STABILITY INEQUALITIES FOR LAWSON CONES

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Dedicated to Xunjing Wei

Abstract

In paper \cite{1}, G. De. Philippis and F. Maggi proved global quadratic stability inequalities and derived explicit lower bounds for the first eigenvalues of the stability operators for all area-minimizing Lawson cones $M_{kh}$, except for those with

$$(k, h), (h, k) \in S = \{(3,5), (2,7), (2,8), (2,9), (2,10), (2,11)\}.$$  

We proved the corresponding inequalities and lower bounds for these Lawson cones $M_{kh}$ with $(h, k), (k, h) \in S$ by using different sub-calibrations from theirs, thus extending their results to all area-minimizing Lawson cones.

1. Introduction

Suppose $h, k \geq 2$ are positive integers. The Lawson cone $M_{kh}$ is the level set

$$M_{kh} = \left\{ z = (x, y) \in \mathbb{R}^k \times \mathbb{R}^h : \frac{|x|}{\sqrt{k-1}} = \frac{|y|}{\sqrt{h-1}} \right\}.$$  

It is known to be area-minimizing (see \cite{2}, \cite{3}, \cite{4}, and \cite{5}) provided

$$h + k \geq 9, \text{ or } (h, k) = (3, 5), (4, 4), (5, 3).$$  

In their paper \cite{1}, G. De. Philippis and F. Maggi proved global quadratic stability inequalities and derived explicit lower bounds for the first eigenvalues of the stability operators for all area-minimizing Lawson cones $M_{kh}$, except for

$$(h, k), (k, h) \in S = \{(3,5), (2,7), (2,8), (2,9), (2,10), (2,11)\}.$$  

They achieved this by exploiting sub-calibrations for Lawson cones. Unfortunately, the sub-calibrations that they used did not work for the cones $M_{kh}$ with $(h, k), (k, h) \in S$. Our main results, Theorem 1 and Theorem 2 in Section 1.1, extend these inequalities to the cones $M_{kh}$ with $(h, k), (k, h) \in S$. We achieve this by carefully choosing sub-calibrations for these Lawson cones in Lemma 2 of Section 2.1. However, our sub-calibrations do not work for other cases in general.
We first review their results and explain their methods, which we mostly follow. Consider a variation with compact support of the Lawson cone $M_{kh}$. Suppose the variation can be realized as the boundary of a set $F$ of finite perimeter. Roughly speaking, their first result controls the volume bounded between the Lawson cone and the variation $\partial F$ by the difference between the area of the variation $\partial F$ and that of the cone $M_{kh}$ up to scaling. Their second result provides lower bounds for the first eigenvalues of the stability operators. For a great discussion of the significance of these results, please refer to Section 1 of [1].

The Lawson cone $M_{kh}$ can be realized as the boundary $\partial K_{kh}$ of the region

$$K_{kh} = \left\{ (x, y) \in \mathbb{R}^k \times \mathbb{R}^h : \frac{|x|}{\sqrt{k-1}} < \frac{|y|}{\sqrt{h-1}} \right\}.$$  

Let $\mathcal{L}^m$ denote the $m$-dimensional Lebesgue measure, $\omega_n$ denote the volume of unit $n$-ball, and $P(A; B)$ denote the perimeter of $A$ in $B$. Their results are as follows.

**Result 1.** (Theorem 5 in [1]) If $R > 0, m = h + k, (h, k) \notin S$ satisfy all the conditions in (1), then

$$\left( \frac{\mathcal{L}^m(K_{kh} \Delta F)}{R^m} \right)^2 \leq C \frac{P(F; H_R) - P(K_{kh}; H_R)}{R^{m-1}},$$

whenever $F$ is a set of locally finite perimeter with symmetric difference $K_{kh} \Delta F \subset \subset H_R = B_R \times B_R$. Possible values of $C$ are

$$C = \frac{2^{12}}{(k-1)^{1/8}} \sqrt{\frac{hk}{m-1}} \left( \frac{h-1}{k-1} \right)^{3/2}, \text{ if } 2 \leq k \leq h, (h, k) \neq (4, 4),$$

Interchange $k, h$ if $2 \leq h \leq k$.

$$C = 128 \omega_4, \text{ if } (k, h) = (4, 4).$$

**Result 2.** (Theorem 2 in [1]) If $R, m, h, k$ are as in Result 1, and

$$\lambda_{k,h}(R) = \inf \left\{ \int_{M_{kh}} |\nabla M_{kh} \varphi|^2 - |\Pi_{M_{kh}}|^2 \varphi^2 d\mathcal{H}^{m-1} : \int_{M_{kh}} \varphi^2 = 1, \text{spt} \varphi \subset \subset B^m_R \right\},$$

then

$$\lambda_{k,h}(R) \geq \frac{c_{k,h}}{R^2}.$$

Possible values of $c_{k,h}$ are

$$c_{k,h} = \frac{1}{2^9} \left( \frac{k-1}{h-1} \right)^{9/4} \left( \frac{m-2}{(h-1)^{1/4}} \right)^{1/2}, \text{ if } 2 \leq k \leq h, (k, h) \neq (4, 4).$$

Interchange $k, h$ if $2 \leq h \leq k$.

$$c_{k,h} = \frac{\sqrt{2}}{16}, \text{ if } (k, h) = (4, 4).$$

As illustrated in Figure 1, their method is based on sub-calibrating the Lawson cones with a unit-length vector field $g$. In other words, the vector field $g$
restricts to the unit normal on $M_{kh}$, and the divergence $\text{div} \ g$ does not change sign in $K_{kh}$ and $K^c_{kh}$, respectively.

After cleverly choosing $g$, they proved that

$$\text{div} \ g(z) \geq c_{k,h} \frac{\text{dist}(z, M_{kh})}{|z|^2},$$

where $\text{dist}$ is the Euclidean distance. Then they exploit inequality (2) to deduce the desired results. For a beautiful discussion of sub-calibrations (also called quantitative calibrations), please refer to their paper [1].

Unfortunately, the sub-calibrations they used did not work for $(h,k), (k,h) \in S$. The main results of this paper extend their stability inequalities to include those $(k,h)$. We achieve this by using sub-calibrations inspired by [5].

### 1.1. Stability Inequalities Extended to $(h,k), (k,h) \in S$.

**Theorem 1.** If $R > 0, m = h + k, (h,k), (k,h) \in S$, then

$$\left( \frac{\mathcal{L}^m(K_{kh}\Delta F)}{R^m} \right)^2 \leq C \frac{P(F; H_R) - P(K_{kh}; H_R)}{R^{m-1}},$$

whenever $F$ is a set of locally finite perimeter with $K_{kh}\Delta F \subset\subset B^k_R \times B^h_R$. A possible value of $C$ is $7^2 \times 12^2 \times 10^{20}$.

**Theorem 2.** If $r, m, h, k$ are as in Theorem 1, and

$$\lambda_{k,h}(R) = \inf \left\{ \int_{M_{kh}} |\nabla M_{kh} \varphi|^2 - |\Pi M_{kh} \varphi|^2 d\mathcal{H}^{m-1} : \int_{M_{kh}} \varphi^2 = 1, \text{spt} \varphi \subset\subset B^m_R \right\},$$

then

$$\lambda_{k,h}(R) \geq \frac{c_{k,h}}{R^2},$$

Possible values of $c_{k,h}$ are

$$c_{3,5} = c_{5,3} = \frac{\sqrt{3}}{21^3},$$
for \( k, h = 7, 8, 9, 10, 11 \).

2. Proof of the Theorems

We now prove, in order, Theorem 2 and Theorem 1. By the symmetry of Lawson cones, it suffices to prove the cases with \( (h, k) \in S \). The following lemma is the basic tool to extract information from the sub-calibrations \( g \).

**Lemma 1.** If \( m \geq 2 \), \( E \) is of locally finite perimeter in \( \mathbb{R}^m \), and \( g \in W^{1,1}_{\text{loc}}(\mathbb{R}^m, \mathbb{R}^m) \),
\[
|g| \leq 1 \text{ on } \mathbb{R}^m, \\
\text{div } g \geq 0, \text{ a.e. on } E^c, \\
\text{div } g \leq 0, \text{ a.e. on } E, \\
g = \nu_E, \mathcal{H}^{m-1} \text{ a.e. on } \partial_{1/2}E,
\]
then \( E \) is a local minimizer of the perimeter in \( \mathbb{R}^m \), with
\[
P(F; A) - P(E; A) = \int_{E \Delta F} |\text{div } g| + \int_{A \cap \partial_{1/2}E} 1 - (g \cdot \nu_E) d\mathcal{H}^{m-1}.
\]

Here \( \mathcal{H}^{m-1} \) is the \( m-1 \)-dimensional Hausdorff measure, \( \nu_E \) is the out-pointing unit normal. If \( |E| \) denote the \( \mathcal{L}^m \)-volume of a set \( E \), then
\[
\partial_{1/2}E = \{ x \in \mathbb{R}^m : \lim_{r \to 0^+} \frac{|E \cap B(x, r)|}{\omega_n r^m} = \frac{1}{2} \},
\]
is defined as the set of points of density 1/2 in \( E \). For proof of Lemma 1 and details about \( \partial_{1/2}E \), please refer to the proof of Proposition 4.1 in [1] and the relevant discussions on page 416 in [1]. Roughly speaking, Lemma 1 can be proved by breaking down the integration definition of perimeter and then using the divergence theorem.

The left hand-side of (3) can be seen as variation of area, so it can provide information for second variation by Taylor expansion and choosing suitable variation \( F \). The key to using this information is to find vector fields \( g \) that satisfy inequality (1) in Section 1.

2.1. Sub-calibrations for \( M_{kh} \) with \( (h, k) \in S \).

**Lemma 1.** For \( E = K_{kh} \), the vector field
\[
g = \frac{\nabla f}{|\nabla f|}
\]
satisfies all the hypothesis in Lemma 1. The function \( f \) for \( (h, k) = (3, 5) \) is
\[
f(x, y) = \begin{cases}
\frac{(h - 1)|x|^2 - (k - 1)|y|^2}{((h - 1)|x|)^{3/2}}, & \text{if } z \in K_{kh}, \\
\frac{4}{((k - 1)|y|)^{3/2}}, & \text{if } z \in K_{kh}^c,
\end{cases}
\]
and the functions $f$ for $(h, k) = (2, k)$ with $k = 7, 8, 9, 10, 11$ are

$$f(x, y) = \begin{cases} \frac{(h - 1)|x|^2 - (k - 1)|y|^2}{4} (h - 1)|x|, & \text{if } z \in K_{kh}, \\ \frac{(h - 1)|x|^2 - (k - 1)|y|^2}{4} (k - 1)|y|, & \text{if } z \in K_{kh}. \end{cases}$$

Moreover, $g$ also satisfy

$$|\text{div } g| \geq \frac{c_{k, h}}{|z|^2} \text{dist}(z, M_{kh}),$$

with values of $c_{k, h}$ the same as in Theorem 2.

The proof of Lemma 2 is left to Section 3. The sub-calibrations we choose work well for $(h, k) \in S$, but do not work for some other Lawson cones. In some sense, these are specifically chosen to cover the cases $(h, k) \in S$.

2.2. Proof of Theorem 2. By Lemma 1, we have

$$P(F; H_R) - P(K_{kh}; H_R) \geq \int_{K_{kh} \Delta F} |\text{div } g|$$

$$\geq c_{k, h} \int_{K_{kh} \Delta F} \frac{\text{dist}(z, M_{kh})}{|z|^2} dz$$

$$\geq \frac{c_{k, h}}{R^2} \int_{K_{kh} \Delta F} \text{dist}(z, M_{kh}) dz.$$

Now, suppose $\varphi \in C^1(M_{kh})$, with $0 \not\in \text{spt } \varphi \subset B_{R}^m$. For $t_0 > 0$ small enough, there exists an open set $F \subset \mathbb{R}^m$ with $\partial F - \{0\}$ a $C^1$ hypersurface and $K_{kh} \Delta F \subset \subset H_R$, such that

$$\partial F - \{0\} = \{z + t \varphi(z) \nu_{K_{kh}}(z) : z \in M_{kh} - \{0\}\}.$$

By second variation and Taylor expansion, we have

$$P(F; H_R) - P(K_{kh}; H_R) = \frac{t^2}{2} \int_{M_{kh}} |\nabla^{M_{kh}} \varphi|^2 - |\Pi_{M_{kh}} |^2 \varphi^2 d\mathcal{H}^{m-1} + O(t^3).$$

Calculating the integral directly by pulling back the volume form on $\mathbb{R}^m$, we have

$$\int_{K_{kh} \Delta F} \text{dist}(z, M_{kh}) dz = (1 + O(t)) \int_{M_{kh}} d\mathcal{H}^{m-1}(z) \int_{0}^{t|\varphi(z)|} sds$$

$$= \frac{t^2}{2} \int_{M_{kh}} \varphi^2 d\mathcal{H}^{m-1} + O(t^3).$$

For details, please refer to Lemma 3.1 in [1]. Putting these two, and letting $t \to 0$, we deduce that

$$\frac{t^2}{2} \int_{M_{kh}} |\nabla^{M_{kh}} \varphi|^2 - |\Pi_{M_{kh}} |^2 \varphi^2 d\mathcal{H}^{m-1} \geq \frac{c_{k, h}}{R^2} \int_{M_{kh}} \varphi^2 d\mathcal{H}^{m-1}.$$

To extend (4) to all $\varphi \in C^1(M_{kh})$, let $\psi_j$ be a sequence of cut-off functions so that $\text{spt } \psi_j \subset B_{2^j/2}^m$ and $\psi_j = 1$ on $B_{1/2}^m$ with $|D \psi_j| \leq C_{m,j}$ everywhere, where $C_m$ is a positive constant depending only on $m$. We know that $\mathcal{H}^{m-1}(M_{kh} \cap
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\[ B^m_r \leq c(m)r^{m-1} \] for some constant \( c(m) \) depending only on \( m \) and \( ||M_{kh}|| \leq C \] for some constant \( C \) depending only on \( k, h \). Combining these estimates, we can see that the integrand on the left hand side of (4) is dominated by \( O(\frac{1}{|z|^2}) \), and thus the integral on the left hand side converges as \( j \to \infty \). Let \( j \to \infty \) and use dominated convergence. We deduce that (4) is true for all \( \varphi \in C^1(M_{kh}) \). q.e.d.

2.3. Proof of Theorem 1. Define

\[ p(z) = \left| \frac{|x|}{\sqrt{k-1}} - \frac{|y|}{\sqrt{h-1}} \right|. \]

By Lemma 1 and Lemma 2, we have

\[ |K_{kh}\Delta F| \leq |(K_{kh}\Delta F) \cap \{ p > \epsilon \}| + |H_R \cap \{ p < \epsilon \}| \]

\[ \leq \int_{(K_{kh}\Delta F) \cap \{ p > \epsilon \}} \frac{p(z)}{\epsilon} \frac{R^2}{|z|^2} dz + |H_R \cap \{ p < \epsilon \}| \]

\[ = \frac{1R^2}{\epsilon} \int_{(K_{kh}\Delta F) \cap \{ p > \epsilon \}} \frac{\text{dist}(z, M_{kh})}{|z|^2} dz + |H_R \cap \{ p < \epsilon \}| \]

\[ \leq \frac{1R^2}{ck,h} \int_{(K_{kh}\Delta F) \cap \{ p > \epsilon \}} |\text{div} g|dz + |H_R \cap \{ p < \epsilon \}| \]

\[ \leq \frac{1R^2}{ck,h} (P(F; H_R) - P(K_{kh}; H_R)) + |H_R \cap \{ p < \epsilon \}|, \]

where \( l = \sqrt{\frac{1}{h-1} + \frac{1}{k-1}} \) by elementary geometry. Now, we need to get a suitable upper bound for \( |H_R \cap \{ p < \epsilon \}| \). We have

\[ |H_R \cap \{ p < \epsilon \}| = \int_{B^h_R} \mathcal{H}^k \left( \left\{ x \in B^k_R : \frac{|y|}{\sqrt{h-1}} - \epsilon < \frac{|x|}{\sqrt{k-1}} < \frac{|y|}{\sqrt{h-1}} + \epsilon \right\} \right) dy \]

\[ \leq \omega_k (k - 1)^{k/2} \int_{B^h_R} \left( \frac{|y|}{\sqrt{h-1}} + \epsilon \right)^k - \left( \frac{|y|}{\sqrt{h-1}} - \epsilon \right)^k dy. \]

We can break down the estimate into two parts, namely

\[ \int_{B^h_{\sqrt{h-1}}} \left( \frac{|y|}{\sqrt{h-1}} + \epsilon \right)^k - \left( \frac{|y|}{\sqrt{h-1}} - \epsilon \right)^k dy \]

\[ = \int_{B^h_{\sqrt{h-1}}} \left( \frac{|y|}{\sqrt{h-1}} + \epsilon \right)^k dy \]

\[ \leq 2^k e^{b+k \omega_h} (h-1)^{k/2}, \]

and

\[ \int_{B^h_{\sqrt{h-1}}} \left( \frac{|y|}{\sqrt{h-1}} + \epsilon \right)^k - \left( \frac{|y|}{\sqrt{h-1}} - \epsilon \right)^k dy \]

\[ = \int_{B^h_{\sqrt{h-1}}} \left( \frac{|y|}{\sqrt{h-1}} + \epsilon \right)^k dy \]

\[ \leq 2^k e^{b+k \omega_h} (h-1)^{k/2}, \]


\[
\frac{1}{(h-1)^{k/2}} \int_{B_h \setminus B_{\sqrt{h-1}}} |y|^k \left( \left( 1 + \frac{\epsilon \sqrt{h-1}}{|y|} \right)^k - \left( 1 - \frac{\epsilon \sqrt{h-1}}{|y|} \right)^k \right) dy \\
\leq \frac{2^k}{(h-1)^{k/2}} \int_{B_h \setminus B_{\sqrt{h-1}}} |y|^k \frac{\epsilon \sqrt{h-1}}{|y|} dy \\
\leq \frac{2^k \epsilon}{(h-1)^{(k-1)/2}} \int_{\sqrt{h-1}}^R r^{k-1} h^{m-1} (S_\epsilon^{h-1}) dr \\
\leq \frac{2^k h \omega_h \epsilon}{(h-1)^{(k-1)/2}} R^{m-1}.
\]

where we use \((1 + t)^k - (1 - t)^k \leq 2^k t\) for \(t \in (0, 1), k \in \mathbb{N}\). Combining the two parts, we have

\[
|H_R \cap \{ p < \epsilon \}| \leq 2^k \omega_h \omega_h (k-1)^{k/2} (h-1)^{h/2} \epsilon \left( \epsilon^{m-1} + \frac{h R^{m-1}}{(h-1)^{(m-1)/2} (m-1)} \right)
\]

Now, note that \(\omega_j < 6\) for all \(2 \leq j \leq 11\), so by substituting the explicit values for \(\epsilon_{k,h}\), we have

\[
|K_{kh} \Delta F| \leq \frac{2 \times 11^5 \sqrt{\pi} R^2}{\epsilon} \left( P(F; H_R) - P(K_{kh}; H_R) \right) \\
+ 2 \times 11^2 \epsilon^{11/2} 2^{3/2} \epsilon^{m-1} + \frac{3}{6} R^{m-1}
\]

(5)

\[
\leq 7 \times 10^{10} \left( \frac{R^2}{\epsilon} (P(F; H_R) - P(K_{kh}; H_R)) + \epsilon (\epsilon^{m-1} + R^{m-1}) \right).
\]

(6)

Let

\[
\alpha = \frac{\xi^m(K_{kh} \Delta F)}{R^m},
\]

\[
\delta = \frac{P(F; H_R) - P(K_{kh}; H_R)}{R^{m-1}}.
\]

Note that \(\alpha \leq R^{-m} \xi^m(H_R) = \omega_h \omega_h \leq 6^2\). If \(\delta \geq 6^2\), then \(\alpha \leq \omega_h \omega_h \leq 6 \sqrt{\delta}\). Thus we assume \(\delta \leq 6^2\). Inequality (6) implies

\[
\alpha \leq 7 \times 10^{10} \left( \frac{R}{\epsilon} \delta + \frac{\epsilon}{R} \left( (\epsilon/R)^{m-1} + 1 \right) \right)
\]

(7)

If \(\epsilon < \frac{1}{\sqrt{35}} R\), then inequality (7) implies

\[
\alpha \leq 7 \times 10^{10} \left( \frac{R}{\epsilon} \delta + 36 \frac{\epsilon}{R} \right).
\]

Note that

\[
\frac{R}{\epsilon} \delta + 36 \frac{\epsilon}{R} \geq 12 \sqrt{\delta}
\]

with equality if and only if \(\epsilon = R \sqrt{\frac{\delta}{36}}\). Since \(\frac{\delta}{36} \leq 1\), we can let \(\epsilon = R \sqrt{\frac{\delta}{36}}\), and deduce that

\[
\alpha \leq 7 \times 12 \times 10^{10} \sqrt{\delta}.
\]
3. Proof of Lemma 2

3.1. Calculating $\text{div } g$ on $K_{kh}$. To make calculations simpler, let $u = (h - 1)|x|^2$, $v = (k - 1)|y|^2$. First, consider the function

$$f(z) = \frac{1}{4}(u - v)u^d.$$ 

We have

$$\partial_i f = \frac{h - 1}{2}x_i((d + 1)u^d - dvu^{d-1}),$$

$$\partial_i \partial_j f = \frac{h - 1}{2}\delta_{ij}((d + 1)u^d - dvu^{d-1}) + (h - 1)^2x_ix_j((d + 1)u^{d-1} - d(d - 1)vu^{d-2}),$$

$$\partial_j f = -\frac{k - 1}{2}y_ju^d,$$

$$\partial_j \partial_i f = -\frac{k - 1}{2}\delta_{ij}u^d,$$

$$\partial_j \partial_k f = -d(h - 1)(k - 1)u^{d-1}x_iy_j.$$ 

This gives

$$|\nabla f|^2 = \frac{h - 1}{4}u((d + 1)u^d - dvu^{d-1})^2 + \frac{k - 1}{4}vu^2d,$$

$$\Delta f = \frac{(h - 1)k}{2}((d + 1)u^d - dvu^{d-1}) - \frac{(k - 1)h}{2}u^d,$$

$$+(h - 1)u((d + 1)u^{d-1} - d(d - 1)vu^{d-2}),$$

$$2\partial_i f(\partial_i f)(\partial_i f) = \frac{(h - 1)^2}{8}u((d + 1)u^d - dvu^{d-1})^3$$

$$+ \frac{(h - 1)^2}{4}u^2((d + 1)u^d - dvu^{d-1})^2((d + 1)u^{d-1} - d(d - 1)vu^{d-2}),$$

$$\partial_j f(\partial_j f)(\partial_j f) = -\frac{(k - 1)^2}{8}vu^{3d},$$

$$\partial_i f(\partial_i f)(\partial_j f) = \frac{(h - 1)(k - 1)}{4}dv^2d((d + 1)u^d - dvu^{d-1}).$$ 

Thus, we have

$$|\nabla f|^3 \text{div } g = \frac{\text{div } \nabla f}{|\nabla f|^2}$$

$$= |\nabla f|^2\Delta f - (\partial_i f)(\partial_i f)(\partial_i f) - (\partial_j f)(\partial_j f)(\partial_j f) - 2(\partial_i f)(\partial_j f)(\partial_k f)$$

$$= (h - 1)(k - 1)u^{3d-2}(u - v)\left( (1 + d) - 1 + d(-1 + h))u^2$$

$$+ d(-2 + d(1 + 2d - 2(1 + d)h) + k)uv + d^3(-1 + h)v^2 \right).$$
3.2. Calculating $\text{div} \ g$ on $K_{kh}^L$. Now, define

$$f(z) = \frac{1}{4}(u - v)v^d,$$

which can be obtained by interchanging $u, v$ and $h, k$ and adding an additional minus sign to $f$ in the previous subsection. Thus, by symmetry or by direct computations, we must have

$$|\nabla f|^2 = \frac{h - 1}{4}uv^{2d} + \frac{k - 1}{4}v(duv^{d-1} - (d + 1)v^d)^2,$$

$$|\nabla f|^3 \text{div} \ g = |\nabla f|^3 \text{div} \frac{\nabla f}{|\nabla f|}$$

$$= \frac{(h - 1)(k - 1)}{8}uv^{3d-2}(d^3(k - 1)u^2$$

$$+ d(-2 + d + 2d^2 + h - 2d(1 + d)kuv$$

$$+ (d + 1)^2(-1 + d(k - 1)v^2)^2).$$

Note that if we set $g = \frac{\nabla f}{|\nabla f|}$, then $g$ is clearly continuous, and smooth except on $M_{kh}$. Calculations can show that the derivative of $g$ is of order $O(|z|^{-1})$ near origin, so $g \in W^{1,1}_\text{loc}(\mathbb{R}^m, \mathbb{R}^m)$.

3.3. The Cases $(2, k)$. We use the basic inequalities $\max\{|x|, |y|\} \leq z \leq \sqrt{2}\max\{|x|, |y|\}$ and $(a^q + b^q)^{1/q} \leq (a^2 + b^2)^{1/2}$ for $a, b > 0, q \geq 2$. Also note that

$$d(z, M_{kh}) = \frac{|\sqrt{u} - \sqrt{v}|}{\sqrt{h + k - 2}}$$

by elementary geometry.

If $u > v$, then choosing $d = 3/2$, we have

$$\text{div} \ g = \frac{1}{25}(-1 + k)u^{5/2}(u - v)(25u^2 + 12(-11 + k)uv + 27v^2)$$

$$\left(\frac{1}{16}u^2(25u^2 + 2(-17 + 2k)uv + 9v^2)\right)^{3/2}.$$ 

Let $p_2(t) = 27t^2 - 48t + 25$. We have $\min_{[0,1]} p_2 = p_2(8/9) = 11/3$.

This gives

$$\text{div} \ g \geq (k - 1)(\sqrt{u} - \sqrt{v})\frac{\sqrt{u} + \sqrt{v}}{\sqrt{u}}\frac{25u^2 + 12(-11 + 7)uv + 27v^2}{(25u^2 + 2(-17 + 2k)uv + 9v^2)^{3/2}}$$

$$\geq (k - 1)(\sqrt{u} - \sqrt{v})\frac{u^2p_3(v/u)}{(25u^2 + 10uv + 9v^2)^{3/2}}$$

$$\geq (k - 1)(\sqrt{u} - \sqrt{v})\frac{1}{44u^4}$$

$$\geq (k - 1)(\sqrt{u} - \sqrt{v})\frac{1}{44|z|^{3/2}}.$$
3.4. The Case

This gives

\[
\frac{(k-1)\sqrt{u} - \sqrt{v}}{2^3\sqrt{11}} |z|^2 \\
\geq \frac{\sqrt{11} \sqrt{u} - \sqrt{v}}{2^3} |z|^2.
\]

If \( u < v \), choosing \( d = 1 \), we have

\[
\text{div } g = \frac{1}{8} (k-1)(u-v) v((k-1) u^2 + (3-4k)uv + 4(-2+k)v^2) + \left(\frac{1}{8} v((k-1)(u-2v)^2 + uv)\right)^{3/2}
\]

Let \( q_2(t) = (k-1)t^2 + (3-4k)t + 4(k-2) \). We know that \( \min_{[0,1]} q_2 = q_2(1) = k-6 \). This gives

\[
|\text{div } g| \geq (k-1) |\sqrt{u} - \sqrt{v}| \frac{\sqrt{u} + \sqrt{v}}{\sqrt{v}} \frac{v^2 q_2(u/v)}{((k-1)4v^2 + u^2)^{3/2}}
\]

\[
\geq (k-1) |\sqrt{u} - \sqrt{v}| \frac{(k-6)(k-1)^2(|z|)^4}{(4(k-1)^3 + (k-1)^2)^{3/2}|z|^{16}}
\]

\[
\geq (k-1) |\sqrt{u} - \sqrt{v}| \frac{(k-6)(k-1)^2(|z|)/\sqrt{v})^4}{(4(k-1)^3 + (k-1)^2)^{3/2}|z|^{16}}
\]

\[
\geq k^2(4(k-1)^3 + (k-1)^2)^{3/2} \frac{|\sqrt{u} - \sqrt{v}|}{|z|^2}
\]

\[
\geq \frac{1}{115} \frac{6^3}{115^2(4 \times 10^3 + 10^2)^{3/2}} |z|^2
\]

\[
\geq \frac{1}{115} \frac{|\sqrt{u} - \sqrt{v}|}{|z|^2},
\]

where we use \( v > u \) if and only if \(|x| < \sqrt{k-1}|y|\) and thus \(|z| < \sqrt{k}|y|\).

This gives

\[
|\text{div } g| \geq \frac{1}{115} \frac{|\sqrt{u} - \sqrt{v}|}{|z|^2} = \frac{1}{115\sqrt{11}} \text{dist}(z, M_{2,k}).
\]

3.4. The Case \((h, k) = (3, 5)\). Choose \( d = 3/4 \). If \( u > v \), we have

\[
\text{div } g = \frac{1}{32} u^{1/4} (u-v) (49u^2 - 72uv + 27v^2)
\]

\[
\left(\frac{1}{32} \sqrt{u}(49u^2 - 10uv + 9v^2)\right)^{3/2}.
\]

Let

\[
p_3(t) = 27t^2 - 72t + 49.
\]

We know that \( \min_{[0,1]} p_3 = p_3(1) = 4 \). This yields

\[
\text{div } g \geq 4\sqrt{2}(\sqrt{u} - \sqrt{v}) \frac{\sqrt{u} + \sqrt{v}}{\sqrt{u}} \frac{u^2 p_3(v/u)}{\left(49 \times 4|x|^4 + 9 \times 16|y|^4\right)^{3/2}}
\]

\[
\geq \frac{u^2 p_3(v/u)}{\left(49 \times 4|x|^4 + 9 \times 16|y|^4\right)^{3/2}}.
\]
\[
\geq 4\sqrt{2}(\sqrt{u} - \sqrt{v}) \frac{4u^2}{\left((49 \times 4)(|x| + |y|)^4\right)^{3/2}} \\
\geq 4\sqrt{2}(\sqrt{u} - \sqrt{v}) \frac{4(|z|/\sqrt{2})^4}{14^3|z|^6} \\
\geq \frac{2\sqrt{2}}{7^3} \frac{\sqrt{u} - \sqrt{v}}{|z|^2}.
\]

If \( u < v \), we have
\[
\text{div } g = \frac{1}{16} \frac{(u - v)v^{1/4}(27u^2 - 123uv + 98v^2)}{\left(\frac{1}{16}\sqrt{v}(9u^2 - 34uv + 49v^2)\right)^{3/2}}.
\]

Let \( q_3(t) = 27t^2 - 123t + 98 \). We have \( \min_{[0,1]} q_3(t) = q_3(1) = 2 \). This gives
\[
|\text{div } g| \geq 4|\sqrt{u} - \sqrt{v}| \frac{\sqrt{u} + \sqrt{v}}{\sqrt{v}} \frac{2v^2 q_3(u/v)}{2v^2 \left(9 \times 4|x|^4 + 49 \times 16|y|^4\right)^{3/2}} \\
\geq 4|\sqrt{u} - \sqrt{v}| \frac{2v^2}{\left(49 \times 16(|x|^4 + |y|^4)\right)^{3/2}} \\
\geq 4|\sqrt{u} - \sqrt{v}| \frac{2 \times 4^2 |y|^4}{28^3|z|^6} \\
\geq 4|\sqrt{u} - \sqrt{v}| \frac{2 \times 4^2 (|z|/\sqrt{3})^4}{28^3|z|^6} \\
\geq \frac{2}{3^27^3} \frac{|\sqrt{u} - \sqrt{v}|}{|z|^2},
\]
where we use \( u < v \Leftrightarrow 2|y|^2 > |x| \) and thus \( |z|^2 < 3|y|^2 \). This yields
\[
|\text{div } g| \geq \frac{2}{3^27^3} \frac{|\sqrt{u} - \sqrt{v}|}{|z|^2} = \frac{\sqrt{3}}{21^3} \frac{\text{dist}(z, M_{5,3})}{|z|^2}.
\]

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