Clifford group and stabilizer states from Chern-Simons theory

Howard J. Schnitzer*
Department of Physics
Brandeis University
Waltham, MA 02454

March 19, 2019

Abstract

The construction of generators of the Clifford group and of stabilizer states from Chern-Simons theory is presented for the Kac-Moody algebras $SU(2)_1$, $U(N)_{N,N(K+N)}$ with $N = 2$ and $K = 1$, and $SU(N)_1$, extending results of Salton, et. al.

---

*schnitzr@brandeis.edu
1 Introduction

We continue the study initiated by Salton, et. al. \cite{1} of entanglement from topology in Chern-Simons theory. This is a topological quantum field theory in an arena where the Euclidean path integral provides a map between geometry and states. For these three-dimensional quantum field theories there is a mapping $Z$, the functional integral, which relates the 3-dimensional manifold $\mathcal{M}$ to a probability amplitude $Z(\mathcal{M})$. If $\mathcal{M}$ has a boundary, the boundary field configuration must be specified. That is, the path integral selects a state $|\mathcal{M}\rangle$ in a Hilbert space $\mathcal{H}_{\partial \mathcal{M}}$ associated to the boundary field configuration. If the boundary consists of several multiply connected components, $\mathcal{H}_{\partial \mathcal{M}} = \bigotimes_{i=1}^{n} \mathcal{H}_{\Sigma_i}$, where $\partial \mathcal{M} = \bigcup_{i=1}^{n} \Sigma_i$. This means that the different components $\Sigma_i$ are not coupled, so that the Euclidean path integral factorizes.

In this paper we focus on the preparation of stabilizer states constructed from Chern-Simons theory defined on the $n$-torus Hilbert space.

In section 2 we discuss the Kac-Moody algebra $SU(2)_1$, while in section 3 it is shown that the unitary Kac-Moody algebra $U(N)_{k,N(N+k)}$ for $N = 2$, $k = 1$ shares the conclusions of section 2. Therefore Chern-Simons theory for both $SU(2)_1$, and $U(2)_{1,6}$ allow one to prepare arbitrary stabilizer states in $\mathcal{H}_{T^2}^\otimes_n$. The Clifford group is then constructed from Clifford gates applied to $|0\rangle^\otimes n$. The Choi-Jamiolkowski isomorphism then allows the preparation of an arbitrary element of the Clifford group \cite{3,4}.

In section 4 the analysis is extended to $SU(N)_1$, while in section 5 related issues are discussed.
SU(2)\textsubscript{1}

The basis of the Kac-Moody algebra for SU(2)\textsubscript{1} is given in terms of Young tableau restricted to a single column, with basis \(a = 0, 1\) where \(a\) denotes the numbers of boxes of the tableau. We present the generators of the Clifford group for SU(2)\textsubscript{1} in terms of this basis, then the fusion matrix is

\[
N_{ab}^c \quad \text{with } a + b = c \mod 2 \tag{2.1}
\]

The modular transformation matrices are [5], with standard normalization

\[
S_{ab} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad a, b = 0, 1 \tag{2.2}
\]

and [6]

\[
T_{ab} = \exp \left[ 2\pi i \left( h_a - \frac{c}{24} \right) \right] \delta_{ab} \tag{2.3}
\]

where \(c = 1\) is the central charge for SU(2)\textsubscript{1} and the conformal dimension

\[
h_a = \frac{C_2(a)}{3} \tag{2.4}
\]

with \(C_2(a)\) the quadratic Casimir operator for the representation. In terms of angular momentum \(j = a/2\), so that

\[
C_2(a) = j(j + 1) = \frac{a}{4}(a + 2) \tag{2.5}
\]

thus

\[
T_{ab} = \exp \left[ 2\pi i \left( \frac{a(a + 2)}{12} - \frac{1}{24} \right) \right] \delta_{ab} \tag{2.6}
\]

The generators of the Clifford group are the Hadamard gate, the phase gate, and the controlled addition gate \(c_{ADD}\), which satisfies [3, 4]

\[
c_{ADD} |a\rangle |b\rangle = |a\rangle |a + b \mod 2\rangle \tag{2.7}
\]
Equation (2.2) shows that $S_{ab}$ is in fact the Hadamard gate: It is convenient to define $\omega = \exp i\pi$. Then in (2.2)

$$\frac{1}{\sqrt{2}}(\omega)_{ab} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

is the Hadamard gate. The phase gate $P_{ab}$ is

$$(\omega^{-1/12})^{a(a+2)/6}T_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} = P_{ab}. \quad (2.9)$$

Given (2.1) and (2.2) we inherit the construction of Figure 3 of Salton et al. [1] to construct the copy tensor, $c_{ADD}$, and a perfect tensor. Therefore for $SU(2)_1$ Chern-Simons theory one can prepare any stabilizer on the $n$-torus Hilbert space

$$H_{T^2}^\otimes = (\mathbb{C}^2)^{\otimes n} \quad (2.10)$$

In section 5 we present this in a broader context.

3 $U(N)_{K,N(K+N)}$ for $N = 2, K = 1$

In order to understand the special case $U(2)_{1,6}$, we first present the representations of the general case, which is described in detail in section 2 of ref. [2], and which is summarized here. The essential feature is that

$$U(N)_{K,N(N+K)} = [SU(N)_K \times U(1)_{N(N+K)}] / \mathbb{Z}_N \quad (3.1)$$

which requires $K$ to be odd for consistency. Representations $(R, Q)$ of $SU(N)_K \times U(1)_{N(N+K)}$ must satisfy

$$Q \equiv r \mod N \quad (3.2)$$
where \( r \) is the number of boxes of the Young tableau associated to \( R \). There is an equivalence relation

\[
(R, Q) \simeq (\sigma(R), Q + N + K)
\]  

(3.3)

where \( \sigma \) is the simple current of \( SU(N)_K \). Applying the simple current \( N \) times, where \( \sigma^N = 1 \), one obtains the equivalence

\[
(R, Q) \simeq (R, Q + N(N + K))
\]  

(3.4)

so that \( Q \) is restricted to the range

\[
0 \leq Q < N(N + K).
\]  

(3.5)

The \( U(N) \) representations \( (R, Q) \) can be characterized by the extended Young tableau \( \mathcal{R} \) with row lengths \( \bar{l}_i \in \mathbb{Z}, (i = 1 \text{ to } N) \). There is exactly one extended tableau \( \mathcal{R} \) which satisfies

\[
0 \leq l_N \leq \ldots \leq l_1 \leq K.
\]  

(3.6)

Hence the primary fields of \( U(N)_{K,N(K+N)} \), where \( K \) is odd, are in one to one correspondence with the Young tableaux \( \mathcal{R} \) with no more than \( N \) rows and \( K \) columns. The number of such tableaux is \( \binom{N+K}{N} \).

The modular transformation matrix for the \( U(N)_{K,N(K+N)} \) character is \([2]\)

\[
S_{AB} = \sqrt{\frac{N}{N + K}} S_{ab} e^{-2 \pi i N Q_B / (N + K)}
\]  

(3.7)

where \( S_{ab} \) is that of \( SU(N)_K \). The subscripts \( A \) or \( a \) indicate whether one refers to \( U(N)_{K,N(K+N)} \) or \( SU(N)_K \). The modular transformation matrix for \( \tau \rightarrow \tau + 1 \) is \([2]\)

\[
T_{AB} = \exp \left[ 2 \pi i \left( h_A - \frac{c}{24} \right) \right] \delta_{AB}
\]  

(3.8)
where the central charge
\[ c = \frac{N(NK + 1)}{K + N} \]  
(3.9)
and
\[ h_A = \frac{\frac{1}{2}C_2(A)}{K + N} \]  
(3.10)
where \( h_A = h(R, Q) \), and
\[ h(R, Q) = h_a + \frac{Q^2}{2N(N + K)} \]  
(3.11a)
\[ = \frac{1}{2} \frac{C_2(R)}{K + N} + \frac{Q^2}{2N(N + K)} \]  
(3.11b)
\[ = \frac{1}{2} \frac{C_2(R, Q)}{K + N} \]  
(3.11c)
and \( Q = r \mod N \) from (3.2). Therefore
\[ T_{AB} = T_{ab} \exp \left[ 2\pi i \left( h_a + \frac{Q^2}{2N(N + K)} - \frac{c}{24} \right) \right] \delta_{AB}. \]  
(3.12)
The fusion matrix is
\[ N_{AB}^c = \sum_D \frac{S_{AD}S_{BD}(S_{CD})^{-1}}{S_{0D}} \]  
(3.13)
\[ = N_{ab}^c \sum_{Q_D} \exp \left[ -2\pi i \frac{(Q_A + Q_B - Q_C) Q_D}{N(N + K)} \right] \]
\[ = N_{ab}^c \delta_{Q_A+Q_B-Q_C} \]
where
\[ a + b = c \mod N, \]
together with (3.2), describes the fusion matrix \( N_{ab}^c \), as well as \( Q \) charge conservation by virtue of (3.2).
Now specialize to $U(2)_{1,6}$, making use of the review of $U(N)_{K,N(K+N)}$ to discuss this case. The fusion matrix $N_{ab}^c$ satisfies $c = a + b \mod 2$, as does the fusion matrix of $SU(2)_1$, where $a$, $b$, and $c$ are the number of boxes of a single column tableau. Then

$$N_{AB}^c = N_{ab}^c \delta_{Q_A+Q_B-Q_C},$$

(3.14)

where now $Q_A = a \mod 2$ from (3.2), so that charge conservation is automatically satisfied. Restricting (3.7) to $N = 2, K = 1$, one can again inherit the construction of Salton, Swingle, and Walter [1] to construct the $C_{ADD}$ gate. The Helmholtz gate and phase gate are essentially that of $SU(2)$, combined with $Q$ conservation.

4 $SU(N)_1$

Representations of $SU(N)_1$ are described by a single column tableau with $0 \leq N - 1$ boxes. The fusion tensor is

$$N_{ab}^c, \quad \text{with} \quad a + b = c \mod N.$$  

(4.1)

Therefore, this case closely parallels that of $SU(2)_1$ in section 2. The $C_{ADD}$ gate will satisfy

$$C_{ADD} |a\rangle |b\rangle = |a\rangle |a + b \mod N\rangle$$

(4.2)

so that with $S_{ab}$, $N_{ab}^c$, and the phase gate, one constructs a basis for the Clifford group.

The modular transformation matrix normalized as in [5] is

$$S_{ab} = \frac{(-i)^N(N-1)/2}{\sqrt{N(N+1)(N-1)/2}} \det M(a, b)$$

(4.3)
where $a, b = 0$ to $N - 1$,

$$M_{ij}(a, b) = \exp \left[ \frac{2\pi i \phi_i(a) \phi_j(b)}{N + 1} \right], \quad (4.4)$$

with $i = 1$ to $N$ and

$$\phi_i(a) = l_i(a) - i - \frac{r(a)}{N} + \frac{1}{2}(N + 1) \quad (4.5)$$

where

$$l_i = \begin{cases} \sum_{j=i}^{N-1} a_j & \text{for } i = 1 \text{ to } N - 1 \\ 0 & \text{for } i = N \end{cases} \quad (4.6)$$

and $r(a) = \sum_{i=1}^{N-1} l_i(a)$ is the total number of boxes in the reduced Young tableau corresponding to the representation $a$.

The modular transformation matrix

$$T_{ab} = \exp \left\{ 2\pi i \left[ \frac{C_2(R)}{2(N + 1)} - \frac{1}{24} \right] \right\} \delta_{ab} \quad (4.7)$$

with the conformal dimension is

$$h_a(R) = \frac{1}{2} \frac{C_2(R)}{N + 1}, \quad (4.8)$$

with the quadratic Casimir operator

$$C_2(R) = X + r(N + 1) - \frac{r^2}{N} \quad (4.9)$$

with $r(a)$ as above, and

$$X = \sum_{i=1}^{N-1} l_i(l_i - 2i), \quad (4.10)$$
where from (4.6)

\[ l_1 \geq l_2 \geq \ldots \geq l_{N-1} \geq 0. \tag{4.11} \]

Note \( C_2(R) \) is quadric in \( a \), vanishing when \( a = 0 \).

Given \( N_{ab}^c \) with \( c = a + b \mod N \), the detailed expression for \( S_{ab} \) is not explicitly needed for the construction of Figure 3 of Salton, Swingle, and Walter [1]. From (4.7) to (4.11) one extracts an overall phase factor to obtain the phase gate

\[
P_{ab} = \exp(i\pi h_a(R))\delta_{ab}. \tag{4.12}\]

This, together with \( C_{ADD} \) and \( S_{ab} \), generates the Clifford group [3, 4].

5 Related issues

In sections 2 to 4 we generalized the results of Theorem 1 of Salton, Swingle, and Walter [1] to \( SU(2)_1 \), \( U(2)_{1,6} \) and \( SU(N)_1 \). The unifying feature which makes this possible is that the fusion tensors are all of the form

\[ N_{ab}^c ; \quad a + b = c \mod N. \tag{5.1} \]

This, together with the modular transformation matrices \( T_{ab} \) and \( S_{ab} \), allows one to construct the phase gate, and the \( C_{ADD} \) gate, while \( S_{ab} \) is (conjectured to be) an appropriate generalization of the Helmholtz gate. Thus, one can repeat the strategy of Figures 3 and 4, ff. of Salton, Swingle, and Walter [1]. In particular, for \( SU(N)_1 \) the computation of Figure 4(a) gives \( N^2 \), that of Figure 4(b) yields \( N^4 \), etc., which means that the entanglement entropy of an arbitrary many torus system is

\[ S(A) = S(B) = S(C) = \log N. \tag{5.2} \]

As a consequence the \( SU(N)_1 \) fusion tensor is equivalent to \( g = 1 \) GHZ state, independent of \( N \) for the states that can be distilled between
A, B, and C for \( \partial M = A \cup B \cup C \) for an arbitrary tripartition of the boundary torii. Similarly, one should expect analogous results for \( Sp(N)_1 \) Chern-Simons theory since the fusion matrix satisfies (5.1).

It is known that a universal topological computer based on \( SU(2)_K \) requires \( K \geq 3 \) \cite{7}. This is exemplified by the work of Freedman, Larsen, and Wang \cite{8} which presents a detailed construction for \( SU(2)_3 \). Then level-rank duality shows that a universal topological quantum computer can be based on \( SU(3)_2 \) \cite{9}. Level-rank duality then suggests that a universal topological quantum computer can be based on \( SU(K)_2 \), where \( K \geq 3 \).

Other applications of entanglement in Chern-Simons theory are discussed in refs. In particular, refs. \cite{10–14} consider stabilizer states in \( U(1) \) Chern-Simons theory.

In that context we follow \cite{12}, where upper-bounds are derived for \( SU(2)_K \), given an \( n \)-component link \( \mathcal{L}^n \subset S^3 \), and two sublinks \( \mathcal{L}^m_A \) and \( \mathcal{L}^{n-m}_{\bar{A}} \subset S^3 \) such that a separating surface \( \Sigma_A|_{\bar{A}} \subset S^3 \) is a connected, compact, oriented two-dimensional surface without boundary, where: (1) \( \mathcal{L}^m_A \) is contained in the handlebody inside \( \Sigma_A|_{\bar{A}} \), (2) \( \mathcal{L}^{n-m}_{\bar{A}} \) is contained in the handlebody outside \( \Sigma_A|_{\bar{A}} \), and (3) \( \Sigma_A|_{\bar{A}} \) does not intersect any of the components of \( \mathcal{L}^n \). Reference \cite{12} presents a trivial upper-bound on the entanglement entropy for \( SU(2)_K \), i.e.

\[
S_{EE}(\mathcal{L}_A^m|\mathcal{L}_{\bar{A}}^{n-m}) \leq \ln(K + 1) \min(m, n - m) \tag{5.3}
\]

and a tighter upper bound

\[
S_{EE}(\mathcal{L}_A^m|\mathcal{L}_{\bar{A}}^{n-m}) \leq \ln \left[ \sum_{u=0}^{K} \frac{1}{S_{0u}^{2 \min(g_{\Sigma})-2}} \right] \tag{5.4}
\]

Specialize to \( SU(2)_1 \), where \( S_{00} = 1/\sqrt{2} \) and \( S_{01} = -1/\sqrt{2} \), so that (5.3) becomes

\[
S_{EE}(\mathcal{L}_A^m|\mathcal{L}_{\bar{A}}^{n-m}) \leq (\ln 2) \min(m, n - m), \tag{5.5}
\]
while (5.4) for \( \min(g_\Sigma) \geq 1 \) becomes

\[
S_{EE}(\mathcal{L}_A^m|\mathcal{L}_A^{n-m}) \leq (\ln 2) \min(g_\Sigma) \tag{5.6}
\]

Further for \( SU(N)_1 \), recalling (5.2) we expect

\[
S_{EE}(\mathcal{L}_A^m|\mathcal{L}_A^{n-m}) \leq (\ln N) \min(g_\Sigma). \tag{5.7}
\]

### 6 Acknowledgements

We thank Isaac Cohen, Alastair Grant-Stuart, and Andrew Rolph for assistance in the preparation of the paper.

### References

[1] G. Salton, B. Swingle, and M. Walter. *Phys. Rev.* D95 (2017), 105007. arXiv: [1611.01516](https://arxiv.org/abs/1611.01516).

[2] S. Naculich and H. Schnitzer. *JHEP* 06 (2007), 023. arXiv: [hep-th/0703089](https://arxiv.org/abs/hep-th/0703089).

[3] D. Gottesman. arXiv: [quant-ph/9807006](https://arxiv.org/abs/quant-ph/9807006).

[4] D. Gottesman. *Chaos, Solitons & Fractals* 10 (1999), 1749. arXiv: [quant-ph/9802007](https://arxiv.org/abs/quant-ph/9802007).

[5] E. Mlawer, S. Naculich, H. Riggs, and H. Schnitzer. *Nucl. Phys.* B352 (1991), 863.

[6] S. Naculich, H. Riggs, and H. Schnitzer. *Phys. Lett.* B246 (1990), 417.

[7] J. Preskill. *Lecture notes, Cal. Tech.*

[8] M. Freedman, M. Larsen, and Z. Wang. *Commun. Math. Phys.* 227 (2002), 605. arXiv: [quant-ph/0001108](https://arxiv.org/abs/quant-ph/0001108).

[9] H. J. Schnitzer. arXiv: [1811.11861](https://arxiv.org/abs/1811.11861).
[10] S. Dong, E. Fradkin, R. Leigh, and S. Nowling. *JHEP* 05 (2008), 016. arXiv: [0802.3231](https://arxiv.org/abs/0802.3231).

[11] V. Balasubramanian, J. Fliss, R. Leigh, and O. Parrikar. *JHEP* 04 (2017), 061. arXiv: [1611.05460](https://arxiv.org/abs/1611.05460).

[12] V. Balasubramanian, M. DeCross, J. Fliss, A. Kar, R. Leigh, and O. Parrikar. *JHEP* 05 (2018), 038. arXiv: [1801.01131](https://arxiv.org/abs/1801.01131).

[13] S. Chun and N. Bao. arXiv: [1707.03525](https://arxiv.org/abs/1707.03525).

[14] S. Dwivedi, V. Singh, S. Dhara, P. Ramadevi, Y. Zhou, and L. Joshi. arXiv: [1711.06474](https://arxiv.org/abs/1711.06474).