Lamarle Formula in 3-Dimensional Lorentz Space

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January 5, 2010

Abstract
The Lamarle Formula, given by Kruppa in [8], is known as a relationship between the Gaussian curvature and the distribution parameter of a ruled surface in the surface theory. The ruled surfaces were investigated in 3 different classes with respect to the character of base curves and rulings, [14], [15]. In this paper on account of these studies, the relationships between the Gaussian curvatures and distribution parameters of spacelike ruled surface, timelike ruled surface with spacelike ruling and timelike ruled surface with timelike ruling are obtained, respectively. These relationships are called as Lorentzian Lamarle formulas. Finally some examples concerning with these relations are given.

Subject Classification: 53B30, 53C50, 14J26
Key Words: Ruled surface, distribution parameter, Gaussian curvature, Lamarle formula.

1 Introduction
The study of ruled surface in $\mathbb{R}^3$ is classical subject in differential geometry. It has again been studied in some areas (i.e. Projective geometry, [13], Computer-aided design, [11], etc.) Also, it is well known that the geometry of ruled surface is very important of kinematics or spatial mechanisms in $\mathbb{R}^3$, [4], [7]. A ruled surface is one which can be generated by sweeping a line through space. Developable surfaces are special cases of ruled surfaces, [10]. Cylindrical surfaces are examples of developable surfaces. On a developable surface at least one of the two principal curvatures is zero at all points. Consequently the Gaussian curvature is zero everywhere too. So it is meaningful for us to study non-cylindrical ruled surfaces.

Lorentz metrics in 3-dimensional Lorentz space $\mathbb{R}^3_1$ is indefinite. In the theory of relativity, geometry of indefinite metric is very crucial. Hence, the theory of ruled surface in Lorentz space $\mathbb{R}^3_1$, which has the metric $ds^2 = dx_1^2 + dx_2^2 - dx_3^2$, attracted much attention. The situation is much more complicated than the Euclidean case, since the ruled surfaces may have a definite metric (spacelike surfaces), Lorentz metric (timelike surfaces) or mixed metric. Some characterizations for ruled surfaces are obtained by [9]. Timelike and spacelike ruled surfaces are defined and the characterizations of timelike and spacelike ruled surfaces are found in [2], [5], [6], [14] and [15].
2 Preliminaries

Let $\mathbb{R}^3_1$ denote the 3–dimensional Lorentz space, i.e. the Euclidean space $E^3$ with standard flat metric given by

$$g = dx_1^2 + dx_2^2 - dx_3^2$$

(2.1)

where $(x_1, x_2, x_3)$ is rectangular coordinate system of $\mathbb{R}^3_1$. Since $g$ is indefinite metric, recall that a vector $\vec{v}$ in $\mathbb{R}^3_1$ can have one of three casual characters: it can be space-like if $g(\vec{v}, \vec{v}) > 0$ or $\vec{v} = 0$, time-like if $g(\vec{v}, \vec{v}) < 0$ and null $g(\vec{v}, \vec{v}) = 0$ and $\vec{v} \neq 0$. Similarly, an arbitrary curve $\vec{\alpha} = \vec{\alpha}(s) \subset \mathbb{R}^3_1$ can locally be space-like, time-like or null (light-like), if all of its velocity vectors $\vec{\alpha}'(s)$ are respectively space-like, time-like or null (light-like). The norm of a vector $\vec{v}$ is given by $|\vec{v}| = \sqrt{|g(\vec{v}, \vec{v})|}$. Therefore, $\vec{v}$ is a unit vector if $g(\vec{v}, \vec{v}) = \pm 1$.

Furthermore, vectors $\vec{v}$ and $\vec{w}$ are said to be orthogonal if $g(\vec{v}, \vec{w}) = 0$, [10]. Let the set of all timelike vectors in $\mathbb{R}^3_1$ be $\Gamma$. For $\vec{u} \in \Gamma$, we call $C(\vec{u}) = \{ \vec{v} \in \Gamma | \langle \vec{v}, \vec{u} \rangle < 0 \}$ as time-conic of Lorentz space $\mathbb{R}^3_1$ including vector $\vec{u}$, [10].

Let $\vec{v}$ and $\vec{w}$ be two time-like vectors in Lorentz space $\mathbb{R}^3_1$. In this case there exists the following inequality

$$|g(\vec{v}, \vec{w})| \geq |\vec{v}| \cdot |\vec{w}|.$$ 

In this inequality if one wishes the equality condition, then it is necessary for $\vec{v}$ and $\vec{w}$ be linear dependent.

If time-like vectors $\vec{v}$ and $\vec{w}$ stay inside the same time-conic then there is a unique non-negative real number of $\theta \geq 0$ such that

$$g(\vec{v}, \vec{w}) = -|\vec{v}| \cdot |\vec{w}| \cosh \theta$$

(2.2)

where the number $\theta$ is called an angle between the timelike vectors, [10].

Let $\vec{v}$ and $\vec{w}$ be spacelike vectors in $\mathbb{R}^3_1$ that span a spacelike subspace. We have that

$$|g(\vec{v}, \vec{w})| \leq |\vec{v}| \cdot |\vec{w}|$$

with equality if and only if $\vec{v}$ and $\vec{w}$ are linearly dependent. Hence, there is a unique angle $0 \leq \theta \leq \pi$ such that

$$g(\vec{v}, \vec{w}) = |\vec{v}| \cdot |\vec{w}| \cos \theta$$

(2.3)

where the number $\theta$ is called the Lorentzian spacelike angle between spacelike vectors $\vec{v}$ and $\vec{w}$, [12].

Let $\vec{v}$ and $\vec{w}$ be spacelike vectors in $\mathbb{R}^3_1$ that span a timelike subspace. We have that

$$|g(\vec{v}, \vec{w})| > |\vec{v}| \cdot |\vec{w}|.$$ 

Hence, there is a unique real number $\theta > 0$ such that

$$g(\vec{v}, \vec{w}) = |\vec{v}| \cdot |\vec{w}| \cosh \theta.$$ 

(2.4)

The Lorentzian timelike angle between spacelike vectors $\vec{v}$ and $\vec{w}$ is defined to be $\theta$, [12].
Let $\vec{v}$ be a spacelike vector and $\vec{w}$ be a timelike vector in $\mathbb{R}^3_1$. Then there is a unique real number $\theta \geq 0$ such that

$$g(\vec{v}, \vec{w}) = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \sinh \theta.$$  \hspace{1cm} (2.5)

The Lorentzian timelike angle between $\vec{v}$ and $\vec{w}$ is defined to be $\theta$. \cite{12}

For any vectors $\vec{v} = (v_1, v_2, v_3)$, $\vec{w} = (w_1, w_2, w_3) \in \mathbb{R}^3_1$, the Lorentzian product $\vec{v} \wedge \vec{w}$ of $\vec{v}$ and $\vec{w}$ is defined as \cite{1}

$$\vec{v} \wedge \vec{w} = (v_3w_2 - v_2w_3, v_1w_3 - v_3w_1, v_1w_2 - v_2w_1).$$  \hspace{1cm} (2.6)

## 3 Ruled Surface in $\mathbb{R}^3_1$

A ruled surface $M \in \mathbb{R}^3_1$ is a regular surface that has a parametrization $\varphi : (I \times \mathbb{R}) \rightarrow \mathbb{R}^3_1$ of the form

$$\varphi(u, v) = \vec{\alpha}(u) + v \vec{\gamma}(u)$$  \hspace{1cm} (3.1)

where $\vec{\alpha}$ and $\vec{\gamma}$ are curves in $\mathbb{R}^3_1$ with $\vec{\alpha}'$ never vanishes. The curve $\vec{\alpha}$ is called the base curve. The rulings of ruled surface are the straight lines $\vec{v} \rightarrow \vec{\alpha}(u) + v \vec{\gamma}(u)$. If consecutive rulings of a ruled surface in $\mathbb{R}^3_1$ intersect, then the surface is said to be developable. All other ruled surfaces are called skew surfaces. If there exists a common perpendicular to two constructive rulings in the skew surface, then the foot of the common perpendicular on the main ruling is called a striction point.

The set of striction points on a ruled surface defines the striction curve, $\vec{\beta}(u)$ as

$$\vec{\beta}(u) = \vec{\alpha}(u) - \frac{g(\vec{\alpha}'(u), \vec{\gamma}'(u))}{\langle \vec{\gamma}', \vec{\gamma}' \rangle} \vec{\gamma}(u).$$  \hspace{1cm} (3.2)

A ruled surface given by \cite{3} is called non-cylindrical if $\vec{\gamma} \wedge \vec{\gamma}'$ is nowhere zero. Thus, the rulings are always changing directions on a non-cylindrical ruled surface. A non-cylindrical ruled surface always has a parameterization of the form

$$\tilde{\varphi}(u, v) = \vec{\beta}(u) + v \vec{\epsilon}(u)$$  \hspace{1cm} (3.3)

where $\|\vec{\epsilon}(u)\| = \frac{\tilde{\varphi}(u)}{\|\tilde{\varphi}(u)\|} = 1$, $\langle \vec{\beta}', \vec{\epsilon}'(u) \rangle = 0$ and $\vec{\beta}(u)$ is striction curve of $\tilde{\varphi}$, \cite{3}

The distribution parameter (or drall) of a non-cylindrical ruled surface given by equation (3.3), is a function $P$ defined by

$$P = \frac{\det \left( \vec{\beta}', \vec{\epsilon}, \vec{\epsilon}' \right)}{\langle \vec{\epsilon}', \vec{\epsilon}' \rangle}.$$  \hspace{1cm} (3.4)

where $\vec{\beta}$ is the striction curve and $\vec{\epsilon}$ is the director curve. Moreover, Gaussian curvature of non-cylindrical ruled surface $\tilde{\varphi}(u, v)$ is

$$K = \frac{LM - N^2}{EG - F^2}.$$  \hspace{1cm} (3.5)
where $E$, $F$ and $G$ are the coefficients of the first fundamental form, whereas $L$, $N$ and $M$ are the coefficients of the second fundamental form, of non-cylindrical ruled surface, [3].

The unit normal vector of non-cylindrical ruled surface $\tilde{\phi}$ is given by

$$\eta(u, v) = \frac{\tilde{\phi}_u \wedge \tilde{\phi}_v}{\|\tilde{\phi}_u \wedge \tilde{\phi}_v\|}.$$  

(3.6)

A surface in the 3–dimensional Minkowski space-time $\mathbb{R}^3_1$ is called a time-like surface if induced metric on the surface is a Lorentzian metric i.e. the normal on the surface is a space-like vector, [15].

In $\mathbb{R}^3_1$, according to the character of the non-null base curve and the non-null ruling, ruled surfaces are classified into three different groups. As a spacelike ruling moves along a spacelike curve it generates a spacelike ruled surface, that will be denoted by $M_1$. Furthermore, the movement of a timelike ruling along a spacelike curve and the movement of a spacelike ruling along a timelike curve generate timelike ruled surfaces. Let us denote these timelike ruled surfaces by $M_2$ and $M_3$, respectively. Now, we will establish Lamarle formula for these ruled surfaces $M_1$, $M_2$, $M_3$ separately.

4 Lamarle Formula for the Spacelike Ruled Surface

Let $M_1$ be a spacelike ruled surface parametrized by

$$\phi_1 : I \times \mathbb{R} \rightarrow \mathbb{R}^3_1$$

$$(u, v) \rightarrow \phi_1(u, v) = \tilde{\alpha}_1(u) + v \tilde{e}_1(u).$$

If we choose $\|\tilde{e}_1\| = 1$, $\tilde{n}_1 = \frac{\tilde{e}_1}{\|\tilde{e}_1\|}$ and $\tilde{\xi}_1 = \frac{\tilde{e}_1 \wedge \tilde{e}_1'}{\|\tilde{e}_1 \wedge \tilde{e}_1'\|}$, we obtain the orthonormal frame field $\{\tilde{e}_1, \tilde{n}_1, \tilde{\xi}_1\}$. Suppose that these orthonormal frame field forms right handed system and is {space, time, space} type. In this case we may write

$$\langle \tilde{e}_1, \tilde{e}_1 \rangle = 1, \quad \langle \tilde{n}_1, \tilde{n}_1 \rangle = -1, \quad \langle \tilde{\xi}_1, \tilde{\xi}_1 \rangle = 1,$$

$$\langle \tilde{e}_1, \tilde{n}_1 \rangle = \langle \tilde{n}_1, \tilde{\xi}_1 \rangle = \langle \tilde{\xi}_1, \tilde{e}_1 \rangle = 0$$  

(4.1)

and

$$\tilde{e}_1 \wedge \tilde{n}_1 = -\tilde{\xi}_1, \quad \tilde{n}_1 \wedge \tilde{\xi}_1 = -\tilde{e}_1, \quad \tilde{\xi}_1 \wedge \tilde{e}_1 = \tilde{n}_1.$$  

(4.2)

The Frenet formulae of this orthonormal frame along $e_1$ become

$$\tilde{e}_1' = \kappa_1 \tilde{n}_1, \quad \tilde{n}_1' = \kappa_1 \tilde{e}_1 + \tau_1 \tilde{\xi}_1, \quad \tilde{\xi}_1' = \tau_1 \tilde{n}_1.$$  

(4.3)

Let $\tilde{\beta}_1(u)$ be a striction curve of spacelike ruled surface $M_1$ given by equation (3.2) in $\mathbb{R}^3_1$. In this case the tangent vector $\tilde{\beta}_1'$ of this curve stays in spacelike plane. Taking the angle $\sigma_1$ to be the angle between $\tilde{\beta}_1'$ and $\tilde{e}_1$ since the tangent vector of striction curve of $M_1$ is

$$\tilde{\beta}_1' = \tilde{e}_1 \cos \sigma_1 + \tilde{\xi}_1 \sin \sigma_1$$

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we find the striction curve of $M_1$ to be 
\[ \vec{\beta}_1 = \int \left( \cos \sigma_1 \vec{e}_1 + \sin \sigma_1 \vec{\xi}_1 \right) du. \]
The spacelike non-cylindrical ruled surface $M_1$ is parametrized by 
\[ \tilde{\varphi}_1 (u, v) = \int \left( \cos \sigma_1 \vec{e}_1 + \sin \sigma_1 \vec{\xi}_1 \right) du + v \vec{e}_1. \]
From the equation (3.4) the distribution parameter of $M_1$ is found to be 
\[ P = \frac{\det \left( \cos \sigma_1 \vec{e}_1 + \sin \sigma_1 \vec{\xi}_1, \vec{e}_1, \kappa_1 \vec{n}_1 \right)}{\langle \kappa_1 \vec{n}_1, \kappa_1 \vec{n}_1 \rangle} = \sin \sigma_1 \frac{\kappa_1}{\kappa_1}. \]
Adopting $\kappa_1 = \frac{1}{\rho_1}$ we get the distribution parameter as follows 
\[ P = \rho_1 \sin \sigma_1. \] (4.4)
Considering equation (4.2) from equation (3.6) we write the unit normal tangent vector of spacelike non-cylindrical ruled surface $M_1$ 
\[ \vec{n}_1 = \sin \sigma_1 \vec{n}_1 + v \kappa_1 \vec{\xi}_1 \frac{v - \sin^2 \sigma_1 + v^2 \kappa_1^2}{\sqrt{|-P^2 + v^2|}}. \] (4.5)
Taking into consideration that $\kappa_1 = \frac{1}{\rho_1}$ and equation (4.4) we obtain 
\[ \vec{n}_1 = \frac{P \vec{n}_1 + v \vec{\xi}_1}{\sqrt{|-P^2 + v^2|}}. \] (4.6)
Furthermore, since the unit normal tangent vector $\eta_1$ of a spacelike surface $M_1$ is timelike we find that $-P^2 + v^2 < 0$, that is $|v| < |P|$. The partial differentiation of $M_1$ with respect to $u$ and $v$ from equation (4.3) are as follows 
\[ \tilde{\varphi}_1_{uu} = \cos \sigma_1 \vec{e}_1 + \sin \sigma_1 \vec{\xi}_1 + v \kappa_1 \vec{n}_1, \]
\[ \tilde{\varphi}_1_{uv} = \kappa_1 \vec{n}_1, \]
\[ \tilde{\varphi}_1_{vv} = 0. \] (4.7)
Therefore, we find the first fundamental form’s coefficients of $M_1$ to be 
\[ E = \langle \tilde{\varphi}_1_{uu}, \tilde{\varphi}_1_{uu} \rangle = \cos^2 \sigma_1 + \sin^2 \sigma_1 - v^2 \kappa_1^2 = 1 - v^2 \kappa_1^2, \]
\[ F = \langle \tilde{\varphi}_1_{uu}, \tilde{\varphi}_1_{uv} \rangle = \cos \sigma_1, \]
\[ G = \langle \tilde{\varphi}_1_{uv}, \tilde{\varphi}_1_{uv} \rangle = 1. \] (4.8)
In addition to these, the second order partial differentials of $M_1$ are found to be 
\[ \tilde{\varphi}_1_{uuv} = (\sigma'_1 \sin \sigma_1 + v \kappa_1^2) \vec{e}_1 + (\kappa_1 \cos \sigma_1 + \tau_1 \sin \sigma_1 + v \kappa_1') \vec{n}_1 + (\sigma'_1 \cos \sigma_1 + v \kappa_1 \tau_1) \vec{\xi}_1, \]
\[ \tilde{\varphi}_1_{uvv} = \kappa_1 \vec{n}_1, \]
\[ \tilde{\varphi}_1_{vvv} = 0. \]
From equation (4.5) and the last equations we get the coefficients of second fundamental of $M_1$ as

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\[ L = \langle \tilde{\varphi}_{1uu}, \tilde{\eta} \rangle = \frac{-\kappa_1 \cos \sigma_1 \sin \sigma_1 - \tau_1 \sin^2 \sigma_1 - \kappa'_1 \sin \sigma_1 + \sigma'_1 \cos \sigma_1 \sqrt{1 + \kappa_1^2 \tau_1^2}}{\sqrt{-\sin^2 \sigma_1 + v^2 \kappa_1^2}}, \]
\[ N = \langle \tilde{\varphi}_{1uv}, \tilde{\eta} \rangle = \frac{-\kappa_1 \sin \sigma_1}{\sqrt{-\sin^2 \sigma_1 + v^2 \kappa_1^2}}, \]
\[ M = \langle \tilde{\varphi}_{1vv}, \tilde{\eta} \rangle = 0. \]  
(4.9)

Considering equation (4.8) and (4.9) together, we give the following theorem for the Gaussian curvature of spacelike ruled surface \( M_1 \).

**Theorem 4.1** Let \( M_1 \) be spacelike non-cylindrical ruled surface in \( \mathbb{R}^3_1 \). The Gaussian curvature of spacelike non-cylindrical ruled surface \( M_1 \) is given in terms of its distribution parameter \( P \) by

\[ K = -\frac{P^2}{(P^2 - v^2)^2} \]  
(4.10)

where \( |v| < |P| \).

**Proof.** Substituting equations (4.8) and (4.9) into equation (3.5) and making appropriate simplifications we find the Gaussian curvature of \( M_1 \) to be

\[ K = -\frac{\kappa_1^3 \sin^3 \sigma_1}{(\sin^2 \sigma_1 - v^2 \kappa_1^2)^2}. \]

Considering \( \kappa_1 = \frac{1}{\rho_1} \) and equation (4.4) completes the proof. □

The relation between Gaussian curvature and the distribution parameter of \( M_1 \) given by equation (4.10) is called **Lorentzian Lamarle formula** for the spacelike non-cylindrical ruled surface \( M_1 \).

The Lorentzian Lamarle formula for the spacelike ruled surface in \( \mathbb{R}^3_1 \) is non-positive. Therefore we give the following corollary.

**Corollary 4.1** Let \( M_1 \) be a spacelike non-cylindrical ruled surface with distribution parameter \( P \) and Gaussian curvature \( K \) in \( \mathbb{R}^3_1 \).

1. Along a ruling the Gaussian curvature \( K(u,v) \to 0 \) as \( v \to \mp\infty \).
2. \( K(u,v) = 0 \) if and only if \( P = 0 \).
3. If the distribution parameter is \( P \) never vanishes then \( K(u,v) \) is continuous and when \( v = 0 \) i.e. at the central point on each ruling, \( K(u,v) \) assumes its maximum value.

**Example 4.1** In 3-dimensional Lorentz space \( \mathbb{R}^3_1 \) let us define a non-cylindrical ruled surface as

\[ \varphi(u,v) = (-v \cosh u, u, -v \sinh u) \]

that is a 2nd type helicoid and a spacelike surface where \(-1 < v < 1\), see Figure 4.1.
The Gaussian curvature of this 2nd type helicoid is \( K = -\frac{1}{(1-v^2)^2}, \quad |v| < 1, \) see Figure 4.2.

5 Lamarle Formula for the Timelike Ruled Surface with Spacelike Base Curve and Timelike Ruling

Let \( M_2 \) be timelike ruled surface with spacelike base curve and timelike ruling in 3-dimensional Lorentz space, \( \mathbb{R}^3_1 \). Thus, this ruled surface is parametrized as follows

\[
\varphi_2 : I \times \mathbb{R} \to \mathbb{R}^3_1 \\
(u, v) \to \varphi_2(u, v) = \vec{a}_2(u) + v \vec{e}_2(u).
\]

Here, taking \( \|\vec{e}_2\| = 1 \), \( \vec{n}_2 = \frac{\vec{e}_2}{\|\vec{e}_2\|} \) and \( \vec{\xi}_2 = \frac{\vec{e}_2 \wedge \vec{e}_2}{\|\vec{e}_2 \wedge \vec{e}_2\|} \), we reach the orthonormal frame field \( \{\vec{e}_2, \vec{n}_2, \vec{\xi}_2\} \). This forms a right handed system and in \{time, space, space\} type. Therefore,

\[
\begin{align*}
-\langle \vec{e}_2, \vec{e}_2 \rangle &= \langle \vec{n}_2, \vec{n}_2 \rangle = \langle \vec{\xi}_2, \vec{\xi}_2 \rangle = 1 \\
\langle \vec{e}_2, \vec{n}_2 \rangle &= \langle \vec{n}_2, \vec{\xi}_2 \rangle = \langle \vec{\xi}_2, \vec{e}_2 \rangle = 0
\end{align*}
\]

(5.1)

and

\[
\begin{align*}
\vec{e}_2 \wedge \vec{n}_2 &= -\vec{\xi}_2, \\
\vec{n}_2 \wedge \vec{\xi}_2 &= \vec{e}_2, \\
\vec{\xi}_2 \wedge \vec{e}_2 &= -\vec{n}_2.
\end{align*}
\]

(5.2)

The differential formulae of this orthonormal system are

\[
\begin{align*}
\vec{e}_2' &= \kappa_2 \vec{n}_2, \\
\vec{n}_2' &= \kappa_2 \vec{e}_2 - \tau_2 \vec{\xi}_2, \\
\vec{\xi}_2' &= \tau_2 \vec{n}_2.
\end{align*}
\]

(5.3)

Now, let the striction curve given by equation (3.2) of timelike ruled surface \( M_2 \) be \( \vec{\beta}_2(u) \). \( \vec{\beta}_2(u) \) is a spacelike curve and the tangent vector of this curve \( \vec{\beta}_2' \) stays in the timelike plane \( \left(\vec{e}_2, \vec{\xi}_2\right) \). Adopting the hyperbolic angle \( \sigma_2 \) between \( \vec{\beta}_2' \) and \( \vec{e}_2 \) we write

\[
\vec{\beta}_2'' = \sinh \sigma_2 \vec{e}_2 + \cosh \sigma_2 \vec{\xi}_2.
\]
From the last equation we write for the striction curve of $M_2$

$$\overline{\beta}_2 = \int (\sinh \sigma_2 \overline{e}_2 + \cosh \sigma_2 \xi_2) \, du.$$ 

Let $M_2$ be timelike non-cylindrical ruled surface with spacelike base curve and timelike ruling in $\mathbb{R}^3_1$. In this case we reparametrize $M_2$ such as

$$\tilde{\varphi}_2(u,v) = \int (\sinh \sigma_2 \overline{e}_2 + \cosh \sigma_2 \xi_2) \, du + v \, \overline{e}_2.$$ 

Considering equation (3.4) we find the distribution parameter of timelike non-cylindrical ruled surface $M_2$ to be

$$P = \det \begin{pmatrix} \sinh \sigma_2 \overline{e}_2 + \cosh \sigma_2 \xi_2, \overline{e}_2, \kappa_2 \overline{n}_2 \end{pmatrix} \langle \kappa_2 \overline{n}_2, \kappa_2 \overline{n}_2 \rangle = \frac{\cosh \sigma_2}{\kappa_2}.$$ 

Taking $\kappa_2 = \frac{1}{\rho_2}$ we rewrite the distribution parameter of $M_2$ as

$$P = \rho_2 \cosh \sigma_2. \quad (5.4)$$

If we consider equation (5.2), from equation (3.6) we see that the timelike non-cylindrical ruled surface’s unit normal vector becomes

$$\overline{n}_2 = \frac{-\cosh \sigma_2 \overline{n}_2 + v \kappa_2 \xi_2}{\sqrt{\left[\cosh^2 \sigma_2 + v^2 \kappa_2^2\right]}}. \quad (5.5)$$

Since $\kappa_2 = \frac{1}{\rho_2}$, from equation (5.4) we find

$$\overline{n}_2 = \frac{-P \overline{n}_2 + v \xi_2}{\sqrt{P^2 + v^2}}. \quad (5.6)$$

From equation (5.3), partial differential of $M_2$ with respect to $u$ and $v$ are

$$\tilde{\varphi}_{2u} = \sinh \sigma_2 \overline{e}_2 + \cosh \sigma_2 \xi_2 + v \kappa_2 \overline{n}_2,$$

$$\tilde{\varphi}_{2v} = \overline{e}_2. \quad (5.7)$$

Considering the last equations with equation (5.1) we find the coefficients of first fundamental form of $M_2$ to be

$$E = \langle \tilde{\varphi}_{2u}, \tilde{\varphi}_{2u} \rangle = -\sinh^2 \sigma_2 + \cosh^2 \sigma_2 + v^2 \kappa_2^2 = 1 + v^2 \kappa_2^2,$$

$$F = \langle \tilde{\varphi}_{2u}, \tilde{\varphi}_{2v} \rangle = -\sinh \sigma_2,$$

$$G = \langle \tilde{\varphi}_{2v}, \tilde{\varphi}_{2v} \rangle = -1. \quad (5.8)$$

Furthermore, if we consider equation, (5.7) we reach that the second order partial differentials of $M_2$

$$\tilde{\varphi}_{2uu} = \left(\sigma_2' \cosh \sigma_2 + v \kappa_2^2 \right) \overline{e}_2 + \left(\kappa_2 \sinh \sigma_2 + \tau_2 \cosh \sigma_2 + v \kappa_2' \right) \overline{n}_2 + \left(\sigma_2' \sinh \sigma_2 - v \kappa_2 \tau_2 \right) \xi_2,$$

$$\tilde{\varphi}_{2uv} = \kappa_2 \overline{n}_2,$$

$$\tilde{\varphi}_{2vv} = 0.$$
From equation (5.5) and the last equations we find the second fundamentals form’s coefficients as follows

\[ L = \langle \tilde{\varphi}_{2uu}, \tilde{\eta}_2 \rangle = -\kappa_2 \sinh \sigma_2 \cosh \sigma_2 - \tau_2 \cosh^2 \sigma_2 - v \kappa_2' \sinh \sigma_2 + v^2 \kappa_2^2 \tau_2, \]
\[ N = \langle \tilde{\varphi}_{2uv}, \tilde{\eta}_2 \rangle = \frac{\kappa_2 \cosh \sigma_2}{\sqrt{\cosh^2 \sigma_2 + v^2 \kappa_2^2}}, \]
\[ M = \langle \tilde{\varphi}_{2vv}, \tilde{\eta}_2 \rangle = 0. \]

(5.9)

Therefore, for the Gaussian curvature of timelike ruled surface \( M_2 \), we give the following theorem.

**Theorem 5.1** Let \( M_2 \) be a timelike non-cylindrical ruled surface with spacelike base curve and timelike ruling in \( \mathbb{R}^3_1 \). Taking \( P \) to be the distribution parameter of \( M_2 \) we see that the Gaussian curvature of \( M_2 \) is

\[ K = \frac{P^2}{(P^2 + v^2)^2}. \]

(5.10)

**Proof.** Substituting equations (5.8) and (5.9) into equation (3.5) we find the Gaussian curvature of \( M_2 \) to be

\[ K = \frac{\kappa_2^2 \cosh^2 \sigma_2}{(\cosh^2 \sigma_2 + \kappa_2^2 v^2)^2}. \]

Here considering \( \kappa_2 = \frac{1}{P^2} \) and equation (5.4) completes the proof. ■

The relation between Gaussian curvature and the distribution parameter of \( M_2 \) given by equation (5.10) is called **Lorentzian Lamarle formula** for the timelike non-cylindrical ruled surface with spacelike base curve and timelike ruling.

The Lamarle formula for the timelike ruled surface in \( \mathbb{R}^3_1 \) is non-negative. So, we give the following corollary.

**Corollary 5.1** Let \( P \) be a distribution parameter and \( K \) be a Gaussian curvature of a timelike non-cylindrical ruled surface \( M_2 \) with spacelike base curve and timelike ruling in \( \mathbb{R}^3_1 \). In this case

1. Along ruling as \( v \to \mp \infty \), \( K(u,v) \to 0 \).
2. \( K(u,v) = 0 \) if and only if \( P = 0 \).
3. If the distribution parameter of \( M_2 \) never vanishes, then \( K(u,v) \) is continuous and as \( v = 0 \) i.e. at the central point on each ruling, \( K(u,v) \) takes its minimum value.

**Example 5.1** In 3–dimensional Lorentz space \( \mathbb{R}^3_1 \),

\[ \varphi(u,v) = (-v \sinh u, u, -v \cosh u) \]

is a 3\(^{rd}\) type helicoid and a timelike non-cylindrical ruled surface with spacelike base curve and timelike ruling, see: Figure 5.1.
The Gaussian curvature of this 3rd type helicoid is \( K = -\frac{1}{(1+v^2)^2} \), see Figure 5.2.

6 Lamarle Formula for Timelike Ruled Surface with Timelike Base Curve and Spacelike Ruling

Suppose that the timelike ruled surface \( M_3 \) with timelike base curve and spacelike ruling in three dimensional Lorentz space \( \mathbb{R}^3_1 \) is parametrized as follows

\[
\varphi_3 : I \times \mathbb{R} \to \mathbb{R}^3_1 \\
(u, v) \to \varphi_3(u, v) = \alpha_3(u) + v \, \epsilon_3(u).
\]

Considering that \( \|\epsilon_3\| = 1 \), \( \bar{n}_3 = \frac{\epsilon_3}{\|\epsilon_3\|} \) and \( \bar{\xi}_3 = \frac{\bar{\epsilon}_3 \wedge \bar{\epsilon}_3}{\|\bar{\epsilon}_3 \wedge \bar{\epsilon}_3\|} \), we reach the orthonormal frame field \( \{\bar{\epsilon}_3, \bar{n}_3, \bar{\xi}_3\} \). This forms a right handed system which is in type \{space, space, time\}. Thus we write

\[
\langle \bar{\epsilon}_3, \bar{\epsilon}_3 \rangle = \langle \bar{n}_3, \bar{n}_3 \rangle = -\langle \bar{\xi}_3, \bar{\xi}_3 \rangle = 1
\]

\[
\langle \bar{\epsilon}_3, \bar{n}_3 \rangle = \langle \bar{n}_3, \bar{\xi}_3 \rangle = \langle \bar{\xi}_3, \bar{\epsilon}_3 \rangle = 0
\]

and cross product is defined to be

\[
\bar{e}_3 \wedge \bar{n}_3 = \bar{\xi}_3, \quad \bar{n}_3 \wedge \bar{\xi}_3 = -\bar{e}_3, \quad \bar{\xi}_3 \wedge \bar{\epsilon}_3 = -\bar{n}_3.
\]

Differential formulae for this orthonormal system is expressed by

\[
\bar{e}_3 = \kappa_3 \, \bar{n}_3, \quad \bar{n}_3' = -\kappa_3 \, \bar{e}_3 + \tau_3 \, \bar{\xi}_3, \quad \bar{\xi}_3' = \tau_3 \, \bar{n}_3.
\]

Let the striction curve of timelike ruled surface given by equation (3.2) be \( \bar{\beta}_3(u) \). This curve is a timelike curve and the tangent vector of this curve stays within the timelike plane \( \left(\bar{\epsilon}_3, \bar{\xi}_3\right) \). Adopting the hyperbolic angle \( \sigma_3 \) to be the angle between \( \bar{\beta}_3' \) and \( \bar{e}_3 \) we may write

\[
\bar{\beta}_3' = \sinh \sigma_3 \, \bar{e}_3 + \cosh \sigma_3 \, \bar{\xi}_3
\]
yielding the striction curve of $M_3$ to be

$$\vec{\beta}_3 = \int \left( \sinh \sigma_3 \vec{e}_3 + \cosh \sigma_3 \vec{\xi}_3 \right) du.$$ 

The timelike non-cylindrical ruled surface $M_3$ with timelike base curve and spacelike ruling is reparametrized by

$$\tilde{\varphi}_3 (u, v) = \int \left( \sinh \sigma_3 \vec{e}_3 + \cosh \sigma_3 \vec{\xi}_3 \right) du + v \vec{e}_3.$$ 

The distribution parameter of this ruled surface is found to be

$$P = \det \left( \sinh \sigma_3 \vec{e}_3 + \cosh \sigma_3 \vec{\xi}_3, \vec{e}_3, \kappa_3 \vec{n}_3 \right) \frac{\cosh \sigma_3}{\kappa_3}.$$ 

The distribution parameter of $M_3$ becomes

$$P = \rho_3 \cosh \sigma_3 \tag{6.4}$$

where $\kappa_3 = \frac{1}{\rho_3}$. Taking equation (6.2) into consideration, we find from equation (6.3) that the unit normal vector of timelike non-cylindrical ruled surface $M_3$ is

$$\vec{n}_3 = -\frac{\cosh \sigma_3 \vec{n}_3 - v \kappa_3 \vec{\xi}_3}{\sqrt{\cosh^2 \sigma_3 - v^2 \kappa_3^2}}. \tag{6.5}$$

Since $\kappa_3 = \frac{1}{\rho_3}$, from equation (6.4) we find

$$\vec{n}_3 = -\frac{P \vec{n}_3 + v \vec{\xi}_3}{\sqrt{P^2 - v^2}}. \tag{6.6}$$

In addition to these, since the unit normal vector $\vec{n}_3$ of timelike ruled surface $M_3$ is spacelike, that is, $P^2 - v^2 > 0$ i.e. $|P| > |v|$. The partial differentials of $M_3$ with respect to $u$ and $v$ (from equation (6.3)) becomes

$$\tilde{\varphi}_{3u} = \sinh \sigma_3 \vec{e}_3 + \cosh \sigma_3 \vec{\xi}_3 + v \kappa_3 \vec{n}_3,$$

$$\tilde{\varphi}_{3v} = \vec{e}_3.$$ 

Considering the last equations together with equation (6.1) the coefficients of first fundamental form of $M_3$ are

$$E = \langle \tilde{\varphi}_{3u}, \tilde{\varphi}_{3u} \rangle = \sinh^2 \sigma_3 - \cosh^2 \sigma_3 + v^2 \kappa_3^2 = -1 + v^2 \kappa_3^2,$$

$$F = \langle \tilde{\varphi}_{3u}, \tilde{\varphi}_{3v} \rangle = \sinh \sigma_3,$$

$$G = \langle \tilde{\varphi}_{3v}, \tilde{\varphi}_{3v} \rangle = 1. \tag{6.8}$$

Furthermore, considering equation (6.7) we find for the second order partial differentials of $M_3$ as

$$\tilde{\varphi}_{3uv} = \sigma_3' \cosh \sigma_3 + v \kappa_3^2 + \kappa_3 \sinh \sigma_3 + \tau_3 \cosh \sigma_3 + v \kappa_3 \vec{n}_3,$$

$$\tilde{\varphi}_{3uu} = \kappa_3 \vec{n}_3,$$

$$\tilde{\varphi}_{3vv} = 0.$$
From equation (6.5) and the last equations, the coefficients of the second order principal form read to be

\[
L = \langle \tilde{\phi}_{3uu}, \vec{n}_3 \rangle = -\kappa_3 \sinh \sigma_3 \cosh^2 \sigma_3 - v \kappa_3 \sinh \sigma_3 + v^2 \kappa_3^3 \tau_3, \\
N = \langle \tilde{\phi}_{3uv}, \vec{n}_3 \rangle = \kappa_3 \cosh \sigma_3 \sqrt{| \cosh^2 \sigma_3 - v^2 \kappa_3^2 |}, \\
M = \langle \tilde{\phi}_{3vv}, \vec{n}_3 \rangle = 0.
\]

(6.9)

Taking equations (6.8) and (6.9) together into account, we can give the following theorem for the Gaussian curvature of timelike ruled surface \(M_3\).

**Theorem 6.1** Let \(M_3\) be a timelike non-cylindrical ruled surface with timelike base curve and spacelike ruling in \(\mathbb{R}^3_1\). Adopting that the distribution parameter of \(M_3\) is \(P\), the Gaussian curvature of \(M_3\) becomes

\[
K = \frac{P^2}{(P^2 - v^2)^2}
\]

(6.10)

where \(P^2 - v^2 > 0\).

**Proof.** Substituting equations (6.8) and (6.9) into equation (3.5) we find the Gaussian curvature of \(M_3\) to be

\[
K = \frac{\kappa_3^2 \cosh^2 \sigma_3}{(\cosh^2 \sigma_3 + \kappa_3^4 v^2)^2}.
\]

Considering equation (6.4) together with \(\kappa_3 = \frac{1}{P_3}\) completes the proof. \(\blacksquare\)

The relation between the Gaussian curvature and the distribution parameter of timelike ruled surface given by equation (6.10) is called Lorentzian Lamarle formula for the timelike non-cylindrical ruled surface with timelike base curve and spacelike ruling.

The Lamarle formula for the timelike ruled surface in \(\mathbb{R}^3_1\) is non–negative. So, we give the following corollary.

**Corollary 6.1** Let \(M_3\) be a timelike non-cylindrical ruled surface with timelike base curve and spacelike ruling in \(\mathbb{R}^3_1\). Considering that \(P\) is the distribution parameter and \(K\) is the Gaussian curvature we see that

1. Along ruling \(v \to \mp\infty\) the Gaussian curvature \(K(u,v) \to 0\).
2. \(K(u,v) = 0\) if and only if \(P = 0\).
3. If the distribution parameter of \(M_3\) never vanishes, then \(K(u,v)\) is continuous and as \(v = 0\) i.e. at the central point on each ruling, \(K(u,v)\) takes its minimum value.

**Example 6.1** Let us parametrize a 1st type helicoid as

\[
\varphi(u,v) = (-v \cos u, -v \sin u, u)
\]
which is timelike non-cylindrical ruled surface with timelike base curve and spacelike ruling in 3-dimensional Lorentz space $\mathbb{R}^3_1$ and here $-1 < v < 1$, see: Figure 6.1.

*Figure 6.1.*

The Gaussian curvature of this 1$^{\text{st}}$ type helicoid is $K = \frac{1}{(1-v^2)^2}$, $|v| < 1$, see Figure 6.2.

*Figure 6.2.*

**Acknowledgement 1** The authors are very grateful to Prof. Dr. Ibrahim Okur for improving presentation of the paper.

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