A generalized Turán problem and a theorem of Chung and Frankl

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Abstract

Let $F_k$ denote the $k$-fan, which is a graph consisting of $k$ triangles sharing a vertex. We determine the maximum number of triangles in $F_k$-free $n$-vertex graphs, provided $n$ is sufficiently large. In the case $k$ is odd, this follows from a hypergraph Turán theorem of Chung and Frankl from 1987. In the case $k$ is even, we adapt their proof. We also found two small mistakes in their proof which we correct.

1 Introduction

One of the most basic problems in extremal Combinatorics is the study of the Turán number $\text{ex}(n, F)$, that is the largest number of edges an $n$-vertex $F$-free graph can have. A natural generalization is to count other subgraphs instead of edges. Given graphs $H$ and $G$, we let $\mathcal{N}(H, G)$ denote the number of copies of $H$ in $G$. The generalized Turán number $\text{ex}(n, H, F)$ is the largest $\mathcal{N}(H, G)$ among $n$-vertex $F$-free graphs $G$.

A particular line of research is to determine for a given graph $H$, what graphs $F$ have the property that $\text{ex}(n, H, F) = O(n)$. This was started by Alon and Shikhelman \cite{1}, who dealt with the case $H = K_3$, and was continued for other graphs in \cite{9, 12}.

The friendship graph or $k$-fan $F_k$ consists of $k$ triangles sharing a vertex $v$. We will call $v$ the center of the $k$-fan. An extended friendship graph consists of $F_k$ for some $k \geq 0$ and any number of additional vertices or edges that do not create any additional cycles. Alon and Shikhelman \cite{1} showed that $\text{ex}(n, K_3, F) = O(n)$ if and only if $F$ is an extended friendship graph. We remark that known results easily imply that if $F$ is not an extended friendship graph, then $\text{ex}(n, K_3, F) = \omega(n)$ and it is also easy to see that adding further edges to $F$ without creating any cycle does not change linearity of $\text{ex}(n, K_3, F)$. Hence the key part of their proof is the following proposition.

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Proposition 1.1 (Alon and Shikhelman [1]). For any $k$ we have $\text{ex}(n, K_3, F_k) < (9k - 15)(k + 1)n$.

They also mention that $\text{ex}(n, K_3, F_k)$ is quadratic in $k$, as shown by $\lfloor n/2k \rfloor$ vertex-disjoint copies of $K_{2k}$. Here we determine $\text{ex}(n, K_3, F_k)$ exactly for sufficiently large $n$.

Theorem 1.2. Let $k \geq 3$. If $n$ is sufficiently large, then

$$\text{ex}(n, K_3, F_k) = \begin{cases} nk(k - 1) + 2\binom{k}{3} & \text{if } k \text{ is odd,} \\ (n - 2k + 1)k(k - 3/2) + 2\binom{k-1}{3} + \binom{k/2}{2} + \binom{k/2-1}{2} & \text{if } k \text{ is even.} \end{cases}$$

The case $k = 2$ is known. It was observed by Liu and Wang [14] that a hypergraph Turán theorem of Erdős and Sós [15] imply the exact result in this case. Let $F_k$ denote the 3-uniform hypergraph consisting of $k$ hyperedges sharing exactly one vertex. Let $\text{ex}_3(n, F_k)$ denote the largest number of hyperedges that an $F_k$-free $n$-vertex 3-uniform hypergraph can contain. Erdős and Sós [15] showed

$$\text{ex}_3(n, F_2) = \begin{cases} n & \text{if } n = 4m, \\ n - 1 & \text{if } n = 4m + 1, \\ n - 2 & \text{if } n = 4m + 2 \text{ or } n = 4m + 3. \end{cases}$$

Given a graph $G$, we let $T(G)$ denote the 3-uniform hypergraph on the vertex set $V(G)$ where $\{u, v, w\}$ form a hyperedge if and only if $uvw$ is a triangle in $G$. The key observation is that if $G$ is $F_k$-free, then $T(G)$ is $F_k$-free. Therefore, $\text{ex}(n, K_3, F_k) \leq \text{ex}_3(n, F_k)$. In the case $k = 2$, the upper bound obtained this way matches the lower bound provided by $\lfloor n/4 \rfloor$ vertex-disjoint copies of $K_4$, and in the case $n = 4m + 3$ we also have a triangle on the remaining vertices. This gives the exact value of $\text{ex}(n, K_3, F_2)$.

The result of Erdős and Sós [15] was extended to arbitrary $k$ by Chung and Frankl [6], after partial results [4, 5, 7].

Theorem 1.3 (Chung and Frankl [6]). Let $k \geq 3$. If $n$ is sufficiently large, then

$$\text{ex}_3(n, F_k) = \begin{cases} nk(k - 1) + 2\binom{k}{3} & \text{if } k \text{ is odd,} \\ (n - 2k + 1)\frac{(2k-1)(k-1)-1}{2} + (2k - 2)\binom{k-1}{2} + \binom{k-2}{2} - \frac{(k-2)(k-4)}{2} + \frac{k}{2} & \text{if } k \text{ is even.} \end{cases}$$

For odd $k$, this completes the proof of the upper bound. However, for even $k$, the construction giving the lower bound in the above theorem is not $T(G)$ for some $F_k$-free graph $G$. Still, the upper bound differs from the lower bound only by an additive constant $c(k)$. We will heavily use the tools provided by Chung and Frankl [6] to obtain the improvement needed in Theorem 1.2. We also found two mistakes in the proof of the case $k$ is even in [6]. One of them does not affect our proof but we give a sketch of the correct proof in Section 4. The other mistake we fix in Lemma 2.4.

In Section 2 we list the statements we use from [6] and prove some further lemmas. In Section 3 we present the proof of Theorem 1.2. We finish the paper with some concluding remarks in Section 4. Among them, we give a sketch of the correction of the proof of Theorem 1.3.
2 Preliminaries

We will present some details of the proof of Theorem 1.3 in [6]. For simplicity, we only talk about graphs and triangles, even though the statements also hold in the more general setting of 3-uniform hypergraphs and hyperedges.

Given an edge $uv$ of a graph $G$, we let $d(u,v)$ denote the number of common neighbors of $u$ and $v$, i.e., the number of triangles containing both $u$ and $v$. We say that $uv$ is heavy if $d(u,v) \geq 2k-1$, light if $d(u,v) \leq k-1$ and medium otherwise. The main use of heavy edges is that they can extend an $F_\ell$ to an $F_k$ greedily. More precisely, we have the following.

**Proposition 2.1.** Assume that $u$ is the center of a copy $F^*$ of $F_\ell$ in $G$ and there are $k-\ell$ heavy edges incident to $u$ that are not contained in $F^*$. Then $u$ is the center of an $F_k$.

**Proof.** Let $v_1, \ldots, v_{k-\ell}$ be the other endpoints of the heavy edges in the statement. We will add to $F^*$ additional triangles $uw_i v_i'$ one by one, such that $v_i'$ is not among the vertices added earlier and is not in $F^*$. We can do this by picking an arbitrary such common neighbor of $u$ and $v_i$. We can find such a common neighbor since there are at most $k-1$ triangles, thus at most $2k-1$ vertices picked earlier or in $F^*$ including $u$, hence at least one of the common neighbors of $u$ and $v_i$ is not among them.

This implies that in an $F_k$-free graph, every vertex is incident to at most $k-1$ heavy edges. More is true:

**Proposition 2.2** (Chung and Frankl [6]). Let $r_1$ and $r_2$ denote the number of heavy and medium edges incident to a vertex $u$. Then we have $r_1 \leq k-1$ and $r_1 + r_2/2 \leq k - 1/2$.

Now we define a weight function $w$. We take a triangle $uvw$ and an edge $uv$. Then the edge $uv$ has a weight $w(uwz, uv)$ in this triangle. As we will later count the weight $W_z = \sum_{uv \in E(G)} w(uwz, uv)$, we will also say that $z$ gets the weight $w(uwz, uv)$ from this triangle. The weight $w$ depends on $d(u,v)$, $d(u,z)$ and $d(v,z)$ as follows.

(i) If none of the three edges in the triangle is heavy or none of the three edges is light, then all the edges have weight $1/3$.

(ii) If there is a heavy edge and two light edges, then the heavy edge has weight $1$, the other edges have weight $0$.

(iii) In the remaining case there is a light edge, a heavy edge, and the third edge is medium or heavy. Then the light edge has weight $0$, the other edges have weight $1/2$.

For convenience, we also say that $w(uwz, uv) = 0$ if $uwz$ is not a triangle. They also use a weight function $w'$ that we will define later. Clearly, the sum of the weights in every triangle is $1$, thus the total weight is equal to the number of triangles in $G$.

Let $N(v)$ denote the set of neighbors of $v$ in $G$. For a set $U \subset V(G)$, $G[U]$ denotes the subgraph of $G$ induced on $U$. We are ready to state the main lemma from [6].

**Lemma 2.3** (Chung and Frankl [6]). Let $k \geq 4$ be even and $G$ be an $F_k$-free graph. Then for every vertex $v \in V(G)$ we have $W_v \leq k(k - 3/2)$. Moreover, $W_v \leq k(k - 3/2) - 1/2$ unless there is an independent set $S \subset N(v)$ such that (a), (b) and (c) hold.
Lemma 2.2 of [6], which states that for any two vertices
triangles containing both \( u \) and \( z \) also get weight 0 from that triangle.

We put all the weight on that edge. This way we ensure that

\[ W' = \sum_{u'z \in E(G)} w'(u'z, uu') \]  

Moreover, the other vertex in \( N(v) \) contained in such a triangle also gets weight 0 from that triangle.

Chung and Frankl [6] claim that “one easily checks through the proofs, that the inequalities concerning the weight function remain valid”. This is incorrect. A key part of the proof

\[ S = N(v) \]  

\[ S \] consists of isolated vertices and one connected component \( D \) with \( 2k - 1 - 2|S| \) vertices and degree sequence \( k - 1, k - 1, \ldots, k - 1, k - 2 \);

(b) the vertices of \( S \) have exactly \( k - 1 \) neighbors in \( N(v) \); and

(c) every edge inside \( N(v) \) is heavy, and all edges incident to \( v \) are light.

Now we are ready to define \( w' \). We fix a vertex \( v \) with \( W_v = k(k - 3/2) \) such that in the above lemma (a), (b) and (c) hold with \( S \neq \emptyset \). We modify \( w \) only on the triangles that contain a vertex from \( S \). As vertices in \( S \) are incident to \( k - 1 \) heavy edges inside \( N(v) \), the triangles that contain a vertex from \( S \) also contain an edge inside \( N(v) \). We put all the weight on that edge. This way we ensure that \( W'_v = 0 \) for every \( v \in S \), where

\[ W'_z = \sum_{u'z \in E(G)} w'(u'z, uu') \]  

Moreover, the other vertex in \( N(v) \) contained in such a triangle also gets weight 0 from that triangle.

To overcome this difficulty, we define two other modified weight functions \( w_1 \) and \( w_2 \). For \( w_1 \), we change the weights in some triangles: If there are two light edges and a medium edge, then the medium edges get weight 3/8 and the light edge gets weight 1/4. If there are two medium edges and a light edge, then the medium edges get weight 3/8 and the light edge gets weight 1/4. For \( w_2 \), we apply all the changes we applied to obtain \( w' \) and \( w_1 \). Note that they do not interact, as first we changed the weight only in some triangles containing a heavy edge, and later we changed the weight only in some triangles without any heavy edges.

Let \( W_1(z) = \sum_{uu'z \in E(G)} w_1(uu'z, uu') \) and \( W_2(z) = \sum_{uu'z \in E(G)} w_2(uu'z, uu') \). If \( s \in S \), we have \( W_2(s) = W'_s = 0 \).

For simplicity we use the graph language below and talk about triangles; everything holds the same way if we replace the triangles by hyperedges. We show that Lemma 2.2 of [6] holds for \( w_1 \) and \( w_2 \). The latter corrects the proof in [6], as this is the only property of the weight function used later in [6].

**Lemma 2.4.** For any \( u, u' \in G \) and \( i \leq 2 \), we have that \( W_i = \sum y w_i(uu'y, uy) \leq k - 1 \). Moreover, \( W_i \leq k - 3/2 \) unless \( d(u, u') = k - 1 \) and \( u' \) gets weight 1 from each of the triangles containing \( uu' \).

**Proof.** If \( uu' \) is light, then the statement is obvious. Assume now that \( uu' \) is heavy. Then the changes to triangles without heavy edges do not have any effect. The heavy edges incident to \( u \) give weight at most 1 to \( u' \), except of course for the edge \( uu' \). Medium edges incident to \( u \) give weight at most 1/2 to \( u' \) and light edges incident to \( u \) give weight 0 to \( u' \). Therefore, \( W_i \leq r_1 - 1 + r_2/2 \leq k - 3/2 \) by Proposition 2.2.
Let us assume that \( uu' \) is medium. Let \( uu'y \) be a triangle. If \( y \in S \), then \( u' \) gets weight at most 1 from this triangle. More precisely, \( u' \) gets weight 1 if \( u \in U, u' \notin U \) and \( u' \) gets weight at most 1/2 if \( u' \in U, u \notin U \). Let \( r \leq k/2 - 1 \) denote the number of triangles containing \( u \) that give weight 1 to \( u' \). Note that these are the only possibilities. Indeed, the triangle \( uu'y \) is not inside \( U \) as it contains a medium edge. We cannot have both \( u \) and \( u' \) outside \( U \) using Proposition 2.2, since \( y \) is incident to \( k - 1 \) heavy edges inside \( U \).

If \( y \notin S \) and \( uy \) is heavy, then \( u' \) gets weight at most 1/2 from this triangle. If \( uy \) is light, then \( u' \) gets weight 1/4 from this triangle. Therefore, we have \( W_i \leq r + (r_1 - r)/2 + 3(r_2 - 1)/8 + r_3/4 \leq k/4 - 1/2 + r_1/2 + 3r_2/8 - 3/8 + r_3/4 \). We have \( r_1 + r_2 + r_3 \leq 2k - 2 \). Dividing this latter inequality by 4 and combining it with the upper bound on \( W_i \), we obtain that \( W_i \leq k/4 - 7/8 + r_1/4 + r_2/8 + (2k - 2)/4 \). Since we have \( r_1/4 + r_2/8 \leq (k - 1)/4 \) by Proposition 2.2, we obtain \( W_i \leq k - 13/8 \), completing the proof.

Recall that if \( S = \emptyset \) in Lemma 2.3, then \( G[N(v)] \) has \( 2k - 1 \) vertices and degree sequence \( k - 1, k - 1, \ldots, k - 1, k - 2 \). For such graphs, Chung and Frankl [6] determined the minimum number of triangles. Let \( G \) be the graph consisting of a complete bipartite graph that has \( k - 1 \) vertices in both parts and misses a matching of size \( k/2 - 1 \) in it, together with another vertex \( z \) that is joined to the end points of the missed matching. Clearly triangles in \( G_i \) are those created by \( z \) and the endpoints of the missed matching, and hence \( N(K_3, G_i) = (k/2 - 1)(k/2 - 2) \).

**Theorem 2.5** (Chung and Frankl [6]). Let \( k \geq 2 \) be an even integer. Suppose \( G \) is a graph on \( 2k - 1 \) vertices with the degree sequence \( k - 1, k - 1, \ldots, k - 1, k - 2 \). Then \( N(K_3, G) \geq (k/2 - 1)(k/2 - 2) \). Moreover, if \( G \) has less than \( k^2/4 - k \) triangles, then it is isomorphic to \( G_i \).

Now we prove a further lemma, determining the maximum number of triangles in such graphs. We need the following result, which relates the number of triangles in a graph \( G \) to that in its complement \( \bar{G} \).

**Proposition 2.6** (Goodman [13]). Let \( G \) be a graph on \( n \) vertices. Then,

\[
N(K_3, G) + N(K_3, \bar{G}) + \frac{1}{2} \sum_v \deg(v)(n - 1 - \deg(v)) = \binom{n}{3}.
\]

Let \( G_0 \) be a graph on \( 2k - 1 \) vertices, \( k \geq 2 \) is an even integer, defined as follows: We take two copies of \( K_{k-1} \) with vertices \( u_1, \ldots, u_{k-1} \) and \( v_1, \ldots, v_{k-1} \) and an additional vertex \( x \). Then we add also the edges \( u_i v_i \) for \( i \leq k/2 - 1 \), the edges \( u_i x \) for \( i \geq k/2 \) and the edges \( v_i x \) for \( i \geq k/2 + 1 \). Thus, each vertex of \( G_0 \) has degree \( k - 1 \) except for \( v_{k/2} \) which has degree \( k - 2 \), and hence, \( |E(G_0)| = k(k - 3)/2 \). It is easy to see that in \( G_0 \) the two copies of \( K_{k-1} \) contain \( 2\binom{k-1}{3} \) triangles and \( x \) is in \( \binom{k/2}{2} + \binom{k/2 - 1}{2} \) triangles. That is,

\[
N(K_3, G_0) = 2\binom{k - 1}{3} + \binom{k/2}{2} + \binom{k/2 - 1}{2}.
\]
Lemma 2.7. Let \( k \geq 2 \) be an even integer. If \( G \) is a graph on \( 2k - 1 \) vertices with the degree sequence \( k - 1, k - 1, \ldots, k - 1, k - 2 \), then \( \mathcal{N}(K_3, G) \leq \mathcal{N}(K_3, G_0) \).

Proof. Let \( G \) be as described in the statement of the lemma. Owing to Proposition 2.6, \( G \) has the maximum number of triangles if and only if its complement has the minimum number of triangles. Now, assume that \( \overline{G} \) has the minimum number of triangles.

Claim 2.8. \( \mathcal{N}(K_3, \overline{G}) = k^2/4 - k + 1 \).

Proof. Note that \( \overline{G} \) has \( 2k - 1 \) vertices and its degree sequence is \( k - 1, k - 1, \ldots, k - 1, k - 2 \). Let \( v \) be the vertex of degree \( k \) in \( \overline{G} \). Removing any one edge of those incident to \( v \) gives a subgraph with the degree sequence \( k - 1, k - 1, \ldots, k - 1, k - 2 \).

If none of the incident edges to \( v \) are contained in a triangle in \( \overline{G} \), then all of the \( k \) neighbors of \( v \) are independent and hence they must be adjacent to all of the remaining \( k - 2 \) vertices in \( \overline{G} \) (as they have degree \( k - 1 \)), causing all the non-neighbors of \( v \) to get degree \( k \), which is impossible. Thus, there is an edge \( e \) incident to \( v \) which is contained in at least one triangle. Take \( H = \overline{G} - e \), and then \( \mathcal{N}(K_3, \overline{G}) \geq \mathcal{N}(K_3, H) + 1 \).

Now, if \( H \) is not isomorphic to \( G_l \), then by the second part of Theorem 2.5 \( H \) has at least \( k^2/4 - k + 1 \) triangles, which implies that \( \mathcal{N}(K_3, \overline{G}) \geq k^2/4 - k + 1 \). If \( H \) is isomorphic to \( G_l \), then the only way of gaining back \( G \) from \( G_l \) by adding only one edge \( e' \) (which should be the case), is to have \( e' \) joining the vertex \( z \) to a vertex outside the missing matching in the bipartite constituent of \( G_l \) (because of the degree sequences). Since any such edge is in exactly \( k/2 - 1 \) triangles, we obtain \( \mathcal{N}(K_3, \overline{G}) = \mathcal{N}(K_3, G_l) + (k/2 - 1) = k^2/4 - k + 1 \).

For the rest of the proof, we apply Proposition 2.6 to obtain what is required.

\[
\mathcal{N}(K_3, G) = \binom{2k-1}{3} - \frac{1}{2} \sum_v \deg(v)((2k-1) - 1 - \deg(v)) - \mathcal{N}(K_3, \overline{G}) \\
= \binom{2k-1}{3} - \frac{1}{2} [2k - 2] ((2k-1) - 1 - (k-1)) \\
- \frac{1}{2} [(k-2) (2k-1) - 1 - (k-2)) - \mathcal{N}(K_3, \overline{G}) \\
\leq \binom{2k-1}{3} - (k-1)^3 - k/2(k-2) - (k^2/4 - k + 1) \\
= k^3/3 - 7k^2/4 + 8k/3 - 1 \\
= \mathcal{N}(K_3, G_0).
\]

3 Proof of Theorem 1.2

Let us present first the constructions that prove the lower bound. If \( k \) is odd, we take two copies of \( K_k \) and \( n - 2k \) other vertices. Then add each edge \( uv \) where \( u \) is in the two copies
of $K_k$ and $v$ is not. Then $u$ is not the center of an $F_k$ as each triangle containing $u$ also contains another vertex from the same copy of $K_k$, and $v$ is not the center of an $F_k$ as there are no $k$ independent edges in its neighborhood.

Let us consider the case $k$ is even and we first recall the definition of $G_0$ from Section 2. We take two copies of $K_{k-1}$ with vertices $u_1, \ldots, u_{k-1}$ and $v_1, \ldots, v_{k-1}$ and an additional vertex $x$. Then we add also the edges $u_iv_i$ for $i \leq k/2-1$, the edges $u_ix$ for $i \geq k/2$ and the edges $v_ix$ for $i \geq k/2+1$. This is the graph $G_0$; to obtain $G$, we take $n-2k+1$ additional vertices and join each of them to each vertex of $G_0$ by an edge. Each vertex of $G_0$ has degree $k-2$, thus $|E(G_0)| = k(k-3/2)$. This implies that the number of triangles in $G$ is $k(k-3/2)(n-2k+1) + \mathcal{N}(K_3, G_0) = k(k-3/2)(n-2k+1) + 2(k^{-3}) + (k/2) + (k/2-1)^2$.

It is left to show that $G$ is $F_k$-free. A vertex outside $G_0$ cannot be the center of an $F_k$, as its neighborhood has $2k-1$ vertices, and a vertex $v$ in $G_0$ cannot be the center of an $F_k$ as each triangle containing $v$ contains at least one of the at most $k-1$ neighbors of $v$ in $G_0$.

Let us continue with the proof of the upper bound and consider an $F_k$-free $n$-vertex graph $G$. If there is no vertex $v$ in $G$ with $W_v = k(k-3/2)$, then the number of triangles is at most $nk(k-3/2) - n/2$, completing the proof. Let us assume $W_v = k(k-3/2)$ and let $U$ denote the neighborhood of $v$. Consider a vertex $v' \notin U$. If $v'$ is in at most $k(k-3/2)$ triangles, then we delete the edges incident to $v'$ and add the edges $uv'$ for each $u \in U$. We claim that the resulting graph does not contain $F_k$. Indeed, otherwise $v'$ is contained in that copy of $F_k$, as all the new edges are incident to $v'$.

If $v'$ is the center, then we could replace $v'$ by $v$ to obtain $F_k$ in $G$. Otherwise, the center $u$ is a neighbor of $v'$, thus $u \in U$. The third vertex of the triangle containing $v'$ is $u' \in U$. Then $uu'$ is a heavy edge by Lemma 2.3, thus $u$ and $u'$ have at least $2k-1$ common neighbors. At least one of them $v''$ is not in the copy of $F_k$, thus we can replace $v'$ by $v''$ to obtain an $F_k$ in $G$, a contradiction. Clearly, the number of triangles does not decrease this way, as we deleted at most $k(k-3/2)$ triangles and created exactly $k(k-3/2)$ triangles.

We repeat this as long as we can find a vertex outside $U$ in at most $k(k-3/2)$ triangles, let $G'$ be the resulting graph. Let $A$ denote the set of vertices in $G'$ with neighborhood $U$ and $B := V(G) \setminus (A \cup U)$. The vertices in $B$ have weight less than $k(k-3/2)$, thus weight at most $k(k-3/2)-1/2$ by Lemma 2.3 and are in more than $k(k-3/2)$ triangles by construction.

Recall that each edge inside $U$ is heavy and edges from $U$ to $A$ are light. Let us call a vertex $u \in U$ nice if $u$ is incident to $k-1$ edges inside $U$. As those edges are heavy, $u$ is not contained in any triangle with two vertices outside $U$ by Proposition 2.1. Observe that the triangles containing an edge inside $U$ and a vertex from $A$ have all the weight on the edge inside $U$, i.e., on the vertex in $A$. Furthermore, only light edges go from nice vertices to vertices of $B$ as well. If both endpoints of an edge $uu'$ inside $U$ are nice and they form a triangle with a vertex of $B$, then $u$ and $u'$ do not get any weight from that triangle either. This means that $\mathcal{N}(K_3, G')$ is at most the number of triangles containing a not nice vertex from $U$ and a vertex of $B$, plus the number of triangles inside $U$ (which is at most $2(k^{-1}) + (k/2) + (k/2-1)$ by Lemma 2.7), plus $|A|k(k-3/2) + |B|(k(k-3/2)-1/2)$.

Let $S$ be the independent set in $G'[U]$ as described in Lemma 2.3. We consider two cases.

**CASE 1.** We have $S = \emptyset$. 


Let \( z \) denote the vertex of \( G'[U] \) with degree \( k-2 \), and \( T \) denote the set of triangles containing \( z \) where the other two vertices are in \( B \). Observe that any two triangles in \( T \) share another vertex besides \( z \). Therefore, the neighborhood of \( z \) inside \( B \) is either a star or a triangle.

**CASE 1.1.** The neighborhood of \( z \) inside \( B \) is a triangle \( z_1z_2z_3 \).

Let \( p_i \) denote the number of common neighbors of \( z \) and \( z_i \) inside \( U \), thus \( p_i \leq k-2 \). Observe that \( p_1 + p_2 + p_3 \leq k-2 \), as a neighbor of \( z \) inside \( U \) is nice, thus cannot have two adjacent neighbors in \( B \). In the \( p_1 + p_2 + p_3 \) triangles containing \( z \), \( z_i \) and a vertex in \( U \), \( z_i \) gets weight at least \( 1/2 \), since the edge opposite to \( z_i \) is heavy. Hence, the vertices in \( U \) get weight at most \((p_1 + p_2 + p_3)/2 \) from these triangles. Therefore, the total weight \( \sum_{v \in V(G')} W_v \) is at most the number of triangles inside \( U \) plus \(|A|k(k-3/2) + |B|(k(k-3/2) - 1/2)\) plus \(3 + (p_1 + p_2 + p_3)/2 \leq (k+4)/2\), using that \( z \) gets weight at most 3 from the triangles \( z_1z_2z \), \( z_2z_3 \) and \( z_2z_3 \). This means we are done unless \(|B| \leq k-4 \). This means that the endpoints of edges inside \( B \) have at most \( k+3 \) common neighbors, thus the edges inside \( B \) are light or medium. This helps us improve our bound.

Observe that we have \( d(z, z_1) \leq k \), since the common neighbors of \( z \) and \( z_1 \) are \( z_2, z_3 \) and some of the \( k-2 \) neighbors of \( z \) in \( U \). Similarly, we have \( d(z, z_2) \leq k \) and we have \( d(z, z_3) \leq k \). Therefore, in the triangles \( z_1z_2z \), \( z_1z_3z \) and \( z_2z_3 \) each vertex gets weight 1/3, hence \( z \) gets weight 1 from these triangles.

Therefore, by replacing 3 with 1 in the above calculation, we obtain that \(|B| \leq k \). Observe that the neighborhoods of \( z_1 \), \( z_2 \) and \( z_3 \) inside \( U \) can share only \( z \). Therefore, one of these vertices, say \( z_1 \) has at most \((2k-2)/3 + 1 = (2k+1)/3 \) neighbors in \( U \). Then \( z_1 \) is in at most \( \left(\frac{k-1}{2}\right) + 2 + \left(\frac{(2k+1)/3}{2}\right) \leq k-3/2 \) triangles, a contradiction.

**CASE 1.2.** The neighborhood of \( z \) inside \( B \) is a star with center \( z' \) and leaves \( z_1, \ldots, z_{r_1} \). Let \( C \) denote the set of other vertices in \( B \), i.e., those vertices of \( B \) that are not adjacent to \( z \).

Let us recall that \( z \) and \( z' \) have \( r_1 \) common neighbors in \( B \) and assume that \( z \) and \( z' \) have \( r_2 \) common neighbors in \( U \). Observe that the total weight \( \sum_{v \in V(G')} W_v \) is at most the number of triangles inside \( U \) plus the sum of the weight of the vertices in \( A \cup B \) plus the weight vertices of \( U \) get from the triangles of the form \( zz'v \).

**Claim 3.1.** \(|C| \leq k-3 \).

**Proof.** If \( zz' \) is light, then for a triangle \( zz'v \), \( z \) and \( v \) get weight 0 if \( v \in U \) and \( z \) gets weight at most 1 if \( v \in B \). Therefore, we are done unless \( \sum_{v \in V(G')} W_v < (n-2k+1)k(k-3/2) - |B|/2 + r_1 \), thus \(|B|< 2r_1 \). We have \(|C| = |B| - r_1 - 1 \), thus \(|C| < r_1 - 1 \leq k-2 \).

If \( zz' \) is medium or heavy, then for a triangle \( zz'v \), \( v \) and \( z \) together get weight at most 1/2 if \( v \in U \) and \( z \) gets weight at most 1/2 if \( v \in B \). This implies that we are done unless \(|B| < r_1 + r_2 \). As we have \( r_1 + 1 \) vertices in \( B \setminus C \), we have that \(|C| < r_2 - 1 \leq k-3 \). ■

Using the above claim, we obtain that the total weight \( \sum_{v \in V(G')} W_v \) is at most the number of triangles inside \( U \) plus the sum of the weight of the vertices in \( A \cup B \) plus \( r_1/3 + r_2/2 \).

Indeed, for a triangle \( zz'v \), \( v \) gets weight at most 1/2 if \( v \in U \) and \( z \) gets weight at most 1/3
if \( v \in B \). This implies that we are done unless \(|B| < 2r_1/3 + r_2\). As we have at least \( r_1 + 1 \) vertices in \( B \), we have \( r_1 < 3r_2 - 3 \).

We are also done if a vertex \( v \in B \) has weight at most \( k(k - 3) \leq k(k - 3/2) - r_1/3 - r_2/2 \).

**Claim 3.2.** There is a vertex \( x \in C \) that has at most \((k - 4)/2\) neighbors in \( U \).

*Proof.* First we will show that \( z_i \) has more than \( k + 1 \) neighbors in \( U \). Assume not, then there are at most \( \binom{k+1}{2} - 2 \) edges inside \( N(z_i) \cap U \). Indeed, this is obvious if \(|N(z_i) \cap U| \leq k \).

If \(|N(z_i) \cap U| = k + 1\), then there are at least two non-edges inside \( N(z_i) \cap U \) incident to \( z \).

Let us count the triangles containing \( z_i \). As \( z_i \) has at most \( k - 2 \) neighbors in \( B \) (the vertices of \( C \) and \( z' \)), there are at most \( \binom{k-2}{2} \) triangles containing \( z_i \) inside \( B \). There is one triangle \( z_i z' z'' \) containing \( z_i \), another vertex from \( B \) and a vertex from \( U \). As \( z_i \) is in at least \( k(k - 3/2) + 1 \) triangles, the above bounds imply that there are at least \( k(k - 3/2) - \binom{k-2}{2} = k^2/2 + k - 3 > \binom{k+1}{2} - 2 \) triangles containing \( z_i \) and two vertices from \( U \). This implies that \( z_i \) has more than \( k + 1 \) neighbors in \( U \).

If there is no triangle containing \( z_i \) and two vertices from \( C \), then \( z_i \) is in at most \( k - 3 \) triangles inside \( B \), thus in at most \( k - 2 \) triangles that do not contain an edge inside \( U \). On the other hand, \( z_i \) cannot be adjacent to each vertex of \( U \), as in that case \( z_i \) could not be in any other triangles. Therefore, \( z_i \) is not adjacent to a vertex \( u \in U \), thus the \( k - 1 \) edges inside \( U \) that are incident to \( u \) do not form triangles with \( z_i \). Thus the total number of triangles containing \( z_i \) is less than \( k(k - 3/2) \), a contradiction.

We obtained that there is a triangle containing \( z_i \) and two vertices from \( C \). The three vertices in the triangle have disjoint neighborhood inside \( U \) and \( z_i \) has at least \( k+2 \) neighbors in \( U \). Thus, the other two vertices have at most \( 2k - 1 - (k + 2) = k - 3 \) neighbors in \( U \) together, thus one of them has at most \( \lfloor (k-3)/2 \rfloor \) neighbors in \( U \), completing the proof.

Observe that each edge incident to \( z_i \) is light for every \( i \leq r_1 \). Indeed, \( d(z_i, z) \leq k - 1 \) since the common neighbors are \( z' \) and some of the \( k - 2 \) neighbors of \( z \) in \( U \). The edges going from \( z_i \) to other vertices of \( U \) are light since those are nice vertices. The edges from \( z_i \) to \( C \) are light since the common neighbors are in \( C \cup \{z'\} \), and the edge \( z_i z' \) is light since the common neighbors are \( z \) plus some vertices in \( C \).

Let \( d_1 \) denote the number of light and \( d_2 \) denote the number of medium and heavy edges from \( x \) to \( C \). We will give an upper bound on \( W_x \). There are at most \( \binom{(k-3)/2}{2} \) triangles containing \( x \) and two vertices from \( U \), they give weight at most \( \binom{(k-4)/2}{2} \) to \( x \).

The triangles containing \( x \) and \( z' \) have the third vertex \( x' \) in \( B \). If \( x' = z_j \), then the triangle gives weight at most \( 1/3 \) to \( x \), since the opposite edge is light. Therefore, the triangles containing \( x \) and \( z' \) give weight at most \( r_1/3 + |C| - 1 \) to \( x \).

The triangles of the form \( xz_iy \) with \( y \in C \) each give weight \( 1/3 \) to \( x \) if \( xy \) is light or medium, and \( 0 \) if \( xy \) is heavy. Observe that \( xy \) can be light at most \( d_1(k - 1) \) times and medium at most \( d_2(2k - 2) \) times.

Finally, consider the triangles containing \( x \) that are inside \( C \). They are of the form \( xyy' \) with \( y, y' \in C \) and give weight at most \( 1/3 \) if both \( xy \) and \( xy' \) are medium or heavy, at most \( 1/2 \) if one of those edges is light and the other is not, and at most \( 1 \) otherwise. The weight
given by the triangles inside $C$ is at most $\left(\frac{d_2}{2}\right)/3 + d_1d_2/2 + \left(\frac{d_2}{2}\right)/2 \leq d_2(k - 4)/6 + d_1(k - 4)/2$. 

Altogether we have that 

$$W_x \leq \left(\frac{k-1}{2}\right) + \frac{r_1}{3} + |C|-1 + \frac{d_1(k-1)}{3} + \frac{d_2(2k-2)}{3} + \frac{d_2(k-4)}{6} + \frac{d_1(k-4)}{2} \leq$$

$$\left(\frac{k-4}{2}\right) + 2k - 7 + \frac{d_15(k-1)}{6} + \frac{d_25(k-1)}{6} \leq (k-4)\left(\frac{k-5}{8} + 2 + \frac{5(k-1)}{6}\right) + 1 =$$

$$\frac{(k-4)(23k+13)}{24} + 1 \leq k(k-3),$$

completing the proof.

**CASE 2.** We have $S \neq \emptyset$. Let $C := U \setminus (D \cup S)$.

**Claim 3.3.** If $v \in C$ is connected to $\ell$ vertices of $S$, then $W_2(v) \leq k(k - 3/2) - (k - 1)\ell$.

**Proof.** Consider the graph $G''$ we obtain the following way. We keep $C \cup B$ with the edges inside, $D \cup S$ with the edges inside and the edges between $B$ and $D \cup S$. We add edges between $v$ and new vertices $x_1, \ldots , x_\ell, y_{ij}^k$ for every $i \leq \ell$ and $j \leq k - 1$. We add furthermore all the edges $x_iy_{ij}^k$. Finally, we add $2k - 2$ further vertices $v_1, \ldots , v_{2k-2}$ connected to each $x_i$ and $y_{ij}^k$. We claim that the resulting graph $G''$ is $F_k$-free. Indeed, consider the center of an $F_k$. It cannot be $v_j$, as its neighborhood consists of $\ell \leq k - 1$ stars with centers $x_i$. It cannot be $x_i$ as its neighborhood consists of $k - 1$ stars with centers $y_{ij}^k$, and cannot be $y_{ij}^k$ as its neighborhood is a star with center $x_i$. It cannot be an element of $\ell$ different from $v$ as its neighborhood is a subgraph of its neighborhood in $G'$. Finally, the center cannot be $v$ as then $k - \ell$ of the triangles in $F_k$ connect edges from $C \cup B$, and then they can be extended to an $F_k$ in $G'$ greedily, using the $\ell$ heavy edges connecting $v$ to $S$ and Proposition 2.1.

In $G''$, we can define a weight function based on the heaviness of the edges in $G''$ analogous to $w_1$. We denote this new weight function by $w^*$ and apply Lemma 2.4 to this weight function. We can also define $W'_x$ for vertices $x$ in $G''$ based on $w^*$ in triangles inside $G''$, analogous to $W$. Clearly, the weight $W^*_v$ of $v$ in $G''$ is the weight that $v$ has from triangles with two vertices in $B$ plus $(k - 1)\ell$. Therefore, $v$ has at most $k(k - 3/2) - (k - 1)\ell$ weight in $W^*$ from triangles with two vertices in $B$ in $G''$, and this is the same as the weight that $v$ gets from those triangles in $G'$ using the weight function $w_1$. We show that $v$ does not get weight in $w_2$ from any other triangles in $G'$. Indeed, such a triangle is $vsx$ with $s \in S$, $x \in B$. The edge $vs$ is heavy and the edge $sx$ is light, thus $v$ does not get any weight $w_2$ from this triangle.

As there are $|S|(k - 1) - 1$ edges between $C$ and $S$, the above claim implies that 

$$\sum_{v \in C} W_2(v) \leq |C|k(k - 3/2) - (|S|(k - 1) - 1)(k - 1).$$

Therefore, we have that 

$$\sum_{v \notin D} W_2(v) \leq (n - 2k + |S| + 1)k(k - 3/2) - |S|(k - 1)^2 + k - 1 = (n - 2k + 1)k(k - 3/2) + \frac{|S|}{2} - |S| + k - 1.$$  \hspace{1cm} (1)
Each vertex in \( D \) is nice, thus triangles containing a vertex \( v \in D \) and a vertex outside \( U \) must contain one of the heavy edges incident to \( v \) and two light edges, thus \( v \) does not get any weight from those triangles. Each triangle inside \( U \) is inside \( D \), thus \( \sum_{v \in D} W_2(v) \) is equal to the number of triangles inside \( D \).

Observe that \(|S| < k/2\). We separate two cases. If \(|S| = k/2 - 1\), then \( G'[D] \) is the complement of the following graph on \( k + 1 \) vertices: a matching of \( k/2 \) edges and one more edge containing the last vertex. It is easy to see that the number of triangles inside \( G \) is 

\[
\frac{(k-2)|S|+2}{3} = \frac{k^3}{6} - \frac{k^2}{2} - \frac{2k}{3} + 2.
\]

Adding that to (1) completes the proof in the case \( k > 6 \).

In the cases \( k = 4 \) and \( k = 6 \), we use that the bound in (1) can be improved by \(|B|/2\). In particular, in the case \( k = 6 \) we need to improve the bound by 1, thus we are done if \(|B| \geq 2\). Otherwise there are no edges inside \( B \), thus vertices of \( C \) have weight 0. This improves the bound by \(|C|k(k-3/2) - (|S|(|S| - 1) - k - 1) \geq 9\). In the case \( k = 4 \), there are at most \( |B| \) edges inside \( B \), thus the vertices in \( C \) have weight at most \( \binom{|B|}{2} \). If \(|B| \leq 3\), then the two vertices of \( C \) have weight at most 3, which improves the bound by (1) by 8. If \(|B| = 4\), then the vertices in \( C \) have weight at most 6, which improves the bound by 2, and \(|B|/2\) also improves the bound by 2. If \(|B| \geq 5\), then the bound is improved by \(|B|/2 \geq 5/2\). This means the number of triangles in \( G' \) is at most the claimed upper bound plus 1/2. Since the number of triangles is an integer, this completes the proof.

If \(|S| < k/2 - 1\), we will use a simple upper bound on the number of triangles. Consider a vertex \( v \in D \) with \( k-1 \) neighbors inside \( D \). There is a set \( D' \) of \(|k-2|S|-1\) other vertices in \( D \), one of them is connected to at least \( 2|S| \) neighbors of \( v \), and \( k-2|S|-2 \) vertices in \( D' \) are connected to at least \( 2|S|+1 \) neighbors of \( v \). This implies that there are at least \((k-2|S|-1)(2|S|+1) - 1 \) edges between the neighbors of \( v \) and \( D' \). Therefore, there are at least \((k-2|S|-1)(2|S|+1) - 1 \) edges missing between neighbors of \( v \), as the neighbors of \( v \) each have at most \( k-1 \) neighbors inside \( D \). This implies that there are at most \( \binom{k-1}{2} - (k-2|S|-1)(2|S|+1) + 1 \) triangles containing \( v \). We have \( 2k-2|S|-2 \) vertices of degree \( k-1 \) and one vertex of degree \( k-2 \), thus we have at most 

\[
\frac{(2k-2|S|-2)|S|+2}{3} = \frac{k^3}{6} - \frac{k^2}{2} + \frac{67k}{6} - 11.
\]

If \(|S| = 1\), then the above expression is equal to \( k^3/3 - (7k^2)/2 + (67k)/6 - 11\). If \(|S| > 1\), the above expression is at most \( \frac{(2k-6)(k-1)-2k|S|+6}{3} \). In both cases, adding these bounds to (1) completes the proof.

## 4 Concluding remarks

There is an oversight in the proof of Theorem 1.3 in [?]. Using our notation, in the case corresponding to our CASE 1.2, it is claimed that Lemma 2.3 implies that the edge \( zz' \) is light. This is not true, the lemma only states that the edges between \( U \) and \( A \) are light,
and indeed $zz'$ can be medium or heavy. This causes two problems. First, there can be more than $k-1$ triples intersecting $U$ in $z$, and second, the triples of the form $\{z, z', u\}$ with $u \in U$ give weight to $u$, not only to $z'$. There seem to be no immediate way to fix this, but our symmetrization approach helps here.

We only give a sketch of the proof. Also, recall that we talked about graphs and triangles instead of hypergraphs and triples for simplicity, but what we use below holds in the more general setting as well.

The problem appears in the case when we are given a vertex $v$ and the graph $G_1$ formed by the edges $uu'$ where $\{u, u', v\}$ is a hyperedge has $2k-1$ vertices and degree sequence $k-1, k-1, \ldots, k-2$. We symmetrize each vertex to $v$ as in our proof, this way we obtain $A$ and $B$.

Let $z$ denote the vertex of degree $k-2$ in $G_1$. The only problematic case is when there are $r_1 > k$ hyperedges of the form $\{z, z', z_i\}$ where $z', z_i \notin V(G_1)$. Here [6] uses the incorrect assumption that $r_1 \leq k-1$. Observe that these triples $\{z, z', z_i\}$ each give weight of at most $1/2$ to $z$ since $zz'$ is not light. Therefore, the extra weight gained by these triangles is at most $r_1/2$. To make up for it, they showed how we also lose weight at least $k/2$ at other places. This may not be enough if $zz'$ is medium or heavy.

However, we also know that we lose weight at least $(r_1+1)/2$, since the vertices $z', z_1, \ldots, z_{r_1}$ are in $B$, thus have weight at most $k(k-3/2) - 1/2$. Consider now the triples $\{u, z, z'\}$ with $u \in V(G_1)$. Observe that $u$ has to be a neighbor of $z$ in $G_1$, otherwise the $k-1$ heavy edges incident to $u$ would create an $F_k$ with this hyperedge, as in Proposition 2.1. Therefore, we have at most $k-2$ such hyperedges, and each gives a weight of $1/2$ to vertices in $G_1$. This means that altogether the extra weight gained is at most $(r_1+k-2)/2$, and the extra weight lost is $(r_1+1+k)/2$, completing the proof. Let us mention that the key part here is that at least $r_1+1$ vertices outside $G_1$ have weight at most $k(k-3/2) - 1/2$, which follows from the symmetrization but would be complicated to show without that.

It is natural to ask what happens if we count larger cliques. The first author [10] showed that $\text{ex}(n, K_2, F_k) = O(n)$ for every $k$ and $r$, but the constants in the upper bound are far from the constants given by either of the two obvious constructions: vertex-disjoint copies of $K_{2k}$ or a straightforward modification of the ones giving the lower bound in Theorem 1.2.

One may consider counting copies of $K_r$ when forbidding $k$ copies of $K_r$ sharing a vertex. The case $k = 2$ was solved by Liu and Wang [14], who showed that for sufficiently large $n$, the extremal graph has two vertices of degree $n-1$ and the balanced complete $(r-3)$-partite graph on the remaining $n-2$ vertices. In the corresponding hypergraph Turán problem, when two hyperedges sharing a vertex are forbidden, there are at most $\binom{n-2}{r-2}$ hyperedges by a theorem of Frankl [8]. In both constructions, we have two vertices contained in every $K_r$ or hyperedge. However, there are much more hyperedges than $K_r$’s in these constructions.

We have determined the largest number of triangles in $F$-free graphs when $F$ is a friendship graph, but not when $F$ is an extended friendship graph. Alon and Shikhelman [1] showed that in that case $c_1|V(F)|^2n \leq \text{ex}(n, K_3, F) \leq c_2|V(F)|^2n$ for absolute constants $c_1$ and $c_2$. Better bounds were obtained for some forests, including exact results for stars [3], paths [2] and forests consisting only of path components of order different from 3 [16].
Asymptotically sharp result was obtained in [11] for any tree such that each of its subtrees satisfy the Erdős-Sós conjecture on the Turán number of trees.

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**References**

[1] N. Alon, C. Shikhelman. Many $T$ copies in $H$-free graphs. *Journal of Combinatorial Theory, Series B*, **121**, 146–172, 2016.

[2] D. Chakraborti, D.Q. Chen. Exact results on generalized Erdős-Gallai problems, *arXiv preprint arXiv:2006.04681*, 2020.

[3] Z. Chase. A Proof of the Gan-Loh-Sudakov Conjecture, *arXiv preprint arXiv:1912.01600*, 2019.

[4] F.R.K. Chung. Unavoidable stars in 3-graphs, *J. Comb. Theory (A)*, **35**, 252–262, 1983.

[5] F.R.K. Chung, P. Erdős. On unavoidable graphs, *Combinatorica*, **3**, 167–176. 1983.

[6] F. R. Chung, P. Frankl. The maximum number of edges in a 3-graph not containing a given star, *Graphs and Combinatorics*, **3**(1), 111–126, 1987.

[7] R.A. Duke, P. Erdős. Systems of finite sets having a common intersection. In: Proceedings, 8th S-E Conf. Combinatorics, Graph Theory and Computing, pp. 247–252. 1977.

[8] P. Frankl. On families of finite sets no two of which intersect in a singleton. *Bull. Austral. Math. Soc.*, **17**, 125–134, 1977.

[9] D. Gerbner. Generalized Turán problems for $K_{2,t}$, *arXiv preprint arXiv:2107.10610*, 2021.

[10] D. Gerbner. A note on the uniformity threshold for Berge hypergraphs, *European Journal of Combinatorics*, **105**, 103561, 2022.

[11] D. Gerbner, A. Methuku, C. Palmer. General lemmas for Berge-Turán hypergraph problems. *European Journal of Combinatorics*, **86**, 103082, 2020.

[12] D. Gerbner, C. Palmer. Counting copies of a fixed subgraph of $F$-free graphs. *European Journal of Combinatorics*, **82**, 103001, 2019.

[13] A. W. Goodman. On sets of acquaintances and strangers at any party. *The American Mathematical Monthly*, **66**(9), 778–783, 1959.
[14] E.L. Liu, J. Wang. The Generalized Turán Problem of Two Intersecting Cliques, *arXiv preprint arXiv:2101.08004*, 2021.

[15] V.T. Sós. Remarks on the connection of graph theory, finite geometry and block designs, *Téorie Combinatorie, 2*, 223–233, 1976.

[16] X. Zhu and Y. Chen. Generalized Turán number for linear forests, *Discrete Mathematics 345*(10), 112997, 2022,