Application of two different algorithms to the approximate long water wave equation with conformable fractional derivative

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ABSTRACT
The current paper devoted on two different methods to find the exact solutions with various forms including hyperbolic, trigonometric, rational and exponential functions of fractional differential equations systems with conformable fractional derivative. We have employed the modified simple equation and \( \exp(-\Phi(z)) \) method here for the approximate long water wave equation. We have adopted here the fractional complex transform accompanied by properties of conformable fractional calculus for reduction of fractional partial differential equation systems to ordinary differential equation systems.

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1. Introduction
Fractional differential equations, containing the fractional differentiation are generalizations of classical differential equations of integer order. Recently, these equations have a great deal of attention and applied in many research fields, such as nonlinear control theory, electromagnetic theory, fluid mechanics, signal processing, electrochemistry and mathematical biology (Kilbas, Srivastava, & Trujillo, 2006; Miller & Ross, 1993). Moreover, various applications of the fractional calculus can be found in plasma physics turbulence, stochastic dynamical system, image processing, fluid dynamics and astrophysics. Fractional partial differential equations are becoming increasingly popular due to their practical applications in various fields of science and engineering. There are two major approaches in the theoretical formulation of initial value problems for fractional differential equations (Dai, Wang, & Liu, 2016; Imran, Khan, Ahmad, Shah, & Nazar, 2017; Saqib, Ali, Khan, Sheikh, & Jan, in press; Sheikh, Ali, Khan, & Saqib, 2016; Shah & Khan, 2016). One of them is based on the interpretation of the initial condition of fractional systems as a distributed initial condition (Agarwal, O’Regan, Hristova, & Cicek, 2017).

It has great importance to obtain exact solutions of these equations such as sub-equation method (Alzaidy, 2013; Mohyud-Din, Nawaz, Azhar, & Akbar, 2017; Ray & Sahoo, 2015; Sahoo & Ray, 2015), tanh method (Ray & Sahoo, 2015), simplest equation method (Taghizadeh, Mirzazadeh, Rahimian, & Akbari, 2013), Jacobi elliptic function expansion method (Tasbozan, Cenesiz, & Kurt, 2016), Kudryashov method (Demiray, Pandir, & Bulut, 2014; Eslami, 2016; Hosseini, Mayeli, & Ansari, 2017; Sonmezoglu, Ekici, Moradi, & Zhou, 2017), trial equation method (Ekici et al., 2016; Odabasi & Misirli, 2015; Pandir, Gurefe, & Misirli, 2013), exp-function method (Bekir, Guner, Aksoy, & Pandir, 2015; Zhang et al., 2010), first integral method (Eslami, Fathi, Mirzazadeh, & Biswas, 2014; Eslami & Rezazadeh, 2016; Mirzazadeh, Eslami, & Biswas, 2014; Younis, 2013), \((G'/G)\)-expansion method (Ray & Sahoo, 2017), modification of the truncated expansion method (Mirzazadeh & Eslami, 2013), functional variable method (Bekir, Guner, Bhrway, & Biswas, 2015), variable separation method (Wang & Dai, 2015; Wang, Zhang, & Dai, 2016), modified tanh-function method (Wang & Dai, 2016; Wang et al., 2016), Laplace transform method (Ali et al., 2016; Ali, Saqib, Khan, & Sheikh, 2016), homotopy analysis method (Kurt, Tasbozan, & Cenesiz, 2016), ansatz method (Zayed & Al-Nowehy, 2017) and so on (Biswas, Bhrway, Abdelkawy, Alshaery, & Hila, 2014; Guner & Eser, 2014; Korkmaz, 2017; Mirzazadeh, 2016; Mirzazadeh & Eslami, 2013; Saqib et al., in press; Sheikh et al., 2016).
In this paper, by use of the properties of fractional calculus, we propose two different methods to seek exact solutions of fractional partial differential equations with conformable derivative. Based on a traveling wave transformation, certain fractional partial differential equation systems can be turned into another fractional ordinary differential equation systems with respect to one new variable.

2. Brief of conformable fractional derivative

Recently, the authors Khalil et al. introduced a new simple and intriguing definition of the fractional derivative called conformable fractional derivative (Khalil, Al Horani, Yousef, & Sababheh, 2014). This derivative is well-behaved and obeys the Leibniz rule and chain rule. Let us review the conformable fractional derivative (Cenesiz, Baleanu, Kurt, & Tasbozan, 2017; Chung 2015).

Definition 1. Suppose \( f : [0, \infty) \to R \) be a function. Then, the conformable fractional derivative of \( f \) of order \( \alpha \), \( 0 < \alpha \leq 1 \), is defined as

\[
(T_\alpha f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}
\]

for all \( t > 0 \). Some useful properties can be listed as follows:

1. **Linearity:** \( T_\alpha (af + bg) = a(T_\alpha f) + b(T_\alpha g) \), for all \( a, b \in R \)
2. **Leibniz rule:** \( T_\alpha (fg) = fT_\alpha (g) + gT_\alpha (f) \)
3. **Let \( f \) be a differentiable and \( \alpha \)-conformable differentiable function and \( g \) be a differentiable function defined in the range of \( f \). Then

\[
T_\alpha (f'g)(t) = t^{1-\alpha} g'(t)f'(g(t)).
\]

Moreover, the following rules are hold.

\[
T_\alpha (f^p) = pf^{p-\alpha}, \quad \text{for all } p \in R
\]

\[
T_\alpha (f) = 0, \quad \text{for all constant functions } f(t) = \lambda
\]

\[
T_\alpha (f/g) = \frac{g(T_\alpha f) - f(T_\alpha g)}{g^2}.
\]

Additively, if \( f \) is differentiable, then \( T_\alpha (f(t)) = t^{1-\alpha} \frac{df}{dt}(t) \).

3. The fractional complex transformation and the modified simple equation method

Consider the fractional differential equation with conformable derivative:

\[
F(u, D_t^\alpha u, D_t^{2\alpha} u, D_t^{3\alpha} u, D_t^{4\alpha} u, \ldots) = 0, \quad 0 < \alpha < 1.
\]

To find the exact solution of Equation (1), the following fractional complex transformation can be introduced:

\[
u(x, t) = U(\xi), \quad \xi = t^{\alpha} \frac{x}{\alpha} - c^{\alpha} \frac{t}{\alpha}.
\]

where \( k \) and \( c \) are constants to be determined. Under the transformation Equation (2), we can rewrite Equation (1) in the following nonlinear ordinary differential equation

\[
Q(U, U', U'', U''', \ldots) = 0.
\]

Then we integrate Equation (3) as many times as possible with respect to \( \xi \) and set the integration constant as zero.

Firstly, according to the modified simple equation method (Kaplan, Bekir, Akbulut, & Aksoy, 2015), the exact solution of Equation (3) can be represented by a polynomial in \( \frac{\psi(\xi)}{\psi(\xi)} \) as follows

\[
U(\xi) = \sum_{n=0}^{m} a_n \left( \frac{\psi(\xi)}{\psi(\xi)} \right)^n
\]

where \( a_n \) (\( n = 0, 1, 2, \ldots, m \)) are unknown constants such that \( a_m \neq 0 \), and \( \psi \) is an unknown function of \( \xi \) to be calculated. Here the positive integer \( m \), called as the balancing number is determined by thinking the homogeneous balance principle between the highest order nonlinear term with the highest order derivative term which appears in Equation (3).

We obtain a polynomial of \( \psi^{-j}(\xi) \) with the derivatives of \( \psi(\xi) \) by substituting Equation (4) into Equation (3). Then, by equating the coefficients of \( \psi^{-j}(\xi) \) to zero, where \( j \geq 0 \), we get a system which can be solved to find \( a_n \) (\( n = 0, 1, 2, \ldots, m \)), \( c \) and \( \psi(\xi) \). Finally, we substitute the values of \( a_n \), \( k \), \( c \) and \( \psi(\xi) \) into Equation (4) to find the exact solution of Equation (1).

4. The fractional complex transformation and the exp (−Φ(ξ)) method

Secondly, we introduce the \( \exp(-\Phi(\xi)) \) method for finding different types of exact solutions to nonlinear fractional differential equations with conformable fractional derivative (Kaplan & Bekir, 2017). To reduce the considering Equation (1) to a nonlinear ordinary differential equation, we follow the same procedure.

According to the \( \exp(-\Phi(\xi)) \) method, we seek the exact solution of Equation (3) in the following form:

\[
U(\xi) = \sum_{n=0}^{m} a_n (\exp(-\Phi(\xi)))^n
\]

where \( a_n \) (\( a_m \neq 0 \)) are constants to be determined later, and \( \Phi(\xi) \) satisfies the following auxiliary ordinary differential equation:

\[
\Phi'(\xi) = \exp(-\Phi(\xi)) + \mu \exp(\Phi(\xi)) + \lambda
\]

One can know that the auxiliary equation Equation (6) has different solutions as follows:

Case 1 (Hyperbolic function solutions): When \( \lambda^2 - 4\mu > 0 \) and \( \mu \neq 0 \),

[...].
\[ \Phi_1(\zeta) = \ln \left( -\sqrt{\frac{\lambda^2 - 4\mu \tanh \left( \frac{\sqrt{\lambda^2 - 4\mu}}{2} (\zeta + C) \right)}{2\mu}} - \lambda \right) \]  

7. \[ \Phi_2(\zeta) = \ln \left( \sqrt{\frac{4\mu - \lambda^2 \tan \left( \frac{\sqrt{4\mu - \lambda^2}}{2} (\zeta + C) \right)}{2\mu}} - \lambda \right) \]  

Case 2 \textbf{(Trigonometric function solutions):} When \( \lambda^2 - 4\mu < 0 \) and \( \mu \neq 0 \),  
\[ \Phi_3(\zeta) = -\ln \left( \frac{\lambda}{\cosh(\lambda(\zeta + C)) + \sinh(\lambda(\zeta + C)) - 1} \right) \]  

Case 3 \textbf{(Hyperbolic function solutions):} When \( \lambda^2 - 4\mu > 0 \), \( \mu = 0 \) and \( \lambda \neq 0 \),  
\[ \Phi_4(\zeta) = \ln \left( \frac{2(\lambda(\zeta + C) + 2)}{\lambda^2(\zeta + C)} \right) \]  

Case 4 \textbf{(Rational function solutions):} When \( \lambda^2 - 4\mu = 0 \), \( \mu \neq 0 \) and \( \lambda = 0 \),  
\[ \Phi_5(\zeta) = \ln(\zeta + C) \]  

Here \( C \) is the integration constant. Also we balance the highest order linear term with the highest order nonlinear term in Equation (5) to find the balancing number \( N \).  

Substituting Equation (5) into Equation (3) and collecting all terms with the same order of \( \exp(-\Phi(\zeta))^n \) \( (n = 0, 1, 2, \ldots) \) together, we get a polynomial in \( \exp(-\Phi(\zeta)) \). Equating each coefficient of this polynomial to zero yields a set of algebraic equations for \( a_n, k, \lambda, \mu \) and \( c \). Solving the equation system, we can construct a variety of exact solutions for Equation (1).  

5. Applications  

5.1. Modified simple equation method  
The fractional approximate long water wave (ALW) equation is known as  
\[ D^\alpha_t u - u D^\alpha_t u - D^\beta_t v + a D^2\alpha_t u = 0, \]  
\[ D^\alpha_t v - D^\beta_t (uv) - a D^2\alpha_t v = 0. \]  

Yan has found three types of travelling wave solutions of this equation via the fractional sub-equation method (Yan, 2015). Also Guner et al. used \( (G'/G) \)-expansion method to establish the exact solutions of Equation (12). By using the transformation:  
\[ u(x,t) = U(\zeta), \quad v(x,t) = V(\zeta), \]  
\[ \zeta = k x^{2\alpha_1} - c t^{2\alpha_2}. \]  

Equation (12) reduces a nonlinear ordinary differential equation system, which reads  
\[ -cU' - kUU' - kv' + ak^2U'' = 0, \]  
\[ -cV' - k(UV)' - ak^2V'' = 0. \]  

Here prime denotes the derivative with respect to \( \zeta \). Then, we can integrate this system once with respect to \( \zeta \)  
\[ -cU - \frac{k}{2} U^2 - kv + ak^2U' = 0, \]  
\[ -cV - kUV - ak^2V' = 0. \]  

If we balance the highest-order derivative term \( U''' \) and the non-linear term \( (U')^2 \) of Equation (14), we obtain the balancing number as \( m = 1 \).  

So, we assume that our solution is in the following form:  
\[ U(\zeta) = a_0 + a_1 \left( \frac{\psi}{\psi} \right), \]  
\[ V(\zeta) = b_0 + b_1 \left( \frac{\psi}{\psi} \right)^2 + b_2 \left( \frac{\psi}{\psi} \right)^3. \]  

By substituting Equation (15) into Equation (14) and collecting all the terms with the same power of \( e^{-\Phi(\zeta)} \), \( (n = -4, -3, \ldots, 0) \). Then by equating each coefficient of the above system to zero yields a set of the following algebraic equations as follows:  

From the first equation, we find:  
\[ \psi^0 : -\frac{k a_0^2}{2} - c a_0 - k b_0 = 0, \]  
\[ \psi^1 : (-k b_1 - c a_1 - k a_0 a_1) \psi' + ak^2 a_1 \psi'' = 0, \]  
\[ \psi^2 : \left( -\frac{1}{2} k a_1^2 - ak^2 a_1 - k b_2 \right) (\psi')^2 = 0, \]  

and from the second equation, we get  
\[ \psi^0 : -k a_0 b_0 - c b_0 = 0, \]  
\[ \psi^1 : (-c b_1 - kab_1 - ka_0 b_1) \psi' - ak^2 b_1 \psi'' = 0, \]  
\[ \psi^2 : (-k a_0 b_2 - c b_2 - k a_1 b_1 + ak^2 b_1^2) (\psi')^2 - 2ak^2 b_1 \psi' = 0, \]  
\[ \psi^3 : (-ka_2 b_2 + 2ak^2 b_2) (\psi')^3 = 0. \]  

Then it is easy to obtain from the first equations of the systems above:  
\[ a_0 = 0, \quad b_0 = 0, \]  
\[ a_0 = -\frac{2c}{k}, \quad b_0 = 0, \]  
\[ a_0 = -\frac{c}{k}, \quad b_0 = \frac{c^2}{2k^2}, \]  

and  
\[ a_1 = 0, \quad b_2 = 0, \]  
\[ a_1 = -2ak, \quad b_2 = -4ak^2 k^3 \]  
\[ a_0 = -\frac{2c}{k}, \quad b_0 = 0, \]  

from the last equations of the systems. Then, we only consider the result: \( a_1 = 2ak, \quad b_2 = -4ak^2 k^3 \)
satisfied. Since \(a_1 \neq 0, b_2 \neq 0\) the first and second cases are omitted.

**Case 1:** \(a_0 = 0, b_0 = 0\).

By substituting these solutions into the remaining equations we get:

\[ b_1 = -4ac. \]

Then we find

\[ \psi(\xi) = C_1 + C_2 \exp \left( \frac{-c \xi}{ak^2} \right). \]

Therefore, we find the exact solutions of the approximate long water wave equation with conformable fractional derivative as

\[
U(\xi) = -\frac{2c}{k} \exp \left( \frac{-c \xi}{ak^2} \right), \\
V(\xi) = -\frac{4c^2 C_2 \exp \left( \frac{-c \xi}{ak^2} \right)}{k^2 \left( C_1 + C_2 \exp \left( \frac{-c \xi}{ak^2} \right) \right)^2},
\]

where \(\xi = \frac{c \xi n}{2} - c \frac{C_2}{C_1} \) (Figures 1 and 2).

**Case 2:** \(a_0 = -\frac{2c}{k}, b_0 = 0\),

By substituting these solutions into the remaining equations we get:

\[ b_1 = 4ac. \]

Then we find

\[ \psi(\xi) = C_1 + C_2 \exp \left( \frac{c \xi}{ak^2} \right). \]

Therefore, we find the exact solutions of the approximate long water wave equation with conformable fractional derivative as

\[
U(\xi) = -\frac{2c}{k} \exp \left( \frac{c \xi}{ak^2} \right), \\
V(\xi) = -\frac{4c^2 C_2 \exp \left( \frac{c \xi}{ak^2} \right)}{k^2 \left( C_1 + C_2 \exp \left( \frac{c \xi}{ak^2} \right) \right)^2}.
\]

5.2 Exp \((-\Phi(\xi))\) method

According to the \(\exp(-\Phi(\xi))\) method, we assume that the exact solutions of the approximate long water wave equation with conformable fractional derivative as

\[
U(\xi) = a_0 + a_1 \exp(-\Phi(\xi)), \\
V(\xi) = b_0 + b_1 \exp(-\Phi(\xi)) + b_2 (\exp(-\Phi(\xi)))^2.
\]

Equation (18) is substituted into Equation (12) and then all the terms with the same power of \(e^{-\Phi(\xi)}\) \((n = -3, \ldots, 0)\) are collected. By equating each coefficient to zero yields a set of the following algebraic equations respectively:

\[
e^{-2\xi} : -\frac{ka^2}{2} - kb_2 - ak^2 a_1 = 0, \\
e^{-3} : -c a_1 - kb_1 - ak^2 a_1 \lambda - ka_0 a_1 = 0, \\
e^{0} : -kb_0 - ca_0 - ak^2 a_1 \mu - \frac{ka_0^2}{2} = 0.
\]

**Figure 1.** Graph of the \(u(x,t)\) corresponding to the values \(\alpha = 0.5, 1\) from left to right when \(C_2 = -1, C_1 = 1, c = 1, k = 2, a = 4\).
Then by solving the set of algebraic equations with the aid of Maple, we get
\[
\begin{align*}
    a_0 &= 2 \frac{\lambda^2 \pm \sqrt{\lambda^4 - 4\mu}}{2} ak, \quad a_1 = 2ak, \quad b_0 = -4a^2k^2\mu, \\
    b_1 &= -4a^2k^2, \quad b_2 = -4a^2k^2, \quad k = k.
\end{align*}
\]  

(19)

and
\[
\begin{align*}
    e^{-3\xi} : -ka_1b_2 + 2ak^2b_2 &= 0, \\
    e^{-2\xi} : 2k^2ab_1 - cb_2 - ka_1b_1 - ka_0b_2 + ak^2b_3 &= 0, \\
    e^{-\xi} : -ka_1b_0 + ak^2b_1 + 2ak^2b_3 - cb_1 - ka_0b_1 &= 0, \\
    e^{3\xi} : -ka_0b_0 - cb_0 + ak^2b_1\mu &= 0.
\end{align*}
\]

Let us discuss the following different cases:

Case 1 (Hyperbolic function solutions): When \(\lambda^2 - 4\mu > 0\) and \(\mu \neq 0\),
\[
    u_1(x,t) = 2 \left( \frac{\lambda^2 \pm \sqrt{\lambda^4 - 4\mu}}{2} \right) ak
    \nonumber
    + 2ak \left( \ln \left( -\sqrt{\lambda^2 - 4\mu} \tanh \left( \frac{\sqrt{\lambda^2 - 4\mu}}{2\mu} (\xi + C) - \lambda \right) \right) \right).
\]

(20)
Figure 4. Graph of the $v(x, t)$ corresponding to the values $\alpha = 0.5, 1$ from left to right when $C_2 = -1, C_1 = 1, c = 1, k = 2, a = 4.$

$$v_1(x, t) = -4a^2k^2\mu$$

$$- 4a^2k^2\lambda \left( \ln \left( \frac{-\sqrt{\lambda^2 - 4\mu} \tanh \left( \frac{\sqrt{\lambda^2 - 4\mu}}{2} \left( \xi + C \right) \right) - \lambda}{2\mu} \right) \right)$$

$$- 4a^2k^2 \left( \ln \left( \frac{-\sqrt{\lambda^2 - 4\mu} \tanh \left( \frac{\sqrt{\lambda^2 - 4\mu}}{2} \left( \xi + C \right) \right) - \lambda}{2\mu} \right) \right)^2$$

where

$$\xi = k \frac{x^2}{a_1} - c \frac{t^2}{a_2}, c = \frac{\lambda\mu - \lambda^2 \left( \frac{1}{2} + \frac{1}{\sqrt{\lambda^2 - 4\mu}} \right) + 2\mu \left( \frac{1}{2} + \frac{1}{\sqrt{\lambda^2 - 4\mu}} \right)}{-\mu + \lambda \left( \frac{1}{2} + \frac{1}{\sqrt{\lambda^2 - 4\mu}} \right)}$$

$C$ is an integration constant.

**Case 2 (Trigonometric function solutions):** When $\lambda^2 - 4\mu < 0$, $\mu_0 = 0$.

$$u_2(x, t) = 2k^2\lambda \left( \ln \left( \frac{\sqrt{\mu - \lambda^2} \tan \left( \frac{\sqrt{\mu - \lambda^2}}{2} \left( \xi + C \right) \right) - \lambda}{2\mu} \right) \right)$$

$$+ 2ak \left( \ln \left( \frac{\sqrt{\mu - \lambda^2} \tan \left( \frac{\sqrt{\mu - \lambda^2}}{2} \left( \xi + C \right) \right) - \lambda}{2\mu} \right) \right)^2$$

$$v_2(x, t) = -4a^2k^2\mu$$

$$- 4a^2k^2\lambda \left( \ln \left( \frac{\sqrt{\mu - \lambda^2} \tan \left( \frac{\sqrt{\mu - \lambda^2}}{2} \left( \xi + C \right) \right) - \lambda}{2\mu} \right) \right)$$

$$- 4a^2k^2 \left( \ln \left( \frac{\sqrt{\mu - \lambda^2} \tan \left( \frac{\sqrt{\mu - \lambda^2}}{2} \left( \xi + C \right) \right) - \lambda}{2\mu} \right) \right)^2$$

where

$$\xi = k \frac{x^2}{a_1} - c \frac{t^2}{a_2}, c = \frac{\lambda\mu - \lambda^2 \left( \frac{1}{2} + \frac{1}{\sqrt{\lambda^2 - 4\mu}} \right) + 2\mu \left( \frac{1}{2} + \frac{1}{\sqrt{\lambda^2 - 4\mu}} \right)}{-\mu + \lambda \left( \frac{1}{2} + \frac{1}{\sqrt{\lambda^2 - 4\mu}} \right)}$$

$C$ is an integration constant.

**Case 3 (Hyperbolic function solutions):** When $\lambda^2 - 4\mu > 0$, $\mu = 0$ and $\lambda \neq 0$.

$$u_3(x, t) = 2k^2 \lambda \left( \ln \left( \frac{\lambda}{2} + \frac{\sqrt{\lambda^2}}{2} \right) \right) ak$$

$$+ 2a \left( \ln \left( \frac{\lambda}{2} + \frac{\sqrt{\lambda^2}}{2} \right) \right)^2$$

$$v_3(x, t) = -4a^2k^2\mu$$

$$- 4a^2k^2 \left( \ln \left( \frac{\lambda}{2} + \frac{\sqrt{\lambda^2}}{2} \right) \right)$$

$$+ 2ak \left( \ln \left( \frac{\lambda}{2} + \frac{\sqrt{\lambda^2}}{2} \right) \right)^2$$

where

$$\xi = k \frac{x^2}{a_1} - c \frac{t^2}{a_2}, c = \frac{\lambda\mu - \lambda^2 \left( \frac{1}{2} + \frac{1}{\sqrt{\lambda^2 - 4\mu}} \right) + 2\mu \left( \frac{1}{2} + \frac{1}{\sqrt{\lambda^2 - 4\mu}} \right)}{-\mu + \lambda \left( \frac{1}{2} + \frac{1}{\sqrt{\lambda^2 - 4\mu}} \right)}$$

$C$ is an integration constant.

**Case 4 (Rational function solutions):** When $\lambda^2 - 4\mu = 0$, $\mu \neq 0$ and $\lambda \neq 0$.

$$u_4(x, t) = \lambda ak + 2ak \left( \ln \left( \frac{-2(\lambda + \xi + C) + 2}{\lambda^2(\xi + C)} \right) \right)$$

$$v_4(x, t) = -4a^2k^2\mu - 4a^2k^2\lambda \left( \ln \left( \frac{-2(\lambda + \xi + C) + 2}{\lambda^2(\xi + C)} \right) \right)$$

$$+ 4a^2k^2 \left( \ln \left( \frac{-2(\lambda + \xi + C) + 2}{\lambda^2(\xi + C)} \right) \right)^2$$

where

$$\xi = k \frac{x^2}{a_1} - c \frac{t^2}{a_2}, c = \frac{\lambda\mu - \lambda^2 \left( \frac{1}{2} + \frac{1}{\sqrt{\lambda^2 - 4\mu}} \right) + 2\mu \left( \frac{1}{2} + \frac{1}{\sqrt{\lambda^2 - 4\mu}} \right)}{-\mu + \lambda \left( \frac{1}{2} + \frac{1}{\sqrt{\lambda^2 - 4\mu}} \right)}$$

$C$ is an integration constant.
Case 5: When $\lambda^2 - 4\mu = 0$, $\mu = 0$ and $\lambda = 0$, $c$ yields an absurd solution. Hence, the case is discarded.

Note that, our solutions are different from the given ones in (Guner, Atik, & Aytugan Kayyryzhanovich, 2017). Also, we can say that the solutions obtained in this paper via two different methods are different from each others. If we compare them, the solution process in the modified simple equation method is difficult from the $\exp(-\Phi(\xi))$ method and it gives more fresh solutions. On the other hand, $\exp(-\Phi(\xi))$ method gives the exact solutions with different forms.

6. Conclusions

In this work, by use of the fractional calculus for conformable fractional derivative, and the modified simple equation and a new $\exp(-\Phi(\xi))$ method is proposed to seek exact solutions of the fractional differential equation systems. The modified simple equation method has proved its validity in that more fresh solutions for fractional partial differential equation systems can be obtained. Since, the auxiliary equation of this method is not a solution of any predefined functions. Moreover, by using the $\exp(-\Phi(\xi))$ method more exact solutions with different forms including solitary wave solutions, periodic wave solutions and rational solutions. As an application the approximate long water wave equation has been considered and abundant exact solutions are verified.

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No potential conflict of interest was reported by the authors.

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