Empirical equilibrium

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Abstract

We introduce empirical equilibrium, the prediction in a game that selects the Nash equilibria that can be approximated by a sequence of payoff-monotone distributions, a well-documented proxy for empirically plausible behavior. Then, we reevaluate implementation theory based on this equilibrium concept. We show that in a partnership dissolution environment with complete information, two popular auctions that are essentially equivalent for the Nash equilibrium prediction, can be expected to differ in fundamental ways when they are operated. Besides the direct policy implications, two general consequences follow. First, a mechanism designer may not be constrained by typical invariance properties. Second, a mechanism designer who does not account for the empirical plausibility of equilibria may inadvertently design implicitly biased mechanisms.

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1 Introduction

*Empirical*: based on, concerned with, or verifiable by observation or experience rather than theory or pure logic.1

In a strategic situation Nash equilibrium predicts that agents choose actions that maximize utility given the behavior of the other agents. Laboratory experiments and empirical data suggest that agents’ behavior generally does not conform to this principle. Their behavior is not random, however. Empirical distributions of play tend to be payoff-monotone, i.e., between two actions, an agent chooses one with weakly higher probability if and only if it has weakly higher expected payoff given the behavior of the other agents. This intuition was articulated by McKelvey and Palfrey (1995) and Goeree et al. (2005) into the regular Quantal Response Equilibrium (QRE) model, whose parametric forms are now ubiquitous in experimental literature.

Even though the experimental and empirical evidence does lead us to reject the rational agents model that entails Nash equilibrium as a perfect representation of strategic situations, it does not force us to dismiss Nash equilibrium altogether. In many applications, the conclusions one obtains retain their policy relevance if the analysis is done with the assumption that Nash equilibrium is a good approximation of behavior. Indeed, the aforementioned empirical regularity of payoff-monotonicity is evidence that incentives are pulling agents in the direction of choosing better actions with higher probability given what the other are doing. Moreover, approximation to a Nash equilibrium is observed in many games in the limited time of a laboratory environment.2

Thus, if one accepts the experimental and empirical evidence on games, the role of Nash equilibrium in applications becomes to identify the behavior that will be approximated, as opposed to the behavior that will exactly happen. Consequently, the only Nash equilibria that are relevant in any analysis one is performing, are those that can possibly be approximated by agents behavior. This leads us to the definition of *empirical equilibrium*, the (approximate) prediction in a game that selects the Nash equilibria that are the limits of payoff-monotone distributions, the well-documented proxy for empirically plausible behavior. The purpose of this paper, besides defining and providing an introspection for this prediction in a game, is to show

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1Google Dictionary search on Nov. 15, 2017.
2Even though it is not universal, it is commonly observed that estimates of the parameter of sophistication in QRE models increase as agents gain experience with games in experiments (c.f., McKelvey and Palfrey, 1995).
that empirical equilibrium analysis is consequential for the so-called (full) implementation theory.

First, we observe that empirical equilibria exist for each finite game (Lemma 1). Second, we show that our choice as basis for empirical plausibility, payoff-monotonicity, is robust. On the one hand, (in a finite game) a Nash equilibrium is an empirical equilibrium if and only if it can be approximated by a sequence of regular QRE (Proposition 1), which are payoff-monotone fixed points of operators satisfying further restrictions. This equivalence provides an explicit connection between our proxy for empirical plausibility and the extensive experimental literature showing how regular QRE rationalizes patterns of behavior in a wide range of strategic situations (see Goeree et al., 2016, for a survey). On the other hand, each Nash equilibrium that is the limit of a sequence of behavior satisfying a weak form of payoff-monotonicity, which is more easily tested in data, is also an empirical equilibrium (Proposition 2).

Implementation theory is the application of game theory that evaluates worst-case scenarios of economic institutions, which, as usual, we refer to as mechanisms. That is, given a certain objective, the designer looks for a mechanism that achieves this objective for all the Nash equilibrium outcomes when the mechanism is operated. In order to understand the implications of accounting for empirical plausibility in this exercise, we study a particular environment that provides us with a test case from which we draw conclusions of general validity.

We consider a symmetric partnership dissolution problem in which two agents who collectively own an object need to decide who receives the object when monetary compensation, chosen out of a finite but fine greed, is possible. In the spirit of the implementation literature with complete information (see Jackson, 2001, for a survey) we assume that an arbitrator who makes a recommendation for this division knows that the agents know each other well but does not know the agents’ preferences on the possible

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3The limits of logistic QRE as the sophistication parameter of this particular quantal response function converges to infinity, which always exist, are empirical equilibria (McKelvey and Palfrey, 1995).

4In Sec. 5 we discuss how the main message of our results survives even if one endorses only a minimal notion of payoff-monotonicity.

5We state Lemma 1 and Propositions 1 and 2 for our two-agent partnership dissolution environment. These results also hold in a general n-agent finite model with complete or incomplete information (c.f., Velez and Brown, 2018).

6In only few exceptions, which we discuss in Sec. 2, the prediction used for the implementation theory exercise is different from Nash equilibrium.
This is a relevant benchmark for the dissolution of a marriage or a long standing partnership. We assume that agents are expected utility maximizers with quasi-linear utility indices. For concreteness, say that agents’ values for the object are $v_l \leq v_h$. We study two prominent mechanisms that operate as follows. First, the arbitrator asks the agents to bid for the object. Then assigns the object to a higher bidder, breaking ties uniformly at random. The transfer from the agent who receives the object to the other agent is determined as follows: the transfer is the winner bid, the winner-bid auction; the transfer is the loser bid, the loser-bid auction.

We characterize the Nash equilibria, both in pure and mixed strategies, of the winner-bid and loser-bid auctions. When agents have equal types, both auctions give, in each Nash equilibrium, equal expected payoff to each agent (Lemma 2). Moreover, there are equilibria that implement this payoff with a deterministic outcome. When valuations are different, the set of efficient Nash equilibrium payoffs of both auctions coincide (Proposition 3). This common set can be placed in a one to one correspondence with the integers $\{v_l/2, ..., v_h/2\}$: each equilibrium has a unique payoff-determinant bid in this set and for each such integer there is an equilibrium with this payoff-determinant bid (Proposition 4). We refer to this set as the Nash range. Between two elements of the Nash range the higher valuation agent prefers the left one (paying less), and the lower valuation agent prefers the right one (being paid more). Interestingly, the Nash equilibrium payoffs of any mechanism that obtains the deterministic outcomes that give equal payoffs to both agents whenever they have equal valuations, necessarily includes the efficient Nash equilibrium payoffs of these two auctions (Lemma 3). Thus, a mechanism designer who bases the analysis on the Nash equilibrium prediction, believes that he or she has to accept a wide range of assignments when

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7The assumption of complete information in our partnership dissolution problem allows us to contrast empirical equilibrium analysis with the well-understood restrictions of Nash implementation in this environment. Our approach to implementation theory, and in general mechanism design, does not impose restrictions on the information structure that the modeler believes is a reasonable description of reality. For instance, one can define and study empirical equilibria in general incomplete information environments, under a Bayesian or non-Bayesian information structure (Velez and Brown, 2018).

8The winner-bid and the loser-bid auctions belong to the family of $\alpha$-auctions studied by Cramton et al. (1987) and McAfee (1992). Note that the partnership dissolution environment differs substantially from a buyer-seller environment in which the payoff of the loser of a first-price auction or a second-price auction does not depend on the price paid by the winner.

9The maximal aggregate loss in an inefficient equilibrium is one unit. The analysis of inefficient equilibria leads to the same conclusions we state here in the introduction for efficient equilibria (Sec. 4.4.1).
valuations are different if he or she insists on equity when valuations are equal, or even just similar (see Sec. 4.4.3). Technically, this is the expression in this environment of the so-called invariance under Maskin monotonic transformations of the Nash equilibrium outcome correspondence (Maskin, 1999).

Finally, in our main results, we characterize the set of empirical equilibrium payoffs of the winner-bid and loser-bid auctions (Theorems 1 and 2). The highlights of these characterizations are the following. With a single exception among all type profiles, these sets are disjoint. The empirical equilibrium payoffs of the winner-bid auction belong to the left half of the Nash range. When \( v_l \) is not too close to the minimal bid, i.e., at least \( 3v_h/8 \), the empirical equilibrium payoffs of the winner-bid auction essentially (up to rounding) are the left fifth of the Nash range. Symmetrically, the empirical equilibrium payoffs of the loser-bid auction belong to the right half of the Nash range.

Thus, empirical equilibrium analysis brings very different news to the mechanism designer. First, we learn that an arbitrator can abide by a principle of equity and at the same time exercise a form of affirmative action that guarantees a special treatment for either low or high value agents. Technically, this proves that a mechanism designer who accounts for empirical plausibility of equilibria is not constrained by invariance to Maskin monotonic transformations (Sec. 4.4.3). Second, and not less important, we learn that even though these auctions have symmetric action spaces and their Nash equilibrium outcomes span the whole spectrum of possible equitable divisions, they likely favor a particular agent in practice. Thus, an arbitrator who uses one of them within a legal system in which this type of affirmative action is forbidden, can be subject to a legitimate challenge supported by theory and empirical data (Sec. 4.4.4).\(^{10}\) Consequently, a mechanism designer who does not account for empirical plausibility of equilibria may be both overly cautious and leave unexplored some possibilities for design and may inadvertently design implicitly biased mechanisms.

The remainder of the paper is organized as follows. Sec. 2 places our contribution in the context of the literature. Sec. 3 introduces the model. Section 4 presents our results. Sec. 5 discusses our definition of empirical equilibrium and open questions. We present all proofs in the Appendix.

\(^{10}\)Indeed, our results produce a series of comparative statics that are supported by experimental evidence (Brown and Velez, 2018).
2 Related literature

Our work builds on the definition of QRE by McKelvey and Palfrey (1995); its redefinition, based on axioms of behavior, as regular QRE by Goeree et al. (2005), which was prompted by the theoretical challenges brought by Haile et al. (2008); and the experimental and theoretical work that has followed (see Goeree et al., 2016, for a survey). Essentially, QRE based studies have two types of results. The majority of them show how behavior in laboratory experiments, which generally differs from Nash equilibria, can be fit to parametric forms of the regular QRE model. Other studies derive comparative statics of logistic QRE behavior as its sophistication parameter diverges and contrast these predictions with empirical evidence (e.g., Anderson et al., 1998, 2001).

Our work differs in a fundamental way from this existing literature on regular QRE. First, we only retain an ordinal testable implication of this model as a basis for empirical plausibility of behavior. We show that this is a robust choice (Sec. 4.2). Then we define and analyze the set of Nash equilibria that can be the limits of this type of behavior. Thus, as a byproduct we are finding properties and comparative statics of all limits, as agents get more sophisticated, of regular QRE behavior, logistic or not. To the length of our knowledge this is the first study to do so. Finally, and most importantly, we find policy relevant implications for the environment we study and conclusions of general validity for implementation theory.

Since we are interested in determining the plausibility of a Nash equilibrium by the existence of plausible behavior arbitrarily close to it, our work is similar in spirit to Harsanyi (1973), who proved that generically, mixed strategy-equilibria are limits of pure-strategy equilibria of information-wise neighboring games; Rosenthal (1989), who aimed at the analysis of payoff-monotone behavior with a particular linear form in two-by-two games; and the definition of logistic limiting equilibrium by McKelvey and Palfrey (1995).

It is worth noting that the logistic limiting equilibrium, and its recent generalizations by Zhang (2016), are empirical equilibria that depend on pa-

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11Haile et al. (2008) noted that in the additive-error structural form of QRE, if one does not restrict error structures, any data set can be rationalized. Thus, unrestricted structural QRE is not falsifiable. Regular QRE, on the other hand, is falsifiable. It is constrained by the behavioral axioms that it endorses. A byproduct of our results is that payoff-monotonicity, an essential component of regular QRE (Goeree et al., 2005), produces sharp restrictions in the limit behavior generated by the model.

12Curiously, our results show that when a mechanism is operated it is possible that most of the pure-strategy equilibria are implausible (Sec. 4). Thus, Harsanyi’s suspicion on mixed-strategy equilibria and confidence in pure-strategy equilibria is somehow unfounded.
Figure 1: (a) Myerson (1978)’s game $\Gamma_1$. Rows label strategies of Player 1 and columns of Player 2. (b) Payoff-monotone distributions (shaded area) and Nash equilibria (large dots) of $\Gamma_1$: $\pi_1(\alpha_1)$ is the probability with which agent 1 plays $\alpha_1$. Equilibrium $(\alpha_1, \beta_1)$ can be approximated by payoff-monotone behavior. Thus, it is an empirical equilibrium of $\Gamma_1$. Equilibrium $(\alpha_2, \beta_2)$ cannot be approximated by payoff-monotone behavior. Thus, it is not an empirical equilibrium of $\Gamma_1$.

Empirical equilibrium is also related to the so-called tremble-based refinements of Nash equilibrium (Selten, 1975; Myerson, 1978, and subsequent literature). Similarly to these refinements, empirical equilibrium selects the Nash equilibria that are plausible in some formal sense. There is a fundamental difference between the two approaches, however. The following example illustrates the usual rational for tremble-based refinements and contrasts it with empirical equilibrium.

**Example 1.** Consider game $\Gamma_1$ proposed by Myerson (1978) in order to motivate his definition of proper equilibrium (Fig. 1 (a)). There are two Nash equilibria in $\Gamma_1$, $(\alpha_1, \beta_1)$ and $(\alpha_2, \beta_2)$. Myerson (1978) argues that $(\alpha_2, \beta_2)$ is not plausible because if “player 1 thought that there was any chance of player 2 using $\beta_1$, then player 1 would certainly prefer $\alpha_1$.” Then he articulates this intuition of how a rational player would play this game into the definition of proper equilibrium, which in turns selects equilibrium $(\alpha_1, \beta_1)$ as the only plausible outcome of this game. From the perspective of empirical equilibrium $(\alpha_2, \beta_2)$ is also implausible. For each distribution of actions of player 2, player 1’s utility from playing $\alpha_1$ is greater than or equal to the utility from playing $\alpha_2$; thus, in a profile of payoff-monotone
distributions of play, agent 1 will always play $\alpha_1$ with probability at least $1/2$ (Fig. 1 (b)); thus, $(\alpha_2, \beta_2)$ cannot be approximated by payoff-monotone behavior. If this game is played and agents behavior is payoff-monotone and approximates a Nash equilibrium, it is necessarily $(\alpha_1, \beta_1)$.

Note that tremble-based refinements are borne out of an intuition about how a game should be played by a utility maximizing agent who makes some conjecture about the behavior of the other agents that includes them failing to perfectly maximize utility. By contrast, empirical equilibrium is borne out of the observation of how agents usually behave, symmetrically applied to all agents. One can either think of these regularities as the expression of bounded rationality, or the existence of unmodeled subtleties that result in agents behaving as if they were boundedly rational (see Goeree et al., 2016, for an extensive discussion).

There are some games, as our example above, in which tremble-based refinements and empirical equilibrium produce similar answers. Moreover, there are some environments in which trembling-hand perfection, properness, and even tremble-based sub-refinements of proper equilibria seem to be good predictions of behavior (e.g., Milgrom and Mollner, 2018). However, there are laboratory experiments and empirical data consistent with payoff-monotonicity, for other strategic situations, that reject the foundation of tremble-based refinements, or any refinement of Nash equilibrium that predicts no weakly dominated action will be played by an agent, as a universal plausibility standard for games. This greatly challenges an implementation theory founded on a refinement that always discards weakly dominated behavior (e.g. Palfrey and Srivastava, 1991; Jackson, 1992; Sjöström, 1994).

A striking example is agents’ persistent and well-documented propensity to use weakly dominated actions in dominant strategy mechanisms (c.f., Kagel et al., 1987; Kagel and Levin, 1993; Harstad, 2000; Chen and Sönmez, 2006; Cason et al., 2006; Andreoni et al., 2007; Hassidim et al., 2016; Rees-Jones, 2017; Li, 2017). Essentially, the intuition behind tremble-based refinements fails because agents do react to incentives, but they seem to react by playing actions with lower utility with lower probability, not by not playing them at all. Thus, weakly dominated actions are persistently played whenever it is possible for the agents to mutually provide the incentives to play them. The first to observe this intuition were McKelvey and Palfrey (1995), who also constructed a game in which a sequence of logistic quantal response equilibria converges to a Nash equilibrium that is not trembling.

\[\text{\textsuperscript{13}In Velez and Brown (2018) we detail how empirical equilibrium analysis rationalizes the subtle differences in behavior observed for different dominant strategy mechanisms.}\]
hand perfect.\textsuperscript{14,15}

Thus, empirical equilibrium produces a conservative selection from the Nash equilibrium set based on empirical regularities, which can be applied to general environments. It is worth emphasizing that we do not claim or believe that each empirical equilibrium will indeed be relevant for a particular environment. The empirical evidence points otherwise. For instance, on the one hand, there are games with multiple strict Nash equilibria, i.e., those where each agent is playing her unique best response, in which agents seem to consistently coordinate on very particular ones (e.g., Van Huyck \textit{et al.}, 1990). On the other hand, each strict Nash equilibrium is an empirical equilibrium (this follows from Proposition 2 and also from Lemma 2.4 in Tumennasan, 2013).

Our work joins the growing literature on mechanism design with (as if) boundedly rational agents. The studies closer to ours aim at characterizing limits of behavior of this type of agents. Tumennasan (2013) studies a form of implementation in the limits, as agents gain sophistication, of pure-strategy anonymous logistic quantal response equilibria. We differ in fundamental ways with this study. First, we do not endorse any particular quantal response form and consider mixed strategies. Second, this author’s definition of implementation requires a strict form of convergence that implies a satisfactory mechanism possesses at least a strict Nash equilibrium for each type. This constrains implementation to social choice correspondences that satisfy a small variation of invariance under Maskin monotonic transformations. By contrast, our work shows that by accounting for empirical plausibility in popular mechanisms, a mechanism designer can go a long way from invariance restrictions (Sec. 4.4.3). Cabrales and Serrano (2011) explicitly model the evolutionary dynamics of strategies, study the limits of these paths given some sets of initial conditions, and obtain conditions under which a social choice function can be dynamically implemented. The result is again that dynamic implementation implies implementation in strict Nash equilibria. Thus, this type of implementation has the same constrains as those in Tumennasan (2013). Other studies generally endorse a form of bounded rationality and aim at finding institutions that perform well when

\textsuperscript{14}Evolutionary game theory also points to the relevance of weakly dominated behavior (see Cabrales and Ponti, 2000, and references therein). Implementation theory based on evolutionary dynamics has similar constraints to traditional mechanism design. See our discussion of Cabrales and Serrano (2011).

\textsuperscript{15}One can also construct games in which a trembling hand perfect equilibrium is not an empirical equilibrium (Velez and Brown, 2018). Limiting equilibria, a subclass of empirical equilibria, are independent of risk dominance selections (Zhang and Hofbauer, 2016).
operated on agents who exhibit these particular patterns of behavior (c.f., Eliaz, 2002; de Clippel, 2014; de Clippel et al., 2017; Kneeland, 2017). Our work differs in that we aim at characterizing limits of boundedly rational behavior and not at designing for the path in which this occurs. Thus, our approach allows us to uncover regularities in boundedly rational behavior when it is disciplined by proximity to a Nash equilibrium (see Sec. 5).

3 Model

We introduce a partnership dissolution environment with complete information. Our definitions are easily adapted to a general \( n \)-agent heterogenous-belief incomplete information environment (c.f. Velez and Brown, 2018).

There are two agents \( N \equiv \{1, 2\} \) who collectively own an object (indivisible good) and need to allocate it to one of them. Monetary compensation is possible. Each agent’s payoff type is characterized by the value that he or she assigns to the object. We assume these type spaces are \( \Theta_1 = \Theta_2 \equiv \{v, v + 2, ..., \overline{v}\} \), where \( v \leq \overline{v} \) are even positive integers. The generic type of agent \( i \) is \( v_i \in \Theta_i \). Let \( \Theta \equiv \Theta_1 \times \Theta_2 \) with generic element \( v \equiv (v_1, v_2) \). The lower and higher values at \( v \) are \( v_l \) and \( v_h \) respectively. We also assume that agents are expected utility maximizers. The expected utility index of agent \( i \) with type \( v_i \) is \( v_i - p \) if receiving the object and paying \( p \) to the other agent; \( p \) if being paid this amount by the other agent and receiving no object. Whenever we make statements in which the identity of the agents is not relevant, we conveniently use neutral notation \( i \) and \( -i \). The set of possible allocations is that in which an agent receives the object and transfers an amount \( p \in \{0, 1, ..., \overline{p}\} \) with \( \overline{p} \geq \overline{v}/2 \), to the other agent. Let \( A \) be the space of these allocations. For an allocation \( a \in A \) the value of agent \( i \)'s utility index at the allotment assigned by \( a \) to this agent is \( u_i(a|v_i) \).

A social choice correspondence (scc) selects a set of allocations for each possible profile of types. The generic scc is \( G : \Theta \rightharpoonup A \).

A mechanism is a pair \( (M, \varphi) \) where \( M \equiv (M_i)_{i \in N} \) is an unrestricted message space and \( \varphi : M \rightarrow \Delta(A) \) is an outcome function. Given a profile of types \( v \in \Theta \), mechanism \( (M, \varphi) \) determines a complete information game \( \Gamma \equiv (M, \varphi, v) \). A (mixed) strategy for agent \( i \) is a probability measure on \( M_i \). Agent \( i \)'s generic strategy is \( \sigma_i \in \Delta(M_i) \). The profile of strategies is \( \sigma \equiv (\sigma_i)_{i \in N} \). We denote the measure that places probability one on \( m_i \in M_i \) by \( \delta_{m_i} \). The expected utility of agent \( i \) with type \( v_i \) in \( \Gamma \) from selecting action

\[ \text{16} \text{We assume positive even valuations in order to simplify notation and to avoid the analysis of trivial cases that do not add to the general message of our results.} \]
when the other agent selects an action as prescribed by $\sigma_{-i}$ is

$$U_\varphi(\delta_{m_i}|\sigma_{-i}; v_i) \equiv \sum_{m_{-i} \in M_{-i}} u(\varphi(m_i, m_{-i})|v_i)\sigma_{-i}(m_{-i}).$$

**Definition 1.** $\sigma \equiv (\sigma_i)_{i \in N}$ is weakly-payoff-monotone for $(M, \varphi, v)$ if for each $i \in N$ and each pair $\{m_i, n_i\} \subseteq M_i$, $U_\varphi(\delta_{m_i}|\sigma_{-i}; v_i) \geq U_\varphi(\delta_{n_i}|\sigma_{-i}; v_i)$ only if $\sigma_i(m_i) \geq \sigma_i(n_i)$; $\sigma$ is payoff-monotone for $(M, \varphi, v)$ if for each $i \in N$ and each pair $\{m_i, n_i\} \subseteq M_i$, $U_\varphi(\delta_{m_i}|\sigma_{-i}; v_i) \geq U_\varphi(\delta_{n_i}|\sigma_{-i}; v_i)$ if and only if $\sigma_i(m_i) \geq \sigma_i(n_i)$.

A profile of strategies $\sigma$ is a Nash equilibrium of $\Gamma$ if for each $i \in N$, each $m_i$ in the support of $\sigma_i$, and each $m_i' \in M_i$, $U_\varphi(\delta_{m_i'}|\sigma_{-i}; v_i) \leq U_\varphi(\delta_{m_i}|\sigma_{-i}; v_i)$.

The set of Nash equilibria of $\Gamma$ is $N(\Gamma)$. The set of Nash equilibrium outcomes of $\Gamma$ are those obtained with positive probability for some Nash equilibrium of $\Gamma$. Agent $i$’s expected payoff in equilibrium $\sigma$ is $\pi_i(\sigma)$.

**Definition 2.** $\sigma \in N(\Gamma)$ is an empirical equilibrium of $\Gamma$ if there is a sequence of payoff-monotone distributions of $\Gamma$, $\{\sigma^\lambda\}_{\lambda \in \mathbb{N}}$, such that as $\lambda \to \infty$, $\sigma^\lambda \to \sigma$.

Given a mechanism $(\varphi, M)$, a quantal response function for agent $i$ with type $v_i \in \Theta_i$ is a function $p_i(\cdot|v_i) : \mathbb{R}^{M_i} \to \Delta(M_i)$. For each $m_i \in M_i$ and each $x \in \mathbb{R}^{M_i}$, $p_{im_i}(x|v_i)$ denotes the value assigned to $m_i$ by $p_i(x|v_i)$. We refer to $p(\cdot|v) \equiv (p_i(\cdot|v_i))_{i \in N}$ as a quantal response function for type $v \in \Theta$.

A quantal response function $p_i$ is regular if it satisfies the following four properties (Goeree et al., 2005):

- **Interiority:** $p_i > 0$.
- **Differentiability:** $p_i$ is a differentiable function.
- **Responsiveness:** for $x \in \mathbb{R}^{M_i}$, $\eta > 0$, and $m \in M_i$, $p_{im}(x + \eta 1_m) > p_{im}(x)$.
- **Monotonicity:** for $x \in \mathbb{R}^{M_i}$ and $\{m, t\} \subseteq M_i$ such that $x_m > x_t$, $p_{im}(x) > p_{it}(x)$.

The (type-independent) logistic quantal response function with parameter $\lambda \geq 0$, denoted by $l^\lambda$, assigns to each $m_i \in M_i$ and each $x \in \mathbb{R}^{M_i}$ the value,

$$l^\lambda_{im_i}(x) \equiv \frac{e^{\lambda x_{m_i}}}{\sum_{n_i \in M_i} e^{\lambda x_{n_i}}}. \quad (1)$$

It can be easily checked that for each $\lambda > 0$, the corresponding logistic quantal response function is regular (McKelvey and Palfrey, 1995).\footnote{$1_m$ denotes the vector in $\mathbb{R}^{M_i}$ that has 1 in component $m$ and 0 otherwise.}
A quantal response equilibrium of $\Gamma \equiv (M, \varphi, v)$ with respect to quantal response function $p(\cdot|v)$ is a fixed point of the composition of $p(\cdot|v)$ and the expected payoff operator in $\Gamma$ (McKelvey and Palfrey, 1995), i.e., a profile of distributions $\sigma \equiv (\sigma_i)_{i \in N}$ such that for each $i \in N$, $\sigma_i = p_i(U_\varphi(\delta_{m_i}|\sigma_{-i}; v_i)_{m_i \in M_i|v_i})$. We refer to a quantal response equilibrium for a regular quantal response function as a regular quantal response equilibrium.

4 Results

4.1 Existence

Consider a finite message space mechanism $(M, \varphi)$ and $v \in \Theta$. Brower’s fixed point theorem implies existence of quantal response equilibria of $(M, \varphi, v)$ for each profile of continuous quantal response functions. It is also straightforward that each sequence of logistic quantal response equilibria of $(M, \varphi, v)$ for which $\lambda \to \infty$, has a convergent subsequence (the simplex is a compact set). These limits are Nash equilibria of $(M, \varphi, v)$ (McKelvey and Palfrey, 1995). Since the logistic quantal response function is continuous and monotone, general existence of empirical equilibria follow.

Lemma 1. For each finite message space mechanism $(M, \varphi)$ and $v \in \Theta$, the set of empirical equilibria of $(M, \varphi, v)$ is non-empty.

4.2 Empirical equilibria are the limits of regular quantal response equilibria

We have defined an empirical equilibrium as a Nash equilibrium that can be approximated by payoff-monotone distributions. In the context of implementation theory it is common to find results stating that certain objectives are not possibly achieved by all Nash equilibria of any simultaneous move mechanism. We will show in the next sections that this conclusion is reversed, for some objectives, when one considers only empirical equilibria. Thus, it is important to determine the extent to which our definition of empirical equilibrium is robust to our choice of proxy for empirically plausible behavior.

One can easily see that each regular QRE is a payoff-monotone distribution. Thus, the experimental literature that has shown how regular QRE provides a good fit to empirical distributions of behavior is in strong support of payoff-monotonicity (Goeree et al., 2016). It is fair to ask why we are not endorsing the whole regular QRE structure as basis of empirical
plausibility. By doing so we may be overly cautious. Indeed, when we impose less restrictions on our basis for empirically plausible behavior, the set of Nash equilibria that can be approximated by this type of behavior weakly enlarges. Thus, in principle, it is possible that a worst-case mechanism designer that expects regular QRE behavior will be observed when a mechanism is operated, and will approximate a Nash equilibrium, may find that some objectives that we find impossible to achieve are within his or her reach. One can argue that ours is a sensible choice, however. Quantal response functions are not observable, there are limitations to test the other building blocks of the regular QRE model with finite data, and there is a well-documented inconsistency of estimated parametric forms of regular QRE on different games (McKelvey and Palfrey, 1995). The following proposition allows us to bypass this discussion altogether: An empirical equilibrium can be equivalently defined as a Nash equilibrium that can be approximated by regular QRE. Thus, a more optimistic mechanism designer that evaluates the worst-case performance of a mechanism on all Nash equilibria that can be approximated by regular QREs finds the same answers that we do by endorsing only payoff-monotonicity.\textsuperscript{18}

**Proposition 1.** Let \((M, \varphi)\) be a finite message space mechanism and \(v \in \Theta\). Then, \(\sigma \in N(M, \varphi, v)\) is an empirical equilibrium of \((M, \varphi, v)\) if and only if there is a sequence of regular quantal response functions \(\{p^\lambda(\cdot|v)\}_{\lambda \in \mathbb{N}}\) and corresponding regular quantal response equilibria \(\{\sigma^\lambda\}_{\lambda \in \mathbb{N}}\) such that as \(\lambda \to \infty\), \(\sigma^\lambda \to \sigma\).

It is worth noting that even though it was known that the limits, as the sophistication parameter of some regular QRE forms diverges to infinity, are Nash equilibria, no previous study defined the set of Nash equilibria that are limits of regular QRE, characterized this set in a policy relevant environment, and studied its invariance properties and its relevance for implementation theory and mechanism design. By Proposition 1, these are all implicit contributions of our study.

It is also relevant to do a more thorough vetting of our basis of empirical plausibility. Experimental studies based on regular QRE do find that this model fits data well.\textsuperscript{19} However, they often may not directly discuss payoff-monotonicity (see the analysis of asymmetric matching pennies games in

\textsuperscript{18}At a technical level, the proof of Proposition 1 reveals that any payoff-monotone distribution can be approximated with an interior payoff-monotone distribution. This fact may also be useful in the characterization of empirical equilibria for some games.

\textsuperscript{19}In some cases with incomplete information good fit necessitates the relaxation of the Bayesian structure of the game (Rogers et al., 2009).
Goeree et al. (2005, 2016) for an exception). Our companion paper (Brown and Velez, 2018) experimentally tests the mechanisms that we analyze in the following sections. We find evidence of both payoff-monotonicity and our main predictions for the performance of these mechanisms when payoff-monotone distributions approximate a Nash equilibrium. We also find that a range of actions that are far from optimal for an agent are usually not played at all. One can argue that this reflects simply the lack of statistical power of feasible experiments to test comparative statics for these low-probability events in realistic environments. However, it is worth noting that our definition of empirical plausibility can be enlarged to include this type of behavior without any change to the set of equilibria that result empirically plausible. More precisely, we can rely on distributions that reveal differences in utility and discard the requirement that differences in utility must induce differences in behavior.

Proposition 2. Let \((M, \varphi)\) be a finite message space mechanism and \(v \in \Theta\). Then, \(\sigma \in N(M, \varphi, v)\) is an empirical equilibrium of \((M, \varphi, v)\) if and only if there is a sequence of weakly-payoff-monotone distributions of \(\Gamma\), \(\{\sigma^\lambda\}_{\lambda \in \mathbb{N}}\), such that as \(\lambda \to \infty\), \(\sigma^\lambda \to \sigma\).

4.3 Extreme-price auctions and Nash equilibrium

The theory of fair allocation has produced a series of principles that an arbitrator may want to adhere to when resolving a partnership dissolution dispute (see Thomson, 2010, for a survey). Two of the most popular are the following: efficiency, i.e., a party who values the object the most should receive it; equity, i.e., no party should prefer the allotment of the other (Foley, 1967). Abstracting from incentives issues imagine that the arbitrator knows the agents’ valuations, \(v \in \Theta\). It is easy to see that if the arbitrator abides by the principles of efficiency and equity, agents’ payoffs should have the form:

\[ v_l/2 + t \] for \(l\) and \[ v_h/2 + (|v_h/2 - v_l/2| - t) \] for \(h\), where \(0 \leq t \leq |v_h/2 - v_l/2|\). In other words, if an arbitrator endorses these two principles, the only that is left for him or her is to determine a division between the agents of the so-called equity surplus, i.e., \(ES(v) \equiv |v_h/2 - v_l/2|\) (Tadenuma and Thomson, 1995b).

Definition 3. The winner-bid auction is the mechanism in which each agent selects a bid in the set \(\{0, 1, 2, ..., p\}\). An agent with the highest bid receives the object. Ties are resolved uniformly at random. The agent who receives

\footnote{In our environment no-envy implies efficiency for deterministic allocations (Svensson, 1983). Our statement here refers also to random assignments.}
the object pays the winner bid to the other agent. The loser-bid auction is the mechanism defined similarly where the payment is the loser bid. We refer to these two auctions as the extreme-price auctions.

The following lemma states that each extreme-price auction achieves, in each Nash equilibrium, the objectives of efficiency and equity when agents have equal valuations. We omit the straightforward proof.

**Lemma 2.** Let \((M, \varphi)\) be an extreme-price auction and \(v \in \Theta\) such that \(v_1 = v_2\). Then for each \(\sigma \in N(M, \varphi, v)\), \(\pi_1(\sigma) = \pi_2(\sigma) = v_1/2 = v_2/2\). Moreover, for each deterministic allocation \(a \in A\) in which each agent receives this common payoff, there is a pure-strategy Nash equilibrium of \((M, \varphi, v)\) whose outcome is this allocation.

The following proposition states that each extreme-price auction essentially achieves, in each Nash equilibrium, the objectives of efficiency and equity for arbitrary valuation profiles.

**Proposition 3.** Let \((M, \varphi)\) be an extreme-price auction and \(v \in \Theta\) such that \(v_1 < v_h\). Let \(\sigma\) be a Nash equilibrium of \((M, \varphi, v)\). Then, there is \(p \in \{v_1/2, ..., v_h/2\}\) such that the support of \(\sigma_l\) belongs to \(\{0, ..., p\}\) and the support of \(\sigma_h\) belongs to \(\{p, ..., \bar{v}\}\). If \(\sigma\) is efficient, the higher value agent receives the object and pays \(p\) to the other agent. Moreover,

1. If \((M, \varphi)\) is the winner-bid auction, then \(p\) is in the support of \(\sigma_h\). If \(\sigma\) is inefficient, i.e., \(\sigma_l(p) > 0\), then \(p = v_1/2\); \(\pi_l(\sigma) + \pi_h(\sigma) \geq v_h - 1\); and \(\pi_h(\sigma) \geq v_h/2 + ES(v) - 1\).

2. If \((M, \varphi)\) is the loser-bid auction, then \(p\) is in the support of \(\sigma_l\). If \(\sigma\) is inefficient, i.e., \(\sigma_h(p) > 0\), then \(p = v_m/2\); \(\pi_l(\sigma) + \pi_h(\sigma) \geq v_h - 1\); and \(\pi_l(\sigma) \geq v_l/2 + ES(v) - 1\).

Proposition 3 states that the Nash equilibria of the extreme-price auctions have a simple structure. In all equilibria, the payoff-determinant bid is in the set \(\{v_1/2, ..., v_h/2\}\). Let us refer to this set of bids as the Nash range. In most of these equilibria agents bids are strictly separated. That is, there is a bid \(p\) in the Nash range such that one agent bids weakly on one side of \(p\) and the other agent bids strictly on the other side of \(p\). In these equilibria, which are strictly separated, outcomes are efficient and equitable. There are inefficient equilibria. For the winner-bid auction, it is possible that both agents bid \(v_l/2\). For the loser-bid auction it is possible that both agents bid \(v_h/2\). In both cases the aggregate welfare loss is at most one unit, i.e., the
size of the minimal difference between bids. This means that if the minimal bid increment is one cent, the maximum that these auctions can lose in aggregate expected utility for any Nash equilibrium is one cent. Thus, one can say that these auctions essentially implement the principles of efficiency and equity in Nash equilibria.  

Proposition 3 allows us to easily characterize Nash equilibrium payoffs for extreme-price auctions.

**Proposition 4.** Let \((M, \varphi)\) be an extreme-price auction and \(v \in \Theta\) such that \(v_l < v_h\).

1. The set of efficient Nash equilibrium payoffs of \((M, \varphi, v)\), i.e., \("(\pi_l(\sigma), \pi_h(\sigma)) : \sigma \in N(M, \varphi, v), \pi_l(\sigma) + \pi_h(\sigma) = v_h\"\), is the set of integer divisions of the equity-surplus, i.e., \("(v_l/2 + ES(v) - t, v_h/2 + t) : t \in \{0, 1, ..., ES(v)\}\"\).

2. The set of inefficient Nash equilibrium payoffs of the winner-bid auction is \("(v_l/2, v_h/2 + ES(v) - \varepsilon) : \varepsilon \in (0, 1]\"\).

3. The set of inefficient Nash equilibrium payoffs of the loser-bid auction is \("(v_l/2 + ES(v) - \varepsilon, v_h/2) : \varepsilon \in (0, 1]\"\).

One can envision an arbitrator having a deliberate choice on the division of the equity surplus in a partnership dissolution problem. For instance the arbitrator may want to guarantee a minimum share of the equity surplus to a given agent. Proposition 4 implies that this is not achieved by the extreme-price auctions if one evaluates them with the Nash equilibrium prediction. The following lemma states that this is a feature not only of these auctions but also of any mechanism that possesses equitable equilibria with deterministic outcomes when agents’ valuations are equal.

**Lemma 3.** Let \((M, \varphi)\) be an arbitrary mechanism. Suppose that for each \(v \in \Theta\) such that \(v_1 = v_2\) and each allocation \(a \in A\) at which each agent gets payoff \(v_1/2 = v_2/2\), there is a Nash equilibrium of \((M, \varphi, v)\) that obtains allocation \(a\) with certainty. Then, for each \(v \in \Theta\) and each \(t \in \{0, 1, ..., ES(v)\}\) there is \(\sigma \in N(M, \varphi, v)\) such that \(\pi_1(\sigma) = v_1/2 + t\) and \(\pi_2(\sigma) = v_2/2 + ES(v) - t\).

Lemma 3 essentially states that if an arbitrator selects a mechanism that obtains with certainty, in Nash equilibria, efficiency and equity whenever  

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\(^{21}\)Our proof of Proposition 3 also reveals that the probability of an inefficient outcome is bounded above by the inverse of the equity surplus, measured in the minimal bid increment.
agents valuations coincide, then any integer division of the equity surplus is an outcome of a Nash equilibrium when valuations are different. The intuition why this is so is the following. If an allocation is an outcome of a Nash equilibrium of a mechanism, this allocation must be the best each agent can achieve given some reports of the other agents; if an agent’s utility changes weakly enlarging the lower contour set of the agent at the equilibrium outcome, then the initial action profile must be also a Nash equilibrium for the second utility profile. One can easily see that this property of the Nash equilibrium outcome correspondence of a mechanism, popularly known as *invariance under Maskin monotonic transformations* (Maskin, 1999), is responsible for the contamination result stated in the lemma.\footnote{Tadenuma and Thomson (1995a) were the first to apply this type of argument to the equitable allocation of an object among \( n \) agents when monetary compensation is possible.}

It is worth noting that Lemma 3 can be generalized to state that if a mechanism obtains efficient and equitable equilibria with certainty when valuations are similar, then it has to obtain a wide range of equity surplus distributions when valuations differ for a wider margin. Thus, one can say that Nash equilibrium predicts that a social planner who wants to obtain equity in most problems, needs to give up the possibility to target specific divisions in problems in which there is a meaningful difference among equitable allocations.

One can relate the design limitations imposed by invariance properties to the multiplicity of Nash equilibria. Thus, it is informative to calculate the prediction of tremble-based refinements for these auctions and to evaluate the plausibility of these predictions. Consider any normal form refinement of Nash equilibrium that dismisses all Nash equilibria that involve an agent playing a weakly dominated action, e.g., trembling hand perfection (Selten, 1975), properness (Myerson, 1978), and undomination (Palfrey and Srivastava, 1991). These theories predict that the unique payoff-determinant bid in the winner-bid auction is \( v_l/2 \) and that the unique payoff-determinant bid in the loser-bid auction is \( v_h/2 \). As discussed in Sec. 2, there is plenty of empirical and experimental evidence showing that agents can persistently play weakly dominated actions in games. Experimental evidence suggests this is so for our particular partnership dissolution environment (Brown and Velez, 2018). However, even without this data challenging these predictions, the general introspection learned from other environments reveals the reasons why one can expect some weakly dominated actions can be persistently observed when these mechanisms are operated.

Consider the winner-bid auction. Let \( b_l > v_l/2 - 1 \). One can easily see
that \( b_l \) is weakly dominated by \( v_l/2 - 1 \). Utility maximization predicts that agent \( l \) bids \( b_l \) only if agent \( h \) never bids to the left of \( b_l \). Thus, it is tempting to think that agent \( l \), considering a small deviation by agent \( h \), will always preemptively bid \( v_l/2 - 1 \) instead. It is not plausible that this has to be always the case, however. This weakly dominated behavior is intuitively plausible when \( b_l \) is close to \( v_l/2 \). There, agent \( l \) is practically bearing no risk. Agent \( l \) would lose very little when agent \( h \) bids below \( b_l \). Consistent with monotonicity agent \( l \) can place more probability to the left of \( b_l \), but still bid \( b_l \) with positive probability.\(^{23}\) This is compounded with a second effect. Agent \( h \) would lose much more than agent \( l \) in case this last agent ends up getting the object and paying \( b_l \) for it. Thus, the probability that agent \( h \) bids close to \( v_l/2 \) should also be very small if agent \( l \) is consistently bidding to the right of \( v_l/2 \) with (enough) positive probability. Thus, not only agent \( l \) may not care much about the risk of buying for \( b_l \), but also if agent \( h \) is taking notice of this behavior, the probability that this happens is also very small, reinforcing the incentive of agent \( l \) to bid \( b_l \). Thus, one can make the case that it is plausible to observe bids accumulating in the interior of the Nash range. It is not clear whether equilibria on the right half of the Nash range can be approximated by this type of behavior, however. Thus, this intuitive analysis does not allow us to determine whether the winner-bid auction necessarily provides an advantage to the high valuation agent. In the next section we show that empirical equilibrium provides a sharp answer to this question.

### 4.4 Empirical equilibrium

#### 4.4.1 Empirical equilibrium payoffs

We characterize the set of empirical equilibrium payoffs of the extreme-price auctions. Recall that by Lemma 2, when valuations are equal, the payoff of each agent is the same in each Nash equilibrium of each extreme-price

\(^{23}\)There is experimental evidence that agent \( l \) consistently chooses buying instead of selling at prices higher than \( v_l/2 \). This is observed when an alternative sequential partnership dissolution mechanism known as divide and choose is operated. In divide and choose an agent proposes a transfer and the other selects to receive the object and do the transfer or to receive the transfer and no object. When the high value agent proposes a transfer \( v_l/2 + t \) where \( t \) is at most one third of the equity surplus, the low valuation agent chooses to receive the object and do the transfer with high probability (Brown and Velez, 2016). In divide and choose there are reciprocity effects that do not allow us to make a direct inference. However, one can imagine agent \( l \) not regretting buying at a price close, but higher than \( v_l/2 \).
auction. Thus, we only need to characterize the payoffs of empirical equilibria when agents’ valuations differ. Since Proposition 4 characterizes Nash equilibrium payoffs for these auctions, it is convenient to describe empirical equilibrium payoffs by the set of conditions for which a Nash equilibrium payoff is an empirical equilibrium payoff.

**Theorem 1.** Let \((M, \varphi)\) be the winner-bid auction, \(v \in \Theta\) such that \(v_l < v_h\), and \(\sigma \in N(M, \varphi, v)\). If \(\sigma\) is efficient, \(\pi(\sigma)\) is the payoff of an empirical equilibrium of \((M, \varphi, v)\) if and only if

1. \(\pi_h(\sigma) = v_h/2 + ES(v)\) when \(ES(v) = 1\), or \(ES(v) = 2\) and \(v_l > 2\);
2. \(\pi_h(\sigma) \geq v_h/2 + ES(v)/2 + v_l/4 - 1/2\) when \(ES(v) > 1\) and \(v_l \leq 3v_h/8\);
3. \(\pi_h(\sigma) \geq v_h/2 + 4ES(v)/5 - 4/5\) when \(ES(v) > 2\), \(v_l > 3v_h/8\), and \(v_l < 7v_h/12 - 7/6\);
4. \(\pi_h(\sigma) > v_h/2 + 4ES(v)/5 - 4/5\) when \(ES(v) > 2\), \(v_l > 3v_h/8\), and \(v_l \geq 7v_h/12 - 7/6\).

If \(\sigma\) is inefficient, \(\pi(\sigma)\) is the payoff of an empirical equilibrium of \((M, \varphi, v)\) except when \(ES(v) = 1\) and \(\pi_h(\sigma) < v_h/2 + 1/2\).

Theorem 1 reveals a surprising characteristic of the empirical equilibria of the winner-bid auction. For simplicity fix \(v_h\) at a certain value. Let \(v_l \leq v_h\). When \(v_l\) is low, i.e., at most \(3v_h/8\), the minimal share of the equity surplus that the higher value agent obtains in an empirical equilibrium is, essentially, at least 50% of the equity surplus (since we assumed \(v_l\) is a positive even number, the exact share depends on rounding, but is never less than 50%). More precisely, for this range of \(v_l\), agent \(h\) receives a payoff that is at least

\[
v_h/2 + ES(v)/2 + v_l/4 - 1/2 = v_h/2 + v_h/4 - 1/2.
\]

This means that \(p\) is the winner bid in an empirical equilibrium of the winner-bid auction for such valuations if and only if \(p \leq v_h/4 + 1/2\) (Fig. 2). Thus, while the maximal bid in an empirical equilibrium is the same for all valuations when \(v_l \leq 3v_h/8\), the minimal percentage of the equity surplus that is assigned to the higher value agent increases from essentially 50% when \(v_l = 2\) to essentially 80% when \(v_l = 3v_h/8\). For higher values of \(v_l\), i.e., \(3v_h/8 < v_l < v_h\), the higher value agent receives, essentially, at least 80% of the equity surplus (Fig. 2).

In summary, Theorem 1 states that the minimal share of the equity surplus that the higher value agent obtains in an empirical equilibrium of
the winner-bid auction depends on the number of possible bids that are to the left of the Nash range. In the extreme case in which there is only one bid to the left of the Nash range, the higher value agent essentially obtains at least 50% of the equity surplus. As the number of bids to the left of the Nash range increases, the minimal share of the equity surplus that is obtained by the higher value agent in an empirical equilibrium increases until it reaches essentially 80% when the number of possible bids to the left of the Nash range is 60% of the number of bids in the Nash range (equivalently, \(v_l \leq 3v_h/8\)). When the number of possible bids to the left of the Nash range is higher than 60% of the number of bids in the Nash range (equivalently, \(v_l > 3v_h/8\)), the minimal share of the equity surplus that is obtained by the higher value agent in an empirical equilibrium remains essentially 80%.

(For low values of the equity surplus, rounding has a significant effect; see Fig. 2).

**Theorem 2.** Let \((M, \varphi)\) be the loser-bid auction, \(v \in \Theta\) such that \(v_l < v_h\), and \(\sigma \in N(M, \varphi, v)\). If \(\sigma\) is efficient, \(\pi(\sigma)\) is the payoff of an empirical equilibrium of \((M, \varphi, v)\) if and only if

1. \(\pi_l(\sigma) \geq v_l/2 + ES(v)\) if \(ES(v) = 1\), or \(ES(v) = 2\) and \(v_h < \overline{p} - 1\);
2. \(\pi_l(\sigma) \geq v_l/2 + ES(v)/2 + (\overline{p} - v_h/2)/2 - 1/2\) if \(ES(v) > 1\) and \(v_h/2 \geq v_l/2 + 5(\overline{p} - v_l/2)/8\).
3. \(\pi_l(\sigma) \geq v_l/2 + 4ES(v)/5 - 4/5\) if \(ES(v) > 2, v_h/2 < v_l/2 + 5(\overline{p} - v_l/2)/8,\) and \(v_h/2 > \overline{p} - 7(\overline{p} - v_l/2)/24 - 7/12\);
4. \(\pi_l(\sigma) > v_l/2 + 4ES(v)/5 - 4/5\) if \(ES(v) > 2, v_h/2 < v_l/2 + 5(\overline{p} - v_l/2)/8,\) and \(v_h/2 \leq \overline{p} - 7(\overline{p} - v_l/2)/24 - 7/12\).

If \(\pi\) is inefficient, \(\pi\) is the payoff of an empirical equilibrium of \((M, \varphi, v)\) except when \(ES(v) = 1\) and \(\pi_l(\sigma) < v_l/2 + 1/2\).
The empirical equilibrium payoffs of the loser-bid auction are symmetric to those of the winner-bid auction. The minimal share of the equity surplus that the lower value agent obtains in an empirical equilibrium of the winner-bid auction depends on the number of possible bids that are to the right of the Nash range. In the extreme case in which there is only one bid to the right of the Nash range, the lower value agent essentially obtains at least 50% of the equity surplus. As the number of bids to the right of the Nash range increases, the minimal share of the equity surplus that is obtained by the lower value agent in an empirical equilibrium increases until it reaches essentially 80% when the number of possible bids to the right of the Nash range is 60% of the number of bids in the Nash range. When the
number of possible bids to the right of the Nash range is higher than 60% 
of the number of bids in the Nash range, the minimal share of the equity 
surplus that is obtained by the lower value agent in an empirical equilibrium 
remains essentially 80%. (For low values of the equity surplus, rounding has 
a significant effect; see Fig. 2).

4.4.2 The intuition

The characterization of empirical equilibrium payoffs of the extreme-price 
auctions reveals that empirical equilibrium makes a delicate selection of Nash 
equilibrria, which is sensitive to the global structure of the game. An empirical 
equilibrium may involve an agent playing a weakly dominated strategy. 
However, not all Nash equilibria, in particular not all Nash equilibria in 
which an agent plays a weakly dominated strategy, are empirical equilibria. Thus, empirical equilibrium somehow discriminates among actions by 
assessing how likely they can be played based on how they compare with 
the other actions. A discussion of the proof of Theorem 1 informs us about 
this feature of empirical equilibrium.

Let \( v \in \Theta \) be such that \( v_l < v_h \). Let \((M, \varphi)\) be the winner-bid auction and consider \( \sigma \in N(M, \varphi, v) \). By Proposition 3 this equilibrium is characterized 
by a winner bid \( p \in \{v_l/2, ..., v_h/2\} \). Suppose that \( p > v_l/2 \) and let \( t \equiv p - 1 - v_l/2 \). Suppose for simplicity that \( t \geq 1 \). One can easily see that agent 
\( l \) always regrets winning the auction when bidding \( p - 1 \). Indeed, bidding \( p - 1 \) 
is weakly dominated for agent \( l \) by each bid in \( \{v_l/2 - 1, v_l/2, ..., p - 2\} \). Let 
\( \sigma^\lambda \) be any monotone quantal response equilibrium of \((M, \varphi, v)\). It follows 
that for each \( b \in \{v_l/2 - 1, v_l/2, ..., p - 2\}, U_\varphi(\delta_h|\sigma^\lambda_l; v_l) \geq U_\varphi(\delta_{p-1}|\sigma^\lambda_l; v_l) \).

This means that there are always at least \( t + 2 \) bids that are at least as good 
as \( p - 1 \) for agent \( l \). This imposes a cap on the probability that this agent 
can place on \( p - 1 \). More precisely,

\[
\sigma^\lambda_l(p - 1) \leq 1/(t + 2). 
\] (2)

Thus, if \( \sigma \) is an empirical equilibrium, by (2), \( \sigma_l(p - 1) \) needs to be the limit 
of a sequence of probabilities bounded above by \( 1/(t + 2) \) and thus,

\[
\sigma_l(p - 1) \leq 1/(t + 2). 
\] (3)

Now, in order for \( \sigma \) to be a Nash equilibrium it has to be the case that agent 
\( h \) has no incentive to bid one unit less than \( p \). This is simply,

\[
v_h - p \geq \sigma_l(p - 1) \left[ \frac{1}{2}(v_h - (p - 1)) + \frac{1}{2}(p - 1) \right] + (1 - \sigma_l(p - 1))(v_h - (p - 1)).
\]
Equivalently, $\sigma_l(p - 1) \geq \frac{1}{(v_h/2 - p + 1)}$. Together with (3) this implies that $p \leq (v_l/2 + v_h/2)/2$, or equivalently

$$\pi_h(\sigma) \geq v_h/2 + ES(v)/2.$$ 

Thus, the reason why empirical equilibrium predicts agent $h$ gets at least half of the equity surplus for any possible valuation is that it is not plausible that agent $l$ will consistently bid above half of the Nash range. These high bids are worse than too many bids to their left for agent $l$. If this agent’s actions are monotone with respect to utility, the maximum probability that he or she will end up placing in a bid on the right half of the Nash range will never be enough to contain the propensity of agent $h$ to lower his or her bid.

When there is more than one bid to the left of the Nash range, the analysis becomes subtler. Let $y \equiv v_l/2 - 3t$. The key to complete this analysis is to prove that in any quantal response equilibrium of $(M, \varphi, v)$ that is close to $\sigma$, all bids $b \in \{\max\{0, y\}, \ldots, p - 1\}$ are weakly better than $p - 1$ for agent $l$. The subtlety here lies in that this is not implied directly by a weak domination relation as in our analysis above. In order to uncover this we need to recursively obtain estimates of agent $h$’s distribution of play, which in turn depend on agent $l$’s distribution of play.

Remarkably, we prove that the set of restrictions that we uncover by means of our analysis are the only ones that need to be satisfied by an empirical equilibrium payoff. Goeree et al. (2005) characterize the set of regular QRE of an asymmetric matching pennies game. At a conceptual level, this exercise is similar to the construction of empirical equilibria. However, the techniques developed by Goeree et al. (2005) are useful only in two-by-two games, where an agent’s distribution of play is described by a single real number. Thus, there is virtually no precedent in the construction of empirical equilibria that are not strict Nash equilibria for games with non-trivial action spaces. We do so as follows. First, we identify for each target payoff an appropriate Nash equilibrium that produces it, say $\sigma$. Then we take a convex combination of a perturbation of $\sigma$ and a logistic quantal response. This defines a continuous operator whose fixed points are the basis of our construction. In a two-limit process we first get close enough to $\sigma$ by placing a weight on it that is high enough so the fixed points of the convex combination operators inherit some key properties of $\sigma$. Then we allow the logistic response to converge to a best response and along this path of convergence we obtain interior distributions that are close to $\sigma$ and are payoff-monotone.
4.4.3 Invariance under Maskin monotonic transformations

The worst-case scenario incarnations of the traditional mechanism design paradigm are constrained by invariance properties that relate equilibria for different types. With complete information, the relevant property is the aforementioned invariance under Maskin monotonic transformations (Maskin, 1999). In our environment this property imposes the type of restrictions on the outcome correspondence of mechanisms stated in Lemma 3, i.e., guaranteeing equity when values are similar implies that when valuations differ most integer divisions of the equity surplus are possible.

It is evident that the empirical equilibrium correspondence of an extreme-price auction is free from these restrictions. Indeed, our analysis reveals that a social planner who accounts for the plausibility of equilibria, realizes that some social goals, which would be ruled out impossible by the traditional analysis, are within his or her reach. For instance, using the loser-bid auction could be sensible for a social planner who is able to exercise some level of affirmative action and chooses to benefit a segment of the population who are likely to have lower valuations for the objects to be assigned.

It is not accurate to simply say that the empirical equilibrium correspondence violates invariance under Maskin monotonic transformations, however. Empirical equilibria may be in mixed strategies or have random outcomes. More importantly, even for mechanisms for which the pure-strategy Nash equilibrium correspondence is well-defined, the pure-strategy deterministic-outcome empirical equilibrium correspondence may not be well-defined.

**Remark 1.** Let \((M, \varphi)\) be an extreme price auction and \(v \in \Theta\). If \(v_l = v_h\), no empirical equilibrium of \((M, \varphi, v)\) has a deterministic outcome.

Thus in order to make a formal statement we need to allow for random outcomes of an scc and accordingly extend the notion of invariance under Maskin monotonic transformations.

**Definition 4.** For each \(a \in A\) and \(\{v, w\} \subseteq \Theta\), \(w\) is a Maskin monotonic transformation of \(v\) at \(a\) if \(w_i \geq v_i\) when agent \(i\) receives the object at \(a\) and \(w_i \leq v_i\) otherwise. Let \(f : \Theta \rightarrow \Delta(A)\). Then, \(f\) is invariant under Maskin monotonic transformations if for each \(v \in \Theta\), each \(a \in A\) such that \(\delta_a \in f(v)\), and each Maskin monotonic transformation of \(v\) at \(a\), say \(w \in \Theta\), we have that \(\delta_a \in f(w)\).

Consider, for instance, \(v \in \Theta\) such that \(v_l < v_h\) and an efficient equilibrium of an extreme-price auction for \(v\), say \(\sigma\). By Proposition 3 this
equilibrium is characterized by a payoff-determinant bid in the Nash range $p \in \{v_l/2, ..., v_h/2\}$. Indeed when $\sigma$ is efficient, agent $h$ receives the object and pays $p$ to the other agent with certainty. One can prove that the Nash equilibrium outcome correspondence of the extreme-price auctions are invariant under Maskin monotonic transformations.\footnote{Our definition of invariance under Maskin monotonic transformations imposes invariance restrictions for equilibria in mixed strategies that generate a deterministic outcome. For this reason our statement here requires a proof, which can be completed along the lines of Lemma 3.} An obvious implication of this property is that if $p$ is in the Nash range for valuations $v$, it is also in the Nash range for valuations $(w_l, v_h)$ where $w_l \leq v_l$. This can be evaluated in Fig. 2 for the winner-bid auction, by simply checking that if $p$ is a payoff-determinant bid for some profile, it must be a payoff-determinant bid when $v_l$ decreases.\footnote{Fig. 2 illustrates empirical equilibrium payoffs for models with different $\overline{p}$. For the winner-bid auction, the equilibria that sustain the payoff determinate bids shown in the figure are available in all these models.} Notice then that the empirical equilibrium correspondence of the winner-bid auction is not invariant under Maskin monotonic transformations: There are triangles in Fig. 2 that have no triangle below them. A symmetric argument can be done for the loser-bid auction.

**Remark 2.** Let $(M, \varphi)$ be an extreme-price auction. The correspondence $\theta \in \Theta \mapsto \{\varphi(\sigma) \in \Delta(A) : \sigma$ is an empirical equilibrium of $(M, \varphi, v)\}$, is not invariant under Maskin monotonic transformations.

Two additional observations are worth noting. First, for a best-case mechanism designer, who bases his or her analysis on the so-called partial implementation theory, empirical equilibrium analysis brings some obvious challenges that are beyond the scope of this paper. Velez and Brown (2018) report good news for the best-case robust mechanism designer, however. Second, empirical equilibrium analysis also allows us to conclude that it is not without loss of generality to restrict our attention to pure-strategy equilibria when a mechanism is operated.

**Remark 3.** Let $v \in \Theta$ such that $v_l < v_h$. The only pure-strategy empirical equilibrium of the winner-bid auction for $v$ is $\sigma_l = \delta_{v_l/2-1}$ and $\sigma_h \equiv \delta_{v_l/2}$. The only pure-strategy empirical equilibrium of the loser-bid auction for $v$ is $\sigma_l \equiv \delta_{v_h/2}$ and $\sigma_h = \delta_{v_h/2+1}$.

Jackson (1992) constructed examples in which including arguably plausible mixed-strategy equilibria in a worst-case scenario analysis would reverse the conclusions one obtains by only analyzing pure-strategy equilibria. Empirical equilibrium analysis goes beyond these observations and provides a
clear framework in which plausibility is built into the prediction of agents’
behavior. It is fair to say then that while empirical mechanism design opens
new possibilities in the design of economic institutions, it also sets the
standards of analysis high by forcing us to consider mixed-strategy equi-
libria. In this context, our complete characterization of empirical equilibria
of extreme-price auctions in Theorems 1 and 2, and the characterization
results in Velez and Brown (2018), show that the technical challenges can
be resolved in policy relevant environments.

4.4.4 Mechanism bias

A mechanism designer who does not account for empirical plausibility of
Nash equilibria may inadvertently bias assignments. For instance, based on
the Nash equilibrium prediction both extreme-price auctions span the whole
range of equity surplus distributions (Proposition 4). However, empirical
equilibrium analysis reveals they consistently select opposite extremes of
the Nash range. This bias could be a deliberate policy choice of a social
planner, as we pointed out above. However, if it is not, it may compromise
the legality or popular support of the institution.

Imagine, for instance, that affirmative action is forbidden by law. The
operation of an extreme-price auction is vulnerable to a legal challenge in
this environment. A critic of this system can present experimental data doc-
umenting its implicit bias (c.f., Brown and Velez, 2018). Most importantly,
empirical equilibrium analysis shows that the realized bias in laboratory ex-
periments is the result of the underlying structure of the system, and not
of the particular circumstances in which experiments were realized. As long
as agents respond to pecuniary incentives, and exhibit noisy behavior that
is guided by ordinal rationality and is self sustaining—two features either
found or implicitly assumed in nearly all empirical economic analyses—the
bias will be present in the operation of these mechanisms.

5 Discussion

We believe that our definition of empirical equilibrium strikes a balance
between optimistic and pessimistic approaches to implementation theory.
There are two components of our definition. The first is payoff-monotonicity.
It turns out that one could have endorsed the regular QRE model or weak
payoff-monotonicity as a basis for empirical plausibility and arrive at the same
definition (Sec. 4.2). One could argue that weak-payoff-monotonicity is still
a considerable restriction on behavior. For instance, it implies that actions
that give the agent equal utility must be played with equal probability. Thus, it is natural to ask whether our results crucially depend on such sharp implications of weak-payoff-monotonicity. Our proofs reveal that the main message of empirical equilibrium analysis is preserved when empirical plausibility is based on even weaker forms of payoff-monotonicity. For instance, consider the assumption that there is \( \alpha \in (0, 1) \) such that an agent plays an action with at least \( \alpha \) times the probability of any action that gives no higher expected payoff. Indeed, the basic result that a mechanisms designer is not constrained by typical invariance properties, is preserved for any such \( \alpha \). Moreover, for reasonably low values of \( \alpha \) one still obtains comparative statics for payoffs and average bids. When \( v_l \geq v_h/2 \), one can check that for \( \alpha \geq (1/2)(ES(v)/(ES(v) - 1)) \), the winning bid in an “\( \alpha \)-payoff-monotonicity-based empirical equilibrium” of the winner-bid auction is on the left half of the Nash range. A symmetric statement holds for the loser-bid auction.

The second component of our definition of empirical equilibrium is convergence. Again we are striking a balance between approaches to implementation theory. On the one hand, one can require that convergence be the result of increasing rationality. This would require us to explicitly model sophistication. One alternative is to use the regular QRE model which provides an alternative equivalent to payoff-monotonicity (Proposition 2). If one insists on empirically plausible equilibria to be the limits of behavior as agents become more sophisticated, one can require that the equilibrium be the limit of regular QRE for a sequence of regular quantal response functions that in the limit only admit best responses as fixed points. It is an open question whether this alternative approach and ours coincide. The following proposition provides a partial answer to this question. It states conditions guaranteeing an empirical equilibrium is the limit of regular QRE for regular quantal response functions that admit only best responses as fixed points in the limit.

---

There are also particular phenomena in games that is difficult to model parsimoniously and may induce violations of weak-payoff-monotonicity under which the main message of our results is preserved. For instance, in the partnership dissolution environment we study, where agents choose numbers, it is common that agents round their bids to the nearest multiple of five (Brown and Velez, 2018).

Let \( p \) the winning bid in such an equilibrium, and \( \sigma^\lambda \) a distribution satisfying the \( \alpha \) modified version of payoff-monotonicity. An argument as in the proof of Theorem 1 shows that agent \( i \)'s expected utility of bids \( v_i/2 - (p - 1 - v_i/2) - 1, \ldots, p - 1 \) is at least the expected utility of \( p - 1 \). This is independent of \( \sigma^\lambda \). Thus, \( \sigma^\lambda(p - 1) \leq 1/(1 + (2(p - 1 - v_i/2) + 1)\alpha) \). This equation jointly with the equilibrium condition \( \sigma(p - 1) \geq 1/(v_h/2 - p + 1) \) allow one to find the bound on \( \alpha \) guaranteeing \( \sigma \leq (v_i/2 + v_h/2)/2 \).
Proposition 5. Let \( v \in \Theta \), \((M, \varphi)\) a finite message space mechanism, and \( \sigma \in N(M, \varphi, v) \). Suppose that there is a sequence of interior payoff-monotone distributions \( \{\sigma^\lambda\}_{\lambda \in \mathbb{N}} \) that converges to \( \sigma \) as \( \lambda \to \infty \). Suppose also that for each \( \lambda \in \mathbb{N} \), each \( i \in N \), and each pair \( \{m_i, n_i\} \subseteq M_i \) outside the support of \( \sigma_i \),

\[
\frac{\sigma^\lambda_i(m_i)}{\sigma^\lambda_i(n_i)} = e^{\lambda U_\varphi(\delta_{m_i} | \sigma^\lambda_{-i}, v_i)}.
\]

Then, there is a subsequence of \( \{\sigma^\lambda\}_{\lambda \in \mathbb{N}} \) which we denote again by \( \{\sigma^\lambda\}_{\lambda \in \mathbb{N}} \), such that there is a sequence of regular quantal response functions \( \{p^\lambda\}_{\lambda \in \mathbb{N}} \) such that for each \( \lambda \in \mathbb{N} \), \( \sigma^\lambda \) is a quantal response equilibrium of \((M, \varphi, v)\) with respect to \( p^\lambda \). Furthermore, if for each \( \lambda \in \mathbb{N} \), \( \gamma^\lambda \) is a quantal response equilibrium of \((M, \varphi, v)\) with respect to \( p^\lambda \) and as \( \lambda \to \infty \), \( \gamma^\lambda \to \gamma \), we have that \( \gamma \in N(M, \varphi, v) \).

It is indeed reassuring to see an explicit process by which behavior converges to an equilibrium. Because of this, we have made explicit in our proofs when the sequences we construct satisfy the conditions in Proposition 5 (the exceptions are the equilibria with interior price-determining bid when \( v_l \) is large). However, we restrain from requiring increasing sophistication as a necessary condition for empirical plausibility. We prefer that empirical equilibrium, as its name suggests, be based only on observables. If behavior is converging to a uniform distribution on all best responses for an agent, it is difficult to see how one can conclude, from data, that the agent is getting more sophisticated.

On the other hand, we are requiring convergence, and this seems to require a leap of faith, for which there is some mixed evidence (c.f., McKelvey and Palfrey, 1995). Technically, we are not claiming that there is universal evidence for convergence and that this will always eventually happen.\(^{28}\) We are finding the consequences for implementation theory of the assumption that this happens. If we examine the objectives and the evolution of mechanism design we can come to terms with this approach. The current paradigm is that we should evaluate a mechanism with the assumption that Nash equilibrium, more precisely that the expected utility rational agents model that entails Nash equilibria, is a good prediction when the mechanism is operated. Data suggests that this prediction is at most approximately correct, however. So our proposal is the gradual departure that assumes behavior will eventually accumulate around a Nash equilibrium. This is already a more

\(^{28}\)This obviously requires that agents react to pecuniary incentives, which is not observed in some laboratory experiments.
realistic foundation for an implementation theory. Of course, one would like to have a better explanation of how and why convergence happens. With this we could get an idea of how to design institutions that guarantee convergence and what to do when this is not possible at all. Whatever satisfying answer to this question we find has to be consistent with data. Thus, empirical equilibria is giving us a window to look at part of the future answer. Alternatively, we could think of empirical equilibrium as being a tractable proxy for mechanism design with boundedly rational agents who exhibit behavior that is noisy, guided by rationality, and self-sustaining. By targeting the limits of this type of self-sustaining behavior we uncover regularities that allow us to discriminate among mechanisms, a challenge that had defied economic theory (see Sec. 10.1.2 in Goeree et al., 2016).

There are obvious challenges, beyond the scope of this paper, that are left open for the design of economic institutions based on empirical equilibrium. Ideally, one would like to have necessary and sufficient conditions on a (probabilistic) social choice correspondence guaranteeing the existence of a mechanism whose empirical equilibrium correspondence, is a selection of, or coincides with it. Some partial answers to this broad question may be easier to obtain. For instance, it would be interesting to determine classes of Nash equilibria that are empirical equilibria. Proposition 2 implies that all strict Nash equilibria are empirical equilibria (Tumennasan (2013) proves that strict equilibria are limits of logistic QRE). Other classes for which it is plausible the analysis could be advanced are undominated Nash equilibria and equilibria in which each agent’s best response has support on the full set of maximizers. It is plausible that the method to construct empirical equilibria that we develop in the proof of Theorem 1, can be applied to other environments of interest.

We have concentrated on implementation theory, the worst-case scenario incarnation of mechanism design. It is evident that empirical equilibrium analysis has implications for other mechanism design approaches. In a companion paper, Velez and Brown (2018), we use empirical equilibrium analysis, in a general incomplete information setting, to further our understanding of an agent’s incentive to choose a dominant strategy, an issue of current interest (c.f. Li, 2017). Our analysis there produces significant news for the design of robust mechanisms à la Bergemann and Morris (2005, 2011). It would be interesting to analyze empirical equilibrium on Bayesian environments and determine its implications for both best-case and worst-case scenario analysis for particular information structures of interest.

Finally, it would also be interesting to generalize the empirical mechanism design approach to sequential mechanisms. An obvious place to start
is the generalizations of quantal response equilibria to these types of games (see Goeree et al., 2016, for a survey).

6 Appendix

For $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the floor of $x$, i.e., the greatest integer that is less than or equal to $x$; $\lceil x \rceil$ denotes the ceiling of $x$, i.e., the smallest integer that is greater than or equal to $x$.

**Proof of Proposition 2.** Payoff-monotone distributions are weakly-payoff monotone. Let $(M, \varphi)$ be a finite message space mechanism and $v \in \Theta$. It is enough to prove that if $\mu$ is weakly-payoff-monotone for $(M, \varphi, v)$, for each $\varepsilon > 0$ there is $\gamma > 0$ that is payoff-monotone for $(M, \varphi, v)$ such that $||\mu - \gamma|| < \varepsilon$ (note that a byproduct of this proof is that an empirical equilibrium is always the limit of interior payoff-monotone distributions). Suppose that $\mu$ is weakly-payoff-monotone for $(M, \varphi, v)$. Let $\lambda > 0$. For each $i \in N$, each $\zeta \in (0, 1)$, and each profile of distributions $\beta \in \Delta(M)$, let

$$f^\zeta(\beta) \equiv (1 - \zeta)\mu_i + \zeta\lambda^1((U_\varphi(\delta_{m_i}|\mu_{-i}; v_i))_{m_i \in M_i}).$$

Let $\gamma^\zeta$ be a fixed point of $f^\zeta$, that exists because $f^\zeta$ is continuous. Let $\{m_i, n_i\} \subseteq M_i$. If $\mu_i(m_i) = \mu_i(n_i)$, then

$$1_{\mu_i}(U_\varphi(\delta_{m_i}|\gamma^\zeta_{-i}; v_i))_{m_i \in M_i} \geq 1_{\mu_i}(U_\varphi(\delta_{m_i}|\gamma^\zeta_{-i}; v_i))_{m_i \in M_i},$$

if and only if $U_\varphi(\delta_{m_i}|\gamma^\zeta_{-i}; v_i) \geq U_\varphi(\delta_{n_i}|\gamma^\zeta_{-i}; v_i)$. That is, $U_\varphi(\delta_{m_i}|\gamma^\zeta_{-i}; v_i) \geq U_\varphi(\delta_{n_i}|\gamma^\zeta_{-i}; v_i)$ if and only if $\gamma^\zeta(m_i) \geq \gamma^\zeta(n_i)$. Suppose then that $\mu_i(m_i) > \mu_i(n_i)$. Since $\mu$ is weakly-payoff-monotone for $(M, \varphi, v)$, $U_\varphi(\delta_{m_i}|\mu_{-i}; v_i) > U_\varphi(\delta_{n_i}|\mu_{-i}; v_i)$. Since as $\zeta \to 0$, $\gamma^\zeta \to \mu$, there is $c > 0$ such that for each $\zeta < c$, $\gamma^\zeta(m_i) > \gamma^\zeta(n_i)$ and $U_\varphi(\delta_{m_i}|\gamma^\zeta_{-i}; v_i) > U_\varphi(\delta_{n_i}|\gamma^\zeta_{-i}; v_i)$. Thus, for each pair $\{m_i, n_i\} \subseteq M_i$, there is $c > 0$ such that for each $\zeta < c$, $U_\varphi(\delta_{m_i}|\gamma^\zeta_{-i}; v_i) > U_\varphi(\delta_{n_i}|\gamma^\zeta_{-i}; v_i)$ if and only if $\gamma^\zeta(m_i) \geq \gamma^\zeta(n_i)$. Since $(M, \varphi)$ has finite message spaces, there is $c > 0$ such that for each $\zeta < c$ and each pair $\{m_i, n_i\} \subseteq M_i$, $U_\varphi(\delta_{m_i}|\gamma^\zeta_{-i}; v_i) > U_\varphi(\delta_{n_i}|\gamma^\zeta_{-i}; v_i)$ if and only if $\gamma^\zeta(m_i) \geq \gamma^\zeta(n_i)$.

Thus, $\sigma \in N(M, \varphi, v)$ is the limit of a sequence of weakly-payoff-monotone distributions for $(M, \varphi, v)$ if and only if it is the limit of a sequence of interior payoff-monotone distributions for $(M, \varphi, v)$. □

**Proof of Propositions 1 and 5.** Let $\sigma \in N(M, \varphi, v)$ be an empirical equilibrium. By our proof of Proposition 2, $\sigma$ is the limit of a sequence
of interior payoff-monotone distributions \( \{\sigma^\lambda\}_{\lambda \in \mathbb{N}} \). Let \( i \in \mathbb{N} \). By passing to a subsequence if necessary we can assume without loss of generality that for each \( \lambda \in \mathbb{N} \), \( \sigma^\lambda(m_1) \leq \sigma^\lambda(m_2) \leq \cdots \leq \sigma^\lambda(m_{|M|}) \). Since as \( \lambda \to \infty \), \( \sigma^\lambda \to \sigma \), there is \( \Lambda \in \mathbb{N} \) such that for each \( \lambda \geq \Lambda \), if \( m_i \in M_i \) is in the support of \( \sigma_i \) and \( n_i \in \mathbb{N} \) is not, then \( \sigma^\lambda_i(n_i) < \sigma^\lambda_i(m_i) \). Thus, by passing to a subsequence if necessary, suppose without loss of generality that the support of \( \sigma_i \) is \( \{m_k, m_{k+1}, \ldots, m_{|M_i|}\} \). Since as \( \lambda \to \infty \), \( \sigma^\lambda \to \sigma \), for each \( m_i \in M_i \), \( U_\varphi(\delta_{m_i}|\sigma^\lambda_{-i}; v_i) \to U_\varphi(\delta_{m_i}|\sigma_{-i}; v_i) \). Since \( \sigma \in N(M, \varphi, v) \), for each \( m_i \) in the support of \( \sigma_i \), \( U_\varphi(\delta_{m_i}|\sigma^\lambda_{-i}; v_i) \to \overline{v} \) where \( \overline{v} \) is the expected payoff of a best response to \( \sigma_{-i} \) for \( i \). Thus, there is a sequence of positive real numbers \( \{\epsilon_\lambda\}_{\lambda \in \mathbb{N}} \) that converges to zero and such that for each \( \lambda \in \mathbb{N} \) and each \( m_i \) in the support of \( \sigma_i \), \( |U_\varphi(\delta_{m_i}|\sigma^\lambda_{-i}; v_i) - \overline{v}| < \epsilon_\lambda \).

Suppose that there is at least one element in \( M_i \) that is not in the support of \( \sigma_i \). Let \( \lambda \in \mathbb{N} \) and

\[
K = \frac{e^{\lambda U_\varphi(\delta_{m_{k-1}}|\sigma^\lambda_{-i}; v_i)}}{\sigma^\lambda_i(m_{k-1})}.
\]

For each \( m_i \in M_i \), let \( y^\lambda(m_i) \equiv \ln(K\sigma^\lambda_i(m_i))/\lambda \), (both \( K \) and \( y^\lambda \) are well-defined because \( \sigma^\lambda_i \) is interior). By definition of \( K \), \( y^\lambda(m_{k-1}) = U_\varphi(\delta_{m_{k-1}}|\sigma^\lambda_{-i}; v_i) \).

Since \( \sigma^\lambda \) is payoff-monotone, and since \( \ln \) is a strictly increasing function, we have that for each pair \( \{m_i, n_i\} \subseteq M_i \), \( y^\lambda(m_i) \geq y^\lambda(n_i) \) if and only if \( U_\varphi(\delta_{m_i}|\sigma^\lambda_{-i}; v_i) \geq U_\varphi(\delta_{n_i}|\sigma^\lambda_{-i}; v_i) \). Thus, one can construct a strictly increasing differentiable function \( f^\lambda : \mathbb{R} \to \mathbb{R} \) such that for each \( m_i \in M_i \), \( f^\lambda(U_\varphi(\delta_{m_i}|\sigma^\lambda_{-i}; v_i)) = y^\lambda(m_i) \).

For each \( \lambda \in \mathbb{N} \), let \( p^\lambda_i \) be composition of \( f^\lambda \) and the logistic quantal response function, i.e., the function that assigns to each \( x \in \mathbb{R}^{M_i} \) the distribution \( \rho^\lambda_i((f^\lambda(x_{m_i}))|_{m_i \in M_i}) \). Since \( f^\lambda \) is strictly increasing and differentiable, \( p^\lambda \) is differentiable, strictly monotone, interior, and responsive. Now, for each \( \lambda \in \mathbb{N} \), and each \( i \in \mathbb{N} \),

\[
p^\lambda_i(U_\varphi(\delta_{m_i}|\sigma^\lambda_{-i}; v_i)|_{m_i \in M_i}) = \frac{e^{\lambda \ln(K\sigma^\lambda_i(m_i))/\lambda}}{\sum_{m_i \in M_i} e^{\lambda \ln(K\sigma^\lambda_i(m_i))/\lambda}} = \sigma^\lambda_i(m_i).
\]

Thus, \( \sigma^\lambda \) is a quantal response equilibrium of \( (M, \varphi, v) \) with respect to \( p^\lambda \).

Suppose now that the additional hypothesis in Proposition 5. Then, for each \( l < k - 1 \),

\[
K\sigma^\lambda_i(m_l)e^{\lambda U_\varphi(\delta_{m_{k-1}}|\sigma^\lambda_{-i}; v_i)} = K\sigma^\lambda_i(m_{k-1})e^{\lambda U_\varphi(\delta_{m_l}|\sigma^\lambda_{-i}; v_i)}.
\]

Equivalently,

\[
y^\lambda(m_l) + U_\varphi(\delta_{m_{k-1}}|\sigma^\lambda_{-i}; v_i) = y^\lambda(m_{k-1}) + U_\varphi(\delta_{m_l}|\sigma^\lambda_{-i}; v_i).
\]

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Thus, $y^\lambda(m_i) = U_\varphi(\delta_{m_i} | \sigma^\lambda_{-i}, v_i)$. Thus, one can construct $f$ so it has slope one outside the interval $(\overline{u} - \varepsilon_\lambda, \overline{u} + \varepsilon_\lambda)$. Let $\gamma^\lambda$ be a quantal response equilibrium of $(M, \varphi, v)$ with respect to $p^\lambda$ and suppose that as $\lambda \to \infty$, $\gamma^\lambda \to \gamma$. We claim that $\gamma \in N(M, \varphi, v)$. Let $x^\lambda \equiv (x^\lambda_i)$ be the list of payoff vectors in $\gamma^\lambda$. Since as $i \leq \lambda$, $\gamma^\lambda \to \gamma$, $x^\lambda \to x$ where $x \equiv (x_i)_{i \in \mathbb{N}}$ is the list of payoff vectors in $\gamma$. Suppose that $x_{im_k}$ is a maximum component of $x_i$ and $x_{im_l}$ is not. There is $\Lambda \in \mathbb{N}$ such that for each $\lambda \geq \Lambda$, $\varepsilon_\lambda/2 < |f^\lambda(x_{im_l}) - f^\lambda(x_{im_k})|/2$. Since $f^\lambda$ has slope one outside an interval of length $\varepsilon_\lambda$, $f^\lambda(x_{im_l}) - f^\lambda(x_{im_k}) \geq (x_{im_k} - x_{im_l})/2$. Thus,

$$0 \leq \frac{p^\lambda_{im_k}(x^\lambda_i)}{p^\lambda_{im_l}(x^\lambda_i)} = \frac{e^{\lambda f(x_{im_l})}}{e^{\lambda f(x_{im_k})}} \leq e^{-\lambda(x_{im_k} - x_{im_l})/2} \to 0.$$

Thus, as $\lambda \to \infty$, $\gamma^\lambda_i(m_i) = p^\lambda_{im_k}(x^\lambda_i) \to 0$.

**Proof of Proposition 3.** In any of the auctions, by bidding $c_h \equiv v_h/2$ the high-value agent guarantees a payoff at least $v_h - c_h = c_h$. By bidding $c_l \equiv v_l/2$ the low-value agent guarantees a payoff at least $c_l = v_l - c_l$. Thus, in a Nash equilibrium the high-value agent’s payoff is at least $c_h$ and the low-value agent’s payoff is at least $c_l$.

We prove the proposition for the *winner-bid auction*, which we denote by $(M, \varphi)$. The proof for the *loser-bid auction* is symmetric. Let $\gamma$ be the probability with which an agent with the high value gets the object when there is a tie. We prove our proposition for the slightly more general mechanism in which $\gamma \geq 1/2$. The alternative tie breaker $\gamma = 1$ may be relevant in experimental settings.

Let $\sigma \in N(M, \varphi, v)$ with $v_l < v_h$. Let $p$ be in the support of $\sigma_l$. We claim that $p \leq c_h$. Suppose without loss of generality that $p$ is the maximal element in the support of $\sigma_l$. Suppose by contradiction that $p \geq c_h + 1$. Since $p > c_l$ and $\gamma > 0$, the expected payoff of any bid $b > p$ for the low-value agent is strictly lower than the expected payoff of $p$. Thus, there is no $b > p$ in the support of $\sigma_l$. Since $p - 1 \geq c_h$,

$$U_\varphi(\delta_{p-1}|\sigma_l; v_h) - U_\varphi(\delta_{p}|\sigma_l; v_h) \geq (1 - \sigma_l(x) - \sigma_l(x - 1)) + \sigma_l(p - 1)(\gamma + (1 - \gamma)((p - 1) - (v_h - p))) + \sigma_l(p)(p - (v_h - p))$$

$$= (1 - \sigma_l(x) - \sigma_l(x - 1)) + \sigma_l(p - 1)(2\gamma - 1 + 2(1 - \gamma)(p - c_h)) + 2\sigma_l(p)(p - c_h) > 0,$$

where the last inequality holds because $\sigma_l$ is a probability distribution, $\gamma \geq 1/2$, and $p - c_h \geq 1$. This contradicts $p$ is in the support of $\sigma_h$.
We claim now that \( p \geq c_l \). Suppose without loss of generality that \( p \) is the minimal element in the support of \( \sigma_h \). Suppose by contradiction that \( p \leq c_l - 1 \). We claim that there is no \( b < p \) in the support of \( \sigma_l \). Suppose by contradiction there is \( b < p \) in the support of \( \sigma_l \). Since \( c_l - p \geq 1 \) and \( b < p \),

\[
U_\varphi(\delta_{p+1} | \sigma_h; v_l) - U_\varphi(\delta_p | \sigma_h; v_l) \geq \sigma_h(p)(v_l - (p+1) - p) = \sigma_h(p)(2(c_l - p) - 1) > 0.
\]

This contradicts \( b \) is in the support of \( \sigma_l \). We claim that \( \sigma_l(p) = 0 \). Suppose by contradiction that \( \sigma_l(p) > 0 \). Then,

\[
U_\varphi(\delta_{p+1} | \sigma_l; v_l) - U_\varphi(\delta_p | \sigma_l; v_l) \geq \sigma_h(p) (\gamma(v_l - (p + 1) - p) + (1 - \gamma)(-1))
\]

\[
= \sigma_h(p) (\gamma(2(c_l - p) - 1) - (1 - \gamma)) \geq 0.
\]

If the inequality above holds strictly, there is a contradiction to \( \sigma_l(p) > 0 \). Since \( \sigma_h(p) > 0 \), \( \gamma \geq 1/2 \), and \( c_l - p \geq 1 \), the expression above is equal to zero only when \( \gamma = 1/2 \) and \( p = c_l - 1 \). Suppose then that \( \gamma = 1/2 \) and \( p = c_l - 1 \). Since for each \( b < p \), \( \sigma_l(b) = 0 \), we have that

\[
U_\varphi(\delta_{p+1} | \sigma_l; v_h) - U_\varphi(\delta_p | \sigma_l; v_h) \geq \frac{1}{2}\sigma_l(p)(-1) + \frac{1}{2}\sigma_l(p)(v_h - c_l - (c_l - 1))
\]

\[
= \sigma_l(p)(c_h - c_l) > 0.
\]

This contradicts \( \sigma_h(p) > 0 \). Thus far we have proved that the support of \( \sigma_l \) belongs to \( (p, +\infty) \). Let \( b \) be the minimum element of the support of \( \sigma_l \). Thus, \( p < b \). If \( b < c_h \), since \( \gamma > 0 \), agent \( h \) would benefit by bidding \( b \) instead of \( p \). Thus, \( b \geq c_h \). Recall that the support of \( \sigma_h \) belongs to \( (-\infty, c_h) \). Since the expected payoff of \( l \) is at least \( c_l \), \( b \leq c_h \), for otherwise agent \( l \), when bidding \( b \), would receive the object with probability one and pay the other agent more than \( c_h > c_l \). Thus, \( b = c_h \). Thus, agent \( l \) would benefit by bidding \( b - 1 \) instead of \( b \), because \( c_h > c_l \) and \( \sigma_h(p) > 0 \). This contradicts \( c_h \) is in the support of \( \sigma_l \). Thus, the support of \( \sigma_l \) is empty. This is a contradiction. Thus, the support of \( \sigma_h \) belongs to \( [c_l, c_h) \).

Let \( p \) be the minimum element of the support of \( \sigma_h \) and \( b \) an element in the support of \( \sigma_l \). We claim that \( b \leq p \). Suppose by contradiction that there is \( b > p \) in the support of \( \sigma_l \). Since \( p \geq c_l \), \( b > c_l \). Thus, since \( \sigma_h(p) > 0 \), agent \( l \) benefits by bidding \( c_l \) instead of \( b \). This contradicts \( b \) is in the support of \( \sigma_l \). Thus, the support of \( \sigma_l \) belongs to \( (-\infty, p) \).

Finally, let \( p \) be the minimum of the support of \( \sigma_h \). If \( \gamma = 1/2 \) and \( \sigma \) is efficient, the support of \( \sigma_l \) belongs to \( \{0, ..., p - 1\} \). Thus, \( \sigma_h = \delta_p \), i.e., agent \( h \) receives the object and pays \( p \) to the other agent. Suppose now that \( \sigma \) is inefficient, i.e., \( \sigma_l(p) > 0 \) and \( \gamma = 1/2 \). We claim that \( p = c_l \) and \( \sigma_l(p) < 1/(c_h - c_l) \). Recall that, \( p \in [c_l, c_h] \). Since \( \sigma_l(p) > 0 \),

\[
0 \geq U_\varphi(\delta_{p-1} | \sigma_h; v_l) - U_\varphi(\delta_p | \sigma_h; v_l) = \sigma_h(p)[p - ((v_l - p)/2 + p/2)]
\]

\[
= \sigma_h(p)(p - c_l).
\]
Since $\sigma_h(p) > 0$, $p = c_l$. Recall that agent $l$ has guaranteed $v_l/2$ in each Nash equilibrium of $(M, \varphi, v)$. If the maximum of the support of $\sigma_l$ is $c_l$, then agent $h$ gets in equilibrium at least what he or she would get by bidding $c_l + 1$. This bid gives agent $h$ an expected payoff of $v_h - (v_l/2 + 1)$. Thus, the aggregate expected payoff is at least $v_h - 1$. \hfill \Box

**Proof of Proposition 4.** We prove the proposition for the winner-bid auction. The proof for the loser-bid auction is symmetric. Let $\sigma \in N(M, \varphi, v)$. By Proposition 3, there is $p \in \{v_l/2, \ldots, v_h/2\}$ that is in the support of $\sigma_h$ such that the support of $\sigma_l$ belongs to $\{0, \ldots, p\}$ and the support of $\sigma_h$ belongs to $\{p, \ldots, \overline{p}\}$. Suppose first that $\sigma$ is efficient. Thus, support of $\sigma_l$ belongs to $\{0, \ldots, p - 1\}$. Thus, $\sigma_h = \delta_p$. Thus, $\pi_l(\sigma) = v_l/2 + t$ and $\pi_h(\sigma) = v_h/2 + ES(v) - t$ for some $t \in \{0, 1, \ldots, ES(v)\}$. One can easily see that for each $t \in \{0, 1, \ldots, ES(v)\}$ the distributions $\sigma_l = \delta_{p-1}$, $\sigma_h = \delta_p$ with $p = v_l/2 + t$ is a Nash equilibrium. Suppose now that $\sigma$ is inefficient. By Proposition 3, $\pi_l(\sigma) = v_l/2$ and $\pi_h(\sigma) \geq v_h/2 + ES(v) - 1$. Since $\sigma$ is inefficient $\pi_h(\sigma) < v_h/2 + ES(v)$. Thus, there is $\varepsilon \in (0, 1]$ such that $\pi_h(\sigma) = v_h/2 + ES(v) - \varepsilon$. Finally, let $\varepsilon \in (0, 1]$ and $\alpha = \varepsilon/ES(v)$. Consider the distributions $\sigma_l = (1 - \alpha)\delta_{v_l/2-1} + \alpha\delta_{v_l/2}$ and $\sigma_h = \delta_{v_l/2}$. Direct calculation yields, $U_{\varphi}(\delta_{v_l/2}\sigma_l; v_h) - U_{\varphi}(\delta_{v_l/2+1}\sigma_l; v_h) = 1 - \alpha ES(v)$. Since $\alpha \in (0, 1/ES(v)]$, agent $h$ weakly prefers to bid $v_l/2$ than $v_l/2 + 1$. One can easily see that this implies $v_l/2$ is a best response for $h$ to $\sigma_l$. One can also easily see that $\sigma_l$ is a best response to $\sigma_h$ for $l$. Moreover, $\pi_h(\sigma) = U_{\varphi}(\delta_{v_l/2}\sigma_l^*; v_h) = v_h/2 + ES(v) - \alpha ES(v) = v_h/2 + ES(v) - \varepsilon$ and $\pi_l(\sigma) = v_l/2$. \hfill \Box

**Proof of Lemma 3.** $(M, \varphi)$ be a mechanism satisfying the property in the statement of the lemma. Let $v \in \Theta$, $0 \leq t \leq ES(v)$, and $a \in A$ an allocation at which $h$’s payoff is $v_h/2 + t$ and $l$’s payoff is $v_l/2 + ES(v) - t$. The aggregate utility at $a$ is $v_h$. Thus, $a$ is efficient and $h$ receives the object at $a$. We claim that $a$ is envy-free. Since $h$ is receiving more than half her value, she does not prefer the allotment of the other agent to her own. Moreover, she transfers the other agent $p_h \equiv v_h/2 - t$. Thus, $l$ weakly prefers her allotment at $a$ to that of $h$ because $v_l/2 - (ES(v) - t) \leq v_l/2 + (ES(v) - t)$. Let $v_l^* = v_l^* \equiv 2p_h$. Then, $v_h \geq v_h^*$ and $v_l^* \geq v_l$. By the property in the statement of the lemma, there is $\sigma \in N(M, \varphi, v^*)$ that obtains $a$ with certainty. We claim that $\sigma \in N(M, \varphi, v)$. We prove first that $\sigma_l$ is a best response to $\sigma_h$ for $l$ with type $v_l$. Let $m_l \in M_l$ be in the support of $\sigma_l$ and $m_l^* \in M_l$. Then $U_{\varphi}(\delta_{m_l}\sigma_h; v_l) = U_{\varphi}(\delta_{m_l^*}\sigma_h; v_l^*) \geq U_{\varphi}(\delta_{m_l^*}\sigma_h; v_l^*) \geq U_{\varphi}(\delta_{m_l}\sigma_h; v_l)$. \hfill \Box

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Where the first equality holds because in equilibrium $\sigma$, $l$ gets paid $v^*/2$ with certainty; the first inequality is the equilibrium condition for $l$ with type $v_l^*$; and the third inequality holds because the expected utility index of an allotment for $l$ with value $v_l$ is less than or equal than the expected utility index of the allotment for $l$ with value $v_l^*$.

Let $m_h \in M_h$ be in the support of $\sigma_h$ and $m_h' \in M_h$. Since $\sigma$ produces $a$ with certainty, $U_\varphi(\delta_{m_h} | \sigma_l; v_h) - U_\varphi(\delta_{m_h'} | \sigma_l; v_h^*) = v_h - v_h^*$. Since the utility index of $h$ is invariant when receiving an amount of money and no object, $U_\varphi(\delta_{m_h} | \sigma_l; v_h) - U_\varphi(\delta_{m_h'} | \sigma_l; v_h^*) \leq v_h - v_h^*$. Thus,

$$U_\varphi(\delta_{m_h} | \sigma_l; v_h) - U_\varphi(\delta_{m_h} | \sigma_l; v_h) \leq U_\varphi(\delta_{m_h'} | \sigma_l; v_h) - U_\varphi(\delta_{m_h} | \sigma_l; v_h^*) \leq 0,$$

where the last inequality is the equilibrium condition for $\sigma$ with types $v^*$. Thus, $\sigma_h$ is a best response to $\sigma_l$ for $h$ with type $v_h$. $\square$

**Proof of necessity in Theorem 1.** We first prove that $h$’s expected payoff in an empirical equilibrium of the winner-bid auction is bounded below by the expression in statement 1 of the theorem. The proof for the loser-bid auction is symmetric.

Let $(M, \varphi)$ be the winner-bid auction and $v \in \Theta$. Let $\sigma \in N(M, \varphi, v)$. For $i \in N$, let $c_i \equiv v_i/2$. By Proposition 3 there is $p \in \{c_l, ..., c_h\}$ separating the supports of $\sigma_l$ and $\sigma_h$, i.e., such that the support of $\sigma_l$ belongs to $\{0, ..., p\}$; the support of $\sigma_h$ belongs to $\{p, ..., \overline{p}\}$; and $p$ belongs to the support of $\sigma_h$. Suppose that $p > c_l + 1$. Since $p$ is a best response to $\sigma_l$ for $h$ with type $v_h$, $h$’s expected payoff of $p$ should be at least the expected payoff of $p - 1$. Thus,

$$v_h - p \geq \sigma_l(p-1) \left[ \frac{1}{2} (v_h - (p-1)) + \frac{1}{2} (p-1) \right] + (1 - \sigma_l(p-1)) (v_h - (p-1)).$$

Equivalently,

$$\sigma_l(p-1) \geq 1/(c_h - p + 1). \quad (4)$$

Let $\sigma$ be an empirical equilibrium of $(M, \varphi, v)$ and let $p$ be its associated separating bid (in the support of $\sigma_l$). Let $\{\sigma^\lambda\}_{\lambda \in \mathbb{N}}$ be a sequence of monotone quantal response equilibria of $(M, \varphi, v)$ that converges to $\sigma$ as $\lambda \to \infty$. For $i \in N$ and $\{b, d\} \subseteq \{0, 1, ..., \overline{p}\}$, let $\Delta_i(b, d)$ be the difference in expected utility for agent $i$ in the *winner-bid auction* between the two situations in which agent $i$ bids strictly to the left of $b$ and bids exactly $d$, conditional on agent $-i$ bidding $b$. Using this notation we have that when $b < d$,

$$U_\varphi(\delta_b | \sigma_{-i}, v_i) - U_\varphi(\delta_d | \sigma_{-i}, v_i) = \sum_{r < b} \sigma_{-i}(r)(d-b) + \sigma_{-i}(b)(d-c_i) + \sum_{b < r \leq d} \sigma_{-i}(r) \Delta_i(r, d). \quad (5)$$

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We prove that the expected payoff for \( h \) given \( \sigma \) satisfies the lower bound in the statement of the theorem.

**Case 1:** \( ES(v) = 1 \). Recall that we require valuations to be positive. Thus, \( c_l > 0 \). Let \( \lambda \in \mathbb{N} \). By (5),

\[
U_\varphi(\delta_{c_l-1}|\sigma^\lambda_h, v_l) - U_\varphi(\delta_{c_l}|\sigma^\lambda_h, v_l) = \sum_{r<c_l-1} \sigma_h(r) + 0 + 0 \geq 0.
\]

By monotonicity (of the quantal response function for which \( \sigma^\lambda \) is a quantal response equilibrium), \( \sigma^\lambda_l(c_l) \leq \sigma^\lambda_l(c_l - 1) \). By convergence, \( \sigma_l(c_l) \leq \sigma_l(c_l - 1) \). Thus, \( \sigma_l(c_l) \leq 1/2 \). We claim that \( p = c_l \). Suppose by contradiction that \( p = c_h = c_l + 1 \). By Proposition 3, the support of \( \sigma_l \) belongs to \( \{0, ..., c_l\} \). Thus,

\[
U_\varphi(\delta_{c_l}|\sigma_l, v_h) - U_\varphi(\delta_{c_h}|\sigma_l, v_h) = (1 - \sigma_l(c_l))(c_h + 1) + \sigma_l(c_l)c_h - c_h > 0.
\]

This contradicts \( \sigma \) is a Nash equilibrium of \( (M, \varphi, v) \). Thus, \( p = c_l \). If \( \sigma \) is efficient, \( \pi_h = v_h/2 + 1 \). In general,

\[
U_\varphi(\delta_p|\sigma_l, v_h) \geq (1 - \sigma_l(c_l))(c_h + 1) + \sigma_l(c_l)c_h \geq v_h/2 + 1/2,
\]

i.e., \( \pi_h(\sigma) \geq v_h/2 + 1/2 \).

**Case 2:** \( ES(v) > 1 \) and \( \sigma \) is efficient. Let \( t \equiv p - c_l - 1 \) and suppose that \( t \geq 1 \). Let \( d \equiv p - 1 \). Then

\[
\Delta_l(d, d) = d - ((v_l - d)/2 + d/2) = t.
\]

Let \( r < d \) and \( n = d - r \). Then,

\[
\Delta_l(r, d) = r - (v_l - d) = c_l + t - n - (2c_l - c_l - t) = 2t - n.
\]

Thus, for each \( \max\{0, c_l - 3t\} \leq b < d \), we have that

\[
U_\varphi(\delta_b|\sigma^\lambda_h, v_l) - U_\varphi(\delta_d|\sigma^\lambda_h, v_l) = \sum_{r<b} \sigma^\lambda_h(r)(d-b) + \sigma^\lambda_h(b)t + \sum_{b<r\leq d} \sigma^\lambda_h(r)\Delta_l(r, d).
\]  

(6)

We claim that there is \( \Lambda \in \mathbb{N} \) such that for each \( \lambda \geq \Lambda \) and each \( \max\{0, c_l - 3t\} \leq b < d \),

\[
U_\varphi(\delta_b|\sigma^\lambda_h, v_l) - U_\varphi(\delta_d|\sigma^\lambda_h, v_l) \geq 0.
\]

(7)

Let \( c_l - 1 \leq b < d \). For each \( d - 2t \leq r, \Delta_l(r, d) \geq 0 \). Thus, by (6), for each \( \lambda \in \mathbb{N} \), \( U_\varphi(\delta_b|\sigma^\lambda_h, v_l) - U_\varphi(\delta_d|\sigma^\lambda_h, v_l) \geq 0 \). By monotonicity, \( \sigma^\lambda_l(b) \geq \sigma^\lambda_l(d) \). By convergence, \( \sigma_l(b) \geq \sigma_l(d) \). We complete the proof of
our claim by proving by induction on \( \eta \in \{1, \ldots, 2t\} \) that the claim holds for \( b = c_l - t - \eta \). Let \( \eta \in \{2, \ldots, 2t\} \) and suppose that for each \( c_l - t - \eta < g < d \), \( \sigma_l(g) \geq \sigma_l(d) \). Let \( b \equiv c_l - t - \eta \). We claim that there is \( \Lambda \in \mathbb{N} \) for which for each \( \lambda \geq \Lambda \), the third term in the right-hand side of (6) is non-negative, i.e., \( \sum_{b < r \leq d} \sigma^\lambda_h(r) \Delta_l(r, d) \geq 0 \). Note that

\[
\sum_{b < r \leq d} \sigma^\lambda_h(r) \Delta_l(r, d) \geq \sum_{1 \leq r \leq \eta - 1} (\sigma^\lambda_h(c_l - t + r) - \sigma^\lambda_h(c_l - t - r))r.
\]

Thus, in order to prove our claim it is enough to show that there is \( \Lambda \in \mathbb{N} \) such that for each \( \lambda \geq \Lambda \) and each \( r \in \{1, \ldots, \eta - 1\} \), \( \sigma^\lambda_h(c_l - t + r) \geq \sigma^\lambda_h(c_l - t - r) \). Let \( r \in \{1, \ldots, \eta - 1\} \). Then,

\[
\Delta_h(c_l - t + r, c_l - t + r) = c_l - t + r - ((v_h - (c_l - t + r))/2 + (c_l - t + r)/2) = -(c_h - c_l + t - r).
\]

Let \( 1 \leq s \leq 2r - 1 \). Then,

\[
\Delta_h(c_l - t + r - s, c_l - t + r) = c_l - t + r - s - (v_h - (c_l - t + r)) = -(2(c_h - c_l + t + r) + s).
\]

Observe that

\[
U_\varphi(\delta_{c_l-t+r}|\sigma_l, v_h) - U_\varphi(\delta_{c_l-t-r}|\sigma_l, v_h) = -\sum_{0 \leq s \leq c_l-t-r} \sigma_l(s)2r + \sigma_l(c_l-t-r)(v_h - (c_l-t+r)) - ((v_h - (c_l-t-r))/2 + (c_l-t-r)/2)) - \sum_{1 \leq s \leq 2r-1} \sigma_l(c_l-t+r-s)\Delta_h(c_l-t+r-s, c_l-t+r) - \sigma_l(c_l-t+r)\Delta_h(c_l-t+r, c_l-t+r).
\]

Thus,

\[
U_\varphi(\delta_{c_l-t+r}|\sigma_l, v_h) - U_\varphi(\delta_{c_l-t-r}|\sigma_l, v_h) \geq -(1 - \sum_{0 \leq s \leq 2r} \sigma_l(c_l-t+r-s))2r + \sigma_l(c_l-t-r)(c_h - c_l + t - r) + \sum_{1 \leq s \leq 2r-1} \sigma_l(c_l-t+r-s)(2(c_h-c_l+t-r)+s) + \sigma_l(c_l-t+r)(c_h-c_l+t-r).
\]

Since \( c_l - t - r \geq c_l - t - \eta + 1 = b + 1 \), for each \( 0 \leq s \leq 2r \), \( b + 1 \leq c_l - t + r - s \). Since \( r \leq \eta - 1 \) and \( \eta \leq 2t \), \( c_l - t + r \leq c_l + t - 1 < d \). By the induction hypothesis, for each \( 0 \leq s \leq 2r \), \( \sigma_l(c_l-t+r-s) \geq \sigma_l(d) = \sigma_l(p - 1) \). By (4), \( \sigma_l(c_l-t+r-s) \geq 1/(c_h-p+1) \). Thus,

\[
U_\varphi(\delta_{c_l-t+r}|\sigma_l, v_h) - U_\varphi(\delta_{c_l-t-r}|\sigma_l, v_h) \geq -(1 - \frac{2r+1}{c_h-p+1})2r + \frac{2r}{c_h-p+1}(2(c_h - c_l + t - r) - (c_h - p + 1)).
\]

Since \( p \geq c_l \) and \( c_h > c_l + 1 \), \( 2(c_h - c_l) - (c_h - p + 1) > 0 \). Thus, \( U_\varphi(\delta_{c_l-t+r}|\sigma_l, v_h) - U_\varphi(\delta_{c_l-t-r}|\sigma_l, v_h) > 0 \). Thus, there is \( \Lambda \in \mathbb{N} \) such that
for each $\lambda \geq \Lambda$, $U_\phi(\delta_{c_l-t+r}|\sigma_1^\lambda, v_h) - U_\phi(\delta_{c_l-t-r}|\sigma_1^\lambda, v_h) > 0$. By monotonicity, $\sigma_1^\lambda(c_l-t+r) \geq \sigma_1^\lambda(c_l-t-r)$. Thus, there is $\Lambda \in \mathbb{N}$ such that for each $\lambda \geq \Lambda$ and each $r \in \{1, \ldots, \eta - 1\}$, $\sigma_1^\lambda(c_l-t+r) \geq \sigma_1^\lambda(c_l-t-r)$. Thus, for each $\lambda \geq \Lambda$, $\sum_{b<r\leq d} \sigma_1^\lambda(r) \Delta_l(r, d) \geq 0$ and

$$U_\phi(\delta_b|\sigma_h, v_l) - U_\phi(\delta_d|\sigma_h, v_l) \geq 0.$$  

By monotonicity $\sigma_1^\lambda(b) \geq \sigma_1^\lambda(d)$. By convergence, $\sigma_1(b) \geq \sigma_1(d)$.

We have proved that when $p-1 \geq c_l + 1$, for each $\max\{c_l - 3t, 0\} \leq b \leq p - 1$,

$$\sigma_1(b) \geq \sigma_1(p - 1). \quad (8)$$

There are two sub-cases

**Case 2.1:** $v_l \leq 3v_h/8$. We claim that $p \leq ES(v)/2 - v_h/4 + 1/2 = v_h/4 + 1/2$. Suppose by contradiction that $p > v_h/4 + 1/2$. Since $p$ and $c_l$ are integers, $p \geq \lceil v_h/4 + 1/2 \rceil + 1$ and $c_l \leq \lceil 3v_h/16 \rceil \leq 3v_h/16$. Then, $p - 1 - c_l \geq \lceil v_h/4 + 1/2 \rceil + 1 - \lceil 3v_h/16 \rceil$. Since $v_h$ is even, $\lceil v_h/4 + 1/2 \rceil \geq v_h/4$ and $p \geq v_h/4 + 1$. Thus, $p - 1 - c_l \geq v_h/16$. Thus, $p - 1 - c_l \geq v_h/16$. Since $ES(v) > 1$, then $v_h \geq 6$ (recall that we assumed positive even valuations). Direct calculation determines that $\lceil v_h/4 + 1/2 \rceil + 1 - \lceil 3v_h/16 \rceil \geq 1$ for $v_h = \{6, \ldots, 14\}$. Since, $t = p - c_l - 1 \geq v_h/4 - c_l \geq v_h/16 = c_h/8$, $c_l - 3t \leq c_l - 3c_h/8 \leq 0$. Thus, $\max\{c_l - 3t, 0\} = 0$. Since $t \geq 1$, by (8), $\sigma_1(p - 1) \leq 1/(c_l + t + 1) = 1/(c_l + p - 1 - c_l + 1) = 1/p$. By (4), $1/(c_h - p + 1) \leq 1/p$. Thus, $p \leq c_h/2 + 1/2 = v_h/4 + 1/2$. This contradicts $p \geq v_h/4 + 1$. Since $\sigma$ is efficient and $\sigma_h(p) > 0$, the support of $\sigma_l$ belongs to $\{0, \ldots, p-1\}$. Thus, $\pi_h(\sigma) \geq v_h - p \geq v_h - (v_h/4 + 1/2) = v_h/2 + v_h/4 - 1/2 = v_h/2 + ES(v)/2 \geq v_h/2 - 1/2$. Note that if $v_l \leq 3v_h/8$ and $ES(v) = 2$, then $v_l = 2$. Thus, statements (a) and (b) in the theorem have no overlap.

**Case 2.2:** $v_l > 3v_h/8$. We claim that $p \leq v_l/2 + (v_h/2 - v_l/2)/5 + 4/5$. Suppose by contradiction that $p > q \equiv \lceil v_l/2 + (v_h/2 - v_l/2)/5 + 4/5 \rceil$. Since $v_l/2$ and $v_h/2 - v_l/2 > 1$ are integers, $q \geq v_l/2 + 1 = c_l + 1$. Moreover, $q = v_l/2 + (v_h/2 - v_l/2)/5 + \varepsilon$ where $\varepsilon \in \{0, 1/5, 2/5, 3/5, 4/5\}$. Let $n \equiv p - q \geq 1$ and $t_q \equiv q - c_l \geq 1$. Then, $t_q = v_l/2 + (v_h/2 - v_l/2)/5 + \varepsilon - c_l = (c_h - c_l)/5 + \varepsilon$. Thus, $c_l - 3t_q = c_l - 3(c_h - c_l)/5 - 3\varepsilon \geq 8(c_h - 3c_h/8)/5 - 3\varepsilon$. Since $v_l > 3v_h/8$, $c_l > 3c_h/8$. Since $c_l$ and $c_h$ are integers, $c_l - 3c_h/8 > 1/8$. Thus, $c_l - 3t_q \geq 1/5 - 3\varepsilon$. Thus, if $\varepsilon \leq 1/5$, $c_l - 3t_q \geq 0$; if $2/5 \leq \varepsilon \leq 3/5$, $c_l - 3t_q \geq -1$; and if $\varepsilon = 4/5$, $c_l - 3t_q \geq -2$. Suppose that $\varepsilon \leq 1/5$. Since $t_q \geq 1$, by (8), $\sigma_1(p - 1) \leq 1/(4t_q + 1)$. By (4), $1/(c_h - p + 1) \leq 1/(4t_q + 1)$. Thus, $1/(c_h - q - n + 1) \leq 1/(4t_q + 1)$. Equivalently, $c_h - 4t_q - n \geq q$. Since $n \geq 1$ and $t_q = (c_h - c_l)/5 + \varepsilon$, $c_h - 4(c_h - c_l)/5 - 4\varepsilon - 1 \geq q$. Equivalently, $q - 4\varepsilon - 1 = c_l + (c_h - c_l)/5 - 5\varepsilon - 1 \geq q$. This is a contradiction.
Suppose that $2/5\varepsilon \leq 3/5$. Since $t_q \geq 1$, by (8), $\sigma_l(p - 1) \leq 1/(4t_q - 1 + 1)$. By (4), $1/(c_h - p + 1) \leq 1/(4t_q)$. Thus, $1/(c_h - q - n + 1) \leq 1/(4t_q)$. Equivalently, $c_h - 4t_q - n + 1 \geq q$. Since $n \geq 1$ and $t_q = (c_h - c_l)/5 + \varepsilon$, $c_h - 4(c_h - c_l)/5 - 4\varepsilon \geq q$. Equivalently, $q - 5\varepsilon = c_l + (c_h - c_l)/5 - 5\varepsilon - 1 \geq q$. This is a contradiction. Finally, suppose that $\varepsilon = 4/5$. Since $t_q \geq 1$, by (8), $\sigma_l(p - 1) \leq 1/(4t_q - 2 + 1)$. By (4), $1/(c_h - p + 1) \leq 1/(4t_q - 1)$. Thus, $1/(c_h - q - n + 1) \leq 1/(4t_q - 1)$. Equivalently, $c_h - 4t_q - n + 1 \geq q$. Since $n \geq 1$ and $t_q = (c_h - c_l)/5 + \varepsilon$, $c_h - 4(c_h - c_l)/5 - 4\varepsilon + 1 \geq q$. Equivalently, $q - 3 = c_l + (c_h - c_l)/5 - 5\varepsilon + 1 \geq q$. This is a contradiction. Since $\sigma$ is efficient and $\sigma_l(p) > 0$, the support of $\sigma_l$ belongs to $\{0, \ldots, p - 1\}$. Thus, $\pi_l(\sigma) \geq v_h - p \geq v_h - (v_l/2 + (v_h/2 - v_l/2)/5 + 4/5) = v_h/2 + 4ES(v)/5 - 4/5$.

Suppose now that $ES(v) = 2$. Recall that $\sigma$ is efficient. Since $v_i > 3v_h/8$, $v_l > 2$. Since $\pi_l(\sigma) \geq v_h/2 + 4ES(v)/5 - 4/5 = v_h/2 + 4/5$, $p \leq c_l + 1$. We prove that $p = c_l$. Suppose by contradiction that $p = c_l + 1$, i.e., $\pi_l(\sigma) = v_h/2 + 1$. By Proposition 3, $\sigma_l = \delta_p$. Let $\{\sigma^\lambda\}_{\lambda \in \mathbb{N}}$ be a sequence of monotone quantal response equilibria such that as $\lambda \to \infty$, $\sigma^\lambda \to \sigma$. Since

$$U_\varphi(\delta_{c_l-2}|\sigma_h|v_i) - U_\varphi(\delta_{c_l+1}|\sigma_h|v_i) = c_l + 1 - c_l = 1,$$

there is $\Lambda \in \mathbb{N}$ such that for each $\lambda \geq \Lambda$, $U_\varphi(\delta_{c_l-2}|\sigma^\lambda_h|v_i) > U_\varphi(\delta_{c_l+1}|\sigma^\lambda_h|v_i)$. By monotonicity, for each $\lambda \geq \Lambda$, $\sigma^\lambda_h(c_l - 2) \geq \sigma^\lambda_h(c_l + 1)$. By convergence, $\sigma_l(c_l - 2) \geq \sigma_l(c_l + 1)$. Notice also that for each $\lambda \in \mathbb{N}$, $U_\varphi(\delta_{c_l-1}|\sigma^\lambda_h|v_i) \geq U_\varphi(\delta_{c_l}|\sigma^\lambda_h|v_i)$. Thus, for each $\lambda \in \mathbb{N}$, $\sigma^\lambda_l(c_l - 1) \geq \sigma^\lambda_l(c_l)$. Now,

$$U_\varphi(\delta_{c_l+1}|\sigma^\lambda_h|v_i) - U_\varphi(\delta_{c_l}|\sigma^\lambda_h|v_i) = \sum_{b < c_l} \sigma^\lambda_l(b)(-1) + \sigma^\lambda_l(c_l)(c_l + 1 - c_l) + \sigma^\lambda_l(c_l + 1)(c_l - (c_l - 1)) \leq -\sigma^\lambda_l(c_l - 2) - \sigma^\lambda_l(c_l - 1) + \sigma^\lambda_l(c_l) + \sigma^\lambda_l(c_l + 1) \leq 0.$$

By monotonicity, for each $\lambda \geq \Lambda$, $\sigma^\lambda_l(c_l) \geq \sigma^\lambda_h(c_l + 1)$. By convergence, $\sigma_l(c_l) \geq \sigma_h(c_l + 1)$. Since $p = c_l + 1$ is in the support of $\sigma_h$, $c_l$ is also in the support of $\sigma_h$. This contradicts that the support of $\sigma_h$ is contained in $\{p, \ldots, \pi\}$.

Finally, suppose that $ES(v) > 2$ and $v_l \geq 7v_h/12 - 7/6$. We claim that $p < v_l/2 + (v_h/2 - v_l/2)/5 + 4/5$. Suppose by contradiction that $p = v_l/2 + (v_h/2 - v_l/2)/5 + 4/5$. Let $y \equiv c_l - 3(p - 1 - c_l)$. Direct calculation yields that since $v_l \geq 7v_h/12 - 7/6$, $y \geq c_h - p$. By (7), there is $\Lambda \in \mathbb{N}$ such that for each $\lambda \geq \Lambda$, and each max$\{0, c_l - 3t\} \leq b < p - 1$, $\sigma^\lambda_l(b) \geq \sigma_l^\lambda(p - 1)$. Since $p$ is in the support of $\sigma_l$ and $p > c_l$, for each $b < p$, $U_\varphi(\delta_b|\sigma_h|v_i) > U_\varphi(\delta_p|\sigma_h|v_i)$. Thus, we can suppose without loss of generality that for each $\lambda \geq \Lambda$ and each $b < p$, $U_\varphi(\delta_b|\sigma^\lambda_h|v_i) > U_\varphi(\delta_p|\sigma^\lambda_h|v_i)$ and consequently $\sigma^\lambda_l(b) \geq \sigma^\lambda_l(p)$. 

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Thus,
\[ U_\varphi(\delta_p I; \sigma; v_h) - U_\varphi(\delta_{p-1} I; \sigma; v_h) = \sum_{b < p - 1} \gamma^h_I(b)(-1) \]
\[ + \gamma^h_I(p - 1)(2c_h - p - c_h) + \gamma^h_I(p)(c_h - p) \]
\[ = \gamma^h_I(p)(c_h - p) - \sum_{0 \leq b < y} \gamma^h_I(b) \]
\[ - \sum_{y \leq b \leq p - 1} \gamma^h_I(b) + (c_h - p + 1)\gamma^h_I(p - 1) \]
\[ \leq \gamma^h_I(p)(c_h - p) - \sum_{0 \leq b < y} \gamma^h_I(p) \]
\[ - \sum_{y \leq b \leq p - 1} \gamma^h_I(p - 1) + (c_h - p + 1)\gamma^h_I(p - 1) \]
\[ \leq \gamma^h_I(p - 1)(c_h - p + 1 - 4(p - 1 - c_h) - 1) = 0, \]
where the last equality follows from direct calculation given that \( p = c_l + (c_h - c_l)/5 + 4/5. \) By monotonicity, for each \( \lambda \geq \Lambda, \sigma^h_I(p - 1) \geq \sigma^h_I(p). \) By convergence, \( \sigma_h(p - 1) \geq \sigma_h(p). \) Thus, \( p - 1 \) is in the support of \( \sigma_h. \) This contradicts that the support of \( \sigma_h \) belongs to \( \{p, ..., \bar{p}\}. \)

**Proof of sufficiency in Theorem 1.** Let \((M, \varphi)\) be an extreme auction and \( v \in \Theta \) such that \( v_l < v_h. \) We prove that each Nash equilibrium of \((M, \varphi, v)\) satisfying the bounds in the statement of the theorem is an empirical equilibrium. We prove it for the winner-bid auction. The proof for the loser-bid auction is symmetric. We consider four cases.

**Case 1.** Let \((\pi_l, \pi_h)\) be such that \( \pi_l = c_l \) and \( c_l + ES(v) > \pi_h = c_h + 1/2 \) if \( ES(v) = 1 \) and \( c_l + ES(v) \geq \pi_h = c_h + ES(v) - 1 \) if \( ES(v) > 1. \) (Note that this case covers all inefficient empirical equilibria.) Let \( 0 < \alpha \leq \min\{1/2, 1/ES(v)\}, \min\{\alpha/3, 1/10, (1 - \alpha)/2(\bar{p} - 1)\} > \varepsilon > 0, \) and \( \kappa > 0. \) For each \( x_h \in \mathbb{R}^M \) and \( x_l \in \mathbb{R}^M, \) let

\[ f^{\varepsilon, \kappa}_l(x_h) = (1 - \varepsilon)\delta_c + \varepsilon f^\kappa_l(x). \]

Let \( \{\kappa_t\}_{t \in \mathbb{N}} \) such that for each \( t \in \mathbb{N}, \kappa_t \in \mathbb{N}, \) be such that as \( t \to \infty, \kappa_t \to \infty. \) Let \( \{\gamma^t\}_{t \in \mathbb{N}} \) be such that for each \( t \in \mathbb{N}, \gamma^t \) is a fixed point of the composition of \((f^{\varepsilon, \kappa}_l, f^{\varepsilon, \kappa}_h)\) with the expected payoff operator, i.e., the continuous mapping that assigns to each profile of mixed strategies \( \sigma, \)

\[ F^{\varepsilon, \kappa}(\sigma) \equiv (f^{\varepsilon, \kappa}_l(U_\varphi(\delta_b I; \sigma_l, v_l)b \in \{0, ..., \bar{p}\}), f^{\varepsilon, \kappa}_h(U_\varphi(\delta_b I; \sigma_h, v_h)b \in \{0, ..., \bar{p}\})). \]

Existence of \( \gamma \) is guaranteed by Brower’s fixed point theorem. Denote \( U_\varphi(\delta_b I; v_l) = u^l_I(b) \) and \( U_\varphi(\delta_b I; v_h) = u^h_I(b). \) Let \( p \in \{c_l - 1, c_l\} \) and \( b < p. \) Then,

\[ u^h_I(p) - u^h_I(b) = \sum_{d < b} \gamma^h_I(d)(-p - b) + \gamma^h_I(p)(c_h - p - b) \]
\[ + \sum_{b < d < p} \gamma^h_I(d)(v_h - p - d) + \gamma^h_I(p)(c_h - p) \]
\[ > -\varepsilon \bar{p} + \gamma^h_I(b)ES(v) + \sum_{b < d < p} \gamma^h_I(d)2ES(v) + \gamma^h_I(p)ES(v) \]
\[ \geq -\varepsilon \bar{p} + (1 - \alpha + \varepsilon) > (1 - \alpha)/2 > 0. \]
Suppose now that \( b > c_l \). Then,
\[
\begin{align*}
    u^t_h(c_l) - u^t_h(b) & = \sum_{d \leq c_l} \gamma^t_h(d)(b - c_l) + \gamma^t_h(c_l)(c_l - (v_h - b)) \\
    & \quad + \sum_{c_l < d < b} \gamma^t_h(d)(d - (v_h - b)) + \gamma^t_h(b)(b - c_l) \\
    & \geq (1 - \alpha + \varepsilon) + (\alpha - 2\varepsilon + \varepsilon)(1 - ES(v)) \\
    & \geq 1 - \alpha ES(v) + \varepsilon + \varepsilon(ES(v) - 1) > 0.
\end{align*}
\]
Moreover,
\[
\begin{align*}
    u^t_l(c_l) - u^t_l(b) & = \sum_{d \leq c_l} \gamma^t_h(d)(b - c_l) + \gamma^t_h(c_l)(c_l - (v_l - b)) \\
    & \quad + \sum_{c_l < d < b} \gamma^t_h(d)(d - (v_l - b)) + \gamma^t_h(b)(b - c_l) \\
    & \geq (1 - \varepsilon)(b - c_l) > 0;
\end{align*}
\]
\[
\begin{align*}
    u^t_l(c_l - 1) - u^t_l(c_l) & = \sum_{d < c_l - 1} \gamma^t_h(d) > 0.
\end{align*}
\]
Now, let \( b < c_l - 1 \).
\[
\begin{align*}
    u^t_l(c_l) - u^t_l(b) & = \sum_{d < b} \gamma^t_h(d)(c_l - b) + \gamma^t_h(b)(v_l - c_l - b) \\
    & \quad + \sum_{b < d < c_l} \gamma^t_h(d)(v_l - c_l - d) + \gamma^t_h(c_l)(c_l - c_l) \\
    & \geq - \sum_{d < b} \gamma^t_h(d)c_l + \gamma^t_h(c_l)(c_l - 1) \\
    & = \gamma^t_h(c_l - 1) \left( 1 - \sum_{d < b} \frac{\gamma^t_h(d)}{\gamma^t_h(c_l - 1)} c_l \right) \\
    & = \gamma^t_h(c_l - 1) \left( 1 - \sum_{d < c_l - 1} \frac{\gamma^t_h(d)}{\gamma^t_h(c_l - 1)} c_l \right) \\
    & = \gamma^t_h(c_l - 1) \left( 1 - c_l \sum_{d < c_l - 1} e^{-\kappa_t(u_h(c_l - 1) - u_h(d))} \right) \\
    & \geq \gamma^t_h(c_l - 1) \left( 1 - c_l \sum_{d < c_l - 1} e^{-\kappa_t(1-\alpha)/2} \right).
\end{align*}
\]
Since as \( t \to \infty, \kappa_t \to \infty \), there is \( T \in \mathbb{N} \) for which for each \( b < c_l - 1 \) and each \( t \geq T \), \( u^t_l(c_l) - u^t_l(b) > 0 \). Since \( t^{\kappa_t} \) is strictly monotone, for each \( i \in N \) and each pair \( \{b, d\} \subseteq \{0, \ldots, \underbar{p}\} \), \( \gamma^t_l(b) \geq \gamma^t_l(d) \) if and only if \( U_\phi(\delta_b) \gamma^t_{L;i}:v_i) \geq U_\phi(\delta_d) \gamma^t_{L;i}:v_i) \). Thus, for a sequence \( \{\varepsilon_t\}_{t \in \mathbb{N}} \) that converges to zero, there is a divergent sequence of natural numbers \( \{\kappa_t\}_{t \in \mathbb{N}} \) and a sequence of distribution profiles \( \{\gamma^t\}_{t \in \mathbb{N}} \) such that (i) for each \( t \in \mathbb{N} \), each \( i \in N \), and each pair \( \{b, d\} \subseteq \{0, \ldots, \underbar{p}\} \), \( \gamma^t_l(b) \geq \gamma^t_l(d) \) if and only if \( U_\phi(\delta_b) \gamma^t_{L;i}:v_i) \geq U_\phi(\delta_d) \gamma^t_{L;i}:v_i) \); (ii) \( \gamma^t \) is a fixed point of \( \mathcal{F}^{\varepsilon_t,\kappa_t} \). Thus, as \( t \to \infty \), \( \gamma^t_l \to (1 - \alpha)\delta_{c_l - 1} + \alpha \delta_{c_l} \) and \( \gamma^t_h \to \delta_{c_l} \). Observe that the constructed sequence satisfies the hypotheses in Proposition 5. Standard arguments show that the limit distribution profile is a Nash equilibrium in which \( \pi_l = v_l/2 \) and \( \pi_h = v_h/2 + ES(v) - \alpha ES(v) \). Thus, this case spans all inefficient empirical equilibria of the winner-bid auction.
Case 2: Let \((\pi_l, \pi_h)\) be such that \(\pi_l = c_l\) and \(\pi_h = c_h + ES(v)\). Let \(\varepsilon > 0\) and \(\kappa > 0\). For each \(x_h \in \mathbb{R}^{M_h}\) and \(x_l \in \mathbb{R}^{M_l}\), let
\[
\begin{align*}
f^{\varepsilon, \kappa}_h(x_h) &= (1 - \varepsilon)\delta_{c_l} + \varepsilon \ell^\kappa(x), \\
f^{\varepsilon, \kappa}_l(x_h) &= (1 - \varepsilon)\delta_{c_l-1} + \varepsilon \ell^\kappa(x).
\end{align*}
\]
Let \(\{\kappa_t\}_{t \in \mathbb{N}}\) such that for each \(t \in \mathbb{N}\), \(\kappa_t \to \infty\). Let \(\{\gamma^t\}_{t \in \mathbb{N}}\) be such that for each \(t \in \mathbb{N}\), \(\gamma^t\) is a fixed point of the composition of \((f^{\varepsilon, \kappa}_h, f^{\varepsilon, \kappa}_l)\) and the expected payoff operator. As \(t \to \infty\), \(\gamma^t_h \to \delta_{c_l}\) and \(\gamma^t_l \to \delta_{c_l-1}\). Let \(\sigma\) be this limit distribution profile (in pure strategies). Then, \(\pi_l(\sigma) = c_l\) and \(\pi_h(\sigma) = c_h + ES(v)\). In order to complete the construction as in Case 1, we need to show that \(\varepsilon\) can be selected small enough such that for a sequence of divergent \(\kappa_s\), eventually the distribution is strictly monotone with respect to expected payoffs. The expected payoff of \(c_l\) given \(\sigma\) for \(h\), is greater than the payoff of any other bid. The expected payoff of \(c_l-1\) given \(\sigma\) for \(l\) is greater than the payoff of any bid higher than \(c_l-1\). Thus, it is only necessary to show that the expected payoff of \(c_l-1\), for the distribution profile \(\gamma^t\) for \(l\), is eventually greater than the payoff of any bid lower than \(c_l-1\). For each \(b < c_l-1\),
\[
u^t_h(c_l - 1) - u^t_h(b) = \sum_{d < b} \gamma^t_l(d)(-c_l - 1 - b) + \gamma^t_l(b)(v_h - c_l + 1 - c_l)
\]
\[
> -\varepsilon \gamma H + 2(1 - \varepsilon).
\]
Thus, the argument in Case 1 can be easily reproduced for the sequence of fixed points.

Case 3: \(c_l = 1, c_h = 3, \pi_l = c_l + 1, \text{ and } \pi_h = c_h + 1\). This is the unique case where \(ES(v) = 2\) and \(v_l \leq 3v_h / 8\). Let \(\varepsilon > 0\) and \(\kappa > 0\). For each \(x_h \in \mathbb{R}^{M_h}\) and \(x_l \in \mathbb{R}^{M_l}\), let
\[
\begin{align*}
f^{\varepsilon, \kappa}_h(x_h) &= (1 - \varepsilon)\delta_{c_l+1} + \varepsilon \ell^\kappa(x), \\
f^{\varepsilon, \kappa}_l(x_h) &= (1/2 - \varepsilon)\delta_{c_l-1} + (1/2 - \varepsilon)\delta_{c_l} + \varepsilon \delta_{c_l+1} + \varepsilon \ell^\kappa(x).
\end{align*}
\]
Let \(\{\kappa_t\}_{t \in \mathbb{N}}\) such that for each \(t \in \mathbb{N}\), \(\gamma^t\) is a fixed point of the composition of \((f^{\varepsilon, \kappa}_h, f^{\varepsilon, \kappa}_l)\) and the expected payoff operator. As \(t \to \infty\), \(\gamma^t_h \to \delta_{c_l}\) and \(\gamma^t_l \to \delta_{c_l-1}\). Let \(\sigma\) be this limit distribution profile. Then, \(\pi_l(\sigma) = c_l + 1\) and \(\pi_h(\sigma) = c_h + ES(v) - 1\). In order to complete the construction as in Case 1, we need to show that \(\varepsilon\) can be selected small enough such that for a sequence of divergent \(\kappa_s\), eventually the distribution
is strictly monotone with respect to expected payoffs. The expected payoff of \( c_l + 1 \) given \( \sigma \) for \( h \), is greater than the payoff of \( c_l - 1 \) and any bid greater than \( c_l + 1 \). The expected payoff of \( c_l - 1 \) and \( c_l \) given \( \sigma \) for \( l \) is greater than the payoff of any bid higher than \( c_l - 1 \). Thus, it is only necessary to show that the expected payoff of \( c_l + 1 \), for the distribution profile \( \gamma^t \) for \( h \), is eventually greater than the payoff of \( c_l \).

\[
 u^t_h(c_l + 1) - u^t_h(c_l) = -\sigma^t_l(0) + \gamma^t_l(1) + 2\sigma^t_l(2) > -(1/2 - \varepsilon + \varepsilon) + (1/2 - \varepsilon) + 2\varepsilon > 0.
\]

Thus, the argument in Case 1 can be easily reproduced for the sequence of fixed points. Notice that the constructed sequence does not satisfy the hypotheses in Proposition 5.

**Case 4:** \( \pi_l = c_l + 1 \) and \( \pi_h = c_h + ES(v) - 1 \) and \( ES(v) > 2 \). Let \( \varepsilon > 0 \) and \( \kappa > 0 \). For each \( x_h \in \mathbb{R}^{M_h} \) and \( x_l \in \mathbb{R}^{M_l} \), let

\[
 f^\varepsilon,\kappa_h(x_h) = (1 - \varepsilon)\delta_{c_l + 1} + \varepsilon l^\kappa(x).
\]

\[
 f^\varepsilon,\kappa_l(x_h) = (1/2 - \varepsilon)\delta_{c_l - 1} + (1/2 - \varepsilon)\delta_{c_l} + 2\varepsilon l^\kappa(x).
\]

Let \( \{\kappa_t\}_{t \in \mathbb{N}} \) such that for each \( t \in \mathbb{N} \), \( \kappa_t \in \mathbb{N} \), be such that as \( t \to \infty \), \( \kappa_t \to \infty \). Let \( \{\gamma^t\}_{t \in \mathbb{N}} \) be such that for each \( t \in \mathbb{N} \), \( \gamma^t \) is a fixed point of the composition of \( (f^\varepsilon,\kappa_h, f^\varepsilon,\kappa_l) \) and the expected payoff operator. As \( t \to \infty \), \( \gamma^t_h \to \delta_{c_l} \) and \( \gamma^t_l \to (1/2)\delta_{c_l-1} + (1/2)\delta_{c_l} \). Let \( \sigma \) be this limit distribution profile. Then, \( \pi_l(\sigma) = c_l + 1 \) and \( \pi_h(\sigma) = c_h + ES(v) - 1 \). In order to complete the construction as in Case 1, we need to show that \( \varepsilon \) can be selected small enough such that for a sequence of divergent \( \kappa \)'s, eventually the distribution is strictly monotone with respect to expected payoffs. The expected payoff of \( c_l + 1 \) given \( \sigma \) for \( h \), is greater than the payoff of any other bid. The expected payoff of \( c_l - 1 \) and \( c_l \) given \( \sigma \) for \( l \) is greater than the payoff of any bid higher than \( c_l \). Thus, it is only necessary to show that the expected payoff of \( c_l + 1 \), for the distribution profile \( \gamma^t \) for \( h \), is eventually greater than the payoff of \( c_l \). Note that the expected payoff of \( c_l - 1 \) is never less than that of \( c_l \) for \( l \). For each \( b < c_l - 1 \),

\[
 u^t_h(c_l - 1) - u^t_h(b) = \sum_{d < b} \gamma^t_l(d)(-(c_l - 1 - b)) + \gamma^t_l(b)(v_h - c_l + 1 - c_h) + \sum_{b < d < c_l - 1} \gamma^t_l(d)(v_h - c_l + 1 - d) + \gamma^t_l(c_l - 1)(c_h - c_l + 1) > -2\varepsilon + 4(1/2 - \varepsilon).
\]

Thus, the argument in Case 1 can be easily reproduced for the sequence of fixed points.
Case 5: \( c_l \leq 3c_h/8, c_l + 1 < p \leq v_h/4 + 1/2, \pi_l = p \) and \( \pi_h = v_h - p \) and \( ES(v) > 2 \). Let \( \varepsilon > 0 \) and \( \kappa > 0 \). For each \( x_h \in \mathbb{R}^{M_h} \) and \( x_l \in \mathbb{R}^{M_l} \), let
\[
f^{\varepsilon, \kappa}_h(x_h) = (1 - \varepsilon)\delta_{c_l+1} + \varepsilon l^\kappa(x).
\]
\[
f^{\varepsilon, \kappa}_l(x_l) = (1/p - \varepsilon) \left( \sum_{b < p} \delta_b \right) + (p - 1)\varepsilon \delta_p + \varepsilon l^\kappa(x).
\]
Let \( \{\kappa_t\}_{t \in \mathbb{N}} \) such that for each \( t \in \mathbb{N}, \kappa_t \in \mathbb{N} \), be such that as \( t \to \infty, \kappa_t \to \infty \). Let \( \{\gamma_t\}_{t \in \mathbb{N}} \) be such that for each \( t \in \mathbb{N}, \gamma_t \) is a fixed point of the composition of \( f^{\varepsilon, \kappa}_h, f^{\varepsilon, \kappa}_l \) and the expected payoff operator. As \( t \to \infty \), \( \gamma_t \to \delta_p \) and \( \gamma_t \to (1/p) \sum_{b < p} \delta_b \). Let \( \sigma \) be this limit distribution profile. Then, \( \pi_l(\sigma) = p \) and \( \pi_h(\sigma) = v_h - p \). In order to complete the construction as in Case 1, we need to show that \( \varepsilon \) can be selected small enough such that for a sequence of divergent \( \kappa \), eventually the distribution is strictly monotone with respect to expected payoffs. The expected payoff of \( p \) given \( \sigma \) for \( h \), is greater than the payoff of any bid greater than \( p \). The expected payoff of each bid less than or equal to \( p - 1 \) given \( \sigma \) for \( l \) is greater than the payoff of any bid greater than or equal to \( p \). The expected payoff of \( p \) given \( \sigma \) for \( l \) is greater than the payoff of any bid greater than \( p \). Thus, it is only necessary to show that the expected payoff of \( p \), for the distribution profile \( \gamma_t \) for \( h \), is eventually greater than the payoff of each \( b < p \). Since \( 2 \leq c_l + 1 < p \leq v_h/4 + 1/2 = c_h + 1/2, (c_h - p + 1)/p \geq 1 \),
\[
u^l_h(p) - u^l_h(p - 1) = -\sum_{d < p - 1} \gamma^l_t(d) + \gamma^l_t(p - 1)(v_h - p - c_h) + \gamma^l_t(p)(c_h - p)
\]
\[
> - (1/p - \varepsilon)(p - 1) - \varepsilon + (1/p - \varepsilon)(c_h - p) + (p - 1)\varepsilon(c_h - p)
\]
\[
= -1 + 1/p + \varepsilon(p - 1) - \varepsilon + (c_h - p + 1)/p - 1/p
\]
\[
- \varepsilon(c_h - p) + (p - 1)\varepsilon(c_h - p) \geq 0.
\]
Moreover, for \( b < p \),
\[
u^l_h(b) - u^l_h(b - 1) = -\sum_{d < b - 1} \gamma^l_t(d) + \gamma^l_t(b - 1)(v_h - b - c_h) + \gamma^l_t(b)(c_h - b)
\]
\[
> - (1/p - \varepsilon)(b - 1) - \varepsilon + 2(1/p - \varepsilon)(c_h - b)
\]
\[
\geq - b/p + 1/p + \varepsilon(b - 1) - \varepsilon + 2(c_h - b + 1)/p - 2/p - 2\varepsilon(c_h - b)
\]
\[
\geq 1 - 2\varepsilon(c_h - b) - \varepsilon - 1/p.
\]
Thus, the argument in Case 1 can be easily reproduced for the sequence of fixed points. Notice that the constructed sequence does not satisfy the hypotheses in Proposition 5. However, this can be avoided whenever \( p <
Thus, for each $t$, let $v_h = v_h - p$ and $\pi_h = v_h - p$ and $ES(v) > 2$. Let $y = c_t - 3(p - 1 - c_t)$. Since $p - 1 \leq c_t + (v_h/2 - v_t/2)/5 + 4/5 - 1$ and $c_t > 3c_h/8$, $y > 0$. Let $n = p - y = 4(p - 1 - c_t)/4$. Since $p < v_t/2 + (v_h/2 - v_t/2)/5 + 4/5$, $1/(4(p - 1 - c_t) + 1) > 1/(c_h - p + 1)$. Thus, $1/n > 1/(c_h - p + 1)$. Let $\sigma$ be defined by: $\sigma_h = \sigma_p$ and $\sigma_t = (1/n) \sum_{y \leq b \leq p} \delta_b$. Then, $U_{\varphi}(\delta_p | \sigma_t; v_h) - U_{\varphi}(\delta_{p-1} | \sigma_t; v_h) = -(n-1)/(n + (1/n)(c_h - p)) > 0$. Thus, $p$ is a unique best response to $\sigma_t$ for $h$. Clearly, $\sigma_t$ is a best response to $\sigma_h$ for $l$. Thus, $\sigma$ is a Nash equilibrium with payoffs $\pi_l(\sigma) = p$ and $\pi_h(\sigma) = v_h - p$. Now, for $y \leq b - 1 < b \leq p - 1$,

$$U_{\varphi}(\delta_b | \sigma_l; v_h) - U_{\varphi}(\delta_{b-1} | \sigma_l; v_h) \geq -(n-2)/n + (1/n)(c_h - b) + (1/n)(c_h - b) > 1.$$ 

Thus, $U_{\varphi}(\delta_b | \sigma_l; v_h)$ strictly increases in the set $b \in \{y, ..., p\}$. Let $r \in \{p, ..., p\}$ be the maximum for which $U_{\varphi}(\delta_r | \sigma_l; v_h) \geq U_{\varphi}(\delta_y | \sigma_l; v_h)$. Thus, $U_{\varphi}(\delta_r+1 | \sigma_l; v_h) < U_{\varphi}(\delta_y | \sigma_l; v_h)$. Now, for each $b < y$,

$$U_{\varphi}(\delta_y | \sigma_l; v_h) - U_{\varphi}(\delta_b | \sigma_l; v_h) \geq \sigma_l(y)(c_h - y) \geq ES(v)/n > 0.$$ 

Thus, there is $\varepsilon > 0$ for which for each distribution profile $\gamma$ such that $||\sigma - \gamma||_\infty \leq \varepsilon$, (i) there is a constant $c > 0$ such that for each $b \neq p$, $U_{\varphi}(\delta_p | \gamma; v_h) - U_{\varphi}(\delta_b | \gamma; v_h) \geq c$; (ii) for each $b \in \{y, ..., r\}$ and each $d \neq b < r$, $U_{\varphi}(\delta_d | \gamma; v_h) - U_{\varphi}(\delta_b | \gamma; v_h) > 0$; (iii) $U_{\varphi}(\delta_b | \gamma; v_h)$ strictly increases in the set $b \in \{y, ..., p\}$; and (iv) for each $b \leq p - 1 < d$, $U_{\varphi}(\delta_b | \gamma; v_h) - U_{\varphi}(\delta_d | \gamma; v_h) > 0$.

Let $\kappa > 0$ and $\eta \equiv \varepsilon/(2(r - y) + 2)$. For each $x_h \in \mathbb{R}^M_h$ and $x_l \in \mathbb{R}^M_h$, let

$$f^\varepsilon_{\varphi}(x_h) = (1 - \varepsilon/2)\delta_p + \eta \sum_{y \leq b \leq r} \delta_b + \eta \varepsilon(x).$$

$$f^{\varepsilon,\kappa}_{l}(x_h) = (1 - \varepsilon/2)\delta_p + \eta \sum_{y \leq b \leq r} \delta_b + \eta \varepsilon(x).$$

Let $\{\kappa_t\}_{t \in \mathbb{N}}$ such that for each $t \in \mathbb{N}$, $\kappa_t \in \mathbb{N}$; and as $t \to \infty$, $\kappa_t \to \infty$. Let $\{\gamma^t\}_{t \in \mathbb{N}}$ be such that for each $t \in \mathbb{N}$, $\gamma^t$ is a fixed point of the composition of $(f^\varepsilon_{\varphi}, f^{\varepsilon,\kappa}_{l})$ and the expected payoff operator. Then, $||\gamma^t - \sigma||_\infty < \varepsilon$. Thus, for each $t \in \mathbb{N}$, $\gamma^t$ satisfies conditions (i)-(iv) above. By (i) and (ii), for each $t \in \mathbb{N}$ and each pair $\{b, d\} \subseteq \{0, ..., M\}$, $\gamma^t_h(b) \geq \gamma^t_h(d)$ if and only if $U_{\varphi}(\delta_b | \gamma^t_h; v_h) - U_{\varphi}(\delta_d | \gamma^t; v_h) > 0$. By (iii) we can reproduce the argument.
in our proof of necessity in Theorem 1 and show that for each \( t \in \mathbb{N} \) and each \( y \leq b < p - 1 \), \( U_\varphi(\delta_b|\gamma_t^h; v_t) - U_\varphi(\delta_{p-1}|\gamma_t^h; v_t) \geq 0 \). Now, by (i), as \( t \to \infty \), \( \gamma_t^h \to \gamma_h \equiv (1 - \varepsilon/2 + \eta)\delta_p + \eta \sum_{y \leq b \leq r} \delta_b \). Direct calculation shows that for each \( b < y \), \( U_\varphi(\delta_{p-1}|\gamma_h; v_t) - U_\varphi(\delta_b|\gamma_h; v_t) = (p - 1 - c_t)\eta \). Thus, there is \( T \in \mathbb{N} \) such that for each \( t \geq T \), if \( d < y \leq b \leq p - 1 \), \( U_\varphi(\delta_b|\gamma^h_t; v_t) - U_\varphi(\delta_d|\gamma^h_t; v_t) > 0 \). This implies that for each \( i \in \mathbb{N} \) and \( \{b, d\} \subseteq \{0, \ldots, \pi\} \), \( \gamma^h_t(b) \geq \gamma^h_t(d) \) if and only if \( U_\varphi(\delta_b|\gamma^h_t; v_t) \geq U_\varphi(\delta_d|\gamma^h_t; v_t) \). Thus, one can complete the proof as in Case 1. Notice that the constructed sequence does not satisfy the hypotheses in Proposition 5.

**Case 7:** \( v_t > 3v_h/8 \), \( v_t < 7v_h/12 - 7/6 \), \( c_t + 1 < p = v_t/2 + (v_h/2 - v_t/2)/5 + 4/5 \), \( \pi_t = p, \pi_h = v_h - p \) and \( ES(v) > 2 \). Let \( y \equiv c_t - 3(p - 1 - c_t) \). Let \( n = p - 1 - y + 1 = 4(p - 1 - c_t) + 1 \). Since \( p - 1 - c_t \geq 1, n \geq 5 \). Direct calculation yields,

\[
\begin{align*}
n &= c_h - p + 1. \\
(9)
\end{align*}
\]

Now, \( (c_h - p) - y = 6((7v_h/12 - 7/6) - v_t)/5 > 0 \). Since \( c_h, p, \) and \( y \) are integers,

\[
(9)
\]

Let \( \sigma_h \equiv \delta_p \) and \( \sigma_t \equiv (1/n)\sum_{y \leq b \leq p - 1} \delta_b \). Clearly, for each \( p \leq b < d \), \( U_\varphi(\delta_b|\sigma_t; v_h) \geq U_\varphi(\delta_d|\sigma_t; v_h) + 1 \). Clearly, for each \( d < y \), \( U_\varphi(\delta_d|\sigma_t; v_h) > U_\varphi(\delta_d|\sigma_t; v_h) \). Let \( y \leq d < p - 1 \). Then,

\[
\begin{align*}
U_\varphi(\delta_{d+1}|\sigma_t; v_h) - U_\varphi(\delta_d|\sigma_t; v_h) &= -(b - y)(1/n) + (1/n)(v_h - (d + 1) - c_h) \\
&= -(b - y + 2)/n + 2(c_h - p + 1)/n - 2/n \\
&\geq 1 - 2/n > 0.
\end{align*}
\]

Let \( r \in \{p, \ldots, \pi\} \) be the maximum for which \( U_\varphi(\delta_r|\sigma_t; v_h) \geq U_\varphi(\delta_\sigma|\sigma_t; v_h) \). Thus, for each \( y \leq b \leq r \) and \( d > r \), \( U_\varphi(\delta_b|\sigma_h; v_h) > U_\varphi(\delta_d|\sigma_h; v_h) \). Clearly, for each \( b < p < d \), \( U_\varphi(\delta_b|\sigma_h; v_h) > U_\varphi(\delta_p|\sigma_h; v_h) > U_\varphi(\delta_d|\sigma_h; v_h) \). Thus, there is \( \zeta > 0 \) such that \( ||\gamma - \sigma||_\infty < \zeta \), then

(a) \( U_\varphi(\delta_b|\gamma_t^h; v_h) > U_\varphi(\delta_d|\gamma_t^h; v_h) \) whenever one of the following four conditions is satisfied (i) \( p \leq b < d \); (ii) \( d < b = y \); (iii) \( y \leq d < p - 1 \) and \( b = d + 1 \); or (iv) \( y \leq b \leq r \) and \( d > r \).

(b) For each \( p < d \), \( U_\varphi(\delta_p|\gamma_t^h; v_h) > U_\varphi(\delta_d|\gamma_t^h; v_h) + 1/2 \).

(c) For each \( b < p < d \), \( U_\varphi(\delta_b|\gamma_h; v_t) > U_\varphi(\delta_p|\gamma_h; v_t) > U_\varphi(\delta_d|\gamma_h; v_t) \).
Let $\kappa > 0$, $\varepsilon > 0$, $\eta \equiv \varepsilon/2(r-y+2)$, $\tau \equiv (y+3/2)\varepsilon/n$ such that $2\max\{\varepsilon, \tau\} < \zeta$. For each $x_h \in \mathbb{R}^{M_h}$ and $x_l \in \mathbb{R}^{M_h}$, let

$$f^\varepsilon,\kappa(x_h) = (1 - \varepsilon/2)\delta_p + \eta \sum_{y \leq b \leq r} \delta_b + \eta l^\kappa(x).$$

$$f^\varepsilon,\kappa(x_l) = (1/n - \tau) \left( \sum_{y \leq b \leq p-1} \delta_b \right) + \varepsilon \left( \sum_{0 \leq b \leq y-1} \delta_b \right) + \varepsilon \delta_p + (\varepsilon/2)l^\kappa(x).$$

Let $\{\kappa_t\}_{t \in \mathbb{N}}$ such that fore each $t \in \mathbb{N}$, $\kappa_t \in \mathbb{N}$; and as $t \rightarrow \infty$, $\kappa_t \rightarrow \infty$. Let $\{\gamma^t\}_{t \in \mathbb{N}}$ be such that for each $t \in \mathbb{N}$, $\gamma^t$ is a fixed point of the composition of $(f^\varepsilon,\kappa_t, f^\varepsilon,\kappa_t)$ and the expected payoff operator. Then, $||\gamma^t - \sigma||_\infty < \zeta$. Thus, for each $t \in \mathbb{N}$, $\gamma^t$ satisfies (a)-(c) above. Now,

$$U_\varphi(\delta_p|\gamma^t_l; v_h) - U_\varphi(\delta_{p-1}|\gamma^t_l; v_h) = \sum_{0 \leq b \leq p-2} \gamma^t_l(b)(-1) + \gamma^t_l(p-1)(v_h - p - c_h) + \gamma^t_l(p)(c_h - p) \geq -\varepsilon / (n-1)(1/n - \tau) - \varepsilon/2 + (1/n - \tau)(c_h - p) + \varepsilon(c_h - p) = \varepsilon(c_h - p - 1/2) + (1/n - \tau)(c_h - p - 1 - n).$$

By (9) and (10), $U_\varphi(\delta_p|\gamma^t_l; v_h) - U_\varphi(\delta_{p-1}|\gamma^t_l; v_h) \geq \varepsilon/2 > 0$. Thus, for each $t \in \mathbb{N}$ and each $\{b, d\} \subseteq \{0, \ldots, p\}$, $\gamma^t_h(b) \geq \gamma^t_h(d)$ if and only if $U_\varphi(\delta_b|\gamma^t_l; v_h) - U_\varphi(\delta_d|\gamma^t_l; v_h)$. Moreover, since for each $t \in \mathbb{N}$ and each $b \neq p$, $U_\varphi(\delta_p|\gamma^t_l; v_h) \geq U_\varphi(\delta_{p-1}|\gamma^t_l; v_h) + \varepsilon/2$, we have that as $t \rightarrow \infty$,

$$\gamma^t_h \rightarrow \gamma_h \equiv (1 - \varepsilon/2 + \eta)\delta_p + \eta \sum_{y \leq b \leq r} \delta_b.$$

The construction can be completed as in Case 6. \hfill \Box

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