CLASSIFICATION OF INDUCTIVE LIMITS OF 1-DIMENSIONAL NCCW COMPLEXES

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ABSTRACT. A classification result is obtained for the $C^*$-algebras that are (stably isomorphic to) inductive limits of 1-dimensional noncommutative CW complexes with trivial $K_1$-group. The classifying functor $\text{Cu}^\sim$ is defined in terms of the Cuntz semigroup of the unitization of the algebra. For the simple $C^*$-algebras covered by the classification, $\text{Cu}^\sim$ reduces to the ordered $K_0$-group, the cone of traces, and the pairing between them. As an application of the classification, it is shown that the crossed products by a quasi-free action $\mathcal{O}_2 \rtimes_\lambda \mathbb{R}$ are all isomorphic for a dense set of positive irrational numbers $\lambda$.

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1. INTRODUCTION

This paper is a contribution to the classification of $C^*$-algebras by means of data invariant under approximately inner automorphisms. The classifying functor used here, denoted by $\text{Cu}^\sim$, is a variation on the Cuntz semigroup functor that looks at the unitizations of the $C^*$-algebras. The $C^*$-algebras being classified are those expressible—up to stable isomorphism—as inductive limits of 1-dimensional noncommutative CW complexes with trivial $K_1$-group. The following theorem is the main result of the paper; see Sections 2 and 3 for the relevant definitions.

**Theorem 1.0.1.** Let $A$ be either a 1-dimensional noncommutative CW complex with trivial $K_1$-group, or a sequential inductive limit of such $C^*$-algebras, or a $C^*$-algebra stably isomorphic to one such inductive limit. Let $B$ be a stable rank 1 $C^*$-algebra. Then for every morphism in the category $\text{Cu}$

$$\alpha: \text{Cu}^\sim(A) \to \text{Cu}^\sim(B)$$

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such that $\alpha([s_A]) \leq [s_B]$, where $s_A \in A^+$ and $s_B \in B^+$ are strictly positive elements, there exists a homomorphism

$$\phi: A \to B$$

such that $\text{Cu}^\sim(\phi) = \alpha$. Moreover, $\phi$ is unique up to approximate unitary equivalence.

The above classification result continues, and extends, the work of Ciuperca and Elliott in [6], and of Ciuperca, Elliott, and Santiago in [7]. In [6] and [7] the classifying functor is the Cuntz semigroup, while the range of the $C^*$-algebra $A$ is the class of approximately interval $C^*$-algebras in [6], and of approximately tree $C^*$-algebras in [7]. These classification results have the following distinctive characteristics:

1. the main theorem classifies homomorphisms from a certain class of domain algebras into arbitrary stable rank 1 $C^*$-algebras (in [26, Theorem 2], the class of codomain algebras is even larger than the stable rank 1 $C^*$-algebras);
2. in contrast to the generality of the codomain algebra, the domain algebra belongs to a relatively special class (up to stable isomorphism): approximately interval $C^*$-algebras, approximately tree $C^*$-algebras, or, as in this paper, inductive limits of 1-dimensional noncommutative CW (NCCW) complexes with trivial $K_1$-group;
3. the isomorphism theorem (i.e., the classification of $C^*$-algebras) is obtained from the classification of homomorphisms by a well-known approximate intertwining argument; in this case both $C^*$-algebras being compared must belong to the class of domain algebras covered by the homomorphism theorem (see Corollary 5.2.3);
4. a certain uniform continuity of the classification is also obtained: if two homomorphisms are close at the level of the invariant, then they are close–up to approximate unitary equivalence and on finite sets–as $C^*$-homomorphisms (see Theorem 3.3.1);
5. neither the domain nor the codomain algebra is required to be simple.

For simple $C^*$-algebras, the connection with the standard Elliott invariant (i.e., $K$-theory and traces) is made as follows: If a $C^*$-algebra $A$ is simple and belongs to the class of domain algebras covered by Theorem 1.0.1, then the invariant $\text{Cu}^\sim(A)$ can be computed in terms of the ordered group $K_0(A)$, the cone of traces of $A$, and the pairing between them. It then follows by our classification that the simple $C^*$-algebras that are (stably isomorphic to) inductive limits of 1-dimensional NCCW complexes with trivial $K_1$-group are classified by their ordered $K_0$-group, their cone of traces, and the pairing between them (a suitable scale is also needed; see Corollary 6.2.4).

Among the simple $C^*$-algebras included in our classification are

1. the Jiang-Su algebra $\mathcal{Z}$ (a finite, nuclear, non-type I $C^*$-algebra with the same Elliott invariant as $\mathbb{C}$);
2. the simple projectionless $C^*$-algebras classified by Razak in [25];
3. the simple $C^*$-algebras with $K_1 = 0$ described by Elliott in [15, Theorem 2.2]; their Elliott invariants exhaust all the pairs $(G, C)$, with $G$ a torsion free countable ordered abelian group and $C$ a non-zero topological cone with a compact base that is a metrizable Choquet simplex, where a pairing $G \times C \to \mathbb{R}$ is given that is weakly unperforated (see [15] for details);
the simple inductive limits of splitting interval algebras classified by Jiang and Su in [20]; the non-simple inductive limits with the same building blocks classified by Santiago in [29] are also included in our classification.

As an application of the classification, we show that the Jiang-Su algebra embeds unitally into any non-elementary unital simple exact \( C^* \)-algebra of stable rank 1, with a unique tracial state, and with strict comparison of positive elements. Furthermore, the embedding is unique up to approximate unitary equivalence. This property characterizes the Jiang-Su algebra up to isomorphism (see Proposition 6.3.1).

In [10], Dean studied the crossed products \( O_2 \rtimes_\lambda \mathbb{R} \) obtained from a quasi-free action of \( \mathbb{R} \) on the Cuntz algebra \( O_2 \). He showed that for \( \lambda \) in a dense \( G_\delta \) subset of \( \mathbb{R}^+ \) containing the rational numbers these crossed products are inductive limits of 1-dimensional NCCW complexes. A careful examination of his result reveals that the building blocks of the inductive limits have trivial \( K_1 \)-group. Thus, such crossed products are included in the classification given here. As an application of the classification, we obtain that for \( \lambda \) in a dense subset of \( \mathbb{R}^+ \setminus \mathbb{Q} \) of second Baire category the crossed products \( O_2 \rtimes_\lambda \mathbb{R} \) are all isomorphic to each other. In fact, they are isomorphic to the unique (up to isomorphism) simple, stable, projectionless \( C^* \)-algebra with a unique trace expressible as an inductive limit of 1-dimensional NCCW complexes with trivial \( K_1 \)-group. In [19], Jacelon shows that this \( C^* \)-algebra is isomorphic to the tensor product with itself and that, furthermore, it is strongly self-absorbing in a suitable sense. This \( C^* \)-algebra is likely to play a significant role in the classification of projectionless simple nuclear \( C^* \)-algebras akin to the role played by the unital strongly self-absorbing \( C^* \)-algebras.

Let us say a few words about the proof of Theorem 1.0.1. It is shown in [7] that the classification of homomorphism into a stable rank 1 \( C^* \)-algebra by means of the functor \( Cu \) has certain permanence properties with respect to the domain algebra. Namely, if the classification is possible for a given collection of domain algebras, it is also possible for their finite direct sums, sequential inductive limits, and for algebras stably isomorphic to the ones in the given collection. In Theorem 3.2.2 below the same permanence results are obtained for \( Cu^\sim \). Furthermore, the operations of adding and removing a unit are added to the list of transformations of the domain algebra. Theorem 1.0.1 is then proved by showing that all 1-dimensional NCCW complexes with trivial \( K_1 \)-group may be gotten by starting with the algebra \( C_0(0,1] \) and combining the operations of direct sums, adding or removing a unit, and passing to stably isomorphic algebras. This reduces proving Theorem 1.0.1 to the case of \( A = C_0(0,1] \). In this case the theorem is essentially a corollary of Ciuperca and Elliott’s classification [6].

The scope of the methods used in this paper is currently limited to domain algebras that have trivial \( K_1 \)-group. This is so because the functors \( Cu \) and \( Cu^\sim \) fail in general to account for the \( K_1 \)-group of the algebra. It seems plausible that a suitable enlargement of these functors with \( K_1 \)-type data may give a classification of all 1-dimensional NCCW complexes and their inductive limits.

This paper is organized as follows: Section 2 contains preliminaries about the Cuntz semigroup and NCCW-complexes; in Section 3 the functor \( Cu^\sim \) is introduced and studied; Section 4 contains proofs of various special cases of Theorem 1.0.1 and serves as a warm up for the proof of the general case; Section 5 is dedicated to the proof of Theorem 1.0.1 in Section 6 a computation is given of \( Cu^\sim(A) \) for \( A \) simple and belonging to the class classified here; the classification of simple \( C^* \)-algebras in terms
of the ordered \( K_0 \)-group and the cone of traces is derived from this computation; in Section 7 the crossed products \( \mathcal{O}_2 \times_\lambda \mathbb{R} \) are discussed.

## 2. Preliminary definitions and results

### 2.1. The functor \( \text{Cu} \)

Let us briefly review the definition of the Cuntz semigroup of a \( C^* \)-algebra and of the functor \( \text{Cu} \) (see [2] for a more detailed exposition). Here we use the positive elements picture of \( \text{Cu} \). In [9], an alternative approach to \( \text{Cu} \) is given which makes use of Hilbert \( C^* \)-modules over the \( C^* \)-algebra rather than positive elements, but we will not rely on it here.

Let \( A \) be a \( C^* \)-algebra. Let \( A^+ \) denote the positive elements of \( A \). Given \( a, b \in A^+ \) we say that \( a \) is Cuntz smaller than \( b \), and denote this by \( a \preceq b \), if \( a b d_n^* \to a \) for some sequence \( (d_n) \) in \( A \). We say that \( a \) is Cuntz equivalent to \( b \) if \( a \preceq b \) and \( b \preceq a \); in this case we write \( a \sim b \).

Let \( \text{Cu}(A) \) denote the set \( (A \otimes K)^+ / \sim \) of Cuntz equivalence classes of positive elements of \( A \otimes K \). For \( a \in (A \otimes K)^+ \) we denote the Cuntz class of \( a \) by \([a]\). The preorder \( \preceq \) defines an order on \( \text{Cu}(A) \):

\[
[a] \leq [b] \text{ if } a \preceq b.
\]

We also endow \( \text{Cu}(A) \) with an addition operation by setting

\[
[a] + [b] := [a' + b'],
\]

where \( a', b' \in (A \otimes K)^+ \) are orthogonal to each other and Cuntz equivalent to \( a \) and \( b \) respectively (the choices of \( a' \) and \( b' \) do not affect the Cuntz class of their sum). We regard \( \text{Cu}(A) \) as an ordered semigroup and we call it the Cuntz semigroup of \( A \).

Let \( \phi: A \to B \) be a homomorphism between \( C^* \)-algebras. Let us continue to denote by \( \phi \) the homomorphism \( \phi \otimes \text{id} \) from \( A \otimes K \) to \( B \otimes K \). (This convention will apply throughout the paper.) The homomorphism \( \phi \) induces an ordered semigroup map \( \text{Cu}(\phi): \text{Cu}(A) \to \text{Cu}(B) \) given by

\[
\text{Cu}(\phi)([a]) := [\phi(a)].
\]

In [9] Theorem 1, Coward, Elliott and Ivanescu show that \( \text{Cu}(\cdot) \) is a functor from the category of \( C^* \)-algebras to a certain sub-category of the category of ordered semigroups.

Let us recall the definition of this sub-category, which we shall denote by \( \text{Cu} \).

The definition of the category \( \text{Cu} \) relies on the compact containment relation. For \( x \) and \( y \) elements of an ordered set, we say that \( x \) is compactly contained in \( y \), and denote this by \( x \ll y \), if for every increasing sequence \( (y_n) \) with \( y \leq \sup_n y_n \), there exists an index \( n_0 \) such that \( x \leq y_{n_0} \). A sequence \( (x_n) \) is called rapidly increasing if \( x_n \ll x_{n+1} \) for all \( n \). An ordered semigroup \( S \) is an object of \( \text{Cu} \) if

- O1 increasing sequences in \( S \) have suprema,
- O2 for every \( x \in S \) there exists a rapidly increasing sequence \( (x_n) \) such \( x = \sup_n x_n \),
- O3 if \( x_i \ll y_i, \ i = 1, 2 \), then \( x_1 + x_2 \ll y_1 + y_2 \),
- O4 \( \sup_n (x_n + y_n) = \sup_n (x_n) + \sup_n (y_n) \), for \( (x_n) \) and \( (y_n) \) increasing sequences.

A map \( \alpha: S \to T \) is a morphism of \( \text{Cu} \) if

- M1 \( \alpha \) is additive and maps 0 to 0,
- M2 \( \alpha \) is order preserving,
- M3 \( \alpha \) preserves the suprema of increasing sequences,
- M4 \( \alpha \) preserves the relation of compact containment.
It is shown in [9, Theorem 2] that the category Cu is closed under sequential inductive limits, and that the functor Cu preserves sequential inductive limits. For later use, we will need the following characterization of inductive limits in the category Cu: Given an inductive system \((S_i, \alpha_{i,j})_{i,j \in \mathbb{N}}\) in the category Cu, and morphisms \(\alpha_{i,\infty}: S_i \to S\) such that \(\alpha_{j,\infty} \circ \alpha_{i,j} = \alpha_{i,\infty}\) for all \(i \leq j\), the ordered semigroup \(S\) is the inductive limit of the \(S_i\)s if

L1 the set \(\bigcup_i \alpha_{i,\infty}(S_i)\) is dense in \(S\) (in the sense that every element of \(S\) is the supremum of a rapidly increasing sequence of elements in \(\bigcup_i \alpha_{i,\infty}(S_i)\)),

L2 for each \(x, y \in S_i\) such that \(\alpha_{i,\infty}(x) \leq \alpha_{i,\infty}(y)\), and \(x' \ll x\), there exists \(j\) such that \(\alpha_{i,j}(x') \leq \alpha_{i,j}(y)\).

The order on Cu\((A)\) is part of its structure and in general is not determined by the addition operation. In one special situation, however, the order coincides with the algebraic order. We say that an element \(e \in Cu(A)\) is compact if \(e \ll e\). The following lemma is a simple consequence of [28, Lemma 7.1 (i)].

**Lemma 2.1.1.** Let \(e \in Cu(A)\) be compact and suppose that \(e \ll [a]\) for some \([a] \in Cu(A)\). Then there exists \([a'] \in Cu(A)\) such that \(e + [a'] = [a]\).

We will sometimes make use of the ordered sub-semigroup W\((A)\) of Cu\((A)\). This ordered semigroup is composed of the Cuntz equivalence classes \([a] \in Cu(A)\) such that \(a \in \bigcup_{n=1}^\infty M_n(A)\). It is known that W\((A)\) is dense in Cu\((A)\), i.e., every element of Cu\((A)\) is the supremum of a rapidly increasing sequence of elements in W\((A)\). (Indeed, if \([a] \in Cu(A)\) then \([(a - \frac{1}{n})_+]\), with \(n = 1, 2, \ldots\), is a rapidly increasing sequence in W\((A)\) with supremum \([a]\).)

In the following sections we will make use of some properties of the Cuntz semigroup that hold specifically for stable rank 1 C*-algebras. Let us recall them here.

The following proposition is due to Coward, Elliott, and Ivanescu ([9, Theorem 3]; see [7, Proposition 1] for the present formulation).

**Proposition 2.1.2.** ([7, Proposition 1]) Let \(A\) be a C*-algebra of stable rank 1 and \(a, b \in A^+\). Then \(a \ll b\) if and only if there is \(x \in A\) such that \(a = x^*x\) and \(xx^* \in bAb\).

Let us say that the ordered semigroup \(S\) has weak cancellation if \(x + z \ll y + z\) implies \(x \ll y\), for \(x, y, z \in S\). If \(S = Cu(A)\) and \([p]\) is the Cuntz class of a projection, then \([p]\) is a compact element of Cu\((A)\) (i.e., \([p] \ll [p]\)). It can be shown from this that if Cu\((A)\) has weak cancellation then \([a] + [p] \ll [b] + [p]\) implies \([a] \ll [b]\). In this case we say that Cu\((A)\) has cancellations of projections. The following proposition is due to Rørdam and Winter (28). A partial case is also due to Elliott (16).

**Proposition 2.1.3.** ([28, Proposition 4.2, Theorem 4.3].) Let \(A\) be a C*-algebra of stable rank 1. Then Cu\((A)\) has weak cancellation. In particular, Cu\((A)\) has cancellation of projections.

2.2. **Noncommutative CW complexes.** Following [13], let us say that the C*-algebra \(A\) is a 1-dimensional noncommutative CW complex (NCCW complex) if it is the pull-back C*-algebra in a diagram of the form

\[
\begin{array}{ccc}
A & \xrightarrow{\pi_1} & C([0, 1], F) \\
\downarrow{\pi_2} & & \downarrow{\text{ev}_0 \oplus \text{ev}_1} \\
E & \xrightarrow{\phi} & F \oplus F.
\end{array}
\]
Here $E$ and $F$ are finite dimensional $C^*$-algebras and $\text{ev}_0$ and $\text{ev}_1$ are the evaluation maps at 0 and 1. As in [10] (and unlike [13]), we do not assume that the homomorphism $\phi$ is unital.

Let us introduce some notation for the data defining $A$. We assume that

$E = M_{e_1}(C) \oplus M_{e_2}(C) \oplus \ldots M_{e_k}(C),$

$F = M_{f_1}(C) \oplus M_{f_2}(C) \oplus \ldots M_{f_l}(C).$

We write $\phi = (\phi_0, \phi_1)$, with $\phi_0, \phi_1 : E \to F$. We identify $K_0(E)$ with $\mathbb{Z}^k$ and $K_0(F)$ with $\mathbb{Z}^l$. We denote by $Z^{\phi_0}$ and $Z^{\phi_1}$ the $l \times k$ integer matrices associated to the maps

$K_0(\phi_0) : \mathbb{Z}^k \to \mathbb{Z}^l$ and $K_0(\phi_1) : \mathbb{Z}^k \to \mathbb{Z}^l$.

**Remark 2.2.1.** For later use, we record the following observations.

(i) The vectors $(e_j)_{j=1}^k$ and $(f_i)_{i=1}^l$ determine the algebras $E$ and $F$ uniquely up to isomorphism. These vectors together with the matrices $Z^{\phi_0}$ and $Z^{\phi_1}$ determine the maps $\phi_0$ and $\phi_1$ up to unitary equivalence. In consequence, the data $(e_j)_{j=1}^k$, $(f_i)_{i=1}^l$, $Z^{\phi_0}$ and $Z^{\phi_1}$ determine $A$ up to isomorphism.

(ii) Given the data $(e_j)_{j=1}^k$, $(f_i)_{i=1}^l$, $Z^{\phi_0}$ and $Z^{\phi_1}$ there is 1-dimensional NCCW complex that attains them if and only if

$Z^{\phi_0}(e_j) \leq (f_i)$ and $Z^{\phi_1}(e_j) \leq (f_i)$.

(iii) Tensoring by $\mathcal{K}$ in (11) we deduce that the matrices $Z^{\phi_0}$ and $Z^{\phi_1}$ alone determine $A$ up to stable isomorphism.

2.3. **Approximate unitary equivalence.** Given two homomorphisms $\phi : A \to B$ and $\psi : A \to B$ let us say that they are approximately unitarily equivalent if there exists a net $(u_\lambda)_{\lambda \in \Lambda}$ of unitaries in $B^*$ such that $u_\lambda^*\phi(x)u_\lambda \to \psi(x)$ for all $x \in A$. Notice that without loss of generality, the index set $\Lambda$ can be chosen to be the pairs $(F, \varepsilon)$, with $F$ finite subset of $A$, $\varepsilon > 0$, and with the order $(F, \varepsilon) \preceq (F', \varepsilon')$ if $F \subseteq F'$ and $\varepsilon' \leq \varepsilon$.

**Proposition 2.3.1.** (i) Let $\phi, \psi : A \to B$ be approximately unitarily equivalent homomorphisms, with $B$ of stable rank 1. Suppose that $\phi(A), \psi(A) \subseteq C$, with $C$ a closed hereditary subalgebra of $B$. Then $\phi$ and $\psi$ are approximately unitarily equivalent as homomorphisms from $A$ to $C$.

(ii) Let $C$ be a stable rank 1 $C^*$-algebra. Let $\phi, \psi : A \oplus B \to C$ be homomorphisms such that $\phi|_A$ is approximately unitarily equivalent to $\psi|_A$ and $\phi|_B$ is approximately unitarily equivalent to $\psi|_B$. Then $\phi$ is approximately unitarily equivalent to $\psi$.

**Proof.** (i) Let $(u_\lambda)_{\lambda \in \Lambda}$ be a net of unitaries such that $u_\lambda^*\phi(x)u_\lambda \to \psi(x)$ for all $x \in A$. Without loss of generality, assume that $\Lambda$ is the pairs $(F, \varepsilon)$ with $F$ finite subset of $A$ and $\varepsilon > 0$. It suffices to assume that $C$ is the closed hereditary subalgebra generated by the images of $\phi$ and $\psi$. For this subalgebra, we may choose an approximate unit $(e_\lambda)_{\lambda \in \Lambda}$ indexed by $\Lambda$ (e.g., for $\lambda = (F, \varepsilon)$ choose $e_\lambda$ such that $\|e_\lambda\phi(x) - \phi(x)\| < \varepsilon$ and $\|e_\lambda\psi(x) - \psi(x)\| < \varepsilon$ for all $x \in F$). Set $z_\lambda = e_\lambda u_\lambda e_\lambda$.

Then

$(2) \quad z_\lambda^*\phi(x)z_\lambda \to \psi(x)$ for all $x \in A$,

$(3) \quad |z_\lambda|\phi(x) \to \phi(x)$ for all $x \in A$. 


Since $C$ is of stable rank 1, for each $\lambda$ there exists a unitary $u_\lambda \in C^\sim$ such that
$$
\|z_\lambda - u_\lambda z_\lambda\| < \varepsilon,
$$
where $\lambda = (F, \varepsilon)$. Then (2) and (3) imply that $u_\lambda^\ast \phi(x)u_\lambda \to \psi(x)$ for all $x \in A$. That is, $\phi$ is approximately unitarily equivalent to $\psi$ as homomorphisms with codomain $C$.

(ii) Let $(u_\lambda)$ a $(v_\kappa)$ be nets of unitaries in $C^\sim$ such that $u_\lambda^\ast \phi(x)u_\lambda \to \psi(x)$ for all $x \in A$ and $v_\kappa^\ast \phi(x)v_\kappa \to \psi(x)$ for all $x \in B$. Find approximate units $(e_\lambda^\phi)$ and $(e_\kappa^\psi)$ for the hereditary subalgebras generated by $\phi(A)$ and $\psi(A)$ respectively. Similarly find $(f_\kappa^\phi)$ and $(f_\kappa^\psi)$, approximate units for the hereditary subalgebras generated by $\phi(B)$ and $\psi(B)$. Set
$$
z_{\lambda,\kappa} = e_\lambda^\phi u_\lambda e_\kappa^\psi + f_\kappa^\psi v_\kappa f_\kappa^\phi.
$$
Then
$$
(4) \quad z_{\lambda,\kappa}^\ast \phi(x)z_{\lambda,\kappa} \to \psi(x) \quad \text{for all } x \in A \oplus B,
$$
$$
(5) \quad |z_{\lambda,\kappa}| \phi(x) \to \phi(x) \quad \text{for all } x \in A \oplus B.
$$
Since $C$ is of stable rank 1, for each $\lambda$ and $\kappa$ there exists a unitary $u_{\lambda,\kappa} \in C^\sim$ such that
$$
\|z_{\lambda,\kappa} - u_{\lambda,\kappa} z_{\lambda,\kappa}\| < \varepsilon, \varepsilon',
$$
where $\lambda = (F, \varepsilon)$ and $\kappa = (F', \varepsilon')$. Then (4) and (5) imply that $u_{\lambda,\kappa}^\ast \phi(x)u_{\lambda,\kappa} \to \psi(x)$ for all $x \in A \oplus B$. \(\square\)

3. The Functor $C^\sim$

3.1. Definition and properties of $C^\sim$. Here we define the functor $C^\sim$ and show that it is well behaved with respect to sequential inductive limits, stable isomorphism, and direct sums, as long as the C*-algebras are assumed to be of stable rank 1. This is sufficient generality for the applications to classification that we will consider later on, but it raises the question of whether the same properties of $C^\sim$ hold for a larger class of C*-algebras.

Let $A$ be a C*-algebra and let $A^\sim$ denote its unitization. The definition of the ordered semigroup $C^\sim(A)$ is analogous to the definition of $K_0(A)$ in terms of the monoid of Murray-von Neumann classes of projections of $A^\sim$ (N.B.: $C^\sim(A)$ is not the enveloping group of the Cuntz semigroup of $A^\sim$.) Just as it is sometimes done when defining $K_0(A)$, we first define $C^\sim(A)$ for $A$ unital and then extend its definition to an arbitrary $A$ by requiring that the sequence
$$
C^\sim(A) \to C^\sim(A^\sim) \to C^\sim(\mathbb{C})
$$
be exact in the middle.

Let $A$ be a unital C*-algebra. We define $C^\sim(A)$ as the ordered semigroup of formal differences $[a] - n[1]$, with $[a] \in Cu(A^\sim)$ and $n \in \mathbb{N}$. That is, $C^\sim(A)$ is the quotient of the semigroup of pairs $([a], n)$, with $[a] \in Cu(A^\sim)$ and $n \in \mathbb{N}$, by the equivalence relation $([a], n) \sim ([b], m)$ if
$$
[a] + m[1] + k[1] = [b] + n[1] + k[1],
$$
for some $k \in \mathbb{N}$. The image of $([a], n)$ in this quotient is denoted by $[a] - n[1]$. An order on $C^\sim(A)$ is set by letting $[a] - n[1] \leq [b] - m[1]$ if for some $k$ the inequality $[a] + m[1] + k[1] \leq [b] + n[1] + k[1]$ holds in $Cu(A)$.

An alternative picture of $C^\sim(A)$ for $A$ unital which we will find convenient to have is as the ordered semigroup of formal differences $[a] - e$, with $[a] \in Cu(A)$ and $e \in Cu(A)$ a compact element. Since for every compact element $e$ there exists $e' \in Cu(A)$ such that $e + e' = n[1]$ for some $n$ (by Lemma 2), this ordered semigroup coincides with the
above defined semigroup of formal differences \([a] - n[1]\). Since the compact elements of \(\text{Cu}(A)\) are intrinsically determined by its order structure, \(\text{Cu}^\sim(A)\) is completely determined by \(\text{Cu}(A)\) if \(A\) is unital. If in addition \(A\) is stably finite (in particular if \(A\) has stable rank 1), then the compact elements of \(\text{Cu}(A)\) are the same as the Cuntz classes \([p]\) with \(p\) a projection (see [3, Theorem 3.5]). Thus, in this case we may view \(\text{Cu}^\sim(A)\) as the ordered semigroup of formal differences \([a] - [p]\), with \(p \in A \otimes \mathcal{K}\) a projection.

Let us now define the ordered semigroup \(\text{Cu}^\sim(A)\) for an arbitrary \(C^*\)-algebra \(A\). Let \(\pi: A^\sim \to \mathbb{C}\) denote the quotient map from the unitization of \(A\) onto \(\mathbb{C}\). This map induces a morphism

\[
\text{Cu}(\pi): \text{Cu}(A^\sim) \to \text{Cu}(\mathbb{C}) \cong \{0, 1, \ldots, \infty\}.
\]

We define \(\text{Cu}^\sim(A)\) as the subsemigroup of \(\text{Cu}^\sim(A^\sim)\) of elements of the form \([a] - n[1]\), where \(\text{Cu}(\pi)([a]) = n < \infty\).

**Remark 3.1.1.** The following facts are readily verified.

(i) If \(A\) is unital, but we ignore this fact and find \(\text{Cu}^\sim(A)\) by unitizing it, we obtain an ordered semigroup (canonically) isomorphic to \(\text{Cu}^\sim(A)\) as defined above.

(ii) As noted above, if \(A\) is unital then \(\text{Cu}^\sim(A)\) is completely determined by \(\text{Cu}(A)\).

This, however, may not be the case if \(A\) is non-unital (see Remark 6.2.5).

There is a canonical map from \(\text{Cu}(A)\) to the positive elements of \(\text{Cu}^\sim(A)\) (i.e., the elements greater than or equal to 0) given by \([a] \mapsto [a] - 0 \cdot [1]\).

**Lemma 3.1.2.** The ordered semigroup \(\text{Cu}(A)\) is mapped surjectively onto the positive elements of \(\text{Cu}^\sim(A)\), and injectively if \(\text{Cu}(A^\sim)\) has cancellation of projections.

Note that if \(A\) has stable rank 1 then \(\text{Cu}(A^\sim)\) has cancellation of projections by Proposition 2.1.3.

**Proof.** Let \([a] \in \text{Cu}(A^\sim)\) be such that \([\pi(a)] = n\) (we continue to denote by \(\pi\) the extension of the quotient map \(\pi: A^\sim \to \mathbb{C}\) to \(A^\sim \otimes \mathcal{K}\)). If \([a] - n[1] \geq 0\) then \([a] + k[1] \geq (n + k)[1]\) for some \(k\), the latter inequality taken in \(\text{Cu}(A^\sim)\). It follows by Lemma 2.1.1 that \([a] + k[1] = (n + k)[1] + [a']\) for some \([a']\). Since \([\pi(a)] = n\), we must have \([\pi(a')] = 0\), i.e., \([a'] \in \text{Cu}(A)\). Thus, \([a] - n[1] = [a'] - 0 \cdot [1]\). This shows that \(\text{Cu}(A)\) is mapped surjectively onto the positive elements of \(\text{Cu}^\sim(A)\).

Suppose that \(\text{Cu}(A^\sim)\) has cancellation of projections. If \([a_1], [a_2] \in \text{Cu}(A)\) are mapped into the same element of \(\text{Cu}^\sim(A)\) then \([a_1] + k[1] = [a_2] + k[1]\) for some \(k\), whence \([a_1] = [a_2]\). \(\square\)

Among the axioms for the category \(\text{Cu}\) given in Subsection 2.1 we have not included that 0 be the smallest element of the ordered semigroups. This suits our purposes here since, unlike for \(\text{Cu}(A)\), 0 may not the smallest element of \(\text{Cu}^\sim(A)\). For example, \(\text{Cu}^\sim(\mathbb{C}) \cong \mathbb{Z} \cup \{\infty\}\), which is easily deduced from the picture of \(\text{Cu}^\sim\) for unital \(C^*\)-algebras.

**Proposition 3.1.3.** Let \(A\) be a \(C^*\)-algebra of stable rank 1. Then \(\text{Cu}^\sim(A)\) is an ordered semigroup in the category \(\text{Cu}\).

**Proof.** We first assume that \(A\) is unital. Notice, from the definition of \(\text{Cu}^\sim(A)\) for unital \(A\), that the map \(x \mapsto x + [1]\) (though clearly not additive) is an ordered set
automorphism of $\text{Cu}^\sim(A)$. Next, notice that for a finite set or increasing sequence $x_i \in \text{Cu}^\sim(A)$, $i = 1, 2, \ldots$, there is a sufficiently large $m \in \mathbb{N}$ such that $x_i + m[1] \geq 0$ for all $i$. But the ordered subsemigroup of positive elements of $\text{Cu}^\sim(A)$ is isomorphic to $\text{Cu}(A)$, by Lemma 3.1.2. So, the axioms of the category $\text{Cu}$ may be routinely verified on $\text{Cu}^\sim(A)$ by translating by a sufficiently large multiple of $[1]$ and using that $\text{Cu}(A)$ is an ordered semigroup in $\text{Cu}$.

Let us now drop the assumption that $A$ is unital. We view $\text{Cu}^\sim(A)$ as an ordered subsemigroup of $\text{Cu}^\sim(A^\sim)$. Suppose that $(a_i - n_i[1])_{i=1}^\infty$ is an increasing sequence in $\text{Cu}^\sim(A)$. Then

\begin{equation}
0 \leq [a_1] \leq [a_2] - (n_2 - n_1)[1] \leq [a_2] - (n_3 - n_1)[1] \leq \ldots
\end{equation}

is an increasing sequence in $\text{Cu}(A^\sim)$ (here we identify $\text{Cu}(A^\sim)$ with the positive elements of $\text{Cu}^\sim(A^\sim)$ by Lemma 3.1.2). Let $[a]$ denote its supremum. Since $\text{Cu}(\pi)$ applied to every term of (6) is equal to $n_1$ we have $\text{Cu}(\pi)([a]) = n_1$. Thus $[a] - n_1[1] \in \text{Cu}^\sim(A)$. Notice that $[a] - n_1[1]$ is the supremum of $(a_i - n_i[1])_{i=1}^\infty$ in $\text{Cu}^\sim(A^\sim)$. Hence $\text{Cu}^\sim(A)$ is closed under the suprema of increasing sequences inside $\text{Cu}^\sim(A^\sim)$. This proves axiom $O_1$ of the category $\text{Cu}$ (see Subsection 2.1).

If $[a] - n_1[1] \in \text{Cu}^\sim(A)$ then $[(a - \varepsilon)_+] - n_1[1] \in \text{Cu}^\sim(A)$ for $0 < \varepsilon < \|a\|$ and

$$[a] - n_1[1] = \sup_{\varepsilon>0}([(a - \varepsilon)_+] - n_1[1]).$$

This proves $O_2$. The axioms $O_3$ and $O_4$ follow from their being valid in $\text{Cu}^\sim(A^\sim)$. □

Let $\phi: A \to B$ be a homomorphism and let $\phi^\sim: A^\sim \to B^\sim$ denote its unital extension. Then $\text{Cu}^\sim(\phi): \text{Cu}^\sim(A) \to \text{Cu}^\sim(B)$ is defined as

\begin{equation}
\text{Cu}^\sim(\phi)([a] - n_1[1]) := [\phi^\sim(a)] - n_1[1].
\end{equation}

**Remark 3.1.4.** Let $\phi: A \to B$ be a homomorphism.

(i) If $A$ and $B$ are unital, and $\phi(1_A) = 1_B$, then using the definition of $\text{Cu}^\sim(A)$ and $\text{Cu}^\sim(B)$ for unital C*-algebras the map $\text{Cu}^\sim(\phi)$ takes the form

$$\text{Cu}^\sim(\phi)([a] - n_1[1]) = [\phi(a)] - n_1[1].$$

(ii) If $A$ is unital, without $B$ nor $\phi$ being necessarily unital, then using the definition of $\text{Cu}^\sim(A)$ for unital C*-algebras the map $\text{Cu}^\sim(\phi)$ takes the form

$$\text{Cu}^\sim(\phi)([a] - n_1[1]) = [\phi(a)] - n_1[1] - \phi(1_A)[1].$$

(iii) As with the functor $\text{Cu}$, if $\phi$ is approximately unitarily equivalent to a homomorphism $\psi$ then $\text{Cu}^\sim(\phi) = \text{Cu}^\sim(\psi)$.

**Proposition 3.1.5.** Restricted to the C*-algebras of stable rank 1, $\text{Cu}^\sim$ is a functor into the category $\text{Cu}$ that preserves sequential inductive limits.

**Proof.** Let $\phi: A \to B$ be a homomorphism, with $A$ and $B$ of stable rank 1. Let us show that $\text{Cu}^\sim(\phi)$ is a morphism in the category $\text{Cu}$. Assume first that $\phi$ is unital. From Remark 3.1.4 (i) we see that $\text{Cu}^\sim(\phi)$ is additive. In particular, we have $\text{Cu}^\sim(\phi)(x + [1]) = \text{Cu}^\sim(\phi)(x) + [1]$ for $x \in \text{Cu}^\sim(A)$. In order to verify the properties of a morphism of $\text{Cu}$ for the map $\text{Cu}^\sim(\phi)$ we can use the same method used for the semigroup $\text{Cu}^\sim(A)$: after translating by a sufficiently large multiple of $[1]$, we reduce the verification of these properties to the restriction of $\text{Cu}^\sim(\phi)$ to the subsemigroup of positive elements.
of Cu\(^\sim\)(A). By Lemma 3.1.2 this subsemigroup may be identified with Cu(A). Finally, on Cu(A) the map Cu\(^\sim\)(\(\phi\)) coincides with Cu(\(\phi\)), which is a morphism in Cu.

Let us drop the assumption that \(A\) and \(B\) are unital. From (i) it is clear that Cu\(^\sim\)(A), viewed as a sub-semigroup of Cu\(^\sim\)(A\(^\sim\)), is preserved by Cu\(^\sim\)(\(\phi\)). A routine verification (left to the reader) shows that the restriction of Cu\(^\sim\)\(\phi\) to Cu\(^\sim\)(A) is a morphism of Cu.

Finally, consider an inductive limit of stable rank 1 C\(^*\)-algebras \(A = \lim\nolimits_\rightarrow A_i\). In order that Cu\(^\sim\)(A) be the inductive limit of the Cu\(^\sim\)(A\(_i\))s we must verify properties L1 and L2 of inductive limits in Cu (see Subsection 2.1). Unitizing in \(A = \lim\nolimits_\rightarrow A_i\) and taking the Cuntz semigroup functor we get Cu\(^\sim\)(A\(_i\)) = \(\lim\nolimits_\rightarrow Cu(A_i^-)\). Hence, for \([a] \in Cu(A^-)\) there is an increasing sequence \((\{a_i\})^\infty_{i=1}\) with supremum \([a]\), and where the \([a_i]\)s come from the ordered semigroups Cu\((A_i^-)\). If \([\pi(a)] = n < \infty\), then \([\pi(a_i)] = n\) for \(i\) large enough. Thus, \([a] - n[1] = \sup_i([a_i] - n[1])\). This proves L1.

Let us prove L2. Let \([a_1] - n_1[1], [a_2] - n_2[1] \in Cu^-\(A_i\) be such that the image of \([a_1] - n_1[1]\) in Cu\(^\sim\)(A) is less than or equal to the image of \([a_2] - n_2[1]\). This is equivalent to the image of \([a_1] + n_2[1]\) in Cu\(^\sim\) being less than or equal to the image of \([a_2] + n_1[1]\). By the continuity of the functor Cu, for every \(\varepsilon > 0\) there is \(j \geq i\) such that the image of \(([a_1] - \varepsilon)_+ + n_2[1]\) in Cu\(^\sim\) is less than or equal to the image of \([a_2] + n_1[1]\) in Cu\((A_j^-)\). That is, the image of \(([a_1] - \varepsilon)_+ - n_1[1]\) in Cu\((A_j^-)\) is less than or equal to the image of \([a_2] - n_2[1]\). This proves L2.

Proposition 3.1.6. Let

\[
\begin{array}{cccc}
0 & \to & I & \xrightarrow{\phi} & A & \xrightarrow{\psi} & A/I & \to & 0 \\
\end{array}
\]

be a short exact sequence of C\(^*\)-algebras, with \(A\) of stable rank 1. We then have that

(i) Im(Cu\(^\sim\)\(\phi\)) = Ker(Cu\(^\sim\)\(\psi\));

(ii) if \(\psi\) splits then Cu\(^\sim\)\(\psi\) is surjective and Cu\(^\sim\)\(\phi\) is injective.

Proof. (i) From \(\psi \circ \phi = 0\) we get that Cu\(^\sim\)\(\psi\) \(\circ\) Cu\(^\sim\)\(\phi\) = 0. Therefore, Im(Cu\(^\sim\)\(\phi\)) \(\subseteq\) Ker(Cu\(^\sim\)\(\psi\)).

Let us prove that Ker(Cu\(^\sim\)\(\psi\)) \(\subseteq\) Im(Cu\(^\sim\)\(\phi\)). Let \([a] - n[1] \in Cu^-\(A\) be mapped to 0 by Cu\(^\sim\)\(\psi\). That is, \([\psi^-\(a\)] - n[1] = 0\) in Cu\((A/I^-)\). By the cancellation of projections in Cu\((A/I^-)\), this implies that \([\psi^-\(a\)] = [1_n]\) in Cu\((A/I^-)\). Since \(\psi^-\(a\) is Cuntz equivalent to a projection, 0 is an isolated point of the spectrum of \(\psi^-\(a\) (see [2, Proposition 5.7]). So, for \(f \in C_0(\mathbb{R}^+)\) strictly positive and chosen suitably \(\psi^-\(f\(a)\) is a projection. Rename \(f\(a)\) as \(a\). Then \(\psi^-\(a\) and \(1_n\) are Cuntz equivalent projections in \(A/I^- \otimes K\). Since \((A/I^-) \otimes K\) is stably finite, \(\psi^-\(a\) and \(1_n\) are Murray-von Neumann equivalent; furthermore, there exists a unital \(u\) of the form \(u = 1 + u'\), with \(u' \in (A/I^-) \otimes K\) such that \(u\psi^-\(a\) = 1_n\). At this point, we go back to the start of the proof, identify \(A^- \otimes K\) with \(A^- \otimes K \otimes M_2\), and choose \(a\) of the form

\[
\begin{pmatrix}
0 & a \\
0 & 0
\end{pmatrix},
\]

Notice then that the unitary \(u\) be chosen in \(((A/I^-) \otimes K \otimes e_{00})^-\). So we can lift

\[
\begin{pmatrix}
u & 0 \\
0 & u^\ast
\end{pmatrix}.
\]
to a unitary $v$ of the form $v = 1 + v'$, with $v' \in A^\sim \otimes K \otimes M_2$. Set $v^*av = a_1 \in M_2(A^\sim) \otimes K$. Then $[a] = [a_1]$ and $\psi^\sim(a_1) = 1_n$, which says that $a_1 \in I^\sim \otimes K \otimes M_2$. Thus, $[a] - n[1] = [a_1] - n[1] \in \text{Im}(\psi^\sim)$. 

(ii) Let $\lambda \colon A/I \to A$ be such that $\lambda(1) = 1$. Then $\psi^\sim(\lambda) = \psi^\sim(1) = \psi^\sim(1)$, which implies that $\psi^\sim(\lambda)$ is surjective.

Let us prove that $\psi^\sim(\phi)$ is injective. Let $a, b \in I^\sim \otimes K$ be positive elements such that $[\pi(a)] = [\pi(b)] = n$ and $[a] - n[1] = [b] - n[1]$ as elements of $\psi^\sim(A)$. We want to show that $[a] - n[1] = [b] - n[1]$ in $\psi^\sim(I)$. By the cancellation of projections in $\psi^\sim(A)$ and $\psi^\sim(I)$, we have that $[a] = [b]$ in $\psi^\sim(A)$ and we want to show that $[a] = [b]$ in $\psi^\sim(I)$.

We may assume without loss of generality that $a = 1_n + a'$ and $b = 1_n + b'$, for $a', b' \in I \otimes K$. Let $\varepsilon > 0$. Since $a \preceq b$ in $A^\sim \otimes K$, there exists $x \in A^\sim \otimes K$ such that $(a - \varepsilon/2)x = x^*x$ and $xx^* \preceq Mb$ for some $M > 0$. Then by [24, Theorem 5], there exists a unitary $u \in (A^\sim \otimes K)\sim$ such that $u^*(a - \varepsilon)u \preceq Mb$.

Recall that we denote by $\psi^\sim$ the unital extension of $\psi$ to $A^\sim$, and that we also denote by $\psi^\sim$ the homomorphism $\psi^\sim \otimes 1$ extending $\psi^\sim$ to $A^\sim \otimes K$. Let us furthermore continue to denote by $\psi^\sim$ rather than $(\psi^\sim)^\sim$—the unital extension of $\psi^\sim$ to $(A^\sim \otimes K)\sim$. We apply the same notational conventions to the homomorphism $\lambda$. Consider the elements

$$u_1 = (\lambda^\sim \circ \psi^\sim)(u),$$

$$a_1 = (uu_1^*)^*a(uu_1^*).$$

We have $(\lambda^\sim \circ \psi^\sim)(uu_1^*) = 1$, which implies that $uu_1^*$ is a unitary in $(I^\sim \otimes K)\sim$. Thus, $a \sim a_1$ in $I^\sim \otimes K$. We also have that

$$(\lambda^\sim \circ \psi^\sim)(a_1) = (\lambda^\sim \circ \psi^\sim)(a) = 1_n.$$ 

This implies that $a_1 = 1_n + a'_1$ for $a'_1 \in I \otimes K$. From $u^*(a - \varepsilon)u \preceq Mb$ and the definition of $u_1$ we get that $u_1^*(a_1 - \varepsilon)u_1 \preceq Mb$. Applying $(\lambda^\sim \circ \psi^\sim)$ on both sides of this last inequality, and using that $(\lambda^\sim \circ \psi^\sim)(u_1) = u_1$, we get that $u_1^*1_nu_1 = 1_n$, i.e., $u_1$ commutes with $1_n$.

For $k = 1, 2, \ldots$, let

$$v_k = 1_n + (u_1 - 1_n)(e_k \otimes 1),$$

where $e_k \in I^+$ is such that $(e_k \otimes 1)a'_1 \to a'_1$ and $(e_k \otimes 1)u_1^*a'_1 \to u_1^*a'_1$ when $k \to \infty$ (e.g., $(e_k)$ is an approximate unit for the hereditary subalgebra generated by the entries of $a'_1$ and $u_1^*a'_1$). Then $v_k$ belongs to $(I^\sim \otimes K)\sim$ for all $k$. We have

$$v_k a_1' = 1_n a_1' + (u_1 - 1_n)(e_k \otimes 1) a_1' \to u_1 a_1'.$$

Here we have used that $(e_k \otimes 1)a'_1 \to a'_1$. Similarly, we deduce that $v_k^*a_1' \to u_1^*a_1'$. We also have that

$$v_k 1_n = 1_n + (u_1 - 1_n)(e_k \otimes 1) 1_n = 1_n + 1_n(u_1 - 1_n)(e_k \otimes 1) = 1_n v_k,$$

for all $k$ (here we have used that $u_1$ commutes with $1_n$). That is, $v_k$ commutes with $1_n$ for all $k$. 

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Since $I \sim \otimes K$ has stable rank 1, for each $k$ there exists a unitary $w_k$ in $(I \sim \otimes K)^\sim$ such that $\|v_k - w_k\| < \frac{1}{k}$. Then $w_k^* a_k^* w_k \to u_1^* a_1^* u_1$ and $w_k^* 1_n w_k \to 1_n$. It follows that $w_k^* a_k w_k \to u_1^* a_1 u_1$. So $(a_1 - \varepsilon)_+ \sim u_1^* (a_1 - \varepsilon)_+ u_1$ in $I \sim \otimes K$. Hence,

$$(a - \varepsilon)_+ \sim (a_1 - \varepsilon)_+ \sim u_1^* (a_1 - \varepsilon)_+ u_1 \leq M_b,$$

where the Cuntz comparisons are all taken in $I \sim \otimes K$. Since $\varepsilon > 0$ is arbitrary, we conclude that $[a] \leq [b]$ in $Cu(I \sim)$. By symmetry, we also have $[b] \leq [a]$. So $[a] = [b]$ as elements of $Cu(I \sim)$.

**Proposition 3.1.7.** Let $A$ be a $C^*$-algebra of stable rank 1. The inclusion $A \hookrightarrow A \otimes K$ (in the top corner) induces an isomorphism of ordered semigroups $Cu^\sim(A) \to Cu^\sim(A \otimes K)$.

**Proof.** Consider the inductive limit

$$A \hookrightarrow M_2(A) \hookrightarrow M_4(A) \hookrightarrow \ldots \hookrightarrow A \otimes K.$$ 

By the continuity of $Cu^\sim$ with respect to sequential inductive limits, it suffices to show that the inclusion in the top corner $A \hookrightarrow M_2(A)$ induces an isomorphism at the level of $Cu^\sim$.

Let us first assume that $A$ is unital. Then using the picture of $Cu^\sim$ for unital $C^*$-algebras and homomorphisms with unital domain (see Remark 3.1.4 (ii)) we see that the inclusion $A \hookrightarrow M_2(A)$ induces an isomorphism in $Cu^\sim$ if it induces an isomorphism in $Cu$. It is known that $A \hookrightarrow M_2(A)$ induces an isomorphism in $Cu$ (see [9, Appendix 6]). So, the desired result follows for $A$ unital.

The non-unital case is reduced to the unital case as follows. Let $A$ be a non-unital $C^*$-algebra. Consider the diagram

$$
\begin{array}{ccccccc}
0 & \rightarrow & A & \rightarrow & A^\sim & \rightarrow & \mathbb{C} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & M_2(A) & \rightarrow & M_2(A^\sim) & \rightarrow & M_2(\mathbb{C}) & \rightarrow & 0,
\end{array}
$$

where the rows form short exact sequences that split and the vertical arrows are the natural inclusions. Passing to the level of $Cu^\sim$ we have

$$
\begin{array}{ccccccc}
0 & \rightarrow & Cu^\sim(A) & \rightarrow & Cu^\sim(A^\sim) & \rightarrow & Cu^\sim(\mathbb{C}) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & Cu^\sim(M_2(A)) & \rightarrow & Cu^\sim(M_2(A^\sim)) & \rightarrow & Cu^\sim(M_2(\mathbb{C})) & \rightarrow & 0.
\end{array}
$$

A diagram chase—as in the proof of the five lemma—using the exactness of the rows of this diagram (in the sense of Proposition 3.1.6 (i) and (ii)), and that the two rightmost vertical arrows are isomorphisms, shows that $Cu^\sim(A) \to Cu^\sim(M_2(A))$ is an isomorphism. \qed

**Proposition 3.1.8.** Let $A$ and $B$ be stable rank 1 $C^*$-algebras. Let $\iota_A : A \to A \oplus B$ and $\iota_B : B \to A \oplus B$ denote the standard inclusions. Then the map $\gamma : Cu^\sim(A) \oplus Cu^\sim(B) \to Cu^\sim(A \oplus B)$, given by

$$\gamma(x + y) := Cu^\sim(\iota_A)(x) + Cu^\sim(\iota_B)(y)$$

is an isomorphism of ordered semigroups.
exists a homomorphism \( \varphi \) of morphisms from \( \alpha \). Observe that the right side depends only on (8) \( \alpha \) such that \( \pi_B : A \oplus B \to B \) is the quotient map.

A simple diagram chase using the exactness of the rows of this diagram (in the sense of Proposition 3.1.6(i) and (ii)) shows that \( \gamma \) is an isomorphism, as desired. \( \square \)

3.2. Classification by \( \text{Cu}^\sim \): permanence properties. Our main focus is the classification of homomorphisms from a given stable rank 1 C*-algebra \( A \) into an arbitrary C*-algebra of stable rank 1. The classifying functor is \( \text{Cu}^\sim \) together with the class \([s_A] \in \text{Cu}^\sim(A)\) of a strictly positive element of \( A \) (this class does not depend on the choice of the strictly positive element). Since we will consider repeatedly such a classification question, we introduce an abbreviated way of referring to it:

In the sequel, given a C*-algebra \( A \), we say that the functor \( \text{Cu}^\sim \) classifies homomorphisms from \( A \) if for any stable rank 1 C*-algebra \( B \), and any morphism in \( \text{Cu} \)

\[ \alpha : \text{Cu}^\sim(A) \to \text{Cu}^\sim(B) \]

such that \( \alpha([s_A]) \leq [s_B] \), with \( s_A \) and \( s_B \) strictly positive elements of \( A \) and \( B \), there exists a homomorphism \( \phi : A \to B \) such that \( \text{Cu}^\sim(\phi) = \alpha \). Moreover, \( \phi \) is unique up to approximate unitary equivalence (by unitaries in \( B^\sim \)).

**Lemma 3.2.1.** Let \( A \) and \( B \) be C*-algebras of stable rank 1, with \( B \) unital. Let \( s_A \in A \) be a strictly positive element. Let \( \alpha : \text{Cu}^\sim(A) \to \text{Cu}^\sim(B) \) be a morphism in the category \( \text{Cu} \) such that \( \alpha([s_A]) \leq [1] \). Then there exists a unique morphism in \( \text{Cu} \)

\[ \tilde{\alpha} : \text{Cu}^\sim(A^\sim) \to \text{Cu}^\sim(B) \]

that extends \( \alpha \) and satisfies \( \tilde{\alpha}([1]) = [1] \).

**Proof.** Let \( \alpha \) be as in the statement of the lemma and suppose the extension \( \tilde{\alpha} \) exists. For \([a] \in W(A^\sim)\) such that \( |\pi(a)| = n < \infty \) we have, by linearity of \( \tilde{\alpha} \), that

\[ \tilde{\alpha}([a] - m[1]) = \alpha([a] - n[1]) + (n - m)[1]. \]

Observe that the right side depends only on \( \alpha \). Since the set of elements \([a] - m[1]\), with \([a] \in W(A^\sim)\) (i.e., \(|\pi(a)| < \infty\), is dense in \( \text{Cu}^\sim(A^\sim) \), the extension \( \tilde{\alpha} \) of \( \alpha \) is unique.

In order to prove the existence of \( \tilde{\alpha} \), let (5) stand for its definition on the subsemigroup \( \tilde{W} := \{[a] - m[1] \mid [a] \in W(A^\sim), m \in \mathbb{Z}\} \). It is clear from (5) that \( \tilde{\alpha} \) is additive on this subsemigroup. Let us prove that \( \tilde{\alpha} \) preserves the order and far below relations on the elements of \( \tilde{W} \), and that it preserves the supremum of increasing sequences in \( \tilde{W} \) with supremum also in \( \tilde{W} \). Translating by a sufficiently large multiple of \([1]\), it suffices to prove these properties for the restriction of \( \tilde{\alpha} \) to \( W(A^\sim) \) (see the proof of Proposition 3.1.5).

So let \([a] \in W(A^\sim)\) and set \( |\pi(a)| = n \). By (8) Theorem 1, we have \( n[1] \leq [a] + n[s_A] \), i.e., \( 0 \leq [a] - n[1] + n[s_A] \) in \( \text{Cu}^\sim(A) \). Hence,

\[ 0 \leq \alpha([a] - n[1]) + n\alpha([s_A]) \leq \alpha([a] - n[1]) + n[1] = \tilde{\alpha}([a]). \]

That is, \( \tilde{\alpha} \) is positive.
Let \([a_1], [a_2] \in W(A^\sim)\) be such that \([a_1] \leq [a_2]\). If \([\pi(a_1)] = [\pi(a_2)]\) then \(\tilde{\alpha}([a_1]) \leq \tilde{\alpha}([a_2])\) by (8) and the fact that \(\alpha\) is order preserving. Suppose that \([\pi(a_1)] < [\pi(a_2)]\). Let \(\varepsilon > 0\). By Lemma 7.1 (i) there is \([c] \in W(A^\sim)\) such that

\[
([a_1 - \varepsilon]_+) + [c] \leq [a_2],
\]

\[
[\pi((a_1 - \varepsilon)_+)] + [\pi(c)] = [\pi(a_2)].
\]

Thus,

\[
\tilde{\alpha}(([a_1 - \varepsilon]_+)) \leq \tilde{\alpha}(([a_1 - \varepsilon]_+) + [c])
\]

\[
= \tilde{\alpha}(([a_1 - \varepsilon]_+) + [c])
\]

\[
\leq \tilde{\alpha}([a_2]).
\]

On the other hand, from (8) we deduce that

\[
\tilde{\alpha}([a]) = \sup_{\varepsilon > 0} \tilde{\alpha}(([a - \varepsilon]_+)),
\]

for \([a] \in W(A^\sim)\). Thus, \(\tilde{\alpha}([a_1]) \leq \tilde{\alpha}([a_2])\), i.e., \(\tilde{\alpha}\) is order preserving on \(W(A^\sim)\).

From (8) and the fact that \(\alpha\) is order preserving the far below relation we get \(\tilde{\alpha}(([a - \varepsilon]_+) \leq \tilde{\alpha}([a])\) for every \(\varepsilon > 0\) and \([a] \in W(A^\sim)\). So if \([b] \leq [a]\) then \([b] \leq ([a - \varepsilon]_+) \leq [a]\) for some \(\varepsilon > 0\), and so \(\tilde{\alpha}([b]) \leq \tilde{\alpha}([a - \varepsilon]_+) \leq \tilde{\alpha}([a])\). Thus, \(\tilde{\alpha}\) preserves the far below relation.

Finally, let \(([a_i])\) be an increasing sequence with supremum \([a] \in W(A^\sim)\). Then \([\pi(a_i)] = [\pi(a)] < \infty\) for \(i\) large enough. Now we deduce that \(\tilde{\alpha}([a]) = \sup_i \tilde{\alpha}([a_i])\) from (8) and the fact that \(\alpha\) is supremum preserving.

We now extend \(\tilde{\alpha}\) to \(Cu^\sim(A^\sim)\) by setting

\[
\tilde{\alpha}([a] - m[1]) := \sup_{\varepsilon > 0} \tilde{\alpha}(([a - \varepsilon]_+ - m[1])).
\]

(Notice that \(([a - \varepsilon]_+ - m[1]) \in \tilde{W}\) for all \(\varepsilon > 0\).) The subsemigroup \(\tilde{W}\) is dense in \(Cu^\sim(A)\) (i.e., every element of \(Cu^\sim(A)\) is the supremum of a rapidly increasing sequence of elements of \(\tilde{W}\)) and belongs to the category \(\text{PreCu}\) (see [1] Definition 2.1). Thus, the properties of \(\tilde{\alpha}\) already established on \(\tilde{W}\) readily extend to \(Cu^\sim(A^\sim)\) (see [1] Theorem 3.3).

Thus, the meaning of “\(Cu\) classifies homomorphisms from \(A\)” in part (i) of the following theorem is the same as the one defined above for \(Cu^\sim\), except with \(Cu\) in place of \(Cu^\sim\).

**Theorem 3.2.2.** Let \(A\) be a \(C^*\)-algebra of stable rank 1.

(i) If \(A\) is unital then the functor \(Cu^\sim\) classifies homomorphisms from \(A\) if and only if \(Cu\) classifies homomorphisms from \(A\).

(ii) The functor \(Cu^\sim\) classifies homomorphisms from \(A\) if and only if it classifies homomorphisms from \(A^\sim\).

(iii) If \(Cu^\sim\) classifies homomorphisms from the stable rank 1 \(C^*\)-algebras \(A_i\), \(i = 1, 2, \ldots\), and \(A = \lim_{\rightarrow} A_i\), then \(Cu^\sim\) classifies homomorphisms from \(A\).

(iv) If \(Cu^\sim\) classifies homomorphisms from \(A\) and \(B\), with \(B\) also of stable rank 1, then it classifies homomorphisms from \(A \oplus B\).

(v) If \(Cu^\sim\) classifies homomorphisms from \(A\), then it classifies homomorphisms from \(A'\) for any \(A'\) stably isomorphic to \(A\).
Proof. (i) Suppose that \( \text{Cu}^\sim \) classifies homomorphisms from \( A \) and let us show that \( \text{Cu} \) classifies homomorphisms from \( A \) too. Consider the uniqueness question first. Let \( B \) be a stable rank 1 C*-algebra and \( \phi, \psi : A \to B \) homomorphisms such that \( \text{Cu}(\phi) = \text{Cu}(\psi) \). From \( [\phi(1)] = [\psi(1)] \) and the fact that \( \text{Cu}(B^\sim) \) has cancellation of projections, we get that \( [1 - \phi(1)] = [1 - \psi(1)] \) in \( \text{Cu}(B^\sim) \). Now using the picture of \( \text{Cu}^\sim \) for homomorphism with unital domain given in Remark 3.1.2 (ii), we get that

\[
\text{Cu}^\sim(\phi)([a] - n[1]) = [\phi(a)] + n[1 - \phi(1)] - n[1] = [\psi(a)] + n[1 - \psi(1)] - n[1] = \text{Cu}^\sim(\psi)([a] - n[1]).
\]

That is, \( \text{Cu}^\sim(\phi) = \text{Cu}^\sim(\psi) \). Since \( \text{Cu}^\sim \) classifies homomorphisms from \( A \), we conclude that \( \phi \) and \( \psi \) are approximately unitarily equivalent, as desired.

Let \( \alpha : \text{Cu}(A) \to \text{Cu}(B) \) be a morphism in the category \( \text{Cu} \) such that \( \alpha([1]) \leq \alpha([s_B]) \). Since \( [\alpha(1)] \) is compact and \( B \) has stable rank 1, there exists a projection \( p \in B \otimes K \) such that \( [\alpha(1)] = [p] \) (see [3, Theorem 3.5]). Moreover, since \( [p] \leq [s_B] \), \( p \) may be chosen in \( B \) (by Proposition 3.1.2). Let us define \( \tilde{\alpha} : \text{Cu}(A) \to \text{Cu}(B) \) by

\[
\tilde{\alpha}([a] - n[1]) := \alpha([a]) + n[1 - p] - n[1].
\]

It is easily verified that \( \tilde{\alpha} \) is well defined, additive, and order preserving. Notice that the restriction of \( \tilde{\alpha} \) to \( \text{Cu}(A) \) (identified with the subsemigroup of positive elements of \( \text{Cu}^\sim(A) \)) coincides with \( \alpha \). Since \( \alpha \) is a morphism in \( \text{Cu} \), it follows that \( \tilde{\alpha} \) is a morphism in \( \text{Cu} \) too (see the first paragraph of the proof of Proposition 3.1.5). Thus, there exists \( \phi : A \to B \) such that \( \tilde{\alpha} = \text{Cu}^\sim(\phi) \). Restricting these morphisms to \( \text{Cu}(A) \) we get \( \text{Cu}(\phi) = \alpha \).

Suppose now that \( \text{Cu} \) classifies homomorphisms from \( A \). Let \( B \) be a C*-algebra of stable rank 1 and let \( \phi, \psi : A \to B \) be homomorphisms such that \( \text{Cu}^\sim(\phi) = \text{Cu}^\sim(\psi) \). Then identifying \( \text{Cu}(A) \) with the elements greater than 0 of \( \text{Cu}^\sim(A) \) we arrive at \( \text{Cu}^\sim(\phi)|_{\text{Cu}(A)} = \text{Cu}^\sim(\psi)|_{\text{Cu}(A)} \), i.e., \( \text{Cu}(\phi) = \text{Cu}(\psi) \). Since \( \text{Cu} \) classifies homomorphisms from \( A \), \( \phi \) and \( \psi \) are approximately unitarily equivalent.

Let \( \alpha : \text{Cu}(A) \to \text{Cu}(B) \) be a morphism in the category \( \text{Cu} \) such that \( \alpha([1]) \leq [s_A] \). Since \( \text{Cu} \) classifies homomorphisms from \( A \), there exists \( \phi : A \to B \) such that \( \alpha|_{\text{Cu}(A)} = \text{Cu}(\phi) \). Notice that \( \alpha \) is uniquely determined by its restriction to \( \text{Cu}(A) \), by means of the formula (10), and similarly for \( \text{Cu}^\sim(\phi) \). Thus, \( \alpha = \text{Cu}^\sim(\phi) \).

(ii) Suppose that \( \text{Cu}^\sim \) classifies homomorphisms from \( A \). Let us prove that \( \text{Cu} \) classifies homomorphisms from \( A^\sim \) (then by (i), \( \text{Cu}^\sim \) also classifies homomorphisms from \( A^\sim \)). We consider the uniqueness question first. Let \( \phi, \psi : A^\sim \to B \) be homomorphisms such that \( \text{Cu}(\phi) = \text{Cu}(\psi) \), with \( B \) of stable rank 1. Since \( \phi(1) \) and \( \psi(1) \) are Cuntz equivalent projections in a stable rank 1 C*-algebra, they must be unitarily equivalent. Thus, after conjugating by a unitary, we have \( \phi(1) = \psi(1) = p \). From \( [\phi(a)] = [\psi(a)] \) for all \( [a] \in \text{Cu}(A^\sim) \) we deduce that \( \text{Cu}^\sim(\phi|_A) = \text{Cu}^\sim(\psi|_A) \). By assumption \( \text{Cu}^\sim \) classifies homomorphisms from \( A \). Thus, \( \phi|_A \) and \( \psi|_A \) are approximately unitarily equivalent. By Proposition 3.1.1 (i), the unitaries implementing this equivalence may be chosen so that they commute with \( p \). Hence, \( \phi \) is approximately unitarily equivalent to \( \psi \).

Let \( \alpha : \text{Cu}(A^\sim) \to \text{Cu}(B) \) be a morphism in \( \text{Cu} \) such that \( \alpha([1]) \leq [s_B] \), with \( B \) of stable rank 1. Let us show that \( \alpha \) is induced by a homomorphism from \( A^\sim \) to \( B \). Since \( [\alpha(1)] \leq [\alpha(1)] \) and \( B \) has stable rank 1, there is a projection \( p \) such that \( [\alpha(1)] = [p] \) (see [3, Theorem 3.5]). Moreover, since \( [p] \leq [s_B] \), \( p \) may be chosen in
It follows from Proposition 3.1.8 that the ranges of \( Cu \) are the desired homomorphisms. Let \( \alpha \) classify homomorphisms from \( \phi \) and \( x \). To achieve this, find \( \phi \): \( A \rightarrow pBp \) be the unital extension of \( \phi \). We have \( Cu(\phi) \sim \alpha \) by Lemma 3.2.1.

Restricting these morphisms to \( Cu(A) \), we get \( Cu(\phi) = \alpha \).

Let us now assume that \( Cu \) classifies homomorphisms from \( A^\sim \) and prove that it classifies homomorphisms from \( A \). Let us consider the uniqueness part first. Let \( \phi, \psi: A \rightarrow B \) be such that \( Cu(\phi) = Cu(\psi) \), with \( B \) of stable rank 1. By Lemma 3.2.1, we have \( Cu(\phi) = Cu(\psi) \), where \( \phi^\sim \) and \( \psi^\sim \) denote the unital extensions of \( \phi \) and \( \psi \) respectively. Since \( Cu \) classifies homomorphisms from \( A^\sim \), \( \phi^\sim \) and \( \psi^\sim \) are approximately unitarily equivalent. Hence, so are \( \phi \) and \( \psi \).

Let \( \alpha: Cu^\sim(A) \rightarrow Cu^\sim(B) \) be a morphism in \( Cu \) such that \( \alpha([s_A]) \leq [s_B] \). By Lemma 3.2.1, there is \( \tilde{\alpha}: Cu^\sim(A^\sim) \rightarrow Cu^\sim(B^\sim) \) that extends \( \alpha \) and satisfies \( \tilde{\alpha}([1]) = [1] \). Let \( \phi: A^\sim \rightarrow B^\sim \) be such that \( Cu^\sim(\phi) = \tilde{\alpha} \). Notice that \( \phi \) must be unital, since \( Cu(\phi)([1]) = [1] \). Hence \( \phi(A) \subseteq B \) and \( Cu^\sim(\phi|A) = \alpha \).

(iii) This is a direct consequence of the continuity of the functor \( Cu^\sim \) with respect to sequential inductive limits (see [17] for a proof for the functor \( Cu \)).

(iv) Let \( \phi, \psi: A \oplus B \rightarrow C \) be homomorphisms that agree on \( Cu^\sim \), with \( C \) of stable rank 1. Composing \( \phi \) and \( \psi \) with the inclusions \( A \xrightarrow{\iota_A} A \oplus B \) and \( B \xrightarrow{\iota_B} A \oplus B \) and applying the functor \( Cu^\sim \) we get \( Cu^\sim(\phi|A) = Cu^\sim(\psi|A) \) and \( Cu^\sim(\phi|B) = Cu^\sim(\psi|B) \). Since \( Cu^\sim \) classifies homomorphisms from \( A \) and \( B \), \( \phi|A \) and \( \psi|A \) are approximately unitarily equivalent, and \( \phi|B \) and \( \psi|B \) too. Thus, \( \phi \) and \( \psi \) are approximately unitarily equivalent by Proposition 2.3.1(ii).

Let us prove the existence part of the classification. Let \( \alpha: Cu^\sim(A \oplus B) \rightarrow Cu^\sim(C) \) be a morphism in the category \( Cu \) such that \( \alpha([s_A + s_B]) \leq [s_C] \). Since \( Cu^\sim \) classifies homomorphisms from \( A \) and from \( B \), there are homomorphisms \( \phi_A: A \rightarrow C \) and \( \phi_B: B \rightarrow C \) that induce \( \alpha \circ Cu^\sim(\iota_A) \) and \( \alpha \circ Cu^\sim(\iota_B) \) at the level of \( Cu^\sim \). Moreover, using that \( [\phi_A(s_A)] + [\phi_B(s_B)] \leq [s_C] \) we can choose \( \phi_A \) and \( \phi_B \) with orthogonal ranges. To achieve this, find \( x \in M_2(C) \) such that \( x^*x = \begin{pmatrix} \phi_A(s_A) & \phi_B(s_B) \\ \phi_B(s_B) & \phi_B(s_B) \end{pmatrix} \) and \( xx^* \in C \). (Here \( C \) is identified with the top left corner of \( M_2(C) \)). Let \( x = v|x| \) be the polar decomposition of \( x \) in \( M_2(C)** \). Then

\[
\phi_A' = v^* \begin{pmatrix} \phi_A & 0 \\ 0 & 0 \end{pmatrix} v \quad \text{and} \quad \phi_B' = v^* \begin{pmatrix} 0 & 0 \\ 0 & \phi_B \end{pmatrix} v
\]

are the desired homomorphisms.

Let us now define \( \phi: A \oplus B \rightarrow C \) by \( \phi := \phi_A \circ \pi_A + \phi_B \circ \pi_B \). We have

\[
Cu^\sim(\phi) \circ Cu^\sim(\iota_A) = Cu^\sim(\phi_A) = \alpha \circ Cu^\sim(\iota_A),
\]
\[
Cu^\sim(\phi) \circ Cu^\sim(\iota_B) = Cu^\sim(\phi_B) = \alpha \circ Cu^\sim(\iota_B).
\]

It follows from Proposition 3.1.8 that the ranges of \( Cu^\sim(\iota_A) \) and \( Cu^\sim(\iota_B) \) span \( Cu^\sim(A \oplus B) \). Thus, \( Cu^\sim(\phi) = \alpha \).
(v) It suffices to show that the functor $C\mu^-$ classifies homomorphisms from $A$ if and only if it classifies homomorphisms from $A \otimes K$. Suppose that $C\mu^-$ classifies homomorphisms from $A$. Since $A \otimes K = \lim_{\to} M_{2i}(A)$, it is enough to show that $C\mu^-$ classifies homomorphisms from $M_{2i}(A)$ and apply part (ii). The proof now proceeds as in [17 Proposition 5 (ii)]. The properties of the functor $C\mu$ that are relevant to the arguments given in [17 Proposition 5 (ii)] hold also for $C\mu^-$. Namely, that the inclusion $A \hookrightarrow M_{2i}(A)$ induces an isomorphism at the level of $C\mu^-$, and that $C\mu^-$ induces the identity on approximately inner homomorphisms.

In order to prove that if $C\mu^-$ classifies homomorphisms from $A \otimes K$, then it classifies homomorphisms from $A$ we proceed as in [17 Proposition 5 (iv)]. Again, the relevant properties of the functor $C\mu^-$ are that the inclusion $A \hookrightarrow A \otimes K$ induces an isomorphism at the level of $C\mu^-$, and that $C\mu^-$ induces the identity on approximately inner homomorphisms.

□

For $C^*$-algebras $A$ and $B$ of stable rank 1, let us write $A \sim B$ if there is a sequence of stable rank 1 $C^*$-algebras $A = A_1, A_2, A_3, \ldots, A_{n-1}, A_n = B$ such that, for each $i$, either $A_i$ is stably isomorphic to $A_{i+1}$, $A_i$ is the unitization of $A_{i+1}$, or $A_{i+1}$ is the unitization of $A_i$. It is clear that $\sim$ is an equivalence relation. Notice that $A \sim B$ implies $K_1(A) \cong K_1(B)$. More importantly, we have the following corollary to the preceding theorem.

**Corollary 3.2.3.** If $A \sim B$ then $C\mu^-$ classifies homomorphisms from $A$ if and only if it classifies homomorphisms from $B$.

### 3.3. Uniform continuity of the classification.

**Theorem 3.3.1.** Let $A$ be a stable rank 1 $C^*$-algebra such that $C\mu^-$ classifies homomorphisms from $A$. Then for every finite subset $F \subseteq A$ and $\varepsilon > 0$ there exists a finite subset $G \subseteq C\mu^-(A)$ such that for any two homomorphisms $\phi, \psi: A \rightarrow B$, with $B$ of stable rank 1, if

$$
\begin{align*}
C\mu^-(\phi)(g') &\leq C\mu^-(\psi)(g) \\
C\mu^-(\psi)(g') &\leq C\mu^-(\phi)(g)
\end{align*}
$$

for all $g', g \in G$ with $g' \ll g$,

then there exists a unitary $u \in B^\sim$ such that

$$
\|u^* \phi(f)u - \psi(f)\| < \varepsilon \text{ for all } f \in F.
$$

**Proof.** Suppose, by contradiction, that there is a pair $(F, \varepsilon)$ such that for every $G \subseteq C\mu^-(A)$ there exist homomorphisms $\phi_G: A \rightarrow B_G$ and $\psi_G: A \rightarrow B_G$ that satisfy (11), but that do not satisfy (12) for any unitary $u \in B_G^\sim$. Let us replace $B_G$ by $B_G^\sim$ and simply assume that $B_G$ is unital. (Notice that (11) continues to hold after doing this.) Let $B$ denote the quotient of the $C^*$-algebra $\prod_G B_G$ of bounded nets $(b_G)$ by the ideal of nets such that $\|b_G\| \rightarrow 0$, where $G$ ranges through the finite subsets of $C\mu^-(A)$. Let us show that $B$ has stable rank 1. We will have this once we show that $\prod G B_G$ has stable rank 1. Any given $b_G \in B_G$ can be approximated by elements of the form $u_G|b_G|$, with $u_G$ unitary. It follows that the elements of the form $(u_G)(|b_G|)$ (i.e., with polar decomposition) form a dense subset of $\prod G B_G$. But the elements with polar decomposition are in the closure of the invertible elements. Thus, $\prod G B_G$ and $B$ have stable rank 1.

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Let $\phi, \psi \colon A \to B$ be the homomorphisms induced by $(\phi_G), (\psi_G) \colon A \to \prod_G B_G$ by passing to the quotient $B$. We will show that on one hand $\phi$ and $\psi$ are not approximately unitarily equivalent, while on the other $\Cu^\sim(\phi) = \Cu^\sim(\psi)$. Since $B$ is a stable rank 1 $\mathcal{C}^*$-algebra, this contradicts the assumption that $\Cu^\sim$ classifies homomorphisms from $A$.

The homomorphisms $\phi$ and $\psi$ cannot be approximately unitarily equivalent for if they were so, then there would be unitaries $(u_G)$ such that $\|u_G^* \phi_G(a) u_G - \psi_G(a)\| \to 0$. But this would contradict our assumption that, for each $G$, $\phi_G$ and $\psi_G$ do not satisfy [12].

Let $a \in M_n(A^\sim)^+$ be such that $[\pi(a)] = n$. For each $G$ that contains $[(a - \varepsilon)_+ - n[1]$ and $[(a - \varepsilon/2)_+] - n[1]$ we have that

$$[(\phi_G^\sim(a) - \varepsilon)_+)] - n[1] \leq [(\psi_G^\sim(a) - \varepsilon/2)_+] - n[1].$$

Since $B_G$ is a $\mathcal{C}^*$-algebra of stable rank 1, $\Cu(B_G)$ has cancellation of projections. So, $[(\phi_G^\sim(a) - \varepsilon)_+] \leq [(\psi_G^\sim(a) - \varepsilon_2)_+]$. Thus, there exists $x_G \in M_n(B_G)$ such that

$$(\phi_G^\sim(a) - 2\varepsilon)_+ = x_G^* x_G$$

and $h(\psi_G(a)) \cdot x_G x_G^* = x_G x_G^*$.

where $h \in C_0(\mathbb{R}^+)^+$ is equal to 1 on $(\varepsilon/2, \infty)$. Let us set $x$ equal to the image of $(x_G) \in \prod_G M_n(B_G)$ in $M_n(B)$. Then $\phi^\sim((a - \varepsilon)_+) = x^* x$ and $h(\psi^\sim(a)) x x^* = x x^*$, which implies that $[\phi^\sim((a - \varepsilon)_+)] \leq [\psi^\sim(a)]$. Since $\varepsilon > 0$ is arbitrary, $[\phi^\sim(a)] \leq [\psi^\sim(a)]$ for all $a \in M_n(A^\sim)^+$, and by the density of $W(A^\sim)$ in $\Cu(A^\sim)$, we get $\Cu^\sim(\phi) \leq \Cu^\sim(\psi)$. By symmetry, we also have that $\Cu^\sim(\phi) \leq \Cu^\sim(\psi)$. Thus, $\Cu^\sim(\phi) = \Cu^\sim(\psi)$. □

4. Special cases of the classification

In this section we discuss a few special cases of Theorem 1.0.1. The case of $A = C(T)$ for $T$ a tree will be used in the proof of Theorem 1.0.1. The other cases are discussed here in order to illustrate the argument used in the general case. (Thus, they will again be dealt with when we prove Theorem 1.0.1 in full generality.)

Our point of departure is Ciupercă and Elliott’s classification of homomorphisms from $C_0(0, 1]$ and $C[0, 1]$ by the functor $\Cu$ (see [6]). By Theorem 3.2.2 (i) and (ii), we then have that $\Cu^\sim$ also classifies homomorphisms from $C_0(0, 1]$ and $C[0, 1]$.

4.1. Trees. By a compact tree we understand a finite, connected, 1-dimensional simplicial complex without loops.

Theorem 4.1.1. Let $T$ be either a compact tree, or a compact tree with a point removed, or a finite disjoint union of such spaces. Then the functor $\Cu^\sim$ classifies homomorphisms from $C_0(T)$.

Proof. By Theorem 3.2.2 (iv), it suffices to consider only trees (with and without a point removed). The proof proceeds by induction on the number of edges, the base case being the trees $(0, 1]$ and $[0, 1]$. Suppose that the theorem is true for all compact trees with less than $n$ edges, and for all such spaces with one point removed. Let $T$ be a compact tree with $n$-edges. Let $x$ be an endpoint of $T$ and consider the space $T_0$ obtained deleting the edge that contains $x$ from $T$. Then $T_0$ is a finite disjoint union of trees with less than $n$ edges and an endpoint removed. By Theorem 3.2.2 (iv) and the induction hypothesis, $\Cu^\sim$ classifies homomorphisms from $C_0(T_0)$. Since $C(T) \cong (C_0(T_0) \oplus C_0(0, 1])^\sim$, the functor $\Cu^\sim$ classifies homomorphisms from $C(T)$.
(here we have used Theorem 3.2.2 (ii) and (iv)). Finally, since \( C(T) \) is the unitization of \( C_0(T \setminus \{y\}) \) for \( y \in T \), \( \Cu^\sim \) also classifies homomorphisms from \( C_0(T \setminus \{y\}) \) for any \( y \in T \). This completes the induction. \( \square \)

4.2. The algebra \( q\mathbb{C} \). Let us show that the functor \( \Cu^\sim \) classifies homomorphisms from the \( \mathbb{C}^* \)-algebra

\[
q\mathbb{C} = \left\{ f \in M_2(C_0(0,1)) \mid f(1) = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}.
\]

This algebra is the unitization of

\[
\left\{ f \in M_2(C_0(0,1)) \mid f(1) = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix} \right\},
\]

which, in turn, is stably isomorphic to \( C_0(0,1) \). Thus \( q\mathbb{C} \simto C_0(0,1) \), and so \( \Cu^\sim \) classifies homomorphisms from \( q\mathbb{C} \) by Corollary 3.2.3.

4.3. Razak’s building blocks. In [25], Razak proves a classification result for simple inductive limits of building blocks of the form \( M_n(\mathbb{C}) \otimes R_{1,n} \), where

\[
R_{1,n} = \left\{ f \in M_n(C[0,1]) \mid f(0) = \lambda 1_{n-1}, \text{ for some } \lambda \in \mathbb{C} \right\}.
\]

Let us see that \( \Cu^\sim \) classifies homomorphisms from \( R_{1,n} \) (whence, also from the inductive limits of algebras of the form \( M_n(\mathbb{C}) \otimes R_{1,n} \)). We have that \( R_{1,n}^\sim \) (i.e., the unitization of \( R_{1,n} \)) is the subalgebra of \( M_n(C[0,1]) \) of functions such that

\[
f(0) = \begin{pmatrix} \lambda 1_{n-1} \\ \mu \end{pmatrix} \text{ and } f(1) = \lambda 1_n,
\]

for some \( \lambda \in \mathbb{C} \). Now notice that \( R_{1,n}^\sim \) is also the unitization of

\[
\left\{ f \in M_n(C_0(0,1)) \mid f(0) = \begin{pmatrix} 0_{n-1} & 0 \\ 0 & \mu \end{pmatrix} \right\}.
\]

This last algebra is stably isomorphic to \( C_0(0,1) \). Thus \( R_{1,n} \simto C_0(0,1) \), and so the functor \( \Cu^\sim \) classifies homomorphisms from \( R_{1,n} \).

4.4. Prime dimension drop algebras. Let \( p \) and \( q \) be relatively prime, with \( q > p \). Consider the 1-dimensional NCCW complex

\[
Z_{p,q} = \{ f \in M_{pq}(C[0,1]) \mid f(0) \in 1_q \otimes M_p, f(1) \in 1_p \otimes M_q \}.
\]

By Remark 2.2.1 (iii), \( Z_{p,q} \) is stably isomorphic to

\[
A_{p,q} = \left\{ f \in M_q(C[0,1]) \mid f(0) = \mu 1_p, \text{ for some } \lambda, \mu \in \mathbb{C} \right\}.
\]

Let us show that \( \Cu^\sim \) classifies homomorphisms from \( A_{p,q} \), and hence also from \( Z_{p,q} \). We have \( A_{p,q} \simto A_{q-p,p} \), since \( A_{p,q} \) and \( A_{q-p,p} \) both have the same unitization. Let us assume that \( 2p < q \) (otherwise, passing to \( A_{q-p,p} \) we have \( 2(q-p) < q \)). Then \( A_{p,q} \) is isomorphic to a full hereditary subalgebra of \( A'_{p,q} \), where \( A'_{p,q} \) is composed of the functions \( f \in M_q(C[0,1]) \) such that

\[
f(0) = \begin{pmatrix} \Lambda \cdot 1_p \\ 0_{q-2p} \end{pmatrix} \text{ and } f(1) = \lambda 1_q,
\]

where \( \Lambda \) is an \( (n-1) \times (n-1) \) invertible matrix.
adding or removing a unit, and passing to a stably isomorphic algebra, that decreases non-zero entry.

Let \( A \) be a 1-dimensional NCCW complex as in (1), and assume that \( A \) is not unital, we unitize it. By Remark 2.2.1 (iii), the algebra \( A_p,q \), in turn stably isomorphic to

\[
A_p,q = \left\{ f \in M_q(C_0(0,1)) \mid f(0) = \left( \begin{array}{c} \lambda \cdot 1_p \\ \mu \end{array} \right) \right\}.
\]

Notice finally that the unitization of \( A_p,q \) is also the unitization of \( A_{p,q} - p \). Therefore, \( A_p,q \sim A_{p,q} - p \). Since \( p \) and \( q \) are relatively prime, the continuation of this process leads to an algebra of the form \( A_{1,d} \). The algebra \( A_{1,d} \) is a full hereditary subalgebra of \( M_d(C_0(0,1)) \). It follows that \( A_{p,q} \sim C_0(0,1) \), and so \( Cu^r \) classifies homomorphisms from \( A_{p,q} \) for \( p \) and \( q \) relatively prime.

5. General case of the classification

5.1. Reduction step. Throughout this section we will assume the notation for NCCW complexes introduced in Subsection 2.2.

Let \( A \) be a 1-dimensional NCCW complex as in (1). Let us say that \( A \) has pure multiplicities if

- P1 \( A \) is unital (i.e., \( \phi_0 \) and \( \phi_1 \) are unital),
- P2 \( E \) is commutative (i.e, \( e_j = 1 \) for all \( j \)),
- P3 the rows of the matrices \( Z^{\phi_0} \) and \( Z^{\phi_1} \) have at most one non-zero entry (i.e., for every matrix block \( M_{f_i}(\mathbb{C}) \) of \( F \), at most one summand \( \mathbb{C} \) of \( E \) embeds in \( M_{f_i}(\mathbb{C}) \)).

**Theorem 5.1.1.** For every 1-dimensional NCCW complex \( A \) there exists \( B \) such that \( A \sim B \) and \( B \) is a 1-dimensional NCCW complex with pure multiplicities.

**Proof.** Let \( A \) be a 1-dimensional NCCW complex as in (1). Properties P1 and P2 are easily arranged as follows: by Remarks 2.2.1 (ii) and (iii) we can always find \( A' \) stably isomorphic to \( A \) and such that the algebra \( E' \) (corresponding to \( A' \)) is commutative. If \( A' \) is not unital, we unitize it.

Let \( A \) be a 1-dimensional NCCW complex as in (1), and assume that \( A \) is unital and \( E \) is commutative. We will show that, after a number of reduction steps, there is \( A', \) 1-dimensional NCCW complex, such that \( A \sim A' \), \( A' \) has properties P1 and P2, and the first row of both matrices \( Z^{\phi_0} \) and \( Z^{\phi_1} \) (associated to \( A' \)) have exactly one non-zero entry.

Suppose that the first row of the matrix \( Z^{\phi_0} \), associated to \( A \), has two distinct non-zero entries. Say (permuting the columns of \( Z^{\phi_0} \) if necessary) that \( Z_{1,1}^{\phi_0} = p \) and \( Z_{1,2}^{\phi_0} = q \), with \( 0 < p \leq q \). Let us describe a sequence of steps involving the operations of adding or removing a unit, and passing to a stably isomorphic algebra, that decreases \( f_1 \) (i.e., the size of the matrix block \( M_{f_1}(\mathbb{C}) \) of \( F \)). Notice that when we start we have \( 1 < p + q \leq f_1 \).

**Case (I):** \( Z_{1,1}^{\phi_1} \neq 0 \).

(1) Remove the unit corresponding to the first summand of \( E \). That is, pass to the subalgebra \( A' \) corresponding to the data \((E', F, \phi_0|_{E'}, \phi_1|_{E'})\), where \( E' \) is the...
subalgebra of $E$ generated by all but the first matrix block of $E$. Notice that $A$ is the unitization of $A'$ (since $e_1 = 1$). This step amounts to setting the entries of the first columns of $Z^{\phi_0}$ and $Z^{\phi_1}$ equal to 0. Rename $A'$ as $A$.

(2) In the algebra resulting from the previous step, the projections of the maps $\phi_0$ and $\phi_1$ to $M_{f_i}(\mathbb{C})$ are not surjective (i.e., the sum of the entries on the first row of $Z^{\phi_0}$ and $Z^{\phi_1}$ is less than $f_1$). Thus, we may cut-down to a full hereditary subalgebra $A'$ with data $(E, F', \phi_0, \phi_1)$, where $f'_1 < f_1$ and $f'_i = f_i$ for $i > 1$. Here $f'_1$ may be chosen equal to

$$\max(\sum_j Z^{\phi_0}_{1,j}, \sum_j Z^{\phi_1}_{1,j}).$$

Since the matrices $Z^{\phi_0}$ and $Z^{\phi_1}$ are unchanged, $A'$ is stably isomorphic to $A$.

Case (II): $Z^{\phi_1}_{1,1} = 0$.

(1) Remove the unit corresponding to the second summand of $E$ (c.f. step (1) of Case (I)). This step amounts to setting the entries of the second column of $Z^{\phi_0}$ and $Z^{\phi_1}$ equal to 0. Notice that for the resulting algebra we have

$$2Z^{\phi_0}_{1,1} + \sum_{j > 1} Z^{\phi_0}_{1,j} \leq f_1.$$  

(2) Pass to a stably isomorphic algebra $A'$ with data $((e'_j), (f'_i), Z^{\phi_0}, Z^{\phi_1})$ where $e'_j = 2$, $e'_j = 1$ for $j > 1$, and $f'_i = f_i$. The numbers $f'_i$ for $i > 1$ are chosen so that these data are attainable (see Remark 2.2.1(ii)). Notice that the inequality (13) guarantees that we may choose $e'_1 = 2$ and $f'_1 = f_1$. Also, $f'_i$ need only be changed for the indices $i$ for which $Z^{\phi_0}_{i,1} \neq 0$. Rename $A'$ as $A$.

(3) Add a unit to the algebra obtained in the previous step if it is not unital.

(4) (a) If the first row of the matrix $Z^{\phi_1}$ has exactly one non-zero entry, remove the unit corresponding to this entry (c.f. step (1)).

(b) If the first row of $Z^{\phi_1}$ has at least two non-zero entries, apply steps (1)-(2) of either Case I or Case II, depending on which is applicable, to the matrix $Z^{\phi_1}$.

(5) After step (4) is completed, $f_1$ is unchanged, and

Case (a): $e_{j_1} = 2$, $Z^{\phi_0}_{1,j_1} \neq 0$ and $Z^{\phi_1}_{1,j} = 0$ for all $j$.

Case (b): $e_{j_2} = e_{j_2} = 2$, $Z^{\phi_0}_{1,j_1} \neq 0$ and $Z^{\phi_1}_{1,j_2} \neq 0$ for some $j_1, j_2$.

These conditions ensure that in either cases (a) or (b) we may find an NCCW complex $A'$ with data $((e'_j), (f'_i), Z^{\phi_0}, Z^{\phi_1})$ such that $e'_j = 1$ for all $j$ and $f'_i$ strictly smaller than $f_1$ (see Remark 2.2.1(ii)). Since the matrices $Z^{\phi_0}$ and $Z^{\phi_1}$ are unchanged, $A'$ is stably isomorphic to $A$.

The steps described above result in a reduction of $f_1$. Thus, after repeating these steps a finite number of times, the first rows of $Z^{\phi_0}$ and of $Z^{\phi_1}$ will have exactly one non-zero entry. Notice that in the process we have possibly enlarged the size of the blocks $M_{f_i}(\mathbb{C})$ for $i > 1$. However, the number of rows of the matrices $Z^{\phi_0}$ and of $Z^{\phi_1}$ remains unchanged. We continue the same process with the second rows of $Z^{\phi_0}$ and $Z^{\phi_1}$. It can be verified that the steps described above, when applied to the second rows, will not affect the property of the first rows of having at most one non-zero entry. Continuing with the rest of the rows we achieve the desired reduction. $\square$
5.2. Proof of Theorem 1.0.1. The following lemma is a straightforward consequence of the Mayer-Vietoris sequence in K-theory applied to the pull-back diagram (1) (see also [15]).

**Lemma 5.2.1.** Let $A$ be a 1-dimensional NCCW complex as in (1). Then $K_1(A) = 0$ if and only if $K_0(\phi_0) - K_0(\phi_1)$ is surjective.

**Corollary 5.2.2.** Let $A$ be a 1-dimensional NCCW complex with pure multiplicities. If $K_1(A) = 0$ then $A \cong C(T)$ where $T$ is a finite disjoint union of trees.

*Proof.* Since $A$ is unital and has pure multiplicities, the $i$-th row of $Z^{\phi_0}$ and of $Z^{\phi_1}$ has exactly one nonzero entry equal to $m_i$. In order that $K_0(\phi_0) - K_0(\phi_1)$ be surjective we must have $m_i = 1$ for all $i$, i.e., $F$ is commutative. So $A$ is commutative, i.e., $A \cong C(T)$ with $T$ a 1-dimensional graph. But $K_1(A) = 0$, so $T$ contains no loops. That is, $T$ is a finite disjoint union of trees. \qed

**Proof of Theorem 1.0.1.** By Theorem 3.2.2, it suffices to show that $\text{Cu}^\sim$ classifies homomorphisms from a 1-dimensional NCCW complex with trivial K$_1$-group. So let $A$ be as in (1) and assume that $K_1(A) = 0$. By Theorem 5.1.1 there is a 1-dimensional NCCW complex $B$ such that $A \sim B$ and $B$ has pure multiplicities. Since the relation $\sim$ does not change the K$_1$-group, $K_1(B) = 0$. So $B \cong C(T)$ with $T$ a finite disjoint union of trees by Corollary 5.2.2. The functor $\text{Cu}^\sim$ classifies homomorphisms from $C(T)$ (by Proposition 4.1.1). Hence, $\text{Cu}^\sim$ classifies homomorphisms from $A$. \qed

**Corollary 5.2.3.** Let $A$ and $B$ be inductive limits of 1-dimensional NCCW complexes with trivial K$_1$-group. Then $A \otimes K \cong B \otimes K$ if and only if $\text{Cu}^\sim(A) \cong \text{Cu}^\sim(B)$. If the isomorphism from $\text{Cu}^\sim(A)$ to $\text{Cu}^\sim(B)$ maps $[s_A]$ into $[s_B]$, where $s_A$ and $s_B$ are strictly positive elements of $A$ and $B$ respectively, then $A \cong B$. Moreover, in this case the isomorphism from $\text{Cu}^\sim(A)$ to $\text{Cu}^\sim(B)$ lifts to an isomorphisms from $A$ to $B$.

*Proof.* See the proof of [7, Corollary 1]. \qed

6. Simple C*-algebras

Here we show that if a C*-algebra $A$ is simple and an inductive limit of 1-dimensional NCCW complexes then $\text{Cu}^\sim(A)$ is determined by $K_0(A)$ and the cone of traces of $A$ (and their pairing). In the case that $A \otimes K$ contains a projection, this is essentially a consequence of Winter’s Z-stability result [34] for simple C*-algebras of finite decomposition rank, and of the computation of $\text{Cu}(A)$ obtained by Brown and Toms in [5] for simple unital Z-stable $A$ (Z denotes the Jiang-Su algebra). If $A \otimes K$ is projectionless, a different route in the computation of $\text{Cu}^\sim(A)$ must be followed, since there is currently no version of Winter’s result for projectionless C*-algebras (although such a result is likely to be true). So, in this case we rely on more ad hoc methods to compute $\text{Cu}(A)$, which we show to be determined solely by the cone of traces of $A$. We then compute $\text{Cu}^\sim(A)$ in terms of $K_0(A)$, the cone of traces of $A$, and their pairing.

Let $A$ be a C*-algebra. Let $T_0(A)$ denote the cone of densely finite, positive, lower semicontinuous traces on $A$. The cone $T_0(A)$ is endowed with the topology of pointwise convergence on elements of the Pedersen ideal of $A$. We shall consider various spaces
of functions on $T_0(A)$.

$$\text{Aff}_+(T_0(A)) := \left\{ f : T_0(A) \to [0, \infty) \mid f is linear, continuous, and f(\tau) > 0 for \tau \neq 0 \right\},$$

$$\text{LAff}_+(T_0(A)) := \left\{ f : T_0(A) \to [0, \infty] \mid \exists (f_n) with f_n \uparrow f and f_n \in \text{Aff}_+(T_0(A)) \right\},$$

$$\text{LAff}_+(T_0(A)) := \left\{ f : T_0(A) \to (-\infty, \infty] \mid f = f_1 - f_2, f_1 \in \text{LAff}_+(T_0(A)), f_2 \in \text{Aff}_+(T_0(A)) \right\}.$$

Since lower semicontinuous traces on $A$ extend (uniquely) to $A \otimes \mathcal{K}$, positive elements $a \in (A \otimes \mathcal{K})^+$, and Cuntz classes $[a] \in \text{Cu}(A)$, give rise to functions on $T_0(A)$:

$$\widehat{a}(\tau) := \tau(a), \text{ for } \tau \in T_0(A),$$

$$[a] := \sup_n \left( a^{\frac{n}{2^k}} \right).$$

**Remark 6.0.4.** The following facts are either known or easily verified.

(i) If $a$ is a positive element in the Pedersen ideal of $A \otimes \mathcal{K}$ then $\widehat{a} \in \text{Aff}_+(T_0(A)).$

(ii) If $a$ is an arbitrary positive element in $A \otimes \mathcal{K}$ then $\widehat{a} \in \text{LAff}_+(T_0(A)).$ (Because $\widehat{a} = \sup_n \left( a - \frac{1}{n} \right)_+$ and $(a - \frac{1}{n})_+$ belongs to the Pedersen ideal of $A \otimes \mathcal{K}$ for all $n > 0.$)

(iii) If $A$ is simple then the range of the map $a \mapsto \widehat{a}$, with $a \in (A \otimes \mathcal{K})^+$, is exactly the space $\text{LAff}_+(T_0(A))$ (see [17, Remarks 5.14 and 6.9]).

### 6.1. Case with projections

**Let $A$ be a simple inductive limit of 1-dimensional NCCW complexes. Assume, furthermore, that $A \otimes \mathcal{K}$ contains at least one non-zero projection $r$. Then $r(A \otimes \mathcal{K})r$ is a full hereditary subalgebra of $A \otimes \mathcal{K}$—by the simplicity of $A \otimes \mathcal{K}$. Thus, by Brown’s theorem, $A$ and $r(A \otimes \mathcal{K})r$ are stably isomorphic and therefore have isomorphic $\text{Cu}^-$-ordered semigroups. Since $r(A \otimes \mathcal{K})r$ is unital, we may use the picture of $\text{Cu}^-$ for unital C*-algebras. Thus, a general element of $\text{Cu}^-(A)$ has the form $[a] - [q]$, where $[a] \in \text{Cu}(r(A \otimes \mathcal{K})r) = \text{Cu}(A)$ and $q \in A \otimes \mathcal{K}$ is a projection.**

Consider the set $K_0(A) \cup \text{LAff}_+(T_0(A)).$ Let us define on this set an ordered semigroup structure. On the subsets $K_0(A)$ and $\text{LAff}_+(T_0(A))$ the addition operation agrees with the addition with which these sets are endowed. For mixed sums we define

$$([p] - [q]) + \alpha := \widehat{[p]} - \widehat{[q]} + \alpha \in \text{LAff}_+(T_0(A)),$$

where $[p] - [q] \in K_0(A)$ and $\alpha \in \text{LAff}_+(T_0(A)).$ The order again restricts to the natural order on the subsets $K_0(A)$ and $\text{LAff}_+(T_0(A)).$ For $[p] - [q]$ and $\alpha$ as before, we define $\alpha \leq [p] - [q]$ if $\alpha \leq \widehat{[p]} - \widehat{[q]}$, and $[p] - [q] \leq \alpha$ if $\widehat{[p]} - \widehat{[q]} < \alpha$.

**Proposition 6.1.1.** Let $A$ be a simple inductive limit of 1-dimensional NCCW complexes. Suppose that $A \otimes \mathcal{K}$ contains at least one non-zero projection. Then the map (14) is an isomorphism of ordered semigroups.

**Proof.** Let $r$ be a non-zero projection in $A \otimes \mathcal{K}$. Then $r(A \otimes \mathcal{K})r$ is simple and unital. Since the decomposition rank is well behaved with respect to inductive limits and hereditary subalgebras, $r(A \otimes \mathcal{K})r$ has decomposition rank at most 1 (see [21] Section 3.3
and Proposition 3.10]). It follows by Winter’s [34] Theorem 5.1 that $r(A \otimes \mathcal{K})r$ absorbs tensorially the Jiang-Su algebra $\mathcal{Z}$. But $A$ and $r(A \otimes \mathcal{K})r$ are stably isomorphic. Thus, $A$ is $\mathcal{Z}$-absorbing too (by [33] Corollary 3.1). It now follows from the computation of $\text{Cu}(A)$ in [5] Theorem 2.5 that

$$[a] \mapsto \begin{cases} [p] & \text{if } [a] = [p], \text{ with } p \text{ a projection in } A \otimes \mathcal{K} \\ [a] & \text{otherwise,} \end{cases}$$

is an ordered semigroup isomorphism from $\text{Cu}(A)$ to $V(A) \cup \text{LAff}_+(T_0(A))$. Here $V(A)$ denotes the semigroup of Murray-von Neumann classes of projections of $A \otimes \mathcal{K}$. Recall that we view $\text{Cu}^\sim(A)$ as the semigroup of formal differences $[a] - [q]$, with $[a] \in \text{Cu}(A)$ and $q \in A \otimes \mathcal{K}$ a projection. A straightforward calculation then shows that (13) is also an isomorphism of ordered semigroups. \hfill \Box

6.2. Projectionless case. Let us now turn to the computation of $\text{Cu}^\sim(A)$ in the stably projectionless case.

**Proposition 6.2.1.** Let $A$ be a simple inductive limit of 1-dimensional NCCW complexes. Suppose that $A \otimes \mathcal{K}$ is projectionless. Then the mapping $[a] \mapsto \hat{[a]}$, from $\text{Cu}(A)$ to $\text{LAff}_+(T_0(A))$, is an isomorphism of ordered semigroups.

**Proof.** Let $A_i$ be 1-dimensional NCCW complexes such that $A = \text{lim}(A_i, \phi_{i,j})$. By [31] Theorem 4.6, the Cuntz semigroup of a 1-dimensional NCCW complex has strict comparison. That is, if $\hat{[a]} \leq (1-\varepsilon)\hat{[b]}$ for some $\varepsilon > 0$, then $[a] \leq [b]$. Since strict comparison passes to inductive limits, $\text{Cu}(A)$ has strict comparison too. It is known that for a simple C*-algebra with strict comparison, the map $[a] \mapsto \hat{[a]}$ is injective on the complement of the subsemigroup of Cuntz classes $[p]$, with $p$ a projection (see [17], [5]). Since in our case $A \otimes \mathcal{K}$ is projectionless, we conclude that $[a] \mapsto \hat{[a]}$ is injective.

In order to prove surjectivity, it suffices to show that for each $[a] \in \text{Cu}(A)$ and $\lambda \in \mathbb{Q}^+$ there exists $[b]$ such that $\lambda \hat{[a]} = \hat{[b]}$ (see [17] Corollary 5.8). In fact, it suffices to show this for $\lambda = \frac{1}{n}$, $n \in \mathbb{N}$. Using that $\text{Cu}(A)$ has strict comparison, this can be reduced to proving the following property of almost divisibility:

(D) For all $x \in \text{Cu}(A)$, $x' \ll x$, and $n \in \mathbb{N}$, there exists $y \in \text{Cu}(A)$ such that $n \hat{y} \leq \hat{x}$ and $\hat{x}' \leq (n+1)\hat{y}$.

The argument to prove that $\text{Cu}(A)$ has this property runs along similar lines as the proof of [32] Theorem 3.4. We sketch it here briefly: First, notice that in order to prove (D) it suffices to verify it for $x$ belonging to a dense subset of $\text{Cu}(A)$, as it then extends easily to all $x$. (Recall that by dense subset we mean one that for every $x$ there is a rapidly increasing sequence of elements in the given subset with supremum $x$.) For $B \subseteq (A \otimes \mathcal{K})^+$ dense and closed under functional calculus, the set $\{[b] \mid b \in B\}$ is dense in $\text{Cu}(A)$ (this is a consequence of Rørdam’s [27] Proposition 4.4). Thus, we may assume that $x = [\phi_{i,\infty}(a)]$, with $a \in (A_i \otimes \mathcal{K})^+$ for some $i$. Let $0 < \varepsilon < \|a\|$. Since $A$ is simple (and non-type I) there exists $j$ such that $\phi_{i,j}(a)$ and $\phi_{i,j}((a - \varepsilon)_+)$ have rank at least $n+1$ on every irreducible representation of $A_j \otimes \mathcal{K}$. Rename $\phi_{i,j}(a)$ as $a$.

**Claim.** There exists a positive element $b \in A_j \otimes \mathcal{K}$ such that

$$n \cdot \text{rank } \pi b \leq \text{rank } \pi a,$$

$$\text{rank } \pi (a - \varepsilon)_+ \leq (n+1) \cdot \text{rank } \pi b,$$
for every irreducible representation $\pi$ of $A_j \otimes K$.

Proof of claim. In order to find $b$, we use that $A_j$ is a 1-dimensional NCCW complex. Let $A_j$ be given by the pull-back diagram $(1)$. Identify $\text{Cu}(A_j)$ with the ordered semigroup of pairs $((n_i)_{i=1}^k, f)$, with $n_i \in \{0, 1, \ldots, \infty\}$ and $f \in \text{LSc}(\{0, 1, \ldots, \infty\})$, such that

$$Z^{\phi_0}(n_i) \leq f(0) \text{ and } Z^{\phi_1}(n_i) \leq f(1).$$

Say $[a]$ corresponds to the pair $((n_i)_{i=1}^k, f)$. Then we can choose $[b]$ as the pair $(((\lfloor \frac{n_i}{\pi} \rfloor)_{i=1}^k, \lfloor \frac{\pi}{n} \rfloor))$. A simple calculation shows that $[b]$ satisfies $(15)$ and $(16)$, as desired.

Let $[b]$ be as in the previous claim. Then $(15)$ and $(16)$ imply that $n[b] \leq [a]$ and $[(a - \varepsilon)\gamma] \leq (n + 1)[b]$. Moving $[a]$ and $[b]$ forward to $\text{Cu}(A)$ we get (D). □

Let us now define a pairing between elements of $\text{Cu}^\sim(A)$ and traces in $T_0(A)$. That is, we define a map from $\text{Cu}^\sim(A)$ to $\text{LAff}_+(T_0(A))$. Let $A$ be a simple, projectionless inductive limit of 1-dimensional NCCW complexes. In the sequel, we identify $\text{Cu}(A)$ with $\text{LAff}_+(T_0(A))$ by the isomorphism given in Proposition 6.2.1. Let $[a] \in \text{Cu}(A^\sim)$ be such that $[\pi(a)] = n < \infty$. Since $[a]$ and $n[1]$ are mapped to the same element in the quotient by $A$, there exists $[b] \in \text{Cu}(A)$ such that $n[1] \leq [a] + [b]$ (see [8, Proposition 1]). Since we are identifying $\text{Cu}(A)$ with $\text{LAff}_+(T_0(A))$, we write $n[1] \leq [a] + \beta$ with $\beta \in \text{LAff}_+(T_0(A))$. Recall that every function of $\text{LAff}_+(T_0(A))$ is the supremum of an increasing sequence of functions in $\text{Aff}_+(T_0(A))$. So, we may choose $\beta$ in $\text{Aff}_+(T_0(A))$ such that $n[1] \leq [a] + \beta$. By Lemma 2.1.1 $n[1]$ sits as a summand of any element above it. Thus, there exists $\gamma \in \text{LAff}_+(T_0(A))$ such that

$$n[1] + \gamma = [a] + \beta.$$  

In summary, for every $[a] \in \text{Cu}(A^\sim)$ such that $[\pi(a)] = n < \infty$ there exist $\beta \in \text{Aff}_+(T_0(A))$ and $\gamma \in \text{LAff}_+(T_0(A))$ such that $(17)$ holds.

**Lemma 6.2.2.** Assume the notation and hypotheses of the preceding paragraph. The assignment

$$(18) \quad [a] - n[1] \mapsto \gamma - \beta.$$  

is a well defined map from $\text{Cu}^\sim(A)$ to $\text{LAff}_+(T_0(A))$.

Proof. Let $[a_1] - n_1[1]$ and $[a_2] - n_2[1]$ be elements of $\text{Cu}^\sim(A)$ such that $[a_1] - n_1[1] \leq [a_2] - n_2[1]$. Let $\beta_1 \in \text{Aff}_+(T_0(A))$ and $\gamma_1 \in \text{LAff}_+(T_0(A))$ be functions such that $(17)$ holds for $[a_1] - n_1[1]$ and define $\beta_2$ and $\gamma_2$ similarly for $[a_2] - n_2[1]$. The following computations are performed in $\text{Cu}^\sim(A)$:

$$[a_1] - n_1[1] \leq [a_2] - n_2[1]$$

$$[a_1] - n_1[1] + \beta_1 + \beta_2 \leq [a_2] - n_2[1] + \beta_1 + \beta_2$$

$$n_1[1] + \gamma_1 - n_1[1] + \beta_2 \leq n_2[1] + \gamma_2 - n_2[1] + \beta_1$$

$$\gamma_1 + \beta_2 \leq \gamma_2 + \beta_1.$$  

That is, $\gamma_1 - \beta_1 \leq \gamma_2 - \beta_2$. This shows at once that the map $(18)$ is well defined and order preserving. □

For $[a] - n[1] \in \text{Cu}^\sim(A)$ let us denote by $(\lfloor [a] - n[1] \rfloor)$ the function $\gamma - \beta$, with $\beta$ and $\gamma$ as in $(17)$. We have shown in the previous lemma that this map is well defined and order preserving. It is also clear from its definition that it is additive. We can
now put an order and an additive structure on $K_0(A) \sqcup \text{LAff}_\sim^\perp(T_0(A))$ just as we did before in the case that $A \otimes \mathcal{K}$ contained a projection. This time we use the map $[a] - n[1] \mapsto ([a] - n[1])\downarrow$ to define mixed sums and the order relation. Moreover, we can define a map from $\text{Cu}^\sim(A)$ to $K_0(A) \sqcup \text{LAff}_\sim^\perp(T_0(A))$ by

$$\text{(19) } [a] - n[1] \mapsto \begin{cases} [p] - n[1] & \text{if } [a] = [p], \text{ with } p \text{ a projection in } A^\sim \otimes \mathcal{K} \\ ([a] - n[1])\downarrow & \text{otherwise.} \end{cases}$$

**Proposition 6.2.3.** Let $A$ be a simple inductive limit of 1-dimensional NCCW complexes. Suppose that $A \otimes \mathcal{K}$ is projectionless. Then the map (19) is an isomorphism of ordered semigroups.

**Proof.** We will prove that (19) defines a bijection from $\text{Cu}^\sim(A)$ to the ordered semigroup $K_0(A) \sqcup \text{LAff}_\sim^\perp(T_0(A))$. The verification that this map is additive and order preserving is left to the reader. (Notice that we have already established that its restrictions to $K_0(A)$ and $\text{LAff}_\sim^\perp(T_0(A))$ are ordered semigroup maps.

Since $A^\sim$ is stably finite, the Cuntz equivalence of projections in $A^\sim \otimes \mathcal{K}$ amounts to their Murray-von Neumann equivalence. Thus, the restriction of the map (19) to the subsemigroup of $\text{Cu}^\sim(A)$ of elements of the form $[p] - n[1]$, with $p$ a projection, is a bijection with $K_0(A)$.

Let us show that the restriction of the map (19) to the elements $[a] - n[1]$, with $[a] \neq [p]$ for any projection $p$, is a bijection with $\text{LAff}_\sim^\perp(T_0(A))$.

Let us show injectivity. Let $[a_1] - n_1[1]$ and $[a_2] - n_2[1]$ be elements of $\text{Cu}^\sim(A)$ such that $([a_1] - n_1[1])\downarrow = ([a_2] - n_2[1])\downarrow$. Choose $\beta_1 \in \text{Aff}_+(T_0(A))$ and $\gamma_1 \in \text{LAff}_\sim^\perp(T_0(A))$ such that $([a_1] - n_1[1])\downarrow = \gamma_1 - \beta_1$ and find $\beta_2$ and $\gamma_2$ similarly for $[a_2] - n_2[1]$. Then

$$\text{(20) } [a_1] + \beta = n[1] + \gamma = [a_2] + \beta,$$

with $\beta = \beta_1 + \beta_2$ and $\gamma = \gamma_2 + \beta_1$.

**Claim.** If $[a_1] \neq [p]$, for any projection $p$, then $([a_1] - \varepsilon)\downarrow + \beta \ll [a_1] + \beta$ for any $0 < \varepsilon < ||a_1||$.

**Proof of claim:** Let $g_\varepsilon \in C_0(0, ||a_1||)^+$ be a function with support $(0, \varepsilon]$. We have

$$([a_1 - \frac{\varepsilon}{2}]\downarrow + [g_\varepsilon(a_1)]) + \beta \ll [a_1] + \beta.$$ 

We cannot have $g_\varepsilon(a_1) = 0$; otherwise 0 would be an isolated point of the spectrum of $a_1$ and this would imply $[a_1] = [p]$ for some projection $p$. Since $[g_\varepsilon(a_1)] \in \text{Cu}(A)$, we view it as a function in $\text{LAff}_+(T_0(A))$. Then, there is $\delta > 0$ such that $(1 + \delta)\beta \ll [g_\varepsilon(a_1)] + \beta$. Since $\beta \ll (1 + \delta)\beta$ (because $\beta$ is continuous), we get that $\beta \ll [g_\varepsilon(a_1)] + \beta$. So,

$$([a_1 - \varepsilon]\downarrow + \beta \ll ([a_1 - \frac{\varepsilon}{2}]\downarrow + [g_\varepsilon(a_1)]) + \beta \ll [a_1] + \beta.$$ 

This proves the claim.

Now from (20) we deduce that $([a_1 - \varepsilon]\downarrow + \beta \ll [a_1] + \beta = [a_2] + \beta$ for any $\varepsilon > 0$. By weak cancellation, this implies $([a_1 - \varepsilon]\downarrow \ll [a_2]$ for any $\varepsilon > 0$. Hence $[a_1] \ll [a_2]$, and by symmetry, $[a_2] \ll [a_1]$. This shows that (18) is injective on the elements $[a] - n[1]$ with $[a] \neq [p]$, for any projection $p \in A^\sim \otimes \mathcal{K}$.

Let us show surjectivity onto $\text{LAff}_\sim^\perp(T_0(A))$. Since the functions in $\text{LAff}_+(T_0(A))$ are attainable by elements in $\text{Cu}(A)$, it is enough to show that $-\gamma$, with $\gamma \in \text{Aff}_+(T_0(A))$, is attainable. For every $\varepsilon > 0$ we have $\gamma \ll (1 + \varepsilon)\gamma$. Thus, there exists a positive
contraction $a$ in $M_m(A)$ for some $m$, and a number $\delta > 0$, such that

$$\gamma \leq [(a - \delta)_+] \leq [a] \leq (1 + \varepsilon)\gamma.$$  

Let $c_\delta \in C_0[0,1]$ be positive on $[0,\delta]$ and 0 elsewhere. Let $c_\delta(a) \in M_m(A^\sim)$ denote the positive element obtained applying functional calculus on $a$ and $1_m$. Then

$$[c_\delta(a)] + [(a - \delta)_+] \leq m[1] \leq [c_\delta(a)] + [a].$$

Passing to $\text{Cu}^\sim(A)$ we have

$$[c_\delta(a)] - m[1] + [(a - \delta)_+] \leq 0 \leq [c_\delta(a)] - m[1] + [a].$$

From this we deduce that $[(c_\delta(a)] - m[1]) \leq -[a - \delta]_+ \leq -\gamma$. Similarly, we have $-(1 + \varepsilon)\gamma \leq ([c_\delta(a)] - m[1])$. In summary, we have found $c_\delta(a) \in M_m(A^\sim)$ such that

$$-(1 + \varepsilon)\gamma \leq ([c_\delta(a)] - m[1]) \leq -\gamma.$$

Notice that the spectrum of $a$ has no gaps, since $A \otimes K$ is projectionless. Thus, $[c_\delta(a)]$ is not the class of a projection in $A^\sim \otimes K$. In order to attain the function $-\gamma$ exactly, we consider the increasing sequence of functions $-(1 + \frac{1}{n})\gamma$, $n = 1, 2, \ldots$, with pointwise supremum $\gamma$. By the previous discussion, between any two consecutive terms of this sequence we can interpolate an element of the form $[c_n] - m_n[1]$, where $[c_n]$ is not the class of a projection. While proving the injectivity of the map \cite{[19]} before, we have in fact shown that if $([a_1] - n_1[1]) \leq ([a_2] - n_2[1])$, and $[a_1]$ is not the class of a projection, then $[a_1] - n_1[1] \leq [a_2] - n_2[1]$. Thus, we have that the sequence $[c_n] - m_n[1]$, $n = 1, 2, \ldots$, is itself increasing. It follows that $([c] - m[1]) = -\gamma$, where $[c] - m[1]$ the supremum of the sequence $[c_n] - m_n[1]$, $n = 1, 2, \ldots$. \hfill \Box

In the following corollary, we combine the computations of $\text{Cu}^\sim(A)$ for both cases, with and without projections, with the classification result of the previous section. The pairing between $K_0(A)$ and $T_0(A)$ alluded to in the statement of this corollary is defined as follows: In the case that $A \otimes K$ contains a projection, the pairing between $K_0(A)$ and $T_0(A)$ is given by

$$([p] - [q], \tau) \mapsto \tau(p) - \tau(q),$$

while, in the stably projectionless case, the pairing is obtained by restricting the pairing between $\text{Cu}^\sim(A)$ and $T_0(A)$ to $K_0(A)$. That is,

$$([p] - n[1], \tau) \mapsto ([p] - n[1])\hat{\tau}(\tau).$$

Alternatively, in the stably projectionless case one may follow the approach used in \cite{[14]} and define the pairing of $[p] - n[1]$ and a lower semicontinuous trace $\tau$ by first finding $A_\tau \subseteq A$, a non-zero hereditary subalgebra on which $\tau$ is bounded, subsequently finding a projection $p' \in \bigcup_n M_n(A_\tau^\sim)$ Murray-von Neumann equivalent to $p$, and setting $([p] - n[1], \tau) := \hat{\tau}(p') - n\hat{\tau}(1)$, where $\hat{\tau}$ denotes the bounded extension of $\tau$ to $A_\tau^\sim$.

**Corollary 6.2.4.** Let $A$ and $B$ be simple $C^*$-algebras that are expressible as inductive limits of 1-dimensional $\text{NCCW}$-complexes with trivial $K_1$-group. Suppose that $K_0(A) \cong K_0(B)$ as ordered groups, $T_0(A) \cong T_0(B)$ as topological cones, and that these isomorphisms are compatible with the pairing between $K_0$ and $T_0$. Then $A \otimes K$ and $B \otimes K$ are isomorphic.

Furthermore, suppose that we have one of the following two cases:
(a) $A$ and $B$ are both unital and the isomorphism from $K_0(A)$ to $K_0(B)$ maps $[1_A]$ to $[1_B]$.

(b) neither $A$ nor $B$ is unital and the isomorphism between $T_0(A)$ and $T_0(B)$ maps the tracial states of $A$ bijectively into the tracial states of $B$.

Then $A$ and $B$ are isomorphic.

Proof. Notice that $A \otimes K$ is projectionless if and only if $B \otimes K$ is projectionless, as this property is equivalent to the $K_0$-groups having trivial cones of positive elements (i.e., trivial order). Thus, either by Proposition 6.1.1 or Proposition 6.2.3 depending on which is applicable, we have that $Cu^\sim(A) \cong Cu^\sim(B)$. Moreover, by (a) or (b), again depending on the case at hand, the isomorphism from $Cu^\sim(A)$ to $Cu^\sim(B)$ maps the class $[s_A]$ into the class $[s_B]$, where $s_A$ and $s_B$ are strictly positive elements of $A$ and $B$ respectively. It follows by Corollary 5.2.3 that $A$ and $B$ are isomorphic.

Remark 6.2.5. In Proposition 6.2.1 we have shown that if a $C^*$-algebra $A$ is stably projectionless and an inductive limit of 1-dimensional NCCW complexes then Cu$(A)$ is determined solely by $T_0(A)$. It is known that among these $C^*$-algebras there are many that have non-trivial $K_0$-group. In fact, it is explained in [15 Theorems 5.2.1 and 5.2.2] how such examples can be obtained as inductive limits of 1-dimensional NCCW complexes with trivial $K_1$-group. It follows that while Cu$^\sim$ is a classifying functor for these $C^*$-algebras, Cu is not, as it fails to account for their $K_0$-groups.

6.3. Embeddings of $Z$. The Jiang-Su algebra is a simple unital exact non-elementary $C^*$-algebra with a unique tracial state and strict comparison of positive elements (i.e., if $[a] \leq (1-\varepsilon)[b]$ for some $\varepsilon > 0$, then $[a] \leq [b]$ for any Cuntz semigroup elements $[a]$ and $[b]$). These properties suffice to compute its Cuntz semigroup (see [5]):

$$Cu(Z) \cong N \sqcup [0, \infty].$$

More generally, if $A$ is a non-elementary simple unital $C^*$-algebra with a unique 2-quasitracial state and with strict comparison of positive elements, then

$$Cu(A) \cong V(A) \sqcup [0, \infty].$$

Now assume that $A$ has stable rank 1. We can then apply the classification result from the previous section with domain $Z$ and with codomain the $C^*$-algebra $A$. Since $Z$ is unital, we can use the functor Cu, instead of Cu$^\sim$, to classify homomorphisms from $Z$ (by Theorem 3.2.2 (i)). Notice that there exists a unique morphism in Cu from Cu$(Z)$ to Cu$(A)$ such that $N \ni 1 \mapsto [1] \in V(A)$ (since it must also map $1 \in [0, \infty]$ to $1 \in [0, \infty]$). It follows that, up to approximate unitary equivalence, there exists a unique unital homomorphism from $Z$ to $A$. This can be turned into a characterization of $Z$.

Proposition 6.3.1. Consider the class $C$ of $C^*$-algebras that are unital, simple, non-elementary, with a unique 2-quasitracial state and with strict comparison of positive elements. Then the Jiang-Su algebra is the unique $C^*$-algebra in $C$ with the property that for any $A \in C$ there exists, up to approximate unitary equivalence, a unique unital embedding from $Z$ to $A$.

Proof. We have already argued in the previous paragraph that $Z$ embeds unitally and in a unique way—up to approximate unitary equivalence—into any $C^*$-algebra in $C$. Suppose that $Z'$ is another $C^*$-algebra in $C$ with this same property. Then there
exist unital embeddings $\phi: Z \to Z'$ and $\psi: Z' \to Z$. By the uniqueness of unital embeddings of $Z$ in $Z$, the endomorphism $\psi \circ \phi$ is approximately inner. Similarly, $\phi \circ \psi$ is also approximately inner. It follows by a standard intertwining argument that $Z' \cong Z$. \hfill $\Box$

Among the notable C*-algebras in the class $\mathcal{C}$ of the previous proposition is $C_r(\mathbb{F}_\infty)$, the reduced C*-algebra of the free group with infinitely many generators. It is well known that $C_r(\mathbb{F}_\infty)$ is simple and has a unique tracial state. Since $C_r(\mathbb{F}_\infty)$ is exact, its 2-quasitraces are traces by Haagerup’s Theorem. Thus, $C_r(\mathbb{F}_\infty)$ has a unique 2-quasi-tracial state. In [11], Dykema, Haagerup, and Rørdam show that $C_r(\mathbb{F}_\infty)$ has stable rank 1. Finally, in [12, Theorem 2.1 (i)], Dykema and Rørdam show that $C_r(\mathbb{F}_\infty)$ has strict comparison of positive elements. (Although they state this result only for projections, their proof is easily adapted to positive elements). Therefore, $C_r(\mathbb{F}_\infty)$ is in $\mathcal{C}$. It is not known whether the C*-algebras $C_r(\mathbb{F}_n)$ have strict comparison of positive elements for $n < \infty$. Notice, however, that since $C_r(\mathbb{F}_\infty)$ embeds unitally in $C_r(\mathbb{F}_n)$ for all $n \geq 2$, we have that $Z$ embeds unitally in $C_r(\mathbb{F}_n)$ for $n = 2, 3, \ldots, \infty$.

7. The crossed products $\mathcal{O}_2 \rtimes_\lambda \mathbb{R}$

Let $\lambda \in \mathbb{R}$. Consider the action of $\mathbb{R}$ on the Cuntz algebra $\mathcal{O}_2$ given by

$$\sigma_\lambda^t(v_1) := e^{2\pi i t} v_1, \quad \sigma_\lambda^t(v_2) := e^{2\pi i \lambda t} v_2,$$

where $v_1$ and $v_2$ are the partial isometries generating $\mathcal{O}_2$. These actions, and the resulting crossed-product C*-algebras $\mathcal{O}_2 \rtimes_\lambda \mathbb{R}$, were first studied by Evans and subsequently by Kishimoto, Kumjian, Dean, and several other authors (see [18], [23], [10]). In [22], Kishimoto shows that if $\lambda < 0$ the crossed product is purely infinite, while if $\lambda > 0$ it is stably finite. In [10], Dean computes $\mathcal{O}_2 \rtimes_\lambda \mathbb{R}$ for $\lambda \in \mathbb{Q}^+$ and uses this to conclude that for $\lambda > 0$ in a dense $G_\delta$ subset of $\mathbb{R}^+$ that contains $\mathbb{Q}^+$ the C*-algebras $\mathcal{O}_2 \rtimes_\lambda \mathbb{R}$ are inductive limits of 1-dimensional NCCW complexes. Here we will show that the 1-dimensional NCCW complexes obtained by Dean all have trivial $K_1$-group. Since for $\lambda > 0$ irrational the C*-algebras $\mathcal{O}_2 \rtimes_\lambda \mathbb{R}$ are simple, stable, projectionless, and with a unique trace, we shall conclude by Corollary [6.2.4] that these C*-algebras are all isomorphic for a dense set of irrational numbers $\lambda > 0$.

Let $p, q > 0$ be relatively prime natural numbers. In [10], Dean shows that there is a simple, stable, AF algebra $A(p, q)$, and an automorphism $\alpha: A(p, q) \to A(p, q)$ such that $\mathcal{O}_2 \rtimes_{p/q} \mathbb{R}$ is isomorphic to the mapping torus $M_{\alpha,p,q}$ of $(A(p, q), \alpha)$. That is,

$$\mathcal{O}_2 \rtimes_{p/q} \mathbb{R} \cong M_{\alpha,p,q} := \{ f \in C([0,1], A(p, q)) \mid f(0) = \alpha(f(1)) \}.$$

A description in terms of generators and relations of the algebra $A(p, q)$, and of the action of the automorphism $\alpha$, is given in [10, Theorem 3.1]. The computation of the K-theory of $A(p, q)$ is sketched at the end of [10] (and carried out in detail for $p = 1, q = 2$). From the discussion given in the final remarks of [10, Section 5], one can gather the following:

1. There is an increasing sequence of finite dimensional algebras $(D_n)_{n=1}^\infty$ such that $A(p, q) = \bigcup_n D_n$.\hfill 29
(2) For each $n$, $K_0(D_n) \cong \mathbb{Z}^q$ and the inclusion $D_n \subseteq D_{n+1}$ induces in $K_0$ a map $\mathbb{Z}^q \to \mathbb{Z}^q$ given by the matrix $A^{q+1}$, with

$$A = \begin{pmatrix}
1 & 1 & \cdots \\
1 & & 1 \\
& & \\
& 1 & \end{pmatrix}.$$ 

The last row of $A$ has 1 in the entries $A_{p-q,q}$ and $A_{q,q}$ and zeroes elsewhere.

(3) For every $n$, $\alpha(D_n) \subseteq D_{n+1}$ and the homomorphism $\alpha|_{D_n}: D_n \to D_{n+1}$ induces in $K_0$ the map $A^q: \mathbb{Z}^q \to \mathbb{A}^q$, with $A$ as above.

The above points may be summarized by the commutative diagram

$$
\begin{array}{ccc}
D_1 & \overset{\alpha|_{D_1}}{\longrightarrow} & D_2 & \overset{\alpha|_{D_2}}{\longrightarrow} & \cdots & A(p,q) \\
\downarrow & & \downarrow & & \downarrow & \\
D_2 & \overset{\alpha}{\longrightarrow} & D_3 & \overset{\alpha}{\longrightarrow} & \cdots & A(p,q)
\end{array}
$$

and the diagram induced in $K_0$,

$$
\begin{array}{ccc}
\mathbb{Z}^q & \overset{A_{q+1}}{\longrightarrow} & \mathbb{Z}^q & \overset{A_{q+1}}{\longrightarrow} & \cdots & \mathbb{Z}^q \\
\downarrow & & \downarrow & & \downarrow & \\
\mathbb{Z}^q & \overset{A^q}{\longrightarrow} & \mathbb{Z}^q & \overset{A^q}{\longrightarrow} & \cdots & \mathbb{Z}^q.
\end{array}
$$

The mapping torus $M_{\alpha,p,q}$ is the inductive limit of the 1-dimensional NCCW complexes $M_{n,p,q} = \{ f \in C([0,1], D_{n+1}) | f(1) \in D_n, f(0) = \alpha(f(1)) \}$. By Lemma 5.2.4, $K_1(M_{n,p,q}) = 0$ if and only if $K_0(\phi_0) - K_0(\phi_1)$ is surjective. In this case, $\phi_0$ is the inclusion of $D_n$ in $D_{n+1}$, while $\phi_1$ is the restriction of $\alpha$ to $D_n$ (with codomain $D_{n+1}$). Identifying $K_0(D_n)$ and $K_0(D_{n+1})$ with $\mathbb{Z}^q$, we have $K_0(\phi_0) = A^{q+1}$ and $K_0(\phi_1) = A^q$. Thus,

$$K_0(\phi_0) - K_0(\phi_1) = A^{q+1} - A^q = A^q(A - I).$$

Since the characteristic polynomial of $A$ is $t^q - t^{q-p} - 1$, we see that $\det(I - A) = -1$ and $\det(-A) = -1$. Thus, $A^{q+1} - A^q$ is an invertible map from $\mathbb{Z}^q$ to $\mathbb{Z}^q$, and so $K_1(M_{p,q}) = 0$. By Theorem 1.0.1 the functor $\text{Cu}^\gamma$ classifies homomorphisms from the NCCW complexes $M_{n,p,q}$, and from their sequential inductive limits. By [10] Theorem 4.10 (or rather, by its proof), these inductive limits include all C* -algebras $O_2 \rtimes_\lambda \mathbb{R}$ for a dense $G_\delta$ set of positive numbers $\lambda$ that includes $\mathbb{Q}^+$.

**Corollary 7.0.2.** The crossed products $O_2 \rtimes_\lambda \mathbb{R}$ are all isomorphic for $\lambda$ belonging to a dense subset of $\mathbb{R}^+ \setminus \mathbb{Q}$ of second Baire category.

**Proof.** By the discussion above, for a $G_\delta$ set $\Lambda \subseteq \mathbb{R}^+$ the crossed products $O_2 \rtimes_\lambda \mathbb{R}$ are inductive limits of 1-dimensional NCCW complexes with trivial $K_1$-group. On the other hand, for $\lambda > 0$ irrational, these crossed products are simple, stable, have trivial $K_0$-group, and tracial cone $\mathbb{R}^+$ (see [23]). It follows by Corollary 6.2.4 that for $\lambda \in \Lambda \setminus \mathbb{Q}$ the crossed products $O_2 \rtimes_\lambda \mathbb{R}$ are all isomorphic. 

□
Remark 7.0.3. Quasifree actions may also be defined on $O_n$ and they have been studied by the authors cited above. However, for the resulting crossed products the $K_1$-group is $\mathbb{Z}/(n - 1)\mathbb{Z}$. So their classification lies beyond the scope of the results obtained here.

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