On vanishing theorems for Higgs bundles

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Abstract

We introduce the notion of Hermitian Higgs bundle as a natural generalization of the notion of Hermitian vector bundle and we study some vanishing theorems concerning Hermitian Higgs bundles when the base manifold is a compact complex manifold. We show that a first vanishing result, proved for these objects when the base manifold was Kähler, also holds when the manifold is compact complex. From this fact and some basic properties of Hermitian Higgs bundles, we conclude several results. In particular we show that, in analogy to the classical case, there are vanishing theorems for invariant sections of tensor products of Higgs bundles. Then, we prove that a Higgs bundle admits no nonzero invariant sections if there is a condition of negativity on the greatest eigenvalue of the Hitchin-Simpson mean curvature. Finally, we prove that invariant sections of certain tensor products of a weak Hermitian-Yang-Mills Higgs bundle are all parallel in the classical sense.

1 Introduction

As it is well known, in complex geometry one has some results on vanishing of holomorphic sections of a holomorphic vector bundle under certain negativity conditions on the Chern mean curvature of the bundle. These results, first proved by Bochner and Yano [17], have been used by Gauduchon [8] and Kobayashi [11] to study some properties of Hermitian vector bundles over compact complex manifolds. In particular, Kobayashi used some of these properties to prove one direction of the Hitchin-Kobayashi correspondence for holomorphic vector bundles over compact Kähler manifolds; namely, Kobayashi proved the polystability of such a bundle if the bundle was Hermitian-Einstein. The other direction of this correspondence has been proved by Donaldson [6, 7] when the base manifold was a compact complex projective manifold, and by Uhlenbeck and Yau [18] when the manifold was compact Kähler. The Hitchin-Kobayashi correspondence plays an important role in Complex Geometry and is the subject of much active research, it has been studied in detail by Lübke and Teleman [12] in the case of holomorphic vector bundles when the base manifold is compact complex, and has been extended to coherent sheaves over compact Kähler manifolds.

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manifolds by Bando and Siu [1].

On the other hand, the notion of a Higgs bundle was introduced by Hitchin [10] and Simpson [14], [15]. They used this concept to construct an extension of the Hitchin-Kobayashi correspondence. In particular, in [10] Hitchin used some vanishing theorems to prove that an irreducible Higgs bundle over a compact Riemann surface of zero degree is stable if and only if it satisfies the Hermitian-Yang-Mills condition. In [14], Simpson extended the result of Hitchin for Higgs bundles over Kähler manifolds of arbitrary dimension and degree, which is indeed the Hitchin-Kobayashi correspondence for Higgs bundles. Following the ideas of Bando and Siu [1], Biswas and Schumacher [2] proved this correspondence also for Higgs sheaves. Now, Bruzzo and Granja Otero [3] proved one of these vanishing results for Higgs bundles over compact Kähler manifolds, and they used it to proved that a Higgs bundle is semistable if it admits an approximate Hermitian-Yang-Mills metric. Finally, Seaman in [13] studied some particular examples of Higgs bundles, which arise in a natural way from bundles of holomorphic forms, and he proved some vanishing theorems for holomorphic forms on these bundles.

This article is organized as follows, in the first section we introduce the notion of a Hermitian Higgs bundle over a compact Hermitian manifold and we make some comments about the basic properties of these objects. In particular, we review the notion of invariant section of a Hermitian Higgs bundle. For Hermitian Higgs bundles we can apply the same operations that are commonly applied to Hermitian vector bundles, and hence make sense to study the same notions that are introduced in Complex Geometry. In this section we review the notion of Hitchin-Simpson curvature and we show that we get a formula for the corresponding mean curvature, which is indeed similar to the classical one. We also introduce the concept of weak Hermitian-Yang-Mills metric, as a natural generalization of the concept of weak Hermitian-Einstein metric introduced by Kobayashi [11]. In the final part of this section we review the classical Weitzenböck formula, a key result that can be used also for Hermitian Higgs bundles.

In the second section we prove some Bochner’s vanishing theorems for Hermitian Higgs bundles over compact complex manifolds. In the first part we show that if the Hitchin-Simpson mean curvature of a Hermitian Higgs bundle is seminegative definite everywhere, every invariant section is parallel in the classical sense, i.e., it is parallel with respect to the Chern connection. If moreover, the Hitchin-Simpson mean curvature is negative definite at some point, then there are no nonzero invariant sections for such a bundle. This result has been proved in [3], when the base manifold is Kähler. Here we modify their proof to cover also the general case. Then we show that, in analogy to the classical case, there is also a vanishing theorem for tensor products of Hermitian Higgs bundles and from this result we get some corollaries. Next, we prove the main theorem of this article. Namely, we prove that if the eigenvalues of the Hitchin-Simpson mean curvature of a Hermitian Higgs bundle satisfy certain negativity condition, then such a bundle admits no nonzero invariant sections. This result is again an extension of a classical result for Hermitian vector bundles. Finally, we prove that if a Hermitian Higgs bundle satisfies the Hermitian-Yang-Mills condition,
then on certain tensor products of this bundle, every invariant section is parallel in the classical sense.

## 2 Hermitian Higgs bundles

We start with some basic definitions. Let $X$ be a compact complex manifold and denote by $\Omega_X^1$ the cotangent bundle to it. Following [10] and [13], a Higgs bundle $\mathcal{E}$ over $X$ is a holomorphic vector bundle $E$ over $X$ together with a map $\phi : E \to E \otimes \Omega_X^1$ such that $\phi \wedge \phi : E \to E \otimes \Omega_X^1$ vanishes. The map $\phi$ is called the Higgs field of $\mathcal{E}$. On Higgs bundles we can apply the same operations that we apply to holomorphic bundles. In particular, the dual of a Higgs bundle is again a Higgs bundle, and tensor products of Higgs bundles are Higgs bundles. If $\mathcal{E}$ is a Higgs bundle we denote its dual by $\mathcal{E}^*$, and if $\mathcal{E}_1$ and $\mathcal{E}_2$ are Higgs bundles over $X$, we denote by $\mathcal{E}_1 \otimes \mathcal{E}_2$ its tensor product. For further details about these basic properties see for instance [4] and [5]. Now, in order to establish the vanishing theorems we need to use the notion of invariant section of a Higgs bundle. Following [3], we say that a section $s$ of a Higgs bundle $\mathcal{E}$ is $\phi$-invariant if $\phi(s) = s \otimes \lambda$ for some holomorphic 1-form $\lambda$ on $X$.

A compact Hermitian manifold is a pair $(X, g)$, where $X$ is a compact complex manifold and $g$ is a Hermitian metric on $X$; a Hermitian Higgs bundle over $(X, g)$ is a pair $(\mathcal{E}, h)$ where $\mathcal{E}$ is a Higgs bundle over $X$ and $h$ is a Hermitian metric on $E$. In other words, a Hermitian Higgs bundle is just a Hermitian vector bundle (in the sense of Kobayashi [11]) such that the corresponding holomorphic bundle is a Higgs bundle. By definition, an invariant section of a Hermitian Higgs bundle $(\mathcal{E}, h)$ is just an invariant section of $\mathcal{E}$. For a pair $(\mathcal{E}, h)$ its dual Hermitian Higgs bundle is the pair $(\mathcal{E}^*, h^*)$ with $h^*$ the usual metric on the holomorphic bundle $E^*$ induced by $h$. Given $(\mathcal{E}_1, h_1)$ and $(\mathcal{E}_2, h_2)$ Hermitian Higgs bundles over $(X, g)$, then $(\mathcal{E}_1 \otimes \mathcal{E}_2, h_1 \otimes h_2)$ is a Hermitian Higgs bundle over $(X, g)$, where $h_1 \otimes h_2$ is the usual Hermitian metric on $E_1 \otimes E_2$.

Let $(\mathcal{E}, h)$ be a Hermitian Higgs bundle over $(X, g)$, as it is well known, there exists a unique connection $D_h$ (the Chern connection), compatible with the holomorphic structure of $E$ and the metric $h$. The curvature of this connection is given by $R_h = D_h \wedge D_h$, it is always a $(1, 1)$ form with coefficients in $\text{End} E$ and is called the Chern curvature. Using the Chern connection $D_h$ and the Higgs field $\phi$, one defines a new connection $D'_h = D_h + \phi + \phi_h$, where $\phi_h$ is the adjoint of the Higgs field with respect to the metric $h$. The connection $D_h$ and its curvature $R_h = D_h \wedge D_h$ are usually called the Hitchin-Simpson connection and curvature of $(\mathcal{E}, h)$ and we say that the pair $(\mathcal{E}, h)$ is Hermitian flat if $R_h$ vanishes. The relation between the Hitchin-Simpson curvature and the Chern curvature is given by (see [3] or [4] for details)

$$R_h = R_h + D'_h(\phi) + D''(\phi_h) + [\phi, \phi_h],$$

(1)

where $D'_h$ and $D''$ are the holomorphic and antiholomorphic parts of the Chern curvature.

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1 Notice that since a Higgs bundle $\mathcal{E}$ is indeed a pair $(E, \phi)$, a Hermitian Higgs bundle can be seen also as a triple $(E, \phi, h)$. 

3
connection and the commutator is an abbreviation for $\phi \wedge \bar{\phi}_h + \bar{\phi}_h \wedge \phi$. Consequently, the $(1,1)$ part of the Hitchin-Simpson curvature is

$$R_{h}^{1,1} = R_h + [\phi, \bar{\phi}_h],$$

and the mean curvature is defined, as usual, as the trace of $R_{h}^{1,1}$ with respect to $g$. To be precise, if $\{z_{\alpha}\}_{\alpha=1}^{n}$ is a local coordinate system of the complex manifold $X$ and $\{e_{j}\}_{j=1}^{r}$ is a local frame field for $E$, with $\{\bar{e}_{j}\}_{j=1}^{r}$ its dual frame, we have the following:

$$g = \sum g_{\alpha\bar{\beta}} dz^\alpha \otimes d\bar{z}^\beta, \quad h = \sum h_{ik} e^i \otimes \bar{e}^k,$$

$$R_{h}^{1,1} = \sum \Omega_{j\alpha\bar{\beta}}^{i} e_i \otimes e^j dz^\alpha \wedge d\bar{z}^\beta.$$ 

Let $K_{j}^{i} = g^{\alpha\beta} \Omega_{j\alpha\bar{\beta}}^{i}$, where $g^{\alpha\beta}$ denotes the components of the inverse of the matrix associated to $g$, then the mean curvature of the Hitchin-Simpson connection is given by:

$$K = \sum K_{j}^{i} e_i \otimes e^j.$$ (3)

It is important to note that the Hitchin-Simpson mean curvature is an endomorphism of $E$ which depends on $h$ and $g$ (we will not write this dependence explicitly in order to simplify the notation). Equivalently, by defining $K_{jk} = \sum h_{ik} K_{j}^{i}$ we can consider the mean curvature as the Hermitian form

$$\hat{K} = \sum K_{jk}^{i} e_i \otimes \bar{e}^k.$$ (4)

From the definitions (3) and (4) it is clear that the endomorphism $K$ and the form $\hat{K}$ are related by the metric $h$. To be precise, if $s$ and $t$ are sections of $E$, we have

$$\hat{K}(s, t) = h(Ks, t).$$ (5)

Now, associated to each Hermitian metric $g$ on $X$, there exists a fundamental 2-form of type $(1,1)$, also called the Kähler form of $X$ (see [16] or [9] for more details), which is defined by

$$\omega = i \sum g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta.$$ If $\omega$ is closed, $g$ is called a Kähler metric of $X$ and the pair $(X, g)$ is called a Kähler manifold. Taking the wedge product of the fundamental form $n$-times we obtain the $(n, n)$ form

$$\omega^n = i^n n!(\det g) dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n.$$ 

We say that a Hermitian Higgs bundle $(\mathcal{E}, h)$ satisfies the weak Hermitian-Yang-Mills condition or that it is weak Hermitian-Yang-Mills if $K = \gamma I$ for some function $\gamma$ (where $I$ is the identity endomorphism of $E$), or equivalently if $\hat{K} = \gamma h$. Using standard identities for the curvature (see for instance [11] or [4]), we see that the property of being weak Hermitian-Yang-Mills is preserved under tensor products and dualization. To be more precise, $(\mathcal{E}^*, h^*)$ is weak

\footnote{It is also used the terminology weak Hermitian-Einstein or weak Hermitian-Yang-Mills-Higgs. We say also that $h$ is weak Hermitian-Yang-Mills.}
Hermitian-Yang-Mills with factor $-\gamma$ if $(\mathcal{E}, h)$ is weak Hermitian-Yang-Mills with factor $\gamma$, and $(\mathcal{E}_1 \otimes \mathcal{E}_2, h_1 \otimes h_2)$ is weak Hermitian-Yang-Mills with factor $\gamma_1 + \gamma_2$ if $(\mathcal{E}_1, h_1)$ and $(\mathcal{E}_2, h_2)$ are weak Hermitian-Yang-Mills with factors $\gamma_1$ and $\gamma_2$, respectively. Notice that in particular from this two results it follows that the Hermitian Higgs bundle $(\mathcal{E} \otimes p \otimes \mathcal{E}^* \otimes q, h \otimes p \otimes h^* \otimes q)$ is weak Hermitian-Yang-Mills with factor $(p - q)\gamma$ if $(\mathcal{E}, h)$ is Hermitian-Yang-Mills with factor $\gamma$.

We say that a Hermitian Higgs bundle satisfies the Hermitian-Yang-Mills condition (or that it is Hermitian-Yang-Mills) if $\gamma$ is constant. Hermitian-Yang-Mills bundles have been studied in detail by Simpson [14], [15] in the case of Kähler manifolds. Indeed, Simpson proved a Hitchin-Kobayashi correspondence for these bundles: a Hermitian Higgs bundle is Hermitian-Yang-Mills if and only if it is polystable.

Let $(\mathcal{E}, h)$ be a Hermitian Higgs bundle over $(X, g)$ and suppose $a = a(x)$ is a real positive function on $X$, then $h' = ah$ defines another Hermitian metric on $\mathcal{E}$, i.e., $(\mathcal{E}, h')$ is another Hermitian Higgs bundle. Since these two metrics are related by a conformal change, $\phi h' = \phi h$ (see [4] for details), and the corresponding Hitchin-Simpson mean curvatures are related by

$$K' = K - \Box (\log a) I,$$

(6)

where $\Box = \sum g^{\alpha \beta} \partial_{\alpha} \partial_{\beta}$. This formula is indeed a generalization of a classical result for Chern mean curvatures, and it is important in the theory; for instance, from (6) it follows that for every weak Hermitian-Yang-Mills metric $h$, there exists a conformal change $a$ such that $ah$ is Hermitian-Yang-Mills (see again [4] for details). Equivalently, we can rewrite the formula (6) in terms of Hermitian forms. Indeed, using (5) we get

$$\hat{K}' = \hat{K} - \Box (\log a) h.$$

(7)

On the other hand, for holomorphic sections of Hermitian vector bundles we have a formula involving the Chern curvature and the metric. If $d'$ and $d''$ denotes the $(1,0)$ and $(0,1)$ parts of $d$, and $s$ is a holomorphic section of the Hermitian vector bundle $(E, h)$, this formula is given by:

$$d'd'' h(s,s) = h(D's, D's) - h(Rs, s),$$

(8)

where $D'$ denotes the $(1,0)$ part of the Chern connection $D$, and $R$ is the corresponding Chern curvature. In terms of local coordinates and local frame fields, $s = \sum s^i e_i$ and we can write (see [11] for more details)

$$D's = \sum \nabla_{\alpha} s^i dz^\alpha e_i, \quad d''s = \sum \nabla_{\beta} s^i d\bar{z}^\beta e_i.$$

Therefore, the identity (8) is equivalent to

$$\partial_{\alpha} \partial_{\beta} h(s,s) = \sum h_{ik} \nabla_{\alpha} s^i \nabla_{\beta} \bar{s}^k - \sum h_{ik} R^i_{\alpha \beta} s^i \bar{s}^k,$$

(9)

where $R^i_{\alpha \beta}$ denotes here the components of the Chern curvature $R$. Now, by defining

$$|D's|^2 = \sum h_{ik} g^{\alpha \beta} \nabla_{\alpha} s^i \nabla_{\beta} \bar{s}^k$$

5
(the usual norm of $D'_s s$ with respect to $h$ and $g$) and taking the trace of the formula (9) with respect to the metric $g$, we obtain finally the Weitzenböck formula:

\[ \Box h(s, s) = |D's|^2 - \hat{K}(s, s) . \quad (10) \]

The Weitzenböck formula plays an important role in the proof of the first vanishing theorem for holomorphic vector bundles, and as we will see, it is also important in the context of Higgs bundles.

### 3 Vanishing theorems

As we said before, in the study of holomorphic vector bundles one has some results on vanishing of holomorphic sections if some specific conditions on the curvature apply. These results, generically called Bochner’s vanishing theorems, play an important role in complex geometry. Some of these vanishing results also holds in the context of Higgs bundles, in that case, we must replace the ordinary mean curvature by the Hitchin-Simpson curvature.

We establish here a first Bochner’s vanishing theorem for Hermitian Higgs bundles over compact Hermitian manifolds. This theorem has been proved first by Bruzzo and Graña-Otero [3] when the manifold was compact Kähler. However, it can be extended to non-Kähler manifolds. We write here the proof presented in [3] adapted to Higgs bundles over compact complex manifolds.

**Theorem 3.1** Let $(\mathcal{E}, h)$ be a Hermitian Higgs bundle over a compact Hermitian manifold $(X, g)$. Then

(i) If the Hitchin-Simpson mean curvature $\hat{K}$ is seminegative definite everywhere on $X$, then every $\phi$-invariant section $s$ of $\mathcal{E}$ is parallel in the classical sense, i.e., $Ds = 0$ with $D$ the Chern connection of $h$, and satisfies

\[ \hat{K}(s, s) = 0 . \]

(ii) If the Hitchin-Simpson mean curvature $\hat{K}$ is seminegative definite everywhere on $X$ and negative definite at some point of $X$, then $\mathcal{E}$ has no nonzero $\phi$-invariant sections.

**Proof:** Let $s$ be a $\phi$-invariant section of $\mathcal{E}$ and assume $\hat{K}$ is seminegative definite everywhere. From the decomposition (11) of the Hitchin-Simpson curvature we have

\[ R s = R s + D' s + D'' s + [\phi, \phi] s . \]

Since $s$ is $\phi$-invariant $[\phi, \phi] s = 0$ and hence, taking the trace with respect to $g$ in the above expression we get $K s = K s$, or equivalently $\hat{K}(s, s) = \hat{K}(s, s)$. Then, using the classical Weitzenböck formula (10) we obtain

\[ \Box h(s, s) = |D's|^2 - \hat{K}(s, s) . \quad (11) \]

Now, since $\hat{K}$ is seminegative definite, the right hand side of (11) is non-negative and by Hopf’s maximum principle (see e.g. [11] or [12]) this implies that $h(s, s)$ is constant and consequently $\Box h(s, s) = 0$. Therefore, necessarily $\hat{K}(s, s) = 0$.
and $D's = 0$, but since $s$ is holomorphic this last condition is equivalent to $Ds = 0$ and (i) follows.

On the other hand, suppose now $s$ is a $\phi$-invariant section of $(E, h)$ and assume this time that $\hat{K}$ is seminegative definite everywhere and negative at some point. Then, from (i) we know that $s$ is parallel with respect to the Chern connection and hence it never vanishes. Using again (i) we have $\hat{K}(s, s) = 0$ and we have a contradiction, because $\hat{K}$ must be negative at some point of $X$. Q.E.D.

As in the classical case, from this first vanishing theorem we have other results involving tensor products. In particular we have the following

**Theorem 3.2** Let $(E_1, h_1)$ and $(E_2, h_2)$ be two Hermitian Higgs bundles over a compact Hermitian manifold $(X, g)$ and let $\hat{K}_1$ and $\hat{K}_2$ be the corresponding Hitchin-Simpson mean curvatures. Let $\phi$ be the Higgs field of $(E_1 \otimes E_2, h_1 \otimes h_2)$ and let $\hat{K}_{1\otimes 2}$ be the Hitchin-Simpson mean curvature of this tensor product. Then

(i) If both $\hat{K}_1$ and $\hat{K}_2$ are seminegative definite everywhere on $X$, then every $\phi$-invariant section $\xi$ of $E_1 \otimes E_2$ is parallel with respect to the Chern connection $D_{1\otimes 2}$ (induced from $D_1$ and $D_2$), i.e., $D_{1\otimes 2}\xi = 0$, and satisfies $\hat{K}_{1\otimes 2}(\xi, \xi) = 0$.

(ii) If both $\hat{K}_1$ and $\hat{K}_2$ are seminegative definite everywhere on $X$ and either one is negative definite at some point of $X$, then $E_1 \otimes E_2$ admits no nonzero $\phi$-invariant sections.

**Proof:** From [4] we know the Hitchin-Simpson mean curvature of the Hermitian Higgs bundle $(E_1 \otimes E_2, h_1 \otimes h_2)$ satisfies

$$\hat{K}_{1\otimes 2} = K_1 \otimes I_2 + I_1 \otimes K_2.$$

Now, similarly to classical case, by choosing orthonormal local frame fields we can represent locally $K_1$ and $K_2$ by diagonal matrices, and hence $\hat{K}_{1\otimes 2}$ becomes also a diagonal matrix whose nonzero elements are sums of the diagonal elements of $K_1$ and $K_2$. That is, if $a_i$ and $b_j$ are the diagonal elements of $K_1$ and $K_2$, the diagonal elements of $\hat{K}_{1\otimes 2}$ are $a_i + b_j$. At this point, since the local frame field is orthonormal, $K_{j\bar{k}} = \sum \delta_{i\bar{k}} K^i_j = K^\bar{i}_j$ and (i) and (ii) follow from Theorem 3.1. Q.E.D.

**Corollary 3.3** Let $(E, h)$ be a Hermitian Higgs bundle over a compact Hermitian manifold $(X, g)$. Let $(E^\otimes p, h^\otimes p)$ be the tensor product of $(E, h)$ $p$-times and let $\psi$ be its Higgs field (constructed from the Higgs field $\phi$ of $E$) and $\hat{K}_{\otimes p}$ its Hitchin-Simpson mean curvature. Then

(i) If $\hat{K}$ is seminegative definite everywhere on $X$, then every $\psi$-invariant section $\xi$ of $E^\otimes p$ is parallel in the classical sense, i.e., $D_{\otimes p}\xi = 0$, and satisfies $\hat{K}_{\otimes p}(\xi, \xi) = 0$.

(ii) If $\hat{K}$ is seminegative definite everywhere on $X$ and negative definite at some point of $X$, then $E^\otimes p$ admits no nonzero $\psi$-invariant sections.
Let $TX$ and $T^*X$ be the tangent and cotangent bundles to $X$, then we can consider the pairs $(TX, g)$ and $(T^*X, g^*)$ as Hermitian Higgs bundles with zero Higgs fields. Since the Hermitian metrics $h$ and $g^*$ on $E$ and $T^*X$ induce a Hermitian metric, say $k$, on the Higgs bundle $\Omega^p(E) = E \otimes \Lambda^p T^*X$, the pair $(\Omega^p(E), k)$ can be considered as a Hermitian Higgs bundle over $(X, g)$. We denote by $\hat{K}_{TX}$ the mean curvature\(^3\) of $(TX, g)$ and by $\hat{K}_{\Omega^p}$ the Hitchin-Simpson curvature of $(\Omega^p(E), k)$. From this we have the following

**Corollary 3.4** Let $(E, h)$ be a Hermitian Higgs bundle over a compact Hermi-
tian manifold $(X, g)$ with Higgs field $\phi$, and denote by $\hat{K}_E$ its Hitchin-Simpson mean curvature. Then

(i) If $\hat{K}_E$ is seminegative definite and $\hat{K}_{TX}$ is semipositive definite everywhere on $X$, then every $\phi$-invariant section $\xi$ of $\Omega^p(E)$ is parallel in the classical sense and satisfies

$$\hat{K}_{\Omega^p}(\xi, \xi) = 0.$$  

(ii) If $\hat{K}_E$ is seminegative definite and $\hat{K}_{TX}$ is semipositive definite everywhere on $X$, and either $\hat{K}_E$ is negative definite or $\hat{K}_{TX}$ is positive definite at some point of $X$, then $\Omega^p(E)$ admits no nonzero $\phi$-invariant sections.

On the other hand, as it is well known (see [11] for details), there exists van-
ishing theorems for Hermitian vector bundles if certain condition for the greatest
eigenvalue of the Chern mean curvature holds, and for Hermitian vector bundles
satisfying the Hermitian-Einstein condition. These results can be extended to
Hermitian Higgs bundles if we use the Hitchin-Simpson mean curvature instead
of the Chern mean curvature. To be precise we have the following results

**Theorem 3.5** Let $(E, h)$ be a Hermitian Higgs bundle over a compact Hermi-
tian manifold $(X, g)$ with fundamental 2-form $\omega$. Let $\lambda_1 \leq \cdots \leq \lambda_r$ be the
eigenvalues of the Hitchin-Simpson mean curvature $K$ of $(E, h)$. If

$$\int_X \lambda_r \omega^n < 0,$$

then $E$ admits no nonzero $\phi$-invariant sections.

**Proof:** In a similar way to the proof of the classical case, we consider a $C^\infty$
function $f$ on $X$ such that $\lambda_r < f$ and

$$\int_X f \omega^n = 0.$$

Clearly $f$ is orthogonal to all constant functions, but since $X$ is compact, every $\Box$-harmonic function is constant and hence $f$ is orthogonal to the $\Box$-harmonic
functions. Therefore, we can consider a $C^\infty$ solution $u$ of the equation $\Box u = f$.

Now, if we define the positive function $a = e^u$ and consider the Hermitian Higgs
bundle $(E, h')$ where $h' = ah$, then $\Box (\log a) = f$ and using [11] we have that $K'$
is diagonal and negative definite. Consequently, also $K'$ is negative definite and
the result follows from Theorem [3.1] Q.E.D.

\(^3\)Notice that the Higgs field of $(TX, g)$ is zero, hence the Hitchin-Simpson mean curvature and the Chern mean curvature coincide for this Hermitian Higgs bundle.
Theorem 3.6 Let \((E, h)\) be a Hermitian Higgs bundle over a compact Hermitian manifold \((X, g)\) and let \(\psi\) be the Higgs field of the Hermitian Higgs bundle \((E^p \otimes \bar{E}^p, h^p \otimes h^* \otimes p)\). If \((E, h)\) satisfies the weak Hermitian-Yang-Mills condition, then every \(\psi\)-invariant section of \((E^p \otimes \bar{E}^p)\) is parallel in the classical sense.

Proof: Let \((E, h)\) be weak Hermitian-Yang-Mills with factor \(\gamma\), then from results of the second section we know that \((E^p \otimes \bar{E}^p, h^p \otimes h^* \otimes p)\) is Hermitian-Yang-Mills with factor \((p - p)\gamma = 0\) and hence the mean curvature vanishes identically. At this point the result follows from Theorem [\ref{thm:WY}]. Q.E.D.

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