The three-body problem is a prototypical example of deterministic chaos, in that tiny perturbations in the initial conditions (or errors in numerical integration) lead to exponentially divergent outcomes. Chaotic systems often ‘forget’ their initial conditions (aside from integrals of motion), although this is by no means guaranteed—indeed, the topology of the chaotic three-body problem does contain islands of regularity. Nonetheless, to a first approximation, it is reasonable to assume that non-hierarchical triples will uniformly explore the phase-space volume accessible to them. In this way, we report a statistical solution to the non-hierarchical three-body problem that is derived using the ergodic hypothesis and that provides closed-form distributions of outcomes (for example, binary orbital elements) when given the conserved integrals of motion. We compare our outcome distributions to large ensembles of numerical three-body integrations and find good agreement, so long as we restrict ourselves to ‘resonant’ encounters (the roughly 50 per cent of scatterings that undergo chaotic evolution). In analysing our scattering experiments, we identify ‘scrambles’ (periods of time in which no pairwise binaries exist) as the key dynamical state that ergodizes a non-hierarchical triple system. The generally super-thermal distributions of survivor binary eccentricity that we predict have notable applications to many astrophysical scenarios. For example, non-hierarchical triple systems produced dynamically in globular clusters are a primary formation channel for black-hole mergers, but the rates and properties of the resulting gravitational waves depend on the distribution of post-disintegration eccentricities.

\[ a = \int \ldots \int \delta(E_b + E_s - E_0)\delta(L_b + L_s - L_0)\,dr\,dp, \quad (1) \]

shaped by the requirements of energy and angular-momentum conservation for both the elliptic orbit of the surviving binary \((E_b, L_b)\) and the hyperbolic orbit between the binary and the escaper \((E_s, L_s)\).
where $G$ is Newton’s gravitational constant. For brevity, we have re-inserted the angular momentum of the escaping star, $L_z^2(U_0, C_0) / (L_0^2 - C_0^2 + (L_0 - C_0)^2)$. $\sigma$ is a phase volume and the integrand of equation (2) is a trivariate outcome distribution representing the differential probability of finding a disintegrating metastable triple in a volume $d\varepsilon_0 d\varepsilon_1 d\varepsilon_2$: the microcanonical ensemble for survivor binaries produced in the non-hierarchical three-body problem (other—angular—binary orbital elements are distributed uniformly). Therefore, specification of the total energy $E_0$ and the total angular momentum $L_0$ suffices to describe the distribution of outcomes in non-hierarchical triple systems, even if this information alone cannot deterministically specify how one individual outcome follows from one set of initial conditions. Conservation of $E_0$ and $L_0$ means that the trivariate outcome distribution in equation (2) can be mapped one to one to the distribution of escape properties. Equation (2) makes fewer simplifying assumptions than did past ergodic analyses of the general three-body problem\cite{520,21}, and its outcome distributions are qualitatively different.

We marginalize over $L_0$ and $C_0$ to compute the distribution of outcome energies, $d\sigma/d\varepsilon_0$. In this and all remaining calculations, we assume that the strong interaction region is a dimensionless multiple of the time-averaged binary size, that is, $R = a_{\text{esc}} (1 + e_0)$, where $a = 1$ is a dimensionless constant (see Extended Data Figs. 1–3 and Supplementary Information for more details). In an $L_0 = 0$ ensemble, this is $d\sigma/d\varepsilon_0 \propto \varepsilon_0^{3/2}$, extending to $|E_0| \rightarrow \infty$. Conversely, when $L_0$ is large, the ergodic energy distribution is slightly steeper, changing roughly as $d\sigma/d\varepsilon_0 \propto |E_0|^{-3/2}$, but only up to a maximum energy of $E_{\text{max}} \propto L_0^2$: larger outcome energies are prohibited by angular-momentum conservation. The energy distribution that we calculate differs from past estimates determined assuming detailed balance\cite{21}, demonstrating that a population of binaries engaging in resonant three-body interactions with a thermal bath of single stars cannot achieve detailed balance, so long as their outcomes are ergodically distributed.

We likewise integrate to find the marginal outcome distributions in angular momentum (which we represent in terms of binary eccentricity $e_0$, as $d\sigma/d\varepsilon_0$) and inclination ($d\sigma/d\varepsilon_0$). In contrast to the usual (although not universal\cite{22}) expectation of a thermal eccentricity distribution, $d\sigma/d\varepsilon_0 = 2\varepsilon_0$, we find a mildly super-thermal eccentricity distribution for large $L_0$: $d\sigma/d\varepsilon_0 \propto \varepsilon_0^{-1}$. This radial orbit bias is a geometric effect arising from the larger average interaction cross-section of a highly eccentric binary, the apocentre of which is twice as large as that of a circular binary of equal energy. In the low-$L_0$ limit, the ergodic distribution of survivor eccentricities is highly super-thermal, with $d\sigma/d\varepsilon_0 \propto \varepsilon_0^{-1} (1 + e_0)^{1/2} / (1 - e_0^2)$ when $L_0 = 0$. There is a strong bias towards producing nearly radial binaries as a consequence of angular-momentum starvation: whereas a low-$L_0$ ensemble of non-hierarchical triples may produce a quasi-circular survivor binary, doing so requires substantial fine-tuning of the angle and velocity of the escapee, and is therefore disfavoured. Similar phase volume considerations explain the strong bias towards prograde ($0 < C_0 < 1$) orbits predicted by equation (2) when marginalized into $d\sigma/dC_0$. More detailed explorations of the ergodic $d\sigma/d\varepsilon_0$, $d\sigma/d\varepsilon_0$ and $d\sigma/dC_0$ distributions are shown in Extended Data Figs. 1, 2, 3, respectively, as well as in Supplementary Information.

Our outcome distribution, $d\sigma/(d\varepsilon_0 d\varepsilon_1 d\varepsilon_2)$, was derived with several assumptions, most notably: (i) the ergodic hypothesis, (ii) instantaneous disintegration and (iii) a specific parameterization of the ‘strong interaction region’ defining the limits of integration. It should therefore be tested against ensembles of numerical scattering experiments. We have explored the ergodicity of non-hierarchical triples in the equal-mass limit by using the FEBODY numerical scattering code to run three ensembles of different binary—single scattering experiments (see Extended Data Table 1). Each ensemble has roughly $N = 10^6$ runs with constant $E_0$ and $L_0$, but otherwise random initial conditions (we initialize our binary—single scatterings with zero impact parameter, so we can
parametrize $L_0$ in terms of the initial binary eccentricity $e_0$. However, many of our scattering experiments do not form resonant three-body systems, but instead resolve abruptly in a prompt exchange, where it is unlikely that the ergodic hypothesis can be applied. Metastable three-body systems generally exhibit intermittent chaos$^{23}$. Long periods of quasi-regular evolution occur during the non-terminal ejection of a single star, but these are then interrupted by brief periods of intensely chaotic evolution when that star returns to the pericentre$^{4,10}$. We hypothesize that the degree of ergodicity in a subset of scattering experiments can be inferred from the number of scrambles, $N_{\text{scram}}$.

We illustrate the development of ergodicity in Fig. 2, which shows topological maps in outcome space. Whereas the full scattering ensemble

![Fig. 2](image)

Fig. 2 | Topological maps of three-body scattering outcomes for run A-a-f. The total number of scrambles is colour-coded (smallest values of $N_{\text{scram}}$ as dark blue, larger $N_{\text{scram}}$ in green and yellow) with logarithmic scaling, as a function of survivor binary eccentricity $e_B$ (a, c, e), energy $E_B$ (b, d, f) and cosine inclination $C_B$. Shown are the cases $N_{\text{scram}} \geq 0$ (a, b), $N_{\text{scram}} \geq 1$ (c, d) and $N_{\text{scram}} \geq 2$ (e, f). Clouds of regularity obscure the underlying chaotic sea in a, b, but have dissipated in e, f, indicating that scrambles are the key dynamical mechanism responsible for ‘ergodicizing’ the comparable-mass three-body problem.

![Fig. 3](image)

Fig. 3 | Marginal distribution of binary energy, $d\sigma/dE_B$, as a function of dimensionless energy, $E_B/E_0$. The dotted lines are ergodic outcome distributions for ensembles with high (purple), medium (blue) and low (green) angular momentum. The data points are binned outcomes from numerical binary–single scattering ensembles ($N \approx 10^5$). Horizontal error bars show bin sizes and vertical error bars indicate 95% Poissonian confidence intervals. a, Full set of results from our numerical scattering experiments. b, Subset of results for $N_{\text{scram}} \geq 1$. c, Subset of results for $N_{\text{scram}} \geq 2$. Detailed balance (black dashed line) is never achieved.
Fig. 4 | Marginal distributions of binary eccentricity and orientation. a, c, e, do/d\(e_b\) against eccentricity \(e_b\). b, d, f, do/d\(C_b\) against the cosine of the binary inclination, \(C_b\). Line styles represent ergodic outcome distributions with the same ensemble angular momenta as in Fig. 3. The data points are binned outcomes from the same numerical scattering ensembles as in Fig. 3, with each row corresponding to the same cuts on \(N_{\text{scram}}\). The eccentricity outcome distributions are notably super-thermal (the thermal distribution, do/d\(e_b\) = 2e, is shown as a black dashed line). The inclination distributions exhibit anisotropic bias towards prograde binaries aligned with \(L_z\) (the isotropic distribution is shown with a black dashed line). Horizontal error bars show bin sizes and vertical error bars indicate 95% Poissonian confidence intervals.

has clear geometrical features indicative of prompt exchanges, these ‘clouds of regularity’ mostly (entirely) disappear if one considers the ~50% of integrations with \(N_{\text{scram}} \geq 2\). With this qualitative argument in mind, we use Figs. 3 and 4 to quantitatively compare the binned results of our scattering experiments to the marginal distributions predicted by the ergodic hypothesis. Horizontal error bars show bin sizes and vertical error bars indicate 95% Poissonian confidence intervals. All three of the marginal distributions that we examine (do/d\(e_b\), do/d\(e_f\) and do/d\(C_b\)) exhibit reasonable (and sometimes very close) agreement between the ergodic theory of equation (2) and our numerical scattering experiments, provided that we examine resonant encounters (\(N_{\text{scram}} \geq 2\)). The marginal distributions for large-\(L_0\) ensembles are very consistent with the numerical experiments. The agreement is slightly worse for our low-\(L_0\) ensemble.

The agreement between ergodic theory and experiment is never exact, even in \(N_{\text{scram}} \geq 2\) subsamples, and in most cases we see data that match analytic predictions to leading order but also exhibit some level of higher-order structure. The nature of these superimposed, second-order structures is not altogether clear, as two explanations seem plausible. First, these could represent islands of regularity in the initial conditions that we explore: regions of parameter space that do not fully forget their initial conditions despite undergoing multiple scrambles. Second, these could represent a failure in the idealized escape criteria, \(R(E_b, L_0)\), that we employ. We only consider very simple definitions of the strong-interaction region, the true shape of which is probably connected to the stability boundary of the triple24. We defer an investigation of these two hypotheses to future work.

Non-hierarchical triples are common, if short-lived, in the astrophysical Universe25. They are responsible for many interesting phenomena. For example, binary–single scattering events in dense star clusters produce blue stragglers26,27, cataclysmic variables28, X-ray binaries29,30 and even binary stellar-mass black holes31. The lattermost of these scenarios may be responsible for most of the black-hole mergers seen by the LIGO experiment21,31. Dynamical formation of these systems in a binary–single scattering is favoured when the surviving binary is drawn from the high-\(e_b\) tail of outcomes. It is therefore notable that (i) we find generic superthermality in the outcomes of comparable-mass scatterings (both from ergodic theory and numerical experiments) and (ii) that our formalism has identified the type of binary–single encounters that are predisposed to produce exotic binaries: low-\(L_0\) scatterings. In the future, it may be possible to apply our formalism to estimate the properties of temporary binaries formed during long, but non-terminal, single-star ejections. High eccentricity binaries formed as ‘intermediate states’ of a three-body resonance may merge during the ejection owing to short-range dissipative
forces, leading to, for example, uniquely eccentric gravitational-wave signals."

**Online content**

Any methods, additional references, Nature Research reporting summaries, source data, extended data, supplementary information, acknowledgements, peer review information; details of author contributions and competing interests; and statements of data and code availability are available at https://doi.org/10.1038/s41586-019-1833-8.

1. Agekyan, T. A. & Anosova, Z. P. A study of the dynamics of triple systems by means of statistical sampling. Astron. Zh. 44, 1261 (1967).
2. Šuvakov, M. & Dmitrašinović, V. Three classes of Newtonian three-body planar periodic orbits. Phys. Rev. Lett. 110, 114301 (2013).
3. Standish, E. M. The dynamical evolution of triple star systems. Astron. Astrophys. 21, 185–191 (1972).
4. Hut, P. & Bahcall, J. N. Binary–single star scattering. I–numerical experiments for equal masses. Astrophys. J. 268, 319–341 (1983).
5. Poincaré, H. Les Méthodes Nouvelles de la Mécanique Céleste (Gauthier-Villars et fils, 1892).
6. Sundman, K. F. Mémoire sur le problème des trois corps. Acta Math. 36, 105–179 (1913).
7. Fermi, E. High energy nuclear events. Prog. Theor. Phys. 5, 570–583 (1950).
8. Monaghan, J. J. A statistical theory of the disruption of three-body systems – I. Low angular momentum. Mon. Not. R. Astron. Soc. 176, 63–72 (1976).
9. Valtonen, M., Myllärä, A., Orlof, V. & Rubinov, A. Dynamics of rotating triple systems: statistical escape theory versus numerical simulations. Mon. Not. R. Astron. Soc. 364, 91–98 (2005).
10. Heggie, D. C. Binary evolution in stellar dynamics. Mon. Not. R. Astron. Soc. 173, 729–787 (1975).
11. Portegies Zwart, S. F. & McMillan, S. L. W. Black hole mergers in the Universe. Astrophys. J. Lett. 528, 17–20 (2000).
12. Rodriguez, C. J., Chatterjee, S. & Rasio, F. A. Binary black hole mergers from globular clusters: masses, merger rates, and the impact of stellar evolution. Phys. Rev. D 93, 084029 (2016).
13. Hong, J. et al. Binary black hole mergers from globular clusters: the impact of globular cluster properties. Mon. Not. R. Astron. Soc. 480, 5645–5656 (2018).
14. Samsing, J., MacLeod, M. & Ramirez-Ruiz, E. The formation of eccentric compact binary inspirals and the role of gravitational wave emission in binary–single stellar encounters. Astrophys. J. 784, 71 (2014).
15. Rodriguez, C. L. et al. Post-Newtonian dynamics in dense star clusters: formation, masses, and merger rates of highly-eccentric black hole binaries. Phys. Rev. D 98, 123005 (2018).
16. Portegies Zwart, S. F. & Boekholt, T. C. N. Numerical verification of the microscopic time reversibility of Newton's equations of motion: fighting exponential divergence. Commun. Nonlinear Sci. Numer. Simul. 61, 160–166 (2018).
17. Hut, P. The topology of three-body scattering. Astron. J. 88, 1549–1559 (1983).
18. Samsing, J. & Ilić, T. Topology of black hole binary–single interactions. Mon. Not. R. Astron. Soc. 476, 1548–1560 (2018).
19. Bohr, N. Neutron capture and nuclear constitution. Nature 137, 344–348 (1936).
20. Monaghan, J. J. A statistical theory of the disruption of three-body systems – II. High angular momentum. Mon. Not. R. Astron. Soc. 177, 583–594 (1976).
21. Nash, P. E. & Monaghan, J. J. A statistical theory of the disruption of three-body systems – III. Three-dimensional motion. Mon. Not. R. Astron. Soc. 184, 119–125 (1978).
22. Geller, A. M., Leigh, N. W. C., Giersz, M., Kremer, K. & Rasio, F. A. In search of the thermal eccentricity distribution. Astrophys. J. 872, 165 (2019).
23. Pomeau, Y. & Manneville, P. Intermittent transition to turbulence in dissipative dynamical systems. Commun. Math. Phys. 74, 189–197 (1980).
24. Mardling, R. A. & Aarseth, S. J. Tidal interactions in star cluster simulations. Mon. Not. R. Astron. Soc. 321, 398–420 (2001).
25. Leigh, N. W. C. & Geller, A. M. The dynamical significance of triple star systems in star clusters. Mon. Not. R. Astron. Soc. 432, 2474–2479 (2013).
26. Leonard, P. J. T. & Fahman, G. G. On the origin of the blue stragglers in the globular cluster NGC 5053. Astron. J. 102, 994 (1991).
27. Leigh, N., Silis, A. & Knigge, C. An analytic model for blue straggler formation in globular clusters. Mon. Not. R. Astron. Soc. 416, 1410–1418 (2011).
28. Ivanova, N. et al. Formation and evolution of compact binaries in globular clusters – I. Binaries with white dwarfs. Mon. Not. R. Astron. Soc. 372, 1043–1059 (2006).
29. Pooley, D. & Hut, P. Dynamical formation of close binaries in globular clusters: cataclysmic variables. Astrophys. J. Lett. 646, 143–146 (2006).
30. Ivanova, N., Heinke, C. O., Rasio, F. A., Belczynski, K. & Fereira, J. M. Formation and evolution of compact binaries in globular clusters – II. Binaries with neutron stars. Mon. Not. R. Astron. Soc. 386, 553–576 (2008).

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**Data availability**
The data that support the findings of this study are available from the corresponding author upon reasonable request.

**Acknowledgements**
We acknowledge discussions with D. Heggie, P. Hut, R. Sari and S. Portegies Zwart, as well as feedback from E. Michaely and O. C. Winter. N.C.S. received financial support from NASA, through Einstein Postdoctoral Fellowship Award number PF5-160145 and the NASA Astrophysics Theory Research Program (grant NNXTAK43G; Principal Investigator, B. Metzger). N.C.S. also thanks the Aspen Center for Physics for its hospitality during early stages of this work. N.W.C.L. acknowledges support by Fondecyt Iniciacion grant number 11180005. We thank the Chinese Academy of Sciences for hosting us as we completed our efforts. We thank M. Valtonen and H. Karttunen, whose book on the three-body problem motivated much of this work.

**Author contributions**
N.C.S. led the analytic work, to which N.W.C.L. contributed significantly. The FEWBODY simulations were performed by N.W.C.L. The comparison between the simulations and the analytic theory was performed jointly by the two authors.

**Competing interests**
The authors declare no competing interests.

**Additional information**
Supplementary information is available for this paper at https://doi.org/10.1038/s41586-019-1833-8.

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**Peer review information**
Nature thanks Erez Michaely and Othon Cabo Winter for their contribution to the peer review of this work.

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Extended Data Fig. 1 | Marginal distribution of binary energies, $d\sigma/dE_B$.
Colours show dimensionless angular momenta $\tilde{L}_0$; upper and lower black dashed lines are asymptotic power laws for $\tilde{L}_0=1$ and $\tilde{L}_0=0$, respectively. 

a, Ergodic outcome distributions using the ‘apocentric escape’ (AE) criterion; that is, assuming that disintegration of metastable triples occurs within a strong interaction region of size $R=\alpha a_B(1+\epsilon_B)$. Here we take $\alpha=2$. Solid lines represent equal-mass scattering ensembles ($m_a=m_b=m_s$) and dotted lines extreme-mass-ratio ensembles ($m_a=m_b=10m_s$). 

b, As in a, but for a ‘simple escape’ (SE) criterion, $R=aa_B$. 

c, Intermediate-mass-ratio scattering ensembles ($m_a=m_b=3m_s$). Solid lines correspond to $\alpha=2$ and dotted lines to $\alpha=5$. 

d, As in c, but for $m_a=m_b=10m_s$. Note that $\tilde{L}_0$ is a dimensionless angular momentum normalized by the circular orbit angular momentum of a binary with energy $E_0$ and masses $m_a$ and $m_b$.
Extended Data Fig. 2 | Marginal distribution of binary eccentricity, \( d\sigma/d\varepsilon_B \).
Line styles and assumptions are as in Extended Data Fig. 1, except for the upper and lower black dashed lines, which here show the \( L_0 = 1 \) and \( L_0 \ll 1 \) limits of the \( d\sigma/d\varepsilon_B \) distribution, respectively (unlike for \( d\sigma/dE_B \), these limits differ significantly in the AE and SE regimes). In comparable-mass AE calculations, mildly super-thermal outcomes arise from geometric effects when \( L_0 = 1 \); by contrast, angular-momentum starvation produces extremely super-thermal outcomes when \( L_0 \ll 1 \). Small \( m \) values foreclose parts of \( \varepsilon_B \) space, as \( L_0 \approx L_\odot \).
Extended Data Fig. 3 | Marginal distribution of binary orientation, $d\sigma/dC_B$.
Assumptions and line styles are as in Extended Data Fig. 1, except that the black dashed lines show (i) an isotropic outcome configuration and (ii) an analytic approximation for $d\sigma/dC_B$, as labelled in a (for an equal-mass triple with $L_0 \approx 0.5$).

For $L_0 \ll 1$, surviving binaries are distributed isotropically (as symmetry dictates). Otherwise, binary orientations $C_B = \mathbf{L}_B \cdot \mathbf{L}_0$ are biased towards prograde outcomes. For extreme mass ratios and large $L_0$, retrograde outcomes may be entirely prohibited.
Extended Data Table 1 | Numerical (binary–single) scattering ensembles used for comparison to analytic theory

| Run | $e_0$ | $\tilde{L}_0$ | $N_0$  | $N_1$  | $N_2$  |
|-----|-------|-------------|-------|-------|-------|
| A   | 0.0   | 1.0         | 116,993 | 56,696 | 39,819 |
| B   | 0.5   | 0.87        | 121,328 | 65,936 | 51,791 |
| C   | 0.9   | 0.44        | 107,992 | 76,051 | 46,852 |

The first two columns show the initial binary eccentricity $e_0$ and the conserved dimensionless angular momentum $\tilde{L}_0$ in each simulated scattering run. The other columns show the number of runs with $N_{\text{scram}} \geq i$, $N_i$. Each run has initial impact parameter $b = 0$, isotropically distributed phase angles and particles of equal mass ($m_a = m_b = m_s$).