Space-time fluctuation of the Kardar-Parisi-Zhang equation in \(d \geq 3\) and the Gaussian free field

Francis Comets\(^1\), Clément Cosco\(^1\), Chiranjib Mukherjee\(^2\)

July 25, 2019

\(^1\)Université de Paris, Laboratoire de Probabilités, Statistique et Modélisation, LPSM (UMR 8001 CNRS, SU, UP) Bâtiment Sophie Germain, 8 place Aurélie Nemours, 75013 Paris comets@lpsm.paris, ccosco@lpsm.paris

\(^2\)Universität Münster, Fachbereich Mathematik und Informatik, Einsteinstraße 62, Münster, D-48149 chiranjib.mukherjee@uni-muenster.de

Abstract

We study the solution \(h_\varepsilon\) of the Kardar-Parisi-Zhang (KPZ) equation in \(\mathbb{R}^d \times [0, \infty), d \geq 3\):

\[
\frac{\partial}{\partial t} h_\varepsilon = \frac{1}{2} \Delta h_\varepsilon + \left[ \frac{1}{2} |\nabla h_\varepsilon|^2 - C_\varepsilon \right] + \beta \varepsilon \frac{d-2}{\varepsilon^2} \xi_\varepsilon, \quad h_\varepsilon(0, x) = 0.
\]

Here \(\beta > 0\) is a parameter called the disorder strength, \(\xi_\varepsilon = \xi \ast \varphi_\varepsilon\) is a spatially smoothened (at scale \(\varepsilon\)) Gaussian space-time white noise and \(C_\varepsilon\) is a divergent constant as \(\varepsilon \to 0\).

When \(\beta\) is sufficiently small and \(\varepsilon \to 0\), \(h_\varepsilon(t, x) - h_{\varepsilon_n}(t, x) \xrightarrow{L^1_{\varepsilon_n}} 0\) for a measurable function \(h\) and \(\xi(\varepsilon, t, x)\) stands for the diffusively rescaled, time-reversed and spatially translated white noise, which possesses the same law as that of \(\xi\). In the present article we quantify the exact (polynomial) rate of the above convergence in this regime and show that, the space-time indexed random field

\[
\left( \varepsilon^{1-\frac{d}{2}} [h_\varepsilon(t, x) - h_{\varepsilon_n}(t, x)] \right)_{x \in \mathbb{R}^d, t > 0} \xrightarrow{\text{law}} (\mathcal{H}(t, x))_{x \in \mathbb{R}^d, t > 0}
\]

converges to a centered Gaussian field

\[
\mathcal{H}(t, x) = \gamma(\beta) \int_0^\infty \int_{\mathbb{R}^d} \rho(\sigma + t, y - x) \xi(\sigma, z) \, d\sigma \, dz
\]

with \(\rho(\sigma, x)\) being the standard heat kernel. The limiting process \(\mathcal{H}\) is also the (real-valued) solution of the non-noisy heat equation \(\partial_t \mathcal{H} = \frac{1}{2} \Delta \mathcal{H}\) with a random initial condition \(\mathcal{H}(0, x)\) given by a Gaussian free field on \(\mathbb{R}^d\). We further obtain convergence of the spatial averages for test functions \(\varphi \in C_\infty(\mathbb{R}^d)\):

\[
\int_{\mathbb{R}^d} \varphi(x) \varepsilon^{1-\frac{d}{2}} [h_\varepsilon(t, x) - h_{\varepsilon_n}(t, x)] \xrightarrow{\text{law}} \int_{\mathbb{R}^d} \varphi(x) \mathcal{H}(t, x).
\]

Keywords: SPDE, Kardar-Parisi-Zhang equation, stochastic heat equation, rate of convergence, Edwards-Wilkinson limit, Gaussian free field, directed polymers, random environment

AMS 2010 subject classifications: Primary 60K35. Secondary 35R60, 35Q82, 60H15, 82D60
1 Introduction and motivation.

1.1 Background.

We consider the Kardar-Parisi-Zhang (KPZ) equation written informally as
\[
\frac{\partial}{\partial t} h = \frac{1}{2} \Delta h + \left[ \frac{1}{2} |\nabla h|^2 - \infty \right] + \xi \quad u: \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R} \quad (1.1)
\]
and driven by a totally uncorrelated Gaussian space-time white noise $\xi$, see below for a precise definition. The above equation enjoys a huge popularity as the default model of stochastic growth in $(d + 1)$-dimensions \cite{24, 35}. For $d = 1$ this equation also becomes relevant for non-equilibrium fluctuations and appears as the scaling limit of front propagation, of exclusion processes and weakly asymmetric interacting particles \cite{3, 33, 11, 13} as well as that of the free energy of the discrete directed polymer \cite{1, 4} at intermediate disorder. However, it should be noted that, for any given time $t > 0$, its solution should locally behave like a Brownian motion indexed by the space variable $x \in \mathbb{R}^d$, which makes the non-linear term ill-defined. Also, since only distribution-valued solutions are expected, one is immediately faced with the trouble of multiplying or squaring distributions in any attempt to provide a precise notion of its solution. The KPZ equation (1.1) in $d = 1$ has now seen a huge upsurge of interest in the recent years and a vast amount of deep mathematical results are now available, see Sasamoto-Spohn \cite{33}, Amir-Corwin-Quastel \cite{2} and Quastel \cite{32} and for a rigorous construction of the notion of a solution, see the seminal papers of Hairer \cite{20, 21} as well as Gubinelli-Imkeller-Perkowski \cite{19} and Gubinelli-Perkowski \cite{18}. A different but related approach is introduced by Kupiainen in \cite{25}, adapting renormalization group techniques to subcritical non-linear parabolic SPDE's.

Before turning to precise statements, it is instructive to briefly dwell on the motivation of the present work on an informal level. We first remark that when $d > 1$, due to the irregular nature of the space-time white noise, a precise construction of a solution to (1.1) does not yield to the aforementioned theories. On the other hand, despite being ill-posed for larger dimensions, the KPZ equation still remains relevant for random surface growth and has its own appeal even in the so-called small disorder regime – a distinguishing attribute of this equation in higher dimensions, see \cite{31} for recent work in the physics literature. Indeed, in the present context we fix a spatial dimension $d \geq 3$. Then, after a suitable mollification of the noise, a renormalized solution $h_\varepsilon$ is closely related to the free energy or the partition function of a directed polymer model. Now when the mollification parameter is turned off, the limiting behavior of these two quantities is dictated by the intrinsic disorder of the system which, in total contrast to the $d = 1$ case, manifests in a non-trivial phase-transition: For large disorder strength, the partition function loses uniform integrability and eventually collapses to zero, while as weak disorder prevails, the latter object converges to a non-degenerate and strictly positive random variable, whose logarithm defines the limiting free energy $h$, a measurable function on the path space of the white noise. A rescaling, time-reversal and translation $\xi^{(\varepsilon, t, x)}$ of the white noise $\xi$ also defines a stationary random field $\{h_\varepsilon(t, x)\}_{t > 0, x \in \mathbb{R}^d}$ so that
More precisely, $h_\varepsilon(t,x) = h(\xi^{(\varepsilon,t,x)})$ also satisfies the smoothed and renormalized KPZ equation itself and the former sequence has a constant law in $\varepsilon, t, x$ (i.e., $h_\varepsilon(t,x)$ is a stationary solution). Since $h_\varepsilon(t,x) - h_\varepsilon(t,x) \to 0$ in probability, the incentive to strive for a quantitative nature of the latter convergence is quite natural, which is also closely related to the finer information one desires to glean from the space-time fluctuation of the centered field $(h_\varepsilon(t,x) - h_\varepsilon(t,x))_{t,x}$. It is imperative to stress that, such fluctuations are inherently different from the ones arising from $\varepsilon$ to necessarily restrict to spatial averaging function $\phi$ oblivious to the local fluctuations of the ambient space-time field. In this context, the goal of the present paper is to develop an alternative approach that enables us to provide an explicit description of the aforementioned limiting field $\varepsilon^{1-d/2}(h_\varepsilon(t,x) - h_\varepsilon(t,x))_{t,x}$. As we will see, results pertaining to its global fluctuations (i.e., spatial averaging) drop out from our analysis as a particular case. Let us now develop the precise mathematical lay out of the necessary objects which will enable us to provide more formal statements of our main results.

### 1.2 The KPZ equation in $d \geq 3$.

Let us now fix an ambient spatial dimension $d \geq 3$ and denote by $\xi$ a space-time white noise in $\mathbb{R}_+ \times \mathbb{R}^d$. More precisely, $\xi$ on $\mathbb{R}_+ \times \mathbb{R}^d$ is a family $\{\xi(\varphi)\}_{\varphi \in S(\mathbb{R}_+ \times \mathbb{R}^d)}$ of Gaussian random variables

$$\xi(\varphi) = \int_0^\infty \int_{\mathbb{R}^d} dt \, dx \, \xi(t,x) \varphi(t,x)$$

with mean 0 and covariance

$$\mathbb{E}[\xi(\varphi_1) \xi(\varphi_2)] = \int_0^\infty \int_{\mathbb{R}^d} \varphi_1(t,x) \varphi_2(t,x) \, dt \, dx.$$

As remarked earlier, the equation (1.1) is a-priori ill-posed, thus we will study its regularized version

$$\frac{\partial}{\partial t} h_\varepsilon = \frac{1}{2} \Delta h_\varepsilon + \left[\frac{1}{2} |\nabla h_\varepsilon|^2 - C_\varepsilon\right] + \beta \varepsilon^{\frac{d+2}{2}} \xi_\varepsilon, \quad h_\varepsilon(0,x) = 0, \quad (1.2)$$

which is driven by the spatially mollified noise

$$\xi_\varepsilon(t,x) = (\xi(t,) \ast \phi_\varepsilon)(x) = \int \phi_\varepsilon(x-y) \xi(t,y) \, dy,$$

with $\ast$ the convolution in space and $\phi_\varepsilon(\cdot) = \varepsilon^{-d} \phi(\cdot/\varepsilon)$ being a suitable approximation of the Dirac measure $\delta_0$ and $C_\varepsilon$ being a suitable divergent (renormalization) constant to be defined below. We will work with a fixed mollifier $\phi : \mathbb{R}^d \to \mathbb{R}_+$ which is smooth and spherically symmetric, with $\text{supp}(\phi) \subset B(0,\frac{1}{2})$ and $\int_{\mathbb{R}^d} \phi(x) \, dx = 1$. Thus, $\{\xi_\varepsilon(t,x)\}$ is a centered Gaussian field with covariance

$$\mathbb{E}[\xi_\varepsilon(t,x) \xi_\varepsilon(s,y)] = \delta(t-s) \varepsilon^{-d} V((x-y)/\varepsilon),$$
where $V = \phi \ast \phi$ is a smooth function supported in $B(0, 1)$.

In order to put our main results into context, it is convenient to relate the smoothened KPZ solution $h_\epsilon$ to the so-called free energy of the continuous directed polymer. For that, let
\begin{equation}
    u_\epsilon = \exp[h_\epsilon]
\end{equation}
denote the Hopf-Cole solution of the linear multiplicative noise stochastic heat equation (SHE) given by
\begin{equation}
    \frac{\partial}{\partial t} u_\epsilon = \frac{1}{2} \Delta u_\epsilon + \beta \frac{d-2}{d} u_\epsilon \xi_\epsilon, \quad u_\epsilon(0, x) = 1,
\end{equation}
where the stochastic integral above is interpreted in the classical Itô-Skorohod sense and we choose
\begin{equation}
    C_\epsilon = \beta^2 (\phi \ast \phi)(0) \varepsilon^{-2} = \beta^2 V(0) \varepsilon^{-2} / 2.
\end{equation}
Then, by the Feynman-Kac formula ([26, Theorem 6.2.5]) we have
\begin{equation}
    u_\epsilon(t, x) = E_x \left[ \exp \left\{ \beta \varepsilon^{d-2} \int_0^t \int_{\mathbb{R}^d} \phi_\epsilon(W_{t-s} - y) \xi(s, y) \, dy \, ds - t C_\epsilon \right\} \right].
\end{equation}
with $E_x$ denoting expectation with respect to the law $P_x$ of a $d$-dimensional Brownian path $W = (W_s)_{s \geq 0}$ starting at $x \in \mathbb{R}^d$, which is independent of the noise $\xi$. Now if we extend $\xi$ also to negative times and set
\begin{equation}
    \xi^{(-t, x)}(s, y) = \varepsilon^{-2} \xi(t - \varepsilon^2 s, \varepsilon y - x)
\end{equation}
to be the diffusively rescaled, time-reversed and spatially translated noise, then $\xi^{(-t, x)}$ is itself a Gaussian white noise and possesses the same law as that of $\xi$. To abbreviate notation, we will also write
\begin{equation}
    \xi^{(-t, x)} = \xi^{(-t, 0)}.
\end{equation}
Plugging (1.7) in (1.6), and using Brownian scaling and time-reversal, we get the a.s. equality
\begin{equation}
    u_\epsilon(t, x) = \mathcal{Z}_{T_\epsilon}(\xi^{(-t, 0)}),
\end{equation}
where
\begin{equation}
    \mathcal{Z}_{T}(x) = \mathcal{Z}_{T}(\xi; x) = E_x \left[ \exp \left\{ \beta \int_0^T \int_{\mathbb{R}^d} \phi(W_s - y) \xi(s, y) \, dy \, ds - \frac{\beta^2 T}{2} V(0) \right\} \right]
\end{equation}
denotes the martingale corresponding to the normalized partition function of the continuous directed polymer in a white noise environment $\xi$, while $\log \mathcal{Z}_{T}$ now stands for its aforementioned free energy. It follows that there exists $\beta_c \in (0, \infty)$ such that $(\mathcal{Z}_{T})_T$ is uniformly integrable for $\beta \in (0, \beta_c)$ and there is a strictly positive non-degenerate random variable $\mathcal{Z}_{\infty}(x)$ so that, a.s. as $T \to \infty$,
\begin{equation}
    \mathcal{Z}_{T}(x) \to \begin{cases} 
        \mathcal{Z}_{\infty}(x) & \text{if } \beta \in (0, \beta_c), \\
        0 & \text{if } \beta \in (\beta_c, \infty).
    \end{cases}
\end{equation}
See [30], or [7] for a general reference. Therefore, if \( u(\xi) \) is any arbitrary representative of the strictly positive random limit \( \mathcal{X}_\infty = \mathcal{X}_\infty(0) \), then, letting \( h = \log u \) and

\[
h_\epsilon(t, x) = h(\xi^{(\epsilon,t,x)}),
\]

we have (recall (1.9)) that for any \( t > 0, x \in \mathbb{R}^d \) and \( \beta \in (0, \beta_c) \),

\[
h_\epsilon(t, x) - h_\epsilon(t, x) \xrightarrow{\mathbb{P}} 0.
\]

It turns out that the limit \( h_\epsilon(t, x) \) is a stationary solution of (1.2) with initial condition \( h(\xi^{(\epsilon,0,x)}) \). In particular, the family \( \{h_\epsilon(t, x)\}_{\epsilon,t,x} \) is constant in law, the law being that of \( \log \mathcal{X}_\infty \) which does not fluctuate in \( \epsilon, t, x \), see Remark 2.6. In this vein, our first main result quantifies the nature of the convergence (1.13) and identifies the limiting distribution of the polynomially rescaled process

\[
\mathcal{H}_\epsilon(t, x) \xrightarrow{\mathbb{D}} 0, x \in \mathbb{R}^d \quad \text{with} \quad \mathcal{H}_\epsilon(t, x) = \epsilon^{1 - \frac{d}{2}} [h_\epsilon(t, x) - h_\epsilon(t, x)]
\]

which, for \( \beta \) small enough, is shown to converge (in the sense of finite dimensional distributions) to an explicit centered Gaussian field \( \mathcal{H}(t, x) \), see Theorem 2.1 below. The latter object is a pointwise solution to the unperturbed heat equation, but with a random initial condition which is a Gaussian free field (GFF) in \( d \geq 3 \) with a perturbed covariance. To this end, we remark that fluctuation results of the averages of the KPZ solution in \( d \geq 3 \) have seen recent progress (see Magnen-Unterberger [28] and Dunlap-Gu-Ryzhik-Zeitouni [14]). However, all these results hold only for the spatial averages of the fluctuations \( \epsilon^{1 - \frac{d}{2}} \int_{\mathbb{R}^d} dx \varphi(x)[h_\epsilon(t, x) - \mathbb{E}[h_\epsilon(t, x)]] \) for a fixed test function \( \varphi \) and the latter quantities converge to the appropriate integrals of the Edwards-Wilkinson equation, see Remark 2.6. We stress that our convergence results hold on the level of processes in \( t > 0 \) and \( x \in \mathbb{R}^d \) (and in particular, pointwise for each fixed \( t, x \)), while the pointwise fluctuations \( h_\epsilon(t, x) - \mathbb{E}[h_\epsilon(t, x)] \) of deterministic centering already diverge at each \( x \in \mathbb{R}^d \) and only spatial mollification of the latter quantity enables oscillations to cancel. On the other hand, our result on the convergence of the process \( \{\mathcal{H}_\epsilon(t, x)\}_{\epsilon > 0, x \in \mathbb{R}^d} \) does imply convergence of spatial averages. Indeed, combined with further technical estimates (e.g. moment estimates, tightness etc.) we prove along the way, the convergence

\[
\int_{\mathbb{R}^d} dx \varphi(x) \mathcal{H}_\epsilon(t, x) \to \int_{\mathbb{R}^d} dx \varphi(x) \mathcal{H}(t, x)
\]

drops out, with \( \mathcal{H}(t, x) \) solving the above random PDE (i.e. heat equation with GFF initial condition), see Theorem 2.4. Moreover, the latter result, together with the earlier Edwards-Wilkinson fluctuation of \( \epsilon^{1 - \frac{d}{2}} \int_{\mathbb{R}^d} dx \varphi(x)[h_\epsilon(t, x) - \mathbb{E}[h_\epsilon(t, x)]] \) then implies that the spatially averaged and deterministically centered fluctuation \( \epsilon^{1 - \frac{d}{2}} \int_{\mathbb{R}^d} dx \varphi(x)[h_\epsilon(t, x) - \mathbb{E}[h_\epsilon(t, x)]] \) of the stationary solution \( h_\epsilon(t, x) = h(\xi^{(\epsilon,t,x)}) \) of the KPZ equation (1.2) itself (we remind the reader
that \( h_\varepsilon(t,x) \) is random, but is constant in law for any \( \varepsilon, t, x \). It follows that the latter averages now converge to the spatial averages of the independent sum \( \mathcal{H} + \mathcal{H} \) of the Edwards-Wilkinson limit \( \mathcal{H} \) and our random limit \( \mathcal{H} \), see Remark 2.7.

Summarizing, the present results pertaining to space-time convergence provide fine information of the underlying deviations of the ambient field which, while including convergence of its averages, remain impervious to investigations of global fluctuations alone for the aforementioned reasons. For these reasons, the method of our proof is also quite independent of the existing literature, for which we develop a new technique by leveraging tools from stochastic analysis (see Section 2.1 for a brief outline of the present approach and comparison with the existing techniques). The present method is robust and quite conceivably can be used in a wider context for studying space-time process fluctuations of a wide class of functionals of multiplicative-noise random PDEs in transient dimensions (cf. Remark 2.8). Let us now turn to a precise description of our main results alluded to in the above discussion.

### 2 Main results.

We will write \( \rho(t,x) = \frac{1}{(2\pi t)^{d/2}} e^{-|x|^2/2t} \) for the standard Gaussian kernel. Throughout the sequel we will write

\[
\gamma^2(\beta) = \beta^2 \int_{\mathbb{R}^d} \text{d}y \ V(y) \ E_y \left[ e^{\beta^2 \int_0^\infty V(W_s) \ ds} \right]. \tag{2.1}
\]

Note that, for \( \beta \) large, \( \gamma^2(\beta) \) diverges.

Here is our first main result. For a sequence of time-space random fields we denote by \( \overset{\text{f.d.m.}}{\rightarrow} \) the convergence in the sense of finite dimensional marginal distributions in time and space.

**Theorem 2.1** (Space-time fluctuations of the KPZ equation). Assume \( d \geq 3 \). Consider the solution \( h_\varepsilon \) to (1.2) with \( h_\varepsilon(0, \cdot) = 0 \) and the corresponding deviations \( \mathcal{H}_\varepsilon \) defined in (1.14). Then there exists \( \beta_0 \in (0, \beta_c) \) such that for \( \beta < \beta_0 \) and as \( \varepsilon \to 0 \),

\[
(\mathcal{H}_\varepsilon(t,x))_{x \in \mathbb{R}^d, t>0} \overset{\text{f.d.m.}}{\rightarrow} (\mathcal{H}(t,x))_{x \in \mathbb{R}^d, t>0}
\]

where

\[
\mathcal{H}(t,x) = \gamma(\beta) \int_0^\infty \int_{\mathbb{R}^d} \rho(\sigma + t, x - z) \xi(\sigma, z) \, d\sigma \, dz,
\]

and \( (\mathcal{H}(t,x))_{x \in \mathbb{R}^d, t\geq 0} \) is a centered Gaussian field with covariance

\[
\text{Cov}(\mathcal{H}(t,x), \mathcal{H}(s,y)) = \gamma^2(\beta) \int_0^\infty \rho(2\sigma + t + s, y - x) \, d\sigma.
\]

In particular, for any \( x \in \mathbb{R}^d \) and \( t > 0 \), as \( \varepsilon \to 0 \),

\[
\mathcal{H}_\varepsilon(t,x) \xrightarrow{\text{law}} \mathcal{N}\left(0, \gamma^2(\beta)\frac{2}{d-2} \frac{d}{(4\pi)^{d/2}} t^{-(d-2)/2}\right).
\]

(2.4)
Remark 2.2 (The limit $\mathcal{H}(t, x)$). The following two observations pertaining to the limiting Gaussian field $(\mathcal{H}(t, x))_{x \in \mathbb{R}^d, t \geq 0}$ in (2.2) are useful.

1. One can define the value $\mathcal{H}(0, x)$ at time 0 through the formula (2.2) with $t = 0$, which is again a mean-zero Gaussian field with covariance structure given by

$$\text{Cov}(\mathcal{H}(0, x), \mathcal{H}(0, y)) = \gamma^2(\beta) \int_0^\infty \rho(2\sigma, x - y) \, d\sigma = \frac{\gamma^2(\beta) \Gamma\left(\frac{d}{2} - 1\right)}{\pi^{d/2} |x - y|^{d-2}},$$

which is a multiple of the Green’s function for Brownian motion in $d \geq 3$. In other words, $\mathcal{H}(0, x)$ is a $d$-dimensional Gaussian free field.

2. Next, we see that

$$\mathcal{H}(t, x) = \gamma(\beta) \int_0^\infty \rho(\sigma + t, \cdot) \ast \xi(\sigma, \cdot)(x) \, d\sigma = \left(\rho(t, \cdot) \ast \gamma(\beta) \int_0^\infty \rho(\sigma, \cdot) \ast \xi(\sigma, \cdot) \, d\sigma\right)(x) = \rho(t, \cdot) \ast \mathcal{H}(0, \cdot)(x).$$

Combining the two observations in Remark 2.2 above, we obtain

**Proposition 2.3.** The limit $\mathcal{H}(t, x)$ is a real-valued solution of the non-noisy heat equation

$$\partial_t \mathcal{H} = \frac{1}{2} \Delta \mathcal{H},$$

but with random initial condition given by the Gaussian free field $\mathcal{H}(0, \cdot)$ with covariance (2.5).

Note that Theorem 2.1 provides convergence of the finite dimensional distributions of the field $\{\mathcal{H}(t, x)\}_{t > 0, x \in \mathbb{R}^d}$. In light of the discussion in Remark 2.2 and Proposition 2.3, it is natural to wonder if a convergence holds for the spatially averaged deviations $\int_{\mathbb{R}^d} d_x \varphi(x) \mathcal{H}(t, x)$ for test functions $\varphi$. Our next result answers this question in the affirmative:

**Theorem 2.4** (Spatially averaged fluctuations of $h_\varepsilon$). For $\beta \in (0, \beta_0)$ as in Theorem 2.1 and for any smooth function $\varphi \in C^\infty_c(\mathbb{R}^d)$ with compact support and any $t > 0$, as $\varepsilon \to 0$,

$$\int_{\mathbb{R}^d} \mathcal{H}_\varepsilon(t, x) \varphi(x) \, dx \xrightarrow{\text{law}} \int_{\mathbb{R}^d} \mathcal{H}(t, x) \varphi(x) \, dx.$$  

Moreover, the above convergence holds jointly for finitely many $\varphi_1, \ldots, \varphi_n \in C^\infty_c(\mathbb{R}^d)$.

As a byproduct of the arguments constituting the proof of Theorem 2.1 and Theorem 2.4, we also obtain a tightness property of the process $\{\mathcal{H}(t, x)\}_{\varepsilon > 0, x \in \mathbb{R}^d}$ which holds in a function space $C^\alpha(\mathbb{R}^d)$ of distributions with “local $\alpha$-regularity” for $\alpha < 0$, see Section 5.4 for a detailed definition. While this result is not directly used in the proof of Theorem 2.1 or Theorem 2.4, it may be of independent interest.
**Proposition 2.5** (Tightness). For any \( \beta \in (0, \beta_0) \) as in Theorem 2.1 and \( t > 0 \), the family \( \{ \mathcal{H}_\epsilon(t, x) \}_{\epsilon > 0, x \in \mathbb{R}^d} \) is tight in the function space \( C^\alpha(\mathbb{R}^d) \) for all \( \alpha < -\frac{d}{2} \). Together with Theorem 2.4, we conclude that \( \{ \mathcal{H}_\epsilon(t, x) \}_{\epsilon > 0, x \in \mathbb{R}^d} \) converges to \( \{ \mathcal{H}(t, x) \}_{x \in \mathbb{R}^d} \) in \( C^\alpha(\mathbb{R}^d) \) for all \( \alpha < -\frac{d}{2} \).

**Remark 2.6** (The random centering in \( \mathcal{H}_\epsilon \)). Note that in our main results, we studied the rescaled deviation \( \mathcal{H}_\epsilon(t, x) = \epsilon^{1-d/2}(h_\epsilon(t, x) - \bar{h}_\epsilon(t, x)) \) which is a random, \( \epsilon \)-dependent centering of \( h_\epsilon \). At this point, we stress that the sequence \( \{ h_\epsilon(t, x) \}_{\epsilon > 0} \) combines three interesting traits:

- It is constant in law for all \( \epsilon, t, x \), with the law determined by that of \( \log Z_\infty \) (see Proposition 3.3 for the spatial correlation structure of \( Z_\infty \)).
- It approximates \( h_\epsilon(t, x) \) pointwise in probability as \( \epsilon \to 0 \), for any fixed \( t, x \). Since it depends on \( \epsilon \), it is not a strong limit, but it can be used similarly which allows one to study the rescaled fluctuations \( \epsilon^{1-d/2}[h_\epsilon(t, x) - \bar{h}_\epsilon(t, x)] \).
- It is a stationary solution of the regularized KPZ equation (1.2) with initial condition \( h_\epsilon(0, x) = h(\xi(0, x)) \). This fact can be observed using the self-consistent property (3.8) and the Feynman-Kac formula.

As already pointed out in Section 1.1-1.2, of course, the scenario is quite different from studying a deterministic centering \( h_\epsilon(t, x) - \mathbb{E}[h_\epsilon(t, x)] \) since in this case the latter quantity does not converge to zero pointwise. Only spatial averaging with respect to smooth test functions \( \varphi \) causes oscillations to cancel and as shown in [28, 14],

\[
\epsilon^{1-d/2} \int_{\mathbb{R}^d} \varphi(x) (h_\epsilon(t, x) - \mathbb{E}[h_\epsilon(t, x)]) \, dx \xrightarrow{\text{law}} \int_{\mathbb{R}^d} \varphi(x) \mathcal{H}(t, x) \, dx,
\]

with \( \mathcal{H} \) solving the stochastic heat equation with additive noise:

\[
\partial_t \mathcal{H} = \frac{1}{2} \Delta \mathcal{H} + \varphi(\beta) \xi, \quad \mathcal{H}(0, x) = 0
\]

where \( \varphi^2(\beta) = \int_{\mathbb{R}^d} V(x) E_x \left[ \exp\{ \beta^2 \int_0^\infty V(W_{2t}) \, dt \} \right] \) is a constant which, like \( \gamma^2(\beta) \), also diverges for \( \beta \) large (compare to (2.1)). For similar results in \( d = 2 \), see [18, 25]. Related questions have been studied by mollifying the noise in both time and space which leads to homogenized diffusion coefficients in (2.9), i.e., \( \frac{1}{2} \Delta \) is replaced by \( \frac{1}{2} \text{div}(a_\beta \nabla) \) where \( a_\beta \neq I_{d \times d} \) (see [29, 17]).

Though being totally independent, our results share some common features with [15], which considers the stochastic homogenization point of view for the heat equation in a random environment.

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1 It is now well understood that \( u_\epsilon \) converges pointwise in law but not in probability (e.g., [30]) and that \( u_\epsilon(t; x) \) and \( u_\epsilon(t; y) \) become asymptotically independent for \( x \neq y \) and \( \beta \) small (see e.g., [33]). Thus, for convergence in probability, spatial averaging w.r.t. a fixed function \( \varphi \) becomes imperative which makes the averaging deterministic in the limit, i.e., \( \int_{\mathbb{R}^d} \varphi(x) u_\epsilon(t, x) \, dx \xrightarrow{\text{law}} \int_{\mathbb{R}^d} \varphi(x) \mathcal{H}(t, x) \, dx \) with \( \mathcal{H}(t, x) = \mathbb{E}[u_\epsilon(t, x)] \) solving the unperturbed heat equation \( \partial_t \mathcal{H} = \frac{1}{2} \Delta \mathcal{H} \). Therefore, meaningful limits of fluctuations via deterministic centering \( u_\epsilon(t, x) - \mathbb{E}[u_\epsilon(t, x)] \) can only be obtained by spatial averaging in \( x \) which again precludes the local picture.
potential (a space-time function). Again, the space dimension is \( d \geq 3 \) and the potential is assumed to be small. The authors derive a (random) corrector, i.e., a pointwise approximation of the solution up to second order as the ratio of the scale of variation of the initial condition to the correlation length of the noise vanishes. Here also, taking a random centering is crucial.

\textbf{Remark 2.7.} While we do not carry out a complete proof, we now allude to the following conceivable implication of Theorems 2.1 and 2.4. Note first that \( \lim_{\varepsilon \to 0} \mathbb{E}[\mathcal{H}_\varepsilon] = 0 \), by Theorem 2.4 and uniform integrability stemming from Proposition 3.5. Therefore, rewriting

\[
\int_{\mathbb{R}^d} \varphi(x) \varepsilon^{1-d/2} [h_\varepsilon(t,x) - \mathbb{E}[h_\varepsilon(t,x)]] \, dx = \int_{\mathbb{R}^d} \varphi(x) \mathcal{H}_\varepsilon(t,x) \, dx + \int_{\mathbb{R}^d} \varphi(x) \mathbb{E}[\mathcal{H}_\varepsilon(t,x)] \, dx + \int_{\mathbb{R}^d} \varphi(x) \varepsilon^{1-d/2} [h_\varepsilon(t,x) - \mathbb{E}[h_\varepsilon(t,x)]] \, dx,
\]

we can conclude that the left-hand side converges to \( \int_{\mathbb{R}^d} \varphi(x) [\mathcal{H}(t,x) + \overline{\mathcal{H}}(t,x)] \, dx \) with \( \mathcal{H} \) solving (2.9) and \( \overline{\mathcal{H}} \) solving (2.9), provided that \( \frac{Z_\infty - Z_T}{Z_T} \) and \( Z_T \) become asymptotically independent fast enough.

\textbf{Remark 2.8.} While we are intrinsically interested in the asymptotic behavior of the rescaled deviation \( \mathcal{H}_\varepsilon(t,x) \) for the KPZ solution \( h_\varepsilon \), the same strategy of our proofs work in more general situation. Indeed, if \( u_\varepsilon \) denotes the solution of (1.4) and \( \beta \in (0, \beta_0) \), then for any function \( \Psi \in C^1(\mathbb{R}; \mathbb{R}) \) with \( \Psi(0) = 0 \) and \( \Psi'(0) = 1 \), we can obtain convergence of the finite dimensional distributions of the spatially indexed process: as \( \varepsilon \to 0 \),

\[
\varepsilon^{-(d-2)/2} \left( \frac{u_\varepsilon(t,x)}{u(\xi_{\varepsilon(t,x)})} - 1 \right) \xrightarrow{\text{f.d.m.}} (\mathcal{H}(t,x))_{x \in \mathbb{R}^d, t > 0}. \tag{2.10}
\]

See Remark 3.3.

The convergence results in Theorem 2.1 and Theorem 2.4 are directly linked with the following behavior of the free energy \( \log Z_T \) of the continuous directed polymer, recall (1.10).

\textbf{Theorem 2.9 (Space-time fluctuations of the free energy).} \textit{Under the assumptions imposed in Theorem 2.1 we have, as \( T \to \infty \),}

\[
T^{\frac{d-2}{4}} \left( \log \mathcal{Z}_{IT}(x \sqrt{T}) - \log \mathcal{Z}_{\infty}(x \sqrt{T}) \right) \xrightarrow{\text{f.d.m.}} (\mathcal{H}(t,x))_{x \in \mathbb{R}^d, t > 0}, \tag{2.11}
\]

\textit{and for any} \( \varphi \in C^\infty_c(\mathbb{R}^d) \),

\[
T^{\frac{d-2}{4}} \int_{\mathbb{R}^d} dx \varphi(x) \left( \log \mathcal{Z}_{IT}(x \sqrt{T}) - \log \mathcal{Z}_{\infty}(x \sqrt{T}) \right) \to \int_{\mathbb{R}^d} dx \varphi(x) \mathcal{H}(t,x). \tag{2.12}
\]

Theorem 2.9 holds the key for the proofs of Theorem 2.1 and Theorem 2.4. In order to provide some guidance to the reader, we now allude to the main strategy for the proof of Theorem 2.9.
2.1 Comparison of proof techniques and central idea of the present approach.

The purpose of the present subsection is to provide a comparative description of the proof techniques employed in the previous approaches as well as sketch the central idea of the present approach. It will also then underline the technical novelty of our work.

2.1.1 Comparison with earlier proofs.

We first remark that the approaches for proving global fluctuations like (2.8) in ([28, 14]) are entirely different from the present method. Indeed [28] is based on applications of intricate renormalization theory which is quite orthogonal to the probabilistic approaches we presently employ. The method of deriving Edwards-Wilkinson limit of the KPZ averages in [14] is based on deriving the same for the corresponding SHE solution \( u_\varepsilon \) (see [17]) which admits the integral representation

\[
\int_{\mathbb{R}^d} \rho(t - s, x - y) u_\varepsilon^T(s, y) \xi(s, y) ds dy
\]

with\( u_0 = u_0(t, x) = \mathbb{E}[u_\varepsilon(t, x)] \) solving the usual heat equation \( \partial_t u = \frac{1}{2} \Delta u \) with initial condition \( u_0 \).

Denoting \( (P_t \varphi)(x) = \int_{\mathbb{R}^d} \rho(t, x - y) \varphi(y) dy \) and rewriting the rescaled averages as

\[
\varepsilon^{(d-2)/2} \int_{\mathbb{R}^d} dx \varphi(x)[u_\varepsilon(t, x) - \mathbb{E}[u_\varepsilon(t, x)] = \beta \int_0^t \int_{\mathbb{R}^d} \varphi(z - \varepsilon y) u_\varepsilon(s, z - \varepsilon y) \xi(s, z) ds dy dz,
\]

the task of proving Edwards-Wilkinson limit as in (2.8) for the (l.h.s) follows from the convergence of the (r.h.s) to a Gaussian law \( \mathcal{N}(0, \sigma^2) \int_{[0, t] \times \mathbb{R}^d} (P_{t-s} \varphi(z))^2 \mathbb{E}(\rho(s, z))^2 ds dz) \). The latter convergence comes from exploiting the asymptotic independence of \( u_\varepsilon(t, x) \) and \( u_\varepsilon(t, y) \) which, as explained earlier (recall Remark 2.6), is closely intertwined with the averaging out phenomena coming from the ubiquitous presence of a fixed test function \( \varphi \).

As remarked earlier, just as the inherent nature of our results, our technique is distinctively different from the above approach. In fact, the present recipe is also quite different from invoking central limit theorem for martingales to get stable and mixing convergence as in [9, 22, 12]. To provide some guidance to the reader, it is useful to sketch the basic idea of the present method.

2.1.2 Central idea of the proof.

To derive Theorem 2.9 we heavily exploit tools from stochastic calculus. The first step is the convergence of the one-dimensional marginal distribution, that is to show that, for \( \beta \in (0, \beta_0) \) and any \( x \in \mathbb{R}^d \), as \( T \to \infty \),

\[
\mathcal{F}_T(x) - \mathcal{F}_\infty(x) \xrightarrow{\text{law}} \mathcal{N}
\]

with \( \mathcal{F}_0 = \mathcal{F}_0(\beta) = \gamma^2(\beta) \frac{1}{4\pi a/2} \),

where \( \gamma^2(\beta) \) is given in (2.1).
Then, we show that this convergence can be extended to $T^{d/2}(\log \mathcal{Z}_T(x) - \log \mathcal{Z}_\infty(x))$ and also to $\mathcal{H}_\varepsilon(t,x)$ through the identification (1.9). The proof of (2.13) proceeds in three further steps.

**Step 1:** The first step is to quantify the decay of correlation of $\mathcal{Z}_T(x)$ and show that

$$\text{Cov}(\mathcal{Z}_\infty(0), \mathcal{Z}_\infty(x)) = C_1 \left( \frac{1}{|x|} \right)^{d-2} \forall |x| \geq 1,$$

and

$$\|\mathcal{Z}_\infty - \mathcal{Z}_T\|_{L^2(P)}^2 \sim C_1 C_2 T^{-(d-2)/2} \mathbb{E}[\mathcal{Z}_\infty^2]$$

for constants $C_1, C_2$ depending on $\beta$.

**Step 2:** To handle the left hand side in (2.13) we now introduce a sequence of processes $G^{(T)} = (G^{(T)}_\tau)_{\tau \geq 1}$ with $G^{(T)}_\tau = T^{d/2} \left( \frac{\mathcal{Z}_\tau - \mathcal{Z}_T}{\mathcal{Z}_T} - 1 \right)$ and observe that, for each $T$, $G^{(T)}$ is a continuous martingale. We can compute its bracket:

$$\langle G^{(T)} \rangle_\tau = T^{d/2} \left( \frac{\mathcal{Z}_\tau}{\mathcal{Z}_T} \right) \int_0^\tau \left( \frac{d}{dt} \langle \mathcal{Z} \rangle_s \right) ds = \int_0^\tau (\sigma T)^{d/2} \left( \frac{d}{dt} \langle \mathcal{Z} \rangle_s \right) \sigma^{d/2} d\sigma$$

(2.16)

Note that, $\mathcal{Z}_T = E_0[\Phi_T(W)]$ with

$$d\langle \mathcal{Z} \rangle_T = \beta^2 E_0^\circ \left[ \Phi_T(W^{(1)}) \Phi_T(W^{(2)}) V(W^{(1)}_T - W^{(2)}_T) \right] dT,$$

where, for fixed Brownian path $W$, $\Phi_T(W) = \exp \left\{ \beta \int_0^T \int_{\mathbb{R}^d} \phi(W_s - y) \xi(s,y) ds dy - \frac{\beta T}{2} V(0) \right\}$ is another martingale. Then in the context of the bracket process (2.19), a key step is to show that, for $\beta \in (0, \beta_0)$,

$$T^{d/2} \left( \frac{d}{dt} \langle \mathcal{Z} \rangle_T \right) - C_0 \mathcal{Z}_T^{2/2} \xrightarrow{L^2(P)} 0$$

(2.17)

with $C_0$ defined in (2.14). Substantial technical work is needed to establish this step, which, loosely speaking, is based on proving asymptotic independence captured by the inherent attractive nature of the polymer measure.

**Step 3:** It can be justified that (2.16) and (2.17) imply that the sequence of angled brackets $\{\langle G^{(T)} \rangle_\tau \}_T$ of the continuous martingales $G^{(T)}$ now converges to a deterministic limit:

$$\langle G^{(T)} \rangle_\tau \to C_0 \int_1^\tau \sigma^{d/2} d\sigma =: g(\tau)$$

It follows that the sequence $G^{(T)}$ itself converges in law to a Brownian motion with time-change given by $g$, i.e., $G^{(T)} \xrightarrow{\text{law}} G$ in $C([1, \infty); \mathbb{R})$ where $G$ is a mean-zero Gaussian process.
with independent increments and variance \( g(\tau) = \frac{2}{T^2} \mathcal{C}_0 [1 - \tau^{\frac{(d-2)}{2}}] \). Then,

\[
\lim_{T \to \infty} T^{d-2} \left( \frac{\mathcal{Z}_T^\infty}{\mathcal{Z}_T} - 1 \right) = \lim_{T \to \infty} \lim_{\tau \to \infty} \left[ G^{(T)}_\tau + \frac{T^{(d-2)}}{\mathcal{Z}_T^T} \left( \mathcal{Z}_T^\infty - \mathcal{Z}_T^T \right) \right]
\]

and thanks to (2.15) and the fact that \( \mathcal{Z}_T^\infty \rightarrow \mathcal{Z}_\infty > 0 \), the second term vanishes in the double limit, so that the left hand side converges to \( \lim_{T \to \infty} \lim_{\tau \to \infty} G^{(T)}_\tau = \lim_{\tau \to \infty} \lim_{T \to \infty} G^{(T)}_\tau \) which has a centered Gaussian law with variance \( \frac{2}{T^2} \mathcal{C}_0 (\beta) = g(\infty) \). The last assertion then also implies (2.13).

The procedure for proving the finite dimensional distributions of the process \( \{\mathcal{H}_x(t, x)\}_{t > 0, x \in \mathbb{R}^d} \) in Theorem 2.1 is based on the above guiding philosophy and an extension of the \( L^2(\mathbb{P}) \) convergence (2.17), while the convergence of the spatial averages in Theorem 2.4 follows from Theorem 2.1 combined with moment bounds originating from existence of negative moments of the free energy log \( \mathcal{Z}_T \) proved in our earlier paper [8].

**Organization of the article:** Let us now comment on how the rest of the article is organized. Section 3 is devoted to proving the convergence of \( \mathcal{H}_x(t, x) \) for fixed \( x \in \mathbb{R}^d \) and \( t > 0 \). A key technical step in its proof is the \( L^2(\mathbb{P}) \) convergence (2.17) which constitutes Section 4. Convergence of finite dimensional distributions of \( \{\mathcal{H}_x(t, x)\}_{x \in \mathbb{R}^d} \) required for Theorem 2.1 and convergence of the averages \( \int_{\mathbb{R}^d} dx \phi(x) \mathcal{H}_x(t, x) \) for Theorem 2.4 as well as that of Proposition 2.5 can be found in Section 5.

### 3 Convergence of marginal distributions.

As mentioned earlier, the first step for the proof of Theorem 2.9 is provided by the following convergences of one dimensional marginal distributions.

**Theorem 3.1.** Fix \( d \geq 3 \) and \( \beta < \beta_0 \) as in Theorem 2.1. Then for all \( x \in \mathbb{R}^d \), and as \( T \to \infty \),

\[
T^{d-2} \left( \frac{\mathcal{Z}_T(x) - \mathcal{Z}_\infty(x)}{\mathcal{Z}_T(x)} \right) \xrightarrow{\text{law}} \mathcal{N} \left( 0, \frac{2}{d-2} \mathcal{C}_0 \right), \tag{3.1}
\]

where \( \mathcal{C}_0(\beta) \) is defined in (2.14).

**Theorem 3.2.** Fix \( d \geq 3 \) and \( \beta < \beta_0 \) as in Theorem 2.1. Then for all \( x \in \mathbb{R}^d \), and as \( T \to \infty \),

\[
T^{\frac{d-2}{4}} \left( \log \mathcal{Z}_T(x) - \log \mathcal{Z}_\infty(x) \right) \xrightarrow{\text{law}} \mathcal{N} \left( 0, 2(d-2)^{-1} \mathcal{C}_0 \right), \tag{3.2}
\]

\( ^2 \)It is well-known that a continuous martingale \( M \) with a deterministic bracket \( \langle M \rangle = \phi \) with \( \phi : \mathbb{R}^+_0 \to \mathbb{R}^+_0 \) being continuous and increasing, is a process with independent and centered Gaussian increments. Moreover, if \( \{M^{(n)}\}_n \) is a sequence of continuous martingales such that the sequence \( \langle M^{(n)} \rangle \) converges in probability to a deterministic function \( \phi \) as above, then \( M^{(n)} \) itself converges in law to a process with independent, centered Gaussian increments. Based on a (conditional) second moment method, but using instead much of the process structure, this argument is an efficient way to prove asymptotic normality, see e.g. [10].
Space-time fluctuations of KPZ in $d \geq 3$ and the GFF

where $C_0(\beta)$ is defined above in (2.14). Moreover, for all $x \in \mathbb{R}^d$, $t > 0$, as $\varepsilon \to 0$,

$$\mathcal{H}_\varepsilon(t,x) \xrightarrow{\text{law}} \mathcal{N}(0, 2(d - 2)^{-1} C_0 t^{-\frac{d-2}{2}}).$$ \hfill (3.3)

The rest of this section is devoted to the proof of Theorem 3.1 and Theorem 3.2.

### 3.1 Rate of decorrelation.

In this section we will provide the following elementary result, which provides an estimate on the asymptotic decorrelation of $u_\varepsilon(t,x)$ and $u_\varepsilon(t,y)$ as $\varepsilon \to 0$ through identification (1.9). This estimate also underlines the fact that smoothing $u_\varepsilon(x)$ w.r.t. any $\varphi \in C^\infty_c(\mathbb{R}^d)$ makes $\int_{\mathbb{R}^d} dx u_\varepsilon(t,x) \varphi(x) \rightarrow \int_{\mathbb{R}^d} dx u(t,x) \varphi(x)$ deterministic, with $\partial_t u = \frac{1}{2} \Delta u$.

**Proposition 3.3.** Let $d \geq 3$ and $\beta$ small enough.

- We have:

  $$\text{Cov}(\mathcal{Z}_\infty(0), \mathcal{Z}_\infty(x)) = \begin{cases} E_{x/\sqrt{2}} \left[ e^{\beta^2 \int_0^\infty V(\sqrt{2}W_s)ds} - 1 \right] & \forall x \in \mathbb{R}^d, \\ C_1 |x|^{-(d-2)} & \forall |x| \geq 1, \end{cases}$$ \hfill (3.4)

  with $C_1 = E_{e_1/\sqrt{2}} \left[ e^{\beta^2 \int_0^\infty V(\sqrt{2}W_s)ds} - 1 \right]$.

- Finally,

  $$\|\mathcal{Z}_\infty - \mathcal{Z}_T\|_2^2 \sim C_1 C_2 E[\mathcal{Z}^2_\infty] T^{-\frac{d-2}{4}} \quad \text{as } T \to \infty.$$ \hfill (3.5)

  with $C_2 = E[(\sqrt{2}/|Z|)^{d-2}]$, where $Z$ is a centered Gaussian vector with covariance $I_d$.

**Remark 3.4.** The random process $(\mathcal{C}_1^{-1/2} \mathcal{Z}_\infty(x); x \in \mathbb{R}^d)$ is quite interesting. It has the same covariance as the Gaussian free field, see (3.9) for $|x| \geq 1$. It is a positive stationary field with covariance equal to the Green function (at distance 1 from the diagonal). We stress that, in this range of values for $x$, the covariance does not not depend on the regularizing function $\phi$, whereas the law does depend.

**Proof.** For any Brownian path $W = (W_s)_{s \geq 0}$ we set

$$\Phi_T(W) = \exp \left\{ \beta \int_0^T \int_{\mathbb{R}^d} \phi(W_s - y) \xi(s,y) \, ds \, dy - \frac{\beta^2 T}{2} V(0) \right\}$$ \hfill (3.6)

and see that, for any $n \in \mathbb{N}$,

$$E \left[ \prod_{i=1}^n \Phi_T(W^{(i)}) \right] = \exp \left\{ \beta^2 \int_0^T \sum_{1 \leq i < j \leq n} V(W_s^{(i)} - W_s^{(j)}) \, ds \right\}.$$ \hfill (3.7)
Then, the first line of equation (3.9) follows from (3.7) with $n = 2$ and the fact that $W_s^{(1)} - W_s^{(2)} \overset{law}{=} \sqrt{2} W_s$. Now, the second line of (3.9) follows by considering the hitting time of the unit ball for $\sqrt{2} W$ and spherical symmetry of $V$.

We now show (3.5) as follows. For two independent paths $W^{(1)}$ and $W^{(2)}$ (which are also independent of the noise $\xi$), we will denote by $\mathcal{F}_T$ the $\sigma$-algebra generated by both paths until time $T$. Then, by Markov property, for $0 < S \leq \infty$,

$$\mathcal{F}_{T+S}(x) = E_x[\Phi_T(W) \mathcal{F}_S \circ \theta_{T,W_T}].$$

(3.8)

where for any $t > 0$ and $x \in \mathbb{R}^d$, $\theta_{t,x}$ denotes the canonical spatio-temporal shift in the white noise environment. Hence,

$$\|\mathcal{L}_\infty - \mathcal{L}_T\|_2^2 = \mathbb{E}\left[E_0^{\otimes 2}\left\{\Phi_T(W^{(1)})\Phi_T(W^{(2)}) \right\} \left(\mathcal{F}_\infty \circ \theta_{T,W_T} - 1\right) \right] = E_0^{\otimes 2} \left[ e^{\beta^2 \int_0^T V(W_s^{(1)} - W_s^{(2)})} \times \text{Cov}(\mathcal{F}_\infty(W^{(1)}_T), \mathcal{F}_\infty(W^{(2)}_T)) \right]$$

$$= E_0^{\otimes 2} \left[ e^{\beta^2 \int_0^T V(W_s^{(1)} - W_s^{(2)})} \times E_0^{\otimes 2} \left[ \text{Cov}(\mathcal{F}_\infty(W^{(1)}_T), \mathcal{F}_\infty(W^{(2)}_T)) | \mathcal{F}_T \right] \right]$$

$$\sim C_1 E_0^{\otimes 2} \left[ e^{\beta^2 \int_0^T V(W_s^{(1)} - W_s^{(2)})} \times \left( \frac{2}{|W_s^{(1)} - W_s^{(2)}|} \right)^{d-2} \right]$$

The asymptotic equivalence in the above display is justified if we note that for $\xi^{(i)}(s,y) = \xi(s + t, y)$,

$$\text{Cov}(\mathcal{F}_\infty(0; \xi), \mathcal{F}_\infty(x, \xi^{(i)})) = \int_{\mathbb{R}^d} \rho(t,y) E_{(y-x)/\sqrt{2}} \left[ e^{\beta \int_0^\infty V(\sqrt{2} W_s) ds} - 1 \right] dy$$

(3.9)

Then (3.5) is proved once we show

$$E_0 \left[ e^{\beta^2 \int_0^T V(\sqrt{2} W_s) ds} \left( \frac{1}{|W_T|} \right)^{d-2} \right] \sim E_0 \left[ e^{\beta^2 \int_0^\infty V(\sqrt{2} W_s) ds} E_0 \left[ \left( \frac{1}{|W_T|} \right)^{d-2} \right] \right]$$

(3.10)

But as $T \to \infty$,

$$\left( \int_0^T V(\sqrt{2} W_s) ds, T^{-1/2} W_T \right) \overset{law}{\to} \left( \int_0^\infty V(\sqrt{2} W_s) ds, Z \right)$$

with $Z \sim N(0, I_d)$ being independent of the Brownian path $W$, and then (3.5) follows from the requisite uniform integrability

$$\sup_{T \geq 1} E_0 \left[ \left( e^{\beta^2 \int_0^T V(\sqrt{2} W_s) ds} \left( \frac{T^{1/2}}{|W_T|} \right)^{d-2} \right)^{1+\delta} \right] < \infty$$

(3.11)
for \( \delta > 0 \). By Hölder’s inequality and Brownian scaling, for any \( p, q \geq 1 \) with \( 1/p + 1/q = 1 \),

\[
(\text{l. h. s.) of } \mathcal{F}_1 \leq E_0 \left[ e^{\eta(1+\delta)\beta^2 \int_0^T V(\sqrt{\mathcal{W}}_s) \, ds} \right]^{1/q} E_0 \left[ \frac{1}{|W_1|^{p(1+\delta)(d-2)}} \right]^{1/p} \\
\leq E_0 \left[ e^{\eta(1+\delta)\beta^2 \int_0^\infty V(\sqrt{\mathcal{W}}_s) \, ds} \right]^{1/q} \left[ \int_{\mathbb{R}^d} \frac{1}{|x|^{p(1+\delta)(d-2)}} e^{-|x|^2/2} \, dx \right]^{1/p} \\
\leq C \left[ \int_0^\infty dr \, t^{d-1} \frac{1}{r^p(1+\delta)(d-2)} e^{-r^2/2} \right]^{1/p}
\]

Then the last integral is seen to be finite provided we choose \( \delta > 0 \) and \( p > 1 \) small enough so that \( 1 < p(1+\delta) \leq \frac{d}{d-2} \).

We will also need the following uniform \( L^p \)-estimate whose proof is based on Proposition 3.3 as well as uniform estimates on moments of \( \log \mathcal{Z}_T \) proved in [8].

**Proposition 3.5.** For all \( 1 < p < 2 \),

\[
\sup_{T>0} E \left[ \left| T^{(d-2)/4} (\log \mathcal{Z}_T - \log \mathcal{Z}_\infty) \right|^p \right] < \infty.
\] (3.12)

**Proof.** Write \( \log \mathcal{Z}_T - \log \mathcal{Z}_\infty \) as \( \log(1 - \frac{\mathcal{Z}_\infty - \mathcal{Z}_T}{\mathcal{Z}_\infty}) \) and decompose the LHS of (3.12) as:

\[
E \left[ \left| T^{(d-2)/4} (\log \mathcal{Z}_T - \log \mathcal{Z}_\infty) \right|^p \right] 1 \left( \left| \frac{\mathcal{Z}_\infty - \mathcal{Z}_T}{\mathcal{Z}_\infty} \right| > \frac{1}{2} \right) \\
+ E \left[ \left| T^{(d-2)/4} \log \left( 1 - \frac{\mathcal{Z}_\infty - \mathcal{Z}_T}{\mathcal{Z}_\infty} \right) \right|^p \right] 1 \left( \left| \frac{\mathcal{Z}_\infty - \mathcal{Z}_T}{\mathcal{Z}_\infty} \right| \leq \frac{1}{2} \right)
\]

which defines the sum of two terms \( A_T + B_T \). For the second term, we can use the upper bound \( |\log(1 - x)| \leq C|x| \) which holds for all \( x \leq 1/2 \), and, combined with Hölder’s inequality with \( 1/(2/p) + 1/q = 1 \), entails

\[
B_T \leq C E \left[ \left| T^{(d-2)/4} \left( \frac{\mathcal{Z}_\infty - \mathcal{Z}_T}{\mathcal{Z}_\infty} \right)^p \right| \right] \leq C E \left[ T^{(d-2)/2} \left( \frac{\mathcal{Z}_\infty - \mathcal{Z}_T}{\mathcal{Z}_\infty} \right)^2 \right]^{p/2} E \left[ \mathcal{Z}_\infty^{-p} \right]^{1/q}. \] (3.13)

In [8], it is proven that \( \mathcal{Z}_\infty \) admits all negative moments for all \( \beta < \beta_L^2 \). Moreover \( T^{(d-2)/2} \left( \frac{\mathcal{Z}_\infty - \mathcal{Z}_T}{\mathcal{Z}_\infty} \right)^2 \) is bounded in \( L^1 \) by (3.5), hence sup\( T B_T \) is finite.

Then, we use the upper bound \( |\alpha + \mu|^p \leq 2^p (|\alpha|^p + |\mu|^p) \) and Hölder’s inequality with \( 1/a + 1/b = 1 \), choosing \( b > 1 \) small enough so that \( bp < 2 \), and obtain:

\[
A_T \leq T^{p(d-2)/4} 2^p \left( E \left[ \left| \log Z_T \right| \right]^{p(1/a)} + E \left[ \left| \log Z_\infty \right| \right]^{p(1/a)} \right) \mathbb{P} \left( \left| \frac{\mathcal{Z}_\infty - \mathcal{Z}_T}{\mathcal{Z}_\infty} \right| > \frac{1}{2} \right)^{1/b} \\
\leq C T^{p(d-2)/4} E \left[ \left| \frac{\mathcal{Z}_\infty - \mathcal{Z}_T}{\mathcal{Z}_\infty} \right|^{pb} \right]^{1/b} \\
\leq C E \left[ T^{(d-2)/2} \left( \frac{\mathcal{Z}_\infty - \mathcal{Z}_T}{\mathcal{Z}_\infty} \right)^2 \right]^{p/2},
\]
where the second inequality comes
\[ \sup_T E \left[ \log \mathcal{Z}_T \right] ^\theta < \infty, \quad \forall \theta \in \mathbb{R}, \] (3.14)
which has been shown for \( \beta \in (0, \beta_0) \) in [8, Theorem 1.3] and Markov’s inequality, and where the third inequality follows from the upper-bound (3.13) with \( p \) replaced by \( pb \) and again invoking (3.14). As above, this implies that \( \sup_T A_T \) is finite, which ends the proof.

3.2 Proof of Theorem 3.1
We start by computing the stochastic differential and bracket of the martingale \( \mathcal{Z}_T \) defined as follows:
\[ d\mathcal{Z}_T = \beta E_0 \left[ \Phi_T(W) (\phi \ast \xi)(T, W_T) \right] dT, \]
\[ d\langle \mathcal{Z} \rangle_T = \beta^2 E_0^2 \left[ \Phi_T(W^{(1)}) \Phi_T(W^{(2)}) V(W_T^{(1)} - W_T^{(2)}) \right] dT, \] (3.15)
\[ = \beta^2 \mathcal{Z}_T^2 \times E_{0, \beta, T}^2 \left[ V(W_T^{(1)} - W_T^{(2)}) \right] dT, \] (3.16)
where \( E_{0, \beta, T}^2 \) is the expectation taken with respect to the product of two independent polymer measures,
\[ P_{0, \beta, T}(dW^{(i)}) = \frac{1}{Z_{\beta, T}} \exp \left\{ \beta \int_0^T \int_{\mathbb{R}^d} \phi(W_s^{(i)} - y) \xi(y, s) ds dy \right\} P(dW^{(i)}) \quad i = 1, 2, \]
with \( Z_{\beta, T} = e^{-\frac{\beta^2}{2} TV(0)} \mathcal{Z}_T \). The proof of Theorem 3.1 splits into two main steps. The first step involves showing the following estimate whose proof constitutes Section 4:

**Proposition 3.6.** There exists \( \beta_0 \in (0, \infty) \), such that for all \( \beta < \beta_0 \), as \( T \to \infty \),
\[ T^\frac{d}{2} \left( \frac{d}{dt} \langle \mathcal{Z} \rangle_T \right)_T - \mathcal{C}_0(\beta) \mathcal{Z}_T^2 \overset{L^2}{\to} 0, \]
where \( \mathcal{C}_0(\beta) \) is defined in (2.14).

For the second step, we define a sequence \( \{G_{\tau}^{(T)}\}_{\tau \geq 1} \) of stochastic processes on time interval \([1, \infty)\), with
\[ G_{\tau}^{(T)} = T^\frac{d-1}{4} \left( \frac{\mathcal{Z}_\tau T}{\mathcal{Z}_T} - 1 \right), \quad \tau \geq 1. \] (3.17)
Then, for all \( T \), \( G^{(T)} \) is a continuous martingale for the filtration \( \mathcal{B}^{(T)} = (\mathcal{B}_\tau^{(T)})_{\tau \geq 1} \), where \( \mathcal{B}_\tau^{(T)} \) denotes the \( \sigma \)-field generated by the white noise \( \xi \) up to time \( \tau T \). Then we need the following result, which provides convergence at the process level:

---

\footnote{In [8, Theorem 1.3] the existence of negative (and positive) moments is stated for \( \log \mathcal{Z}_\infty \). However, exactly the same proof yields a uniform (in \( T \)) estimate for \( \log \mathcal{Z}_T \).}
**Theorem 3.7.** For $\beta < \beta_0$, as $T \to \infty$, we have convergence

$$G^{(T)} \xrightarrow{\text{law}} G$$

on the space of continuous functions on $[1, \infty)$, equipped with the topology of uniform convergence on compact intervals, where $G$ is a mean zero Gaussian process with independent increments and variance

$$g(\tau) = \frac{2}{d-2} \mathcal{C}_0(\beta) \left[ 1 - \tau^{-\frac{d-2}{2}} \right].$$

**Proof of Theorem 3.7 (Assuming Proposition 3.6):** From the definition (3.17) we compute the bracket of the square-integrable martingale $G^{(T)}$,

$$\langle G^{(T)} \rangle_T = \frac{T^{\frac{d-2}{2}}}{2_T} \langle Z^2 \rangle_T = \frac{T^{\frac{d-2}{2}}}{2_T} \int_1^T \left( \frac{d}{dt} \langle Z \rangle_s \right) ds$$

by replacing the variables $s = \sigma T$. Then,

$$\langle G^{(T)} \rangle_T - g(\tau) = \int_1^T \left[ \frac{(\sigma T)^{\frac{d}{2}}}{2_T} \left( \frac{d}{dt} \langle Z \rangle_{\sigma T} \right) - \mathcal{C}_0 \right] \sigma^{-d/2} d\sigma$$

$$= \int_1^T \left[ \frac{\sigma^{\frac{d}{2}}}{2_{\sigma T}} \left( \frac{d}{dt} \langle Z \rangle_{\sigma T} \right) - \mathcal{C}_0 \right] \sigma^{-d/2} d\sigma + \frac{\mathcal{C}_0}{2_T} \int_1^T \left[ \langle Z \rangle_{\sigma T}^2 - \langle Z \rangle_T^2 \right] \sigma^{-d/2} d\sigma$$

$$=: I_1 + I_2$$

As $T \to \infty$ the last integral vanishes in $L^1$ and $I_2$ vanishes in probability. For $\varepsilon \in (0, 1]$, introduce the event

$$A_\varepsilon = \left\{ \sup \{ Z_t; t \in [0, \infty) \} \vee \sup \{ Z_t^{-1}; t \in [0, \infty) \} \leq \varepsilon^{-1} \right\}$$

and observe that $\lim_{\varepsilon \to 0} \mathbb{P}(A_\varepsilon) = 1$ since $Z_t$ is continuous, positive with a positive limit. So, we can estimate the expectation of $I_1$ by

$$\mathbb{E}[I_{A_\varepsilon} | I_1] \leq \frac{\tau}{\varepsilon^6} \sup_{t \geq T} \left\{ \left\| \frac{t^{\frac{d}{2}}}{\varepsilon^6} \left( \frac{d}{dt} \langle Z \rangle_t \right) - \mathcal{C}_0 Z_t^2 \right\|_1 \right\},$$

which vanishes by Proposition 3.6. Thus, $\langle G^{(T)} \rangle \to g$ in probability. Since for the sequence of continuous martingales $G^{(T)}$ the brackets converge pointwise to a deterministic limit $g$, we derive that the sequence $G^{(T)}$ itself converges in law to a Brownian motion with time-change given by $g$, that is, the process $G$ defined in the statement of Theorem 3.7 (see [23, Theorem 3.11 in Chapter 8]), which is proved now.
Concluding the proof of Theorem 3.1. Write

\[ T^{\frac{d-2}{4}} \left( \frac{\mathcal{Z}_\infty}{\mathcal{Z}_T} - 1 \right) = G^{(T)} \]

\[ = G_\tau^{(T)} + T^{\frac{d-2}{4}} \left[ \frac{\mathcal{Z}_\infty}{\mathcal{Z}_T} - \frac{\mathcal{Z}_\tau}{\mathcal{Z}_T} \right], \]

and consider the last term. By (3.5), the numerator has $L^2$-norm tending to 0 as $\tau \to \infty$ uniformly in $T \geq 1$ whereas the denominator has a positive limit. Then, the last term vanishes in the double limit $T \to \infty, \tau \to \infty$, and therefore

\[ \lim_{T \to \infty} T^{\frac{d-2}{4}} \left( \frac{\mathcal{Z}_\infty}{\mathcal{Z}_T} - 1 \right) = \lim_{T \to \infty} \lim_{\tau \to \infty} G^{(T)}(\tau), \]

which is the Gaussian law with variance $g(\infty) = \frac{2}{d-2} \xi_0$ by Theorem 3.7. Hence,

\[ T^{\frac{d-2}{4}} \left( \frac{\mathcal{Z}_T(x) - \mathcal{Z}_\infty(x)}{\mathcal{Z}_T(x)} \right) \xrightarrow{\text{law}} N\left(0, \frac{2}{d-2} \xi_0\right) \text{ as } T \to \infty. \]

We now provide the proof of Theorem 3.2 whose presented arguments will be used again several times through Section 5.

3.3 Proof of Theorem 3.2

Note that the convergences (3.2) and (3.3) are also equivalent by the identification (1.9) and we will therefore only prove (3.2). Now by Itô's formula, equations (3.15) and (3.16) imply that

\[ \log \mathcal{Z}_T = N_T - \frac{1}{2} \langle N \rangle_T, \]

where

\[ N_T = \beta \int_0^T \int_{\mathbb{R}^d} E_{0,\beta,t}(\phi(y - W_t))\xi(t,y) \, dy \, dt \]

\[ \langle N \rangle_T = \beta^2 \int_0^T E_{0,\beta,s}^2 \left[ V(W_s^{(1)} - W_s^{(2)}) \right] \, ds, \]

with $N$ being a martingale. Then, Proposition 3.6 shows that, in probability and as $T \to \infty$,

\[ T^{d/2} \frac{d}{dT} \langle N \rangle_T \to \xi_0. \]

Mimicking the proof of Theorem 3.7, we further get that the bracket of the rescaled martingale $N^{(\tau)} : \tau \to T^{(d-2)/4} (N_T - N_0)$ converges in probability as $T \to \infty$ to the deterministic
function \(g(\tau)\), where \(g\) is given by (3.19), implying thereby the convergence \(N(T) \xrightarrow{\text{law}} G\). Moreover, convergence of the bracket also implies that as \(T \to \infty\),

\[ T^{(d-2)/4}(\langle N \rangle_{\tau T} - \langle N \rangle_T) \xrightarrow{\mathcal{F}} 0, \]

so that the bracket part in (3.20) vanishes under the scaling limit. Putting things together, we obtain that for all \(\tau \geq 1\),

\[ T^{(d-2)/4}(\log \mathcal{Z}_{\tau T} - \log \mathcal{Z}_T) \xrightarrow{\text{law}} G(\tau). \]  

(3.21)

Then, we write:

\[ T^{(d-2)/4}(\log \mathcal{Z}_{\infty} - \log \mathcal{Z}_T) = T^{(d-2)/4}(\log \mathcal{Z}_{\infty} - \log \mathcal{Z}_{\tau T}) + T^{(d-2)/4}(\log \mathcal{Z}_{\tau T} - \log \mathcal{Z}_T), \]

where, by Proposition 3.5, the second term vanishes in \(L^p\)-norm as \(\tau \to \infty\), uniformly in \(T \geq 1\), for all \(p < 2\). Therefore, convergence (3.22) follows from (3.21) and the last display, by letting \(T \to \infty\) and \(\tau \to \infty\) as in the conclusion of the proof of Theorem 3.1.

Remark 3.8. Note that with the following lemma, we can see that Theorem 3.1 and Theorem 3.2 are in fact equivalent (choose for example \(\Psi(x) = \log(1 + x)\) to go from Theorem 3.1 to Theorem 3.2). We can also obtain a general version of the two theorems which is Corollary 3.10.

Lemma 3.9. If \(X \xrightarrow{\text{d}} N(0, \sigma^2)\) in distribution, then for any \(\Psi \in C^1(\mathbb{R})\) with \(\Psi(0) = 0\), \(\Psi'(0) = 1\) and \(a > 0\), we have \(\varepsilon^{-a}\Psi(\varepsilon X) \to N(0, a^2)\).

Corollary 3.10. Fix \(d \geq 3\) and \(\beta < \beta_0\) as in Theorem 2.1 and \(\Psi\) as in Lemma 3.9. Then for all \(x \in \mathbb{R}^d\), \(t > 0\), as \(\varepsilon \to 0\),

\[ \varepsilon^{-\frac{(d-2)}{2}} \left( \Psi\left( \frac{u_{t, x}(t, x)}{u(t, x)} \right) - 1 \right) \xrightarrow{\text{law}} N\left(0, \frac{2}{d-2} \varepsilon_0 t^{\frac{d-2}{2}} \right). \]  

(3.22)

Proof. The proof follows from Theorem 3.1, identification (1.9) and the above lemma.

4 Proof of Proposition 3.6

This section is entirely devoted to the proof of Proposition 3.6. Denote for short by \(L_T\) the quantity of interest,

\[ L_T := T^{d/2} \left( \frac{d}{dt} \langle \mathcal{Z} \rangle_T \right) - \mathcal{E}_0 \mathcal{Z}_T^2 \]

\[ = E^{\otimes 2}_0 \left[ \Phi_T(W^{(1)}) \Phi_T(W^{(2)}) \left( T^2 V(W^{(1)}_T - W^{(2)}_T) - \mathcal{E}_0 \right) \right], \]

and proceed in two steps. The first step is devoted to determining the value of the appropriate constant \(\mathcal{E}_0\) below.
Space-time fluctuations of KPZ in $d \geq 3$ and the GFF

4.1 The first moment.

We first want to show that:

**Proposition 4.1.** There exists $\beta_1 \in (0, \infty)$ such that for all $\beta < \beta_1$, if we choose

$$C_0(\beta) = \frac{\beta^2}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} dy \, V(\sqrt{2}y) \, E_y \left[ e^{\beta^2 \int_0^\infty V(\sqrt{2}W_s) \, ds} \right], \tag{4.1}$$

then $\mathbb{E}(L_T) \to 0$ as $T \to \infty$.

**Remark 4.2.** By a simple change of variables, one can check that the definition of $C_0$ in (4.1) corresponds indeed to the one of Proposition 3.6.

The rest of Section 4.1 is devoted to the proof of Proposition 4.1. For any $t > s \geq 0$ and $x, y \in \mathbb{R}^d$, we will denote by $P_{s,x}^y$ the law (and by $E_{s,x}^y$ the corresponding expectation) of the Brownian bridge starting at $x$ at time $s$ and conditioned to reach $y$ at time $t > s$. Recall that

$$\rho(t, x) = (2\pi t)^{-d/2} e^{-|x|^2/2t},$$

denotes the standard Gaussian kernel.

We note that

$$\mathbb{E}(\mathcal{L}_T) = E_0 \left[ e^{\beta^2 \int_0^T V(W_t^{(1)} - W_t^{(2)}) \, dt} \left( T_{\mathbb{R}}^4 V(W_T^{(1)} - W_T^{(2)}) - \mathcal{C}_0 \right) \right]$$

and

$$E_0 \left[ e^{\beta^2 \int_0^T V(\sqrt{2}W_t) \, dt} T_{\mathbb{R}}^4 V(\sqrt{2}W_T) \right] = \int_{\mathbb{R}^d} V(\sqrt{2}y) E_{T,y}^{T,0} \left[ e^{\beta^2 \int_0^T V(\sqrt{2}W_t) \, dt} \right] T_{\mathbb{R}}^4 \rho(T, y) \, dy.$$

Now, we fix a sequence

$$m = m(T) \quad \text{such that} \quad m \to \infty \quad \text{and} \quad m = o(T) \quad \text{as} \quad T \to \infty,$$

which allows us to prove Proposition 4.1 in two steps:

**Proposition 4.3.** For small enough $\beta$, as $T \to \infty$,

$$\lim_{T \to \infty} \sup_{y \in \mathbb{R}^d} \left| E_{T,y}^{T,0} \left[ e^{\beta^2 \int_0^T V(\sqrt{2}W_t) \, dt} \right] - \mathcal{T}_1(y) \right| = 0,$$

where

$$\mathcal{T}_1(y) = E_{T,y}^{T,0} \left[ e^{\beta^2 \int_{[0,m] \cup [T-m,T]} V(\sqrt{2}W_t) \, dt} \right].$$
Proposition 4.4. For small enough $\beta$, for all fixed $y \in \mathbb{R}^d$ and as $T \to \infty$,

$$\mathcal{T}_1(y) \sim E_{0,0}^{T,y} \left[ e^{\beta^2 \int_{[0,m]} V(\sqrt{2}W_t) \, dt} \right] E_{0,0}^{T,y} \left[ e^{\beta^2 \int_{[T-m,T]} V(\sqrt{2}W_t) \, dt} \right]$$

$$\to E_0 \left[ e^{\beta^2 \int_0^\infty V(\sqrt{2}W_t) \, dt} \right] E_y \left[ e^{\beta^2 \int_0^\infty V(\sqrt{2}W_t) \, dt} \right].$$

We will provide some auxiliary results which will be needed to prove Proposition 4.3 and Proposition 4.4. First, we state a simple consequence of Girsanov’s theorem:

Lemma 4.5. For any $s < t$ and $y, z \in \mathbb{R}^d$, the Brownian bridge $P_{0,y}^{t,z}$ is absolutely continuous w.r.t. $P_{0,y}$ on the $\sigma$-field $\mathcal{F}_{[0,s]}$ generated by the Brownian path until time $s < t$, and

$$\frac{dP_{0,y}^{t,z}}{dP_{0,y}} \bigg|_{\mathcal{F}_{[0,s]}} = \frac{\rho(t-s, z-W_s)}{\rho(t, z-y)} \left( \frac{t}{t-s} \right)^{d/2} \exp \left\{ \frac{|z-y|^2}{2t} \right\}.$$  \hspace{1cm} (4.2)

We will need the following version of Khas’minskii’s lemma [34, p.8, Lemma 2.1] for the Brownian bridge:

Lemma 4.6. If $E_0 \left[ 2\beta^2 \int_0^\infty V(\sqrt{2}W_s) \, ds \right] < 1$, then

$$\sup_{z,x \in \mathbb{R}^d, t>0} E_{0,x}^{t,z} \left[ \exp \left\{ \beta^2 \int_0^t V(\sqrt{2}W_s) \, ds \right\} \right] < \infty.$$

Proof. By Girsanov’s theorem, for any $s < t$, $\alpha \in \mathbb{R}^d$ and $A \in \mathcal{F}_{[0,s]}$,

$$P_{0,x}^{t,z}(A) = E_x^{(\alpha)} \left[ \frac{\rho(\alpha)(t-s; z-W_s)}{\rho(\alpha)(t, z-x)} 1_A \right]$$  \hspace{1cm} (4.3)

where $E^{(\alpha)}$ (resp. $P^{(\alpha)}$) refers to the expectation (resp. the probability) with respect to Brownian motion with drift $\alpha$ and transition density

$$\rho(\alpha)(t, z) = \frac{1}{(2\pi t)^{d/2}} \exp \left\{ -\frac{|z-t\alpha|^2}{2t} \right\}.$$  

With $\alpha = (z-x)/t$ and $s = t/2$, applying (4.3), we get

$$P_{0,x}^{t,z}(A) \leq 2^{d/2} P^{(\alpha)}(A).$$
Replacing $A$ by $e^{2\beta^2 \int_0^{t/2} V(\sqrt{2} W_s) \, ds}$, we have

$$
\sup_{z,x \in \mathbb{R}^d, t > 0} E_{0,x}^{t,z} \left[ \exp \left\{ 2\beta^2 \int_0^{t/2} V(\sqrt{2} W_s) \, ds \right\} \right] \leq 2^{d/2} \sup_\alpha E^{(\alpha)} \left[ \exp \left\{ 2\beta^2 \int_0^{t/2} V(\sqrt{2} W_s) \, ds \right\} \right] 
$$

$$
\leq 2^{d/2} \frac{1}{1 - a} < \infty,
$$

where the second upper bound follows from Khas’minskii’s lemma provided we have

$$
2\beta^2 \sup_{x,\alpha} E^{(\alpha)} \left[ \int_0^\infty V(\sqrt{2} W_s) \, ds \right] \leq a < 1.
$$

But since the expectation in the above display is equal to $\int_0^\infty ds \int_{\mathbb{R}^d} dz V(\sqrt{2} z) \rho^{(\alpha)}(s, z - x)$ and is maximal for $x = 0$ and $\alpha = 0$, the requisite condition reduces to

$$
2\beta^2 E_0 \left[ \int_0^\infty V(\sqrt{2} W_s) \, ds \right] < 1,
$$

which is satisfied by our assumption. Finally, the lemma follows from the observation

$$
\exp \left\{ 2\beta^2 \int_0^{t/2} V(\sqrt{2} W_s) \, ds \right\} \leq \frac{1}{2} \left\{ \exp \left\{ 2\beta^2 \int_0^{t/2} V(\sqrt{2} W_s) \, ds \right\} + \exp \left\{ 2\beta^2 \int_0^{t/2} V(\sqrt{2} W_s) \, ds \right\} \right\}
$$

combined with time reversibility of Brownian motion.

Recall that $V = \phi * \phi$ is bounded and has support in a ball of radius 1 around the origin, and therefore, for some constant $c, c' > 0$, and any $a > 0$,

$$
P_0 \left[ \int_m^\infty ds \, V(\sqrt{2} W_s) > a \right] \leq \frac{c}{a} \int_m^\infty \frac{ds}{s^{3/2}} \int_{B(0,1)} dy V(\sqrt{2} y) \exp \left\{ - \frac{|y|^2}{2s} \right\} \leq c' \frac{\|V\|_\infty}{am^{3/2}} \to 0
$$

as $m \to \infty$, implying

**Lemma 4.7.** For any $a > 0$, $\lim_{T \to \infty} P_0 \left[ \int_m^\infty ds \, V(\sqrt{2} W_s) > a \right] = 0$.

By Lemma 4.3, we also have

**Lemma 4.8.** For any $a > 0$,

$$
\lim_{T \to \infty} \sup_{z \in \mathbb{R}^d} P_{0,z}^{T,z} \left[ \int_m^{T-m} V(\sqrt{2} W_s) \, ds > a \right] = 0.
$$
Proof of Proposition 4.3. Note that it is enough to show that, for any \( a > 0 \) and \( y \in \mathbb{R}^d \),
\[
\lim_{T \to \infty} \sup_{y \in \mathbb{R}^d} E_{0,0}^{T,y} \left[ e^{\beta^2 \int_0^T V(\sqrt{2}W_t) \, dt} \cdot \mathbf{1}\left( \int_m^{T-m} V(\sqrt{2}W_s) \, ds > a \right) \right] = 0. \tag{4.4}
\]
Indeed, letting \( A = \left\{ \int_m^{T-m} V(\sqrt{2}W_s) \, ds > a \right\} \), we have
\[
\left| E_{0,0}^{T,y} \left[ e^{\beta^2 \int_0^T V(\sqrt{2}W_t) \, dt} - T_1 \right] - E_{0,0}^{T,y} \left[ e^{\beta^2 \int_{[0,m]} V(\sqrt{2}W_t) \, dt} \left( e^{\beta^2 \int_m^{T-m} V(\sqrt{2}W_t) \, dt} - 1 \right) \right] \right| \\
\leq 2 E_{0,0}^{T,y} \left[ e^{\beta^2 \int_0^T V(\sqrt{2}W_t) \, dt} \cdot 1_A \right] + a e^a E_{0,0}^{T,y} \left[ e^{\beta^2 \int_0^T V(\sqrt{2}W_t) \, dt} \right],
\]
where we decomposed the RHS of the first line on the two events \( A \) and \( A^c \) and we used that \( |e^a - 1| \leq ae^a \) in the second line. From this last display, we see that Proposition 4.3 is obtained by choosing \( a \) arbitrary small, Lemma 4.6 and (4.4).

Finally, we observe that convergence (4.4) follows from Hölder’s inequality, Lemma 4.6 and Lemma 4.8, which ends the proof.

We now turn to the proof of

Proof of Proposition 4.4. Condition on the position of the Brownian bridge at time \( T/2 \), then use reversal property of the Brownian bridge and change of variable \( z \to \sqrt{T}z \), to get:
\[
T_1(y) = \int_{\mathbb{R}^d} E_{0,0}^{T/2,z} \left[ e^{\beta^2 \int_{[0,m]} V(\sqrt{2}W_t) \, dt} \right] E_{T/2,z}^{T/2} \left[ e^{\beta^2 \int_{(m,T]} V(\sqrt{2}W_t) \, dt} \right] \frac{\rho(T/2, z) \rho(T/2, y - z)}{\rho(T, y)} \, dz \\
= \int_{\mathbb{R}^d} E_{0,0}^{T/2,z\sqrt{T}} \left[ e^{\beta^2 \int_0^m V(\sqrt{2}W_t) \, dt} \right] E_{0,y}^{T/2,z\sqrt{T}} \left[ e^{\beta^2 \int_0^m V(\sqrt{2}W_t) \, dt} \right] \frac{\rho(1/2, z) \rho(1/2, z - y/\sqrt{T})}{\rho(1, y/\sqrt{T})} \, dz.
\]

We now claim that, for fixed \( z \),
\[
E_{0,y}^{T/2,z\sqrt{T}} \left[ e^{\beta^2 \int_0^m V(\sqrt{2}W_t) \, dt} \right] \sim E_y \left[ e^{\beta^2 \int_0^\infty V(\sqrt{2}W_t) \, dt} \right]. \tag{4.5}
\]
Then, by dominated convergence theorem applied to the above integral, where the expectations in the integrand are bounded thanks to Lemma 4.6 we obtain that:
\[
T_1(y) \sim \int_{\mathbb{R}^d} E_0 \left[ e^{\beta^2 \int_0^\infty V(\sqrt{2}W_t) \, dt} \right] E_y \left[ e^{\beta^2 \int_0^\infty V(\sqrt{2}W_t) \, dt} \right] \frac{\rho(1/2, z) \rho(1/2, z)}{\rho(1, 0)} \, dz \\
= E_0 \left[ e^{\beta^2 \int_0^\infty V(\sqrt{2}W_t) \, dt} \right] E_y \left[ e^{\beta^2 \int_0^\infty V(\sqrt{2}W_t) \, dt} \right].
\]
To prove (4.5), we use Lemma 4.5:

$$E_{0,y}^{T/2, z, \sqrt{T}} \left[ e^{2\beta_0 \int_0^T V(\sqrt{\mathcal{W}_t}) \, dt} \right] = \frac{1}{\rho(T/2, z, \sqrt{T} - y)} E_y \left[ e^{2\beta_0 \int_0^T V(\sqrt{\mathcal{W}_t}) \, dt} \rho(T/2 - m, z\sqrt{T} - \sqrt{2W_m}) \right]$$

By monotone convergence and the fact that $m = o(T)$, we obtain:

$$P\text{-a.s.} \quad e^{2\beta_0 \int_0^T V(\sqrt{\mathcal{W}_t}) \, dt} \to e^{2\beta_0 \int_0^\infty V(\sqrt{\mathcal{W}_t}) \, dt} \quad \text{and} \quad e^{-\frac{|z - \sqrt{2T}W_m|^2}{1 - 2m/T}} \to e^{-2z^2}.$$ 

Then, we have the following uniform integrability property for small $\delta > 0$ and small $\beta$:

$$E_y \left[ \left( e^{2\beta_0 \int_0^T V(\sqrt{\mathcal{W}_t}) \, dt} e^{-\frac{|z - \sqrt{2T}W_m|^2}{1 - 2m/T}} \right)^{1+\delta} \right] \leq E_y \left[ e^{(1+\delta)2\beta_0 \int_0^\infty V(\sqrt{\mathcal{W}_t}) \, dt} \right] < \infty.$$ 

Hence,

$$E_{0,y}^{T/2, z, \sqrt{T}} \left[ e^{2\beta_0 \int_0^T V(\sqrt{\mathcal{W}_t}) \, dt} \right] \to \frac{e^{-2z^2}}{\rho(1/2, z) \pi^{d/2}} E_y \left[ e^{2\beta_0 \int_0^\infty V(\sqrt{\mathcal{W}_t}) \, dt} \right] = E_y \left[ e^{2\beta_0 \int_0^\infty V(\sqrt{\mathcal{W}_t}) \, dt} \right].$$

\[ \blacksquare \]

### 4.2 Second moment.

The goal of this section is to show

**Proposition 4.9.** There exists $\beta_0 \in (0, \infty)$, such that for all $\beta < \beta_0$, $E(\mathcal{L}_T^2) \to 0$.

For this result we will proceed as in the proof of Proposition 4.1. It is enough to show that $\limsup_{T \to \infty} E(\mathcal{L}_T^2) \leq 0$. Computing second moment, we get an integral over four independent Brownian paths:

$$E(\mathcal{L}_T^2) = E_0 \sum_{i \in \{1, 3\}} \left( T^2 V(W_T^{(i)} - W_T^{(i+1)}) - C_0 \right) e^{2\beta_0 \sum_{1 \leq i < j \leq 4} \int_0^T V(W_t^{(i)} - W_t^{(j)}) \, dt}$$

$$= E_0 \sum_{i \in \{1, 3\}} \left\{ e^{2\beta_0 \int_0^T V(W_t^{(i)} - W_t^{(i+1)}) \, dt} \left( T^2 V(W_T^{(i)} - W_T^{(i+1)}) - C_0 \right) \right\} \times e^{2\beta_0 \sum_{i \in \{1, 3\}} \int_0^T V(W_t^{(i)} - W_t^{(j)}) \, dt}$$

(4.6)

where the sum $\sum^*$ is considered for 4 pairs $(i, j), 1 \leq i < j \leq 4$ different from $(1, 2)$ and $(3, 4)$. 
Throughout the rest of the article, for notational convenience, we will write
\[
H_m = e^{\beta^2 \sum_{1 \leq i < j \leq 4} \int_0^m V(W^{(i)}_t - W^{(j)}_t) \, dt},
\]
\[
H_{T-m,T} = \prod_{i \in \{1, 3\}} \left\{ e^{\beta^2 \int_{T-m}^T V(W^{(i)}_t - W^{(i+1)}_t) \, dt} \left( T^{d/2} V(W^{(i)}_T - W^{(i+1)}_T) - \mathcal{C}_0 \right) \right\}. \tag{4.7}
\]
We will now estimate each term in the expectation in (4.6). Proposition 4.10 stated below enables us to neglect the contributions of \( \int_{m}^{T-m} V(W^{(i)}_t - W^{(j)}_t) \, dt \) for all \( i, j \) and \( \int_{T-m}^{T} V(W^{(i)}_t - W^{(j)}_t) \, dt \) for all \( (i, j) \neq (1, 2), (3, 4) \). More precisely, we want to show that

**Proposition 4.10.** For \( m = m(T) \) as above, there exists a constant \( C > 0 \) such that, for small enough \( \beta \), as \( T \to \infty \),
\[
\mathbb{E} \mathcal{L}_T^2 = \mathcal{T}_2 + o(1),
\]
where
\[
\mathcal{T}_2 = E_0^{\otimes 4} \left[ H_m \, H_{T-m,T} \right]. \tag{4.8}
\]

Then, Proposition 4.9 will be a consequence of

**Proposition 4.11.** For small enough \( \beta \), we have as \( T \to \infty \):
\[
\mathcal{T}_2 = E_0^{\otimes 4} \left[ e^{\beta^2 \sum_{1 \leq i < j \leq 4} \int_0^\infty V(W^{(i)}_t - W^{(j)}_t) \, dt} \right] \times \left[ E_0^{\otimes 2} \left( e^{\beta^2 \int_{T-m}^T V(W^{(1)}_t - W^{(2)}_t) \, dt} V(W^{(1)}_T - W^{(2)}_T) - \mathcal{C}_0 \right) \right]^2 + o(1). \tag{4.9}
\]

As a result, \( \mathcal{T}_2 \to 0 \).

### 4.3 Proof of Proposition 4.10

By the proof of Proposition 4.4, we see that it is enough to prove that
\[
\sup_{T > 0} E_0^{\otimes 4} \left[ e^{\beta^2 \sum_{1 \leq i < j \leq 4} \int_0^T V(W^{(i)}_t - W^{(j)}_t) \, dt} \prod_{i \in \{1, 3\}} \left| T^{d/2} V(W^{(i)}_T - W^{(i+1)}_T) - \mathcal{C}_0 \right| \right] < \infty, \tag{4.10}
\]
and that for all \( a > 0 \),
\[
\lim_{T \to \infty} E_0^{\otimes 4} \left[ e^{\beta^2 \sum_{1 \leq i < j \leq 4} \int_0^T V(W^{(i)}_t - W^{(j)}_t) \, dt} \prod_{i \in \{1, 3\}} \left| T^{d/2} V(W^{(i)}_T - W^{(i+1)}_T) - \mathcal{C}_0 \right| 1_A \right] = 0, \tag{4.11}
\]
where the event \( A \) is defined as
\[
A = \left\{ \int_m^{T-m} V(W^{(i)}_t - W^{(j)}_t) \, dt \geq a \text{ for some } 1 \leq i < j \leq 4, \right\}
\]
\[
\cup \left\{ \int_{T-m}^{T} V(W^{(i)}_t - W^{(j)}_t) \, dt \geq a \text{ for some } (i, j) \neq (1, 2), (3, 4) \right\}.
\]
To prove the above properties, we first estimate, using that \( V, \mathcal{C}_0 \geq 0 \),
\[
\prod_{i \in \{1, 3\}} \left| T^{d/2} V(W_T^{(i)} - W_T^{(i+1)}) - \mathcal{C}_0 \right| \leq \prod_{i \in \{1, 3\}} \left| T^{d/2} V(W_T^{(i)} - W_T^{(i+1)}) + \mathcal{C}_0 \right|
\]
and expand the last product. Then, we observe that for \( \beta \) small enough, Hölder’s inequality and Lemma 4.7 directly give:
\[
\mathcal{C}_0^2 \sup_{T > 0} E_0^{\otimes 4} \left[ e^{\beta^2 \sum_{1 \leq i < j \leq 4} \int_0^T V(W_t^{(i)} - W_t^{(j)}) \, dt} \right] < \infty,
\]
and
\[
\mathcal{C}_0^2 \lim_{T \to \infty} E_0^{\otimes 4} \left[ e^{\beta^2 \sum_{1 \leq i < j \leq 4} \int_0^T V(W_t^{(i)} - W_t^{(j)}) \, dt} \, 1_A \right] = 0.
\]
Moreover, switching from free Brownian motion to the Brownian bridge,
\[
E_0^{\otimes 4} \left[ e^{\beta^2 \sum_{1 \leq i < j \leq 4} \int_0^T V(W_t^{(i)} - W_t^{(j)}) \, dt} \prod_{i \in \{1, 3\}} \left( T^{d/2} V(W_T^{(i)} - W_T^{(i+1)}) \right) 1_A \right]
\]
\[
= \int_{(\mathbb{R}^d)^4} dy \prod_{i \in \{1, 3\}} \left( T^{d/2} V(y_i - y_{i+1}) \rho(T, y_i) \rho(T, y_{i+1}) \right)
\]
\[
\times \prod_{i=1}^4 E_{0, 0}^{T, \zeta_1} \left[ e^{\beta^2 \sum_{1 \leq i < j \leq 4} \int_0^T V(W_t^{(i)} - W_t^{(j)}) \, dt} \right] ,
\]
where the same equality also holds without the indicator \( 1_A \). A similar, even simpler, decomposition holds for
\[
\mathcal{C}_0 \ E_0^{\otimes 4} \left[ e^{\beta^2 \sum_{1 \leq i < j \leq 4} \int_0^T V(W_t^{(i)} - W_t^{(j)}) \, dt} \left( T^{d/2} V(W_T^{(1)} - W_T^{(2)}) \right) 1_A \right]
\]
Since \( \rho(T, y_{i+1} - r) \leq CT^{-d/2} \) and \( V \) is compactly supported, we finally obtain (4.10) and (4.11) by Hölder’s inequality, Lemma 4.6 and Lemma 4.8.

4.4 Proof of Proposition 4.11
If we denote by \( \mathcal{F}_{[0, T/2]} \) the \( \sigma \)-algebra generated by all four Brownian paths until time \( T/2 \), then, using Markov’s property,
\[
\mathcal{T}_2 = E_0^{\otimes 4} [H_m H_{T-m, T}] = E_0^{\otimes 4} \left[ E_0^{\otimes 4} \left( H_m H_{T-m, T} \left| (W_T^{(i)})_{i=1}^4 \right. \right) \right]
\]
\[
= E_0^{\otimes 4} \left[ E_0^{\otimes 4} \left( H_m H_{T-m, T} \left| \mathcal{F}_{[0, T/2]} \right. \left| \left( W_{T/2}^{(i)} \right)_{i=1}^4 \right) \right) \right]
\]
\[
= E_0^{\otimes 4} \left[ H_m \left| \left( W_{T/2}^{(i)} \right)_{i=1}^4 \right. \right] E_0^{\otimes 4} \left( H_{T-m, T} \left| \left( W_{T/2}^{(i)} \right)_{i=1}^4 \right) \right].
\]
We will prove that there exists a constant $C < \infty$, such that:

\[
\begin{align*}
(\text{i}) \sup_{T>0} E_0^{\otimes 4} \left\{ H_m \left( W_{T/2}^{(i)} \right)_{i=1}^4 \right\} &\leq C, \\
(\text{ii}) \sup_{T>0} E_0^{\otimes 4} \left\{ H_{T-m,T} \left( W_{T/2}^{(i)} \right)_{i=1}^4 \right\} &\leq C, \\
(\text{iii}) E_0^{\otimes 4} \left\{ H_m \left( W_{T/2}^{(i)} \right)_{i=1}^4 \right\} &\xrightarrow{\text{law}} E_0^{\otimes 4} [H_{\infty}], \quad \text{as } T \to \infty.
\end{align*}
\]

where $H_{\infty}$ is defined as $H_m$ with the time interval $[0, m]$ replaced by $[0, \infty)$, recall (4.1). Let us first conclude the proof of Proposition 4.11 assuming the above three assertions. The difference of the two first terms in (4.9) writes:

\[
\mathcal{T}_2 = E_0^{\otimes 4} [H_{\infty}] E_0^{\otimes 4} [H_{T-m,T}]
\]

\[
= E_0^{\otimes 4} \left[ \left( E_0^{\otimes 4} \left\{ H_m \left( W_{T/2}^{(i)} \right)_{i=1}^4 \right\} - E_0^{\otimes 4} [H_{\infty}] \right) E_0^{\otimes 4} \left\{ H_{T-m,T} \left( W_{T/2}^{(i)} \right)_{i=1}^4 \right\} \right],
\]

which goes to 0 as $T \to \infty$ by (i)-(iii), proving (4.9). Finally, computations of Section 3.1 ensure that:

\[
\left[ E_0^{\otimes 2} \left\{ e^{2 \beta f_{T-m}^V (W_{s}^{(1)} - W_{s}^{(2)})} \right\} \right] \xrightarrow{T \to \infty} 0.
\]

We now owe the reader the proofs of (i)-(iii). To prove (i), we use Hölder’s inequality to get

\[
E_0^{\otimes 4} \left\{ H_m \left( W_{T/2}^{(i)} \right)_{i=1}^4 \right\} \leq \prod_{1 \leq i < j \leq 4} E_0^{\otimes 4} \left[ e^{6 \beta^2 f_0^m V(W_t^{(i)} - W_t^{(j)})} \right] \left( W_{T/2}^{(i)} \right)_{i=1}^4 \right\}^{1/6}
\]

\[
= \prod_{1 \leq i < j \leq 4} \left[ e^{6 \beta^2 f_0^m V(V_0^{(i)} - V_0^{(j)})} \right]^{1/6}
\]

\[
\leq \sup_{T, z} \left[ e^{6 \beta^2 f_0^m V(V_0^{(i)} - V_0^{(j)})} \right] < \infty,
\]

by Lemma 4.6. For (ii), we note that by Markov’s property,

\[
E_0^{\otimes 4} \left\{ H_{T-m,T} \left( W_{T/2}^{(i)} \right)_{i=1}^4 \right\} = \prod_{i \in \{1, 3\}} E_{W_{T/2}^{(i)} - W_{T/2}^{(i+1)}} \left[ e^{2 \beta f_{T/2-m}^V (V_0^{(i)} - V_0^{(j)})} \right] \left( V_0^{(i)} - V_0^{(j)} \right).
\]

We have:

\[
\mathcal{E}_0 E_{W_{T/2}^{(i)} - W_{T/2}^{(i+1)}} \left[ e^{2 \beta f_{T/2-m}^V (V_0^{(i)} - V_0^{(j)})} \right] \leq \mathcal{E}_0 \sup_z E_z \left[ e^{2 \beta f_{T/2-m}^V (V_0^{(i)} - V_0^{(j)})} \right] < \infty,
\]
while, for some constant $C' > 0$,
\[
E_{W_T^{(i)} - W_T^{(i+1)}} \left[ e^{\beta^2 \int_{T/2 - m}^{T/2} V(\sqrt{2}W_t) \, dt} \, T^{d/2} V (\sqrt{2}W_T^{(i)}) \right] 
\leq C' \int_{\mathbb{R}^d} \, dz \, E_{W_T^{(i)} - W_T^{(i+1)}} \left[ e^{\beta^2 \int_{T/2 - m}^{T/2} V(\sqrt{2}W_t) \, dt} \, V (\sqrt{2}z) \right] 
\leq C' \sup_{T,y,z} E_{0,y} \left[ e^{\beta^2 \int_{0}^{T/2} V(\sqrt{2}W_t) \, dt} \right] \int_{B(0,1)} \, d\sigma \, V (\sqrt{2}z) < \infty,
\]
again by Lemma 4.6.

Finally, to prove (iii), we fix any bounded continuous test function $f : \mathbb{R} \to \mathbb{R}$, so that
\[
E_{\left( W_T^{(i)} \right)_{i=1}^4} \left[ f (\left( H_m \right)_{i=1}^4) \right] = \int \, dy \, f \left( E_{0,y} \left[ H_m \right] \right) \prod_{i=1}^4 \rho (T/2, y_i) 
= \int \, dz \, f \left( E_{0,y} \left[ \sqrt{T} \right] \right) \prod_{i=1}^4 \rho (1/2, z_i). \tag{4.12}
\]
Now, letting $T \to \infty$, we get similarly to [4.5] that $E_{0,y} \left[ \sqrt{T} \right] \to E_{0} \left[ H_\infty \right]$. By dominated convergence, the RHS of (4.12) converges to $f \left( E_{0} \left[ H_\infty \right] \right)$, implying (iii).

\section{Proof of Theorem 2.1}

The proof of Theorem 2.1 builds on that of Theorem 3.1. For the reader's convenience, we split it into two tasks in each of the next two sections.

\subsection{Convergence of finite dimensional distributions of the spatially indexed process \{\mathcal{H}_x(t, x)\}_{x \in \mathbb{R}^d}.}

We first show that,

**Proposition 5.1.** For $\beta \in (0, \beta_0)$, any fixed $t > 0$, $k \in \mathbb{N}$ and $x_1, \ldots, x_k \in \mathbb{R}^d$, the joint distributions of $(\mathcal{H}_x(x_1, t), \ldots, \mathcal{H}_x(x_k, t))$ converges to that of $(\mathcal{H}(x_1, t), \ldots, \mathcal{H}(x_k, t))$.

For the rest of this section, we will write, for any $\sigma > 0$,
\[
L_T^{(\sigma)} (x) \equiv T^{d/2} \left( \frac{d}{dt} \langle \mathcal{Z}(0), \mathcal{Z}(r) \rangle \sigma_T - \gamma^2 \rho(2\sigma, x) \mathcal{Z}_{\sigma T}(0) \mathcal{Z}_{\sigma T}(r) \right), \tag{5.1}
\]
where $\gamma^2 = \gamma^2 (\beta)$ is defined in (2.1).

The key step for the proof of Proposition 5.1 is
Recall the square integrable martingale
\[ d\mathcal{Z}_T(r) = \beta E_r \left[ \Phi_T(W) \right] dT, \]
where we again remind the reader that \( \mathcal{Z}(0, r) \rightarrow \infty \).

The proof of Proposition 5.3 splits into two main steps. Again, we fix \( m = m(T) \) such that \( m \rightarrow \infty \) and \( m = o(T) \) as \( T \rightarrow \infty \).

**Lemma 5.4.** Under the assumptions imposed in Proposition 5.3, we have, for any \( x \in \mathbb{R}^d \),
\[ \lim_{T \rightarrow \infty} \sup_{y \in \mathbb{R}^d} \left| E_{2\sigma T, y} \left[ e^{\beta^2 \int_0^{T \sigma} V(W_{dt}) \, dt} \right] - E_{0, r} \left[ e^{\beta^2 \int_{\lfloor o(T) \rfloor} \sigma T \cdot \lfloor o(T) \rfloor} V(W_{dt}) \, dt \right] \right| = 0. \]
Proof. The proof of Lemma 5.5 follows exactly the same line of arguments as in Proposition 4.3.

The next result will then conclude the proof of Proposition 5.3.

Lemma 5.5. Under the assumptions imposed in Proposition 5.3, we have, for any $x \in \mathbb{R}^d$ and for $r = x/\sqrt{T}$,

$$
\sup_{y \in \mathbb{R}^d} \left| E_{0,r}^{\sigma T,x-y} \left[ e^{\beta^2 \int_0^T V(W_{2t}) \, dt} \right] - E_r \left[ e^{\beta^2 \int_0^T V(W_{2t}) \, dt} \right] E_y \left[ e^{\beta^2 \int_0^T V(W_{2t}) \, dt} \right] \right| \xrightarrow{T \to \infty} 0.
$$

Proof. Note that when $x \neq 0$, the integrals over $[0,m]$ vanish in the limit and the claim reduces to showing

$$
\sup_{y} \left| E_{0,r}^{\sigma T,x-y} \left[ e^{\beta^2 \int_0^T V(W_{2t}) \, dt} \right] - E_r \left[ e^{\beta^2 \int_0^T V(W_{2t}) \, dt} \right] E_y \left[ e^{\beta^2 \int_0^T V(W_{2t}) \, dt} \right] \right| \to 0,
$$

which is straightforward to check.

We will now show

Lemma 5.6. Under the assumptions imposed in Proposition 5.3, we have, for any $x \in \mathbb{R}^d$,

- As $T \to \infty$,
  $$
  \mathbb{E} \left[ \mathcal{L}^{(\sigma)}(x)^2 \right] = \mathcal{T}_2^{(\sigma)}(x) + o(1),
  $$
  where
  $$
  \mathcal{T}_2^{(\sigma)}(x) = \left( E_{0,x}^{\otimes 2} \otimes E_{0,r}^{\otimes 2} \right) \left[ H_m H_{\sigma T-m,\sigma T} \right],
  $$
  with $H_m$ and $H_{\sigma T-m,\sigma T}$ defined in (4.7).

- As $T \to \infty$,
  $$
  \left| \mathcal{T}_2^{(\sigma)}(x) - \left[ E_{0,x}^{\otimes 2} \left( e^{\beta^2 \int_0^T V(W^{(i)}_t-W^{(i)}_t)} \right) \right]^2 \times \left[ E_{0,x}^{\otimes 2} \left( e^{\beta^2 \int_0^T V(W^{(i)}_t-W^{(i)}_t)} \, dt \left[ T^{d/2} V(W^{(i)}_T - W^{(i)}_T) - \gamma^2 \rho(2\sigma, x) \right] \right) \right]^2 \right| \to 0.
  $$

In particular, $\mathcal{T}_2^{(\sigma)}(x) \to 0$ and $\mathcal{L}_T^{(\sigma)}(x) \to 0$ in $L^2$.

Proof. Again, it suffices to consider the case $x \neq 0$. Note that

$$
\mathbb{E} \left[ \mathcal{L}_T^{(\sigma)}(x)^2 \right] = \left( E_{0,r}^{\otimes 2} \otimes E_{0,y}^{\otimes 2} \right) \left[ \prod_{j \in \{1,3\}} \left\{ e^{\beta^2 \int_0^T V(W^{(i)}_t-W^{(i)}_t)} \, dt \left[ T^{d/2} V(W^{(i)}_T - W^{(i)}_T) - \gamma^2 \rho(2\sigma, x) \right] \right\} \right] \times e^{\beta^2 \sum_{j \in \{1,3\}} \int_0^T V(W^{(i)}_t-W^{(i)}_t)}.
$$
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Then a repetition of the exactly same arguments as in Proposition 3.10 prove (5.5). To deduce (5.6), shortening notations $E_{0;r}^\otimes 4 \otimes E_{0;r}^\otimes 4$ into $E_{0;r;0;r}\otimes E_{0;r;0;r}'$, observe that

$$\mathcal{F}_2^{(e)}(x) = E_{0;r;0;r}^\otimes 4 \left[ E_{0;r;0;r}^\otimes 4 \left\{ H_m \left( W_{\sigma T/2}^{(i)} \right)_{i=1}^4 \right\} \right]$$

Hence, the left hand side in (5.6) can be rewritten as

$$\mathcal{F}_2^{(e)}(x) - \left( E_{0,0} \left[ e^{\beta^2 \int_0^\infty V(W_{2t}) \, dt} \right] \right)^2 E_{0;r;0;r}^\otimes 4 \left( H_{\sigma T-m,\sigma T} \right)$$

$$= E_{0;r;0;r}^\otimes 4 \left\{ E_{0;r;0;r}^\otimes 4 \left[ H_m \left( W_{\sigma T/2}^{(i)} \right)_{i=1}^4 \right] - \left( E_{0,0} \left[ e^{\beta^2 \int_0^\infty V(W_{2t}) \, dt} \right] \right)^2 \right\} \times E_{0;r;0;r}^\otimes 4 \left( H_{\sigma T-m} \left( W_{\sigma T/2}^{(i)} \right)_{i=1}^4 \right)$$

which tends to 0, as can be seen from the proof of Proposition 4.10.

Proof of Proposition 5.1. The proof follows the same line of arguments as in Section 3.3. In particular, it is enough to show that for all $t > 0$,

$$T^{(d-2)/4} \left( \log \mathcal{Z}_{\infty}(\sqrt{T}x) - \log \mathcal{Z}_{\Sigma}(\sqrt{T}x) \right) \xrightarrow{\text{law}} \mathcal{H}(t,x),$$

where the convergence holds jointly for finitely many $x$’s. Note that again by Itô’s formula,

$$\log \mathcal{Z}_T(x) = N_T(x) - \frac{1}{2} \langle N(x) \rangle_T,$$

is a martingale with cross-bracket

$$\langle N(x), N(y) \rangle_T = \beta^2 \int_0^T \int_{\mathbb{R}^d} E_{x,\beta,t} \left[ \phi(y-W_t) \right] \xi(t,y) \, dy \, dt,$$

where we remind the reader that $E_{x,\beta,t}$ denotes the expectation w.r.t. the polymer measure, while $E_{x,\beta,t}\otimes E_{y,\beta,t}$ denotes the same w.r.t. the product polymer measure (on the same environment) defined on two independent Brownian paths starting at $x$ and $y$.

We now use the multidimensional version of the functional central limit for martingales ([22, Theorem 3.11]) combined with Proposition 5.2 to conclude that the sequence of rescaled martingales

$$\langle N^{(T)}(x) : \tau \rightarrow T^{(d-2)/4} \left( N_T(\sqrt{T}x) - N_{\Sigma T}(\sqrt{T}x) \right), \quad \tau \geq t, \quad (5.7)$$

converge in the sense of finite dimensional distributions to a Gaussian field. For all $r = xT^{1/2}$ and $r' = yT^{1/2}$,

$$\langle N^{(T)}(x), N^{(T)}(y) \rangle_T = T^{(d-2)/2} \beta^2 \int_0^{\tau T} E_{r,\beta,t} \otimes E_{r',\beta,t} \left[ V(W_s^{(1)} - W_s^{(2)}) \right] \, ds$$

$$= \int_0^{\tau} T^{d/2} \beta^2 E_{r,\beta,T} \otimes E_{r',\beta,\sigma T} \left[ V(W_s^{(1)} - W_s^{(2)}) \right] \, d\sigma$$
and we claim that
\[
\lim_{T \to \infty} \sup_{x,y} \left\| \langle N^{(T)}(x), N^{(T)}(y) \rangle_T - \gamma^2 \int_t^T \rho(2\sigma, y - x) \, d\sigma \right\|_1 = 0. \tag{5.8}
\]

Note that for \( \tau = \infty \), the second term in the norm in (5.8) equals \( \text{Cov}(\mathcal{H}(t, x), \mathcal{H}(t, y)) \), so that, assuming (5.8), this concludes the proof of Proposition 5.1 by the same arguments as exposed in Section 3.3.

We now prove the uniform limit (5.8). By (5.2) and Cauchy-Schwarz inequality,
\[
\mathbb{E} \left| \langle N^{(T)}(x), N^{(T)}(y) \rangle_T - \gamma^2 \int_t^T \rho(2\sigma, y - x) \, d\sigma \right| \\
= \mathbb{E} \left| \int_t^T \frac{1}{2\sigma_T(r)} \mathcal{F}^{(T)} \left( \frac{d}{dt} \langle \mathcal{F}(r), \mathcal{F}(r') \rangle_{\sigma_T} \right) \right| d\sigma \\
\leq \int_t^T \left\| \frac{1}{2\sigma_T(r)} \mathcal{F}^{(T)} \left( \frac{d}{dt} \langle \mathcal{F}(r), \mathcal{F}(r') \rangle_{\sigma_T} \right) \right\|_2 d\sigma.
\]

Recall that \( \sup_T \mathbb{E}[\log \mathcal{Z}_T^p] < \infty \) for all \( p < 0 \) (see (3.14)). Hence, convergence (5.8) is obtained from Proposition 5.2 and dominated convergence applied to the last line of the above display. Note that domination can be justified by the following bound for all \( \sigma \geq t \):
\[
\left\| T^{d/2} \left( \frac{d}{dt} \langle \mathcal{F}(r), \mathcal{F}(r') \rangle_{\sigma_T} \right) \right\|_2 \leq C\sigma^{-d/2}, \tag{5.9}
\]
where \( C \) is some finite constant independent of \( x, y, T \) and \( \sigma \) (independence with respect to \( x, y \) will be useful for the proof of Theorem 2.4). It can be directly obtained by observing that
\[
\sup_{T > 0} \sup_{x,y \in \mathbb{R}^d} \left\| T^{d/2} \left( \frac{d}{dt} \langle \mathcal{F}(r), \mathcal{F}(r') \rangle \right) \right\|_2 < \infty, \tag{5.10}
\]
which we get by computing the second moment:
\[
\mathbb{E} \left[ T^d \left( \frac{d}{dt} \langle \mathcal{F}(0), \mathcal{F}(r) \rangle \right)^2 \right] \\
= \left( \mathbb{E}_{0,T}^r \otimes \mathbb{E}_{0,T}^r \right) \left[ e^{\beta^2 \sum_{i<j \leq 4} t_i^T dV(W_t^{(i)} - W_t^{(j)})} \prod_{j \in \{1,3\}} T^{d/2} dV(W_t^{(i)} - W_t^{(i+1)}) \right] \\
= \int_{(\mathbb{R}^d)^4} \prod_{i \in \{1,3\}} \left( T^{d/2} V(y_i - y_{i+1}) \rho(T, y_i) \rho(T, y_{i+1} - r) \right) \\
\times \bigotimes_{i \in \{1,3\}} \left( \mathbb{E}_{0,0}^{T,y_i} \otimes \mathbb{E}_{0,r}^{T,y_{i+1}} \right) \left[ e^{\beta^2 \sum_{i \leq j \leq 4} t_i^T dV(W_t^{(i)} - W_t^{(j)})} \right] dt.
\]
By Lemma 4.6 and Hölder’s inequality, the expectation over the Brownian bridges in the last display is uniformly bounded for small enough $\beta$. Since $\rho(T, y_{i+1} - r) \leq C T^{-d/2}$ and $V$ is compactly supported, we obtain (5.10).

5.2 Proof of Theorem 2.1  Convergence of space-time finite dimensional distributions of $\{\mathcal{H}_\varepsilon(t, x)\}_{t > 0, x \in \mathbb{R}^d}$.

Note that to derive Theorem 2.1 it suffices to show that for $\beta \in (0, \beta_0)$ the following convergence of the joint distribution of a finite vector holds:

$$(\mathcal{H}_\varepsilon(t, x), \mathcal{H}_\varepsilon(s, y), \ldots, \mathcal{H}_\varepsilon(T, y_{i+1} - r), \ldots) \to (\mathcal{H}(t, x), \mathcal{H}(s, y), \ldots).$$

The proof actually follows closely the lines of arguments as that of Proposition 5.1. To avoid repetition we will only sketch the argument quite briefly.

Recall that with $\xi(\varepsilon, t; s, y) = \varepsilon^{(d+2)/2} \xi(t - \varepsilon^2 s, \varepsilon y)$ we have $u_\varepsilon(t, x) = Z_{t/\varepsilon^2}(\xi(\varepsilon, t); x/\varepsilon)$ and therefore, for all $s \leq t$,

$$(u_\varepsilon(s, y), u_\varepsilon(t, x)) = \left( Z_{t/\varepsilon^2} \left( \frac{y}{\varepsilon}, \xi(\varepsilon, t) \right), Z_{t/\varepsilon^2} \left( \frac{x}{\varepsilon}, \xi(\varepsilon, t) \right) \right)$$

(5.11)

where for any $0 \leq A \leq B$ and $X \in \mathbb{R}^d$, we wrote

$$Z_{A,B}(X; \xi) = E[\Phi_{A,B}(W)|W_A = X],$$

$$\Phi_{A,B}(W) = \exp \left\{ \beta \int_A^B \int_{\mathbb{R}^d} \phi(W_s - y)\xi(s, y) \, ds \, dy - \frac{\beta^2}{2}(B - A)V(0) \right\}. \quad (5.12)$$

In this section we will also write

$$T = \varepsilon^{-2}, u = t - s \in [0, t) \quad \text{and} \quad \sigma > u.$$

Starting from (5.11), we can restrict ourselves to $y = 0$ by shift invariance and compute as before:

$$\left( \frac{d}{dt} \left( \mathcal{Z}_{T, u}(0, \cdot), \mathcal{Z}_{0, (\sqrt{T}x, \cdot)} \right) \right)_{T \sigma} = \beta^2 (E_{T, u, 0} \otimes E_{0, \sqrt{T}x}) \left[ \Phi_{T, u, T \sigma}(W^{(1)}) \Phi_{0, T \sigma}(W^{(2)}) V(W^{(1)}_{T \sigma} - W^{(2)}_{T \sigma}) \right].$$

As in (5.11) if we now write

$$\mathcal{L}_{\sigma,u}^T(x) = T^{d/2} \left[ \beta^2 (E_{T, u, 0} \otimes E_{0, \sqrt{T}x}) \left[ \Phi_{T, u, T \sigma}(W^{(1)}) \Phi_{0, T \sigma}(W^{(2)}) V(W^{(1)}_{T \sigma} - W^{(2)}_{T \sigma}) \right] - \gamma^2 \rho(2\sigma - u, x) \mathcal{Z}_{T, u, T \sigma}(0) \mathcal{Z}_{0, T \sigma}(\sqrt{T}x), \right.$$

(5.13)

then we again need to show
Proposition 5.7. Under the same assumption as in Theorem 2.1, for any $\sigma > u \geq 0$ and $x \in \mathbb{R}^d$

- As $T \to \infty$, \[ \mathbb{E} [\mathcal{L}_{T}^{\sigma,u}(x)] \to 0. \]  

- As $T \to \infty$, \[ \mathbb{E} [\mathcal{L}_{T}^{\sigma,u}(x)^2] \to 0. \]

Therefore, $L_{T}^{\sigma,u}(x) \overset{L^2(\mathbb{P})}{\longrightarrow} 0$.

Proof. We will first carry out

Proof of (5.14): With $L_{T}^{(\sigma-u)}(\cdot)$ defined in (5.1), we will show that

\[ \mathbb{E}[L_{T}^{\sigma,u}(x)] = \int dz \rho(u, z-x) \mathbb{E}[L_{T}^{(\sigma-u)}(z)] + R_{T}, \]  

(5.16)

where

\[ R_{T} = -\gamma^2 \int dz \rho(u, z-x) (\rho(2\sigma-u, x) - \rho(2\sigma-2u, z)) \mathbb{E} \left[ \mathcal{Z}_{0,T(\sigma-u)}(0) \mathcal{Z}_{0,T(\sigma-u)}(\sqrt{T}z) \right], \]

and that

\[ \sup_{x \in \mathbb{R}^d, T > 0} \mathbb{E} [|L_{T}^{(\sigma-u)}(z)|] < \infty. \]  

(5.17)

By dominated convergence, Proposition 5.3 and (5.17) justify that the first term of the RHS of (5.16) vanishes as $T \to \infty$. Then, the covariance computation in (3.4) implies that the expectation in the summand of $R_{T}$ goes to 1 for all $z \neq 0$. By dominated convergence and the Chapman-Kolmogorov property, we therefore obtain that $R_{T}$ also vanishes as $T \to \infty$, which in turn implies (5.14).

To derive (5.17), we appeal to (5.1) and (5.4), so that

\[ \mathbb{E}[|L_{T}^{\sigma,u}(z)|] \leq \int_{\mathbb{R}^d} dy V(y) E_{0,z \sqrt{T}}^{2(\sigma-u)T} \left[ e^{\beta^2 f_0^{(\sigma-u)T} V(W_{2t}) dt} \right] T^{d/2} \rho(2(\sigma-u)T, y - z \sqrt{T}) \]

\[ + \gamma^2 \rho(2\sigma, z) E[\mathcal{Z}^{2}_{(\sigma-u)T}] \]

\[ \leq C \int_{\mathbb{R}^d} dy V(y) + \gamma^2 (4\pi\sigma)^{-d/2} \sup_{T} \mathbb{E}[\mathcal{Z}^{2}_{(\sigma-u)T}] < \infty. \]

For the second upper bound in the above display, we used that in the first summand the first expectation is uniformly bounded thanks to Lemma 4.6, while $T^{d/2} \rho((2(\sigma-u)T, y - z \sqrt{T}) \leq C'$ and for the second summand we used $\sup_{T} \mathbb{E}[\mathcal{Z}^{2}_{(\sigma-u)T}] < \infty$ for $\beta \in (0, \beta_0)$ and moreover $\rho(2\sigma, z) \leq (4\pi\sigma)^{-d/2}$. Hence, (5.17) is justified.
To prove (5.16), we note that (recall the definition of $\Phi_{A,B}(\cdot)$ in (5.12)),
\[
\mathbb{E}[\mathcal{Z}_T^{\sigma,u}(x)] = T^{d/2} \beta^2 (E_{T,u,0} \otimes E_{0,\sqrt{T}x}) \left[ \mathbb{E} \left[ \Phi_{T,u,T\sigma}(W^{(1)})\Phi_{0,T\sigma}(W^{(2)}) \right] V(W^{(1)}_{T\sigma} - W^{(2)}_{T\sigma}) \right]
- \gamma^2 \rho(2\sigma - u, x)\mathbb{E} \left[ \mathcal{Z}_{T,u,T\sigma}(0)\mathcal{Z}_{u,\sqrt{T}x} \right]
= T^{d/2} \beta^2 (E_{T,u,0} \otimes E_{0,\sqrt{T}x}) \left[ e^{\beta^2 \int_{T\sigma}^T V(W^{(1)}_s - W^{(2)}_s) \, ds} V(W^{(1)}_{T\sigma} - W^{(2)}_{T\sigma}) \right]
- \gamma^2 \rho(2\sigma - u, x) \int dz \rho(u, z - x)\mathbb{E} \left[ \mathcal{Z}_{0,T(\sigma-u)}(0)\mathcal{Z}_{0,T(\sigma-u)}(\sqrt{T}z) \right]
\]
where the second identity follows from covariance structure of the white noise, the Markov property (3.8) and the diffusive scaling of the heat kernel. Thus, (5.16) follows from the above display and the following computation:
\[
(E_{T,u,0} \otimes E_{0,\sqrt{T}x}) \left[ e^{\beta^2 \int_{T\sigma}^T V(W^{(1)}_s - W^{(2)}_s) \, ds} V(W^{(1)}_{T\sigma} - W^{(2)}_{T\sigma}) \right]
= \int \rho(u, z - x) (E_{0,\otimes 0} \otimes E_{0,\sqrt{T}x}) \left[ e^{\beta^2 \int_0^{T(\sigma-u)} V(W^{(1)}_s - W^{(2)}_s) \, ds} V(W^{(1)}_{T(\sigma-u)} - W^{(2)}_{T(\sigma-u)}) \right].
\]

We will now sketch the proof of

**Proof of (5.15):** Note that
\[
\mathbb{E}[\mathcal{Z}_T^{\sigma,u}(x)^2] = \left( E_{(T,u,0);(0,\sqrt{T}x)} \otimes E_{(T,u,0);(0,\sqrt{T}x)} \right) \left[ \prod_{i \in \{1,2\}} e^{\beta^2 \int_{T\sigma}^T V(W^{(i)}_s - W^{(i+1)}_s) \, ds} \right]
\times \left\{ T^{d/2} V(W^{(i)}_{T\sigma} - W^{(i+1)}_{T\sigma}) - \rho(2\sigma - u, x)\gamma^2 \right\}
\times \exp \left\{ \beta^2 \sum_{i < j} \int_{T\sigma}^T V(W^{(i)}_s - W^{(i+1)}_s) \, ds \right\} e^{\beta^2 \int_{0}^{T\sigma} V(W^{(2)}_s - W^{(4)}_s) \, ds}
\]
(5.18)
We can again repeat exactly the same arguments as in the proof of Lemma 5.6 to derive the above result. We refrain from spelling out the details. \[\]

**Proof of Theorem 2.1:** To derive Theorem 2.1 it suffices to carry out the same line of arguments as the proof of Proposition 5.1. In particular, by property (5.11) for general vector length, it is enough to show that for all $t > 0$ and all $u_1, \ldots, u_n \in [0, t)$, $x_1, \ldots, x_n \in \mathbb{R}^d$,
\[
\left( \log \mathcal{Z}_{T_{u_1}},T_{t}(\sqrt{T}x_1), \ldots, \log \mathcal{Z}_{T_{u_n}},T_{t}(\sqrt{T}x_n) \right) \overset{\text{law}}{\longrightarrow} (\mathcal{H}(t - u_1, x_1), \ldots, \mathcal{H}(t - u_n, x_n)) \quad \text{as } T \to \infty.
\]
(5.19)
Again, we define

\[ N_{U,T}(X) = \beta \int_U^T \int_{\mathbb{R}^d} E_{X,U,\sigma} [\phi(y - W_\sigma)] \xi(\sigma,y) \, d\sigma \, dy, \]

where \( E_{X,A,B} \) denotes the expectation with respect to the polymer measure \( P_{X,A,B}(\cdot) = \frac{1}{Z_{A,B}(X)} E_{W,U} = X [\Phi_{A,B}(W) \, 1] \) conditional on the Brownian path to start at \( X \in \mathbb{R}^d \) at time \( U \), and \( \Phi_{A,B}(\cdot) \) is defined in (5.12). Then,

\[ \log Z_{U,T}(x) = N_{U,T}(x) - \frac{1}{2} \langle N_{U,\cdot}(x) \rangle_T, \quad (5.20) \]

where the bracket satisfies

\[ \frac{d}{dt} \langle N_{U,\cdot}(Y), N_{0,\cdot}(X) \rangle_T = E_{U,Y,T} \otimes E_{0,X,T} \left[ V \left(W_T^{(1)} - W_T^{(2)} \right) \right], \quad (5.21) \]

so that the following result holds:

**Lemma 5.8.** Let \( t \geq 0 \) and define, for \( u \leq t \leq \tau \), the martingales

\[ (N^{(t)}(u,x)) : \tau \to T^{(d-2)/4} \left( N_{T_u,T\tau}(\sqrt{T}x) - N_{T_u,T\tau}(\sqrt{T}x) \right). \]

Then, for all \( u_1, \ldots, u_n \in [0,t), x_1, \ldots, x_n \in \mathbb{R}^d \),

- As \( T \to \infty \),
  \[ \langle N^{(t)}(u_1,x_1), N^{(t)}(u_2,x_2) \rangle_T \xrightarrow{L^1} \gamma^2 \int_t^\tau \rho(2\sigma - (u_1 + u_2), x_1 - x_2) \, d\sigma. \quad (5.22) \]

- Moreover,
  \[ \text{Cov} \left( N^{(t)}(u_1,x_1), N^{(t)}(u_2,x_2) \right) = \gamma^2 \int_t^\infty \rho(2\sigma - (u_1 + u_2), x_1 - x_2) \, d\sigma. \quad (5.23) \]

**Proof.** First observe that for \( u_2 \leq u_1 \),

\[ \left( N^{(t)}_{\tau}(u_1,x_1), N^{(t)}_{\tau}(u_2,x_2) \right) \overset{\text{law}}{=} \left( N^{(\tau-u_2)}_{\tau-u_2}(u_1 - u_2, x_1), N^{(\tau-u_2)}_{\tau-u_2}(0, x_2) \right), \]

so that it suffices to prove convergence (5.22) when \( u_2 = 0 \), and this is obtained from (5.21), (5.13) and (5.15) in the same way as (5.8) was proved. (5.23) is obtained by a simple covariance computation.

Now, convergence (5.19) follows as in the proof from Section 3.3 that is by appealing to the multi-dimensional CLT for martingales which will give that for all \( t > 0 \), the martingales \( N^{(t)}(u,x) \) will converge jointly in \( u \) and \( x \) to a Gaussian processes with covariance structure given by (5.22), then neglecting the bracket part in (5.20) in the scaling limit and finally letting \( \tau \to \infty \) (observe that the RHS of (5.23) equals the RHS of (5.22) when \( \tau = \infty \)).
5.3 Proof of Theorem 2.4

Recall that we need to show that for all \( t > 0 \),
\[
\int_{\mathbb{R}^d} \mathcal{H}_\varepsilon(t, x) \varphi(x) dx \overset{\text{law}}{\rightarrow} \int_{\mathbb{R}^d} \mathcal{H}(t, x) \varphi(x) dx,
\]  
(5.24)
as \( T \to \infty \), where the convergence holds jointly for finitely many test functions \( \varphi \in C_c^\infty \). Once again, it suffices to prove that
\[
\int_{\mathbb{R}^d} T^{(d-2)/4} \left( \log Z_{\infty}(\sqrt{T} x) - \log Z_{T}(\sqrt{T} x) \right) \varphi(x) dx \overset{\text{law}}{\rightarrow} \int_{\mathbb{R}^d} \mathcal{H}(t, x) \varphi(x) dx,
\]
(5.25)
holds jointly for finitely many \( \varphi \)'s. As in the proof of Proposition 5.1, we decompose \( \log Z_T(x) \) in \( N_T(x) - \frac{1}{2} \langle N(x) \rangle_T \), so that convergence (5.25) reduces to studying the joint convergence as \( T \to \infty \) of the following family of martingales:
\[
\langle N^{(T)}(\varphi) \rangle : \tau \to \int_{\mathbb{R}^d} N^{(T)}_\tau(x) \varphi(x) dx,
\]
where \( N^{(T)}(x) \) is defined by (5.7). For all test functions \( \varphi_1 \) and \( \varphi_2 \) in \( C_c^\infty \), we compute the cross-bracket and find that
\[
\langle N^{(T)}(\varphi_1), N^{(T)}(\varphi_2) \rangle_{\tau} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \langle N^{(T)}(x), N^{(T)}(y) \rangle_{\tau} \varphi_1(x) \varphi_2(y) dx dy
\]
\[
\rightarrow \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( \gamma^2 \int_t^\tau \rho(2\sigma, y-x) d\sigma \right) \varphi_1(x) \varphi_2(y) dx dy.
\]
where the convergence is in \( L^1 \)-norm as \( T \to \infty \) and comes from uniform convergence (5.8).

The proof is again concluded by the multidimensional functional central limit for martingales ([23, Theorem 3.11]) and the observation that the last integral in the above display, for \( \tau = \infty \), is the covariance function of the joint Gaussian variables \( \int_{\mathbb{R}^d} \mathcal{H}(t, x) \varphi_1(x) dx \) and \( \int_{\mathbb{R}^d} \mathcal{H}(t, x) \varphi_2(x) dx \).

5.4 Proof of Proposition 2.5

It is enough to show that \( \{ \mathcal{H}_\varepsilon(t, x) \}_{\varepsilon > 0, x \in \mathbb{R}^d} \) forms a tight family, since the convergence part of the proposition will follow from tightness and uniqueness of the limit established in Theorem 2.4. To prove tightness, we appeal to the following tightness criterion which was recently established in [16] and was shown to hold in a function space \( C^\alpha_{\text{loc}}(\mathbb{R}^d) \) of distributions with “local \( \alpha \)-Hölder regularity” for \( \alpha \in \mathbb{R} \). Loosely speaking, this means that a distribution \( f \in C^\alpha_{\text{loc}}(\mathbb{R}^d) \) if and only if for any smooth test function \( \varphi \in C_c^\infty(\mathbb{R}^d) \) with compact support and \( x \in \mathbb{R}^d \), \( \lambda^{-d} \langle f, \varphi(\lambda^{-1}(\cdot-x)) \rangle \leq C\lambda^\alpha \) for \( \lambda \sim 0 \). In [16, Theorem 1.1] it was shown there that there is a finite family of smooth and compactly supported functions \( \phi \) and \( (\psi^{(i)})_{1 \leq i \leq 2^d} \) so that the following condition holds:
Theorem 5.9. Let \((f_\varepsilon)_\varepsilon\) be a family of random linear forms on \(C^r_c(\mathbb{R}^d)\), let \(p \in [1, \infty)\) and \(\beta \in \mathbb{R}\) such that \(|\beta| < r\). Suppose there is an absolute constant \(C < \infty\) such that the following two conditions hold:

\[
\sup_{x \in \mathbb{R}^d, \varepsilon > 0} E\left[\left|f_\varepsilon(\phi(\cdot - x))\right|^p\right]^{1/p} \leq C, \tag{5.26}
\]

and for all \(i = 1, \ldots, 2^d - 1\) and \(n \in \mathbb{N}\),

\[
\sup_{x \in \mathbb{R}^d, \varepsilon > 0} E\left[\left|\psi^{(i)}(\frac{\cdot - x}{2^{-n}})\right|^p\right]^{1/p} \leq C 2^{-nd} 2^{-n\beta}. \tag{5.27}
\]

Then the family \((f_\varepsilon)_\varepsilon\) is tight in \(C^\alpha_{\text{loc}}(\mathbb{R}^d)\) for every \(\alpha < \beta - \frac{d}{p}\).

Concluding the proof of Proposition 2.5: We will check the requisite conditions \((\ref{5.26})\) and \((\ref{5.27})\) for \(f_\varepsilon(y) = \mathcal{H}(t,y) = e^{1-\frac{y}{2}[h_\varepsilon(t,y) - h(\xi_{(\varepsilon^t,y)})]}\) and \(\beta = 0\) and \(p \in (1, 2)\). First remark that in this context \(f_\varepsilon(y)\) is stationary in the \(y\)-variable. Next to check the above two requirements, it suffices to show that for any smooth function with compact support (say, in a ball of radius 1),

\[
\sup_{x \in \mathbb{R}^d, \varepsilon > 0} \left(\mathbb{E}\left[\left|\int_{\mathbb{R}^d} f_\varepsilon(y)\psi\left(\frac{y - x}{2^{-n}}\right)\,dy\right|^p\right]^{1/p}\right) \leq C 2^{-nd}. \tag{5.28}
\]

Suppose that \(f_\varepsilon\) is stationary for all \(\varepsilon > 0\). Let \(q > 1\) verify \(p^{-1} + q^{-1} = 1\). By Hölder’s inequality, the LHS in the above display is bounded above by

\[
\sup_{x \in \mathbb{R}^d, \varepsilon > 0} \mathbb{E}\left[\int_{B(x,2^{-n})} |f_\varepsilon(y)|^p\,dy\right]^{1/p} \left(\int_{\mathbb{R}^d} \psi\left(\frac{y - x}{2^{-n}}\right)^q\,dy\right)^{1/q} \leq \sup_{x \in \mathbb{R}^d} |B(x,2^{-n})|^{1/p} \sup_{\varepsilon > 0} \mathbb{E}\left[|f_\varepsilon(0)|^p\right]^{1/p} 2^{-nd/q} \|\psi\|_q \leq \|\psi\|_q \sup_{\varepsilon > 0} \mathbb{E}\left[|f_\varepsilon(0)|^p\right]^{1/p} 2^{-nd},
\]

where we have used stationarity of \(f_\varepsilon\) in second line. Hence, we observe that tightness of \((f_\varepsilon)_\varepsilon\) would follow from \(L^p(\mathbb{P})\)-boundedness of \((f_\varepsilon)_\varepsilon\) for \(p \in (1, 2)\). To this end, we appeal to the \(L^p(\mathbb{P})\)-boundedness of \(T^{d-2/4}\log \mathcal{Z}_T - \log \mathcal{Z}_\infty\) for all \(p \in (1, 2)\) (recall Proposition 3.5), which, together with the identity \((\ref{1.3})\) in turn implies that \(f_\varepsilon\) enjoys the same property, implying tightness of \((\mathcal{H}_\varepsilon(t,x))_{x \in \mathbb{R}^d, \varepsilon > 0}\) in the space \(C^\alpha_{\text{loc}}(\mathbb{R}^d)\) for all \(\alpha < -d/2\). Thus, Proposition 2.5 is proved.

Acknowledgements: The authors were partly supported by the French Agence Nationale de la Recherche under grant ANR-17-CE40-0032.
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