Conservation laws for linear equations on quantum Minkowski spaces

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Abstract
The general, linear equations with constant coefficients on quantum Minkowski spaces are considered and the explicit formulae for their conserved currents are given. The proposed procedure can be simplified for ∗-invariant equations.
The derived method is then applied to Klein-Gordon, Dirac and wave equations on different classes of Minkowski spaces. In the examples also symmetry operators for these equations are obtained. They include quantum deformations of classical symmetry operators as well as an additional operator connected with deformation of the Leibnitz rule in non-commutative differential calculus.

1 Introduction
The investigation of conservation laws and invariants of motion for given action or equation of motion is an important part of classical mechanics and field theory. The general problem is solved by Noether theorem in which symmetry of the action yields conserved currents and integrals of motion. When we restrict our study to linear equations there is known method of Takahashi and Umezawa [1] which allows us to construct invariants for classical field-theoretic models. We have shown previously that it can be extended to discrete and mixed models on commutative spaces [2, 3, 4]. The equations of this kind appear also in realizations of generators and Casimir operators of quantum algebras on commutative spaces, namely for κ-deformed algebras [5, 6, 7]. The aim of this paper is to extend this method to linear equations on quantum Minkowski

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spaces. We have chosen to work within the framework of class of Minkowski spaces endowed with the action of quantum Poincaré groups, which were introduced by Woronowicz and Podleś in [8, 9, 10] and use the differential calculus developed by Podleś in [11]. Let us however notice that once the explicit formula of Leibnitz rule for exterior derivatives for other quantum spaces and differential calculi is given, one can extend the proposed procedure. The example of this type is the braided differential calculus on q-Minkowski space introduced in the work by Ogievetsky et al. [12] and formally developed by Majid [13, 14].

Let us sketch briefly the steps of the procedure:

- in derivation of the conserved currents Leibnitz rule is used. In case of non-commutative differential calculus it is deformed similarly as in the discrete calculus [2, 4]. We introduce modified Leibnitz rule

- the special operator Γ is built from derivatives acting on the right- and left-hand side. In quantum case we must use derivatives and their conjugations

- for hermitian equation operators the hermitian currents can be derived. As the explicit form of scalar product on quantum Minkowski space is not known we shall use throughout the paper the notion of *-invariant equation operator for which the construction can be simplified

- to obtain different solutions for given equation one needs its symmetry operators. Their algebras are well known in classical models; we show in examples the symmetry operators for Klein-Gordon, Dirac and wave equations without discussion of their algebraic properties. It was however shown in example of Klein-Gordon equation on quantum Minkowski space with $Z = 0$ that the algebra closes [15].

## 2 Modification of Leibnitz rule in differential calculi on quantum Minkowski spaces

In the paper we shall work within the framework of differential calculi on quantum Minkowski spaces, in general case introduced and investigated in [11]. Let us remind the fundamental rules of commutation for partial derivatives and variables:

\[
\partial^i (x^j) = g^{ji} \\
\partial^i \partial^k = R_{ij}^{kl} \partial^j \partial^l \\
\partial^i x^k = g^{ik} + (R_{ik}^{ab} x^a - (RZ)_{ik}^b) \partial^b \\
(R - 1)^{ij}_{kl} [x^k x^l - Z_{kl}^{ab} x^a + T^{kl}] = 0
\]

where R-matrix fulfills quantum Yang-Baxter equation $R_{23}R_{12}R_{23} = R_{12}R_{23}R_{12}$ and the condition $R^2 = 1$. The metric tensor $g$ appearing in the above formulae is R-symmetric, that means $Rg = g$.

The functions on quantum Minkowski space are understood as formal power series of monomials of variables. For the product of two such arbitrary functions the partial derivatives
obey the following Leibnitz rule [11]:
\[ \partial^i fg = (\partial^i f)g + (\zeta_j^i f)\partial^j g \]  
(5)
with the transformation operator \( \zeta \) fulfilling the equality:
\[ \zeta_j^i (fg) = (\zeta_j^m f)(\zeta_m^n g) \]  
(6)
and connected with the operator \( \rho \) from [11] via the equation:
\[ \zeta_j^i = g^{ia} \rho_a g_{bj} \]  
(7)
Formula (3) yields the explicit form of transformation operator acting on the monomial of the first order:
\[ \zeta_j^i x^k = R_{aj}^{ik} x^a - (RZ)^{ik}_j \]  
(8)
and using (6) can be easily extended to arbitrary function on quantum Minkowski space. It is interesting point that (4) implies analogous Leibnitz rule to be valid also for variables, namely:
\[ x^i (fg) = (x^i f)g = (x^i f)act g + (\tilde{\zeta}^i_j f)x^j g \]  
(9)
where the action of variable on monomial is due to the selfinteraction of variables, is determined by properties of quantum Minkowski space (8) and looks as follows:
\[ (x^i[k_1,...,k_n])act = \sum_{l=1}^{n} (\tilde{\zeta}^i_j [k_1,...,k_{l-1}])v^{jk_l} [k_{l+1},...,k_n] \]  
(10)
The initial and last term are of the form:
\[ l = 1 \quad v^{ik_1} [k_2,...,k_n] \quad l = n \quad \tilde{\zeta}^i_j [k_1,...,k_{n-1}]v^{jk_n} \]  
(11)
and appearing in the formulae (3,10) transformation operator \( \tilde{\zeta} \) is defined for arbitrary function by its action on \( x^k \) and multiplicity property:
\[ \tilde{\zeta}_j^i x^k = R_{aj}^{ik} x^a + (Z - RZ)^{ik}_j \]  
(12)
\[ \tilde{\zeta}_j^i (fg) = (\tilde{\zeta}_j^m f)(\tilde{\zeta}_m^n g) \]  
(13)
In the case where the selfinteraction of variables determined by tensor \( v = RT - T \) vanishes, the Leibnitz rule for variables reduces to the formula:
\[ x^i (fg) = (x^i f)g = (\tilde{\zeta}_j^i f)x^j g \]  
(14)
Finally we can also write both Leibnitz rules (3,9) in the vector form:
\[ \vec{\partial} fg = (\vec{\partial} f)g + (\zeta f)\vec{\partial} g \]  
(15)
\[ \vec{x} fg = (\vec{x} f)g = (\vec{x} f)act g + (\zeta f)\vec{x} g \]  
(16)
In the next sections we shall consider linear equations of motion and derive for them the conservation laws. We have solved similar problem in discrete models on commutative spaces [2, 3, 4]. The common feature of discrete and non-commutative models is the
deformation of Leibnitz rules for partial derivatives. In both cases the transformation operators appear on the right-hand side of formulae. Their form is different, depending on the kind of model - in discrete models it is simply shift operator in given direction while on quantum spaces it is described by \((6,8)\). Thus the main obstacle in extending the Takahashi-Umezawa method is the fact that one of the operators on the right-hand side of \((5)\) acts simultaneously on the first and second function in the product. In the discrete case we modified Leibnitz rule using the inverse transformation operator which was simply the back-shift operator on the lattice. Investigating the special case of Klein-Gordon equation on quantum Minkowski spaces with \(Z = 0\) we also obtained the inverse operator \(\zeta^-\), showed how it is connected with the transformation operator via \(*\)-operation and applied it to modification of Leibnitz rule. This modification is also possible in the general case which we now investigate. It is easy to conjecture from the proof of Proposition 1.1 of [11] that the operator:

\[
\zeta_j^{-1} := \ast \zeta_j^\ast
\]  

(17)
is the inverse transformation operator fulfilling:

\[
\zeta_j^{-m}\zeta_m^i = \delta_j^i
\]  

(18)
The explicit form of the inverse operator for arbitrary function results from its multiplicity property:

\[
\zeta_j^{-i}(fg) = (\zeta_j^{-m}f)(\zeta_m^{-i}g)
\]  

(19)
and from its action on monomials of the first order:

\[
\zeta_j^{-i}x^k = g^{-ki}x^a + Z^{ki} j
\]  

(20)
Now using the formula \((18)\) and changing the product on the left-hand side of Leibnitz rule \((\ref{Leibnitz})\) we obtain its modification:

\[
\partial_i[\zeta_i^{-a}fg] = (\partial_i^\ast a)f + f\partial^a g
\]  

(21)
where we use the following notation for conjugated derivative

\[
\partial_i^\ast a := -\partial_i^{-a} = -\partial_i^\ast \zeta_i^\ast
\]  

(22)
The conjugated derivative \(\partial_i^\ast\) can be described using the \(*\)-operation. Let us notice that similarly to the special case from \([15]\) the Leibnitz rules for conjugated derivative and the operator \(\ast(-\partial)\ast\) are identical:

\[
\partial_i^\ast a(fg) = (\partial_i^{-a}f)\zeta_i^{-a}g + f\partial_i^{-a}g
\]  

(23)
\[
\ast(-\partial^a) \ast (fg) = (-\ast \partial^{-i} f)\zeta_i^{-a}g + f(-\ast\partial^a \ast g)
\]  

(24)
It is easy to check that the action of both operators on monomials of the first order is the same:

\[
\partial_i^{-a}x^k = -g^{ka}
\]  

(25)
\[
\ast(-\partial^a) \ast x^k = -g^{ka}
\]  

(25)

To this aim we have used the definition of \(\partial_i^\ast\) given in \((24)\) and the property of metric tensor \(g^{ij}\) from \([11]\).

Now Leibnitz rules for both operators \((23, 24)\) and their equality for monomials of the first order imply the identity of \(\partial_i^{-a}\) and \(\ast(-\partial^a)\ast\) for arbitrary monomials by virtue of mathematical induction principle. Therefore they act in the same way on all functions on quantum Minkowski space, thus are identical:

\[
\partial_i^{-a} = \ast(-\partial^a)\ast
\]  

(26)
3 The conservation laws for linear equations of motion on quantum Minkowski spaces

3.1 Equations of motion on quantum Minkowski space

Lately a number of equations of motion on quantum spaces were studied in the literature. They include the Klein-Gordon and Dirac equations and their solutions investigated by Podleś [11] as well as equations considered on q-Minkowski space from [16, 17, 18, 19, 20, 21] built within the framework of braided differential calculus [12, 13, 14, 23]. In addition some quantum models on non-commutative spaces, in which deformation of commutation relations is motivated by Heisenberg principle and classical gravity, were studied by Doplicher et al. in [24, 25].

In this section we shall consider general, linear equation on quantum Minkowski space defined by (4). These operators include the Klein-Gordon and Dirac operators from [11]. We construct the operator of equation using partial derivatives fulfilling (2,3) in the following way:

\[ \Lambda(\partial)\Phi = 0 \]  (27)

\[ \Lambda(\partial) = \Lambda_0 + \sum_{l=1}^N \Lambda_{\mu_1...\mu_l} \partial^{\mu_1}...\partial^{\mu_l} \]  (28)

As the derivatives in (28) R-commute we have the following property of the coefficients (which may be matrices) with respect to permutation of indices:

\[ R^{\mu_k\mu_{k+1}}_{\nu\rho} \Lambda_{\mu_1...\mu_{k+1}...\mu_l} = \Lambda_{\mu_1...\nu\rho...\mu_l} \]  (29)

where \( l = 1, ..., N \) and \( k = 1, ..., l - 1 \); what means that they are R-symmetric with respect to permutations.

In analogy with the classical field theory we consider constant coefficients and this implies that they obey the following equations:

\[ \zeta^\mu_j \Lambda_{\mu_1...\mu_{k-1}\mu_k...\mu_l} = \Lambda_{\mu_1...\mu_{k-1}j...\mu_l} \]  (30)

\[ \partial^j \Lambda_{\mu_1...\mu_l} = 0 \]  (31)

where \( l = 1, ..., N \) \( k = 1, ..., l \) \( j = 1, 2, 3, 4 \).

Let us notice that formulae (18,30) imply also:

\[ \zeta^\mu_j \Lambda_{\mu_1...\mu_{k-1}\mu_k...\mu_l} = \Lambda_{\mu_1...\mu_{k-1}j...\mu_l} \]  (32)

for \( l = 1, ..., N \) \( k = 1, ..., l \) \( j = 1, 2, 3, 4 \) while (26, 31) and the condition of \( * \)-invariance of the equation operator (65) yields for \( * \)-invariant equations:

\[ \partial^j \Lambda_{\mu_1...\mu_l} = 0 \]  \( l = 1, ..., N \) \( j = 1, 2, 3, 4 \)  (33)

3.2 The operators \( \Gamma \) and \( \hat{\Gamma} \)

In order to derive the conservation law for equation (27) we need an operator \( \Gamma \) which in the classical procedure of Takahashi-Umezawa fulfills the equality:

\[ \sum_{\mu} (\hat{\partial}^\mu + \partial^\mu)\Gamma_\mu(\partial, \hat{\partial}) = \Lambda(\partial) - \Lambda(-\hat{\partial}) \]  (34)
In the above formula the partial derivatives obey the rule of classical differential calculus, so derivatives acting on the left- and right-hand side commute. This is not the case in non-commutative differential calculi, in which the derivatives do not commute according to (2). Additionally we shall deal with conjugated derivatives introduced by modification of Leibnitz rule. The set of derivatives and their conjugations in the sense of (22) becomes commutative only in special case $R = \tau$.

As we consider the general case we should replace the equality (34) with the following condition for the operator $\Gamma$:

$$\sum_{\mu} (-\frac{\epsilon_{\mu}}{\partial} + \partial^\mu) \circ \Gamma_{\mu}(\partial, \frac{\epsilon_{\mu}}{\partial}) = \Lambda(\partial) - \Lambda(\frac{\epsilon_{\mu}}{\partial}) \quad (35)$$

where the operator $\Lambda(\frac{\epsilon_{\mu}}{\partial})$ looks as follows:

$$\Lambda(\frac{\epsilon_{\mu}}{\partial}) = \Lambda_0 + \sum_{l=1}^{N-1} \sum_{k=0}^{l} \frac{\epsilon_{\mu}}{\partial} \partial^{\mu_1} ... \partial^{\mu_l} \Lambda_{\mu_1...\mu_l} \quad (36)$$

We introduced the notation for the product "$\circ$" to underline the way it acts on monomials of derivatives:

$$(-\frac{\epsilon_{\mu}}{\partial} + \partial^\mu) \circ [\nu_1, ..., \nu_l]a(x)[\rho_1, ..., \rho_k] := -[\nu_1, ..., \nu_l, \mu]\partial^\mu a(x)[\rho_1, ..., \rho_k] + [\nu_1, ..., \nu_l]\partial^\mu a(x)[\rho_1, ..., \rho_k] \quad (37)$$

where we have denoted the monomials of derivatives as follows:

$$[\rho_1, ..., \rho_k] := \partial^{\rho_1} ... \partial^{\rho_k} \quad (38)$$

$$[\nu_1, ..., \nu_l] := \frac{\epsilon_{\nu_1}}{\partial} ... \frac{\epsilon_{\nu_l}}{\partial} \quad (39)$$

**Proposition 3.1** The unique solution of (35) in class of polynomials of derivatives $\frac{\epsilon_{\mu}}{\partial}$ and $\partial$ is of the form:

$$\Gamma_{\mu}(\partial, \frac{\epsilon_{\mu}}{\partial}) = \Lambda_{\mu} + \sum_{l=1}^{N-1} \sum_{k=0}^{l} \frac{\epsilon_{\mu}}{\partial} \partial^{\mu_1} ... \partial^{\mu_k} \Lambda_{\mu_1...\mu_k\mu_{k+1}...\mu_l} \partial^{\mu_{k+1}} ... \partial^{\mu_l} \quad (40)$$

**Proof:**

The technique we use is very similar to that applied in proof of analogous proposition for discrete models [3]. We denote the monomials of derivatives as described above (38,39). Now the modified Leibnitz rule (21) implies that in order to solve (35) we should consider the general polynomial of order N-1 with functional coefficients of the following form:

$$\Gamma_{\mu}(\partial, \frac{\epsilon_{\mu}}{\partial}) = a_{\mu}^0 + \sum_{l=1}^{N-1} \sum_{k=0}^{l} \frac{\epsilon_{\mu}}{\partial} \partial^{\mu_1} ... \partial^{\mu_k} a_{\mu_1...\mu_k\mu_{k+1}...\mu_l}[\mu_{k+1}, ..., \mu_l] \quad (41)$$
We apply the condition (35) to the general from of the solution (41) to derive the equations for coefficients $a_{\mu_{1}\ldots\mu_{l}}^{k}$:

$$\sum_{\mu}(-\partial^{\mu} + \partial^{\mu}) \circ \Gamma_{\mu}(\partial, \partial) = (42)$$

$$-\sum_{l=1}^{N-1} \sum_{k=0}^{l} \sum_{\mu} [\mu_{1}, \ldots, \mu_{k}, \mu] a_{\mu_{1}\ldots\mu_{l}}^{k} [\mu_{k+1}, \ldots, \mu]$$

$$+ \sum_{l=1}^{N-1} \sum_{k=0}^{l} [\mu_{1}, \ldots, \mu_{k}] \sum_{\mu} (\partial^{\mu} a_{\mu_{1}\ldots\mu_{l}}^{k}) [\nu, \mu_{k+1}, \ldots, \mu]$$

$$+ \sum_{l=1}^{N-1} \sum_{k=0}^{l} [\mu_{1}, \ldots, \mu_{k}] \sum_{\mu} (\partial^{\mu} a_{\mu_{1}\ldots\mu_{l}}^{k}) [\mu_{k+1}, \ldots, \mu]$$

$$- \sum_{\mu} [\mu] a_{\mu}^{0} + \sum_{\mu} (\zeta^{\mu}_{\nu} a_{\mu}^{0}) [\nu] + \sum_{\mu} (\partial^{\mu} a_{\mu}^{0}) =$$

$$\Lambda(\partial) - \Lambda(\partial^{\dagger})$$

We compare the coefficients of monomials of the same type on both sides of the above condition and obtain the following set of equations for functions $a_{\mu_{1}\ldots\mu_{l}}^{k}$:

$$a_{\mu}^{0} = \Lambda_{\mu}$$

$$\partial^{\mu} a_{\mu_{1}\ldots\mu_{l}}^{0} + \zeta^{\mu}_{\nu} a_{\mu_{1}\ldots\mu_{l}}^{0} = \Lambda_{\mu}$$

$$- a_{\mu_{1}\ldots\mu_{l}}^{k} + \zeta^{\mu}_{\nu} a_{\mu_{1}\ldots\mu_{l}}^{k+1} + \partial^{\mu} a_{\mu_{1}\ldots\mu_{l}}^{k+1} = 0$$

with $l = 1, \ldots, N-1$, $k = 0, \ldots, l-1$.

Starting from (43,44,45) we get the unique solution for coefficients $a_{\mu_{1}\ldots\mu_{l}}^{0}$:

$$a_{\mu_{1}\ldots\mu_{l}}^{0} = \Lambda_{\mu_{1}\ldots\mu_{l}}$$

This R-symmetric, constant in the sense of (30,31) solution for initial coefficients allows us to evaluate the remaining ones using (46).

We start from equations for $l = N-1$ and conclude that they reduce to the following set:

$$- a_{\mu_{1}\ldots\mu_{N-1}}^{k} + \zeta^{\mu}_{\nu} a_{\mu_{1}\ldots\mu_{N-1}}^{k+1} = 0$$

As we know the explicit expressions for $a_{\mu_{1}\ldots\mu_{N}}^{0}$ from (48) we are able to derive from the above equation and from the properties of coefficients of the equation (30,31) the solution for $a_{\mu_{1}\ldots\mu_{N}}^{1}$ which has the following form:

$$a_{\mu_{1}\ldots\mu_{N}}^{1} = \Lambda_{\mu_{1}\ldots\mu_{N}}$$

Using the explicit form of coefficients of type $a_{\mu_{1}\ldots\mu_{N}}^{1}$ we perform the next step of solution of the set (49) and solve it for $k = 1$ applying the same method as before, namely rewriting
the set of equations, putting the solution (50) into it and then using (32). After subsequent calculations for \( k = 1, ..., N - 1 \) we obtain the general form of coefficients \( a_{\mu_1, ..., \mu_N}^k \):

\[
a_{\mu_1, ..., \mu_N - 1, \mu_1, ..., \mu_N - 1}^k = \Lambda_{\mu_1, ..., \mu_k, \mu_{k+1}, ..., \mu_{N-1}}
\]

(51)

Now from the form of coefficients \( a_{\mu_1, ..., \mu_N}^k \) we conclude that they all fulfill the condition (31) so for \( l = N - 2 \) we also obtain the set of equations in which the part with the full divergence vanishes:

\[
\begin{align*}
-a_{\mu_1, ..., \mu_{N-2}}^k + \zeta_{\mu_{k+1}} a_{\mu_1, ..., \mu_{k+1}, \mu_{k+2}, ..., \mu_{N-2}}^{k+1} &= 0 \\
0 &\leq k < N - 2
\end{align*}
\]

(52)

The method of solving equations (52) follows the calculations done for \( l = N - 1 \). We start from the known coefficient of type \( a^0 \) described by (48) and derive the remaining ones using the subsequent equations from (52).

The result as before are expressions constant in the sense of (30,31) of the form:

\[
a_{\mu_1, ..., \mu_{N-2}}^k = \Lambda_{\mu_1, ..., \mu_k, \mu_{k+1}, ..., \mu_{N-2}}
\]

(53)

It is obvious that the next steps are analogous so we shall not present them in detail.

The general and unique solution of the set (43 - 47) looks as follows:

\[
a_{\mu_1, ..., \mu_l}^k = \Lambda_{\mu_1, ..., \mu_k, \mu_{k+1}, ..., \mu_l}
\]

(54)

The derivation of the explicit formulae for unique solution of the coefficients of the operator \( \Gamma_{\mu} \) concludes the proof of Proposition 3.1.

From the above proof of Proposition 3.1 we conclude that the unique solution of the equation (35) can also be derived for equations in which the coefficients obey the requirement (30), but the condition (31) is weakened as follows:

\[
\sum_{\mu_k} \partial^{\mu_k} \Lambda_{\mu_1, ..., \mu_k, \mu_{k+1}, ..., \mu_l} = 0 \quad l = 1, ..., N \quad k = 1, ..., l
\]

(55)

**Corollary 3.2** The unique solution of (35) in class of polynomials of derivatives \( \partial^{\downarrow} \) and \( \partial \) for the equation operator \( \Lambda \) fulfilling (30,55) is of the form:

\[
\Gamma_{\mu}(\partial, \partial^{\downarrow}) = \Lambda_{\mu} + \sum_{l=1}^{N-1} \sum_{k=0}^{l} \partial^{\downarrow} \mu_1 \cdots \partial^{\downarrow} \mu_k \Lambda_{\mu_1, ..., \mu_k, \mu_{k+1}, ..., \mu_l} \partial^{\mu_{k+1}} \cdots \partial^{\mu_l}
\]

(56)

In contrast with classical case where it is sufficient to know the operator \( \Gamma_{\mu} \) to construct the conserved currents, we must additionally modify \( \Gamma_{\mu} \) due to the deformation of Leibnitz rule (5).

We introduce the operator \( \hat{\Gamma}_{\mu} \) in the form:

\[
\hat{\Gamma}_{\mu}(\partial, \partial^{\downarrow}) = \zeta_{\mu} \Lambda_{\mu} + \sum_{l=1}^{N-1} \sum_{k=0}^{l} \partial^{\downarrow} \mu_1 \cdots \partial^{\downarrow} \mu_k \zeta_{\mu} \Lambda_{\mu_1, ..., \mu_k, \mu_{k+1}, ..., \mu_l} \partial^{\mu_{k+1}} \cdots \partial^{\mu_l}
\]

(57)

As we see the modification consists of introducing the inverse transformation operator \( \zeta^{\downarrow} \) in the monomials between derivatives acting on the left- and right-hand side.
3.3 The conservation laws and conserved currents

In this section we derive the conservation laws for linear equations with constant coefficients described by (27,28). To this aim it is necessary to have the solutions of initial equation and of its conjugation (22). Having these two functions we can formulate the following proposition which describes the explicit form of conserved currents for equation (27).

Proposition 3.3 Let us assume that function $\Phi$ is an arbitrary solution of equation (27) with coefficients fulfilling (30,31), that means:

$$\Lambda(\partial)\Phi = 0$$

(58)

and function $F$ solves the conjugated equation:

$$FA(\partial)\Phi = 0$$

(59)

Then

$$J_\mu = F\hat{\Gamma}_\mu(\partial, \partial^{\dagger})\Phi$$

(60)

where the operator $\hat{\Gamma}_\mu$ is defined by (57), is a current which obeys the conservation law in given differential calculus on quantum Minkowski space:

$$\sum_\mu \partial^\mu J_\mu = 0$$

(61)

Proof:

In the straightforward proof of the proposition we use the modified Leibnitz rule (21) and the properties of operator $\Gamma_\mu$ (35) as well as properties of coefficients of equation (30,31):

$$\sum_\mu \partial^\mu J_\mu =$$

(62)
\[ F \left( \Lambda(\partial) - \Lambda(\partial^\dagger) \right) \Phi = 0 \]

Thus the conservation law for arbitrary linear equation with constant coefficients is valid provided functions \( F \) and \( \Phi \) are solutions of corresponding equations.

**Corollary 3.4** If function \( \Phi \) is an arbitrary solution of equation (27) with coefficients fulfilling (30), and function \( F \) solves its conjugation (22):

\[ F(\Lambda(\partial)\Phi) = 0 \]

then the current of the form:

\[ J_\mu = F\hat{\Gamma}_\mu(\partial, \partial^\dagger)\Phi \]

with \( \hat{\Gamma} \) given by (54) is conserved:

\[ \sum \partial_\mu J_\mu = 0 \]

**Proof:**

It results from the proof of the Proposition 3.3 and from Corollary 3.2 describing operator fulfilling (34) for equations obeying weakend condition (30). Let us observe that in the special case of Klein-Gordon equation on quantum Minkowski space with \( Z = 0 \) we were able to connect the solution of conjugated equation \( F \) with \( \Phi^\ast \)-transformation of \( \Phi \). This possibility was due to the fact that Klein-Gordon operator is an \( \ast \)-invariant one. We shall now check the conditions of \( \ast \)-invariance for the operator of equation of the form (27). Taking into account \((i\partial)^\ast = i\partial\) from (34) we see that after \( \ast \)-operation we obtain:

\[ \Lambda(\partial)^\ast = \Lambda_0^\ast + \sum_{l=1}^N (\partial^{\mu_1})^\ast \cdots (\partial^{\mu_l})^\ast \Lambda^\ast_{\mu_1 \cdots \mu_l} = \]

\[ = \Lambda_0^\ast + \sum_{l=1}^N (-1)^l \partial^{\mu_1} \cdots \partial^{\mu_l} \Lambda^\ast_{\mu_1 \cdots \mu_l} = \Lambda(\partial) \]

Comparing the coefficients of the \( \Lambda(\partial)^\ast \) with coefficients of the initial operator \( \Lambda(\partial) \) we conclude that the following proposition is valid:

**Proposition 3.5** The operator of equation (27) is \( \ast \)-invariant:

\[ \Lambda(\partial)^\ast = \Lambda(\partial) \]

iff the coefficients fulfill the conditions:

\[ \Lambda_0^\ast = \Lambda_0 \quad \Lambda^\ast_{\mu_1 \cdots \mu_l} = \Lambda_{\mu_1 \cdots \mu_l}(-1)^l \quad l = 1, \ldots, N \]
For equations fulfilling $*$-invariance condition we can express the solution of conjugated equation (36) in terms of solutions of initial equation (27,28). Therefore the conserved currents for such equations can be constructed using only the latter solution.

**Proposition 3.6** For linear equation with constant coefficients fulfilling the conditions (30,31,65) the current of the form:

$$ J_\mu = \Phi \hat{\Gamma}_\mu(\partial, \partial^\dagger)\Phi $$

where $\Phi$ is an arbitrary solution of (27,28) is conserved:

$$ \sum_\mu \partial^\mu J_\mu = 0 $$

**Proof:**

The Proposition 3.3 implies that we need only to show:

$$ \Phi^* \Lambda(\partial^\dagger) = 0 $$

provided $\Phi$ is an arbitrary solution of $\Lambda(\partial)\Phi = 0$. Let us check it using the property of conjugated derivative (26):

$$ \Phi^* \Lambda(\partial^\dagger) = $$

$$ \sum_{l=0}^N \Phi^* \frac{\partial^\mu_1}{\partial} \cdots \frac{\partial^\mu_l}{\partial} \Lambda_{\mu_1 \cdots \mu_l} = $$

$$ \sum_{l=0}^N \Phi^*(-1)^l \frac{\partial}{\partial} \cdots \frac{\partial}{\partial} \Lambda_{\mu_1 \cdots \mu_l} = $$

$$ \sum_{l=0}^N (-1)^l \Lambda_{\mu_1 \cdots \mu_l} \Phi(-1)^l \frac{\partial}{\partial} \cdots \frac{\partial}{\partial} \frac{\partial^\mu_1}{\partial} \cdots \frac{\partial^\mu_l}{\partial} \Phi = 0 $$

When two solutions of equation of motion (27) are known one can construct the current according to the following corollary which is the result of Propositions 3.3 and 3.6:

**Corollary 3.7** For linear equation with coefficients fulfilling (30,31,65) the current of the form:

$$ J_\mu = i(\Phi')^* \hat{\Gamma}_\mu(\partial, \partial^\dagger)\Phi - i\Phi^* \hat{\Gamma}_\mu(\partial, \partial^\dagger)\Phi' $$

where $\Phi$ and $\Phi'$ are arbitrary solutions of (27,28) is conserved:

$$ \sum_\mu \partial^\mu J_\mu = 0 $$
4 Applications

We have developed simple method of derivation of conserved currents for linear equations on a class of quantum Minkowski spaces. This procedure can be applied to equations with coefficients constant in the sense of (30,31) or fulfilling weakened conditions (30,55). Now we shall apply the presented technique to a few equations on different quantum Minkowski spaces. Some of these equations were studied earlier in [11] where also their solutions were constructed.

Following the classical field theory we shall obtain different solutions of equation of motion using the symmetry transformation operators. In the examples we show that they are quantum deformations of classical operators plus the transformation operator (6,8). The algebraic and possible co-algebraic properties of the set of symmetry operators are still to be investigated, nevertheless we wish to point out that in the special case studied in [15] they form closed algebra and we hope to obtain their co-algebraic structure from Leibnitz rules of symmetry operators determined by Leibnitz rules for derivatives and variables (34).

4.1 Klein-Gordon equation

Klein-Gordon equation on quantum Minkowski space in the sense of (4) was introduced by Podleś in [11] where also its solutions were studied. It looks as follows:

\[(\Box + m^2)\Phi = 0\]  
(72)

with d’Alembert’s operator built using exterior partial derivatives from non-commutative differential calculus (2,3,4):

\[\Box = g_{ab}\partial^a\partial^b = \partial^a\partial^b g_{ab}\]  
(73)

The consistency conditions which allow us to write coefficients of equation in front or after the differential operators coincide with requirements studied in the previous section (30,31).

In our construction we shall consider currents connected with symmetry transformations of solutions of Klein-Gordon equation. The special case for \(Z = 0\) was solved earlier in [15] where also algebraic properties of symmetry transformation operators were investigated.

Let us now assume that \(R, Z\) and \(T\) are arbitrary matrices and tensors allowed in calculus on quantum Minkowski space and check the commutator of d’Alembert’s operator with variable \(x^k\):

\[[\Box, x^k] = 2\partial^k\]  
(74)

Taking into account the property (2) rewritten as follows:

\[(R - 1)^{kl}_{\ \ab}\partial^a\partial^b = 0\]

we can easily construct the symmetry transformation operator analogous to angular momentum operator from classical field theory:

\[M^{kl} = i(R - 1)^{kl}_{\ \ab}x^a\partial^b\]  
(75)
As these operators commute with Klein-Gordon equation operator they transform solution of Klein-Gordon equation into another solution. The set of symmetry operators can be also completed with momentum operators \( [11] \):

\[
P^l = i\partial^l
\]

In addition, using properties of the transformation operator \( \zeta \) given in Proposition 3.1 of [11] one concludes that there is an additional symmetry operator for Klein-Gordon equation:

\[
[\Box, \zeta^a_b] = 0
\]

due to the property of \( R \)-matrix:

\[
g^{ab}R^{dc}_{ba} = R^{jd}_{ak}g^{kc}
\]

Now we should construct the \( \hat{\Gamma} \) operator for our equation using the general formula (57). It has the form identical with the one obtained in [15]:

\[
\hat{\Gamma}_\mu (\partial, \partial^{-\dagger}) = \partial - g_{aj} \zeta^-_{\mu j} + g_{ja} \partial^a
\]

The results (70,71) presented earlier imply that the currents:

\[
\begin{align*}
J_{kl}^\mu &= i\Phi^* \hat{\Gamma}_\mu (\partial, \partial^{-\dagger}) M^{kl}\Phi - i(M^{kl}\Phi)^* \hat{\Gamma}_\mu (\partial, \partial^{-\dagger}) \Phi \\
J_{k}^\mu &= i\Phi^* \hat{\Gamma}_\mu (\partial, \partial^{-\dagger}) P^k\Phi - i(P^k\Phi)^* \hat{\Gamma}_\mu (\partial, \partial^{-\dagger}) \Phi \\
J_{a}^{\mu b} &= i\Phi^* \hat{\Gamma}_\mu (\partial, \partial^{-\dagger}) \zeta^a_b \Phi - i(\zeta^a_b \Phi)^* \hat{\Gamma}_\mu (\partial, \partial^{-\dagger}) \Phi
\end{align*}
\]

are conserved quantities by virtue of Corollary 3.7:

\[
\sum_\mu \partial^\mu J_{kl}^\mu = 0 \quad \sum_\mu J_{k}^\mu = 0 \quad \sum_\mu \partial^\mu J_{a}^{\mu b} = 0
\]

### 4.2 Dirac equation on quantum Minkowski space with \( R = \tau \)

Let us remind the Dirac equation from [11]:

\[
(i\gamma^\mu \partial^\mu + m)\Psi = 0
\]

In our case when \( R = \tau \) the \( \gamma \) matrices fulfill the following condition:

\[
\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}
\]

Similarly to the Klein-Gordon operator the coefficients \( \gamma_\mu \) (which are now matrices) obey the consistency conditions ([30],[31]).

For the Dirac equation we know symmetry operators analogous to momentum from [11]:

\[
P^l = i\partial^l
\]
In order to construct the angular momentum operator we check the commutator of the Dirac operator with the scalar angular momentum (75):
\[
[i\gamma_{\mu}\partial^{\mu}, M^{kl}] = i\gamma_{\rho}\delta^{\rho k}\partial^{l} - i\gamma_{\rho}\delta^{\rho l}\partial^{k} + i\gamma_{\rho}Z_{\mu}^{jk}\partial^{\mu}\partial^{\rho}\partial^{l} \quad (86)
\]

The additional term appearing on the right-hand side implies that the spinorial part of the angular momentum must be extended:
\[
M_{spin}^{kl} = ix^{k}\partial^{l} - ix^{l}\partial^{k} + \frac{1}{2}(\gamma^{k}, \gamma^{l}) + \frac{1}{2}(Z_{\rho}^{k}\partial^{l} - Z_{\rho}^{l}\partial^{k})\gamma^{\rho} \quad (87)
\]

This symmetry operator contains the scalar part and spin part built using Dirac matrices of identical form as in the classical commutative models and is extended with part depending on $Z$ matrices due to commutation rule (3).

Following the general method we construct the $\hat{\Gamma}$ operator for (83) using (57):
\[
\hat{\Gamma}_{\mu}(\partial, \partial) = i\zeta_{\mu} - j_{\mu}\gamma_{j} \quad (88)
\]

Thus the currents built using symmetry operators (85,87) according to the Corollary 3.7:
\[
J_{\mu}^{kl} = i\Psi^{\ast} \zeta_{\mu} - j_{\mu}\gamma_{j} \quad (89)
\]
\[
J_{\mu}^{l} = i\Psi^{\ast} \zeta_{\mu} - j_{\mu}\gamma_{j} \quad (90)
\]

are conserved:
\[
\sum_{\mu}\partial^{\mu}J_{\mu}^{kl} = 0 \quad \sum_{\mu}J_{\mu}^{l} = 0 \quad (91)
\]

Let us notice that we have not included the symmetry operator connected with the transformation operator because in the case when $R = \tau$ it can be expressed by momenta (85).

4.3 Wave equation on quantum Minkowski spaces with $Z = 0$

Let us now assume that the mass in the Klein-Gordon equation (72) is equal to zero. The result is the wave equation of the form:
\[
\square \Phi = 0 \quad (92)
\]

We shall study the symmetry operators for class of Minkowski spaces (4) with $Z = 0$. Similarly to the classical field theory the set of symmetry operators of the Klein-Gordon equation can be extended by additional operators [27].

It is easy to check the following commutation relations:
\[
[\square, \frac{1}{2}g_{ab}(R_{\mu}^{ab} - \delta_{\mu}^{ab}) x^{k}\partial^{l}] = 2\square \quad (93)
\]
\[
[\square, \frac{1}{2}g_{ab}(R_{\mu}^{ab} - \delta_{\mu}^{ab}) x^{k} x^{l}] = 2g_{ab}g_{\mu} + 2g_{ab}(R_{\mu}^{ab} - \delta_{\mu}^{ab}) x^{k}\partial^{l} \quad (94)
\]
Let us denote:
\[ D := \frac{i}{2} g^{ab} (R^{ab}_{\; kl} + \delta^{ab}_{\; kl}) x^k \partial^l \]  
\[ \hat{x}^2 := \frac{1}{2} g^{ab} (R^{ab}_{\; kl} + \delta^{ab}_{\; kl}) x^k x^l \]  
These operators allow us to construct additional symmetry operators, namely:
- the dilatation operator \( D \) given by (95)
- the conformal boosts \( K^m \):
\[ K^m = i \hat{x}^2 \partial^m - 2 Dx^m \]  
Acting on arbitrary solution of the wave equation (92) they produce another solution of this equation:
\[ \Box D \Phi = (D + 2i) \Box \Phi = 0 \]
\[ \Box K^m \Phi = (K^m - 2ix^m) \Box \Phi = 0 \]

The \( \hat{\Gamma}_\mu \) operator for wave equation is given by (78)
\[ \hat{\Gamma}_\mu (\partial, \overset{\leftrightarrow}{\partial}) = \overset{\leftrightarrow}{\partial} a \ g_{aj} \overset{\leftrightarrow}{\partial} j - \overset{\leftrightarrow}{\partial} j \ g_{ja} \partial^a \]  
Using this operator and symmetry operators from the set:
\[ P^l = i \overset{\leftrightarrow}{\partial}^l \]
\[ M^{kl} = i (R - 1)^{kl}_{\; ab} x^a \partial^b \]
\[ \zeta^a_b = \frac{1}{2} g^{ab} (R + 1)^{ab}_{\; kl} x^k \partial^l \]
\[ D = i \hat{x}^2 \partial^m - 2 Dx^m \]  
we can construct the full set of conserved currents for wave equation:
\[ J^{kl}_\mu = i \Phi * \hat{\Gamma}_\mu (\partial, \overset{\leftrightarrow}{\partial}) M^{kl} \Phi - i (M^{kl} \Phi) * \hat{\Gamma}_\mu (\partial, \overset{\leftrightarrow}{\partial}) \Phi \]  
\[ J^l_\mu = i \Phi * \hat{\Gamma}_\mu (\partial, \overset{\leftrightarrow}{\partial}) P^l \Phi - i (P^l \Phi) * \hat{\Gamma}_\mu (\partial, \overset{\leftrightarrow}{\partial}) \Phi \]  
\[ J^a_{\mu \; b} = i \Phi * \hat{\Gamma}_\mu (\partial, \overset{\leftrightarrow}{\partial}) \zeta^a_b \Phi - i (\zeta^a_b \Phi) * \hat{\Gamma}_\mu (\partial, \overset{\leftrightarrow}{\partial}) \Phi \]  
\[ J^D_\mu = i \Phi * \hat{\Gamma}_\mu (\partial, \overset{\leftrightarrow}{\partial}) D \Phi - i (D \Phi) * \hat{\Gamma}_\mu (\partial, \overset{\leftrightarrow}{\partial}) \Phi \]  
\[ J^I_\mu = i \Phi * \hat{\Gamma}_\mu (\partial, \overset{\leftrightarrow}{\partial}) K^I \Phi - i (K^I \Phi) * \hat{\Gamma}_\mu (\partial, \overset{\leftrightarrow}{\partial}) \Phi \]  
which are conserved according to Corollary 3.7.
Similarly to Klein-Gordon and Dirac equation in the case \( R = \tau \) the transformation operator \( \zeta \) is an symmetry operator described by momenta, so it can be then excluded from the set of independent symmetry operators.
5 Final remarks

The presented extension of Takahashi-Umezawa procedure gives explicit formulae for construction of conserved currents for linear equations of motion on quantum Minkowski spaces. There is an interesting technical analogy between non-commutative differential calculus and discrete calculus on commutative spaces - the appearance of the transformation operator in Leibnitz rules, which was the main obstacle in our construction.

For all presented equations this operator becomes an additional symmetry operator. This fact is trivial for all equations fulfilling (30) with \( R = \tau \) due to Proposition 3.1 of [11]. In this case however it can be shown that the transformation operator can be expressed via momentum operator [26].

The general case must be further studied as two questions arise:
- whether the transformation operator is also the symmetry operator for a certain class of equations considered above,
- can the transformation operator be expressed by deformations of classical symmetry operators as in the case \( R = \tau \).

Let us notice that in the algebraic structure of the example studied in [15] the transformation operator is not necessary to close algebra, however without this operator one cannot close the co-algebra.

We wish to point out some other open questions.

In classical theory (as well as in discrete models [2, 3, 4]) the consequence of conservation laws for equations is existence of conserved quantities. They were constructed using integrals on continuous and discrete space-time obeying Stokes-type theorem. Once the integral calculus on quantum space-time [1] compatible with the differential calculus [2, 3] will be developed we shall be able to derive conserved quantities for arbitrary linear models.

Our aim is also the extension of the presented method to other types of non-commutative differential calculi, it should be interesting for braided differential calculus studied by Majid in [23, 13, 14]. The promising feature is existence of integral calculus which could be applied in further construction of integrals of motion [4, 28, 29].

The other interesting problem is systematic study of symmetry operators, their algebraic and co-algebraic structure for equations on quantum Minkowski spaces.

We hope to come back to these questions in the subsequent paper.

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