On some properties of seminormed fuzzy integrals

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Abstract

We give solutions to Problems 2.21, 2.31 and 2.32, which were posed Borzová-Molnárová, Halčinová and Hutník in [The smallest semicopula-based universal integrals I: properties and characterizations, Fuzzy Sets and Systems (2014), \url{http://dx.doi.org/10.1016/j.fss.2014.09.0232014}].

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1 Introduction

Let \((X, \mathcal{A})\) be a measurable space, where \(\mathcal{A}\) is a \(\sigma\)-algebra of subsets of a non-empty set \(X\), and let \(\mathcal{S}\) be the family of all measurable spaces. The class of all \(\mathcal{A}\)-measurable functions \(f: X \to [0, 1]\) is denoted by \(\mathcal{F}(X, \mathcal{A})\). A capacity on \(\mathcal{A}\) is a non-decreasing set function \(\mu: \mathcal{A} \to [0, 1]\) with \(\mu(\emptyset) = 0\) and \(\mu(X) = 1\). We denote by \(\mathcal{M}(X, \mathcal{A})\) the class of all capacities on \(\mathcal{A}\).

Suppose that \(S: [0,1]^2 \to [0,1]\) is a semicopula (also called a \(t\)-seminorm), i.e., a non-decreasing function in both coordinates with the neutral element equal to 1. It is clear that \(S(x,y) \leq x \land y\) and \(S(x,0) = 0 = S(0,x)\) for all \(x, y \in [0, 1]\), where \(x \land y = \min(x,y)\) (see [1,2,5]). We denote the class of all semicopulas by \(\mathcal{S}\).

There are three important examples of semicopulas: \(M, \Pi\) and \(S_L\), where \(M(a, b) = a \land b\), \(\Pi(a,b) = ab\) and \(S_L(a,b) = (a+b-1) \lor 0\); \(S_L\) is called the \(\text{Lukasiewicz}\) \(t\)-norm [7]. Hereafter, \(a \lor b = \max(a,b)\). A function \(S^*: [0,1]^2 \to [0,1]\) is called a \(t\)-coseminorm if \(S^*(x,y) = 1 - S(1-x,1-y)\) for some semicopula \(S\).

A generalized Sugeno integral is defined by

\[
I_S(\mu, f) := \sup_{t \in [0,1]} S\left(t, \mu\left(\{f \geq t\}\right)\right),
\]

where \(\{f \geq t\} = \{x \in X : f(x) \geq t\}\), \((X, \mathcal{A}) \in \mathcal{S}\) and \((\mu, f) \in \mathcal{M}(X,\mathcal{A}) \times \mathcal{F}(X,\mathcal{A})\). In the literature the functional \(I_S\) is also called \(\text{seminormed fuzzy integral}\) [8,6,9]. Replacing semicopula \(S\) with \(M\), we get the Sugeno integral [11]. Moreover, if \(S = \Pi\), then \(I_\Pi\) is called the Shilkret integral [10].

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The paper is devoted to the solution of Problems 2.21, 2.31 and 2.32, which were posed by Borzová-Molnárová, Halčinová and Hutník [4] (see also [8]). In Section 2 we present our main results and some related results. In the next section we give conditions equivalent to comonotone ◦-additivity of the generalized Sugeno integral $I_S$.

2 Weak subadditivity of integral $I_S$

In this section we present necessary and sufficient conditions for validity of the inequality

$$I_S(\mu, f + a) \leq I_S(\mu, f) + a,$$

where $f + a \in [0, 1]$. Borzová-Molnárová et al. [4] proposed a sufficient condition only, which turns out to be too restrictive.

Theorem 2.1. The inequality (1) is satisfied for all $a \in [0, 1]$ and $(\mu, f) \in M(X, A) \times F(X, A)$ such that $f + a \in [0, 1]$, iff

$$S(c + a, b) \leq S(c, b) + a$$

for all $a, b, c \in [0, 1]$ such that $a + c \in [0, 1]$.

Proof. We first prove that the inequality (1) holds true under the assumption (2). The inequality can be rewritten as follows

$$\sup_{t \in [0, 1]} S\left(t, \mu(\{f + a \geq t\})\right) \leq \sup_{t \in [0, 1]} S\left(t, \mu(\{f \geq t\})\right) + a.$$  (3)

The inequality (3) is obvious for $a \in \{0, 1\}$, so we assume that $a \in (0, 1)$. Observe that $\mu(\{f + a \geq t\}) = 1$ for $t \in [0, a]$. Since $S(a, 1) = a$ for all $a$, the left-hand side of (3) takes the form

$$\sup_{t \in [0, 1]} S\left(t, \mu(\{f + a \geq t\})\right) = \max\left[\sup_{t \in [0, a]} S(t, 1), \sup_{t \in (a, 1]} S\left(t, \mu(\{f + a \geq t\})\right)\right]$$

$$= a \lor \sup_{t \in (0, 1-a]} S\left(t + a, \mu(\{f \geq t\})\right)$$

$$= \sup_{t \in [0, 1-a]} S\left(t + a, \mu(\{f \geq t\})\right).$$  (4)

From (3) and (4) we get

$$\sup_{t \in [0, 1-a]} S\left(t + a, \mu(\{f \geq t\})\right) \leq \sup_{t \in [0, 1-a]} S\left(t, \mu(\{f \geq t\})\right) + a,$$  (5)
The proof is complete.

that $f$: $S(a, b, c)$ for all $A \notin \{\emptyset, X\}$, where $0 < b < 1$, $0 < c \leq 1 - a$. Thus

$$
\sup_{t \in [0,1-a]} S\left(t + a, \mu\{f \geq t\}\right) = a \lor (S(c + a, b)),
$$

$$
\sup_{t \in [0,1-a]} S\left(t, \mu\{f \geq t\}\right) + a = S(c, b) + a = a \lor (S(c, b) + a).
$$

Combining this with (5) we obtain

$$
a \lor (S(c + a, b)) \leq a \lor (S(c, b) + a) \quad (6)
$$

for all $(a, b, c) \in C$, as $C = \{(a, b, c): 0 < a < 1, 0 < b < 1, 0 < c \leq 1 - a\}$.

It is obvious that $x \lor y \leq x \lor z$ implies $(y \leq z)$ or $(x \geq y \geq z)$ for all $x, y, z$, so from (6) it follows that

$$
(S(c + a, b) \leq S(c, b) + a) \quad \text{or} \quad (a \geq S(c + a, b) \text{ and } S(c + a, b) > S(c, b) + a). \quad (7)
$$

Hence we get

$$
S(c + a, b) \leq S(c, b) + a, \quad (8)
$$

for all $a, b, c \in C$, as the second sentence in (7) leads to a contradiction. Let us notice that (5) also holds for $a = 0$, $a = 1$ and $c = 0$, as well as $b = 0$, $b = 1$. Now we show that if $S(c + a, b) \leq S(c, b) + a$, then inequality (11) is satisfied for all $(\mu, f) \in M(X,A) \times F(X,A)$ such that $f + a \in [0,1]$. Applying (2) we obtain

$$
I_S(\mu, f + a) = \sup_{t \in [0,1-a]} S\left(t + a, \mu\{f \geq t\}\right) \leq \sup_{t \in [0,1-a]} S\left(t, \mu\{f \geq t\}\right) + a = I_S(\mu, f) + a.
$$

The proof is complete. \qed

**Remark 2.1.** From (2) it follows that $S_L(x, y) \leq S(x, y)$ for all $x, y \in [0,1]$; to see this put $c + a = 1$ in (2).

We show that the sufficient condition guaranteeing validity of (11) (see the inequality (3) in Borzová-Molnárová et al. [4]) is also necessary but for a slightly different problem (see Corollary 2.1(a) below).

**Theorem 2.2.** The inequality

$$
I_{S_1}(\mu, (f + a)1_A) \leq I_{S_2}(\mu, f1_A) + I_{S_3}(\mu, a1_A) \quad (9)
$$

holds for all $a \in [0,1]$, $S_i \in \mathcal{S}$, $A \in \mathcal{A}$, and $(\mu, f) \in M(X,A) \times F(X,A)$ such that $f + a \in [0,1]$ iff the inequality

$$
S_1(x + y, z) \leq S_2(x, z) + S_3(y, z) \quad (10)
$$

is satisfied for all $x, y, z \in [0,1]$ such that $x + y \in [0,1]$. 
Proof. We claim that (9) follows from (10). Indeed

\[
\sup_{t \in [0,1]} S_1(t, \mu(A \cap \{f + a \geq t\})) = \sup_{t \in [0,a]} S_1(t, \mu(A)) \vee \sup_{t \in (a,1]} S_1(t, \mu(A \cap \{f + a \geq t\}))
\]

\[
= S_1(a, \mu(A)) \vee \sup_{t \in [0,1-a]} S_1(t + a, \mu(\{f \geq t\}))
\]

\[
\leq \sup_{t \in [0,1-a]} \left( S_2(t, \mu(A \cap \{f \geq t\})) + S_3(a, \mu(A \cap \{f \geq t\})) \right)
\]

\[
\leq \mathbf{I}_{S_2}(\mu, f \mathbb{1}_A) + \mathbf{I}_{S_3}(\mu, a \mathbb{1}_A).
\]

Putting \( f = b \mathbb{1}_A \in [0,1] \) in (9), we obtain (10), which completes the proof.

Applying Theorem 2.2 to \( S_1 = S_2 \) and \( S_3 = S, S_3 = \wedge \) or \( S_3 = \Pi \) we get

Corollary 2.1. (a) \( S(x + y, z) \leq S(x, z) + S(y, z) \) for all \( x, y, z \) iff for all \( a, f, A, \mu \)

\[
\mathbf{I}_S(\mu, (f + a) \mathbb{1}_A) \leq \mathbf{I}_S(\mu, f \mathbb{1}_A) + \mathbf{I}_S(\mu, a \mathbb{1}_A).
\]

(b) \( S(x + y, z) \leq S(x, z) + (y \wedge z) \) for all \( x, y, z \) iff for all \( a, f, A, \mu \)

\[
\mathbf{I}_S(\mu, (f + a) \mathbb{1}_A) \leq \mathbf{I}_S(\mu, f \mathbb{1}_A) + (a \wedge \mu(A)),
\]

(c) \( S(x + y, z) \leq S(x, z) + yz \) for all \( x, y, z \) iff for all \( a, f, A, \mu \)

\[
\mathbf{I}_S(\mu, (f + a) \mathbb{1}_A) \leq \mathbf{I}_S(\mu, f \mathbb{1}_A) + a \mu(A).
\]

3 Some other properties of integral \( \mathbf{I}_S \)

In this section, the solution to Problems 2.31 and 2.32 of Borzowá-Molnárová, Halčinová and Hutník [4] is provided. We say that \( f, g: X \to [0,1] \) are comonotone on \( A \in \mathcal{F} \), if \( (f(x) - f(y))(g(x) - g(y)) \geq 0 \) for all \( x, y \in A \). Clearly, if \( f \) and \( g \) are comonotone on \( A \), then for any real number \( t \) either \( \{f \geq t\} \subset \{g \geq t\} \) or \( \{g \geq t\} \subset \{f \geq t\} \).

Problem 1 (Problem 2.31). Let \( \mu \in \mathcal{M}_{(X,A)} \) and \( f, g \in \mathcal{F}_{(X,A)} \) be comonotone functions. Characterize all the semicopulas \( S \) for which

\[
\mathbf{I}_S(\mu, f \vee g) = \mathbf{I}_S(\mu, f) \vee \mathbf{I}_S(\mu, g).
\]

Theorem 3.1. The equality (11) holds for any semicopula \( S \in \mathcal{E} \).
Proof. Since \( f, g \) are comonotone, \( \{ f \geq t \} \subset \{ g \geq t \} \) or \( \{ g \geq t \} \subset \{ f \geq t \} \) for all \( t \). Hence \( \mu(\{ f \vee g \geq t \}) = \mu(\{ f \geq t \}) \cup \mu(\{ g \geq t \}) \) and
\[
I_S(\mu, f \vee g) = \sup_{t \in [0,1]} S\left(t, \mu(\{ f \geq t \}) \cup \mu(\{ g \geq t \})\right)
\]
\[
= \sup_{t \in [0,1]} \left\{ S\left(t, \mu(\{ f \geq t \})\right) \cup S\left(t, \mu(\{ g \geq t \})\right) \right\}
\]
\[
= \sup_{t \in [0,1]} S\left(t, \mu(\{ f \geq t \})\right) \cup \sup_{t \in [0,1]} S\left(t, \mu(\{ g \geq t \})\right)
\]
\[
= I_S(\mu, f) \cup I_S(\mu, g).
\]
The proof is complete. \( \square \)

**Problem 2** (Problem 2.32). Let \( S \in \mathcal{S} \) be fixed. To describe all the commuting binary operators with the integral \( I_S \) (under the condition of comonotonicity of functions \( f, g \)), i.e., to find all operators \( \circ : [0,1]^2 \rightarrow [0,1] \) such that
\[
I_S(\mu, f \circ g) = I_S(\mu, f) \circ I_S(\mu, g)
\]
(12)
for all \( f, g : X \rightarrow [0,1] \) comonotone functions.

We give an answer to this problem for some class of operators \( \circ \).

**Theorem 3.2.** Let \( S \in \mathcal{S} \) be fixed and let \( \circ \) be either a semicopula or \( t \)-coseminorm. The equality (12) holds for all comonotone functions \( f, g \in \mathcal{F}(X, A) \) if \( \circ = \vee \).

**Proof.** Put \( f = 1_A = g \) and \( A \in \mathcal{F} \). Thus \( (f \circ g)(x) \) equals 1, if \( x \in A \) and 0 otherwise, which implies that \( \mu(\{ f \circ g \geq t \}) = \mu(A) \) for \( t \in (0,1] \) and \( \mu(\{ f \circ g \geq 0 \}) = 1 \). By (12)
\[
\mu(A) = \mu(A) \circ \mu(A).
\]
In consequence, we have \( x = x \circ x \) for all \( x \). If \( \circ \) is a semicopula then \( \circ = \wedge \). Indeed, if \( x \leq y \), then
\[
x = x \circ x \leq x \circ y \leq x \wedge y = x
\]
so \( x \circ y = x \wedge y \). Similar arguments apply to the case of \( x \geq y \). Obviously, if \( \circ \) is a \( t \)-coseminorm then \( \circ = \vee \).

Now we show that if \( \circ = \wedge \), then the equality (12) is not satisfied for all \( S \in \mathcal{S} \). Suppose that \( S = \Pi \) and \( X = \{a, b\} \) with \( \mu(\{a\}) = 0.5 \), \( \mu(\{b\}) = 0.5 \) and \( \mu(X) = 1 \). Furthermore, we assume that \( f(a) = 1 \), \( f(b) = 0.4 \), \( g(a) = 0.8 \) and \( g(b) = 0.6 \). The functions \( f, g \) are comonotone on \( X \). An elementary algebra shows that
\[
I_\Pi(\mu, f) = 0.4 \vee (1 \cdot 0.5) = 0.5,
\]
\[
I_\Pi(\mu, g) = 0.6 \vee (0.8 \cdot 0.5) = 0.6,
\]
\[
I_\Pi(\mu, (f \wedge g)) = 0.4 \vee (0.8 \cdot 0.5) = 0.4,
\]
so $I_\Pi(\mu, (f \land g)) < I_\Pi(\mu, f) \land I_\Pi(\mu, g)$. The equality (12) holds for $\circ = \lor$, which is a consequence of Theorem 3.1.

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