Explicitly Non-Standard Uniserial Modules

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Abstract

A new construction is given of non-standard uniserial modules over certain valuation domains; the construction resembles that of a special Aronszajn tree in set theory. A consequence is the proof of a sufficient condition for the existence of non-standard uniserial modules; this is a theorem of ZFC which complements an earlier independence result.

Introduction

This paper is a sequel to [ESh]. Both papers deal with the existence of non-standard uniserial modules over valuation domains; we refer to [ESh] for history and motivation. While the main result of the previous paper was an independence result, the main results of this one are theorems of ZFC, which complement and extend the results of [ESh].

We are interested in necessary and sufficient conditions for a valuation domain $R$ to have the property that there is a non-standard uniserial $R$-module of a given type $J/R$. (Precise definitions are given below.) The question is interesting only when $R$ is uncountable, and since additional complications arise for higher cardinals, we confine ourselves to rings of cardinality $\aleph_1$. Associated to any type $J/R$ is an invariant, denoted $\Gamma(J/R)$, which is a member of a Boolean algebra $D(\omega_1)$ (equal to $\mathcal{P}(\omega_1)$ modulo the filter of closed unbounded sets). For example, if $R$ is an almost maximal valuation domain, then $\Gamma(J/R) = 0$ for all types $J/R$; but there are natural and easily defined examples where $\Gamma(J/R) = 1$.

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It is a fact that

\[ \text{if } \Gamma(J/R) = 0, \text{ then there is no non-standard uniserial } R\text{-module of type } J/R \]

(cf. [ESh] Lemma 5). In [ESh] we showed that the converse is independent of ZFC + GCH; the consistency proof that the converse fails involved the construction of a valuation domain \( R \) associated with a stationary and co-stationary subset of \( \omega_1 \) — that is, \( 0 < \Gamma(J/R) < 1 \). The existence of such sets requires a use of the Axiom of Choice; no such set can be explicitly given. Thus — without attempting to give a mathematical definition of “natural” — we could say that for natural valuation domains, \( R \), it is the case that for every type \( J/R \), \( \Gamma(J/R) \) is either 0 or 1. For natural valuation domains, it turns out that the converse is true: if there is no non-standard uniserial \( R \)-module of type \( J/R \), then \( \Gamma(J/R) = 0 \). This is a consequence of the following result which is proved below (for all valuation domains of cardinality \( \aleph_1 \)):

\[ \text{if } \Gamma(J/R) = 1, \text{ then there is a non-standard uniserial } R\text{-module of type } J/R. \]

(Theorem [2]). This vindicates a conjecture made by Barbara Osofsky in [O1], (9), p. 164. (See also the Remark following Theorem [2].)

The proof of Theorem [2] divides into several cases; the key new result which is used is a construction of a non-standard uniserial module in the essentially countable case; this construction is done in ZFC and is motivated by the construction of a special Aronszajn tree. (See Theorem [7].) Moreover, the uniserial constructed is “explicitly non-standard” in that there is an associated “special function” which demonstrates that it is non-standard. This special function continues to serve the same purpose in any extension of the universe, \( V \), of set theory, so the module is “absolutely” non-standard. In contrast, this may not be the case with non-standard uniserials constructed using a prediction (diamond) principle. (See the last section.)

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Preliminaries

For any ring \( R \), we will use \( R^* \) to denote the group of units of \( R \). If \( r \in R \) we will write \( x \equiv y \pmod{r} \) to mean \( x - y \in rR \).

A module is called uniserial if its submodules are linearly ordered by inclusion. An integral domain \( R \) is called a valuation domain if it is a uniserial \( R \)-module. If \( R \) is a valuation domain, let \( Q \) denote its quotient field; we assume \( Q \neq R \). The residue field
of $R$ is $R/P$, where $P$ is the maximal ideal of $R$. \cite{FS} is a general reference for modules over valuation domains.

If $J$ and $A$ are $R$-submodules of $Q$ with $A \subseteq J$, then $J/A$ is a uniserial $R$-module, which is said to be \textit{standard}. A uniserial $R$-module $U$ is said to be \textit{non-standard} if it is not isomorphic to a standard uniserial.

Given a uniserial module $U$, and a non-zero element, $a$, of $U$, let $\text{Ann}(a) = \{ r \in R : ra = 0 \}$ and let $D(a) = \cup \{ r^{-1}R : r \text{ divides } a \text{ in } U \}$. We say $U$ is of \textit{type} $J/A$ if $J/A \cong D(a)/\text{Ann}(a)$. This is well-defined in that if $b$ is another non-zero element of $U$, then $D(a)/\text{Ann}(a) \cong D(b)/\text{Ann}(b)$. For example, $U$ has type $Q/R$ if and only if $U$ is divisible torsion and the annihilator ideal of every non-zero element of $U$ is principal. (But notice that there is no $a \in U$ with $\text{Ann}(a) = R$.) It is not hard to see that if $U$ has type $J/A$, then $U$ is standard if and only if it is isomorphic to $J/A$. We will only consider types of the form $J/R$; it is a consequence of results of \cite{BFS} that the question of the existence of a non-standard uniserial $R$-module of type $J_1/A$ can always be reduced to the question of the existence of a non-standard uniserial of type $J/R$ for an appropriate $J$.

¿From now on we will assume that $R$ has cardinality $\aleph_1$. We always have

\[(*) \quad J = \cup_{\sigma < \omega_1} r^{-1}_\sigma R \]

for some sequence of elements $\{ r_\sigma : \sigma < \omega_1 \}$ such that for all $\tau < \sigma$, $r_\tau | r_\sigma$. If $J$ is countably generated, then $U$ is standard, so generally we will be assuming that $J$ is not countably generated; then it has a set of generators as in $(*)$, where, furthermore, $r_\sigma$ does not divide $r_\tau$ if $\tau < \sigma$.

If $\delta \in \lim(\omega_1)$, let

\[\text{(**)} \quad J_\delta \overset{\text{def}}{=} \cup_{\sigma < \delta} r^{-1}_\sigma R.\]

By results in \cite{BS} every uniserial module $U$, of type $J/R$, is described up to isomorphism by a family of units, $\{ e_\tau^\sigma : \sigma < \tau < \omega_1 \}$ such that

\[(\dagger) \quad e_\tau^\delta e_\sigma^\tau \equiv e_\delta^\sigma \pmod {r_\sigma} \]

for all $\sigma < \tau < \delta < \omega_1$. Indeed, $U$ is a direct limit of submodules $a_\sigma R$ where $\text{Ann}(a_\sigma) = r_\sigma R$; then $a_\sigma R \cong r^{-1}_\sigma R/R$ and $U$ is isomorphic to a direct limit of the $r^{-1}_\sigma R/R$, where the morphism from $r^{-1}_\sigma R/R$ to $r^{-1}_\tau R/R$ takes $r^{-1}_\sigma$ to $e_\tau^\sigma r^{-1}_\sigma$ if $a_\sigma = e_\tau^\sigma r^{-1}_\sigma r_\tau a_\tau$.

If $U$ is given by $(\dagger)$, then $U$ is standard if and only if there exists a family $\{ c_\sigma : \sigma < \omega_1 \}$ of units of $R$ such that

\[(\dagger\dagger) \quad c_\tau \equiv e_\tau^\sigma c_\sigma \pmod {r_\sigma} \]
for all $\sigma < \tau < \omega_1$. Indeed, if the family $\{c_\sigma; \sigma < \omega_1\}$ satisfying (††) exists, then multiplication by the $c_\sigma$ gives rise to isomorphisms from $r_\sigma^{-1}R/R$ to $a_\sigma R$, which induce an isomorphism of $J/R$ with $U$.

**Essentially Countable Types**

**Definition.** Suppose $J = \bigcup_{\sigma<\omega_{1}}r_{\sigma}^{-1}R$ as in (*). Call the type $J/R$ essentially uncountable if for every $\sigma < \omega$ there exists $\tau > \sigma$ such that $r_{\sigma}R/r_{\tau}R$ is uncountable. Otherwise, $J/R$ is essentially countable; this is equivalent to saying that there is a $\gamma < \omega_1$ such that for all $\gamma < \sigma < \omega_1$, $r_{\gamma}R/r_{\sigma}R$ is countable. Say that $J/R$ is strongly countable if for all $\sigma < \omega_1$, $R/r_{\sigma}R$ is countable; clearly, a strongly countable type is essentially countable.

It is easily seen that the notions of being essentially or strongly countable are well-defined, that is, independent of the choice of the representation (†). If the residue field of $R$ is uncountable, then, except in trivial cases, the types $J/R$ have to be essentially uncountable; but if the residue field is countable, the question is more delicate.

**Proposition 1** If the residue field of $R$ is uncountable, then every type $J/R$ such that $J$ is not countably generated is essentially uncountable.

**Proof.** Let $J = \bigcup_{\sigma<\omega_{1}}r_{\sigma}^{-1}R$ as in (*). It suffices to prove that if $\sigma < \tau$, then $r_{\sigma}R/r_{\tau}R$ is uncountable. But $r_{\sigma}R/r_{\tau}R \cong R/tR$ where $t = r_{\tau}r_{\sigma}^{-1} \in P$. So we have $tR \subseteq P \subseteq R$, and hence $(R/tR)/(P/tR) \cong R/P$, the residue field of $R$. Since $R/P$ is uncountable, so is $R/tR$. □

**Theorem 2** For any countable field $K$ there are valuation domains $R_1$ and $R_2$, both of cardinality $\aleph_1$ with the same residue field $K$ and the same value group, whose quotient fields, $Q_1$ and $Q_2$, respectively, are generated by $\aleph_1$ but not countably many elements, and such that $Q_1/R_1$ is essentially uncountable and $Q_2/R_2$ is strongly countable.

**Proof.** Let $G$ be the ordered abelian group which is the direct sum $\oplus_{\alpha<\omega_{1}}\mathbb{Z}\alpha$ ordered anti-lexicographically; that is, $\sum_{\alpha} n_{\alpha}\alpha > 0$ if and only if $n_{\beta} > 0$, where $\beta$ is maximal such that $n_{\beta} \neq 0$. In particular, the basis elements have their natural order and if $\alpha < \beta$, then $k\alpha < \beta$ in $G$ for any $k \in \mathbb{Z}$. Let $G^+ = \{g \in G; g \geq 0\}$.

Let $\hat{R} = K[[G]]$, that is, $\hat{R} = \{\sum_{g \in \Delta} k_gX^g; k_g \in K, \Delta$ a well-ordered subset of $G^+\}$, with the obvious addition and multiplication (cf. [O1, p. 156]). Given an element $y = \sum_{g \in \Delta} k_gX^g$ of $\hat{R}$, let $\text{supp}(y) = \{g \in \Delta; k_g \neq 0\}$; let $p\text{-supp}(y) = \{\alpha \in \omega_1; \exists g \in \text{supp}(y)$ whose projection on $\mathbb{Z}\alpha$ is non-zero}. Define $v(y)$ = the least element of $\text{supp}(y)$. If $X \subseteq G$, then $y/X$ is defined to be $\sum_{g \in X \cap \Delta} k_gX^g$. Let $y|\nu = y|\{g \in G^+; g < \nu\}$.
Let $R_1 = \{y \in \hat{R} : p\text{-supp}(y) \text{ is finite}\}$. Then $R_1$ is a valuation domain since $p\text{-supp}(xy^{-1}) \subseteq p\text{-supp}(x) \cup p\text{-supp}(y)$. Let $R_2$ be the valuation subring of $R_1$ generated by \{\(X^g : g \in G\)\}. We have $Q_j = \bigcup_{\alpha < \omega} X^{-\alpha}R_j$ for $j = 1, 2$. Now $Q_1/R_1$ is essentially uncountable since for all $\beta > \alpha$, $X^{\alpha}R_1/X^{\beta}R_1$ contains the $2^{\aleph_0}$ elements of the form

\[ \sum_{n \in \omega} \zeta(n)X^{(n+1)\alpha} \]

(with $p\text{-supp} = \{\alpha\}$) where $\zeta$ is any function: $\omega \rightarrow 2$. $R_1$ has cardinality $2^{\aleph_0}$; if $2^{\aleph_0} > \aleph_1$, to get an example of cardinality $\aleph_1$, choose a valuation subring of $R_1$ which contains all the monomials $X^g$ ($g \in G$) and $\aleph_1$ of the elements $\sum_{n \in \omega} \zeta(n)X^{(n+1)\alpha}$ for each $\alpha$.

We claim that $Q_2/R_2$ is essentially countable. Let $K[G]$ be the subring of $\hat{R}$ generated by $\{X^g : g \in G^+\}$; thus $K[G]$ consists of the elements of $\hat{R}$ with finite support; we shall refer to them as polynomials. $R_2$ consists of all elements of the form $xy^{-1}$ where $x$ and $y$ are polynomials and $v(x) \geq v(y)$. We claim that $R_2/X^{\beta}R_2$ is countable for any $\beta < \omega_1$. There are uncountably many polynomials, but we have to show that there are only countably many truncations $xy^{-1}|\beta$.

Given polynomials $x$ and $y$ with $v(x) \geq v(y)$, there is a finitely generated subgroup $\bigoplus_{1 \leq i \leq d} \mathbb{Z}\sigma_i$ of $G$ (with $\sigma_1 < \sigma_2 < \ldots < \sigma_d$) such that $x$ and $y$ are linear combinations of monomials $X^g$ with $g \in \bigoplus_{1 \leq i \leq d} \mathbb{Z}\sigma_i$. More precisely, there exist $k, r \in \omega$ and a $(k+r)$-tuple $(a_1, \ldots, a_{k+r})$ of elements of $K$ and $k + r$ linear terms $t_j$ of the form

\[ t_j = \sum_{i=1}^d n_{ij}v_i \]

($n_{ij} \in \mathbb{Z}$, $v_i$ variables) such that if we let $t_j(\sigma)$ denote $\sum_{i=1}^d n_{ij}\sigma_i$, then

\[ (z) \quad x = \sum_{j=1}^k a_jX^{t_j(\sigma)} \quad \text{and} \quad y = \sum_{j=k+1}^{r} a_jX^{t_j(\sigma)}. \]

Finally, there is $q \leq d$ such that $\sigma_q$ is maximal with $\sigma_q < \beta$.

Now, consideration of the algorithm for computing $xy^{-1}$ shows that, for fixed $(a_1, \ldots, a_{k+r})$ and $t_j$, there are linear terms

\[ s_\ell = \sum_{i=1}^d m_{i\ell}v_i \]

($m_{i\ell} \in \mathbb{Z}$, $\ell \in \omega$) and elements $c_\ell \in K$ such that for any strictly increasing sequence $\sigma = \langle \sigma_1, \ldots, \sigma_d \rangle$, if $x$ and $y$ are as in (z), then

\[ xy^{-1} = \sum_{\ell \in \omega} c_\ell X^{s_\ell(\sigma)}. \]
For any \( q \leq d \), only certain of the \( s_\ell \) involve only variables \( v_i \) with \( i \leq q \) (i.e. \( m_\ell = 0 \) if \( i > q \)); say these are the \( s_\ell \) with \( \ell \in T \) (\( T \subseteq \omega \)). If \( \sigma \) is such that \( \sigma_i < \beta \) iff \( i \leq q \), then
\[
xy^{-1}|\beta = \sum_{\ell \in T} q_\ell X^{s_\ell(\sigma)}.
\]
There are only countably many choices for \( q, d, k \) and \( r \) in \( \omega \), \( (a_1, \ldots, a_{k+t}) \in K^{k+r} \), and for \( \sigma_1 < \ldots < \sigma_q < \beta \). Therefore, there are only countably many possibilities for the truncations \( xy^{-1}|\beta \). \( \square \)

By the first part of the following, the type \( Q_2/R_2 \) of the previous theorem must be strongly countable; on the other hand, there are types which are essentially countable but not strongly countable.

**Proposition 3** (i) If \( Q/R \) is essentially countable, then it is strongly countable.

(ii) For any countable field \( K \), there is a valuation domain \( R \) with residue field \( K \) which has a type \( J/R \) which is essentially countable but not strongly countable.

**Proof.** (i) Since \( Q/R \) is essentially countable, we can write \( Q = \bigcup_{\sigma < \omega_1} r_\sigma^{-1}R \) such that for all \( \sigma < \tau \), \( r_\sigma R/r_\tau R \) is countable. We claim that \( R/r_0 R \) is countable, which clearly is equivalent to \( R/r_\tau R \) countable for all \( \tau < \omega_1 \). Suppose not. There is a \( \sigma < \omega_1 \) such that \( r_\sigma = r_0^2 t \) for some \( t \in R \) (since \( r_0^{-2} \in Q \)). But then \( r_0 R/r_\sigma R \cong R/r_0 R \), which is uncountable since \( R/r_0 R \) is uncountable, and this contradicts the choice of the \( r_\sigma \).

(ii) Let \( G = \bigoplus_{\alpha \leq \omega_1} \mathbb{Z} \alpha \), ordered anti-lexicographically. Let \( \hat{R} = K[[G]] \) (cf. proof of Theorem [2]), and let \( R \) be the smallest valuation subring of \( \hat{R} \) containing all the monomials \( X^g \) (\( g \in G \)). Let \( J = \bigcup_{\alpha < \omega_1} r_\alpha^{-1}R \) where \( r_\alpha = X^{\alpha+\omega_1} \). Then the proof that \( r_\alpha R/r_\beta R \cong R/X^{\beta-\alpha}R \) is countable for all \( \alpha < \beta \) is the same as in Theorem [2]. But \( R/r_0 R = R/X^{\omega_1}R \) is clearly uncountable. \( \square \)

**Remark.** More generally, referring to a dichotomy in [O1, Prop. 7, p. 155], if the type \( J/R \) is essentially countable and falls into Case (A), then \( J/R \) is strongly countable; if it falls into case (B), then it is not strongly countable.

**Gamma Invariants**

A subset \( C \) of \( \omega_1 \) is called a **cub** — short for closed unbounded set — if \( \sup C = \omega_1 \) and for all \( Y \subseteq C \), \( \sup Y \in \omega_1 \) implies \( \sup Y \in C \). Call two subsets, \( S_1 \) and \( S_2 \), of \( \omega_1 \) equivalent iff there is a cub \( C \) such that \( S_1 \cap C = S_2 \cap C \). Let \( \bar{S} \) denote the equivalence class of \( S \). The inclusion relation induces a partial order on the set, \( D(\omega_1) \), of equivalence classes, i.e., \( \bar{S}_1 \leq \bar{S}_2 \) if and only if there is a cub \( C \) such that \( S_1 \cap C \subseteq S_2 \cap C \). In fact, this induces a Boolean algebra structure on \( D(\omega_1) \), with least element, 0, the equivalence class of sets disjoint from a cub; and greatest element, 1, the equivalence class of sets...
containing a cub. We say \( S \) is stationary if \( \tilde{S} \neq 0 \), i.e., for every cub \( C \), \( C \cap S \neq \emptyset \). We say \( S \) is co-stationary if \( \omega_1 \setminus S \) is stationary.

Given \( R \) and a type \( J/R \), where \( J \) is as in (*), define \( \Gamma(J/R) \) to be \( \tilde{S} \), where

\[
S = \{ \delta \in \lim(\omega_1): R/\cap_{\sigma<\delta} r_\sigma R \text{ is not complete} \}
\]

where the topology on \( R/\cap_{\sigma<\delta} r_\sigma R \) is the metrizable linear topology with a basis of neighborhoods of 0 given by the submodules \( r_\sigma R \) \( (\sigma < \delta) \). This definition is independent of the choice of the representation of \( J \) as in (*) — see [ESh].

For any limit ordinal \( \delta < \omega_1 \), let

\[
\mathcal{T}_{J/R}^\delta = \{ \langle u_\sigma: \sigma < \delta \rangle: \forall \sigma < \tau < \delta (u_\sigma \in R^*, \text{ and } u_\tau - u_\sigma \in r_\sigma R) \};
\]

that is, \( \mathcal{T}_{J/R}^\delta \) consists of sequences of units which are Cauchy in the metrizable topology on \( R/\cap_{\sigma<\delta} r_\sigma R \). Let \( \mathcal{L}_{J/R}^\delta \) consist of those members of \( \mathcal{T}_{J/R}^\delta \) which have limits in \( R \), i.e.

\[
\mathcal{L}_{J/R}^\delta = \{ \langle u_\sigma: \sigma < \delta \rangle \in \mathcal{T}_{J/R}^\delta: \exists u_\delta \in R^* \text{ s.t. } \forall \sigma < \delta (u_\delta - u_\sigma \in r_\sigma R) \}.
\]

Note that \( \Gamma(J/R) = \tilde{S} \) where

\[
S = \{ \delta \in \lim(\omega_1): \mathcal{T}_{J/R}^\delta \neq \mathcal{L}_{J/R}^\delta \}.
\]

If \( J \) is not countably generated, then \( \neg \text{CH} \) implies that \( \Gamma(J/R) = 1 \), since the completion of \( R/\cap_{\sigma<\delta} r_\sigma R \) has cardinality \( 2^{\aleph_0} > \aleph_1 \). An \( \omega_1 \)-filtration of \( R \) by subrings is an increasing chain \( \{ N_\alpha: \alpha \in \omega_1 \} \) of countable subrings of \( R \) such that \( R = \bigcup_{\alpha \in \omega_1} N_\alpha \), and for limit \( \alpha \), \( N_\alpha = \bigcup_{\beta < \alpha} N_\beta \).

Define \( \Gamma'(J/R) = \tilde{E}' \) where

\[
E' = \{ \delta \in \lim(\omega_1): \exists \langle u_\sigma: \sigma < \delta \rangle \in \mathcal{T}_{J/R}^\delta \text{ s.t. } \forall f \in R^* \exists \sigma < \delta \text{ s.t. } u_\sigma f \notin N_\delta \pmod{r_\sigma} \}.
\]

Again, it can be shown that the definition does not depend on the choice of \( \{ r_\nu: \nu < \omega_1 \} \) or of \( \{ N_\alpha: \alpha < \omega_1 \} \). Notice that \( \Gamma'(J/R) \leq \Gamma(J/R) \) since if \( \mathcal{T}_{J/R}^\delta = \mathcal{L}_{J/R}^\delta \), then we can let \( f \) be a limit of \( \langle u_\sigma^{-1}: \sigma < \delta \rangle \).

In [ESh], Theorem 7 it is proved that if \( \Gamma'(J/R) \neq 0 \), then there is a non-standard uniserial \( R \)-module.

**Theorem 4** Suppose \( J/R \) is essentially countable. Then

(i) \( \Gamma'(J/R) = 0 \);

(ii) \( \Gamma(J/R) = 1 \).
Proof. Without loss of generality we can assume that \( J = \bigcup_{\sigma < \omega_1} r_\sigma^{-1} R \) where \( r_\sigma R / r_\sigma R \) is countable for all \( \sigma < \omega_1 \).

(i) We can also assume that the \( \omega_1 \)-filtration of \( R \) by subrings, \( R = \bigcup_{\alpha < \omega} N_\alpha \), has the property that for all \( \alpha \), \( N_\alpha \) contains a complete set of representatives of \( r_\sigma R / r_\sigma R \) for each \( \sigma < \alpha \). For any \( \delta \in \text{lim}(\omega_1) \), and any \( \langle u_\sigma : \sigma < \delta \rangle \) in \( T_\delta^J / R \), let \( f = u_0^{-1} \). To show that \( \delta \notin E' \), it suffices to show that \( u_\sigma f \in N_\delta \) (mod \( r_\sigma \)) for all \( \sigma < \delta \). Now \( u_\sigma f = u_\sigma u_0^{-1} \equiv 1 \) (mod \( r_0 \)), since \( u_\sigma \equiv u_0 \) (mod \( r_0 \)), by definition of \( T_\delta^J / R \). Say \( u_\sigma u_0^{-1} - 1 = y \in r_0 R \). By the assumption on \( N_\delta \), there exists \( a \in N_\delta \) such that \( y \equiv a \) (mod \( r_\sigma \)). Then \( u_\sigma f = u_\sigma u_0^{-1} \equiv 1 + y \equiv 1 + a \) (mod \( r_\sigma \)), and \( 1 + a \in N_\delta \) since \( N_\delta \) is a subring of \( R \).

(ii) To show that \( \Gamma(J / R) = 1 \), it suffices to show that for all limit ordinals \( \delta < \omega_1 \), \( R / \bigcap_{\nu < \delta} r_\nu R \) is not complete. Assuming that it is complete, we shall obtain a contradiction by showing that \( r_\delta R / r_\delta R \) is uncountable. Fix a ladder on \( \delta \), i.e., a strictly increasing sequence \( \langle \nu_n : n \in \omega \rangle \) whose sup is \( \delta \). For each function \( \zeta : \omega \to 2 \), define \( u_\zeta = \langle u_\zeta^\nu : \sigma < \delta \rangle \in T_\delta^J / R \) as follows: if \( \nu_m < \sigma \leq \nu_{m+1} \), then

\[
u_\sigma = \sum_{i \leq m} \zeta(i) r_{\nu_i}.
\]

Clearly \( u_\sigma \in r_0 R \), and if \( \tau > \sigma \), where \( \nu_k < \tau \leq \nu_{k+1} \), then \( m \leq k \) and

\[
u_\tau - \nu_\sigma = \sum_{i = m+1}^k \zeta(i) r_{\nu_i} \subset r_{\nu_{m+1}} R \subset r_\sigma R.
\]

Since \( R / \bigcap_{\nu < \delta} r_\nu R \) is assumed to be complete, for each \( \zeta \) there is an element \( u_\zeta \in R \) which represents the limit of \( \langle u_\zeta^\nu : \sigma < \delta \rangle \) in \( R / \bigcap_{\nu < \delta} r_\nu R \). To obtain a contradiction, we need only show that if \( \eta \neq \zeta \), then \( u_\zeta - u_\eta \notin r_\delta R \). Without loss of generality there exists \( m \) such that \( \zeta|m = \eta|m \) and \( \zeta(m) = 0 \), \( \eta(m) = 1 \), then

\[
u_\eta - \nu_{\nu_{m+1}} \in r_{\nu_{m+1}} R; \quad \text{and}
\]

\[
u_\zeta - \nu_{\nu_{m+1}} \in r_{\nu_{m+1}} R;
\]

but \( u_\eta - u_\zeta = r_{\nu_m} \notin r_{\nu_{m+1}} R \), so \( u_\zeta - u_\eta \notin r_{\nu_{m+1}} R \supseteq r_\delta R \). □

### Special Aronszajn trees

This section contains standard material from set theory. (See, for example, [J, §22] or [Dr] Ch. 7, §3.] It is included simply to provide motivation for the notation and proof in the next section.
A tree is a partially ordered set \((T, <)\) such that the predecessors of any element are well ordered. An element \(x\) of \(T\) is said to have height \(\alpha\), denoted \(\text{ht}(x) = \alpha\), if the order-type of \(\{y \in T: y < x\}\) is \(\alpha\). The height of \(T\) is defined to be \(\sup\{\text{ht}(x) + 1: x \in T\}\).

If \(T\) is a tree, a branch of \(T\) is a maximal linearly ordered initial subset of \(T\); the length of a branch is its order type. If \(T\) is a tree, let \(T_\alpha = \{y \in T : \text{ht}(y) = \alpha\}\). We say that a tree \(T\) is a \(\kappa\)-Aronszajn tree if \(T\) is of height \(\kappa\), \(|T_\alpha| < \kappa\) for every \(\alpha < \kappa\), and \(T\) has no branch of length \(\kappa\).

A tree \(T\) of height \(\omega_1\) is a special Aronszajn tree if \(T_\alpha\) is countable for all \(\alpha < \omega_1\) and for each \(\alpha < \omega_1\) there is a function \(f_\alpha : T_\alpha \to \mathbb{Q}\) such that

\[
\text{(S)} \quad \text{whenever } x \in T_\alpha \text{ and } y \in T_\beta \text{ and } x < y, \text{ then } f_\alpha(x) < f_\beta(y).
\]

Notice that a special Aronszajn tree is an \(\omega_1\)-Aronszajn tree, since an uncountable branch would give rise to an uncountable increasing sequence of rationals.

König’s Lemma implies that there is no \(\omega\)-Aronszajn tree. However, there is an \(\omega_1\)-Aronszajn tree:

**THEOREM 5** There is a special Aronszajn tree.

**PROOF.** Let \(<^\alpha \omega\) denote the set of all functions from \(\{\beta \in \omega_1: \beta < \alpha\}\) to \(\omega\). We shall construct \(T_\alpha\) and \(f_\alpha\) by induction on \(\alpha < \omega_1\) such that \(T_\alpha\) is a countable subset of \(<^\alpha \omega\) and the partial ordering is inclusion, i.e., if \(x \in T_\alpha\) and \(y \in T_\beta\) then \(x < y\) if and only if \(\alpha < \beta\) and \(y/\alpha = x\). Finally, \(T\) will be defined to be \(\bigcup_{\alpha < \omega_1} T_\alpha\).

Let \(T_0 = \{\emptyset\}, f_0(\emptyset) = 0, T_1 = \{0\}, \omega,\) and \(f_1 : T_1 \to \mathbb{Q}\) be onto \((0, \infty)\). Suppose now that \(T_\alpha\) and \(f_\alpha\) have been defined for all \(\alpha < \delta\) such that for all \(\sigma < \rho < \delta\):

\[
\text{(\ast)} \quad \text{for any } \epsilon > 0, \text{ and } x \in T_\sigma \text{ there is } y \in T_\rho \text{ such that } x < y \text{ and } f_\rho(y) < f_\sigma(x) + \epsilon.
\]

There are two cases. In the first case, if \(\delta\) is a successor ordinal, \(\delta = \tau + 1\), let

\[T_\delta = \{x \cup \{(\tau, n)\} : n \in \omega, x \in T_\tau\}.
\]

Define \(f_\delta\) so that for every \(x \in T_\tau\),

\[
\{f_\delta(x \cup \{(\tau, n)\}) : n \in \omega\} = \{r \in \mathbb{Q} : r > f_\tau(x)\}.
\]

Clearly (\ast) continues to hold.

In the second case, \(\delta\) is a limit ordinal. Choose a ladder \(\langle \nu_n : n \in \omega\rangle\) on \(\delta\). For each \(\sigma < \delta, x \in T_\sigma\) and \(k > 0\), by inductive hypothesis (\ast) there exists a sequence \(\langle y_n : n \in \omega\rangle\) s.t. \(\nu_n > \sigma\) such that \(y_n \in T_{\nu_n}\), \(x < y_n < y_m\) for \(n < m\) and \(f_{\nu_m}(y_n) < f_\sigma(x) + (1/k - 1/n)\).

Let \(y[\sigma, x, k] = \bigcup_{n \in \omega} y_n \in <^\delta \omega\). Let \(T_\delta\) consist of one such \(y[\sigma, x, k]\) for each \(\sigma, x, k\).

Define \(f_\delta(y[\sigma, x, k]) = f_\sigma(x) + 1/k\). Then it is clear that (\ast) still holds. \(\square\)
Special Uniserial Modules

**Definition.** Suppose $U$ is a uniserial module of type $J/R$ where $J = \bigcup_{\sigma < \omega_1} r^{-1}_{\sigma} R$ as in (*). For each $\sigma > \omega_1$, fix an element $a_{\sigma}$ of $U$ such that $\text{Ann}(a_{\sigma}) = r_{\sigma} R$ (so that the submodule $a_{\sigma} R$ of $U$ is isomorphic to $R/r_{\sigma} R \cong r_{\sigma}^{-1} R/R$). Let $I_{\sigma}$ be the set of all $R$-module isomorphisms $\varphi: a_{\sigma} R \to r_{\sigma}^{-1} R/R$. We say that $\{f_{\sigma}: \sigma \in \omega_1\}$ is a special family of functions for $U$ if for each $\sigma < \omega_1$, $f_{\sigma}: I_{\sigma} \to \mathbb{Q}$ such that whenever $\sigma < \rho$ and $\varphi \in I_{\rho}$ extends $\psi \in I_{\sigma}$, then $f_{\sigma}(\psi) < f_{\rho}(\varphi)$.

**Lemma 6** If $U$ has a special family of functions, then $U$ is non-standard.

**Proof.** Suppose there is an isomorphism $\theta: U \to J/R$. Then for every $\sigma < \omega_1$, $\theta$ restricts to an isomorphism $\varphi_{\sigma}$ of $a_{\sigma} R$ onto $r_{\sigma}^{-1} R/R$. But then $\langle f_{\sigma}(\varphi_{\sigma}): \sigma < \omega_1 \rangle$ is an uncountable strictly increasing sequence of rationals, a contradiction. □

With this lemma as justification, we will say that $U$ is explicitly non-standard if $U$ has a special family of functions.

If the uniserial module uniserial module $U$, of type $J/R$, is described up to isomorphism by a family of units, $\{e_{\rho}^\sigma: \sigma < \rho < \omega_1\}$ as in (**), then it is clear that $U$ is explicitly non-standard if and only if for every $\sigma < \omega_1$, there is a function

$$f_{\sigma}: (R/r_{\sigma} R)^* \longrightarrow \mathbb{Q}$$

such that

\[\text{whenever } \sigma < \rho \text{ and } c_{\sigma}, c_{\rho} \in R^* \text{ satisfy } c_{\rho} \equiv c_{\sigma} e_{\rho}^\sigma \pmod{r_{\sigma}}, \text{ then } f_{\sigma}(c_{\sigma}) < f_{\rho}(c_{\rho}).\]

(Here, and hereafter, we abuse notation and regard $f_{\rho}$ and $f_{\sigma}$ as functions on $R^*$.)

Note that we have a tree, $T$, such that $T_{\sigma} = (R/r_{\sigma} R)^*$ and the partial ordering is given by:

$$c_{\sigma} + r_{\sigma} R < c_{\rho} + r_{\rho} R \iff \sigma < \rho \text{ and } c_{\rho} \equiv c_{\sigma} e_{\rho}^\sigma \pmod{r_{\sigma}}.$$  

Assume $\sigma < \rho$. Each $c_{\sigma}$ has at least one successor of height $\rho$, namely $c_{\sigma} e_{\rho}^\sigma$, and if $r_{\sigma} R/r_{\rho} R$ is countable, then $c_{\sigma}$ has only countably many successors of height $\rho$. For each $c_{\rho} \in T_{\rho}$, its unique predecessor in $T_{\sigma}$ is $c_{\rho}(e_{\rho}^\sigma)^{-1}$. (Here again we abuse notation and write, for example, $c_{\rho}$ for an element of $T_{\rho}$ instead of $c_{\rho} + r_{\rho} R$.)

Without loss of generality we can assume that the $r_{\sigma}$ are such that for all $\sigma < \omega_1$, $r_{\sigma} R/r_{\sigma+1} R$ is infinite. (Just choose a subsequence of the original $r_{\sigma}$’s if necessary.) Thus for all $\sigma < \omega_1$, there is an infinite subset $W_{\sigma}$ of $R^*$ such that for all $u \neq v \in W_{\sigma}$, $u \equiv 1 \pmod{r_{\sigma}}$ and $u \neq v \pmod{r_{\sigma+1}}$. 


Theorem 7 If $J/R$ is an essentially countable type, then there is an explicitly non-standard uniserial $R$-module of type $J/R$.

Proof. We will first give the construction in the case when $J/R$ is strongly countable, and afterward indicate the modifications needed for the general case. Thus $T_\sigma$ is assumed countable for all $\sigma < \omega_1$.

We will define, by induction on $\delta$, $e_\sigma^\rho$ for $\sigma < \tau < \delta$ as in (†) and, at the same time, the maps $f_\sigma: (R/r_\sigma R)^* \to \mathbb{Q}$ for $\sigma < \delta$. We will do this so that ($\S$) holds and the following condition is satisfied for all $\sigma < \rho < \omega_1$:

\[\text{for any } \epsilon > 0, \ m \in \omega, \ c_\sigma^j \in \mathcal{T}_\sigma, \text{ and } c_\rho^j \in \mathcal{T}_\rho \ (j = 1, \ldots, m)\]

\[\text{such that } c_\sigma^j < c_\rho^j, \text{ there exists } u \in R^* \text{ such that } u \equiv 1 \pmod{r_\sigma} \text{ and } f_\rho(uc_\rho^j) < f_\sigma(c_\rho^j) + \epsilon \text{ for all } j = 1, \ldots, m.\]

Note that the $c_\sigma^j$ determine the $c_\sigma^j - c_\rho^j \equiv c_\rho^j(e_\rho^\sigma)^{-1} \pmod{r_\sigma}$ — and $uc_\rho^j$ is another successor of $c_\rho^j$ of height $\rho$. For any given $\epsilon > 0$, $\sigma < \rho < \delta$, $m \in \omega$, and $c_\rho^j \in R^*$ ($j = 1, \ldots, m$), there exist infinitely many $u$ as in ($\star \star \sigma, \rho$), since we can decrease $\epsilon$ as much as we like.

Suppose we have defined $e_\rho^\sigma$ and $f_\sigma$ for all $\sigma < \rho < \delta$ satisfying the inductive hypotheses. Let $\langle u_n : n \in \omega \rangle$ enumerate representatives of all the elements of $(R/r_\delta R)^*$. Also, let $\langle \theta_q : q \in \omega \rangle$ enumerate all instances of ($\star \star \sigma, \delta$), for all $\sigma < \delta$, with each instance repeated infinitely often. More precisely, we enumerate (with infinite repetition) all tuples of the form

\[\langle \epsilon = \frac{1}{n}, \ \sigma, \ c_\delta^j + r_\delta R; j = 1, \ldots, m \rangle\]

with $n \in \omega \setminus \{0\}, \ \sigma < \delta$, and $c_\delta^j \in R^*$.

We will define $f_\delta$ as the union of a chain of functions $f_{\delta,k}$ into $\mathbb{Q}$, each with a finite domain. When $k$ is even we will concentrate on insuring that the domain of $f_\delta$ will be $T_\delta$; and when $k$ is odd, we will work at satisfying the conditions ($\star \star \sigma, \delta$).

Suppose first that $\delta = \tau + 1$ and define $e_\tau^\delta = 1$ and $e_\sigma^\delta = e_\sigma^\tau$ for $\sigma < \tau$. Suppose that $f_{\delta,i}$ has been defined for $i < k$, and assume first that $k$ is even. Let $n$ be minimal such that $u_n \notin \text{dom}(f_{\delta,k-1})$. Let $\text{dom}(f_{\delta,k}) = \text{dom}(f_{\delta,k-1}) \cup \{u_n\}$ and let $f_{\delta,k}(u_n)$ be any rational greater than $f_\tau(u_n(e_\tau^\delta)^{-1})$ ($= f_\tau(u_n)$).

Now suppose $k$ is odd; say $k = 2q + 1$. It’s easy to see that it’s enough to construct $f_\delta$ to satisfy ($\star \star \tau, \delta$). So if $\theta_q$ is an instance of ($\star \star \sigma, \delta$) for $\sigma < \tau$, let $f_{\delta,k} = f_{\delta,k-1}$. Otherwise, suppose $\theta_q$ is the instance of ($\star \star \tau, \delta$) given by

\[\frac{1}{n} \ c_\delta^j + r_\delta R; j = 1, \ldots, m.\]
Since \( W \) is infinite (see above), there is a unit \( u \) such that \( u \equiv 1 \pmod{r_\tau} \) and \( uc_\delta^j \notin \text{dom}(f_{\delta,k-1}) \) for \( j = 1, \ldots, m \). Then define \( f_{\delta,k} \) to be the extension of \( f_{\delta,k-1} \) with domain \( \text{dom}(f_{\delta,k-1}) \cup \{ uc_\delta^j : j = 1, \ldots, m \} \) such that

\[
f_{\delta,k}(uc_\delta^j) = f_\tau(c_\delta^j(e_\tau^j)^{-1}) + \frac{1}{2n}.
\]

Now we consider the case where \( \delta \) is a limit ordinal. Fix a ladder \( \langle \nu_n : n \in \omega \rangle \) on \( \delta \). We are going to define units \( e_\nu^\delta \) by induction such that \( e_\nu^\delta \equiv e_\nu^\delta e_\nu^\delta (\text{mod } r_\nu) \) whenever \( n < m < \omega \). This will easily determine the sequence \( \langle e_\sigma^\delta : \sigma < \delta \rangle \) such that for all \( \sigma < \tau < \delta \), \( e_\sigma^\delta \equiv e_\sigma^\delta e_\tau^\delta (\text{mod } r_\tau) \); then \((\dagger)\) will be satisfied for \( \langle e_\sigma^\delta : \sigma < \tau \leq \delta \rangle \).

For simplicity of notation, let \( e_n \) denote \( e_{\nu_n}^\delta \). Suppose we’ve already defined \( f_{\delta,k-1} \) and \( e_k \) such that for all \( x \in \text{dom}(f_{\delta,k-1}) \),

\[
f_{\nu_k}(xe_k^{-1}) < f_{\delta,k-1}(x).
\]

(Recall that if \( x \in T_\delta \), then \( xe_k^{-1} \) is the unique predecessor of \( x \) in \( T_{\nu_k} \).) If \( k \) is even, we proceed as in the even case above (when \( \delta \) is a successor). If \( k = 2q + 1 \) and \( \theta_q \) is

\[
\langle \frac{1}{n}, \sigma, c_\delta^j + r_\delta R : j = 1, \ldots, m \rangle
\]

we can assume — since each instance is repeated infinitely often — that \( \sigma < \nu_k \). Thus \( e_\sigma^\delta = e_{\nu_k}^\delta e_\sigma^\delta \) is defined. Note that \( c_\delta^j(e_\sigma^\delta)^{-1} \equiv c_\delta^j e_k^{-1}(e_\sigma^\delta)^{-1} (\text{mod } r_\sigma) \), so we can apply \((**x)\) [with \( c_\nu^j = c_\delta^j e_k^{-1} \)] and obtain a unit \( w \equiv 1 \pmod{r_\sigma} \) such that for all \( j = 1, \ldots, m \)

\[
f_{\nu_k}(wc_\delta^j e_k^{-1}) < f_{\sigma}(c_\delta^j(e_\sigma^\delta)^{-1}) + \frac{1}{2n}.
\]

Moreover, since there are infinitely many such \( w \), we can choose one so that the elements \( wc_\delta^j (j = 1, \ldots, m) \) do not belong to \( \text{dom}(f_{\delta,k-1}) \). Let these be the new elements of the domain of \( f_{\delta,k} \) and define

\[
f_{\delta,k}(wc_\delta^j) = f_{\sigma}(c_\delta^j(e_\sigma^\delta)^{-1}) + \frac{1}{2n}.
\]

Now we will define \( e_{k+1} \) (for \( k \) odd or even). For each \( x \in \text{dom}(f_{\delta,k}) \) we have committed ourselves to \( f_{\delta}(x) = f_{\delta,k}(x) \) and to the predecessor of \( x \) in \( T_{\nu_k} \) \( (= xe_k^{-1}) \); we need to choose \( e_{k+1} \) so that \( x \) and its predecessor, \( xe_k^{-1} \), in \( T_{\nu_k+1} \) satisfy \((\S)\).

Let \( e' = e_k(\nu_k+1)^{-1} \). The desired element \( e_{k+1} \) will have the form \( ue' \) for some unit \( u \equiv 1 \pmod{r_{\nu_k}} \). Choose \( e' < f_{\delta,k}(x) - f_{\nu_k}(xe_k^{-1}) \) for each \( x \in \text{dom}(f_{\delta,k}) \). Apply \((**x)\) to this \( e' \) and \( xe_k^{-1} \in T_{\nu_k}, xe_k^{-1} \in T_{\nu_k+1} \) \( (x \in \text{dom}(f_{\delta,k})) \). (Note that \( xe_k^{-1} < xe_k^{-1} \) by choice of \( e' \).) This gives us \( v \equiv 1 \pmod{r_{\nu_k}} \) such that for all \( x \)

\[
f_{\nu_k+1}(vx e^{-1}) < f_{\nu_k}(xe_k^{-1}) + e' < f_{\delta,k}(x).
\]
Then we let $e_{k+1} = v^{-1}e'$, and we have completed the inductive step.

This completes the proof in the strongly countable case. We turn now to the general (essentially countable) case. In this case, $R/r_0 R$ may be uncountable; let $Z$ be a complete set of representatives of $(R/r_0 R)^*$. Fix $z_0 \in Z$. We first define, by induction on $\sigma$, $f_\sigma(c_\sigma)$ — or, more precisely, $f_\sigma(c_\sigma + r_\sigma R)$ — for all $c_\sigma \in R^*$ such that $c_\sigma \equiv z_0 e_0^\sigma \pmod{r_0}$. We do the construction exactly as in the previous strongly countable case; this will work since there are only countably many cosets $c + r_\sigma R$ such that $c \equiv z_0 e_0^\sigma \pmod{r_0}$ since $r_0 R/r_\sigma R$ is countable.

Having done this, the $e_\sigma^\sigma$ are determined. We claim that there is no family $\{c_\sigma: \sigma < \omega_1\}$ satisfying (††). Indeed, suppose we had such a family. Let $z \in Z$ be such that $c_0 \equiv z \pmod{r_0}$. Then for all $\sigma < \omega_1$, $c_\sigma \equiv ze_0^\sigma \pmod{r_0}$. Hence the family $\{z_0 z^{-1}c_\sigma: \sigma < \omega_1\}$ satisfies (††) and also satisfies $z_0 z^{-1}c_\sigma \equiv z_0 e_0^\sigma \pmod{r_0}$; but this is impossible by construction. □

**Consequences**

Now we consider some of the general consequences, for the question of the existence of non-standard uniserials, of the results of the previous sections. First of all, we can construct non-standard uniserial modules associated to any residue field of cardinality $\leq \aleph_1$.

**Proposition 8** (i) For any countable field $K$, there exists a valuation domain $R$ of cardinality $\aleph_1$ with residue field $K$ such that there is an explicitly non-standard uniserial module of type $Q/R$.

(ii) For any field $K$ of cardinality $\leq \aleph_1$, there exists a valuation domain $R$ of cardinality $\aleph_1$ with residue field $K$ such that there is a non-standard uniserial module of type $Q/R$.

**Proof.** Part (i) is an immediate consequence of Theorem 2 and Theorem 4. Part (ii) follows from (i) in the case of a countable $K$ and from the Osofsky construction in the case of an uncountable $K$ (cf. [O1]; see also [ESh, Theorem 11]). □

The following improves [ESh, Corollary 15], in that it is a theorem of ZFC rather than a consistency result. It shows that the condition $\Gamma'(J/R) > 0$ is not necessary for the existence of a non-standard uniserial of type $J/R$.

**Proposition 9** There is a valuation domain $R$ of cardinality $\aleph_1$ such that $\Gamma'(Q/R) = 0$ and there is a non-standard uniserial $R$-module of type $Q/R$. 
Proof. Let $R$ be such that $Q/R$ is essentially countable (cf. Theorem 2). By Theorem (i), $\Gamma'(Q/R) = 0$, but there is a non-standard uniserial $R$-module of type $Q/R$ by Theorem 7.  □

The following sums up some old results which we want to combine with results proved here.

**Theorem 10** Suppose that $R$ is a valuation domain of cardinality $\aleph_1$.

(i) If CH does not hold and $J/R$ is an essentially uncountable type, then there is a non-standard uniserial $R$-module of type $J/R$.

(ii) If CH holds and $\Gamma(J/R) = 1$, then there is a non-standard uniserial $R$-module of type $J/R$.

Proof. Part (i) is Theorem 8 of [ESh]. Part (ii) is because the weak diamond principle, $\Phi_{\omega_1}(\omega_1)$, is a consequence of CH (see [DSv]) and this implies that there exists a non-standard uniserial of type $J/R$ when $\Gamma(J/R) = 1$ (see [ESh, Proposition 3] or [FrG]). □

Now we can completely handle the cases when either CH fails, or $\Gamma = 1$.

**Theorem 11** If CH does not hold, then for every valuation domain $R$ of cardinality $\aleph_1$, and every type $J/R$ such that $\Gamma(J/R) \neq 0$, there is a non-standard uniserial $R$-module of type $J/R$.

Proof. Use Theorem 7 for the essentially countable case, and Theorem 10(i) otherwise. □

Remark. This result shows that CH is needed for the independence result in [ESh, Thm. 14].

**Theorem 12** For every valuation domain $R$ of cardinality $\aleph_1$ and every type $J/R$, if $\Gamma(J/R) = 1$, then there is a non-standard uniserial $R$-module of type $J/R$.

Proof. If CH fails, use the previous theorem. If CH holds, use Theorem 10(ii). □

Remark. Osofsky’s original conjecture ([O1], (9), p. 164), restricted to valuation domains of cardinality $\aleph_1$, said — in our notation — that there is a non-standard uniserial $R$-module of type $J/R$ if and only if $\Gamma(J/R) = 1$. This is now seen to be true assuming $\neg$CH. On the other hand, it cannot be true in this form assuming CH, since CH implies the weak diamond principle for some co-stationary subsets of $\omega_1$ (cf. [EM, VI.1.10]). Indeed, as in the proof of [ESh, Prop. 3] it is possible to construct $R$ with a type $J/R$ where $\Gamma(J/R) = S$ and $\Phi_{\omega_1}(S)$ holds; so there is a non-standard uniserial $R$-module of type $J/R$ (cf. [ESh, proof of Prop. 3]). On the other hand, to construct such an $R$
one has to begin with the stationary and co-stationary set $S$, so such rings will not be “natural”, i.e. will not be ones ordinarily met in algebraic contexts.

Recall, from [ESh], that it is in the case when the hypotheses of the previous theorems fail — i.e., when CH holds and $\Gamma(J/R) < 1$ (and non-zero) — that the independence phenomena occur.

**Absoluteness**

Finally, let us briefly discuss absoluteness. Consider Theorem 11; if CH fails and $\Gamma(J/R) \neq 0$, we always have a non-standard uniserial module of type $J/R$, but there are two separate constructions involved. In one case, when $J/R$ is essentially countable, we construct an explicitly non-standard uniserial. If the universe of set theory is extended to a larger universe (with the same $\aleph_1$) this module remains non-standard because the special family of functions remains a special family for $U$ in the extension of the universe. In the essentially uncountable case we use the fact that $\Gamma'(J/R) = 1$ ( [ESh, Theorem 8]) and construct our non-standard uniserial $U$ as in [ESh, Theorem 7]. In this case too $U$ remains non-standard in an extension of the universe (preserving $\aleph_1$). The reason here is more subtle; relative to a fixed $\omega_1$-filtration of $R$ by subrings, $N_\alpha$, the $e_\sigma^\rho$ we construct satisfy the following property for every $\delta \in \lim(\omega_1)$ and every $c \in R^*$:

$$(\#_{c,\delta}) \quad \forall(c_\sigma: \sigma < \delta) \in \delta N_\delta^* [\exists \sigma < \delta (\forall t \in R(c - c_\sigma e_\delta^\delta \neq r_\sigma t))]$$

It is a theorem of ZFC that if $(\#_{c,\delta})$ holds for $U$ (defined by the $e_\sigma^\rho$) for all $c, \delta$, then $U$ is non-standard. Now $(\#_{c,\delta})$ is, by a coding argument, a $\Pi^1_1$ statement (with parameters in the ground model) about $\omega$. Hence, by a theorem of Mostowski (cf. [Dr, Thm. 7.13, p. 160]), it remains true in an extension of the universe, so $U$ remains non-standard.

On the other hand, in the proof of Theorem 12, there is one additional case: when $J/R$ is essentially uncountable and $\Gamma'(J/R) = 0$ (so CH holds). In this case the existence of a non-standard uniserial is proved using the weak diamond principle, which is a consequence of CH. Here the $U$ we construct may not remain non-standard in an extension of the universe. Consider for example that $R$ is constructed as in [ESh, Theorem 14], but with $\Gamma(J/R) = 1$. If $P$ is the forcing defined in the proof there, then $P$ is proper, so it preserves $\aleph_1$ and, moreover, in the $P$-generic extension $U$ is standard. (Of course, in the generic extension we can construct another non-standard module.)
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