Polynomial properties on large symmetric association schemes

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Abstract

We show that if a symmetric association scheme of class $d$ has larger size than the Moore bound for diameter $d-1$, then it has the $P$-polynomial property. We also prove its dual theorem for the $Q$-polynomial property. Moreover these theorems are generalized for connected regular graphs with at most $d+1$ distinct eigenvalues, and spherical finite set with at most $d$ distances.

Key words: $P$-polynomial association scheme, $Q$-polynomial association scheme, upper bound, Moore graph, tight spherical design.

1 Introduction

There are two well-known upper bounds for the size of symmetric association schemes $(X, \{R_i\}_{i=0}^d)$.

(1) If $P_i(0)$ is distinct from $P_i(1), P_i(2), \ldots, P_i(d)$, then

$$|X| \leq M(k_i, d) = 1 + k_i \sum_{j=0}^{d-1} (k_i - 1)^j.$$

(2) If $Q_i(0)$ is distinct from $Q_i(1), Q_i(2), \ldots, Q_i(d)$, then

$$|X| \leq N(m_i, d) = \binom{m_i + d - 1}{d} + \binom{m_i + d - 2}{d - 1}.$$

The notation of parameters of an association scheme is defined in Section 2.

There is a dual property between two bounds as follows.

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The assumption in (1) means the graph \((X, R_i)\) is connected, and the bound \((1)\) is called the Moore bound. If \((X, R_i)\) attains the bound \((1)\), then \((X, R_i)\) is called a Moore graph, and the scheme \((X, \{R_i\}_{i=0}^d)\) has the \(P\)-polynomial property with respect to \(R_i\).

On the other hand, the bound \((2)\) is known as an absolute bound for primitive association schemes \cite[Theorem 4.9, p. 85]{1}. Indeed we need only the assumption of \((2)\) instead of the primitivity. Since a primitive idempotent \(E_i\) is positive semi-definite, we regard \(E_i\) as the spherical embedding of \((X, \{R_i\}_{i=0}^d)\). The assumption in \((2)\) means the spherical embedding \(E_i\) is injective. If the spherical embedding \(E_i\) attains the upper bound \((2)\), then \(E_i\) is a tight spherical \(2d\)-design on \(S^{m_i-1}\), and \((X, \{R_i\}_{i=0}^d)\) has the \(Q\)-polynomial property \cite{3}.

The main theorems in the present paper are the following.

(i) If \(P_i(0)\) is distinct from \(P_i(1), P_i(2), \ldots, P_i(d)\) and \(|X| > M(k_i, d-1)\) holds, then \((X, \{R_i\}_{i=0}^d)\) has the \(P\)-polynomial property with respect to \(A_i\).

(ii) If \(Q_i(0)\) is distinct from \(Q_i(1), Q_i(2), \ldots, Q_i(d)\) and \(|X| > N(m_i, d-1)\) holds, then \((X, \{R_i\}_{i=0}^d)\) has the \(Q\)-polynomial property with respect to \(E_i\).

Note that \(P_i(0), P_i(1), \ldots, P_i(d)\) are the eigenvalues of \((X, R_i)\). A connected \(k\)-regular graph with at most \(d+1\) distinct eigenvalues has the same upper bound as \((1)\). On the other hand, \(Q_i(1), Q_i(2), \ldots, Q_i(d)\) correspond to the inner products of the spherical embedding with respect to \(E_i\). A finite set on the unit sphere \(S^{m_i-1}\) with at most \(d\) distances has the same upper bound as \((2)\). For these concepts, we prove some generalizations of the above theorems (i) and (ii).

2 Preliminaries

In this section, we introduce some basic definitions and known results.

A symmetric association scheme of class \(d\) is a pair \(X = (X, \{R_i\}_{i=0}^d)\), where \(X\) is a finite set and each \(R_i\) is a nonempty subset of \(X \times X\) satisfying the following:

1. \(R_0 = \{(x, x) \mid x \in X\}\),
2. \(X \times X = \bigcup_{i=0}^d R_i\) and \(R_i \cap R_j\) is empty if \(i \neq j\),
3. \(R_i = R_i^i\) for any \(i \in \{0, 1, \ldots, d\}\), where \(R_i^i = \{(y, x) \mid (x, y) \in R_i\}\),
4. for all \(i, j, k \in \{0, 1, \ldots, d\}\), there exist integers \(p_{ij}^k\) such that for all \(x, y \in X\) with \((x, y) \in R_k\),
\[ p_{ij}^k = \{z \in X \mid (x, z) \in R_i, (z, y) \in R_j\}\].

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The integers $p_{ij}^k$ are called the intersection numbers. The $i$-th adjacency matrix $A_i$ of $X$ is the matrix indexed by $X$ with the entry

$$(A_i)_{xy} = \begin{cases} 
1 & \text{if } (x, y) \in R_i, \\
0 & \text{otherwise}.
\end{cases}$$

The Bose–Mesner algebra of $X$ is the algebra generated by the adjacency matrices $\{A_i\}_{i=0}^{d}$ over the complex field $\mathbb{C}$. Then $\{A_i\}_{i=0}^{d}$ is a natural basis of the Bose–Mesner algebra. The Bose–Mesner algebra has a second basis $\{E_i\}_{i=0}^{d}$ such that

1. $E_0 = |X|^{-1}J$, where $J$ is the all-ones matrix,

2. $I = \sum_{i=0}^{d} E_i$, where $I$ is the identity matrix,

3. $E_i E_j = \delta_{ij} E_i$, where $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$.

The basis $\{E_i\}_{i=0}^{d}$ is called the primitive idempotents of $X$. The parameters $P_i(j)$, $Q_i(j)$, $p_{ij}^k$, $q_{ij}^k$ are appeared in the following expressions:

$$A_i = \sum_{j=0}^{d} P_i(j)E_j, \quad E_i = \frac{1}{|X|} \sum_{j=0}^{d} Q_i(j)A_j,$$

$$A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k, \quad E_i \circ E_j = \frac{1}{|X|} \sum_{k=0}^{d} q_{ij}^k E_k,$$

where $\circ$ denotes the Hadamard product, that is, the entry-wise matrix product. The numbers $q_{ij}^k$ are called the Krein parameters. The Krein parameters are nonnegative real numbers (the Krein condition) [12] [11] page 69]. Throughout this paper, we use the notation $v = |X|$, $k_i = P_i(0)$, $m_i = Q_i(0)$ for $0 \leq i \leq d$, $k = k_1$, and $m = m_1$.

A symmetric association scheme is called a $P$-polynomial scheme (or a metric scheme) with respect to the ordering $\{A_i\}_{i=0}^{d}$ if for each $i \in \{0, 1, \ldots, d\}$, there exists a polynomial $v_i$ of degree $i$ such that $P_i(j) = v_i(P_i(j))$ for any $j \in \{0, 1, \ldots, d\}$. We say a symmetric association scheme is a $P$-polynomial scheme with respect to $A_1$ if it has the $P$-polynomial property with respect to some ordering $A_0, A_1, A_{12}, A_{13}, \ldots, A_{1d}$. Dually a symmetric association scheme is called a $Q$-polynomial scheme (or a cometrix scheme) with respect to the ordering $\{E_i\}_{i=0}^{d}$ if for each $i \in \{0, 1, \ldots, d\}$, there exists a polynomial $v_i^*$ of degree $i$ such that $Q_i(j) = v_i^*(Q_i(j))$ for any $j \in \{0, 1, \ldots, d\}$. Moreover a symmetric association scheme is called a $Q$-polynomial scheme with respect to $E_1$ if it has the $Q$-polynomial property with respect to some ordering $E_0, E_1, E_{12}, E_{13}, \ldots, E_{1d}$.

We say $A_j$ (resp. $E_j$) is a component of an element $M$ of the Bose–Mesner algebra if $A_j \circ M \neq 0$ (resp. $E_j M \neq 0$). Let $N_{i, h}$ (resp. $N_{i, h}^*$) denote the set
of indices $j$ such that $A_j$ (resp. $E_j$) is a component of $A^h_i$ (resp. $E^h_i$) but not of $A^l_i$ (resp. $E^l_i$) for each $0 \leq l \leq h - 1$. Here $A^h$ means $A \circ \cdots \circ A$ to $h$ factors. The following lemmas are used later.

**Lemma 2.1** ([7]). Let $\mathcal{X}$ be a symmetric association scheme of class $d$. Suppose $P_i(0)$ is distinct from $P_i(1), P_i(2), \ldots, P_i(d)$. Then the following are equivalent.

1. $\mathcal{X}$ is a $P$-polynomial scheme with respect to $A_i$.
2. $N_{i,d}$ is nonempty.

**Lemma 2.2** ([6]). Let $\mathcal{X}$ be a symmetric association scheme of class $d$. Suppose $Q_i(0)$ is distinct from $Q_i(1), Q_i(2), \ldots, Q_i(d)$. Then the following are equivalent.

1. $\mathcal{X}$ is a $Q$-polynomial scheme with respect to $E_i$.
2. $N^*_i,d$ is nonempty.

**Remark 2.3.** Lemma 2.1 (2) is equivalent to that the graph $(X, R_i)$ is of diameter $d$. Lemma 2.1 is also proved in [5, Lemma 3.2, p. 229]. Lemma 2.2 (2) is equivalent to that $E_i$ has Schur diameter $d$. Lemma 2.2 is commented in [4, p. 230]

**Lemma 2.4.** Let $\mathcal{X}$ be a symmetric association scheme of class $d$. Suppose $Q_i(0)$ is distinct from $Q_i(1), Q_i(2), \ldots, Q_i(d)$. Then we have $\bigcup_{j=0}^d N^*_{i,j} = \{0, 1, \ldots, d\}$, where the union is disjoint.

**Proof.** It is immediate from the equation

$$v^d E_i - Q_i(1)E_0 \circ \cdots \circ E_i - Q_i(d)E_0 \over Q_i(0) - Q_i(1) = I = \sum_{j=0}^d E_j. \quad \square$$

**Lemma 2.5** (Lemma 4.7, p.84 in [1]). Let $A$ be a matrix of rank $m$. Then the rank of $A^h$ is at most $\binom{m+h-1}{h}$.

3 Large symmetric association schemes

3.1 $P$-polynomial property

Let $(X, \{R_i\}_{i=0}^d)$ be a symmetric association scheme of class $d$. For each $i \in \{1, 2, \ldots, d\}$, we define the graph $G_i = (X, R_i)$. If $G_i$ is connected, then $G_i$ is a $k_i$-regular graph of diameter at most $d$. Indeed if the diameter is more than $d$, then $A_i$ needs at least $d + 2$ distinct eigenvalues.

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Suppose \( k_i = P_i(0) \) is distinct from \( P_i(1), P_i(2), \ldots, P_i(d) \), that is, \( G_i \) is connected. Then we have a well known bound
\[
|X| \leq M(k_i, d) = 1 + k_i \sum_{j=0}^{d-1} (k_i - 1)^j
\] (3.1)
for each \( i \). The following is the main theorem in the present subsection.

**Theorem 3.1.** Let \((X, \{R_j\}_{j=0}^d)\) be a symmetric association scheme. Suppose \( P_i(0) \) is distinct from \( P_i(1), P_i(2), \ldots, P_i(d) \), and \(|X| > M(k_i, d-1)\). Then \((X, \{R_j\}_{j=0}^d)\) is a \( P \)-polynomial association scheme with respect to \( A_i \).

**Proof.** Since \(|X| > M(k_i, d-1)\), the graph \( G_i = (X, R_i) \) is of diameter \( d \). By Lemma 2.1, \((X, \{R_j\}_{j=0}^d)\) is a \( P \)-polynomial scheme with respect to \( A_i \).

We can find a lot of examples satisfying the assumption in Theorem 3.1 from [2].

**Example 3.2.** Every connected strongly regular graph which is not a complete graph satisfies the assumption in Theorem 3.1 because of \( v > M(k, 1) = 1 + k \).

**Example 3.3.** Generalized polygons (see [2] for the definition) satisfies the assumption in Theorem 3.1. They have the parameters \( a_1 = \cdots = a_{d-1} = 0 \), \( c_1 = \cdots = c_{d-1} = 1 \), and hence the size is greater than \( M(k, d-1) \).

**Example 3.4.** When \( d \) is fixed, there exists a natural number \( n_0 \) such that Johnson scheme \( J(n, d) \) satisfies the assumption in Theorem 3.1 for every \( n \geq n_0 \). Indeed \( v = \binom{n}{d} \) is a polynomial in \( n \) of degree \( d \), and \( M(k, d-1) = 1 + d(n-d) \sum_{j=0}^{d-2} d(n-d-1)^j \) is a polynomial in \( n \) of degree \( d-1 \), where \( k = d(n-d) \). For instance when \( d = 3 \), we have \( \binom{n}{3} > M(k, 2) \) holds for \( n \geq 51 \).

**Example 3.5.** When \( d \) is fixed, there exists a natural number \( q_0 \) such that Hamming scheme \( H(d, q) \) satisfies the assumption in Theorem 3.1 for all \( q \geq q_0 \). Indeed \( v = q^d \) is a polynomial in \( q \) of degree \( d \), and \( M(k, d-1) = 1 + d(q-1) \sum_{j=0}^{d-2} d(q-1-1)^j \) is a polynomial in \( q \) of degree \( d-1 \), where \( k = d(q-1) \). For instance when \( d = 3 \), we have \( q^3 > M(k, 2) \) holds for \( n \geq 7 \).

Besides the above schemes we have the following examples from the table in [2].
3.2 $Q$-polynomial property

First we show an absolute bound for a symmetric association scheme.

**Theorem 3.6.** Let $(X, \{R_i\}_{i=0}^d)$ be a symmetric association scheme. Suppose $Q_i(0)$ is distinct from $Q_i(1), Q_i(2), \ldots, Q_i(d)$. Then we have

$$|X| \leq N(m_i, d) = \frac{(m_i + d - 1)}{d} + \frac{(m_i + d - 2)}{d - 1}.$$  \hspace{1cm} (3.2)

**Proof.** Since $E_0$ is a component of $E_i^{(0)}$ and the Krein condition, every component of $E_i^{(h-2)}$ is also a component of $E_i^{(0)}$. This implies that

$$\sum_{j \in N^*_i, h} m_j = \sum_{j \in N^*_i, h} \text{Rank } E_j \leq \text{Rank } E_i^{(0)} - \text{Rank } E_i^{(h-2)}$$

for $h \in \{2, 3, \ldots, d\}$. By Lemmas 2.4 and 2.5 we obtain

$$|X| = \sum_{j=0}^d m_j = \sum_{h=0}^d \sum_{j \in N^*_i, h} m_j$$

$$\leq \text{Rank } E_0 + \text{Rank } E_i + \sum_{h=2}^d (\text{Rank } E_i^{(0)} - \text{Rank } E_i^{(h-2)})$$

$$\leq \left(\frac{m_i + d - 1}{d}\right) + \left(\frac{m_i + d - 2}{d - 1}\right).$$

**Remark 3.7.** An association scheme $(X, \{R_i\}_{i=0}^d)$ is said to be primitive, if the graph $(X, R_i)$ is connected for every $i \in \{1, \ldots, d\}$. The same bound as (3.2) is proved in [1, Theorem 4.9, p. 85] for primitive association schemes. Indeed a primitive association scheme satisfies that $Q_i(0)$ is distinct from $Q_i(1), Q_i(2), \ldots, Q_i(d)$ for every $i \in \{1, 2, \ldots, d\}$.  

The following is the main theorem in the present subsection.

**Theorem 3.8.** Let \((X, \{R_j\}_{j=0}^d)\) be a symmetric association scheme. Suppose \(Q_i(0)\) is distinct from \(Q_i(1), Q_i(2), \ldots, Q_i(d)\). If \(|X| > N(m_i, d - 1)\), then \((X, \{R_j\}_{j=0}^d)\) is a Q-polynomial scheme with respect to \(E_i\).

**Proof.** Assume \((X, \{R_j\}_{j=0}^d)\) is not a Q-polynomial scheme with respect to \(E_i\), that is, \(N_{i,d}^*\) is empty. By Lemma 2.4, we have \(\cup_{j=0}^{d-1} N_{i,j}^* = \{0, 1, \ldots, d\}\). Therefore

\[
|X| = \sum_{j=0}^d m_j = \sum_{h=0}^{d-1} \sum_{j \in N_{i,h}^*} m_j
\]

\[
\leq \text{Rank } E_0 + \text{Rank } E_i + \sum_{h=2}^{d-1} \left( \text{Rank } E_i^{gh} - \text{Rank } E_i^{(h-2)} \right)
\]

\[
\leq \left( \frac{m_i + d - 2}{d - 1} \right) + \left( \frac{m_i + d - 3}{d - 2} \right) = N(m_i, d - 1).
\]

This is a contradiction. \(\square\)

**Example 3.9.** Every connected strongly regular graph which is not a complete graph satisfies the assumption in Theorem 3.8 because of \(v > N(m, 1) = 1 + m\).

**Example 3.10** (see [3] for details). Let \(X \subset S^{m-1}\) be a \(d\)-distance set and a spherical \(t\)-design, where \(S^{m-1}\) denotes the unit sphere in \(\mathbb{R}^m\) centered at the origin. If \(t \geq 2d - 2\) holds, then \(X\) has a Q-polynomial association scheme. From the inequality \(t \geq 2d - 2\), it follows that \(|X| \geq N(m, d - 1)\) from an absolute bound for spherical designs. However if \(|X| = N(m, d - 1)\) holds, then \(X\) becomes a tight spherical \(t\)-design, and hence it is a \((d - 1)\)-distance set. In conclusion, when \(t \geq 2d - 2\) holds we have \(|X| > N(m, d - 1)\).

**Example 3.11.** When \(d\) is fixed, there exists a natural number \(n_0\) such that Johnson scheme \(J(n, d)\) satisfies the assumption in Theorem 3.8 for every \(n \geq n_0\). Indeed \(v = \binom{n}{d}\) is a polynomial in \(n\) of degree \(d\), and \(N(m, d - 1) = \binom{n+d-3}{d-1} + \binom{n+d-4}{d-2}\) is a polynomial in \(n\) of degree \(d - 1\), where \(m = n - 1\). For instance when \(d = 3\), we have \(\binom{n}{3} > N(m, 2)\) holds for \(n \geq 7\).

**Example 3.12.** When \(d\) is fixed, there exists a natural number \(q_0\) such that Hamming scheme \(H(d, q)\) satisfies the assumption in Theorem 3.8 for every \(q \geq q_0\). Indeed \(v = q^d\) is a polynomial in \(q\) of degree \(d\), and \(N(m, d - 1) = \binom{d(q-2)}{d-1} + \binom{d(q-3)}{d-2}\) is a polynomial in \(q\) of degree \(d - 1\), where \(m = d(q - 1)\). For instance when \(d = 3\), we have \(q^3 > N(k, 2)\) holds for \(n \geq 4\).

Besides the above examples, we can find examples in [9].
corresponding to $\theta \in \gamma \leq s$

Let $G$ be a connected \(d\)-regular graph with at most \(d + 1\) distinct eigenvalues, and for Theorem 3.8, that is a finite set on \(S^{m-1}\) with at most \(d\) distances, namely the number of Euclidean distances (or inner products) between distinct vectors is at most \(d\). In this section, we show some properties of such objects having large sizes.

### 4 Generalized concepts

We deal with more generalized concepts of \(P\)- or \(Q\)-polynomial association schemes from the viewpoint of Theorems 3.1 and 3.8. Namely for Theorem 3.1, a generalized concept is a connected \(k\)-regular graph with at most \(d + 1\) distinct eigenvalues, and for Theorem 3.8 that is a finite set on \(S^{m-1}\) with at most \(d\) distances, namely the number of Euclidean distances (or inner products) between distinct vectors is at most \(d\). In this section, we show some properties of such objects having large sizes.

#### 4.1 Connected \(k\)-regular graph

Let \(G = (V, E)\) be a connected \(k\)-regular graph, and \(A\) the adjacency matrix. Suppose \(G\) has at most \(d + 1\) of distinct eigenvalues. If \(G\) is of diameter at least \(d + 1\), then \(I, A, A^2, \ldots, A^{d+1}\) are linearly independent, and hence \(G\) has at least \(d + 2\) distinct eigenvalues. Therefore the diameter of \(G\) is less than \(d + 1\). Thus \(G\) has the same bound as (3.1):

\[
|V| \leq M(k, d) = 1 + k \sum_{j=0}^{d-1} (k-1)^j.
\]

Let \(k = \theta_0 > \theta_1 > \theta_2 > \cdots > \theta_s\) be distinct eigenvalues of \(G\), where \(s \leq d\). Let \(E_i\) be the orthogonal projection matrix onto the eigenspace corresponding to \(\theta_i\). Let \(G(j) = \{(x, y) \mid x, y \in V, \partial(x, y) = j\}\) and \(G_{x}(j) = \{y \mid y \in V, \partial(x, y) = j\}\), where \(\partial(x, y)\) is the path distance of \(x \in V\) and \(y \in V\). We define

\[
K_i = \prod_{j=1,\ldots,s, j \neq i} \frac{\theta_0 - \theta_j}{\theta_i - \theta_j}.
\]

**Theorem 4.1.** Let \(G = (V, E)\) be a connected \(k\)-regular graph with at most \(d + 1\) distinct eigenvalues. Suppose \(|V| > M(k, d - 1)\) holds. Then we have the following.

1. \(G\) has \(d + 1\) distinct eigenvalues.
For \((x, y) \in G(d)\), the \((x, y)\)-entry of \(E_i\) is \(-K_i\) for each \(i \in \{1, \ldots, d\}\).

(3) For each \(x \in V\), the number of entries \(-K_i\) in the \(x\)-th row of \(E_i\) is at least \(|V| - M(k, d - 1)\).

**Proof.** (1): Since \(|V| > M(k, d - 1)\), \(G\) is of diameter \(d\) and has \(d + 1\) distinct eigenvalues.

(2): Define
\[
f_i(t) = \prod_{j=1,2,\ldots,d,j\neq i} \frac{t - \theta_j}{\theta_i - \theta_j}
\]
for each \(i \in \{1,2,\ldots,d\}\). Then we have
\[
f_i(A) = K_i E_0 + E_i.
\]
Since the degree of \(f_i(t)\) is \(d - 1\), for \((x, y) \in G(d)\) the \((x, y)\)-entry of \(f_i(A)\) is equal to 0. Therefore the \((x, y)\)-entry of \(E_i\) is \(-K_i\).

(3): For each \(x\), we have \(|\bigcup_{j=0}^{d-1} G_x(j)| \leq M(k, d - 1)\). Therefore \(|G_x(d)| \geq |V| - M(k, d - 1)\), this implies that the number of entries \(-K_i\) in the \(x\)-th row of \(E_i\) is at least \(|V| - M(k, d - 1)|). 

**Remark 4.2.** If a graph \(G\) which satisfies the assumption in Theorem 4.1 has the structure of an association scheme, then we have \(Q_i(d) = -K_i\) for each \(i \in \{1,2,\ldots,d\}\). This implies that \(G\) has a \(P\)-polynomial property [7, 10].

### 4.2 Spherical \(d\)-distance set

Let \(X\) be a finite set in \(S^{m-1}\) with at most \(d\) distances. The set \(X\) has usual inner products \(1 = \theta_0^* > \theta_1^* > \cdots > \theta_s^*\), where \(s \leq d\). Then we have an absolute upper bound:
\[
|X| \leq N(m, d) = \binom{m + d - 1}{d} + \binom{m + d - 2}{d - 1}, \tag{4.1}
\]
see [3]. We naturally obtain the graph \(G_i = (X, R_i)\) from \(X\), where \(R_i = \{(x, y) \mid x, y \in X, \langle x, y \rangle = \theta_i\}\) for each \(i \in \{1,2,\ldots,d\}\), and \(\langle x, y \rangle\) is the usual inner product of \(x\) and \(y\). Let \(A_i\) be the adjacency matrix of \(G_i\). For each \(i \in \{1,2,\ldots,d\}\) the value
\[
K_i^* = \prod_{j=1,2,\ldots,s,j\neq i} \frac{\theta_0^* - \theta_j^*}{\theta_i^* - \theta_j^*}
\]
is called the Larman–Rogers–Seidel ratio [11, 8]

**Theorem 4.3.** Let \(X\) be a finite set in \(S^{m-1}\) with at most \(d\) distances. Suppose \(|X| > N(m, d - 1)|\) holds. Then we have the following.
(1) $X$ has $d$ distances.

(2) $-K^*_i$ is an eigenvalue of $A_i$ for each $i \in \{1, 2, \ldots, d\}$.

(3) The multiplicity of $-K^*_i$ is at least $|X| - N(m, d-1)$.

Proof. (1): Since $|X| > N(m, d-1)$ holds, $X$ has $d$ distances.

(2),(3): Define $f^*_i(x) = \prod_{j=1, 2, \ldots, d, j \neq i} \frac{t - \theta^*_j}{\theta^*_i - \theta^*_j}$ for each $i \in \{1, 2, \ldots d\}$. Let $M = (\langle x, y \rangle)_{x,y \in X}$ be the Gram matrix of $X$. Note that every diagonal entry of $M$ is 1. Then we have

$$f^*_i(M^o) = K^*_i I + A_i,$$

where the multiplication is the Hadamard product. For each $x \in X$ and $m$ variables $\xi = (\xi_1, \ldots, \xi_m)$, we consider a polynomial $f_i(\langle x, \xi \rangle)$. By Lemma 2.2 in \cite{11}, the rank of $K^*_i I + A_i = (f_i(\langle x, y \rangle))_{x,y \in X}$ is bounded above by $N(m, d-1)$, which is the dimension of the linear spaces of the restrictions of $m$-variable polynomials of degree $d-1$ to $S^{m-1}$. Therefore the matrix (4.2) has the zero eigenvalue with multiplicity at least $|X| - N(m, d-1)$. Thus $A_i$ has the eigenvalue $-K_i$ with multiplicity at least $|X| - N(m, d-1)$.

Remark 4.4. If $X \subset S^{m-1}$ which satisfies the assumption in Theorem 4.3 has the structure of an association scheme, then we have $P_i(d) = -K^*_i$ for each $i \in \{1, 2, \ldots, d\}$. This implies that $X$ has a $Q$-polynomial property \cite{6, 10}.

Remark 4.5. If $|X| \geq 2N(m, d-1)$ holds, then $K^*_i$ is an integer by the generalized Larman–Rogers–Seidel theorem \cite{11}.

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