Cosmological Dynamics of a Dirac-Born-Infeld field

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(Dated: March 31, 2010)

We analyze the dynamics of a Dirac-Born-Infeld (DBI) field in a cosmological set-up which includes a perfect fluid. Introducing convenient dynamical variables, we show the evolution equations form an autonomous system when the potential and the brane tension of the DBI field are arbitrary power-law or exponential functions of the DBI field. In particular we find scaling solutions can exist when powers of the field in the potential and warp-factor satisfy specific relations. A new class of fixed-point solutions are obtained corresponding to points which initially appear singular in the evolution equations, but on closer inspection are actually well defined. In all cases, we perform a phase-space analysis and obtain the late-time attractor structure of the system. Of particular note when considering cosmological perturbations in DBI inflation is a fixed-point solution where the Lorentz factor is a finite large constant and the equation of state parameter of the DBI field is $w = -1$. Since in this case the speed of sound $c_s$ becomes constant, the solution can be thought to serve as a good background to perturb about.

PACS numbers: pacs: 98.80.Cq

I. INTRODUCTION

The inflationary paradigm remains to date the most successful explanation for the origin of the observed temperature fluctuations of the cosmic microwave background (CMB) (see, e.g., [1, 2] for reviews). However, establishing its origin in fundamental theory has not been quite as successful and so far for many it remains a fascinating paradigm in search of an underlying theory. The favourite candidate for this is String theory and for a nice review of the construction of inflation models in string theory, see [3].

One interesting model, recently proposed from string theory is Dirac-Born-Infeld (DBI) inflation [4–8], where inflation is driven by the motion of a D3-brane in a warped throat region of a compact internal space. In this model, since the inflaton is the position of a D-brane with a DBI action, its kinetic term is inevitably non-canonical. In addition to this kinetic term, its effective action includes a potential arising from the quantum interaction between D-branes, with the brane tension encoding geometrical information about the throat region of the compact space. Because of these novel ingredients, the predictions of DBI inflation are quite different from the ones from the standard slow-roll inflation models and it has led to an intense period of research into the scenario, including work into the background dynamics and linear perturbations [9–16].

From the phenomenological viewpoint, one of the main reasons why this model has attracted attention is because of its sizable equilateral type of primordial non-Gaussianity, first pointed by [4] and explored further in [4–6, 17–34]. Such a large signal can not be obtained in standard slow-roll inflation, hence it opens up the possibility of distinguishing this model from that of the standard slow-roll inflation—although not necessarily distinguishing it from more general slow-roll models going beyond single field inflation as they may also lead to large signals. Furthermore, because of the dependence of the effective four dimensional string tension on the nature of the compact internal space, it is possible to place further constraints on the parameters related with compactifications such as flux numbers. For the most up to date observational constraints on and consequences of DBI inflation, see [35–48]. Recently the idea of low scale inflation arising from the DBI action has been invoked to explain the late time acceleration we associate with dark energy [49–52].

Given the results mentioned above and inspired by the possibility of inflation being an attractor solution in DBI models, we believe there is a need to understand the late-time attractor structure for as general a DBI set-up as possible. It is well known that for a canonical scalar field with a potential, scaling solutions can exist where the ratio of the kinetic and potential energy of the scalar field maintain the same ratio, and understanding the stability of these solutions is important in determining the nature of the late-time solutions [53–64]. Among the earlier work, the phase-space analysis proposed by [57] is particularly powerful because it allows us to make use of suitable dimensionless dynamical variables, in order to establish the global stability of such scaling solutions. For related work which analyzes the dynamics in DBI models with general inflationary potentials, see [65–68].

Now, this method for analysing the stability of the solutions has only been applied to the case where the DBI field has a quadratic potential and the associated D-brane is in the anti-de Sitter throat [69]. The late-time attrac-
tor structure for the case without a potential, known as tachyon cosmology, has been studied extensively.\cite{tachyon1, tachyon2} and the case with another degree of freedom, has been studied in\cite{tachyon3, tachyon4}.

In this paper, in order to make the cosmological application of scaling solutions in DBI models more complete and as a natural extension of \cite{tachyon5}, we obtain the late-time attractor structure of the system including a perfect fluid plus a DBI field whose potential and brane tension are arbitrary power-law or exponential functions of the DBI field.

The rest of the paper is arranged as follows. In section II we present the model and basic equations. Then, in section III we consider the models where the potential and brane tension are arbitrary exponential functions of the DBI field. Special emphasis is given to a new set of fixed-point solutions which at first site appear singular in the equations of motion, but on closer inspection are well defined. This is followed in section IV with an analysis of the models where the potential and brane tension are arbitrary power-law functions of the DBI field. Finally, we summarise in section V.

II. BASIC EQUATIONS

We consider a DBI field, $\phi$, with the following effective action \cite{DBI1}:

$$S = - \int d^4x \sqrt{-g} \left( \frac{1}{f(\phi)} \left( \sqrt{1 - 2f(\phi)X} - 1 \right) - V(\phi) \right),$$

where

$$X = -\frac{1}{2} g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi. \quad (2)$$

$V(\phi)$ is a potential that arises from quantum interactions between a D3-brane associated with $\phi$ and other D-branes. Although a quadratic potential was considered in \cite{DBI1}, discussions on the exact form of the potential are still ongoing. $f(\phi)$ is the inverse of the D3-brane tension and contains geometrical information about the throat in the compact internal space. Current proposals for the form of $f(\phi)$ include $f(\phi) = \alpha \phi^{-4}$ ($\alpha$ constant), for the case of an AdS throat. Another case considered in the literature is a constant function $f(\phi) = \alpha$ \cite{DBI2}. In this paper, we will keep $f(\phi)$ and $V(\phi)$ as general non-negative functions, i.e. $V(\phi) \geq 0$ and $f(\phi) \geq 0$.

In a spatially flat Friedmann-Robertson-Walker (FRW) metric with scale factor $a(t)$, $X = \dot{\phi}^2/2$ for a homogeneous field $\phi$, and it can be shown that the energy density and pressure of the DBI field are given by

$$\rho_\phi = \frac{\gamma^2}{\gamma + 1} \dot{\phi}^2 + V(\phi), \quad p_\phi = \frac{\gamma}{\gamma + 1} \dot{\phi}^2 - V(\phi), \quad (3)$$

where a dot denotes a derivative with respect to the cosmic time, $t$, and

$$\gamma \equiv \frac{1}{\sqrt{1 - f(\phi)\dot{\phi}^2}}, \quad (4)$$

which characterises the motion of the brane and serves as the Lorentz factor. As in usual special relativity $\gamma \geq 1$ because $f(\phi)$ is non-negative.

If we also take into account a perfect fluid with equation of state $p_m = w_m \rho_m$ ($-1 < w_m < 1$), the basic cosmological background equations are given by

$$H^2 = \frac{1}{3} \left[ \frac{\gamma^2}{\gamma + 1} \dot{\phi}^2 + V(\phi) + \rho_m \right], \quad (5)$$

$$\dot{\rho}_m + 3H \frac{3H}{\gamma^2} \dot{\phi} + \frac{V_\phi}{\gamma^3} + \frac{f(\phi) (\gamma + 2)(\gamma - 1)}{2f} \dot{\phi}^2 = 0, \quad (6)$$

$$\dot{\rho}_m + 3H(1 + w_m)\rho_m = 0, \quad (7)$$

where the subscript $\phi$ means differentiation with respect to the field, $H \equiv \frac{\dot{a}}{a}$ is the Hubble parameter and we have set $8\pi G = 1$ for simplicity. Eqs. (5) - (7) close the system that determines the dynamics.

In order to obtain the late-time attractor behaviour of the system we introduce the following set of dynamical variables:

$$x = \frac{\gamma \dot{\phi}}{\sqrt{3(\gamma + 1)H}}, \quad y = \frac{\sqrt{V}}{\sqrt{3H}}, \quad \dot{\gamma} = \frac{1}{\gamma^3}. \quad (8)$$

By defining $x$ and $y$ in this way it is clear that in the limit of $\lambda \rightarrow 1$, we recover the dynamical variables for a canonical scalar field originally proposed in Ref. \cite{attractor1}. By doing this, we can intuitively think of $x$ corresponding to the contribution of the kinetic energy, and $y$ to the potential energy of the field. For another dynamical degree of freedom which arises due to the introduction of the function $f(\phi)$ in this set-up, we adopt $\dot{\gamma}$ as our third dynamical variable. This particular choice is intended to make the phase space being compact.

In terms of these variables, the Friedmann constraint \cite{DBI3} can be expressed as

$$x^2 + y^2 + \Omega_m = 1, \quad (9)$$

where $\Omega_m \equiv \rho_m/(3H^2)$. The energy fraction and equation of state of the DBI field are given by

$$\Omega_\phi = x^2 + y^2, \quad (10)$$

$$w_\phi = \frac{\dot{x}x^2 - \dot{y}y^2}{x^2 + y^2}. \quad (11)$$

Similarly, for a general potential $V(\phi)$ and brane tension $f(\phi)$, by introducing $\lambda$ and $\mu$ which are defined by

$$\lambda \equiv -\frac{V_\phi}{V}, \quad \mu \equiv -\frac{f(x)}{f}, \quad (12)$$

we can express

$$\dot{x} = \lambda x - \mu y, \quad \dot{y} = \mu x + \lambda y. \quad (13)$$

These equations are integrable for arbitrary non-negative functions of $V(\phi)$ and $f(\phi)$, and we can investigate particular solutions.
Eqs. (13) and (16) can be written as

\[
x_{N} = \frac{1}{2} \sqrt{3\gamma(1 + \gamma)} \lambda y^2 + \frac{3}{2} x [(1 + \gamma)(x^2 - 1) + (1 + w_m)(1 - x^2 - y^2)],
\]

(13)

\[
y_{N} = -\frac{1}{2} \sqrt{3\gamma(1 + \gamma)} \lambda xy + \frac{3}{2} y [(1 + \gamma)x^2 + (1 + w_m)(1 - x^2 - y^2)],
\]

(14)

\[
\gamma_{N} = \frac{\tilde{\gamma}(1 - \tilde{\gamma}^2)}{\sqrt{1 + \gamma}} \left[3 \sqrt{1 + \gamma} + \sqrt{3\gamma} \frac{1}{x} (\mu x^2 - \lambda y^2)\right],
\]

(15)

\[
\lambda_{N} = -\sqrt{3\gamma(1 + \gamma)} \lambda^2 (\Gamma - 1) x,
\]

(16)

\[
\mu_{N} = -\sqrt{3\gamma(1 + \gamma)} \mu^2 (\Sigma - 1) x,
\]

where \( N \equiv \ln a, x_{N} \equiv \frac{dx}{da} \), etc... and we have introduced \( \Gamma \) and \( \Sigma \) through

\[
\Gamma \equiv \frac{V'V_{\phi\phi}}{V_{\phi}}, \quad \Sigma \equiv \frac{f_{\phi\phi}}{f_{\phi}}.
\]

Although Eqs. (13) - (16) hold for a general potential and brane tension, if both are given as exponential functions of \( \phi, V \propto \exp[-\lambda \phi], f \propto \exp[-\mu \phi], \) then from Eqs. (13) and (16), \( \lambda \) and \( \mu \) become constant and Eqs. (13) - (16) form the closed autonomous system. We will see the late-time attractor structure of this case in Sec. [V].

There is an important new feature which emerges in the DBI case and is not present for the case of the canonical scalar field. In the latter case only an exponential potential can truly lead to an autonomous system, but as we shortly show, this fact does not apply to the DBI field. By combining the degree of freedoms related with \( V(\phi) \) and \( f(\phi) \), we will be able to obtain an autonomous system for a wider class of potentials and brane tensions (see also [69]).

### III. MODELS WITH A GENERAL POWER-LAW POTENTIAL AND BRANE TENSION

Here, we consider models where the scalar potential and the brane tension are arbitrary non-negative power-law functions of the DBI field

\[
V(\phi) = \sigma |\phi|^p, \quad f(\phi) = \nu |\phi|^r,
\]

(18)

with constants \( \sigma > 0 \) and \( \nu > 0 \).

Guo and Ohta [69] analysed such a model for the case \( p = 2 \) and \( r = -4 \) but as we will now show it can be addressed as an autonomous system for more general combinations (the exception being \( r + p = 0 \) which requires a separate treatment as we will see).

#### A. Autonomous System

In order to represent Eqs. (13) - (17) by an autonomous set of equations, we first introduce the variables \( \tilde{\lambda} \) and \( \tilde{\mu} \) defined by

\[
\tilde{\lambda} = -\frac{V_{\phi}}{f_{\phi}VV_{\phi}}, \quad \tilde{\mu} = -\frac{f_{\phi}}{f_{\phi}VV_{\phi}V_{\phi}},
\]

(19)

where \( q \equiv -1/(p + r) \). Initially we assume \( p + r \neq 0 \), and will address that special case later. Given the definitions in Eqs. (19), we can easily verify that \( \tilde{\lambda} \) and \( \tilde{\mu} \) are constants given by

\[
\tilde{\lambda} = -\frac{-\varepsilon p}{\sigma^q V^q}, \quad \tilde{\mu} = -\frac{-\varepsilon r}{\sigma^q V^q} = \frac{x}{p} \lambda,
\]

(20)

where \( \varepsilon \) is 1 for \( \phi > 0 \) and is \(-1\) when \( \phi < 0 \). The requirement that \( \lambda > 0 \) is equivalent to demanding \( \phi < 0 \) for \( p > 0 \), and \( \phi > 0 \) for \( p < 0 \). We, therefore, restrict our analysis to \( \lambda > 0 \) without loss of generality. Then, for physically interesting cases, we can also restrict our solutions to those where \( \phi \geq 0 \) (\( x \geq 0 \)).

From the form of \( V(\phi) \) and \( f(\phi) \) in Eq. (18) it follows that \( \tilde{\lambda} \) and \( \tilde{\mu} \) are related with \( \lambda \) and \( \mu \) through

\[
\lambda = \left(\frac{1 - \tilde{\gamma}}{\gamma(1 + \gamma)x^2}\right)^q \tilde{\lambda}, \quad \mu = \left(\frac{1 - \tilde{\gamma}}{\gamma(1 + \gamma)x^2}\right)^q \tilde{\mu}.
\]

(21)

Therefore since \( \lambda_N \) and \( \mu_N \) can then be expressed in terms of \( x_N, y_N, \gamma_N, \) defined in Eqs. (13) - (16), we only need to solve those three equations.

In terms of the dynamical variables we have defined above, the basic equations can be expressed as

\[
x_{N} = \frac{\sqrt{3(1 + \gamma)}}{2} \lambda \gamma_{N}^{\frac{q}{q - 1}} x^q (1 - \gamma) \frac{y^{2q + 2}}{x^{2q}},
\]

\[
+ \frac{3}{2} x [(1 + \gamma)(x^2 - 1) + (1 + w_m)(1 - x^2 - y^2)],
\]

(22)

\[
y_{N} = -\frac{\sqrt{3(1 + \gamma)}}{2} \lambda \gamma_{N}^{\frac{q}{q - 1}} x^q (1 - \gamma) \frac{y^{2q + 1}}{x^{2q - 1}}
\]

\[
+ \frac{3}{2} y [(1 + \gamma)x^2 + (1 + w_m)(1 - x^2 - y^2)],
\]

(23)

\[
\gamma_{N} = \gamma(1 - \gamma^2) \times \left[3 + \frac{\sqrt{3}}{1 + \gamma} \lambda \gamma_{N}^{\frac{q}{q - 1}} (1 - \gamma) \frac{y^{2q}}{x^{2q + 1}} \left(\frac{x}{p} x^2 - y^2\right)\right].
\]

(24)

Equipped with the basic equations forming an autonomous system, we will perform the stability analysis to obtain the late-time attractor structure in the following.
B. Standard Fixed-point Solutions

Here we obtain fixed-point solutions of the dynamical system given by Eqs. (22)-(24). The fact that there can be terms involving \( x, y \) and \( \dot{\gamma} \) in the denominators of these equations means that we have to treat the cases where these terms vanish with considerable care. In this subsection we make sure we are working in regimes where any possible ambiguities involving possible divisions by zero or ratios of ‘0/0’ are avoided. In order to distinguish between them, we call the fixed-point solutions considered in this subsection to be ‘standard’ fixed-point solutions. We will address the other cases in subsection III D.

From Eq. (24), the requirement that \( \dot{\gamma} \) is a constant at the standard fixed points implies three possible scenarios:

\[
\begin{align*}
(a) & \quad \dot{\gamma} = 0, \\
(b) & \quad \dot{\gamma} = 1, \\
(c) & \quad 3 = \sqrt{\frac{3}{1 + \gamma}} \gamma^{\frac{q}{2}} \gamma^{-q} (1 - \gamma) q y^{2q} x^{2q-1} \left( -\frac{r}{p} x^2 + y^2 \right) .
\end{align*}
\]

(25)

We will investigate each scenario in turn in what follows.

1. Case \( a : \dot{\gamma} = 0 \)

This result is only valid as long as \( q \leq 1/2 \). In this case, since the Lorentz factor \( \gamma \) tends to infinity, we shall refer to this class of fixed-points as the “ultra-relativistic” solutions. Substituting \( \dot{\gamma} = 0 \) into Eqs. (22)-(24) and eliminating the terms that obviously vanish we obtain

\[
x_N = \frac{\sqrt{3}}{2} \lambda \gamma^{\frac{q}{2}} \gamma^{-q} \frac{y^{2q+2} x^{2q}}{x^{2q}} \\
y_N = -\frac{\sqrt{3}}{2} \lambda \gamma^{\frac{q}{2}} \gamma^{-q} \frac{y^{2q+1} x^{2q-1}}{x^{2q-1}} \\
\dot{\gamma}_N = \sqrt{3} \lambda \gamma^{\frac{q}{2}} \gamma^{-q} \frac{y^{2q} x^{2q+1}}{x^{2q+1}} \left( \frac{r}{p} x^2 + y^2 \right) .
\]

(26)-(27)

Due to the freedom in choosing the sign and the magnitude of \( q \), it is clear that there are situations where some of the terms in these equations are ill-defined. We, therefore, classify our analysis in terms of the value of \( q \).

For \( 0 \leq q \leq 1/2 \), we find the following standard fixed points:

\[
\begin{align*}
(a1) & \quad (x, y, \dot{\gamma}) = (1, 0, 0) \\
(a2) & \quad (x, y, \dot{\gamma}) = (x_0, 0, 0) \quad \text{with } 0 < x_0 < 1, \ w_m = 0 .
\end{align*}
\]

(29)-(30)

We refer to the point \( (a1) \) as the ultra-relativistic kinetic dominated solution, and to the family of solutions (one for each \( x_0 \) \( (a2) \)) as the ultra-relativistic kinetic-fluid scaling solution. For \( q \leq -1/2 \), we find the following standard fixed-point solution

\[
\begin{align*}
(a3) & \quad (x, y, \dot{\gamma}) = (0, 1, 0) ,
\end{align*}
\]

(32)

which represents the ultra-relativistic potential dominated solutions.

There are no \( \dot{\gamma} = 0 \) solutions for the case, \(-1/2 < q < 0 \), but for the special case of \( q = 1/2 \), there exist two other standard fixed points in the system of equations (20)-(27):

\[
\begin{align*}
(a4) & \quad (x, y, \dot{\gamma}) = \left( \sqrt{\frac{\lambda (\sqrt{\lambda^2 + 12} - \lambda)}{6}}, \sqrt{\frac{-\lambda + \sqrt{12 + \lambda^2}}{2\sqrt{3}}}, 0 \right) \\
(a5) & \quad (x, y, \dot{\gamma}) = \left( \sqrt{\frac{-3(1 + w_m)^3}{\lambda^2 w_m}}, \frac{\sqrt{3}(1 + w_m)}{\lambda}, 0 \right) \quad \text{with } w_m < 0, \lambda \geq \frac{\sqrt{3}(1 + w_m)}{-w_m} ,
\end{align*}
\]

(33)

which we call the ultra-relativistic kinetic-potential scaling solutions, and the ultra-relativistic kinetic-potential-fluid scaling solutions, respectively.

It is worth commenting on the solutions \( (a1) - (a5) \). \( (a1) \) is well known and corresponds to the case where the kinetic energy of the DBI field dominates over the potential energy, leading to an effective equation of state for the DBI field in Eq. (11) which mimics that of dust \( (w_\phi = 0) \). The existence of the solution \( (a2) \) where this ultrarelativistic kinetic dominated DBI field scales with matter was previously obtained by Ahn. et. al [50, 52] who also obtained the solutions \( (a3) \) and \( (a4) \). In fact \( (a3) \) is the solution actively investigated in the context of DBI inflation, for example as seen in Refs. [4-8]. The solution \( (a5) \) is of a new type, where the ratio of the kinetic and potential terms remain constant in the ultra-relativistic limit. The particular case of this solution for \( p = 2 \) and \( r = -4 \) was first discovered in [69], although here we have shown that this type of solution exists as long as \( p + r = -2 \).

2. Case \( b : \dot{\gamma} = 1 \)

In this case the Lorentz factor \( \gamma = 1 \) and the DBI field mimics the behaviour of a canonical scalar field. We shall refer to this class of fixed-points as the “standard” solutions. Substituting \( \dot{\gamma} = 1 \) into Eqs. (22) - (24), we obtain

\[
\begin{align*}
(x, y) &= \left( \sqrt{\frac{\lambda (\sqrt{\lambda^2 + 12} - \lambda)}{6}}, \sqrt{\frac{-\lambda + \sqrt{12 + \lambda^2}}{2\sqrt{3}}}, 0 \right) \\
(x, y) &= \left( \sqrt{\frac{-3(1 + w_m)^3}{\lambda^2 w_m}}, \frac{\sqrt{3}(1 + w_m)}{\lambda}, 0 \right) \quad \text{with } w_m < 0, \lambda \geq \frac{\sqrt{3}(1 + w_m)}{-w_m} ,
\end{align*}
\]
\[ x, N = \frac{\sqrt{6}}{2} \lambda (1 - \tilde{\gamma}) y^{q + 2} \]
\[ + \frac{3}{2} \xi^2 [2(x^2 - 1) + (1 + w_m)(1 - x^2 - y^2)], \]  \tag{34}
\[ y, N = -\frac{\sqrt{6}}{2} \lambda (1 - \tilde{\gamma}) y^{q + 1} \]
\[ + \frac{3}{2} \xi y [2x^2 + (1 + w_m)(1 - x^2 - y^2)], \]  \tag{35}
\[ \tilde{\gamma}, N = \sqrt{6} \lambda (1 - \tilde{\gamma})^{q + 1} \]
\[ \frac{y^{2q}}{x^{2q + 1}} \left( \frac{x^2 - y^2}{p} \right). \]  \tag{36}

Following our previous approach we classify the standard fixed points of this system based on the range of values the parameter \( q \) can take. We find that for \( q \geq 0 \), there exists a standard fixed point

\[ (b1) \quad (x, y, \tilde{\gamma}) = (1, 0, 1), \]  \tag{37}

which is the standard kinetic energy dominated solution.

For the special case of \( q = 0 \) \((p + r \rightarrow \infty)\), we find two extra interesting standard fixed-points:

\[ (b2) \quad (x, y, \tilde{\gamma}) = \left( \frac{\tilde{\lambda}}{\sqrt{6}}, \frac{\sqrt{6 - \lambda^2}}{\sqrt{6}}, 1 \right), \]
with \( \tilde{\lambda} < \sqrt{6}, \)  \tag{38}
and

\[ (b3) \quad \left( \frac{\sqrt{3}(1 + w_m)}{\sqrt{2\lambda}}, \frac{\sqrt{3}(1 - w_m^2)}{\sqrt{2\lambda}}, 1 \right), \]
with \( \tilde{\lambda} = \sqrt{3(1 + w_m)} \) and \( -1 \leq w_m \leq 1, \)  \tag{39}

which we call the standard kinetic-potential scaling solutions, and the standard kinetic-potential-fluid scaling solutions, respectively.

In reviewing these solutions, recall that the DBI field behaves just as the usual canonical scalar field when \( \tilde{\gamma} = 1 \), hence the fixed point solutions found in Case (b) are already well known. For example the properties of (b2) and (b3) are identical with that of a standard power-law inflationary solution and scaling solution respectively, obtained with an exponential potential in the presence of a canonical scalar field  \([52, 53]\). In fact (b2) was obtained previously in the context of DBI Inflation in  \([51, 52]\).

3. Case c : \[ 3 = \frac{4}{1 + \tilde{\gamma}} \frac{\sqrt{3}}{\sqrt{2\lambda}} \tilde{\gamma}^{\frac{2}{3} - q} (1 - \tilde{\gamma})^q \frac{y^{2q}}{x^{2q + 1}} \left( -x^2 + y^2 \right). \]  \tag{43}

Here \( \tilde{\gamma} \) is a constant which is different from 0 or 1. In this case, since the Lorentz factor \( \gamma > 1 \) and is constant as defined in \([4]\), we shall refer to this class of fixed point solutions as the “relativistic” ones.

As there are no values of \( q \) which permit standard fixed-point solutions \( x = y = 0 \) in equations (22)-(24), we therefore begin by exploring the possibilities of either \( x \) or \( y \) being 0 in this case.

For the case of \( x \neq 0, \ y = 0 \), we find the following standard fixed points in the system which exists only for \( q = 0 \):

\[ (c1) \quad (x, y, \tilde{\gamma}) = \left( 1, 0, \frac{3}{\tilde{\mu}^2 - 3} \right), \]
with \( \tilde{\mu} < -\sqrt{6} \) \tag{40}
and

\[ (c2) \quad (x, y, \tilde{\gamma}) = \left( -\frac{\sqrt{3(1 + w_m)}}{\sqrt{w_m \tilde{\mu}}}, 0, w_m \right), \]
with \( 0 < w_m < 1, \) \( \tilde{\mu} < -\frac{\sqrt{3(1 + w_m)}}{\sqrt{w_m}} \) \tag{41}
where we refer to (c1) as the relativistic kinetic dominated solution, and to (c2) as the relativistic kinetic-fluid scaling solution.

For the case \( x = 0, \ y \neq 0 \), a standard fixed-point solution exists but only for \( q = -1/2 \)

\[ (c3) \quad (x, y, \tilde{\gamma}) = \left( 0, 1, \frac{\sqrt{3}}{\sqrt{\lambda^2 + 3}} \right), \]  \tag{42}
which we call the relativistic potential dominated late-time solution.

For the remaining cases with \( x \neq 0, \ y \neq 0 \) it follows that combining Eqs. (22)-(24) for general \( q \), yields the condition

\[ 3 = \frac{4}{1 + \tilde{\gamma}} \frac{4}{\sqrt{3}} \tilde{\gamma}^{\frac{2}{3} - q} (1 - \tilde{\gamma})^q \frac{y^{2q}}{x^{2q + 1}} \left( x^2 + y^2 \right). \]  \tag{43}

Comparing this with Eq. (25), we see that in Case (c) the condition for a fixed point for general non-zero \( x \) and \( y \) requires \( p = -r \) (or \( q \rightarrow -\infty \)) a limit we have decided not analyse in this section.

Summarising the results of Case (c) we note that the fixed-point solutions (c1) and (c2) are completely new, while (c3) can also be derived as a special case of line 5 of Table I in  \([52]\). Of particular note for cosmology is (c3) which is the concrete example of an inflationary solution with constant \( \tilde{\gamma} \) which differs from 1 and 0. Phenomenologically this is a very interesting solution when considering cosmological perturbations in DBI inflation.

In Table 1, we have provided a breakdown of the 11 standard fixed-point solutions obtained in these class of models corresponding to cases (a) - (c).
C. Stability Analysis for standard fixed-points

We now turn our attention to carrying out a stabili-
ysis analysis for the standard fixed-points obtained in the
previous section. Calling these points in general \( x_c, y_c \)
and \( \gamma_c \), we consider small fluctuations about them given by

\[
x = x_c + \delta x, \quad y = y_c + \delta y, \quad \gamma = \gamma_c + \delta \gamma_c
\]

and consider solutions of the form \( \delta x \propto e^{wN}, \delta y \propto e^{wN} \)
and \( \delta \gamma \propto e^{wN} \). As before we consider each case in turn
and examine the dynamical behaviour of the system close
to their fixed-point positions on the phase plane.

1. Case a : \( \gamma = 0 \)

Expanding Eqs. (22)-(23) around the ultra-relativistic
kinetic energy dominated solution (a1), the ultra-
relativistic kinetic-fluid scaling solutions (a2) and the
ultra-relativistic potential dominated solution (a3) we
obtain the following eigenvalues

\[
(a1): \quad w_1 = 3, \quad w_2 = \frac{3}{2}, \quad w_3 = -3w_m
\]

\[
(a2): \quad w_1 = 3, \quad w_2 = \frac{3}{2}, \quad w_3 = 0
\]

\[
(a3): \quad w_1 = \frac{3}{2}, \quad w_2 = -3(1 + w_m), \quad w_3 = 3
\]

The presence of positive eigenvalues in each case indicates
that all three solutions are always unstable.

For the ultra-relativistic kinetic-potential scaling solution
(a4), we obtain the following eigenvalues

\[
w_{1,2} = \frac{3}{8}(4w_m + 8 + \lambda^2 - \sqrt{4\lambda^2 + \lambda^2})
\]

\[
\times \left[ -1 \pm \sqrt{1 - A(\lambda, w_m)} \right]
\]

\[
w_3 = \frac{1}{2} \left( -\lambda + \sqrt{12 + \lambda^2} \right) (\lambda + \mu)
\]

where \( A(\lambda, w_m) = 8 \left( -12 - \sqrt{12 + \lambda^2} \right) \)

\[
\times \frac{-6w_m - 6 - \lambda^2 + \lambda \sqrt{12 + \lambda^2}}{9(-\lambda^2 + \lambda \sqrt{12 + \lambda^2} - 4(2 + w_m)^2)}.
\]

It can be seen that \( w_{1,2} \) are non-positive if the condition
\( w_m \geq -1 + (\lambda/6)(\sqrt{\lambda^2 + 12} - \lambda) \) is satisfied.

Rewriting we see that this condition becomes \( 0 < \lambda < \sqrt{3(1 + w_m)} \sqrt{-w_m} \) if \( w_m < 0 \). Therefore, considering
also \( w_3 \) we see that these solutions are stable if the addi-
tional condition \( \lambda + \mu < 0 \) is also satisfied. However from
Eq. (20) we know that \( \lambda + \mu = \frac{\sigma}{2}(r + p) \) and given that
the condition for the existence of the solution to (a4) is
\( q = 1/2 \), we see that this solution is stable only for the case
\( p > 0 \) where \( r + p = -2 \).

A similar analysis of the ultra-relativistic kinetic-
potential-fluid scaling solution (a5) yields

\[
w_{1,2} = \frac{3}{4}(1 - w_m) \left[ -1 \pm \sqrt{1 + B(\lambda, w_m)} \right]
\]

\[
w_3 = 3(1 + w_m) \frac{\lambda + \mu}{\lambda}
\]

where \( B(\lambda, w_m) = \frac{8(w_m \lambda^2 + 3(1 + w_m)^2)}{\lambda^2 w_m^2 (1 + w_m)} \).

In this case \( w_m < 0 \) and \( B < 0 \) for \( \lambda \geq \frac{\sqrt{3}}{2}(1 + w_m)/\sqrt{-w_m} \), hence \( w_{1,2} \) are negative whenever these solu-
tions exist. Of course the bound on \( \lambda \) is the com-
plement of that arising above for the stability of (a4), hence
it follows that only one of the two solutions (a4) or (a5)
can be stable for a given value of \( \lambda \). The other condition
required for stability is once again that \( w_3 \leq 0 \), which as
before corresponds to, \( \lambda + \mu < 0 \) or \( p > 0 \) (recall we are
assuming \( \lambda > 0 \)). It is worth noting that \( \lambda = |p|/\sqrt{\sigma \nu} \)
for \( q = 1/2 \).

2. Case b : \( \gamma = 1 \)

For the standard kinetic energy dominated solution
(b1) the eigenvalues are

\[
w_1 = 3(1 - w_m), \quad w_2 = 3, \quad w_3 = -6
\]

for \( q \neq 0 \) and

\[
w_1 = 3(1 - w_m), \quad w_2 = 3 - \frac{\sqrt{6}}{2} \lambda, \quad w_3 = -6 - \sqrt{6} \mu
\]

for \( q = 0 \), which clearly indicates that these are unsta-
table solutions. For the standard kinetic-potential scaling
solutions (b2), (also \( q = 0 \)), we find the eigenvalues

\[
w_1 = \frac{\lambda^2 - 6}{2}, \quad w_2 = \lambda^2 - 3(1 + w_m), \quad w_3 = -\lambda \lambda + \mu, \quad w_4 = -\lambda \lambda + \mu
\]

suggesting these solutions are stable when \( \lambda < \sqrt{3(1 + w_m)} \) and \( \lambda + \mu > 0 \). It is worth noting that \( \lambda = \|p\| \) and \( \lambda + \mu = -\|p\|/(pq) \) for \( q = 0 \). Then, the
condition \( \lambda + \mu > 0 \) corresponds to \( p < 0 \) for \( q \to 0 \) and
\( p > 0 \) for \( q \to 0 \).

Similarly, for the standard kinetic-potential-fluid scaling
solutions (b3) we obtain

\[
w_{1,2} = \frac{3}{4}(1 - w_m)
\]
\[ w_3 = -3(1 + w_m) \frac{\lambda + \mu}{\lambda}. \] 

(53)

Since \( \lambda \geq \sqrt{3(1 + w_m)} \) whenever these solutions exist, \( w_1 \) and \( w_2 \) are always negative. Therefore, these solutions are stable if the condition \( \lambda + \mu > 0 \) is satisfied. Note also from the condition above for the stability of (b2), if the conditions for both (b2) and (b3) occur, only (b3) will be the stable late time attractor, in other words the standard kinetic-potential-fluid scaling solution will dominate over the standard kinetic potential scaling solution.

3. Case c : \[ 3 = \sqrt{\frac{3(1 + \gamma)}{1 + \gamma}} \frac{\lambda}{\mu} \frac{1}{\gamma} (1 - \gamma)^{q} x^{2} + y^{2} \]

For the relativistic kinetic energy dominated solution (c1), expanding around this fixed point yields the following eigenvalues

\[ w_1 = 3(-w_m + \gamma) \]

\[ w_2 = 3 + 3\gamma - \sqrt{3\lambda \gamma (1 + \gamma)} \]

\[ w_3 = \frac{3(-1 + \gamma)}{2}, \] 

(54)

where \( w_3 \) is clearly always negative. The parameter space for which this solution is stable is \( \lambda \geq \sqrt{3(1 + \gamma)/\gamma} \) and \( w_m \geq \gamma \). This is natural since if the potential is steep enough, the kinetic term easily dominates the potential term and if \( w_m \geq \gamma (= w_{\phi}) \), the energy density of the fluid decreases faster than that of the DBI field even though it is dominated by the kinetic term. It is also worth noting that since \( \tilde{\gamma} = 3/(\tilde{\mu}^{2} - 3) \) for this fixed-point, the stability conditions are given in terms of \( \tilde{\mu} \) as \( \lambda > \tilde{\mu}^{2} \) and \( w_m > 3/(\tilde{\mu}^{2} - 3) \). We can go a little further using Eq. (20) implying \( \lambda = -\epsilon p \). It follows that the condition for stability is \( \frac{\epsilon p}{\tilde{\mu}^{2}} < 1 \) with \( \frac{\epsilon}{p} < 0 \), so \( r \) and \( p \) have to have the opposite sign whilst satisfying \( q = 0 \).

For the relativistic kinetic-fluid scaling solutions (c2), a similar analysis produces

\[ w_{1,2} = \frac{3}{4}(1 - w_m) \]

\[ \times \left[ -1 \pm \sqrt{1 - \frac{8(3(1 + w_m)(\tilde{\mu}^{2}w_m - 3(1 + w_m)))}{\tilde{\mu}^{2}(1 + w_m)}} \right] \]

\[ w_3 = \frac{3(\lambda + \tilde{\mu})(1 + w_m)}{2\tilde{\mu}}. \] 

(55)

Since \( \tilde{\mu}^{2} > 3(1 + w_m)/w_m \) for these solutions, either \( w_1 \) or \( w_3 \) is always positive, which means these solutions are unstable.

The relativistic potential dominated solution (c3) yields the following eigenvalues

\[ w_1 = 0, \quad w_2 = -3, \quad w_3 = -3(1 + w_m). \] 

(56)

As this has a zero eigenvalue, we say this is a ‘marginally stable’ solution in the sense that there is no instability growing exponentially, although it could be unstable to higher orders in the perturbation. Obviously, the stability of this point is weaker than that of the fixed-point with three negative eigenvalues.

To help clarify all the possible standard fixed-point solutions and their stability the information just provided is summarised in Table I below.

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D. Fixed-points arising from solutions that initially appear singular

As mentioned earlier, for models with a general power-law potential and warp factor, in addition to the usual standard fixed points, it is necessary to check if the points where the denominator is 0 in the right hand side of Eqs. (22)-(24) can also be late-time attractor solutions for the system, a situation which does not arise in the case with a canonical scalar field [57].

Even in the case that the point itself leads to a singularity, it doesn’t necessarily mean the system is ill defined. For example it could be that the solutions approach the point exponentially slowly (i.e. like \( \exp[-N] \)), hence it would take an infinite time to reach the singularity and physically there is no ill behaviour in the system. In particular as long as the ratio of the singular terms (loosely called ‘0/0’) tends to a constant value then the system can be analyzed for the stability of these fixed points. As standard techniques can be applied to judge the stability of such a point, we also call them fixed-points in what follows. In the following, depending on the value of \( q \) we show there are 6 kinds of fixed-point where ‘0/0’ is finite in the phase space. As in the previous analysis, we have excluded the special case with \( p = -r \) here.

1. Standard potential dominated solutions

First, we consider the point (a1): \((x, y, \tilde{\gamma}) = (0, 1, 1)\) which in the case of a canonical field would simply corresponds to a standard slow-roll inflationary solution. However, in this case \( x = 0 \) leads to a singularity in Eqs. (22)-(24) for \( q > -1/2 \) and \( \tilde{\gamma} = 1 \) does the same for
TABLE I: Summary of the standard fixed-points and their stability in the models where the potential and brane tension are arbitrary non-negative power-law functions of the DBI field. Notice that we have restricted $\lambda > 0$ and $x \geq 0$. We have expressed the conditions for the stability in terms of $(p, r) \not= (\lambda, \bar{\mu})$ as these are the more fundamental quantities.

$q < 0$, hence we have to tread carefully in analysing the system.

Since the coordinate $y = 1$ does not result in any ill-defined behaviour in Eqs. (22)-(24), we can consider the reduced system in which we determine the leading order behaviour of $x$ and $\bar{\gamma}$ around the point $(a1)$. Writing $x = 0 + \delta x$, $\bar{\gamma} = 1 - \delta \bar{\gamma}$ and keeping only the leading order terms, we obtain

$$
\delta x_{,N} = -\frac{\delta x}{\sqrt{2}} \left( 3\sqrt{2} - \lambda \sqrt{6} (\delta \bar{\gamma})^q \right),
$$

(57)

$$
\delta \bar{\gamma}_{,N} = -2\delta \bar{\gamma} \left( 3\sqrt{2} - \lambda \sqrt{6} (\delta \bar{\gamma})^q \right),
$$

(58)

If we introduce $\beta \equiv \frac{\langle \delta \gamma \rangle^{q+1}}{\langle \delta \gamma \rangle^{q+1}}$, for $q \not= -1/2$ and $q > 0$, this appears to be of the form of ‘0/0’ and requires a careful analysis to properly understand the behaviour of this system. In this case from Eqs. (57) - (58), it becomes

$$
(\frac{\delta x}{\langle \delta \gamma \rangle^q})^{q+1} = ce^{-3N} + \frac{\sqrt{6}}{\sqrt{6} \lambda},
$$

(59)

where $c$ is an integration constant. At late times ($N \to \infty$) we see that even though both terms tend to zero, the ratio $\langle \delta \gamma \rangle^q / \langle \delta \gamma \rangle^{q+1}$ approaches the constant given by $\langle \delta \gamma \rangle^q / \langle \delta \gamma \rangle^{q+1} = \sqrt{6}/\lambda$. Then, it is possible to sensibly discuss whether the point $(a1) = (0, 1, 1)$ is stable or not in terms of the remaining two-dimensional system obtained by substituting $(1 - \delta \bar{\gamma})^q / \langle \delta \gamma \rangle^{q+1} = \sqrt{6}/\lambda$ back into Eqs. (22)-(23). It is worth noting that as the coefficient of $N$ in the exponential function in Eq. (59) is $-3$ which is negative, this solution is stable along the direction of $\beta$.

The eigenvalues $w_1$, $w_2$ and $w_3$ corresponding to evolution of the perturbations in $x$, $y$ and $\bar{\gamma}$ respectively are obtained from Eqs. (57), (57) and (58):

$$
w_1 = 0 = w_3, \quad w_2 = -3(1 + w_m).
$$

(60)

To be specific the results $w_1 = 0 = w_3$ arise because at leading order, the analytic solution including $(\delta \gamma)^q / \langle \delta \gamma \rangle^{q+1} = \sqrt{6}/\lambda$ leads to a vanishing right hand side for Eqs. (57) and (58). The fact that at leading order all eigenvalues are non-positive, implies that for $q < -1/2$ and $q > 0$ there are solutions which tend to $(a1)$ regardless of the values of $\bar{\lambda}$, $\bar{\mu}$ and $w_m$ even though it is not a fixed-point solution in a usual sense.

But this zero eigenvalue implies that the stability of this point is weaker than the point which has negative eigenvalues for all three directions. As in the case (c3), we say this solution is ‘marginally stable’. Of course, to obtain the strict stability of this solution we would have to go to higher order.

2. Ultra-relativistic potential dominated solutions

Next, we consider the point $(a2): (x, y, \bar{\gamma}) = (0, 1, 0)$ whose behaviour is the same as $(a3)$ except for the fact that $x = 0$ is singular for $q > -1/2$ and $\bar{\gamma} = 0$ for $q > 1/2$ in Eqs. (22)-(23).

Following the arguments used for $(a1)$ and writing $x = 0 + \delta x$, $\bar{\gamma} = 0 + \delta \bar{\gamma}$ we obtain to leading order:
\[
\delta x_{N} = \frac{\delta x}{2} \left( -3 + \sqrt{3} \lambda \frac{(\delta \gamma)^{2-q}}{(\delta x)^{2q+1}} \right), \tag{61}
\]
\[
\delta \gamma_{N} = -\delta \gamma \left( -3 + \sqrt{3} \lambda \frac{(\delta \gamma)^{2-q}}{(\delta x)^{2q+1}} \right). \tag{62}
\]

For \(-1/2 < q < 1/2\), \(\beta \equiv \frac{(\delta x)^{2q+1}}{(\delta \gamma)^{2-q}}\) once again appears ill defined, however we can solve Eqs. (61) - (62) to give
\[
\frac{(\delta x)^{2q+1}}{(\delta \gamma)^{2-q}} = ce^{-3N} + \sqrt{3} \lambda, \tag{63}
\]
where \(c\) is an integration constant. At late times \((N \to \infty)\) we see that even though both terms tend to zero, the ratio \((\delta \gamma)^{1/2-q}/(\delta x)^{2q+1} \to \sqrt{3}/\lambda\).

As in the case of \((\alpha 1)\), it is possible to discuss whether the point \((\alpha 2) = (0, 1, 0)\) is stable or not in terms of the remaining two-dimensional system obtained by substituting the particular solution \(\tilde{\gamma}^{1/2-q}/x^{2q+1} = \sqrt{3}/\lambda\) back into Eqs. (22) - (23). It is worth noting that as the coefficient of \(N\) in the exponential function in Eq. (63) is \(-3\) (i.e. negative), this solution is stable along the direction of \(\beta\).

The corresponding eigenvalues for the perturbations in \(x, y\) and \(\tilde{\gamma}\) follow from Eqs. (61), (23) and (62) and are given by
\[
w_1 = 0 = w_3, \quad w_2 = -3(1 + w_m). \tag{64}
\]

Note that the eigenvalues for \((\alpha 2)\) are identical to those of \((\alpha 1)\) hence to obtain the full stability of the system, as in that case, we would have to go to higher order to obtain the strict stability of the system.

3. Ultra-relativistic kinetic dominated solutions

Next, we consider the point \((\alpha 3)\): \((x, y, \tilde{\gamma}) = (1, 0, 0)\) whose property is the same as \((\alpha 1)\) except that \(y = 0\) is singular for \(q < 0\) and \(\tilde{\gamma} = 0\) is singular for \(q > 1/2\) in Eqs. (22)-(24).

In this case, since the coordinate \(x = 1\) does not result in any ill-defined behaviour in Eqs. (22)-(24), we can consider the reduced system in which we determine the leading order behaviour of \(y\) and \(\tilde{\gamma}\) around the point \((\alpha 3)\). Writing \(y = 0 + \delta y, \tilde{\gamma} = 0 + \delta \tilde{\gamma}\) and keeping only leading order terms, we obtain
\[
\delta y_{N} = \sqrt{\frac{3}{2}} \delta y \left( \sqrt{3} \lambda \frac{(\delta \gamma)^{2-q}}{(\delta y)^{2q}} \right), \tag{65}
\]
\[
\delta \tilde{\gamma}_{N} = \sqrt{\frac{3}{2}} \delta \tilde{\gamma} \left( \sqrt{3} \frac{\lambda}{p} \frac{(\delta \gamma)^{2-q}}{(\delta y)^{2q}} \right). \tag{66}
\]

Following the earlier examples, for \(q < 0\) and \(q > 1/2\), we introduce \(\beta \equiv \frac{(\delta y)^{-2q}}{(\delta \gamma)^{2-q}}\) and proceed to show that it has a well behaved non-trivial behaviour of its own. Solving Eqs. (65) - (66) we find
\[
\frac{(\delta y)^{-2q}}{(\delta \gamma)^{2-q}} = ce^{-\frac{3}{2} N} + \sqrt{3} \lambda \left( 2q + (2q - 1) \frac{r}{p} \right), \tag{67}
\]
where \(c\) is an integration constant. At late times \((N \to \infty)\) we see that even though both terms tend to zero, the ratio \((\delta \gamma)^{1/2-q}/(\delta y)^{-2q}\) approaches the constant given by \((\delta \gamma)^{1/2-q}/(\delta y)^{-2q} = \sqrt{3}/[\lambda (2q + (2q - 1)r/p)]\). Since both \(\delta y\) and \(\delta \tilde{\gamma}\) are positive, for this solution to be physical, for a given \(q, r/p\) should satisfy
\[
2q + (2q - 1) \frac{r}{p} > 0, \tag{68}
\]
which is equivalently \(p > 0\) with \(r < -2\) or \(p < 0\) with \(r > -2\).

As in the previous cases, it is possible to discuss whether the point \((\alpha 3) = (1, 0, 0)\) stable or not in terms of the remaining two-dimensional system obtained by substituting \(\tilde{\gamma}^{1/2-q}/\gamma^{2q} = \sqrt{3}/[\lambda (2q + (2q - 1)r/p)]\) back into Eqs. (22) - (23). It is worth noting that as the coefficient of \(N\) in the exponential function in Eq. (67) is \(-3/2\) which is negative, this solution is stable along the direction of \(\beta\).

The eigenvalues for the perturbations in \(x, y\) and \(\tilde{\gamma}\) follow from Eqs. (22) - (24) and (65) and (66) to give
\[
w_1 = -3w_m, \quad w_2 = \frac{3(2q - 1)(1 + \frac{r}{p})}{2[2q + (2q - 1) \frac{r}{p}]} = \frac{3}{2} \left[ 1 + \frac{p}{r + 2} \right], \tag{69}
\]
\[
w_3 = \frac{6q(1 + \frac{r}{p})}{[2q + (2q - 1) \frac{r}{p}]} = \frac{6}{2 + r}, \tag{70}
\]
where in Eqs. (69) - (71) we have made use of the fact \(q = \frac{-1}{r + p}\). Although \(w_2\) and \(w_3\) diverge for \(r = -2\), from Eq. (68) this case is not physical.

From these eigenvalues, we see that for both cases \(q < 0\) and \(q > 1/2\), stable solutions require \(w_m > 0\) and \(r < -2\). Note that in Eq. (68) this implies \(p > 0\) for stable solutions. For \(q < 0\), the additional constraints are \(r/p > -1\), whereas for the case \(q > 1/2\) it is \(-(2p + 1) < r/p < -1\).

4. Ultra-relativistic kinetic-fluid scaling solutions

In this case the behaviour is identical to \((\alpha 3)\), for the special case with \(w_m = 0\). However, there is another important point \((\alpha 4)\): \((x, y, \tilde{\gamma}) = (x_0, 0, 0)\) with \(x_0\) satisfying \(0 < x_0 < 1\) and whose property is the same as \((\alpha 2)\) in that there are a family of solutions depending on the value of \(x_0\) chosen. Again \(y = 0\) appears singular for \(q < 0\) and \(\tilde{\gamma} = 0\) for \(q > 1/2\) in Eqs. (22)-(24).

Following the arguments used for \((\alpha 3)\) and writing \(x = x_0, y = 0 + \delta y, \tilde{\gamma} = 0 + \delta \tilde{\gamma}\), we obtain to leading order
\[
\delta y_{N} = \frac{\sqrt{3}}{2} \delta y \left( \sqrt{3} - \lambda x_0^{-2q} \frac{(\delta \gamma)^{2-q}}{(\delta y)^{-2q}} \right), \tag{72}
\]
\[ \delta \gamma, N = \sqrt{3} \delta \gamma \left( \sqrt{3} + \frac{r}{p} \lambda_x^{1-2q} (\delta \gamma)^{2-q} (\delta y)^{-2q} \right), \] (73)

For \( q < 0 \) and \( q > 1/2 \), again introducing \( \beta \equiv (\delta y)^{-2q} \), we can solve Eqs. (72)-(73) to give

\[ \frac{(\delta y)^{-2q}}{(\delta \gamma)^{1/2-q}} = ce^{-\frac{2}{3} N} + \frac{3}{2} \lambda x_0^{1-2q} \left( 2q + (2q - 1) \frac{r}{p} \right), \] (74)

where \( c \) is an integration constant. Once again at late times \( (N \to \infty) \) we see that even though both terms tend to zero, the ratio \( (\delta \gamma)^{1/2-q}/(\delta y)^{-2q} \) approaches the constant given by \( (\delta \gamma)^{1/2-q}/(\delta y)^{-2q} = \sqrt{3}/[\lambda x_0^{1-2q}(2q + (2q - 1)r/p)] \). Since both \( \delta y \) and \( \delta \gamma \) are positive, for this solution to be physical, for a given \( q, r/p \) should satisfy Eq. (68). It is worth noting that the coefficient of \( N \) in the exponential function in Eq. (74) is \(-3/2\) which being negative implies this solution is stable along the direction of \( \beta \).

The eigenvalues for the perturbations in \( x, y \) and \( \tilde{\gamma} \) turn out to be identical to those for \( (a3) \) in Eqs. (70) and (71), except for the case of \( w_1 \). In particular from Eqs. (22), (74) and (73) we obtain

\[ w_1 = 0 \] (75)
\[ w_2 = \frac{3(2q - 1)(1 + \frac{r}{p})}{2[2q + (2q - 1) \frac{r}{p}]} = \frac{3}{2} \left[ 1 + \frac{p}{r + 2} \right] \] (76)
\[ w_3 = \frac{6q(1 + \frac{r}{p})}{[2q + (2q - 1) \frac{r}{p}]} = \frac{6}{2 + r}. \] (77)

Again, although \( w_2 \) and \( w_3 \) diverge for \( r = -2 \), this case is excluded as \( \beta \) is then no longer a finite constant.

\( w_1 \) shows that although there is a zero eigenvalue along the \( x \) direction, when \( w_2 < 0 \), there are solutions approaching the point \( (a4) \). From the discussions for \( (a3) \), we see that for both cases \( q < 0 \) and \( q > 1/2 \), stable solutions require \( r < -2, p > 0 \). For \( q < 0 \), the additional constraints are \( r/p > 1 \), whereas for the case \( q > 1/2 \) it is \(-2/p + 1 < r/p < -1\).

5. Standard fluid dominated solutions

Next, we consider the point \( (a5) \): \((x, y, \tilde{\gamma}) = (0, 0, 1) \) which in the case of a canonical field corresponds to the usual fluid dominated solution. However, here \( x = 0 \) is singular for \( q > -1/2 \), \( y = 0 \) and \( \tilde{\gamma} = 1 \) are singular for \( q < 0 \) in Eqs. (22)-(24). Since all the coordinates \( x = 0, y = 0, \tilde{\gamma} = 1 \) can result in ill defined behaviour in Eqs. (22)-(24), the stability analysis for this point is more complicated than in the previous examples in this subsection.

To investigate the behaviour of perturbations in \( x, y \) and \( \tilde{\gamma} \) around \( (a5) \), we write \( x = 0 + \delta x, y = 0 + \delta y, \tilde{\gamma} = 1 - \delta \tilde{\gamma} \) and keep only the leading order terms, to obtain

\[ \delta x, N = \frac{\delta x}{2} \left[ -3(1 - w_m) + \sqrt{6\lambda} \left( \frac{\delta x}{1-2q} \right)^{2q+2} \right], \] (78)
\[ \delta y, N = \frac{\delta y}{2} \left[ 3(1 + w_m) - \sqrt{6\lambda} \left( \frac{\delta y}{1-2q} \right)^{2q+1} \right], \] (79)
\[ \delta \tilde{\gamma}, N = -2\delta \tilde{\gamma} \left[ 3 + \sqrt{\frac{3}{2}} \left( \frac{\delta y}{1-2q} \right)^{2q+1} \left( \frac{r}{p} - \left( \frac{\delta y}{1-2q} \right)^{2q} \right) \right]. \] (80)

We can in fact make considerable progress by solving for combinations of the variables as in the previous example. Introducing \( \theta = (\delta x)^{-2q+1} \equiv (\delta y)^{-2q+1} \) and \( \beta = (\delta y)^{q} \), from Eqs. (78)-(80) we obtain

\[ \frac{\theta, N}{\theta} = -\frac{3}{2} \left( 1 - w_m \right) + \sqrt{\frac{3}{2}} \lambda \theta \left[ -2q \left( 1 + \frac{r}{p} \right) + \beta \right], \] (81)
\[ \frac{\beta, N}{\beta} = 6 - \sqrt{6} \lambda \theta (1 + \beta), \] (82)

These equations pick up two of the three degrees of freedom of the system given by Eqs. (78) - (80), and stability in this two-dimensional system is necessary for the stability in the full three-dimensional system. We find this two-dimensional system has three fixed-points characterised by

\[ (\theta, \beta) = (0, 0), \] (83)
\[ \left( \frac{3(1 - w_m)}{-2q\sqrt{6\lambda} \left( 1 + \frac{r}{p} \right)}, 0 \right), \] (84)
\[ \left( \frac{3(1 + w_m)}{\sqrt{6\lambda} \left( 2q \left( 1 + \frac{r}{p} \right) + 1 \right)}, \frac{4q \left( 1 + \frac{r}{p} \right) + (1 - w_m)}{1 + w_m} \right). \] (85)

It turns out that the eigenvalues corresponding to fluctuations about Eq. (83) are \(-3(1 - w_m)/2\) and 6 whilst those corresponding to Eq. (84) are \(3(1 - w_m)\) and \(3(1 - w_m) + 4q(1 + r/p)/[2q(1 + r/p)]\), indicating that both solutions are unstable.

Turning to the stability of the fixed-point corresponding to Eq. (85) we first of all note that in order for the solutions to be physical requires a number of conditions be satisfied. For \( \theta \) and \( \beta \) to be finite constants requires \( q < 0 \) since \( \delta x \) and \( \delta y \) have the same time dependence. Further, since \( \delta x, \delta y, \delta \tilde{\gamma} \) are all positive, \( \theta \) and \( \beta \) must be positive, which leads to the conditions

\[ 1 + 2q \left( 1 + \frac{r}{p} \right) > 0, \] (86)
\[ 4q \left( 1 + \frac{r}{p} \right) + (1 - w_m) > 0. \] (87)
which are equivalently $p > 4/(1 - w_m)$ or $p < 0$.

Now, by considering small perturbations around this point of the form $\delta \theta = A \epsilon^{w \lambda} + B \epsilon^{w \alpha} + \delta \beta = C \epsilon^{w \lambda} + D \epsilon^{w \alpha}$, we obtain the following eigenvalues,

$$w_{a,b} = \frac{3(2q(3 + w_m)(1 + \frac{r}{p}) + (1 - w_m))}{4(1 + 2q(1 + \frac{r}{p}))} \times \left[ -1 \pm \sqrt{1 - C(\lambda, w_m, \frac{r}{p})} \right]$$

where $C(\lambda, w_m, \frac{r}{p})$ is

$$8(1 + 2q(1 + \frac{r}{p}))(1 + w_m)(4q(1 + \frac{r}{p}) + (1 - w_m))$$

$$\frac{(2q(3 + w_m)(1 + \frac{r}{p}) + (1 - w_m))^2}{(2q(3 + w_m)(1 + \frac{r}{p}) + (1 - w_m))^2}.$$  \hspace{1cm} (88)

and $(+, -)$ is associated with $(w_a, w_b)$ respectively. Given the conditions $(85)$ and $(87)$ are satisfied for physically relevant situations, it follows that stable eigenvalues $(w_{a,b} \leq 0)$ are obtained if

$$2q(3 + w_m)(1 + \frac{r}{p}) + (1 - w_m) \geq 0,$$  \hspace{1cm} (89)

which is equivalent to $p > 2(3 + w_m)/(1 - w_m)$ or $p < 0$.

In fact $(89)$ is more restrictive than $(80)$ and $(87)$ as long as we are in the physically acceptable range $-1 < w_m < 1$.

If the condition $(89)$ is satisfied, the terms including $\delta x$, $\delta y$, $\delta \gamma$ in Eqs. $(73)-(80)$ are all finite and constant. Then it is possible to sensibly discuss whether the point $(\alpha 5) = (0,0,1)$ is stable or not in terms of the remaining one-dimensional system obtained by substituting $(85)$ back into say Eq. $(22)$.

Then we find the eigenvalue for the perturbations in $x$

$$w_1 = \frac{3q(1 + w_m)(1 + \frac{r}{p})}{1 + 2q(1 + \frac{r}{p})} = -\frac{3(1 + w_m)}{p - 2}.$$  \hspace{1cm} (90)

Notice that we have already obtained the stability for $\theta$ and $\beta$ directions, the stability based on the eigenvalues for the directions of $x$, $y$, and $\gamma$ have the same information. Actually, the eigenvalues corresponding to the perturbation in $y$ and $\gamma$ are given by

$$w_2 = \frac{3q(1 + w_m)(1 + \frac{r}{p})}{1 + 2q(1 + \frac{r}{p})} = -\frac{3(1 + w_m)}{p - 2},$$  \hspace{1cm} (91)

$$w_3 = -\frac{3(1 + w_m)(1 + \frac{r}{p})}{1 + 2q(1 + \frac{r}{p})},$$  \hspace{1cm} (92)

which are consistent with

$$w_1 = w_2,$$  \hspace{1cm} (93)

$$(-2q + 1)w_1 + 2qw_2 + qw_3 = 0.$$  \hspace{1cm} (94)

derived from $\theta$ and $\beta$ are constant. Although $w_1$, $w_2$ and $w_3$ diverge for $p = 2$, from Eq. $(85)$, this case is excluded.

We can summarise the constraints on the allowed parameters if the solution $(\alpha 5)$ is to be stable. From $q < 0$ we have $r + p > 0$ and from $w_1 < 0$ we have $p > 2$ and $1 + r/p > 0$. Then, from Eq. $(89)$, we have $p > 2(3 + w_m)/(1 - w_m) > 2$ and $w_m < (p - 6)/(p + 2)$.

6. Ultrarelativistic fluid dominated solutions

Next, we consider the point $(\alpha 6):(x,y,\gamma) = (0,0,0)$ which corresponds to a new type of fluid dominated solution. We don’t consider as one of the standard fixed points because of the apparently singular behaviour of the point in various limits. For example in Eqs. $(22)-(24)$, $x = 0$ is singular for $q > -1/2$, $y = 0$ is singular for $q < 0$ and $\gamma = 0$ for $q > 1/2$. It complicates the analysis for this point but the procedure itself is similar to the case with $(\alpha 5)$.

Writing $x = 0 + \delta x$, $y = 0 + \delta y$, $\gamma = 0 + \delta \gamma$ and keeping only the leading terms, we obtain

$$\delta x, N = \frac{\delta x}{2} \left[ 3w_m + \sqrt{3} \lambda \left( \frac{\delta y}{(\delta x)^2 + 1} \right)^2 \right],$$  \hspace{1cm} (95)

$$\delta y, N = \frac{\delta y}{2} \left[ 3(1 + w_m) - \sqrt{3} \lambda \left( \frac{\delta y}{(\delta x)^2} \right)^2 \right],$$  \hspace{1cm} (96)

$$\delta \gamma, N = \delta \gamma \left[ 3 + \sqrt{3} \lambda \left( \frac{\delta y}{(\delta x)^2} \right)^2 \right] + \left( \frac{p - (\delta y)^2}{(\delta x)^2} \right).$$  \hspace{1cm} (97)

Following the previous case $(\alpha 5)$, we introduce $\theta = \frac{(\delta y)^2}{(\delta x)^2 + 1} + \beta = \frac{(\delta y)^2}{(\delta x)^2}$. Eqs. $(95)-(97)$ then yield:

$$\frac{\theta}{\theta} = \frac{3}{2} (1 + w_m) + \frac{\sqrt{3} \lambda \theta}{-2q \left( \frac{1 + \frac{r}{p}}{p} \right) + \frac{r}{p}},$$  \hspace{1cm} (98)

$$\frac{\beta}{\beta} = 3 - \sqrt{3} \lambda \theta (1 + \beta),$$  \hspace{1cm} (99)

which has three fixed points given by

$$\left( \theta, \beta \right) = (0,0), \hspace{1cm} \left( \frac{\sqrt{3} \lambda (1 + w_m)}{\lambda \left( \frac{2q}{2q + 1} \right) - w_m}, 0 \right), \hspace{1cm} \left( \frac{\sqrt{3} (1 + w_m)}{\lambda \left( \frac{2q}{2q + 1} \right) - w_m}, \frac{(2q - 1) \left( \frac{1 + \frac{r}{p}}{p} \right)}{1 + w_m} \right).$$  \hspace{1cm} (100)

For the eigenvalues corresponding to Eq. $(100)$ we obtain $3(1 + w_m)/2$ and $3$, which means this fixed-point is unstable.

Considering the fixed-point Eq. $(100)$ first, the conditions for $\theta$ to be a finite constant are, $q > 1/2$ or $q < 0$. 

If \(0 \leq q \leq 1/2\), then in the definition of \(\theta\), all the terms \(\delta x, \delta y\) and \(\delta \gamma\) would appear in the numerator, with no apparent singularity present, implying the solution would not be of the form being discussed in this section. In addition to this, as \(\delta x, \delta y, \delta \gamma\) are all positive, \(\theta\) must be positive, which gives the following condition

\[
2q \left( 1 + \frac{r}{p} \right) - \frac{r}{p} > 0, \tag{103}
\]

which is \(r < -2\) with \(p > 0\), or \(r > -2\) with \(p < 0\).

Small perturbations around the fixed point Eq. (101), lead to the following eigenvalues for \(\delta \theta\) and \(\delta \beta\),

\[
w_\theta = -\frac{3}{2}(1 + w_m) \tag{104}
\]

\[
w_\beta = 3 - \frac{3(1 + w_m)}{2q(1 + \frac{r}{p}) - \frac{r}{p}}, \tag{105}
\]

where from Eq. (103), \(w_\beta < 0\) is satisfied as long as

\[
(1 + w_m) > 2q(1 + \frac{r}{p}) - \frac{r}{p} > 0, \tag{106}
\]

which is \(-2 - p(1 + w_m) < r\) with \(p > 0\), or \(-2 - p(1 + w_m) > r\) with \(p < 0\).

Having shown these new fixed points exist and have well defined behaviour through their combination in Eq. (101), we perturb about them in Eqs. (22) - (24) to obtain the following eigenvalues for the perturbations in \(x, y\) and \(\gamma\)

\[
w_1 = \frac{3}{2} w_m, \tag{107}
\]

\[
w_2 = \frac{3}{2}(1 + w_m) \left( 1 - \frac{1}{2q(1 + \frac{r}{p}) - \frac{r}{p}} \right), \tag{108}
\]

\[
w_3 = 3 \left( 1 + \frac{\frac{r}{p}(1 + w_m)}{2q(1 + \frac{r}{p}) - \frac{r}{p}} \right), \tag{109}
\]

where we have used the relation \(q = -1/(p + r)\). Although \(w_1, w_2\) and \(w_3\) diverge for \(r = -2\), from Eq. (103), this case is excluded.

Notice that although we have shown eigenvalues for 3 directions, only two of them are independent of the \(\theta\) direction. This can be seen by the fact that \(w_1, w_2\) and \(w_3\) are related through:

\[
(1 - 2q)w_1 + 2qw_2 + \left( \frac{1}{2} - q \right)w_3 = 0. \tag{110}
\]

We can summarise the constraints on the parameters if the solutions approaching the point \((\alpha 6)\) given by Eq. (101) are stable. From \(q > 1/2\) or \(q < 0\) (or equivalently \(0 > p + r > -2\) or \(p + r > 0\)) and \(w_2 < 0\), we have \(r < -2\), which means \(-2 - p(1 + w_m) < r < -2\) with \(p > 0\) from Eqs. (103) and (106). From \(w_1 < 0\) and \(w_3 < 0\) we have \(2/r < w_m < 0\). As \(p > 0\), \(0 > 1 > r/p > -2/p\) for \(q > 1/2\) and \(1 + r/p > 0\) for \(q < 0\).

We now turn to consider the fixed-point given by (102). In this case only \(q > 1/2\) is allowed as \(\delta x, \delta y, \delta \theta\) have the same time dependence. Furthermore, as \(\beta\) is a nonzero finite constant, in addition to (103), another condition given by

\[
(2q - 1) \left( 1 + \frac{r}{p} \right) - w_m > 0, \tag{111}
\]

which is \(r < -2 - p(1 + w_m)\) with \(p > 0\) or \(r > -2 - p(1 + w_m)\) with \(p < 0\) must be satisfied. In fact this extra condition is more restrictive than the one given by Eq. (103).

By considering small perturbations around this point, we obtain the following eigenvalues for \(\delta \theta\) and \(\delta \beta\),

\[
w_\theta = -\frac{3}{2}(1 + w_m), \tag{112}
\]

\[
w_\beta = -\frac{3q(2q - 1)(1 + \frac{r}{p}) - w_m}{2q(1 + \frac{r}{p}) - \frac{r}{p}}, \tag{113}
\]

which are both negative once the conditions given by Eqs. (104) and (111) are satisfied.

By substituting (102) into Eq. (22) we obtain the following eigenvalue for the perturbations in \(x\)

\[
w_1 = \frac{3(1 + \frac{r}{p})(-1 + 2q)(1 + w_m)}{2[2q(1 + \frac{r}{p}) - \frac{r}{p}]}, \tag{114}
\]

where we have used the relation \(q = -1/(p + r)\).

Notice that we have already obtained the stability for the \(\theta\) and \(\beta\) directions, the stability based on the eigenvalues for the directions of \(x, y\) and \(\gamma\) have the same information. Actually, the eigenvalues corresponding to the perturbation in \(y\) and \(\gamma\) are given by

\[
w_2 = \frac{3(1 + \frac{r}{p})(-1 + 2q)(1 + w_m)}{2[2q(1 + \frac{r}{p}) - \frac{r}{p}]}, \tag{115}
\]

\[
w_3 = \frac{6(1 + \frac{r}{p})(1 + w_m)}{2[2q(1 + \frac{r}{p}) - \frac{r}{p}]}, \tag{116}
\]

\[
= -\frac{3(p + r)(1 + w_m)}{(r + 2)}, \tag{117}
\]

which are consistent with

\[
w_1 = w_2, \tag{118}
\]

\[
(-2q + 1)w_1 + 2qw_2 + \left( \frac{1}{2} - q \right)w_3 = 0, \tag{119}
\]

derived from the fact that \(\theta\) and \(\beta\) are constant. Although \(w_1, w_2\) and \(w_3\) diverge for \(r = -2\), from Eq. (103), this case is excluded.
We can summarise the constraints on the parameters if the solutions approaching the point (a6) given by Eq. (102) are stable. From \( q > 1/2 \) (or equivalently \( 0 > p + r > -2 \)) and \( w_{1,2,3} < 0 \), we have \( r < -2 \), which means \( -2 - p(1 + w_m) > r \) with \( p > 0 \) from Eq. (111). If we interpret this as the constraint on \( w_m \), together with \( q > 1/2 \), \( w_m < 0 \) should be also satisfied. As \( p > 0 \), \( 0 > 1 + r/p > -2/p \) for \( q > 1/2 \).

It is worth mentioning that for the fluid dominated solution (a6) realised both through Eqs. (101) and (102), \( w_m < 0 \) is required. Such a restriction implies that we cannot be considering the usual matter or radiation fluid as the background for (a6) to be stable under, although it is consistent for example with that of a cosmological constant.

We finish this subsection by providing a summary of the solutions (a1) to (a6) and their stability criteria in Table II below.

| \( x \) | \( y \) | \( \gamma \) | \( \Omega_x \) | Valid \( q \) | Stability |
|-------|-----|-----|-----|--------|---------|
| (a1)  | 0   | 1   | 1   | \( q < -1/2 \) and \( q > 0 \) | marginally stable |
| (a2)  | 0   | 1   | 0   | \( -1/2 < q < 1/2 \) | marginally stable |
| (a3)  | 1   | 0   | 0   | \( q < 0 \) | \( w_m > 0 \), \( r < -2 \), \( p > 0 \), \( \frac{\lambda}{\mu} > -1 \) |
| (a4)  | 0   | 0   | \( x_0 \) | \( 0 < x_0 < 1 \) | \( q > 0 \) | \( w_m > 0 \), \( r < -2 \), \( p > 0 \), \( -1 - \frac{q}{\sqrt{2}} < \frac{\lambda}{\mu} < -1 \) |
| (a5)  | 0   | \( 1 \) | \( 0 \) | \( q > 0 \) | \( w_m = 0 \), \( r < -2 \), \( p > 0 \), \( -1 - \frac{q}{\sqrt{2}} < \frac{\lambda}{\mu} < -1 \) |
| (a6)  | 0   | 0   | \( 0 \) | \( q < 0 \) | \( 0 > w_m > \frac{q}{\sqrt{2}} \), \( p > 0 \), \( -2 > r \), \( -2 > r \), \( -2/p > -2/p \) |

TABLE II: This table summarises the validity and stability of the fixed-points a1 to a6. Notice that we have restricted ourselves to the regime \( \lambda > 0 \) and \( x \geq 0 \). For the condition for stability of (a6) for \( q > 1/2 \), \( w_m > 2/r \) holds only for \( r > -2 - p(1 + w_m) \).

### E. Late-time behaviour

Now that we have analyzed the stability of the fixed-point solutions, we are in a position to discuss the late-time attractor structure of the autonomous system described by Eqs. (22)-(24). It is clear from the solutions presented in TABLE I and II that the precise structure of the solutions to the dynamical system depends on the value of \( q \). We have seen that for the particular cases of \( q = 1/2 \), \( q = 0 \), \( q = -1/2 \) there are non-trivial “scaling solutions” where \( x \), \( y \) and \( \gamma \) are finite constants depending on the model parameters \( \lambda \), \( \mu \), and the equation of state of the fluid \( w_m \). Therefore, in this subsection, first we will discuss the late-time attractor structure of the system with special emphasis on these three cases, and then we will move on to a discussion of the other cases.

1. The case with \( q = 1/2 \)

TABLE II shows that in the case of \( \lambda + \mu < 0 \) (or equivalently \( p > 0 \)), if \( w_m \geq 0 \), the standard fixed-point (a4) (ultra-relativistic kinetic-potential scaling solutions) is stable for \( w_m > -1 + \lambda (\sqrt{\lambda} - 12 - \lambda)/6 \). It is also stable for \( w_m < 0 \) if \( \lambda < \sqrt{\lambda} (1 + w_m)/\sqrt{-w_m} \), which also corresponds to the requirement that \( p > 0 \) with \( p + r = -2 \). On the other hand, if \( \lambda \geq \sqrt{\lambda} (1 + w_m)/\sqrt{-w_m} \) with \( w_m < 0 \)

(or equivalently \( p \geq \sqrt{3\lambda p (1 + w_m)/\sqrt{-w_m}} > 0 \)), then the standard fixed-point (a5) (ultra-relativistic kinetic-potential-fluid scaling solution) is the stable solution. Clearly, these two fixed-point solutions are candidates for the late-time attractor solution to the system as long as the model parameters satisfy the conditions mentioned above. On the other hand, TABLE III shows that for \( q = 1/2 \), the fixed-point associated with (a1) (standard potential dominated solutions) is marginally stable. As (a1) has a zero eigenvalue, to specify which of the three possibilities (a4), (a5) or (a1) is likely to win out will require that we go to higher order, although it is likely to be (a4) or (a5) as one of these is definitely stable when the conditions \( \lambda + \mu < 0 \) and \( q = 1/2 \) are satisfied. It is worth mentioning that for the cosmologically sensible cases, we expect \( w_m > 0 \). Then, only (a4) can be the late-time attractor.

Alternatively, for \( \lambda + \mu > 0 \) (or equivalently \( p < 0 \)), as both points (a4) and (a5) have positive eigenvalues, we can expect (a1) to be the late-time attractor. These findings have been confirmed numerically as can be seen in Fig. III

As we have already mentioned, (a4) and (a5) are new solutions found only in the case of the DBI field. (a4) corresponds to the ultrarelativistic version of the usual canonical power-law inflationary solution. It is well
known that power-law inflation is strongly constrained by the observation of the spectral index of primordial perturbations (for example, see [82]). Although it is difficult to have DBI driven power-law inflation, we can in principle use it to explain the current acceleration of the universe. On the other hand, although (a5) is also unique to the DBI field, because the solution requires \( w_m < 0 \) to be satisfied, it is not so realistic given that we are usually considering \( w_m \) to be either matter or radiation.

However, since (a1) corresponds to the usual accelerating solution where the scalar field rolls slowly down its potential and is not unique to the DBI field, this solution can in principle be used to explain both an early stage of inflation and the present dark energy dominated period of acceleration.

In Fig. 2, we summarise the late-time attractor structure with \( q = 1/2 \) and \( w_m \geq 0 \).

Here, we consider the late-time attractor structure in the models where the potential and the brane tension are power-law functions of the DBI field with \( q = 1/2 \) and \( w_m \geq 0 \). We show the late-time attractor solution for a given model parameter in \( \lambda - \mu \) space with \( \lambda > 0 \). In the case with \( w_m < 0 \), the region where (a5) is the late-time attractor appears for \( \lambda + \mu < 0 \) and \( \lambda \geq \sqrt{3(1 + w_m)}/\sqrt{-w_m} \). It is worth noting that \( \lambda \) and \( \mu \) are related with the parameters of the potential and brane tension as \( \lambda = |p|/\sqrt{\sigma \nu} \) and \( \mu = -(2 + p)p/(p\sqrt{\sigma \nu}) \).
for \( q = 0 \), the fixed-point \((\alpha 2)\) (ultrarelativistic potential dominated solutions) is marginally stable.

Using arguments similar to those applied for the case with \( q = 1/2 \), we expect that for \( \tilde{\lambda} + \tilde{\mu} > 0 \) (or equivalently \( 1 + r/p > 0 \)), one of \((\beta 2), (\beta 3), (\gamma 1)\) will be the late-time attractor solution of this system as long as their stability conditions are satisfied. This is because we have seen that the stability properties of these three points are given by three negative eigenvalues which is a stronger condition than for \((\alpha 2)\).

On the other hand for \( \tilde{\lambda} + \tilde{\mu} < 0 \) (or equivalently \( 1 + r/p < 0 \)), the points \((\beta 2), (\beta 3)\) and \((\gamma 1)\) have positive eigenvalues, so we expect \((\alpha 2)\) will become the late-time attractor. This has once again been confirmed numerically in Fig. 3.

Since \((\beta 2)\) and \((\beta 3)\) correspond to the well known power-law inflationary and scaling solutions respectively, the cosmology based on these solutions has already been well studied \([5, 57]\). As in the ultra-relativistic case, the power-law inflationary solution \((\beta 2)\) can explain the present acceleration, while the scaling solution \((\beta 3)\) can play a very important role in classifying the late-time attractor structure of the system.

As we have mentioned earlier, \((\gamma 1)\) is a new solution specific to the DBI field, but we do not consider it further because it can neither explain the current acceleration of the Universe, nor can it accommodate a fluid.

\((\alpha 2)\) is also a solution specific to the DBI field and in fact a very interesting new type of inflationary solution. These models are however tightly constrained. They have been shown to give too large a degree of primordial non-Gaussianity \([4, 8]\).

In Fig. 4, we summarise the late-time attractor structure with \( q = 0 \) and \( w_m > 0 \).

3. The case with \( q = -1/2 \)

Here, unlike the previous cases with \( q = 1/2 \) and \( q = 0 \), when \( q = -1/2 \), the stable solutions with three negative eigenvalues for \( \tilde{\lambda} + \tilde{\mu} > 0 \) (or equivalently \( p > 0 \)), are given by the four fixed-points ((\(\alpha 3\) - \(\alpha 6\)) shown in TABLE 11).

\((\alpha 3)\) (ultrarelativistic kinetic dominated solutions) is stable for \( w_m \geq 0 \) with \( p > 4 \), while \((\alpha 4)\) (Ultrarelativistic kinetic-fluid scaling solutions) is stable for \( w_m = 0 \) with \( p > 4 \). \((\alpha 5)\) (Standard fluid dominated solutions) is stable for \( (p - 6)/(p + 2) > w_m > -1 \), with \( p > 2 \), and \((\alpha 6)\) (Ultrarelativistic fluid dominated solutions) is stable for \( 0 > w_m > 2/(2 - p) \), with \( 4/(-w_m) > p > 4 \).

These four fixed-points are candidates to be the late-time attractor solutions to the system as long as the model parameters satisfy the conditions mentioned above. When a region of parameter space allows more than one stable fixed-point, which one of them becomes the late-time attractor depends on the initial values of \( x, y \) and \( \tilde{\gamma} \). On the other hand, from TABLE 11 we see that in this case the fixed-point \((\alpha 3)\) (the relativistic potential dominated solution) is marginally stable. Therefore \((\alpha 3)\) is also a candidate late-time attractor solution. However, as we saw before, when a marginally stable solution also exists, we expect that for \( \tilde{\lambda} + \tilde{\mu} > 0 \) (or equivalently \( p > 0 \)), one of the four fixed points \((\alpha 3)-(\alpha 6)\) will be the late-time attractor solution of the system in the region of the phase space when their stability conditions are satisfied.
Turning our attention instead to consider the regime $\lambda + \bar{\mu} < 0$ (or equivalently $p < 0$), we see that the fixed-points $(\alpha 3)$ - $(\alpha 6)$ now have positive eigenvalues, and so we expect $(\alpha 3)$ to become the late-time attractor, a result that we have confirmed numerically in Fig. 5.

Cosmology based on the existence of the solutions $(\alpha 3)$ and $(\alpha 4)$ are peculiar to the DBI field in the sense that the field behaves like dust even though the kinetic term completely dominates the potential term. However, although interesting in its own right, the solutions do not completely dominate the potential term. However, all the field behaves like dust even though the kinetic term $\alpha \gamma$ is negligible role in its evolution.

On the other hand, the solution $(\alpha 3)$ appears to have some interesting features specific to the DBI field. In this case, $\gamma$ or equivalently the sound speed $c_s$ can be a constant ranging between 0 and 1. In such a case, if this solution is realised in the very early Universe, it opens up the possibility of a large but still allowed degree of non-Gaussianity being obtained. Furthermore, since the current prediction of non-Gaussianity from this type of inflation model is based on the assumption that $c_s$ is constant, this new solution serves as a good concrete background about which to consider cosmological perturbations.

4. The case with general $q \neq 0, 1/2$ or $-1/2$

To complete the classification of the late-time attractor structure in the models where the potential and the brane tension are power-law functions of the DBI field, we now consider the case where $q$ is different from $1/2, 0$ or $-1/2$. For three cases we have earlier seen that the question of whether marginally stable solutions can be the late-time attractor or not depends upon the stability of the other allowed fixed-points. In particular those fixed points with three negative eigenvalues, means that these marginally stable fixed-point solutions can not be the late-time attractor as their stability is weaker than that of the fixed-points. However, if all other fixed-points have positive eigenvalues, then the marginally stable fixed-point solutions can turn out to be the late-time attractor.

Significantly, since this criteria also holds for the cases considered here, we will apply it without showing the numerical plots except the special case with $0 < q < 1/2$.

For $q > 1/2$, for $\lambda + \bar{\mu} < 0$ (or equivalently $p > 0$), there are four stable fixed-point solutions $(\alpha 3) - (\alpha 6)$. In addition to these, the fixed-point solution $(\alpha 1)$ is marginally stable.

As in the previous cases, for $\lambda + \bar{\mu} < 0$, one of the fixed-point solutions $(\alpha 3) - (\alpha 6)$ can be the late-time attractor depending on the values of parameters $p, r, w_m$ or the initial values of $x, y, \gamma$, while for $\lambda + \bar{\mu} > 0$ (or equivalently $p < 0$), $(\alpha 1)$ is the late-time attractor.

For $0 < q < 1/2$, there are two marginally stable fixed-point solutions $(\alpha 1)$ and $(\alpha 2)$. As there are no other fixed-points whose stability around the points are charac-
When \( q < -1/2 \), for \( \tilde{\lambda} + \tilde{\mu} > 0 \) (or equivalently \( p > 0 \)), the story is same as in the case with \( 0 > q > -1/2 \), there are four stable fixed-point solutions \((\alpha 3) - (\alpha 6)\). But in this case, an additional marginally stable solution is \((\alpha 1)\). Once again, one of \((\alpha 3) - (\alpha 6)\) can be the late-time attractor solutions depending on the values of parameters \( p, r, w_m \) or the initial values of \( x, y, \tilde{\gamma}, \tilde{\chi} \), while for \( \lambda + \tilde{\mu} < 0 \)

(See Figs. 6 and 7)

We summarise the possible late-time attractor solutions for a given set of \( q, p, r \) based on the discussion in this subsection in TABLE III.

IV. MODELS WITH EXPONENTIAL POTENTIAL AND BRANE TENSION

In this section we consider the models where both the potential and brane tension are exponential functions of the DBI field,
TABLE III: This table summarises which fixed-points can be the late-time attractors for the models where the potential and brane-tension are power-law functions of the DBI field for a given $q$, $p$ and $r$. “or” means there may be more than one possible late-time attractor and which is realised depends on $w_m$ and the initial value of $x$, $y$, $\tilde{\gamma}$, which has been described in TABLE II and TABLE III in detail.

| $q > \frac{1}{4}$ | $a(1)$ | $(a3)$ or $(a4)$ or $(a5)$ or $(a6)$ | $1 + \frac{x}{y} > 0$ |
| $q = \frac{1}{4}$ | $(a1)$ | $(a4)$ or $(a5)$ | $1 + \frac{x}{y} < 0$ |
| $q > \frac{1}{2}$ | $(a1)$ | | |
| $q = \frac{1}{2}$ | | | |
| $q < \frac{1}{2}$ | | | |
| $q < 0$ | $(a3)$ or $(a4)$ or $(a5)$ or $(a6)$ | | |
| $q > 0$ | $(a4)$ or $(a6)$ | | |
| $q = 0$ | $(a2)$ | | |
| $0 > q > -\frac{1}{2}$ | $(a3)$ or $(a4)$ or $(a5)$ or $(a6)$ | | |
| $-\frac{1}{2} > q$ | $(a2)$ | | |

A. Autonomous System

In this case, from the discussions in Sec. II, in terms of constants $\lambda$ and $\mu$ given by Eq. (120), Eqs. (13) - (15) constitute an autonomous system.

These equations correspond exactly to Eqs. (22)-(24) with $q = 0$ under the identification $\lambda \rightarrow \tilde{\lambda} (= |p|)$ and $\mu \rightarrow \tilde{\mu}(= r|p|/p)$. It is as expected because the exponential function can be regarded as the power law function $\phi^p$ with $p \rightarrow \infty$. This means that all the fixed points associated with $q = 0$ are found in the case of the exponential functions as well. However, as we shall now show, when $\lambda + \mu = 0$, there is an additional fixed point found in the exponential case. As in the previous section, without loss of generality, we restrict our discussion to the case $\lambda > 0$ and $x \geq 0$.

B. Fixed-point Solutions

From our discussion in the power-law case for the potential and brane-tension with $q = 0$, the fixed-point solutions obtained there are also fixed point solutions for the exponential case. This means that the eight fixed-point solutions $(a1), (a2), (b1) - (b3), (c1), (c2), (a2)$ obtained in Sec. III are also fixed-point solutions with the replacement $\lambda(= |p|) \rightarrow \lambda$ and $\tilde{\mu}(= r|p|/p) \rightarrow \mu$.

The additional solution not present in the power law case follows when

$$\mu = -\lambda. \quad (121)$$

As we will now show a class of new solutions then exist for $0 \leq \tilde{\gamma} \leq 1$ and non-zero $x$ and $y$.

Strictly speaking, since in this case $V(\phi)$ and $f(\phi)$ are related through the constraint $fV = \sigma \nu$, this is a two-dimensional system as opposed to a three-dimensional one. Actually $\tilde{\gamma}$ can be completely specified once $x$ and $y$ are given, through the relation

$$\tilde{\gamma} = \frac{y^2}{3\sigma\nu x^2 + y^2}. \quad (122)$$

However, for simplicity we continue to solve explicitly the evolution equation of $\tilde{\gamma}$ to obtain the fixed-points. Of
course, from Eq. (122), \( \dot{\gamma} \) becomes constant when \( x \) and \( y \) become constant, which provides a consistency check for the method. In fact the constraint Eq. (122), leads to some more interesting constraints on the DBI field. Substituting it into Eq. (11) and defining \( r \equiv y^2/x^2 \), the equation of state of the DBI field is given by

\[
w_{\phi}(r) = \frac{r(1 - 3\sigma \nu - r)}{(3\sigma \nu + r)(1 + r)}, \quad \text{with } 0 \leq r, \tag{123}
\]

which clearly asymptotes to 0 as \( r \to 0 \), while \(-1\) as \( r \to \infty \). In fact the function decreases monotonically and is constrained as

\[-1 \leq w_{\phi} < 0. \tag{124}\]

To find the fixed-points in the system, substituting Eq. (121) into Eq. (15), and requiring \( \dot{\gamma} \) remain constant implies,

\[1 - x^2 - y^2 = 1 - \frac{3(1 + \dot{\gamma})}{\sqrt{\gamma \lambda}}. \tag{125}\]

Substituting Eqs. (121) and (125) into Eq. (13), leads to the following two constant solutions for \( x \):

\[x = \frac{\sqrt{\gamma \lambda}}{\sqrt{3(1 + \dot{\gamma})}}, \quad \text{or} \quad x = \frac{\sqrt{3(1 + w_m)}}{\sqrt{3(1 + \dot{\gamma})}}. \tag{126}\]

The first of these solutions leads to \( x^2 + y^2 = 1 \), which from Eqs. (9) and (10) implies that during scaling \( \Omega_m = 0 \) and \( \Omega_\phi = 1 \). It corresponds to the relativistic kinetic-potential scaling solutions given by

\[\dot{\gamma} = \left(\frac{\sqrt{\gamma \lambda}}{\sqrt{3(1 + \dot{\gamma})}}\right)^{-1} \left(1 - \frac{\dot{\gamma} \lambda^2}{3(1 + \dot{\gamma})} \right), \tag{127}\]

with \( 0 < \dot{\gamma} < 1 \), \( \lambda < \frac{3}{\dot{\gamma}}, \mu = -\lambda \).

From Eqn. (11) it implies \( w_{\phi} = \frac{2 \lambda^2}{3} - 1 \) which means that the solution is accelerating if \( \lambda^2 \dot{\gamma} < 2 \). Since for this solution, the DBI field completely dominates the fluid, this solution exists irrespective of the value of \( w_m \).

Although somewhat complicated, the constraint Eq. (122), allows us to determine the actual values for \( \dot{\gamma}, x \) and \( y \) for this fixed-point solution in terms of the parameters \( \lambda, \sigma, \nu, w_m \). For example, for \( \sigma \nu > 2/3 \) they are given by

\[
\begin{align*}
\dot{\gamma} &= \frac{6}{\lambda^2 + \sqrt{36 + 12(-1 + 3\sigma \nu)\lambda^2 + \lambda^4}}, \tag{128} \\
x^2 &= \frac{2\lambda^2}{6 + \lambda^2 + \sqrt{36 + 12(-1 + 3\sigma \nu)\lambda^2 + \lambda^4}}, \tag{129} \\
y^2 &= 1 - x^2, \tag{130}
\end{align*}
\]

while for \( \sigma \nu < 2/3 \), a related but slightly different expression holds.

The second solution in Eq. (126) corresponds to the relativistic kinetic-potential-fluid scaling solution and is characterized by

\[
\begin{align*}
\dot{\gamma} &= \left(\frac{\sqrt{3(1 + w_m)}}{\sqrt{\gamma(1 + \dot{\gamma})}}\right)^{-1} \left(\frac{3(1 + w_m)(\dot{\gamma} - w_m)}{\lambda^2 \dot{\gamma}(1 + \dot{\gamma})} \right), \\
\text{with } 0 < \dot{\gamma} < 1, \quad w_m < 0, \\
\lambda &= \frac{3(1 + w_m)}{\dot{\gamma}}, \quad \mu = -\lambda. \tag{131}
\end{align*}
\]

From Eqs. (10) and (11) we see that it leads to \( \Omega_\phi = \frac{3(1 + w_m)}{2\phi} \) and \( w_{\phi} = w_m \). In other words the DBI field equation of state tracks that of the background matter. Since the property of this solution is completely affected by the nature of the fluid, the existence condition of this solution depends on the value of \( w_m \). It is worth noting that when we take into account the constraint Eq. (122), then from Eq. (127), \( w_{\phi} < 0 \). Therefore, although it cannot be seen from Eq. (131) explicitly, this solution exists only for the case with \( w_m < 0 \).

As in the case of (c4), we can solve explicitly for \( \dot{\gamma}, x \) and \( y \) in terms of \( \lambda, \sigma, \nu, w_m \) by making use of the constraint given by Eq. (122). For example, for the case with \( \sigma \nu > 1/3 \),

\[
\begin{align*}
\dot{\gamma} &= \frac{-2w_m}{(3\sigma \nu - 1)(1 + w_m)} \times \left[ \frac{1}{1 - \frac{4w_m}{1 - 3\sigma \nu} + 1} \right], \tag{132} \\
y^2 &= \frac{18\sigma \nu (1 + w_m)}{(3\nu - 1)(1 + w_m)\lambda^2} \times \left[ \frac{1}{2 + (3\nu - 1) \left(1 + \sqrt{1 + \frac{1}{(3\sigma \nu - 1)^2(1 + w_m)}\lambda^2}\right)} \right], \tag{133} \\
x^2 &= \frac{1 + w_m}{\sqrt{\gamma - w_m}} y^2, \tag{134}
\end{align*}
\]

while for \( \sigma \nu < 2/3 \), a slightly different expression holds.

In summary we see that there are ten fixed-point solutions for the case where both the potential and brane tension are exponential functions of the DBI field. In TABLE IV, we summarise the two solutions (c4) and (c5) which appear when the particular condition \( \lambda = -\mu \) is satisfied. The other solutions correspond to the case \( q = 0 \) and are given in TABLE IV.
Existence $\lambda < \sqrt{3(1+w_m)/\gamma}$

and $\gamma (1+\tilde{\gamma})$ then the stability conditions are automatically satisfied. Although we keep $\gamma$, from the constraint given by Eq. (122), it is specified in terms of $\lambda$, $\sigma$, $\nu$, $w_m$ through Eq. (128) for (c4) and Eq. (132) for (c5). In addition to these two, the solutions (a1), (a2), (b1) – (b3), (c1), (c2), (a2) obtained in Sec. [11] are also fixed-point solutions with the replacement $\lambda(=|p|)\rightarrow \lambda$ and $\tilde{\mu}(=r|p|/p)\rightarrow \mu$. Notice that we have restricted ourselves to $\lambda > 0$ and $x \geq 0$.

C. Stability Analysis

We turn now to study the stability of the fixed-point solutions obtained in the previous subsection. Given that eight of the fixed-point solutions ((a1), (a2), (b1) – (b3), (c1), (c2), (a2)) were also obtained and their stability determined in Sec. [11] we do not repeat that analysis here. It is basically the same as in that case except we make the replacement $\lambda(=|p|)\rightarrow \lambda$ and $\tilde{\mu}(=r|p|/p)\rightarrow \mu$. Here, we concentrate on the two extra solutions (c4) and (c5) in Eqs. (124) and (131). For the fixed-point (c4) (relativistic kinetic-potential scaling solutions), having perturbed about the solution to linear order, we obtain for the corresponding eigenvalues:

$$w_1 = \lambda^2\tilde{\gamma} - 6$$

$$w_2 = \lambda^2\tilde{\gamma} - 3(1 + w_m)$$

$$w_3 = 0.$$  (135)

$w_3$ is zero because this corresponds to the $\tilde{\gamma}$ direction and at this fixed point, the RHS of Eq. (15) becomes 0. As mentioned earlier, this is to be expected because there is a constraint on $\tilde{\gamma}$ and only two of the three degrees of freedom are actually independent for $\lambda = -\mu$. Therefore the existence of two negative eigenvalues will guarantee the stability of this fixed-point solution.

Since $\lambda < \sqrt{3(1 + \tilde{\gamma})/\tilde{\gamma}} < \sqrt{6}\tilde{\gamma}$, $w_1$ is always negative. Therefore, solution (c5) is stable if $w_2 < 0$, that is, $\lambda < \sqrt{3(1 + w_m)/\gamma}$. Recall that the condition for inflation in this case is $\lambda^2\tilde{\gamma} < 2$.

It is worth mentioning that by adopting the constraint given by Eq. (122), actually $\tilde{\gamma}$ is completely specified in terms of $\lambda$, $\sigma$, $\nu$. For example if $\sigma \nu > 2/3$, it can be shown that this solution is stable if

$$6(3 + (3\sigma \nu - 1)\lambda^2)]w_m + 18 - 6(1 - 3\sigma \nu)\lambda^2 + \lambda^4 > \sqrt{36 - 12(1 - 3\sigma \nu)\lambda^2 + 3\lambda^4}.$$  (136)

Similarly, for the fixed-point (c5) (relativistic kinetic-potential-fluid scaling solutions) we obtain:

$$w_{1,2} = \frac{3}{4}(1 - w_m)$$

$$\times \left[-1 \pm \sqrt{1 - \frac{8(1 + w_m)(w_m - \tilde{\gamma}^2)(3 + 3w_m - \lambda^2\tilde{\gamma})}{\lambda^2(1 + w_m)\tilde{\gamma}}} \right]$$

$$w_3 = 0.$$  (137)

The same reasoning as applied to (c4) suggests that even though $w_3 = 0$, the solution (c5) is stable if the real parts of both $w_1$ and $w_2$ are negative. This requires $w_m < \tilde{\gamma}^2$, which is automatically satisfied if the existence condition $w_m < 0$ is satisfied, although it would imply an unusual form of background matter being considered.

As in the case of (c4), by adopting the constraint given by Eq. (122), actually $\tilde{\gamma}$ is completely specified in terms of $\lambda$, $\sigma$, $\nu$, $w_m$. It can be shown that if $\sigma \nu > 1/3$, the existence condition $\lambda \geq \sqrt{3(1 + w_m)/\gamma}$ which is the most stringent condition for this solution to be stable can be expressed as

$$[6(3 + (3\sigma \nu - 1)\lambda^2)]w_m + 18 - 6(1 - 3\sigma \nu)\lambda^2 + \lambda^4$$

$$\leq \sqrt{36 - 12(1 - 3\sigma \nu)\lambda^2 + 3\lambda^4}.$$  (138)

Clearly, Eqs. (136) and (138) show that for $\sigma \nu > 2/3$, there is no overlap in the parameter space between where (c4) and (c5) are the late-time attractors. In Table IV we summarise the stability conditions of these two fixed-point solutions.

There has been related work looking at the fixed-point solutions (c4) and (c5). Actually, (c4) was previously obtained in [50, 52], while the existence of the scaling solution (c5) was pointed out by Martin and Yamaguchi in [49]. The authors’ stability analysis followed that of [54], which although providing a proof that the fixed-point solution is attractive, it did not explain how the the solution could be realised from all initial values. In our approach we have gone into more details, following the analysis of [57]. In particular by making the phase space compact, we have been able to establish all the fixed-points in the compact phase space. As we will now see it...
then becomes possible to discuss the late-time attractor structure, something we now turn our attention to in the following subsection. The solution (c5) has recently led to a number of papers investigating its cosmology.

D. Late-time behaviour

In Sec. III Fig. 4 shows the late-time attractor structure with \( q = 0 \) and \( w_m > 0 \). From the discussion at the beginning of this section, except the special case with \( \mu = -\lambda \), Fig. 4 shows also the late-time attractor structure in the models where the potential and brane tension are exponential functions of the DBI field with the replacement \( \tilde{\lambda} = |p| \rightarrow \lambda \) and \( \tilde{\mu} = r|p|/p \rightarrow \mu \). In the case with \( \mu > -\lambda \), for \( \lambda < \sqrt{3(1 + w_m)} \), the fixed-point (b2) (standard kinetic-potential scaling solutions) is the late-time attractor, whereas for \( \lambda \geq \sqrt{3(1 + w_m)} \), the fixed point (b3) (standard kinetic-potential-fluid scaling solutions) is a late-time attractor. For \( \mu < -\sqrt{3(1 + w_m)}/\sqrt{w_m} \) with \( w_m > 0 \), in addition to (b3), there is a possibility that the fixed-point (c1) (relativistic kinetic dominated solutions) is also a late-time attractor. Because, they are all stable locally, which of these solutions actually wins out depends on the initial value of \( x, y \) and \( \dot{\gamma} \).

On the other hand, from the discussions in Sec. III when \( \mu < -\lambda \), the fixed-point solution (a2) (ultra-relativistic potential dominated one) is the late-time attractor. For the case \( \mu = -\lambda \), as we have just seen, the additional fixed-points solutions (c4) and (c5) are also possible late-time attractors, although the conditions under which they become attractors do not overlap in the available parameter space, so they do not compete with one another for overall stability.

Of course when considering the overall stability for the case of \( \mu = -\lambda \), we need to also include a discussion of the fixed-point solutions which can be late-time attractors for the general case \( \mu \neq \lambda \). Since (c1) and (a2) do not satisfy the constraint given by Eq. (122), these two fixed-points can not be the late-time attractor for \( \mu = -\lambda \). It can be shown that (c4) and (c5) reduce to (b2) and (b3), in the limit of \( \dot{\gamma} \rightarrow 1 \).

It follows that only (c4) and (c5) are the late-time attractors for \( \mu = -\lambda \) and once we specify the values of the parameters, \( \lambda, \sigma, \nu \) and \( w_m \) we can judge which of these two fixed-points will be the late-time attractor.

E. Power-law models with \( p + r = 0 \)

Here, for completeness, it is appropriate to mention what happens when the models \( V(\phi) \) and \( f(\phi) \) are given by

\[
V(\phi) = \sigma|\phi|^p, \quad f(\phi) = \nu|\phi|^{-p}.
\]

From (139), as in the exponential potential case with \( \mu = -\lambda \) discussed in section IV.B, there is a constraint given by Eq. (122), implying that only two of the three variables are independent. This in turn implies that we cannot make use of the degree of freedom corresponding to \( f(\phi) \) to construct an autonomous system to solve Eqs. (13) - (15) as they stand. So if there is a constraint of the form \( f(\phi)V(\phi) = const. \), the only case we can obtain an autonomous system is when \( V \sim \exp[-\lambda \phi] \) and \( f \sim \exp[\lambda \phi] \) (or equivalently \( p \rightarrow \infty \)).

We can of course still make some progress. For example for the case of a canonical scalar field, not described as an autonomous system, the late-time behaviour with a power-law potential has been determined in [83, 85]. We adopt a similar procedure here for the case described by Eq. (139), by regarding these cases as limits of the models where the potential and brane tension are exponential functions of the DBI field satisfying \( \mu = -\lambda \). The point is that in terms of \( \lambda \) and \( \mu \), the evolution equations for \( x, y, \dot{\gamma} \) are still given by Eqs. (13), (14) and (15). The difference from the cases with the exponential potential and brane tension is that \( \lambda \) and \( \mu \) are not constant for the power-law cases but are given by

\[
\lambda = -\mu = -\varepsilon \phi \frac{1}{|\phi|}.
\]

As in the other cases, we can generally restrict \( \lambda > 0 \) which is equivalent to considering only \( \phi < 0 \) for \( p > 0 \) and \( \phi > 0 \) for \( p < 0 \). Therefore, in the late-time limit, \( \phi \rightarrow 0 \) for \( p > 0 \) while \( \phi \rightarrow \infty \) for \( p < 0 \). This means that the late-time asymptotic value of \( \lambda \) is \( \lambda \rightarrow +\infty \) for \( p > 0 \), while \( \lambda \rightarrow +0 \) for \( p < 0 \).

First, let us consider the case with \( p < 0 \) (\( \lambda \rightarrow 0 \)). In this case, from Table IV and Fig. 4 the only possible late-time attractor solution is (c4) (relativistic kinetic-potential scaling solutions). In the limit \( \lambda \rightarrow 0 \), from Eqs. (128) - (130), it asymptotes to \( (x, y, \dot{\gamma}) = (0, 1, 1) \), that is, standard potential dominated solutions. Therefore, the DBI-field behaves like a cosmological constant at late-time.

Next, let us consider the cases with \( p > 0 \) (\( \lambda \rightarrow \infty \)). In these cases, since \( \lambda \) is very large, from Table IV and Fig. 4 the only possible late-time attractor solution is (c5) (relativistic kinetic-potential-fluid scaling solutions). In the limit \( \lambda \rightarrow \infty \), from Eqs. (132) - (134), it asymptotes to \( (x, y, \dot{\gamma}) = (0, 0, \dot{\gamma}) \), that is fluid dominated solution and the DBI field becomes irrelevant compared to the fluid. It is worth noting that from Eq. (132) \( \dot{\gamma} \) is completely specified once \( \sigma, \nu, w_m \) are fixed.

V. SUMMARY

Successful models of inflation arising within string theory from the DBI action have generated a great deal of interest recently. With their non-canonical kinetic terms, non-trivial potentials and brane tensions, they have led to a number of fascinating results including the prediction of distinctive non-Gaussian fluctuations in the CMB.
Although most models investigated to date have had specific functional forms for the potential and brane tension, in this paper we have decided to broaden the class of models being discussed and so have analysed the dynamics associated with more general forms for these potential and brane tension functions, including in the analysis the presence of a background perfect fluid in a flat FRW universe. Following the approach developed in [57], we have introduced a suitable set of dynamical variables $x$, $y$ and $\tilde{\gamma}$ in Eq. (5) which has allowed us to determine the phase-space portrait of the system. In particular, we have established the late time behaviour of these systems, demonstrating where appropriate the attractor nature of the solutions.

In Sec. [III] we have considered the models where the potential and brane tension are given by power-law functions of the DBI field $V(\phi) = \sigma |\phi|^p$, $f(\phi) = \nu |\phi|^r$. The standard fixed-point solutions of this system are summarised in TABLE [I] where we see that the late-time attractor nature of the solutions depends on $q = -1/(p+r)$. The interesting cases of scaling where the ratio of the kinetic to potential energies of the DBI field is a constant are found to exist only for $q = 1/2$ ((a4), (a5)) and $q = 0$ ((b2), (b3)). This is because if we require $x$ and $y$ to be nonzero constants, there are only two possibilities, that is, $\tilde{\gamma} = 0$ (leading to scaling with $q = 1/2$) and $\tilde{\gamma} = 1$ (leading to scaling with $q = 0$). These scaling solutions are then shown to be stable for certain regions of the parameter space. In addition to these, we have also explicitly demonstrated the existence and stability of an interesting inflationary solution (c3) specific to the DBI field with constant $\tilde{\gamma}$ which differs from $\tilde{\gamma} = 1$ and $\tilde{\gamma} = 0$.

The DBI system is rich. For example the evolution equations (22–23) can appear singular when some of $x$, $y$ or $\tilde{\gamma}$ either tend to zero or unity, which one is singular depends on the value of $q$. On the face of it, the equations appear to be ill-defined, but in practice it turns out that these points can actually be late-time attractor solutions for the system. In section [III] we obtain these fixed points and determine their stability. These are summarised in Table [I].

Having established all the fixed-points and their stability for a given set of parameters, we have gone on to determine which of these solutions will be the late-time attractor. Particular care is required when considering the cases where the eigenvalues associated with the perturbations vanish, implying that the solution is marginally stable. Whether these are the late time attractors for the system depends upon the stability of the other allowed fixed-points. In particular those fixed points with three negative eigenvalues, means that these marginally stable fixed-point solutions can not be the late-time attractor as their stability is weaker than that of the fixed-points. However, if all other fixed-points have positive eigenvalues, then the marginally stable fixed-point solution can turn out to be the late-time attractor. We have summarised the possible late-time attractor solutions for a given set of $q$, $p$, $r$ in Table III.

In Sec. [IV] we have considered the models where the potential and brane tension are exponential functions of the DBI field $V(\phi) = Ce^{-\lambda \phi}$, $f(\phi) = De^{-\nu \phi}$. This system has a similar dynamical structure to that of the power-law models with $q = 0$. However, there are additional scaling solutions present in the exponential case because we can construct an autonomous system for the case with $\lambda = -\mu$, leading to solutions where $\tilde{\gamma}$ is a constant between 0 and 1 ((e4),(e5)). In this case, we find that once we specify the values of the parameters, $\lambda$, $\sigma$, $\nu$ and $w_m$ we can judge which of these two fixed-points will be the late-time attractor. The stability of these two fixed-points are summarised in Table IV. We have also shown that the special case $V(\phi) = \sigma |\phi|^p$, $f(\phi) = \nu |\phi|^{-p}$ can be discussed in terms of the limiting behaviour $\lambda \to 0$ or $\lambda \to \infty$ in the models with an exponential potential and brane tension satisfying $\lambda = -\mu$.

There is an overlap between elements of this work and other published material. For example, we are able to reproduce a number of results obtained earlier in [69] where the authors considered the case with a massive potential and AdS throat. It turns out to be a special case of $q = -1/2$ discussed in Sec. III in this paper.

The existence of the scaling solution (e5) was originally pointed out by Martin and Yamaguchi in [14]. The authors stability analysis followed that of [54], which although providing a proof that the fixed-point solution is attractive, it did not explain how the the solution could be realised from all initial values. In our approach we have gone into more details, following the analysis of [57]. In particular by making the phase space compact, we have been able to establish all the fixed-points in the compact phase space, allowing us to then properly discuss the late-time attractor structure of the solution.

Of most interest to us though is the existence and stability of a class of cosmologically relevant solutions. We have found that a fixed point solution (c3) (relativistic potential dominated solutions) where $\tilde{\gamma}$ is a constant satisfying $0 < \sqrt{3}/\sqrt{\lambda^2 + 3} < 1$ can be the late-time attractor for $q = -1/2$ with $r/p < -1$ (see also [22]). Although it can not be applied to the AdS throat ($r = -4$) as $r/p > -1$, we believe this solution is very interesting and important. In calculations of primordial perturbations of the DBI inflation models, properly incorporating the time dependence of the sound speed $c_s$ is a complicated issue [3, 56], and in fact it is usually assumed to be constant. (For a recent approach to relaxing this assumption, see [22].) Since in our case $c_s = \tilde{\gamma}$ is constant for the fixed-point solution (c3), this serves as a good background to use when we consider cosmological perturbations in DBI inflation.

Acknowledgments

We would like to thank Kazuya Koyama, Jerome Martin, Shinji Mukohyama, Ryo Saito, David Wands, Masahide Yamaguchi and Jun’ichi Yokoyama for interesting...
ing discussions. We also would like to thank Eric Linder for detailed discussions concerning some of the solutions. S. M. is grateful to the RESCEU, the University of Tokyo for their hospitality when this work was completed. S. M. is supported by JSPS Postdoctoral Fellowships for Research Abroad. E. J. C. is grateful to the Royal Society for financial support.

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