Research Article

An Application of Possibilistic Moments of Nonlinear Type of Fuzzy Numbers in Supply Chain Management

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The supply chain (SC) network is prone to disturbance due to various uncertainties associated with their subsystems. The COVID-19 outbreak has exposed the global vulnerability of the supply chain network. The current pandemic has severely affected almost every SC network because its members are situated at the international level. One of the reasons for SC network failure is the deterministic assumptions of different parameters. A realistic SC network model requires the use of the uncertain value of the parameters, which can be further captured by fuzzy numbers. This paper discusses the possibilistic moment of several nonlinear types of fuzzy numbers that are important for SC network modeling. We give closed-form possibilistic moments’ expression for various types of fuzzy numbers that are very similar to the moment’s properties in probability theory and stochastic process. We then illustrate the application of proposed fuzzy numbers by solving an inventory model. This paper also provides results related to the EPQ inventory model in a fuzzy possibilistic setup.

1. Introduction

The COVID-19 pandemic has exposed the vulnerability of many organizations, especially the supply chain network that is heavily dependent on China. The improper management of the supplier chain network has left many companies without an adequate supply of raw materials or with unsold finished products. One of the reasons for supply chain failure is the traditional consideration of deterministic parameters, despite real-world supply chain management actually containing uncertainty in the model parameters. The uncertainties in different supply chain activities from raw material suppliers to the final customers make the supply chain network imprecise [1]. Thus, many researchers have implemented the fuzzy set theory to model various aspects of supply chain management. Some of the examples of fuzzy set application in supply chain management are supply chain innovation enabling process [2], the supplier selection decision [3, 4], the vehicle routing determination process [5], the domino effect of supply chain resilience [6], quality design process [7], etc. According to Zardani et al. [1], the system designers can use fuzzy set concepts to capture the uncertainty in a real-world environment. One of the major advantages of using fuzzy set theory is its simplicity and ability to deal with a nonlinear function of arbitrary complexity [1]. In light of this observation, this paper presents a possibility framework for supply chain decision-making problems. First, the paper provides the possibilistic moment of different fuzzy numbers and then illustrates the application of proposed fuzzy numbers through the economic production quantity (EPQ) inventory model.

The fuzzy set theory has been applied in decision-making, often in the discipline of management. The interest in fuzzy set theory is growing because of its proximate use in capturing real-world problems. In the literature, fuzzy algebra-based possibility theory has been used in various decision-making environments [8–11]. Carlsson and Fuller [9] studied possibility theory and provided closed-form
expression for possibilistic mean, possibilistic variance, and possibilistic covariance of fuzzy numbers. Fuller and Majender [12] provided the concept of crisp weighted possibilistic moments of fuzzy numbers. Thavaneswaran et al. [13] and Thavaneswaran et al. [11] extended those concepts and provided expressions for crisp possibilistic \( r \)-th moments of fuzzy numbers. Thavaneswaran et al. [11] applied the crisp possibilistic \( r \)-th moments to dynamic time-series models and the option pricing models. Gu and Xuan [14] introduce a new method of the ranking method based on possibility theory, and some examples are given to illustrate the advantages of the method.

In recent times, possibility theory has been applied to many decision-making processes. Appadoo et al. [15] derived possibilistic moments of a special type of fuzzy numbers. They showed that the linear trapezoidal and the linear triangular fuzzy numbers are special cases of the nonlinear fuzzy numbers. They also provided examples of weighted possibilistic moments for a nonlinear class of trapezoidal fuzzy numbers using some form of nonnegative weight function. Wang et al. [16] successfully applied possibility theory to a fuzzy control chart. Tsaur [17] used possibilistic mean-standard deviation theory in a portfolio selection model. An illustrative numerical example is provided towards the end of the paper to illustrate the proposed portfolio selection model. Rabbani et al. [18] evaluated the sustainability performance of suppliers using the interval-valued fuzzy group decision theory. They provided separation measure matrices using possibility theory. Verma et al. [19] used possibility theory to determine an optimal maintenance policy. They provided a detailed algorithm for the replacement maintenance interval optimization model. Borges et al [20], presented a scenario-based real option valuation method using possibility distributions. Chen and Tan [21] applied possibility theory to the portfolio selection with an uncertain time horizon. Zhang et al. [22] introduced the concepts of possibilistic efficient portfolios using possibility distributions. They applied possibility theory to the multiperiod portfolio selection problem in which the degree of portfolio diversification is measured using the possibilistic entropy. Li et al. [23] used possibilistic moments in a portfolio selection model with value at risk constraint. An empirical study based on the Shanghai Stock Exchange validated the results.

Some of the most essential studies into fuzzy inventory models have recently been conducted. Masae et al. [24] presented a systematic review of order picking routing models in warehouses. Ali et al. [25] considered a multi-objective inventory optimization model with intuitionistic fuzziness in the model parameters. They used a fuzzy goal programming-based approach to solve the proposed inventory model. In a recent article, Gautam et al. [26] performed an extensive literature review that considered the effect of imperfect quality items in inventory replenishment or production models. The findings offered state-of-the-art insights to scientists and business professionals in supply chain management. Similarly, Janssen et al. [27] provided a comprehensive literature review of perishable inventory models from January 2012 to December 2015. Ali et al. [28] developed a supply chain optimization model that minimizes the combination of transportation, deliveries, and ordering costs under uncertain situations. Ali et al. [25] used both a multiobjective vendor selection problem under fuzzy environment and a fuzzy goal programming approach in their model formulation. The proposed model in Ali et al. [25] can handle realistic vendor selection problems in a fuzzy setup and can serve as a useful approach for multicriteria decision-making in supply chain management. Medaglia et al. [29] provide intuitive methods to construct membership functions with desired shapes that can be very useful in supply chain applications models. Medasani et al. [30] provide an overview of various techniques for generating membership functions that can be easily tuned. Medasani et al. [30] also discuss the suitability of these membership function generation techniques to some particular situations. Ye and Li [31] consider a fuzzy single-period product inventory control in a distributed supply chain using the weighted possibilistic mean value method to rank the model parameters considered in the model.

The motivation of this research is to reconsider the EPQ model using fuzzy sets theory assisted by possibility theory to capture the uncertainty embedded in the model. The EPQ model is illustrated in Figure 1. This paper is an extension of the work carried out in [32, 33]. The economic production quantity (EPQ) deterministic inventory model with finite replenishment rate is one of the most commonly inventory model, used by many industries. The EPQ inventory model extends the economic order quantity (EOQ) model rests on crisp value assumptions of different parameters such as the setup cost, the holding cost, demand rate, and production rate. Due to unpredictable market conditions, these assumptions may not hold in practice. This is mainly due to those weaknesses of the current EPQ deterministic inventory models that call upon more realistic fuzzy and stochastic inventory model formulations. The main contributions of this paper are as follows:

(i) Generally, inventory model parameters may be imprecise. Therefore, we use the possibility theory to calculate the order quantity.

(ii) The fuzzy model discussed in this paper can be easily modified to account for multi-items inventory models with imprecise parameters.

(iii) We derived various closed-form expressions for possibilistic moments of different types of fuzzy numbers.

(iv) The fuzzy inventory model discussed in this paper is straightforwardly understandable and can be extended to accommodate nonlinear types of fuzzy numbers.

(v) The proposed fuzzy inventory model does not imply the rejection of other fuzzy inventory formulations, but rather complements the existing fuzzy inventory model formulation.

This study follows an organizational structure. Section 1 provides a detailed literature review on the application of possibility theory and a summary of the preliminaries and
notations that will be used in the sequel. Section 2 discusses possibility theory along the line of [9]. In Section 3, we provide a mathematical derivation of possibilistic moments of two nonlinear classes of fuzzy numbers along with the necessary theorems and illustrative examples. Two applications of possibility theory are provided in Section 4 to discuss the applicability of this paper’s theory to weighted functions. Finally, the conclusion and remarks are given in Section 5.

1.1. Preliminaries and Notation. In this section, we provide some fuzzy definitions necessary to follow the discussion in the paper. Detail descriptions may be found in [35–39].

**Definition 1** (see [15]). A fuzzy number $A = [a_1, a_2, a_3, a_4]_{O(m,n)}$, $a_1 < a_2 < a_3 < a_4$ is said to be $O(m,n)$-trapezoidal-type fuzzy number ($O(m,n)$-Tr.F.N.) if its membership function is given as

$$
\mu(x) = \begin{cases} 
0, & x \leq a_1, \\
1 - \left( \frac{a_2 - x}{a_2 - a_1} \right)^m, & a_1 \leq x \leq a_2, \\
1, & a_2 \leq x \leq a_3, \\
1 - \left( \frac{a_4 - x}{a_4 - a_3} \right)^n, & a_3 \leq x \leq a_4, \\
0, & x \geq a_4.
\end{cases}
$$

(1)

Alternatively, following [36], defining the $\alpha$-cut (interval of confidence at level-$\alpha$) as $A(\alpha) = [a_1(\alpha), a_2(\alpha)]$, we characterize $O(m,n)$-Tr.F.N. $[a_1, a_2, a_3, a_4]_{O(m,n)}$ as

$$
A(\alpha) = [a_2 - (a_2 - a_1)(1 - \alpha)^{1/m}, a_3 - (a_3 - a_4)(1 - \alpha)^{1/n}], \quad \forall \alpha \in (0, 1),
$$

(2)

by setting $1 - ((a_2 - x)/(a_2 - a_1))^m = \alpha$ and $1 - ((a_3 - x)/(a_3 - a_4))^n = \alpha$, respectively.

An $O(m,n)$-Tr.F.N. is said to be symmetrical if it satisfied the following two conditions:

(a) $a_2 - a_1 = a_4 - a_3$

(b) $m = n$

It is important to note that $(O(m,n)$-Tr.F.N.) can be easily tuned to desired shapes.

**Definition 2.** Let $A = (x; \mu, \sigma, \alpha)$ be a Gaussian fuzzy number; then, the membership function can be written as

$$
\mu_A(x) = \exp \left( - \frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2 \right),
$$

(3)

where $\mu$ is the mean value and $\sigma^2$ is the variance. The confidence interval will be $A(\alpha) = [a_1(\alpha), a_2(\alpha)]$ and setting will be $\exp\left( - (1/2) \left( (x - \mu)/\sigma^2 \right)^2 \right) = \alpha$; we get $a_1(\alpha) = \mu - \sigma \sqrt{(-2 \ln \alpha)}$ and $a_2(\alpha) = \mu + \sigma \sqrt{(-2 \ln \alpha)}$. Figure 1: Production vs. inventory level with uniform replenishment [34].
Given the above expressions, the α representation of the Gaussian fuzzy number is

\[
A(\alpha) = [a_1(\alpha), a_2(\alpha)] = [\mu - \sigma \sqrt{(-2 \ln \alpha)}, \mu + \sigma \sqrt{(-2 \ln \alpha)}], \quad \forall \alpha \in (0, 1].
\]  

\[4\]

2. Possibility Theory

In this section, we discussed the possibilistic moment properties of fuzzy numbers along the line of [9, 41].

2.1. Moment Properties of Fuzzy Numbers. As in [9], lower possibilistic mean \(E_L(A)\), upper possibilistic mean \(E_R(A)\), possibilistic mean \(E(A)\), and interval value possibilistic mean IVPM(A) = \([E_L(A), E_R(A)]\) of a fuzzy number A defined by \([a_1(\alpha), a_2(\alpha)]\), \(\alpha \in [0, 1]\) are given below. It is important to note here that, in order to derive the results in this section, we implicitly use the following results:

\[
\text{Possibility}[A \leq a_1(\alpha)] = \pi(-\infty, a_1(\alpha)) = \lim_{\alpha \rightarrow 0} A(\alpha) = \alpha,
\]

\[
\text{Possibility}[A \geq a_2(\alpha)] = \pi(a_2(\alpha), \infty) = \lim_{\alpha \rightarrow 1} A(\alpha) = \alpha.
\]

(5)

For more detail, an interested reader may refer to [9]

\[
\text{IVPM}(A) = \left[ \frac{\int_0^1 \text{Pos}[A \leq a_1(\alpha)] a_1(\alpha) d\alpha}{\int_0^1 \text{Pos}[A \leq a_1(\alpha)] d\alpha}, \frac{\int_0^1 \text{Pos}[A \geq a_2(\alpha)] a_2(\alpha) d\alpha}{\int_0^1 \text{Pos}[A \geq a_2(\alpha)] d\alpha} \right].
\]

(8)

According to Carlsson and Fuller [9], the zero centered possibilistic variance \(\text{Var}(A)\) of a fuzzy number A is defined as

\[
\text{Var}(A) = \int_0^1 \frac{1}{2} a(a_2(\alpha) - a_1(\alpha))^2 d\alpha.
\]

(9)

The possibilistic covariance between fuzzy numbers A and B is defined as

\[
\text{Cov}(A, B) = \int_0^1 \frac{1}{2} a(a_2(\alpha) - a_1(\alpha))(b_2(\alpha) - b_1(\alpha)) d\alpha.
\]

(10)

Zhang and Nie [42] and Zhang and Xiao [43] defined the lower possibilistic variance of a fuzzy number A as

\[
\text{Var}_L(A) = \int_0^1 \text{Pos}[A \leq a_1(\alpha)] (E_L(A) - a_1(\alpha))^2 d\alpha
\]

and the upper possibilistic variance of a fuzzy number A as

\[
\text{Var}_R(A) = \int_0^1 \text{Pos}[A \geq a_2(\alpha)] (E_R(A) - a_2(\alpha))^2 d\alpha.
\]

(12)

Similar to the definition of \(E(A)\), the crisp possibilistic variance is as the arithmetic mean of its lower and upper possibilistic variances, i.e.,
as A crisp possibilistic variance and covariance of a fuzzy number $A$ lower and upper possibilistic covariances between two fuzzy numbers $A$ and $B$ as given in expressions (14) and (15):

$$\text{Var}(A) = \frac{1}{2} \left[ \int_0^1 \text{Pos}[A \leq a_1(\alpha)] (E_L(A) - a_1(\alpha))^2 \, d\alpha + \int_0^1 \text{Pos}[A \geq a_2(\alpha)] (E_R(A) - a_2(\alpha))^2 \, d\alpha \right]$$

(13)

Zhang and Nie [42] and Zhang and Xia [43] defined the lower and upper possibilistic covariances between two fuzzy numbers $A$ and $B$ as in [22, 42–44]:

$$\text{Cov}_L(A, B) = \frac{\int_0^1 \text{Pos}[A \leq a_1(\alpha), B \leq b_1(\alpha)] (E_L(A) - a_1(\alpha)) (E_L(B) - b_1(\alpha)) \, d\alpha}{\int_0^1 \text{Pos}[A \leq a_1(\alpha)] \, d\alpha}$$

$$= 2 \int_0^1 a (E_L(A) - a_1(\alpha)) (E_L(B) - b_1(\alpha)) \, d\alpha,$$

$$\text{Cov}_R(A, B) = \frac{\int_0^1 \text{Pos}[A \geq a_2(\alpha), B \geq b_2(\alpha)] (E_L(A) - a_2(\alpha)) (E_R(B) - b_2(\alpha)) \, d\alpha}{\int_0^1 \text{Pos}[A \geq a_2(\alpha)] \, d\alpha}$$

$$= 2 \int_0^1 a (E_R(A) - a_2(\alpha)) (E_R(B) - b_2(\alpha)) \, d\alpha,$$

and the crisp possibilistic covariances between fuzzy numbers $A$ and $B$ as the arithmetic mean of their lower and upper possibilistic covariances, i.e.,

$$\text{Cov}(A, B) = \frac{\int_0^1 \text{Pos}[A \leq a_1(\alpha), B \leq b_1(\alpha)] (E_L(A) - a_1(\alpha)) (E_L(B) - b_1(\alpha)) \, d\alpha}{2 \int_0^1 \text{Pos}[A \leq a_1(\alpha)] \, d\alpha}$$

$$+ \frac{\int_0^1 \text{Pos}[A \geq a_2(\alpha), B \geq b_2(\alpha)] (E_L(A) - a_2(\alpha)) (E_R(B) - b_2(\alpha)) \, d\alpha}{2 \int_0^1 \text{Pos}[A \geq a_2(\alpha)] \, d\alpha}$$

(16)

We use the following results as in [22, 42–44]:

$$\text{Pos}[A \leq a_1(\alpha), B \leq b_1(\alpha)] = \sup_{u \leq a_1(\alpha), v \leq b_1(\alpha)} \min\{A(u), B(v)\} = a,$$

$$\text{Pos}[A \geq a_2(\alpha), B \geq b_2(\alpha)] = \sup_{u \geq a_2(\alpha), v \geq b_2(\alpha)} \min\{A(u), B(v)\} = a.$$  

(17)

Alternatively, Fuller and Majlender [12] introduced the crisp possibilistic variance and covariance of a fuzzy number $A$ as

$$\text{Var}(A) = \int_0^1 a((E(A) - a_1(\alpha))^2 + (E(A) - a_2(\alpha))^2) \, d\alpha.$$  

(18)

$$\text{Cov}(A, B) = \int_0^1 a((E(A) - a_1(\alpha))(E(B) - b_1(\alpha)) + (E(A) - a_2(\alpha))(E(B) - b_2(\alpha))) \, d\alpha.$$  

(19)
Remark 1. We can define the lower possibilistic variance of fuzzy number \( A \) as follows:
\[
\text{Var}_L(A) = \frac{\int_0^1 \alpha \text{Pos}[A \leq a_1(\alpha)] (E_L(A) - a_1(\alpha))^2 \, d\alpha}{\int_0^1 \alpha \text{Pos}[A \leq a_1(\alpha)] d\alpha},
\]
where the upper possibilistic variance of fuzzy number \( A \) is as follows:
\[
\text{Var}_U(A) = \frac{\int_0^1 \alpha \text{Pos}[A \geq a_2(\alpha)] (E_U(A) - a_2(\alpha))^2 \, d\alpha}{\int_0^1 \alpha \text{Pos}[A \geq a_2(\alpha)] d\alpha}.
\]

Remark 2. An alternative definition for the \( r \)-th possibilistic moments of a fuzzy number can be formulated as follows:
\[
E_r(A) = \int_0^1 \alpha [(E_L(A) - a_1(\alpha))^r + (E_U(A) - a_2(\alpha))^r] \, d\alpha.
\]

Remark 3. Along the lines of [42, 43], we can also defined the lower and upper possibilistic \( r \)-th moments of fuzzy number \( A \) as follows:
\[
E'_L(A) = \frac{\int_0^1 \alpha (E_L(A) - a_1(\alpha))^r \, d\alpha}{\int_0^1 \alpha \text{Pos}[A \leq a_1(\alpha)] d\alpha},
\]
\[
E'_U(A) = \frac{\int_0^1 \alpha (E_U(A) - a_2(\alpha))^r \, d\alpha}{\int_0^1 \alpha \text{Pos}[A \geq a_2(\alpha)] d\alpha}.
\]

In this section, we will discuss moments of Gaussian fuzzy numbers and \( O(m, n) - \text{Tr.F.N.} \).

2.2. Moment of \( O(m, n) - \text{Tr.F.N.} \). Using the mathematical properties that linked the Gamma function \( \Gamma(u) \) with the Beta function \( B(u; v) \) such as \( B(u; v) = \int_0^1 t^{u-1} (1 - t)^{v-1} \, dt \), it can easily mathematically shown that the symmetry property also holds true for the beta function such that \( B(u; v) = B(v; u) \). The mathematical relationship that exists between the beta function and the gamma function is that \( B(u; v) = (\Gamma(u) \Gamma(v) / \Gamma(u+v)), \Gamma(u) = (u-1) \), and \( B(u; v) = ((u-1)! (v-1)!/(u+v-1)!)) \). Now, if \( A \) is an \( O(m, n) - \text{Tr.F.N.} \) whose \( a \)-cuts is given by \( A(a) = [a_2 - (a_2-a_1) \alpha, a_3 - (a_3-a_2)(1-\alpha)] \), then we have the lower possibilistic mean \( E_L(A) \) and the upper possibilistic mean \( E_U(A) \) as
\[
E_L(A) = a_2 - 2(a_2-a_1) B \left(2, \frac{1}{m} + 1 \right),
\]
\[
E_U(A) = a_3 - 2(a_3-a_2) B \left(2, \frac{1}{n} + 1 \right).
\]

The interval value possibilistic mean and possibilistic mean are
\[
\text{IVPM}(A) = \left[ a_2 - 2(a_2-a_1) B \left(2, \frac{1}{m} + 1 \right), a_3 - 2(a_3-a_2) B \left(2, \frac{1}{n} + 1 \right) \right].
\]

If \( m = n \), then the interval value possibilistic mean and possibilistic mean are
\[
\text{IVPM}(A) = \left[ E_L(A), E_U(A) \right] = \left[ a_2 - 2(a_2-a_1) B \left(2, \frac{1}{m} + 1 \right), a_3 - 2(a_3-a_2) B \left(2, \frac{1}{n} + 1 \right) \right].
\]

The possibilistic mean is
\[
E(A) = \frac{a_2 + a_3}{2} - \left[ (a_2-a_1) B \left(2, \frac{1}{m} + 1 \right) + (a_3-a_2) B \left(2, \frac{1}{n} + 1 \right) \right].
\]

If \( m = n \), then the possibilistic mean of \( A \) is
The zero centered variance of $A$ is

$$E(A) = \frac{a_2 + a_3 - B(2, \frac{1}{m} + 1)}{2}((a_2 - a_1) + (a_3 - a_4)).$$

(28)

The zero-centered variance of $A$ is when $m = n$ is

$$\text{Var}(A) = \frac{(a_3 - a_2)^2}{4} + (a_3 - a_2)(a_2 - a_1)B\left(2, \frac{1}{m} + 1\right) - (a_3 - a_2)(a_3 - a_4)B\left(2, \frac{1}{n} + 1\right) + \frac{(a_2 - a_1)^2}{2} - (a_2 - a_1)(a_3 - a_4)B\left(2, \frac{2}{m} + 1\right) + \frac{(a_3 - a_4)^2}{2}B\left(2, \frac{2}{n} + 1\right).$$

(29)

The zero-centered variance of $A$ is when $m = n$ is

$$\text{Var}(A) = \frac{(a_3 - a_2)^2}{4} + ((a_3 - a_2)(a_2 - a_1) - (a_3 - a_2)(a_3 - a_4))B\left(2, \frac{1}{m} + 1\right) + \frac{(a_2 - a_1)^2}{2} - (a_2 - a_1)(a_3 - a_4) + \frac{(a_3 - a_4)^2}{2}B\left(2, \frac{2}{m} + 1\right).$$

(30)

Thus, we have the following possibilistic expressions for a fuzzy number, where $m = 2$ and $n = (1/2)$ in expressions (24)–(29), respectively:

$$E_L(A) = a_2 - 2(a_2 - a_1)\frac{\Gamma(2)\Gamma(1 + (1/2))}{\Gamma(3 + (1/2))} = \frac{8}{15}a_1 + \frac{7}{15}a_2,$$

$$E_R(A) = a_3 - 2(a_3 - a_4)\frac{\Gamma(2)\Gamma(1 + 2)}{\Gamma(3 + 2)} = \frac{5}{6}a_3 + \frac{1}{6}a_4,$$

$$E(A) = \frac{4}{15}a_1 + \frac{7}{30}a_2 + \frac{5}{12}a_3 + \frac{1}{12}a_4,$$

$$\text{IVPM}(A) = \left[\frac{8}{15}a_1 + \frac{7}{15}a_2, \frac{5}{6}a_3 + \frac{1}{6}a_4\right].$$

(31)

The possibilistic variance of a fuzzy number, where $m = 2$ and $n = (1/2)$, is given by

$$\text{Var}(A) = \frac{1}{12}(a_1 - a_2)^2 + \frac{1}{4} (a_2 - a_3)^2 + \frac{1}{60} (a_3 - a_4)^2 + \frac{4}{15} (a_1 - a_2)(a_2 - a_3) + \frac{4}{63} (a_1 - a_2)(a_3 - a_4) + \frac{1}{12} (a_2 - a_3)(a_3 - a_4).$$

(32)

To find more details on possibility theory, any interested reader may refer to the works of Carlsson and Fuller [9], Appadoo et al. [41], and Georgescu [45].
3. Possibilistic Properties of Gaussian Fuzzy Numbers

In this section, we discuss certain moments of fuzzy numbers and their applications, such as skewness and kurtosis.

**Theorem 1.** Consider the Gaussian fuzzy number of the form given below such that \( A(\alpha) = [\mu - \sigma \sqrt{(-2 \ln \alpha)}, \mu + \sigma \sqrt{(-2 \ln \alpha)}] \). Then, the lower possibilistic mean, upper possibilistic mean, possibilistic mean, and possibilistic variance are given below:

\[
E_L(A) = \mu - \frac{1}{2} \sigma \sqrt{\pi},
\]

\[
E_R(A) = \mu + \frac{1}{2} \sigma \sqrt{\pi},
\]

\[
E(A) = \mu,
\]

\[
\text{Var}(A) = \sigma^2.
\]

**Proof.** The proof of (33) is easy and is omitted. \( \square \)

**Remark 4.** Consider \( A \) and \( B \) are two Gaussian fuzzy numbers as in Definition 2 such that the \( \alpha \)-cuts for \( A \) and \( B \) are as follows:

\[
A(\alpha) = [\mu - \sigma \sqrt{(-2 \ln \alpha)}, \mu + \sigma \sqrt{(-2 \ln \alpha)}],
\]

\[
B(\alpha) = [\mu_1 - \sigma_1 \sqrt{(-2 \ln \alpha)}, \mu_1 + \sigma_1 \sqrt{(-2 \ln \alpha)}],
\]

where \( 0 \leq \alpha \leq 1, \mu \neq \mu_1, \) and \( \sigma \neq \sigma_1 > 0 \). It can easily be shown that the possibilistic covariance between those two fuzzy numbers is

\[
\text{Cov}(A, B) = \int_0^1 a \left( (a_2(\alpha) - a_1(\alpha))(b_2(\alpha) - b_1(\alpha)) \right) d\alpha = \sigma \sigma_1.
\]

**Remark 5.** Note that if we used the definition of crisp possibilistic variance and crisp possibilistic covariance of [42], then

\[
\text{Var}(A) = \int_0^1 a \left( \left( \frac{1}{2} \int_0^1 a a_1(\alpha) d\alpha - a_1(\alpha) \right)^2 + \left( \frac{1}{2} \int_0^1 a a_2(\alpha) d\alpha - a_2(\alpha) \right)^2 \right) d\alpha,
\]

\[
\text{Cov}(A, B) = \int_0^1 a \left( \left( \frac{1}{2} \int_0^1 a a_1(\alpha) d\alpha - a_1(\alpha) \right) \left( \frac{1}{2} \int_0^1 a b_1(\alpha) d\alpha - b_1(\alpha) \right) \right) d\alpha.
\]

Then,

\[
\text{Var}(A) = \int_0^1 a \left( \left( \frac{1}{2} \int_0^1 a a_1(\alpha) d\alpha - a_1(\alpha) \right)^2 + \left( \frac{1}{2} \int_0^1 a a_2(\alpha) d\alpha - a_2(\alpha) \right)^2 \right) d\alpha,
\]

\[
\text{Cov}(A, B) = \int_0^1 a \left( \left( \frac{1}{2} \int_0^1 a a_1(\alpha) d\alpha - a_1(\alpha) \right) \left( \frac{1}{2} \int_0^1 a b_1(\alpha) d\alpha - b_1(\alpha) \right) \right) d\alpha.
\]

Remark 6. Let \( A \) be a Gaussian fuzzy number as in Definition 2; then, for \( 0 \leq \alpha \leq 1 \), we have \( a_1(\alpha) = \mu - \sigma \sqrt{(-2 \ln \alpha)} \) and \( a_2(\alpha) = \mu + \sigma \sqrt{(-2 \ln \alpha)} \). It can easily be shown that the \( r \)-th possibilistic moments of \( A \) are given by \( E_r(A) = [\int \left( (\sigma)^r (\mu - \sigma \sqrt{(-2 \ln \alpha)}) \right) d\alpha, \int \left( (\sigma)^r (\mu + \sigma \sqrt{(-2 \ln \alpha)}) \right) d\alpha] \). Then, the first four moments of the Gaussian fuzzy number are as follows:

\[
E_1(A) = 0, \quad E_2(A) = \sigma^2, \quad E_3(A) = 0, \quad \text{and} \quad E_4(A) = 2\sigma^4.
\]

Notice that all the odd powers are zero which is also true for the Gaussian probability distribution in probability theory. As a result, it is observed that the possibilistic skewness is equal to zero and possibilistic kurtosis is equal to 2.

**Theorem 2.** Let \( A \) be a Gaussian fuzzy number of the form given in Definition 2 whose \( \alpha \)-cuts are

\[
A(\alpha) = [a_1(\alpha), a_2(\alpha)]
\]

\[
= [\mu - \sigma \sqrt{(-2 \ln \alpha)}, \mu + \sigma \sqrt{(-2 \ln \alpha)}],
\]

where \( a_1(\alpha) = \mu - \sigma \sqrt{(-2 \ln \alpha)} \) and \( a_2(\alpha) = \mu + \sigma \sqrt{(-2 \ln \alpha)} \) for \( 0 \leq \alpha \leq 1 \). If we use the \( r \)-th possibilistic moments formula given by

\[
E_r(A) = \int_0^1 a ((E_L(A) - a_1(\alpha))^r + (E_R(A) - a_2(\alpha))^r) d\alpha,
\]

where we explicitly used...
\[ E_L (A) = \mu - \frac{1}{2} \sigma \sqrt{\pi}, \]  
\[ E_R (A) = \mu + \frac{1}{2} \sigma \sqrt{\pi}, \]  
\[ \text{then} \]
\[ E_r (A) = \frac{((-1)^r + 1)}{2} \sum_{k=0}^{r} \left( \frac{r!}{(r-k)!k!} \right) (-\sigma)^k \left( \frac{1}{2} \sigma \sqrt{\pi} \right)^{r-k} \Gamma \left( \frac{k+1}{2} \right). \]  
\[ (41) \]

**Proof**

\[ E_r (A) = \int_0^1 a \left( (E_L (A) - a_1 (a))^r + (E_R (A) - a_2 (a))^r \right) da \]
\[ = \frac{((-1)^r + 1)}{2} \left( \sum_{k=0}^{r} \frac{r!}{(r-k)!k!} \left( \frac{1}{2} \sigma \sqrt{\pi} \right)^{r-k} \right) \int_0^\infty e^{-2u} (-\sigma \sqrt{2u})^k du \]
\[ = \frac{((-1)^r + 1)}{2} \left( \sum_{k=0}^{r} \frac{r!}{(r-k)!k!} (-\sigma)^k \left( \frac{1}{2} \sigma \sqrt{\pi} \right)^{r-k} \Gamma \left( \frac{k+1}{2} \right) \right). \]  
\[ (42) \]

It can be observed that \( E_2 (A) = -(1/4) \sigma^2 \pi + \sigma^2 \), \( E_3 (A) = 0 \), \( E_4 (A) = -(3/16) \sigma^4 \pi^2 + 2 \sigma^4 \), \( E_5 (A) = 0 \), \( E_6 (A) = -(5/64) \sigma^6 \pi^3 - (15/16) \sigma^6 \pi^2 + (15/8) \sigma^6 \pi + 6 \sigma^6 \), and \( E_7 (A) = 0 \). Note that all odd possibilistic moments are zeros. The possibilistic skewness and possibilistic kurtosis are as follows:
\[ S(A) = \frac{E_3 (A)}{\sqrt{E_2 (A)}} = 0, \]  
\[ (43) \]
\[ K (A) = \frac{E_4 (A)}{(E_2 (A))^2} = \frac{-(3/16) \sigma^4 \pi^2 + 2 \sigma^4}{-(1/4) \sigma^2 \pi^2 + \sigma^2} = 3.2451. \]

**Theorem 3.** Let \( A \) be a Gaussian fuzzy number of the form as in Definition 2 whose \( \alpha \)-cuts are
\[ A (\alpha) = [a_1 (\alpha), a_2 (\alpha)] = [\mu - \sigma \sqrt{(-2 \ln \alpha)}, \mu + \sigma \sqrt{(-2 \ln \alpha)}], \]  
\[ (44) \]
where \( a_1 (\alpha) = \mu - \sigma \sqrt{(-2 \ln \alpha)} \) and \( a_2 (\alpha) = \mu + \sigma \sqrt{(-2 \ln \alpha)} \) for \( 0 \leq \alpha \leq 1 \). If we use the \( r^{th} \) possibilistic moments formula given by
\[ E_r (A) = \int_0^1 a (a_1 (\alpha) + a_2 (\alpha)) da, \]  
\[ (45) \]
then
\[ E_r (A) = \sum_{k=0}^{r} \left( \frac{((-1)^{r-k} + 1)}{2} \right) \frac{r!}{(r-k)!k!} (-\sigma)^k \left( \frac{1}{2} \sigma \sqrt{\pi} \right)^{r-k} \Gamma \left( \frac{k+1}{2} \right). \]  
\[ (46) \]

**Proof**
\[ E_1(A) = \int_0^1 a \alpha^2 (a) + \alpha^3 (a) \, da = \int_0^1 a \left( (\mu - \sigma \sqrt{2} \sqrt{-\ln(a)}) + (\mu + \sigma \sqrt{2} \sqrt{-\ln(a)}) \right) \, da \]
\[ = \int_0^\infty e^{-2a} \left( \sum_{k=0}^r \frac{r!}{(r-k)!k!} \mu (\sigma \sqrt{2} u^{1/2})^{r-k} \right) \, du 
\]
\[+ \int_0^\infty e^{-2a} \left( \sum_{k=0}^r \frac{r!}{(r-k)!k!} \mu (\sigma \sqrt{2} u^{1/2})^{r-k} \right) \, du 
\]
\[= \sum_{k=0}^r \left( \frac{(-1)^{-k} + 1}{2} \right) \frac{r!}{(r-k)!k!} \mu^k (a)^{-1/2} \left( \frac{r-k+1}{2} \right). \tag{47} \]

The possibilistic kurtosis is given by
\[ K(A) = \frac{(2\sigma^4 + 6\mu^2 \sigma^2 + \mu^4) - 4\mu(3\mu \sigma^2 + \mu^3) + 6\mu^2(\sigma^2 + \mu^2)}{(\sigma^2)^2} - 4\mu^3 + \mu^4 = 2, \tag{48} \]
and the possibilistic skewness is \( S(A) = 0. \)

**Theorem 4.** Let \( A \) be an \( O(m, n) - Tr.T.N. \), whose \( \alpha \)-cuts are explicitly given by
\[ A(\alpha) = [a_2 - (a_2 - a_1)(1 - \alpha)^{1/m}, a_3 - (a_3 - a_4)(1 - \alpha)^{1/n}], \tag{49} \]
where \( a_1(\alpha) = [a_2 - (a_2 - a_4)(1 - \alpha)^{1/m}] \) and \( a_2(\alpha) = [a_3 - (a_3 - a_4)(1 - \alpha)^{1/n}] \) for \( 0 \leq \alpha \leq 1. \) If we use the \( \eta^\alpha \) possibilistic moments formula given by
\[ E_\alpha(A) = \int_0^1 a (\alpha) \alpha^2 (\alpha) + \alpha^3 (\alpha) \, da, \tag{50} \]
then
\[ E_{\eta}(A) = \int_0^1 a (\alpha) (\alpha + 2) \alpha^3 (\alpha) \, da. \tag{51} \]

**Proof.** The proof is obvious and is omitted.

For a \( Tr.F.N. \) with \( m = n = 1, \) we have the following results:

\[ E_2(A) = \frac{1}{4} (a_2^2 + a_3^2) + \frac{1}{6} (a_2 a_1 + a_3 a_4) + \frac{1}{12} (a_1^2 + a_4^2), \tag{52} \]
\[ E_3(A) = \frac{1}{5} (a_2^3 + a_3^3) + \frac{1}{20} (3a_2^2 a_1 + a_1^3 + 3a_3^2 a_4 + a_4^3) + \frac{1}{10} (a_2 a_1^2 + a_3 a_4^2), \tag{53} \]
\[ E_4(A) = \frac{1}{15} (a_2 a_1^2 + 2a_3 a_4 + a_3 a_4 + 2a_2 a_1) + \frac{1}{10} (a_2 a_1^2 + a_3 a_4^2) + \frac{1}{6} (a_2 + a_4^2) + \frac{1}{30} (a_1^2 + a_4^2). \tag{54} \]

Note that if we set \( a_1 = a - \alpha, a_2 = a, a_3 = b, \) and \( a_4 = b + \beta \) in (52)–(54), we obtained the same results as Thavaneswaran et al. [13] and Thavaneswaran et al. [11] as follows:
\[ E_2(A) = \frac{1}{4} (a_2^2 + a_3^2) + \frac{1}{6} (a_2a_1 + a_3a_4) + \frac{1}{12} (a_1^2 + a_4^2) \]
\[ = \frac{(a^2 + b^2)}{2} + \frac{b^2 - 2a}{3} + \frac{a^2 + b^2}{12}, \]
\[ E_3(A) = \frac{1}{5} (a_2^2 + a_3^2) + \frac{1}{20} (3a_2a_1 + a_3^3 + 3a_1a_4 + a_4^3) + \frac{1}{10} (a_2a_1 + a_3a_4) \]
\[ = \frac{b^3 - a^3}{20} + \frac{aa^3 + b^2}{4} + \frac{b^3 - a^2a + a^3 + b^3}{2}, \]
\[ E_4(A) = \frac{1}{15} (a_2a_1^3 + 2a_3a_4 + a_3a_4^3 + 2a_1a_4) + \frac{1}{10} (a_2a_1^2 + a_3^2a_4) + \frac{1}{6} (a_2^2 + a_4^2) + \frac{1}{30} (a_1^2 + a_4^2) \]
\[ = \frac{a^4 + b^4}{30} + \frac{b^4 - aa^4}{5} + \frac{a^4 + b^4 + a^4 + b^4}{2} + \frac{2b^4 - 2a^4}{3}. \] (55)

**Theorem 5.** Let A be an O(m, n) - Tr.T.N, whose α-cuts are explicitly given by

\[ A(\alpha) = [a_2 - (a_2 - a_1)(1 - \alpha)^{1/m}, a_3 - (a_3 - a_4)(1 - \alpha)^{1/n}], \] (56)

where \( a_1(\alpha) = [a_2 - (a_2 - a_1)(1 - \alpha)^{1/m}] \) and \( a_2(\alpha) = [a_3 - (a_3 - a_4)(1 - \alpha)^{1/n}] \) for \( 0 \leq \alpha \leq 1 \) and

\[ E(A) = \left( \frac{a_2 + a_4}{2} - \frac{m^2(a_2 - a_1)}{(1 + m)(1 + 2m)} - \frac{n^2(a_3 - a_4)}{(n + 1)(2n + 1)} \right). \] (57)

We use the \( r^{th} \) possibilistic moments formula given by

\[ E_r(A) = \int_0^1 a((a_1(\alpha) - E(A))^r + (a_2(\alpha) - E(A))^r) \, da \]

\[ + \left( \sum_{k=0}^{r-k} \frac{r!}{(r-k)!k!} \left( a_2 - \left( \frac{(a_2 + a_3)}{2} - \frac{m^2(a_2 - a_1)}{(1 + m)(1 + 2m)} - \frac{n^2(a_3 - a_4)}{(n + 1)(2n + 1)} \right) \right)^{r-k} a_1 \Gamma((k/m) + 1) \right) \]

\[ + \left( \sum_{k=0}^{r-k} \frac{r!}{(r-k)!k!} \left( a_3 - \left( \frac{(a_2 + a_3)}{2} - \frac{m^2(a_2 - a_1)}{(1 + m)(1 + 2m)} - \frac{n^2(a_3 - a_4)}{(n + 1)(2n + 1)} \right) \right)^{r-k} a_3 \Gamma((k/n) + 1) \right) \] (58)

**Proof.** The proof is easy and is omitted. \( \Box \)

**Theorem 6.** Let A be an O(m, n) - Tr.T.N, whose α-cuts are explicitly given by

\[ A(\alpha) = [a_2 - (a_2 - a_1)(1 - \alpha)^{1/m}, a_3 - (a_3 - a_4)(1 - \alpha)^{1/n}], \] (59)

where \( a_1(\alpha) = [a_2 - (a_2 - a_1)(1 - \alpha)^{1/m}] \) and \( a_2(\alpha) = [a_3 - (a_3 - a_4)(1 - \alpha)^{1/n}] \) for \( 0 \leq \alpha \leq 1 \). If we use the following formula for the \( r^{th} \) possibilistic moments

\[ E_r(A) = \int_0^1 a((E_L((a_1(\alpha)) a_1(\alpha)))^r + (E_R(A) - a_2(\alpha))^r) \, da, \] (60)

where

\[ E_L(A) = a_2 - \left( \frac{2m^2(a_2 - a_1)}{(1 + m)(1 + 2m)} \right); \]

\[ E_R(A) = a_3 - \left( \frac{2n^2(a_3 - a_4)}{(n + 1)(2n + 1)} \right), \] (61)

then we have the \( r^{th} \) moments as
\[ E_r(A) = \left( \sum_{k=0}^{r} \frac{r!}{(r-k)!k!} \left( \frac{2m^2(a_3-a_2)}{(1+m)(1+2m)} \right)^{r-k} (a_2-a_1)^k \Gamma(2,(k/m) + 1) \right) \Bigg/ \Gamma((k/m) + 3) \]
\[ + \left( \sum_{k=0}^{r} \frac{r!}{(r-k)!k!} \left( \frac{2r^2(a_4-a_3)}{(n+1)(2n+1)} \right)^{r-k} (a_3-a_4)^k \Gamma(2,(k/n) + 1) \right) \Bigg/ \Gamma((k/n) + 3) \].

**Proof.**

\[ E_r(A) = \int_0^1 a (E_L(A) - a_1(a))' + (E_R(A) - a_2(a))' \, da \]
\[ = \int_0^1 a \left( \left( a_2 - \frac{2m^2(a_3-a_2)}{(1+m)(1+2m)} \right) - (a_2 - (a_2-a_1)(1-\alpha)^{1/m}) \right)^r \, da \]
\[ + \left( \left( a_3 - \frac{2n^2(a_4-a_3)}{(n+1)(2n+1)} \right) - (a_3 - (a_3-a_4)(1-\alpha)^{1/n}) \right)^r \, da. \]

Hence, the result

\[ E_r(A) = \left( \sum_{k=0}^{r} \frac{r!}{(r-k)!k!} \left( \frac{2m^2(a_3-a_2)}{(1+m)(1+2m)} \right)^{r-k} (a_2-a_1)^k \Gamma(2,(k/m) + 1) \right) \Bigg/ \Gamma((k/m) + 3) \]
\[ + \left( \sum_{k=0}^{r} \frac{r!}{(r-k)!k!} \left( \frac{2r^2(a_4-a_3)}{(n+1)(2n+1)} \right)^{r-k} (a_3-a_4)^k \Gamma(2,(k/n) + 1) \right) \Bigg/ \Gamma((k/n) + 3) \].

### 3.1. Fuzzy Possibilistic Economic Production Quantity Inventory Model

In this section, the following fuzzy numbers are used to develop the fuzzy production inventory model, where \( \bar{D} \) is the fuzzy yearly demand, \( \bar{C}_o \) is the fuzzy annual ordering cost per run, \( \bar{d} \) is the fuzzy daily demand rate, and \( \bar{C}_h \) is the fuzzy holding cost. For further details, see [32, 33]:
\[ C_0(\alpha) = [C_1(\alpha), C_2(\alpha)], \]
\[ D(\alpha) = [D_1(\alpha), D_2(\alpha)], \]
\[ C_h(\alpha) = [h_1(\alpha), h_2(\alpha)], \]
\[ d(\alpha) = [d_1(\alpha), d_2(\alpha)]. \]

It is important to note here that the only parameter that we allow to be crisp in our model formulation is the daily production rate \( p \). Thus, the fuzzy total production inventory cost can be written in terms of its \( \alpha \)-level set as
\[ TC(\alpha) = C_0(\alpha)\frac{D(\alpha)}{q} + C_h(\alpha)\frac{q}{2}\left[1 - \frac{d(\alpha)}{p}\right]. \]  

Using equation (3), we can rewrite the \( \alpha \)-level set of \( TC(\alpha) \), for \( 0 \leq \alpha \leq 1 \), as follows:
\[ [TC_1(\alpha), TC_2(\alpha)] = [C_{01}(\alpha), C_{02}(\alpha)]\left(\frac{D_1(\alpha)}{q}, \frac{D_2(\alpha)}{q}\right) + [C_{h1}(\alpha), C_{h2}(\alpha)]\frac{q}{2}\left[1 - \frac{d_2(\alpha)}{p}, 1 - \frac{d_1(\alpha)}{p}\right]. \]

We make the assumption at \( 0 < (d_1(\alpha)/p) < 1 \) and \( 0 < (d_2(\alpha)/p) < 1 \) along with \((d_2(\alpha)/p) > (d_1(\alpha)/p)\). Simplifying expression (67) yields the following \( \alpha \)-level set for fuzzy total cost as
\[ [TC_1(\alpha), TC_2(\alpha)] = \begin{bmatrix}
\left(\frac{C_{01}(\alpha)D_1(\alpha)}{q}\right) + \frac{C_{h1}(\alpha)q}{2}\left(1 - \frac{d_2(\alpha)}{p}\right) \\
\left(\frac{C_{02}(\alpha)D_2(\alpha)}{q}\right) + \frac{C_{h2}(\alpha)q}{2}\left(1 - \frac{d_1(\alpha)}{p}\right)
\end{bmatrix}, \]  

where
\[ TC_1(\alpha) = \left(\frac{C_{01}(\alpha)D_1(\alpha)}{q}\right) + \frac{C_{h1}(\alpha)q}{2}\left(1 - \frac{d_2(\alpha)}{p}\right), \]
\[ TC_2(\alpha) = \left(\frac{C_{02}(\alpha)D_2(\alpha)}{q}\right) + \frac{C_{h2}(\alpha)q}{2}\left(1 - \frac{d_1(\alpha)}{p}\right). \]

The lower possibilistic total production inventory cost is given as
\[ E_L(TC) = \int_{\alpha} \text{Poss}[TC \leq TC_1(\alpha)]\left((\frac{C_{01}(\alpha)D_1(\alpha)}{q}) + \frac{C_{h1}(\alpha)q}{2}(1 - \frac{d_2(\alpha)}{p})\right) d\alpha, \]
\[ \int_{\alpha} \text{Poss}[TC \leq TC_1(\alpha)] d\alpha \]

The upper possibilistic total production inventory cost is given as
\[ E_R(TC) = \int_{\alpha} \text{Poss}[TC \geq TC_2(\alpha)]\left((\frac{C_{02}(\alpha)D_2(\alpha)}{q}) + \frac{C_{h2}(\alpha)q}{2}(1 - \frac{d_1(\alpha)}{p})\right) d\alpha, \]
\[ \int_{\alpha} \text{Poss}[TC \geq TC_2(\alpha)] d\alpha \]
The weighted average of (70) and (71) lead to $E_{LR}(TC)$ as follows:

$$
E_{LR}(TC) = \int_0^1 \left( \frac{C_{o1}(\alpha)D_1(\alpha)}{q} + \frac{C_{h1}(\alpha)q}{2} \left( 1 - \frac{d_2(\alpha)}{p} \right) \right) \, d\alpha. \tag{72}
$$

(a) In order to find the lower possibilistic optimal economic production quantity, we set

$$
d \frac{\int_0^1 \text{Poss}[TC \leq TC_1(\alpha)] \left( \frac{(C_{o1}(\alpha)D_1(\alpha)/q) + ((C_{h1}(\alpha)q/2)(1 - (d_2(\alpha)/p))) \, d\alpha}{\int_0^1 \text{Poss}[TC \leq TC_1(\alpha)] \, d\alpha} \right)}{dq} = 0 \tag{73}
$$

and then solve for $q$. That value of $q$ that minimizes (73) is the lower possibilistic production quantity which will be denoted by $q_1$.

(b) In order to find the upper possibilistic optimal economic production quantity, we set

$$
d \frac{\int_0^1 \text{Poss}[TC \geq TC_2(\alpha)] \left( \frac{(C_{o2}(\alpha)D_2(\alpha)/q) + ((C_{h2}(\alpha)q/2)(1 - (d_1(\alpha)/p))) \, d\alpha}{\int_0^1 \text{Poss}[TC \geq TC_2(\alpha)] \, d\alpha} \right)}{dq} = 0 \tag{74}
$$

and then solve for $q$. That value of $q$ that minimizes (74) is the upper possibilistic production quantity which will be denoted by $q_2$.

(c) In order to find the possibilistic optimal economic production quantity, we set

$$
d \frac{\int_0^1 \left( \frac{C_{o1}(\alpha)D_1(\alpha)}{q} + \frac{C_{h1}(\alpha)q}{2} \left( 1 - \frac{d_2(\alpha)}{p} \right) \right) \, d\alpha}{dq} = 0 \tag{75}
$$

and then solve for $q$. That value of $q$ that minimizes (75) is the possibilistic optimal economic production quantity which will be denoted by $q^\ast$.

Remark 7. It is important to remark here that $q^\ast \neq ((q_1 + q_2)/2)$.

Below, we give a theorem to find fuzzy total cost for the economic production quantity inventory model with finite replenishment rate in a fuzzy possibilistic setup. In what follows, we shall assume that $C_o$, $D$, $C_h$, and $d$ are Tr.F.N.’s, where $C_o = (c_1, c_2, c_3, c_4)$, $D = (D_1, D_2, D_3, D_4)$, $C_h = (h_1, h_2, h_3, h_4)$, and $d = (d_1, d_2, d_3, d_4)$. Paraindent after equation (76).
\[
C_q(\alpha) = [C_{q1}(\alpha), C_{q2}(\alpha)] = [c_1 + \alpha(c_2 - c_1), c_4 + \alpha(c_3 - c_4)],
\]
\[
D(\alpha) = [D_1(\alpha), D_2(\alpha)] = [D_1 + \alpha(D_2 - D_1), D_4 + \alpha(D_3 - D_4)],
\]
\[
C_h(\alpha) = [C_{h1}(\alpha), C_{h2}(\alpha)] = [h_1 + \alpha(h_2 - h_1), h_4 + \alpha(h_3 - h_4)],
\]
\[
d(\alpha) = [d_1(\alpha), d_2(\alpha)] = [d_1 + \alpha(d_2 - d_1), d_4 + \alpha(d_3 - d_4)],
\]
\[
TC_L(q) = \left(\frac{1}{6} \frac{3c_2D_2 + c_1D_1 + c_1D_2 + c_1D_1}{q}\right) + \left(\frac{1}{12} \frac{-3d_1h_2 - h_1d_3 - d_4h_2 - h_1d_4 + 4ph_2 + 2ph_1}{p}\right),
\]
\[
TC_R(q) = \left(\frac{1}{6} \frac{3c_3D_3 + D_4c_3 + c_4D_4}{q}\right) - \left(\frac{1}{12} \frac{3h_3d_2 + h_4d_3 + c_4D_4}{p}\right),
\]

and we have the following values for lower \( q_1 \) as

\[
TC_{LR}(q) = \frac{1}{12} \left(\frac{3c_2D_2 + c_2D_1 + c_1D_2 + c_1D_1}{q}\right) + \left(\frac{1}{24} \frac{-3d_1h_2 - h_1d_3 - d_4h_2 - h_1d_4 + 4ph_2 + 2ph_1}{p}\right),
\]

\[
Theorem 7. For \( 0 \leq \alpha \leq 1 \), let the fuzzy ordering cost \( C_q(\alpha) \), fuzzy demand \( D(\alpha) \), fuzzy holding cost \( C_h(\alpha) \), and fuzzy demand rate \( d(\alpha) \) be given by

\[
q_1 = \sqrt{\frac{2p(3c_2D_2 + c_1D_1 + c_1D_2 + c_1D_1)}{(-3d_1h_2 - h_1d_3 - d_4h_2 - h_1d_4 + 4ph_2 + 2ph_1)}},
\]

\[
The upper possibilistic production lot size \( q_2 \) is

\[
q_2 = \sqrt{\frac{2p(3c_2D_1 + D_4c_3 + c_4D_4)}{(4h_3p + 2h_4p - 3h_3d_2 - h_3d_3 - h_4d_2 - h_4d_1 - h_4d_1)}}.
\]

The possibilistic production lot size \( q^* \) is

\[
q^* = \sqrt{\frac{2p[(3c_2D_2 + c_1D_1 + c_1D_2 + c_1D_1) + (3c_3D_3 + D_4c_3 + c_4D_4) - (3h_3d_2 + h_4d_2 + h_4d_1 + h_4d_1 - 4h_3p - 2h_4p)]}{((-3d_1h_2 - h_1d_3 - d_4h_2 - h_1d_4 + 4ph_2 + 2ph_1)})}.
\]

Using (77), the total cost in a possibilistic setup at \( q_1 \) is

\[
TC_L(q = q_1) = \frac{3c_2D_2 + c_2D_1 + c_1D_2 + c_1D_1}{6\sqrt{2p(3c_2D_2 + c_1D_1 + c_1D_2 + c_1D_1)/(-3d_1h_2 - h_1d_3 - d_4h_2 - h_1d_4 + 4ph_2 + 2ph_1))}} + \left(\frac{1}{12} \frac{2p(3c_2D_2 + c_2D_1 + c_1D_2 + c_1D_1)}{(-3d_1h_2 - h_1d_3 - d_4h_2 - h_1d_4 + 4ph_2 + 2ph_1)}\right).
\]
Using (78), the total cost in a possibilistic setup at \( q_2 \) is

\[
TC_R(q = q_2) = \frac{3c_3D_3 + D_4c_3 + c_4D_4}{6\sqrt{2p(3c_3D_3 + D_4c_3 + c_4D_4)/(4h_3p + 2h_4p - 3h_3d_2 - h_3d_1 - h_4d_1)}}
\]

\[
+ \left( \frac{1}{12} \left( \frac{2p(3c_3D_3 + D_4c_3 + c_4D_4)}{(4h_3p + 2h_4p - 3h_3d_2 - h_3d_1 - h_4d_1)} \right) \right)
\]

\[
\left( \frac{(3h_3d_2 + h_3d_1 + h_4d_1 - 4h_3p - 2h_4p)}{p} \right)
\]

Using (79), the total cost in a possibilistic setup at \( q^* \) is

\[
= \frac{(3c_2D_2 + c_2D_1 + c_1D_2 + c_1D_1 + 3c_4D_3 + D_4c_3 + c_4D_4)}{12\sqrt{2p((3c_2D_2 + c_2D_1 + c_1D_2 + c_1D_1) + (3c_4D_3 + D_4c_3 + c_4D_4))(-3h_3d_2 - h_3d_1 - d_4h_2 - h_3d_4 + 4ph_2 + 2ph_1)}}
\]

\[
+ \frac{1}{24} \left( \frac{2p((3c_2D_2 + c_2D_1 + c_1D_2 + c_1D_1) + (3c_4D_3 + D_4c_3 + c_4D_4)}{(-3h_3d_2 - h_3d_1 - d_4h_2 - h_3d_4 + 4ph_2 + 2ph_1)} \right)
\]

\[
\left( \frac{-3d_2h_2 - h_1d_3 - d_4h_2 - h_3d_4 + 4ph_2 + 2ph_1}{p} \right)
\]

\[
\left( \frac{3h_3d_2 + h_3d_1 + h_4d_1 - 4h_3p - 2h_4p}{p} \right)
\]

\[
(84)
\]

\[
(85)
\]

4. Findings and Practical and Managerial Implications

The fuzzy possibilistic inventory model derived in this paper can be considered as a more generalized version of the classical economic inventory model. We have determined the optimal order quantity in a fuzzy sense using the concept of possibility theory. Although the problem considered in this paper is geared towards a single-item and single-period inventory model, it can be easily extended to multiple periods with multiple items’ inventory models. These fuzzy possibilistic inventory models can be easily developed to accommodate other inventory models with various products. The solution of the fuzzy model does not require excessive computation; moreover, the fuzzy possibilistic approach can be extended to include budgetary, space, and weight constraints.

Most of the methods in the literature are unfortunately limited in that sense. The possibilistic inventory model can be modified to compute future demand in a vague sense and thus generate dynamic reordering points in case of unexpected stock out of products. It is vital to reduce inventory costs during the ongoing COVID-19 pandemic; thus, it is our view that fuzzy inventory models can address the sudden rise in inventory cost, with this paper serving as an attempt to do so. It has been heavily documented that inventories consume a large part of budget, space, overheads, and maintenance. Therefore, we need a sophisticated and practical solution-based approach to address the rise in inventory costs.

5. Conclusion

The supply chain network contains many subsystems, and each subsystem usually includes uncertainties. Thus, the overall network becomes uncertain and imprecise. This paper advocates the use of fuzzy numbers to model the uncertainty underlying the SC network. In this paper, we derive the possibilistic moments of some nonlinear types of fuzzy numbers, and those proposed fuzzy numbers can be
regarded as an extension of previous results for the linear fuzzy numbers. These new derivations and concepts are consistent with the definition of moments in probability theory. Furthermore, we have been able to give an alternative definition for possibilistic variance. We also define lower possibilistic $r^{th}$ moments and upper possibilistic $r^{th}$ moments of fuzzy numbers, which have numerous applications in supply chain management. Finally, a fuzzy EPQ inventory model is developed to demonstrate the application of proposed fuzzy sets. The proposed fuzzy inventory model does not reject other fuzzy inventory formulations, but rather complements those already commonly used. Future studies should also consider a modified version of the EPQ model for perishable products, making the model even more appealing for supply chain professionals. This approach can motivate new research in the other functional areas of the supply chain network as well. Another important aspect of this research is that, in the post-COVID era, most of the parameters in inventory models will be arbitrary and include a considerable degree of randomness. Researchers have started to realize the complicated natures of those inventory systems. Most inventory parameters fluctuate from one period to the next, and the classical deterministic inventory model will no longer be applicable. This research considers a fuzzy inventory model compared to a deterministic one, enabling the inventory personnel to adapt to demand fluctuations and thus replenish stock as required. In our view, this research partly addresses this gap by considering fuzzy inventory parameters.

**Data Availability**

No dataset were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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