This class of deformed multimode oscillator algebras comprises multimode Biedenharn-Macfarlaine [3], Aric-Coon [6], two-(p,q) parameter [11], Fermi, generalized Green’s [12][13], as well as anyonic [8] and Pusz-Woronowicz (with or without the hard-core condition) oscillators covariant under the $SU_q(n)$ ($SU_q(n|m)$) algebra (superalgebra) [7].

Non-isomorphic (non-equivalent) algebras are classified by different matrix elements given by (7), i.e. with the functions $g_i(n_1, n_2, \ldots n_n) = |\varphi_i(n_i)|e^{\sum_j(c_{ij} + c_{ij}^*)n_j}$. It is important to mention that there are mappings of Bose algebra which do not preserve the relation $N_i^{(a)} = N_i^{(b)}$, given by (4). Moreover, there are mappings of Bose algebra for which the number operators $N_i^{(a)}$ do not even exist. Such an example is the exchange algebra presented in [14].

Finally, we construct and discuss the number operators $N_i^{(a)}$. Let $\varphi_i(n_i)$ be bijective mappings. Then, using equation (3) we have

$$b_i = a_i e^{-\sum_j c_{ij} N_j \sqrt{N_{\varphi_i(n_i)}}}, \quad \forall i \in S$$

$$N_i = b_i^+ b_i.$$  \hspace{1cm} (9)

The spectra of $N_i^{(a)}$ and $N_i^{(b)}$ coincide. In this case,

$$a_i^+ a_i = g_i(N_1, N_2, \ldots N_i, \ldots) = |\bar{\varphi}_i(N_i)|e^{\sum_{j \neq i} (c_{ij} + c_{ij}^*)N_j}.$$  \hspace{1cm} (10)

Let us denote $a_i^+ a_i = x_i$, and remark that $x_i$ commute with any $N_j$ and among
themselves. Then the \( N_i \) operators can be written as
\[
N_i = N_i(x_1, \ldots x_n) = \sum_{k=0}^{\infty} \sum_{(i_1 \ldots i_k)} \frac{1}{k!} \left( \frac{\partial^k N_i}{\partial x_{i_1} \ldots \partial x_{i_k}} \right) x_0(x_{i_1} x_{i_2} \ldots x_{i_k})
\]
(11)
\[N_i(0, 0, \ldots 0) = 0, \quad \forall i \in S.\]

The coefficients in the Taylor expansion can be obtained from (10)
\[
g_i(N_1, N_2, \ldots N_i, \ldots) = x_i, \quad \forall i \in S
\]
(12)
namely from
\[
\left( \frac{\partial^k g_i(N_1, N_2, \ldots N_i, \ldots)}{\partial x_{j_1} \ldots \partial x_{j_k}} \right)_{x=0} = \delta_{k1}\delta_{ji}.
\]
(13)

These equations give a set of recurrence relations for the coefficients in the Taylor expansion of \( N_i \) as a function of the variables \( x_j, \ j \in S \). For example,
\[
c^{(i)}_0 \equiv (N_i)_{x=0} = 0
\]
\[
c^{(i)}_j \equiv \left( \frac{\partial N_i}{\partial x_j} \right)_{x=0} = \frac{1}{(\frac{\partial g_i}{\partial N_i})_0} \delta_{ij}
\]
\[
c^{(i)}_{jk} \equiv \frac{1}{2} \left( \frac{\partial^2 N_i}{\partial x_j \partial x_k} \right)_{x=0} = -\frac{1}{2} \left( \frac{\partial^2 g_i}{\partial x_j \partial x_k} \right)_{x=0} c^{(i)}_j c^{(j)}_k
\]
(14)

Specially
\[
c^{(i)}_j = \frac{1}{(\frac{\partial g_i}{\partial N_i})_0} \delta_{ij}
\]
\[
c^{(i)}_{ij} = -\frac{1}{2} (c_{ij} + c^{*}_{ji}) c^{(i)}_i c^{(j)}_j
\]
\[
c^{(i)}_{jk} = 0, \quad j \neq i, \quad k \neq i
\]
(15)

where \( g_i(N_1, N_2, \ldots N_i, \ldots) \) is given by equation (10). We also mention the following result.

If there exists a continuous mapping defined by (3) from Bose oscillators to deformed oscillators, then the number operators exist and can be written in the form
\[
N_i = x_i[c^{(i)}_i + \sum_j c^{(i)}_j x_j + \ldots]
\]
(16)
where $x_j = a_j^+ a_j$.

For example, let us consider the n-mode Pusz-Woronowicz oscillator algebra (of Bose type, covariant under the SU_q(n) quantum algebra), $q \in \mathbb{R}$:

\[
a_i a_j = q^{\text{sgn}(j-i)} a_j a_i
\]

\[
a_i a_j^+ = q a_j^+ a_i, \quad i \neq j
\]

\[
a_i a_i^+ = 1 + (q^2 - 1) \sum_j \theta_{ij} a_j^+ a_j + q^2 a_i^+ a_i.
\] (17)

There exist mappings to the Bose algebra (1) and (2) with $c_{ij} = \theta_{ij} \ln q$, $\varphi(N) = \frac{2^N-1}{q^2-1}$ (see Table 1). Hence, we can calculate the coefficients in the expansion (16). Using equation (15) we obtain

\[
c^{(i)}_j = \frac{q^2-1}{\ln q} \delta_{ij}
\]

\[
c_{ij}^{(i)} = -\frac{1}{2} \left( \frac{q^2-1}{\ln q} \right)^2 (\delta_{ij} + \theta_{ij}).
\] (18)

When $q^2 \to 1$, then $c^{(i)}_i \to 1$, whereas $c_{ij}^{(i)} \to 0$, etc., and $N_i \to a_i^+ a_i$. However, when $q^2 \to 0$, all coefficients diverge and the expansion (16) is not valid. Note that when $q = 0$, the mapping (9) becomes singular, but one can still define the corresponding number operators. For $q = 0$, the Pusz-Woronowicz algebra (17) reduces to

\[
a_i a_j = 0, \quad i < j
\]

\[
a_i a_j^+ = 0, \quad i \neq j
\]

\[
a_i a_i^+ = 1 - \sum_j \theta_{ij} a_j^+ a_j.
\] (19)

Therefore we also present the number operators in another form that holds for an arbitrary algebra having number operators, and holds even for $q = 0$ (12):

\[
N_i = a_i^+ a_i + \sum_{k=1}^{\infty} \sum_{\pi \in S_k} \sum_{j_1...j_k} d_{\pi(j_1...j_k),j_1...j_k} a_{\pi(j_k)}^+ a_{\pi(j_k)}^+ a_i a_j^+ a_i a_j^+ a_j...
\] (20)
where $S_k$ denotes the permutation group and $d_{\pi(j_1...j_k),j_1...j_k}$ are the coefficients of the expansion.

For the Pusz-Woronowicz algebra the number operators in the above form (see also [7]) are given by

$$N_i = \sum_{k=0}^{\infty} \sum_{j=0}^{k} c_{k,j} \sum_{p_1,...,p_j} \theta_{ip_1}..., \theta_{ip_j} a_{p_j}^+ a_{p_1}^+ (a_i^+)^{k-j+1} (a_i)_{k-j+1}^1 a_{p_1}...a_{p_j}$$

(21)

with the conditions $c_{0,0} = 1, c_{0,-j} = 0, j > 0,$ and with the recurrence relations

$$c_{k+1,k+1-j} = \frac{(1-q^2)(1-q^{2j})}{1-q^{2(j+1)}} c_{k,k+1-j} + q^{2j}(1-q^2)c_{k,k-j}.$$ 

(22)

Starting with $c_{0,0} = 1,$ we find that

$$c_{k,1} = c_{k,k} = (1-q^2)^k$$

$$c_{k,0} = \frac{(1-q^2)^{k+1}}{1-q^{2(k+1)}}.$$ 

(23)

In the limit $q^2 \to 1,$ the number operator is $N_i = a_i^+ a_i.$ In the limit $q = 0,$ all coefficients are $c_{k,j} = 1, \forall k, j.$ This result is similar to that found by Greenberg [9] for quons with $q = 0.$ Finally, for $\varphi_i(N_i) = \frac{1}{2}(-)^{N_i}$, the number operators, $N_i^{(a)} \neq N_i^{(b)},$ are simply

$$N_i^{(a)} = a_i^+ a_i = b_i^+ b_i[1 - \theta(n_i - 1)].$$

(24)

We conclude that we unify and generalize the results for the states in the Fock space and the number operators for the mappings of multimode Bose algebra presented in Table 1.
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Table 1: Parameters $c_{ij}$ and $\varphi_i(N_i)$ for the various deformed algebras. We use definition $[x]_p = \frac{p^x - 1}{p - 1}$.

| type of oscillators                          | $c_{ij}$                                                                 | $\varphi_i(N_i)$ |
|---------------------------------------------|-------------------------------------------------------------------------|------------------|
| Bose                                        | $i\pi \lambda_{ij}$, $\lambda_{ij} - \lambda_{ji} \in 2\mathbb{Z}$, $\forall i, j$ | $N_i$            |
| Green’s oscillators [12][13]                | $i\pi \lambda_{ij}$, $\lambda_{ij} - \lambda_{ji} \in \mathbb{Z}$, $\forall i, j$ | $[N_i]_{-1}$    |
| Anyonic-type [14]                           | $i\pi \lambda_{ij}$, $\lambda_{ij} - \lambda_{ji} \in \mathbb{R}$, $\forall i, j$ | $[N_i]_{\pm 1}$ |
| Anyons [8]                                  | $i\lambda \theta_{ij}$, $\lambda \in \mathbb{R}$, $\theta_{ij}$ is angle | $[N_i]_{\pm 1}$ |
| Pusz-Woronowicz (Bose) [2]                  | $\theta_{ij} \ln q$, $q \in \mathbb{R}$                               | $[N_i]_{q^2}$    |
| Pusz-Woronowicz (Fermi) [2]                 | $\theta_{ij} \ln(-q)$, $q \in \mathbb{R}$                            | $[N_i]_{-1}$     |
| $SU_q(n|m)$-covariant (Bose) oscillators [7] | $\theta_{ij} \ln q$, $q \in \mathbb{R}$, $i \leq n$                  | $[N_i]_{q^2}$    |
| $SU_q(n|m)$-covariant (Fermi) oscillators [7]| $\theta_{ij} \ln(-q)$, $q \in \mathbb{R}$, $n + 1 \leq i$             | $[N_i]_{-1}$     |
| Biedenharn-Macfarlane [3]                   | $-\frac{1}{2} \delta_{ij} \ln q_i$, $q_i \in \mathbb{C}$             | $[N_i]_{q_i}$    |
| Arik-Coon [4]                               | $0$, $q_i \in \mathbb{C}$                                            | $[N_i]_{q_i}$    |