Implicit Real Vector Automata

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Goal: Representing symbolically polyhedra in $\mathbb{R}^n$, i.e., sets of the form
\[
\left\{ \vec{x} \in \mathbb{R}^n \mid B_i \left( \vec{a}_i.\vec{x} \left\{ \leq \right\} b_i \right) \right\},
\]
with $\forall i : \vec{a}_i \in \mathbb{Z}^n$, $b_i \in \mathbb{Z}$.

Wish list:

- Concise data structure.
- Efficient manipulation algorithms: $\cap$, $\cup$, $\setminus$, $\times$, ... 
- Efficient decision procedures: $\subseteq$, $=$, empty?, ... 
- Possible conversions to and from other representations.
Closed Convex Polyhedra

In the particular case of closed convex polyhedra

\[ \bigwedge_i \vec{a}_i \cdot \vec{x} \leq b_i, \]

suitable representations can be:

- a set of bounding constraints,
- sets of vertices and extremal rays, or
- a combination of both.

![Diagram showing bounding constraints and vertices with extremal rays]
Representing Non-Convex Polyhedra

Classical solutions:

- Represent a polyhedron by a formula.
- Decompose a polyhedron into an explicit union of convex polyhedra.

Main drawback: Set comparison operations become difficult.

Better approach: Look for a canonical representation of polyhedra.
Real Vector Automata

Principles:

- Vectors in $\mathbb{R}^n$ are encoded as infinite words over the alphabet \{0, 1, \ldots, r - 1, \star\}, where $r > 1$ is a base, and $\star$ a separator symbol.

  Example ($r = 2$): $Enc\left(\frac{1}{3}\right) = 0^+ \star (01)^\omega$.

- A **Real Vector Automaton (RVA)** representing a set $S \subseteq \mathbb{R}^n$ is an infinite-word automaton accepting the encodings of the elements of $S$.

Key properties:

- All the sets definable in $\langle \mathbb{R}, \mathbb{Z}, <, + \rangle$ can be represented by weak deterministic RVA.

- These automata are easily manipulated algorithmically, and can be minimized into a canonical form.

- But the representation of linear constraints is not efficient enough . . .
Definition: Let $D$ be a convex subset of $\mathbb{R}^n$. A set $S \subseteq D$ is conical in $D$ if there exists $\vec{v} \in D$ such that

$$\forall \vec{x} \in D - \vec{v}, \lambda \in ]0, 1]: \vec{x} \in S - \vec{v} \iff \lambda \vec{x} \in S - \vec{v}.$$ 

Property of weak deterministic RVA representing polyhedra: For each state $q$ that belongs to a non-trivial strongly connected component, the language $0^* L(q)$ encodes a conical set.
**Application to Polyhedra**

**Theorem:** Let $P$ be a polyhedron in $\mathbb{R}^n$. For every $\vec{x} \in \mathbb{R}^n$ and sufficiently small convex neighborhood $D$ of $\vec{x}$, $P$ is conical in $D$.

![Graphical representation of the polyhedron and its properties]

**Application:**

- The conical structure of arbitrarily small convex neighborhoods of points partitions $\mathbb{R}^n$ into a finite equivalence relation $\sim$.
- The equivalence classes of $\sim$ correspond to the characteristic elements of $P$ (vertices, edges, faces, . . .).
- An incidence relation $\prec$ between those elements can be defined.
The Representation of Polyhedra by RVA

Properties:

- Non-trivial strongly connected components correspond to characteristic elements.

- The acyclic structures between components represent the incidence relation.

Illustration:
Improving the Conciseness of RVA

Principles:

• A characteristic element of a polyhedron is characterised by:
  – an affine space \( \{\vec{v}_0 + \lambda_1 \vec{v}_1 + \ldots + \lambda_p \vec{v}_p \mid \forall i : \lambda_i \in \mathbb{R}\} \), with \( \vec{v}_0, \vec{v}_1, \ldots, \vec{v}_p \in \mathbb{Q}^n \).
  (Note: By a technical argument, one can assume w.l.o.g. \( \vec{v}_0 = \vec{0} \).)
  – its successors by the incidence relation, and
  – its polarity (in/out).

• Real Vector Automata can be represented implicitly by:
  – Replacing their non-trivial strongly connected components by canonical algebraic descriptions of vector spaces ("implicit states"),
  – Keeping an acyclic transition relation for representing the incidence relation, and
  – Encoding the polarity of elements by a Boolean acceptance condition.
Problem: In a RVA, the structure of transitions leaving a strongly connected component can be as large as the component itself.

Solution: At each component, perform a variable change operation: The coordinate system \((\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_m)\) becomes \((\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_p, \vec{z}_1, \vec{z}_2, \ldots, \vec{z}_{m-p})\), where

- The dimension of the corresponding characteristic element is \(p\),
- \(\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_p\) move inside that element, and
- \(\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_{m-p}\) explore the directions along which one can leave the element.

Remark: The vectors \(\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_{m-p}\) can be chosen canonically.
Encoding Directions

Problem: Encoding the directions in which one can leave an implicit state, in a canonical way.

Solution:

1. Apply a normalization function to the direction vectors.

   We use

   \[ \text{norm}(\vec{v}) = \frac{1}{2d} \vec{v}, \]

   with \( \vec{v} = (v_1, v_2, \ldots, v_q) \) and \( d = \max_i |v_i| \).
2. Encode a normalized vector by:

• A prefix identifying the face of the hypercube to which it belongs.

  Example \((q = 3)\):

  \[
  \begin{align*}
  \{\frac{1}{2}\} \times [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}] & \quad \rightarrow \quad +1 \\
  [-\frac{1}{2}, \frac{1}{2}] \times \{\frac{1}{2}\} \times [-\frac{1}{2}, \frac{1}{2}] & \quad \rightarrow \quad +2 \\
  [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}] \times \{\frac{1}{2}\} & \quad \rightarrow \quad +3 \\
  \{\frac{1}{2}\} \times [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}] & \quad \rightarrow \quad -1 \\
  [-\frac{1}{2}, \frac{1}{2}] \times \{-\frac{1}{2}\} \times [-\frac{1}{2}, \frac{1}{2}] & \quad \rightarrow \quad -2 \\
  [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}] \times \{-\frac{1}{2}\} & \quad \rightarrow \quad -3
  \end{align*}
  \]

• A suffix encoding the vector inside this face (i.e., in a restricted domain of the form \([-\frac{1}{2}, \frac{1}{2}]^{q-1}\)).
Examples of Implicit Real Vector Automaton

\[ x - 2z = 0 \]
\[ y - 3z = 0 \]
Overview of Main Results

• The representation of a polyhedron can easily be minimized into a canonical form.

• Membership of a given vector can be decided by following a single path in a IRVA.

• An algorithm for computing the product of two IRVA is available.

• The following operations are currently being investigated:
  – Projection/determinization,
  – Import/export to and from other structures,
  – Visualization of sets,
  – Efficient ray tracing computations,
  – Efficient handling of large dimensions,
  – . . .