Analyses of a Yang-Mills Field over the Three-Level Quantum Systems

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Utilizing a number of results of Dittmann, we investigate the nature of the Yang-Mills field over the eight-dimensional convex set, endowed with the Bures metric, of three-level quantum systems. Adopting a numerical strategy, we first decompose the field into self-dual and anti-self-dual components, by implementing the octonionic equations of Corrigan, Devchand, Fairlie and Nuyts. For each of these three fields, we obtain approximations to: (1) the Yang-Mills functional; (2) certain quantities studied by Bilge, Dereli, and Koçak in their analysis of self-dual Yang-Mills fields in eight dimensions; and (3) other measures of interest.

Keywords: Bures metric, Yang-Mills fields, three-level quantum systems, action, numerical integration

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The Bures metric, defined on the nondegenerate density matrices, has been the object of considerable study [1–5]. It is the minimal member of the nondenumerable family of monotone metrics [6]. Other members of this family of particular note are the “Bogoliubov-Kubo-Mori” (BKM) metric [8], the maximal monotone metric, as well as the “Morozova-Chentsov” [10] and “quasi-Bures” [11] ones, the last yielding the minimax/maximin asymptotic redundancy in universal quantum coding [12]. Interestingly, these (operator) monotone metrics correspond in a direct fashion to certain “measures of central tendency”, with, for example, the Bures metric corresponding to the arithmetic mean, \((x + y)/2\), of numbers \(x\) and \(y\) [6,7]. All these monotone metrics constitute various extensions to the quantum domain of the (unique) Fisher information metric in the classical realm where the objects of study are probability distributions (rather than density matrices) [13] (cf. [14]). However, the only one of these monotone metrics that can be extended to the boundary of pure states yielding the standard Fubini-Study metric on this boundary is the Bures (minimal monotone) one [6, sec. IV].

Dittmann has shown that “the connection form (gauge field) related to the generalization of the Berry phase to the mixed states proposed by Uhlmann satisfies the source-free Yang-Mills equation \(*D*D\omega\), where the Hodge operator is taken with respect to the Bures metric on the space of finite-dimensional density matrices” [15] (cf. [16]). (Let us also indicate here that the BKM metric has been shown to be the unique monotone Riemannian metric for which two “natural flat” connections — the “exponential” and “mixture” ones — are dual [8].) Here we report our efforts to evaluate — using numerical (lattice-gauge-like [17]) methods — the action functional for this Yang-Mills field over the eight-dimensional convex set \((M)\) equipped with the Bures metric of \(3 \times 3\) density matrices, as well as for the two self-dual components of this field. (Using an exact version for the \(2 \times 2\) density matrices of the methodology described below, we have found a value of precisely three for the action of the corresponding Yang-Mills field. The Chern-Simons functional is zero.)

To proceed, we exploit our recent work [18,19] in determining the elements of the Bures metric using a certain Euler angle parameterization of the \(3 \times 3\) density matrices [20]. (Earlier, Dittmann had noted [15] “that in affine coordinates (e. g. using the Pauli matrices for \(n = 2\)) the [Bures] metric becomes very complicated for \(n > 2\)” and no good parameterization seems to be available for general \(n\”.) This allows us, among other things, to employ as our parameter space four (noncontiguous) eight-dimensional hyper rectangles (each having three sides of length \(\pi\), three sides of length \(\pi\) in length and one \(\cos^{-1}\frac{1}{\sqrt{3}}\), rather than less analytically convenient ones, such as that discussed by Bloore [21, Fig. 3], involving spheroids and “tetrapaks”. (The four hyperrectangles are not adjacent due to the fact that two of the six Euler angles used in the parameterization have disconnected ranges. This situation — which results in different normalization factors only — has only come to our attention relatively recently, and serves as an erratum to earlier analyses [22,23], which had relied upon work of Marinov [24], without taking into account a correction [25] to [24] (cf. [26]).)

To be specific, we seek to implement the formula [27, (1.1.10)] for the Yang-Mills functional \(YM(A)\) of the appropriate connection \(A\)

\[
YM(A) = \frac{1}{4\pi^2} \int_M |F_A|^2 dV_g.
\] (1)
Here $F_A$ is the curvature of $A$ and, for our purposes, $dV_q$ is the volume form of the Bures metric $g$ on the eight-dimensional manifold $M$, composed of the three-level quantum systems. More detailedly \[ (2) \]

$$|F_A|^2 = \sum_{i,j,k,l} < F_{aij}, F_{akl} > g^{ij}g^{kl},$$

where $(g_{ij})$ is the metric tensor of $g$ and $(g^{ij})$ is its inverse. (Formulas for $V_q$, $(g_{ij})$ and $(g^{ij})$, using the Euler-angle parameterization \[ (23) \], have been derived in \[ (18) \] (cf. \[ (17) \]. Let us indicate here that since we directly adopt formulas from a number of different sources, our overall notation may not be as uniform as fully desirable, though hopefully readily enough comprehensible to the informed reader.)

To achieve the decomposition $F_A = F_A^+ + F_A^-$ of $F_A$ into the sum of “anti-self-dual” ($F_A^+$) and “self-dual” ($F_A^-$) parts \[ (28) \] \[ (29) \], we use the formulas first given by Corrigan et al \[ (30) \], and presented again numerous times (e.g. \[ (32) \] \[ (34) \]). The components of the anti-self-dual field satisfy a set of seven linear equations (corresponding to an eigenvalue 1) \[ (31) \] \[ (39) \] and those of the self-dual field, a set of twenty-one linear equations (corresponding to an eigenvalue of -3) \[ (30) \] \[ (40) \]. The set of seven equations has been viewed as more fundamental in nature than the set of twenty-one equations by Figueroa-O’Farrill, who is also somewhat critical of the application of the terms “self-dual” and “anti-self-dual” in this octonionic setting \[ (11) \].

For the required connection form $A$, Dittmann has presented the general formula \[ (13) \] \[ (9) \]

$$A = \frac{1}{L + R}(W^*T - T^*W) \quad (3)$$

Here the elements of $T$ lie in the tangent space to the principal $U(\mathcal{H})$-bundle, the manifold of invertible normalized (Tr$W^*W = 1$) Hilbert-Schmidt operators, while $L$ is the operator (depending on $W$) of left multiplication by the density matrix $\rho = WW^*$ and $R$ the corresponding operator of right multiplication. Also, $\hat{L}$ and $\hat{R}$ are the counterpart operators for $\hat{\rho} = W^*\rho$. Since the Euler-angle parameterization of the $3 \times 3$ density matrices presented in \[ (20) \] is of the (“Schur-Schatten”) form,

$$\rho = UD\hat{U}^*, \quad (4)$$

$U$ being unitary, $U^*$ its conjugate transpose, and $D$ the diagonal matrix composed of the three eigenvalues of $\rho$, one can immediately express $W$ as $UD^{1/2}U^*$, with $W = W^*$. In the non-trivial task of implementing formula \[ (33) \] \[ (7) \] for the three-level quantum systems, we relied upon the implicit relation (equivalence) between two formulas for the Bures metric for $n$-level quantum systems \[ (33) \] \[ (2) \] \[ (16) \],

$$g = \frac{1}{2}\Tr d\rho \frac{1}{L + R} d\rho, \quad (5)$$

and

$$g = \frac{1}{2} \sum_{i,j} a_{ij} \Tr d\rho p^{i-1} d\rho p^{j-1}. \quad (6)$$

(The somewhat involved formula for the coefficients $a_{ij}$, functions of elementary invariants, is stated in Proposition 3 of \[ (33) \].) In other words, we used the formula for the connection

$$A = \frac{1}{2} \sum_{i,j} a_{ij} p^{i-1} S p^{j-1}, \quad (7)$$

where $S = W^*T - T^*W$. (We were able to express some of the individual components of $A$ in relatively simple forms, but others remain, at this stage, quite cumbersome in nature.)

Let us note that in a previous similarly-motivated study \[ (13) \] to that here, we pursued many of the same questions, but somewhat misguided employed for our analyses the curvature ($R$) of the Bures Riemannian metric, rather than the curvature ($F_A$) of the gauge field $A$, so the interpretation to be given to the results there, in retrospect, is not altogether clear. (By embedding the spin connection in the gauge connection, one can construct a self-dual gauge field directly from a self-dual metric \[ (30) \].) In that analysis \[ (13) \], numerical evidence led us to conclude that for the $8 \times 8$ skew-symmetric matrices of the $R_{abcd}$, holding $c, d$ constant, one of the four pairs of imaginary eigenvalues is always degenerate, that is $(0,0)$ cf. \[ (23) \] \[ (25) \].

2
In [19], we pursued both a Monte Carlo numerical integration scheme, randomly selecting points lying in the ranges of the eight parameters, as well as a scheme in which the points were systematically selected as the grid points of a hyperrectangular lattice. Here we have chosen to focus on a lattice scheme, employing $1,024 = 4 \times 256$ grid points (cf. [17]). The coordinates of the nodes of the lattice were chosen in the natural (symmetric) manner, so as to divide each individual side of the four hyperrectangles into three intervals, the middle interval being twice the length of the two end ones, which under wrapping (identifying the end points) form a single interval equal in length to the middle one. Based upon this lattice scheme, we approximated the value of the Yang-Mills functional (1) for the Yang-Mills field $F_A$ itself to be 137.653, for the “anti-self-dual” part $F_A^+$ to be 222.806, and for the “self-dual” part $F_A^-$ to be 176.194 (Table I).

| quantity                  | $F_A$  | $F_A^+$ | $F_A^-$ |
|---------------------------|--------|---------|---------|
| Yang-Mills functional     | 137.653| 222.806 | 176.194 |
| $\int_M (F, F)^2$         | 37.2769| 15.0131 | 1.1757  |
| $\int_M (F^2, F^2)$       | 162.468| 97.6351 | 4.7715  |
| $\int_M (\text{tr} F^2, \text{tr} F^2)$ | 129.285| 72.9629 | 8.5455  |
| $H_4$                     | $-1.35 \cdot 10^{-11}$ | 4.15623 | 35.6896 |
| $H_2$                     | $-1.74 \cdot 10^{-11}$ | 5.4683  | 71.7151 |
| $H_4 - H_2/2$             | $-4.77 \cdot 10^{-12}$ | 1.42208 | -.167908 |

TABLE I. Lattice approximations, based on $1,024 = 4 \times 2^8$ grid points, to various quantities of interest for the eight-dimensional Yang-Mills field $F_A$ over the three-level quantum systems, and for its anti-self-dual ($F_A^+$) and self-dual ($F_A^-$) constituents.
The explanation for the Yang-Mills functionals being larger for both $F_A^+$ and $F_A^-$ than for $F_A$ itself, in apparent contradiction to the general assertion in \[28\] sec.2, may be that the topological types of $F_A^+$ and $F_A^-$ are different from that of $F_A$. (I thank Y.-H. Gao for suggesting this possibility.) It should be noted, though, that in more than four dimensions, the Yang-Mills functional itself need not be a minimum of the action \[39\] sec. 2.4 \[40\].

Motivated by the work of Bilge, Dereli and Ko¸cak (BDK) \[34\] concerning $SO(n)$-bundles in eight-dimensional spaces, we computed the integrals over $M$ of $(F_A, F_A^+)^2$, $(F_A^2, F_A^3)$ and $(\text{tr} F^2, \text{tr} F^3)$, and also similarly for the self-dual and anti-self-dual parts of $F_A$. BDK considered a generic action density of the form

$$a(F_A, F_A^+) + b(F_A^2, F_A^3) + c(\text{tr} F^2, \text{tr} F^3).$$

(The action density proposed by Grossman, Kephardt and Stasheff \[11\] corresponds to the choice $a = c = 0$.) BDK derived a topological bound

$$\int_M (F_A, F_A^+)^2 \geq k \int_M p_1(E)^2$$

that is achieved by “strongly (anti)self-dual” Yang-Mills fields (those conforming to the set of twenty-one — not the set of seven — equations of Corrigan et al), where $p_1$ is the first Pontrjagin class of the $SO(n)$ Yang-Mills bundle $E$ and $k$ is a constant. This is to be compared with a topological bound obtained earlier \[11\]

$$\int_M (F_A^2, F_A^3)^2 \geq k' \int_M p_2(E).$$

We also approximated for the three fields $(F_A, F_A^+)$ and $F_A^-$ the quantities \[12\] eq. (A1)], where (using $\tilde{F}$ to generically represent $F_A$, $F_A^-$ and $F_A^+$)

$$H_4 = \int_M \epsilon_{\mu \nu \rho \sigma \tau \lambda \kappa \eta} \text{tr}(\tilde{F}_{\mu \nu} \tilde{F}_{\rho \sigma} \tilde{F}_{\tau \lambda} \tilde{F}_{\kappa \eta})$$

and

$$H_2 = \int_M \epsilon_{\mu \nu \rho \sigma \tau \lambda \kappa \eta} \text{tr}(\tilde{F}_{\mu \nu} \tilde{F}_{\rho \sigma}) \cdot \text{tr}(\tilde{F}_{\tau \lambda} \tilde{F}_{\kappa \eta}),$$

for which $H_4 - H_2/2$ is proportional to a certain topological invariant, $q_{24}$ \[12\] eq. (A1)] ($\epsilon_{\mu \nu \rho \sigma \tau \lambda \kappa \eta}$ is the completely antisymmetric Levi-Civita symbol). Let us note that Table I indicates that for the Yang-Mills field $F_A$, the terms $H_4$ and $H_2$ are both essentially zero. ($H_i$ is proportional to the $i$-th Chern number. “The Chern numbers are those obtained by integrating characteristic polynomials of degree dim $M$ over the entire manifold $M$” \[13\], Def. (2.5.2).)

In contrast to the Yang-Mills functional \[1\], we see from Table I that (following the work of BDK \[34\]) the integrals over the Riemannian manifold composed of the three-level quantum systems endowed with the Bures metric of $(F, F)^2$, $(F^2, F^3)$ and $(\text{tr} F^2, \text{tr} F^3)$ are largest for the Yang-Mills field $F_A$ \[13\], smaller for the anti-self-dual part $F_A^-$ and smallest for the self-dual component $F_A^+$. In the context of $SO(n)$-bundles and 2-forms in $2m$ dimensions represented by $2m \times 2m$ skew-symmetric real matrices, BDK derived the result that

$$(F_A, F_A^+)^2 \geq \frac{2}{3}(\text{tr} F_A^2, \text{tr} F_A^3).$$

We see from Table I that for our $SU(3)$-bundle, on the other hand, this result does not appear to hold for all three fields.

For analyses of non-self-dual Yang-Mills fields (in four dimensions) see \[15\] \[18\] and (more generally, in $2n$ dimensions) \[19\] (cf. \[20\]).

The computational demands in deriving (with the use of MATHEMATICA) the results reported above in Table I have been considerable — due, in particular, to the still somewhat cumbersome formulas for certain of the elements of the Bures metric and the term $W^* T - T^* W$ (occurring in the formula \[3\] for the connection $A$) that we have derived \[18\] \[19\], using the parameterization in \[20\]. To employ instead of the 1,024 = $4 \times 2^8$ point lattice, a 26,244 = $4 \times 3^8$ one, for example, presently seems too much for us to reasonably accomplish. (Actually, it appears that since the results for the four noncontiguous hyperrectangles are all identical, we would only have to undertake 6,561 = $3^8$ independent evaluations.) However, we have been able to refine the lattice in two of the eight directions, so that we
deal with $2,304 = 4 \times 3^2 \times 2^6$ points. If we choose these two directions (more or less arbitrarily) to correspond to one of the six Euler angles ($\alpha$ in the notation of [18,19]) and one of the spherical angles parameterizing the eigenvalues ($\theta_1$), then we obtain the outcomes reported in Table II.

| quantity                  | $F_A$   | $F_A^+$ | $F_A^-$ |
|---------------------------|---------|---------|---------|
| Yang-Mills functional     | 197.458 | 324.53  | 256.239 |
| $\int_M (F, F)^2$         | 37.9854 | 15.3356 | 1.20481 |
| $\int_M (F^2, F^2)$       | 162.107 | 98.5983 | 4.92389 |
| $\int_M (\text{tr} F^2, \text{tr} F^2)$ | 132.557 | 74.8023 | 8.77179 |
| $H_4$                     | -3.08 \times 10^{-14} | 4.26952 | 36.6592 |
| $H_2$                     | 4.13 \times 10^{-14}  | 5.28644 | 73.5915 |
| $H_4 - H_2/2$             | -2.37 \times 10^{-14} | 1.50635 | -1.36522 |

**TABLE II.** Lattice approximations, based on $2,304 = 4 \times 3^2 \times 2^6$ grid points, to various quantities of interest for the eight-dimensional Yang-Mills field $F_A$ over the three-level quantum systems, and for its anti-self-dual ($F_A^+$) and self-dual ($F_A^-$) constituents
We see (quantitative, not really qualitative) changes in Table II from the values reported in Table I, with quite substantial increases in the case of the Yang-Mills functionals, and comparatively modest differences in the other sets of indices [34].

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