SPANNING SIMPLICIAL COMPLEXES OF UNI-CYCLIC MULTIGRAPHS

IMRAN AHMED, SHAHID MUHMOOD

ABSTRACT. A multigraph is a nonsimple graph which is permitted to have multiple edges, that is, edges that have the same end nodes. We introduce the concept of spanning simplicial complexes $\Delta_s(G)$ of multigraphs $G$, which provides a generalization of spanning simplicial complexes of associated simple graphs. We give first the characterization of all spanning trees of a uni-cyclic multigraph $U_{r,n,m}$ with $n$ edges including $r$ multiple edges within and outside the cycle of length $m$. Then, we determine the facet ideal $I_F(\Delta_s(U_{r,n,m}))$ of spanning simplicial complex $\Delta_s(U_{r,n,m})$ and its primary decomposition. The Euler characteristic is a well-known topological and homotopic invariant to classify surfaces. Finally, we device a formula for Euler characteristic of spanning simplicial complex $\Delta_s(U_{r,n,m})$.

Key words: multigraph, spanning simplicial complex, Euler characteristic.
2010 Mathematics Subject Classification: Primary 05E25, 55U10, 13P10, Secondary 06A11, 13H10.

1. Introduction

Let $\mathcal{G} = \mathcal{G}(V,E)$ be a multigraph on the vertex set $V$ and edge-set $E$. A spanning tree of a multigraph $\mathcal{G}$ is a subtree of $\mathcal{G}$ that contains every vertex of $\mathcal{G}$. We represent the collection of all edge-sets of the spanning trees of a multigraph $\mathcal{G}$ by $s(\mathcal{G})$. The facets of spanning simplicial complex $\Delta_s(\mathcal{G})$ is exactly the edge set $s(\mathcal{G})$ of all possible spanning trees of a multigraph $\mathcal{G}$. Therefore, the spanning simplicial complex $\Delta_s(\mathcal{G})$ of a multigraph $\mathcal{G}$ is defined by

$$\Delta_s(\mathcal{G}) = \langle F_k \mid F_k \in s(\mathcal{G}) \rangle,$$

which gives a generalization of the spanning simplicial complex $\Delta_s(G)$ of an associated simple graph $G$. The spanning simplicial complex of a simple connected finite graph was firstly introduced by Anwar, Raza and Kashif in [1]. Many authors discussed algebraic and combinatorial properties of spanning simplicial complexes of various classes of simple connected finite graphs, see for instance [1], [5], [6] and [9].

Let $\Delta$ be simplicial complex of dimension $d$. We denote $f_i$ by the number of $i$-cells of simplicial complex $\Delta$. Then, the Euler characteristic of $\Delta$ is given
by
\[ \chi(\Delta) = \sum_{i=0}^{d-1} (-1)^i f_i, \]
which is a well-known topological and homotopic invariant to classify surfaces, see [4] and [7].

The uni-cyclic multigraph \( \mathcal{U}_{n,m}^r \) is a connected graph having \( n \) edges including \( r \) multiple edges within and outside the cycle of length \( m \). Our aim is to give some algebraic and topological characterizations of spanning simplicial complex \( \Delta_s(\mathcal{U}_{n,m}^r) \).

In Lemma 3.1, we give characterization of all spanning trees of a uni-cyclic multigraph \( \mathcal{U}_{n,m}^r \) having \( n \) edges including \( r \) multiple edges and a cycle of length \( m \). In Proposition 3.2, we determine the facet ideal \( I_F(\Delta_s(\mathcal{U}_{n,m}^r)) \) of spanning simplicial complex \( \Delta_s(\mathcal{U}_{n,m}^r) \) and its primary decomposition. In Theorem 3.3, we give a formula for Euler characteristic of spanning simplicial complex \( \Delta_s(\mathcal{U}_{n,m}^r) \).

2. Basic Setup

A simplicial complex \( \Delta \) on \([n] = \{1, \ldots, n\}\) is a collection of subsets of \([n]\) satisfying the following properties.
(1) \( \{j\} \in \Delta \) for all \( j \in [n] \);
(2) If \( F \in \Delta \) then every subset of \( F \) will belong to \( \Delta \) (including empty set).

The elements of \( \Delta \) are called faces of \( \Delta \) and the dimension of any face \( F \in \Delta \) is defined as \( |F| - 1 \) and is written as \( \dim F \), where \( |F| \) is the number of vertices of \( F \). The vertices and edges are 0 and 1 dimensional faces of \( \Delta \) (respectively), whereas, \( \dim \emptyset = -1 \). The maximal faces of \( \Delta \) under inclusion are said to be the facets of \( \Delta \). The dimension of \( \Delta \) is denoted by \( \dim \Delta \) and is defined by \( \dim \Delta = \max \{ \dim F \mid F \in \Delta \} \).

If \( \{F_1, \ldots, F_q\} \) is the set of all the facets of \( \Delta \), then \( \Delta = \langle F_1, \ldots, F_q \rangle \).

A simplicial complex \( \Delta \) is said to be pure, if all its facets are of the same dimension.

A subset \( M \) of \([n]\) is said to be a vertex cover for \( \Delta \) if \( M \) has non-empty intersection with every \( F_k \). \( M \) is said to be a minimal vertex cover for \( \Delta \) if no proper subset of \( M \) is a vertex cover for \( \Delta \).

**Definition 2.1.** Let \( \mathcal{G} = \mathcal{G}(V, E) \) be a multigraph on the vertex set \( V \) and edge-set \( E \). A spanning tree of a multigraph \( \mathcal{G} \) is a subtree of \( \mathcal{G} \) that contains every vertex of \( \mathcal{G} \).

**Definition 2.2.** Let \( \mathcal{G} = \mathcal{G}(V, E) \) be a multigraph on the vertex set \( V \) and edge-set \( E \). Let \( s(\mathcal{G}) \) be the edge-set of all possible spanning trees of \( \mathcal{G} \). We define a simplicial complex \( \Delta_s(\mathcal{G}) \) on \( E \) such that the facets of \( \Delta_s(\mathcal{G}) \) are exactly the elements of \( s(\mathcal{G}) \), we call \( \Delta_s(\mathcal{G}) \) as the spanning simplicial complex of \( \mathcal{G} \) and given by

\[ \Delta_s(\mathcal{G}) = \langle F_k \mid F_k \in s(\mathcal{G}) \rangle. \]
Definition 2.3. A uni-cyclic multigraph $U^r_{n,m}$ is a connected graph having $n$ edges including $r$ multiple edges within and outside the cycle of length $m$.

Let $\Delta$ be simplicial complex of dimension $d$. Then, the chain complex $C_*(\Delta)$ is given by

$$0 \rightarrow C_d(\Delta) \xrightarrow{\partial_d} C_{d-1}(\Delta) \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_1} C_1(\Delta) \xrightarrow{\partial_1} C_0(\Delta) \xrightarrow{\partial_0} 0.$$  

Each $C_i(\Delta)$ is a free abelian group of rank $f_i$. The boundary homomorphism $\partial_d : C_d(\Delta) \rightarrow C_{d-1}(\Delta)$ is defined by

$$\partial_d(\sigma^d_\alpha) = \sum_{i=0}^d (-1)^i \sigma^d_\alpha|_{v_0, v_1, \ldots, \hat{v_i}, \ldots, v_d}.$$  

Of course,

$$H_i(\Delta) = Z_i(\Delta)/B_i(\Delta) = \text{Ker} \partial_i/\text{Im} \partial_{i+1},$$

where $Z_i(\Delta) = \text{Ker} \partial_i$ and $B_i(\Delta) = \text{Im} \partial_{i+1}$ are the groups of simplicial $i$-cycles and simplicial $i$-boundaries, respectively. Therefore,

$$\text{rank } H_i(\Delta) = \text{rank } Z_i(\Delta) - \text{rank } B_i(\Delta).$$

One can easily see that rank $B_d(\Delta) = 0$ due to $B_d(\Delta) = 0$. For each $i \geq 0$ there is an exact sequence

$$0 \rightarrow Z_i(\Delta) \rightarrow C_i(\Delta) \xrightarrow{\partial_i} B_{i-1}(\Delta) \rightarrow 0.$$  

Moreover,

$$f_i = \text{rank } C_i(\Delta) = \text{rank } Z_i(\Delta) + \text{rank } B_{i-1}(\Delta).$$

Therefore, the Euler characteristic of $\Delta$ can be expressed as

$$\chi(\Delta) = \sum_{i=0}^d (-1)^i f_i = \sum_{i=0}^d (-1)^i (\text{rank } Z_i(\Delta) + \text{rank } B_{i-1}(\Delta))$$

$$= \sum_{i=0}^d (-1)^i \text{rank } Z_i(\Delta) + \sum_{i=0}^d (-1)^i \text{rank } B_{i-1}(\Delta).$$

Changing index of summation in the last sum and using the fact that rank $B_{-1}(\Delta) = 0 = \text{rank } B_d(\Delta)$, we get

$$\chi(\Delta) = \sum_{i=0}^d (-1)^i \text{rank } Z_i(\Delta) + \sum_{i=0}^d (-1)^{i+1} \text{rank } B_i(\Delta)$$

$$= \sum_{i=0}^d (-1)^i (\text{rank } Z_i(\Delta) - \text{rank } B_i(\Delta)) = \sum_{i=0}^d (-1)^i \text{rank } H_i(\Delta).$$

Thus, the Euler characteristic of $\Delta$ is given by

$$\chi(\Delta) = \sum_{i=0}^d (-1)^i \beta_i(\Delta),$$

where $\beta_i(\Delta) = \text{rank } H_i(\Delta)$ is the $i$-th Betti number of $\Delta$, see [4] and [7].

3. Topological Characterizations of $\Delta_t(U^r_{n,m})$

Let $U^r_{n,m}$ be a uni-cyclic multigraph having $n$ edges including $r$ multiple edges within and outside the cycle of length $m$. We fix the labeling of the edge set $E$ of $U^r_{n,m}$ as follows:

$$E = \{e_{11}, \ldots, e_{1t_1}, \ldots, e_{rt_r}, e_{r+11}, \ldots, e_{r+m1}, e_{(m+1)1}, \ldots, e_{(m+1)t_{m+1}}, \ldots, e_{(m+r')1}, \ldots, e_{(m+r')t_{m+r'}}, e_1, \ldots, e_v\},$$

where $e_{it_i}$ are the multiple edges of $i$-th edge of cycle with $1 \leq i \leq r'$ while $e_{r+11}, \ldots, e_{m1}$ are the single edges of the cycle and $e_{jt_j}$ are the multiple edges of the $j$-th edge outside the cycle with $m+1 \leq j \leq m+r''$, moreover, $e_1, \ldots, e_v$ are single edges appeared outside the cycle.
We give first the characterization of \( s(\mathcal{U}_{n,m}^r) \).

**Lemma 3.1.** Let \( \mathcal{U}_{n,m}^r \) be the uni-cyclic multigraph having \( n \) edges including \( r \) multiple edges and a cycle of length \( m \) with the edge set \( E \), given above. A subset \( E(T_{wi}) \subset E \) will belong to \( s(\mathcal{U}_{n,m}^r) \) if and only if \( T_{wi} = \{ e_{1i_1}, \ldots, e_{ri_r}, e_{(r'+1)i_1}, \ldots, e_{(m+1)i_{(m+1)'}}, \ldots, e_{(m+r'\prime)i_{(m+r'\prime)'}, e_{1}, \ldots, e_{v}} \} \) for some \( i_h \in \{1, \ldots, t_h \} \) with \( 1 \leq h \leq r' \), \( m+1 \leq h \leq m+r'' \) and \( i_w \in \{1, \ldots, t_w \} \) with \( 1 \leq w \leq r' \) or \( i_w = 1 \) with \( r'+1 \leq w \leq m \) for some \( w = h \) and \( i_w = i_h \) appeared in \( T_{wi} \).

**Proof.** By cutting down method [3], the spanning trees of \( \mathcal{U}_{n,m}^r \) can be obtained by removing exactly \( t_h - 1 \) edges from each multiple edge such that \( 1 \leq h \leq r' \), \( m+1 \leq h \leq m+r'' \) and in addition, an edge from the resulting cycle need to be removed. Therefore, the spanning trees will be of the form \( T_{wi} = \{ e_{1i_1}, \ldots, e_{ri_r}, e_{(r'+1)i_1}, \ldots, e_{(m+1)i_{(m+1)'}, e_{(m+r'\prime)i_{(m+r'\prime)'}, e_{1}, \ldots, e_{v}} \} \) for some \( i_h \in \{1, \ldots, t_h \} \) with \( 1 \leq h \leq r' \), \( m+1 \leq h \leq m+r'' \) and \( i_w \in \{1, \ldots, t_w \} \) with \( 1 \leq w \leq r' \) or \( i_w = 1 \) with \( r'+1 \leq w \leq m \) for some \( w = h \) and \( i_w = i_h \) appeared in \( T_{wi} \). \( \square \)

In the following result, we give the primary decomposition of facet ideal \( I_\Delta(\Delta_s(\mathcal{U}_{n,m}^r)) \).

**Proposition 3.2.** Let \( \Delta_s(\mathcal{U}_{n,m}^r) \) be the spanning simplicial complex of uni-cyclic multigraph \( \mathcal{U}_{n,m}^r \) having \( n \) edges including \( r \) multiple edges within and outside the cycle of length \( m \). Then,

\[
I_\Delta(\Delta_s(\mathcal{U}_{n,m}^r)) = \\
\left( \bigcap_{1 \leq i \leq v} (x_a) \right) \bigcap \left( \bigcap_{1 \leq i \leq r'} (x_{i1}, \ldots, x_{it_i}, x_{k1}) \right) \bigcap \left( \bigcap_{r'+1 \leq k < l \leq m} (x_{k1}, x_{l1}) \right) \\
\bigcap \left( \bigcap_{1 \leq i < h \leq r'} (x_{i1}, \ldots, x_{it_i}, x_{b1}, \ldots, x_{bt_i}) \right) \bigcap \left( \bigcap_{m+1 \leq j \leq m+r''} (x_{j1}, \ldots, x_{jt_j}) \right),
\]

where \( t_i \) with \( 1 \leq i \leq r' \) is the number of multiple edges appeared in the \( i \)-th edge of the cycle and \( t_j \) with \( m+1 \leq j \leq m+r'' \) is the number of multiple edges appeared in the \( j \)-th edge outside the cycle.

**Proof.** Let \( I_\Delta(\Delta_s(\mathcal{U}_{n,m}^r)) \) be the facet ideal of the spanning simplicial complex \( \Delta_s(\mathcal{U}_{n,m}^r) \). From (3, Proposition 1.8), minimal prime ideals of the facet ideal \( I_\Delta(\Delta) \) have one-to-one correspondence with the minimal vertex covers of the simplicial complex \( \Delta \). Therefore, in order to find the primary decomposition of the facet ideal \( I_\Delta(\Delta_s(\mathcal{U}_{n,m}^r)) \); it is sufficient to find all the minimal vertex covers of \( \Delta_s(\mathcal{U}_{n,m}^r) \).

As \( e_a, a \in [v] \) is not an edge of the cycle of uni-cyclic multigraph \( \mathcal{U}_{n,m}^r \) and does not belong to any multiple edge of \( \mathcal{U}_{n,m}^r \). Therefore, it is clear by definition of minimal vertex cover that \( \{e_a\}, 1 \leq a \leq v \) is a minimal vertex cover of \( \Delta_s(\mathcal{U}_{n,m}^r) \). Moreover, a spanning tree is obtained by removing exactly \( t_h - 1 \) edges from each multiple edge with \( h = 1, \ldots, r', m+1, \ldots, m+r'' \) and
in addition, an edge from the resulting cycle of $\mathcal{U}_{n,m}^r$. We illustrate the result into the following cases.

**Case 1.** If at least one multiple edge is appeared in the cycle of $\mathcal{U}_{n,m}^r$. Then, we cannot remove one complete multiple edge and one single edge from the cycle of $\mathcal{U}_{n,m}^r$ to get spanning tree. Therefore, $(x_{i_1}, \ldots, x_{i_{k_1}}, x_{k_1})$ with $1 \leq i \leq r'$, $r' + 1 \leq k \leq m$ is a minimal vertex cover of the spanning simplicial complex $\Delta_s(\mathcal{U}_{n,m}^r)$ having non-empty intersection with all the spanning trees of $\mathcal{U}_{n,m}^r$.

Moreover, two single edges cannot be removed from the cycle of $\mathcal{U}_{n,m}^r$ to get spanning tree. Consequently, $(x_{k_1}, x_{i_1})$ for $r' + 1 \leq k < l \leq m$ is a minimal vertex cover of $\Delta_s(\mathcal{U}_{n,m}^r)$ having non-empty intersection with all the spanning trees of $\mathcal{U}_{n,m}^r$.

**Case 2.** If at least two multiple edges are appeared in the cycle of $\mathcal{U}_{n,m}^r$. Then, two complete multiple edges cannot be removed from the cycle of $\mathcal{U}_{n,m}^r$ to get spanning tree. Consequently, $(x_{i_1}, \ldots, x_{i_{k_1}}, x_{k_1})$ for $1 \leq i < b \leq r'$ is a minimal vertex cover of $\Delta_s(\mathcal{U}_{n,m}^r)$ having non-empty intersection with all the spanning trees of $\mathcal{U}_{n,m}^r$.

**Case 3.** If at least one multiple edge appeared outside the cycle of $\mathcal{U}_{n,m}^r$. Then, one complete multiple edge outside the cycle of $\mathcal{U}_{n,m}^r$ cannot be removed to get spanning tree. So, $(x_{j_1}, \ldots, x_{j_l})$ for $m + 1 \leq j \leq m + r''$ is a minimal vertex cover of $\Delta_s(\mathcal{U}_{n,m}^r)$ having non-empty intersection with all the spanning trees of $\mathcal{U}_{n,m}^r$. This completes the proof.

We give now formula for Euler characteristic of $\Delta_s(\mathcal{U}_{n,m}^r)$.

**Theorem 3.3.** Let $\Delta_s(\mathcal{U}_{n,m}^r)$ be spanning simplicial complex of uni-cyclic multigraph $\mathcal{U}_{n,m}^r$ having $n$ edges including $r$ multiple edges and a cycle of length $m$. Then, $\dim(\Delta_s(\mathcal{U}_{n,m}^r)) = n - \sum_{i=1}^{r'} t_i - \sum_{j=m+1}^{m+r''} t_j + r - 2$ and the Euler characteristic of $\Delta_s(\mathcal{U}_{n,m}^r)$ is given by

$$
\chi(\Delta_s(\mathcal{U}_{n,m}^r)) = \sum_{i=0}^{n-1} (-1)^i \left[ \binom{n}{i+1} - \prod_{i=1}^{r'} \binom{n-a+r'-m}{i+1-m} \right] \\
- \sum_{j=2}^{\beta} \left( \binom{\beta}{j} - \left( \sum_{m+1 \leq i_1 < \ldots < i_j \leq m+r''} \prod_{k=i_1}^{i_j} \binom{\alpha}{k} \right) \right) \sum_{l=j}^{\beta} (-1)^{l-j} \binom{\beta}{l-j} \binom{n-a+r'-m-l}{i+1-m-l} \\
- \sum_{j=2}^{\alpha+\beta} \left( \binom{\alpha+\beta}{j} - \left( \sum_{1 \leq i_1 < \ldots < i_j \leq m+r''} \prod_{k=i_1}^{i_j} \binom{\alpha}{k} \right) \right) \sum_{l=j}^{\alpha+\beta} (-1)^{l-j} \binom{\alpha+\beta}{l-j} \binom{n-l}{i+1-l} 
$$

with $\alpha = \sum_{i=1}^{r'} t_i$ and $\beta = \sum_{j=m+1}^{m+r''} t_j$ such that $r'$ and $r''$ are the number of multiple edges appeared within and outside the cycle, respectively.

**Proof.** Let $E = \{e_{11}, \ldots, e_{1t_1}, \ldots, e_{r't}, e_{(r'+1)t}, \ldots, e_{m1}, e_{(m+1)t}, \ldots, e_{(m+r''t), m+r''}, e_{1}, \ldots, e_{v}\}$ be the edge set of uni-cyclic multigraph $\mathcal{U}_{n,m}^r$ having $n$ edges including $r$ multiple edges and a cycle of length $m$ such that $r'$ and $r''$ are the number of multiple edges appeared within and outside the cycle, respectively.
One can easily see that each facet $T_{w_i}$ of $\Delta_s(U_{n,m})$ is of the same dimension $n - \sum_{i=1}^{r'} t_i - \sum_{j=m+1}^{m+r''} t_j + r - 2 = n - \alpha - \beta + r - 2$ with $\alpha = \sum_{i=1}^{r'} t_i$ and $\beta = \sum_{j=m+1}^{m+r''} t_j$, see Lemma 3.1.

By definition, $f_i$ is the number of subsets of $E$ with $i + 1$ elements not containing cycle and multiple edges. There are $\prod_{i=1}^{r'} t_i(n - \alpha + r' - m)$ number of subsets of $E$ containing cycle but not containing any multiple edge within the cycle. There are

$$\binom{n - \alpha + r' - m - \beta}{i + 1 - m - \beta} = \left(\frac{\beta}{\beta - 1}\right) \left[\binom{n - \alpha + r' - m - (\beta - 1)}{i + 1 - m - (\beta - 1)} - \binom{n - \alpha + r' - m - \beta}{i + 1 - m - \beta}\right]$$

subsets of $E$ containing cycle and $\beta$ multiple edges of $U_{n,m}$ outside the cycle but not containing any multiple edge within the cycle. There are

$$\left(\frac{\beta}{\beta - 1}\right) \sum_{l=\beta-1}^{\beta} (-1)^{l-(\beta-1)} \binom{\beta - (\beta - 1)}{l - (\beta - 1)} \binom{n - \alpha + r' - m - l}{i + 1 - m - l}$$

subsets of $E$ containing cycle and $\beta - 1$ multiple edges of $U_{n,m}$ outside the cycle but not containing any multiple edge within the cycle. Continuing in similar manner, the number of subsets of $E$ containing cycle and two edges from a multiple edge outside the cycle but not containing any multiple edge within the cycle is given by

$$\binom{\beta-2}{0} \binom{n-\alpha+r'-m-2}{i+1-m-2} - \binom{\beta-2}{2} \binom{n-\alpha+r'-m-3}{i+1-m-3} + \binom{\beta-2}{2} \binom{n-\alpha+r'-m-4}{i+1-m-4} + \cdots$$

$$+ (-1)^{\beta-2} \binom{\beta-2}{2} \binom{n-\alpha+r'-m-\beta}{i+1-m-\beta} = \sum_{l=2}^{\beta} (-1)^{l-2} \binom{\beta-2}{l-2} \binom{n-\alpha+r'-m-l}{i+1-m-l}.$$
not containing multiple edges within the cycle)—(number of subsets of \(E\) with \(i+1\) elements containing cycle and \(\beta-1\) multiple edges outside the cycle but not containing multiple edges within the cycle)—\(\cdots\)—(number of subsets of \(E\) with \(i+1\) elements containing cycle and two edges from a multiple edge outside the cycle but not containing multiple edges within the cycle)]

\[
\begin{align*}
&= \prod_{i=1}^{r'} t_i \left[\binom{n-a+r'-m}{i+1-m} - \beta \sum_{l=\beta} (\binom{\beta}{l-1} (\binom{n-a+r'-m-l}{i+1-m-l})
\right. \\
&\left. - \beta \sum_{l=\beta-1} (\binom{\beta}{l-1} (\binom{n-a+r'-m-l}{i+1-m-l})
\right. \\
&\left. - \cdots - \left(\binom{\beta}{2} - \sum_{m+1 \leq i_1 < i_2 \leq m+r'} \prod_{k=i_1}^{i_2} t_k\right)\right) \sum_{l=\beta} (\binom{\beta}{l-1} (\binom{n-a+r'-m-l}{i+1-m-l})
\right. \\
&\left. - \sum_{j=2}^{\beta} \left(\binom{\beta}{j} - \sum_{m+1 \leq i_1 < \cdots \leq i_j \leq m+r'} \prod_{k=i_1}^{i_j} t_k\right)\right) \sum_{l=j}^{\beta} (\binom{\beta}{l-1} (\binom{n-a+r'-m-l}{i+1-m-l})
\right. \\
&\left. \right]
\end{align*}
\]

Therefore, we compute

\[
\begin{align*}
f_i &= \text{(number of subsets of } E \text{ with } i+1 \text{ elements)—(number of subsets of } E \text{ with } i+1 \text{ elements containing cycle but not containing multiple edges)—(number of subsets of } E \text{ with } i+1 \text{ elements containing } \alpha + \beta \text{ multiple edges)—(number of subsets of } E \text{ with } i+1 \text{ elements containing } \alpha + \beta - 1 \text{ multiple edges)—\cdots—(number of subsets of } E \text{ with } i+1 \text{ elements containing two edges from a multiple edge of } U_{r_{n,m}}^{i+1})
\end{align*}
\]

\[
\begin{align*}
&= \binom{n}{i+1} - \prod_{i=1}^{r'} t_i \left[\binom{n-a+r'-m}{i+1-m}
\right. \\
&\left. - \sum_{j=2}^{\beta} \binom{\beta}{j} - \sum_{m+1 \leq i_1 < \cdots \leq i_j \leq m+r'} \prod_{k=i_1}^{i_j} t_k\right) \sum_{l=j}^{\beta} (\binom{\beta}{l-1} (\binom{n-a+r'-m-l}{i+1-m-l})
\right. \\
&\left. \right]
\end{align*}
\]
Example 3.4. Let $E = \{e_{11}, e_{12}, e_{13}, e_{21}, e_{31}, e_{41}, e_{42}\}$ be the edge set of unicyclic multigraph $\mathcal{U}_{7,3}$ having 7 edges including 2 multiple edges and a cycle of length 3, as shown in Figure 1. By cutting-down method, we obtain $s(\mathcal{U}_{7,3}^2) = \{\{e_{21}, e_{31}, e_{41}\}, \{e_{11}, e_{21}, e_{41}\}, \{e_{12}, e_{21}, e_{41}\}, \{e_{13}, e_{21}, e_{41}\}, \{e_{11}, e_{31}, e_{41}\}, \{e_{12}, e_{31}, e_{41}\}, \{e_{13}, e_{31}, e_{41}\}, \{e_{11}, e_{21}, e_{31}\}, \{e_{12}, e_{21}, e_{31}\}, \{e_{13}, e_{21}, e_{31}\}, \{e_{11}, e_{42}\}, \{e_{12}, e_{42}\}, \{e_{13}, e_{42}\}\}$. By definition, $f_i$ is the number of subsets of $E$ with $i + 1$ elements not containing cycle and multiple edges. Since, $\{e_{11}\}, \{e_{12}\}, \{e_{21}\}, \{e_{31}\}, \{e_{41}\}$ and $\{e_{42}\}$ are subsets of $E$ containing one element. It implies that $f_0 = 7$. There are $\{e_{11}, e_{21}\}, \{e_{11}, e_{31}\}, \{e_{11}, e_{41}\}, \{e_{11}, e_{42}\}, \{e_{12}, e_{21}\}, \{e_{12}, e_{31}\}, \{e_{12}, e_{41}\}, \{e_{12}, e_{42}\}, \{e_{13}, e_{21}\}, \{e_{13}, e_{31}\}, \{e_{13}, e_{41}\}, \{e_{13}, e_{42}\}, \{e_{21}, e_{31}\}, \{e_{21}, e_{41}\}, \{e_{21}, e_{42}\}, \{e_{31}, e_{41}\}, \{e_{31}, e_{42}\} \subseteq E$ containing two elements but not containing cycle and multiple edges. So, $f_1 = 17$. We know that the spanning trees of $\mathcal{U}_{7,3}^2$ are 2-dimensional facets of the spanning simplicial complex $\Delta_s(\mathcal{U}_{7,3}^2)$. Therefore, $f_2 = 14$. Thus,

$$\chi(\Delta_s(\mathcal{U}_{7,3}^2)) = f_0 - f_1 + f_2 = 7 - 17 + 14 = 4.$$ 

Now, we compute the Euler characteristic of $\Delta_s(\mathcal{U}_{7,3}^2)$ by using Theorem 3.3. We observe that, $n = 7$, $r' = 1$, $r'' = 1$, $r = 2$, $m = 3$, $\alpha = 3$, $\beta = 2$ and $0 \leq i \leq d$, where $d = n - \alpha - \beta + r - 2 = 2$ is the dimension of $\Delta_s(\mathcal{U}_{7,3}^2)$. By substituting these values in Theorem 3.3 we get

$$f_i = \binom{7}{i+1} - 3 \left[ \binom{2}{i+1-3} - \binom{2}{i+1-5} \right] - \left[ \binom{5}{2} - \binom{3}{1}\binom{2}{1} \right] \left[ \binom{3}{0}\binom{5}{i+1-2} - \binom{3}{1}\binom{4}{i+1-3} \right] +$$

$$\binom{3}{2}\binom{5}{i+1-4} - \binom{3}{3}\binom{2}{i+1-4} - \binom{5}{3}\binom{2}{0}\binom{4}{i+1-3} - \binom{2}{1}\binom{3}{i+1-4} + \binom{2}{2}\binom{2}{i+1-5} -$$

$$\binom{5}{3}\binom{2}{i+1-4} - \binom{2}{1}\binom{3}{i+1-5} - \binom{5}{3}\binom{2}{i+1-5}.$$

Alternatively, we compute $(f_0, f_1, f_2) = (7, 17, 14)$ and $\chi(\Delta_s(\mathcal{U}_{7,3}^2)) = 4$.

We compute now the Betti numbers of $\Delta_s(\mathcal{U}_{7,3}^2)$. The facet ideal of $\Delta_s(\mathcal{U}_{7,3}^2)$ is given by
\[ I_x(\Delta_5(U_{7,3}^2)) = \langle x_{F_21,31,41}, x_{F_{11,31,41}}, x_{F_{11,21,41}}, x_{F_{12,31,41}}, x_{F_{12,41,31}}, x_{F_{13,31,41}}, x_{F_{13,21,41}}, x_{F_{21,31,42}}, x_{F_{11,31,42}}, x_{F_{11,21,42}}, x_{F_{12,31,42}}, x_{F_{12,41,32}}, x_{F_{13,31,42}}, x_{F_{13,21,42}} \rangle. \]

We consider the chain complex of \( \Delta_5(U_{7,3}^2) \)

\[ 0 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0 \]

The homology groups are given by \( H_i = \frac{\text{Ker}(\partial_i)}{\text{Im}(\partial_{i+1})} \) with \( i = 0, 1, 2 \).

Therefore, the \( i \)-th Betti number of \( \Delta_5(U_{7,3}^2) \) is given by

\[ \beta_i = \text{rank}(H_i) = \text{rank}(\text{Ker}(\partial_i)) - \text{rank}(\text{Im}(\partial_{i+1})) \] with \( i = 0, 1, 2 \).

Now, we compute rank and nullity of the matrix \( \partial_i \) of order \( f_i \times f_{i+1} \) with \( i = 0, 1, 2 \).

The boundary homomorphism \( \partial_2 : C_2(\Delta_5(U_{7,3}^2)) \rightarrow C_1(\Delta_5(U_{7,3}^2)) \) can be expressed as

\[ \partial_2(F_{21,31,41}) = F_{31,41} - F_{21,41} + F_{21,31}; \quad \partial_2(F_{11,31,41}) = F_{31,41} - F_{11,41} + F_{11,31}; \]
\[ \partial_2(F_{12,31,41}) = F_{31,41} - F_{12,41} + F_{12,31}; \quad \partial_2(F_{13,31,41}) = F_{31,41} - F_{13,41} + F_{13,31}; \]
\[ \partial_2(F_{21,32,41}) = F_{32,41} - F_{21,41} + F_{21,32}; \quad \partial_2(F_{12,32,41}) = F_{32,41} - F_{12,41} + F_{12,32}; \]
\[ \partial_2(F_{13,32,41}) = F_{32,41} - F_{13,41} + F_{13,32}; \quad \partial_2(F_{21,33,41}) = F_{33,41} - F_{21,41} + F_{21,33}; \]
\[ \partial_2(F_{12,33,41}) = F_{33,41} - F_{12,41} + F_{12,33}; \quad \partial_2(F_{13,33,41}) = F_{33,41} - F_{13,41} + F_{13,33}. \]

The boundary homomorphism \( \partial_1 : C_1(\Delta_5(U_{7,3}^2)) \rightarrow C_0(\Delta_5(U_{7,3}^2)) \) can be written as

\[ \partial_1(F_{11,21}) = e_{21} - e_{11}; \quad \partial_1(F_{11,31}) = e_{31} - e_{11}; \quad \partial_1(F_{11,41}) = e_{41} - e_{11}; \quad \partial_1(F_{12,21}) = e_{21} - e_{12}; \quad \partial_1(F_{12,31}) = e_{31} - e_{12}; \quad \partial_1(F_{12,41}) = e_{41} - e_{12}; \]
\[ \partial_1(F_{13,21}) = e_{21} - e_{13}; \quad \partial_1(F_{13,31}) = e_{31} - e_{13}; \quad \partial_1(F_{13,41}) = e_{41} - e_{13}; \]
\[ \partial_1(F_{21,31}) = e_{31} - e_{21}; \quad \partial_1(F_{22,31}) = e_{41} - e_{21}; \quad \partial_1(F_{23,31}) = e_{41} - e_{31}. \]

Then, by using MATLAB, we compute rank of \( \partial_2 = 3 \); nullity of \( \partial_2 = 3 \); rank of \( \partial_1 = 6 \); nullity of \( \partial_1 = 11 \).

Therefore, the Betti numbers are given by \( \beta_0 = \text{rank}(\text{Ker}(\partial_0)) - \text{rank}(\text{Im}(\partial_1)) = 7 - 6 = 1 \);
\[ \beta_1 = \text{rank}(\text{Ker}(\partial_1)) - \text{rank}(\text{Im}(\partial_2)) = 11 - 11 = 0 \);
\[ \beta_2 = \text{rank}(\text{Ker}(\partial_2)) - \text{rank}(\text{Im}(\partial_3)) = 3 - 0 = 3. \]

Alternatively, the Euler characteristic of \( \Delta_5(U_{7,3}^2) \) is given by

\[ \chi(\Delta_5(U_{7,3}^2)) = \beta_0 - \beta_1 + \beta_2 = 1 - 0 + 3 = 4. \]

**References**

[1] I. Anwar, Z. Raza and A. Kashif, Spanning Simplicial Complexes of Uni-Cyclic Graphs, *Algebra Colloquium*, 22 (2015), no.4, 707-710.

[2] S. Faridi, The Facet Ideal of a Simplicial Complex, *Manuscripta*, 109 (2002), 159-174.

[3] F. Harary, *Graph Theory*, Reading, MA: Addison-Wesley, 1994.

[4] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.

[5] A. Kashif, I. Anwar and Z. Raza, On the Algebraic Study of Spanning Simplicial Complex of r-cycles Graphs \( G_{n,r} \), *ARS Combinatoria*, 115 (2014), 89-99.

[6] Y. Pan, R. Li and G.Zhu, Spanning Simplicial Complexes of n-Cyclic Graphs with a Common Vertex, *International Electronic Journal of Algebra*, 17 (2015), 180-187.
[7] J. J. Rotman, *An Introduction to Algebraic Topology*, Springer-Verlag, New York, 1988.
[8] R. H. Villarreal, *Monomial Algebras*. Dekker, New York, 2001.
[9] G. Zhu, F. Shi and Y. Geng, Spanning Simplicial Complexes of \( n \)-Cyclic Graphs with a Common Edge, *International Electronic Journal of Algebra*, **15** (2014), 132-144.

COMSATS Institute of Information Technology, Lahore, Pakistan
*E-mail address: drimranahmed@ciitlahore.edu.pk*

COMSATS Institute of Information Technology, Lahore, Pakistan
*E-mail address: shahid_nankana@yahoo.com*