A Theorem on First-Order Interaction Vertices for Free $p$-Form Gauge Fields

Marc Henneaux$^{a,b}$ and Bernard Knaepen$^{a,c}$

$^a$ Physique Théorique et Mathématique, Université Libre de Bruxelles, Campus Plaine C.P. 231, B–1050 Bruxelles, Belgium

$^b$ Centro de Estudios Científicos de Santiago, Casilla 16443, Santiago 9, Chile

$^c$ DAMTP, Silver Street, Cambridge CB3 9EW, UK

Abstract

The complete proof of a theorem announced in [1] on the consistent interactions for (non-chiral) exterior form gauge fields is given. The theorem can be easily generalized to the analysis of anomalies. Its proof amounts to computing the local BRST cohomology $H^0(s|d)$ in the space of local $n$-forms depending on the fields, the ghosts, the antifields and their derivatives.

$^1$henneaux@ulb.ac.be, bknaepen@ulb.ac.be
1 Introduction

Geometric attempts to generalize the Yang-Mills construction to $p$-form gauge fields with $p > 1$ have led to no-go results that indicate that this goal cannot be achieved while maintaining spacetime locality [2, 3, 4]. In fact, self-interactions of $p$-form gauge fields are so constrained that one can completely list them, even if one drops any a priori geometric interpretation of the $p$-forms as connections for extended objects. This task was explicitly performed in [1], where the following question was analyzed.

Consider the free action,

$$I = \int d^n x \sum_a \left( \frac{-1}{2(p_a + 1)!} H^a_{\mu_1 \ldots \mu_{p_a+1}} H^{a\mu_1 \ldots \mu_{p_a+1}} \right), \quad (1.1)$$

for a system of (non-chiral) exterior form gauge fields $B^a_{\mu_1 \ldots \mu_{p_a}}$ of degree $\geq 2$. Here, the $H^a$’s are the “field strengths” or “curvatures”,

$$H^a = \frac{1}{(p_a + 1)!} H^a_{\mu_1 \ldots \mu_{p_a+1}} dx^{\mu_1} \ldots dx^{\mu_{p_a+1}} = dB^a, \quad (1.2)$$

$$B^a = \frac{1}{p_a!} B^a_{\mu_1 \ldots \mu_{p_a}} dx^{\mu_1} \ldots dx^{\mu_{p_a}}. \quad (1.3)$$

We assume throughout that the spacetime dimension satisfies the condition $n > p_a + 1$ for each $a$ so that all the $p_a$-forms have local degrees of freedom. The action (1.1) is invariant under the abelian gauge transformations,

$$B^a \to B^a + d\Lambda^a, \quad (1.4)$$

where $\Lambda^a$ are arbitrary $p_a - 1$ forms. The equations of motion, obtained by varying the fields $B^a_{\mu_1 \ldots \mu_{p_a}}$, are given by,

$$\partial_\rho H^{a\rho \mu_1 \ldots \mu_{p_a}} = 0 \iff d\overline{H}^a = 0, \quad (1.5)$$

where $\overline{H}^a$ is the dual of $H^{a\mu_1 \ldots \mu_{p_a}}$.

The question addressed in [1] was: what are the consistent (local) interactions that can be added to the free action (1.1)? Interaction terms are said to be consistent if their preserve the number (but not necessarily the form) of the independent gauge symmetries.
Of course, one can always add to (1.1) gauge-invariant interaction terms constructed out of the curvature components and their derivatives,

$$\int f(H^{(k)}_{\mu_1 \ldots \mu_{p_k+1}}, \partial_\nu H^{(k)}_{\mu_1 \ldots \mu_{p_k+1}}, \ldots, \partial_\nu \ldots \partial_\gamma H^{(k)}_{\mu_1 \ldots \mu_{p_k+1}}) d^n x.$$  (1.6)

Being strictly gauge-invariant, these terms actually do not deform the gauge symmetries. One may, however, also search for interaction terms that deform not only the action, but also the gauge transformations. These turn out to be extremely scarce, as the following theorem indicates:

**Theorem 1.1** Besides the obvious gauge-invariant interactions, the only consistent interaction vertices that can be added to (1.1) have the Noether form,

$$V = \sum_{(A)} g(A) V(A)$$  (1.7)

where the $g(A)$ are the coupling constants and the $V(A)$ read

$$V(A) = \int j^{(t)} \wedge B^{(t)}.$$  (1.8)

Here, $j^{(t)}$ are gauge-invariant conserved $(n - p_t)$-forms, $d j^{(t)} \approx 0$, and therefore, are exhausted by the exterior polynomials in the curvature forms $H^{(k)}$ and their duals $\bar{H}^{(k)}$.

Because $j^{(t)}$ must have exactly form-degree $n - p_t$, so that the form degree of the integrand of (1.8) matches the spacetime dimension $n$, there may be no vertex of the type (1.7) for given spacetime dimension and form-degrees of the exterior form gauge fields. For example, a set of 2-form gauge fields admits gauge symmetry-deforming non-trivial interactions only in $n = 4$ dimensions and these are of the Freedman-Townsend type. Other examples of vertices of the form (1.8) involving $p$-form gauge fields of different form degrees are provided by the Chapline-Manton interactions. The analysis of also enabled one to exhibit new symmetry-deforming interactions, but again only in special dimensions (see also these interactions have been further analysed in [14].)

In (1.8), the $j^{(t)}$ are exterior polynomials in $H^{(k)}$ and $\bar{H}^{(k)}$ with coefficients that can involve $dx^\mu$. If one imposes Lorentz invariance, bare $dx^\mu$’s cannot appear. Note also that if $(n - 1)$-forms are included, an infinite number of
couplings (1.8) may in general be constructed since arbitrary powers of the
duals (which are zero forms) can appear.

The vertices (1.7) have a number of remarkable properties:

1. First, while the strictly gauge-invariant vertices may involve derivatives
   of the individual components $H^{(k)}_{\mu_1...\mu_{p+1}}$ of the curvatures, the vertices
   (1.8) are very special: they can be expressed as polynomials in the exterior product
   ("exterior polynomials") in the (undifferentiated) forms $B^{(k)}$, $H^{(k)}$ and $\bar{H}^{(k)}$. This is not an extra requirement. Rather, this
   property follows directly from the demand that (1.7) defines a consistent interaction.

2. If the vertices (1.7) do not involve the duals $\bar{H}^{(k)}$, one recovers the fa-
miliar Chern-Simons terms [17]. These are off-shell gauge-invariant up
   to a total derivative and so, do not deform the gauge transformations. Vertices (1.7) involving the duals are only on-shell gauge-invariant up to
   a total derivative. These vertices do deform the gauge transformations.

3. Although the vertices (1.7) deform the gauge symmetries when they
   involve the duals $\bar{H}^{(k)}$, they do not modify the algebra of the gauge
   transformations (to the first order in the coupling constants considered
   here) because they are linear in the $p$-form potentials. This is in sharp
   contrast with the Yang-Mills construction, which yields a vertex of the
   form $\bar{H}^a \wedge B^b \wedge B^c$. There is thus no room for an analog of the Yang-
   Mills vertex for exterior forms of degree $\geq 2$. How the result is amended
   in the presence of 1-forms will be discussed at the end.

4. The fact that the gauge transformations remain abelian to first-order
   in the coupling constant is not in contradiction with [16]. Indeed, we
   focus here only on symmetries of the equations of motion that are also
   symmetries of the action. Furthermore, the non-abelian structure un-
   covered in [15] concerns symmetries associated with non-trivial global
   features of the spacetime manifold, which are rigid symmetries [19].

The above theorem was stated and discussed in [1] but a complete demon-
stration of it was not given. The purpose of this paper is to fill this gap. As
we shall see, the proof has an interest in itself since it illustrates various
cohomologies arising in local field theory.
We conclude this introduction by observing that the interaction vertices are in general not duality-invariant, in the sense that an interaction vertex that is available in one version of the theory may not be so in the dual version where some of the $p$-form potentials are traded for “dual” $(n - p - 2)$-form potentials.

2 Consistent interactions and Local BRST Cohomology

Our approach to the problem of constructing consistent interaction vertices for a gauge theory is based on the BRST symmetry. As shown in [20, 21], the question boils down to computing the local BRST cohomological group at ghost number zero in the algebra of local $n$-forms depending on the fields, the ghosts, the antifields and their derivatives. These groups are denoted by $H^0(s|d)$. The cocycle condition reads,

$$sa + db = 0,$$

(2.1)

where $a$ (respectively $b$) is a local $n$-form (respectively $(n-1)$-form) of ghost number zero (respectively one). Trivial solutions of (2.1) are of the form,

$$a = sm + dn$$

(2.2)

where $m$ (respectively $n$) is a local $n$-form (respectively, $(n-1)$-form) of ghost number $-1$ (respectively $0$). One often refers to (2.1) as the “Wess-Zumino consistency condition” [22].

If $a$ is a solution of (2.1), its antifield-independent part defines a consistent interaction; and conversely, given a consistent interaction, one can complete it by antifield-dependent terms to get a BRST cocycle (2.1). As explained in [20, 21], it is necessary to include the antifields in the analysis of the cohomology in order to cover symmetry-deforming interactions.

In the case at hand, the gauge symmetries are reducible and the following set of antifields is required [23, 24],

$$B^{*a\mu_1...\mu_p}, B^{*a\mu_1...\mu_{p-1}}, \ldots, B^{*a\mu_1}, B^{*a}.$$  

(2.3)

The Grassmann parity and the antighost number of the antifields $B^{*a\mu_1...\mu_p}$ associated with the fields $B^a_{\mu_1...\mu_p}$ are equal to 1. The Grassmann parity and
the antighost number of the other antifields is determined according to the following rule. As one moves from one term to the next one to its right in (2.3), the Grassmann parity changes and the antighost number increases by one unit. Therefore the parity and the antighost number of a given antifield \( B^{*a\mu_1\ldots\mu_{pa-j}} \) are respectively \( j + 1 \) modulo 2 and \( j + 1 \).

Reducibility also imposes the following set of ghosts,

\[
C^a_{\mu_1\ldots\mu_{pa-1}}, \ldots, C^a_{\mu_1\ldots\mu_{pa-j}}, \ldots, C^a.
\]  

(2.4)

These ghosts carry a degree called the pure ghost number. The pure ghost number of \( C^a_{\mu_1\ldots\mu_{pa-1}} \) and its grassmann parity are equal to 1. As one moves from one term to the next one to its right in (2.4), the Grassmann parity changes and the ghost number increases by one unit up to \( pa \).

We denote by \( \mathcal{P} \) the algebra of spacetime forms with coefficients that are polynomials in the fields, antifields, ghosts and their derivatives.

The action of \( s \) in \( \mathcal{P} \) is the sum of two parts, namely, the “Koszul-Tate differential \( \delta \)” and the “longitudinal exterior derivative \( \gamma \):

\[
s = \delta + \gamma,
\]

(2.5)

where we have,

\[
\delta B^a_{\mu_1\ldots\mu_{pa}} = 0,
\]

(2.6)

\[
\delta C^a_{\mu_1\ldots\mu_{pa-j}} = 0,
\]

(2.7)

\[
\delta B_{1}^{*a} + d\overline{\mathcal{P}}^a = 0,
\]

\[
\delta B_{2} + d\overline{B}_{1}^{*a} = 0,
\]

\[
\vdots
\]

\[
\delta B_{pa+1}^{*a} + d\overline{B}_{pa}^{*a} = 0,
\]

(2.8)

and,

\[
\gamma B^{*a\mu_1\ldots\mu_{pa+1-j}} = 0,
\]

(2.9)

\[
\gamma B^a + dC^a_1 = 0,
\]

(2.10)

\[
\gamma C^a_1 + dC^a_2 = 0,
\]

(2.11)

\[
\vdots
\]

\[
\gamma C^a_{pa-1} + dC^a_{pa} = 0,
\]

(2.12)

\[
\gamma C^a_{pa} = 0.
\]

(2.13)
In the above equations, \( C^a_\gamma \) is the \((p_a - j)\)-form whose components are \( C^a_{\mu_1 \cdots \mu_{p_a - j}} \). Furthermore, we have systematically denoted (as above) the duals by an overline to avoid confusion with the *-notation of the antifields. The actions of \( \delta \) and \( \gamma \) on the individual components of the antifields (2.3), ghosts (2.4) and their derivatives are easily read off from the above formulas (recalling that \( \delta(dx^\mu) = \gamma(dx^\mu) = 0, [\partial_\mu, \delta] = 0, [\partial_\mu, \gamma] = 0 \)).

3 General procedure for working out BRST cohomology

In order to prove the theorem, we shall solve the BRST cocycle condition by proceeding as in the Yang-Mills case \([25, 26]\). To that end, one expands the cocycles and the cocycle condition according to the antighost number. Thus, if \( a \) is a BRST cocycle (modulo \( d \)), then its various components in the expansion,

\[
a = a_0 + a_1 + a_2 + \cdots + a_k, \quad \text{antigh}(a_i) = i,
\]

must fulfill the chain of equations,

\[
\begin{align*}
\gamma a_0 + \delta a_1 + db_0 &= 0, \\
\vdots \\
\gamma a_{k-1} + \delta a_k + db_{k-1} &= 0, \\
\gamma a_k + db_k &= 0.
\end{align*}
\]

The last equation in this chain no longer involves the differential \( \delta \) and can be easily solved. The idea, then, is to start the resolution of the cocycle condition from \( a_k \) and to work one’s way up until one reaches \( a_0 \), which is the quantity of physical interest. [Recall that \( a_0 \) defines a consistent deformation of the Lagrangian. And conversely, if \( a_0 \) is a consistent deformation of the Lagrangian, then one may complete it by terms of positive antighost number, as in \([24]\), so as to construct a BRST cocycle \( a \). Furthermore, trivial BRST cocycles (in the cohomological sense) correspond to trivial deformations (i.e., deformations that can be absorbed through redefinitions of the field variables) \([20, 21]\). The reconstruction of the cocycle \( a \) from \( a_0 \) stops at some antifield number \( k \) because \( a_0 \) is polynomial in the derivatives (see the argument in \([20]\) section 3).]
Before doing this, we shall introduce some useful notations and give a few solutions.

In the analysis of the BRST cohomology, it turns out that two combinations of the fields and antifields play a central rôle. The first one combines the field strengths and the duals of the antifields and is denoted $\tilde{H}^a$,

$$\tilde{H}^a = \mathcal{H}^a + \sum_{j=1}^{p_a+1} \mathcal{B}^a_j.$$  \hspace{1cm} (3.5)

The second one combines the $p_a$-forms and their associated ghosts and is denoted $\tilde{B}^a$,

$$\tilde{B}^a = B^a + C^a_1 + \ldots + C^a_{p_a}.$$  \hspace{1cm} (3.6)

It is easy to see that both $\tilde{H}^a$ and $\tilde{B}^a$ have a definite Grassmann parity respectively given by $n - p_a + 1$ and $p_a$ modulo 2. On the other hand, exterior products of $\tilde{H}^a$ or $\tilde{B}^a$ (including the $\tilde{H}^a$ and $\tilde{B}^a$ themselves) are not homogeneous in form degree and ghost number. To isolate a component of a given form degree $k$ and ghost number $g$, we enclose the product in brackets $[\ldots]^{k,g}$. The component in $[A]^{k,g}$ which has definite antighost number $l$ is denoted $[A]^{k,g}_l$.

Since products of $\tilde{B}^a$ very frequently appear in the rest of the analysis, we introduce the following notations,

$$Q^{a_1\ldots a_m}_k = \tilde{B}^{a_1}\ldots \tilde{B}^{a_m}_k$$ and $$Q^{a_1\ldots a_m}_{k,g} = [\tilde{B}^{a_1}\ldots \tilde{B}^{a_m}]^{k,g}_g.$$  \hspace{1cm} (3.7)

We shall not write explicitly the wedge product from now on ($dx^0 dx^1$ can clearly only mean $dx^0 \wedge dx^1$).

We also define the three “mixed operators”: $\Delta = \delta + d$, $\tilde{\gamma} = \gamma + d$ and $\tilde{s} = s + d$.

Using those definitions we have the following relations:

$$\Delta \tilde{H}^a = 0, \quad \Delta \tilde{B}^a = 0, \quad \Delta H^a = 0$$  \hspace{1cm} (3.8)

$$\tilde{\gamma} \tilde{H}^a = 0, \quad \tilde{\gamma} \tilde{B}^a = H^a, \quad \tilde{\gamma} H^a = 0$$  \hspace{1cm} (3.9)

$$\tilde{s} \tilde{H}^a = 0, \quad \tilde{s} \tilde{B}^a = H^a, \quad \tilde{s} H^a = 0.$$  \hspace{1cm} (3.10)

Eq. $\tilde{\gamma} \tilde{B}^a = H^a$ is known in the literature as the “horizontality condition” \cite{13}.
It is easy to construct solutions of the Wess-Zumino consistency condition out of the variables $H^a, \tilde{H}^a, \tilde{B}^a$. For example, in ghost number zero,

$$a^{\mathbf{n},0} = [P_b(H^a, \tilde{H}^a)\tilde{B}^b]^{\mathbf{n},0},$$

(3.11)
is a solution of (2.1). This can be seen by applying $\tilde{s}$ to $P_b(H^a, \tilde{H}^a)\tilde{B}^b$.

One gets $\tilde{s}(P_b\tilde{B}^b) = (-)^{\mathbf{p}}P_b(\tilde{s}\tilde{B}^b) = (-)^{\mathbf{p}}P_bH^b$ and thus, $s[P_b\tilde{B}^b]^{\mathbf{n},0} + d[P_b\tilde{B}^b]^{\mathbf{n},1} = [s(P_b\tilde{B}^b)]^{\mathbf{n},1} = [P_bH^b]^{\mathbf{n},1} = 0$ (no ghost occurs in $P_bH^b$). We shall prove in this article the remarkable property that all antifield dependent solutions of the Wess-Zumino consistency condition in ghost number 0 are in fact of the form (3.11) (modulo antifield independent terms). According to the discussion at the beginning of Section 2, this is equivalent to proving Theorem 1.1 since $a_0^{\mathbf{n},0} = [P_b(H^a, \tilde{H}^a)\tilde{B}^b]_0^{\mathbf{n},0} = P_b(H^a, \bar{H}^a)B^b$ is of the required form.

4 Some useful lemmas

In order to construct the general solution of the (mod $d$) BRST cocycle condition along the lines indicated in the previous section, we shall need a few lemmas.

**Lemma 4.1** Let $a_k$ be a solution of $\gamma a_k + db_k = 0$, with non-vanishing antighost number $k$. Then one has $a_k = a'_k + \gamma m_k + dn_k$ where $a'_k$ is annihilated by $\gamma$, $\gamma a'_k = 0$.

**Proof:** The proof proceeds as in the Yang-Mills case: one analyses the descent equation associated with $\gamma a_k + db_k = 0$. In [27] we have listed all the non-trivial descents without taking into account the antifields. However the results are unchanged even if one includes the antifields since their contributions to non-trivial descents can always be absorbed by trivial terms (the proof of this statement is identical to the one in the Yang-Mills case [26]). Therefore, if $a_k$ involves the antifields, the descent associated with it is necessarily trivial so that one can find a different representative $a'_k$ in the same class of $H(\gamma|d)$ as $a_k$ which is annihilated by $\gamma$. □
Lemma 4.2 The general solution of $\gamma a_k = 0$ is given by,

$$a_k = \sum_I P^I_k \omega^I + \gamma c_k,$$

where the $\omega^I$ are polynomials in the undifferentiated "last" ghosts of ghosts $C^a_{pa}$ and the $P^I_k$ are spacetime $n$-forms with coefficients that are polynomials in the field strengths, their derivatives, the antifields and their derivatives (these variables will be denoted $\chi$ in the sequel).

Proof: The proof of this lemma is quite standard. One redefines the variables into three sets obeying respectively $\gamma x^i = 0$, $\gamma y^\alpha = z^\alpha$, $\gamma z^\alpha = 0$. The variables $y^\alpha$ and $z^\alpha$ form "contractible pairs" and the cohomology is then generated by the (independent) variables $x^i$. In our case, the $x^i$ are given by $dx^\mu$, the fields strengths components, the antifields and their derivatives as well as the last (undifferentiated) ghosts of ghosts. A complete proof of the lemma in the absence of antifields can be found in [27]. Here we simply note that the antifields are automatically part of the $x^i$ variables since they are all $\gamma$-closed and do not appear in the $\gamma$ variations. \qed

Using the conventions (3.7) and dropping the trivial term, we can write the cocycle (4.1) as,

$$a_k = \sum_m P_{k}^{a_1...a_m} [\tilde{B}^{a_1}...\tilde{B}^{a_m}]^{0,l} = \sum_m P_{k}^{a_1...a_m} Q_{0,l}^{a_1...a_m},$$

with $l = \sum_m p_{a_m}$.

Lemma 4.3 Let $\alpha$ be an antifield independent $\gamma$-cocycle that takes the form

$$\alpha = R_1(H^{a_s}, C^a_{par}) R_2(H^{b_s}, C^b_{ps}), \quad p_s > p_{ar},$$

where $R_1$ (respectively $R_2$) is an exterior polynomial in the curvature form $H^{a_r}$ (respectively $H^{b_s}$) and the last ghost of ghost $C^a_{par}$ (respectively $C^b_{ps}$) such that $p_s > p_{ar}$. Assume that $R_1$ contains no constant term and is trivial in $H(\gamma|d)$,

$$R_1 = \gamma U_1 + dV_1.$$  

Then, $\alpha$ is also trivial in $H(\gamma|d)$.

Proof: This result was proved in [27]. Since $R_1$ is trivial, it is the obstruction to the lift of a $\gamma$-cocycle $\beta_1$ through the descent equations of $H(\gamma|d)$. Because of the condition $p_s > p_{ar}$, $\alpha$ then also appears as the obstruction to the lift of the $\gamma$-cocycle $\beta_1 R_2$ indicating that $\alpha$ is trivial in $H(\gamma|d)$. \qed
Thus, we now have all the necessary tools required to solve the Wess-Zumino consistency condition (2.3). Consider first the case where the expansion of $a$ (which has total ghost number 0) reduces to $a_0$ (no antifields). Then, $a \equiv a_0$ fulfills $\gamma a_0 + db_0 = 0$. This equation was investigated in detail in [27], where

Theorem

Lemma 4.4 Let $a$ be a cochain with form-degree $p$ and ghost number $g$, $a \equiv \{a\}^p,g$, and let $a = a_0 + \ldots + a_k$ be its expansion according to the antighost number, $a_i = [a_i]^p,g$. Assume that the last term $a_k$ takes the form $a_k = [P]^q_{k} \gamma m$ where $P$ is an exterior polynomial in $H$ and $H$ and where $\chi \equiv \chi^p,q,k,g$ is an exterior polynomial in $H$ and $C^a_{p,q,k}$ which is trivial in $H(\gamma|d)$, $\chi(H,C) = \gamma m + dn$. Then one can redefine $a_k$ away by adding $s$-exact terms modulo $d$ to $a$,

$$a = su + dv + \text{terms of antighost number} < k. \quad (4.4)$$

Proof: One has $P(\tilde{H}, H) = [P]^q_{0} + \ldots + [P]^q_{k} + \ldots + [P]^n,g,q^{-k}$ and $\tilde{s}\tilde{P} = 0$. One has also by assumption, $\chi \equiv \chi^p,q,k,g = \gamma m^p,q,k,g = \gamma m^p,q,k,g = \gamma n^p,q,k,g = \gamma n^p,q,k,g = 0, \ldots$ and $\tilde{m} = m^p,q,k,g = m^p,q,k,g = m^p,q,k,g = m^p,q,k,g = m^p,q,k,g = m^p,q,k,g = \ldots + m^p,q,k,g = 0, \ldots$ one gets,

$$\chi^p,q,k,g = \chi^p,q,k,g = \chi^p,q,k,g = \chi^p,q,k,g = \chi^p,q,k,g = \chi^p,q,k,g = \ldots \chi^p,q,k,g = \chi^p,q,k,g = \ldots \chi^p,q,k,g = \chi^p,q,k,g = \ldots$$

Thus, $\tilde{s}((-1)^{s}P \tilde{m}) = a_k - Pdm^p,q,k,g - 1.1$ If we project this equation on the form degree $p$ of $a_k$, one finds the equation,

$$su^{p,q-1} + du^{p-1,q} = a_k - [P]^q_{k-1} \gamma m^p,q,k,g = 1, \quad (4.5)$$

where we have set $u^{p,q-1} \equiv ((-1)^{s}P \tilde{m})^{p,q-1}$ and $u^{p-1,q} \equiv ((-1)^{s}P \tilde{m})^{p-1,q}$. Thus,

$$a_k = su^{p,q-1} + du^{p-1,q} + \text{terms of antighost number} < k, \quad (4.6)$$

which is the desired result. □

5 Proof of theorem

We now have all the necessary tools required to solve the Wess-Zumino consistency condition (2.3). Consider first the case where the expansion of $a$ (which has total ghost number 0) reduces to $a_0$ (no antifields). Then, $a \equiv a_0$ fulfills $\gamma a_0 + db_0 = 0$. This equation was investigated in detail in [27], where
it was shown that it has only two types of solutions: those for which one can assume that \( b_0 = 0 \), which are the strictly gauge-invariant terms; and those for which no redefinition yields \( b_0 = 0 \) (“semi-invariant terms”), which are exhausted by the Chern-Simons terms. Both types of solutions preserve the form of the gauge symmetries and are in agreement with the theorem; we can thus turn to the case where \( a \) involves the antifields, \( k \neq 0 \).

By lemma 4.1, one can assume that the last term \( a_k \) in the expansion of \( a \) is annihilated by \( \gamma \). Indeed, the (allowed) redefinition \( a \rightarrow a - sm_k - dn_k \) (see Lemma 4.1) enables one to do so. Then, the next to last equation in the chain (1.2) implies \( d\gamma b_{k-1} \), i.e., by the algebraic Poincaré lemma, \( \gamma b_{k-1} + dc_{k-1} = 0 \) for some \( c_{k-1} \) (the cohomology of \( d \) is trivial in form-degree \( n - 1 \)).

Now, two cases must be considered: either \( k > 1 \), in which case lemma 4.1 implies again that one can assume \( \gamma b_{k-1} = 0 \) through redefinitions. Or \( k = 1 \), in which case \( b_{k-1} \equiv b_0 \) does not involve the antifields and may lead to a non trivial descent. This second possibility arises only if \( H(\gamma) \) does not vanish in pureghost number one since \( a_k \equiv a_1 \) must be a non-trivial element of \( H^k(\gamma) \) or else can be eliminated through a redefinition. In the absence of 1-forms, \( H^1(\gamma) \) vanishes (lemma 4.2), so we can assume \( k > 1 \). The case \( k = 1 \) will be discussed in section 6 where we allow for the presence of 1-forms.

If \( k > 1 \), one can expand the elements \( a_k \) and \( b_{k-1} \) according to lemma 4.2:

\[
\begin{align*}
  a_k &= \sum P_k^I \omega^I, & b_{k-1} &= \sum Q_{k-1}^I \omega^I
\end{align*}
\]  
(\( \gamma \)-trivial terms can be eliminated). The next to last equation in the chain (1.2) then implies

\[
\delta P_k^I + dQ_{k-1}^I = 0,
\]

which indicates that \( P_k^I \) is a cocycle of the cohomology \( H(\delta|d) \).

This cohomology, which is related to the so-called invariant characteristic cohomology, was completely worked out in [9]. It was shown that all its representatives can be written as the \([\cdot]^{n-k} \) component of an exterior polynomial in \( H^a \) and \( \tilde{H}^a \),

\[
P_k^I = [P^I(H^a, \tilde{H}^a)]^{n-k}, \quad (k > 1).
\]

It is because of this property that antifield dependent solutions of the Wess-Zumino consistency condition, which belong a priori to the algebra generated by all the variables and their individual, successive derivatives, turn out to be expressible in terms of the forms \( H^a \), \( \tilde{H}^a \) and \( B^a \) only.
Relation (5.3) implies that the term \( a_k \) of highest antighost number in the expansion of \( a \) is up to trivial terms of the form,

\[
    a_k = [P^I(H^a, \tilde{H}^a)]^{n-k} \omega^I,
\]

where the pureghost number of the \( \omega^I \) must be equal to \( k \) in order to obtain a BRST cocycle in ghost number 0.

The question is now: can we construct from the known higher-order component \( a_k \) the components \( a_j \) of lower antighost numbers in order to obtain a solution of the Wess-Zumino consistency condition?

As we have seen in Section 3 this is always possible when the \( \omega^I \) are linear in the ghosts of ghosts and the resulting BRST cocyle is then given by (3.11).

We are now going to show that when the \( \omega^I \) in \( a_k \) are at least quadratic in the ghosts of ghosts then one encounters an obstruction in the construction of the corresponding solution of the Wess-Zumino consistency condition.

To proceed we exhibit explicitly in \( a_k \) the \( \tilde{B}^a \) which correspond to the forms of lowest degree occurring in \( a_k \) and denote them by \( \tilde{B}^a_{i1} \). The form degree in question is called \( p \). The other \( \tilde{B}^a \) are denoted \( \tilde{B}^a_{2} \). Thus we write \( a_k \) as,

\[
    a_k = [P_{a_1...a_{r} b_1...b_s}]^{n-k}[\tilde{B}^a_{i1} \ldots \tilde{B}^a_{r1} \tilde{B}^b_{12} \ldots \tilde{B}^b_{s2}]^{0,k}.
\]

Of course, \( k > p \) (\( a_k \) is at least quadratic in the \( \tilde{B} \)). In fact, \( k > p + 1 \) since there is no 1-form in the problem.

A direct calculation then shows that the equations \( \gamma a_j + \delta a_{j+1} + db_j = 0 \) determining \( a_{k-1}, a_{k-2}, \ldots \) have a solution up to \( a_{k-p} \). These solutions are,

\[
    a_{k-j} = [P_{a_1...a_{r} b_1...b_s}]^{n-j-k+j}[\tilde{B}^a_{i1} \ldots \tilde{B}^a_{r1} \tilde{B}^b_{12} \ldots \tilde{B}^b_{s2}]^{j,k-j},
\]

for \( 0 \leq j \leq p \).

Unless \( a_k \) is trivial (i.e., can be removed by the addition of exact terms to \( a \)), there is however an obstruction in the construction of \( a_{k-p-1} \). To discuss this obstruction, one needs to know the ambiguity in the \( a_{k-j} \) \( (0 \leq j \leq p) \). One easily verifies that it is given by \( a_{k-j} \rightarrow a_{k-j} + m_0 + m_1 + \ldots + m_{j-1} \) where \( m_0 \) satisfies \( \gamma m_0 = 0 \), \( m_1 \) satisfies \( \gamma m_1 + \delta n_1 + db_1 = 0 \), \( \gamma n_1 = 0 \), \( m_2 \) satisfies \( \gamma m_2 + \delta n_2 + db_2 = 0 \), \( \gamma n_2 + \delta l_2 + dc_2 = 0 \), \( \gamma l_2 = 0 \), etc. However, none of these ambiguities except \( m_0 \) in \( a_{k-p} \) can play a role in the construction of a non-trivial solution. To see this, we note that \( \delta, \gamma \) and \( d \) conserve the
polynomial degree of the variables of any given sector. We can therefore work at fixed polynomial degree in the variables of all the different $p$-forms. Since $n_1$, $l_2$, etc. are $\gamma$-closed terms which can be lifted at least once, they have the generic form $\bar{R}[H, \bar{H}]Q$ where $Q$ has to contain a ghost of ghost of degree $p_A < p$. Because we work at fixed polynomial degree, the presence of such terms imply that $P_{a_1...a rb_1...b_s}$ has to depend on $H^A$ (a dependence on $\bar{H}^A$ is not possible since by assumption $k > p$). However, $a_k$ is then of the form described in Lemma 4.4 and can be eliminated from $a$ by the addition of trivial terms and the redefinition of the terms of antighost numbers < $k$. Therefore we may now assume that $a_k$ does not contain $H^A$ and that the only ambiguity in the definitions of the $a_{k-j}$ is $m_0$ in $a_{k-p}$.

Since $k > p$, we have to substitute $a_{k-p}$ in the equation $\gamma a_{k-p-1} + \delta a_{k-p} + db_{k-p-1} = 0$. We then get,

$$\gamma a_{k-p-1} + \delta [P_{a_1...a rb_1...b_s}]^{n-p,-k+p}[\bar{B}_{1}^{a_1}... \bar{B}_{1}^{a_r} \bar{B}_{2}^{b_1}... \bar{B}_{2}^{b_s}]^{p,k-p} + \delta m_0 + db_{k-p-1} = 0,$$

which can be written as,

$$\gamma a'_{k-p-1} + db'_{k-p-1} + \delta m_0 + \epsilon \rho r[P_{a_1...a rb_1...b_s}]^{n-p-1,-k+p+1}H_1^{a_1}Q_{0,k-p}^{a_2...a_r b_1...b_s} = 0. \quad (5.9)$$

By acting with $\gamma$ on the above equation we obtain $d\gamma b'_{k-p-1} = 0 \Rightarrow \gamma b'_{k-p-1} + db'_{k-p-1} = 0$ which means that $b'_{k-p-1}$ is a $\gamma$ mod $d$ cocycle. Because we have excluded 1-forms from the discussion, $k - p - 1 > 0$ so that we may assume that $b'_{k-p-1}$ is strictly annihilated by $\gamma$. Accordingly, $db_{k-p-1} = [d\beta_{a_2...a_r b_1...b_s}(\chi)]Q_{0,g+q-p}^{a_2...a_r b_1...b_s} + \gamma m_{0,p-k-p}$. Equation (5.9) then reads,

$$(-)^{\epsilon \rho r}[P_{a_1...a rb_1...b_s}]^{n-p-1,-k+p+1}H_1^{a_1} + \delta \alpha_{a_2...a_r b_1...b_s}(\chi) + d\beta_{a_2...a_r b_1...b_s}(\chi) = 0, \quad (5.10)$$

where we have set $m_0 = \alpha_{a_2...a_r b_1...b_s}(\chi)Q_{0,k-p}^{a_2...a_r b_1...b_s}$. Eq. (5.10) implies,

$$[P_{a_1...a rb_1...b_s}]^{n-p-1,-q+p+1}H_1^{a_1} = 0, \quad (5.11)$$

\footnote{By sector we mean the variables corresponding to a given $p$-form and its associated antifields and ghosts.}
since $\delta$ and $d$ both increase the number of derivatives of $\chi$. Let us first note that $P_{a_1 \ldots a_r b_1 \ldots b_s}$ cannot depend on $\tilde{H}_1^c$ because in that case we would have $k - p - 1 \leq 0$ which contradicts our assumption that there is no 1-form (indeed, the component of form-degree $n$ of a polynomial in $H^a$ and $\tilde{H}^a$ which depends on $\tilde{H}_1^c$ has maximum antighost number $p + 1$). Therefore, $P_{a_1 \ldots a_r b_1 \ldots b_s}$ will satisfy (5.11) only if it is of the form, $P_{a_1 \ldots a_r b_1 \ldots b_s} = R_{ca_1 \ldots a_r b_1 \ldots b_s} H_1^c$ with $R_{ca_1 \ldots a_r b_1 \ldots b_s}$ symmetric in $c \leftrightarrow a_1$ (resp. antisymmetric) if $H_1$ is anticommuting (resp. commuting). However, using Lemma 4.4 we conclude once more that in that case $a_k$ can be absorbed by the addition of trivial terms and a redefinition of the components of lower antighost number of $a$. This ends our proof of the statement that for a system of $p$-forms with $p \geq 2$ all the antifield dependent solutions of the Wess-Zumino consistency conditions in ghost number 0 are of the form (3.11).

6 Presence of 1-forms

If 1-forms are present in the system of $p$-forms considered, the solutions in Theorem 1.1 are still valid. However, new solutions of the Wess-Zumino consistency condition appear, so the list is no longer exhaustive.

The first set of new solutions, related to the Noether conserved currents of the theory, arise because $H^1(\gamma)$ no longer vanishes. Although the term $b_{k-1} \equiv b_0$ which appears in (5.1) may lead to a non-trivial descent, one can show that (5.2) still holds [24, 28] so that $P^I \equiv P^a$ has to be an element of $H^a_{\gamma}(\delta|d)$. This cohomology is isomorphic to the set $a^\Delta$ of non-trivial global symmetries of the theory. The corresponding solutions of the Wess-Zumino consistency condition can then be written as,

$$a = k^a_{\Delta}(j^\Delta B_1^a + a^\Delta C_1^a), \quad (6.1)$$

where the $j^\Delta$ are the Noether currents corresponding to the $a^\Delta$ and satisfy $\delta a^\Delta + dj^\Delta = 0$. The dimension of this set of solutions is infinite since one can construct infinitely many conserved currents $j^\Delta$ [3]. This feature is characteristic of free lagrangians. Although these solutions define consistent interactions to first order in the deformation parameter, it is expected that most of them are obstructed at the second order. Furthermore, they are severely constrained by Lorentz invariance.
The second set of new solutions of the Wess-Zumino consistency condition arise because the condition $k - p - 1 > 0$ under (5.9) may no longer hold. Indeed, if $p = 1$ and $k = 2$ then we have $k - p - 1 = 0$. As above, the term $b'_k - b'_0$ appearing in (5.9) may now lead to a non-trivial descent in $H(\gamma|d)$. According to the analysis of [27], equation (5.11) is then replaced by,

$$(-)^p r [P_{a_1...a_r b_1...b_s} (H^a, \tilde{H}^a)]_{10}^n - 2 H^a_{11} + V_{a_2...a_r b_1...b_s} (H^a) = 0.$$  \hspace{1cm} (6.2)

The only solution of the above equation for $P^I$ is $P^I \equiv k_{abc} \tilde{H}^a_1$ with $k_{abc}$ completely antisymmetric [26, 28]. The corresponding BRST cocycles are given by,

$$a = k_{abc} [\tilde{H}^a_1 \tilde{B}^b_1 \tilde{B}^c_1]_0.$$  \hspace{1cm} (6.3)

They give rise to the famous Yang-Mills vertex since $a_0 = k_{abc} \tilde{H}^a_1 \tilde{B}^b_1 \tilde{B}^c_1$.

In particular, the above discussion confirms that it is not possible to construct a Lagrangian with coloured $p$-forms ($p > 1$) since vertices of the form $a_0 \sim \tilde{H}^a BA$ (where $A$ is a 1-form potential) do not exist. This fact is well appreciated in the litterature.

7 Comments and conclusions

In this paper we have provided the complete proof of the Theorem given in [1] on the consistent deformations of non-chiral free $p$-forms. The same techniques can be used to study solutions of the Wess-Zumino consistency condition at other ghost numbers (e.g., candidate anomalies) [28]. For instance, one can show that if all the exterior gauge fields have form degree $\geq 3$, Theorem 1.1 is also valid for candidate anomalies (the gauge potential being replaced by the corresponding ghosts of pure ghost number 1).

The same methods have also been extended recently to cover chiral $p$-forms [29].

8 Acknowledgements

This work is supported in part by the “Actions de Recherche Concertées” of the “Direction de la Recherche Scientifique - Communauté Française de Belgique”, by IISN - Belgium (convention 4.4505.86) and by Proyectos FONDECYT 1970151 and 7960001 (Chile).
Bernard Knaepen is supported by a post-doc grant from the “Wiener-Anspach” foundation.

References

[1] M. Henneaux and B. Knaepen, *Phys. Rev. D* **56** (1997) 6076, [hep-th/9706113](http://arxiv.org/abs/hep-th/9706113) (v3).

[2] R. Nepomechie, *Nucl. Phys. B**212** (1983) 301.

[3] C. Teitelboim, *Phys. Lett. B**167** (1986) 63.

[4] T. Damour, S. Deser and J. McCarthy, *Phys. Rev. D**47** (1993) 1541, [gr-qc/9207003](http://arxiv.org/abs/gr-qc/9207003).

[5] M. Henneaux, B. Knaepen and C. Schomblond, *Commun. Math. Phys* **186** (1997) 137.

[6] M. Henneaux, *Phys. Lett. B**368** (1996) 83.

[7] D. Freedman and P.K. Townsend, *Nucl. Phys. B**177** (1981) 282.

[8] G.F. Chapline and N.S. Manton, *Phys. Lett. B**120** (1983) 105.

[9] H. Nicolai and P.K. Townsend, *Phys. Lett. B**98** B (1981) 257.

[10] A.H. Chamseddine, *Nucl. Phys. B**185** (1981) 403.

[11] A.H. Chamseddine, *Phys. Rev. D**24** (1981) 3065.

[12] E. Bergshoeff, M. de Roo, B. de Wit and P. van Nieuwenhuizen, *Nucl. Phys. B**195** (1982) 97.

[13] L. Baulieu, in *Perspectives in Particles and Fields*, Cargèse 1983, M. Levy, J.-L. Basdevant, D. Speiser, J. Weyers, M. Jacob and R. Gastmans eds, NATO ASI Series B126, Plenum Press, New York (1983).

[14] F. Brandt and N. Dragon, *Nonpolynomial gauge invariant interactions of 1-form and 2-form gauge potentials*, in *Theory of Elementary Particles*, pp. 149-154, H. Dorn, D. Lüst, G. Weigt (eds.) (Wiley-VCH, Weinheim, 1998), [hep-th/9709021](http://arxiv.org/abs/hep-th/9709021).
[15] F. Brandt and U. Theis, *Nucl. Phys.* **B550** (1999) 495.

[16] F. Brandt, J. Simón and U. Theis, *Exotic gauge theories from tensor calculus*, hep-th/9910177.

[17] S. Deser, R. Jackiw and S. Templeton, *Phys. Rev. Lett.* **48** (1982) 975.

[18] I.V. Lavrinenko, H. Lu, C.N. Pope and K.S. Stelle, *Superdualities, Brane Tensions and Massive IIA/IIB Duality*, hep-th/9903057.

[19] E. Cremmer, B. Julia, H. Lu and C.N. Pope, *Nucl. Phys.* **B535** (1998) 242.

[20] G. Barnich and M. Henneaux, *Phys. Lett.* **B311** (1993) 123.

[21] M. Henneaux, *Contemp. Math.* **219** (1998) 93.

[22] J. Wess and B. Zumino, *Phys. Lett.* **37B** (1971) 95.

[23] I.A. Batalin and G.A. Vilkovisky, *Phys. Rev.* **D28** (1983) 2567.

[24] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems*, Princeton University Press, (1992).

[25] G. Barnich and M. Henneaux, *Phys. Rev. Lett.* **72** (1994) 1588.

[26] G. Barnich, F. Brandt and M. Henneaux, *Commun. Math. Phys.* **174** (1995) 93.

[27] M. Henneaux and B. Knaepen, *Nucl. Phys. B* **548** (1999) 491.

[28] B. Knaepen, *Cohomologie BRST locale des théories de p-formes*, PhD thesis, Université Libre de Bruxelles, 1999, hep-th/9912021.

[29] Xavier Bekaert, Marc Henneaux and Alexander Sevrin, *Deformations of chiral two-forms in six dimensions*, hep-th/9909094.