A POLYHEDRAL APPROACH TO COMPUTING BORDER BASES

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ABSTRACT. Border bases can be considered to be the natural extension of Gröbner bases that have several advantages. Unfortunately, to date the classical border basis algorithm relies on (degree-compatible) term orderings and implicitly on reduced Gröbner bases. We adapt the classical border basis algorithm to allow for calculating border bases for arbitrary degree-compatible order ideals, which is independent from term orderings. Moreover, the algorithm also supports calculating degree-compatible order ideals with preference on contained elements, even though finding a preferred order ideal is NP-hard. Effectively we retain degree-compatibility only to successively extend our computation degree-by-degree.

The adaptation is based on our polyhedral characterization: order ideals that support a border basis correspond one-to-one to integral points of the order ideal polytope. This establishes a crucial connection between the ideal and the combinatorial structure of the associated factor spaces.

1. INTRODUCTION

Gröbner bases are fundamental tools in commutative algebra to model and perform important operations on ideals such as intersection, membership test, elimination, projection, and many more. More precisely, the Gröbner bases framework makes these operations computationally accessible allowing to perform actual computations on ideals.

1.1. Comparing border bases and Gröbner bases. Unfortunately, Gröbner bases are not always well suited to perform these operations, in particular when the actual ideals under consideration are inferred from measured data. In fact, Gröbner bases do not react smoothly to small error in the input data. Border bases are a natural generalization of Gröbner bases that are believed to deform smoothly in the input (cf. [26]) and that have been used for computations with numerical data (cf., e.g., [17,1]).

In [17] a numerically stable version of the Buchberger–Möller algorithm has been derived which heavily relies on the usage of border bases in order to ensure numerical stability. An alternative algorithm based on border bases with similar stability behavior was investigated in [1]. Both algorithms calculate bases of ideals of points which can be used to derive polynomial models from empirical datasets.

It is well known that every Gröbner basis with respect to a degree-compatible term ordering can be extended to a border basis (see [20], p. 281ff) but not every border basis is an extension of a Gröbner basis. Moreover, not every order ideal supports a border basis even if it has the right cardinality. An example illustrating these two cases is presented in [20, Example 6].

While the border basis algorithm in [20], which is a specification of Mourrain’s generic algorithm [28], allows for computing border bases of zero-dimensional ideals for order ideals supported by a degree-compatible term ordering, it falls short to provide a border basis for more general order ideals: The computed border basis is supported by a reduced Gröbner basis. The alternative algorithm presented in [20, Proposition 5] which can potentially compute arbitrary border bases requires the a priori knowledge of the order ideal that might support a border basis so that the order ideal has to be guessed.
in advance. As we cannot expect this prior knowledge, the algorithm does not solve the problem of characterizing those order ideals for which a border basis does exist. Further, as pointed out in [20, p. 284], the basis transformation approach of this algorithm is unsatisfactory as it significantly relies on Gröbner basis computations.

It can be favorable though to be able to compute border bases for order ideals that do not necessarily stem from a degree-compatible term ordering. It has been an open question to characterize those order ideals of a given zero-dimensional ideal $I$ that support a border basis. We will answer this question for degree-compatible order ideals by resorting to polyhedral combinatorics, mentioned in Subsection 1.2. The restriction to degree-compatible order ideals is due to the design of the algorithm to proceed degree-wise and the authors believe that this restriction can be overcome as well.

The preference of border bases over Gröbner bases partly arises from the iterative generation of linear syzygies, inherent in the border basis algorithm, which allows for successively approximating the basis degree-by-degree. Moreover, the border basis algorithm is a linear algebra algorithm with a tiny exception, the final step, which does not contribute to the inherent complexity though as its running time is polynomial in the input size.

1.2. Polynomial method. In discrete mathematics and combinatorial optimization, polynomial systems have been used to formulate combinatorial problems such as the graph coloring problem, the stable set problem, and the matching problem (see e.g., [13] for an extensive list of references). This well-known method, which Alon referred to as the polynomial method (cf. [3, 4]) recently regained strong interest. This emphasizes the alternative view of the border basis algorithm as a proof system which successively uncovers hidden information by making it explicit. In [14, Section 2.3] and [13, 12] infeasibility of certain combinatorial problems, e.g., 3-colorability of graphs is established using Hilbert’s (complex) Nullstellensatz and the authors provide an algorithm Nu1LA to establish infeasibility by using a linear relaxation. The core of the algorithm is identical to the $L$-stable span procedure used in the border basis algorithm, which intimately links both procedures. The difference is of a technical nature: whereas Nu1LA establishes infeasibility, the classical border basis algorithm as presented in [20] computes the actual border bases of the ideal. Another recent link between border bases and the Sherali–Adams closure (introduced in [33]), a method for convexification, was investigated in [31]. The authors show that the Sherali–Adams procedure can be understood as a weaker version of the $L$-stable span procedure, which is an essential part of the border basis algorithm. As a consequence a new tighter relaxation than the Sherali–Adams closure is derived by exploiting the stronger $L$-stable span procedure.

1.3. Applications of border bases. Surprisingly, it turns out that there are deep connections to other mathematical disciplines, and border bases represent the combinatorial structure of the ideal under consideration in a canonical way. Although the use of border basis as a concise framework is quite recent, introduced in [19, 20, 21] and [25, Section 6.4], the concept of border basis is rather old and appeared in different fields of mathematics including computer algebra, discrete optimization, logic, and cryptography (independently) under different names. Originally, border bases were introduced in computer algebra as a generalization of Gröbner bases to address numerical instabilities. Border bases have been successfully used since for solving zero-dimensional systems of polynomial equations (see, e.g., [5, 27, 28]), which in particular include those with solutions in $0/1$ and thus a large variety of combinatorial problems.

Border bases have been also used to solve sparse quadratic systems of equations thus giving rise to applications in cryptography in a natural way. Such systems arise from crypto systems (such as AES, BES, HFE, DES, CTC variants, etc.) when rewriting the S-boxes as polynomial equations. The celebrated XL, XSL, MutantXL attacks, which are based on relinearization methods, are essentially equivalent to the reformulation-linearization-technique (RLT) of Sherali and Adams [33] and use a version of the Nullstellensatz to break ciphers. In fact the XL algorithm (see e.g., [22, 8]) in its classical
form is actually identical to a level \( d \) Sherali–Adams closure of the associated system and therefore it is a border bases computation at its core. Motivated by the success of the aforementioned methods, border bases have also been used in crypto analysis and coding theory, see [6].

Another application of border bases is the modeling of dynamic systems from measured data (see e.g., [17, 23, 1]) where the better numerical stability can be advantageous.

Our contribution. A perennial problem in various applications is that the classical computation of a border basis depends on a degree-compatible term ordering and hence finds only the border bases supported by an order ideal induced by the ordering. These border bases do represent only a rather small fraction of all possible border bases (see Example 2.6). Further, in this case the border basis contains a reduced Gröbner basis, and for example the theoretically nice numerical/symmetry properties are often lost, as the vector space basis of \( K[X]/I \) (where \( I \) is a zero-dimensional ideal) might not be optimally suited for numerical computation. Moreover, for example when solving large systems of polynomial equations, it is desirable to guide the solution process by having a vector space basis that does actually have a nice interpretation attached. These bases often cannot be obtained by a degree-compatible term ordering.

Techniques from commutative algebra, and in particular border bases (and alike) have been of great value to discrete mathematics and combinatorial optimization. This time we will proceed the other way around. We will apply the combinatorial optimization toolbox to the combinatorial structure of zero-dimensional ideals and we solve the aforementioned problems by freeing the computation of a border basis from the term ordering. Our contribution is the following:

Polyhedral characterization of all border bases: We provide a complete, polyhedral characterization of all degree-compatible border bases of any zero-dimensional ideal \( I \): we associate an order ideal polytope \( P \) to \( I \) that characterizes all its degree-compatible border bases. The integral 0/1 points in the order ideal polytope are in one-to-one correspondence with degree-compatible order ideals supporting a border basis of \( I \). (An order ideal is degree-compatible if it is a degree-wise complement of \( I \), see Definition 2.3.) This explicitly establishes the link between the combinatorial structure of the basis of the factor space and the structure of the ideal. Whether an order ideal supports a border basis is solely determined by the combinatorial structure of the order ideal polytope. A related result for Gröbner bases of a vanishing ideal of generic points was established in [29].

Computation of border bases not coming from a term ordering: We outline an algorithm based on the classical border basis algorithm as defined in [20] and show how to compute border bases for arbitrary degree-compatible order ideals without relying on a specific degree-compatible term ordering. Recall that not every order ideal supports a border basis and guessing the order ideal in advance is permissive. It is also not advised to search for an admissible order ideal with brute force as the combinatorial structure of the ideal can be so complicated that it is not even possible to easily guess a (non-term ordering induced) feasible order ideal, not even to mention to find one that has a preferred structure (see also the next point).

Finding preferred order ideals: Our approach is also able to compute order ideals maximizing a prespecified preference function (and compute their border bases). We will show that computing a preference-optimal order ideal supporting a border basis is NP-hard in general, which is surprising, as choosing the order ideal is merely a basis transformation. The NP-hardness does not come from the hardness of computing the \( L \)-stabilized span as the problem remains NP-hard in cases where the \( L \)-stabilized span is small enough to be determined efficiently.

Computational feasibility: We provide computational tests that demonstrate the feasibility of our method.

Applications: The structure of ideals and counting of border bases: Having the order ideal polytope available for a zero-dimensional ideal \( I \), it is possible to examine the structure of the ideal
based on its border bases. One straight-forward application is counting the number of degree-compatible border bases for a zero-dimensional ideal $I$.

Order ideals and determining those with maximum score do appear in a very natural way in combinatorial optimization as the so called maximum weight closure problem (cf. e.g., [30]) and they have a variety of applications, e.g., in open-pit mining where any feasible production plan is indeed an order ideal; clearly we can only mine the lower levels after having mined the upper ones. A good survey as well as an introduction to the problem can be found in [18].

Other, more involved applications might arise, e.g., in computational biology where the structure of boolean network is inferred from the Gröbner fan. Due to their higher numerical stability, border bases might be a better choice here and the order ideal polytope allows for a detailed study of the underlying combinatorial structure of the networks.

Outline. We start with the necessary preliminaries in Section 2 and recall the classical border basis algorithm in Subsection 2.1. In Section 3 we introduce the order ideal polytope and establish the one-to-one correspondence between the 0/1 points of this polytope and degree-compatible border bases. We also derive an equivalent characterization that is better suited for actual computations. In Section 4 we then show how the results from Section 3 can be used to compute admissible degree-compatible border bases using a preference function for the structure of the order ideal. We also establish the NP-hardness of the optimization variant of this problem. As an immediate application, we also outline how to count all admissible border bases of a zero-dimensional ideal. We conclude with computational results in Section 6 and a few final remarks in Section 7.

2. Preliminaries

We consider a polynomial ring $K[X]$ over the field $K$ with indeterminates $X = \{x_1, \ldots, x_n\}$. For convenience we define $x^m := \prod_{j \in [n]} x_j^{m_j}$ for $m \in \mathbb{N}^n$ and let $T^n := \{x^m | m \in \mathbb{N}^n\}$ be the monoid of terms. For any $d \in \mathbb{N}$ we let $T^n_{\leq d} := \{x^m \in T^n | \|m\|_1 \leq d\}$ be the set of monomials of total degree at most $d$. For a polynomial $p \in K[X]$ with $p = \sum_{i=1}^t a_i x^{m_i}$ we define the support of $p$ to be $\text{supp}(p) := \{x^{m_i} | i \in [t]\}$ and similarly, for a set of polynomials $P \subseteq K[X]$ we define the support of $P$ to be $\text{supp}(P) := \bigcup_{p \in P} \text{supp}(p)$. Given a term ordering $\sigma$, the leading term $\text{LT}_\sigma(p)$ of a polynomial $p$ is $\text{LT}_\sigma(p) := t$ with $t \in \text{supp}(p)$ such that for all $t' \in \text{supp}(p) \setminus \{t\}$ we have $t >_\sigma t'$; the leading coefficient $\text{LC}_\sigma(p)$ of $p$ is the coefficient of $\text{LT}_\sigma(p)$. We drop the index $\sigma$ if the ordering is clear from the context. Recall that the degree of a polynomial $p \in K[X]$ is $\text{deg}(p) := \max_{x^{m} \in \text{supp}(p)} \|m\|_1$. The leading form $\text{LF}(p)$ of a polynomial $p = \sum_{i=1}^t a_i x^{m_i} \in K[X]$ is defined to be $\text{LF}(p) = \sum_{i=1, \|m_i\|_d \leq d} a_i x^{m_i}$ where $d = \text{deg}(p)$, i.e., we single out the part with maximum degree. Both $\text{LF}$ and $\text{LT}$ generalize to sets in the obvious way, i.e., for a set of polynomials $P$ we define $\text{LF}(P) := \{\text{LF}(p) | p \in P\}$ and $\text{LT}(P) := \{\text{LT}(p) | p \in P\}$.

In the following we will frequently switch between considering polynomials $M = \{p_1, \ldots, p_s\}$, the generated ideal, and the generated vector space whose coordinates are indexed by the monomials in the support of $M$. We denote the ideal generated by $M$ as $\langle M \rangle_K[X]$ and the vector space generated by $M$ as $\langle M \rangle_K$. For $n \in \mathbb{N}$ we define $[n] := \{1, \ldots, n\}$. All other notation is standard as to be found in [9, 24]; we have chosen the border basis specific notation to be similar to the one in [20] where the border basis algorithm was first introduced in its current form.

Central to our discussion will be the notion of an order ideal:

Definition 2.1. Let $\mathcal{O}$ be a finite subset of $T^n$. If for all $t \in \mathcal{O}$ and $t' \in T^n$ such that $t' | t$ we have $t' \in \mathcal{O}$, i.e., $\mathcal{O}$ is closed under factors, then we call $\mathcal{O}$ an order ideal. Furthermore, the border $\partial \mathcal{O}$ of a non-empty order ideal $\mathcal{O}$ is the set of terms $\partial \mathcal{O} := \{x_j t | j \in [n], t \in \mathcal{O}\} \setminus \mathcal{O}$. As an exception, we set $\partial \emptyset := \{1\}$ for the empty order ideal.
Recall that an ideal \( I \subseteq K[\mathbb{X}] \) is zero-dimensional, if and only if \( K[\mathbb{X}]/I \) is finite dimensional. The \( \mathcal{O} \)-border basis of a zero-dimensional ideal \( I \) is a special set of polynomials:

**Definition 2.2.** Let \( \mathcal{O} = \{t_1, \ldots, t_u\} \) be an order ideal with border \( \partial \mathcal{O} = \{b_1, \ldots, b_v\} \). Further let \( \mathcal{G} = \{g_1, \ldots, g_v\} \subseteq K[\mathbb{X}] \) be a (finite) set of polynomials and let \( I \subseteq K[\mathbb{X}] \) be a zero-dimensional ideal. Then the set \( \mathcal{G} \) is an \( \mathcal{O} \)-border basis of \( I \) if:

1. The polynomials in \( \mathcal{G} \) have the form \( g_j = b_j - \sum_{i=1}^{\nu} \alpha_{ij} t_i \) for \( j \in [v] \) and \( \alpha_{ij} \in K \);
2. \( \langle \mathcal{G} \rangle_{K[\mathbb{X}]} = I \);
3. \( K[\mathbb{X}] = I \oplus \langle \mathcal{O} \rangle_K \) as vector spaces.

If there exists an \( \mathcal{O} \)-border basis of \( I \) then the order ideal \( \mathcal{O} \) supports a border basis of \( I \).

Note that the condition \( \langle \mathcal{G} \rangle_{K[\mathbb{X}]} = I \) is a consequence of \( \mathcal{O} \subseteq I \), the particular form of the elements in \( \mathcal{G} \), and \( K[\mathbb{X}] = I \oplus \langle \mathcal{O} \rangle_K \). More precisely, \( \langle \mathcal{G} \rangle_{K[\mathbb{X}]} + \langle \mathcal{O} \rangle_K \) is closed under multiplication by the \( x_i \), and hence it is an ideal. As it contains \( 1 \), it must be the whole ring, and hence \( \langle \mathcal{G} \rangle_{K[\mathbb{X}]} = I \) by the modular law. See \[21\] Proposition 4.4.2 for another proof.

In particular, an order ideal \( \mathcal{O} \) supports an \( \mathcal{O} \)-border basis of \( I \) if and only if \( K[\mathbb{X}] = I \oplus \langle \mathcal{O} \rangle_K \). Moreover, for any given order ideal \( \mathcal{O} \) and ideal \( I \) the \( \mathcal{O} \)-border basis of \( I \) is unique as \( b_j \) has a unique representation in \( K[\mathbb{X}] = I \oplus \langle \mathcal{O} \rangle_K \) for all \( j \in [v] \). Furthermore, as \( K[\mathbb{X}] = I \oplus \langle \mathcal{O} \rangle_K \) it follows that \( |\mathcal{O}| = \dim(\langle \mathcal{O} \rangle_K) \) is invariant for all choices of \( \mathcal{O} \). The requirement for \( I \) being zero-dimensional is a consequence of this condition as well, as \( \partial \mathcal{O} \) (and in consequence \( \mathcal{O} \)) as part of the output of our computation should be finite. Clearly, as a vector space, \( I \) has a degree filtration, i.e., \( I = \bigcup_{i \in \mathbb{N}} I_{\leq i} \) where \( I_{\leq i} := \{ p \in I \mid \deg(p) \leq i \} \). For a set of monomials \( \mathcal{O} \) we define \( \mathcal{O}^{=i} := \{ m \in \mathcal{O} \mid \deg(m) = i \} \) (and similarly for \( \leq \) instead of \( = \)). In the following we consider special types of order ideals, i.e., those that essentially preserve the filtration:

**Definition 2.3.** Let \( I \subseteq K[\mathbb{X}] \) be a zero-dimensional ideal and let \( \mathcal{O} \subseteq \mathbb{T}^n \) be an order ideal. We say that \( \mathcal{O} \) is degree-compatible (to \( I \)) if

\[
\dim \langle \mathcal{O}^{=i} \rangle_K = \dim \langle \mathbb{T}^n_{\leq i} \rangle_K - \dim \langle I_{\leq i-1} \rangle
\]

for all \( i \in \mathbb{N} \).

Thus, the \( \mathcal{O} \)-border basis of a zero-dimensional ideal \( I \) with respect to any degree-compatible order ideal \( \mathcal{O} \) has a pre-determined size for each degree \( i \in \mathbb{N} \). Intuitively, the degree-compatible order ideals are precisely those that correspond to degree-compatible orderings on the monomials. The important difference is that the orderings do not have to be term orderings. The definition above only requires local compatibility with multiplication as \( \mathcal{O} \) is an order ideal and thus downwardly closed and degree-compatible, i.e., if \( p, q \) are polynomials and \( \deg(p) < \deg(q) \) then \( p \leq q \). The following example shows that the requirements of Definition 2.3 are not automatically satisfied by all order ideals \( \mathcal{O} \). We also give an example of a degree-compatible ideal not coming from a term ordering.

Recall that the order ideal associated to a Gröbner basis of an ideal consists of all monomials not divisible by any leading term in the Gröbner basis.

**Example 2.4** (A non-degree-compatible order ideal). With the degree lexicographic ordering on two variables, the set of polynomials

\[
x_1^3 + x_1x_2, x_1^2x_2, x_1x_2^2, x_2^3
\]

is a Gröbner basis of an ideal with associated order ideal consisting of all monomials of degree at most 2:

\[
\{1, x_1, x_2, x_1^2, x_1x_2, x_2^2\}.
\]
The element $x_2^2 + x_1x_2$ of the ideal enables us to replace $x_1x_2$ with $x_1^2$ and thus obtain another order ideal of the ideal, which is not degree-compatible:

$$\{1, x_1, x_2, x_1^2, x_2^2\}.$$

**Example 2.5** (A degree-compatible order ideal not coming from a term ordering). The homogeneous ideal having a Gröbner basis in the degree lexicographic term ordering

$$x_2^2 + x_1x_2 + x_1^2x_2^2$$

has the associated order ideal

$$\{1, x_1, x_2, x_1^2, x_2^3\}.$$ Via a basis change, we obtain another order ideal

$$\{1, x_1, x_2, x_1^2, x_2^2, x_2^3\},$$

which cannot come from a term ordering. The reason is that every Gröbner basis of the ideal must contain a polynomial with degree-two leading term, which therefore must be a multiple of the first generator $x_2^2 + x_1x_2 + x_1^2$ modulo higher-degree terms. The degree-two leading term cannot be $x_1x_2$ because if e.g., $x_1 < x_2$ then the term $x_2^2$ is larger then $x_1x_2$. So the leading term must be either $x_1^2$ or $x_2^2$, which excludes the order ideal.

**Example 2.6** (Generic ideal). Let $k$ and $n$ be positive integers and let $\{a_{ij}\}_{i \in \mathbb{N}, j \in [k]}$ be algebraically independent real numbers over $\mathbb{Q}$. Let $I$ be the ideal of polynomials in the variables $x_1, \ldots, x_n$ which are zero on the points $(a_{1j}, \ldots, a_{nj})$ for $j \in [k]$. Thus, the ideal is zero-dimensional, and $K[x]/I$ has dimension $k$.

Every $k$ distinct monomials form a complementary basis of $I$, since they are linearly independent on the $k$ points $(a_{1j}, \ldots, a_{nj})$. An equivalent formulation of linear independence is that the determinant of the matrix formed by the values of the monomials on these points is non-zero. The determinant is indeed non-zero, as it is a non-trivial polynomial of the algebraic independent $a_{ij}$ with integer coefficients.

In particular, every order ideal of size $k$ is an order ideal of $I$. The degree-compatible order ideals are the ones where the monomials have the least possible degree, i.e., consisting of all monomials of degree less than $l$ and $k - \binom{n+l-1}{l-1}$ monomials of degree $l$, where $l$ is the smallest non-negative integer satisfying $k \leq \binom{n+l}{l}$, i.e., there are at least $k$ monomials of degree at most $l$.

### 2.1. Computing border bases for term ordering induced $\sigma$.

Without proofs, we recall the classical border basis algorithm introduced in [20] as it will serve as a basis for our algorithm. The interested reader is referred to [21] [19] for a general introduction to border bases and to [20] in particular for an introduction of the border basis algorithm.

The classical border basis algorithm calculates border bases of zero-dimensional ideals with respect to an order ideal $\sigma$ which is induced by a degree-compatible term ordering $\sigma$.

First, the border basis algorithm heavily relies on the following neighborhood extension:

**Definition 2.7.** (cf. [20] Definition 7.1 and paragraph preceding Proposition 13]) Let $V$ be a vector space. We define the neighborhood extension of $V$ to be

$$V^+ := V + VX_1 + \cdots + VX_n.$$

For a finite set $W$ of polynomials, its neighborhood extension is

$$W^+ = W \cup WX_1 \cup \cdots \cup WX_n.$$
Note that for a given set of polynomials \( W \) such that \( \langle W \rangle_K = V \) we have \( \langle W^+ \rangle_K = \langle W \rangle_K^+ = V^+ \) as multiplication with \( x_i \) is a \( K \)-vector space homomorphism. It thus suffices to perform the neighborhood extension on a set of generators \( W \) of \( V \).

Let \( F \) be a finite set of polynomials and let \( L \subseteq \mathbb{T}^n \) be an order ideal, then \( F \cap \langle L \rangle_K = \{ f \in F \mid \text{supp}(f) \subseteq L \} \), i.e., \( F \cap \langle L \rangle_K \) contains only those polynomials that lie in the vector space generated by \( L \). Clearly, for \( L = \mathbb{T}^n \subseteq \mathbb{Z}^n \) we have \( \langle F \rangle_K \cap L = \langle F \rangle_K \). Using the neighborhood extension we can define:

**Definition 2.8.** (cf. [20, Definition 10]) Let \( L \) be an order ideal and let \( F \) be a finite set of polynomials such that \( \text{supp}(F) \subseteq L \). The set \( F \) is \( L \)-stabilized if \( \langle F^+ \rangle_K \cap L = \langle F \rangle_K \). The \( L \)-stable span \( F_L \) of \( F \) is the smallest vector space \( G \) containing \( F \) satisfying \( G^+ \cap L = G \).

A straightforward construction of the \( L \)-stable span of \( F \) is to inductively define the following increasing sequence of vector spaces:

\[
F_0 := \langle F \rangle_K \quad \text{and} \quad F_{k+1} := F_k^+ \cap \langle L \rangle_K \quad \text{for } k > 0.
\]

The union \( \bigcup_{k \geq 0} F_k \) is the \( L \)-stable span \( F_L \) of \( F \).

The set \( L \) represents our computational universe and we will be in particular concerned with finite sets \( L \subseteq \mathbb{T}^n \). Note that \( L \)-stability is a property of the vector space and does not depend on the basis:

**Remark 2.9.** The \( L \)-stable span of a finite set \( F \) depends only on the generated vector space \( \langle F \rangle_K \), as \( \langle F^+ \rangle_K = \langle F \rangle_K \).

In the following we will explain how the \( L \)-stable span can be computed explicitly for \( L = \mathbb{T}^n \subseteq \mathbb{Z}^n \). We will use a modified version of Gaussian elimination as a tool, which allows us to extend a given basis \( V \) with a set \( W \) as described in the following:

**Lemma 2.10.** [20, Lemma 12] Let \( V = \{ v_1, \ldots, v_r \} \subseteq K[\mathbb{X}] \setminus \{ 0 \} \) be a finite set of polynomials such that \( \text{LT}(v_i) \neq \text{LT}(v_j) \) whenever \( i, j \in [r] \) with \( i \neq j \) and \( \text{LC}(v_i) = 1 \) for all \( i \in [r] \). Further let \( G = \{ g_1, \ldots, g_s \} \) be a finite set of polynomials. Then Algorithm 2.17 computes a finite set of polynomials \( W \subseteq K[\mathbb{X}] \) with \( \text{LC}(w) = 1 \) for all \( w \in W \), \( \text{LT}(u_1) \neq \text{LT}(u_2) \) for any distinct \( u_1, u_2 \in V \cup W \), and \( \langle V \cup W \rangle_K = \langle V \cup G \rangle_K \). (\( V, W \) may be empty.)

**Algorithm 2.11** (Gaussian Elimination for polynomials—GaussEl).

**Input:** \( V, G \) as in Lemma 2.10

**Output:** \( W \subseteq K[\mathbb{X}] \) as in Lemma 2.10

1. Let \( H := G \) and \( \eta := 0 \).
2. If \( H = \emptyset \) then return \( W := \{ v_{r+1}, \ldots, v_{r+\eta} \} \) and stop.
3. Choose \( f \in H \) and remove it from \( H \). Let \( i := 1 \).
4. If \( f = 0 \) or \( i > r + \eta \) then go to step 7.
5. If \( \text{LT}(f) = \text{LT}(v_i) \) then replace \( f \) with \( f - \text{LC}(f) \cdot v_i \). Set \( i := 1 \) and go to step 4.
6. Set \( i := i + 1 \). Go to step 4.
7. If \( f \neq 0 \) then put \( \eta := \eta + 1 \) and let \( \nu_{r+\eta} := f / \text{LC}(f) \). Go to step 2.

We can now compute the \( L \)-stable span using the Gaussian elimination algorithm 2.11:

**Lemma 2.12.** [20, Proposition 13] Let \( L = \mathbb{T}_{\leq d}^n \) and \( F \subseteq K[\mathbb{X}] \) be a finite set of polynomials supported on \( L \). Then Algorithm 2.13 computes a vector space basis \( V \) of \( F_L \) with pairwise different leading terms.

**Algorithm 2.13** (L-stable span computation—LStabSpan).

**Input:** \( F, L \) as in Lemma 2.12

**Output:** \( V \) as in Lemma 2.12

1. \( V := \text{GaussEl}(\emptyset, F) \).
2. \( W' := \text{GaussEl}(V, V^+ \setminus V) \).
(3) \( W := \{ w \in W' \mid \text{supp}(w) \subseteq L \} = \{ w \in W' \mid \deg(w) \leq d \} \).
(4) If \(|W| > 0\) set \( V := V \cup W \) and go to step 2.
(5) Return \( V \).

The last ingredient that we need in order to formulate the border basis algorithm is the final reduction algorithm. This algorithm basically transforms the output of the border basis algorithm to bring it into the desired form of a border basis by applying linear algebra steps only. It interreduces the elements so that they only have support in the leading term and \( \mathcal{O} \).

**Lemma 2.14.** [20, Proposition 17] Let \( F \) be a system of generators of a zero-dimensional ideal \( I \), let \( L \) be an order ideal, and let \( V \) be a vector space basis of \( F_L \) with pairwise different leading terms and \( \mathcal{O} := L \setminus \text{LT}(V) \) such that \( \partial \mathcal{O} \subseteq L \). Then Algorithm 2.15 computes the \( \mathcal{O} \)-border basis \( \mathcal{G} \) of \( I \).

It is easy to see that \( \langle L \rangle_K = F_L \oplus \langle \mathcal{O} \rangle_K \), so the algorithm will do a form of Gaussian elimination on the basis \( V \) to obtain a new basis \( V_R \) with all non-leading terms supported on \( \mathcal{O} \). Of course, this new basis will contain an \( \mathcal{O} \)-border basis.

**Algorithm 2.15** (Final Reduction Algorithm—FinalRed).

**Input:** \( V, \mathcal{O} \) as in Lemma 2.14
**Output:** \( \mathcal{G} \) as in Lemma 2.14

(1) Let \( V_R := \emptyset \).
(2) If \( V = \emptyset \) then go to step 8.
(3) Let \( v \in V \) such that \( v \) has minimal leading term. Put \( V := V \setminus \{v\} \).
(4) Let \( H := \text{supp}(v) \setminus (\text{LT}(v) \cup \mathcal{O}) \).
(5) If \( H = \emptyset \) then append \( v / \text{LC}(v) \) to \( V_R \) and go to step 2.
(6) For each \( h \in H \) choose \( w_h \in V_R \) and \( c_h \in K \) such that \( \text{LT}(w_h) = h \) and \( h \notin \text{supp}(v - c_hw_h) \).
(7) Set \( v := v - \sum_{h \in H} c_hw_h \); append \( v / \text{LC}(v) \) to \( V_R \) and go to step 2.
(8) Return \( \mathcal{G} := \{ v \in V_R \mid \text{LT}(v) \in \partial \mathcal{O} \} \).

We will now formulate the border basis algorithm.

**Proposition 2.16.** [20, Proposition 18] Let \( F \subseteq K[X] \) be a finite set of polynomials that generates a zero-dimensional ideal \( I = \langle F \rangle_K[X] \). Then Algorithm 2.17 computes the \( \mathcal{O} \)-border basis \( \mathcal{G} \) of \( I \).

**Algorithm 2.17** (Border basis algorithm—BBasis).

**Input:** \( F \) as in Proposition 2.16
**Output:** \( \mathcal{G} \) as in Proposition 2.16

(1) Let \( d := \max_{f \in F} \deg(f) \).
(2) \( V = \{ v_1, \ldots, v_r \} := \text{LStabSpan}(F, \mathbb{T}_n^{\leq d}) \).
(3) Let \( \mathcal{O} := \mathbb{T}_n^{\leq d} \setminus \{ \text{LT}(v_1), \ldots, \text{LT}(v_r) \} \).
(4) If \( \partial \mathcal{O} \not\subseteq \mathbb{T}_n^{\leq d} \) then set \( d := d + 1 \) and go to step 2.
(5) Return \( \mathcal{G} := \text{FinalRed}(V, \mathcal{O}) \).

It is worthwhile to note that step 4 in Algorithm 2.17 is essentially for testing if the \( L \)-stable span is large enough to support an \( \mathcal{O} \)-border basis.

The rationale for searching such a large span is due to the following proposition that serves as a stopping criterion. It is obvious from its Corollary 2.19 that the span is indeed minimal in the sense that it is the smallest span that contains a degree-compatible order ideal that supports a border basis (and thus all such degree-compatible order ideals).

**Proposition 2.18.** [20, Proposition 16] Let \( L \) be an order ideal. Further let \( \tilde{I} \) be an \( L \)-stabilized generating vector subspace of a zero-dimensional ideal \( I \subseteq K[X] \), i.e., \( \tilde{I}^+ \cap \langle L \rangle_K = \tilde{I} \) and \( \langle \tilde{I} \rangle_{K[X]} = I \). If \( \mathcal{O} \) is an order ideal such that \( \langle L \rangle_K = \tilde{I} \oplus \langle \mathcal{O} \rangle_K \) and \( \partial \mathcal{O} \subseteq L \) then \( \mathcal{O} \) supports a border basis of \( I \).
We obtain the following corollary:

**Corollary 2.19.** Let $\bar{I}$ be an $T_{\leq d}^n$-stabilized vector space satisfying $\bar{I} + \langle T_{\leq d-1}^n \rangle_K = \langle T_{\leq d}^n \rangle_K$. Then $\langle \bar{I} \rangle_{K[X]} \cap \langle T_{\leq d}^n \rangle_K = \bar{I}$.

**Proof.** We apply Proposition 2.18 with the choice $L := T_{\leq d}^n$, $I := \langle \bar{I} \rangle_{K[X]}$ and $\mathcal{O} := T_{\leq d}^n \setminus \text{LT}(\bar{I})$ where the leading terms are with respect to any degree-compatible term ordering. Clearly, $\langle T_{\leq d}^n \rangle_K = \bar{I} \oplus \langle \mathcal{O} \rangle_K$. The condition $\bar{I} + \langle T_{\leq d-1}^n \rangle_K = \langle T_{\leq d}^n \rangle_K$ ensures that $\mathcal{O}$ consists of monomials of degree less than $d$, so $\partial \mathcal{O} \subseteq T_{\leq d}^n$. Hence the proposition applies, and we obtain $K[X] = I \oplus \langle \mathcal{O} \rangle_K$. Together with $\langle T_{\leq d}^n \rangle_K = \bar{I} \oplus \langle \mathcal{O} \rangle_K$ this gives $I \cap \langle T_{\leq d}^n \rangle_K = \bar{I}$. \hfill $\square$

The border basis algorithm decomposes into two components. The first one is the calculation of the $L$-stable span for $L = T_{\leq d}^n$ for sufficiently large $d$ (this is the main ‘work’) and the second component is the extraction of a border basis via the [final reduction algorithm](#). Effectively, after using any degree-compatible term ordering in order to compute the $L$-stable span one can choose a different ordering with respect to which the basis can be transformed. This approach is very much in spirit of the FGLM algorithm, Steinitz’s exchange lemma, and related basis transformation procedures. In Remark 2.9 it is shown that $L$-stability does not depend on the basis and thus any sensible basis transformation that results in an admissible order ideal $\mathcal{O}$ is allowed. In the following we will characterize all admissible order ideals.

### 3. The order ideal polytope

We will now introduce the *order ideal polytope* $P$ (a 0/1 polytope) that characterizes all order ideals that support a border basis (for a given zero-dimensional ideal $I$) in an abstract fashion independent of particular vector space bases for the stable span approximation. Its role will be crucial for the [later computation](#) of border bases for general degree-compatible order ideals. In the first subsection, we introduce the polytope in an abstract, invariant way highlighting its main property that its integral points are in bijection with degree-compatible order ideals supporting a border basis (Theorem 3.2). In the second subsection, we give a more direct reformulation targeted to actual computations.

Given $d \in \mathbb{N}$ in advance for which Algorithm 2.17 stops, LStabSpan computes the actual border basis (and some unused extra polynomials). The computation of the border basis is performed by Gaussian elimination applied to the matrices obtained within LStabSpan. The order ideal for which we effectively compute the border basis is solely determined by the pivoting rule when doing the elimination step. Degree-compatible term orderings ensure that we obtain an order ideal for which a border basis exists (given the correct $d$). If we would now pivot arbitrarily, which effectively means permuting columns, it is not clear that, first, the resulting ideal is an order ideal and, second, that it does actually support a border basis. If we remain in the setting of degree-compatible order ideals (which means that we only permute columns of monomials with same degree) then we face the combinatorial problem that when permuting a column such that it will end up being an element of $\mathcal{O}$ we also have to ensure that all its divisors will also end up in $\mathcal{O}$. On the other hand, due to the degree-compatibility constraint (see Definition 2.3) we must choose an exactly determined amount of elements for each degree that will end up in $\mathcal{O}$. We will show now how to transform this combinatorial problem of *faithful pivoting* into a polyhedral setting. We obtain a 0/1 polytope $P$, the *order ideal polytope* that characterizes all admissible degree-compatible order ideals that support a border basis for the ideal at hand.

#### 3.1. Theoretical point of view.
Definition 3.1. Let \( I \) be a zero-dimensional ideal. Its order ideal polytope \( P(I) \) is given by the system of inequalities in Figure 3.1.

The order ideal polytope is actually a finite dimensional polytope as all the \( z_m \) are 0 when the degree of \( m \) is large enough. Indeed, for large \( i \), we have \( \langle \mathbb{T}^n_{=i} \rangle_K \cong I^{\leq i}/I^{\leq i-1} \) and hence Condition 3.7 gives \( z_m = 0 \) for every \( m \) of degree \( i \).

We are ready to relate the order ideal polytope with order ideals. From now on, let \( \Lambda(I) \) denote the set of degree-compatible order ideals of a zero-dimensional ideal \( I \).

Theorem 3.2. Let \( I \) be a zero-dimensional ideal. There is a bijection between the set \( \Lambda(I) \) of its degree-compatible order ideals and the set of integral points of the order ideal polytope of \( I \). The bijection is given by

\[
\xi : z \in P(I) \cap \mathbb{Z}^n \to \mathcal{O}(z) := \{ m \in \mathbb{T}^n \mid z_m = 1 \}.
\]

Proof. In fact, we will see that the order ideal polytope is defined exactly to this end.

First, the integral solutions \( z \) of Figure 3.1 are exactly the 0/1 points, i.e., the characteristic vectors of sets of terms \( \mathcal{O}(z) := \{ m \in \mathbb{T}^n \mid z_m = 1 \} \).

Second, it is easy to see that Figure 3.1 means that \( \mathcal{O}(z) \) is indeed an order ideal, as whenever \( m_1 \mid m_2 \) and \( m_2 \in \mathcal{O}(z) \), i.e., \( z_{m_2} = 1 \), then it follows that \( z_{m_1} = 1 \), i.e., \( m_1 \in \mathcal{O}(z) \) as well.

In the third step we will provide an algebraic characterization of Conditions 3.2 and 3.3. Clearly, Figure 3.1 can be rewritten to

\[
|\mathcal{O}(z)^{=i}| = \dim \langle \mathbb{T}^n_{=i} \rangle_K - \dim I^{\leq i}/I^{\leq i-1}.
\]

We will now show that Figure 3.1 is equivalent to

\[
\langle \mathcal{O}(z)^{=i} \rangle_K \cap I^{\leq i}/I^{\leq i-1} = \{0\},
\]

i.e., the image of \( \mathcal{O}(z)^{=i} \) is linearly independent in the factor \( \langle \mathbb{T}^n_{=i} \rangle_K/(I^{\leq i}/I^{\leq i-1}) \).

Similarly as above, Figure 3.1 can be rewritten to

\[
|U \cap \mathcal{O}(z)^{=i}| \leq \dim \langle U \cup I^{\leq i}/I^{\leq i-1} \rangle_K - \dim I^{\leq i}/I^{\leq i-1},
\]

i.e., the size of \( U \cap \mathcal{O}(z)^{=i} \) is at most the dimension of the vector space generated by the image of \( U \) in the factor \( \langle \mathbb{T}^n_{=i} \rangle_K/(I^{\leq i}/I^{\leq i-1}) \). This is obviously necessary for the image of \( \mathcal{O}(z)^{=i} \) to be linearly independent in the factor. (Here the size of \( U \) does not matter.)

For sufficiency choose \( U := \mathcal{O}(z)^{=i} \). Then the dimension of the vector space generated by the image of \( \mathcal{O}(z)^{=i} \) is at least \( |\mathcal{O}(z)^{=i}| \), so the image of \( \mathcal{O}(z)^{=i} \) is independent.
So far we have proved that the integral points of the order ideal polytope correspond bijectively to order ideals \( \mathcal{O} \) with the properties

\[
|\mathcal{O}(z)^{\leq i}| = \dim \langle T^n_{=i} \rangle_K - \dim I^{\leq i} / I^{\leq i-1}
\]

and

\[
\langle \mathcal{O}(z)^{\leq i} \rangle_K \cap I^{\leq i} / I^{\leq i-1} = \{0\}
\]

for all \( i \).

Lastly, we will show now that this is equivalent to \( |\mathcal{O}(z)^{\leq i}| = \dim \langle T^n_{=i} \rangle_K - \dim I^{\leq i} / I^{\leq i-1} \) and \( I \oplus \langle \mathcal{O}(z) \rangle_K = \langle T^n \rangle_K \) and thus the assertion follows.

By dimensionality it follows, \( |\mathcal{O}(z)^{\leq i}| = \dim \langle T^n_{=i} \rangle_K - \dim I^{\leq i} / I^{\leq i-1} \) and \( \langle \mathcal{O}(z)^{\leq i} \rangle_K \cap I^{\leq i} / I^{\leq i-1} = \{0\} \) for all \( i \) together are equivalent to \( I^{\leq i} / I^{\leq i-1} \oplus \langle \mathcal{O}(z)^{\leq i} \rangle_K = \langle T^n_{=i} \rangle_K \) for all \( i \). For brevity, we will omit the phrase 'for all \( i \)'. Using the filtration argument, the latter is equivalent to \( I^{\leq i} \oplus \langle \mathcal{O}(z)^{\leq i} \rangle_K = \langle T^n_{=i} \rangle_K \). Via a dimension argument on embeddings, this is further equivalent to \( I \oplus \langle \mathcal{O}(z) \rangle_K = \langle T^n \rangle_K \) and \( \dim I^{\leq i} + \dim \langle \mathcal{O}(z)^{\leq i} \rangle_K = \dim \langle T^n_{=i} \rangle_K \). Finally, filtrating the dimension by degree shows that the latter is equivalent to \( I \oplus \langle \mathcal{O}(z) \rangle_K = \langle T^n \rangle_K \) and \( \dim I^{\leq i} / I^{\leq i-1} + \dim \langle \mathcal{O}(z)^{\leq i} \rangle_K = \dim \langle T^n_{=i} \rangle_K \). □

### 3.2. Computational point of view

Throughout this subsection we assume that \( M = \bigcup_{i \in \mathbb{N}} M_i \subseteq K[\mathbb{X}] \) is a finite set of polynomials with degree-filtration \( \{M_i \mid i \in \mathbb{N}\} \) that generates a zero-dimensional ideal \( \langle M \rangle_{K[\mathbb{X}]} \subseteq K[\mathbb{X}] \) such that all \( p \in M_i \) have degree \( i \). Furthermore, for each \( i \in \mathbb{N} \) we have an enumeration \( M_i = \{p_{ij} \mid j \in [k_i]\} \) with \( k_i \in \mathbb{N} \). As seen in Subsection 2.1, an important component of the border basis algorithm is the computation of \( L \)-stable spans with respect to some computational universe \( L \subseteq \mathbb{T}^n \). Computing a border basis with respect to a different order ideal \( \mathcal{O} \) is merely a basis transformation of the vector space obtained from the \text{LStabSpan} procedure. In the following we assume that \( L \) is of the form \( L = \mathbb{T}^n_{=d} \), where \( d \in \mathbb{N} \) is such that Algorithm 2.17 stops. In view of Remark 2.9 and Definition 2.3 we have that for a given computational universe \( L \) either all border bases (supported by degree-compatible order ideals) are contained in \( L \) or none. Furthermore we assume that \( M \) is \( L \)-stabilized and has in particular a convenient form. Effectively one might want to think of \( M \) being the output of the \text{LStabSpan} procedure, which is then brought into the following reduced form:

**Definition 3.3.** Let \( M \) be a finite set of polynomials of degree at most \( \ell \) for some \( \ell \in \mathbb{N} \). Then \( M \) is in canonical form if the leading term of any element of \( M \) does not occur in the other elements. Here we can freely choose leading terms of the polynomials with the only constraint that they have to be maximal-degree terms.

We give a visual interpretation of the definition. The coefficient matrix \( A \in K^{M \times \mathbb{T}^n_{\leq \ell}} \) of \( M \) is the matrix where the rows are the elements of \( M \), the columns are all the monomials of degree at most \( \ell \), and the entries are the coefficients of the terms in the elements of \( M \). We use the convention that for terms \( t_1, t_2 \) with \( \deg(t_1) > \deg(t_2) \) we put column \( t_1 \) to the left of column \( t_2 \). Similarly, we put leading terms of a polynomial to the left of the other terms. Now \( M \) is in canonical form if the matrix \( A \) has the structure as depicted in Figure 3.2, i.e., it consists of degree blocks and each degree block is maximally interreduced.

The degree blocks correspond to the leading forms of the polynomials in \( M \). Any finite set can be brought into canonical form by applying Gaussian elimination and column permutations of terms with same degree if necessary. In particular, the output of the \text{LStabSpan} procedure can be easily brought into this form. The following lemma summarizes the basic properties of a set \( M \) in canonical form:

**Lemma 3.4.** Let \( M \) be in canonical form and \( L \)-stabilized with \( L = \mathbb{T}^n_{=d} \). Let \( \mathbb{T}^n_{=d} \subseteq \langle M \rangle_K + \mathbb{T}^n_{=d-1} \). Then the following hold for all \( i \in [d] \):

1. \( \langle M \rangle^{\leq i}_{K[\mathbb{X}]} / \langle M \rangle^{\leq i-1}_{K[\mathbb{X}]} \cong \langle \text{LF}(M_i) \rangle_K \)
\[ A = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \cdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & \cdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & \cdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & \cdots & \ddots & \ddots & \ddots & \vdots \end{pmatrix} \]

Figure 3.2. canonical form

(2) \( \langle M \rangle^\leq_i = \left( \bigcup_{j \leq i} M_j \right) \)

(3) \( \langle M \rangle^<_i = 0 \) and thus \( \langle M \rangle^<_i \subseteq \left( \bigcup_{0 \leq j < i} M_j \right) \)

Proof. We first show that \( \langle M \rangle^<_i = 0 \) for all \( i \in [d] \). Let \( i \in [d] \) be arbitrary and observe that each nonzero element \( p \in M_i \) has degree \( i \). As \( M \) is in canonical form, the polynomials in \( M_i \) are interreduced (see the matrix in Figure 3.2 for Definition 3.3) and thus we also obtain each nonzero element \( p \in M_i \) has degree \( i \).

By Corollary 2.19, \( \langle M \rangle^{\leq_i} \cap (L)_K = \langle M \rangle_K \). Hence \( \langle M \rangle^\leq_i = \langle M \rangle^\leq_i \) for \( i \in [d] \). Now the statements of the lemma are obvious consequences of \( M \) being in canonical form.

The following lemma provides us a practical way to compute the sizes of the degree components of degree-compatible order ideals, which are the same for all order ideals of a given ideal.

Lemma 3.5. Let \( M \) be in canonical form and \( L \)-stabilized with \( L = T^n_{\leq d} \) and \( d = \max_{m \in \partial} \deg(m) \). Let \( T^n = \langle M \rangle_K + T^n_{\leq d-1} \). Further let \( O \) be an order ideal of \( \langle M \rangle_K \). Then \( O \) is degree-compatible if and only if

\[ |O^{\equiv i}| = \dim \langle L^{\equiv i} \rangle_K - \dim \langle LF(M_i) \rangle_K \]

for every \( i \in [d] \).

Proof. In view of Definition 2.3 it suffices to observe that \( I^{\equiv i} / I^{\equiv i-1} \cong \langle LF(M_i) \rangle_K \) by Lemma 3.4 (1) where \( I := \langle M \rangle_K \).

We are ready to provide a reformulation of the definition of order ideal polytopes, which is better suited for actual computations, partly as it only involves direct matrix operations via replacing dimensions with ranks of subsets:

Lemma 3.6. Let \( M \) be \( L \)-stabilized and in canonical form with \( L = T^n_{\leq d} \) and \( \langle M \rangle_{\leq d} \). Then the order ideal polytope \( P(M, L) \subseteq [0, 1]^L \) of \( \langle M \rangle_K \) is given by the system of inequalities in Figure 3.3

Proof. Let \( I := \langle M \rangle_K \). We successively transform the defining inequalities of the order ideal polytope in Figure 3.1 into the desired form of Figure 3.3. The reformulation is mostly based on \( I^{\equiv i} / I^{\equiv i-1} \cong \langle LF(M_i) \rangle_K \) from Lemma 3.4 (1).

First, as \( I^{\equiv d} / I^{\equiv d-1} \cong \langle T^n \rangle_{K} \), we can remove the variables \( z_m \) with \( m \) degree at least \( d \) together with the inequalities involving them. These variables are always zero.
Indeed, using this we can rewrite the inequality as

\[ z_{m_1} \geq z_{m_2} \quad \forall m_1, m_2 \in L : m_1 \mid m_2 \]  
\[ \sum_{m \in L^{z_i}} z_m = \text{dim } \langle L^{z_i} \rangle_K - \text{dim } \langle LF(M_i) \rangle_K \quad \forall i \in [d-1] \]  
\[ \sum_{m \in U} z_m \geq |U| - \text{rk}(\bar{U}) \quad \forall i \in [d-1], U \subseteq L^{z_i} : |U| = \text{dim } \langle LF(M_i) \rangle_K \]  
\[ z_m \in [0, 1] \quad \forall m \in L \]

Second, we replace all occurrences of \( I^{z_i}/I^{z_i-1} \) with \( \langle LF(M_i) \rangle_K \) (or simply \( LF(M_i) \)). This almost results in the inequality system of Figure 3.3, with the only difference that instead of Condition 3.8 we have

\[ \sum_{m \in U} z_m \leq \text{dim } \langle U' \cup LF(M_i) \rangle_K - \text{dim } \langle LF(M_i) \rangle_K \quad \forall i \in [d-1], U \subseteq L^{z_i} : |U'| = \text{dim } \langle L^{z_i} \rangle_K - \text{dim } \langle LF(M_i) \rangle_K, \]

where we have deliberately replaced \( U \) with \( U' \).

We will show that the difference of Conditions 3.7 and 3.8 is equal to Condition 3.9 with the choice \( U' := L^{z_i} \setminus U \), which has size \( |U'| = \text{dim } \langle L^{z_i} \rangle_K - \text{dim } \langle LF(M_i) \rangle_K \). This will finish the proof.

Let \( U \subseteq L^{z_i} \) as above and compute the difference of Condition 3.7 and Condition 3.8. We obtain

\[ \sum_{m \in L^{z_i} \setminus U} z_m \leq \text{dim } \langle L^{z_i} \rangle_K - \text{dim } \langle LF(M_i) \rangle_K - |U| + \text{rk}(\bar{U}). \]

It is easy to see that \( |U| = \text{dim } \frac{\langle L^{z_i} \rangle_K}{\langle L^{z_i} \setminus U \rangle_K} \). We claim that it suffices to show that

\[ \text{rk}(\bar{U}) = \text{dim } \frac{\langle LF(M_i) \cup (L^{z_i} \setminus U) \rangle_K}{\langle L^{z_i} \setminus U \rangle_K}. \]

Indeed, using this we can rewrite the inequality as

\[ \sum_{m \in L^{z_i} \setminus U} z_m \leq \text{dim } \langle L^{z_i} \rangle_K - \text{dim } \langle LF(M_i) \rangle_K - \text{dim } \frac{\langle L^{z_i} \rangle_K}{\langle L^{z_i} \setminus U \rangle_K} + \text{dim } \frac{\langle LF(M_i) \cup (L^{z_i} \setminus U) \rangle_K}{\langle L^{z_i} \setminus U \rangle_K} \]

\[ = \text{dim } \langle LF(M_i) \cup (L^{z_i} \setminus U) \rangle_K - \text{dim } \langle LF(M_i) \rangle_K, \]

which is (3.9) for \( U' := L^{z_i} \setminus U \) as claimed.

We will show now that \( \text{rk}(\bar{U}) = \text{dim } \frac{\langle LF(M_i) \cup (L^{z_i} \setminus U) \rangle_K}{\langle L^{z_i} \setminus U \rangle_K} - \text{dim } \frac{\langle L^{z_i} \rangle_K}{\langle L^{z_i} \setminus U \rangle_K} \). Let \( B \) denote the matrix obtained when writing the elements in \( LF(M_i) \) as rows and let \( L^{z_i} \) index the columns. Clearly, \( \langle L^{z_i} \rangle_K = \bigoplus_{\ell \in L^{z_i}} K e_\ell \) as a vector space and \( \langle U' \rangle_K = \bigoplus_{U' \subseteq U} K e_\ell \) as a sub vector space with \( U' := L^{z_i} \setminus U \).

We obtain \( \frac{\langle L^{z_i} \rangle_K}{\langle U' \rangle_K} \cong \bigoplus_{\ell \in L^{z_i} \setminus U} K e_\ell \). Note that \( \langle LF(M_i) \rangle_K \subseteq \langle U' \rangle_K \) and thus we can consider \( \langle LF(M_i) \cup U' \rangle_K \subseteq \langle L^{z_i} \rangle_K / \langle U' \rangle_K \). Now dim \( \frac{\langle LF(M_i) \cup U' \rangle_K}{\langle U' \rangle_K} = \text{rk}(\bar{U}) \) where \( \bar{U} \) is
obtained from $B$ by removing the columns in $U'$. We obtain
\[ \text{rk}(\tilde{U}) = \dim \frac{\langle LF(M_i) \cup U' \rangle_K}{\langle U' \rangle_K} = \dim \frac{\langle LF(M_i) \cup (L^{-i} \setminus U) \rangle_K}{\langle L^{-i} \setminus U \rangle_K} \]
and thus the result follows. □

4. Computing border bases using the order ideal polytope

In the following we explain how Theorem 3.2 can be used to actually compute border bases for general degree-compatible order ideals. We cannot expect to be able to compute a border basis for any degree-compatible order ideal, simply as such a basis does not necessarily exist. Having the order ideal polytope at hand something slightly more subtle can be done: By choosing a linear objective function $c \in \mathbb{Z}^n$ and optimizing it over the order ideal polytope, we can actually search for an order ideal with preferred monomials in its support (see Subsection 4.2). Having the order ideal polytope available we can also count the number of degree-compatible border bases that exist for a specific ideal. Before we can address this application though, we will first show how to obtain an $\mathcal{O}$-border basis for $\mathcal{O} \in \Lambda(I)$ where $I \subseteq K[\mathbb{X}]$ is a zero-dimensional ideal.

4.1. Computing border bases for $\mathcal{O} \in \Lambda(I)$. As the computation of the $L$-stable span of a set of generators $M$ is independent of the actual chosen vector space basis (see Remark 2.9), we can adapt the classical border basis algorithm (Algorithm 2.17) to compute border bases for general degree-compatible order ideal. We first determine the right computational universe $L = T_{\leq d}$ for some $d \in \mathbb{N}$ such that the associated $L$-stable span $M$ contains all border bases. In a second step we optimize over the order ideal polytope $P(M, L)$ and then perform the corresponding basis transformation. We will first formulate the generalized border basis algorithm by adding two steps after (3) in Algorithm 2.17 to the classical border basis algorithm, formulate the missing parts, and then prove its correctness:

Algorithm 4.1 (Generalized border basis algorithm—BBasis).

**Input:** $F$ a finite generating set of a zero-dimensional ideal.

**Output:** $\mathcal{G}$ a border basis of the ideal.

1. Let $d := \max_{f \in F} \{\deg(f)\}$ and put $L := T_{\leq d}$.
2. $V = \{v_1, \ldots, v_r\} := \text{LStabSpan}(F, L)$.
3. If $\pi_{=d} \not\subseteq \text{LT}(V)$ then set $d := d + 1$ and put $L := \pi_{\leq d}$ and go to step (2).
4. Choose $\mathcal{O} \in \Lambda((V)_{K[\mathbb{X}]})$ ($\Leftrightarrow z \in P(V, L) \cap \mathbb{Z}^r$ and $\mathcal{O} = \mathcal{O}(z)$).
5. Let $\mathcal{G} := \text{BasisTransformation}(V, \mathcal{O})$.

Note that step (3) is a convenient way to quickly check whether $V$ is already $L$-stabilized and if it contains all degree-compatible order ideals. In this augmented algorithm we added the steps (4) and (5). The first step will be extensively discussed in Subsection 4.2 as there are various ways to determine $\mathcal{O} \in \Lambda((V)_{K[\mathbb{X}]})$ and this is precisely one of the main features, i.e., to choose the order ideal more freely. Note that by Theorem 3.2 we already know that $\mathcal{O}$ does support an $\mathcal{O}$-border basis of $\langle F \rangle_{K[\mathbb{X}]}$ (as $\langle F \rangle_{K[\mathbb{X}]} = \langle V \rangle_{K[\mathbb{X}]}$) and our task is now to actually extract this basis from $V$. This extraction is performed in step (5).

Let $\tilde{A}$ be a matrix representing a set of polynomials $M$ where the columns correspond to the monomials in some fixed ordering. Let the head (short: Head(a)) of a row $a$ of $\tilde{A}$ be the left-most monomial in the matrix representation whose coefficient is non-zero. Note that the notion of head replaces the notion of leading term of a polynomial as we do not (necessarily) have a term ordering anymore. The
main idea is to reorder the columns of $V$ and then to bring $V$ into a reduced row echelon form such that no $m \in O$ is head of a row of the resulting matrix — a classical basis transformation:

**Lemma 4.2.** Let $L = \mathbb{T}^n_{\leq d}$ with $\ell \in \mathbb{N}$, let $V$ be a finite set of polynomials satisfying $\langle V \rangle_K = \langle V \rangle_{K[X]} \cap \langle L \rangle_K$ and let $\mathcal{O} = \{t_1, \ldots, t_\mu \}$ be an order ideal with $\partial \mathcal{O} \subseteq L$ and $\mathcal{O} \in \Lambda(\langle V \rangle_{K[X]}).$ Then Algorithm 4.3 returns the $\mathcal{O}$-border basis $\mathcal{G}$ of $\langle V \rangle_{K[X]}$.

**Proof.** First, the algorithm finds $\ell$ from $M$. As $\mathcal{O} \in \Lambda(\langle V \rangle_{K[X]}$ we have that $\mathcal{O}$ supports a border basis and in particular we have $K[X] = \langle V \rangle_{K[X]} \oplus \langle \mathcal{O} \rangle_K$, and hence $\langle L \rangle_K = \langle V \rangle_K \oplus \langle \mathcal{O} \rangle_K$ by the modular law. We will now show that Condition (1) of Definition 2.2 is satisfied. This in turn follows from the fact that the algorithm creates every element $g_j$ of $\mathcal{G}$ to have the form

$$g_j = b_j - \sum_{i=1}^\mu \alpha_{ij} t_i$$

with $\alpha_{ij} \in K$. By construction $b_j \in \partial \mathcal{O}$ and thus $\mathcal{G}$ is an $\mathcal{O}$-border basis of $\langle V \rangle_{K[X]}$. □

**Algorithm 4.3** (Basis transformation algorithm—BasisTransformation).

- **Input:** $V, \mathcal{O}$ as in Lemma 4.2
- **Output:** $\mathcal{G}$ as in Lemma 4.2
  1. Set $\ell := \max_{m \in M} \deg(m)$.
  2. Permute the columns of $V$ such that $t$ is right of $m$ in the matrix representation of $V$ for all $m \in \mathbb{T}^n_{\leq \ell}$ and $t \in \mathcal{O}$.
  3. Reduce $V$ using Gaussian elimination: like Algorithm 2.11 but use Head instead of LT and the coefficient of the head instead of LC. Let $\mathcal{G}'$ be the result.
  4. Let $\mathcal{G} := \{g \in \mathcal{G}' : \text{Head}(g) \in \partial \mathcal{O}\}$.
  5. Return $\mathcal{G}$.

Note that Algorithm 4.3 works for any order ideal that supports a border basis of $\langle V \rangle_{K[X]}$, i.e., also those that are not necessarily degree-compatible. When the order ideal is known to be degree-compatible, it is enough to do the permutations in each degree block in the first step, and then use Algorithm 2.11 in the second step.

We will show now that Algorithm 4.1 computes an $\mathcal{O}$-border basis for $\mathcal{O} \in \Lambda(I)$.

**Proposition 4.4.** Let $F = \{f_1, \ldots, f_l\} \subseteq K[X]$ be a finite set of polynomials that generates a zero-dimensional ideal $I = \langle F \rangle_{K[X]}$. Then Algorithm 4.1 computes the $\mathcal{O}$-border basis $\mathcal{G}$ of $I$ for any (chosen) $\mathcal{O} \in \Lambda(I)$.

**Proof.** Whenever we reach step (4) in Algorithm 4.1 we have that $V$ is $L$-stabilized for some $L = \mathbb{T}^n_{\leq d}$ with $d \in \mathbb{N}$ and it contains all degree-compatible order ideals supporting a border basis, i.e., all $\mathcal{O} \in \Lambda(I)$. Observe that $I = \langle F \rangle_{K[X]} = \langle V \rangle_{K[X]}$, and thus, by Lemma 4.2 it follows that $\mathcal{G}$ is indeed an $\mathcal{O}$-border basis of $\langle F \rangle_{K[X]}$. Note that step (3) ensures $\partial \mathcal{O} \subseteq L$ via $\langle V \rangle^d_{L} / \langle V \rangle^{d-1}_{L} \cong \langle \mathbb{T}^n_{=d} \rangle_K = \langle L=\mathbb{T}^n_{=d} \rangle_K$.

An improved version of the border basis algorithm has been also considered in [20]. Basically, the improvement can be traced back to considering more restricted computational universes $L$ that arise from choosing $L$ to be the smallest order ideal that contains the support of the initial system and then successively extending it using the $+$ operation. This improvement due to the restriction of the computational universe $L$ cannot work in our setting anymore: Suppose that $V$ is $L$-stabilized with respect to some computational universe $L \neq \mathbb{T}^n_{\leq d}$ for all $d \in \mathbb{N}$ (i.e., $L$ is not obtained by bounding the total degree of the monomials in $\mathbb{T}$) and contains the order ideal that is induced by the chosen degree-compatible term ordering in the classical border basis algorithm. Then the associated polytope $P(V, L)$

[15]
would only contain a subset of all possible degree-compatible order ideals, as we might be lacking monomials that we need to represent certain alternative choices of \( \mathcal{O} \). If a subset of all admissible degree-compatible order ideals is sufficient, then the same optimizations can be applied though.

4.2. **Computing preferred border bases.** Let \( V \) and \( L \) be as obtained after step (3) in Algorithm 4.1. As shown in Theorem 3.2 and Lemma 3.6, the order ideal polytope \( P(V, L) \) characterizes all degree-compatible order ideals that support a border basis of \( (V)_{K[x]} \). Every \( z \in P(V, L) \cap \mathbb{Z}^L \) induces an order ideal \( \mathcal{O}(z) \) which supports an \( \mathcal{O}(z) \)-border basis of \( (V)_{K[x]} \). This also shows that we cannot expect that every order ideal \( \mathcal{O} \) supports an \( \mathcal{O} \)-border basis of \( (V)_{K[x]} \) as the characterization is one-to-one. The natural question is therefore how to specify which order ideal should be computed, i.e., which monomials we would like to be contained in \( \mathcal{O}(z) \) and which ones we would rather not. As the coordinates of \( z \) are in direct correspondence with the monomials in \( L \) we can define:

**Definition 4.5.** A preference is a vector \( c \in \mathbb{Z}^L \) which assigns a weight to each monomial \( m \in L \). If \( z \in P(V, L) \cap \mathbb{Z}^L \), then \( cz \) is the score or weight of \( z \).

As \( P(V, L) \subseteq [0, 1]^n \) is a polytope we can optimize over \( P(V, L) \cap \mathbb{Z}^L \) and compute an element \( z_0 \in P(V, L) \cap \mathbb{Z}^L \) that has maximal score, i.e., we can compute \( z_0 \in P(V, L) \cap \mathbb{Z}^L \) such that

\[
    cz_0 = \max \{ cz \mid z \in P(V, L) \cap \mathbb{Z}^L \}.
\]

In this sense a preference is an indirect way of specifying an order ideal. For certain choices of \( c \in \mathbb{Z}^L \) though it can be hard to compute an order ideal that maximizes the score as we will show now.

5. **Complexity of finding preferred order ideals**

In this section, we show that finding a weight optimal, border basis supporting order ideal of a zero-dimensional ideal given by generators is NP-hard (Theorem 5.3). Note that this also translates to large classes of other choice functions as \( P(V, L) \cap \mathbb{Z}^L \) is a 0/1 polytope and thus its extremal points are given by an inequality description. So any choice function that implicitly asks for a \( k \)-clique (which we will use in our reductions) can be replaced by the corresponding linear function and hardness also follows in this case. The hardness for computing a weight optimal order ideal is unexpected in the sense that we merely ask for a basis transformation. On the other hand it highlights the crucial role of order ideals in describing the combinatorial structure of the ideal.

As an immediate consequence it follows that it is rather unlikely that we can obtain a good characterization of the integral hull \( \text{conv}(P(V, L) \cap \mathbb{Z}^L) \) and we will not be able to compute degree-compatible order ideals that support a border basis and have maximum score efficiently unless \( P = NP \). This shows that not only computing the necessary liftings of the initial set of polynomials via the \( \text{LStabSpan} \) procedure is hard but also actually determining an optimal choice of an order ideal once an \( L \)-stable span has been computed. As mentioned before, this is in some sense surprising as the actual interference has been already performed at that point and we are only concerned with choosing a nice basis. From a practical point of view this is not too problematic as, although NP-hard, computing a weight optimal order ideal is no harder than actually computing the \( \text{LStabSpan} \) in general. For bounds on the degree \( d \in \mathbb{N} \) needed to compute border bases see, e.g., [13] Lemma 2.4; the border basis algorithm generates the Nullstellensatz certificates and is therefore subject to the same bounds. Further, state-of-the-art mixed integer programming solvers that can solve the optimization problem such as \texttt{scip} ([2]), \texttt{cplex} ([10]), or \texttt{gurobi} ([16]) can handle instance sizes far beyond the point for which the actual border bases can be computed. Very good solutions can also be generated using simple local search schemes starting from a feasible order ideal derived from a degree-compatible term ordering.
rewrite the optimization problem as follows:

Therefore, if \(c \in \mathbb{Z}^L\) be a preference vector. Define a graph \(\Gamma := (V, E)\) with \(V := L \cup \{s, t\}\) and \(\tilde{E} := \{(u, v) \mid u, v \in L \text{ and } v \mid u\}\), i.e., whenever \(v \mid u\) we add an arc from \(u\) to \(v\). In fact, it is enough to have an arc when \(u = vx\) for some variable \(x\), i.e., to consider the transitive reduction of \(\tilde{E}\). Define

\[
E := \tilde{E} \cup \{(s, u) \mid u \in L, c_u > 0\} \cup \{(u, t) \mid u \in L, c_u < 0\}.
\]

Further we set all the capacities of the arcs with both vertices in \(L\) to \(\infty\), for any arc \((s, u)\) with \(u \in L\) we set the capacity to \(c_u\), and for any arc \((u, t)\) with \(u \in L\) we set the capacity to \(|c_u|\); let \(\kappa(u, v)\) denote the capacity of the arc \((u, v) \in E\). An example is depicted in Figure 5.1.

For \(U, W \subseteq V\), we define \(C(U, W) := \sum_{(u, w) \in U \times W} \kappa(u, w)\). A \((s, t)\)-cut \((S, \bar{S})\) is a partition \(S \cup \bar{S} = V\) of the vertices of \(V\) with \(s \in S\) and \(t \in \bar{S}\) and the weight of the cut is \(C(S, \bar{S})\).

We would like to compute an order ideal contained in \(L\) with maximum score:

\[
\max \left\{ \sum_{u \in \mathcal{O}} c_u \mid \mathcal{O} \subseteq L \text{ order ideal} \right\}.
\]

Observe that, if \((S, \bar{S})\) is a cut in \(\Gamma\) of finite weight, there exist no arc \((u, v) \in \tilde{E}\) with \(u \in S\) and \(v \in \bar{S}\). Therefore, \((S, \bar{S})\) is a cut in \(\Gamma\) of finite weight if and only if \(S \setminus \{s\}\) is an order ideal. We can therefore rewrite the optimization problem as follows:

\[
\max \left\{ \sum_{u \in \mathcal{O}} c_u \mid \mathcal{O} \subseteq L \text{ order ideal} \right\} = \max\{C(\{s\}, \mathcal{O}) - C(\mathcal{O}, \{t\}) \mid \mathcal{O} \subseteq L \text{ order ideal}\}
\]

\[
\quad = \max\{C(\{s\}, L) - C(\{s\}, L \setminus \mathcal{O}) - C(\mathcal{O}, \{t\}) \mid \mathcal{O} \subseteq L \text{ order ideal}\}
\]

\[
\quad = C(\{s\}, L) - \min\{C(\{s\}, L \setminus \mathcal{O}) + C(\mathcal{O}, \{t\}) \mid \mathcal{O} \subseteq L \text{ order ideal}\}
\]

\[
\quad = C(\{s\}, L) - \min\{C(\{s\} \cup \mathcal{O}, (L \setminus \mathcal{O}) \cup \{t\}) \mid \mathcal{O} \subseteq L\}.
\]

The last line asks for a minimum weight cut in the graph \(\Gamma\). Note that we can indeed drop the condition that \(\mathcal{O}\) has to be an order ideal as it is guaranteed implicitly by all finite weight cuts as explained above.

It is well-known that the computation of a minimum cut can be performed in time polynomial in the number of vertices and arcs (see [34]). Thus we can indeed compute an order ideal \(\mathcal{O}\) of maximum score efficiently in this case.
5.2. NP-hard with constraints. So far we did not include the additional requirements as specified by the order ideal polytope (see Figure 3.3), in order to obtain degree-compatible order ideals that do actually support a border basis of the ideal \( I \) under consideration. Unfortunately, when including these additional requirements, the problem of computing an order ideal of maximum score is NP-hard as we will show in the following.

We will show NP-hardness by a reduction from the \textsc{Max-Clique} problem, which is well known to be \textsc{NP}-complete (see, e.g., \cite{15} or \cite{11 GT22}). Given a graph \( \Gamma = (V, E) \), recall that a clique \( C \) is a subset of \( V \) such that for all distinct \( u, v \in C \) we have \((u, v) \in E\).

\textbf{Max-Clique:} Let \( \Gamma = (V, E) \) be a graph. Determine the maximum size of a clique \( C \) contained in \( \Gamma \).

We will in particular use the following variant:

\textbf{k-Clique:} Let \( \Gamma = (V, E) \) be a graph. Determine whether \( \Gamma \) contains a clique \( C \) of size \( k \).

Note that if \( \Gamma \) contains a clique of size \( k \) for some \( k \in \mathbb{N} \), then so does it for any \( 1 \leq l \leq k \). Further the maximum size of a clique is bounded by \( |V| \). It follows that already testing whether \( \Gamma \) contains a clique \( C \) of size \( k \) for some given \( k \in \mathbb{N} \) has to be \textsc{NP}-complete, as otherwise, we could solve the \textsc{Max-Clique} problem with \( O(\log_2 |V|) \) calls of the algorithm that solves the test variant via binary search. In \cite{18} Discussion after Definition 3.2 \ it was indicated that determining a maximum weight order ideal of a pre-defined size is \textsc{NP}-hard by a reduction from \textsc{Max-Clique}. While already indicating hardness due to one additional cardinality constraint, this is a slightly different problem than the one that we are facing here: in addition to the size constraint, the order ideal has to be degree-compatible. Thus we have a cardinality/degree-constraint for each degree of the monomials (Constraints (3.7) and (3.9) in Figure 3.3). Further, the degree constraints are not completely independent of each other. So it could be \textit{a priori} perfectly possible that this particular restriction of the problem can be actually solved efficiently, which is not the case as we will see soon.

We consider the following optimization problem:

\textbf{Max-Bounded Order Ideal:} Let \( L = T^n_{\leq d} \) for some \( d \in \mathbb{N} \) and \( c \in \mathbb{Z}^L \) be a preference. Further let \( d_i \in \mathbb{N} \) for \( i \in [d-1] \). Determine an order ideal \( \varnothing_0 \subseteq L \) with \( |\varnothing_0^{=i}| = d_i \) for all \( i \in [d-1] \) and maximum score, i.e.,

\[
c(\xi^{-1}(\varnothing_0)) = \max \{ c(\xi^{-1}(\varnothing)) | \varnothing \subseteq L \text{ order ideal and } |\varnothing^{=i}| = d_i \text{ for all } i \in [d-1] \}.\]

By a reduction from \textbf{k-Clique} we obtain:

\textbf{Theorem 5.1.} \textsc{Max-Bounded Order Ideal} is \textsc{NP}-hard.

\textbf{Proof.} Let \( \Gamma = (V, E) \) be an arbitrary simple graph and choose \( k \in [\lfloor |V| \rfloor] \). We consider the polynomial ring \( K[x_v | v \in V] \). Let \( L = T^{|V|}_{\leq 2} \). Further choose \( d_1 = k \) and \( d_2 = k(k+1)/2 \). Define

\[
c_m = \begin{cases} 1, & \text{if } m = x_u x_v \text{ and } (u, v) \in E; \\ 0, & \text{otherwise.} \end{cases}
\]

A vertex \( v \in V \) is represented by the degree-one monomial \( x_v \) and each edge \((u, v) \in E\) is represented by the degree-two monomial \( x_u x_v \). In some sense we extended the graph \( (V, E) \) to the complete graph \( K_{|V|} \) that has \(|V|(|V| - 1)/2 \) edges as we consider all possible degree-two monomials that correspond to the edges; those edges that do not belong to \( E \) have weight 0 though.

The order ideals satisfying the size constraints are those consisting of the monomials \( 1, x_{v_1}, \ldots, x_{v_i} \) for \( 1 \leq i, j \leq k \) for some distinct vertices \( v_1, \ldots, v_k \). The score of the order ideal is the number of edges in the subgraph spanned by the \( x_{v_i} \).

Thus when maximizing \( c \) we ask for a \( k \)-vertex subgraph with maximal number of edges. The subgraph is given by the degree-one monomials contained in the order ideal, which we denote by \( \varnothing \),
i.e.,

\[ C := \{ v \in V \mid x_v \in \mathcal{O}\}. \]

So the maximum score is equal to \( k(k - 1)/2 \) if and only if \( \Gamma \) contains a clique of size \( k \). Thus we can solve \([k, \text{CLIQUE}]\) efficiently if we can solve \([\text{MAX-BOUNDED ORDER IDEAL}]\) efficiently and so the latter has to be \( \text{NP-hard} \).

Finally, it remains to show that for every graph \( \Gamma = (V, E) \) and \( k \in [|V|] \) there exists a system of polynomials \( F_{|V|, k} \subseteq K[x_v \mid v \in V] \) spanning a zero-dimensional ideal such that solving the \([\text{MAX-ORDER IDEAL OF IDEAL}]\) problem for \( F_{|V|, k} \) solves the \([k, \text{CLIQUE}]\) problem for \( \Gamma \). For this, we construct an ideal encoding all \( k \)-cliques of the complete graph on \( n \) vertices: Let \( n \in \mathbb{N} \) and \( k \in [n] \) and define

\[ F_{n,k} := \{ v_j \mid j \in [n-k] \} \cup \mathbb{T}_3^n. \]

We consider the ideal generated by \( F_{n,k} \). We show that its order ideals are in one-to-one correspondence with the \( k \)-element subsets of the set of \( n \) variables \( x_1, \ldots, x_n \) as stated in the following lemma.

**Lemma 5.2.** Let \( n \in \mathbb{N} \) and \( k \in [n] \). Then \( F_{n,k} \) generates a zero-dimensional ideal such that \( \mathcal{O} \in \Lambda\left(F_{n,k}\right)_{K[\mathbb{X}]} \) if and only if \( \mathcal{O} = 1 \subseteq \mathbb{T}_1^n \) with \( |\mathcal{O}| = 1 \) if \( k \), \( \mathcal{O} = 2 \) = \{ \( xy \mid x, y \in \mathcal{O} = 1 \) \}, and \( \mathcal{O} = \ell = \emptyset \) for all \( \ell \geq 3 \).

**Proof.** We will first characterize the vector space \( K[x_1, \ldots, x_n]/\left(F_{n,k}\right)_{K[\mathbb{X}]} \). Observe that the coefficient matrix \( A := (v_j)_{j \in [n-k]} \) is actually a Vandermonde matrix and in particular every square submatrix of \( A \) is invertible. Furthermore, the polynomials \( v_j \) with \( j \in [n-k] \) are homogeneous of degree one. Thus, for any \( k \) variables of \( \{x_1, \ldots, x_n\} \), without loss of generality say, \( x_1, \ldots, x_k \), we have that \( \{x_1, \ldots, x_k, v_1, \ldots, v_{n-k}\} \) is a basis for the homogeneous polynomials of degree one. This is just another way of saying that removing the columns belonging to \( x_1, \ldots, x_k \) from \( A \), the resulting square submatrix is invertible. It follows

\[ \langle \mathcal{O} \rangle_{K} \cong K[x_1, \ldots, x_n]/\left(F_{n,k}\right)_{K[\mathbb{X}]} \cong K[x_1, \ldots, x_k, v_1, \ldots, v_{n-k}]/\left(F_{n,k}\right)_{K[\mathbb{X}]} \cong K[x_1, \ldots, x_k, v_1, \ldots, v_{n-k}]. \]

As the substitution preserves degrees, homogeneity, etc., it follows, that any degree-compatible order ideal has to have \( |\mathcal{O}| = 1 \) if \( k \), \( |\mathcal{O}| = 2 \) = \{ \( xy \mid x, y \in \mathcal{O} = 1 \) \}, and \( |\mathcal{O}| = \emptyset \) for all \( \ell \geq 3 \).

The other direction follows immediately as each order ideal \( \mathcal{O} \) with \( |\mathcal{O}| = 1 \) if \( k \), \( |\mathcal{O}| = 2 \) = \{ \( xy \mid x, y \in \mathcal{O} = 1 \) \}, and \( |\mathcal{O}| = \emptyset \) for all \( \ell \geq 3 \) is actually a degree-compatible order ideal such that \( K[x_1, \ldots, x_n] = \left(F_{n,k}\right)_{K[\mathbb{X}]} \oplus \langle \mathcal{O} \rangle_{K} \) as vector spaces, by the argumentation above.

Note that the order ideals of \( F_{n,k} \) indeed correspond to the \( k \)-cliques of the complete graph on \( n \) vertices: If \( \mathcal{O} \in \Lambda(F_{n,k}) \), then \( \mathcal{O} = 1 \) = \{ \( x_{i_1}, \ldots, x_{i_k} \) \} and \( x_{i_j}x_{i_i} \in \mathcal{O} = 2 \) if and only if \( x_{i_j}, x_{i_i} \in \mathcal{O} = 1 \). If we now remove all elements of the form \( x_{i_j}^2 \) with \( x_{i_j} \in \mathcal{O} = 1 \), and there are \( k \) of those, then

\[ |\mathcal{O}| = 2 \setminus \{ x_{i_j}^2 \mid x_{i_j} \in \mathcal{O} = 1 \} = \frac{k(k-1)}{2}, \]

the size of a \( k \)-clique. We are ready to state the main result of this section. Consider the following problem:

**MAX-ORDER IDEAL OF IDEAL:** Let \( M \subseteq K[\mathbb{X}] \) be a system of polynomials generating a zero-dimensional ideal and \( c \in \mathbb{Z}_{\geq 0}^n \) be a preference on the monomials. Compute an order ideal \( \mathcal{O} \) supporting an \( \mathcal{O} \)-border basis of \( \langle M \rangle_{K[\mathbb{X}]} \) with maximum score with respect to \( c \).
Theorem 5.3. \textbf{Max-Order Ideal of Ideal} is NP-hard.

\textit{Proof.} The proof is by a reduction from the NP-hard \textbf{k-Clique} along the lines of the proof of Theorem 5.1. Let $\Gamma = (V,E)$ be a graph with $n := |V|$ and $k \in \lfloor n \rfloor$ be an instance of \textbf{k-Clique}. We consider $M := F_{n,k}$ and define $c \in \mathbb{Z}_{\leq 3}$ via

$$c_m = \begin{cases} 1, & \text{if } m = x_u x_v \text{ and } (u,v) \in E; \\ 0, & \text{otherwise.} \end{cases}$$

By Lemma 5.2, we have that the degree-compatible order ideals of $\langle M \rangle_{K[X]}$ are in one-to-one correspondence with the $k$-cliques of the complete graph on $n$ vertices. Similarly to the proof of Theorem 5.1, the score of an order ideal is just the number of edges between the corresponding $k$ vertices of the graph, so the maximum score is $k(k - 1)/2$ if and only if the graph contains a $k$-clique. Thus, we obtain a test whether $\Gamma$ contains a clique of size $k$. \qed

6. Computational results

We performed a few computational tests to verify the practical feasibility of our method. All computations were performed with CoCoA 4.7.5 ([7]) and scip 1.1.0 ([2]) on a 2 GHz Dual Core Intel machine with 2 GB of main memory. We performed computations on various sets of systems of polynomial equations. The employed methodology was as follows. We first computed a border basis using the classical border basis algorithm. From the last run of the algorithm we extracted the $L$-stabilized span and brought it into canonical form. We generated the constraints (3.7) from the order ideal that we obtained; from the $L$-stabilized span in matrix form, we generated the constraints (3.8). As we needed to get access to the $L$-stable span computed in the last round of the border basis algorithm, we had to use an implementation of the border basis algorithm in CoCoA-L which is slower in terms of speed compared to a C or C++ implementation. We then transcribed these constraints into the CPLEX LP format which served as input for scip. For the optimization we chose various preference functions. One was a random function, and the other one was chosen with the intent to make the optimization particularly hard by giving monomials deep in the order ideal negative weights and assigning positive weights for the outer elements.

In all cases the optimization, i.e., the computation of the weight optimal order ideal was performed in less than a second, whereas the actual calculation of the initial border bases was significantly more time consuming. As indicated before, this is not unexpected as the computation of the $L$-stable span is significantly more involved than computing a weight optimal order ideal (in the worst case double exponential vs. single exponential). When computationally feasible we also counted all feasible order ideals with scip, which basically means enumerating all feasible solutions. This in fact is equivalent to optimizing all potential preference functions simultaneously and thus emphasizes the computational feasibility.

We considered 7 systems of polynomials of various complexity in terms of number of variables and order ideal degree. We would have liked to test significantly larger instances but we were not able to compute the initial border basis (or more precisely the $L$-stable span), neither with our implementation nor with the C/C++ implementation of CoCoA 5 (BBasis5). This shows once more that the limiting factor is the actual computation of the $L$-stable span.

In Table 1 we report our results.

7. Concluding remarks

We provided a way to characterize all degree-compatible order ideals that support a border basis for a given zero-dimensional ideal $I$ by borrowing from combinatorial optimization and in particular polyhedral theory. We established a one-to-one correspondence of the integral points of a certain polytope, the order ideal polytope, and those degree-compatible order ideals that support a border
Table 1. Computational results. The first column contains the considered polynomial system. The second column contains the degree vector of the order ideal, i.e., \((\dim I^{\leq i}/I^{\leq i-1})_i\) starting with \(i = 1\) and \(I^{\leq 0} := 0\). The third column contains the average time (in seconds) needed to optimize a random preference over the order ideal polytope (we performed 20 runs for each system). The fourth column contains the time (in seconds) needed to count all admissible degree-compatible order ideals and the last column contains the actual number of admissible degree-compatible order ideals. The ‘*’ indicates that the counting had been stopped after 300 seconds. The number of order ideals reported in this case is the number that have been counted up to that point in time.

| polynomial system | order ideal signature | optimization [s] | counting [s] | # order ideals |
|-------------------|-----------------------|-----------------|--------------|---------------|
| \(x^2 + xy^2 + y^3\) | \((1,3,1,1,1)\) | < 0.01 | 0.02 | 3 |
| vanishing ideal of the points \((0,0,0,1), (1,0,0,2), (3,0,0,2), (5,0,0,3), (-1,0,0,4), (4,4,4,5), (0,0,7,6)\). | \((1,4,2)\) | < 0.01 | 0.02 | 45 |
| \(x + y + z - u - v, x^2 - x, y^2 - y, z^2 - z, u^2 - u, v^2 - v\) | \((1,4,5)\) | < 0.01 | 0.35 | 1,260 |
| \(x + y + z - u - v, x^3 - x, y^3 - y, z^2 - z, u^2 - u, v^2 - v\) | \((1,4,7,6)\) | 0.02 | 51.50 | 106,820 |
| \(x + y + z - u - v, x^3 - x, y^3 - y, z^3 - z, u^2 - u, v^2 - v\) | \((1,4,8,9)\) | 0.02 | 53.00 | 108,900 |
| \(x + y + z - u - v, x^3 - x, y^3 - y, z^3 - z, u^3 - u, v^2 - v\) | \((1,4,9,12,9)\) | 0.08 | 300.00* | > 1,349,154 |
| \(x + y + z - u - v + a, x^2 - x, y^2 - y, z^2 - z, u^2 - u, v^2 - v, a^2 - a\) | \((1,5,9)\) | < 0.01 | 8.68 | 30,030 |

basis of \(I\). This connection in particular links the ideals to their combinatorial structure of the factor spaces.

Using this polytope we adapted the classical border basis algorithm in order to be able to compute border basis for general degree-compatible order ideal based on a preference ordering on terms contained therein. Effectively, the algorithm can be used for any integral point contained in the order ideal polytope.

We also showed that computing a border basis for a preference on the monomials one might want to have included in the order ideal is NP-hard and thus we cannot expect to be able to efficiently compute preferred order ideals in general although it is merely a basis transformation. On the other hand, this is not restricting the applicability of our method in any practical application because the preceding computation of the \(L\)-stable span dominates in terms of computational complexity. We finally presented a few computational results showing the applicability of our method for actual computations.
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