Supersymmetric Completion of an $R^2$ Term in Five-Dimensional Supergravity

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Abstract

We analyze the structure of a particular higher derivative correction of five-dimensional ungauged and gauged supergravity with eight supercharges. Specifically, we determine all the purely bosonic terms which are connected by the supersymmetry transformation to the mixed gauge-gravitational Chern-Simons term, $W \wedge \text{tr} R \wedge R$. Our construction utilizes the superconformal formulation of supergravity. As an application, we determine the condition for the supersymmetric anti-de Sitter vacuum with this term. We check that it gives precisely the same condition as the $a$-maximization in four-dimensional superconformal field theory on the boundary, as predicted by the AdS/CFT correspondence.
§1. Introduction

Five-dimensional (5d) supergravity with the minimal number\(^\ast\) of supersymmetries, which was first presented in Refs.1) and 2), has applications in our attempt to obtain an understanding of the fundamental properties of nature. The theory itself and the theory coupled to a four-dimensional brane are natural starting points of studies if one wants to consider a supersymmetric model with one large extra dimension. It can also be used for studying the superconformal field theory in four dimensions, via the celebrated anti-de Sitter/conformal field theory (AdS/CFT) correspondence. Also, recently, rich exact solutions of the theory, including black rings, have been obtained. Thus, it has been found that this theory can be used as a testbed for the dynamical study of gravity.

The Lagrangian of 5d supergravity, however, cannot be thought of as giving ultraviolet-complete definition of the theory, since it is not renormalizable. Thus it must be embedded in a consistent theory of quantum gravity, say a suitable compactification of string theory, and the supergravity Lagrangian should be regarded as an effective description of the low energy limit of such theories. Therefore, the Lagrangian should contain various higher derivative terms, such as a curvature-squared term, \(R^2\), with a small coefficient. Thus, it is of utmost importance to determine how local supersymmetry governs the higher derivative terms.

Similar problems have been previously studied in six-dimensional\(^3\) and four-dimensional cases. (See the excellent review in Ref.4) and references therein.) In those works, the superconformal formalism for supergravity was used to facilitate the analysis. This was useful because the formalism is fully off-shell, and thus the analysis of the higher derivative terms can be done without modifying the supersymmetry transformation laws.

The first objective of this paper is to construct a supersymmetric \(R^2\) term in 5d \(\mathcal{N} = 2\) supergravity using the superconformal formalism developed in Refs.5)–8). More specifically, we study the supersymmetric completion of a distinctive class of higher derivative terms in five dimensions. This class is represented by the mixed gravitational Chern-Simons term

\[
W \wedge \text{tr} R \wedge R,
\]

where \(W\) is a \(U(1)\) gauge field and \(R\) is the curvature two-form constructed from the metric. While we spell out how one can determine all of the terms in the supersymmetric completion, we explicitly determine only the purely bosonic terms. The terms we obtain should suffice in the study of purely bosonic backgrounds and their properties under supersymmetry transformations. Thus we believe that our results will be of great use.

\(^{\ast}\) Minimal supersymmetry in five dimensions has eight supercharges. This form of supersymmetry is usually called \(\mathcal{N} = 2\) in the supergravity literature and \(\mathcal{N} = 1\) in the field theory literature. We call it \(\mathcal{N} = 2\), following the supergravity convention.
Although we have not been able to prove that it is the only possible form of the four-derivative correction, we believe the uniqueness of the result, judging from the structure of the formalism. Some of the bosonic terms in the completion were known before using the compactification of the $R^4$ term in M-theory down to five dimensions. Our result is consistent with theirs.

As an application, we study how the condition for the maximally supersymmetric AdS solution is modified by the completed $R^2$ term. We see that it can be mapped to the $a$-maximization of the conformal field theory on the boundary.

This paper is organized as follows. In §2, we give a brief introduction to the superconformal formalism in five dimensions. In §3, we construct the supersymmetric completion of the $W \wedge \text{tr} R \wedge R$ term using the material reviewed in §2. Then, the analysis of the AdS solution with the $R^2$ term is presented in §4. Finally, we compare the result to the $a$-maximization of the boundary theory in §5. Section 6 gives a summary and discussion.

In this paper, we mostly follow the convention of Ref.6). One difference is that we explicitly impose constraints to express dependent gauge fields in terms of composites of independent ones. This greatly simplifies the supersymmetry transformation laws and the expression for the supercovariant curvatures. In the hope of making this paper accessible to all readers, we provide appendices detailing the notation and definitions: Appendix A contains our conventions, Appendix B collects definitions and useful formulae regarding the Weyl multiplet, and Appendix C compares various conventions for vector multiplets in the literature.

§2. Brief review of the superconformal formalism

Ordinary supergravity theories are invariant under general coordinate transformations, local Lorentz transformations and local supertranslations. Thus, they are, in a sense, the gauge theory of the super-Poincaré group. For this reason, they are called Poincaré supergravities. The construction of such theories was originally done by following the Noether procedure, which yields on-shell actions.

The superconformal approach to Poincaré supergravity starts from the construction of theories which are gauge invariant under a much larger group, the superconformal group in respective dimensions. Then, by imposing constraints, these theories are identified as gravitational theories, so-called conformal supergravities. The enlargement of the local symmetry greatly facilitates the determination of the multiplet structure and the construction of the invariant action. As we are not interested in conformal supergravities themselves, we arrange one of the scalar fields to take a non-zero vacuum expectation value (VEV), spontaneously
breaking the conformal supergravity down to the Poincaré supergravity.

This off-shell approach is particularly suited to the construction of the higher derivative terms in the supergravity theories. This is because the superconformal multiplet contains the auxiliary fields, and hence the supersymmetry transformation law is independent of the action. To find a higher derivative term in the on-shell approach, one needs to consider the modification of the action and of the supersymmetry transformation simultaneously. This makes it quite difficult to carry out. We see below that having a local superconformal invariance makes the analysis of the symmetry of the AdS background surprisingly transparent.

The four main ingredients required to write down the action are the following:

1. the structures of various superconformal multiplets,
2. the “embedding formulae” which create new multiplets from existing multiplets,
3. the “invariant action formulae” to form the Lagrangian density,
4. and the gauge fixing down to the Poincaré supergravity.

We review each of these in turn.

2.1. **Superconformal multiplets**

The superconformal algebra relevant for 5d \( \mathcal{N} = 2 \) supergravity is the supergroup \( F(4) \), with the generators

\[
P_a, \quad Q_i, \quad M_{ab}, \quad D, \quad U_{ij}, \quad S^i, \quad K_a, \quad (2.1)
\]

where \( a, b, \ldots \) are Lorentz indices, and \( i, j, \ldots (= 1, 2) \) are for the SU(2) doublets. Here we have suppressed the spinor indices. The operators \( P_a \) and \( M_{ab} \) are the usual Poincaré generators, \( D \) is the dilatation, \( U_{ij} \) is the SU(2) generator, \( K_a \) represents special conformal boosts, \( Q_i \) is the \( \mathcal{N} = 2 \) supersymmetry, and \( S_i \) is the conformal supersymmetry. The charge of the field with respect to the dilatation \( D \) is called its Weyl weight. We introduce the gauge fields

\[
e^a_{\mu}, \quad \psi^i_{\mu}, \quad \omega^{ab}_{\mu}, \quad b_\mu, \quad V^{ij}_{\mu}, \quad \phi^{i}_{\mu}, \quad f^a_{\mu}, \quad (2.2)
\]
corresponding, respectively, to the generators above where \( \mu, \nu, \ldots \) are the world vector indices and \( \psi^i_{\mu} \) and \( \phi^{i}_{\mu} \) are SU(2)-Majorana spinors. The definitions of the covariant derivatives and curvatures are given in Appendix B. We first write down Yang-Mills theory for this gauge group \( F(4) \). Up to this point, the generators \( P_a \) and \( M_{ab} \) have not been related to the diffeomorphism, and the transformation law for various gauge fields follows from the structure constants of \( F(4) \).

Next, we impose the so-called conventional constraints to identify the generators \( P_a \) and \( M_{ab} \) as those of the general coordinate transformation and the local Lorentz transformation:

\[
\hat{R}_{\mu\nu}^a(P) = 0, \quad \gamma^\mu \hat{R}_{\mu\nu}^i(Q) = 0, \quad \hat{R}_\mu^a(M) = 0 \quad (2.3)
\]
where the hat denotes supercovariantization, and the curvature with respect to a generator \( X_A \) is written as \( \hat{R}_{\mu
u}^A(X) \). These allow us to express the \( M \), \( S \) and \( K \) gauge fields \( \omega_{\mu}^{ab} \), \( \phi_{\mu}^i \) and \( f_{\mu}^a \) in terms of composite fields constructed out of other gauge fields. The local \( P, Q \) transformation law needs to be modified to preserve the constraints (2.3), after which \( e_{\mu}^a \) and \( \omega_{\mu}^{ab} \) can be identified with the usual fünfbein and the spin connection, respectively. The \( \{Q, Q\} \) commutator is also modified from that of \( F(4) \), and it is presented below in (2.3). As argued above, the \( P \) transformation becomes essentially the general coordinate transformation \( \delta_{GC}(\xi^A) \):

\[
\delta_P(\xi) = \delta_{GC}(\xi^A) - \delta_A(\xi^A h^A_\chi).
\]

(2.4)

On a covariant quantity \( \Phi \) with only flat indices, \( \delta_P(\xi) \) acts as the full covariant derivative:

\[
\delta_P(\xi)\Phi = \xi^a \left( \partial_a - \delta_A(h^a_\chi) \right) \Phi \equiv \xi^a \hat{D}_a \Phi.
\]

(2.5)

It is sometimes called the covariant general coordinate transformation.

Next we summarize the structure of the multiplets we use. Their properties are listed in Table I of Appendix A.

2.1.1. The Weyl multiplet

We add auxiliary fields \( \nu^{ab}, \chi^i \) and \( D \) to the set of gauge fields above to obtain an irreducible Weyl multiplet, which consists of 32 bosonic plus 32 fermionic component fields, \( e_{\mu}^a, \psi_{\mu}^i, V_{\mu}^{ij}, b_{\mu}, v^{ab}, \chi^i, D \),

(2.6)

where \( v^{ab} \) is antisymmetric in \( a \) and \( b \), \( \chi^i \) is an \( SU(2) \)-Majorana spinor, and \( D \) is a scalar. The \( Q, S \) and \( K \) transformation laws for the Weyl multiplet are as follows [with \( \delta \equiv \bar{\epsilon}^i Q_i + \bar{\eta}^a S_a + \xi_K^a K_a \equiv \delta_Q(\varepsilon) + \delta_S(\eta) + \delta_K(\xi_K^a)]:

\[
\begin{align*}
\delta e_{\mu}^a &= -2i\bar{\varepsilon}^{a} \gamma^a \psi_{\mu}, \\
\delta \psi_{\mu}^i &= D_{\mu} \varepsilon^i + \frac{1}{2} v^{ab} \gamma_{\mu ab} \varepsilon^i - \gamma_{\mu} \bar{\eta}^i, \\
\delta b_{\mu} &= -2i\bar{\varepsilon}^i \phi_{\mu} - 2i\bar{\eta} \psi_{\mu} - 2\xi_{\mu}, \\
\delta V_{\mu}^{ij} &= -6i\bar{\varepsilon}^{(i} \psi_{\mu}^{j)} + 4i\bar{\varepsilon}^{(i} \gamma \cdot v_{\mu}^{j)} - 4i \bar{\varepsilon}^{(i} \gamma_{\mu} \chi^{j)} + 6i \bar{\eta}^{(i} \psi_{\mu}^{j)} , \\
\delta v^{ab} &= -4i \bar{\varepsilon}^i \gamma_{ab} \chi - \frac{2}{3} i \bar{\varepsilon} \hat{R}_{ab}(Q), \\
\delta \chi^i &= D \varepsilon^i - 2\gamma^c \gamma_{ab} \varepsilon^i D_{ab} v_{bc} + \gamma \cdot \hat{R}(U)^i_{j} \varepsilon^j - 2\gamma^a \varepsilon^i \varepsilon_{a b c d e} b_{b c} v_{d e} + 4\gamma \cdot v_{\chi}^{i}, \\
\delta D &= -i \bar{\varepsilon} \hat{\Phi}_{\chi} - 8i \bar{\varepsilon} \hat{R}_{ab}(Q) v^{ab} + i\bar{\eta} \chi,
\end{align*}
\]

(2.7)

As discussed in Ref.6), the second and third constraints in (2.3) are avoidable and we can keep the gauge fields \( f_{\mu}^a \) and \( \phi_{\mu}^i \) independent. Here, we impose the constraints to obtain simpler transformation laws.
where the derivative $\mathcal{D}_\mu$ is covariant only with respect to the homogeneous transformations $M_{ab}, D$ and $U^{ij}$. The dot product $\gamma \cdot T$ for a tensor $T_{ab...}$ generally represents the contraction $\gamma^{ab...} T_{ab...}$. When the $SU(2)$ indices are suppressed in bilinear terms of spinors, the northwest-southeast contraction, $\bar{\psi}^a \gamma^{a1...an} \lambda = \bar{\psi}^i \gamma^{a1...an} \lambda_i$, is understood. The algebra of the $Q$ and $S$ transformations takes the form

$$
\begin{align*}
[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)] &= \delta_p (2i\bar{\varepsilon}_1 \gamma a \varepsilon_2) + \delta_M (2i\bar{\varepsilon}_1 \gamma^{abcd} \varepsilon_2 v_{ab}) + \delta_U (-4i\bar{\varepsilon}_1 \gamma \cdot v \varepsilon_2) \\
&+ \delta_S (\cdots) + \delta_K (\cdots), \\
[\delta_S(\eta), \delta_Q(\varepsilon)] &= \delta_D (-2i\bar{\varepsilon} \eta) + \delta_M (2i\bar{\varepsilon} \gamma^{ab} \eta) + \delta_U (-6i\bar{\varepsilon}^{(i} \eta^{j)}) \\
&+ \delta_K (\cdots),
\end{align*}
$$

where the translation $\delta_p (\xi^a)$ is defined in (2.4). We summarize the useful formulae for the supercovariant derivatives and curvatures in Appendix [B].

One particular point which is relevant in the analysis given in subsequent sections is that the third constraint in (2.3) makes the supercovariant curvature $\hat{R} (M)_{abcd}$ traceless. Thus, for a background in which the nontrivial component of the Weyl multiplet is only the fünfbein, $\hat{R}_{\mu\nu}^{ab} (M)$ is the Weyl tensor of the metric, i.e.

$$
\hat{R}_{\mu\nu}^{ab} (M) = R_{\mu\nu}^{ab} + \frac{4}{3} R_{[a \epsilon_{\mu}]}^{[a \epsilon_{b}] b} - \frac{1}{6} c_{[a \epsilon_{\mu}]}^{[a \epsilon_{b}]} R,
$$

where $R_{abcd}$ is the ordinary curvature tensor constructed from the metric.

### 2.1.2. Vector multiplet

The vector multiplet consists of

$$
W_I^\mu, \quad M^I, \quad \Omega^I, \quad Y_I^{ij},
$$

where the index $I$ labels the generators $T_I$ of the gauge group $G$. Here, $W_I^\mu$ are the gauge fields, $M^I$ are the scalar fields in the vector multiplet, $\Omega^I$ are the $SU(2)$-Majorana gaugini, and $Y_I^{ij}$ are $SU(2)$-triplet auxiliary fields. We set $W_\mu \equiv W_I^\mu T_I$, and similarly for other components. The $Q$ and $S$ transformation laws of the vector multiplet are then given by

$$
\begin{align*}
\delta W_\mu &= -2i\bar{\varepsilon} \gamma_\mu \Omega + 2i\bar{\varepsilon} \psi_\mu M, \\
\delta M &= 2i\bar{\varepsilon} \Omega, \\
\delta \Omega^i &= -\frac{1}{4} \gamma \cdot \hat{\mathcal{F}} (W) \varepsilon^i - \frac{1}{2} \hat{\mathcal{D}} M \varepsilon^i + Y^i j \varepsilon^j - M \eta^i, \\
\delta Y^{ij} &= 2i\bar{\varepsilon}^{(i} \hat{\mathcal{D}} \Omega^{j)} - i\bar{\varepsilon}^{(i} \gamma \cdot v \Omega^{j)} - \frac{i}{4} \bar{\varepsilon}^{(i} \chi^{j)} M - 2ig\bar{\varepsilon}^{(i} [M, \Omega^{j)]} - 2i\bar{\eta}^{(i} \Omega^{j)}.
\end{align*}
$$

The transformation law of the gauge field $W_\mu$ above shows that the superconformal group and the gauge group $G$ are not separate but have non-zero structure functions, $f_P Q^G$ and
\( f_{QQ}^G \), between them. For consistency, it is thus required that the commutator of two \( Q \) transformations is modified to

\[
[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)] = (\text{R.H.S. of (2.8)}) + \delta_G(-2i\bar{\varepsilon}_1\varepsilon_2M).
\] (2.13)

Thus, the supercovariant curvature is given by

\[
\hat{F}_{\mu
u}(W) = 2\partial_{[\mu}W_{\nu]} - g[W_{\mu}, W_{\nu}] + 4i\bar{\psi}_{[\mu}\gamma_{\nu]}\Omega - 2i\bar{\psi}_{\mu}\psi_{\nu}M.
\] (2.14)

2.1.3. Hypermultiplet

The hypermultiplet in 5D consists of scalars \( A^i_\alpha \), spinors \( \zeta_\alpha \) and auxiliary fields \( F^i_\alpha \). They carry the index \( \alpha (= 1, 2, \ldots, 2r) \) of \( USp(2r) \). The scalars satisfy the reality condition \( A^i_\alpha = -(A^i_\beta)^* \), and the spinors \( \zeta_\alpha \) are \( USp(2r) \)-Majorana. A subgroup \( G' \) of the gauge group \( G \) can act on the index \( \alpha \) as a subgroup of \( USp(2r) \). The \( Q \) and \( S \) transformations of \( A^i_\alpha \) and \( \zeta_\alpha \) are given by

\[
\delta A^i_\alpha = 2i\bar{\varepsilon}^i\zeta_\alpha,
\]
\[
\delta \zeta_\alpha = \bar{\mathcal{D}}A^0_\alpha \varepsilon^j - \gamma \cdot v \varepsilon^j A^0_\alpha \varepsilon^j - gM_\alpha A^0_\alpha \eta^j + 3A^0_\alpha \eta^j,
\] (2.15)

where \( \bar{\mathcal{D}} \) and \( M_\alpha \) include the ‘central charge’ gauge transformation \( Z \). The quantity \( g \) is the coupling constant, and the notation \( X_\alpha Y \) represents generator of the gauge transformation,

\[
(X_\alpha Y)^\alpha = X^I t^\alpha_\beta Y^\beta + X^0 Z Y^\alpha,
\] (2.16)

where \( X \) takes values in a Lie algebra, \( Y \) takes values in its representation, and \( t^\alpha_\beta \) is the representation matrix. The closure of the algebra thus determines the ‘central charge’ gauge transformation of \( A^i_\alpha \) via \( F^i_\alpha \), though we set \( Z = 0, (F^i_\alpha = 0) \) in this paper. (Interested readers are referred to Refs.5 and 6) for details.)

2.1.4. Linear Multiplet

A linear multiplet consists of

\[
L^{ij}, \quad \varphi^i, \quad E_a, \quad N,
\] (2.17)

where \( L^{ij} \) is symmetric in \( i \) and \( j \) and is real, \( \varphi^i \) is \( SU(2) \)-Majorana, \( E_a \) is a vector, and \( N \) is a scalar.

The \( Q \) and \( S \) transformation laws of the linear multiplet are given by

\[
\delta L^{ij} = 2i\bar{\varepsilon}^{(i}\varphi^{j)},
\]
\[
\delta \varphi^i = -\bar{\mathcal{D}}L^{ij} \varepsilon_j + \frac{1}{2} \gamma^a \varepsilon^i E_a + \frac{1}{2} \varepsilon^i N
\]
are given by $f$ homogeneous polynomial $2.2$. Embedding formulae from the lowest component, $L$ of a linear multiplet can be carried out by repeated supersymmetric transformations starting the above transformation law with suitable choices of $\phi$ opposite bosonic field $L_I, J,...$ where a scalar function with the subscripts $E$.

The algebra closes if $E^a$ satisfies the following $Q$- and $S$-invariant constraint:

$$\hat{D}_a E^a + g M_s N + 4ig \hat{\Omega}_s \varphi + 2g Y^i j L_{ij} = 0. \quad (2.19)$$

An important property concerning the linear multiplet is that any symmetric, real composite bosonic field $L_{ij}$, which is invariant under $S$ transformations, automatically leads to the above transformation law with suitable choices of $\varphi^i, E^a$ and $N$. Thus, the construction of a linear multiplet can be carried out by repeated supersymmetric transformations starting from the lowest component, $L_{ij}$.

2.2. Embedding formulae

Vector multiplets can be embedded into a linear multiplet given an arbitrary quadratic homogeneous polynomial $f(M)$ of the first components $M^f$ of the vector multiplets. They are given by

$$L_{ij}(V) = Y^i_j f_I - i \hat{\Omega}_i^I \Omega^j_I f_{IJ},$$

$$\varphi_i(V) = -\frac{1}{4} \chi_i f$$

$$+ \left( \hat{\Phi} \Omega_i^I - \frac{1}{2} \gamma \cdot v \Omega_i^I - g [M, \Omega]^I \right) f_I$$

$$+ \left( -\frac{1}{4} \gamma \cdot \hat{\Phi}^I(W) \Omega^I + \frac{1}{2} \hat{\Phi} M^I \Omega^I - Y^I \Omega^I \right) f_{IJ},$$

$$E_a(V) = \hat{D}^b \left( 4v_{ab} f + \hat{F}_{ab}^I(W) f_I + i \hat{\Omega}_i^I \gamma_{ab} \Omega^j_I f_{IJ} \right)$$

$$+ \left( -2ig [\hat{\Omega}, \gamma_{ab} \Omega]^I + g [M, \hat{D}_a M]^I \right) f_I$$

$$+ \left( -2ig \hat{\Omega}_i^I \gamma_{ab} [M, \Omega]^J + \frac{1}{8} \epsilon_{abce} \hat{F}^{bcI}(W) \hat{F}^{dej}(W) \right) f_{IJ},$$

$$N(V) = -\hat{D}^a \hat{D}_a f + \left( -\frac{1}{2} D - 3v^2 \right) f$$

$$+ \left( -2\hat{\Phi}_{ab}(W) v_{ab} + i \chi \Omega^I + 2ig [\hat{\Omega}, \Omega]^I \right) f_I$$

$$+ \left( -\frac{1}{4} \hat{\Phi}_{ab}^I(W) \hat{F}_{abc}(W) + \frac{1}{2} \hat{D}_a M^I \hat{D}_a M^J$$

$$+ 2i \hat{\Omega}_i^I \hat{\Phi} \Omega^J - i \hat{\Omega}_i^J \gamma \cdot \Omega^I + Y_{ij}^I Y^j_{ij} \right) f_{IJ}, \quad (2.20)$$

where a scalar function with the subscripts $I, J, \ldots$ represent for its repeated derivative with respect to $M^I, \ldots$. For example, $f_{IJ} \equiv \partial_I \partial_J f$. We often use the notation $v^2$ for $v_{ab} v^{ab}$ in this paper. One can also form a linear multiplet from two hypermultiplets.
2.3. Invariant action formulae

We can construct an invariant action from a pair of vector and linear multiplets as

\[
e^{-1} \mathcal{L}(V \cdot L) \equiv Y^{ij} \cdot L_{ij} + 2i\bar{\Omega} \cdot \varphi + 2i\bar{\psi}_i^a \gamma_i \Omega_j \cdot L_{ij}
\]

\[
- \frac{1}{2} W_a \cdot \left( E^a - 2i\bar{\psi}_b \gamma^{ba} \varphi + 2i\bar{\psi}_b^{(i} \gamma^{j)ab} \psi_c^j L_{ij} \right)
\]

\[
+ \frac{1}{2} M \cdot \left( N - 2i\bar{\psi}_b \gamma^b \varphi - 2i\bar{\psi}_c^{(i} \gamma^{j)ab} \psi_c^j L_{ij} \right),
\]

where we have restricted our consideration to the case that \( L_{ij} \) is neutral, for simplicity.

As we have seen, one can form a linear multiplet from two vector multiplets \( V^I \) and \( V^J \) by using the embedding formula. Then, the invariant action formula above can combine it with another vector multiplet \( V^K \) to form an action. The resulting action is completely symmetric in \( I, J \) and \( K \). Thus, we obtain an invariant action \( \mathcal{L}_V \) given a gauge-invariant cubic function \( \mathcal{N} = c_{IJK} M^I M^J M^K \). For brevity, we consider the case \( G = U(1)^{n_v+1} \). Then the bosonic term is

\[
e^{-1} \mathcal{L}_V \big|_{\text{bosonic}} = \mathcal{N} \left( -\frac{1}{2} D + \frac{1}{4} R(M) - 3v^2 \right) + \mathcal{N}_I \left( -2v^a F_{ab}(W) \right)
\]

\[
+ \mathcal{N}_{IJ} \left( \frac{1}{4} F_{ab}^I(W) F_{ab}^J(W) + \frac{1}{2} D^a M^I D_a M^J + Y_{ij} Y^{ij} \right)
\]

\[
- e^{-1} \frac{1}{24} \epsilon^{\lambda \mu \nu \rho} \mathcal{N}_{IJK} W_\lambda W_\mu F_{\rho \sigma}^I(W) F^J_{\rho \sigma}(W).
\]

(Note the appearance of the gauge Chern-Simons interaction, \( W \wedge F \wedge F \), which came from the \( W_a \cdot E^a \) term in the invariant action formula. The strength of the Chern-Simons interaction directly gives the function \( \mathcal{N} \). Thus, it governs the entire vector-multiplet Lagrangian.)

For the hypermultiplets, the combination of the embedding into the linear multiplet and the \( V \cdot L \) action formula gives an action with the following bosonic terms:

\[
e^{-1} \mathcal{L}_H \big|_{\text{bosonic}} = D^a A^i_a D_a A^i + A^i_a (g M)^2 A^i_a + \mathcal{A}^2 \left( \frac{1}{8} D + \frac{8}{16} R(M) - \frac{1}{4} v^2 \right) + 2g Y_{ij} A^i_a A^j_a,
\]

where \( \mathcal{A}^2 \equiv A^i_a A^i_a = A^i_a d_{\alpha}^a A^i_a \) with the metric \( d_{\alpha}^a \) arranged to be \( \delta_{\beta}^\alpha \) for a compensator. We have already eliminated the auxiliary fields \( F_{i\alpha} \) using their equations of motion.

2.4. Gauged supergravity

Let us now consider a system coupled to \( n_v+1 \) conformal vector multiplets, \( I = 0, \ldots, n_v \), and one conformal hypermultiplet, \( A^i_\alpha \) \( (i, \alpha = 1, 2) \), as a compensator. We let its action be \( \mathcal{L}_0 = \mathcal{L}_H - \frac{1}{2} \mathcal{L}_V \). The equation of motion for \( D \) gives \( \mathcal{A}^2 + 2\mathcal{N} = 0 \), while the scalar curvature appears in the Lagrangian in the form

\[
(\frac{3}{16} \mathcal{A}^2 - \frac{1}{8} \mathcal{N}) R(M).
\]

(2.24)
Thus, we can make the Einstein-Hilbert term canonical by fixing the dilatational gauge transformation \( D \) via the condition \( \mathcal{A}^2 = -2 \). It also fixes \( \mathcal{N} = c_{IJK} M^I M^J M^K = 1 \) via the equation of motion for \( D \). Thus, the scalars parametrize a ‘very special’ manifold.\(^1\)

It is known that the AdS background requires gauged supergravity, which is obtained by introducing a charged compensator. Therefore, let us consider a model with charges \( G^I A^a_i = P_I (i \sigma^3)^{\alpha \beta} A^a_{ij} \) and fix the \( U \)-gauge transformation by \( A^a_{i} = \delta^a_i \). Under this fixing, only the combination

\[
\delta_G^I(A^I) = \delta_G(A^I) + \delta_U \left( A^I P_I (i \sigma^3)^{ij} \right)
\]

(2.25)
of \( U(1) \) gauge and \( U \) transformations survives. In this model, the vectors \( W^I_{\mu} \) are coupled to the hypermultiplet via

\[
\mathcal{D}_{\mu} A^a_{i} = \partial_{\mu} A^a_{i} - W^I_{\mu} P_I (i \sigma^3)^{\alpha \beta} A^a_{ij} + A^a_{i} V^{ij}_{\mu},
\]

(2.26)

where \( V^{ij}_{\mu} \) is the gauge field for the \( U \) transformation. The equation of motion for \( V^{ij}_{\mu} \) and the condition \( A^a_{i} = \delta^a_i \) require

\[
V^{ij}_{\mu} = P_I (i \sigma^3)^{ij} W^I_{\mu},
\]

(2.27)

which is preserved under the transformations \( \delta_G^I(A) \). Thus, any \( SU(2)_R \) doublet becomes effectively charged with respect to the vectors \( W^I_{\mu} \) through (2.27), with the charges \( P_I \).

The auxiliary fields \( v_{ab} \) and \( Y^{I}_{ij} \) are determined to be

\[
v_{ab} = -\mathcal{N}_I F^I_{ab} / 4 \mathcal{N}, \quad Y^{I}_{ij} = 2 (\mathcal{N}^{-1})^{IJ} P_I (i \sigma^3)^{ij},
\]

(2.28)

where \( (\mathcal{N}^{-1})^{IJ} \) is the inverse of \( \mathcal{N}_{IJ} \). Then, the scalar potential \( V \) is given by

\[
V = -4 (\mathcal{N}^{-1})^{IJ} P_I P_J - 2 (P_I M^I)^2.
\]

(2.29)

By changing the convention to that employed by Günaydin, Sierra and Townsend in Refs.1) and 2) via the dictionary given in in Appendix C and using various identities of the very special geometry, it can be shown that

\[
V = 3 g^{xy} \partial_x h^I \partial_y h^J P_I P_J - 4 (P_I h^I)^2,
\]

(2.30)

which is the usual form presented in the supergravity Lagrangian in the on-shell formalism.

The above procedure reproduces the structure of 5d \( \mathcal{N} = 2 \) gauged supergravity in the on-shell formalism, as should be the case. We use the action \( \mathcal{L}_H - \frac{1}{2} \mathcal{L}_V \) as the zeroth-order term, where we add the \( R^2 \) term to be constructed below.

\(^{10}\) We usually take the \( D \) gauge field \( b_{\mu} \) to be zero through \( K \) gauge fixing.
§3. Construction of a supersymmetric $R^2$ term

3.1. Strategy

Before moving on, we need to make a few comments on the physical interpretation of the higher derivative terms, in particular in the off-shell formalism. Firstly, if we naively apply the variational method to obtain the equation of motion from a higher derivative theory, it results in a differential equation which is higher than second order. This means that giving the value and the first derivative of a field does not suffice as initial values. In other words, there are ‘extra modes’ in addition to the modes of the two-derivative Lagrangian. This is inevitable if we take the Lagrangian as giving an ultra-violet definition.

However, we regard our Lagrangian to be the effective low-energy description in a derivative expansion with a small expansion parameter $\alpha'$. Thus, the solution to the equation of motion should take the form of a perturbative expansion in $\alpha'$, and, in particular, its $\alpha' \to 0$ limit should exist. Such solutions are known to be determined by the value and the first derivative of a field at $t = 0$, just as in the case with two-derivative Lagrangian, making the ‘extra modes’ mentioned above unphysical. (The details can be found, for example, in Refs.12 and 13.)

Secondly, it is readily checked that the auxiliary fields would appear with physical kinetic terms and begin to propagate when one constructs higher derivative terms in the off-shell formalism. It is known, however, that the auxiliary fields can be eliminated perturbatively in $\alpha'$ (see e.g. the introduction of Ref.14) to produce many higher derivative terms in the physical fields. The resulting Lagrangian is to be understood as explained in the previous paragraph. Thus, the would-be propagating auxiliary fields are just the ‘extra modes’ associated with the higher derivative terms, and they are not to be regarded as physical fields.

The third comment is of a slightly different nature. In the higher derivative theory of gravity, one can redefine the metric as

$$g_{\mu\nu} \to g_{\mu\nu} + a R g_{\mu\nu} + b R_{\mu\nu} + \cdots,$$

(3.1)

with $a$ and $b$ small parameters. This leaves the leading-order Einstein-Hilbert term intact, while changing the form of the higher-order derivative terms. For example, it can be used to arbitrarily shift the coefficients of $R^{\mu\nu} R_{\mu\nu}$ and $R^2$, while that of $R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}$ cannot be shifted. It should also change the supersymmetry transformation law. The physics described by the Lagrangian, of course, remains the same under the redefinition. We need to use a redefinition to compare our results to those in literature.

Below, we construct a very specific higher derivative term, whose form is not preserved
by (3.1). This is because we use a very specific form of the supersymmetry transformation dictated by the superconformal formalism. Change in the conventional constraints (2.3) also induces a field redefinition among the fields in the Weyl multiplets without altering physical contents of the theory. Our choice of the constraint $\hat{R}^a_\mu = 0$ is a convenient one because it greatly reduces the number of higher derivative terms to consider by forbidding the appearance of the terms like $\hat{R}_{ab}\hat{R}^{ab}$ or $\hat{R}^2$.

With these preliminary remarks, we set out to construct a supersymmetric curvature-squared term in 5d $\mathcal{N} = 2$ supergravity. More precisely, we obtain the supersymmetric completion of the mixed gauge-gravitational Chern-Simons term,

$$\epsilon^{abcdef}W_a^I R_{bcfg}R_{de}^{fg}. \quad (3.2)$$

We recall that the gauge Chern-Simons term in (2.22) arises from the $W_a \cdot E^a$ term in the $V \cdot L$ invariant action formula. Judging from the similarity of the roles played by the gauge curvature $F_{ab}^I$ and the metric curvature $R_{abcd}$, a natural guess would be to first embed the Weyl multiplet into a vector multiplet $V^{cd}[W]$ with extra antisymmetric Lorentz indices $c$ and $d$, and then to construct a linear multiplet from the $L(V^I, V^J)$ embedding formula. However, we have found that this method is not significantly better than the direct construction of the linear multiplet. Therefore, our strategy is as follows:

1. embed the Weyl multiplet to the linear multiplet;
2. use the $L(V \cdot L)$ invariant action formula;
3. gauge-fix down to the Poincaré supergravity;

We believe that the combination we determine is the most general form of supersymmetric coupling between the Weyl multiplet and vector multiplets, although we do not have a definite proof. The following is a rough argument. Suppose that a supersymmetric combination of four-derivative terms contains a term of the form $f(M, \hat{R}(M)^2)\hat{R}(M)^2$. Then $f(M, \hat{R}(M)^2)$ must be of Weyl weight 1. If this is not a linear combination of $M^I$, its supersymmetric completion will contain a term of the form $c_I(M, \hat{R}(M)^2)W^I \wedge \text{tr}\hat{R}(M) \wedge \hat{R}(M)$, with non-trivial function $c_I$ of zero Weyl weight. This term is not gauge invariant unless $c_I$ is constant. Thus $f(M)$ is necessarily of the form $c_I M^I$ for constant $c_I$. This is in stark contrast with the situation in four-dimensional $\mathcal{N} = 2$ supergravity, where one can use an arbitrary holomorphic function $F(X^I, W^2)$ constructed from the vector multiplet scalar $X^I$ and the scalar $W^2$ constructed from the square of the Weyl multiplet.

The strong restriction in five dimensions, of course, already appears in the two-derivative Lagrangian. Indeed, the structure of the four-dimensional $\mathcal{N} = 2$ vector multiplet is determined by a holomorphic function $F(X^I)$, but in five dimensions, the corresponding object
\( N \) must be a purely cubic function. This restriction comes from the gauge invariance of the gauge Chern-Simons terms, just as in the case considered above.

3.2. Embedding and an invariant action

The linear multiplet should have \( E_a \equiv \epsilon_{abcdef} R^{bfg}(M) R^{defg}(M) \) to be used in the invariant action formula in order to obtain the gravitational Chern-Simons term (3.2). The supertransformation law (B.9) for \( \hat{R}(Q) \) reveals that we need the following structure:

\[
E_a \equiv R(M)^2 \quad \leftarrow \quad \phi \equiv R(M) R(Q) \quad \leftarrow \quad L_{ij} \equiv R(Q)^2. \tag{3.3}
\]

Thus, \( L_{ij} \) is of Weyl-weight 3 and an \( SU(2)_R \) triplet, constructed solely from the Weyl multiplet. Hence \( L_{ij} \) should be given by

\[
L_{ij}[W^2] = i \tilde{R}_{ab}(i) \tilde{R}^{abj}(Q) + A_1 i \tilde{\chi}^{(i} \chi^{j)} + A_2 v^{ab} \tilde{R}_{abij}(U) \tag{3.4}
\]

for suitable coefficients \( A_{1,2} \). This quantity must be invariant under \( S \) transformations to be the lowest component of a linear multiplet. The transformation

\[
\delta_S(\eta) L_{ij}[W^2] = 8 i \eta^{(i} \tilde{\chi}^{j)}(Q) v^{ab} - 8 i \eta^{(i} \tilde{\gamma}^{ab} \tilde{\chi}^{j)} A_1 + \left( 6 i \eta^{(i} \tilde{\chi}^{j)}(Q) v^{ab} - \frac{i}{2} \eta^{(i} \tilde{\gamma}^{ab} \chi^{j)} \right) A_2 \tag{3.5}
\]

fixes \( A_2 = -4/3 \) and \( A_1 = 1/12 \). Then, the embedding formula is determined by a straightforward but tedious and lengthy repeated application of the supersymmetry transformation:

\[
E_a[W^2] = -\frac{1}{8} \epsilon_{abcdef} \tilde{R}^{bfg}(M) \tilde{R}_{defg}(M) + \frac{1}{6} \epsilon_{abcdef} \tilde{R}^{bcij}(U) \tilde{R}_{deij}(U) + \hat{D}^a \left( -\frac{3}{4} v^{ab} D + 2 \tilde{R}_{abcd}(M) v^{cd} - \frac{2}{3} \epsilon_{abcdef} v^{ef} \hat{D} f v^{de} - 4 \epsilon_{abcdef} v^{ef} \hat{D}^a v^{bc} + \frac{16}{3} v^{ac} v^{cd} v^{db} + \frac{4}{3} v^{ab} v^2 \right),
\]

\[
N[W^2] = \frac{1}{6} D^2 + \frac{1}{4} \hat{R}^{abcd}(M) \tilde{R}_{abcd}(M) - \frac{2}{3} \hat{R}_{abij}(U) \tilde{R}^{abij}(U) - \frac{2}{3} \hat{R}_{abcd}(M) v^{ab} v^{cd} + \frac{16}{3} v^{ab} \hat{D}^b \hat{D}^c v^{ac} + \frac{8}{3} \hat{D}^a v^{bc} \hat{D}_a v_{bc} + \frac{8}{3} \hat{D}^a v^{bc} \hat{D}_b v_{ca} - \frac{4}{3} \epsilon_{abcdef} v^{ab} v^{cd} \hat{D} f v^{ef} + 8 v^{ab} v^{bc} v^{cd} v^{da} - 2(v^{ab} v^{ab})^2. \tag{3.6}
\]

Here, we have omitted the terms trilinear in fermions in the expression of \( \phi^i \) and the terms including fermions in the expressions of \( E_a \) and \( N \). The first non-trivial check comes from
the constraint (2.19), indicating that the divergence of $E_a$ vanishes. This can hold because
the divergence of the first line in $E_a$ vanishes, by the Bianchi identity, while the second
and third term vanish if we use the identity $\hat{D}^a\hat{D}^bA_{ab} = 0$ for a $K$-invariant, $SU(2)$-singlet,
antisymmetric tensor $A_{ab}$. Another non-trivial check is the $K$-invariance of $E_a$ and $N$, and
we can see that $E_a$ and $N$ are invariant under $K$ transformations.

We form an invariant action for off-shell conformal supergravity from the linear multiplet
constructed above, using the $V \cdot L$ formula. The bosonic term is

$$
\mathcal{L}(V \cdot L[W^2])|_{\text{bosonic}} = c_I Y_{ij} L^{ij}[W^2] - \frac{1}{2} c_I W_a^i E^a[W^2] + \frac{1}{2} c_I M^I N[W^2]
$$

$$
= -\frac{4}{3} c_I Y_{ij} v^{ab} \hat{R}_{ab}(U) + \frac{1}{16} \epsilon_{abcdef} c_I W^{ai} \hat{R}^{bcfg}(M) \hat{R}^{de} fg(M) - \frac{1}{12} \epsilon_{abcdef} c_I W^{ai} \hat{R}^{bc} jk(U) \hat{R}^{dejk}(U)
$$

$$
+ \frac{1}{8} c_I M^I \hat{R}^{abcd}(M) \hat{R}_{abcd}(M) - \frac{1}{3} c_I M^I \hat{R}_{abcd}(U) \hat{R}^{abcd}(U)
$$

$$
+ \frac{1}{12} c_I M^I D^2 + \frac{1}{6} c_I \hat{F}^{Iab} v_{ab} D - \frac{1}{3} c_I M^I \hat{R}_{abcd}(M) v^{ab} v^{cd}
$$

$$
- \frac{1}{2} c_I \hat{F}^{Iab} \hat{R}_{abcd}(U) v^{cd} + \frac{8}{3} c_I M^I v_{ab} \hat{D}^b \hat{D}^{Ia} v^{ac} + \frac{4}{3} c_I M^I \hat{D}^a v^{bc} \hat{D}^c v_{bc}
$$

$$
+ \frac{4}{9} c_I M^I \hat{D}^a v^{bc} \hat{D}_b v_{ca} - \frac{2}{3} c_I M^I \epsilon_{abcdef} v^{ab} v^{cd} \hat{D}^f v^{ef} + \frac{2}{3} c_I \hat{F}^{Iab} \epsilon_{abcdef} v^{ef} \hat{D}^c v^{de}
$$

$$
+ c_I \hat{F}^{Iab} \epsilon_{abcdef} \hat{D}^a v^{ef} - \frac{4}{3} c_I \hat{F}^{Iab} v^{ac} v^{cd} v_{db} - \frac{1}{3} c_I \hat{F}^{Iab} v_{ab} v^{cd} v^{da}
$$

$$
+ 4 c_I M^I v_{ab} v^{bc} v^{cd} v^{da} - c_I M^I (v_{ab} v^{ab})^2
$$

(3.7)

for constants $c_I$. Note that the term containing the second-order supercovariant derivative
of $v$ depends on the Ricci tensor through the $K$-gauge field given in (B.1), because $\hat{D}_a v_{bc}$
includes the terms $\sim b_a v_{bc}$ and $\sim \omega_{a[b} v_{c]d}$, and the supercovariant derivative of $b_a$ and $\omega_{a[b}$
yields $f_{ab} \ [\text{see (B.2), (B.4) and (B.6)}]$. The result is

$$
v_{ab} \hat{D}^b \hat{D}_c v^{ac} = v_{ab} \hat{D}^b \hat{D}_c v^{ac} - \frac{2}{3} v^{ac} v_{cb} R_a^b - \frac{1}{12} v_{ab} v^{ab} R
$$

(3.8)

modulo terms including fermions.

3.3. On-shell Poincaré supergravity

We consider $c_I$ to represent a small perturbation and add the resulting formula $\mathcal{L}_1 \equiv
\mathcal{L}(V \cdot L[W^2]), (3.7)$, to the zeroth-order terms $\mathcal{L}_0 = \mathcal{L}_H - \frac{1}{2} \mathcal{L}_V$, fix the gauge $A_i^a = \delta_i^a$, and eliminate auxiliary fields. For the components in the Weyl multiplet, the equations of
motion including fermionic fields can be obtained as follows. First, from the analysis of the
Weyl-weight, we can see that the omitted terms in (3.6) are independent of the auxiliary
field $D$. Thus, the full equation of motion for $D$ can be computed, and it is given by

$$
\frac{3}{4} (\mathcal{A}^2 + 2 N) + c_I (M^I D + F^I_{ab} v^{ab} + i \hat{\Omega}^I \chi) = 0.
$$

(3.9)
Then, by taking superconformal transformations of this equation, we obtain the full equations of motion for other components in the Weyl multiplet.

The first-order correction to the Lagrangian is obtained by substituting the zeroth-order solution for the auxiliary fields (2.27) and (2.28) into (3.7). Note that the appearance of the $D^2$ term in (3.7) changes the role of $D$, as can be seen from (3.9). That is, it is no longer a Lagrange multiplier enforcing the constraint $N = 1$, but, instead, it gives a steep potential $\left(\left(N - 1\right)^2/(c_I M^I)\right)$ minimized at $N = 1$. Another important point is that, from (2.27), the gauge field $V_{ij}^{\mu}$ for the generator $U$ is identified with a suitable combination of the gauge fields $W_I^{\mu}$ for the gauged supergravity.

Thus, the supersymmetric completion $L_1$ of the $W \wedge \text{tr} R \wedge R$ term in the on-shell Poincaré supergravity becomes

$$L_1 = \epsilon^{abde} \left( \frac{1}{16} c_I W_a^I R_{be}^{f g} R_{de fg} - \frac{1}{6} c_I P_J P_K W_a^I \hat{F}_b^J \hat{F}_c^K \right)$$

$$+ \frac{1}{8} c_I M^I \left( R_{abcd} R_{abcd} - \frac{4}{3} R_{ab} R_{ab} + \frac{1}{6} R^2 \right)$$

$$- \frac{2}{3} c_I P_J P_K M^I \hat{F}_a^J \hat{F}_b^K + \frac{4}{3} \left( c_I (N^{-1})^{IJ} P_J \right) \left( N_I F_{ab}^I \right) \left( F^{Iab} \right) + \cdots , \quad (3.10)$$

where we have only kept terms necessary for our subsequent analysis. Note that we used (2.10) to express the supercovariant curvature $\hat{R}_{abcd}(M)$ in terms of the metric curvature $R_{abcd}$. For ungauged supergravity ($P_I = 0$), our result is consistent with those obtained by dimensional reduction from M-theory.9,10 The important point here is that the supersymmetric completion of the $W \wedge \text{tr} R \wedge R$ term in gauged supergravity not only introduces an $MR^2$ term in the action but also modifies the gauge kinetic and gauge Chern-Simons terms.

### § 4. Condition for the supersymmetric AdS solution

As an application of the $R^2$ term constructed in the previous section, let us study how it modifies the condition for the supersymmetric AdS solution. One merit of the superconformal formalism presented above is that it allows us to study the supersymmetry condition in a manner that is largely independent of the action itself.

Let the metric of the AdS space be

$$L^2 \left( u^2 \eta_{\alpha \beta} dx^\alpha dx^\beta - \frac{du^2}{u^2} \right), \quad (4.1)$$

where $\alpha, \beta = 0, 1, 2, 3$, $\eta = \text{diag}(+, -, -, -)$, $u = x^4$ and $L$ is the curvature radius. We further suppose that any field with non-zero spin is zero. We start with the fact that in such a background, the equation

$$D_\mu \varepsilon - \frac{i}{2L} \gamma_\mu \varepsilon = 0 \quad (4.2)$$
has eight linearly-independent solutions. Here, $D_\mu$ denotes the derivative covariant with respect to local Lorentz transformations, and $\varepsilon$ is a spinor without the $SU(2)$-Majorana condition. If the $i = 1$ component of an $SU(2)$-Majorana spinor $\varepsilon^i$ satisfies (4.2), then the $i = 2$ component instead satisfies

$$D_\mu \varepsilon^{i=2} + \frac{i}{2L} \gamma_\mu \varepsilon^{i=2} = 0.$$  (4.3)

Thus, to express it covariantly under $SU(2)_R$, one needs to introduce a unit three-vector $\vec{q}$ so that

$$D_\mu \varepsilon^i - \frac{1}{2L} \gamma_\mu i(\vec{q} \cdot \vec{\sigma})^i j \varepsilon^j = 0.$$  (4.4)

The supersymmetry transformation of the gravitino (2.7) can then be made zero by choosing

$$\eta^i = \frac{1}{2L} (i\vec{q} \cdot \vec{\sigma})^i j \varepsilon^j.$$  (4.5)

The supersymmetric transformation which remains after the gauge fixing is

$$\delta_Q' (\varepsilon) = \delta_Q (\varepsilon) + \delta_S \left( \frac{1}{2L} (i\vec{q} \cdot \vec{\sigma}) \varepsilon \right).$$  (4.6)

The vanishing of $\delta_Q \chi^i$ implies that $D = 0$.

Next, the vanishing of the gaugino transformation (2.12) requires

$$Y^I_{i j} \varepsilon^j - \frac{1}{2L} M^I (i\vec{q} \cdot \vec{\sigma})^i j \varepsilon^j = 0$$  (4.7)

for all $I$. This relation is satisfied for the maximal number of $\varepsilon^i$ if and only if

$$Y^I_{i j} = \frac{1}{2L} (i\vec{q} \cdot \vec{\sigma})_{i j} M^I.$$  (4.8)

We can set $i\vec{q} \cdot \vec{\sigma} = i\sigma^3$ without loss of generality. The vanishing of the transformation of the hyperino $\delta \zeta^\alpha = 0$ under the gauge fixing $A^\alpha_i \propto \delta^\alpha_i$ determines the curvature radius as

$$L = \frac{3}{2} (P_i M^I)^{-1}.$$  (4.9)

Another interesting condition comes from the $[\delta_Q', \delta_Q']$ commutator. From (2.8), (2.9) and (2.13), it is

$$[\delta_Q'(\varepsilon), \delta_Q'(\varepsilon')] = \delta_U \left( -\frac{6}{L} \varepsilon^i (i\sigma^3)^{i j} \varepsilon^j \right) + \delta_G (-2i M^I \varepsilon \varepsilon')$$

$$= \delta_U \left( 2P_i M^I (i\sigma^3)^{i j} \varepsilon^j \right) + \delta_G (-2i M^I \varepsilon \varepsilon')$$

$$= \delta_G' (-2i M^I \varepsilon \varepsilon'),$$  (4.10)

where $\delta_G'$ is the surviving gauge transformation under the condition $A^\alpha_i \propto \delta^\alpha_i$ defined in (2.25). This implies that $\delta_G'(M^I)$ should leave the scalar VEVs invariant if we consider additional charged matter fields.
The reader can check that the analysis up to this point does not use any specific property of the action. Thus it is applicable to any $d = 5\,$ $\mathcal{N} = 2$ supergravity Lagrangian with arbitrarily complicated higher derivative terms.

Now, let us write down the condition (4.8) for our Lagrangian $L_0 + L_1$. To the first order in $c_I$, $Y^I_{ij}$ is given by the same expression as in (2.28),

$$Y^I_{ij} = 2(N^{-1})^I J (i\sigma^3)_{ij}.$$  

Substituting this into (4.8), we obtain

$$P_I = \frac{1}{4} N_{IJ} M^J / L = \frac{3}{2} c_{IK} M^J M^K / L.$$  

This is the attractor equation in 5d gauged supergravity first found in Ref.15). By multiplying this equation by $M^I$ we find the condition $N = c_{IK} M^I M^K = 1$ again. One can check that it solves the modified equations of motion which follows from $L_0 + L_1$. The correction to the potential $(N - 1)^2$ does not shift the VEVs of the scalars, since the solution before considering higher derivative corrections satisfies $N = 1$, minimizing the added potential. Note that higher terms with respect to the hatted curvature $\hat{R}_{abcd}(M)$ do not change the AdS solution, since the AdS background gives $\hat{R}_{abcd}(M) = 0$.

§5. Comparison to the $a$-maximization

5.1. Brief review of $a$-maximization

The $a$-maximization is a powerful technique to uncover the dynamics of $\mathcal{N} = 1$ superconformal field theories (SCFT) in four dimensions.\(^\text{16}\) It determines the $R$-symmetry $R_{SC}$ entering into the four-dimensional superconformal group as a linear combination $r^I G_I$ of the $U(1)$ symmetries $G_I$ of the theory. The AdS dual of the $a$-maximization has been studied,\(^\text{17,18}\) and it was found that the dual is precisely the supersymmetry condition for the AdS solution. The investigations in Refs.17) and 18) are restricted to the vanishing $U(1)$-gravity-gravity anomaly, corresponding to the vanishing of the $W \wedge \text{tr} R \wedge R$ contribution. This is because its supersymmetric completion was not known at that time. The aim of this section is to extend the analysis of Refs.17) and 18) to the case with non-zero $W \wedge \text{tr} R \wedge R$.

Let us denote by $G_I$, ($I = 0, 1, \ldots, n_V$) the conserved $U(1)$ charges of the theory. $G_I$ also acts on the supercharges. We denote them by $P_I$:

$$[G_I, Q_\alpha] = \hat{P}_I Q_\alpha.$$  

The anomaly among global $U(1)$ symmetries can described through the descent construc-
tion using the Chern-Simons term in five dimensions,
\[ \int \frac{1}{24\pi^2} \hat{c}_{IJK} W^I \wedge F^J \wedge F^K, \]  
(5.2)
where \( W^I \) is the gauge field for \( G_I \), and \( F^I = F^I_{\mu\nu} dx^\mu \wedge dx^\nu / 2 \) is the curvature two-form. The constants \( \hat{c}_{IJK} \) are given by
\[ \hat{c}_{IJK} = \text{tr} G_I G_J G_K, \]
(5.3)
where the trace is over the labels of the Weyl fermions of the theory. It is known that under the AdS/CFT correspondence, the Chern-Simons interaction (5.2) is present in the Lagrangian in five dimensions. Similarly, the \( U(1) \)-gravity-gravity anomaly characterized by
\[ \hat{c}_I = \text{tr} G_I \]
(5.4)
is manifested as the mixed gauge-gravitational Chern-Simons term
\[ \int \frac{1}{192\pi^2} \hat{c}_I W^I \wedge \text{tr} R \wedge R, \]
(5.5)
where \( R \) is the curvature two-form constructed from the metric.\footnote{The coefficients in (5.2) and (5.5) are dictated by the index theorem, and they can be found in any textbook on anomalies (see, e.g., Ref.19)).}

The anticommutator of the supertranslation \( Q_\alpha \) and the special superconformal transformation \( S^\alpha \) contains a particular combination of global symmetries:
\[ \{ Q_\alpha, S^\alpha \} \sim r^I G_I. \]
(5.6)
We normalize \( r^I \) so that the charge of the supercharge under \( r^I G_I \) be 1, that is, \( r^I \hat{P}_I = 1 \). We denote the superconformal R-symmetry by \( R_{SC} = r^I G_I \).

\( R_{SC} \) can be used to study various physical properties of the theory under consideration. The relation we need is that involving the central charges of the theory. In four dimensions, there are two of them, \( a \) and \( c \), which can be expressed in terms of the superconformal R-symmetry as follows:
\[ a = \frac{3}{32} (3 \text{tr} R_{SC}^3 - \text{tr} R_{SC}), \quad c = \frac{1}{32} (9 \text{tr} R_{SC}^3 - 5 \text{tr} R_{SC}). \]
(5.7)
The basic problem here is the identification of the superconformal R-symmetry \( R_{SC} = r^I G_I \), which can be done with the \( a \)-maximization.\footnote{\( Q \) and \( S \)-supersymmetry here should not be confused with \( Q \) and \( S \)-supersymmetry in the superconformal tensor formalism in five dimensions. Here \( Q \) and \( S \) are those of the four-dimensional superconformal group. In effect the combination of \( Q \) and \( S \) in five dimensions preserved in the AdS background corresponds to both \( Q \) and \( S \) in four dimensions.}

Let \( Q_F = t^I G_I \) be a global symmetry
which commutes with the supercharges, i.e. $t^I \hat{P}_I = 0$. The triangle diagram with one $Q_F$ and two $R_{SC}$ insertions can be mapped, using the superconformal transformation, to the triangle diagram with $Q_F$ and two energy-momentum tensor insertions. The coefficient was calculated precisely and yields the relation

$$9 \, \text{tr} \, Q_F R_{SC} R_{SC} = \text{tr} \, Q_F. \quad (5.8)$$

Another requirement is that $\text{tr} \, Q_F Q_F R_{SC}$ be negative definite. This comes from the positivity of the two-point function of the currents. Let us introduce the trial $a$-function $a(s)$ for a trial R-charge $R(s) = s^A Q_A$ by generalizing (5.7):

$$a(s) = \frac{3}{32} (3 \text{tr} R(s)^3 - \text{tr} R(s)). \quad (5.9)$$

The conditions in (5.8) imply that $a(s)$ is locally maximized under the condition $P_I s^I = 1$, at the point $s^I = r^I$, which gives $R_{SC}$. This is the origin of the terminology of $a$-maximization.

For our purposes, it is convenient to rewrite $a(s)$ as

$$a(s) = \frac{3}{32} (3 \hat{c}_{IJK} s^I s^J s^K - \hat{c}_I s^I) = \frac{3}{32} (3 \hat{c}_{IJK} - \hat{c}_I \hat{P}_J \hat{P}_K) s^I s^J s^K, \quad (5.10)$$

where we have used $\hat{P}_I s^I = 1$.

### 5.2. Analysis in the AdS Space

Let us suppose that the dual theory in the AdS has the Lagrangian $\mathcal{L}_0 + \mathcal{L}_1$. We now re-derive $a$-maximization using the AdS/CFT prescription and the supergravity analysis presented in §4.

First, (2.27) shows that the gravitino has charge $\pm P_I$ with respect to the gauge fields $W^I_\mu$. This fact is translated into $[G_I, Q_\alpha] = P_I Q_\alpha$ under the AdS/CFT duality, which allows us to identify $P_I$ with $\hat{P}_I$, which was defined in (5.1).

Next, by comparing the gauge Chern-Simons term (5.2) corresponding to the anomaly and the gauge Chern-Simons term in our Lagrangian $\mathcal{L}_0 + \mathcal{L}_1$, we get

$$\hat{c}_{IJK} = 12 \pi^2 c_{IJK} - 16 \pi^2 c_{(I} P_{J} P_{K)}, \quad (5.11)$$

whereas the comparison of (5.5) and the gravitational Chern-Simons term in $\mathcal{L}_0 + \mathcal{L}_1$ yields

$$\hat{c}_I = -48 \pi^2 c_I. \quad (5.12)$$

Thus we see that $c_{IJK}$ entering the Lagrangian is given by

$$c_{IJK} = \frac{1}{12 \pi^2} (\hat{c}_{IJK} - \frac{1}{3} \hat{c}_{(I} P_{J} P_{K)}). \quad (5.13)$$
Then, the supersymmetry condition for the AdS space is given by
\[ c_{IJK} \langle M^I \rangle \langle M^K \rangle \propto P_I, \]  
(5.14)
where we indicated the scalar VEV at the AdS solution by enclosing in brackets. Using a Lagrange multiplier, the condition above is equivalent to

the extremization of \( P_I M^I \) under the constraint \( c_{IJK} M^I M^J M^K = 1. \)  
(5.15)

Let us define \( t^I = M^I / P_I M^I. \) Then this can be further translated as

the extremization of \( c_{IJK} t^I t^J t^K \) under the condition \( P_I t^I = 1. \)  
(5.16)

The important point here is that \( a(t) = \frac{27 \pi^2}{8} c_{IJK} t^I t^J t^K. \) Thus, we have found that the supersymmetry condition for the AdS space is given by the extremization of the same function \( a(t). \)

Finally, we would like to relate the value \( \langle t^I \rangle = \langle M^I \rangle / (P_I \langle M^I \rangle) \) at the extrema and \( r^I \) entering into \( R_{SC} = r^I G_I. \) The 4d supercharge corresponds to \( \delta'_Q \) defined in (4.6). Thus, \( R_{SC} \) should be a linear combination of \( U(1) \) generators in the \([\delta'_Q, \delta'_Q]\) commutator (4.10), which implies that \( r^I \propto \langle M^I \rangle. \) From the normalization of \( r^I, \) we should have \( r^I = \langle t^I \rangle. \) Thus, we find that

\[ r^I \] is the value of \( s^I \) which extremizes \( a(s) \) under the condition \( P_I s^I = 1, \)  
(5.17)

which is precisely the statement of the \( a \)-maximization procedure. One can also check that the condition \( \text{tr} Q_F Q_F R_{SC} < 0 \) is equivalent to the positivity of the metric of the scalar fields at \( M^I = \langle M^I \rangle, \) just as is the case with \( c_I = 0, \) analyzed in Refs.17) and 18).

As a final exercise in this paper, let us calculate the central charge \( a \) from the bulk AdS perspective. The method to obtain the central charge \( a \) and \( c \) in higher derivative gravity theory was pioneered in Refs.20) and 21), and was extended to the general Lagrangian in Ref.22). In the latter paper, the formula for the central charge \( a \) and \( c \) for the boundary CFT with the bulk Lagrangian

\[ e^{-1} \mathcal{L} = \frac{1}{2} \left( \frac{12}{L^2} - \frac{80 \alpha + 16 \beta + 8 \gamma}{L^4} - R + \alpha R^2 + \beta R^2_{\mu \nu} + \gamma R^2_{\mu \nu \rho \sigma} \right) \]  
(5.18)

is determined to be

\[ a = \pi^2 L^3 (1 - 40 \alpha - 8 \beta - 4 \gamma), \]
\[ c = \pi^2 L^3 (1 - 40 \alpha - 8 \beta + 4 \gamma), \]  
(5.19)
where the parametrization of the cosmological constant in (5.18) is chosen so that the resulting AdS space has curvature radius $L$.

Our Lagrangian $\mathcal{L}_0 + \mathcal{L}_1$ corresponds to the case

$$\left(\alpha, \beta, \gamma\right) = \frac{1}{4} c_I M^I \left(\frac{1}{6}, -\frac{4}{3}, 1\right). \quad (5.20)$$

Thus we obtain

$$a = \pi^2 L^3 = \frac{27}{8} \pi^2 (M_I P^I)^{-3} = \frac{27}{8} \pi^2 c_{IJK} t^I t^J t^K = \frac{3}{32} \left(3\hat{c}_{IJK} t^I t^J t^K - \hat{c}_I t^I\right), \quad (5.21)$$

which is the identical result obtained in (5.7).

§6. Summary and discussion

In this paper, we have seen how the superconformal formalism can be used to construct the supersymmetric completion of the mixed gravitational Chern-Simons term, $c_I W^I \wedge \text{tr} \mathcal{R} \wedge \mathcal{R}$, in 5d $\mathcal{N} = 2$ gauged supergravity. In addition to the known term $c_I M^I R^2$, we have identified a new contribution in the supersymmetric completion, namely the modification to the gauge kinetic and Chern-Simons terms.

We also analyzed how the BPS equation for the maximally supersymmetric AdS solution is modified by the supersymmetric higher derivative term constructed above. It was shown that it correctly reproduces the $a$-maximization of the boundary CFT in the case that the mixed $U(1)$-gravity-gravity anomaly exists.

Regarding the outlook for future studies, there is a great opportunity for research using our new $R^2$ term in supergravity. As discussed in the introduction, such a term naturally arises when one compactifies string theory down to five dimensions. Thus, it would be interesting to see, for example, how these terms affect the entropy of the five-dimensional black rings and black holes.

Another interesting problem would be to study the effects of these terms on the many exact supersymmetric solutions to five-dimensional supergravity that were recently derived. They were found by exploiting the BPS equation fully. As we now have the full supersymmetry transformation law including the higher derivative correction, we should be able to extend their analysis to our case. We hope to revisit these problems in the future.

Acknowledgements

YT would like to acknowledge various helpful discussions with M. Günaydin and S. Matsuura. He would also like to express his sincere gratitude to the members of the Aspen
Center for Physics, where most of this work was done. The authors would like to thank a referee for insightful comments which greatly helped to improve this paper.

The work of KO is supported by the Japan Society for the Promotion of Science (JSPS) under the Post-doctoral Research Program. YT was partially supported by JSPS Research Fellowships for Young Scientists when the authors started this work. He is now supported by the United States DOE Grant DE-FG02-90ER40542.

Appendix A

Notation

We summarize our notational conventions in this appendix. Firstly, the components of various multiplets and their basic properties are summarized in Table I. The gamma matrices $\gamma^a$ satisfy $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$ and $(\gamma^a)^\dagger = \eta_{ab}\gamma^b$, where $\eta^{ab} = \text{diag}(+, -, -, -, -)$. $\gamma_{a\ldots b}$ represents an antisymmetrized product of gamma matrices:

$$\gamma_{a\ldots b} = \gamma_{[a \ldots} \gamma_{b]}, \quad (A.1)$$

where the square brackets denote complete antisymmetrization with weight 1. Similarly $(\ldots)$ denote complete symmetrization with weight 1. We choose the Dirac matrices to satisfy

$$\gamma^{a_1 \ldots a_5} = \epsilon^{a_1 \ldots a_5}, \quad (A.2)$$

where $\epsilon^{a_1 \ldots a_5}$ is a totally antisymmetric tensor with $\epsilon^{01234} = 1$.

The $SU(2)$ index $i$ ($i=1,2$) is raised and lowered with $\epsilon_{ij}$, where $\epsilon_{12} = \epsilon^{12} = 1$, in the northwest-southeast (NW-SE) convention:

$$A^i = \epsilon^{ij} A_j, \quad A_i = A^j \epsilon_{ji}. \quad (A.3)$$

The charge conjugation matrix $C$ in 5D has the properties

$$C^T = -C, \quad C^\dagger C = 1, \quad C \gamma_a C^{-1} = \gamma_a^T. \quad (A.4)$$

Our five-dimensional spinors satisfy the $SU(2)$-Majorana condition

$$\bar{\psi}^i \equiv \psi_i^\dagger \gamma^0 = \psi_i^{iT} C, \quad (A.5)$$

where the spinor indices are omitted. When the $SU(2)$ indices are suppressed in the bilinear terms of spinors, the NW-SE contraction is understood, e.g. $\bar{\psi}\gamma^{a_1 \ldots a_n} \lambda = \bar{\psi}^i \gamma^{a_1 \ldots a_n} A_i$. Changing the order of the spinors in a bilinear leads to the following signs:

$$\bar{\psi}\gamma^{a_1 \ldots a_n} \lambda = (-1)^{(n+1)(n+2)/2} \bar{\lambda}\gamma^{a_1 \ldots a_n} \psi. \quad (A.6)$$
### Table I. Multiplets in 5D superconformal gravity.

| field | type       | remarks                  | SU(2) | Weyl-weight |
|-------|------------|--------------------------|-------|-------------|
| $e_{\mu}^a$ | boson    | fünfbein                 | 1     | -1          |
| $\psi_{\mu}^i$ | fermion | SU(2)-Majorana            | 2     | $-\frac{1}{2}$ |
| $b_{\mu}$ | boson     | real                     | 1     | 0           |
| $V_{\mu}^{ij}$ | boson   | $V_{\mu}^{ij} = V_{\mu}^{ji} = (V_{\mu}^{ij})^*$ | 3     | 0           |
| $v_{ab}$ | boson     | real, antisymmetric      | 1     | 1           |
| $\chi^i$ | fermion   | SU(2)-Majorana            | 2     | $\frac{3}{2}$ |
| $D$    | boson     | real                     | 1     | 2           |

**dependent gauge fields**

| field | type       | remarks                  | SU(2) | |
|-------|------------|--------------------------|-------|---|
| $\omega_{\mu}^{ab}$ | boson   | spin connection         | 1     | 0     |
| $\phi_{\mu}^i$ | fermion   | SU(2)-Majorana           | 2     | $\frac{1}{2}$ |
| $f_{\mu}^a$ | boson     | real                     | 1     | 1     |

**Vector multiplet**

| field | type       | remarks                  | SU(2) | |
|-------|------------|--------------------------|-------|---|
| $W_{\mu}$ | boson   | real gauge field         | 1     | 0     |
| $M$    | boson     | real                     | 1     | 1     |
| $\Omega^i$ | fermion | SU(2)-Majorana           | 2     | $\frac{3}{2}$ |
| $Y_{ij}$ | boson     | $Y_{ij} = Y_{ji} = (Y_{ij})^*$ | 3     | 2     |

**Hypermultiplet**

| field | type       | remarks                  | SU(2) | |
|-------|------------|--------------------------|-------|---|
| $\mathcal{A}_a^i$ | boson   | $\mathcal{A}_a^i = e^{ij} \mathcal{A}_j^\beta \rho_{\beta \alpha} = -(\mathcal{A}_a^i)^*$ | 2     | $\frac{3}{2}$ |
| $\zeta^\alpha$ | fermion   | $\bar{\zeta}^\alpha \equiv (\zeta_\alpha)^T \gamma_0 = \zeta^\alpha T C$ | 1     | 2     |
| $\mathcal{F}_a^i$ | boson   | $\mathcal{F}_a^i = -(\mathcal{F}_a^i)^*$ | 2     | $\frac{5}{2}$ |

**Linear multiplet**

| field | type       | remarks                  | SU(2) | |
|-------|------------|--------------------------|-------|---|
| $L^{ij}$ | boson   | $L^{ij} = L^{ji} = (L_{ij})^*$ | 3     | 3     |
| $\varphi^i$ | fermion | SU(2)-Majorana           | 2     | $\frac{7}{2}$ |
| $E_a$   | boson     | real, constrained by (2.19) | 1     | 4     |
| $N$     | boson     | real                     | 1     | 4     |

If the SU(2) indices are not contracted, the sign switches. We often use the Fierz identity, which in 5D reads

$$\psi^i \bar{\chi}^j = -\frac{1}{4}(\bar{\chi}^j \psi^i) - \frac{1}{4} (\bar{\chi}^j \gamma^a \psi^i) \gamma_a + \frac{1}{8} (\bar{\chi}^j \gamma^{ab} \psi^i) \gamma_{ab}. \quad (A.7)$$
Appendix B
Definitions and Useful Formulae for the Weyl Multiplet

In this appendix, we summarize useful formulae for the Weyl multiplet. Firstly, the solution to the constraints \((2.3)\) is given by the following:

\[
\omega_{\mu}^{ab} = \epsilon_{\mu}^{0ab} + i(2\bar{\psi}_\mu \gamma^a \psi^b) + 2\bar{e}_\mu^a \eta^b,
\]

with \(\omega_{\mu}^{ab} = -2\epsilon^{\nu[a} \partial_{[\mu} e_{\nu]}^b \right) + \epsilon^{b[a} \epsilon^{d]} \epsilon_{\mu}^c \partial_{\rho} e_{\sigma c} \),

\[
\phi^i = \left( -\frac{1}{3} \epsilon_{\mu}^{a} \gamma^b + \frac{1}{24} \gamma_{\mu} \gamma^{ab} \right) \hat{R}_{ab}^i(Q),
\]

\[
f_{\mu}^a = \left( \frac{1}{6} \delta_{\mu}^{a} \delta^b - \frac{1}{48} \epsilon_{\mu}^a \epsilon^b \right) \hat{R}_{ab}^i(M).
\]

Here \(\hat{R}_{ab}^i(M) \equiv \hat{R}_{ab}^{i\alpha}(M) e_\nu^\alpha\), and the primes on the curvatures indicate that \(\hat{R}_{ab}^{i\alpha}(Q) = \hat{R}_{ab}^{i\alpha}(Q)|_{\phi_\mu = 0}\) and \(\hat{R}_{ab}^{i\alpha}(M) = \hat{R}_{ab}^{i\alpha}(M)|_{f_{\mu}^a = 0}\). The transformation laws of their dependent gauge fields can be obtained by using those of the independent fields of the Weyl multiplet, in principle. The explicit \(K\)-transformation laws of the gauge field \(\omega_{\mu}^{ab}\),

\[
\delta_K(\xi^K) \omega_{\mu}^{ab} = -4\epsilon^K_{\mu} e_\mu^b,
\]

are needed to check the \(K\)-invariance of the embedding formulae in \((3.6)\).

We used two types of covariant derivatives in the main text. The first one is the ‘unhatted’ derivative \(D_\mu\), which is covariant only with respect to the homogeneous transformations \(M_{ab}, D, U^{ij}\) and the \(G\) transformation for non-singlet fields under the Yang-Mills group \(G\). The other is the ‘hatted’ derivative \(\hat{D}_\mu\), which denotes the fully superconformal covariant derivative. With \(h_{\mu}^A\) denoting the gauge fields of the transformation \(X_A\), they are defined as

\[
D_\mu = \partial_\mu - \sum_{X_A=M_{ab},D,U^{ij},G} h_{\mu}^A X_A, \quad \hat{D}_\mu = \hat{\partial}_\mu - \sum_{X_A=Q,S,K} h_{\mu}^A X_A.
\]

Below we give the explicit forms of the covariant derivatives appearing in Eq.\((2.7)\) for convenience:

\[
D_\mu \epsilon^i = \left( \partial_\mu - \frac{1}{4} \omega_{\mu}^{ab} \gamma_{ab} + \frac{1}{2} b_\mu \right) \epsilon^i - V_{\mu}^i \epsilon^j,
\]

\[
D_\mu \eta^i = \left( \partial_\mu - \frac{1}{4} \omega_{\mu}^{ab} \gamma_{ab} - \frac{1}{2} b_\mu \right) \eta^i - V_{\mu}^i \eta^j,
\]

\[
D_\mu \xi^K = \left( \partial_\mu - b_\mu \right) \xi^K - \omega_{\mu}^{ab} \xi^K_{ab},
\]

\[
\hat{D}_\mu v_{ab} = \partial_\mu v_{ab} + 2 \omega_{\mu[a} v_{b]c} - b_\mu v_{ab} + \frac{i}{8} \bar{\psi}_\mu \gamma_{ab} \chi + \frac{3}{2} i \bar{\psi}_\mu \hat{R}_{ab}(Q),
\]

\[
\hat{D}_\mu \chi^i = D_\mu \chi^i - D_\mu \psi^i + 2 \gamma^i_{ab} \psi^j_\mu \hat{D}_\mu \psi^j_\mu - \gamma^i_{ab} \psi^j_\mu \epsilon_{abcde} v^{bcde} - 4 \gamma^i_{ab} \psi^j_\mu \chi - 4 \gamma^i_{ab} \psi^j_\mu \chi^j - 4 \chi^i_{ab} \psi^j_\mu \chi^j,
\]

\[
D_\mu \chi^i = \left( \partial_\mu - \frac{1}{4} \omega_{\mu}^{ab} \gamma_{ab} - \frac{3}{2} b_\mu \right) \chi^i - V_{\mu}^i \chi^j.
\]
The superconformally covariant curvatures $\hat{R}_{\mu
u}^A$ are defined as the commutator of the covariant derivatives:

$$[\hat{D}_a, \hat{D}_b] = - \sum_{A=Q^i, M_{ab}, D, U, \iota, S, K_a} \hat{R}_{ab}^A X_A \quad (B.5)$$

They are given explicitly by the following expressions:

$$\begin{align*}
\hat{R}_{\mu\nu}^a(P) &= 2\partial_{[\mu}e_{\nu]}^a - 2\omega_{[\mu}^{ab}e_{\nu]}b + 2b_{[\mu}e_{\nu]}^a + 2i\bar{\psi}_\mu \gamma^a \psi_\nu, \\
\hat{R}_{\mu\nu}^i(Q) &= 2\partial_{[\mu}\psi_{\nu]}^i - \frac{1}{2}\omega_{[\mu}^{ab}\gamma_{ab}\psi_{\nu]}^i + b_{[\mu}\psi_{\nu]}^i - 2V_{[\mu j}^i\psi_{\nu]}^j + \gamma_{a[b}\psi_{\nu]}v_{ab} - 2\gamma_{[\mu} \phi_{\nu]}^i, \\
\hat{R}_{\mu\nu}^{ab}(M) &= 2\partial_{[\mu}\omega_{\nu]}^a - 2\omega_{[\mu}^{a\omega}e_{\nu]}b - 4i\bar{\psi}_\mu \gamma^{ab}\phi_\nu + 2i\bar{\psi}_\mu \gamma^{abcd}\psi_\nu v_{cd} \\
&\quad + 4i\bar{\psi}_\mu \gamma^a \hat{R}_{ij}^b(Q) + 2i\bar{\psi}_\mu \gamma_{ij} \hat{R}^{ab}(Q) + 8f_{[\mu}^{a[e}v_{ba]}, \\
\hat{R}_{\mu\nu}(D) &= 2\partial_{[\mu}b_{\nu]} + 4i\bar{\psi}_\mu \phi_\nu + 4f_{[\mu}], \\
\hat{R}_{\mu\nu}^{ij}(U) &= 2\partial_{[\mu}V_{\nu]}^i - 2V_{[\mu k}^i V_{\nu]}^k + 12i\bar{\psi}_\mu [\gamma^i \psi_\nu^j] - 4i\bar{\psi}_\mu \bar{\psi}_\nu \chi^j + \bar{\psi}_\mu [\bar{\psi}_\nu \chi^j], \\
\hat{R}_{\mu\nu}^i(S) &= 2\partial_{[\mu}\phi_{\nu]}^i - \frac{1}{2}\omega_{[\mu}^{ab}\gamma_{ab}\phi_{\nu]} - b_{[\mu}\phi_{\nu]}^i - 2V_{[\mu j}^i\phi_{\nu]}^j + 2f_{[\mu}^a \gamma_a \phi_{\nu]}^i + \cdots, \\
\hat{R}_{\mu\nu}^a(K) &= 2\partial_{[\mu}f_{\nu]}^a - 2\omega_{[\mu}^{a\omega}f_{\nu]}b - 2b_{[\mu}f_{\nu]}^a + 2i\phi_{[\mu}^a \gamma \phi_\nu + \cdots. \quad (B.6)
\end{align*}$$

Here, the dots in the $S^i$ and $K^a$ curvature expressions denote terms containing other curvatures.

To compute the $Q$-variation of the covariant derivatives of some fields, the following formula is useful:

$$[\delta_Q, \hat{D}_a] = -\delta_Q([\delta_Q \psi_{\mu}^i]_{\text{cov}}) - \delta_S([\delta_Q \phi_{\mu}^i]_{\text{cov}}) + \cdots \quad (B.7)$$

Here the fermionic terms are omitted and $[\cdots]_{\text{cov}}$ denotes the covariant part of the variations, namely,

$$\begin{align*}
[\delta_Q \psi_{\mu}^i]_{\text{cov}} &= \frac{1}{2} \gamma_{abc} v^{bc} \varepsilon^i, \\
[\delta_Q \phi_{\mu}^i]_{\text{cov}} &= \frac{1}{3} \left( \hat{R}_{ab}^i j(U) \gamma^b - \frac{1}{8} \gamma_a \gamma \cdot \hat{R}_{ij}(U) \right) \varepsilon^j \\
&\quad - \frac{1}{12} \left( 3 \hat{D}_a \cdot v \varepsilon^i + \gamma_{abcd} \hat{D}_b v^{cd} \varepsilon^i + \gamma_{ab} \hat{D}_c v^{eb} \varepsilon^i - 2 \gamma_{[bc} \varepsilon^i \hat{D}_b v_{ca} - 3 \varepsilon^i \hat{D}_b v_{ba} \\
&\quad - \gamma_{abcd} \varepsilon^i v^{bc} v^{de} + 4 v_{ab} v_{cd} \gamma_{[bc} \varepsilon^i + 16 v_{ab} v^{bc} \gamma_a \varepsilon^i + 5 v_{bc} v^{bc} \gamma_a \varepsilon^i \right). \quad (B.8)
\end{align*}$$

Using this, we can verify that the variations of the supercovariant curvatures do not contain any term non-covariant with respect to the superconformal transformations.

Finally, we present the explicit forms of the variations of the supercovariant curvatures $\hat{R}^i(Q)$ and $\hat{R}^{ij}(U)$:

$$\delta \hat{R}_{ab}^i(Q) = -\frac{1}{4} \hat{R}_{ab}^{cd}(M) \gamma_{cd} \varepsilon^i - \frac{1}{3} \hat{R}_{ab}^{ij}(U) \varepsilon^j + \frac{1}{12} \gamma \cdot \hat{R}_{ij}(U) \gamma_{ab} \varepsilon^j$$

25
\[
\delta \hat{R}_{ab}^{ij}(U) = -6i\varepsilon^{(i\hat{R}_{ab}^{j})}(S) + 4i\varepsilon^{(i\gamma \cdot v\hat{R}_{ab}^{j})(Q) + \frac{i}{2}\varepsilon^{(i\gamma [a\hat{D}_{b}]\chi^{j})},
\]

\[
-\frac{i}{4}\varepsilon^{(i\gamma_{abcd}\chi^{j})}v^{cd} - \frac{i}{2}\varepsilon^{(i\gamma_{c[a}\chi^{j])}v^{b]}c}
\]

\[
+ 6i\bar{\eta}^{(i\hat{R}_{ab}^{j})}(Q) - \frac{i}{2}\bar{\eta}^{(i\gamma_{abcd}\chi^{j})},
\]

The ellipsis in (B.9) represents terms trilinear in fermions in \(\delta_{Q}\hat{R}(Q)\). No term of \(\delta_{S}\hat{R}(Q)\) is omitted.

### Appendix C

#### Conventions for Vector Multiplets

Here, we summarize the conventions for the vector multiplets in the original on-shell formalism of Günaydin, Sierra and Townsend,\(^1,2\) in the superconformal formalism of Fujita, Kugo and Ohashi,\(^5,6\) and in the formalism of Bergshoeff et al.,\(^7,8\) for convenience. The multiplets are labeled as follows:

| scalar | gaugini | vector |
|--------|---------|--------|
| GST :  | \(h^{I}\), \(\chi_{I}^{i}\), \(A_{I}^{\mu}\) |
| FKO :  | \(M^{I}\), \(Q_{I}^{i}\), \(W_{I}^{\mu}\) |
| Berg. :| \(\sigma^{I}\), \(\psi_{I}^{i}\), \(A_{I}^{\mu}\) |

All groups denote the Chern-Simons coefficients by \(c_{IJK}\). The gauge fields are to be identified according to

\[
A_{\mu}^{\text{GST}} = W_{\mu}^{\text{FKO}} = A_{\mu}^{\text{Berg.}},
\]

while the symbols for the Chern-Simons are related by

\[
c_{IJK}^{\text{GST}} = - \left(\frac{3}{2}\right)^{3/2} c_{IJK}^{\text{FKO}}, \quad c_{IJK}^{\text{Berg.}} = 3c_{IJK}^{\text{FKO}}.
\]

The scalars are related by

\[
h^{I} = -\sqrt{\frac{2}{3}}M^{I}, \quad \sigma^{I} = -M^{I},
\]

and the very special real manifold is defined, respectively, by

\[
e_{IJK}^{\text{FKO}} M^{I} M^{J} M^{K} = 1, \quad e_{IJK}^{\text{GST}} h^{I} h^{J} h^{K} = 1, \quad e_{IJK}^{\text{Berg.}} \sigma^{I} \sigma^{J} \sigma^{K} = -3.
\]
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