Soliton Solutions on Noncommutative Orbifold $T^{2N}/G$

Hui Deng *, Bo-Yu Hou †, Guo-Fang Shi ‡
Kang-Jie Shi §, Rui-Hong Yue ¶, Hua-Hui Xiong ¶

Institute of Modern Physics, Northwest University,
Xi'an, 710069, P. R. China

May 11, 2021

Abstract

In this paper, we construct the common eigenstates of "translation" operators $\{U_s\}$ and establish the generalized $Kq$ representation [1, 2] on integral noncommutative torus $T^{2N}$. We then study the finite rotation group $G$ in noncommutative space as a mapping in the $Kq$ representation and prove a Blocking Theorem. We finally obtain the complete set of projection operators on the integral noncommutative orbifold $T^{2N}/G$ in terms of the generalized $Kq$ representation. Since projectors are soliton solutions on noncommutative space in the limit $\alpha'B_{ij} \rightarrow \infty(\Theta_{ij}/\alpha' \rightarrow 0)$, we thus obtain all soliton solutions on that orbifold $T^{2N}/G$.

PACS: 11.10.Nx

Keywords: Soliton, Projection operator, Noncommutative orbifold $T^{2N}/G$.

*Email: hdeng_phy@yahoo.com.cn
†Email: byhou@nwu.edu.cn
‡Email: shiguofang@eyou.com
§Email: kjshi@nwu.edu.cn
¶Email: rhyue@nwu.edu.cn
∥Email: jimharry@eyou.com
1 Introduction

The idea about the noncommutative space-time coordinates occurred rather long ago [3]. A lot of work develops the idea extensively from the perspectives of both mathematics and physics[4, 5, 6]. In the past few years, it has been shown that some noncommutative gauge theories can be embedded in string theories[7, 8, 9]. Seiberg and Witten[9] pointed out that in the presence of constant $B$- field, the effective action of string theory can equivalently be described in terms of noncommutative gauge theory. Another intriguing finding about noncommutative field theory is the $UV/IR$ mixing arising from noncommutativity of spacetime[23, 24]. Noncommutative geometry can also be applied to condensed matter physics. The currents and density of a system of electrons in a strong magnetic field may be described by a noncommutative quantum field theory [10, 11, 12]. The connection between a finite quantum Hall system and a noncommutative Chern-Simon Matrix model first proposed by [13] was further elaborated in papers [14, 15]. Many papers are concentrated on the research for the related questions about the quantum Hall effect [16]-[22].

Solitons in various noncommutative theories have played a central role in understanding the physics of noncommutative theories and certain aspects of string theories. One of the reasons why the quantum Hall effect has received so many concerns is that the quantum Hall effect provides us with a practical objection embodying rich relations among soliton and noncommutative theory and string theory [17, 21, 22, 25]. In string theory, the existence and form of these classical solutions known as solitons are fairly independent of the details of the theory, making them useful to probe the string behavior. Soliton solutions and study of integrable systems in context of noncommutative space have attracted a lot of interests[26]-[30]. Noncommutative solitons are comprehended as D-branes in string field theory with a background $B$ field. Many of Sen’s conjectures [31, 32] regarding tachyon condensation in string field theory have been beautifully confirmed using properties of noncommutative solitons.

String theory demands that there are some compactified dimensions in the target space. Connes, Douglas and Schwartz studied the question of compactification on noncommutative tori of two kinds of Matrix models in terms of noncommutative geometry [8]. To study the soliton solution on noncommutative compactified space is instructive. Although Derrick’s theorem forbids the existence of soliton solution in 2+1 dimensional commutative scalar
field theory \cite{33}, however Gopakumar, Minwalla and Strominger found projectors can be used to construct the soliton solutions in the noncommutative space \cite{34}. Thus to study projection operators on various noncommutative space is endowed by direct physical meaning. Reiffel \cite{35} constructed the complete set of projection operators on the noncommutative torus. On the basis Boca studied the projection operators on noncommutative orbifold \cite{36} obtaining important results and showed the well-known example of projection operator for $T^2/Z_4$ in terms of the elliptic function. Soliton solutions in noncommutative gauge theory were introduced by Polychronakos in \cite{37}. Martinec and Moore in their important article deeply studied soliton solutions namely projectors on a wide variety of orbifolds, and the relation between physics and mathematics in this area \cite{38}. Gopakumar, Headrick and Spradlin gave a rather apparent method to construct the multi-soliton solution on noncommutative integrable torus with generic $\tau$ \cite{39}. We have shown projection operators of a manifest covariant form on noncommutative orbifold $T^2/Z_4$ \cite{41}.

The noncommutative spaces in string theory origin from the D-brane on which the bottom of open string moves. In general D-brane is high dimensional. So the study of soliton solutions in compactified high dimensional spaces is very helpful for us to understand the properties of D-brane. In solid physics the 3-dimensional coordinates space together with the dual momenta space constitutes a 6-dimensional noncommutative phase space. Since the spacial objects studied in solid physics possess periodicities, the phase space must be also a lattice space. There has been some work to do with the lattice model on high dimensional space. For example, Dai and Song defined hypercubic group in any dimensions and carried out detailed calculation on the structure and representation of the four dimensional cubic group $O_4$ and its double group \cite{40}.In this paper we study the general solutions of projectors in high dimensional noncommutative space $T^{2N}/G$ in the integral case. Concretely, for the case of $N = 3$, what we are studying is the projector invariant under the rotation which keeps the six-dimensional phase-space lattice invariant and might be useful in solid state physics.

This paper is organized as following: in Section 2 we define operators in the noncommutative space $T^{2N}/G$, where $G$ is a finite subgroup of rotation group $SO(2N)$. We can prove that in general the problem in high dimensional noncommutative space doesn’t degenerate into the case of direct product of two dimensions. Thus it is necessary to directly deal with the questions in the high dimensional case. In Section 3, we establish com-
mon eigenstates \( \{|\vec{B}, \vec{q}\rangle\} \) of the wrapping operators \( \{U_s\} \), namely the generalized \( Kq \) representation in the high dimensional case. In Section 4, we explore the properties of base vectors such as orthogonality, completeness and quasi-periodicity. In Section 5, we study the transformation of \( |\vec{B}, \vec{q}\rangle \) under action of rotation group \( G \) and then prove the Blocking Theorem and related propositions. Finally, in the integral case we construct the complete set of projection operators in the noncommutative orbifold \( T^{2N}/G \) in Section 6.

2 Operators On the Noncommutative \( T^{2N}/G \)

Give a set of hermiltian operators \( \{x_i\}, i = 1, 2, \cdots, 2N \)

\[
[x_i, x_j] = i\theta_{ij},
\]

where \( \{\theta_{ij}\} \) is an antisymmetric real matrix. The noncommutative space \( R^{2N} \) is formed by the Tailor series of \( x_i \). We can construct unitary operators, under action of which, \( x_j \) undergoes "translation" transformation

\[
U_s = e^{iC_{js}x_j}, \quad s = 1, 2, \cdots, 2N,
\]

\[
x_j \rightarrow x'_j = U_s^{-1}x_jU_s = x_j + d_{js},
\]

where \( d_{js} = [x_j, iC_{js}x_{j'}] = -C_{js}\theta_{jj'} \). In this paper we only consider the case in which the translations don’t degenerate, namely \( \det d_{js} \neq 0 \). We have

\[
\det \theta_{jj'} \neq 0, \quad \det C_{js} \neq 0.
\]

The "translations" in terms of \( \{U_s\} \) generate a set of lattice in the \( R^{2N} \) space. We define the set of all the Tailor series which commute with \( \{U_s\} \) as noncommutative torus \( T^{2N} \). It is verified that they are composed of the Laurent series of \( \{u_s\}, s = 1, 2, \cdots, 2N \), where

\[
u_s = e^{ic_{js}x_j}, \quad s = 1, 2, \cdots, 2N,
\]

commuting with all the \( U_s \)

\[
[U_s, u_s'] = 0, \quad s, s' = 1, 2, \cdots, 2N
\]
Now we consider a finite subgroup $G$ of $SO(2N)$, $G = \{R_k\}$:

$$R_k : x_j \rightarrow x'_j = (R_k)_{jl} x_l = R_k^{-1} x_j R_k$$

$$U_s \rightarrow U'_s = e^{iC_{js}x'_j} = e^{iC_{js}(R_k)_{jl}x_l}. \quad (8)$$

In this paper we study the case of any $s$, we always have

$$U'_s \equiv R_k^{-1}U_s R_k = \eta_{sk} \prod_{s'} (U_{s'})^{K_{ss'}} = K_{ss'} \in \mathbb{Z} \quad (integer) \quad (9)$$

$$\det(K_{ss'}) = 1.$$ 

Namely under action of $R_k$, $U_s$ changes according to an integer matrix $K$. The group $G$ doesn’t change the noncommutative torus. The set of operators which commute with $\{U_s\}$ and are invariant under action of $\{R_k\}$ forms noncommutative orbifold $T^{2N}/G$. When all the $U_s$ commute with each other, we call the noncommutative torus $T^{2N}$ integral. In this paper we study the projectors $P$ on the integral noncommutative orbifold $T^{2N}/G$, namely the operators satisfying:

$$P^2 = P, \quad (10)$$

$$[U_s, P] = 0, \quad (11)$$

$$[R_k, P] = 0. \quad (12)$$

### 3 The Common Eigenvectors of the Wrapping Operators $\{U_s\}$

In this section, we construct common eigenstates of the wrapping operators $\{U_s\}$ on the integral $T^{2N}$. Set

$$C_{js}x_j = y_s, \quad (13)$$

we have

$$U_s = e^{iy_s}. \quad (14)$$

Due to the mutual commutativity of $\{U_s\}$, we get

$$[y_s, y_{s'}] = 2\pi i l_{ss'}, \quad (15)$$
where \( l_{ss'} \) is an integer number. It can be verified that in the case of \( \det(l_{ss'}) \neq 0, \ l_{ss'} \in \mathbb{Z} \), which is the consequence of (4), we can always expand \( y_s \) as the linear combination of the operators \( \{\hat{p}_l, \hat{q}_l\} \), \( l = 1, 2, \cdots N \)

\[
y_s = \sum_{l=1}^{N} \alpha_{ls} \hat{p}_l + \sum_{l=1}^{N} \beta_{ls} \hat{q}_l \equiv \vec{\alpha}_s \cdot \vec{p} + \vec{\beta}_s \cdot \vec{q},
\]

such that \( \alpha_{sl} \) are integer numbers, \( \beta_{sl} \) are rational numbers times \( 2\pi \) and \( \{\hat{p}_l, \hat{q}_l\} \) satisfies the following canonical commutation relations:

\[
[\hat{q}_j, \hat{p}_{j'}] = i\delta_{jj'}, \quad [\hat{q}_j, \hat{q}_{j'}] = [\hat{p}_j, \hat{p}_{j'}] = 0.
\]

The \( N \)-dimensional vectors \( \vec{\alpha}_s \) and \( \vec{\beta}_s \) are defined via (16). Eq. (15) and (16) imply

\[
\vec{\beta}_s \cdot \vec{\alpha}_{s'} - \vec{\beta}_{s'} \cdot \vec{\alpha}_s = 2\pi l_{ss'}, \ l_{ss'} \in \mathbb{Z}.
\]

Since \( \alpha_{sj} \) are integers, from (4) it can be shown that in the \( N \)-dimensional space for the \( 2N \) vectors \( \{\vec{\alpha}_s\} \) we can always find \( N \) vectors \( \{\vec{a}_j\} \) which satisfy the following conditions,

\[
\vec{a}_j = \sum_{s=1}^{2N} Z_{sj} \vec{\alpha}_s, \quad \vec{\alpha}_s = \sum_{j=1}^{N} \bar{Z}_{js} \vec{a}_j;
\]

where \( \bar{Z}_{js}, Z_{sj} \in \mathbb{Z} \) (standing for integer), \( s = 1, 2, \cdots, 2N, j = 1, 2, \cdots, N \).

Note that the solution of \( Z_{sj} \) in (20) is not unique, we can arbitrarily choose one from them and define

\[
\vec{\gamma}_j = \sum_{s=1}^{2N} Z_{sj} \vec{\beta}_s, \quad (21)
\]

We then have from (18),(20) and (21)

\[
\vec{\beta}_s \cdot \vec{a}_j - \vec{\gamma}_j \cdot \vec{\alpha}_s = 2\pi \bar{l}_js; \quad \vec{\gamma}_j \cdot \vec{a}_{j'} - \vec{\gamma}_{j'} \cdot \vec{a}_j = 2\pi \bar{l}_{jj'},
\]

with \( \bar{l}_{js}, \bar{l}_{jj'} \in \mathbb{Z} \).
Now we start constructing the common eigenstates of \( \{ U_s \} \). Consider the lattice vectors generated by \( \{ \vec{a}_j \} \) in the N-dimensional space:

\[
\vec{m} = \vec{r}_{\{m_j\}} = \sum_{j=1}^{N} m_j \vec{a}_j, \quad m_j \in \mathbb{Z}.
\]

(23)

From (19) any vector generated by linear combination of \( \vec{\alpha}_s \) with integral coefficients is located on the lattice. Now we set the common eigenstates of \( \{ U_s \} \) as follows,

\[
| \vec{B}, \vec{q} \rangle = \sum_{\vec{m}} e^{i(\vec{m}^T A \vec{m} + \vec{m} \cdot \vec{B})} | \vec{q} + \vec{m} \rangle,
\]

(24)

where \( \vec{B}, \vec{q}, \vec{m} \) are all N-dimensional vectors, and \( A \) is an \( N \times N \) real matrix, the sum goes over all the lattice sites. \( | \vec{q} + \vec{m} \rangle \) is common eigenstate of the coordinate operators \( \{ \hat{q}_j \} \),

\[
\hat{q}_j | \vec{q} + \vec{m} \rangle = (\vec{q} + \vec{m})_j | \vec{q} + \vec{m} \rangle.
\]

(25)

Note:

\[
(\vec{m})_j = \sum_{l=1}^{N} m_l (\vec{a}_l)_j \neq m_j.
\]

We require

\[
U_s | \vec{B}, \vec{q} \rangle = \lambda_s \left( \vec{B}, \vec{q} \right) | \vec{B}, \vec{q} \rangle
\]

(26)

obtaining

\[
e^{-i(\vec{\alpha}_s \cdot \vec{\beta}_s)} e^{i\vec{\alpha}_s \cdot \hat{p}} e^{i\vec{\beta}_s \cdot \hat{q}} | \vec{B}, \vec{q} \rangle
\]

(27)

\[
e^{-i(\vec{\alpha}_s \cdot \vec{\beta}_s)} \sum_{\vec{m}} e^{i(\vec{m}^T A \vec{m} + \vec{m} \cdot \vec{B})} e^{i\vec{\beta}_s \cdot (\vec{q} + \vec{m})} | \vec{q} + \vec{m} - \vec{\alpha}_s \rangle
\]

(28)

Since \( \vec{m} - \vec{\alpha}_s \) is also at the lattice site due to (19), we may rewrite the right-hand side of (28) as

\[
U_s | \vec{B}, \vec{q} \rangle = e^{-i(\vec{\alpha}_s \cdot \vec{\beta}_s)} \sum_{\vec{m}'} e^{i((\vec{m}' + \vec{\alpha}_s)^T A (\vec{m}' + \vec{\alpha}_s) + (\vec{m}' + \vec{\alpha}_s) \cdot \vec{B})} e^{i\vec{\beta}_s \cdot (\vec{q} + \vec{m}' + \vec{\alpha}_s)} | \vec{q} + \vec{m}' \rangle
\]

\[
= \lambda_s \sum_{\vec{m}} e^{i(\vec{m}^T A \vec{m} + \vec{m} \cdot \vec{B})} | \vec{q} + \vec{m} \rangle.
\]
Naturally we demand
\[ e^{i(\vec{\alpha}_s^T (A+A^T) \vec{m} + \vec{\beta}_s \cdot \vec{m})} = \lambda_s e^{i(\vec{\alpha}_s^T A \vec{\alpha}_s - \vec{\alpha}_s \cdot \vec{B} - \vec{\beta}_s \cdot \vec{q} + \frac{1}{2} \vec{\alpha}_s \cdot \vec{\beta}_s)} \]
(29)
to be valid for all \( \vec{m} \) belonging to the lattice vectors in (23). The rhs of the above equation is independent of the lattice sites, requiring the condition
\[ \vec{\alpha}_s^T (A + A^T) \vec{m} + \vec{\beta}_s \cdot \vec{m} \in 2\pi \mathbb{Z} \]
should be satisfied for all \( \vec{m} \). Since \( \vec{m} \) may be generated by the integral coefficient linear combination of \( \vec{\alpha}_s \), this means
\[ \vec{\alpha}_s^T (A + A^T) \vec{\alpha}_s' + \vec{\beta}_s \cdot \vec{\alpha}_s' \in 2\pi \mathbb{Z} \]
for all \( s, s' \). If the \( N \times N \) matrix \( A \) satisfies the above condition, then (26) holds with
\[ \lambda_s = \lambda(s, \vec{B}, \vec{q}) = e^{i(-\vec{\alpha}_s^T A \vec{\alpha}_s + \vec{\alpha}_s \cdot \vec{B} - \vec{\beta}_s \cdot \vec{q} - \frac{1}{2} \vec{\alpha}_s \cdot \vec{\beta}_s)} \].
Define \( g_s \) by
\[ -\vec{\alpha}_s^T A \vec{\alpha}_s - \frac{1}{2} \vec{\alpha}_s \cdot \vec{\beta}_s = \pi g_s, \quad g_s \in \mathbb{Z} \]
(32)
\[ \lambda_s = (-1)^{g_s} e^{i(\vec{\alpha}_s \cdot \vec{B} + \vec{\beta}_s \cdot \vec{q})}. \]
(33)
When \( A = A^T \), (see (39) below) we have \( g_s \in \mathbb{Z} \) from (31). In order to solve matrix \( A \) satisfying Eq.(31), we consider the following equation:
\[ \vec{\alpha}_j^T (A + A^T) \vec{\alpha}_{j'} + \vec{\gamma}_j \cdot \vec{\alpha}_{j'} = 2\pi T_{jj'} \quad T_{jj'} \in \mathbb{Z} \]
(34)
For the convenience, we define \( N \times N \) matrices \( H \) and \( a \) via
\[ H_{jj'} = \vec{\alpha}_j^T (A + A^T) \vec{\alpha}_{j'}, \]
(35)
\[ a_{kj} \equiv (\vec{\alpha}_j)_k, \]
(36)
\[ H = a^T (A + A^T) a. \]
(37)
The crucial point is that \( H \) must be symmetric. Since we have \( \det a \neq 0 \) and there exists the inverse of \( a \), so long as we could find symmetric matrix \( H \) satisfying the following equation:
\[ H_{jj'} + \vec{\gamma}_j \cdot \vec{\alpha}_{j'} = 2\pi T_{jj'}. \]
(38)
then Eq. (34) can be solved. Actually, we may set

$$2A = (a^{-1})^T Ha^{-1} \equiv X.$$  

Due to

$$XT = (a^{-1})^T H^T a^{-1} = X;$$

Eq. (34) can be verified. Next we solve matrix $H$:

1. When $j \leq j'$, for an arbitrary integer matrix elements $T_{jj'}$. Let

$$H_{jj'} = 2\pi T_{jj'} - \vec{\gamma}_j \cdot \vec{a}_{j'}$$

Eq. (38) is certainly satisfied.

2. When $j > j'$, we take

$$H_{jj'} = H_{j'j} = 2\pi T_{j'j} - \vec{\gamma}_{j'} \cdot \vec{a}_j.$$

Due to (22)

$$H_{jj'} + \vec{\gamma}_j \cdot \vec{a}_{j'} = 2\pi T_{j'j} - \vec{\gamma}_{j'} \cdot \vec{a}_j + \vec{\gamma}_j \cdot \vec{a}_{j'}$$

$$= 2\pi T_{j'j} + 2\pi \bar{l}_{jj'} = 2\pi \left(T_{j'j} + \bar{l}_{jj'}\right)$$

Therefore, when $j > j'$ we set

$$T_{jj'} = T_{j'j} + \bar{l}_{jj'},$$

which is an integer.

Eq. (38) then holds for all cases. That is

$$\vec{a}_j^T \left(A + A^T \right) \vec{a}_{j'} + \vec{\gamma}_j \cdot \vec{a}_{j'} \in 2\pi \mathbb{Z}.$$  

From (19), we obtain

$$\vec{a}_j^T \left(A + A^T \right) \vec{a}_s + \vec{\gamma}_j \cdot \vec{a}_s \in 2\pi \mathbb{Z},$$

By (22) we have

$$\vec{\alpha}_s^T \left(A + A^T \right) \vec{a}_j + \vec{\beta}_s \cdot \vec{a}_j \in 2\pi \mathbb{Z},$$

We further have

$$\vec{\alpha}_{s'}^T \left(A + A^T \right) \vec{a}_s + \vec{\alpha}_{s'} \cdot \vec{\beta}_s \in 2\pi \mathbb{Z}.$$
4 The Properties of Eigenvector $|\vec{B}, \vec{q}\rangle$

4.1 Periodicity

In an N-dimensional space we define N vectors $\{\vec{b}_j\}$ dual to $\{\vec{a}_j\}$, satisfying

$$\vec{a}_j \cdot \vec{b}_{j'} = 2\pi \delta_{jj'}.$$  \hfill (41)

Due to

$$|\vec{B}, \vec{q}\rangle = \sum_{\vec{m}} e^{i(\vec{m}^T A \vec{m} + \vec{m} \cdot \vec{B})} |\vec{q} + \vec{m}\rangle,$$  \hfill (42)

and from (23) we have

$$|\vec{B} + \vec{b}_j, \vec{q}\rangle = |\vec{B}, \vec{q}\rangle.$$  \hfill (43)

We also have

$$|\vec{B} - \vec{\gamma}_j, \vec{q} + \vec{a}_j\rangle = \sum_{\vec{m}} e^{i(\vec{m}^T A \vec{m} + \vec{m} \cdot (\vec{B} - \vec{\gamma}_j))} |\vec{q} + \vec{a}_j + \vec{m}\rangle.$$  \hfill (44)

On the other hand, since $\vec{a}_j$ is on the lattice formed by $\vec{m}$, we have

$$|\vec{B}, \vec{q}\rangle = \sum_{\vec{m} + \vec{a}_j} e^{i(\vec{m} + \vec{a}_j)^T A (\vec{m} + \vec{a}_j) + i(\vec{a}_j + \vec{m}) \cdot \vec{B}} |\vec{q} + \vec{m} + \vec{a}_j\rangle$$

$$= \sum_{\vec{m}} e^{i\vec{m}^T A \vec{m} + i\vec{m} \cdot \vec{B} + i\vec{a}_j^T (A+A^T)\vec{m} + i\vec{a}_j A \vec{a}_j + i\vec{a}_j \cdot \vec{B}} |\vec{q} + \vec{m} + \vec{a}_j\rangle.$$  \hfill (45)

From (34), we have

$$\vec{a}_j^T (A + A^T) \vec{m} + \vec{\gamma}_j \vec{m} \in 2\pi \mathbb{Z}.$$  

Comparing (44) and (45), we have

$$|\vec{B}, \vec{q}\rangle = e^{i(\vec{a}_j^T A \vec{a}_j + \vec{a}_j \cdot \vec{B})} |\vec{B} - \vec{\gamma}_j, \vec{q} + \vec{a}_j\rangle.$$  \hfill (46)

So the eigenstate $|\vec{B}, \vec{q}\rangle$ has two sets of quasi-periodicity relations. In the view of 2N-dimensional space, the two sets of quasi-periodic vectors are respectively

$$\{\vec{b}_j, 0\} \text{ and } \{-\vec{\gamma}_j, \vec{a}_j\}.$$  \hfill (47)
Similarly, we can prove that for \( s = 1, 2, \cdots, 2N \)
\[
\left| \vec{B}, q \right> = e^{i(\vec{A} \cdot \vec{\alpha}_s + \vec{\alpha}_s \cdot \vec{B})} \left| \vec{B} - \vec{\beta}_s, \vec{q} + \vec{\alpha}_s \right>.
\]

### 4.2 Orthogonal Relation

Consider the unit cell \( V_a \) composed of \( \{\vec{a}_j\} \) and the unit cell \( V_b \) composed of \( \{\vec{b}_j\} \).

\[
< \vec{B}', q' | \vec{B}, q > = \sum_{m \bar{m}} e^{-i m^T A m' - i \bar{m}^T B' + i m^T A \bar{m} - i \bar{m}^T B} \delta^N (q - q' + \bar{m} - \bar{m}').
\]

When \( q \) and \( q' \) are both inside \( V_a \), only when \( m = m', \delta^N (q - q' + \bar{m} - \bar{m}') \neq 0 \), then
\[
< \vec{B}', q' | \vec{B}, q > = \sum_{\bar{m}} e^{i \bar{m}^T (B - B')} \delta^N (q - q')
= \delta^N (q - q') \sum_{\{m_j \in \mathbb{Z}\}} e^{i \sum_j m_j \vec{a}_j \cdot (\vec{B} - B')}.
\]

Considering the case that \( \vec{B} \) and \( \vec{B}' \) are both inside \( V_b \), we have
\[
\vec{B} - \vec{B}' = \sum_j \vec{b}_j (c_j - c'_j), \quad -1 < c_j - c'_j < 1, \quad |\vec{a}_j \cdot (\vec{B} - \vec{B}')| < 2\pi.
\]

The right-hand side of Eq.(51) becomes
\[
\delta^N (q - q') \prod_j (2\pi) \delta \left( \vec{a}_j \cdot (\vec{B} - \vec{B}') \right)
= \delta^N (q - q') \prod_j \delta (c_j - c'_j).
\]

Therefore in the unit cell \( V_a \times V_b \), the different eigenstates \( \left| \vec{B}, q \right> \) are orthogonal.
4.3 Completeness

Consider the integration over unit cell $V'_2 = V_a \times V_b$
\[
\mathcal{L} = \int_{V'_2} d^N \vec{B} \left< \vec{B}, \vec{q} \right| \left< \vec{B}, \vec{q} \right|
\]
\[
= \int_{V'_2} d^N \vec{B} \sum_{\vec{m}' \vec{m}} e^{-i \vec{m}'^T A \vec{m} - i \vec{m}' \cdot \vec{B}} |\vec{q} + \vec{m} \rangle \langle \vec{q} + \vec{m}'|.
\]

Let $\vec{B} = \sum_j \vec{b}_j c_j$, we have
\[
e^{i(\vec{m} - \vec{m}') \cdot \vec{B}} = e^{2\pi i \sum_j (m_j - m'_j) c_j}.
\]
The integration with respect to $\vec{B}$ inside the unit cell $V_b$ can be realized through changing it into the integration with respect to $c_j \in [0, 1)$,
\[
\int_{V_b} d^N B e^{i(\vec{m} - \vec{m}') \cdot \vec{B}} = \begin{cases} 0 & \vec{m} \neq \vec{m}' \\ \upsilon_b & \vec{m} = \vec{m}' \end{cases}
\]

where $\upsilon_b$ is the volume of $V_b$ in Eq.(53). Then we have
\[
\mathcal{L} = \int_{V_a} \upsilon_b d^N \vec{q} \sum_{\vec{m}} |\vec{q} + \vec{m} \rangle \langle \vec{q} + \vec{m}|
\]
\[
= \upsilon_b \int_{-\infty}^{\infty} \prod_j dq_j |\vec{q} \rangle \langle \vec{q}| = \upsilon_b id.
\]

We then get the completeness condition:
\[
id = \frac{1}{\upsilon_b} \int_{V'_2} d^N \vec{B} d^N \vec{q} |\vec{B}, \vec{q} \rangle \langle \vec{B}, \vec{q}|
\]

Since we have the other set of periodic vectors (47) we can also rewrite the completeness condition and orthogonal condition (52) in $V_2 = V'_a \times V_b$, where $V'_a$ is the unit cell composed of quasi-periodic vectors (48).

4.4 Degenerate Lattice

Now we study the degeneracy of eigenvalue $\lambda_s(\vec{B}, \vec{q})$ of $U_s$ in the space $\left( \vec{B}, \vec{q} \right)$. Consider a 2N-dimensional vector $\left( \vec{\alpha}_s \quad \vec{\beta}_s \right) \equiv \vec{w}_s$. From (4) $\det(C_{js}) \neq $
0, we know that \( \{ \vec{y}_s \} \) are linearly independent of each other. We can always find the dual 2N-dimensional vectors \( \vec{\tau}_s \) satisfying
\[
\vec{w}_s \cdot \vec{\tau}_{s'} = 2\pi \delta_{ss'}, \quad s, s' = 1, 2, \cdots, 2N
\]
then when
\[
\left( \begin{array}{c}
\triangle \vec{B} \\
\triangle \vec{q}
\end{array} \right) = \sum_s \vec{\tau}_s n_s, \quad n_s \in \mathbb{Z},
\]
we have
\[
\tilde{\alpha}_s \cdot \triangle \vec{B} + \tilde{\beta}_s \cdot \triangle \vec{q} = \vec{w}_s \cdot \sum_{s'} \vec{\tau}_{s'} n_{s'} = \sum_{s'} 2\pi \delta_{ss'} n_{s'} = 2\pi n_s.
\]

Then from (33), \( \left| \vec{B}, \vec{q} \right> \) and \( \left| \vec{B} + \triangle \vec{B}, \vec{q} + \triangle \vec{q} \right> \) are degenerate for the eigenvalues of all \( \{ U_s \} \). Consider the unit cell \( \sigma_{00} \) made up of \( \{ \vec{\tau}_s \} \) in the \( \left( \vec{B}, \vec{q} \right) \) space. The eigenstates of \( \left| \vec{B}, \vec{q} \right> \) and \( \left| \vec{B}', \vec{q}' \right> \) which respectively correspond to two different points \( \left( \vec{B}, \vec{q} \right) \) and \( \left( \vec{B}', \vec{q}' \right) \) inside \( \sigma_{00} \) are not degenerate. Let us check this in the following
\[
\left( \begin{array}{c}
\vec{B}' - \vec{B} \\
\vec{q}' - \vec{q}
\end{array} \right) = \left( \begin{array}{c}
\triangle \vec{B} \\
\triangle \vec{q}
\end{array} \right) = \sum_s \vec{\tau}_s \nu_s, \quad -1 < \nu_s < 1,
\]
\[
\lambda_s \left( \vec{B}', \vec{q}' \right) = e^{i(\tilde{\alpha}_s \cdot \triangle \vec{B} + \tilde{\beta}_s \cdot \triangle \vec{q})} = e^{i\vec{w}_s \cdot \sum_{s'} \vec{\tau}_{s'} \nu_{s'}} = e^{2\pi i \nu_s}.
\]
in the region of (58) when and only when \( \nu_s = 0 \), the right-hand side of 59 equals to 1.

We call the lattice generated by \( \vec{\tau}_s \) in the \( \left( \vec{B}, \vec{q} \right) \) space as degenerate lattice and the unit cell is the nondegenerate region of \( \left| \vec{B}, \vec{q} \right> \) with respect to \( \{ U_s \} \).

### 4.5 The Degree of Degeneracy in the Space of \( \left( \vec{B}, \vec{q} \right) \)

In the 2N-dimensional \( \left( \vec{B}, \vec{q} \right) \) space we have three lattices:
1. The lattice $L_1$ generated by
$$
\left( \begin{array}{c}
\tilde{\beta}_s \\
-\tilde{\alpha}_s 
\end{array} \right) = J \left( \begin{array}{c}
\bar{\alpha}_s \\
\bar{\beta}_s 
\end{array} \right) = J \vec{\omega}_s,
$$
here
$$
J = \left( \begin{array}{cc}
0 & I \\
-I & 0 
\end{array} \right).
$$

2. The lattice $L_2$ generated by
$$
\left( \begin{array}{c}
\vec{b}_j \\
0
\end{array} \right) \text{ and } \left( \begin{array}{c}
-\vec{\gamma}_j \\
\vec{a}_j
\end{array} \right).
$$

3. The lattice $L_3$ generated by $\vec{\tau}_s$.

Obviously $L_3$ has to contain $L_2$ because two associated states connected by the periodic vectors at most differ by a phase factor, therefore the eigenvalues with respect to $U_s$ must be the same, and the periodic vectors have to be vectors belonging to $L_3$, we denote the vectors in $L_3$ by
$$
\vec{t}_s = \sum_{s=1}^{2N_s} \vec{\tau}_s n_s, \quad n_s \in \mathbb{Z}.
$$

On the other hand, we can prove that $J \vec{\omega}_s$ belongs to the periodic lattice vectors as follows:

$$
e^{i(\vec{a} \cdot \vec{p} + \vec{\beta} \cdot \vec{q})} \bigg| \vec{B}, \vec{q} \biggangle = e^{-\frac{i}{2}(\vec{a} \cdot \vec{B})} e^{i\vec{a} \cdot \vec{p}} e^{i\vec{\beta} \cdot \vec{q}} \sum_{\vec{m}} e^{i(\vec{m}^T \vec{A} \vec{m} + \vec{m} \cdot \vec{B})} |\vec{q} + \vec{m}\rangle = e^{-\frac{i}{2}(\vec{a} \cdot \vec{B})} e^{i\vec{a} \cdot \vec{p}} \sum_{\vec{m}} e^{i\vec{m}^T \vec{A} \vec{m} + i\vec{m} \cdot (\vec{B} + \vec{\beta})} |\vec{q} + \vec{m}\rangle = e^{-\frac{i}{2}(\vec{a} \cdot \vec{B})} e^{i\vec{\beta} \cdot \vec{q}} \sum_{\vec{m}} e^{i\vec{m}^T \vec{A} \vec{m} + i\vec{m} \cdot (\vec{B} + \vec{\beta})} |\vec{q} + \vec{m} - \vec{a}\rangle = e^{-\frac{i}{2}(\vec{a} \cdot \vec{B})} e^{i\vec{\beta} \cdot \vec{q}} \bigg| \vec{B} + \vec{\beta}, \vec{q} - \vec{a} \bigg\rangle \quad (60)
$$

However, when taking $\vec{a} = \vec{a}_s$, $\vec{B} = \vec{B}_s$, on the left hand side of (60) equals to $U_s \bigg| \vec{B}, \vec{q} \bigg\rangle = \lambda_s \bigg| \vec{B}, \vec{q} \bigg\rangle$, so $\left( \begin{array}{c}
\vec{\beta}_s \\
-\vec{\alpha}_s
\end{array} \right)$ belongs to periodic lattice too, $L_1$ is naturally contained in $L_2$.

$$
L_1 \subset L_2 \subset L_3. \quad (61)
$$

In order to calculate how many points of $L_2$ there exist in a unit cell of $L_1$ and how many points of $L_3$ in a unit cell of $L_2$, we compare the volumes of three kinds of unit cells and respectively set them to be $v_1, v_2$ and $v_3$. 

14
corresponding to the three kind of unit cells. Let $\Gamma$ and $W$ be the matrices defined as $\Gamma_{s's'} = (\vec{r}_s)_{s'}$, $W_{s's'} = (\vec{w}_s)_{s'}$. From (56) we have

$$\Gamma_{s's'} W_{s's'} = \vec{r}_{s1} \cdot \vec{w}_{s2} = 2\pi \delta_{s1s2}$$

to give out

$$\Gamma^T W = 2\pi I_{2N \times 2N}$$

and

$$\det \Gamma \cdot \det W = (2\pi)^{2N} \det I = (2\pi)^{2N}$$

Consequently, we have

$$v_3 = |\det \Gamma| = (2\pi)^{2N} |\det W|^{-1}$$

In order to calculate the determinant of $W$, we calculate $W^T J W$ by (18)

$$W^T J W = \begin{pmatrix} \tilde{\alpha}_1^T & \tilde{\beta}_1^T \\ \vdots & \vdots \\ \tilde{\alpha}_{2N}^T & \tilde{\beta}_{2N}^T \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \tilde{\alpha}_1 & \cdots & \tilde{\alpha}_{2N} \\ \tilde{\beta}_1 & \cdots & \tilde{\beta}_{2N} \end{pmatrix}$$

$$= \begin{pmatrix} -\tilde{\beta}_1 \cdot \tilde{\alpha}_1 + \tilde{\alpha}_1 \cdot \tilde{\beta}_1 & -\tilde{\beta}_1 \cdot \tilde{\alpha}_2 + \tilde{\alpha}_1 \cdot \tilde{\beta}_2 & \cdots & -\tilde{\beta}_1 \cdot \tilde{\alpha}_{2N} + \tilde{\alpha}_1 \cdot \tilde{\beta}_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -\tilde{\beta}_{2N} \cdot \tilde{\alpha}_1 + \tilde{\alpha}_{2N} \cdot \tilde{\beta}_1 & -\tilde{\beta}_{2N} \cdot \tilde{\alpha}_2 + \tilde{\alpha}_{2N} \cdot \tilde{\beta}_2 & \cdots & -\tilde{\beta}_{2N} \cdot \tilde{\alpha}_{2N} + \tilde{\alpha}_{2N} \cdot \tilde{\beta}_{2N} \end{pmatrix}$$

$$= -2\pi \begin{pmatrix} l_{11} & l_{12} & \cdots & l_{1,2N} \\ \vdots & \vdots & \ddots & \vdots \\ l_{2N,1} & l_{2N,1} & \cdots & l_{2N,2N} \end{pmatrix}.$$ 

So we have

$$\det (W^T J W) = (\det W)^2 \det J = (-2\pi)^{2N} \det (l_{ss'}) \equiv (-2\pi)^{2N} \det L,$$

$$|\det J| = 1 \implies |\det W| = (2\pi)^N \sqrt{|\det L|},$$

The volumes of three kinds of unit cells are given as follows

$$v_3 = \frac{(2\pi)^N}{\sqrt{|\det L|}},$$

(64)
\[ v_2 = |\det \begin{pmatrix} b & -\gamma \\ 0 & a \end{pmatrix}| = |\det b \det a|, \]

with \( b_{ij} = (\vec{b}_j) \). From (41) we have

\[ v_2 = (2\pi)^N. \]

\[ v_1 = |\det JW| = |\det W| = (2\pi)^N \sqrt{|\det L|}. \]

Thus we can obtain the ratios by volume among the three kinds of unit cells.

\[ \frac{v_1}{v_2} = \sqrt{|\det L|}, \]

\[ \frac{v_2}{v_3} = \frac{(2\pi)^N}{(2\pi)^N \frac{1}{\sqrt{|\det L|}}} = \sqrt{|\det L|}, \]

\[ \sqrt{|\det L|} = |PfL| = N_d. \]

This is the degree of degeneracy in \( V_1 \).

5 Transformation of \( |\vec{B}, \vec{q}\rangle \) under Group \( G \) and Blocking Theorem

In this section we discuss the transformation of the common eigenstates \( |\vec{B}, \vec{q}\rangle \) of \( \{U_s\} \) under rotation \( G \). When \( U_s = e^{iy_s}, R_k \in G \) in consistent with (9),

\[ R_k^{-1} y_s R_k = y'_s = \sum_{s'} K_{ss'} y_{s'} \quad K_{ss'} \in \mathbb{Z}, \]

\[ \det K = 1. \]

Thus, we have

\[ U'_s = R_k^{-1} U_s R_k = \eta_{sk} \prod_{s'} U_{s's'}, \]

\[ \eta_{sk} = (-1)^{h_{sk}} = \pm 1. \]

From

\[ U_s |\vec{B}, \vec{q}\rangle = \lambda_s |\vec{B}, \vec{q}\rangle. \]
\[ \lambda_s = (-1)^{g_s} e^{i(\bar{\alpha}_s \cdot \vec{B} + \bar{\beta}_s \cdot \vec{q})} \equiv (-1)^{\mu_s} e^{i\mu_s} , \quad (69) \]

we obtain

\[ U_s R_k | \vec{B}, \vec{q} \rangle = R_k \left( R_k^{-1} U_s R_k \right) | \vec{B}, \vec{q} \rangle \]
\[ = R_k (-1)^{h_{sk}} \prod_{s'} \lambda_s^{K_{ss'}} | \vec{B}, \vec{q} \rangle \]
\[ = (-1)^{h_{sk} + \sum_{s'} g_{ss'} K_{ss'}} e^{i \sum_{s'} K_{ss'} (\bar{\alpha}_{s'} \cdot \vec{B} + \bar{\beta}_{s'} \cdot \vec{q})} R_k | \vec{B}, \vec{q} \rangle . \quad (70) \]

So \( R_k | \vec{B}, \vec{q} \rangle \) also belongs to the set of eigenstates of \( \{ U_s \} \), the eigenvalue of which is

\[ \lambda_s' = (-1)^{h_{sk} + \sum_{s'} g_{ss'} K_{ss'}} e^{i \sum_{s'} K_{ss'} (\bar{\alpha}_{s'} \cdot \vec{B} + \bar{\beta}_{s'} \cdot \vec{q})} \equiv (-1)^{g_s} e^{i\mu_s} . \quad (71) \]

Define 2N-dimensional vectors \( \vec{z}, \vec{g}, \vec{h} \) as

\[ \{ \begin{align*}
(\vec{z})_j &= B_j , \\
(\vec{z})_{N+j} &= q_j , \\
(\vec{g})_s &= g_s , \\
(\vec{h})_s &= h_{sk} ,
\end{align*} \]
\[ \{ \begin{align*}
&j = 1, 2, \cdots , N \\
&s = 1, 2, \cdots , 2N
\end{align*} \]

Recall the \( 2N \times 2N \) matrix \( W \) is defined via \( W_{s's'} = (\vec{w}_s)_s' \), i.e.

\[ \{ \begin{align*}
W_{js} &= (\vec{\alpha}_s)_j \\
W_{j+N,s} &= (\vec{\beta}_s)_j ,
\end{align*} \]
\[ \{ \begin{align*}
&j = 1, 2, \cdots , N \\
&s = 1, 2, \cdots , 2N
\end{align*} \]

Thus, we have

\[ K_{ss'} \left( \bar{\alpha}_{s'} \cdot \vec{B} + \bar{\beta}_{s'} \cdot \vec{q} \right) \equiv K_{ss'} \left( \alpha_{js'} B_j + \beta_{js'} q_j \right) \]
\[ = K_{ss'} W_{s's'} z_{s'} \]
\[ = \left( KW^T \vec{z} \right)_s , \quad (72) \]
In Eq. (71)
\[
\mu_s' = (KW^T \vec{z})_s + \pi \left( \vec{h}_k + K \vec{g} - \vec{g} \right)_s
\]
\[
= \left[ WT \left( (W^T)^{-1} KW^T \vec{z} + \vec{\Delta}_k \right) \right]_s
\]
\[
\equiv \left[ WT \left( K' \vec{z} + \vec{\Delta}_k \right) \right]_s,
\]
(73)
where \( \vec{\Delta}_k = \pi (W^T)^{-1} \left( \vec{h}_k + (K - 1) \vec{g} \right) \) is a constant vector related to \( R_k \).
\( \vec{\Delta}_k \) doesn’t depend on \( (\vec{B}, \vec{q}) \). We know that the states are complete in the periodic unit cell \( V_2 \).
The common eigenstates of \( \{ U_s \} \) with certain eigenvalue can be spanned by \( \left\{ \left| (\vec{B}_0, \vec{q}_0) + \vec{t}_l \right> \right\}_{l=1,2,\cdots,N_d} \), where \( (\vec{B}_0, \vec{q}_0) \) is inside the nondegenerate little unit cell \( \sigma_{00} \) which is made by \( \{ \vec{\tau}_s \} \) and \( \vec{t}_l \) are degenerate lattice vectors in the periodic unit cell \( V_2 \).
\[
\vec{t}_l = \sum_{s=1}^{2N} \vec{\tau}_s n_{ls}, \quad (l = 1, 2, \cdots, N_d, \ n_{ls} \in \mathbb{Z})
\]
(74)
For the common eigenstates \( \left| \vec{B}, \vec{q} \right> \) of \( \{ U_s \} \) with same eigenvalue, their \( \left( \vec{B}, \vec{q} \right) \) at most differ by a degenerate lattice vector \( \vec{t}_l \). So, we obtain from (70)
\[
R_k \left| (\vec{B}_0, \vec{q}_0) + \vec{t}_l \right> = \sum_{l'=1}^{N_d} A_{l'l} \left| (\vec{B}_0', \vec{q}_0') + \vec{t}_{l'} \right>.
\]
(75)
Here \( (\vec{B}_0, \vec{q}_0) \) and \( (\vec{B}_0', \vec{q}_0') \) both belong to the little unit cell \( \sigma_{00} \). Summation goes over the degenerate lattice sites in the periodic unit cell \( V_2 \).
Let
\[
\left( \vec{B}_0, \vec{q}_0 \right) = \vec{z}_0, \quad \left( \vec{B}_0', \vec{q}_0' \right) = \vec{z}_0'.
\]
Since \( \lambda_s = (-1)^{g_s} e^{i(W\vec{z}_0)} \), \( \lambda_s' = (-1)^{g_s} e^{i(W\vec{z}_0')} \), thus from (73) we have
\[
\vec{z}_0' = K' \vec{z}_0 + \vec{\Delta}_k - \vec{t},
\]
(76)
where \( \vec{t} \) is a degenerate vector of \( L_3 \)
\[
W^T \vec{t} = 2\pi n,
\]
(77)
We call the mapping from $\tilde{z}_0$ to $\tilde{z}_0$ as mapping induced by $R_k$ denoted by $\tilde{K}$ in the nondegenerate little unit cell $\sigma_{00}$. Consider an arbitrary degenerate vector $\tilde{t}$, $\tilde{t} = K'\tilde{t} = (W^T)^{-1}KW^t = 2\pi (W^T)^{-1}K\tilde{n} = 2\pi (W^T)^{-1}\tilde{n}' \implies W^T\tilde{t} = 2\pi \tilde{n}'$, namely $\tilde{t}$ still belongs to the set of degenerate lattice vectors. So mapping $K' : \tilde{z}' = K'\tilde{z}$ change lattice $L_3$ into lattice $L_3$. The original whole $\left(\tilde{B}, \tilde{q}\right)$ space $X_z = \{\tilde{z} = \tilde{z}_0 + \tilde{t} | \tilde{z}_0 \in \sigma_{00}, \tilde{t} \in L_3\}$ can be built up by little unit cell $\sigma_{00}$ according to degenerate lattice $L_3$. The linear mapping $\tilde{K}'$ on $X_z$ defined as $K' : z \rightarrow \tilde{z}' = K'\tilde{z} + \tilde{\Delta}_k$ maps $X_z$ into $X_{z'}$ and make $\sigma_{00}$ mapped into $\sigma'_{00}$. Note that the original $X_z$ can be built up by $\sigma_{00}$ according to degenerate lattice $L_2$, and

$$\text{det } K' = \text{det } \left((W^T)^{-1}KW^T\right) = \text{det } K = 1$$  \hspace{1cm} (78)

Since $\tilde{t} = K'\tilde{t} \in L_3$ the mapping $K'$ maps degenerate lattice $L_3$ into $L_3$, so the space $\left(\tilde{B}, \tilde{q}\right)$ namely $X_z$ can also be built up by $\sigma'_{00}$ according to degenerate lattice $L_3$. Based on this fact we infer that when $\sigma_{00}$ is generated by a finite number of $2N - 1$ dimensional hyper planes (for example let $\sigma_{00}$ be hyper polyhedron built by $\{\tilde{t}_s\}$), $\sigma_{00}$ and $\sigma'_{00}$ can be made up of the same little blocks $\sigma^j_i$. This is because two kinds of periodic configurations respectively corresponding to $\sigma_{00}$ and $\sigma'_{00}$ are both able to constitute $X_z$. The interfaces of $\sigma_{00}$ and $\sigma'_{00}$ in the configurations form and these little blocks certainly can build up both $\sigma_{00}$ and $\sigma'_{00}$. The equations (75) and (76) imply that the mapping $\left(\tilde{B}_0, \tilde{q}_0\right) \rightarrow \left(\tilde{B}'_0, \tilde{q}'_0\right)$ can be realized by two steps: first map $\sigma_{00}$ into $K'\sigma_{00} = K'\sigma_{00} + \tilde{\Delta}_k = \sigma'_{00}$, next according to some definite block $\sigma^j_i$ drag it back to $\sigma_{00}$ by different degenerate vectors $\tilde{t}_j$.

Setting the preimage of $\sigma^j_i$ to be $\tilde{\sigma}^j_i$, we have

$$\sigma^j_i = K'\tilde{\sigma}^j_i + \tilde{\Delta}_k + \tilde{t}_j = K' \left(\tilde{\sigma}^j_i + K'^{-1} \left(\tilde{\Delta}_k + \tilde{t}_j\right)\right),$$

where the $\sigma^j_i$ are smaller blocks inside $\sigma_{00}$, namely $\sigma^j_i$ can be obtained from $\tilde{\sigma}^j_i$ through a linear mapping $\tilde{K}'(\cdot) : \tilde{z} \rightarrow \tilde{z}' = K' \times (\tilde{z} + \tilde{e}_j), \left(\tilde{e}_j = K'^{-1} \left(\tilde{\Delta}_k + \tilde{t}_j\right)\right)$. From (78) we know this mapping is an volume-preserving
mapping. So we obtain a result: the effect of \( R_k \) acting on the state vector \( \left( \vec{B}_0, \vec{q}_0 \right) + \vec{t}_1 \) is to introduce a mapping \( \left( \vec{B}_0, \vec{q}_0 \right) \to \left( \vec{B}'_0, \vec{q}'_0 \right) \) as (75). This mapping can be described as follows: First cut \( \sigma_{00} \) into finite blocks \( \tilde{\sigma}_1^j \), which are then respectively linearly mapped into \( \sigma_1^j \), the shift vector \( \vec{c}_j \) may depend on \( j \) and be different.

Next, through the following Lemma 1 and Lemma 2 we will prove a Blocking Theorem about the mapping \( (B_0, q_0) \xrightarrow{R \in G} (B'_0, q'_0) \).

**Lemma 1** For the mapping \( \tilde{K}^j \vec{z} = K' \times (\vec{z} + \vec{c}) \), \( K' \neq I \) we can always divide \( X_z \) namely the space of \( \left( \vec{B}, \vec{q} \right) \) into a finite number of sectorial regions \( V^j \). Under the mapping different sectors exchange with each other and the image of each block doesn't overlap itself. The meaning of "overlap" is that there exist a common \( 2N \)-dimensional little continuous regions \( V^j \cap V'^j \).

**Proof.** First let the fixed point of the mapping be at the origin of coordinates.

Matrix \( K'(2N \times 2N) \) has the following property:

\[
(K')^{n_k} = I.
\]

\( (G \) is a finite group, \( R_{n_k} = \text{id} \)\) The rank of \( G \) is finite and \( K' \) is a \( 2N \)-dimensional representation of a cyclic group. Based on the theorem about finite group that if the representation is reducible, surely completely reducible, and a cyclic group only has one-dimensional irreducible representation, after reduction Matrix \( K' \) goes to

\[
\begin{pmatrix}
\lambda_1 & & \\
& \lambda_2 & \\
& & \ddots \\
& & & \lambda_{2N}
\end{pmatrix}
\]

The corresponding invariant sub-space is one-dimensional, namely for eigenvector \( \psi_j, K' \psi_j = \lambda_j \psi_j, \lambda_j^{n_k} = 1 \), so \( \lambda_j \) is a \( n_k \) order root of unity. If \( \lambda_j \) equals to 1, \( \psi_j \) can be chosen as a real vector, which is an invariant direction under action of \( K' \). If \( \lambda_j \) is an imaginary number \( e^{i\omega_j} \), \( \psi_j \) can be decomposed into
\( \phi_1 + i \phi_2 \), here \( \phi_1 \) and \( \phi_2 \) are real vectors.

\[
K'(\phi_1 + i \phi_2) = (\cos \omega + i \sin \omega)(\phi_1 + i \phi_2) = (\phi_1 \cos \omega - \phi_2 \sin \omega) + i (\phi_2 \cos \omega + \phi_1 \sin \omega). \quad (79)
\]

Since \( K' \) is real, we have

\[
\Rightarrow K' \phi_1 = \phi_1 \cos \omega - \phi_2 \sin \omega, \quad K' \phi_2 = \phi_2 \cos \omega + \phi_1 \sin \omega.
\]

This is a rotation in the plane spanned by \( \phi_1 \) and \( \phi_2 \). The complex conjugate of (79) implies that the eigenvalue \( \lambda = e^{-i \omega j} \) corresponding to \( \psi^* \) corresponds to the same plane too, which is an invariant plane because \( K'' \) always maps \( \phi_1 \) and \( \phi_2 \) to their linear combination. Since \( K' \) is completely reducible, we can prove that we always have a co-subspace which is also invariant.

Let the invariant plane be \( X' \) and the co-subspace be \( X'' \). Any vector \( \tilde{z} \in X_z \) can be uniquely decomposed as

\[
\tilde{z} = \tilde{z}_1 + \tilde{z}_2, \quad \tilde{z}_1 \in X', \tilde{z}_2 \in X''
\]

The above discussion means

\[
K' \tilde{z} = K'(\tilde{z}_1 + \tilde{z}_2) = K' \tilde{z}_1 + K' \tilde{z}_2,
\]

where \( K' \tilde{z}_1 \in X', K' \tilde{z}_2 \in X'' \). Let \( \tilde{c} = \tilde{c}_1 + \tilde{c}_2, \tilde{c}_1 \in X', \tilde{c}_2 \in X'' \), we have

\[
K' \tilde{z} = K' (\tilde{z}_1 + \tilde{c}_1) + K' (\tilde{z}_2 + \tilde{c}_2)
\]

with

\[
K' (\tilde{z}_1 + \tilde{c}_1) \in X', K' (\tilde{z}_2 + \tilde{c}_2) \in X''.
\]

Thus the mapping \( \tilde{K}' \) induces a map in \( X' \), which is a shift \( \tilde{c}_1 \) that follows a rotation by equation (79). We can always divide \( X' \) into \( n_k \) sector by lines in \( X' \), such that after mapping each sector doesn’t overlap itself. These sectors together with \( X'' \) form division in \( X_z \), which has the desired property. Concretely, let

\[
X' = D_1 \cup D_2 \cdots \cup D_{n_k}, \tilde{z} = \tilde{z}_1 + \tilde{z}_2, \tilde{z}_1 \in D_l
\]

\[
X_z = D_1 \oplus X'' \cup D_2 \oplus X'' \cdots \cup D_{n_k} \oplus X''
\]
\[
K'(\vec{z}_1 + \vec{c}_1) \in D_m, m \neq l \\
\implies K'(\vec{z}) = K'(\vec{z}_1 + \vec{c}_1) + K'(\vec{z}_2 + \vec{c}_2) = \vec{r}_1 + \vec{r}_2.
\]

\[
\vec{r}_1 \in D_m, \vec{r}_2 \in X'' \implies \vec{r}_1 + \vec{r}_2 \in D_m \oplus X''.
\]

which doesn’t overlap \(D_l \oplus X''\). The lemma is valid. ■

Lemma 2 For group \(G\) (or its any subgroup \(G^j\)), to consider any figure point set of \(\sigma_{00}\), we can always get a new figure which contains the original figure (point set), and the figure is invariant under action of \(G\).

Proof. Let any point \(P_1\) be mapped into \(P_1, P_2, \ldots, P_{|G|}\), then the point set \(\{P_1, P_2, \ldots, P_{|G|}\}\) doesn’t change. So an arbitrary point set \(S_1\) is mapped by \(R_k \in G\) into \(S_1, S_2, \ldots, S_{|G|}\), then \(\bigcup S_j\) is an invariant point set. ■

Theorem 1 (Blocking Theorem) We can always divide the little unit cell \(\sigma_{00}\) into a finite number of blocks \(\sigma^q_3\) such that they exchange positions with each other under transformation \(R_k \in G\): 

\[
\left(\vec{B}_0, \vec{q}_0\right) \longrightarrow \left(\vec{B}_0', \vec{q}_0'\right)
\]

and make their image not overlap themselves for any \(R_k \neq id\).

Proof. Consider the mapping induced by \(R_k\), which maps the subset of \(\sigma_{00}: \widetilde{\sigma}^j_1 \rightarrow \sigma^j_1, \{\widetilde{\sigma}^j_1\}\) and \(\{\sigma^j_1\}\) can both build up \(\sigma_{00}\). Meanwhile the mapping \(\tilde{K}^{(j)}:\widetilde{\sigma}^j_1 \rightarrow \sigma^j_1\) is a linear mapping, which is in general not homogeneous. Thanks to Lemma 1, we can divide \(\widetilde{\sigma}^j_1\) into several little sections \(\widetilde{\sigma}^p_2\), the image of each little section doesn’t overlap itself under \(\tilde{K}^{(j)}\). This kind of division together with their image (namely \(\sigma^p_2\) the division of \(\sigma^j_1\)) form a figure by the boundaries of the regions. We do this procedure for every \(R \in G\), and finally get a figure. Based on Lemma 2, we add up the divisions induced by every group element to obtain the final figure which is invariant under action of any group element \(R_k \in G\). At last we obtain a division of \(\sigma_{00}\), named as \(\sigma^q_3\), which only exchange their position under the action of \(G\), since the whole figure keep invariant under \(G\) and the image of any little block under \(R_k \in G\) doesn’t overlap itself, which is because each block included in the figure certainly belongs to one block of the division \(\{\widetilde{\sigma}^p_2\}\) corresponding to certain definite \(R_k\), thus \(R_k\) must shift its position due to Lemma 1. ■

Based on the Blocking Theorem, we have the following corollary:

Corollary 1 If the mapping \(\sigma^q_3 \rightarrow \sigma^q_3\) given by \(R_k_1\) and \(R_k_2\) are the same, then \(R_{k_1} = R_{k_2}\).
Proof. If not, the mapping given by $R_{k_1}$ and $R_{k_2}^{-1}$ will make $\sigma_3^q$ overlap itself, however it is possible only when $R_{k_1} \cdot R_{k_2}^{-1} = 1$. The conclusion follows. ■

Corollary 2 We can always divide $\sigma_{00}$ into a collection of $|G|$ little blocks $\sigma_3^q$ ($|G|$ is the rank of group $G$), i.e. $\{S^l\} (l = 1, 2, \cdots , |G|), \bigcup_0^l S^l = \sigma_{00}$, under $R_k \in G$, $S^1$ can be mapped into any set $S^k, k = 1, 2, \cdots , |G|$.

Proof. Arbitrarily to take a little block $\sigma_3^q$, each transformation induced by the group element of $G$ can change it into $\sigma_3^q = \sigma_3^{q_1}, \sigma_3^{q_2}, \cdots , \sigma_3^{q_{|G|}}$. There is no superposition for these blocks due to corollary 1. Using the above method, pick out $|G|$ $\sigma_3^q$ and deal with the rest similarly. Since the number of the blocks is finite, the selection can always be finished. Therefore the $S^1$ can be made up of the first blocks picking out every time, the corollary is obtained. ■

Corollary 3 Let $\sigma_4 \subset \sigma_3^{q_j}$ be a continuous region belonging to $X_z$, its dimension is $2N$, if the mapping by both $R_{k_1}$ and $R_{k_2}$ send $\sigma_4$ to the same continuous region $\sigma'_4 \subset \sigma_3^{q_j}$, then we have

$$R_{k_1} \left( \vec{B}_0 + \vec{t} \right) = R_{k_2} \left( \vec{B}_0 + \vec{t} \right).$$

(80)

Proof. Due to Lemma 1, $R_{k_1} = R_{k_2}$. ■

6 The Complete Set of the Projectors of $T^{2N}/G$

Next we will construct the complete set of the projectors on $T^{2N}/G$. Let $\hat{O}$ be an operator on $T^{2N}$ and $\hat{O}$ commutates with $\{U_s\}$. Let $|\psi\rangle$ be the common eigenstate of $\{U_s\}$, then $\hat{O} |\psi\rangle$ is also their common eigenstate.

$$U_s \hat{O} |\psi\rangle = \hat{O} U_s |\psi\rangle = \lambda_s \hat{O} |\psi\rangle,$$

namely $\hat{O} |\psi\rangle$ is also the eigenstate of $U_s$ with the same eigenvalue as for $|\psi\rangle$. So we have

$$\hat{O} \left( \vec{B}_0 + \vec{t} \right) = \sum_{l'} M_{l't} \left( \vec{B}_0 + \vec{t}_{l'} \right),$$

(81)

23
$M_{v'l}^0$ is a function with respect to $(\vec{B}_0, \vec{q}_0)$. Obviously,

$$\hat{A}\hat{B}\left| (\vec{B}_0, \vec{q}_0) + \vec{t}_v \right\rangle = \hat{A} \sum_{v'} M_{v'l}^B \left| (\vec{B}_0, \vec{q}_0) + \vec{t}_v \right\rangle = \sum_{v'} M_{v'l}^B M_{v''v'}^A \left| (\vec{B}_0, \vec{q}_0) + \vec{t}_v \right\rangle = (M^A M^B)_{v'l} \left| (\vec{B}_0, \vec{q}_0) + \vec{t}_v \right\rangle.$$  

(82)

Let $M^P \left( \vec{B}_0, \vec{q}_0 \right)$ be the corresponding $N_d \times N_d$ matrix, then for every projector of $T^{2N}$, we have

$$P \rightarrow M^P \left( \vec{B}_0, \vec{q}_0 \right),$$

$$(M^P)^2 = M^P.$$  

Since $\left| (\vec{B}_0, \vec{q}_0) + \vec{t}_v \right\rangle$ is orthogonal and complete, we may rewrite (55) as

$$id = \frac{1}{v} \sum_{l=1}^{N_d} \int_{\sigma_0} d^N \vec{B}_0 d^N \vec{q}_0 \left| (\vec{B}_0, \vec{q}_0) + \vec{t}_l \right\rangle \left\langle (\vec{B}_0, \vec{q}_0) + \vec{t}_l \right\|.$$  

Thus any operator $\hat{O}$ can be expressed by their corresponding $M_{v'l}^0 \left( \vec{B}_0, \vec{q}_0 \right)$ as

$$\hat{O} = \frac{1}{v} \sum_{l} \int_{\sigma_0} d^N \vec{B}_0 d^N \vec{q}_0 \hat{O} \left| (\vec{B}_0, \vec{q}_0) + \vec{t}_l \right\rangle \left\langle (\vec{B}_0, \vec{q}_0) + \vec{t}_l \right|$$

$$= \frac{1}{v} \sum_{l'} \int_{\sigma_0} d^N \vec{B}_0 d^N \vec{q}_0 M_{v'l}^0 \left( \vec{B}_0, \vec{q}_0 \right) \left| (\vec{B}_0, \vec{q}_0) + \vec{t}_{l'} \right\rangle \left\langle (\vec{B}_0, \vec{q}_0) + \vec{t}_{l'} \right|.$$  

(83)

Conversely, for a given $M_{v'l}^0 \left( \vec{B}_0, \vec{q}_0 \right)$ we can construct an operator $\hat{O}$ via the right-hand side of (83). One can show that the operator satisfies $[\hat{O}, U_s] = 0, (s = 1, 2, \cdots, 2N)$ and the equation (81).

Next we study the relation between the $M \left( \vec{B}_0, \vec{q}_0 \right)$ matrices before and after action of rotation $R$. Recall (75), we have

$$|\psi\rangle = R^{-1} \hat{O} R \left| (\vec{B}_0, \vec{q}_0) + \vec{t}_l \right\rangle = R^{-1} \hat{O} A_{v'l} \left| (\vec{B}_0, \vec{q}_0) + \vec{t}_l \right\rangle \left| (\vec{B}_0, \vec{q}_0)' + \vec{t}_v \right\rangle$$

$$= \sum_{l'v''} A_{v'l} M_{v'v''} \left( \vec{B}_0, \vec{q}_0 \right) \left| (\vec{B}_0, \vec{q}_0)' + \vec{t}_v \right\rangle R^{-1} \left| (\vec{B}_0, \vec{q}_0) + \vec{t}_v \right\rangle.$$  

(84)
Let
\[ R^{-1} \left| \vec{B}'_0, q'_0 \right> + \vec{t}_l \right> = \sum_k A_{R^{-1}} (\vec{B}'_0, q'_0) \left| \vec{B}''_0, q''_0 \right> + \vec{t}_k \]

Because when \( \hat{O} = id \), we have
\[ R^{-1}IR \left| \vec{B}_0, q_0 \right> + \vec{t}_l \right> = \sum_{kl'} A_{R^{-1}} (\vec{B}'_0, q'_0) \left| \vec{B}''_0, q''_0 \right> + \vec{t}_k \]
\[ \Rightarrow (\vec{B}''_0, q''_0) = (\vec{B}_0, q_0) . \]

So, Eq. (84) becomes
\[ |\psi> = \left[ A^{-1} (\vec{B}_0, q_0) M (\vec{B}''_0, q''_0) A (\vec{B}_0, q_0) \right]_{kl'} \left| \vec{B}_0, q_0 \right> + \vec{t}_k , \quad (85) \]
namely, when \( \hat{O} \) corresponds to \( M \), \( R^{-1}\hat{O}R \) corresponds to
\[ \tilde{M} (\vec{B}_0, q_0) = A^{-1} (\vec{B}_0, q_0) M (\vec{B}''_0, q''_0) A (\vec{B}_0, q_0) . \]

We have
\[ M (\vec{B}''_0, q''_0) = A (\vec{B}_0, q_0) \tilde{M} (\vec{B}_0, q_0) A^{-1} (\vec{B}_0, q_0) . \quad (86) \]

Then in order to make \( \hat{O} \) invariant under \( R \), we need the sufficient and necessary condition
\[ \tilde{M} = M \quad (87) \]
as a matrix function of \( (\vec{B}_0, q_0) \). Namely,
\[ A (\vec{B}_0, q_0) M (\vec{B}_0, q_0) A^{-1} (\vec{B}_0, q_0) = M (\vec{B}_0, q_0) . \quad (88) \]

In the following, we begin to construct the complete set of the operators \( P \):

First of all, in a little region \( S^1 \) of \( \sigma_{00} \), one matrix \( M(\vec{B}_0, q_0) \) satisfying \( M^2 = M \) can certainly be generated by the general formula \( M(\vec{B}_0, q_0) = \)
$T^{-1}(\vec{B}_0, \vec{q}_0)M_0(\vec{B}_0, \vec{q}_0)T(\vec{B}_0, \vec{q}_0)$(ref.[41]), where $M_0(\vec{B}_0, \vec{q}_0)$ is an arbitrary diagonal matrix with diagonal elements taking 0 or 1, and $T$ is an arbitrary invertible matrix. $T$ and $M_0$ are both functions with respect to $(\vec{B}_0, \vec{q}_0)$. Through the equation (88) we assign $M$ in the region $S^k$ as

$$M \left( \vec{B}_0', \vec{q}_0 \right) = A_{R_k} \left( \vec{B}_0, \vec{q}_0 \right) M \left( \vec{B}_0, \vec{q}_0 \right) A_{R_k}^{-1} \left( \vec{B}_0, \vec{q}_0 \right),$$  \hspace{1cm} (89)

where $(\vec{B}_0', \vec{q}_0')$ is the image of $(\vec{B}_0, \vec{q}_0)$ induced by $R_k \in G$, $A$ is the matrix corresponding to $R_k$. By corollary 2, this construction can cover the whole cell $\sigma_{00}$. We thus get $M$ corresponding to the whole $\sigma_{00}$.

The $M(\vec{B}_0, \vec{q}_0)$ in the whole $\sigma_{00}$ keeps invariant under action of any group element $R \in G$. The following is the proof:

**Proof.** Let

$$\tilde{M} \left( \vec{B}_0', \vec{q}_0 \right) = A \left( \vec{B}_0, \vec{q}_0 \right) M \left( \vec{B}_0, \vec{q}_0 \right) A^{-1} \left( \vec{B}_0, \vec{q}_0 \right).$$  \hspace{1cm} (90)

where $R$ maps $(\vec{B}_0, \vec{q}_0)$ to $(\vec{B}_0', \vec{q}_0')$.

1. If $(\vec{B}_0, \vec{q}_0) \in S^1$, according to the construction, the conclusion is obviously valid.

2. If $(\vec{B}_0, \vec{q}_0) \notin S^1$, set $(\vec{B}_0, \vec{q}_0) \in S', \left( \vec{B}_0, \vec{q}_0 \right) = R' \left( \vec{B}, \vec{q} \right)$ where $R' \left( \vec{B}, \vec{q} \right)$ is the mapping from point $(\vec{B}, \vec{q})$ in the $S^1$ to $(\vec{B}_0, \vec{q}_0)$, then under $R$, from (90) we have

$$\tilde{M} \left( \vec{B}_0', \vec{q}_0 \right) = A_R \left( \vec{B}_0, \vec{q}_0 \right) M \left( \vec{B}_0, \vec{q}_0 \right) A_{R_k}^{-1} \left( \vec{B}_0, \vec{q}_0 \right) = A_R \left( \vec{B}_0, \vec{q}_0 \right) A_{R'} \left( \vec{B}, \vec{q} \right) M \left( \vec{B}, \vec{q} \right) A_{R'}^{-1} \left( \vec{B}, \vec{q} \right) A_{R_k}^{-1} \left( \vec{B}_0, \vec{q}_0 \right).$$  \hspace{1cm} (91)

Since under action of the group elements of $G$, the transformation of $\left( \vec{B}_0, \vec{q}_0 \right) + t_i$ should form a representation of $G$, we have (75)

$$A_R \left( \vec{B}_0, \vec{q}_0 \right) A_{R'} \left( \vec{B}, \vec{q} \right) = A_{(RR')} \left( \vec{B}, \vec{q} \right).$$

26
The mapping induced by \((RR')\) in the \(\sigma_{00}\) just send \((\tilde{B}, \tilde{q}) \rightarrow (\tilde{B}_0', \tilde{q}_0')\):
\[
RR' \left| \left( \tilde{B}, \tilde{q} \right) + \vec{t}_1 \right> = RA_{R'} \left( \tilde{B}, \tilde{q} \right)_{t_2l_1} \left| \left( \tilde{B}_0, \tilde{q}_0 \right) + \vec{t}_{l_2} \right>
= A_{R'} \left( \tilde{B}, \tilde{q} \right)_{t_2l_1} A_R \left( \tilde{B}_0, \tilde{q}_0 \right)_{t_3l_2} \left| \left( \tilde{B}_0', \tilde{q}_0' \right) + \vec{t}_{l_3} \right> = A_{(RR')} \left( \tilde{B}, \tilde{q} \right)_{t_3l_1} \left| \left( \tilde{B}_0', \tilde{q}_0' \right) + \vec{t}_{l_3} \right>.
\]

So, we get
\[
A_R \left( \tilde{B}_0, \tilde{q}_0 \right) A_{R'} \left( \tilde{B}, \tilde{q} \right) = A_{(RR')} \left( \tilde{B}, \tilde{q} \right) = A_{R} \left( \tilde{B}_0, \tilde{q}_0 \right) A_{R'}^{-1} \left( \tilde{B}_0', \tilde{q}_0' \right) \left( \tilde{B}, \tilde{q} \right) \left( \tilde{B}_0, \tilde{q}_0 \right) A_{R'} \left( \tilde{B}_0', \tilde{q}_0' \right) = A_{R} \left( \tilde{B}_0, \tilde{q}_0 \right) A_{R'}^{-1} \left( \tilde{B}_0', \tilde{q}_0' \right) \left( \tilde{B}, \tilde{q} \right) = A_{R} \left( \tilde{B}_0, \tilde{q}_0 \right) A_{R'}^{-1} \left( \tilde{B}_0', \tilde{q}_0' \right) \left( \tilde{B}, \tilde{q} \right) = A_{R} \left( \tilde{B}_0, \tilde{q}_0 \right) A_{R'}^{-1} \left( \tilde{B}_0', \tilde{q}_0' \right) \left( \tilde{B}, \tilde{q} \right)
\]

From \((91)\) we have
\[
\tilde{M} \left( \tilde{B}_0', \tilde{q}_0' \right) = A_{(RR')} \left( \tilde{B}, \tilde{q} \right) M \left( \tilde{B}, \tilde{q} \right) A_{R} \left( \tilde{B}_0, \tilde{q}_0 \right) A_{R'}^{-1} \left( \tilde{B}_0', \tilde{q}_0' \right) \left( \tilde{B}, \tilde{q} \right) \left( \tilde{B}_0', \tilde{q}_0' \right) \left( \tilde{B}, \tilde{q} \right)
\]

From \((92)\) we see that \(RR'\) maps \(S^1\) to \(S'\) which implies \(\tilde{M} \left( \tilde{B}_0', \tilde{q}_0' \right) = M \left( \tilde{B}_0', \tilde{q}_0' \right) \). Thus based on the above construction, the right hand side of \((94)\) equals to \(M \left( \tilde{B}_0', \tilde{q}_0' \right) \). Since \(M\) constructed like this keeps invariant under action of any \(R \in G\), the corresponding operator formed as \((83)\) is also invariant under \(G\). Due to \(M^2 = M\) in the region \(S^1\), we get \(\tilde{M}^2 = \tilde{M}\). By construction, therefore \(M^2 = M\) also holds in the other regions \(S^k\).

So, we achieve the complete set of projectors on the \(T^{2N}/G\).

7 Discussion

When we take
\[
M \left( \tilde{B}_0, \tilde{q}_0 \right)_{t_2l_1} = \frac{< \left( \tilde{B}_0, \tilde{q}_0 \right) + \vec{t}_{l_2} | \Omega > < \Omega | \left( \tilde{B}_0, \tilde{q}_0 \right) + \vec{t}_{l_1} >}{\sum_{l=1}^{N_d} < \left( \tilde{B}_0, \tilde{q}_0 \right) + \vec{t}_l | \Omega > < \Omega | \left( \tilde{B}_0, \tilde{q}_0 \right) + \vec{t}_l >}
\]

and when \(R_k | \Omega \rangle = e^{i \omega_k} | \Omega \rangle\), \(M \left( \tilde{B}_0, \tilde{q}_0 \right)\) satisfies all the requirements about the reduced matrix \(M(\tilde{B}_0, \tilde{q}_0)\) of the projector on the \(T^{2N}/G\). This is the
GHS construction [39]. We can construct the closed solution of the projectors in terms of hyper elliptic function, we will study this question in the sequel.

8 Acknowledgments

This work is supported by the National Natural Science Foundation of China granted by No.10175050.

References

[1] H. Bacry, A. Grassman and J. Zak, Phys. Rev. B12(1975) 1112.

[2] J. Zak, In solid State Physics, edited by H. Ehrenreich, F. Seitz and D. Turnbull (Academic, New York, 1972), Vol. 27.

[3] H. S. Snyder, quantized space time, Phys. Rev. 71(1947)38, the electromagnetic field in quantized space time, phys. Rev. 72(1947)68.

[4] A. Connes, Non-commutative Geometry, Academic Press, 1994.

[5] G. Landi, "An introduction to non-commutative space and their geometry", hep-th/9701078; J. Varilly, "An introduction to non-commutative Geometry", physics/9709045.

[6] J. Madore, "An introduction to non-commutative Differential Geometry and its physical Applications", Cambridge University press 2nd edition, 1999.

[7] E. Witten, "Noncommutative Geometry and String Field Theory", Nucl. Phys. B268 (1986) 253.

[8] A. Connes, M. Douglas, A. Schwartz, "Matrix theory compactification on Tori", JHEP 9802 (1998) 003, hep-th/9711162; M. dougals, C. Hull, JHEP 9802 (1998) 008, hep-th/9711165.
[9] Nathan. Seiberg and Edward. Witten, "String theory and noncommutative geometry", JHEP 9909 (1999) 032, hep-th/9908142; V. Schomerus, "D-branes and Deformation Quantization", JHEP 9906 (1999) 030.

[10] S. S. Gubser, M. Rangamani, "D-brane Dynamics and the Quantum Hall Effect", JHEP 0105 (2001) 041, hep-th/0012155.

[11] Alexios P. Polychronakos, "Quantum Hall states as matrix Chern-Simons theory", JHEP 0104 (2001) 011, hep-th/0103013.

[12] S. Hellerman, M. V. Raamsdonk, "Quantum Hall Physics equals Noncommutative Field Theory", JHEP 0110 (2001) 039, hep-th/0103179.

[13] Alexios P. Polychronakos, "Quantum Hall states as matrix Chern-Simons theory", JHEP 0104 (2001) 011, hep-th/0103013.

[14] A. P. Polychronakos, "Quantum Hall states on the cylinder as unitary matrix Chern-Simons theory", JHEP 0106 (2001) 070, hep-th/0106011.

[15] B. Morariu, A. P. Polychronakos, Finite Noncommutative Chern-Simons with a Wilson Line and the Quantum Hall Effect", JHEP 0107 (2001) 006, hep-th/0106072.

[16] L. Susskind, "The Quantum Hall Fluid and Non-Commutative Chern Simons Theory", hep-th/0101029.

[17] M. Fabinger, "Higher-Dimensional Quantum Hall Effect in String Theory", JHEP 0205 (2002) 037, hep-th/0201016.

[18] J. Hu, S.C. Zhang, "Collective excitations at the boundary of a 4D quantum Hall droplet", cond-mat/0112432.

[19] D. Karabali, V.P. Nair, "Quantum Hall Effect in Higher Dimensions", Nucl.Phys. B641 (2002) 533-546, hep-th/0203264.

[20] Y.X. Chen, B.Y. Hou, B.Y. Hou, "Non-commutative geometry of 4-dimensional quantum Hall droplet", Nucl.Phys. B638 (2002) 220-242, hep-th/0203095.

[21] B. Freivogel, L. Susskind, N. Toumbas, "A Two Fluid Description of the Quantum Hall Soliton", hep-th/0108076.
[22] S. Hellerman, L. Susskind, ”Realizing the Quantum Hall System in String Theory”, hep-th/0107200.

[23] A. Matusis, L. Susskind, N. Toumbas, ”The IR/UV Connection in the Non-Commutative Gauge Theories”, JHEP 0012 (2000) 002, hep-th/0002075.

[24] S. Minwalla, M. V. Raamsdonk, N. Seiberg, ”Noncommutative Perturbative Dynamics”, JHEP 0002 (2000) 020, hep-th/9912072.

[25] B.A.Bernevig, J. Brodie, L. Susskind, ”How Bob Laughlin Tamed the Giant Graviton from Taub-NUT space N.Toumbas”, JHEP 0102 (2001) 003, hep-th/0010105.

[26] J. Harvey, ” Komaba Lectures on Noncommutative Solitons and D-branes, hep-th/0102076; J. A. Harvey, P. Kraus and F.Larsen, JHEP 0012 (2000) 024, hep-th/0010060; M. Hamanaka and S. Terashima, ”On exact noncommutative BPS solitons”, JHEP 0103 (2001) 034, hep-th/0010221.

[27] D. J. Gross and N. A. Nekrasov, ” Solitons in noncommutative Gauge Theory”, hep-th/0010090; M. R. Douglas and N. A. Nekrasov, ”Noncommutative Field Theory”, hep-th/0106048.

[28] B. Y. Hou, D.T. Peng, K.J. Shi, R.H. Yue, ”Solitons on Noncommutative Torus as Elliptic Calogero Gaudin Models, Branes and Laughlin wave function ”, hep-th/0204163.

[29] B. Y. H, D.T. Peng, ”Elliptic Algebra and Integrable Models for Solitons on Noncommutative Torus”, Int. J. Mod. Phys. B16 (2002) 2079-2088.

[30] B.Y. Hou, K. J. Shi, Z. Y. Yang, ”Solitons on Noncommutative Orbifold $T^2/Z_N$”, Lett. Math. Phys. 61 (2002) 205-220, hep-th/0204102.

[31] A.Sen, ”Tachyon condensation on the brane antibrane system”, JHEP 08, 012(1998), hep-th/9805170.

[32] A.Sen, ”Tachyon condensation in string theory”, JHEP 0003,(2000) 0002, hep-th/9912249.

[33] G.Derrick, ”Comments on Nonlinear Wave Equations as Models for Elementary Particles”, J. Math. Phys. 5, 1252(1965).
[34] R. Gopakumar, S. Minwalla and A. Strominger, ”Noncommutative Soliton”, JHEP005 (2000) 048, hep-th/0003160.

[35] M. Rieffel, Pacific J. Math. 93 (1981) 415.

[36] F. P. Boca, Comm. Math. Phys. 202 (1999) 325.

[37] A. P. Polychronakos, ”Flux tube solutions in noncommutative gauge theories”, Phys.Lett. B495 (2000) 407-412, hep-th/0007043.

[38] E. J. Martinec and G. Moore, ”Noncommutative Solitons on Orbifolds”, hep-th/0101199.

[39] R. Gopakumar, M. Headrick, M. Spradin, ”on Noncommutative Multisolitons”, hep-th/0103256.

[40] J. Dai and X. C. Song, ”Structure and representation theory for the double group of the four-dimensional cubic group”, Jour. Math. Phys. 42 (2001) 2213.

[41] H. Deng, B-Y Hou K. J. Shi, Z. Y. Yang and R. H. Yue, ”Soliton Solutions on Noncommutative Orbifold $T^2/Z_4$”, hep-th/0305212.