A BASIS FOR THE POLYNOMIAL EIGENFUNCTIONS OF DEFORMED CALOGERO-MOSER-SUTHERLAND OPERATORS

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ABSTRACT. We construct a linear basis for the polynomial eigenfunctions of a family of deformed Calogero-Moser-Sutherland operators naturally associated with hypergeometric polynomials. In our construction the eigenfunctions are obtained as linear combinations of polynomials which generalise the (super) Schur polynomials. As a byproduct, we obtain explicit series representations for the super Jack polynomials.

1. Introduction

The main purpose of this paper is to construct and study a particular linear basis for the polynomial eigenfunctions of a certain family of partial differential operators which, following Sergeev and Veselov [SV05], we will refer to as deformed Calogero-Moser-Sutherland (CMS) operators. In our construction the eigenfunctions are expressed as linear combinations of particular polynomials which generalise the so-called (super) Schur polynomials; see e.g. Fulton and Pragacz [FP95]. Our main motivation for the construction is that it leads to rather simple and explicit formulæ; see Section 7 for concrete examples.

To give a precise definition of the family of deformed CMS operators we consider, and to better describe our results, we start by discussing a simple and well known result from the theory of polynomials in one variable: suppose that we are given a sequence of polynomials

\[ p_0(x), p_1(x), \ldots, p_n(x), \ldots, \]

where each polynomial \( p_n \) is such that it has precisely degree \( n \) and is an eigenfunction of a second order ordinary differential operator

\[ \mathcal{L} = \alpha(x) \frac{\partial^2}{\partial x^2} + \beta(x) \frac{\partial}{\partial x} \]

for some fixed polynomials \( \alpha \) and \( \beta \). It is then a straightforward exercise to verify that \( \alpha \) is of at most degree two and that \( \beta \) is of at most degree one, i.e.,

\[ \alpha(x) = \alpha_2 x^2 + \alpha_1 x + \alpha_0, \quad \beta(x) = \beta_1 x + \beta_0 \]

for some (real) coefficients \( \alpha_k \) and \( \beta_k \); see e.g. Bochner [Boc29]. Examples of such sequences of polynomials are given by the classical orthogonal Hermite-, Laguerre- and Jacobi polynomials, as well as the Bessel polynomials, all of which can be expressed in terms of hypergeometric functions. For a comprehensive discussion of the classical orthogonal polynomials see for example Andrews et al. [AAR99], and for the Bessel polynomials the book by Grosswald [Gro78]. It is interesting to note...
that, as long as $\alpha$ is not identically zero, we can always reduce to one of these four cases by an affine transformation of the variable $x$; see e.g. Bochner (loc. cit.).

This type of (complete) sequences of polynomials have a natural many-variable generalisation within the theory of symmetric polynomials. In fact, Lassalle [Las91b, Las91c, Las91a] and Macdonald [Mac] introduced and studied a many-variable generalisations of the classical orthogonal Hermite-, Laguerre- and Jacobi- polynomials as eigenfunctions of partial differential operators

$$L_n = \sum_{k=0}^{2} \alpha_k D_{\tilde{n}}^k + \sum_{\ell=0}^{1} \beta_{\ell} E_{\tilde{n}}^\ell$$

which can be obtained from the corresponding ordinary differential operators $L$ by replacing each term $x^k \partial^2 / \partial x^2$ by

$$D_{\tilde{n}}^k = \sum_{i=1}^{n} x_i \frac{\partial^2}{\partial x_i^2} + 2\theta \sum_{i \neq j} x_i \frac{\partial}{\partial x_j}$$

and each term $x^\ell \partial / \partial x$ by

$$E_{\tilde{n}}^\ell = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$$

for $k = 0, 1, 2$ and $\ell = 0, 1$, respectively. These many-variable polynomials have subsequently been extensively studied in the literature. In particular, by Baker and Forrester [BF97] and van Diejen [vD97]. We also mention Heckman and Opdam's closely related root system generalisation of the Jacobi polynomials; see e.g. their paper [HO87]. For additional related references see e.g. the book by Dunkl and Xu [DX01].

In this paper we consider a further natural generalisation of the ordinary differential operators $L$ in two sequences of independent variables $x = (x_1, \ldots, x_n)$ and $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_{\tilde{n}})$. More precisely, we consider the partial differential operators

$$L_{n,\tilde{n}} = \sum_{k=0}^{2} \alpha_k D_{n,\tilde{n}}^k + \sum_{\ell=0}^{1} \beta_{\ell} E_{n,\tilde{n}}^\ell$$

obtained from $L$ by replacing each term $x^k \partial^2 / \partial x^2$ by

$$D_{n,\tilde{n}}^k = \sum_{i=1}^{n} x_i \frac{\partial^2}{\partial x_i^2} - \theta \sum_{I=1}^{\tilde{n}} \tilde{x}_I \frac{\partial^2}{\partial \tilde{x}_I^2}$$

$$+ 2\theta \sum_{i \neq j} x_i \frac{\partial}{\partial x_j} - 2 \sum_{I \neq J} \tilde{x}_I \frac{\partial}{\partial \tilde{x}_J}$$

$$- 2 \sum_{i,j} \frac{1}{x_i - \tilde{x}_j} \left( x_i \frac{\partial}{\partial x_j} + \theta \tilde{x}_j \frac{\partial}{\partial \tilde{x}_j} \right) + k(1 + \theta) \sum_{i=1}^{n} x_i^{k-1} \frac{\partial}{\partial x_i},$$

and each term $x^\ell \partial / \partial x$ by

$$E_{n,\tilde{n}}^\ell = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} + \sum_{I=1}^{\tilde{n}} \tilde{x}_I \frac{\partial}{\partial \tilde{x}_I}$$

for $k = 0, 1, 2$ and $\ell = 0, 1$, respectively. This type of partial differential operators were introduced and studied by Chalykh et al. [CFV98] for $\tilde{n} = 1$ and by Sergeev [Serg01, Serg02] for arbitrary $\tilde{n}$. In special cases they have subsequently
been extensively studied by Sergeev and Veselov [SV04,SV05]. As mentioned above, following Sergeev and Veselov we will refer to these partial differential operators $L_n, \tilde{L}_n$ as deformed Calogero-Moser-Sutherland (CMS) operators. We mention that the operators studied in the papers by Sergeev, as well as in the former of the two papers by Sergeev and Veselov, are constructed using certain deformations of so-called generalised root systems; see Serganova [Ser96]. For the classical series of such deformed generalised root systems the resulting deformed CMS operators can also be obtained by specialising to particular polynomials $\alpha$ and $\beta$ in $L_n, \tilde{L}_n$; see Table 1 below and the paper [HL07] for specific examples. However, it is interesting to note that the operator $L_n, \tilde{L}_n$ corresponding to the Bessel polynomials (see Table 1) can not be directly defined in terms of root systems.

| $\alpha(x)$ | $\beta(x)$ | Eigenfuncs. of $L$ | Type of potential | Root system |
|-------------|-------------|--------------------|------------------|-------------|
| $x^2$       | 0           | $x^n$ (Monomials)  | Trigonometric    | $A(n-1, \tilde{n}-1)$ |
| 1           | $-2x$       | $H_n(x)$ (Hermite pols.) | Rational | $A(n-1, \tilde{n}-1)$ |
| $x$         | $a+1-x$    | $L_n(x)$ (Laguerre pols.) | Rational | $B(n, \tilde{n})$ |
| $(1-x^2)$   | $b-a-(a+b+2)x$ | $P_n^{(a,b)}(x)$ (Jacobi pols.) | Trigonometric | $BC(n, \tilde{n})$ |
| $x^2$       | $b+ax$     | $y_n^{(a,b)}(x)$ (gen. Bessel pols.) | – | – |

Table 1. Special cases of the deformed CMS operators $L_n, \tilde{L}_n$. In each case, the two rightmost columns refers to the type of potential and root system in the construction used by Sergeev and Veselov (loc. cit.).

As indicated above, the main purpose of this paper is to construct and study a particular linear basis for the polynomial eigenfunctions of each of the deformed CMS operators $L_n, \tilde{L}_n$. The eigenfunctions are in this construction expressed as linear combinations of certain polynomials generalising, or more accurately, deforming the (super) Schur polynomials; see e.g. Fulton and Pragacz [FP95] for a discussion of the super Schur polynomials. Our main motivation for using these polynomials is that they lead to rather simple and explicit formulae. The type of series representations we construct were first obtained by Langmann [Lan01] (see also [Lan06]) for eigenfunctions of an operator $L_n$ with only $\alpha_2$ and $\beta_1$ non-zero. His results have subsequently been generalised to all ‘ordinary’ CMS operators $L_n$, as well as their deformed counterparts $L_n, \tilde{L}_n$; see the paper [HL07] and references therein. The present work is in many ways a natural continuation of this latter paper. In particular, we obtain complete proofs of a number of results which are only sketched or mentioned in [HL07]. On the other hand, certain of the results obtained in
the present paper can be inferred from results in [HL07]. However, the point of view in this latter paper, and also in the papers by Langmann (loc. cit.), is that of quantum many-body systems of (deformed) Calogero-Sutherland type; see e.g. Calogero [Cal71] and Sutherland [Sut72]. In all cases were such an overlap of results occur our approach thus provides an alternative and independent derivation, making no reference to their relation with such quantum many-body systems.

We conclude this introduction by presenting a brief outline of the paper. The first two sections are of an introductory nature. We begin in Section 2 by giving a brief review of some basic facts from the theory of symmetric functions. We also recall a definition of the Jack polynomials and prove a short technical lemma related to the complete symmetric polynomials. In Section 3 we recall the definition of the so-called super Jack polynomials and review some of their basic properties. In addition, we establish a particular triangular structure in the expansion of the super Jack polynomials in ordinary monomials. The main results of the paper are obtained in Sections 4-6. In Section 4 we establish particular identities which relates each of the deformed CMS operators above to their adjoints for a scalar product naturally associated with the super Jack polynomials. In Section 5 we define and study certain polynomials $f_{a}^{(m, \tilde{m})}(x, \tilde{x}; \theta)$ labeled by two non-negative integers $(m, \tilde{m})$, an integer vector $a \in \mathbb{Z}^{m+\tilde{m}}$, and the parameter $\theta$. We prove that they essentially coincide with the (super) Schur polynomials for $\theta = 1$ and $\tilde{m} = 0$. We also study the structure of their expansion in terms of super Jack polynomials. As a consequence, we obtain a simple characterisation of the linear span of a certain natural subset of the polynomials $f_{a}^{(m, \tilde{m})}(x, \tilde{x}; \theta)$. In Section 6 we construct and study polynomial eigenfunctions of the deformed CMS operators $L_{n, \tilde{n}}$ as linear combinations of the polynomials $f_{a}^{(m, \tilde{m})}(x, \tilde{x}; \theta)$. An important aspect of our construction is that the resulting eigenfunctions can be normalised such that they are independent of the choice of parameters $(m, \tilde{m})$. As we then discuss, this freedom in choosing the values of $(m, \tilde{m})$ can be used to minimise the complexity of the series representation of a given eigenfunction, in many cases significantly below that of the canonical choice $(m, \tilde{m}) = (n, \tilde{n})$. We also obtain a simple characterisation of the linear span of the eigenfunctions we construct. In Section 7 we deduce the explicit series expansion of the super Jack polynomials in terms of the polynomials $f_{a}^{(m)}(x, \tilde{x}; \theta)$. We also study in some detail certain particularly simple special cases of this series expansion. We conclude the paper in Section 8 by a brief discussion of some open problems.

2. Symmetric functions and Jack polynomials

In this section we briefly recall some basic facts and definitions concerning symmetric functions and Jack polynomials. We also prove a short technical lemma on the so-called complete symmetric polynomials. With a few minor exceptions we follow the notation of Macdonald [Mac95] to which the reader is referred for further details.

Consider the algebra $\mathbb{C}[x_1, \ldots, x_n]$ of polynomials in $n$ independent variables $x = (x_1, \ldots, x_n)$ with complex coefficients. The subalgebra of all symmetric polynomials is denoted $\Lambda_n$. It is graded by the degree of the polynomials, i.e.,

$$
\Lambda_n = \bigoplus_{k \geq 0} \Lambda_n^k
$$
with $\Lambda^k_n$ the homogeneous component of $\Lambda_n$ of degree $k$. Let $n \geq m$ and consider the homomorphism
\[ \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_m] \]
which sends each of the variables $x_{m+1}, \ldots, x_n$ to zero and the remaining variables $x_i$ to themselves. Let $\rho_{n,m}$ and $\rho_{k,n,m}$ denote the restriction of this homomorphism to $\Lambda_n$ and $\Lambda^k_n$, respectively. The inverse limit
\[ \Lambda^k = \varprojlim \Lambda^k_n \]
of the linear spaces $\Lambda^k_n$ relative to the homomorphisms $\rho_{k,m,n}$ can now be formed, and the algebra of symmetric functions can be defined as the direct sum
\[ \Lambda = \bigoplus_{k \geq 0} \Lambda^k. \]

A partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ is any sequence of non-negative integers in decreasing order, i.e.,
\[ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_i \geq \cdots, \]
containing only a finite number of non-zero terms. These non-zero terms $\lambda_i$ are called the parts of $\lambda$ and the number of parts the length of $\lambda$, in the following denoted $\ell(\lambda)$. The sum $|\lambda| := \lambda_1 + \lambda_2 + \cdots$ of its parts is referred to as the weight of $\lambda$. If $|\lambda| = n$ it is said that $\lambda$ is a partition of $n$. We will for simplicity not distinguish two partitions differing only by a string of zeros at the end. A partition $\lambda$ can be identified with its diagram, which consists of the points $(i, j) \in \mathbb{Z}^2$ such that $1 \leq j \leq \lambda_i$. The partition $\lambda'$, obtained by reflection in the main diagonal, is called the conjugate of $\lambda$. On the set of the partitions of a given non-negative integer $n$ is the so-called dominance order defined by
\[ \mu \leq \lambda \iff \mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i, \quad \forall i \geq 1. \]

More generally, we will write $a \leq b$ for any two integer vectors $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ such that
\[ a_1 + \cdots + a_i \leq b_1 + \cdots + b_i \]
for all $i = 1, \ldots, n$.

Throughout the paper we will write $x^a = x_1^{a_1} \cdots x_n^{a_n}$ for any integer vector $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$. With this notation in mind we recall the definition of the following linear basises for $\Lambda_n$:

1. Monomial symmetric polynomials: defined for each partition $\lambda$ of length $\ell(\lambda) \leq n$ by
\[ m_\lambda(x_1, \ldots, x_n) = \sum_{\alpha} x^\alpha \]
where the sum extends over all distinct permutations $\alpha$ of $\lambda$.

2. The power sums: defined for each $r \geq 1$ by
\[ p_r(x_1, \ldots, x_n) = x_1^r + \cdots + x_n^r, \]
and for all partitions $\lambda$ such that $\ell(\lambda') \leq n$,
\[ p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots. \]
(3) Elementary symmetric polynomials: can be defined for all $r \leq n$ by the expansion
\[
\prod_{i=1}^{n}(1 + x_it) = \sum_{r=1}^{n} e_r(x_1, \ldots, x_n)t^r.
\]
For all partitions $\lambda$ such that $\ell(\lambda') \leq n$,
\[
e_\lambda = e_{\lambda_1}e_{\lambda_2} \cdots.
\]

(4) Complete symmetric polynomials: can be defined for all non-negative integers $r$ by the expansion
\[
\prod_{i=1}^{n}(1 - x_it)^{-1} = \sum_{r \geq 0} h_r(x_1, \ldots, x_n)t^r.
\]
Furthermore, for each partition $\lambda$ such that $\ell(\lambda) \leq n$,
\[
h_\lambda = h_{\lambda_1}h_{\lambda_2} \cdots.
\]

(5) 'Modified' complete symmetric polynomials: can be defined for all non-negative integers $r$ and each real number $\theta$ by the expansion
\[
\prod_{i=1}^{n}(1 - x_it)^{-\theta} = \sum_{r \geq 0} g_r(x_1, \ldots, x_n; \theta)t^r.
\]
In addition, for all partitions $\lambda$ such that $\ell(\lambda) \leq n$,
\[
g_\lambda = g_{\lambda_1}g_{\lambda_2} \cdots.
\]

It is a well known fact that these polynomials $m_\lambda(x_1, \ldots, x_n)$, $p_\lambda(x_1, \ldots, x_n)$, $e_\lambda(x_1, \ldots, x_n)$, $h_\lambda(x_1, \ldots, x_n)$ and $g_\lambda(x_1, \ldots, x_n)$, under the restrictions on the partitions $\lambda$ stated above, all form linear basises for $\Lambda_n$; see e.g. Sections I.2 and VI.10 in Macdonald [Mac95]. In addition, since these polynomials are stable under the homomorphisms $\rho_{n,m}$, the symmetric functions $m_\lambda$, $p_\lambda$, $e_\lambda$, $h_\lambda$ and $g_\lambda$ can be defined and form linear basises for $\Lambda$.

In later parts of the paper we make use of the fact that the complete symmetric polynomials in three variables can be expressed as a very particular quotient. This fact is established in the following:

**Lemma 2.1.** Set $h_r = 0$ for all $r < 0$. Let $x$, $y$ and $z$ be three independent variables. Then, for each non-negative integer $k$,
\[
\frac{x^k(z - y) + y^k(x - z) + z^k(y - x)}{(y - x)(x - z)(z - y)} = -h_{k-2}(x, y, z).
\]

**Proof.** Let
\[
g_k(x, y, z) = \frac{x^k(z - y) + y^k(x - z) + z^k(y - x)}{(y - x)(x - z)(z - y)}
\]
and observe that
\[\sum_{k \geq 0} q_k(x, y, z)t^k = \frac{1}{(y - x)(x - z)} \sum_{k \geq 0} (xt)^k + \frac{1}{(y - x)(z - y)} \sum_{k \geq 0} (yt)^k + \frac{1}{(x - z)(z - y)} \sum_{k \geq 0} (zt)^k\]
\[= \frac{1}{(y - x)(x - z)(1 - xt)} + \frac{1}{(y - x)(z - y)(1 - yt)} + \frac{1}{(x - z)(z - y)(1 - zt)}.\]

This latter sum can be written as a single fraction with nominator
\[(z - y)(1 - yt)(1 - zt) + (x - z)(1 - xt)(1 - zt) + (y - x)(1 - xt)(1 - yt) = -t^2(y - x)(x - z)(z - y).\]

It follows that
\[\sum_{k \geq 0} q_k(x, y, z)t^k = -t^2(1 - xt)^{-1}(1 - yt)^{-1}(1 - zt)^{-1}\]
\[= -\sum_{k \geq 0} h_k(x, y, z)t^{k+2}.\]

We proceed to recall a definition of the Jack polynomials suitable for our purposes. In doing so we use the inverse \(\theta = 1/\alpha\) of the parameter \(\alpha\) used by Macdonald [Mac95]. We recall that for each 'square' \(s = (i, j)\) in the diagram of a partition \(\lambda\) the so-called arm-length \(a(s)\) and leg-length \(l(s)\) are given by \(a(s) = \lambda_i - j\) and \(l(s) = \lambda'_j - i\), respectively. We let
\[b_\lambda(\theta) = \prod_{s \in \lambda} \frac{a(s) + \theta l(s) + \theta}{a(s) + \theta l(s) + 1}.\]

Following Macdonald (loc. cit.) we let \(\square_n^{1/\theta}\) be the so-called Laplace-Beltrami operator given by
\[\square_n^{1/\theta} = \frac{1}{2\theta} D_n^2 - (n - 1) E_n^1 = \frac{1}{2\theta} \sum_i x_i^2 \frac{\partial^2}{\partial x_i^2} + \sum_{i \neq j} \frac{x_i x_j}{x_i - x_j} \frac{\partial}{\partial x_i}.\]

It is clear that \(\square_n^{1/\theta}\) is stable under the restriction homomorphisms \(\rho_{n,m}\), i.e., that
\[\rho_{n,m} \circ \square_n^{1/\theta} = \square_m^{1/\theta} \circ \rho_{n,m}\]
for all \(m \leq n\). It follows that its inverse limit
\[\square_n^{1/\theta} = \lim \square_n^{1/\theta}\]
is a well defined operator on \(\Lambda\). Jack's symmetric functions \(P_\lambda\) can now be defined by the following result due to Macdonald (see e.g. Example 3 in Section VI.5 of [Mac95]):
Theorem 2.1 (Macdonald). If $\theta$ is not a negative rational number or zero there exist for each partition $\lambda$ a unique eigenfunction $P_{\lambda}$ of the operator $\Box^{1/\theta}$ such that

$$P_{\lambda} = m_{\lambda} + \sum_{\mu < \lambda} u_{\lambda\mu}m_{\mu}$$

for some coefficients $u_{\lambda\mu}$.

By setting all variables $x_i = 0$ for $i > n$, where $n$ is some positive integer, we obtain the corresponding Jack polynomials $P_{\lambda}(x_1, \ldots, x_n)$ in $n$ variables $x = (x_1, \ldots, x_n)$.

Let $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ be two infinite sequences of independent variables. A fundamental object in the theory of Jack’s symmetric functions is the infinite product

$$\Pi(x, y; \theta) = \prod_{i,j} (1 - x_i y_j)^{-\theta}.$$ 

We let $\Pi_{n,m}(x, y; \theta)$ denote its restriction to a finite number of variables $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$. The following well-known result is due to Stanley [Sta89]:

**Proposition 2.1** (Stanley). If $\theta$ is not a negative rational number or zero, then the infinite product $\Pi(x, y; \theta)$ has the following expansion in terms of Jack’s symmetric functions:

$$\Pi(x, y; \theta) = \sum_{\lambda} b_{\lambda}(\theta) P_{\lambda}(x; \theta)P_{\lambda}(y; \theta)$$

where the sum extends over all partitions.

### 3. The super Jack polynomials

In this section we recall the definition of the so-called super Jack polynomials, as stated by Kerov et al. [KOO98]. We also recall certain related results due to Sergeev and Veselov [SV04,SV05]. In addition, we establish a particular triangular structure in the expansion of the super Jack polynomials in ordinary monomials, and we derive the analogue of Proposition 2.1 for the super Jack polynomials.

Let $x = (x_1, \ldots, x_n)$ and $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)$ be two sequences of independent variables. Following Sergeev and Veselov [SV04] we let $\Lambda_{n,\tilde{n},\theta}$ be the subalgebra of $\mathbb{C}[x_1, \ldots, x_n, \tilde{x}_1, \ldots, \tilde{x}_\tilde{n}]$ consisting of all polynomials $p(x, \tilde{x})$ which, in addition to being separately symmetric in the variables $x$ and $\tilde{x}$, satisfy the condition

$$\left( \frac{\partial}{\partial x_i} + \theta \frac{\partial}{\partial \tilde{x}_I} \right) p(x, \tilde{x}) = 0$$

on each hyperplane $x_i = \tilde{x}_I$ with $i = 1, \ldots, n$ and $I = 1, \ldots, \tilde{n}$. In addition, for each non-negative integer $k$ we let $\Lambda_{n,\tilde{n},\theta}^k$ be the homogeneous component of $\Lambda_{n,\tilde{n},\theta}$ of degree $k$. It is clear that the ‘deformed’ power sums

$$p_{r,\theta}(x, \tilde{x}) = x_1^r + \cdots + x_n^r - \theta^{-1} (\tilde{x}_1^r + \cdots + \tilde{x}_{\tilde{n}}^r)$$

for $r \geq 1$ are contained in $\Lambda_{n,\tilde{n},\theta}$. We let $\varphi_{n,\tilde{n}} : \Lambda \to \Lambda_{n,\tilde{n},\theta}$ be the algebra homomorphism defined by

$$\varphi_{n,\tilde{n}}(p_r) = p_{r,\theta}(x, \tilde{x})$$
for all \( r \geq 1 \). Note that since the the power sums \( p_r \) are free generators of \( \Lambda \),
this uniquely determines \( \varphi_{n,\tilde{n}} \). The super Jack polynomials can now be defined as follows:

**Definition 3.1** (Kerov et al.). For each partition \( \lambda \) the super Jack polynomial \( SP_{\lambda}(x, \tilde{x}) \) is defined by

\[
SP_{\lambda}(x, \tilde{x}) = \varphi_{n,\tilde{n}}(P_{\lambda}).
\]

As a direct consequence of this definition and the fact that a Jack polynomial \( P_{\lambda}(x) \), if non-zero, is homogeneous of degree \( |\lambda| \) we obtain the following:

**Lemma 3.1.** If a super Jack polynomial \( SP_{\lambda} \) is non-zero it is homogeneous of degree \( |\lambda| \).

We let \( H_{n,\tilde{n}} \) be the set of partitions contained in the fat \((n,\tilde{n})\)-hook, i.e., the set of partitions \( \lambda = (\lambda_1, \lambda_2, \ldots) \) such that \( \lambda_{n+1} \leq \tilde{n} \). For any partition \( \lambda \in H_{n,\tilde{n}} \) we let

\[
n\lambda = (\lambda_1, \ldots, \lambda_n), \quad n\lambda = (\lambda_{n+1}, \lambda_{n+2}, \ldots).
\]

From Sergeev and Veselov [SV05] we now recall the following:

**Theorem 3.1** (Sergeev and Veselov). Assume that \( \theta \) is not a negative rational number or zero. Then the kernel of \( \varphi_{n,\tilde{n}} \) is spanned by the Jack’s symmetric functions \( P_{\lambda} \) indexed by the partitions \( \lambda \notin H_{n,\tilde{n}} \). Moreover, the super Jack polynomials \( SP_{\lambda}(x, \tilde{x}) \), indexed by the partitions \( \lambda \in H_{n,\tilde{n}} \), form a linear basis for \( \Lambda_{n,\tilde{n},\theta} \).

We proceed to establish a particular triangular structure in the expansion of the super Jack polynomials in ordinary monomials. This result will play an important role in later parts of the paper.

**Lemma 3.2.** Assume that \( \theta \) is not a negative rational number or zero and let \( \lambda \in H_{\tilde{n}} \). The super Jack polynomial \( SP_{\lambda}(x, \tilde{x}; \theta) \) is then a linear combination of monomials \( x^a\tilde{x}^b \) with \((a, b) \leq (n\lambda, n\lambda')\). Moreover, the leading term is given by

\[
(-1)^{|n\lambda|} b_{n,\lambda'} (\theta^{-1}) (x^{n\lambda} \tilde{x}^{n\lambda'}). \tag{5}
\]

It has been shown by Sergeev and Veselov [SV05] that (5) is the leading term of \( SP_{\lambda} \) in the lexicographic order; see the proof of their Theorem 2. We recall that for two integer vectors \( a, b \in \mathbb{Z}^m \), where \( m \) is some positive integer, the integer vector \( a \) is said to be of lower order than \( b \) in the lexicographic order if \( a \neq b \) and the first non-zero term in \( b - a \) is positive. Note that this is the case if \( a < b \). Consequently, Lemma 3.2 generalises the result of Sergeev and Veselov. We will obtain a proof of the lemma by extending their argument. The idea is to first deduce an explicit expansion of the super Jack polynomials in terms of Jack- and so-called skew Jack polynomials. The proof of Lemma 3.2 is then obtained by well-known properties of these latter polynomials.

**Proof of Lemma 3.2.** In order for the proof to be self-contained we start by recalling the definition of the skew Jack symmetric functions, as stated in Section VI.7 of Macdonald [Mac95]. To any Jack symmetric function \( P_{\lambda}(x; \theta) \) is associated

\[
Q_{\lambda}(x; \theta) = b_{\lambda}(\theta) P_{\lambda}(x; \theta).
\]

We define the scalar product \( \langle \cdot, \cdot \rangle \) on \( \Lambda \) (antilinear in its second argument) by setting

\[
\langle P_{\lambda}, Q_{\mu} \rangle = \delta_{\lambda\mu}.
\]
for all partitions \( \lambda \) and \( \mu \). For any two partitions \( \lambda \) and \( \mu \) the skew Jack symmetric functions \( P_{\lambda/\mu} \) can now be defined by requiring

\[
\langle P_{\lambda/\mu}, Q_{\nu} \rangle = \langle P_{\lambda}, Q_{\mu}Q_{\nu} \rangle
\]

for all partitions \( \nu \). The corresponding skew Jack polynomials \( P_{\lambda/\mu}(x_1, \ldots, x_n) \) are obtained by setting all variables \( x_i = 0 \) for \( i > n \).

We proceed to prove the statement. To this end, we let \( x = (x_1, x_2, \ldots) \) and \( \tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots) \) be two infinite sequences of independent variables. It is a well known fact that

\[
P_{\lambda}(x, \tilde{x}) = \sum_{\mu \subseteq \lambda} P_{\lambda/\mu}(x)P_{\mu}(\tilde{x});
\]

see e.g. Macdonald (loc. cit.). We let \( \omega_{\theta} \) denote the automorphism of \( \Lambda \) defined by

\[
\omega_{\theta}(p_r) = (-1)^{r-1}\theta p_r
\]

for all \( r \geq 1 \), and recall from Macdonald (loc. cit.) that

\[
\omega_{\theta-1}(P_{\lambda}(x; \theta)) = Q_{\lambda}(x; \theta^{-1}) = b_{\lambda}(\theta^{-1})P_{\lambda}(x; \theta^{-1}).
\]

We observe that acting with the homomorphism \( \varphi_{n, \tilde{n}} \) on a Jack’s symmetric function \( P_{\lambda}(x, \tilde{x}) \) is equivalent to first acting with the automorphism \( \omega_{\theta-1} \) in the variables \( \tilde{x} \), followed by changing the sign of all variables \( \tilde{x} \), and finally setting all variables \( x_i = 0 \) and \( \tilde{x}_I = 0 \) for \( i > n \) and \( I > \tilde{n} \), respectively. It follows that

\[
SP_{\lambda}(x, \tilde{x}; \theta) = \sum_{\mu \subseteq \lambda} (-1)^{|\mu|}b_{\mu}(\theta^{-1})P_{\lambda/\mu}(x; \theta)P_{\mu}(\tilde{x}; \theta^{-1})
\]

with \( x = (x_1, \ldots, x_n) \) and \( \tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_{\tilde{n}}) \). We recall that a skew Jack polynomial \( P_{\lambda/\mu}(x) \) has an expansion in ‘ordinary’ Jack polynomials of the form

\[
P_{\lambda/\mu}(x) = \sum_{\nu \subseteq \lambda} c_{\lambda}^{\nu}_{\mu}P_{\nu}(x),
\]

where the coefficients \( c_{\lambda}^{\nu}_{\mu} \) are non-zero only if \( |\mu| + |\nu| = |\lambda| \); see e.g. Macdonald (loc. cit.). We insert this expansion in (7) and fix a non-zero term \( P_{\nu}(x; \theta)P_{\mu}(\tilde{x}; \theta^{-1}) \) appearing in the resulting expression. Since \( \nu \subseteq \lambda \) and \( P_{\nu}(x) \) vanishes unless \( \ell(\nu) \leq n \) we have that \( \nu \leq n\lambda \). For any integer \( a \) we let \( (a) = \max(a, 0) \). Since \( P_{\lambda/\mu} = 0 \) unless \( 0 \leq \lambda_i' - \mu_i' \leq n \) for all \( i \geq 1 \) (see e.g. Macdonald (loc. cit.)), we have that \( \mu_i' \geq (\lambda_i' - n) = n\lambda_i' \) for all \( i \geq 1 \). It follows that

\[

\nu_1 + \cdots + \nu_n + \mu_1' + \cdots + \mu_1' = |\lambda| - \mu_{i+1}' - \mu_{i+1}' \cdots \\

\leq |\lambda| - n\lambda_{i+1}' - n\lambda_{i+1}' \cdots \\

= n\lambda_1' + \cdots + n\lambda_n' + n\lambda_i' + \cdots + n\lambda_i'
\]

for each \( i \geq 1 \). Theorem 2.1 and the fact that each symmetric monomial \( m_{\mu}(x) \) is a linear combination of monomials \( x^a \) with \( a \leq \mu \) thus implies that the super Jack polynomial \( SP_{\lambda}(x, \tilde{x}) \) indeed is a linear combination of monomials \( x^a\tilde{x}^b \) with \( (a, b) \leq (n\lambda, n\lambda') \). There remains only to verify that (6) is the leading term in the expansion of \( SP_{\lambda}(x, \tilde{x}) \) in such monomials. However, using the fact that \( \mu_i' \geq n\lambda_i' \) we infer from (7) and Theorem 2.1 that the monomial \( x^{n\lambda}\tilde{x}^{n\lambda'} \) can appear only in the term \( P_{\lambda/\mu}(x)P_{n\lambda}(\tilde{x}) \). That it indeed appears in this term, and with the coefficient given in the statement, follows from (6). □
With \( y = (y_1, \ldots, y_m) \) and \( \tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_\tilde{m}) \) two sequences of independent variables we now obtain the analogue of Proposition 2.1 for the super Jack polynomials.

**Proposition 3.1.** Assume that \( \theta \) is not a negative rational number or zero and let
\[
\Pi_{n,\tilde{n},m,\tilde{m}}(x, \tilde{x}, y, \tilde{y}; \theta) = \prod_{i,j}(1 - x_i y_j) \prod_{I,J}(1 - \tilde{x}_I \tilde{y}_J)^{\theta} \prod_{I,J}(1 - \tilde{x}_I \tilde{y}_J)^{-1/\theta}.
\]
Then
\[
\Pi_{n,\tilde{n},m,\tilde{m}}(x, \tilde{x}, y, \tilde{y}; \theta) = \sum_\lambda b_\lambda(\theta) SP_\lambda(x, \tilde{x}) SP_\lambda(y, \tilde{y})
\]
where the sum is over all partitions \( \lambda \in H_{n,\tilde{n}} \cap H_{m,\tilde{m}} \).

**Proof.** To prove the statement we compute the action of the homomorphism \( \varphi = \varphi_{n,\tilde{n}} \) on the infinite product \( \Pi(x, y; \theta) \) in both the variables \( x \) and \( y \).

In the proof of Lemma 3 in [SV05] Sergeev and Veselov established that
\[
(8) \quad \varphi_x(\Pi(x, y; \theta)) = \prod_j (1 - x_j y_j)^{-\theta} \prod_{I,J}(1 - \tilde{x}_I \tilde{y}_J),
\]
where the suffix \( x \) indicates that the homomorphism \( \varphi \) acts in the variables \( x \). To deduce the effect of a subsequent application of \( \varphi_y \) we essentially repeat their computation. We first observe that since \( \varphi \) is a homomorphism, it is sufficient to compute its action on
\[
\prod_j (1 - x_j y_j)^{-\theta} \quad \text{and} \quad \prod_j (1 - \tilde{x}_I \tilde{y}_J)
\]
for some fixed \( i \) and \( I \). For the first product (8) directly implies that
\[
\varphi_y \left( \prod_j (1 - x_j y_j)^{-\theta} \right) = \prod_j (1 - x_j y_j)^{-\theta} \prod_{J}(1 - x_j \tilde{y}_J).
\]
To treat the second product we let \( \sigma_\theta \) be the automorphism of \( \Lambda \) defined by
\[
\sigma_\theta(p_r) = -\theta^{-1} p_r
\]
for all \( r \geq 1 \). With an infinite number of independent variables \( \tilde{y} = (\tilde{y}_1, \tilde{y}_2, \ldots) \) we thus obtain
\[
\varphi_y \left( \prod_j (1 - \tilde{x}_I \tilde{y}_J) \right) = \prod_{j=1}^n (1 - \tilde{x}_I \tilde{y}_J) \sigma_\theta(\tilde{y}) \left( \prod_j (1 - \tilde{x}_I \tilde{y}_J) \right)
\]
\[
= \prod_{j=1}^n (1 - \tilde{x}_I \tilde{y}_J) \sigma_\theta(\tilde{y}) \left( \exp \prod_j (1 - \tilde{x}_I \tilde{y}_J) \right)
\]
\[
= \prod_{j=1}^n (1 - \tilde{x}_I \tilde{y}_J) \sigma_\theta(\tilde{y}) \left( \exp \sum_{r \geq 1} \frac{\tilde{x}_J p_r(\tilde{y})}{r} \right)
\]
\[
= \prod_{j=1}^n (1 - \tilde{x}_I \tilde{y}_J) \exp \left( -\theta^{-1} \sum_{r \geq 1} \frac{\tilde{x}_J p_r(\tilde{y})}{r} \right)
\]
\[
= \prod_{j=1}^n (1 - \tilde{x}_I \tilde{y}_J) \prod_j (1 - \tilde{x}_I \tilde{y}_J)^{-1/\theta}.
\]
Setting all variables $\tilde{y}_J = 0$ for $J > \tilde{n}$ the proof now follows from Proposition 2.1 and the definition of the super Jack polynomials. □

4. Identities and adjoints for deformed CMS operators

The main purpose of this section is to generalise the well known identity

(9) \[ D_{n,x}^2 \Pi_{n,n}(x,y) = D_{n,y}^2 \Pi_{n,n}(x,y), \]

where the suffix’s $x$ and $y$ indicates that the operator acts in the variables $x$ and $y$, respectively, to similar identities for each of the operators $L_n, \tilde{n}$. We mention that the identity (9) can be obtained as a direct consequence of the expansion of $\Pi_{n,n}$ in terms of Jack polynomials (stated in Proposition 2.1) and the fact that the Jack polynomials are eigenfunctions of $D_{n,x}^2$ (see e.g. Theorem 3.1 in Stanley [Sta89]). In addition, it is a well known fact that the identity (9) is equivalent to the self-adjointness of $D_{n,x}^2$ for a particular scalar product on $\Lambda_n$ (for which the Jack polynomials are pairwise orthogonal); see Statement 3.11 and Example 3 in Section VI.3 of Macdonald [Mac95]. In this section we also establish a natural generalisation of this fact to the operators $L_n, \tilde{n}$.

To simplify the exposition in this and following sections we will at this point change our notation somewhat. We set $x_{n+I} = \tilde{x}_I$ for $I = 1, \ldots, \tilde{n}$, and let $p$ denote the ‘parity’ function defined by

\[ p(i) = \begin{cases} 0 & \text{if } 0 \leq i \leq n \\ 1 & \text{if } n < i \leq n + \tilde{n}. \end{cases} \]

We also collect the (non-negative) integers $n$ and $\tilde{n}$ in a vector $\vec{n} = (n, \tilde{n})$, and similarly for $m$ and $\tilde{m}$. In addition, we let $|\vec{n}| = n + \tilde{n}$ and $|\vec{m}| = m + \tilde{m}$. The operators $E_{n,\tilde{n}}^\ell$ and $D_{n,\tilde{n}}^k$ can now be rewritten in the following simple form:

\[ E_{n,\tilde{n}}^\ell = \sum_{i=1}^{|\vec{n}|} x_i^\ell \frac{\partial}{\partial x_i} \]

and

\[ D_{n,\tilde{n}}^k = \sum_{i=1}^{|\vec{n}|} (-\theta)^{p(i)} x_i^k \frac{\partial^2}{\partial x_i^2} - 2 \sum_{i \neq j} (-\theta)^{1-p(j)} \frac{x_i^k}{x_i - x_j} \frac{\partial}{\partial x_i} \]

\[ + k \sum_{i=1}^{|\vec{n}|} \left( 1 - (-\theta)^{1-p(i)} \right) x_i^{k-1} \frac{\partial}{\partial x_i}. \]

Similarly, setting $y_{m+J} = \tilde{y}_J$ for $J = 1, \ldots, \tilde{m}$ and introducing a corresponding ‘parity’ function

\[ q(j) = \begin{cases} 0 & \text{if } 0 \leq j \leq m \\ 1 & \text{if } m < j \leq m + \tilde{m}, \end{cases} \]

the function $\Pi_{n,\tilde{n},m,\tilde{m}}(x, \tilde{x}, y, \tilde{y}; \theta)$ can be rewritten as

\[ \Pi_{\vec{n},\vec{m}}(x,y; \theta) = \prod_{i,j} (1 - x_i y_j)^{(-\theta)^{1-p(i)-q(j)}}. \]
We proceed to associate a particular operator in the variables $x$ to each of the operators $E_n^\ell$ and $D_n^k$. In doing so we make use of the short hand notation $n_\theta = \theta n - \tilde{n}$ and $m_\theta = \theta m - \tilde{m}$. For $\ell = 0, 1$ we let
\[
E_{n,m}^\ell = \sum_{j=1}^{\tilde{m}} y_j^{2-\ell} \frac{\partial}{\partial y_j} + (1-\ell)n_\theta p_{1,\theta}(y),
\]
and for $k = 0, 1, 2$ we let
\[
\tilde{D}_{n,m}^k = \sum_{j=1}^{\tilde{m}} (-\theta)^q(j) y_j^{1-k} \frac{\partial^2}{\partial y_j^2} - 2 \sum_{j \neq l} (-\theta)^{1-q(l)} \frac{y_j^{3-k}}{y_j - y_l} \frac{\partial}{\partial y_j} + \sum_{j=1}^{\tilde{m}} C_{j,k} y_j^{3-k} \frac{\partial}{\partial y_j} + P_k(y)
\]
with the constants
\[
C_{j,k} = 2(n_\theta - m_\theta) + (2-k) \left((-\theta)^q(j) - (-\theta)^{1-q(j)}\right) + k \left(1 - (-\theta)^{1-q(j)}\right)
\]
and where the polynomials
\[
P_k(y) = (1 - \delta_{k,2})n_\theta(n_\theta + 1)p_{2-k,\theta}(y) + \delta_{k,0} \theta n_\theta(p_{1,\theta}(y) - p_{2,\theta}(y)).
\]
We note that the polynomials $P_k$ are contained in $\Lambda^{2-k}_{\tilde{m},\theta}$. We are now ready to state and prove identities which generalise \((10)\) to the operators $E_n^\ell$ and $D_n^k$.

Lemma 4.1. For $\ell = 0, 1$,
\[(10)\]
\[
E_{n,x}^\ell \Pi_{n,m}(x, y) = \tilde{E}_{n,m,n}^\ell \Pi_{n,m}(x, y),
\]
and for $k = 0, 1, 2$,
\[(11)\]
\[
D_{n,x}^k \Pi_{n,m}(x, y) = \tilde{D}_{n,m,n}^k \Pi_{n,m}(x, y).
\]

Proof. The specific number of variables $x$ and $y$ is in most parts of the proof of little importance. For simplicity of notation, we therefore suppress the index’s $n$ and $\tilde{m}$. In addition, we assume that, whenever they occur, $\ell = 0, 1$ and $k = 0, 1, 2$.

We start by proving the identity \((10)\). To this end we observe that
\[(12)\]
\[
\Pi^{-1}(x, y) \frac{\partial}{\partial x_i} \Pi(x, y) = -\sum_j (-\theta)^{1-p(i)-q(j)} \frac{y_j}{1-x_i y_j}
\]
and similarly for $\partial/\partial y_j$. This, together with the fact that
\[
\frac{x_i y_j^{2-\ell} - x_i^\ell y_j}{1-x_i y_j} = (\ell - 1)y_j,
\]
implies the equalities
\[
\Pi^{-1}(x, y) \left( E_x^\ell - E_y^{2-\ell} \right) \Pi(x, y) = \sum_{i,j} (-\theta)^{1-p(i)-q(j)} \frac{x_i y_j^{2-\ell} - x_i^\ell y_j}{1-x_i y_j}
\]
\[
= (\ell - 1) \sum_{i,j} (-\theta)^{1-p(i)-q(j)} y_j
\]
\[
= (1-\ell)(n_\theta - \tilde{n})p_{1,\theta}(y),
\]
from which \((10)\) immediately follows.
We thus conclude that the first sum in (13) equals
\[ \sum_j (-\theta)^{1-q(j)} \left( (-\theta)^{1-p(i)-q(j)} - 1 \right) \frac{x_i^2 y_j^2}{(1-x_i y_j)^2} \]
and observe that
\[ \sum_j (-\theta)^{2-p(i)-q(j)-q(l)} \frac{x_i^2 y_j y_l}{(1-x_i y_j)(1-x_i y_l)}. \]
This, together with (12) and the corresponding results for \((-\theta)^{p(i)} y_j^2 \partial^2 / \partial y_j^2\) and \(\partial / \partial y_j\), imply
\[ \Pi^{-1}(x, y) (D_x^2 - \tilde{D}_y^2) \Pi(x, y) \]
\[ = \sum_i \sum_{j \neq l} (-\theta)^{2-p(i)-q(j)-q(l)} \left( \frac{x_i^2 y_j y_l}{(1-x_i y_j)(1-x_i y_l)} - 2 \frac{x_i y_j^2}{(y_j - y_l)(1-x_i y_l)} \right) \]
\[ + \sum_{i \neq j} \sum_l (-\theta)^{2-p(i)-p(j)-q(l)} \left( 2 \frac{x_i^2 y_l}{(x_i - x_j)(1-x_i y_l)} - \frac{x_i^2 y_j y_l}{(1-x_i y_l)(1-x_j y_l)} \right) \]
\[ + 2 \sum_{i, j} (-\theta)^{1-p(i)-q(j)} \left( n_{ij} - m_{ij} + (-\theta)^{1-p(i)} - (-\theta)^{1-q(j)} \right) \frac{x_i y_j}{1-x_i y_j}. \]

We now rewrite the first two sums in this expression such that their dependence on the variables \(x\) appear only through terms of the form \(x_i y_j / (1-x_i y_j)\). To rewrite the first sum we use the fact that
\[ \frac{x_i y_j}{(y_j - y_l)(1-x_i y_l)} = \frac{y_l}{(y_j - y_l)(1-x_i y_l)} + \frac{x_i y_j}{1-x_i y_j} - \frac{y_l}{y_j - y_l}, \]
and the fact that the resulting sum
\[ 2 \sum_{j \neq l} (-\theta)^{2-p(i)-q(j)-q(l)} \frac{y_l}{(y_j - y_l)(1-x_i y_l)} \]
\[ = \sum_{j \neq l} (-\theta)^{2-p(i)-q(j)-q(l)} \left( \frac{y_l}{(y_j - y_l)(1-x_i y_j)} - \frac{y_j}{(y_j - y_l)(1-x_i y_l)} \right). \]

Setting \(x = x_i, y = 1/y_j\) and \(z = 1/y_l\) in Lemma 2.1 we find that
\[ \frac{x_i^2 y_j y_l}{(1-x_i y_j)(1-x_i y_l)} = \frac{y_l}{(y_j - y_l)(1-x_i y_j)} + \frac{y_j}{(y_j - y_l)(1-x_i y_l)} = 1. \]

We thus conclude that the first sum in (13) equals
\[ -2 \sum_i \sum_{j \neq l} (-\theta)^{2-p(i)-q(j)-q(l)} \frac{x_i y_j}{1-x_i y_j}. \]

The second sum in (13) can be similarly rewritten to yield
\[ 2 \sum_{i, j \neq i} (-\theta)^{2-p(i)-p(j)-q(l)} \frac{y_l}{1-x_i y_l}. \]
Inserting these latter expressions for the first two sums in (13) we obtain
\[
\Pi^{-1}(x, y) \left( D_x^2 - D_y^2 \right) \Pi(x, y) = 2 \sum_{i,l} (-\theta)^{1-p(i)-q(l)} (n_\theta - m_\theta) \frac{x_i y_l}{1 - x_i y_l} + 2 \sum_{i,j,l} (-\theta)^{2-p(i)-p(j)-q(l)} \frac{x_i y_l}{1 - x_i y_l} - 2 \sum_{i,j,l} (-\theta)^{2-p(i)-q(j)-q(l)} \frac{x_i y_l}{1 - x_i y_l}.
\]

The identity (11) for \( k = 2 \) now follows from the fact that
\[
\sum_j (-\theta)^{-p(j)} = -n_\theta, \quad \sum_j (-\theta)^{-q(j)} = -m_\theta.
\]

For the remaining values of \( k \) the identity (11) can now be obtained by first verifying that
\[
D^k = \frac{1}{k+1} \left[ E^0, D^{k+1} \right], \quad \bar{D}^k = \frac{1}{k+1} \left[ \bar{D}^{k+1}, E^0 \right],
\]
and then observing that
\[
\left[ E^0_x, D^{k+1}_x \right] \Pi(x, y) = \left[ \bar{D}^{k+1}_y, E^0_y \right] \Pi(x, y).
\]

By combining the identities (11) and (11) we can obtain such identities for all deformed CMS operators (11). Indeed, if we for each such operator \( \mathcal{L}_{\vec{n}} \) let
\[
\bar{\mathcal{L}}_{\vec{n}, \vec{m}} = \sum_{k=0}^{2} \alpha_k \bar{D}^k_{\vec{n}, \vec{m}} + \sum_{\ell=0}^{1} \beta_\ell E^\ell_{\vec{n}, \vec{m}}
\]
then we obtain the following:

**Proposition 4.1.** For all deformed CMS operators \( \mathcal{L}_{\vec{n}} \),
\[
(14) \quad \mathcal{L}_{\vec{n}, \vec{m}} \Pi(x, y) = \bar{\mathcal{L}}_{\vec{n}, \vec{m}} \Pi(x, y).
\]

**Remark 4.1.** In certain special cases these identities have been obtained before. For \( \mathcal{L}_\vec{n} = D^2_\vec{n} \) it is implicit in Stanley’s paper [Sta89], see also Chapter VI in Macdonald [Mac95]. Sergeev and Veselov [SV05] obtained an identity closely related to our identity (11) for \( D^2_\vec{n} \) with \( \vec{n} \) arbitrary and \( \vec{m} = (m, 0) \) for some positive integer \( m \). We also mention that similar identities have been obtained from the point of view of quantum Calogero-Sutherland models, see Gaudin [Gau92], Serban [Ser97] and the paper [HL07].

For the remainder of this section we set \( \vec{m} = \vec{n} \). We will show that the resulting operators \( \bar{\mathcal{L}}_{\vec{n}} := \bar{\mathcal{L}}_{\vec{n}, \vec{n}} \) are the adjoints of \( \mathcal{L}_{\vec{n}} \) for a particular scalar product on \( \Lambda_{\vec{n}, \theta} \).

To obtain the precise statement we proceed in analogy with the proof of Statement 2.13 in Section VI.2 of Macdonald.

**Definition 4.1.** We define the scalar product \( \langle \cdot, \cdot \rangle_{\vec{n}} \) on \( \Lambda_{\vec{n}, \theta} \) (antilinear in its second argument) by setting
\[
(15) \quad \langle SP_\lambda, SP_\mu \rangle_{\vec{n}} = b^{-1}_\lambda \delta_{\lambda\mu}
\]
for all \( \lambda, \mu \in H_{\vec{n}} \).

As we now prove, for \( \vec{m} = \vec{n} \) Proposition 4.1 is equivalent to the following:

**Proposition 4.2.** For all \( f, g \in \Lambda_{\vec{n}, \theta} \),
\[
(16) \quad \langle \mathcal{L}_{\vec{n}} f, g \rangle_{\vec{n}} = \langle f, \bar{\mathcal{L}}_{\vec{n}} g \rangle_{\vec{n}}.
\]
Proof. As indicated above, we prove the statement by establishing its equivalence to Proposition 4.1 for $\tilde{m} = \tilde{n}$. With

$$SQ_\lambda = b_\lambda SP_\lambda$$

it is clear that

$$⟨SP_\lambda, SQ_\mu⟩_\tilde{n} = \delta_{\lambda\mu}$$

for all $\lambda, \mu \in H_\tilde{n}$. For each pair of partitions $\lambda, \mu \in H_\tilde{n}$ we define constants $e_{\lambda\mu}$ and $\tilde{e}_{\lambda\mu}$ by

$$L_\tilde{n}SP_\lambda = \sum_{\mu \in H_\tilde{n}} e_{\lambda\mu} SP_\mu$$

and

$$\tilde{L}_\tilde{n}SQ_\lambda = \sum_{\mu \in H_\tilde{n}} \tilde{e}_{\lambda\mu} SQ_\mu,$$

respectively. Inserting these two expressions into the identity (14) and using Proposition 3.1 we obtain

$$\sum_{\lambda, \mu \in H_\tilde{n}} e_{\lambda\mu} SP_\mu(x)SP_\lambda(y) = \sum_{\lambda, \mu \in H_\tilde{n}} \tilde{e}_{\lambda\mu} SP_\lambda(x)SP_\mu(y).$$

Hence, the identity (14) is equivalent to

$$e_{\lambda\mu} = \tilde{e}_{\mu\lambda}$$

for all $\lambda, \mu \in H_\tilde{n}$. On the other hand, using the definition of the scalar product $⟨·, ·⟩_\tilde{n}$ to compute the left- and right-hand sides of (16) for $f = SP_\lambda$ and $g = SQ_\mu$ with $\lambda$ and $\mu$ running through all partitions in $H_\tilde{n}$, we find that also (16) is equivalent to (18). Clearly, this implies (16). □

It is easily verified that $E^1_1 = E^1_{\tilde{n}}$ and that $\tilde{D}^2_\tilde{n} = D^2_\tilde{n}$. As a special case of Proposition 4.2 we thus obtain the following:

**Corollary 4.1.** Suppose that $\alpha(x) = \alpha_2 x^2$ and $\beta(x) = \beta_1 x$ for some coefficients $\alpha_2$ and $\beta_1$, respectively. Then, the corresponding deformed CMS operator $L_\tilde{n}$ is self-adjoint for the scalar product $⟨·, ·⟩_\tilde{n}$ on $\Lambda_{\tilde{n}, \theta}$.

**Remark 4.2.** The scalar product obtained by setting $\tilde{m} = \tilde{n} = 0$ was used by Macdonald [Mac95] and the corollary can in this case be inferred from his Statement 3.11 in Section VI.3.

5. A linear basis for the algebra $\Lambda_{\tilde{n}, \theta}$

In this section we give a precise definition of the polynomials $f^{(\tilde{m})}_a$ and deduce some of their basic properties. We prove, in particular, that a subset of these polynomials, parametrised by the partitions in $H_\tilde{n} \cap H_\tilde{m}$, span the same subspace of $\Lambda_{\tilde{n}, \theta}$ as the corresponding super Jack polynomials.

The polynomials $f^{(\tilde{m})}_a$ will be indexed by integer vectors $a \in \mathbb{Z}^{[\tilde{m}]}$ and before stating their definition we introduce some notation related to this fact. We will on a number of occasions relate such polynomials $f^{(\tilde{m})}_a$ to super Jack polynomials indexed by partitions in $H_\tilde{m}$. For that we define $\varphi = \varphi_{\tilde{m}}$ to act on partitions $\lambda \in H_\tilde{m}$ as

$$\varphi(\lambda) = (m_{\lambda, m}]'. $$
Let $M$ for some positive integer $s$ and (21) to the so-called Jacobi-Trudi identity, originally due to Jacobi [Jac41]; see Proposition 5.1.

$h$ stated above is equivalent to the one given in [HL07].

Definition 5.2 (Pragacz and Thorup) introduced by Pragacz and Thorup [PT92]; see also Fulton and Pragacz [FP95].

In special cases they stand in a simple relation to the so-called super Schur polynomials, (21) is then defined as the determinant

\[
\text{det } (\bar{a}, \bar{b}) = (a_1, \ldots, a_{\bar{m}}) \in \mathbb{Z}^{\bar{m}} \text{ we let } |a| = a_1 + \cdots + a_{\bar{m}}. \text{ In addition, we will make extensive use of the partial order on } \mathbb{Z}^{\bar{m}} \text{ defined by the equivalence}
\]

\[
a \succeq b \Leftrightarrow a_k \leq b_k \text{ for all } k = 1, \ldots, \bar{m}, \quad \forall i = 1, \ldots, |\bar{m}|.
\]

By comparing definitions it is straightforward to verify that this partial order is related to the partial order defined by (2) as follows:

**Lemma 5.1.** Let $a, b \in \mathbb{Z}^{\bar{m}}$ be such that $|a| = |b|$. Then $a \succeq b$ if and only if $b \preceq a$.

We are now ready to state the following:

**Definition 5.1.** We define the polynomials $f_a^{(\bar{m})}$, $a \in \mathbb{Z}^{\bar{m}}$, by the expansion

\[
\prod_{j=l}^n (1 - y_l/y_j)^{-(\theta_l-\theta_j)-q(j)} \prod_{i,l} (1 - x_i y_l)^{-(\theta_l-\theta_i)-q(i)} = \sum_a f_a^{(\bar{m})}(x)y^a,
\]

valid for $\min_i (|x_i^{-1}|) > |y_1| > \cdots > |y_{\bar{m}}|$, and where the sum is over integer vectors $a \in \mathbb{Z}^{\bar{m}}$.

**Remark 5.1.** These polynomials were defined by Langmann [Lan01] for the special cases $\bar{m} = \bar{n} = (N, 0)$ with $N$ any positive integer, while the more general definition stated above is equivalent to the one given in [HL07].

Before studying the polynomials $f_a^{(\bar{m})}$ in general it is instructive to consider the special cases for which $\theta = 1$ and $\bar{m} = (M, 0)$ for some positive integer $M$. In these cases they stand in a simple relation to the so-called super Schur polynomials, introduced by Pragacz and Thorup [PT92]; see also Fulton and Pragacz [FP95].

**Definition 5.2 (Pragacz and Thorup).** For each non-negative integer $k$, the polynomial $s_k(x, \bar{x})$ is defined by the expansion

\[
\prod_{i=1}^{n} (1 - x_i y_l)^{-1} \prod_{l=1}^{\bar{n}} (1 + \bar{x}_l y_l) = \sum_k s_k(x, \bar{x})y^k.
\]

Let $\lambda$ be a partition and let $l = \ell(\lambda)$. The so-called super Schur polynomial $S_\lambda(x, \bar{x})$ is then defined as the determinant

\[
S_\lambda(x, \bar{x}) = \text{det } (s_{\lambda, -i+j}(x, \bar{x}))_{1 \leq i, j \leq l}.
\]

**Remark 5.2.** Note that if $\bar{n} = 0$ the polynomials $s_k$ reduce to the complete symmetric polynomials $h_k$, the super Schur polynomials to the ‘ordinary’ Schur polynomials and (21) to the so-called Jacobi-Trudi identity, originally due to Jacobi [Jac41]; see also Section I.3 in Macdonald [Mac95].

**Proposition 5.1.** Let $\lambda$ be a partition. Suppose that $\theta = 1$ and that $\bar{m} = (M, 0)$ for some positive integer $M \geq \ell(\lambda)$. Then

\[
f_{\lambda}^{(\bar{m})}(x, \bar{x}) = S_\lambda(x, \bar{x}).
\]

**Proof.** For each $a \in \mathbb{N}^l$ we let $s_a = s_{a_1} \cdots s_{a_l}$. In addition, we let $\delta = (l - 1, l - 2, \ldots, 1, 0)$ and observe that

\[
\text{det } (s_{\lambda, -i+j}(x, \bar{x}))_{1 \leq i, j \leq l} = \text{det } (s_{\lambda, \delta_i+\delta_j}(x, \bar{x}))_{1 \leq i, j \leq l} = \sum_{w \in S_l} \epsilon(w)s_{\lambda+\delta-w\delta},
\]
where \( S_l \) refers to the permutation group of \( l \) objects and \( \epsilon(w) \) denotes the sign of the permutation \( w \). Let \( R_{ij} \) denote the so-called raising operator defined by \( R_{ij} a = a + e_i - e_j \) for each \( a \in \mathbb{Z}_l \). In \( \mathbb{C}[x_1^\pm 1, \ldots, x_l^\pm 1] \) we then have that
\[
\sum_{w \in S_l} \epsilon(w) x_i^{\lambda + \delta - w \delta} = x_i^{\lambda + \delta} \prod_{i < j} (x_i^{x_i} - x_j^{x_j}) = \prod_{i < j} (1 - x_i/x_j)x^\lambda = \prod_{i < j} (1 - R_{ij})x^\lambda.
\]
By applying the linear map \( \mathbb{C} \in \mathbb{R} \) + \( \mathbb{P} \) polynomials \( S \), we let \( \epsilon \) refer to the permutation group of \( l \) objects and \( \epsilon(w) \) the sign of the permutation \( w \).

Comparing coefficients of \( y^a \) for any Laurent polynomial \( a(x) \) if \( f(x) \) of weight \( |\lambda| \), we obtain
\[
\lambda \in \Lambda_{\alpha, \beta} \text{ defined by } \lambda(a) \mapsto s_a \text{ for all } a \in \mathbb{Z}_l \text{ we find that } S_\lambda = \prod_{i < j} (1 - R_{ij})s_\lambda.
\]

On the other hand,
\[
\sum_{a \in Z^M} f(\lambda)(x, -\bar{x})y^a = \prod_{j<1} (1 - y_i/y_j) \prod_{j<i} (1 - x_i y_j)^{-1} \prod_{j<i} (1 + \bar{x}_i y_j) = \prod_{j<i} (1 - y_i/y_j) \sum_{a \in \mathbb{N}^M} s_a(x, \bar{x})y^a.
\]
Comparing coefficients of \( y^a \) we obtain
\[
f(\lambda)(x, -\bar{x}) = \prod_{i < j} (1 - R_{ij})s_\lambda(x, \bar{x}) = S_\lambda(x, \bar{x}).
\]

We proceed to prove that the set of polynomials \( f(\lambda) \) and the set of super Jack polynomials \( SP_\lambda \), both indexed by the partitions \( \lambda \in H_\alpha \cap H_\beta \) of a given weight \( |\lambda| \), span the same linear subspace of \( \Lambda_\alpha \). The main facts required for the proof are contained in the following:

**Lemma 5.2.** For any non-negative integer \( k \) there exists a transition matrix \( M = (M_{\alpha, \beta}) \), defined by the equalities
\[
f_\alpha^{(\lambda)} = \sum_{\mu} M_{\alpha, \mu} SP_\mu,
\]
from the polynomials \( f_\alpha^{(\lambda)} \), indexed by the integer vectors \( a \in \mathbb{Z}^{|\lambda|} \) such that \( |a| = k \), to the super Jack polynomials \( SP_\mu \), indexed by the partitions \( \mu \in H_\alpha \cap H_\beta \) of weight \( |\mu| = k \). The entries \( M_{\alpha, \mu} \) of \( M \) are non-zero only if \( \varphi(\mu) \leq a \). In addition, if \( a = \varphi(\lambda) \) for some partition \( \lambda \in H_\alpha \cap H_\beta \),
\[
M_{\alpha, \lambda} = (-1)^{|a|} b_\lambda(\theta) b_{|\mu|}(\theta^{-1}).
\]

**Proof.** For any Laurent polynomial \( g = g(y) \) in the variables \( y = (y_1, \ldots, y_{|\lambda|}) \) and integer vector \( a \in \mathbb{Z}^{|\lambda|} \) we let \([g]_a \) denote the coefficient of \( y^a \) in \( g(y) \). It follows from the definition of the polynomials \( f_\alpha^{(\lambda)} \) and Proposition 3.1 that
\[
f_\alpha^{(\lambda)}(x) = \sum_{\mu \in H_\alpha \cap H_\beta} b_\lambda(\theta) \prod_{j<1} (1 - y_i/y_j)^{\varphi(\mu)^{j-\xi(i-\mu)}} SP_\mu(y) \left[S \alpha \right]_a SP_\mu(x).
\]
Expanding the products in binomial series we conclude that the entries
\[
M_{\alpha, \mu} = b_\lambda(\theta) \prod_{j<1} \sum_{\nu_{\mu}} (-1)^{\nu_{\mu}} \left[SP_\mu(y) \right]_a \nu_{\mu} (\nu_{\mu} (\epsilon_j - \epsilon_i)) \nu_{\mu} \cdot
\]
We recall from Lemma 3.1 that a super Jack polynomial $SP_\mu$, if non-zero, is homogeneous of degree $|\mu|$. Lemmas 3.2 and 3.4 thus imply that such a super Jack polynomial $SP_\mu(y)$ is a linear combination of monomials $x^b$ with $b \geq \varphi(\mu)$. Since
\[ a + \sum_{j<l} \mu_{jl}(e_j - e_l) \preceq a \]
for all non-negative integers $\mu_{ij}$, it follows that the entries $M_{a\mu}$ are non-zero only if $|\mu| = |a|$ and $\varphi(\mu) \preceq a$. Clearly, this establishes both the existence, as well as the stated triangular structure, of the transition matrix $M$. Suppose that the integer vector $a = \varphi(\lambda)$ for some partition $\lambda \in H_n \cap H_m$. It is then clear that the only term which contributes to $M_{a\lambda}$ is the leading term $[SP_\lambda(y)]_{\varphi(\lambda)}$ of $SP_\lambda$. The statement thus follows from Lemma 3.2. \hfill \Box

Corollary 5.1. A polynomial $f_a^{(\bar{m})}, a \in \mathbb{Z}^{\bar{m}}$, is non-zero only if $a \geq 0$. In that case, $f_a^{(\bar{m})} \in \Lambda_{\bar{n},\theta}$ and it is homogeneous of degree $|a|$. 

Proof. We observe that if $a$ violates the condition $a \geq 0$ there exist no partition $\mu \in H_m$ such that $\varphi(\mu) \preceq a$. The statement is thus a direct consequence of Lemmas 3.1 and 5.2. \hfill \Box

Using Lemma 5.2 we now prove the following:

Proposition 5.2. Assume that $\theta$ is not a negative rational number or zero. Then, as $\lambda$ runs through all partitions of a given weight $|\lambda| = k$ in $H_n \cap H_{\bar{m}}$, the corresponding polynomials $f_{\varphi(\lambda)}^{(\bar{m})}$ form a linear basis for the linear space
\[ \mathbb{C} \langle SP_\lambda : \lambda \in H_n \cap H_{\bar{m}}, |\lambda| = k \rangle \subseteq \Lambda_{\bar{n},\theta}^k. \]
In particular, if $m \geq n$ and $\bar{m} \geq \bar{n}$, then they form a linear basis for $\Lambda_{\bar{n},\theta}^k$.

Proof. We let $K = (K_{\lambda\mu})$ be the restriction of the transition matrix $M$ defined in Lemma 5.2 to the polynomials $f_a^{(\bar{m})}$, indexed by the integer vectors $a = \varphi(\lambda)$ for some partition $\lambda \in H_n \cap H_{\bar{m}}$ of weight $|\lambda| = k$. We note that $K$ is a square matrix, whereas $M$ is not. We also note that, in the terminology of Section I.6 in Macdonald [Mac95], $K$ is a strictly upper triangular matrix in the sense that $K_{\lambda\mu} = 0$ unless $\varphi(\lambda) \succeq \varphi(\mu)$. It follows from Lemma 5.2 and the definition of $b\lambda(\theta)$ that all diagonal elements of $K$ (under the stated condition on $\theta$) are all well defined and non-zero. It is readily verified that the inverse of such a matrix exists and is of the same form, c.f. Statement 6.1 in Macdonald (loc. cit.). For all $\lambda \in H_n \cap H_{\bar{m}}$ we thus have
\[ SP_\lambda = \sum_{\mu} (K^{-1})_{\lambda\mu} f_{\varphi(\mu)}^{(\bar{m})} \]
and the statement follows. \hfill \Box

6. POLYNOMIAL EIGENFUNCTIONS OF DEFORMED CMS OPERATORS

For each of the deformed CMS operators (1) we construct in this section sets of polynomial eigenfunctions, parametrised by the partitions in $H_n \cap H_{\bar{m}}$, and expressed as linear combinations of the polynomials $f_a^{(\bar{m})}$. Under a certain condition of non-degeneracy on their eigenvalues we prove that these polynomial eigenfunctions span the same subspace of $\Lambda_{\bar{n},\theta}$ as the super Jack polynomials labeled by the partitions in $H_n \cap H_{\bar{m}}$. 

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The first step in our construction is to compute the action of the operators $L_{\tilde{\bar{n}}}$ on the polynomials $f_a^{(\tilde{\bar{m}})}$. Before proceeding to the computation we sketch our approach. We observe that the generating function for the polynomials $f_a^{(\tilde{\bar{m}})}$, as defined by (20), is related to the function $\Pi_{\tilde{\bar{n}},\tilde{\bar{m}}}(x, y)$ as follows:

$$\prod_{j<l}(1 - y_l/y_j)^{-(\theta)^{-q(j)-q(l)}} \prod_{i,j}(1 - x_i y_j)^{(-\theta)^{1-q(j)-q(l)}} = G_{\tilde{\bar{m}}}(y)\Pi_{\tilde{\bar{n}},\tilde{\bar{m}}}(x, y)$$

with

$$G_{\tilde{\bar{m}}}(y) = \prod_{j<l}(1 - y_l/y_j)^{-(\theta)^{1-q(j)-q(l)}}.$$ 

We also observe that Proposition 4.1 implies the following identity:

$$L_{\tilde{\bar{n}},x}G_{\tilde{\bar{m}}}(y)\Pi_{\tilde{\bar{n}},\tilde{\bar{m}}}(x, y) = (G_{\tilde{\bar{m}}}L_{\tilde{\bar{n}},\tilde{\bar{m}}}G_{\tilde{\bar{m}}}^{-1})_y G_{\tilde{\bar{m}}}(y)\Pi_{\tilde{\bar{n}},\tilde{\bar{m}}}(x, y).$$

By first expanding the function $G_{\tilde{\bar{m}}}(y)\Pi_{\tilde{\bar{n}},\tilde{\bar{m}}}(x, y)$ as in (20), then computing the right-hand side of this identity, and finally comparing coefficients with the left-hand side, we can thus obtain the action of the operators $L_{\tilde{\bar{n}}}$ on the polynomials $f_a^{(\tilde{\bar{m}})}$.

The difficult part of this computation will be to obtain the explicit form of the operators $G_{\tilde{\bar{m}}}L_{\tilde{\bar{n}},\tilde{\bar{m}}}G_{\tilde{\bar{m}}}^{-1}$. However, the computation is considerably simplified if we first consider only the operators $E_{\ell}^{\tilde{\bar{n}}}$ and $D_k^{\tilde{\bar{n}}}$. 

Before proceeding we mention that a detailed discussion of the approach sketched above in the case of the ‘ordinary’ CMS operators $L_{\bar{n}}$, as well as a comparison with a construction of their eigenfunctions in terms of more elementary bases for the symmetric polynomials, can be found in [Hal07].

We find it convenient to introduce the following notation: to each $a \in \mathbb{Z}^{\tilde{\bar{m}}}$ we associate the ‘shifted’ integer vector $a^+ = a + s$ with the shift $s = (s_1, \ldots, s_{\tilde{\bar{m}}})$ given by

$$s_j = (-\theta)^{1-q(j)} (-\theta)^{q(j) (j - 1) - n} (-\theta)^{-q(j) (m + \tilde{\bar{n}})} + m.$$ 

In addition, for each non-negative integer $\nu$ and $k = 0, 1, 2$ we let

$$E_{\ell}^{k\nu} = (2 - k + \nu)e_\ell - \nu e_j.$$ 

We are now ready to prove the following:

**Lemma 6.1.** Let $a \in \mathbb{Z}^{\tilde{\bar{m}}}$ be such that $a \geq 0$. Then

(22a) \quad $E_{\ell}^1 f_a^{(\tilde{\bar{m}})} = |a| f_a^{(\tilde{\bar{m}})}$, 

(22b) \quad $E_{\ell}^0 f_a^{(\tilde{\bar{m}})} = \sum_{j=1}^{\tilde{\bar{m}}} (a_j^+ - 1) f_a^{(\tilde{\bar{m}})}$. 


Furthermore, with \( k = 0, 1, \)

\[
(23a) \quad D_n^2 f^i_a(m) = \left( \sum_{j=1}^{[m]} (-\theta)^q(j) a_j (a_j - 1 + 2s_j) + 2|a| \right) f^i_a(m) \\
+ 2(\theta - 1) \sum_{j<k} (-\theta)^{1-q(j)-q(l)} \sum_{\nu=1}^{s} \nu f^{i(\bar{m})}_{a-E_{2\nu}},
\]

\[
(23b) \quad D_n^k f^i_a(m) = \sum_j (-\theta)^q(j) \left( a_j^+ - 2\delta_{k0} + k \left( (-\theta)^{-q(j)} - 1 \right) \right) \left( a_j^+ - 1 \right) f^{i(\bar{m})}_{a-(2-k)e_j} \\
+ (\theta - 1) \sum_{j<k} (-\theta)^{1-q(j)-q(l)} \sum_{\nu=0}^{s} (2\nu + 2 - k) f^{i(\bar{m})}_{a-E_{2\nu}}.
\]

**Proof.** We will throughout the proof assume that, whenever they occur, \( \ell = 0, 1 \) and \( k = 0, 1, 2 \). For simplicity of notation we let \( v_{ij} = (-\theta)^{-q(j)} \) for all \( j = 1, \ldots, [m] \) and suppress the index’s \( i \) and \( m \). In addition, we let \( M_G \) denote the operator of multiplication by the function \( G \).

In order to prove (22) we note that the similarity transformation of a partial derivative \( \partial/\partial y_j \) by \( M_G \) is given by

\[
M_G \frac{\partial}{\partial y_j} M_G^{-1} = \frac{\partial}{\partial y_j} - \theta \sum_{l>j} v_{jl} \frac{y_l}{y_j} - \theta \sum_{l<j} v_{lj} \frac{1}{y_j - y_l}.
\]

Multiplying by \( y_j^{2-\ell} \), taking the sum over \( j \), and interchanging summation index’s in the resulting second sum, we obtain

\[
M_G \bar{E}_x^l M_G^{-1} = \sum_j y_j^{2-\ell} \frac{\partial}{\partial y_j} + (1-\ell) \sum_j \left( n_{\theta} - \theta \sum_{l<j} v_{lj} \right) v_{j} y_j = \sum_j y_j^{2-\ell} \frac{\partial}{\partial y_j} + (1-\ell) \sum_j a_j y_j.
\]

It follows from Lemma 11 and the definition of the polynomials \( f_a \) that

\[
\sum_a \left( E_x^l f_a(x) \right) y^a = \sum_a f_a(x) G(y) \bar{E}_y^l G^{-1}(y) y^a.
\]

By first computing the right-hand side and then comparing coefficients of the monomials \( y^a \) with the left-hand side we thus obtain (22).

We proceed to prove (23). Using the identity

\[
(24) \quad \frac{y_j^{n-1} y_i}{y_j - y_l} = \frac{y_j^n}{y_j - y_l} - y_j^{n-1},
\]

valid for any integer \( n \), it is readily verified that the similarity transformation of an operator \( D^k_y \) by \( M_G \) is given by

\[
(25) \quad M_G D^k_y M_G^{-1} = \sum_j v_j^{1-k} y_j^{4-k} \frac{\partial^2}{\partial y_j^2} + \sum_j \left( C_{j,k} + 2\theta \sum_{l>j} v_{jl} \right) y_j^{3-k} \frac{\partial}{\partial y_j} \\
+ G(y) \bar{D}_y^k G^{-1}(y).
\]
Repeatedly applying the identity (21), and using the fact that $s_j v_j^{-1} = n_\theta - \theta \sum_{l < j} v_l$, we deduce through a straightforward but somewhat lengthy computation

$$G(y) \hat{D}_y^k G^{-1}(y) = \theta \sum_{j \neq l} \left( kv_l (1 - v_j) \frac{y_j^{2-k} y_l}{y_j - y_l} - (\theta v_j v_l - 1) v_l \frac{y_j^{2-k} y_l^2}{(y_j - y_l)^2} \right)$$

$$- \theta^2 \sum_{j \neq l, l', l''} v_j v_l v_{l'} v_{l''} \frac{y_j^{2-k} y_l y_{l'} y_{l''}}{(y_j - y_l)(y_{l'} - y_l)} - 2\theta n_\theta \sum_{j \neq l} v_j v_l \frac{y_j^{2-k} y_l}{y_j - y_l}$$

$$+ \sum_{l}(s_l(s_j + 1) - n_\theta v_l(n_\theta v_j + 1) + k(v_l - 1)(s_l - n_\theta v_l)) v_j^{-1} y_j^{2-k} + P_k(y).$$

Since $2(v_l f - v_j g) = (v_l + v_j)(f + g) + (v_l - v_j)(f - g)$ for any two functions $f$ and $g$, we have

$$\theta \sum_{j \neq l} (\theta v_j v_l - 1) v_l \frac{y_j^{2-k} y_l^2}{(y_j - y_l)^2} = \frac{\theta}{2} \sum_{j < l} (\theta v_j v_l - 1)(v_l + v_j) \frac{y_j^{2-k} y_l^2 + y_j^2 y_l^{2-k}}{(y_j - y_l)^2}$$

$$+ \sum_{j < l} (\theta v_j v_l - 1)(v_l - v_j) \frac{y_j^{2-k} y_l^2 - y_j^2 y_l^{2-k}}{(y_j - y_l)^2}.$$ 

By a straightforward computation using the fact that $2v_l (1 - v_j) = v_l (1 - v_j) (1 - \theta v_j)$ we thus obtain

$$\theta \sum_{j \neq l} \left( kv_l (1 - v_j) \frac{y_j^{2-k} y_l}{y_j - y_l} - (\theta v_j v_l - 1) v_l \frac{y_j^{2-k} y_l^2}{(y_j - y_l)^2} \right)$$

$$= -\frac{\theta}{2} \sum_{j < l} (\theta v_j v_l - 1)(v_l + v_j) \frac{y_j^{2-k} y_l^2 + y_j^2 y_l^{2-k}}{(y_j - y_l)^2}$$

$$- \delta_{k,2} \theta \sum_{j < l} \left( \frac{1}{2}(\theta v_j v_l + 1)(v_l + v_j) - (1 + \theta) v_j v_l \right).$$

Rewriting the sum such that its summand is symmetric in the summation index’s and setting $x = 1/y_j$, $y = 1/y_l$ and $z = 1/y_{l'}$ in Lemma 2.1 we deduce

$$\theta^2 \sum_{j \neq l, l', l''} v_j v_l v_{l'} v_{l''} \frac{y_j^{2-k} y_l y_{l'} y_{l''}}{(y_j - y_l)(y_{l'} - y_l)} = \delta_{k,2} 2\theta^2 \sum_{j < l < l'} v_j v_l v_{l'}.$$

Using the fact that

$$2 \sum_{j \neq l} v_j v_l \frac{y_j^{2-k} y_l}{y_j - y_l} = \sum_{j \neq l} v_j v_l (\delta_{k,0} y_j y_l - \delta_{k,2}),$$

as well as the definition of the polynomials $P_k(y)$, we thus obtain

$$G(y) \hat{D}_y^k G^{-1}(y) = (1 - \delta_{k,2}) \sum_{j} (s_j(s_j + 1) + k(v_j - 1)s_j) v_j^{-1} y_j^{2-k}$$

(26)

$$- \frac{\theta}{2} \sum_{j < l} (\theta v_j v_l - 1)(v_l + v_j) \left( \frac{y_j^{2-k} y_l^2 + y_j^2 y_l^{2-k}}{(y_j - y_l)^2} - \delta_{k,2} \right).$$
We now make the Laurent series expansion
\[ \frac{y_j^{-k} y_i^2 + y_j y_i^{2-k}}{(y_j - y_i)^2} - \delta_{k2} = \sum_{\nu=0}^{\infty} (2\nu + 2 - k) \frac{y_i^{\nu+2-k}}{y_j^{\nu}}, \]
valid for \( \min \{|x_i^{-1}| > |y_1| > \cdots > |y_{mj}| \}. \) Inserting this Laurent series expansion, as well as (26), into (25) and using the fact that such that \( a \) for some \( C \)

Corollary 6.1. Set
\[ \text{proof of (22) above.} \]

it is straightforward to compute \( G(y)D_0^kG^{-1}(y)y^\alpha \) for all integer vectors \( a \in \mathbb{Z}^{[m]} \)
such that \( a \geq 0 \). It is then readily verified that (22) follows from Lemma 4.1, as in the proof of (22) above.

By Corollary 5.1 and Proposition 5.2, we thus obtain the following:

**Corollary 6.1.** Set \( \Lambda_{n,\theta}^N = \{0\} \) for all \( N < 0 \). Then, for \( \ell = 0, 1, \)

\[ E_n^\ell : \Lambda_{n,\theta}^N \to \Lambda_{n,\theta}^{N-(1-\ell)} \]

and, for \( k = 0, 1, 2, \)

\[ D_n^k : \Lambda_{n,\theta}^N \to \Lambda_{n,\theta}^{N-(2-k)} \]

for all non-negative integers \( N \).

**Remark 6.1.** We say that a linear operator \( L \) on \( \Lambda_{n,\theta} \) is homogeneous of degree \( k \) if \( LA_{n,\theta}^N \subseteq \Lambda_{n,\theta}^{N-k} \) for all \( N \geq 0 \). Corollary 6.1 then states that each operator \( E_n^\ell \) and \( D_n^k \) is homogeneous of degree \( \ell - 1 \) and \( k - 2 \), respectively. In addition, using Proposition 4.2, it is readily inferred that \( E_n^\ell \) and \( D_n^k \) are similarly homogeneous of degree \( 3 - \ell \) and \( 4 - k \), respectively.

As mentioned above, the polynomial eigenfunctions we construct for the deformed CMS operators (1) will be labeled by the partitions in \( H_{\alpha} \cap H_{\beta} \). For a given such partition \( \lambda \) the corresponding eigenfunction will be obtained as a linear combination of polynomials \( f_{\lambda}^{(n)} \) indexed by integer vectors of the form \( b = \varphi(\lambda) - a \geq 0 \), where the integer vectors \( a \) will be contained in certain subsets of \( \{a \in \mathbb{Z}^{[m]} : a \geq 0\} \).

Although it is difficult to give a precise characterisation of these subsets, it will become evident below that they are contained in sets of integer vectors \( \mathcal{C}_{[m]}(\alpha, \beta) \) which are determined in a simple manner by the two polynomials \( \alpha \) and \( \beta \), and which in many cases are significantly smaller than \( \{a \in \mathbb{Z}^{[m]} : a \geq 0\} \).

**Definition 6.1.** Fix a non-negative integer \( n \). Associate to each pair of polynomials \( \alpha \) and \( \beta \) the index sets \( I_{\alpha} = \{k : \alpha_k \neq 0\} \) and \( I_{\beta} = \{\ell : \beta_\ell \neq 0\} \). For \( k = 0, 1, 2 \) let

\[ A_n^{(k)} = \{a \in \mathbb{Z}^n : a \geq 0, |a| = (2-k)\nu \text{ for some } \nu \in \mathbb{N}\}, \]

and for \( \ell = 0, 1 \) let

\[ B_n^{(\ell)} = \{a \in \mathbb{N}^n : |a| = (1-\ell)\nu \text{ for some } \nu \in \mathbb{N}\}. \]

We then define \( \mathcal{C}_n = \mathcal{C}_n(\alpha, \beta) \) to be the set of integer vectors of the form

\[ a = \sum_{k \in I_{\alpha}} a^{(k)} + \sum_{\ell \in I_{\beta}} b^{(\ell)} \]

for some \( a^{(k)} \in A_n^{(k)} \) and \( b^{(\ell)} \in B_n^{(\ell)} \).
Remark 6.2. We note that in most special cases this definition can be considerably simplified, e.g., for \( \alpha(x) = x^2 \) and \( \beta = 0 \), corresponding to the deformed CMS operator \( \mathcal{L}_n = D_n^2 \), we have \( \mathcal{C}_n = A_n^{(2)} \). Hence, in this special case, \( \mathcal{C}_n \) is the set of all integer vectors \( a \geq 0 \) such that \( |a| = 0 \).

At this point we fix the two polynomials \( \alpha \) and \( \beta \) and consider the resulting deformed CMS operator \( \mathcal{L}_n \), as defined by (1). It is clear from Lemma 6.1 and Corollary 5.1 that \( \mathcal{L}_n \) maps each polynomial \( f_a^{(m)} \) to a linear combination of polynomials \( f_{a-b}^{(m)} \) with \( b \in \mathcal{C}_{|a|} \) such that \( b \leq a \). This suggests that for each partition \( \lambda \in H_n \cap H_{\bar{m}} \) there exists an eigenfunction \( P^{(\bar{m})}_\lambda \) of \( \mathcal{L}_n \) of the form

\[
P^{(\bar{m})}_\lambda = \sum_a u_\lambda(a) f^{(m)}(\varphi(\lambda) - a),
\]

where the sum is over integer vectors \( a \in \mathcal{C}_{|\bar{m}|} \) such that \( 0 \leq a \leq \varphi(\lambda) \). Indeed, using Lemma 6.1 it is straightforward to verify that this is the case if the coefficients \( u_\lambda(a) \) satisfy the recurrence relation

\[
(E_{\bar{m}}(\varphi(\lambda)) - E_{\bar{m}}(\varphi(\lambda) - a)) u_\lambda(a) = \alpha_0 \sum_j (-\theta)^{q(j)} a_j^+ (a_j^+ + 1) u_\lambda(a + 2e_j)
\]

\[
+ \sum_j a_j^+ \left( \beta_0 + \alpha_1 (-\theta)^{q(j)} \left( a_j^+ + (-\theta)^{-q(j)} \right) \right) u_\lambda(a + e_j)
\]

\[
+ (\theta - 1) \sum_{j < l} (-\theta)^{1-q(j)-q(l)} \sum_{k=0}^2 \alpha_k \sum_{\nu=0}^\infty (2\nu + 2 - k) u_\lambda(a + E_{ji}^{k\nu})
\]

with

\[
E_{\bar{m}}(a) = \alpha_2 \sum_{j=1}^{\bar{m}} (-\theta)^{q(j)} a_j (a_j - 1 + 2s_j) + (2\alpha_2 + \beta_1) |a|.
\]

In addition, the corresponding eigenvalues are given by \( E_{\bar{m}}(\varphi(\lambda)) \). We note that if \( E_{\bar{m}}(\varphi(\lambda)) \neq E_{\bar{m}}(\varphi(\lambda) - a) \) for all \( a \in \mathcal{C}_{|\bar{m}|} \) such that \( 0 < a \leq \varphi(\lambda) \) then the recursions relation (28) uniquely determines the coefficients \( u_\lambda(a) \) once the leading coefficient \( u_\lambda(0) \) has been fixed. We also note that since \( 0 \leq a \leq \varphi(\lambda) \), the sum over \( \nu \) in (28) truncates after a finite number of terms.

Definition 6.2. We say that a partition \( \lambda \in H_{\bar{m}} \) is \( \bar{m} \)-admissible if \( E(\varphi(\lambda)) \neq E(\varphi(\lambda) - a) \) for all \( a \in \mathcal{C}_{|\bar{m}|} \) such that \( 0 < a \leq \varphi(\lambda) \).

A polynomial \( P^{(\bar{m})}_\lambda \) with the leading coefficient \( u_\lambda(0) \) fixed to some non-zero constant and remaining coefficients determined by the recursion relation (28) is thus a well-defined eigenfunction of the deformed CMS operator \( \mathcal{L}_n \) if the partition \( \lambda \) is \( \bar{m} \)-admissible. An important aspect of our construction is that these eigenfunctions can be normalised such that they are independent of \( \bar{m} \). To give a first indication of this fact we proceed to show that, although the expression for \( E_{\bar{m}}(a) \) above depends on \( \bar{m} \), the eigenvalues \( E_{\bar{m}}(\varphi(\lambda)) \) of the eigenfunctions \( P^{(\bar{m})}_\lambda \) do not. For the convenience of the reader we provide a complete proof of this fact although a proof has been previously given in [HL07]; see Lemma 4.1.
Lemma 6.2. For all partitions $\lambda \in H_\bar{n} \cap H_{\bar{m}}$,
\[
\mathcal{E}_\bar{m}(\varphi(\lambda)) = \mathcal{E}_{(\ell(\lambda),0)}(\lambda) = \alpha_2 \sum_{j} \lambda_j (\lambda_j + 1 + 2\theta(n - j + 1) - 2\bar{n}) + \beta_1 \sum_{j} \lambda_j,
\]
where the two sums run over all parts of $\lambda$.

Proof. By the definition of the map $\varphi$ we have
\[
\mathcal{E}(\varphi(\lambda)) - (2\alpha_2 + \beta_1)|\lambda| = \alpha_2 \sum_{j=1}^{m} \lambda_j (\lambda_j - 1 + 2s_j) - \alpha_2 \theta \sum_{j=1}^{\bar{m}} m\lambda_j' (m\lambda_j' - 1 + 2s_{m} + j).
\]
For any partition $\mu$ it is easily verified that
\[
\sum_{i} i\mu'_i = \frac{1}{2} \sum_{i} \mu_i(\mu + 1),
\]
where the two sums run over the parts of $\mu'$ and $\mu$, respectively. The definition of the ‘shift’ $s$ thus imply
\[
\theta \sum_{j=1}^{\bar{m}} m\lambda_j' (m\lambda_j' - 1 + 2s_{m} + j) = - \sum_{j} m\lambda_j (m\lambda_j - 1 + 2\theta(n - m - J + 1) - 2\bar{n})
\]
where the latter sum is over the parts of $m\lambda$. We now obtain the statement by using again the definition of the shift $s$.

For the remainder of this section we set the leading coefficients
\[
(29) \quad u_\lambda(0) = (-1)^{|m\lambda|} b^{-1}_{m\lambda'} (\theta^{-1})
\]
for all polynomials $P^{(m)}_{\lambda}$ with $\lambda \in H_\bar{n} \cap H_{\bar{m}}$ an $\bar{m}$-admissible partition. This is motivated by the following:

Proposition 6.1. Let $k \in \mathbb{N}^2$ and let $\lambda \in H_\bar{n} \cap H_{\bar{m}} \cap H_{\bar{k}}$ be $\bar{m}$- and $\bar{k}$-admissible. Then $P^{(k)}_{\lambda} = P^{(\bar{k})}_{\lambda}$.

Proof. To prove the statement we will make use of the fact that any super Jack polynomial
\[
(30) \quad SP_{\mu} = \sum_{a} c_{\mu a} f^{(\bar{m})}_{a}
\]
for some coefficients $c_{\mu a}$ and where the sum is over integer vectors $a \in \mathbb{Z}^{[\bar{m}]}$ such that $|a| = |\mu|$ and $a \preceq \varphi(\mu)$. This fact can be inferred either from Lemma 5.2 (c.f. the proof of Proposition 5.2) or directly from Theorem 7.1 below. We let $C_{\bar{m}}(\mu)$ be the set of integer vectors $a < \varphi_{\bar{m}}(\mu)$ such that $\varphi_{\bar{m}}(\mu) - a \in \mathcal{C}_{[\bar{m}]}$. From (29) and Lemma 6.1 we deduce that
\[
(31) \quad \mathcal{L}_{\bar{n}} SP_{\mu} = \sum_{|a|=|\mu|} \mathcal{E}(a) c_{\mu a} f^{(\bar{m})}_{a} + \sum_{b \in C_{\bar{m}}(\mu)} c'_{\mu b} f^{(\bar{m})}_{b}
\]
for some coefficients $c'_{\mu b}$. To proceed we consider the two cases $\alpha_2 = 0$ and $\alpha_2 \neq 0$ separately. We first assume that $\alpha_2 = 0$. In this case $\mathcal{L}_{\bar{m}}(a) = \mathcal{E}_{\bar{m}}(\varphi(\mu))$ for all integer vectors $a \in \mathbb{Z}^{[\bar{m}]}$ such that $|a| = |\mu|$. Hence,
\[
\mathcal{L}_{\bar{n}} SP_{\mu} = \mathcal{E}_{\bar{m}}(\varphi(\mu)) SP_{\mu} + \sum_{b \in C_{\bar{m}}(\mu)} c'_{\mu b} f^{(\bar{m})}_{b}.
\]
It is easily inferred from Definition 6.1 that if \( b \in C_m \) then any integer vector \( a \in \mathbb{Z}^{|m|} \) such that \(|a| = |b|\) and \( a \leq b \) is contained in \( C_m \). It thus follows from Lemma 5.2 that

\[
\mathcal{L}_m SP_\mu = \mathcal{E}(\varphi(\mu))SP_\mu + \sum_{\nu \in C_m(\mu)} d_{\mu \nu} SP_\nu
\]

for some coefficients \( d_{\mu \nu} \). We turn now to the case \( \alpha_2 \neq 0 \). In this case all integer vectors \( a \neq \varphi(\mu) \) in (31) are contained in \( C_m \). It follows that

\[
\mathcal{L}_m SP_\mu = \mathcal{E}(\varphi(\mu))SP_\mu + \sum_{b \in C_m(\mu)} c''_{\mu b} f^{(\tilde{m})}_b
\]

for some coefficients \( c''_{\mu b} \). Referring again to Lemma 5.2 we thus find that (32) holds true also for \( \alpha_2 \neq 0 \). It is clear that if we replace \( m \) by \( \tilde{m} \) in the discussion above we obtain (32) with \( C_m(\mu) \) replaced by \( C_k(\mu) \). Since \( \lambda \) is both \( \tilde{m} \) - and \( k \) -admissible, it follows that with \( C(\lambda) \) either of the sets \( C_m(\lambda) \), \( C_k(\lambda) \) or \( C_m(\lambda) \cap C_k(\lambda) \) there exist a unique eigenfunction \( P_\lambda \) of \( \mathcal{L}_m \) such that

\[
P_\lambda = SP_\lambda + \sum_{\mu \in C(\lambda)} u_{\lambda \mu} SP_\mu
\]

for some coefficients \( u_{\lambda \mu} \). Since \( C_m(\lambda) \cap C_k(\lambda) \) is contained in both \( C_m(\lambda) \) and \( C_k(\lambda) \), these eigenfunctions must all coincide. It thus follows from (27), (29) and Lemma 5.2 that

\[
P^{(\tilde{m})}_\lambda = P^{(\tilde{m})}_\lambda = b_\lambda(\theta)P_\lambda.
\]

Given a partition \( \lambda \in H_\lambda \) we thus obtain different series representations for the same eigenfunction of \( \mathcal{L}_m \) by varying the value of \( \tilde{m} \) such that \( \lambda \in H_\lambda \) is \( \tilde{m} \) -admissible. It is interesting to note that the complexity of the resulting series representation is in many cases highly dependent on the specific value we choose for \( \tilde{m} \). To give a simple and concrete example of this fact we consider the deformed CMS operator \( \mathcal{L}_m = D^2_3 \) for \( \tilde{m} = (2, 1) \). Suppose that we are interested in the eigenfunction indexed by the partition \( \lambda = (1^3) \equiv (1, 1, 1) \). We observe that \( \mathcal{C}_3(x^2, 0) = A_3(2) \) consists of all integer vectors \( a \in \mathbb{Z}^3 \) such that \( a \geq 0 \) and \(|a| = 0\). In the discussion below we assume that the parameter \( \theta \) is such that the partition \( \lambda \) is \( \tilde{m} \) -admissible for all values of \( \tilde{m} \) under consideration. That such values of \( \theta \) exist follows from Proposition 5.2 below. If we set \( \tilde{m} = \tilde{m} = (2, 1) \) then we obtain a series representation of the form

\[
P^{(2,1)}_{(1^3)} = u_{(1^3)}(0)f^{(2,1)}_{(1^3)} + u_{(1^3)}((-1, 0, 1))f^{(2,1)}_{(1^3)}((-1, 0, 1))f^{(2,1)}_{(2,0,1)} + u_{(1^3)}((-2, 0))f^{(2,1)}_{(3,-1,1)} + u_{(1^3)}((-2, 1, 1))f^{(2,1)}_{(3)}.
\]

On the other hand, with \( \tilde{m} = (0, 1) \) the same eigenfunction is given by the series

\[
P^{(0,1)}_{(1^3)} = u_{(1^3)}(0)f^{(0,1)}_{(3)}
\]

containing only the polynomial

\[
f^{(0,1)}_{(3)} = -(\frac{-1}{\theta})x_1 x_2 \bar{x}_1 - (\frac{-1}{\theta})x_1 + x_2 \bar{x}_1^2 - (\frac{-1}{\theta})\bar{x}_1^2.
\]

In this case there is clearly a large difference in the complexity of the series representation obtained for \( \tilde{m} = (2, 1) \) and \( (0, 1) \), respectively. In general, the 'simplest'
series representation for a given eigenfunction is obtained by choosing \( \bar{m} \) such that \(|\bar{m}|\) is minimised. This reflects the fact that the complexity of an eigenfunction depends to a large extent on the partition to which it corresponds, and to a lesser extent on the number of variables \( \bar{m} \). For a further discussion of the complexity of this type of series representations for the eigenfunctions of (deformed) CMS operators, and their dependence on the value of \( \bar{m} \), see [HL07], in particular the discussion following Theorem 4.1.

In analogy with Proposition 5.2 we proceed to prove that the eigenfunctions \( P_{\lambda}(\bar{m}) \), indexed by the partitions \( \lambda \in H_{\bar{n}} \cap H_{\bar{m}} \), span the same linear subspace of \( \Lambda_{\bar{n},\theta} \) as the corresponding super Jack polynomials.

**Theorem 6.1.** Assume that \( \theta \) is not a negative rational number or zero and that all partitions in \( H_{\bar{n}} \cap H_{\bar{m}} \) are \( \bar{m} \)-admissible. Then, the corresponding eigenfunctions \( P_{\lambda}(\bar{m}) \) of the deformed CMS operator \( L_{\bar{n}} \) form a linear basis for the linear space \( C\langle SP_{\lambda} : \lambda \in H_{\bar{n}} \cap H_{\bar{m}} \rangle \subseteq \Lambda_{\bar{n},\theta} \).

In particular, if \( m \geq n \) and \( \bar{m} \geq \bar{n} \), then they form a linear basis for \( \Lambda_{\bar{n},\theta} \).

*Proof.* Fix a partition \( \lambda \in H_{\bar{n}} \cap H_{\bar{m}} \) and consider the eigenfunctions \( \left( P_{\mu}(\bar{m}) \right) \), indexed by the partitions \( \mu \in H_{\bar{n}} \cap H_{\bar{m}} \) such that \( \varphi(\mu) \preceq \varphi(\lambda) \). Observe that \( \varphi(\lambda) - a \preceq \varphi(\lambda) \) for all \( a \in \mathbb{C}|\bar{m}| \). It follows from (27) and Lemma 5.2 that there exist a well defined strictly upper triangular transition matrix \( N = (N_{\lambda\mu}) \) (c.f. the proof of Proposition 5.2) from the eigenfunctions \( \left( P_{\mu}(\bar{m}) \right) \) to the super Jack polynomials \( (SP_{\mu}) \), also indexed by the partitions \( \mu \in H_{\bar{n}} \cap H_{\bar{m}} \) such that \( \varphi(\mu) \preceq \varphi(\lambda) \). Furthermore, the specific form chosen for the leading coefficients \( u_{\lambda}(0) \) together with Lemma 5.2 and the definition of \( b_{\lambda}(\theta) \) implies that all of its diagonal elements are non-zero (under the stated assumption on \( \theta \)). Hence, its inverse \( N^{-1} \) exist and is of the same form. For each partition \( \mu \in H_{\bar{n}} \cap H_{\bar{m}} \) such that \( \varphi(\mu) \preceq \varphi(\lambda) \) we thus have that

\[ SP_{\mu} = \sum_{\nu} (N^{-1})_{\nu\mu} P_{\nu}(\bar{m}). \]

Since \( \lambda \) can be fixed to any partition in \( H_{\bar{n}} \cap H_{\bar{m}} \) the statement follows. \( \square \)

We conclude this section by establishing two sufficient conditions for all partitions in \( H_{\bar{n}} \cap H_{\bar{m}} \) to be \( \bar{m} \)-admissible.

**Proposition 6.2.** Assume that at least one of the coefficients \( \alpha_2 \) and \( \beta_1 \) are non-zero. All partitions in \( H_{\bar{n}} \cap H_{\bar{m}} \) are then \( \bar{m} \)-admissible if either of the following two conditions are satisfied:

1. The coefficient \( \alpha_2 \) is zero,
2. \( \theta \) is not a negative rational number, \( \bar{m} \leq 1 \), and the coefficients \( \alpha_1, \alpha_0 \) and \( \beta_0 \) are zero.
Proof. For any partition \( \lambda \in H_{m} \cap H_{n} \) and integer vector \( a \in \mathcal{C}_{m} \) such that \( 0 \prec a \preceq \varphi(\lambda) \),

\[
\mathcal{E}_{\bar{m}}(\varphi(\lambda)) - \mathcal{E}_{\bar{n}}(\varphi(\lambda) - a) = 2\alpha_{2} \sum_{j=1}^{\bar{m}} (-\theta)^{\bar{q}(j)} a_{j} ((\varphi(\lambda))_{j} + s_{j})
\]

\[
- \alpha_{2} \sum_{j=1}^{\bar{m}} (-\theta)^{\bar{q}(j)} a_{j} (a_{j} + 1) + (2\alpha_{2} + \beta_{1})|a|.
\]

Suppose that \( \alpha_{2} = 0 \). Then \( |a| > 0 \) and Condition (1) is clearly sufficient for all partitions in \( H_{m} \cap H_{n} \) to be \( \bar{m} \)-admissible. Suppose instead that \( \alpha_{2} \neq 0 \). Assume furthermore that \( \bar{n} \leq 1 \) and that \( \alpha_{1}, \alpha_{0} \) and \( \beta_{0} \) are zero. Then it is readily verified that the integer vector

\[
a = - \sum_{j<l} \nu_{jl}(e_{j} - e_{l})
\]

for some non-negative integers \( \nu_{jl} \) not all zero. We thus have

\[
\sum_{j=1}^{\bar{m}} (-\theta)^{\bar{q}(j)} a_{j} ((\varphi(\lambda))_{j} + s_{j}) = - \sum_{1 \leq j < l \leq m} \nu_{jl}(\lambda_{j} + s_{j} - \lambda_{l} - s_{l})
\]

\[
- \sum_{j=1}^{m} \nu_{jm+1}(\lambda_{j} + s_{j} + \theta((\varphi(\lambda))_{m+1} + s_{m+1})).
\]

In addition,

\[
\sum_{j=1}^{\bar{m}} (-\theta)^{\bar{q}(j)} a_{j} (a_{j} + 1) = \sum_{j=1}^{m} a_{j} (a_{j} + 1) - \theta a_{m+1}(a_{m+1} + 1).
\]

The definition of the ‘shift’ \( s \) and the fact that \( a_{m+1} = \sum_{j=1}^{m} \nu_{jm+1} \) thus imply

\[
\mathcal{E}_{\bar{m}}(\varphi(\lambda)) - \mathcal{E}_{\bar{n}}(\varphi(\lambda) - a) = -K(\lambda, a) - \theta L(\lambda, a)
\]

with

\[
K(\lambda, a) = \sum_{j=1}^{m} a_{j} (a_{j} + 1) + 2 \sum_{1 \leq j < l \leq m} \nu_{jl} \lambda_{j} - \lambda_{l} + 2 \sum_{j=1}^{m} \nu_{jm+1} \lambda_{j}
\]

and

\[
L(a, \lambda) = 2 \sum_{1 \leq j < l \leq m} \nu_{jl}(l - j) + 2 \sum_{j=1}^{m} \nu_{jm+1}((\varphi(\lambda))_{m+1} - \frac{1}{2}a_{m+1} + m - j).
\]

Since \( a \preceq \varphi(\lambda), (\varphi(\lambda))_{m+1} - a_{m+1} \geq 0 \). It follows that both \( K(\lambda, a) \) and \( L(a, \lambda) \) are strictly positive integers. Clearly, this implies that also Condition (2) is sufficient for all partitions in \( H_{n} \cap H_{\bar{n}} \) to be \( \bar{m} \)-admissible. \( \square \)

7. Series expansions of the super Jack polynomials

A recursion relation closely related to (28) is in [HL07] solved by iteration. Rather than deriving a similar solution of (28), we concentrate in this section on the special case \( L_{\bar{n}} = D_{\bar{n}}^{n} \), corresponding to \( \alpha(x) = x^{2} \) and \( \beta(x) = 0 \). In particular, we exhibit the explicit solution to the recursion relation (28) for this special case and prove that the resulting eigenfunctions (27), up to a constant of proportionality,
Theorem 7.1. Let $\lambda \in H_\tilde{m} \cap H_\tilde{n}$ be an $\tilde{m}$-admissible partition. Then

$$b_\lambda SP_\lambda = f_{\varphi(\lambda)}^{\tilde{m}} + \sum_{s=1}^{\infty} 2^s (\theta - 1)^s \sum_{j_1 < l_1} (-\theta)^{1-q(j_1)} - q(l_1) \sum_{\nu_1=1}^{\infty} \nu_1$$

$$\times \cdots \times \sum_{j_s < l_s} (-\theta)^{1-q(j_s)} - q(l_s) \sum_{\nu_s=1}^{\infty} \nu_s$$

$$\times \prod_{r=1}^{s} \left( \mathcal{E}_\tilde{m}(\varphi(\lambda)) - \mathcal{E}_\tilde{m}(\varphi(\lambda) - \sum_{t=r}^{s} \nu_t E_{j_t,l_t}) \right)^{-1} f_{\varphi(\lambda)}^{\tilde{m}} - \sum_{r=1}^{s} \nu_r E_{j_r,l_r}$$

with

$$E_{jl} = e_l - e_j.$$

Proof. Setting $\alpha_2 = 1$, and the remaining coefficients $\alpha_k$ and $\beta_l$ to zero, in (27) we find that $f_{\varphi(\lambda)}^{\tilde{m}}$ is an eigenfunction of the differential operator $D_a^2$ if the coefficients $u_\lambda(a)$ satisfy the recursion relation

$$\left( \mathcal{E}_\tilde{m}(\varphi(\lambda)) - \mathcal{E}_\tilde{m}(\varphi(\lambda)) \right) u_\lambda(a) = 2(\theta - 1) \sum_{j < l} (-\theta)^{1-q(j)} - q(l) \sum_{\nu=1}^{\infty} \nu u_\lambda(a + \nu(e_l - e_j)).$$

We set $u_\lambda(0) = 1$ and write $\delta_a(b)$ for the Kronecker delta of two integer vectors $a, b \in \mathbb{Z}^{\tilde{m}}$, i.e., $\delta_a(b)$ equals 1 if $a = b$ and zero otherwise. By iterating (35) we thus deduce that each coefficient

$$u_\lambda(a) = \delta_{\varphi(\lambda)}(a) + \sum_{s=1}^{\infty} 2^s (\theta - 1)^s \sum_{j_1 < l_1} \nu_1 (-\theta)^{1-q(j_1)} - q(l_1)$$

$$\times \sum_{j_2 < l_2} \nu_2 (-\theta)^{1-q(j_2)} - q(l_2) \times \cdots$$

$$\times \sum_{j_s < l_s} \nu_s (-\theta)^{1-q(j_s)} - q(l_s) \delta_{\varphi(\lambda)}(a + \sum_{r=1}^{s-1} \nu_r E_{j_r,l_r})$$

$$= \delta_{\varphi(\lambda)}(a) + \sum_{s=1}^{\infty} 2^s (\theta - 1)^s \sum_{j_1 < l_1} (-\theta)^{1-q(j_1)} - q(l_1) \sum_{\nu_1=1}^{\infty} \nu_1$$

$$\times \cdots \times \sum_{j_s < l_s} (-\theta)^{1-q(j_s)} - q(l_s) \sum_{\nu_s=1}^{\infty} \nu_s$$

$$\times \prod_{r=1}^{s} \left( \mathcal{E}_\tilde{m}(\varphi(\lambda)) - \mathcal{E}_\tilde{m}(\varphi(\lambda) - \sum_{t=r}^{s} \nu_t E_{j_t,l_t}) \right)^{-1} \delta_{\varphi(\lambda)}(a + \sum_{r=1}^{s} \nu_r E_{j_r,l_r}).$$
Inserting this into (27) we obtain the right-hand side of (34). It remains to show that the eigenfunction equals \( b_\lambda S_\lambda \). It follows from Lemma 5.2 that the right-hand side of (34) is a linear combination of super Jack polynomials \( S_\mu \) with \( \varphi(\lambda) - \varphi(\mu) \in \mathcal{C}_0 \) and leading term \( b_\lambda S_\lambda \). Moreover, the assumption that \( \lambda \) is \( \bar{m} \)-admissible implies that all terms other than the leading one have eigenvalues different from \( E_{\bar{m}}(\varphi(\lambda)) \). Consequently, these terms must be zero. \( \square \)

Remark 7.1. It is important to note that Corollary 5.1 implies that the series expansion (30) of the super Jack polynomials only contains a finite number of non-zero terms and thus is well defined.

We proceed to further study a number of particularly simple special cases of the series expansion (30) of the super Jack polynomials. In particular, it is interesting to consider this series expansion for \( \theta = 1 \). In that case \( b_\lambda = 1 \), and the right-hand side of (30) contains only the leading term \( f_{\varphi(\lambda)}^{(\bar{m})} \). Setting \( \bar{m} = (m,0) \), and using Proposition 5.1 as well as case (2) of Proposition 6.2 we thus recover the following:

Corollary 7.1. For \( \theta = 1 \) and each partition \( \lambda \in H_{\bar{n}} \),

\[
S_\lambda(x, -\tilde{x}) = S_\lambda(x, \tilde{x}).
\]

In addition, allowing arbitrary integer vectors \( \bar{m} \) we obtain the following generalisation of Proposition 5.1

Corollary 7.2. For \( \theta = 1 \) and each \( \bar{m} \)-admissible partition \( \lambda \in H_{\bar{n}} \cap H_{\bar{m}} \),

\[
f_{\varphi(\lambda)}^{(\bar{m})}(x, -\tilde{x}) = S_\lambda(x, \tilde{x}).
\]

We note that, in general, the series expansion (30) contains not only polynomials \( f_a^{(\bar{m})} \) parametrised by an integer vector of the form \( a = \varphi(\lambda) \) for some partition \( \lambda \in H_{\bar{n}} \cap H_{\bar{m}} \), i.e., the sum is not only over the polynomials singled out in Proposition 5.1 for providing a linear basis for a natural subspace of \( \Lambda_{\bar{n}, \theta} \). However, if \( |\bar{m}| \leq 2 \) it follows from Corollary 5.1 that the sum in fact run only over precisely the \( f_a^{(\bar{m})} \) corresponding to a partition in \( H_{\bar{n}} \cap H_{\bar{m}} \) under the map \( \varphi \). In addition, for \( |\bar{m}| \leq 2 \) the polynomials \( f_a^{(\bar{m})} \) themselves are particularly simple. More precisely, for \( |\bar{m}| = 1 \) we have the following:

Proposition 7.1. Suppose that \( |\bar{m}| = 1 \). Let \( \lambda \in H_{\bar{n}} \cap H_{\bar{m}} \). Then

\[
b_\lambda S_\lambda = f_{\varphi(\lambda)}^{(\bar{m})}.
\]

Moreover, for each (positive) integer \( a \),

\[
\begin{align*}
f_{(1,0)}^{(\bar{m})}(x, \tilde{x}; \theta) &= \sum_{r=0}^{a} (-1)^r e_r(\tilde{x}) g_{a-r}(x; \theta), \\
f_{(0,1)}^{(\bar{m})}(x, \tilde{x}; \theta) &= \sum_{r=0}^{a} (-1)^r e_r(x) g_{a-r}(\tilde{x}; 1/\theta).
\end{align*}
\]

Proof. We first observe that \( \mathcal{C}_{\bar{m}} \) contains only the zero vector. Consequently, all partitions \( \lambda \in H_{\bar{m}} \) are trivially \( \bar{m} \)-admissible. The first part of the statement is thus immediate from Theorem 7.1. To prove the second part of the statement we first note that, by definition,

\[
\prod_i (1 - \tilde{x}_i y_i) = \sum_{a \geq 0} f_{(1,0)}^{(a)}(x, \tilde{x}; \theta) y_1^a.
\]
On the other hand,
\[
\prod_i (1 - \bar{x}_i \bar{y}_1) = \sum_{r \geq 0} (-1)^r e_r(\bar{x}) \bar{y}_1 \sum_{s \geq 0} g_s(x; \theta) \bar{y}_1^s.
\]

The formula for the polynomials \( f_r^{(0,0)} \) is now obtained by comparing coefficients in the two expansions above. The formula for the polynomials \( f_r^{(0,1)} \) can be verified in a similar manner. \( \square \)

Furthermore, for \(|\bar{m}| = 2\) the following statement holds true:

**Proposition 7.2.** Suppose that \(|\bar{m}| = 2\). Let \( \lambda \in H_{\bar{m}} \cap H_{\bar{m}} \) be an \( \bar{m} \)-admissible partition. Then
\[
b_{\lambda} S P_{\lambda} = f_{m_{\bar{m}}}^{(r)} + \sum_{\nu} 2^{\ell(\nu)} (\theta - 1)^{\ell(\nu)} (-\theta)^{\frac{1}{2}(m_{\bar{m}} - \bar{m})} \nu_1 \cdots \nu_{\ell(\nu)}
\begin{align*}
& \times \prod_{r=1}^{\ell(\nu)} ( E_{m_{\bar{m}}} (\varphi(\lambda)) - E_{m_{\bar{m}}} (\varphi(\lambda) - |\nu|_{r_1}) )^{-1} f_{m_{\bar{m}}}^{(\bar{m})} \\
& \text{where the sum is over all positive integer vectors } \nu = (\nu_1, \ldots, \nu_{\ell(\nu)}) \text{ such that } |\nu| \leq (\varphi(\lambda))_2, \text{ and where we have used the notation}
\end{align*}
\[
|\nu|_r = \nu_r + \cdots + \nu_s
\]

for all positive integers \( r \leq \ell(\nu) \). Moreover, for each (positive) integer vector \( a = (a_1, a_2) \),
\[
f_{a_{\bar{m}}}^{(\bar{m})} = \sum_{t=0}^{a_2} \binom{-\theta}{t} \frac{1}{2}(m_{\bar{m}} - \bar{m}) p_{(a_1+t,a_2-t)}
\]

with
\[
\begin{align*}
p_{a}^{(2,0)} &= \sum_{r=0}^{a_1} (-1)^r e_r(\bar{x}) g_{a_1-r}(x; \theta) \sum_{s=0}^{a_2} (-1)^s e_s(\bar{x}) g_{a_2-s}(x; \theta), \\
p_{a}^{(1,1)} &= \sum_{r=0}^{a_1} (-1)^r e_r(\bar{x}) g_{a_1-r}(x; \theta) \sum_{s=0}^{a_2} (-1)^s e_s(x) g_{a_2-s}(x; 1/\theta), \\
p_{a}^{(0,2)} &= \sum_{r=0}^{a_1} (-1)^r e_r(x) g_{a_1-r}(\bar{x}; 1/\theta) \sum_{s=0}^{a_2} (-1)^s e_s(x) g_{a_2-s}(\bar{x}; 1/\theta).
\end{align*}
\]

**Proof.** The first part of the statement is just a reformulation of Theorem [11.1]. To prove the second part of the statement we first consider the case \( \bar{m} = (2, 0) \). The corresponding polynomials \( f_{a}^{(\bar{m})} \) are then defined by the expansion
\[
(1 - y_2/y_1)^{-\theta} \prod_i (1 - \bar{x}_i y_1)(1 - \bar{x}_i y_2) = \sum_a f_{a}(x, \bar{x}) y^a.
\]

On the other hand,
\[
\prod_i (1 - \bar{x}_i y_1)(1 - \bar{x}_i y_2) = \sum_{r \geq 0} (-1)^r e_r(\bar{x}) y_1^r \sum_{s \geq 0} (-1)^s e_s(\bar{x}) y_1^s \sum_{p \geq 0} g_p(x; \theta) y_1^p \sum_{q \geq 0} g_q(x; \theta) y_2^q.
\]

The formula for the polynomials \( f^{(2,0)}_a \) is now obtained by expanding the factor \((1 - y_2/y_1)^{-\theta}\) and comparing coefficients in the two expansions above. The two remaining cases \( \vec{m} = (1,1) \) and \((0,2)\) are proved similarly.

Finally, we note that the expansions just obtained are non-trivial and interesting already in the special case \( \theta = 1 \). For example, from Corollary 7.2 and Proposition 7.2 we deduce the following expansion for the super Schur polynomials labeled by hook partitions:

\[
S_{(a_1,1+a_2)}(x,\tilde{x}) = \sum_{t=0}^{a_2} (-1)^{a_2-t} \left( \sum_{r=0}^{a_1+t} e_r(x) h_{a_1+t-r}(x) \sum_{s=0}^{a_2-t} e_s(x) h_{a_2-t-s}(\tilde{x}) \right),
\]

where we have used the fact that both the elementary symmetric polynomials \( e_r \), as well as the complete symmetric polynomials \( h_r \), are homogeneous of degree \( r \).

For the 'ordinary' Schur polynomials, i.e., for \( \tilde{x} = 0 \) this reduces to the well known formula

\[
s_{(a_1,1+a_2)}(x) = \sum_{t=0}^{a_2} (-1)^{a_2-t} e_{a_2-t}(x) h_{a_1+t}(x);
\]

see e.g. Example 9 in Section I.3 of Macdonald [Mac95].

8. Concluding remarks

We have in this paper studied the polynomial eigenfunctions of the deformed CMS operators (1) in terms of series expansion in the polynomials \( f^{(\vec{m})}_a \). In particular, we have obtained (under a certain condition of non-degeneracy on their eigenvalues) a linear basis for these eigenfunctions. In addition, we have demonstrated in the special case of the super Jack polynomials that these series expansions in the polynomials \( f^{(\vec{m})}_a \) are rather explicit. In conclusion, we briefly discuss two related papers and some remaining problems.

As mentioned in the introduction, the present paper is closely related to a recent paper by Langmann and the author [HL07]. In this latter paper, eigenfunctions of both 'ordinary' CMS- as well as deformed CMS operators are obtained from the point of view of quantum many-body systems of Calogero-Sutherland type. Many of the results obtained here are results of questions raised in this paper. We also mention a recent paper by Langmann [Lan07], which offers an alternative interpretation of his original construction of the polynomial eigenfunctions of the operator \( L_n \) corresponding to the so-called Sutherland model.

In Section 6 we showed that the eigenfunctions \( P^{(\vec{m})}_\lambda \) can be normalised such that they are independent of the specific value of \( \vec{m} \). For the polynomials \( f^{(\vec{m})}_a \) the situation is more complicated. In fact, it is largely an open problem to characterise their dependence on \( \vec{m} \). However, certain properties can be inferred from the results of the present paper. For example, it follows from Lemma 5.2 that the polynomials \( f^{(\vec{m})}_{\vec{\varphi}(\lambda)} \), corresponding to a given partition \( \lambda \), have the same leading term when expanded in super Jack polynomials. Also, it is easily inferred from Definition 5.1 that with a fixed integer vector \( a = (a_1,\ldots,a_{|\vec{m}|}) \) the corresponding polynomial \( f^{(\vec{m})}_a \) is invariant under the replacement of \( \vec{m} = (m,\tilde{m}) \) by \((m,\tilde{m}+N)\) for any positive integer \( N \). We note, however, that it is easily verified in special cases that the polynomials \( f^{\vec{m}}_a \) can be distinctly different for different values of \( \vec{m} \); c.f. the discussion following Proposition 6.1.
As noted in the discussion preceding Proposition 7.1, we have in many instances worked with an 'overcomplete' set of polynomials $f^{(m)}_a$. It would be desirable to be able to rewrite the corresponding expression such that the only involve a set of linearly independent polynomials $f^{(m)}_a$, e.g. those singled out in Proposition 5.2. In fact, already Lemma 5.2 provides enough information to obtain the structure of the expansion of an arbitrary polynomial $f^{(m)}_a$ in these latter polynomials. However, it provides very little insight into the explicit form of the coefficients in such an expansion.

Finally, we observe that in many statements in Sections 6 and 7 it is required that a certain partition be $\bar{m}$-admissible. This condition encodes the level of non-degeneracy required of the eigenvalues $\xi(\lambda)$. In Proposition 6.2 we proved two sufficient conditions for all partitions in $H_{\bar{n}} \cap H_{\bar{m}}$ to be $\bar{m}$-admissible. In particular, for $\alpha_2 = 0$ this is always the case. On the other hand, for $\alpha_2 \neq 0$ a number of special cases remain to be investigated. An important such case, corresponding to the super Jack polynomials, is that for which only $\alpha_2$ is non-zero and $\bar{m}$ is arbitrary.

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References

[AAR99] G.E. Andrews, R. Askey and R. Roy, Special functions, Cambridge university press, 1999.

[BF97] T.H. Baker and P.J. Forrester, The Calogero-Sutherland model and generalized classical polynomials, Commun. Math. Phys. 188 (1997), 175–216.

[Boc29] S. Bochner, Über Sturm-Liouvilleche Polynomsysteme, Math. Z. 29 (1929), 730–736.

[Cal71] F. Calogero, Solution of the one-dimensional N-body problems with quadratic and/or inversely quadratic pair potentials, J. Math. Phys. 12 (1971), no. 3, 419–436.

[CFV98] O.A. Chalykh, M.V. Feigin, and A.P. Veselov, New integrable generalizations of Calogero-Moser quantum problem, J. Math. Phys. 39 (1998), 695–703.

[DX01] C.F. Dunkl and Y. Xu, Orthogonal polynomials of several variables, Encyclopedia of mathematics and its applications, vol. 81, Cambridge university press, 2001.

[FP95] W. Fulton and P. Pragacz, Schubert varieties and degeneracy loci, Lecture notes in mathematics, vol. 1689, Springer-Verlag, 1995.

[Gau92] M. Gaudin, Conjugaison $\lambda \leftrightarrow \lambda^{-1}$ de l’hamiltonien de Calogero-Sutherland, Sacley Preprint SPht/92-158, 1992.

[Gro78] E. Grosswald, Bessel polynomials, Lecture notes in mathematics, vol. 698, Springer-Verlag, 1978.

[Hal07] M. Hallnäs, An explicit formula for symmetric polynomials related to the eigenfunctions of Calogero-Sutherland models, SIGMA 3 (2007), 037, 17 pages.

[HL07] M. Hallnäs and E. Langmann, Quantum Calogero-Sutherland type models and generalised classical polynomials, arXiv [math-ph/0703090] 2007.

[HO87] G.J. Heckman and E.M. Opdam, Root systems and hypergeometric functions I, Compositio Math. 64 (1987), 329–352.

[Jac41] C.G. Jacobi, De functionibus alternantibus..., Crelle’s J. 22 (1841), 360–371.

[KOO98] S. Kerov, A. Okounkov, and G. Olshanski, The boundary of the Young graph with Jack edge multiplicities, Internat. Math. Res. Notices 4 (1998), 173–199.

[Lan01] E. Langmann, Algorithms to solve the Sutherland model, J. Math. Phys. 41 (2001), 4148–4158.

[Lan06] E. Langmann, A method to derive explicit formulas for an elliptic generalization of the Jack polynomials, Jack, Hall-Littlewood and Macdonald polynomials (V.B. Kuznetsov and
S. Sahi, eds.), Contemporary mathematics, vol. 417, American mathematical society, 2006.

[Lan07] Singular eigenfunctions of Calogero-Sutherland type systems and how to transform them into regular ones, SIGMA (2007), 031, 18 pages.

[Las91b] Polynômes de Jacobi généralisés, C. R. Acad. Sci. Paris, Série I 312 (1991), 425–428.

[Las91c] Polynômes de Laguerre généralisés, C. R. Acad. Sci. Paris, Série I 312 (1991), 725–728.

[Las91a] M. Lassalle, Polynômes de Hermite généralisés, C. R. Acad. Sci. Paris, Série I 313 (1991), 579–582.

[Mac] I.G. Macdonald, Hypergeometric functions, unpublished manuscript.

[Mac95] I.G. Macdonald, Symmetric functions and Hall polynomials, second ed., Oxford university press, 1995.

[PT92] P. Pragacz and A. Thorup, On a Jacobi-Trudi identity for supersymmetric polynomials, Adv. Math. 95 (1992), 8–17.

[Ser97] D. Serban, Some properties of the Calogero-Sutherland model with reflections, J. Phys. A: Math. Gen. 30 (1997), 4215–4225.

[Ser96] V. Serganova, On generalizations of root systems, Comm. Algebra, 24 (1996), 4281–4299.

[Serg01] A.N. Sergeev, Supernanals of the Calogero operators and Jack polynomials, J. Nonlinear Math. Phys. 8 (2001), 59–64.

[Serg02] Calogero operator and Lie superalgebras, Theor. Math. Phys. 131 (2002), 747–764.

[SV04] A.N. Sergeev and A.P. Veselov, Deformed quantum Calogero-Moser problems and Lie superalgebras, Commun. Math. Phys. 245 (2004), 249–278.

[SV05] Generalised discriminants, Calogero-Moser-Sutherland operators and super-Jack polynomials, Adv. Math. 192 (2005), 341–375.

[Sta89] R.P. Stanley, Some combinatorial properties of Jack symmetric functions, Adv. Math. 77 (1989), 76–115.

[Sut72] B. Sutherland, Exact result for a quantum many-body problem in one dimension. II, Phys. Rev. A 5 (1972), no. 3, 1372–1376.

[vD97] J.F. van Diejen, Confluent hypergeometric orthogonal polynomials related to the rational quantum Calogero system with harmonic confinement, Commun. Math. Phys. 188 (1997), 467–497.

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