Counterexamples to the local-global divisibility over elliptic curves

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Abstract
Let $p \geq 5$ be a prime number. We find all the possible subgroups $G$ of $\text{GL}_2(\mathbb{Z}/p\mathbb{Z})$ such that there exists a number field $k$ and an elliptic curve $E$ defined over $k$ such that the $\text{Gal}(k(E[p])/k)$-module $E[p]$ is isomorphic to the $G$-module $(\mathbb{Z}/p\mathbb{Z})^2$ and there exists $n \in \mathbb{N}$ such that the local-global divisibility by $p^n$ does not hold over $E(k)$.

1 Introduction

Let $k$ be a number field and let $\mathcal{A}$ be a commutative algebraic group defined over $k$. Several papers have been written on the following classical question, known as the Local-Global Divisibility Problem.

PROBLEM: Let $P \in \mathcal{A}(k)$. Assume that for all but finitely many valuations $v$ of $k$, there exists $D_v \in \mathcal{A}(k_v)$ such that $P = qD_v$, where $q$ is a positive integer. Is it possible to conclude that there exists $D \in \mathcal{A}(k)$ such that $P = qD$?

By Bézout’s identity, to get answers for a general integer it is sufficient to solve it for powers $p^n$ of a prime. In the classical case of $\mathcal{A} = \mathbb{G}_m$, the answer is positive for $p$ odd, and negative for instance for $q = 8$ (and $P = 16$) (see for example [AT], [Tro]).

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For general commutative algebraic groups, Dvornicich and Zannier gave a cohomological interpretation of the problem (see [DZ1] and [DZ3]) that we shall explain. Let $\Gamma$ be a group and let $M$ be a $\Gamma$-module. We say that a cocycle $Z : \Gamma \to M$ satisfies the local conditions if for every $\gamma \in \Gamma$ there exists $m_\gamma \in M$ such that $Z_\gamma = \gamma(m_\gamma) - m_\gamma$. The set of the class of cocycles in $H^1(\Gamma, M)$ that satisfy the local conditions is a subgroup of $H^1(\Gamma, M)$. We call it the first local cohomology group $H^1_{\text{loc}}(\Gamma, M)$. Dvornicich and Zannier [DZ1, Proposition 2.1] proved the following result.

**Proposition 1.** Let $p$ be a prime number, let $n$ be a positive integer, let $k$ be a number field and let $A$ be a commutative algebraic group defined over $k$. If $H^1_{\text{loc}}(\text{Gal}(k(A[p^n]))/k), A[p^n]) = 0$, then the local-global divisibility by $p^n$ over $A(k)$ holds.

The converse of Proposition 1 is not true, but in the case when the group $H^1_{\text{loc}}(\text{Gal}(k(A[p^n]))/k), A[p^n])$ is not trivial we can find an extension $L$ of $k$ such that $L \cap k(A[p^n]) = k$, and the local-global divisibility by $p^n$ over $A(L)$ does not hold (see [DZ3] Theorem 3 for the details).

Many mathematicians got criterions for the validity of the local-global divisibility principle for many families of commutative algebraic groups, as algebraic tori ([DZ1] and [III]), elliptic curves ([Cre1], [Cre2], [DZ1], [DZ2], [DZ3], [GR1], [Pal1], [Pal2], [PRV1], [PRV2]), and very recently polarized abelian surfaces ([GR2]) and GL$_2$-type varieties ([GR3]).

In this paper we are interested in the family of the elliptic curves. Let $p$ be a prime number, let $k$ be a number field and let $E$ be an elliptic curve defined over $k$. Dvornicich and Zannier [DZ3, Theorem 1] found a very interesting criterion for the validity of the local-global divisibility by a power of $p$ over $E(k)$, in the case when $k \cap \mathbb{Q}(\zeta_p) = \mathbb{Q}$.

In a joint work with Paladino and Viada (see [PRV1], and Section 2) we refined this criterion, by proving that if $k$ does not contain $\mathbb{Q}(\zeta_p + \zeta_p)$ and $E(k)$ does not admit a point of order $p$, then for every positive integer $n$, the local-global divisibility by $p^n$ holds over $E(k)$. In another joint work with Paladino and Viada [PRV2] we improved our previous criterion and the new criterion allowed us to show that if $k = \mathbb{Q}$ and $p \geq 5$, for every positive integer $n$ the local-global divisibility by $p^n$ holds for $E(\mathbb{Q})$. 
Very recently, Lawson and Wutrich [LW] found a very strong criterion for the triviality of $H^1(\text{Gal}(k(\mathcal{E}[p^n])/k), \mathcal{E}[p^n])$ (then for the validity of the local-global principle by $p^n$ over $\mathcal{E}(k)$, see Proposition 1), but still in the case when $k \cap \mathbb{Q}(\zeta_p) = \mathbb{Q}$.

Finally, Dvornicich and Zannier [DZ2] and Paladino ([Pal1]) studied the case when $p = 2$ and Paladino ([Pal2]) and Creutz ([Cre1]) studied the case when $p = 3$.

Then, we have a fairly good understanding of the local-global divisibility by a power of $p$ over $\mathcal{E}(k)$ either when $p \in \{2, 3\}$ or $k$ does not contain $\mathbb{Q}(\zeta_p + \overline{\zeta}_p)$ and $\mathcal{E}(k)$ does not admit a point of order $p$. In this paper we study the cases not treated, yet. We get the following results:

**Theorem 2.** Let $p \geq 5$ be a prime number, let $k$ be a number field and let $\mathcal{E}$ be an elliptic curve defined over $k$. Suppose that there exists a positive integer $n$ such that the local-global divisibility by $p^n$ does not hold over $\mathcal{E}(k)$. Let $G_1 = \text{Gal}(k(\mathcal{E}[p])/k)$. Then one of the following holds:

1. $G_1$ is cyclic of order dividing $p - 1$ generated by an element with an eigenvalue equal to 1;

2. $p \equiv 2 \mod (3)$ and $G_1$ is isomorphic to a subgroup of $S_3$ of order multiple of 3;

3. $G_1$ is contained in a Borel subgroup and it is generated by an element $\sigma$ of order $p$ and an element $g$ of order dividing 2 such that $\sigma$ and $g$ have the same eigenvector for the eigenvalue 1.

Moreover, there exist number fields $L_1$, $L_2$, $L_3$ and elliptic curves $\mathcal{E}_1$ defined over $L_1$, $\mathcal{E}_2$ defined over $L_2$, $\mathcal{E}_3$ defined over $L_3$ such that for every $i \in \{1, 2, 3\}$, the Gal($L_i(\mathcal{E}_i[p])/L_i$)-module $\mathcal{E}_i[p]$ is isomorphic to the $G_1$-module $\mathcal{E}[p]$ of the case $i$, and the local-global divisibility by $p^2$ does not hold over $\mathcal{E}(L_i)$.

Clearly case 1. of Theorem 2 corresponds to the case when $\mathcal{E}(k)$ has a point of order $p$ defined over $k$. Recall that if $k$ is a number field and $\mathcal{E}$ is an elliptic curve defined over $k$, then for every $\tau \in \text{Gal}(k(\mathcal{E}[p])/k)$, the
determinant of $\tau$ is the $p$th cyclotomic character. Then the cases 2. and cases 3. of Theorem 2 correspond to the case when $\mathbb{Q}(\zeta_p + \overline{\zeta}_p) \subseteq k$.

By the main result of [PRV1] and Theorem 2, we have the following corollary.

**Corollary 3.** Let $p \geq 5$ be a prime number, let $k$ be a number field and let $E$ be an elliptic curve defined over $k$. If $p \equiv 1 \mod (3)$ and $E$ does not admit any point of order $p$ over $k$, then for every positive integer $n$ the local-global divisibility by $p^n$ holds over $E(k)$. If $p \equiv 2 \mod (3)$, $E$ does not admit any point of order $p$ over $k$ and $[k(E[p]) : k] \neq 3$ and 6, then for every positive integer $n$ the local-global divisibility by $p^n$ holds over $E(k)$.

**Proof.** If $k$ does not contain $\mathbb{Q}(\zeta_p + \overline{\zeta}_p)$, just apply the main result of [PRV1]. If $p \equiv 1 \mod (3)$ and $k$ contains $\mathbb{Q}(\zeta_p + \overline{\zeta}_p)$, if there exists $n \in \mathbb{N}$ such that the local-global divisibility by $p^n$ does not hold over $E(k)$, then either case 1. or case 3. of Theorem 2 holds. Then $E$ admits a point of order $p$ defined over $k$.

If $p \equiv 2 \mod (3)$, $E$ does not admit any point of order $p$ over $k$, and there exists a positive integer $n$ such that the local-global divisibility by $p^n$ does not hold over $E(k)$, then case 2. of Theorem 2 holds. Hence $k(E[p])/k$ is either an extension of degree 3 or an extension of degree 6.

□

2 Known results

In the following proposition we put together the main results of [PRV1] and [PRV2] and we use some results of [GR2].

**Proposition 4.** Let $k$ be a number field and let $E$ be an elliptic curve defined over $k$. Let $p$ be a prime number and, for every $m \in \mathbb{N}$, let $G_m$ be $\text{Gal}(k(E[p^m])/k)$. Suppose that there exists $n \in \mathbb{N}$ such that $H^1_{\text{loc}}(G_n, E[p^n]) \neq 0$. Then one of the following cases holds:
Known results

1. If \( p \) does not divide \(|G_1|\) then either \( G_1 \) is cyclic of order dividing \( p - 1 \), generated by an element fixing a point of order \( p \) of \( \mathcal{E} \), or \( p \equiv 2 \pmod{3} \) and \( G_1 \) is a group isomorphic either to \( S_3 \) or to a cyclic group of order 3;

2. If \( p \) divides \(|G_1|\) then \( G_1 \) is contained in a Borel subgroup and it is either cyclic of order \( p \), or it is generated by an element of order \( p \) and an element of order 2 distinct from \(-Id\).

Proof. Suppose first that \( p \) does not divide \(|G_1|\). By the argument in [DZ3, p. 29], we have that \( G_1 \) is isomorphic to its projective image. By [Ser, Proposition 16], then \( G_1 \) is either cyclic, or dihedral or isomorphic to one of the following groups: \( A_4, S_4, A_5 \).

Suppose that the last case holds. Then \( G_1 \) should contain a subgroup isomorphic to \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) and so \(-Id\). This contradicts the fact that \( G_1 \) is isomorphic to its projective image.

Suppose that \( G_1 \) is dihedral. Then \( G_1 \) is generated by \( \tau \) and \( \sigma \) with \( \sigma \) of order 2 and \( \sigma \tau = \tau^{-1} \sigma \). In particular all the elements of \( G_1 \) have determinant either 1 or \(-1\). Suppose that there exists \( i \in \mathbb{N} \) such that \( \tau^i \) has order dividing \( p - 1 \) and distinct from 1. By [GR2, Theorem 2], \( \tau^i \) has at least an eigenvalue equal to 1. Then, since \( \tau^i \) has determinant \(-1\), the unique possibility is that \( \tau^i \) has order 2 and so it commutes with \( \sigma \). Thus we get a contradiction because we should have \(-Id \in G_1\). Then \( \tau \) has determinant 1 and order dividing \( p + 1 \). In particular it has two eigenvalues over \( \mathbb{F}_{p^2} \): \( \lambda \) and \( \lambda^p \). By [GR2, Proposition 17, Lemma 18] (or see [CS, Section 3]) we have that \( \tau \) has order 3. Then 3 divides \( p + 1 \) and \( G_1 \) is isomorphic to \( S_3 \).

Finally suppose that \( G_1 \) is cyclic. If \( G_1 \) is generated by an element of order dividing \( p - 1 \), by [GR2, Theorem 2] we have that such an element has an eigenvalue equal to 1. On the other hand if the generator of \( G_1 \) has order not dividing by \( p - 1 \), again by [GR2, Proposition 17, Lemma 18] we get that such an element has order 3 and 3 divides \( p + 1 \).

Suppose now that \( p \) divides \(|G_1|\). Since \( p \) divides the order of \( G_1 \), by [Ser, Proposition 15] and the fact that \( G_1 \) is isomorphic to its projective
image, we have that $G_1$ is contained in a Borel subgroup. In particular the $p$-Sylow subgroup $N$ of $G$ is normal. Suppose that $G/N$ is not cyclic. Then $G_1$ is not isomorphic to its projective image. Thus $G_1$ is generated by $\sigma$ an element of order $p$, which generates $N$, and $g$ an element of order dividing $p - 1$. Suppose that $g$ has the eigenvalues distinct from 1. Then by [GR2, Theorem 2] (in particular observe that, by [GR2, Remark 16], it is not necessary the hypothesis $H^1(G_1, E[p]) = 0$, $H^1_{\text{loc}}(G_m, E[p^m]) = 0$ for every $m \in \mathbb{N}$ and so we get a contradiction. Then $g$ has an eigenvalue equal to 1. Suppose that $g$ has order $\geq 3$. Then its determinant has order $\geq 3$ and so, since the determinant is the $p$th cyclotomic character, $k$ does not contain $\mathbb{Q}(\zeta_p + \overline{\zeta_p})$. Then if $g$ and $\sigma$ does not fix the same point of order $p$, by [PRV1, Theorem 1] we get a contradiction. On the other hand, since $p$ divides the order of $G_1$, $k(E[p]) \neq k(\zeta_p)$. Then by [PRV2, Theorem 3], we get a contradiction.

Finally, then either $G_1$ is cyclic generated by an element $\sigma$ of order $p$ or $G_1$ is generated by an element of order $p$ generating a normal subgroup $N$ of $G_1$ and an element $g$ of order $\leq 2$ and which is not $-Id$.

\[\Box\]

3 The image of the Galois action over the torsion points

In this section we recall some important theorems on the Galois action over the torsion points on an elliptic curve over a number field. In [GR1] we proved the following lemma, which is a direct consequence of very interesting results of Greicius [Gre] and Zywina [Zyw].

**Lemma 5.** Given a prime number $p$, a positive integer $n$ and a subgroup $G$ of $GL_2(\mathbb{Z}/p^n\mathbb{Z})$, there exists a number field $k$ and an elliptic curve $E$ defined over $k$ such that there are an isomorphism $\phi: \text{Gal}(k(E[p^n])/k) \to G$ and a $\mathbb{Z}/p^n\mathbb{Z}$-linear homomorphism $\tau: E[p^n] \to (\mathbb{Z}/p^n\mathbb{Z})^2$ such that, for all $\sigma \in \text{Gal}(k(E[p^n])/k)$ and $v \in E[p^n]$, we have $\phi(\sigma)v = \tau(\sigma(v))$. 
The prime to $p$ case

**Proof.** See [GR1, Lemma 11].

The following corollary is an immediate consequence of the previous lemma and [DZ3, Theorem 3].

**Corollary 6.** Given a prime number $p$, a positive integer $n$ and a subgroup $G$ of $\text{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$ such that $H^1_{\text{loc}}(G, (\mathbb{Z}/p^n\mathbb{Z})^2) \neq 0$, there exists a number field $k$ and an elliptic curve $E$ defined over $k$ such that there exist isomorphisms $\phi: \text{Gal}(k(E[p^n])/k) \to G$ and $\tau: E[p^n] \to (\mathbb{Z}/p^n\mathbb{Z})^2$ such that, for all $\sigma \in \text{Gal}(k(E[p^n])/k)$ and $v \in E[p^n]$, we have $\phi(\sigma)(\tau(v)) = \tau(\sigma(v))$. Then $H^1_{\text{loc}}(\text{Gal}(k(E[p^n])/k), E[p^n])$ is isomorphic to $H^1_{\text{loc}}(G, (\mathbb{Z}/p^n\mathbb{Z})^2)$ and there exists a finite extension $L$ of $k$ such that $L \cap k(E[p^n]) = k$ and the local-global divisibility by $p^n$ does not hold over $E(L)$.

4 The prime to $p$ case

By Proposition 4 we get that if $k$ is a number field and $E$ is an elliptic curve defined over $k$ such that $\text{Gal}(k(E[p])/k)$ has order prime to $p$ and the local-global divisibility by a certain power of $p$ does not hold over $E(k)$, then we have substantially two cases to study: the case when $p \equiv 2 \mbox{ mod } (3)$ and $\text{Gal}(k(E[p])/k)$ is isomorphic to a subgroup of $S_3$ and the case when $\text{Gal}(k(E[p])/k)$ is cyclic of order dividing $p - 1$ generated by an element fixing a point of order $p$. We do this in the following subsections.

4.1 The case when $p \equiv 2 \mbox{ mod } (3)$ and the Galois group is a subgroup of $S_3$

Let $p \equiv 2 \mbox{ mod } (3)$ be a prime number. In [GR2, Section 5] we already found a subgroup $G$ of $\text{GL}_2(\mathbb{Z}/p^2\mathbb{Z})$ such that $H^1_{\text{loc}}(G, (\mathbb{Z}/p^2\mathbb{Z})^2) \neq 0$ and the quotient of $G$ by the subgroup $H$ of the elements congruent to the identity modulo $p$ is a cyclic group of order 3. We use this, the following lemma and the following proposition to extend the example to a group $G'$ containing $G$ such that $G'/H$ is isomorphic to $S_3$. 

□
The case when \( p \equiv 2 \mod (3) \) and the Galois group is a subgroup of \( S_3 \).

**Lemma 7.** Let \( p \) be a prime number, let \( m \) be a positive integer, let \( V \) be \((\mathbb{Z}/p^2\mathbb{Z})^{2m}\), let \( G \) be a subgroup of \( \text{GL}_{2m}(\mathbb{Z}/p^2\mathbb{Z}) \) and let \( H \) be the subgroup of \( G \) of the elements congruent to the identity modulo \( p \). Then we have the following exact sequence:

\[
0 \to H^1(G/H, V[p]) \to H^1(G, V[p]) \to H^1(H, V[p])^{G/H} \to H^2(G/H, V[p]).
\]

Moreover the exact sequence

\[
0 \to V[p] \to V \to V[p] \to 0
\]

(the first map is the inclusion and the second map the multiplication by \( p \)) induces the following exact sequence:

\[
H^0(G, V[p]) \to H^1(G, V[p]) \to H^1(G, V) \to H^1(G, V[p]).
\]

**Proof.** Since \( H \) is a normal subgroup of \( G \), the exact sequence \((1.1)\) is just the inflation-restriction sequence for \( H \).

Consider the exact sequence of \( G \)-modules:

\[
0 \to V[p] \to V \to V[p] \to 0,
\]

where the first map is the inclusion and the second is the multiplication by \( p \). Then it induces the following exact sequence of \( G \)-modules

\[
H^0(G, V[p]) \to H^1(G, V[p]) \to H^1(G, V) \to H^1(G, V[p]).
\]

\(\square\)

**Proposition 8.** Let \( p \) be a prime number, let \( m \) be a positive integer, let \( V \) be \((\mathbb{Z}/p^2\mathbb{Z})^{2m}\) and let \( G \) be a subgroup of \( \text{GL}_{2m}(\mathbb{Z}/p^2\mathbb{Z}) \) such that:

1. \( G \) has an element \( \delta \) not fixing any element of \( V \);

2. Let \( H \) be the subgroup of \( G \) of the element congruent to the identity modulo \( p \). \( H \) is isomorphic, as \( G/H \)-module, to a non trivial \( G/H \)-submodule of \( V[p] \);
The case when \( p \equiv 2 \mod (3) \) and the Galois group is a subgroup of \( S_3 \).

3. For every \( h \in H \) distinct from the identity, the endomorphism \( h - \text{Id} : V/V[p] \rightarrow V/V[p] \) is an isomorphism;

4. \( G/H \) has order not divisible by \( p \).

Then \( H^1_{\text{loc}}(G, V) \neq 0 \).

Proof. Consider the inflation-restriction sequence (see Lemma 7)

\[
0 \rightarrow H^1(G/H, V[p]) \rightarrow H^1(G, V[p]) \rightarrow H^1(H, V[p])^{G/H} \rightarrow H^2(G/H, V[p]).
\]

By Hypothesis 4., \( H^1(G/H, \mathcal{A}[p]) \) and \( H^2(G/H, \mathcal{A}[p]) \) are trivial. Then the restriction \( H^1(G, V[p]) \rightarrow H^1(H, V[p])^{G/H} \) is an isomorphism. Since the action of \( H \) over \( V[p] \) is trivial and \( H \) is an abelian group of exponent \( p \), we have that \( H^1(H, V[p])^{G/H} \) is isomorphic to \( \text{Hom}_{\mathbb{Z}/p\mathbb{Z}[G/H]}(H, V[p]) \). By hypothesis 2., \( H \) is isomorphic to a non trivial \( G/H \)-submodule of \( V[p] \).

Then there exists \( \phi : H \rightarrow V[p] \) an injective homomorphism of \( \mathbb{Z}/p\mathbb{Z}[G/H] \)-modules. Let \([Z]\) be in \( H^1(G, V[p]) \) such that its image in \( H^1(H, V[p])^{G/H} \) is the class of \( \phi \). In particular observe that since \( \phi \) is an injective homomorphism, \([Z]\) \( \neq 0 \).

Now observe that \( H^0(G, V[p]) = 0 \) by hypothesis 1.. Then, by Lemma 7 we have the following exact sequence of \( G \)-modules

\[
0 \rightarrow H^1(G, V[p]) \rightarrow H^1(G, V) \rightarrow H^1(G, V[p]).
\]

Let us call \([W] \in H^1(G, V)\) the image of \([Z] \in H^1(G, V[p])\) defined above by the injective map \( H^1(G, V[p]) \rightarrow H^1(G, V) \). Since \([Z]\) \( \neq 0 \), the same holds for \([W]\). Moreover, since \( G/H \) is not divisible by \( p \), the restriction \( H^1(G, V) \rightarrow H^1(H, V) \) is injective. Since, by 3., the image by the restriction of \([W]\) over \( H^1(H, V) \) is in \( H^1_{\text{loc}}(H, V) \), we have that \([W]\) is a non-trivial element of \( H^1_{\text{loc}}(G, V) \).

\(\square\)
Corollary 9. Let \( p \) be an odd prime such that \( p \equiv 2 \mod (3) \). Let \( G \) be the subgroup of \( \text{GL}_2(\mathbb{Z}/p^2\mathbb{Z}) \), such that \( G \) is generated by 

\[
\tau = \begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix}
\]

of order 3, \( \sigma \) of order 2 such that \( \sigma \tau \sigma^{-1} = \tau^2 \) and 

\[
H = \left\{ \begin{pmatrix} 1 + p(a - 2b) & 3p(b - a) \\ -pb & 1 - p(a - 2b) \end{pmatrix} \right., \ a, b \in \mathbb{Z}/p^2\mathbb{Z} \right\}.
\]

Then \( H^1_{\text{loc}}(G, (\mathbb{Z}/p^2\mathbb{Z})^2) \neq 0 \).

Proof. Observe that conditions 1. and 4. of Proposition 8 hold for \( G \). Moreover condition 3. holds by [GR2, Section 5]. Observe that \( G/H \) is isomorphic to \( S_3 \) and recall that \( S_3 \) has a unique irreducible representation of dimension 2 over \( \mathbb{F}_p \). Then condition 2. of Proposition 8 is equivalent to prove that \( H \) is stable by the conjugation by \( \tau \) and \( \sigma \). In [GR2, Section 5] we proved that the conjugation by \( \tau \) sends \( H \) to \( H \). A straightforward computation shows that if \( \sigma \) has order 2 in \( G/H \) and \( \sigma \tau \sigma^{-1} = \tau^2 \), then there exists \( \alpha, \beta \in \mathbb{F}_p \) such that

\[
\sigma = \begin{pmatrix} \alpha - 2\beta & 3(\beta - \alpha) \\ \beta & 2\beta - \alpha \end{pmatrix}.
\]

Another straightforward computation shows that \( \sigma H \sigma^{-1} = H \) (since \( H \) has dimension 2 as \( \mathbb{F}_p \)-vector space, it is sufficient to verify this over a basis and it is a trivial verification at least over the element of \( H \) with \( \alpha = a \) and \( \beta = b \)). Then, by Proposition 8 we have \( H^1_{\text{loc}}(G, (\mathbb{Z}/p^2\mathbb{Z})^2) \neq 0 \).

\[\square\]

4.2 The case when the Galois group is cyclic of order dividing \( p - 1 \)

By using some results of the previous subsection, we study the second case.
Lemma 10. Let $p$ be a prime number and let $V$ be $(\mathbb{Z}/p^2\mathbb{Z})^2$. Let $\lambda \in (\mathbb{Z}/p^2\mathbb{Z})^*$ of order dividing $p - 1$ and let $G$ be the following subgroup of $\text{GL}_2(V)$:

$$G = \left\langle g = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}, h(1, 0) = \begin{pmatrix} 1 + p & 0 \\ 0 & 1 - p \end{pmatrix}, h(0, 1) = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \right\rangle.$$ 

Then $H^1_{\text{loc}}(G, V) \neq 0$.

Proof. Observe that the subgroup $H$ of $G$ of the elements congruent to the identity modulo $p$ is the group generated by $h(1, 0)$ and $h(0, 1)$. Since $G/H$ has order not divisible by $p$, $H^1(G/H, V[p]) = 0$ and $H^2(G/H, V[p]) = 0$. Then, from the exact sequence (4.1) in Lemma 7 we get an isomorphism from $H^1(G/H, V[p])$ to $H^1(H, V[p])^{G/H}$. Since $H$ acts like the identity over $V[p]$ and $H$ are abelian groups with exponent $p$, $H^1(H, V[p])^{G/H} = \text{Hom}_{\mathbb{Z}/p\mathbb{Z}[G/H]}(H, V[p])$. Observe that $gh(0, 1)g^{-1} = h(0, 1)^\lambda$ and $g(p, 0) = \lambda(p, 0)$. Then we can define a non-trivial homomorphism $\phi$ from $H$ to $V[p]$ by sending $h(0, 1)$ to $(p, 0)$ and $h(1, 0)$ to $(0, 0)$ and extending it by linearity. Let $Z$ be a cocycle representing the class $[Z]$ in $H^1(G, V[p])$ corresponding to $\phi$. By the exact sequence (4.2) in Lemma 7 we get an homomorphism from $H^1(G, V[p])$ to $H^1(G, V)$. Let $W$ be a cocycle representing the class $[W]$ in $H^1(G, V[p])$ image of $[Z]$ for such homomorphism. Let us show that $[W] \in H^1_{\text{loc}}(G, V)$ and $[W] \neq 0$. Since $G/H$ has order not divisible by $p$, it is sufficient to prove that the image by the restriction of $[W]$ to $H^1(H, V)$ is in $H^1_{\text{loc}}(H, V)$. For all $a, b$ integers put $h(a, b) = ah(1, 0) + bh(0, 1)$. Then, by definition of $[Z]$, $h(a, b)$ is sent to $(bp, 0)$. An easy calculation shows that for every $a, b$, there exists $x, y$ in $\mathbb{Z}/p^2\mathbb{Z}$ such that $(h - Id)(x, y) = (bp, 0)$. This proves that $[W] \in H^1_{\text{loc}}(G, V)$.

Finally observe that for every $x, y$ in $\mathbb{Z}/p^2\mathbb{Z}$ such that $(h(1, 0) - Id)(x, y) = (0, 0)$, we have $x \equiv 0 \mod (p)$ and $y \equiv 0 \mod (p)$. On the other hand for every $x, y$ in $\mathbb{Z}/p^2\mathbb{Z}$ such that $(h(1, 0) - Id)(x, y) = (p, 0)$, we have $y \equiv 1 \mod (p)$. Thus $[W] \neq 0$. 

□
5 The case when $p$ divides the order of the Galois group

By Proposition 4 we get that if $k$ is a number field and $E$ is an elliptic curve defined over $k$ such that $\text{Gal}(k(E[p])/k)$ has order divisible by $p$ and the local-global divisibility by a certain power of $p$ does not hold over $E(k)$, then $\text{Gal}(k(E[p])/k)$ is contained in a Borel subgroup and it is generated by an element $\sigma$ of order $p$ and an element $g$ of order dividing 2 and distinct from $-\text{Id}$. In the following subsections we study first the case when $g$ and $\sigma$ fix the same element of order $p$, then the case when $g$ and $\sigma$ do not fix any element of order $p$.

5.1 The case when $g$ and $\sigma$ fix the same vector

In this section we prove the following result.

**Lemma 11.** Let $V$ be $(\mathbb{Z}/p^2\mathbb{Z})^2$ and let $G$ be the following subgroup of $\text{GL}_2(\mathbb{Z}/p^2\mathbb{Z})$:

$$G = \left\{ g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma = \begin{pmatrix} 1+p & 1 \\ 2p & 1+p \end{pmatrix}, h = \begin{pmatrix} 1+p & 0 \\ 0 & 1-p \end{pmatrix} \right\}.$$  

Then $H^1_{\text{loc}}(G, V) \neq 0$.

**Proof.** Let $H$ be the subgroup of $G$ of the elements congruent to 1 modulo $p$. Let $g$ and $\sigma$ be the classes of $g$ and $\sigma$ modulo $H$. We have that $H^1(G/H, V[p]) \neq 0$. In fact we can define a cocycle $Z: G/H \to V[p]$, which is not a coboundary, by sending for every integer $i_1, i_2, Z_{\overline{g}_{\overline{i}_1}\overline{g}_{\overline{i}_2}} = (pi_2(i_2 - 1)/2, (-1)^{i_1}pi_2)$. Since $H$ is normal, we have an injective homomorphism (the inflation) from $H^1(G/H, V[p])$ to $H^1(G, V[p])$. By abuse of notation we still call $Z$ a cocycle representing the image of the class of $Z$ in $H^1(G, V[p])$. Moreover, see Lemma 4 in particular the sequence (4.2), we have a homomorphism from $H^1(G, V[p])$ to $H^1(G, V)$. It sends the class of $Z$ in $H^1(G, V[p])$ in the class of the cocycle $W$ representing the class $[W] \in H^1(G, V)$. We shall prove that $[W] \in H^1_{\text{loc}}(G, V)$ and $[W] \neq 0$.

First of all let us observe that for every $a, b, c, d \in \mathbb{Z}/p^2\mathbb{Z}$, we have

$$\begin{pmatrix} 1+ap & 1+bp \\ cp & 1+dp \end{pmatrix}^p = \begin{pmatrix} 1+p & 0 \\ 0 & 1 \end{pmatrix}.$$
The case when \( g \) and \( \sigma \) fix the same vector

To verify this write
\[
\begin{pmatrix}
1 + ap & 1 + bp \\
cp & 1 + dp
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} ap & 1 + bp \\ cp & dp \end{pmatrix}
\]
and observe that
\[
\begin{pmatrix} ap & 1 + bp \\ cp & dp \end{pmatrix}^2 \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \mod (p), \quad \begin{pmatrix} ap & 1 + bp \\ cp & dp \end{pmatrix}^4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]
Thus the subgroup \( H \) of \( G \) of the elements congruent to the identity modulo \( p \) is
\[
H = \left\langle \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 + p & 0 \\ 0 & 1 - p \end{pmatrix} \right\rangle.
\]
Now observe that, since \( H \) and \( \langle \sigma, H \rangle \) are normal in \( G \), for every \( \tau \in G \) there exist integers \( i_1, i_2, i_3 \) and \( h \in H \) such that \( \tau = g^{i_1} \sigma^{i_2} h^{i_3} \). By definition of \( W \), \( W_\tau = (p(i_2 - 1), (-1)^{i_1} p i_2) \). If \( i_2 \equiv 0 \mod (p) \), then clearly \( W_\tau = (0, 0) \) and so \( W_\tau = (\tau - Id)((0, 0)) \). Then we can suppose \( i_2 \not\equiv 0 \mod (p) \). It is simple to prove by induction on \( i_2 \) that
\[
\sigma^{i_2} = \begin{pmatrix} 1 + ap & i_2 + bp \\ 2i_2p & 1 + cp \end{pmatrix}
\]
for certain \( a, b, c \in \mathbb{Z}/p^2\mathbb{Z} \). Moreover \( \sigma^{i_2} h^{i_3} \) has still the coefficient at the top on the right congruent to \( i_2 \) modulo \( p \) and the low coefficient on the left equal to \( 2i_2p \). From these remarks is an easy exercise to prove that there exist \( \alpha \) and \( \beta \in \mathbb{Z}/p^2\mathbb{Z} \) such that \( W_\tau = (\tau - Id)((\alpha, \beta)) \). Then \([W]\) is in \( H^1_{\text{loc}}(G, V) \).

Finally let us observe that \( W \) is not a coboundary. Let \( \alpha, \beta \in \mathbb{Z}/p^2\mathbb{Z} \) such that \( W_\sigma = (0, p) = (\sigma - Id)((\alpha, \beta)) \). Then \( \alpha \not\equiv 0 \mod (p) \). On the other hand let \( h \) be in \( H \) be
\[
h = \begin{pmatrix} 1 + p & 0 \\ 0 & 1 - p \end{pmatrix}.
\]
Then \( W_h = (0, 0) \) and so for every \( \alpha, \beta \in \mathbb{Z}/p^2\mathbb{Z} \) such that \( (h - id)((\alpha, \beta)) = (0, 0) \), we have \( \alpha \equiv 0 \mod (p) \). Hence \( W \) is not a coboundary.
The case when $g$ and $\sigma$ does not fix the same vector and proof of Theorem

Remark 12. Since we shall use it in the next subsection, we make the following remark. In the proof of the previous lemma, we observed that for every $a, b, c, d \in \mathbb{Z}/p^2\mathbb{Z}$, we have
\[
\left( \begin{array}{cc}
1 + ap & 1 + bp \\
 c p & 1 + dp
\end{array} \right)^p = \left( \begin{array}{c}
1 \\
0
\end{array} \right).
\]
In a similar way, for every integer $m \geq 2$, and every $a_m, b_m, c_m, d_m \in \mathbb{Z}/p^m\mathbb{Z}$, we have
\[
\left( \begin{array}{cc}
1 + a_m p & 1 + b_m p \\
 c_m p & 1 + d_m p
\end{array} \right)^{p^{m-1}} = \left( \begin{array}{c}
1 \\
0
\end{array} \right).
\]

Corollary 13. Let $V$ be $(\mathbb{Z}/p^2\mathbb{Z})^2$ and let $G$ be the following subgroup of $\text{GL}_2(\mathbb{Z}/p^2\mathbb{Z})$:
\[
\widetilde{G} = \left\langle \sigma = \left( \begin{array}{cc}
1 + p & 1 \\
 2p & 1 + p
\end{array} \right), h = \left( \begin{array}{cc}
1 + p & 0 \\
 0 & 1 - p
\end{array} \right) \right\rangle.
\]
Then $H^1_{\text{loc}}(\widetilde{G}, V) \neq 0$

Proof. Observe that $\widetilde{G}$ is a subgroup of index 2 of the group $G$ of Lemma 11. Since $p \neq 2$ the restriction $H^1_{\text{loc}}(G, V) \rightarrow H^1_{\text{loc}}(\widetilde{G}, V)$ is injective. Thus, since $H^1_{\text{loc}}(G, V) \neq 0$, $H^1_{\text{loc}}(\widetilde{G}, V) \neq 0$.

5.2 The case when $g$ and $\sigma$ does not fix the same vector and proof of Theorem

In this section we study the case when $g$ and $\sigma$ have a common eigenvector, but they do not fix the same vector.

Lemma 14. Let $n \in \mathbb{N}$ and let $G$ be a subgroup of $\text{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$. Let $H$ be the subgroup of $G$ of the elements congruent to the identity modulo $p$. Suppose that $G/H$ is contained in a Borel subgroup and it is generated by an element $g$ of order 2 and an element $\sigma$ of order $p$ such that $\sigma$ and $g$ do not fix the same element of order $p$. Let $V_n$ be $(\mathbb{Z}/p^n\mathbb{Z})^2$. Then $H^1_{\text{loc}}(G, V_n) = 0$. 
The case when $g$ and $\sigma$ does not fix the same vector and proof of Theorem 2

**Proof.** By replacing $V$ with $V_n$, by observing that $H^0(G, V_n[p^{n-1}]) = 0$ because the group generated by $g$ and $\sigma$ do not fix any element of $V_n[p^{n-1}]$, and by copying the proof of Lemma 17 we get the following exact sequence

$$0 \to H^1(G, V_n[p]) \to H^1(G, V_n) \to H^1(G, V_n[p^{n-1}]). \quad (5.1)$$

Suppose that $H^1_{\text{loc}}(G, V_n) \neq 0$. Then $H^1_{\text{loc}}(G, V_n)[p] \neq 0$ and let $Z$ be a cocycle representing a non-trivial class $[Z] \in H^1_{\text{loc}}(G, V_n)[p]$. Let us observe that $[Z]$ is in the kernel of $H^1(G, V_n) \to H^1(G, V_n[p^{n-1}])$ (here we generalize the proof of [GR2, Lemma 13]). Since $[Z]$ has order $p$, then $pZ$ is a coboundary and so there exists $v \in V_n$ such that, for every $\tau \in G$, $pZ\tau = \tau(v) - v$. Let us observe that $v \in V_n[p^{n-1}]$. Since for every $\tau$, $\tau(v) - v \in V_n[p^{n-1}]$, we have $v \in \cap_{\tau \in G} \ker(p^{n-1}(\tau - Id))$. Since $G$ does not fix any element of order $p$ the unique possibility is that $v \in V_n[p^{n-1}]$. Then (see the sequence (5.1)) $[Z]$ is in the image of $H^1(G, V_n[p]) \to H^1(G, V_n)$. By abuse of notation we call $[Z]$ the class in $H^1(G, V_n[p])$ sent to $[Z]$.

Consider now the inflation-restriction sequence

$$0 \to H^1(G/H, V_n[p]) \to H^1(G, V_n[p]) \to H^1(H, V_n[p])^{G/H}. \quad (5.2)$$

Let us observe that $H^1(G/H, V_n[p]) = 0$. Let $W: G/H \to V_n[p]$ be a cocycle. Since $\sigma$ and $g$ are contained in a Borel subgroup, $g$ has order 2, and $g$ and $\sigma$ do not fix any non-zero element of $V_n[p^{n-1}]$, we can choose a basis of $V_n$ such that $(p^{n-1}, 0)$ is fixed by $\sigma$ and $g((p^{n-1}, 0)) = (-p^{n-1}, 0)$ and $(0, p^{n-1})$ is sent to $(p^{n-1}, p^{n-1})$ by $\sigma$ and fixed by $g$. Observe that, since summing a coboundary to $W$ does not change its class, we can suppose that $W_{\sigma} = (0, p^{n-1})$. Then, for every integer $i$, $W_{\sigma^i} = (p^{n-1}i(i - 1)/2, p^{n-1}(i - 1))$.

Observe that since $g$ has order 2, $W_{g^2} = W_g + gW_g = (0, 0)$. In particular there exists $a \in \mathbb{Z}/p^n\mathbb{Z}$ such that $W_g = (p^{n-1}a, 0)$, which is fixed by $\sigma$. Thus $W_{g^1} = gW_{\sigma} = (p^{n-1}, -p^{n-1})$. On the other hand $g\sigma g^{-1} = \sigma^{-1}$ and so $W_{g^{-1}} = (-p^{n-1}, -p^{n-1})$. We then get a contradiction. Thus, by the sequence (5.2), we get that to every class of $H^1(G, V_n[p])$ we can associate a class in $H^1(H, V_n[p])^{G/H}$. Since $H$ acts like the identity over $V_n[p]$, we have that $H^1(H, V_n[p])^{G/H}$ is a subgroup of $\text{Hom}(H, V_n[p])$. In particular we can associate to $[Z] \in H^1(G, V[p])$ defined above a homomorphism from $H$ to $V_n[p]$. We now need the following lemma.
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Lemma 15. Let \( \tau \) be in \( H \) and let \( \sigma_n \in G \) be such that \( \sigma_n \) is sent to \( \sigma \) by the projection of \( G \) over \( G/H \). Then there exists \( \tau_d, \tau_l \in H, \lambda \in \mathbb{N}, \) such that \( \tau_d \) is diagonal, \( \tau_l \) is lower unitriangular and \( \tau = \tau_d \tau_l \sigma_n^{p\lambda} \). Then \( H \) is generated by its subgroups of the diagonal matrices, its subgroup of the lower unitriangular matrices and \( \sigma_n^p \).

Proof. We remark that \( \sigma_n^p \in H \). In fact \( \sigma_n^p \equiv Id \mod (p) \).

We first show that every \( \tau \in H \) can be written as a product of a lower triangular matrix \( \tau_L \in H \) and a power of \( \sigma_n^p \). Since \( \tau \in H, \tau \equiv Id \mod (p) \) and so there exist \( e, g, m, r \in \mathbb{Z}/p^n\mathbb{Z} \) such that

\[
\tau = \begin{pmatrix}
1 + pe & pg \\
pm & 1 + pr
\end{pmatrix},
\]

We prove by induction that for every integer \( i \geq 1 \), there exists \( \lambda_i \in \mathbb{Z}/p^n\mathbb{Z} \) such that

\[
\tau \sigma_n^{p\lambda_i} = \begin{pmatrix}
1 + pe_i & p^i g_i \\
pmi & 1 + pr_i
\end{pmatrix}
\]

for certain \( e_i, g_i, m_i, r_i \in \mathbb{Z}/p^n\mathbb{Z} \). Set \( \lambda_{i+1} = p\lambda_i - p^i g_i \). Observe that this element exists because \( i \geq 1 \). By Remark 12 we have

\[
\sigma_n^{-p^i g_i} = \begin{pmatrix}
1 + p^{i+1} a_{i+1} & p^i + p^{i+1} b_{i+1} \\
p^{i+1} c_{i+1} & 1 + p^{i+1} d_{i+1}
\end{pmatrix}^{-g_i}
\]

\[
= \begin{pmatrix}
1 + p^{i+1} a'_{i+1} & -p^i g_i + p^{i+1} b'_{i+1} \\
p^{i+1} c'_{i+1} & 1 + p^{i+1} d'_{i+1}
\end{pmatrix},
\]

for certain \( a'_{i+1}, b'_{i+1}, c'_{i+1}, d'_{i+1} \in \mathbb{Z}/p^n\mathbb{Z} \). By a short computation

\[
\tau \sigma_n^{p\lambda_{i+1}} = \tau \sigma_n^{p\lambda_i} \sigma_n^{-p^i g_i}
\]

\[
= \begin{pmatrix}
1 + pe_i & p^i + p^i g_i \\
pmi & 1 + pr_i
\end{pmatrix} \begin{pmatrix}
1 + p^{i+1} a'_{i+1} & -p^i g_i + p^{i+1} b'_{i+1} \\
p^{i+1} c'_{i+1} & 1 + p^{i+1} d'_{i+1}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 + pe_{i+1} + p^{i+1} g_{i+1} \\
pmi & 1 + pr_{i+1}
\end{pmatrix},
\]
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for certain $e_{i+1}, g_{i+1}, m_{i+1}, r_{i+1} \in \mathbb{Z}/p^n\mathbb{Z}$. Then (5.3) is verified for $\lambda_{i+1}$ that satisfies $p\lambda_{i+1} = p\lambda_i - p^ig_i$. In particular for $i = n$ we have

$$\tau\sigma_{\frac{p^\lambda}{n}} = \begin{pmatrix} 1 + pe_n & 0 \\ pm_n & 1 + pr_n \end{pmatrix}.$$ 

Then setting $\tau_L = \tau\sigma_{\frac{p^\lambda}{n}}$ and $\lambda = -\lambda_n$, we have shown that $\tau$ can be written as a product of a lower triangular matrix $\tau_L \in H$ and the power $\sigma_{\frac{p^\lambda}{n}}$ of $\sigma_n$.

Observe that, to conclude the proof, it is sufficient to show that $\tau_L$ can be written as the product of a diagonal matrix $\tau_d \in H$ and a lower unitriangular matrix $\tau_l \in H$. Let $g_n \in G$ be an element of order 2 whose projection to $G/H$ is $g$. Since $H$ is normal in $G$, $g_n\tau_Lg_n^{-1} \in H$. Then $g_n\tau_Lg_n^{-1}\tau_L^{-1} \in H$. Moreover by a simple computation, we have

$$g_n\tau_Lg_n^{-1}\tau_L^{-1} = \begin{pmatrix} 1 & 0 \\ -2pm_n/(pe_n + 1) & 1 \end{pmatrix}.$$ 

Thus

$$(g_n\tau_Lg_n^{-1}\tau_L^{-1})^{-\frac{1}{2}(pe_n+1)/(pr_n+1)} = \begin{pmatrix} 1 & 0 \\ pm_n/(pr_n + 1) & 1 \end{pmatrix} \in H.$$

Call such a matrix $\tau_l$ and observe that

$$\tau_L = \begin{pmatrix} 1 + pe_n & 0 \\ 0 & 1 + pr_n \end{pmatrix} \begin{pmatrix} 1 & 0 \\ pm_n/(pr_n + 1) & 1 \end{pmatrix}.$$ 

Call the diagonal matrix $\tau_d$. Since $\tau_L, \tau_l \in H$, also $\tau_d \in H$, proving the assumption.

☐

Let $\tau \in H$. By Lemma 15 there exists $\tau_l \in H$ a lower unitriangular matrix, $\tau_D \in H$ a diagonal matrix and $\lambda \in \mathbb{Z}$ such that $\tau = \tau_l\tau_D\sigma_{\frac{p^\lambda}{n}}$ and consider the homorphism associated to $[Z] \in H^1(G, V_n[p])$. Since the cocycle $Z$ have values in $V_n[p]$, in particular $Z_{\sigma_n} \in V_n[p]$ and, by properties of cocycle, $Z_{\sigma_n} = (0, 0)$. On the other hand, since $g_n\tau_Dg_n^{-1} = \tau_D$, we have that there exists $b \in \mathbb{Z}/p^n\mathbb{Z}$ such that $Z_{\tau_D} = (0, p^{n-1}b)$. Since if $p^{n-1}b$ is distinct from 0, $(0, p^{n-1}b)$ generates $V[p]$ as $G/H$-module and $g_n\tau_1g_n^{-1} = \tau_1^{-1}$,
there exists \(a \in \mathbb{Z}/p^n\mathbb{Z}\) such that \(Z_{\tau} = (p^{n-1}a, 0)\). Observe that for every \((\alpha, \beta) \in V_n, (\tau - Id)(\alpha, \beta) = (p^{n-1}a, 0)\) only if \(p^{n-1}a = 0\). Then if the image of \(Z\) satisfies the local conditions over \(V_n\), the homomorphism associated to \(Z\) is trivial and so \(Z\) is a coboundary.

\[\square\]

**Proof of Theorem 2.** Let \(k\) be a number field and let \(E\) be an elliptic curve defined over \(k\). Set, as before, for every \(m \in \mathbb{N}, G_m\) the group \(\text{Gal}(k(E[p^m])/k)\). By Proposition 4 and Lemma 14 either case 1., or case 2., or case 3. holds.

By Corollary 6 and Lemma 10 we can find a number field \(L_1\) and an elliptic curve \(E_1\) defined over \(L_1\) such that \(\text{Gal}(k(E_1[p])/k)\) is cyclic of order dividing \(p - 1\) and its generator has an eigenvalue equal to 1.

Suppose that \(p \equiv 2 \mod (3)\). By Corollary 6 and Corollary 9 we can find a number field \(L_2\) and an elliptic curve \(E_2\) defined over \(L_2\) such that \(\text{Gal}(k(E_2[p])/k)\) is isomorphic to a subgroup of \(S_3\) of order multiple of 3 and the local-global divisibility by \(p^2\) does not hold over \(E(L_2)\).

Finally, by Corollary 6, Lemma 11 and Corollary 13, we can find a number field \(L_3\) and an elliptic curve \(E_3\) defined over \(L_3\) such that \(\text{Gal}(k(E_3[p])/k)\) is contained in a Borel subgroup and generated by \(\sigma\) of order \(p\) and \(g\) of order 2, fixing the same point of \(E_3[p]\) of order \(p\).

\[\square\]

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