Massive Dual Spin 2 Revisited

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Abstract

We reconsider a massive dual spin 2 field theory in four spacetime dimensions. We obtain the Lagrangian that describes the lowest order coupling of the field to the four-dimensional curl of its own energy-momentum tensor. We then find some static solutions for the dual field produced by other energy-momentum sources and we compare these to similar static solutions for non-dual “finite range” gravity. Finally, through use of a nonlinear field redefinition, we show the theory is the exact dual of the Ogievetsky-Polubarinov model for a massive spin 2 field.

In tribute to Peter George Oliver Freund (1936-2018)

1 Introduction

We reconsider research first pursued long ago in collaboration with Peter Freund at the University of Chicago [1], in a continuing effort to tie up some loose ends. Inde est, quod.

In the winter of 1979-1980, while thinking about strings as a post-doctoral fellow in Yoichiro Nambu’s theory group at The Enrico Fermi Institute, one of us (TLC) proposed gauge theories for massless fields that were neither totally symmetric nor totally antisymmetric Lorentz representations, and also discussed the effects of adding mass terms for such fields [2]. Peter found this to be very interesting and worth further consideration, perhaps because of his previous work on massive scalars [3] and finite range gravity [4].

The simplest example is $T_{\lambda\mu\nu}$ with permutation symmetries $T_{\lambda\mu\nu} = -T_{\mu\lambda\nu}$ and $T_{\lambda\mu\nu} + T_{\mu\nu\lambda} + T_{\nu\lambda\mu} = 0$. While the theory has remarkable differences between the massless and massive cases in any number of spacetime dimensions, in 4D Minkowski space the model has a particularly striking mass discontinuity, with no propagating modes when massless but with five modes corresponding to angular momentum $J = 2\hbar$ when massive [2].

A consistent field equation for the massive 4D model was subsequently proposed in [1], namely,

$$(\Box + m^2) T_{\lambda\mu\nu} = \kappa \left( 2\varepsilon_{\lambda\mu\alpha\beta} \partial^\alpha \Theta^\beta_{\nu} + \varepsilon_{\nu\mu\alpha\beta} \partial^\alpha \Theta^\beta_{\lambda} - \varepsilon_{\nu\lambda\alpha\beta} \partial^\alpha \Theta^\beta_{\mu} \right), \tag{1}$$

where $\Theta_{\mu\nu}$ is any conserved, symmetric tensor, e.g. the energy-momentum tensor, and $\kappa$ is a dimensionful parameter with units $1/m^2$, most naturally related to Newton’s constant and the Hubble length. This field equation implies that the trace and all divergences of $T_{\lambda\mu\nu}$ decouple, i.e. they are free fields, and therefore they may be consistently set to zero leaving a model with unadulterated spin 2 particles of mass $m$.

That said, [1] stopped short and did not provide a Lagrangian that led to the field equation (1) for the case where $\Theta_{\mu\nu}$ depends only on $T_{\lambda\mu\nu}$ itself. This shortcoming is remedied here, albeit only to lowest order in $\kappa$. However, there seem to be no fundamental principles to prevent extension of the result to all orders in $\kappa$, thereby emulating the theory of a self-coupled dual scalar as discussed in a companion paper [5].
Properties of the massless free field theory

There are three increasingly compact forms for the massless $T_{[\lambda\mu][\nu]}$ free field Lagrangian density, involving three terms, two terms, and one term, respectively.

$$\mathcal{L} = -\frac{1}{4} \left( R_{[\lambda\mu][\nu\rho]} R^{[\lambda\nu][\rho\mu]} - 4 R_{\lambda\nu} R^{\lambda\nu} + R^2 \right)$$

$$= -\frac{1}{6} \left( F_{[\lambda\mu\nu]\rho} F^{[\lambda\nu\mu]\rho} - 3 F_{[\mu\nu]} F^{[\mu\nu]} \right)$$

$$= +\frac{1}{36} K_{\mu\nu} K^{\nu\mu} \quad .$$

(2)

The first two forms are legitimate for any number of spacetime dimensions \[^2\] but the last is specific to 4D. This is evident in the flat spacetime definitions

$$R_{[\lambda\mu][\nu\rho]} = \partial_\rho T_{[\lambda\mu][\nu]} - \partial_\nu T_{[\lambda\mu][\rho]} \quad ,$$

$$R_{\lambda\nu} = g^{\mu\rho} R_{[\lambda\mu][\nu\rho]} \quad ,$$

$$R = g^{\lambda\nu} R_{\lambda\nu} \quad .$$

(3)

and finally, for flat 4D spacetime \[^3\]

$$K_{\mu}^\nu = F_{[\alpha\beta\gamma][\mu]} \varepsilon^{\alpha\beta\gamma\nu} = 3 (\partial_\alpha T_{[\beta\gamma][\mu]}) \varepsilon^{\alpha\beta\gamma\nu} \quad .$$

(5)

This last definition immediately leads to the kinematic identities

$$K_{\mu}^\nu = 0 \quad , \quad \partial^\nu K_{\mu\nu} = 0 \quad ,$$

(6)

where the first of these is a consequence of $T_{[\lambda\mu][\nu]} + T_{[\mu\nu][\lambda]} + T_{[\nu\lambda][\mu]} = 0$.

Gauge invariance of the theory is most easily seen in 4D for the bilinear-in-$K$ form of $\mathcal{L}$. A gauge transformation on the $T_{[\lambda\mu][\nu]}$ field is \[^2\]

$$\delta T_{[\lambda\mu][\nu]} = \partial_\lambda S_{\mu\nu} - \partial_\mu S_{\lambda\nu} + \partial_\nu A_{\lambda\mu} - \partial_\lambda A_{\mu\nu} + 2 \partial_\nu A_{\mu\lambda} \quad ,$$

(7)

where $S_{\mu\nu}$ is any local, differentiable, symmetric tensor field, and $A_{\mu\nu}$ is any local, differentiable, antisymmetric tensor field. This in turn leads to the gauge transformation of the $F_{[\alpha\beta\gamma][\mu]}$ field strength \[^2\]

$$\delta F_{[\alpha\beta\gamma][\mu]} = -2 \partial_\mu (\partial_\alpha A_{\beta\gamma} + \partial_\beta A_{\gamma\alpha} + \partial_\gamma A_{\alpha\beta}) \quad ,$$

(8)

and thence to the gauge transformation of $K_{\mu\nu}$, but only in 4D,

$$\delta K_{\mu\nu} = \partial_\mu \omega_\nu \quad ,$$

(9)

where $\omega_\nu = -6 \left( \partial_\alpha A_{\beta\gamma} \right) \varepsilon_{\alpha\beta\gamma\nu}$ is a differentiable, divergenceless pseudo-vector, but otherwise arbitrary. This form of $\delta K_{\mu\nu}$ manifestly maintains the trace and divergence conditions for $K_{\mu\nu}$ as given in (6).

Gauge invariance of the action in 4D, modulo surface contributions, then follows immediately from the divergence condition in (6). For any 4-volume $V_4$,

$$\delta \int_{V_4} K_{\mu\nu} K^{\nu\mu} d^4 x = 2 \int_{V_4} K_{\mu\nu} \delta K^{\nu\mu} d^4 x = 2 \int_{V_4} K_{\mu\nu} \partial^\nu \omega^\mu d^4 x = 2 \int_{\partial V_4} \omega^\mu K_{\mu\nu} n^\nu d^3 x \quad ,$$

(10)

where $n^\nu$ is the local normal vector on the boundary $\partial V_4$. For gauge transformations that vanish on $\partial V_4$ the action is therefore invariant.

\[^3\]Moreover, in 4D spacetime

$$R_{[\lambda\mu][\nu\rho]} R^{[\lambda\mu][\nu\rho]} - 4 R_{\lambda\nu} R^{\lambda\nu} + R^2 = -\frac{1}{4} \varepsilon^{\lambda\mu\nu\delta} \varepsilon_{\alpha\beta\gamma\rho} R_{[\lambda\mu][\nu\rho]} R_{[\alpha\beta][\gamma\delta]}$$

If $R_{[\lambda\mu][\nu\rho]}$ were the Riemann curvature, such that $R_{[\lambda\mu][\nu\rho]} = R_{[\nu\rho][\lambda\mu]}$, this would be the Euler density. But as defined here, $R_{[\lambda\mu][\nu\rho]} \neq R_{[\nu\rho][\lambda\mu]}$, so $\mathcal{L}$ is not a total divergence and its action is not just topological.
There are no physical states for the massless free field theory in 4D

The bilinear-in-\(K\) form for the action is the most transparent way to see the only solutions of the free field equations in 4D are just gauge transformations. With \(K_{\mu \nu}\) defined by \([5]\), the bulk field equations that follow from \(\int K_{\mu \nu} K^{\nu \rho} d^4x\) are simply

\[
\epsilon_{\alpha \beta \lambda \mu} \partial^\lambda K^{\mu \nu} = 0 ,
\]

along with the conditions \([5]\). That is to say,

\[
\partial^\lambda K^{\mu \nu} - \partial^\mu K^{\lambda \nu} = 0 ,
\]

and therefore \(K_{\mu \nu}\) must have the form \([9]\). So on-shell \(K_{\mu \nu}\) is just a gauge transformation. Thus there are no physical degrees of freedom for the massless, free \(T_{[\lambda \mu] \nu}\) field in 4D.

Moreover, on-shell conservation of energy-momentum for the flat space 4D theory is as easy as one might expect for fields which are local gauge transformations. Consider

\[
\partial^\nu K_{\mu \beta} = K_{\nu \beta} - \frac{1}{2} \delta^\nu_\mu K_{\alpha \beta}^\alpha .
\]

On-shell in 4D \(K_{\alpha \beta} = \partial_\alpha \omega_\beta\) with \(\partial^\mu \omega_\mu = 0\), hence

\[
\partial^\nu (K_{\beta \nu} K^{\beta \nu}) = (\partial_\nu \partial_\mu \omega_\beta) \partial^\nu \omega_\beta = (\partial_\mu \partial_\nu \omega_\beta) \partial^\beta \omega_\nu = \frac{1}{2} \partial_\mu (K_{\alpha \beta} K^{\beta \alpha}) ,
\]

and therefore the tensor \([13]\) is conserved on-shell, \(\partial_\nu \partial^\nu = 0\). Indeed, \(\partial^\nu \omega_\nu\) encodes the physics of the massless \(T_{[\lambda \mu] \nu}\) model in 4D because both the energy and spatial momentum densities are given on-shell by spatial derivatives, namely, \(\partial_\nu = \vec{\nabla} \cdot \left( - \frac{1}{2} \left( \vec{\omega} \cdot \vec{\omega} \right) \right)\) and \(\partial_\nu = \partial_\nu (\omega_\nu \partial_\nu \omega) + \vec{\nabla} \cdot \left( \vec{\omega} \partial_\nu \vec{\omega} \right)\). Consequently, fields that vanish sufficiently rapidly as \(r \rightarrow \infty\) (i.e. \(\lim_{r \rightarrow \infty} r \omega^\alpha = 0\) carry neither net energy nor net momentum, as would be expected for configurations that are just gauge transformations.

Also, by definition the tensor \(\partial^\mu \omega_\nu\) is not manifestly symmetric because in general \(K_{\mu \nu} \neq K_{\nu \mu}\). Rather,

\[
K_{\mu \nu} = K_{\nu \mu} + 3 \epsilon_{\mu \nu \alpha \beta} F^{[\alpha \beta \lambda]} ,
\]

But this is remedied for the massless theory by imposing the Lorentz covariant gauge condition

\[
F^{[\alpha \beta \lambda]} = \partial^\lambda T^{[\alpha \beta]} - \partial^\nu T^{[\beta \lambda]} - \partial^\beta T^{[\alpha \lambda]} = 0 ,
\]

a condition preserved by gauge transformations so long as \(\omega_\alpha = \partial_\alpha \psi\) with \(\square \psi = 0\). In this class of gauges, both \(K_{\mu \nu} = K_{\nu \mu}\) and \(\partial^\mu \omega_\nu = \partial^\nu \omega_\mu\).

Moreover, in this class of gauges there are some obvious similarities with Galileon theory \([2]\) in that the \(K\)-tensor is a Hessian matrix,

\[
K_{\mu \nu} = \partial_\mu \partial_\nu \psi ,
\]

albeit traceless. This may then be combined with the expansion of a determinant in terms of traces for general \(4 \times 4\) traceless matrices,

\[
\det \left( \delta^\beta_\alpha + \kappa K^\beta_\alpha \right) = 1 - \frac{1}{2} \kappa^2 K^{\mu}_\mu K^\lambda_\lambda + \frac{1}{3} \kappa^3 K^{\mu}_\lambda K^{\nu}_\mu K^\lambda_\nu + \frac{1}{8} \kappa^4 \left( (K^{\mu}_\mu K^\lambda_\lambda)^2 - 2 K^{\mu}_\lambda K^{\nu}_\mu K^{\lambda}_\nu K^{\lambda}_\rho \right) ,
\]

an expansion that is valid in any gauge.

The discussion given above, concerning conservation of the energy-momentum tensor for the massless free field theory, does not apply to the massive theory that we shall discuss next. For the massive free field theory, the gauge invariance of the model is broken by explicit mass terms. In that case, energy and momentum are still conserved, of course, but a more detailed proof is required to demonstrate this fact.
Properties of the massive free field theory

Consider the massive $T_{[\lambda\mu][\nu]}$ theory. Recall the free-field Lagrangian density \[ L = -\frac{1}{6} \left( F_{[\lambda\mu][\nu]} F^{[\lambda\mu][\nu]} - 3 F_{[\mu\nu]} F^{[\mu\nu]} \right) + \frac{1}{2} m^2 \left( T_{[\lambda\mu][\nu]} T^{[\lambda\mu][\nu]} - 2 T_{\lambda} T^{\lambda} \right), \] where the trace of the field is $T_{\lambda} = g_{\mu\nu} T^{[\lambda\mu][\nu]}$, and the field strength and its trace are as defined in (4). Once again, it is slightly more compact to write $L$ using $K^{\nu\mu}$ as defined in (5), to obtain for the Lorentz metric case
\[ \frac{1}{6} K_{\alpha\beta} K^{\beta\alpha} = -F_{[\alpha\beta][\gamma]} F^{[\alpha\beta][\gamma]} + 3 F_{[\alpha\beta]} F^{[\alpha\beta]} . \] The field equations that follow from varying the action for $L$ are the usual Klein-Gordon equation for $T_{[\lambda\mu][\nu]}$, i.e. the on-shell condition
\[ (\square + m^2) T_{[\lambda\mu][\nu]} = 0 , \] along with the secondary Fierz-Pauli conditions
\[ T^{[\mu\nu]} = 0 , \quad \partial_{\lambda} T_{[\lambda\mu][\nu]} = 0 , \quad \partial_{\nu} T^{[\mu\nu]} = 0 . \] We shall call these last three equations the massive half-shell conditions, and we designate relations that hold subject to one or more of these conditions by the half-shell symbol "\[ \approx \]". Similarly, we designate relations that hold subject to the Klein-Gordon relation (21) as well as one or more of (22a,22b,22c) by the fully on-shell (i.e. full-shell) symbol "\[ \equiv \]".

So, as a consequence of (22a,22b,22c) we have
\[ F_{[\alpha\beta][\gamma]} \approx 0 , \quad K_{\mu\nu} \approx K_{\nu\mu} , \quad \partial^{\mu} K_{\mu\nu} \approx 0 . \] Energy-momentum tensors

Consider
\[ \theta^{\mu\nu} = K_{\mu\alpha} K^{\alpha\nu} - 36 m^2 T_{[\mu\beta][\nu\gamma]} T^{[\mu\beta][\nu\gamma]} - \delta^{\mu\nu} \left( \frac{1}{2} K_{\alpha\beta} K^{\beta\alpha} - 9 m^2 T_{[\alpha\beta][\gamma]} T^{[\alpha\beta][\gamma]} \right) . \] Note that on-shell,
\[ \theta_{\mu\nu} \approx \theta_{\nu\mu} , \quad \partial^{\mu} \theta_{\mu\nu} \approx 0 , \] where the latter conservation is established in detail in an Appendix. This energy-momentum tensor has the interesting feature that the $m^4$ term drops out of the 4D trace.

\[ \theta^{\mu\nu} = -K_{\alpha\beta} K^{\beta\alpha} . \] Also, by direct calculation, any $\delta^{\mu\nu} (\cdots)$ terms in $\theta_{\mu\nu}$ do not contribute in the field equation (11). Moreover, any such $\delta^{\mu\nu} (\cdots)$ terms would contribute nothing to an interaction written as $K_{\mu\nu} \theta^{\rho\mu}$ because $K_{\mu\nu}$ is traceless. Similarly, any conformal improvement to $\theta_{\mu\nu}$ of the form $(\partial_{\mu} \partial_{\nu} - \eta_{\mu\nu} \square) (\cdots)$ will contribute nothing to either the field equation (11) or to the bulk action obtained by integrating $K_{\mu\nu} \theta^{\rho\mu}$ because of the vanishing trace and divergence kinematic conditions on $K_{\mu\nu}$. But note that, in principle, boundary contributions to the action are possible from coupling to a conformal improvement term of this form following an obvious integration by parts.
Interaction Lagrangian to lowest order

The field equation may be written more compactly as

\[ (\Box + m^2) T_{\alpha\beta}^\nu = \kappa P_{\lambda\mu\nu,\alpha\beta} \partial^\alpha \Theta^\beta \gamma \]

where we have defined a symmetrizing tensor

\[ P_{\lambda\mu\nu,\alpha\beta} = 2\varepsilon_{\lambda\mu\alpha} \eta_{\nu\beta} + \varepsilon_{\mu\nu\alpha} \eta_{\lambda\beta} - \varepsilon_{\nu\lambda\alpha} \eta_{\mu\beta} . \]

Note that any \( \delta \gamma \) (\( \cdots \)) or \( \delta \beta \) (\( \cdots \)) terms in \( \Theta^\gamma \beta \) will give no contribution to the field equation because \( \varepsilon^{\lambda\mu\nu} \kappa T_{[\alpha\beta]} = 0 \) and \( \varepsilon^{\lambda\mu\alpha} \partial_{\gamma} \delta_{\beta} \) (\( \cdots \)) = 0.

Ignoring terms of \( O(\kappa^2) \), to obtain the field equations to \( O(\kappa) \) the previous massive field energy-momentum tensor must be augmented by adding a manifestly conserved, symmetric “improvement”:

\[ \Theta^\gamma \beta = \theta^\gamma \beta - 36 \kappa T_{[\gamma b]}^\nu \partial^\alpha \left( \Box + m^2 \right) \left( T_{[\beta c]}^\nu T_{[b c]}^\nu \right) - \delta^\gamma \beta \partial^\alpha \left( T_{[\gamma b]}^\nu T_{[b c]}^\nu \right) \]

A Lagrangian which gives the field equation to \( O(\kappa) \) is then obtained by adding to the free field massive Lagrangian \( \mathcal{L}_{\text{int}}(\tau) \) \( O(\kappa) \) interactions suggested by the form \( K^\alpha \beta \Theta^\gamma _\beta \), namely,

\[ \mathcal{L}_{\text{int}} = -\frac{1}{3} \kappa K^\alpha _\beta K^\gamma _\beta K^\alpha (36\kappa T_{[\gamma b]}^\nu \partial^\alpha \left( \Box + m^2 \right) \left( T_{[\beta c]}^\nu T_{[b c]}^\nu \right) - \delta^\gamma \beta \partial^\alpha \left( T_{[\gamma b]}^\nu T_{[b c]}^\nu \right)) \]

\[ = -\frac{1}{3} \kappa K^\alpha _\beta K^\gamma _\beta K^\alpha - 108\kappa \epsilon^{\lambda\mu\alpha\beta} \partial^\alpha \left( \Box + m^2 \right) \left( T_{[\gamma b]}^\nu T_{[\beta c]}^\nu \right) - \delta^\gamma \beta \partial^\alpha \left( T_{[\gamma b]}^\nu T_{[b c]}^\nu \right)) . \]

The resulting action due to \( \mathcal{L}_{\text{int}} \) is of course

\[ \mathcal{A}_{\text{int}} = \int \mathcal{L}_{\text{int}} d^4 x . \]

So then, by varying \( T_{[\gamma b]}^\nu \) in \( \mathcal{A}_{\text{int}} \) the contributions to the field equations follow from

\[ \delta \mathcal{A}_{\text{int}} = -\kappa \int \left( \delta K^\alpha _\beta \right) K^\gamma _\beta K^\alpha (\Box + m^2) x + \delta \mathcal{T}_{[\gamma b]}^\nu \partial^\alpha \left( \Box + m^2 \right) \left( T_{[\beta c]}^\nu T_{[b c]}^\nu \right) \]

\[ - 108\kappa \epsilon^{\lambda\mu\alpha\beta} \partial^\alpha \left( \Box + m^2 \right) \left( T_{[\gamma b]}^\nu T_{[\beta c]}^\nu \right) - \delta^\gamma \beta \partial^\alpha \left( T_{[\gamma b]}^\nu T_{[b c]}^\nu \right) d^4 x . \]

However, upon integrating by parts the terms in the last line give no contributions to the bulk field equations at \( O(\kappa) \) because of the \( O(\kappa^0) \) on-shell conditions, \( 22 \) and \( 24 \). These terms might be important at \( O(\kappa^2) \), but they have no effect at \( O(\kappa) \).

Rewriting the \( K \) trilinear variation

\[ -\kappa \int \left( \delta K^\alpha _\beta \right) K^\gamma _\beta K^\alpha d^4 x = -3\kappa \int \partial_{\gamma} \delta T_{[\beta c]}^\nu \epsilon^{\alpha\beta\gamma} x + \kappa \int \delta \mathcal{T}_{[\gamma b]}^\nu \epsilon^{\lambda\mu\alpha\beta} \partial^\alpha \left( K^\gamma _\beta K^\nu \right) d^4 x , \]

the \( O(\kappa) \) variation of the interaction is therefore

\[ \delta \mathcal{A}_{\text{int}} = 3\kappa \int \left( \delta \mathcal{T}_{[\gamma b]}^\nu \right) \epsilon^{\lambda\mu\alpha\beta} \partial^\alpha \left( K^\gamma _\beta K^\nu \right) - 36 \left( \Box + m^2 \right) \left( T_{[\gamma b]}^\nu T_{[\beta c]}^\nu \right) \]

\[ \delta \mathcal{T}_{[\gamma b]}^\nu \epsilon^{\lambda\mu\alpha\beta} \partial^\alpha \left( T_{[\gamma b]}^\nu T_{[\beta c]}^\nu \right) d^4 x + O(\kappa^2) . \]

That is to say,

\[ \delta \mathcal{A}_{\text{int}} \epsilon^{\lambda\mu\alpha\beta} \partial^\alpha \Theta_{\beta}^\nu d^4 x + O(\kappa) . \]

This variation therefore gives precisely the RHS of the field equation \( 26 \) to lowest non-trivial order in \( \kappa \).

\[ \delta \mathcal{A}_{\text{int}} = \kappa \int \left( \delta \mathcal{T}_{[\gamma b]}^\nu \right) P_{\lambda\mu\nu,\alpha\beta} \partial^\alpha \Theta^\beta \gamma d^4 x + O(\kappa^2) . \]

This is the final result, \( \mathcal{A}_{\text{int}} \), which gives precisely the RHS of the field equation to lowest non-trivial order in \( \kappa \).
Other forms of the field equations and their static solutions

Given the proposed field equation for \( T_{\lambda\rho;\nu} \) the on-shell equation for the \( T \)-field strength \( K_{\mu\nu} \) is

\[
(\Box + m^2) K_{\mu\nu} = -18\kappa \Box \Theta_{\mu\nu} + 6\kappa (\eta_{\mu\nu} \Box - \partial_\mu \partial_\nu) \Theta ,
\]

(35)

where \( \Theta = \Theta^\lambda \). If \( \partial^\rho \Theta_{\mu\nu} = 0 \) and \( \Theta_{\mu\nu} = \Theta_{\nu\mu} \), the RHS of (35) is conserved, symmetric, and manifestly traceless in 4D, so the trace and divergences of \( K_{\mu\nu} \) are free fields and may be consistently set to zero. Moreover, while \( K_{\mu\nu} + K_{\nu\mu} \) couples to \( \Theta_{\mu\nu} = \Theta_{\nu\mu} \), the antisymmetric part \( K_{\mu\nu} - K_{\nu\mu} \) is also a free field and again may be consistently set to zero.

The field equation (35) is almost familiar. Were it not for the manifestly conserved trace term, \( (\partial_\mu \partial_\nu - \eta_{\mu\nu} \Box) \Theta \), an obvious inference from (35) would be that a more conventional form of massive gravity, such as that in [4], would be related to the on-shell dual theory just by the identification \( K_{\mu\nu} \propto \Box h_{\mu\nu} \), where

\[
(\Box + m^2) h_{\mu\nu} = \kappa \Theta_{\mu\nu} .
\]

(36)

The trace term invalidates this identification, in general. But there are situations where such an identification is essentially correct. This is especially true for static configurations.

In fact, it may be somewhat surprising that static energy-momentum sources do produce dual fields, given that the source on the RHS of (35) is a total divergence. This is perhaps more easily seen from (35). Static sources do indeed produce \( K_{00} \) fields. In that case,

\[
(\nabla^2 - m^2) K_{00} = -18\kappa \nabla^2 \Theta_{00} + 6\kappa \nabla^2 \Theta .
\]

(37)

For either traceless \( \Theta_{\mu\nu} \) or stress-free matter, this is equivalent to the static case of more conventional massive gravity, as given by (36), but with \( K_{00} \propto \nabla^2 h_{00} \).

For example, suppose the energy density is given by a static isotropic radial electric field around a small ball of charge, with either \( H_{00} \) or \( h_{00} \) due just to the electric field energy, ignoring any fields produced by the mass density of the ball. In that case \( \Theta = 0 \), classically, and it is not difficult to determine \( H_{00} \) and \( h_{00} \) fields outside the charged ball with homogeneous Dirichlet boundary conditions at spatial infinity. The only difference in the functional form of \( H_{00} \) and \( h_{00} \) for these exact solutions is an extra \( 1/r^4 \) term in \( H_{00} \). Explicitly, solving the equations

\[
-\frac{d^2}{dr^2} h + m^2 h = \frac{A}{r^3} , \quad -\frac{d^2}{dr^2} h + m^2 h = \frac{B}{r^3} ,
\]

(38)

for \( H (r) = rH_{00} (r) \) and \( h (r) = rh_{00} (r) \), where \( A \) and \( B \) are constants proportional to the source’s electric charge squared, leads to

\[
H_{00} (r) = C_1 \exp \left( \frac{-mr}{r} \right) - \frac{1}{24} \frac{Am^3}{r^3} - \frac{1}{12} \frac{A}{r^4} + \frac{1}{24} \frac{Am^3}{r} (\text{Shi} (mr) \cosh mr - \text{Chi} (mr) \sinh mr) ,
\]

(39)

\[
h_{00} (r) = C_2 \exp \left( \frac{-mr}{r} \right) - \frac{1}{2} \frac{B}{r^2} + \frac{1}{2} \frac{Bm}{r} (\text{Shi} (mr) \cosh mr - \text{Chi} (mr) \sinh mr) ,
\]

(40)

where the \( C \)'s are constants of integration determined by boundary conditions at the surface of the charged ball. The same special functions appear in both cases, namely,

\[
\text{Shi} (mr) = \int_0^{mr} \frac{\sinh (t)}{t} dt = mr + \frac{1}{18} m^3 r^3 + O (r^5) ,
\]

(41)

\[
\text{Chi} (mr) = \gamma + \ln (mr) + \int_0^{mr} \frac{\cosh (t) - 1}{t} dt = \gamma + \ln (mr) + \frac{1}{4} m^2 r^2 + \frac{1}{96} m^4 r^4 + O (r^5) .
\]

(42)

A discussion of the phenomenological differences between the two types of static fields for this example, and for other more realistic source terms, will be given elsewhere.
Relation to the Ogievetsky-Polubarinov model

The analysis of the previous section suggests that a field redefinition may provide additional insight for the dual theory. It does.

Adding and substracting $m^2 \Theta_{\mu\nu}$ to the RHS of (35) and moving $(\Box + m^2) \Theta_{\mu\nu}$ to the LHS gives

\[ (\Box + m^2) H_{\mu\nu} = \kappa \Theta_{\mu\nu} + \frac{\kappa}{3m^2} (\eta_{\mu\nu} \Box - \partial_{\mu} \partial_{\nu}) \Theta, \]  

(43)

where the field redefinition is simply

\[ H_{\mu\nu} = \frac{1}{18m^2} (K_{\mu\nu} + 18\kappa \Theta_{\mu\nu}) . \]  

(44)

From the first kinematic constraint in (6), the trace $H = H_{\mu\mu}$ is then constrained to be

\[ H = \frac{\kappa}{m^2} \Theta . \]  

(45)

This constraint on the trace is consistent with (43) because, given that field equation, the difference $H - \kappa \Theta/m^2$ is a free field. Similarly, the divergence and antisymmetric parts of the $H$-field are free and consistently set to zero.

In general, (44) is a nonlinear field redefinition, given that $\Theta_{\mu\nu}$ depends on the dual field. But in the weak-field limit, outside any non-$T$-field source of energy-momentum, the $H_{\mu\nu}$ field is just proportional to $K_{\mu\nu}$, hence proportional to the $T$-field strength, an expected relation that characterizes free field (or weak-field) duality.

More importantly, the field equation (43) is not the conventional one in (36). Rather, (43) is the field equation of the Ogievetsky-Polubarinov model for a pure spin 2 massive field [8]. This may explain why [1] encountered difficulties and could not obtain a “perfect” dualization connecting (1) and (36). The latter two equations are not dual to one another. Rather, the interacting massive $T$-theory is the exact dual of the Ogievetsky-Polubarinov model.

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Appendix

On-shell conservation of $\theta_{\mu\nu}$ may be established as follows.

[Lemma 1]

\[ \partial^\mu \left( K_{\mu}^\lambda K_{\lambda}^\nu - \frac{1}{2} \delta_{\mu}^{\nu} K_{\alpha\beta} K^{\beta\alpha} \right) \approx 3K_{\mu}^{\lambda} \varepsilon_{\alpha\beta\mu\nu} \Box_{\alpha\beta \lambda} . \]  

(A1)

Proof:

\[ \partial^\mu \left( K_{\mu}^\lambda K_{\lambda}^\nu \right) \approx K_{\mu}^{\lambda} \partial^\mu K_{\lambda}^\nu = K_{\mu}^{\lambda} \varepsilon_{\alpha\beta\gamma\nu} \partial^\mu F_{\alpha\beta\gamma \lambda} \]  

using (23) and (5)

\[ = K_{\mu}^{\lambda} (\varepsilon_{\alpha\beta\gamma\mu} \partial^\nu + 3\varepsilon_{\alpha\beta\mu\nu} \partial^\gamma) F_{\alpha\beta\gamma \lambda} \]  

syzygy in 4D [9]

\[ \approx K_{\mu}^{\lambda} \partial^\nu K_{\lambda}^\nu + 3K_{\mu}^{\lambda} \varepsilon_{\alpha\beta\mu\nu} \Box_{\alpha\beta \lambda} \]  

using (5) and (22b,22c)

So (A1) is established. Thus we are led to

[Lemma 2]

\[ K_{\mu}^{\lambda} \varepsilon_{\alpha\beta\mu\nu} \approx 6F_{\alpha\beta\nu \lambda} . \]  

(A2)

Proof:

\[ K_{\mu}^{\lambda} \varepsilon_{\alpha\beta\mu\nu} \approx K_{\mu}^{\lambda} \varepsilon_{\alpha\beta\mu\nu} = F_{\alpha\beta\nu \lambda} \]  

using (23) and (5)

\[ = \delta_{\alpha\beta\nu} F_{\alpha\beta\nu \lambda} = 6F_{\alpha\beta\nu \lambda} \]  

using $\varepsilon_{\alpha\beta\mu} \varepsilon_{\alpha\beta\mu} = -\delta_{\alpha\beta\mu}$ in 4D
So (A2) is also established. Now, combining (A1) and (A2) along with (21) gives immediately

[Lemma 3]

$$\partial^\mu \left( K^\lambda_{\mu} K^\nu_{\lambda} - \frac{1}{2} \delta^\nu_{\mu} K^{\alpha}_{\alpha} K^{\beta}_{\beta} \right) \simeq -18m^2 F^{[\alpha \beta \nu] \lambda} T_{[\alpha \beta] \lambda}.$$  

(A3)

This leads to a final

[Lemma 4]

$$F^{[\alpha \beta \nu] \lambda} T_{[\alpha \beta] \lambda} \simeq \partial^\nu \left( \frac{1}{2} T_{[\alpha \beta] \gamma} T^{[\alpha \beta] \gamma} \right) - 2 \partial^\mu \left( T_{[\mu \beta] \gamma} T^{[\nu \beta] \gamma} \right).$$  

(A4)

Proof:

$$F^{[\alpha \beta \nu] \lambda} T_{[\alpha \beta] \lambda} = \left( \partial_{\nu} T^{[\alpha \beta] \lambda} + 2 \partial^\alpha T^{[\beta \nu] \lambda} \right) T_{[\alpha \beta] \lambda} \quad \text{definition of } F^{[\alpha \beta \nu] \lambda}$$

$$\simeq \partial^\nu \left( \frac{1}{2} T_{[\alpha \beta] \gamma} T^{[\alpha \beta] \gamma} \right) + 2 \partial^\mu \left( T_{[\mu \beta] \gamma} T^{[\nu \beta] \gamma} \right) \quad \text{using (22) and renaming indices}$$

So (A3) is established. Combining (A3) and (A4) we then obtain

$$\partial^\mu \left( K^\lambda_{\mu} K^\nu_{\lambda} - \frac{1}{2} \delta^\nu_{\mu} K^{\alpha}_{\alpha} K^{\beta}_{\beta} \right) \simeq -18m^2 \partial^\nu \left( \frac{1}{2} \delta^\nu_{\mu} T_{[\alpha \beta] \gamma} T^{[\alpha \beta] \gamma} - 2 T_{[\mu \beta] \gamma} T^{[\nu \beta] \gamma} \right).$$  

(A5)

That is to say, $\partial^\mu \theta_{\mu} \simeq 0$ with $\theta_{\mu}$ given by (24).

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