Quantum Mechanics as a Classical Field Theory

A. C. de la Torre and A. Daleo
Departamento de Física, Universidad Nacional de Mar del Plata
Funes 3350, 7600 Mar del Plata, Argentina
dltorre@mdp.edu.ar
(April 1, 2022)

Abstract

The formalism of quantum mechanics is presented in a way that its interpretation as a classical field theory is emphasized. Two coupled real fields are defined with given equations of motion. Densities and currents associated to the fields are found with their corresponding conserved quantities. The behavior of these quantities under a galilean transformation suggest the association of the fields with a quantum mechanical free particle. An external potential is introduced in the Lagrange formalism. The description is equivalent to the conventional Schrödinger equation treatment of a particle. We discuss the attempts to build an interpretation of quantum mechanics based on this scheme. The fields become the primary ontology of the theory and the particles appear as emergent properties of the fields. These interpretations face serious problems for systems with many degrees of freedom.

KEY WORDS: quantum mechanics, field theory, interpretation.

PACS 03.65.Bz 03.65.Ca
I. INTRODUCTION

After almost one century that Planck and Einstein made the first quantum postulates [1,2] and after 70 years that the mathematical formalism of quantum mechanics was established [3], the challenge posed by quantum mechanics is still open. Until now, no completely satisfactory interpretation of quantum mechanics has been found and we can still say today that “nobody understands quantum mechanics” [4]. The lack of interpretation was compensated by the development of an extremely precise and esthetic mathematical formalism; we do not know what quantum mechanics is but we know very well how it works. The development of the very successful axiomatic formalism had the consequence that many physicists where satisfied with the working of quantum mechanics and did no longer tried to understand it. This attitude was favored by the establishment of an orthodox instrumentalist “interpretation” which, if we are allowed to put it in a somewhat oversimplified manner, amounts to say “thou shall not try to understand quantum mechanics”. Only a few authorities like Einstein, Schrödinger, Planck, could dare not to accept the dogma and insist in trying to understand quantum mechanics [5]. Fortunately the situation has changed and today it is an acceptable subject to search for an interpretation of quantum mechanics. The roots for this change are found in the pioneering work of Einstein Podolsky and Rosen [6] which pointed out to some peculiar correlations in the theory; followed by the work of Bell [7] that established conditions for the existence of those correlations in nature and finally, the experimental evidence for their existence [8]. Although no definite interpretation of quantum mechanics has been found, we have made some progress in the understanding of the quantum world. There exist correlations among the observables of a quantum system that can not be explained by some classical effects [9]. These correlations appear always among noncommuting observables and also appear in some cases among commuting ones as is the case, for instance, in nonlocal or noseparable states of quantum systems. It is no longer possible to think that the observables assume values independent of the context, that is, of the values assigned to other (commuting) observables [10].

It is impossible to assign the established features of the quantum systems to a classical particle. It is impossible to develop an image for a particle having those properties. Therefore if we want to understand quantum mechanics and if we keep in mind the image of a particle, we are in troubles. In this work we will see that it is possible to develop an image, not of a particle but of a field, that is compatible with the properties of a quantum system. Guided by our sense perception we may have the tendency to assign somehow a higher priority to the particles than to the fields. Indeed we have a clear sense perception for macroscopic particles but we have no “feeling” for the fields (only after a long intellectual development did we realize that “light” is a field). So usually we think of particles as having ontological entity, that is, as really existent, whereas fields are some mathematical construction associated to particles like, for instance, the electric or gravitational fields associated to charged or massive particles. Classical electromagnetism give us however some indication that this hierarchy may be incorrect. True, electric fields are a property of charged particles but we find, through Maxwell equations, that time varying electric fields can exist without charged particles associated to them. We have therefore no deep reason to think that particles are “more fundamental” than fields. It could even be the other way round: we will see that quantum mechanics is simpler if we consider that the fundamental entities are the fields instead of taking the particles as primary objects. Quantum mechanics, if considered as a
field theory, is no more weird than electrodynamics but if we take it as a theory of particles, very strange, unnatural and contradictory things must be introduced. For instance we must assign to a particle, a paradigm of a localized thing, a nonlocal quality. Movement and position become incompatible although movement is the change of position for all images that we can make of a particle. After having understood the mathematical and physical properties of the fields, we can recognize that some of its features can be associated to some properties of particles. In this way we may recover the particles as being emergent properties of the fields. One further advantage of considering nonrelativistic quantum mechanics as a field theory is that this paves the way towards relativistic quantum field theory.

In this work we will see a scheme that avoids the counterintuitive structures that appear when the quantum behavior of particles is presented. The main idea of the work is to present quantum mechanics as a field theory with acceptable features, no more abstract than the ones found in classical electromagnetism. We will see however that this revival of interpretations of quantum mechanics that assign ontological reality to the fields must face some severe problems. In section II, the mathematical structure of a field theory is presented and associated, in section III, to a free physical system. A potential is easily introduced in section IV and contact with the conventional formalism of quantum mechanics is made in section V. In section VI the possibility to build an interpretation of quantum mechanics based on this scheme is discussed.

II. MATHEMATICAL FEATURES

Let us assume a physical system represented by two coupled real fields. From their equations of motion we can find all relevant properties of the fields, and associate them to physical concepts. Let \( A(x, t) \) and \( B(x, t) \) denote the fields in the simplest case of a one dimensional space and time. These fields have no external sources like the electric charges and currents for the electromagnetic fields, but each field acts as a source for the polarization of the other field. This coupling through the polarization becomes clear in a discrete simulation of these fields in a lattice \([11]\), where particles and antiparticles associated to the fields are created at neighboring sites in each step of time evolution. Let the time evolution of the fields be determined by the equations

\[
\begin{align*}
\frac{\partial}{\partial t} A(x, t) &= -\frac{\partial^2}{\partial x^2} B(x, t), \\
\frac{\partial}{\partial t} B(x, t) &= \frac{\partial}{\partial x} A(x, t).
\end{align*}
\]  

(1)

We will use \( \partial^n_x A(x, t) \) to denote the \( n \)-th order partial derivative with respect to \( x \) and similarly for the time derivatives. In these equations we have suppressed a constant in order to take the fields \( A \) and \( B \), as well as the space-time variables, as dimensionless. For the moment we are only interested in the mathematical structure of the fields. All constants and dimensions needed to make contact with physical reality can be introduced later. These two equations play the similar rôle as Maxwell equations for the electric and magnetic fields, but are however much simpler. By direct substitution we can prove the following result:

If \( A(x, t) \) and \( B(x, t) \) are solutions of the Eqs. 1 then \( \partial^n_x A(x, t) \) and \( \partial^n_x B(x, t) \) \( \forall n \) are also solutions. The same can be said for the time derivatives \( \partial^m_t A(x, t) \) and \( \partial^m_t B(x, t) \) \( \forall m \).

Since the Eqs. 1 are linear, all linear combinations of solutions are also solutions. Therefore if \( A(x, t) \) and \( B(x, t) \) are solutions of the Eqs. 1, then \( \sum n \lambda_n \partial^n_x A(x, t) \) and \( \sum n \lambda_n \partial^n_x B(x, t) \) are also solutions. As a special case we may choose \( \lambda_n = L^n / n! \) and the summations become the Taylor expansion of \( A(x + L, t) \) and \( B(x + L, t) \). Therefore the
solutions of Eqs. 1 are invariant under space translations \( x \rightarrow x + L \). In the same way we can prove that the solutions are also invariant under a time translation \( t \rightarrow t + T \). These two results also follow directly from inspection of Eqs. 1. Another symmetry of the solutions that can be proven by direct substitution is that, if \( A(x, t) \) and \( B(x, t) \) are solutions of Eqs. 1 then

\[
A'(x, t) = cA(x, t) - sB(x, t), \\
B'(x, t) = sA(x, t) + cB(x, t),
\]

(2)

where \( c \) and \( s \) are arbitrary real constants, are also solutions. An interesting question is whether the constants \( c \) and \( s \) can become functions \( c(x, t) \) and \( s(x, t) \). Indeed, it can be easily proven that \( A'(x, t) \) and \( B'(x, t) \) given below are also solutions.

\[
A'(x, t) = \cos \left( \frac{\pi}{2}(x - \frac{v}{c}t) \right) A(x - vt, t) - \sin \left( \frac{\pi}{2}(x - \frac{v}{c}t) \right) B(x - vt, t), \\
B'(x, t) = \sin \left( \frac{\pi}{2}(x - \frac{v}{c}t) \right) A(x - vt, t) + \cos \left( \frac{\pi}{2}(x - \frac{v}{c}t) \right) B(x - vt, t),
\]

(3)

where \( v \) is an arbitrary constant. This solution is interesting because it shows how the fields behave in a galilean transformation \( x \rightarrow x' = x - vt, t \rightarrow t' = t \), if we require that the equations of motion in Eqs. 1 remain invariant. A similar situation is found in electrodynamics where the requirement of invariance of Maxwell equations under a Lorentz transformation mixes the electric and magnetic fields. In this nonrelativistic case we ask for invariance under a galilean transformation since Eqs. 1 are clearly not invariant under Lorentz transformation because of the different treatment of time and space variables. The equations above, considered as an active transformation, show how to boost the fields at an initial time \((t = 0)\) with a velocity \( v \).

We will see now that Eqs. 1 fully determine the time evolution of the fields. That is, given the fields at some time, say \( t = 0 \), we can determine the fields for all other times. In order to see this, we first derive an expression for all time derivatives of the fields in terms of their space derivatives. The result for the \( A \) field is:

\[
\partial^n_t A = \begin{cases} 
(-1)^{\frac{n}{2}} \partial_x^{2n} A & n = 0, 2, 4, \cdots \\ 
-(-1)^{\frac{n-1}{2}} \partial_x^{2n} B & n = 1, 3, 5, \cdots 
\end{cases}
\]

(4)

The result for \( B \) is similar, except for a sign change when \( n \) is odd, and can be obtained from the above equation by the replacement \( A \rightarrow B \) and \( B \rightarrow -A \). The proof of these equations follows by iterative time derivatives of Eqs. 1. Now we can use these time derivatives in a Taylor expansion around the value \( t = 0 \). That is, for the \( A \) field we have,

\[
A(x, t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \partial^n_t A|_{t=0} = \sum_{n=0, 2, 4, \cdots}^{\infty} \frac{t^n}{n!} (-1)^{\frac{n}{2}} \partial_x^{2n} A(x, 0) - \sum_{n=1, 3, 5, \cdots}^{\infty} \frac{t^n}{n!} (-1)^{\frac{n-1}{2}} \partial_x^{2n} B(x, 0). 
\]

(5)

We can nicely express this result if we notice that the sums above correspond to the series expansion of sine and cosine. Therefore we obtain the formal expression, also for the \( B \) field,

\[
A(x, t) = \cos \left( t \partial^2_x \right) A(x, 0) - \sin \left( t \partial^2_x \right) B(x, 0), \\
B(x, t) = \sin \left( t \partial^2_x \right) A(x, 0) + \cos \left( t \partial^2_x \right) B(x, 0).
\]

(6)
Now we want to find some conserved properties of the fields. These conserved properties are candidates for a physical interpretation. In the search for conserved quantities, we must eliminate the space dependence. To do this we can integrate the fields and functions of them in all space. In order to have such integrals mathematically well defined, we impose the supplementary condition that the fields should be confined. This means that the fields and all their derivatives must vanish at infinity.

\[ \partial^n_x A(\pm \infty, t) = \partial^n_x B(\pm \infty, t) = 0, \forall n = 0, 1, 2, \ldots \]  

(7)

Actually we need the stronger condition that they should vanish faster than any power of \( x \).

Before we identify the conserved quantities, we need a couple of definitions. Given two fields \( A(x, t) \) and \( B(x, t) \), we define their associated \( n \)th order density \( M_n(x, t) \) and \( n \)th order current \( P_n(x, t) \), (shortly \( n \)-density and \( n \)-current) by

\[ M_n(x, t) = (\partial^n_x A)^2 + (\partial^n_x B)^2, \]

\[ P_n(x, t) = \partial^n_x A \partial^{n+1}_x B - \partial^n_x B \partial^{n+1}_x A, \quad n = 0, 1, 2 \ldots \]  

(8)

The reason for the names chosen is that if \( A(x, t) \) and \( B(x, t) \) are solutions of Eqs. 1 then these quantities obey a continuity equation,

\[ \partial_t M_n(x, t) + 2 \partial_x P_n(x, t) = 0. \]  

(9)

This can be proven directly performing the derivatives and replacing from Eqs. 1. We can now prove that the space integrated \( n \)-densities and \( n \)-currents are conserved. That is, the quantities \( M_n \) and \( P_n \) defined as

\[ M_n = \int_{-\infty}^{\infty} M_n(x, t) \, dx \]

\[ P_n = \int_{-\infty}^{\infty} P_n(x, t) \, dx, \]  

(10)

are such that

\[ \partial_t M_n = 0, \quad \partial_t P_n = 0. \]  

(11)

These equations can be proven performing the time derivative of the products in Eqs. 8 and replacing them with Eqs. 1 and finally doing appropriate integrations by parts with vanishing border terms due to Eq. 7. However, the first of these equations follows easier from the continuity equation:

\[ \partial_t M_n = \int_{-\infty}^{\infty} \partial_t M_n(x, t) \, dx = -2 \int_{-\infty}^{\infty} \partial_x P_n(x, t) \, dx = -2 \int_{-\infty}^{\infty} P_n(x, t) \, dx = 0, \]  

(12)

where, again, the last term vanishes because of Eq. 7. For later reference, it is convenient to perform \( n \) integrations by parts of the densities and currents in Eqs. 10. The border terms in all these integrations vanish and we get the results

\[ M_n = (-1)^n \int_{-\infty}^{\infty} (A \partial^{2n}_x A + B \partial^{2n}_x B) \, dx \]

\[ P_n = (-1)^n \int_{-\infty}^{\infty} (A \partial^{2n+1}_x B - B \partial^{2n+1}_x A) \, dx. \]  

(13)

III. PHYSICAL FEATURES
In the last section we have identified several mathematical features of the fields \(A(x, t)\) and \(B(x, t)\) determined by Eqs. 1. Now we want to investigate what physical systems these fields may describe. A natural thing to do is to associate the conserved properties of the fields with some permanent features of the physical system.

We have seen that if we know the fields at a particular time, then the solutions are determined for all times. However there is a large class of possible initial conditions, producing different solutions to the field equations. We will identify these different solutions with different states of the same physical system. All these solutions share the property of invariance under space and time translation \(x \to x + L, \ t \to t + T\). That is, it is irrelevant where and when the system is considered. Such a system is free, that is, noninteracting with any external system.

The fields have an inescapable and inherent time dependence. That is, there are no static \(A(x, t)\) and \(B(x, t)\) fields. Clearly, if the fields have no time dependence, that is if \(\partial_t A = \partial_t B = 0\), the only solution consistent with the asymptotic conditions of Eq. 7 is \(A(x, t) = B(x, t) = 0\). In electromagnetism we have static fields that are generated by static electric charges and currents. Our field equations have no such sources and therefore we do not have static solutions. The fields \(A\) and \(B\) must have then a time dependence. There are however solutions, with time varying fields, such that the n-densities and n-currents are time independent. Let us call them stationary solutions. From Eqs. 8 we can find that the condition \(\partial_t M_n(x, t) = 0\) is satisfied if

\[
\begin{align*}
\partial_t A(x, t) &= EB(x, t), \\
\partial_t B(x, t) &= -EA(x, t),
\end{align*}
\]

(14)

where \(E\) is some constant; and \(\partial_t \mathcal{P}_n(x, t) = 0\) is valid if

\[
\begin{align*}
\partial^2_x A(x, t) &= -EA(x, t), \\
\partial^2_x B(x, t) &= -EB(x, t).
\end{align*}
\]

(15)

To reach this last condition we have used Eqs. 1. These two sets of equations are the sufficient (necessary?) conditions for the stationary states and are consistent with Eqs. 1. Indeed, the second set can be obtained using the first one in Eqs. 1. Each stationary solution is then characterized by a real constant \(E\) (we will see later that \(E > 0\)). These stationary solutions are linearly independent because linear combinations of stationary solutions are not stationary solutions. Indeed, if \(E_1\) and \(E_2\) characterize the solutions \((A_1, B_1)\) and \((A_2, B_2)\), one can prove that \((c_1A_1 + c_2A_2, c_1B_1 + c_2B_2)\) is not a stationary solution (although it is of course a solution to Eqs. 1). In order to study the properties of these solutions it is very useful to notice, from Eqs. 15, that the differential operator \(\partial^2_x\), when applied to the stationary solutions, can be simply replaced by the constant \(-E\). With this we can immediately give the stationary state in terms of some initial stationary state, replacing \(\partial^2_x \to -E\) in Eqs. 6:

\[
\begin{align*}
A(x, t) &= \cos (Et) A(x, 0) + \sin (Et) B(x, 0), \\
B(x, t) &= -\sin (Et) A(x, 0) + \cos (Et) B(x, 0).
\end{align*}
\]

(16)

In the “\(A, B\) plane”, the stationary states, at fixed \(x\), rotate in a circle of radius \(M_0\) with constant angular velocity \(E\). With respect to time, the stationary solutions oscillate with
(time) frequency $Et$ and with respect to space they oscillate with (space) frequency $\sqrt{Ex}$, as can be seen from the solutions of Eqs. 15. Here we must impose the condition $E > 0$ in order to avoid divergent solutions.

In the stationary states, the n-densities and n-currents have no time dependence. What about their spatial dependence? For the n-currents the answer is immediately given by the continuity equation 9

$$\partial_t \mathcal{M}_n(x, t) = 0 \implies \partial_x \mathcal{P}_n(x, t) = 0.$$  (17)

The n-currents in the stationary states are therefore constants independent of time and space. To find the value of these constants we can use the replacement $\partial_x^2 \rightarrow -E$ in Eqs. 8 to find

$$\mathcal{P}_n = E \mathcal{P}_{n-1} \implies \mathcal{P}_n = E^n \mathcal{P}_0.$$  (18)

The n-densities are, as seen, time independent but in general they may depend on $x$. With the same replacement $\partial_x^2 \rightarrow -E$ we can prove that (dropping explicitly the time variable $t$)

$$\mathcal{M}_n(x) = \begin{cases} E^n \mathcal{M}_0(x) & n = 0, 2, 4, \cdots \\ E^{n-1} \mathcal{M}_1(x) & n = 1, 3, 5, \cdots. \end{cases}$$  (19)

Furthermore, $\mathcal{M}_1$ can be given in terms of $\mathcal{M}_0$ because $\partial_x \mathcal{M}_1 = \partial_x ((\partial_x A)^2 + (\partial_x B)^2) = 2 \partial_x A \partial_x^2 A + 2 \partial_x B \partial_x^2 B = -E(2A \partial_x^2 A + 2B \partial_x^2 B) = -E \partial_x ((A)^2 + (B)^2) = -E \partial_x \mathcal{M}_0$; therefore $\mathcal{M}_1 = E(C - \mathcal{M}_0)$ with an arbitrary constant $C$. This constant must be such that $C > \max \mathcal{M}_0$ to guarantee positivity of $\mathcal{M}_1$.

These stationary solutions oscillate for all time and in all space. They are similar to the electromagnetic plane wave solutions to Maxwell’s equations. The same happens here as with the electromagnetic plane waves: these solutions can only be considered to represent approximately a physical system. Strictly speaking these solutions are unphysical because they imply an infinitely extended system and the conserved quantities of Eqs. 10 are meaningless. However they are useful to represent approximate situations and, most important, these solutions are linearly independent and therefore can be used in superposition in order to construct wave packets with any desirable shape and extension. There is however an important difference between the wave packets solutions to Maxwell equations and the propagating “wave packets” solutions to our Eqs. 1. There exist electromagnetic pulses or packets of arbitrary shape that propagate without changing the shape. This is not possible in our case. Indeed, we can show that our Eqs. 1 do not admit solutions of the type $A = f(x + vt)$, $B = g(x + vt)$ where $f$ and $g$ are arbitrary (differentiable) functions. Only sine and cosines are acceptable, and these are the stationary solutions already mentioned. Therefore the shape of our field distributions must change during the time evolution. In other words, the fields representing a quantum system described by Eqs. 1 are dispersive: localization and shape of the distributions are not a permanent feature of quantum systems.

We will see now, for arbitrary states, how the conserved quantities behave under the transformations of the fields. Let us first consider the transformation described by Eqs. 2. If the fields $A(x, t)$ and $B(x, t)$ have the n-densities and n-currents $\mathcal{M}_n(x, t)$ and $\mathcal{P}_n(x, t)$ and their corresponding conserved quantities $M_n$ and $P_n$, then the transformed fields $A'(x, t)$ and $B'(x, t)$ given by Eqs. 2 will correspond to $\mathcal{M}'_n(x, t) = (c^2 + s^2) \mathcal{M}_n(x, t)$ and $\mathcal{P}'_n(x, t) = (c^2 +
\(s^2)\mathcal{P}_n(x,t)\) and also, \(M'_n = (c^2 + s^2)M_n\) and \(P'_n = (c^2 + s^2)P_n\). Therefore this transformation produces just a change of scale that can be canceled by a numerical factor multiplying the fields. Furthermore, if the constants are such that \(c^2 + s^2 = 1\) then the transformation is irrelevant for the physical quantities represented by the n-densities and n-currents and their conserved quantities.

The next transformation of Eqs. 3 gives much more information about the physical nature of the conserved quantities. This transformation represents either the observation of the same physical system from a reference frame moving with velocity \(-v\) or alternatively, the same physical system “boosted” with a velocity \(v\). Let us consider first the 0-density. Although the fields are mixed in this transformation, this density remains unchanged in shape and is just boosted with velocity \(v\).

\[
M'_0(x,t) = A'^2(x,t) + B'^2(x,t) = M_0(x - vt, t) \tag{20}
\]

This suggests that the 0-density \(M_0(x,t)\) represents the \textit{space localization} of the physical system described by the fields \(A(x,t)\) and \(B(x,t)\). We can always scale the field such that the conserved quantity associated to the 0-density takes the value \(M_0 = 1\). With this normalization, the 0-density can be thought as a distribution of localization. Let us consider now the 0-current \(\mathcal{P}_0(x,t)\). Using the transformation of Eqs. 3 we obtain, after straightforward manipulations,

\[
\mathcal{P}'_0(x,t) = \mathcal{P}_0(x - vt, t) + \frac{v}{2}M_0(x - vt, t) \tag{21}
\]

and the corresponding conserved quantities, assuming the normalization \(M_0 = 1\),

\[
P'_0 = P_0 + \frac{v}{2}. \tag{22}
\]

For the 1-density we get:

\[
M'_1(x,t) = M_1(x - vt, t) + v\mathcal{P}_0(x - vt, t) + \left(\frac{v}{2}\right)^2 M_0(x - vt, t) \tag{23}
\]

and the corresponding conserved quantities are

\[
M'_1 = M_1 + vP_0 + \left(\frac{v}{2}\right)^2. \tag{24}
\]

The conserved quantities \(P_0\) and \(M_1\) can be consistently associated to the momentum and kinetic energy of the system (with mass \(m = 1/2\)) because under a boost of velocity \(v\) they are changed accordingly. Indeed if \(P_0 = \frac{1}{2}u\) and \(M_1 = \frac{1}{4}u^2\) then \(P'_0 = \frac{1}{2}(u + v)\) and \(M'_1 = \frac{1}{4}(u + v)^2\). For the remaining n-densities and n-currents we can also find their transformation properties in a boost but there are no remaining particle observables to which they can be related. Furthermore, one must be cautious in the identification of the conserved properties of the quantum system with the conserved properties of a particle. For instance, the relation between kinetic energy and momentum valid for classical particles is not always true for the quantum system. One can prove that in general it is \(M_1 \geq P_0^2\) and not \(M_1 = P_0^2\) as one would expect for a particle of mass \(m = 1/2\). Of course, the
identification of a delocalized quantum system, for instance in a stationary state, with a localized classical particle becomes very suspect.

We can obtain further confirmation that the system is moving with momentum $P_0$ by calculating the time derivative of the center of the localization distribution defined as

$$X = \int_{-\infty}^{\infty} x M_0(x, t) \, dx .$$

(25)

The center of the distribution moves with constant velocity $2P_0$:

$$\partial_t X = \int_{-\infty}^{\infty} x \partial_t M_0(x, t) \, dx = -2 \int_{-\infty}^{\infty} x \partial_x P_0 \, dx$$

$$= -2x P_0 |_{-\infty}^{\infty} + 2 \int_{-\infty}^{\infty} P_0 \, dx = 2P_0 ,$$

(26)

where we have used the continuity equation and that the border term of the integration by parts vanish because the vanishing in Eq. 7 is faster than any power of $x$. We can use the relation between the velocity of the center of the distribution and the conserved momentum in order to define the mass of the free quantum system: $m = P_0 / \partial_t X$. Notice that with a similar calculation we reach the conclusion that the center of the distributions, corresponding to the $n$-densities $M_n(x, t)$, all move with constant “velocities” equal to $2P_n$. The fact that these velocities are not all equal is indicative of the dispersive nature in the evolution of the quantum system.

We summarize: the fields $A(x, t)$ and $B(x, t)$ determined by Eqs. 1 describe a free quantum system localized according to the density distribution $M_0(x, t) = A^2(x, t) + B^2(x, t)$ moving with a constant drift velocity $2P_0 = 2 \int (A \partial_x B - B \partial_x A ) \, dx$ and constant kinetic energy $M_1 = \int (\partial_x A)^2 + (\partial_x B)^2 \, dx$. Referring to a similar classical system, we can denote this system as a free quantum particle. The shape of the distributions change in time, indicating that localization is not a permanent property of the free quantum system. When the distribution $M_0(x, t)$ is localized in a small region, the system is in a particle-like state. If the distribution $M_0(x, t)$ is delocalized in a large region, the system is in a wave-like state.

IV. EXTERNAL INTERACTION

The field equations given in Eqs. 1 can be derived using the Euler-Lagrange equations applied to the following lagrangian density, a functional of the fields and of their space and time derivatives,

$$\mathcal{L} = B \partial_t A - A \partial_t B - (\partial_x A)^2 - (\partial_x B)^2 .$$

(27)

As usual, terms like $\partial_B \partial_x A - \partial_A \partial_x B$ or $B \partial_x A + A \partial_x B$ and other total differentials can be added to this lagrangian density but they are irrelevant because they make a vanishing contribution to the field equations. From this lagrangian density we can find the canonical field momenta associated to the fields $A(x, t)$, $B(x, t)$. It turns out that $A(x, t)$ and $B(x, t)$ are reciprocally the canonical momenta of each other (within a minus sign). With these momenta and with the lagrangian density we obtain the canonical hamiltonian density

$$\mathcal{H} = (\partial_x A)^2 + (\partial_x B)^2 .$$

(28)

Here again, total differential terms can be added that have no contribution when integrated in all space because of Eq. 7. Notice that $\mathcal{H} = M_1(x, t)$ and therefore the integrated hamiltonian density is conserved. We should not naively attempt to derive the equations of
motion from the hamiltonian formalism using the last expression. This fails because we are dealing here with a nonstandard lagrangian linear in the “velocities”. For this lagrangian we can not solve the “velocities” in terms of the canonical momenta. We will not treat here this constrained hamiltonian field theory. We have brought here the hamiltonian density, that can be interpreted as an energy density, only for the purpose of guiding us in the introduction of an interaction in our system.

The free system of Eqs. 1 is translation invariant and all regions of space are equivalent. Let us break this translation symmetry. Let us assume now that there are some regions of space where the energy of the system is changed by the action of an external field denoted by $V(x)$. We call this a potential field. The presence or localization of the system in the regions where the potential field $V(x)$ is nonvanishing, will change the energy of the system by an amount given by the product of the potential with the density of localization of the system in such a region. This last density is given by $M_0(x,t)$. Therefore we introduce the interaction of the system with an external potential $V(x)$ just changing the hamiltonian density to the new expression

$$H = (\partial_x A)^2 + (\partial_x B)^2 + V(x)\left(A^2 + B^2\right).$$

With this new hamiltonian, that includes the interaction, we get a new lagrangian density

$$L = B \partial_t A - A \partial_t B - (\partial_x A)^2 - (\partial_x B)^2 - V(x)\left(A^2 + B^2\right),$$

and with the Euler-Lagrange equations we obtain the equations of motions for the fields describing a quantum particle in a potential,

$$\partial_t A(x,t) = - \partial_x^2 B(x,t) + V(x)B(x,t),$$

$$\partial_t B(x,t) = \partial_x^2 A(x,t) - V(x)A(x,t).$$

An analysis similar to the one performed for the free system could be now done. These equations are invariant under the transformation Eq. 2 and an observer in a moving frame will see the potential moving, $V(x-vt)$, as expected. The time evolution can be determined in a similar way as before. A similar study of stationary states can be done with the replacement $\partial_x^2 - V(x) \rightarrow -E$

V. CONVENTIONAL FORMALISM

Once that we have seen the advantages of the description of quantum mechanics as a classical field theory we must make contact with the conventional formalism. For this we just have to define a complex field $\Psi(x,t) = A(x,t) + iB(x,t)$ from the equations of motion for the fields $A$ and $B$, given in Eqs. 31, we obtain the corresponding equation for $\Psi$. Furthermore we introduce constants in order to recover physical dimensions for space, time, mass and energy, and we obtain thereby Schrödinger’s equation, in all its glamour:

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x,t)}{\partial x^2} + V(x)\Psi(x,t).$$

All the formal expressions found before have a corresponding representation in terms of the complex field. For instance $\mathcal{M}_0(x,t) = |\Psi(x,t)|^2$. The conserved quantities (when $V = 0$) of Eqs. 13 correspond to the expectation value of even and odd powers of an operator
\[ M_n = \langle \Psi, (-i\partial_x)^{2n}\Psi \rangle \]
\[ P_n = \langle \Psi, (-i\partial_x)^{2n+1}\Psi \rangle , \]  

suggesting the association of that operator with the conserved momentum.

In the derivation of Schrödinger equation, or of the coupled equations in Eqs. 31, from the lagrangian Eq. 30, or its equivalent in terms of \( \Psi \) and \( \Psi^* \), the two fields, either \( A \) and \( B \) or \( \Psi \) and \( \Psi^* \), must be varied independently. It is not possible to write a lagrangian in terms of only one field that leads to Schrödinger’s equation. Two fields must be taken. The necessity of two fields is inherent to the formalism and therefore the use of a complex field for quantum mechanics (or the equivalent two fields \( A \) and \( B \)) is different from other cases in physics, for instance in electrodynamics, where complex quantities are just a convenience. This necessity of two fields is more clearly stated in the approach presented here where the elegance of the Lagrange formalism becomes a very convenient way for introducing an external potential.

Another advantage of the formalism in terms of the two fields \( A \) and \( B \) instead of one complex field \( \Psi \) is the possibility of an enlightening analogy with classical mechanics. The phase space of a classical particle (in a line) is two dimensional and is spanned by the dynamic variables \( X \) and \( P \) whereas the “phase space” for the quantum system is also two dimensional and is spanned by the dynamic fields \( A \) and \( B \). This parallel can be continued noticing that the canonical transformation corresponding to a rotation in phase space

\[ X' = cX - sP \]
\[ P' = sX + cP , \]  

with \( c^2 + s^2 = 1 \), is equivalent to the transformation of the fields in Eqs. 2. Furthermore \( A \) and \( B \) as well as \( X \) and \( P \) are canonical conjugate of each other (within a minus sign).

V. INTERPRETATION

Quantum mechanics loses most of the strange and awkward aspects when it is seen as a field theory and becomes a simple subject in comparison with the unsurpassable conceptual difficulties found when we look at it as a theory for particles. The advantages of presenting quantum mechanics as a field theory become obvious. Encouraged by this, we want to face now the question of whether this is just a formal advantage or if this trick can become an interpretation of quantum mechanics.

To turn this into an interpretation we must decide to choose the fields as the primary ontology, that is, the fields are the really existent objects that the theory must describe, instead of taking the particles as the primary objects to be described by quantum mechanics. In a simplified manner, we may choose between two extreme interpretations: either the coupled fields \( A(x,t) \) and \( B(x,t) \) really exist and are determined by field equations with quite reasonable properties, no more strange than the electromagnetic fields, or on the contrary, the particles really exist and the fields are a mathematical construction corresponding to the probability of finding, or of localization, of such particles that must have strange schizophrenic properties of being at different places at the same time, and that sometimes appear as point like particles but some other times as extended waves. In the first option we just have fields as the primary ontology and “the particles” are not existent objects but are a name, learned from the classical physics of macroscopic object, that we may use to denote a set of emergent properties of the fields. In this interpretation, what we usually call “an
electron” is not a point like particle that is detected somewhere according to a “probability cloud” but is the cloud itself.

There have been several attempts of interpretations based on different choices for the primary ontology. Without many details we just mention some of them [12]. The well known probability interpretation of the wave function proposed by Max Born favors a particle ontology. In this case the field $\Psi$ does not carry energy and is not really existent in physical space. Opposite to it, we can find Schrödinger interpretation proposing that only the wave function has objective existence. In between, we can find Madelung hydrodynamical interpretation, no longer considered probably because it does not clearly states what is precisely the fluid being described by Schrödinger equation. A hybrid interpretation was proposed by L. de Broglie with his “double solution” suggesting a mixed particle and field ontology. Another idea originated by L. de Broglie is based on a particle ontology with a pilot wave determining its motion. This interpretation was successfully taken by D. Bohm in his causal quantum mechanics (Bohmian mechanics) that has received much attention in the lasts years. We can safely say that no one of the mentioned interpretations is fully satisfactory because they all face some difficulties, and probably also no one of them is totally excluded at present. It is desirable to present all these alternatives instead of taking, as is unfortunately often done, an instrumentalist position that implies an evasion from the problem.

Since the scheme of this work clearly favors Schrödinger interpretation, we must recall the most serious problem faced in this choice. For a single quantum system, the formalism presented can be interpreted in the lines shown by Schrödinger. However for a compound system with many degrees of freedom we must introduce more variables upon which the fields depend. The fields no longer “live” in physical space but in a $3N$ dimensional phase space. Furthermore, if we want to give to the fields an ontological character then we must solve somehow the ambiguity in the fields apparent in Eqs. 2. Is there a way to fix the phase and define the fields without ambiguity? or can we tolerate the ambiguity and define the primary ontology as an equivalence class of fields? Another item that requires much thought is the determination of what are precisely the observable properties of the fields and the meaning of the measurement process. The fields are, apparently, not directly observables but this would not be a serious difficulty for the interpretation, under the consideration that any measurement is a complex process that involves macroscopic apparatus. The electron, as a particle in the conventional interpretation of quantum mechanics (whatever that is), is also, as is the case for the fields, not a direct observable.

VI. CONCLUSIONS

It has been shown that the consideration of quantum mechanics as a classical field theory can have significant advantages. In this way, very confusing concepts like, for instance, the probability of observing, or of existence, of a particle somewhere, or the incompatibility between position and momentum, are avoided. Quantum mechanics, as a field theory, is not stranger than classical electrodynamics. The extension of this work to the realistic three dimensional space is trivial and can be performed by the substitutions $x \rightarrow r$ and $\partial_x \rightarrow \nabla$.

The ideas presented in this work suggest a revival of the interpretations of quantum mechanics based on a field ontology, similar to the original interpretation proposed by Schrödinger. These interpretations have, however, serious difficulties in the case of a system with many degrees of freedom because the dimension of the space where the fields have
support is not equal to the dimension of physical space.

We would like to acknowledge helpful discussions with H. Mártiln and P. Sisterna. This work has received partial support from “Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Argentina.
REFERENCES

[1] M. Planck. “Zur Theorie des Gesetzes der Energieverteilung im Normalspektrum”. Verhandlungen der Deutschen Physikalischen Gesellschaft, 2, 237-245 (1900).

[2] A. Einstein. “Über einen die Erzeugung und Verwandlung des Lichtes betreffenden heuristischen Gesichtspunkt”. Annalen de Physik (4), 17, 132-148 (1905); A. Einstein. “Theorie der Lichterzeugung und Lichtabsorption”. Annalen de Physik (4), 20, 199-206 (1906);

[3] J. von Neumann, Mathematische Grundlagen der Quantenmechanik Springer, Berlin (1932).

[4] R. Feynman. Character of Physical Law. MIT Press (1967).

[5] For the historical development of quantum mechanics see M. Jammer. The Philosophy of Quantum Mechanics. Wiley, New York (1974).

[6] A. Einstein, B. Podolsky, N. Rosen. “Can quantum mechanical description of physical reality be considered complete”, Phys. Rev. 47, 777-780 (1935).

[7] J. S. Bell. “On the Einstein Podolsky Rosen paradox”, Physics 1, 195-200 (1964).

[8] J. F. Clauser, A. Shimony. “Bell’s theorem: experimental tests and implications”, Rep. Prog. in Phys. 41, 1881-1927 (1978).

[9] A. C. de la Torre, J. L. Iguain. “Manifest and Concealed Correlations in Quantum Mechanics”, Eur. J. of Phys. 19, 563-568, (1998)

[10] For a didactical treatment of correlations and contextuality see: A. Peres. Quantum Theory: Concepts and Methods. Kluwer Acad. Pub. (1995).

[11] A. C. de la Torre. “A one-dimensional lattice model for a quantum mechanical free particle”, Mar del Plata Preprint. quant-ph/9905031

[12] For more details see Chapter 2 of Ref.5 above or Chapter 8 of J. T. Cushing. Quantum Mechanics: Historical Contingency and the Copenhagen Hegemony. The University of Chicago Press, Chicago 1994. or also relevant chapters of Tian Yú Cao. Conceptual Developments of 20th Century Field Theories. Cambridge University Press, Cambridge 1997.