ON FACTORIZATION OF $p$-ADIC MEROMORPHIC FUNCTIONS

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Abstract

In this paper, we study primeness and pseudo primeness of $p$-adic meromorphic functions. We also consider left (resp. right) primeness of these functions. We give, in particular, sufficient conditions for a meromorphic function to satisfy such properties. Finally, we consider the problem of permutability of entire functions.

1 Introduction

For every prime number $p$, we denote by $\mathbb{Q}_p$ the field of $p$-adic numbers, and we denote by $\mathbb{C}_p$ the completion of an algebraic closure of $\mathbb{Q}_p$, which is endowed with the usual $p$-adic absolute value. Given $a \in \mathbb{C}_p$ and $r > 0$, $d(a, r)$ and $d(a, r^-)$ are respectively the disks $\{x \in \mathbb{C}_p / |x - a| \leq r\}$ and $\{x \in \mathbb{C}_p / |x - a| < r\}$; and $C(a, r)$ is the circle $\{x \in \mathbb{C}_p / |x - a| = r\} = d(a, r) \setminus d(a, r^-)$. It is easily seen that, two disks have a non-empty intersection if and only if they are nested (i.e, one of them is contained in the other).

We denote by $\mathcal{A}(\mathbb{C}_p)$ the $\mathbb{C}_p$-algebra of entire functions in $\mathbb{C}_p$ and $\mathcal{M}(\mathbb{C}_p)$ the field of meromorphic functions in $\mathbb{C}_p$, i.e, the field of fractions of $\mathcal{A}(\mathbb{C}_p)$. Let $f(x) = \sum_{n \geq 0} a_n x^n$ be a $p$-adic entire function. For all $r > 0$, we denote by

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$|f|(r) = \max_{n \geq 0} |a_n|r^n$, the maximum modulus of $f$. This is extended to meromorphic functions $h = f/g$ by $|h|(r) = |f|(r)/|g|(r)$.

An element $f \in \mathcal{A}(\mathbb{C}_p)$ (resp. $f \in \mathcal{M}(\mathbb{C}_p)$) is said to be transcendental if it is not a polynomial (resp. rational function). Thus, a $p$-adic transcendental meromorphic function admits infinitely many zeros or infinitely many poles or both. It should also be noted that a transcendental entire function $f$ has no exceptional value or Picard value in $\mathbb{C}_p$, so for every $\beta \in \mathbb{C}_p$, the function $f - \beta$ has infinitely many zeros.

Recall also that, if $D$ is a disk and $f$ an analytic function in $D$, then $f(D)$ is a disk. Moreover if $f'$ has no zero in $D$, there exist a disk $d \subset D$ such that the restriction $f|_d : d \rightarrow f(d)$ is bi-analytic. This means that $f|_d$ is an analytic bijection and that its reciprocal function $f|_d^{-1}$ is also analytic [4].

Let $F, f, g$ be $p$-adic meromorphic functions such that $F = f \circ g$. We say that $f$ and $g$ are respectively left and right factors of $F$. The function $F$ is said to be prime (resp. pseudo-prime, resp. left-prime, resp. right-prime) if every factorization of $F$ of the above form implies that either $f$ or $g$ is a linear rational function (resp. $f$ or $g$ is a rational function, resp. $f$ is a linear rational function whenever $g$ is transcendental, $g$ is linear whenever $f$ is transcendental). If the factors are restricted to entire functions, the factorization is said to be in the entire sense. The article by Bézivin and Boutabaa [2] is to our knowledge the only work dedicated to the study of factorization of $p$-adic meromorphic functions. They show, in particular, that if an entire function $F$ is prime (resp pseudo-prime) in $\mathcal{A}(\mathbb{C}_p)$, then $F$ is prime (resp pseudo-prime) in $\mathcal{M}(\mathbb{C}_p)$. This is false in the field $\mathbb{C}$ of complex numbers. Indeed, Ozawa in [6] gives examples of complex entire functions that are prime in $\mathcal{A}(\mathbb{C})$ without being prime in $\mathcal{M}(\mathbb{C})$.

In this article, we provide sufficient conditions for a $p$-adic meromorphic function to be prime or pseudo-prime. We give examples of meromorphic functions satisfying these conditions.

We also show that almost all $p$-adic transcendental entire functions are prime, in the sense that it is most often enough to add or multiply these functions by an affinity to obtain a prime entire function.

Finally, we briefly discuss the question of permutability of entire functions. Or,
in other words, one wonders: when do we have \( f \circ g = g \circ f \) for \( p \)-adic entire functions \( f \) and \( g \)?

Our method is based on the distribution of zeros and poles of the considered functions as well as the properties of their maximum modulus.

## 2 PRIMALITY AND PSEUDO PRIMALITY

Let us first prove the following result which gives sufficient conditions for a \( p \)-adic meromorphic function to be pseudo-prime.

**Theorem 2.1.** Let \( F \) be a transcendental meromorphic function in \( \mathbb{C}_p \) whose all poles are simple except a finite number of them. Suppose moreover that, for every \( \beta \in \mathbb{C}_p \), all the zeros of the function \( F - \beta \) are simple except a finite number of them. Then \( F \) is pseudo-prime.

To prove Theorem 2.1 we need the following lemma, whose proof is given in [5]

**Lemma 2.2.** Let \( f_1, f_2 \in \mathcal{A}(\mathbb{C}_p) \) such that \( f_1'f_2 - f_1f_2' \equiv c; \ c \in \mathbb{C}_p \). So, if one of the functions \( f_1, f_2 \) is not affine then \( c = 0 \) and \( f_1/f_2 \) is a constant.

We also need the following lemma, whose proof is given in [2]

**Lemma 2.3.** Let \( F, f, g \) be three meromorphic functions in \( \mathbb{C}_p \). Suppose that \( f \) is not a rational function and that \( F = f \circ g \). Then \( g \) entire.

**Proof of Theorem 2.1.**

Suppose that \( F \) is not pseudo-prime. Hence there exist two transcendental meromorphic functions \( f \) and \( g \) such that \( F = f \circ g \). Then, by Lemma 2.3, the function \( g \) is entire. Let us write \( f \) in the form \( f = f_1/f_2 \) where \( f_1, f_2 \) are entire functions with no common zeros. As \( f \) is transcendental, we see that at least one of the functions \( f_1, f_2 \) is transcendental. So by Lemma 2.2, \( f_1'f_2 - f_1f_2' \) is a non-constant entire function, hence admits at least a zero \( \alpha \). Let us distinguish the two following cases:

1. If \( f_2(\alpha) = 0 \), then \( f_1(\alpha) \neq 0 \) and \( f_2'(\alpha) = 0 \), so \( \alpha \) is a multiple zero of \( f_2 \) and all element of the set \( g^{-1}(\{\alpha\}) \) are multiple zeros of \( f_2 \circ g \) and are multiple poles of
Since \( g \in \mathcal{A}(\mathbb{C}_p) \setminus \mathbb{C}_p[X] \), then the set \( g^{-1}(\{\alpha\}) \) is infinite and \( F \) has infinitely many multiple poles, which is a contradiction.

2. If \( f_2(\alpha) \neq 0 \), then \( f'(\alpha) = (f_1 f_2 - f_1 f_2')(\alpha)/f_2(\alpha) = 0 \). Let \( \beta = f(\alpha) \), then \( \alpha \) is a multiple zero of \( f - \beta \). But the equation \( g(x) = \alpha \) admits infinitely many solutions and for every such solution \( \omega \) we have:

\[
\begin{align*}
(F - \beta)(\omega) &= F(\omega) - \beta = f \circ g(\omega) - \beta = f(\alpha) - \beta = 0, \\
(F - \beta)'(\omega) &= (g'(\omega)) = f'(g(\omega)) \times g'(\omega) = f'(\alpha) \times g'(\omega) = 0.
\end{align*}
\]

Then \( F - \beta \) has a infinitely many zeros, which is a contradiction again. Hence \( F \) is pseudo-prime.

Theorem 2.1. provides, in particular, necessary conditions for the pseudo-primeness of \( p \)-adic entire functions, which are summarized in the following corollary:

**Corollary 2.4.** Let \( F \in \mathcal{A}(\mathbb{C}_p) \setminus \mathbb{C}_p[X] \) be such that, for every \( \beta \in \mathbb{C}_p \), all the zeros of the function \( F - \beta \) are simple except a finite number of them. Then \( F \) is pseudo-prime.

The following results give more information about the left factor in any factorization of a \( p \)-adic entire function that satisfies the above conditions.

**Theorem 2.5.** Let \( F \in \mathcal{A}(\mathbb{C}_p) \setminus \mathbb{C}_p[X] \) be such that for every \( \beta \in \mathbb{C}_p \), only finitely many zeros of \( F - \beta \) are multiple. Then \( F \) is left-prime.

**Proof.** Suppose that \( F = f \circ g \) where \( g \in \mathcal{A}(\mathbb{C}_p) \setminus \mathbb{C}_p[X] \) and \( f \in \mathcal{A}(\mathbb{C}_p) \). By Corollary 2.4, we know that \( F \) is pseudo-prime. So \( f \) is a polynomial. Suppose that \( \deg f \geq 2 \), then \( f' \) has at least one zero \( \alpha \). Since \( g \in \mathcal{A}(\mathbb{C}_p) \setminus \mathbb{C}_p[X] \), the set \( W = \{x \in \mathbb{C}_p \mid g(x) = \alpha\} \) is infinite.

Let \( \beta = f(\alpha) \). Then, for every \( \omega \in W \), we have:

\[
\begin{align*}
((F - \beta)(\omega)) &= F(\omega) - \beta = f \circ g(\omega) - \beta = f(\alpha) - \beta = 0, \\
(F - \beta)'(\omega) &= (g'(\omega)) = f'(g(\omega)) \times g'(\omega) = f'(\alpha) \times g'(\omega) = 0.
\end{align*}
\]

This means that \( F - \beta \) has infinitely many multiple zeros, a contradiction. Hence the left-primeness of \( f \) is proven.

**Theorem 2.6.** Let \( F \in \mathcal{A}(\mathbb{C}_p) \setminus \mathbb{C}_p[X] \) be such that for every \( \beta \in \mathbb{C}_p \), the function \( F - \beta \) has at most one multiple zero. Then \( F \) is prime.
Proof. By Theorem 2.5, we already see that $F$ is left-prime. So, it remains to show the right-primeness of $F$. For that, suppose that $F(z) = f \circ g$, where $f \in \mathcal{A}(\mathbb{C}_p) \setminus \mathbb{C}_p[X]$ and $g \in \mathcal{A}(\mathbb{C}_p)$. From Corollary 2.4, we know that $F$ is pseudo-prime. So $g$ is a polynomial. Suppose that $\deg g = d \geq 2$. We have $F'(z) = f'(g(z))g'(z)$. Since $f \in \mathcal{A}(\mathbb{C}_p) \setminus \mathbb{C}_p[X]$, the function $f'$ has infinitely many zeros. So we may choose an element $w \in \mathbb{C}_p$ such that $f'(w) = 0$ and $g - w$ has only simple zeros $\gamma_1, ..., \gamma_d$. Then for $i = 1, ..., d$, we have $\left\{ \begin{array}{l} F(\gamma_i) = f(w) = \beta \\ (F - \beta)'(\gamma_i) = 0 \end{array} \right.$, which means $\gamma_1, ..., \gamma_d$ are multiple zeros of $F - \beta$, a contradiction. Hence $F(z)$ is right-prime.

Theorem 2.7. Let $F$ be a $p$-adic transcendental meromorphic function that admits at most finitely many poles. Suppose that for every $\beta \in \mathbb{C}_p$, only finitely many zeros of $F - \beta$ are multiple. Then $F$ is right-prime.

To prove this theorem, we need the following lemma whose proof is given in Bézivin [3]. It is more general than Lemma 2.2.

Lemma 2.8. Let $n \geq 1$ and let $f_1, \ldots, f_n$ be $p$-adic entire functions such that the wronskian $W(f_1, \ldots, f_n)$ is a non-zero polynomial. Then $f_1, \ldots, f_n$ are polynomials.

Proof of Theorem 2.7.

Suppose that $F(z) = f \circ g$, where $f \in \mathcal{M}(\mathbb{C}_p) \setminus \mathbb{C}_p(X)$ and $g \in \mathcal{M}(\mathbb{C}_p)$. By Theorem 2.1 the function $F$ is pseudo-prime. Then $g$ is a polynomial function. Suppose that $\deg g \geq 2$. Let us write $f$ in the form $f = f_1/f_2$ where $f_1, f_2$ are entire functions with no common zeros. We have $f'(z) = W(f_2, f_1/f_2)(g(z))g'(z)$. As $f$ is transcendental and has a finite number of poles, we see that $f_1$ is transcendental and $f_2$ is polynomial. As $f \in \mathcal{M}(\mathbb{C}_p) \setminus \mathbb{C}_p(X)$, it follows by Lemma 2.8 that $W(f_2, f_1) \in \mathcal{A}(\mathbb{C}_p) \setminus \mathbb{C}_p(X)$ and admits then infinitely many zeros $\{w_n\}$ (which are not zeros of $f_2$). For every integer $n$, big enough, the equation $g(z) = w_n$ admits at least two distinct roots, which are also common roots of $\left\{ \begin{array}{l} F(z) = f(w_n) \\ F'(z) = 0 \end{array} \right.$.

Then we have a contradiction and $F(z)$ is right-prime. □
Theorem 2.9. Let $f(x) = \sum_{n \geq N} a_n x^n$ be a $p$-adic entire function such that $a_n \neq 0$, for every $n \geq N$. Suppose that there exists an integer $n_0 \geq N$ such that $(|a_n/a_{n+1}|)_{n \geq n_0}$ is a strictly increasing unbounded sequence. Then the function $f$ is pseudo-prime.

To prove Theorem 2.9 we need the following lemma whose proof is given in [4].

Lemma 2.10. Let $f$ be a $p$-adic entire function defined by $f(x) = \sum_{n \geq N} a_n x^n$. Let $\mu$ and $\nu$ be, respectively, the smallest integer and the largest one such that $|f|(r) = |a_\mu|r^\mu = |a_\nu|r^\nu$. Then:

(i) $\mu$ is the number of zeros of $f$ in the disk $d(0, r^{-})$;

(ii) $\nu$ is the number of zeros of $f$ in the disk $d(0, r)$;

(iii) $\nu - \mu$ is the number of zeros of $f$ in the circle $\mathcal{C}(0, r)$.

Proof of Theorem 2.9.

Let us, first set $r_n = |a_n/a_{n+1}|$, $\forall n \geq 0$. Quit to replace $n_0$ by a greater integer, we may assume that $n_0$ is such that: $r_{n_0} > \max\{r_0, \ldots, r_{n_0-1}\}$. In the sequel, we will show that $f$ has only simple zeros in $\mathbb{C}_p \setminus d(0, r_{n_0})$. Indeed, let $r > r_{n_0}$. We distinguish two cases:

i) $r = r_n$ for some $n \geq n_0 + 1$. As the sequence $(r_k)_{k \geq n_0}$ is strictly increasing, we have $r_{n-1} < r_n < r_{n+1}$. We then have:

$$|a_{n+1}|r_n^{n+1}/|a_n|r_n^n = |a_{n+1}/a_n|r_n = 1,$$

which means that $|a_n|r_n^n = |a_{n+1}|r_n^{n+1}$.

Moreover for every integer $\ell$, $0 \leq \ell \leq n - 1$, we have:

$$|a_\ell|r_\ell^n/|a_n|r_n^n = |a_\ell/a_n|1/r_n^{n-\ell} = |a_\ell/a_{\ell+1}| \ldots |a_{n-1}/a_n|1/r_n^{n-k} < 1.$$

Finally, for every integer $\ell$, $\ell > n + 1$, we have:

$$|a_\ell|r_\ell^n/|a_{n+1}|r_n^{n+1} = |a_\ell/a_{n+1}|r_\ell^{n-\ell}$$

$$< |a_\ell/a_{n+1}||a_{n+1}/a_{n+2}| \ldots |a_{\ell-1}a_\ell| = 1.$$

Hence $|f|(r_n) = \max_{\ell \geq 0} |a_\ell|r_\ell^n$ is reached for the two values $\ell = n$ and $\ell = n + 1$. This implies, by Lemma 2.10., that $f$ has only one zero in the circle $\mathcal{C}(0, r_n)$.

ii) Suppose now that $r$ is different from $r_n$ for all $n > n_0$. Let $n \geq n_0$ be the sole integer such that $r_n < r < r_{n+1}$. We then have, for every integer $\ell$, $0 \leq \ell \leq n$:

$$|a_\ell|r_\ell^\ell = \frac{|a_\ell|}{a_{\ell+1}} |a_{\ell+1}/a_{\ell+2}| \ldots |a_n/a_{n+1}|r_\ell^\ell \leq |a_{n+1}|r_n^{n+1-\ell}r_\ell^\ell < |a_{n+1}|r_n^{n+1}.$$
Moreover for every integer \( \ell > n + 1 \), we have:

\[
\frac{|a_\ell|r^\ell}{|a_{n+1}|r^{n+1}} = \frac{|a_\ell|}{|a_{n+1}|}r^{\ell - n - 1} \leq \frac{|a_\ell|}{|a_{n+1}|} \frac{a_{n+1}}{a_{n+2}} \ldots \frac{a_{\ell - 1}}{a_\ell} = 1.
\]

Then \( |f|(r) = \max_{\ell \geq 0} |a_\ell|r^\ell \) is reached only for \( \ell = n + 1 \). This implies, by Lemma 2.10., that \( f \) has no zero in the circle \( C(0, r) \). It follows that all the zeros of \( f \) in \( \mathbb{C}_p \setminus d(0, r_0) \) are simple. Now, for every \( \beta \in \mathbb{C}_p \), there exists \( r_\beta > 0 \) such that:

\[
|f - \beta|(r) = |f|(r), \ \forall r > \max(r_0, r_\beta).
\]

It follows that \( f - \beta \) has only simple zeros in \( \mathbb{C}_p \setminus d(0, \max(r_0, r_\beta)) \). This means that all the possible multiple zeros of \( f - \beta \) lie in \( d(0, \max(r_0, r_\beta)) \) and are therefore finitely many. Using Corollary 2.4, we complete the proof of Theorem 2.9. \( \square \)

**Corollary 2.11.** Let \( f \) be a \( p \)-adic entire function satisfying the conditions of Theorem 2.9., then for any non-zero polynomial \( P \), the \( p \)-adic meromorphic function \( g = f/P \) is pseudo-prime.

**Proof.** Note first that we may suppose that \( f \) and \( P \) have no common zeros. It is clear that \( g \) has finitely many poles. Now let \( \beta \in \mathbb{C}_p \). We see that the zeros of \( g - \beta \) are the same as those of \( f - \beta P \). Moreover, there exists \( r_\beta > 0 \) such that for every \( r > r_\beta \) we have:

\[
|f - \beta P|(r) = |f|(r).
\]

It follows, by Theorem 2.9, that the function \( f - \beta P \) (and therefore also \( g - \beta \)) has at most a finite number of multiple zeros. Thus by Theorem 2.1, the meromorphic function \( g \) is pseudo-prime. \( \square \)

**Corollary 2.12.** Let \( f(x) = \sum_{n \geq N} a_n x^n \) and \( g(x) = \sum_{n \geq N} b_n x^n \) be two \( p \)-adic entire functions such that \( a_n b_n \neq 0, \forall n \geq N \). Let \( n_0 \geq N \) be an integer such that the sequences \( (|a_n/a_{n+1}|)_{n \geq n_0} \) and \( (|b_n/b_{n+1}|)_{n \geq n_0} \) are strictly increasing and unbounded. Suppose moreover that \( \lim_{r \to +\infty} (|f|(r)/|g|(r)) = +\infty \). Then the meromorphic function \( h = f/g \) is pseudo-prime.

**Proof.** Note first that we may suppose that the entire functions \( f \) and \( g \) have no common zeros. By Theorem 2.9, we see that the entire functions \( f \) and \( g \) have at most finitely many multiple zeros. Hence the meromorphic function \( h = f/g \) has at most finitely many multiple zeros and poles.

Now let \( \beta \in \mathbb{C}_p \). We see that the zeros of \( h - \beta \) are the same as those of \( f - \beta g \). Moreover, as \( \lim_{r \to +\infty} (|f|(r)/|g|(r)) = +\infty \), there exists \( r_\beta > 0 \) such that for
every $r > r_\beta$ we have: $|f - \beta g|(r) = |f|(r)$. It follows, by Theorem 2.9, that the function $f - \beta g$ (and therefore also $h - \beta$) has at most a finite number of multiple zeros. Thus, by Theorem 2.1, the meromorphic function $h$ is pseudo-prime.

\[ \square \]

In what follows, we will provide examples of meromorphic p-adic functions satisfying the conditions of Corollary 2.12. Let’s first recall that, given a real number $x$, we call integer part of $x$ and we denote by $E(x)$ the unique integer such that $E(x) \leq x < E(x) + 1$. It is easily shown that:

**Lemma 2.13.** For all real numbers $x$ and $y$, we have:

$$E(x - y) \leq E(x) - E(y) \leq E(x - y) + 1.$$ 

**Proposition 2.14.** Let $N$ be an integer $\geq 3$ and let $\alpha, \beta \in \mathbb{C}_p$ be such that $|\beta| < |\alpha| < 1$. Let $f$ and $g$ the functions defined by $f(x) = \sum_{n=0}^{\infty} \alpha^{E(n/N)} x^n$ and $g(x) = \sum_{n=0}^{\infty} \beta^{E(n/N)} x^n$. The meromorphic function $h = f/g$ is a pseudo-prime.

**Proof.** We have $f(x) = \sum_{n=0}^\infty a_n x^n$, where $a_n = \alpha^{E(n/N)}$ for every $n \geq 0$.

We easily check that, $\lim_{n \to \infty} |a_n|^r = \lim_{n \to \infty} |\alpha^{E(n/N)}|^r = 0$, for every $r > 0$; which means that $f$ is an entire function in $\mathbb{C}_p$.

Let us now show that, if $n_0$ is an integer $\geq (2N^{-2}/(N - 1))^{1/(N-2)}$, the sequence $(|a_n/a_{n+1}|)_{n \geq n_0}$ is strictly increasing.

For every $n \geq 0$, we have: $|a_n/a_{n+1}| = (1/|\alpha|)^{E(n+1/N)-E(n/N)}$.

As the real function $x \mapsto (1/|\alpha|)^x$ is strictly increasing, it follows by Lemma 2.10. that:

$$\left(\frac{1}{|\alpha|}\right)^{E((n+1/N) - (\frac{2}{N})^N)} \leq \frac{a_n}{a_{n+1}} \leq \left(\frac{1}{|\alpha|}\right)^{E((n+1/N) - (\frac{2}{N})^N) + 1} \quad (2.1)$$

In the same way, we have:

$$\left(\frac{1}{|\alpha|}\right)^{E((\frac{2}{N})^N - (\frac{2}{N})^N)} \leq \frac{a_n + 1}{a_{n+2}} \leq \left(\frac{1}{|\alpha|}\right)^{E((\frac{2}{N})^N - (\frac{2}{N})^N) + 1} \quad (2.2)$$

It follows from these last two inequalities that:

$$\frac{a_{n+1}}{a_{n+2}} - \frac{a_n}{a_{n+1}} \geq \left(\frac{1}{|\alpha|}\right)^{E((\frac{2}{N})^N - (\frac{2}{N})^N)} - \left(\frac{1}{|\alpha|}\right)^{E((\frac{2}{N})^N - (\frac{2}{N})^N) + 1} \quad (2.3)$$
But, by Lemma 2.10., we have:
\[
E\left(\left(\frac{n+2}{N}\right)^N - \left(\frac{n+1}{N}\right)^N\right) - E\left(\left(\frac{n+1}{N}\right)^N - \left(\frac{n}{N}\right)^N\right) - 1 \geq \]
\[
E\left(\left(\frac{n+2}{N}\right)^N - \left(\frac{n+1}{N}\right)^N\right) - \left[\left(\frac{n+1}{N}\right)^N - \left(\frac{n}{N}\right)^N\right] - 1.
\]
As:\n\[
\left(\frac{n+2}{N}\right)^N - \left(\frac{n+1}{N}\right)^N = \frac{1}{N^N} \sum_{i=0}^{N-1} (n+2)^i(n+1)^{N-i-1}
\]
we have:
\[
\left(\frac{n+1}{N}\right)^N - \left(\frac{n}{N}\right)^N = \frac{1}{N^N} \sum_{i=0}^{N-1} n^i(n+1)^{N-i-1},
\]
we see that:
\[
\left(\left(\frac{n+2}{N}\right)^N - \left(\frac{n+1}{N}\right)^N\right) - \left(\left(\frac{n+1}{N}\right)^N - \left(\frac{n}{N}\right)^N\right) = \frac{1}{N^N} \sum_{i=0}^{N-1} (n+1)^{N-i-1}[((n+2)^i - n^i].
\]
Hence:
\[
\left(\left(\frac{n+2}{N}\right)^N - \left(\frac{n+1}{N}\right)^N\right) - \left(\left(\frac{n+1}{N}\right)^N - \left(\frac{n}{N}\right)^N\right) = \frac{2}{N^N} \sum_{i=0}^{N-1} (n+1)^{N-i-1} \sum_{j=0}^{i-1} (n+2)^j n^{i-j-1}.
\]
It follows that:
\[
\left(\left(\frac{n+2}{N}\right)^N - \left(\frac{n+1}{N}\right)^N\right) - \left(\left(\frac{n+1}{N}\right)^N - \left(\frac{n}{N}\right)^N\right) \geq \frac{2 n^{N-2}}{N^N} \sum_{i=0}^{N-1} i,
\]
then that:
\[
\left(\left(\frac{n+2}{N}\right)^N - \left(\frac{n+1}{N}\right)^N\right) - \left(\left(\frac{n+1}{N}\right)^N - \left(\frac{n}{N}\right)^N\right) \geq \frac{2 n^{N-2}}{N^N} \sum_{i=0}^{N-1} i,
\]
and finally:
\[
\left(\left(\frac{n+2}{N}\right)^N - \left(\frac{n+1}{N}\right)^N\right) - \left(\left(\frac{n+1}{N}\right)^N - \left(\frac{n}{N}\right)^N\right) \geq \frac{n^{N-2}(N-1)}{N^{N-1}}.
\]
It follows that for every \( n \geq n_0 \), we have:
\[
\left(\left(\frac{n+2}{N}\right)^N - \left(\frac{n+1}{N}\right)^N\right) - \left(\left(\frac{n+1}{N}\right)^N - \left(\frac{n}{N}\right)^N\right) \geq 2.
\]
Consequently for every \( n \geq n_0 \), we have:
\[
E\left(\left(\frac{n+2}{N}\right)^N - \left(\frac{n+1}{N}\right)^N\right) - E\left(\left(\frac{n+1}{N}\right)^N - \left(\frac{n}{N}\right)^N\right) - 1 > 0.
\]
It follows that for every \( n \geq n_0 \), we have:
\[
\frac{a_{n+1}}{a_{n+2}} > \frac{a_n}{a_{n+1}}.
\]
We complete the proof by applying Theorem 2.9.

In what follows, one aims to show that, given a \( p \)-adic transcendental entire function \( f \), it is always easy to transform it into a prime entire function. For this, most often, we have just to add, or multiply \( f \) by an affinity.
Theorem 2.15. Let \( f \) be a \( p \)-adic transcendental entire function. Then the set 
\( \{ a \in \mathbb{C}_p; f(x) - ax \text{ is not prime} \} \) is at most a countable set.

To prove Theorem 2.15 we need the following lemma.

Lemma 2.16. Let \( f \in \mathcal{A}(\mathbb{C}_p) \backslash \mathbb{C}_p[X] \). Then there exists a countable set \( E \subset \mathbb{C}_p \) such that for every \( a \in \mathbb{C}_p \backslash E \) and every \( b \in \mathbb{C}_p \), the function \( f(x) - (ax + b) \) has at most one multiple zero.

Proof. Let \( Z(f'') \) be the set of zeros of \( f'' \). Since \( \mathbb{C}_p \) is a separable space, there exists a countable family of disks \( (d_i)_{i \geq 1} \) such that \( \mathbb{C}_p \setminus Z(f'') = \bigcup_{i \geq 1} d_i \) and, for every \( i \geq 1 \), the restriction \( f_i' \) of \( f' \) to \( d_i \) is a bi-analytic function on \( d_i \). Then we have \( \mathbb{C}_p = (\bigcup_{i \geq 1} D_i) \cup f'(Z(f'')) \).

Let \( g \) be the function defined on \( \mathbb{C}_p \) by \( g(x) = f(x) - xf'(x) \). It should be noted that even if the family \( (d_i)_{i \geq 1} \) is chosen so that the disks \( d_i \) are pairwise disjoint, there is no guarantee that the family \( (D_i)_{i \geq 1} \) retains this property. In other words some of the disks \( D_i \) could be nested. To take account of this fact, let us set:

\[
\Gamma = \{(i, j) \in (\mathbb{N}^*)^2 / D_i \subsetneq D_j \text{ and } g \circ (f_i')^{-1} \neq g \circ (f_j')^{-1} \text{ in } D_i\};
\]
\[
\Delta_{ij} = \{x \in D_i / g \circ (f_i')^{-1}(x) = g \circ (f_j')^{-1}(x), \forall (i, j) \in \Gamma \text{ and } \Delta = \bigcup_{(i, j) \in \Gamma} \Delta_{ij}.\]

Then \( E = \mathbb{C}_p \backslash [(\bigcup_{i \geq 1} D_i) \setminus (f'(Z(f''))) \cup \Delta] \) is a countable subset of \( \mathbb{C}_p \). Indeed it is easily seen that \( E \subset (f'(Z(f''))) \cup \Delta. \)

Let \( a \in \mathbb{C}_p \backslash E \). Let \( I_a = \{ i \in \mathbb{N}^* / a \in D_i \} \). Hence the disks \( D_i \), for \( i \in I_a \), are nested. We easily show that the set of multiple zeros of the function \( f(x) - (ax + b) \) is equal to \( A = \{(f_i')^{-1}(a) / i \in I_a \text{ and } f \circ (f_i')^{-1}(a) = a(f_i')^{-1}(a) + b\} \).

Suppose that \( (f_i')^{-1}(a) \) and \( (f_j')^{-1}(a) \) are two distinct elements of \( A \). We may assume that \( D_i \nsubseteq D_j \). The fact that each of these elements is a solution of the equation \( f(x) - ax = b \) implies that: \( f((f_i')^{-1}(a)) - (f_i')^{-1}(a)f'((f_i')^{-1}(a)) = f((f_j')^{-1}(a)) - (f_j')^{-1}(a)f'((f_j')^{-1}(a)) \) or, in other words, \( g \circ (f_i')^{-1}(a) = g \circ (f_j')^{-1}(a) \). As \( a \notin \Delta \), we deduce that \( g \circ (f_i')^{-1}(x) = g \circ (f_j')^{-1}(x), \forall x \in D_i \).

By derivation of this last equation, we have: \( (f_i')^{-1}(x) = (f_j')^{-1}(x), \forall x \in D_i \) and particularly \( (f_i')^{-1}(a) = (f_j')^{-1}(a) \), which is a contradiction. Hence the set \( A \) admits at most one element. This completes the proof Lemma 2.16. \( \square \)
Proof of Theorem 2.15

Let \( E \) be the countable set of Lemma 2.16. and let \( a \in \mathbb{C}_p \setminus E \). Since \( f \) is a transcendental entire function, we see that the function \( h(x) = f(x) - ax \) is also a transcendental entire function. Lemma 2.16 then ensures that, for every \( b \in \mathbb{C}_p \), the function \( h(x) - b = f(x) - ax - b \) admits at most one multiple zero. Theorem 2.6 finally enables us to conclude that the function \( f(x) - ax = h(x) \) is prime.

Theorem 2.17. Let \( f \) be a \( p \)-adic transcendental entire function. Then the set \( \{ a \in \mathbb{C}_p; f(x)(x - a) \text{ is not prime} \} \) is at most a countable set.

To prove Theorem 2.17 we need the following lemma.

Lemma 2.18. Let \( f \in \mathcal{A}(\mathbb{C}_p) \setminus \mathbb{C}_p[X] \). Then there exists a countable set \( E' \subset \mathbb{C}_p \) such that for every \( a \in \mathbb{C}_p \setminus E' \) and every \( b \in \mathbb{C}_p \), the function \( (x - a)f(x) - b \) has at most one multiple zero.

Proof. The procedure is very similar to that used in the proof of Lemma 2.16. We easily check that an element \( \zeta \) of \( \mathbb{C}_p \) is a multiple zero of the function \( (x - a)f(x) - b \) if and only if \( \{ a = g(\zeta), b = h(\zeta) \} \), where \( g \) and \( h \) are meromorphic functions, defined in \( \mathbb{C}_p \), by:

\[
\begin{align*}
g(x) &= x + f(x)/f'(x) \\
h(x) &= (x - g(x))f(x)
\end{align*}
\]  

Let \( S(g') \) be the set of zeros and poles of \( g' \) and let \( (d_i)_{i \geq 1} \) be a countable family of disks such that \( \mathbb{C}_p \setminus S(g') = \bigcup_{i \geq 1} d_i \) and that, for every \( i \geq 1 \), the restriction \( g_i \) of \( g \) to \( d_i \) is a bi-analytic function on \( d_i \). Then we have: \( \mathbb{C}_p = (\bigcup_{i \geq 1} D_i) \cup g(S(g')) \), where \( D_i = g(d_i) \).

As noted before, the disks \( D_i \) are not necessarily pairwise disjoint. In other words, some of them could be nested. To take account of this fact, let us set:

\[
\begin{align*}
\Gamma &= \{(i, j) \in (\mathbb{N}^*)^2/D_i \subseteq D_j \text{ and } h \circ g_{i}^{-1} \neq h \circ g_{j}^{-1} \text{ in } D_i \}; \\
\Delta_{ij} &= \{x \in D_i/h \circ g_{i}^{-1}(x) = h \circ g_{j}^{-1}(x), \forall (i, j) \in \Gamma \text{ and } \Delta = \bigcup_{(i, j) \in \Gamma} \Delta_{ij} \}.
\end{align*}
\]

Then \( E' = \mathbb{C}_p \setminus [(\bigcup_{i \geq 1} D_i) \setminus ((g(S(g'))) \cup \Delta \cup (\bigcup_{i \geq 1} Z(f \circ g_{i}^{-1}))) \] is a countable subset of \( \mathbb{C}_p \). Indeed, it is clear that \( E' \subset (g(S(g'))) \cup \Delta \cup (\bigcup_{i \geq 1} Z(f \circ g_{i}^{-1})) \).
Let $a \in \mathbb{C}_p \setminus E'$. Let $I_a = \{i \in \mathbb{N}^* / a \in D_i\}$. Hence the disks $D_i$, for $i \in I_a$, are nested. We easily show, by relation (1), that the set of multiple zeros of the function $(x - a)f(x) - b$ is equal to $A = \{g_i^{-1}(a) / b = h \circ g_i^{-1}(a) \text{ for } i \in I_a\}$.

Suppose that $g_i^{-1}(a)$ and $g_j^{-1}(a)$ are two distinct elements of $A$. Hence, we have: $h(g_i^{-1}(a)) = h(g_j^{-1}(a)) = b$. Assuming that $D_i \subset \not\subset D_j$ and using the fact that $a \notin \Delta$, we have:

$$h \circ g_i^{-1}(x) = h \circ g_j^{-1}(x), \forall x \in D_i$$

(2)

From relation (1), we have:

$$h' = -fg'$$

(3)

By derivation of relation (2) and using relation (3), we obtain:

$$f \circ g_i^{-1}(x) = f \circ g_j^{-1}(x), \forall x \in D_i$$

(4)

Particularly, we have $f \circ g_i^{-1}(a) = f \circ g_j^{-1}(a)$. But, since $a \in E'$, we have $f \circ g_i^{-1}(a) = f \circ g_j^{-1}(a) \neq 0$. Using this, we deduce from relation (1) that:

$$g_i^{-1}(a) = g_j^{-1}(a),$$

a contradiction. Hence the set $A$ admits at most one element. This completes the proof Lemma 2.18.

Proof of Theorem 2.17

Let $E$ be the countable set of Lemme 2.18 and let $a \in \mathbb{C}_p \setminus E'$. Since $f$ is a transcendental entire function, we see that the function $h(x) = f(x)(x - a)$ is also a transcendental entire function. Lemma 2.18 then ensures that, for every $b \in \mathbb{C}_p$, the function $h(x) - b = f(x)(x - a) - b$ admits at most one multiple zero. Theorem 2.6 finally enables us to conclude that the function $f(x)(x - a) = h(x)$ is prime.

3 PERMUTABILITY OF ENTIRE FUNCTIONS

In this part, we consider the question of permutability of $p$-adic meromorphic functions. In other words, given two $p$-adic meromorphic functions $f$ and $g$, under what conditions can we have $f \circ g = g \circ f$? The very particular situation that we are going to study gives an idea of the extreme difficulty of exploring this problem in a general way. More precisely, we will show the following result:

Theorem 3.1. Let $P$ and $f$ be respectively a non-constant polynomial and a transcendental entire function on $\mathbb{C}_p$ such that $P \circ f = f \circ P$. Then the polynomial $P$ is of one of the following forms:
i) $P(x) = x$,  
ii) $P(x) = ax + b$, where $a, b \in \mathbb{C}_p$ such that $a$ is an $n$-th root of unity for some non-zero integer $n$.

To prove this theorem, we need the following lemmas:

**Lemma 3.2.** Let $f \in \mathcal{A}(\mathbb{C}_p)$. Then the two following assertions are equivalent:

i) $f$ is a polynomial,

ii) there exist $c > 0$ and $d \geq 0$ such that $|f(r)| \leq cr^d$, $r \to +\infty$.

**Proof.** Suppose that $f$ is a polynomial of degree $n$: $f(x) = a_0 + \ldots + a_n x^n$, $a_n \neq 0$. Since, for $r$ large enough, we have $|f(r)| = |a_n| r^n$, just take $c = |a_n|$ and $d = n$ to have the desired inequality.

Conversely, if there are $c > 0$ and $d \geq 0$ such that $|f(r)| \leq cr^d$ as $r \to +\infty$, we have for every integer $n > d$: $|f^{(n)}(r)| \leq |f(r)|/r^n \leq cr^{d-n} \to 0$. So $f^{(n)} \equiv 0$ and $f$ is a polynomial.

**Lemma 3.3.** Let $f \in \mathcal{A}(\mathbb{C}_p)$ and suppose that there exist real numbers $\alpha, \beta > 0$ and an integer $n \geq 2$ such that, $|f(\alpha r^m)| < \beta (|f(r)|)^n$, when $r \to +\infty$. Then $f$ is a polynomial.

**Proof.** We will construct real numbers $c, d > 0$ such that $|f(r)| \leq cr^d$, $r \to +\infty$.

Let $r_0$ be a real number such that $r_0 > \max\{1, \alpha^{1/(1-n)}\}$. Let us choose $c, d > 0$ such that $|f(r_0)| < cr_0^d$, $\beta < \alpha^d/c^{n-1}$, and let us set $r_k+1 = \alpha r_k^n$ for $k \geq 0$.

The sequence $(r_k)$ is strictly increasing and tends to $+\infty$. We prove by induction that: $|f(r_k)| < cr_k^d, \forall k \geq 0$. Indeed, the property holds for $k = 0$. Suppose that it holds for some integer $k \geq 0$. Then we have:

$|f(r_{k+1})| = |f(\alpha r_k^n)| < \beta (|f(r_k)|)^n < \beta c^n r_k^{dn} = c(\beta c^{n-1})r_k^{dn}$

$< c\alpha^d r_k^{dn} = c(\alpha^n r_k^n)^d = cr_k^{d+1},$

which means that this inequality is true for $k + 1$.

For every $r \in [r_k, r_{k+1}]$, there exists $t \in [0; 1]$ such that $r = r_k^{\frac{1}{d}} r_{k+1}^{\frac{1-t}{d}}$. As $\log r \mapsto \log |f|(r)$ is a convex function, we deduce that:
\[ |f|(r) \leq (|f|(r_k))^t (|f|(r_{k+1}))^{1-t} = (|f|(r_k))^t (|f|\sigma_k)^{1-t} < (|f|(r_k))^t (\beta (|f|(r_k))^n)^{1-t} < (cr_k^t (\beta c^n)^{1-t})^{1-t} < cr_k^t (\sigma_k c^n)^{1-t} = c(r_k^t (\sigma_k c^n)^{1-t}) = c(r_k^t (\sigma_k c^n)^{1-t}) = cr^d. \]

Hence \(|f|(r) < cr^d, \forall r \geq r_0.\) By Lemma 3.2, \(f\) is then a polynomial. \(\square\)

**Proof of Theorem 3.1.**

Let us set \(P(x) = a_0 + \cdots + a_kx^k,\) where \(a_0, \ldots, a_k\) are elements of \(\mathbb{C}_p\) and \(a_k \neq 0.\) Then: \((f \circ P)(x) = (P \circ f)(x) = a_0 + \cdots + a_k(f(x))^k.\) Suppose that \(k \geq 2.\) Then, for \(r > 0,\) we have \(|f|(|P|(r)) = |P|(|f|(r)).\) It follows that, for sufficiently large \(r > 0,\) we have \(|f|(|a_k|^r) = |a_k||f|(r)|^k < 2|a_k||f|(r)^k.\) Lemma 3.3 then implies that \(f\) is a polynomial, which is a contradiction. Hence, we have \(k = 1\) and \(P\) is of the form \(P(x) = ax + b,\) where \(a, b \in \mathbb{C}_p\) and \(a \neq 0.\) Two cases can then arise:

\(i)\) \(a = 1.\) Then we have \(b = 0.\) Indeed suppose that \(b \neq 0.\) Then we have, for every \(x \in \mathbb{C}_p, f(x + b) = f(x) + b.\) It follows that \(f'(x + b) = f'(x),\) for every \(x \in \mathbb{C}_p.\) Let \(\zeta\) be a zero of \(f'\) such that \(|\zeta| > |b|.\) Then \(\zeta, \zeta + b, \zeta + 2b, \ldots\) are infinitely many zeros of \(f'\) included in the disk \(d(0; |\zeta|),\) a contradiction. Hence in this case we have \(P(x) = x.\)

\(ii)\) \(a \neq 1.\) Let \(\sigma\) be the affine application \(\sigma(t) = t + b(a - 1)^{-1},\) hence its inverse \(\sigma^{-1}\) is given by \(\sigma^{-1}(t) = t + b(1 - a)^{-1}.\) Let \(F\) be the function given by \(F = \sigma \circ f \circ \sigma^{-1}.\) It is easily seen that: \(F(ax) = aF(x), \forall x \in \mathbb{C}_p.\)

If \(F(x) = \sum_{n \geq 0} b_n a^n x^n,\) we have \(F(ax) = \sum_{n \geq 0} b_n a^n x^n.\) It follows that:

\(\sum_{n \geq 0} (a^n - a)b_n x^n = 0,\) and hence \(b_0 = 0\) and \((a^n - a)b_{n+1} = 0, \forall n \geq 1.\)

Suppose that there exist two relatively prime integers \(m, n \geq 2\) such that \(b_{n+1} \neq 0,\) and \(b_{m+1} \neq 0\) we would have \(a^n = a_m = 1,\) and therefore \(a = 1,\) which excluded. Hence \(F\) has the form \(F(X) = \sum_{k \geq 1} b_{nk+1} x^{nk+1},\) where \(n\) is the smallest positive integer such that \(a^n = 1.\) It follows that:

\(f(x) = \sum_{k \geq 1} b_{nk+1} (x + b/(a - 1))^{nk+1} + b/(1 - a).\) \(\square\)
Corollary 3.4. Let $f \in \mathcal{A}(\mathbb{C}_p) \setminus \mathbb{C}_p[X]$, $f(x) = \sum_{n \geq 0} a_n x^n$. Suppose that there exist two relatively prime integers $m, l$ such that $a_m a_l \neq 0$. Then the only non-constant polynomial $P$ such that $P \circ f = f \circ P$ is $P(x) = x$.

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