703: $\mathcal{O}(N_f \alpha^2)$ Radiative Corrections in Low-Energy Electroweak Processes

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Abstract. Of the three best-measured electroweak observables, $\alpha$, $G_\mu$, and $M_Z$, the first two are extracted from low-energy processes. Both $G_\mu$ and $M_Z$ are now known to an accuracy of about 2 parts in $10^5$ and there is a proposal to improve the measurement of the muon lifetime by a factor of 10 in an experiment at Brookhaven. Yet calculations of electroweak radiative corrections currently do no better than a few parts in $10^3$–$10^4$ and therefore cannot exploit to available experimental precision. We report on the calculation of the $\mathcal{O}(N_f \alpha^2)$ corrections to Thomson scattering and the muon lifetime from which $\alpha$ and $G_\mu$, respectively, are obtained. The $\mathcal{O}(N_f \alpha^2)$ corrections are expected to be a dominant gauge-invariant subset of 2-loop corrections.

We have studied the $\mathcal{O}(N_f \alpha^2)$ corrections to Thomson scattering and muon decay [1, 2], and present some of our results below. These are corrections that come from 2-loop diagrams containing a massless fermion loop where $N_f$ is the number of fermions. Because, $N_f$ is quite large, the $\mathcal{O}(N_f \alpha^2)$ corrections are expected to be a dominant subset of 2-loop graphs contributing at about the $1.5 \times 10^{-4}$ level. Moreover, since $N_f$ provides a unique tag, the complete set of $\mathcal{O}(N_f \alpha^2)$ corrections contributing to a particular physical process will form a gauge-invariant set.

The vast majority of $\mathcal{O}(N_f \alpha^2)$ diagrams, including box diagrams, can be reduced to expressions involving a master integral of the form

\[
\int \frac{d^n p}{i \pi^2} \frac{1}{[p^2][p^2 + M^2]^k} \int \frac{d^n q}{i \pi^2} \frac{1}{[q^2][q^2 + (q + p)^2]^m} = \frac{\pi^{n-4}}{(M^2)^{k-j+l+m-n}} \Gamma \left( l + m - \frac{n}{2} \right) \Gamma \left( \frac{n}{2} - l \right) \Gamma \left( \frac{n}{2} - m \right) \times \frac{\Gamma(n-j-l-m)\Gamma(k+j+l+m-n)}{\Gamma \left( \frac{n}{2} \right) \Gamma(k)\Gamma(l)\Gamma(m)\Gamma(n-l-m)}
\]

using techniques described in [1].

There are a few diagrams in which the internal fermion mass cannot be set to zero initially but they can be treated via a mass expansion along the lines given in ref.[3].

Dimensional regularization is employed throughout and we find that simpler results are often obtained by carrying the full analytic dependence on $n$, the number of space-time dimensions, rather than expanding up to finite
terms in $\epsilon = 2 - n/2$. A case in point is the $Z$-$\gamma$ mixing at zero momentum transfer that contributes to Thomson scattering and for which the diagrams are shown in Fig.1.

![Diagrams](image1)

**Fig. 1.** $\mathcal{O}(N_f \alpha^2)$ corrections to the $Z$-$\gamma$ mixing, $\Pi^{(2)}_{Z\gamma}(0)$.

The result is

$$
\Pi^{(2)}_{Z\gamma}(0) = \left(\frac{g^2}{16\pi^2}\right)^2 8 s_{\theta_W} c_{\theta_W} M_Z^2 \frac{(\pi M_W^2)^{n-4}}{n} \Gamma(4 - n) \Gamma\left(2 - \frac{n}{2}\right) \Gamma\left(\frac{n}{2}\right)
$$

where $s_{\theta_W}$ and $c_{\theta_W}$ are $\sin \theta_W$ and $\cos \theta_W$ respectively and we assume one generation of massless fermions.

The diagrams contributing at $\mathcal{O}(N_f \alpha^2)$ to the photon vacuum polarization are shown in Fig.2.

![Diagrams](image2)

**Fig. 2.** $\mathcal{O}(N_f \alpha^2)$ corrections to the photon vacuum polarization, $\Pi^{(2)}_{\gamma\gamma}(0)$.

The result of calculation of these diagrams yields
\[ \Pi^{(2)}_{\gamma \gamma}(0) = \left( \frac{g^2s_\theta}{16\pi^2} \right)^2 \frac{8(n+2)}{3n}(\pi M_W^2)^{n-4} \Gamma(4-n) \Gamma\left(2 - \frac{n}{2}\right) \Gamma\left(\frac{n}{2} - 1\right) \]

\[ + \frac{g^2s_\theta}{16\pi^2} \frac{4}{27c_\theta^2}(44s_\theta^4 - 27s_\theta^2 + 9) \]

\[ \times \frac{(n-6)}{n}(\pi M_Z^2)^{n-4} \Gamma(5-n) \Gamma\left(2 - \frac{n}{2}\right) \Gamma\left(\frac{n}{2} - 1\right) \]

\[ - \frac{g^2s_\theta}{16\pi^2} \frac{4(n-2)}{3n}(\pi M_W^2)^{n-4} \n\Gamma\left(2 - \frac{n}{2}\right) \Gamma\left(\frac{n}{2} - 1\right) \]

\[ \times \sum_f Q_f^2 (m_f^2)^{n-2} \]

\[ - \frac{g^2s_\theta}{16\pi^2} \frac{8}{3}(\pi M_W^2)^{n-4} \n\Gamma\left(2 - \frac{n}{2}\right) \sum_f Q_f t_{3f}(m_f^2)^{n-2} \]

\[ - \frac{g^2s_\theta}{16\pi^2} \frac{16(n-2)}{3n}(\pi M_Z^2)^{n-2} \Gamma\left(3 - \frac{n}{2}\right) \Gamma\left(2 - \frac{n}{2}\right) \]

\[ \times \sum_f Q_f^2 \left( \frac{\beta_{2f} + \beta_{3f}}{2} \right) (m_f^2)^{n-2} \]

\[ + \frac{g^2s_\theta}{16\pi^2} \frac{16n}{3(n-2)}(\pi M_Z^2)^{n-2} \Gamma\left(3 - \frac{n}{2}\right) \Gamma\left(2 - \frac{n}{2}\right) \]

\[ \times \sum_f Q_f^2 \beta_{LF} \beta_{RF}(m_f^2)^{n-2} \]

\[ + \frac{g^2s_\theta}{16\pi^2} \frac{4s_\theta^2(5n^2 - 33n + 34)}{3n(n-5)}(\pi M_Z^2)^{n-4} \Gamma\left(3 - \frac{n}{2}\right) \Gamma\left(2 - \frac{n}{2}\right) \]

\[ \times \sum_f Q_f^4 (m_f^2)^{n-4} \]

(1)

where \( \beta_{LF} \) and \( \beta_{RF} \) are the left- and right-handed couplings of the \( Z^0 \) to a fermion, of charge, \( Q_f \), weak isospin, \( t_{3f} \), and mass, \( m_f \), given by

\[ \beta_{LF} = \frac{t_{3f} - s_\theta^2 Q_f}{c_\theta}, \quad \beta_{RF} = -\frac{s_\theta^2 Q_f}{c_\theta}. \]

Contributions that are suppressed by factors \( m_f^2/M_W^2 \) relative to the leading terms have been dropped. The first term on the right hand side of eq.(1) comes from diagrams Fig.2a–e that contain an internal \( W \) boson, the second comes from diagrams Fig.2f&g containing an internal \( Z^0 \). The last term in eq.(1) comes from diagrams Fig.2h&i and is pure QED in nature. Contributions for which the fermion mass can be safely set to zero without affecting the final result were obtained using the methods described in ref.[1]. The terms in which the fermion mass appears are obtained using the asymptotic expansion of ref.[3]. Note that setting \( m_f = 0 \) in Fig.2c–g does not immediately cause
any obvious problems in the computation because the diagram still contains one non-vanishing scale. A certain amount of care is thus required to identify situations in which the fermion mass cannot be discarded.

Although the calculation of the photon vacuum polarization is somewhat lengthy one has the non-trivial check that longitudinal part vanishes. Dependence on the fermion mass, \( m_f \), is eliminated in all diagrams, with the exception of Fig.2h&i, by fermion mass counterterms. The diagrams of Fig.2h&i obviously become non-perturbative when the internal fermions are light quarks. In that case the hadronic contribution is treated in the usual manner by writing

\[
\Pi^{(f)}(0) = \text{Re} \Pi^{(f)}(\bar{q}^2) - [\text{Re} \Pi^{(f)}(\bar{q}^2) - \Pi^{(f)}(0)]
\]

with \( \bar{q}^2 \) being chosen to be sufficiently large that perturbative QCD can be used. The term in square brackets on the rhs can be obtained in the usual way using dispersion relations and we have calculated for \( |\bar{q}^2| \gg m_f^2 \)

\[
\Pi^{(2QED)}(\bar{q}^2) = -\sum_f \left( \frac{g^2}{16\pi^2} \right)^2 Q_f^4 s^4_\phi(\pi \bar{q}^2)^{n-4} \times \left\{ 8(n^2 - 7n + 16) \frac{\Gamma \left( 2 - \frac{n}{2} \right)^2 \Gamma \left( \frac{n}{2} \right)^3 \Gamma \left( \frac{n}{2} - 2 \right)}{\Gamma(n)\Gamma(n-1)} \right. \\
+ \left. 24 \frac{(n^2 - 4n + 8)}{(n-1)(n-4)} \frac{\Gamma(4-n)\Gamma \left( \frac{n}{2} \right)^2 \Gamma \left( \frac{n}{2} - 2 \right)}{\Gamma \left( \frac{3n}{2} - 2 \right)} \right\}. \quad (2)
\]

Despite appearances, the expression on the right hand side of eq.(2) has only a simple pole with a constant coefficient at \( n = 4 \) that can be canceled by local counterterms. The leading logarithmic expressions can be found in ref.[4, section 8-4-4] where the authors invite the “foolhardy reader” to check that the finite parts are transverse. Here we have gone further and demonstrated this property in the exact result. Eq.(2) could also be obtained by applying analytic continuation relations for the hypergeometric functions, \( _2F_1 \) and \( _3F_2 \), to formulas given by Broadhurst et al. [5].

References

[1] R. Akhoury, P. Malde and R. G. Stuart, hep-ph/9707520.
[2] P. Malde and R. G. Stuart, in preparation.
[3] A. I. Davydychev and J. B. Tausk, Nucl. Phys. B 397 (1993) 123.
[4] C. Itzykson and J.-B. Zuber, Quantum Field Theory, McGraw-Hill (1980).
[5] D. J. Broadhurst, J. Fleischer and O. V. Tarasov, Z. Phys. C 60 (1993) 287.