Extremal regular graphs of given chromatic number

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Abstract

We define an extremal \((r|\chi)\)-graph as an \(r\)-regular graph with chromatic number \(\chi\) of minimum order. We show that the Turán graphs \(T_{ak,k}\), the antihole graphs and the graphs \(K_k \times K_2\) are extremal in this sense. We also study extremal Cayley \((r|\chi)\)-graphs and we exhibit several \((r|\chi)\)-graph constructions arising from Turán graphs.

Keywords: Extremal graphs; Turán graphs; Reed’s conjecture.

Mathematics Subject Classifications: 05C35, 05C15.

1 Introduction

An \(r\)-regular graph is a simple finite graph such that each of its vertices has degree \(r\). Regular graphs are one of the most studied classes of graphs; especially those with symmetries such as Cayley graphs. Let \(\Gamma\) be a finite group and let \(X = \{x_1, x_2, \ldots, x_t\}\) a generating set for \(\Gamma\) such that \(X = X^{-1}\) with \(1_\Gamma \notin X\); a Cayley graph \(\text{Cay}(\Gamma, X)\) has vertex set consisting of the elements of \(\Gamma\) and two vertices \(g\) and \(h\) are adjacent if \(gx_i = h\) for some \(1 \leq i \leq t\). Cayley graphs are regular but there exist non-Cayley vertex-transitive graphs. The Petersen graph is a classic example of this fact.

The girth of a graph is the size of its shortest cycle. An \((r, g)\)-graph is an \(r\)-regular graph of girth \(g\). An \((r, g)\)-cage is an \((r, g)\)-graph of smallest possible order. The diameter of a graph is the largest length between shortest paths of any two vertices. An \((r, D)\)-graph is an \(r\)-regular graph of diameter \(D\).

While the cage problem asks for the constructions of cages, the degree-diameter problem asks for the construction of \((r; D)\)-graphs of maximum order. Both of them are open and active problems (see [3, 6]) in which, frequently, it is considered the restriction to Cayley graphs, see [4, 5].

In this paper, we study a similar problem using a well-known parameter of coloration instead of girth or diameter. A \(k\)-coloring of a graph \(G\) is a partition

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of its vertices into \( k \) independent sets. The \textit{chromatic number} \( \chi(G) \) of \( G \) is the smallest number \( k \) for which there exists a \( k \)-coloring of \( G \).

We define an \( (r|\chi) \)-\textit{graph} as an \( r \)-regular graph of chromatic number \( \chi \). In this work, we investigate the \( (r|\chi) \)-graphs of minimum order. We also consider the case of Cayley \( (r|\chi) \)-graphs.

The remainder of this paper is organized as follows: In Section 2 we show the existence of \( (r|\chi) \)-graphs, we define \( n(r|\chi) \) as the order of the smallest \( (r|\chi) \)-graph, and similarly, we define \( c(r|\chi) \) as the order of the smallest Cayley \( (r|\chi) \)-graph. We also exhibit lower and upper bounds on the orders of the extremal graphs. We show that the Turán graphs \( T_{ak,k} \), antihole graphs (the complements of cycles) and \( K_k \times K_2 \) are Cayley \( (r|\chi) \)-graphs of order \( n(r|\chi) \) for some \( r \) and \( \chi \). To prove that \( K_k \times K_2 \) are extremal we use instances of the Reed’s Conjecture for which it is true. In Section 3 we only consider non-Cayley graphs. We give another upper bound for \( n(r|\chi) \) and we exhibit two families of \( (r|\chi) \)-graphs with a few number of vertices which are extremal for some values of \( r \) and \( \chi \). Finally, in Section 4 we study the small values \( 2 \leq r \leq 10 \) and \( 2 \leq \chi \leq 6 \). We obtain a full table of extremal \( (r|\chi) \)-graphs except for the pair \((6|6)\).

### 2 Cayley \( (r|\chi) \)-graphs

It is known that for any graph \( G \), \( 1 \leq \chi(G) \leq \Delta + 1 \) where \( \Delta \) is the maximum degree of \( G \). Therefore, for any \( (r|\chi) \)-graph we have that

\[
1 \leq \chi \leq r + 1.
\]

Suppose that \( G \) is a \((r|1)\)-graph. Hence \( G \) is the empty graph, then \( r = 0 \). Therefore, the extremal graph is the trivial graph. We can assume that \( 2 \leq \chi \leq r + 1 \).

Next, we prove that for any \( r \) and \( \chi \) such that \( 2 \leq \chi \leq r + 1 \), there exists a Cayley \((r|\chi)\)-graph \( G \).

We recall that the \((n,k)\)-Turán graph \( T_{n,k} \) is the complete \( k \)-partite graph on \( n \) vertices whose partite sets are as nearly equal in cardinality as possible, i.e., it is formed by partitioning a set of \( n = ak + b \) vertices (with \( 0 \leq b < k \)) into the partition of independent sets \( \{V_1, V_2, \ldots, V_b, V_{b+1}, \ldots, V_k\} \) with order \( |V_i| = a + 1 \) if \( 1 \leq i \leq b \) and \( |V_i| = a \) if \( b + 1 \leq i \leq k \). Every vertex in \( V_i \) has degree \( a(k-1) + b - 1 \) for \( 1 \leq i \leq b \) and every vertex in \( V_i \) has degree \( a(k-1) + b \) for \( b + 1 \leq i \leq k \). The \((n,k)\)-Turán graph has chromatic number \( k \), and size (see [1])

\[
\left\lfloor \frac{(k-1)n^2}{2k} \right\rfloor.
\]

**Lemma 2.1.** The \((ak,k)\)-Turán graph \( T_{ak,k} \) is a Cayley graph.

**Proof.** Let \( \Gamma \) be the group \( \mathbb{Z}_a \times \mathbb{Z}_k \) and \( X = \{(i,j): 0 \leq i < a, 0 < j < k\} \). Then, the graph \( \text{Cay}(\Gamma, X) \) is isomorphic to \( T_{ak,k} \). \( \square \)
Before to continue, we recall some definitions. Given two graphs $H_1$ and $H_2$, the cartesian product $H_1 \Box H_2$ is defined as the graph with vertex set $V(H_1) \times V(H_2)$ and two vertices $(u, u')$ and $(v, v')$ are adjacent if either $u = v$ and $u'$ is adjacent with $v'$ in $H_2$, or $u' = v'$ and $u$ is adjacent with $v$ in $H_1$. The following proposition appears in [10].

**Proposition 2.2.** The cartesian product of two Cayley graphs is a Cayley graph.

On the other hand, the chromatic number of $H_1 \Box H_2$ is the maximum between $\chi(H_1)$ and $\chi(H_2)$, see [2]. Now we can prove the following theorem.

**Theorem 2.3.** For any $r$ and $\chi$ such that $2 \leq \chi \leq r + 1$, there exists a Cayley $(r|\chi)$-graph.

**Proof.** Let $r = a(\chi - 1) + b$ where $a \geq 1$ and $0 \leq b < \chi - 1$. Consider the Cayley graph $H_1 = T_{a\chi, \chi}$. The graph $H_1$ has chromatic number $\chi$ and it is an $a(\chi - 1)$-regular graph of order $a\chi$.

Additionally, consider the graph $H_2 = T_{b+1, b+1} = K_{b+1}$. The graph $H_2$ has chromatic number $b + 1 < \chi$ and it is a $b$-regular graph of order $b + 1$.

Therefore, the graph $G = H_1 \Box H_2$ is a Cayley graph by Proposition 2.2 such that it has chromatic number

$$\max\{\chi(H_1), \chi(H_2)\} = \chi,$$

regularity $r$ and order $a\chi(b + 1)$.

Now, we define $n(r|\chi)$ as the order of the smallest $(r|\chi)$-graph and $c(r|\chi)$ as the order of the smallest Cayley $(r|\chi)$-graph. Hence,

$$r + 1 \leq n(r|\chi) \leq c(r|\chi) \leq a\chi(b + 1)$$

where $r = a(\chi - 1) + b$ with $a \geq 1$ and $0 \leq b < \chi - 1$.

To improve the lower bound we consider the $(n, \chi)$-Turán graph $T_{n, \chi}$. Suppose $G$ is an $(r|\chi)$-graph. Let $\varsigma$ be a $\chi$-coloring of $G$ resulting in the partition $(V_1, V_2, \ldots, V_\chi)$ with $|V_i| = a_i$ for $1 \leq i \leq \chi$. Then the largest possible size of $G$ occurs when $G$ is a complete $\chi$-partite graph with partite sets $(V_1, V_2, \ldots, V_\chi)$ and the cardinalities of these partite sets are as equal as possible. This implies that

$$\frac{nr}{2} \leq \left\lfloor \frac{(\chi - 1)n^2}{2\chi} \right\rfloor \leq \frac{(\chi - 1)n^2}{2\chi},$$

since $G$ has size $rn/2$. After some calculations we get that

$$\frac{r\chi}{\chi - 1} \leq n.$$

**Theorem 2.4.** For any $2 \leq \chi \leq r + 1$,

$$\left\lfloor \frac{r\chi}{\chi - 1} \right\rfloor \leq n(r|\chi) \leq c(r|\chi) \leq \frac{r - b}{\chi - 1}\chi(b + 1)$$

where $\chi - 1|r - b$ with $0 \leq b < \chi - 1$. 

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An $(r|\chi)$-graph $G$ of $n(r|\chi)$ vertices is called extremal $(r|\chi)$-graph. Similarly, a Cayley $(r|\chi)$-graph $G$ of $2(r|\chi)$ vertices is called extremal Cayley $(r|\chi)$-graph. When $\chi - 1 | r$ the lower bound and the upper bound of Theorem 2.4 are equal. We have the following corollary.

**Corollary 2.5.** The Cayley graph $T_{a\chi,\chi}$ is an extremal $(a(\chi - 1)|\chi)$-graph.

In the remainder of this paper we exclusively work with $b \neq 0$, that is, when $\chi - 1$ is not a divisor of $r$.

### 2.1 Antihole graphs

A **hole graph** is a cycle of length at least four. An **antihole graph** is the complement $G^c$ of a hole graph $G$. Note that a hole graph and its antihole graph are both connected if and only if their orders are at least five. In this subsection we prove that antihole graphs of order $n$ are extremal $(r|\chi)$-graphs for any $n$ at least six. There are two cases depending on the number of vertices.

1. $G = C_n^c$ for $n = 2k$ and $k \geq 3$.
   The graph $G$ has regularity $r = 2k - 3$ and chromatic number $\chi = k$. Any $(2k - 3|k)$-graph has an even number of vertices and at least $\frac{r\chi}{\chi - 1} = \frac{(2k - 3)k}{k - 1} = 2k - \frac{k - 1}{k - 1}$ vertices.
   If $k > 2$, then $\frac{k - 1}{k - 1} < 2$. Therefore we have the following result:
   
   $n(2k - 3, k) = c(2k - 3, k) = 2k$

   for all $k \geq 3$.

2. $G = C_n^c$ for $n = 2k - 1$ and $k \geq 4$.
   The graph $G$ has regularity $r = 2k - 4$ and chromatic number $\chi = k$. Any $(2k - 4|k)$-graph has at least $\frac{r\chi}{\chi - 1} = \frac{(2k - 4)k}{k - 1} = 2k - 2 - \frac{2}{k - 1}$ vertices.
   If $k - 1 > 2$, we have that $\frac{2}{k - 1} < 1$. Therefore
   
   $2k - 2 \leq n(2k - 4, k) \leq c(2k - 4, k) \leq 2k - 1$

   for all $k \geq 4$.

Suppose that $G$ is a $(2k - 4|k)$-graph of $2k - 2$ vertices. Then $G = ((k - 1)K_2)^c$, i.e., $G$ is the complement of a matching of $k - 1$ edges. Then $\chi(G) = k - 1$, a contradiction. Therefore

$n(2k - 4, k) = c(2k - 4, k) = 2k - 1$

for all $k \geq 4$.

Therefore, we have the following theorem.
Theorem 2.6. The antihole graphs of order $n \geq 6$ are extremal $(n-3|\lceil \frac{n}{2} \rceil)$-graphs.

A hole graph is also considered a 2-factor since it is a spanning 2-regular graph. For short, we denote the disjoint union of $j$ cycles of length $i$ as $jC_i$.

Let $G$ be an union of cycles

$$a_3C_3 \cup a_4C_4 \cup \ldots \cup a_{2t}C_{2t}$$

for $a_i \geq 0$ with $i \in \{3, 4, \ldots, 2t\}$. Note that the complement $G^c$ of $G$ is the join of the complement of cycles.

Theorem 2.7. The graph $(a_3C_3 \cup a_4C_4 \cup \ldots \cup a_{2t}C_{2t})^c$ is extremal if $a_5 + a_7 + \cdots + a_{2t-1} + 1 < a_3$.

Proof. Let $G^c = (a_3C_3 \cup a_4C_4 \cup \ldots \cup a_{2t}C_{2t})^c$. The graph $G^c$ has order $n = 3a_3 + 4a_4 + \cdots + 2ta_{2t}$, regularity $r = n - 3$ and chromatic number $\chi = a_3 + 2a_4 + 3a_5 + 3a_6 + \cdots + ta_{2t-1} + ta_{2t}$ since the chromatic numbers of $C_3^c$, $C_4^c$, $C_5^c$, ..., $C_{i-1}^c$ are 1, 2, 3, ..., $\lceil i/2 \rceil$ respectively.

Any $(r|\chi)$-graph has at least $\frac{r\chi}{\chi - 1} = r + \frac{r}{\chi - 1} = n - \frac{3\chi - n}{\chi - 1}$ vertices for $r = n - 3$.

If $\frac{3\chi - n}{\chi - 1} < 1$ then $G^c$ is extremal, that is, when

$$2\chi + 1 < n,$$

i.e. when

$$a_5 + a_7 + \cdots + a_{2t-1} + 1 < a_3.$$

Moreover, we have the following results.

Theorem 2.8. Since $C_n^c$ is extremal then

1. When $n$ is even, if $G = (a_3C_3 \cup a_4C_4 \cup \ldots \cup a_{2t}C_{2t})^c$ is a graph of order $n$ such that $a_5 + a_7 + \cdots + a_{2t-1} = a_3$, then $G$ is extremal.

2. When $n$ is odd, if $G = (a_3C_3 \cup a_4C_4 \cup \ldots \cup a_{2t}C_{2t})^c$ is a graph of order $n$ such that $a_5 + a_7 + \cdots + a_{2t-1} = a_3 + 1$, then $G$ is extremal.

Corollary 2.9. Since the antihole graphs of order $n \geq 8$ are $(r|\chi)$-graphs, then there exist many non-isomorphic extremal $(r|\chi)$-graphs (not necessarily Cayley).

For instance, there are three extremal $(5, 4)$-graphs, namely, $C_8^c$, $(2C_4)^c$ and $(C_3 \cup C_5)^c$. See also Table 1.
2.2 The case of $r = \chi$

In this subsection, we discuss the case of $r = \chi = k$, i.e., the \((k|k)\)-graphs of minimum order. We have the following bounds so far:

\[
\left\lceil \frac{k^2}{k-1} \right\rceil = k + 1 \leq n(k|k) \leq 2k.
\]

We prove that the upper bound is correct except for $k = 4$ and maybe for $k = 6, 8, 10, 12$. To achieve it, we assume that there exist \((k|k)\)-graphs of order $n \leq 2k - 2$, that is

\[
\left\lceil \frac{n}{2} \right\rceil < k = \chi. \tag{1}
\]

Now, we use a bound for the chromatic number arising from the Reed’s Conjecture, see [9]. We recall the clique number $\omega(G)$ of a graph $G$ is the largest $k$ for which $G$ has a complete subgraph of order $k$.

**Conjecture 2.10.** For every graph $G$,

\[
\chi(G) \leq \left\lceil \frac{\omega(G) + 1 + \Delta(G)}{2} \right\rceil.
\]

It is known that the conjecture is true for graphs satisfying Equation (1) see [7]. It follows that $k \leq \omega(G) + 1$ for any \((k|k)\)-graph $G$ of order $n \leq 2k - 2$, that is, $\omega(G) = k$ or $\omega(G) = k - 1$.

**Case 1:** $\omega(G) = k$.

Let $H_1$ be a clique of $G$ and $H_2 = G \setminus V(H_1)$. There is a set of $k$ edges from $V(H_1)$ and $V(H_2)$. Therefore, if $t = n - k \leq k - 2$ is the order of $H_2$ and $m = (kt - k)/2$ is the number of edges in $H_2$, then

\[
m \leq \binom{t}{2}.
\]

We obtain that $k \leq t$, a contradiction.

**Case 2:** $\omega(G) = k - 1$.

Let $H_1$ be a clique of $G$ and $H_2 = G \setminus V(H_1)$. There is a set of $2(k - 1)$ edges from $V(H_1)$ to $V(H_2)$. Therefore, if $t = n - (k - 1) \leq k - 1$ is the order of $H_2$ and $m = (kt - 2(k - 1))/2$ is the number of edges in $H_2$, then

\[
m \leq \binom{t}{2}.
\]

We obtain that $k \leq t + 1$, hence, $k = t + 1$ and $n$ has to be $2k - 2$. Since every vertex $v$ in $V(H_2)$ has degree $k$ in $G$, $v$ has at least two neighbours in $H_1$. By symmetry, $G$ is the union of two complete graphs $K_{k-1}$ with the addition of two perfect matchings between them. Its complement is a \((k - 3)\)-regular bipartite graph. Any perfect matching of $G^c$ induce a \((k - 1)\)-coloring in $G$, a contradiction.
We have the following results.

**Lemma 2.11.** For any \( k \geq 3 \),
\[
2k - 1 \leq n(k|k) \leq c(k|k) \leq 2k.
\]

If \( k \) is odd then the order of any \( k \)-regular graph is even, therefore:

**Corollary 2.12.** For any \( k \geq 3 \) an odd number, \( n(k|k) = c(k|k) = 2k \).

We have that \( C_7^k \) is the extremal \( (4|4) \)-graph. Next, assume that \( k \geq 6 \) is an even number and there exists a \( (k|k) \)-graph \( G \) of \( n = 2k - 1 \) vertices. Owing to the fact that \( \chi(G) \leq n - \alpha(G) + 1 \) where \( \alpha(G) \) is the independence number of \( G \), we get that \( \alpha(G) \leq k \).

In [7] was proved that the Reed’s conjecture holds for graphs of order \( n \) satisfying \( \chi > \frac{n + 3 - \alpha(G)}{2} \). In the case of the graph \( G \), we have that
\[
\frac{n + 3 - \alpha(G)}{2} \leq \frac{k}{2} + 1 < k.
\]

It follows that \( \omega(G) \leq k \leq \omega(G) + 1 \). Newly, we have two cases:

**Case 1:** \( \omega(G) = k \).

As we saw before, let \( H_1 \) be a clique of \( G \) and \( H_2 = G \setminus V(H_1) \). There is a set of \( k \) edges from \( V(H_1) \) and \( V(H_2) \). Therefore, if \( t = k - 1 \) is the order of \( H_2 \) and \( m = (kt - k)/2 \) is the number of edges in \( H_2 \), then
\[
m \leq \left( \frac{t}{2} \right).
\]

We obtain that \( k \leq t \), a contradiction.

**Case 2:** \( \omega(G) = k - 1 \).

In [8] was proved that every graph satisfies
\[
\chi \leq \left\{ \omega, \Delta - 1, \left[ \frac{15 + \sqrt{96k + 25}}{4} \right] \right\}.
\]

Hence, for the graph \( G \) we have that \( k \leq \left[ \frac{15 + \sqrt{96k + 25}}{4} \right] \). After some calculations we get that \( k = 6, 8, 10, 12 \), otherwise, \( k > \left[ \frac{15 + \sqrt{96k + 25}}{4} \right] \).

Finally, we have the following theorem.

**Theorem 2.13.** For any \( k \geq 3 \) such that \( k \notin \{4, 6, 8, 10, 12\} \),
\[
n(k|k) = c(k|k) = 2k.
\]

Moreover, if \( k = 4 \) then \( n(k|k) = c(k|k) = 2k - 1 \) and if \( k \in \{6, 8, 10, 12\} \) then
\[
2k - 1 \leq n(k|k) \leq c(k|k) \leq 2k.
\]

We point out that if there exists an extremal \( (k|k) \)-graph \( G \) of \( 2k - 1 \) vertices for \( k \in \{6, 8, 10, 12\} \), then \( G \) has clique number \( \omega = k - 1 \), a clique \( H_1 \) of order \( \omega \) for which \( G \setminus V(H_1) \) has \( \frac{k}{2} - 1 \) edges, \( G \) is Hamiltonian-connected and it has independence number \( \alpha(G) \) such that \( \alpha(G) \in \{k/4, \ldots, k/2 + 1\} \), see [8].
3 Non-Cayley constructions

In this section we improve the upper bound of $n(r|\chi)$ given on Theorem 2.3 by exhibiting a construction of graphs not necessarily Cayley. We assume that $r$ is not a multiple of $\chi - 1$, therefore $2 \leq \chi \leq r$. Additionally, we show two more constructions which are tight for some values.

3.1 Upper bound

To begin with, take the Turán graph $T_{n,\chi}$, for $n = a\chi + b$, $0 < b < \chi$ with $r = a(\chi - 1) + b$ and the partition $(V_1, V_2, \ldots, V_{b+1}, \ldots, V_{\chi})$ such that $|V_i| = a + 1$ if $1 \leq i \leq b$ and $|V_i| = a$ if $b + 1 \leq i \leq \chi$. Every vertex in $V_i$ for $1 \leq i \leq b$ has degree $r - 1$ and every vertex in $V_i$ for $b + 1 \leq i \leq \chi$ has degree $r$.

Next, we define the graph $G_{n,\chi}$ as the graph formed by two copies $G_1$ and $G_2$ of $T_{n,\chi}$ with the addition of a matching between the vertices of degree $r - 1$ of $G_1$ and the vertices of degree $r - 1$ of $G_2$ in the natural way. In consequence, the graph $G_{n,\chi}$ is an $r$-regular graph of order $2n$ and chromatic number $\chi$. To obtain its chromatic number, suppose that $T_{n,\chi}$ has the vertex partition $V_i$, then the vertices of $V_i$ have the color $i$ in $G_1$ and the vertices of $V_i$ are colored $i + 1$ mod $\chi$ in $G_2$. Hence $\chi = \chi(G_1) \leq \chi(G_{n,\chi}) \leq \chi$ and then $\chi(G_{n,\chi}) = \chi$.

**Theorem 3.1.** For $2 \leq \chi \leq r + 1$, then

\[
\left\lceil \frac{r\chi}{\chi - 1} \right\rceil \leq n(r|\chi) \leq \min \left\{ 2 \left\lceil \frac{r\chi}{\chi - 1} \right\rceil, \frac{r - b}{\chi - 1} \chi(b + 1) \right\},
\]

where $\chi - 1 | r - b$ with $0 \leq b < \chi$.

3.2 The graph $T_{n,\chi}^*$

In this subsection we give a better construction for some values of $r$ and $\chi$. Consider the $(a\chi + b, \chi)$-Turán graph $T_{a\chi + b, \chi}$ such that $\chi > b \geq 0$ and partition $(V_1, \ldots, V_{\chi - b}, \ldots, V_{\chi})$ for $\chi \geq 3$, $|V_i| = a_i = a \geq 2$ with $i \in \{1, \ldots, \chi - b\}$ and $|V_i| = a_i = a + 1 \geq 3$ with $i \in \{\chi - b + 1, \ldots, \chi\}$.

We claim that $a$ is even or $\chi - b$ is even. To prove it, assume that $a$ and $\chi - b$ are odd. Hence, if $b$ is even, then $\chi$ is odd, $n = a\chi + b$ is odd and $r$ is odd, a contradiction. If $b$ is odd, then $\chi$ is even, $n = a\chi + b$ is odd and $r$ is odd, newly, a contradiction.

Now, we define the graph $T_{n,\chi}^*$ of regularity $r = a(\chi - 1) + b + 1$ as follows: If $\chi - b$ is even, the removal of a perfect matching between $X_i$ and $X_{i+1}$ for all $i \in \{1, 3, \ldots, \chi - b - 1\}$ of $T_{n,\chi}$ produces $T_{n,\chi}^*$. If $\chi - b \geq 3$ is odd then $a$ is even, therefore, the removal of a perfect matching between $X_i$ and $X_{i+1}$ for all $i \in \{4, 6, \ldots, \chi - b - 1\}$ and a perfect matching between $V_i'$ and $V_i''$, $V_i'$ and $V_i'''$, and $V_i''$ and $V_i'''$ where $V_i \setminus V_i' = V_i''$ is a set of $a/2$ vertices for $i \in \{1, 2, 3\}$, of $T_{n,\chi}$ produces $T_{n,\chi}^*$. 
The graphs $T_{n,\chi}^{*}$ improve the upper bound given in Theorem 3.1 for some numbers $n$ and $\chi$:

$$r \chi - 1 = a \chi + b - \frac{\chi - b}{\chi - 1} \leq a \chi + b.$$ 

Hence, if $\frac{\chi - b}{\chi - 1} < 1$, the construction gives extremal graphs, that is, when

$$1 < b.$$ 

**Theorem 3.2.** Let $\chi \geq 3$, $\chi \geq b > 1$ and $a \geq 2$. Then the graph $T_{a \chi + b, \chi}$ defined above is an extremal $(a(\chi - 1) + b - 1|\chi)$-graph when $\chi - b$ is even or $a > 2$ is even.

### 3.3 The graph $G_{a,c,t}$

Consider the $(at, t)$-Turán graph $T_{at,t}$ with partition $(V_1, \ldots, V_t)$. Now, we define the graph $G_{a,c,t}$ with $1 \leq c < a$ as follows: consider two parts of $(V_1, \ldots, V_t)$, e.g. $V_1$ and $V_2$, and $c$ vertices of these two parts $\{u_1, \ldots, u_c\} \subseteq V_1$ and $\{v_1, \ldots, v_c\} \subseteq V_2$.

The removal of the edges $u_i v_j$ for $i, j \in \{1, \ldots, c\}$ when $i \neq j$ (all the edges between $\{u_1, \ldots, u_c\}$ and $\{v_1, \ldots, v_c\}$ except for a matching) and the addition of the edges $u_i v_j$ for $i, j \in \{1, \ldots, c\}$ when $i \neq j$ (all the edges between the vertices $u_i$ and all the edges between the vertices $v_i$) results in the graph $G_{a,c,t}$.

The graph $G_{a,c,t}$ is a $a(t-1)$-regular graph of order $at$. Its chromatic number is $t + c - 1$ because the partition

$$(V_1 \setminus \{u_2, \ldots, u_c\}, V_2 \setminus \{v_1, \ldots, v_{c-1}\}, V_2, \ldots, V_t, \{u_2, v_1\}, \ldots, \{u_c, v_{c-1}\})$$

is a proper coloring with $t + c - 1$ colors. Moreover, the graph $G_{a,c,t}$ has a clique of $t + c - 1$ vertices, namely, the vertices $\{u_1, \ldots, u_c, x_2, \ldots, x_t\}$ where $x_i \in V_i$ for $i \in \{3, \ldots, t\}$ and $x_2 \in V_2 \setminus \{v_1, \ldots, v_c\}$.

The graphs $G_{a,c,t}$ improve the upper bound given in Theorem 2.4

$$\frac{t + c - 1}{t + c - 2} a(t - 1) = at - a \frac{c - 1}{t + c - 2} \leq at.$$ 

Hence, if $a \frac{c - 1}{t + c - 2} < 1$, the construction gives extremal graphs, that is, when

$$(a - 1)(c - 1) < t - 1.$$ 

**Theorem 3.3.** Let $a, t \geq 2$ and $a > c \geq 1$. The graph $G_{a,c,t}$ defined above is an extremal $(a(t - 1)|at)$-graph when $(a - 1)(c - 1) < t - 1$.

### 4 Small values

In this section we exhibit extremal $(r|\chi)$-graphs of small orders. These exclude the extremal graphs given before. Table [I] shows the extremal $(r|\chi)$-graphs for $2 \leq r \leq 10$ and $2 \leq \chi \leq 6$. 

9
4.1 Extremal \((5|3)\)-graph

Suppose that \(G\) is an extremal \((5|3)\)-graph of order 8, i.e., its order equals the lower bound given in Theorem 2.4. Then its complement is 2 regular. That is, \(G^c\) is \(C_8\) or \(C_5 \cup C_3\) or \(C_4 \cup C_4\). By Theorem 2.8, the complement of \(C_8\) or \(C_5 \cup C_3\) or \(C_4 \cup C_4\) has chromatic number 4. Since \(G\) is 5-regular, a \((5|3)\)-graph of order 9 does not exist and therefore 10 is the best possible. The graph \(G_{5,2,2}\) is an extremal \((5|3)\)-graph with 10 vertices.

4.2 Extremal \((7|\chi)\)-graphs for \(\chi = 3, 6\)

Let \(G\) be an extremal \((7|3)\)-graph. Its order is at least 11. Since its degree is odd, its order is at least 12. The graph \(T_{12,3}\) is an extremal \((7|3)\)-graph.

Now, suppose that \(G\) is an extremal \((7|6)\)-graph. \(G\) has at least 9 vertices. Newly, because it has an odd regularity, \(G\) has at least 10 vertices. If this is the case, its complement is a 2 regular graph. The graph \((2C_5)^c\) has chromatic number 6. It is unique and it is Cayley.

4.3 Extremal \((9|3)\)-graph

Any \((9|3)\)-graph has 14 vertices, i.e., its order equals the lower bound given in Theorem 2.4. Suppose that there exist at least one of degree 14. Let \((V_1, V_2, V_3)\) a partition by independent sets. Some of the parts, \(V_1\), has at least five vertices. Since the graph is 9-regular, \(V_1\) has exactly 5 vertices. The induced graph of \(V_2\) and \(V_3\) is a bipartite regular graph of an odd number of vertices, a contradiction. Then, any \((9|3)\)-graph has at least 16 vertices.

| \(r \setminus \chi\) | 2  | 3  | 4  | 5  | 6  |
|---------------------|----|----|----|----|----|
| 2                   | \(T_{4,2}\) | \(T_{3,3}\) | -  | -  | -  |
| 3                   | \(T_{6,2}\) | \(C_6^c\) | \(T_{4,4}\) | -  | -  |
| 4                   | \(T_{8,2}\) | \(T_{6,3}\) | \(C_5^c\) | \(T_{5,5}\) | -  |
| 5                   | \(T_{10,2}\) | \(G_{5,2,2}\) | \(C_8^c, (2C_4)^c, (C_3 \cup C_5)^c\) | \(K_5 \times K_2\) | \(T_{6,6}\) |
| 6                   | \(T_{12,2}\) | \(T_{9,3}\) | \(T_{8,4}\) | \(C_6^c, (C_4 \cup C_5)^c\) | -  |
| 7                   | \(T_{14,2}\) | \(T_{12,3}^*\) | \(T_{10,4}^*\) | \(C_{10}^c, (C_4 \cup C_6)^c, (C_3 \cup C_7)^c\) | \((2C_5)^c\) |
| 8                   | \(T_{16,2}\) | \(T_{12,3}\) | \(G_{4,2,3}\) | \(T_{10,5}\) | \(C_{11}^c, (C_4 \cup C_7)^c, (C_5 \cup C_6)^c\) |
| 9                   | \(T_{18,2}\) | \(T_{16,3}^*\) | \(T_{12,4}\) | \(T_{12,5}^*\) | \(C_{12}^c, (2C_6)^c, (3C_4)^c, (C_3 \cup C_4 \cup C_5)^c, (C_5 \cup C_6)^c\) |
| 10                  | \(T_{20,2}\) | \(T_{15,3}\) | \(T_{14,4}\) | \(T_{13,5}^{*}\) | \(T_{12,6}^*\) |

Table 1: Extremal \((r|\chi)\)-graphs.
Consider the graph $T_{16,3}$ with partition $(U,V,W)$ and the sets partition are $U = \{u_1, u_2, u_3, u_4, u_5\}$, $V = \{v_1, v_2, v_3, v_4, v_5\}$, $W = \{w_1, w_2, w_3, w_4, w_5, w_6\}$. The removal of the edges

\[
\{ w_1v_1, v_1u_1, u_1w_4, w_2v_2, v_2u_2, u_2w_5, w_3v_3, v_3u_3, u_3w_6, u_4v_4, v_4u_5, w_5v_5, v_5w_5 \}
\]

is the graph $T_{16,3}^{**}$ which is the extremal $(9|3)$-graph.

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