The aim of this note is to prove

**Theorem 1.** Let $G/K$ be an irreducible symmetric space of compact type with $K$ connected and $G$ acting effectively on $G/K$, and let $\rho$ be a real exact $G$-invariant $(1,1)$-form on the complexification $G^C/K^C_1$. Then there exists a $G$-invariant Kähler metric on $G^C/K^C$ whose Ricci form is $\rho$.

**Remarks:**
1. The assumptions on $G$ and $K$ are fulfilled for simply connected symmetric spaces. In general, these assumptions guarantee that $G$ is connected ([6], Thm. V.4.1), that $G/K$ is globally symmetric ([6], Ex. VII.10), and that $K = \text{Ad}_G(K)$ (as $\text{Ad}_G(K) = K/K \cap Z(G)$).
2. The Kähler form obtained in Theorem 1 is exact.

The above result has been proved in [9] for symmetric spaces of rank 1 and in [2] for compact groups, i.e. for the case when $G = K \times K$ and $K$ acts diagonally. For hermitian symmetric spaces and $\rho = 0$, Theorem 1 has also been known [4].

The proof given here is quite different from that given for group manifolds in [2]. We show that the complex Monge-Ampère equation on $G^C/K^C$ reduces, for $G$-invariant functions, to a real Monge-Ampè re equation on the dual symmetric space $G^*/K$. We also show that the Monge-Ampère operator on non-compact symmetric spaces has a radial part, i.e. it is equal, for $K$-invariant functions, to another Monge-Ampère operator on the maximal abelian subspace of $G^*/K$. These facts, together with the theorem on $K$-invariant real Monge-Ampère equations proved in [3], yield Theorem 1.

1. **RIEMANNIAN SYMMETRIC SPACES OF NON-COMPACT TYPE**

Here we recall some facts about the geometry of Riemannian symmetric spaces. The standard reference for this section is [6].

Let $G/K$ be a symmetric space of compact type with $G$ acting effectively and $K$ connected, and let $G^*/K$ be its dual\(^2\). If $\mathfrak{g}$, $\mathfrak{g}^*$ and $\mathfrak{k}$ denote the Lie algebras of $G$, $G^*$ and $K$, then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{g}^* = \mathfrak{k} \oplus i\mathfrak{p}$, where $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{p}$. The restriction of the Killing form to $i\mathfrak{p}$ is positive definite and induces the Riemannian metric of $G^*/K$. Moreover, the Riemannian exponential mapping provides a diffeomorphism between $\mathfrak{p}$ and $G^*/K$. This can be viewed as the map:
\[
p \mapsto e^{i\rho} K, \tag{1.1}\]

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\(^1\)The complexification of a compact connected Lie group $G$ is the connected group $G^C$ whose Lie algebra is the complexification of the Lie algebra of $G$ and which satisfies $\pi_1(G^C) = \pi_1(G)$.

\(^2\)Since $K$ is connected and $K = \text{Ad}_G(K)$ (Remark 1), there exists a $G^*$ such that $(G^*, K)$ is a symmetric pair corresponding to $(\mathfrak{g}^*, \mathfrak{k})$. 

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where \( p \in \mathfrak{p} \) and \( e \) is the group-theoretic exponential map for \( G^* \). Thus we have two \( K \)-invariant Riemannian metrics on \( \mathfrak{p} \simeq \mathbb{R}^n \): the Euclidean one given by the Killing form, and the negatively curved one given by the diffeomorphism (1.1).

Let \( \mathfrak{a} \) be a maximal abelian subspace of \( \mathfrak{p} \) and \( l \) its centraliser in \( \mathfrak{k} \). Let \( \Sigma \) the set of restricted roots and \( \Sigma^+ \) the set of restricted positive roots. For each \( \alpha \in \Sigma \), let \( \mathfrak{p}_\alpha \) (resp. \( \mathfrak{e}_\alpha \)) denote the subspace of \( \mathfrak{p} \) (resp. of \( \mathfrak{k} \)) where each \((\text{ad } H)^2\), \( H \in \mathfrak{a} \), acts with eigenvalue \( \alpha(H)^2 \). We have the direct decompositions

\[
\mathfrak{p} = \mathfrak{a} + \sum_{\alpha \in \Sigma^+} \mathfrak{p}_\alpha, \quad \mathfrak{k} = l + \sum_{\alpha \in \Sigma^+} \mathfrak{e}_\alpha. \tag{1.2}
\]

Let \( \mathfrak{a}^+ \) be an open Weyl chamber and let \( \mathfrak{p}' \) be the union of \( K \)-orbits of points in \( \mathfrak{a}^+ \). Any \( K \) orbit in \( \mathfrak{p}' \) is isomorphic to \( K/L \) where the Lie algebra of \( L \) is \( l \). Moreover, we have the diffeomorphism:

\[
\mathfrak{a}^+ \times K/L \to \mathfrak{p}', \quad (h, k) \mapsto \text{Ad}(k)h. \tag{1.3}
\]

We now wish to write the two \( K \)-invariant metrics on \( \mathfrak{p} \) in coordinates given by this diffeomorphism. Let \( \sum dr_i^2 \) be the Killing metric on \( \mathfrak{a}^+ \) (the \( r_i \) can be viewed as \( K \)-invariant functions on \( \mathfrak{p}' \)). For each \( \mathfrak{e}_\alpha \), choose a basis \( X_{\alpha,m} \) (\( m \) runs from 1 to twice the multiplicity of \( \alpha \)) of vectors orthonormal for the Killing form and denote by \( \theta_{\alpha,m} \) the corresponding basis of invariant 1-forms on \( K/L \). We have

**Proposition 1.1.** Let \( g_0 \) be the Euclidean metric on \( \mathfrak{p} \), given by the restriction of the Killing form, and let \( g \) be the negatively curved symmetric metric on \( \mathfrak{p} \) given by the diffeomorphism (1.1). Then, under the diffeomorphism (1.3) the metrics \( g_0 \) and \( g \) can be written in the form

\[
\sum_i dr_i^2 + \sum_{(\alpha,m)} F(\alpha(r))\theta_{\alpha,(m)}^2, \tag{1.4}
\]

where \( F(z) = z^2 \) for \( g_0 \), and \( F(z) = \sinh^2(z) \) for \( g \).

**Proof.** Since all these metrics are \( K \)-invariant, it is enough to compute them at points of \( \mathfrak{a}^+ \). Let \( H \) be such a point and let \((h, \rho), h \in \mathfrak{a}, \rho \in \mathfrak{t}, K/L \) at \((H, [1])\). The vector \( \rho \) can be identified with an element of \( \sum \mathfrak{e}_\alpha \subset \mathfrak{t} \). The corresponding (under (1.3)) tangent vector at \( H \in \mathfrak{p}' \) is \( h + [\rho, H] \). Computing the Killing form of this vector yields the formula (1.4) with \( F(z) = z^2 \) for \( g_0 \). The formula for \( g \) follows from a similar computation, using the expression for the differential of the map (1.1) given in [6], Theorem IV.4.1. \( \square \)

2. **Monge-Ampère equation on symmetric spaces**

Let \((M, g)\) be a Riemannian manifold and \( u : M \to \mathbb{R} \) a smooth function. Then the Hessian of \( u \) is the symmetric \((0,2)\)-tensor \( D^2u \) where \( D \) is the Levi-Civita connection of \( g \). In local coordinates \( x_i \), \( D^2u \) is represented by the matrix

\[
H_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_k \Gamma^k_{ij} \frac{\partial u}{\partial x_k}. \tag{2.1}
\]

We say that the function \( u \) is \( g \)-convex (resp. strictly \( g \)-convex), if \( D^2u \) is non-negative (resp. positive) definite. The Monge-Ampère equation on the manifold \((M, g)\) is then

\[
\mathcal{M}_g(u) := (\det g)^{-1} \det D^2u = f \tag{2.2}
\]
where \( f \) is a given function.

Let \((G^*/K, g)\) be a symmetric space of non-compact type given by a Cartan decomposition \( g^* = \mathfrak{k} + \mathfrak{i} \mathfrak{p} \). As in the previous section, we identify \( M = G^*/K \) with \( \mathfrak{p} \) and denote by \( g_0 \) the (flat) metric given by restricting the Killing form to \( \mathfrak{p} \). We have:

**Theorem 2.1.** Let \( M \simeq \mathfrak{p} \) be a symmetric space of noncompact type and let \( u \) be a \( K \)-invariant (smooth) function on \( M \). Then

1. \( u \) is \( g \)-convex if and only if \( u \) is \( g_0 \)-convex (i.e. convex in the usual sense on \( \mathfrak{p} \)).
2. The following equality of Monge-Ampère operators holds:

\[
M_g(u) = F \cdot M_{g_0}(u),
\]

where \( F : M \to \mathbb{R} \) is a positive \( K \)-invariant smooth function depending only on \( M \).

We have proved in [3] a theorem on the existence and regularity of \( K \)-invariant solutions to Monge-Ampère equations on \( \mathbb{R}^n \). From this we immediately obtain

**Corollary 2.2.** Let \((G^*/K, g)\) be an irreducible symmetric space of noncompact type and let \( f \) be a positive smooth \( K \)-invariant function on \( G^*/K \). Then the Monge-Ampère equation (2.2) has a global smooth \( K \)-invariant strictly \( g \)-convex solution. \(\square\)

We shall now prove Theorem 2.1. In fact we shall prove it in the following, more general situation. Suppose that we are given a \( K \)-invariant metric on \( \mathfrak{p} \) whose pullback under (1.3) can be written as (cf. (1.4)):

\[
\sum_i dr_i^2 + \sum_{(\alpha, m)} F_{(\alpha, m)}(\alpha(r)) \theta^2_{(\alpha, m)},
\]

where \( F_{(\alpha, m)} : \mathbb{R} \to \mathbb{R} \) are smooth functions vanishing at the origin such that \( z^{-1} \frac{dF_{(\alpha, m)}}{dz} \) is smooth and positive everywhere. Proposition 1.1 implies that the symmetric metric on \( G^*/K \) is of this form. We claim that Theorem 2.1 holds for any metric \( g \) of the form (2.3).

In order to simplify the notation, let us write \( j \) for the index \( (\alpha, m) \) and \( \alpha_j \) for \( \alpha \) if \( j = (\alpha, m) \). The metric \( g \) can be now written as

\[
\sum_i dr_i^2 + \sum_j F_j(\alpha_j(r)) \theta^2_j.
\]

We recall the following formula:

\[
2Ddu = L_{\nabla u} g,
\]

where \( L \) is the Lie derivative and \( \nabla u \) is the gradient of \( u \) with respect to the metric \( g \). On the other hand, for any \((0, 2)\)-tensor \( g \) and vector fields \( X, Y, Z \), we have:

\[
(L_X g)(Y, Z) = X.g(Y, Z) - g([X, Y], Z) - g(Y, [X, Z]).
\]

We now compute \( L_{\nabla u} g \) on \( \mathfrak{p}' \) with respect to the basis vector fields \( \partial / \partial r_i \), \( X_j \), where \( X_j \) are dual to \( \theta_j \). Here \( u \) is a \( K \)-invariant function. The gradient of \( u \) is just \( \sum \frac{\partial u}{\partial r_i} \frac{\partial}{\partial r_i} \), in particular it is independent of the functions \( F_j \). It follows
immediately that \((L \nabla ug)(\partial/\partial r_i, X_j) = 0\) and that the matrix \((L \nabla ug)(X_j, X_k)\) is equal to \(\nabla u.g(X_j, X_k)\) and hence it is diagonal with the \((jj)-\text{entry equal to}\)
\[
\nabla u (F_j(\alpha_j(r))) = \frac{dF_j}{dz}
\]

\(\mid z=\alpha_j(r)\).

Here \(\nabla u = \sum \frac{\partial u}{\partial r_i} \frac{\partial}{\partial r_i}\) is the gradient of \(u\) restricted to the Euclidean space \(a = \mathbb{R}^n\) in coordinates \(r_i\), and viewed as a map from \(\mathbb{R}^n\) to itself.

Theorem 2.1 with the more general metric (2.3) follows easily with the function \(F\) given explicitly by
\[
F = \prod \frac{\alpha_j(r)}{F_j(\alpha_j(r))} \left( \prod \left( \frac{1}{2} \frac{dF_j}{dz} \right)_\mid z=\alpha_j(r) \right).
\]

Observe that the assumptions on the \(F_j\) guarantee that \(F\) extends to a smooth positive function on \(p\).

3. Proof of the Main Theorem

Let \(G/K\) be a locally symmetric space of compact type with \(K\) connected. There is a canonical isomorphism between \(G^C/K^C\) and \(G \times_K p\) (i.e. the tangent bundle of \(G/K\)) given by the map:
\[
G \times p \rightarrow G^C \rightarrow G^C/K^C, \quad (g, p) \mapsto ge^{ip}.
\]

This isomorphism can be viewed in many ways: as an example of Mostow fibration [8], as given via Kähler reduction of \(G^C \simeq G \times g\) by the group \(K\) [5], or as given by the adapted complex structure construction [7] which provides a canonical diffeomorphism between the tangent bundle of \(G/K\) and a complexification of \(G/K\). In any case it provides a fibration
\[
\pi : G^C/K^C \rightarrow G/K.
\]

The fibers of this projection can be identified with \(p\) via the map (3.1). In particular, the fiber over [1] is given by the \(K^C\)-orbits of elements \(e^{ip}, p \in p\). We shall relate \(G\)-invariant plurisubharmonic functions on \(G^C/K^C\) to convex functions on this fiber (see [1] for a different approach to this).

For a function \(w\) on a complex manifold one defines its Levy form \(Lw\) to be the Hermitian \((0, 2)\) tensor given in local coordinates as
\[
\frac{\partial^2 w}{\partial z_k \partial \bar{z}_l} dz_k \otimes d\bar{z}_l.
\]

This form does not depend on the choice of local coordinates. We shall compute this form for a \(G\)-invariant function \(w\) on \(G^C/K^C\). It is enough to compute it at points \(e^{ip}, p \in p\). First of all, we choose local holomorphic coordinates at such a point:

**Lemma 3.1.** In a neighbourhood of a point \(e^{ip}, p \in p\), complex coordinates are provided by the map \(p^C \rightarrow G^C \rightarrow G^C/K^C, \quad (a + ib) \mapsto e^{a+ib}e^{ip}\).

**Proof.** We have to show that \(e^{a+ib}\) does not belong to the isotropy group \(e^{ip}K^Ce^{-ip}\) or, equivalently, that for \(u \in p^C\), \((\text{ad} e^{-ip}) u \notin p^C\). We have
\[
(\text{ad} e^{-ip}) u = e^{\text{ad}(-ip)} u = \cosh(\text{ad}(-ip)) u + \sinh(\text{ad}(-ip)) u,
\]
where the first term of the sum lies in \(p^C\) and the second one in \(t^C\). To show that the first term does not vanish recall that \((\text{ad}(-ip))^2\) has all eigenvalues nonnegative. □
We now have:

**Lemma 3.2.** In the complex coordinates \( z = a + ib \) given by the previous lemma, the Levy form (3.3) of a \( G \)-invariant function \( w \) satisfies the equation:

\[
\left( \frac{\partial^2 w}{\partial z_k \partial \bar{z}_l} \right)_{a=0 \atop b=0} = \frac{1}{4} \frac{\partial^2}{\partial b_k \partial b_l} w(e^{ib} e^{ip})_{b=0}, \tag{3.5}
\]

**Proof.** The polar decomposition of \( G^C \) implies that \( e^{a+ib} \) can be uniquely written as \( g e^{iy} \), where \( g \in G \) and \( y \in \mathfrak{g} \). Any \( G \)-invariant function on \( G^C/K^C \) in a neighbourhood of \( e^{ip} \) is a function of \( y \) only. On the other hand, as \( e^{2iy} = (e^x e^{iy})^* (e^x e^{iy}) = e^{-a+ib} e^{a+ib} \), it follows from the Campbell-Hausdorff formula that \( y = b + [b, a]/2 + \text{higher order terms} \). Hence the matrix of second derivatives in (3.3) at \( e^{ip} \) (i.e. at \( a = 0, b = 0 \)) is the same as the matrix of second derivatives of

\[
(a, b) \mapsto e^{i(b+i[b, a])} e^{ip} \tag{3.6}
\]

at \( a = 0, b = 0 \). We shall now show that for a \( G \)-invariant function \( w \) on \( G^C/K^C \), this matrix of second derivatives is equal to the right-hand side of (3.5).

The Campbell-Hausdorff formula implies that up to order 2 in \( a, b \), we have \( e^{i(b+i[b, a])} = e^{ib} e^{i[b, a]} \). Set \( c = [b, a]/2 \), which is a point in \( \mathfrak{k} \). We are going to show that modulo terms of order 2 in \( c \) (hence of order 4 in \( a, b \)), \( e^{ic} e^{ip} \) is equal to \( e^{p} e^{ib} e^{iq} \), where \( p \in \mathfrak{g} \) and \( q \in \mathfrak{k} \) are both linearly dependent on \( c \). We note that this proves the lemma, as

\[
e^{ib} e^{p} e^{ib} e^{iq} = e^{p} e^{ib+O(3)} e^{ip} e^{iq} = e^{p} e^{ib+O(3)} e^{ip}
\]

in \( G^C/K^C \), where \( O(3) \) denotes terms of order 3 and higher in \( a, b \).

We find \( q \) from the equation \( \cosh \text{ad}(ip)(q) = c \), which can be solved uniquely as \( \cosh \text{ad}(ip) \) is symmetric and positive-definite on \( \mathfrak{k} \subset \mathfrak{g} \). We then put \( p = -i \sinh \text{ad}(ip)(q) \). We observe that \( p \in \mathfrak{g} \) and \( e^{p} e^{ic} = e^{ip} e^{iq} e^{ip} \), thanks to (3.4). Moreover, modulo terms quadratic in \( c \), \( e^{p} e^{ip} \) and, consequently:

\[
e^{p} e^{ip} e^{iq} = e^{ic} e^{p} e^{ic} e^{ip} e^{iq} = e^{ic} (e^{ip} e^{iq} e^{ip}) e^{ip} e^{iq} = e^{ic} e^{ip},
\]

again modulo terms quadratic in \( c \). This finishes the proof of the lemma. \( \square \)

According to this lemma, we have to compute \( \frac{\partial^2}{\partial b_k \partial b_l} w(e^{ib} e^{ip})_{b=0} \). Now, since \( e^{ib} e^{ip} \in G^* \), \( e^{ib} e^{ip} = k e^{iz} \), where \( z = z(b) \in \mathfrak{p} \) and \( k \in K \). As \( w \) is \( G \)-invariant, \( w(e^{ib} e^{ip}) = w(e^{iz}) \) and therefore

\[
\frac{\partial^2}{\partial b_k \partial b_l} w(e^{ib} e^{ip})_{b=0} = \frac{\partial^2}{\partial b_k \partial b_l} w(e^{iz})_{b=0}.
\]

Thus we compute the matrix of second derivatives of a function defined on \( \exp(ip) \) in the coordinates given by \( b \mapsto e^{ib} e^{ip} \mapsto e^{iz} \). These, however, are the geodesic coordinates at the point \( e^{ip} \) in the symmetric space \( K \backslash G^* \) (being translations of geodesics at \( [1] \)), and hence the matrix of second derivatives in these coordinates is equal to the Riemannian Hessian (2.1) for the symmetric metric on \( K \backslash G^* \).

We obtain

**Theorem 3.3.** Let \( w \) be a smooth \( G \)-invariant function on \( X = G^C/K^C \) and let \( \tilde{w} \) be its restriction to the fiber \( S = \exp(ip) \) of (3.2) over [1]. Let \( g \) denote the
symmetric metric on $S \simeq K / G^\ast$. Then $w$ is (strictly) plurisubharmonic if and only if $\bar{w}$ is (strictly) $g$-convex. Moreover, the following equality holds:
\[
\bar{\partial} \bar{\partial} \log \det Lw = \bar{\partial} \bar{\partial} \log M_g(\bar{w}),
\]
where $\tilde{u} : X \to \mathbb{R}$ is a $G$-invariant function such that $\tilde{u}$ is a given $K$-invariant function $u$ on $S$.

We are now ready to prove Theorem 1. Recall that $X = G^C / K^C$ is a Stein manifold and so if $\rho$ is an exact $(1, 1)$ form on $X$, then $\rho = -i \partial \bar{\partial} h$ for some function $h$. If $\rho$ is $G$-invariant, then we can assume that $h$ is $G$-invariant. We can restrict $h$ to the fiber $S$ defined in the last theorem and thanks to Corollary 2.2 we can find a strictly $g$-convex $K$-invariant smooth solution $\tilde{u}$ to the equation (2.2) with $f = e^h$, where the metric $g$ is the symmetric metric on $S \simeq K / G^\ast$. We can extend this solution via $G$-action to a $G$-invariant function $u$ on $X$. Theorem 3.3 implies now that $u$ is strictly plurisubharmonic and that the Ricci form of the Kähler metric with potential $u$ is $\rho$.

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