A self-sustained traversable scale-dependent wormhole

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A self-sustained traversable wormhole is obtained as a vacuum solution of a scale-dependent gravitational theory. Comparison with other approaches towards wormhole self-sustainability are presented, with emphasis on the running of the gravitational coupling and on a possible effective description of gravity near the Planck scale.

I. INTRODUCTION

Wormholes are bridges between different universes or different parts of the same universe. They were first recognized by Flamm, who found that the Schwarzschild solution of can be thought as representing a wormhole 1. Later, Einstein and Rosen 2 developed a model for an elementary particle consisting in a throat connecting two sheets. After the “mass without mass”–like elaborations of Wheeler and Misner in their Geometrodynamics 3, the field experimented a renaissance due to the work of Morris and Thorne 4, 5. In the last 20 years, the book by Visser 6 has been considered as an appropriate and authoritative reference in the field. Very recently, Lobo 7 has also authoritative updated on the state–of–the–art on wormhole physics. Therefore, after having a look at this timeline, one can conclude that we are living a new renaissance on wormholes and related physics.

Let us focus our interest in traversable wormholes. As it is well known, the problem with them is that they violate the classical energy conditions, serving primarily as useful probes of the foundations of General Relativity. The kind of matter which allows traversable wormholes is called exotic. As a consequence, quantum effects must be considered in order to solve the traversability problem. As the full theory of quantum gravity is still lacking, many different works have been devoted to get some insight into the underlying physics (for an incomplete list check 8–23 and for a review see 24). Despite the fact that in those works the authors discuss different aspects of quantum gravity, most of them have the common feature that the resulting effective gravitational action acquires a scale–dependence. This behaviour is observed through the couplings of the effective action: they change from fixed values to scale–dependent quantities, i.e. \( \{G_0, \Lambda_0\} \mapsto \{G_k, \Lambda_k\} \), where \( G_0 \) is Newton’s coupling and \( \Lambda_0 \) is the cosmological coupling. Indeed, there is some evidence which supports that this scaling behaviour is consistent with Weinberg’s Asymptotic Safety program 25–32. This effective action which appears when running couplings are assumed has been studied in three–dimensional space–times in the context of black hole physics 33–37 as well as in four dimensions 38–39. In these works, the corresponding scale–dependent couplings take into account a quantum effect in the sense that this approach admits corrections to the classical black hole backgrounds.

From the point of view of Semiclassical General Relativity, self–consistent solutions to the semiclassical Einstein’s equations corresponding to a Lorentzian wormhole coupled with a quantum scalar field have been considered by Hochberg et al. 40 and by Khusnutdinov and Sushkov 41. Regarding self–sustainability,Garatini fixed the attention on wormholes which are totally supported by their own quantum fluctuations (see the original works 42–44 and also 45 and references therein for a recent account of these kind of wormholes). By studying the one–loop contribution of the gravitons to the total energy, which is quite similar to computing the Casimir energy on a fixed background, he found a self–consistent source for a traversable wormhole 42. An important feature of this self–sustainability lies in the fact that a renormalized energy–dependent Newton’s gravitational constants appears as a consequence of considering effective Einstein’s equations coming from the fluctuations of the Einstein tensor. Therefore, in this sense, an effective action description of self–sustainability would be possible. This description, within a scale–dependent gravitational setting, is the purpose of the present work.

The work is organized as follows: In Sect. II we give a brief review on traversable wormholes in Einstein’s gravity. Section III summarizes the gravitational scale–dependent setting which is employed in Sect. IV to obtain a self–sustained wormhole with the Schwarzschild spatial part of the metric. Finally, discussion and concluding remarks are given in Sects. V and VI respectively.

II. TRAVERSABLES WORMHOLES IN EINSTEIN’S GRAVITY

Let us consider a Morris–Thorne wormhole 44, which is one of the simplest traversable wormholes. It can be described by a static and spherically symmetric line element
given by
\[ ds^2 = -e^{2A(r)}dt^2 + \frac{dr^2}{1 - \frac{B(r)}{r}} + r^2 d\Omega^2. \]  
(1)

For a static observer, the only nonzero components of the stress-energy tensor are
\[ T^t_t = \rho(r), \quad T^r_r = -\tau(r), \quad T^\theta_\theta = T^\phi_\phi = P(r), \]  
(2)

where \( \rho(r) \) is the total density of mass–energy, \( \tau(r) \) is the tension per unit of area in the radial direction and \( P(r) \) is the pressure in lateral directions.

With the above parametrization of the line element and the choice (2) for the matter content, the Einstein’s field equations lead to
\[ \rho = \frac{B'}{8\pi r^2}, \]  
(3)
\[ \tau = \frac{B/r - 2(r-B)A'}{8\pi r^2}, \]  
(4)
\[ P = \frac{r}{2}\left(\rho - \tau\right)A' - \tau. \]  
(5)

The former equations suggest that, for a suitable choice of the functions \( A(r) \) and \( B(r) \), we can obtain the matter content for our wormhole. However, the functions \( A(r) \) and \( B(r) \) are not arbitrary but they must fulfill some constraints in order to obtain a traversable wormhole. For example, if there is no cutoff in the stress–energy we must demand that \[ \frac{B}{r} \to 0, \]  
(6)
\[ A(r) \to 0, \]  
(7)
as \( r \to \infty \). Furthermore, the requirement that a traversable wormhole does not possess any horizon corresponds to demand \( A(r) \) to be finite everywhere.

As an example, we will briefly comment on two types of traversable wormholes of ultrastatic \[ \text{(1)} \]  
\[ \text{(2)} \] type. First, the prototype of traversable wormhole, which is the Ellis–Bronnikov one \[ \text{(3)} \]  
\[ \text{(4)} \]. This wormhole has a shape function given by \( B(r) = r_0^2/r \) (with \( r_0 \) constant) and \( A(r) = 0 \). Note that, although these wormholes were thought to be unstable \[ \text{(5)} \]  
\[ \text{(6)} \], rotation might possibly stabilize them \[ \text{(7)} \]. Even more, Bronnikov et al. have shown \[ \text{(8)} \] that a perfect fluid with negative density and a source-free radial electric or magnetic field (for a certain class of fluid equations of state) allows linear stability for the Ellis–Bronnikov solution under both spherically symmetric perturbations and axial perturbations of arbitrary multipolarity (see also Bronnikov’s study on Chapter 7 of \[ \text{(9)} \]). Very recently, and in analogy with black holes \[ \text{(10)} \], uniqueness theorems for the Ellis–Bronnikov wormhole supported by a phantom scalar field has been proven both in four \[ \text{(11)} \] and in higher–dimensional cases \[ \text{(12)} \]. Concerning the study of gravitational lensing of wormholes, due to their astrophysical importance, the deflection of light for Ellis–Bronnikov wormholes was initially computed in \[ \text{(13)} \]. Other authors have extended the study of these kind of signatures both in non–rotating \[ \text{(14)} \] \[ \text{(15)} \] and rotating Ellis–Bronnikov wormholes \[ \text{(16)} \].

Second, let us consider a wormhole with \( A(r) = 0 \) and \( B(r) = r_0 \). As pointed out by Morris and Thorne \[ \text{(17)} \], the parameter \( \xi = \frac{r}{r_0} - 1 \) quantifies the amount of exotic material needed to sustain the wormhole. In this particular case, although the exotic material decays rapidly with radius, \( \xi \) is positive and huge. In this sense, the authors of Ref. \[ \text{(18)} \] point out that this situation, which implies the use of exotic material throughout all the wormhole, is extremely unpleasing. Given this unpleasant feature of these wormholes when interpreted within General Relativity, in this work we will show that they are an exact vacuum solution of a particular scale–dependent gravity. Therefore, no exotic matter but a modified gravitational theory is implemented in order to obtain a self–sustained wormhole solution of this type.

### III. SCALE–DEPENDENT GRAVITY

As commented in the introduction, one possible way of introducing an effective gravitational theory beyond General Relativity is by promoting both the Newton and the cosmological constants to scale–dependent quantities.

In the following, the scale–setting presented will follow closely the spirit and concept of Ref. \[ \text{(19)} \].

The scale–dependent Einstein–Hilbert effective action reads
\[ \Gamma[g_{\mu\nu}, k] = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi G_k} (R - 2\Lambda_k) + \mathcal{L}^M_k \right], \]  
(8)
where \( G_k \) and \( \Lambda_k \) stand for the scale–dependent gravitational and cosmological coupling, respectively, and \( \mathcal{L}^M_k \) is the Lagrangian density for the matter content.

After performing variations with respect to the metric field \( g_{\mu\nu} \), we obtain the modified Einstein’s field equations
\[ G_{\mu\nu} + \mu g_{\mu\nu} = 8\pi G_k T^{\text{eff}}_{\mu\nu}, \]  
(9)
where \( T^{\text{eff}}_{\mu\nu} \) is the effective energy–momentum tensor, defined as
\[ T^{\text{eff}}_{\mu\nu} := (T^M_k)_{\mu\nu} - \frac{1}{8\pi G_k} \Delta t_{\mu\nu}. \]  
(10)

In Eq. \[ \text{(11)} \], \( (T^M_k)_{\mu\nu} \) is the matter energy–momentum tensor and \( \Delta t_{\mu\nu} \) is given by
\[ \Delta t_{\mu\nu} = G_k (g_{\mu\nu} \Box - \nabla_{\mu} \nabla_{\nu}) G_k^{-1}. \]  
(11)

As discussed previously in Ref. \[ \text{(12)} \], the renormalization scale \( k \) is not constant anymore. Therefore, the
stress energy tensor is likely not conserved. This kind of
problem has been considered in the context of renormal-
ization group improvement of black holes in asymptotic
safety scenarios (see, for instance 65–68 and references
therein).

One can circumvent this problem by applying the vari-
tional scale–setting procedure described in Ref. 38,
where the Eqs. (19) are complemented by an equation ob-
tained performing variations with respect to the scale–
field, \( k(x) \):

\[
\frac{d}{dk} \Gamma[g_{\mu\nu}, k] = 0. \tag{12}
\]

However, if the precise beta functions of the problem are
not known, Eqs. (8) and (12) do not provide enough in-
formation in order to find both \( g_{\mu\nu}(x) \) and \( k(x) \). One
can solve this problem by considering that the couplings
\( \{G_k, \Lambda_k\} \) depend explicitly on space–time coordinates,
a dependence which is inherited from the space-time de-
pendence of \( k(x) \). Promoting Newton’s coupling to a space–dependent field, \( G(x) \), and find-
ing wormhole solutions for this modified theory, is the
purpose of the following section. Note that this scale–
dependent gravity corresponds to an effective Brans–
Dicke theory but without a kinetic term. In this sense,
\( G(x) \) does not have dynamics.

IV. SELF–SUSTAINED SCALE–DEPENDENT
SOLUTION

The modified vacuum Einstein’s equations without cos-
mological term are given by \( S_{\mu\nu} = G_{\mu\nu} + \Delta t_{\mu\nu} = 0 \),
where

\[
S^I_\mu = -2G^2(r)B^I(r) + 4r(r-B(r))(G'(r))^2 +
+ G(r) \left( 3B(r) + r(-4 + B'(r)) \right) G'(r)
+ 2r(-r + B(r))G''(r) \tag{13}
\]

\[
S^r_\mu = G(r) \left( -B(r) + 2r(r-B(r))A'(r) \right)
+ r(-r + B(r)) (2 + rA'(r)) G'(r) \tag{14}
\]

\[
S^0_\mu = S^0_\phi = B(r) \left( -4r^2G^2(r)^2 - G^2(r) \right) \left( 1 + r \left( A'(r) \right) \right)
+ 2r(A'(r))^2 + 2rA''(r) \right) + rG(r) \left( 1 + 2rA'(r) \right) \times
\times G'(r) + 2G''(r) \right) \right) \right) r \left( 4r^2(G'(r))^2 + G^2(r) \right) \left( 1 +
\times \left( rA'(r) \right) \left( 2rA'(r) - B'(r) \right) + 2r^2A''(r) \right) + rG(r) \times
\times \left( -2 - 2rA'(r) + B'(r) \right) G'(r) + 2rG''(r) \right) \right) \tag{15}
\]

Note that the equations are highly coupled. However, the
following protocol can be implemented in order to look
for some solutions. First, solve for \( G'(r) \) from Eq. (14).
Second, substitute \( G'(r) \) in Eq. (13) and then solve for
\( G''(r) \). With these algebraic identities, Eq. (14) results in

\[
A'(r) \left( -B(r) (6 + rA'(r) (1 + rA'(r))) \right)
+ r(rA'(r) (2 + rA'(r) - B'(r)) - 2(-8 + B'(r))) \right)
+ 8B'(r) + 2(r - B(r))(4 + rA'(r)) A''(r) = 0. \tag{16}
\]

Surprisingly, Eq. (16) does not contain \( G(r) \). Even more,
one possible solution is given, by inspection, by

\[
A(r) = A_0 \tag{17}
B(r) = B_0. \tag{18}
\]

Note that, given this choice for the metric, all the equa-
tions (where \( G(r) \) is the only unknown) can be consist-
tsily solved leading to

\[
G(r) = \frac{G_0}{1 - \frac{r}{r_0}}. \tag{19}
\]

where \( G_0 \) is the classical Newton’s constant. It is
worth noticing that, as the redshift is constant \( (A_0) \),
the radial tidal acceleration felt by an observer trying
to traverse the wormhole is zero. On the contrary, the
transversal tidal acceleration essentially depends on the
velocity with which the observer traverses the wormhole
[4, 6, 7].

V. DISCUSSION

At this point, a number of comments are in order.
First, note that the spatial part of the obtained worm-
hole is similar to that of a Schwarzschild wormhole (the
Schwarzschild redshift is somehow incorporated in the ef-
fective \( G(r) \)). Second, in the context of scale–dependent
gravity, the wormhole throat, \( B_0 \), can be inter-
preted as the so–called running parameter, which controls
the strength of the scale–dependence. [33–35, 37, 39]. In other
words, when the running parameter is turned off, \( B_0 \to 0 \),
the classical solution is recovered, and \( G(r) \to G_0 \)
Even more, this limit corresponds to Minkowski space-
time, as can be easily checked. In this sense, the solution
here presented can be considered to be self–sustained by
a scale–dependent gravitational theory where the effec-
tive Newton’s constant is given by Eq. (19). Third, as
in general the scale–dependent effects are assumed to be
weak [37], it is reasonable to treat the running param-
eter, which we recall is encoded in \( B_0 \), as small with
respect to the other scales entering the problem. There-
fore, the effects of the running of \( G(r) \) [71] are expected to
be noticeable only near the throat. Specifically, as
\( [B_0] = L \) and \( [G_0] = [L]^{1/2} \) when \( c = h = 1 \), we get
that \( B_0 < \sqrt{G_0} = \ell_p \). Then, provided scale–dependent
gravity can be considered as an effective model for quantum gravity in some sense, the (trans)-planckian bound obtained for $B_0$ is consistent and, even more, it is in agreement with [42, 43]. Within this interpretation, no violation of energy conditions appears, since we are dealing with a vacuum spacetime, but a modified gravity emerges. In fact, as pointed out in Ref. [2]: “in the context of modified theories of gravity, it is shown that the higher-order curvature terms, interpreted as a gravitational fluid, can effectively sustain wormhole geometries, while the matter threading the wormhole can be imposed to satisfy the energy conditions”. In our case, the matter content which Ref. [7] refers to is the vacuum and the new gravity is not given by higher-order curvature terms but by the scale-dependence. Therefore, scale-dependent gravity provides a possible realization of the previous claim. Concerning the self-sustainability of the wormhole note that, one hand, in the approach of Refs. [42, 43], the effective Einstein’s equations are given by

$$G_{\mu\nu} = -\langle\Delta G_{\mu\nu}(\bar{g}_{\mu\nu}, h_{\mu\nu})\rangle^{ren},$$

where $-\langle\Delta G_{\mu\nu}(\bar{g}_{\mu\nu}, h_{\mu\nu})\rangle^{ren}$ is an effective energy–momentum tensor which appears as a consequence of a one-loop renormalization procedure over a fixed wormhole background given by $g_{\mu\nu}$ ($\bar{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$). Moreover, given the fact that an arbitrary mass scale, $\mu$, emerges unavoidably in any regularization scheme, a scale–dependent running gravitational coupling appears. The specific running obtained in Ref. [12] reads

$$G(\mu) = \frac{G_0}{1 + KG_0 \ln(\mu/\mu_0)},$$

where $K$ is a constant related to the background geometry and $\mu_0$ is the normalization point. On the other hand, within our approach, the effective Einstein’s equations are given by

$$G_{\mu\nu} = -\Delta t_{\mu\nu},$$

where the effective energy–momentum tensor appears when the scale–dependence can not be avoided anymore. Therefore, one can conclude that the scale–dependence of the gravitational coupling provides an effective mechanism for the inclusion of quantum effects in the context of wormholes. In this sense, the obtained solution can be also taken to be self-sustainable, but this time due to the effect of the running of the Newton’s gravitational coupling.

VI. CONCLUDING REMARKS

In this work we have constructed the first wormhole solution in the context of scale-dependent gravity. Interestingly, the obtained geometry is a vacuum solution of the modified Einstein’s equations and, therefore, no violations of the energy conditions appear. The width of the wormhole’s throat has been shown to correspond to the running parameter, which measures deviations from General Relativity. Even more, this parameter controls the running of the Newton’s coupling, which appears to be redshifted instead being constant as in the usual case. We have noted that this wormhole is self–sustained in the sense that the obtained effective gravity is the only responsible of its sustainability. In this sense, the model here presented can be thought as an effective description of previously considered self–sustained wormholes, which is confirmed by their (trans)-planckian size. Therefore, following [42, 43], we conclude that the obtained traversability has to be regarded as in “principle” rather than in “practice”. Finally, in order to propose some astrophysical signatures of the wormhole here presented, a study of its stability is mandatory. We leave this and other topics for a future work.

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