Abstract

The problem of $f$-divergence estimation is important in the fields of machine learning, information theory, and statistics. While several nonparametric divergence estimators exist, relatively few have known convergence properties. In particular, even for those estimators whose MSE convergence rates are known, the asymptotic distributions are unknown. We establish the asymptotic normality of a recently proposed ensemble estimator of $f$-divergence between two distributions from a finite number of samples. This estimator has MSE convergence rate of $O\left(\frac{1}{T}\right)$, is simple to implement, and performs well in high dimensions. This theory enables us to perform divergence-based inference tasks such as testing equality of pairs of distributions based on empirical samples. We experimentally validate our theoretical results and, as an illustration, use them to empirically bound the best achievable classification error.

1 Introduction

This paper establishes the asymptotic normality of a nonparametric estimator of the $f$-divergence between two distributions from a finite number of samples. For many nonparametric divergence estimators the large sample consistency has already been established and the mean squared error (MSE) convergence rates are known for some. However, there are few results on the asymptotic distribution of non-parametric divergence estimators. Here we show that the asymptotic distribution is Gaussian for the class of ensemble $f$-divergence estimators [1], extending theory for entropy estimation [2, 3] to divergence estimation. $f$-divergence is a measure of the difference between distributions and is important to the fields of machine learning, information theory, and statistics [4]. The $f$-divergence generalizes several measures including the Kullback-Leibler (KL) [5] and Rényi-$\alpha$ [6] divergences. Divergence estimation is useful for empirically estimating the decay rates of error probabilities of hypothesis testing [7], extending machine learning algorithms to distributional features [8, 9], and other applications such as text/multimedia clustering [10]. Additionally, a special case of the KL divergence is mutual information which gives the capacities in data compression and channel coding [11]. Mutual information estimation has also been used in machine learning applications such as feature selection [11], fMRI data processing [12], clustering [13], and neuron classification [14]. Entropy is also a special case of divergence where one of the distributions is the uniform distribution. Entropy estimation is useful for intrinsic dimension estimation [15], texture classification and image registration [16], and many other applications.

However, one must go beyond entropy and divergence estimation in order to perform inference tasks on the divergence. An example of an inference task is detection: to test the null hypothesis that the divergence is zero, i.e., testing that the two populations have identical distributions. Prescribing a p-value on the null hypothesis requires specifying the null distribution of the divergence estimator. Another statistical inference problem is to construct a confidence interval on the divergence based on...
densities and define $L$ Bold face type is used in this paper for random variables and random vectors. Let distributions or implement the Bayes classifier.

probability between two population distributions, without having to construct estimates for these then apply the theory to the practical problem of empirically bounding the Bayes classification error ensemble estimator of entropy such as the one given in [3]. We verify the theory by simulation. We theory developed to accomplish this can also be used to derive a central limit theorem for a weighted series expansions) that the random variables and their squares are asymptotically uncorrelated. The of interchangeable random variables and then showing (by concentration inequalities and Taylor

in distribution to the standard normal distribution. This is accomplished by constructing a sequence of interchangeable random variables and then showing (by concentration inequalities and Taylor series expansions) that the random variables and their squares are asymptotically uncorrelated. The theory developed to accomplish this can also be used to derive a central limit theorem for a weighted ensemble estimator of entropy such as the one given in [3]. We verify the theory by simulation. We

Asymptotic normality has been established for certain appropriately normalized divergences be-

Recent work has focused on deriving convergence rates for divergence estimators. Nguyen et al [23]. Singh and Póczos [24], and Krishnamurthy et al [25] each proposed divergence estimators that achieve the parametric convergence rate ($O\left(\frac{1}{T}\right)$) under weaker conditions than those given in [1]. However, solving the convex problem of [23] can be more demanding for large sample sizes than the estimator given in [1] which depends only on simple density plug-in estimates and an offline convex optimization problem. Singh and Póczos only provide an estimator for Rényi-$\alpha$ divergences that requires several computations at each boundary of the support of the densities which becomes difficult to implement as $d$ gets large. Also, this method requires knowledge of the support of the densities which may not be possible for some problems. In contrast, while the convergence results of the estimator in [1] requires the support to be bounded, knowledge of the support is not required for implementation. Finally, the estimators given in [25] estimate divergences that include functionals of the form $\int f_1^\alpha(x)f_2^\beta(x)d\mu(x)$ for given $\alpha,\beta$. While a suitable $\alpha,\beta$ indexed sequence of divergence functionals of the form in [25] can be made to converge to the KL divergence, this does not guarantee convergence of the corresponding sequence of divergence estimates, whereas the estimator in [1] can be used to estimate the KL divergence. Also, for some divergences of the specified form, numerical integration is required for the estimators in [25], which can be computationally difficult. In any case, the asymptotic distributions of the estimators in [23][24][25] are currently unknown.

Asymptotic normality has been established for certain appropriately normalized divergences be-

between a specific density estimator and the true density [26, 27, 28]. However none of these works study the convergence rates of their estimators nor do they derive the asymptotic distributions.

1.1 Related Work

Estimators for some $f$-divergences already exist. For example, Póczos & Schneider [8] and Wang et al [18] provided consistent $k$-nn estimators for Rényi-$\alpha$ and the KL divergences, respectively. Consistency has been proven for other mutual information and divergence estimators based on plug-in histogram schemes [19][20][21][22]. Hero et al [16] provided an estimator for Rényi-$\alpha$ divergence but assumed that one of the densities was known. However none of these works study the convergence rates of their estimators nor do they derive the asymptotic distributions.

Bold face type is used in this paper for random variables and random vectors. Let $f_1$ and $f_2$ be densities and define $L(x) = \frac{f_1(x)}{f_2(x)}$. The conditional expectation given a random variable $Z$ is $E_Z$.

2 The Divergence Estimator

Moon and Hero [1] focused on estimating divergences that include the form [4]

$$G(f_1,f_2) = \int g \left( \frac{f_1(x)}{f_2(x)} \right) f_2(x)dx,$$

for a smooth, function $g(f)$. (Note that although $g$ must be convex for [1] to be a divergence, the estimator in [1] does not require convexity.) The divergence estimator is constructed us-
ing $k$-nn density estimators as follows. Assume that the $d$-dimensional multivariate densities $f_1$ and $f_2$ have finite support $S = [a, b]^d$. Assume that $T = N + M_2$ i.i.d. realizations \{${\bf X}_1, \ldots, \bf X_N, \bf X_{N+1}, \ldots, \bf X_{N+M_2}$\} are available from the density $f_2$ and $M_1$ i.i.d. realizations \{${\bf Y}_1, \ldots, \bf Y_{M_1}$\} are available from the density $f_1$. Assume that $k_1 \leq M_i$. Let $\rho_{2,k_2}(i)$ be the distance of the $k_2$th nearest neighbor of $\bf X_i$ in \{${\bf X}_{N+1}, \ldots, \bf X_T$\} and let $\rho_{1,k_1}(i)$ be the distance of the $k_1$th nearest neighbor of $\bf X_i$ in \{${\bf Y}_1, \ldots, \bf Y_{M_1}$\}. Then the $k$-nn density estimate is \cite{29}:

$$\hat{f}_{i,k_1}(X_j) = \frac{k_i}{M_i c \rho_{1,k_1}(j)},$$

where $c$ is the volume of a $d$-dimensional unit ball.

To construct the plug-in divergence estimator, the data from $f_2$ are randomly divided into two parts \{${\bf X}_1, \ldots, \bf X_N$\} and \{${\bf X}_{N+1}, \ldots, \bf X_{N+M_2}$\}. The $k$-nn density estimate $\hat{f}_{2,k_2}$ is calculated at the $N$ points \{${\bf X}_1, \ldots, \bf X_N$\} using the $M_2$ realizations \{${\bf X}_{N+1}, \ldots, \bf X_{N+M_2}$\}. Similarly, the $k$-nn density estimate $\hat{f}_{1,k_1}$ is calculated at the $N$ points \{${\bf X}_1, \ldots, \bf X_N$\} using the $M_1$ realizations \{${\bf Y}_1, \ldots, \bf Y_{M_1}$\}. Define $\hat{L}_{k_1,k_2}(x) = \frac{\hat{f}_{1,k_1}(x)}{\hat{f}_{2,k_2}(x)}$. The functional $G(f_1, f_2)$ is then approximated as

$$\hat{G}_{k_1,k_2} = \frac{1}{N} \sum_{i=1}^{N} g \left( \hat{L}_{k_1,k_2}(X_i) \right).$$

The principal assumptions on the densities $f_1$ and $f_2$ and the functional $g$ are that: 1) $f_1$, $f_2$, and $g$ are smooth; 2) $f_1$ and $f_2$ have common bounded support sets $S$; 3) $f_1$ and $f_2$ are strictly lower bounded. The full assumptions (A.0) – (A.5) are given in the appendices in ref 17. Moon and Hero \cite{1} showed that under these assumptions, the MSE convergence rate of the estimator in Eq. 2 to the quantity in Eq. 1 depends exponentially on the dimension $d$ of the densities. However, Moon and Hero also showed that an estimator with the parametric convergence rate $O(1/T)$ can be derived by applying the theory of optimally weighted ensemble estimation as follows.

Let $I = \{i_1, \ldots, i_L\}$ be a set of index values and $T$ the number of samples available. For an indexed ensemble of estimators \{${\bf \hat{E}}_I$\}, the weighted ensemble estimator with weights $w = \{w(1), \ldots, w(L)\}$ satisfying $\sum_{l \in I} w(l) = 1$ is defined as $\hat{E}_w = \sum_{l \in I} w(l) \hat{E}_l$. The key idea to reducing MSE is that by choosing appropriate weights $w$, we can greatly decrease the bias in exchange for some increase in variance. Consider the following conditions on \{${\bf \hat{E}}_I$\} \cite{3}:

- **C.1** The bias is given by

  $$\text{Bias} \left( \hat{E}_I \right) = \sum_{i \in J} c_i \psi_i(l) T^{-i/2d} + O \left( \frac{1}{\sqrt{T}} \right),$$

  where $c_i$ are constants depending on the underlying density, $J = \{i_1, \ldots, i_I\}$ is a finite index set with $I < L$, $\min(J) > 0$ and $\max(J) \leq d$, and $\psi_i(l)$ are basis functions depending only on the parameter $l$.

- **C.2** The variance is given by

  $$\text{Var} \left( \hat{E}_I \right) = c_v \left( \frac{1}{T} \right) + o \left( \frac{1}{T} \right).$$

**Theorem 1.** \cite{3} Assume conditions C.1 and C.2 hold for an ensemble of estimators \{${\bf \hat{E}}_I$\}. Then there exists a weight vector $w_0$ such that

$$\mathbb{E} \left[ \left( \hat{E}_{w_0} - E \right)^2 \right] = O \left( \frac{1}{T} \right).$$

The weight vector $w_0$ is the solution to the following convex optimization problem:

$$\min_{w} \|w\|^2 \text{ subject to } \sum_{l \in I} w(l) = 1, \gamma_w(i) = \sum_{l \in I} w(l) \psi_i(l) = 0, \ i \in J.$$
Theorem 2. Assume that assumptions (A.0) – (A.5) hold and let \( M = O(M_1) = O(M_2) \) and \( k(l) = \sqrt{M} \) with \( l \in I \). The asymptotic distribution of the weighted ensemble estimator \( \hat{G}_w \) is given by

\[
\lim_{M,N \to \infty} \Pr \left( \frac{\hat{G}_w - \mathbb{E} \left[ \hat{G}_w \right]}{\sqrt{\text{Var} \left[ \hat{G}_w \right]}} \leq t \right) = \Pr(S \leq t),
\]

where \( S \) is a standard normal random variable.

3 Asymptotic Normality of the Estimator

The following theorem shows that the appropriately normalized ensemble estimator \( \hat{G}_w \) converges in distribution to a normal random variable.
where \( S \) is a standard normal random variable. Also \( \mathbb{E} \left[ \hat{G}_w \right] \to G(f_1, f_2) \) and \( \text{Var} \left[ \hat{G}_w \right] \to 0. \)

The results on the mean and variance come from [1]. The proof of the distributional convergence is outlined below and is based on constructing a sequence of interchangeable random variables \( \{Y_{M,i}\}_{i=1}^{N} \) with zero mean and unit variance. We then show that the \( Y_{M,i} \) are asymptotically uncorrelated and that the \( Y_{M,i}^2 \) are asymptotically uncorrelated as \( M \to \infty \). This is similar to what was done in [30] to prove a central limit theorem for a density plug-in estimator of entropy. Our analysis for the ensemble estimator of divergence is more complicated since we are dealing with a functional of two densities and a weighted ensemble of estimators. In fact, some of the equations we use to prove Theorem 2 can be used to prove a central limit theorem for a weighted ensemble of entropy estimators such as that given in [3].

3.1 Proof Sketch of Theorem 2

The full proof is included in the appendices. We use the following lemma from [30, 31]:

**Lemma 3.** Let the random variables \( \{Y_{M,i}\}_{i=1}^{N} \) belong to a zero mean, unit variance, interchangeable process for all values of \( M \). Assume that \( \text{Cov}(Y_{M,1}, Y_{M,2}) \) and \( \text{Cov}(Y_{M,1}^2, Y_{M,2}^2) \) are \( O(1/M) \). Then the random variable

\[
S_{N,M} = \left( \sum_{i=1}^{N} Y_{M,i} \right) / \sqrt{\text{Var} \left[ \sum_{i=1}^{N} Y_{M,i} \right]}
\]

(4)

converges in distribution to a standard normal random variable.

This lemma is an extension of work by Blum et al [32] which showed that if \( \{Z_i; i = 1, 2, \ldots\} \) is an interchangeable process with zero mean and unit variance, then \( S_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} Z_i \) converges in distribution to a standard normal random variable if and only if \( \text{Cov}[Z_1, Z_2] = 0 \) and \( \text{Cov}[Z_1^2, Z_2^2] = 0 \). In other words, the central limit theorem holds if and only if the interchangeable process is uncorrelated and the squares are uncorrelated. Lemma 3 shows that for a correlated interchangeable process, a sufficient condition for a central limit theorem is for the interchangeable process and the squared process to be asymptotically uncorrelated with rate \( O(1/M) \).

For simplicity, let \( M_1 = M_2 = M \) and \( \hat{L}_{k(t)} := \hat{L}_{k(t), k(l)} \). Define

\[
Y_{M,i} = \frac{\sum_{l \in i} w(l)g \left( \hat{L}_{k(t)}(X_{i_l}) \right) - \mathbb{E} \left[ \sum_{l \in i} w(l)g \left( \hat{L}_{k(t)}(X_{i_l}) \right) \right]}{\sqrt{\text{Var} \left[ \sum_{l \in i} w(l)g \left( \hat{L}_{k(t)}(X_{i_l}) \right) \right]}}
\]

Then from Eq. [4] we have that

\[
S_{N,M} = \left( G_w - \mathbb{E} \left[ G_w \right] \right) / \sqrt{\text{Var} \left[ G_w \right]}. 
\]

Thus it is sufficient to show from Lemma 3 that \( \text{Cov}(Y_{M,1}, Y_{M,2}) \) and \( \text{Cov}(Y_{M,1}^2, Y_{M,2}^2) \) are \( O(1/M) \). To do this, it is necessary to show that the denominator of \( Y_{M,i} \) converges to a nonzero constant or to zero sufficiently slowly. It is also necessary to show that the covariance of the numerator is \( O(1/M) \). Therefore, to bound \( \text{Cov}(Y_{M,1}, Y_{M,2}) \), we require bounds on the quantity \( \text{Cov} \left[ g \left( \hat{L}_{k(t)}(X_{i_l}) \right), g \left( \hat{L}_{k(t)}(X_{i_{l'}}) \right) \right] \) where \( l, l' \in l \).

Define \( M(Z) := Z - \mathbb{E} Z, \hat{f}_{k(t)}(Z) := \hat{L}_{k(t)}(Z) - \mathbb{E} \hat{L}_{k(t)}(Z), \) \( \hat{e}_{k(t)}(Z) := \hat{L}_{k(t)}(Z) - \mathbb{E} \hat{L}_{k(t)}(Z) \). Assuming \( g \) is sufficiently smooth, a Taylor series expansion of \( g \left( \hat{L}_{k(t)}(Z) \right) \) around \( \mathbb{E} \hat{L}_{k(t)}(Z) \) gives

\[
g \left( \hat{L}_{k(t)}(Z) \right) = \sum_{i=0}^{\lambda - 1} \frac{g^{(i)} \left( \mathbb{E} \hat{L}_{k(t)}(Z) \right)}{i!} \hat{f}_{k(t)}^i(Z) + \frac{g^{(\lambda)} \left( \xi Z \right)}{\lambda!} \hat{f}_{k(t)}^\lambda(Z),
\]
where $\xi Z \in \left( \mathbb{E} Z \hat{\mathbf{F}}_{k(l)}(Z), \hat{\mathbf{F}}_{k(l)}(Z) \right)$. We use this expansion to bound the covariance. The expected value of the terms containing the derivatives of $g$ is controlled by assuming that the densities are lower bounded. By assuming the densities are sufficiently smooth, an expression for $\mathbf{F}^q_{k(l)}(Z)$ in terms of powers and products of the density error terms $\mathbf{e}_{1,k(l)}$ and $\mathbf{e}_{2,k(l)}$ is obtained by expanding $\mathbf{L}_{k(l)}(Z)$ around $\mathbb{E} \mathbf{L}_{k(l)}(Z)$ and applying the binomial theorem. The expected value of products of these density error terms is bounded by applying concentration inequalities and conditional independence. Then the covariance between $\mathbf{F}^q_{k(l)}(Z)$ terms is bounded by bounding the covariance between powers and products of the density error terms by applying Cauchy-Schwarz and other concentration inequalities. This gives the following lemma which is proved in the appendices.

**Lemma 4.** Let $l, l' \in \mathcal{I}$ be fixed, $M_1 = M_2 = M$, and $k(l) = l\sqrt{M}$. Let $\gamma_1(x), \gamma_2(x)$ be arbitrary functions with $1$ partial derivative wrt $x$ and $\sup_x |\gamma_i(x)| < \infty$, $i = 1, 2$ and let $\mathbf{L}_{k(l)}$ be the indicator function. Let $X_i$ and $X_j$ be realizations of the density $f_2$ independent of $\hat{\mathbf{F}}_{1,k(l)}, \hat{\mathbf{F}}_{2,k(l)}, \hat{\mathbf{F}}_{1,k(l')}, \hat{\mathbf{F}}_{2,k(l')}$ and independent of each other when $i \neq j$. Then

$$Cov \left[ \gamma_1(X_i) \mathbf{F}^q_{k(l)}(X_i), \gamma_2(X_j) \mathbf{F}^r_{k(l')}(X_j) \right] = \sum_{\substack{1 \leq j \leq \min(l, l') \leq M \\bar{1} \leq \min(j, j') \leq M \\bar{1} \leq \min(l, l') \leq M}} \left[ o(1),_{\bar{1} \leq r = 1} c_8 \left( \gamma_1(x), \gamma_2(x) \right) \left( \frac{1}{M} \right)^2 + o \left( \frac{1}{M} \right)^2, \right] \quad i = j, \quad \bar{1} \neq \bar{1}.$$  

Note that $k(l)$ is required to grow with $\sqrt{M}$ for Lemma 4 to hold. Define $h_{l,g}(X) = g \left( \mathbb{E} X \mathbf{L}_{k(l)}(X) \right)$. Lemma 4 can then be used to show that

$$Cov \left[ g \left( \mathbf{L}_{k(l)}(X_i) \right), g \left( \mathbf{L}_{k(l')}(X_j) \right) \right] = \left\{ \begin{array}{ll} o(1), & i = j \vspace{0.1cm} \\
_{\bar{1} \leq r = 1} c_8 \left( h_{l,g}(X), h_{l,g}(X) \right) \left( \frac{1}{M} \right)^2 + o \left( \frac{1}{M} \right)^2, & \bar{1} \neq \bar{1}. \end{array} \right.$$  

For the covariance of $Y_{M,i}^2$ and $Y_{M,j}^2$ assume WLOG that $i = 1$ and $j = 2$. Then for $l, l', j, j'$ we need to bound the term

$$Cov \left[ M \left( g \left( \mathbf{L}_{k(l)}(X_1) \right) \right), M \left( g \left( \mathbf{L}_{k(l')}(X_1) \right) \right) \right].$$

For the case where $l = l'$ and $j = j'$ we can simply apply the previous results to the functional $d(x) = (M \left( g(x) \right))^2$. For the more general case, we need to show that

$$Cov \left[ \gamma_1(X_1) \mathbf{F}^q_{k(l)}(X_1), \gamma_2(X_2) \mathbf{F}^r_{k(l')}(X_2) \right] = O \left( \frac{1}{M} \right).$$

To do this, bounds are required on the covariance of up to eight distinct density error terms. Previous results can be applied by using Cauchy-Schwarz when the sum of the exponents of the density error terms is greater than or equal to 3. When the sum is equal to 3, we use the fact that $k(l) = O(k(l'))$ combined with Markov’s inequality to obtain a bound of $O(1/M)$. Applying Eq. 6 to the term in Eq. 5 gives the required bound to apply Lemma 3.

### 3.2 Broad Implications of Theorem 2

To the best of our knowledge, Theorem 2 provides the first results on the asymptotic distribution of an $f$-divergence estimator with MSE convergence rate of $O(1/T)$ under the setting of a finite number of samples from two unknown, non-parametric distributions. This enables us to perform inference tasks on the class of $f$-divergences (defined with smooth functions $g$) on smooth, strictly lower bounded densities with finite support. Such tasks include hypothesis testing and constructing confidence intervals on the error exponents of the Bayes probability of error for a classification problem. This greatly increases the utility of these divergence estimators.

Although we focused on a specific divergence estimator, we suspect that our approach of showing that the components of the estimator and their squares are asymptotically uncorrelated can be adapted to derive central limit theorems for other divergence estimators that satisfy similar assumptions (smooth $g$, and smooth, strictly lower bounded densities with finite support). We speculate that this would be easiest for estimators that are also based on $k$-nearest neighbors such as in [23] and [13]. It is also possible that the approach can be adapted to other plug-in estimator approaches such as in [24] and [25]. However, the qualitatively different convex optimization approach of divergence estimation in [24] may require different methods.
4 Experiments

We first apply the weighted ensemble estimator of divergence to simulated data to verify the central limit theorem. We then use the estimator to obtain confidence intervals on the error exponents of the Bayes probability of error for the Iris data set from the UCI machine learning repository [33, 34].

4.1 Simulation

To verify the central limit theorem of the ensemble method, we estimated the KL divergence between two truncated normal densities restricted to the unit cube. The densities have means \( \mu_1 = 0.7 + 1_d \), \( \mu_2 = 0.3 + 1_d \) and covariance matrices \( \sigma_i I_d \) where \( \sigma_1 = 0.1 \), \( \sigma_2 = 0.3 \). \( I_d \) is a \( d \)-dimensional vector of ones, and \( 1_d \) is a \( d \)-dimensional identity matrix. We show the Q-Q plot of the normalized optimally weighted ensemble estimator of the KL divergence with \( d = 6 \) and 1000 samples from each density in Fig. 1. The linear relationship between the quantiles of the normalized estimator and the standard normal distribution validates Theorem 2.

4.2 Probability of Error Estimation

Our ensemble divergence estimator can be used to estimate a bound on the Bayes probability of error [7]. Suppose we have two classes \( C_1 \) or \( C_2 \) and a random observation \( x \). Let the \( a \) priori \( \) class probabilities be \( w_1 = Pr(C_1) > 0 \) and \( w_2 = Pr(C_2) = 1 - w_1 > 0 \). Then \( f_1 \) and \( f_2 \) are the densities corresponding to the classes \( C_1 \) and \( C_2 \), respectively. The Bayes decision rule classifies \( x \) as \( C_1 \) if and only if \( w_1 f_1(x) > w_2 f_2(x) \). The Bayes error \( P_e^* \) is the minimum average probability of error and is equivalent to

\[
P_e^* = \int \min(w_1 f_1(x), w_2 f_2(x)) p(x) dx
\]

where \( p(x) = w_1 f_1(x) + w_2 f_2(x) \). For \( a, b > 0 \), we have

\[
\min(a, b) \leq a^\alpha b^{1-\alpha}, \forall \alpha \in (0, 1)
\]

Replacing the minimum function in Eq. [7] with this bound gives

\[
P_e^* \leq w_1^{\alpha} w_2^{1-\alpha} c_\alpha(f_1||f_2),
\]

where \( c_\alpha(f_1||f_2) = \int f_1^\alpha(x) f_2^{1-\alpha}(x) dx \) is the Chernoff \( \alpha \)-coefficient. The Chernoff coefficient is found by choosing the value of \( \alpha \) that minimizes the right hand side of Eq. [8]

\[
c^*(f_1||f_2) = c_{\alpha^*}(f_1||f_2) = \min_{\alpha \in (0,1)} \int f_1^\alpha(x) f_2^{1-\alpha}(x) dx.
\]

Thus if \( \alpha^* = \arg\min_{\alpha \in (0,1)} c_\alpha(f_1||f_2) \), an upper bound on the Bayes error is

\[
P_e^* \leq w_1^{\alpha^*} w_2^{1-\alpha^*} c^*(f_1||f_2).
\]
Table 1: Estimated 95% confidence intervals for the bound on the pairwise Bayes error and the misclassification rate of a QDA classifier with 5-fold cross validation applied to the Iris dataset. The right endpoint of the confidence intervals is nearly zero when comparing the Setosa class to the other two classes while the right endpoint is much higher when comparing the Versicolor and Virginica classes. This is consistent with the QDA performance and the fact that the Setosa class is linearly separable from the other two classes.

| Estimated Confidence Interval | Setosa-Versicolor | Setosa-Virginica | Versicolor-Virginica |
|------------------------------|-------------------|------------------|----------------------|
| QDA Misclassification Rate   | (0, 0.0013)       | (0, 0.0002)      | (0, 0.0726)          |

Equation 9 includes the form in Eq. 1 \( g(x) = x^\alpha \). Thus we can use the optimally weighted ensemble estimator described in Sec. 2 to estimate a bound on the Bayes error. In practice, we estimate \( c_\alpha(f_1||f_2) \) for multiple values of \( \alpha \) (e.g. 0.01, 0.02, ..., 0.99) and choose the minimum.

We estimated a bound on the pairwise Bayes error between the three classes (Setosa, Versicolor, and Virginica) in the Iris data set \([33, 34]\) and used bootstrapping to calculate confidence intervals. We compared the bounds to the performance of a quadratic discriminant analysis classifier (QDA) with 5-fold cross validation. The pairwise estimated 95% confidence intervals and the misclassification rates of the QDA are given in Table 1. Note that the right endpoint of the confidence interval is less than 1/50 when comparing the Setosa class to either of the other two classes. This is consistent with the performance of the QDA and the fact that the Setosa class is linearly separable from the other two classes. In contrast, the right endpoint of the confidence interval is higher when comparing the Versicolor and Virginica classes which are not linearly separable. This is also consistent with the QDA performance. Thus the estimated bounds provide a measure of the relative difficulty of distinguishing between the classes, even though the small number of samples for each class (50) limits the accuracy of the estimated bounds.

5 Conclusion

In this paper, we established the asymptotic normality for a weighted ensemble estimator of \( f \)-divergence using \( d \)-dimensional truncated \( k \)-nn density estimators. To the best of our knowledge, this gives the first results on the asymptotic distribution of an \( f \)-divergence estimator with MSE convergence rate of \( O(1/T) \) under the setting of a finite number of samples from two unknown, non-parametric distributions. Future work includes simplifying the constants in front of the convergence rates given in [1] for certain families of distributions, deriving Berry-Esseen bounds on the rate of distributional convergence, extending the central limit theorem to other divergence estimators, and deriving the nonasymptotic distribution of the estimator.

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A Assumptions

We use the same assumptions on the densities and the functional as in [1] and [17]. They are

- (A.0): Assume that \( k_i = k_0 M_i^\beta \) with \( 0 < \beta < 1 \), that \( M_2 = \alpha_{frac} T \) with \( 0 < \alpha_{frac} < 1 \).
- (A.1): Assume there exist constants \( e_0, \epsilon_\infty \) such that \( 0 < e_0 \leq f_i(x) \leq \epsilon_\infty < \infty \), \( \forall x \in S \).
- (A.2): Assume that the densities \( f_i \) have continuous partial derivatives of order \( d \) in the interior of \( S \) that are upper bounded.
- (A.3): Assume that \( g \) has derivatives \( g^{(j)} \) of order \( j = 1, \ldots, \max\{\lambda, d\} \) where \( \lambda \beta > 1 \).
- (A.4): Assume that \( |g^{(j)}(f_1(x)/f_2(x))|, j = 0, \ldots, \max\{\lambda, d\} \) are strictly upper bounded for \( e_0 \leq f_i(x) \leq \epsilon_\infty \).
• (A.5): Let $\epsilon \in (0, 1)$, $\delta \in (2/3, 1)$, and $C(k) = \exp (-3k^{1-\delta})$. For fixed $\epsilon$, define $p_{i,i} = (1 - \epsilon)\epsilon^{k_{i,i}-1}/M$, $p_{u,i} = (1 + \epsilon)\epsilon^{k_{i,i}-1}/M$, $q_{i,i} = \epsilon^{k_{i,i}-1}/M$, and $q_{u,i} = (1 + \epsilon)\epsilon^{-1}$. Define $g$ uniformly over the $D$-dimensional unit ball. Assume that for $U(L) = g(L)$, $g^{(3)}(L)$, and $g^{(3)}(L)$, the random variable $\text{Cov}[U(L)C(k)]$ converges to a standard normal random variable.

Densities for which assumptions (A.0) – (A.5) hold include the truncated Gaussian distribution and the Beta distribution on the unit cube. Functions for which the assumptions hold include $g(L) = -\ln L$ and $g(L) = L^2$.

### B Proof of Theorem 2

We use Lemma 5 which is proved in [30] and restate it here:

**Lemma 5.** Let the random variables $\{Y_{M,i}\}_{i=1}^N$ belong to a zero mean, unit variance, interchangeable process for all values of $M$. Assume that $\text{Cov}(Y_{M,1}, Y_{M,2})$ and $\text{Cov}(Y_{M,1}^2, Y_{M,2}^2)$ are $O(1/M)$. Then the random variable

$$S_{N,M} = \frac{\sum_{i=1}^N Y_{M,i}}{\sqrt{\text{Var} \left[ \sum_{i=1}^N Y_{M,i} \right]}}$$

(10)

converges in distribution to a standard normal random variable.

For simplicity, let $M_1 = M_2 = M$ and $\hat{L}_{k(l),k(l)} := \hat{L}_{k(l),k(l)}$. Define

$$Y_{M,i} = \frac{\sum_{l \in I} w(l)g \left( \hat{L}_{k(l),k(l)}(X_i) \right) - \mathbb{E} \left[ \sum_{l \in I} w(l)g \left( \hat{L}_{k(l),k(l)}(X_i) \right) \right]}{\sqrt{\text{Var} \left[ \sum_{l \in I} w(l)g \left( \hat{L}_{k(l),k(l)}(X_i) \right) \right]}}.$$

Then from Eq. (10), we have that

$$S_{N,M} = \frac{\hat{G}_w - \mathbb{E} \left[ G_w \right]}{\sqrt{\text{Var} \left[ G_w \right]}}.$$

Thus it is sufficient to show from Lemma 5 that $\text{Cov}(Y_{M,1}, Y_{M,2})$ and $\text{Cov}(Y_{M,1}^2, Y_{M,2}^2)$ are $O(1/M)$. To do this, it is necessary to show that the denominator of $Y_{M,i}$ converges to a nonzero constant or to zero sufficiently slowly. Note that the numerator and denominator of $Y_{M,i}$ are, respectively,

$$\sum_{l \in I} w(l)g \left( \hat{L}_{k(l),k(l)}(X_i) \right) - \mathbb{E} \left[ \sum_{l \in I} w(l)g \left( \hat{L}_{k(l),k(l)}(X_i) \right) \right] = \sum_{l \in I} w(l) \left( g \left( \hat{L}_{k(l),k(l)}(X_i) \right) - \mathbb{E} \left[ g \left( \hat{L}_{k(l),k(l)}(X_i) \right) \right] \right),$$

(11)
Some preliminary work is required before we can directly tackle this quantity. Define the binomial theorem, where

\[ \text{Cov} \left( g \left( \hat{L}_{k(t)}(Z) \right), g \left( \hat{L}_{k'(t')}(Z) \right) \right) \]

Therefore, to bound \( \text{Cov}(Y_{M,1}, Y_{M,2}) \), we require bounds on the quantity \( \text{Cov} \left[ g \left( \hat{L}_{k(t)}(X_i) \right), g \left( \hat{L}_{k'(t')}(X_j) \right) \right] \).

Some preliminary work is required before we can directly tackle this quantity. Define \( \mathcal{M}(Z) := Z - EZ \hat{f}_{k(t)}(Z) := \hat{L}_{k(t)}(Z) - EZ \hat{L}_{k(t)}(Z), \) and \( \hat{e}_{1,k(t)}(Z) := \hat{f}_{1,k(t)}(Z) - EZ \hat{f}_{1,k(t)}(Z) \). By forming a Taylor series expansion of \( g \left( \hat{L}_{k(t)}(Z) \right) \) around \( EZ \hat{L}_{k(t)}(Z) \), we get

\[ g \left( \hat{L}_{k(t)}(Z) \right) = \sum_{i=0}^{\lambda-1} \frac{g^{(i)}(EZ \hat{L}_{k(t)}(Z))}{i!} \hat{F}_{k(t)}(Z) + \frac{g^{(\lambda)}(EZ \hat{L}_{k(t)}(Z))}{\lambda!} \hat{F}_{k(t)}(Z), \]

where \( \xi_Z \in \left( EZ \hat{f}_{k(t)}(Z), \hat{F}_{k(t)}(Z) \right) \). Let \( \Psi(Z) = g^{(\lambda)}(EZ \hat{L}_{k(t)}(Z)) / \lambda! \) and

\[ p_i^{(t)} := \mathcal{M} \left( g \left( \hat{E} X_i \hat{L}_{k(t)}(X_i) \right) \right), \]

\[ q_i^{(t)} := \mathcal{M} \left( g \left( \hat{E} X_i \hat{L}_{k(t)}(X_i) \right) \right), \]

\[ r_i^{(t)} := \mathcal{M} \left( \sum_{j=2}^{\lambda-1} \frac{g^{(j)}(EZ \hat{L}_{k(t)}(X_i))}{j!} \hat{F}_{k(t)}(X_i) \right), \]

\[ s_i^{(t)} := \mathcal{M} \left( \Psi(X_i) \hat{F}_{k(t)}(X_i) \right). \]

Then

\[ \text{Cov} \left[ g \left( \hat{L}_{k(t)}(X_i) \right), g \left( \hat{L}_{k'(t')}(X_j) \right) \right] = \mathbb{E} \left[ \left( p_i^{(t)} + q_i^{(t)} + r_i^{(t)} + s_i^{(t)} \right) \left( p_j^{(t')} + q_j^{(t')} + r_j^{(t')} + s_j^{(t')} \right) \right]. \tag{13} \]

To obtain expressions for \( \hat{F}_{k(t)}(Z) \), we expand \( \hat{L}_{k(t)}(Z) \) around \( EZ \hat{f}_{1,k(t)}(Z) \) and \( EZ \hat{f}_{2,k(t)}(Z) \):

\[ \hat{f}_{1,k(t)}(Z) = \frac{EZ \hat{f}_{1,k(t)}(Z)}{EZ \hat{f}_{2,k(t)}(Z)} + \frac{\hat{e}_{1,k(t)}(Z)}{EZ \hat{f}_{2,k(t)}(Z)} - EZ \hat{f}_{1,k(t)}(Z) \frac{\hat{e}_{2,k(t)}(Z)}{EZ \hat{f}_{2,k(t)}(Z)} \]

\[ - \frac{\hat{e}_{1,k(t)}(Z) \hat{e}_{2,k(t)}(Z)}{EZ \hat{f}_{2,k(t)}(Z)} + EZ \hat{f}_{1,k(t)}(Z) - \frac{\hat{e}^2_{2,k(t)}(Z)}{2 EZ \hat{f}_{2,k(t)}(Z)} \]

\[ + \frac{\hat{e}_{1,k(t)}(Z) \hat{e}^2_{2,k(t)}(Z)}{2 EZ \hat{f}_{2,k(t)}(Z)} \]

\[ = \frac{EZ \hat{f}_{1,k(t)}(Z)}{EZ \hat{f}_{2,k(t)}(Z)} + h(\hat{e}_{1,k(t)}(Z), \hat{e}_{2,k(t)}(Z)). \tag{14} \]

Let \( h(Z) = h(\hat{e}_{1,k(t)}(Z), \hat{e}_{2,k(t)}(Z)) \). Thus \( \hat{F}_{k(t)}(Z) = \frac{EZ \hat{f}_{1,k(t)}(Z)}{EZ \hat{f}_{2,k(t)}(Z)} - EZ \hat{L}_{k(t)}(Z) + h(Z) \). By the binomial theorem,

\[ \hat{F}_{k(t)}^{q}(Z) = \sum_{j=0}^{q} a_{q,j} \left( \frac{EZ \hat{f}_{1,k(t)}(Z)}{EZ \hat{f}_{2,k(t)}(Z)} - EZ \hat{L}_{k(t)}(Z) \right)^{q-j} h^{j}(Z), \tag{15} \]
where \( a_{q,j} \) is the binomial coefficient. Using a Taylor series expansion of \( \frac{1}{k} \) about \( \mathbb{E}[\hat{f}_{2,k_2}(Z)] \),

\[
\mathbb{E}[\frac{1}{\hat{f}_{2,k_2}(Z)}] = \mathbb{E}[\frac{1}{\mathbb{E}[\hat{f}_{2,k_2}(Z)]} - \frac{\hat{e}_{2,k_2}}{(\mathbb{E}[\hat{f}_{2,k_2}(Z)])^2} + \frac{\hat{e}_{2,k_2}^2}{2\zeta_2 Z}]
\]

\[
= \frac{1}{\mathbb{E}[\hat{f}_{2,k_2}(Z)]} + \left( \text{Var}_{\mathbb{E}}[\hat{f}_{2,k_2}(Z)] \right) \frac{1}{2\zeta_2 Z}
\]

\[
= \frac{1}{\mathbb{E}[\hat{f}_{2,k_2}(Z)]} + c_{3,2}(Z)\left( \frac{1}{k_2} \right).
\]  

(16)

where \( \xi_{2,Z} \in \left( \mathbb{E}[\hat{f}_{2,k_2}(Z), \hat{f}_{2,k_2}(Z)] \right) \) from the mean value theorem and we use the fact that the variance of the kernel density estimate converges to zero with rate \( \frac{1}{M^2} \). Sricharan et al. [3] showed that for a truncated uniform kernel density estimator with bandwidth \( \left( k/M \right)^{1/d} \),

\[
\mathbb{E}[\hat{f}_{1,k(l)}(Z)] = f_{1}(Z) + \sum_{j=1}^{d} c_{1,j,k(l)}(Z) \left( \frac{k(l)}{M} \right)^{j/d} + o \left( \frac{k(l)}{M} \right) = f_{1}(Z) + c_{1,1}(Z,k(l), M) = f_{1}(Z) + o(1).
\]

It can then be shown that the \( k-n \) density estimator converges to a truncated uniform kernel density estimator [3]. Thus the result holds for the \( k-n \) density estimator as well. Combining this with Eq. (16) gives

\[
\left( \frac{\mathbb{E}[\hat{f}_{1,k(l)}(Z)]}{\mathbb{E}[\hat{f}_{2,k(l)}(Z)]} - \mathbb{E}[\hat{f}_{k_1,k_2}(Z)] \right)^{q}
\]

\[
= \left( \mathbb{E}[\hat{f}_{1,k(l)}(Z)] c_{3,2}(Z) \left( \frac{1}{k(l)} \right) \right)^{q}
\]

\[
= \left( \frac{f_{1}(Z) c_{3,2}(Z) \left( \frac{1}{k(l)} \right) + \sum_{j=1}^{d} c_{1,j,k(l)} \left( \frac{k(l)}{M} \right)^{j/d} \left( \frac{1}{k(l)} \right) + o \left( \frac{1}{M} \right) \right)}{1_{(q=1)} c_{3}(Z) \left( \frac{1}{k(l)} \right) + 1_{(q \geq 2)} O \left( \frac{1}{k(l)^q} \right) + o \left( \frac{1}{M} \right)} \right) = b_{q,k(l)}(Z).
\]  

(17)

Combining Eqs. (14), (15), and (18) gives

\[
\hat{F}_{k(l)}^q(Z) = b_{q,k(l)}(Z) + b_{1,q\geq 2}^{(q \geq 3)}(Z) a_{q,1} h(Z) + 1_{(q \geq 2)} b_{q-2,k(l)}^{(q \geq 3)}(Z) a_{q,2} h^2(Z)
\]

\[
+ 1_{(q \geq 3)} b_{q-3,k(l)}^{(q \geq 4)}(Z) O(h^3(Z))
\]  

(19)

where

\[
h(Z) = \frac{\hat{e}_{1,k(l)}(Z)}{\mathbb{E}[\hat{f}_{2,k(l)}(Z)]} - \frac{\mathbb{E}[\hat{f}_{1,k(l)}(Z)]}{\mathbb{E}[\hat{f}_{2,k(l)}(Z)]} \frac{\hat{e}_{2,k(l)}(Z)}{\mathbb{E}[\hat{f}_{2,k(l)}(Z)]^2} - \frac{\hat{e}_{1,k(l)}(Z) \hat{e}_{2,k(l)}(Z)}{\mathbb{E}[\hat{f}_{2,k(l)}(Z)]^2}
\]

\[
+ \frac{\mathbb{E}[\hat{f}_{1,k(l)}(Z)]}{2 \mathbb{E}[\hat{f}_{2,k(l)}(Z)]} \hat{e}_{2,k(l)}(Z) + o \left( \frac{\hat{e}_{2,k(l)}(Z)}{\mathbb{E}[\hat{f}_{2,k(l)}(Z)]} \right),
\]

\[
h^2(Z) = \frac{\hat{e}_{1,k(l)}(Z)}{\mathbb{E}[\hat{f}_{2,k(l)}(Z)]} + \frac{\mathbb{E}[\hat{f}_{1,k(l)}(Z)]^2}{\mathbb{E}[\hat{f}_{2,k(l)}(Z)]^2} \frac{\hat{e}_{2,k(l)}(Z)}{\mathbb{E}[\hat{f}_{2,k(l)}(Z)]} + o \left( \frac{\hat{e}_{1,k(l)}(Z) \hat{e}_{2,k(l)}(Z)}{\mathbb{E}[\hat{f}_{2,k(l)}(Z)]} + \frac{\hat{e}_{2,k(l)}(Z)}{\mathbb{E}[\hat{f}_{2,k(l)}(Z)]} \right),
\]

\[
O(h^3(Z)) = O \left( \frac{\hat{e}_{1,k(l)}(Z) + \hat{e}_{2,k(l)}(Z) + \hat{e}_{3,k(l)}(Z)}{\mathbb{E}[\hat{f}_{2,k(l)}(Z)]} \right).
\]

We now obtain bounds on the expected value of products of the \( \hat{e}_{1,k(l)} \) terms:
Lemma 6. Let \( l, l' \in \tilde{I} \) be fixed, \( M_1 = M_2 = M \), and \( k(l) = l\sqrt{M} \). Let \( \gamma(z) \) be an arbitrary function with \( \sup_z |\gamma(z)| < \infty \). Let \( Z \) be a realization of the density \( f_2 \) independent of \( \hat{f}_{i,l(k)} \) and \( \hat{f}_{i,k(l')} \) for \( i = 1, 2 \). Then,

\[
E \left[ \gamma(Z) \hat{e}_{i,l(k)}^q(Z) \right] = \begin{cases} 
1_{\{q=2\}} \left( e_{2,i}^q(\gamma(z)) \left( \frac{1}{k(l)} \right) + o \left( \frac{1}{k(l)} \right) \right) + 1_{\{q\geq 3\}} O \left( \frac{1}{k(l)\bar{z}} \right), & q > 2 \quad \text{for} \ q = 1,
\end{cases}
\]

where we use the fact that \( \sup \hat{e}_{i,l(k)} = O \left( \frac{2}{k(l)} \right) \).

\[
E \left[ \gamma(Z) \hat{e}_{i,l(k)}^q(Z) \hat{e}^{-r}_{i,k(l')}^q(Z) \right] = \begin{cases} 
O \left( \frac{1}{k(l)\bar{z}} \right), & q + r \geq 2 \quad \text{otherwise},
\end{cases}
\]

\[
E \left[ \gamma(Z) \hat{e}_{i,l(k)}^q(Z) \hat{e}^{-r}_{i,k(l')}^q(Z) \right] \hat{e}_{i,k(l')}^r(Z) = \begin{cases} 
0, & q + q' = 1 \text{ or } r + r' = 1,
\end{cases}
\]

\[
E \left[ \gamma(Z) \hat{F}_{k(l)}^q(Z) \right] = 1_{\{q=1\}} O \left( \frac{1}{k(l)} \right) + 1_{\{q\geq 2\}} O \left( \frac{1}{k(l)\bar{z}} \right).
\]

Proof. For \( i = 2 \), Eq. (20) is given and proved as Lemma 5 in [2] where the density estimator is a truncated uniform kernel density estimator with bandwidth \( (k(l)/M)^{1/4} \). The proof uses concentration inequalities to bound \( \mathbb{E} \mathbb{Z} \hat{e}_{i,k(l)}^q(Z) \) in terms of \( k(l) \). Then since the truncated uniform kernel density estimator converges to the \( k \)-nn estimator, it holds for the \( k \)-nn estimator as well. For \( i = 1 \), the proof follows the same procedure but results in a different constant.

Equation (21) is proved in a similar manner. Let \( S_l(X) := \{ Y \in S : ||X - Y||_2 \leq (k(l)/M)^{1/4/2} \} \), \( V_i(X) := \int_{S_l(X)} dz \), \( U_{i,l}(X) := \Pr(Z \in S_l(X)) \) where \( Z \) is drawn from \( f_i \), and \( 1_{i,l}(X) \) denote the number of samples from the \( i \)th distribution that fall in \( S_l(X) \); i.e. the number of samples from \( \{Y_1, \ldots, Y_M\} \) if \( i = 1 \) or \( \{X_{N+1}, \ldots, X_{N+M}\} \) if \( i = 2 \) that fall in \( S_l(X) \). The uniform kernel density estimator is then

\[
\hat{f}_{i,l(k)}(X) = \frac{1_{i,l}(X)}{MV_i(X)}.
\]

Let \( z_{i,l}(X) \) denote the event \( (1 - p_{l,k(l)})MU_{i,l}(X) < 1_{i,l}(X) < (1 + p_{l,k(l)})MU_{i,l}(X) \), where \( p_{l,k(l)} = 1/(k(l)/4)^{d/2} \). It can be shown [2] using standard Chernoff inequalities that \( \Pr(z_{i,l}(X)) = O \left( e^{-p_{l,k(l)}^{d/2}} \right) \) and that under the event \( z_{i,l}(X) \), \( \hat{e}_{i,l(k)} \) and \( \hat{e}_{i,k(l')} \) are \( (1/(k(l)/4)^{d/2}) \). Thus

\[
E \left[ \gamma(Z) \hat{e}_{i,l(k)}^q(z) \hat{e}^{-r}_{i,k(l')}^q(z) \right] = E \left[ \gamma(Z) 1_{z_{i,l}(X) \cap z_{i,l'}(X)} \hat{e}_{i,l(k)}^q(z) \hat{e}^{-r}_{i,k(l')}^q(z) \right] + E \left[ \gamma(Z) 1_{z_{i,l}(X) \cap z_{i,l'}(X)} \hat{e}_{i,l(k)}^q(z) \hat{e}^{-r}_{i,k(l')}^q(z) \right] = O \left( \frac{1}{k(l)\bar{z}} \right),
\]

where we use the fact that \( \delta \) can be chosen arbitrarily close to 1.

For Eq. (22) note that due to conditional independence and Eq. (21)

\[
E \left[ \gamma(Z) \hat{e}_{i,l(k)}^q(z) \hat{e}^{-r}_{i,k(l')}^q(z) \hat{e}_{i,k(l')}^r(z) \right] = E \left[ \gamma(Z) E_Z \left[ \hat{e}_{i,l(k)}^q(z) \hat{e}^{-r}_{i,k(l')}^q(z) \hat{e}_{i,k(l')}^r(z) \right] \right] = O \left( \frac{1}{k(l)\bar{z}} \right).
\]

\[\square\]
Equation 23 is obtained by applying Eqs. 20 and 21 to Eq. 19.

Lemma 4 provides bounds on the covariance between the $\hat{F}_q^{i}(Z)$ terms and we restate it here along with its proof:

**Lemma 7.** Let $l, l' \in \mathcal{F}$ be fixed, $M_1 = M_2 = M$, and $k(l) = l \sqrt{M}$. Let $\gamma_1(x), \gamma_2(x)$ be arbitrary functions with 1 partial derivative wrt $x$ and $\sup_x |\gamma_i(x)| < \infty$, $i = 1, 2$. Let $X_i$ and $X_j$ be realizations of the density $f_2$ independent of $\hat{f}_{1,k(l)}$, $\hat{f}_{1,k(l')}$, $\hat{f}_{2,k(l)}$, and $\hat{f}_{2,k(l')}$ and independent of each other when $i \neq j$. Then

$$\text{Cov} \left[ \gamma_1(X_i) \hat{F}_q^{i}(X_i), \gamma_2(X_j) \hat{F}_q^{j}(X_j) \right] = \begin{cases} o(1), & i = j \\ 1_{(q = 1, r = 1)} (\gamma_1(x), \gamma_2(x)) \left( \frac{1}{M} \right) + o \left( \frac{1}{M} \right), & i \neq j. \end{cases}$$

**Proof.** Throughout the following, assume that $X$ and $Y$ are realizations of the density $f_2$ independent of each other and $\hat{f}_{1,k(l)}$, $\hat{f}_{1,k(l')}$, $\hat{f}_{2,k(l)}$, and $\hat{f}_{2,k(l')}$. First consider the case where $i = j$. By Cauchy-Schwarz and Eq. 20

$$\text{Cov} \left[ \gamma_1(X) \hat{e}_i^{q,k(l)}(X), \gamma_2(X) \hat{e}_i^{q,k(l')}(X) \right] = O \left( \frac{1}{M^{q+r}} \right).$$

By Eq. 21 and Eq. 22

$$\text{Cov} \left[ \gamma_1(X) \hat{e}_i^{q,k(l)}(X), \gamma_2(X) \hat{e}_i^{q,k(l')}(X) \right] = O \left( \frac{1}{M^{q+r}} \right).$$

Applying Eqs. 24 and 25 to Eq. 19 completes the proof for this case.

We’ll now prove the case where $i \neq j$. Define $\Psi(l, l') = \left\{ \|X - Y\|_1 \geq 2 \left( \frac{\max(k(l), k(l'))}{M} \right)^{\frac{1}{2}} \right\}$.

For a fixed pair of points $\{X, Y\} \in \Psi(l, l')$,

$$\text{Cov} \left[ \hat{e}_i^{q,k(l)}(X), \hat{e}_i^{q,k(l')}(Y) \right] = 1_{(q = r = 1)} \left( -f_i(X)f_i(Y) \right) \frac{1}{M} + o \left( \frac{1}{M} \right).$$

This can be shown in the same way as in the proof of Lemma 6 in [3] for a truncated uniform kernel density estimator. This is done by recognizing that for $\{X, Y\} \in \Psi(l, l')$, the functions $1_{i,l}(X)$ and $1_{i,l'}(Y)$ are distributed jointly as a multinomial random variable with parameters $M$, $U_i,l(X)$, $U_i,l'(Y)$ and $1 - U_i,l(X) - U_i,l'(Y)$. Equation 26 is then established by using the concentration inequality for the high probability event of $\hat{z}_{i,l}(X) \cap \hat{z}_{i,l'}(Y)$ and then relating the functions $1_{i,l}(X)$ and $1_{i,l'}(Y)$ to two binomial random variables with parameters $\{U_i,l(X), M - q\}$ and $\{U_i,l'(Y), M - r\}$, respectively. Note that the relationship holds whether $l = l'$ or $l \neq l'$. For fixed $\{X, Y\} \in \Psi(l, l')^C$, Cauchy-Schwarz and Eq. 20 give

$$\text{Cov} \left[ \hat{e}_i^{q,k(l)}(X), \hat{e}_i^{q,k(l')}(Y) \right] = O \left( \frac{1}{k(l)^{\frac{1}{2}} k(l')^{\frac{1}{2}}} \right).$$

From Eqs. 26 and 27 we have that

$$\text{Cov} \left[ \gamma_1(X) \hat{e}_i^{q,k(l)}(X), \gamma_2(Y) \hat{e}_i^{q,k(l')}(Y) \right] = 1_{(q = r = 1)} c_{7,i,l}(\gamma_1(x), \gamma_2(x)) \left( \frac{1}{M} \right) + o \left( \frac{1}{M} \right).$$

This is proved in the same way as in the proof of Lemma 8 in [3] by splitting the covariances into the cases where $\{X, Y\} \in \Psi(l, l')$ and $\{X, Y\} \in \Psi(l, l')^C$. For the first case, the bound falls clearly from Eq. 26. For the second case, the bound holds with Eq. 27 since $\int_{\Psi(l, l')^C} dy = o\left( \max(k(l), k(l')) \right)$.

Now let $E_0 = \{s, q, t, r \geq 1\}$, $E_{i,1} = \{s = 0, q \geq 2, t \geq 1, r \geq 1\} \cup \{s = 1, q \geq 1, t = 0, r \geq 2\}$, and $E_{i,2} = \{s \geq 2, q = 0, t \geq 1, r \geq 1\} \cup \{s \geq 1, q \geq 1, t \geq 2, r = 0\}$. For fixed $X, Y$, we have by Eqs. 20 and 21 and conditional independence when $E_0$, $E_{i,1}$, or $E_{i,2}$ hold that

$$\text{Cov} \left[ \gamma_1(X) \hat{e}_{1,k(l)}^{q}(X), \gamma_2(Y) \hat{e}_{1,k(l')}^{q}(Y) \right]$$
Combining Eqs. 29 and 30 gives

Now

By Eqs. 20, 26, and 27 this gives (when \(s, t \geq 1\))

Now

where

Combining Eqs. 29 and 30 gives

Similarly,

and so

Assume now that neither \(E_0\), \(E_{1,1}\), nor \(E_{1,2}\). If either \(q, r = 0\) or \(s, t = 0\) and the remaining exponents are nonzero, then the left hand side of Eq. 31 reduces to Eq. 28. For the other cases, suppose that \(s, q = 0\) and \(l, r \geq 2\) as an example. Then we have that

```python
Cov \left[ \gamma_1(X), \gamma_2(Y) \hat{e}_{1,k(l)}(X) \hat{e}_{1,k'(l')}(Y) \hat{e}_{2,k(l)}(X) \hat{e}_{2,k'(l')}(Y) \right] = \mathbb{E} \left[ \gamma_1(X) \gamma_2(Y) \hat{e}_{1,k(l)}(X) \hat{e}_{1,k'(l')}(Y) \hat{e}_{2,k(l)}(X) \hat{e}_{2,k'(l')}(Y) \right] \\
- \mathbb{E} \left[ \gamma_1(X) \hat{e}_{1,k(l)}(X) \hat{e}_{2,k(l)}(X) \right] \mathbb{E} \left[ \gamma_2(Y) \hat{e}_{1,k'(l')}(Y) \hat{e}_{2,k'(l')}(Y) \right] \\
= \mathbb{E} \left[ \gamma_1(X) \gamma_2(Y) \hat{e}_{1,k(l)}(X) \hat{e}_{1,k'(l')}(Y) \hat{e}_{2,k(l)}(X) \hat{e}_{2,k'(l')}(Y) \right] \\
- \mathbb{E} \left[ \gamma_1(X) \hat{e}_{1,k(l)}(X) \hat{e}_{2,k(l)}(X) \right] \mathbb{E} \left[ \gamma_2(Y) \hat{e}_{1,k'(l')}(Y) \hat{e}_{2,k'(l')}(Y) \right] \\
= \mathbb{E} \left[ \gamma_1(X) \gamma_2(Y) \hat{e}_{1,k(l)}(X) \hat{e}_{1,k'(l')}(Y) \hat{e}_{2,k(l)}(X) \hat{e}_{2,k'(l')}(Y) \right] \\
- \mathbb{E} \left[ \gamma_1(X) \hat{e}_{1,k(l)}(X) \hat{e}_{2,k(l)}(X) \right] \mathbb{E} \left[ \gamma_2(Y) \hat{e}_{1,k'(l')}(Y) \hat{e}_{2,k'(l')}(Y) \right] \\
= 0.
```
Now let \( A \) be any arbitrary functional which satisfies

\[
\Psi(\xi) = (\xi_1 - \xi_2)^2
\]

Lemma 8. Assume that \( U(x) \) is any arbitrary functional which satisfies

\[
(i) \quad \mathbb{E} \left[ \sup_{L \in \mathcal{P}_p} \left| U \left( \frac{L}{P_u} \right) \right| \right] = G_1 < \infty, \\
(ii) \quad \sup_{L \in \mathcal{P}_p} \left| U \left( \frac{L}{P_p} \right) \right| \mathcal{C}(k_1) \mathcal{C}(k_2) = G_2 < \infty, \\
(iii) \quad \mathbb{E} \left[ \sup_{L \in \mathcal{P}_p} \left| U \left( \frac{L}{P_u} \right) \right| \mathcal{C}(k_1) \right] = G_3 < \infty, \\
(iv) \quad \mathbb{E} \left[ \sup_{L \in \mathcal{P}_p} \left| U \left( \frac{L}{P_u} \right) \right| \mathcal{C}(k_2) \right] = G_4 < \infty.
\]

Let \( Z \) be \( X_i \) for some fixed \( i \in \{1, \ldots, N\} \) and \( \xi_Z \) be any random variable which almost surely lies in \((L(Z), \hat{L}_{k_1,k_2}(Z))\). Then \( \mathbb{E}[U(\xi_Z)] < \infty \).

Proof. This is a version of Lemma 9 in [3] modified to apply to functionals of the likelihood ratio. Because of assumption A.1, it is sufficient to show that the conditional expectation \( \mathbb{E}[U(\xi_Z) | X_1, \ldots, X_N] < \infty \).

First, some properties of \( k\)-NN density estimators are required. Let \( S_{k,i}(Z) = \{Y : d(Z,Y) \leq d_{Z,i}^{(k)}\} \) where \( d_{Z,i}^{(k)} \) is the distance to the \( k \)th nearest neighbor of \( Z \) from the corresponding set of samples. Then let \( P_i(Z) = \int_{S_{k,i}(Z)} \hat{f}_i(x) dx \) which has a beta distribution with parameters \( k_i \) and \( M_i - k_i + 1 \). Let \( A_i(Z) \) be the event that \( P_i(Z) < \left( \frac{1}{\sqrt{k_i} + 1} \right)^{k_i+1} M_i \). It has been shown that \( \text{Pr}(A_i(Z)) = \Theta(\mathcal{C}(k_i)) \) and that under \( A_i(Z) \) it has also been shown that under \( A_i(Z)^C \) \( q_{l,i} \) is \( q_{u,i} \).

\[
q_{l,i} < \hat{f}_{l,k_i}(Z) < q_{u,i}
\]

Let \( A(Z) = A_1(Z) \cap A_2(Z) \) and note that \( A_1(Z) \) and \( A_2(Z) \) are independent events. Thus since \( \hat{L}_{k_1,k_2}(Z) = \frac{\hat{f}_{i,k_1}(Z)}{\hat{f}_{i,k_2}(Z)} \), we have that under \( A(Z) \),

\[
\hat{P}_i(Z) < \hat{L}_{k_1,k_2}(Z) < \hat{P}_u(Z).
\]

Now let \( Q_1(Z) = A_1(Z)^C \cap A_2(Z)^C \), \( Q_2(Z) = A_1(Z)^C \cap A_2(Z) \), and \( Q_3(Z) = A_1(Z) \cap A_2(Z)^C \). Then due to independence and the fact that the \( Q_i(Z) \)'s are disjoint,

\[
A(Z)^C = A_1(Z)^C \cup A_2(Z)^C = Q_1(Z) \cup Q_2(Z) \cup Q_3(Z),
\]

\[
\Rightarrow \text{Pr}(A(Z)^C) = \text{Pr}(Q_1(Z)) + \text{Pr}(Q_2(Z)) + \text{Pr}(Q_3(Z)) \leq C(k_1)C(k_2) + C(k_1) + C(k_2).
\]
Then under $Q_1(Z)$, $Q_2(Z)$, and $Q_3(Z)$, respectively,

$$
\frac{q_{i,1}}{q_{i,2}} < \hat{l}_{k_1,k_2}(Z) < \frac{q_{u,1}}{q_{u,2}},
$$

$$
\frac{q_{i,1} p_{2}(Z)}{p_{i,2}} < \hat{l}_{k_1,k_2}(Z) < \frac{q_{u,1} p_{2}(Z)}{p_{i,2}},
$$

$$
\frac{p_{i,1}}{P_{1}(Z) q_{u,2}} < \hat{l}_{k_1,k_2}(Z) < \frac{p_{u,1}}{P_{1}(Z) q_{i,2}}.
$$

Conditioning on $X_1, \ldots, X_N$ gives

$$
\mathbb{E} \left[ |U(\xi)| \right] = \mathbb{E} \left[ 1_A(Z) |U(\xi)| \right] + \mathbb{E} \left[ 1_{Q_1(Z)} |U(\xi)| \right] + \mathbb{E} \left[ 1_{Q_2(Z)} |U(\xi)| \right] + \mathbb{E} \left[ 1_{Q_3(Z)} |U(\xi)| \right]
$$

$$
\leq Pr(A(Z)) \mathbb{E} \left[ \sup_{L \in (p_i, p_u)} \left| U \left( \frac{L p_2(Z)}{P_{1}(Z)} \right) \right| \right] + Pr(Q_1(Z)) \mathbb{E} \left[ \sup_{L \in \left( \frac{q_{i,1}}{q_{u,2}}, \frac{q_{u,1}}{q_{i,2}} \right)} \left| U \left( \frac{L P_{2}(Z)}{P_{1}(Z)} \right) \right| \right]
$$

$$
+ Pr(Q_1(Z)) \mathbb{E} \left[ \sup_{L \in \left( \frac{q_{i,1}}{q_{u,2}}, \frac{q_{u,1}}{q_{i,2}} \right)} \left| U \left( \frac{L}{P_{1}(Z)} \right) \right| \right]
$$

$$
\leq \mathbb{E} \left[ \sup_{L \in (p_i, p_u)} \left| U \left( \frac{L p_2(Z)}{P_{1}(Z)} \right) \right| \right] + \mathbb{E} \left[ \sup_{L \in \left( \frac{q_{i,1}}{q_{u,2}}, \frac{q_{u,1}}{q_{i,2}} \right)} \left| U \left( \frac{L P_{2}(Z)}{P_{1}(Z)} \right) \right| \right]
$$

$$
+ \mathbb{E} \left[ \sup_{L \in \left( \frac{q_{i,1}}{q_{u,2}}, \frac{q_{u,1}}{q_{i,2}} \right)} \left| U \left( \frac{L}{P_{1}(Z)} \right) \right| \right] \mathcal{C}(k_1) \mathcal{C}(k_2)
$$

$$
= G_1 + G_2 + G_3 + G_4 < \infty.
$$

\[\square\]

The next lemma gives the last result necessary to bound the covariance of $Y_{M,1}$ and $Y_{M,2}$.

**Lemma 9.** Let $l, l' \in \mathbb{I}$ be fixed, $M_1 = M_2 = M$, $k(l) = l\sqrt{M}$, and $\hat{L}_{k(l)} = \hat{L}_{k(l), k(l)}$. Let $X_i$ and $X_j$ be realizations of the density $f_2$ independent of $\hat{f}_{1,k_1}$ and $\hat{f}_{2,k_2}$ and independent of each other when $i \neq j$. Then

$$
Cov \left[ g \left( \hat{L}_{k(l)}(X_i) \right), g \left( \hat{L}_{k(l')}(X_j) \right) \right]
$$

$$
= \begin{cases} 
\mathbb{E} \left[ p_i^{(l)} p_i^{(l')} \right] + o(1), & i = j \\
ct\left( g' \left( \mathbb{E} X \hat{L}_{k(l)}(x) \right) \right) \left( \frac{1}{M} \right) + o \left( \frac{1}{M} \right), & i \neq j.
\end{cases}
$$

(32)

**Proof.** Consider the case where $i = j$. Then applying Lemma 7 to Eq. 15 gives

$$
Cov \left[ g \left( \hat{L}_{k(l)}(X_i) \right), g \left( \hat{L}_{k(l')}(X_i) \right) \right] = \mathbb{E} \left[ p_i^{(l)} p_i^{(l')} \right] + o(1).
$$

Note that $\mathbb{E} \left[ p_i^{(l)} p_i^{(l')} \right] = O(1)$ since $p_i^{(l)} = \mathcal{M} \left( g \left( \mathbb{E} X \hat{L}_{k(l)}(X_i) \right) \right) = \mathcal{M} \left( g \left( L(X_i) \right) \right) + o(1)$.
Now let $i \neq j$. Since $X_i$ and $X_j$ are independent, $\mathbb{E} \left[ p_i^{(l)} \left( p_j^{(l')} + q_j^{(l')} + r_j^{(l')} + s_j^{(l')} \right) \right] = 0$. Applying Lemma 7 gives

\[
\mathbb{E} \left[ q_i^{(l)} q_j^{(l')} \right] = c_8 \left( g^\prime \left( \mathbb{E}_X \hat{L}_{k(l)}(x) \right), g^\prime \left( \mathbb{E}_X \hat{L}_{k(l')}(x) \right) \right) \left( \frac{1}{M} \right) + o \left( \frac{1}{M} \right),
\]

\[
\mathbb{E} \left[ q_i^{(l)} r_j^{(l')} \right] = o \left( \frac{1}{M} \right),
\]

\[
\mathbb{E} \left[ r_i^{(l)} r_j^{(l')} \right] = o \left( \frac{1}{M} \right).
\]

We use Cauchy-Schwarz and Lemma 6 to get

\[
\mathbb{E} \left[ g^\prime \left( \mathbb{E}_X \hat{L}_{k(l)}(X_i) \right) \hat{F}_{k(l)}(X_i) \left( \Psi(X_j) \hat{F}_{k(l')}^\lambda(X_j) \right) \right]
\]

\[
\leq \sqrt{\mathbb{E} \left[ \Psi^2(X_j) \right] \mathbb{E} \left[ \left( g^\prime \left( \mathbb{E}_X \hat{L}_{k(l)}(X_i) \right) \hat{F}_{k(l)}(X_i) \right)^2 \hat{F}_{k(l')}^{2\lambda}(X_j) \right]}
\]

\[
\leq \sqrt{\mathbb{E} \left[ \Psi^2(X_j) \right] \mathbb{E} \left[ \left( g^\prime \left( \mathbb{E}_X \hat{L}_{k(l)}(X_i) \right) \hat{F}_{k(l)}(X_i) \right)^4 \hat{F}_{k(l')}^{4\lambda}(X_j) \right]}
\]

\[
= \sqrt{\mathbb{E} \left[ \Psi^2(X_j) \right] \mathbb{E} \left[ O \left( \frac{1}{k(l)^2} \right) O \left( \frac{1}{k(l')^{2\lambda}} \right) \right]}
\]

\[
= \sqrt{\mathbb{E} \left[ \Psi^2(X_j) \right] o \left( \frac{1}{k(l)^{\lambda/2}} \right)}.
\]

Lemma 8 and assumption (A.5) implies that $\mathbb{E} \left[ \Psi^2(X_j) \right] = O(1)$ and from assumption (A.3), $o \left( \frac{1}{k(l)^{\lambda/2}} \right) = o \left( \frac{1}{M} \right)$. This implies that $\mathbb{E} \left[ q_i^{(l)} s_j^{(l')} \right] = o \left( \frac{1}{M} \right)$. Similarly, $\mathbb{E} \left[ r_i^{(l)} s_j^{(l')} \right] = o \left( \frac{1}{M} \right)$ and $\mathbb{E} \left[ s_i^{(l)} s_j^{(l')} \right] = o \left( \frac{1}{M} \right)$. Combining these results with Eq. 13 completes the proof.

Applying Lemma 9 to Eqs. 11 and 12 shows that $\text{Cov}(Y_{M,i}, Y_{M,j}) = O \left( \frac{1}{M} \right)$.

For the covariance of $Y_{M,i}^2$ and $Y_{M,j}^2$, we only need to consider the numerator since we previously showed that $\text{Var} \left[ \sum_{l \in l} w(l) g \left( \hat{L}_{k(l)}(X_i) \right) \right] = O(1) + o(1)$. Assume WLOG that $i = 1$ and $j = 2$ and let $h_l = \mathbb{E} \left[ g \left( \hat{L}_{k(l)}(X_1) \right) \right]$. The numerator of the covariance is then

\[
\sum_{l \in l} \sum_{l' \in l'} \sum_{i \in i} \sum_{j' \in j'} \text{Cov} \left[ \left( g \left( \hat{L}_{k(l)}(X_1) \right) - h_l \right) \left( g \left( \hat{L}_{k(l')} (X_1) \right) - h_{l'} \right), \right.
\]

\[
\left. \left( g \left( \hat{L}_{k(l)}(X_2) \right) - h_l \right) \left( g \left( \hat{L}_{k(l')} (X_2) \right) - h_{l'} \right) \right]
\]

\[
= \sum_{l \in l} \sum_{l' \in l'} \sum_{i \in i} \sum_{j' \in j'} \text{Cov} \left[ \left( p_1^{(l)} + q_1^{(l)} + r_1^{(l)} + s_1^{(l)} \right) \left( p_1^{(l')} + q_1^{(l')} + r_1^{(l')} + s_1^{(l')} \right), \right.
\]

\[
\left. \left( p_2^{(l)} + q_2^{(l)} + r_2^{(l)} + s_2^{(l)} \right) \left( p_2^{(l')} + q_2^{(l')} + r_2^{(l')} + s_2^{(l')} \right) \right]
\]

Let $d_l(x) = (g(x) - h_l)^2$. Then for the case where $l = l'$ and $j = j'$, we have

\[
\text{Cov} \left[ d_l \left( \hat{L}_{k(l)}(X_1) \right), d_j \left( \hat{L}_{k(l)}(X_2) \right) \right] = O \left( \frac{1}{M} \right).
\]

This follows from Lemma 9.
For the general case, note that due to the independence of $X_1$ and $X_2$, 
\[
\text{Cov} \left[ p_1^{(l)}, p_2^{(j)} \left( g \left( \hat{L}_{k(j)}(X_2) \right) - h_j \right) \left( g \left( \hat{L}_{k(j')}^{(l)}(X_2) \right) - h_{j'} \right) \right] = 0, 
\]
\[
\text{Cov} \left[ g \left( \hat{L}_{k(l)}(X_1) \right) - h_l \right) \left( g \left( \hat{L}_{k(j')}^{(l)}(X_1) \right) - h_{j'} \right) , p_2^{(j)} p_2^{(j')} \right] = 0. 
\]
To bound the remaining terms, we require the following Lemma:

**Lemma 10.** Let $\gamma_1(x)$, $\gamma_2(x)$ be arbitrary functions with 1 partial derivative wrt $x$ and $\sup_x |\gamma_i(x)| < \infty$, $i = 1, 2$. Let $l, l', j, j' \in \mathbb{I}$ be fixed, $M_1 = M_2 = M$, $k(l) = l\sqrt{M}$. Let $X$ and $Y$ be realizations of the density $f_x$ independent of $\hat{f}_i(k(l))$, $\hat{f}_i(k(l'))$, $\hat{f}_i(k(j))$, and $\hat{f}_i(k(j'))$, $i = 1, 2$. If $q, r, s, t \geq 0$ and the cases $\{t = 0, r = 0\}$ or $\{q = 0, s = 0\}$ do not hold, then
\[
\text{Cov} \left[ \gamma_1(X) \hat{F}_{k(l)}^s(X), \gamma_2(Y) \hat{F}_{k(j')}^r(Y) \right] = O \left( \frac{1}{M} \right). 
\]

**Proof.** Under certain conditions, by Cauchy-Schwarz and Eq.[22] we have
\[
\text{Cov} \left[ \gamma_1(X) \hat{e}_{1,k(l)}(X) \hat{e}_{2,k(l')}^{r}(X) \hat{e}_{1,k(l')}^{s}(X) \right], \gamma_2(Y) \hat{e}_{1,k(j)}(Y) \hat{e}_{2,k(j')}^{r}(Y) \hat{e}_{1,k(j')}^{s}(Y) 
\]
\[
\leq E \left[ \gamma_1(X) \gamma_2(Y) \text{Var}_X \left[ \hat{e}_{1,k(l)}(X) \hat{e}_{2,k(l')}^{r}(X) \hat{e}_{1,k(l')}^{s}(X) \right] \right] 
\]
\[
\times \text{Var}_Y \left[ \hat{e}_{1,k(j)}(Y) \hat{e}_{2,k(j')}^{r}(Y) \hat{e}_{1,k(j')}^{s}(Y) \hat{e}_{1,k(j')}^{s}(Y) \right] 
\]
\[
= O \left( \frac{1}{M} \right). 
\]
Note that the exponents $q, s, r, t$ are not the same as in the statement of the lemma. The conditions under which this expression holds are as follows: (1) There must be at least one positive exponent on both sides of the arguments in the covariance. (2) $\{s + s' + t + t' \neq 1\} \cap \{q + q' + r + r' \neq 1\}$. If neither case holds, this reduces to Eq. [28]. If only one holds, then the covariance is zero. Note that if $s + q + s' + q' + t + r + t' + r' \geq 4$, Eq. [33] becomes $O \left( \frac{1}{M^4} \right)$. Now consider the case where $\{s + s' + t + t' = 3\} \cap \{s, s', s, t, t' \leq 1\} \cap \{q, q', r, r' = 0\} \cup \{q + q' + r + r' = 3\} \cap \{q, q', r, r' \leq 1\} \cap \{s, s', t, t' = 0\}$. Assume WLOG that $s, s', t = 1$. Then Eq. [33] becomes $O \left( \frac{1}{M^4} \right)$ which does not decay fast enough to use Lemma 5. However, we can use the fact that $k(l) = O(k(l'))$ to obtain a bound of $O \left( \frac{1}{M^4} \right)$. By Markov's inequality and Eqs. [20] and [21] for fixed $\nu > 0$,
\[
\Pr \left[ \left| \hat{e}_{1,k(l)}(X) - \hat{e}_{1,k(l')}^{s}(X) \right| > \nu \right] \leq \frac{E \left[ \left( \hat{e}_{1,k(l)}(X) - \hat{e}_{1,k(l')}^{s}(X) \right)^4 \right]}{\nu^4} = O \left( \frac{1}{M} \right). 
\]
Let $H$ be the event that $\left| \hat{e}_{1,k(l)}(X) - \hat{e}_{1,k(l')}^{s}(X) \right| \leq 1$. This gives
\[
\text{Cov} \left[ \gamma_1(X) \hat{e}_{1,k(l)}(X) \hat{e}_{1,k(l')}^{s}(X), \gamma_2(Y) \hat{e}_{1,k(j)}^{s}(Y) \right] 
\]
\[
= E \left[ \gamma_1(X) \gamma_2(Y) \hat{e}_{1,k(l)}(X) \hat{e}_{1,k(l')}^{s}(X) \hat{e}_{1,k(j)}^{s}(Y) \right] 
\]
\[
= E \left[ H \gamma_1(X) \gamma_2(Y) \hat{e}_{1,k(l)}(X) \hat{e}_{1,k(l')}^{s}(X) \hat{e}_{1,k(j)}^{s}(Y) \right] 
\]
\[
+ E \left[ H^c \gamma_1(X) \gamma_2(Y) \hat{e}_{1,k(l)}(X) \hat{e}_{1,k(l')}^{s}(X) \hat{e}_{1,k(j)}^{s}(Y) \hat{e}_{1,k(j)}^{s}(Y) \right] 
\]
\[
\leq E \left[ H \gamma_1(X) \gamma_2(Y) \hat{e}_{1,k(l)}^{2}(X) \hat{e}_{1,k(j')}^{s}(Y) \right] + E \left[ H \gamma_1(X) \gamma_2(Y) \hat{e}_{1,k(l)}^{s}(X) \hat{e}_{1,k(l')}^{s}(X) \hat{e}_{1,k(j)}^{s}(Y) \right] 
\]
\[
+ E \left[ H^c \gamma_1(X) \gamma_2(Y) \hat{e}_{1,k(l)}^{s}(X) \hat{e}_{1,k(l')}^{s}(X) \hat{e}_{1,k(j)}^{s}(Y) \hat{e}_{1,k(j')}^{s}(Y) \right] 
\]
\[
= O \left( \frac{1}{M} \right). 
\]
The final step for the first two terms comes from Eq. 28. The final step for the third term comes from the fact that $Pr(H^O) = O \left( \frac{1}{N} \right)$ and the fact that $\mathbb{E} [ \gamma_1(X) \gamma_2(Y) \mathbf{e}_{1,k(l)}^{(X)} \mathbf{e}_{1,k(l')}^{(X)} \mathbf{e}_{1,k(l)}^{(Y)} ] = o(1)$ by Eq. 33. Applying Eqs. 28, 31, and 33 to Eq. 19 completes the proof.

From Lemma 10 it is clear that
\[
\text{Cov} \left[ \left( \mathbf{p}_1^{(l)} + \mathbf{q}_1^{(l)} + \mathbf{r}_1^{(l)} + \mathbf{s}_1^{(l)} \right) \left( \mathbf{p}_1^{(l')} + \mathbf{q}_1^{(l')} + \mathbf{r}_1^{(l')} + \mathbf{s}_1^{(l')} \right), \\
\left( \mathbf{p}_2^{(j)} + \mathbf{q}_2^{(j)} + \mathbf{r}_2^{(j)} + \mathbf{s}_2^{(j)} \right) \left( \mathbf{p}_2^{(j')} + \mathbf{q}_2^{(j')} + \mathbf{r}_2^{(j')} + \mathbf{s}_2^{(j')} \right) \right] = O \left( \frac{1}{M} \right)
\]
\[
\implies \text{Cov} \left[ Y_{M,i}^2, Y_{M,j}^2 \right] = O \left( \frac{1}{M} \right).
\]
Then by Lemma 5 $\mathbf{S}_{N,M} = \frac{G_{w} - \mathbb{E}[G_{w}]}{\sqrt{\text{Var}[G_{w}]}$ converges in distribution to a standard normal random variable.

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