HYPERBOLIC AND BI-HYPERBOLIC SOLUTIONS IN THE PLANAR RESTRICTED \((N + 1)\)-BODY PROBLEM

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Abstract. Consider the planar restricted \((N + 1)\)-body problem with trajectories of the \(N(\geq 2)\) primaries forming a collision-free periodic solution of the \(N\)-body problem, for any positive energy \(h\) and directions \(\theta_{\pm} \in [0, 2\pi]\), we prove that starting from any initial position \(x\) at any initial time \(t_x\), there are hyperbolic solutions \(\gamma_{\pm}\rvert_{t_x, \pm\infty}\) satisfying \(\gamma_{\pm}(t_x) = x\) and
\[
\lim_{t \rightarrow \pm\infty} \frac{\gamma_{\pm}^{-1}(t)}{|\gamma_{\pm}^{-1}(t)|} = e^{i\theta_{\pm}(\bmod 2\pi)}, \quad \lim_{t \rightarrow \pm\infty} \dot{\gamma}_{\pm}(t) = \pm \sqrt{2h}e^{i\theta_{\pm}(\bmod 2\pi)}.
\]
Moreover we also prove the existence of a bi-hyperbolic solution \(\gamma_{\pm}\) satisfying
\[
\lim_{t \rightarrow \pm\infty} \frac{\gamma(t)}{|\gamma(t)|} = e^{i\theta_{\pm}(\bmod 2\pi)}, \quad \lim_{t \rightarrow \pm\infty} \dot{\gamma}(t) = \pm \sqrt{2h}e^{i\theta_{\pm}(\bmod 2\pi)}.
\]

1. Introduction

The \(N\)-body problem studies motion of \(N\) point masses, \(m_i > 0, i \in \mathbb{N} := \{1, \ldots, N\}\), under Newton’s universal gravitational law. The trajectories of the masses \(q(t) = (q_i(t))_{i \in \mathbb{N}}\) satisfy
\[
\ddot{q}_i(t) = -\sum_{j \in \mathbb{N}\setminus\{i\}} \frac{m_j(q_i(t) - q_j(t))}{|q_i(t) - q_j(t)|^3}, \quad \forall i \in \mathbb{N}.
\]

By introducing an additional massless body into the system, we get the so-called restricted \((N + 1)\)-body problem, where the motion of the massless body is governed by the gravitational forces of the \(N\) positive masses (will be referred as primaries), while the massless body does not produce any gravitational force, so the motion of the primaries are not affected by its presence. As a result, the trajectory \(z(t)\) of the massless body satisfies
\[
\ddot{z}(t) = \partial_z U(z(t), t) = -\sum_{i \in \mathbb{N}} \frac{m_i(z(t) - q_i(t))}{|z(t) - q_i(t)|^3},
\]
where \(U(z, t)\) is a time-dependent potential function
\[
U(z, t) = \sum_{i \in \mathbb{N}} \frac{m_i}{|z - q_i(t)|}.
\]

The restricted \((N + 1)\)-body problem can be seen as certain limit of the \((N + 1)\)-body problem, when one of the masses is much smaller than the others. Hence it is a good model in describing the
motion of comets and satellites in gravitational systems, and has been studied by many authors including Poincaré and Birkhoff.

Solutions of (2) may also experience singularities at finite time. However if we assume there is no collision between the primaries, then all these singularities must be binary collisions between the massless body and one of the primaries. As is well-known such singularities can be regularized (see [23], [29] and [35]), so solutions can be extended through these binary collisions as elastic bounce. Hence the maximum domain of any solution of (2) will be the whole time interval \( \mathbb{R} \). Then following Chazy, a complete classification of all the solutions of (2) according to the final behaviors of the massless body can be given as below.

- **hyperbolic solution**: \(|z(t)| \to \infty\) and \(|\dot{z}(t)| \to c > 0\), as \( t \to \pm \infty \);
- **parabolic solution**: \(|z(t)| \to \infty\) and \(|\dot{z}(t)| \to 0\), as \( t \to \pm \infty \);
- **bounded solution**: \( \limsup_{t \to \pm \infty} |z(t)| < \infty \);
- **oscillatory solution**: \( \limsup_{t \to \pm \infty} |z(t)| < \infty \) and \( \liminf_{t \to \pm \infty} |z(t)| < \infty \).

**Remark 1.1.** \( z \rvert_\mathbb{R} \) will be called bi-hyperbolic, if both \( z \rvert_{[0,\infty)} \) and \( z \rvert_{(-\infty,0]} \) are hyperbolic solutions.

In this paper, we will study (bi)-hyperbolic solutions of the planar restricted \((N+1)\)-body problem, and we set \( \mathbb{R}^2 = \mathbb{C} \) for the rest of the paper.

**Theorem 1.1.** Assume \( q(t) = (q_i(t))_{i \in \mathbb{N}} \) is a planar collision-free periodic solution of (1), i.e.,

\[
 q(t) = q(t + T) \in \mathbb{C}^N, \quad \forall t \in \mathbb{R}, \quad \text{for some } T > 0;
\]

\[
 \rho_0 = \inf\{|q_i(t) - q_j(t)| : t \in \mathbb{R}, \ i \neq j \in \mathbb{N}\} > 0.
\]

Given arbitrarily an energy constant \( h > 0 \) and asymptotic directions \( \theta_{\pm} \in [0, 2\pi) \), for any \( x \in \mathbb{C} \) and \( t_x \in \mathbb{R} \), there are hyperbolic solutions \( \gamma_{\pm} : [t_x, \pm \infty) \to \mathbb{C} \) of (2) satisfies \( \gamma_{\pm}(t_x) = x \) and

\[
 \lim_{t \to \pm \infty} \gamma_{\pm}(t)/|\gamma_{\pm}(t)| = e^{i\theta_{\pm}(mod 2\pi)}, \quad \lim_{t \to \pm \infty} \dot{\gamma}_{\pm}(t) = \pm \sqrt{2he^{i\theta_{\pm}(mod 2\pi)}}.
\]

Moreover \( \gamma_{\pm}(t) \) must be collision-free, whenever \( t \neq t_x \).

**Remark 1.2.** For any \( t \in \mathbb{R} \), we say \( \gamma(t) \in \mathbb{C} \) is collision-free, if \( \gamma(t) \neq q_i(t) \), for all \( i \in \mathbb{N} \).

Next we will compare the above result with some previous results of the \( N \)-body problem.

**Definition 1.1.** Let \( q \rvert_{[t_0, \pm \infty)} \) be a solution of (1). We say it is a hyperbolic solution, if

\[
 |q_i(t) - q_j(t)| \approx O(|t|), \quad \text{as } t \to \pm \infty, \quad \forall \{i \neq j\} \subset \mathbb{N},
\]

it is a parabolic solution, if

\[
 |q_i(t) - q_j(t)| \approx O(|t|^{\frac{3}{2}}), \quad \text{as } t \to \pm \infty, \quad \forall \{i \neq j\} \subset \mathbb{N},
\]

and it is a hyperbolic-elliptic solution, if there is a \( K \subset \mathbb{N} \) with \( 2 \leq |K| < N \), such that

\[
 |q_i(t) - q_j(t)| \leq C_1 < \infty, \quad \forall |t| \geq t_0 \quad \text{and} \quad \forall \{i \neq j\} \subset K;
\]

\[
 |q_i(t) - q_j(t)| \approx O(|t|), \quad \text{as } t \to \pm \infty, \quad \forall (i, j) \in \mathbb{N} \times (\mathbb{N} \setminus K) \text{ with } i \neq j.
\]

**Remark 1.3.** Recall that the energy of a solution of the \( N \)-body is a constant. In particular the energy of a hyperbolic solution must be positive, and of a parabolic solution must be zero. In contrary, the energy of a hyperbolic-elliptic solution can be any real number.
In [26] Maderna and Venturelli showed that for any $h > 0$, starting from any initial configuration, there is a $h$-energy hyperbolic solution with any prescribed non-collision limiting shape at infinity (a different proof of this result was given in [24] recently). A similar result for parabolic solutions was obtained by Maderna and Venturelli in [25], when the energy is zero and the limiting shapes at infinity are minimal central configurations. While hyperbolic and parabolic solutions only affect dynamics on positive and zero energy surfaces, hyperbolic-elliptic solutions will affect dynamics at any energy surface as there is no restriction on energies of these solutions. However so far it is unclear whether similar results hold for hyperbolic-elliptic solutions.

Since hyperbolic solutions of the restricted $(N+1)$-body problem could be seen as certain limit of hyperbolic-elliptic solutions of the $(N+1)$-body problem with a single escaping mass, Theorem 1.1 may also help us understand hyperbolic-elliptic solutions of the unrestricted problem.

Under different coordinates, solutions mentioned above form stable manifolds (when time goes to positive infinity) or unstable manifolds (when time goes to negative infinity) of different invariant subsets at infinity parameterized by the limiting energy and shape or direction (see [27], [30] and [15]). The geometric and topological properties of these manifolds have great influences on the global dynamics. A basic question is: with given limiting energies and shapes or directions at infinity corresponding to negative and positive infinite time, does the corresponding stable and unstable manifolds intersect? The intersection of these stable and unstable manifolds is equivalent to the existence of solutions with the given limiting energies and shapes as time goes to negative and positive infinity.

For the $N$-body problem, this is a difficult problem with very few results. Up to our knowledge there is only some partial result in [15] for hyperbolic solutions. Here we obtain the following result for the restricted $(N+1)$-body problem.

**Theorem 1.2.** Under the assumption (**1**), given arbitrarily an energy constant $h > 0$ and a pair of asymptotic directions $\theta_\pm \in [0, 2\pi)$. There is a bi-hyperbolic solution $\gamma : \mathbb{R} \to \mathbb{C}$ of (2) satisfying
\[
\lim_{t \to \pm \infty} \gamma(t)/|\gamma(t)| = e^{i\theta_\pm (\text{mod } 2\pi)}, \quad \lim_{t \to \pm \infty} \dot{\gamma}(t) = \pm \sqrt{2h}e^{i\theta_\pm (\text{mod } 2\pi)}.
\]
Moreover $\gamma|_\mathbb{R}$ has at most one collision.

For the $N$-center problem, which is a simplified model of the restricted $(N+1)$-problem with the primaries fixed, there are much more results. In [21] and [22], (bi)-hyperbolic solutions of the planar and spatial $N$-center problem were studied correspondingly. More recently using variational methods, [6] and [3] studied bi-parabolic solutions and bi-hyperbolic solutions for the spatial $N$-center problem correspondingly, and [4] studied bi-parabolic solutions in the planar $N$-center problem.

Our proofs of Theorem 1.1 and Theorem 1.2 are variational in nature. Since the work of Chenciner and Montgomery [14], variational methods have had a lot of success in the $N$-body problem, first in proving periodic solutions (see [16], [9], [39] and the references within), and recently in proving parabolic and hyperbolic solutions (see [25] and [26] given above). The success is similar in the $N$-center problem. Besides references mentioned above, periodic solutions were studied in [33], [38], [7] and [12], while homoclinic and heteroclinic solutions were studied in [13].

Compare to the $N$-body problem and the $N$-center problem, very few results were obtained using variational methods in the restricted $(N+1)$-body problem. Up to our knowledge, so far there are only a few results for periodic solutions, see [1], [20] and [11], and no result seems to be available for parabolic or hyperbolic solutions.

The main difficulties in proving (bi)-hyperbolic solutions of the restricted $(N+1)$-body problems using variational methods are as below. First, energy is not conserved in this case as the
system is time periodic, and the energy conservation plays an important role in controlling the asymptotic behaviors at infinity. Second, to obtain a bi-hyperbolic solution as a minimizer, we need to find some proper topological constraint. Without it the minimizer is likely to be a trivial solution staying at infinity all the time. On the other hand, if the topological constraint was not chosen properly then the corresponding minimizer may be a collision-ejection solution that is not regularizable, as indicated by Gordon’s result [19] for the Kepler problem.

Our paper is organized as following: Section 2 contains results that are essential for applying action minimization methods to the restricted (\(N+1\))-body problem; Section 3 and Section 4 are devoted to the proofs of Theorem 1.1 and Theorem 1.2 correspondingly; Appendix A shows how to lower the action of a collision-ejection solution near an isolated collision by local deformation; Appendix B contains proofs of two propositions essential to control the asymptotic behavior of the massless body at infinity.

Notations. The following notations will be used throughout the paper.

1. for any \( t \in \mathbb{R}, \{ t \} \in (0,1) \) represents the fractional part of \( t \).
2. \( C_i, i = 1,2,\ldots, \) are finite positive constants that vary in different properties and proofs.
3. For any \( x \in \mathbb{C} \) and \( \delta > 0 \), \( B_\delta(x) = \{ y \in \mathbb{C} : |y-x| \leq \delta \} \) and \( B_\delta = \{ y \in \mathbb{C} : |y| \leq \delta \} \).
4. \( H^1([t_1,t_2],\mathbb{C}) \) denotes the set of all Sobolev paths defined on \([t_1,t_2] \).
5. For any \( x, y \in \mathbb{C} \) and \( t_1 < t_2 \in \mathbb{R}, \)
   \[ H^1_{t_1,t_2}(x,y) = \{ \gamma \in H^1([t_1,t_2],\mathbb{C}) : \gamma(t_1) = x, \gamma(t_2) = y \}. \]
6. For any \( \gamma \in H^1_{t_1,t_2}(x,y), \Delta(\gamma) = \{ t \in [t_1,t_2] : \gamma(t) = q_i(t), \text{ for some } i \in \mathbb{N} \} \).
7. For any \( \gamma \in H^1([t_1,t_2],\mathbb{C}) \) and \( \xi \in H^1([\tau_1,\tau_2],\mathbb{C}) \) with \( \gamma(t_2) = \xi(\tau_2) \) and \( \{ t_2 \} = \{ \tau_1 \} \), \((\gamma \ast \xi)_{|t_1,t_2+\tau_2-\tau_1} = \gamma_{|t_1,t_2} \ast \xi_{|\tau_1,\tau_2}\) represents the concatenation of \( \gamma \) and \( \xi \) as
   \[ (\gamma \ast \xi)(t) = \begin{cases} \gamma(t), & \text{when } t \in [t_1,t_2]; \\ \xi(t-t_2+\tau_1), & \text{when } t \in [t_2,t_2+\tau_2-\tau_1]. \end{cases} \]

2. Properties of action minimizers

In this section we prove some basic results that will be needed when applying action minimization methods to the restricted (\(N+1\))-body problem. Some ideas are similarly to those used in the \(N\)-body problem.

For any constant \( h \), a collision-free critical point of the action functional
\[ \mathcal{A}_h(\gamma; t_1,t_2) = \int_{t_1}^{t_2} L(\gamma(t), \dot{\gamma}(t), t) + h dt; \quad L(\gamma, \dot{\gamma}, t) = \frac{1}{2} |\dot{\gamma}(t)|^2 + U(\gamma(t), t), \]
is a solution of the restricted problem (2).

Definition 2.1. We say \( \gamma \in H^1_{t_1,t_2}(x,y) \) is a fixed-end local minimizer of \( \mathcal{A}_h \), if \( \exists \varepsilon > 0 \), such that
\[ \mathcal{A}_h(\gamma; t_1,t_2) \leq \mathcal{A}_h(\xi; t_1,t_2), \quad \forall \xi \in H^1_{t_1,t_2}(x,y), \text{ satisfying } ||\xi - \gamma||_{H^1} \leq \varepsilon. \]

We say \( \gamma \in H^1_{t_1,t_2}(x,y) \) a fixed-end minimizer of \( \mathcal{A}_h \), if
\[ \mathcal{A}_h(\gamma; t_1,t_2) = \wp_h(x,y; t_1,t_2) := \inf \{ \mathcal{A}_h(\gamma) : \gamma \in H^1_{t_1,t_2}(x,y) \}. \]

For simplicity, a fixed-end (local) minimizer will be called a (local) minimizer in the following.

Proposition 2.1. If \( \gamma \in H^1([t_1,t_2],\mathbb{C}) \) is a local minimizer of \( \mathcal{A}_h \) with finite action value, then its set of collision moments \( \Delta(\gamma) \) is an isolated subset in \([t_1,t_2]\).
Proof. As for any $t \in \Delta(\gamma)$, $U(\gamma(t), t) = \infty$, the Lebesgues measure of $\Delta(\gamma)$ must be zero. Therefore $[t_1, t_2] \setminus \Delta(\gamma)$ is the union of some (at most countable) open intervals.

Choose a $t_0 \in \Delta(\gamma) \cap (t_1, t_2)$ (if $t_1$ or $t_2$ $\in \Delta(\gamma)$, the proof is similar and will be omitted). Then $\gamma(t_0) = q_{i_0}(t_0)$, for some $i_0 \in \mathbb{N}$. By [4], we can find a $\delta > 0$ small enough, such that

$$\forall t \in [t_0 - \delta, t_0 + \delta], \begin{cases} |\gamma(t) - q_i(t)| \leq \rho_0/2, & \text{if } i = i_0; \\ |\gamma(t) - q_i(t)| \geq \rho_0/2, & \text{if } i \in \mathbb{N} \setminus \{i_0\}. \end{cases}$$

(5)

Notice that for any $[\tau_1, \tau_2] \subset [t_0 - \delta, t_0 + \delta] \setminus \Delta(\gamma)$, $\gamma|_{[\tau_1, \tau_2]}$ is a smooth solution of (2), as it is a collision-free local minimizer of $A_h$. Then $E_{i_0} \in C^1([t_0 - \delta, t_0 + \delta] \setminus \Delta(\gamma), \mathbb{R})$, where

$$E_{i_0}(t) = \frac{1}{2} |\gamma(t) - \dot{q}_{i_0}(t)|^2 - \frac{m_{i_0}}{|\gamma(t) - q_{i_0}(t)|}. \quad (6)$$

Moreover we have the following lemma, whose proof will be postponed.

**Lemma 2.1.** $E_{i_0}(t)$ can be extended continuously to $[t_0 - \delta, t_0 + \delta]$.

By a contradiction argument, let’s assume $t_0$ is not isolated in $\Delta(\gamma)$. Set $I(t) = |\gamma(t) - q_{i_0}(t)|^2$, for any $t$, the we can find a sequence of times $\{\tau_n(\neq t_0)\}_{n \in \mathbb{N}}$ satisfying

$$\lim_{n \to \infty} \tau_n = t_0 \quad \text{and} \quad I(\tau_n) = 0, \quad \forall n \in \mathbb{N}.$$ 

We may further assume $\{\tau_n\}_{n \in \mathbb{N}}$ is monotone and $(\tau_{2k-1}, \tau_{2k}) \cap \Delta(\gamma) = \emptyset$, $\forall k \in \mathbb{N}$. Then for each $k \in \mathbb{N}$, there is a $s_k \in (\tau_{2k-1}, \tau_{2k})$, such that

$$I(s_k) = 0, \quad \forall k \in \mathbb{N}. \quad (7)$$

By [5] and Lemma 2.1, for any $t \in [t_0 - \delta, t_0 + \delta] \setminus \Delta(\gamma)$, there is a $C_1$, such that

$$\dot{I}(t) = 2 \frac{m_{i_0}}{|\gamma(t) - q_{i_0}(t)|} + 4E_{i_0}(t) - 2(\gamma(t) - q_{i_0}(t), \sum_{i \neq i_0} \frac{m_i(\gamma(t) - q_i(t))}{|\gamma(t) - q_i(t)|^3} + \dot{q}_{i_0}(t))$$

$$\geq 2m_{i_0} I^{-\frac{1}{2}}(t) - C_1. \quad (8)$$

Since $I(t_0) = 0$, we can find a $\delta_0 \in (0, \delta)$, such that

$$\dot{I}(t) > 0, \quad \forall t \in [t_0 - \delta_0, t_0 + \delta] \setminus \Delta(\gamma) \quad (9)$$

As $\lim_{k \to \infty} s_k = t_0$, we have $\dot{I}(s_k) > 0$, for $k$ large enough, which is a contradiction to (7). \qed

**Proof of Lemma 2.1.** Since $\max\{|\dot{q}_{i_0}(t)|, |\dot{q}_{i_0}(t)|: t \in [t_1, t_2]\} < C_2$, for some constant $C_2$,

$$\int_{t_0 - \delta}^{t_0 + \delta} |\dot{\gamma} - \dot{q}_{i_0}|^2 dt \leq \int_{t_0 - \delta}^{t_0 + \delta} |\dot{\gamma}|^2 dt + 2C_2 \int_{t_0 - \delta}^{t_0 + \delta} |\dot{\gamma}| dt + 2\delta C_2^2$$

$$\leq \int_{t_0 - \delta}^{t_0 + \delta} |\dot{\gamma}|^2 dt + 2C_2 \left( 2\delta \int_{t_0 - \delta}^{t_0 + \delta} |\dot{\gamma}|^2 dt \right)^{\frac{1}{2}} + 2\delta C_2^2$$

$$\leq 2A(\gamma; t_1, t_2) + 4C_2 \sqrt{\delta} A(\gamma; t_1, t_2) + 2\delta C_2^2 < \infty.$$ 

By the definition of $E_{i_0}(t)$,

$$\int_{t_0 - \delta}^{t_0 + \delta} |E_{i_0}(t)| dt \leq \frac{1}{2} \int_{t_0 - \delta}^{t_0 + \delta} |\dot{\gamma} - \dot{q}_{i_0}|^2 dt + A(\gamma; t_1, t_2) < \infty.$$
Meanwhile for any $t \in [t_1, t_2] \setminus \Delta(\gamma)$,
\[
\dot{E}_{i_0} = \langle \dot{\gamma} - \dot{q}_{i_0}, \dot{\gamma} - \dot{q}_{i_0} \rangle + \frac{m_{i_0}(\gamma - q_{i_0})}{|\gamma - q_{i_0}|^3}, \dot{\gamma} - \dot{q}_{i_0} = -(\dot{q}_{i_0} + \sum_{i \neq i_0} m_i(\gamma - q_i)/|\gamma - q_i|^3, \dot{\gamma} - \dot{q}_{i_0}).
\]

By (\ref{ineq:energy}), $|\dot{E}_{i_0}| \leq C_3|\dot{\gamma} - \dot{q}_{i_0}|$, for some finite $C_3$. Then the following implies $\dot{E}_{i_0} \in L^1([t_0 - \delta, t_0 + \delta], \mathbb{R})$
\[
\int_{t_0 - \delta}^{t_0 + \delta} |\dot{\gamma} - \dot{q}_{i_0}| \, dt \leq \left(2\delta \int_{t_0 - \delta}^{t_0 + \delta} |\dot{\gamma} - \dot{q}_{i_0}|^2 \, dt \right)^{1/2}. 
\]
As a result, $E_{i_0} \in W^{1, 1}([t_0 - \delta, t_0 + \delta], \mathbb{R}) \subset C([t_0 - \delta, t_0 + \delta], \mathbb{R})$. □

By the above result, let’s assume $\gamma \in H^1([t_0 - \delta, t_0 + \delta], \mathbb{R})$ is a collision-ejection solution of (\ref{eq:kepler}), i.e., $\gamma(t)$ is collision-free and satisfies (\ref{eq:kepler}), $\forall t \neq t_0$, and $\gamma(t_0) = q_{i_0}(t_0)$, for some $i_0 \in \mathbb{N}$. Then we have the following two propositions, whose proofs can be found in Appendix A.

**Proposition 2.2.** There exist $\sigma_\pm \in \mathbb{C}$ and $f^\pm \in C^0([t_0 - \delta, t_0 + \delta], \mathbb{C}) \cap C^1([t_0 - \delta, t_0 + \delta] \setminus \{t_0\}, \mathbb{C})$ with $f^\pm(t) \simeq o(|t - t_0|^{2/3})$ and $\dot{f}^\pm(t) \simeq o(|t - t_0|^{-1/3})$, when $t \to t_0$, such that the following hold.

(a). $\sigma_\pm = \lim_{t \to t_0^\pm} \frac{\gamma(t) - q_{i_0}(t)}{\gamma(t) - q_{i_0}(t)}$.

(b). When $t \in (t_0, t_0 + \delta]$,
\[
\gamma(t) - q_{i_0}(t) = \left(\frac{9}{2} m_{i_0}\right)^{1/2} (t - t_0)^{3/2} \sigma_+ + f^+(t);
\]
\[
\dot{\gamma}(t) - \dot{q}_{i_0}(t) = \frac{9}{3} \left(\frac{9}{2} m_{i_0}\right)^{1/2} (t - t_0)^{-1/2} \sigma_+ + \dot{f}^+(t).
\]

(c). When $t \in [t_0 - \delta, t_0)$,
\[
\gamma(t) - q_{i_0}(t) = \left(\frac{9}{2} m_{i_0}\right)^{1/2} (t - t_0)^{3/2} \sigma_- + f^-(t);
\]
\[
\dot{\gamma}(t) - \dot{q}_{i_0}(t) = \frac{9}{3} \left(\frac{9}{2} m_{i_0}\right)^{1/2} (t - t_0)^{-1/2} \sigma_- + \dot{f}^-(t).
\]

**Proposition 2.3.** When $\sigma_- \neq \sigma_+$, for any $\varepsilon > 0$ small enough, there are two collision-free paths $\eta^\pm \in H^1_{t_0 - \delta, t_0 + \delta}(\gamma(t_0 - \delta), \gamma(t_0 + \delta))$ satisfying $A(\eta^\pm; t_0 - \delta, t_0 + \delta) < A(\gamma; t_0 - \delta, t_0 + \delta)$ and
\[
|\eta^\pm(t) - \gamma(t)| \leq \varepsilon, \quad \forall t \in [t_0 - \delta, t_0 + \delta].
\]
Moreover for $\delta$ small enough, we have
\[
\text{Arg}(\eta^+(t_0 + \delta) - q_{i_0}(t_0 + \delta)) - \text{Arg}(\eta^+(t_0 - \delta) - q_{i_0}(t_0 - \delta)) \in (0, 2\pi);
\]
\[
\text{Arg}(\eta^-(t_0 + \delta) - q_{i_0}(t_0 + \delta)) - \text{Arg}(\eta^-(t_0 - \delta) - q_{i_0}(t_0 - \delta)) \in (-2\pi, 0).
\]

**Remark 2.1.** A similar result as the above proposition was obtained in \cite{7} for the perturbed Kepler problem. However it does not imply our result, as the trajectories of the primaries were given a priori, so can not be deformed.

**Proposition 2.4.** If $\gamma \in H^1([t_1, t_2], \mathbb{C})$ is a local minimizer of $A_h$, then $\gamma|_{(t_1, t_2)}$ is a collision-free solution of (\ref{eq:kepler}).

**Proof.** This follows immediately from Proposition 2.1 and 2.3. □

**Proposition 2.5.** For any $x, y \in \mathbb{C}$ and $t_1 < t_2$, there is a $\gamma \in H^1_{t_1, t_2}(x, y)$ satisfying $A_h(\gamma; t_1, t_2) = \psi_h(x, y; t_1, t_2)$. 

\[\text{Arg}(\eta^+(t_0 + \delta) - q_{i_0}(t_0 + \delta)) - \text{Arg}(\eta^+(t_0 - \delta) - q_{i_0}(t_0 - \delta)) \in (0, 2\pi);
\]
\[
\text{Arg}(\eta^-(t_0 + \delta) - q_{i_0}(t_0 + \delta)) - \text{Arg}(\eta^-(t_0 - \delta) - q_{i_0}(t_0 - \delta)) \in (-2\pi, 0).
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**Proof.** This follows immediately from Proposition 2.1 and 2.3. □

**Proposition 2.5.** For any $x, y \in \mathbb{C}$ and $t_1 < t_2$, there is a $\gamma \in H^1_{t_1, t_2}(x, y)$ satisfying $A_h(\gamma; t_1, t_2) = \psi_h(x, y; t_1, t_2)$. 

\[\text{Arg}(\eta^+(t_0 + \delta) - q_{i_0}(t_0 + \delta)) - \text{Arg}(\eta^+(t_0 - \delta) - q_{i_0}(t_0 - \delta)) \in (0, 2\pi);
\]
\[
\text{Arg}(\eta^-(t_0 + \delta) - q_{i_0}(t_0 + \delta)) - \text{Arg}(\eta^-(t_0 - \delta) - q_{i_0}(t_0 - \delta)) \in (-2\pi, 0).
\]
Proof. As $A_h$ is weakly lower semi-continuous in $H^1_{t_1,t_2}(x,y)$, by a standard argument from the direct method of calculus of variation, the existence of such a $\gamma$ follows once we can show $A_h$ is coercive in $H^1_{t_1,t_2}(x,y)$, i.e., if a sequence $\{\gamma_n \in H^1_{t_1,t_2}(x,y)\}$ satisfies $\|\gamma_n\|_{H^1} \to \infty$, as $n \to \infty$, then $A_h(\gamma_n) \to \infty$ as well.

By a contradiction argument, let’s assume $\|\gamma_n\|_{H^1} \to \infty$, as $n \to \infty$ and there is a finite $C_1$, such that $A_h(\gamma_n) \leq C_1$, $\forall n$. Then the Cauchy-Schwartz inequality implies

$$\int_{t_1}^{t_2} |\gamma_n| dt \leq (t_2 - t_1)^{\frac{1}{2}} \left( \int_{t_1}^{t_2} |\gamma_n|^2 dt \right)^{\frac{1}{2}} \leq \sqrt{2(t_2 - t_1)}C_1.$$ 

As a result,

$$|\gamma_n(t)| \leq |x| + \int_{t_1}^{t_2} |\gamma_n(\tau)| d\tau \leq |x| + \sqrt{2(t_2 - t_1)}C_1, \forall t \in [t_1, t_2].$$

Then the following inequality holds for all $n$, which is a contradiction.

$$\|\gamma_n\|_{H^1}^2 = \int_{t_1}^{t_2} |\gamma(t)|^2 + |\dot{\gamma}(t)|^2 dt \leq (|x| + \sqrt{2(t_2 - t_1)}C_1)^2(t_2 - t_1) + 2C_1. \quad \Box$$

By \cite{5} or the periodicity of $q(t)$, we can find an $R_0 > 0$, such that

$$\sup\{|q_i(t)|: t \in \mathbb{R}, i \in \mathbb{N}\} = R_0 - 1. \quad (12)$$

Lemma 2.2. Given any $(x, y) \in \mathbb{C} \times (\mathbb{C} \setminus B_{R_0}^2)$ and $t_1 < t_2$, there is a $C_1$ with

$$\sup\{\psi_h(x, z; t_1, t_2) : z \in B_{1/2}(y)\} \leq C_1. \quad \text{Proof. By Proposition \cite{2.5}, there is a } \gamma \in H^1_{t_1,t_2}(x,y) \text{ with } A_h(\gamma; t_1, t_2) = \psi_h(x, y; t_1, t_2). \text{ Then }$$

$$\tau_0 = \min\{t \in [t_1, t_2] : \gamma(t) \in B_{1/2}(y)\} < t_2.$$

Notice that for any $z \in B_{1/2}(y)$, $|z| \geq |y| - |y - z| \geq R_0 - 1/2$, then there is a $C_2 > 0$, such that

$$U(z, t) \leq C_2, \forall z \in B_{1/2}(y) \text{ and } t \in \mathbb{R}. \quad (13)$$

Now for any $z \in B_{1/2}(y)$, we define a new path

$$\xi(t) = \begin{cases} \gamma(t), & \text{if } t \in [t_1, \tau_0], \\ \gamma(\tau_0) + \frac{t - \tau_0}{\tau_2 - \tau_0}(z - \gamma(\tau_0)), & \text{if } t \in [\tau_0, t_2]. \end{cases}$$

Since $\xi(t) \in B_{1/2}(y), \forall t \in [\tau_0, t_0]$, by \cite{13},

$$\psi_h(x, z; t_1, t_2) - \psi_h(x, y; t_1, t_2) \leq \int_{t_1}^{t_2} L(\xi, \dot{\xi}, t) \leq \int_{\tau_0}^{t_2} L(\xi, \dot{\gamma}, t) dt$$

$$= \int_{\tau_0}^{t_2} \frac{1}{2} |\dot{\gamma}(t)|^2 + U(\xi(t), t) dt \leq \frac{|z - \gamma(\tau_0)|^2}{2(t_2 - \tau_0)} + C_2(t_2 - \tau_0)$$

$$\leq \frac{1}{2(t_2 - \tau_0)} + C_2(t_2 - \tau_0).$$

As a result, $C_1 = \psi_h(x, y; t_1, t_2) + C_2(t_2 - \tau_0) + 2^{-1}(t_2 - \tau_0)^{-1}. \quad \Box$

Proposition 2.6. Fix arbitrarily an $x \in \mathbb{C}$ and a $t_1 \in \mathbb{R}$, $(y, t_2) \mapsto \psi_h(x, y; t_1, t_2)$ is locally Lipschitz continuous in $(\mathbb{C} \setminus B_{R_0}^2) \times \{t_2 \in \mathbb{R} : t_2 > t_1\}$.\]
Proof. Since $h$ is a constant, it is enough to prove the result for $h = 0$.

Let’s fix an arbitrary $t_2 > t_1$, and prove the local Lipschitz continuity of $\phi_0$ with respect to $y$ first. Choose a $y_0 \in \mathbb{C} \setminus B_{R_0}^\circ$, and a $\delta \in (0, 1/4]$ small enough. For any $y \in B_\delta(y_0)$, by Proposition 2.3, there is a $\gamma_y \in H_{t_1,t_2}^1(x,y)$ satisfying $A_0(\gamma_y; t_1, t_2) = \psi_0(x,y; t_1, t_2)$. Let

$$\tau_y = \min \{ t_0 \in [t_1,t_2] : \gamma_y(t) \in B_{1/2}(y_0), \forall t \in [t_0,t_2] \}.$$ We claim

$$\tau = \sup \{ \tau_y : y \in B_\delta(y_0) \text{ with } \tau_y > t_1 \} < t_2.$$ Notice that we only need to consider the cases, when $\tau_y > t_1$, and in these case, we must have $|\gamma_y(\tau_y) - y_0| = 1/2$. Then by Lemma 2.2, there is a $C_1 > 0$, such that

$$C_1 \geq A_0(\gamma_y; t_1, t_2) \geq \frac{1}{2} \int_{\tau_y}^{t_2} |\gamma_y| dt \geq \frac{1}{2} \int_{\tau_y}^{t_2} |\gamma_y| dt \geq \frac{|y - \gamma_y(\tau_y)|^2}{2(t_2 - \tau_y)} \geq \frac{(\gamma_y(\tau_y) - y_0| - |y_0 - y|)^2}{2(t_2 - \tau_y)} \geq \frac{(\frac{1}{2} - \delta)^2}{2(t_2 - \tau_y)}.$$ As a result $t_2 - \tau_y \geq (1 - 2\delta)^2/(8C_1) > 0$, and this implies (14).

Next we show that there must be an $s_y \in [\tau, t_2]$ with

$$|\gamma_y(s_y)| \leq \sqrt{2C_1(t_2 - t_1)/(t_2 - \tau)}.$$ Otherwise $\int_{\tau}^{t_2} |\gamma_y| dt > \sqrt{2C_1(t_2 - t_1)}$, which is a contradiction to

$$\int_{\tau}^{t_2} |\gamma_y| dt \leq \int_{t_1}^{t_2} |\gamma_y| dt \leq \sqrt{2(t_2 - t_1)} A_0(\gamma_y; t_1, t_2) \leq \sqrt{2C_1(t_2 - t_1)}.$$ Meanwhile since $\forall t \in [\tau, t_2)$, $\gamma_y(t)$ satisfies (2), and $\gamma_y(t) \in B_{1/2}(y_0) \subset (\mathbb{C} \setminus B_{R_0-1/2})$, there is a $C_2 > 0$, such that

$$|\gamma_y(t)| \leq |\nabla U(\gamma_y(t), t)| \leq C_2.$$ Therefore for any $t \in [\tau, t_2]$ and $y \in B_\delta(y_0)$,

$$|\gamma_y(t)| \leq |\gamma_y(s_y)| + \int_{\tau}^{t_2} |\gamma_y(s)| ds \leq \sqrt{2C_1(t_2 - t_1)/t_2 - \tau} + C_2(t_2 - \tau) = C_3.$$ Let $\delta_0 \equiv \text{min}\{\delta, t_2 - t_1\}$. For any $y_0 \in B_{\delta_0}(y_0)$, define a new path $\xi \in H_{t_2-\Delta y,t_2}^1(\gamma_y(t_2-\Delta y), y_0)$ as below, where $\Delta y = |y - y_0|$,

$$\xi_y(t) = \gamma_y(t_2 - \Delta y) + \frac{t + \Delta y - t_2}{\Delta y}(y_0(t_2 - \Delta y)),$$ for $t \in [t_2 - \Delta y, t_2]$. By the triangle inequality and (16),

$$t \int_{t_2 - \Delta y}^{t_2} |\xi_y|^2 dt \leq \frac{1}{\Delta y} \left( \int_{t_2 - \Delta y}^{t_2} |\xi_y| dt \right)^2 \leq \frac{|y - \gamma_y(t_2 - \Delta y)|^2}{\Delta y} \leq \frac{(|y - y_0| + |y_0 - \gamma_y(t_2 - \Delta y)|^2)}{\Delta y} \leq \frac{\left( \Delta y + t_2 - \Delta y |\gamma_y| dt \right)^2}{\Delta y} \leq (C_3 + 1)^2 \Delta y.$$
As there is a $C_4 > 0$, such that $U(z, t) \leq C_4$, for any $z \in \mathbb{C} \setminus B^0_{\eta_0-1/2}$ and $t \in \mathbb{R}$,
\[
\psi_0(x, y; t_1, t_2) - \psi_0(x, y_0; t_1, t_2) \leq \int_{t_1-\Delta y}^{t_2} \frac{1}{2} \dot{\xi}_y^2 + U(\xi(t), t) \, dt \leq \frac{1}{2}(C_3 + 1)^2 \Delta y + C_4 \Delta y.
\]
Meanwhile we can also define a path $\eta_y \in H^1_{t_2-\Delta y, t_2}(\gamma_y(t_2 - \Delta y), y)$ as
\[
\eta_y(t) = \gamma_y(t_2 - \Delta y) + \frac{t + \Delta y - t_2}{\Delta y} (y - \gamma_y(t_2 - \Delta y)), \quad t \in [t_2 - \Delta y, t_2].
\]
Then a similar argument as above shows
\[
\psi_0(x, y_0; t_1, t_2) - \psi_0(x, y; t_1, t_2) \leq \int_{t_1-\Delta y}^{t_2} \frac{1}{2} |\eta_y|^2 + U(\eta_y(t), t) \, dt \leq \frac{1}{2}(C_3 + 1)^2 \Delta y + C_4 \Delta y.
\]
These results imply that for any $y \in B_\delta(y_0)$,
\[
|\psi_0(x, y; t_1, t_2) - \psi_0(x, y_0; t_1, t_2)| \leq \frac{(C_3 + 1)^2 + 2C_4}{2} |y - y_0| = C_5 |y - y_0|.
\]
This finishes our proof of the local Lipschitz continuity of $\phi_0$ with respect to $y$.

To show the local Lipschitz continuity of $\phi_0$ with respect to $t_2$. Let $\gamma_y \in H^1_{t_1, t_2}(x, y)$ be a minimizer of $A_0$. Now choose a $\Delta t > 0$ small enough, then similar arguments as above show
\[
|\gamma_y(t)| \leq C_3 \text{ and } U(\gamma_y(t), \tau) \leq C_4, \quad \forall t \in [t_2 - \Delta t, t_2] \text{ and } \tau \in \mathbb{R}.
\]
This implies
\[
|y - \gamma_y(t_2 - \Delta t)| \leq \int_{t_2-\Delta t}^{t_2} |\gamma_y(t)| \, dt \leq C_3 \Delta t;
\]
\[
A_0(\gamma_y; t_2 - \Delta t, t_2) = \int_{t_2-\Delta t}^{t_2} \frac{1}{2} |\gamma_y(t)|^2 + U(\gamma_y(t), t) \, dt \leq \frac{1}{2}(C_3^2 + C_4) \Delta t.
\]
Combining these inequalities with the Lipschitz continuity with respect to $y$, we get
\[
|\psi_0(x, y; t_1, t_2) - \psi_0(x, y; t_1, t_2 - \Delta t)|
\leq A_0(\gamma_y; t_2 - \Delta t, t_2) + |\psi_0(x, \gamma_y(t_2 - \Delta t); t_1, t_2 - \Delta t) - \psi_0(x, y; t_1, t_2 - \Delta t)|
\leq \frac{1}{2} (C_3^2 + C_4) \Delta t + C_5 |y - \gamma_y(t_2 - \Delta t)| \leq \frac{1}{2} (C_3^2 + C_4 + C_3C_5) \Delta t.
\]
On the other hand,
\[
|\psi_0(x, y; t_1, t_2 + \Delta t) - \psi_0(x, y; t_1, t_2)|
\leq |\psi_0(\gamma_y(t_2 - \Delta t), y; t_2, t_2 + \Delta t)| + |\psi_0(x, \gamma_y(t_2 - \Delta t); t_1, t_2) - \psi_0(x, y; t_1, t_2)|
\leq |\psi_0(\gamma_y(t_2 - \Delta t), y; t_2, t_2 + \Delta t)| + C_6 |y - \gamma_y(t_2 - \Delta t)|
\leq |\psi_0(\gamma_y(t_2 - \Delta t), y; t_2, t_2 + \Delta t)| + C_3C_6 \Delta t.
\]
Now define a new path $\tilde{\gamma}_y \in H^1_{t_2, t_2 + \Delta t}(\gamma_y(t_2 - \Delta t), y)$ as $\tilde{\gamma}_y(s) = \gamma_y(s - \Delta t), \forall s \in [t_2, t_2 + \Delta t]$. With (17), we get
\[
|\psi_0(\gamma_y(t_2 - \Delta t), y; t_2, t_2 + \Delta t)| \leq A_0(\tilde{\gamma}_y; t_2, t_2 + \Delta t) \leq \frac{1}{2}(C_3^2 + C_4) \Delta t.
\]
As a result,
\[
|\psi_0(x, y; t_1, t_2 + \Delta t) - \psi_0(x, y; t_1, t_2)| \leq \frac{1}{2} (C_3^2 + C_4 + C_3C_6) \Delta t.
\]
This proves the local Lipschitz continuity of $\phi_0$ with respect to $t_2$. \qed
To control the asymptotic energy of the massless body as it goes to infinity, instead of minimizer with the same starting and ending time, we need to consider minimizers with varying starting and ending time. However as the system is $T$-periodic the time difference must belong to $T\mathbb{Z}$. For simplicity, we will only state and prove these results for $T = 1$ (it is easy to generalize the following results to arbitrary $T > 0$). Notice that

In the rest of the section, we assume $q(t)$ is 1-periodic, i.e., $T = 1$ in \[14\].

**Definition 2.2.** For any $x, y \in \mathbb{C}$ and $s_1, s_2 \in [0, 1)$, we say $\gamma \in H^1_{t_1, t_2}(x, y)$ is a free-time minimizer of $A_h$, if $\{t_i\} = s_i$, $i = 1, 2$, and $A_h(\gamma; t_1, t_2) = \phi_h(x, y; s_1, s_2)$, where

$$
\phi_h(x, y; s_1, s_2) := \inf_{\gamma} \{A_h(\gamma) : \gamma \in H^1_{\tau_1, \tau_2}(x, y) \text{ with } \tau_1 < \tau_2, \{\tau_i\} = s_i, i = 1, 2\};
$$

**Remark 2.2.** Since the Lagrangian $L$ is 1-periodic, if $\gamma|_{[t_1, t_2]}$ is a free-time minimizer of $A_h$, then so is $\gamma|_{[\tau_1, \tau_2]}$, for any $t_1 \leq \tau_1 < \tau_2 \leq t_2$.

**Proposition 2.7.** When $h > 0$, for any $x \neq y \in \mathbb{C}$ and $s_1, s_2 \in [0, 1)$, there exist $t_1 < t_2$ with $\{t_i\} = s_i$, $i = 1, 2$ and a $\gamma \in H^1_{t_1, t_2}(x, y)$ satisfying $A_h(\gamma; t_1, t_2) = \phi_h(x, y; s_1, s_2)$. Moreover $\gamma|_{(t_1, t_2)}$ is a collision-free solution of \[2\].

**Proof.** First let’s assume $s_1 < s_2$, then

$$
\phi_h(x, y; s_1, s_2) = \inf_{n \in \mathbb{Z}^+ \cup \{0\}} \psi_h(x, y; s_1, s_2 + n).
$$

Since $h > 0$, $\psi_h(x, y; s_1, s_2 + n) \geq h(s_2 - s_1 + n) \to \infty$, when $n \to \infty$. As a result,

$$
\phi_h(x, y; s_1, s_2) = \psi_h(x, y; s_1, s_2 + n_0), \text{ for some } n_0 \in \mathbb{Z}^+ \cup \{0\}.
$$

By Proposition 2.5 there is a $\gamma \in H^1_{s_1, s_2 + n_0}(x, y)$ satisfying

$$
A_h(\gamma; s_1, s_2 + n_0) = \psi_h(x, y; s_1, s_2 + n_0) = \phi_h(x, y; s_1, s_2),
$$

and $\gamma|_{(s_1, s_2 + n_0)}$ is a collision-free solution of \[2\].

When $s_1 \geq s_2$, $\phi_h(x, y; s_1, s_2) = \inf_{n \in \mathbb{Z}^+} \psi_h(x, y; s_1, s_2 + n)$, and the rest of the proof is the exactly the same as above. \(\square\)

**Proposition 2.8.** When $h > 0$, for any $x \neq y$ and $s \in [0, 1)$, there is a $s^* \in [0, 1)$, such that

$$
\phi_h(x, y; s, s^*) = \inf \{\phi_h(x, y; s, t) : t \in [0, 1)\},
$$

**Proof.** As the infimum in the above equation is finite, there is a sequence $\{s_n \in [0, 1)\}_{n \in \mathbb{Z}^+}$ with

$$
\lim_{n \to \infty} \phi_h(x, y; s, s_n) = \inf \{\phi_h(x, y; s, t) : t \in [0, 1)\} \leq C_1.
$$

By Proposition 2.7, for each $n \in \mathbb{Z}^+$, there is a $k_n \in \mathbb{Z}^+ \cup \{0\}$ and a $\gamma_n \in H^1_{s, s_n + k_n}(x, y)$ satisfying

$$
A_h(\gamma_n; s, s_n + k_n) = \phi_h(x, y; s, s_n).
$$

We claim $\{k_n\}_{n \in \mathbb{Z}^+}$ must have a finite upper bound. Otherwise

$$
\phi_h(x, y; s, s_n) = A_h(\gamma_n; s, s_n + k_n) \geq h k_n \to \infty, \text{ as } n \to \infty,
$$

which is a contradiction to \[18\].

This implies the existence of a finite $t^* \in \mathbb{R}$ with (after passing to a subsequence) $s_n + k_n \to t^*$, as $n \to \infty$. Since $\gamma_n(s) = x, \forall n$ and

$$
\int_s^{s_n + k_n} |\gamma_n'|^2 \, dt \leq 2 A_h(\gamma_n; s, s_n + k_n) = 2 \phi_h(x, y; s, s_n) \leq 2 C_1,
$$
By a similar argument as in the proof of Proposition 2.5, we can show \( \{\gamma_n|_{s,s_n+k_n}\}_{n \in \mathbb{Z}^+} \) is a bounded sequence under the \( H^1 \)-norm. Then there is a \( \gamma \in H^1_{s,t^*}(x,y) \), such that (after passing to a proper subsequence) \( \gamma_n \) will converge weakly to \( \gamma \) with respect to the \( H^1 \)-norm, as \( n \to \infty \). The weakly lower semi-continuity of \( A_h \) then implies
\[
\phi_h(x,y; s, s^*) \leq A_h(\gamma; s, t^*) \leq \liminf_{n \to \infty} A_h(\gamma_n; s_n, s_n + k_n) = \inf \{\phi_h(x,y; s, t) : t \in [0,1]\},
\]
where \( s^* = \{t^*\} \). The desired result then follows from the fact that
\[
\inf \{\phi_h(x,y; s, t) : t \in [0,1]\} \leq \phi_h(x,y; s, s^*).
\]

3. Existence of hyperbolic solutions

In this section, we assume \( q(t) \) is a 1-periodic solution of (1), and give a detailed proof of Theorem 1.1 until the last proof, where the cases with \( T \neq 1 \) will be treated by a blow-up argument. Moreover only the proof of \( \gamma^+|_{[t_x, \infty)} \) will be given, as it is the same for \( \gamma^-|_{[-\infty,t_x]} \).

In the following it is more convenient to rewrite the potential function \( U \) given in (3) as
\[
U(z, t) = \frac{m}{|z|} + W(z, t), \text{ where } m = \sum_{i=1}^{N} m_i.
\]

A direct computation shows
\[
W(z, t) \sim O(m|z|^{-2}), \text{ as } |z| \to \infty.
\]
Therefore we can find a constant \( R_1 \geq R_0 \), and positive constants \( \alpha_1, \alpha_2 \), such that
\[
|W(z, t)| \leq \frac{\alpha_1}{|z|^2} \leq \frac{m}{|z|^2}, \quad |\partial_z W(z, t)| \leq \frac{\alpha_2}{|z|^3} \leq \frac{m}{|z|^3}, \quad \forall |z| \geq R_1.
\]

Let \( \gamma(t) = r(t)e^{i\theta(t)} \in C^2([t_1, t_2], \mathbb{C}) \) be a collision-free solution of (2), we have the following results, which are useful in controlling the asymptotic behavior of the massless body near infinity.

**Proposition 3.1.** If \( r(t_1) \geq R_1 \) and \( \dot{r}(t_1) \geq \sqrt{6m/r(t_1)} \), then \( \dot{r}(t) > \frac{1}{2} \dot{r}(t_1) \), \( \forall t \in [t_1, t_2] \).

**Proposition 3.2.** If \( r(t_1) \geq R_1 \) and \( \dot{r}(t) \geq v_0 \), \( \forall t \geq t_1 \), for some constant \( v_0 > 0 \), then
(a). \( \sup \{ |\omega(t)| : t \in [t_1, t_2] \} \leq |\omega(t_1)| + \frac{\alpha_2}{\omega_0(t_1)} \), where \( \omega(t) = \gamma(t) \wedge \dot{\gamma}(t) \);  
(b). when \( t_2 = \infty \), \( \lim_{t \to \infty} \theta(t) = \theta_0 \), for some \( \theta_0 \in \mathbb{R} \).

Proofs of the above two propositions will be given in Appendix B.

**Lemma 3.1.** For any \( x, y \in \mathbb{C} \) satisfying \( \sqrt{2}R_1 \leq |x|, |y| \leq R \), the following results hold.
(a). \( \psi_0(x,y,t_1,t_2) \leq 16 \frac{R^2}{|r_2-r_1|} + 12m \frac{t_2-t_1}{R} \), \( \forall t_1 < t_2 \). 
(b). For any \( h > 0 \), there exist positive constants \( \beta_1, \beta_2 \) independent of \( R \), such that
\[
\sup \{ \phi_h(x,y;s_1,s_2) : s_1, s_2 \in [0,1] \} \leq \beta_1 R + \beta_2.
\]

**Proof.** (a). Choose a point \( z \in \mathbb{C} \) with \( |z| = R \). Let \( \vec{\alpha}x, \vec{\alpha}y \) and \( \vec{\alpha}z \) denote the three rays starting from the origin and passing the points \( x, y \) and \( z \) correspondingly. Moreover we assume \( \vec{\alpha}z \) divide the angle between \( \vec{\alpha}x \) and \( \vec{\alpha}y \) into two equal angles with each angle less than or equal to \( \pi/2 \).
Define a straight line segment \( \xi(t) = (1 - \lambda(t))x + \lambda(t)z, \ t \in [t_1, \frac{t_1 + t_2}{2}] \) with

\[
\lambda(t) = \begin{cases} 
\mu \left[ 1 - \left( \frac{t - t_1}{\tau - t_1} \right)^2 \right], & \text{if } t \in [t_1, \tau]; \\
\mu + (1 - \mu) \left( \frac{t - \tau}{\frac{t_1 + t_2}{2} - \tau} \right)^2, & \text{if } t \in [\tau, \frac{t_1 + t_2}{2}],
\end{cases}
\]

satisfying \( \lambda(t_1) = 0 \) and \( \lambda(\frac{t_1 + t_2}{2}) = 1 \). For this, we need to let

\[
\mu = \frac{|x|}{|x| + |z|} \in (0, 1), \quad \tau = t_1 + \frac{\mu^{3/2}}{\mu^{3/2} + (1 - \mu)^{3/2}} \frac{t_2 - t_1}{2} \in (t_1, \frac{1}{2}(t_1 + t_2)).
\]

By the triangle inequality, for any \( t \in [t_1, \frac{t_1 + t_2}{2}] \),

\[
|\xi(t)| = |(1 - \lambda(t))x + \lambda(t)z| \geq |(1 - \lambda(t))|x| - \lambda(t)|z| = |\mu - \lambda(t)||x| - |\mu - \lambda(t)||z|
\]

Since \( |x|, |z| \geq \sqrt{2}R_1, |\xi(t)| \geq R_1, \forall t \in [t_1, \frac{t_1 + t_2}{2}] \). Then (21) and the above inequality imply

\[
U(\xi(t), t) \leq \frac{2m}{|\xi(t)|} \leq \frac{2m}{(|x| + |z|)|\mu - \lambda(t)|} \leq \frac{2m}{|z| \cdot |\mu - \lambda(t)|}.
\]

Using this, we get

\[
\int_{t_1}^{\tau} L(\xi, \dot{\xi}, t) \, dt = \int_{t_1}^{\tau} \frac{1}{2} |\dot{\xi}(t)|^2 + U(\xi(t), t) \, dt \leq \frac{|z - x|^2}{2} \int_{t_1}^{\tau} |\dot{\lambda}(t)|^2 \, dt + \frac{2m}{|z|} \int_{t_1}^{\tau} \frac{1}{|\mu - \lambda(t)|} \, dt
\]

\[
= \frac{2}{9} |z - x|^2 \frac{\mu^2}{(\tau - t_1)^2} \int_{t_1}^{\tau} \left( \frac{t - \tau}{\tau - t_1} \right)^{-\frac{3}{2}} \, dt + \frac{2m}{|z|} \int_{t_1}^{\tau} \left( \frac{\tau - t}{\tau - t_1} \right)^{-\frac{3}{2}} \, dt
\]

\[
= \frac{2}{3} |z - x|^2 \frac{\mu^2}{\tau - t_1} + \frac{6m \mu^2}{|z|} \tau - t_1
\]

\[
= \frac{4}{3} \mu^2 \left( \mu^3 + (1 - \mu)^3 \right) \frac{|z - x|^2}{t_2 - t_1} + \frac{3m}{\mu^3 + (1 - \mu)^3} \mu^\frac{1}{2} \frac{t_2 - t_1}{|z|};
\]

\[
\int_{\tau}^{\frac{t_1 + t_2}{2}} L(\xi, \dot{\xi}, t) \, dt = \int_{\tau}^{\frac{t_1 + t_2}{2}} \frac{1}{2} |\dot{\xi}(t)|^2 + U(\xi(t), t) \, dt
\]

\[
\leq \frac{|z - x|^2}{2} \int_{\tau}^{\frac{t_1 + t_2}{2}} |\dot{\lambda}(t)|^2 \, dt + \frac{2m}{|z|} \int_{\tau}^{\frac{t_1 + t_2}{2}} \frac{1}{|\mu - \lambda(t)|} \, dt
\]

\[
= \left( \frac{2}{9} |z - x|^2 \frac{(1 - \mu)^2}{(\frac{t_1 + t_2}{2} - \tau)^2} + \frac{2m}{|z|} \frac{1}{1 - \mu} \right) \int_{\tau}^{\frac{t_1 + t_2}{2}} \left( \frac{t - \tau}{\frac{t_1 + t_2}{2} - \tau} \right)^{-\frac{3}{2}} \, dt
\]

\[
= \frac{2}{3} |z - x|^2 \frac{(1 - \mu)^2}{\frac{t_1 + t_2}{2} - \tau} + \frac{6m \frac{t_1 + t_2}{2} - \tau}{|z|} \frac{1 - \mu}{1 - \mu}
\]

\[
= \frac{4}{3} (1 - \mu)^\frac{1}{2} \left( \mu^3 + (1 - \mu)^3 \right) \frac{|z - x|^2}{t_2 - t_1} + \frac{3m}{\mu^3 + (1 - \mu)^3} \mu^\frac{1}{2} \frac{t_2 - t_1}{|z|}.
\]

These two estimates imply

\begin{equation}
\int_{t_1}^{\frac{t_1 + t_2}{2}} L(\xi, \dot{\xi}, t) \, dt \leq \frac{4}{3} f_1(\mu) f_2(\mu) \frac{|z - x|^2}{t_2 - t_1} + \frac{3m f_1(\mu) t_2 - t_1}{f_2(\mu) |z|},
\end{equation}
where \( f_1(\mu) = \mu^{\frac{1}{2}} + (1 - \mu)^{\frac{1}{2}} \), \( f_2(\mu) = \mu^{\frac{3}{2}} + (1 - \mu)^{\frac{3}{2}} \).

Since \( \mu \in (0, 1) \),

\[
f_1(\mu) = 2 \left( \frac{1}{2} \mu^{\frac{1}{2}} + \frac{1}{2} (1 - \mu)^{\frac{1}{2}} \right) \leq 2 \left( \frac{1}{2} \mu + \frac{1}{2} (1 - \mu) \right)^{\frac{1}{2}} = \sqrt{2};
\]

\[
1 \geq f_2(\mu) = 2 \left( \frac{1}{2} \mu^{\frac{3}{2}} + \frac{1}{2} (1 - \mu)^{\frac{3}{2}} \right) \geq 2 \left( \frac{1}{2} \mu + \frac{1}{2} (1 - \mu) \right)^{\frac{3}{2}} \geq \frac{\sqrt{2}}{2}.
\]

Plug these into (22), we have

\[
\int_{t_1}^{t_1+t_2} L(\xi,\dot{\xi}) \, dt \leq \frac{4\sqrt{2}}{3} \frac{|z-x|^2}{t_2-t_1} + 6m \frac{t_2-t_1}{R} \leq \frac{16\sqrt{2}}{3} \frac{R^2}{t_2-t_1} + 6m \frac{t_2-t_1}{R}.
\]

A similarly straight line segment \( \zeta |_{t_1+t_2} \) going from \( z \) to \( y \) can be found, which satisfies

\[
\int_{t_1}^{t_1+t_2} L(\zeta,\dot{\zeta},t) \, dt \leq \frac{16\sqrt{2}}{3} \frac{R^2}{t_2-t_1} + 6m \frac{t_2-t_1}{R}.
\]

As a result,

\[
\psi_0(x,y;t_1,t_2) \leq \int_{t_1}^{t_1+t_2} L(\xi,\dot{\xi},t) \, dt + \int_{t_1}^{t_1+t_2} L(\zeta,\dot{\zeta},t) \, dt \leq 16 \frac{R^2}{t_2-t_1} + 12m \frac{t_2-t_1}{R} + h(t_2-t_1)
\]

(b). Let \( t_1 = s_1 \), we can always find a \( t_2 \in [s_1 + R, s_1 + R + 1] \) satisfying \( \{t_2\} = s_2 \). Then \( R \leq t_2 - t_1 = t_2 - s_1 \leq R + 1 \). Hence

\[
\phi_{h}(x,y;s_1,s_2) \leq \psi_{h}(x,y;t_1,t_2) \leq 16 \frac{R^2}{t_2-t_1} + 6m \frac{t_2-t_1}{R} + h(t_2-t_1)
\]

\[
\leq 16R + 6m(1+R^{-1}) + hR + h
\]

\[
\leq (16 + h)R + 6m + \frac{3\sqrt{2}m}{R_1} + h.
\]

\( \square \)

For the rest of the section, we fix arbitrarily a positive constant \( h \), a position \( x \in \mathbb{C} \), a moment \( t_x \) (without loss of generality we further assume \( t_x \in [0, 1] \)) and an angle \( \theta_+ \in [0, 2\pi) \). Set

(23) \( A_{\theta_+} (R) := \{ re^{i\theta} \in \mathbb{C} : r \geq R \text{ and } \theta = \theta_+ (\text{mod } 2\pi) \}, \) for \( R > 0 \).

By Proposition 2.7 and 2.8 for any \( y \in A_{\theta_+} (|x| + 1) \), there is a \( \gamma_y \in H^1_{t_x,t_y} (x, y) \) satisfying

(24) \( A_h(\gamma_y; t_x, t_y) = \phi_h(x,y;t_x,\{t_y\}) = \inf_{s \in [0,1]} \phi_h(x,y;t_x,s) \).

Lemma 3.2. For any \( R_2 \geq \max \{2\sqrt{2}R_1, |x| + 1 \} \) and \( y \in A_{\theta_+} (R_2) \), let

(25) \( \tau_y := \min \{ t \in [t_x, t_y] : |\gamma_y(t)| = R_2 \}. \)

The following results hold.

(a). \( t_x < \inf \{ \tau_y : y \in A_{\theta_+} (R_2) \} \leq \sup \{ \tau_y : y \in A_{\theta_+} (R_2) \} < \infty \).

(b). When \( R_2 \) is large enough, \( \dot{r}_y(t) \geq \sqrt{3m/2R_2}, \forall t \in [\tau_y,t_y], \) where \( r_y(t) = |\gamma_y(t)| \).
Lemma 3.3. Above lemma always holds. Notice that inequality first. Fix an \[ n \in \mathbb{Z}^+ \] by Proposition 2.6
\[
\sup\{ \psi_h(x, z; t_x, t) : z \in \partial B_{R_2} \text{ and } t \in [n, n_0 + 1] \} \leq C_1.
\]
Since \[ \gamma_y|_{t_x, t_y} \] is a free-time minimizer, so is \[ \gamma_y|_{\tau} \]. Then
\[
(26) \quad h(\tau_y - t_x) \leq A_h(\gamma_y; t_x, \tau_y) = \phi_h(x, \gamma_y(\tau_y); t_x, \{\tau_y\}) \leq \psi_h(x, \gamma_y(\tau_y); t_x, n_0 + \{\tau_y\}) \leq C_1.
\]
This implies
\[
\sup\{ \tau_y : y \in \Lambda_{\theta^+}(R_2) \} \leq t_x + C_1/h < \infty.
\]
Meanwhile by the Cauchy-Schwarz inequality,
\[
(\tau_y - t_x) \int_{t_x}^{\tau_y} |\dot{\gamma}_y|^2 \, dt \geq \left( \int_{t_x}^{\tau_y} |\dot{\gamma}_y| \, dt \right)^2 \geq |\gamma(\tau_y) - x| \geq (R_2 - |x|)^2.
\]
Combining this with (26), we get
\[
C_1 \geq A_h(\gamma_y; t_x, \tau_y) \geq \frac{1}{2} \int_{t_x}^{\tau_y} |\dot{\gamma}_y|^2 \, dt \geq \frac{(R_2 - |x|)^2}{2(\tau_y - t_x)}.
\]
Assuming the first inequality in the statement does not hold, then we can find a sequence \[ \{y_n \in \Lambda_{\theta^+}(R_2)\}_{n \in \mathbb{Z}^+} \] with the corresponding \[ \{\tau_{y_n}\}_{n \in \mathbb{Z}^+} \] satisfying \[ \lim_{n \to \infty} \tau_{y_n} = t_x \]. Then
\[
C_1 \geq A_h(\gamma_{y_n}; t_x, \tau_{y_n}) \geq \frac{(R_2 - |x|)^2}{2(\tau_{y_n} - t_x)} \to \infty, \quad \text{as } n \to \infty,
\]
which is absurd.
(b) For each \[ y \in \Lambda_{\theta^+}(R_2) \], let \[ a_y = \max\{ t \in [t_x, \tau_y) : r_y(t) = R_2/2 \} \]. Then
\[
r_y(t) \geq R_2/2 \geq R_1, \quad \forall t \in [a_y, \tau_y].
\]
Now it is enough to show \[ \dot{r}_y(t_0) \geq \sqrt{6m/r_y(t_0)}, \] for some \[ t_0 \in [a_y, \tau_y] \], as Proposition 3.1 implies
\[
\dot{r}_y(t) \geq \frac{1}{2} \dot{r}(t_0) \geq \frac{1}{2} \sqrt{6m/r_y(t_0)} \geq \sqrt{3m/2R_2^{1/4}}, \quad \forall t \in [t_0, t_y],
\]
where the last inequality follows from \[ r_y(t_0) \leq r_y(\tau_y) = R_2 \].
By a contradiction argument, let’s assume \[ \dot{r}_y(t) < \sqrt{6m/r_y(t)} \], \[ \forall t \in [a_y, \tau_y] \]. Then
\[
\frac{R_2}{2} = r_y(\tau_y) - r_y(a_y) = \int_{a_y}^{\tau_y} \dot{r}_y(t) \, dt < \int_{a_y}^{\tau_y} \sqrt{6m/r_y(t)} \, dt \leq \sqrt{12m/R_2}(\tau_y - a_y).
\]
Notice that \[ r_y(t) \geq R_2/2, \forall t \in [a_y, \tau_y] \]. As a result,
\[
A_h(\gamma_y; a_y, \tau_y) \geq h(\tau_y - a_y) > \frac{h}{4\sqrt{3m}}R_2^2.
\]
However with Lemma 3.1, they imply
\[
\frac{h}{4\sqrt{3m}}R_2^2 < A_h(\gamma_y; a_y, \tau_y) = \phi_h(\gamma_y(a_y), \gamma_y(\tau_y); \{a_y\}, \{\tau_y\}) \leq \beta_1R_2 + \beta_2,
\]
which is absurd, when \[ R_2 \] is large enough. □

In the rest of the section, we fix an \[ R_2 \geq \max\{2\sqrt{2}R_1, |x| + 1\} \] large enough, such that the above lemma always holds.

**Lemma 3.3.** For any \( \tau > \sup\{ \tau_y : y \in \Lambda_{\theta^+}(R_2) \} \), the following results hold.
Lemma 3.2 implies $|\gamma_y(t)|$ is strictly increasing, when $t \in [t_y, \tau]$. Hence it is enough to show $|\gamma_y(\tau)| \leq R$, for some finite constant $R$ independent of $y$. By the Cauchy-Schwartz inequality,

$$|\gamma_y(\tau)| \leq |\gamma_y(\tau_y)| + |\gamma_y(\tau) - \gamma_y(\tau_y)| \leq R_2 + \int_{\tau_y}^{\tau} |\dot{\gamma}_y| \, dt \leq R_2 + \left(\tau - \tau_y\right) \int_{\tau_y}^{\tau} |\dot{\gamma}_y|^2 \, dt^{\frac{1}{2}}.$$

At the same time, Lemma 3.1 implies

$$\int_{\tau_y}^{\tau} |\dot{\gamma}_y|^2 \, dt \leq 2A_h(\gamma_y; \tau_y, \tau) = 2\phi_h(\gamma_y(\tau_y), \gamma_y(\tau); \{\tau_y\}, \{\tau\}) \leq 2\beta_1 |\gamma_y(\tau)| + 2\beta_2.$$

These two estimates implies $|\gamma_y(\tau)| \leq R_2 + \sqrt{2\tau} (\beta_1 |\gamma_y(\tau)| + \beta_2)^\frac{1}{2}$, which is equivalent to

$$|\gamma_y(\tau)|^2 - 2(2R_2 + \beta_1) |\gamma_y(\tau)| + 2R_2^2 - 2\beta_2 \tau \leq 0.$$

Solving this inequality, we get

$$|\gamma_y(\tau)| \leq R_2 + \beta_1 \tau + (\beta_1^2 \tau^2 + 2\beta_1 \tau R_2 + 2\beta_2)\frac{1}{2} := R.$$

(b). Let $a = \inf \{\tau_y : y \in \Lambda^+_\theta(R_2) \) and $\tau_y \geq \tau\}, b = \sup \{\tau_y : y \in \Lambda^+_\theta(R_2) \) and $\tau_y \geq \tau\}$. By Lemma 3.2 $t_x < a < b < \infty$. As $|\gamma_y(\tau_y)| = R_2$, $\forall y \in \Lambda^+_\theta(R_2)$ with $t_y \geq \tau$. By Proposition 2.6

$$\mathcal{A}_h(\gamma_y; t_x, \tau_y) = \psi_h(x, \gamma_y(\tau_y); t_x, \tau_y) \leq \sup_{t \in [a, b], z \in \partial B_{R_2}} \psi_h(x, z; t_x, t) \leq C_2.$$

Meanwhile as $\sqrt{2}R_1 \leq R_2 = |\gamma_y(\tau_y)| \leq |\gamma_y(\tau)| \leq R$, Lemma 3.1 implies

$$\mathcal{A}_h(\gamma_y; t_x, \tau, \tau_y) = \phi_h(\gamma_y(\tau_y), \gamma_y(\tau); \{\tau_y\}, \{\tau\}) \leq \beta_1 R + \beta_2.$$

Combining these two estimates, we get

$$\int_{t_x}^{\tau} |\dot{\gamma}_y|^2 \, dt \leq 2\mathcal{A}_h(\gamma_y; t_x, \tau) \leq 2(\beta_1 R + \beta_2 + C_2).$$

On the other hand, property (a) implies

$$\int_{t_x}^{\tau} |\gamma_y(t)|^2 \, dt \leq (\tau - t_x) R^2.$$

As a result, for any $y \in \Lambda^+_\theta(R_2)$ with $t_y \geq \tau$,

$$\|\gamma_y|_{t_x, \tau}\|_{H^1} \leq \int_{t_x}^{\tau} |\dot{\gamma}_y|^2 + |\gamma_y|^2 \, dt \leq (\tau - t_x) R^2 + 2(\beta_1 R + \beta_2 + C_2).$$

The next result is a simple exercise from real analysis, which will be needed.

**Proposition 3.3.** Given a sequence $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{C}^2([T_1, T_2], \mathbb{R})$, whose $C^2$-norm is bounded. If there is a $f \in \mathcal{C}^2([T_1, T_2], \mathbb{R})$ with $\|f_n - f\|_{L^\infty} \to 0$, as $n \to \infty$, then $\|f_n - f\|_{C^1} \to 0$, as $n \to \infty$. 
Now let’s choose a sequence of points \( \{y_n \in \Lambda_{\theta_n}(R_2)\}_{n \in \mathbb{N}} \) with \(|y_n| \to \infty\), as \( n \to \infty \). For each \( n \), let \( \gamma_{y_n} \in H^1_{t_n,t_n}(x,y_n) \) be a free-time minimizer satisfying
\[
A_h(\gamma_{y_n}; t_n, t_n) = \phi_h(x, y_n; t_n, \{t_n\}) = \inf_{s \in [0,1]} \phi_h(x, y_n; t_n, s).
\]

**Proposition 3.4.** Under the above notations, the following results hold.

(a). \( t_n \to \infty \), as \( n \to \infty \).

(b). There exists a \( \gamma \in H^1([t_0, \infty), \mathbb{C}) \), such that (after passing to a proper subsequence) \( \gamma_{y_n} \to \gamma \) strongly in \( L^\infty \)-norm and weakly in \( H^1_{loc} \)-norm, as \( n \to \infty \).

(c). \( \forall t > t_x, A_h(\gamma; t_x, t) = \phi_h(x, \gamma(t); t_x, \{t\}), \) and \( \gamma|_{(t_x, \infty)} \) is a collision-free solution of (26).

(d). \( \gamma_{y_n} \to \gamma \) in \( C^1_{loc} \)-norm, as \( n \to \infty \), i.e., for any compact subset \( I \subset (t_x, \infty) \), \( \|\gamma_{y_n} - \gamma\|_{C^1} \to 0 \), as \( n \to \infty \).

**Remark 3.1.** For the rest of the section, we set \( \gamma_n = \gamma_{y_n}, t_n = t_{y_n} \) and \( \tau_n = \tau_{y_n} \) (given as in (25)).

**Proof.** (a). Since for \( n \) large enough, \( |\gamma_n(t_n)| \) and \( |\gamma_n(t_n)| = |y_n| \geq \sqrt{R_1} \), by Lemma 3.1

\[
\beta_1 |y_n| + \beta_2 \geq \phi_h(\gamma_n(t_n), \gamma_n(t_n); t_n), \{t_n\} \geq \frac{1}{2} \int_{\tau_n}^{t_n} |\gamma_n|^2 dt \geq \frac{1}{2(t_n - \tau_n)} \left( \int_{\tau_n}^{t_n} |\gamma_n| dt \right)^2 \geq \frac{|y_n - \gamma_n(t_n)|^2}{2(t_n - \tau_n)} \geq \frac{(|y_n| - |\gamma_n(t_n)|)^2}{2} \geq \frac{|y_n|^2 - 2R_2|y_n| + R_2^2}{2t_n}.
\]

This then implies
\[
t_n \geq \frac{|y_n|^2 - 2R_2|y_n| + R_2^2}{2(\beta_1 |y_n| + \beta_2)} \geq \frac{|y_n| - 2R_2 + R_2^2 |y_n|^{-1}}{2\beta_1 + 2\beta_2 |y_n|^{-1}} \to \infty, \text{ as } |y_n| \to \infty.
\]

(b). For any \( \tau > t_x \) large enough, by Lemma 3.3, \( \{\gamma_n|_{[t_x, \tau]}\}_{n \in \mathbb{N}} \) is bounded under the \( H^1 \)-norm, so (after passing to a proper subsequence) it converges to a \( \xi \in H^1([t_x, \infty), \mathbb{C}) \) weakly in \( H^1 \)-norm and strongly in \( L^\infty \)-norm.

Since \( t_n \to \infty \), when \( n \to \infty \), by a diagonal argument, we can find a \( \gamma \in H^1([t_x, \infty), \mathbb{C}) \), such that (after passing to a proper subsequence) \( \gamma_n \to \gamma \) strongly in \( L^\infty \)-norm and weakly in \( H^1 \)-norm. Moreover the weakly lower semi-continuity of \( A_h \) implies
\[
(27) \quad A_h(\gamma; t_x, \tau) \leq \liminf_{n \to \infty} A_h(\gamma_n; t_x, \tau), \quad \forall \tau > t_x.
\]

(c). Notice that is enough to prove the result for all \( t \) large enough. By a contradiction argument, let’s assume there is a \( t > t_x \) large, such that
\[
A_h(\gamma; t_x, t) > \phi_h(x, \gamma(t); t_x, \{t\}).
\]

Then by Proposition 2.7 there is a \( \xi \in H^1_{t_x,t_x}(x, \gamma(t)) \) with \( \{t\} = \{t\} \) (see Figure 1), such that
\[
A_h(\xi; t_x, \tau) = \phi_h(x, \gamma(t); t_x, \{t\}) \leq A_h(\gamma; t_x, t) - 3\varepsilon,
\]

for some \( \varepsilon > 0 \) small enough. Then by (27), when \( n \) is large enough,
\[
(28) \quad A_h(\gamma_n; t_x, t) \geq A_h(\gamma; t_x, t) - \varepsilon \geq A_h(\xi; t_x, \tau) + 2\varepsilon.
\]

Since \( \xi(\tau) = \gamma(t) \) must be collision-free, for \( \tau \) large enough, so is \( \xi|_{[t_x, \tau]} \) for \( \tau' \) close enough to \( \tau \).

Then \( A_h \) is continuous in a small neighborhood of \( \xi|_{[t_x, \tau]} \) in \( H^1([t_x, \tau], \mathbb{C}) \) as \( \gamma_n(t) \to \gamma(t) = \xi(t) \), when \( n \to \infty \). For each \( n \) large enough, we can find a \( \eta_n \in H^1_{t_x,t_x}(\xi(t), \gamma_n(t)) \) with
\[
|A_h(\xi; t_x, \tau) - A_h(\eta_n; t_x', \tau)| \leq \varepsilon.
\]
Then \( \zeta_{n|t_x,\tau} = \xi_{|t_x,\tau} \ast \eta_{n|\tau,\tau} \in H^1_{t_x,\tau}(x, \gamma_n(t)) \) and
\[
A_h(\xi; t_x, \tau) - A_h(\zeta_{n|t_x,\tau}) = A_h(\xi; \tau', \tau) - A_h(\eta_{n|\tau',\tau}) \geq -\varepsilon.
\]
Combining this with (28), we get the following inequality, which is absurd.
\[
\phi_h(x, \gamma_n(t); t_x, \{t\}) = A_h(\gamma_n; t_x, t) \geq A_h(\xi; t_x, \tau) + 2\varepsilon \geq A_h(\zeta_{n|t_x,\tau}) + \varepsilon.
\]

(d). Given an arbitrary compact subset \( I \subset (t_x, \infty) \), by property (c), \( \gamma|_I \) is a collision-free solution of (2). As a result,
\[
|\dot{\gamma}(t)| = \left| \sum_{i \in \mathbb{N}} \frac{m_i(\gamma(t) - q_i(t))}{|\gamma(t) - q_i(t)|^3} \right| \leq \sum_{i \in \mathbb{N}} \frac{m_i}{|\gamma(t) - q_i(t)|} \leq C_1, \ \forall t \in I.
\]
Since each \( \gamma|_I \) is also a free-time minimizer, \( \gamma|_I \) is a collision-free solution of (2) as well. By property (a), \( \|\gamma|_I - \gamma|_I\|_{L^\infty} \to 0 \), as \( n \to \infty \). Then for \( n \) large enough,
\[
|\ddot{\gamma}(t)| = \left| \sum_{i \in \mathbb{N}} \frac{m_i(\gamma_n(t) - q_i(t))}{|\gamma_n(t) - q_i(t)|^3} \right| \leq \sum_{i \in \mathbb{N}} \frac{m_i}{|\gamma_n(t) - q_i(t)|} \leq 2C_1, \ \forall t \in I.
\]
The desired result now follows directly from Proposition 3.3. \( \square \)

For \( \gamma \) and \( \gamma_n \) from above, in polar coordinate, we let \( \gamma(t) = r(t)e^{i\theta(t)} \) and \( \gamma_n(t) = r_n(t)e^{i\theta_n(t)} \).

**Proposition 3.5.** \( \lim_{t \to \infty} \theta(t) = \theta_+ (\text{mod} \ 2\pi) \).

**Proof.** Notice that Proposition 3.3 implies
\[
\lim_{n \to \infty} |\gamma_n(t) - \gamma(t)| = 0, \ \lim_{n \to \infty} |\dot{\gamma}_n(t) - \dot{\gamma}(t)| = 0, \ \forall t > t_x.
\]
Meanwhile by a direct computation,
\[
|\dot{r} - \dot{r}_n| = \left| \frac{\langle \gamma_n, \dot{\gamma}_n \rangle - \langle \gamma, \dot{\gamma} \rangle}{|\gamma_n|} \right| = \left| \frac{|\langle \gamma|\gamma_n, \dot{\gamma}_n \rangle - \langle \gamma_n|\gamma, \dot{\gamma} \rangle|}{|\gamma_n||\gamma|} \right|
\]
\[
\leq \frac{|\langle \gamma|\gamma_n, \dot{\gamma}_n - \dot{\gamma} \rangle + |\langle \gamma|\gamma_n - \gamma, \dot{\gamma} \rangle + |\langle \gamma|\gamma_n - \gamma, \dot{\gamma} \rangle - |\langle \gamma|\gamma_n, \dot{\gamma} \rangle|}{|\gamma_n||\gamma|}
\]
\[
\leq |\gamma| \cdot |\gamma_n| \cdot |\dot{\gamma}_n - \dot{\gamma}| + |\gamma| \cdot |\gamma_n - \gamma| \cdot |\dot{\gamma}| + |\gamma - \gamma_n| \cdot |\gamma| \cdot |\dot{\gamma}|
\]
Then by (29),
\[
\lim_{n \to \infty} |\dot{r}_n(t) - \dot{r}(t)| = 0, \ \forall t > t_x.
\]
Recall \( \tau_n = \tau_{y_n} = \min \{ t \in [t_x, t_{y_n}] : |\gamma_{y_n}(t)| = R_2 \} \). By Lemma 3.2, \( \tau = \sup \{ \tau_n : n \geq 1 \} < \infty \).

From now on we always assume \( t_n = t_{y_n} > \tau \). Then again by Lemma 3.2

\[
(31) \quad \dot{r}_n(t) \geq v_0 = (3m/2)^{1/3} R_2^{-1/3}, \quad \forall t \geq \tau \text{ and } \forall n.
\]

Then (30) implies

\[
(32) \quad \dot{r}(t) \geq v_0, \quad \forall t \geq \tau.
\]

As \( |\gamma(\tau)| = \lim_{n \to \infty} |\gamma_n(\tau)| \geq R_2 \), Proposition 3.2 implies,

\[
\lim_{t \to \infty} \theta(t) = \theta_0, \quad \text{for some } \theta_0 \in \mathbb{R}.
\]

What’s left now is to show \( \theta_0 = \theta_+ (\mod 2\pi) \). For this, we need an uniform upper bound of the angular momenta \( \omega(t) \) and \( \omega_n(t) \).

First by Proposition 3.2 for all \( t \geq \tau \),

\[
|\omega(t)| \leq |\omega(\tau)| + \frac{\alpha_2}{v_0 R_2} \leq |\omega(\tau)| + \frac{\alpha_2}{v_0 R_2} ;
\]

\[
|\omega_n(t)| \leq |\omega_n(\tau)| + \frac{\alpha_2}{v_0 r_n(\tau)} \leq |\omega_n(\tau)| + \frac{\alpha_2}{v_0 R_2}.
\]

Meanwhile notice that

\[
|\omega - \omega_n| = |\gamma \wedge \dot{\gamma} - \gamma \wedge \dot{\gamma}_n + \gamma \wedge \dot{\gamma}_n - \gamma_n \wedge \dot{\gamma}_n| \leq |\gamma| \cdot |\dot{\gamma} - \dot{\gamma}_n| + |\gamma_n| \cdot |\gamma - \gamma_n|.
\]

Then (30) implies, \( \lim_{n \to \infty} |\omega(\tau) - \omega_n(\tau)| = 0 \). Combining this with (33), we get

\[
(34) \quad |\omega(t)|, |\omega_n(t)| \leq C_1, \quad \forall t \geq \tau \text{ and } \forall n.
\]

For any \( s_2 > s_1 \geq \tau \), using (32) and (34), we get

\[
(35) \quad |\theta(s_2) - \theta(s_1)| \leq \int_{s_1}^{s_2} |\dot{\theta}(t)| dt \leq \int_{s_1}^{s_2} \frac{|\omega(t)|}{r^2(t)} dt \leq \frac{C_1}{v_0^2} \int_{s_1}^{s_2} \left( t - s_1 + \frac{r(s_1)}{v_0} \right)^{-2} dt.
\]

By (31) and (34), a similar computation as above shows that for any \( n \),

\[
(36) \quad |\theta_n(s_2) - \theta_n(s_1)| \leq \frac{v_0^{-1} C_1}{r_n(s_1)}.
\]

By a contradiction argument, let’s assume \( \theta_0 \neq \theta_+ (\mod 2\pi) \), then there is a \( \varepsilon > 0 \), such that

\[
(37) \quad |\theta_0 - \theta_+ - 2k\pi| \geq 6\varepsilon, \quad \forall k \in \mathbb{Z}.
\]

As \( \tau(t) \to \infty \), when \( t \to \infty \), there must be a \( s_0 \geq \tau \) with \( r(s_0) \geq C_1/(v_0 \varepsilon) \). Then (35) implies

\[
(38) \quad |\theta(t) - \theta(s_0)| \leq \varepsilon, \quad \forall t \geq s_0.
\]

Since \( \gamma_n(s_0) \to \gamma(s_0) \), when \( n \to \infty \). For each \( n \) large enough, there is a \( j_n \in \mathbb{Z} \), such that

\[
(39) \quad |\theta(s_0) - \theta_n(s_0) - 2j_n\pi| \leq \varepsilon.
\]

Notice that \( r_n(s_0) \geq \frac{r(s_0)}{2} \geq \frac{C_1}{2v_0^2 \varepsilon} \), for \( n \) large enough. Then (36) implies

\[
(40) \quad |\theta_n(t) - \theta_n(s_0)| \leq 2\varepsilon, \quad \forall t \geq s_0.
\]

Since \( y_n = \gamma_n(t_n) \in \Lambda_{\theta_+}(R_2) \), for each \( n \), there is a \( k_n \in \mathbb{Z} \), such that

\[
(41) \quad \theta_n(t_n) = \theta_+ = 2k_n\pi.
\]
Meanwhile as $t_n \to \infty$, when $n \to \infty$, we have
\begin{equation}
|\theta_0 - \theta(t_n)| \leq \varepsilon, \quad \text{for } n \text{ large enough.}
\end{equation}

Then by \[38, 39, 40, 41, 42\], for $n$ large enough, we have
\begin{align*}
|\theta_0 - \theta_+ - 2k_n \pi - 2j_n \pi| &= |\theta_0 - \theta_n(t_n) - 2j_n \pi| \\
&= |\theta_0 - \theta(t_n) + \theta(t_n) - \theta(s_0) + \theta(s_0) - \theta_n(s_0) - 2j_n \pi + \theta_n(s_0) - \theta_n(t_n)| \\
&\leq 5\varepsilon,
\end{align*}
which is a contradiction to \[37\].

**Proposition 3.6.** $\lim_{t \to \infty} \gamma(t) = \sqrt{2h} e^{i\theta_+}$.

**Proof.** First we claim $\lim_{t \to \infty} \dot{r}(t) = v_1$, for some $v_1 > 0$.

Since $r(t) \geq R_2 \geq R_1, \forall t \geq \tau$, \[34\] and the first equation in \[93\] imply
\begin{equation}
|\ddot{r}(t)| \leq \frac{2m}{r^2(t)} + \frac{|\omega^2(t)|}{r^3(t)} \leq \frac{2m}{r^2(t)} + \frac{C^2_i}{r^3(t)}.
\end{equation}

Meanwhile by \[32\], $r(t) \geq v_0(t - \tau) + r(\tau), \forall t \geq \tau$. As a result, $\int_{\tau}^{\infty} |\ddot{r}(t)| \, dt < \infty$. This implies $\lim_{t \to \infty} \dot{r}(t)$ exists and is finite. Moreover \[32\] shows it is also positive. This proves the claim.

Since $|\omega(t)| \leq C_1, \forall t \geq \tau$ and $r(t) \to \infty$, as $t \to \infty$,
\begin{equation}
\lim_{t \to \infty} r(t) \dot{\theta}(t) = \lim_{t \to \infty} (\omega(t)/r(t)) = 0.
\end{equation}

Then Proposition 3.5 and the above claim imply
\begin{equation}
\lim_{t \to \infty} \dot{\gamma}(t) = \lim_{t \to \infty} (\dot{r}(t) + ir(t) \dot{\theta}(t)) e^{i\theta(t)} = v_1 e^{i\theta_+(\mod 2\pi)}.
\end{equation}

Now we only need to show $v_1 = \sqrt{2h}$. By a contradiction argument, assume $v_1 \neq \sqrt{2h}$. Notice that $\sqrt{2h}$ is the unique global minimum of the following continuous function
\begin{equation}
f : (0, \infty) \to \mathbb{R}; \quad v \mapsto v/2 + h/v.
\end{equation}

Then $\delta = (f(v_1) - \sqrt{2h})/5 > 0$, and for $\varepsilon \in (0, \delta)$ small enough, $f(v_1 + \varepsilon) - \sqrt{2h} \geq 4\delta$.

As $\lim_{t \to \infty} |\dot{\gamma}(t)| = v_1$, we can find an $s_1 > \tau$ large enough, such that
\begin{equation}
v_1 - \varepsilon \leq |\dot{\gamma}(t)| \leq v_1 + \varepsilon, \forall t \geq s_1.
\end{equation}

Then for any $s_2 > s_1$, $|\gamma(s_2) - \gamma(s_1)| \leq \int_{s_1}^{s_2} |\dot{\gamma}(t)| \, dt \leq (v_1 + \varepsilon)(s_2 - s_1)$, which implies
\begin{equation}
s_2 - s_1 \geq \frac{|\gamma(s_2) - \gamma(s_1)|}{(v_1 + \varepsilon)}.
\end{equation}

Using the above estimates, we get
\begin{equation}
A_h(\gamma; s_1, s_2) \geq \int_{s_1}^{s_2} \frac{1}{2} |\dot{\gamma}|^2 + h \, dt \geq \left( \frac{1}{2} (v_1 - \varepsilon)^2 + h \right) \frac{|\gamma(s_2) - \gamma(s_1)|}{v_1 + \varepsilon}
\end{equation}
\begin{align*}
&\geq \left( \frac{1}{2} (v_1 + \varepsilon) + \frac{h}{v_1 + \varepsilon} - 2\varepsilon \right) |\gamma(s_2) - \gamma(s_1)| \\
&= (f(v_1 + \varepsilon) - 2\varepsilon) |\gamma(s_2) - \gamma(s_1)| \geq (\sqrt{2h} + 4\delta - 2\varepsilon) |\gamma(s_2) - \gamma(s_1)| \\
&\geq (\sqrt{2h} + 2\delta) |\gamma(s_2) - \gamma(s_1)|.
\end{align*}

For any $s_2 > s_1$ as above, we can always find an $s \in [0, 1)$, such that
\begin{equation}
s_2^* = s_1 + (2h)^{-\frac{1}{2}} |\gamma(s_2) - \gamma(s_1)| + s \text{ satisfying } \{s_2^*\} = \{s_2\}.
\end{equation}
We will show $\xi \in H^1_{s_1, s_2^*}(\gamma(s_1), \gamma(s_2))$ given as below satisfying $A_h(\xi; s_1, s_2^*) < A_h(\gamma; s_1, s_2)$.

$$
\xi(t) = \gamma(s_1) + \frac{t - s_1}{s_2^* - s_1}(\gamma(s_2) - \gamma(s_1)), \forall t \in [s_1, s_2^*].
$$

However this is a contradiction to $A_h(\gamma; s_1, s_2) = \phi_h(\gamma(s_1), \gamma(s_2); \{s_1\}, \{s_2\})$.

To estimate $\int |\dot{\xi}|^2 \, dt$, consider the following strictly increasing continuous function,

$$
g : [0, \infty) \to \mathbb{R}; \quad v \mapsto (1 + v) + (1 + v)^{-1}.
$$

As $|\gamma(s_2) - \gamma(s_1)| \to \infty$, when $s_2 \to \infty$,

$$
g \left( \frac{\sqrt{2h}}{|\gamma(s_2) - \gamma(s_1)|} \right) \leq 2 + \frac{\sqrt{2h}}{\delta}, \text{ for } s_2 \text{ large enough.}
$$

As a result, for $s_2$ large enough,

$$
\int_{s_1}^{s_2} \frac{1}{2} |\dot{\xi}|^2 + h \, dt = \left( \frac{1}{2} \frac{|\gamma(s_2) - \gamma(s_1)|^2}{(s_2^* - s_1)^2} + h \right) \left( \frac{|\gamma(s_2) - \gamma(s_1)|}{\sqrt{2h}} + s \right)
$$

$$
= \left[ \left( 1 + \frac{\sqrt{2h}}{|\gamma(s_2) - \gamma(s_1)|} \right) + \left( 1 + \frac{\sqrt{2h}}{|\gamma(s_2) - \gamma(s_1)|} \right)^{-1} \right] \frac{h}{2} |\gamma(s_2) - \gamma(s_1)|
$$

$$
\leq (\sqrt{2h} + \delta)|\gamma(s_2) - \gamma(s_1)|.
$$

To estimate $\int U(\xi(t), t) \, dt$, we set $\theta_+ = 0$ for simplicity. For any $s_2 > s_1$ with $s_1$ large enough,

$$
|\theta(s_2)|, |\theta(s_1)| \leq \pi/12.
$$

By further assuming $s_2 - s_1$ is large enough, we can get

$$
|\gamma(s_2) - \gamma(s_1)| > |\gamma(s_1)|.
$$

Consider the triangle with $\gamma(s_1), \gamma(s_2)$ and the origin as the three vertices, and $\sigma_i, i = 1, 2, 3,$ as the three angles (see Figure 2). By (45), $0 \leq \sigma_1 \leq \pi/6$. Then (46) implies $0 \leq \sigma_3 \leq \pi/6$. As a result, $2\pi/3 \leq \sigma_2 \leq \pi$.

Meanwhile for any $t \in [s_1, s_2^*]$, there is a triangle with $\gamma(s_1), \xi(t)$ and the origin as the three vertices, and $\sigma_i, i = 2, 4, 5$ as the three angles (see Figure 2). Then $2\pi/3 \leq \sigma_2 \leq \pi$ implies $0 \leq \sigma_4, \sigma_5 \leq \pi/3$ and as a result, for any $t \in [s_1, s_2^*],\n
$$
|\xi(t)| = |\xi(t) - \gamma(s_1)| \cos \sigma_4 + |\gamma(s_1)| \cos \sigma_5 \geq \frac{1}{2} \left( \frac{t - s_1}{s_2^* - s_1} |\gamma(s_2) - \gamma(s_1)| + |\gamma(s_1)| \right) \geq R_1.
$$
Using this, we get

$$\int_{s_1}^{s_2} |U(\xi(t), t)| dt \leq \int_{s_1}^{s_2} \frac{2m}{\xi(t)} dt \leq \int_{s_1}^{s_2} \frac{4m(s_2^s - s_1)}{|\gamma(s_2) - \gamma(s_1)|} \left( t - s_1 + \frac{(s_2^s - s_1)|\gamma(s_1)|}{|\gamma(s_2) - \gamma(s_1)|} \right)^{-1} dt$$

$$= \int_{s_1}^{s_2} \frac{4m}{\sqrt{2h}} \left( 1 + \frac{\sqrt{2hs}}{|\gamma(s_2) - \gamma(s_1)|} \right) dt$$

$$= \frac{4m}{\sqrt{2h}} \left( 1 + \frac{\sqrt{2hs}}{|\gamma(s_2) - \gamma(s_1)|} \right) \log \left( 1 + \frac{|\gamma(s_2) - \gamma(s_1)|}{|\gamma(s_1)|} \right)$$

Combining the above inequality with (43) and (44), we get

$$\mathcal{A}_h(\gamma; s_1, s_2) - \mathcal{A}_h(\xi; s_1, s_2) \geq \delta|\gamma(s_2) - \gamma(s_1)| - \frac{4m}{\sqrt{2h}} \left( 1 + \frac{\sqrt{2hs}}{|\gamma(s_2) - \gamma(s_1)|} \right) \log \left( 1 + \frac{|\gamma(s_2) - \gamma(s_1)|}{|\gamma(s_1)|} \right) > 0,$$

for $s_2$ large enough, which is absurd. \(\square\)

Up to now, we have proved the existence of the desired hyperbolic solution $\gamma^+|_{[t_x, \infty)}$, when $T = 1$. By a similar argument, we can obtain the corresponding $\gamma^-|_{(-\infty, t_x]}$, when $T = 1$. To generalize the result to $T \neq 1$, we use a blow-up argument similar to the one that has been used quite a lot in the variational study of the N-body problem (see [37] and [16]).

Proof of Theorem 1.1. Recall that $q(t) = (q_i(t))_{i \in \mathbb{N}}$ is a collision-free $T$-periodic solution of the N-body problem [1]. Define $q^\lambda(t) = (q_i^\lambda(t))_{i \in \mathbb{N}}$ as

$$q_i^\lambda(t) = \lambda^{\frac{i}{2}} q_i(t) = \lambda^{\frac{i}{2}} q_i(s/\lambda), \quad i \in \mathbb{N}, \quad \text{where } \lambda = T^{-1}.$$

Then $q^\lambda(t)$ is a collision-free $T$-periodic solution of [1], as a direct computation shows

$$\ddot{q}_i^\lambda(s) = -\sum_{j \in \mathbb{N}\setminus\{i\}} \frac{m_j (q_j^\lambda(s) - q_i^\lambda(s))}{|q_i^\lambda(s) - q_j^\lambda(s)|^3}, \quad \forall i \in \mathbb{N}.$$

By previous results of this section, for $h^\lambda = \lambda^{-\frac{i}{2}} h$ and $\theta_+ \in [0, 2\pi)$, there is a $\gamma^\lambda(s), s \in [\lambda t_x, \infty)$, with $\gamma^\lambda(\lambda t_x) = \lambda^{\frac{i}{2}} x$, which is a solution of

$$\ddot{\gamma}^\lambda(s) = -\sum_{i \in \mathbb{N}} \frac{m_i (\gamma(s) - q_i^\lambda(s))}{|\gamma(s) - q_i^\lambda(s)|^3},$$

and satisfies

$$\lim_{s \to \infty} \frac{\gamma^\lambda(s)}{|\gamma^\lambda(s)|} = e^{i\theta_+ (\text{mod } 2\pi)}, \quad \lim_{s \to \infty} \dot{\gamma}^\lambda(s) = \sqrt{2h^\lambda} e^{i\theta_+ (\text{mod } 2\pi)}.$$

Then $\gamma^+(t) = \lambda^{-\frac{i}{2}} \gamma^\lambda(\lambda t)$ is a solution of [2] satisfying $\gamma^+(t_x) = x$ and

$$\lim_{t \to \infty} \gamma^+(t)/|\gamma^+(t)| = \lim_{s \to \infty} \gamma^\lambda(s)/|\gamma^\lambda(s)| = e^{i\theta_+ (\text{mod } 2\pi)}$$

and

$$\lim_{t \to \infty} \dot{\gamma}^+(t) = \lim_{t \to \infty} \lambda^{\frac{i}{2}} \dot{\gamma}^\lambda(s) = \lambda^{\frac{i}{2}} \sqrt{2h^\lambda} e^{i\theta_+ (\text{mod } 2\pi)} = \sqrt{2he^{i\theta_+ (\text{mod } 2\pi)}}.$$

The proof of $\gamma^-|_{(-\infty, t_x]}$ is exactly the same and will be omitted. \(\square\)
4. Existence of bi-hyperbolic solutions

Like in the previous section we assume $T = 1$ throughout this section except the last proof.

**Definition 4.1.** For any $x, y \in \mathbb{C} \setminus B^0_{R_0}$ and $\gamma \in H^1_{t_1,t_2}(x,y)$ with $\gamma([t_1,t_2]) \cap B_{R_0} \neq \emptyset$, let

\[
\begin{align*}
  s_y &= s_y(\gamma) = \max \{t \in [t_1,t_2] : |\gamma(t)| = R_0\}; \\
  s_x &= s_x(\gamma) = \min \{t \in [t_1,t_2] : |\gamma(t)| = R_0\}.
\end{align*}
\]

We say such a $\gamma$ is tied with $m_{i_0}$ and $m_{i_1}$ ($\{i_0 \neq i_1\} \subset \mathbb{N}$), if for any $\xi \in H^1_{s_x,s_y}(\gamma(s_x), \gamma(s_y))$ with $\xi([s_x,s_y]) \subset \mathbb{C} \setminus B^0_{R_0}$, there is no fixed-end homotopy between $\gamma|[s_x,s_y]$ and $\xi|[s_x,s_y]$ avoiding the trajectories of $m_{i_0}$ and $m_{i_1}$, i.e., there is no continuous paths $P(t, \tau), (t, \tau) \in [s_x, s_y] \times [0,1]$, in $H^1([s_x,s_y], \mathbb{C})$ satisfying $P(t,0) = \gamma(t)$, $P(t,1) = \xi(t)$, $\forall t \in [s_x,s_y]$, and

\[
P(t,\tau) \in \mathbb{C} \setminus \{q_{i_0}(t), q_{i_1}(t)\}, \forall (t,\tau) \in [s_x,s_y] \times [0,1].
\]

**Remark 4.1.** The named tied was used as in the 3-dim space $\mathbb{C} \times \mathbb{R}$, the trajectories $q_{i_0}|[s_x,s_y], q_{i_1}|[s_x,s_y]$ and $\gamma|[s_x,s_y]$ can be seen as three braids. It has been used by Gordon [19] and Montgomery [28].

For the rest of the section, we shall always assume $x, y \in \mathbb{C} \setminus B^0_{R_0}$ unless otherwise stated.

**Definition 4.2.** For any $t_1 < t_2 \in \mathbb{R}$ and $s_1, s_2 \in [0,1)$, we define

\[
\begin{align*}
  \hat{\Gamma}^{i_0,i_1}_{t_1,t_2}(x,y) &= \{ \gamma \in H^1_{t_1,t_2}(x,y) : \gamma \text{ is tied with } m_{i_0} \text{ and } m_{i_1} \}; \\
  \Gamma^{i_0,i_1}_{t_1,t_2}(x,y) &= \text{the weak closure of } \hat{\Gamma}^{i_0,i_1}_{t_1,t_2}(x,y) \text{ in } H^1_{t_1,t_2}(x,y); \\
  \tilde{\Xi}^{i_0,i_1}_{s_1,s_2}(x,y) &= \bigcup_{\{t_1\} = s_1,i = 1,2} \hat{\Gamma}^{i_0,i_1}_{t_1,t_2}(x,y); \\
  \Xi^{i_0,i_1}_{s_1,s_2}(x,y) &= \bigcup_{\{t_1\} = s_1,i = 1,2} \Gamma^{i_0,i_1}_{t_1,t_2}(x,y).
\end{align*}
\]

**Remark 4.2.** An arbitrary pair of indices $\{i_0 \neq i_1\} \subset \mathbb{N}$ will be fixed for the rest of the section, and for simplicity, they will be omitted from the notations introduced in the above definition.

The next result is a simple corollary of the above definition, but will be useful in our proof.

**Lemma 4.1.** For any $\gamma \in H^1_{t_1,t_2}(x,y)$, if there is a $t_0 \in (t_1,t_2)$, such that $\gamma(t_0) = q_{i_0}(t_0)$ or $q_{i_1}(t_0)$, then $\gamma \in \Gamma_{t_1,t_2}(x,y)$.

We further introduce the following two functions:

\[
\begin{align*}
  \Psi_h(x,y;t_1,t_2) &= \inf \{A_h(\gamma;t_1,t_2) : \gamma \in \Gamma_{t_1,t_2}(x,y)\}, \\
  \Phi_h(x,y;s_1,s_2) &= \inf \{A_h(\gamma) : \gamma \in \Xi_{s_1,s_2}(x,y)\}.
\end{align*}
\]

In the previous section, we assume $x \neq y$ in several lemmas and propositions to give a positive lower bound of $\int_0^1 |\dot{\gamma}| dt$, $\forall \gamma \in H^1_{t_1,t_2}(x,y)$. Here it is not needed, because of the following result.

**Lemma 4.2.** For any $x, y \in \mathbb{C} \setminus B^0_{R_0}$ and $\gamma \in \Gamma_{t_1,t_2}(x,y)$, $\int_{t_1}^{t_2} |\dot{\gamma}(t)| dt \geq 2$.

**Proof.** By the definition $\Gamma_{t_1,t_2}(x,y)$, there must be a $t_0 \in (t_1,t_2)$ with $|\gamma(t_0)| \leq R_0 - 1$, then

\[
\int_{t_1}^{t_2} |\dot{\gamma}(t)| dt \geq |\gamma(t_1) - \gamma(t_0)| + |\gamma(t_0) - \gamma(t_2)| \geq 2.
\]

\[
\square
\]

**Proposition 4.1.** There is a $\gamma \in \Gamma_{t_1,t_2}(x,y)$ satisfying $A_h(\gamma;t_1,t_2) = \Psi_h(x,y;t_1,t_2)$. Moreover (a), for any $t_0 \in (t_1,t_2) \setminus \Delta(\gamma)$, there is a $\delta > 0$ small enough, such that $\gamma|_{[t_0-\delta,p_0+\delta]}$ is a collision-free local minimizer and a solution of (2).
(b). \( \Delta(\gamma) = \emptyset \) or \( \{t_0\} \subset (t_1, t_2) \) and in the latter case, \( \gamma(t_0) = q_{j_0}(t_0) \) with \( j_0 \in \{i_0, i_1\} \) and the following limits exist

\[
\lim_{t \to t_0^+} \frac{1}{2} |\dot{\gamma}(t) - \dot{q}_{j_0}(t)|^2 - \frac{m_1}{|\gamma(t) - q_{j_0}(t)|}, \quad \lim_{t \to t_0^-} |\gamma(t) - q_{j_0}(t)|.
\]

(c). \( \gamma|_{[t_1, t_2]} \) is a solution of (23) with at most one collision, and if there is a collision, it is regularizable.

**Proof.** By Lemma 4.2, the existence of such a \( \gamma \in \Gamma_{t_1, t_2}(x, y) \) follows from the same argument given in the proof of property (a) in Proposition 2.5. In particular \( \mathcal{A}_h(\gamma; t_1, t_2) \) must be finite.

(a). Since \( \mathcal{A}_h(\gamma; t_1, t_2) \) is finite, the Lebesgue measure of \( \Delta(\gamma) \) must be zero with \( (t_1, t_2) \setminus \Delta(\gamma) \) being the union of at most countably many open intervals. Then \( \forall t_0 \in (t_1, t_2) \setminus \Delta(\gamma) \), there is a \( \delta > 0 \) small enough, such that \( [\tau_0 - \delta, \tau_0 + \delta] \subset (t_1, t_2) \setminus \Delta(\gamma) \) with \( \gamma|_{[\tau_0 - \delta, \tau_0 + \delta]} \) being collision-free.

As a result, we can find a \( \varepsilon > 0 \) small enough, such that \( \forall \xi \in H^1_{t_0 - \delta, t_0 + \delta} (\gamma(\tau_0 - \delta), \gamma(\tau_0 + \delta)) \) satisfying \( \|\xi - \gamma\|_{H^1} \leq \varepsilon \), the following new path still belongs to \( \Gamma_{t_1, t_2}(x, y) \).

\[
\tilde{\gamma}|_{[t_1, t_2]} = \gamma|_{[\tau_0 - \delta, \tau_0 + \delta]} \ast \xi|_{[\tau_0 - \delta, \tau_0 + \delta]} \ast \gamma|_{[\tau_0 + \delta, t_2]} \in \Gamma_{t_1, t_2}(x, y).
\]

This means \( \gamma|_{[\tau_0 - \delta, \tau_0 + \delta]} \) is a collision-free local minimizer and a solution of (23), as otherwise

\[
\mathcal{A}_h(\tilde{\gamma}; t_1, t_2) < \mathcal{A}_h(\gamma; t_1, t_2) = \Psi_h(x, y; t_1, t_2),
\]

which is absurd.

(b). By the above property (a), \( \gamma(t), t \in [t_1, t_2] \setminus \Delta(\gamma) \), satisfies (23). Then similar arguments as in the proof of Proposition 2.1 show that \( \Delta(\gamma) \) must be isolated in \( [t_1, t_2] \).

Choose an arbitrary \( \tau_0 \in \Delta(\gamma) \). Then \( \gamma(\tau_0) = q_{j_0}(\tau_0) \) for some \( j_0 \in \mathbb{N} \). First we will show \( j_0 \in \{i_0, i_1\} \). Otherwise we can find \( \delta, \varepsilon > 0 \) small enough, such that \( \forall \xi \in H^1_{\tau_0 - \delta, \tau_0 + \delta} (\gamma(\tau_0 - \delta), \gamma(\tau_0 + \delta)) \) satisfying \( \|\gamma|_{[\tau_0 - \delta, \tau_0 + \delta]} - \xi|_{[\tau_0 - \delta, \tau_0 + \delta]}\|_{H^1} \leq \varepsilon \), the new path \( \tilde{\gamma} \) given as in (52) is still contained in \( \Gamma_{t_1, t_2}(x, y) \). Meanwhile by Proposition 2.3, one of such a \( \xi \) satisfies

\[
\mathcal{A}_h(\tilde{\gamma}; \tau_0 - \delta, \tau_0 + \delta) < \mathcal{A}_h(\gamma; \tau_0 - \delta, \tau_0 + \delta).
\]

This then leads to (53), which is absurd.

Next we will show \( |\Delta(\gamma)| \leq 1 \). By a contradiction argument, assume there are two different collision moments \( \tau_0 \) and \( \tau_1 \). Then \( \gamma(\tau_k) = q_{j_k}(\tau_k) \) with \( j_k \in \{i_0, i_1\} \), for \( k = 1, 2 \). By Proposition 2.3, there is a \( \xi \in H^1_{\tau_0 - \delta, \tau_0 + \delta} (\gamma(\tau_0 - \delta), \gamma(\tau_0 + \delta)) \) satisfying \( \|\gamma|_{[\tau_0 - \delta, \tau_0 + \delta]} - \xi|_{[\tau_0 - \delta, \tau_0 + \delta]}\|_{H^1} \leq \varepsilon \) and (54), for \( \delta, \varepsilon > 0 \) small enough. Let \( \tilde{\gamma} \) be a path given as in (52), then (53) holds. However by Lemma 4.1, \( \tilde{\gamma} \in \Gamma_{t_1, t_2}(x, y) \), which is absurd.

What’s left now is to prove (51). Without loss of generality, let’s assume \( \gamma(\tau_0) = q_{i_0}(\tau_0) \). Then the first limit in (51) follows directly from Lemma 2.1. For the second limit, by Proposition 2.2, the following two one-sided limits exist

\[
\sigma_\pm = \lim_{t \to \tau_0^\pm} \frac{\gamma(t) - q_i(t)}{|\gamma(t) - q_i(t)|}.
\]

Then it is enough show \( \sigma_- = \sigma_+ \). Assume this is not true. By Proposition 2.3, for \( \delta, \varepsilon > 0 \) small enough, there are two paths \( \eta^\pm \in H^1_{\tau_0 - \delta, \tau_0 + \delta} (\gamma(\tau_0 - \delta), \gamma(\tau_0 + \delta)) \) satisfying (10), (11) and

\[
\mathcal{A}_h(\eta^\pm; \tau_0 - \delta, \tau_0 + \delta) < \mathcal{A}_h(\gamma; \tau_0 - \delta, \tau_0 + \delta).
\]

Let \( \gamma^\pm|_{[t_1, t_2]} = \gamma|_{[\tau_0 - \delta, \tau_0 + \delta]} \ast \xi^\pm|_{[\tau_0 - \delta, \tau_0 + \delta]} \ast \gamma|_{[\tau_0 + \delta, t_1]} \). Then \( \mathcal{A}_h(\gamma^\pm; t_1, t_2) < \mathcal{A}_h(\gamma; t_1, t_2) \). Since one of \( \gamma^\pm \) must belong to \( \Gamma_{t_1, t_2}(x, y) \), we get a contradiction.
(c). By property (c), $\gamma_{[t_1,t_2]}$ contains at most one collision moment. If $t_0$ is such a collision moment, then $t_0 \in (t_1, t_2)$ and the two limits in (51) hold. This implies $\gamma_{[t_1,t_2]}$ is regularizable, where the details can be found in [10].

\[ \Box \]

**Lemma 4.3.** For any $t_1 < t_2$ and $x,y \in \mathbb{C} \setminus B_{R_0}^o$, there is a $C_1 = C_1(x,y,t_1,t_2,h)$, such that

\[ \sup \left\{ \Psi_h(x',y';t_1,t_2) : (x',y') \in B_{1/2}(x) \times B_{1/2}(y) \text{ with } |x'|, |y'| \geq R_0 \right\} \leq C_1. \]

**Proof.** By Proposition 4.1, there is a $\gamma \in \Gamma_{t_1,t_2}(x,y)$ with $\mathcal{A}_h(\gamma; t_1,t_2) = \Psi_h(x,y;t_1,t_2)$. Then

\[ \tau_1 = \max \{ \tau \in [t_1,t_2] : \gamma(\tau) \in B_{1/2}(x), \forall \tau \in [t_1,\tau] \} ; \]

\[ \tau_2 = \min \{ \tau \in [t_1,t_2] : \gamma(\tau) \in B_{1/2}(y), \forall \tau \in [\tau,t_2] \}, \]

are well-defined. Moreover $t_1 < \tau_1 < \tau_2 < t_2$ and $|\gamma(\tau_1) - x| = |\gamma(\tau_2) - y| = 1/2$.

For any $x' \in B_{1/2}(x) \cap (\mathbb{C} \setminus B_{R_0})$ and $y' \in B_{1/2}(y) \cap (\mathbb{C} \setminus B_{R_0})$, we can define a new path

\[ \xi(t) = \begin{cases} \gamma(\tau_2) + \frac{t-\tau_2}{\tau_2-\tau_1}(y' - \gamma(t_2)), & \text{if } t \in [\tau_2,t_2], \\ \gamma(t), & \text{if } t \in [\tau_1,\tau_2], \\ x' + \frac{t-t_1}{\tau_1-t_1}(\gamma(\tau_1) - x'), & \text{if } t \in [t_1,\tau_1]. \end{cases} \]

Notice that $\xi \in \Gamma_{t_1,t_2}(x',y')$. Moreover $\xi(t) \in B_{1/2}(x), \forall t \in [t_1,\tau_1]$ and $\xi(t) \in B_{1/2}(y), \forall t \in [\tau_2,t_2]$. By (12), there is a $C_2 = C_2(x,y)$ with $U(\xi(t), t) \leq C_2$, for all $t \in [t_1,\tau_1] \cup [\tau_2,t_2]$. As a result,

\[ \Psi_h(x',y';t_1,t_2) \leq \mathcal{A}_h(\xi; t_1,t_2) - \mathcal{A}_h(\gamma; t_1,t_2) + \Psi_h(x,y;t_1,t_2) \]

\[ \leq \int_{t_1}^{\tau_1} L(\xi, \dot{\xi}, t) \; dt + \int_{\tau_1}^{t_2} L(\xi, \dot{\xi}, t) \; dt + \Psi_h(x,y;t_1,t_2) \]

\[ \leq \frac{|\gamma(\tau_1) - x'|^2}{2(\tau_1 - t_1)} + \frac{|\gamma(\tau_2) - y'|^2}{2(t_2 - \tau_2)} + C_2(t_2 - t_1) + \Psi_h(x,y;t_1,t_2) \]

\[ \leq \frac{1}{2(\tau_1 - t_1)} + \frac{1}{2(t_2 - \tau_2)} + C_2(t_2 - t_1) + \Psi_h(x,y;t_1,t_2) := C_1. \]

The next three propositions are similarly to Proposition 2.6 and 2.7, so are their proofs.

**Proposition 4.2.** $\Psi_h$ is locally Lipschitz continuous in $\{(x,y,t_1,t_2) : x,y \in \mathbb{C} \setminus B_{R_0}^o, t_2 > t_1\} \subset \mathbb{C}^2 \times \mathbb{R}^2$.

**Proof.** With Lemma 4.3, this can be proven by a similar argument as Proposition 2.6.

**Proposition 4.3.** When $h > 0$, for any $s_1,s_2 \in [0,1)$ and $x,y \in \mathbb{C} \setminus B_{R_0}^o$. There exist $t_1 < t_2 \in \mathbb{R}$ satisfying $\{t_i\} = s_i, i = 1,2$, and a $\gamma \in \Gamma_{t_1,t_2}(x,y)$, such that

\[ \mathcal{A}_h(\gamma; t_1,t_2) = \Psi_h(x,y;t_1,t_2) = \Phi_h(x,y;s_1,s_2). \]

**Proof.** With Proposition 4.1, this can be proven by the same arguments as Proposition 2.7.

For any $x,y \in \mathbb{C} \setminus B_{R_0}^o$ and $t_1 < t_2$, we set

\[ \Omega_{t_1,t_2}(x,y) = \{ \xi \in H_{t_1,t_2}^1(x,y) : |\xi(t)| \geq R_0, \forall t \in [t_1,t_2] \}. \]

**Proposition 4.4.** For any $h > 0$ and $x,y \in \mathbb{C} \setminus B_{R_0}^o$, there is a $\gamma_{xy} \in \Gamma_{t_{xy},t_y}(x,y)$ satisfying:

(a). $\mathcal{A}_h(\gamma_{xy}; t_x,t_y) = \Phi_h(x,y;\{t_x\},\{t_y\}) = \inf \{ \Phi_h(x,y;s_1,s_2) : s_1, s_2 \in [0,1) \}$;
(b). for any \( \tau_1 \in [t_x, s_x] \) and \( \tau_2 \in [s_y, t_y] \) (with \( s_x \) and \( s_y \) given as in (48)),
\[
A_h(\gamma_{xy}; \tau_1, \tau_2) = \Phi_h(\gamma_{xy}(\tau_1), \gamma_{xy}(\tau_2); \{\tau_1\}, \{\tau_2\}).
\]

(c). for any \( [\tau_1, \tau_2] \subseteq [t_x, s_x] \) or \( [s_y, t_y] \),
\[
A_h(\gamma_{xy}; \tau_1, \tau_2) = \inf\{A_h(\xi; \tau_1, \tau_2) : \xi \in \Omega_{\tau_1, \tau_2}(\gamma_{xy}(\tau_1), \gamma_{xy}(\tau_2))\}.
\]

Proof. (a). By Proposition 4.3 we can find a sequence of paths \( \{\gamma_n \in \Gamma_{t_n, \tau_n}(x, y)\}_{n \in \mathbb{N}} \) satisfying
\[
\lim_{n \to \infty} A_h(\gamma_n; t_n, \tau_n) = \Phi_h(x, y; \{t_n\}, \{\tau_n\}) = \inf\{\Phi_h(x, y; s_1, s_2); s_1, s_2 \in [0, 1]\}.
\]
Since the above infimum must be finite, \( \{\tau_n - t_n\}_{n \in \mathbb{N}} \) is bounded. Otherwise the following holds, which is absurd.
\[
A_h(\gamma_n; t_n, \tau_n) \geq h(\tau_n - t_n) \to \infty, \quad \text{as } n \to \infty.
\]
As we may assume \( t_n \in [0, 1], \forall n \), the rest of property (a) can be proven just like Proposition 2.8.

(b). For simplicity, set \( \gamma = \gamma_{xy} \). By a contradiction argument, let’s assume the desired result does not hold. Then there exist \( t_1 < t_2 \) with \( \{t_1\} = \{\tau_i\}, i = 1, 2 \), and a \( \xi \in \Gamma_{t_1, t_2}(\gamma_{xy}(\tau_1), \gamma_{xy}(\tau_2)) \) satisfying \( A_h(\xi; t_1, t_2) < \Phi_h(\gamma(\tau_1), \gamma(\tau_2); \{\tau_1\}, \{\tau_2\}) \).
Then
\[
\tilde{\gamma} \mid_{[t_1, t_1 + t_2 - t_1 + t_y - t_2]} \gamma = \xi \mid_{[t_1, t_2]} * \gamma \mid_{[t_2, t_2 + t_2 - t_1 - t_y + t_2]} \in \Xi_{\{t_1, t_2\}}(\gamma(t_x), \gamma(t_y))
\]
and \( A_h(\tilde{\gamma}; t_1, t_1 + t_2 - t_1 + t_y - t_2) < A_h(\gamma; t_x, t_y) = \Phi_h(x, y; \{t_n\}, \{\tau_n\}) \), which is absurd.

(c). This follows directly from Definition 4.1 and (55). \( \square \)

A lemma similar to Lemma 3.1 will be needed. However because of the topological constraint imposed in Definition 4.1, we only consider paths from \( \Omega \) as defined in (55).

Lemma 4.4. For any \( x, y \in \mathbb{C} \) satisfying \( \sqrt{2}R_1 \leq |x|, |y| \leq R \), the following results hold.

(a). For any \( t_1 < t_2 \), \( \inf\{A_h(\gamma; t_1, t_2) : \gamma \in \Omega_{t_1, t_2}(x, y)\} \leq 16\frac{R^2}{t_2 - t_1} + 12m\frac{t_2 - t_1}{R} \).

(b). For any \( h > 0 \), there exist positive constant \( \beta_1, \beta_2 \) independent of \( R \), such that \( \inf\{A_h(\gamma; \tau_1, \tau_2) : \gamma \in \Omega_{\tau_1, \tau_2}(x, y)\} \leq \beta_1 R + \beta_2 \).

Proof. This can be proven exactly the same as Lemma 3.1. \( \square \)

For the rest of the section, let’s fix arbitrarily an \( h > 0 \), a pair of angles \( \theta_{\pm} \in [0, 2\pi) \) and set
\[
\Lambda_{\theta_{+}}(R) = \{(r_1e^{i\theta_1}, r_2e^{i\theta_2}) : r_1, r_2 \geq R, \theta_1 = \theta_{-}(\text{mod } 2\pi)\} \cup \{(r_1e^{i\theta_1}, r_2e^{i\theta_2}) : r_1, r_2 \geq R, \theta_2 = \theta_{+}(\text{mod } 2\pi)\}.
\]
For any \( (x, y) \in \Lambda_{\theta_{+}}(R_0) \), let \( \gamma_{xy} \in \Gamma_{t_x, t_y}(x, y) \) be the minimizer given by Proposition 4.4.

Lemma 4.5. For any \( (x, y) \in \Lambda_{\theta_{+}}(R_0) \) with \( R_0 \geq 2\sqrt{2}R_1 \), let
\[
(56) \quad \tau_y = \min\{t \in [s_y, t_y] : |\gamma_{xy}(t)| = R_2\}, \quad \tau_x = \max\{t \in [t_x, s_x] : |\gamma_{xy}(t)| = R_2\},
\]
with \( s_x, s_y \) defined as in (48). The following results hold.

(a). \( 0 < \inf\{s_y - s_x : (x, y) \in \Lambda_{\theta_{+}}(R_2)\} \leq \sup\{s_y - s_x : (x, y) \in \Lambda_{\theta_{-}}(R_2)\} < \infty \).

(b). \( 0 < \inf\{\tau_y - s_y : (x, y) \in \Lambda_{\theta_{+}}(R_2)\} \leq \sup\{\tau_y - s_y : (x, y) \in \Lambda_{\theta_{-}}(R_2)\} < \infty \).

(c). \( 0 < \inf\{s_x - \tau_x : (x, y) \in \Lambda_{\theta_{+}}(R_2)\} \leq \sup\{s_x - \tau_x : (x, y) \in \Lambda_{\theta_{-}}(R_2)\} < \infty \).

(d). Set \( r_{xy}(t) = |\gamma_{xy}(t)|, \) when \( R_2 \) is large enough,
\[
(57) \quad r_{xy}(t) \geq \sqrt{3m/2R_2^{-2}}, \forall t \in [\tau_y, t_y]; \quad \dot{r}_{xy}(t) \leq -\sqrt{3m/2R_2^{-2}}, \forall t \in [t_x, \tau_x].
\]
Proof. (a). Fixed an $n_0 \geq 2$, by Proposition 4.2
\[
\sup\{\Psi_h(x_0, y_0; s_1, s_2 + n_0) : x_0 \in \partial B_{R_2}, y_0 \in \partial B_{R_2}, s_1, s_2 \in [0, 1]\} \leq C_1.
\]
Then by Proposition 4.4, for any $[\tau_1, \tau_2] \subset [\tau_x, \tau_y]$ (notice that $[s_x, s_y] \subset [\tau_x, \tau_y]$, see Figure 3),
\[
A_h(\gamma_{xy}; \tau_1, \tau_2) \leq A_h(\gamma_{xy}; \tau_x, \tau_y) = \Phi_h(\gamma_{xy}(\tau_x), \gamma_{xy}(\tau_y); \{\tau_x\}, \{\tau_y\})
\leq \Psi_h(x, y; \{\tau_x\}, \{\tau_y\} + n_0) \leq C_1.
\]
This gives us the following estimates, which imply the third inequality in (a),
\[
s_y - s_x \leq A_h(\gamma_{xy}; s_x, s_y)/h \leq A_h(\gamma_{xy}; \tau_x, \tau_y)/h \leq C_1/h.
\]
Meanwhile by Lemma 4.2 and the Cauchy-Schwartz inequality,
\[
(s_y - s_x) \int_{s_x}^{s_y} |\hat{\gamma}_{xy}|^2 dt \geq \left( \int_{s_x}^{s_y} |\hat{\gamma}_{xy}| dt \right)^2 \geq 4.
\]
Therefore
\[
\frac{2}{s_y - s_x} \leq \frac{1}{2} \int_{s_x}^{s_y} |\hat{\gamma}_{xy}|^2 dt \leq A_h(\gamma_{xy}; s_x, s_y) \leq C_1.
\]
This implies the first inequality in (a), while the second inequality in (a) is trivial.

(b) & (c). Since $[\tau_x, s_x], [s_y, \tau_y] \subset [\tau_x, \tau_y]$ and
\[
\int_{s_y}^{\tau_y} |\hat{\gamma}_{xy}| dt, \int_{s_x}^{\tau_x} |\hat{\gamma}_{xy}| dt \geq R_2 - R_0 > 0,
\]
the proofs are similar to those given in part (a) and will be omitted.

(d). With Lemma 4.4 this can be proven by a similar argument as the proof of property (b) in Lemma 3.2. \[\Box\]

Like in the previous section, we fix an $R_2 \geq 2\sqrt{2}R_1$ for the rest of the section, such that the above lemma always holds. Moreover since the system is 1-periodic, we may further assume
\[
s_x \in [0, 1), \forall (x, y) \in A_{\theta_+}^0(R_2).
\]
Under the above assumption, Lemma 4.5 implies
\[(60) \sup\{|\tau_x|, |\tau_y| : (x, y) \in \Lambda^\theta_+(R_2)\} < \infty.\]

**Lemma 4.6.** Given an arbitrary \(\tau \geq \sup\{|\tau_x|, |\tau_y| : (x, y) \in \Lambda^\theta_+(R_2)\},\) the following results hold.

(a). For any \((x, y) \in \Lambda^\theta_+(R_2)\) with \([-\tau, \tau] \subset [t_x, t_y]\), there is a finite \(R = R(\tau, R_2)\) independent of \((x, y),\) such that \(|\gamma_{xy}(t)| \leq R, \forall t \in [-\tau, \tau]\).

(b). \(\{\gamma_{xy}[-\tau, \tau] : (x, y) \in \Lambda^\theta_{\theta-}(R_2)\) with \([-\tau, \tau] \subset [t_x, t_y]\) is a bound subset of \(H^1([-\tau, \tau], \mathbb{C})\).

**Proof.** Throughout this proof, we only consider \((x, y) \in \Lambda^\theta_+(R_2)\) with \([-\tau, \tau] \subset [t_x, t_y]\).

(a). Since \(|\gamma_{xy}(t)| \leq R_2, \forall t \in [\tau_x, \tau_y].\) By property (d) in Lemma 4.5, it is enough to show \(|\gamma_{xy}(\pm \tau)| \leq R,\) for some \(R\) independent of \((x, y).\) Meanwhile by property (c) in Proposition 4.4 and property (b) in Lemma 4.4
\[(61) \int_{\tau_x}^{\tau_y} |\gamma_{xy}|^2 dt \leq 2A_h(\gamma_{xy}; \tau_y, \tau) \leq 2\beta_1 R + 2\beta_2;\]
\[(62) \int_{-\tau}^{\tau_x} |\gamma_{xy}|^2 dt \leq 2A_h(\gamma_{xy}; -\tau, \tau_x) \leq 2\beta_1 R + 2\beta_2.\]

The desired result then follows from the same argument as in property (a) of Lemma 3.3.

(b). It is enough to show \(\int_{-\tau}^{\tau_x} |\gamma_{xy}|^2 dt \leq C_1\) for some \(C_1\) independent of \(\gamma_{xy}.\) By Lemma 4.5
\[0 < a^- = \inf\{\tau_x : (x, y) \in \Lambda^\theta_{\theta+}(R_2)\} < a^+ = \sup\{\tau_x : (x, y) \in \Lambda^\theta_{\theta-}(R_2)\} < \infty;\]
\[0 < b^- = \inf\{\tau_y : (x, y) \in \Lambda^\theta_{\theta+}(R_2)\} < b^+ = \sup\{\tau_y : (x, y) \in \Lambda^\theta_{\theta-}(R_2)\} < \infty.\]

Then Proposition 4.2 implies
\[\sup\{\Psi_h(z_1, z_2; t_1, t_2) : z_1, z_2 \in \partial B_{R_2}, t_1 \in [a^-, a^+], t_2 \in [b^-, b^+]\} \leq C_2.\]

By Proposition 4.4
\[A_h(\gamma_{xy}; \tau_x, \tau_y) = \Phi_h(\gamma_{xy}(\tau_x), \gamma_{xy}(\tau_y); \{\tau_x\}, \{\tau_y\}) \leq \Psi_h(\gamma_{xy}(\tau_x), \gamma_{xy}(\tau_y); \{\tau_x\}, \{\tau_y\}).\]

Since \(|\gamma_{xy}(\tau_x)| = |\gamma_{xy}(\tau_y)| = R_2,\) the above two estimates imply
\[\int_{\tau_x}^{\tau_y} |\gamma_{xy}|^2 dt \leq 2A_h(\gamma_{xy}; \tau_x, \tau_y) \leq 2\Psi_h(\gamma_{xy}(\tau_x), \gamma_{xy}(\tau_y); \{\tau_x\}, \{\tau_y\}) \leq 2C_2.\]
Combining this with (61) and (62), we get the desired result. \(\square\)

For the rest of the section, let’s choose a sequence \(\{(x_n, y_n) \in \Lambda^\theta_+(R_2)\}_{n \in \mathbb{N}}\) satisfying
\[\lim_{n \to \infty} |x_n| = \lim_{n \to \infty} |y_n| = \infty.\]

For each \(n,\) by Proposition 4.4 there is a \(\gamma_n \in \Gamma_{t_x, t_y}(x_n, y_n),\) such that
\[A_h(\gamma_n; t_x, t_y) = \Phi_h(x_n, y_n; \{t_x\}, \{t_y\}) = \inf\{\Phi_h(x_n, y_n; s_1, s_2) : s_1, s_2 \in [0, 1]\}.\]

**Proposition 4.5.** Under the above notations.

(a). \(t_{x_n} \to -\infty\) and \(t_{y_n} \to \infty,\) as \(n \to \infty.\)

(b). There is a \(\gamma \in H^1(\mathbb{R}, \mathbb{C}),\) such that (after passing to a proper subsequence) \(\gamma_n \to \gamma\) strongly in \(L^{\infty}_{\text{loc}}\)-norm and weakly in \(H^1_{\text{loc}}\)-norm, as \(n \to \infty.\)
(c). For any \( t_0 > 0 \) large enough, \( \gamma|_{[-t_0, t_0]} \in \Gamma_{-t_0, t_0}(\gamma(-t_0), \gamma(t_0)) \) and
\[
A_h(\gamma; -t_0, t_0) = \Phi_h(\gamma(-t_0), \gamma(t_0); \{-t_0\}, \{t_0\}).
\]

(d). \( \gamma(t) \) is a solution of \( \{2\} \) with at most one regularizable collision, and there is a \( \tau_0 > 0 \) large enough, such that for any compact interval \( I \subset (-\infty, \tau_0) \) or \( (\tau_0, \infty) \), \( \| \gamma_n|_I - \gamma|_I \|_{C^1} \to 0 \), as \( n \to \infty \).

**Proof.** (a). We give the details for \( t_{y_n} \to \infty \), while \( t_{x_n} \to \infty \) can be proven similarly.

By Proposition 4.4, Lemma 4.4 and the Cauchy-Schwartz inequality,
\[
\beta_1 |y_n| + \beta_2 \geq A_h(\gamma_n; \tau_{y_n}, t_{y_n}) \geq \frac{1}{2} \int_{\tau_{y_n}}^{t_{y_n}} |\gamma_n'|^2 \, dt \geq \frac{1}{2(t_{y_n} - \tau_{y_n})} \left( \int_{\tau_{y_n}}^{t_{y_n}} |\gamma_n'| \, dt \right)^2.
\]
This implies
\[
t_{y_n} \geq \frac{|y_n|^2 - 2R_2|y_n| + R_2^2}{2(\beta_1 |y_n| + \beta_2)} \geq \frac{|y_n|^2 - 2R_2 + R_2^2 |y_n|^{-1}}{2\beta_1 + 2\beta_2 |y_n|^{-1}} \to \infty, \text{ and } |y_n| \to \infty.
\]

(b). With property (a), it follows from a similarly argument as in the proof of property (b) in Lemma 3.4.

(c). Choose an arbitrary \( t_0 > \sup\{\tau_{x_n}, |\tau_{y_n}| : n \in \mathbb{N}\} \). By \( \|\cdot\|_{0} \), the supremum must be finite. Then for \( n \) large enough, \( \gamma_n|_{[-t_0, t_0]} \in \Gamma_{-t_0, t_0}(\gamma_n(-t_0), \gamma_n(t_0)) \). Then property (b) implies \( \gamma|_{[-t_0, t_0]} \in \Gamma_{-t_0, t_0}(\gamma(-t_0), \gamma(t_0)) \). To obtain the rest of the property, by a contradiction argument, let’s assume there is a \( t_0 \) large enough, such that
\[
A_h(\gamma; -t_0, t_0) < \Phi_h(\gamma(-t_0), \gamma(t_0); \{t_0\}, \{t_0\}).
\]

By Proposition 4.4, there is a \( \xi \in H^1_{t_1, t_2}(\gamma(-t_0), \gamma(t_0)) \) with \( \{t_1\} = \{-t_0\} \) and \( \{t_2\} = \{t_0\} \) (see Figure 4), and an \( \varepsilon > 0 \) small enough, such that
\[
A_h(\xi; t_1, t_2) = \Phi_h(\gamma(-t_0), \gamma(t_0); \{-t_0\}, \{t_0\}) < A_h(\gamma; -t_0, t_0) - 4\varepsilon.
\]

By property (b) above and the lower semi-continuity of \( A_h \), for \( n \) large enough,
\[
A_h(\gamma_n; -t_0, t_0) \geq A_h(\gamma; -t_0, t_0) - \varepsilon \geq A_h(\xi; t_1, t_2) + 3\varepsilon.
\]

Notice that \( \xi(t_1) = \gamma(-t_0) \) and \( \xi(t_2) = \gamma(t_0) \) are collision-free, for \( t_0 \) large enough. Then so is \( \gamma|_{[t_1, \tau_1]} \) and \( \gamma|_{[\tau_2, t_2]} \), for \( \tau_i \) close enough to \( t_i, \ i = 1, 2 \). As a result, \( A_h \) is continuous in a small neighborhood of \( \xi|_{[t_1, \tau_1]} \) in \( H^1_{t_1, \tau_1}(\xi(t_1), \xi(\tau_1)) \), and of \( \xi|_{[\tau_2, t_2]} \) in \( H^1_{\tau_2, t_2}(\xi(\tau_2), \xi(t_2)) \). Since
\[
\gamma_n(-t_0) \to \gamma(-t_0) = \xi(t_1), \ \gamma_n(t_0) \to \gamma(t_0) = \xi(t_2), \ \text{ when } n \to \infty.
\]

For each \( n \) large enough, there is a \( \eta_n \in H^1_{t_1, \tau_1}(\gamma_n(-t_0), \xi(\tau_1)) \) and \( \tilde{\eta}_n \in H^1_{\tau_2, t_2}(\xi(t_2), \gamma_n(t_0)) \) with
\[
|A_h(\xi; t_1, t_1) - A_h(\eta_n; t_1, \tau_1)|, |A_h(\xi; \tau_2, t_2) - A_h(\tilde{\eta}_n; \tau_2, t_2) | \leq \varepsilon.
\]

Let \( \zeta_n|_{[t_1, t_2]} = \eta_n|_{[-t_0, \tau_1]} \ast \xi|_{[\tau_1, \tau_2]} \ast \tilde{\eta}_n|_{[\tau_2, t_2]} \). Then
\[
A_h(\xi; t_1, t_2) - A_h(\zeta_n; t_1, t_2) \geq -2\varepsilon.
\]

Combining this with \( \{64\} \), we get
\[
A_h(\gamma_n; -t_0, t_0) - A_h(\zeta_n; t_1, t_2) \geq \varepsilon.
\]
This is absurd, as \( \zeta|_{[t_1,t_2]} \in \Xi(-t_0,\{t_0\}) \) and Proposition 4.4 implies

\[
A_h(\gamma_n;\{-t_0\},\{t_0\}) = \Phi_h(\gamma_n(t_0),\{t_0\}).
\]

(d). With property (c), Proposition 4.1 shows \( \gamma |_{R} \) is a solution of (2) with at most one regularizable collision. Then we can find a \( \tau > 0 \) large enough, such that \( \gamma(t) \) is collision-free, for all \( t \in (-\infty,\tau_0] \cup [\tau_0,\infty) \). The result then follows from similar arguments as in the proof of property (d) in Proposition 3.4.

With the above results, similar arguments as in the proofs of Proposition 3.5 and 3.6 can show Proposition 4.6.

\[
\lim_{t \to \pm \infty} \gamma(t) = e^{i\theta \pm (mod \ 2\pi)} \quad \text{and} \quad \lim_{t \to \pm \infty} \dot{\gamma}(t) = \pm \sqrt{2}he^{i\theta \pm (mod \ 2\pi)}.
\]

Proof of Theorem 1.2. When \( T = 1 \), the desired result follows from previous results obtained in this section. When \( T \neq 1 \), it follows from a blow-up argument as in the proof of Theorem 1.1 given in the end of Section 3.

5. Appendix A: asymptotic analysis near an isolated collision

We give proofs of Proposition 2.2 and 2.3 in this appendix. For this it is more convenient to consider the relative position between the massless body and the primary \( m_{i_0} \), which is

\[
(65) \quad \zeta(t) = \gamma(t) - q_{i_0}(t), \ \forall t \in [t_0 - \delta, t_0 + \delta].
\]

Then the Lagrangian \( L \) becomes

\[
L(\zeta, \dot{\zeta}, t) = \frac{1}{2}(\dot{\zeta}(t) + \dot{q}_{i_0}(t))^2 + \sum_{i \neq i_0} m_i \frac{m_{i_0}}{|\zeta(t)|} + \frac{m_i}{|\zeta(t) + q_{i_0}(t) - q_{i}(t)|} \ dt.
\]

Proof of Proposition 2.2. Since \( \gamma(t) \) is a collision-ejection solution of (2), \( \zeta(t) \) satisfies

\[
(67) \quad \ddot{\zeta}(t) = -\frac{m_{i_0} \dot{\zeta}(t)}{|\zeta(t)|^3} - \dot{q}_{i_0}(t) - \sum_{i \neq i_0} \frac{m_i (\zeta(t) + q_{i_0}(t) - q_{i}(t))}{|\zeta(t) + q_{i_0}(t) - q_{i}(t)|^3}, \ \forall t \neq t_0.
\]

Notice that there is a \( C_1 > 0 \), such that for any \( t \in [t_0 - \delta, t_0 + \delta] \),

\[
\left| \dot{q}_{i_0}(t) + \sum_{i \neq i_0} \frac{m_i (\zeta(t) + q_{i_0}(t) - q_{i}(t))}{|\zeta(t) + q_{i_0}(t) - q_{i}(t)|^3} \right| \leq C_1.
\]
This means $\zeta(t)$ is a collision-ejection solution of the Kepler problem with bounded perturbation. Then the desired results follow from Sperling. □

For the proof of Proposition 2.3, we need some results of the Kepler problem

\begin{equation}
\ddot{\xi}(s) = -m_0 \frac{\xi(s)}{\lvert \xi(s) \rvert^3},
\end{equation}

which is the Euler-Lagrangian equation of the following Lagrangian action functional

\begin{equation}
\hat{A}(\xi; s_1, s_2) = \int_{s_1}^{s_2} \hat{L}(\xi, \dot{\xi}) \, ds, \quad \text{where} \quad \hat{L}(\xi, \dot{\xi}) = \frac{1}{2} \lvert \dot{\xi} \rvert^2 + \frac{m_0}{\lvert \xi \rvert}.
\end{equation}

Recall that for $\sigma_\pm \in \mathbb{C}$ with $\lvert \sigma_\pm \rvert = 1$, the following $\bar{\zeta}(s)$ is a parabolic homothetic collision-ejection solution of the Kepler problem (68) with an isolated collision at the moment $t_0$,

\begin{equation}
\bar{\zeta}(s) = \begin{cases} 
\left(\frac{2}{3} m_0 \right)^{\frac{3}{2}} \lvert s \rvert \frac{2}{5} \sigma_-, & \text{if } s \leq t_0; \\
\left(\frac{2}{3} m_0 \right)^{\frac{3}{2}} \lvert s \rvert \frac{2}{5} \sigma_+, & \text{if } s \geq t_0.
\end{cases}
\end{equation}

**Proposition 5.1.** When $\sigma_- \neq \sigma_+$, for any $T > 0$, there are two collision-free solutions of (68), $\xi^\pm \in C^\infty([-T, T], \mathbb{C} \setminus \{0\})$ satisfying $\hat{A}(\xi^\pm; -T, T) < \hat{A}(|\zeta; -T, T)$ and

(a). $\xi^+(T) = \bar{\zeta}(T)$ and $\text{Arg}(\xi^+(T)) - \text{Arg}(\xi^-(T)) \in (0, 2\pi);$ 
(b). $\xi^-(T) = \bar{\zeta}(T)$ and $\text{Arg}(\xi^-(T)) - \text{Arg}(\xi^-(T)) \in (-2\pi, 0)$.

**Remark 5.1.** As the angular momentum of a collision-free solution of the Kepler problem is a non-zero constant, $\text{Arg}(\xi^+(t))$ is strictly increasing and $\text{Arg}(\xi^-(t))$ is strictly decreasing, as $t$ goes from $-T$ to $T$.

**Proof.** This is a well-known result with several different proofs, see [36, 18, 38] or [10]. □

A blow-up argument will be used in order to apply this result.

**Definition 5.1.** For any $\lambda > 0$, define the $\lambda$-blow-up of $\zeta \rvert_{[t_0-\delta, t_0+\delta]}$ and $q_i \rvert_{[t_0-\delta, t_0+\delta]}$, $i \in \mathbb{N}$, as

\begin{equation}
\zeta^\lambda(s) = \lambda^{\frac{2}{3}} \zeta(t_0 + \lambda^{-1} s), \quad q_i^\lambda(s) = \lambda^{\frac{2}{3}} q_i(t_0 + \lambda^{-1} s), \quad \forall s \in [-\lambda \delta, \lambda \delta].
\end{equation}

The time is shifted only to simplify notations. As a result, $\zeta^\lambda(s), s \in [-\lambda \delta, \lambda \delta] \setminus \{0\}$, satisfies

\begin{equation}
\ddot{\zeta}^\lambda(s) + \ddot{q}_i^\lambda(s) = -m_0 \frac{\zeta^\lambda(s)}{[\zeta^\lambda(s)]^3} - \sum_{i \neq i_0} \frac{m_i (\zeta^\lambda(s) + q_i^\lambda(s) - q_i^\lambda(s))}{[\zeta^\lambda(s) + q_i^\lambda(s) - q_i^\lambda(s)]^3},
\end{equation}

which is the Euler-Lagrangian equation of the action functional

\begin{equation}
\mathcal{A}^\lambda(\zeta^\lambda; s_1, s_2) = \int_{s_1}^{s_2} \mathcal{L}^\lambda(\zeta^\lambda(s), \dot{\zeta}^\lambda(s), s) \, ds,
\end{equation}

where

\begin{equation}
L^\lambda(\zeta^\lambda(s), \dot{\zeta}^\lambda(s), s) = \frac{1}{2} \lvert \dot{\zeta}^\lambda(s) + \dot{q}_i^\lambda(s) \rvert^2 + \frac{m_0}{[\zeta^\lambda(s)]^3} + \sum_{i \neq i_0} \frac{m_i}{[\zeta^\lambda(s) + q_i^\lambda(s) - q_i^\lambda(s)]^3}.
\end{equation}

**Lemma 5.1.** For any $\delta_1 \in (0, \delta]$, $\mathcal{A}^\lambda(\zeta^\lambda; -\lambda \delta_1, \lambda \delta_1) = \lambda^{\frac{2}{3}} \mathcal{A}(\zeta; t_0 - \delta_1, t_0 + \delta_1)$.

**Proof.** By a direct computation,

\begin{equation}
L^\lambda(\zeta^\lambda(s), \dot{\zeta}^\lambda(s), s) = \lambda^{-\frac{2}{3}} L(\zeta(t), \dot{\zeta}(t), t).
\end{equation}

The result then follows from the fact that $t = \lambda^{-1} s + t_0$, as it implies $ds = \lambda dt$. □
Lemma 5.2. For any $T > 0$, as $\lambda$ goes to infinity, 
(a) $\zeta^{\lambda}(s)$ converges uniformly to $\tilde{\zeta}(s)$ on $[-T, T]$;  
(b) $\zeta^{\lambda}(s)$ converges uniformly to $\tilde{\zeta}(s)$ on any compact subset of $[-T, T] \setminus \{0\}$; 

Proof. It follows from Proposition 2.2 and direct computations. □

Proof of Proposition 2.3 Fix a $T > 0$ for the rest of the proof. Let $\xi$ be either $\xi^+$ or $\xi^-$ obtained in Proposition 5.1. Then there is an $\varepsilon > 0$, such that 

$$\bar{A}(\tilde{\zeta}; -T, T) - \bar{A}(\tilde{\zeta}; -T, T) = 6\varepsilon_0.$$  

As $\sigma = \frac{\bar{\xi}(\pm T)}{\xi(\pm T)} = \frac{\xi(\pm T)}{\bar{\xi}(\pm T)}$, we may further assume $\arg(\sigma) = \arg(\xi(\pm T))$. By Proposition 5.1, 

$$\arg(\sigma) - \arg(\bar{\xi}) \in \begin{cases} (0, 2\pi), & \text{if } \xi = \xi^-; \\ (-2\pi, 0), & \text{if } \xi = \xi^+. \end{cases}$$

By Lemma 5.2, $\{\zeta^{\lambda}\}_{\lambda > 2T/\delta}$ is bound in $H^1([-2T, 2T], \mathbb{C})$, so we can find a sequence of positive numbers $\{\lambda_n \to \infty\}_{n=1}^\infty$ with $\zeta^{\lambda_n}\}_{[-2T, 2T]}$ converging weakly to $\zeta^{\lambda}\}_{[-2T, 2T]}$, as $n$ goes to infinity. The weakly lower semi-continuity of $\bar{A}$ then implies 

$$\lim_{n \to \infty} \bar{A}(\zeta^{\lambda_n}; -2T, 2T) \geq \bar{A}(\zeta; -2T, 2T).$$

As a result, for $n$ large enough, 

$$\bar{A}(\zeta^{\lambda_n}; -2T, 2T) \geq \bar{A}(\zeta; -2T, 2T) - \varepsilon_0.$$  

By Lemma 5.2 we can also find a sequence of positive numbers $\{\delta_n \to 0\}_{n=1}^\infty$, such that 

$$\zeta^{\lambda_n}(s) - \zeta(s) \leq \delta_n, \quad \|\zeta^{\lambda_n}(s) - \zeta(s)\| \leq \delta_n, \quad \forall s \in [-2T, 2T] \setminus [-T, T].$$

For each $n$, we define a path $\xi^{\lambda_n} \in H^1([-\delta/\lambda_n, \delta/\lambda_n], \mathbb{C})$ as 

$$\xi^{\lambda_n}(s) = \begin{cases} \zeta^{\lambda_n}(s), & \text{when } s \in [2T, \lambda_0], \\ \frac{2\lambda_n}{\lambda_0} (\zeta(s) - \zeta^{\lambda_n}(s)) + \zeta^{\lambda_n}(s), & \text{when } s \in [2T - \delta_n, 2T], \\ \tilde{\xi}(s), & \text{when } s \in [T, 2T - \delta_n], \\ \tilde{\zeta}(s), & \text{when } s \in [-T, T], \\ \tilde{\zeta}(s), & \text{when } s \in [-2T + \delta_n, -T], \\ \frac{\sigma + 2T - \delta_n}{\delta_n} (\zeta(s) - \zeta^{\lambda_n}(s)) + \tilde{\zeta}(s), & \text{when } s \in [-2T, -2T + \delta_n], \\ \zeta^{\lambda_n}(s), & \text{when } s \in [-\lambda_0 \delta, -2T], \\ \end{cases}$$

and a corresponding path $\eta^{\lambda_n} \in H^1([t_0 - \delta, t_0 + \delta], \mathbb{C})$ as 

$$\eta^{\lambda_n}(t) = \lambda_n^{-\frac{2}{3}} \xi^{\lambda_n}(\lambda_n(t - t_0)), \quad \forall t \in [t_0 - \delta, t_0 + \delta].$$

We will show that for large $n$, $(\eta^{\lambda_n} + q_{t_0})|_{[t_0 - \delta, t_0 + \delta]}$ are the desired paths we are looking for.

Notice that $\xi^{\lambda_n}(s)$ and $\zeta^{\lambda_n}(s)$ are correspondingly the $\lambda_n$-blown-up of $\eta^{\lambda_n}(t)$ and $\zeta(t)$. By (79), for any $n$ large enough, we have the following inequality, which verifies (10). 

$$|\eta^{\lambda_n}(t) - \zeta(t)| = |\eta^{\lambda_n}(t) + q_{t_0}(t) - \gamma(t)| \leq \varepsilon, \quad \forall t \in [t_0 - \delta, t_0 + \delta].$$

In particular this implies that, for $\varepsilon$ small enough, 

$$\eta^{\lambda_n}(t) + q_{t_0}(t) \neq q_{t_0}(t), \quad \forall t \in [t_0 - \delta, t_0 + \delta].$$
At the same time Proposition 5.1 and (79) also implies $|\xi^{\lambda_n}(s)| \neq 0$, $\forall s \in [-\lambda_0\delta, \lambda_n\delta]$. Then
\[
\eta^{\lambda_n}(t) \neq 0, \quad \forall t \in [t_0 - \delta, t_0 + \delta].
\]
As a result, for $n$ large enough, $(\eta^{\lambda_n} + q_{i_0})|_{[t_0 - \delta, t_0 + \delta]}$ must be collision-free.

Meanwhile by Proposition 2.2, $\lim_{\delta \to 0} \zeta(t_0 \pm \delta)/|\zeta(t_0 \pm \delta)| = \sigma_{\pm}$. Therefore we may assume
\[
\lim_{\delta \to 0} \arg(\zeta(t_0 \pm \delta)) = \arg(\sigma_{\pm}).
\]
Then by (76), for $\delta$ small enough,
\[
\arg(\zeta(t_0 + \delta)) - \arg(\zeta(t_0 - \delta)) \in \begin{cases} (0, 2\pi), & \text{if } \xi = \xi^+; \\ (-2\pi, 0), & \text{if } \xi = \xi^-.
\end{cases}
\]
Since $\xi^{\lambda_n}(\pm \lambda_0\delta) = \zeta^{\lambda_n}(\pm \lambda_n\delta)$, we may assume
\[
\arg(\xi^{\lambda_n})(\pm \lambda_0\delta) = \arg(\zeta^{\lambda_n})(\pm \lambda_n\delta).
\]
This then implies
\[
\arg(\eta^{\lambda_n})(t_0 \pm \delta) = \arg(\zeta)(t_0 \pm \delta).
\]
Combining this with (82), we get
\[
\arg(\eta^{\lambda_n}(t_0 + \delta)) - \arg(\eta^{\lambda_n}(t_0 - \delta)) \in \begin{cases} (0, 2\pi), & \text{if } \xi = \xi^+; \\ (-2\pi, 0), & \text{if } \xi = \xi^-.
\end{cases}
\]
This implies $\eta^{\lambda_n}|_{[t_0 - \delta, t_0 + \delta]} + q_{i_0}|_{[t_0 - \delta, t_0 + \delta]}$ satisfies (11), for $n$ large enough.

We claim that for $n$ large enough,
\[
\mathcal{A}^{\lambda_n}(\xi^{\lambda_n}; -\lambda_0\delta, \lambda_0\delta) < \mathcal{A}^{\lambda_n}(\zeta^{\lambda_n}; -\lambda_n\delta, \lambda_n\delta).
\]
Then (83) and Lemma 5.1 imply
\[
\mathcal{A}(\eta^{\lambda_n}; t_0 - \delta, t_0 + \delta) < \mathcal{A}(\zeta; t_0 - \delta, t_0 + \delta).
\]
As a result, our proof is finished once (83) is established. To show this, notice that
\[
\begin{align*}
\mathcal{A}^{\lambda_n}(\zeta^{\lambda_n}; -\lambda_0\delta, \lambda_0\delta) - \mathcal{A}^{\lambda_n}(\xi^{\lambda_n}; -\lambda_0\delta, \lambda_0\delta) \\
= \mathcal{A}^{\lambda_n}(\xi^{\lambda_n}; -2T, 2T) - \mathcal{A}^{\lambda_n}(\xi^{\lambda_n}; -2T, 2T) \\
\geq \bar{A}(\zeta^{\lambda_n}; -2T, 2T) - \bar{A}(\xi^{\lambda_n}; -2T, 2T) + \int_{-2T}^{2T} \langle \dot{\xi}^{\lambda_n} - \dot{\xi}^{\lambda_n}, \dot{q}_{i_0}^{\lambda_n} \rangle \ ds \\
- \int_{-2T}^{2T} \sum_{i \neq i_0} \frac{m_i}{\xi^{\lambda_n} + q_{i_0}^{\lambda_n} - q_i^{\lambda_n}} \ ds.
\end{align*}
\]
Meanwhile by (75) and (79),
\[
\bar{A}(\zeta; -2T, 2T) - \bar{A}(\xi^{\lambda_n}; -2T, 2T) \geq 6\varepsilon_0 - \bar{A}(\xi^{\lambda_n}; -2T, -2T + \delta_n) - \bar{A}(\xi^{\lambda_n}; 2T - \delta_n, 2T).
\]
As (78) and (79) imply
\[
|\dot{\xi}^{\lambda_n}(s)| \leq \begin{cases} 1 + |\dot{\xi}^{\lambda_n}(s)|, & \text{if } s \in [2T - \delta_n, 2T]; \\
1 + |\dot{\xi}(s)|, & \text{if } s \in [-2T, -2T + \delta_n],
\end{cases}
\]
By Lemma 5.2 for $n$ large enough, there exist positive $C_1$ and $C_2$, such that
\[
|\xi^{\lambda_n}(s)| \leq C_1, \quad \forall s \in [-2T, -2T + \delta_n] \cup [2T - \delta_n, 2T];
\]
\[
|\xi^{\lambda_n}(s)| \leq C_2^{-1}, \quad \forall s \in [-2T, -2T + \delta_n] \cup [2T - \delta_n, 2T].
\]
As a result, for $n$ large enough,
\[
\mathcal{A}(\xi^{\lambda_n}; -2T, -2T + \delta_n) + A(\xi^{\lambda_n}; 2T - \delta_n, 2T)
\]
\[
\leq \int_{-2T}^{-2T+\delta_n} 1/2 |\xi^{\lambda_n}(s)|^2 + \frac{m_i}{|\xi^{\lambda_n}(s)|} ds \leq 2 \left( \frac{1}{2} C_1^2 + m_i C_2 \right) \delta_n \leq \varepsilon_0.
\]
Plug the above inequality into (85), we get
\[
\mathcal{A}(\xi^{\lambda_n}; -2T, 2T) - \mathcal{A}(\xi^{\lambda_n}; -2T, 2T) \geq 5\varepsilon_0.
\]
Then (77) implies
\[
A(\xi^{\lambda_n}; -2T, 2T) - A(\xi^{\lambda_n}; -2T, 2T) \geq 4\varepsilon_0.
\]
Since $\rho_1 = \min\{|\gamma(t) - q_i(t)| : \forall t \in [t_0 - \delta, t_0 + \delta] \text{ and } i \neq i_0 \} > 0$, $\forall i \neq i_0$, (80) implies,
\[
|\eta^{\lambda_n}(t) + q_i(t) - q_i(t)| \geq |\zeta(t) + q_i(t) - q_i(t)| - |\eta^{\lambda_n}(t) - \zeta(t)| \geq \rho_1 - \varepsilon, \quad \forall t \in [t_0 - \delta, t_0 + \delta].
\]
As a result, when $0 < \varepsilon \leq \rho_1/2$, for $n$ large enough, we have
\[
\int_{-2T}^{2T} \sum_{i \neq i_0} m_i \left|\xi^{\lambda_n}(s) + q_i^{\lambda_n}(s) - q_i^{\lambda_n}(s) \right| ds = \lambda_n^2 \int_{-2T}^{2T} \sum_{i \neq i_0} m_i \left|\eta^{\lambda_n}(t) + q_i(t) - q_i(t) \right| dt
\]
\[
\leq \frac{1}{\lambda_n^2} \int_{-2T}^{2T} \sum_{i \neq i_0} m_i \frac{\delta_n}{\rho_1} dt \leq \frac{8T(\sum_{i \neq i_0} m_i)}{\rho_1} \lambda_n^{-\frac{2}{\theta}} \leq \varepsilon_0.
\]
Plug (87) and (88) into (84), we get that when $n$ is large enough,
\[
\mathcal{A}^{\lambda_n}(\xi^{\lambda_n}; -\delta/\lambda_n, \lambda_n, \delta/\lambda_n) - \mathcal{A}^{\lambda_n}(\xi^{\lambda_n}; -\delta/\lambda_n, \lambda_n, \delta/\lambda_n) \geq 3\varepsilon_0 + \int_{-2T}^{2T} (\dot{\xi}^{\lambda_n}(s) - \dot{\xi}^{\lambda_n}(s), \dot{q}^{\lambda_n}(s)) ds.
\]
As a result, it is enough to show that when $n$ is large enough,
\[
\int_{-2T}^{2T} |\dot{\xi}^{\lambda_n}(s)| \cdot |\dot{q}^{\lambda_n}(s)| ds + \int_{-2T}^{2T} |\dot{\xi}^{\lambda_n}(s)| \cdot |\dot{q}^{\lambda_n}(s)| ds \leq 2\varepsilon_0.
\]
By the definition of $\lambda_n$-blow up,
\[
\int_{-2T}^{2T} |\dot{\xi}^{\lambda_n}(s)| \cdot |\dot{q}^{\lambda_n}(s)| ds = \lambda_n^2 \int_{-2T}^{2T} |\dot{\xi}(t)| \cdot |\dot{q}_i(t)| dt
\]
Since $q_i[t_0 - \delta, t_0 + \delta]$, is a collision-free solution of (2),
\[
C_3 = \sup\{|\dot{q}_i(t)| : t \in [t_0 - \delta, t_0 + \delta] \} < \infty.
\]
Meanwhile by Proposition 2.2 there exist two positive constants $C_4, C_5$,
\[
|\dot{\xi}(t)| \leq C_4 |t - t_0|^{-\frac{1}{2}} + C_5, \quad \forall t \neq t_0 \text{ small enough}.
\]
As a result, for $n$ large enough,
\[
\int_{-2T}^{2T} |\dot{\xi}^{\lambda_n}(s)| \cdot |\dot{q}_{i_0}^{\lambda_n}(s)| \, ds \leq \frac{1}{\lambda_n^2} C_3 \int_{t_0 - \frac{2T}{\lambda_n}}^{t_0 + \frac{2T}{\lambda_n}} C_4 |t - t_0|^{-\frac{1}{3}} + C_5 \, dt
\]
\[
= C_3(3C_4(2T)^{\frac{2}{3}} \lambda_n^{-\frac{1}{3}} + 4C_5T\lambda_n^{-\frac{2}{3}}) \leq \varepsilon_0.
\]

By (79) and the first inequality in (86), we can find a $C_6$, such that for $n$ large enough,
\[
\int_{-2T}^{2T} |\dot{\xi}^{\lambda_n}(s)| \, ds \leq C_6.
\]
Since $|\dot{q}_{i_0}^{\lambda_n}(s)| = \lambda_n^{-\frac{2}{3}} |\dot{q}_{i_0}(t_0 + \lambda_n^{-1}s)| \leq \lambda_n^{-\frac{1}{3}} C_3$, $\forall s \in [-2T, 2T]$, for $n$ large enough, we have
\[
\int_{-2T}^{2T} |\dot{\xi}^{\lambda_n}(s)| \cdot |\dot{q}_{i_0}^{\lambda_n}(s)| \, ds \leq \lambda_n^{-\frac{1}{3}} C_3 C_6 \leq \varepsilon_0.
\]
This finishes our proof.

6. **Appendix B: The Perturbed Kepler problem**

In this appendix, instead of the restricted $(N + 1)$-body problem, we will prove two propositions of the perturbed Kepler problem (92), that imply Proposition 3.1 and 3.2, as we believe these results will be useful in more general setting.

(92)
\[
\ddot{z} = -\frac{mz}{|z|^3} + F(z, t), \quad F \in C^0(\mathbb{C} \times \mathbb{R}, \mathbb{R}).
\]

Let $z|_{t_1, t_2}$, is a solution of (92) for the rest of the appendix. Rewrite it in polar coordinates with $z(t) = r(t)e^{i\theta(t)}$, we get

(93)
\[
\begin{cases}
\ddot{r} - r\dot{\theta}^2 = -m/r^2 + F_1(r, \theta, t); \\
r\ddot{\theta} + 2r\dot{\theta} = F_2(r, \theta, t),
\end{cases}
\]
where $F_1(r, \theta, t) = \text{Re}(F(re^{\theta}, t)e^{-i\theta})$, and $F_2(r, \theta, t) = \text{Im}(F(re^{\theta}, t)e^{-i\theta})$.

**Proposition 6.1.** Assume there are two positive constants $R$ and $C$, such that

(94)
\[
|F(x, t)| \leq C/|x|^2, \quad \forall x \in \mathbb{C} \setminus B_R^{10}.
\]
If $r(t_1) \geq R$ and $\dot{r}(t_1) \geq \sqrt{\frac{3(m+C)}{r(t_1)}} > 0$, then $\dot{r}(t) > \frac{1}{2}\dot{r}(t_1)$, $\forall t \in [t_1, t_2]$.

**Proof.** Assume the desired result does not hold, then there is a $\tau_1 \in (t_1, t_2]$, such that
\[
\dot{r}(\tau_1) = \dot{r}(t_1)/2 \quad \text{and} \quad \dot{r}(t) > \frac{1}{2}\dot{r}(t_1)/2, \quad \forall t \in [t_1, \tau_1).
\]
By (94), for $i = 1, 2$,

(95)
\[
|F_i(r, \theta, t)| \leq C/r^2, \quad \forall r \geq R.
\]
Let $g(t) = \frac{1}{2}\dot{r}(t)^2 - \frac{m+C}{r(t)}$. Then
\[
\dot{g} = \dot{r}(\dot{r} + \frac{m+C}{r^2}).
\]
Combining this with (93) and (95), we get
\[ \dot{r} = r \dot{\theta}^2 - \frac{m}{r^2} + F_1 \geq - \frac{m + C}{r^2}, \quad \forall t \in [t_1, \tau]. \]

As a result, \( \dot{g}(t) \geq 0, \forall t \in [t_1, \tau] \). This implies \( g(\tau_1) \geq g(t_1) \) and
\[ \frac{\dot{r}^2(t_1)}{4} = \dot{r}^2(\tau_1) > 2g(\tau_1) \geq 2g(t_1) \geq \frac{\dot{r}^2(t_1)}{r(t_1)}, \]
which is a contradiction to the condition that \( \dot{r}(t_1) \geq \sqrt{3(m + C)/r(t_1)} \).

Proposition 6.2. Assume there are two positive constants \( R \) and \( C \), such that
\[ |F(x, t)| \leq C/|x|^3, \quad \forall x \in \mathbb{C} \setminus B_R^1. \]
If \( r(t_1) \geq R \) and \( \dot{r}(t) \geq v_0, \forall t \geq t_1, \) for some constant \( v_0 > 0 \), then
\[ (a). \sup\{ |\omega(t)| : t \in [t_1, t_2]\} \leq |\omega(t_1)| + \frac{C}{v_0 r(t_1)}, \text{ where } \omega(t) = z(t) \wedge \dot{z}(t); \]
\[ (b). \text{ when } t_2 = \infty, \lim_{t \to \infty} \theta(t) = \theta_0, \text{ for some } \theta_0 \in \mathbb{R}. \]

Proof. (a). By (92) and (96),
\[ |\dot{\omega}(t)| = |z(t) \wedge \dot{z}(t)| = |z(t) \wedge F(z, t)| \leq \frac{C}{r(t)^2}. \]
For any \( t \in [t_1, t_2] \), since \( r(t) \geq v_0(t - t_1) + r(t_1) \),
\[ |\omega(t)| \leq |\omega(t_1)| + \int_{t_1}^{t} \frac{C}{r^2(\tau)} d\tau \leq |\omega(t_1)| + \int_{t_1}^{t} \frac{C/v_0^2}{(\tau - t_0 + \frac{r(t_1)}{v_0})^2} d\tau \]
\[ = |\omega(t_1)| + \frac{C}{v_0^2} \left( \frac{v_0}{r(t_1)} - \frac{v_0}{v_0(t - t_1) + r(t_1)} \right) \]
\[ \leq |\omega(t_1)| + \frac{C}{v_0 r(t_1)}. \]
(b). Notice that in polar coordinates \( \omega(t) = r^2(t) \dot{\theta}(t) \). Hence
\[ \dot{\theta}(t) = \frac{\omega(t)}{r^2(t)} \leq \frac{v_0 r(t_1)|\omega(t)| + C}{v_0^3 r(t_1)} \cdot \frac{1}{(t - t_1 + r(t_1)/v_0)^2}. \]
Therefore for any \( t > t_1 \),
\[ |\theta(t) - \theta(t_1)| \leq \int_{t_1}^{t} |\dot{\theta}(\tau)| d\tau \leq \frac{|\omega(t_1)|}{v_0 r(t_1)} + \frac{C}{v_0^3 r^2(t_1)}. \]

Since \( \dot{r}(t) \geq v_0 > 0, \forall t \geq t_0, \) \( r(t) \to \infty \) as \( t \to \infty \). As a result, the upper bound given in the above inequality goes to zero, as \( t_1 \) goes to infinity. By Cauchy’s criterion for convergence, \( \theta(t) \) converges to a finite \( \theta_0 \), as \( t \) goes to infinity.

By (20), Proposition 3.1 and 3.2 follows directly from the above two results correspondingly.
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