Smooth approximations and CSPs
over finitely bounded homogeneous structures

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ABSTRACT
We introduce the novel machinery of smooth approximations, and apply it to confirm the CSP dichotomy conjecture for first-order reducts of the random tournament, and to give new short proofs of the conjecture for various homogeneous graphs including the random graph (STOC’11, ICALP’16), and for expansions of the order of the rationals (STOC’08). Apart from obtaining these dichotomy results, we show how our new proof technique allows to unify and significantly simplify the previous results from the literature. For all but the last structure, we moreover characterize for the first time those CSPs which are solvable by local consistency methods, again using the same machinery.

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1 INTRODUCTION AND RESULTS
1.1 Constraint Satisfaction Problems
The Constraint Satisfaction Problem, or CSP for short, over a relational structure $A$ in a finite signature is the computational problem of deciding whether a given finite relational structure $B$ in the same signature can be homomorphically mapped to $A$. The structure $A$ is known as the template or constraint language of the CSP, and the CSP of the particular structure $A$ is denoted by CSP$(A)$.

CSPs form a class of computational problems that are of interest for practitioners and theoreticians alike.

On the one hand, CSPs generalize numerous flavours of satisfiability problems that are of interest in practice. Fundamental problems in operations research like SAT solving or combinatorial optimization can readily be formulated as constraint satisfaction problems with $A$ being a structure over $\{0, 1\}$ or over the rational numbers. Moreover, planning/scheduling problems over various qualitative calculi used in temporal and spatial reasoning can be seen as constraint satisfaction problems with an infinite template.

Finite-domain CSPs, i.e., CSPs where the template is a finite structure, have been proved to either be in P or NP-complete [37, 63, 64], that is, the class of finite-domain CSPs does not contain any NP-intermediate problem, which are known to exist if P does not equal NP [51]. Moreover, the tractability border can be algorithmically decided, and it can be expressed using algebraic conditions on the set of polymorphisms of the template (definitions are given in Section 2; for the remainder of the introduction, the reader can think of polymorphisms as generalized automorphisms or endomorphisms of the template).

However, finite-domain CSPs form a proper subset of all CSPs, and in fact most of the problems that are of interest for applications can only be formulated as CSPs over infinite templates. The present paper provides a new algebraic approach to CSPs with an infinite template within the scope of the so-called Bodirsky-Pinsker conjecture. We create a general algebraic machinery that allows us: on the one hand, to study the borderline between polynomial-time tractability and NP-hardness, and in particular reprove all the complexity results known so far and obtain new ones; on the other hand, to study for the first time systematically the applicability of certain local consistency methods, more precisely the logic Datalog, and to obtain new results about Datalog-(in)expressibility of CSPs within the scope of the Bodirsky-Pinsker conjecture. We describe below the extent of these contributions after introducing the setting and the state-of-the-art in the area.

1.2 Finitely bounded homogeneous structures
The algebraic approach that underpins the proofs of the finite-domain complexity dichotomy (and virtually all theoretical work
on finite-domain CSPs) does not require the template to be finite, but also works under the assumption of \(\omega\)-categoricity. And although every computational decision problem is polynomial-time Turing-equivalent to the CSP of some infinite template [12], for a large and natural class of \(\omega\)-categorical templates, which considerably expands the class of finite templates, a similar conjecture as for finite-domain CSPs has been formulated by Bodirsky and Pinsker in 2011 (see [33]): the following is a modern formulation taking into account recent progress from [1, 2, 4–6].

**Conjecture 1.** Let \(A\) be a CSP template which is a first-order reduct of a countable finitely bounded homogeneous structure \(B\). Then one of the following holds.

- The polymorphism clone \(\text{Pol}(A)\) of \(A\) has a uniformly continuous minion homomorphism to the clone of projections \(\mathcal{P}\), and \(\text{CSP}(A)\) is NP-complete.
- The polymorphism clone \(\text{Pol}(A)\) has no uniformly continuous minion homomorphism to the clone of projections \(\mathcal{P}\), and \(\text{CSP}(A)\) is in P.

The conjectured P/NP-complete dichotomy has been demonstrated for numerous subclasses: for example for all CSPs in the class MMSNP [19], as well as for the CSPs of the first-order reducts of \((\mathbb{Q}; <)\) [17], of any countable homogeneous graph [21] (including the random graph [30]), of any unary structure [23], of the random poset [48, 49], of the homogeneous binary branching C-relation [15]; moreover, for the CSPs of representations of some relation algebras [18] as well as CSPs from the spatial reasoning formalism RCC5 [10].

It is easy to see from the definitions that the CSP of any template within the range of Conjecture 1 is in NP; moreover, the results from [4] imply that if \(A\) is such a template and \(\text{Pol}(A)\) does have a uniformly continuous minion homomorphism to \(\mathcal{P}\), then its CSP is NP-hard. Resolving Conjecture 1 therefore involves the investigation of the consequences of the absence of a uniformly continuous minion homomorphism from \(\text{Pol}(A)\) to \(\mathcal{P}\). Only one general non-trivial algebraic consequence, the existence of a pseudo-Siggers polymorphism, is known [5, 6]; it remains however open whether this implies membership of the CSP in P.

### 1.3 Reduction to the finite case

The above-mentioned complexity classifications all use the concept of **canonical functions** from [29, 34] (see also [32]). However, there are two regimes for these proofs.

In the first regime, which encompasses all the above-mentioned classifications except the ones for the first-order reducts of \((\mathbb{Q}; <)\) and of the homogeneous binary branching C-relation, canonical functions are used in order to reduce the problem to a CSP of a finite template. Originally, ad hoc algorithms were given, which were later subsumed by the general reduction from [22]. Roughly speaking, if \(A\) is a first-order reduct of a finitely bounded homogeneous structure \(B\), then a function in \(\text{Pol}(A)\) is called canonical with respect to \(B\) if it acts as \(\text{Aut}(B)\) on \(n\)-tuples, for all \(n \geq 1\); by \(\omega\)-categoricity, each of these actions has only finitely many orbits, and hence the action is on a finite set. It is a consequence of the presence of a Ramsey expansion for the structure \(B\) that canonical functions are, in a sense, not too rare: canonical functions with respect to the expansion are **locally interpolated** by \(\text{Pol}(A)\). It is an open problem whether every finitely bounded homogeneous structure \(B\) has a finitely bounded homogeneous Ramsey expansion (see [9, 28, 34, 40, 47, 56]). Until the recent proofs of Conjecture 1 for MMSNP and first-order reducts of unary structures, which established a primitive basis for the present work that we outline in Section 1.4 below, the classifications in the first regime followed the following strategy: they first identified some relations which, if preserved by \(\text{Pol}(A)\), implied a uniformly continuous minion homomorphism from \(\text{Pol}(A)\) to \(\mathcal{P}\), and hence NP-completeness of \(\text{CSP}(A)\). If none of the identified relations was preserved by \(\text{Pol}(A)\), then it was shown that the canonical functions in \(\text{Pol}(A)\) satisfied non-trivial **identities**, putting \(\text{CSP}(A)\) into P by the general result from [22]. In particular, the border between polynomial-time tractability and NP-hardness of the CSP is explained purely in terms of polymorphisms that are canonical with respect to the ground structure: more precisely, \(\text{CSP}(A)\) is in P if \(\text{Pol}(A)\) contains a pseudo-Siggers operation (modulo \(\text{Aut}(B)\)) that is canonical with respect to \(B\); otherwise \(\text{Pol}(A)\) has a uniformly continuous minion homomorphism to \(\mathcal{P}\) and \(\text{CSP}(A)\) is NP-hard.

In the second regime, canonical functions are also used to obtain a description of "good" polymorphisms, which are however themselves **not** canonical. This is not a deficiency in the proof: it is known that the tractability border for first-order reducts of \((\mathbb{Q}; <)\) cannot be explained by the satisfaction of non-trivial identities within the clone of polymorphisms that are canonical with respect to \((\mathbb{Q}; <)\). Therefore, the reduction to finite-domain CSPs from [22] cannot be used in this case, and different polynomial-time algorithms are featured in these classifications.

### 1.4 Smooth approximations

The current proof techniques suffer from several problems. First, in both regimes, they rely on a classification of all the first-order reducts of the given base structure \(B\) up to first-order interdefinability, and do a proof by case distinction for each of those. This is not an approach that can scale, as even within the scope of Conjecture 1 the number of such reducts can grow arbitrarily large, and it is even an open question to know whether this number is always finite (the question is a special case of Thom’s conjecture [60]).

Secondly, the proofs rely on identifying, for each structure \(B\), a different list of relations whose invariance under \(\text{Pol}(A)\) implies NP-hardness, and whose non-invariance should imply containment of \(\text{CSP}(A)\) in P. The non-invariance of the relations is however a **local** condition: it expresses that there exists a finite subset \(S\) of the domain on which some polymorphisms violate the relations, but the polymorphisms might well leave these relations invariant on the complement of \(S\). In order to derive significant structural information (in particular, information sufficient to prove tractability of \(\text{CSP}(A)\)) from this sort of local information, the proofs need several local-to-global arguments which are often long and highly depend on the structure \(B\) under consideration.

We develop here a different strategy which still uses canonical functions, but avoids the two problems mentioned above by a unified algebraic reasoning capable of comparing canonical with arbitrary polymorphisms independently of any concrete structure.
It is in fact based on the general comparison of two polymorphism clones $\mathcal{C}, \mathcal{D}$ with $\mathcal{C} \subseteq \mathcal{D}$.

- In our first result, the loop lemma of approximations (Theorem 11), we assume that $\mathcal{C}$ acts on the orbits of the largest permutation group $\mathcal{G}_\mathcal{D}$ within $\mathcal{D}$ as the projections. We moreover assume that $\mathcal{D}$ is a model-complete core and has no algebraicity. The lemma then roughly states that there exists an $\mathcal{C}$-invariant equivalence relation $\sim$ such that $\mathcal{C}$ acts on the $\sim$-classes as the projections, and such that either $\sim$ is preserved by uniformly continuous clone homomorphisms characterizes bounded width for structures within the scope of the Bodirsky-Pinsker conjecture. As a matter of fact, to our knowledge, every single dichotomy result that was published so far within the scope of the Bodirsky-Pinsker conjecture can be reproved using various combinations of our methods. As a proof of concept, we show in this paper how to prove the conjecture for first-order reducts of the universal homogeneous tournament – this result is new. Basically the very same proof can then be used to repro the dichotomies for the first-order reducts of the universal homogeneous graph (known as the $\mathcal{Q}$-free graph) and the universal homogeneous $K_n$-free graph for all $n \geq 3$. We also outline the dichotomy proof for expansions of the order of the rational numbers using smooth approximations. Our new proofs of these results are considerably shorter than the original ones: once our general theory is laid down, the arguments that are specifically needed for these proofs only cover a few pages, while the original proofs add up to over 70 pages of specific arguments [17, 21, 30]. Moreover, they follow the same unifying principles, thus showing the versatility and potential of our approach for Conjecture 1: this shows nicely in our new proof of the complexity dichotomy for the homogeneous graphs which, reusing the strategy of the universal homogeneous tournament from Section 4, is achieved in only a couple of pages (most of which are rather superfluous once the transfer principle of the proof is explained) and requires no more creativity, nourishing hope for a general resolution of the conjecture.

**Theorem 2.** Conjecture 1 is true for first-order reducts of the universal homogeneous tournament, of the random graph, of the universal homogeneous $K_n$-free graph for all $n \geq 3$, and for expansions of the rational numbers.

As mentioned above, in the case of the first-order expansions of $\mathcal{Q}$-free graphs, one cannot use the reduction to finite-domain CSPs from [22] and therefore one needs to exhibit specific algorithms. In the original proof and its recent refinement [17, 27], four distinct subroutines are needed, one for each minimal operation giving tractability (the so-called operations $\min, \max, \mi, \ll$), together with two “master” algorithms using these routines (one for each of the so-called operations $\pp$ and $\ll$). The advantage of our proof via a reduction to a finite-domain CSP is that we do not need to exhibit these subroutines, which come instead from the finite-domain dichotomy theorem (or from Schaefer’s boolean dichotomy theorem [59], in this special case). In particular, our approach is robust against the explosion of the number of cases that one would have to consider in a more general setting.

### 1.6 Bounded width

For fixed parameters $k \leq l$, the local consistency algorithm works as follows: given an instance, one derives the strongest possible constraints on $k$ variables that can be seen by looking at $l$ variables at a time. If one derives an empty constraint, then one can safely reject the instance. We say that the local consistency algorithm solves a CSP if whenever no empty constraint is derived, then the instance has a solution. A template $A$ has bounded width if for some $k, l$, the local consistency algorithm correctly solves $A$ (in which case the width of $A$ is the lexicographically smallest pair $(k, l)$ [41]. Since this algorithm runs in polynomial time, CSPs of templates with bounded width are in $P$.

For finite structures, an algebraic characterization of templates with bounded width is known [3], but it is also known that no condition that is preserved by uniformly continuous clone homomorphisms characterizes bounded width for structures within the scope of the Bodirsky-Pinsker conjecture [27]. Using the method
of smooth approximations, we provide a characterization of templates with bounded width for reducts of the structures mentioned above, with the exception of \((\mathbb{Q}, <)\) for which a characterization of bounded width is already known [27]. This result is entirely new and constitutes major progress towards an understanding of bounded width in general, and is the first result of its kind in that it does not rely on a previous detailed complexity classification for the CSPs. It provides further evidence that the approximation approach developed here is a fundamental tool that can be used to investigate some of the most important questions in the area of infinite-domain CSPs, especially so since the structure of our proofs for characterizing bounded width is exactly the same as the one for characterizing membership in \(P\). Our proof for the case of homogeneous graphs, building on the proofs for the random tournament and our \(P/NP\)-complete classifications, becomes almost a footnote.

**Theorem 3.** Let \(\mathcal{A}\) be a model-complete core that is a first-order reduct of \(\mathcal{B}\), where \(\mathcal{B}\) is the universal homogeneous tournament, the random graph, or the universal \(K_n\)-free graph for some \(n \geq 3\). Then the following are equivalent:

- \(\mathcal{A}\) has bounded width;
- for every \(m \geq 3\), \(\mathcal{A}\) has an \(m\)-ary pseudo-WNU polymorphism modulo \(\text{Aut}(\mathcal{B})\);
- for every \(m \geq 3\), \(\mathcal{A}\) has an \(m\)-ary pseudo-WNU polymorphism modulo \(\text{Aut}(\mathcal{B})\) that is canonical with respect to \(\mathcal{B}\).

Using Theorem 3, Wrona’s results on relational width [61, 62] have recently been improved in [54], where a collapse of the bounded width hierarchy was shown for first-order reducts of the structures in the scope of Theorem 3. Moreover, smooth approximations have also been used in [54] to solve the Datalog-rewritability problem for monadic disjunctive Datalog, thereby solving an open problem from [7] and providing another example of the versatility of our approach.

### 1.7 Outline

After providing the necessary notions and definitions in Section 2, we prove our general results on smooth approximations outlined in Section 1.4; this will be achieved in Section 3. We apply these results to obtain the \(P/NP\)-complete dichotomy, as well as the bounded width classification for the random tournament in Section 4. In Section 5, we reprove the \(P/NP\)-complete dichotomy for first-order reducts of the order of the rationals. Many proofs have been omitted due to space restrictions and can be found in the long version [55].

# 2 PRELIMINARIES

## 2.1 CSPs

If \(\mathcal{A}\) is a relational structure in a finite signature, called a CSP *template*, then \(\text{CSP}(\mathcal{A})\) is the set of all finite structures \(\mathcal{C}\) in the same signature with the property that there exists a homomorphism from \(\mathcal{C}\) into \(\mathcal{A}\). This set can be viewed as a computational problem where we are given a finite structure \(\mathcal{C}\) in that signature, and we have to decide whether \(\mathcal{C} \in \text{CSP}(\mathcal{A})\).

We will tacitly assume that all relational structures that appear in this article, as well as their signatures, are at most countably infinite. In the following, we introduce notions from model theory as well as universal algebra; for more background we refer to [46] in the former and to [45] in the latter case.

### 2.2 Model theory

A relational structure \(\mathcal{B}\) is *homogeneous* if every partial isomorphism between finite induced substructures of \(\mathcal{B}\) extends to an automorphism of the entire structure. For example, the linear order of the rationals \((\mathbb{Q}, <)\) is homogeneous; it is moreover *universal* for the class of finite linear orders in the sense that it contains any such order as an induced substructure. Homogeneity and universality define \((\mathbb{Q}, <)\) up to isomorphism. Similarly, there exist a unique universal homogeneous tournament, undirected loopless graph (called the *random graph*), and undirected loopless \(K_n\)-free graph for all \(n \geq 3\). A relational structure \(\mathcal{B}\) is called *finitely bounded* if its age, i.e., its finite induced substructures up to isomorphism, is given by a finite set \(\mathcal{F}\) of forbidden finite substructures: that is, its age consists precisely of those finite structures in its signature which do not embed any member of \(\mathcal{F}\). All of the above-mentioned universal homogeneous structures are finitely bounded. An element of \(\mathcal{F}\) is called *minimal* if no induced substructure is an element of \(\mathcal{F}\); this is the case if and only if it must be present in any set \(\mathcal{F}'\) which serves as a description of the age of \(\mathcal{B}\) as above. A structure \(\mathcal{B}\) is *Ramsey* if its age satisfies a certain combinatorial regularity property; we will not need the definition, but only a consequence from [34] which we are going to cite at the end of this section. Except for \((\mathbb{Q}, <)\), the structures above are not Ramsey, but they have a finitely bounded homogeneous *Ramsey expansion*: a linear order can be added freely (i.e., in a way that the new age consists of the old age ordered in all possible ways) so they become homogeneous, finitely bounded, and Ramsey (this follows, e.g., from the Nešetřil-Rödl theorem [57]). A structure \(\mathcal{B}\) is *transitive* if its automorphism group has only one orbit in its action on the domain of \(\mathcal{B}\). All of the above structures are transitive.

A *first-order reduct* of a relational structure \(\mathcal{B}\) is a relational structure \(\mathcal{A}\) on the same domain whose relations are first-order definable without parameters in \(\mathcal{B}\). Every first-order reduct \(\mathcal{A}\) of a finitely bounded homogeneous structure is \(\omega\)-*categorical*, i.e., its automorphism group \(\text{Aut}(\mathcal{A})\) has finitely many orbits in its componentwise action on \(n\)-tuples of elements of its domain, for all finite \(n \geq 1\). Permutation groups with the latter property are called *oligomorphic*. An \(\omega\)-categorical structure \(\mathcal{B}\) has no *algebraicity* if none of its elements is first-order definable using other elements as parameters; this is the case if all stabilizers of its automorphism group by finitely many elements have only infinite orbits (except for the orbits of the stabilized elements). We also say that arbitrary permutation groups with the latter property have no algebraicity; such groups have the property that all orbits under their action on \(n\)-tuples contain two disjoint tuples, for all \(n \geq 1\). First-order reducts of structures without algebraicity have no algebraicity.

A structure \(\mathcal{A}\) is a *model-complete core* if for all endomorphisms of \(\mathcal{A}\) and all finite subsets of its domain there exists an automorphism of \(\mathcal{A}\) which agrees with the endomorphism on the subset. If \(\mathcal{A}\) is any \(\omega\)-categorical structure, then there exists an \(\omega\)-categorical model-complete core \(\mathcal{A}'\) with the same CSP as \(\mathcal{A}\); this structure \(\mathcal{A}'\) is unique up to isomorphism [8].
A formula is *primitive positive*, in short *pp*, if it contains only existential quantifiers, conjunctions, equalities, and relational symbols. If $A$ is a relational structure, then a relation is *pp-definable* in $A$ if it can be defined by a pp-formula in $A$. For any $\omega$-categorical model-complete core $A$, all orbits of the action of $\text{Aut}(A)$ on $n$-tuples are pp-definable, for all $n \geq 1$ (this follows from [26]). In this paper, pp-definability of the binary disequility relation $\neq : \{(x, y) \in A^2 | x \neq y\}$ on a set $A$ will play a role; we will use this single notation for various sets $A$ which will however be clear from context.

2.3 Universal algebra

A *polymorphism* of a relational structure $A$ is a homomorphism from some finite power $A^n$ of the structure to $A$. The set of all polymorphisms of $A$ is called the *polymorphism clone* of $A$ and is denoted by $\text{Pol}(A)$. Polymorphism clones are special cases of *functions clones*, i.e., sets of finitary operations on a fixed set which contain all projections and which are closed under arbitrary composition.

The domain of a function clone $C$ is the set $C$ where its functions are defined; we also say that $C$ acts on $C$. The clone $C$ then also naturally acts on any power $C^I$ of $C$ (componentwise), on any invariant subset $S$ of $C$ (by restriction), and on the classes of any invariant equivalence relation $\sim$ on an invariant subset $S$ of $C$ (by its action on representatives of the classes). We write $C \curvearrowright C^I$, $C \curvearrowright S$, and $C \curvearrowright S/\sim$ for these actions, and call any such pair $(S, \sim)$ a *subfactor* of $C$. A subfactor $(S, \sim)$ of $C$ is *minimal* if $\sim$ has at least two equivalence classes and any $C$-invariant subset of $S$ intersecting two $\sim$-classes equals $S$.

For a permutation group $G$, an operation $f$ on its domain, and $n \geq 1$, we say that $f$ is $n$-*canonical* with respect to $G$ if it acts on the $G$-orbits of $n$-tuples; in other words, orbit-equivalence on $n$-tuples is invariant under $f$. It is *canonically related* to $G$ if it is $n$-canonical for all $n \geq 1$. If $B$ is a structure, we also say that $f$ is canonical with respect to $B$ if it is canonical with respect to $\text{Aut}(B)$. We say that a function clone $C$ is $n$-canonical (or canonical, respectively) with respect to $A$ if all of its functions are. In that case, we write $C^n \downarrow A$ for the action of $C$ on the set $C^n \downarrow A$ consisting of $A$-orbits of $n$-tuples of the domain $C$ of $C$. If $C$ is oligomorphic then $C^n \downarrow A$ is a clone on a finite set. Note that if $A$-ary $f$ is canonical with respect to $A$ if for all tuples $a_1, \ldots, a_k$ of the same length and all $a_1, \ldots, a_k \in A$ there exists $\beta \in G$ such that $f(a_1, \ldots, a_k) = \beta \circ f(a_1, \ldots, a_k)$. We say that it is diagonally canonical if the same holds in case $a_1 = \cdots = a_k$.

A function $f$ is *idempotent* if $f(x, \ldots, x) = x$ for all values $x$ of its domain; a function clone is idempotent if all of its functions are. A function is *essentially unary* if it depends on at most one of its variables, and *essential* otherwise.

If $G = \text{Pol}(A)$ is a polymorphism clone, then the unary functions in $C$ are precisely the endomorphisms of $A$, denoted by $\text{End}(A)$. The unary bijective functions in $C$ whose inverse is also contained in $C$ are precisely the automorphisms of $A$, denoted by $\text{Aut}(A)$. For an arbitrary function clone $C$, we write $C_e$ for the permutation group of unary functions in $C$ which have an inverse in $C$. We say that $C$ has no algebraicity if $C_e$ has no algebraicity, and we say it is oligomorphic if $C_e$ is.

An identity is a formal expression

$$s(x_1, \ldots, x_n) = t(y_1, \ldots, y_m)$$

where $s$ and $t$ are abstract terms of function symbols, and $x_1, \ldots, x_n, y_1, \ldots, y_m$ are the variables that appear in these terms. The identity is of *height* $1$, and called an *h1 identity*, if the terms $s$ and $t$ contain precisely one function symbol, i.e., no nesting of function symbols is allowed, and no term may be just a variable. A cyclic identity is an identity of the form $f(x_1, \ldots, x_n) = f(x_2, \ldots, x_n, x_1)$, and a weak near-unanimity (WNU) identity is of the form $w(x, \ldots, x, y) = \cdots = w(y, x, \ldots, x)$.

The majority identities are given by $m(x, y, x) = m(x, y, x) = m(y, x, x) = x$, the *minority* identities by $m(x, y, y) = m(x, y, x) = m(y, x, x) = y$, and the *Siggers* identity by $s(x, y, y, x, y, z) = s(y, x, z, y, z)$. Each set of identities also has a pseudo-variant obtained by composing each term appearing in the identities with a distinct unary function symbol: for example, the *pseudo-Siggers* identity is given by $e \circ s(x, y, x, y, z, y) = f \circ s(y, y, z, y, z)$.

A set $\Theta$ of identities is *satisfied* in a function clone $C$ if the function symbols of $\Theta$ can be assigned functions in $C$ in such a way that all identities of $\Theta$ become true for all possible values of their variables of the domain. A set of identities is called *trivial* if it is satisfied in the *projection clone* $C$ consisting of the projection operations on the set $\{0, 1\}$. Otherwise, the set is called *non-trivial*. A function is called a cyclic weak near-unanimity (WNU) / majority / minority / Siggers operation if it satisfies the identity of the same name. For the pseudo-variants of these identities, e.g., the pseudo-Siggers identity, and a set of unary functions $F$, we also say that a function $s$ is a pseudo-Siggers operation modulo $F$ if satisfaction of the pseudo-Siggers identity is witnessed by $s$ and unary functions from $F$.

A map $\xi : C \rightarrow D$ between function clones is called a *clone homomorphism* if it preserves arities, maps the $i$-th $n$-ary projection in $C$ to the $i$-th $n$-ary projection in $D$ for all $1 \leq i \leq n$, and satisfies $\xi(f \circ (g_1, \ldots, g_n)) = \xi(f) \circ (\xi(g_1), \ldots, \xi(g_n))$ for all $m, n \geq 1$, all $n$-ary $f \in C$, and all $m$-ary $g_1, \ldots, g_n \in C$. This is the case if and only if the map $\xi$ preserves identities, i.e., whenever some functions in $C$ witness the satisfaction of some identity in $C$, then their images under $\xi$ witness the satisfaction of the same identity in $D$. It is known that for $\omega$-categorical structures $A$ and $B$, if there exists a bijective clone homomorphism $\xi$ between $\text{Pol}(A)$ and $\text{Pol}(B)$ such that both $\xi$ and its inverse are *uniformly continuous*, then CSP($A$) and CSP($B$) are polynomial-time equivalent [31]. The exploitation of this fact is often called the *algebraic approach* to CSPs. If Conjecture 1 holds, then this same fact holds without the continuity assumption for the class of structures within the range of Conjecture 1.

A map $\xi : C \rightarrow D$ is called a *minion homomorphism* if it preserves arities and composition with projections; the latter meaning that for all $n, m \geq 1$, all $n$-ary $f \in C$, and all $m$-ary projections $p_1, \ldots, p_m \in C$, we have $\xi(f \circ (p_1, \ldots, p_n)) = \xi(f) \circ (\xi(p'_1), \ldots, \xi(p'_m))$, where $p'_i$ is the $m$-ary projection in $D$ onto the same variable as $p_i$, for all $1 \leq i \leq n$. This is the case if and only if the map $\xi$ preserves all identities in the sense above.

The existence of clone and minion homomorphisms $C \rightarrow D$ is connected to the satisfaction of non-trivial identities in a polymorphism clone $C$. Namely, there exists a clone homomorphism...
\( C \rightarrow \mathcal{P} \) if and only if every set of identities satisfied in \( C \) is trivial, in which case we say that \( C \) is \textit{equationally trivial}. Similarly, there exists a minion homomorphism \( C \rightarrow \mathcal{P} \) if and only if every set of \( h_1 \) identities satisfied in \( C \) is trivial. We say that \( C \) is \textit{equationally affine} if it has a clone homomorphism to a clone \( \mathcal{M} \) of affine maps over a finite module.

If \( C, \mathcal{P} \) are function clones with \( C \) acting on a domain \( C \) and \( \mathcal{P} \) on a finite domain, then a clone (or minion) homomorphism \( \xi : C \rightarrow \mathcal{P} \) is \textit{uniformly continuous} if for all \( n \geq 1 \) there exists a finite subset \( F \) of \( C^n \) such that \( \xi(f) = \xi(g) \) for all \( n \)-ary \( f, g \in C \) which agree on \( F \).

If \( \mathcal{F} \) is a set of functions on a fixed set \( C \), then \( \overline{\mathcal{F}} \) denotes the closure of \( \mathcal{F} \) in the topology of pointwise convergence: that is, a function \( g \) is contained in \( \overline{\mathcal{F}} \) if for all finite subsets \( F \) of \( C \), there exists a function in \( \overline{\mathcal{F}} \) which agrees with \( g \) on \( F \). For functions \( f, g \) on the same domain, and \( \mathcal{G} \) a permutation group on this domain, we say that \( f \) \textit{locally interpolates} \( g \) modulo \( \mathcal{G} \) if \( g \in \{ f \circ (a_1, \ldots, a_k) \mid a_1, \ldots, a_k \in \mathcal{G} \} \). For function clones \( C, \mathcal{G} \) on this domain, we say that \( C \) \textit{locally interpolates} \( \mathcal{G} \) modulo \( \mathcal{G} \) if every function in \( C \) locally interpolates some function in \( \mathcal{G} \) modulo \( \mathcal{G} \). If \( C \) is the function clone of those functions in \( C \) which are canonical with respect to \( \mathcal{G} \), and \( \mathcal{G} \) is the automorphism group of a homogeneous Ramsey structure in a finite signature, then \( \overline{C} \) locally interpolates \( \mathcal{G} \) modulo \( \mathcal{G} \) [32, 34]. Similarly, we define \textit{diagonal interpolation} by \( g \in \{ \beta \circ f(a, \ldots, a) \mid \beta, a \in \mathcal{G} \} \). If \( \mathcal{G} \) is the automorphism group of a homogeneous Ramsey structure in a finite signature, then every function diagonally interpolates a diagonally canonical function modulo \( \mathcal{G} \).

We say that a function clone \( C \) is a \textit{model-complete core} if its unary functions coincide with \( \overline{\mathcal{F}} \). If \( C = \text{Pol}(\mathcal{A}) \) for an \( \omega \)-categorical \( \mathcal{A} \), then this is the case if and only if \( \mathcal{A} \) is a model-complete core.

If \( \mathcal{A} \) is a relational structure and \( R \) is a relation on its domain which is pp-definable in \( \mathcal{A} \), then \( R \) is invariant under \( \text{Pol}(\mathcal{A}) \); conversely, invariant relations are pp-definable under the assumption that \( \mathcal{A} \) is \( \omega \)-categorical [26].

### 3 SMOOTH APPROXIMATIONS

We first give precise definitions of approximation notions for equivalence relations of varying strength, including the smooth approximations which appeared in the outline in Section 1.4. Then we prove the three results announced there: the loop lemma (Section 3.1), the fundamental theorem (Section 3.2), and the existence of weakly commutative functions (Section 3.3).

**Definition 4** (Smooth approximations). Let \( A \) be a set, \( n \geq 1 \), and let \( \sim \) be an equivalence relation on a subset \( S \) of \( A^n \). We say that an equivalence relation \( \eta \) on some set \( S' \) with \( S \subseteq S' \subseteq A^n \) approximates \( \sim \) if the restriction of \( \eta \) to \( S \) is a (possibly non-proper) refinement of \( \sim \); we call \( \eta \) an \textit{approximation} of \( \sim \).

For a permutation group \( \mathcal{G} \) acting on \( A \) and leaving the \( \sim \)-classes invariant as well as \( \eta \), we say that the approximation \( \eta \) is

- very smooth if orbit-equivalence with respect to \( \mathcal{G} \) is a (possibly non-proper) refinement of \( \eta \) on \( S \);
- smooth if each equivalence class \( C \) of \( \sim \) intersects some equivalence class \( C' \) of \( \eta \) such that \( C \cap C' \) contains a \( \mathcal{G} \)-orbit.

- presmooth if each equivalence class \( C \) of \( \sim \) intersects some equivalence class \( C' \) of \( \eta \) such that \( C \cap C' \) contains two disjoint tuples in the same \( \mathcal{G} \)-orbit.

Note that trivially, any very smooth approximation is smooth. Smoothness clearly implies presmoothness if \( \mathcal{G} \) has no algebraic-see our remark in the preliminaries, which will appear as an assumption in the general results to follow. Presmooth approximations are useless in that we will need at least smoothness in the fundamental theorem; however presmooth approximations can be obtained in a general setting. In the following, we observe general conditions which allow us to upgrade an approximation in strength.

**Definition 5** ("Primitivity"). Let \( A \) be a set and \( n \geq 1 \). A permutation group \( \mathcal{G} \) acting on \( A \) is \( n \)-"primitive" if for every orbit \( O \subseteq A^n \) of \( \mathcal{G} \), every \( \mathcal{G} \)-invariant equivalence relation on \( O \) containing \( (a, b) \) with \( a, b \) disjoint is full. A function clone \( C \) is \( n \)-"primitive" if \( \mathcal{G}_C \) is.

We remark that a permutation group is called \textit{primitive} if it preserves no non-trivial equivalence relations on its domain. Our notion is a suitable adaptation for \( \mathcal{G} \) acting on \( A^n \) which excludes the obvious reasons of failure of primitivity, namely orbit-equivalence (any primitive permutation group must act transitively) and equivalences given by equalities of components.

**Example 6.** The automorphism group of the universal homogeneous tournament \( \mathbb{T} \) is \( n \)-"primitive" for all \( n \geq 1 \). Indeed, let \( n \geq 1 \), let \( O \) be an orbit of \( n \)-tuples under \( \text{Aut}(\mathbb{T}) \) and let \( \sim \) be an equivalence relation containing \((a, b)\) with \( a, b \) disjoint. Without loss of generality, \( O \) can be assumed to be an orbit of injective tuples. Let \( c, d \) be arbitrary tuples in \( O \). Consider the digraph \( \mathbb{X} \) over \( 3n \) elements \( x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n \) such that the subgraphs induced by \((x_1, \ldots, x_n, y_1, \ldots, y_n)\) and \((y_1, \ldots, y_n, z_1, \ldots, z_n)\) are isomorphic to the structure induced by \((a_1, \ldots, a_n, b_1, \ldots, b_n)\) in \( \mathbb{T} \), and such that the subgraph induced by \((x_1, \ldots, x_n, z_1, \ldots, z_n)\) is isomorphic to the structure induced by \((c_1, \ldots, c_n, d_1, \ldots, d_n)\) in \( \mathbb{T} \). By universality of \( \mathbb{T} \), there is an embedding of \( \mathbb{X} \) into \( \mathbb{T} \) and by homogeneity one can assume that the embedding maps \((x_1, \ldots, x_n, z_1, \ldots, z_n)\) to \((c_1, \ldots, c_n, d_1, \ldots, d_n)\). By transitivity of \( \sim \), one obtains that \( c \sim d \), so that \( \sim \) is full.

**Lemma 7.** Let \( A \) be a set, \( n \geq 1 \), and let \( \sim \) be an equivalence relation on a subset \( S \) of \( A^n \). Let \( \mathcal{G} \) be an \( n \)-"primitive" permutation group acting on \( A \) and leaving the \( \sim \)-classes invariant. Then any approximation \( \eta \) of \( \sim \) which is presmooth is also smooth with respect to \( \mathcal{G} \).

**Proof.** Pick any equivalence class \( C \) of \( \sim \), and let \( C' \) be an \( n \)-equivalence class therein such that \( C \cap C' \) contains two disjoint tuples \( a, b \) in the same \( \mathcal{G} \)-orbit \( O \). We have \( O \subseteq C \) since \( \mathcal{G} \) preserves the \( \sim \)-classes. The support of \( \eta \) contains \( S \) and hence also \( O \); moreover, the restriction of \( \eta \) to \( O \) is invariant under \( \mathcal{G} \) and relates two disjoint tuples. Hence this restriction is full, so \( C' \) contains \( O \) as well, which yields \( C \cap C' \supseteq O \).

The following lemma, which allows us to obtain very smooth approximations from presmooth ones in our applications, might be better appreciated after perusal of Sections 3.1 and 3.2.

**Lemma 8.** Let \( A \) be a set and \( n \geq 1 \). Let \( \mathcal{G} \) be a function clone on \( A \) leaving \( \sim \) invariant, and let \( \mathcal{G} \) be a permutation group acting on \( A \) which is \( n \)-"primitive" and such that \( \mathcal{G} \) is \( n \)-canonical with respect to
\( \mathcal{G} \) be a minimal subfactor of \( \mathcal{C} \) such that \( \mathcal{G} \) is \( \sim \)-invariant. Then any \( \mathcal{C} \)-invariant approximation \( \eta \) of \( \sim \) is smooth with respect to \( \mathcal{G} \).

Proof. Consider the set \( T \subseteq A^n \) consisting of those \( a \in A^n \) which are \( \eta \)-equivalent to some disjoint tuple \( b \in A^n \) in the same \( \mathcal{G} \)-orbit. This set is invariant under \( \mathcal{C} \) since \( \sim \), \( \mathcal{G} \)-orbit-equivalence with respect to \( \mathcal{G} \), and \( \eta \) are. We claim that \( S \subseteq T \). Indeed, since \( \eta \) is a presmooth approximation of \( \sim \), we find a \( a \neq b \) in \( T \cap S \) that are not \( \sim \)-equivalent; here, we use in particular that \( \sim \) has at least two classes by the minimality of the subfactor \( (S, \sim) \). Since \( T \) is invariant under \( \mathcal{C} \), we must therefore have \( S \subseteq T \), for otherwise the restriction of the action \( \mathcal{C}^n / \mathcal{G} \) on \( S \sim \) to \( (S \cap T) / \sim \) would contradict the minimality of \( (S, \sim) \). Thus, for every \( a \in S \), there is \( b \) in the same \( \mathcal{G} \)-orbit as \( a \), disjoint from \( a \), and such that \( a \sim b \). By the \( n \)\(^{-}\)primitivity\(^{-}\)of \( \mathcal{G} \), it follows that \( \eta \) is full restricted to the \( \mathcal{G} \)-orbit of \( a \). Since \( a \) was chosen arbitrarily, it follows that \( \eta \) is very smooth. \( \square \)

### 3.1 The loop lemma

**Definition 9 (Naked set).** Let \( A \) be a set, \( n \geq 1 \), and let \( \mathcal{C} \) be a function clone on \( A \). An \( n \)-ary naked set of \( \mathcal{C} \) is a subfactor of \( (S, \sim) \) of \( \mathcal{C} \) such that \( \sim \) has at least two equivalence classes and such that \( \mathcal{C} \) acts on \( S/\sim \) by projections.

Note that any minimal naked set, i.e., one where the subfactor \( (S, \sim) \) is minimal, has the property that \( \sim \) has precisely two classes. The existence of a naked set of an oligomorphic function clone \( \mathcal{C} \) is equivalent to the existence of a uniformly continuous clone homomorphism from \( \mathcal{C} \) to \( \mathcal{P} \) [31, 42]. The loop lemma of approximations, which has its root in [19, Proposition 44], examines under certain conditions the consequence of a larger function clone \( \mathcal{D} \) not \( \sim \)-invariant under \( \mathcal{C} \).

In the proof we use the following classical result by Balatov [36]: if \( \mathcal{D} \) is an idempotent clone acting on a finite set and preserving an undirected graph that contains a cycle of odd length but no loop, then \( \mathcal{D} \) is graphically trivial. Here, by a cycle we mean any sequence of \( x_0, \ldots, x_\ell \) of vertices of the graph such that \( x_0 = x_\ell \) and such that \( x_i \) and \( x_{i+1} \) are adjacent for all \( i \in \{0, \ldots, \ell - 1\} \); the number \( \ell \) is the length of the cycle, and a cycle of length 1 is a loop.

Now let \( A \) be a set and \( n \geq 1 \), let \( \mathcal{C} \) be a function clone on \( A \), and let \( \mathcal{D} \) be an oligomorphic permutation group on \( A \) such that \( \mathcal{D} \) acts idempotently on the \( \mathcal{G} \)-orbits of \( n \)-tuples. Every binary symmetric relation on \( A^n \) is such that \( \mathcal{D} \) acts on \( A^n/\mathcal{G} \) by projections.

**Definition 10.** Let \( A \) be a set, \( n \geq 1 \), and let \( \sim \) be a \( \sim \)-invariant relation on \( A \). A pair \( (a, b) \) is called \( \mathcal{D} \)-unstable if every \( \mathcal{D} \)-invariant binary symmetric relation \( R \subseteq (A^n)^2 \) containing a pair \( (a, b) \in \mathcal{S}^2 \) such that \( a \neq b \) is disjoint and such that \( a \sim b \) also contains a pseudo-loop modulo \( \mathcal{G} \), i.e., a pair \( (c, c') \) where \( c, c' \in \mathcal{S} \) belong to the same \( \mathcal{G} \)-orbit.

**Theorem 11 (The loop lemma of approximations).** Let \( A \) be a set and \( n \geq 1 \). Let \( \mathcal{C} \subseteq \mathcal{D} \) be oligomorphic function clones on \( A \), where \( \mathcal{D} \) is a model-complete core without algebraicity and \( \mathcal{C} \) is \( n \)-canonical with respect to \( \mathcal{G} \). Suppose that \( \mathcal{C}^n / \mathcal{G} \) is an idempotent function clone on a finite set, and thus \( \mathcal{C}^n / \mathcal{G} \) is graphically trivial if it has a 1-ary minimal naked set (see, e.g., [38, Proposition 4.14]). Any such naked set \( (S', \sim') \) induces an \( n \)-ary minimal naked set \( (S, \sim) \) of \( \mathcal{C} \) with \( \mathcal{G} \)-invariant \( \sim \)-classes: the \( \sim \)-classes are simply the unions of the orbits of the \( \sim' \)-classes. In the following, a witness is a \( n \)-ary minimal naked set \( (S, \sim) \) of \( \mathcal{C} \) with \( \mathcal{G} \)-invariant \( \sim \)-classes; note that by minimality, \( \sim \) has precisely two classes.

Suppose that for any witness \( (S, \sim) \) is not \( \mathcal{D} \)-unstable, i.e., there exists an \( \mathcal{D} \)-invariant symmetric binary relation \( R \) on \( A^n \) that relates two disjoint \( n \)-tuples in the two different classes of \( \sim \), and which does not have a pseudo-loop modulo \( \mathcal{G} \). We call such any triple \( (S, \sim, R) \) an attempt (at (1)). An attempt \( (S, \sim, R) \) is good if there is \( R/\mathcal{G} \) contains no cycle of odd length, and bad otherwise. Note that for any attempt \( (S, \sim, R) \), the support of \( R \), i.e., the set of all \( y \in A^n \) such that \( R(x, y) \) holds, is \( \mathcal{D} \)-invariant and \( \sim \)-stable.

We first claim that if \( (S, \sim, R) \) is a good attempt, then (1) holds for \( (S, \sim, R) \). To this end, it suffices to show in that situation there is no \( R \)-path of even length connecting two elements of distinct \( \sim \)-classes; here, by an \( R \)-path of length \( \ell \) we mean a sequence \( x_0, \ldots, x_{\ell - 1} \) such that \( R(x_i, x_{i+1}) \) for all \( i \in \{0, \ldots, \ell - 1\} \); we say that such \( R \)-path connects the elements \( x_0 \) and \( x_{\ell - 1} \). Clearly, the support of \( R \) contains the \( \mathcal{D} \)-orbit of \( y \) and hence \( \mathcal{D} \)-orbit of \( x_{\ell - 1} \), which is a subset of the \( \mathcal{D} \)-orbit of \( x_0 \), and thus \( \mathcal{D} \)-invariant. Since \( \mathcal{D} \) has algebraicity, there exists an element \( \mathcal{G} \) which sends \( a \) to a tuple \( a' \) disjoint from \( a \) while fixing \( b \). This tuple \( a' \) belongs to the same \( \mathcal{D} \)-orbit as \( a \), and hence \( \mathcal{D} \)-invariant. This proves (1).

So suppose for contradiction that \( (S, \sim, R) \) is good and there exists an \( R \)-path of even length \( 2k \) connecting two elements of different classes of \( \sim \). Let \( v \) be the midpoint of such a path. Consider the set \( T \) of those elements in \( S \) that are connected to some element of the \( \mathcal{G} \)-orbit of \( v \) by an \( R \)-path of length \( k \). Since \( \mathcal{D} \) is a model-complete core, the \( \mathcal{G} \)-orbit of \( v \) is invariant under \( \mathcal{D} \), and hence also under
'C'. Moreover, since R is invariant under 'C', so is \( R^k \), and it follows that T is 'C'-invariant. Since T contains elements in both classes of \( \sim \), it equals S, by the minimality of \((S, \sim)\). Let a, b \in S be so that \( R(a, b) \) holds. Then the existence of an \( R \)-path of length \( k \) from \( a \) to some element in the orbit of \( v \) together with the existence of an \( R \)-path of length \( k \) from \( b \) to some element in the orbit of \( v \) shows that \( R/\mathcal{G}_G \) contains a cycle of length \( 2k + 1 \), a contradiction to goodness.

It remains to show that there exists a good attempt. Striving for a contradiction, suppose that all attempts are bad. Then there exists a \( \mathcal{D} \)-invariant symmetric relation \( Q \subseteq (A^n)^2 \) such that \( Q/\mathcal{G}_G \) has an odd cycle but no loop: any relation \( Q \) of an attempt \((S, \sim, Q)\) has this property. Pick such a relation \( Q \) with minimal support; it exists since \( \mathcal{G}_G \) is oligomorphic and hence there is only a finite number of binary relations on \( A^n \) which are invariant under it. Let \( 2\ell + 1 \) be the length of the shortest odd cycle of \( Q/\mathcal{G}_G \), and let \( M' \) consist of those elements of the support of \( Q/\mathcal{G}_G \) which belong to a cycle of length \( 2\ell + 1 \). Then \( \mathbb{E}^n/\mathcal{G}_G \) acts on \( M' \), and since \( Q/\mathcal{G}_G \cap (M')^2 \) contains an odd cycle but no loop, there exists a 1-ary minimal naked set \((S', \sim', v')\) within \( M' \), by our remarks preceding this theorem and in the first paragraph of this proof. Let \((S, \sim)\) be the corresponding witness as explained in the first paragraph, and pick \( R \) so that \((S, \sim, R)\) is an attempt; by assumption, it is bad. Let \( a, b \) be disjoint, in distinct \( \sim \)-classes, and such that \( R(a,b) \) holds. Note that by the minimality of the support of \( Q \), we have that any element in this support is connected to some element from the orbit of \( a \) by a \( Q \)-path of length \( 2\ell \): otherwise the relation \( P \subseteq Q \) defined by intersecting \( Q \) with those pairs of elements connected by a \( Q \)-path of length \( 2\ell \) to some element in the orbit of \( a \) is \( \mathcal{D} \)-invariant, has a smaller support, and is such that \( P/\mathcal{G}_G \) has an odd cycle (since the orbit of \( a \) is contained in \( M' \)) but no loop. Since the \( \mathcal{G}_G \)-orbit of \( b \) belongs to \( M' \), we have that \( b \) belongs to the support of \( Q \), and hence there exists a \( Q \)-path of length \( 2\ell \) connecting \( b \) to some element from the orbit of \( a \). Let \( v \) be the midpoint of such a path, let \( X \) be the set of those elements connected to some element of the orbit of \( v \) by a \( Q \)-path of length \( \ell \), and set \( T := R \cap X^2 \). Then \((S, \sim, T)\) is still an attempt, and therefore bad, meaning that \( T/\mathcal{G}_G \) has an odd cycle. Since \( R \) has no pseudo-loop, \( T/\mathcal{G}_G \) has no loop. But the support of \( T \) is properly contained in that of \( Q \), because otherwise the shortest odd cycle of \( Q/\mathcal{G}_G \) would be at most \( \ell + 1 \). This contradicts the assumed minimality of the support of \( Q \). \( \square \)

The following variant of the loop lemma deals with the easier case when \( \mathbb{E}^n/\mathcal{G}_G \) is equationally non-trivial.

**Theorem 12** (The second loop lemma of approximations). Let \( A \) be a set and \( n \geq 1 \). Let \( \mathcal{C} \subseteq \mathcal{D} \) be oligomorphic function clones on \( A \), where \( \mathcal{D} \) is a model-complete core without algebraicity and \( \mathcal{C} \) is \( n \)-canonical with respect to \( \mathcal{G}_G \). Suppose that \( \mathbb{E}^n/\mathcal{G}_G \) is equationally non-trivial. Let \((S, \sim)\) be a minimal subfactor of \((C \sim A^n) \) with \( \mathcal{G}_G \)-invariant \( \sim \)-classes. Then one of the following holds:

1. \( \sim \) is approximated by a \( \mathcal{D} \)-invariant equivalence relation which is presmooth with respect to \( \mathcal{G}_G \);
2. \((S, \sim)\) is \( \mathcal{D} \)-unstable.

**Proof.** The proof is as in Theorem 11 above, except that no bad attempt can possibly exist given that \( \mathbb{E}^n/\mathcal{G}_G \) is equationally non-trivial; cf. our remark preceding Theorem 11. Therefore either there exists no \( R \) such that \((S, \sim, R)\) is an attempt, and hence item (2) holds, or there exists a good attempt, which yields (1). \( \square \)

### 3.2 The fundamental theorem

Our fundamental theorem allows us to lift an action of a function clone to a larger clone. It will be applied in situations where the first case of the loop lemma holds.

**Theorem 13** (The fundamental theorem of smooth approximations). Let \( A \) be a set. Let \( \mathcal{C} \subseteq \mathcal{D} \) be function clones on \( A \), and let \( \mathcal{G} \) be a permutation group on \( A \) such that \( \mathcal{D} \) locally interpolates \( \mathcal{C} \) modulo \( \mathcal{G} \). Let \((S, \sim)\) be a subfactor of \( \mathcal{C} \) with \( \mathcal{G} \)-invariant \( \sim \)-classes.

- If \( \sim \) has a \( \mathcal{D} \)-invariant very smooth approximation \( \eta \) with respect to \( \mathcal{G} \), then there exists a clone homomorphism from \( \mathcal{D} \) to \( \mathcal{C} \sim S/\sim \).
- If \( \sim \) has a \( \mathcal{D} \)-invariant smooth approximation \( \eta \) with respect to \( \mathcal{G} \), then there exists a miniton homomorphism from \( \mathcal{D} \) to \( \mathcal{C} \sim S/\sim \).

Moreover, if \( \sim \) has finitely many classes, then the above homomorphism is uniformly continuous.

**Proof.** We first prove the statement for very smooth approximations. Our first claim then is that whenever \( f \in \mathcal{D} \) locally interpolates \( f^* \in \mathcal{C} \) modulo \( \mathcal{G} \) and \( u \in S^n \), where \( k \) is the arity of \( f \), then \( f(u)(\eta \circ \sim) f^*(u) \). Indeed, there exist \( a \in \mathcal{G} \) and a tuple \( \beta \in S^k \) such that \( f(u) = a \circ f(\beta(u)) \) (where \( \beta(u) \) is calculated componentwise). Since \( \eta \) is very smooth with respect to \( \mathcal{G} \), we have that \( \beta(u) \) and \( u \) are \( \eta \)-related in every component. Thus, with \( f \) preserving \( \eta \), we have that \( f(u) \) and \( f(\beta(u)) \) are \( \eta \)-related. Since \( a \circ f(\beta(u)) f^*(u) \in S \), and since the \( \sim \)-classes are invariant under \( \mathcal{G} \), moreover get that \( f(u) \) and \( f(\beta(u)) \) are \( \sim \)-related, proving our claim.

Consequently, whenever \( f^*, f'' \in \mathcal{C} \) are locally interpolated by \( f \in \mathcal{D} \), then they act in the same way on \( S/\sim \). We can thus define a mapping \( \xi \) which sends any function \( f \) in \( \mathcal{D} \) to the action on \( S/\sim \) of any function \( f^* \) in \( \mathcal{C} \) which is locally interpolated by \( f \).

It remains to prove that \( \xi \) is a clone homomorphism. Let \( f \in \mathcal{D} \) be of arity \( k \), let \( g \) be a \( k \)-tuple of functions in \( \mathcal{D} \) of equal arity \( m \), and let \( u \in S^m \). For every function of the tuple \( g \), we pick a function in \( \mathcal{C} \) locally interpolated by it, and we collect these functions into a \( k \)-tuple \( g' \).

By the above there exists \( v \in S^k \) such that the \( k \)-tuple \( g(u) \) (calculated componentwise) is \( \eta \)-related to \( v \) in every component, and such that \( v \sim \) related to \( g'(u) \) in every component. Then

\[
f(g(u)) \sim f(v) (\eta \circ \sim) f'(u) \sim f'(g'(u)) .
\]

whence \( \xi \) is a clone homomorphism.

If \( \xi \) has only finitely many classes, then the action of any function in \( \mathcal{D} \) on a fixed set of representatives of these classes determines the value of the function under \( \xi \), and hence \( \xi \) is uniformly continuous. This completes the proof for very smooth approximations.

Let us consider the case where the approximation \( \eta \) is smooth but not very smooth. Observe that the first claim above stating that \( f(u)(\eta \circ \sim) f^*(u) \) still holds for those tuples \( u \) whose components belong to \( \mathcal{G} \)-orbits entirely contained in \( \eta \)-equivalence classes; in fact, with this additional assumption on \( u \), no smoothness whatsoever is needed for the argument. Adding smoothness it follows,
as before, that whenever \( f', f'' \in \mathcal{C} \) are locally interpolated by \( f \in \mathcal{D} \), then they act in the same way on \( S/\sim \). We can thus define the mapping \( \xi \) in the same manner; \( \xi \) is uniformly continuous if \( \sim \) has finitely many equivalence classes. It remains to prove that \( \xi \) is a minion homomorphism. Let \( f \in \mathcal{D} \) be \( k \)-ary, let \( g \) be a \( k \)-tuple of projections in \( \mathcal{D} \) of equal arity \( m \), and let \( a \in S^m \) be a tuple whose components belong to \( \mathcal{D} \)-orbits entirely contained in \( \eta \)-equivalence classes. Then the tuple \( g(u) \) still has this property since all components of \( g \) are projections, and hence \( f(g)(u) (q \circ \sim) f'(g)(u) \) by our observation at the beginning of this paragraph. It follows that any function in \( \mathcal{C} \) locally interpolated by \( f(g) \) modulo \( \mathcal{D} \) acts like \( f'(g) \) on \( S/\sim \), and hence \( \xi(f(g)) = \xi(f'(g)) \). The proof is complete. \( \square \)

The action \( \mathcal{C} \wr S/\sim \) can be viewed as a clone homomorphism \( \phi \) which sends every function \( f \in \mathcal{C} \) to its action on \( S/\sim \). In the general setting where \( \mathcal{C} \subseteq \mathcal{D} \) and \( \mathcal{D} \) locally interpolates \( \mathcal{C} \), and where the action \( \mathcal{C} \wr S/\sim \) is idempotent, any clone homomorphism \( \xi \) from \( \mathcal{D} \) to \( \mathcal{C} \wr S/\sim \) extending \( \phi \) is necessarily unique: it can only be defined as in the proof of Theorem 13. Thus, (very) smooth approximations give a sufficient condition for the only possible extension \( \xi \) to be well-defined. In \([10, \text{Lemma } 5.1]\), it is proved that in the special case where \( \mathcal{C} \) consists of those operations in \( \mathcal{D} \) that are canonical with respect to the automorphism group of a homogeneous Ramsey structure, if the extension \( \xi \) as in the proof of Theorem 13 is well-defined, then it is automatically a minion homomorphism, although a similar statement is not known to be true for clone homomorphisms. A sufficient condition for \( \xi \) to be well-defined is given in \([10, \text{Theorem } 5.4]\) in the case where \( \mathcal{C} \wr S/\sim \) consists solely of projections. We require, however, the full strength of Theorem 13, which does not impose restrictions on \( \mathcal{C} \) or the action \( \mathcal{C} \wr S/\sim \).

### 3.3 Weakly commutative functions

The following result concerns the consequences of the second case of the loop lemma when we cannot apply the fundamental theorem of the previous section.

**Lemma 14** (Weakly commutative functions). Let \( A \) be a set, \( n \geq 1 \), and let \( \mathcal{D} \) be an oligomorphic polymorphism clone on \( A \) that is a model-complete core. Let \( \sim \) be an equivalence relation on a set \( S \subseteq A^n \) with \( \mathcal{D} \)-invariant classes, and such that \((S, \sim)\) is \( \mathcal{D} \)-unstable. Then \( \mathcal{D} \) contains a binary operation \( f \) with the property that \( f(a, b) \sim f(b, a) \) holds for all \( a, b \in A^n \) such that \( f(a, b), f(b, a) \) are in \( S \) and disjoint.

**Proof.** Let us call a pair \((a, b)\) of elements of \( A^n \) troublesome if there exists a binary \( h \in \mathcal{D} \) such that \( h(a, b), h(b, a) \in S \) are disjoint but \( h(a, b) \not\sim h(b, a) \). Note that if \((a, b)\) is not troublesome and \( g \in \mathcal{D} \), then \( g(a, b), g(b, a) \) is not troublesome either. Moreover, if \( a \) and \( b \) are in the same orbit under \( \mathcal{D} \), then \((a, b)\) is not troublesome: since \( \mathcal{D} \) is a model-complete core, \( h(a, b) \) and \( h(b, a) \) are also in the same orbit for any \( h \in \mathcal{D} \); since the classes of \( \sim \) are closed under \( \mathcal{D} \), it follows that \( h(a, b) \sim h(b, a) \).

Observe that if \( u, v \in S \) are disjoint and such that \( u + v \), then there exist \( g \in \mathcal{D} \) and \( \alpha \in \mathcal{D}_\mathcal{C} \) such that \( \alpha(g(u, v), g(v, u)) = g(u, v) \). Indeed, these are obtained by application of our assumption to the relation \( R = \{(g(u, v), g(v, u)) | g \in \mathcal{D}\} \), since \( R \) contains the pair \((u, v)\) by virtue of the first binary projection.

Therefore, if \((a, b)\) is troublesome, then setting \( d(x, y) := g(h(x, y), h(y, x)) \), where \( g, h \) have the properties above, gives us \( \alpha \circ d(a, b) = \alpha \circ g(h(a, b), h(b, a)) = g(h(b, a), h(a, b)) = d(b, a) \).

In conclusion, for every pair \((a, b)\) troublesome or not \(-\) there exists a binary function \( d \in \mathcal{D} \) such that \( d(a, b), d(b, a) \) is not troublesome: if \((a, b)\) is troublesome one can take \( d \) to be the operation we just described; if \((a, b)\) is not troublesome then the first projection works.

Let \( ((a_i, b_i))_{i \in \omega} \) be an enumeration of all pairs of tuples in \( A^n \). We build by induction on \( i \in \omega \) an operation \( f_i \in \mathcal{D} \) such that \( f_i(a, b), f_i(b, a) \) is not troublesome for any \( j < i \). For \( i = 0 \) there is nothing to show, so suppose that \( f_i \) is built. Let \( d \in \mathcal{D} \) be an operation such that

\[
(d(f_i(a_i, b_i), f_i(b_i, a_i)), d(f_i(b_i, a_i), f_i(a_i, b_i)))
\]

is not troublesome and let \( f_{i+1}(x, y) := d(f_i(x, y), f_i(y, x)) \). By definition \( f_{i+1}(a_i, b_i), f_{i+1}(b_i, a_i) \) is not troublesome, and moreover \( f_{i+1} \) also satisfies the desired property for \( j < i \) by the remark in the first paragraph.

By a standard compactness argument using the oligomorphy of \( \mathcal{D} \) (essentially from \([31]\)) and the fact that the polymorphism clone is topologically closed, we may assume that the sequence \( (f_i)_{i \in \omega} \) converges to a function \( f \). This function \( f \) satisfies the claim of the proposition. \( \square \)

Note that if \( f \) satisfies the weak commutativity property of Lemma 14, then any operation in

\[
\{\tilde{\phi} \circ f(a, \alpha) | \alpha \in \mathcal{D}_\mathcal{C} \}
\]

also satisfies the same property, since the equivalence classes of \( \sim \) are invariant under \( \mathcal{D} \). This implies that in applications, we will be able to assume that \( f \) is diagonally canonical.

### 4 THE RANDOM TOURNAMENT

Let \( T = (T, \rightarrow) \) be the universal homogeneous tournament, i.e., the unique (up to isomorphism) homogeneous structure on a countable set \( T \) with a single binary relation \( \rightarrow \) such that for all \( x, y \in T \), either \((x, y) \) is in \( \rightarrow \) (which we henceforth denote by \( x \rightarrow y \)) or \( y \rightarrow x \).

#### 4.1 The P/\text{NP}-complete dichotomy

We first prove the following.

**Theorem 15.** Let \( A \) be a first-order reduct of \( T \) that is a model-complete core. Then precisely one of the following holds:

- \( \text{Po}(A) \) has a uniformly continuous clone homomorphism to \( \mathcal{D} \);
- \( \text{Po}(A) \) contains a ternary operation that is canonical with respect to \( T \) and pseudo-cyclic modulo \( \text{Aut}(T) \).

If \( A \) is a CSP template, i.e., has a finite signature, then this yields a complexity dichotomy: in the first case CSP(\( A \)) is \( \text{NP}\)-complete by \([31]\), and in the second case CSP(\( A \)) is in \( \text{P} \) by \([22]\). Moreover, the following lemma, which is not hard to prove using \([56]\), states that the class of first-order reducts of \( T \) is essentially closed under taking model-complete cores. This allows us to derive the following corollary, which states that the two cases in Theorem 15 match...
with the two cases of Conjecture 1, showing that Theorem 2 holds for the universal homogeneous tournament.

**Lemma 16.** Let $A$ be a first-order reduct of $T$. Then the model-complete core of $A$ is either a one-element structure, or is again a first-order reduct of $T$. Moreover, if $A$ is a model-complete core that is a first-order reduct of $T$ and not of $(T; \equiv)$, then the range of every endomorphism of $A$ intersects every orbit of $\text{Aut}(T; \prec)$.

**Corollary 17.** Let $A$ be a CSP template that is a first-order reduct of $T$. Then precisely one of the following holds:

- $\text{Pol}(A)$ has a uniformly continuous minion homomorphism to $\mathcal{P}$, and $\text{CSP}(A)$ is NP-complete.
- $\text{Pol}(A)$ has no uniformly continuous minion homomorphism to $\mathcal{P}$, contains a ternary operation that is pseudo-cyclic modulo $\text{Aut}(\mathcal{T})$, and $\text{CSP}(A)$ is in $P$.

**Proof.** Let $A'$ be the model-complete core of $A$. By Lemma 16 we have that $A'$ is either a first-order reduct of $T$, or an one-element structure. If the latter is the case, then $\text{Pol}(A)$ contains a constant function, and clearly the second statement holds. In the former case, Theorem 15 applies to $A'$. In the first case of that theorem, $\text{Pol}(A')$ has in particular a uniformly continuous minion homomorphism to $\mathcal{P}$, and hence so does $\text{Pol}(A)$ by [4]. This situation moreover implies NP-hardness of $\text{CSP}(A')$ by [4]. In the second case of the theorem, $\text{Pol}(A)$ does not have a uniformly continuous minion homomorphism to $\mathcal{P}$ by [1, 2], and $\text{Pol}(A)$ has, as $\text{Pol}(A')$, a ternary pseudo-cyclic operation modulo $\text{Aut}(\mathcal{T})$ (see, for example, the proof of Corollary 6.2 in [6]). Moreover, $\text{CSP}(A') = \text{CSP}(A)$ is in $P$ by [22].

Our strategy for proving Theorem 15 is the following. First, we note that if $A$ is a first-order reduct of $(T; \equiv)$, or equivalently, if $\text{Aut}(T)$ is the full symmetric group, then the result follows easily (either $\text{Pol}(A)$ only contains essentially unary functions, and the first case of Theorem 15 holds, or it contains a binary injection, and in particular the action $\mathcal{G}_A^T \acts \{ \rightarrow, \leftarrow \}$ is essentially unary operations [58]. The latter implies $\mathcal{G}_A^T \acts \mathcal{G}_A^A \acts \mathcal{G}_A^inj$). Hence, we will have $\mathcal{G}_A^T \acts \mathcal{G}_A^inj$ when we want to apply the theory of smooth approximations.

In that situation, it will turn out that $\mathcal{G}_A^inj$ is equationally trivial as well, which uses the above-mentioned fact that $\mathcal{G}_A^T \acts \{ \rightarrow, \leftarrow \}$ is by essentially unary functions.

We then apply the loop lemma of smooth approximations (Theorem 11) to $(\mathcal{G}_A^T \acts U) \subseteq (\text{Pol}(A) \acts U)$, where $U$ is the set of injective $k$-tuples for some $k \geq 1$ large enough so that $\mathcal{G}_A^inj \acts U/\text{Aut}(A)$ is equationally trivial; such $k \geq 1$ exists by a general compactness argument using the existence of a binary injection in $\text{Pol}(A)$. The hypotheses of Theorem 11 are met (with $A := U$, $n := 1$), since $T$ and hence also $\text{Pol}(A) \acts U$ have no algebraicity, since $A$ and hence also $\text{Pol}(A) \acts U$ are model-complete cores, and since the functions of $\mathcal{G}_A^inj$ act on the orbits of injective $k$-tuples with respect to $\mathcal{G}(\text{Pol}(A)) = \text{Aut}(A)$, by definition. By the theorem, we are in one of two cases.

In the first case, there is a naked set $(S, \sim)$ for $\mathcal{G}_A^T \acts U$ that has a presmooth $(\text{Pol}(A) \acts U)$-invariant approximation. This leads, via the fundamental theorem of smooth approximations (Theorem 13), to $\text{Pol}(A)$ having a uniformly continuous clone homomorphism to $\mathcal{P}$, putting us into the first case of Theorem 15. To apply Theorem 13 with function clones $(\mathcal{G}_A^T \acts U) \subseteq (\text{Pol}(A) \acts U)$ and permutation group $\text{Aut}(T) \acts U$, we have to show that $\mathcal{G}_A^inj \acts U$ locally interposes $\mathcal{G}_A^inj \acts U$ modulo $\text{Aut}(T) \acts U$; this we do by showing that $\text{Pol}(A)$ locally interposes $\mathcal{G}_A^T$ and using $\mathcal{G}_A^T \acts \mathcal{G}_A^A \acts \mathcal{G}_A^inj$. We also need to observe that $\text{Aut}(T)$ leaves the $\sim$-classes invariant, which is clear since they are unions of $\text{Aut}(A)$-orbits by definition. Finally, Lemma 8 applies since $\sim$ is invariant under $\text{Pol}(A)$ and the universal homogeneous tournament is $n$-“primitive” for all $n \geq 1$ (Example 6), so that the approximation is very smooth.

The second case of the loop lemma leads to a contradiction, since using the weakly commutative function from Lemma 14 we can produce a function in $\mathcal{G}_A^T$ satisfying a non-trivial identity. The proof of Theorem 15 is complete.

### 4.2 Bounded width

We show that if $A$ is a first-order reduct of $T$ that is a model-complete core, then it has bounded width if and only if $\mathcal{G}_A^T$ is not equationally affine, i.e., does not have a clone homomorphism to any clone of affine maps over a finite module.

**Theorem 18.** Let $A$ be a first-order reduct of $T$ that is a model-complete core. Then precisely one of the following holds:
• Pol(\(A\)) has a uniformly continuous clone homomorphism to the clone of affine maps over a finite module;
• Pol(\(A\)) contains for all \(n \geq 3\) an \(n\)-ary operation that is canonical with respect to \(T\) and a pseudo-WNU modulo Aut(\(T\)).

If \(A\) is a CSP template, i.e., has a finite signature, this gives us indeed a mathematical criterion for bounded width: in the first case \(A\) does not have bounded width by results from \([31, 52]\), and in the second case it does by \([22]\). It also proves Theorem 3 for the structure \(\mathbb{B} = T\): the first item (bounded width) of that theorem implies that the second item of Theorem 18 holds, which is identical to the third item of Theorem 3. This in turn implies the second item of Theorem 3 trivially. Finally, the second item of Theorem 3 implies the second item of Theorem 18 (being incompatible with the first item), which implies the first item of Theorem 3.

The following corollary is a description of bounded width without the assumption that \(A\) is a model-complete core.

**Corollary 19.** Let \(A\) be a CSP template that is a first-order reduct of \(T\). If Pol(\(A\)) has a uniformly continuous mininum homomorphism to the clone of affine maps over a finite module, then \(A\) does not have bounded width. Otherwise, Pol(\(A\)) contains operations of all arities \(\geq 3\) that are pseudo-WNU modulo Aut(\(T\)), and \(A\) has bounded width.

**Proof.** The first statement is an immediate consequence of general results from \([31, 52]\). For the second statement, let \(A'\) be the model-complete core of \(A\). By Lemma 16 we have that \(A'\) is either a first-order reduct of \(T\) or a one-element structure. If the latter is the case, then Pol(\(A\)) contains a constant function, and the conclusion of our statement holds. We may thus assume that \(A'\) is a first-order reduct of \(T\). The assumption of Pol(\(A\)) not having a uniformly continuous continuous minor homomorphism to the clone of affine maps over a finite module implies that the same is true for Pol(\(A'\)) by \([4]\), and hence the second item of Theorem 18 applies to \(A'\). It follows that \(A'\) and hence also \(A\) has bounded width by \([22]\). Moreover, the identities in Pol(\(A'\)) provided by that item lift to Pol(\(A\)) (see, for example, the proof of Corollary 6.2 in \([6]\)), proving the statement. \(\square\)

The proof strategy for Theorem 18 is similar as for Theorem 15. As before, we may assume that \(A\) is not a first-order reduct of \((T; =)\), since otherwise the result holds trivially. It is known if \(\mathcal{C}_A^T,\text{inj}\) is not equationally affine, then the second item of Theorem 18 applies, by analogous results for finite structures (\([53]\) and \([50, Theorem 2.8]\)) and standard lifting techniques (see \([35]\)).

Otherwise, we show in several steps that Pol(\(A\)) has a uniformly continuous clone homomorphism to the clone of affine maps over a finite module. To this end, we first establish that if \(\mathcal{C}_A^T,\text{inj}\) is equationally affine, then so is \(\mathcal{C}_A^{T,\text{inj}} \sim \langle \rightarrow, \rightarrow \rangle\); moreover, \(\mathcal{C}_A^{T,\text{inj}}\) is locally interpolated by Pol(\(A\)), and \(\mathcal{C}_A^{T,\text{inj}}\) is equationally affine as well. We then prove that \(\mathcal{C}_A^{T,\text{inj}}\) acts by functions from a clone \(\mathcal{M}\) of affine maps over a finite module is approximated by a Pol(\(A\))-invariant equivalence relation, then Pol(\(A\)) has a uniformly continuous clone homomorphism to \(\mathcal{M}\) by Theorem 13. Otherwise, Theorem 12 and Lemma 14 imply that Pol(\(A\)) has a weakly commutative binary operation. We then prove that \(\mathcal{C}_A^{T,\text{inj}} \sim \langle \leftrightarrow, \rightarrow \rangle\) contains a majority operation, which contradicts the fact that \(\mathcal{C}_A^{T,\text{inj}} \sim \langle \leftrightarrow, \rightarrow \rangle\) is equationally affine.

### 5 TEMPORAL CSPS

We now outline a short proof of the complexity classification from \([16, 17]\) for first-order expansions of \((Q; <)\) using the theory of smooth approximations. Using Cameron’s classification of first-order reducts of \((Q; <)\) \([39]\), it is easy to either derive a proof of the P/NP-complete dichotomy for first-order reducts from the one for expansions, or alternatively to show that every proper first-order reduct \(A\) of \((Q; <)\) (i.e., every \(A\) that is not a reduct of \((Q; =)\)) and such that \((Q; <)\) is not a reduct of \(A\) is such that CSP(\(A\)) is NP-hard \([17]\). Either way, we rather choose to present the result only for expansions so as to keep the argument concise, in order to showcase the most important part of our proof.

**Theorem 20** (see \([17]\)). Let \(A\) be a first-order expansion of \((Q; <)\) that is a model-complete core. Then one of the following holds.

• Pol(\(A\), 0) has a uniformly continuous clone homomorphism to \(\mathcal{P}\), and CSP(\(A\)) is in P;
• Pol(\(A\), 0) has no such clone homomorphism, and CSP(\(A\)) is NP-complete;

In the situation of the first item, NP-completeness follows by the general theory \([31]\). In order to show polynomial-time solvability of CSP(\(A\)) in the absence of a uniformly continuous clone homomorphism from Pol(\(A\), 0) to \(\mathcal{P}\), we apply twice the combination of Theorem 11, Theorem 13, and Lemma 14.

As in Section 4, we introduce various subclones of Pol(\(A\)):

• \(\mathcal{C}_A^Q\) is the clone of functions in Pol(\(A\)) that are canonical with respect to \(Q\);
• \(\mathcal{C}_A^{f\text{in}}\) is the clone of functions in Pol(\(A\), 0) preserving the equivalence relation \(\Theta\) on the set of non-negative rationals whose blocks are \(\{0\}\) and \(Q_{>0}\), the set of positive rationals.

If \(\mathcal{C}_A^{f\text{in}}\) is not equationally trivial (a case that in fact does not happen but is covered by our theory), then CSP(\(A\)) is in P by \([22]\). Otherwise, by the loop lemma (Theorem 11), either some naked set for \(\mathcal{C}_A^{f\text{in}}\) is pseudo-smoothly approximated (and very smoothly approximated, by Lemma 8) and there is a uniformly continuous clone homomorphism from Pol(\(A\), 0) to \(\mathcal{P}\) by the fundamental theorem (Theorem 13), or some naked is Pol(\(A\))-unstable and Lemma 14 gives the existence of a weakly commutative operation in Pol(\(A\)).

We then apply the same procedure to \(\mathcal{C}_A^{f\text{in}} \sim Q/\Theta\); if \(\mathcal{C}_A^{f\text{in}} \sim Q/\Theta\) is not equationally trivial, then CSP(\(A\)) is in P, which we obtain by recasting two known algorithms for temporal CSPs as polynomial-time Turing reductions to CSP(\(A^{f\text{in}}\)), where \(A^{f\text{in}}\) is a finite-domain structure whose polymorphisms contain \(\mathcal{C}_A^{f\text{in}} \sim Q/\Theta\). Otherwise, either \(\Theta\) is very smoothly approximated and Pol(\(A\), 0) admits a uniformly continuous clone homomorphism to \(\mathcal{P}\) (Theorem 13), or by a combination of Lemma 14 and the operation obtained in the previous paragraph, Pol(\(A\), 0) contains an operation acting as a semilattice operation on the orbits of Aut(\(Q\), 0), which we show leads to a contradiction to \(\mathcal{C}_A^{f\text{in}} \sim Q/\Theta\) being equationally trivial.

The P/NP-complete complexity dichotomy for CSPs of first-order reducts of \((Q; <)\) has been refined in \([27]\), where the descriptive
complexity of such CSPs is investigated. We observe that the positive results about membership in fixed-point logic (with or without counting, and with or without the rank operator) in [27] (see Proposition 3.6, Proposition 3.13, Proposition 3.23, Proposition 4.10 in [35] for the most recent exposition) can also be derived from our approach by directly lifting the membership of CSP($\mathcal{A}^{fin}$) in FP/FPR.

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