The asymptotic behaviour of the Hawking energy along null asymptotically flat hypersurfaces

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Abstract

In this work we obtain the limit of the Hawking energy of a large class of foliations along general null hypersurfaces Ω satisfying a weak notion of asymptotic flatness. The foliations are not required to be either geodesic or approaching large spheres at infinity. The limit is obtained in terms of a reference background geodesic foliation approaching large spheres and a positive function, constant along the null generators on Ω, which describes the relation between the two foliations at infinity. The integrand in the limit expression has interesting covariance and invariance properties with respect to change of background foliation. The standard result that the Hawking energy tends to the Bondi energy under suitable circumstances is recovered in this framework.

1 Introduction

For any closed spacelike surface S with spherical topology embedded in a four dimensional spacetime, the Hawking energy is defined by

\[ m_H(S) = \sqrt{\frac{|S|}{16\pi}} \left( 1 - \frac{1}{16\pi} \int_S \vec{H}^2 \eta_S \right), \]  

where \( \vec{H} \) is the mean curvature of \( S \) and \( |S| \) is the area of \( S \).

The Hawking energy was introduced by Hawking [13] and under certain circumstances it gives a measure of the total energy contained in the surface \( S \). However, it is well-known that this is not always the case. For instance, in the Minkowski spacetime any surface embedded in a spacelike hyperplane has negative Hawking energy, except for round spheres where it vanishes, while there are spacelike surfaces embedded in the time-cylinder over a two-sphere with positive Hawking energy. For surfaces embedded in the past null cone of a point (also in Minkowski) the Hawking energy turns out to be identically zero. In more general spacetimes, the small sphere limit of the Hawking energy has been studied by Horowitz and Schmidt [15] who found that, for suitably constructed surfaces embedded in the future null cone of a point \( p \), the leading term of the Hawking energy agrees with the energy-density at the vertex or, if the spacetime is vacuum at \( p \), by a suitable time-component of the Bel-Robinson tensor. Hence the Hawking energy enjoys interesting positivity properties in this limit.
In the opposite regime of very large spheres in an asymptotically flat spacetime the Hawking energy also has interesting properties, at it is known that for suitably round spheres at infinity, the Hawking energy approaches the ADM or Bondi energies. More precisely, in the asymptotically flat spacelike context, the Hawking energy of the surfaces of constant coordinate radius \( r \) in the asymptotic region has a limit when \( r \to \infty \) which agrees with the ADM energy of the hypersurface \([13]\). In the asymptotically hyperboloidal case, the approximate Bondi spheres have the same property, the limit now being the Bondi energy (see \([10]\) for details). When the surfaces are embedded in a null hypersurface intersecting null infinity on a cross section, the limit of the Hawking energy is again the Bondi energy provided the surfaces approach large spheres in the sense that (see \([25], [2]\) and Sauter’s Ph.D. thesis \([26]\) where this result is explicitly quoted):

\[
K_\infty := \lim_{r \to \infty} \frac{|S_r|}{4\pi} K(r) = 1, \tag{2}
\]

where \( \{S_r\} \) is the collection of surfaces along which the limit is taken, \( |S_r| \) is the area of \( S_r \) and \( K(r) \) its Gauss curvature. Since \( K_\infty \) is the Gauss curvature of the surface at infinity after a suitable rescaling, the condition above states that the surfaces approach large round spheres at infinity, in agreement with the behaviour in the asymptotically Euclidean and hyperboloidal cases.

It turns out, however, that understanding the behaviour of the Hawking energy at infinity when the condition of round spheres is not imposed is much more subtle. The aim of this paper is to carry out such an analysis for surfaces embedded in a asymptotically flat null hypersurface (we give below the precise definition). This problem is interesting for several reasons. First of all, it is relevant in order to help clarifying the physical meaning of the Hawking energy, which, as already said, is related to an energy in some circumstances but not in others. From a more practical point of view, the Hawking energy has become a very valuable tool for various problems in geometric analysis. The underlying reason is that the Hawking energy enjoys interesting monotonicity properties for specific flows of surfaces. In order to become truly useful, this monotonicity needs to be complemented with a good behaviour of the Hawking energy at infinity, so that its asymptotic value can be related to the ADM (or Bondi) energies of the spacetime. Whenever the flow can be proved to approach large round spheres, the results above suffice, but often this is not the case and understanding the behaviour of the Hawking energy at infinity under general circumstances becomes a useful piece of information.

To be more specific, the Hawking energy has played a fundamental role in the proof by Huisken & Ilmanen \([16]\) of the Penrose inequality in the time-symmetric, asymptotically euclidean case in four spacetime dimensions. The key fact behind their proof is a monotonicity formula for the Hawking energy under the inverse mean curvature flow, first discovered by Geroch \([12]\) and extended by Huisken and Ilmanen to a suitably weak setting that guarantees global existence of the flow. On the horizon \( S_H \) (a connected outermost minimal surface in this case), the Hawking energy agrees with \( \sqrt{|S_H|}/16\pi \). At infinity, the authors were able to prove that the flow makes the surfaces sufficiently round so as to guarantee that the limit of the Hawking energy is not larger than the ADM mass, thus establishing the Penrose inequality \( M_{\text{ADM}} \geq \sqrt{|S_H|}/16\pi \).

Monotonicity of the Hawking energy has been studied in various contexts, both as codimension one flows within spacelike \([18]\) or null hypersurfaces \([26]\) or as codimension-two flows in a spacetime setting \([14, 4]\) where the notion of uniformly expanding flows where introduced and sufficient conditions for monotonicity of the Hawking energy were found. Spacetimes flows under which the Hawking energy is monotonic have received renewed interest recently \([5]\), where additional sufficient conditions for monotonicity have been found and the role of so-called time flat surfaces
has been emphasized (see [5] for the relationship with the previous spacetime monotonicity results). All these results show that monotonicity of the Hawking energy is a versatile property which can be accommodated to many circumstances.

However, the limit at infinity of the flows turns out to be much more problematic. This was first realized by A. Neves [23] who studied inverse mean curvature flows in Riemannian, asymptotically hyperbolic 3-dimensional manifolds with scalar curvature bounded below by a negative constant. It turns out that the flow does not guarantee convergence of the surfaces to sufficiently round spheres, which, in general, prevents comparison of the limit of the Hawking energy and the total mass of the hypersurface. A similar difficulty is faced for flows along null hypersurfaces [26].

It is therefore interesting to know, in as much generality as possible, what is the limit of the Hawking energy at infinity without assuming that the surfaces \( \{ S_r \} \) approach large spheres at infinity. The null case is particularly interesting because it allows for a very neat description of spacelike surfaces embedded in the null hypersurfaces as graphs with respect to a background foliation that can be chosen conveniently. We exploit this fact in order to obtain an explicit and simple expression for the limit of the Hawking energy at infinity for a very general flow of spacelike surfaces. Our main result is as follows (see Section 3 for definitions)

**Theorem 1.** Let \( \Omega \) be a past asymptotically flat null hypersurface. Let \( \{ S_r \} \) be a foliation defined as the level sets of a function \( r : \Omega \to \mathbb{R} \) satisfying \( k(r) = -1 \) where \( k \) is future, tangent to the null generators of \( \Omega \) and geodesic. Let \( \ell \) be the unique null vector orthogonal to \( S_r \) and satisfying \( \langle \ell, k \rangle = -2 \). Assume that \( \{ S_r \} \) approaches large spheres with round limit metric \( \tilde{q} \). Consider any flow of surfaces \( \{ S_{r^*} \} \), \( r^* = \text{const.} \) defined by

\[
r = \phi r^* + \tau + \xi^*
\]

where \( \phi, \tau \) are Lie constant along \( k \) with \( \phi > 0 \) everywhere and \( \xi \) satisfies \( \xi = o_1(1) \cap o_2^X(1) \), \( k(\xi) = o_1^X(\tau^{-1}) \). Then, the limit of the Hawking energy along \( \{ S_{r^*} \} \) is

\[
\lim_{r^* \to \infty} m_H(S_{r^*}) = \frac{1}{8\pi \sqrt{16\pi}} \left( \sqrt{\int_{S^2} \phi^2 \eta_\phi} \right) \int_{S^2} \left( \Delta_\phi \theta_k^{(1)} - (\theta_k^{(1)} + \theta_\ell^{(1)}) - 4 \text{div}_\phi(s_\ell^{(1)}) \right) \frac{1}{\phi} \eta_\phi,
\]

where

\[
\theta_k^{(1)} = \lim_{r \to \infty} (\theta_k(S_r) r^2 + 2r), \quad \theta_\ell^{(1)} = \lim_{r \to \infty} (\theta_\ell(S_r) r^2 - 2r), \quad s_\ell^{(1)} = \lim_{r \to \infty} (r s_\ell)
\]

and \( \theta_k(S_r), \theta_\ell(S_r) \) are the null expansions of \( S_r \) and \( s_\ell(X) := \frac{1}{2} \langle \ell, \nabla X k \rangle, X \in \mathcal{X}(S_r) \) is the connection of the normal bundle of \( S_r \).

For definiteness, we work with past null hypersurfaces in this paper. The changes required to deal with future null hypersurfaces are indicated in Remark 6 below.

In addition to this theorem we also find an interesting covariance property of the integrand in (4) under changes of background foliation. This is part of the content of Theorems 3, 4 and 5 below, and which may be of independent interest.

We conclude by noting that the Hawking energy is also useful as a tool to control the Bondi mass of a spacetime. For null hypersurfaces this idea has been exploited by Alexakis and Shao [1], who prove bounds for the Bondi energy-momentum for vacuum, null hypersurfaces close to the shear-free outgoing null cones in Schwarzschild spacetime. According to these authors obtaining the limit of the Hawking energy is particularly difficult, fundamentally due to the lack of a neat
formula for the change of the mass aspect function under change of geodesic foliation. Recall that the mass aspect function \([8, 9]\) is a specific geometric quantity \(\mu_S\) defined on surfaces with the property that

\[
m_H(S) = \sqrt{\frac{|S|}{16\pi}} \int \mu_S \eta_S.
\]

Theorems \([3, 4]\) and \([5]\) do provide a simple transformation formula at infinity for an integrand of the Hawking energy for foliations approaching large spheres, so a more general application of this result may well have interesting applications also in the context of Alexakis and Shao’s work.

As a more general application of our results, flows along null hypersurfaces are likely to play a role in many attempts to prove the Penrose inequality at null infinity, cf. \([17]\), \([3]\), \([19]\), \([21]\), \([7]\), \([22]\). The general methods developed in this paper to deal with general foliations in null hypersurfaces are thus likely to be useful in that context as well.

The paper is organized as follows. In Section 2 we recall the geometry of graphs over a background foliation in a null hypersurface. For the sake of generality, we work here in arbitrary dimension and with general foliations transverse to the null generator, even though for the rest of the paper only dimension four and geodesic background foliations are used. In Section 3 we introduce our definition of asymptotically flat null hypersurface and consider some simple consequences. We also introduce the notion of energy flux decay condition which imposes additional decay on some components of the Einstein tensor and which is used later on in Section 7. In Section 4 the definition of approaching large spheres is introduced. Section 5 is devoted to find the limit of the Hawking energy for geodesic foliations. Given a background foliation approaching large spheres we consider first what happens for other geodesic foliation starting at the same initial surface (Theorem 2) and then the case when the foliation has the same null generator but starts on a different surface (Theorem 4). We find, in particular, an interesting covariance property of the integrand arising in the limit expression of the Hawking mass under changes of background foliation (Theorems 3 and 4). The case of non-geodesic foliations is treated in Section 6. We devote Section 7 to recover the well-known result that the Hawking energy tends to the Bondi energy when the foliation approaches large spheres. This involves recalling known properties of the relationship between the conformal group of the sphere and the Lorentz group in Minkowski, with a subtlety that arises when the hypersurfaces extend to past null infinity. We also compare our results with the analysis of the limit of the Hawking energy in the so-called null quasi-spherical gauge by Bartnik \([2]\).

2 Setup

Let \((\mathcal{M}, g)\) be a time-oriented spacetime of dimension \(n + 1\), \(n \geq 3\). We consider a smooth, connected, null hypersurface \(\Omega\) embedded in \((\mathcal{M}, g)\). Let \(k\) be a smooth, nowhere zero, future directed null vector field tangent to \(\Omega\) (i.e. a null generator). Since the integral curves of \(k\) are geodesics, there exists \(Q_k \in \mathcal{F}(\Omega)\) such that \(\nabla_k k = Q_k k\), where \(\nabla\) is the Levi-Civita covariant derivative of \((\mathcal{M}, g)\). We make the assumption that there is an embedded spacelike connected hypersurface \(S_0\) in \(\Omega\) (with embedding \(\Phi_0\)) such that any integral curve of \(-k\) intersects \(S_0\) precisely once. This implies the existence of a smooth map \(\pi : \Omega \to S_0\) (we identify \(S_0\) with its image, the meaning being clear from the context) which sends \(p \in \Omega\) to the intersection of the integral curve \(\gamma^k_p\) of \(-k\) passing through \(p\) with \(S_0\). The map \(\pi\) is clearly a submersion. We choose the parameter \(\lambda\) of the curve \(\gamma^k_p\) so that \(\gamma^k_p(0) = p\).
Given $k$, $S_0$ and a constant $r_0$, a scalar function $r \in \mathcal{F}(\Omega)$ is defined by $k(r) = -1$ and $r(p) = r_0$ for all $p \in S_0$. Let $(r_-, r_+)$ be the range of the function $r$ restricted to the curve $\gamma_p$. We also assume that the open interval $(R_- := \sup_{S_0} r_-, R_+ := \inf_{S_0} r_+)$ is non-empty. The function $r$ having nowhere zero gradient, the level sets $S_{r_1} = \{r = r_1\}$ are either empty or smooth, embedded (not necessarily connected) hypersurfaces. The collection $\{S_r\}$ is a foliation of $\Omega$. For $r_1 \in (R_-, R_+)$ the hypersurfaces $S_{r_1}$ are in fact connected and diffeomorphic to $S_0$.

At any $p \in \Omega$ let $\ell|_p \in T_p M$ be the unique null vector field satisfying $\langle k, \ell\rangle|_p = -2$ and $\langle \ell, X\rangle|_p = 0$ for any $X \in T_p S_{r(p)}$. $S_r$ is endowed with an induced metric $\gamma_{S_r}$, with two null second fundamental forms $K^k(X, Y) := \langle \nabla_X k, Y\rangle$, $K^\ell(X, Y) := \langle \nabla_X \ell, Y\rangle$ $X, Y \in \mathcal{X}(S_r)$ and with a normal bundle connection one-form $s_\ell(X) := -\frac{1}{2}\langle \nabla_X \ell, k\rangle$.

In order to obtain the limit of the Hawking energy as described in the introduction, we need to relate the geometry of different spacelike surfaces embedded in $\Omega$. Consider a spacelike embedded hypersurface $S$ in $\Omega$ with embedding $\Phi : S \rightarrow \Omega$ and let $p \in S$. This hypersurface is uniquely defined by a diffeomorphism $\Psi : S \rightarrow \Psi(S) \subset S_0$ and a function $F \in \mathcal{F}(S)$ as follows. For all $p \in S$ define $F(p) = r(\Phi(p))$ and $\Psi(p) = (\pi \circ \Phi)(p)$. Conversely, a function $F \in \mathcal{F}(S)$ with image in $(r_-, r_+)$ and a diffeomorphism $\Psi$ as above defines an embedding

$$\Phi : S \rightarrow \Omega$$

$$p \rightarrow \gamma^k_p(\lambda = F(p)).$$

We want to relate the intrinsic and extrinsic geometry of $S$ at $p$ with the geometry of the surface $S_{r=F(p)}$. Since this is all local we can assume $\Psi(S) = S_0$, which makes the presentation simpler. We extend $F$ to a function on $\Omega$ defined by $F(q) = F((\Psi^{-1} \circ \pi)(q))$. We keep the same symbol for the extension. It is clear that $k(F) = 0$. For the extrinsic geometry of $S$ (we again identify $S$ with its image) we define at $p \in S$, the null normal $\ell_S|_p$ by the conditions $\langle \ell_S, k\rangle|_p = -2$ and $\langle \ell_S, X\rangle|_p = 0$ for all $X \in T_p S$. The null second fundamental forms $K^\ell_S$, $K^k$ and the normal connection one-form $s_\ell_S$ are defined similarly as before. The following Proposition is known (see e.g. [26]) when the background foliation $\{S_r\}$ is geodesic (i.e. $Q^k = 0$). Although this is the situation we will require later, we include for the sake of generality the non-geodesic case as well. Our proof also follows a somewhat different approach.

**Proposition 1.** Let $p \in S$, then the map

$$T_F : T_p S_{r=F(p)} \rightarrow T_p S$$

$$X \rightarrow X^\prime := X - X(F)k$$

is a well-defined isomorphism. The induced metric $\gamma_S$, null second fundamental forms $K^\ell_S$, $K^k$ and normal bundle connection $s_\ell_S$ of $S$ are given by

$$\gamma_S|_p(X^\prime, Y^\prime) = \gamma(X, Y),$$

$$K^k(X^\prime, Y^\prime)|_p = K^k(X, Y)$$

$$s_\ell_S(X^\prime)|_p = s_\ell(X) - K^k(X, \text{grad } F) + X(F)Q_k$$

$$K^\ell_S(X^\prime, Y^\prime)|_p = K^\ell(X, Y) + |DF|^2 K^k(X, Y) + 2X(F)s_\ell(Y) + Y(F)s_\ell(X)$$

$$- 2X(F)K^k(X, \text{grad } F) - 2Y(F)K^k(Y, \text{grad } F) - 2\text{Hess } F + 2Q_kX(F)Y(F)$$

where $\gamma, K^k, K^\ell, s_\ell, \text{grad }, \text{Hess}$ and $|DF|^2 = \langle \text{grad } F, \text{grad } F\rangle$ refer to the surface $S_{r=F(p)}$ and are evaluated at $p$. 

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The proof is based on the following simple identity that may be useful in other contexts.

**Lemma 1.** Let \( S \) be an embedded spacelike surface with embedding \( \Psi : S \rightarrow \mathcal{M} \). Select a pair of null normal vector fields \( \{ k, \ell \} \) along \( S \) satisfying \( \langle k, \ell \rangle = -2 \). For any vector field \( \xi \) on a spacetime neighbourhood of \( S \) write its deformation tensor as \( \mathcal{L}_g := a_\xi \). Then

\[
a_\xi(X, Y) = -\langle \xi, \ell \rangle K^k(X, Y) - \langle \xi, k \rangle K^\ell(X, Y) + (\nabla^S_X \xi)(Y) + (\nabla^S_Y \xi)(X), \quad X, Y \in \mathfrak{X}(S)
\]

where \( \nabla^S \) is the Levi-Civita covariant derivative of \( S \), \( \xi := \Psi^* (\xi) \) and \( \xi := g(\xi, \cdot) \).

**Remark 1.** Given \( \xi \) merely along \( S \), this result can be applied to any extension of \( \xi \) to a neighbourhood of \( S \), the result being independent of the extension.

**Proof.** Decompose \( \xi \) in tangential and normal parts \( \xi = \xi_\parallel + \xi_\perp \) and \( \xi_\perp \) in the null basis \( \{ k, \ell \} \) \( \xi = -\frac{1}{2} \langle \xi, \ell \rangle k - \frac{1}{2} \langle \xi, k \rangle \ell + \xi_\parallel \) so that

\[
a_\xi(X, Y) = \langle \nabla_X \xi, Y \rangle + \langle \nabla_Y \xi, X \rangle
\]

\[
= -\langle \xi, \ell \rangle K^k(X, Y) - \langle \xi, k \rangle K^\ell(X, Y) + \langle \nabla_X \xi_\parallel, Y \rangle + \langle \nabla_Y \xi_\parallel, X \rangle
\]

\[
= -\langle \xi, \ell \rangle K^k(X, Y) - \langle \xi, k \rangle K^\ell(X, Y) + \langle \nabla^S_X \xi_\parallel, Y \rangle + \langle \nabla^S_Y \xi_\parallel, X \rangle
\]

\[
= -\langle \xi, \ell \rangle K^k(X, Y) - \langle \xi, k \rangle K^\ell(X, Y) + (\nabla^S_X \xi_\parallel)(Y) + (\nabla^S_Y \xi_\parallel)(X).
\]

\( \square \)

**Proof of the Proposition.** \( T_F \) is well-defined provided \( X - X(F)k \) is tangent to \( S \). This follows because \( S \) is defined by \( r - F = 0 \) (note that \( d(r - F) \neq 0 \) everywhere) and \( \mathcal{L}_{X - X(F)k}(r - F) = -X(F) - X(F)k(r) = 0 \). \( T_F \) is obviously injective, hence an isomorphism. Properties \( 5 \) and \( 6 \) are immediate (and well-known). For the remaining parts we note the decomposition

\[
\ell_S|_p = \ell + |DF|^2k - 2 \text{ grad } F|_p, \quad p \in S,
\]

which holds because the right-hand side is null, satisfies \( \langle \ell_S, k \rangle = -2 \) and is orthogonal to \( X' = T_F(X) \), for all \( X \in T_p S \). To show \( 7 \) we compute

\[
s_{\ell_S}(X') = -\frac{1}{2} \langle \nabla_X \ell_S, k \rangle = \frac{1}{2} \langle \nabla_X k, \ell_S \rangle = \frac{1}{2} \langle \nabla_X (k - X(F) \nabla_k k), \ell_S \rangle
\]

\[
= s_\ell(X) - K^k(X, \text{ grad } F) + X(F)Q_k.
\]

For the null extrinsic curvature \( K^s \) we use Lemma 11. First observe that the right-hand side of \( 9 \) makes sense for all \( p \in \Omega \), so it defines an extension of \( \ell_S \) which remains null and satisfying \( \langle \ell_S, k \rangle = -2 \). Extend also \( Y \in T_p S_{r=F(p)} \) to a neighbourhood under the condition that remains tangent to the foliation \( \{ S_r \} \). This induces an extension of \( Y' \) which remains orthogonal to \( \ell_S \). Note

\[
[k, Y'] = [k, Y] - k(Y(F))k = [k, Y] - [k, Y](F) = ([k, Y])'
\]

which shows that \( [k, Y'] \) is tangent to \( S \) at \( p \) (and we used that \( [k, Y] |_p \) is tangent to \( S_{r=F(p)} \)). We apply Lemma 11 on the surface \( S_{r=F(p)} \) and to the vector field \( \ell_S \). Concerning the deformation
Corollary 1. Let \( K \), \( S \), and \( Y \) be denoted respectively as \( \theta, S \), and \( Y \). The following corollary is a trivial consequence of how the null second fundamental forms and the normal bundle connection transforms under a boost in \( \{ \ell_S, k \} \).

\[
\langle \nabla_X \ell_S, Y \rangle = \langle \nabla_{X'} \ell_S, Y' \rangle + Y(F)k
\]
\[
= \langle \nabla_X \ell_S, Y \rangle + X(F)\langle \nabla_k \ell_S, Y' \rangle + Y(F)\langle \nabla_X \ell_S, k \rangle + X(F)Y(F)\langle \nabla_k \ell_S, k \rangle
\]
\[
= K^\ell_S(X', Y') - X(F)\langle \nabla_k X', \ell_S \rangle - 2Y(F)\ell_S(X') - X(F)Y(F)\langle \nabla_k \ell_S \rangle
\]
\[
= K^\ell_S(X', Y') - X(F)[[Y', Y] + \nabla Y', k] - 2Y(F)s_{\ell_S}(X') + 2Q_kX(F)Y(F)
\]
\[
= K^\ell_S(X', Y') - 2X(F)s_{\ell_S}(X') - 2Y(F)s_{\ell_S}(X') + 2Q_kX(F)Y(F).
\]

where in the last equality we used that \([k, Y']_p \) is tangent to \( S \). Hence

\[
a_{\ell_S}(X, Y) = 2K^\ell_S(X', Y') - 4X(F)s_{\ell_S}(Y') - 4Y(F)s_{\ell_S}(X') + 4Q_kX(F)Y(F)
\]
\[
= 2K^\ell_S(X', Y') - 4X(F)s_{\ell}(Y') - 4Y(F)s_{\ell}(X') + 4X(F)K^k(X, \nabla F) + 4Y(F)K^k(Y, \nabla F) - 4Q_kX(F)Y(F) \tag{10}
\]

after using (7) in the second equality. Now Lemma 1 gives

\[
a_{\ell_S}(X, Y) = 2|DF|^2K^k(X, Y) + 2K^\ell(X, Y) - 4\text{Hess } F. \tag{11}
\]

Solving for \( K^\ell_S(X', Y') \) in (10) and (11) yields the result.

The following corollary is a trivial consequence of how the null second fundamental forms and the normal bundle connection transforms under a boost in \( \{ \ell_S, k \} \).

Corollary 1. Let \( S \) as before and for all \( p \in S \) \( k'|_p = \alpha(p)k|_p \) and \( \ell'_S|_p = \frac{1}{\alpha(p)}\ell_S \), where \( \alpha : S \to \mathbb{R} \) is a smooth positive function. Then

\[
s_{\ell'_S}(X') = s_{\ell}(X) - K^k(X, \nabla F) + X(F)Q_k - \frac{1}{\alpha}X(\alpha) \tag{12}
\]

\[
K^{k'}(X', Y') = \alpha K^k(X, Y)
\]

\[
K^{\ell'_S}(X', Y')|_p = \frac{1}{\alpha} \left( K^{\ell}(X, Y) + |DF|^2K^k(X, Y) + 2X(F)s_{\ell}(Y) + 2Y(F)s_{\ell}(X) - 2\text{Hess } F \right.
\]
\[
- 2X(F)K^k(X, \nabla F) - 2Y(F)K^k(Y, \nabla F) + 2Q_kX(F)Y(F) \right).
\]

The trace of \( K^k \) and \( K^\ell \) on \( S_r \) with the induced metric define the null expansions of \( S_r \) and are denoted respectively as \( \theta_k \) and \( \theta_\ell \). The relationship between the null expansions \( \theta_k, \theta_{\ell_S} \) of a graph \( S \) with the corresponding ones at the level set \( S_{r=F(p)} \) follow from Proposition 1.

Corollary 2. Let \( S, k' \) and \( \ell'_S \) as in Corollary 1. The null expansions \( \theta_k' \) and \( \theta_\ell' \) at \( p \in S \) and the null expansions \( \theta_k, \theta_{\ell_S} \) of \( S_{r=F(p)} \) at \( p \) are related by

\[
\theta_k' = \alpha \theta_k \tag{13}
\]

\[
\theta_{\ell'_S} = \frac{1}{\alpha} \left( \theta_\ell + |DF|^2\theta_k + 4s_\ell(\nabla F) - 4K^k(\nabla F, \nabla F) - 2\Delta F + 2Q_k|DF|^2 \right) \tag{14}
\]

where \( \Delta F \) is the Laplacian of \( S_r \) with the induced metric.

Another useful identity that will play a role later is the evolution equation for the connection of the normal bundle. For geodesic flows this identity is known, see e.g. [II]. Although again this case is all we shall need in this paper, we state and prove the result in full generality (i.e. for arbitrary \( k \)).
Proposition 2. With the same notation as above, let $X \in \mathfrak{X}(\Omega)$ be a vector field satisfying 
$[k, X] = 0$ and tangent to $S_0$. Then

\[
k(s_\ell(X)) = -X(Q_k) - s_\ell(X)\theta_k + (\text{div}_{S_r} K^k)(X) - D_X\theta_k - \text{Ein}^g(k, X)
\]

where $\text{Ein}^g$ is the Einstein tensor of $(\mathcal{M}, g)$.

Proof. Since $s_\ell(X) = \frac{1}{2} (\nabla_X k, \ell)$, we compute

\[
k(s_\ell(X)) = \frac{1}{2} \langle \nabla_X \nabla_X k, \ell \rangle + \frac{1}{2} \langle \nabla_X k, \nabla_\ell \theta \rangle
\]

where in the second equality we used the definition of the curvature operator $\text{Riem}^g$ and $[k, X] = 0$ and in the third one we used $\nabla_k k = Q_k k$. For the last term we use the immediate decomposition

\[
\nabla_\ell \theta = -2s^\ell - Q_\ell \theta
\]

where $s^\ell$ is the vector metrically related to $s_\ell$. Hence $\frac{1}{2} \langle \nabla_X k, \nabla_\ell \theta \rangle = -K^k(X, s^\ell) - Q_k s_\ell(X)$ and (16) becomes

\[
k(s_\ell(X)) = -X(Q_k) + \frac{1}{2} \text{Riem}^g(\ell, k, k, X) - K^k(X, s^\ell).
\]

To elaborate this further we use the Codazzi identity applied to $S_r$ along $k$ [24]

\[
\text{Riem}^g(Y, X, Z, k) = (D_Y K^k)(X, Z) - (D_X K^k)(Y, Z) + s_\ell(Y) K^k(X, Z) - s_\ell(X) K^k(Y, Z)
\]

where $X, Y, Z \in \mathfrak{X}(S_r)$ and $D$ is the covariant derivative of $S_r$. Taking trace on $S_r$ in the first and third indices and using that

\[
\text{tr}_{S_r} (\text{Riem}^g(\cdot, X, \cdot, k)) = \text{Ric}^g(X, k) + \frac{1}{2} \text{Riem}^g(k, X, \ell, k) = \text{Ein}^g(X, k) + \frac{1}{2} \text{Riem}^g(k, X, \ell, k),
\]

which follows from the fact that $-\frac{1}{2}(k \otimes \ell + \ell \otimes k)$ is the metric of $(T_p S_r)_{\perp}$, we obtain the contracted Codazzi identity along $k$

\[
\text{Ein}^g(X, k) + \frac{1}{2} \text{Riem}^g(\ell, k, k, X) = (\text{div}_{S_r} K^k)(X) - D_X\theta_k + K^k(s^\ell, X) - s_\ell(X)\theta_k
\]

Eliminating $\text{Riem}^g(\ell, k, k, X)$ in (17)-(18) yields the result.

\[
3 \text{ Null Asymptotic flatness of } \Omega \text{ and asymptotic behaviour}
\]

The previous section involved general properties of $\Omega$ of local nature and valid in any spacetime dimension. We now impose global conditions and restrict to dimension four. First of all we assume that $\Omega$ admits a global cross section $S_0$ (i.e. a smooth embedded spacelike surface intersected precisely once by every inextendible curve along the null generators) of spherical topology $S_0$. 

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We also assume that for one (and hence any) choice of geodesic null generator \( k \) (i.e. satisfying \( \nabla_k k = 0 \)) the corresponding integral curve starting at \( p \in S_0 \) has maximal domain \(( -\infty, \lambda_+(p))\), i.e. the null generators are past complete. After possibly removing portions of \( \Omega \) lying to the future of \( S_0 \) we can assume that \( \Omega \) is foliated by the level sets \( \{ S_r \} \) of the function \( r \in F(\Omega) \) defined by \( k(r) = -1, \ r|_{S_r} = r_0 \) and that all these level sets are diffeomorphic to \( S_0 \) (so that in particular \( \Omega = S_0 \times (R_\infty, \infty)) \). The function \( r \) is called level set function of \( k \). If we change the selection of null geodesic generator \( k \), the set of points to be removed is different, but since we are only interested in the past of \( S_0 \) this is irrelevant, and we keep the same name \( \Omega \). A null hypersurface \( \Omega \) satisfying these properties is called extending to past null infinity.

In order to define asymptotic flatness we need to impose decay of various objects at infinity. First note that covariant tensor fields \( T \) on \( \Omega \) completely orthogonal to \( k \) (i.e. satisfying \( T(k, \cdots) = T(\cdots, k) = 0 \)) are in one-to-one correspondence with smooth collections of covariant tensor fields \( T(r) \) on each level set \( S_r \). We call such tensors transversal. An immediate example is the first fundamental form \( K^K_k \) defines a transversal tensor denoted simply by \( K^K \) (this is compatible with the notation already used in the previous section). Similarly we have the transversal tensors \( K^\ell \) and \( s^\ell \). A transversal tensor field \( T \) is positive definite iff \( T(r) \) is positive definite for all \( r \). A transversal tensor field \( T \) is called Lie constant iff \( \mathcal{L}_k T = 0 \).

A vector field \( X \in \mathfrak{X}(S_0) \) can be extended uniquely to \( \mathfrak{X}(\Omega) \) by \( [k, X] = 0 \). A local basis \( \{ X_A \} \) on \( S_0 \) extended this way defines a basis of each level set \( S_r \). By \( \{ X_A \} \) we always mean any such basis. A transversal tensor field \( T \) on \( \Omega \) is \( O(1) \) iff \( T_{A_1 \cdots A_q} := T(X_{A_1}, \cdots, X_{A_q}) \) is uniformly bounded. We write \( T = O_n(r^{-q}), \ q \in \mathbb{R}, \ n \in \mathbb{N} \) iff

\[
r^q T = O(1), \quad r^{q+1} \mathcal{L}_k T = O(1), \quad \cdots, \quad r^{q+n} \prod_{k=1}^n \mathcal{L}_k T = O(1).
\]

We also write \( T = o(r^{-q}) \) iff \( \lim_{r \to \infty} r^q T(r)_{A_1 \cdots A_q} = 0 \) and \( T = o_n(r^{-q}) \) iff \( r^{i+q}(\mathcal{L}_k)^i T = o(1) \) for all \( i = 0, 1, \cdots, n \). Given a transversal tensor (field) \( T \) the tensor \( \mathcal{L}_{X_A} T \) is also transversal. We write \( T = o_n^X(r^{-q}) \) iff

\[
r^q \prod_{A_1} \prod_{A_2} \mathcal{L}_{X_{A_1}} \cdots \mathcal{L}_{X_{A_2}} T = o(1) \quad \forall i = 0, 1, \cdots, n
\]

It is clear that all these definitions are independent of the choice of \( \{ X_A \} \).

**Definition 1.** Let \( (\mathcal{M}, g) \) be a four-dimensional spacetime. A null hypersurface \( \Omega \) is past asymptotically flat if it extends to past null infinity and there exists a choice of cross section \( S_0 \) and null geodesic generator \( k \) with corresponding level set function \( r \) with the following properties:

(i) There exist two symmetric 2-covariant transversal and Lie constant tensor fields \( \hat{q} \) and \( h \) such that \( \hat{\gamma} := \gamma - (r - r_0)^2 \hat{q} - (r - r_0)h \) is \( \hat{\gamma} = o_1(r) \cap o_2^X(r) \)

(ii) There exists a transversal, Lie constant one-form \( s^{(1)}_\ell \) such that \( \hat{s}_\ell := s_\ell - \frac{s^{(1)}_\ell}{r - r_0} \) is \( \hat{s}_\ell = o_1(r^{-1}) \).

(iii) There exist Lie constant functions \( \theta^{(0)}_\ell \) and \( \theta^{(1)}_\ell \) such that \( \hat{\theta}_\ell := \theta_\ell - \frac{\theta^{(0)}_\ell}{r - r_0} - \frac{\theta^{(1)}_\ell}{(r - r_0)^2} \) is \( \hat{\theta}_\ell = o(r^{-2}) \).

(iv) The scalar \( Riem^g(X_A, X_B, X_C, X_D) \) along \( \Omega \) is such that \( \lim_{r \to \infty} \frac{1}{r^2} Riem^g(X_A, X_B, X_C, X_D) \) exists while its double trace satisfies \( 2Ein^g(k, \ell) - Scal^g - \frac{1}{2} Riem^g(\ell, k, \ell, k) = o(r^{-2}) \).
This definition of asymptotic flatness is weaker than most existing definitions in the literature. Sometimes it will be convenient to supplement it with a stronger notion where additional decay for some components of the Einstein tensor and for the remainder tensor $\tilde{\gamma}$ is assumed. Specifically, we say that an asymptotically flat null hypersurface $\Omega$ satisfies the energy flux decay condition if

$$\text{Ein}^g(\ell, X_A)|_\Omega = o(r^{-2}), \quad \mathcal{L}_k \tilde{\gamma} = o^X(1).$$

The name is motivated by the analogous role of density flux that the Einstein tensor component $\text{Ein}^g(\ell, X_A)$ plays in the constraint equations for null hypersurfaces (see e.g. [20]).

Given a past asymptotically flat null hypersurface $\Omega$ with a choice of level set function $r$, it is convenient to define $\tau := r - r_0$. The following Proposition determines the asymptotic expansion of $K^k$ and provides an explicit expression for $\theta^{(1)}$.

**Proposition 3.** Let $\Omega$ be a past asymptotically flat null hypersurface with a choice of affinely parametrized null generator $k$ and corresponding level set function $r$. Let $\gamma(r)^{AB}$ be the inverse of $\gamma(r)$. Then

$$\gamma(r)^{AB} = \frac{1}{r^2} \hat{q}^{AB} - \frac{1}{r^3} \hat{h}^{AB} + o(r^{-3}),$$

$$K^k_{AB} = -\hat{q}_{AB} \tau - \frac{1}{2} h_{AB} + o(1),$$

$$\theta_\ell = \frac{2K^k}{\tau} + \frac{\theta^{(1)}_\ell}{\tau^2} + o(r^{-2}).$$

$\hat{q}^{AB}$ is the inverse of $\hat{q}_{AB}$, indices in hatted tensors are raised and lowered with these metrics and $K_{\hat{q}}$ is the Gauss curvature of $\hat{q}_{AB}$.

**Proof.** Expression (19) is an immediate consequence of item (i) in the definition of asymptotic flatness, namely

$$\gamma_{AB} = \hat{q}_{AB} \tau^2 + h_{AB} \tau + o_1(\tau) \cap o_2^X(\tau).$$

For expression (20) we use the standard identity

$$k(\gamma_{AB}) = k(X_A, X_B) = \langle \nabla_k X_A, X_B \rangle + \langle X_A, \nabla_k X_B \rangle = 2K^k_{AB},$$

the last step following from $[k, X_A] = 0$. Inserting (22) yields (20) immediately. Concerning the expansion for $\theta_\ell$ we invoke the Gauss identity for $(S, \gamma(r))$, namely [24]

$$\text{Riem}^g(X_A, X_B, X_C, X_D) = \text{Riem}^{(r)}_{ABCD} + \langle K_{BC}, K_{AD} \rangle - \langle K_{BD}, K_{AC} \rangle$$

where $K$ is the second fundamental form vector. Decomposing $K = -\frac{1}{2}(K^k\ell + K^\ell k)$ we have

$$\text{Riem}^g(X_A, X_B, X_C, X_D) = \text{Riem}^{(r)}_{ABCD} - \frac{1}{2}(K^k_{BC} K^\ell_{AD} + K^k_{AD} K^\ell_{BC}) + \frac{1}{2}(K^k_{AC} K^\ell_{BD} + K^k_{BD} K^\ell_{AC}),$$

The decomposition (22) implies that $\text{Riem}^{(r)}_{ABCD} = \tau^2 \text{Riem}^g_{ABCD} + O(\tau)$ and given that $K^k_{AB} = -\hat{q}_{AB} \tau + O(1)$, it follows from item (iv) in Definition 1 that $K^k_{AB}$ is of the form

$$K^\ell_{AB} = \tau K^{\ell}_{(0)AB} + o(\tau).$$
with $K^\ell_{(0)}$ a transverse Lie constant symmetric tensor on $\Omega$. Taking trace in the $AC$ and $BD$ indices in (23) and using the fact that $g^\flat = \gamma(r)^{AB}X_A \otimes X_B - \frac{1}{r}(k \otimes \ell + \ell \otimes k)$ yields

$$2\text{Ein}^g(k, \ell) - \text{Scal}^g - \frac{1}{2}\text{Riem}^g(\ell, k, \ell, k) = 2\mathcal{K}_{\gamma(r)} + \theta_k \theta_k - K^k_{AB}K^\ell_{AB}.$$ 

Now, $\mathcal{K}_{\gamma(r)} = \frac{1}{r^2}K_q + o(r^{-2})$ and the decompositions (19)-(20), (24) and the trace condition in item (iv) of Definition 1 imply

$$0 = \frac{1}{r^2} \left(2\mathcal{K}_{\hat{q}} - \theta^{(0)}_k\right) + o(r^{-2})$$

which, together with item (iii) in Definition 1 proves (21).

**Remark 2.** Note that the expansion for $K^k$ only depends on item (i) in the definition of asymptotic flatness. The expression for $\theta^{(0)}_k$ depends on items (i), (iii) and (iv).

**Remark 3.** We can raise the index to the tensor $K^k(r)$ with the contravariant metric $\gamma^{AB}$. Combining the asymptotic expansions (19) and (20) yields

$$K^k(r)^A_B = -\frac{1}{r}\delta^A_B + \frac{1}{2}\hat{h}^A_B \frac{1}{r^2} + o(r^{-2})$$

and taking trace

$$\theta_k = -\frac{2}{r} + \frac{1}{2}(\text{tr}_q \hat{h}) \frac{1}{r^2} + o(r^{-2}) := -\frac{2}{r} + \theta^{(1)}_k \frac{1}{r^2} + o(r^{-2}).$$

It will be convenient to endow each level set $\{S_r\}$ with a covariant derivative independent of $r$. The natural choice is $\hat{q}$ which is Lie constant, a metric on each $S_r$ and gives the leading term of the asymptotic expansion of $\gamma(r)$. Denote by $\hat{D}$ the covariant derive of $\hat{q}$. In the following lemma we find asymptotic expansion of the difference tensor

$$D_X Y - \hat{D}_X Y = Q(X, Y)$$

and apply it to relate the Laplacians in the metrics $\gamma(r)$ and $\hat{q}$ of a function. This will be needed later when relating two different foliations on $\Omega$.

**Lemma 2.** The difference tensor $Q$ admits the decomposition

$$Q^C_{AB} = \frac{1}{2}(\hat{D}_A \hat{h}^C_B + \hat{D}_B \hat{h}^C_A - \hat{D}^C h_{AB}) \frac{1}{r^2} + O(r^{-2}).$$

Moreover, if $F$ is a Lie constant function on $\Omega$ then

$$\triangle_\gamma F = \triangle_\hat{q} F \frac{1}{r^2} + \left(-\hat{h}^{AB} \hat{D}_A \hat{D}_B F - (\hat{D}_A \hat{h}^{CA})F_C + (\hat{D}^C \theta^{(1)}_k)F_C\right) \frac{1}{r^3} + o(r^{-3})$$

**Proof.** We use the general formula for the difference tensor of Levi-Civita covariant derivatives, see e.g. [27],

$$Q^C_{AB} = \frac{1}{2}\gamma^{CD}(\hat{D}_A \gamma_{DB} + \hat{D}_B \gamma_{DA} - \hat{D}_D \gamma_{AB}).$$
Given that \( \hat{D}_A \gamma_{DB} = (\hat{D}_A h_{DB}) \tau + O(1) \) and \( \gamma^{CD} = \frac{1}{r^2} q^{CD} + O(\tau^{-3}) \), expression (27) follows. For the Laplacian we use

\[
\triangle \gamma F = \gamma^{AB} D_A D_B F = \gamma^{AB} \left( \hat{D}_A \hat{D}_B F - Q^{C}_{AB} \hat{D}_C F \right)
\]

\[
= \left( \frac{1}{r^2} q^{AB} - \frac{1}{r^2} \hat{h}^{AB} + o(\tau^{-3}) \right) \left( \hat{D}_A \hat{D}_B F - \frac{1}{2} (\hat{D}_A \hat{h}^C B + \hat{D}_B \hat{h}^C A - \hat{D}^C h_{AB}) F_C \right) \frac{1}{r^2} + O(\tau^{-2})
\]

\[
= \frac{1}{r^2} \triangle q F + \left( -\hat{h}^{AB} \hat{D}_A \hat{D}_B F - (\hat{D}_A \hat{h}^C A) F_C + \frac{1}{2} \hat{D}^C (\hat{q}^{AB} h_{AB}) F_C \right) \frac{1}{r^2} + o(\tau^{-3})
\]

which is (28) after recalling that \( \text{tr} \hat{h} = 2 \theta_k^{(1)} \). \( \square \)

We conclude this section by showing that the leading term \( s_{\ell}^{(1)} \) of \( s_\ell \) is fully determined in terms of the rest of objects whenever the energy flux decay condition is assumed.

**Proposition 4.** Let \( \Omega \) be a past asymptotically flat null hypersurface and assume that the energy flux decay condition holds. Then

\[
s_{\ell}^{(1)} A = \hat{D}_A \theta_k^{(1)} - \frac{1}{2} \hat{D}_B \hat{h}^B A.
\]

**Proof.** From item (ii) in Definition 1 and the decomposition (26) we have

\[
\mathcal{L}_k s_A = \frac{s_{\ell}^{(1)} A}{r^2} + o(r^{-2}), \quad \theta_k s_{\ell} A = -\frac{2s_{\ell}^{(1)} A}{r^2} + o(r^{-2}) \quad \text{and} \quad \hat{D}_A \theta_k = \frac{\hat{D}_A \theta_k^{(1)}}{r^2} + o(r^{-2}) \quad (29)
\]

Concerning the divergence of \( K_{B}^{kA} \) in the metric \( \gamma(r) \) we use that the leading term in (25) is covariantly constant and then replace the \( D \)-covariant derivate by the \( \hat{D} \)-derivative and use \( Q_{AB} = O(\tau^{-1}) \) to obtain

\[
D_B K_{C}^{kB} = \frac{1}{2r^2} \hat{D}_B \hat{h}^B C + o(\tau^{-2}) = \frac{1}{2r^2} \hat{D}_B \hat{h}^B C + o(\tau^{-2}).
\]

Thus, identity (15) (with \( Q_k = 0 \)) becomes

\[
\frac{1}{r^2} \left( s_{\ell}^{(1)} A + \frac{1}{2} \hat{D}_B \hat{h}^B A - \hat{D}_A \theta_k^{(1)} \right) + o(\tau^{-2}) = 0
\]

and the result follows. \( \square \)

### 4 Background foliation approaching large spheres

As discussed in the introduction, the Hawking energy has the interesting and well-known property of approaching the Bondi energy when the surfaces approach large spheres in a suitable sense. Our general limiting expressions for the Hawking energy will of course have to recover this fact. To that aim, it is useful to restrict the choice of affinely parametrized null generator \( k \) and corresponding level set function \( r \) so that the geometry of the level sets \( S_r \) approaches, after rescaling, the standard metric of unit radius on the sphere, denoted by \( \hat{q} \).
**Definition 2.** Let Ω be null and past asymptotically flat with a choice of affine null generator k and level set function r. The foliation \{S_r\} is said to approach large spheres iff the leading term \(\hat{\eta}\) in the expansion (22) of γ is the standard metric of a unit two-sphere.

Our definition of approaching large spheres is equivalent to demanding that the rescaled metric \(\frac{1}{s} \gamma(r)\), has a limit \(\hat{\eta}\) when \(r \to \infty\). In [26], the definition of approaching large spheres is defined more generally for exhaustions \{S_s\} of Ω with all elements diffeomorphic to each other by demanding that the rescaled metric \(\frac{1}{s} \hat{\gamma}_s\) has a limit when \(s \to \infty\) and defines a metric \(\hat{\eta}\) of constant unit curvature. It is clear that both definitions agree for the geodesic foliations \{S_r\} that we are using in this paper.

Our aim is to consider very general exhaustions \{S_{r'}\} on Ω and obtain the limit of the Hawking energy along them by referring all objects to an affine background foliation approaching large spheres. It is important to note that, since all Riemannian metrics on a manifold \(\simeq \mathbb{S}^2\) are conformal to the standard metric \(\hat{\eta}\), there always exists a (non-unique) choice of affine null generator k in an asymptotically flat Ω with corresponding background foliation \{S_r\} approaching large spheres (cf. Remark [4] below). Tensors raised and lowered with the metric \(\hat{\eta}_{AB}\) and its inverse \(\hat{\eta}^{AB}\) will have a circle on top, so that for instance (31) reads

\[
\gamma(r)^{AB} = \frac{1}{r^2} \hat{\eta}^{AB} - \frac{1}{r^3} \hat{H}^{AB} + o(r^{-3}).
\]

Note also that since \(\hat{\eta}\) has constant unit curvature, the null expansion \(\theta_{\ell}\) has asymptotic behaviour

\[
\theta_{\ell} = \frac{2}{r} + \frac{\theta_{\ell}^{(1)}}{r^2} + o(r^{-2}).
\]

We will consider three types of foliations \{S_{r'}\} and then combine them to obtain the general case treated in Theorem [1]. Given a null basis \{k', \ell'\} orthogonal to a section S in Ω and satisfying \(\langle k', \ell' \rangle = -2\), the mean curvature H of S decomposes as \(H = -\frac{1}{2}(\theta_{k'}\ell' + \theta_{\ell'}k')\) and the Hawking energy is

\[
m_H(S) = \sqrt{\frac{|S|}{16\pi}} \left(1 + \frac{1}{16\pi} \int_S \theta_{k'}\theta_{\ell'}\eta_{S'}\right)
\]

so our aim will be to compute the limit of the areas \(|S_{r'}|\) and of \(\theta_{k'}\theta_{\ell'}\eta_{S_{r'}}\) of the new foliation \{S_{r'}\} in terms of the background foliation geometry. The following section deals with the case when \{S_{r'}\} is any geodesic foliation (not necessarily approaching large spheres).

### 5 Limit of the Hawking energy for geodesic foliations

In this section we assume that Ω is null and past asymptotically flat and endowed with a foliation \{S_r\} associated to an affinely parametrized null generator k and approaching large spheres. By definition, a geodesic foliation in Ω is a foliation \{S_{r'}\} by cross sections with defining function \(r' \in \mathcal{F}(\Omega)\) such that the (unique) null generator \(k'\) satisfying \(k'(r') = -1\) is affinely parametrized. Thus, there exists a positive function \(\phi \in \mathcal{F}(\Omega)\), Lie constant along k such that \(k' = \phi k\). Consequently, the level set functions \(r\) and \(r'\) are necessarily related by \(r = \tau + \phi(r' - r_0)\) where \(\tau \in \mathcal{F}(\Omega)\) is a Lie constant function. We first consider the case when the two foliations \{S_r\} and \{S_{r'}\} have the same starting surface, i.e. that \(S_{r'=r_0} = S_{r=r_0}\), which fixes \(\tau = r_0\). Thus, each of the surfaces \{S_{r'}\} can be described by a graph function \(r = r_0 + \phi(r' - r_0)\).
Our strategy is to use the general expressions in Section 2 for the geometry of a graph \( r = F \) in a background foliation, insert the resulting expressions in (32) and take the limit when \( r' \to \infty \). We start with the following Proposition.

**Proposition 5.** Let \( \Omega \) be a past asymptotically flat null hypersurface endowed with an affinely parametrized background foliation \( \{ S_r \} \) with generator \( k \) and approaching large spheres. Let \( \phi > 0 \) be a Lie constant function and define the geodesic foliation \( \{ S_{r'} \} \) by the graph functions

\[
r = F_{r'} := r_0 + \phi r', \quad r' := r' - r_0.
\]

Let \( k' = \phi k \) and \( \ell' \) be the null normal orthogonal to \( \{ S_{r'} \} \) satisfying \( \langle k', \ell' \rangle = -2 \). Then the induced metric \( \gamma(r') \), volume form \( \eta_{S_{r'}} \), null expansions \( \theta_{k'}, \theta_{\ell'} \) and the connection of the normal bundle \( s_{r'} \) can be expressed in terms of the background geometry as

\[
\gamma_{AB} := \gamma'(X_A', X_B') = \phi^2 q_{AB\tau^2} + \phi h_{AB\tau} + o(\tau^2) \tag{33}
\]

\[
\eta_{S_{r'}} = \left( \phi^2 \tau^2 + \phi \theta_k(1) \tau + o(\tau) \right) \eta_q \tag{34}
\]

\[
\theta_{k'} = -\frac{2}{\phi} + \frac{\theta_k(1)}{\phi} + o(\tau^{-2}) \tag{35}
\]

\[
\theta_{\ell'} = -\frac{2}{\phi^2}(\Delta_q \log \phi - 1) + \frac{1}{\phi^3} + \left( \frac{\theta_k(1)}{\phi^5} - \frac{4h_{AB} \phi A \phi B}{\phi^4} + \frac{\theta_k(1)|\nabla \phi|_q^2}{\phi^5} + \frac{4q_{AB} \phi_A (s_B)}{\phi^4} \right)
\]

\[
+ 2h_{AB} \hat{\nabla}_A \hat{\nabla}_B \phi \frac{1}{\phi^4} + 2\hat{\nabla}_A \hat{\nabla}_B \phi \frac{1}{\phi^4}
\]

\[
+ 2\hat{\nabla}_A \hat{\nabla}_B \phi \frac{1}{\phi^4} + 2\hat{\nabla}_A \hat{\nabla}_B \phi \frac{1}{\phi^4}
\]

\[
\left( \frac{s_k(1)}{\phi} - \frac{\phi L A}{2\phi^2} \right) \frac{1}{\phi^3} + o(\tau^{-1}) \tag{36}
\]

\[
s_{r'} = \frac{\phi}{\phi^2} \frac{L A}{\phi^3} \frac{1}{\phi^3} + o(\tau^{-1}) \tag{37}
\]

where \( X_A' := X_A - X_A(F_{r'})k \).

**Proof.** As already mentioned \( k' \) is the generator of the foliation \( \{ S_{r'} \} \). For any point \( p \in S_{r'} \), the level set passing through \( p \) has \( \tau = F_{r'}(p) - r_0 = \phi(p) \tau' \). Thus \( \tau \) and the background expansion \( \hat{q} \rightarrow \hat{q} \) gives

\[
\gamma_{AB}'|_{p} = \gamma_{AB}|_{r=r_0+\phi(p)\tau'} = \hat{q}_{AB\tau^2} + h_{AB\tau} + \hat{\gamma}|_{\tau=\phi\tau'}
\]

which is (33). (34) follows by taking determinants and using the standard identity

\[
\det(M + sB) = (\det M)(1 + s \text{ tr } (M^{-1}B) + O(s^2)),
\]

valid for any invertible matrix \( M \). For \( \theta_{k'} \) we use the fact that \( \theta_k \) is a property of \( \Omega \) and not of the surface embedded in \( \Omega \) passing through that point, cf. Corollary 2. Thus

\[
\theta_{k'} = \phi \theta_k|_{\tau=\phi\tau'} = -\frac{2}{\phi} + \frac{\theta_k(1)}{\phi} + \frac{1}{\phi^2} + o(\tau^{-2}), \tag{38}
\]

as claimed in (35). To compute \( \theta_{\ell'} \) we use expression (14) with \( Q_k = 0, a = \phi \) and graph function \( F = F_{r'} = r_0 + \phi \tau' \). As before, the right-hand side has to be evaluated at \( \tau = \phi \tau' \). We work out
each term separately:

\[
\begin{align*}
\theta_t &= \frac{2}{\rho} + \frac{\theta_t^{(1)}}{\rho^2} + o(\rho^{-2}) = \frac{2}{\phi} + \frac{\theta_t^{(1)}}{\phi^2} + o(\rho^{-2}), \\
\theta_k |D F_r|^2 &= \theta_k \gamma^{AB} \nabla_A F_r \nabla_B F_r \\
&= \left( -\frac{2}{\phi} + \frac{\theta_k^{(1)}}{\phi^2} + o(\rho^{-2}) \right) \left( \frac{\hat{q}^{AB}}{\phi^2} \frac{1}{\rho^2} - \frac{\hat{h}^{AB}}{\phi^3} + o(\rho^{-4}) \right) (\phi, \theta_F)(\phi, \theta_F) \\
&= -\frac{2|\nabla \phi|^2}{\phi^3} \frac{1}{\rho} + \left( \frac{2\hat{h}^{AB} \phi_{,A} \phi_{,B}}{\phi^4} + \frac{\theta_k^{(1)} |\nabla \phi|^2}{\phi^3} \right) \frac{1}{\rho^2} + o(\rho^{-2}).
\end{align*}
\]

For the Laplacian of \(F_r\) we use (28) which gives, using \(\nabla F_r = (\nabla \phi)^{\prime} \), \(\text{Hess}_q F_r = (\text{Hess}_q \phi)^{\prime} \) and \(\Delta_q F_r = (\Delta_q \phi)^{\prime} \),

\[
\Delta \gamma F_r = \frac{\Delta_q \phi}{\phi^2} \frac{1}{\rho} + \left( \frac{-\hat{h}^{AB} \nabla_A \nabla_B \phi}{\phi^3} - \frac{\nabla_A \hat{h}^{CA} \phi_{,C}}{\phi^3} + \frac{\nabla_C \theta_k^{(1)} (\phi_{,C})}{\phi^3} \right) \frac{1}{\rho^2} + o(\rho^{-2}).
\]

For the term \(s_\ell(\text{grad } F_r)\)

\[
\begin{align*}
s_\ell(\text{grad } F_r) &= \gamma^{AB} s_{\ell A} \nabla_B F_r = \left( \frac{\hat{q}^{AB}}{\phi^2} \frac{1}{\rho^2} + o(\rho^{-2}) \right) \left( \frac{s_{\ell A}}{\phi} \frac{1}{\rho} + o(\rho^{-1}) \right) \phi, \theta_F \\
&= \frac{\hat{q}^{AB} s_{\ell A}}{\phi^3} \frac{\phi_{,B}}{\rho^2} + o(\rho^{-2})
\end{align*}
\]

and finally \(K^k(\text{grad } F_r, \text{grad } F_r)\) is, inserting (19) and (20),

\[
\begin{align*}
K^k(\text{grad } F_r, \text{grad } F_r) &= \gamma^{AB} \gamma^{CD} K^k_{BD} \phi_{,A} \phi_{,B} \rho^{2} = \frac{1}{\phi^3 \rho^2} \left( \hat{q}^{AB} - \frac{1}{\phi^2} \hat{h}^{AB} + o(\rho^{-1}) \right) \times \left( \hat{q}^{CD} - \frac{1}{\phi^2} \hat{h}^{CD} + o(\rho^{-1}) \right) \left( -\hat{q}_{BD} \phi^{\prime} - \frac{1}{2} \hat{h}_{BD} + o(1) \right) \phi, \theta_F \\
&= -\frac{|\nabla \phi|^2}{\phi^3} \frac{1}{\rho} + \frac{3 \hat{h}^{LA} \phi_{,A} \phi_{,L}}{2 \phi^4} \frac{1}{\rho^2} + o(\rho^{-2}).
\end{align*}
\]

Putting things together leads to (36) after using \(\Delta_q \phi = |\nabla \phi|^2 \frac{1}{\phi^2} = \Delta_q \log \phi\).

Finally, if we substitute in expression (12) the asymptotic expansion of item (ii) in Definition (11) and the expansion of \(K^k\) of (25), we have

\[
\begin{align*}
s_{\ell A}^{(1)} &= -\frac{\phi_{,A}}{\phi} + \frac{\phi_{,A}}{\phi} \frac{s_{\ell A}}{\phi} \frac{1}{\rho} + \frac{\phi_{,A} \hat{h}^{LL}}{\phi} \frac{1}{\rho^2} + o(\rho^{-1}) = \left( \frac{s_{\ell A}}{\phi} - \frac{\phi_{,A} \hat{h}^{LL}}{2 \phi^2} \right) \frac{1}{\rho} + o(\rho^{-1}),
\end{align*}
\]

where the term in \(\frac{1}{\rho}\) is \(s_{\ell A}^{(1)}\).
Remark 4. The foliation \( \{ S_r \} \) has limit metric \( \lim_{r' \to \infty} \theta_r' = \phi^2 q := \hat{q} \). Using the formula for the scalar curvature of a conformal metric it follows

\[
K_{\hat{q}} = \frac{1 - \Delta_{\hat{q}} \log \phi}{\phi^2}
\]

so that the expansion of \( \theta' \) is \( \theta' = \frac{2K_{\hat{q}}}{\phi} + o(\mathcal{F}^{-1}) \) in agreement with Proposition 3. Note that the transformation law \( \hat{q} \to \phi^2 q' \) for the leading term in the metric \( \gamma(r) \) under change of foliation \( k' = \phi k \) holds irrespectively of whether the background foliation approaches large spheres or not. Since as mentioned above, any metric on \( S^2 \) is conformal to \( \hat{q} \), it follows that any asymptotically flat \( \Omega \) admits a background foliation approaching large spheres.

We can now evaluate the limit of the Hawking energy for the foliation \( \{ S_r \} \):\

**Theorem 2.** Let \( \Omega \) be a past asymptotically flat null hypersurface endowed with an affinely parametrized background foliation \( \{ S_r \} \) with generator \( k \) and approaching large spheres. Let \( \Psi > 0 \) be a Lie constant function and define the geodesic foliation \( \{ S_r \} \) by the graph functions

\[
r = F_r := r_0 + \frac{1}{\Psi'} \theta', \quad \mathcal{F} := r' - r_0
\]

The limit of the Hawking energy along the foliation \( \{ S_r \} \) is, in terms of the background geometry,

\[
\lim_{r' \to \infty} m_H(S_r) = \frac{1}{8\pi \sqrt{16\pi}} \left( \int_{S^2} \frac{1}{\Psi^2} \eta_q \right) \int_{S^2} \left( \Delta_q \theta_k^{(1)} - (\theta^{(1)}_k + \theta^{(1)}_\ell) - 4 \text{div}(s_k^{(1)}) \right) \Psi \eta_q
\]

**Proof.** Set \( \phi = \Psi^{-1} \) so that we can use the expressions in Proposition 3. We need to compute \( \theta_k \theta_{\ell} \eta_{S_{r'}} \). Denoting by \( \theta^{(1)}_k \) the coefficient of the term \( \frac{1}{\mathcal{F}} \) in \( \theta' \) we immediately find, from (34) and (36),

\[
\theta_k \theta_{\ell} \eta_{S_{r'}} = \left( 4(\Delta_{\hat{q}} \log \phi - 1) + (-2K_{\hat{q}} \theta_k^{(1)} \phi - 2\theta_{\ell}^{(1)} \phi^2) \frac{1}{\mathcal{F}} + o(\mathcal{F}^{-1}) \right) \eta_{\hat{q}}
\]

and hence

\[
1 + \frac{1}{16\pi} \int_{S_{r'}} \theta_k \theta_{\ell} \eta_{S_{r'}} = 1 + \frac{1}{16\pi} \int_{S^2} \left( 4(\Delta_{\hat{q}} \log \phi - 1) + (-2K_{\hat{q}} \theta_k^{(1)} \phi - 2\theta_{\ell}^{(1)} \phi^2) \frac{1}{\mathcal{F}} + o(\mathcal{F}^{-1}) \right) \eta_{\hat{q}}
\]

\[
= \frac{1}{16\pi} \int_{S^2} \left( -2K_{\hat{q}} \theta_k^{(1)} \phi - 2\theta_{\ell}^{(1)} \phi^2 \right) \eta_{\hat{q}} + o(\mathcal{F}^{-1}).
\]

Concerning the area term

\[
|S_r| = \int_{S_{r'}} \eta_{S_{r'}} = \left( \int_{S^2} \phi^2 \eta_{\hat{q}} \right) \mathcal{F}^2 + O(\mathcal{F}) \implies \sqrt{|S_r|} = \mathcal{F} \int_{S^2} \phi^2 \eta_{\hat{q}} + O(1)
\]

so that the limit of the Hawking energy is

\[
\lim_{r' \to \infty} m_H(S_r) = \frac{1}{16\pi \sqrt{16\pi}} \left( \int_{S^2} \phi^2 \eta_{\hat{q}} \right) \int_{S^2} \mathcal{F} \eta_{\hat{q}}.
\]

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We now compute $\mathcal{I}$. Using the explicit form of $\mathcal{K}_q$ and $\theta_{\ell}^{(1)}$ it follows

$$\mathcal{I} = \frac{2\theta_k^{(1)}}{\phi^2} \bigtriangleup_q \phi - \frac{4\theta_k^{(1)}}{\phi^3} |\nabla \phi q|^2 - \frac{2}{\phi} (\theta_{k}^{(1)} + \theta_{\ell}^{(1)}) + \frac{8\dot{h}^{AB}\phi_A\phi_B}{\phi^3} - \frac{8\dot{q}^{AB}\phi_A s_{\ell}^{(1)}}{\phi^2}$$

$$- \frac{4\dot{h}^{AB}\dot{\nabla}_A \dot{\nabla}_B \phi}{\phi^2} - \frac{4\nabla_A \dot{h}^{AB} \phi_B}{\phi^2} + \frac{4\nabla_B \theta_{(1)}^{(1)} \phi_B}{\phi^2}$$

$$\nabla_A \left( - \frac{4}{\phi^2} \dot{h}^{AB} \phi_B + \frac{2\theta_k^{(1)}}{\phi^2} \nabla_A \phi + \frac{8s_{\ell}^{(1)}}{\phi} \right) + \frac{2}{\phi} \nabla_A \theta_{(1)}^{(1)} \phi_A - \frac{2}{\phi} (\theta_{k}^{(1)} + \theta_{\ell}^{(1)}) - \frac{8}{\phi} \text{div}_q (s_{\ell}^{(1)})$$

$$= \nabla_A \left( - \frac{4}{\phi^2} \dot{h}^{AB} \phi_B + \frac{2\theta_k^{(1)}}{\phi^2} \nabla_A \phi + \frac{8s_{\ell}^{(1)}}{\phi} \right) - \frac{2}{\phi} \Delta_q \theta_{(1)}^{(1)} - \frac{2}{\phi} (\theta_{k}^{(1)} + \theta_{\ell}^{(1)}) - \frac{8}{\phi} \text{div}_q (s_{\ell}^{(1)})$$

and (41) becomes (40) after using the Gauss identity and $\phi = \Psi^{-1}$.

Remark 5. It is interesting that all terms involving derivatives of $\phi$ (or $\Psi$) combine themselves into a divergence and drop out after integration. The behaviour of the limit of the Hawking energy under change of geodesic foliation is hence much simpler than one might have expected a priori.

Given a past asymptotically flat null hypersurface, there are many possible choices of geodesic background foliations approaching large spheres. Any two such foliations are related by $\Omega = \phi \phi'$ with $\phi$ satisfying

$$\Delta_q \log \phi + \phi^2 = 1$$

so that the Gaussian curvature (39) of $\hat{q}$ is also one. In this case the limit of the Hawking energy of the foliation $\{S_{r'}\}$ can be computed in two different ways, namely refering $\{S_{r'}\}$ to the background foliation $\{S_{r}\}$ and using Theorem 2 or considering $\{S_{r'}\}$ itself as a background foliation (so that the result would be (40) with $\Psi = 1$ and $\theta_{k}^{(1)}$, $\theta_{\ell}^{(1)}$, $s_{\ell}^{(1)}$ all referred to the foliation $\{S_{r'}\}$). It is clear that both results must agree. This requires a kind of covariance property of the integral in (40). Remarkably, this covariance occurs already at the level of the integrand, as we show next. All geometric objects referred to the geodesic foliation $\{S_{r'}\}$ will carry a prime.

Theorem 3. Let $\Omega$ be a past asymptotically flat null hypersurface endowed with an affinely parametrized background foliation $\{S_{r}\}$ with generator $k$ and approaching large spheres. Let $\phi > 0$ be a Lie constant function and define the geodesic foliation $\{S_{r'}\}$ by the graph functions

$$r = F_{r'} := r_0 + \phi \phi', \quad \phi' := r' - r_0,$$

with $\phi$ satisfying (42). Then

$$\left( \Delta_q \theta_{k}^{(1)} - (\theta_{k}^{(1)} + \theta_{\ell}^{(1)}) - 4 \text{div}_q s_{\ell}^{(1)} \right) \frac{1}{\phi} \eta_q = \left( \Delta_q' \theta_{k}^{(1)}' - (\theta_{k}^{(1)}' + \theta_{\ell}^{(1)}') - 4 \text{div}_q s_{\ell}^{(1)} \right) \eta_q,$$

As a consequence we have the necessary invariance of the limit of the Hawking energy

$$\lim_{r' \to \infty} m_H (S_{r'}) = \frac{-1}{16\pi} \int_{S^2} \left( (\theta_{k}^{(1)} + \theta_{\ell}^{(1)}) \right) \eta_q = \frac{1}{16\pi} \int_{S^2} \left( \Delta_q \theta_{k}^{(1)} - (\theta_{k}^{(1)} + \theta_{\ell}^{(1)}) - 4 \text{div}_q s_{\ell}^{(1)} \right) \frac{1}{\phi} \eta_q.$$
Proof. The general expressions \((33)-(35)\) give the explicit form of the geometric objects of \(\{S_r\}\) in terms of the background foliation \(\{S_r\}\), namely

\[
q'_{AB} = \phi^2 q_{AB}, \quad \theta_k^{(1)'} = \frac{\theta_k^{(1)}}{\phi}, \quad \theta_{\ell}^{(1)'} = \frac{\theta_{\ell}^{(1)}}{\phi^3} - \frac{4 \hat{h}^A B \phi A \phi_B}{\phi^5} + \frac{\theta_k^{(1)} |\nabla \phi|^2}{\phi^5} + \frac{4 q_{AB} \phi A (s_{B})_1}{\phi^3} + 2 \frac{\hat{h}^A \nabla_A \hat{h}^A \phi}{\phi^4} + 2 \frac{\nabla^C \theta_{(k)} \phi_C}{\phi^4} + s_{(1)}^{(1)} A = \frac{s_{(1)}^{(1)} A}{\phi} - \frac{\hat{h} \hat{h}^L}{2 \phi^2}.
\]

The Laplacian in two-dimensions is conformally covariant so that \(\Delta q_{k} \theta_k^{(1)'} = \frac{1}{\phi^2} \Delta q_{k} \theta_k^{(1)'}\), and hence

\[
\Delta q_{\ell} \theta_k^{(1)'} = \frac{1}{\phi^2} \Delta q_{\ell} \theta_k^{(1)'} = \frac{\Delta q_{\ell} \theta_k^{(1)}}{\phi^3} - \frac{\theta_k^{(1)} \Delta q \phi}{\phi^4} - \frac{2 \nabla^A \theta_{(k)} \phi_A}{\phi^4} + \frac{2 \theta_{k}^{(1)} |\nabla \phi|^2}{\phi^5}.
\]

The divergence of a one-form in two-dimensions is also conformally covariant \(\text{div}_q s_{\ell}^{(1)} = \frac{1}{\phi^2} \text{div}_q s_{\ell}^{(1)}\) and

\[
\text{div}_q s_{\ell}^{(1)} = \frac{\nabla A s_{\ell}^{(1)}}{\phi^3} - \frac{s_{(1)}^{(1)}}{\phi^4} + \frac{1}{\phi^3} \frac{\nabla A \phi}{\phi^4} \left( \frac{\Delta q \phi}{\phi} - \frac{|\nabla \phi|^2}{\phi^2} \right) - \frac{\theta_{k}^{(1)}}{\phi^3} + \frac{\theta_{\ell}^{(1)}}{\phi^3} - \frac{4 \theta_{k}^{(1)}}{\phi^3} \text{div}_q s_{\ell}^{(1)}.
\]

Putting things together many terms cancel out and we find

\[
\Delta q_{\ell} \theta_k^{(1)'} - (\theta_k^{(1)'} + \theta_{\ell}^{(1)'} - 4 \text{div}_q s_{\ell}^{(1)}) = \frac{\Delta q_{\ell} \theta_k^{(1)}}{\phi^3} - \frac{\theta_k^{(1)}}{\phi^3} \left( \frac{\Delta q \phi}{\phi} - \frac{|\nabla \phi|^2}{\phi^2} \right) - \frac{\theta_{k}^{(1)}}{\phi^3} + \frac{\theta_{\ell}^{(1)}}{\phi^3} - \frac{4 \theta_{k}^{(1)}}{\phi^3} \text{div}_q s_{\ell}^{(1)}.
\]

Using the large sphere equation \((42)\)

\[
\frac{\Delta q \phi}{\phi} - \frac{|\nabla \phi|^2}{\phi^2} = \Delta q \log \phi = 1 - \phi^2 \text{ we find}
\]

\[
\Delta q_{\ell} \theta_k^{(1)'} - (\theta_k^{(1)'} + \theta_{\ell}^{(1)'} - 4 \text{div}_q s_{\ell}^{(1)}) = \left( \Delta q_{\ell} \theta_k^{(1)} - (\theta_k^{(1)} + \theta_{\ell}^{(1)} - 4 \text{div}_q s_{\ell}^{(1)} \right) \frac{1}{\phi^3}.
\]

Since the volume forms are related by \(\eta_q = \phi^2 \eta_q\), the result follows.

We have considered so far the limit of the Hawking energy for geodesic foliations with fixed initial surface \(S_0\). The second step is to consider geodesic foliations with a different initial surface. Let us fix a geodesic background foliation \(\{S_r\}\) approaching large spheres and, as usual, let \(k\) be the associated null generator \(k\) satisfying \(k(r) = -1\). Any geodesic foliation with the same null generator is defined by the equation \(r' = \text{const}\), where \(r'\) is any solution of \(k(r') = -1\). Hence \(k(r - r') = 0\) and the function \(\tau := r - r'\) is Lie constant. This function can be interpreted as the graph function of the initial surface \(S_{r'=r_0}\) in the original foliation \(\{S_{r}\}\). If \(\{S_{r}\}\) starts at a larger initial value \(r_0\), the graph function of \(S_{r_0}\) is given by an appropriate constant shift of \(\tau\), namely \(\tau + r'_0 + r_0\). The following theorem gives the limit of the Hawking energy for the foliation \(\{S_{r}\}\) and shows that the integrand is also covariant (in fact invariant) for this change of foliation.

**Theorem 4.** Let \(\Omega\) be a past asymptotically flat null hypersurface endowed with an affinely parametrized background foliation \(\{S_r\}\) with generator \(k\) and approaching large spheres. Let \(\tau\) be a Lie constant function and define the geodesic foliation \(\{S_{r}^{\tau}\}\) by the graph functions

\[
r = F_{\tau} := \tau + r',
\]

(44)
Moreover, the limits of the Hawking energies along the two foliations coincide and read

\[ \lim_{r \to \infty} m_H(S_r) = \lim_{r' \to \infty} m_H(S_{r'}) = -\frac{1}{16\pi} \int_{\mathbb{S}^2} (\theta_k^{(1)} + \theta_k^{(1)}) \eta q = -\frac{1}{16\pi} \int_{\mathbb{S}^2} (\theta_k^{(1)'} + \theta_k^{(1)'} \eta q'. \] (46)

Proof. We write, as before, \( \tau = r - r_0 \) and \( \tau' = r' - r_0 \), so that \( \tau = \tau' + \tau \). Changing parameters in the first fundamental form

\[ \gamma_{AB} = \hat{q}_{AB} \tau^2 + h_{AB} \tau + \Psi_{AB} = \hat{q}_{AB}(\tau + \tau')^2 + O(\tau') = \hat{q}_{AB} \tau^2 + O(\tau') \]

it follows \( \hat{q}' = \hat{q} \). Similarly, \( (26) \) gives

\[ \theta_k = \frac{\theta_k^{(1)}}{\tau} + \frac{\theta_k^{(1)}}{\tau' \tau} + o(\tau^{-2}) = \frac{-2}{\tau + \tau'} + \frac{\theta_k^{(1)}}{(\tau + \tau')^2} + o(\tau^{-2}) = \frac{-2}{\tau} + \frac{\theta_k^{(1)} + 2\tau}{\tau' \tau} + o(\tau^{-2}) \]

so that \( \theta_k^{(1)'} = \theta_k^{(1)} + 2\tau \). For \( \theta_{\ell} \) we use \( (14) \) with \( \alpha = 1, Q_k = 0 \) and \( F = F_{\nu} \). First we change parameter in the expansion of the first terms in the right-hand side

\[ \theta_{\ell} = \frac{2}{\tau} + \frac{\theta_{\ell}^{(1)}}{\tau} + o(\tau^{-2}) = \frac{2}{\tau} + \frac{\theta_{\ell}^{(1)}}{(\tau + \tau')^2} + o(\tau^{-2}) = \frac{2}{\tau} + \frac{\theta_{\ell}^{(1)} - 2\tau}{\tau' \tau} + o(\tau^{-2}). \]

Since \( DF_{\nu} = D\tau \) independent of \( r \) and \( \gamma^{AB} = O(\tau^{-2}) \), cf. \( (12) \), we have \( |DF_{\nu}|^2 = O(\tau^{-2}) \) and the term \( \theta_k^{(1)} DF_{\nu}^2 = O(\tau^{-3}) \). The same argument shows that \( s_{\ell}(\text{grad } F) = O(\tau^{-3}) \) and \( K^k(\text{grad } F, \text{grad } F) = O(\tau^{-3}) \). The Laplacian term can be computed, using \( (28) \), as

\[ \Delta_\gamma F_{\nu} = \frac{\Delta q^T}{\tau'^2} + o(\tau^{-2}) = \frac{\Delta q^T}{\tau'^2} + o(\tau^{-2}). \]

Inserting also these into expression \( (14) \) yields

\[ \theta_{\nu} = \frac{2}{\tau} + \frac{\theta_{\ell}^{(1)} - 2\tau}{\tau'} - \frac{\Delta q^T}{\tau'^2} + o(\tau^{-2}) = \frac{2}{\tau} + \frac{\theta_{\ell}^{(1)} - 2\tau - 2\Delta q^T}{\tau'^2} + o(\tau^{-2}), \]

as claimed. Note in particular that \( \theta_{\ell}^{(1)'} = \theta_{\ell}^{(1)} - 2\tau - 2\Delta q^T \). The expansion of the connection one-form \( s_{\ell}^{(1)} \) is obtained from \( (12) \) with \( \alpha = 1 \) and \( Q_k = 0 \). The term \( -K^k(X_A, \text{grad } F_{\nu}) \) is \( -K^k(X_A, \text{grad } F_{\nu}) = \frac{1}{\tau} X_A + O(\tau^{-2}) \) after using \( (25) \). Given the expansion of \( s_{\ell} \) in item (ii) of Definition \( 1 \) we conclude

\[ s_{\ell}^{(1)}(1) = \frac{s_{\ell}^{(1)} + \tau A}{\tau} + o(\tau^{-1}) = \frac{1}{\tau} = \frac{s_{\ell}^{(1)}}{\tau} + o(\tau^{-1}) \quad \implies \quad s_{\ell}^{(1)} = s_{\ell}^{(1)} + \tau A. \]
Inserting $q^r$, $\theta_k^{(1)}$, $\theta^r_{(1)}$ and $s_k^{(1)}$ in $(\Delta q^r \theta_k^{(1)})' - (\theta_k^{(1)})' + \theta^r_{(1)} - 4 \nabla q^r A(s_k^{(1)})_1 \eta q^r$ all terms in $\tau$ cancel out and the invariance (45) is established. For the last statement we use the fact that both foliations $\{S_r\}$ and $\{S_{r'}\}$ are geodesic foliations approaching large spheres. So, both can be taken as background foliations and in each case we can apply Theorem 2 with $\Psi = 1$. Using the Gauss identity, the equalities
\[
\lim_{r \to \infty} m_H(S_r) = -\frac{1}{16\pi} \int_{S^2} (\theta_k^{(1)} + \theta^r_{(1)}) \eta q^r \quad \text{and} \quad \lim_{r' \to \infty} m_H(S_{r'}) = -\frac{1}{16\pi} \int_{S^2} (\theta_k^{(1)} + \theta^r_{(1)}) \eta q^r
\]
hold. Invoking the invariance (45) the remaining equality $\lim_{r' \to \infty} m_H(S_r) = \lim_{r \to \infty} m_H(S_{r'})$ follows.

6 Limit of the Hawking energy for non-geodesic foliations

Our aim in this paper is to obtain the limit of the Hawking energy along very general foliations $\{S_{r'}\}$. In the previous section we dealt with the general case when the foliations are geodesic. In order to go into more general settings we need to consider vector fields $k'$ not affinely parametrized.

We will however assume that $k'$ is nowhere zero even at the limit at infinity. More specifically, we assume that there exists a function $r' \in F(\Omega)$ satisfying $k'(r') = -1$ and a geodesic background foliation (not necessarily approaching large spheres) defined as the level sets of a function $r \in F(\Omega)$ such that $\xi := r - r'$ decays at infinity in an appropriate way. In other words, the foliation $\{S_{r'}\}$ is assumed to approach at infinity a geodesic foliation $\{S_r\}$ at an appropriate rate. Conversely, given a geodesic background foliation and a function $\xi \in F(\Omega)$ satisfying $\xi = o_1(1)$ we can define a function $r' := r - \xi \in F(\Omega)$. The level sets of this function are smooth surfaces at least for points at large enough $r$. This is because $dr'(k) = k(r - \xi) = -1 - k(\xi) \neq 0$ because $k(\xi)$ decays at infinity. Thus, for $r$ bigger than some (possible large) value $R_1$, the level sets $r' = \text{const}$ define a foliation $\{S_{r'}\}$. Each surface on this foliation is transverse to $k$ and hence spacelike. The null generator $k'$ satisfying $k'(r') = -1$ is given by $k' = 1 + k(\xi)$ because
\[
k'(r') = \frac{1}{1 + k(\xi)} k(r') = \frac{1}{1 + k(\xi)} k(r - \xi) = -1.
\]

It is clear that the foliation $\{S_{r'}\}$ is not geodesic in general. Given a value $r'$ large enough, the surface $S_{r'}$ is a graph on the background foliation $\{S_r\}$. The graph function $r = F_{r'}$ is given by $F_{r'}(p) = r' + \xi(p)$ for all $p \in S_{r'}$. As usual, we extend the graph function to $\Omega$ by Lie dragging along $k$. Note that $F_{r'}$ extended this way is not $r' + \xi$, but both agree on $S_{r'}$. Thus we can safely abuse notation an write the graph simply as $F_{r'} = r' + \xi$. The following theorem gives the limit of the Hawking energy for the foliation $\{S_{r'}\}$.

**Theorem 5.** Let $\Omega$ be a past asymptotically flat null hypersurface endowed with an affinely parametrized background foliation $\{S_r\}$ with generator $k$. Let $\xi \in F(\Omega)$ satisfy $\xi = o_1(1) \cap o_2^X(1)$, and $k(\xi) = a_1^X(\tau^{-1})$. Define the foliation $\{S_{r'}\}$ by the graph functions
\[
r = F_{r'} := r' + \xi.
\]

Then the null expansions and the connection one-form of $\{S_{r'}\}$ have, to leading orders, the same form as for the background foliation, i.e.
\[
\theta_k^{(1)} = \frac{2k}{r^2} + \theta_k^{(1)} + o(\tau^{-2}), \quad \theta^r_{(1)} = \frac{2K q^r}{r^2} + \theta^r_{(1)} + o(\tau^{-2}) \quad s_{r'} = \frac{(s_k^{(1)})_A}{r^3} + o(\tau^{-1}),
\]

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where $\tau := \tau - \tau_0$, $\tau' := \tau' - \tau_0$ and all objects refer to the background foliation. The limit of the Hawking energy along $\{S_{\tau'}\}$ is the same as the limit along $\{S_{\tau}\}$ and reads

$$\lim_{r \to \infty} m_H(S_{\tau}) = \lim_{r' \to \infty} m_H(S_{\tau'}) = \frac{-1}{8\pi \sqrt{16\pi}} \sqrt{|\hat{S}|} \int_{\hat{S}} \left( K_{\hat{q}} \hat{\theta}^{(1)}_k + \hat{\theta}^{(1)}_{r} \right) \eta_{\hat{q}}, \quad (49)$$

where $|\hat{S}|$ is the area of $(S_0, \hat{q})$ and $\eta_{\hat{q}}$ the corresponding volume form.

Proof. The asymptotic expansion for $\theta_k$ is obtained by simply changing the parameter of the foliation $\tau = \tau' + \xi$

$$\theta_k = \frac{-2}{\tau} + \frac{\hat{\theta}^{(1)}_k}{\tau^2} + o(\tau^{-2}) = \frac{-2}{\tau} + \frac{\hat{\theta}^{(1)}_k}{(\tau' + \xi)^2} + o(\tau'^{-2}) = \frac{-2}{\tau} + \frac{\hat{\theta}^{(1)}_k}{\tau'^2} + o(\tau'^{-2})$$

where $\xi = o_1(1)$ has been used in the last equality. Given that $k' = \left( \frac{1}{1 + k(\xi)} \right) k$ and $k(\xi) = o(\tau^{-1})$, the null expansion along $k'$ is

$$\theta_{k'} = \left( \frac{1}{1 + k(\xi)} \right) \theta_k = \frac{-2}{\tau} + \frac{\hat{\theta}^{(1)}_k}{\tau'^2} + o(\tau'^{-2}),$$

which is the first expression in $(48)$. We next compute $\theta_{\ell'}$ from $(13)$ with $\alpha = \frac{1}{1 + k(\xi)}$. Changing parameters in the first term of the right hand side,

$$\theta_{\ell} = \frac{2K_{\hat{q}}}{\tau'} + \frac{\hat{\theta}^{(1)}_{\ell}}{(\tau' + \xi)^2} + o(\tau'^{-2}) = \frac{2K_{\hat{q}}}{\tau'} + \frac{\hat{\theta}^{(1)}_{\ell}}{(\tau' + \xi)^2} - \frac{2\xi}{\tau'^2} + o(\tau'^{-2}) = \frac{2K_{\hat{q}}}{\tau'} + \frac{\hat{\theta}^{(1)}_{\ell}}{\tau'^2} + o(\tau'^{-2})$$

For the terms involving the graph function, it is immediate to check that $|DF_{\tau'}|^2 \theta_k = o(\tau'^{-3})$, $s_\ell(\text{grad } F_{\tau'}) = o(\tau'^{-3})$, $K^k(\text{grad } F_{\tau'}, \text{grad } F_{\tau'}) = o(\tau'^{-3})$. For the Laplacian term $(28)$ gives

$$\triangle_{\gamma} F_{\tau'} = \frac{\triangle_{\gamma} F_{\tau'}}{\tau'} + o(\tau'^{-2}) = \frac{\triangle_{\gamma} F_{\tau'}}{\tau'^2} + o(\tau'^{-2}) = \frac{\triangle_{\gamma} \xi}{\tau'^2} + o(\tau'^{-2}) = o(\tau'^{-2}),$$

because $\xi = o_2^X(1)$. Finally, $k(\xi) = o(\tau'^{-1})$ implies $\frac{1}{\alpha} = 1 + k(\xi) = 1 + o(\tau'^{-1})$ and $(14)$ is simply

$$\theta_{\ell'} = (1 + o(\tau'^{-1})) \left( \frac{2K_{\hat{q}}}{\tau'} + \frac{\hat{\theta}^{(1)}_{\ell}}{\tau'^2} + o(\tau'^{-2}) \right) = \frac{2K_{\hat{q}}}{\tau'} + \frac{\hat{\theta}^{(1)}_{\ell}}{\tau'^2} + o(\tau'^{-2})$$

as stated in the Theorem. The connection one-form $s_{\ell'}$ is obtained from $(12)$ with $Q_k = 0$ and $\alpha = \frac{1}{1 + k(\xi)}$. Given that $\alpha_{A} = o(\tau'^{-1})$ (because $k(\xi) = o_1^X(\tau'^{-1})$) and $s_\ell(\text{grad } F_{\tau'}) = o(\tau'^{-2})$, we conclude

$$s_{\ell'A} = \frac{(s_{\ell'}^1)^A}{\tau'} + o(\tau'^{-1}).$$

We next compute the limit of the Hawking energy along $\{S_{\tau'}\}$. The metric in $S_{\tau'}$ is

$$\gamma(\tau')_{AB} = \hat{q}_{AB} \tau'^2 + o(\tau'^2) = \hat{q}_{AB} (\tau' + \xi)^2 + o(\tau'^2) = \hat{q}_{AB} \tau'^2 + o(\tau'^2),$$

so that in particular the rescaled limit metric and corresponding volume forms remain unchanged, $\hat{q}' = \hat{q}$ and $\eta_{\hat{q}'} = \eta_{\hat{q}}$. This, together with the expansions $(48)$, already implies that the limit of the Hawking energy along $\{S_{\tau'}\}$ and along $\{S_{\tau}\}$ are the same. To obtain expression $(49)$ we need
the volume form of $S_{\nu}$. As with the metric $\gamma(r)$ or with the null expansion $\theta_k$ it suffices to change parameter in the volume form $\eta_{S_{\nu}}$ which is given by (33) with $\phi = 1$

$$\eta_{S_{\nu}} = \left( (\tau^\prime + \xi)^2 + \theta_k^{(1)}(\tau^\prime + \xi) + o(\tau^\prime) \right) \eta_\theta = (\tau^2 + \theta_k^{(1)}\tau + o(\tau)) \eta_\theta.$$  \hspace{1cm} (50)

It is immediate to check that the product $\theta_k \vartheta_{\nu} \eta_{S_{\nu}}$ is

$$\theta_k \vartheta_{\nu} \eta_{S_{\nu}} = \left( -4K_\theta + (-2K_\theta \theta_k^{(1)} - 2\theta_k^{(1)}) \frac{1}{\tau^2} + o(\tau^{-1}) \right) \eta_{\theta}$$

so that, using Gauss-Bonnet $\int_S K_\theta \eta_{\theta} = 4\pi$,

$$1 + \frac{1}{16\pi} \int_{S_{\nu}} \theta_k \vartheta_{\nu} \eta_{S_{\nu}} = \frac{1}{16\pi} \int_S (-2K_\theta \hat{\theta}_k^{(1)} - 2\hat{\theta}_k^{(1)}) \eta_{\theta} \frac{1}{\tau^2} + o(\tau^{-1}).$$

On the other hand $|S_{\nu}| = \int_S (\tau^2 + \hat{\theta}_k^{(1)}\tau + o(\tau)) \eta_\theta = |\hat{\tau}|\tau^2 + o(\tau^2) \Rightarrow \sqrt{|S_{\nu}|} = \sqrt{|\hat{\tau}|\tau^2 + o(\tau^2)}$ and

$$m_H(S_{\nu}) = \frac{1}{\sqrt{16\pi}} \left( \sqrt{|\hat{\tau}|\tau^2 + o(\tau^2)} \right) \left( \frac{1}{16\pi} \int_S (-2K_\theta \hat{\theta}_k^{(1)} - 2\hat{\theta}_k^{(1)}) \eta_{\theta} \frac{1}{\tau^2} + o(\tau^{-1}) \right)$$

$$= \frac{1}{8\pi \sqrt{16\pi}} \sqrt{|\hat{\tau}|} \int_S \left( K_\theta \hat{\theta}_k^{(1)} + \hat{\theta}_k^{(1)} \right) \eta_{\theta} + o(1).$$

\[ \square \]

We are ready to obtain our main Theorem 6 by simply combining the previous results. In fact, we state and prove a slightly more complete theorem that provides two different expressions for the limit.

**Theorem 6 (General Hawking energy limit).** Let $\Omega$ be a past asymptotically flat null hypersurface endowed with an affinely parametrized background foliation $\{S_\nu\}$ with generator $k$ that tends to large spheres. Define the foliation $\{S_{\nu}\}$ by the graph functions

$$r = F_{\nu} := r_0 + \frac{1}{\Psi}(r^* - r_0) + \tau + \xi,$$  \hspace{1cm} (51)

with $\Psi > 0$, $\tau$ Lie constant functions on $\Omega$ and $\xi = o_1(1) \cap o_2^X(1)$ and $k(\xi) = o_1^X(\tau^{-1})$. The limit of the Hawking energy along $\{S_{\nu}\}$ is

$$\lim_{r^* \to \infty} m_H(S_{\nu}) = \frac{-1}{8\pi \sqrt{16\pi}} \left( \int_{S^2} \eta_{\theta} \right) \int_{S^2} \left( K_\theta \hat{\theta}_k^{(1)*} + \hat{\theta}_k^{(1)*} \right) \eta_{\theta}$$

$$= \frac{1}{8\pi \sqrt{16\pi}} \left( \int_{S^2} \frac{1}{\Psi^2} \eta_{\theta} \right) \int_{S^2} \left( \hat{\Delta} \theta_k^{(1)} - (\theta_k^{(1)} + \theta_k^{(1)}) - 4\text{div}(s_{\ell}^{(1)}) \right) \Psi \eta_{\theta},$$  \hspace{1cm} (52)

where $\hat{\theta}_k^{(1)*}$ and $\hat{\theta}_k^{(1)*}$ refer either to the foliation $\{S_{\nu}\}$ or to the geodesic foliation $\{S_{\nu}\}$ defined by the graph functions $r = F_{\nu} := r_0 + \frac{1}{\Psi}(r - r_0) + \tau$, and $\hat{\theta}_k^{(1)}$, $\hat{\theta}_k^{(1)}$ and $s_{\ell}^{(1)}$ refer to the background foliation $\{S_\nu\}$.
Proof. The strategy is to pass from the background foliation to \( \{ S_r \} \) in three steps. The geometric elements of each foliation use the same symbol as the foliation, so the meaning of each quantity should be clear. Consider first a foliation defined by the levet sets of \( r' := r - \tau \). Theorem 4 gives

\[
\left( \triangle q \theta_k^{(1)} - (\theta_k^{(1)} + \theta_\ell^{(1)}) - 4\operatorname{div}_q(s_\ell^{(1)}) \right) \eta_q = \left( \triangle q \theta_k^{(1)}' - (\theta_k^{(1)}' + \theta_\ell^{(1)}') - 4\operatorname{div}_q(s_\ell^{(1)})' \right) \eta_q'.
\]

Consider next the foliation defined by the level sets of \( \eta \), which, upon using (6) and \( \eta \) of the foliation is spherical \( \hat{q} \) and the limit of the Hawking energy along \( \{ \hat{q} \} \) is related to \( \{ q \} \). It is proved in Theorem 2 that the limit of the Hawking energy is

\[
\lim_{r'' \to \infty} m_H(S_{r''}) = \frac{1}{8\pi \sqrt{16\pi}} \left( \int_{S^2} \frac{1}{\Psi^2} \eta_q \right) \int_{S^2} \left( \triangle q \theta_k^{(1)} - (\theta_k^{(1)} + \theta_\ell^{(1)}) - 4\operatorname{div}_q(s_\ell^{(1)}) \right) \Psi \eta_q,'
\]

which, upon using (6) and \( \eta = \eta_q' \), implies

\[
\lim_{r'' \to \infty} m_H(S_{r''}) = \frac{1}{8\pi \sqrt{16\pi}} \left( \int_{S^2} \frac{1}{\Psi^2} \eta_q \right) \int_{S^2} \left( \triangle q \theta_k^{(1)} - (\theta_k^{(1)} + \theta_\ell^{(1)}) - 4\operatorname{div}_q(s_\ell^{(1)}) \right) \Psi \eta_q. \tag{53}
\]

Now, \( \{ S_{r''} \} \) is geodesic but does not necessarily tend to large spheres. The final foliation \( \{ S_{r''} \} \) is related to \( \{ S_{r''} \} \) by \( r'' = r'' + \Psi \xi \). Since \( \Psi \xi \) satisfies the hypotheses of Theorem 5 we conclude that the Hawking energy has the same limit along \( \{ S_{r''} \} \) and along \( \{ S_{r''} \} \). In combination with (53) this proves the second equality in (52). For the first equality we simply note that the rescaled limit metric of \( \{ S_{r''} \} \) is \( \hat{q} = \Psi^{-2}q \) and apply again Theorem 5.

Remark 6. In this paper we have considered null hypersurfaces extending to past null infinity. Obviously similar results apply for asymptotically flat null hypersurfaces extending to future null infinity. By repeating the arguments before, the following result is obtained: consider a future directed geodesic null vector \( k \) tangent to \( \Omega \) and define the function \( r \in F(\Omega) \) by \( \hat{k}(r) = 1 \) with \( r = r_0 \) on some initial cross section. The level sets \( \{ S_r \} \) define a foliation which allows to construct a transversal future directed null normal \( \ell \) satisfying \( \langle k, \ell \rangle = -2 \). If the rescaled asymptotic metric of the foliation is spherical \( \hat{q} \), the expansions of the null second fundamental forms and connection one-form take the form (note the change of signs with respect to the past null case)

\[
\theta_k = \frac{2}{r} + \frac{\theta_k^{(1)}}{r^2} + o(r^{-2}), \quad \theta_\ell = \frac{-2}{r} + \frac{\theta_\ell^{(1)}}{r^2} + o(r^{-2}) \quad s_{\ell A} = \frac{s_{\ell A}^{(1)}}{r} + o(r^{-1})
\]

and the limit of the Hawking energy along \( \{ S_{r''} \} \) (with the same definition as in Theorem 6) is

\[
\lim_{r'' \to \infty} m_H(S_{r''}) = \frac{1}{8\pi \sqrt{16\pi}} \left( \int_{S^2} \frac{1}{\Psi^2} \eta_q \right) \int_{S^2} \left( -\triangle q \theta_k^{(1)} + (\theta_k^{(1)} + \theta_\ell^{(1)}) - 4\operatorname{div}_q(s_\ell^{(1)}) \right) \Psi \eta_q.
\]
7 The large sphere equation and the Bondi energy-momentum

As mentioned in the introduction, the limit of the Hawking energy when the foliation approaches large spheres is the Bondi energy. In this section we want to recover this fact from our general expressions. Recall first that the conformal group of the two-sphere is defined as the set of diffeomorphisms \( \Phi: (S^2, \hat{q}) \mapsto (S^2, \hat{q}) \) satisfying \( \Phi^* (\hat{q}) = \Theta^2 \hat{q} \), \( \Theta \in \mathcal{F}(S^2, \mathbb{R}^+) \) (i.e. the set of conformal diffeomorphisms). We restrict ourselves to the connected component of the identity of this group. It is well-known (see e.g. [25]) that this group is isomorphic to the connected component of the identity of Lorentz group of Minkowski space \( M^{1,3} \), and also isomorphic to the Möbius group of the Riemann sphere

\[
F: S^2 \mapsto S^2
\]

\[
z \mapsto F(z) = \frac{\alpha z + \beta}{\gamma z + \delta} \quad \left( \frac{\alpha \beta}{\gamma \delta} \right) \in SL(2, \mathbb{C})
\]

(55)

where \( z \in \mathbb{C} \cup \{\infty\} \simeq S^2 \). In these coordinates, the standard metric on the sphere is \( \hat{q} = \frac{4}{(1 + z\bar{z})^2} dzd\bar{z} \) and the \( l = 1 \) spherical harmonics read

\[
Y_1^1 = \frac{z + \bar{z}}{1 + z\bar{z}}, \quad Y_2^1 = \frac{z - \bar{z}}{i(1 + z\bar{z})}, \quad Y_3^1 = \frac{z\bar{z} - 1}{1 + z\bar{z}}.
\]

(56)

For a vector \( a \in \mathbb{R}^3 \) we write \( a \cdot Y_i^1 := \sum_{i=1}^3 a^i Y_i^1 \). These properties allow us to obtain easily the general solution to the large sphere equation (42).

Proposition 6 (Solution of the large sphere equation). A smooth function \( \phi: S^2 \mapsto \mathbb{R}^+ \) solves equation (42) if and only if there exists \( a = (a^1, a^2, a^3) \in \mathbb{R}^3 \) such that

\[
\Psi := \frac{1}{\phi} = \sqrt{1 + |a|^2 + a \cdot Y_1^1}.
\]

(57)

Proof. In terms of \( \Psi := \frac{1}{\phi} \), equation (42) becomes

\[
\Psi^2 + (\triangle_{\hat{q}} \Psi) - |\nabla \Psi|^2_{\hat{q}} = 1.
\]

(58)

We first show that (57) solves this equation. Applying a rotation to \( S^2 \) we can assume without loss of generality that \( a = (0, 0, c) \) and hence \( \Psi = \sqrt{1 + c^2} + c Y_3^1 \). Thus \( \triangle_{\hat{q}} \Psi = c \triangle_{\hat{q}} Y_3^1 = -2c Y_3^1 = -2\Psi + 2\sqrt{1 + c^2} \), and \( |\nabla \Psi|^2_{\hat{q}} = (1 + z\bar{z})^2 \partial_z \Psi \partial_{\bar{z}} \Psi = \frac{4c^2 z\bar{z}}{(1 + z\bar{z})^2} = -\left( \Psi^2 + 1 - 2\sqrt{1 + c^2} \Psi \right) \), and (58) holds after immediate cancellations.

To show the converse we recall that equation (12) is the statement that the Gauss curvature \( K_{\phi^2 \hat{q}} = 1 \). This means that there exist coordinates \( z' \in \mathbb{C} \cup \{\infty\} \) where \( \phi^2 \hat{q}' = \frac{4dz'd\bar{z}'}{(1 + z'\bar{z}')^2} \). We can assume without loss of generality that the map \( F(z) = z' \) is orientation preserving. Since it is also an element of the conformal group, it must be an element of the Möbius group (55). Performing the pull-back of \( \hat{q} \)

\[
\phi^2 \hat{q} = \frac{4|\frac{\partial F}{\partial z}|^2}{(1 + |F|^2)^2} dzd\bar{z}.
\]
Thus \( \phi = \frac{1 - |z|^2}{1 + |F|^2} \). Since \( \frac{\partial F}{\partial z} = \frac{1}{(1 + |z|^2)} \) if follows

\[
\Psi = \frac{1}{\phi} = \frac{|\alpha z + \beta|^2 + |\gamma z + \delta|^2}{(1 + |z|^2)}.
\]

Expanding in terms of \( l = 1 \) spherical harmonics yields

\[
\Psi = \frac{|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2}{2} + \text{Re} (\overline{\alpha} \beta + \overline{\gamma} \delta) Y_1^1 + \text{Im} (\overline{\alpha} \beta + \overline{\gamma} \delta) Y_2^1 + \frac{|\alpha|^2 - |\beta|^2 + |\gamma|^2 - |\delta|^2}{2} Y_3^1.
\]

It is straightforward to check that this expression is of the form \( \Psi = \sqrt{1 + |a|^2} + a \cdot Y^1 \) with \( a \in \mathbb{R}^3 \).

\( \square \)

Remark 7. The Bondi energy-momentum is a vector in an abstract Minkowski space. Let us recall the construction for the sake of completeness and because of a subtlety that arises in the case of past null hypersurfaces. The Lorentz transformation \( x'^{\mu} = \Lambda(F)^{\mu}_{\nu} x^{\nu} \) associated to the Möbius transformation \( F \) has a time direction \( \partial_t' \) given by

\[
u : = \partial_t' = \Lambda(F)^{0}_{0} \partial_t - \sum_{i=1}^{3} \Lambda(F)^{0}_{i} \partial_{x^i}.
\]

The explicit map \( \Lambda(F) \) can be found e.g. in page 17 of [25]. Comparing \( \Lambda^0_i(F) \) with the expresion for \( a^i \) above it follows

\[
u = \sqrt{1 + |a|^2} \partial_t - a^i \partial_{x^i}.
\]

The construction of \( \Lambda(F) \) in [25] is performed with the unit sphere lying at the intersection of the hyperplane \( t = 1 \) and the future null cone of the origin. It is hence adapted to future directed null hypersurfaces extending to future null infinity. In this paper we have considered null hypersurfaces extending to past null infinity. This case is obtained from the previous one by a time inversion, which has the efect that the observer \( \nu \) has the form

\[
u = \sqrt{1 + |a|^2} \partial_t + a^i \partial_{x^i}
\]

in terms of the coefficients \( a^i \) in the conformal factor \( \phi \). Summarizing, to a background foliation \( \{S_r\} \) of \( \Omega \) approaching large spheres with asymptotic rescaled metric \( \tilde{q} \) one can assign an asymptotic inertial reference frame \( \{t, x^i\} \) in an (abstract) Minkowski spacetime. Given another such foliation \( \{S_{r'}\} \) with asymptotic rescaled metric \( \phi^2 \tilde{q} \), one associates an asymptotic inertial observer with time direction \( u^\mu = (\sqrt{1 + |a|^2}, a^i) \) in the basis \( \partial_{x^i} \) above.

We can now recover the result that the Hawking energy approaches the Bondi energy for spherical foliations.

Corollary 3. Let \( \Omega \) be a past asymptotically flat null hypersurface endowed with an affinely parametrized background foliation \( \{S_r\} \) with generator \( k \) that tends to large spheres. Consider another foliation associated to the parameter \( r'^* \) so that \( r = r_0 + \phi(r^* - r_0) + \tau + \xi \), as in Theorem 6, where \( \phi > 0 \) satisfies the large sphere equation (42). Let \( u^\mu \in \mathcal{M}_{1,3} \) be the asymptotic inertial observer associated to this foliation. Then

\[
\lim_{r^* \to \infty} m_H(S_{r^*}) = -P^\mu_B u^\nu \eta_{\mu\nu} := F^\mu_B,
\]
where $\eta_{\mu\nu}$ is the Minkowski metric and the Bondi four-momentum vector $P_B$ reads

$$E_B := P_B^0 := -\frac{1}{16\pi} \int_{S^2} (\theta_k^{(1)} + \theta_l^{(1)}) \eta_{\hat{q}}$$

$$P_B^i := \frac{1}{16\pi} \int_{S^2} \left( -\Delta_q \theta_k^{(1)} + (\theta_k^{(1)} + \theta_l^{(1)}) + 4 \text{div}_q s_{\ell}^{(1)} \right) Y^1_i \eta_{\hat{q}}, \quad i \in \{1, 2, 3\}. \quad (60)$$

If, in addition, the energy flux decay condition of Proposition 4 is satisfied, then the Bondi three-momentum simplifies to

$$P_B^i = \frac{1}{16\pi} \int_{S^2} \left( \Delta_q \theta_k^{(1)} + (\theta_k^{(1)} + \theta_l^{(1)}) \right) Y^1_i \eta_{\hat{q}}, \quad i \in \{1, 2, 3\}. \quad (61)$$

Proof. We can use expression (52) with $\Psi$ as in (57) so that

$$\lim_{r^* \to \infty} m_H(S_{r^*}) = \frac{1}{16\pi} \int_{S^2} \left( \Delta_q \theta_k^{(1)} - (\theta_k^{(1)} + \theta_l^{(1)}) - 4 \text{div}_q s_{\ell}^{(1)} \right) \left( \sqrt{1 + |a|^2} + \sum_{i=1}^{3} a^i Y^1_i \right) \eta_{\hat{q}}$$

$$= \left( -\frac{1}{16\pi} \int_{S^2} (\theta_k^{(1)} + \theta_l^{(1)}) \eta_{\hat{q}} \right) \sqrt{1 + |a|^2} +$$

$$+ \sum_{i=1}^{3} \left( \frac{1}{16\pi} \int_{S^2} \left( \Delta_q \theta_k^{(1)} - (\theta_k^{(1)} + \theta_l^{(1)}) - 4 \text{div}_q s_{\ell}^{(1)} \right) Y^1_i \eta_{\hat{q}} \right) a^i = -\eta(u, P_B), \quad (62)$$

with $u = (\sqrt{1 + |a|^2}, a^1, a^2, a^3)$ and $P_B^i$ as given in the statement of the Corollary.

When the energy flux decay condition holds, we have from Proposition 4 $s_{\ell}^{(1)} A = \hat{\nabla} A \theta_k^{(1)} - \frac{1}{2} \hat{\nabla} B h_{B A}^{AB}$, and the integral (62) becomes

$$\int_{S^2} \left( -3 \Delta_q \theta_k^{(1)} - (\theta_k^{(1)} + \theta_l^{(1)}) + 2 \hat{\nabla} A \hat{\nabla} B h_{AB}^{AB} \right) \Psi_a \eta_{\hat{q}}$$

where $\Psi_a := \sqrt{1 + |a|^2} + a \cdot Y^1$. Integrating by parts the last term and using that $\text{Hess}_q \Psi_a = -(a \cdot Y^1) \hat{q} = \frac{1}{2} (\Delta_q \Psi_a) \hat{q}$ yields

$$\int_{S^2} 2 \hat{\nabla} A \hat{\nabla} B h_{AB}^{AB} \Psi_a \eta_{\hat{q}} = \int_{S^2} 2 \theta_k^{(1)} \Delta_q \Psi_a \eta_{\hat{q}} = \int_{S^2} 2 (\Delta_q \theta_k^{(1)}) \Psi_a \eta_{\hat{q}}$$

where in the first equality we used $\text{tr}_q \hat{h} = 2 \theta_k^{(1)}$ and in the second we performed another integration by parts. Arguing as before, the expression (61) for $P_B^i$ follows. \qed

Remark 8. An analogous result can be obtained for the case of asymptotically flat null hypersurfaces $\Omega$ approaching future null infinity. Using the general expression in Remark 4 for the limit of the Hawking energy in this case and using the fact that $\Psi_a = \sqrt{1 + |a|^2} + a \cdot Y$ corresponds now to the asymptotic observer with four velocity $u^\alpha = (\sqrt{1 + |a|^2}, -a^1, -a^2, -a^3)$, the Bondi energy-momentum vector $P_B$ satisfying $\lim_{r^* \to \infty} M_H(S_{r^*}) = -\eta(u, P_B) := E_B^u$ is

$$E_B := P_B^0 := -\frac{1}{16\pi} \int_{S^2} (\theta_k^{(1)} + \theta_l^{(1)}) \eta_{\hat{q}}$$

$$P_B^i := \frac{1}{16\pi} \int_{S^2} \left( -\Delta_q \theta_k^{(1)} + (\theta_k^{(1)} + \theta_l^{(1)}) - 4 \text{div}_q s_{\ell}^{(1)} \right) Y^1_i \eta_{\hat{q}}.$$
The energy flux decay condition in this case implies (i.e. the analogous on Proposition 4)

\[ s^{(1)}_k = -\mathcal{D}_A \theta^{(1)}_k - \frac{1}{2} \mathcal{D}_B \hat{h}^B_A \]

and the Bondi momentum simplifies to

\[ P^i_B = \frac{1}{16\pi} \int_{S^2} \left( \triangle \theta^{(1)}_k + (\theta^{(1)}_k + \theta^{(1)}_\ell) \right) Y_i^1 \eta_4, \quad i \in \{1, 2, 3\}. \]

Note that in this case

\[ E^u_B := -\eta_{\alpha\beta} u^\alpha P^\beta_B = \frac{1}{16\pi} \int_{S^2} \left( \triangle \theta^{(1)}_k + (\theta^{(1)}_k + \theta^{(1)}_\ell) \right) (u^0 - u^i Y_i^1) \eta_4. \]

Remark 9. The relationship between the limit of the Hawking energy and the Bondi four-momentum for foliations approaching large spheres has been investigated in [25] and [2] (see also Definition 4.2 in [26]). As a useful check, it is convenient to see how the results in this paper fit with the results in [2]. The setup there involves so-called null quasi-spherical coordinates which are adapted to a foliation by future outgoing null hypersurfaces \( \{\mathcal{N}_z\} \), each of them foliated by codimension-two spacelike surfaces \( S_{z,r_B} \) (we change Bartnik’s notation \( r \) to \( r_B \) to avoid conflict with our notation above). Each \( S_{z,r_B} \) has induced metric isometric to the standard sphere of radius \( r_B \). In fact, the null quasi-spherical coordinates \( \{z, r_B, \theta, \phi\} \) are such that the surface \( \{z = \text{const}, r_B = \text{const}\} \) has induced metric \( r_B^2 (d\theta^2 + \sin^2 \theta d\phi^2) := r_B^2 \hat{q} \), which selects the diffeomorphism of \( S_{r_B,z} \) with the standard unit sphere \( (\mathbb{S}^2, \hat{q}) \). Under asymptotic conditions along the null hypersurface involving the shear and its angular derivative, Bartnik shows among various other things that the Bondi energy-momentum is well-defined and agrees with the limit of the Hawking energy along the quasi-spherical foliation \( S_{z,r_B} \). More precisely, defining the mass aspect function \( m = \frac{1}{2} r_B \left( 1 - \frac{1}{4} \hat{H}^2 r_B^2 \right) \) of the sphere \( S_{z,r_B} \), so that \( m_H(z, r_B) = \frac{1}{4\pi} \int_{S^2_{z,r_B}} m \eta_4 \) (recall that \( \hat{H} \) is the mean curvature vector of the surface), Bartnik shows that \( \lim_{r_B \to \infty} m = m_0 \) with \( m_0 \in C^\infty(\mathbb{S}^2) \) and that, under sufficient decay of suitable components of the Einstein tensor which include the energy flux decay condition of this paper,

\[ E_B = \lim_{r_B \to \infty} \frac{1}{4\pi} \int_{S^2_{z,r_B}} m \eta_4 = \frac{1}{4\pi} \int_{S^2} m_0 \eta_4, \]

\[ P^i_B = \lim_{r_B \to \infty} \frac{1}{4\pi} \int_{S^2_{z,r_B}} m Y_i^1 \eta_4 = \frac{1}{4\pi} \int_{S^2} m_0 Y_i^1 \eta_4. \]

The null quasi-spherical gauge is such that \( \theta_k - \frac{2}{r_B} \) is automatically a divergence. Thus, for our results to fit with his it is necessary to select \( r_0 \) in the geodesic background foliation \( \{S_r\} \) (which is of the form \( r = r_B + \xi \)) so that \( \theta^{(1)}_k + 2r_0 \) is a divergence. An explicit computation shows that, in terms of our notation, \( m_0 = \frac{1}{4}(\theta^{(1)}_\ell - \theta^{(1)}_k - 4r_0) \). Bartnik’s result is recovered from (64) because, with the shorthand \( \Psi_u := u_0 - u^i Y_i^1 \),

\[ E^u_B = \frac{1}{16\pi} \int_{S^2} \left( \triangle \theta^{(1)}_k + (\theta^{(1)}_k + \theta^{(1)}_\ell) \right) \Psi_u \eta_4 \]

\[ = \frac{1}{16\pi} \int_{S^2} \left( \triangle \theta^{(1)}_k + (\theta^{(1)}_k + \theta^{(1)}_\ell) - (\theta^{(1)}_k + (\theta^{(1)}_k + 2r_0) \right) \Psi_u \eta_4 \]

\[ = \frac{1}{16\pi} \int_{S^2} (\theta^{(1)}_\ell - \theta^{(1)}_k - 4r_0) \Psi_u \eta_4 = \frac{1}{4\pi} \int_{S^2} m_0 \Psi_u \eta_4. \]
where in the second expression we added zero in the form \(0 = \int_{S^2} \left( -(\nabla_q + 2)(\theta_k^{(1)} + 2r_0) \right) \Psi_u \eta_q \).

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