Electromagnetic Flow of Leptons in Heavy Ion Collisions

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We investigate the azimuthal angular correlation between the lepton transverse momentum \( P_\perp \) and the impact parameter \( b_\perp \) in non-central heavy ion collisions, where the leptons are produced through two photon scattering. Among the Fourier harmonic coefficients, a significant \( v_2 \) asymmetry is found for the typical kinematics at RHIC and LHC with a mild dependence on the \( P_\perp \). whereas \( v_2 \) power suppressed by the lepton mass over \( P_\perp \). This unique prediction, if confirmed from the experiments, shall provide a crucial information on the production mechanism for the dilepton in two photon processes and help to identify the hadrons’ flow phenomena in heavy ion collisions as well.

I. INTRODUCTION

The flow phenomena of final state particles in heavy ion collisions is one of the most important observations that signals the collective modes of the quark-gluon plasma created in these collisions [1–6]. They are defined as the anisotropy of final state hadrons in the transverse plane with respect to the impact parameter of the collision [7], e.g., in terms of \( \cos(n\phi) \) where \( \phi \) is the azimuthal angle between the hadron’s momentum \( \vec{p}_h \) and the impact parameter \( \vec{b}_\perp \). In this paper, we study the electromagnetic flow of leptons in heavy ion collisions, where the lepton pair is produced in a pure electromagnetic process of \( \gamma \gamma \rightarrow \ell^+\ell^- \). Here, the flow is used in a broad sense, and it refers to the anisotropic angular distribution of final state particles with respect to the impact parameter in heavy ion collisions, which is not limited to that only caused by the final state interactions. Although the electromagnetic flow discussed in this work may strongly resembles the conventional hadronic flow in experimental measurements, its underlying physics mechanism is from the initial state interactions. The comparison of the anisotropy between the leptons and hadrons will provide a unique perspective for the flow phenomena in heavy ion collisions.

Di-lepton production through the QED processes in heavy ion collisions has a long history, mainly in the so-called Ultra-peripheral collisions (UPC) [8–17]. More recently, experiments at RHIC and LHC have pushed these measurements toward peripheral and central collisions. Significant deviations from the UPC case have been reported [18–22], where the total transverse momentum of the lepton pair \( q_\perp \) increases with the centrality. Especially, from the ATLAS measurement it reaches to a value around 100 MeV in the most central collisions at the LHC, whereas it is about 40 MeV for UPC case [23]. These developments have generated quite an interest in heavy ion community. If it is confirmed that the observed effects indeed come from the medium interactions with the lepton pair, this shall lead to a potential probe to the electromagnetic property of the hot medium [18, 19, 23]. Therefore, the key step is to quantify the initial state contributions from the two-photon processes. To do that, we have to go beyond the previous calculations which only apply to the dilepton production in UPC events [23–28].

On the other hand, this extension is not straightforward, since we have to compute the joint transverse momentum and impact parameter dependence for the incoming photon fluxes of the colliding nuclei. By definition, these two distributions are Fourier conjugate to each other [29, 30]. Different assumptions and models have been introduced [23–30]. Among these calculations, the so-called QED calculation seems to suggest that the observed \( P_\perp \)-broadening effects may solely come from the initial state effects due to different geometry of the collisions [24]. However, the predicted azimuthal \( \cos(2\phi) \) asymmetry between the total transverse momentum \( q_\perp \) and the impact parameter \( b_\perp \) remains to be confirmed in experiments [31]. This asymmetry depends on simultaneously determining the transverse momentum and impact parameter information and needs further studies.

The proposed flow measurement in this paper is different from those in Refs. [24, 26, 27], because it does not depend on the total transverse momentum of the lepton pair \( q_\perp \). Therefore, the photon fluxes only depend on the impact parameter \( b_\perp \) and can be rigorously computed through the classic electromagnetic treatments, like the Jackson method [32]. Similar to the case of the transverse momentum dependent photon flux [26], the impact parameter dependent photon flux predicts a significant linear polarization along the impact parameter direction. This will generate a \( \cos(4\phi) \) azimuthal asymmetries between the lepton’s transverse momentum and the impact parameter \( b_\perp \).

The flow of leptons can be measured through the azimuthal angular correlations between the lepton and hadrons, similar to what have been done for hadron flow. The measurements will provide important information on the production mechanism for the dilepton in peripheral collisions.
and central collisions. In addition, once the flow phenomena of these leptons has been established, in return, we can utilize the unique property of the lepton’s flow to identify the underlying mechanism for the hadron flow.

The rest of our paper is organized as follows. In Sec. II, we briefly discuss the impact parameter dependent photon flux and its polarization for a moving ion. In Sec. III, we derive the lepton flow in the two photon process due to the incoming photons’ polarizations. Numeric results will be shown for RHIC and LHC experiments in the relevant kinematics. We conclude our paper in Sec. IV.

II. POLARIZATION IN INCOMING PHOTON FLUX

When a heavy ion moves in an ultra-relativistic speed, e.g., along the \( \hat{z} \) direction, it coherently generates associated electromagnetic (EM) fields when the wave length of the EM field is comparable with nucleus radius. These EM fields can be described as an effective photon flux [33–35]. As illustrated in Fig. 1, not only the intensity but also the polarization of the photon flux depend on the impact parameter \( \vec{b}_\perp \), where \( b_\perp \) represents the transverse distance relative to the center of the moving nucleus.

From the perspective of the classical electrodynamics, as pointed out in Refs. [32, 36] and shown in Fig. 1, the electric field \( \vec{E} \) generated by the relativistically moving charged nucleus is linearly polarized along the impact parameter \( \vec{b}_\perp \) direction, and the corresponding magnetic field \( \vec{B} \) is perpendicular to the electric field in the transverse plane. Therefore, a physical gauge choice of the polarization vector \( \vec{e}_\perp = \vec{b}_\perp \) with \( b_\perp \) the unit vector of \( \vec{b}_\perp \) can be used to a quantum field theory calculation, especially when the polarization of the equivalent photon plays an important role.

The above physics is appropriately captured by introducing the photon distribution from the nucleus. Following the example of the generalized parton distribution (GPD) in nucleon [37], we introduce the generalized photon distribution. Similar to that of quark/gluon GPD [38], the photon GPD can be interpreted as the impact parameter dependent photon distribution. This is equivalent to the photon flux discussed in the literature. The photon GPD is defined through the following matrix,

\[
x f_{\gamma}^{\alpha\beta}(x; b_\perp) = \int d^2k_\perp d^2q_\perp \int d^2r_\perp \delta(x - q_\perp) \frac{\delta(x-r_\perp)}{2} F^\mu\nu(x, b_\perp).
\]

where \( F^\mu\nu \) represent the EM field strength. The photon GPD can be parameterized as

\[
x f_{\gamma}^{\alpha\beta}(x; b_\perp) = \frac{\delta\alpha\beta}{2} x f_\gamma(x; b_\perp) + \left( \frac{b_\perp^\alpha b_\perp^\beta}{b_\perp^2} - \frac{\delta\alpha\beta}{2} \right) x h_\gamma(x; b_\perp).
\]

Here, \( f_\gamma(x, b_\perp) \) is the normal polarization averaged impact parameter dependent photon distribution, and \( h_\gamma(x, b_\perp) \) is conventionally referred to as the helicity flip photon GPD, similar to the helicity flip gluon GPD [39–41]. When \( x \) is sufficiently small (\( x < \frac{1}{2m_p^2} \)), photon distribution is dominated by these coherently generated due to the \( Z^2 \) enhancement where \( Z \) is the nuclear charge number. By treating the external electromagnetic filed of a relativistic nucleus as a classical Coulomb potential, the associated coherent photon distributions can be readily computed in terms of the nuclear charge form factor, \( x h_\gamma(x, b_\perp) \) as

\[
x h_\gamma(x, b_\perp) = x f_\gamma(x; b_\perp) = 4Z^2 \alpha \int \frac{d^2q_\perp}{(2\pi)^2} \frac{\delta(q_\perp - b_\perp)}{q^2} F_A(q^2),
\]

where \( q^2 = q^2_\perp + \frac{1}{4} \cdot \frac{2}{2} \cdot F_A(q^2) \) and \( F_A \) represents the EM form factor for the nucleus, and \( m_p \) being proton mass. One finds that for a given \( b_\perp \), coherent photons are fully linearly polarized due to the fact that \( h_\gamma = f_\gamma \) in the above equation, see also the discussions in Ref. [10]. This relation essentially is the consequence of the property of highly boosted Coulomb field: the direction of electric field generated by a spherically symmetric charge source distribution is parallel to the impact parameter. The similar relation between unpolarized photon TMD and linearly polarized photon TMD was also established in Ref. [26]. In Ref. [42], the above photon flux \( f_\gamma(x, b_\perp) \) has been applied to understand the dilepton production in peripheral collisions, and it was found that the photon flux at small impact parameter of \( b_\perp < R_A \) plays a significant role in the non-UPC events in heavy ion collisions. In the following, we will derive the lepton’s flow from the helicity flip photon GPD \( h_\gamma(x, b_\perp) \).
III. FLOW OBSERVABLES FOR LEPTON IN TWO-PHOTON PROCESSES

One naturally expects that the helicity flip photon GPD could introduce a $\cos 4\phi$ modulation in the azimuthal distribution of di-lepton produced in two photon processes as the linearly polarized photon TMD does due to the similar photon polarization tensor structure. However, the $\cos 4\phi$ azimuthal asymmetries induced by the helicity flip photon GPD and the linearly polarized photon TMD are different types. The angle $\phi$ here refers to the azimuthal angle between leading lepton transverse momentum and the impact parameter of heavy ion collisions. On the other hand, the $\cos 4\phi$ azimuthal asymmetry investigated in the previous work [26, 27] describes the correlation between lepton transverse momentum and the total transverse momentum of lepton pair.

The dominant channel for di-lepton production in peripheral and ultraperipheral heavy ion collisions are the Breit-Wheeler process $\gamma(x_1 P) + \gamma(x_2 \bar{P}) \rightarrow \ell^+ (l_1) + \ell^- (l_2)$ where leptons are produced nearly back-to-back in the transverse plane. The differential cross section for the lepton will be normally azimuthal angular symmetric. However, the helicity-flip photon will contribute to an azimuthal asymmetry. Because the photon polarizations are correlated to the individual impact parameters ($b_{1\perp}$ and $b_{2\perp}$) of the incoming nuclei and these impact parameters are constrained by the impact parameter of the collisions: $\mathbf{b}_1 = \mathbf{b}_{1\perp} - \mathbf{b}_{2\perp}$, the final state lepton’s transverse momentum will be correlated with the collisions impact parameter $b_{\perp}$. Following Ref. [26], the azimuthal angle dependence can be computed from the lowest order QED which gives the following square of the amplitude

$$|M|^2 = 2e^4 \left[ \left(\frac{u}{t} + \frac{t}{u}\right) - 2 \cos 2(\phi_1 + \phi_2) \right]$$

with $u$ and $t$ being the usual Mandelstam variables, the azimuthal angles $\phi_1$ and $\phi_2$ being the angles of $\mathbf{b}_{1\perp}$ and $\mathbf{b}_{2\perp}$ with respect to the transverse momentum of measured lepton. Therefore, in addition to the usual isotropic term in the cross section, there is a nonzero $v_4$ in the leading contributions as follows

$$\frac{d\sigma}{d^2P_1 dy_1 dy_2 db_{\perp}} = \frac{2\alpha^2}{Q^4} [A + C \cos 4\phi]$$

where $P_\perp = (l_{1\perp} - l_{2\perp})/2$ is approximately equal to the leading lepton’s transverse momentum in the back-to-back configuration, $y_1$ and $y_2$ are leptons’ rapidities, respectively. $Q$ is the invariant mass of the lepton pair. The coefficients $A$ and $C$ read

$$A = \frac{Q^2 - 2P_{\perp}^2}{P_{\perp}^2} \int d^2b_{1\perp} db_{2\perp} \delta^2(\mathbf{b}_1 - \mathbf{b}_{1\perp} + \mathbf{b}_{2\perp})$$

$$\times x_1 f_\gamma(x_1, b_{1\perp}^2) x_2 f_\gamma(x_2, b_{2\perp}^2),$$

$$C = -2 \int d^2b_{1\perp} d^2b_{2\perp} \delta^2(\mathbf{b}_1 - \mathbf{b}_{1\perp} + \mathbf{b}_{2\perp})$$

$$\times \left[ 2 \left( (\mathbf{b}_{2\perp} \cdot \mathbf{b}_{1\perp}) (\mathbf{b}_{1\perp} \cdot \mathbf{b}_{1\perp}) - \mathbf{b}_{1\perp} \cdot \mathbf{b}_{2\perp} \right)^2 - 1 \right]$$

$$\times x_1 h_\gamma(x_1, b_{1\perp}^2) x_2 h_\gamma(x_2, b_{2\perp}^2) .$$

The above expressions are very similar to those in Ref. [26]. We have also computed for the $v_2$ flow, and found that they are power suppressed by the lepton mass in terms of $m_\ell / P_{\perp}$. This is because the $\cos(2\phi)$ asymmetry requires helicity flip in the QED process of $\gamma\gamma \rightarrow \ell^+ \ell^-$, which is suppressed by the lepton mass. The lepton mass is too small to have any observational effects for $v_2$ for the typical kinematics at RHIC and LHC.

The incoming photons longitudinal momenta fraction are fixed by the external kinematics according to $x_1 = \sqrt{P_{\perp}^2/s (e^{y_1} + e^{y_2})}$ and $x_2 = \sqrt{P_{\perp}^2/s (e^{-y_1} + e^{-y_2})}$ where we have neglected the lepton mass to determine $x_{1,2}$ for the typical kinematics at RHIC and LHC. Note that the initial photon transverse momenta have been integrated out since the calculation is formulated in collinear factorization. As a consequence, there is no Sudakov effects resulting from final state soft photon radiations.

![FIG. 2. Estimates of the cos 4$\phi$ asymmetry as the function of $b_{\perp}$ in Au-Au collisions at $\sqrt{s} = 200$ GeV (the upper plot) and in Pb-Pb collisions at $\sqrt{s} = 5.02$ TeV (the lower plot). The dilepton rapidities are integrated over the regions [-1,1].](image-url)
tor [43],
\[ F(|\vec{k}|) = \frac{4\pi\rho^0}{|\vec{k}|^2 A \alpha^2 k^2 + 1} \times \left[ \sin(|\vec{k}| R_A) - |\vec{k}| R_A \cos(|\vec{k}| R_A) \right], \tag{8} \]
where \( R_A = 1.1 A^{1/3} \text{fm} \), and \( \alpha = 0.7 \text{fm} \). This parametrization numerically is very close to the Woods-Saxon distribution. The numerical results for the computed azimuthal asymmetries in the different kinematical regions for different collisions species are presented in Figs. 2 and 3. Here the azimuthal asymmetries, i.e. the average value of \( \cos(4\phi) \) are defined as,
\[ \langle \cos(4\phi) \rangle = \frac{\int \frac{da}{dP_{\perp}} \cos(4\phi) dP_{\perp} S}{\int \frac{da}{dP_{\perp}} dP_{\perp} S}. \tag{9} \]

In Fig. 2, we show the asymmetries as the functions of the impact parameter \( b_\perp \) in the heavy ion collisions at RHIC and LHC, respectively. As one can see, the general trend is that the asymmetry increases with \( b_\perp \) until it reaches a maximal value when \( b_\perp \) is slightly larger than \( 2R_A \). The maximal value of the asymmetry \( -2\langle \cos(4\phi) \rangle \) ranges from 26\% to 34\% depending on the center mass energy, lepton transverse momenta regions and collision species. After reaching its maximal value, the asymmetry slowly decreases with \( b_\perp \), but remain sizable till the impact parameter is very large.

In Fig. 3, we plot the asymmetries as functions of the lepton’s transverse momentum \( P_{\perp} \). Clearly, at both RHIC and LHC, the asymmetries do not change dramatically with \( P_{\perp} \). This is very much different from hadron’s \( v_4 \) in non-central heavy ion collisions, where the transverse momentum dependence is one of characteristic feature of the medium flow. Of course, in the current case, we can not go to very small transverse momentum for the leptons. This is because we need to keep back-to-back kinematics for the dilepton and large transverse momentum for the leptons to guarantee the dominance from the QED two-photon scattering and the factorization formalism in Eq. (6). Nevertheless, the transverse momentum dependence of lepton flow is a unique feature which can be utilized to probe the flow of hadrons through the azimuthal angular correlations between the leptons and hadrons.

IV. CONCLUSIONS

In summary, we have studied the electromagnetic flow of the leptons in heavy ion collisions, where the leptons are produced in the pure QED process of \( \gamma\gamma \rightarrow \ell^+\ell^- \). Our study shows that there is a significant size of \( v_4 \) flow, whereas the elliptic flow \( v_2 \) vanishes due to small lepton mass. The flow observables are defined as the azimuthal angular asymmetries of the lepton’s transverse momentum respect to the impact parameter of non-central heavy ion collisions. The asymmetries evaluated in various kinematic regions and for different collisions species are shown to be rather sizable. The experiment confirmation of \( v_1 \)-flow in Figs. 2 and 3 will help to identify the production mechanism of the lepton pair at low transverse momentum in heavy ion collisions. Any deviation will indicate other production channel. Once this is established, we can utilize the lepton pair to probe the EM property of the quark-gluon plasma.

As mentioned in the introduction, the EM flow of the leptons can be measured through the azimuthal angular correlations between the lepton and hadrons. The comparison of the flows between the lepton and hadrons shall provide useful information on the underlying physics of the flow phenomena in heavy ion collisions. In particular, because the lepton flow comes from the initial EM field of the colliding nuclei in non-central collisions while the hadron flow comes from the collective modes in the quark-gluon plasma, a detailed study of both flow observables will lead to a deeper understanding of the interface of the initial geometry and later interactions of the medium. For the most central collisions, the fluctuation dominates both flows, and the comparison of the flows will provide a perfect probe to distinguish the EM and QCD effects in heavy ion collisions. In principle, these interesting physical phenomena can be investigated at RHIC and the LHC soon.

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