Differential Projective Modules Over Algebras with Radical Square Zero

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Abstract
Let $Q$ be a finite connected quiver and $\Lambda$ be the radical square zero algebra of $Q$. We give a bijection between the reduced differential projective modules over $\Lambda$ and the representations of the opposite quiver of $Q$. If $Q$ has oriented cycles and $Q$ is not a basic cycle, we prove that the algebra of dual numbers over $\Lambda$ is not virtually Gorenstein.

Keywords Gorenstein projective module · Differential module · Algebra with radical square zero · Virtually Gorenstein algebra

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1 Introduction

Given a ring $\Lambda$, a differential $\Lambda$-module is a pair $(M, d_M)$, where $M$ is a $\Lambda$-module and $d_M$ is an endomorphism of $M$ such that $d_M^2 = 0$. If $M, N$ are differential $\Lambda$-modules, a differential $\Lambda$-module map is a $\Lambda$-module map $f : M \to N$ such that $f d_M = d_N f$. A $\Lambda$-module map $f : M \to N$ is null homotopic if there exists a $\Lambda$-module map $r : M \to N$ such that $f = rd_M + d_N r$.

Let $M$ be a differential $\Lambda$-module. The shift $\Sigma(M)$ of $M$ is $(M, -d_M)$. We say $M$ is contractible if the identity map of $M$ is null homotopic, $M$ is reduced if $M$ has no nonzero contractible direct summand.

By the term differential projective $\Lambda$-module, we mean a differential $\Lambda$-module whose underlying $\Lambda$-module is projective. Finitely generated differential projective modules are called perfect differential modules in [21].

The theory of complexes has been extensively studied in decades. However, few people investigate differential modules in detail. L. L. Avramov, R.-O. Buchweitz and S. Iyengar [4] study the projective class as well as free class and flat class for differential modules. C.
M. Ringel and P. Zhang [21] investigate the perfect differential modules over path algebras, they prove that the homology functor gives a bijection from the isoclasses of reduced perfect differential modules to the isoclasses of finitely generated modules over path algebras. J. Wei [24] studies the Gorenstein homological theory for differential modules and extends this bijection to arbitrary hereditary rings.

Let $k$ be a field and $Q$ be a finite quiver. Let $kQ/J^2$ be the radical square zero algebra of $Q$. The category $C_1(P)$ of all differential projective $kQ/J^2$-modules is a Frobenius category [16]. The stable category $K_1(P)$ of $C_1(P)$ is a triangulated category [23]. We denote by $C_1^e(P)$ the full subcategory of $C_1(P)$ consisting of reduced objects.

In this paper, we study the differential projective modules over $kQ/J^2$. It turns out that the top functor gives a bijection between the isoclasses of reduced differential projective $kQ/J^2$-modules and the isoclasses of $kQ^{op}$-modules, where $Q^{op}$ is the opposite quiver of $Q$.

More precisely, we have the following; compare [7, 21].

**Theorem A** Let $k$ be a field and $Q$ be a finite quiver. There is a functor

$$
T : C_1^e(P) \to \text{Mod} kQ^{op}
$$

such that

1. $T$ is full, dense, and detects isomorphisms;
2. $T$ commutes with small coproducts;
3. $T$ vanishes on null-homotopic maps.

Moreover, for any $M$ and $N$ in $C_1^e(P)$, there is an isomorphism

$$
\text{Hom}_{K_1(P)}(M, N) \cong \text{Hom}_{kQ^{op}}(TM, TN) \bigoplus \text{Ext}^1_{kQ^{op}}(TM, T\Sigma N).
$$

The study of differential modules is related to the Gorenstein homological theory. M. Auslander and M. Bridger [1] introduce the notion of modules of G-dimension zero over two-sided Noetherian rings. These modules are also called totally reflexive modules [5]. E. E. Enochs and O. M. G. Jenda [13, 14] extend their ideas and introduce the notion of Gorenstein projective modules, Gorenstein injective modules and Gorenstein flat modules for arbitrary rings. In particular, totally reflexive modules are just finitely generated Gorenstein projective modules for two-sided Noetherian rings.

Gorenstein projective modules over algebras with radical square zero have been studied by many experts. X.-W. Chen [11] shows that a connected algebra with radical square zero is either selfinjective or CM-free. An algebra is said to be CM-free if every totally reflexive module over this algebra is projective. C. M. Ringel and B.-L. Xiong [20] extend this result to arbitrary Gorenstein projective modules.

However, Gorenstein projective modules over algebras with radical cubic zero are quite complicated. Y. Yoshino [25] studies a class of commutative local Artin algebras with radical cubic zero, over these algebras the simple module has no right approximations by totally reflexive modules. The totally reflexive modules over the algebra $S_n = k[X, Y_1, \cdots, Y_n]/(X^2, Y_iY_j)$ are studied by D. A. Rangel Tracy [19], where $n \geq 2$ and $1 \leq i, j \leq n$. It turns out that there is a bijection between totally reflexive modules over $S_n$ and finite dimensional modules over the free $k$-algebra of $n$ variables.

The algebras in [19, 25] are not virtually Gorenstein algebras. Recall that an algebra $\Lambda$ is said to be virtually Gorenstein if the right perpendicular class of Gorenstein projective $\Lambda$-modules coincides with the left perpendicular class of Gorenstein injective
Differential Projective Modules... A-modules [8, 9]. A lot of algebras such as Gorenstein algebras and algebras of finite representation type are virtually Gorenstein. However, few examples of non-virtually Gorenstein algebras are known.

In this paper, we provide a new class of non-virtually Gorenstein algebras with radical cubic zero. In fact, we give a compact generator for the stable category of Gorenstein projective modules over these algebras and show that the compact generator is infinite dimensional.

**Theorem B** If $Q$ is a finite connected quiver with oriented cycles and $Q$ is not a basic cycle, then the algebra of dual numbers over $kQ/J^2$ is not virtually Gorenstein.

Recall that a finite quiver $Q$ is said to be a basic cycle if the number of vertices is equal to the number of arrows in $Q$ and all arrows form an oriented cycle.

The rest of the paper is organized as follows. Section 2 contains preliminaries which are needed in the sequel. The proof of Theorem A is given in Section 3. In Section 4, a compact generator for the homotopy category of differential projective modules is constructed. In Section 5, we investigate the relationship between the Gorenstein projective modules and the differential modules and then give the proof of Theorem B.

Throughout this paper, all modules are left modules unless otherwise stated.

**2 Preliminaries**

We refer to [3, III.1] for quivers and their representations.

A quiver is a quadruple $(Q_0, Q_1; s, t)$, where $Q_0$ is the set of vertices, $Q_1$ is the set of arrows, and $s, t: Q_1 \to Q_0$ are the source map and the target map, respectively. If $Q_0$ and $Q_1$ are both finite, then the quiver is said to be finite.

Let $k$ be a field. The path algebra $kQ$ of a finite quiver $Q$ is the $k$-vector space generated by all paths in $Q$ whose multiplication is concatenation of paths. Every path algebra $kQ$ is hereditary; it is finite dimensional if and only if $Q$ is acyclic.

Let $kQ_0$ be the subalgebra of $kQ$ generated by all trivial paths. Then $kQ_1$ is a $kQ_0$-bimodule. The path algebra $kQ$ is isomorphic to the tensor algebra of $kQ_1$ over $kQ_0$. We use this fact to describe $kQ$-modules.

A $kQ$-module is a pair $(M, \phi_M)$, where $M$ is a $kQ_0$-module and $\phi_M$ is a $kQ_0$-module map from $kQ_1 \otimes_{kQ_0} M$ to $M$. If $M, N$ are $kQ$-modules, a $kQ$-module map from $M$ to $N$ is a $kQ_0$-module map $f: M \to N$ such that $f\phi_M = \phi_N(1 \otimes f)$.

Let $J$ be the arrow ideal of $kQ$; it is the ideal generated by all arrows in $Q$. For an ideal $I$ of $kQ$ satisfying $J^n \subseteq I \subseteq J^2$ for some $n \geq 2$, the quotient algebra $kQ/I$ is finite dimensional. In particular, the radical square zero algebra $kQ/J^2$ is finite dimensional.

Given a module $M$, the radical $\text{rad } M$ of $M$ is the intersection of all maximal submodules of $M$, and the top $\text{top } M$ of $M$ is the quotient module $M/\text{rad } M$. For any module map $f: M \to N$, let $T(f)$ be the induced map between their tops. The map $f$ is said to be radical if $\text{Im } f \subseteq \text{rad } N$, that is, $T(f)$ is zero. Denote by $\text{Rad}(M, N)$ the subspace of $\text{Hom}_A(M, N)$ of radical maps.
Lemma 2.1 Let $M$ be a projective $kQ/J^2$-module. Then the following hold.

1. $\text{rad } M = \text{Im } \phi_M$;
2. $\text{rad}^2 M = 0$;
3. $kQ_1 \otimes_{kQ_0} \text{rad } M = \text{Ker } \phi_M$;
4. $\overline{\phi}_M : kQ_1 \otimes_{kQ_0} \text{top } M \cong \text{rad } M$.

Proof Observe that (1)–(4) hold for the regular module $kQ/J^2$. Since $M$ is a projective module over $kQ/J^2$, it is a direct summand of direct sums of copies of $kQ/J^2$. Since taking radicals, kernels, images and quotients commute with small coproducts, then (1)–(4) also hold for $M$.

Lemma 2.2 Let $M, N$ be $kQ/J^2$-modules where $M$ is projective. Then there is a short exact sequence

$$\text{Rad}(M, N) \subset \rightarrow \text{Hom}_{kQ/J^2}(M, N) \rightarrow T \rightarrow \text{Hom}_{kQ_0}(\text{top } M, \text{top } N).$$

Proof Since $M$ is projective, for any map $f' : M/\text{rad } M \rightarrow N/\text{rad } N$ there is a map $f : M \rightarrow N$ such that $T(f) = f'$. Then $T$ is surjective. Since the kernel of $T$ is $\text{Rad}(M, N)$, we get the desired short exact sequence.

Let $kQ_1^*$ be the $k$-dual of $kQ_1$. For any $kQ_0$-module $X, Y$, we denote $E(X, Y) = \text{Hom}_{kQ_0}(kQ_1^* \otimes_{kQ_0} X, Y)$.

Lemma 2.3 For any projective $kQ/J^2$-modules $M, N$, there is an isomorphism

$$\gamma : \text{Rad}(M, N) \rightarrow \text{Hom}_{kQ_0}(\text{top } M, \text{rad } N) \rightarrow E(\text{top } M, \text{top } N).$$

Moreover, if $f, g, h$ are composable maps where $g$ is radical, then

$$\gamma(hgf) = T(h)\gamma(g)(1 \otimes T(f)).$$

Proof Since $N$ is projective, the square radical of $N$ is zero. By Lemma 2.2 the first map is an isomorphism. Since $N$ is projective, we have $kQ_1 \otimes \text{top } N$ and $\text{rad } N$ are isomorphic by Lemma 2.1. Since $kQ_1^*$ is a finitely generated projective $kQ_0$-module, $\text{Hom}_{kQ_0}(kQ_1^*, \text{top } N)$ is isomorphic to $kQ_1 \otimes_{kQ_0} \text{top } N$. By adjunction the second map is an isomorphism. Then $\gamma$ is an isomorphism.

If $f, g, h$ are composable maps where $g$ is radical, we can check that

$$\gamma(hg) = T(h)\gamma(g), \quad \gamma(gf) = \gamma(g)(1 \otimes T(f)).$$

Then $\gamma(hgf) = T(h)\gamma(g)(1 \otimes T(f))$. 

For a finite quiver $Q$, the opposite quiver $Q^{\text{op}}$ has the same underlying graph as $Q$ while the orientations are all reversed. Denote by $a^*$ the corresponding arrow in $Q^{\text{op}}$ for each arrow $a$ in $Q$.

If $X, Y$ are $kQ^{\text{op}}$-modules, there is a map

$$\phi : \text{Hom}_{kQ_0}(X, Y) \rightarrow \text{Hom}_{kQ_0}(kQ^* \otimes_{kQ_0} X, Y)$$

$$\theta \mapsto \theta \phi_X - \phi_Y (1 \otimes \theta)$$

for any $\theta \in \text{Hom}_{kQ_0}(X, Y)$. The image of $\phi$ is denoted by $E_0(X, Y)$. 

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We need the following; see [15, 7.2] and [22, 1.4 Corollary 2].

**Lemma 2.4** For any \( kQ^{\text{op}} \)-modules \( X, Y \), there is an isomorphism
\[
\text{Ext}^1_{kQ^{\text{op}}}(X, Y) \cong E(X, Y)/E_0(X, Y).
\]

**Lemma 2.5** Let \( X \) be a \( kQ^{\text{op}} \)-module. Then \( X \) is finitely presented if and only if the functors \( \text{Hom}_{kQ^{\text{op}}}(X, -) \) and \( \text{Ext}^1_{kQ^{\text{op}}}(X, -) \) commute with small coproducts.

### 3 Differential Projective Modules

In this section, we show that taking the top is a functor from the category of reduced differential projective \( kQ/J^2 \)-modules to the category of \( kQ^{\text{op}} \)-modules. Then we study some properties of this functor.

A differential module \( M \) is called contractible if its identity map is null homotopic, and \( M \) is called reduced if it has no nonzero contractible direct summands.

**Lemma 3.1** Let \( \Lambda \) be an artin algebra and \( M \) be a differential projective \( \Lambda \)-module.

1. \( M \) is reduced if and only if \( dM \) is a radical map.
2. There is a decomposition \( M = M' \oplus M'' \) such that \( M' \) is reduced and \( M'' \) is contractible. Moreover, this decomposition is unique up to isomorphism.

**Proof** The statements are known for complexes. We give an outline of proof here. For more details, see [17, Appendix B].

Let \( C(M) \) be the cokernel of \( dM \). Then \( dM \) is a radical map if and only if the canonical map \( M \to C(M) \) is a projective cover. Let \( M_1 \) be a projective cover of \( C(M) \). There is a module \( M_2 \) such that \( M = M_1 \oplus M_2 \) and \( M_2 \subseteq \text{Im} dM \). Since the composite \( M_1 \to M \xrightarrow{d} M \to M_2 \) is surjective, there is a copy \( N_2 \) of \( M_2 \) inside \( M_1 \). Let \( M'' \) be the differential module \( M_2 \oplus N_2 \). Then \( M'' \) is contractible and the projection \( M \to M'' \) is split. We get a decomposition \( M = M' \oplus M'' \) such that \( dM' \) is a radical map and \( M'' \) is contractible. \( \square \)

Let \( k \) be a field and \( Q \) be a finite quiver. Let \( C_1(P) \) be the category of differential projective \( kQ/J^2 \)-modules and \( C_1^{\text{re}}(P) \) be the full subcategory of reduced objects. For differential modules \( M, N \), recall that \( \text{Htp}(M, N) \) is the space of null-homotopic maps and \( \text{Rad}(M, N) \) is the space of radical maps.

**Lemma 3.2** For any \( M, N \) in \( C_1^{\text{re}}(P) \), we have
\[
\text{Htp}(M, N) \subseteq \text{Rad}(M, N) \subseteq \text{Hom}_{C_1^{\text{re}}(P)}(M, N).
\]

**Proof** Since \( M \) and \( N \) are reduced, by Lemma 3.1 \( dM \) and \( dN \) are radical maps. If \( f : M \to N \) is radical, then \( fdM = 0 = dNf \) and thus \( f \) is a differential map. If \( f = rdM + dNr \) for some \( kQ/J^2 \)-module map \( r \), then \( rdM \) and \( dNr \) are radical maps. It follows that \( f \) is a radical map. \( \square \)
The following is a corollary of Lemma 2.3.

**Corollary 3.3** Let \( f : M \to N \) be a \( kQ/J^2 \)-module map between reduced differential projective \( kQ/J^2 \)-modules.

1. \( f \) is a differential map if and only if \( T(f) \gamma(dM) = \gamma(dN)(1 \otimes T(f)) \);
2. \( f \) is a radical map if and only if \( T(f) = 0 \);
3. \( f \) is null homotopic if and only if \( T(f) = 0 \) and there exists a \( kQ_0 \)-module map \( \psi : \text{top} M \to \text{top} N \) such that \( \gamma(f) = \psi \gamma(dM) + \gamma(dN)(1 \otimes \psi) \).

There is a functor \( T : C^\text{re}_1(P) \to \text{Mod } kQ^{\text{op}} \).

For any object \( M \) in \( C^\text{re}_1(P) \), set \( T(M) = (\text{top } M, \gamma(dM)) \). For any morphism \( f \) in \( C^\text{re}_1(P) \), the induced map \( T(f) \) of \( f \) is a \( kQ^{\text{op}} \)-module map by Corollary 3.3(1).

There is another functor \( F_1 : \text{Mod } kQ^{\text{op}} \to C^\text{re}_1(P) \).

For any object \( X \) in \( \text{Mod } kQ^{\text{op}} \), set \( F_1(X) = kQ/J^2 \otimes_{kQ_0} X \) with differential
\[
d(p \otimes x) = \sum_{\alpha \in Q_1} p\alpha \otimes \alpha^* x
\]
for \( p \in kQ/J^2, x \in X \). For any morphism \( \psi \) in \( \text{Mod } kQ^{\text{op}} \), set \( F_1(\psi) = 1 \otimes \psi \).

**Lemma 3.4** The composite \( T \circ F_1 \) is isomorphic to the identity functor.

**Proof** For any \( X \) in \( \text{Mod } kQ^{\text{op}} \), there is a \( kQ^{\text{op}} \)-module map
\[
\psi : X \to TF_1(X) = F_1(X)/\text{rad } F_1(X)
\]
such that \( \psi(x) = 1 \otimes x + \text{rad } F_1(X) \) for any \( x \in X \). It is routine to check that \( \psi \) is a natural isomorphism. \( \square \)

Let \( \sigma \) be the \( k \)-algebra isomorphism of \( kQ^{\text{op}} \) induced by \( \sigma(w) = (-1)^{\ell(w)} w \) for any path \( w \) in \( Q^{\text{op}} \), where \( \ell(w) \) is the length of \( w \). For any \( kQ^{\text{op}} \)-module \( X \), denote by \( \sigma X \) the twisted module of \( X \). Then \( \sigma X \) is equal to \( X \) as \( k \)-vector spaces, and the action \( \circ \) is given by \( w \circ x = \sigma(w)x \) for \( w \in kQ^{\text{op}}, x \in X \).

The double twist \( \sigma (\sigma X) \) is the same as \( X \). However, the twisted module \( \sigma X \) and the original module \( X \) need not be isomorphic. The following is an example.

**Example 3.5** Let \( k \) be a field and \( Q \) be the following quiver.

\[\begin{array}{ccc}
1 & \xrightarrow{\alpha} & 2 \\
\downarrow{\gamma} & & \downarrow{\beta} \\
3
\end{array}\]

Let \( X \) be the \( kQ \)-module with \( X_1 = X_2 = X_3 = k, X_\alpha = X_\beta = X_\gamma = 1_k \), where \( 1_k \) is the identity map. Then \( \sigma X_1 = \sigma X_2 = \sigma X_3 = k, \sigma X_\alpha = \sigma X_\beta = \sigma X_\gamma = 1_k \). If the characteristic of \( k \) is not equal to 2, then \( X \) and \( \sigma X \) are not isomorphic.
Let $M$ be a differential module. The shift $\Sigma(M)$ of $M$ is equal to $M$ as modules, the differential of $\Sigma(M)$ is the negative of the differential of $M$.

**Lemma 3.6** For any $M, N$ in $C_1^e(\mathcal{P})$, there is an isomorphism

$$\text{Rad}(M, N) \cong \text{Ext}_{kQ^{op}}^1(TM, T\Sigma N).$$

Here, we write $\text{Rad}(M, N) = \text{Rad}(M, N)/\text{Htp}(M, N)$.

**Proof** Note that $\sigma T(M) \cong T\Sigma(M)$. By Lemma 2.3 there is an isomorphism

$$\gamma: \text{Rad}(M, N) \cong \text{Ext}^1_{kQ^{op}}(TM, TN).$$

By Corollary 3.3(3) the image of $\text{Htp}(M, N)$ is $E_0(TM, T\Sigma N)$. Then the statement follows by Lemma 2.4.

The homotopy category $K_1^e(\mathcal{P})$ of differential projective $kQ/J^2$-modules is the stable category of $C_1(\mathcal{P})$. For any $M, N \in C_1(\mathcal{P})$, we have

$$\text{Hom}_{K_1^e(\mathcal{P})}(M, N) = \text{Hom}_{C_1^e(\mathcal{P})}(M, N)/\text{Htp}(M, N).$$

We now restate and prove Theorem A.

**Theorem 3.7** Let $k$ be a field and $Q$ be a finite quiver. There is a functor

$$T: C_1^e(\mathcal{P}) \to \text{Mod}_{kQ^{op}}$$

such that

1. $T$ is full, dense, and detects isomorphisms;
2. $T$ commutes with small coproducts;
3. $T$ vanishes on null-homotopic maps.

Moreover, for any $M$ and $N$ in $C_1^e(\mathcal{P})$, there is an isomorphism

$$\text{Hom}_{K_1^e(\mathcal{P})}(M, N) \cong \text{Hom}_{C_1^e(\mathcal{P})}(M, N)/\text{Htp}(M, N).$$

**Proof**

1. Let $M$ and $N$ be in $C_1^e(\mathcal{P})$ and let $f: M \to N$ be a $kQ^{op}$-module map. Since $M$ is projective, there is a $kQ/J^2$-module map $f': M \to N$ such that $g = T(f')$. Since $g$ is a $kQ^{op}$-module map, by Corollary 3.3(1) $f$ is a differential map. This shows that $T$ is full. By Lemma 3.4 the composite $T \circ F_1$ is isomorphic to the identity functor, then $T$ is dense.

   It remains to show that $T$ detects isomorphisms. Let $f: M \to N$ be a morphism in $C_1^e(\mathcal{P})$ such that $T(f)$ is an isomorphism. Let $g$ be the inverse of $T(f)$.

   Since $N$ is projective, there is a morphism $h: N \to M$ such that $T(h) = g$. Then $T(1_M) = T(h),$ $T(1_N) = T(fh),$ where $1_M$ and $1_N$ are the identity maps. Then both $1_M - hf$ and $1_N - fh$ are radical maps. Since $\text{rad}^2(M) = 0$, $\text{rad}^2(N) = 0$, we have $(1_M - hf)^2 = 0$, $(1_N - fh)^2 = 0$. Then $hf$ and $fh$ are isomorphisms. It follows that $f$ is an isomorphism.

2. For any $M$ in $C_1^e(\mathcal{P})$, the object $T(M)$ is isomorphic to $kQ_0 \otimes_{kQ/J^2} M$ as $k$-vector spaces. It follows that $T$ commutes with small coproducts.

3. By Lemma 3.2 every null-homotopic map is radical. Since $T$ vanishes on radical maps, it follows that $T$ vanishes on null-homotopic maps.
Let $M$ and $N$ be in $C_1^{re}(P)$. Since $T$ is full, there is a short exact sequence

\[ \text{Rad}(M, N) \hookrightarrow \text{Hom}_{C_1^{re}(P)}(M, N) \twoheadrightarrow \text{Hom}_{kQ^{op}}(TM, TN). \]

Note that $\text{Htp}(M, N) \subseteq \text{Rad}(M, N)$. This yields a short exact sequence

\[ \text{Rad}(M, N) \hookrightarrow \text{Hom}_{kQ_k}(P)(M, N) \twoheadrightarrow \text{Hom}_{kQ^{op}}(TM, TN). \]

By Lemma 3.6 we have $\text{Rad}(M, N) \cong \text{Ext}^1_{kQ^{op}}(TM, T\Sigma N)$. Since $kQ_0$ is semisimple, the previous sequence splits. This finishes our proof.

We have the following corollary; compare [24, Corollary 4.10].

**Corollary 3.8** The functor $T$ gives a bijection from the isoclasses of the objects in the category of reduced differential projective $kQ/J^2$-modules to the isoclasses of the objects in the category of $kQ^{op}$-modules which carries indecomposable objects to indecomposable objects.

**Example 3.9** Let $Q$ be the following quiver.

\[ Q: \quad 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \]

The indecomposable $kQ^{op}$-modules are

- $S(1) = k \leftarrow 0 \leftarrow 0$,
- $S(2) = 0 \leftarrow k \leftarrow 0$,
- $S(3) = 0 \leftarrow 0 \leftarrow k$,
- $P(2) = k \leftarrow k \leftarrow 0$,
- $I(2) = 0 \leftarrow k \leftarrow k$,
- $P(3) = k \leftarrow k \leftarrow k$.

By the bijection, the indecomposable objects in $C_1^{re}(P)$ are

- $F_1(S(1)) = k \rightarrow k \rightarrow 0$,
- $F_1(S(2)) = 0 \rightarrow k \rightarrow k$,
- $F_1(S(3)) = 0 \rightarrow 0 \rightarrow k$,
- $F_1(P(2)) = k \rightarrow k^2 \rightarrow k$,
- $F_1(I(2)) = 0 \rightarrow k \rightarrow k^2$,
- $F_1(P(3)) = k \rightarrow k^2 \rightarrow k$.

Here, we omit the differentials of these modules.

## 4 Comparison with Koszul Duality

In this section, we compare our functors with the Koszul duality functors. Then we give an explicit compact generator for the homotopy category of differential projective modules.

Let $k$ be a field and $Q$ be a finite quiver. Then $kQ/J^2$ is a Koszul algebra and its quadratic dual is $kQ^{op}$. We recall the DG Koszul duality for them; see [7, 12].

Let $A$ be the DG algebra $kQ/J^2$ concentrated in degree zero, and let $A^!$ be the graded algebra $kQ^{op}$ with $\deg Q_0 = 0$ and $\deg Q_1 = 1$ with zero differential. For DG modules, there is an adjoint pair

\[ F: \{\text{dg } A^!\text{-modules}\} \rightleftarrows \{\text{dg } A\text{-modules}\}: G \]

where $F(X) = A \otimes_{kQ_0} X$ with differential

\[ d(p \otimes x) = \sum_{\alpha \in Q_1} p\alpha \otimes \alpha^* x + p \otimes d_X(x) \]
for $p \in A$, $x \in X$, and $G(M) = \text{Hom}_{kQ_0}(A^1, M)$ with differential

$$d(f) = d_M \circ f - (-1)^{\deg f} \sum_{\alpha \in Q_1} \alpha f(\alpha^* \cdot -)$$

for homogenous $f \in \text{Hom}_{kQ_0}(A^1, M)$. Here, $\otimes$ and $\text{Hom}$ are the tensor product and the $\text{Hom}$ of DG modules, respectively.

In ungraded case, we consider the twisted differential $kQ^{op}$-modules rather than the differential $kQ^{op}$-modules. Recall the $k$-algebra isomorphism $\sigma$ of $kQ$. A twisted differential $kQ^{op}$-module is a pair $(X, d_X)$, where $X$ is a $kQ^{op}$-module and $d_X : X \to \sigma X$ is a twisted differential $kQ^{op}$-module map such that $\sigma d_X \circ d_X = 0$. If $X, Y$ are twisted differential $kQ^{op}$-modules, a twisted differential $kQ^{op}$-module map $f : X \to Y$ is a $kQ^{op}$-module map $f : X \to Y$ such that $\sigma f \circ d_X = d_Y \circ f$.

Denote by $C_{tw}^1(\text{Mod} \ kQ^{op})$ be the category of twisted differential $kQ^{op}$-modules. Let $X$ be a twisted differential $kQ^{op}$-module. The shift $\Sigma_1 X$ of $X$ is $(\sigma X, -\sigma d_X)$. The homology group $H(X)$ of $X$ is the quotient group $\text{Ker} d_X/\text{Im} \sigma d_X$. Observe that $H(\Sigma_1 X)$ is isomorphic to $\sigma H(X)$.

In ungraded case, there is an adjoint pair

$$F_1 : C_{tw}^1(\text{Mod} kQ^{op}) \rightleftharpoons C_1(\text{Mod} kQ/J^2) : G_1$$

where $F_1(X) = kQ/J^2 \otimes_{kQ_0} X$ with differential

$$d(p \otimes x) = \sum_{\alpha \in Q_1} p\alpha \otimes \alpha^* x + p \otimes d_X(x)$$

for $p \in kQ/J^2$, $x \in X$, and $G_1(M) = \text{Hom}_{kQ_0}(kQ^{op}, M)$ with twisted differential

$$d(f) = d_M \circ f \circ \sigma - \sum_{\alpha \in Q_1} \alpha f(\alpha^*. \sigma(-))$$

for $f \in \text{Hom}_{kQ_0}(kQ^{op}, M)$. The restriction of $F_1$ is used in Lemma 3.4.

**Lemma 4.1** For any $M$ in $C^{re}(P)$, there are isomorphisms

$$T(M) \simeq \text{Hom}_{C_1(P)}(F_1(kQ^{op}), M) \simeq HG_1(M).$$

**Proof** First, by Theorem 3.7 there are isomorphisms

$$\text{Hom}_{C_1(P)}(F_1(kQ^{op}), M) \cong \text{Hom}_{kQ^{op}}(kQ^{op}, TM) \cong TM.$$

Indeed, they are $kQ^{op}$-module isomorphisms. Second, there is an isomorphism

$$\text{Hom}_{kQ/J^2}(F_1(kQ^{op}), M) \cong \text{Hom}_{kQ_0}(kQ^{op}, M) = G_1(M).$$

We have a twisted differential on $\text{Hom}_{kQ/J^2}(F_1(kQ^{op}), M)$. Note that $f$ lies in $\text{Ker} d$ if and only if $f$ is a differential map, and $f$ lies in $\text{Im} \sigma d$ if and only if $f$ is null homotopic. Then we are done by taking homology. 

Let $\mathcal{T}$ be a triangulated category admitting small coproducts. An object $S$ in $\mathcal{T}$ is called compact [18] if for any set $\{T_\lambda\}_{\lambda \in L}$ of objects in $\mathcal{T}$, the natural map

$$\bigoplus_{\lambda \in L} \text{Hom}_\mathcal{T}(S, T_\lambda) \to \text{Hom}_\mathcal{T}(S, \bigoplus_{\lambda \in L} T_\lambda)$$

is an isomorphism.
**Proposition 4.2** Let $M$ be in $C^{re}_1(\mathcal{P})$. Then the following hold.

1. $M$ is finite dimensional if and only if $TM$ is finite dimensional.
2. $M$ is compact in $K_1(\mathcal{P})$ if and only if $TM$ is finitely presented.

**Proof**

(1) Since $M$ is projective and $TM$ is the top of $M$, then $M$ is finite dimensional if and only if $TM$ is finite dimensional.

(2) Suppose that $M$ is compact. Let $\{X_\lambda\}_{\lambda \in L}$ be a set of $kQ^{op}$-modules. Since $T$ is dense by Theorem 3.7, every $X_\lambda$ is isomorphic to $TM_\lambda$ for some reduced differential projective $kQ/J^2$-module $M_\lambda$.

By Theorem 3.7, there is a natural isomorphism

$$\bigoplus_{\lambda \in L} \text{Ext}^i_{kQ^{op}}(TM, X_\lambda) \cong \text{Ext}^i_{kQ^{op}}(TM, \bigoplus_{\lambda \in L} X_\lambda)$$

for $i = 0, 1$. It follows from Lemma 2.5 that $TM$ is finitely presented.

Suppose that $TM$ is finitely presented. Let $\{M_\lambda\}_{\lambda \in L}$ be a set of differential projective $kQ/J^2$-modules. We may assume that every $M_\lambda$ is reduced.

By Lemma 2.5 and Theorem 3.7 there is a natural isomorphism

$$\bigoplus_{\lambda \in L} \text{Hom}_{K_1(\mathcal{P})}(M, M_\lambda) \cong \text{Hom}_{K_1(\mathcal{P})}(M, \bigoplus_{\lambda \in L} M_\lambda).$$

Then $M$ is a compact object in $K_1(\mathcal{P})$. \qed

Denote by $K^{cf}_{1}(\mathcal{P})$ the full subcategory of compact objects and by $K^{fd}_{1}(\mathcal{P})$ the full subcategory of finite dimensional objects.

**Corollary 4.3** Let $Q$ be a finite quiver, then $K^{fd}_{1}(\mathcal{P}) \subseteq K^{cf}_{1}(\mathcal{P})$. The equality holds if and only if $Q$ is acyclic.

**Proof** Since $Q$ is a finite quiver, all finite-dimensional $kQ^{op}$-modules are finitely presented. These two kinds of modules coincide if and only if $Q$ is acyclic. Then the statements follows from Proposition 4.2. \qed

By Theorem 3.7 we have the following; compare [21, Theorem 2].

**Corollary 4.4** The bijection in Corollary 3.8 carries finite-dimensional objects to finite-dimensional objects and carries compact objects to finitely presented objects.

In particular, if $Q$ is the $n$-loop quiver with $n \geq 2$, then the bijection between finite-dimensional objects is given in [19, Theorem 3.6].

A triangulated category $\mathcal{T}$ is said to be compactly generated [18] if $\mathcal{T}$ admits small coproducts, and there exists a set $S$ of objects in $\mathcal{T}$ such that

1. Given $T \in \mathcal{T}$, if $\text{Hom}_\mathcal{T}(\Sigma^n S, T) = 0$ for every $S \in S$ and $n \in \mathbb{Z}$, then $T \cong 0$;
2. Every object $S \in \mathcal{S}$ is compact.

In particular, if $\mathcal{S} = \{S_0\}$, then $S_0$ is called a compact generator for $\mathcal{T}$. 

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Theorem 4.5 The object $F_1(kQ^{\text{op}})$ is a compact generator for $\mathcal{K}_1(\mathcal{P})$.

Proof By Lemma 3.4 $TF_1(kQ^{\text{op}}) \cong kQ^{\text{op}}$, it is finitely presented. By Proposition 4.2(2) we know that $F_1(kQ^{\text{op}})$ is compact.

Suppose $\text{Hom}_{\mathcal{K}_1(\mathcal{P})}(F_1(kQ^{\text{op}}), M) = 0$ for $M \in \mathcal{K}_1(\mathcal{P})$. Then $M = M' \oplus M''$ such that $M'$ is reduced and $M''$ is contractible. We have $T(M') \cong 0$ by Lemma 4.1. Since $M'$ is projective, then $M' = 0$ and thus $M = M''$. It follows that $M \cong 0$ in $\mathcal{K}_1(\mathcal{P})$. Therefore, $F_1(kQ^{\text{op}})$ is a compact generator for $\mathcal{K}_1(\mathcal{P})$.

5 Gorenstein Projective Modules

In this section, we relate differential modules over a ring $\Lambda$ to Gorenstein projective modules over the algebra of dual numbers over $\Lambda$. Then we give a class of non-virtually Gorenstein algebras.

Given a ring $\Lambda$, a complex $P$ of projective $\Lambda$-modules is said to be totally acyclic if $P$ is acyclic and $\text{Hom}_\Lambda(P, T)$ is acyclic for every projective $\Lambda$-module $T$. A $\Lambda$-module $M$ is said to be Gorenstein projective if there exists a totally acyclic complex $P$ of projective $\Lambda$-modules such that $M$ is isomorphic to some coboundary of $P$. Similarly, a complex $I$ of injective $\Lambda$-modules is said to be totally acyclic if $I$ is acyclic and $\text{Hom}_\Lambda(T, I)$ is acyclic for every injective $\Lambda$-module $T$. A $\Lambda$-module $M$ is said to be Gorenstein injective if there exists a totally acyclic complex $I$ of injective $\Lambda$-modules such that $M$ is isomorphic to some cocycle of $I$.

Denote by $\text{Mod} \Lambda$ the category of all $\Lambda$-modules. Let $\text{GProj} \Lambda$ be the full subcategory of Gorenstein projective $\Lambda$-modules and $\text{Proj} \Lambda$ be the full subcategory of projective $\Lambda$-modules. Recall the following facts; see [10, 14] for more details.

1. $\text{GProj} \Lambda$ is a Frobenius category whose projective objects are just $\text{Proj} \Lambda$.
2. $\text{GProj} \Lambda$ is a triangulated category admitting small coproducts.
3. $\Lambda$ is a quasi-Frobenius ring if and only if $\text{GProj} \Lambda = \text{Mod} \Lambda$.
4. If the left global dimension of $\Lambda$ is finite, then $\text{GProj} \Lambda = \text{Proj} \Lambda$.

Let $\Lambda[\varepsilon] = \Lambda[T]/(T^2)$ be the ring of dual numbers over $\Lambda$. Note that differential modules over $\Lambda$ are just modules over $\Lambda[\varepsilon]$. In particular, if $\Lambda$ is an algebra over a field $k$, then $\Lambda[\varepsilon]$ is isomorphic to the tensor product algebra $\Lambda \otimes_k k[\varepsilon]$.

By [24, Theorem 1.1] a differential $\Lambda$-module $(M, d_M)$ is Gorenstein projective if and only if the underlying $\Lambda$-module $M$ is Gorenstein projective. Then we know that every differential projective $\Lambda$-module is Gorenstein projective since every projective module is Gorenstein projective.

An artin algebra $\Lambda$ is called Gorenstein if the injective dimension of $\Lambda$ are finite on both sides [2]. Selfinjective algebras and algebras of finite global dimension are Gorenstein.

An artin algebra $\Lambda$ is called virtually Gorenstein if for a module $M$, the functor $\text{Ext}^i_\Lambda(–, M)$ vanishes on all Gorenstein projective $\Lambda$-modules if and only if the functor $\text{Ext}^i_\Lambda(M, –)$ vanishes on all Gorenstein injective $\Lambda$-modules [8]. It is known that Gorenstein algebras are virtually Gorenstein. There are many virtually Gorenstein algebras which are not Gorenstein.

A Gorenstein projective module $M$ is said to be reduced if $M$ has no nonzero projective direct summands.
We need the following; see [2, Proposition 2.2] and [8, Theorem 8.2].

**Lemma 5.1** Let $\Lambda$ and $\Gamma$ be finite-dimensional algebras over a field $k$.

1. $\Lambda \otimes_k \Gamma$ is selfinjective if and only if $\Lambda$ and $\Gamma$ are selfinjective.
2. $\Lambda \otimes_k \Gamma$ is Gorenstein if and only if $\Lambda$ and $\Gamma$ are Gorenstein.
3. $\Lambda$ is virtually Gorenstein if and only if every reduced and compact object in the stable category of Gorenstein projective $\Lambda$-modules is finite dimensional.

**Proposition 5.2** Let $k$ be a field and $Q$ be a finite connected quiver.

1. If $Q$ is not a basic cycle, then the Gorenstein projective differential $kQ/J^2$-modules coincide with the differential projective $kQ/J^2$-modules.
2. Otherwise, every differential $kQ/J^2$-module is Gorenstein projective.

**Proof** (1) If $Q$ is not a basic cycle, by [20, Theorem 2] every Gorenstein projective modules over $kQ/J^2$ is projective. Then the Gorenstein projective differential $kQ/J^2$-modules are just the differential projective $kQ/J^2$-modules.

(2) If $Q$ is a basic cycle, then $kQ/J^2$ is selfinjective. By Lemma 5.1 $kQ/J^2[\varepsilon]$ is selfinjective. Then every differential $kQ/J^2$-module is Gorenstein projective.

The following contains Theorem B.

**Theorem 5.3** Let $k$ be a field and $Q$ be a finite connected quiver.

1. If $Q$ is an acyclic quiver or a basic cycle, then $kQ/J^2[\varepsilon]$ is Gorenstein.
2. Otherwise, $kQ/J^2[\varepsilon]$ is not virtually Gorenstein.

**Proof** (1) If $Q$ is acyclic, $kQ/J^2$ has finite global dimension. If $Q$ is a basic cycle, $kQ/J^2$ is selfinjective. Then by Lemma 5.1 $kQ/J^2[\varepsilon]$ is Gorenstein.

(2) Since $Q$ is not a basic cycle, by Proposition 5.2(1) the homotopy category of differential projective $kQ/J^2$-modules is exactly the stable category of Gorenstein projective $kQ/J^2[\varepsilon]$-modules. Since $Q$ has oriented cycles, the reduced compact generator in Theorem 4.5 is infinite dimensional. By Lemma 5.1 we infer that $kQ/J^2[\varepsilon]$ is not virtually Gorenstein.

**Remark 5.4** By [20, Theorem 2] every algebra $\Lambda$ with radical square zero is actually virtually Gorenstein. This shows that the tensor product of virtually Gorenstein algebras may be non-virtually Gorenstein.

For a field $k$ of characteristic 2, the algebra $\Lambda[\varepsilon]$ is isomorphic to the group algebra $\Lambda C_2$ of $\Lambda$ over the cyclic group $C_2$ of order 2. Suppose $\Lambda = kQ/J^2$ where $Q$ is a finite connected quiver with oriented cycles and $Q$ is not a basic cycle. Then the group algebra $\Lambda C_2$ is not virtually Gorenstein; compare [6, Proposition 3.1].

**Example 5.5** Let $k$ be a field and $Q, \tilde{Q}$ be the following quivers.

\[
\begin{align*}
Q: & \quad \begin{array}{c}
\xrightarrow{\alpha} \\
1 \xrightarrow{\beta} 2
\end{array} \\
\tilde{Q}: & \quad \begin{array}{c}
\xrightarrow{\epsilon_1} \\
1 \xrightarrow{\beta} 2 \xrightarrow{\epsilon_2}
\end{array}
\end{align*}
\]
Let $I$ be the ideal of $k\tilde{Q}$ generated by
\[\alpha^2, \beta\alpha, \epsilon_1^2, \epsilon_2^2, \alpha\epsilon_1 - \epsilon_1\alpha, \beta\epsilon_1 - \epsilon_2\beta.\]
Then $k\tilde{Q}/I$ and $kQ/J^2[\varepsilon]$ are isomorphic. By Theorem 5.3 the algebra $k\tilde{Q}/I$ is not virtually Gorenstein.

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