MODULI THEORY ASSOCIATED TO HOCHSCHILD PAIRS

ISAMU IWANARI

ABSTRACT. Let $C$ be an $A$-linear stable $\infty$-category and let $(\mathcal{H}H^\bullet(C/A),\mathcal{H}H_\bullet(C/A))$ be the pair of the Hochschild cohomology spectrum (Hochschild cochain complex) and the Hochschild homology spectrum (Hochschild chain complex). The purpose of this paper is to establish a moduli-theoretic interpretation of the algebraic structure on the Hochschild pair $(\mathcal{H}H^\bullet(C/A),\mathcal{H}H_\bullet(C/A))$ of $C$. The notions of cyclic deformations and equivariant deformations (of the Hochschild chain complex) associated to deformations of $C$ play a central role.

1. Introduction

Since [7] Gerstenhaber developed the deformation theory of associative algebras by means of Hochschild cohomology [10]. The Hochschild cohomology of an associative algebra governs the deformation theory of algebra. The relationship between deformations of associative algebras and Hochschild cohomology has been fundamental and successful in many branches of mathematics: e.g., deformation quantizations of Poisson manifolds [17]. The algebraic structure on the Hochschild cochain complex (which computes Hochschild cohomology) plays an important role in recent developments in deformations of associative algebras based on Hochschild cohomology. The Hochschild cochain complex admits a structure of an algebra over the little disk operad (see e.g. Introduction of [15] and references therein for many proofs of the existence of an algebraic structure). The Hochschild cohomology of associative algebras is invariant under Morita equivalences. Based on the invariance, Hochschild cohomology is generalized to those of abelian categories, suitable enriched categories and stable $\infty$-categories, etc. Moreover, deformation theories of abelian categories and stable $\infty$-categories have been developed by using Hochschild cohomology theories of them [20], [16], [23, X]. In particular, in [23] the deformation theory of stable $\infty$-categories is formulated in the local version of derived geometry over $E_2$-algebras, i.e., algebras over the little disk operad $E_2$, and it is shown that the deformation theory is controlled by the $E_2$-algebra of Hochschild cochain complex.

Let $A$ be a commutative ring spectrum (or simply we may suppose that $A$ is an ordinary commutative ring). Let $C$ be an $A$-linear small stable idempotent-complete $(\infty, 1)$-category/$\infty$-category. Let $\mathcal{H}H^\bullet(C/A)$ and $\mathcal{H}H_\bullet(C/A)$ denote the Hochschild cohomology spectrum and the Hochschild homology spectrum of $C$ over $A$, respectively (in differential graded setting, they are the Hochschild cochain complex and the Hochschild chain complex). We refer to them simply as Hochschild cohomology and Hochschild homology, respectively though these are not graded abelian groups. The Hochschild homology $\mathcal{H}H_\bullet(C/A)$ admits an $S^1$-action given by the Connes operator. Moreover, there exists a certain module action of the $E_2$-algebra $\mathcal{H}H^\bullet(C/A)$ on $\mathcal{H}H_\bullet(C/A)$. These algebraic structures on the Hochschild pair $(\mathcal{H}H^\bullet(C/A),\mathcal{H}H_\bullet(C/A))$ is defined as an algebra over a topological operad $KS$ having two colors, that is called Kontsevich-Soibelman operad [18]. For the construction of this algebra structure we refer the reader to [11] and references therein (we will use the construction in [11], cf. Section 8.1 for a quick review). The algebra $(\mathcal{H}H^\bullet(C/A),\mathcal{H}H_\bullet(C/A))$ over $KS$ can be thought of as an analogue of the pair $(\oplus_{p\geq 0} \Lambda^p T_M,\oplus_{q\geq 0} \Omega^q_M)$ of multivector fields and differential forms of a differential (or algebraic) manifold $M$. Needless to say, in algebraic geometry and differential geometry, the structures on $(\oplus_{p\geq 0} \Lambda^p T_M,\oplus_{q\geq 0} \Omega^q_M)$ given by Cartan calculus are important. For example, the algebraic description of local period maps for a family of algebraic manifolds due to Griffiths [8] has been fundamental in Torelli problems.

The purpose of this paper is to provide a moduli-theoretic interpretation of the algebraic structure on the Hochschild pair $(\mathcal{H}H^\bullet(C/A),\mathcal{H}H_\bullet(C/A))$. To get a feeling of our results and moduli problems we consider, we will describe the formulation in an informal way. For precise presentations, we refer the reader to the text. Let $A$ be a connective noetherian commutative differential graded (dg) algebra.
over a field $k$ of characteristic zero. Let $C = C_A$ be an $A$-linear small stable idempotent-complete $\infty$-category. We may regard $C_A$ as a family of stable $\infty$-categories over $\text{Spec} A$. We denote by $\text{St}_R$ the $\infty$-category of $R$-linear stable idempotent-complete $\infty$-categories. For an augmented commutative dg $A$-algebra $R \to A$, we let $\text{Def}_C(R)$ denote the $\infty$-groupoid of deformations of $C$ to $R$ that consists of $(C_R \in \text{St}_R, \phi : C_R \otimes R A \simeq C)$ where $C_R \otimes_R A$ is the base change to $\text{St}_A$. Let $\text{Art}^{\text{tr}}_A$ be the $\infty$-category of trivial square-zero extensions of $A$. The assignment $[R \to A] \mapsto \text{Def}_C(R)$ defines the deformation functor

$$\text{Def}_C : \text{Art}^{\text{tr}}_A \to S$$

associated to $C$, where $S$ is the $\infty$-category of $\infty$-groupoids/spaces. Let $\mathcal{HH}_*(C/A)$ be the Hochschild homology of $C$ that is a dg $A$-module. It admits an action of $S^1$, which corresponds to Connes operator. Let $\text{Def}^{S^1}(\mathcal{HH}_*(C/A)) : \text{Art}^{\text{tr}}_A \to S$ be the deformation functor which assigns $R \to A$ to the $\infty$-groupoid $\text{Def}^{S^1}(\mathcal{HH}_*(C/A))(R)$ of $S^1$-equivariant deformations $(N_R, N \otimes_R A \simeq \mathcal{HH}_*(C/A))$ of $\mathcal{HH}_*(C/A)$ such that $N_R$ is a dg $R$-module endowed with $S^1$-action.

Suppose that we are given a deformation $C_R$ of $C$ to $\text{St}_R$. Then the relative Hochschild homology $\mathcal{HH}_*(C_R/R)$ is a deformation of $\mathcal{HH}_*(C/A)$ to $R$. The moduli-theoretic assignment $C_R \mapsto \mathcal{HH}_*(C_R/R)$ determines a natural transformation

$$M_C : \text{Def}_C \to \text{Def}^{S^1}(\mathcal{HH}_*(C/A)).$$

There is a categorical correspondence between dg Lie algebras and pointed formal stacks [23, X], [6], [9] (cf. Section 3.3). In particular, to a dg Lie algebra $L$ over $A$ one can associate a pointed formal stack $\mathcal{F}_L$ that is defined as a functor $\mathcal{F}_L : \text{Art}^{\text{tr}}_A \to S$. This beautiful correspondence allows us to use dg Lie algebras in the study of deformation theories, i.e., local moduli theory. Let $\mathcal{G}_C$ be the dg Lie algebra over $A$ associated to the $E_2$-algebra $\mathcal{HH}^*(C/A)$ whose underlying complex of $\mathcal{G}_C$ is $\mathcal{HH}^*(C/A)$ [1]. Let $\text{End}^L(\mathcal{HH}_*(C/A))$ be the endomorphism dg Lie algebra which has the $S^1$-action arising from that of $\mathcal{HH}_*(C/A)$. Let $\text{End}^L(\mathcal{HH}_*(C/A))^{S^1}$ denote the homotopy fixed points. We associate formal stacks $\mathcal{F}_{\mathcal{G}_C}$ and $\mathcal{F}_{\text{End}^L(\mathcal{HH}_*(C/A))^{S^1}}$. There are canonical morphisms

$$\text{Def}_C \to \mathcal{F}_{\mathcal{G}_C} \quad \text{and} \quad \text{Def}^{S^1}(\mathcal{HH}_*(C/A)) \to \mathcal{F}_{\text{End}^L(\mathcal{HH}_*(C/A))^{S^1}}$$

in the functor category $\text{Fun}(\text{Art}^{\text{tr}}_A, S)$. Intuitively, they exhibit $\mathcal{F}_{\mathcal{G}_C}$ and $\mathcal{F}_{\text{End}^L(\mathcal{HH}_*(C/A))^{S^1}}$ as formal stacks closest to $\text{Def}_C$ and $\text{Def}^{S^1}(\mathcal{HH}_*(C/A))$, respectively [23]. For instance, each $\text{Def}^{S^1}(\mathcal{HH}_*(C/A))(R)$ is fully faithful when we regard spaces as $\infty$-groupoids. Moreover, $\text{Def}^{S^1}(\mathcal{HH}_*(C/A)) \to \mathcal{F}_{\text{End}^L(\mathcal{HH}_*(C/A))^{S^1}}$ is an equivalence under a mild condition. In this way, we may think that $\mathcal{G}_C$ and $\text{End}^L(\mathcal{HH}_*(C/A))^{S^1}$ control deformations of $C$ and $S^1$-equivariant deformations of $\mathcal{HH}_*(C/A)$.

The algebraic input is an action of the dg Lie algebra $\mathcal{G}_C$ on $\mathcal{HH}_*(C/A)$. This action is determined by the algebra $(\mathcal{HH}_*(C/A), \mathcal{HH}_*(C/A))$ over $\text{KS}$ in a purely algebraic way. The corresponding dg Lie algebra map $\mathcal{G}_C \to \text{End}^L(\mathcal{HH}_*(C/A))^{S^1}$ gives rise to a morphism of formal stacks

$$\zeta_C : \mathcal{F}_{\mathcal{G}_C} \to \mathcal{F}_{\text{End}^L(\mathcal{HH}_*(C/A))^{S^1}}.$$

In this situation, the simplest version of the main result can be stated as follows:

**Theorem 1.1.** The diagram

$$\begin{array}{ccc}
\text{Def}_C & \xrightarrow{M_C} & \text{Def}^{S^1}(\mathcal{HH}_*(C/A)) \\
\downarrow & & \downarrow \\
\mathcal{F}_{\mathcal{G}_C} & \xrightarrow{\zeta_C} & \mathcal{F}_{\text{End}^L(\mathcal{HH}_*(C/A))^{S^1}}
\end{array}$$

commutes up to canonical homotopy.

The morphism $M_C : \text{Def}_C \to \text{Def}^{S^1}(\mathcal{HH}_*(C/A))$ has a clear moduli-theoretic meaning. On the other hand, $\mathcal{F}_{\mathcal{G}_C} \to \mathcal{F}_{\text{End}^L(\mathcal{HH}_*(C/A))^{S^1}}$ is obtained by the $S^1$-equivariant Lie algebra action of $\mathcal{G}_C$ on $\mathcal{HH}_*(C/A)$ which comes from the algebraic structure on the Hochschild pair $(\mathcal{HH}_*(C/A), \mathcal{HH}_*(C/A))$. Therefore, we may think that the above result reveals a moduli-theoretic aspects of a certain portion of $(\mathcal{HH}_*(C/A), \mathcal{HH}_*(C/A))$. 
Cyclic deformations. An important observation is that $M_C$ factors as the sequence

$$\text{Def}_C \xrightarrow{M_C} \text{Def}^C(\mathcal{H}_C(C/A)) \xrightarrow{N_C} \text{Def}^{S^1}(\mathcal{H}_C(C/A))$$

such that $\text{Def}^C(\mathcal{H}_C(C/A))$ are refinements of $\text{Def}^{S^1}(\mathcal{H}_C(C/A))$. This factorization and $\text{Def}^C(\mathcal{C})$ are crucial in this work though they do not appear in the statement of Theorem 1.1. Moreover, the refined diagram not only provides an appropriate framework which allows us to prove main results but also amplifies Theorem 1.1 with respect to practical uses. Recall that $\mathcal{H}_C(C/A)$ has an $S^1$-action corresponding to Connes operator. The functor $\text{Def}^{S^1}(\mathcal{H}_C(C/A))$ describes $S^1$-equivariant deformations of $\mathcal{H}_C(C/A)$. The functor $\text{Def}^C(\mathcal{H}_C(C/A))$ describes what we call cyclic deformations of $\mathcal{H}_C(C/A)$. The notion of cyclic deformation is the key to revealing the moduli-theoretic meaning, and it also has practical significance. Since this notion is relatively new, we briefly introduce the definition of cyclic deformations. For an augmented $A$-algebra $R \to A$, a cyclic deformation of $\mathcal{H}_C(C/A)$ to $R$ is a pair $(N, N \otimes_{R\otimes A^S} A \simeq \mathcal{H}_C(C/A))$ such that $N$ is a $(R \otimes A S^1)$-module endowed with an $S^1$-action which commutes with the canonical action on $R \otimes A S^1$. Here $R \otimes A S^1$ is the tensor of $R$ by $S^1$ in the $\infty$-category of commutative dg algebras over $A$ so that $R \otimes A S^1 \simeq R \otimes_{R \otimes A} R$. Let $\text{Def}^C(\mathcal{H}_C(C/A)) : \mathcal{A}_{tsz} \to \mathcal{S}$ denote the deformation functor which assigns to $R \in \mathcal{A}_{tsz}$ to the $\mathcal{S}$-groupoid of cyclic deformations of $\mathcal{H}_C(C/A)$. Each deformation of $C_R$ of $C$ in $\text{Def}_C(R)$ maps to a cyclic deformation $\mathcal{H}_C(C_R/A)$ of $\mathcal{H}_C(C/A)$ that belongs to $\text{Def}^C(C)(R)$ (it is important to notice that the associated cyclic deformation is not $\mathcal{H}_C(C_R/R)$ but $\mathcal{H}_C(C_R/A)$). The base change $\otimes_{R \otimes A} R$ sends $\mathcal{H}_C(C_R/A)$ to $\mathcal{H}_C(C_R/R)$, that is an $S^1$-equivariant deformation of $\mathcal{H}_C(C/A)$ to $R$. The transition

$$C_R \rightsquigarrow \{\mathcal{H}_C(C_R/A) \cap S^1\} \rightsquigarrow \{\mathcal{H}_C(C_R/R) \cap S^1\}$$

induces the above factorization (the final procedure forgets the $S^1$-action).

The functor $\text{Def}^C(\mathcal{H}_C(C/A))$ has a natural transformation to a functor $\mathcal{F}^C \otimes \text{End}(\mathcal{H}_C(C/A))$ that is defined in an algebraic way (cf. Notation 8.8, Construction 8.6). We will give several different presentations of $\mathcal{F}^C \otimes \text{End}(\mathcal{H}_C(C/A))$ which are related to one another via Koszul-type dualities. These different descriptions enable us to prove Theorem 1.1 and the following generalization (cf. Theorem 8.23, Remark 8.25):

**Theorem 1.2.** There exists a diagram in the functor category $\text{Fun}(\mathcal{A}_{tsz}, \mathcal{S})$

$$\begin{array}{ccc}
\text{Def}_C & \xrightarrow{M_C} & \text{Def}^C(\mathcal{H}_C(C/A)) \\
F_{\mathcal{G}_C} & \xrightarrow{T_C} & \mathcal{F}^C \otimes \text{End}(\mathcal{H}_C(C/A)) \\
\text{J}_C & \xrightarrow{J_C^C} & \mathcal{F}^C \otimes \text{End}(\mathcal{H}_C(C/A)) \\
\end{array}$$

which commutes and is an extension of the diagram in Theorem 1.1.

It should be worth emphasizing that the upper row has a moduli-theoretic interpretation while the lower row is defined in a purely algebraic way. In fact, the lower row admits a description in terms of dg Lie algebras. See Proposition 8.15, Proposition 8.24, Remark 8.7, and Remark 8.21. Let $\mathcal{G}_C$ denote the cotensor by $S^1$, i.e., $\mathcal{G}_C \simeq \mathcal{G}_C \times_{\mathcal{G}_C} \mathcal{G}_C$. The morphism $T_C : F_{\mathcal{G}_C} \to F_{\mathcal{A}_{tsz}}$ is induced by the Lie algebra action $\mathcal{G}_C \to \text{End}^L(\mathcal{H}_C(C/A))$ determined by the Hochschild pair $(\mathcal{H}_C(C/A), \mathcal{H}_C(C/A))$, which extends the action of $\mathcal{G}_C$ on $\mathcal{H}_C(C/A)$. Informally, Theorem 1.2 especially means that a moduli-theoretic interpretation of the Lie algebra action of $\mathcal{G}_C$ on $\mathcal{H}_C(C/A)$ is given by $M_C : \text{Def}_C \to \text{Def}^C(\mathcal{H}_C(C/A))$ determined by the assignment $C_R \rightsquigarrow \mathcal{H}_C(C_R/A)$. Likewise, an algebraic incarnation of $\text{Def}_C \to \text{Def}^{S^1}(\mathcal{H}_C(C/A))$ is the composite $F_{\mathcal{G}_C} \to F_{\text{End}^L(\mathcal{H}_C(C/A)) S^1}$ induced by the $S^1$-equivariant Lie algebra map $\mathcal{G}_C \xrightarrow{\text{diagonal}} \mathcal{G}_C \to \text{End}^L(\mathcal{H}_C(C/A))$. The situation may be depicted as the following table:

| algebra | moduli |
|---------|--------|
| $\mathcal{G}_C \rightsquigarrow \mathcal{H}_C(C/A)$ | $C_R \rightsquigarrow \mathcal{H}_C(C_R/R)$; equivariant deformation |
| $\mathcal{G}_C \rightsquigarrow \mathcal{H}_C(C/A)$ | $C_R \rightsquigarrow \mathcal{H}_C(C_R/R)$; cyclic deformation |
Applications. Let us mention what are uses of our results. One use is to applications to the study of Hochschild homology and Hochschild cohomology of stable ∞-categories. The relationship between moduli-theoretic methods in derived geometry and algebraic structures of Hochschild pair is useful. In fact, our main result has already been applied to the study of Hochschild homology: in [12] we use the pair (\(\mathcal{HH}^\wedge(C/A), \mathcal{HH}_\bullet(C/A)\)) and main results of this paper to prove that \(\mathcal{HH}^\wedge(C/A)\) admits an equivariant deformation to the derived loop space \(LX = \text{Map}(S^1, \text{Spec } A)\) in a natural way. Moreover, we show that the associated periodic cyclic homology/complex \(\mathcal{HP}_\bullet(C/A)\) has a D-module structure. It is worth mentioning that the data of cyclic deformations plays an essential role in these applications. The results of this paper will be used in our future works.

Another use is to the study of the period map in noncommutative algebraic geometry. The negative cyclic homology gives a Hodge-type filtration on the periodic cyclic homology of a stable ∞-category (cf. [14]). Theorem 1.2 can be used to construct and study the period map of a family of stable ∞-categories which is a map into the classifying (moduli) space of Hodge-type filtrations on the periodic cyclic homology. Combining with the recent progress on the study of \(\mathcal{HH}^\wedge_\bullet(C/A)\), one may expect applications of algebraic structures on \(\mathcal{HH}^\wedge_\bullet(C/A)\) to the period map in noncommutative algebraic geometry.

This paper is organized as follows: Section 2 collects conventions and some of the notation that we will use. In Section 3, we will review some of background material and will formulate adequate basis for our main goal. In Section 4 we will give a brief guide to subsequent sections for the reader’s convenience. In Section 5, we give an axiomatic formulation of deformation problems and their Koszul duals. In Section 6 we apply the axiomatic formulation defined in Section 5 to deformations of categories and cyclic deformations of modules. The main result is Proposition 6.8. In Section 7 we give detail analysis of the interaction between Koszul duality with changes of operads and with Hochschild homology. In Section 8 we prove main results of this paper.

2. Notation and Convention

(\(\infty, 1\))-categories. Throughout this paper we use the language of \((\infty, 1)\)-categories. We use the theory of quasi-categories as a model of \((\infty, 1)\)-categories. We assume that the reader is familiar with this theory. We will use the notation similar to that used in [11], [12]. A quasi-category is a simplicial ∞-categories as a set which satisfies the weak Kan condition of Boardman-Vogt. Following [21], we shall refer to quasi-categories. Our main references are [21] and [22]. To an ordinary category, we can assign an ∞-category by taking its nerve, and therefore when we treat ordinary categories we often omit the nerve \(N(\cdot)\) and directly regard them as ∞-categories.

Here is a list of (some) of the conventions and notation that we will use:

- \(S\): the ∞-category of (small) spaces.
- \(\mathbb{Z}\): the ring of integers, \(\mathbb{R}\) denotes the set of real numbers which we regard as either a topological space or a ring.
- \(\Delta^n\): the category of linearly ordered non-empty finite sets (consisting of \([0], [1],\ldots, [n] = \{0,\ldots, n\},\ldots\)
- \(\Delta^n\): the standard \(n\)-simplex
- \(N\): the simplicial nerve functor (cf. [21, 1.1.5])
- \(S\): ∞-category of small spaces/∞-groupoids. We denote by \(\hat{S}\) the ∞-category of large spaces (cf. [21, 1.2.16]).
- \(C^\infty\): the largest Kan subcomplex of an ∞-category \(C\)
- \(C^{\text{op}}\): the opposite ∞-category of an ∞-category. We also use the superscript “\(\text{op}\)” to indicate the opposite category for ordinary categories and enriched categories.
- \(\text{Sp}\): the stable ∞-category of spectra.
- \(\text{Fun}(A, B)\): the function complex for simplicial sets \(A\) and \(B\). If \(A\) and \(B\) are ∞-categories, we regard \(\text{Fun}(A, B)\) as the functor category.
- \(\text{Map}_{\infty}(C, C')\): the mapping space from an object \(C \in C\) to \(C' \in C\) where \(C\) is an ∞-category. We usually view it as an object in \(S\) (cf. [21, 1.2.2]).
- \(\text{Fin}_*\): the category of pointed finite sets \((0), (1),\ldots, (n),\ldots\) where \((n) = \{\ast, 1,\ldots, n\}\) with the base point \(\ast\). We write \(\Gamma\) for \(N(\text{Fin}_*)\). \((n)^\circ = (n)\backslash\ast\). Notice that the (nerve of) Segal’s gamma category is the opposite category of our \(\Gamma\).
Operads and Algebras. We will use operads. We employ the theory of $\infty$-operads which is thoroughly developed in [22]. The notion of $\infty$-operads gives one of the models of colored operads. Here is a list of (some) of the notation about $\infty$-operads and algebras over them that we will use:

- Let $\mathcal{M}^\otimes \to \mathcal{O}^\otimes$ be a fibration of $\infty$-operads. We denote by $\text{Alg}_{/\mathcal{O}^\otimes}(\mathcal{M}^\otimes)$ the $\infty$-category of algebra objects (cf. [22, 2.1.3.1]). We often write $\text{Alg}_{/\mathcal{O}^\otimes}(\mathcal{M})$ for $\text{Alg}_{/\mathcal{O}^\otimes}(\mathcal{M}^\otimes)$.

- $\text{CAlg}(\mathcal{M}^\otimes)$: $\infty$-category of commutative algebra objects in a symmetric monoidal $\infty$-category $\mathcal{M}^\otimes \to \text{N}(\text{Fin}_*) = \Gamma$. When the symmetric monoidal structure is clear, we usually write $\text{CAlg}(\mathcal{M})$ for $\text{CAlg}(\mathcal{M}^\otimes)$.

- $\text{Mod}^\otimes_R(\mathcal{M}^\otimes) \to \Gamma$: symmetric monoidal $\infty$-category of $R$-module objects, where $\mathcal{M}^\otimes$ is a symmetric monoidal $\infty$-category such that (1) the underlying $\infty$-category admits a colimit for any simplicial diagram, and (2) its tensor product functor $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$ preserves colimits of simplicial diagrams separately in each variable. Here $R$ belongs to $\text{CAlg}(\mathcal{M}^\otimes)$ cf. [22, 3.3.3, 4.5.2]. We write $\text{Mod}^\otimes_R(\mathcal{M}^\otimes)$ for the underlying $\infty$-category. When $\mathcal{M}^\otimes$ is the symmetric monoidal $\infty$-category $\text{Sp}^\otimes$ of spectra, we write $\text{Mod}^\otimes_R(\mathcal{M}^\otimes)$.

- $\text{CAlg}_{/R}^\otimes$: $\infty$-category of commutative algebra objects in the symmetric monoidal $\infty$-category $\text{Mod}^\otimes_R$, where $R$ is a commutative ring spectrum, that is, an object of $\text{CAlg}(\text{Sp})$. We write $\text{CAlg}_{/R}$ for $\text{CAlg}_{/R}^\otimes(\mathcal{M}_{/R}) \simeq \text{CAlg}(\text{Sp}_{/R})$. When $R$ is the Eilenberg-MacLane spectrum $HC$ with a commutative ring $C$, then we write $\text{CAlg}_C$ for $\text{CAlg}_{/HC}$. If $F$ is a field of characteristic zero, the $\infty$-category $\text{CAlg}_F$ is equivalent to the $\infty$-category obtained from the model category of commutative differential graded $F$-algebras by inverting quasi-isomorphisms (cf. [22, 7.1.4.11]). Therefore, we often think of a commutative differential graded (dg) algebra as an object of $\text{CAlg}_F$, and refer to an object of $\text{CAlg}_F$ as a commutative dg algebra. We denote by $\text{CAlg}^\otimes_{/F}$ the full subcategory of $\text{CAlg}_F$ which consists of connective commutative dg algebras. For $A \in \text{CAlg}^\otimes_{/F}$, we say that $A$ is noetherian if $H^n(A)$ is a usual noetherian ring and if $H^{-n}(A)$ is trivial when $n >> 0$ and is of finite type over $H^0(A)$ for any $n$.

- $E^\otimes_n$: the $\infty$-operad of little $n$-cubes (cf. [22, 5.1]). For a symmetric monoidal $\infty$-category $\mathcal{C}^\otimes$, we write $\text{Alg}_n(\mathcal{C})$ or $\text{Alg}_{E^\otimes_n}(\mathcal{C})$ for the $\infty$-category of algebra objects over $\mathcal{E}^\otimes_n$ in $\mathcal{C}^\otimes$. We refer to an object of $\text{Alg}_n(\mathcal{C})$ as an $E^\otimes_n$-algebra in $\mathcal{C}$. If we denote by $A^\otimes_n$ the associative operad ([22, 4.1.1]), there is the standard equivalence $A^\otimes_n \simeq E^\otimes_n$ of $\infty$-operads. We usually identify $\text{Alg}_1(\mathcal{C})$ with the $\infty$-category $\text{Alg}_{A^\otimes_n}(\mathcal{C})$, that is, the $\infty$-category of associative algebras in $\mathcal{C}$. We write $\text{Alg}_{n,C}(\mathcal{C})$ for the $\infty$-category $\text{Alg}_n(\mathcal{C})/1_C$ of augmented objects where $1_C$ denotes the unit algebra.

- $L\mathcal{M}^\otimes$: the $\infty$-operad defined in [22, 4.2.1.7]. Roughly, an algebra over $L\mathcal{M}^\otimes$ is a pair $(A, M)$ such that $A$ is a unital associative algebra and $M$ is a left $A$-module. For a symmetric monoidal $\infty$-category $\mathcal{C}^\otimes \to \Gamma$, we write $\text{LMod}(\mathcal{C})$ or $\text{LMod}(\mathcal{C})$ for $\text{Alg}_{L\mathcal{M}^\otimes}(\mathcal{C}^\otimes)$. There is the natural inclusion of $\infty$-operads $A^\otimes_n \to L\mathcal{M}^\otimes$. This inclusion determines $\text{LMod}(\mathcal{C}) \to \text{Alg}_{A^\otimes_n}(\mathcal{C}) \simeq \text{Alg}_1(\mathcal{C})$ which sends $(A, M)$ to $A$. For $A \in \text{Alg}_1(\mathcal{C})$, we define $\text{LMod}_A(\mathcal{C})$ to be the fiber of $\text{LMod}(\mathcal{C}) \to \text{Alg}_1(\mathcal{C})$ over $A$ in $\text{Cat}^\infty$.

- $\mathcal{R}M^\otimes$, $\mathcal{B}M^\otimes$: these $\infty$-operads are variants of $L\mathcal{M}^\otimes$ which are used to define structures of right modules over associative algebras and bimodules over pairs of algebras [22, 4.2.1.36, 4.3.1]. Informally, an algebra over $\mathcal{R}M^\otimes$ is a pair $(A, M)$ such that $A$ is a unital associative algebra and $M$ is a right $A$-module. In a similar vein, an algebra over $\mathcal{B}M^\otimes$ is a triple $(A, M, B)$ such that $A$ and $B$ are unital associative algebras, and $M$ is an $A-B$-bimodule. For a symmetric monoidal $\infty$-category $\mathcal{C}^\otimes \to \Gamma$, we write $\text{RMod}(\mathcal{C})$ (or simply $\text{RMod}(\mathcal{C})$ and $\text{BMod}(\mathcal{C})$ (or $\text{BMod}(\mathcal{C})$) for $\text{Alg}_{\mathcal{R}M^\otimes}(\mathcal{C})$ and $\text{Alg}_{\mathcal{B}M^\otimes}(\mathcal{C})$, respectively. There is a canonical functor $\text{BMod}(\mathcal{C}) \to \text{Alg}_1(\mathcal{C}) \times \text{Alg}_1(\mathcal{C})$ which sends $(A, M, B)$ to $(A, B) \in \text{Alg}_1(\mathcal{C}) \times \text{Alg}_1(\mathcal{C})$.

- Unless otherwise stated, $k$ is a base field of characteristic zero.

Group actions. Let $G$ be a group object in $\mathcal{S}$ (see e.g. [21, 7.2.2.1] for the notion of group objects). The main example in this paper is the circle $S^1$. Let $\mathcal{C}$ be an $\infty$-category. For an object $C \in \mathcal{C}$, a $G$-action on $C$ means a lift of $C \in \mathcal{C}$ to $\text{Fun}(BG, \mathcal{C})$, where $BG$ is the classifying space of $G$. A $G$-equivariant morphism means a morphism in $\text{Fun}(BG, \mathcal{C})$. We often identify $\text{Fun}(BG, \mathcal{C})$ as the limit (“$G$-invariants”) of the trivial $G$-action on $C$ and write $C^G$ for $\text{Fun}(BG, \mathcal{C})$ (e.g., $\text{Mod}_A^{S^1}$). (We remark that when we regard $G$ as a group object, $C^G$ is not the cotensor with the space $G$.)
3. Background

In this section, we review some theories which we will use.

3.1. $\infty$-categories of stable $\infty$-categories. Let $\mathcal{St}$ be the $\infty$-category of small stable idempotent-complete $\infty$-categories whose morphisms are exact functors. This $\infty$-category is compactly generated. Let $\mathcal{C}$ be a small stable idempotent-complete $\infty$-category and let $\text{Ind}(\mathcal{C})$ denote the $\infty$-category of $\text{Ind}$-objects [21, 5.3.5]. Then $\text{Ind}(\mathcal{C})$ is a compactly generated stable $\infty$-category. The inclusion $\mathcal{C} \to \text{Ind}(\mathcal{C})$ identifies the essential image with the full subcategory $\text{Ind}(\mathcal{C})^\omega$ spanned by compact objects in $\text{Ind}(\mathcal{C})$. Given $\mathcal{C}, \mathcal{C}' \in \mathcal{St}$, if we write $\text{Fun}^e(\mathcal{C}, \mathcal{C}')$ for the full subcategory spanned by exact functors, the left Kan extension [21, 5.3.5.10] gives rise to a fully faithful functor $\text{Fun}^e(\mathcal{C}, \mathcal{C}') \to \text{Fun}_L^e(\text{Ind}(\mathcal{C}), \text{Ind}(\mathcal{C}'))$ whose essential image consists of those functors $F$ such that the essential image of $F$ are contained in $\mathcal{C}'$. Here $\text{Fun}^e((\text{Ind}(\mathcal{C}), \text{Ind}(\mathcal{C}')) \subset \text{Fun}(\text{Ind}(\mathcal{C}), \text{Ind}(\mathcal{C}'))$ denotes the full subcategory consisting of those functors which preserve small colimits.

We let $\text{Pr}_L^t$ denote the $\infty$-category of presentable $\infty$-categories such that mapping spaces are spaces of functors which preserve small colimits (i.e., left adjoint functors) [21, 5.5.3]. It has a closed symmetric monoidal structure whose internal Hom/mapping objects are given by $\text{Fun}^e(\mathcal{C}, \mathcal{C}') \subset \text{Fun}(\text{Ind}(\mathcal{C}), \text{Ind}(\mathcal{C}'))$ denotes the full subcategory consisting of those functors which preserve small colimits and preserve compact objects. There exists a sequence

$$\mathcal{St} \to \text{Cgt}_{\mathcal{St}}^L \subset \text{Pr}_{\mathcal{St}}^L$$

where the left arrow is an equivalence given by $\text{Ind}$-construction $\mathcal{C} \mapsto \text{Ind}(\mathcal{C})$. The subcategory $\text{Cgt}_{\mathcal{St}}^L \subset \text{Pr}_{\mathcal{St}}^L$ is closed under the tensor product so that $\text{Cgt}_{\mathcal{St}}^L$ and $\mathcal{St}$ inherit symmetric monoidal structures from that of $\text{Pr}_{\mathcal{St}}^L$. The stable $\infty$-category of compact spectra is a unit object in $\mathcal{St}$. Given two objects $\mathcal{C}$ and $\mathcal{C}'$ of $\mathcal{St}$, the tensor product $\mathcal{C} \otimes \mathcal{C}'$ is naturally equivalent to the full subcategory $\text{Ind}(\mathcal{C}) \otimes \text{Ind}(\mathcal{C}')$ spanned by compact objects. The tensor product functor $\otimes : \mathcal{St} \times \mathcal{St} \to \mathcal{St}$ preserves small colimits separately in each variable since $\text{Fun}^e(\mathcal{C}, \mathcal{C}')$ gives rise to a fully faithful functor $\text{Fun}^e(\mathcal{C}, \mathcal{C}') \to \text{Fun}_L^e(\text{Ind}(\mathcal{C}), \text{Ind}(\mathcal{C}'))$.

For more details, we refer the readers to [4, Section 3], [22, 4.8].

Let $A \in \text{CAlg}(\text{Sp})$. Let $\text{Mod}_{A}^\otimes \in \text{CAlg}(\text{Pr}_{\mathcal{St}}^L)$ be the symmetric monoidal $\infty$-category of $A$-modules in $\text{Sp}$. Let $\text{Perf}_A^\otimes (\mathcal{St})$ be the symmetric monoidal $\infty$-category of $\text{Perf}_A^\otimes$-modules in $\mathcal{St}$. This symmetric monoidal $\infty$-category is presentable, and the tensor product functor preserves small colimits separately in each variable. We refer to an object of the underlying $\infty$-category $\text{Mod}_{\text{Perf}_A^\otimes}(\mathcal{St})$ as an $A$-linear small stable $\infty$-category. For ease of notation, put $\mathcal{St}_A^\otimes = \text{Mod}_{\text{Perf}_A^\otimes}(\mathcal{St})$ and $\mathcal{St}_A : = \text{Mod}_{\text{Perf}_A^\otimes}(\mathcal{St})$. We refer the reader to [11] for the description of $\mathcal{St}_A^\otimes$ by means of spectral categories. We write $(\text{Pr}_{A}^L)^\otimes$ for $\text{Mod}_{\text{Mod}_{A}^\otimes}^\otimes (\text{Pr}_{\mathcal{St}}^L)$. We refer to an object of the underlying $\infty$-category $\text{Pr}_{A}^L$ as an $A$-linear stable presentable $\infty$-categories.

For $B \in \text{Alg}_1(\text{Mod}_A)$, we denote by $\text{LMod}_B(\text{Mod}_A)$ (resp. $\text{RMod}_A(\text{Mod}_A)$) (or simply $\text{LMod}_B$ and (resp. $\text{RMod}_A$)) the $\infty$-category of left $B$-modules (resp. right $B$-module spectra) (there is a canonical equivalence $\text{LMod}_B(\text{Mod}_A) \simeq \text{LMod}_B(\text{Sp})$ induced by the forgetful functor $\text{Mod}_A \to \text{Sp}$). There is a symmetric monoidal functor $\Theta_A : \text{Alg}_1(\text{Mod}_A) \to (\text{Pr}_{A}^L)_{\text{Mod}_A}$/ which carries $B : \text{Mod}_A \to \text{LMod}_B(\text{Mod}_A)$ such that $\alpha$ is determined by $\text{Sp} \otimes_A^L \text{Mod}_A \to \text{LMod}_B(\text{Mod}_A)$ which carries the sphere spectrum to $B$. The homomorphism $f : B \to C$ maps to the base change functor $f^* : \text{LMod}_B \to \text{LMod}_C$. It is a fully faithful left adjoint functor, see [22, 4.8.5.11]. Since $\Theta_A$ is a symmetric monoidal functor, by Dunn additivity theorem [22, 5.1.2], it determines $\text{Alg}_n(\text{Mod}_A) \simeq \text{Alg}_{n-1}(\text{Alg}_1(\text{Mod}_A)) \to \text{Alg}_{n-1}(\text{Pr}_{A}^L)$, that is, an associative monoidal $\infty$-category. For $B \in \text{Alg}_1(\text{Mod}_A)$, we denote by $\text{Perf}_B$ the smallest stable idempotent-complete subcategory of $\text{LMod}_B$ which contains $B$ (it is more appropriate to write $\text{LPerf}_B$ instead of $\text{Perf}_B$). The full subcategory $\text{Perf}_B$ coincides with the full subcategory of compact objects so that $\text{Perf}_B \to \text{LMod}_B$ is extended to an equivalence $\text{Ind}(\text{Perf}_B) \to \text{LMod}_B$. Moreover, if $B$ is a commutative algebra (which comes from $\text{CAlg}(\text{Mod}_A)$), $\text{Perf}_B$ can be identified with the full subcategory of dualizable objects in the symmetric monoidal $\infty$-category $\text{Mod}_B \simeq \text{LMod}_B$. When $B \in \text{Alg}_1(\text{Mod}_A)$.
is promoted to $\text{Alg}_2(\text{Mod}_A)$, the tensor product functor $\otimes : \text{LMod}_B \times \text{LMod}_B \to \text{LMod}_B$ induces $\text{Perf}_B \times \text{Perf}_B \to \text{Perf}_B$ so that $\text{Perf}_B$ inherits a monoidal structure from that of $\text{LMod}_B$. The functor $\Theta_A : \text{Alg}_1(\text{Mod}_A) \to (\text{Mod}_A^\Lambda_{\text{L,cpt}})_{\text{Mod}_A}$ factors as $\text{Alg}_1(\text{Mod}_A) \to \text{LMod}_B^\Lambda_{\text{L,cpt}}(\text{Mod}_A) \to (\text{Mod}_A^\Lambda_{\text{L,cpt}})_{\text{Mod}_A}$. It gives rise to a symmetric monoidal functor $\text{Alg}_1(\text{Mod}_A) \to (\text{St}_A)_{\text{Perf}_A} \simeq \text{LMod}_B^\Lambda_{\text{L,cpt}}(\text{Mod}_A)$. This functor sends $B$ to $\text{Perf}_A \to \text{Perf}_B$. We denote the symmetric monoidal functor $\text{Alg}_1(\text{Mod}_A) \to \text{St}_A$ (induced by forgetting the functor from $\text{Perf}_A$) by $\text{Perf}_{-}$. 

3.2. Let $\mathcal{A}$ be a small $\infty$-category. Let $\mathcal{P}(\mathcal{A})$ denote the functor category $\text{Fun}(\mathcal{A}^{\text{op}}, \mathcal{S})$ where the $\mathcal{S}$ is the $\infty$-category of spaces/$\infty$-groupoids. There is the Yoneda embedding $\mathcal{h}_A : \mathcal{A} \to \mathcal{P}(\mathcal{A})$. Let $\mathcal{P}_\Sigma(\mathcal{A}) \subset \mathcal{P}(\mathcal{A})$ be the full subcategory spanned by those functors $\mathcal{A}^{\text{op}} \to \mathcal{S}$ which preserve finite products [21, 5.5.8.8]. The $\infty$-category $\mathcal{P}_\Sigma(\mathcal{A})$ is compactly generated, and $\mathcal{P}_\Sigma(\mathcal{A}) \subset \mathcal{P}(\mathcal{A})$ is characterized as the smallest full subcategory which contains the essential image of the Yoneda embedding and is closed under sifted colimits.

Suppose that $\mathcal{A}$ admits finite coproducts and a zero object $0$. Consider the set of morphisms $S = \{ \mathcal{h}_A(0) \cup M \mathcal{h}_A(0) \to \mathcal{h}_A(0 \cup M \mathcal{h}_A(0)) \}$ where $M \in \mathcal{A}$ such that the pushout $0 \cup M \mathcal{h}_A(0)$ exists in $\mathcal{A}$. Let $\mathcal{P}_\Sigma^A(\mathcal{A})$ be the presentable $\infty$-category obtained from $\mathcal{P}_\Sigma(\mathcal{A})$ by inverting morphisms in $S$ (see e.g. [21, 5.5.4] for the localization). The $\infty$-category $\mathcal{P}_\Sigma^A(\mathcal{A})$ can be regarded as the full subcategory of $\mathcal{P}_\Sigma(\mathcal{A})$ spanned by $S$-local objects. In other words, $\mathcal{P}_\Sigma^A(\mathcal{A})$ is the full subcategory which consists of functors $F$ such that the canonical morphism $F(0 \cup M \mathcal{h}_A(0)) \to \ast \times F(M) \ast$ is an equivalence for any $M \in \mathcal{A}$ such that $0 \cup M \mathcal{h}_A(0)$ exists in $\mathcal{A}$. Note that any object of the essential image of the Yoneda embedding $\mathcal{A} \to \mathcal{P}_\Sigma(\mathcal{A})$ is $S$-local.

Let $\mathcal{D}$ be a presentable $\infty$-category. Let $\text{Fun}^L(\mathcal{P}_\Sigma^A(\mathcal{A}), \mathcal{D})$ be the full subcategory of $\text{Fun}(\mathcal{P}_\Sigma^A(\mathcal{A}), \mathcal{D})$ which consists of colimit-preserving functors (i.e., left adjoint functors). Let $\text{Fun}^S(\mathcal{A}, \mathcal{D})$ be the full subcategory of $\text{Fun}(\mathcal{A}, \mathcal{D})$ spanned by those functors $f$ which preserve finite coproducts and carry pushouts of the form $0 \cup M \mathcal{h}_A(0) \to f(0 \cup M \mathcal{h}_A(0))$.

3.3. Formal stacks. Let $A$ be a connective commutative dg algebra $A$ over a field $k$ of characteristic zero. We use a correspondence between formal stacks and dg Lie algebras over $A$, that is proved by Hennion [9]. In Gaitsgory and Rozenblyum [6], a similar correspondence is established in the Ind-coherent setting. These are generalizations of the correspondence between dg Lie algebras and formal moduli problems over a field of characteristic zero in Lurie [23, X].

Let $\text{Lie}_A$ be the $\infty$-category of dg Lie algebras: there are several approaches to define it. One approach is to obtain it from the model category of dg Lie algebras by inverting quasi-isomorphisms. Another one is to consider the $\infty$-category of algebras in $\text{Mod}_A$ over the Lie operad $\text{Lie}$. Let $\text{Art}^{\text{ez}}_A$ be the full subcategory of $\text{CAlg}^{\leq 0}_{\text{A}//\text{A}} := (\text{CAlg}^{\leq 0}_A)_{\text{A}/\text{A}}$, which is spanned by trivial square zero extensions $A = A \oplus 0 \hookrightarrow A \oplus M$ such that $M$ is a connective $A$-module of the form $\bigoplus_{1 \leq i \leq n} A^{[d_i]}$. By abuse of notation, we often write $R$ for an object $A \to R \to A$ of $\text{CAlg}^{\leq 0}_{\text{A}//\text{A}}$. Similarly, we often omit the augmentations from the notation. Let $\text{TSZ}_A$ denote the opposite category of $\text{Art}^{\text{ez}}_A$. We define the $\infty$-category $\hat{\text{St}}^*_A$ of formal stacks over $A$ to be $\mathcal{P}_\Sigma^A(\text{TSZ}_A)$. We often regard $\text{TSZ}_A$ as a full subcategory of $\hat{\text{St}}^*_A$. By definition, $\hat{\text{St}}^*_A$ is the full subcategory of $\text{Fun}(\text{Art}^{\text{ez}}_A, \mathcal{S})$ so that we think of a formal stack as a functor $\text{Art}^{\text{ez}}_A \to \mathcal{S}$.

Let $\text{Free}_{\text{Lie}} : \text{Mod}_A \to \text{Lie}_A$ be the free Lie algebra functor which is a left adjoint to the forgetful functor $\text{Lie}_A \to \text{Mod}_A$. Let $\text{Mod}^{	ext{ez}}_A \subset \text{Mod}_A$ be the full subcategory that consists of objects of the form $\bigoplus_{1 \leq i \leq n} A^{[d_i]} (d_i \leq -1)$. Let $\text{Lie}^{	ext{ez}}_A$ be the full subcategory of $\text{Lie}_A$, which is the essential image of the restriction of the free Lie algebra functor $\text{Mod}^{	ext{ez}}_A \to \text{Lie}_A$. According to [9, 1.2.2], the inclusions $\text{Mod}^{	ext{ez}}_A \hookrightarrow \text{Mod}_A$ and $\text{Lie}^{	ext{ez}}_A \hookrightarrow \text{Lie}_A$ are extended to equivalences $\mathcal{P}_\Sigma^A(\text{Mod}^{	ext{ez}}_A) \to \text{Mod}_A$ and $\mathcal{P}_\Sigma^A(\text{Lie}^{	ext{ez}}_A) \to \text{Lie}_A$ in an essentially unique way (cf. Section 3.2).

\[ Ch^* : \text{Lie}_A \xrightarrow{\text{equiv}} (\text{CAlg}_{\text{A}//\text{A}})^{\text{op}} : \mathbb{D}_\infty \]
be the adjoint pair where the left adjoint $Ch^\bullet$ is the “Chevalley-Eilenberg cochain functor” which carries $L \in Lie_A$ to the Chevalley-Eilenberg cochain complex $Ch^\bullet(L)$ (i.e., the $A$-linear dual of Chevalley-Eilenberg chain complex), see e.g. [9, 1.4], [23, X, 2.2]. Thanks to [9, 1.5.6], this adjoint pair induces an adjoint pair

$$F : Lie_A \rightleftarrows \hat{St}_A : L.$$  

Moreover, if $A$ is noetherian (cf. section 3) both $F$ and $L$ are categorical equivalences. The left adjoint $F$ is defined as follows. The restriction of the functor $Ch^\bullet$ to $Lie_A$ induces $Lie_A^f \rightarrow TSZ \subset (CAlg_{A/\Lambda})^{op}$ such that $Lie_A^f \rightarrow TSZ \rightarrow \hat{St}_A$ belongs to $Fun^{st}(Lie_A^f, \hat{St}_A)$. There exists an essentially unique left adjoint functor $F : Lie_A \simeq P_{\mathbb{A}}(Lie_A^f) \rightarrow \hat{St}_A$ which extends $Lie_A^f \rightarrow \hat{St}_A$ (cf. the final equivalence in Section 3.2). The right adjoint $L$ is induced by the composition with $Lie_A^f \rightarrow TSZ$ where we think of $Lie_A$ as the full subcategory of $Fun((Lie_A^f)^{op}, S)$. If $A$ is noetherian, $F$ and $L$ are reduced to a pair of mutually inverse functors $Ch^\bullet : Lie_A^f \simeq TSZ : \mathbb{D}_\infty$. For $L \in Lie_A$, we usually write $FL$ for the image $F(L)$.

3.4. Koszul duals of $E_n$-algebras. Let $n \geq 1$ be a natural number and let $Alg_n^+(Mod_A)$ be the $\infty$-category of $E_n$-algebras in $Mod_A$. We write $Alg_n^+(Mod_A)$ for the $\infty$-category $Alg_n(Alg_n^+(Mod_A))$ of augmented $E_n$-algebras. We review the Koszul duals of augmented $E_n$-algebras. Let $p_B : B \rightarrow A$ and $p_C : C \rightarrow A$ be augmented $E_n$-algebras. Let $Map_{Alg_n^+(Mod_A)}(B \otimes_A C, A) \rightarrow Map_{Alg_n^+(Mod_A)}(B, A) \times Map_{Alg_n^+(Mod_A)}(C, A)$ be the map induced by the compositions with $B = B \otimes_A A \rightarrow B \otimes_B C$ and $C = A \otimes_A C \rightarrow B \otimes_B C \rightarrow A$. We denote by $Pair(p_B, p_C)$ or simply $Pair(B, C)$ the fiber product $Map_{Alg_n^+(Mod_A)}(B \otimes_A C, A) \times Map_{Alg_n^+(Mod_A)}(B, A) \times Map_{Alg_n^+(Mod_A)}(C, A) \{ (p_B, p_C) \}$. We shall refer to $Pair(B, C)$ as the space of pairing of $p_B : B \rightarrow A$ and $p_C : C \rightarrow A$. The functor $Alg_n^+(Mod_A)^{op} \rightarrow S$ given by $C \mapsto Pair(B, C)$ is representable by an object $D_{E_n}(B) \in Alg_n^+(Mod_A)$. We shall call $D_{E_n}(B)$ the $E_n$-Koszul dual of $B \rightarrow A$. For ease of notation, we write $D_n$ for $D_{E_n}$. There is a universal/tautological pairing $B \otimes_A D_n(B) \rightarrow A$ which corresponds to the identity map $id \in Map_{Alg_n^+(Mod_A)}(D_n(B), D_n(B))$. The Koszul dual $D_n(B)$ can also be interpreted as a centralizer of $B \rightarrow A$, see [22, 5.3.1]. Thanks to the construction in [23, X, 4.4.6] or [22, 5.2.5.5] or Construction 5.5, the assignment $B \mapsto D_n(B)$ is promoted to a $(E_n$-Koszul duality) functor $D_n : Alg_n^+(Mod_A)^{op} \rightarrow Alg_n^+(Mod_A)$ whose right adjoint is $D_n : Alg_n^+(Mod_A) \rightarrow Alg_n^+(Mod_A)^{op}$. The Koszul duals can be described in terms of bar constructions. Let $Bar : Alg_n^+(Mod_A) \rightarrow Mod_A$ be the functor given by $[B \rightarrow A] \mapsto A \otimes_B A$, which we refer to as the bar construction of augmented algebras. Then $A \otimes_B A$ admits a structure of a coaugmented coalgebra in a suitable way, and $Bar$ is promoted to $Bar : Alg_n^+(Mod_A) \rightarrow Alg_n^+(Mod_A)^{op}$ (here we abuse notation by using the same symbol). Interating bar construction we have induces $Bar^n : Alg_n^+(Mod_A) \rightarrow Alg_n^+(Mod_A)^{op}$. According to [23, X, 4.4.20], for $B \in Alg_n^+(Mod_A)$, the $E_n$-Koszul dual $D_n(B)$ is equivalent to the $A$-linear dual $Hom_A(B, A)$. We remark that there exists a Koszul duality functor in more general setting. Let $M$ be a symmetric monoidal presentable $\infty$-category whose tensor product $M \otimes M \rightarrow M$ preserves small colimits separately in each variable. If we replace $Mod_A$ by $M$, there exists an $E_n$-Koszul duality functor $(Alg_n(M)^{op})^{op} \rightarrow (Alg_n(M)/A)$ which carries an augmented algebra $B \rightarrow 1$ to $D_n(B) \in Alg_n(M)/A$ where 1 is a unit algebra (see [22, 5.2.5.8]).

3.5. Universal enveloping algebras. We will define an adjoint pair

$$U_n : Lie_A \rightleftarrows Alg_n^+(Mod_A) : res_{E_n/Lie}$$

where the left adjoint $U_n$ sends $L$ to a universal enveloping $E_n$-algebra of $L$. We assume that $A$ is noetherian. We first define the right adjoint. Let $Y_n^U : Alg_n^+(Mod_A) \rightarrow Fun Alg_n^+(Mod_A), S)$ be the functor defined as the composite $Alg_n^+(Mod_A) \rightarrow Fun Alg_n^+(Mod_A)^{op}, S) \rightarrow Fun Alg_n^+(Mod_A), S)$ where the left functor is the Yoneda embedding and the right functor is induced by the composition with $D_n$. Namely, $Y_n^U(R)$ is given by $B \mapsto Map_{Alg_n^+(Mod_A)}(D_n(B), R)$. 


Lemma 3.1. The functor $\text{CAlg}_k^+ = \text{CAlg}(\text{Mod}_A)/A \to \text{Alg}_n^+(\text{Mod}_A)$ be the forgetful functor. Composition with this functor induces $\text{Res}_{n/\infty} : \text{Fun}(\text{Alg}_n^+(\text{Mod}_A), S) \to \text{Fun}(\text{Art}^\text{Art}_A, S)$. Thus we obtain

$$\text{res}_{n/\infty} : \text{Alg}_n^+(\text{Mod}_A)/Y^\text{tsz}_n \to \text{Fun}(\text{Alg}_n^+(\text{Mod}_A), S) \to \text{Fun}(\text{Art}^\text{Art}_A, S).$$

The essential image of this composite is contained in $\hat{\text{St}}_A^* \subset \text{Fun}(\text{Art}^\text{Art}_A, S)$. To see this, it will suffice to verify that

1. For a finite product $B_1 \times \cdots \times B_n$ in $\text{Art}^\text{Art}_A$, a finite coproduct $\mathbb{D}_n(B_1) \sqcup \cdots \sqcup \mathbb{D}_n(B_n)$ in $\text{Alg}_n^+(\text{Mod}_A) = \text{its E}_n$-Koszul dual $\mathbb{D}_n(B_1 \times \cdots \times B_n)$.

2. For $B$ in $\text{Art}^\text{Art}_A$, $\mathbb{D}_n(A \times B) \cong \mathbb{D}_n(A) \sqcup_{\mathbb{D}_n(B)} \mathbb{D}_n(A) \cong A \sqcup_{\mathbb{D}_n(B)} A$.

These facts follows from [23, X, 4.4.5, 4.5.6] when $A$ is a field of characteristic zero. The general case for $A \in \text{CAlg}_k^{>0}$ follows from Remark 3.2 below. Consequently, the composition $\text{Alg}_n^+(\text{Mod}_A) \to \hat{\text{St}}_A^*$ and the equivalence $\hat{\text{St}}_A^* \cong \text{Lie}_A(A$ is noetherian) gives us

$$\text{res}_{E_n/Lie} : \text{Alg}_n^+(\text{Mod}_A) \to \text{Lie}_A.$$

**Lemma 3.1.** The functor $\text{res}_{E_n/Lie}$ admits a left adjoint functor $U_n : \text{Lie}_A \to \text{Alg}_n^+(\text{Mod}_A)$.

**Proof.** Note first that both $\text{Alg}_n^+(\text{Mod}_A)$ and $\text{Lie}_A$ are presentable $\infty$-categories, so that by the adjoint functor theorem, it is enough to prove that $\text{Alg}_n^+(\text{Mod}_A) \to \hat{\text{St}}_A^*$ preserves small limits and filtered colimits. Note that $Y^\text{tsz}_n$ preserves small limits, and the inclusion $\hat{\text{St}}_A^* \subset \text{Fun}(\text{Art}^\text{Art}_A, S)$ preserves small limits (since it admits a left adjoint). Thus $\text{Alg}_n^+(\text{Mod}_A) \to \hat{\text{St}}_A^*$ preserves small limits. Taking into account the definition $\mathcal{P}_n^\text{Art}_n(TSZ_A) = \hat{\text{St}}_A^*$, we see that $\hat{\text{St}}_A^* : \text{Fun}(\text{Art}^\text{Art}_A, S)$ is stable under filtered colimits. Thus it will suffice to show that $\text{res}_{n/\infty}$ preserves filtered colimits. It is enough to show that $\mathbb{D}_n(B)$ is a compact object in $\text{Alg}_n^+(\text{Mod}_A)$ for any $B \to A$ in $\text{Art}^\text{Art}_A$. But when $B \to A$ is $A \oplus M \to A$, $\mathbb{D}_n(B)$ is a free augmented $E_n$-algebra object $\text{Free}_{E_n}(M^\vee[-n])$ (see Remark 3.2). Thus our claim follows from the compactness of $M$ in $\text{Mod}_A$. 

We obtain an adjoint pair

$$U_n : \text{Lie}_A \rightleftarrows \text{Alg}_n^+(\text{Mod}_A) : \text{res}_{E_n/Lie}.$$

By composing forget : $\text{Alg}_n^+(\text{Mod}_A) \rightleftarrows \text{Alg}_n(\text{Mod}_A)$, we also have

$$U_n : \text{Lie}_A \rightleftarrows \text{Alg}_n(\text{Mod}_A) : \text{res}_{E_n/Lie}.$$

where by abuse of notation we use the same symbols $U_n$ and $\text{res}_{E_n/Lie}$. We shall refer to $U_n$ as the universal enveloping $E_n$-algebra functor.

**Remark 3.2.** Let $B(A)$ be an augmented $E_n$-algebra over $A$ and let $B \to \mathbb{D}_n\mathbb{D}_n(B)$ be the biduality morphism (induced by the universality). Let $M$ be an $A$-module spectrum. Suppose that $M$ is of the form $\oplus_{1 \leq i \leq m} A^{\oplus r_i}[d_i]$ ($d_i \leq -n$). Let $\text{Free}_{E_n} : \text{Mod}_A \to \text{Alg}_n^+(\text{Mod}_A)$ denote the free functor, that is a left adjoint of the forgetful functor. We set $B = \text{Free}_{E_n}(M)$. We show that the biduality map $B \to \mathbb{D}_n\mathbb{D}_n(B)$ is an equivalence. Then (the proof of) [23, DAG X 4.5.6] shows that there is a canonical equivalence $\mathbb{D}_n(\text{Free}_{E_n}(M)) \cong A \oplus M^\vee[-n]$ where $A \oplus M^\vee[-n]$ indicates the trivial square zero extension. Consequently, we have the canonical morphism $\text{Free}_{E_n}(M) \to \mathbb{D}_n\mathbb{D}_n(\text{Free}_{E_n}(M)) \cong \mathbb{D}_n(A \oplus M^\vee[-n])$. It will suffice to prove that $\text{Free}_{E_n}(M) \to \mathbb{D}_n(A \oplus M^\vee[-n])$ is an equivalence. When $A$ is $k$, then the desired equivalence follows from [23, 4.4.5, 4.5.6]. We will describe how to deduce the general case from the case when $A = k$. Let $\text{Free}_k : \text{Mod}_k \to \text{Alg}_k^+(\text{Mod}_k)$ be the free functor (i.e., a left adjoint of the forgetful functor), and we write $\text{Free}_A$ for the above $\text{Free}_{E_n}$. Let $\text{D}_{n,k} : \text{Alg}_n^+(\text{Mod}_k)^{op} \to \text{Alg}_n^+(\text{Mod}_k)$ be the Koszul duality functor. Note canonical equivalences

$$\text{Free}_A(\oplus_{1 \leq i \leq m} A^{\oplus r_i}[d_i]) \cong \text{Free}_k(\oplus_{1 \leq i \leq m} k^{\oplus r_i}[d_i]) \otimes_k A \cong \mathbb{D}_{n,k}(k \oplus (\oplus_{1 \leq i \leq m} k^{\oplus r_i}[d_i])^\vee[-n]) \otimes_k A.$$
equivalence in $\Mod_A$. It is obtained from
\[ \mathbb{D}_{n,k}(k \oplus N) \otimes_k A \simeq \mathcal{H}om_k(\text{Bar}^+_n(k \oplus N), k) \otimes_k A \]
\[ \simeq \mathcal{H}om_k(\text{Bar}^+_n(k \oplus N), A) \]
\[ \simeq \mathcal{H}om_A(\text{Bar}^+_n(A \otimes_k (k \oplus N)), A) \]
\[ \simeq \mathbb{D}_n(A \otimes_k (k \oplus N)) \]
where $\text{Bar}_k$ and $\text{Bar}_A$ indicate the bar construction of augmented algebras in $\Mod_k$ and $\Mod_A$, respectively. Here we denote by $\mathcal{H}om_k$ and $\mathcal{H}om_A$ the internal Hom objects in $\Mod_k$ and $\Mod_A$, respectively. The second equivalence (key point) follows from the fact that the homology of $\text{Bar}^+_n(k \oplus N)$ is a finite dimensional $k$-vector space in each degree. Namely, each degree of $\mathcal{H}om_k(\text{Bar}^+_n(k \oplus N), k) \simeq \text{Free}_\mathbb{A}(N^\vee[-n])$ is finite dimensional: it can be deduced from the description of free $E_n$-algebras in terms of configuration spaces of Euclidean spaces and the estimate of the amplitude and the finite dimensionality of homology groups of configuration spaces (see e.g. [23, 4.1.15]). The first and final equivalences are the presentation of Koszul duals in terms of iterated bar constructions (see Section 3.4).

**Proposition 3.3.** The followings hold.

1. The left adjoint functor $U_n : \text{Lie}_A \to \text{Alg}^+_n(\Mod_A)$ is an essentially unique colimit-preserving functor which extends the composite
\[ \text{Lie}^f_A \xrightarrow{\Omega} (\text{CAlg})^{op} \xrightarrow{\mathbb{D}_n} \text{Alg}^+_n(\Mod_A). \]

2. The composite $\text{res}_{E_n/Lie} : \text{Alg}^+_n(\Mod_A) \to \text{Lie}_A \xrightarrow{\text{forget}} \Mod_A$ is equivalent to the functor defined by the formula $[B \to A] \mapsto \text{Ker}(B \to A)[n-1]$.

**Proof.** We prove (1). By the equivalences $\text{Fun}^L(\mathcal{P}_\Sigma^f(\text{Lie}^f_A), \text{Alg}^+_n(\Mod_A)) \simeq \text{Fun}^{st}(\text{Lie}^f_A, \text{Alg}^+_n(\Mod_A))$ and $\mathcal{P}_\Sigma^f(\text{Lie}^f_A) \simeq \text{Lie}_A$, it will suffice to show that $\text{Lie}^f_A \hookrightarrow \text{Lie}_A \xrightarrow{U_n} \text{Alg}^+_n(\Mod_A)$ is equivalent to $\mathbb{D}_n \circ \text{Ch}^*_{\text{Lie}_A}$. Unwinding the definition of $\text{res}_{E_n/Lie}$, we see that for $C \in \text{Alg}^+_n(\Mod_A)$, $\text{res}_{E_n/Lie}(C) \in \mathcal{P}_\Sigma^f(\text{Lie}^f_A) \subset \text{Fun}(\text{Lie}^f_A, S)$ (given by $L \mapsto \text{Map}_{\text{Lie}_A}(L, \text{res}_{E_n/Lie}(C))$) is equivalent to the functor given by $L \mapsto \text{Map}_{\text{Alg}^+_n(\Mod_A)}(\mathbb{D}_n(\text{Ch}^*_L(L)), C)$. It follows that the restriction of $U_n$ to $\text{Lie}^f_A$ is equivalent to $\mathbb{D}_n \circ \text{Ch}^*_{\text{Lie}_A}$.

Next, we prove (2). To observe (2), we first consider
\[ (\text{Mod}_A)^{op} \xrightarrow{\text{Free}_{\text{Lie}}} (\text{Lie}_A)^{op} \xrightarrow{\text{Ch}^*} \text{CAlg}^+_A. \]
Taking into account the universal property of $\mathcal{P}_\Sigma^{st}$ (see Section 3.2), $(\text{Mod}_A)^{op} \to (\text{Lie}_A)^{op} \to \text{Art}^\text{tsz}$ induces colimit-preserving functors of presentable $\infty$-categories
\[ \mathcal{P}_\Sigma^{st}(\text{Mod}_A) \to \mathcal{P}_\Sigma^{st}(\text{Lie}_A) \to \mathcal{P}_\Sigma^{st}(\text{TSZ}_A) = \text{St}^*_A \]
where the left functor can be identified with $\text{Free}_{\text{Lie}} : \text{Mod}_A \to \text{Lie}_A$, and the final equality is the definition of $\text{St}^*_A$. Note that the right adjoint functors of this sequence are the restriction of the sequence
\[ \text{Fun}(\text{Art}^\text{tsz}, S) \to \text{Fun}(\text{Lie}_A^{op}, S) \to \text{Fun}(\text{L}^{op}, S) \]
given by the composition with $(\text{Mod}_A)^{op} \to (\text{Lie}_A)^{op} \to \text{Art}^\text{tsz}$. In order to verify that $\text{Alg}^+_n(\Mod_A) \to \text{Lie}_A \to \text{Mod}_A$ sends $B \to A$ to $\text{Ker}(B \to A)[n-1]$, by (1) and left Kan extension to $\mathcal{P}_\Sigma^{st}(\text{Mod}_A)$ it is enough to prove that the composite
\[ (\text{Mod}_A)^{op} \xrightarrow{\text{Free}_{\text{Lie}}} (\text{Lie}_A)^{op} \xrightarrow{\text{Ch}^*} (\text{CAlg}^+_A)^{op} \xrightarrow{\text{forget}} \text{Alg}^+_n(\Mod_A)^{op} \xrightarrow{\text{Bar}} \text{Alg}^+_n(\Mod_A) \]
is equivalent to the “shifted free functor” given by $M \mapsto \text{Free}_{E_n}(M[1-n])$. The composite $(\text{Mod}_A)^{op} \to (\text{Lie}_A)^{op} \to \text{CAlg}^+_A$ is equivalent to $M \mapsto [A \oplus M^\vee[-1] \to A]$ (see [9, 1.4.11]). Thus, we are reduced to proving that $\text{Art}^\text{tsz} \to \text{Alg}^+_n(\Mod_A)^{op} \xrightarrow{\text{Bar}} \text{Alg}^+_n(\Mod_A)^{op}$ is equivalent to the functor given by $[A \oplus N \to A] \mapsto \text{Free}_{E_n}(N^\vee[-n])$. This is a consequence of the canonical equivalence $\mathbb{D}_n(A \oplus N) \simeq \text{Free}_{E_n}(N^\vee[-n])$ where $N$ is a connective $A$-module of the form $\bigoplus_{1 \leq i \leq n} A[i]^{op}$ (see Remark 3.2). □
Remark 3.4. The functor $U_1 : \text{Lie}_A \rightarrow \text{Alg}_1^+(\text{Mod}_A)$ defined above is equivalent to the standard definition of the universal enveloping algebras $U : \text{Lie}_A \rightarrow \text{Alg}_1^+(\text{Mod}_A)$ (see e.g. [23, X, 2.1.7]; [9] for the universal enveloping algebras of dg Lie algebras). We first consider the case when $A = k$.

The result from [23, 3.3.2] says that $Ch^* : (\text{Lie}_k)^{op} \rightarrow \text{Alg}_k^+(\text{Mod}_k)$ is equivalent to $\mathbb{D}_1 \circ U : (\text{Lie}_k)^{op} \rightarrow \text{Alg}_k^+(\text{Mod}_k)$. If $L \in \text{Lie}_k^+$, it follows from [23, 3.1.5] that the canonical map $U(L) \rightarrow \mathbb{D}_1 \circ U(U(L))$ is an equivalence. Thus, there exists natural equivalences $\mathbb{D}_1 \circ U \simeq \mathbb{D}_1 \circ \mathbb{D}_1 \circ U \simeq U$ between functors $\text{Fun}(\text{Lie}_k^+, \text{Alg}_k^+(\text{Mod}_k))$ (here we omit the forgetful functor $\text{CAlg}_k^+ \rightarrow \text{Alg}_k^+(\text{Mod}_k)$ from the notation). Since $U$ is a left adjoint, we deduce from Proposition 3.3 (1) that $U \simeq U_1$ in $\text{Fun}(\mathbb{P}_k^+(\text{Lie}_k^+), \text{Alg}_1^+(\text{Mod}_k)) \simeq \text{Fun}(\text{Lie}_k^+, \text{Alg}_1^+(\text{Mod}_k))$. Next we consider the general case. The proof of [23, 3.3.2] reveals that it holds in the general case. According to Remark 3.2, $U(L) \rightarrow \mathbb{D}_1 \circ \mathbb{D}_1 (U(L))$ is an equivalence for $L \in \text{Lie}_A^+$. The general case follows from these observations and the argument in the case $A = k$.

3.6. Hochschild homology and Hochschild cohomology. Let $\text{St}_A$ denote $\text{Mod}_{\text{perf}}^A(\text{St})$. Let

$$\mathcal{H}_h(\ast) : \text{St}_A \rightarrow \text{Mod}_A^S = \text{Fun}(BS^1, \text{Mod}_A)$$

be the symmetric monoidal functor which carries $C \in \text{St}_A$ to the Hochschild homology $A$-module spectrum $\mathcal{H}_h(C/A)$ over $A$. We refer the reader to [11, Section 6, 6.14] for the construction of the Hochschild chain/homology functor $\mathcal{H}_h(\ast) : \text{St}_A \rightarrow \text{Mod}_A^S$ which is a model of $\text{Alg}_A$ by inverting weak equivalences in $\text{Alg}_A$. We will describe the construction of $\mathcal{H}_h(\ast) : \text{St}_A \rightarrow \text{Mod}_A^S$ which assigns to $A$ the factorization homology $\mathcal{H}_A$. Moreover, we will use:

Lemma 3.5. For ease of notation, we let $h : \text{CAlg}(\text{Mod}_A) \simeq \text{CAlg}(\text{Alg}_1(\text{Mod}_A)) \rightarrow \text{CAlg}(\text{Mod}_A^S)$ denote the functor induced by $\mathcal{H}_h(\ast) : \text{Alg}_1(\text{Mod}_A) \rightarrow \text{Mod}_A^S$. (The equivalence $\text{CAlg}(\text{Mod}_A) \simeq \text{CAlg}(\text{Alg}_1(\text{Mod}_A))$ follows from Dunn additivity theorem.) Let $i : \text{CAlg}(\text{Mod}_A) \rightarrow \text{CAlg}(\text{Mod}_A^S)$ denote the functor which carries $A$ to $\text{Alg}_1(A)$ instead of $\text{Mod}_A^S$. Then $h : \text{CAlg}(\text{Mod}_A) \rightarrow \text{CAlg}(\text{Mod}_A^S)$ is equivalent to the functor $\otimes_A S^1$ given by the tensor with $S^1 \in S$, and there is a natural transformation $\sigma : h \rightarrow i$ induced by the $S^1$-equivariant map $S^1 \rightarrow \ast$ into the contractible space.

Proof. We will describe the construction of $\sigma : h \rightarrow i$. To this end, we describe $h : \text{CAlg}(\text{Mod}_A) \simeq \text{CAlg}(\text{Alg}_1(\text{Mod}_A)) \rightarrow \text{CAlg}(\text{Mod}_A^S)$ in terms of cyclic sets. For this purpose, we briefly review the Hochschild cyclic objects: the construction of $\mathcal{H}_h(\ast) : \text{Alg}_1(\text{Mod}_A) \rightarrow \text{Mod}_A^S$ in [11]. We use the theory of symmetric spectra. The readers who do not know symmetric spectra are invited to skip the construction on the first reading. We use the notation and terminology in [11, Section 6]. We refer the reader to loc. cit. for details. The symmetric monoidal functor $\mathcal{H}_h(\ast)$ is obtained from the composite of symmetric monoidal functors

$$\text{Alg}_1(\text{Sp}^\Sigma(\mathbb{A})^c) \xrightarrow{\mathcal{H}_h(\ast)} \text{Fun}(\mathbb{A}^c, \text{Sp}^\Sigma(\mathbb{A})^c) \xrightarrow{\mathcal{H}_h(\ast)} \text{Fun}(\mathbb{A}^c, \text{Sp}^\Sigma(\mathbb{A})^c[W^{-1}]) \xrightarrow{L_i} \text{Fun}(BS^1, \text{Sp}^\Sigma(\mathbb{A})^c[W^{-1}])$$

by inverting weak equivalences in $\text{Alg}_1(\text{Sp}^\Sigma(\mathbb{A})^c)$ (see the construction before [11, Lemma 6.11]). Here $\mathbb{A}$ is a cofibrant commutative symmetric spectrum which is a model of $A$, $\text{Sp}^\Sigma(\mathbb{A})^c$ is the category of cofibrant $\mathbb{A}$-module symmetric spectra, and $\mathbb{A}$ is the cyclic category. We denote by $\text{Sp}^\Sigma(\mathbb{A})^c[W^{-1}]$ the (symmetric monoidal) $\infty$-category obtained from $\text{Sp}^\Sigma(\mathbb{A})^c$ by inverting weak equivalences (see [22, 1.3.4, 4.1.7, 4.1.8] for localizations with respect to weak equivalences). There is a canonical functor $\text{Sp}^\Sigma(\mathbb{A})^c \rightarrow \text{Sp}^\Sigma(\mathbb{A})^c[W^{-1}]$ that induces the middle functor. The third (right) functor is the symmetric monoidal functor determined by left Kan extensions along the groupoid completion $\mathbb{A}^c \rightarrow BS^1$. There are relationships $\text{Sp}^\Sigma(\mathbb{A})^c[W^{-1}] \simeq \text{Mod}_A$ and $\text{Alg}_1(\text{Sp}^\Sigma(\mathbb{A})^c[W^{-1}]) \simeq \text{Alg}_1(\text{Mod}_A)$. The first functor $\mathcal{H}_h(\ast)$ carries $\mathbb{R} \in \text{Alg}_1(\text{Sp}^\Sigma(\mathbb{A})^c)$ to the Hochschild cyclic object $\mathcal{H}_h(\mathbb{R}) : \mathbb{A}^c \rightarrow \text{Sp}^\Sigma(\mathbb{A})^c$ (see e.g. [11, Definition 6.18] for the formula: in loc. cit., spectral categories are used in the definition...
instead of $\text{Alg}_1(\text{Sp}^P(\Delta^r))$. Consider the composite functor $\text{CAlg}(\text{Sp}^P(\Delta^r)) \cong \text{CAlg}(\text{Alg}_1(\text{Sp}^P(\Delta^r))) \to \text{Fun}(\Delta^{op}, \text{CAlg}(\text{Sp}^P(\Delta^r)))$ induced by $\mathcal{H}(\ast)$. Let $\Delta^1 / \partial \Delta^1$ be the standard simplicial model of the circle $S^1$. The simplicial set $\Delta^1 / \partial \Delta^1 : \Delta^{op} \to \text{Sets}$ is extended to a cyclic set $C : \Lambda^{op} \to \text{Sets}$ (see [19, 6.1.9]). Then $\text{CAlg}(\text{Sp}^P(\Delta^r)) \to \text{Fun}(\Delta^{op}, \text{CAlg}(\text{Sp}^P(\Delta^r)))$ is given by the formula

$$\mathcal{R} \mapsto \mathcal{H}(\mathcal{R}), \ast := [\Delta^{op} \ni [p] \mapsto C([p]) \otimes \mathcal{R} \in \text{CAlg}(\text{Sp}^P(\Delta^r))]$$

where $C([p]) \otimes \mathcal{R}$ is the tensor product with the finite set $C([p])$ in $\text{CAlg}(\text{Sp}^P(\Delta^r))$, that is, the $(p+1)$-fold (homotopy) coproduct $(p+1)$-fold smash product $\wedge^{p+1} \mathcal{R}$.

To define $h \mapsto i$, we use the “contraction” of $C$ to the terminal cyclic set. Let $C_r$ denote the cyclic object which is the constant functor whose value is the set consisting of one element $\ast$ (that is, the terminal/final cyclical set). The (unique) map $C \to C_r$ determines the natural transformation $\sigma_r : \mathcal{H}(\ast) \to \text{const}$ between functors $\text{CAlg}(\text{Sp}^P(\Delta^r)) \to \text{Fun}(\Delta^{op}, \text{CAlg}(\text{Sp}^P(\Delta^r)))$ where const$(\mathcal{R})$ is the constant cyclic object with value $\mathcal{R}$. For any $\mathcal{R}$ and $p \geq 0$, $\mathcal{H}(\mathcal{R})_p = \wedge^{p+1} \mathcal{R} \to \mathcal{R}$ induced by $\sigma_r$ is the multiplication map $\wedge^{p+1} \mathcal{R} \to \mathcal{R}$.

By inverting weak equivalences, we have the diagram

$$\begin{align*}
\text{CAlg}(\text{Sp}^P(\Delta^r))[W^{-1}] & \longrightarrow \text{Fun}(\Delta^{op}, \text{CAlg}(\text{Sp}^P(\Delta^r)))[W^{-1}] \\
\cong & \\
\text{CAlg}(\text{Alg}_1(\text{Sp}^P(\Delta^r))[W^{-1}]) & \longrightarrow \text{CAlg}(\text{Fun}(\Delta^{op}, \text{CAlg}(\text{Sp}^P(\Delta^r))[W^{-1}]))
\end{align*}$$

which commutes up to canonical homotopy, where the horizontal arrows are defined by taking Hoehschild cyclic objects $\mathcal{H}(\ast)$, and $[W^{-1}]$ indicates the localization of weak equivalences. Let

$$h_A : \text{CAlg}_A \cong \text{CAlg}(\text{Alg}_1(\text{Sp}^P(\Delta^r))[W^{-1}]) \to \text{CAlg}(\text{Fun}(\Delta^{op}, \text{CAlg}(\text{Sp}^P(\Delta^r))[W^{-1}])) \cong \text{Fun}(\Delta^{op}, \text{CAlg}_A)$$

denote the functor obtained by the composition with the lower horizontal functor. (By the universal property of $[W^{-1}]$, this functor is given by $R \to [\Delta^{op} \ni [p] \mapsto C([p]) \otimes R \in \text{CAlg}_A]$. Let $\text{const}_\infty$ denote the functor $\text{CAlg}_A \to \text{Fun}(\Delta^{op}, \text{CAlg}_A)$ which carries $R$ to the constant functor with value $R$. Then $\sigma_r$ determines $\sigma_r : h_A \to \text{const}_\infty$. The composition of $\sigma_r$ and $\text{Fun}(\Delta^{op}, \text{CAlg}_A) \to \text{Fun}(BS^1, \text{CAlg}_A)$ induced by $L : \text{Fun}(\Delta^{op}, \text{Mod}_A) \to \text{Fun}(BS^1, \text{Mod}_A)$ determines $\sigma : h \to i$ between functors $\text{CAlg}_A \to \text{Fun}(BS^1, \text{CAlg}_A) \cong \text{CAlg}(\text{Mod}_A)$.

Let $\text{LS} : \text{Fun}(\Delta^{op}, \mathcal{S}) \to \text{Fun}(BS^1, \mathcal{S})$ be a left adjoint functor of the functor $\text{Fun}(BS^1, \mathcal{S}) \to \text{Fun}(\Delta^{op}, \mathcal{S})$ induced by the composition with $\Delta^0 \to BS^1$. Then $\text{LS}$ sends $C$ to $S^1$ with an $S^1$-action. Indeed, we note that the cyclic set $C : \Lambda^{op} \to \text{Sets} \subset C$ is represented by $[0] \in \Lambda$. The left adjoint functor $\text{LS}$ is an essentially unique colimit-preserving functor which extends $\Lambda \to (BS^1)^{op} \to \text{Fun}(BS^1, \mathcal{S})$ where the second functor is Yoneda embedding. If we denote the unique object of $BS^1$ by $\ast$, it follows that $\text{LS}(C)$ is corepresented by $\ast$. That is, $\text{LS}(C)$ amounts to $\text{Map}_{BS^1}(\ast, \ast) = S^1$ with the $S^1$-action determined by the multiplication $S^1 \times S^1 \to S^1$.

For a cyclic object $E : \Delta^{op} \to \mathcal{M}$ in an $\infty$-category $\mathcal{M}$, $F : BS^1 \to \mathcal{M}$ is a left Kan extension of $E$ along $\Delta^{op} \to BS^1$ if and only if the composite $\ast = \Delta^0 \to BS^1 \to \mathcal{M}$ is a colimit of the restriction $E|_{\Delta^{op}} : \Delta^{op} \to \mathcal{M}$ (see e.g. [11, Lemma 6.9 (ii)]). Since the restriction $C|_{\Delta^{op}}$ is $S^1 / \partial S^1$, it follows that the underlying space of $\text{LS}(C)$ is $S^1$ = $[\Delta^1 / \partial \Delta^1]$. Note that $\lim_{\to \Delta^r \in \Delta^{op}} (C([p]) \otimes R) \cong \lim_{\to \Delta^r \in \Delta^{op}} (C([p]) \otimes R) \cong S^1 \otimes R \cong S^1 \otimes R \in \text{CAlg}_A$. We then see that there is a canonical equivalence $h(R) \cong \text{LS}(C) \otimes R$ in $\text{CAlg}(\text{Mod}_A)$ because the underlying map $\lim_{\to \Delta^r \in \Delta^{op}} (C([p]) \otimes R) \to S^1 \otimes R$ is an equivalence. In particular, the $S^1$-action on $h(R)$ is induced by that of $\text{LS}(C)$ $\cong S^1$. We may regard $\sigma : h \to i$ as the natural transformation informally given by $h(R) \to R \otimes S^1 \to R \otimes \ast = i(R)$ induced by the $S^1$-equivariant map $S^1 \to \ast$. $\square$

We review the Hoehschild cohomology of $\mathcal{C} \in \text{St}_A$. Let $\text{Ind}(\mathcal{C})$ be the Ind-category that belongs to $\text{Pr}^A$. Moreover, it is compactly generated. We denote by $\theta_A : \text{Alg}_2(\text{Mod}_A) \to \text{Alg}_1(\text{Pr}^A)$ the functor informally given by $B \mapsto L\text{Mod}^B_A$ (Section 3.1). By definition, the endomorphism algebra object $\mathcal{E}nd_A(\text{Ind}(\mathcal{C})) \in \text{Alg}_1(\text{Pr}^A)$ endowed with a tautological action on $\text{Ind}(\mathcal{C})$ is a final object of $L\text{Mod}(\text{Pr}^A) \times_{\text{Pr}^A} \{\text{Ind}(\mathcal{C})\}$. There exists a final object of $\text{Alg}_2(\text{Mod}_A) \times_{\text{Alg}_1(\text{Pr}^A)} L\text{Mod}(\text{Pr}^A) \times_{\text{Pr}^A} \{\text{Ind}(\mathcal{C})\}$. To see this, note first that there exists a right adjoint $\text{Alg}_2(\text{Pr}^A) \to \text{Alg}_2(\text{Mod}_A)$ of $\theta_A$, see
Consider the case where the key steps to the construction of the left commutative square in Theorem 1.2 is to prove that is an equivalence after the restriction along the functor Artsz is reduced to the Koszul duality result which says that rise to natural transformations (Proposition 6.8).

Type data associated to the deformation defines the “duality functor” with Hochschild chain functor. Arguably, the key equivalences are given by the commutative diagram (Section 3.4), informally.

Before proceeding to Section 5–8, we will highlight several points in view of Theorem 1.2 in an informal way.

(i) Deformation problems we will study have the following form or its variant. Let be an augmented algebra object of , that is, an object of where is the unit algebra. The -category of deformations of to is (cf. Section 3.4). Let .

The vertical arrows in Theorem 1.2 are given by descent type data (Koszul dual type data) associated to deformations. Roughly, data is defined as follows. Suppose that is a -bi-module. Then is a right -module. Since can be thought of as a -module (see Section 3.4), is a right -module, that is, a left -module. This is the descent/Koszul type data associated to the deformation . The assignment defines the “duality functor”

which extends the to monoidal by let . The duality functors are functorial in the sense that gives rise to natural transformations

where is the induced functor . One of the key steps to the construction of the left commutative square in Theorem 1.2 is to prove that is an equivalence after the restriction along the functor given by . It is reduced to the Koszul duality result which says that is an equivalence after the restriction (cf. Proposition 6.8).

(ii) We need the interaction of Koszul duality with the change of operads including the Lie operad, the -operad (the associative operad), the -operad and the -operad (commutative operad) and with Hochschild chain functor. Arguably, the key equivalences are given by the commutative diagram of -equivariant associative algebras

\[
\begin{align*}
U_1(D_\infty(R)S^1) & \xrightarrow{\simeq} D_1(R \otimes_A S^1) \\
\mathcal{HH}_*(U_2(D_\infty(R))/A) & \xrightarrow{\simeq} \mathcal{HH}_*(D_2(R)/A)
\end{align*}
\]
for $R \in \text{Art}_{\mathcal{M}}^0$ (see Proposition 7.11). These equivalences allow us to describe the bottom row of the diagram in Theorem 1.2 in terms of dg Lie algebras (which coherently commutes with moduli-theoretic construction). We find that they are controlled by the algebraic structure on $(\mathcal{H}^\bullet(C/A), \mathcal{H}^\bullet(C/A))$.

In particular, the Koszul dual of the cyclic deformation of $\mathcal{H}^\bullet(C/A)$ associated to a deformation of an $A$-linear stable $\infty$-category $\mathcal{C}$ is determined by the $S^1$-equivariant action of $\mathcal{G}_C^S$ on $\mathcal{H}^\bullet(C/A)$ that arises from $(\mathcal{H}^\bullet(C/A), \mathcal{H}^\bullet(C/A))$ (cf. Proposition 8.15). The Lie algebra theoretic presentation of the transition from cyclic deformations to $S^1$-equivariant deformations is the restriction along the diagonal map $\mathcal{G}_C \to \mathcal{G}_C^S$ (cf. Remark 8.22).

5. Deformations in abstract contexts

We will define the formalism of deformations of an object $M$ in a monoidal $\infty$-category $\mathcal{M}^\otimes$. The formalism will be given in Section 5.2. For this purpose, we first recall the relative tensor product of bimodules in Section 5.1.

Next, we turn to consider the Koszul duality functors. In Section 5.3, we review the notion of pairing of $\infty$-categories. Under a good condition, the pairing of $\infty$-categories determines a “duality functor.” In Section 5.4 and Section 5.5, we will give examples of pairings and duality functors used in subsequent sections.

5.1. Tensor products of bimodules. Let $\mathcal{M}^\otimes \to \text{As}^\otimes$ be a monoidal $\infty$-category. (All examples of $\mathcal{M}$ in this paper come from symmetric monoidal $\infty$-categories.) Suppose that it has geometric realizations/collimits of simplicial objects, and the tensor product functor $\otimes: \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ preserves geometric realizations of simplicial objects. Let $1$ be a unit object of $\mathcal{M}$.

Consider the canonical projection $\text{BMod}(\mathcal{M}) \to \text{Alg}_1(\mathcal{M}) \times \text{Alg}_1(\mathcal{M})$ which carries a $A$-$B$-bimodule $A_B$ to $(A, B)$. We denote by $\text{BMod}_B(\mathcal{M})$ the fiber of $(\pi_1, \pi_2)$ over $(A, B)$. Let

$$\text{BMod}(\mathcal{M}) \times \pi_2, \text{Alg}_1(\mathcal{M}), \pi_1 \text{BMod}(\mathcal{M}) \to \text{BMod}(\mathcal{M})$$

be the relative tensor product functor. We refer the readers to [22, 4.4.2] for the construction of relative tensor products. If $A_B$ and $B_N$ are an $A$-$B$-bimodule and a $B$-$C$-bimodule, respectively, then it sends $(A_B, B_N)$ to an $A$-$C$-bimodule $A_B \otimes_B B_N$ whose underlying object in $\mathcal{M}$ is the tensor product $M \otimes_B N$ obtained by bar construction. The forgetful functors induce canonical equivalence $\text{BMod}_1(\mathcal{M}) \to \text{LMod}(\mathcal{M})$, $\text{1BMod}(\mathcal{M}) \to \text{RMod}(\mathcal{M})$, and $\text{1BMod}_1(\mathcal{M}) \to \mathcal{M}$. In particular, the relative tensor product functor of bimodules induces

$$\text{RMod}(\mathcal{M}) \times \pi_2, \text{Alg}_1(\mathcal{M}), \pi_1 \text{LMod}(\mathcal{M}) \to \mathcal{M}$$

which carries $(M_B, B_N)$ to $M \otimes_B N$. We write $\text{LMod}(\mathcal{M})$ for $\text{LMod}(\mathcal{M}, \mathcal{M}(1))$ where $\text{LMod}(\mathcal{M}) \to \mathcal{M}$ is the forgetful functor. By [22, 4.7.1.40], there is an equivalence $\text{LMod}(\mathcal{M})_1 \simeq \text{Alg}_1(\mathcal{M})_1 = \text{Alg}_1^+(\mathcal{M})$ which commutes with the projection to $\text{Alg}_1(\mathcal{M})$ up to canonical homotopy. Thus, the relative tensor product functor determines the “reduction functor”

$$r_M: \text{RMod}(\mathcal{M}) := \text{RMod}(\mathcal{M}) \times \pi_2, \text{Alg}_1(\mathcal{M}) \to \text{LMod}(\mathcal{M})_1 \to \mathcal{M}$$

which carries $(N_B, B, B \to 1)$ to $N_B \otimes_B 1$.

5.2. Abstract deformation functors. We continue to assume that $\mathcal{M}^\otimes \to \text{As}^\otimes$ is a monoidal $\infty$-category such that it has geometric realizations/collimits of simplicial objects, and the tensor product functor $\otimes: \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ preserves geometric realizations of simplicial objects. Let $M$ be an object of $\mathcal{M}$.

Definition 5.1. We will define the notion of deformations of $M$ in a general situation. Set $\text{RMod}(\mathcal{M}) = \{M\} \times_M \text{RMod}(\mathcal{M})$ where $\text{RMod}(\mathcal{M}) \to \mathcal{M}$ is the reduction functor $r_M$ (cf. Section 5.1). Let $\text{RMod}(\mathcal{M})^+$ be the subcategory spanned by coCartesian morphisms over $\text{Alg}_1^+(\mathcal{M})$ whose objects are the same as those of $\text{RMod}(\mathcal{M})$. Consider the left fibration

$$\text{Def}(\mathcal{M}) := \{M\} \times_M \text{RMod}(\mathcal{M})^+ \to \text{Alg}_1^+(\mathcal{M}).$$

This left fibration is classified by the functor $\text{Def}(\mathcal{M}): \text{Alg}_1^+(\mathcal{M}) \to \mathcal{S}$ informally given by

$$[B \to 1] \mapsto \text{RMod}_B(\mathcal{M}) \simeq \times_M \{M\}.$$
where $\text{RMod}_B(M) \to M$ is given by the reduction functor that sends $N_B$ to $N_B \otimes_B 1$. We refer to $\text{Def}_M(M)$ as the deformation functor of $M$ over $\text{Alg}^+_1(M)$. We call $\text{Def}_M(M)(B \to 1)$ the space of deformations of $M$ along $B \to 1$ (or simply to $B \in \text{Alg}^+_1(M)$). For ease of notation, we often write $\text{Def}_M(M)(B \to 1)$ for $\text{Def}_M(M)(B \to 1)$.

An object of the space of deformations of $M$ to $B \in \text{Alg}^+_1(M)$ is described as $(M' \in \text{RMod}_B(M), B \to 1, M \simeq M' \otimes_B 1)$. We will define a change of domains of deformation functors.

**Definition 5.2.** Consider the context in Definition 5.1. Let $N \to \text{As}^\otimes$ be another monoidal ∞-category such that it has geometric realizations/colimits of simplicial objects, and the tensor product functor $\otimes : N \times N \to N$ preserves geometric realizations of simplicial objects. Let $F : N \to M \to \text{As}^\otimes$ be a monoidal functor. Let $\text{Alg}^+_1(N) \to \text{Alg}^+_1(M)$ be the induced functor. Then the base change $\text{Def}_{\text{Alg}^+_1(M)}(\text{Alg}^+_1(M)) \to \text{Alg}^+_1(N)$ of the left fibration is classified by $\text{Alg}^+_1(N) \to \text{Alg}^+_1(N) \to \text{Alg}^+_1(N)$ to $S$. We will refer to the composite $\text{Alg}^+_1(N) \to S$ as the deformation functor of $M$ over $\text{Alg}^+_1(N)$.

5.3. In order to define Koszul duality functors, following [23, X], we review the notion of pairings. Let $C$ and $D$ be ∞-categories. Let $\lambda : M \to C \times D$ be a right fibration. Following [23, X, 3.1.1], we refer to the right fibration to $C \times D$ as a pairing of ∞-categories $C$ and $D$. Let $C^{\text{op}} \times D^{\text{op}} \to S$ or a functor which corresponds to the right fibration $\lambda$. We say that $\lambda$ is a left representable (resp. right representable) if $C^{\text{op}} \to \text{Fun}(D^{\text{op}}, S)$ (resp. $D^{\text{op}} \to \text{Fun}(C^{\text{op}}, S)$) factors as $C^{\text{op}} \xrightarrow{\gamma} \text{D} \xrightarrow{\psi} \text{Fun}(D^{\text{op}}, S)$ (resp. $D^{\text{op}} \xrightarrow{\varphi} C \xrightarrow{\lambda} \text{Fun}(C^{\text{op}}, S)$). Here $\gamma$ and $\psi$ are Yoneda embeddings. We refer to $\text{D}_\lambda$ and $\text{D}_\lambda'$ as the duality functor associated to $\lambda$. For example, Koszul duality functors can be described-defined by duality functors associated to pairings.

**Lemma 5.3.** Let $f : C_1 \to C_2$ and $g : D_1 \to D_2$ be functors between ∞-categories and let

$$
\begin{array}{ccc}
M & \to & N \\
\downarrow \lambda & & \downarrow \mu \\
C_1 \times D_1 & \to & C_2 \times D_2
\end{array}
$$

be a commutative diagram in $\text{Cat}_{\infty}$ such that the vertical arrows are pairing of ∞-categories. We refer to the commutative diagram as a morphism of pairings. Suppose that both $\lambda$ and $\mu$ are left representable. The diagram determines a natural transformation

$$
g \circ \text{D}_\lambda \to \text{D}_\mu \circ f.
$$

**Proof.** Indeed, consider the functors $S_\lambda : C_1^{\text{op}} \times D_1^{\text{op}} \to S$ and $S_\mu : C_2^{\text{op}} \times D_2^{\text{op}} \to S$ corresponding to $\lambda$ and $\mu$. The diagram induces a natural transformation $S_\lambda \to S_\mu \circ (f \times g)$. Let $T_\lambda : C_1^{\text{op}} \to \text{Fun}(D_1^{\text{op}}, S)$ and $T_\mu : C_2^{\text{op}} \to \text{Fun}(D_2^{\text{op}}, S)$ be functors determined by $S_\lambda$ and $S_\mu$, respectively. Let $\psi : \text{Fun}(D_1^{\text{op}}, S) \to \text{Fun}(D_1^{\text{op}}, S)$ be the functor induced by the composition with $g$. Let $\phi$ be a left adjoint to $\psi$ (if necessary we replace $S$ by the ∞-category of spaces in a larger universe). The natural transformation $S_\lambda \to S_\mu \circ (f \times g)$ induces $T_\lambda \to \psi \circ T_\mu \circ f$. It gives rise to $\phi \circ T_\lambda \to T_\mu \circ f$. Since the left adjoint $\phi$ is a left Kan extension of $h C_2 \circ g : D_1 \to D_2 \to \text{Fun}(D_2^{\text{op}}, S)$, we have $h C_2 \circ g \simeq h C_2 \circ g$. By the induced equivalence $\phi \circ T_\lambda \simeq h C_2 \circ g \circ D_\lambda$ and $T_\mu \circ f \simeq h C_2 \circ D_\mu \circ f$, we obtain $h C_2 \circ g \circ D_\lambda \to h C_2 \circ D_\mu \circ f$.

**Remark 5.4.** Suppose that we are given a composite of morphisms of pairings

$$
\begin{array}{ccc}
M & \to & N \\
\downarrow \lambda & & \downarrow \nu \\
C_1 \times D_1 & \to & C_2 \times D_2 & \xrightarrow{h \times 1} & C_3 \times D_3.
\end{array}
$$

In this situation, morphisms of pairings give rise to natural transformations $\xi_1 : g \circ D_\lambda \to D_\mu \circ f$ and $\xi_2 : i \circ D_\mu \to D_\nu \circ h$. Moreover, the big square induces $\kappa : i \circ g \circ D_\lambda \to D_\nu \circ h \circ f$. By the construction, $\kappa$ factors as

$$
i \circ g \circ D_\lambda \xrightarrow{\xi_1} i \circ D_\mu \xrightarrow{\xi_2} D_\nu \circ h \circ f.
$$
5.4. We give a first example of pairings and duality functors which is used in subsequent sections.

Construction 5.5. (i) Let $\mathcal{M}^\otimes \to \mathcal{A}^\otimes$ be a monoidal $\infty$-category. Suppose that the underlying $\infty$-category $\mathcal{M}$ has geometric realizations/colimits of simplicial diagrams, and the tensor product functor $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$ preserves geometric realizations of simplicial objects. By using the $\infty$-category of bimodules, we will define a pairing which determines a Koszul duality functor $\text{Alg}_1^+(\mathcal{M})^{\text{op}} \to \text{Alg}_1^+(\mathcal{M})$ in good cases (see Example 5.6).

Let $\mathbf{1}$ be a unit object of $\mathcal{M}$. We let $\text{BMod}(\mathcal{M})$ denote the $\infty$-category of bimodules objects in $\mathcal{M}^\otimes$ (see Section 3, [22, 4.3]). There are canonical functors

$$\text{BMod}(\mathcal{M}) \to \text{LMod}(\mathcal{M}), \quad \text{BMod}(\mathcal{M}) \to \text{RMod}(\mathcal{M})$$

which forget the right module structures and left module structures, respectively. These functors induce

$$\text{LMod}(\mathcal{M}) \times_{\mathcal{M}} \{\mathbf{1}\} \xrightarrow{\delta^L} \text{BMod}(\mathcal{M}) \times_{\mathcal{M}} \{\mathbf{1}\} \xrightarrow{\delta^R} \text{RMod}(\mathcal{M}) \times_{\mathcal{M}} \{\mathbf{1}\}.$$  

Here projections to $\mathcal{M}$ are forgetful functors. There exists an endomorphism algebra object of $\mathbf{1}$, that is the unit algebra $\mathbf{1} \in \text{Alg}_1(\mathcal{M})$ (we abuse notation by writing $\mathbf{1}$ for the unit algebra). If we regard $\mathbf{1}$ as the endomorphism algebra, there is a canonical equivalence $\text{LMod}(\mathcal{M}) \times_{\mathcal{M}} \{\mathbf{1}\} \simeq \text{Alg}_1(\mathcal{M})/\mathbf{1}$ (see [22, 4.7.1.40]). By reversing module actions in the operadic level, we have an equivalence $\text{LMod}(\mathcal{M}) \rightarrow \text{RMod}(\mathcal{M})$ which commutes with $(-)^{\text{op}}: \text{Alg}_1(\mathcal{M}) \rightarrow \text{Alg}_1(\mathcal{M})$ which carries $A$ to the opposite algebra $A^{\text{op}}$ (cf. [22, 4.6.3]). The equivalence $\text{LMod}(\mathcal{M}) \rightarrow \text{RMod}(\mathcal{M})$ carries a left $A$-module $M$ to the right $A^{\text{op}}$-module $M$. It is left over to $\text{RMod}(\mathcal{M}) \times_{\mathcal{M}} \{\mathbf{1}\} \simeq \text{LMod}(\mathcal{M}) \times_{\mathcal{M}} \{\mathbf{1}\} \simeq \text{Alg}_1(\mathcal{M})/\mathbf{1}$ that lies over the equivalence $(-)^{\text{op}}: \text{Alg}_1(\mathcal{M}) \rightarrow \text{Alg}_1(\mathcal{M})$. Set $\text{Alg}_1^+(\mathcal{M}) = \text{Alg}_1(\mathcal{M})/\mathbf{1}$. We obtain

$$\phi: \text{LMod}(\mathcal{M}) \times_{\mathcal{M}} \{\mathbf{1}\} \simeq \text{Alg}_1^+(\mathcal{M}) \quad \text{and} \quad \psi: \text{RMod}(\mathcal{M}) \times_{\mathcal{M}} \{\mathbf{1}\} \simeq \text{Alg}_1^+(\mathcal{M})$$

We obtain

$$p_\mathcal{M} \times q_\mathcal{M}: K(\mathcal{M}) := \text{BMod}(\mathcal{M}) \times_{\mathcal{M}} \{\mathbf{1}\} \xrightarrow{(p_\mathcal{M} \times q_\mathcal{M})} \text{Alg}_1^+(\mathcal{M}) \times \text{Alg}_1^+(\mathcal{M}).$$

This functor is a right fibration. The canonical projection $\text{BMod}(\mathcal{M}) \to \text{Alg}_1(\mathcal{M}) \times \text{Alg}_1(\mathcal{M})$ is a Cartesian fibration such that a morphism $f$ in $\text{BMod}(\mathcal{M})$ is a Cartesian morphism exactly when the image of $f$ in $\mathcal{M}$ is an equivalence. The induced functor $\text{BMod}(\mathcal{M}) \times_{\mathcal{M}} \{\mathbf{1}\} \to \text{Alg}_1(\mathcal{M}) \times \text{Alg}_1(\mathcal{M})$ is also a Cartesian fibration. Let $e: s \to t$ be a morphism in $\text{LMod}(\mathcal{M}) \times_{\mathcal{M}} \{\mathbf{1}\}$ and let $\overline{e} \in \text{BMod}(\mathcal{M}) \times_{\mathcal{M}} \{\mathbf{1}\}$ be an object lying over $t$. We denote by $e'$ the image of $e$ in $\text{Alg}_1(\mathcal{M}) \times \text{Alg}_1(\mathcal{M})$, and let $\overline{e}' : \overline{e} \to \overline{e}$ be a Cartesian morphism (with respect to the projection to $\text{Alg}_1(\mathcal{M}) \times \text{Alg}_1(\mathcal{M})$) lying over $e$. We easily see that $\overline{e}'$ is a $(p \times q)$-Cartesian morphism. To verify that $p \times q$ is a right fibration, it will suffice to prove that each fiber of $p \times q$ is an $\infty$-groupoid. Indeed, by [22, 4.3.2.7, 4.8.4.6, 4.8.5.16], for $(A, B) \in \text{Alg}_1(\mathcal{M}) \times \text{Alg}_1(\mathcal{M})$, there are canonical equivalences

$$\text{A BMod}_B(\mathcal{M}) \simeq \text{LMod}_A(\text{RMod}_B(\mathcal{M})) \simeq \text{LMod}_A(\mathcal{M}) \otimes_{\mathcal{M}} \text{RMod}_B(\mathcal{M}) \simeq \text{LMod}_A(\mathcal{M}) \otimes_{\mathcal{M}} \text{LMod}_{B^{\text{op}}}(\mathcal{M}) \simeq \text{LMod}_{A \otimes B^{\text{op}}}(\mathcal{M}).$$

which commutes with the projections to $\mathcal{M}$ up to canonical homotopy. We have

$$\text{A BMod}_B(\mathcal{M}) \times_{\mathcal{M}} \{\mathbf{1}\} \simeq \text{LMod}_{A \otimes B^{\text{op}}}(\mathcal{M}) \times_{\mathcal{M}} \{\mathbf{1}\} \simeq \text{Map}_{\text{Alg}_1(\mathcal{M})}(A \otimes B^{\text{op}}, \mathbf{1}).$$

If we regard an object of $(\text{LMod}(\mathcal{M}) \times_{\mathcal{M}} \{\mathbf{1}\}) \times (\text{RMod}(\mathcal{M}) \times_{\mathcal{M}} \{\mathbf{1}\})$ as $(\epsilon_A : A \to \mathbf{1}, \epsilon_{B^{\text{op}}} : B^{\text{op}} \to \mathbf{1}) \in \text{Alg}_1^+(\mathcal{M}) \times \text{Alg}_1^+(\mathcal{M})$, the fiber of $p \times q$ over $(\epsilon_A, \epsilon_{B^{\text{op}}})$ is naturally equivalent to $\text{Map}_{\text{Alg}_1(\mathcal{M})}(A \otimes B^{\text{op}}, \mathbf{1}) \times_{\text{Map}(A, 1) \times \text{Map}(B, 1)} \{((\epsilon_A, \epsilon_{B^{\text{op}}}))\}$, that is an $\infty$-groupoid. As a byproduct of the argument, the composite right fibration $\text{BMod}(\mathcal{M}) \times_{\mathcal{M}} \{\mathbf{1}\} \to \text{Alg}_1^+(\mathcal{M}) \times \text{Alg}_1^+(\mathcal{M})$ corresponds to the functor $\text{Alg}_1^+(\mathcal{M}) \times \text{Alg}_1^+(\mathcal{M}) \to \mathcal{S}$ informally given by

$$(\epsilon_A : A \to \mathbf{1}, \epsilon_B : B \to \mathbf{1}) \mapsto \text{Map}_{\text{Alg}_1(\mathcal{M})}(A \otimes B, \mathbf{1}) \times_{\text{Map}(A, 1) \times \text{Map}(B, 1)} \{((\epsilon_A, \epsilon_B))\}).$$

(ii) Let $\mathcal{N}^\otimes \to \mathcal{A}^\otimes \simeq \mathcal{E}_1^\otimes$ be another monoidal $\infty$-category which has geometric realizations/colimits of simplicial objects, and the tensor product functor preserves geometric realizations of simplicial objects.
Let $F : \mathcal{M}^\otimes \to \mathcal{N}^\otimes$ be a monoidal functor which preserves geometric realizations of simplicial objects. The symmetric monoidal functor $F$ gives rise to a commutative diagram

$$
\begin{array}{c}
\text{BMod}(\mathcal{M}) \times_{\mathcal{M}} \{1\} \ar[r]^-{p \times q} \ar[d]^-{p \times q} & \text{BMod}(\mathcal{N}) \times_{\mathcal{N}} \{1\} \ar[d]^-{p' \times q'} \\
\text{LMod}(\mathcal{M}) \times_{\mathcal{M}} \{1\} \times \text{RMod}(\mathcal{M}) \times_{\mathcal{M}} \{1\} \ar[r]^-{f \times f'} & \text{LMod}(\mathcal{N}) \times_{\mathcal{N}} \{1\} \times \text{RMod}(\mathcal{N}) \times_{\mathcal{N}} \{1\}
\end{array}
$$

where the horizontal arrows are induced by $F$, and the vertical arrows are right fibrations, that is, pairings of $\infty$-categories. Suppose that both pairings are left representable so that there are duality functors

$$
\mathbb{D}_{p \times q} : \text{Alg}_1^+(\mathcal{M})^{\text{op}} \simeq (\text{LMod}(\mathcal{M}) \times_{\mathcal{M}} \{1\})^{\text{op}} \to \text{RMod}(\mathcal{M}) \times_{\mathcal{M}} \{1\} \simeq \text{Alg}_1^+(\mathcal{M}),
$$

$$
\mathbb{D}_{p' \times q'} : \text{Alg}_1^+(\mathcal{N})^{\text{op}} \simeq (\text{LMod}(\mathcal{N}) \times_{\mathcal{N}} \{1\})^{\text{op}} \to \text{RMod}(\mathcal{N}) \times_{\mathcal{N}} \{1\} \simeq \text{Alg}_1^+(\mathcal{N}).
$$

We write $\mathbb{D}_{1,\mathcal{M}}$ and $\mathbb{D}_{1,\mathcal{N}}$ for $\mathbb{D}_{p \times q}$ and $\mathbb{D}_{p' \times q'}$, respectively. By Lemma 5.3, it gives rise to a natural transformation

$$
f_r \circ \mathbb{D}_{1,\mathcal{M}} \to \mathbb{D}_{1,\mathcal{N}} \circ f_t.
$$

**Example 5.6.** Let $\mathcal{A}^\otimes$ be a symmetric monoidal presentable $\infty$-category such that the tensor product functor $\mathcal{A} \times A \to \mathcal{A}$ preserves small colimits in each variable. Let $\text{Alg}_n^\otimes(\mathcal{A})$ denote the symmetric monoidal $\infty$-category of $\mathcal{E}_n$-algebra objects in $\mathcal{A}$. The underlying $\infty$-category $\text{Alg}_n(\mathcal{A})$ admits sifted colimits (e.g., geometric realizations of simplicial objects), and the tensor product $\text{Alg}_n(\mathcal{A}) \times \text{Alg}_n(\mathcal{A}) \to \text{Alg}_n(\mathcal{A})$ preserves sifted colimits since the forgetful functor $\text{Alg}_n(\mathcal{A}) \to \mathcal{A}$ preserves sifted colimits (cf. [22, 3.2.3.2]). Suppose that $\mathcal{M}^\otimes$ is $\text{Alg}_n^\otimes(\mathcal{A})$ ($n \geq 0$). By convention, $\mathcal{A}^\otimes = \text{Alg}_0^\otimes(\mathcal{A})$. Then we can apply Construction 5.5 to $\mathcal{M}^\otimes$. We obtain

$$
P(\mathcal{M}) := p_{\mathcal{M}} \times q_{\mathcal{M}} : K(\mathcal{M}) := \text{BMod}(\mathcal{M}) \times_{\mathcal{M}} \{1\} \to \text{Alg}_1^+(\mathcal{M}) \times \text{Alg}_1^+(\mathcal{M}).
$$

This pairing is left representable (and right representable). Let $\mathcal{M}^\otimes = \text{Alg}_n^\otimes(\mathcal{A})$ and consider $D : \text{Alg}_1^+(\text{Alg}_n(\mathcal{A}))^{\text{op}} \to \text{Fun}(\text{Alg}_1^+(\text{Alg}_n(\mathcal{A}))^{\text{op}}, \mathcal{S})$ determined by $P(\text{Alg}_n(\mathcal{A}))$ (cf. Section 5.3). By Dunn additivity theorem [22], $\text{Alg}_1^+(\text{Alg}_n(\mathcal{A})) \simeq \text{Alg}_{n+1}(\mathcal{A})$. As discussed in the proof of [23, X, Lemma 3.1.5], the image of $f : B \to 1_A \in \text{Alg}_{n+1}(\mathcal{A})$ in $\text{Fun}(\text{Alg}_{n+1}(\mathcal{A})^{\text{op}}, \mathcal{S})$ under $D$ is represented by a centralizer $Z(f)$ of $f : B \to 1_A$ with the associated augmentation $Z(f) : B \otimes Z(f) \to 1_A$ (see [22, 5.3.1.15] for the existence of a centralizer). This shows that $P(\text{Alg}_n(\mathcal{A}))$ is left representable, which defines

$$
\mathbb{D}_{n+1,\mathcal{A}}^+ : \text{Alg}_{n+1}^+(\mathcal{A})^{\text{op}} \to \text{Alg}_{n+1}^+(\mathcal{A}).
$$

For example, when $\mathcal{A}^\otimes = \text{Mod}_\mathcal{B}^\otimes$, $\mathbb{D}_{n+1,\mathcal{A}}$ is $\mathcal{D}_n : \text{Alg}_{n+1}^+(\text{Mod}_\mathcal{B})^{\text{op}} \to \text{Alg}_{n+1}^+(\text{Mod}_\mathcal{B})^{\text{op}}$ in Section 3.4.

**5.5. Koszul duals from deformation functors.** We define a Koszul dual associated to a deformation.

**Construction 5.7.** (i) Consider the situation in Section 5.2. Given a deformation $M' \in \text{RMod}_B(\mathcal{M}) \times_{\mathcal{M}} \{M\}$ of $M$, we define the module action of the Koszul dual of $B$ on $M$. For this purpose, we first define a pairing of $\mathbb{D} \circ f_M(\mathcal{M})$ and $\{M\} \times_{\mathcal{M}} \text{LMod}^+(\mathcal{M}) := \{M\} \times_{\mathcal{M}} \text{LMod}(\mathcal{M}) \times_{\text{Alg}_1(\mathcal{M})} \text{Alg}_1^+(\mathcal{M})$.

Recall $K(\mathcal{M}) = \text{BMod}(\mathcal{M}) \times_{\mathcal{M}} \{1\}$ from Example 5.6. There is the canonical functor $K(\mathcal{M}) \to \text{LMod}(\mathcal{M}) \times_{\mathcal{M}} \{1\} \simeq \text{Alg}_1^+(\mathcal{M})$. Define

$$
P_M(\mathcal{M}) := \mathbb{D} \circ f_M(\mathcal{M}) \times_{\text{Alg}_1^+(\mathcal{M})} K(\mathcal{M}).
$$

Next, we will define $\mathcal{P}_M(\mathcal{M}) \to \{M\} \times_{\mathcal{M}} \text{LMod}^+(\mathcal{M})$. We first consider the composite

$$
\mathcal{P}_M(\mathcal{M}) = \mathbb{D} \circ f_M(\mathcal{M}) \times_{\text{Alg}_1^+(\mathcal{M})} K(\mathcal{M}) \to \{M\} \times_{\mathcal{M}} \text{RMod}(\mathcal{M}) \times_{\text{Alg}_1(\mathcal{M})} \text{BMod}(\mathcal{M}) \times_{\mathcal{M}} \{1\}
$$

$$
\to \{M\} \times_{\mathcal{M}} \text{RMod}(\mathcal{M}) \times_{\text{Alg}_1(\mathcal{M})} \text{BMod}(\mathcal{M})
$$

$$
\to \{M\} \times_{\mathcal{M}} \text{RMod}(\mathcal{M}),
$$

where the first functor and the second functor are the forgetful functors, and the third functor is determined by the relative tensor product functor $\text{RMod}(\mathcal{M}) \times_{\text{Alg}_1(\mathcal{M})} \text{BMod}(\mathcal{M}) \to \text{RMod}(\mathcal{M})$ (cf. Section 5.1). Let $\text{RMod}(\mathcal{M}) \xrightarrow{\sim} \text{LMod}(\mathcal{M})$ be the equivalence obtained by the reversing module actions in the operadic level. The composition with this equivalence yields $\mathcal{P}_M(\mathcal{M}) \to \{M\} \times_{\mathcal{M}} \text{LMod}(\mathcal{M})$. If an object of $\mathcal{P}_M(\mathcal{M})$ is defined as the pair of $(M' \in \text{RMod}_B(\mathcal{M}), M \simeq M' \otimes_B 1)$ and a $B$-$C$-bimodule.
1, then its image in \( \{M\} \times _M \text{LMod}(\mathcal{M}) \) is the left \( C^{\text{op}} \)-module corresponding to the right \( C \)-module \( M' \otimes _B 1 \) determined by the right \( C \)-module 1. Consider the composite \( \mathcal{P}_M(\mathcal{M}) \to \{M\} \times _M \text{LMod}(\mathcal{M}) \to \text{Alg}_1(\mathcal{M}) \). This composite is naturally promoted to \( \mathcal{P}_M(\mathcal{M}) \to K(\mathcal{M}) \to \text{RMod}(\mathcal{M}) \times _M \{1\} \simeq \text{LMod}(\mathcal{M}) \times _M \{1\} \simeq \text{Alg}_1(\mathcal{M}) \)
where the first functor and the second functor are projections. Thus, we have \( \omega_M(\mathcal{M}) : \mathcal{P}_M(\mathcal{M}) \to \{M\} \times _M \text{LMod}(\mathcal{M}) \times _{\text{Alg}_1(\mathcal{M})} \text{Alg}_1(\mathcal{M}) = \{M\} \times _M \text{LMod}(\mathcal{M}). \)

The projection \( \mathcal{P}_M(\mathcal{M}) \to \text{Def}_M(\mathcal{M}) \) and \( \omega_M(\mathcal{M}) \) determine \( \mathcal{P}_M(\mathcal{M}) : \mathcal{P}_M(\mathcal{M}) \to \text{Def}_M(\mathcal{M}) \times (\{M\} \times _M \text{LMod}(\mathcal{M})). \)

This is a right fibration, up to an equivalence, whose promotion of a morphism to a Cartesian morphism is induced by the promotion of the restriction of scalars. The fiber of this right fibration over the pair \( (M', B \to 1, M \simeq M' \otimes _B 1) \) and a left \( C^{\text{op}} \)-module \( M \) is the space equivalent to the fiber of the induced morphism

\[
\text{Def}_M(\mathcal{M})(B \to 1) \times _{\text{Alg}_1(\mathcal{M})} \text{Map}_{\text{Alg}_1(\mathcal{M})}(B \otimes C^{\text{op}}, 1) \\
\rightarrow \text{Def}_M(\mathcal{M})(B \to 1) \times (\{M\} \times _M \text{LMod}(\mathcal{M}) \times _{\text{Alg}_1(\mathcal{M})} \{C^{\text{op}} \to 1\})
\]
over the point determined by the pair (keep in mind that \( B \text{BMod}_C(\mathcal{M}) \times _M \{1\} \simeq \text{Map}_{\text{Alg}_1(\mathcal{M})}(B \otimes C^{\text{op}}, 1) \), cf. Construction 5.5). When it is left representable pairing of \( \infty \)-categories, it gives rise to

\[
\mathbb{D}_{M,M^\otimes} : \text{Def}_M(\mathcal{M})^{\text{op}} \to \{M\} \times _M \text{LMod}(\mathcal{M})
\]

(cf. Example 5.8).

(ii) Let \( \mathcal{N} \otimes \Rightarrow \mathcal{A} \otimes \simeq \mathcal{E}_1 \otimes \) be another monoidal \( \infty \)-category which has geometric realizations/collimits of simplicial objects, and the tensor product functor preserves geometric realizations of simplicial objects. Let \( F : \mathcal{M} \otimes \to \mathcal{N} \otimes \) be a monoidal functor. We consider the induced morphism of pairings. Suppose that \( \text{RMod}^{\text{op}}(\mathcal{M}) \to \text{RMod}^{\text{op}}(\mathcal{N}) \) induced by \( F \) carries \( \text{co} \)-Cartesian morphisms over \( \text{Alg}_1(\mathcal{M}) \) to \( \text{co} \)-Cartesian morphisms over \( \text{Alg}_1(\mathcal{N}) \). Then \( F \) induces \( \text{Def}_F(\mathcal{M}) : \text{Def}_M(\mathcal{M}) \to \text{Def}_F(\mathcal{N}) \). Moreover, \( F \) also induces \( K(\mathcal{M}) \to K(\mathcal{N}) \) lying over \( \text{Alg}_1(\mathcal{M}) \to \text{Alg}_1(\mathcal{N}) \) in the natural way. Consequently, \( F \) determines \( \mathcal{P}_M(\mathcal{M}) \to \mathcal{P}_F(\mathcal{N}) \). Note that \( F \) induces \( F(M_B \otimes F(1)) F(1) \Rightarrow F(M_B \otimes B 1) \) for any \( M_B \in \text{RMod}_B(\mathcal{M}) \). Thus, we obtain a morphism of pairings

\[
\mathcal{P}_M(\mathcal{M}) \twoheadrightarrow \mathcal{P}_F(\mathcal{N}) \\
\downarrow \\
\text{Def}_M(\mathcal{M}) \times (\{M\} \times _M \text{LMod}(\mathcal{M})) \twoheadrightarrow \text{Def}_F(\mathcal{N}) \times (\{F(M)\} \times _N \text{LMod}(\mathcal{N})).
\]

where the lower horizontal functors are determined by \( \text{Def}_F(\mathcal{M}) \) and \( F : \{M\} \times _M \text{LMod}^{\text{op}}(\mathcal{M}) \to \{F(M)\} \times _N \text{LMod}^{\text{op}}(\mathcal{N}) \) induced by \( F \). When both pairings are left representable, it follows from Lemma 5.3 that there is a natural transformation \( F \circ \mathbb{D}_{M,M^\otimes} \Rightarrow \mathbb{D}_{F(\mathcal{M}), N^\otimes} \circ \text{Def}_F(\mathcal{M}). \)

**Example 5.8.** Consider the situation in Example 5.6. That is, \( \mathcal{A} \otimes \) is a symmetric monoidal presentable \( \infty \)-category such that the tensor product functor \( \mathcal{A} \times \mathcal{A} \to \mathcal{A} \) preserves small colimits in each variable. Suppose that \( \mathcal{M} \otimes \) is \( \mathcal{A} \otimes \). Then the right fibration \( \mathcal{P}_M(\mathcal{M}) \) is left representable so that it gives rise to

\[
\mathbb{D}_{M,M^\otimes} : \text{Def}_F(\mathcal{M})^{\text{op}} \to \{M\} \times _M \text{LMod}(\mathcal{M}).
\]

Indeed, \( \mathbb{D}_{M,M^\otimes} \) sends \( (M_B \in \text{RMod}_B(\mathcal{M}), M_B \otimes B 1 \simeq M) \) to \( M \in \text{LMod}_{D_1(B)}(\mathcal{M}) \) whose \( D_1(B) \)-module structure is defined as follows. If we think of \( 1 \) as an object of \( B \text{-}\mathbb{D}_1(B)^{\text{op}} \)-bimodule determined by \( B \otimes \mathbb{D}_1(B) \to 1 \), then the “integral kernel” \( 1 \) defines \( \text{RMod}_B \to \text{RMod}_{D_1(B)^{\text{op}}} \) given by \( M_B \mapsto M_B \otimes B 1 \). We can regard \( M = M_B \otimes B 1 \) as a right \( D_1(B)^{\text{op}} \)-module, that is, a \( D_1(B) \)-module.

**Remark 5.9.** We record a simple version of the above construction for the future reference. Consider the situation in Example 5.6. Set \( \mathcal{P}(\mathcal{A}) = \text{RMod}^{\text{op}}(\mathcal{A}) \times _{\text{Alg}_1(\mathcal{A})} K(\mathcal{A}) \) and let \( \mathcal{P}(\mathcal{A}) \to \text{RMod}^{\text{op}}(\mathcal{A}) \times \text{LMod}(\mathcal{A}) \) be a pairing defined similarly to the above. It gives rise to the duality functor

\[
\text{RMod}^{\text{op}}(\mathcal{A})^{\text{op}} \xrightarrow{\mathbb{D}} \text{LMod}(\mathcal{A})
\]
lying over $\mathcal{D}_{1,A} : \text{Alg}_1^+(\mathcal{A})^{op} \to \text{Alg}_1^+(\mathcal{A})$.

6. Deformations of categories and cyclic deformations

Using the formalism in Section 5, we define the deformation functors that describe the deformation problem of a stable $\infty$-category and that of a module endowed with $S^1$-action. Also, based on Section 5, to a deformation of a stable $\infty$-category $\mathcal{C}$, we associate the Koszul dual of the deformation, that is a certain module structure on $\mathcal{C}$. Similarly, we define the Koszul dual of a deformation of module endowed with $S^1$-action. These are carried out in Section 6.1 and Section 6.2. In Section 6.3, we study how Hochschild chain functor $\mathcal{H}H(-/A)$ relates two deformation functors and their Koszul duals. The main result is Proposition 6.8.

6.1. Deformations of categories. We start with the definition of deformations of stable $\infty$-categories. Let $\mathcal{M}^\otimes = \mathcal{S}^\otimes_A$. This symmetric monoidal $\infty$-category is compactly generated, and the tensor product $\otimes : \mathcal{S}^\otimes_A \times \mathcal{S}^\otimes_A \to \mathcal{S}^\otimes_A$ preserves small colimits separately in each variable (cf. Section 3.1).

Let $\mathcal{C}$ be an object of $\mathcal{S}^\otimes_A$, that is, an $A$-linear (small) stable $\infty$-category. We will define the deformation functor of $\mathcal{C}$ over $\text{Alg}_1^+(\mathcal{S}^\otimes_A)$. We apply Definition 5.1 to $\mathcal{M}^\otimes = \mathcal{S}^\otimes_A$. Then we obtain the left fibration $\text{Def}_c(\mathcal{C}) \to \text{Alg}_1^+(\mathcal{S}^\otimes_A)$. It is classified by the functor $\text{Def}_c(\mathcal{S}^\otimes_A) : \text{Alg}_1^+(\mathcal{S}^\otimes_A) \to \mathcal{S}^\otimes_A$.

Next, we consider the symmetric monoidal functor $\text{Alg}_1(\text{Mod}_{\mathcal{S}^\otimes_A}) \to \mathcal{S}^\otimes_A$ given by $B \mapsto \text{Perf}_{\mathcal{S}^\otimes_A}$ (cf. Section 3.1). It determines $\text{Alg}_2^+(\text{Mod}_{\mathcal{S}^\otimes_A})$ defined as the composite

$$
\text{Def}^E_\mathcal{C} : \text{Alg}_2^+(\text{Mod}_{\mathcal{S}^\otimes_A}) \to \text{Alg}_1^+(\mathcal{S}^\otimes_A) \to \mathcal{S}^\otimes_A
$$

(cf. Definition 5.2). It sends $[B \to A] \in \text{Alg}_2^+(\text{Mod}_{\mathcal{S}^\otimes_A})$ to the space of deformations $\{C\} \times_{\mathcal{S}^\otimes_A} \text{RMod}_{\mathcal{S}^\otimes_A}(\mathcal{S}^\otimes_A)^\otimes$. We refer to $\text{Def}^E_\mathcal{C}$ as the $E_2$-deformation functor of $\mathcal{C}$. An object of $\text{Def}^E_\mathcal{C}(B)$ is described as $(C' \in \text{RMod}_{\mathcal{S}^\otimes_A}(\mathcal{S}^\otimes_A), C \simeq C' \otimes_{\text{Perf}_{\mathcal{S}^\otimes_A}} \mathcal{S}^{\otimes_A})$.

Next, we define the Koszul dual arising from a deformation of $\mathcal{C}$. To this end, we apply Construction 5.7 and Example 5.8 to $\mathcal{M}^\otimes = \mathcal{S}^\otimes_A$. We have the pairing $\mathcal{P}_c(\mathcal{S}^\otimes_A) : \mathcal{P}_c(\mathcal{S}^\otimes_A) \to \text{Def}_c(\mathcal{S}^\otimes_A) \times \{\{C\} \times_{\mathcal{S}^\otimes_A} \text{LMod}^+(\mathcal{S}^\otimes_A)\}$. Here $\text{LMod}^+(\mathcal{S}^\otimes_A) = \text{LMod}(\mathcal{S}^\otimes_A) \times_{\text{Alg}_1(\mathcal{S}^\otimes_A)} \text{Alg}_2^+(\mathcal{S}^\otimes_A)$ and we use the notation in Construction 5.7. The projection $\mathcal{P}_c(\mathcal{S}^\otimes_A) \to \text{Def}_c(\mathcal{S}^\otimes_A)$, the forgetful functor $\text{BMod}(\text{Alg}_1(\text{Mod}_{\mathcal{S}^\otimes_A})) \to \text{LMod}(\text{Alg}_1(\text{Mod}_{\mathcal{S}^\otimes_A}))$, and $p_{\mathcal{S}^\otimes_A} : K(\mathcal{S}^\otimes_A) \to \text{Alg}_1^+(\mathcal{S}^\otimes_A)$ induce

$$
\mathcal{P}_c(\mathcal{S}^\otimes_A) \times_{K(\mathcal{S}^\otimes_A)} K(\text{Alg}_1(\text{Mod}_{\mathcal{S}^\otimes_A})) \to \text{Def}_c(\mathcal{S}^\otimes_A) \times_{\text{Alg}_1^+(\mathcal{S}^\otimes_A)} \text{LMod}(\text{Alg}_1(\text{Mod}_{\mathcal{S}^\otimes_A})) \times_{\text{Alg}_1(\text{Mod}_{\mathcal{S}^\otimes_A})} \{A\}
$$

(see Construction 5.5 for $p_{\mathcal{S}^\otimes_A}$). Consider $Q : \mathcal{P}_c(\mathcal{S}^\otimes_A) \to \text{Alg}_1^+(\mathcal{S}^\otimes_A)$ defined as the composite

$$
\mathcal{P}_c(\mathcal{S}^\otimes_A) \to K(\mathcal{S}^\otimes_A) \to \text{LMod}^+(\mathcal{S}^\otimes_A) \times_{\text{Alg}_1(\mathcal{S}^\otimes_A)} \text{Alg}_2^+(\mathcal{S}^\otimes_A)
$$

(see Construction 5.5 for $p_{\mathcal{S}^\otimes_A}$, keep in mind that it is not same with $\mathcal{P}_c(\mathcal{S}^\otimes_A) \to K(\mathcal{S}^\otimes_A) \to \text{Alg}_1^+(\mathcal{S}^\otimes_A)$). The morphism $\mathcal{P}_c(\mathcal{S}^\otimes_A) \to \{\{C\} \times_{\mathcal{S}^\otimes_A} \text{LMod}^+(\mathcal{S}^\otimes_A)\}$ naturally commutes with $Q$ and the forgetful functor. Let $\mathcal{P}_c(\mathcal{S}^\otimes_A) \times_{Q,\text{Alg}_1^+(\mathcal{S}^\otimes_A)} \text{Alg}_1^+(\mathcal{S}^\otimes_A)$ be the fiber product induced by $Q$. Then we have

$$
\mathcal{P}_c(\mathcal{S}^\otimes_A) \times_{K(\mathcal{S}^\otimes_A)} K(\text{Alg}_1(\text{Mod}_{\mathcal{S}^\otimes_A})) \xrightarrow{\text{id} \times q_{\mathcal{S}^\otimes_A}} \mathcal{P}_c(\mathcal{S}^\otimes_A) \times_{Q,\text{Alg}_1^+(\mathcal{S}^\otimes_A)} \text{Alg}_1^+(\text{Mod}_{\mathcal{S}^\otimes_A})
$$

Set $\mathcal{D}_c : = \mathcal{D}_c(\mathcal{S}^\otimes_A) \times_{\text{Alg}_1^+(\mathcal{S}^\otimes_A)} \text{Alg}_2^+(\text{Mod}_{\mathcal{S}^\otimes_A})$ and

$$
\text{LMod}^E(\mathcal{A}) := \{\{C\} \times_{\mathcal{S}^\otimes_A} \text{LMod}^+(\mathcal{S}^\otimes_A) \times_{\text{Alg}_1^+(\mathcal{S}^\otimes_A)} \text{Alg}_2^+(\mathcal{S}^\otimes_A)\}.
$$

Write $\mathcal{P}_c := \mathcal{P}_c(\mathcal{S}^\otimes_A) \times_{K(\mathcal{S}^\otimes_A)} K(\text{Alg}_1(\text{Mod}_{\mathcal{S}^\otimes_A}))$. We obtain the pairing

$$
\mathcal{P}_c : \mathcal{P}_c \to \mathcal{D}_c \times \text{LMod}^E(\mathcal{A})
$$

and the associated duality functor

$$
\mathcal{D}_c : (\mathcal{D}_c)^{op} \to \text{LMod}^E(\mathcal{A})^{op}.
$$

Given a deformation of $\mathcal{C}$, we call the image under $\mathcal{D}_c$ the Koszul dual of the deformation. Taking into account Example 5.8 and $\text{Alg}_2(\text{Mod}_{\mathcal{S}^\otimes_A}) \to \text{Alg}_1^+(\mathcal{S}^\otimes_A)$, we see that the duality functor $\mathcal{D}_c$ carries $(C' \in \text{RMod}_{\mathcal{S}^\otimes_A}(\mathcal{S}^\otimes_A), C \simeq C' \otimes_{\text{Perf}_{\mathcal{S}^\otimes_A}} \mathcal{S}^{\otimes_A})$ to the left $\text{Perf}_{\mathcal{S}^\otimes_A}^{op}$-module $\mathcal{C}$ (together with the
augmentation \( \mathbb{D}_2(B) \to A \) defined as follows. The universal pairing \( B \otimes_A \mathbb{D}_2(B) \to A \) induces a morphism \( \text{Perf}^\otimes_{\mathbb{D}_2(B)} \simeq \text{Perf}^\otimes_{A/\mathbb{D}_2(B)} \to \text{Perf}^\otimes_{A/\mathbb{D}_2(B)} \) in \( \text{Alg}_1(\text{St}_A) \) where \( \text{Perf}^\otimes_{\mathbb{D}_2(B)} \) and \( \text{Perf}^\otimes_{A/\mathbb{D}_2(B)} \) are regarded as objects of \( \text{Alg}_1(\text{St}_A) \). Thus, it determines a structure of \( \text{Perf}^\otimes_{A/\mathbb{D}_2(B)} \) as a \( \text{Perf}^\otimes_{\mathbb{D}_2(B)} \)-bimodule on \( \text{Perf}^\otimes_{A/\mathbb{D}_2(B)} \). Here \( (\text{Perf}^\otimes_{\mathbb{D}_2(B)})^\otimes = \text{Perf}^\otimes_{\mathbb{D}_2(B)} \) is the opposite algebra of \( \text{Perf}^\otimes_{\mathbb{D}_2(B)} \) in \( \text{Alg}_1(\text{St}_A) \). Therefore, \( C \simeq C' \otimes_{\text{Perf}^\otimes_{A/\mathbb{D}_2(B)}} \text{Perf}^\otimes_{A/\mathbb{D}_2(B)} \) inherits a right \( (\text{Perf}^\otimes_{\mathbb{D}_2(B)})^\otimes \)-module structure (i.e., a left \( \text{Perf}^\otimes_{\mathbb{D}_2(B)} \)-module) from that of \( \text{Perf}^\otimes_{A/\mathbb{D}_2(B)} \). From this construction and the fully faithful embedding \( \text{Alg}_2(\text{Mod}_A) \hookrightarrow \text{Alg}_1(\text{St}_A) \), the pairing \( \mathbf{P} \) is left representable.

### 6.2. Cyclic deformations of modules with \( S^1 \)-actions

Let \( \text{Mod}^{S^1}_A = \text{Fun}(B S^1, \text{Mod}_A) \) and let \( H \) be an object of \( \text{Mod}^{S^1}_A \). Given \([B \to A] \in \text{Alg}^+_1(\text{Mod}^{S^1}_A)\), a deformation of \( H \) to \( B \) is defined as an object of \( \text{RMod}_B(\text{Mod}^{S^1}_A) \times \text{Mod}^{S^1}_A \) \{H\} where \( \text{RMod}_B(\text{Mod}^{S^1}_A) \to \text{Mod}^{S^1}_A \) is the reduction functor (cf. Section 5.1). This deformation problem plays a central role in this paper.

Suppose that \( \mathcal{M} \) is \( \text{Mod}^{S^1}_A = \text{Fun}(B S^1, \text{Mod}_A) \) endowed with the symmetric monoidal structure induced by that of \( \text{Mod}^{S^1}_A \). The \( A \)-module \( A \) endowed with the trivial \( S^1 \)-action is a unit object in \( \text{Mod}^{S^1}_A \).

Applying Definition 5.1 to \( \mathcal{M} = \text{Mod}^{S^1}_A \), we have the left fibration

\[
\text{Def}_H^\mathcal{M} := \text{Def}_H(\text{Mod}^{S^1}_A) \to \text{Alg}^+_1(\text{Mod}^{S^1}_A),
\]

which is classified by \( \text{Def}_H^\mathcal{M} := \text{Def}_H(\text{Mod}^{S^1}_A) : \text{Alg}^+_1(\text{Mod}^{S^1}_A) \to \mathcal{S} \). This deformation functor carries \([B \to A] \to \{H\} \times \text{Mod}^{S^1}_A \) \text{RMod}_B(\text{Mod}^{S^1}_A)^\mathcal{M} \) to \( \text{Def}_H^\mathcal{M}(B \to A) \) (simply denoted by \( \text{Def}_H^\mathcal{M}(B) \)) as the space of cyclic deformations of \( H \) to \( B \).

We apply Construction 5.7 to \( \mathcal{M} = \text{Mod}^{S^1}_A \). Then it gives rise to the pairing

\[
\mathcal{P}_H(\text{Mod}^{S^1}_A) \to \text{Def}_H^\mathcal{M} \times \{\mathcal{H}\} \times \text{LMod}^{+}(\text{Mod}^{S^1}_A).
\]

We write \( \text{LMod}^{+}(\text{Mod}^{S^1}_A) = \{\mathcal{H}\} \times \text{Mod}^{S^1}_A \times \text{LMod}^{+}(\text{Mod}^{S^1}_A) \). Here \( \text{LMod}^{+}(\text{Mod}^{S^1}_A) = \text{LMod}(\text{Mod}^{S^1}_A) \times \text{Alg}^+_1(\text{Mod}^{S^1}_A) \).

By Example 5.8, we have the duality functor associated to the pairing

\[
\mathbb{D}_H^\mathcal{M} := \mathbb{D}_{H,\text{Mod}^{S^1}_A} : (\text{Def}_H^\mathcal{M})^\mathcal{M} \to \text{LMod}^{+}(\text{Mod}^{S^1}_A) \mathcal{M}. \]

Given a deformation of \( H \), we call the image under \( \mathbb{D}_H^\mathcal{M} \) the Koszul dual of the deformation. Given a deformation \( (H_B \in \text{RMod}_B(\text{Mod}^{S^1}_A), H_B \otimes_B A \simeq H) \in \text{Def}_H^\mathcal{M}(B) \), the Koszul dual of the deformation is defined as follows. Consider \( A \) to be an object of \( \text{B-\mathbb{D}_1}(B)^\mathcal{M} \) determined by \( B \otimes \mathbb{D}_1(B) \to A \), then the “integral kernel” \( A \) defines \( \text{RMod}_B(\text{Mod}^{S^1}_A) \to \text{RMod}_B(\text{Mod}^{S^1}_A) \) given by \( H_B \mapsto H_B \otimes_B A \).

We can regard \( H \simeq H_B \otimes_B A \) as a right \( \mathbb{D}_1(B)^\mathcal{M} \)-module, that is, a left \( \mathbb{D}_1(B) \)-module in \( \text{Mod}^{S^1}_A \).

### 6.3. From deformations of categories to cyclic deformations

Let \( h = \mathcal{H}_\bullet(\mathcal{C}) : \mathcal{C}_\mathcal{C} \to (\text{Mod}^{S^1}_A)^\mathcal{C} \) be the symmetric monoidal Hochschild chain functor (see Section 3.6). We observe that the functor \( h \) carries a deformation of an \( A \)-linear stable \( \infty \)-category \( \mathcal{C} \) to a cyclic deformation of \( \mathcal{H}_\bullet(\mathcal{C}/A) \) (Construction 6.5). This is the main motivation for the notion of cyclic deformations. Proposition 6.6 provides a comparison between the image of the Koszul dual of a deformation of \( \mathcal{C} \) (i.e., the image under \( \mathbb{D}_\mathcal{C} \), see Section 6.1) under \( h \) (more precisely, \( \text{LMod}^\mathcal{M}(h) \), see Construction 6.5) and the Koszul dual of the induced cyclic deformation of \( \mathcal{H}_\bullet(\mathcal{C}/A) \) (i.e., the image under \( \mathbb{D}_\mathcal{C}^\mathcal{M}(\mathcal{H}_\bullet(\mathcal{C}/A)) \), see Section 6.2).

We start with an exchange of the duality functors arising from \( h \).

**Definition 6.1.** We define an exchange of the duality functors \( \mathbb{D}_{2,\text{Mod}_A} \) and \( \mathbb{D}_{1,\text{Mod}^{S^1}_A} \), which is induced by \( \text{Alg}_1(\text{Mod}_A) \to \text{St}_A \to \text{Mod}^{S^1}_A \) (for ease of notation, we write \( h \) also for the composite). By
Construction 5.5, Example 5.5 and Dunn additivity theorem, \( h \) induces the diagram

\[
K(\text{Alg}_1(\text{Mod}_A)) \xrightarrow{} K(\text{Mod}_A^{S^1}) \\
\text{Alg}_2^+ (\text{Mod}_A) \times \text{Alg}_2^+ (\text{Mod}_A) \xrightarrow{} \text{Alg}_1^+ (\text{Mod}_A^{S^1}) \times \text{Alg}_1^+ (\text{Mod}_A^{S^1})
\]

which commutes up to canonical homotopy. The horizontal bottom arrow is determined by the product of \( \text{Alg}_1^+ (h) : \text{Alg}_2^+ (\text{Mod}_A) \to \text{Alg}_1^+ (\text{Mod}_A^{S^1}) \) induced by \( h \). Both vertical arrows are left representable pairings so that it induces duality functors \( \text{d}^! \) for ease of notation, we often write

\[ \text{Alg}_2^+ (\text{Mod}_A) \times \text{Alg}_2^+ (\text{Mod}_A) \xrightarrow{} \text{Alg}_1^+ (\text{Mod}_A^{S^1}) \times \text{Alg}_1^+ (\text{Mod}_A^{S^1}) \]

\text{Alg}_2^+ (\text{Mod}_A) \xrightarrow{} \text{Alg}_1^+ (\text{Mod}_A^{S^1}) \]

\text{Alg}_2^+ (\text{Mod}_A)

which commutes up to canonical homotopy. The horizontal bottom arrow is determined by the product of \( \text{Alg}_1^+ (h) : \text{Alg}_2^+ (\text{Mod}_A) \to \text{Alg}_1^+ (\text{Mod}_A^{S^1}) \) induced by \( h \). Both vertical arrows are left representable pairings so that it induces duality functors \( \text{d}^! \)

\[ \text{Alg}_2^+ (\text{Mod}_A) \times \text{Alg}_2^+ (\text{Mod}_A) \xrightarrow{} \text{Alg}_1^+ (\text{Mod}_A^{S^1}) \times \text{Alg}_1^+ (\text{Mod}_A^{S^1}) \]

\text{Alg}_2^+ (\text{Mod}_A)

which commutes up to canonical homotopy. The horizontal bottom arrow is determined by the product of \( \text{Alg}_1^+ (h) : \text{Alg}_2^+ (\text{Mod}_A) \to \text{Alg}_1^+ (\text{Mod}_A^{S^1}) \) induced by \( h \). Both vertical arrows are left representable pairings so that it induces duality functors \( \text{d}^! \)

\[ \text{Alg}_2^+ (\text{Mod}_A) \times \text{Alg}_2^+ (\text{Mod}_A) \xrightarrow{} \text{Alg}_1^+ (\text{Mod}_A^{S^1}) \times \text{Alg}_1^+ (\text{Mod}_A^{S^1}) \]

\text{Alg}_2^+ (\text{Mod}_A)

which commutes up to canonical homotopy. The horizontal bottom arrow is determined by the product of \( \text{Alg}_1^+ (h) : \text{Alg}_2^+ (\text{Mod}_A) \to \text{Alg}_1^+ (\text{Mod}_A^{S^1}) \) induced by \( h \). Both vertical arrows are left representable pairings so that it induces duality functors \( \text{d}^! \)
Construction 6.5. We will construct a morphism of pairings defined in Section 6.1 and Section 6.2. We suppose that $H = \mathcal{H}_\bullet(C/A) = \mathcal{H}_\bullet$ in Section 6.2. It yields an exchange of duality functors $D_C$ and $D^\vee_{\mathcal{H}_\bullet}$. According to Lemma 6.3, $h = \mathcal{H}_\bullet(-/A) : St_A \to Mod_A^{S_1}$ induces $Def_c(St_A) \to Def_c(St_A)$. Thus, invoking Construction 5.5 (ii), we obtain a morphism of pairings

\[ P_C(St_A) \to P_H(\text{Mod}^{S_1}_A) \]

\[ Def_c(St_A) \times ([C] \times_{St_A} \text{LMod}^+(St_A)) \to Def_c(St_A) \times ([C] \times_{St_A} \text{LMod}^+(St_A)). \]

(see Section 6.1 and Section 6.2 for these pairings). On the other hand, taking into account definitions of $P_C$, $Def_c$ and $\text{LMod}^{E_2}(A)_C$ described as fiber products (see Section 6.1), we see that projections induce a morphism of pairings in the natural way:

\[ P_C \to P_C(St_A) \to Def_c \times \text{LMod}^{E_2}(A)_C \to Def_c(St_A) \times ([C] \times_{St_A} \text{LMod}^+(St_A)). \]

The composition of two morphisms of pairings gives

\[ Def_c \times \text{LMod}^{E_2}(A)_C \to Def_c(St_A) \times ([C] \times_{St_A} \text{LMod}^+(St_A)). \]

We write $Def(f)$ and $\text{LMod}'(h)$ for the induced maps $Def_c \to Def_c(St_A)$ and $\text{LMod}^{E_2}(A)_C \to \text{LMod}^+(\text{Mod}^{S_1}_A)_{\mathcal{H}_\bullet}$, respectively. By Lemma 5.3, the composite induces a natural transformation

\[ \text{LMod}'(h) \circ D_C \to D^\vee_{\mathcal{H}_\bullet} \circ Def(f) \]

which fills the square

\[ (Def_c)^{op} \xrightarrow{Def(f)} (Def_c(St_A))^{op} \xrightarrow{Def_c(St_A)^{op}} \text{LMod}^{E_2}(A)_C \to \text{LMod}^+(\text{Mod}^{S_1}_A)_{\mathcal{H}_\bullet}. \]

Remark 6.6. We clarify the relationship between $\text{LMod}'(h) \circ D_C \to D^\vee_{\mathcal{H}_\bullet} \circ Def(f)$ and $\text{Alg}^{\dagger}_1(h) \circ D_2 \to D_1^{S_1} \circ \text{Alg}^{\dagger}_2(h)$ (see Definition 6.1). The consequence is stated in Lemma 6.7. Consider the square of pairings which is described as the commutative diagram

\[ \begin{array}{ccc}
K(\text{Alg}_1(\text{Mod}_A)) & \to & K(\text{Mod}^{S_1}_A) \\
\text{Alg}_2^+(\text{Mod}_A) \times \text{Alg}_2^+(\text{Mod}_A) & \xrightarrow{p_1 \times p_2} & \text{Alg}_1^+(\text{Mod}^{S_1}_A) \times \text{Alg}_1^+(\text{Mod}^{S_1}_A) \\
\text{Def}_c \times \text{LMod}^{E_2}(A)_C & \xrightarrow{\text{Def}_c(\text{St}_A)^{op}} & \text{Def}_c(\text{St}_A) \times ([C] \times_{St_A} \text{LMod}^+(St_A)) \\
\text{Def}_c \times \text{LMod}^{E_2}(A)_C & \xrightarrow{\text{Def}_c(\text{St}_A)^{op}} & \text{Def}_c(\text{St}_A) \times ([C] \times_{St_A} \text{LMod}^+(St_A)). \end{array} \]
in \( \hat{\text{Cat}}_{\infty} \). Here \( p_1, p_2, q_1 \) and \( q_2 \) are canonical projections. It gives rise to the diagram

\[
\begin{array}{ccc}
\text{LMod}^{\text{op}}(A) & \to & \text{LMod}^+(\text{Mod}_A^S) \\
\downarrow & & \downarrow \\
\text{Alg}_2^+(\text{Mod}_A) & \to & \text{Alg}_2^+(\text{Mod}^S) \\
\end{array}
\]

The bottom square is filled by the natural transformation \( \text{Alg}_2^+(\text{Mod}_A) \to \text{LMod}^{\text{op}}(A) \). From the description of \( \text{D}_C \) and \( \text{D}_C \) in Section 6.1 and Section 6.2, we observe that \( \text{LMod}^{\text{op}}(A) \to \text{LMod}^+(\text{Mod}_A^S) \) and \( p_2 \circ \text{D}_C \to \text{D}_2 \circ p_1 \) are invertible. Applying the observation in Remark 5.4 to the above square of pairings, we obtain the diagram

\[
\begin{array}{ccc}
\text{LMod}^{\text{op}}(A) & \to & \text{LMod}^+(\text{Mod}_A^S) \\
\downarrow & & \downarrow \\
\text{Alg}_2^+(\text{Mod}_A) & \to & \text{Alg}_2^+(\text{Mod}^S) \\
\end{array}
\]

which commutes up to homotopy. Note that \( \text{LMod}^{\text{op}}(A) \to \text{LMod}^+(\text{Mod}_A^S) \) is invertible if and only if \( \text{LMod}(h) \circ \text{D}_C \to \text{D}_2 \circ \text{p}_1 \) is invertible. Thus, taking into account the diagram we conclude that \( \text{LMod}^{\text{op}}(A) \to \text{LMod}^+(\text{Mod}_A^S) \) is invertible if \( \text{Alg}_2^+(\text{Mod}_A) \to \text{Alg}_2^+(\text{Mod}^S) \) is invertible.

We record the observation:

**Lemma 6.7.** If \( \text{Alg}_2^+(\text{Mod}_A) \to \text{LMod}^{\text{op}}(A) \) is invertible, then \( \text{LMod}^{\text{op}}(A) \to \text{LMod}^+(\text{Mod}_A^S) \) is invertible.

**Proposition 6.8.** Consider the composite \( \text{Art}_{A}^{\text{tsz}} \to \text{CA}_{A}^{\text{tsz}} \to \text{Alg}_2^+(\text{Mod}_A) \), the induced base change \( \text{D} \times \text{Alg}_2^+(\text{Mod}_A) \to \text{Art}_{A}^{\text{tsz}} \), and the projection \( \text{p} : \text{D} \to \text{Alg}_2^+(\text{Mod}_A) \). Then \( \text{LMod}(h) \circ \text{D}_C \to \text{D}_2 \circ \text{p}_1 \) is invertible.

**Proof.** Unfolding the definitions, \( \text{LMod}(h) \circ \text{D}_C \to \text{D}_2 \circ \text{p}_1 \) is a \( \text{H}_{A}(\text{D}(R)/A) \)-module \( \text{H}_{A}(C/A) \), and \( \text{D}(x) \circ \text{D}(h)(C/R) \) is a \( \text{H}_{A}(\text{D}(R)/A) \)-module \( \text{H}_{A}(C/A) \). Moreover, the morphism \( \text{LMod}(h) \circ \text{D}_C \to \text{D}_2 \circ \text{p}_1 \) is the restriction map along \( \eta : \text{H}_{A}(\text{D}(R)/A) \to \text{D}(\text{H}_{A}(C/A)) \) which is determined by the pairing \( \text{H}_{A}(\text{D}(R)/A) \times \text{H}_{A}(\text{D}(R)/A) \to \text{H}_{A}(\text{D}(R)/A) \) induced by the universal pairing \( A \times \text{D}(R)/A \to \text{H}_{A}(\text{D}(R)/A) \) (cf. Definition 6.1, Remark 6.1, Remark 6.6). By Lemma 6.7, it suffices to prove that \( \eta \) induces an equivalence of underlying complexes. We write \( \text{Bar} \) for the bar construction functor \( \text{Alg}_2^+(\text{Mod}_A) \to \text{Alg}_2^+(\text{Mod}^S) \). Let \( (-)^{\vee} \) denote the \( A \)-linear dual functor which carries \( M \to M^{\vee} \). Using \( \text{D}_n \simeq (-)^{\vee} \circ \text{Bar}^{\text{op}} \), the map \( \eta \) can be identified with the canonical map \( \text{H}_{A}(\text{D}(R)/A) \to \text{H}_{A}(\text{D}(R)/A)^{\vee} \simeq \text{H}_{A}(\text{D}(R)/A)^{\vee} \) where the latter equivalence follows from the compatibility of \( \text{H}_{A}(\text{D}(R)/A) \) with respect to sifted colimits (see the proof of Lemma 6.3). Suppose that \( A \) is the field \( k \) for the moment. By applying the standard reduced bar resolution to \( R \in \text{Art}_{A}^{\text{tsz}} \) we see that the complex \( \text{Bar}(R) \to \text{H}_{A}(\text{D}(R)/A) \) is connected and finite dimensional in each degree. Under this condition, by duality theorem [2, 4.1.1, 4.1.3], the canonical map \( \text{H}_{A}(\text{D}(R)/A) \to \text{H}_{A}(\text{D}(R)/A)^{\vee} \) is an equivalence. Next, we consider the case when \( A \) is an arbitrary connective dg algebra over \( k \). For \( R \in \text{Art}_{A}^{\text{tsz}} \), the standard reduced bar resolution shows also that \( \text{Bar}^{S}(R) \) is connected and finite dimensional in each degree. Moreover, according to the formula of the standard Hochschild complexes, \( \text{H}_{A}(\text{Bar}(R)/k) \) is connected and finite dimensional in each degree. By the base change along \( k \to A \) and these finiteness properties of \( \text{Bar}^{S}(R) \) and \( \text{H}_{A}(\text{Bar}(R)/k) \), we deduce the general case from the case of \( A = k \). \( \square \)
7. Various algebras and Hochschild homology

In this section, we study a comparison of Koszul duals of Hochschild homology (Proposition 7.1). Combined with Proposition 6.8 and Proposition 3.3, we prove Proposition 7.11 and Proposition 7.12. The square of equivalences in Proposition 7.11 plays an important role in the next section.

7.1. We will construct morphisms \( f_C : U_1(\mathbb{D}_\infty(C)) \to \mathbb{D}_1(C) \), which are natural in \( C \in \text{CAlg}^+_A \), and prove that \( f_C \) is an equivalence when \( C = R \otimes_A S^1 \) such that \( R \in \text{Art}^\text{tsz}_A \).

**Proposition 7.1.** Suppose that \( C = R \otimes_A S^1 \simeq R \otimes_{R \otimes_A R} R \) such that \( R \in \text{Art}^\text{tsz}_A \). Then \( f_C : U_1(\mathbb{D}_\infty(C)) \to \mathbb{D}_1(C) \) is an equivalence.

**Definition 7.2.** We will define \( f_C : U_1(\mathbb{D}_\infty(C)) \to \mathbb{D}_1(C) \). Consider the adjoint pair

\[
l : \text{Fun}((\text{Art}^\text{tsz}_A)^\text{op}, \text{Alg}^+_1(\text{Mod}_A)) \rightleftarrows \text{Fun}((\text{CAlg}^+_A)^\text{op}, \text{Alg}^+_1(\text{Mod}_A)) : r
\]

where \( r \) is induced by the composition with the inclusion \((\text{Art}^\text{tsz}_A)^\text{op} \to (\text{CAlg}^+_A)^\text{op}\), and the left adjoint \( l \) carries a functor to its left Kan extension along \((\text{Art}^\text{tsz}_A)^\text{op} \to (\text{CAlg}^+_A)^\text{op}\) (note that \( \text{Alg}^+_1(\text{Mod}_A) \) has small colimits). Then the counit map of the adjunction applied to \( \mathbb{D}_1 : (\text{CAlg}^+_A)^\text{op} \to \text{Alg}^+_1(\text{Mod}_A) \) (we usually omit the forgetful functor \((\text{CAlg}^+_A)^\text{op} \to (\text{Alg}^+_1(\text{Mod}_A))^\text{op}\)) gives us a natural transformation \( l \circ r(\mathbb{D}_1) \to \mathbb{D}_1 \). For each \( C \in \text{CAlg}^+_A \), we write \( f_C : l \circ r(\mathbb{D}_1)(C) \to \mathbb{D}_1(C) \) for the morphism obtained by the evaluation.

Recall that \( r(\mathbb{D}_1) \) is naturally equivalent to the functor \( U_1 \circ \mathbb{D}_\infty : (\text{Art}^\text{tsz}_A)^\text{op} \to \text{Alg}^+_1(\text{Mod}_A) \) (cf. Proposition 3.3 (1)). Thus, for \( C \in \text{CAlg}^+_A \), there exist equivalences

\[
l \circ r(\mathbb{D}_1)(C) \simeq \lim_{R \in (\text{Art}^\text{tsz}_A)^\text{op}} U_1 \circ \mathbb{D}_\infty(R) \simeq \lim_{L \in (\text{Lie}_A)^\text{op}} U_1(L) \simeq U_1(\mathbb{D}_\infty(C))
\]

where the second equivalence follows from the equivalence \((\text{Art}^\text{tsz}_A)^\text{op} \simeq \text{Lie}^I_A\) and the universality of \( \mathbb{D}_\infty(C) \) (determined by the adjoint pair \((\text{Ch}^*, \mathbb{D}_\infty)\), see Section 3.3), and the third one is an equivalence since \( \mathbb{D}_\infty(C) \) is a (sifted) colimit of \((\text{Lie}^I_A)^\text{op}/\mathbb{D}_\infty(C) \to \text{Lie}_A\). Thus, we have

\[
f_C : U_1(\mathbb{D}_\infty(C)) \longrightarrow \mathbb{D}_1(C).
\]

The rest of Section 7.1 is devoted to the proof of Proposition 7.1. We define a closely related map \( g_C \).

**Definition 7.3.** We will define \( g_C \). Given \( C \in \text{CAlg}^+_A \) we consider the morphism \( p_C : C \to \text{Ch}^*(\mathbb{D}_\infty(C)) \) in \( \text{Alg}^+_1(\text{Mod}_A) \) determined by the counit map of the adjoint pair \((\text{Ch}^*, \mathbb{D}_\infty)\). Since there is a natural equivalence \( \text{Ch}^* \simeq \mathbb{D}_1 \circ U_1 \) between functors \( \text{Lie}_A \to \text{Alg}^+_1(\text{Mod}_A) \) (we omit the forgetful functor \( \text{CAlg}^+_A \to \text{Alg}^+_1(\text{Mod}_A) \)), it follows that \( p_C \) can be identified with \( C \to \mathbb{D}_1(U_1(\mathbb{D}_\infty(C))) \), which is determined by the pairing

\[
C \otimes_A U_1(\mathbb{D}_\infty(C)) \to \text{Ch}^*(\mathbb{D}_\infty(C)) \otimes_A U_1(\mathbb{D}_\infty(C)) \simeq \mathbb{D}_1(U_1(\mathbb{D}_\infty(C))) \otimes U_1(\mathbb{D}_\infty(C)) \to A
\]

induced by \( C \to \text{Ch}^*(\mathbb{D}_\infty(C)) \simeq \mathbb{D}_1(U_1(\mathbb{D}_\infty(C))) \) and the universal pairing \( \mathbb{D}_1(U_1(\mathbb{D}_\infty(C))) \circ U_1(\mathbb{D}_\infty(C)) \to A \). Thus, we also define a morphism \( g_C : U_1(\mathbb{D}_\infty(C)) \to \mathbb{D}_1(C) \).

**Lemma 7.4.** For any \( C \in \text{CAlg}^+_A \), \( f_C \) is equivalent to \( g_C \) in \( \text{Map}_{\text{Alg}^+_1(\text{Mod}_A)}(U_1(\mathbb{D}_\infty(C)), \mathbb{D}_1(C)) \).

**Proof.** For \( C, C' \in \text{CAlg}^+_A \), there is an equivalence

\[
\text{Map}_{\text{Alg}^+_1(\text{Mod}_A)}(U_1(\mathbb{D}_\infty(C)), \mathbb{D}_1(C)) \simeq \text{Map}_{\text{Alg}^+_1(\text{Mod}_A)}(C', U_1(\mathbb{D}_\infty(C)))
\]

given by the adjoint pair \((\mathbb{D}_1, \mathbb{D}_\infty)\). The right hand side is equivalent to \( \text{Map}_{\text{Alg}^+_1(\text{Mod}_A)}(C', \text{Ch}^*(\mathbb{D}_\infty(C))) \). When \( C = C' \), by the definitions \( p_C : C \to \text{Ch}^*(\mathbb{D}_\infty(C)) \simeq \mathbb{D}_1(U_1(\mathbb{D}_\infty(C))) \) corresponds to \( g_C \) through equivalences of mapping spaces. Thus, we will prove that \( f_C \) naturally corresponds to \( p_C \) through the equivalences. We note that \( f_C \) is given by the morphism \( U_1(\mathbb{D}_\infty(C)) \simeq \lim_{R \in (\text{Art}^\text{tsz}_A)^\text{op}/C} \mathbb{D}_1(R) \to \mathbb{D}_1(C) \), determined by the canonical morphism \( C \to \lim_{R \in (\text{Art}^\text{tsz}_A)^\text{op}/C} R \). Here limit by \( R \in (\text{Art}^\text{tsz}_A)^\text{op}/C \).
maps to (the component of) $C \to \lim_{R \in (\text{Art}_{\mathcal{A}})^{op}} R$ in $\text{Map}_{\text{Alg}_+^+}(\text{Mod}_A)\left(C, \lim_{R \in (\text{Art}_{\mathcal{A}})^{op}} R\right)$ through equivalences

$$\text{Map}_{\text{Alg}_+^+}(\text{Mod}_A)\left(\lim_{R \in (\text{Art}_{\mathcal{A}})^{op}} D_1(R), D_1(C)\right) \simeq \lim_{R \in (\text{Art}_{\mathcal{A}})^{op}} \text{Map}_{\text{Alg}_+^+}(\text{Mod}_A)\left(D_1(R), D_1(C)\right)$$

$$\simeq \text{Map}_{\text{Alg}_+^+}(\text{Mod}_A)(C, \lim_{R \in (\text{Art}_{\mathcal{A}})^{op}} D_1(R))$$

$$\simeq \text{Map}_{\text{Alg}_+^+}(\text{Mod}_A)(C, \lim_{R \in (\text{Art}_{\mathcal{A}})^{op}} \lim_{R \in \mathcal{A}} R).$$

On the other hand, taking into account

$C \to \text{Ch}^\bullet(\mathbb{D}_\infty(C)) \simeq \text{Ch}^\bullet(\lim_{L \in \text{Lie}_A^{op}} L) \simeq \text{Ch}^\bullet(\lim_{R \in (\text{Art}_{\mathcal{A}})^{op}} \mathbb{D}_\infty(R)) \simeq \lim_{R \in (\text{Art}_{\mathcal{A}})^{op}} \text{Ch}^\bullet(\mathbb{D}_\infty(R))$

$p_C$ can be identified with the canonical map $C \to \lim_{R \in (\text{Art}_{\mathcal{A}})^{op}} R$. It follows that $p_C : C \to \text{Ch}^\bullet(\mathbb{D}_\infty(C))$ corresponds to $f_C$. \hfill \square

We will prove Proposition 7.1. For this purpose, we start with Lemma 7.5, Corollary 7.6 and Remark 7.10.

**Lemma 7.5.** Let $L \in \text{Lie}_k$ and let $L^{S_1}$ be the cotensor of $L$ by $S_1$. Denote by $(L \otimes_k A)^{S_1}$ the cotensor in $\text{Lie}_A$. Then there exists a canonical equivalence $L^{S_1} \otimes_k A \simeq (L \otimes_k A)^{S_1}$.

**Corollary 7.6.** Let $R_k$ be an object of $\text{Art}_{k^{op}}$ and $\mathbb{D}_{\infty,k}(R_k)^{S_1}$ is the cotensor by $S_1$ where $\mathbb{D}_{\infty,k} : (\text{CAAlg}_k)^{op} \to \text{Lie}_k$ is the Koszul duality functor $\mathbb{D}_{\infty}$ in the case of $A = k$. Let $U_{1,k} : \text{Lie}_k \to \text{Alg}_+^+(\text{Mod}_k)$ be the universal enveloping algebra functor over $k$. Then there exists an equivalence

$$U_1(\mathbb{D}_{\infty}(R_k \otimes_k A)^{S_1}) \simeq U_{1,k}(\mathbb{D}_{\infty,k}(R_k)^{S_1}) \otimes_k A$$

in $\text{Alg}_+^+(\text{Mod}_A)$.

**Proof.** If we put $R_k = k \oplus M$, then $\mathbb{D}_{\infty,k}(R_k)$ is a free Lie algebra generated by $M^\vee[-1]$ in $\text{Lie}_k$, and $\mathbb{D}_{\infty}(R_k \otimes_k A)$ is a free Lie algebra generated by $(M^\vee \otimes_k A)[-1]$ in $\text{Lie}_A$ (see e.g. [23, X]) so that there is a canonical equivalence $\mathbb{D}_{\infty,k}(R_k) \otimes_k A \simeq \mathbb{D}_{\infty}(R_k \otimes_k A)$. By Lemma 7.5, $(\mathbb{D}_{\infty,k}(R_k) \otimes_k A)^{S_1} \simeq \mathbb{D}_{\infty,k}(R_k)^{S_1} \otimes_k A$. Consequently, we have

$$U_1(\mathbb{D}_{\infty}(R_k \otimes_k A)^{S_1}) \simeq U_1((\mathbb{D}_{\infty,k}(R_k) \otimes_k A)^{S_1}) \simeq U_1(\mathbb{D}_{\infty,k}(R_k)^{S_1} \otimes_k A) \simeq U_{1,k}(\mathbb{D}_{\infty,k}(R_k)^{S_1}) \otimes_k A.$$

This completes the proof. \hfill \square

**Proof of Lemma 7.5.** It is enough to prove that the canonical map $L^{S_1} \otimes_k A \to (L \otimes_k A)^{S_1}$ is an equivalence in $\text{Mod}_A$. Since $\text{Mod}_A$ is stable, it follows that $(-)^{S_1} : \text{Mod}_k \to \text{Mod}_k$ given by $C \mapsto C^{S_1} = C \times C \times C$ preserves small colimits (in fact, $(-)^{S_1}$ is equivalent to the functor given by $C \mapsto C \times C[-1]$). In particular, $L^{S_1} \otimes_k A \to (L \otimes_k A)^{S_1}$ is an equivalence. \hfill \square

We assume that $A$ is the base field $k$ in Lemma 7.7, Lemma 7.8, and Lemma 7.9. See Lemma 7.10 for the reduction to the case of $A = k$.

**Lemma 7.7.** For $R \in \text{Art}_{k^{op}} = \text{Art}_{k^{op}}$, we let $u : U_1(\mathbb{D}_{\infty}(R \otimes_A S^1)) \to \mathbb{D}_1 \mathbb{D}_1(U_1(\mathbb{D}_{\infty}(R \otimes_A S^1)))$ be the unit map associated to the adjoint pair $(\mathbb{D}_1, \mathbb{D}_1)$. This morphism is an equivalence in $\text{Alg}_+^+(\text{Mod}_A)$.

**Proof.** Thanks to [23, X, 3.1.15], if $U_1(\mathbb{D}_{\infty}(R \otimes_k S^1))$ is coconnective (i.e., $H_i(\text{Ker}(U_1(\mathbb{D}_{\infty}(R \otimes_k S^1)))) \to A) = 0$ for $i \geq 0$ and $H_i(U_1(\mathbb{D}_{\infty}(R \otimes_k S^1)))$ is finite dimensional in each degree, then $u$ is an equivalence. Thus, it is enough to show that $U_1(\mathbb{D}_{\infty}(R \otimes_k S^1))$ satisfies these conditions. There is a distinguished triangle

$$\mathbb{D}_{\infty}(R)[-1] \to \mathbb{D}_{\infty}(R \otimes_k S^1) \simeq \mathbb{D}_{\infty}(R)^{S_1} \to \mathbb{D}_{\infty}(R)$$

in the homotopy (triangulated) category of $\text{Mod}_A$. The right morphism is induced by a point $* \to S^1$. Thus, it admits a section $\mathbb{D}_{\infty}(R) \to \mathbb{D}_{\infty}(R \otimes_k S^1)$ determined by the map $S^1 \to *$. It follows that
there exists an equivalence $D_\infty(R)[-1] \oplus D_\infty(R) \simeq D_\infty(R \otimes_k S^1)$ in $\text{Mod}_k$. The (dg) Lie algebra $D_\infty(R) \in \text{Lie}_k$ is a free Lie algebra generated by some perfect complex $N$ in $\text{Mod}_k$ such that $H_i(N) = 0$ for $i > 0$ (cf. [23, X, Section 2]). Using this description and Poincaré-Birkhoff-Witt theorem we see that $U_1(D_\infty(R \otimes_k S^1))$ is coconnective, and $H_i(U_1(D_\infty(R \otimes_k S^1)))$ is finite dimensional in each degree. □

Lemma 7.8. Suppose that $R$ belongs to $\text{Art}^+_A$. Let $v : R \otimes_A S^1 \to C \text{h}^*(D_\infty(R \otimes_A S^1))$ be the canonical morphism associated to the adjoint pair $(C \text{h}^*, D_\infty)$. Then $v$ is an equivalence. In particular, the induced morphism $D_1(C \text{h}^*(D_\infty(R \otimes_A S^1))) \to D_1(R \otimes_A S^1)$ is an equivalence in $\text{Alg}^+_k(\text{Mod}_A)$.

Proof. In this proof, we write $k$ for $A$ since we assume $A = k$. We denote by $\text{Mod}^+_k$ the full subcategory of $\text{Mod}_k$ spanned by connective objects, i.e., objects $P$ such that $H_i(P) = 0$ for $0 > i$. We denote by $\text{Mod}^+_k(n)$ the full subcategory of $\text{Mod}^+_k$ spanned by those objects $P$ such that $H_i(P) = 0$ for $i > n$. By definition, $\text{CAlg}^+_k(n) = \text{CAlg}(\text{Mod}^+_k(n))$. In this proof, we put $\text{CAlg}^+_k = \text{CAlg}(\text{Mod}^+_k)$. Let $\text{CAlg}^+_k(n)$ be the full subcategory of $\text{CAlg}^+_k$ spanned by those objects whose underlying complexes belong to $\text{Mod}^+_k(n)$. The faithfully embedding $\text{Mod}^+_k(n) \to \text{Mod}^+_k$ has a left adjoint, that is the truncation functor $\tau_{\leq n} : \text{Mod}^+_k \to \text{Mod}^+_k(n)$. The faithfully functor $\text{CAlg}^+_k(n) \to \text{CAlg}^+_k(n)$ admits a left adjoint $\text{CAlg}^+_k(n) \to \text{CAlg}^+_k(n)$ which extends the truncation functor $\tau_{\leq n} : \text{Mod}^+_k \to \text{Mod}^+_k(n)$ in the natural way. Thus, there exists an adjoint pair

$$\tau_{\leq n} : \text{CAlg}^+_k(n) \rightleftarrows \text{CAlg}^+_k(n)$$

where we slightly abuse notation by writing $\tau_{\leq n}$ for the left adjoint (cf. [6, Volume 1, Chap.2, 1.2]). Since the standard t-structure on $\text{Mod}_k$ (determined by the homological amplitudes) is left complete (see [22, 1.2.1.17] for this notion), it follows that there exists a canonical equivalence $R \otimes_k S^1 \simeq \lim_{\leq n} \tau_{\leq n}(R \otimes_k S^1)$ in $\text{CAlg}_k$. We write $A_n$ for $\tau_{\leq n}(R \otimes_k S^1)$. By [21, 4.4.2.9], $\text{CAlg}^+_k(n) \to \text{CAlg}_k(n)$ preserves sequential limits so that $R \otimes_k S^1 \simeq \lim_{\leq n} A_n$ is promoted to an equivalence in $\text{CAlg}^+_k$ from that in $\text{CAlg}^+_k$. Next, we will show that each $A_n$ belongs to $\text{Art}^+_k$. Observe that each homology group of $R \otimes_k S^1$ is finite dimensional. Indeed, the underlying chain complex of $R \otimes_k S^1$ is equivalent to the standard Hochschild chain complex. Since $R$ is a trivial square zero extension $k \oplus M$ (such that $M$ is a connective perfect complex over the field $k$), we deduce from this presentation that $H_i(R \otimes_k S^1)$ is a finite dimensional $k$-vector space for any $i \in \mathbb{Z}$. Moreover, $H_0(R)$ is a usual finite dimensional algebra. Since the reduced ring $H_0(R \otimes_{R \otimes_k R} R)_{\text{red}}$ is isomorphic to the derived tensor product $H_0(R)_{\text{red}} \otimes_{H_0(R)_{\text{red}}} H_0(R)_{\text{red}}$ $H_0(R)_{\text{red}} \simeq k$, it follows that $H_0(R)$ is a usual artin algebra with residue field $k$. Thus, each $A_n$ belongs to $\text{Art}^+_k$. Consequently, there is an equivalence $A_n \simeq \text{CAlg}^+_k(D_\infty(A_n))$ induced by the adjoint pair $(\text{CAlg}^+, D_\infty)$. Next, we will show that $\lim_n D_\infty(A_n) \simeq D_\infty(R \otimes_k S^1)$. By the adjoint pair $(\text{CAlg}^+, D_\infty)$, for $B \in \text{Art}^+_k$ there are canonical equivalences

$$\text{Map}_{\text{Lie}_k}(D_\infty(B), D_\infty(R \otimes_k S^1)) \simeq \text{Map}_{\text{CAlg}^+_k}(R \otimes_k S^1, \text{CAlg}^+(D_\infty(B))) \simeq \text{Map}_{\text{CAlg}^+_k}(R \otimes_k S^1, B).$$

Thus, the formal stack $\text{F}_{D_\infty(R \otimes_k S^1)}$ in $\text{St}^+_k$ determined by $D_\infty(R \otimes_k S^1)$ can be identified with the functor $F : \text{Art}^+_k \to \text{S}$ given by $B \mapsto \text{Map}_{\text{CAlg}^+_k}(R \otimes_k S^1, B)$. Similarly, $\text{F}_{D_\infty(A_n)}$ can be identified with the functor $F_n : \text{Art}^+_k \to \text{S}$ given by $B \mapsto \text{Map}_{\text{CAlg}^+_k}(A_n, B)$. Suppose that $B$ is the trivial square zero extension $k \oplus k[m]$. Then

$$\text{Map}_{\text{Lie}_k}(D_\infty(B), D_\infty(R \otimes_k S^1)) \simeq \text{Map}_{\text{Lie}_k}(\text{Free}_{\text{Lie}_k}(k[-m-1]), D_\infty(R \otimes_k S^1)) \simeq \Omega_{\text{Lie}_k}^\infty(D_\infty(R \otimes_k S^1)[m+1])$$

where $\text{Free}_{\text{Lie}_k} : \text{Mod}_k \to \text{Lie}_k$ is the free Lie algebra functor, that is a left adjoint to the forgetful functor. Here $\Omega_{\text{Lie}_k}^\infty$ is the composite $L_{\text{Lie}} \text{forget} \text{forget} \text{Sp}$ $\text{Sp} \simeq \text{S}$. The left equivalence is induced by $\text{Free}_{\text{Lie}_k}(k[-m-1]) \simeq D_\infty(k \oplus k[m])$ (see [23, X, Section 2]). Similarly, $\text{Map}_{\text{Lie}_k}(D_\infty(B), D_\infty(A_n)) \simeq \Omega_{\text{Lie}_k}^\infty(D_\infty(A_n)[m+1])$. To see $\lim_n D_\infty(A_n) \simeq D_\infty(R \otimes_k S^1)$, it is enough to prove that for any $m \geq 0$, the induced morphism $l_{m,n} : \Omega_{\text{Lie}_k}^\infty(D_\infty(A_n)[m+1]) \to \Omega_{\text{Lie}_k}^\infty(D_\infty(R \otimes_k S^1)[m+1])$ is an equivalence if $n \geq m$. By equivalences, $l_{m,n}$ can be identified with

$$l_{m,n} : \text{Map}_{\text{CAlg}^+_k}(A_n, k \oplus k[m]) \to \text{Map}_{\text{CAlg}^+_k}(R \otimes_k S^1, k \oplus k[m])$$

induced by the composition with $R \otimes_k S^1 \to A_n$. Taking into account the adjoint pair $\tau_{\leq n} : \text{CAlg}^+_k \rightleftarrows \text{CAlg}^+_k(n)$, we conclude that $l_{m,n}$ is an equivalence if $n \geq m$. Hence $D_\infty(R \otimes_k S^1) \simeq \lim_n D_\infty(A_n)$. Now
the left adjoint functor $Ch^*: Lie_k \to \mathrm{CAlg}^+_k$ gives rise to equivalences

$$Ch^*(\mathbb{D}_\infty(R \otimes_k S^1)) \simeq Ch^*(\lim_n(\mathbb{D}_\infty(A_n))) \simeq \lim_n Ch^*(\mathbb{D}_\infty(A_n))$$

in $\mathrm{CAlg}^+_k$. By the canonical equivalences $A_n \simeq Ch^*(\mathbb{D}_\infty(A_n))$, we obtain $\lim_n A_n \simeq R \otimes_k S^1$. This completes the proof.}

\textbf{Lemma 7.9.} The morphism $g_C: U_1(\mathbb{D}_\infty(R \otimes A S^1)) \to \mathbb{D}_1(R \otimes A S^1)$ is an equivalence.

\textbf{Proof.} Note that the morphism $g_C$ factors as

$$U_1(\mathbb{D}_\infty(R \otimes A S^1)) \to \mathbb{D}_1(1,1) \to \mathbb{D}_1(1,1) \to U_1(\mathbb{D}_\infty(R \otimes A S^1)) \to \mathbb{D}_1(R \otimes A S^1)$$

where the left morphism is that of Lemma 7.7, the right morphism is that of Lemma 7.8, and the middle equivalence comes from $\mathbb{D}_1 \circ U_1 \simeq Ch^*$. Our claim follows from Lemma 7.7 and Lemma 7.8.

\textbf{Lemma 7.10.} Proposition 7.1 follows from the case when $A = k$.

\textbf{Proof.} To observe this, we set $R = k \otimes_k A$ (recall that any object $R \in \mathrm{Art}^A_{tsz}$ is of this form). Suppose that $\mathbb{D}_{1,k}(R_k \otimes_k S^1)$ is coconnective and is finite dimensional in each degree. Taking into account the description of $\mathbb{D}_{1,k}(-)$ as the $k$-linear dual $(\mathrm{Bar}(-))^\vee$, we see that there exists a canonical equivalence $\mathbb{D}_{1,k}(R_k \otimes_k S^1) \otimes_k k \simeq \mathbb{D}_{1,k}((R_k \otimes_k A) \otimes A S^1)$ (note also that the noetherian dg algebra $A$ has a bounded amplitude in $\mathrm{Mod}_k$). Corollary 7.6 allows us to identify $U_1(\mathbb{D}_\infty(R \otimes A S^1)) \to \mathbb{D}_1(R \otimes A S^1)$ in Proposition 7.1 with the composite

$$U_1(\mathbb{D}_\infty((R_k \otimes_k A) \otimes A S^1)) \simeq U_1(\mathbb{D}_\infty(R_k \otimes_k A)S^1) \simeq U_{1,k}(\mathbb{D}_\infty,k(R_k)S^1) \otimes_k k A \to \mathbb{D}_{1,k}(R_k \otimes_k S^1) \otimes_k k A$$

where the final morphism is induced by $U_{1,k}(\mathbb{D}_\infty,R_k S^1) \to \mathbb{D}_{1,k}(R_k \otimes_k S^1)$ (that is a morphism in Proposition 7.1 in the case of $A = k$). Lemma 7.9 shows that $U_{1,k}(\mathbb{D}_\infty,k(R_k)S^1) \simeq U_{1,k}(\mathbb{D}_\infty,R_k S^1)$ and, in the proof of Lemma 7.7 we prove that $U_{1,k}(\mathbb{D}_\infty,R_k S^1)$ is coconnective and is finite dimensional in each degree. It follows that $\mathbb{D}_{1,k}(R_k \otimes_k S^1)$ is coconnective and is finite dimensional in each degree. Therefore, to verify Proposition 7.1, it will suffice to prove the case of $A = k$.

\textbf{Proof of Proposition 7.1.} (In this proof, $A$ is arbitrary.) According to Lemma 7.10, we may and will assume $A = k$. By Lemma 7.4, we are reduced to proving that $g_C$ is an equivalence. Lemma 7.9 shows that $g_C$ is an equivalence.

\textbf{7.2.}

\textbf{Proposition 7.11.} We continue to abuse notation by writing $h$ for both functors $\mathrm{CAlg}^+_{tsz} \to \mathrm{Alg}^+_1(\mathrm{Mod}_{tsz}^A)$ and $\mathrm{Alg}^+_1(\mathrm{Mod}_{tsz}^A) \to \mathrm{Alg}^+_1(\mathrm{Mod}_{tsz}^S)$ induced by $\mathcal{H}_*: (-)/A: \mathrm{Alg}_1(\mathrm{Mod}_A) \to \mathrm{Mod}_1^A$. Here we denote simply by $\mathbb{D}_1^S \circ h$ or $\mathbb{D}_1 \circ h$ the composite

$$1 \leftarrow (\mathrm{Art}^A_{tsz})^\op \to (\mathrm{CAlg}^+_{tsz}^A)^\op \to (\mathrm{CAlg}^+_A(\mathrm{Mod}_{tsz}^S))^\op \to (\mathrm{Alg}_1^+_A(\mathrm{Mod}_{tsz}^S))^\op \to \mathrm{Alg}^+_1(\mathrm{Mod}_{tsz}^S).$$

We denote simply by $h \circ \mathbb{D}_2$

$$(\mathrm{Art}^A_{tsz})^\op \to (\mathrm{CAlg}^+_A)^\op \to (\mathrm{Alg}_2^+_A(\mathrm{Mod}_{tsz}^S))^\op \to \mathrm{Alg}^+_1(\mathrm{Mod}_{tsz}^S).$$

Then there exist an equivalence $\delta: h \circ U_2 \circ \mathbb{D}_\infty \simeq U_1 \circ \mathbb{D}_\infty \circ h$ in $\mathrm{Fun}((\mathrm{Art}^A_{tsz})^\op, \mathrm{Alg}^+_1(\mathrm{Mod}_{tsz}^S))$ and the square diagram of equivalences

$$
\begin{array}{ccc}
U_1 \circ \mathbb{D}_\infty \circ h & \simeq & \mathbb{D}_1^S \circ h \\
\delta & \simeq & \lambda \\
h \circ U_2 \circ \mathbb{D}_\infty & \simeq & h \circ \mathbb{D}_2
\end{array}
$$

that commutes up to canonical homotopy.
Proof. By Proposition 6.8 and its proof (see Remark 6.6), there is a canonical equivalence \( \lambda : h \circ \mathbb{D}_2 \cong h \) in \( \text{Fun}(\text{Art}^{\text{ttx}})^{\text{op}}, \text{Alg}^+_1(\text{Mod}^S_1)) \). Note that \( \mathbb{D}_2 : (\text{Art}^{\text{ttx}})^{\text{op}} \to \text{Alg}^+_1(\text{Mod}^A_1) \) is naturally equivalent to \( U_2 \circ \mathbb{D}_\infty : (\text{Art}^{\text{ttx}})^{\text{op}} \to \text{Alg}^+_2(\text{Mod}^A_1) \) (see Proposition 3.3). It follows immediately that there exists a canonical equivalence \( \mu : h \circ U_2 \circ \mathbb{D}_\infty \cong h \circ \mathbb{D}_2 \) in \( \text{Fun}((\text{Art}^{\text{ttx}})^{\text{op}}, \text{Alg}^+_1(\text{Mod}^S_1)) \). Let \( U_1 \circ \mathbb{D}_\infty \to \mathbb{D}_1 \) be the morphism in \( \text{Fun}((\text{CAlg}^+_1)^{\text{op}}, \text{Alg}^+_1(\text{Mod}^A_1)) \) defined in Section 7.1. It induces a morphism in \( \text{Fun}((\text{Fun}(BS^1, \text{CAlg}^+_1)^{\text{op}}), \text{Fun}(BS^1, \text{Alg}^+_1(\text{Mod}^A_1))) \) for which we write \( U_1 \circ \mathbb{D}_\infty \to \mathbb{D}_1 \) by abuse of notation. Here we identify with \( \text{CAlg}^+_1(\text{Mod}^S_1) \toganf \text{Alg}^+_1(\text{Mod}^A_1) \) (see Example 5.6 for the notation \( \mathbb{D}^S_1 \)) with \( \text{Fun}(BS^1, (\text{CAlg}^+_1)^{\text{op}}) \to \text{Fun}(BS^1, \text{Alg}^+_1(\text{Mod}^A_1)) \) induced by \( (\text{CAlg}^+_1)^{\text{op}} \toganf \text{Alg}^+_1(\text{Mod}^A_1) \in \text{Alg}^+_1(\text{Mod}^A_1) \) in the natural way, see Remark 7.14. The composition with \( \text{Art}^{\text{ttx}}_A \toganf \text{CAlg}^+_1(\text{Mod}^S_1) \simeq \text{Fun}(BS^1, \text{CAlg}^+_1(\text{Mod}^A_1)) \) (that is equivalent to \( \otimes^S_1 \)) induces a natural transformation \( \nu : U_1 \circ \mathbb{D}_\infty \circ \mu \to \mathbb{D}_1 \circ \mu \) between functors \( \text{Art}^{\text{ttx}}_A \to \text{Alg}^+_1(\text{Mod}^A_1) \). According to Proposition 7.1, \( \nu \) is an equivalence in \( \text{Fun}((\text{Art}^{\text{ttx}}_A)^{\text{op}}, \text{Alg}^+_1(\text{Mod}^A_1)) \). We take an inverse \( \nu^{-1} \) and let \( \delta \) be the composite \( \nu^{-1} \circ \lambda \circ \mu \). \( \square \)

Using the equivalence \( \delta \) in Proposition 7.11, we prove the following:

**Proposition 7.12.** Let \( L \) be a dg Lie algebra over \( A \), that is, an object of \( \text{Lie}_A \). Let \( L^S_1 \) denote the cotensor of \( L \) by \( S^1 \). Then there exists a natural equivalence \( h \circ U_2 = \mathcal{H}_*(-/A) \circ U_2 \to \mathcal{H}_*(-/S^1) \) between functors \( \text{Lie}_A \to \text{Alg}^+_1(\text{Mod}^S_1) \). In particular,

\[
\mathcal{H}_*(U_2(L)/A) \to \mathcal{H}_*(U_1(L^S_1)),
\]

for any \( L \in \text{Lie}_A \).

**Proof.** Consider the projection \( (\text{Lie}^f_A)/L \simeq (\text{Art}^{\text{ttx}}_A)^{\text{op}} \times_{\text{Lie}^f_A(L)} (\text{Lie}^f_A)/_{L} \overset{\text{pt}_L}{\to} (\text{Art}^{\text{ttx}}_A)^{\text{op}} \) whose composite is given by the formula \( [M \to L] \to \text{Ch}^*(M) \). The composition of this functor and \( \delta : h \circ U_2 \circ \mathbb{D}_\infty \to U_1 \circ \mathbb{D}_\infty \circ h \) induces

\[
\text{lim}_M \text{H}_\bullet(U_2(M)/A) \to \text{lim}_M U_1(M^S_1)
\]

where \( \mathbb{D}_\infty(\text{Ch}^*(M) \otimes S^1) \) is naturally identified with \( \mathbb{D}^S_1(\text{Ch}^*(M))^S_1 \simeq M^S_1 \). Since \( L \) is a sifted colimit of \( (\text{Lie}^f_A)/L \to \text{Lie}_A \), and both \( U_2 \) and \( \mathcal{H}_\bullet(-/A) \) preserve sifted colimits, it follows that there exists a canonical equivalence \( \text{lim}_M U_2(M)/A \to \mathcal{H}_\bullet(U_2(L)/A) \simeq \mathcal{H}_\bullet(U_2(L)/A) \). Note that the canonical morphism \( \text{lim}_M U_1(M^S_1) \to L^S_1 \) is an equivalence (see Lemma 7.13). The functor \( U_1 \) also preserves colimits so that we have \( \text{lim}_M U_1(M^S_1) \simeq U_1(L^S_1) \). We conclude that \( \mathcal{H}_\bullet(-/A) \circ U_2 \) and \( U_1 \circ (-)^S_1 \) are left Kan extensions of their restrictions \( (\mathcal{H}_\bullet(-/A) \circ U_2)|_{\text{Lie}^f_A} \) and \( (U_1 \circ (-)^S_1)|_{\text{Lie}^f_A} \), respectively. Thus, \( \delta \) induces a natural equivalence \( \mathcal{H}_\bullet(-/A) \circ U_2 \to U_1 \circ (-)^S_1 \). \( \square \)

**Lemma 7.13.** The canonical morphism \( \text{lim}_M U_1(M^S_1) \to L^S_1 \) is an equivalence.

**Proof.** To see this, it is enough to show that the functor \( (-)^S_1 : \text{Mod}^S_1 \to \text{Mod}^A_1 \) given by \( C \to C^S_1 \) commutes with sifted colimits (because \( \text{Lie}_A \to \text{Mod}^A_1 \) is conservative and preserves sifted colimits). Since \( C^S_1 = C \times_{C \times C} C \) is defined as a finite limit and \( \text{Mod}^A_1 \) is a stable \( \infty \)-category, \( (-)^S_1 \) preserves small colimits. \( \square \)

**Remark 7.14.** There is no harm to use \( \mathbb{D}_1 \) instead of \( \mathbb{D}^S_1 \) because it is easy to see that for \( B \in \text{Alg}^+_1(\text{Mod}^A_1) \) the underlying object \( \mathbb{D}^S_1(B) \) in \( \text{Alg}^+_1(\text{Mod}^A_1) \) (which forgets the \( S^1 \)-action) is naturally equivalent to \( \mathbb{D}_1(B) \) when we regard \( B \) as an object of \( \text{Alg}^+_1(\text{Mod}^A_1) \). For example, it can be shown by using the description of \( \mathbb{D}^S_1 \) in terms of bar construction or by applying [23, X. 4.14.18] to our situation. Moreover, the composite functor \( \text{CAlg}^+(\text{Mod}^S_1)^{\text{op}} \to \text{Alg}^+_1(\text{Mod}^A_1)^{\text{op}} \overset{\mathbb{D}^S_1}{\to} \text{Alg}^+_1(\text{Mod}^A_1) \) can naturally be identified with the functor

\[
\text{CAlg}^+(\text{Mod}^S_1)^{\text{op}} \simeq \text{Fun}(BS^1, \text{CAlg}^+(\text{Mod}^A_1)^{\text{op}}) \overset{\text{Fun}(BS^1, \mathbb{D}_1)}{\to} \text{Fun}(BS^1, \text{Alg}^+_1(\text{Mod}^A_1))
\]
induced by \( \text{CAlg}^+(\text{Mod}_A)^{\text{op}} \) \( \text{forget} \quad \text{Alg}^+(\text{Mod}_A)^{\text{op}} \) \( \cong \) \( \text{Alg}^+(\text{Mod}_A) \). To see this, by [23, X, 4.4.20], \( \mathbb{D}_1^S \) is the composite of the bar construction \( \text{Bar}_{\text{Mod}_A^S} : \text{Alg}_1^+(\text{Mod}^S_1) = \text{Alg}_1((\text{Mod}^S_1)^{\text{op}}) \) \( \rightarrow \) \( \text{Alg}_1((\text{Mod}^S_1)^{\text{op}})^{\text{op}} =: \text{CoAlg}_1((\text{Mod}^S_1)^{\text{op}}) \) (see [23, X, 4.3.1–4.3.3]) and the (weak) dual functor \((-)^\vee : \text{CoAlg}_1(\text{Mod}^S_1) \rightarrow \text{Alg}_1(\text{Mod}^S_1)^{\text{op}} \) where \((-)^\vee \) carries each object \( M \) to its dual object in the symmetric monoidal infinite category \( \text{Mod}^S_1 \) \( \text{(CoAlg}_1(\text{Mod}^S_1)^{\text{op}} \) indicates the infinite category of coaugmented coalgebra objects in \( \text{Mod}^S_1 \)). Then it suffices to prove that \( \text{CAlg}^+(\text{Mod}^S_1)^{\text{op}} \) \( \text{forget} \quad \text{Alg}^+(\text{Mod}^S_1)^{\text{op}} \) \( \cong \) \( \text{CoAlg}_1^+(\text{Mod}^S_1) \) is equivalent to the composite functor

\[
\text{Fun}(\text{BS}^1, \text{CAlg}^+(\text{Mod}_A)) \rightarrow \text{Fun}(\text{BS}^1, \text{Alg}_1^+(\text{Mod}_A)) \rightarrow \text{Fun}(\text{BS}^1, \text{CoAlg}_1^+(\text{Mod}_A))
\]

induced by \( \text{CAlg}^+(\text{Mod}_A)^{\text{op}} \) \( \rightarrow \) \( \text{Alg}^+(\text{Mod}_A)^{\text{op}} \) \( \text{Bar}_{\text{Mod}_A^S} : \text{Alg}_1((\text{Mod}_A)^{\text{op}})^{\text{op}} = \text{CoAlg}_1((\text{Mod}_A)^{\text{op}}) \) \( \rightarrow \text{Alg}_1((\text{Mod}_A)^{\text{op}})^{\text{op}} = \text{CoAlg}_1((\text{Mod}_A)^{\text{op}}) \) where \( \text{Bar}_{\text{Mod}_A^S} \) is the bar construction functor. Let

\[
\text{Bar}_{\text{CAlg}_1^S} : \text{Alg}_1(\text{CAlg}(\text{Mod}^S_1)^{\text{op}}) \rightarrow \text{Alg}_1(\text{CAlg}(\text{Mod}^S_1)^{\text{op}})^{\text{op}}
\]

be the bar construction for augmented \( \mathbb{E}_1 \)-algebra objects in \( \text{CAlg}_1^S = \text{CAlg}(\text{Mod}^S_1) \). By Dunn additivity theorem, there exists a canonical (second) equivalence \( \text{CAlg}^+(\text{Mod}^S_1) \simeq \text{CAlg}(\text{Mod}^S_1)^{\text{op}} \simeq \text{Alg}_1((\text{Mod}^S_1)^{\text{op}}) \). By [23, X, 4.4.19], there exists a natural equivalence between \( \text{Bar}_{\text{Mod}_A^S} \) \( \text{forget} \) and the composite functor

\[
\text{CAlg}^+(\text{Mod}^S_1) \simeq \text{Alg}_1(\text{CAlg}(\text{Mod}^S_1)^{\text{op}}) \rightarrow \text{Alg}_1(\text{CAlg}(\text{Mod}^S_1)^{\text{op}})^{\text{op}} \rightarrow \text{CoAlg}_1^+(\text{Mod}^S_1)
\]

where the middle functor is \( \text{Bar}_{\text{CAlg}_1^S} \) and the right functor is the forgetful functor induced by \( \text{Alg}_1((\text{Mod}^S_1)^{\text{op}}) \rightarrow (\text{Mod}^S_1)^{\text{op}} \). In order to verify our assertion, it is enough to prove that \( \text{Bar}_{\text{CAlg}_1^S} \) is equivalent to the functor obtained from the bar construction \( \text{Bar}_{\text{CAlg}_1^S} : \text{Alg}_1(\text{CAlg}(\text{Mod}_A)^{\text{op}}) \rightarrow \text{Alg}_1((\text{Mod}_A)^{\text{op}})^{\text{op}} \) (defined in a similar way) by taking \( \text{Fun}(\text{BS}^1, -) \). To this end, consider the adjoint pair of bar construction and cobar construction functors

\[
\text{Bar}_{\text{CAlg}_1^S} : \text{CAlg}^+(\text{Mod}^S_1) \simeq \text{Alg}_1((\text{CAlg}^+(\text{Mod}^S_1)^{\text{op}})^{\text{op}} : \text{CoBar}_{\text{CAlg}_1^S}.
\]

Since the symmetric monoidal infinite category \( \text{CAlg}^+(\text{Mod}^S_1) \) is coCartesian, thus \( \text{Alg}_1((\text{CAlg}^+(\text{Mod}^S_1)^{\text{op}})^{\text{op}} \) can be identified with a full subcategory of \( \text{Fun}(\Delta, \text{CAlg}^+(\text{Mod}^S_1)) \). Then if we regard an object of \( \text{Alg}_1((\text{CAlg}^+(\text{Mod}^S_1)^{\text{op}})^{\text{op}} \) as a cosimplicial object \( c : \Delta \rightarrow \text{CAlg}^+(\text{Mod}^S_1) \), \( \text{CoBar}_{\text{CAlg}_1^S} \) is given by totalizations/limits, that is, it carries \( c \) to the limit in \( \text{CAlg}^+(\text{Mod}^S_1) \). Therefore, the left adjoint \( \text{Bar}_{\text{CAlg}_1^S} \) is equivalent to the functor induced by

\[
\text{CAlg}^+(\text{Mod}^S_1) \rightarrow \text{Fun}(\Delta^1, \text{CAlg}(\text{Mod}^S_1)) \rightarrow \text{Fun}(\Delta^+, \text{CAlg}(\text{Mod}^S_1)) \rightarrow \text{Fun}(\Delta, \text{CAlg}(\text{Mod}^S_1))
\]

where the left functor is the natural fully faithful functor, the middle functor is a left adjoint to the restriction functor along \( \Delta^1 = \{[-1] \rightarrow [0]\} \rightarrow \Delta^+ \), and the right functor is induced by the restriction along \( \Delta \subset \Delta^+ \). More precisely, \( \text{Bar}_{\text{CAlg}_1^S} \) can be identified with \( (\text{CAlg}^+(\text{Mod}^S_1)^{\text{op}})^{\text{op}} \rightarrow \text{Fun}(\Delta, \text{CAlg}(\text{Mod}^S_1)^{\text{op}}) \) induced by the composite. Using this description of \( \text{Bar}_{\text{CAlg}_1^S} \), we conclude that \( \text{Bar}_{\text{CAlg}_1^S} \) is equivalent to the functor obtained from \( \text{Bar}_{\text{CAlg}_1^S} \) by taking \( \text{Fun}(\text{BS}^1, -) \).

**Remark 7.15.** There is another method to prove that \( \mathcal{H}(U_2(L)/A) \) is equivalent to \( U_1(L^S) \). The nonabelian Poincaré duality for the factorization homology from Lie algebras and the description of \( U_n(L) \) as the Chevalley–Eilenberg chain complex \( \text{Ch}(\Omega^n(L)) \) (\( \Omega^n_0 \) indicates the \( n \)-fold based loop) [1, Proposition 5.13, Remark 5.14] show that

\[
\int_{S^1 \times \mathbb{R}} U_2(L) \simeq \text{Ch}(\Omega(L^S)) \simeq U_1(L^S).
\]

However, it is not clear to the author that this equivalence makes the square diagram in Proposition 7.11 commute (that is the reason that I do not directly apply this equivalence). The commutativity of the square in Proposition 7.11 will be used to prove that the Lie algebra actions constructed from the
Hochschild pair \( (\mathcal{H}^\bullet(C/A), \mathcal{H}_\bullet(C/A)) \) determines the lower arrow in Theorem 1.2. More specifically, it is necessary for the consistency of identifications which appear in arguments in Proposition 8.15 and Remark 8.14.

8. Moduli and Lie algebra actions

In this section, we prove Theorem 1.2. We first review an algebraic structure on the Hochschild pair \( (\mathcal{H}^\bullet(C/A), \mathcal{H}_\bullet(C/A)) \) (cf. Section 8.1). The Hochschild cohomology (cochain complex) \( \mathcal{H}^\bullet(C/A) \) is an \( E_2 \)-algebra. It gives rise to the dg Lie algebra \( \mathcal{G}_C \) whose underlying complex is equivalent (quasi-isomorphic) to the shifted Hochschild cohomology \( \mathcal{H}^\bullet(C/A)[1] \). To \( (\mathcal{H}^\bullet(C/A), \mathcal{H}_\bullet(C/A)) \) we associate the Lie algebra actions of \( \mathcal{G}_C \) and \( \mathcal{G}_C^{S^1} \) on \( \mathcal{H}_\bullet(C/A) \). It turns out that these actions determine the morphisms in lower horizontal arrows in Theorem 1.2. In Section 8.2, we will construct the left square in the diagram in Theorem 1.2. To do this, we will use Proposition 6.8 and the content of Section 6.3. We also interpret the action of \( \mathcal{G}_C^{S^1} \) on \( \mathcal{H}_\bullet(C/A) \) as the lower horizontal arrow (see Proposition 8.15). In Section 8.3, we will construct and discuss the right square in the diagram in Theorem 1.2. The Koszul dualities in Proposition 6.6 play a pivotal role. In Section 8.4, based on Section 8.2 and 8.3, we obtain the main result: Theorem 8.23 (and Proposition 8.24). The upper horizontal arrows in Theorem 8.23 have moduli-theoretic interpretations (see Remark 8.7 and Remark 8.21).

8.1. We briefly review the algebra of \( (\mathcal{H}^\bullet(C/A), \mathcal{H}_\bullet(C/A)) \) encoded by Kontsevich–Soibelman operad \( \mathbf{KS} \). For details, we refer the reader to [11] where we construct an algebra \( (\mathcal{H}^\bullet(C/A), \mathcal{H}_\bullet(C/A)) \) over \( \mathbf{KS} \). We do not recall the operad \( \mathbf{KS} \). Instead, we will give an equivalent formulation which is suitable for our purpose. According to [11, Theorem 1.2], a \( \mathbf{KS} \)-algebra \( (\mathcal{H}^\bullet(C/A), \mathcal{H}_\bullet(C/A)) \) in \( \text{Mod}_A \) is equivalent to giving the following triple: an \( E_2 \)-algebra \( \mathcal{H}^\bullet(C/A) \in \text{Alg}_2(\text{Mod}_A) \), an \( A \)-module with \( S^1 \)-action \( \mathcal{H}_\bullet(C/A) \in \text{Mod}_A^{S^1} \) and a left \( \mathcal{H}_\bullet(\mathcal{H}^\bullet(C/A)/A) \)-module \( \mathcal{H}_\bullet(C/A) \) in \( \text{Mod}_A^{S^1} \) that is an object of \( \text{LMod}_{\mathcal{H}_\bullet(\mathcal{H}^\bullet(C/A)/A)(\text{Mod}_A^{S^1})} \) lying over \( \mathcal{H}_\bullet(C/A) \in \text{Mod}_A^{S^1} \). In other words, the data of triple amounts to giving an object of

\[
\text{Alg}_2(\text{Mod}_A) \times \text{Alg}_1(\text{Mod}_A^{S^1}) \times \text{LMod}(\text{Mod}_A^{S^1})
\]

where \( \text{Alg}_2(\text{Mod}_A) \to \text{Alg}_1(\text{Mod}_A^{S^1}) \) is induced by \( \mathcal{H}_\bullet(\text{res}) \). We sketch the construction of a left \( \mathcal{H}_\bullet(\mathcal{H}^\bullet(C/A)/A) \)-module \( \mathcal{H}_\bullet(C/A) \). Recall the adjoint pair \( (\theta_A, E_A) \) between \( \text{Alg}_2(\text{Mod}_A) \) and \( \text{Alg}_1(\text{Pr}_A^1) \) from Section 3.6. Consider the associated counit map

\[
\text{LMod}_{\mathcal{H}^\bullet(C/A)} = \theta_A \circ E_A(\text{End}(\text{Ind}(C))) \to \text{End}(\text{Ind}(C))
\]

in \( \text{Alg}_1(\text{Pr}_A^1) \) and the tautological left action of \( \text{End}(\text{Ind}(C)) \) on \( \text{Ind}(C) \). We obtain a left \( \text{LMod}_{\mathcal{H}^\bullet(C/A)} \)-module \( \text{Ind}(C) \) in \( \text{Pr}_A^1 \). Passing to full subcategories of compact objects we have a left \( \text{Perf}_{\mathcal{H}^\bullet(C/A)} \)-module \( C \) in \( \text{St}_A \). The map \( \text{LMod}(\text{St}_A) \to \text{LMod}(\text{Mod}_A^{S^1}) \) induced by \( \mathcal{H}_\bullet(\text{res}) \) gives rise to a left \( \mathcal{H}_\bullet(\mathcal{H}^\bullet(C/A)/A) \)-module \( \mathcal{H}_\bullet(C/A) \).

Construction 8.1. We construct actions of \( \mathcal{G}_C \) and \( \mathcal{G}_C^{S^1} \) on \( \mathcal{H}_\bullet(C/A) \) out of the left \( \mathcal{H}_\bullet(\mathcal{H}^\bullet(C/A)/A) \)-module \( \mathcal{H}_\bullet(C/A) \), that is defined as a morphism \( U_1(\mathcal{G}_C) \to \text{End}(\mathcal{H}_\bullet(C/A))^{S^1} \) in \( \text{Alg}_1(\text{Mod}_A) \) (see Section 3.5). The counit map of the adjoint pair \( (U_2, \text{res}_{E_2/Lie}) \) induces \( U_2(\mathcal{G}_C) = U_2(\text{res}_{E_2/Lie}(\mathcal{H}_\bullet)) \to \mathcal{H}_\bullet \). By applying \( \mathcal{H}_\bullet(\text{res}) \), it gives rise to \( \mathcal{H}_\bullet(U_2(\mathcal{G}_C)/A) \to \mathcal{H}_\bullet(\mathcal{H}^\bullet(C/A)/A) = \mathcal{H}_\bullet(\mathcal{H}^\bullet / A) \). Consider the sequence in \( \text{Alg}_1(\text{Mod}_A^{S^1}) \):

\[
U_1(\mathcal{G}_C) \to U_1(\mathcal{G}_C^{S^1}) \simeq \mathcal{H}_\bullet(\text{res}_{E_2/Lie}(\mathcal{G}_C)) \to \mathcal{H}_\bullet(\mathcal{H}_\bullet)^{A_{\mathcal{C}}} \to \text{End}(\mathcal{H}_\bullet(C/A)) = \text{End}(\mathcal{H}_\bullet)
\]

where the first morphism is determined by the cotensor by \( S^1 \to * \), the second morphism is the equivalence which comes from Proposition 7.1.12, and \( A_{\mathcal{C}} \) corresponds to the left \( \mathcal{H}_\bullet(\mathcal{H}^\bullet(C/A)) \)-module \( \mathcal{H}_\bullet(C/A) \). Since the \( S^1 \)-action on \( U_1(\mathcal{G}_C) \) is trivial, the composite of the sequence amounts to a morphism

\[
A_{\mathcal{C}}^{E_2} : U_1(\mathcal{G}_C) \to \text{End}(\mathcal{H}_\bullet(C/A))^{S^1}
\]

in \( \text{Alg}_1(\text{Mod}_A) \). By the adjoint pair \( (U_1, \text{res}_{E_2/Lie}) \), this morphism gives us

\[
A_{\mathcal{C}}^{E_2} : \mathcal{G}_C \to \text{End}(\mathcal{H}_\bullet(C/A))^{S^1}
\]
in $\text{Lie}_A$. Here $\text{End}(\mathcal{H}_*(\mathcal{C} / A))$ is the endomorphism algebra object which is defined as an object of $\text{Alg}_1(\text{Mod}_{A}^S)$, and $\text{End}^L(\mathcal{H}_*(\mathcal{C} / A)) \in \text{Lie}_A^S$ is the dg Lie algebra obtained from $\text{End}(\mathcal{H}_*(\mathcal{C} / A)) \in \text{Alg}_1(\text{Mod}_{A}^S)$. Let $\text{End}^L(\mathcal{H}_*(\mathcal{C} / A))^S \in \text{Lie}_A$ be the homotopy fixed points of the $S^1$-action. Similarly, $U_1(\mathcal{G}_C^S) \to \text{End}(\mathcal{H}_*(\mathcal{C} / A))$ gives rise to
\[
\hat{\eta}_C^S : \mathcal{G}_C^S \longrightarrow \text{End}^L(\mathcal{H}_*(\mathcal{C} / A))
\]
in $\text{Lie}_A^S = \text{Fun}(BS^1, \text{Lie}_A)$.

8.2. The main object of this Section 8.2 is the left square in Theorem 1.2 (see Proposition 8.12, Remark 8.21, Proposition 8.4). We begin by considering right fibrations $\text{LMod}^{\partial E_Z}(A)_{C} \to \text{Alg}_2^+(\text{Mod}_{A})$ and $\text{LMod}^{\partial}(\text{Mod}_A^S)_{\eta_* \to \text{Alg}_1^+(\text{Mod}_A^S)}$ defined in Section 6.1 and Section 6.2. We observe that in both cases, the functor $\text{Alg}_2^+(\text{Mod}_A)^{op \to S}$ and $\text{Alg}_1^+(\text{Mod}_A^S)^{op \to S}$ corresponding right fibrations are representable.

**Lemma 8.2.** The right fibration $\text{LMod}^{\partial E_Z}(A)_{C} \to \text{Alg}_2^+(\text{Mod}_{A})$ is equivalent to
\[
\text{Alg}_2^+(\text{Mod}_{A}) \times_{\text{Alg}_1^+(\text{Pr}_{A})} \text{Alg}_1^+(\text{Pr}_{A}) \times_{\text{End}_A(\text{Ind}(\mathcal{C}))} \text{Alg}_2^+(\text{Mod}_{A})
\]
where $\theta_{A} : \text{Alg}_2^+(\text{Mod}_{A}) \to \text{Alg}_2^+(\text{Mod}_{A})$ is obtained by applying $\text{Alg}_1^+(\text{Mod}_{A}) \to (\text{Pr}_{A})_{\text{Mod}_{A}}$ formally given by $B \mapsto [\text{Mod}_{A} \to \text{LMod}_{B}]$, cf. Section 3.1 (note that $\text{Alg}_2^+(\text{Mod}_{A}) \cong \text{Alg}_1^+(\text{Mod}_{A}))$, and $\text{End}_A(\text{Ind}(\mathcal{C}))$ is the endomorphism functor category (that is, the endomorphism algebra object of $\text{Ind}(\mathcal{C})$ in $\text{Pr}_{A}$). Moreover, it is equivalent to the right fibration
\[
\text{Alg}_2^+(\text{Mod}_{A}) / \text{A} \oplus \mathcal{H}_*(\mathcal{C} / A) \to \text{Alg}_2^+(\text{Mod}_{A})
\]
where $A \oplus \mathcal{H}_*(\mathcal{C} / A) \to A$ is the augmented Hochschild cohomology complex associated to the Hochschild cohomology complex $\mathcal{H}_*(\mathcal{C} / A) \in \text{Alg}_1^+(\text{Mod}_{A})$. Namely, if we let $\xi : \text{Alg}_2^+(\text{Mod}_{A})^{op \to S}$ be a functor that corresponds to $u$, then $\xi$ is represented by the augmented Hochschild cohomology $A \oplus \mathcal{H}_*(\mathcal{C} / A) \to A$.

**Proof.** We first note that Ind-construction provides a symmetric monoidal equivalence $\text{St} \sim \text{Cgt}^{\text{Lcpt}}_{\text{St}}$ (cf. Section 3.1). Moreover, if $\mathcal{D}_B$ is a left module over the monoidal $\infty$-category $\text{LMod}_B^S$ (with $B \in \text{Alg}_2^+(\text{Mod}_{A})$), by passing to the full subcategories of compact objects, the module action functor $\text{Mod}_B \times \mathcal{D}_B \to \mathcal{D}_B$ induces $\text{Perf}_B \times \mathcal{D}_B^\partial \to \mathcal{D}_B^\partial$. Therefore, there is an equivalence
\[
\text{LMod}_{\text{Perf}_B}^\partial(\text{St}) \times_{\text{LMod}_{\text{Perf}_B}^\partial(\text{St})} \{\mathcal{C}\} \cong \text{LMod}_{\text{Mod}_B}^\partial(\text{Pr}_{A}) \times_{\text{Pr}_{A}} \{\text{Ind}(\mathcal{C})\}.
\]
By the definition of endomorphism algebra objects, the functor $\text{Alg}_1^+(\text{Pr}_{A})^{op \to S}$ corresponding to $\text{End}_A(\text{Ind}(\mathcal{C})) \in \text{Alg}_1^+(\text{Pr}_{A})$. This proves the first assertion.

Next, we will prove the second assertion. By [22, 4.8.5.11], the functor $\theta_{A} : \text{Alg}_2^+(\text{Mod}_{A}) \to \text{Alg}_1^+(\text{Pr}_{A})$ is fully faithful. By the definition of $\mathcal{H}^\bullet := \mathcal{H}_*(\mathcal{C} / A)$ (see Section 3.6 or [11, Definition 5.4]), $\mathcal{H}^\bullet$ is the image of $\text{End}_A(\text{Ind}(\mathcal{C}))$ under the right adjoint $\text{Alg}_1^+(\text{Pr}_{A}) \to \text{Alg}_2^+(\text{Mod}_{A})$ of $\theta_{A}$. The functor $\text{Alg}_2^+(\text{Mod}_{A}) / \eta_* \to \text{Alg}_1^+(\text{Pr}_{A}) / \text{LMod}_{\eta_*}$ induced by $\theta$ is fully faithful. Moreover, the counit map $\text{LMod}_{\mathcal{H}^\bullet} \to \text{End}_A(\text{Ind}(\mathcal{C}))$ induces an equivalence in $\text{Cat}^{\infty}$:
\[
\text{Alg}_2^+(\text{Mod}_{A}) \times_{\text{Alg}_1^+(\text{Pr}_{A})} \text{Alg}_1^+(\text{Pr}_{A}) / \text{LMod}_{\eta_*} \to \text{Alg}_2^+(\text{Mod}_{A}) \times_{\text{Alg}_1^+(\text{Pr}_{A})} \text{Alg}_1^+(\text{Pr}_{A}) / \text{End}_A(\text{Ind}(\mathcal{C})).
\]
Consequently, we see that $u$ is equivalent to the right fibration $\text{Alg}_2^+(\text{Mod}_{A}) / A \oplus \mathcal{H}^\bullet \to \text{Alg}_2^+(\text{Mod}_{A})$. This implies the second assertion.

A similar (but easier) argument also shows:

**Lemma 8.3.** The right fibration $\text{LMod}^{\partial}(\text{Mod}_A^S)_{\eta_*} \to \text{Alg}_1^+(\text{Mod}_A^S)$ is equivalent to
\[
\text{Alg}_1^+(\text{Mod}_A^S) / A \oplus \text{End}(\mathcal{H}^\bullet) \to \text{Alg}_1^+(\text{Mod}_A^S)
\]
where $A \oplus \text{End}(\mathcal{H}^\bullet)$ indicates the augmented endomorphism algebra object $A \oplus \text{End}(\mathcal{H}^\bullet) \to A$. □
According to Lemma 8.2 and Lemma 8.3, LMod^E_2(A)_C \to \text{Alg}^+_2(\text{Mod}_A) and LMod^+(\text{Mod}^{s_1})_{\mathcal{H}_*} \to \text{Alg}^+_2(\text{Mod}^{s_1}_A) are classified by representable functors \( h_{A \oplus \mathcal{H}_*} : \text{Alg}^+_2(\text{Mod}_A)^{op} \to \mathcal{S} \) and \( h_{A \oplus \text{End}(\mathcal{H}_*)} : \text{Alg}^+_2(\text{Mod}^{s_1}_A)^{op} \to \mathcal{S} \), respectively. Consider LMod^E_2(A)_C \to LMod^+(\text{Mod}^{s_1})_{\mathcal{H}_*} lying over \( \text{Alg}^+_2(\text{Mod}_A) \to \text{Alg}^+_2(\text{Mod}^{s_1}_A) \) (see Remark 6.6). By abuse of notation, we often write \( h \) for \( \text{Alg}^+_2(h) \). It gives rise to a natural transformation

\[ \mathcal{T}_{E_2} : h_{A \oplus \mathcal{H}_*} \to h_{A \oplus \text{End}(\mathcal{H}_*)} \circ h \]

**Proposition 8.4.** The natural transformation \( \mathcal{T}_{E_2} \) is determined by the left \( \mathcal{H}_*(\mathcal{H}^*/A) \)-module \( \mathcal{H}_* \) in Section 8.1, that is, a morphism \( A_C : \mathcal{H}_*(\mathcal{H}^*/A) \to \text{End}(\mathcal{H}_*) \) in \( \text{Alg}_1(\text{Mod}^{s_1}_A) \) in the natural way (see the proof).

**Proof.** By the construction of the left \( \mathcal{H}_*(\mathcal{H}^*/A) \)-module \( \mathcal{H}_* \), the left module action of monoidal \( \infty \)-category \( \text{Perf}^\oplus_{\mathcal{H}_*} \), on \( C \) is induced by the canonical monoidal functor \( \text{LMod}^\oplus_{\mathcal{H}_*} \to \text{End}(\mathcal{C}) \) and the tautological (left) action of \( \text{End}(\mathcal{Ind}(\mathcal{C})) \) on \( \mathcal{Ind}(\mathcal{C}) \) (cf. Section 8.1). By the restriction, we have a left module action of the \( A \)-linear monoidal \( \infty \)-category \( \text{Perf}^\oplus_{\mathcal{H}_*} \), on \( C \), which is universal: given \( B \in \text{Alg}^+_2(\text{Mod}_A) \), any left action of \( \text{Perf}^\oplus_{\mathcal{H}_*} \) on \( C \) is uniquely given by the monoidal functor \( \text{Perf}^\oplus_{\mathcal{H}_*} \to \text{Perf}^\oplus_{\mathcal{H}_*} \) induced by a unique morphism \( B \to \mathcal{H}_* \) (namely, \( \text{Map}_{\text{Alg}_2(\text{Mod}_A)}(B, \mathcal{H}_*) \approx \text{LMod}_{\text{Perf}^\oplus_{\mathcal{H}_*}(\text{St}_A) \times \text{St}_A(\{C\}) \), see Lemma 8.2). Passing to Hochschild chain functor, we have \( \mathcal{H}_*(\mathcal{H}^*/A) \to \text{End}(\mathcal{H}_*) \) in \( \text{Alg}_1(\text{Mod}^{s_1}_A) \), which is nothing but a part of data of the algebraic structure on the Hochschild pair (see Section 8.1). In this way, the natural transformation is given by

\[ h_{A \oplus \mathcal{H}_*}(B) = \text{Map}_{\text{Alg}_2(\text{Mod}_A)}(B, \mathcal{H}_*) \to \text{Map}_{\text{Alg}_1(\text{Mod}^{s_1}_A)}(h(B), h(\mathcal{H}_*)) \to \text{Map}_{\text{Alg}_1(\text{Mod}^{s_1}_A)}(h(B), \text{End}(\mathcal{H}_*)) \]

where the second arrow is given by the composition with \( h(\mathcal{H}_*) = \mathcal{H}_*(\mathcal{H}^*/A) \to \text{End}(\mathcal{H}_*) \). Finally, taking into account augmentations, the claim follows. \( \square \)

**Definition 8.5.** We define a natural transformation \( \mathcal{T}_{E_2} : h_{A \oplus \mathcal{H}_*} \circ D_2 \to h_{A \oplus \text{End}(\mathcal{H}_*)} \circ h \circ D_2 \) by composing \( \mathcal{T}_{E_2} \) with \( D_2 : \text{Alg}^+_2(\text{Mod}_A)^{op} \to \text{Alg}^+_2(\text{Mod}_A) \).

**Construction 8.6.** Consider the second cube in Remark 6.6 which defines a square diagram of right fibrations. The base changes of right fibrations induce a diagram of right fibrations over \( \text{Alg}^+_2(\text{Mod}_A)^{op} \). It corresponds to the diagram in \( \text{Fun}(\text{Alg}^+_2(\text{Mod}_A), \mathcal{S}) \):

\[ \text{Def}_{E_2}^C \xrightarrow{M_{E_2,\circ}} \text{Def}_{C}^{E_2,\circ}(\mathcal{H}_*) \]

\[ \text{Def}_{E_2}^C \]

\[ \text{Def}_{C}^{E_2,\circ}(\mathcal{H}_*) \]

such that each triangle commutes up to canonical homotopy, where we write \( h \) for \( \text{Alg}^+_2(h) \). The functor \( \text{Def}_{E_2,\circ}(\mathcal{H}_*) \) corresponds to \( \text{Def}_{E_2,\circ}(\text{Alg}^+_2(\text{Mod}_A)) \to \text{Alg}^+_2(\text{Mod}_A) \to \text{Alg}^+_2(\text{Mod}_A)^{op} \). If we take the restriction along the forgetful functor \( \text{Art}^{sz}_{A} \to \text{Alg}^+_2(\text{Mod}_A) \), it follows from Proposition 6.8 that \( h_{A \oplus \text{End}(\mathcal{H}_*)} \circ D_2 \circ h \to h_{A \oplus \text{End}(\mathcal{H}_*)} \circ h \circ D_2 \) becomes an equivalence after the restriction. Let us consider the diagram in \( \text{Fun}(\text{Art}^{sz}_{A}, \mathcal{S}) \) induced by the restriction along \( \text{Art}^{sz}_{A} \to \text{Alg}^+_2(\text{Mod}_A) \):

\[ \text{Def}_{C} \]

\[ \text{Def}_{C}^{E_2} \]

\[ \text{Def}_{C}^{E_2,\circ}(\mathcal{H}_*) \]

\[ \text{Def}_{C}^{E_2,\circ}(\mathcal{H}_*) \]

where \( (-)|_{\text{Art}^{sz}_{A}} \) indicates the restriction along \( \text{Art}^{sz}_{A} \to \text{Alg}^+_2(\text{Mod}_A) \), and \( \text{Def}_{C} = \text{Def}_{C}^{E_2} \mid_{\text{Art}^{sz}_{A}} \) and \( \text{Def}_{C}^{E_2,\circ}(\mathcal{H}_*) = \text{Def}_{C}^{E_2,\circ}(\mathcal{H}_*) \mid_{\text{Art}^{sz}_{A}} \).
Remark 8.7. Let us consider the moduli-theoretic meaning of $M^\triangleleft_C$. By definition, $\text{Def}^\triangleleft_C(H_\bullet): \text{Art}_{A}^{\text{tz}} \to S$ is given by

$$\text{Art}_{A}^{\text{tz}} \ni [R \to A] \mapsto \text{RMod}_{\mathcal{H}_\bullet(R/A)}(\text{Mod}_{A}^{S^1}) \cong (\text{Mod}_{A}^{S^1})^\triangleright \ni \{H_\bullet\} \in S.$$  

By Lemma 3.5, $\text{Art}_{A}^{\text{tz}} \to \text{Alg}_{A}(\text{Mod}_{A}^{S^1})$ given by $h|_{\text{Art}_{A}^{\text{tz}}}$ is equivalent to the functor determined by the tensor with $S^1$, that is, the functor given by $R \mapsto R \otimes_{A} S^1$. Namely, the space $\text{Def}^\triangleleft_C(H_\bullet)(R)$ parametrizes cyclic deformations of $\mathcal{H}_\bullet(C/A)$ to $R \otimes_{A} S^1$ (cf. Section 6.2). We refer to $\text{Def}^\triangleleft_C(H_\bullet)$ to the deformation functor of cyclic deformations of $H_\bullet$ over $\text{Art}_{A}^{\text{tz}}$. Let $\mathcal{C} = (C' \in \text{RMod}_{\text{Perf}}_{R}(\text{St}_{A}), R \to A, C \simeq C' \otimes_{\text{Perf}_{A}} \text{Perf}_{A})$ be a deformation of $C$ to $R$, that is, an object of $\text{Def}_{C}(R)$. Then $M^\triangleleft_C$ sends $\mathcal{C}$ to the cyclic deformation of $\mathcal{H}_\bullet(C/A)$:

$$(\mathcal{H}_\bullet(C'/A) \in \text{RMod}_{R \otimes_{A} S^1}(\text{Mod}_{A}^{S^1}), R \otimes_{A} S^1 = \mathcal{H}_\bullet(R/A) \to A, H_\bullet \simeq \mathcal{H}_\bullet(C'/A) \otimes_{R \otimes_{A} S^1} A),$$

where the final equivalence follows from Lemma 6.3.

Notation 8.8. We write $\mathcal{F}_{\mathcal{G}_{C}}: \text{Art}_{A}^{\text{tz}} \to S$ for $h_{A \otimes \text{End}(\mathcal{H}_\bullet(C/A))} \circ \mathbb{D}S_{1} \circ h|_{\text{Art}_{A}^{\text{tz}}}$. For ease of notation we usually write $\mathcal{F}_{\mathcal{G}_{C}}^{\triangleleft}$ for $\mathcal{F}_{A \otimes \text{End}(\mathcal{H}_\bullet(C/A))}^{\triangleleft}$.

Notation 8.9. We write $\mathcal{F}_{A \otimes H_\bullet}$ for $h_{A \otimes H_\bullet} \circ \mathbb{D}S_{1} \circ h|_{\text{Art}_{A}^{\text{tz}}}$.  

Notation 8.10. We write $\mathcal{G}_{C}$ for the image $\text{res}_{E_{2}/\text{Lie}}(H_\bullet) = \text{res}_{E_{2}/\text{Lie}}(\mathcal{H}_\bullet(C/A))$ of $\mathcal{H}_\bullet(C/A)$ under the right adjoint $\text{Alg}(\text{Mod}_{A}) \to \text{Lie}_{A}$. Let $\mathcal{F}_{\mathcal{G}_{C}}$ denote the formal stack associated to $\mathcal{G}_{C}$, that is given by the functor given by $R \mapsto \text{Map}_{\text{Lie}_{A}}(\mathbb{D}_{\infty}(R), \mathcal{G}_{C})$ (cf. Section 3.3).

Lemma 8.11. There is a canonical equivalence $\mathcal{F}_{\mathcal{G}_{C}} \simeq \mathcal{F}_{A \otimes H_\bullet}$.

Proof. This follows from the definition of $\mathcal{G}_{C}$ (see Section 3.5). □

Proposition 8.12. We use the notation in Construction 8.6, Notation 8.8, 8.9, 8.10, and Lemma 8.11. Then we have the diagram in $\text{Fun}(\text{Art}_{A}^{\text{tz}}, S)$:

$$\begin{array}{ccc}
\text{Def}_{C} & \xrightarrow{M^\triangleleft} & \text{Def}^\triangleleft_C(H_\bullet) \\
\downarrow J_{C} & & \downarrow J_{C} \\
\mathcal{F}_{\mathcal{G}_{C}} & \xrightarrow{\mathcal{F}_{\triangleleft}} & \mathcal{F}_{A \otimes \text{End}(H_\bullet)},
\end{array}$$

which commutes up to canonical homotopy. By abuse of notation, the lower horizontal arrow denotes the composite $\mathcal{F}_{\mathcal{G}_{C}} \simeq h_{A \otimes \mathcal{H}_\bullet(C)} \circ \mathbb{D}S_{1}|_{\text{Art}_{A}^{\text{tz}}} \xrightarrow{\mathcal{F}_{\triangleleft}} h_{A \otimes \text{End}(H_\bullet)} \circ h \circ \mathbb{D}S_{1}|_{\text{Art}_{A}^{\text{tz}}} \simeq h_{A \otimes \text{End}(H_\bullet)} \circ \mathbb{D}S_{1} \circ h|_{\text{Art}_{A}^{\text{tz}}} = \mathcal{F}_{A \otimes \text{End}(H_\bullet)}^{\triangleleft}$.

Proof. Combine Construction 8.6 and Lemma 8.11. □

The lower horizontal arrow $\mathcal{F}_{\mathcal{G}_{C}} \to \mathcal{F}_{A \otimes \text{End}(H_\bullet)}^{\triangleleft}$ is induced by $\mathcal{F}_{C}$ (determined by the algebra of Hochschild pair) and the duality $h_{A \otimes \text{End}(H_\bullet)} \circ h \circ \mathbb{D}S_{1}|_{\text{Art}_{A}^{\text{tz}}} \simeq h_{A \otimes \text{End}(H_\bullet)} \circ \mathbb{D}S_{1} \circ h|_{\text{Art}_{A}^{\text{tz}}}$ (cf. Proposition 8.4). To pursue the relationship with the algebraic structure of the Hochschild pair, we prove Proposition 8.15 below.

Lemma 8.13. Let $E^L$ be an object of $\text{Lie}_{A}^{S^1}$. Let $M$ be a dg Lie algebra, that is, $M \in \text{Lie}_{A}$. We define $\mathcal{F}^{\triangleleft}_{E^L}(\text{Lie}_{A}^{op}) \to S$ by the formula $L \mapsto \text{Map}_{\text{Lie}_{A}}(L^{S^1}, E^L)$. Then there exists a canonical equivalence of spaces

$$\text{Map}_{\text{Lie}_{A}}(M^{S^1}, E^L) \simeq \text{Map}_{\text{Fun}(\text{Art}_{A}^{\text{tz}}, S)}(\mathcal{F}_{M}, \mathcal{F}^{\triangleleft}_{E^L}).$$

Given $M^{S^1} \to E^L$, the corresponding morphism $\mathcal{F}_{M} \to \mathcal{F}^{\triangleleft}_{E^L}$ in $\text{Fun}((\text{Lie}_{A}^{op}, S)$ is given by

$$\mathcal{F}_{M}(L) = \text{Map}_{\text{Lie}_{A}}(L, M) \to \text{Map}_{\text{Lie}_{A}}^{\triangleright}(L^{S^1}, M^{S^1}) \to \text{Map}_{\text{Lie}_{A}}(L^{S^1}, E^L) = \mathcal{F}^{\triangleleft}_{E^L}(L),$$

where the first arrow is induced by the cotensor with $S^1$, and the second arrow is induced by the composition with $M^{S^1} \to E^L$.  

Remark 8.14. Suppose that \( E^L = \text{End}^L(\mathcal{H}_\bullet) \). Then there exists an equivalence \( \mathcal{F}^\gamma_{E^L} \simeq \mathcal{F}^\gamma_{\text{End}(\mathcal{H}_\bullet)} \) as functors \( \text{Art}^{\text{etz}} \simeq (\text{Lie}_A^L)^{\text{op}} \to \mathcal{S} \) since \( U_1(\mathbb{D}_\infty(R)^{S_1}) \simeq \mathbb{D}_1(\text{h}(R)) \) for \( R \in \text{Art}^{\text{etz}} \) (see Proposition 7.11). Namely, \( \mathcal{F}^\gamma_{\text{End}(\mathcal{H}_\bullet)} \) is given by

\[
L \mapsto \text{Map}_{\text{Lie}_A^{S_1}}(L^{S_1}, \text{End}^L(\mathcal{H}_\bullet))
\]
as a functor \((\text{Lie}_A^L)^{\text{op}} \to \mathcal{S} \).

Proof. We first observe that we may assume that \( M \) belongs to \( \text{Lie}_A^L \). Let \((-)^{S_1} : \text{Lie}_A \to \text{Lie}_A^{S_1}\) be the functor given by cotensor with \( S^1 \). By (the proof of) Lemma 7.13, \((-)^{S_1}\) preserves sifted colimits. For any \( M \in \text{Lie}_A^L \), \( M \) is a (sifted) colimit of \((\text{Lie}_A^L)_M \to \text{Lie}_A \simeq \mathcal{P}^{S_1}(\text{Lie}_A^L)\). The inclusion \( \text{Lie}_A \simeq \mathcal{P}^{S_1}_2(\text{Lie}_A^L) \to \text{Fun}(\text{Art}^{\text{etz}}, \mathcal{S}) \) carries \( M \) to the functor \( \mathcal{F}_M : \text{Art}^{\text{etz}} \simeq (\text{Lie}_A^L)^{\text{op}} \to \mathcal{S} \) corepresented by \( M \) (cf. Section 3.3). An object of \( \text{Lie}_A^L \) is compact and projective in \( \text{Lie}_A^L \), that is, the functor it corepresents preserves sifted colimits. It follows that \( \text{Lie}_A^L \to \text{Fun}(\text{Art}^{\text{etz}}, \mathcal{S}) \) preserves sifted colimits. Consequently, we may and will assume that \( M \) belongs to \( \text{Lie}_A^L \).

The functor \((-)^{S_1}\) gives rise to the adjoint pair

\[
\tau : \text{Fun}((\text{Lie}_A)^{S_1}, \mathcal{S}) \xleftarrow{\sim} \text{Fun}((\text{Lie}_A^{S_1})^{\text{op}}, \mathcal{S})
\]
where the left arrow \( \leftarrow \) is the right adjoint functor given by the construction with \((-)^{S_1}\). The right arrow \( \tau \) is the left adjoint given by left Kan extensions along \((-)^{S_1} : (\text{Lie}_A)^{\text{op}} \to (\text{Lie}_A^{S_1})^{\text{op}}\). Let \( \mathcal{F}^\gamma_{E^L} : (\text{Lie}_A)^{\text{op}} \to \mathcal{S} \) denote the image of \( \mathcal{H}_{E^L} \) under the right adjoint (where \( \mathcal{H}_{E^L} \) is the functor represented by \( E^L \)). Let \( \mathcal{H}_M : (\text{Lie}_A)^{\text{op}} \to \mathcal{S} \) denote the functor represented by \( M \). Then by the adjoint pair, \( \text{Map}(\mathcal{H}_M, \mathcal{F}^\gamma_{E^L}) \simeq \text{Map}(\tau(\mathcal{H}_M), \mathcal{H}_{E^L}) \). Note that \((-)^{S_1} : \text{Lie}_A \to (\text{Lie}_A^{S_1})^{\text{op}} \) commutes with the left adjoint \( \tau \) through the Yoneda embeddings. Thus, \( \tau(\mathcal{H}_M) \) is equivalent to the functor \( \mathcal{H}_M^{S_1} \) (represented by \( M^{S_1} \)). It follows that \( \tau(\mathcal{H}_M, \mathcal{H}_{E^L}) \simeq \text{Map}_{\text{Lie}_A^{S_1}}(M^{S_1}, E^L) \).

To prove our assertion, it suffices to obtain \( \text{Map}_{\text{Fun}(\text{Lie}_A)^{\text{op}}, \mathcal{S})}(\mathcal{H}_M, \mathcal{F}^\gamma_{E^L}) \simeq \text{Map}_{\text{Fun}(\text{Art}^{\text{etz}}, \mathcal{S})}(\mathcal{F}_M, \mathcal{F}^\gamma_{E^L}) \).

The composition with \( \text{Art}^{\text{etz}} \simeq (\text{Lie}_A^L)^{\text{op}} \to (\text{Lie}_A^{S_1})^{\text{op}} \) induces \( \text{Fun}(\text{Lie}_A)^{\text{op}}, \mathcal{S}) \to \text{Fun}(\text{Art}^{\text{etz}}, \mathcal{S}) \). The image of \( \mathcal{H}_M \) under this functor is \( \mathcal{F}_M \). The image of \( \mathcal{F}^\gamma_{E^L} \) is \( \mathcal{F}^\gamma_{E^L} \). Consider the adjoint pair \( \gamma : \text{Fun}(\text{Art}^{\text{etz}}, \mathcal{S}) \xleftarrow{\sim} \text{Fun}(\text{Lie}_A^{S_1}, \mathcal{S}) \) such that the right adjoint is induced by the composition with \( \text{Art}^{\text{etz}} \simeq (\text{Lie}_A^L)^{\text{op}} \to (\text{Lie}_A^{S_1})^{\text{op}} \). It is enough to show that \( \gamma(\mathcal{F}_M) \simeq \mathcal{H}_M \) for \( M \in \text{Lie}_A^L \). The left adjoint \( \gamma \) is the left Kan extension of \( \text{Lie}_A^L \subset \text{Lie}_A \to \text{Fun}(\text{Lie}_A^{S_1}, \mathcal{S}) \) along \( \text{Lie}_A^L \simeq (\text{Art}^{\text{etz}})^{\text{op}} \to \text{Fun}(\text{Art}^{\text{etz}}, \mathcal{S}) \) where \( \gamma \) is the Yoneda embedding. Thus, \( \gamma(\mathcal{F}_M) \simeq \mathcal{H}_M \) when \( M \in \text{Lie}_A^L \). The final assertion is straightforward to check from the construction of the equivalence. \( \square \)

Proposition 8.15. The morphism \( \zeta : \mathcal{F}_\mathcal{G}_c \to \mathcal{F}^\gamma_{\text{End}(\mathcal{H}_\bullet)} \) in Proposition 8.12 is determined by the morphism \( \mathcal{A}_c^L : \mathcal{G}_c^{S_1} \to \text{End}^L(\mathcal{H}_\bullet) \) in Construction 8.1 through the equivalence in Lemma 8.13 (see also Remark 8.14).

Proof. By the definition of \( \zeta \) and Proposition 8.4, \( \zeta : \mathcal{F}_\mathcal{G}_c \to \mathcal{F}^\gamma_{\text{End}(\mathcal{H}_\bullet)} \) is given by

\[
\alpha : \mathcal{F}_\mathcal{G}_c(-) \simeq \text{Map}_{\text{Alg}_2(\text{Mod}_A)}(\mathbb{D}_2(-), \mathcal{H}_\bullet) \xrightarrow{h} \text{Map}_{\text{Alg}(\text{Mod}_A)}(h \circ \mathbb{D}_2(-), \text{h}(\mathcal{H}_\bullet)) \rightarrow \text{Map}_{\text{Alg}(\text{Mod}_A)}(h \circ \mathbb{D}_2(-), \text{End}(\mathcal{H}_\bullet)) \simeq \mathcal{F}^\gamma_{\text{End}(\mathcal{H}_\bullet)}(-)
\]
where the second arrow is induced by \( \mathcal{A}_c : h(\mathcal{H}_\bullet) = \mathcal{H}_{\text{Alg}}(\mathcal{H}_\bullet / A) \to \text{End}(\mathcal{H}_\bullet) \) (see Remark 8.4). This sequence indicates the sequence in \( \text{Fun}(\text{Art}^{\text{etz}}, \mathcal{S}) \), and \((-)\) means the “argument”. By the adjoint pair \( U_2 : \text{Lie}_A \simeq \text{Alg}_2(\text{Mod}_A) \) and the natural equivalence \( U_2 \circ \mathbb{D}_\infty \simeq \mathbb{D}_2 \) between functors \( \text{Art}^{\text{etz}})^{\text{op}} \to \text{Alg}_2(\text{Mod}_A) \) (see Proposition 3.3 (1)), the first equivalence is defined as the composite \( \text{Map}_{\text{Lie}_A}(\mathbb{D}_\infty(-), \mathcal{G}_c) \xrightarrow{\mathbb{G}_c^L} \text{Map}_{\text{Alg}_2(\text{Mod}_A)}(U_2(\mathbb{D}_\infty(-)), U_2(\mathcal{G}_c)) \rightarrow \text{Map}_{\text{Alg}_2(\text{Mod}_A)}(\mathbb{D}_2(-), \mathcal{H}_\bullet) \) where the second arrow is given by the composition with the counit map \( U_2(\mathcal{G}_c) \to \mathcal{H}_\bullet \) and \( U_2 \circ \mathbb{D}_\infty \simeq \mathbb{D}_2 \). Thus,
\( \alpha \) is equivalent to
\[
\beta : \text{Map}_{\text{Alg}_2(\text{Mod}_A)}(\mathbb{D}_2(\mathcal{T}), \mathcal{H}^*) \simeq \text{Map}_{\text{Lie}}(\mathbb{D}_\infty(-), \mathcal{G}_C)
\]
\[
\xrightarrow{\text{hst}_2} \text{Map}_{\text{Alg}_1(\text{Mod}_A)}(h \circ U_2 \circ \mathbb{D}_\infty(-), h(U_2(\mathcal{G}_C)))
\]
\[
\rightarrow \text{Map}_{\text{Alg}_1(\text{Mod}_A)}(h \circ U_2 \circ \mathbb{D}_\infty(-), \text{End}(\mathcal{H}_*)),
\]
\[
\simeq \text{Map}_{\text{Alg}_1(\text{Mod}_A)}(h \circ \mathbb{D}_2(-), \text{End}(\mathcal{H}_*))
\]
where the third arrow is induced by \( h(U_2(\mathcal{G}_C)) \rightarrow h(\mathcal{H}_*) \rightarrow \text{End}(\mathcal{H}_*) \), and the fourth arrow is induced by \( U_2 \circ \mathbb{D}_\infty \simeq \mathbb{D}_2 \). Next we use the natural equivalence \( h \circ U_2 = \mathcal{H}_* \circ U_2 \simeq U_1 \circ (-)^{S1} \) between functors \( \text{Lie}_A \rightarrow \text{Alg}_1^+ (\text{Mod}_A^{S1}) \) in Proposition 7.12. Then using this natural equivalence, we see that \( \beta \) is equivalent to
\[
\gamma : \text{Map}_{\text{Alg}_2(\text{Mod}_A)}(\mathbb{D}_2(-), \mathcal{H}^*) \simeq \text{Map}_{\text{Lie}}(\mathbb{D}_\infty(-), \mathcal{G}_C)
\]
\[
\xrightarrow{U_1 \circ (-)^{S1}} \text{Map}_{\text{Alg}_1(\text{Mod}_A)}(U_1(\mathbb{D}_\infty(-)^{S1}), U_1(\mathcal{G}_C^{S1}))
\]
\[
\rightarrow \text{Map}_{\text{Alg}_1(\text{Mod}_A)}(U_1(\mathbb{D}_\infty(-)^{S1}), \text{End}(\mathcal{H}_*))
\]
\[
\simeq \text{Map}_{\text{Alg}_1(\text{Mod}_A)}(h \circ \mathbb{D}_2(-), \text{End}(\mathcal{H}_*))
\]
where the third arrow is induced by \( U_1(\mathcal{G}_C^{S1}) \simeq h(U_2(\mathcal{G}_C)) \rightarrow h(\mathcal{H}_*) \rightarrow \text{End}(\mathcal{H}_*) \), and the fourth arrow is induced by the natural equivalence \( U_1 \circ (-)^{S1} \simeq h \circ U_2 \circ \mathbb{D}_\infty \simeq h \circ \mathbb{D}_2 \) between functors \( \text{Art}_A^\text{tsz} \rightarrow \text{Alg}_1(\text{Mod}_A^{S1}) \) (see Proposition 7.11). Note that \( \gamma \) is equivalent to \( \text{Map}_{\text{Lie},A}(\mathbb{D}_\infty(-), \mathcal{G}_C) \rightarrow \text{Map}_{\text{Lie}_A}(\mathbb{D}_\infty(-)^{S1}, \mathcal{G}_C^{S1}) \rightarrow \text{Map}_{\text{Lie}_A}(\mathbb{D}_\infty(-)^{S1}, \text{End}(\mathcal{H}_*)) \) where the second functor is induced by \( \hat{A}_2^1 \). Now our assertion follows from Lemma 8.13. \( \square \)

### 8.3
We will construct the right square in Theorem 1.2 (see Proposition 8.20, Remark 8.21).

The functors \( h \) and \( i \) in Lemma 3.5 are extended to functors \( h^+, i^+ : \text{CAlg}_A^+ \rightarrow \text{CAlg}_A^+ (\text{Mod}_A^{S1}) \), and \( \sigma : h \rightarrow i \) is extended to \( \sigma^+ : h^+ \rightarrow i^+ \) in the natural way. We denote by \( h|_{\text{Art}_A^{\text{tsz}}} \rightarrow i|_{\text{Art}_A^{\text{tsz}}} \) a natural transformation between functors \( \text{Art}_A^{\text{tsz}} \rightarrow \text{Alg}_1^+ (\text{Mod}_A^{S1}) \), which is obtained from \( h^+ \rightarrow i^+ \) by the compositions with \( \text{Art}_A^{\text{tsz}} \rightarrow \text{CAlg}_A^+ \) and \( \text{CAlg}_A^+ (\text{Mod}_A^{S1}) \rightarrow \text{Alg}_1^+ (\text{Mod}_A^{S1}) \).

**Construction 8.16.** We construct the square diagram in \( \text{Fun}(\text{Art}_A^{\text{tsz}}, \mathcal{S}) \). Recall the diagram such that the vertical arrows are right fibrations
\[
\begin{array}{ccc}
(\text{Def}_A^{\text{tsz}}(\mathcal{H}_*))^{op} & \xrightarrow{\mathbb{D}_2(\mathcal{H}_*)} & \text{LMod}^+ (\text{Mod}_A^{S1})_{\mathcal{H}_*} \\
\downarrow & & \downarrow \\
\text{Alg}_1^+ (\text{Mod}_A^{S1})^{op} & \xrightarrow{\mathbb{D}_1^{S1}} & \text{Alg}_1^+ (\text{Mod}_A^{S1})
\end{array}
\]
from the second cube in Remark 6.6. Taking into account the corresponding functors \( \text{Alg}_1^+ (\text{Mod}_A^{S1}) \rightarrow \mathcal{S} \) and Lemma 8.3, we have a natural transformation \( \text{Def}_A^{\text{tsz}}(\mathcal{H}_*) \circ J_{\mathcal{H}_*}^\triangleright \circ h_{A@ \text{End}(\mathcal{H}_*)} \circ \mathbb{D}_1^{S1} \) (compare Construction 8.6). The composition with \( h|_{\text{Art}_A^{\text{tsz}}} \rightarrow i|_{\text{Art}_A^{\text{tsz}}} \) yields the diagram in \( \text{Fun}(\text{Art}_A^{\text{tsz}}, \mathcal{S}) \)
\[
\begin{array}{ccc}
(\text{Def}_A^{\text{tsz}}(\mathcal{H}_*)) \circ h|_{\text{Art}_A^{\text{tsz}}} & \rightarrow & \text{Def}_A^{\text{tsz}}(\mathcal{H}_*) \circ i|_{\text{Art}_A^{\text{tsz}}} \\
\downarrow & & \downarrow \\
\text{h}_{A@ \text{End}(\mathcal{H}_*)} \circ \mathbb{D}_1^{S1} \circ h|_{\text{Art}_A^{\text{tsz}}} & \rightarrow & \text{h}_{A@ \text{End}(\mathcal{H}_*)} \circ \mathbb{D}_1^{S1} \circ i|_{\text{Art}_A^{\text{tsz}}}
\end{array}
\]
which commutes up to canonical homotopy. Note that \( \text{Def}_A^{\text{tsz}}(\mathcal{H}_*) = \text{Def}_A^{\text{tsz}}(\mathcal{H}_*)|_{\text{Art}_A^{\text{tsz}}} = \text{Def}_A^{\text{tsz}}(\mathcal{H}_*) \circ h|_{\text{Art}_A^{\text{tsz}}} \). Note also that \( \mathcal{F}_A^{\triangleright}(\mathcal{H}_*) = \text{h}_{A@ \text{End}(\mathcal{H}_*)} \circ \mathbb{D}_1^{S1} \circ h|_{\text{Art}_A^{\text{tsz}}} \). The left vertical arrow is \( J_{\mathcal{H}_*}^\triangleright \) in Proposition 8.12.
Remark 8.17. The functor $\text{Def}^{\sim} (\mathcal{H}_*) \circ i|_{\text{Art}^{\text{tsz}}_A} : \text{Art}^{\text{tsz}}_A \to \mathcal{S}$ is informally given by

$$\text{Art}^{\text{tsz}}_A \ni \{R \to A\} \mapsto \text{LMod}_{i(R)}(\text{Mod}^{S^1}_A)^{\sim} \times_{(\text{Mod}^{S^1}_A)^{\sim}} \{\mathcal{H}_*\} \in \mathcal{S}$$

where $\text{LMod}_{i(R)}(\text{Mod}^{S^1}_A) \to \text{Mod}_A$ is the base change induced by $R \to A$. The right-hand side is the space of $S^1$-equivariant deformations of $\mathcal{H}_* \circ (C/A)$ to $R$. Namely, it describes the $S^1$-equivariant deformation problem.

Notation 8.18. We write $\text{Def}^{S^1}(\mathcal{H}_*)$ for $\text{Def}^{\sim} (\mathcal{H}_* \circ (C/A)) \circ i|_{\text{Art}^{\text{tsz}}_A}$.

Lemma 8.19. Let $\text{End}^{L}(\mathcal{H}_*)$ denote the dg Lie algebra endowed with $S^1$-action (i.e., an object of $\text{Lie}^{S^1} \simeq \text{Fun}(BS^1, \text{Lie}_A)$) associated to $\text{End}(\mathcal{H}_*) \in \text{Alg}_1(\text{Mod}^{S^1}_A)$ (that is, $\text{End}^{L}(\mathcal{H}_*) = \text{res}_{E_1/Lie}(\text{End}(\mathcal{H}_*))$). Let $\text{End}^{L}(\mathcal{H}_*)^{S^1}$ be the $S^1$-invariants (homotopy fixed points). Then there exists a canonical equivalence $\mathfrak{h} : \text{End}^{L}(\mathcal{H}_*) \circ \text{End}^{L}(\mathcal{H}_*)^{S^1}$ is given by $\mathcal{F}_{\text{End}^{L}(\mathcal{H}_*)^{S^1}}$ where $\mathcal{F}_{\text{End}^{L}(\mathcal{H}_*)^{S^1}}$ is the formal stack associated to $\text{End}^{L}(\mathcal{H}_*)^{S^1}$ (cf. Section 3.3).

Proof. Observe that $\mathfrak{h} \circ \text{End}^{L}(\mathcal{H}_*) \circ i|_{\text{Art}^{\text{tsz}}_A}$ is given by

$$[R \to A] \mapsto \text{Map}_{\text{Alg}_1(\text{Mod}^{S^1}_A)}(\mathcal{D}_1(R), \text{End}(\mathcal{H}_*))$$

Here we think of $\mathcal{D}_1(R)$ as an unaugmented $E_1$-algabra in $\text{Mod}_A$ endowed with the trivial $S^1$-action. Since $\mathcal{D}_1(R) \simeq U_1(\mathcal{D}_1(R))$ for $R \in \text{Art}^{\text{tsz}}_A$ (see Proposition 3.3 (1)), it follows that $\mathfrak{h} \circ \text{End}^{L}(\mathcal{H}_*) \circ i|_{\text{Art}^{\text{tsz}}_A} : \text{Art}^{\text{tsz}}_A \to \mathcal{S}$ is given by

$$[R \to A] \mapsto \text{Map}_{\text{Alg}_1(\text{Mod}^{S^1}_A)}(\mathcal{D}_1(R), \text{End}(\mathcal{H}_*))$$

It follows that $\mathfrak{h} \circ \text{End}^{L}(\mathcal{H}_*) \circ i|_{\text{Art}^{\text{tsz}}_A} \simeq \mathcal{F}_{\text{End}^{L}(\mathcal{H}_*)^{S^1}}$.

Proposition 8.20. There exists a diagram in $\text{Fun}(\text{Art}^{\text{tsz}}_A, \mathcal{S})$

$$\begin{array}{ccc}
\text{Def}^{\sim} (\mathcal{H}_* \circ (C/A)) & \xrightarrow{N_C} & \text{Def}^{S^1}(\mathcal{H}_* \circ (C/A)) \\
\mathcal{F}_{\text{Art}^{\text{tsz}}_A(\mathcal{H}_* \circ (C/A))} & \xrightarrow{J^{S^1}_{\mathcal{H}_* \circ (C/A)}} & \mathcal{F}_{\text{End}^{L}(\mathcal{H}_* \circ (C/A))^{S^1}}
\end{array}$$

which commutes up to canonical homotopy.

Proof. Consider the second square diagram in Construction 8.16. By Lemma 8.19 and definitions, we can interpret the square diagram in Construction 8.16 as the desired one.

Remark 8.21. We consider the moduli-theoretic meaning of $N_C$. Recall that the space $\text{Def}^{\sim} (\mathcal{H}_*)(R)$ parametrizes cyclic deformations of $\mathcal{H}_* \circ (C/A)$ to $R \otimes A S^1$, that is, $\text{LMod}_{\mathcal{H}_* \circ (C/A)}(\text{Mod}^{S^1}_A)^{\sim} \times_{(\text{Mod}^{S^1}_A)^{\sim}} \{\mathcal{H}_*\}$. Let

$$(\mathcal{H}' \in \text{RMMod}_{R \otimes A S^1}(\text{Mod}^{S^1}_A), R \otimes A S^1 = \mathcal{H}_* \circ (C/A) \to A, \mathcal{H}_* \simeq \mathcal{H}' \otimes_{R \otimes A S^1} A),$$

be an object of $\text{Def}^{\sim} (\mathcal{H}_*)(R)$. The image under $N_C$ is

$$(\mathcal{H}' \otimes_{R \otimes A S^1} R \in \text{RMMod}_R(\text{Mod}^{S^1}_A), R \to A, \mathcal{H}_* \simeq (\mathcal{H}' \otimes_{R \otimes A S^1} R) \otimes_R A).$$

Namely, $N_C$ is given by the base change along $R \otimes A S^1 \to R$ for each $R \in \text{Art}^{\text{tsz}}_A$. 
**Remark 8.22.** By the definition (see Construction 8.16), $\mathcal{F}_{A\oplus \text{End}(\mathcal{H}_*)} \rightarrow \mathcal{F}_{\text{End}^L(\mathcal{H}_*)}^{s_1}$ is given by

$$\mathcal{F}_{A\oplus \text{End}(\mathcal{H}_*)}^{s_1}(-) \cong \text{Map}_{\text{Lie}_A}(\mathbb{D}_\infty(-)^{s_1}, \text{End}^L(\mathcal{H}_*))$$

$$\rightarrow \text{Map}_{\text{Lie}_A}(\mathbb{D}_\infty(-), \text{End}^L(\mathcal{H}_*))$$

$$\cong \text{Map}_{\text{Lie}_A}(\mathbb{D}_\infty(-), \text{End}^L(\mathcal{H}_*)^{s_1}) = \mathcal{F}_{\text{End}^L(\mathcal{H}_*)}^{s_1}(-)$$

where the second arrow is induced by the diagonal map $\mathbb{D}_\infty(-) \rightarrow \mathbb{D}_\infty(-)^{s_1}$ (see also Lemma 8.13 for the first equivalence).

**8.4.**

**Theorem 8.23.** There exists a diagram in $\text{Fun}(\text{Art}_{A^\text{Az}}^{\infty}, S)$

$$\text{Def}_C \xrightarrow{M^C_G} \text{Def}^C(\mathcal{H}_*(C/A)) \xrightarrow{N_C} \text{Def}^{s_1}(\mathcal{H}_*(C/A))$$

which commutes up to canonical homotopy.

**Proof.** Combine the diagram in Proposition 8.12 and the diagram in Proposition 8.20. $\Box$

**Proposition 8.24.** The composition $\mathcal{F}_{\mathcal{G}_C} \xrightarrow{\mathcal{T}_C} \mathcal{F}_{A\oplus \text{End}(\mathcal{H}_*(C/A))} \rightarrow \mathcal{F}_{\text{End}^L(\mathcal{H}_*(C/A))}^{s_1}$ corresponds to $A^L_C : \mathcal{G}_C \rightarrow \text{End}^L(\mathcal{H}_*(C/A))^{s_1}$ (cf. Construction 8.1) through $\mathcal{G}_C \simeq \text{Lie}_A$.

**Proof.** We write $\mathcal{H}_*$ for $\mathcal{H}_*(C/A)$. Consider the square diagram

$$\text{Map}_{\text{Lie}_A^{s_1}}(\mathcal{G}_C^{s_1}, \text{End}^L(\mathcal{H}_*)) \xrightarrow{\cong} \text{Map}(\mathcal{F}_{\mathcal{G}_C}, \mathcal{F}_{A\oplus \text{End}(\mathcal{H}_*)}^{s_1})$$

The upper horizontal equivalence comes from in Lemma 8.13. The lower horizontal equivalence comes from $\text{Lie}_A \simeq \hat{S}_A^\text{tr}$ and $\text{Map}_{\text{Lie}_A^{s_1}}(\mathcal{G}_C, \text{End}^L(\mathcal{H}_*)) \simeq \text{Map}_{\text{Lie}_A}(\mathcal{G}_C, \text{End}^L(\mathcal{H}_*)^{s_1})$. The left vertical arrow is induced by the composition with the diagonal map $\mathcal{G}_C \rightarrow \mathcal{G}_C^{s_1}$. The right vertical arrow is induced by the composition with $\mathcal{F}_{A\oplus \text{End}(\mathcal{H}_*)} \rightarrow \mathcal{F}_{\text{End}^L(\mathcal{H}_*)}^{s_1}$. Unfolding the construction of the upper horizontal equivalence in Lemma 8.13 (see also Remark 8.14), we observe that this square commutes up to homotopy. According to Proposition 8.15, $\mathcal{T}_C$ corresponds to $A^L_C$ in Construction 8.1 (via upper horizontal equivalence) so that $\mathcal{F}_{\mathcal{G}_C} \xrightarrow{\mathcal{T}_C} \mathcal{F}_{A\oplus \text{End}(\mathcal{H}_*)} \rightarrow \mathcal{F}_{\text{End}^L(\mathcal{H}_*)}^{s_1}$ corresponds to $A^L_C : \mathcal{G}_C \rightarrow \text{End}^L(\mathcal{H}_*)^{s_1}$, as desired. $\Box$

**Remark 8.25.** Let us observe the diagram in Theorem 8.23. Roughly, the upper row has a moduli-theoretic description (Remark 8.21 and Remark 8.21), while the lower row is defined in terms of (Lie) algebras (Proposition 8.4 and Proposition 8.15). Suppose that we are given a deformation $\mathcal{C}_R \in \text{Def}(\mathcal{C}_R)$ of $\mathcal{C}$ to $R \in \text{Art}_{A^\text{Az}}^{\infty}$ (informally, we think of $\mathcal{C}_R$ as an object of $\mathcal{S}_R$). Taking into account the modular interpretations of $\mathcal{F}_{A\oplus \text{End}(\mathcal{H}_*(C/A))}$ and $\mathcal{F}_{\text{End}^L(\mathcal{H}_*(C/A))}^{s_1}$ (Lemma 8.13), the images of $\mathcal{C}_R$ under the above diagram can informally be depicted as follows:

$$\mathcal{C}_R \xrightarrow{\text{cyclic deformation}} \mathcal{H}_*(\mathcal{C}_R/C/A) \xrightarrow{\text{equivariant deformation}} \mathcal{H}_*(\mathcal{C}_R/C/R)$$

$\{\mathbb{D}_\infty(R) \xrightarrow{p} \mathcal{G}_C\} \xrightarrow{\{\mathbb{D}_\infty(R)^{s_1} \xrightarrow{\hat{A}^{L}_C}_{\text{op}} \text{End}^L(\mathcal{H}_*(C/A))\}} \{\mathbb{D}_\infty(R) \xrightarrow{\hat{A}^{L}_C}_{\text{op}} \text{End}^L(\mathcal{H}_*(C/A))\}$
Here $p : \mathbb{D}_\infty(R) \to \mathcal{G}_C$ indicates the image of the deformation under $J_C$, that is an object of the space $\mathcal{F}_{\mathcal{G}_C}(R)$. The middle and right items on the lower row is defined by compositions of the sequence of morphisms $\mathbb{D}_\infty(R) \xrightarrow{\iota} \mathbb{D}_\infty(R)^{S^1} \xrightarrow{\iota^S_{S^1}} \mathcal{G}^{S^1} \xrightarrow{\Lambda_C} \text{End}_{\Lambda_C}^L(\mathcal{H}_C(A))$ in $\text{Lie}_{\Lambda_C}$ where $\iota$ is the diagonal map.

We also use $\mathcal{H}_C^*(\mathcal{C}/R) \otimes \mathcal{H}_C^*(\mathcal{C}/A) \simeq \mathcal{H}_C^*(\mathcal{C}/R)$.

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Mathematical Institute, Tohoku University, Sendai, Miyagi, 980-8578, Japan

Email address: isamu.iwanari.a2@tohoku.ac.jp