Geometry-dependent interplay of long- and short-range interactions in ultracold fermionic gases: models for condensed matter and astrophysics

B Deb\textsuperscript{1}, G Kurizki\textsuperscript{2,5} and I E Mazets\textsuperscript{3,4}

\textsuperscript{1} Department of Materials Science, Indian Association for the Cultivation of Science, Jadavpur, Kolkata-700032, India
\textsuperscript{2} Weizmann Institute of Science, 76100 Rehovot, Israel
\textsuperscript{3} Ioffe Physico-Technical Institute, St Petersburg 194021, Russia
\textsuperscript{4} Atominstutut der Österreichischen Universitätten, TU Vienna, Vienna A-1020, Austria
E-mail: gershon.kurizki@weizmann.ac.il

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Abstract. We study the two mechanisms of the interplay of long- and short-range interactions in different geometries of ultracold fermionic atomic or molecular gases. We show that in the range of validity of the one-dimensional (1D) approximation, both mechanisms yield similar superconductivity. We show that electromagnetically induced isotropic dipole–dipole interactions in a spin-polarized non-degenerate fermionic gas can cause an extremely exothermic phase transition, analogous to the isothermal collapse in gravitationally interacting star clusters. This collapse may result in fragmentation of the gas into a hot ‘halo’ and a highly degenerate ‘core’. Possible realization is envisaged in microwave-illuminated fermionic molecular gases at microkelvin temperatures.

\textsuperscript{5} Author to whom any correspondence should be addressed.
1. Introduction

Recent advancements in cooling fermionic gases below the degeneracy temperature in magnetic [1] and optical traps [2] and lattices [3], as well as tuning their interaction [4], allow us to consider these gases as candidates for laboratory explorations of diverse models ranging from condensed matter physics to astrophysics. Here, we focus on the comparison of three-dimensional (3D) as well as low-dimensional Fermi gases to their unexplored long-range interacting counterparts.

Studies of 1D or quasi-one dimensional (Q1D) interacting fermionic systems are important for modelling unconventional (possibly high-$T_c$) superconductors. The problem of 1D fermions interacting via a contact potential is exactly solvable [5] through the use of the Bethe ansatz [6]. By contrast, long-range 1D interacting fermions have not been studied so far. Yet such studies may serve as important precursors to a satisfactory theory of high-$T_c$ superconductivity if one can overcome the difficulty of their generalization to 2D.

Interacting 1D Fermi systems cannot be characterized by elementary quasi-particles because the coupling of quasi-particles to other collective excitations related to spin and density fluctuations is large even for arbitrarily small interactions. This makes perturbation theory inapplicable and renders the 1D system strongly correlated, even for weak interactions. Therefore, Landau’s Fermi liquid theory that is based on the concept of quasi-particles breaks down for 1D systems. The collective bosonic modes related to spin and density fluctuations are the basic physical attributes that characterize a 1D Fermi system. A unique feature of 1D interacting fermions is spin–charge separation, which means that the collective modes of spin and density excitations propagate with different velocities [7]. When both the spin and density excitations are gapless, the system is described by the Luttinger liquid theory [7]. By contrast, when collective spin excitations (spinons) have a gap while density excitations (phonons) remain gapless, the 1D system becomes superconductive/superfluid [8]. The ground-state properties of the 1D Fermi gas are described by the exactly solvable Gaudin–Yang model [5] that expresses the ground-state energy, being an important extension of the Lieb–Liniger model [9].

Here, we compare the characteristics of 1D BCS pairing due to long-range quasi-static (when the resonant wavelength is much larger than the system size) dipole–dipole interactions and short-range (contact) interactions, traditionally considered in the theory of superconductivity.

Another promising line of research of ultracold gases is inspired by analogies of these gases to astrophysical systems, e.g. degenerate Fermi gases (DFGs) viewed as analogues of white dwarf stars [10] and other self-gravitating objects [11]–[13].
Here, we point out a fascinating possibility of this kind: the creation of highly degenerate Fermi gases with highly enhanced phase-space density using electromagnetically induced dipole–dipole interactions at a constant temperature. It has been theoretically demonstrated that, provided that the electromagnetic (EM) illumination of the gaseous sample is made essentially isotropic, the dipole–dipole interaction resembles gravitational attraction \((\propto 1/r)\) at near-zone separations (well within the EM wavelength) [14]. Such interactions have been predicted to yield a novel self-trapping (self-gravitating) regime in atomic BEC [14]. Here, we investigate the properties of a cold, trapped Fermi gas cloud in thermal equilibrium with a bosonic ‘reservoir’, when this cloud is omni-directionally irradiated by EM beams that induce ‘self-gravity’ in the Fermi gas.

In contrast to the short-range van der Waals forces accounted for by a contact-interaction pseudo-potential, the DDI is essentially a long-range potential
\[
V_{3D}(r) = \frac{D_1 D_2 r^2 - 3(D_1 r)(D_2 r)}{4\pi \varepsilon_0 r^5},
\]
where \(r\) is the interatomic separation vector and \(D_1\) and \(D_2\) are the dipole moments (electric or magnetic) of interacting atoms or molecules. The particles are of the same chemical species, but may differ in their electronic and/or nuclear spin orientation.

The paper is organized as follows. In section 2, we study the effects of pairing due to the long-range dipole–dipole interactions in fermionic systems with a strong radial confinement that makes the dynamics effectively 1D. We show that, despite different dependences of the gap on the Fermi momentum, the excitation spectrum of a fermionic system with the pairing provided by the long-range interactions is quite similar in the truly 1D regime to the spectrum of a system with the standard BCS pairing by contact interactions. In section 3, we consider the ‘gravothermal’ collapse in a 3D system of spin-polarized, isotropically confined fermions with the electromagnetically induced gravity-like \((\propto 1/r)\) attraction. We show that a system of fermions with \(1/r\) attraction can undergo a strongly exothermic transition from a non-degenerate state to a highly degenerate stage, the heat being taken away by a reservoir consisting of bosonic particles. The conclusions are presented in section 4.

2. 1D dipole–dipole BCS pairing

We assume that the radial motion of the particles is frozen, so that all of them occupy the ground state of the radial motion in the confining potential, so that the transverse profile of the density is
\[
n_{\perp}(x, y) = (\pi w_r^2)^{-1} \exp\left[\frac{(x^2 + y^2)}{w_r^2}\right].
\]

Here \(w_r = \sqrt{\hbar/(m\omega_r)}\) is the typical size of the profile of the ground-state wavefunction of the radial motion of particles trapped in the \((x, y)\)-plane by a harmonic potential with the fundamental frequency \(\omega_r\). Then the effective Hamiltonian of the 1D motion of the fermions is
\[
\hat{H}_{1D} = \sum_{i=1}^{2} \int dz \hat{\psi}_i^\dagger(z) \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} \right) \hat{\psi}_i(z) + \frac{1}{2} \int dz \int dz' \hat{\psi}_1^\dagger(z) \hat{\psi}_2^\dagger(z) [g_{1D}\delta(z - z') + V_{1D}(z - z') \hat{\psi}_2(z) \hat{\psi}_1(z)],
\]
alkaline atom has hyperfine splitting much larger than that of the first excited (P
potential \((\text{such a moderate interaction strength, the first Born approximation is valid, and, for the 1D
moments are perpendicular to
of truly 3D nature.}

\text{interactions are absent in ultracold fermions in the same spin state due to Fermi statistics. The
long-range nature of the DDI could give rise to the dimerization of particles in the same internal (spin) state (1D analogue of p-wave pairing in the strong coupling regime). However, this may occur only for such a strong DDI that the 1D approximation breaks down, and the collisions are of truly 3D nature.}

\text{The dimensionless potential (4) consists of the singular and regular parts,}

\begin{equation}
V_{1D}(z - z') = U_0 \left\{ 2w_i \delta(z - z') + 2^{-3/2} \left[ \Gamma \left( \frac{1}{2}, \frac{z^2}{2w_i^2} \right) - \frac{3z^2}{2w_i^2} \Gamma \left( \frac{3}{2}, \frac{z^2}{2w_i^2} \right) \right] \exp \left( \frac{z^2}{2w_i^2} \right) \right\},
\end{equation}

\begin{equation}
\Gamma(a, \xi) \text{ being the incomplete gamma-function. The prefactor } U_0 \text{ will be equal to}
\frac{D_1 D_2}{(4\pi \varepsilon_0 w_i^3)} \text{ if the dipole moments are directed along } z \text{ or to } -\frac{D_1 D_2}{(8\pi \varepsilon_0 w_i^3)} \text{ if the dipole moments are perpendicular to } z. \text{ To maintain the 1D approximation, we require } U_0 \lesssim \hbar \omega_i. \text{ For such a moderate interaction strength, the first Born approximation is valid, and, for the 1D collision studies, we can apply, instead of the } T\text{-matrix, simply the Fourier transform of the potential (5):

\begin{equation}
\tilde{V}_{1D}(k_z) = U_0 k_z^2 w_i^2 \exp \left( \frac{1}{2} k_z^2 w_i^2 \right) \Gamma \left( 0, \frac{1}{2} k_z^2 w_i^2 \right).
\end{equation}

\text{The dimensionless potential } \tilde{V}_{1D}(k_z)/U_0 \text{ is zero for } k_z = 0, \text{ then grows monotonically and asymptotically approaches } 2 \text{ for } k_z \to \infty. \text{ In other words, we obtain net attraction, necessary for BCS pairing, for negative } U_0. \text{ Note also that 1D approximation also implies that the ground-state kinetic momentum distribution is confined within } \pm \sqrt{2}/w_i; \text{ otherwise occupation of the first excited state of transversal motion becomes energetically profitable.}

\text{Unlike the previously considered case of DDI between the particles in the same spin state, where the prefactor in front of the singular part of the potential is always positive, the use of two distinct spin states opens new perspectives: If the polarizabilities of the particles in the } |1\rangle \text{ and } |2\rangle \text{ states have the same sign, the situation is like the previously considered one: the repulsive delta-functional part of the DDI 1D potential is added to the mainly attractive regular part. However, if the polarizability signs are opposite, then the situation is reversed, and the DDI potential then allows for BCS pairing. For example, in the case of fermionic alkaline atoms, two polarizations of opposite sign can be attained by the following method. The laser frequency is tuned off-resonance to the } D_1 \text{ line of the atom. The ground } (S_{1/2}) \text{ state of the alkaline atom has hyperfine splitting much larger than that of the first excited } (P_{1/2}) \text{ state, so that in the roughest approximation the } P_{1/2} \text{ state can be regarded as degenerate. Tuning the laser frequency } \omega_L \text{ somewhere around the frequency difference between the } S \text{ and } P \text{ states unperturbed by hyperfine interactions and choosing the } |1\rangle \text{ and } |2\rangle \text{ states belonging to the lower and upper hyperfine levels of the ground state, respectively, one obtains red detuning for } |1\rangle \text{ and blue detuning for } |2\rangle, \text{ and, hence, two polarizabilities of opposite sign.}
The momentum-dependent gap obtained from equation (9) for the 1D system of particles interacting via DDI with \( U_0 = -1.0 \hbar \omega_r \). The 1D number density \( n_{1D} = 0.56/w_r \). On this plot and the subsequent ones, the units on the axis are dimensionless, \( w_r \) and \( \hbar \omega_r \) being the length and energy units, respectively.

Figure 1. The momentum-dependent gap obtained from equation (9) for the 1D system of particles interacting via DDI with \( U_0 = -1.0 \hbar \omega_r \). The 1D number density \( n_{1D} = 0.56/w_r \). On this plot and the subsequent ones, the units on the axis are dimensionless, \( w_r \) and \( \hbar \omega_r \) being the length and energy units, respectively.

The excitation spectra of 1D fermionic systems with contact interaction are obtained exactly within the Gaudin–Yang approach [5]. However, the underlying Bethe ansatz cannot be constructed for systems with long-range interactions. Therefore, we resort to the BCS gap equation, known to reproduce qualitatively the exact Gaudin–Yang results,

\[
\Delta_k = -\frac{1}{4\pi} \int dk' \frac{[\hbar g_{1D} + \tilde{V}_{1D}(k-k')]\Delta_{k'}}{\sqrt{\eta_{k}^{2} + \Delta_{k}^{2}}},
\]

where \( \eta_{k} = \hbar^2 k^2/(2m) - \mu \), \( \mu \) being the chemical potential, supplied by the 1D number density definition

\[
n_{1D} = \frac{1}{2\pi} \int dk \left( 1 - \frac{\eta_{k}}{\sqrt{\eta_{k}^{2} + \Delta_{k}^{2}}} \right).
\]

Equation (7) contains both the short- and long-range potentials. To investigate the pure effect of the long-range DDI potential, we set \( g_{1D} = 0 \), thus arriving at

\[
\Delta_k = -\frac{1}{4\pi} \int dk' \tilde{V}_{1D}(k-k')\Delta_{k'}/\sqrt{\eta_{k'}^{2} + \Delta_{k'}^{2}}.
\]

We solved equation (9) numerically, by iterations. The largest gap was obtained for the values of \( U_0 \) close to \( \hbar \omega_r \), as expected. Note that in this case the system is beyond the weak-coupling regime.

In figure 1, we show a typical dependence of the gap on the kinetic momentum. In contrast to the case of contact interaction, the gap is not constant but demonstrates dispersion. As seen from figure 2, this dispersion can hardly be revealed in an experiment, since the spectrum of the superfluid fermionic system with DDI can be excellently fit by a spectrum \( \epsilon(k) = \sqrt{\eta_{k}^{2} + \Delta_{sr}^{2}} \) of a system with short-range (contact) interactions \( g_{1D} \delta(z-z') \) only, with suitably chosen coupling constant \( g_{1D} < 0 \). In the latter case, the gap \( \Delta_{sr} \) is determined from
Figure 2. The excitation spectra of superconducting systems interacting via DDI (solid line) and contact interactions with $g_{1D} = -1.0 \hbar \omega_r$ (dashed line). Other parameters are the same as in figure 1.

Figure 3. The gap at the minimum of the excitation spectrum as a function of the 1D number density for DDI with $U_0 = -1.0 \hbar \omega_r$ (filled circles and solid line) and contact interactions with $g_{1D} = -1.0 \hbar \omega_r$ (crosses and dashed line).

the equation $\frac{1}{4\pi} |g_{1D}| \int \frac{dk}{q} (\eta^2 + \Delta_{st}^2)^{-1/2} = 1$, which is obtained from equation (7) by setting $\tilde{V}_{1D}(k-k') \equiv 0$. It is worth noting that the dependence of the gap size on the 1D number density of fermions (figure 3) is significantly different for the two cases.

3. Isotropic dipole–dipole ‘gravothermal’ collapse

Let us start this section by summarizing the conditions for the emergence of electromagnetically induced ‘gravity’ [16] in 3D isotropically trapped gases. Consider first a circularly polarized plane wave of frequency $\omega_L$, wavenumber $q = \omega_L/c$ and intensity $I$, propagating in the $z$-direction. The expression (taking into account effects of finite propagation distance or retardation) for the time-averaged interaction potential of two particles separated by $r$ is [14]

$$V_{dd}(\mathbf{q}, \mathbf{r}) = U_0 \left[ -\frac{2}{3} P_2(\cos \theta) y_2(qr) + \frac{4}{3} y_0(qr) \right] \cos qz. \quad (10)$$
energy density and it vanishes in the quasi-static limit and arises due to retardation (propagation-distance) effects; its ratio to the first term is expressed obtained: the quasi-static dipole–dipole interaction in the near zone. The second term (chemical potential, are discussed in various textbooks characterized by thermodynamical parameters. to invoke the local density approximation wherein the cloud is treated as a continuous medium, considerably smaller than the wavelength of the microwave radiation which induces the dipole–interaction of equation (11) between the fermions (whereas the particles comprising the bosonic ‘bath’ are unaffected by the microwave radiation). We assume that the fermionic cloud has a size \( R \) that is considerably smaller than the wavelength of the microwave radiation which induces the dipole–dipole interactions, \( qR \ll 1 \). The electromagnetically induced potential in eq... the near-zone limit \( qr \approx \) isotropic and arises due to retardation (propagation-distance) effects; its ratio to the first term is \( \sim (qr)^2 \) and it vanishes in the quasi-static limit \( q \to 0 \).

If one averages equation (10) over all possible EM beam orientations relative to \( r \), i.e. over an isotropic distribution of orientations of the wavevector \( q \), a remarkable isotropic potential is obtained:

\[
\bar{V}_{iso}(r) = \frac{1}{4\pi} \int_{(4\pi)} d\Omega_q \, V_{dd}(q, r) = U_0 \left[ \frac{2}{3} \gamma_2(qr) j_2(qr) + \frac{4}{3} \gamma_0(qr) j_0(qr) \right],
\]

where \( j_j(qr) = \sqrt{\pi/(2qr)} J_{j_{\text{isotropic}}}(qr) \) is the spherical Bessel function of the first kind [17]. The potential \( \bar{V}_{iso} \) reduces, in the near-zone limit \( qr \ll 1 \), to \( \bar{V}_{iso} \approx -u/\pi \), where \( u = 22U_0/(15q) = 11\alpha^2 q^2 I/(60\pi c_0^2 \epsilon_0^2) \). This near-zone behaviour has been called electromagnetically induced ‘gravity’ [14]. In the far zone \( (qr \gg 1) \), we find \( \bar{V}_{iso} \approx -\frac{15}{4} u \sin(2qr)/(2qr^2) \), exhibiting diminishing periodic spatial oscillations. The elimination of the \( 1/r^3 \) contribution due to the directional averaging was first noted by Thirunamachandran [18], but no practical scheme for its implementation had been proposed until it was found by our group that the perfectly isotropic interaction of equation (11) can be obtained in a configuration consisting of a finite number of EM beams [14].

Consider now a gas of spin-polarized fermions in thermal contact with a large ‘bath’ of bosons within a spherical trap, illuminated by microwaves that give rise to the interaction potential (11) between the fermions (whereas the particles comprising the bosonic ‘bath’ are unaffected by the microwave radiation). We assume that the fermionic cloud has a size \( R \) that is considerably smaller than the wavelength of the microwave radiation which induces the dipole–dipole interactions, \( qR \ll \pi/2 \). The electromagnetically induced potential in equation (11) can then be approximated by an attractive potential \( \propto 1/r \). The spatial scale on which the fermionic density profile changes is taken to be large compared to the interparticle distance. This allows us to invoke the local density approximation wherein the cloud is treated as a continuous medium, characterized by thermodynamical parameters.

The thermodynamic properties of a Fermi gas in equilibrium, having a spatially varying chemical potential, are discussed in various textbooks [19]. Its local number density \( \rho(r) \), energy density \( \epsilon(r) \) and entropy density \( s(r) \) are given in our case by the self-consistent coupled expressions

\[
\rho(r) = \lambda_T^{-3} f_{3/2}(Z),
\]

\[
\epsilon(r) = \frac{3}{2} \lambda_T^{-3} k_B T f_{3/2}(Z),
\]

\[
s(r) = \lambda_T^{-3} k_B \left[ \frac{5}{2} f_{3/2}(Z) - \log Z f_{3/2}(Z) \right],
\]

\[
Z(r) = \exp[\mu_{loc}(r)/(k_B T)],
\]

\[
\mu_{loc}(r) = E_r - U_u(r) - U_{\text{EM}}(r).
\]
Here $\lambda_T = \hbar \sqrt{2\pi/(mk_B T)}$ is the thermal de Broglie wavelength, $m$ being the particle mass, $f_\nu$ is the Fermi integral defined as

$$f_\nu(Z) = [\Gamma(\nu)]^{-1} \int_0^\infty (Z^{-1} e^x + 1)^{-1} x^{\nu-1} \mathrm{d}x$$

(17)

and $Z(r)$ in equation (15) is the fugacity. The local chemical potential $\mu_{\text{loc}}(r)$ (equation (16)) depends on the Fermi energy $E_F$, the parabolic potential $U_\text{tr}(r) = \frac{1}{2} m \omega^2_{\text{tr}} r^2$ of the trap, and the isotropic self-consistent potential seen by each particle at point $r$ due to its electromagnetically induced interaction with all other particles,

$$U_{\text{EM}}(r) = \int \mathrm{d}^3 r' \tilde{V}_{\text{iso}}(|r - r'|) \rho(r').$$

(18)

Note that $\mu_{\text{loc}}(r)$ does not include a short-range scattering term, in accordance with our assumption that the cold fermions are in the same spin state, so that s-wave scattering is precluded by the Fermi–Dirac statistics.

We have numerically solved and investigated the set of equations (12)–(16) by the following iterative procedure. For the given input parameters, $T$, $\tilde{V}_{\text{iso}}$ and $E_F$, we have used a suitable initial approximation for the density distribution to calculate the self-consistent potential according to equation (18), by taking the fast Fourier transform of the density profile, multiplying it by the analytically known Fourier transform of $\tilde{V}_{\text{iso}}$ and then applying the inverse fast Fourier transform. The resulting approximation for $U_{\text{EM}}(r)$ has then been used in the next iteration of $\rho(r)$ via equation (12), and so on, until convergence has been reached. The resulting density profile $\rho(r)$ yields the total number of particles $N = \int \mathrm{d}^3 r \rho$. The total energy $E = \int \mathrm{d}^3 r \epsilon$ and the total entropy $S = \int \mathrm{d}^3 r s$ have also been calculated. Although our boundary conditions differ from those of [12], our findings are qualitatively similar (cf figure 10 of [12]).

In figure 4, we present an isothermal line of the degeneracy parameter (the density of fermions at the trap centre times $\lambda^3_T$) as a function of $u$, which is proportional to the EM intensity, for a fixed number of particles $N$. Such isothermal lines are physically relevant for a Fermi system immersed in a large cloud of bosons, as in sympathetic cooling. If the number of bosons is much larger than the number of fermions, the bosonic subsystem (which is unaffected by the radiation) acts as a reservoir which keeps the fermions at an approximately constant temperature. Then, by changing the EM intensity slowly enough to ensure effective heat exchange between the two subsystems, we follow an isothermal line such as the one plotted in figure 4.

There are three branches in this plot:

1. **The lower branch** starts at $u = 0$ and persists up to a certain value $u = u_{\text{crit}}$. This branch corresponds to a trapped Maxwell–Boltzmann (non-degenerate) gas, interacting via the long-range (‘gravitational’) potential $\tilde{V}_{\text{iso}}$. In [20], it was shown that a system of gravitationally interacting particles confined in a spherical box of radius $R_{\text{box}}$ becomes dynamically and thermodynamically unstable when $G m^2 = 2.52 k_B T R_{\text{box}} / N$, where $G$ is the Newtonian gravitational constant. In our case there is no artificial boundary condition, since the gas is confined in a trap, so that we should replace $R_{\text{box}}$ by the size of the non-degenerate fermionic cloud $R \approx \sqrt{k_B T/(m \omega^2_{\text{tr}})}$. Since in our case $u$ is equivalent to $G m^2$...
Figure 4. Degree of degeneracy at the trap centre (on a logarithmic scale) versus ‘gravitational constant’ $u$ (proportional to the EM intensity) for $N = 10^4$ fermions at $T = 100\hbar\omega_T/k_B$, computed from equations (12)–(16). The isothermal collapse is seen to occur at the point A ($u_{\text{crit}} = 0.014\,k_BT\lambda_T$). Point C: $u_{\text{min}} = 0.0076\,k_BT\lambda_T$.

in the Newtonian gravitation theory, we may estimate the critical EM intensity from the expression

$$u_{\text{crit}} \approx 2.52\frac{k_BT^2}{N},$$

(19)

implying that for $u > u_{\text{crit}}$, the non-degenerate regime is no longer possible: if $u$ exceeds $u_{\text{crit}}$, then the cloud of the size $R$ is no longer stable against the Jeans instability [21].

2. The intermediate branch corresponds to the entropy minimum and is thus unstable, as is usual for bistable systems.

3. The upper branch is semi-infinite and exists for $u > u_{\text{min}}$, where $u_{\text{min}} < u_{\text{crit}}$ (point C in figure 4). The upper branch corresponds to a ‘white dwarf’ regime: the Fermi gas is contracted by the gravity-like forces to a highly degenerate ‘core’ with the size $R_c \lesssim q^{-1}$, and further contraction is stopped by the Fermi-degeneracy pressure. The size $R_c$ of the ‘white dwarf core’ can be estimated from the formula

$$R_c = \frac{107\,\hbar^2}{muN^{1/3}},$$

(20)

which is the analogue of the well-known astrophysical relation between the mass and radius of a white dwarf or a neutron star [13, 21]. We stress that $R_c$ is smaller than (but independent of) $R$, the radius determined by the trap for $u = 0$.

A spontaneous jump from the lower to the upper branch due to the onset of instability should occur at $u = u_{\text{crit}}$, analogously to the isothermal collapse of [12]. In figure 5, we present a density (local-degeneracy) profile of a fermionic cloud just beyond the transition point B, i.e. for $u$ slightly exceeding $u_{\text{crit}}$. The density and thus the degeneracy at $r \lesssim 2\lambda_T$ are then hugely (by several orders of magnitude) increased when compared to the point A on the lower branch. A large amount of energy must then be released in this collapse. It could be carried away by the
bosons comprising a thermal reservoir. In order to avoid significant increase in the temperature of the reservoir, one may employ standard laser cooling techniques [22].

The proposed mechanism, which amounts to compression at a constant temperature $T$, well above the Fermi temperature $T_F$, should be contrasted with the present methods of achieving the Fermi degeneracy regime in atomic vapours: these are based on sympathetic cooling of fermionic atoms below $T_F$, either by bosonic atoms [10, 23] or by their fermionic counterparts (in a different magnetic state) [1, 2].

Which kind of fermionic particle is suitable for the realization of the ‘gravothermal’ collapse? The requirements for the attainment of equation (19), along with the near-zone condition $q R \lesssim \pi/2$, preclude the use of optically driven fermionic atoms, since this would require a sub-wavelength optical trap where a few thousands of atoms would be compressed to very high densities, $>10^{17}$ cm$^{-3}$. These requirements may be considerably eased in gases of fermionic molecules (e.g. $^6$Li $^{87}$Rb or $^{40}$K $^{87}$Rb), provided they are cooled down (by Feshbach-resonance photoassociation) to the microkelvin range and to their rovibronic ground state (by stimulated Raman adiabatic passage (STIRAP)) [24]. The prospects appear to be realistic, in view of the rapid progress in this area, culminating in the recent realization of a molecular BEC [25]. The EM radiation can be tuned to a strong dipolar transition between rotational molecular levels in the microwave region (the transition dipole moment being of the order of 0.1 Debye and the transition wave number $q \approx 3$ cm$^{-1}$). This allows us to have large numbers of molecules in the trap. For example, for $N = 10^4$ fermionic molecules ($m \sim 10^{-22}$ g) in a trap with $\omega_t \sim 10^3$ s$^{-1}$ at $T \approx 1$ $\mu$K, in accordance with the parameters of figure 4, we get the density of $10^{12}$ cm$^{-3}$ and $u_{\text{crit}} \sim 10^{29}$ J cm, which corresponds to microwave radiation with a spectral intensity of about 1 W (cm$^2$ Hz)$^{-1}$ within the bandwidth exceeding the trap frequency (in our example the numerical estimation $u_{\text{crit}} \approx 0.014 k_B T \lambda_T$ and the analytic estimation of equation (19) are in agreement within 30%). The heating rate due to scattering of microwave photons is practically negligible. An increase of the density at the trap centre by an order of magnitude is followed by the energy release of about $k_B T$ per molecule. If the molecules are immersed in a laser-cooled gas of $N_{\text{at}} \sim 10^6$ bosonic atoms used as a thermostat, the temperature
can be kept almost constant while slowly increasing the microwave intensity from zero to the value corresponding to $u_{\text{crit}}$.

If we further increase the EM intensity so that $u$ exceeds its critical value, the amount of energy released is too large to be taken by the finite bosonic reservoir without significant rise in the temperature. Then a realistic scenario, consistent with the parameter values above, would be an outburst of hot bosonic atoms, followed by fragmentation of the cloud of fermionic molecules into a non-degenerate hot ‘halo’ and a ‘core’ with hugely enhanced Fermi degeneracy.

4. Conclusions

We have found that the superconducting properties of fermionic systems with contact and (quasi-) static dipole–dipole interactions differ, but these differences become drastic only for high values of $U_0$, when the collisions become 3D, and tightly localized bound states appear for the combined potential of DDI and lateral confinement. Since DDI coexists with short-range interactions, they can enhance or cancel each other, depending on their relative sign. Our estimations show that for the fermionic isotope of chromium the contact interactions dominate over the magnetic DDI. This makes the detection of the effects specific for long-range interactions even more difficult than in the idealized case presented in figures 1–3. One may also consider trapped polar molecules, interacting via their electric dipole moments induced by an external electric field. However, in this case, van der Waals forces are much stronger than for atoms, due to very large polarizability of polar molecules, and a complicated interplay of van der Waals and induced dipole–dipole interactions is again expected.

We suggest that optimal candidates for the study of these properties are cold fermionic molecules driven by microwave radiation near one of their rotational resonances. The confinement of such molecules within the near-zone region of the microwave radiation corresponds to molecular densities $\sim 10^{12}$ cm$^{-3}$. An isotropically illuminated near-zone spherical sample of cold fermionic molecules is shown here to bear analogy to certain self-gravitating objects that have been extensively investigated in astrophysics [11]–[13]. The coarse-grained distribution function for a classical gas of bodies that interact via long-range forces and do not experience close collisions (e.g. a cluster of stars interacting via gravity) was postulated to be of the Fermi–Dirac type, since, in the collisionless regime, the coarse-grained distribution function cannot increase and thus has an upper limit determined by the initial conditions [11]. The equilibrium states of collisionless stellar clusters have been studied, and evidence has been found, among other effects, for isothermal gravitational collapse that is accompanied by large release of heat into the envelope (‘halo’) and is stopped by the onset of Fermi degeneracy in a dense ‘core’ formed at the centre of the cluster [12]. This stable core is similar to a white dwarf—a star whose self-gravity is balanced by the pressure of non-relativistic DFG of electrons [13].

Our gravity-like interaction has an effective cut-off at separations exceeding the EM wavelength. This may give rise to equilibrium states other than those of genuine self-gravitating systems, unless the EM wavelength is much larger than the cloud size. Thus we have to localize the gas in a rather small volume. Instead of the boundary conditions at a large (arbitrary) radius, used in [12] to preclude thermal runaway of particles to infinity, we have a parabolic trapping potential. Notwithstanding this difference, our analysis is a case study of one of the possible phase transitions in self-gravitating clusters, whose comprehensive theory is given in [12].
The conceptual significance of the mechanism proposed in section 3 is that it would constitute an electromagnetically induced isothermal phase transition capable of converting a classical (Maxwell–Boltzmann) gas into a quantum (Fermi-degenerate) one, in contrast to the familiar cooling processes or isentropic compression of cold gases. The envisaged dramatic effects are expected to provide an impetus to the experimental realization of cold fermionic molecules, a line of research that has been overshadowed so far by efforts aimed at producing molecular BEC.

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References

[1] DeMarco B and Jin D S 1999 Science 285 1703
[2] O’Hara K M et al 2002 Science 298 2179
Granade S R, Gehm M E, O’Hara K M and Thomas J E 2002 Phys. Rev. Lett. 88 120405
[3] Kohl M et al 2005 Phys. Rev. Lett. 94 080403
[4] Moritz H et al 2005 Phys. Rev. Lett. 94 210401
[5] Gaudin M 1967 Phys. Lett. A 24 55
Yang C N 1967 Phys. Rev. Lett. 19 1312
[6] Bethe H A 1931 Z. Phys. 71 205
[7] Giamarchi T 2004 Quantum Physics in One Dimension (Oxford: Clarendon)
[8] Ovchinnikov A A 1970 Sov. Phys.—JETP 30 1160
Krivnov V Y and Ovchinnikov A A 1975 Sov. Phys.—JETP 40 781
[9] Lieb E H and Liniger W 1963 Phys. Rev. 130 1605
Lieb E H 1963 Phys. Rev. 130 1616
[10] Truscott A G et al 2001 Science 291 2570
[11] Lynden-Bell D 1967 Mon. Not. R. Astron. Soc. 136 101
[12] Chavanis P-H and Sommeria J 1998 Mon. Not. R. Astron. Soc. 296 569
Chavanis P-H 2002 Phys. Rev. E 65 056123
[13] Shapiro S L and Teukolsky S A 1983 Black Holes, White Dwarfs and Neutron Stars (New York: Wiley)
[14] O’Dell D, Giovanazzi S, Kurizki G and Akulin V M 2000 Phys. Rev. Lett. 84 5687
Giovanazzi S, O’Dell D and Kurizki G 2001 Phys. Rev. A 63 031603
Giovanazzi S, Kurizki G, Mazets I E and Stringari S 2001 Europhys. Lett. 56 1
Kurizki G, Giovanazzi S, O’Dell D and Artemiev A I 2002 Dynamics and Thermodynamics of Systems with Long-Range Interactions (Springer Lecture Notes in Physics vol 602) (Berlin: Springer) p 369
[15] Olshanii M 1998 Phys. Rev. Lett. 81 938
[16] Artemiev A I, Mazets I E, Kurizki G and O’Dell D 2004 Int. J. Mod. Phys. B 18 2027
[17] Abramowitz M and Stegun I A (ed) 1964 Handbook of Mathematical Functions (Washington, DC: National Bureau of Standards)
[18] Thirunamachandran T 1980 Mol. Phys. 40 393
Craig D P and Thirunamachandran T 1984 Molecular Quantum Electrodynamics (London: Academic)
[19] Landau L D and Lifshitz E M 1999 Statistical Physics. Part I (Oxford: Butterworth Heinemann)
[20] Chavanis P H 2002 Astron. Astrophys. 381 340

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[21] Chandrasekhar S 1959 *An Introduction to the Study of Stellar Structure* (New York: Dover)
[22] Wallis H 1995 *Phys. Rep.* **255** 204
[23] Schreck F *et al* 2001 *Phys. Rev. Lett.* **87** 080403
    Hadzibabic Z *et al* 2002 *Phys. Rev. Lett.* **88** 160401
    Roati G, Riboli F, Modugno G and Inguscio M 2002 *Phys. Rev. Lett.* **89** 150403
    Modugno G *et al* 2002 *Science* **297** 2240
[24] Bergmann K, Theuer H and Shore B W 1998 *Rev. Mod. Phys.* **70** 1003
    Vitanov N V, Fleischhauer M, Shore B W and Bergmann K 2001 *Adv. At. Mol. Opt. Phys.* **46** 55
[25] Zwierlein M W *et al* *Phys. Rev. Lett.* **91** 250401