The Dirichlet problem for the minimal hypersurface equation with Lipschitz continuous boundary data on domains of a Riemannian manifold

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Abstract

Given a $C^2$-domain $\Omega \subset M$ with compact boundary, where $M$ is an arbitrary complete Riemannian manifold, we search for smallness conditions on the boundary data for which the Dirichlet problem for the minimal hypersurface equation is solvable. We obtain an extension to Riemannian manifolds of an existence result of G. H. Williams (J. Reine Angew. Math. 354:123-140, 1984).

1 Introduction

Let $M^n$, $n \geq 2$, be a complete Riemannian manifold and let $\Omega \subset M$ be a $C^2$-domain with compact boundary. We consider the Dirichlet problem

$$\begin{cases}
  \mathcal{M}(u) := \text{div} \left( \frac{\text{grad} u}{\sqrt{1+|\text{grad} u|^2}} \right) = 0 \text{ in } \Omega, \\
  u \in C^2(\Omega) \cap C^0(\overline{\Omega}), \\
  u|_{\partial\Omega} = f
\end{cases} \tag{1}$$

where $f \in C^0(\partial\Omega)$ is given a priori, grad and div are the gradient and divergence in $M$. If $u$ is a solution of (1) then the graph of $u$ is a minimal hypersurface of $M \times \mathbb{R}$.

When $M = \mathbb{R}^n$ and $\Omega$ is bounded, it is well known that problem (1) is solvable for any $f \in C^0(\partial\Omega)$ if and only if $\Omega$ is mean convex (Theorem 1 of [6]). The existence part of this result has been extended and generalized to the Riemannian setting (see for instance [3],[4],[9],[11]). If the boundary
data \( f \) is restricted in some way, problem (\( \Pi \)) may be solvable even if \( \Omega \) is non mean convex. It was shown in Theorem 1 of [1] - which is an extension to Riemannian manifold of the classical result of H. Jenkins and J. Serrin (Theorem 2 of [6]) - that if \( f \in C^2(\partial \Omega) \) and

\[
osc(f) = \sup_{\partial \Omega} f - \inf_{\partial \Omega} f \leq \mathcal{C}(|Df|, |D^2f|, |A|, \text{Ric}_M),
\]

where \( |A| \) denotes the norm of the second fundamental for of \( \partial \Omega \) and \( \mathcal{C} \) is a function which has an explicit form (Section 2, p. 78 of [1]), then problem (\( \Pi \)) is solvable. However, at least in Euclidean spaces, such smallness condition on the boundary data is not the least. In fact, it seems that the least restrictive condition was given by G. H. Williams in Theorem 1 of [13]. He shows that for a non mean convex bounded \( C^2 \)-domain \( \Omega \subset \mathbb{R}^n \) and \( f \in C^{0,1}(\partial \Omega) \) with Lipschitz constant

\[
\text{Lip}(f) = K \in [0, \frac{1}{\sqrt{n-1}}),
\]

the problem (\( \Pi \)) is solvable if \( osc(f) \) is sufficiently small, \( osc(f) < \varepsilon(n,K,\Omega) \) (Corollary 1 of [13]). He also shows that if \( K > \frac{1}{\sqrt{n-1}} \) then there is a positive boundary data \( f \) with \( \text{Lip}(f) = K \) such that (\( \Pi \)) has no classical solution (Theorem 4 of [13]).

Williams’ results (existence and non existence) were extended to unbounded domain \( \Omega \subset \mathbb{R}^2 \) by N. Kutev and F. Tomi in [8] and J. Ripoll and F. Tomi gave, in the specific case \( \Omega \subset \mathbb{R}^2 \), Williams’ condition in a more explicit form (see Theorem 1 of [11]). Schulz and Williams [12] and Bergner [2] generalized Williams’ result to prescribed mean curvature (in Euclidean spaces). Here, our main objective is to obtain an extension of Williams’ existence theorem [13] to Riemannian manifolds. In order to state our main result, we establish some notation.

Let \( \nu \) be the unit normal vector field to \( \partial \Omega \) which point to \( \Omega \). Let \( \mathcal{H} \) be the mean curvature of \( \partial \Omega \) with respect to \( \nu \) and set

\[
\partial^- \Omega = \text{clos} \{ x \in \partial \Omega; \mathcal{H}(x) < 0 \}
\]

and

\[
\mathcal{H}_{\text{inf}} = \inf \{ \mathcal{H}(x); x \in \partial^- \Omega \}.
\]

Given \( x \in \partial^- \Omega \), let \( R(x) \) the maximal radius of the normal sphere contained in \( M - \Omega \) which is tangent to \( \partial \Omega \) at \( x \) and set

\[
R = \inf \{ R(x); x \in \partial^- \Omega \}.
\]
Since $\partial^{-}\Omega$ is compact, $R > 0$.

Let $r \in (0, R)$. Given $x \in \partial^{-}\Omega$ set

$$x^* = \exp_{x} r (-\nu(x)),$$

(6)

$B_r(x^*)$ the normal ball with center at $x^*$ and radius $r$ and consider the normal sphere $\Sigma = \partial B_r(x^*)$. Let $\eta$ be the unit normal vector field to $\Sigma$ which points to $M - B_r(x^*)$. Let $\lambda_i, i = 1, ..., n-1$, be the principal curvatures of $\Sigma$ with respect to $\eta$ and set

$$\lambda(x) = \min \{ \lambda_i(p), \ i = 1, ..., n-1, p \in \Sigma \}.$$  

(7)

Let $II_{\partial\Omega}$ be the second fundamental form of $\partial\Omega$ relatively to $\nu$ and let $II_{\Sigma}$ be the second fundamental form of $\Sigma$ relatively to $\eta$. Notice that, at $x$, $\nu(x) = \eta(x)$, since $T_x\partial\Omega = T_x\Sigma$. Set

$$\kappa(x) = \min \{ II_{\partial\Omega}(v) - II_{\Sigma}(v) ; \ v \in T_x\partial\Omega, \ |v| = 1 \}.$$  

(8)

Now, consider the real numbers

$$\lambda_r := \inf \{ \lambda(x) ; x \in \partial^{-}\Omega \}$$

(9)

and

$$\kappa_r := \inf \{ \kappa(x) ; x \in \partial^{-}\Omega \}.$$  

(10)

Notice that $\lambda_r < 0$ and, since $r < R$, we have $\kappa_r > 0$.

Finally, consider $\varrho > 0$ as the biggest number such that $\exp_{\partial\Omega} : \partial\Omega \times [0, \varrho) \longrightarrow \overline{\Omega}$ is a diffeomorphism and set

$$\Omega_{\varrho} = \exp_{\partial\Omega} (\partial\Omega \times [0, \varrho))$$

(11)

We obtain the following extension of Williams’ existence result.
Theorem 1 Let $M^n$, $n \geq 2$, be a complete Riemannian manifold, $\Omega \subset M$ be a $C^2$-domain with compact boundary and assume that $\text{Ric}_{\Omega_e} \leq 0$.

i) If $\partial^e \Omega = \emptyset$, then the Dirichlet problem (1) has a bounded solution for any $f \in C^0(\partial \Omega)$.

ii) If $\partial^e \Omega \neq \emptyset$, assume additionally that $\text{Ric}_{\Omega_e} \geq -(n-1)S_{\text{int}}^2$. Then, given $r \in (0, R)$, $a \in (0, \sqrt{\frac{r}{(n-1)|\lambda_r|}})$ and $K \in [0, a\sqrt{\kappa_r/r})$, where $R$, $\lambda_r$ and $\kappa_r$ are given by (3), (9) and (10) respectively, there is $0 < \delta_0$ such that, for all $\delta \in (0, \delta_0)$ there is $\epsilon = \epsilon (r, a, K, \Omega, \text{Ric}_{\Omega_e}) > 0$ such that, if $f \in C^0(\partial \Omega)$ satisfies

$$|f(z) - f(x)| \leq Kd(z, x),\ x \in \partial^e \Omega,\ z \in B_\delta(x) \cap \partial \Omega$$

and $\text{osc}(f) < \epsilon$, then the Dirichlet problem (1) has a bounded solution.

Moreover, if $\Omega$ is bounded, the solutions mentioned in the items i) and ii) are unique.

Remark 2 For $r \in (0, R)$ and $a \in (0, \sqrt{\frac{r}{(n-1)|\lambda_r|}})$ we have $a\sqrt{\kappa_r/r} < \frac{1}{\sqrt{n-1}}$ (see Lemma 2). When $M = \mathbb{R}^n$ we have $|\lambda_r| = 1/r$ and, therefore,

$$\sqrt{\frac{r}{(n-1)|\lambda_r|}} = \frac{r}{\sqrt{n-1}} \ \text{and} \ \frac{1}{r}\sqrt{\frac{R-r}{R}} \leq \frac{\kappa_r}{r}.$$ 

It follows that, given $K \in \left(0, \frac{1}{\sqrt{n-1}}\right)$ there are $r \in (0, R)$ and $a < \frac{r}{\sqrt{n-1}}$ such that $K < a\sqrt{\kappa_r/r}$. Therefore, Theorem 1 extends to Riemannian manifolds the existence results of G. H. Williams given in the Corollary 1 of [13].

Remark 3 If the domain is strictly mean convex, the hypothesis on the Ricci curvature is not necessary.

2 Barriers

Consider the set

$$S = \left\{ v \in C^0(\Omega) ; v \text{ is subsolution of } \mathfrak{M}, v(z) \leq f(z), \forall z \in \partial \Omega, \sup_\Omega v \leq \sup_{\partial \Omega} f \right\}. \tag{12}$$

Note that $S \neq \emptyset$ ($v = \inf_{\partial \Omega} f \in S$) and that any function of $S$ is bounded from above by $w = \sup_{\partial \Omega} f$. The function

$$u(z) = \sup \{v(z) ; v \in S\},\ z \in \Omega \tag{13}$$

4
is then well defined. From Perron’s method it follows that $u \in C^2(\Omega)$ and $\mathcal{M}(u) = 0$ (see [5] and Section 2 of [10]). We will prove that $u \in C^0(\overline{\Omega})$ and $u|_{\partial \Omega} = f$. Our main work is to construct barriers relatively to the points $x \in \partial^{-}\Omega$.

Given $r \in (0, R)$ and $x \in \partial^{-}\Omega$, let $x^*$ and $\Sigma$ be as defined in (6) and set

$$d(z) = d(z, x^*), \; z \in M,$$

where $d$ is the Riemannian distance in $M$. Denote by $\rho(x)$ the largest positive number such that

$$\exp_x^* : B_{r+\rho(x)}(0) \subset T_x M \longrightarrow \exp_x^* \left( B_{r+\rho(x)}(0) \right)$$

is a diffeomorphism (for Hadamard manifold $\rho(x) = \infty$) and set

$$A_r^{+, \rho(x)} : = \exp_x^* \left( B_{r+\rho(x)}(0) - B_r(0) \right)$$

$$= \{ z \in M - B_r(x^*) ; \; r \leq d(z) < r + \rho(x) \}.$$ (14)

Now, consider the number

$$\rho := \min \{ \rho, \inf \{ \rho(x) ; x \in \partial^{-}\Omega \} \},$$ (15)

where $\rho$ is given in (11).

In all results of this section, we are considering the following context:

Let $r \in (0, R)$ and let $x \in \partial^{-}\Omega$ be an arbitrary but fixed point. Let $x^*$, $\Sigma$ be as defined in (6) and let $A_r^{+, \rho}$ be defined by (14) and (15). At $z \in A_r^{+, \rho}$, consider an orthonormal referential frame $\{ E_i \}, \; i = 1, \ldots, n$, where $E_n = \nabla d$.

**Lemma 4** Assume $\text{Ric}_{\Omega_\rho} > -(n - 1) S_{\text{inf}}^2$, where $\Omega_\rho$ and $S_{\text{inf}}$ are given by (11) and (4), respectively. Given $\psi \in C^2([r, \infty))$, consider $w \in C^2(A_r^{+, \rho})$ given by

$$w(z) = (\psi \circ d)(z).$$ (16)

We have $\mathcal{M}(w) \leq 0$ in $A_r^{+, \rho} \cap \Omega$ if

$$\psi'' + \left( \psi' + [\psi']^3 \right) (n - 1) |\lambda_r| \leq 0,$$ (17)

where $\lambda_r$ is given by (9).
Proof. Straightforward calculus give us that \( M(w) \leq 0 \) in \( A_r^{+\rho} \cap \Omega \) if

\[
\psi'' + \left( \psi' + [\psi']^3 \right) \Delta d \leq 0,
\]

where \( \Delta \) is the Laplacian in \( M \). Since \( \text{Ric}_{\Omega} > -(n - 1) \mathfrak{f}_{\text{inf}}^2 \), there is

\[
0 < k < |\mathfrak{f}_{\text{inf}}| \leq |\lambda_r| 
\]

such that

\[
\text{Ric}_{\Omega} \geq -(n - 1) k^2. \tag{18}
\]

Define \( f : [0, \rho] \to (0, +\infty) \) by

\[
f(t) = k \sinh \left( \coth^{-1} \left( \frac{|\lambda_r|}{k} \right) + kt \right),
\]

We have

\[
\frac{f''(t)}{f(t)} = k^2, \quad t \in [0, \rho]. \tag{19}
\]

Let \( H_t \) be the mean curvature of \( P_t := \{ z \in A_r^{+\rho}; d(z) = r + t \} \subset A_r^{+\rho} \) with respect to the normal given by \( E_n = \nabla d \). As \( P_0 = \Sigma \), it follows that

\[
H_0 \geq \lambda_r \geq -\frac{f'(0)}{f(0)}. \tag{20}
\]

Let \( \gamma : [0, \rho] \to A_r^{+\rho} \) be the arc length geodesic such that \( \gamma(0) \in P_0 \) and \( \gamma'(t) = \nabla d(\gamma(t)) \). We have from (18) and (19) that

\[
\text{Ric}_M (\gamma'(t), \gamma'(t)) \geq -(n - 1) \frac{f''(t)}{f(t)} \tag{21}
\]

for all \( t \in [0, \rho] \). Since \( \nabla d \) is an extension of \( \eta \) to \( A_r^{+\rho} \) and, in presence of \( (20), (21) \), it follows from Theorem 5.1 of \( [7] \) that

\[
-H_t(\gamma(t)) = \frac{\Delta d(\gamma(t))}{(n - 1)} \leq \frac{f'(t)}{f(t)}.
\]

Then

\[
\Delta d \leq (n - 1) \frac{f'(t)}{f(t)} = (n - 1) k \coth \left( \coth^{-1} \left( \frac{|\lambda_r|}{k} \right) + kt \right) \leq (n - 1) |\lambda_r|.
\]

\[\blacksquare\]
Lemma 5  Given

\[ 0 < a < \sqrt{\frac{r}{(n-1)|\lambda_r|}} \]  \hspace{1cm} (22)

set

\[ s_0 = \frac{1 + \sqrt{1 - 4(n-1)^2\lambda_r^2(a^2 - r^2)}}{2(n-1)|\lambda_r|}. \]  \hspace{1cm} (23)

Then, the function \( w(z) = a \cosh^{-1} \left( \frac{d(z)}{r} \right) \) satisfies \( \mathfrak{M}(w) \leq 0 \) in \( A_r^{\mu} \cap \Omega \), where

\[ \mu = \min \{ \rho, s_0 - r \}. \]  \hspace{1cm} (24)

Proof. Let \( \psi(s) = \alpha \cosh^{-1} \left( \frac{s}{r} \right) \), \( s := d(z) > r \), where \( \alpha > 0 \) is to be determined. We have

\[ \psi'(s) = \frac{\alpha}{(s^2 - r^2)^{1/2}}, \quad \psi''(s) = \frac{-\alpha s}{(s^2 - r^2)^{3/2}} \]

and then, from (17) of Lemma 4, since \( w = \psi \circ d \), \( \mathfrak{M}(w) \leq 0 \) if

\[ \frac{-\alpha s}{(s^2 - r^2)^{3/2}} + \left[ \frac{\alpha}{(s^2 - r^2)^{1/2}} + \frac{\alpha^3}{(s^2 - r^2)^{3/2}} \right] (n-1)|\lambda_r| \leq 0, \]

that is, if

\[ -s + \left[ s^2 - r^2 + \alpha^2 \right] (n-1)|\lambda_r| \leq 0, \]

that is, if

\[ (n-1)|\lambda_r| s^2 - s + (n-1)|\lambda_r| (\alpha^2 - r^2) \leq 0. \]  \hspace{1cm} (25)

In order to get the desirable neighborhood, we need that for \( s \) near \( r \), \( s > r \), the inequality (25) to be strict and that is the case if

\[ \alpha < \sqrt{\frac{r}{(n-1)|\lambda_r|}}. \]

(notice that \( \sqrt{\frac{r}{(n-1)|\lambda_r|}} < \sqrt{\frac{1 + 4(n-1)^2\lambda_r^2r^2}{2(n-1)|\lambda_r|}} \)). Then, taking \( \alpha = a \) satisfying (22), it follows that for \( s \in [r, s_0] \), where \( s_0 > r \) is given by (23), the inequality (25) is true, and this concludes the proof. \( \blacksquare \)
Lemma 6  Let $a \in (0, \sqrt{r \over (n-1) \lambda(x)}$ and $0 < \varepsilon < r \xi_x$ be given, where $\xi_x$ is
given by (10). Then, there exist $\delta_1 > 0$ such that $w(z) = \cosh^{-1}\left(\frac{d(z)}{r}\right)$ as
defined in Lemma 5 satisfies
\[ w(z) \geq ar^{-1}(r \xi_x - \varepsilon)^{1/2}d(z, x) + o(d(z, x)), \quad z \in B_{\delta_1}(x) \cap \partial \Omega. \quad (26) \]

Proof. Let $g : M \rightarrow \mathbb{R}$ given by $g(z) = d(z)^2 - r^2$. Then $\Sigma = g^{-1}(0)$. Given $v \in T_x \partial \Omega = T_x \Sigma, \quad |v| = 1$, let $Y$ be an extension of $v$ which is tangent
to $\Sigma$, that is, $Y \in \mathfrak{X}(\Sigma)$ and set $X = \nabla g |\nabla g|^{-1}$. Note that $X$ is an extension
to $M$ of the unit normal vector field $\eta$, that is, $X|_{\Sigma} = \eta$. As $Y(g|_{\Sigma}) = 0$ and
$\nabla g|_{\Sigma} = 2r \eta$, setting $\nabla$ the Riemannian connection of $M$, it follows that on $\Sigma$,
\[
\text{Hess}_g(Y, Y) = - \langle \nabla Y Y, \nabla g \rangle = - \langle \nabla Y Y, \eta \rangle |\nabla g| \]
\[
= -2r \langle \nabla Y Y, \eta \rangle = -2r \left\langle [\nabla Y Y]^T + B(Y, Y), \eta \right\rangle \]
\[
= -2r \left\langle B(Y, Y), \eta \right\rangle = -2r II_\Sigma(Y) \]
where $II_\Sigma$ is the second fundamental form relatively to $\Sigma$ with respect to $\eta$. Therefore, at $x$ we have
\[
\text{Hess}_g (v, v) = -2r II_\Sigma (v). \quad (27) \]

Let $\alpha : [0, l] \rightarrow \partial \Omega, \quad l > 0$, be an arc length parametrized and simple curve
(in the induced metric), such that $\alpha(0) = x, \alpha(l) \neq x$ and $\alpha'(0) = v$. Let $\sigma$
the arc length parameter and define
\[
\xi : [0, l] \rightarrow \mathbb{R} \]
by $\xi(\sigma) = g(\alpha(\sigma))$. We have
\[
\xi'(\sigma) = 2d(\alpha(\sigma)) \langle \nabla d(\alpha(\sigma)), \alpha'(\sigma) \rangle =
\langle \nabla g(\alpha(\sigma)), \alpha'(\sigma) \rangle \]
and, relatively to $\partial \Omega$,
\[
\xi''(\sigma) = \langle \nabla_{\alpha'(\sigma)} \nabla g(\alpha(\sigma)), \alpha'(\sigma) \rangle + \langle \nabla g(\alpha(\sigma)), \nabla_{\alpha'(\sigma)} \alpha'(\sigma) \rangle \]
\[
= \langle \nabla_{\alpha'(\sigma)} \nabla g(\alpha(\sigma)), \alpha'(\sigma) \rangle +
+ \langle \nabla g(\alpha(\sigma)), [\nabla_{\alpha'(\sigma)} \alpha'(\sigma)]^T + B(\alpha'(\sigma), \alpha'(\sigma)) \rangle \]
\[
= \langle \nabla_{\alpha'(\sigma)} \nabla g(\alpha(\sigma)), \alpha'(\sigma) \rangle +
+ |\nabla g(\alpha(\sigma))| \left\langle X(\alpha(\sigma)), [\nabla_{\alpha'(\sigma)} \alpha'(\sigma)]^T + B(\alpha'(\sigma), \alpha'(\sigma)) \right\rangle. \]
\[8\]
As $X$ is normal to $\partial \Omega$ at $x = \alpha(0)$, that is $X(x) = \eta(x) = \nu(x)$, we have
\[
\xi''(0) = (\nabla_\nu \nabla g(x), v) + |\nabla g(x)| \langle \nu(x), B(v,v) \rangle \\
= \text{Hess} g(v,v) + 2r \text{I}_\partial \Omega(v),
\]
where $\text{I}_\partial \Omega$ is the second fundamental form relatively to $\partial \Omega$ with respect to $\nu$ at $x$. Then, from (27), we obtain
\[
\xi''(0) = 2r (\text{I}_\partial \Omega(v) - \text{I}_\Sigma(v)) > 0,
\]
being the inequality in (28) consequence of the fact that $0 < r < R$ and from the comparison principle. On the other hand, as $\xi(0) = \xi'(0) = 0$, setting
\[
2C := 2r (\text{I}_\partial \Omega(v) - \text{I}_\Sigma(v)) ,
\]
we can write
\[
\xi(\sigma) = \frac{1}{2} \xi''(0) \sigma^2 + \vartheta(\sigma) = C \sigma^2 + \vartheta(\sigma)
\]
where
\[
\lim_{\sigma \to 0} \frac{\vartheta(\sigma)}{\sigma^2} = 0.
\]
Given $0 < \varepsilon < C$, there is $0 < \tau \leq l$ such that, for $0 < \sigma \leq \tau$, we have $-\varepsilon \sigma^2 < \vartheta(\sigma) < \varepsilon \sigma^2$. It follows that
\[
d(\alpha(\sigma))^2 = \xi(\sigma) + r^2 \\
\geq C \sigma^2 + r^2 - \varepsilon \sigma^2 = (C - \varepsilon) \sigma^2 + r^2 \geq 0
\]
and then
\[
d(\alpha(\sigma)) \geq \sqrt{(C - \varepsilon) \sigma^2 + r^2}, 0 \leq \sigma \leq \tau.
\]
(29)

As $\sigma$ is the arc length parameter, it follows that $\sigma \geq d(\alpha(\sigma), x)$. Then, from (29), for $0 < \sigma \leq \tau$,
\[
a \cosh^{-1} \left( \frac{d(\alpha(\sigma))}{r} \right) \geq a \cosh^{-1} \left( \frac{\sqrt{(C - \varepsilon) d(\alpha(\sigma), x)^2 + r^2}}{r} \right).
\]
(30)

Let
\[
\overline{\delta} := \sup \{d(\alpha(\sigma), x), \sigma \in [0, \tau]\}.
\]
Setting \( t = d(\alpha(\sigma), x) \), we have \( t \to 0 \) when \( \sigma \to 0 \). For \( t \in [0, \delta] \), set
\[
h(t) = a \cosh^{-1}\left( \frac{\sqrt{(C - \varepsilon) t^2 + r^2}}{r} \right).
\] (31)

Since \( h(0) = 0 \) and \( h'(0) = \frac{2}{r} \sqrt{C - \varepsilon} \), (31) can be rewritten as
\[
h(t) = h(0) + h'(0) t + \theta(t) = \frac{a}{r} t \sqrt{C - \varepsilon} + \theta(t),
\]
with
\[
\lim_{t \to 0} \frac{\theta(t)}{t} = 0.
\]

Thus, replacing in (31), we have
\[
a \cosh^{-1}\left( \frac{d(\alpha(\sigma))}{r} \right) \geq \left( \frac{a}{r} \sqrt{C - \varepsilon} \right) d(\alpha(\sigma), x) + \theta(d(\alpha(\sigma), x)) \] (32)
where
\[
\lim_{\sigma \to 0} \frac{\theta(d(\alpha(\sigma), x))}{d(\alpha(\sigma), x)} = 0.
\]

Let \( \kappa(x) \), \( \kappa_r \) as defined in (8) and (10) respectively. We have \( 0 < \kappa(x) \) since \( 0 < r < R \) and \( \{ v \in T_x \partial \Omega, |v| = 1 \} \) is compact. Moreover, as \( \partial^{-} \Omega \) is compact, it follows that \( 0 < \kappa_r \). Note that \( 0 < r \kappa_r \leq r \kappa(x) \leq C \). Thus, given \( 0 < \varepsilon < r \kappa_r \), it follows from (32) that there is \( \delta_1 > 0 \) (which does not depend on \( x \in \partial^{-} \Omega \)), such that, for all \( z \in B_{\delta_1}(x) \cap \partial \Omega \), we obtain
\[
a \cosh^{-1}\left( \frac{d(z)}{r} \right) \geq \left( \frac{a}{r} \sqrt{r \kappa(x) - \varepsilon} \right) d(z, x) + o(d(z, x)) \]
\[
\geq \left( \frac{a}{r} \sqrt{r \kappa_r - \varepsilon} \right) d(z, x) + o(d(z, x))
\]
that is,
\[
w(z) \geq \left( \frac{a}{r} \sqrt{r \kappa_r - \varepsilon} \right) d(z, x) + o(d(z, x)), z \in B_{\delta_1}(x) \cap \partial \Omega.
\]
Lemma 7 Let \( a \in (0, \sqrt{\frac{r}{(n-1)[\lambda_r]}}) \). Then

\[
0 < a \sqrt{\frac{\kappa(x)}{r}} < \frac{1}{\sqrt{n-1}}.
\]

where \( \kappa(x) \) is given by (33).

Proof. Notice that

\[
a \sqrt{\frac{\kappa(x)}{r}} = \sqrt{\frac{a^2}{r} \kappa(x)} \leq \sqrt{\frac{\kappa(x)}{(n-1)[\lambda_r]}} = \frac{1}{\sqrt{n-1}} \sqrt{\frac{\kappa(x)}{[\lambda_r]}}. \tag{33}
\]

On the other hand, there is \( v^* \in T_x \partial \Omega \) such that

\[
\lambda(x) \leq II_{\Sigma}(v^*) < II_{\partial \Omega}(v^*) < 0
\]

and then

\[
0 < -II_{\partial \Omega}(v^*) < -II_{\Sigma}(v^*) \leq |\lambda(x)|.
\]

It follows that

\[
0 < \frac{-II_{\partial \Omega}(v^*)}{|\lambda(x)|} < \frac{-II_{\Sigma}(v^*)}{|\lambda(x)|} \leq 1,
\]

that is

\[
0 < \frac{II_{\partial \Omega}(v^*) - II_{\Sigma}(v^*)}{|\lambda(x)|} < 1.
\]

Therefore, as \( \kappa(x) = \inf \{ II_{\partial \Omega}(v) - II_{\Sigma}(v) \mid v \in T_x \partial \Omega, \ |v| = 1 \} \),

\[
\sqrt{\frac{\kappa(x)}{|\lambda(x)|}} < 1
\]

and then, from (33), as \( \lambda_r \leq \lambda(x) < 0 \),

\[
a \sqrt{\kappa_r r^{-1}} \leq a \sqrt{\frac{\kappa(x)}{r}} r^{-1} = \frac{a}{r} \sqrt{r \kappa(x)} \leq \frac{1}{\sqrt{n-1}}.
\]
3 Proof of Theorem \[1\]

**Proof.** We use the Perron method.

Consider the solution of Perron \[13\]. In order to show that \( u \in C^0 (\overline{\Omega}) \), \( u|_{\partial \Omega} = f \), we will take into account, first, the barriers for the non mean convex points of \( \partial \Omega \) given by Lemma \[6\].

Let \( r \in (0, R) \), \( a \in (0, \sqrt{\frac{r}{(n-1)|\lambda_r|}}) \) and \( K \in [0, a\sqrt{\mathcal{K}_r}r^{-1}) \). Then, there is \( 0 < \varepsilon < r \mathcal{K}_r \) such that

\[
K < \frac{a}{r} \sqrt{r \mathcal{K}_r} - \varepsilon \leq \frac{a}{r} \sqrt{r \mathcal{K}(x)} - \varepsilon
\]

for all \( x \in \partial \Omega \). Then, given \( x \in \partial \Omega \), by the Lemma \[6\] there is \( \delta_1 > 0 \) such that

\[
w_x(z) = a \cosh^{-1}\left(\frac{d(z)}{r}\right),
\]

as in Lemma \[5\] satisfies

\[
w_x(z) \geq ar^{-1}(r \mathcal{K}_r - \varepsilon)^{1/2} d(z, x) + o\left(d(z, x)\right)
\]

\[
\geq K d(z, x), \quad z \in B_{\delta_1}(x) \cap \partial \Omega.
\]

(34)

Set

\[
\delta_0 = \min\{\delta_1, \mu\},
\]

where \( \mu \) is given by \[24\]. Then, given \( \delta \in (0, \delta_0) \), define

\[
\epsilon = \inf\{w_x(z) ; \ z \in \partial B_\delta(x) \cap \partial \Omega, \forall x \in \partial^- \Omega\}.
\]

It follows that \( \epsilon > 0 \) since \( r < R \). Let

\[
\omega^-_x(z) = f(x) - w_x(z), \quad \omega^+_x(z) = f(x) + w_x(z), \quad x \in \partial^- \Omega, \ z \in B_\delta(x) \cap \overline{\Omega}.
\]

From \[34\], since

\[
|f(z) - f(x)| \leq K d(z, x), \quad x \in \partial^- \Omega, \ z \in B_\delta(x) \cap \partial \Omega
\]

we have

\[
\omega^-_x(z) \leq f(z) \leq \omega^+_x(z), \quad x \in \partial^- \Omega, \ z \in B_\delta(x) \cap \partial \Omega
\]

and, since \( osc(f) < \epsilon \),

\[
\omega^-_x(z) < \inf_{\partial \Omega} f, \sup_{\partial \Omega} f < \omega^+_x(z), \quad x \in \partial^- \Omega, \ z \in \partial B_\delta(x) \cap \Omega.
\]
Then, setting
\[
W^-(x)(z) = \begin{cases} \max \{ \omega^-(x)(z), \inf_{\partial \Omega} f \} & \text{if } z \in B_{\delta}(x) \cap \Omega \\ \inf_{\partial \Omega} f, & \text{if } z \in \Omega - B_{\delta}(x) \cap \Omega \end{cases}
\]
and
\[
W^+(x)(z) = \begin{cases} \min \{ \omega^+(x)(z), \sup_{\partial \Omega} f \} & \text{if } z \in B_{\delta}(x) \cap \Omega \\ \sup_{\partial \Omega} f, & \text{if } z \in \Omega \setminus B_{\delta}(x) \cap \Omega \end{cases}
\]
we have from Lemma 5 that \(W^-, W^+\) are subsolution and supersolution of \(\mathcal{M}\) in \(\Omega\), respectively, \(W^-(x) \in S, W^+ \leq \sup_{\partial \Omega} f\), with \(W^-(x) = W^+(x) = f(x)\).

At the mean convex points, we proceed as follows: given \(x \in \partial \Omega \setminus \partial^\circ \Omega\), since \(x\) is a mean convex point and \(\partial \Omega\) is of class \(C^2\), there is a neighborhood \(U\) of \(x\) in \(\partial \Omega\) such that \(U = B \cap \partial \Omega\), where \(B \subset \Omega\) is a mean convex \(C^2\)-domain. The Dirichlet problem (11) is solvable on \(B\) for arbitrary continuous boundary data. We observe that, here, at the mean convex point, the hypothesis on the Ricci curvature is necessary only at the points where \(\mathcal{H}(x) = 0\). We can then choose \(g^\pm_x \in C^2(B) \cap C^0(\overline{B})\) satisfying \(\mathcal{M}(g^\pm_x) = 0\) in \(B\), such that \(g^+_x(x) = f(x)\),

\[
g^-_x(z) \leq f(z) \leq g^+_x(z), \quad z \in U
\]
and
\[
g^-_x(z) < \inf_{\partial \Omega} f, \quad \sup_{\partial \Omega} f < g^+_x(z), \quad z \in \partial B \setminus U.
\]
Then,
\[
\mathcal{G}^-_x(z) = \begin{cases} \max \{ g^-_x(z), \inf_{\partial \Omega} f \} & \text{if } z \in \overline{B} \\ \inf_{\partial \Omega} f, & \text{if } z \in \Omega \setminus \overline{B} \end{cases}
\]
and
\[
\mathcal{G}^+_x(z) = \begin{cases} \min \{ g^+_x(z), \sup_{\partial \Omega} f \} & \text{if } z \in \overline{B} \\ \sup_{\partial \Omega} f, & \text{if } z \in \Omega \setminus \overline{B} \end{cases}
\]
are subsolution and supersolution of \(\mathcal{M}\) in \(\Omega\), respectively, \(\mathcal{G}^-_x \in S, \mathcal{G}^+_x \leq \sup_{\partial \Omega} f\), with \(\mathcal{G}^-_x(x) = \mathcal{G}^+_x(x) = f(x)\).

Thus, for each point \(x \in \partial \Omega\) we got the barriers and, then, the solution of Perron (13) is such that \(u \in C^0(\Omega), u|_{\partial \Omega} = f\). ■
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