Algebraic winged pair correlations of dilute active Brownian particles

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We study the pair correlation of active Brownian particles at low density using experiments of self-propelled Janus particles, numerical simulations, and analytical calculations. We observe a winged pair correlation: while particles accumulate in front of an active particle as expected, the depletion wake consists of two depletion wings. In the limit of very soft particles, we obtain a closed equation for the pair correlation, allowing us to characterize the depletion wings. In particular, we unveil two regimes at high activity where the wings adopt a self-similar profile and decay algebraically. Our results suggest a wealth of possible behaviors for the pair correlation in active systems with velocity-orientation coupling or alignment interactions.

The pair correlation function has played a pivotal role in our understanding of the structure of equilibrium liquids [1]; the same is to be expected for active liquids, such as bacterial colonies [2, 3], flocks of starlings [4, 5] or colloidal rollers [6–8], or assemblies of Janus particles [9, 10]. It has indeed been used to quantify the order in bacterial colonies [2] and to infer the interactions in bird flocks [4, 5]. However, its broad utilization is precluded by the lack of analytical results on its general form, notably little is known about its regions of positive and negative sign, even in the homogeneous phase and in the absence of alignment interactions [11–13].

The theoretical prediction of the pair correlation is indeed an important challenge, which has been undertaken for models without alignment, such as active Brownian or Ornstein-Uhlenbeck particles (ABP or AOUP). So far, even for these minimal models, the study of the pair correlation focused mostly on the explanation of the phase separation that occurs at large activity [14–16]. Two main approaches have been followed. The first consists in computing the angular average of the pair correlation due to an activity represented by a persistent translational noise. An attractive term has been found, and interpreted as a higher tendency to phase separate [13, 17, 18]. However, this approach does not, by nature, retain the angular dependence of the correlations. The second approach consists of a quantitative prediction of the polar pair correlation using a closure of the many body Smoluchowski equation leading to the effective velocity, whose decrease with density may explain the phase separation [11, 12, 19–21]. This method requires a numerical solution of nonlinear equations, and does not provide explicit predictions. Finally, up to now there is no analytical characterization of the pair correlation.

In this Letter, we address the global shape of the pair correlation of ABP in the dilute and homogeneous regime [15]. We first observe a winged pair correlation in experiments of self-propelled Janus particles: while particles accumulate in front of an active particle as expected, the depletion wake consists of two depletion wings (Fig. 1). This winged structure is reproduced in numerical simulations of ABP. We then resort to a linearized Dean equation to obtain a closed equation for the pair correlation. Solving this equation under different limits, we unveil three different regimes for the correlations, which we organize on a phase diagram. In two regimes, the wings adopt a self-similar profile and decay algebraically.

First, we measure the pair correlation in a minimal experimental system: Janus particles propelled by an AC electric field [22, 23] (see Appendix), whose dynamics is

![Figure 1. Pair correlation in experiments of Janus particles and numerical simulations. (a) Experimental image, indicating the detected positions $r_i(t)$ and orientations $\theta_i(t)$ of the Janus particles (red dots and arrows). The blue axes represent the local frame used to compute the pair correlation. Scale bar: 10 µm. (b) Pair correlation $B(r)$ (Eq. (2)). (c) Pair correlation in numerical simulations of active Brownian particles ($\phi = 0.04$ and $\epsilon : D : D_\epsilon : U = 50 : 0.05 : 1 : 20$).](image-url)
similar to that of ABP \cite{10, 24} (Fig. 1(a)). We use a low area fraction \( \phi \simeq 0.04 \) to focus on two-body effects and remain in the homogeneous phase \cite{14-16}. The density field for the positions \( \mathbf{r}_i(t) \) and orientations \( \mathbf{\theta}_i(t) \) is defined as \( f(\mathbf{r}, \mathbf{\theta}, t) = \sum_i \delta(\mathbf{r}_i(t) - \mathbf{r}) \delta(\mathbf{\theta}_i(t) - \mathbf{\theta}) \), from which we define the polar pair correlation as

\[
C(\mathbf{r}, \mathbf{\theta}, \mathbf{\theta}') = \frac{\langle f(0, \mathbf{\theta}) f(\mathbf{r}, \mathbf{\theta}') \rangle}{\rho/(2\pi)^2} - \delta(\mathbf{\theta} - \mathbf{\theta'}) - 1, \tag{1}
\]

where \( \rho = 4\phi/(\pi a^2) \) is the density, with \( a \) the diameter of the particles; the correlation of a particle with itself is removed in the second term, and the \( r \to \infty \) limit is removed in the third. From rotational invariance, \( C(\mathbf{r}, 0, \mathbf{\theta}') \) contains all the information in \( C(\mathbf{r}, \mathbf{\theta}, \mathbf{\theta}') \). We focus on the density of particles in the reference frame of a given particle (blue axes in Fig. 1(a) \cite{2}), which retains the polar character of the correlations:

\[
B(\mathbf{r}) = \frac{1}{2\pi} \int_0^{2\pi} C(\mathbf{r}, 0, \mathbf{\theta}') d\mathbf{\theta}'. \tag{2}
\]

Note that if translational and rotational degrees of freedom decouple, for instance in absence of propulsion, the isotropic pair correlation is recovered: \( C(\mathbf{r}, \mathbf{\theta}, \mathbf{\theta}') = B(\mathbf{r}) = h(r) \) \cite{1}. The polar pair correlation \( B(\mathbf{r}) \) in our experiments is shown in Fig. 1(b). As expected, the correlation is positive in front of the particle, indicating an accumulation of other particles. However, while a depletion wake is expected behind the active particle, as in active microhccology \cite{25, 26} or in driven binary mixtures \cite{27}, the depletion concentrates in two “wings” on the sides of the particle.

To assess the generic character of the depletion wings, we compare the experimental correlations to numerical simulations of ABP interacting via a pairwise harmonic potential \( V(r) = \frac{\epsilon}{2} (1 - r/a)^2 \) for \( r < a \) (see Appendix). The position and orientation of the particle \( i \) follow

\[
\dot{\mathbf{r}}_i = -\gamma^{-1} \nabla_i \sum_{j \neq i} V(\mathbf{r}_i - \mathbf{r}_j) + U \hat{\mathbf{e}}_\theta_i + \sqrt{2D} \mathbf{\eta}_i, \tag{3}
\]

\[
\dot{\mathbf{\theta}}_i = \sqrt{2D} \nu_i, \tag{4}
\]

where \( U \) is the propulsion velocity, \( D \) is the translational diffusion coefficient, \( D_r \) is the rotational diffusion coefficient, \( \hat{\mathbf{e}}_\theta \) is the unit vector with polar angle \( \theta \) and \( \mathbf{\eta} \) and \( \nu \) are normalized Gaussian white noises. The diameter \( a \) of the particles and the friction coefficient \( \gamma \) can be set to one through a rescaling; the parameters \( \epsilon, U, D \) and \( D_r \) are given in arbitrary units and only their relative values are important. The pair correlation obtained in the simulation with the experimental parameters is shown in Fig. 1(c); the depletion wings observed in the experiments are reproduced.

We explore the possible behaviors of the pair correlation with the simulations by varying the rotational diffusion coefficient \( D_r \) (Fig. 2(a)-(d)). While the correlation decays quickly at large rotational diffusion, the characteristic depletion wings appear, their length increases and their curvature decreases as \( D_r \) decreases.

To rationalize these different regimes, we consider the lengthscales and dimensionless parameters of the problem. In the dilute limit considered here, the structure of the pair correlation beyond the size of a particle is controlled by the parameters \( U, D \) and \( D_r \), which combine into three lengthscales

\[
\ell_U = \frac{D}{U}; \quad \ell_U = \frac{U}{D_r}; \quad \ell_\theta = \frac{U}{D}, \tag{5}
\]

whose relative values are set by the Péclet number

\[
\text{Pe} = \frac{U}{\sqrt{DD_r}} = \frac{\ell_U}{\ell_U} = \frac{\ell_\theta}{\ell_U}. \tag{6}
\]

The Péclet number is \( \text{Pe} \simeq 90 \) in the experiments and varies between 10 and 316 in Fig. 2(a)-(d). For spherical particles the coefficients \( D \) and \( D_r \) are related through \( D_r \sim D/a^2 \); here we regard them as independent parameters to disentangle the effects of translational and rotational diffusion. The form of the pair correlation thus depends on the Péclet number and the relative value of the observation length \( r \) with the three lengthscales, so that the experiments and numerical simulations can be placed on a parameter plane (blue arrows in Fig. 2(e)).

To account for the observed correlations and characterize the shape of the depletion wings, we resort to a linearized Dean equation for the density field \( f(\mathbf{r}, \mathbf{\theta}, t) \) \cite{26-31}, which is valid for weak interactions. At low density, the pair correlation satisfies (see Appendix):

\[
[2D\nabla^2 + D_r(\partial^2_\theta + \partial^2_\psi) + U(\hat{\mathbf{e}}_\theta - \hat{\mathbf{e}}_\psi) \cdot \nabla] C = -\frac{2}{\gamma} \nabla^2 V. \tag{7}
\]

Solving this equation numerically (see Appendix), we obtain an excellent agreement with the numerical simulations (Fig. 2(a)-(d)). We now show that this equation captures the structure of the pair correlation by examining its limiting regimes \( U \to 0, D_r \to 0 \), and \( D \to 0 \).

For small propulsion velocity \( U \), which corresponds to the left side of the phase diagram, Eq. (7) can be solved perturbatively. To order one in \( U \), we get a dipolar correction to the equilibrium radial correlation \( h_{\text{eq}}(r) \): \( B(\mathbf{r}, \mathbf{\theta}) = h_{\text{eq}}(r) + U \cos(\theta) B_1(r) \), where the first Fourier coefficient \( B_1(r) \) decays exponentially over a length \( \ell_\theta \) (see exact expression in Appendix). This prediction is compared to the numerical simulations in Fig. 3(a)-(c); a quantitative agreement with the prediction is obtained, without any adjustable parameter. Note that at order \( U^2 \), an attractive correction to the equilibrium pair correlation \( h_{\text{eq}}(r) \) is found (see Appendix), which is compatible with the results obtained when the activity is introduced in the form of a colored noise \cite{13, 17, 18}.

We turn to the large activity regime, and focus first on the limit \( D_r = 0 \), where the wings are entirely deployed (Fig. 3(d)); this limit corresponds to the bottom
the function $G$ can be obtained numerically. This prediction is in very good agreement with the numerical simulations (Fig. 3(f),(g)). Last, the transition with the scaling form obtained in the limit $D_r = 0$ can be obtained by matching the widths of the two profiles, $x \sim \ell_r^{1/3} y^{2/3}$ (Eq. (9)) and $x \sim \ell_p^{1/3} y^{2/3}$ (Eq. (10)), leading to $y \sim \sqrt{\ell_r \ell_p} = \ell_U$.

The structure of the correlations is now characterized (Fig. 2(e)). For $\text{Pe} < 1$, the activity generates a dipolar correction to the equilibrium correlations for lengths $r < \ell_U$. For $\text{Pe} > 1$, the dipolar correction crosses-over to depletion wings at a scale $\ell_r$; the shape of the wings is given by the scaling forms (9) for $\ell_r < r < \ell_U$ and (10) for $\ell_U < r < \ell_p$; the wings decay exponentially beyond $\ell_p$. The behavior of the correlations is summarized on the phase diagram, Fig. 2(e).

For weak interactions, we have obtained a quantitative agreement between our predictions derived from Eq. (7) and the numerical simulations. Moreover, the left hand side of Eq. (7), which controls the structure of the correlations, is also present in the two-body Smoluchowski equation [20], which is valid in the dilute limit for any interaction strength. We conclude that the structure of the correlations predicted here holds in the dilute limit for any interaction strength, which we check with numerical simulations of hard particles (see Appendix).

We finally note that, in practice, $D$ and $D_r$ are not taken as independent parameters. For spherical particles, where $D \sim a^2 D_r$, the Péclet number defined here takes the more common forms $\text{Pe} = a U / D = U / (a D_r)$, and $\ell_U \sim a$. At small Péclet, the dipolar correction, which decays over a length $\ell_U$, should thus be barely observable. At large Péclet, depletion wings with the shape
(10) should be observed for $a < r < \ell_p = a Pe$. Some theoretical studies assume $D = 0$ [32–34] and define the Péclet number as $Pe' = U/(aD_r)$; here also depletion wings with the shape (10) should be observed below $\ell_p$.

We have unveiled two regimes where active polar particles without alignment interactions have a pair correlation with a self-similar shape and an algebraic decay characterized by anomalous exponents. In presence of velocity-orientation coupling [35, 36] or alignment interactions [6], we may expect distinct scaling laws to appear in the correlations. As we have shown, the correlations have a rich structure even without three-body interactions, it would therefore be instructive to measure them in a dilute configuration first, and then to see how they evolve as density increases.

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EXPERIMENTS

Experimental system

The experimental system is the one used in Ref. [23]. We used a solution of Janus particles of diameter \( a = 3.17 \pm 0.32 \mu\text{m} \) sandwiched between two ITO electrodes separated by a spacer of size \( H = 50 \mu\text{m} \). The solvant is a sodium chloride solution of concentration \( 10^{-4} \text{ mol L}^{-1} \). The use of sodium chloride diminishes the temporal variation of the system, which enables long observation required for calculating pair correlations. For the correlations presented in the main text, we applied an electric field of frequency \( f = 5 \text{ kHz} \) and amplitude \( 2 \times 10^6 \text{ V}_{\text{pp}} \text{m}^{-1} \) (voltage \( 10 \text{ V}_{\text{pp}} \)). In this range of frequency, the Janus particles move in the direction of the uncoated hemisphere due to the induced-charge electrophoresis [22] and do not exhibit attractive interactions [23]. Moreover no polar order is observed [22]; this justifies the theoretical modeling by Active Brownian Particles.

Videos of the system were captured by a CMOS camera (Baumer, LXG-80, 3000 × 2400 pixels, 8 bit grayscale) at the framerate 10 fps mounted on an inverted microscope (Olympus, IX70) equipped with a 40× objective lens (LUCPLFLN, NA=0.60). A green filter is inserted between the sample and the halogen lamp to increase the contrast of the hemispheres of the Janus particles. The acquisition length is 14 minutes (8400 frames).

Image analysis

Particles were detected using the Hough Circle Transform algorithm implemented in the OpenCV library [37]. The positions of the particles are the centers of the circles. For each detected circle, we compute the center of mass of the pixels within it (the weights are the pixels’ values). The orientation of the particle is defined as the direction of the vector between the center of mass and the center of the circle. We will later give an estimate of the precision of this measure.

Experimental parameters

We now estimate the parameters of the video used for the correlations in the article.

Particle diameter. We captured the images with the resolution 0.12 μm px\(^{-1}\). In the obtained images of the particle with the diameter \( a = 3.17 \mu\text{m} \), the particle diameter appears \( a \approx 26 \text{ px} \).

Density. There are on average 487 particles in a 3000 × 2400 image. This gives \( \rho a^2 \approx 487 \frac{26}{3000} \frac{26}{2400} \approx 0.05 \). The area fraction is \( \phi = (\pi/4)\rho a^2 \approx 0.04 \).

Velocity. We use Trackpy [38] to obtain the trajectories. The instantaneous velocities can be obtained by applying a Savitzky–Golay filter [39]. From the instantaneous velocities, we find an average velocity \( U = 56 \pm 7 \text{ px s}^{-1} \) (standard deviation given for different particles, see Fig. S4 (a)). Thus, \( U \approx 6.7 \mu\text{m s}^{-1} \), that is to say of the order of two particle diameters per second.

Alternatively, one can compute the mean square displacement as a function of time (Fig. S4 (b)). At the lowest order in time, \( \langle \Delta x^2 \rangle = (U \Delta t)^2 \). We find \( U \approx 55 \text{ px s}^{-1} \approx 6.6 \mu\text{m s}^{-1} \). This is consistent with the previous result.

Rotational diffusion. The rotational diffusion constant \( D_r \) can be measured by looking at the mean square angle as a function of time. We expect

\[
\langle \Delta \theta^2 \rangle = \theta_{\text{err}}^2 + 2D_r \Delta t
\]

at short time (at long time, particles collide). \( \theta_{\text{err}} \) is the typical error we make in the detection of orientations. Looking at Fig. S4 (d), we find \( D_r \approx 0.12 \text{ s}^{-1} \) and \( \theta_{\text{err}} \approx 0.22 \text{ rad} \approx 13^\circ \).
Another way to go is to compute the velocity autocorrelation. For an Active Brownian Particle, the velocity $\mathbf{V}(t)$ is given by

$$\mathbf{V}(t) = U \hat{\mathbf{e}}_{\theta(t)} + \sqrt{2D\eta(t)} \quad \dot{\theta}(t) = \sqrt{2D_r \nu_i(t)} \quad (S12)$$

where $\eta$ and $\nu$ are unit Gaussian white noises. A quick computation shows that the velocity autocorrelation decays exponentially,

$$\langle \mathbf{V}(t) \cdot \mathbf{V}(t + \Delta t) \rangle = U^2 e^{-D_r \Delta t}. \quad (S13)$$

We focus on the short time regime and measure $D_r = 0.11 \text{ s}^{-1}$ (Fig. S4 (c)). This is consistent with the previous result. Note that the typical time scale over which collisions occurs is $[U^2 \rho]^{-1/2} \simeq 2 \text{ s}$, this is the reference we use when talking about “short time”.

The persistence length is $U/D_r \simeq 57 \mu \text{m} \simeq 18a$.

**Translational diffusion.** This is the hardest quantity to evaluate. We give an estimate based on the Stokes-Einstein relation,

$$D = \frac{k_B T}{6 \pi \eta (a/2) \beta}, \quad (S14)$$

$\eta = 1.0 \cdot 10^{-3} \text{ Pa s}^{-1}$ is the viscosity of water, $T \simeq 300 \text{ K}$ is the temperature, and $\beta$ is a correction factor due to the proximity of the bottom electrode. Assuming that Faxen’s law (Eq. (7-4.28) of Ref [40]) is valid for $h$ (distance to the wall) of the order of $a$, we obtain $\beta \simeq 3$ (one checks that the thermal fluctuations are negligible). At the end of the day,

$$D \simeq \frac{300 \times 1.38 \cdot 10^{-23}}{6 \pi \times 1.0 \cdot 10^{-3} \times 1.585 \cdot 10^{-6} \times 3} \simeq 0.05 \mu \text{m}^2 \text{s}^{-1} \simeq (0.07a)^2 / \text{s}. \quad (S15)$$

One notes that $a^2 D_r / D \simeq 24$: the rotational diffusion is dominant compared to the thermal diffusion. This is due to our choice of diameter $a \simeq 3 \mu \text{m}$. For much smaller diameters, the thermal diffusion would be so important that particles would swim upwards. For much larger diameters, the rotational diffusion would be negligible and particles would swim straight.
Correlations

Once we have detected the positions and the orientations, we consider a given frame. For each particle far from the edges of the image, we consider every other particle and compute its position in the reference frame of the orientation of the first particle. We put the result in bins of size $\Delta x = \Delta y = 0.1 \alpha$. After processing all the frames and normalizing the bins, we obtain the correlation that we plot on Fig 1b of the article (recalled on Fig. S5a).

We compare the cuts of the correlations from experiments to the ones from numerical simulations (Fig. S5) and observe that they are qualitatively similar. The experiments are thus well described by the numerics. According to the phase diagram (Fig 2e), the correlations should follow the scaling law given in Eq. (10). The experimental statistics are insufficient to conclude (Fig. S5c). In the numerical simulations, a deviation from the predicted scaling laws is observed (Fig. S5f); it may be due to the density, which is too high to be in the dilute regime (see Sec. ).

NUMERICAL SIMULATIONS

Description

We consider $N$ particles ($N = 5000$) in a square periodic box of size $L = \sqrt{N} \rho$ with $\rho$ the density ($L \sim 300$ at $\rho = 0.05$). Initially the positions $r_i$ and the orientations $\theta_i$ of the particles are assigned uniformly at random.

We use stochastic molecular dynamics and consider the following Langevin equations

\[
\dot{r}_i = -\nabla_i \sum_{j \neq i} V(r_i - r_j) + U \hat{e}_\theta_i + \sqrt{2D} \eta_i, \tag{S16}
\]

\[
\dot{\theta}_i = \sqrt{2D} \nu_i. \tag{S17}
\]
$U$ is the velocity, $D$ the translational diffusion and $D_r$ the rotational diffusion. The mobility $\gamma$ is set to 1. $\eta_i$ and $\nu_i$ are Gaussian white noises with unit variance. During a time increment $\Delta t$ ($\Delta t = 0.05$ for the simulations presented in the article), the term $\sqrt{2D_r} \nu_i$ generates an increment $\nu_i \sqrt{2D_r} \Delta t$ with $\nu_i$ a random number generated from a standard normal distribution. Similarly for $\sqrt{2D} \eta_i$.

We use the following soft-sphere potential, where the diameter $a$ of a particle and the potential strength $\epsilon$ are set to unity,

$$V(r) = \begin{cases} \frac{1}{2}(1 - \|r\|)^2 & \text{if } \|r\| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(S18)

We let the system evolve for a time $t_{eq} \sim 500$ before starting to record the correlations. $B(x)$ is then recorded with a spatial resolution $\Delta x = 0.1$, with one measure every $\Delta t_c = 5$, during an overall time period $T \sim 5 \cdot 10^5$. The results are averaged over 100 realizations of the system.

Simulations of dilute hard particles

The theoretical approach that we use is a linearized Dean equation, a framework which is valid for weak interactions. Consequently, the simulations that we compare to our theoretical results have relative values of potential strength and velocity $\epsilon : U = 1 : 10$. This means that the particles are able to interpenetrate. In this weak interaction regime, we are able to test quantitatively the scalings that we obtain.

That being said, it is interesting to test the scaling exponents that we obtain on simulations in the regime of strong interactions. We do it in Fig S6 and find that indeed the cuts of the correlation function obey the predicted scalings both in the limit of low rotational diffusion and in the limit of low translational diffusion. This leads us to state that the scaling exponents that we found are robust. However, the limit curves are different from our prediction in the weak interaction regime.

Fig S6c,d corresponds to a density $\rho = 0.02$. Indeed, for $\rho = 0.05$ three-body effects are present, leading to a deviation from the scaling form, similarly to what is observed in Fig. S5f.

THEORETICAL COMPUTATIONS

Equation for the correlations

Dean equation for Active Brownian Particles

The Langevin equations for Active Brownian Particles read,

$$\dot{r}_i = -\nabla_i \sum_{j \neq i} V(r_i - r_j) + U \hat{e}_{\theta_i} + \sqrt{2D} \eta_i,$$

(S19)

$$\dot{\theta}_i = \sqrt{2D_r} \nu_i$$

(S20)

$D$ is the translational diffusivity, $D_r$ is the rotational diffusivity, $\eta_i$ et $\nu_i$ are Gaussian white noises of unit variance. In this supplementary material, we set the particle diameter $a = 1$ and the friction coefficient $\gamma = 1$.

We define the density in position-orientation space $f(r, \theta, t)$ as

$$f(r, \theta, t) = \frac{1}{N} \sum_{i=1}^{N} \sum_{m=-\infty}^{\infty} f_i(r, \theta + 2m\pi, t)$$

(S21)

$$f_i(r, \theta, t) = \delta(r_i(t) - r)\delta(\theta_i(t) - \theta).$$

(S22)

We consider a smooth and fastly decaying test function $\varphi(r, \theta)$. By definition of $f_i$,

$$\varphi(r_i(t), \theta_i(t)) = \int \int_{-\infty}^{\infty} d\theta f_i(r, \theta, t) \varphi(r, \theta).$$

(S23)
The differentials are computed from Eqs. (S19) and (S20) (we assume translational diffusion. Correlation \(B(r)\) for \(\rho = 0.05, \epsilon : D : D_r : U = 50 : 0.1 : 0 : 10\). (b) Rescaled cuts with the exponents predicted in the article. (c) No translational diffusion. Correlation \(B(r)\) for \(\rho = 0.02, \epsilon : D : D_r : U = 50 : 0 : 0.1 : 10\). (d) Rescaled cuts with the exponents predicted in the article. The reader should note that (a) and (c) exhibit a non trivial structure at short range due to the hard interactions.

Then, the time derivative of \(\varphi(r_i(t), \theta_i(t))\) can be written in two different ways:

\[
\frac{d}{dt}\varphi(r_i(t), \theta_i(t)) = \int dr \int_{-\infty}^{\infty} d\theta \frac{\partial f_i}{\partial t}(r, \theta, t) \varphi(r, \theta) = \int dr \int_{-\infty}^{\infty} d\theta f_i(r, \theta, t)(dt)^{-1} d\varphi(r, \theta),
\]

with the differential \(d\varphi\) given by the Itô formula [41],

\[
d\varphi = \nabla \varphi \cdot d\mathbf{r}_i + \frac{\partial \varphi}{\partial \theta} d\theta_i + \frac{1}{2} \nabla^2 \varphi(d\mathbf{r}_i)^2 + \frac{1}{2} \frac{\partial^2 \varphi}{\partial \theta^2} d\theta_i^2 + \frac{\partial}{\partial \theta} \nabla \varphi \cdot d\mathbf{r}_i d\theta_i
\]

\[
= \nabla \varphi \cdot \left\{ -\nabla_i \sum_j V(r_i - r_j) + U \hat{e}_{\theta_i} \right\} dt + D dt \nabla^2 \varphi + D_r dt \frac{\partial^2 \varphi}{\partial \theta^2}.
\]

The differentials are computed from Eqs. (S19) and (S20) (we assume \(\nabla V(\mathbf{0}) = 0\)). Performing integrations by part and recalling that \(\varphi\) is arbitrary, one obtains

\[
\frac{\partial f_i}{\partial t} = D\nabla^2 f_i + D_r \frac{\partial^2 f_i}{\partial \theta^2} + \nabla \left( f_i \sum_{j=1}^{N} \nabla V(r - r_j(t)) \right) - U e_\theta \cdot \nabla f_i - \sqrt{2D} \nabla f_i \cdot \mathbf{\eta}_i - \sqrt{2D_r} \frac{\partial}{\partial \theta}(f \nu_i).
\]

Using Eq. (S21) and rearranging the noises like Dean [28], we finally obtain the following Dean equation for \(f(r, \theta, t)\),

\[
\frac{\partial}{\partial t} f(r, \theta, t) = -\nabla \mathbf{J}(r, \theta, t) - \frac{\partial}{\partial \theta} K(r, \theta, t)
\]

with the currents

\[
\mathbf{J}(r, \theta, t) = -D \nabla f(r, \theta, t) - f(r, \theta, t) \int_0^{2\pi} d\theta (\nabla V * f)(r, \theta, t) + f(r, \theta, t) U \hat{e}_\theta - f^{1/2}(r, \theta, t) \mathbf{\eta}(r, \theta, t)
\]

\[
K(r, \theta, t) = -D_r \frac{\partial}{\partial \theta} f(r, \theta, t) - f^{1/2}(r, \theta, t) \nu(r, \theta, t).
\]
The spatial convolution is defined by \( (f * g)(r) = \int dr' f(r') g(r - r') \). \( \eta \) and \( \nu \) are Gaussian white noises with correlations

\[
\langle \eta^\alpha(r, \theta, t) \eta^\beta(r, \theta, t) \rangle = 2D \delta^{\alpha \beta} \delta(r - r') \delta(t - t'), \tag{S31}
\]
\[
\langle \nu(r, \theta, t) \nu(r, \theta, t) \rangle = 2D_r \delta(r - r') \delta(t - t'). \tag{S32}
\]

**Linearized Dean equation**

The Dean equation for Active Brownian Particles is non linear with multiplicative noise. It is thus very difficult to tackle. Our approximation consists in linearizing it around an homogeneous density \( \rho \) which is the average density of particles. We write

\[
f(r, \theta, t) = \frac{\rho}{2\pi} + \sqrt{\frac{\rho}{2\pi}} \phi(r, \theta, t). \tag{S33}
\]

The field \( \phi \) is assumed to be of order 1 in \( \rho \).

At the lowest order, the Dean equation (S28) becomes linear with additive noise.

\[
\frac{\partial \phi}{\partial t} = \left[ D V^2 + D_r \frac{\partial^2}{\partial \theta^2} - U \hat{e}_\theta \cdot \nabla \right] \phi + \frac{\rho}{2\pi} \int_0^{2\pi} d\theta' \langle \nabla V * \phi \rangle(\theta') + \nabla \cdot \eta + \frac{\partial \nu}{\partial \theta}. \tag{S34}
\]

**Equation for the correlations**

We define the following correlations

\[
C(r_1, r_2, \theta_1, \theta_2) = \langle \phi(r_1, \theta_1) \phi(r_2, \theta_2) \rangle, \tag{S35}
\]
\[
C(r, \theta, \theta') = \frac{1}{\rho} \{ C(0, r, \theta') - \delta(r) \delta(\theta - \theta') \}. \tag{S36}
\]

One checks that the definition of \( C(r, \theta, \theta') \) is consistent with Eq. (1) the article. We use Itô calculus to compute the time evolution of \( C \).

\[
C(r_1, r_2, \theta_1, \theta_2, t + \delta t) - C(r_1, r_2, \theta_1, \theta_2, t) = \langle \phi(r_1, \theta_1, t) \delta \phi(r_2, \theta_2, t) \rangle + \langle \delta \phi(r_1, \theta_1, t) \phi(r_2, \theta_2, t) \rangle + \langle \delta \phi(r_1, \theta_1, t) \delta \phi(r_2, \theta_2, t) \rangle. \tag{S37}
\]

Computing the terms from the linearized Dean equation (S34), one shows that

\[
\partial_t C(r_1, r_2, \theta_1, \theta_2) = \left[ D \nabla_1^2 + \nabla_2^2 + D_r (\partial^2_{\theta_1} + \partial^2_{\theta_2}) - U (\hat{e}_{\theta_1} \cdot \nabla_1 + \hat{e}_{\theta_2} \cdot \nabla_2) \right] C(r_1, r_2, \theta_1, \theta_2)
\]
\[
+ \frac{\rho}{2\pi} \int_0^{2\pi} d\theta' \left[ \nabla_1^2 V * C(r_1, r_2, \theta', \theta_2) + \nabla_2^2 V * C(r_1, r_2, \theta_1, \theta') \right] + \left[ 2D \nabla_1 \nabla_2 + 2D_r \partial_{\theta_1} \partial_{\theta_2} \right] \delta(r_1 - r_2) \delta(\theta_1 - \theta_2).
\]
\[
\tag{S38}
\]

We use the invariance of the system by translation and write the equation in terms of \( C(r, \theta, \theta') \),

\[
\partial_t C(r, \theta, \theta') = \left[ 2D \nabla^2 + D_r (\partial^2_{\theta} + \partial^2_{\theta'}) + U (\hat{e}_{\theta} - \hat{e}_{\theta'}) \cdot \nabla \right] C(r, \theta, \theta') + 2 \nabla^2 V(r)
\]
\[
+ \frac{\rho}{2\pi} \int_0^{2\pi} d\theta'' \nabla^2 V * [C(r, \theta, \theta'') + C(r, \theta'', \theta')]. \tag{S39}
\]

The conventions are such that the pair correlation function \( C(r, \theta, \theta') \) is normalized with respect to the density \( \rho \). Focusing on the low density regime \( (\rho \rightarrow 0) \), we can neglect the convolution of \( C \) with the potential \( V \). The equation that we consider is

\[
\partial_t C(r, \theta, \theta') = \left[ 2D \nabla^2 + D_r (\partial^2_{\theta} + \partial^2_{\theta'}) + U (\hat{e}_{\theta} - \hat{e}_{\theta'}) \cdot \nabla \right] C(r, \theta, \theta') + 2 \nabla^2 V(r). \tag{S40}
\]
Note that for a passive system, the solution of this equation is the direct correlation function (the one involved in the Ornstein-Zernike equation). In the following, we are only interested in the stationary correlations which satisfy the following linear partial differential equation.

\[ [2D\nabla^2 + D_r(\partial_\theta + \partial_{\theta'}) + U(\hat{e}_\theta - \hat{e}_{\theta'}) \cdot \nabla] C(r, \theta, \theta') = -2\nabla^2 V(r). \]  

(S41)

By rotational invariance, \( C(r, \theta, \theta') = C(R_{-\theta} \cdot r, 0, \theta' - \theta) \). Like in the article, we define the “profile seen by a particle in its reference frame”,

\[ B(r) = \frac{1}{2\pi} \int_0^{2\pi} C(r, 0, \theta') d\theta'. \]  

(S42)

**Small activity**

At small activity, \( U \ll 1 \), one can expand \( C \) in power of \( U \).

\[ C(x, \theta, \theta') = C^{(0)}(x) + UC^{(1)}(x, \theta, \theta') + U^2 C^{(2)}(x, \theta, \theta') + \ldots \]  

(S43)

Let us write Eq (S41) in Fourier space, using the convention \( \hat{C}(k, \theta, \theta', t) = \int d\mathbf{r} e^{-i(kr)} C(r, \theta, \theta', t) \).

\[ \left[ -2Dk^2 + D_r(\partial_\theta + \partial_{\theta'}) \right] \hat{C}(k, \theta, \theta') = 2k^2\hat{V} - iUk \cdot (\hat{e}_\theta - \hat{e}_{\theta'}) \hat{G}(k, \theta, \theta'). \]  

(S44)

The passive correlation (\( U = 0 \)) doesn’t depend on the angles, the order 0 of the equation above leads to

\[ \hat{C}^{(0)}(k) = \frac{-\hat{V}(k)}{D}. \]  

(S45)

This is the usual Random Phase Approximation solution for direct correlations. At order 1, the equation to solve and its solution are

\[ \left[ -2Dk^2 + D_r(\partial_\theta^2 + \partial_{\theta'}^2) \right] \hat{C}^{(1)}(k, \theta, \theta') = -ik \cdot (\hat{e}_\theta - \hat{e}_{\theta'}) \hat{C}^{(0)}(k), \]  

(S46)

\[ \hat{C}^{(1)}(k, \theta, \theta') = ik \cdot (\hat{e}_\theta - \hat{e}_{\theta'}) \hat{D}^{(1)}(k), \]  

(S47)

\[ \hat{D}^{(1)}(k) = \frac{\hat{C}^{(0)}(k)}{2Dk^2 + D_r}. \]  

(S48)

Next, we solve the second order,

\[ \left[ -2Dk^2 + D_r(\partial_\theta^2 + \partial_{\theta'}^2) \right] \hat{C}^{(2)}(k, \theta, \theta') = 2k^2(1 - \hat{e}_\theta \cdot \hat{e}_{\theta'}) \hat{D}^{(1)}(k), \]  

(S49)

\[ \hat{C}^{(2)}(k, \theta, \theta') = -\hat{D}^{(1)}(k) \frac{D}{Dk^2 + D_r} + k^2 \hat{D}^{(1)}(k) \hat{e}_\theta \cdot \hat{e}_{\theta'}. \]  

(S50)

At the end of the day, the expansions at order \( U^2 \) of \( \hat{C}(k, \theta, \theta') \) and \( \hat{B}(k) \) are

\[ \hat{C}(k, \theta, \theta') = \hat{V}(k) \frac{D}{D} \left\{ \begin{array}{l} \hat{D}^{(0)}(k) \frac{1}{1 - \frac{D^2}{2Dk^2 + \ell_U^2}} - \frac{D}{D(2Dk^2 + \ell_U^2)} + \frac{U^2}{D^2} \hat{e}_\theta \cdot \hat{e}_{\theta'} \left( \frac{1}{2Dk^2 + \ell_U^2} - \frac{1}{k^2 + \ell_U^2} \right) \end{array} \right\}, \]  

(S51)

\[ \hat{B}(k) = \frac{\hat{V}(k)}{D} \left\{ \begin{array}{l} \hat{D}^{(0)}(k) \frac{1}{1 - \frac{D^2}{2Dk^2 + \ell_U^2}} - \frac{U}{2\pi D(2Dk^2 + \ell_U^2)} i k_x \end{array} \right\} \]  

(S52)

with the typical length scale \( \ell_U = \sqrt{D_r/D} \).

Our goal is now to look at large distances, that is to say, small wave number \( k \). We assume that \( \hat{V}(k) \) is regular at 0 (short-range potential) and we make the substitution \( \hat{V}(k) \mapsto \hat{V}(0) \). We define the function \( \hat{G}(k) \) and its inverse Fourier transform \( G(r) \) as

\[ \hat{G}(k) = \frac{1}{2Dk^2 + \ell_U^2} \Rightarrow G(r) = \frac{1}{4\pi} K_0 \left( \frac{\|r\|}{\sqrt{2\ell_U}} \right), \]  

(S53)
with \( K_0 \) the modified Bessel function of the second kind of order 0. In real space, the expansion of \( B(r) \) becomes,

\[
B(r) \sim -\frac{\tilde{V}(0)}{D} + U^2 \frac{\tilde{V}(0)}{D^3} G(r) - U \frac{\tilde{V}(0)}{2\pi D^2} \frac{\partial G(r)}{\partial x}
\]

\[
B(r, \theta) \sim -\frac{\tilde{V}(0)}{D} + U^2 \frac{\tilde{V}(0)}{4\pi D^3} K_0 \left( \frac{r}{\sqrt{2\ell_U}} \right) + U \frac{\tilde{V}(0) \sqrt{D}}{2\pi \sqrt{2D^{3/2}}} K_1 \left( \frac{r}{\sqrt{2\ell_U}} \right) \cos \theta
\]

(54)

(55)

\( K_1 \) is the modified Bessel function of the second kind of order 1. We detail the meaning of the terms. The first one in the passive correlation. The second term, scaling as \( U^2 \) is a positive contribution to the isotropic part of the correlations. As stated in the article, this has been argued, in the literature, to account for the motility induced phase separation. The last term is the dipolar correlation, at order \( U \), on which we focused in the article. Eq. (55) corresponds to the prediction in Fig. 3b-c. At large distance, the Bessel functions decay exponentially,

\[
K_0(r) \sim K_1(r) \sim \sqrt{\frac{\pi}{2r}} e^{-r}.
\]

(56)

Both the dipolar contribution (\( \cos \theta \)) and the additional isotropic part thus decay exponentially over the length scale \( \ell_U \).

**No rotational diffusion**

We now focus on the limit of no rotational diffusion \( D_r = 0 \). We easily obtain the Fourier transform \( C(r, \theta, \theta') \) from Eq. (S41),

\[
[2D\nabla^2 + U(\hat{e}_\theta - \hat{e}_{\theta'}) \cdot \nabla] C(r, \theta, \theta') = -2\nabla^2 V(r),
\]

(57)

\[
\tilde{C}(\mathbf{k}, \theta, \theta') = \frac{-2k^2 \tilde{V}(\mathbf{k})}{2Dk^2 - i\ell_U \mathbf{k} \cdot (\hat{e}_\theta - \hat{e}_{\theta'})}.
\]

(58)

This solution is very similar to the one for a binary mixture of particles forced respectively by \( U\hat{e}_\theta \) and \( U\hat{e}_{\theta'} \) [27]. We integrate over \( \theta' \) to obtain \( B \), this gives,

\[
\tilde{B}(\mathbf{k}) = -\frac{k^2 \tilde{V}(\mathbf{k})}{\pi D} \int_0^{2\pi} \frac{d\theta'}{2k^2 - i\ell_r \mathbf{k} \cdot (\hat{e}_\theta - \hat{e}_{\theta'})} = \frac{-2k^2 \tilde{V}(\mathbf{k})}{D \sqrt{(2k^2 - i\ell_r^{-1}k_x)^2 + \ell_r^{-2}k_y^2}}
\]

(59)

with the characteristic length scale \( \ell_r = D/U \). \( k_x = \mathbf{k} \cdot \hat{e}_x, k_y = \mathbf{k} \cdot \hat{e}_y \) where \( \hat{e}_x \) and \( \hat{e}_y \) are the unit vector along the horizontal and vertical axes of the plane.

We consider the limit of large distance, that is to say \( k \to 0 \). We assume that \( \tilde{V}(\mathbf{k}) \) is regular so that we can replace it by \( \tilde{V}(0) \). Furthermore, an analysis of leading terms gives \( k_y^2 \sim k_x^3 \). We keep up these leading terms:

\[
\tilde{B}(\mathbf{k}) \sim \frac{-2\tilde{V}(0)k_x^2}{D \sqrt{\ell_r^{-2}k_y^2 - 4i\ell_r^{-1}k_x^3}}.
\]

(60)

We can inverse Fourier transform with respect to \( k_y \):

\[
B(k_x, y) = -\frac{2\tilde{V}(0)\ell_r}{\pi D} k_x^2 K_0 \left( \left| y \right| \sqrt{-4i\ell_r k_x^2} \right).
\]

(61)

We then Fourier transform with respect to \( k_x \): \( B(x, y) = (2\pi)^{-1} \int dk_x e^{ik_x x} B(k_x, y) \). We perform the changes of variables \( q = -\ell_x y^2 \ell_r^{-1/3} k_x, w = xu^{1/3}(\ell_r y^2)^{-1/3} \) and obtain

\[
B(x, y) \sim \frac{\tilde{V}(0)}{D} \frac{1}{y^2} F \left( \frac{x}{|y|^{1/3} |y|^{2/3}} \right),
\]

(62)

\[
F(w) = \frac{1}{\pi^2} \int_{-\infty}^{\infty} e^{-iwq} q^2 K_0 \left( 2\sqrt{|q|^3} \right) dq.
\]

(63)

This is the scaling form mentioned in the article, and used in Fig. 3e.
No translational diffusion

We set $D = 0$. Looking at distances large compared to the particle diameter, we replace $\tilde{V}(k)$ by $\tilde{V}(0)$. The equation we consider is

$$[(\partial^2_\gamma + \partial^2_{\gamma'}) + i\ell_p k \cdot (\hat{e}_\theta - \hat{e}_{\theta'})] \tilde{C}(k, \theta, \theta') = \frac{2 \tilde{V}(0)}{D_r}.$$  \hfill (S64)

with the persistence length $\ell_p = U/D_r$.

At small $\ell_p k$ (large distance compared to $\ell_p$), the following development can be obtained,

$$\tilde{C}(k, \theta, \theta') = -\frac{\tilde{V}(0)}{D_r} \left\{ 1 + i\ell_p k \cdot (\hat{e}_\theta - \hat{e}_{\theta'}) + \ell_p^2 k^2 \hat{e}_\theta \cdot \hat{e}_{\theta'} + O\left((\ell_p k)^3\right) \right\}.$$  \hfill (S65)

This is a hint that $\tilde{C}(k, \theta, \theta')$ is analytical around $k = 0$, meaning that $C(r, \theta, \theta')$ decays fastly (e.g. exponentially) at distances large compared to $\ell_p$.

We now consider distances below $\ell_p$, but still large compared to the particle diameter. We define the angles $\gamma$ and $\gamma'$ in the reference frame of $k$: $k \cdot \hat{e}_\theta = k \cos \gamma$ and $k \cdot \hat{e}_{\theta'} = k \cos \gamma'$, where $k$ is the norm of $k$. We obtain

$$\left[(\partial^2_\gamma + \partial^2_{\gamma'}) + \frac{i}{2} \ell_p k (\cos \gamma - \cos \gamma')\right] \tilde{C}(k, \gamma, \gamma') = \frac{2 \tilde{V}(0)}{D_r} k^2.$$  \hfill (S66)

and we study it in the regime $\ell_p k \gg 1$. A numerical resolution at constant $k$ (see Fig. S7) shows that $\tilde{C}(k, \gamma, \gamma')$ concentrates around the two points $(\gamma, \gamma') = (0, 0)$ and $(\pi, \pi)$. We focus on $(0, 0)$ around which the equation reads

$$\left[(\partial^2_\gamma + \partial^2_{\gamma'}) - \frac{i}{2} \ell_p k(\gamma^2 - \gamma'^2)\right] \tilde{C}(k, \gamma, \gamma') = \frac{2 \tilde{V}(0)}{D_r}.$$  \hfill (S67)

We realize that we can inject the following scalings

$$\tilde{C}(k, \gamma, \gamma') \sim \frac{2 \tilde{V}(0)}{D_r} \ell_p k^{3/2} \tilde{H}_r \left(\gamma (\ell_p k)^{1/4}, \gamma' (\ell_p k)^{1/4}\right).$$  \hfill (S68)

From Eq. (S67), the function $\tilde{H}(u, v)$, for $u$ and $v$ unbounded, is independent of $k$ and is the solution of the linear partial differential equation

$$\left[\partial^2_u + \partial^2_v - \frac{i}{2} (u^2 - v^2)\right] \tilde{H}(u, v) = 1.$$  \hfill (S69)

Around $(\gamma, \gamma') = (\pi, \pi)$, the scalings are

$$\tilde{C}(k, \gamma, \gamma') \sim \frac{2 \tilde{V}(0)}{D_r} \ell_p k^{3/2} \tilde{H}(\gamma - \pi)(\ell_p k)^{1/4}, \gamma' - \pi)(\ell_p k)^{1/4}. \hfill (S70)$$

with $H^*$ the complex conjugate of $H$.

The scaling for $\tilde{B}$ around $\gamma = 0$ is

$$\tilde{B}(k, \gamma) = \frac{1}{2\pi} \int_0^{2\pi} d\gamma' \tilde{C}(k, \gamma', \gamma') \sim \frac{\tilde{V}(0)}{\pi D_r} \ell_p k^{5/4} \tilde{H}_r \left(\gamma (\ell_p k)^{1/4}\right),$$  \hfill (S71)

$$H_B(u) = \int_{-\infty}^{\infty} dv \tilde{H}(u, v).$$  \hfill (S72)

And around $\gamma = \pi$, one checks that

$$\tilde{B}(k, \pi - \gamma) \sim \frac{\tilde{V}(0)}{\pi D_r} \ell_p k^{5/4} \tilde{H}_r^* \left((\pi - \gamma)(\ell_p k)^{1/4}\right).$$  \hfill (S73)
Figure S7. Numerical resolution of Eq. (S66) with \( \tilde{V}(0) = U = D_r = 1 \), for \( k = 10^4 \). (a) Real part of \( C \). (b) Imaginary part of \( C \). The solution concentrates around \( (\gamma, \gamma') = (0,0) \) and \( (\pi, \pi) \). The numerical resolution consists in discretizing \( \gamma \) and \( \gamma' \), and then solving the linear system corresponding to Eq. (S66).

with \( H^*_B \) the complex conjugate of \( H_B \). We now switch from polar coordinates \( (k, \gamma) \) to cartesian coordinates \( (k_x, k_y) \). We approximate

\[
k_x = k \cos \gamma \simeq \begin{cases} +k & \text{if } \gamma \simeq 0 \\ -k & \text{if } \gamma \simeq \pi \end{cases}, \quad k_y = k \sin \gamma \simeq \begin{cases} k \gamma & \text{if } \gamma \simeq 0 \\ k(\pi - \gamma) & \text{if } \gamma \simeq \pi \end{cases}
\]

(S74)

As we consider small angles (\(|\gamma| \ll 1\) or \(|\pi - \gamma| \ll 1\)), the values of \( k_x \) and \( k_y \) are such that \( k_y \ll k_x \).

Using the two expressions Eqs (S71) and (S73), we obtain a scaling form for the Fourier transform \( \tilde{B} \).

\[
\tilde{B}(k_x \geq 0, k_y) = \frac{\tilde{V}(0)}{\pi D_r} \left( \ell_p |k_x| \right)^{5/4} H^+_B \left( \frac{\ell_p^{1/4} k_y}{|k_x|^{3/4}} \right)
\]

(S75)

with \( H^+_B = \tilde{H}_B \) used when \( k_x > 0 \) and \( H^-_B = H^*_B \) used when \( k_x < 0 \). Finally, we perform the Fourier inversion

\[
B(x, y) = \frac{1}{2\pi} \int dk_x e^{ik_y x} \frac{1}{2\pi} \int dk_y e^{ik_y y} \tilde{B}(k_x, k_y).
\]

(S76)

first with respect to \( k_y \), then with respect to \( k_x \). Using the appropriate changes of variables, we obtain a scaling form for \( B(x, y) \),

\[
B(x, y) = \frac{\tilde{V}(0)}{\pi D_r} \frac{\ell_p^4}{y^4} G \left( \frac{\ell_p^{1/3} x}{|y|^{4/3}} \right),
\]

(S77)

\[G(w) = \frac{1}{2\pi} \int_0^\infty dz e^{iwz} z^2 H^+_B(z^{3/4}) + \frac{1}{2\pi} \int_{-\infty}^0 dz e^{iwz} z^2 H^-_B(|z|^{3/4})
\]

(S78)

\( H^+_B \) is the inverse Fourier transform of \( \tilde{H}^+_B \), one has \( H_B(a) = (H^+_B)^*(-a) \). This form corresponds to Eq. (10) of the article. The rescaled cuts are plotted on Fig. 3g with the prediction (gray line) computed from a rescaling of the numerical solution of Eq. (7).

**NUMERICAL INTEGRATION OF THE THEORETICAL EQUATION**

Equation in terms of three parameters

We consider the time-dependent equation (S40) for the correlations \( C \) at low density,

\[
\partial_t C(r, \theta, \theta') = \left[ 2D \nabla^2 + D_r (\partial_{\theta}^2 + \partial_{\theta'}^2) + U (\hat{e}_\theta - \hat{e}'_{\theta'}) \cdot \nabla \right] C(r, \theta, \theta') + 2 \nabla^2 V(r).
\]

(S79)
In polar coordinates, we write $r = r e_φ$. The later equation depends on four coordinates (plus time): $(r, φ, θ, θ')$. By performing a rotation of angle $θ$, the symmetries allow to reduce the problem to three parameters $(r, α, β)$,

$$α = φ - θ \quad \beta = θ' - θ.$$  \hspace{1cm} \text{(S80)}

We write $x = r e_α = (x, y)$, $C$ is then a function only of $x$ and $β$. Its time evolution is given by

$$\partial_t C(x, β) = \left[ 2D\nabla^2 + D_e L_{\text{angles}} + U \left((1 - \cos β) \frac{∂}{∂x} - \sin β \frac{∂}{∂y}\right) \right] C(x, β) + 2\nabla^2 V(x),$$  \hspace{1cm} \text{(S81)}

$$\nabla^2 = \frac{∂^2}{∂x^2} + \frac{∂^2}{∂y^2}$$  \hspace{1cm} \text{(S82)}

$$L_{\text{angles}} = \left( y^2 \frac{∂^2}{∂x^2} + x^2 \frac{∂^2}{∂y^2} - 2xy \frac{∂^2}{∂x∂y} - x \frac{∂}{∂x} - y \frac{∂}{∂y}\right) + 2 \left(-y \frac{∂}{∂x} + x \frac{∂}{∂y}\right) \frac{∂}{∂β} + 2 \frac{∂^2}{∂β^2}$$  \hspace{1cm} \text{(S83)}

It is important to note that $B$ is given by the integration over $β$,

$$B(x) = \int_0^{2π} dβ C(x, β).$$  \hspace{1cm} \text{(S84)}

**Numerical integration**

We consider the domain $(x, y, β) \in [-x_{\text{max}}, x_{\text{max}}] \times [0, y_{\text{max}}] \times [-π, π]$. We discretize it with steps $Δx$ in $x$, $Δy = Δx$ in $y$ and $Δβ$ in $β$.

We start from $C(x, y, β, t = 0) = 0$ and integrate Eq. (S81) in time using an explicit Euler scheme with time step $Δt$.

The differential operators are evaluated using finite differences valid at order $(Δx)^2$ and $(Δβ)^2$,

$$\frac{∂}{∂x} C(x, y, β) = \frac{C(x + Δx) - C(x - Δx)}{2Δx},$$  \hspace{1cm} \text{(S85)}

$$\frac{∂^2}{∂x^2} C(x, y, β) = \frac{C(x + Δx) + C(x - Δx) - 2C(x)}{(Δx)^2},$$  \hspace{1cm} \text{(S85)}

and so on.

The potential is

$$V(x) = \begin{cases} \frac{1}{2}(1 - ||x||)^2 & \text{if } ||x|| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$  \hspace{1cm} \text{(S86)}

and its Laplacian is evaluated on the grid.

The boundary conditions are as follow:

- Periodic boundary conditions for $β$;
- $C(-x_{\text{max}}, y, β) = C(+x_{\text{max}}, y, β) = C(x, y_{\text{max}}, β) = 0$;
- We use the symmetry relation $C(x, -y, -β) = C(x, y, β)$ to impose the additional points $C(x, -Δy, β) = C(x, Δy, -β)$.

The numerical integration in time converges to the stationary solution at sufficiently large time. When the increment on $C$ over a time step $Δt$ is small enough, we output the stationary solution $C^{eq}(x, β)$ and its integral over $β$, $B^{eq}(x)$. This method is used on Fig 2a-d.

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