On Resolving Hop Domination in Graphs

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Abstract. A set $S$ of vertices in a connected graph $G$ is a resolving hop dominating set of $G$ if $S$ is a resolving set in $G$ and for every vertex $v \in V(G) \setminus S$ there exists $u \in S$ such that $d_G(u, v) = 2$. The smallest cardinality of such a set $S$ is called the resolving hop domination number of $G$. This paper presents the characterizations of the resolving hop dominating sets in the join, corona and lexicographic product of two graphs and determines the exact values of their corresponding resolving hop domination number.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Resolving hop dominating set, resolving hop domination number, join, corona, lexicographic product

1. Introduction

Domination in graphs was first introduced by C. Berge in 1958 [3]. There are now many studies involving domination and its variations. Natarajan and Ayyaswamy [9] introduced and studied the concept of hop domination in graphs. Hop domination in graphs were also studied in [6, 10, 11].

Slater [12] introduced and studied the concept of resolving set. Resolving sets and resolving dominating sets were studied in [1, 2, 4, 7, 8].

This paper combines the idea of resolving and hop domination sets by introducing the concept of resolving hop domination in graphs. Resolving hop dominating sets in graphs can have real world applications. One possible application is in the minimization problem with specific conditions. For example, a company that makes electric cars with a smart navigation feature, wants to build the least number of charging stations in a given city, such that any car with a low charge, from any area, can reach a charging station before

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DOI: https://doi.org/10.29020/nybg.ejpam.v14i3.4055

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running out of either or both its remaining charge and auxiliary power. However, fully charging a car takes time, which at some point can overwhelm a station’s capacity. To reduce the chance of this from happening, the company may require, as much as possible, that no such two cars from different areas arrive at the same station at about the same time. Assuming that both the remaining low charge and auxiliary power can each cover the same travel distance \( d \), the graph-theoretic model for this scenario could be that vertices represent the areas, and adjacency of vertices represent a connected route of distance \( d \). Resolving hop domination in graphs can be used to determine the minimum number of charging stations and where to build them in such a manner that cars from different areas have relatively distinct distances from these stations.

In this study, we only consider graphs that are finite, simple, undirected and connected. Readers are referred to [5] for elementary Graph Theory concepts.

Let \( G = (V(G), E(G)) \) be a graph. \( N_G(v) = \{u \in V(G) : uv \in E(G)\} \) is a neighborhood of \( v \). An element \( u \in N_G(v) \) is called a neighbor of \( v \). \( N_G[v] = N_G(v) \cup \{v\} \) is a closed neighborhood of \( v \). The degree of \( v \), denoted by \( deg_G(v) \), is equal to \( |N_G(v)| \). For \( S \subseteq V(G) \), \( N_G(S) = \bigcup_{v \in S} N_G(v) \) and \( N_G[S] = \bigcup_{v \in S} N_G[v] \).

A connected graph \( G \) is said to be point determining if distinct vertices have distinct neighborhoods, that is, \( N_G(a) \neq N_G(b) \) whenever \( a, b \in V(G) \) and \( a \neq b \).

A connected graph \( G \) of order \( n \geq 3 \) is totally point determining if for any two distinct vertices \( u \) and \( v \) of \( G \), \( N_G(u) \neq N_G(v) \) and \( N_G[u] \neq N_G[v] \).

A vertex \( x \) of a graph \( G \) is said to resolve two vertices \( u \) and \( v \) of \( G \) if \( d_G(x, u) \neq d_G(x, v) \). For an ordered set \( W = \{x_1, \ldots, x_k\} \subseteq V(G) \) and a vertex \( v \) in \( G \), the \( k \)-vector 

\[
r_G(v/W) = (d_G(v, x_1), d_G(v, x_2), \ldots, d_G(v, x_k))
\]

is called the representation of \( v \) with respect to \( W \). The set \( W \) is a resolving set for \( G \) if and only if no two vertices of \( G \) have the same representation with respect to \( W \). The metric dimension of \( G \), denoted by \( dim(G) \), is the minimum cardinality over all resolving sets of \( G \). A resolving set of cardinality \( dim(G) \) is called basis.

A set \( S \subseteq V(G) \) of vertices of \( G \) is a dominating set if every \( u \in V(G) \setminus S \) is adjacent to at least one vertex \( v \in S \). The domination number of a graph \( G \), denoted by \( \gamma(G) \), is given by \( \gamma(G) = \min\{|S| : S \text{ is a dominating set of } G\} \).

A set \( S \subseteq V(G) \) is a hop dominating set of \( G \) if for every \( v \in V(G) \setminus S \), there exists \( u \in S \) such that \( d_G(u, v) = 2 \). The minimum cardinality of a hop dominating set of \( G \), denoted by \( \gamma_h(G) \), is called the hop domination number of \( G \). Any hop dominating set with cardinality equal to \( \gamma_h(G) \) is called a \( \gamma_h \)-set.

A vertex \( v \) in \( G \) is a hop neighbor of vertex \( u \) in \( G \) if \( d_G(u, v) = 2 \). The set \( N_G(u, 2) = \{v \in V(G) : d_G(v, u) = 2\} \) is called the open hop neighborhood of \( u \). The closed hop neighborhood of \( u \) in \( G \) is given by \( N_G[u, 2] = N_G(u, 2) \cup \{u\} \). The open hop neighborhood of \( X \subseteq V(G) \) is the set \( N_G(X, 2) = \bigcup_{u \in X} N_G(u, 2) \). The closed hop neighborhood of \( X \) in \( G \) is the set \( N_G[X, 2] = N_G(X, 2) \cup X \).
A set $S \subseteq V(G)$ is a locating set of $G$ if for every two distinct vertices $u$ and $v$ of $V(G) \setminus S$, $N_G(u) \cap S \neq N_G(v) \cap S$. The locating number of $G$, denoted by $\ln(G)$, is the smallest cardinality of a locating set of $G$. A locating set of $G$ of cardinality $\ln(G)$ is referred to as a ln-set of $G$. A set $S \subseteq V(G)$ is a strictly locating set of $G$ if it is a locating set of $G$ and $N_G(u) \cap S \neq S$ for all $u \in V(G) \setminus S$. The strictly locating number of $G$, denoted by $\sln(G)$, is the smallest cardinality of a strictly locating set of $G$. A strictly locating set of $G$ of cardinality $\sln(G)$ is referred to as a sln-set of $G$.

A set $S \subseteq V(G)$ is a resolving hop dominating set of $G$ if $S$ is both a resolving set and a hop dominating set. The minimum cardinality of a resolving hop dominating set of $G$, denoted by $\gamma_{Rh}(G)$, is called the resolving hop domination number of $G$. Any resolving hop dominating set with cardinality equal to $\gamma_{Rh}(G)$ is called a $\gamma_{Rh}$-set.

2. Preliminary Results

Remark 1. For any connected graph $G$ of order $n \geq 2$, $2 \leq \gamma_{Rh}(G) \leq n$. Moreover, $\gamma_{Rh}(P_2) = 2$ and $\gamma_{Rh}(K_n) = n$.

Proposition 1. For any connected graph $G$ of order $n \geq 2$. Then, $\gamma_{Rh}(G) = n$ if and only if $G = K_n$.

Proof: If $G = K_n$, then $\gamma_{Rh}(G) = n$. Suppose $\gamma_{Rh}(G) = n$ and $G \neq K_n$. Then there exists $x, y \in V(G)$ such that $d(x, y) = 2$. Let $S = V(G) \setminus \{y\}$. Then $S$ is a resolving hop dominating set of $G$. Hence, $\gamma_{Rh}(G) \leq |S| = n - 1$, a contradiction.

Remark 2. Let $G$ be a connected graph and $S \subseteq V(G)$. Then for any two distinct vertices $x, y \in V(G) \setminus S$ with $N_G(x, 2) \cap S \neq N_G(y, 2) \cap S$, we have $r_G(x/S) \neq r_G(y/S)$.

Remark 3. Every resolving hop dominating set of a connected graph $G$ is a resolving set of $G$. Thus, $\dim(G) \leq \gamma_{Rh}(G)$.

Proposition 2. Let $G$ be a connected graph of order 4. Then $\gamma_{Rh}(G) = 2$ if and only if $G = C_4$ or $G = P_4$.

Proof: If $G = C_4$ or $P_4$, then $\gamma_{Rh}(G) = 2$. Suppose that $\gamma_{Rh}(G) = 2$. Let $W = \{x_1, x_2\}$ be a $\gamma_{Rh}$-set of $G$. Since $W$ is a hop dominating set, possible representations of distinct vertices $u, v \in V(G) \setminus W$ are $(1,2)$, $(2,1)$ or $(2,2)$. Clearly $(2,2)$ cannot be a representation of vertex $u$ or $v$ since $G$ is of order 4. Thus we consider the following cases:

Case 1. $r_G(u/W) = (1, 2)$ and $r_G(v/W) = (2, 1)$

Case 2. $r_G(u/W) = (2, 1)$ and $r_G(v/W) = (1, 2)$

For case 1, $ux_1, vx_2 \in E(G)$ and either $x_1, x_2 \in E(G)$ or $uv \in E(G)$ or both $x_1, x_2, uv \in E(G)$. Hence, $G = [u, x_1, x_2, v]$ or $G = [x_1, u, v, x_2]$ or $G = [u, x_1, x_2, v, u]$. Thus, $G$ is either a path $P_4$ or a cycle $C_4$. Similarly, if case 2 holds, then $G = P_4$ or $G = C_4$. 

Proposition 3. Let $n$ be a positive number.

(i) For a path $P_n$ on $n$ vertices, $n > 1$
Theorem 1. [7, 8] Let \( G \) and \( H \) be non-trivial connected graphs. A set \( W \subseteq V(G+H) \) is a resolving set of \( G+H \) if and only if \( W = W_G \cup W_H \) where \( W_G \subseteq V(G) \) and \( W_H \subseteq V(H) \) are locating sets of \( G \) and \( H \), respectively, where \( W_G \) or \( W_H \) is a strictly locating set.

Theorem 2. Let \( G \) and \( H \) be non-trivial connected graphs. A set \( W \subseteq V(G+H) \) is a resolving hop dominating set of \( G+H \) if and only if \( W = W_G \cup W_H \) where \( W_G \) and \( W_H \) are strictly locating sets of \( G \) and \( H \), respectively.

Proof: Suppose that \( W \) is a resolving hop dominating set of \( G+H \). Then \( W \) is a resolving set of \( G+H \). By Theorem 1, \( W = W_G \cup W_H \) where \( W_G \subseteq V(G) \) and \( W_H \subseteq V(H) \) are locating sets of \( G \) and \( H \), respectively. Suppose \( W_G \) or \( W_H \) is not strictly locating set, say \( W_G \) is not strictly locating. Then there exists \( v \in V(G) \setminus W_G \) such that \( N_G(v) \cap W_G = W_G \). Hence, \( v \in V(G+H) \setminus W \) and \( d_{G+H}(v, w) = 1 \) for all \( w \in W \). This contradicts the assumption that \( W \) is a hop dominating set of \( G+H \). Similarly, if \( W_H \) is not strictly locating, then a contradiction follows. Hence, \( W_G \) and \( W_H \) are both strictly locating.

For the converse, suppose that \( W = W_G \cup W_H \) where \( W_G \subseteq V(G) \), \( W_H \subseteq V(H) \) and both \( W_G \) and \( W_H \) are strictly locating sets of \( G \) and \( H \), respectively. Since \( W_G \) and \( W_H \) are locating sets by Theorem 1, \( W \) is a resolving set of \( G+H \). Let \( v \in V(G+H) \setminus W \). If \( v \in V(G) \), then \( v \notin W_G \). Since \( W_G \) is strictly locating there exists \( u \in W_G \setminus N_G(v) \). Hence, \( d_{G+H}(v, u) = 2 \). Similarly, if \( v \in V(H) \), then \( v \notin W_H \) and there exists \( w \in W_G \setminus N_G(v) \). Thus, \( d_{G}(v, w) = 2 \). Therefore \( W \) is a hop dominating set of \( G+H \).

Accordingly, \( W \) is a resolving hop dominating set of \( G+H \).

The next result follows immediately from Theorem 2.

Corollary 1. Let \( G \) and \( H \) be non-trivial connected graphs. Then

\[
\gamma_{Rh}(G+H) = sln(G) + sln(H).
\]
4. On Resolving Hop Domination in the Corona of Graphs

The corona of two graphs $G$ and $H$, denoted by $G \circ H$, is the graph obtained by taking one copy of $G$ of order $n$ and $n$ copies of $H$, and then joining every vertex of the $i$th copy of $H$ to the $i$th vertex of $G$. For $v \in V(G)$, denote by $H^v$ the copy of $H$ whose vertices are attached one by one to the vertex $v$. Subsequently, denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle \{v\} \rangle + H^v, v \in V(G)$.

**Theorem 3.** [7, 8] Let $G$ and $H$ be non-trivial connected graphs. Then $W \subseteq V(G \circ H)$ is a resolving set of $G \circ H$ if and only if $W \cap V(H^v) \neq \emptyset$ for all $v \in V(G)$ and $W = A \cup B$, where $A \subseteq V(G)$, and $B = \cup \{B_v : v \in V(G)\}$ and $B_v$ is a locating set of $H^v$.

**Theorem 4.** Let $G$ and $H$ be non-trivial connected graphs. Then $W \subseteq V(G \circ H)$ is a resolving hop dominating set of $G \circ H$ if and only if $W \cap V(H^v) \neq \emptyset$ for every $v \in V(G)$ and $W = A \cup B \cup D$ where $A \subseteq V(G)$,

$$B = \cup \{B_v : v \in V(G) \cap N_G(A)\} \text{ and } B_v \text{ is a locating set of } H^v$$

$$D = \cup \{D_u : u \in V(G) \setminus N_G(A)\} \text{ and } D_u \text{ is a strictly locating set of } H^u.$$

**Proof:** Suppose $W$ is a resolving hop dominating set of $G \circ H$. Then by Theorem 3, $W \cap V(H^v) \neq \emptyset$ for each $v \in V(G)$. Let $A = W \cap V(G)$,

$$B_v = W \cap V(H^v) \text{ for each } v \in V(G) \cap N_G(A) \text{ and }$$

$$D_u = W \cap V(H^u) \text{ for each } u \in V(G) \setminus N_G(A).$$

Set $B = \cup B_v$ and $D = \cup D_u$. Then $W = A \cup B \cup D$ where $A \subseteq V(G)$. By Theorem 3, $B_v$ and $D_u$ are locating sets of $H^v$ and $H^u$, respectively. Let $x \in V(G \circ H) \setminus D_u$. Then $x \in V(G \circ H) \setminus W$. Since $W$ is a hop dominating set of $G \circ H$, there exists $y \in W$ such that $d_{G \circ H}(x, y) = 2$. Since $u \in V(G) \setminus N_G(A)$, $y \in V(H^u) \cap D_u$. Hence, $y \in D_u \setminus N_{H^u}(x)$. Thus, $N_{H^u}(x) \cap D_u \neq D_u$, showing that $D_u$ is strictly locating.

For the converse, suppose that $W \cap V(H^v) \neq \emptyset$ for every $v \in V(G)$ and $W = A \cup B \cup D$ where $A$, $B$ and $D$ satisfy the given conditions. Let $x \in V(G \circ H) \setminus W$ and let $v \in V(G)$ such that $x \in V(v + H^v)$. Suppose $x = v$. Then $v \notin A$. Let $u \in V(G) \setminus N_G(v)$. Since $W \cap V(H^u) \neq \emptyset$, there exists $y \in W \cap V(H^u)$ and $d_{G \circ H}(x, y) = 2$. Suppose $x \neq v$. If $v \in N_G(A)$, then there exists $z \in A \cap N_G(v)$. Hence, $z \in W$ and $d_{G \circ H}(x, z) = 2$. Suppose $v \notin N_G(A)$. Then $x \in V(H^v) \setminus D_v$. Since $D_v$ is strictly locating there exists $y \in D_v \setminus N_{H^v}(x)$. Thus, $y \in W$ and $d_{G \circ H}(x, y) = 2$. This shows that $W$ is a hop dominating set of $G \circ H$. Since $B_v$ or $D_v$ is a locating set for each $v \in V(G)$, by Theorem 3, $W$ is a resolving set of $G \circ H$.

Accordingly, $W$ is a resolving hop dominating set of $G \circ H$. \qed

**Corollary 2.** Let $G$ be a non-trivial graph of order $m$ and $H$ be any graph. Then the following statements hold.

(i) $\gamma_{Rh}(G \circ H) \leq m(1 + ln(H))$. 


Let \( A_1 = V(G) \) and let \( B_v \) be an \( ln \)-set of \( H \) for each \( v \in V(G) \). Then \( W_1 = A_1 \cup (\bigcup_{v \in V(G)} B_v) \) is a resolving hop dominating set of \( G \circ H \) by Theorem 4. Hence,
\[
\gamma_{Rh}(G \circ H) \leq |W_1| = |V(G)| + |V(G)||B_v| = m(1 + ln(H)).
\]

(ii) Suppose that \( sln(H) = ln(H) \). Set \( A_2 = \emptyset \) and let \( D_u \) be an \( sln \)-set of \( H \) for each \( u \in V(G) \). Then \( W_2 = A_2 \cup (\bigcup_{u \in V(G)} D_u) \) is a resolving hop dominating set of \( G \circ H \) by Theorem 4. Thus,
\[
\gamma_{Rh}(G \circ H) \leq |W_2| = |A_2| + |V(G)||D_u| = m(sln(H)).
\]

Now, let \( W_0 = A_0 \cup (\bigcup_{u \in V(G)} S_0 B_v) \cup (\bigcup_{u \in S_0} D_u) \) be a \( \gamma_{Rh} \)-set of \( G \circ H \). By Theorem 4, \( A_0 \subseteq V(G) \), \( S_0 = \{x \in V(G) : x \notin N_G(A_0)\} \), \( B_v \) is a locating set of \( H^v \) for each \( v \in V(G) \setminus S_0 \) and \( D_u \) is a strict locating set of \( H^u \) for each \( u \in S_0 \). Thus,
\[
\gamma_{Rh}(G \circ H) = |W_0|
= |A_0| + |V(G) \setminus S_0| |B_v| + |S_0| |D_u|
\geq |V(G) \setminus S_0| ln(H) + |S_0| sln(H)
= (|V(G)| - |S_0|) sln(H) + |S_0| sln(H)
= m(sln(H)).
\]

Therefore, \( \gamma_{Rh}(G \circ H) = m(sln(H)). \) \( \square \)

5. On Resolving Hop Domination in the Lexicographic Product of Graphs

The \textit{lexicographic product} of two graphs \( G \) and \( H \), denoted by \( G[H] \), is the graph with vertex-set \( V(G[H]) = V(G) \times V(H) \) such that \((u_1, u_2)(v_1, v_2) \in E(G[H])\) if either \( u_1v_1 \in E(G) \) or \( u_1 = v_1 \) and \( u_2v_2 \in E(H) \).

**Theorem 5.** \([7,8]\) Let \( G \) and \( H \) be non-trivial connected graphs with \( \Delta(H) \leq |V(H)| - 2 \). Then \( W = \bigcup_{x \in S} \{x \} \times T_x \), where \( S \subseteq V(G) \) and \( T_x \subseteq V(H) \) for each \( x \in S \), is a resolving set of \( G[H] \) if and only if

(i) \( S = V(G) \);

(ii) \( T_x \) is a locating set for every \( x \in V(G) \);

(iii) \( T_x \) or \( T_y \) is a strictly locating set of \( H \) whenever \( x \) and \( y \) are adjacent vertices of \( G \) with \( N_G[x] = N_G[y] \); and

(iv) \( T_x \) or \( T_y \) is a \((\text{locating})\) dominating set of \( H \) whenever \( x \) and \( y \) are nonadjacent vertices of \( G \) with \( N_G(x) = N_G(y) \).
Theorem 6. Let $G$ and $H$ be non-trivial connected graphs with $\triangle(H) \leq |V(H)| - 2$. Then $W = \bigcup_{x \in S} \{x\} \times T_x$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a resolving hop dominating set of $G[H]$ if and only if

(i) $S = V(G)$;

(ii) $T_x$ is a locating set for every $x \in V(G)$;

(iii) $T_x$ or $T_y$ is a strictly locating set of $H$ whenever $x$ and $y$ are adjacent vertices of $G$ with $N_G[x] = N_G[y]$;

(iv) $T_x$ or $T_y$ is a (locating) dominating set of $H$ whenever $x$ and $y$ are nonadjacent vertices of $G$ with $N_G(x) = N_G(y)$; and

(v) $T_x$ is a strictly locating set of $H$ for each $x \in S \setminus N_G(S,2)$.

Proof: Suppose $W$ is a resolving hop dominating set of $G[H]$. Then $W$ is a resolving set. By Theorem 5, (i) to (iv) hold. Let $x \in S \setminus N_G(S,2)$. If $T_x = V(H)$, then $T_x$ is a strictly locating set of $H$. So suppose that $T_x \neq V(H)$ and let $a \in V(H) \setminus T_x$. Since $W$ is hop dominating and $(x,a) \notin W$, there exists $(y,b) \in W$ such that $d_{G[H]}((x,a),(y,b)) = 2$. The condition $x \in S \setminus N_G(S,2)$ would imply that $y = x$ and $b \in (V(H) \setminus N_H(a)) \cap T_x$. Hence, $T_x$ is a strictly locating set of $H$.

Conversely, suppose that $W$ satisfies (i) to (v). By Theorem 5, $W$ is a resolving set. Let $(x,a) \in V(G[H]) \setminus W$. Since $S = V(G)$, $a \in V(H) \setminus T_x$. If $x \in N_G(S,2)$, then there exists $z \in N_G(x,2)$. Let $b \in T_x$. Then $(z,b) \in W \cap N_G((x,a),2)$. Suppose $x \in S \setminus N_G(S,2)$. By (v), $T_x$ is a strictly locating set of $H$. Hence, there exists $p \in [V(H) \setminus N_H(a)] \cap T_x$. This implies that $(x,p) \in W \cap N_G((x,a),2)$. Therefore, $W$ is a hop dominating set of $G[H]$.

Accordingly, $W$ is a resolving hop dominating set of $G[H]$. □

Corollary 3. Let $G$ and $H$ be non-trivial connected graphs. Then

$$\gamma_{Rhb}(G[H]) \leq |V(G)|sln(H).$$

If $G$ is totally point determining graph and $\gamma(G) \neq 1$, then

$$\gamma_{Rhb}(G[H]) = |V(G)|ln(H).$$

Proof: Let $S = V(G)$ and let $T_x$ be an $sln$-set of $H$. By Theorem 6, $W = \bigcup_{x \in S} \{x\} \times T_x$ is a resolving hop dominating set of $G[H]$. It follows that

$$\gamma_{Rhb}(G[H]) \leq |W| = |V(G)||T_x| = |V(G)|sln(H).$$

Next, suppose that $G$ is totally point determining graph and $\gamma(G) \neq 1$. Let $S = V(G)$ and let $R_x$ be an $ln$-set of $H$ for each $x \in S$. Since $\gamma(G) \neq 1$, $x \in N_G(S,2)$ for each $x \in S$. By Theorem 6, $W = \bigcup_{x \in S} \{x\} \times R_x$ is a resolving hop dominating set of $G[H]$. It follows that

$$\gamma_{Rhb}(G[H]) \leq |W| = |V(G)||R_x| = |V(G)|ln(H).$$
Now, if \( W_0 = \bigcup_{x \in S_0} \{x\} \times T_x \) is a \( \gamma_{Rh} \)-set of \( G[H] \), then \( S_0 = V(G) \) and \( T_x \) is a locating set of \( H \) for each \( x \in V(G) \) by Theorem 6. Hence,
\[
\gamma_{Rh}(G[H]) = |W_0| = |V(G)||T_x| \geq |V(G)||n(H)|.
\]
Therefore, \( \gamma_{Rh}(G[H]) = |V(G)||n(H)| \).
\( \square \)

**Corollary 4.** Let \( G \) and \( H \) be non-trivial connected graphs. If \( G \) is totally point determining and \( \gamma(G) = 1 \), then
\[
\gamma_{Rh}(G[H]) = sln(H) + (|V(G)| - 1)n(H).
\]

**Proof:** Let \( D_G = \{v \in V(G) : \{v\} \) is a dominating set of \( G \} \). Since \( G \) is totally point determining, it follows that \( |D_G| = 1 \). Set \( S = V(G) \). Let \( T_v \) be an \( sln \)-set of \( H \) for \( v \in D_G \) and let \( T_x \) be an \( ln \)-set of \( H \) for each \( x \in V(G) \). Then by Theorem 6, \( W = \bigcup_{x \in S \setminus \{v\}} \{x\} \times T_x \) is a resolving hop dominating set of \( G[H] \). Hence,
\[
\gamma_{Rh}(G[H]) \leq |W_0| = (|V(G)| - 1)n(H) + sln(H).
\]
Suppose now that \( W^* = \bigcup_{x \in S \setminus \{v\}} \{x\} \times R_x \) is a \( \gamma_{Rh} \)-set of \( G[H] \). Then there exists a unique vertex \( v \) such that \( \{v\} \) is a dominating set of \( G \). By Theorem 6, \( S^* = V(G) \), \( R_v \) is a strictly locating set of \( H \) and \( R_x \) is a locating set of \( H \) for each \( x \in V(G) \). Thus,
\[
\gamma_{Rh}(G[H]) = |W^*| = |R_v| + \sum_{x \in S^\setminus\{v\}} |R_x| \geq s\ln(H) + (|V(G)| - 1)n(H).
\]
Therefore, \( \gamma_{Rh}(G[H]) = s\ln(H) + (|V(G)| - 1)n(H) \).
\( \square \)

**Corollary 5.** Let \( H \) be a non-trivial connected graph and let \( n \geq 2 \) be an integer. Then
\[
\gamma_{Rh}(K_n[H]) = n(s\ln(H)).
\]

**Proof:** Let \( G = K_n \). Then \( v \) is a dominating vertex of \( G \) for each \( v \in V(G) \). Thus, if \( W_0 = \bigcup_{x \in S_0} \{x\} \times T_x \) is a \( \gamma_{Rh} \)-set of \( G[H] \), then \( S_0 = V(G) \) and \( T_x \) is an \( s\ln \)-set of \( H \) for each \( x \in S_0 \), by Theorem 6. Hence,
\[
\gamma_{Rh}(K_n[H]) = |W_0| = |V(K_n)||s\ln(H) = n(s\ln(H)). \]
\( \square \)

**Acknowledgements**

This research is funded by the Department of Science and Technology - Accelerated Science and Technology Human Resource Development Program (DOST-ASTHRDP), Philippines.
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