LAGRANGE ANCHOR FOR
BARGMANN-WIGNER EQUATIONS

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ABSTRACT. A Poincaré invariant Lagrange anchor is found for the non-Lagrangian relativistic wave equations of Bargmann and Wigner describing free massless fields of spin $s > 1/2$ in four-dimensional Minkowski space. By making use of this Lagrange anchor, we assign a symmetry to each conservation law.

INTRODUCTION

The notions of symmetry and conservation law are of paramount importance for classical and quantum field theory. For Lagrangian theories both the notions are tightly connected to each other due to Noether’s first theorem. Beyond the scope of Lagrangian dynamics, this connection has remained unclear, though many particular results and generalizations are known (see [1] for a review). In our recent works [2, 3] a general method has been proposed for connecting symmetries and conservation laws in not necessarily Lagrangian field theories. The key ingredient of the method is the notion of a Lagrange anchor introduced earlier [4] in the context of quantization of (non-)Lagrangian dynamics. Geometrically, the Lagrange anchor defines a map from the vector bundle dual to the bundle of equations of motion to the tangent bundle of the configuration space of fields such that certain compatibility conditions are satisfied. The existence of the Lagrange anchor is much less restrictive for the equations than the requirement to be Lagrangian or admit an equivalent Lagrangian reformulation.

The theory of massless higher-spin fields is an area of particular interest for application of the Lagrange anchor construction. Here one can keep in mind Vasiliev’s higher-spin equations in the form of unfolded representation [5, 6, 7]. The unfolded field equations are not Lagrangian even at the free level and their quantization by the conventional methods is impossible. Finding a Lagrange anchor for these equations can be considered as an important step towards the consistent quantum theory of higher-spin fields. In our recent paper [8], a general construction for the Lagrange anchor was proposed for unfolded equations that admit an equivalent Lagrangian formulation.

In this paper, the general concept of Lagrange anchor is exemplified by the Bargmann-Wigner equations for free massless fields of spin $s \geq 1/2$ in the four-dimensional Minkowski space [9]. The choice
of the example is not accidental. First of all, it has long been known that the model admits infinite sets of symmetries and conservation laws. These have been a subject of intensive studies by many authors during decades, see e.g. [10, 11, 12, 13, 14, 15, 16, 17] and references therein. However, a complete classification has been obtained only recently, first for the conservation laws [18] and then for the symmetries [19]. As the field equations are non-Lagrangian for \( s > 1/2 \), there is no immediate Noether’s correspondence between symmetries and conservation laws. The rich structure of symmetries and conservation laws in the absence of a Lagrangian formulation makes this theory an appropriate area for testing the concept of Lagrange anchor.

1. THE LAGRANGE ANCHOR IN FIELD THEORY

In this section we give a brief exposition of the Lagrange anchor construction. A more detailed discussion can be found in [4].

Consider a collection of fields \( \phi^i(x) \) whose dynamics are governed by a system of PDEs

\[
T_a(x, \phi^i(x), \partial_\mu \phi^i(x), \ldots) = 0.
\]

Here \( x \)'s denote local coordinates on a space-time manifold \( X \) and indices \( i \) and \( a \) numerate the components of fields and field equations. As we do not assume the field equations (1) to come from the least action principle, the indices \( i \) and \( a \) may run through different sets. In what follows we accept Einstein’s convention on summation by repeated indices.

Instead of working with the set of PDEs (1) it is convenient for us to introduce a single linear functional

\[
T[\xi] = \int_X dx \xi^a T_a
\]

of the test functions \( \xi^a = \xi^a(x) \) with compact support. Then \( \phi^i(x) \) is a solution to (1) iff \( T[\xi] = 0 \) for all \( \xi \)'s.

Consider now the linear space of the variational vector fields of the form

\[
V[\xi] = \int_X dx V^i(\xi) \frac{\delta}{\delta \phi^i(x)};
\]

where \( V^i(\xi) = \hat{V}^i_a \xi^a(x) \) and

\[
\hat{V}^i_a = \sum_{q=0}^p V^i_{a, \mu_1, \ldots, \mu_q}(x, \partial_{\mu_1} \phi(x), \ldots) \partial_{\mu_1} \ldots \partial_{\mu_q}
\]
is a matrix differential operators with coefficients being smooth functions of space-time coordinates, fields and their partial derivatives up to some finite order. Action of the variational vector fields on local functionals of \( \phi \)'s is defined by the usual rules of variational calculus.

The variational vector field (2) is called the Lagrange anchor if for any \( \xi_1 \) and \( \xi_2 \) there exist a test function \( \xi_3 \) such that the following condition is satisfied:

\[
V[\xi_1]T[\xi_2] - V[\xi_2]T[\xi_1] = T[\xi_3] .
\]

Clearly, if exists, the function \( \xi_3 \) is given by a bilinear differential operator acting on \( \xi_1 \) and \( \xi_2 \):

\[
\xi_3^a = C^a(\xi_1, \xi_2) .
\]

The coefficients of the operator \( C \) may depend on space-time coordinates \( x \), fields \( \phi \) and their derivatives.

The defining condition (3) means that the left hand side vanishes whenever \( \phi \)'s satisfy the field equations (1).

The Lagrangian equations \( \delta S/\delta \phi^i(x) = 0 \) admit an identical (or canonical) Lagrange anchor determined by the operator \( \hat{V}^i_j = \delta^i_j \). The defining condition (3) reduces to commutativity of variational derivatives

\[
\frac{\delta^2 S}{\delta \phi^i(x) \delta \phi^j(x')} = \frac{\delta^2 S}{\delta \phi^j(x') \delta \phi^i(x)} .
\]

If the Lagrange anchor is invertible in the class of differential operators, then the operator \( \hat{V}^{-1} \) have the sense of an integrating multiplier in the inverse problem of calculus of variations. In this case, one can define the local action functional \( S[\phi] \) such that \( \delta S/\delta \phi^i = \hat{V}_i^{-1}(T) \).

The classification of Lagrange anchors for the equations of evolutionary type was obtained in [20]. In particular, it was shown that all the stationary and strongly integrable (we explain the notion of integrability below) Lagrange anchors for determined systems of evolutionary equations are in one-to-one correspondence with the Poisson structures that are preserved by evolution. Let us illustrate this fact by the example of autonomous system of ODEs in normal form

\[
\dot{y}^i = F^i(y) .
\]

Consider the following ansatz for the Lagrange anchor:

\[
V[\xi] = \int dt V^{ij}(y(t))\xi_j(t) \frac{\delta}{\delta y^i(t)} .
\]

Here \( V^{ij}(y) \) is a contravariant tensor on the space of \( y \)'s. Verification of the defining condition (3) yields

\[
V^{ij} + V^{ji} = 0 , \quad F^k \partial_k V^{ij} + V^{ik} \partial_k F^{ji} - V^{jk} \partial_k F^{ij} = 0 ,
\]
that is, \( V^{ij}(y) \) must be an \( F \)-invariant bivector field on the phase space of the system. The corresponding bidifferential operator (4) is given by

\[
\xi^3_k = \partial_k V^{ij} \xi^1_i \xi^2_j.
\]

(In this particular case it does not involve derivatives of \( \xi_1 \) and \( \xi_2 \).)

One more important notion related to the Lagrange anchor is that of integrability. The Lagrange anchor is said to be strongly integrable if the following two conditions are satisfied:

\[
[V[\xi_1], V[\xi_2]] = V[C(\xi_1, \xi_2)],
\]

\[
C^a(\xi_1, C(\xi_2, \xi_3)) + V[\xi_1]C^a(\xi_2, \xi_3) + cycle(\xi_1, \xi_2, \xi_3) = 0.
\]

The first condition means that the variational vector fields \( V[\xi] \) form an integrable distribution in the configuration space of fields. If the Lagrange anchor is injective, that is, \( V[\xi] = 0 \) implies \( \xi = 0 \), then the second relation follows from the first one due to the Jacobi identity for the commutator of vector fields. Taken together relations (7) define what is known in mathematics as the Lie algebroid with anchor \( V \) and bracket \( C \), see e.g. [21].

The canonical Lagrange anchor is strongly integrable since \( C = 0 \) in this case. The integrability condition for (5) requires the bivector \( V = V^{ij}(y)\partial_i \wedge \partial_j \) to satisfy the Jacobi identity

\[
V^{in} \partial_n V^{jk} + cycle(i, j, k) = 0.
\]

It should be noted, that the strong integrability condition is not a part of the definition of Lagrange anchor. In many cases it can be considerably relaxed or even omitted. So, in general, the concept of Lagrange can not be substituted by that of Lie algebroid. A lot of examples of non-canonical Lagrange anchors for non-Lagrangian and non-Hamiltonian theories can be found in [2, 4, 8, 22, 23, 24, 25].

2. The generalization of Noether theorem for non-Lagrangian theories

A vector field \( j^\mu(x, \phi^i, \partial_\mu \phi^i, \ldots) \) on \( X \) is called a conserved current if its divergence is proportional to the equations of motion (1), i.e.,

\[
\partial_\mu j^\mu = \sum_{q=0}^{p} \Psi^{a_1 \ldots a_q}(x, \phi^i(x), \partial_\mu \phi^i(x), \ldots) \partial_{a_1} \ldots \partial_{a_q} T_a.
\]

The right hand side is defined by some differential operator \( \Psi \) called the characteristic of the conserved current \( j \). Two conserved currents \( j \) and \( j' \) are considered to be equivalent if \( j^\mu - j'^\mu = \partial_\nu i^{\nu \mu} \text{ (mod } T_a) \) for some bivector \( i^{\mu \nu} = -i^{\nu \mu} \). Similarly, two characteristics \( \Psi \) and \( \Psi' \) are said to be equivalent if they correspond to equivalent currents. These equivalences can be used to simplify the form of characteristics.
Namely, one can see that in each equivalence class of $j$ there is a representative with $\Psi$ being the zero order differential operator $\Psi^a$. For such a representative equation (8) can be written as

$$T[\Psi] = \int_X \partial_\mu j^\mu.$$  

It can be shown that there is a one-to-one correspondence between equivalence classes of conserved currents and characteristics [2].

Given a Lagrange anchor, one can assign to any characteristic $\Psi$ a variational vector field $V[\Psi]$. The main observation made in [2] was that $V[\Psi]$ generates a symmetry of the field equations (1):

$$\begin{align*}
\delta_\varepsilon \phi^i &= \varepsilon V^i(\Psi), \\
\delta_\varepsilon T[\xi] &= \varepsilon V[\Psi] T[\xi] = \varepsilon T[C(\Psi, \xi) - V[\xi] \Psi],
\end{align*}$$

with $\varepsilon$ being an infinitesimal constant parameter. These relations follow immediately from the definitions of the Lagrange anchor (3) and characteristic (9) upon substitution $\xi_1 = \Psi$.

Recall that any characteristic $\Psi$ of Lagrangian equations $\delta S/\delta \phi^i(x) = 0$ generates a symmetry $\delta_\varepsilon \phi^i = \varepsilon \Psi^i$ of the action functional and thus the equations of motion. This statement constitutes the content of Noether’s first theorem [11] on correspondence between symmetries and conservations laws. One the other hand, this correspondence is a simple consequence of a more general relation (10) if one takes the canonical Lagrange anchor $V^i(\xi) = \xi^i$ for Lagrangian equations. From this perspective, the assignment

$$\Psi \mapsto V[\Psi]$$

(11)

can be regarded as a generalization of the first Noether’s theorem to the case of non-Lagrangian PDEs. In general, the map (11) from the space of characteristics (= conservation laws) to the space of symmetries is neither surjective nor injective. The symmetries from the image of this map are called characteristic symmetries.

In the particular case of strongly integrable Lagrange anchor the space of characteristics can be endowed with the structure of Lie algebra. The corresponding Lie bracket reads

$$\{\Psi_1, \Psi_2\}^a = V[\Psi_1] \Psi_2^a - V[\Psi_2] \Psi_1^a + C^a(\Psi_1, \Psi_2).$$

Furthermore, the anchor map (11) defines a homomorphism from the Lie algebra of characteristics to the Lie algebra of symmetries

$$[V[\Psi_1], V[\Psi_2]] = V[\{\Psi_1, \Psi_2\}].$$

The bracket (12) generalizes the Dickey bracket of conserved currents [26] known in Lagrangian dynamics.
3. THE LAGRANGE ANCHOR AND CHARACTERISTIC SYMMETRIES FOR THE BARGMANN-WIGNER EQUATIONS

In this section we illustrate the general concept of Lagrange anchor by the example of Bargmann-Wigner’s equations. These equations describe free massless fields of spin $s > 0$ on $d = 4$ Minkowski space. The equations read

$$T_{\alpha_1 \cdots \alpha_{2s-1}}^\dot{\alpha} := \partial^\dot{\alpha} \varphi_{\alpha_1 \cdots \alpha_{2s-1}} = 0,$$

where $\varphi_{\alpha_1 \cdots \alpha_{2s}}(x)$ is a symmetric, complex-valued spin-tensor field on $\mathbb{R}^{3,1}$. We use the standard notation of the two-component spinor formalism [9], e.g.

$$\partial^\dot{\alpha} \dot{\alpha} = (\sigma^\mu)_{\dot{\alpha} \dot{\beta}} \partial/\partial x^\mu, \quad \mu = 0, 1, 2, 3, \alpha, \dot{\alpha} = 1, 2,$$

and the spinor indices are rised/lowered with $\varepsilon_{\alpha \beta}, \varepsilon_{\dot{\alpha} \dot{\beta}}$ and the inverse $\varepsilon^{\alpha \beta}, \varepsilon^{\dot{\alpha} \dot{\beta}}$.

To make contact with the general definitions of the previous section let us mention that the indices of equations and fields are given by the multi-indices $a = (\dot{\alpha}, \alpha_1, \dots, \alpha_{2s-1})$ and $i = (\alpha_1, \dots, \alpha_{2s})$. It is well known that the Bargmann-Wigner equations are non-Lagrangian unless $s = 1/2$.

In [25], it was shown that the Bargmann-Wigner equations admit the following Pioncaré-invariant and strongly integrable Lagrange anchor:

$$V(\xi)_{\alpha_1 \cdots \alpha_{2s}} = i^{2s} (\alpha_{2s} \cdots \alpha_2) \partial_{\alpha_{2s} \cdots \alpha_2} \bar{\varphi}_{\alpha_1 \cdots \alpha_{2s}}.$$

The round brackets mean symmetrization. This Lagrange anchor is unique (up to equivalence) if the requirements of (i) field-independence, (ii) Pioncaré-invariance and (iii) locality are imposed. Being independent of fields, the Lagrange anchor is integrable with $C = 0$.

Let $\Psi$ be a characteristic of a conserved current $j$ such that

$$\partial^\dot{\alpha} j_{\dot{\alpha}} = \Psi^\dot{\alpha}_{\alpha_1 \cdots \alpha_{2s-1}} T_{\alpha_1 \cdots \alpha_{2s-1}} + c.c.\ .$$

Then the Lagrange anchor (13) takes this characteristic to the symmetry

$$(14) \quad \delta_\varepsilon \varphi_{\alpha_1 \cdots \alpha_{2s}} = \varepsilon V(\Psi)_{\alpha_1 \cdots \alpha_{2s}},$$

where $V(\Psi)$ is defined by (13).

Applying (14) to the characteristics obtained and classified in [18], we get all the characteristic symmetries. Since the Lagrange anchor is strongly integrable, characteristic symmetries form an infinite dimensional Lie subalgebra in the Lie algebra of all symmetries. This subalgebra was previously unknown. For low spins ($s = 1/2, 1$) the Lie algebra of characteristic symmetries contains a finite dimensional subalgebra which is isomorphic to the Lie algebra of conformal group. The elements of
this subalgebra correspond to conserved currents that are expressible in terms of the energy-momentum tensor.

**CONCLUSION**

We have presented a Poincaré invariant Lagrange anchor for the Bargmann-Wigner equations. By making use this Lagrange anchor we have established a systematic connection between the symmetries and conservation laws of the equations. The Lagrange anchor, being independent of fields, is strongly integrable. As a consequence the symmetries associated with the conservation laws (characteristic symmetries) form an infinite-dimensional subalgebra in the full Lie algebra of symmetries. The physical meaning of this subalgebra remains unclear for us at the moment.

The Lagrange anchor (13) may be used for quantization of the Bargmann-Wigner equations. At the free level the corresponding generalized Schwinger-Dyson equations and probability amplitude was found in [25]. It can also be a good starting point for constructing the Lagrange anchor for Vasiliev’s equations and development of a quantum theory of higher-spin interactions.

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