Research article

On transitivity and connectedness of Cayley graphs of gyrogroups

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ABSTRACT

In this work, we explore edge direction, transitivity, and connectedness of Cayley graphs of gyrogroups. More specifically, we find conditions for a Cayley graph of a gyrogroup to be undirected, transitive, and connected. We also show a relationship between the cosets of a certain type of subgyrogroups and the connected components of Cayley graphs. Some examples and applications regarding these findings are provided.

1. Introduction

Gyrogroups, a generalization of groups whose associativity is replaced by a more general one, have been one of the fast growing areas in Mathematics recently. The structure was introduced by A.A. Ungar in the attempt to find a proper structure to Einstein’s velocity model; see [1] for more details. Since then, many properties, including geometric, algebraic, and topological properties of gyrogroups have been studied.

For a group, regarding the combinatorial property, its Cayley graph is considered as a combinatorial representation of that group, giving a visualization to its algebraic structure. The vertices of a Cayley graph of a group \( G \) are the elements of \( G \) and there is a directed edge from a vertex \( u \) to a vertex \( v \), denoted \( u \rightarrow v \), if \( v = su \) for some \( s \in G \). We will call this graph a left Cayley graph or an L-Cayley graph. Another definition is defined by changing the edge condition to \( u \rightarrow v \) if \( v = us \) for some \( s \in G \), giving rise to a right Cayley graph or an R-Cayley graph. It is then natural to think about the same combinatorial structures of gyrogroups since they are a generalization of groups. Some preliminary properties and examples of L-Cayley graphs of gyrogroups have been studied in [2]. In that study, a property on connectedness of Cayley graphs of gyrogroups has been proved.

In this study, we further explore these combinatorial structures of gyrogroups. More precisely, we study properties of transitivity of L-Cayley graphs of gyrogroups, edge direction, transitivity, connectedness and connected components of R-Cayley graphs of gyrogroups.

Outline of the paper: We give necessary definitions and background knowledge in Section 2, including the definitions of L-Cayley graphs and R-Cayley graphs of gyrogroups. In Section 3, we give some sufficient conditions for an L-Cayley graph of a gyrogroup to be transitive, together with an example. Section 4 is devoted to R-Cayley graphs of gyrogroups. In that section, we give a sufficient and necessary condition of an R-Cayley graph to be undirected, give a sufficient condition for it to be transitive, and show a connection of the cosets of L-subgyrogroups with the connected components of the graph. A few examples illustrating these results are also given. In Section 5, a sufficient condition for an R-Cayley graph of a gyrogroup to be isomorphic to a group graph is given as an application of results in Section 4. Lastly, in Section 6, we provide necessary and sufficient conditions for a left (right) Cayley graph of a gyrogroup to be a cycle, hence transitive in this specific case.

2. Background

Included in this section are necessary backgrounds. We give important definitions and algebraic identities regarding gyrogroups as well as the definitions of two types of Cayley graphs of gyrogroups. For more detailed knowledge of gyrogroups, we recommend readers to see [1, 3, 4]. For basic knowledge of graph theory, we refer the reader to [5].

An undirected graph is connected if two distinct vertices are joined by a path. Let \( D \) be a directed graph. The underlying graph \( D \) of \( D \) is the undirected graph with the same vertex set as \( D \) and for all vertices \( u, v \), \((u, v)\) is an edge in \( D \) if and only if \((u, v)\) or \((v, u)\) is an edge in \( D \). A directed graph is connected if its underlying graph is connected.
Table 1. Addition table for the gyrogroup $G_{15}$.

| @ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 1 |
| 2 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 1 | 2 |
| 3 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 1 | 2 | 3 |
| 4 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 1 | 2 | 3 | 4 |
| 5 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 1 | 2 | 3 | 4 | 5 |
| 6 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 1 | 2 | 3 | 4 | 5 | 6 |
| 7 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 8 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 9 | 9 | 10 | 11 | 12 | 13 | 14 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 10 | 10 | 11 | 12 | 13 | 14 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 11 | 11 | 12 | 13 | 14 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 12 | 12 | 13 | 14 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| 13 | 13 | 14 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| 14 | 14 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |

Let $G$ be an undirected group (respectively, a directed graph). A graph automorphism of $G$ is a bijection $\sigma$ from the vertex set of $G$ to itself such that for all vertices $u, v$, $[u, v]$ is an edge in $G$ if and only if $[(\sigma(u), \sigma(v))]$ is an edge in $G$. An undirected graph (respectively, a directed graph) $G$ is vertex-transitive if for all vertices $u, v$, there is an automorphism $\sigma$ of $G$ such that $\sigma(u) = v$.

Let $(G, \oplus)$ be a groupoid. Sometimes, we will simply call it $G$ when there is no possible confusion. An automorphism $f$ on $G$ is a bijection from $G$ to itself with the property that $f(g_1 \oplus g_2) = f(g_1) \oplus f(g_2)$ for all $g_1, g_2 \in G$. The set of all automorphisms on $G$ is denoted by $\text{Aut}(G, \oplus)$.

Definition 2.1 (Definition 2.7 of [1]). Let $(G, \oplus)$ be a non-empty groupoid. We say that $G$ is a gyrogroup if the following hold:

1. There is a unique identity element $e \in G$ such that $e \oplus x = x \oplus e$ for all $x \in G$.
2. For each $x \in G$, there exists a unique inverse element $\ominus x \in G$ such that $\ominus x \oplus x = e = x \ominus (\ominus x)$.
3. For all $x, y \in G$, there exists an automorphism $\text{gyr}[x, y] \in \text{Aut}(G, \oplus)$ such that $x \ominus (y \ominus z) = (x \ominus y) \ominus \text{gyr}[x, y]z$ (left gyroassociative law)

for all $z \in G$.

4. For all $x, y \in G$, $\text{gyr}[x \ominus y, y] = \text{gyr}[x, y]$. (left loop property)

Example 2.2 ([Example 8, p. 432 of [3]]). An example of a finite gyrogroup of order 15 is $G_{15} = \{0, 1, 2, \ldots, 14\}$ whose operation is given by Table 1. Its gyration table is described by Table 2. In cycle notation, the four non-identity gyroautomorphisms of $G_{15}$ can be expressed as in (1):

$$A = (1 \ 7 \ 5 \ 10 \ 6)(2 \ 3 \ 8 \ 11 \ 14),$$
$$B = (1 \ 6 \ 10 \ 5 \ 7)(2 \ 14 \ 11 \ 8 \ 3),$$
$$C = (1 \ 10 \ 7 \ 6 \ 5)(2 \ 11 \ 3 \ 14 \ 8),$$
$$D = (1 \ 5 \ 6 \ 7 \ 10)(2 \ 8 \ 14 \ 3 \ 11).$$

With the absence of associativity, the gyrogroup $G_{15}$ is not a group. For all elements $a, b, c$ in $G_{15}$, the gyroautomorphism $\text{gyr}[a, b]$ is given by the gyrator identity, which is true for every gyrogroup:

$$\text{gyr}[a, b]c = \ominus(a \ominus b) \ominus (a \ominus (b \ominus c)).$$

(gyrator identity)

In this work, we consider finite gyrogroups. Inspired by the solution of the equation $x \ominus a = b$, Ungar introduced a second binary operation in $G$ called the gyrogroup coadition or coaddition $\ominus$, defined by

$$a \ominus b = a \ominus \text{gyr}(a, b)b$$

for all $a, b \in G$. We write $a \ominus b$ for $a \ominus b$. Then, the solution to the equation $x \ominus a = b$ is $x = b \ominus a$. 


Many identities regarding the gyrogroup addition and coaddition have been discovered and can be found together with the proofs in [1]. We list some identities needed later in this work here.

**Theorem 2.3 ([1]).** Let $G$ be a gyrogroup. For all $a, b, c \in G$, the following properties hold:

1. if $a \oplus b = a \oplus c$, then $b = c$;  
2. $e a \oplus (a \oplus b) = b$;  
3. $(a \oplus b) \oplus a = a$;  
4. $(a \oplus b) \oplus b = a$;  
5. $gyr(a, b)(c) = c$ (right gyroassociative law).

The fourth named author has thoroughly studied algebraic properties of gyrogroups analogous to those of groups; among the work, the following definitions and theorems are important to our work. We encourage readers to see [3] for more explanations and motivations.

**Definition 2.4.** A non-empty subset $H$ of a gyrogroup $(G, \oplus)$ is a subgyrogroup of $G$ if $(H, \oplus)$ is a gyrogroup and $gyr(a, b)(H) = H$ for all $a, b \in H$. It is called an L-subgyrogroup of $G$ if $gyr(a, h)(H) = H$ for all $a \in G, h \in H$.

**Theorem 2.5.** If $H$ is an L-subgyrogroup of a gyrogroup $G$, then the set $\{ g \oplus H \mid g \in G \}$ forms a partition of $G$.

From Theorem 2.5, when $H$ is an L-subgyrogroup of a gyrogroup $G$, we will call each $g \oplus G$, a left coset.

**Theorem 2.6 (Theorem 21 of [6], Lagrange’s Theorem).** If $H$ is an L-subgyrogroup of a finite gyrogroup $G$, then $|H|$ divides $|G|$.

Writing $[G : H]$ as the number of left cosets of $H$ in $G$, we have the following corollary as a consequence of Theorem 2.6.

**Corollary 2.7.** If $H$ is an L-subgyrogroup of a finite gyrogroup $G$, then $|G| = |G : H||H|$.

In the last part of this section, we turn to a combinatorial representation of a gyrogroup analogous to that of group, a Cayley graph. The following definitions are gyrogroup versions of Cayley graphs.

**Definition 2.8.** Let $G$ be a gyrogroup and let $S$ be a subset of $G$ not containing the identity element $e$. The L-Cayley graph or left-Cayley graph of $G$ generated by $S$, denoted by $L$-Cay($G, S$), is a directed graph whose vertices are the gyrogroup elements, and for any two vertices $u$ and $v, u \rightarrow v$ if $v = s \oplus u$ for some $s \in S$. We will conflate the gyrogroup elements and the vertices of the graph whenever there are no confusions.

In the same sense, the R-Cayley graph or right-Cayley graph of $G$ generated by $S$, denoted by $R$-Cay($G, S$), is a directed graph whose vertex set is $G$ and, for any two vertices $u$ and $v, u \rightarrow v$ if $v = u \oplus s$ for some $s \in S$.

If a Cayley graph has the property that $v \rightarrow u$ whenever $u \rightarrow v$ for all vertices $u, v$, then we say that the graph is undirected. In this case, we may draw each edge with arrows on both ends or drop the arrows entirely.

If $S$ is the empty set, then each type of Cayley graphs is the union of disjoint vertices, each corresponding to an element of that gyrogroup. We do not allow the identity element $e$ to be in $S$ to avoid the presence of loops in the graph, and from this time forward, we will assume this condition without mentioning it. Many examples of Cayley graphs of gyrogroups are given in succeeding sections.

![Fig. 1. (Left) the L-Cayley graph $L$-Cay($G_a, \{1, 3\}$). (Right) the L-Cayley graph $L$-Cay($G_b, \{1, 2, 3\}$).](image-url)
Table 3. The gyroaddition table (left) and the gyration table (right) for $G_6 = \{0, 1, 2, 3, 4, 5, 6, 7\}$. The gyroautomorphism $A$ is given by $A = (1 6)(2 5)$.

| @ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 0 |
| 2 | 2 | 3 | 0 | 1 | 6 | 7 | 4 | 5 |
| 3 | 3 | 5 | 6 | 0 | 7 | 1 | 2 | 4 |
| 4 | 4 | 2 | 1 | 7 | 0 | 6 | 5 | 3 |
| 5 | 5 | 4 | 7 | 6 | 1 | 0 | 3 | 2 |
| 6 | 6 | 7 | 4 | 5 | 2 | 3 | 0 | 1 |
| 7 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |

| gyr | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----|---|---|---|---|---|---|---|---|
| 0 | I | I | I | I | I | I | I | I |
| 1 | I | A | A | A | A | I | I | I |
| 2 | I | A | A | A | A | I | I | I |
| 3 | I | A | A | I | I | A | A | I |
| 4 | I | A | A | I | I | A | A | I |
| 5 | I | A | A | I | I | A | A | I |
| 6 | I | I | I | A | A | A | I | I |
| 7 | I | I | I | I | I | I | I | I |

Proof. Note that, from the assumption, $|G|$ is even by Lagrange’s theorem for groupoids, proved in [7, Theorem 5.7]. Since $\{s\}$ is symmetric and contains only one element, the L-Cayley graph is undirected by Theorem 3.2 and each vertex has degree 1. Hence, the graph is a disjoint union of $|G|/2$ edges, which means that it is transitive.

Continuing from the discussion at the end of Example 3.4, we state the main theorem of this section.

Theorem 3.6. Let $G$ be a finite groupoid and let $S$ be a symmetric subset of $G$. If $gyr[g, s]$ is the identity map for all $g \in G, s \in S$, then $L$-Cayley$(G, S)$ is transitive.

Proof. The idea of this theorem is that the condition on $gyr[g, s]$ makes the right additions by any element of $G$ automorphisms on $L$-Cayley$(G, S)$. First, we note that since $S$ is symmetric, the Cayley graph is undirected. Now, let $u$ and $v$ be two vertices in $L$-Cayley$(G, S)$, i.e., two elements in $G$. Then $v = u \oplus g$ for some $g \in G$. Suppose that $u$ and $z$ are adjacent in $L$-Cayley$(G, S)$, i.e., $u = s \oplus z$, for some $s \in S$. Adding both sides on the right by $g$ gives

$$u \oplus g = (s \oplus z) \oplus g = s \oplus (z \oplus g[yr[z, s][g]]) = s \oplus (z \oplus g).$$

which implies that $u \oplus g$ and $z \oplus g$ are adjacent. Hence, the map $\phi : L$-Cayley$(G, S) \rightarrow L$-Cayley$(G, S)$ sending $x$ to $x \oplus g$ is an automorphism that maps $u$ to $v$. Therefore, $L$-Cayley$(G, S)$ is transitive.

The converse of Theorem 3.6 is not true. As discussed in Example 3.4, a certain right addition by an element on the transitive Cayley graph $L$-Cayley$(G, \{1, 3\})$ is not an automorphism on the graph. The following is an example of the groupoid $G_{16}$, a groupoid with 16 elements.

Example 3.7. Introduced in [4, p. 41], the groupoid $G_{16}$ (called $K_{16}$ in the book) has its addition and gyration tables as shown in Tables 4 and 5, respectively. Let $S = \{1, 2, 3\}$. From the gyration table, $gyr[g, s]$ is the identity map for all $g \in G, s \in S$. By Theorem 3.6, the L-Cayley graph $L$-Cayley$(G_{16}, S)$ is transitive; it is a disjoint union of four copies of a complete graph with four vertices, as shown in Fig. 2. Observe, for example, that the right addition by 1 acts on $L$-Cayley$(G_{16}, S)$ by flipping each copy of the complete graph exchanging the top and bottom pairs of vertices. Picking two vertices, say 1 and 7 = 1 \oplus 6, an automorphism $\phi$ on the L-Cayley graph sending 1 to 7 is the right addition by 6.

4. Right-Cayley graphs

In this section, we provide sufficient and necessary conditions for an R-Cayley graph of a groupoid to be undirected, give a sufficient condition for the Cayley graph to be transitive, and explore a connection of the cosets of L-subgyrogroup with the connected components.
| gyr | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-----|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|
| 0   | I | I | I | I | I | I | I | I | I | I | I   | I   | I   | I   | I   | I   |
| 1   | I | I | I | I | I | I | I | I | I | I | I   | I   | I   | I   | I   | I   |
| 2   | I | I | I | I | I | I | I | I | I | I | I   | I   | I   | I   | I   | I   |
| 3   | I | I | I | I | I | I | I | I | I | I | I   | I   | I   | I   | I   | I   |
| 4   | I | I | I | I | I | I | I | A | A | A | A   | A   | A   | A   | A   | A   |
| 5   | I | I | I | I | I | I | I | A | A | A | A   | A   | A   | A   | A   | A   |
| 6   | I | I | I | I | I | I | I | A | A | A | A   | A   | A   | A   | A   | A   |
| 7   | I | I | I | I | I | I | I | A | A | A | A   | A   | A   | A   | A   | A   |
| 8   | I | I | I | I | I | I | I | A | A | A | A   | A   | A   | A   | A   | A   |
| 9   | I | I | I | I | I | A | A | A | A | I | I   | I   | I   | I   | A   | A   |
| 10  | I | I | I | I | I | A | A | A | A | I | I   | I   | I   | I   | A   | A   |
| 11  | I | I | I | I | I | A | A | A | A | I | I   | I   | I   | I   | A   | A   |
| 12  | I | I | I | I | I | A | A | A | A | I | I   | I   | I   | I   | A   | A   |
| 13  | I | I | I | I | I | A | A | A | A | I | I   | I   | I   | I   | A   | A   |
| 14  | I | I | I | I | I | A | A | A | A | I | I   | I   | I   | I   | A   | A   |
| 15  | I | I | I | I | I | A | A | A | A | I | I   | I   | I   | I   | A   | A   |

**Table 5.** The gyration table of the gyrogroup \( G_{16} \). The gyroautomorphism \( A \) is given by \( A = (8 \, 9 \, 10 \, 11 \, 12 \, 13 \, 14 \, 15) \).

**Proof.** Suppose that there is a directed edge from \( u \) to \( v \) in \( \text{R-Cay}(G, S) \). Then \( v = u \oplus s \) for some \( s \in S \). By the right cancellation law and the definition of coaddition, we have \( u = v \ominus s = v \oplus \text{gyr}(v, s)(\ominus s) = v \ominus s' \) for some \( s' \in S \), which implies that there is a directed edge from \( v \) to \( u \). Hence, \( \text{R-Cay}(G, S) \) is undirected.

Now suppose that \( \text{R-Cay}(G, S) \) is undirected. Given elements \( g \in G \) and \( s \in S \), we obtain that \( g \ominus s \) is a vertex in the graph. Since \( (g \ominus s) \ominus = g \), there is a directed edge \( g \ominus s \rightarrow g \ominus s' \). The undirected condition implies the existence of a directed edge \( g \ominus s \rightarrow g \ominus s' \), which means that \( g \ominus s' = g \ominus s = g \oplus \text{gyr}(g, s)(\ominus s) \) for some \( s' \in S \). Using the general left cancellation law and Item 6 in Theorem 2.3, we have \( \text{gyr}(g, s) = \Theta s' \), which is an element in \( S \), as desired. \( \square \)

**Example 4.4.** Since \( \text{gyr}(g_1, g_2)(8, 9) = (8, 9) \) for all \( g_1, g_2 \in G_{16} \), the right Cayley graph \( \text{R-Cay}(G_{16}, (8, 9)) \) is an undirected graph as shown in Fig. 5. Observe, for instance, that \( 14 \rightarrow 4 \) because \( 4 = 14 \ominus 8 \), whereas \( 4 \rightarrow 14 \) because \( 14 = 4 \ominus 9 \) but \( 8 \) and \( 9 \) are not inverses of each other. This is different from the left Cayley graph, where a bidirected edge arises from an element of \( S \) and its inverse. Note that \( (8, 9) \) right-generates the L-subgyrogroup \( \{0, 1, 8, 9\} \).

### 4.2. Transitivity

In this subsection, we give a sufficient condition for an R-Cayley graph to be transitive.
Theorem 4.5. Let \( G \) be a finite gyrogroup and let \( S \) be a symmetric subset of \( G \) such that \( \text{gyr}(g, g')(S) = S \) for all \( g, g' \in G \). Then \( R\text{-Cay}(G, S) \) is transitive.

Proof. We note that, by Theorem 3.3, \( R\text{-Cay}(G, S) \) is undirected. A similar idea as in Theorem 3.6 is applied here; the condition on gyrations causes the left additions by any element of \( G \) to be automorphisms on \( R\text{-Cay}(G, S) \). Let \( u \) and \( v \) be two vertices in \( R\text{-Cay}(G, S) \), i.e., two elements in \( G \). Then \( v = g \oplus u \) for some \( g \in G \). Suppose that \( w \) and \( z \) are adjacent in \( R\text{-Cay}(G, S) \), i.e., \( w = z \oplus s \), for some \( s \in S \). Adding both sides on the left by \( g \) yields

\[
g \oplus w = g \oplus (z \oplus s) = (g \oplus z) \oplus \text{gyr}(g, z)(s) = (g \oplus z) \oplus s',
\]

for some \( s' \in S \). Hence, \( g \oplus w \) and \( g \oplus z \) are adjacent. This implies that the map \( \phi : x \mapsto g \oplus x \) is an automorphism on \( R\text{-Cay}(G, S) \) sending \( u \) to \( v \). Therefore, \( R\text{-Cay}(G, S) \) is transitive. \( \square \)

The right Cayley graph \( R\text{-Cay}(G_{16}, [8, 9, 10, 11]) \) in Example 4.4 is transitive since \([8, 9, 10, 11]\) satisfies the gyration condition in Theorem 4.5. We present other two examples as follows.

Example 4.6. The subset \( S = \{1, 2, 3\} \) of \( G_{16} \) has the property that for all \( g, g' \in G_{16}, s \in S \), \( \text{gyr}(g, g')(s) = I(s) = s \), which means that it has the required gyration condition stated in Theorem 4.5. The \( R\text{-Cay} \) graph \( R\text{-Cay}(G_{16}, S) \) is isomorphic to the \( L\text{-Cayley} \) graph \( \text{L-Cay}(G_{16}, S) \), shown in Fig. 2, with the same vertex labeling but different edge labeling.

Example 4.7. Consider the subset \( S' = \{8, 9, 10, 11\} \) of \( G_{16} \). Since the non-identity automorphism \( A \) swaps 8 with 9 and 10 with 11, and these four elements are self-inverse, the subset \( S' \) is symmetric and has the required gyration condition: \( \text{gyr}(g, g')(S) = S \) for all \( g, g' \in G_{16} \). The \( R\text{-Cay} \) graph \( R\text{-Cay}(G_{16}, S') \) is shown in Fig. 6. It is easily seen to be transitive. The left addition by 15 is an automorphism on the graph exchanging the two connected components, and swapping the inner and outer cycles; in particular, it sends 5 to 8, 15 to 0, and 4 to 9.

4.3. Connectedness

In this subsection, we show a relationship between the cosets of L-subgyrogroups and the connected components of \( R\text{-Cay} \) graphs.

Theorem 4.8. Let \( G \) be a finite gyrogroup and let \( S \) be a symmetric subset of \( G \) such that it right-generates an L-subgyrogroup \( H \) and \( \text{gyr}(g, h)(S) = S \) for all \( g \in G, h \in H \). Then two vertices \( u \) and \( v \) are in the same connected component of \( R\text{-Cay}(G, S) \) if and only if \( u \) and \( v \) are in the same left coset of \( H \).

Proof. First, note that from Theorem 4.3, \( R\text{-Cay}(G, S) \) is undirected. Suppose that \( u \) and \( v \) are two vertices in the same connected component. Then there is a path connecting \( u \) and \( v \), i.e.,

\[
u = ((\cdots ((v \oplus s_1) \oplus s_2) \oplus \cdots \oplus s_{n-2}) \oplus s_{n-1}) \oplus s_n,
\]

for some \( s_1, s_2, \ldots, s_n \in S \). Keep moving the parentheses to the right starting from the outer most one yields

\[
u = ((\cdots ((v \oplus s_1) \oplus s_2) \oplus \cdots \oplus s_{n-2}) \oplus s_{n-1}) \oplus s_n
\]

\[
= ((\cdots ((v \oplus s_1) \oplus s_2) \oplus \cdots \oplus s_{n-2}) \oplus s_{n-1}) \oplus s_n
\]

\[
= ((\cdots ((v \oplus s_1) \oplus s_2) \oplus \cdots \oplus s_{n-2}) \oplus s_{n-1}) \oplus s_n
\]

where \( g = ((\cdots ((v \oplus s_1) \oplus s_2) \oplus \cdots \oplus s_{n-2}) \oplus s_{n-1}) \oplus s_n \) in \( S \). Hence, \( u \) and \( v \) are in the same \( H \)-coset.

Conversely, suppose that \( u \) and \( v \) are in the same left coset. Then \( \oplus v \oplus u \in H \) and

\[
u = v \oplus ((\cdots (s_1 \oplus s_2) \oplus \cdots \oplus s_{n-1}) \oplus s_n)
\]

for some \( s_1, s_2, \ldots, s_n \in S \). Now, keep moving the parentheses to the left starting from the outer most one yields

\[
u = ((\cdots ((v \oplus s_1) \oplus s_2) \oplus \cdots \oplus s_{n-1}) \oplus s_n)
\]

\[
= ((\cdots ((v \oplus s_1) \oplus s_2) \oplus \cdots \oplus s_{n-1}) \oplus s_n)
\]

\[
= ((\cdots ((v \oplus s_1) \oplus s_2) \oplus \cdots \oplus s_{n-1}) \oplus s_n)
\]

where \( h_1 = ((\cdots (s_1 \oplus s_2) \oplus \cdots \oplus s_{n-1}) \oplus s_n) \). Hence, \( u \) and \( v \) are in the same connected component in the \( R\text{-Cay} \) graph. \( \square \)

Examples 4.4 and 4.7 are examples of this theorem. We have four left cosets and hence four connected components in Example 4.4, whereas there are two left cosets represented by two connected components in Example 4.7.

5. Isomorphisms between Cayley graphs of gyrogroups and Cayley graphs of groups

In this section, we give an application of our results in Section 4. More specifically, we provide a sufficient condition for an \( R\text{-Cay}\) graph of a gyrogroup to be a group graph. The result is analogous to
Lemma 5.1. A graph is vertex-transitive if and only if its connected components are all vertex-transitive and are pairwise isomorphic.

Lemma 5.2. If a vertex-transitive graph $G$ is a group graph, then the disjoint union of $n$ copies of $G$, called $nG$, is a group graph. More specifically, if $G$ is isomorphic to a Cayley graph $\text{Cay}(G, S)$ of a group $G$, then $nG$ is isomorphic to $\text{Cay}(G \times \mathbb{Z}_n, S \times \{0\})$.

We are now in a position to state the main result of this section.

Theorem 5.3. Let $G$ be a finite gyrogroup and let $S$ be a symmetric subset of $G$. Suppose that $S$ right-generates an L-subgyrogroup $H$ with the following properties:

1. $H$ is a group, and
2. $\text{gyr}(g, h)(S) = S$ for all $g, h \in H$.

Then the R-Cayley graph $\text{R-Cay}(G, S)$ is a group graph isomorphic to the Cayley graph $\text{Cay}(H \times \mathbb{Z}_n, S \times \{0\})$, where $n$ is the index of $H$ in $G$ (the number of the connected components of $G$).

Proof. By Theorem 4.8, the graph $G$ is a disjoint union of connected components, each corresponding to a left coset of $H$. Since $H$ is a group, each connected component of $G$ is a group graph isomorphic to the Cayley graph $\text{Cay}(H, S)$. Then, $G = \bigcup_{i=1}^{n} G_i$, where $G_i \cong \text{Cay}(H, S)$. By Lemma 5.1, $G$ is vertex-transitive. Finally, Lemma 5.2 implies that $G = \bigcup_{i=1}^{n} G_i \cong \text{Cay}(H \times \mathbb{Z}_n, S \times \{0\})$.

Example 5.4. In Example 4.7, the set $S = \{8, 9, 10, 11\}$ right-generates the L-subgyrogroup $\{0, 1, 2, 3, 8, 9, 10, 11\}$, which is a group isomorphic to the dihedral group of order 8, $D_8$. The R-Cayley graph $\text{R-Cay}(G_{16}, S)$ is a group graph isomorphic to $\text{Cay}(D_8 \times \mathbb{Z}_2, S \times \{0\})$.

6. More on transitivity of Cayley graphs

In this section, we provide necessary and sufficient conditions for the left and right Cayley graphs of a gyrogroup to be undirected cycles, hence transitive.

Theorem 6.1. Let $G$ be a finite gyrogroup and let $S$ be a symmetric subset of $G$. Then the left Cayley graph $\text{L-Cay}(G, S)$ is a union of undirected cycles if and only if $S$ has two elements—either they are inverses of each other or the two are self-inverse. In particular, $\text{L-Cay}(G, S)$ is an undirected cycle if and only if $S$ has two elements and right-generates $G$.

Proof. First, note that $\text{L-Cay}(G, S)$ is undirected since $S$ is symmetric ($a \rightarrow s \oplus u \rightarrow \oplus (s \oplus u = a)$). Suppose that $\text{L-Cay}(G, S)$ is a union of undirected cycles. Then every vertex has degree 2, implying that $S$ has two elements. Since $S$ is symmetric, either those two elements are inverses of each other or each of them is self-inverse.

Conversely, since $S$ is symmetric and has two elements, each vertex in $\text{L-Cay}(G, S)$ has degree 2, implying that it is a union of undirected cycles.

We note that Theorem 6.1 is not, in general, true for right Cayley graphs; a counterexample is shown in Fig. 4. In the case of right Cayley graphs, we need to impose a condition on $S$ as in Theorem 4.3.

Theorem 6.2. Let $G$ be a finite gyrogroup and let $S$ be a symmetric subset of $G$. Then the right Cayley graph $\text{R-Cay}(G, S)$ is a union of undirected cycles if and only if $S$ has two elements—either they are inverses of each other or the two are self-inverse and $\text{gyr}(g, s)(S) = S$ for all $g, s \in S$. In particular, $\text{R-Cay}(G, S)$ is an undirected cycle if and only if $S$ has two elements and right-generates $G$.

Proof. Suppose that $\text{R-Cay}(G, S)$ is a union of undirected cycles. Then every vertex has degree 2, implying that $S$ has two elements. Moreover, Theorem 4.3 implies that $\text{gyr}(g, s)(S) = S$ for all $g, s \in S$.

Conversely, the cardinality of $S$ suggests that each vertex of $\text{R-Cay}(G, S)$ has the in-degree and out-degree at most 2. Theorem 4.3 shows that the property $\text{gyr}(g, s)(S) = S$ induces the undirectedness on $\text{R-Cay}(G, S)$. This combined with the fact that $S$ has only two elements implies that every vertex of $\text{R-Cay}(G, S)$ has degree 2 and so the graph is a union of cycles.

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