Topological gauge theories
with antisymmetric tensor matter fields

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Abstract
A new type of topological matter interactions involving second–rank antisymmetric tensor matter fields with an underlying \(N_T \geq 1\) topological supersymmetry are proposed. The construction of the 4–dimensional, \(N_T = 1\) Donaldson–Witten theory, the \(N_T = 1\) super–BF model and the \(N_T = 2\) topological B–model with tensor matter are explicitly worked out.

1 Introduction
It has been known for a long time that in quantum field and string theories besides totally symmetric tensor and tensor–spinor fields also second–rank antisymmetric tensor fields play an important role. As a significant example, the Green–Schwarz anomaly cancellation mechanism\[1\] underlies, among others, a coupling of a two–form gauge potential with a Chern–Simons form.

All the known couplings in Minkowski space–time involving antisymmetric tensor fields may be put into two categories. Depending on whether they transform as gauge or as matter fields, one distinguishes
(i) tensor gauge couplings,
\[
S_{\text{gauge}} \propto \int_M d^4 x \left( \partial^a \tilde{B}_{ac}(\partial_b \tilde{B}^{bc}) \right), \quad \tilde{B}_{ab} = \frac{i}{2} \epsilon_{abcd} B^{cd}, \quad \tilde{\tilde{B}}_{ab} = -B_{ab},
\]
(ii) (conform invariant) tensor matter couplings,
\[
S_{\text{matter}} \propto \int_M d^4 x \left( \partial^a \varphi_{ac}(\partial_b \varphi^{bc})^\dagger \right), \quad \varphi_{ab} = T_{ab} + i \tilde{T}_{ab},
\]
where \(\varphi_{ab}\) is an antisymmetric complex tensor field involving the tensor matter field \(T_{ab}\) and satisfying the complex self–duality condition \(\varphi_{ab} = i \varphi_{ab}\).

In the first case, the action \([1]\) possesses a first–stage reducible gauge symmetry \(\delta_G B_{ab} = \partial_a \omega_b \)\[2\]. Such antisymmetric tensor gauge fields appear quite naturally in extended supergravity theories \([3]\) and in effective low–energy tensor gauge theories derived from string models \([4]\), e.g., the axion/dilaton complex in Calabi–Yau compactifications of type–II superstrings \([5]\).

In the second case, gauge symmetry is lost, but the action \([4]\) exhibits an invariance under the (global) chiral symmetry \(\delta_C T_{ab} = \alpha \tilde{T}_{ab}\), i.e., \(T_{ab}\) transforms as an ordinary matter field. Antisymmetric tensor matter fields arise in extended conformal supergravity theories \([5]\) and in 2D conformal quantum field theories (CQFT’s) \([6]\).

Besides of this, there is a renewed interest in antisymmetric tensor fields due to their connection to a large class of Schwarz type topological models, namely the BF–models \([8]\), which are exactly solvable QFT’s. Generally, topological quantum field theories (TQFT’s) \([9]\) are characterized by observables depending only on the global features of the manifold on which

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they are defined and provide novel representations of certain topological invariants. The most familiar examples, which have been widely studied during the last years, are the Donaldson–Witten theory \[10\], the Chern–Simons gauge theory \[11\] in \(D = 3\) and the topological sigma models \[12\] in \(D = 2\), which constitute quantum field theoretic representations of the theory of Donaldson invariants \[13\], of knot and link invariants \[14\] and of Gromov invariants \[15\], respectively. Moreover, TQFT’s have significantly enhanced our understanding of CQFT’s in \(D = 2\) and they promised new insights into string theories \[16\].

In the Schwarz type topological models the antisymmetric tensor fields appear always as gauge fields. The aim of the present paper is to construct, rather differently, Witten type topological models which include antisymmetric tensors fields as matter fields. More precisely, we consider extensions of the 4D, \(N_T = 1\) Donaldson–Witten (DW) theory, the \(N_T = 1\) super–BF model \[17\] and the \(N_T = 2\) topological B–model, constructed by Marcus \[18\], respectively, which involve a coupling of the gauge field \(A_\mu\) to the (anti)self–dual parts \(T^\pm_\mu\nu\) of a second–rank antisymmetric matter field \(T_\mu\nu\). These models allow, in principle, also the inclusion of a quartic tensor self–interaction term. Both types of interaction terms, when put on a general curved 4-dimensional gravitational background with Euclidean signature, may be regarded as a non–abelian generalization and \(N_T \geq 1\) supersymmetric extension of the abelian axial gauge model for antisymmetric tensor matter fields in Minkowski space–time introduced by Avdeev and Chizhov \[19\].

The search for a new type of topological matter action is motivated as follows:

One of the possible constructions of DW theory consists in twisting the action of Euclidean \(N = 2\) super Yang–Mills (SYM) theory with global automorphism group \(Sp(2) \otimes U(1)\) (R–symmetry) and Euclidean rotation group \(SO(4) \cong SU(2)_L \otimes SU(2)_R\) by replacing \(SU(2)_L\) through the diagonal subgroup of \(SU(2)_L \otimes Sp(2)\) and coupling the theory to Euclidean gravity \[17\]. Due to its independence on the gauge coupling constant \(e\) there exists the possibility to study the observables of the theory from both the perturbative and non–perturbative point of view, i.e., either in the weak or in the strong coupling limit, \(e \to 0\) or \(e \to \infty\), respectively. Perhaps the most important outcome of both approaches is the existence of a totally unexpected relation between two different moduli spaces in 4D topology, one defined by the anti–selfdual instanton equations \[20\] and another one defined by the abelian Seiberg–Witten monopol equations \[21\]. These moduli problems can be naturally generalized including also spinor fields, namely by twisting the \(N = 2\) SYM coupled to \(N = 2\) matter hypermultiplets in various representations of the gauge group \[22\]. Since the global R–symmetry group of \(N = 2\) supersymmetric gauge theories is at most \(U(2) \cong Sp(2) \otimes U(1)\), the twist in these more general cases is essentially unique. The moduli space associated to that generalized DW theory is determined by the non–abelian monopol equations.

To construct fundamentally different, \(N_T > 1\) topological theories, one needs at least \(N = 4\) Euclidean SYM. Since the \(R\)–symmetry group of \(N = 4\) supersymmetric gauge theories is \(SU(4)\) there exist three non–equivalent ways of twisting the Euclidean rotation group with the \(R\)–symmetry group. One of them, the \(N_T = 2\) topological A–model, constructed by Yamron \[23\], was studied by Vafa and Witten in order to perform a strong coupling test of S–duality \[24\]. Another one, the \(N_T = 2\) topological B–model leads to a theory whose moduli space is dominated by flat complexified gauge fields \(A_\mu \pm iV_\mu\); it can be regarded as a deformation of the \(N_T = 1\) super–BF model \[23\]. The remaining one, also constructed by Yamron \[23\], is the \(N_T = 1\) half–twisted theory which provides another example of a DW theory with matter, now for the particular case when the spinor fields are in the adjoint representation of the gauge group. The latter theory bears a strong resemblance to the non–abelian generalization of the Seiberg–Witten monopol theory. However, twisted \(N = 4\) SYM does not lead to a topological matter action having a \(N_T = 2\) supersymmetry. This is due to the fact that \(N = 4\) SYM cannot be coupled to a \(N = 4\) matter hypermultiplet.
In continuing earlier studies [26] of topological gauge theories we pursued further the idea of constructing TQFT’s with matter leading, in particular, to a new topological tensor matter action with extended, $N_T = 2$, supersymmetry.

The outline of the paper is as follows: In Sect. 2 we briefly review the DW theory and then we construct a tensor matter action with $N_T = 1$ topological supersymmetry; it is shown that its tensorial structure is uniquely fixed by gauge and local Weyl invariance. In Sect. 3 we generalize the previous construction for complexified gauge fields, $A_\mu \pm iV_\mu$, and pass to the $N_T = 1$ super–BF model with matter. In Sect. 4, by a suitable deformation of the super–BF model, in the version of [25], we arrive at the $N_T = 2$ topological B–model with matter and underlying complexified supersymmetry $Q \pm i\bar{Q}$. In the Appendix it is proven, in accordance with [18, 25] and contrary to some statement in [27, 28], that the on–shell conditions in the formulation of the B–model cannot be completely lifted by using an appropriate set of auxiliary fields.

Throughout the paper we use the following conventions: Greek letters $\mu, \nu, \ldots$ denote world indices and lower case latin letters $a, b, \ldots$ are flat $SO(4)$ tangent space indices.

2 Donaldson–Witten theory coupled to tensor matter fields

Let us first consider the DW theory whose moduli space is the space of anti–selfdual instantons. In order to complete the construction of that theory — which has been described in the Introduction — we must specify its configuration space. It consists of the gauge potential $A_\mu$, the Grassmann–odd self–dual tensor, vector and scalar fields $\chi_{\mu\nu}, \psi_\mu$ and $\eta$, respectively, and the Grassmann–even scalar fields $\phi$ and $\bar{\phi}$. For the closure of the topological superalgebra it is necessary to introduce the bosonic auxiliary self–dual tensor field $B_{\mu\nu}$. All the fields are in the adjoint representation, i.e., taking their values in the Lie algebra $\text{Lie}(G)$ of some compact (semisimple) gauge group $G$. Throughout this paper we adopt the convention to choose the generators $T^i \in \text{Lie}(G)$ always anti–Hermitean.

The action of DW theory, with a $N_T = 1$ off–shell equivariantly nilpotent topological supersymmetry $Q$, adopting the notation of Ref. [25], can be cast into the $Q$–exact form

$$S_{DW} = Q\Psi_{DW},$$

with the gauge fermion (see, also, footnote 3 below)

$$\Psi_{DW} = \frac{i}{e^2} \int d^4x \sqrt{g} \text{tr} \left\{ \frac{1}{2} \chi^{\mu\nu} F_{\mu\nu} + i \frac{1}{4} \chi^{\mu\nu} B_{\mu\nu} - \psi^\mu D_\mu \bar{\phi} + i \frac{1}{2} \eta [\bar{\phi}, \phi]\right\},$$

where $F_{\mu\nu} = \partial_\mu A_\nu + [A_\mu, A_\nu]$ and $D_\mu = \partial_\mu + [A_\mu, \cdot]$ are the field strenght and the covariant derivative in the adjoint representation, respectively; $e$ is the usual YM coupling constant.

In (4) the gauge fermion has been chosen in a Feynman type gauge, thereby the first term enforces the localization into the moduli space and the third term ensures that pure gauge degrees of freedom are projected out; the remaining terms belong to the non–minimal sector and could be droped (getting a Landau type gauge).

The off–shell equivariantly nilpotent $Q$–transformations take the form

$$Qg_{\mu\nu} = 0, \quad Q\phi = 0,$$

$$QA_\mu = \psi_\mu, \quad Q\psi_\mu = D_\mu \phi,$$

$$Q\bar{\phi} = \eta, \quad Q\eta = [\bar{\phi}, \phi],$$

$$Q\chi_{\mu\nu} = B_{\mu\nu}, \quad QB_{\mu\nu} = [\chi_{\mu\nu}, \phi].$$

Therefore, the topological supercharge $Q$ squares to zero only modulo field–dependent gauge transformations,

$$Q^2 = \delta_G(\phi),$$
which are defined by \( \delta_G(\omega) A_\mu = D_\mu \omega \) and \( \delta_G(\omega) X = [X, \omega] \) for all the other fields. Hence, all the local symmetries of the action, apart from the ordinary gauge invariance, have been fixed.

Spelling out the action (3) explicitly one obtains, recalling that \( \chi_{\mu} \) and \( B_\mu \) are self–dual, \(^3\)

\[
S_{\text{DW}} = \frac{i}{e^2} \int d^4 x \sqrt{g} \text{tr} \left\{ \frac{1}{2} B^\mu{}_{\rho\nu} F_{\mu \rho\nu} - \chi^{\mu \nu} D_\mu \psi_\nu + \frac{i}{4} B^{\mu \nu} B_{\mu \nu} - \frac{i}{4} \phi \{ \chi^{\mu \nu}, \chi_{\mu \nu} \} \right. \\
\left. - D^\mu \tilde{\phi} D_\mu \phi + \bar{\phi} \{ \psi^\mu, \psi_\mu \} + \psi^\mu D_\mu \eta + \frac{i}{2} \bar{\phi} \{ \phi, \psi \} - \frac{i}{2} \phi \{ \eta, \eta \} \right\}. \quad (7)
\]

The \( Q \)–exactness of the action (3) is common to all Witten type topological theories and has striking consequences on the general features of cohomological gauge theories. It means that the physical observables, in particular the partition functions themself, have no dependence on the metric \( g_{\mu \nu} \) and on the coupling constant \( e \).

Now, we describe the inclusion of a new type of interaction into topological gauge theories involving antisymmetric tensor matter fields. The way of constructing such tensor interactions is governed by the following strategy:

First, we generalize the coupling (2) such that it might be interpreted as a \( \varphi^4 \)–type theory for antisymmetric tensor matter fields leading to the non–abelian extension \(^2\) of the Avdeev–Chizhov (AC) model \(^{19}\) in Minkowski space–time:

\[
S_{\text{AC}}(\alpha) = \int_M d^4 x \left\{ (D^a \varphi_{ab})(D_b \varphi^{bc}) + \alpha (\varphi_{ac} \varphi^{bc})(\varphi^{ad} \varphi^{\dagger}_{bd}) \right\}. \quad (8)
\]

Here, \( D^a \varphi_{ab} = \partial^a \varphi_{ab} - \varphi_{ab} A^a \) is the covariant derivative of \( \varphi_{ab} \) which belongs to some finite (complex) representation of \( \text{Lie}(G) \). Remind that the gauge potential \( A_a \) is anti–Hermitian. For notational simplicity we also dropped the group index of the matter fields. \( \alpha \) is the coupling constant of the quartic self–interaction.

The action (8) is invariant under the following gauge transformations,

\[
\delta_G(\omega) A_\mu = D_\mu \omega, \quad \delta_G(\omega) \varphi_{ab} = \varphi_{ab} \omega, \quad \delta_G(\omega) \varphi^{\dagger}_{ab} = -\omega \varphi^{\dagger}_{ab},
\]

where the choice of a complex representation allows for a non–trivial mixing between the chiral components (\( T_{ab}, \tilde{T}_{ab} \)) of the complex tensor field \( \varphi_{ab} = T_{ab} + i \tilde{T}_{ab} \). Let us recall that Minkowski space–time does not allow for self–dual fields \( \varphi_{ab}, \tilde{\varphi}_{ab} \equiv (i/2) e_{abcd} \varphi^{cd} \) due to \( \tilde{\varphi}_{ab} = -\varphi_{ab} \).

Second, we perform in (8) a Wick rotation to the Euclidean space as a result of which \( \varphi_{ab} \) becomes two times the anti–selfdual part of \( T_{ab} \). Then, after appropriate rescaling of \( e \) and \( \alpha \), we put the resulting action, denoted by \( S(\alpha) \), on a general Riemannian 4–manifold endowed with a vierbein \( e_\mu{}^a \) and a spin connection \( \omega_{\mu}{}^{ab} \),

\[
S(\alpha) = \frac{1}{e^2} \int d^4 x \sqrt{g} \left\{ (\nabla^\mu T^-_{\mu \rho})(\nabla_\mu T^+_{\mu \rho}) + \alpha (T^-_{\mu \rho} T^+_{\rho \sigma})(T^-_{\mu \sigma} T^+_{\rho \sigma}) \right\}, \quad (9)
\]

where \( T^\pm_{\mu \nu} \) are the (anti)self–dual parts of the tensor matter field \( T_{\mu \nu} \),

\[
T^+_{\mu \nu} = \frac{1}{2} (T_{\mu \nu} \pm \tilde{T}_{\mu \nu}), \quad T^-_{\mu \nu} = \frac{1}{2} \sqrt{-g} e_{\mu \nu \rho \sigma} T^{\rho \sigma}, \quad \tilde{T}_{\mu \nu} = T_{\mu \nu}.
\]

\(^3\)The various factors of \( i \) are due to some subleties in the formulation of topological gauge theories (see, the remarks at the end of Sect. 3). Formally, they can be avoided completely when the fields \( \chi_{\mu \nu}, \tilde{\phi}, \eta \) and \( B_\mu \) in the DW theory and, later on, \( \tilde{\eta} \) and \( Y \) in the super–BF and the B–model (see, Sects. 3 and 4), are redefined by multiplying them with \(-i\). Then, the ghost number symmetry group changes into \( SO(2) \) instead of being \( SO(1, 1) \). The Euclideanized amplitudes are defined, as usual, by \( \exp(-S) \).
with the Levi–Civita tensor density being normalized as
\[
\sqrt{g} e_{\mu
u\rho\sigma} e^{abcd} = e_{[\mu} a \cdots e_{\rho]} d, \quad e_{\mu} a e_{\nu} b g^{\mu\nu} = \delta^{ab}, \quad e_{\mu} a e_{\nu} b \delta_{ab} = g_{\mu\nu}.
\]
As usual, the gauge and metric covariant derivative of \( A_\mu \) is defined by
\[
\nabla_\mu = D_\mu + \frac{1}{2} \omega_\mu^{ab} \sigma_{ab}, \quad \omega_\mu^{ab} = -\left( \partial_\mu e_{\nu}^a - \Gamma_\mu^\rho e_\rho^a \right) e^{\nu b}, \quad \partial_\mu e_{\nu}^a + \omega_\mu^{ab} e_{\nu \mid b} = 0,
\]
with \( \sigma_{ab} \) being the generators of the holonomy group \( SO(4) \); the Levi–Civita connection \( \Gamma_\mu^\rho \) is determined, as usual, by requiring covariant constancy of the metric and absence of torsion. With these definitions one gets
\[
\nabla^\mu T_{\mu\nu}^\pm = \frac{1}{\sqrt{g}} D^\mu (\sqrt{g} T_{\mu\nu}^\pm), \quad D^\mu T_{\mu\nu}^- = \partial^\mu T_{\mu\nu}^- - T_{\mu\nu}^- A_\mu, \quad D^\mu T_{\mu\nu}^+ = \partial^\mu T_{\mu\nu}^+ + A^\mu T_{\mu\nu}^+,
\]
where \( \Gamma_\mu^\nu = \partial_\mu \ln \sqrt{g} \) has been taken into account. Now, it is easy to verify that the action (9) is invariant under the following gauge transformations:
\[
\delta_G(\omega) A_\mu = D_\mu \omega, \quad \delta_G(\omega) T_{\mu\nu}^- = T_{\mu\nu}^- \omega, \quad \delta_G(\omega) T_{\mu\nu}^+ = -\omega T_{\mu\nu}^+.
\]
Besides the gauge symmetry, the action (9) possesses also a discrete symmetry under Hermitian conjugation, stemming from the CP invariance of the original Avdeev–Chizhov action. Under this conjugation \( A_\mu \) and \( T_{\mu\nu}^\pm \) transform into \( -A_\mu \) and \( T_{\mu\nu}^\mp \).

Although, for non–trivial \( A_\mu \neq 0 \) and without any restriction of the holonomy group of the underlying 4–manifold, the coupling in (9) is no more conformally, the action (9) exhibits still an invariance under the following local rescalings of the metric \( g_{\mu\nu} \) and the tensor matter fields \( T_{\mu\nu}^\pm \):
\[
\delta_W(\sigma) g_{\mu\nu} = -2 \sigma g_{\mu\nu}, \quad \delta_W(\sigma) T_{\mu\nu}^\pm = \sigma T_{\mu\nu}^\pm,
\]
with \( \delta_W(\sigma) \sqrt{g} = -\left( \sqrt{g}/2 \right) g_{\mu\nu} \delta_W(\sigma) g_{\mu\nu} = 4 \sigma \sqrt{g} \). Hence, this action satisfies, by construction, one of the important properties of cohomological gauge theories, namely local scale (or Weyl) invariance.

Third, without spoiling its gauge and local scale invariance, a \( N_T = 1 \) supersymmetric extension of the action (9) is obtained by introducing Grassmann–odd (anti)self–dual tensor fields \( \lambda^\pm_\mu \) and vector fields \( \xi^\nu_\mu \) as the superpartners of \( T_{\mu\nu}^\pm \) and \( \nabla^\mu T_{\mu\nu}^\pm \), respectively, and the Grassmann–even symmetric tensor field \( \zeta_\mu^\nu \) as the superpartner of the gauge invariant expression \( T_{\mu\nu}^- T_{\mu\nu}^\nu \). For the closure of the topological superalgebra it is necessary to introduce the bosonic auxiliary vector and symmetric tensor fields \( Y_\mu^\pm \) and \( G_\mu^\nu \), respectively. This supersymmetric action, denoted by \( S_T(\alpha) \), can be cast, analogous to (9), in a \( Q \)–exact form,
\[
S_T(\alpha) = Q \Psi_T(\alpha),
\]
with the following gauge and locally scale invariant matter fermion (cf., Eqs. (19)),
\[
\Psi_T(\alpha) = \frac{1}{2e^2} \int d^4 x \sqrt{g} \left\{ \xi^\nu_\mu (\nabla^\mu T_{\mu\nu}^+ - Y_\nu^+) + (\nabla_\mu T_{\mu\nu}^- - Y_\nu^-) \zeta_\mu^\nu + \alpha \zeta_\nu^\mu (T_{\mu\rho} T_{\nu\rho} - G_\mu^\nu) \right\}.
\]
The off–shell equivalently nilpotent \( Q \)–transformations of the matter fields are given by
\[
Q T_{\mu\nu}^- = \lambda_{\mu\nu}^-, \quad Q \lambda_{\mu\nu}^- = T_{\mu\nu}^- \phi, \quad Q \lambda_{\mu\nu}^+ = -T_{\mu\nu}^- \psi^\mu, \quad Q T_{\mu\nu}^+ = \lambda_{\mu\nu}^+, \quad Q \lambda_{\mu\nu}^+ = T_{\mu\nu}^+ \psi^\mu, \quad Q \zeta_\mu^\nu = \lambda_{\mu\rho}^+ T_{\rho\nu}^\nu + G_\mu^\nu, \quad Q G_\mu^\nu = -\lambda_{\mu\rho}^- T_{\rho\nu}^\nu + T_{\rho\nu}^\nu \lambda_{\mu\rho}^-, \quad Q Y_\nu^+ = -\phi \zeta_\nu^\mu - \nabla_\mu \lambda_{\mu\nu}^+ - \psi^\mu T_{\mu\nu}^+ + \nabla_\mu \lambda_{\mu\nu}^- - \psi^\mu T_{\mu\nu}^-.
\]
with $Q$ satisfying the topological superalgebra (3). The gauge transformations of $\lambda^\pm_{\mu\nu}$, $\zeta^\pm_{\mu}$ and $Y^\pm_{\mu}$ agree with those of $T^\pm_{\mu\nu}$ (c.f., Eqs. (11)), and $G_{\mu\nu}$ and $\zeta_{\mu\nu}$ are gauge invariant.

Performing the $Q$-transformation, thereby making use of Eqs. (3) and (14), the action (12) becomes

$$S_T(\alpha) = \frac{1}{e^2} \int d^4 x \sqrt{g} \left\{ (\nabla^\mu T^-_{\mu\rho})(\nabla_\nu T^\sigma_{\nu\rho}) - Y^-_{\mu} Y^+_{\mu} + (\nabla^\mu \lambda^\pm_{\mu\nu} - T^\pm_{\mu\nu} \psi^\mu) \xi^\nu_{\pm} - \xi^\nu_{\mu} \phi^\nu_{\pm} - \xi^\nu_{\nu} (\nabla^\mu \lambda^\mu_{\mu\nu} + \psi^\mu T^\nu_{\mu\nu}) + \alpha \left( (T^-_{\nu\rho} T^\sigma_{\nu\rho})(T^-_{\mu\sigma} T^\nu_{\mu\nu}) - \xi^\mu_{\nu} (\lambda^\mu_{\mu\nu} T^\nu_{\nu\rho} - T^-_{\mu\rho} \lambda^\mu_{\nu\rho}) - \frac{1}{2} G_{\mu\nu} G^{\mu\nu} \right) \right\}. \quad (15)$$

It should be stressed that, unlike the DW theory, the $Q$-transformations (14) are not obtained from a $N = 2$ supersymmetric tensor matter action via a topological twist. Let us shortly comment on why such a ‘detour’, if possible at all, was actually not necessary. This is simply due to the fact that the scalar field $\phi$, entering into the DW theory, is $Q$-inert. Therefore, by choosing a suitable gauge and locally scale invariant fermion $\Psi_T(\alpha)$ and off–shell equivariantly nilpotent transformation rules for the matter fields the resulting action $S_T(\alpha)$ is very alike the action $S_{DW}$. However, this is no longer the case for $N_T = 2$ topological matter, whose construction is rather involved and, obviously, quite special (see, Sect. 4 below).

The action (8) of the non-abelian Avdeev–Chizhov model has several remarkable features [29]. Above all, it is worth noting that its tensorial structure is completely fixed. Namely, due to the following purely algebraic relations,

$$g^{\alpha\beta} T^-_{\mu\rho} F^{\mu\nu} T^+_{\nu\sigma} = 0, \quad T^-_{\mu\rho} (C^\mu_{\mu\nu\sigma} + g^{\mu\sigma} R^{\mu\nu}) T^+_{\nu\sigma} = 0 \quad (16)$$

and

$$(T^-_{\mu\rho} T^\sigma_{\rho\sigma})(T^-_{\nu\sigma} T^\mu_{\nu\sigma}) = 4(T^-_{\mu\rho} T^\mu_{\nu\sigma})(T^-_{\nu\rho} T^\nu_{\nu\sigma}) = 4(T^-_{\mu\rho} T^\mu_{\nu\sigma})(T^-_{\nu\sigma} T^\nu_{\nu\sigma}), \quad (17)$$

an unique gauge and local scale invariant kinetic and quartic self–interaction term is singlet out (this explains why the action (8) deserves our interest). In addition, these relations forbid the existence of mass and cubic self–interacting terms.

The equivalence of the three possible self–interaction terms (17) can simply be proven by using the identity $e^\sigma_d \sqrt{g} \epsilon_{\mu\rho\sigma \epsilon} e^{abcd} = e^\mu_{(a} e^\nu_{b} e^\rho_{c} e^{d)}$. In the same way one verifies the relation

$$g^{\alpha\sigma} T^-_{\mu\rho} \left[ \nabla^\mu, \nabla^\nu \right] T^+_{\nu\sigma} = 0,$$

which guarantees the uniqueness of the kinetic term. From this relation one derives

$$g^{\rho\sigma} T^-_{\mu\rho} F^{\mu\nu} T^+_{\nu\sigma} = 0, \quad T^-_{\mu\rho} (R_{\mu\rho\sigma} - g^{\mu\sigma} R^{\mu\nu}) T^+_{\nu\sigma} = 0, \quad g^{\mu\nu} g^{\rho\sigma} T^-_{\mu\rho} R T^+_{\nu\sigma} = 0, \quad (18)$$

where $R_{\mu\nu} = g^{\rho\sigma} R_{\mu\rho\sigma}$ and $R = g^{\mu\nu} R_{\mu\nu}$ are the Ricci tensor and the Ricci scalar, respectively, $R_{\mu\rho\sigma} = \partial_\mu \Gamma^\nu_{\rho\sigma} + \Gamma^\mu_{\nu\rho} \Gamma^\lambda_{\rho \lambda}$ being the Riemannian curvature tensor. Decomposing $R_{\mu\rho\sigma}$ into its irreducible parts,

$$R_{\mu\rho\sigma} = C_{\mu\rho\sigma} - \frac{1}{6} R (g_{\mu\nu} g_{\rho\sigma} - g_{\mu\sigma} g_{\rho\nu}) + \frac{1}{2} (g_{\mu\nu} R_{\rho\sigma} - g_{\mu\sigma} R_{\nu\rho} - g_{\nu\rho} R_{\mu\sigma} + g_{\rho\sigma} R_{\mu\nu}),$$

where the conformal Weyl tensor $C_{\mu\rho\sigma}$ is completely traceless, from (15) one obtains immediately the relations (16). The fact, that $C_{\mu\rho\sigma}$ appears in (16) only in combination with $R_{\mu\nu}$ is not surprising if one remembers the definition of the Euler number,

$$\chi = \frac{1}{32\pi^2} \int d^4 x \sqrt{g} \left\{ C_{\mu\rho\sigma} C_{\mu\rho\sigma} - 2 R^{\mu\nu} R_{\mu\nu} + \frac{2}{3} R^2 \right\},$$

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and, in addition, takes into account the last of the relations (18).

Let us remark that the second of Eqs. (18) along with our choice of the matter fermion (13) ensures also that a coupling of DW theory to tensorial matter does not spoil the local scale invariance. Indeed, it is simple to check that the actions (7) and (15) are left invariant by the following local Weyl symmetry,

\[
\begin{align*}
\delta_W(\sigma)g_{\mu\nu} &= -2\sigma g_{\mu\nu}, \\
\delta_W(\sigma)\phi &= -2\sigma\phi, \\
\delta_W(\sigma)T^\pm_{\mu\nu} &= \sigma T^\pm_{\mu\nu}, \\
\delta_W(\sigma)Y^\pm_\mu &= -\sigma Y^\pm_\mu, \\
\delta_W(\sigma)\lambda_\mu^\pm &= \sigma\lambda_\mu^\pm, \\
\delta_W(\sigma)\xi^\pm_\mu &= -\sigma\xi^\pm_\mu,
\end{align*}
\]

where we have only written down the non–trivial scale transformations. Obviously, this symmetry commutes with the topological supersymmetry Q, i.e., it holds \([\delta_W(\sigma), Q] = 0\).

Since the total action, \(S_{DW} + S_T(\alpha)\), is \(Q\)-exact, and because \(g_{\mu\nu}\) is \(Q\)-inert, it follows immediately that the metric variation of that action is also \(Q\)-exact. Therefore, the full stress tensor \(T^\mu_\nu(\alpha)\) derived from this action is \(Q\)-exact, as well, which is sufficient to ensure that physical observables have no dependence on the metric of the underlying manifold. Since this action is locally scale invariant, the trace of the stress tensor becomes, on–shell, the divergence of a current,

\[g^{\mu\nu}T^\nu_\mu(\alpha) = \frac{2\delta}{\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}}(S_{DW} + S_T(\alpha)) \approxeq \partial_\mu j^\mu(\alpha).\]

Moreover, since topological gauge theories should not involve arbitrary parameters — at least, as long as they do not enter into the \(Q\)-transformations — physical observables should also not depend on the coupling constant \(\alpha\) of the self–interaction term. Indeed, the \(Q\)-exactness of the action (12) — which, as already emphasized above, has far–reaching consequences — makes it possible to use field–theoretical arguments to conclude that the \(\alpha\)-dependent term in (13) is irrelevant and can be omitted. Therefore, in the following considerations we shall take into account only the kinetic term of the tensor matter action.

3 \(N_T = 1\) super–BF model with tensor matter

In the previous section we extended the DW theory to a topological model with tensor matter. Its moduli space remains to be dominated by instantons. But, the evaluation of the partition function in the weak coupling limit, which is expected to go still through as in Ref. [10], will now receive contributions to the ratios of determinants of the kinetic operators from the even and odd integer spin fields \((A_\mu, \chi_\mu, \psi_\mu)\) of the gauge multiplet as well as from the even and odd integer spin fields \((T^\pm_{\mu\nu}, \lambda_\mu^\pm, \xi^\pm_\mu)\) of the matter multiplet.

Now, as a preliminary stage, we will generalize the previous construction for the \(N_T = 1\) super–BF model whose moduli space is the space of flat complexified gauge fields \(A_\mu = iV_\mu\).

The idea behind of this intention is that, in principle, such an generalization allows also for introducing an extended, \(N_T = 2\), supersymmetry. For that purpose, we have to assume that, apart from the complexified gauge field, all the Grassmann–odd fields are complexified ones, as well. In Sect. 4, when turning to the \(B\)–model, it will become obvious that such an extension, roughly speaking, amounts to introduce an extended, \(N_T = 2\), complexified supersymmetry \(Q = i\bar{Q}\). Usually, extended supersymmetries arise when \(N_T = 1\) theories are formulated on manifolds with reduced holonomy groups, e.g., DW theory on Kähler manifolds. Therefore, the complexified supersymmetry \(Q = i\bar{Q}\) which we encounter here is of a different kind as both, \(Q\) and \(\bar{Q}\), must have the same ghost number.

The \(N_T = 1\) super–BF model was described in detail in [17] using the Batalin–Vilkovsky formalism. However, due to some redundancy in that description it is possible to find a more
simpler formulation of the model with a reduced field content \[25\]. It consists of the gauge field \(A_\mu\), the vector field \(V_\mu\), the complex Grassmann–odd tensor, vector and scalar fields \(\chi_{\mu\nu}, \psi_\mu, \bar{\psi}_\mu\) and \(\eta, \bar{\eta}\), respectively, and the Grassmann–even scalar fields \(\phi\) and \(\bar{\phi}\). Moreover, in order to ensure off–shell equivariantly nilpotency of the topological supersymmetry \(Q, Q^2 = \delta_G(\phi)\), it is necessary to introduce the bosonic auxiliary tensor and scalar fields \(B_{\mu\nu}\) and \(Y\). (Let us point out that \(\chi_{\mu\nu}\) and \(B_{\mu\nu}\) are not self–dual.)

Adopting that formulation, the action of the model can be written in a \(Q\)–exact form,

\[ S_{BF} = Q\Psi_{BF}, \]

where the gauge fermion (again choosen in a Feynman type gauge) is given by

\[ \Psi_{BF} = \frac{i}{e^2} \int d^4x \sqrt{g} \left\{ \frac{1}{2} \tilde{\chi}^{\mu\nu} F_{\mu\nu} + i \frac{1}{4} \chi^{\mu\nu} B_{\mu\nu} + \chi^{\mu\nu} D_\mu V_\nu - \psi^\mu D_\mu \bar{\phi} + i \frac{1}{2} \bar{\eta} [\bar{\phi}, \phi] + V^\mu D_\mu \bar{\eta} + i \frac{1}{2} \bar{\eta} Y \right\}, \]

and the supersymmetry transformations \(Q\) are defined as follows:

\[ Qg_{\mu\nu} = 0, \quad Q\phi = 0, \]
\[ QA_\mu = \psi_\mu, \quad Q\psi_\mu = D_\mu \phi, \]
\[ QV_\mu = \bar{\psi}_\mu, \quad Q\bar{\psi}_\mu = [V_\mu, \phi], \]
\[ Q\phi = \eta, \quad Q\eta = [\bar{\phi}, \phi], \]
\[ QB = Y, \quad QY = [\eta, \phi], \]
\[ Q\chi_{\mu\nu} = B_{\mu\nu}, \quad QB_{\mu\nu} = [\chi_{\mu\nu}, \phi]. \]

The reduced configuration space of the model suggests the existence of a discrete symmetry under Hermitean conjugation, namely \[24\]

\[ (A_\mu, \psi_\mu, \bar{\psi}_\mu, \phi, V_\mu, \bar{\phi}, \eta, \bar{\eta}, Y, \chi_{\mu\nu}, B_{\mu\nu}) \rightarrow (-A_\mu, i\bar{\psi}_\mu, -i\psi_\mu, -\phi, -V_\mu, \bar{\phi}, -i\bar{\eta}, i\eta, Y, i\chi_{\mu\nu}, -B_{\mu\nu}), \]
\[ \epsilon_{\mu\nu\rho\sigma} \rightarrow -\epsilon_{\mu\nu\rho\sigma}, \]

so that one could expect the occurrence of an extended, \(N_T = 2\), topological supersymmetry \(Q \pm i\bar{Q}\). However, carrying out in (21) the \(Q\)–transformation explicitly, which yields

\[ S_{BF} = \frac{i}{e^2} \int d^4x \sqrt{g} \left\{ \frac{1}{2} \tilde{B}^{\mu\nu} F_{\mu\nu} + B^{\mu\nu} D_\mu V_\nu + i \frac{1}{4} B^{\mu\nu} B_{\mu\nu} - i \frac{1}{4} \bar{\phi} \{\chi^{\mu\nu}, \chi_{\mu\nu}\} \right\} \]
\[ - D^\mu \bar{\phi} D_\mu \bar{\phi} + \bar{\phi} \{\psi_\mu, \psi_\mu\} - \bar{\chi}^{\mu\nu} D_\mu \psi_\nu + \psi_\mu D_\eta \]
\[ - \chi^{\mu\nu} D_\mu \bar{\psi}_\nu + \bar{\psi}_\mu D_\mu \bar{\eta} - \chi^{\mu\nu} [V_\mu, \psi_\nu] - \psi_\mu [V_\mu, \bar{\eta}] \]
\[ - i \frac{1}{2} \bar{\phi} \{\bar{\eta}, \bar{\eta}\} + i \frac{1}{2} [\bar{\phi}, \phi]^2 - i \frac{1}{2} \bar{\phi} \{\bar{\eta}, \eta\} + V^\mu D_\mu Y + i \frac{1}{2} Y^2 \}, \]

it is easily seen that this is only partly the case. Therefore, this action possesses really only a simple, \(N_T = 1\), topological supersymmetry.

Let us now turn to the construction of the matter action. For that purpose we incorporate into (1) the vector field \(V_\mu\) in such a way that the invariance under Hermitean conjugation is preserved. This is simply achieved by replacing in \(\nabla^\mu T_{\mu\nu}^\pm\) the gauge field \(A_\mu\) through \(A_\mu \mp iV_\mu\),

\[4\]This symmetry is identical with the Hermitean conjugation introduced in Ref. \[13\] after an appropriate redefinition of the original fields through complexified ones (see also remarks at the end of this Section).
respectively. The \(\alpha\)-independent part of the resulting action, denoted by \(S_C\) (\(C\) standing for ‘complexified’), takes the form

\[
S_C = \frac{1}{e^2} \int d^4 x \sqrt{g} \left\{ \left( \nabla^\mu T_{\mu \rho} - i T_{\mu \rho} V^\mu \right) (\nabla_\nu T^\nu_{\rho \mu} - i V_\nu T^\nu_{\rho \mu}) \right\},
\]

(25)

and is, in fact, invariant under Hermitian conjugation \((A_\mu, V_\mu, T_{\mu \rho}^\pm) \rightarrow (-A_\mu, -V_\mu, T_{\mu \rho}^\pm)\).

In order to get the \(N_T = 1\) supersymmetric extension of (24) we make the ansatz

\[
S_{CT} = Q \Psi_{CT},
\]

(26)

and choose the following gauge and locally scale invariant matter fermion

\[
\Psi_{CT} = \frac{1}{2 e^2} \int d^4 x \sqrt{g} \left\{ \xi^\nu (\nabla^\mu T^\mu_{\nu} - i V^\mu T^\mu_{\nu} - Y_\nu^\pm) + (\nabla^\mu T^\mu_{\nu} - i T^\mu_{\nu} V^\mu - Y_\nu^-) \xi^\nu_+ \right\},
\]

(27)

where the \(Q\)-transformations are given by

\[
\begin{align*}
Q T^\mu_{\nu} & = \lambda^\mu_{\nu}, \\
Q \lambda^\nu_{\mu} & = T^\nu_{\mu} \phi, \\
Q \xi^\nu & = \nabla^\mu T^\mu_{\nu} - i T^\mu_{\nu} V^\mu + Y^-_{\nu}, \\
Q Y^\nu_+ & = \xi^\nu \phi - \nabla^\mu \lambda^\mu_{\nu} + i \lambda^\nu_{\mu} V^\mu + T^\mu_{\nu} (\psi^\mu + i \bar{\psi}^{\mu}), \\
Q T_\nu^+ & = \lambda^\nu_+, \\
Q \lambda_{\nu}^\mu & = - \phi T^\nu_{\mu}, \\
Q \xi^\nu_+ & = \nabla^\mu T^\mu_{\nu} - i V^\mu T^\mu_{\nu} + Y_\nu^+, \\
Q Y^\nu_+ & = - \phi \xi^\nu_+ - \nabla^\mu \lambda^\mu_{\nu} + i V^\mu \lambda^\mu_{\nu} - (\psi^\mu - i \bar{\psi}^{\mu}) T^\mu_{\nu}. 
\end{align*}
\]

(28)

With that choice the action (26) takes the form

\[
S_{CT} = \frac{1}{e^2} \int d^4 x \sqrt{g} \left\{ \left( \nabla^\mu T^\mu_{\rho} - i T^\mu_{\rho} V^\mu \right) (\nabla_\nu T^\nu_{\rho \mu} - i V_\nu T^\nu_{\rho \mu}) - Y^\nu_+ Y^\nu_- \right\}
\]

\[
+ \left( \nabla^\mu \lambda^\mu_{\nu} - i \lambda^\nu_{\mu} V^\mu \right) \xi^\nu_+ - \xi^\nu \left( \nabla^\mu \lambda^\mu_{\nu} - i V^\mu \lambda^\mu_{\nu} \right)
\]

\[
- T^\mu_{\nu} (\psi^\mu + i \bar{\psi}^{\mu}) \xi^\nu_+ - \xi^\nu \left( \psi^\mu + i \bar{\psi}^{\mu} \right) T^\mu_{\nu},
\]

(29)

and one can easily verify that, as promised, this action exhibits a discrete symmetry under Hermitian conjugation, namely

\[
(A_\mu, \psi_\mu, \bar{\psi}_\mu, \phi, V_\mu, T^\pm_{\mu \nu}, \lambda^\pm_{\mu \nu}, Y^\pm_{\mu \nu}, \xi^\pm_\nu) \rightarrow (-A_\mu, i \bar{\psi}_\mu, -i \psi_\mu, -\phi, -V_\mu, T^\mp_{\mu \nu}, +\lambda^\mp_{\mu \nu}, Y^\mp_{\mu \nu}, +\xi^\mp_\nu),
\]

(30)

which is clearly compatible with (23). Also this symmetry suggests the presence of a hidden \(N_T = 2\) supersymmetry \(Q \pm i \bar{Q}\). However, in order to expose such a second supersymmetry \(\bar{Q}\) the action (24) must be deformed by adding further terms to the gauge fermion (21) so that its partly discrete symmetry under Hermitian conjugation (24) becomes completely manifest. This will be the subject of the next section.

Finally, there are several points worth noting about the appearance of some ‘wrong’ signs in the symmetry (23): First, because all of the (real parts of the) fields of the BF–model are represented by anti–Hermitian matrices, there is an extra minus sign in these transformations, e.g., \(A_\mu \pm i V_\mu \rightarrow -(A_\mu \pm i V_\mu)\). Second, in order to ensure that the transformation \(\chi_{\mu \nu} \pm i \bar{\chi}_{\mu \nu} \rightarrow i(\bar{\chi}_{\mu \nu} \pm i \chi_{\mu \nu})\) of the complexified tensor fields agrees, formally, with that of the complexified vector fields, \(\psi_\mu \pm i \bar{\psi}_\mu \rightarrow i(\bar{\psi}_\mu \pm i \psi_\mu)\), it is necessary that Hermitian conjugation is combined with a simultaneous replacement \(\epsilon_{\mu \nu \rho \sigma} \rightarrow -\epsilon_{\mu \nu \rho \sigma}\), which reverses the orientation of the 4–manifold.
Third, the transformations $\bar{\phi} \to \phi$, $\eta \pm i\bar{\eta} \to -i(\bar{\eta} \pm i\eta)$ and $Y \to Y$ have apparently a wrong sign. These extra sign changes can be traced back to some subtleties in the formulation of cohomological gauge theories. The DW theory was originally derived from the Wick–rotated $N = 2$ SYM with compact $R$–symmetry group $U(2) \cong Sp(2) \otimes U(1)$, with $U(1) \cong SO(2)$ being the ghost number symmetry [10]. But, in this approach, the sign of the kinetic term of one of the two original scalars, $\phi$ and $\bar{\phi}$, must be changed, so that the twisted theory has an $SO(1,1)$ ghost number symmetry [10]. One can simply overcome this difficulty if the ‘problematic’ scalar field $\phi$ is replaced by $\phi \to i\phi$. Therefore, under Hermitean conjugation the scalars $\phi$ and $\bar{\phi}$ transform like a real and a purely imaginary field, respectively. The situation is quite similar for the super–BF model. Recently, it has been shown that one can completely sidestep this problem by twisting directly the Euclidean $N = 2$ SYM with non–compact $R$–symmetry group $Sp(2) \otimes SO(1,1)$ instead of $Sp(2) \otimes SO(2)$ [18].

4 $N_T = 2$ topological B–model with tensor matter

The purpose of this Section is to deform the action (24) of the super–BF model — according to the proposal [25] of Blau and Thompson — to that of the B–model and to reveal the second topological supersymmetry $\hat{Q}$ of the matter action (29). We begin by shortly reviewing the structure of the B–model [18]. This model is obtained by a certain twist of $N = 4$ SYM, namely breaking down the $R$–symmetry group $SU(4)$ to $SO(4) \cong Sp(2)_A \otimes Sp(2)_B \otimes U(1)$ and by changing the action of the rotation group $SO(4) \cong SU(2)_L \otimes SU(2)_R$ of the Euclidean space by replacing $SU(2)_L$ and $SU(2)_R$ through the diagonal subgroup of $SU(2)_L \otimes Sp(2)_A$ and of $SU(2)_R \otimes Sp(2)_B$, respectively.

The $N = 4$ SYM is believed to be exactly finite and conformal invariant, even non–perturbatively [31]. Furthermore, it is believed that the S–duality [32] in $N = 4$ SYM, which includes a discrete $Z_2$ symmetry corresponding to an interchange of electric and magnetic charges along with an interchange of weak and strong coupling, $e \to 1/e$, is exact. (It is natural to conjecture that the twisted theory has also an S–duality symmetry.)

For $N = 4$ SYM it is possible to introduce a further coupling constant $\theta$ by adding to the YM action a topological term (owing to the absence of a chiral anomaly in $N = 4$ SYM it is impossible to shift away the $\theta$–term by means of a chiral rotation). In the presence of a non–zero $\theta$–angle the original $Z_2$ symmetry $e \to 1/e$ is extended to a full $SL(2,Z)$ symmetry acting on $\tau = \theta/2\pi + 4\pi i/e^2$. Thereby, one expects that under an inversion $\tau \to -1/\tau$ of this coupling the gauge group $G$ is exchanged with its dual group Dual$(G)$. Moreover, as pointed out by ’t Hooft [33], one can consider topological non–trivial gauge transformations of $G/\text{Center}(G)$ with discrete magnetic ’t Hooft flux through certain two–cycles of the 4–manifold.

The action of the B–model in the presence of a non–zero $\theta$–term can be cast into the form

$$S_{BT}(\theta) = Q\Psi_{BT} - i\theta k = \bar{Q}\bar{\Psi}_{BT} - i\bar{\theta} \bar{k}, \quad k = \frac{1}{32\pi^2} \int d^4x \sqrt{g} \text{tr}\left\{ \bar{F}^{\mu\nu}F_{\mu\nu} \right\}, \quad (31)$$

where $k$ is the instanton number, i.e., its $\theta$–independent part is $Q$– and $\bar{Q}$–exact, but not $Q\bar{Q}$–exact, $S_{BT}(0) \neq Q\bar{Q}\Omega_{BT}$ (notice that under the Hermitean conjugation, Eqs. (23), $k$ transforms into $-k$ whereas $\theta$ remains inert).

The gauge fermion $\Psi_{BT}$ in (31) is an appropriate extension of the gauge choice (24),

$$\Psi_{BT} = \frac{i}{e^2} \int d^4x \sqrt{g} \text{tr}\left\{ \frac{1}{2} \chi^{\mu\nu}(F_{\mu\nu} - [V_\mu, V_\nu]) + \frac{i}{4} \chi^{\mu\nu}(B_{\mu\nu} - iD_{[\mu} V_{\nu]} - \psi^\mu D_\mu \phi - \bar{\psi}^\mu [V_\mu, \bar{\phi}] + \frac{i}{2} \eta [\bar{\phi}, \phi] + V^\mu D_\mu \bar{\eta} + \frac{i}{2} \bar{\eta} Y \right\}, \quad (32)$$
whereas

\[ \tilde{\Psi}_{BT} = \frac{i}{e^2} \int d^4x \sqrt{g} \text{tr} \left\{ \frac{1}{2} \chi^\mu\nu (F_{\mu\nu} - [V_\mu, V_\nu]) + \frac{i}{4} \tilde{\chi}^\mu\nu (B_{\mu\nu} - iD_{[\mu} V_{\nu]} ) \right. 
\]

\[ \left. - \tilde{\psi}^\mu D_\mu \tilde{\phi} + \psi^\mu [V_\mu, \tilde{\phi}] + \frac{i}{2} \tilde{\eta} [\tilde{\phi}, \phi] - V^\mu D_\mu \eta - \frac{i}{2} \tilde{\eta} Y \right\}. \]

is obtained from (32) by applying the following discrete Z_2 symmetry of the B–model (32) (see Eq. (38) below),

\[ (A_\mu, \psi_\mu, \bar{\psi}_\mu, \phi, V_\mu, \bar{\phi}, \bar{\eta}, \eta, \bar{Y}, Y, \chi_{\mu\nu}, B_{\mu\nu}) \rightarrow (A_\mu, \bar{\psi}_\mu, \psi_\mu, \phi, -V_\mu, \bar{\phi}, \eta, \bar{\eta}, Y, \bar{Y}, \chi_{\mu\nu}, B_{\mu\nu}). \]  

(34)

The on–shell equivariantly nilpotent Q– and \bar{Q}–transformations, being interchanged by the Z_2 symmetry (34), are the following:

\[ Q\phi = 0, \quad \bar{Q}\phi = 0, \]
\[ QA_\mu = \psi_\mu, \quad \bar{Q}A_\mu = \bar{\psi}_\mu, \]
\[ Q\psi_\mu = D_\mu \phi, \quad \bar{Q}\bar{\psi}_\mu = \bar{D}_\mu \phi, \]
\[ QV_\mu = \psi_\mu, \quad \bar{Q}V_\mu = -\psi_\mu, \]
\[ Q\bar{\psi}_\mu = [V_\mu, \phi], \quad \bar{Q}\psi_\mu = -[V_\mu, \phi], \]
\[ Q\bar{\phi} = \eta, \quad \bar{Q}\phi = \bar{\eta}, \]
\[ Q\eta = [\bar{\phi}, \phi], \quad \bar{Q}\bar{\eta} = [\bar{\phi}, \phi], \]
\[ Q\bar{\eta} = Y, \quad \bar{Q}\eta = -Y, \]
\[ QY = [\bar{\eta}, \phi], \quad \bar{Q}Y = -[\eta, \phi], \]
\[ Q\chi_{\mu\nu} = B_{\mu\nu} + iD_{[\mu} V_{\nu]}, \quad \bar{Q}\bar{\chi}_{\mu\nu} = B_{\mu\nu} - iD_{[\mu} V_{\nu]}, \]
\[ QB_{\mu\nu} = [\chi_{\mu\nu}, \phi] - iD_{[\mu} \bar{\psi}_{\nu]} - i[\psi_{[\mu}, V_{\nu]}, \quad \bar{Q}B_{\mu\nu} = [\bar{\chi}_{\mu\nu}, \phi] - iD_{[\mu} \bar{\psi}_{\nu]} + i[\bar{\psi}_{[\mu}, V_{\nu}]. \]  

(35)

Here, both operators Q and \bar{Q} square to zero modulo field–dependent gauge transformations \delta_G(\phi) but anticommute only on–shell on \chi_{\mu\nu} and \phi_{\mu\nu},

\[ Q^2 = \delta_G(\phi), \quad \{Q, \bar{Q}\} = 0, \quad \bar{Q}^2 = \delta_G(\phi). \]  

(36)

In the Appendix we reanalyse in detail a statement of Lozano (27), namely that the action of the B–model in the presence of a non–zero \theta–term can also be cast into the following form:

\[ S_L(\tau) = Q\Psi_L + 2\pi i \tau k = \bar{Q}\bar{\Psi}_L + 2\pi i \bar{\tau} k, \quad \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}, \]

(37)

\[ \bar{\tau} \text{ being the complex conjugate of } \tau, \text{ with } Q \text{ and } \bar{Q} \text{ satisfying the superalgebra (33) off–shell.} \]

We have not been able to confirm that result. A closer analysis (see Appendix) shows that, on the one hand, it is always possible to introduce off–shell equivariantly nilpotent Q– and \bar{Q}–transformations, but then \( S_L(\tau) \) cannot be cast into the form (37). On the other hand, it is always possible to express \( S_L(\tau) \) in the form (37), but then the Q– and \bar{Q}–transformations close only on–shell.
In the Feynman type gauge \[\text{(22)}\] the \(\theta\)-independent part of the action \[\text{(31)}\] reads

\[
S_{\text{BT}}(0) = \frac{i}{e^2} \int d^4x \sqrt{g} \left\{ \frac{1}{2} \bar{B}^{\mu\nu}(F_{\mu\nu} - [V_{\mu}, V_{\nu}]) + \frac{i}{4} D^{[\mu} V^{\nu]} D_{[\mu} V_{\nu]} \right. \\
+ \frac{i}{4} B^{\mu\nu} B_{\mu\nu} - \frac{i}{4} \bar{\phi}\{\chi^{\mu\nu}, \chi_{\mu\nu}\} - D^\mu \bar{\phi} D_\mu \phi \\
+ \bar{\phi}\{\bar{\psi}_\mu, \bar{\psi}_\mu\} - [V^\mu, \bar{\phi}][V_\mu, \phi] + \bar{\phi}\{\psi^\mu, \psi_\mu\} \\
- \bar{\chi}_{\mu\nu} D_\mu \bar{\psi}_\nu + \bar{\chi}^{\mu\nu}[V_\mu, \bar{\psi}_\nu] + \psi^\mu D_\mu \eta - \psi^\mu [V_\mu, \bar{\eta}] \\
- \bar{\chi}_{\mu\nu} D_\mu \bar{\psi}_\nu - \chi^{\mu\nu}[V_\mu, \psi_\nu] + \bar{\psi}_\mu D_\mu \bar{\eta} + \bar{\psi}^\mu [V_\mu, \eta] \\
- \frac{i}{2} \bar{\phi}\{\bar{\eta}, \bar{\eta}\} + \frac{i}{2}(\bar{\bar{\phi}}, \phi)^2 - \frac{i}{2} \phi\{\eta, \eta\} + V^\mu D_\mu Y + \frac{i}{2} Y^2 \right\}.
\]

(38)

By construction, this action is invariant under Hermitean conjugation \[\text{(23)}\] and may be regarded, formally, as deformation of the action \[\text{(24)}\] of the \(N_T = 1\) super–BF model. By choosing the Landau type gauge this action and the supersymmetry transformations \[\text{(35)}\] coincide precisely with those of Ref. \[\text{(25)}\].

After having deformed the action of the super–BF model to a point in the deformation space where it possesses an extended, \(N_T = 2\), supersymmetry, let us now show that at this point the matter action \[\text{(26)}\] can be cast also into a \(Q\)-exact form,

\[
S_{\text{CT}} = \bar{Q}\Psi_{\text{CT}},
\]

(39)

with the following choice of the matter fermion,

\[
\Psi_{\text{CT}} = -\frac{i}{2e^2} \int d^4x \sqrt{g} \left\{ \xi_-^\nu (\nabla^\mu T^+_{\mu\nu} - i V^\mu T^+_{\mu\nu} - Y^+_\nu) - (\nabla^\mu T^-_{\mu\nu} - i T^-_{\mu\nu} V^\mu - Y^-\nu) \xi^\nu_+ \right\}.
\]

(40)

Indeed, introducing the \(\bar{Q}\)-transformations of the matter fields according to

\[
\bar{Q} T^\mu_{\mu\nu} = -i \lambda^-_{\mu\nu}, \\
\bar{Q} \lambda^-_{\mu\nu} = i T^\mu_{\mu\nu} \phi, \\
\bar{Q} \xi_\nu^- = i \nabla^\mu T^-_{\mu\nu} + T^-_{\mu\nu} V^\mu + i Y^-_\nu, \\
\bar{Q} Y^-_\nu = -i \xi^-_\nu \phi + i \nabla^\mu \lambda^-_{\mu\nu} + \lambda^-_{\mu\nu} V^\mu + T^-_{\mu\nu}(\bar{\psi}_\mu + i \psi^\mu), \\
\bar{Q} T^+_{\mu\nu} = i \lambda^+_{\mu\nu}, \\
\bar{Q} \lambda^+_{\mu\nu} = i \phi T^+_{\mu\nu}, \\
\bar{Q} \xi^+_\nu = -i \nabla^\mu T^+_{\mu\nu} - V^\mu T^+_{\mu\nu} - i Y^+_\nu, \\
\bar{Q} Y^+_\nu = -i \phi \xi^+_\nu - i \nabla^\mu \lambda^+_{\mu\nu} - V^\mu \lambda^+_{\mu\nu} - (\bar{\psi}_\mu + i \psi^\mu) T^+_{\mu\nu},
\]

(41)

it is easy seen that they are equivariently nilpotent and anticommute with the \(Q\)-transformations \[\text{(28)}\], i.e., \(Q\) and \(\bar{Q}\) obey the topological superalgebra \[\text{(23)}\] off–shell. Then, spelling out \[\text{(39)}\] in detail, one recovers precisely the orginal action \[\text{(24)}\], namely

\[
S_{\text{CT}} = \frac{1}{e^2} \int d^4x \sqrt{g} \left\{ (\nabla^\mu T^\mu_{\nu\rho} - i T^\mu_{\nu\rho} V^\mu)(\nabla_\rho T^\nu_{\mu\rho} - i V_\rho T^\nu_{\mu\rho}) - Y^\mu Y^-_\mu \right. \\
+ (\nabla^\mu \lambda^\rho_{\nu\mu} - i \lambda^-_{\nu\mu} V^\mu) \xi^\nu_+ - \xi^-_\nu (\nabla^\mu \lambda^\rho_{\nu\mu} - i \lambda^-_{\nu\mu} V^\mu) \\
- T^-_{\mu\nu}(\bar{\psi}_\mu + i \psi^\mu) \xi^\nu_+ - \xi^-_\nu \phi \xi^\nu_+ - \xi^-_\nu (\psi - i \bar{\psi}_\mu) T^+_{\mu\nu} \right\}.
\]

Furthermore, it is simple to verify that the actions \[\text{(38)}\] and \[\text{(28)}\] are invariant under a local rescaling of the metric \(\delta_W(\sigma)g_{\mu\nu} = -2\sigma g_{\mu\nu}\) and local Weyl transformations of the fields. The
latter are given by

\[
\begin{align*}
\delta W(\sigma)\phi &= -2\sigma\bar{\phi}, & \delta W(\sigma)T^\pm_{\mu\nu} &= \sigma T^\pm_{\mu\nu}, \\
\delta W(\sigma)\eta &= -2\sigma\eta, & \delta W(\sigma)\lambda^\pm_{\mu\nu} &= \sigma\lambda^\pm_{\mu\nu}, \\
\delta W(\sigma)Y &= -2\sigma Y, & \delta W(\sigma)Y^\pm_\mu &= -\sigma Y^\pm_\mu, \\
\delta W(\sigma)\bar{\eta} &= -2\sigma\bar{\eta}, & \delta W(\sigma)\xi^\pm_\mu &= -\sigma\xi^\pm_\mu,
\end{align*}
\]

with the properties \([\delta W(\sigma), Q] = 0 = [\delta W(\sigma), \bar{Q}]\), where again we have only written down the non–trivial transformations.

This finishes our construction of the topological B–model involving antisymmetric tensor matter fields.

In this context, let us mention that recently a fourth, conformal twist of \(N = 4\) SYM has been proposed leading to a conformal invariant deformation of the B–model whose action is local scale invariant and has two Weyl invariant topological supersymmetries, \(Q\) and \(\bar{Q}\), [28]. Moreover, it has been conjectured that this model could have a dual holographic description in the 5–dimensional de Sitter space. We suppose that it should be possible to couple this model to antisymmetric tensor matter fields, too.

## 5 Concluding remarks

Motivated by the question whether, at least from a purely algebraic point of view, a topological model with matter having \(N_T = 2\) supersymmetry can be constructed, we have proposed a new type of matter interactions involving antisymmetric tensor fields. These interactions may be regarded as supersymmetric extensions of a \(\varphi^4\)–type theory for antisymmetric tensor matter fields, firstly considered in [19, 29], on a general, curved 4–dimensional Euclidean gravitational background. Such tensorial matter interactions have been explicitly worked out for the DW theory, the \(N_T = 1\) super–BF model and the \(N_T = 2\) topological B–model.

In that paper we have focused primarily on the algebraic aspects of how topological gauge theories involving tensor matter fields can be constructed. Many other aspects remain still to be clarified. Among the interesting questions which deserve further investigations let us only mention the following:

(i) What are the unitarity properties of the independent propagating degrees of freedom associated with antisymmetric tensor matter fields in Euclidean space?

(ii) What are the relevant equations of the moduli problem in the presence of tensor matter fields?

(iii) Are there new topological observables associated with tensor matter fields?

(iv) Does, analogous to the DW theory with matter hypermultiplet, the ghost–number anomaly of the topological B–model change when the tensor matter multiplet is coupled?

(v) How behave antisymmetric tensor matter fields in topological gauge theories under renormalization?

(vi) Is it possible to couple tensorial matter also to the \(N_T = 2\) topological A–model whose underlying supersymmetries, \(Q\) and \(\bar{Q}\), have different ghost numbers and therefore different cohomologies?

(vii) Is it possible to construct a \(N = 2\) — or even a \(N = 4\) — supersymmetric extension of the (non–abelian) Avdeev–Chizhov model from which tensor matter interactions, like the ones introduced in this paper, could be obtained via a topological twist?

(viii) Recently, Berhadsky, Sadov and Vafa [14] have shown that all the topologically twisted \(N = 4\) gauge theories appear quite naturally as world–volume theories of Dirichlet \(p\)–brane instantons in string theory. With respect to this the perhaps most interesting question is whether
topological tensor matter interactions could also appear in some low–energy effective tensor gauge theories derived from string models.

A  Lozano’s formulation of the B–model

In this Appendix it will be shown that in Lozano’s action (see, Ref. [27], Eq. (7.17)),

$$S_L(\tau) = Q\bar{\Psi}_L + 2\pi i\tau k = \bar{Q}\Psi_L + 2\pi i\tau k \equiv S_L(\theta = 0) + i\theta k,$$  

(A.1)

develops the topological supercharges $Q$ and $\bar{Q}$ do not provide an off–shell formulation of the B–model. First of all, to be in line with the convention used here, let us perform the following redefinitions of Lozano’s fields (denoted by a subscript L):

$$[Q]_L = Q,$$  

$$[A_\mu]_L = -iA_\mu,$$  

$$[B]_L = -i\sqrt{2}\phi,$$  

$$[\bar{Q}]_L = \bar{Q},$$

$$[V_\mu]_L = \frac{i}{\sqrt{2}}V_\mu,$$  

$$[C]_L = -i\sqrt{2}\phi,$$

$$[\chi^\pm_{\mu\nu}]_L = \frac{i}{2}\chi^\pm_{\mu\nu},$$  

$$[\psi_\mu]_L = -i\psi_\mu,$$  

$$[\eta]_L = -2i\eta,$$

$$[P]_L = -4iY,$$  

$$[\bar{\psi}_\mu]_L = -i\bar{\psi}_\mu,$$  

$$[\bar{\eta}]_L = -2i\bar{\eta},$$

$$[N^\pm_{\mu\nu}]_L = iB^\pm_{\mu\nu},$$

where the overall factor of $i$ is due to the different choice of the group generators, $[T^i]_L = -iT^i$.

After carrying out these redefinitions in Eqs. (7.11) and (7.16) of Ref. [27] and coupling the resulting action $S_L(\theta = 0) + i\theta k$ to Euclidean gravity, which requires a non–minimal $R_{\mu\nu}$–dependent term $R^{\mu\nu}V_\mu V_\nu$, for the \(\theta\)–independent part one obtains

$$S_L(\theta = 0) = \frac{1}{e^2} \int d^4x \sqrt{-g} \text{tr} \left\{ \frac{1}{4}(F^{\mu\nu} - [V^\mu, V^\nu])(F_{\mu\nu} - [V_\mu, V_\nu]) + \frac{1}{4}D^{[\mu}V^{\nu]}D_{[\mu}V_{\nu]} + 2\chi^{\mu\nu}D_\mu \psi_\nu + 2\chi^{\mu\nu} [V_\mu, \bar{\psi}_\nu] - 2\chi^{\mu\nu} [V_\mu, \psi_\nu] - 2\psi^\mu D_\mu \psi - 2\bar{\psi}^\mu D_\mu \bar{\psi} - \frac{1}{2}B^\mu_{\nu\rho}B^{\nu\rho}_\mu + 2\chi^{\mu\nu}D_\mu \bar{\psi}_\nu - 2\chi^{\mu\nu} [V_\mu, \bar{\psi}_\nu] - 2\bar{\psi}^\mu D_\mu \bar{\psi} + 2\bar{\psi}^\mu [V_\mu, \bar{\eta}] + \frac{1}{2}D^\mu \bar{\phi} D_\mu \bar{\phi} + 2D^\mu \bar{\phi} D_\mu \phi - 2[V^\mu, \bar{\phi}]V_\mu, \phi) - 2[\bar{\phi}, \phi]^2 - 2V^\mu D_\mu Y - 2Y^2 \right\},$$

(A.2)

where the $R_{\mu\nu}$–dependence is cancelled by the $Y$–dependent terms.

Obviously, the action $S_L(\theta = 0) + i\theta k$ is invariant under a discrete $Z_2$ symmetry acting on both the fields and the coupling constants [27],

$$(A_\mu, \psi_\mu, \bar{\psi}_\mu, \phi, V_\mu, \bar{\phi}, \eta, \bar{\eta}, Y, \chi^{\mu\nu}, B^{\pm}_{\mu\nu}) \rightarrow (A_\mu, -\psi_\mu, -\bar{\psi}_\mu, \phi, -V_\mu, \bar{\phi}, -\eta, -\bar{\eta}, -Y, -\chi^{\mu\nu}, -B^{\pm}_{\mu\nu})$$

$$\epsilon_{\mu\nu\rho\sigma}(\theta) \rightarrow (\epsilon_{\mu\nu\rho\sigma}, \theta).$$

(A.3)

Now, decomposing $F_{\mu\nu}$ into its self–dual and anti–selfdual parts, then the action $S_L(\theta = 0)$ can be expressed either as a $Q$–exact or as a $\bar{Q}$–exact term, in both cases modulo a term depending only on the instanton number $k$,

$$S_L(\theta = 0) = Q\Psi_L - \frac{8\pi^2 k}{e^2} = \bar{Q}\Psi_L + \frac{8\pi^2 k}{e^2}.$$  

(A.4)
Here the gauge fermions are given by
\[
\Psi_L = \frac{1}{e^2} \int d^4x \sqrt{g} \left\{ \frac{1}{2} \chi^\mu_\nu (F^+_{\mu\nu} - [V_\mu, V_\nu]^+ - B^+_{\mu\nu}) + 2\bar{\psi}^\mu D_\mu \phi + 2\bar{\phi}^\mu [V_\mu, \phi] \right. \\
+ \left. \frac{1}{2} \chi^\mu_\nu ((D_\mu V_\nu) - B^-_{\mu\nu}) - 2\eta [\bar{\phi}, \phi] + V^\mu D_\mu \eta - 2\bar{\eta} Y \right\},
\]
\[
\bar{\Psi}_L = \frac{1}{e^2} \int d^4x \sqrt{g} \left\{ \frac{1}{2} \chi^\mu_\nu (F^-_{\mu\nu} - [V_\mu, V_\nu]^- + B^-_{\mu\nu}) + 2\bar{\psi}^\mu D_\mu \bar{\phi} - 2\bar{\phi}^\mu [V_\mu, \bar{\phi}] \\
- \frac{1}{2} \chi^\mu_\nu ((D_\mu V_\nu)^+ - B^+_{\mu\nu}) - 2\bar{\eta} [\bar{\phi}, \phi] - V^\mu D_\mu \bar{\eta} + 2\eta Y \right\}.
\]

Then, recasting within the full action $S_L(\theta = 0) + i\theta k$ the topological part in terms of the modular coupling $\tau = \theta/2\pi + 4\pi i/e^2$ one gets directly the action (A.1) with a topological term depending only upon $\tau$.

The $Q$– and $\bar{Q}$–transformations, being interchanged by the $Z_2$ symmetry (A.3), $Q \leftrightarrow -\bar{Q}$, are given by

\[
\begin{align*}
Q \phi &= 0, & \bar{Q} \bar{\phi} &= 0, \\
QA_\mu &= \psi_\mu, & QA_\mu &= \bar{\psi}_\mu, \\
Q \psi_\mu &= D_\mu \phi, & Q \bar{\psi}_\mu &= D_\mu \phi, \\
Q V_\mu &= \bar{\psi}_\mu, & Q V_\mu &= -\psi_\mu, \\
Q \bar{\psi}_\mu &= [V_\mu, \phi], & Q \psi_\mu &= -[V_\mu, \phi], \\
Q \bar{\phi} &= \eta, & Q \bar{\phi} &= \bar{\eta}, \\
Q \eta &= [\bar{\phi}, \phi], & Q \bar{\eta} &= [\phi, \bar{\phi}], \\
Q Y &= [\eta, \phi], & Q \bar{Y} &= -[\eta, \phi], \\
Q \chi^{\mu\nu}_+ &= F^{\mu\nu}_+ - [V_\mu, V_\nu]^+ + B^+_{\mu\nu}, & Q \bar{\chi}^{\mu\nu}_- &= F^{\mu\nu}_- - [V_\mu, V_\nu]^- - B^-_{\mu\nu}, \\
Q B^{\mu\nu}_+ &= \chi^{\mu\nu}_+ + (D_\mu \psi_\nu - [\psi_\mu, V_\nu]^+)^+, & Q \bar{B}^{\mu\nu}_- &= \bar{\chi}^{\mu\nu}_- + (D_\mu \bar{\psi}_\nu + [\bar{\psi}_\mu, V_\nu])^-^-; \\
Q \chi^{\mu\nu}_- &= (D_\mu V_\nu)^-_+ + B^-_{\mu\nu}, & Q \bar{\chi}^{\mu\nu}_+ &= -(D_\mu V_\nu)^+_+ - B^+_{\mu\nu}, \\
Q B^{\mu\nu}_- &= \chi^{\mu\nu}_- + (D_\mu \bar{\psi}_\nu - [\bar{\psi}_\mu, V_\nu])^-^-; & Q \bar{B}^{\mu\nu}_+ &= -\bar{\chi}^{\mu\nu}_+ + (D_\mu \psi_\nu + [\psi_\mu, V_\nu])^+_+.
\end{align*}
\]

Here, the operators $Q$ and $\bar{Q}$ are both equivariantly nilpotent and anticommute on–shell for $\chi^{\mu\nu}_\pm$ and $B^{\mu\nu}_\pm$ and off–shell for all the other fields, i.e., unlike to the claim of Ref. [27], they do not provide an off–shell realization of the topological superalgebra,

\[
Q^2 = \delta_G(\phi), \quad \{Q, \bar{Q}\} = 0, \quad \bar{Q}^2 = \delta_G(\phi).
\]

Here, some remarks are in order: First, in the case under consideration it is impossible to cast $S_L(\theta = 0)$ into the $Q$– and $\bar{Q}$–exact form $Q \bar{Q} \Omega_L$ for some gauge boson $\Omega_L$. Second, whenever $Q$ and $\bar{Q}$ are interchanged by a $Z_2$ symmetry the action $S_L(\theta = 0)$ can be cast into the form (A.4). In the present case this is only possible when $Q$ and $\bar{Q}$ anticommute on–shell. Third, it is impossible to replace in (A.4) $Q$ and $\bar{Q}$ by off–shell equivariantly nilpotent operators because such a replacement would come into conflict with the $Z_2$ symmetry (A.3). Indeed, if we replace in (A.5) the $\bar{Q}$–transformations for $\chi^{\mu\nu}_\pm$ and $B^{\mu\nu}_\pm$ according to

\[
\begin{align*}
\bar{Q} \chi^{\mu\nu}_+ &= F^{\mu\nu}_+ - [V_\mu, V_\nu]^+ - iB^-_{\mu\nu}, \\
\bar{Q} B^{\mu\nu}_- &= i[\chi^{\mu\nu}_-, \phi] - i(D_\mu \bar{\psi}_\nu + [\bar{\psi}_\mu, V_\nu])^-; \\
\bar{Q} \chi^{\mu\nu}_- &= -(D_\mu V_\nu)^+ + iB^+_{\mu\nu}, \\
\bar{Q} B^{\mu\nu}_+ &= i[\chi^{\mu\nu}_+, \phi] - i(D_\mu \psi_\nu - [\psi_\mu, V_\nu])^+;
\end{align*}
\]
leaving all the other transformations unaltered, it is easily seen that the modified \( Q \)– and \( \overline{Q} \)–transformations provide an off–shell realization of the superalgebra (A.6). However, the old \( Q \)– and the new \( \overline{Q} \)–transformations are no longer related to each other through a \( Z_2 \) symmetry and, therefore, the modified \( \overline{Q} \)–transformations are not any more a symmetry of \( S_L(\theta = 0) \! \) .

Finally, to close our analysis, let us state the relation between the action and the super-symmetry transformations given in Sect. 4 and those presented above. After integrating out in (38) and (A.2) the auxiliary field \( B_{\mu \nu} \) and redefining the Blau–Thompson’s fields (denoted by a subscript BT) according to:

\[
\begin{align*}
[Q]_{\text{BT}} &= \zeta (Q - \overline{Q}), & [A_{\mu}]_{\text{BT}} &= A_{\mu}, & [\phi]_{\text{BT}} &= -i\phi, \\
[\overline{Q}]_{\text{BT}} &= \zeta (Q + \overline{Q}), & [V_{\mu}]_{\text{BT}} &= V_{\mu}, & [\overline{\phi}]_{\text{BT}} &= 2\overline{\phi}, \\
[\chi_{\mu \nu}]_{\text{BT}} &= -2\zeta \chi_{\mu \nu}, & [\psi_{\mu}]_{\text{BT}} &= \zeta (\psi_{\mu} - \overline{\psi}_{\mu}), & [\eta]_{\text{BT}} &= 2\zeta (\eta - \overline{\eta}), \\
[Y]_{\text{BT}} &= -2iY, & [\overline{\psi}_{\mu}]_{\text{BT}} &= \zeta (\overline{\psi}_{\mu} + \psi_{\mu}), & [\overline{\eta}]_{\text{BT}} &= 2\zeta (\eta + \overline{\eta}),
\end{align*}
\]

with the abbreviation \( \zeta = (1 - i)/2 \), one finds that \( S_{\text{BT}}(\theta = 0) = -S_L(\theta = 0) \). Furthermore, one verifies that the transformations (35) match precisely those given in (A.5). Hence, the formulations of the B–model proposed by Blau and Thompson [23] and by Lozano [27] are equivalent on–shell, but they differ from each other after introducing of \( B_{\mu \nu} \). But, in neither cases the on–shell condition (A.6) can be completely lifted by introducing off–shell formulations for the topological supersymmetries \( Q \) and \( \overline{Q} \) using auxiliary fields.

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