COMPOSITION OPERATORS AND ENDOMORPHISMS

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Abstract. If $b$ is an inner function, then composition with $b$ induces an endomorphism, $\beta$, of $L^\infty(T)$ that leaves $H^\infty(T)$ invariant. We investigate the structure of the endomorphisms of $B(L^2(T))$ and $B(H^2(T))$ that implement $\beta$ through the representations of $L^\infty(T)$ and $H^\infty(T)$ in terms of multiplication operators on $L^2(T)$ and $H^2(T)$. Our analysis, which is based on work of R. Rochberg and J. McDonald, will wind its way through the theory of composition operators on spaces of analytic functions to recent work on Cuntz families of isometries and Hilbert $C^*$-modules.

1. Introduction

Our objective in this note is to link the venerable theory of composition operators on spaces of analytic functions to the representation theory of $C^*$-algebras. The theory of composition operators is full of equations that involve operators that intertwine various types of representations. In certain situations the equations can be recast in terms of “covariance equations” that are familiar from the theory of $C^*$-algebras, their endomorphisms and their representations; doing this yields both new theorems and new understanding of known results.

We are inspired in particular by papers by Richard Rochberg [15] and John McDonald [14]. In [15, Theorem 1], Rochberg performs calculations which may be seen from a more contemporary perspective as identifying certain Cuntz families of isometries and Hilbert $C^*$-modules at the heart of what he is studying. In [14], McDonald built upon Rochberg’s work and proved, among other things, that the canonical transfer operator associated to composition with a finite Blaschke product leaves the Hardy space $H^2(T)$ invariant. This note is in large part the result of trying to recast [15, Theorem 1] in the setting of $C^*$-algebras and endomorphisms using McDonald’s observation on transfer operators [14, Lemma 2].

The classical Lebesgue and Hardy spaces on the unit circle $T$ will be denoted by $L^p(T)$ and $H^p(T)$ respectively. Normalized Lebesgue measure on $T$ will be denoted $m$. The orthogonal projection from $L^2(T)$ onto $H^2(T)$ will be denoted by $P$. The usual exponential orthonormal basis for $L^2(T)$ will be denoted by $\{e_n\}_{n \in \mathbb{Z}}$, i.e., $e_n(z) := z^n$. We write $(\cdot, \cdot)$ for the inner product of $L^2(T)$. The multiplication operator on $L^2(T)$ determined by a function $\varphi \in L^\infty(T)$ will be denoted $\pi(\varphi)$ and the Toeplitz operator on $H^2(T)$ determined by $\varphi$ will be denoted by $\tau(\varphi)$, i.e., $\tau(\varphi)$ is the restriction of $P\pi(\varphi)P$ to $H^2(T)$.

Our use of the notation $\pi$ and $\tau$ is nonstandard. More commonly, one writes $M_f$ for the multiplication operator determined by $f$ and $T_f$ for the Toeplitz operator determined by $f$, but for the purposes of this note, we have found the standard notation to be a bit awkward. In any case, the map $\pi$ is a $C^*$-representation of $DC$ and SS were partially supported by the University of Iowa Department of Mathematics NSF VIGRE grant DMS-0602242.
\( L^\infty(\mathbb{T}) \) on \( L^2(\mathbb{T}) \) that is continuous with respect to the weak-* topology on \( L^\infty(\mathbb{T}) \) and the weak operator topology on \( B(L^2(\mathbb{T})) \), and \( \tau \) is a (completely) positive linear map from \( L^\infty(\mathbb{T}) \) to \( B(H^2(\mathbb{T})) \) with similar continuity properties.

We fix throughout an inner function \( b \) which at times will further be assumed to be a finite Blaschke product. Composition with \( b \), that is, the map \( \varphi \mapsto \varphi \circ b \), is known to induce a \(*\)-endomorphism \( \beta \) of \( L^\infty(\mathbb{T}) \) that is continuous with respect to the weak-* topology on \( L^\infty(\mathbb{T}) \). When \( b \) is a finite Blaschke product this statement is fairly elementary; if \( b \) is an arbitrary inner function, it is somewhat more substantial. We give an operator-theoretic proof in Corollary 3.2. When \( \beta \) leaves a subspace of \( L^\infty(\mathbb{T}) \) invariant, we will continue to use the notation \( \beta \) for its restriction to the subspace. The central focus of our analysis is

**Problem 1.1.** Describe all \(*\)-endomorphisms \( \alpha \) of \( B(L^2(\mathbb{T})) \) such that

\[
\alpha \circ \pi = \pi \circ \beta
\]

and describe all \(*\)-endomorphisms \( \alpha_+ \) of \( B(H^2(\mathbb{T})) \) such that

\[
\alpha_+ \circ \tau = \tau \circ \beta.
\]

If an endomorphism \( \alpha \) of \( B(L^2(\mathbb{T})) \) satisfies (1.1), the pair \((\pi, \alpha)\) is called a **covariant representation** of the pair \((L^\infty(\mathbb{T}), \beta)\). As \( \pi \) will be fixed throughout this note, the first part of our problem is thus to identify all endomorphisms \( \alpha \) of \( B(L^2(\mathbb{T})) \) that yield a covariant representation \((\pi, \alpha)\) of \((L^\infty(\mathbb{T}), \beta)\). Equation (1.2) is a hybrid version of (1.1), but as we shall see, it may be interpreted as describing certain covariant representations of the Toeplitz algebra, i.e., of the \( C^*\)-algebra \( \mathcal{T} \) generated by all the Toeplitz operators \( \tau(\varphi), \varphi \in L^\infty(\mathbb{T}) \). It is not clear **a priori** that any endomorphisms satisfying (1.1) or (1.2) exist. They do, however, as we shall show in Theorem 3.3, where Rochberg’s work plays a central role. Then, in Corollary 3.5, we show how Rochberg’s analysis yields a complete description of all solutions to (1.2). Identifying all solutions to (1.1) is more complicated, and it is here that we must assume that \( b \) is a finite Blaschke product. The set of solutions to (1.1) is described in Theorem 5.2 under this restriction.

In solving Problem 1.1 we obtain many new proofs of known results. We do not take any position on the matter of which proofs are simpler or more elementary. Our more modest goal is to separate what can be derived through elementary Hilbert space considerations from what requires more specific function-theoretic analysis. In this respect, we were inspired by the work of Helson and Lowdenslager [8], Halmos and others who cast Hardy space theory in Hilbert space terms and, in particular, showed that Beurling’s theorem about invariant subspaces of the shift operator can be proved with elementary Hilbert space methods. Indeed, as we shall see, our main Theorem 3.3 is a straightforward corollary of Beurling’s theorem and requires no more technology than Helson and Lowdenslager’s approach to that result. This paper, therefore, has something of a didactic component. When we reprove or reinterpret a known result, we call attention to it and give references to alternative approaches.

2. Preliminaries and Background

It is well known that when \( H \) is a Hilbert space, \( B(H) \) is the dual space of the space of trace class operators on \( H \). The weak-* topology on \( B(H) \) is often called the **ultraweak** topology. We adopt that terminology here. The ultraweak topology
is different from the weak operator topology, but the two coincide on bounded subsets of $B(H)$. It follows that our representation $\pi$ is continuous with respect to the weak-$\star$ topology on $L^\infty(T)$ and either the weak operator topology or the ultraweak topology on $B(L^2(T))$.

As indicated earlier, it is straightforward to see that composition with a finite Blaschke product induces an endomorphism of $L^\infty(T)$. It is less clear that composition with an arbitrary inner function has this property. There are two reasons for this. The first is that the boundary values of a general inner function $b$ are only defined on a set $F \subseteq T$ with $m(T \setminus F) = 0$. The second is that an element of $L^\infty(T)$ is an equivalence class of measurable functions containing a bounded representative, where two functions are equivalent if and only if they differ on a null set. Thus we want to know that if we extend $b$ arbitrarily on $T \setminus F$, mapping to $T$, and if $\varphi$ and $\psi$ differ at most on a null set, then so do $\varphi \circ b$ and $\psi \circ b$. A little reflection reveals that for this to happen, it is necessary and sufficient that the following assertion be true:

If $b$ is an inner function whose domain on $T$ is the measurable set $F$, then for every null set $E$ of $T$, $b^{-1}(E)$ is a null set of $F$.

This fact is well known, but exactly who deserves credit for first proving it is unclear to us. The short note by Kametani and Ugahei [10] proves it in the case that $b(0) = 0$. This implies the general case, as Lebesgue null sets of $T$ are preserved by conformal maps of the disc, and every inner function $b$ can be written $b = \alpha \circ b_1$ with $b_1$ an inner function fixing the origin and $\alpha$ a conformal map of the disc. In Corollary 3.2, we will give a proof of this assertion from the abstract Hilbert space perspective that we are promoting. We will need the following lemma. To emphasize the distinction between a measurable function $f$ and its equivalence class modulo the relation of being equal almost everywhere, we temporarily write $[f]$ for the latter.

Lemma 2.1. Let $\theta$ be a Lebesgue measurable function from $T$ to $T$. Suppose $\Theta$ is defined on trigonometric polynomials $p$ by the formula $\Theta(p) = p \circ \theta$. Then

1. $\Theta$ has a unique extension to a $\ast$-homomorphism from $C(T)$ into $L^\infty(T)$, and it is given by the formula $\Theta(\varphi) = [\varphi \circ \theta], \varphi \in C(T)$.

2. If $\Theta$ is continuous with respect to the weak-$\ast$ topology of $L^\infty(T)$ restricted to $C(T)$ and the weak-$\ast$-topology on $L^\infty(T)$, then for each Lebesgue null set $E$ of $T$, $m(\theta^{-1}(E)) = 0$, and thus $\Theta$ extends uniquely to a $\ast$-endomorphism of $L^\infty(T)$ satisfying $\Theta([\varphi]) = [\varphi \circ \theta]$ for all $[\varphi] \in L^\infty(T)$. The map $\Theta$ is completely determined by $[\theta]$.

Proof. For the first assertion it suffices to note that if $p$ is a trigonometric polynomial, then since $\theta$ is assumed to map $T$ to $T$, $\|p \circ \theta\|_{L^\infty} = \sup_{z \in T} |p(\theta(z))| \leq \sup_{z \in T} |p(z)| = \|p\|_{C(T)}$.

For the second assertion, fix the Lebesgue null set $E$, and choose a $G_\delta$ set $E_0$ containing $E$ such that $E_0 \setminus E$ has measure zero. So, if $\{f_n\}_{n \geq 0}$ is a sequence in $C(T)$ such that $f_n \downarrow 1_{E_0}$ pointwise, then $[f_n]$ converges to $[1_{E_0}] = [1_E]$ weak-$\ast$. But also, $f_n \circ \theta \downarrow 1_{E_0} \circ \theta = 1_{\theta^{-1}(E_0)}$ pointwise. Therefore, $[f_n \circ \theta]$ converges to $[1_{E_0} \circ \theta] = [1_{\theta^{-1}(E_0)}]$ weak-$\ast$. As $E$ is a null set, so is $E_0$, and the $[f_n]$ converge to 0 weak-$\ast$. Our hypothesis then implies that the $[\Theta(f_n)] = [f_n \circ \theta]$ converge to 0 weak-$\ast$. We write the characteristic function (or indicator function) of a set $E$ as $1_E$. ①
weak-, proving that \( m(\theta^{-1}(E_0)) = 0 \). As \( \theta^{-1}(E) \subseteq \theta^{-1}(E_0) \) it follows that \( \theta^{-1}(E) \) is also a null set, as desired. The remaining assertions are immediate. \( \square \)

Because of this lemma, if \( b \) is an inner function that may be defined only on a subset \( F \) of \( T \) with \( m(T \setminus F) = 0 \), it does no harm to extend \( b \) to all of \( T \) by setting \( b(z) = 1 \) for all \( z \in T \setminus F \).

Next, we want to say a few words about \(*\)-endomorphisms of \( B(H) \), where \( H \) is a separable Hilbert space. Our discussion largely follows Section 2 of [1]. A Cuntz family on \( H \) is an \( N \)-tuple of isometries \( \{S_i\}_{i=1}^N \) on \( H \) with mutually orthogonal ranges that together span \( H \); here the number \( N \) may be a positive integer or \( \infty \). A Cuntz family \( S = \{S_i\}_{i=1}^N \) on \( H \) determines a map \( \alpha_S : B(H) \to B(H) \) via

\[
\alpha_S(T) = \sum_{i=1}^N S_iTS_i^*, \quad T \in B(H).
\]

(If \( N = \infty \), this sum is convergent in the strong operator topology.) The map \( \alpha_S \) is readily seen to be a \(*\)-endomorphism of \( B(H) \); multiplicativity is deduced from the fact that a tuple \( S = \{S_i\}_{i=1}^N \) is a Cuntz family if and only if the Cuntz relations

\[
S_iS_j = \delta_{ij}I, \quad 1 \leq i, j \leq N,
\]

and

\[
\sum_{i=1}^N S_iS_i^* = I
\]

are satisfied. (These relations are named after J. Cuntz, who made a penetrating analysis of them in [2].)

Significantly, every \(*\)-endomorphism \( \alpha \) of \( B(H) \), with \( H \) separable, is of the form \( \alpha_S \) for some Cuntz family \( S \). We recall the details. Fixing a \(*\)-endomorphism \( \alpha \), define \( E = \{S \in B(H) \mid ST = \alpha(T)S, T \in B(H)\} \). A short calculation shows that for any \( S_1 \) and \( S_2 \) in \( E \) the product \( S_2^*S_1 \) commutes with all elements of \( B(H) \), and is hence a scalar. We may thus define an inner product \( \langle \cdot, \cdot \rangle \) on \( E \) by the formula

\[
\langle S_1, S_2 \rangle I = S_2^*S_1, \quad S_1, S_2 \in E,
\]

and \( E \) with this inner product is a Hilbert space. It is readily checked that any orthonormal basis \( S = \{S_i\}_{i=1}^N \) for \( E \) is a Cuntz family satisfying \( \alpha = \alpha_S \), so it is enough to know that \( E \) has an orthonormal basis - that is, that \( E \neq \{0\} \). This follows from the fact that a \(*\)-endomorphism of \( B(H) \), when \( H \) is separable, is necessarily ultraweakly continuous and that an ultraweakly continuous unital representation of \( B(H) \) is necessarily unitarily equivalent to a multiple of the identity representation of \( B(H) \). That multiple is the dimension of \( E \).

The correspondence between endomorphisms and Cuntz families is not quite one-to-one. However, as Laca observed [11 Proposition 2.2], if \( S = \{S_i\}_{i=1}^N \) and \( \tilde{S} = \{\tilde{S}_i\}_{i=1}^N \) are two Cuntz families such that \( \alpha_S = \alpha_{\tilde{S}} \), then there is a unitary matrix \( (u_{ij}) \) so that \( \tilde{S}_i = \sum_{j} u_{ij}S_j \), and conversely. The reason is that \( S \) and \( \tilde{S} \) are both orthonormal bases for the same Hilbert space \( E \). (More concretely, one may just check that the scalars \( u_{ij} = S_i^*\tilde{S}_i \) have the desired properties.)

Our goal, then, is to describe the collection of Cuntz families \( S = \{S_i\}_{i=1}^N \) on \( L^2(T) \) and \( R = \{R_i\}_{i=1}^N \) on \( H^2(T) \) such that \( (\pi, \alpha_S) \) is a covariant representation.
of \((L^\infty(\mathbb{T}), \beta)\) and \((\tau, \alpha_R)\) is a covariant representation of \((\Xi, \beta)\) in the sense of equations (1.1) and (1.2).

\begin{equation}
\sum_{i=1}^{N} S_i \pi(\varphi) S_i^* = \pi(\beta(\varphi)), \quad \varphi \in L^\infty(\mathbb{T}),
\end{equation}

and

\begin{equation}
\sum_{i=1}^{N} R_i \tau(\varphi) R_i^* = \tau(\beta(\varphi)), \quad \varphi \in L^\infty(\mathbb{T}).
\end{equation}

Finally, we adopt the following notation for Blaschke products. If \(w\) is a nonzero point of the open unit disc \(\mathbb{D}\) then \(b_w\) will denote the function

\[b_w(z) := \frac{|w|}{w} \frac{w - z}{1 - wz},\]

and \(b_0(z) := z\). If \(a_1, a_2, \ldots, a_N\) is a finite list of not-necessarily-distinct numbers in \(\mathbb{D}\), then we will write \(b = \prod_{j=1}^{N} b_{a_j}\) for the Blaschke product with zeros at \(a_1, a_2, \ldots, a_N\), i.e., multiplicity will be taken into account.

3. Rochberg’s Observation

Our analysis hinges on an observation that we learned from R. Rochberg’s paper [15]. A preliminary remark on isometries in abstract Hilbert space is useful. If \(V\) is an isometry on a Hilbert space \(H\), and \(D\) is the subspace \(H \ominus VH\), it is easy to check that the spaces \(D, VH, V^2D, \ldots\) are mutually orthogonal, and that \((\bigoplus_{k \geq 0} V^k D)^\perp = \bigcap_{j \geq 0} V^j H\), so that \(H = \bigoplus_{k \geq 0} V^k D\) if and only if \(V\) is pure in the sense that \(\bigcap_{j \geq 0} V^j H = \{0\}\).

If \(H = H^2(\mathbb{T})\) and \(V\) is the isometry \(\tau(b) = \pi(b)|_{H^2(\mathbb{T})}\) induced by a nonconstant inner function \(b\), it turns out that \(V\) is pure, and that in fact \(D\) is a complete wandering subspace for the unitary \(\pi(b)\) in the sense of (3.2) below. This is a minor modification of a point made in [15] Theorem 1].

Lemma 3.1. Let \(b\) be a nonconstant inner function, and let \(\mathcal{D} := H^2(\mathbb{T}) \ominus \pi(b)H^2(\mathbb{T})\). Then

\begin{equation}
H^2(\mathbb{T}) = \bigoplus_{k \geq 0} \pi(b)^k \mathcal{D},
\end{equation}

and

\begin{equation}
L^2(\mathbb{T}) = \bigoplus_{n \in \mathbb{Z}} \pi(b)^n \mathcal{D}.
\end{equation}

Proof. As we have just observed, equation (3.1) follows once we know that the space \(K := \bigcap_{n=0}^{\infty} \pi(b)^n H^2(\mathbb{T})\) is the zero subspace. But as \(\pi(b)\) commutes with \(\pi(z)\), the space \(K\) is invariant for the unilateral shift \(\tau(z) = \pi(z)|_{H^2(\mathbb{T})}\). If \(K \neq \{0\}\), by Beurling’s theorem there is an inner function \(\theta\) with \(K = \pi(\theta)H^2(\mathbb{T})\). As \(\pi(b)K = K\) by definition, we see that \(\pi(b)\pi(\theta)H^2(\mathbb{T}) = \pi(\theta)H^2(\mathbb{T})\), and applying \(\pi(\theta^{-1})\) to both sides we conclude that \(\pi(b)H^2(\mathbb{T}) = H^2(\mathbb{T})\). But \(b\) is nonconstant, so by the uniqueness assertion in Beurling’s theorem (see [7] Theorem 3)], \(\pi(b)H^2(\mathbb{T})\) is a proper subspace of \(H^2(\mathbb{T})\). This contradiction shows that \(K = \{0\}\), and (3.1) follows.
Since $\pi(b)$ is a unitary on $L^2(\mathbb{T})$, it is immediate from (3.1) that the spaces $\pi(b)^n D$, $n \in \mathbb{Z}$, are mutually orthogonal. Letting $L = \bigvee_{k \in \mathbb{Z}} \pi(b)^k D$, it is clear from (3.1) that $L = \bigvee_{k \geq 0} \pi(b)^{-k} H^2(\mathbb{T})$, and thus that $L$ is invariant under $\pi(z)$. By Helson and Lowdenslager’s generalization of Beurling’s theorem (see [3], Section 1) or [7, Theorem 3]), either there is a unimodular $\theta \in L^\infty(\mathbb{T})$ with $L = \pi(\theta) H^2(\mathbb{T})$ or there is a measurable $E \subseteq \mathbb{T}$ satisfying $L = \pi(1_E) L^2(\mathbb{T})$. In the first case, as clearly $\pi(b)L = L$, we conclude that $\pi(\theta)\pi(b)H^2(\mathbb{T}) = \pi(\theta)H^2(\mathbb{T})$, and applying $\pi(\theta^{-1})$ to both sides we conclude that $\pi(b)H^2(\mathbb{T}) = H^2(\mathbb{T})$, which contradicts the fact that $b$ is not constant. Thus there is $E \subseteq \mathbb{T}$ with $L = \pi(1_E) L^2(\mathbb{T})$, and the fact that $L$ contains $H^2(\mathbb{T})$ implies $E = \mathbb{T}$, so $L = L^2(\mathbb{T})$ as desired. □

**Corollary 3.2.** If $b$ is an arbitrary inner function and if $\beta$ is defined on trigonometric polynomials $p$ by the formula $\beta(p) := p \circ b$, then $\beta$ extends to a $*$-endomorphism of $L^\infty(\mathbb{T})$ that is continuous with respect to the weak* topology.

**Proof.** Lemma 3.1 implies that $\pi(b)$ is unitarily equivalent to a multiple of the bilateral shift - the multiple being dim$(D)$. Thus there is a Hilbert space isomorphism $W$ from $L^2(\mathbb{T})$ to $L^2(\mathbb{T}) \otimes D$ such that $\pi(b) = W^{-1}(\pi(z) \otimes I_D)W$. So, for every trigonometric polynomial $p$,

$$\pi(\beta(p)) = p(\pi(b)) = W^{-1} p(\pi(z) \otimes I_D)W.$$

Since $\pi$ is a homeomorphism with respect to the weak* topology on $L^\infty(\mathbb{T})$ and the ultraweak topology restricted to the range of $\pi$, it is evident that $b$ and $\beta$ satisfy the hypotheses of Lemma 2.1 and the desired result follows. □

Of course, the proof just given recapitulates parts of the well-known theory of the functional calculus for unitary operators.

**Theorem 3.3.** Let $b$ be a non-constant inner function. If $\{v_i\}^N_{i=1}$ is an orthonormal basis for $D = H^2(\mathbb{T}) \oplus \pi(b)H^2(\mathbb{T})$, then there is a unique Cuntz family $S = \{S_i\}^N_{i=1}$ on $L^2(\mathbb{T})$ satisfying

$$S_i(e_n) = v_i b^n, \quad 1 \leq i \leq N, \quad n \in \mathbb{Z}.$$

The endomorphism $\alpha_S$ determined by $S$ as in (2.1) satisfies $\alpha_S \circ \pi = \pi \circ \beta$, where $\beta$ is the endomorphism $\varphi \mapsto \varphi \circ b$ of $L^\infty(\mathbb{T})$.

Each $S_i$ is reduced by $H^2(\mathbb{T})$, and if $R_i$ is the restriction of $S_i$ to $H^2(\mathbb{T})$, then $R = \{R_i\}^N_{i=1}$ is a Cuntz family on $H^2(\mathbb{T})$ with the property that $\alpha_R \circ \tau = \tau \circ \beta$.

The proof of Lemma 3.1 showed that $D = H^2(\mathbb{T}) \oplus \pi(b)H^2(\mathbb{T})$ is nonzero, so it has an orthonormal basis; its dimension $N$ may be finite or infinite. It is well known that $N$ is finite if and only if $b$ is a finite Blaschke product. (See Remark 3.6)

**Proof.** Lemma 3.1 implies that if $v$ is any unit vector in $D$, the set $\{vb^n : n \in \mathbb{Z}\}$ is an orthonormal set of vectors in $L^2(\mathbb{T})$. It follows that for any $1 \leq i \leq N$, there is a unique isometry $S_i$ on $L^2$ satisfying $S_i(e_n) = v_i b^n$ for all $n \in \mathbb{Z}$.

Lemma 3.1 also implies that if $v$ and $w$ are any orthogonal unit vectors in $D$, the closed linear spans of $\{vb^n : n \in \mathbb{Z}\}$ and $\{wb^n : n \in \mathbb{Z}\}$ are orthogonal. It follows that the isometries in the tuple $S = \{S_i\}^N_{i=1}$ just defined have orthogonal ranges. Let $K$ denote the closed linear span of the ranges of the operators $\{S_i\}^N_{i=1}$. By construction, for all $n \in \mathbb{Z}$ we have $v_i b^n \in K$ for all $1 \leq i \leq N$, and thus
As Lemma 3.1 implies that a basis of equation (2.4) is satisfied for all $\alpha$, every trigonometric polynomial is a Cuntz family on $H^2(\mathbb{T})$. The fact that $H^2(\mathbb{T})$ is invariant under each $S_i$ is immediate from the definition (3.3). As Lemma 3.1 implies that $\{v_i b^n : 1 \leq i \leq N, n < 0\}$ is an orthonormal basis of $H^2(\mathbb{T})^\perp$, it is also clear from (3.3) that $H^2(\mathbb{T})^\perp$ is invariant under each $S_i$, so each $S_i$ is reduced by $H^2(\mathbb{T})$. The fact that $R$ is a Cuntz family on $H^2(\mathbb{T})$ satisfying $\alpha R \circ \tau = \tau \circ \beta$ is then immediate.

Recall that $\mathcal{S}$ is the $C^*$-algebra generated by all the Toeplitz operators $\{\tau(\varphi) | \varphi \in L^\infty(\mathbb{T})\}$. We shall write $\mathcal{S}(C(\mathbb{T}))$ for $C^*$-subalgebra generated by the Toeplitz operators with continuous symbols, i.e., $\mathcal{S}(C(\mathbb{T}))$ is the $C^*$-subalgebra of $B(H^2(\mathbb{T}))$ generated by $\{\tau(\varphi) | \varphi \in C(\mathbb{T})\}$. It is well known that $\mathcal{S}(C(\mathbb{T})) = \{\tau(\varphi) + k | \varphi \in C(\mathbb{T}), k \in \mathbb{R}\}$, where $\mathbb{R}$ denotes the algebra of compact operators on $H^2(\mathbb{T})$ [4, 7.11 and 7.12].

**Corollary 3.4.** If $b$ is an inner function, then the map $\tau(\varphi) \to \tau(\varphi \circ b)$, $\varphi \in L^\infty(\mathbb{T})$, extends to a $*$-endomorphism of $\mathcal{S}$ that we will continue to denote by $\beta$. Further, $\beta$ leaves $\mathcal{S}(C(\mathbb{T}))$ invariant if and only if $b$ is a finite Blaschke product. Thus, if $i$ denotes the identity representation of $\mathcal{S}$ on $H^2(\mathbb{T})$, then any solution $\alpha_+$ of equation (2.5) (equivalently, any solution $R := \{R_i\}_{i=1}^N$ to equation (2.5)) yields a covariant representation $(i, \alpha_+)$ of $(\mathcal{S}, \beta)$ and $(i, \alpha_+)$ preserves $(\mathcal{S}(C(\mathbb{T})), \beta)$ if and only if $b$ is a finite Blaschke product.

As elementary as this result seems to be, we do not know how to prove it without recourse to Theorem 3.3.

**Proof.** The existence of a solution $\alpha_+$ to equation (2.5) guarantees that the map $\tau(\varphi) \to \tau(\varphi \circ b)$, $\varphi \in L^\infty(\mathbb{T})$, extends to a $*$-endomorphism of $\mathcal{S}$, because $\alpha_+$ is a $C^*$-endomorphism of a larger $C^*$-algebra, namely $B(H^2(\mathbb{T}))$. Thus Theorem 3.3 shows that composition with $b$ extends to $\mathcal{S}$. If $b$ is a finite Blaschke product then composition with $b$ leaves $C(\mathbb{T})$ invariant, i.e., $\beta$ leaves $C(\mathbb{T})$ invariant. Since the solution $\alpha_+$ to equation (2.5) is of the form $\alpha R$ where the Cuntz family $R$ is finite, $\alpha_+$ leaves $\mathcal{S}$ invariant and, therefore, it leaves $\mathcal{S}(C(\mathbb{T}))$ invariant when $b$ is a finite Blaschke product. Conversely, if $\beta$ leaves $\mathcal{S}(C(\mathbb{T}))$ invariant, then letting $\varphi(z) = z$, we see that $\tau(b) = \alpha_+(\tau(\varphi)) = \tau \circ \beta(\varphi)$ must be of the form $\tau(f) + k$, for some compact operator $k$ and some continuous function $f$. But then $\tau(b - f) = k$, and so, by [4, 7.15], $b = f$ is continuous, and hence a finite Blaschke product.

Rochberg’s analysis and Laca’s result [11, Proposition 2.2] together yield the following.
Corollary 3.5. A Cuntz family $R = \{R_i\}_{i=1}^N$ in $B(H^2(\mathbb{T}))$ satisfies the equation $\alpha_R \circ \tau = \tau \circ \beta$ if and only if there is an orthonormal basis $\{v_i\}_{i=1}^N$ for $D = H^2(\mathbb{T}) \oplus \pi(b)H^2(\mathbb{T})$ so that the $R_i$ may be expressed in terms of it as in Theorem 3.3.

Proof. Theorem 3.3 asserts that if $R$ is a Cuntz family in $B(H^2(\mathbb{T}))$ of the indicated form, then $\alpha_R \circ \tau = \tau \circ \beta$. For the converse, suppose $R := \{R_i\}_{i=1}^N$ is a Cuntz family in $B(H^2(\mathbb{T}))$ so that $\alpha_R \circ \tau = \tau \circ \beta$. Then, as we saw in Corollary 3.4, $\alpha_R$ leaves the Toeplitz algebra $\mathcal{T}$ invariant. Choose any orthonormal basis $\{v_i\}_{i=1}^N$ for $D$ and let $\tilde{R} = \{\tilde{R}_i\}_{i=1}^N$ be corresponding Cuntz family on $H^2(\mathbb{T})$ obtained from Theorem 3.3. Then the equation $\alpha_R \circ \tau = \tau \circ \beta$ is also satisfied, by Theorem 3.3. It follows that $\alpha_R$ and $\alpha_{\tilde{R}}$ agree on $\mathcal{T}$. Since $\mathcal{T}$ is ultraweakly dense in $B(H^2(\mathbb{T}))$ (because $\mathcal{T}$ contains $\mathcal{R}$) and since $\alpha_R$ and $\alpha_{\tilde{R}}$ are ultraweakly continuous maps of $B(H^2(\mathbb{T}))$, $\alpha_R = \alpha_{\tilde{R}}$ on all of $B(H^2(\mathbb{T}))$. Thus by [11, Proposition 2.2], there is a unitary $N \times N$ scalar matrix $(u_{ij})$ such that $R_i = \sum_j u_{ij} \tilde{R}_j$. But $\{(\sum_j u_{ij}v_j)\}_{i=1}^N$ is also an orthonormal basis of $D$, and so the $R_i$’s have the desired form. \(\square\)

Remark 3.6. It was previously remarked that the nonconstant inner function $b$ is a finite Blaschke product if and only if the space $D = H^2(\mathbb{T}) \oplus \pi(b)H^2(\mathbb{T})$ has finite dimension. In fact, if $b$ is a finite Blaschke product, then $D$ has dimension equal to the number of zeros of $b$ and its elements are rational functions with poles located in a finite set outside the closed unit disc. This may be seen by writing

\begin{equation}
(3.5) \quad b(z) = \prod_{j=1}^N b_{\alpha_j},
\end{equation}

where the $\alpha_i$ are the not-necessarily-distinct zeros of $b$. One can check that the functions $\{w_i\}_{i=1}^N$ constructed from partial products of $b$ by way of

\begin{equation}
\begin{aligned}
w_j(z) &= \frac{(1 - |\alpha_j|^2)^{1/2}}{1 - \overline{\alpha}_jz} \prod_{k=1}^{j-1} b_{\alpha_k}, \\
1 \leq j \leq N,
\end{aligned}
\end{equation}

(the product $\prod_{k=1}^{j-1} b_{\alpha_k}$ is interpreted as 1 when $j = 1$), form an orthonormal basis for $D$ (see [18, p. 305]). We call this the canonical orthonormal basis for $D$. Note that the elements of the canonical basis are nonzero on $\mathbb{T}$ and hence invertible elements of $C(\mathbb{T})$. The analysis in [18] shows that if $b$ is not a finite Blaschke product, then $D$ is infinite dimensional. Alternatively, one may use the simple corollary of Beurling’s theorem that asserts that $\pi(\theta_1)H^2(\mathbb{T}) \subseteq \pi(\theta_2)H^2(\mathbb{T})$ if and only if the quotient $\theta_1/\theta_2$ is an inner function. (See [4] page 11 ff.) The point from this perspective is: if $b$ is not a finite Blaschke product, then $b$ has infinitely many inner factors, say $b = \prod_{n=1}^\infty b_n$, and from these, one can construct an infinite increasing sequence of closed subspaces of $D$.

To identify all the solutions to equation (1.1) in Problem 1.1, we need to restrict attention to finite Blaschke products. For this reason and to get a clearer picture of the Cuntz isometries implementing $\alpha$ and $\alpha_+$ we emphasize:

From now on, $b$ will denote a finite Blaschke product.

In [16, Theorem 1], Ryff shows that if $\varphi$ is analytic on the disc $\mathbb{D}$ and maps $\mathbb{D}$ into $\mathbb{D}$, then composition with $\varphi$ induces a bounded operator on all the $H^p$ spaces. The principal ingredient in his proof is Littlewood’s subordination theorem. In [16, Theorem 3], Ryff shows further that composition with $\varphi$ is an isometry on
Let $b$ be a finite Blaschke product and define $\Gamma_b$ on trigonometric polynomials $p$ by $\Gamma_b(p) := p \circ b$. Then $\Gamma_b$ extends in a unique way to a bounded operator on $L^2(\mathbb{T})$ that leaves $H^2(\mathbb{T})$ invariant.

Moreover, letting $\Gamma_b$ now denote the extension, the following are equivalent:

1. $\Gamma_b$ is an isometry.
2. $b(0) = 0$.
3. $\Gamma_b$ is reduced by $H^2(\mathbb{T})$.

Proof. Fix an element $w$ of the canonical basis for $\mathcal{D} = H^2(\mathbb{T}) \ominus \pi(b)H^2(\mathbb{T})$. By Theorem 3.3 there is a unique isometry $S$ on $L^2(\mathbb{T})$ satisfying

\[ S(e_n) = wb^n, \quad n \in \mathbb{Z}. \]

As observed in Remark 3.6, $w$ is an invertible element of $C(\mathbb{T})$, so the operator $\pi(w)$ is invertible. The relation (3.6) then implies that for any trigonometric polynomial $p$ we have

\[ \pi(w^{-1})S(p) = \Gamma_b(p), \]

so the bounded operator $\pi(w^{-1})S$ is an extension of $\Gamma_b$ to all of $L^2(\mathbb{T})$. Uniqueness of the extension follows from the density of the trigonometric polynomials in $L^2(\mathbb{T})$. The fact that $\Gamma_b(e_n) = b^n$ is in $H^2(\mathbb{T})$ for every $n \geq 0$ implies that this extension leaves $H^2(\mathbb{T})$ invariant.

If $b(0) = 0$, then $b(z) = zb_1(z)$, where $b_1$ is in $H^2(\mathbb{T})$. It follows that for any $n > m$ we have $(b^n, b^m) = (z^{n-m}b_1^{n-m}, 1) = 0$, so that the family $\{b^n\}_{n \in \mathbb{Z}}$ is orthonormal. Since $\Gamma_b(e_n) = b^n$ for all $n \in \mathbb{Z}$, we conclude that $\Gamma_b$ is an isometry. Conversely, if $\Gamma_b$ is an isometry,

\[ b(0) = (b, e_0) = (\Gamma_b(e_1), \Gamma_b(e_0)) = (e_1, e_0) = 0. \]

This establishes the equivalence of (1) and (2).

It will be useful later to deduce the equivalence of (2) and (3) from the assertion that if a vector $\xi \in L^2(\mathbb{T})$ has the property that the pointwise product $\xi \otimes b$ is in $H^2(\mathbb{T})$, then $\Gamma_b^*\xi$ is in $H^2(\mathbb{T})$ if and only if $\langle \xi b \rangle(0) = 0$. To prove this assertion, note that $\Gamma_b^*\xi$ is in $H^2(\mathbb{T})$ if and only if $\langle \Gamma_b^*\xi, z^{-n} \rangle = 0$ for all $n > 0$, and this is equivalent to

\[ 0 = \langle \xi, \Gamma_b(z^{-n}) \rangle = \langle \xi, b^{-n} \rangle = (\xi b^n, 1) = ((\xi b)b^{n-1}, 1) = (\xi b)(0)b^{n-1}(0), \quad n > 0, \]

which is equivalent to $\langle \xi b \rangle(0) = 0$. It follows from this assertion that $\Gamma_b^*\xi \in H^2(\mathbb{T})$ for all $\xi \in H^2(\mathbb{T})$ if and only if $b(0) = 0$. \hfill $\square$

All of our proofs to this point have used only elementary operator theory. To go further, we require more detailed information about finite Blaschke products.

4. The Master Isometry

The zeros of our finite Blaschke product $b$ will be written $\alpha_1, \alpha_2, \ldots, \alpha_N$, and we abbreviate $b_{\alpha_i}$ by $b_i$. As it was in Corollary 3.7, the bounded operator of composition by $b$ on $L^2(\mathbb{T})$ is denoted $\Gamma_b$.

Although all Cuntz families $\{S_i\}_{i=1}^N$ that we constructed in Theorem 3.3 are closely linked to $\Gamma_b$, $\Gamma_b$ is not quite the operator we want to work with. It turns out

$H^p$ if and only if $\phi$ is an inner function that vanishes at the origin. The following consequence of Theorem 3.3 is a variation on this theme with a very elementary proof.
that there is a single isometry $C_b$, built canonically from $\Gamma_b$, that has the property that every Cuntz family $S = \{ S_i \}_{i=1}^N$ satisfying equation (2.4) can be expressed in terms of $C_b$. More remarkably, $C_b$ is reduced by $H^2(T)$. We call this isometry the master isometry determined by $b$ (or by the endomorphism $\beta$ induced by $b$).

Much of the material below is contained in results already in the literature (see in particular [14] and [19]). But many calculations are done under the additional hypothesis that $b(0) = 0$, which we want specifically to avoid. In the interest of keeping our treatment self-contained, we present all of the details.

**Lemma 4.1.** Define

$$J_0(z) := \frac{b'(z)z}{N b(z)}. \quad (4.1)$$

Then the restriction of $J_0$ to $T$ is a positive continuous function and, in particular, is bounded away from zero.

**Proof.** Of course, $J_0$ is a rational function. What needs proof is that on $T$, $J_0$ is positive, non-vanishing and has no poles. If $\alpha_j \neq 0$, then

$$\frac{b_j'(z)}{b_j(z)} = \frac{1}{z} \frac{1 - |\alpha_j|^2}{|\alpha_j - z|^2},$$

while if $\alpha_j = 0$, $\frac{b_j'(z)}{b_j(z)} = \frac{1}{z}$. In either case, $\frac{b_j'(z)}{b_j(z)}$ is strictly positive on $T$. A short calculation shows that $J_0(z) = \frac{1}{N} \sum_{j=1}^N \frac{b_j'(z)}{b_j(z)}$, so the result follows. \hfill \Box

The next lemma follows [14] Lemma 1] closely.

**Lemma 4.2.** There is an increasing homeomorphism $\theta : [0, 2\pi] \to [\theta(0), \theta(0) + N \cdot 2\pi]$, where $e^{i\theta(0)} = b(1)$, such that

1. $b(e^{it}) = e^{i\theta(t)}$.
2. The derivative of $\theta$ on $(0, 2\pi)$ is $\frac{b'(e^{it})}{b(e^{it})} e^{it} \geq 0$.
3. If $(t_{j-1}, t_j) = \theta^{-1}((\theta(0) + (j - 1) \cdot 2\pi, \theta(0) + j \cdot 2\pi))$, $j = 1, 2, \ldots, N$, and if $A_j := \{ e^{it} | t_{j-1} \leq t \leq t_j \}$, then $\bigcup_{j=1}^N A_j = T$, except for a finite set of points, and $b$ maps each $A_j$ diffeomorphically onto $T \setminus \{b(1)\}$.
4. If $\sigma_j : T \setminus \{b(1)\} \to A_j$ denotes the inverse of the restriction of $b$ to $A_j$, then as $s$ ranges over $(\theta(0) + 2\pi(j - 1), \theta(0) + 2\pi j)$, $e^{is}$ ranges over $T \setminus \{b(1)\}$ and

$$\sigma_j(e^{is}) = e^{i\theta^{-1}(s)}.$$  

**Proof.** Each $b_j$ is analytic in a neighborhood of the closed unit disc and maps $T$ homeomorphically onto $T$ in an orientation preserving fashion. If the plane is slit along the ray through the origin and $b_j(1)$, then one can define an analytic branch of $\log z$ in the resulting region. On $T \setminus \{b_j(1)\}$, $\log b_j(e^{it}) = i\theta_j(t)$ for a smooth function $\theta_j(t)$ defined initially on $(0, 2\pi)$, and mapping to $(\theta_j(0), \theta_j(0) + 2\pi)$. Further, if one differentiates the defining equation for $\theta_j$, one finds that $i\theta'_j(t) = \frac{b'_j(e^{it})}{b_j(e^{it})} e^{it}$, so $\theta'_j$ is strictly positive, as was shown in the preceding lemma. Hence $\theta_j$ is strictly increasing. Since $b_j(e^{i\theta}) = b_j(1) = b_j(e^{i\theta_j(0)})$, $\theta_j$ extends to a homeomorphism from $[0, 2\pi]$ onto $[\theta_j(0), \theta_j(0) + 2\pi]$. If $\theta$ is defined on $[0, 2\pi]$ by the formula $\theta(t) = \sum_{j=1}^N \theta_j(t)$, then $\theta$ is a strictly increasing homeomorphism from $[0, 2\pi]$ onto $[\theta(0), \theta(0) + N \cdot 2\pi]$ such that $b(e^{it}) = e^{i\theta(t)}$. The remaining assertions are now clear. \hfill \Box
Definition 4.3. The (canonical) transfer operator determined by the Blaschke product $b$ is defined on measurable functions $\xi$ by the formula

$$\mathcal{L}(\xi)(z) := \frac{1}{N} \sum_{b(w)=z} \xi(w).$$

Of course, an alternate formula for $\mathcal{L}$ is $\mathcal{L}(\xi)(z) = \frac{1}{N} \sum_{j=1}^{N} \xi(\sigma_j(z))$, when $z \in T \setminus \{b(1)\}$. It is clear that $\mathcal{L}$ carries measurable functions to measurable functions, preserves order, and is unital. Because $b$ is a local homeomorphism, $\mathcal{L}$ carries $C(T)$ into itself. It is not difficult to see that $\mathcal{L}$ is a bounded linear operator on $L^2(T)$. However, we present a proof of this that connects $\mathcal{L}$ with the adjoint of $\Gamma_b$. For this purpose, note that by Lemma 4.1, $\pi(J_0)$ is a bounded, positive, invertible operator on $L^2(T)$ with inverse $\pi(J_0^{-1})$.

Theorem 4.4.

(4.2) $$\mathcal{L}\pi(J_0)^{-1} = \Gamma_b^*$$

Proof. For $\xi$ and $\eta$ in $L^2(T)$,

$$(\Gamma_b^*\xi, \eta) = (\xi, \Gamma_b\eta) = \int_0^{2\pi} \xi(z)\overline{\eta(b(z))} \, dm(z) = \sum_{j=1}^{N} \int_{A_j} \xi(z)\overline{\eta(b(z))} \, dm(z).$$

From the first and third assertions of Lemma 4.2

$$\int_{A_j} \xi(z)\overline{\eta(b(z))} \, dm(z) = \int_{t_j}^{t_{j-1}} \xi(e^{i\theta})\overline{\eta(e^{i\theta(t)})} \, dt.$$  

Changes the variable to $s = \theta(t)$, the third and fourth assertions of Lemma 4.2 imply

$$\int_{t_j}^{t_{j-1}} \xi(e^{i\theta})\overline{\eta(e^{i\theta(t)})} \, dt = \int_{\theta(t_{j-1})}^{\theta(t_j)} \xi(e^{i\theta^{-1}(s)})\overline{\eta(e^{i\theta^{-1}(s)})} (\theta^{-1})'(s) \, ds.$$  

Calculating $(\theta^{-1})'(s) = (\theta'(t))^{-1} = (\theta'(\theta^{-1}(s)))^{-1}$ and using the second assertion of Lemma 4.2 we deduce

$$\int_{\theta(t_{j-1})}^{\theta(t_j)} \xi(e^{i\theta^{-1}(s)})\overline{\eta(e^{i\theta^{-1}(s)})} (\theta^{-1})'(s) \, ds = \int_{\theta(t_{j-1})}^{\theta(t_j)} \xi(e^{i\theta^{-1}(s)})\overline{\eta(e^{i\theta^{-1}(s)})} \frac{b(e^{i\theta^{-1}(s)})}{b'(e^{i\theta^{-1}(s)})e^{i\theta^{-1}(s)}} \, ds.$$  

But by the fourth statement of Lemma 4.2 $e^{i\theta^{-1}(s)} = \sigma_j(e^{is})$, when $s \in (\theta(0) + 2\pi(j - 1), \theta(0) + 2\pi j)$. So the last integral is

$$\int_{\theta(t_{j-1})}^{\theta(t_j)} \xi(\sigma_j(e^{is}))\overline{\eta(e^{is})} \frac{b(\sigma_j(e^{is}))}{b'(\sigma_j(e^{is}))\sigma_j(e^{is})} \, ds.$$
As $e^{is}$ sweeps out $\mathbb{T}\setminus \{b(1)\}$ as $s$ ranges over each interval $(\theta(t_{j-1}), \theta(t_j)) = (\theta(0) + 2\pi(j - 1), \theta(0) + 2\pi j)$, we conclude that

$$
(\Gamma_b \xi, \eta) = \sum_{j=1}^{N} \int_{\theta(0) + 2\pi(j-1)}^{\theta(0) + 2\pi j} \xi(\sigma_j(e^{is}))\eta(e^{is}) b'(\sigma_j(e^{is})) ds
$$

$$
= \frac{1}{N} \sum_{j=1}^{N} \int_{\mathbb{T}} \xi(\sigma_j(z)) \frac{N b(\sigma_j(z))}{b'(\sigma_j(z))} \overline{\eta(z)} dz \int_{\mathbb{T}} \theta(0) + 2\pi \left(\sum_{j=1}^{N} \int_{\mathbb{T}} \xi(\sigma_j(z)) (J_0(\sigma_j(z)))^{-1} \overline{\eta(z)} dm(z)
$$

$$
= (\mathcal{L}(\pi(J_0)^{-1} \xi), \eta),
$$

showing that $\Gamma_b = \mathcal{L}(J_0)^{-1}$. \hfill \box

**Notation 4.5.**

$$
J(z) := \exp \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \ln(J_0(e^{it})) dt \right]
$$

Of course $J$ is the unique outer function that is positive at 0 and satisfies the equation $|J(z)| = J_0(z)$ for all $z \in \mathbb{T}$. (See [7, Theorem 5] and the surrounding discussion.) Significantly, $J$ does not vanish on $\mathbb{D}$ and is in $H^\infty(\mathbb{T})$; note that $J_0$ is not even in $H^2(\mathbb{T})$ except in trivial cases. We will work primarily with $J^{1/2}$, which is $\exp \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \ln(J_0(e^{it})) dt \right]$. Note that $J^{1/2}$ and $J^{-1/2}$ are both in $H^\infty(\mathbb{T})$.

**Lemma 4.6.** For all $\varphi \in L^\infty(\mathbb{T})$,

$$
\mathcal{L}(\pi(\varphi) \Gamma_b) = \pi(\mathcal{L}(\varphi)).
$$

In particular, $\mathcal{L}$ is a left inverse for $\Gamma_b$.

**Proof.** Take $\xi \in L^2(\mathbb{T})$ and $\varphi \in L^\infty(\mathbb{T})$ and calculate:

$$
\mathcal{L}(\pi(\varphi) \Gamma_b(\xi))(z) = \frac{1}{N} \sum_{b(w) = z} \pi(\varphi)(\Gamma_b(\xi))(w) = \frac{1}{N} \sum_{b(w) = z} \varphi(\omega) \xi(b(w))
$$

$$
= \frac{1}{N} \sum_{b(w) = z} \varphi(\omega) \xi(z) = \mathcal{L}(\varphi)(z) \xi(z)
$$

$$
= (\pi(\mathcal{L}(\varphi)) \xi)(z). \hfill \box
$$

**Lemma 4.7.** Set

$$
C_b := \pi(J^{1/2}) \Gamma_b.
$$

Then $C_b$ is an isometry on $L^2(\mathbb{T})$ and

$$
C_b^* = \mathcal{L}(J^{-1/2}).
$$

Further, if $\{S_i\}_{i=1}^{N}$ is the Cuntz family constructed in Theorem 3.3 using an orthonormal basis $\{v_i\}_{i=1}^{N}$ for $H^2(\mathbb{T}) \oplus \pi(b)H^2(\mathbb{T})$, then $S_i = \pi(v_i J^{-1/2}) C_b$ for all $1 \leq i \leq N$. 

Proof. The key is the relation $L\pi(J_n^{-1}) = \Gamma_n^*$ from Theorem [12]. We just compute:

$$C_n \pi C_b = \Gamma_b^* \pi (J_2^\pi) \pi (J_2^\pi) \Gamma_b = \Gamma_b^* \pi (J_2^\pi) \Gamma_b = \Gamma_b^* \pi (J_0) \Gamma_b = L\Gamma_b = I.$$  

and

$$C_n^* = \Gamma_b^* \pi (J_2^\pi) = L\pi (J_0)^{-1} \pi (J_2^\pi) = L\pi (J_0^\frac{1}{2}).$$

For the final assertion, simply observe that the definition of $S_i$ (using $\{v_i\}_{i=1}^N$) shows that $S_i = \pi(v_i)\Gamma_b$. As $C_b = \pi(J_0^\frac{1}{2})\Gamma_b$, we conclude

$$S_i = \pi(v_i)\pi(J_0^\frac{1}{2})\pi(J_0^\frac{1}{2})\Gamma_b = \pi(v_iJ_0^\frac{1}{2})C_b.$$

\[\square\]

**Proposition 4.8.** $H^2(T)$ reduces $C_b$ and $C_b$ implements $L$ in the sense that

$$C_n^* \pi(\varphi) C_b = \pi(L(\varphi)),$$

for all $\varphi \in L^\infty(T)$.

Proof. Since $\Gamma_b$ and $\pi(J_0^\frac{1}{2})$ leave $H^2(T)$ invariant, so does $C_b = \pi(J_0^\frac{1}{2})\Gamma_b$. On the other hand, $C_n^* = L\pi(J_0^\frac{1}{2})$ by Lemma [4.7], so one way to show that $H^2(T)$ reduces $C_b$ is to show that $L$ leaves $H^2(T)$ invariant. McDonald did this in [13] Lemma 2.

We can also prove this directly: fixing $\eta \in H^2(T)$, we must show that $L\eta \in H^2(T)$. By Theorem [4.4], we have that $L = \Gamma_b^* \pi(J_0)$, so it suffices to show that the vector $\xi = \pi(J_0)\eta \in L^2(T)$ is mapped into $H^2(T)$ by $\Gamma_b^*$. By the definition (4.3) of $J_0$ we have

$$b(z)\xi(z) = b(z)J_0(z)\eta(z) = zb'(z)\eta(z),$$

showing that $b\xi$ is in $H^2(T)$ and that $(b\xi)(0) = 0$. Thus $\Gamma_b^*\xi \in H^2(T)$ by the argument given in the proof of Corollary [4.4].

Equation (4.3) follows from Lemmas [4.7] and [4.8] because

$$C_n^* \pi(\varphi) C_b = L\pi(J_0^{-\frac{1}{2}})\pi(\varphi)\pi(J_0^\frac{1}{2})\Gamma_b = L\pi(\varphi)\Gamma_b = \pi(L(\varphi)).$$

\[\square\]

We shall denote the restriction of $C_b$ to $H^2(T)$ by $C_{b^+}$.

**Theorem 4.9.**

1. If $T$ is a bounded operator on $L^2(T)$, then $T$ satisfies

$$T \pi(\varphi) = \pi(\beta(\varphi))T, \quad \varphi \in L^\infty(T),$$

if and only if $T = \pi(m)C_{b^+}$ for some function $m \in L^\infty(T)$.

2. If $T$ is a bounded operator on $H^2(T)$, then

$$T \tau(\varphi) = \tau(\beta(\varphi))T, \quad \varphi \in H^\infty(T),$$

if and only if $T = \tau(m)C_{b^+}$ for some function $m \in H^\infty(T)$.

Further, if $T = \pi(m)C_{b^+}$, $m \in L^\infty(T)$, (resp. if $T = \tau(m)C_{b^+}$, $m \in H^\infty(T)$) then $\|T\| = (\|L(|m|^2)\|_\infty)^\frac{1}{2}$, and $T$ is an isometry if and only if $L(|m|^2) = 1$ a.e.
Proof. We begin by proving assertion (1). If \( T = \pi(m)C_b \) for some \( m \in L^\infty(\mathbb{T}) \), a short calculation shows that \( T \) satisfies (4.4). The formula (1.3) then implies

\[
T^*T = C_b^* \pi(m)\pi(m)C_b = \pi(\mathcal{L}(|m|^2)),
\]

and \( \|T\| = (\|\mathcal{L}(|m|^2)\|_\infty)^{1/2} \) follows as \( \pi \) is faithful. The fact that \( T \) is isometric if and only if \( \mathcal{L}(|m|^2) = 1 \) a.e. is immediate.

Suppose conversely that \( T \) is an operator on \( L^2(\mathbb{T}) \) satisfying (4.4). Define \( m := \pi(J^{-1/2})T(1) \), where 1 is the constant function that is identically equal to 1. Note that a priori we have that \( m \in L^2(\mathbb{T}) \), but not that \( m \in L^\infty(\mathbb{T}) \).

The hypothesis (4.7) and the definition of \( m \) imply that

\[
T\varphi = T\pi(\varphi)1 = \pi(\beta(\varphi))\pi(J^{1/2})m = mC_b(\varphi), \quad \varphi \in L^\infty(\mathbb{T}).
\]

If more generally \( \varphi \in L^2(\mathbb{T}) \), there is a sequence \( \varphi_n \) in \( L^\infty(\mathbb{T}) \) such that \( \varphi_n \to \varphi \) in \( L^2(\mathbb{T}) \). Boundedness of \( C_b \) implies that the sequence of vectors \( C_b\varphi_n \) is convergent in \( L^2(\mathbb{T}) \) with limit \( C_b\varphi \), and boundedness of \( T \) together with (1.3) implies that the sequence \( mC_b\varphi_n \) is convergent in \( L^2(\mathbb{T}) \) with limit \( T\varphi \). By passing to a subsequence as necessary we may assume that \( C_b\varphi_n \to C_b\varphi \) pointwise a.e., and \( mC_b\varphi_n \to T\varphi \) pointwise a.e. and deduce \( T\varphi = mC_b\varphi \). We conclude that

\[
T\varphi = mC_b(\varphi), \quad \varphi \in L^2(\mathbb{T}).
\]

Fix an orthonormal basis \( \{v_i\}_{i=1}^N \) for \( H^2(\mathbb{T}) \cap \pi(\mathcal{B})H^2(\mathbb{T}) \) and set \( S_i = \pi(v_i J^{-1/2})C_b \). By Lemma (4.7) and Theorem (3.3) \( \{S_i\}_{i=1}^N \) is a Cuntz family, so for any \( \xi \in L^2(\mathbb{T}) \) we have

\[
m\xi = m \sum_{j=1}^N S_j S_j^* \xi = m \sum_{j=1}^N \pi(v_j J^{-1/2})C_b S_j^* \xi
= \sum_{j=1}^N v_j J^{-1/2} m C_b S_j^* \xi
= \sum_{j=1}^N v_j J^{-1/2} T S_j^* \xi \quad \text{by (4.7)}.
\]

Thus multiplication by \( m \) is the operator \( \sum_{j=1}^N \pi(v_j J^{-1/2})T S_j^* \) on \( L^2(\mathbb{T}) \). As this operator is bounded we deduce that \( m \in L^\infty(\mathbb{T}) \) as desired.

The proof of assertion (2) is similar, but it is important to keep track of the differences. If \( T = \tau(m)C_{b+} \) for some \( m \in H^\infty(\mathbb{T}) \), it is easily seen that (1.3) is satisfied, since \( \tau(m) \) and \( \tau(\varphi) \) commute when \( m \) and \( \varphi \) are in \( H^\infty(\mathbb{T}) \), and since \( C_{b+} \) is the restriction of \( C_b \) to a reducing subspace. Furthermore,

\[
T^*T = C_{b+}^* \tau(m)^* \tau(m)C_{b+} = PC_b^* P\pi(m)P\pi(m)PC_bP|_{H^2(\mathbb{T})}.
\]

Since \( H^2(\mathbb{T}) \) is invariant under \( \pi(m) \) and reduces \( C_b \), we deduce

\[
T^*T = PC_b^* \pi(m)C_bP|_{H^2(\mathbb{T})} = P\pi(\mathcal{L}(|m|^2))P|_{H^2(\mathbb{T})} = \tau(\mathcal{L}(|m|^2)).
\]

Thus \( \|T^*T\| = \|\tau(\mathcal{L}(|m|^2))\| \) follows as \( \tau \) is faithful, which proves the formula for the norm of \( T \). Also, it shows that \( T \) is an isometry if and only if \( \mathcal{L}(|m|^2) = 1 \) a.e.

Suppose conversely that \( T \) on \( H^2(\mathbb{T}) \) satisfies equation (1.5) and set \( m := \tau(J^{-1/2})T(1) \); we know \( m \in H^2(\mathbb{T}) \) and wish to deduce that \( m \in H^\infty(\mathbb{T}) \). The
fact that $J^{-\frac{1}{2}} \in H^\infty(\mathbb{T})$ and the properties of $C_{b+}$ show that

$$T \varphi = T\tau(\varphi)1 = \tau(\beta(\varphi))\tau(J^{1/2})m = mC_{b+}(\varphi)$$

for all $\varphi \in H^\infty(\mathbb{T})$ and hence all $\varphi \in H^2(\mathbb{T})$. With $S_i = \pi(v_i J^{-\frac{1}{2}})C_b$ as before, we note that $H^2(\mathbb{T})$ reduces $S_i$, by Theorem 3.3 and we set $R_i := S_i|_{H^2(\mathbb{T})}$. Theorem 3.3 asserts that $\{R_i\}_{i=1}^N$ is a Cuntz family of isometries on $H^2(\mathbb{T})$. Since $H^2(\mathbb{T})$ reduces $C_b$, we have for any $\xi \in H^2(\mathbb{T})$

$$m\xi = m \sum_{j=1}^N R_j R_j^* \xi = m \sum_{j=1}^N \pi(v_j J^{-1/2})C_b R_j^* \xi = \sum_{j=1}^N v_j J^{-1/2} mC_{b+} R_j^* \xi$$

$$= \sum_{j=1}^N \pi(v_j J^{-1/2})TR_j^* \xi.$$ 

As $v_j J^{-1/2} \in H^\infty$ for each $j$, the conclusion is that multiplication by $m$ is the bounded operator $\sum_{j=1}^n \pi(v_j J^{-1/2})TR_j^*$ on $H^2(\mathbb{T})$. Thus $m \in H^\infty(\mathbb{T})$. \hfill \box

We have called $C_b$ the master isometry. One explanation of our use of the definite article is that when one builds the Deaconu-Renault groupoid $G$ determined by $b$, viewed as a local homeomorphism of $\mathbb{T}$, then $C_b$ appears as the image of a special isometry $S$ in the groupoid $C^*(G)$ under a representation that gives rise to the Cuntz families we consider here. We have not seen any compelling reason to bring this technology into this note - nevertheless, $C^*(G)$ and $S$ are lying in the background and may prove useful in the future. For further information about the use of groupoids and $C^*$-algebras generated by local homeomorphisms, see [9].

One should not infer from our use of the definite article that $C_b$ is uniquely determined by the abstract properties that we have shown it has. More precisely, suppose that $V$ is an isometry on $L^2(\mathbb{T})$ that implements $\mathcal{L}$ in the sense of (4.3), interacts with $\pi$ as in (4.4), and is reduced by $H^2(\mathbb{T})$. By Theorem 1.9 $V$ must be of the form $V = \pi(m)C_b$ for some $m \in L^\infty(\mathbb{T})$ satisfying $\mathcal{L}(\|m\|^2) = 1$. The assumption that $V$ is reduced by $H^2(\mathbb{T})$, together with the assumption that $V$ implements $\mathcal{L}$ imply that $|m| = 1$ a.e.; it may further be shown that $m$ is an inner function with the property that $\mathcal{L}(\overline{m})$ is constant. But in general we have been unable to deduce more about $m$ than this. (We note that $m$ need not be constant. If $b(z) = z^2$, then $V = \pi(m)C_b$ will have all of the indicated properties if $m(z) = z^k$ for any odd positive integer $k$. What happens when $b$ is a more general Blaschke product remains to be investigated.)

5. HILBERT MODULES AND ORTHONORMAL BASES

The endomorphism $\beta$ of $L^\infty(\mathbb{T})$ and the transfer operator $\mathcal{L}$ may be used to endow $L^\infty(\mathbb{T})$ with the structure of a Hilbert $C^*$-module over $L^\infty(\mathbb{T})$. We will exploit this structure in order to solve Problem 1.1. We do not need much of the general theory about these modules. Rather, we only need to expose enough so that the formulas we use make good sense. Excellent references for the basics of the theory are [12] [13].

Suppose $A$ is a $C^*$-algebra and that $E$ is a right $A$-module. Then $E$ is called a Hilbert $C^*$-module over $A$ in case $E$ is endowed with an $A$-valued sesquilinear form $\langle \cdot, \cdot \rangle : E \times E \to A$ that is subject to the following conditions.

1. $\langle \cdot, \cdot \rangle$ is conjugate linear in the first variable, so $\langle \xi \cdot a, \eta \cdot b \rangle = a^* \langle \xi, \eta \rangle b$. 

(2) For all \( \xi \in E \), \( \langle \xi, \xi \rangle \) is a positive element in \( A \) that is 0 if and only if \( \xi = 0 \).

(3) \( E \) is complete in the norm defined by the formula

\[
\| \xi \|^2_E := \| \langle \xi, \xi \rangle \|_A.
\]

Of course, it takes a little argument to prove that \( \| \cdot \|_E \) is a norm on \( E \).

In the application of Hilbert modules that we have in mind, our \( C^* \)-algebra \( A \) will be unital, and we will denote the unit by \( 1 \). A vector \( v \in E \) is called a unit vector if \( \langle v, v \rangle = 1 \). Note that this says more than simply \( \|v\| = 1 \). A family \( \{v_i\}_{i \in I} \) of vectors in \( E \) is called an orthonormal set if \( \langle v_i, v_j \rangle = \delta_{ij} 1 \). Further, if linear combinations of vectors from \( \{v_i\}_{i \in I} \) (where the coefficients are from \( A \)) are dense in \( E \) then we say that \( \{v_i\}_{i \in I} \) is an orthonormal basis for \( E \). In this event, every vector \( \xi \in E \) has the representation

\[
\xi = \sum_{i \in I} v_i \cdot \langle v_i, \xi \rangle,
\]

where the sum converges in the norm of \( E \). In general, a Hilbert \( C^* \)-module need not have an orthonormal basis. Also, in general, two orthonormal bases need not have the same cardinal number. Nevertheless, two orthonormal bases \( \{v_i\}_{i \in I} \) and \( \{w_j\}_{j \in J} \) are linked by a unitary matrix over \( A \) in the usual way:

\[
w_j = \sum_{i \in I} v_i \cdot \langle v_i, w_j \rangle = \sum_{i \in I} v_i \cdot u_{ij}.
\]

So if the cardinality of \( I \) is \( n \) and the cardinality of \( J \) is \( m \), then \( U = (u_{ij}) \) is a unitary matrix in \( M_{nm}(A) \), i.e., \( U U^* = 1_n \) in \( M_n(A) \), while \( U^* U = 1_m \) in \( M_m(A) \). And conversely, any such matrix transforms the orthonormal basis \( \{v_i\}_{i \in I} \) for \( E \) into an orthonormal basis \( \{w_j\}_{j \in J} \) for \( E \) via this formula. In our application of these notions, the coefficient algebra \( A \) will be commutative so, as is well known, all unitary matrices are square and, therefore, any two orthonormal bases have the same number of elements.

We shall view \( L^\infty(T) \) as right module over \( L^\infty(T) \) via the formula

\[
\xi \cdot a := \xi \beta(a), \quad a, \xi \in L^\infty(T),
\]

where the product on the right hand side is the usual pointwise product in \( L^\infty(T) \). Also, we shall use \( \mathcal{L} \) to endow \( L^\infty(T) \) with the \( L^\infty(T) \)-valued inner product defined by the formula

\[
\langle \xi, \eta \rangle := \mathcal{L}(\overline{\xi} \eta), \quad \xi, \eta \in L^\infty(T).
\]

Using the fact that \( \mathcal{L} \circ \beta = \text{id}_{L^\infty(T)} \) (Lemma 4.6), it is straightforward to see that \( L^\infty(T) \) is a Hilbert \( C^* \)-module over \( L^\infty(T) \), which we shall denote by \( L^\infty(T)_\mathcal{L} \). The only thing that may be seem problematic is the fact that \( L^\infty(T)_\mathcal{L} \) is complete in the norm defined by the inner product. However, a moment’s reflection reveals that the norm is equivalent to the \( L^\infty(T) \)-norm, which is complete. We remark that (5.2) and (5.3) make sense when the functions in \( L^\infty(T) \) are restricted to lie in \( C(T) \), and \( C(T) \) also is a Hilbert module over \( C(T) \) in this structure, but we will focus on the \( L^\infty(T) \) case in what follows.

Vectors \( \{m_i\}_{i=1}^N \) in \( L^\infty(T)_\mathcal{L} \) form an orthonormal basis for \( L^\infty(T)_\mathcal{L} \) if and only if

\[
\mathcal{L}(\overline{m_i} m_j) = \langle m_i, m_j \rangle = \delta_{ij} 1,
\]
where \( 1 \) is the constant function 1, and
\[
f = \sum_{i=1}^{N} m_i \cdot \langle m_i, f \rangle = \sum_{i=1}^{N} m_i \beta(\mathcal{L}(m_i f)), \quad f \in L^\infty(\mathbb{T})\mathcal{L}.
\]

We have intentionally used \( N \), the order of the Blaschke product \( b \), as the upper limit in these sums because \( L^\infty(\mathbb{T})\mathcal{L} \) has an orthonormal basis with \( N \) elements, viz. \( \{ \sqrt{N} A_i \}_{i=1}^{N} \), where the \( A_i \)'s are the arcs in Lemma 12 and because any two orthonormal bases for \( L^\infty(\mathbb{T})\mathcal{L} \) have the same number of elements, as we noted above.

**Remark 5.1.** As a map on \( L^\infty(\mathbb{T}) \), \( \mathbb{E} := \beta \circ \mathcal{L} \) is the conditional expectation onto the range of \( \beta \). Indeed, \( \mathbb{E} \) is a weak-* continuous, positivity preserving, idempotent unital linear map on \( L^\infty(\mathbb{T}) \). So it is the restriction to \( L^\infty(\mathbb{T}) \) of an idempotent and contractive linear map on \( L^1(\mathbb{T}) \) that preserves the constant functions. Hence \( \mathbb{E} \) is a conditional expectation by the corollary to [3, Theorem 1]. Of course, the range of \( \mathbb{E} \) consists of functions in the range of \( \beta \) by definition. On the other hand, if \( f = \beta(g) \) for some function \( g \in L^\infty(\mathbb{T}) \), then \( \mathbb{E}(f) = \beta \circ \mathcal{L} \circ \beta(g) = \beta(g) = f \), since \( \mathcal{L} \) is a left inverse for \( \beta \). Thus, to say that \( \{ m_i \}_{i=1}^{N} \) is an orthonormal basis for \( L^\infty(\mathbb{T})\mathcal{L} \) is to say that
\[
f = \sum_{i=1}^{N} m_i \mathbb{E}(m_i f), \quad f \in L^\infty(\mathbb{T}).
\]

In light of the discussion in Section 2, the following describes all solutions to (1.1).

**Theorem 5.2.** If a Cuntz family \( S = \{ S_i \}_{i=1}^{N} \) on \( B(L^2(\mathbb{T})) \) gives rise to a covariant representation \( (\pi, \alpha_S) \) of \( (L^\infty(\mathbb{T}), \beta) \), then there is an orthonormal basis \( \{ m_i \}_{i=1}^{N} \) for \( L^\infty(\mathbb{T})\mathcal{L} \) such that
\[
S_i = \pi(m_i) C_b, \quad 1 \leq i \leq N.
\]
Further, if \( \{ m_i \}_{i=1}^{N} \) is any family of functions in \( L^\infty(\mathbb{T}) \) such that the operators \( S_i \) defined by (5.4) form a Cuntz family \( S \) such that \( (\pi, \alpha_S) \) is a covariant representation of \( (L^\infty(\mathbb{T}), \beta) \), then \( \{ m_i \}_{i=1}^{N} \) is an orthonormal basis for \( L^\infty(\mathbb{T})\mathcal{L} \). Conversely, if \( \{ m_i \}_{i=1}^{N} \) is any orthonormal basis for \( L^\infty(\mathbb{T})\mathcal{L} \) and \( S_i \) is defined by (5.4) for \( 1 \leq i \leq N \), then \( S = \{ S_i \}_{i=1}^{N} \) is a Cuntz family and \( (\pi, \alpha_S) \) is a covariant representation of \( (L^\infty(\mathbb{T}), \beta) \).

**Proof.** Suppose \( \{ S_i \}_{i=1}^{N} \) is a Cuntz family on \( L^2(\mathbb{T}) \) that satisfies equation (2.2). If both sides of this equation are multiplied on the right by \( S_j \), then one finds from equation (2.2) that \( S_j \pi(\cdot) = \pi \circ \beta(\cdot) S_j \) for each \( j \). By Theorem 4.9, for each \( j \) there is \( m_j \in L^\infty(\mathbb{T}) \) satisfying \( S_j = \pi(m_j) C_b \) and \( 1 = \mathcal{L}(|m_j|^2) = \langle m_j, m_j \rangle \). The fact that \( S \) satisfies equation (2.2) then yields
\[
\delta_{i,j} I_{L^2(\mathbb{T})} = S_i^* S_j = C_b \pi(\overline{m_i} m_j) C_b = \pi(\mathcal{L}(\overline{m_i} m_j)) = \pi(\langle m_i, m_j \rangle).
\]
Since \( \pi \) is faithful, \( \langle m_i, m_j \rangle = \delta_{i,j} 1 \), where \( 1 \) is the constant function 1. Thus, \( \{ m_i \}_{i=1}^{N} \) is an orthonormal set in \( L^\infty(\mathbb{T})\mathcal{L} \). We now show that the \( \{ m_i \}_{i=1}^{N} \) span
$L^\infty(T)_\mathcal{L}$. If $f \in L^\infty(T)_\mathcal{L}$ satisfies $(f, m_i) = 0$ for all $i$, then we have
\[
(\pi(f)C_b)^* = C_b^* \pi(J) \left( \sum_{i=1}^N S_i S_i^* \right) = C_b^* \pi(J) \left( \sum_{i=1}^N \pi(m_i)C_b S_i^* \right)
\]
\[
= \sum_{i=1}^N C_b^* \pi(J) \pi(m_i)C_b S_i^*
\]
\[
= \sum_{i=1}^N \pi((f, m_i))S_i^* \quad \text{by (4.3)}
\]

and thus $\pi(f)C_b = 0$, which in turn implies $f J^{\frac{1}{2}} = \pi(f)J^{\frac{1}{2}} = \pi(f)C_b 1 = 0$, and thus $f = 0$. This shows that $\{m_i\}_{i=1}^N$ is an orthonormal basis for $L^\infty(T)_\mathcal{L}$.

For the converse assertion, suppose $\{m_i\}_{i=1}^N$ is any orthonormal basis for $L^\infty(T)_\mathcal{L}$, and set $S_i := \pi(m_i)C_b$. Then from (4.3) we deduce

\[
S_i^*S_j = C_b^* \pi(m_im_j)C_b = \pi(\{m_i, m_j\}) \quad \text{by (4.3)}
\]

\[
= \delta_{i,j} = 1 = \delta_{i,j} 1_{L^2(T)}.
\]

So the relations (2.2) are satisfied. To verify the Cuntz identity (2.3), note first that equation (2.2) shows that the sum $\sum_{i=1}^N S_i S_i^*$ is a projection. To show that $\sum_{i=1}^N S_i S_i^* = 1$, it suffices to show that $\sum_{i=1}^N S_i S_i^*$ acts as the identity operator on a dense subset of $L^2(T)$. So fix $f \in L^\infty(T)$ and observe that we may write
\[
(5.5) \quad \sum_{i=1}^N S_i S_i^* f = \sum_{i=1}^N S_i S_i^* \pi(f) 1 = \sum_{i=1}^N S_i S_i^* \pi(f) \Gamma_b 1 = \sum_{i=1}^N S_i S_i^* \pi(f J^{-\frac{1}{2}}) C_b 1.
\]

Since $S_i = \pi(m_i)C_b$, the last sum in (5.5) is
\[
\sum_{i=1}^N \pi(m_i)C_b C_b^* \pi(m_i)\pi(f J^{-\frac{1}{2}}) C_b 1 = \sum_{i=1}^N \pi(m_i)C_b \pi(L(m_i f J^{-\frac{1}{2}})) 1,
\]

where we have used (4.3). But by Theorem 4.9 the right hand side of this equation is
\[
\sum_{i=1}^N \pi(m_i)\pi(\beta(L(m_i f J^{-\frac{1}{2}}))) C_b 1 = \sum_{i=1}^N \pi(m_i)C_b \pi(L(m_i f J^{-\frac{1}{2}})) C_b 1 = \pi(f J^{-\frac{1}{2}}) C_b 1,
\]

because $\{m_i\}_{i=1}^N$ is an orthonormal basis for $L^\infty(T)_\mathcal{L}$, by hypothesis. As $C_b 1 = \pi(J^{\frac{1}{2}}) \Gamma_b 1 = \pi(J^{\frac{1}{2}}) 1$ it follows that $\pi(f J^{-\frac{1}{2}}) C_b 1 = f$, and thus $\sum_{i=1}^N S_i S_i^* f = f$.

We conclude that $S = \{S_i\}_{i=1}^N$ is a Cuntz family.

To see that this family implements $\beta$, simply note that
\[
\pi(\beta(\varphi)) = \pi(\beta(\varphi)) \sum_{i=1}^N S_i S_i^* = \sum_{i=1}^N S_i \pi(\varphi) S_i^*
\]

since the $S_i$ satisfy equation (4.4).

\[\square\]

**Corollary 5.3.** If $\{v_i\}_{i=1}^N$ is an orthonormal basis for the Hilbert space $H^2(T) \otimes \pi(b)H^2(T)$, then the functions $\{v_i J^{-\frac{1}{2}}\}_{i=1}^N$ form an orthonormal basis for the Hilbert module $L^\infty(T)_\mathcal{L}$.
Proof. By Lemma [4,7] the Cuntz isometries coming from \{v_i\}_{i=1}^N via Theorem [3,3] have the form \(\pi(v_i J^{-\frac{1}{2}})C_b\). Therefore by Theorem [5,2] the functions \(\{v_i J^{-\frac{1}{2}}\}_{i=1}^N\) form an orthonormal basis for \(L^\infty(\mathbb{T})_L\).

Corollary 5.4. If \(S^{(1)}\) and \(S^{(2)}\) are two Cuntz families in \(B(L^2(\mathbb{T}))\) satisfying
\[
\alpha_{S^{(i)}} \circ \pi = \pi \circ \beta, \quad i = 1, 2,
\]
then there is a unitary matrix \((u_{ij})\) in \(M_N(L^\infty(\mathbb{T}))\) such that
\[
S_j^{(2)} = \sum_{i=1}^N S_i^{(1)} \pi(u_{ij}),
\]
\(j = 1, 2, \cdots, N\). Conversely, if \(S^{(1)}\) and \(S^{(2)}\) are Cuntz families on \(L^2(\mathbb{T})\) that are linked by equation [5.0], then \(\alpha_{S^{(1)}}\) implements \(\beta\) if and only if \(\alpha_{S^{(2)}}\) implements \(\beta\). Further, \(\alpha_{S^{(1)}} = \alpha_{S^{(2)}}\) on \(B(L^2(\mathbb{T}))\) if and only if \((u_{ij})\) is a unitary matrix of constant functions.

Proof. By Theorem [5,2] we may suppose there are orthonormal bases \(\{m_i^{(1)}\}_{i=1}^N\) and \(\{m_i^{(2)}\}_{i=1}^N\) for \(L^\infty(\mathbb{T})_L\) that define \(S^{(1)}\) and \(S^{(2)}\). In this event, there is a unitary matrix \((u_{ij})\) in \(M_N(L^\infty(\mathbb{T}))\) so that
\[
m_j^{(2)} = \sum_{i=1}^N m_i^{(1)} \cdot u_{ij}.
\]
But then we may use [5.4] to derive [5.6] as follows:
\[
S_j^{(2)} = \pi(m_j^{(2)})C_b = \sum_{i=1}^N \pi(m_i^{(1)})\pi(\beta(u_{ij}))C_b = \sum_{i=1}^N \pi(m_i^{(1)})C_b \pi(u_{ij})
\]
\[= \sum_{i=1}^N S_i^{(1)} \pi(u_{ij}).\]
The same equation proves the converse assertion and the last assertion follows from Laca’s Proposition 2.2 in [11].

We conclude with a new look at Rochberg’s [15, Theorem 1] and related work of McDonald [14]. Because of the complex conjugates that appear in the formula for the inner product on \(L^\infty(\mathbb{T})_L\), it is somewhat surprising that \(\langle m_i, f \rangle \in H^\infty(\mathbb{T})\) whenever \(f \in H^\infty(\mathbb{T})\) and \(m_i\) comes from an orthonormal basis for \(H^2(\mathbb{T}) \ominus \pi(b)H^2(\mathbb{T})\).

Theorem 5.5. Let \(\{v_i\}_{i=1}^N\) be an orthonormal basis for \(D = H^2(\mathbb{T}) \ominus \pi(b)H^2(\mathbb{T})\) and let \(m_i = v_i J^{-\frac{1}{2}}\), so that \(\{m_i\}_{i=1}^N\) is an orthonormal basis for \(L^\infty(\mathbb{T})_L\) by Corollary [5.3]. Then a function \(f \in L^\infty(\mathbb{T})\) lies in \(H^\infty(\mathbb{T})\) if and only if \(\langle m_i, f \rangle\) lies in \(H^\infty(\mathbb{T})\) for all \(i\). Further, \(f\) lies in the disc algebra \(A(\mathbb{D})\) if and only if \(\langle m_i, f \rangle\) lies in \(\mathbb{D}(\mathbb{D})\) for all \(i\).

Proof. By Remark [3,6] we know the functions \(m_i\) are in the disc algebra. It is thus immediate from
\[
f = \sum_{i=1}^n m_i \cdot \langle m_i, f \rangle, \quad f \in L^\infty(\mathbb{T}),
\]
and the fact that β preserves both \( H^\infty(\mathbb{T}) \) and \( A(\mathbb{D}) \) that if the coefficients \( \langle m_i, f \rangle \) all lie in \( H^\infty(\mathbb{T}) \) or \( A(\mathbb{D}) \) then \( f \) will also.

Conversely, fix \( f \in H^\infty(\mathbb{T}) \) and any \( v \in \mathcal{D} \). We must show that \( \langle vJ^{-1/2}, f \rangle \) is in \( H^\infty(\mathbb{T}) \). Note that \( \langle vJ^{-1/2}, f \rangle \) is in \( L^\infty \) so it suffices to show that this function is in \( H^2(\mathbb{T}) \). To this end, fix a positive integer \( k \), and compute

\[
\langle vJ^{-1/2}, f \rangle, e_{-k} = (\mathcal{L}(vJ^{-1/2}f), e_{-k})
= (\Gamma_b(J_0vJ^{-1/2}f), e_{-k})
= (J_0vJ^{-1/2}f, b^{-k})
= (J^{1/2}f, b^{-k}) \text{ as } J_0 = |J| = J^{1/2}J^{1/2}
= (J^{1/2}fb^k, v).
\]

Since \( J^{1/2}f \in H^\infty \) and \( k > 0 \) the function \( J^{1/2}fb^k \) is in \( \pi(b)H^2(\mathbb{T}) \), so as \( v \in \mathcal{D} \) we conclude \( \langle J^{1/2}fb^k, v \rangle = 0 \). As \( k > 0 \) was arbitrary, \( \langle vJ^{-1/2}, f \rangle \) is in \( H^2(\mathbb{T}) \), as desired. If \( f \) is further assumed to be in \( A(\mathbb{D}) \), as \( \mathcal{L} \) maps \( C(\mathbb{T}) \) into itself we conclude \( \langle vJ^{-1/2}, f \rangle \in C(\mathbb{T}) \cap H^2(\mathbb{T}) = A(\mathbb{D}) \). □

In our notation, Rochberg’s Theorem 1 in [15] asserts that if \( \{v_i\}_{i=1}^N \) is the canonical orthonormal basis for \( \mathbb{D} \), then for any \( f \in A(\mathbb{D}) \), there are uniquely determined \( f_1, f_2, \ldots, f_N \in A(\mathbb{D}) \) satisfying

\[
f(z) = \sum_{i=1}^N v_i(z)\beta(f_i)(z), \quad z \in \mathbb{D}, \tag{5.8}
\]

and that moreover for each \( 1 \leq i \leq N \) the linear map \( f \to f_i \) thus determined on \( A(\mathbb{D}) \) is continuous in the norm of \( A(\mathbb{D}) \).

We recover this theorem by applying Theorem 5.5 to the canonical basis \( \{v_i\}_{i=1}^N \) and the function \( J^{-1/2}f \in A(\mathbb{D}) \): it asserts that (5.8) holds with the functions \( f_i = \langle m_i, J^{-1/2}f \rangle \in A(\mathbb{D}) \). The norm continuity of the \( f_i \) in \( f \) is immediate from this formula. Our Theorem 5.5 provides a slightly stronger uniqueness statement: if \( f \in A(\mathbb{D}) \), assuming only that the \( f_i \) are in \( L^\infty(\mathbb{T}) \), multiplying both sides of (5.8) by \( J^{-1/2} \), applying \( (m_i, -) \), and using the fact that \( \{m_i\}_{i=1}^N \) is an orthonormal basis for \( L^\infty(\mathbb{T}) \), one finds that \( f_i \) must be given by the formula above.

Rochberg [15] and McDonald [14] establish more information about the \( f_i \) using the special structure of the canonical orthonormal basis of \( \mathbb{D} \). Our analysis does not seem to contribute anything new to their refinements. On the other hand, our results are explicitly independent of the choice of basis and connect to the structure of the Hilbert module \( L^\infty(\mathbb{T})_{\mathcal{L}} \).

**Remark 5.6.** The reader may have noticed that if \( m \in L^\infty(\mathbb{T}) \) and if \( T = \pi(m)C_b \), then from the calculations in Theorem 4.4 the norm of \( T \) is the norm of \( m \) calculated in \( L^\infty(\mathbb{T})_{\mathcal{L}} \). This is not an accident. The Hilbert module \( L^\infty(\mathbb{T})_{\mathcal{L}} \) becomes a left module over \( L^\infty(\mathbb{T}) \) via the formula \( a \cdot \xi := a\xi, a \in L^\infty(\mathbb{T}), \xi \in L^\infty(\mathbb{T})_{\mathcal{L}} \). This makes \( L^\infty(\mathbb{T})_{\mathcal{L}} \) what is known as a C*-correspondence or Hilbert bimodule over \( L^\infty(\mathbb{T}) \). Further, if \( \psi : L^\infty(\mathbb{T})_{\mathcal{L}} \to B(L^2(\mathbb{T})) \) is defined by the formula

\[
\psi(m) = \pi(m)C_b, \quad m \in L^\infty(\mathbb{T})_{\mathcal{L}},
\]

then the pair \( (\pi, \psi) \) turns out to be what is known as a Cuntz-Pimsner covariant representation of the pair \( (L^\infty, L^\infty(\mathbb{T})_{\mathcal{L}}) \). This means, in particular, that
ψ(m)∗ψ(m) = π((m, m)), as we noted in Theorem 4.9. Further, the pair (π, ψ) extends to a C∗-representation of the so-called Cuntz-Pimsner algebra of \( L^{∞}(\mathbb{T})_L \), \( O(L^{∞}(\mathbb{T})_L) \). We have not made use of this here, but it strikes us as worthy of further investigation. See [5] for further information about Cuntz-Pimsner algebras and their representations.

References

[1] W. Arveson, Continuous analogues of Fock space, Mem. Amer. Math. Soc. v. 80, #409, 1989.
[2] J. Cuntz, Simple C∗-algebras generated by isometries, Comm. Math. Phys. 57 (1977), 173–185.
[3] R. Douglas, Contractive projections on an \( \mathbb{L}_1 \) space, Pac. J. Math. 15 (1965), 443–462.
[4] R. Douglas, Banach Algebra Techniques in Operator Theory, 2nd ed., Graduate Text in Mathematics 179, Springer, New York, 1998.
[5] N. Fowler, P. Muhly and I. Raeburn, Representations of Cuntz-Pimsner algebras, Indiana U. Math. J. 52 (2003), 569–605.
[6] H. Hamada and Y. Watatani, Toeplitz-composition C∗-algebras for certain finite Blaschke products, preprint (arXiv: 0809.3061).
[7] H. Helson, Lectures on Invariant Subspaces, Academic Press, New York and London, 1964.
[8] H. Helson and D. Lowdenslager, Invariant subspaces, Proc. International Conf. on Linear Spaces, 1960, pp. 251–262, Macmillan (Pergamon) New York, 1961.
[9] M. Ionescu and P. Muhly, Groupoid methods in wavelet analysis, in Group Representations, Ergodic Theory, and Mathematical Physics: A Tribute to George W. Mackey, Contemporary Mathematics 449, Amer. Math. Soc., Providence, 2008, pp. 193–208.
[10] S. Kametani and T. Ugaerhi, A remark on Kawakami’s extension of Löwner’s lemma, Proc. Imp. Acad. Tokyo 18 (1942), 14–15.
[11] M. Laca, Endomorphisms of \( B(H) \) and Cuntz algebras, J. Operator Th. 30 (1993), 85 – 108.
[12] E. C. Lance, Hilbert C∗-modules, London Math. Soc. Lect. Note Series 210, Cambridge Univ. Press, Cambridge, 1995.
[13] V. Manuilov and E. Troitsky, Hilbert C∗-modules, Translation of Mathematical Monographs, Vol. 226, Amer. Math. Soc., 2005.
[14] J. McDonald, Adjoints of a class of composition operators, Proc. Amer. Math. Soc. 131 (2003), 601 – 606.
[15] R. Rochberg, Linear maps of the disk algebra, Pac. J. Math. 44 (1973), 337–354.
[16] J. Ryff, Subordinate \( H^p \) functions, Duke Math. J. 33 (1966), 347–354.
[17] M. Takesaki, Theory of Operator Algebras I, Springer-Verlag, New York, 1979.
[18] J. Walsh, Interpolation and Approximation by Rational Functions in the Complex Domain, American Mathematical Society Colloquium Publications, vol. 20, American Mathematical Society, Providence, R. I., 1956.

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