Blind Deconvolution with Additional Autocorrelations via Convex Programs

Philipp Walk*, Peter Jung†, Götz E. Pfander‡ and Babak Hassibi*
*Department of Electrical Engineering, Caltech, Pasadena, CA 91125
Email: {pwalk,hassibi}@caltech.edu
†Communications & Information Theory, Technical University Berlin, 10587 Berlin
Email: peter.jung@tu-berlin.de
‡Philipps-University Marburg, Mathematics & Computer Science
Email: pfander@mathematik.uni-marburg.de

Abstract

In this work we characterize all ambiguities of the linear (aperiodic) one-dimensional convolution on two fixed finite-dimensional complex vector spaces. It will be shown that the convolution ambiguities can be mapped one-to-one to factorization ambiguities in the $z$-domain, which are generated by swapping the zeros of the input signals. We use this polynomial description to show a deterministic version of a recently introduced masked Fourier phase retrieval design. In the noise-free case a (convex) semidefinite program can be used to recover exactly the input signals if they share no common factors (zeros). Then, we reformulate the problem as deterministic blind deconvolution with prior knowledge of the autocorrelations. Numerically simulations show that our approach is also robust against additive noise.

I. INTRODUCTION

Blind deconvolution problems occur in many signal processing applications, as in digital communication over wire or wireless channels. Here, the channel (system), usually assumed to be linear time invariant, has to be identified or estimated at the receiver. Once, the channel can be probed sufficiently often and the channel parameter stay constant over a longer period, pilot signals can be used for this purpose. However, in some cases one also has to estimate or equalize the channel blindly. Blind channel equalization and estimation methods were already developed in the 90ties, see for example in [1–3] for the case where the receiver has statistical channel knowledge, for example second order or higher moments. If no statistical knowledge of the data and the channel is available, for example, for fast fading channels, one can still ask under which conditions on the data and the channel a blind channel identification is possible. Necessary and sufficient conditions in a multi-channel setup where first derived in [4, 5] and continuously further developed, see e.g. [6] for a nice summary. All these techniques are of iterative nature which are therefor difficult to analyze. Most of the algorithms often suffer from instabilities in the presence of noise and overall the performance is inadequate for many applications. To overcome these difficulties, we will propose in this work a convex program for simultaneous reconstruction of the channel and data signal. We show that this program is always successful in the noiseless setting and we numerically demonstrate its stability under noise. The blind reconstruction can hereby be re-casted as a phase retrieval problem if we have additional knowledge of the autocorrelation of the data and the channel at the receiver, which was shown by Jaganathan and one of the authors in [7]. The uniqueness of the phase retrieval problem can then be shown by constructing an explicit dual certificate in the noise free case by translating the ideas of [7] to a purely deterministic setting. We show that the convex program derived in [7] holds indeed for every signal and channel of fixed dimensions as long as the corresponding $z$-transforms have no common zeros, which is known to be a necessary condition for blind deconvolution [4]. Before we propose the new blind deconvolution setup we will define and analyse all ambiguities of (linear) convolutions in finite dimensions.

II. AMBIGUITIES OF CONVOLUTION

The convolution defines a product and it is therefore obvious that this comes with factorization ambiguities. But, so far, the authors couldn’t find a mathematical rigorous and complete characterization and definition of all
convolution ambiguities in the literature. Even in the case of autocorrelations, as investigated in phase retrieval problems, the definition of ambiguities seems at least not consistent, see for example [8, 9] or even a recent work [10]. To obtain well-posed blind deconvolution problems of finite dimensional vectors, we have to precisely define all ambiguities of convolutions over the field $\mathbb{C}$ in the finite dimensions $L_1$ respectively $L_2$. Only if we exclude all non-trivial ambiguities we obtain identifiability of the inputs $(x_1, x_2) \in \mathbb{C}^{L_1} \times \mathbb{C}^{L_2}$ from their aperiodic or linear convolution product $y \in \mathbb{C}^{L_1+L_2-1}$, given component-wise for $k \in \{0, 1, \ldots, L_1+L_2-2\} =: \{L_1+L_2-1\}$ as

$$y_k = (x_1 \ast x_2)_k := \min\{L_1-1, k\} \cdot x_{1,1} x_{2, k-1}. \tag{1}$$

A first analytic characterization of such identifiable classes, also for general bilinear maps, in the time domain $\mathbb{C}^{L_1} \times \mathbb{C}^{L_2}$ was obtained in [11–13]. However, before we define the convolution ambiguities, we will define first the scaling ambiguity in $\mathbb{C}^{L_1} \times \mathbb{C}^{L_2}$ which is the intrinsic ambiguity of scalar multiplication $m: \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ mapping any pair $(a, b)$ to the product $m(a, b) := ab$. Obviously, this becomes the only ambiguity if any bilinear map, as the convolution, is defined for trivial dimensions $L_1 = L_2 = 1$. We have therefore the following definition.

**Definition 1** (Scaling Ambiguities). Let $L_1, L_2$ be positive integers. Then the scalar multiplication $m$ in $\mathbb{C}$ induces a scaling equivalence relation on $\mathbb{C}^{L_1} \times \mathbb{C}^{L_2}$ defined by

$$(x_1, x_2) \sim_m (\tilde{x}_1, \tilde{x}_2) \iff \exists \lambda \in \mathbb{C}: \tilde{x}_1 = \lambda x_1, \tilde{x}_2 = \lambda^{-1} x_2. \tag{2}$$

We call $[(x_1, x_2)]_m := \{(\tilde{x}_1, \tilde{x}_2) \mid (x_1, x_2) \sim_m (\tilde{x}_1, \tilde{x}_2)\}$ the scaling equivalence class of $(x_1, x_2)$.

**Remark.** The scaling ambiguity can be easily generalized over any field $\mathbb{F}$.

We identify $x \in \mathbb{C}^N$ with its one-sided or unilateral $z$–transform or transfer function, given by

$$X(z) = (Zx)(z) := \sum_{k=0}^{N-1} x_k z^{-k} = \sum_{k=F}^{D} x_k z^{-k}, \tag{3}$$

where $D$ denotes the largest (degree of $X$) and $F$ the smallest non-zero coefficient index of $x$. The transfer function in (3) is also called and FIR filter or all-zero filter, i.e., the only pole is attained at $z = 0$, and if the first coefficient is not vanishing all zeros are finite (lying in a circle of finite radius), see Figure 2 and Figure 3. Here, $X \in \mathbb{C}[z]$ defines a polynomial over $z^{-1}$ and therefore we will not distinguish in the sequel between polynomial and unilateral $z$–transform. The set of all finite degree polynomials $\mathbb{C}[z]$ defines with the polynomial multiplication · (algebraic convolution)

$$Y(z) = X_1(z) \cdot X_2(z) := \sum_{l=0}^{L_1-1} x_{1,l} z^{-l} \cdot \sum_{k=0}^{L_2-1} x_{2,k} z^{-k} = \sum_{k=0}^{\min\{L_1-1, k\}} \left( \sum_{l=0}^{L_2-2} x_{1,l} x_{2, k-l} \right) z^{-k} \tag{4}$$

a ring, called the polynomial ring. Since $\mathbb{C}$ is an algebraically closed field we have, up to a unit $u \in \mathbb{C}$, a unique factorization of $X \in \mathbb{C}[z]$ of degree $D$ in primes $P_k(z) := z^{-1} - \zeta_k^{-1}$ (irreducible polynomials of degree one), i.e.,

$$X(z) = x_F \prod_{k=1}^{D} (z^{-1} - \zeta_k^{-1}), \tag{5}$$

is determined by the $D$ zeros $\zeta_k$ of $X$ and the unit $x_F$. Hence, for finite-length sequences (vectors), the linear convolution (2) can be represented with the $z$–transform $Z$ one-to-one in the $z$–domain as the polynomial multiplication (4), see for example the classical text books [14] or [15]. This allows us to define the set of all convolution

\[
\begin{align*}
\text{Root-Domain} & \quad \mathbb{C}_{L_1} \times \mathbb{C}_{L_2} \\
\text{z-Domain} & \quad \mathbb{C}_{L_1}[z] \times \mathbb{C}_{L_2}[z] \\
\text{Time-Domain} & \quad \mathbb{C}_{L_1} \times \mathbb{C}_{L_2}
\end{align*}
\]

Figure 1. Zero/root representation of the convolution.
ambiguities precisely in terms of their factorization ambiguities in the \( z \)-domain, see Figure \( \square \) where we denoted by \( \mathbb{C}_L[z] \) polynomials of degree \(< L \). Note, the convolution ambiguities are described in the root-domain therefor by a partitioning map \( \Pi \) of the roots (zeros). This brings us to the following definition.

**Definition 2** (Convolution Ambiguities). Let \( L_1, L_2 \) be positive integers. Then the linear convolution \(*: \mathbb{C}^{L_1} \times \mathbb{C}^{L_2} \to \mathbb{C}^{L_1+L_2-1} \) defines on the domain \( \mathbb{C}^{L_1} \times \mathbb{C}^{L_2} \) a equivalence relation \( \sim_* \) given by

\[
(x_1, x_2) \sim_* (\tilde{x}_1, \tilde{x}_2) :\Leftrightarrow \tilde{x}_1 * \tilde{x}_2 = x_1 * x_2.
\]

For each \((x_1, x_2)\) we denote by \( X_1(z) \) and \( X_2(z) \) its \( z \)-transforms of degree \( D_1 \) respectively \( D_2 \). Moreover we denote by \( z_{F_1} \) respectively \( z_{F_2} \) the first non-zero coefficients of \( x_1 \) respectively \( x_2 \) and by \( \{\zeta_k\}_{k=1}^{D_1+D_2} \subset \mathbb{C} \cup \{\infty\} \) the zeros of the product \( X_1 X_2 \). Then the pair

\[
(x_1, x_2) = (z^{-1}(\tilde{X}_1), z^{-1}(\tilde{X}_2)),
\]

with

\[
\tilde{X}_1 = z_{F_1} z_{F_2} \prod_{k \in P} (z^{-1} - \zeta_k^{-1}) \quad \text{and} \quad \tilde{X}_2 = \prod_{k \in [D] \setminus P} (z^{-1} - \zeta_k^{-1}),
\]

where \( P \) is some subset of \([D]\) such that \( D - L_2 + 1 \leq |P| \leq L_1 - 1 \), is called a left-scaled non-trivial convolution ambiguity of \((x_1, x_2)\). The set of all convolution ambiguities of \((x_1, x_2)\) is then the equivalence class defined by the finite union of the scaling equivalence classes of all left-scaled non-trivial convolution ambiguities given by

\[
[(x_1, x_2)]_* := \bigcup_n \left[ (x_1^{(n)}, x_2^{(n)}) \right]_m.
\]

We will call \((\tilde{x}_1, \tilde{x}_2) \in [(x_1, x_2)]_* \) a scaling convolution ambiguity or trivial convolution ambiguity of \((x_1, x_2)\) if \((\tilde{x}_1, \tilde{x}_2) \in [(x_1, x_2)]_m \) and in all other cases a non-trivial convolution ambiguity of \((x_1, x_2)\).

Remark. The naming trivial and non-trivial is borrowed from the polynomial language, where a trivial polynomial is a polynomial of degree zero, represented by a scalar (unit), and a non-trivial polynomial is given by a polynomial of degree greater than zero. Hence, the factorization ambiguity of a trivial polynomial corresponds to the scaling or trivial convolution ambiguity and the factorization ambiguity of a non-trivial polynomial corresponds to the non-trivial convolution ambiguity. We want to emphasize at this point, that the \( z \)-domain (polynomial) picture is known and used for almost a century in the engineering, control and signal processing community. Hence this factorization of convolutions is certainly not surprising, but by the best knowledge of the authors, not rigorous defined in the literature. For a factorization of the auto-correlation in the \( z \)-domain see for example [16], [9, Sec.3.], and the summarizing text book about phase retrieval [8]. A complete one-dimensional ambiguity analysis for the auto-correlation problem was recently obtained by [10]. A very similar, but not a full characterization of the ambiguities of the phase retrieval problem was given by [17]. Both works extend the results in [18]. Let us mentioned at last, that the non-trivial ambiguities for multi-dimensional convolutions are almost not existence, by the observation that multivariate polynomials, chosen randomly, are irreducible with probability one, i.e., a factorization ambiguity is then not possible, see for example [14]. This is in contrast to a random chosen univariate polynomial, which has full degree and no multiplicities of the zeros (factors), and obtains therefore the maximal amount of non-trivial ambiguities, see upper bound in \( \square \).

a) On the Combinatorics of Ambiguities: The determination of the amount \( M \) of left-scaled non-trivial convolution ambiguities of some \((x_1, x_2) \in \mathbb{C}^{L_1 \times L_2} \) is a hard combinatorial problem. The reason are the multiplicities of the zeros of \( X_1 X_2 \). If a zero \( \zeta_k \) has multiplicity \( m_k \geq 1 \), then we have \( m_k + 1 \) possible assignments of the \( m_k \) equal \( \zeta_k \) to \( X_1 \), i.e., we can choose one of the factors \( \{1, 1 - \zeta_k, (1 - \zeta_k)^2, \ldots, (1 - \zeta_k)^{m_k}\} \) as long as \( m_k \leq L_1 - 1 \). Hence, if all zeros are equal, we only have \( \min\{D + 1, L_1, L_2\} \) different choices to assign zeros for \( X_1 \). Contrary, if all zeros are distinct, then we end up with \( 2^D \) different zero assignments for \( X_1 \), which yields to

\[
\min\{D + 1, L_1, L_2\} \leq M \leq 2^D.
\]

In Figure \( \square \) we plotted for arbitrary polynomials \( X_1 \) and \( X_2 \) their zeros in the \( z \)-domain, where we assumed one common zero. Since the polynomials have finite degree, the only pole is located at the origin. Every permutation of the zeros yields then to an ambiguity.
b) Ambiguities of Autocorrelations: A very well investigated special case of blind deconvolution is the reconstruction of the signal of $x \in \mathbb{C}^N$ by its linear or aperiodic autocorrelation, see for example [10], given as the convolution of $x$ with its conjugate-time-reversal $x^{-}$ defined component-wise by $(x^{-})_k = x_{N-1-k}$ for $k \in [N]$. To transfer this in the $z$-domain we need to define on the polynomial ring $\mathbb{C}[z]$ an involition $(\cdot)^*$, given for any polynomial $X \in \mathbb{C}[z]$ of degree $N - 1$ by

$$X^*(z) := z^{1-N} \overline{X(1/z)}.$$  \hspace{1cm} (10)

Then, the autocorrelation $a = x \ast x^{-}$ in the time-domain transfers to $A = XX^*$ in the $z$-domain. All non-trivial correlation ambiguities are then given by assigning for the conjugated-zero-pairs $(\zeta_k, \overline{\zeta_k})$ of $A$ one zero to $\tilde{X}$. Since we do not have more than $N - 1$ different zeros for $\tilde{X}$, we have not more than $2^{N-1}$ different factorization ambiguities, see Figure 3. The scaling ambiguities reduce by $\lambda \tilde{X} = 1$ to a global phase scaling $e^{i\phi}$ for any $\phi \in \mathbb{R}$.

c) Well-posed Blind Deconvolution Problems: To guarantee a unique solution of a deconvolution problem up to a global scalar [13, Def.1], we have to resolve all non-trivial convolution ambiguities, which demands therefore a unique disjoint structure on the zeros of $X_1$ and $X_2$. The most prominent structure for a unique factorization is given by the spectral factorization (phase retrieval) for minimum phase signals, i.e., for signals $X$ having all its zeros inside the unit circle (a zero on the unit circle has even multiplicity and a swapping of its conjugated pair has therefore no effect). Another structure for a blind deconvolution would be to demand that $X_1$ has all its zeros inside the unit circle and $X_2$ has all its zeros strictly outside the unit circle. In fact, every separation would be valid, as long as it is practical realizable for an application setup. In this spirit, the condition that $X_1$ and $X_2$ do no have a common zero is equivalent with the statement that a unique separation is possible. This is the weakest and hence a necessary structure we have to demand on the zeros, which was already exploited in [4]. However, the challenge is still to find an efficient and stable reconstruction algorithm, which have to come with a price of further structure and constrains. But, instead of designing further constraints on the zeros, one can also demand further measurements of $x_1$ and $x_2$. In the next section we will introduce an efficient recovery algorithm given by a convex program with the knowledge of additional autocorrelation measurements.

III. Blind Deconvolution with Additional Autocorrelations via SDP

Since the autocorrelation of a signal $x \in \mathbb{C}^N$ does not contain enough information to obtain a unique recovery, as shown in the previous section, the idea is to use cross-correlation informations of the signal by partitioning $x$ in two disjoint signals $x_1 \in \mathbb{C}^{L_1}$ and $x_2 \in \mathbb{C}^{L_2}$, which yield $x$ if stacked together. This approach was first investigated in [19] and called vectorial phase retrieval. The same approach was obtained independently by one of the authors in [7, Thm. III.1], which steamed from a generalization of a phase retrieval design in [20, Thm.4.1.4.], from three masked Fourier magnitude-measurements in $N$ dimension, to a purely correlation measurement design between arbitrary vectors $x_1 \in \mathbb{C}^{L_1}$ and $x_2 \in \mathbb{C}^{L_2}$. To solve the phase retrieval problem via a semi-definite program (SDP),
the autocorrelation or equivalent the Fourier magnitude-measurements has to be represented as linear mappings on positive-semidefinite rank–1 matrices. This is know as the lifting approach or in mathematical terms as the tensor calculus. The above partitioning of \( \mathbf{x} \) yields to a block structure of the positive-semidefinite matrix

\[
\mathbf{xx}^* = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix} \begin{pmatrix} \mathbf{x}_1^\top & \mathbf{x}_2^\top \end{pmatrix} = \begin{pmatrix} \mathbf{x}_1 \mathbf{x}_1^* & \mathbf{x}_1 \mathbf{x}_2^* \\ \mathbf{x}_2 \mathbf{x}_1^* & \mathbf{x}_2 \mathbf{x}_2^* \end{pmatrix}.
\]

(11)

The linear measurement \( \mathcal{A} \) are then given component-wise by the inner products with the sensing matrices \( \mathbf{A}_{i,j,k} \), defined below, which correspond to the \( k \)th correlation components of \( \mathbf{x}_i \ast \mathbf{x}_j^* \) for \( i, j \in \{1, 2\} \). Hence, the autocorrelations and cross-correlations can be obtain from the same object \( \mathbf{x} \mathbf{x}^* \). Let us define the \( N \times N \) down-shift and \( N \times L \) embedding matrix as

\[
\mathbf{T}_N = \begin{pmatrix} 0 & \cdots & 0 \\
1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1 \end{pmatrix}, \quad \Pi_{N,L} = \begin{pmatrix} \mathbb{1}_{L,L} \\
0_{N-L,L} \end{pmatrix},
\]

(12)

where \( \mathbb{1}_{L,L} \) denotes the \( L \times L \) identity matrix and \( 0_{N-L,L} \), the \( (N-L) \times L \) zero matrix. Then, the \( L_i \times L_j \) rectangular shift matrices\(^1\) are defined as

\[
\mathbf{T}_{L_j,L_i}^{(k)} := \mathbf{T}_{N,L}^T \mathbf{T}_{L_j,L_i}^{k-L_j+1} \mathbf{T}_{N,L_i},
\]

(13)

for \( k \in \{0, \ldots, L_i + L_j - 2\} = [L_i + L_j - 1] \), where we set \( \mathbf{T}_{N}^l := \mathbf{T}_{N}^{(k)} \) if \( l < 0 \). Then, the correlation between \( \mathbf{x}_i \in \mathbb{C}^{L_i} \) and \( \mathbf{x}_j \in \mathbb{C}^{L_j} \) is given component-wise\(^2\) as

\[
(a_{i,j})_k := (\mathbf{x}_i^* \mathbf{x}_j^*)_k = \left( \mathbf{x}_i, (\mathbf{T}_{L_j,L_i}^{(k)})^T \mathbf{x}_j \right) = \left( \mathbf{x}_j^T \mathbf{T}_{L_i,L_j}^{(k)} \mathbf{x}_i \right) = \mathbf{x}_j^* \mathbf{T}_{L_i,L_j}^{(k)} \mathbf{x}_i = \text{tr}(\mathbf{T}_{L_i,L_j}^{(k)} \mathbf{x}_i \mathbf{x}_i^*).
\]

Hence, this defines the linear maps \( \mathcal{A}_{i,j,k} (\mathbf{X}) := \text{tr}(\mathbf{A}_{i,j,k} \mathbf{X}) \) for \( k \in [L_i + L_j - 1] \) with sensing matrices

\[
\begin{align*}
\mathbf{A}_{1,1,k} &= \begin{pmatrix} \mathbf{T}_{L_1,L_1}^{(k)} & 0_{L_1,L_2} \\
0_{L_2,L_1} & 0_{L_2,L_2} \end{pmatrix}, & k & \in [2L_1 - 1] \\
\mathbf{A}_{2,2,k} &= \begin{pmatrix} 0_{L_1,L_1} & \mathbf{T}_{L_2,L_2}^{(k)} \\
0_{L_2,L_1} & 0_{L_2,L_2} \end{pmatrix}, & k & \in [2L_2 - 1] \\
\mathbf{A}_{1,2,k} &= \begin{pmatrix} 0_{L_1,L_1} & \mathbf{T}_{L_2,L_1}^{(k)} \\
0_{L_2,L_1} & 0_{L_2,L_2} \end{pmatrix}, & k & \in [L_1 + L_2 - 1] \\
\mathbf{A}_{2,1,k} &= \begin{pmatrix} 0_{L_1,L_1} & \mathbf{T}_{L_1,L_2}^{(k)} \\
0_{L_2,L_1} & 0_{L_2,L_2} \end{pmatrix}, & k & \in [L_1 + L_2 - 1].
\end{align*}
\]

(14)\(\hspace{1cm}\)

(15)\(\hspace{1cm}\)

(16)\(\hspace{1cm}\)

(17)\(\hspace{1cm}\)

Stacking all the \( \mathcal{A}_{i,j,k} \) together gives the measurement map \( \mathcal{A} \). Hence, the \( 4N - 4 \) linear measurements are

\[
\mathbf{b} := \mathcal{A}(\mathbf{x} \mathbf{x}^*) = \begin{pmatrix} \mathbf{A}_{1,1}(\mathbf{x} \mathbf{x}^*) \\
\mathbf{A}_{2,2}(\mathbf{x} \mathbf{x}^*) \\
\mathbf{A}_{1,2}(\mathbf{x} \mathbf{x}^*) \\
\mathbf{A}_{2,1}(\mathbf{x} \mathbf{x}^*) \end{pmatrix} = \begin{pmatrix} a_{1,1} \\
a_{2,2} \\
a_{1,2} \\
a_{2,1} \end{pmatrix}.
\]

(18)

Note, since the cross-correlation \( a_{1,2} \) is the conjugate-time-reversal of \( a_{2,1} \), i.e., \( a_{1,2} = \overline{a_{2,1}} \), we only need \( 3N - 3 \) correlation measurements to determine \( \mathbf{b} \).

\(^1\)Note, it holds not \( \mathbf{T}_{L_j,L_i}^{(k)} = \mathbf{T}_{L_i,L_j}^{(k)} \) unless \( L_i = L_j \), cause the involution in the vector-time domain is \( \mathbf{x} \ast \mathbf{x}^\top \).

\(^2\)We use here the vector definition and hence the time-reversal \( \mathbf{x}^\top \) of the signal \( \mathbf{x} \) is a flipping of the vector coefficient indices in \( \mathbb{C}^{[L_i]} \) and not a flipping at the origin 0 as defined for sequences. The scalar product is given as \( \langle \mathbf{a}, \mathbf{b} \rangle := \sum_k a_k b_k \).
A. Unique Factorization of Self-Reciprocal Polynomials

To prove our main result in Theorem\(^1\), we need a unique factorization of self-reciprocal polynomials in irreducible self-reciprocal polynomials, where we call a polynomial \( X \) self-inversive if \( X^* = e^{i\phi} X \) for some \( \phi \in [0, 2\pi) \) and self-reciprocal\(^1\) if \( \phi = 0 \), see for example [21] and reference therein. The term self-reciprocal refers to the conjugate-symmetry of the coefficients, given by

\[
x = x^* \in \mathbb{C}^N,
\]

(19)

which can be used as the definition of a self-reciprocal polynomial by its coefficients. In fact, it was shown by some of the authors in [22] and [23], that the autocorrelation of conjugate-symmetric vectors is stable up to a global sign. As for the unique factorization \(^5\) of any polynomial \( X \in \mathbb{C}[z] \) of degree \( D \geq 1 \) in \( D \) irreducible polynomials (primes) \( P_k(z) = 1 - \zeta_k z^{-1} \), up to a unit \( u \in \mathbb{C} \setminus \{0\} \), we can ask for a unique factorization of any self-reciprocal polynomial \( S \in \mathbb{C}[z] \) in irreducible self-reciprocal polynomials \( S_k \), i.e., \( S_k \) can not be further factored in self-reciprocal polynomials of smaller degree. To see this, we first use the definition of a self-reciprocal factor \( S \) of degree \( D \), which demands that each zero \( \zeta \) comes with its conjugate-inverse pair \( 1/\zeta = : \zeta^* \). If \( \zeta \) lies on the unit circle, then we have \( \zeta = \zeta^* \) and the multiplicities of these zeros can be even or odd. Let us assume we have \( T \) zeros on the unit circle, then we get the factorization

\[
S(z) = u \prod_{k=1}^{D-T} (1 - \zeta_k z^{-1})(1 - \bar{\zeta}_k z^{-1}) \prod_{k=D-T+1}^{D} (1 - \zeta_k z^{-1}),
\]

where the phase \( \phi \) of the unit \( u \in \mathbb{C} \) is determined by the phases \( \phi_k \) of the conjugate-inverse zeros. To see this we derive

\[
S^*(z) = z^{-D} u \prod_{k=1}^{D-T} (1 - \bar{\zeta}_k z)(1 - \zeta_k z) \prod_{k=D-T+1}^{D} (1 - \bar{\zeta}_k z)
\]

\[
= u \prod_{k} \frac{\zeta_k}{\bar{\zeta}_k} (1 - \frac{1}{\zeta_k} z^{-1})(1 - \zeta_k z^{-1}) \prod_{k} (\bar{\zeta}_k)(1 - \frac{1}{\zeta_k} z^{-1}).
\]

If we set for the zeros \( \zeta_k = p_k e^{i\phi_k} \) and unit \( u = \rho e^{i\phi} \) we get

\[
= e^{-i(2\phi + \sum_{k=1}^{D} \phi_k - T\pi)} S(z) \quad \overset{!}{=} S(z).
\]

Hence, it must hold for the phase \( \phi = (T\pi - \sum_{k=1}^{D} \phi_k)/2 \). Moreover, for every prime \( P_k \) of \( S \) also \( P_k^* \) is a prime of \( S \). Hence, if \( P_k \not\perp P_k^* \) then \( S_k := P_k P_k^* \) is a self-reciprocal factor of \( S \) of degree two. If \( P_k = P_k^* \), then \( S_k := P_k \) is already a self-reciprocal factor of \( S \) of degree one. However, the conjugate-inverse factor pairs \((1 - \zeta_k z^{-1})(1 - \bar{\zeta}_k z^{-1})\) are not self-reciprocal, but self-inversive. We have to scale them with \( e^{-i\phi_k} \) to obtain a self-reciprocal factor \( S_k := P_k P_k^* \), i.e., we have to set \( P_k(z) := \rho_k^{-1/2}(1 - \zeta_k z^{-1}) \). Similar, for the primes on the unit circle, we set \( S_k(z) := e^{-i(\pi + \phi_k)/2}(1 - e^{i\phi_k} z^{-1}) \). Hence, we can write \( S \) as a factorization of irreducible self-reciprocal polynomials \( S_k \), i.e., self-reciprocal polynomials which are not further factored in self-reciprocal polynomials of smaller degree,

\[
S = \prod_{k=1}^{D-T} P_k P_k^* \prod_{k=D-T+1}^{D} S_k.
\]

Let us define the greatest self-reciprocal factor/divisor (GSD).

**Definition 3** (Greatest Self-Reciprocal Divisor). Let \( X \in \mathbb{C}[z] \) be a non-zero polynomial. Then the greatest self-reciprocal divisor \( S \) of \( X \) is the self-reciprocal factor with largest degree. It is unique up to a real-valued trivial factor \( c \in \mathbb{R} \).

\(^1\)In the literature there also called conjugate-self-reciprocal to distinguish them from the real case \( \mathbb{F} = \mathbb{R} \). For \( \mathbb{F} = \mathbb{R} \) or \( \mathbb{F} = \mathbb{Z} \) they are also called palindromic polynomials or simply palindromes (Coding Theory).
Let us denote by $C \mid X$ that $C$ is a factor/divisor of the polynomial $X$ and by $C \not\mid X$ that $C$ is not. Then $C \mid X$ and $C \mid Y$ is equivalent to the assertion that $C$ is a common factor of $X$ and $Y$. For any polynomial $X \in \mathbb{C}[z]$, which factors in $X = SR$, it holds

$$S \text{ self-reciprocal } \Rightarrow S \mid X \text{ and } S \mid X^*, \tag{21}$$

since it holds by the self-reciprocal property of $S$

$$X^* = S^*R^* = SR^*, \tag{22}$$

which proofs that $S \mid X$ and $S \mid X^*$. For the reverse we can only show this for the greatest common divisor (GCD).

Lemma 1. For $X \in \mathbb{C}[z]$ it holds

$$G \text{ is GSD of } X \iff G \text{ is GCD of } X \text{ and } X^*. \tag{23}$$

Proof: The $\Rightarrow$ follows from (21) since a GSD is trivially also a self-reciprocal factor of $X$ and therfor a factor of $X^*$. To see the other direction, we denote by $G$ the GCD of $X$ and $X^*$, which factorize as

$$X = GR \quad \text{and} \quad X^* = GQ, \tag{24}$$

where $R$ and $Q$ are the co-factors of $X$ respectively $X^*$. Then we get

$$X^* = G^*R^* = GQ. \tag{25}$$

Let us assume $G$ is not self-reciprocal, i.e., $G \not\mid G^*$, then we can still factorize $G$, as any polynomial, in the greatest self-reciprocal factor $S$ and a non-self-reciprocal factor $N$. Note, it might also hold the trivial case $0 \mid S = c \in \mathbb{R}$. Moreover, if the multiplicity of at least one zero in $S$, not lying on the unit circle, is larger than one, then $N$ might contain this zero (if the corresponding conjugate-inverse zero is missing in $G$). It is clear, that $N$ can not contain more than $(D - T)/2$ such isolated factors, lets call the product of all them $I_1$ and $I_2$ resp. $N_2$ the co-factors, i.e., $S = I_1I_2$ and $N = I_1N_2$. Hence, $I_1$ is the GCD of $S$ and $N$. Then (24) becomes

$$X = SNR \quad \text{and} \quad X^* = SN^*R^*, \tag{26}$$

which yields to

$$G \mid X^* \iff SN \mid SN^*R^* \iff N \mid N^*R^* \Rightarrow N \mid I_1I_2^*R^*. \tag{27}$$

Then $I_1^* \not\mid N$, since, if any factor $I_1^* \subset I_1^*$ would be a factor of $N$, then also $I_1^* \mid N^*$ and hence $I_1 \mid N$ and therefore $I_1I_1^* \mid N$, which would be a non-trivial self-reciprocal factor and contradicts the definition of $N$. By the same reason $N_2^* \not\mid N$ since any non-trivial factor of $N_2^*$ would result in a non-trivial self-reciprocal factor of $N$ which is again a contradiction. Hence $N \mid R^*$, i.e., we have $R^* = NT$ which yields to

$$R = (R^*)^* = (NT)^* = N^*T^*. \tag{28}$$

On the other hand it holds also

$$GQ \mid X^* \mid SN^*R^* = SN^*NT = GN^*T \quad \Rightarrow \quad Q = N^*T.$$

Hence $N^* \mid R$ and $N^* \mid Q$ and by (24) also $N^* \mid X$ and $N^* \mid X^*$, which is a contradiction, since $G$ is the GCD of $X$ and $X^*$. Hence the assumption is wrong and it must hold $G = G^*$. To see that $G$ is also the GSD, assume $G$ would be self-reciprocal and contain $G$ as factor, then $G$ would be by (21) a common factor which is greater then $G$ and hence contradicts again with $G$ to be the GCD. \hfill \blacksquare

Let us define the anti-self-reciprocal polynomial $A$ by the property $A = -A^*$, where $i$ is the trivial anti-self-reciprocal factor. Hence, for any self-reciprocal factor $S$ we get by $A = iS$ an anti-self-reciprocal factor. Hence, if we factorize $X$ in the GSD $G$ and the co-factor $R$, we obtain with the identity $-i \cdot i = 1$ the factorization

$$X = iGR, \tag{29}$$

Actually, also $i c$ for any $c \in \mathbb{R}$ would be a trivial anti-self-reciprocal factor. But since we are interested in the factorization of a anti-self-reciprocal polynomial in a self-reciprocal $S$ and a trivial anti-self-reciprocal $i$, we can assign the $c$ either to $i$ or to $S$. \hfill 4
where \( \tilde{R} = -iR \) does not contain non-trivial self-reciprocal or anti-self-reciprocal factors. With this we can show the following result.

**Lemma 2.** Let \( X \in \mathbb{C}[z] \) be a polynomial of order \( L \geq 0 \) with no infinite zeros and let \( G, R \) be polynomials with \( X = GR \), where \( G \) is the GSD of degree \( D \leq L \) and \( R \) its co-factor. Then the only non-zero polynomial \( H \) of order \( \leq L \), which yields to a anti-self-reciprocal product \( XH^* \), i.e., fulfills
\[
XH^* + X^*H = 0
\] (30)
is given by \( H = iRS \) for any self-reciprocal polynomial \( S \) of degree \( \leq D \).

**Proof:** Since \( X \) factors in the GSD \( G \) and the co-factor \( R \) we have by Lemma 1 that \( G \) is the GCD of \( X \) and \( X^* \), which gives \( X^* = GR^* \). Inserting this factorization in (30) yields to
\[
G(RH^* + R^*H) = 0 \iff RH^* + R^*H = 0.
\] (31)
Since \( R \) and \( R^* \) do not have a common factor by definition of \( G \), but have degree \( L - D \), which is less or equal then \( H \) and \( H^* \) (degree \( \leq L \)), the only solution is of the form
\[
H = iRS,
\] (32)
where \( S \) is any self-reciprocal polynomial of degree \( \leq D \).

**Remark.** This result (32) can be seen as a special case of the Sylvester Criterion for the polynomials \( A = X_1 \) and \( B = X_2^* \) in (30), where \( G \) is the GCD of \( A \) and \( B \). Hence \( H \) and \( H^* \) have GCD \( S \) of degree \( D \), which must be by Lemma 1 the GSD with the co-factors \( iR \) respectively \( -iR^* \), see Appendix B.

**B. Main Result**

Let us denote by \( \mathcal{C}_0^{L} := \{ x \in \mathbb{C}^L \mid x_0 \neq 0 \neq x_{L-1} \} \). Then in [7, Thm.III.1] and extended by the author (purely deterministic) it holds the following theorem.

**Theorem 1.** Let \( L_1, L_2 \) be positive integers and \( x_1 \in \mathbb{C}_0^{L_1}, x_2 \in \mathbb{C}_0^{L_2} \) such that their \( z \)-transforms \( X_1(z) \) and \( X_2(z) \) do not have any common factors. Then \( x_1^T = (x_1^T, x_2^T) \in \mathbb{C}^N \) with \( N = L_1 + L_2 \) can be recovered uniquely up to global phase from the measurement \( b \in \mathbb{C}^{4N-4} \) defined in (18) by solving the feasible convex program
\[
\text{find } X \in \mathbb{C}^{N \times N} \quad \text{s.t. } A(X) = b
\] (33)
which has \( X^\# = xx^* \) as the unique solution.

**Remark.** The condition that the first and last coefficient does not vanish, guarantee that \( X_i \) and \( X_i^* \) have no zeros at the origin and are both of degree \( L_i \). Since the correlation is conjugate-symmetric, we only need to measure one cross correlation, since we have \( x_1^* \overline{x_2^*} = (\overline{x_2^*} \overline{x_1^*})^\# \). Hence we can omit the last \( N - 1 \) measurements in \( b \) and achieve recovery from only \( 3N - 3 \) measurements. In fact, if we set \( \tilde{x}_2 = \overline{x_2^*} \) and demand \( X_1 \) and \( \tilde{X}_2 = X_2 \) to be co-prime, then the theorem gives recovery up to global phase of \( x_1 \) and \( \tilde{x}_2 \) from its convolution
\[
x_1^* \tilde{x}_2
\] (34)
by knowing additionally the auto-correlations \( a_{1,1} \) and \( \tilde{a}_{2,2} \), since it holds by conjugate-symmetry of the autocorrelations, that \( a_{2,2} = x_2^* \overline{x_2^*} = x_2 \overline{x_2} = (\overline{x_2^*} \overline{x_2^*})^\# = \overline{x_2^*} \overline{x_2} = \tilde{a}_{2,2} \).

**Proof of Theorem 7**. In [7] the authors could show that the feasible convex program is solvable by constructing a unique dual certificate which implies to show the uniqueness condition only on the tangent space \( T_x \) of \( xx^* \), for a detailed proof see Appendix A.

**Lemma 3.** The feasible convex problem given in (33) has the unique solution \( X^\# = xx^* \) if the following conditions are satisfied
1) There exists a dual certificate \( W \in \mathcal{A}^* \) such that
(i) \( \text{Wx} = 0 \)

(ii) \( \text{rank(} \text{W} \text{)} = N - 1 \)

(iii) \( \text{W} \geq 0 \)

2) For all \( \text{H} \in T_x := \{ \text{xh}^* + \text{hx}^* \mid \text{h} \in \mathbb{C}^N \} \) it holds

\[
\text{A(} \text{H} \text{)} = 0 \Rightarrow \text{H} = 0.
\]

Indeed, the conditions in [1] are satisfied for the dual certificate \( \text{W} := \text{S}^* \text{S} \), where \( \text{S} \) is the \( N \times N \) Sylvester matrix of the polynomials \( -z^{L_1} \text{X}_1 \) and \( z^{L_2} \text{X}_2 \) given in (78). To see the first two conditions \([\text{i} \) and (ii)] we use the Sylvester Theorem [2] in Appendix [B] which states, that the only non-zero vector in the one-dimensional nullspace of the Sylvester matrix \( \text{S} \) is given by \( \text{x}^T = (x_1^T, x_2^T) \) (up to scalar), i.e., by (81) we have

\[
\text{Sx} = \begin{pmatrix} x_2^* \times x_1 - x_1^* \times x_2 \\ 0 \end{pmatrix} = 0_N,
\]

where the difference of the cross-convolutions vanishes due to the commutation property of the convolution. Since the dimension of the nullspace is 1 we have \( \text{rank(} \text{S} \text{)} = N - 1 \), which shows \([\text{ii}] \) with \( \text{rank(} \text{S}^* \text{S} \text{)} = \text{rank(} \text{S} \text{)} \).

The positive-semi-definiteness in \([\text{iii}] \) is given by definition of \( \text{W} = \text{S}^* \text{S} \), since for any matrix \( \text{A} \in \mathbb{C}^{N \times N} \) it holds \( \text{A}^* \text{A} \geq 0 \). To see that \( \text{W} = \text{S}^* \text{S} \) is in the range of \( \text{A}^* \), we have to set \( \lambda \) accordingly, since \( \text{S}^* \text{S} \) corresponds to the correlations of the \( x_1 \) and \( x_2 \) by (83) to (85), it turns out that the \( \lambda \) can be decompose in terms of our four measurements, see Appendix [B-A].

Hence, it remains to show the uniqueness condition [2] in Lemma [3] which is the new result in this work. For that we have to show for any \( \text{H} \in T_x \) given by \( \text{H} = \text{xh}^* + \text{hx}^* \) for some \( \text{h} \in \mathbb{C}^N \) that it follows \( \text{H} = 0 \) from

\[
\text{A(} \text{H} \text{)} = \text{A(} \text{hx}^* \text{)} + \text{A(} \text{xh}^* \text{)} = 0.
\]

Here \( \text{H} \) produce a sum of different correlations which have to vanish. As we split \( \text{x} \) in \( \text{x}_1 \) and \( \text{x}_2 \) we can also split \( \text{h} \) in \( \text{h}_1 \) and \( \text{h}_2 \). Then we can use the block structure in \( \text{xh}^* \) and \( \text{hx}^* \) to split condition (37) in

\[
\text{A(} \text{H} \text{)} = \begin{pmatrix} \text{x}_1^* \times \text{h}_1 \\ \text{x}_2^* \times \text{h}_2 \\ \text{x}_1 \times \text{h}_2 \\ \text{x}_2 \times \text{h}_1 \\ \text{x}_1 \times \text{h}_1 \\ \text{x}_2 \times \text{h}_2 \end{pmatrix} = 0.
\]

Let us translate the four equations in (38) to the \( z \)-domain:

\[
\text{X}_1 \text{H}_1 + \text{X}_1^* \text{H}_1 = 0 \quad \text{(39)}
\]

\[
\text{X}_2 \text{H}_2 + \text{X}_2^* \text{H}_2 = 0 \quad \text{(40)}
\]

\[
\text{X}_1^* \text{H}_2 + \text{X}_2 \text{H}_1 = 0 \quad \text{(41)}
\]

where we omitted the last one, which is redundant to (41). Let us assume, \( \text{X}_1 = \text{G}_1 \text{R}_1 \) and \( \text{X}_2 = \text{G}_2 \text{R}_2 \) where \( \text{G}_1 \) and \( \text{G}_2 \) are the GSDs of \( \text{X}_1 \) respectively \( \text{X}_2 \) and \( \text{R}_1 \) and \( \text{R}_2 \) their co-factors, then we can find by Lemma [2] self-reciprocal factors \( \text{S}_1 \) and \( \text{S}_2 \) such that \( \text{H}_1 = i \text{R}_1 \text{S}_1 \) and \( \text{H}_2 = i \text{R}_2 \text{S}_2 \) which are the only solutions for (39) and (40). But, then it follows for the second equation (41).

\[
0 = \text{X}_1 \text{H}_2^* + \text{X}_2^* \text{H}_1 = -i\text{G}_1 \text{R}_1 \text{R}_2^* \text{S}_2 + i\text{G}_2 \text{R}_2 \text{R}_1^* \text{S}_1 \quad \Leftrightarrow \quad \text{G}_1 \text{R}_1 \text{R}_2^* \text{S}_2 = \text{G}_2 \text{R}_2 \text{R}_1^* \text{S}_1 \quad \Leftrightarrow \quad \text{G}_1 \text{S}_2 = \text{G}_2 \text{S}_1
\]

Here, only the self-reciprocal polynomials \( \text{S}_1 \) and \( \text{S}_2 \) of degree \( \leq D_1 \) respectively \( \leq D_2 \) can be chosen freely. Since \( \text{X}_1 \) and \( \text{X}_2 \) do not have a common factor also \( \text{G}_1 \) and \( \text{G}_2 \) do not have a common factor and the \( D \times D \) Sylvester matrix \( \text{S}_{z^{-1} \text{G}_1,-z^{-1} \text{G}_2} \) has rank \( D_1 + D_2 - 1 = D - 1 \) and again as in (83) the only solutions for (42) are given by \( \text{S}_1 = c \text{G}_1 \) respectively \( \text{S}_2 = c \text{G}_2 \) for some \( c \in \mathbb{R} \) (note, \( \text{S}_1 \) and \( \text{S}_2 \) must be self-reciprocal, hence only real units).

This would imply \( \text{H}_1 = ic \text{R}_1 \text{G}_1 = ic \text{X}_1 \) and \( \text{H}_2 = ic \text{R}_2 \text{G}_2 = ic \text{X}_2 \), which gives \( \text{h} = ic \text{x} \) and result in

\[
\text{H} = \text{xh}^* + \text{hx}^* = -ic \text{x}^* \times ic \text{x}_1 = 0.
\]

Remark. To guarantee a unique solution of (33) we need both auto-correlations, since only then we obtain both constraints (39) and (40), which yielding to the constraints \( \text{H}_1 = i \text{R}_1 \text{S}_1 \) and \( \text{H}_2 = i \text{R}_2 \text{S}_2 \). If one of them is missing, we can constuct by (42) non-zero \( \text{H} \)'s satisfying (37), and hence violating the uniqueness in [2].
Figure 4. MSE/dim in dB for deconvolution of $x_1 \in \mathbb{C}^{L_1}$ and $x_2 \in \mathbb{C}^{L_2}$ for various dimensions with additive Gaussian noise on the convolution and autocorrelations.

IV. SIMULATION AND ROBUSTNESS

If we obtain only noisy correlation measurements, i.e., disturbed by the noise vectors $n_{1,1}, n_{2,2}, n_{1,2}$ as

$$b = y + n = \begin{pmatrix} a_{1,1} \\ a_{2,2} \\ a_{1,2} \\ a_{2,1} \end{pmatrix} + \begin{pmatrix} n_{1,1} \\ n_{2,2} \\ n_{1,2} \\ n_{1,2} \end{pmatrix},$$

we can search for the least-square solutions in (33), given as

$$X^\# := \arg\min_{X \geq 0} \| b - A(X) \|_2^2.$$  \hspace{1cm} (45)

Extracting form $X^\#$ via SVD the best rank–1 approximation $x^\#$ gives the normalized MSE of the reconstruction

$$\text{MSE} := \min_{\phi \in [0, 2\pi)} \left\| x - e^{i\phi} x^\# \right\|^2 / \|x\|^2.$$ \hspace{1cm} (46)

We plotted the normalized MSE in Figure 4 over the received SNR (rSNR), given by

$$\text{rSNR} := \frac{\mathbb{E}[\|y\|^2_2]}{\mathbb{E}[\|n\|^2_2]}.$$ \hspace{1cm} (47)

Since, the noise is i.i.d. Gaussian we get for $\text{rSNR} = \mathbb{E}[\|y\|^2_2]/(N\sigma^2)$ where $\sigma$ is the noise variance.

Surprisingly, the least-square solution $X^\#$ seems also to be the smallest rank solution, i.e., numerically a regularization with the trace norm of $X$, to promote a low-rank solution, does not yield to better results or even lower rank solutions. Although, the authors can not give an analytic stability result of the above algorithm, the reconstruction from noisy observations gives reasonable stability, as can be seen in Figure 4. Here, we draw $x_1$ and $x_2$ from an i.i.d. Gaussian distribution with unit variance. If the magnitude of the first or last coefficients is less than 0.1 we dropped them from the simulation trial, this ensures full degree polynomials, as demanded in the Theorem 1. As dimension grows, computation complexity increase dramatically and stability decreases significant. Nevertheless, the MSE per dimension scales nearly linear with the noise power in dB. Noticeable is the observation, that unequal dimension partitioning of $N$ yields to a better performance.
V. CONCLUSION

We characterized the ambiguities of convolution by exploiting their polynomial factorizations. As an application we could derandomize a $4N - 4$ auto and cross-correlation setup in [7] by only assuming a co-prime structure in $x$ and full degrees of the polynomials. Moreover, we can provide a convex recovery algorithm which numerically also performs robust against additive noise.

Acknowledgments. We would like to thank Kishore Jaganathan, Fariborz Salehi and Michael Sandbichler for helpful discussions. A special thank goes to Richard Künig for discussing the dual certificate construction in dual problems in more detail. This work was partially supported by the DFG grant JU 2795/3 and WA 3390/1. We also like to thank the Hausdorff Institute of Mathematics for providing for some of the authors resources at the Trimester program in spring 2016 on “Mathematics of Signal Processing” where part of the work have been prepared.

APPENDIX A
PROOF OF LEMMA 3

Usually, in the math literature, SDP problems are formulated for symmetric objects on symmetric cones over the real field $\mathbb{R}$. This is due to the fact that minimizing or maximizing an objective function is only possible for real-valued functions. Nevertheless, there is an extension to the complex case, which is sometimes called complex SDP problems. Let $\mathcal{A} : \mathbb{C}^{N \times N} \to \mathbb{C}^M$ be a linear map given by sensing matrices $\{A_m\}_{m=0}^{M-1} \subset \mathbb{C}^{N \times N}$ (not necessarily Hermitian or symmetric). Moreover, we define the linear objective function $\text{tr}(CX)$ by a Hermitian matrix $C \in H_N := \{A \in \mathbb{C}^{N \times N} \mid A = A^*\}$. Then the primal complex optimization problem is given by

$$\min_{X \in \mathbb{C}^{N \times N}, X \neq 0} \text{tr}(CX) \quad \text{such that} \quad \mathcal{A}(X) = b,$$  \tag{48}

But $\mathcal{A}$ is not convex, since $b$ is not real-valued, if $A_m$ is not Hermitian. To obtain convex conditions, we can just split imaginary and real part of $b$ by setting

$$\mathcal{A}_{R,m}(X) = \frac{A_m(X) + A_m^*(X)}{2} = \frac{\text{tr}((A_m + A_m^*)X)}{2} = \text{tr}(A_{R,m}X) = \frac{b_m + b_m^{-1}}{2} = \text{Re}(b_m),$$

$$\mathcal{A}_{I,m}(X) = \frac{A_m(X) - A_m^*(X)}{2i} = \frac{\text{tr}((A_m - A_m^*)X)}{2i} = \text{tr}(A_{I,m}X) = \frac{b_m - b_m^{-1}}{2} = \text{Im}(b_m),$$ \tag{50}

for all $m \in [M]$. Hence, we yield $2M$ real-valued convex measurements $\tilde{\mathcal{A}}$ with the Hermitian sensing matrices $A_{I,m}$ and $A_{R,m}$. This gives finally the equivalent primal complex convex optimization problem (primal complex SDP problem)

$$\min_{X \neq 0} \text{tr}(CX) \quad \text{such that} \quad \tilde{\mathcal{A}}(X) = \begin{pmatrix} \mathcal{A}_{R}(X) \\ \mathcal{A}_{I}(X) \end{pmatrix} = \begin{pmatrix} \text{Re}(b) \\ \text{Im}(b) \end{pmatrix},$$ \tag{51}

This complex SDP can be rewritten as a standard SDP over real-valued positive-semidefinite matrices in $S_N$, see for example [24, Sec.4]. We therefor can assume the duality properties of the real SDP problems for the complex SDP as well. The dual convex optimization problem is then given by

$$\max_{c,d \in \mathbb{R}^M} \begin{pmatrix} c \\ d \end{pmatrix} \cdot \begin{pmatrix} \text{Re}(b) \\ \text{Im}(b) \end{pmatrix} \quad \text{s.t.} \quad \sum_{m=0}^{M-1} c_mA_{R,m} + d_mA_{I,m} + S = C, S \succeq 0.$$ \tag{52}

If $C = 0$ the primal optimization problem (48) becomes a primal feasible problem since any $X$ would yield the same objective value zero, which is equivalent to no objective function and hence to a primal complex feasible SDP problem:

$$\text{find } X \geq 0 \quad \text{such that} \quad \tilde{\mathcal{A}}(X) = \begin{pmatrix} \text{Re}(b) \\ \text{Im}(b) \end{pmatrix},$$ \tag{53}

which is equivalent to

$$\text{find } X \geq 0 \quad \text{such that} \quad \mathcal{A}(X) = b.$$ \tag{54}
Then the dual complex feasible problem is given by, [7, Sec.VI (12)],
\[
\max_{\lambda \in \mathbb{C}^M} - \sum_m (\lambda_m b_m + \overline{\lambda_m} b_m) \quad \text{such that} \quad \sum_m (\lambda_m A_m + \overline{\lambda_m} A_m^*) \geq 0,
\]
(55)
which can be obtained by setting \(-2\lambda_m = c_m - id_m\) and \(C = 0\) in (52), since it holds
\[
-(\lambda_m b_m + \overline{\lambda_m} b_m) = \frac{c_m - id_m}{2} (\text{Re}(b_m) + i \text{Im}(b_m)) + \frac{c_m + id_m}{2} (\text{Re}(b_m) - i \text{Im}(b_m))
\]
(56)
\[
= c_m \text{Re}(b_m) + d_m \text{Im}(b_m)
\]
(57)
and
\[
\lambda_m A_m + \overline{\lambda_m} A_m^* = (c_m A_m - id_m A_m + c_m A_m^* + id_m A_m^*)/2 = c_m A_{R,m} + d_m A_{I,m}.
\]
(58)
The set of matrices
\[
\text{range}(\mathcal{A}^*) = \left\{ W = \sum_{m=0}^{M-1} (\lambda_m A_m + \overline{\lambda_m} A_m^*) \mid \lambda \in \mathbb{C}^M \right\},
\]
(59)
is the range space of \(\mathcal{A}^*\), which is indeed the set of Hermitian matrices spanned by the Hermitian sensing matrices \(A_{R,m}\) and \(A_{I,m}\). Note, the real dimension is less or equal to \(2M\).

We will now proof the central lemma for the uniqueness of the complex SDP program used in Theorem 1.

Proof of Lemma 3: Note, that we have the equivalence
\[
\mathcal{A}(X) = \left( \begin{array}{c} \text{Re}(b) \\ \text{Im}(b) \end{array} \right) \iff X = b
\]
(60)
and therefor the range is equal, i.e., \(\text{range}(\mathcal{A}^*) = \text{range}(\mathcal{A}^{**})\). One can insert the problem (54) directly into Matlab toolbox, since it will be interpreted as the convex problem (53) with real-valued constraints. We will use the version (54) since it is more natural for the proof. Let us assume \(X^\# \geq 0\) is a feasible solution of the primal problem (54), i.e.,
(a) \(\forall m \in [M]: \text{tr}(A_m X^\#) = b_m\)

If we can show that \(X^\# = xx^*\) is the only feasible solution then we have shown the unique solution. Let us further assume \(X^\# \geq 0\) is a solution of the dual complex feasible problem (55), i.e.,
(b) \(W = \sum_m \lambda_m A_m + \overline{\lambda_m} A_m^* \geq 0\)

then by the KKT conditions, strong duality see for example [25, Thm.5.1], the solutions are the same if the duality gap is zero\(^5\) i.e.

3. \(\text{tr}(WX^\#) = 0\) (Complementary slackness)

By definition, \(xx^*\) is a primal feasible solution. If we can construct a dual certificate \(W\), which satisfy (b), and which fulfills \(3\), then \(xx^*\) is an optimal solution. Since (55) is a feasible problem every feasible solution is an optimal solution \((C = 0)\). But then for every primal feasible solution \(X^\#\) there must exists a dual certificate \(W\) satisfying the slackness property (3). We will use this condition to relax the uniqueness condition. To ensure uniqueness of the primal feasible solution \(xx^*\) we have to show that no other primal feasible (optimal) solution \(X^\# \geq 0\) exist. This is equivalent to show that for any \(X^\# = xx^* + H \geq 0\) given by any \(H \in H_N\) it holds
\[
\mathcal{A}(X^\#) = b \Rightarrow X^\# = xx^*,
\]
(61)
which is by linearity of \(\mathcal{A}\) equivalent to
\[
\forall H \in H_N \text{ with } xx^* + H \geq 0 \text{ it holds: } \mathcal{A}(H) = 0 \Rightarrow H = 0.
\]
(62)
To relax this to a more tractable condition we use an orthogonal decomposition of the set of Hermitian matrices \(H_N\), in an orthogonal sum, given by the tangent space at \(xx^*\) to the manifold of Hermitian rank–1 matrices, defined as
\[
T_x := \{ xh^* + hx^* \mid h \in \mathbb{C}^N \}
\]
(63)
\(^5\)This works with every \(C \in H_N\) defining the objective function \(\text{tr}(CX)\) (Note, that the dual certificate \(W\) has to be also include \(C\). For the feasible problem we have \(C = 0\).
and its orthogonal complement $T_x^\perp$, i.e. $T_x \oplus T_x^\perp = H_N$ (note, $H_N$ is a real vector space). Let $X^# \succeq 0$ be a feasible primal solution, then we can write

$$X^# = xx^* + H = xx^* + H_{T_x} + H_{T_x^\perp} \succeq 0$$

(64)

for some $H \in H_N$. Then it holds

$$H_{T_x^\perp} \perp T_x \iff \forall h \in \mathbb{C}^N: H_{T_x^\perp} \perp \langle xh^* + hx^* \rangle$$

(65)

$$\iff \forall h \in \mathbb{C}^N: \langle xh^* \rangle + \langle H_{T_x^\perp}hx^* \rangle = 0.$$  

(66)

Since $H_{T_x^\perp}$ is Hermitian this is equivalent to

$$\iff \forall h \in \mathbb{C}^N: 2 \text{Re}(\langle xh^* \rangle) = 0.$$  

(67)

But this holds for all $h \in \mathbb{C}^N$ and hence also for $h = H_{T_x^\perp}x$ which implies

$$H_{T_x^\perp}x = 0,$$  

(68)

since $\text{tr}(hh^*) = \text{Re}(\text{tr}(hh^*)) \succeq 0$. It holds

$$H_{T_x^\perp} \succeq 0 \iff \forall z \in \mathbb{C}^N: z^*H_{T_x^\perp}z \geq 0.$$  

(69)

We can decompose $z \in \mathbb{C}^N$ for any $x \in \mathbb{C}^N$ in an orthogonal sum $\text{span}(x) \oplus \text{span}(x)^\perp$ such that there exists $\lambda \in \mathbb{C}$ and $z_1 \in \text{span}(x)^\perp$ with $z = x + z_1$. Hence,

$$H_{T_x^\perp} \succeq 0 \iff \forall \lambda \in \mathbb{C}, z_1 \in \text{span}(x)^\perp: (\lambda x + z_1)^*H_{T_x^\perp}(\lambda x + z_1) \geq 0,$$  

(70)

which is by (68) equivalent to

$$H_{T_x^\perp} \succeq 0 \iff \forall z_1 \in \text{span}(x)^\perp: z_1^*H_{T_x^\perp}z_1 \geq 0.$$  

(71)

But since we know that $X^\# \succeq 0$ we get for all $z_1 \in \text{span}(x)^\perp$ with (64)

$$0 \leq z_1^*X^#z_1 = z_1^*xx^*z_1 + z_1^*(xh^* + hx^*)z_1 + z_1^*H_{T_x^\perp}z_1 = z_1^*H_{T_x^\perp}z_1,$$  

(72)

which proofs the positive-semi-definiteness of $H_{T_x^\perp}$ by (71). Since $X^\#$ is a feasible primal solution there must exists a dual certificate $W^# \succeq 0$ with $\text{tr}(W^#X^#) = 0$. If we can show that the dual certificate $W$ for $xx^*$ is the dual certificate for every feasible primal solution $X^#$, then the only feasible solution is $xx^*$ and we are done. To do so, take a feasible $X^# = xx^* + H$. Then

$$A(H) = 0$$  

(73)

which is equivalent to

$$\forall m \in [M]: \text{tr}(A_m H) = 0$$  

(74)

and also $\text{tr}(A_m^*H) = 0$ since $H = H^*$. Then we can take an arbitrary $\lambda \in \mathbb{C}^M$ which defines $W$ and get

$$0 = \sum_m \lambda_m \text{tr}(A_m H) + \overline{\lambda_m} \text{tr}(A_m^* H) = \text{tr}((\sum_m \lambda_m A_m + \overline{\lambda_m}A_m^*)H) = \text{tr}(WH) = \text{tr}(WH_{T_x}) + \text{tr}(WH_{T_x^\perp}).$$  

(75)

By condition (1) in the Lemma we have $Wx = 0$ and hence it follows

$$\text{tr}(WH_{T_x}) = 0.$$  

(76)

But since $Wx = 0$ by condition (1) and $H_{T_x}x = 0$ by (68) both matrices share a one-dimensional subspace of their nullspaces. But $W \succeq 0$ with (iii) and $H_{T_x} \succeq 0$ by (70) it follows $\text{range}(W) \subseteq \text{kern}(H_{T_x})$, which implies $\text{kern}(H_{T_x}) = \mathbb{C}^N$ since $\text{rank}(W) = N - 1$ by (ii) and therefor $H_{T_x} = 0$. This gives the three conditions in 1) of the Lemma. Hence the uniqueness condition (61) relaxes to condition 2) as

$$\forall H_{T_x} \in T_x: A(H_{T_x}) = 0 \Rightarrow H_{T_x} = 0.$$  

(77)

Then $X^# = xx^*$ is the unique solution of (54) and hence (48).
APPENDIX B
SYLVESTER MATRIX

The $N \times N$ Sylvester matrix of two vectors $a \in \mathbb{C}_{L_1+1}^L$, $b \in \mathbb{C}_{L_2+1}^L$ with $N = L_1 + L_2$ play the crucial role in our analysis and are defined for $L_1 \leq L_2$ as

$$S_{a,b} := \begin{pmatrix}
L_1 & L_2 \\
\begin{pmatrix} b_0 & 0 & \ldots & 0 \\
b_1 & b_0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
b_{L_2} & b_{L_2-1} & \ldots & b_{L_2-(L_1-1)} \\
0 & b_{L_2} & \ldots & b_{L_2-(L_1-2)} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & b_{L_2} \\
\end{pmatrix} & \begin{pmatrix} a_0 & 0 & \ldots & 0 & 0 \\
a_1 & a_0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{L_1} & a_{L_1-1} & \ldots & a_0 & 0 \\
0 & a_{L_1} & \ldots & a_1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & a_{L_1} \\
\end{pmatrix}
\end{pmatrix}$$

(78)

where the first $L_1$ columns are down-shifts of the vector $b$ and the last $L_2$ columns are down-shifts of the vector $a$, see for example [7, Sec.VII] or [26, Def.7.2],[27, (1.84)] (here they define the transpose version and for polynomials $a(z) := \sum_k a_{L_1-k} \lambda^k = a(z)$ and $b(z) := \sum_k b_{L_2-k} \lambda^k = b(z)$ with degree $L_1$ respectively $L_2$, which has no effect on the resultant (determinant) or rank). The resultant of the polynomials $a$ and $b$ is the determinant of the Sylvester matrix $S_{a,b}$. SYLVESTER showed that the two polynomials have a common factor (non-trivial, i.e., not a constant) if and only if $\det(S_{a,b}) \neq 0$, which is equivalent of having full rank, i.e., $\text{rank}(S_{a,b}) = N := L_1 + L_2$. This can be generalized to the degree of the greatest common factor (GCD), see [27, Thm.1.8].

**Theorem 2.** Let $a, b \in \mathbb{C}[z]$ with degree $L_1$ and $L_2$ generating the Sylvester matrix $S_{a,b}$, then the greatest common factor of $a, b$ has degree

$$D = L_1 + L_2 - \text{rank}(S_{a,b}).$$

(79)

Multiplying the polynomials $X_1$ and $X_2$ in Theorem 1 by $z^{-1}$ is equivalent to adding a zero to the coefficient vectors $x_1$ and $x_2$, hence we set

$$a = -x_1^0 := \begin{pmatrix} -x_1 \\
0
\end{pmatrix}, \quad b = x_2^0 := \begin{pmatrix} x_2 \\
0
\end{pmatrix}.$$  

(80)

Then the nullspace of $S := S_{a,b} = S_{-x_1^0,x_2^0}$, which dimension is given by Theorem 2 as $D$, determines the set of convolution equivalences, since we have (see also Appendix B-A)

$$S \begin{pmatrix} \tilde{x}_1 \\
\tilde{x}_2
\end{pmatrix} = \begin{pmatrix} x_2 * \tilde{x}_1 - x_1 * \tilde{x}_2 \\
0
\end{pmatrix} = 0_N$$

(81)

with vectors $\tilde{x}_1 \in \mathbb{C}^{L_1}$, $\tilde{x}_2 \in \mathbb{C}^{L_2}$. Hence, if the polynomials $X_1$ and $X_2$ do not have a common factor ($a$ and $b$ have common factor $z$ of degree $D = 1$), then by Theorem 2 the rank of $S$ is $N - 1$, i.e., there exists only one pair $(\tilde{x}_1, \tilde{x}_2) \in \mathbb{C}^{L_1} \times \mathbb{C}^{L_2}$ up to a global scaling, for which their convolutions are equal, i.e.,

$$x_1 * \tilde{x}_2 = x_2 * \tilde{x}_1 \iff \tilde{x}_1 = \lambda x_1, \tilde{x}_2 = \lambda x_2 \text{ for some } \lambda \in \mathbb{C}.$$  

(82)

Usually this result is written in the polynomial or $z$-domain as

$$x_1 * \tilde{x}_2 = x_2 * \tilde{x}_1 \iff X_1 \tilde{X}_2 = X_2 \tilde{X}_1.$$  

(83)

where $\tilde{X}_1$ and $\tilde{X}_2$ are polynomials of degree $\leq L_1$ respectively $\leq L_2$. Hence, if $X_1$ and $X_2$ are co-prime the only possible polynomials are $\tilde{X}_1 = \lambda X_1$ and $\tilde{X}_2 = \lambda X_2$, up to a unit $\lambda$ (trivial polynomial), which becomes the scalar factor for $x$. Hence the nullspace of $S$ is one-dimensional and therefor $\text{rank}(S) = N - 1$. 


A. Dual Certificate Construction

To show that $S^*S$ is a dual certificate, we have to define $\lambda \in \mathbb{C}^{4N-4}$ such that by (85) we get

$$S^*S = \sum_{m=0}^{4N-5} \lambda_m A_m + \lambda_m A^*_m = \sum_{i,j=1}^{L_1+L_2-2} \lambda_{i,j,k} A_{i,j,k} + \lambda_{i,j,k} A^T_{i,j,k},$$

(84)

where we split again $\lambda^T = (\lambda^T_{1,1}, \lambda^T_{2,2}, \lambda^T_{1,2}, \lambda^T_{2,1})$ in four blocks corresponding to the $A_{i,j,k}$ in (14)-(17). To derive the $\lambda_{i,j}$ we need to write $S^*S$ in block structure. Let us define the lower banded Toeplitz matrix generated by $x_i$ as

$$C_{X_i} = \sum_{m=0}^{L_i-1} x_i, m T_{N-1}^m,$$

(85)

where $T_{N-1}^m$ is the $m$th $(N-1) \times (N-1)$ shift-matrix (elementary Toeplitz matrices) defined in (12). To apply this on $x_j \in \mathbb{C}^{L_j}$ we have to embed $x_j$ in $N-1$ dimensions with the $(N-1) \times L_j$ embedding matrix as defined in (12) by

$$C_{X_i} \cdot x_j = x_i * x_j.$$

(86)

Here, the upper index $j$ refers to the embedding dimension $L_j$. We then obtain the matrix notation for the linear convolution ($\Pi$) between $x_i \in \mathbb{C}^{L_i}$ and $x_j \in \mathbb{C}^{L_j}$ as

$$C_{X_i} x_j = x_i * x_j.$$

(87)

Hence, the Sylvester matrix $S = S_\cdot' x_i'^0 x_i^0$ is the concatenation of the two lower banded matrices $C_{x_0}' C_{x_0}^*$ and $C_{x_i}' C_{x_i}^*$ and we get for any $\tilde{x}_1 \in \mathbb{C}^{L_1}$ and $\tilde{x}_2 \in \mathbb{C}^{L_2}$ the convolution products embedded in $N = L_1 + L_2$ dimensions as

$$S \cdot' \cdot x_i'^0 x_i^0 \cdot (\tilde{x}_1 \tilde{x}_2) = (C_{X_0}^* C_{X_0} C_{X_1}^* C_{X_1}) \cdot (\tilde{x}_1 \tilde{x}_2) = \begin{pmatrix} \tilde{x}_1 & \tilde{x}_2 \\ 0 & 0 \end{pmatrix}.$$

(88)

If we consider the product of the adjoint Sylvester matrix with itself we get

$$S^* \cdot' \cdot S_\cdot'^0 x_i'^0 x_i^0 = \begin{pmatrix} C_{X_0}^* C_{X_0} C_{X_1}^* C_{X_1} & C_{X_0} C_{X_1}^* C_{X_0} C_{X_1}^* \\ C_{X_0} C_{X_1}^* C_{X_0} C_{X_1}^* & C_{X_0}^* C_{X_0} C_{X_1}^* C_{X_1}^* \end{pmatrix}.$$

(89)

Since we have for $i,j \in \{1,2\}$

$$C_{X_i}^* C_{X_i} = \sum_{m=0}^{L_i-1} \frac{L_i-1}{m} \cdot x_i, m T_{N-1}^m.$$

(90)

we get for each of the four blocks in (89) denoted by $i,i',j,j'$ as

$$C_{X_i}^* C_{X_{i'}} = \sum_{m=0}^{L_i-1} \frac{L_i-1}{m} \cdot x_i, m T_{N-1}^m \cdot x_{i'}, m T_{N-1}^m.$$

(91)

Let us emphasize that $l,m$ are limited by $\pm L_1$ resp. $\pm L_2$, and since we consider the $L_1$ resp. $L_2$ embeddings, the zeros on the $l$-th diagonal in $T_{N-1}^m$ can be ignored. By substituting $k = l - m$ we get

$$C_{X_i}^* C_{X_{i'}} = \sum_{k=-L_1+1}^{L_1-1} \sum_{m=0}^{L_i-1} \frac{L_i-1}{m} \cdot x_i, m T_{N-1}^m \cdot x_{i'}, k+m T_{N-1}^m \Pi_{N,L_j} \sum_{l=0}^{L_i-1} \frac{T_{N-1}^m \Pi_{N,L_j} T_{N-1}^m \Pi_{N,L_j}}{T_{N-1}^m \Pi_{N,L_j} T_{N-1}^m \Pi_{N,L_j}}.$$

(92)

where the inner sum is the correlation between $x_{i'}$ and $x_i$ at index $k \in [L_1 + L_1' - 1]$. Hence we get for the autocorrelations $i = i' \in \{1,2\}$ the diagonal blocks in (89)

$$C_{x_2}^* C_{x_2} = \sum_{k=0}^{2L_2-2} (x_2 \cdot x_2) T_{N,L_1} T_{N,L_1} \cdot k \Pi_{N,L_j} T_{N,L_j} \cdot k \Pi_{N,L_j} \cdot k \Pi_{N,L_j} = \sum_{k=0}^{2L_2-2} (a_{2,2} k \cdot T_{N,L_1} T_{N,L_1} \cdot k \Pi_{N,L_j} T_{N,L_j} \cdot k \Pi_{N,L_j} \cdot k \Pi_{N,L_j}.$$

(93)

$$C_{x_1}^* C_{x_1} = \sum_{k=0}^{2L_1-2} (x_1 \cdot x_1) T_{N,L_2} T_{N,L_2} \cdot k \Pi_{N,L_1} T_{N,L_1} \cdot k \Pi_{N,L_1} \cdot k \Pi_{N,L_1} = \sum_{k=0}^{2L_1-2} (a_{1,1} k \cdot T_{N,L_2} T_{N,L_2} \cdot k \Pi_{N,L_1} T_{N,L_1} \cdot k \Pi_{N,L_1} \cdot k \Pi_{N,L_1}.$$

(94)
where $T_{L_i,L_j}$ was defined in \([13]\), but with the difference that $i \neq j$. Since the dimensions of the $L_j \times L_j$ block matrices in \([89]\) are not fitting with the autocorrelations $x_i^* \overline{x_i}$ on the diagonal we have to cut respectively zero-pad the $\lambda_{i,i}$ correspondingly. Let us assume w.l.o.g. that $L_1 \leq L_2$. Then we set

$$
\lambda_{1,1} := \frac{1}{2} \left( (a_{2,2})_k \right)_{k=L_2-L_1}^{L_2+L_1-2} \in \mathbb{C}^{2L_2-1},
\lambda_{2,2} := \frac{1}{2} \left( \begin{array}{c} 0_{L_2-L_1} \\ a_{1,1}^T \end{array} \right) \in \mathbb{C}^{2L_2-1},
$$

which gives by the conjugate-symmetry of the autocorrelations

$$
\sum_{i=1}^{2} \sum_{k=0}^{2L_2-2} \lambda_{i,i,k} A_{i,i,k} + \overline{\lambda_{i,i,k}} A_{i,i,k}^T = 2 \sum_{i=1}^{2} \sum_{k} \lambda_{i,i,k} A_{i,i,k},
$$

where the transpose of $A_{i,i,k}$ is equivalent to a time-reversal of $\lambda_{i,i,k}$, i.e. $\lambda_{i,i,k} = \lambda_{i,i,2L_1-2-k}$. For the anti-diagonal in \([89]\) we have

$$
-C_{x_1}^* C_{x_2}^0 = - \sum_{k=0}^{L_1+L_2-2} (x_2 * x_1)_k \Pi_{N,L_2}^T T_{N,L_2}^{k-L_1+1} \Pi_{N,L_1} = - \sum_{k} (a_{2,1})_k T_{k,L_2,L_1},
$$

$$
-C_{x_2}^* C_{x_1}^0 = - \sum_{k=0}^{L_1+L_2-2} (x_1 * x_2)_k \Pi_{N,L_1}^T T_{N,L_1}^{k-L_2+1} \Pi_{N,L_2} = - \sum_{k} (a_{1,2})_k T_{k,L_1,L_2},
$$

denoting the time-reversal of the cross-correlations. Hence we set similar

$$
\lambda_{1,2} := -\frac{1}{2} a_{2,1}^- \in \mathbb{C}^{N-1}
$$

$$
\lambda_{2,1} := -\frac{1}{2} a_{1,2}^- \in \mathbb{C}^{N-1}.
$$

Since the off-diagonal matrices satisfy $A_{2,1,k}^* = A_{2,1,k}^T = A_{1,2,L_1+L_2-2-k}$ for $k \in [L_1 + L_2 - 1]$ we have again the transpose is equivalent to a time reversal of $\lambda_{2,1}$ and since $\overline{\lambda_{2,1}} = \lambda_{1,2}$ we have

$$
\sum_{k} \lambda_{1,2,k} A_{1,2,k} + \overline{\lambda_{1,2,k}} A_{1,2,k}^T = -\frac{1}{2} \sum_{k} (a_{2,1}^-)_k A_{1,2,k} + (a_{2,1}^-)_k A_{1,2,k}^T
$$

$$
= -\frac{1}{2} \sum_{k} (a_{2,1}^-)_k A_{1,2,k} + (a_{1,2}^-)_k A_{2,1,k}^T (a_{1,2}^-)_k A_{2,1,k}^T,
$$

$$
\sum_{k} \lambda_{2,1,k} A_{2,1,k} + \overline{\lambda_{2,1,k}} A_{2,1,k}^T = -\frac{1}{2} \sum_{k} (a_{1,2}^-)_k A_{2,1,k} + (a_{1,2}^-)_k A_{2,1,k}^T
$$

$$
= -\frac{1}{2} \sum_{k} (a_{1,2}^-)_k A_{2,1,k} + (a_{2,1}^-)_k A_{1,2,k}^T.
$$

Hence, adding \([96]\), \([101]\) and \([102]\) yields $W = S^* S$. 

REFERENCES

[1] L. Tong, G. Xu, and T. Kailath, “A new approach to blind identification and equalization of multipath channels”, in *25th Asilomar Conf.*, 1991, pp. 856–860.

[2] Z. Ding, R. A. Kennedy, B. Anderson, and C. R. Johnson, “Ill-convergence of Godard blind equalizers in data communication systems”, *IEEE Trans. Commun.*, vol. 39, no. 9, pp. 1313 –1327, 1991.

[3] L. Tong, G. Xu, B. Hassibi, and T. Kailath, “Blind channel identification based on second-order statistics: a frequency-domain approach”, *IEEE Trans. Inf. Theory*, vol. 41, no. 1, pp. 329–334, 1995.

[4] G. Xu, H. Liu, L. Tong, and T. Kailath, “A least-squares approach to blind channel identification”, *IEEE Trans. Signal Process.*, vol. 43, no. 12, 2982–2993, 1995.

[5] M. Gürelli and C. Nikias, “EVAM: an eigenvector-based algorithm for multichannel blind deconvolution of input colored signals”, *IEEE Trans. Signal Process.*, vol. 43, no. 1, pp. 134 –149, 1995.

[6] K. Abed-Meraim, W. Qiu, and Y. Hua, “Blind system identification”, *Proc. IEEE*, vol. 85, no. 8, pp. 1310 –1322, 1997.

[7] K. Jaganathan and B. Hassibi, “Reconstruction of signals from their autocorrelation and cross-correlation vectors, with applications to phase retrieval and blind channel estimation”, 2016. eprint: arXiv:1610.02620.

[8] N. Hurt, *Phase Retrieval and Zero Crossings: Mathematical Methods in Image Reconstruction*. Kluwer Academic Publishers, 1989.

[9] Y. M. Bruck and L. Sodin, “On the ambiguity of the image reconstruction problem”, *Opt. Commun.*, vol. 30, no. 3, 304–308, 1979.

[10] R. Beinert and G. Plonka, “Ambiguities in one-dimensional discrete phase retrieval from Fourier magnitudes”, *J. Fourier. Anal. Appl.*, vol. 21, pp. 1169–1198, 2015.

[11] P. Walk and P. Jung, “Compressed sensing on the image of bilinear maps”, in *IEEE ISIT*, 2012, pp. 1291 –1295. eprint: arXiv:1205.4933.

[12] S. Choudhary and U. Mitra, “On identifiability in bilinear inverse problems”, in *IEEE ICASSP*, 2013, 4325–4329.

[13] —, “Fundamental limits of blind deconvolution part I: Ambiguity kernel”, *Arxiv*, 2014. eprint: 1411.3810.

[14] S. Lang, *Algebra*, S. Axler, G. F.W., and K. A. Ribet, Eds. Springer, 2002.

[15] A. V. Oppenheim, R. W. Schafer, and J. A. Buck, *Discrete-time signal processing*. Prentice Hall, 1999.

[16] M. H. Hayes, L.-H. Lim, and A. V. Oppenheim, “Signal reconstruction from phase or magnitude”, *IEEE Trans. Signal Process.*, vol. 28, p. 672, 1980.

[17] K. Jaganathan, S. Oymak, and B. Hassibi, “Sparse phase retrieval: uniqueness guarantees and recovery algorithms”, *Arxiv.org*, 2013. eprint: abs/1311.2745.

[18] Y. Lu and M. Vetterli, “Sparse spectral factorization: unicity and reconstruction algorithms”, in *IEEE ICASSP*, Prague, 2011.

[19] O. Raz, N. Dudovich, and B. Nadler, “Vectorial phase retrieval of 1-d signals”, *IEEE Trans. Signal Process.*, vol. 61, no. 7, pp. 1632–1643, 2013.

[20] K. Jaganathan, “Convex programming-based phase retrieval: theory and applications”, PhD thesis, California Institute of Technology, 2016.

[21] R. S. Vieira, “On the number of roots of self-inversive polynomials on the complex unit circle”, *Arxiv*, 2015.

[22] P. Walk and P. Jung, “Stable recovery from the magnitude of symmetrized fourier measurements”, in *IEEE ICASSP*, 2014, pp. 1813 –1816.

[23] P. Walk, P. Jung, and G. E. Pfander, “On the stability of sparse convolutions”, *Appl. Comput. Harmon. Anal.*, vol. 42, pp. 117–134, 2017. eprint: arXiv:1409.6874.

[24] J. S. Geronimo and H. J. Woerdeman, “Positive extensions, Fejer-Riesz factorization and autoregressive filters in two variables”, *Annals of Mathematics*, vol. 160, pp. 839–906, 2004.

[25] R. M. Freund, *Introduction to semidefinite programming (SDP)*. Massachusetts Institute of Technology, 2004.

[26] K. O. Geddes, S. R. Czapor, and G. Labahn, *Algorithms for Computer Algebra*. Springer, 1992.

[27] S. Barnett, *Polynomials and Linear Control Systems*. Marcel Dekker Inc, 1983.