Toeplitz Determinants, Random Growth and Determinantal Processes

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Abstract

We summarize some of the recent developments which link certain problems in combinatorial theory related to random growth to random matrix theory.

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1. Introduction

Let \( \sigma \) be a permutation from \( S_N \). We say that \( \sigma(i_1), \ldots, \sigma(i_m), i_1 < \cdots < i_m \), is an increasing subsequence of \( \sigma \) if \( \sigma(i_1) < \cdots < \sigma(i_m) \). The number \( m \) is the length of the subsequence. The length of the longest increasing subsequence in \( \sigma \) is denoted by \( \ell_N(\sigma) \). If we pick \( \sigma \) from \( S_N \) uniformly at random \( \ell_N(\sigma) \) becomes a random variable. Ulam’s problem, [29], is the study of the asymptotic properties as \( N \to \infty \) of this random variable in particular its mean. It turns out that there is a surprisingly rich mathematical structure around this problem as we hope will be clear from the presentation below. It has been known for some time that \( \mathbb{E}[\ell_N] \sim 2\sqrt{N} \) as \( N \to \infty \), [30], [16]. We refer to [2] for some background to the problem. A Poissonized version of the problem can be obtained by letting \( N \) be an independent Poisson random variable with mean \( \alpha \). This gives a random variable \( L(\alpha) \) with distribution

\[
\mathbb{P}[L(\alpha) \leq n] = \sum_{N=0}^{\infty} \frac{e^{-\alpha} \alpha^N}{N!} \mathbb{P}[\ell_N \leq n].
\] (1.1)

Since \( \mathbb{P}[\ell_N \leq n] \) is a decreasing function of \( N \), [29], asymptotics of the left hand side of (1.1) can be used to obtain asymptotics of \( \mathbb{P}[\ell_N \leq n] \) (de-Poissonization).

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The random variable $L(\alpha)$ can be realized geometrically using Hammersley’s picture, [5]. Consider a Poisson process with intensity 1 in the square $[0, \gamma]^2$, $\gamma = \sqrt{\alpha}$. An up/right path is a sequence of Poisson points $(x_1, y_1), \ldots, (x_m, y_m)$ in the square such that $x_i < x_{i+1}$ and $y_i < y_{i+1}$, $i = 1, \ldots, m - 1$. The maximal number of points in an up/right path has the same distribution as $L(\alpha)$. A sequence of points realizing this maximum is called a maximal path. It is expected from heuristic arguments, see below, that the standard deviation of $L(\alpha)$ should be of order $\gamma^{1/3} = \alpha^{1/6}$. The proof that this is true, [3], and that we can also understand the law of the fluctuations is the main recent result that will be discussed below. Also, the deviations of a maximal path from the diagonal $x = y$ should be of order $\gamma^{2/3}$. This last statement is proved in [11].

A generalization of the random variable $L(\alpha)$ can be defined in the following way. Let $w(i, j)$, $(i, j) \in \mathbb{Z}_+^2$, be independent geometric random variables with parameter $q$. An up/right path $\pi$ from $(1, 1)$ to $(M, N)$ is a sequence $(1, 1) = (i_1, j_1), (i_2, j_2), \ldots, (i_m, j_m) = (M, N), m = M + N - 1$, such that either $i_{r+1} - i_r = 1$ and $j_{r+1} = j_r$, or $i_{r+1} = i_r$ and $j_{r+1} - j_r = 1$. Set

$$G(M, N) = \max_{\pi} \sum_{(i,j) \in \pi} w(i, j),$$

where the maximum is taken over all up/right paths $\pi$ from $(1, 1)$ to $(M, N)$. Alternatively, we can define $G(M, N)$ recursively by

$$G(M, N) = \max(G(M - 1, N), G(M, N - 1)) + w(M, N).$$

Some thought shows that if we let $q = \alpha/N^2$ then $G(N, N)$ converges in distribution to $L(\alpha)$ as $N \to \infty$, [12], so we can view $G(N, N)$ as a generalization of $L(\alpha)$. We can think of [12] as a directed last-passage site percolation problem. Since all paths $\pi$ have the same length, if $w(i, j)$ were a bounded random variable we could relate [12] to the corresponding first-passage site percolation problem, with a min instead of a max. in [12]. The random variable $G(M, N)$ connects with many different problems, a corner growth model, zero-temperature directed polymers, totally asymmetric simple exclusion processes and domino tilings of the Aztec diamond, see [10], [12] and references therein. It is also related to another growth model, the (discrete) polynuclear growth (PNG) model, [13], [21] defined as follows. Let $h(x, t) \in \mathbb{N}$ denote the height above $x \in \mathbb{Z}$ at time $t \in \mathbb{N}$. The growth model is defined by the recursion

$$h(x, t + 1) = \max(h(x - 1, t), h(x, t), h(x + 1, t)) + a(x, t),$$

where $a(x, t)$, $(x, t) \in \mathbb{Z} \times \mathbb{N}$, are independent random variables. If we assume that $a(x, t) = 0$ whenever $x - t$ is even, and that the distribution of $a(x, t)$ is geometric with parameter $q$, then setting $w(i, j) = a(i - j, i + j - 1)$, we obtain $G(i, j) = h(i - j, i + j - 1)$. The growth model [14] has some relation to the so called Kardar-Parisi-Zhang (KPZ) equation, [15], and is expected to fall within the so called KPZ-universality class. The exponents $1/3$ and $2/3$ discussed above are the conjectured exponents for $1 + 1$-dimensional growth models in this class.
2. Orthogonal polynomial ensembles

Consider a probability (density) on $\Omega^N$, $\Omega = \mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ or $\{0,1,\ldots,M\}$ of the form

$$u_N(x) = \frac{1}{Z_N} \Delta_N(x)^2 \prod_{j=1}^M w(x_j),$$

(2.5)

where $\Delta_N(x) = \prod_{1 \leq i < j \leq N} (x_i - x_j)$ is the Vandermonde determinant, $w(x)$ is some non-negative weight function on $\Omega$ and $Z_N$ is a normalization constant. We call such a probability an orthogonal polynomial ensemble. We can think of this as a finite point process on $\Omega$. Let $d\mu$ be Lebesgue or counting measure on $\Omega$ and let $p_n(x) = \kappa_n x^n + \ldots$ be the normalized orthogonal polynomials with respect to the measure $w(x)d\mu(x)$ on $\Omega$. The correlation functions $\rho_{m,N}(x_1,\ldots,x_m)$ of the point process are given by determinants, we have a so called determinantal point process,

$$\rho_{m,N}(x_1,\ldots,x_m) = \frac{N!}{(N-m)!} \int_{\Omega^{N-m}} \cdots \int_{\Omega} u_N(x) d\mu(x_{m+1}) \cdots d\mu(x_N)$$

(2.6)

where the kernel $K_N$ is given by

$$K_N(x,y) = \frac{\kappa_{N-1} p_N(x)p_{N-1}(y) - p_{N-1}(x)p_N(y)}{x-y} (w(x)w(y))^{1/2}.$$

(2.7)

A computation shows that for bounded $f : \Omega \to \mathbb{C}$,

$$\mathbb{E}\left[\prod_{j=1}^N (1 + f(x_j))\right] = \sum_{k=0}^N \frac{1}{k!} \int_{\Omega} \cdots \int_{\Omega} f(x_j) \det(K_N(x_i,x_j))_{1 \leq i,j \leq k} d\mu(x)$$

(2.8)

where the last determinant is the Fredholm determinant of the integral operator on $L^2(\Omega,d\mu)$ with kernel $f(x)K_N(x,y)$. In particular, we can compute hole or gap probabilities, e.g. the probability of having no particle in an interval $I \subseteq \Omega$ by taking $f = -\chi_I$, minus the characteristic function of the interval $I$. If $x_{\text{max}} = \max x_j$ denotes the position of the rightmost particle it follows that

$$\mathbb{P}[x_{\text{max}} \leq a] = \det(I - K_N)_{L^2((a,\infty),d\mu)}.$$

(2.9)

As $N \to \infty$ we can obtain limiting determinantal processes on $\mathbb{R}$ or $\mathbb{Z}$ with kernel $K$, i.e. the probability (density) of finding particles at $x_1,\ldots,x_m$ is given by $\det(K(x_i,x_m))_{1 \leq i,j \leq m}$. We will be interested in the limit process around the rightmost particle. This is typically given by the Airy kernel, $x,y \in \mathbb{R}$,

$$A(x,y) = \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x-y}.$$

(2.10)
This limit, the Airy point process has a rightmost particle almost surely and its position has the distribution function

$$F_2(\xi) = \det(I - A)_{L^2(\xi, \infty)},$$

(2.11)

known as the Tracy-Widom distribution. Hole probabilities as functions of the endpoints of the intervals satisfy systems of differential equations. For example, we have

$$F_2(\xi) = \exp\left(-\int_{\xi}^{\infty} (x - \xi)u(x)^2 \, dx\right),$$

(2.12)

where $u$ solves the Painlevé-II equation $u'' = xu + 2u^3$ with boundary condition $u(x) \sim Ai(x)$ as $x \to \infty$.

An example of a measure of the form (2.5) comes from the Gaussian Unitary Ensemble (GUE) of random matrices. The GUE is a Gaussian measure on the space $H_{N} \cong \mathbb{R}^{N^2}$ of all $N \times N$ Hermitian matrices. It is defined by $d\mu_{\text{GUE},N}(M) = Z_N^{-1} \exp(-\text{tr} M^2) dM$, where $dM$ is the Lebesgue measure on $H_{N}$ and $Z_N$ is a normalization constant. The corresponding eigenvalue measure has the form (2.5) with $w(x) = \exp(-x^2)$ and $\Omega = \mathbb{R}$. Hence the $p_n$s are multiples of the ordinary Hermite polynomials. The largest eigenvalue $x_{\text{max}}$ will lie around $\sqrt{2}N$. This is related to the fact that the largest zero of $p_N(x)$ lies around $\sqrt{2}N$. The local asymptotics of $p_N(x) \exp(-x^2/2)$ around this point, $x = \sqrt{2N} + \xi/N^{1/6} \sqrt{2}$, is given by the Airy function, $Ai(\xi)$. This asymptotics, some estimates, (2.9) and (2.11) give the following result,

$$\lim_{N \to \infty} P_{\text{GUE},N}\left(\sqrt{2N}x_{\text{max}} - 2N^{1/3} \leq \xi\right) = F_2(\xi)$$

(2.13)

as $N \to \infty$.

3. Some theorems

The previous section may seem unrelated to the first but as the next theorems will show the problem of understanding the distribution of $L(\alpha)$ and $G(M, N)$ fits nicely into the machinery of sect. 2.

**Theorem 3.1.** \[10\]. Take $\Omega = \mathbb{N}$, $M \geq N$ and $w(x) = (M - N + x)$ in (2.5). Then $G(M, N)$ is distributed exactly as $x_{\text{max}}$.

The corresponding orthogonal polynomials are the Meixner polynomials, a classical family of discrete orthogonal polynomials, and we refer to the measure obtained as the Meixner ensemble. It is an example of a discrete orthogonal polynomial ensemble. \[12\]. By computing the appropriate Airy asymptotics of the Meixner polynomials we can use (2.13) to prove the next theorem.

**Theorem 3.2.** \[10\]. Let $\gamma \geq 1$ be fixed and set $\omega(\gamma, q) = (1 - q)^{-1}(1 + \sqrt{q})^2 - 1$ and $\sigma(\gamma, q) = (1 - q)^{-1}(q/\gamma)^{1/6}(\sqrt{\gamma} + \sqrt{q})^{2/3}(1 + \sqrt{q\gamma})^{2/3}$. Then,

$$\lim_{N \to \infty} P_G(G([\gamma N], N) - \omega(\gamma, q)N^{1/3} \leq \xi) = F_2(\xi).$$

(3.14)
Thus \( G(\gamma N, N) \) fluctuates like the largest eigenvalue of a GUE matrix. As discussed above by setting \( q = \alpha / N^2 \) we can obtain \( L(\alpha) \) as a limit of \( G(N, N) \) as \( N \to \infty \). By taking this limit in theorem 3.1, and using the fact that the measure has determinantal correlation functions, we see that \( L(\alpha) \) behaves like the rightmost particle in a determinantal point process on \( \mathbb{Z} \) given by the discrete Bessel kernel, \[ B_\alpha(x,y) = \sqrt{\alpha} J_x(\sqrt{2\alpha}) J_{x+1}(\sqrt{2\alpha}) - J_{x+1}(\sqrt{2\alpha}) J_y(\sqrt{2\alpha}), \] for \( x,y \in \mathbb{Z} \). This gives

\[
P[L(\alpha) \leq n] = \det(I - B^\alpha)_{\ell^2 \{n,n+1,\ldots\}}.
\]

Once we have this formula we see that all we need is the classical asymptotic formula \( \alpha^{1/6} J_{\sqrt{2\alpha}}(\sqrt{2\alpha}) \to \text{Ai}(\xi) \) as \( \alpha \to \infty \) uniformly in compact intervals, and some estimates of the Bessel functions in order to get a limit theorem for \( L(\alpha) \):

**Theorem 3.3.** As \( \alpha \to \infty \),

\[
P\left[ L(\alpha) - 2\sqrt{\alpha} \alpha^{1/6} \leq \xi \right] \to F_2(\xi).
\]

Note the similarity with \( (2.13) \), just replace \( N \) by \( \sqrt{\alpha} \). This result was first proved in \( [3] \) by another method, see below. De-poissonizing we get a limit theorem for \( \ell_N(\pi) \), see \( [3] \).

### 4. Rewriting Toeplitz determinants

The Toeplitz determinant of order \( n \) with generating function \( f \in L^1(\mathbb{T}) \) is defined by

\[
D_n(f) = \det(\hat{f}_{i-j})_{1 \leq i,j \leq n},
\]

where \( \hat{f}_k = (2\pi)^{-1} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-ik\theta} d\theta \) are the complex Fourier coefficients of \( f \). Consider the generating function

\[
f(z) = \prod_{\ell=1}^{M} \frac{1}{z} (1 + a_\ell z)(1 + b_\ell z),
\]

where \( a_\ell, b_\ell \) are complex numbers. The elementary symmetric polynomial \( e_m(a) \), \( a = (a_1, \ldots, a_M) \) is defined by \( \prod_{j=1}^{M} (1 + a_j z) = \sum_{|m| \leq \infty} e_m(a) z^m \). A straightforward computation shows that when \( f \) is given by \( (4.19) \) then

\[
\hat{f}_{i-j} = \sum_{m=0}^{\infty} e_{m-j}(a) e_{m-k}(b).
\]
Insert this into the definition (4.18) and use the Heine identity,

\[
\frac{1}{n!} \int_{\Omega^n} \det(\phi_i(x_j))_{1 \leq i, j \leq n} \det(\psi_i(x_j))_{1 \leq i, j \leq n} d^n \mu(x)
\]
\[= \det(\int_{\Omega} \phi_i(x) \psi_j(x) d\mu(x))_{1 \leq i, j \leq n},\]

(4.20)
to see that

\[
D_n(f) = \sum_{m_1 > m_2 > \cdots > m_n \geq 0} \det(e_{m_i - j}(a))_{1 \leq i, j \leq n} \det(e_{m_i - j}(b))_{1 \leq i, j \leq n}.
\]

(4.21)

Here we have removed the \(n!\) by ordering the variables. These determinants are again symmetric polynomials, the so called Schur polynomials. Let \(\lambda = (\lambda_1, \lambda_2, \ldots)\) be a partition and let \(\lambda' = (\lambda_1', \lambda_2', \ldots)\) be the conjugate partition, \([23]\). Set \(m_i = \lambda_1' + n - i, i = 1, \ldots, n\) and \(\lambda_i' = 0\) if \(i > n\), so that \(\lambda'\) has at most \(n\) parts, \(\ell(\lambda') \leq n\), which means that \(\lambda_1 \leq n\). Then the Schur polynomial \(s_\lambda(a)\) is given by

\[
s_\lambda(a) = \det(e_{\lambda'_i - i + j}(a))_{1 \leq i, j \leq n} = \det(e_{m_i - j}(a))_{1 \leq i, j \leq n},
\]

(4.22)

the Jacobi-Trudi identity. Hence,

\[
D_n(f) = \sum_{\lambda : \lambda_1 \leq n} s_\lambda(a)s_\lambda(b),
\]

(4.23)

and we have derived Gessel’s formula, \([7]\). If we let \(n \to \infty\) in the right hand side we obtain \(\prod_{i,j=1}^M (1 - a_i b_j)^{-1}\) by the Cauchy identity, \([23]\). In the case when all \(a_i, b_j \in [0, 1]\), \(s_\lambda(a) s_\lambda(b) \geq 0\), and we can think of

\[
\prod_{i,j=1}^M (1 - a_i b_j) s_\lambda(a) s_\lambda(b)
\]

(4.24)
as a probability measure on all partitions \(\lambda\) with at most \(n\) parts, the Schur measure, \([19]\). In this formula we can insert the combinatorial definition of the Schur polynomial, \([23]\),

\[
s_\lambda(a) = \sum_{T : \text{sh}(T) = \lambda} a_1^{m_1(T)} \cdots a_M^{m_M(T)},
\]

(4.25)

where the sum is over all semi-standard Young tableaux \(T, [23]\), with shape \(\lambda\), and \(m_i(T)\) is the number of \(i\)'s in \(T\).

A connection with the random variables in section 1 is now provided by the Robinson-Schensted-Knuth (RSK) correspondence, \([23]\). This correspondence maps an \(M \times M\) integer matrix to a pair of semi-standard Young tableaux \((T, S)\) with entries from \(\{1, 2, \ldots, M\}\). If we let the random variables \(w(i, j)\) be independent geometric with parameter \(a_i b_j\) then the RSK-correspondence maps the measure we get on the integer matrix \((w(i, j))_{1 \leq i, j \leq M}\) to the Schur measure \((4.24)\). Also, the RSK-correspondence is such that \(G(M, M) = \lambda_1\), the length of the first row. If we
put $a_j = 0$ for $N < j \leq M$ and $a_i = b_i = \sqrt{q}$ for $1 \leq i \leq N$ in the Schur measure and set $x_j = \lambda_j + N - j$, $1 \leq j \leq N$, we obtain the result in Theorem 3.1.

In the limit $M = N \to \infty$, $q = \alpha/N^2$, in which case $G(N,N)$ converges to $L(\alpha)$, the Schur measure converges to the so called Plancherel measure on partitions, 30, 12. In the variables $\lambda_i - i$ this measure is a determinantal point process on $\mathbb{Z}$ given by the kernel $B^\alpha$, (3.15). This result was obtained independently in [4], which also gives a description in terms of different coordinates. See also [18] for a more direct geometric relation between GUE and the Plancherel measure. In this limit the Toeplitz determinant formula (4.23) gives

$$
P[L(\alpha) \leq n] = e^{-\alpha} D_n(e^{2\sqrt{\alpha} \cos \theta}).$$

This variant of Gessel’s formula was the starting point for the original proof of Theorem 3.3 in [3]. The right hand side of (4.26) can be expressed in terms of the leading coefficients of the orthogonal polynomials on $\mathbb{T}$ with respect to the weight $\exp(2\sqrt{\alpha} \cos \theta)$. These orthogonal polynomials in turn can be obtained as a solution to a matrix-valued Riemann-Hilbert problem (RHP), and the asymptotics of this RHP as $\alpha \to \infty$ can be analyzed using the powerful asymptotic techniques developed by Deift and Zhou, [6]. This approach leads to the formula (2.12) for the limiting distribution.

Write $f = \exp(g)$ and insert the definition of the Fourier coefficients into the definition (4.18). By the Heine identity we obtain an integral formula for the Toeplitz determinant,

$$
D_n(f) = \frac{1}{(2\pi)^n n!} \int_{[-\pi,\pi]^n} \prod_{1 \leq \mu < \nu \leq n} |e^{i\theta_\mu} - e^{i\theta_\nu}|^2 \prod_{\mu=1}^n e^{g(e^{i\theta_\mu})} d^n \theta
$$

In the last integral $dU$ denotes normalized Haar measure on the unitary group $U(n)$ and the identity is the Weyl integration formula. The limit of (4.28) as $\alpha \to \infty$ is then a so called double scaling limit in a unitary matrix model, [20]. The formula (4.28) can also be obtained by considering the integral over the unitary group, see [22].

Another way to obtain the Schur measure is via families of non-intersecting paths which result from a multi-layer PNG model, [13]. The determinants in the measure then come from the Karlin-McGregor theorem or the Lindström-Gessel-Viennot method.

5. **A curiosity**

Non-intersecting paths can also be used to describe certain tilings, e.g. domino tilings and tilings of a hexagon by rhombi. By looking at intersections with appropriate lines one can obtain discrete orthogonal polynomial ensembles. In the case of tilings of a hexagon by rhombi, which correspond to boxed planar partitions, [5],
the Hahn ensemble, i.e. \((2.5)\) with \(\Omega = \{0, \ldots, M\}\) and a weight giving the Hahn polynomials, is obtained, \([13]\). The computation leading to this result also gives a proof of the classical MacMahon formula, \([25]\), for the number of boxed planar partitions in an \(abc\) cube, i.e. the number of rhombus tilings of an \(abc\)-hexagon. In terms of Schur polynomials the result is

\[
\sum_{\mu; \ell(\mu) \leq c} s_{\mu^*}^{(1^a)} s_{\mu^*}^{(1^b)} = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}, \tag{5.28}
\]

where the right hand side is MacMahon’s formula. (Here \(1^a\) means \((1, \ldots, 1)\) with \(a\) components.) Comparing this formula with the formula \((4.23)\) we find

\[
(-1)^{a+b} D_n \left( \prod_{\ell=1}^{a} (1 - e^{-i\theta}) \right)^{\frac{b}{2}} \prod_{\ell=1}^{b} (1 - e^{i\theta}) = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2} \tag{5.29}
\]

It has been conjectured by Keating and Snaith, \([14]\), that the following result should hold for the moments of Riemann’s \(\zeta\)-function on the critical line,

\[
\lim_{T \to \infty} \frac{1}{(\log T)^{k^2}} \frac{1}{T} \int_0^T |\zeta(1/2 + it)|^{2k} dt = f_{\text{CUE}}(k)a(k), \tag{5.30}
\]

where \(a(k)\) is a constant depending on the primes,

\[
f_{\text{CUE}}(k) = \lim_{n \to \infty} \frac{1}{nk^2} \int_{U(n)} |Z(U, \theta)|^{2k} dU = \prod_{j=0}^{k-1} \frac{j!}{(j+k)!} \tag{5.31}
\]

and \(Z(U, \theta) = \det(I - U e^{-i\theta})\) is the characteristic polynomial of the unitary matrix \(U\). If we take \(a = b = k\) in \((5.28)\) and use \((4.27)\) we find

\[
\int_{U(n)} |Z(U, \theta)|^{2k} dU = \prod_{i=1}^{k} \prod_{j=1}^{k} \prod_{\ell=1}^{n} \frac{i+j+\ell-1}{i+j+\ell-2} = \prod_{j=0}^{n-1} \frac{j!(j+2k)!}{(j+k)!^2} \tag{5.32}
\]

as computed in \([14]\) by different methods. Letting \(n \to \infty\) we obtain the last expression in \((5.31)\). Hence, we see that the formula \((5.32)\) has a curious combinatorial interpretation via MacMahon’s formula.

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