Parametric Evaluations of the Rogers Ramanujan Continued Fraction

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Abstract
In this article with the help of the inverse function of the singular moduli we evaluate the Rogers Ramanujan continued fraction and his first derivative.

1 Introductory Definitions and Formulas
For $|q| < 1$, the Rogers Ramanujan continued fraction (RRCF) (see [6]) is defined as

$$R(q) := \frac{q^{1/5}}{1 + \frac{1}{1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \cdots}}}} \quad (1)$$

We also define

$$(a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k) \quad (2)$$

$$f(-q) := \prod_{n=1}^{\infty} (1 - q^n) = (q; q)_\infty \quad (3)$$

Ramanujan give the following relations which are very useful:

$$\frac{1}{R(q)} - 1 - R(q) = \frac{f(-q^{1/5})}{q^{1/5}f(-q^{5})} \quad (4)$$

$$\frac{1}{R^6(q)} - 11 - R^6(q) = \frac{f^6(-q)}{q^6f(-q^5)} \quad (5)$$

From the Theory of Elliptic Functions (see [6],[7],[10]),

$$K(x) = \int_{0}^{\pi/2} \frac{1}{\sqrt{1 - x^2 \sin(t)^2}} dt \quad (6)$$

is the Elliptic integral of the first kind. It is known that the inverse elliptic nome $k = k_r$, $k_r^2 = 1 - k_r^2$ is the solution of the equation

$$\frac{K(k_r)}{K(k)} = \sqrt{r} \quad (7)$$
where $r \in \mathbb{R}^+$. When $r$ is rational then the $k_r$ are algebraic numbers. We can also write the function $f$ using elliptic functions. It holds (see [10]):

$$f(-q)^8 = \frac{2^{8/3}}{\pi^4} q^{-1/3} (k_r)^{2/3} (k'_r)^{8/3} K(k_r)^4$$  \hfill (8)

also holds

$$f(-q^2)^6 = \frac{2k_r k'_r K(k_r)^3}{\pi^3 q^{1/2}}$$  \hfill (9)

From [5] it is known that

$$R'(q) = 1/5 q^{-5/6} f(-q)^4 R(q) \sqrt{R(q)^5 - 11 - R(q)^5}$$  \hfill (10)

Consider now for every $0 < x < 1$ the equation

$$x = k_r,$$

which, have solution

$$r = k^{(-1)}(x)$$  \hfill (11)

Hence for example

$$k^{(-1)}\left(\frac{1}{\sqrt{2}}\right) = 1$$

With the help of $k^{(-1)}$ function we evaluate the Rogers Ramanujan continued fraction.

2 Propostions

The relation between $k_{25r}$ and $k_r$ is (see [6] pg. 280):

$$k_r k_{25r} + k'_r k'_{25r} + 2 \cdot 4^{1/3} (k_r k_{25r} k'_r k'_{25r})^{1/3} = 1$$  \hfill (12)

For to solve equation (12) we give the following

Proposition 1.

The solution of the equation

$$x^6 + x^3(-16 + 10x^2)w + 15x^4w^2 - 20x^3w^3 + 15x^2w^4 + x(10 - 16x^2)w^5 + w^6 = 0$$  \hfill (13)

when we know the $w$ is given by

$$\frac{y^{1/2}}{w^{1/2}} = \frac{w^{1/2}}{x^{1/2}} =$$

$$= \frac{1}{2} \sqrt{4 + \frac{2}{3} \left( \frac{L^{1/6}}{M^{1/6}} - 4 \frac{M^{1/6}}{L^{1/6}} \right)^2 + \frac{1}{2} \sqrt{2} \left( \frac{L^{1/6}}{M^{1/6}} - 4 \frac{M^{1/6}}{L^{1/6}} \right)}$$  \hfill (14)
where
\[ w = \sqrt{\frac{L(18 + L)}{6(64 + 3L)}} < 1 \] (15)

and
\[ M = \frac{18 + L}{64 + 3L} \] (16)

If happens \( x = k_r \) and \( y = k_{25r} \), then \( r = k^{(-1)}(x) \) and \( w^2 = k_{25r}k_r \), \( (w')^2 = k'_{25r}k'_r \).

**Proof.**
The relation (14) can be found using Mathematica. See also [6].

**Proposition 2.**
If \( q = e^{-\pi \sqrt{x}} \) and
\[ a = a_r = \left( \frac{k'_r}{k'_{25r}} \right)^2 \sqrt{\frac{k_r}{k_{25r}}} M_5(r)^{-3} \] (17)

Then
\[ a_r = R(q)^{-5} - 11 - R^5(q) \] (18)

Where \( M_5(r) \) is root of: \((5x - 1)^5(1 - x) = 256(k_r)^2(k'_r)^2x \).

**Proof.**
Suppose that \( N = n^2\mu \), where \( n \) is positive integer and \( \mu \) is positive real then it holds that
\[ K[n^2 \mu] = M_n(\mu)K[\mu] \] (19)

Where \( K[\mu] = K(k_\mu) \)
The following formula for \( M_5(r) \) is known
\[ (5M_5(r) - 1)^5(1 - M_5(r)) = 256(k_r)^2(k'_r)^2M_5(r) \] (20)

Thus if we use (5) and (8) and the above consequence of the Theory of Elliptic Functions, we get:
\[ R^{-5}(q) - 11 - R^5(q) = \frac{f^5(-q)}{qf^5(-q^r)} = a = a_r \]

See also [4],[5].

### 3 The Main Theorem

From Proposition 2 and relation \( w^2 = k_{25r}k_r \) we get
\[ w^5 - k_r^2w = \frac{k'_r(k_r^2 - 1)}{a_rM_5(r)^3} \] (21)
Combining (13) and (21), we get:

\[-10k_r^4 + 26k_r^6 + a_r M_5(r)^3 k_r^6 - 16k_r^8] + [-k_r^3 - 6a_r M_5(r)^3 k_r^3 + k_r^5 - 6a_r M_5(r)^3 k_r^5]w +

+ [a_r M_5(r)^3 k_r^2 + 15a_r M_5(r)k_r^4]w^2 - 20a_r M_5(r)^3 k_r^3 w^3 + 15a_r M_5(r)^3 k_r^2 w^4 = 0

(22)

Solving with respect to \(a_r M_5(r)^3\), we get

\[a_r M_5(r)^3 = \frac{16k_r^6 - 26k_r^4 - wk_r^3 + 10k_r^2 + wk_r}{k_r^4 - 6k_r^2 w - 20k_r^2 w + 15w^2 k_r^2 - 6k_r w + 15w^4 + w^2}

(23)

Also we have

\[\frac{K(k_{25r})}{K(k_r)} = M_5(r) = \frac{1}{m} = \left(\frac{k_{25r}}{k_r} + \frac{\sqrt{K(k_{25r})}}{k_r'} - \frac{\sqrt{k_{25r} k_r'}}{k_r k_r'}\right)^{-1}

= \left(\frac{w}{k_r} + \frac{w'}{k_r'} - \frac{ww'}{k_r k_r'}\right)^{-1}

The above equalities follow from (\[6\] pg. 280 Entry 13-xii) and the definition of \(w\). Note that \(m\) is the multiplier.

Hence for given \(0 < w < 1\) we find \(L \in \mathbb{R}\) and we get the following parametric evaluation for the Rogers Ramanujan continued fraction

\[R \left(e^{-\pi \sqrt{r(L)}}\right)^5 - 11 - R \left(e^{-\pi \sqrt{r(L)}}\right)^5 = a_r = \frac{16k_r^6 - 26k_r^4 - wk_r^3 + 10k_r^2 + wk_r}{k_r^4 - 6k_r^2 w - 20k_r^2 w + 15w^2 k_r^2 - 6k_r w + 15w^4 + w^2}\left(\frac{w}{k_r} + \frac{w'}{k_r'} - \frac{ww'}{k_r k_r'}\right)^3

(24)

Thus for a given \(w\) we find \(L\) and \(M\) from (15) and (16). Setting the values of \(M, L, w\) in (14) we get the values of \(x\) and \(y\) (see Proposition 1). Hence from (24) if we find \(k^{(-1)}(x) = r\) we know \(R(e^{-\pi \sqrt{r}})\). The clearer result is:

**Main Theorem.**

When \(w\) is a given real number, we can find \(x\) from equation (14). Then for the Rogers Ramanujan continued fraction holds

\[R \left(e^{-\pi \sqrt{k^{(-1)}(x)}}\right)^5 - 11 - R \left(e^{-\pi \sqrt{k^{(-1)}(x)}}\right)^5 = a_r = \frac{16x^6 - 26x^4 - wx^3 + 10x^2 + wx}{x^4 - 6x^3 w - 20xw^3 + 15w^2x^2 - 6xw + 15w^4 + w^2 \times}

\times \left(\frac{w}{x} + \frac{w'}{\sqrt{1 - x^2}} - \frac{ww'}{x(1 - x^2)}\right)^3

(25)
Note. In the case of $x = k_r$, then $k^{-1}(x) = r$ and we have the classical evaluation with $k_{25r}$ (see [12]).

Theorem 1. (The first derivative)

$$R'(e^{-\pi \sqrt{k^{-1}(x)}}) = \frac{2^{4/3}x^{1/2}(1-x^2)}{5w^{1/6}w'^{2/3}} \left( \frac{w}{x} + \frac{w'}{\sqrt{1-x^2}} - \frac{ww'}{x\sqrt{1-x^2}} \right)^{1/2} \times$$

$$\times R\left(e^{-\pi \sqrt{k^{-1}(x)}}\right) K^2(x)e^{\pi \sqrt{k^{-1}(x)}} \frac{\pi^2}{\pi^2}$$

(26)

Proof.

Combining (8) and (10) and Proposition 2 we get the proof.

We see how the function $k^{-1}(x)$ plays the same role in other continued fractions. Here we consider also the Ramanujan’s Cubic fraction (see [4]), which is completely solvable using $k_r$.

Define the function:

$$G(x) = \frac{x}{\sqrt{2\sqrt{x} - 3x + 2x^{3/2} - 2\sqrt{x}\sqrt{1-3\sqrt{x} + 4x - 3x^{3/2} + x^2}}}
(27)$$

Set for a given $0 < w_3 < 1$

$$x = G(w_3)
(28)$$

Then as in Main Theorem, for the Cubic continued fraction $V(q)$, holds (see [4]):

$$t = V\left(e^{-\pi \sqrt{k^{-1}(x)}}\right) = \frac{(1-x^2)^{1/3}w_3^{1/4}}{2^{1/3}x^{1/3}(1-\sqrt{w_3})}
(29)$$

Observe here that again we only have to know $k^{-1}(x)$.

If happens $x = k_r$, for a certain $r$, then

$$k_{9r} = \frac{w_3}{k_r}
(30)$$

and if we set

$$T = \sqrt{1 - 8V(q)^3},
(31)$$

then holds

$$(k_r)^2 = x^2 = \frac{(1-T)(3+T)^3}{(1+T)(3-T)^3}
(32)$$

which is solvable always in radicals quartic equation. When we know $w_3$ we can find $k_r$ from $x = G(w_3)$ and hence $t$.

The inverse also holds: If we know $t = V(q)$ we can find $T$ and hence $k_r = x$. 

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The \( w_3 \) can be found by the degree 3 modular equation which is always solvable in radicals:

\[
\sqrt{k_r k_r'} + \sqrt{k_9 k_9'} = 1
\]

Let now

\[
V(q) = z \Leftrightarrow q = V^{(-1)}(z)
\]  \( \text{ (33)} \)

if

\[
V_i(t) := \left[ \frac{1 - \sqrt{1 - 8t^4}}{1 + \sqrt{1 - 8t^4}} \right]^3 \left( \frac{3 + \sqrt{1 - 8t^4}}{3 - \sqrt{1 - 8t^4}} \right)^3
\]  \( \text{ (34)} \)

then

\[
V_i(V(e^{-\pi\sqrt{q}})) = k_x
\]  \( \text{ (35)} \)

or

\[
V(e^{-\pi\sqrt{q}}) = V_i^{(-1)}(k_r)
\]

\[
V(e^{-\pi\sqrt{k^{(-1)}(x)}}) = V_i^{(-1)}(x)
\]

or

\[
e^{-\pi\sqrt{k^{(-1)}(x)}} = V^{(-1)}(V_i^{(-1)}(x)) = (V_i \circ V)^{(-1)}(x)
\]

and

\[
k^{(-1)}(V_i(V(q))) = \frac{1}{\pi^2} \log(q)^2 = r
\]  \( \text{ (36)} \)

Setting now values into (36) we get values for \( k^{(-1)}(\cdot) \). The function \( V_i(\cdot) \) is an algebraic function.

4 Some Evaluations of the Rogers Ramanujan Continued Fraction

Note that if \( x = k_r, r \in \mathbb{Q} \) then we have the classical evaluations with \( k_r \) and \( k_{25r} \).

Evaluations.

1) \[
R(e^{-2\pi}) = -\frac{1}{2} - \frac{\sqrt{5}}{2} + \sqrt{\frac{5 + \sqrt{5}}{2}}
\]

\[
R'(e^{-2\pi}) = 8 \sqrt{\frac{2}{5}} \left( 9 + 5\sqrt{5} - 2\sqrt{50 + 22\sqrt{5}} \right) \frac{e^{2\pi}}{\pi^3 \Gamma \left( \frac{5}{4} \right)}^4
\]

2) Assume that \( x = \frac{1}{\sqrt{2}} \), hence \( k^{(-1)} \left( \frac{1}{\sqrt{2}} \right) = 1 \). From (16) which for this \( x \) can be solved in radicals, with respect to \( w \), we find

\[
w = \sqrt{\frac{2}{5}} \left( \sqrt{5} - 1 \right) - \frac{1}{2} \sqrt{7\sqrt{5} - 15}
\]
Hence from
\[ w' = \sqrt{1 - \frac{w^4}{x^2}} \sqrt{1 - x^2} \]
we get
\[ w' = \left( \frac{1 + 21\sqrt{-30 + 14\sqrt{5}} - 9\sqrt{-150 + 70\sqrt{5}}}{\sqrt{2}} \right)^{1/4} \]
Setting these values to (25) we get the value of \( a_r \) and then \( R(q) \) in radicals.

The result is
\[ R(e^{-\pi})^{-5} - 11 - R(e^{-\pi})^5 = -\frac{1}{8} \left( 3 + \sqrt{5} - \sqrt{-30 + 14\sqrt{5}} \right) \left[ 1 - \sqrt{5} + \sqrt{-30 + 14\sqrt{5}} + 
+ 2^{3/8} \left( -3 + \sqrt{5} - \sqrt{-30 + 14\sqrt{5}} \right) \left( 1 + 21\sqrt{-30 + 14\sqrt{5}} - 9\sqrt{-150 + 70\sqrt{5}} \right)^{1/4} \right]^3 \times 
\times [\sqrt{-1574 + 704\sqrt{5} - 655\sqrt{-30 + 14\sqrt{5}} + 293\sqrt{-150 + 70\sqrt{5}}]^{-1} \]

3) Set \( w = 1/64 \) and \( a = 1359863889, b = 36855 \), then

\[ x = 9 (\sqrt{a} + b)^{5/6} \left[ 491526^{1/3} (\sqrt{a} + b)^{1/6} - 960 (\sqrt{a} + b)^{5/6} + 2 \cdot 6^{2/3} (\sqrt{a} + b)^{3/2} - 
- 2 \cdot 6^{5/6} (\sqrt{a} + b)^{1/6} \sqrt{-86980957248 + 36855 \cdot 2^{2/3} 1^{1/6} \sqrt{453287963} \cdot (36855 + 
+ \sqrt{a})^{2/3} - 2358720\sqrt{\sqrt{a} + 150958080} \cdot 6^{1/3} (\sqrt{a} + b)^{1/3} + 
+ 4096 \cdot 2^{1/3} 5^{3/6} \sqrt{453287963} (\sqrt{a} + b)^{1/3} + 453025819 \cdot 6^{2/3} (\sqrt{a} + b)^{2/3}] + 
+ 384 \cdot 2^{2/3} 3^{1/6} \sqrt{-2358720 - 64\sqrt{\sqrt{a} + 8192} \cdot 6^{1/3} (\sqrt{a} + b)^{1/3} + 
+ 12285 \cdot 6^{2/3} (\sqrt{a} + b)^{2/3} + 2^{2/3} 3^{1/6} \sqrt{453287963} (\sqrt{a} + b)^{2/3}]^{-1} \]

4) For
\[ w = \sqrt{\frac{277}{108} + \frac{13\sqrt{385}}{108}} \]
we get
\[ x = \sqrt{\frac{277}{12} + \frac{13\sqrt{385}}{12}} \]

Hence
\[ R \left( \exp \left[ -\pi \cdot k^{(-1)} \left( \sqrt{\frac{277}{12} + \frac{13\sqrt{385}}{12}} \right)^{1/2} \right] \right) = \]
\[
= \left( -\frac{8071}{18} + \frac{1075\sqrt{55}}{18} + \frac{1}{18}\sqrt{5(25740148 - 3470530\sqrt{55})} \right)^{1/5}
\]

5) Set \( q = e^{-\pi \sqrt{r_0}} \), then from

\[ V(e^{-\pi \sqrt{r_0}}) = V_i^{(-1)}(k_{r_0}) = V_0 \]

and from

\[ V(q^{1/3}) = \sqrt[3]{V(q)^2} - V(q) + 2V(q)^2 \]

We can evaluate all

\[ V(q_0(n)) = b_0(n) = \text{Algebraic function of } r_0 \]

where

\[ q_0(n) = e^{-\pi \sqrt{r_0}/3^n} \]

and

\[ V_i(V(q_0(n))) = V_i(b_0(n)) = k_{r_0/9^n} \]

hence

\[ k^{(-1)}(V_i(b_0(n))) = \frac{r_0}{g_0} \]

An example for \( r_0 = 2 \) is

\[ V(e^{-\pi \sqrt{2}}) = -1 + \sqrt{\frac{3}{2}} \]

\[ V(e^{-\pi \sqrt{2}/3}) = \frac{1}{2^{1/3}} \left(-1 + \sqrt[3]{\frac{3}{2}}\right)^{1/3} \]

\[ V(e^{-\pi \sqrt{2}/9}) = \rho_3^{1/3} \]

Where \( \rho_3 \) can be evaluated in radicals but for simplicity we give the polynomial form.

\[ -1 - 72x - 6408x^2 + 50048x^3 + 51264x^4 - 4608x^5 + 512x^6 = 0 \]

Then respectively we get the values

\[ k^{(-1)} \left(-49 + 35\sqrt{2} + 4\sqrt[3]{3(99 - 70\sqrt{2})}\right) = 2/9 \] \( \cdots \) \( (37) \)

\[ k^{(-1)} \left(V_i(\rho_3^{1/3})\right) = 2/81 \] \( \cdots \) \( (38) \)
Hence
\[ k^{(-1)} \left( V_i(b_0(n)) \right) = r_0/g^a \] (39)
and
\[ R \left( e^{-\pi \sqrt{r_0/3^n}} \right)^5 - 11 - R \left( e^{-\pi \sqrt{r_0/3^n}} \right)^5 = \]
\[ = \frac{16x_n^6 - 26x_n^4 - w_nx_n^3 + 10x_n^2 + w_nx_n}{x_n^4 - 6x_n^3w_n - 20x_nw_n^3 + 15w_n^2x_n^2 - 6x_nw_n + 15w_n^4 + w_n^2} \times \]
\[ \times \frac{w_n + w'_n}{x_n} - \frac{w_nw'_n}{x_n \sqrt{1 - x_n^2}} \] (40)

Where \( x_n = V_i(b_0(n)) \) is known. The \( w_n \) are given from (13) (in this case we don’t find a way to evaluate \( w_n \) in radicals, but as a solution of (13)).

6) Set now
\[ w_0 = -64 + a + \sqrt{4096 + a(88 + a)} \]
\[ \frac{6\sqrt{a}}{6\sqrt{a}} \]
then
\[ x_0 = \frac{-64 + a + \sqrt{4096 + a(88 + a)}}{\sqrt{6} \left( -4 + \sqrt{-2 + \frac{16}{a^{1/3}} + a^{1/3}a^{1/3} + a^{1/3}} \right) a^{1/6}} \]
\[ R \left( e^{-\pi \sqrt{k^{(-1)}(x_0)}} \right)^5 - 11 - R \left( e^{-\pi \sqrt{k^{(-1)}(x_0)}} \right)^5 := A(a) \]

where the \( A(a) \) is a known algebraic function of \( a \) and can calculated from the Main Theorem. Setting arbitrary real values to \( a \) we get algebraic evaluations of the RRCCF as in evaluation 4.

If we set
\[ g(x) := \frac{-64 + x + \sqrt{4096 + x(88 + x)}}{\sqrt{6} \left( -4 + x^{1/6} \sqrt{-2 + \frac{16}{x^{1/3}} + x^{1/3} + x^{1/3}} \right) x^{1/6}} \]
and if we manage to write \( k_r \) in the form \( g(a_r) \) for a certain \( a_r \) i.e. \( V_i(V(e^{-\pi \sqrt{r}})) = k_r = g(a_r) \), then
\[ R \left( e^{-\pi \sqrt{r}} \right)^5 - 11 - R \left( e^{-\pi \sqrt{r}} \right)^5 = A(a_r) = A \left( g^{(-1)}(k_r) \right) \]
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