DISTRIBUTED ESTIMATION FOR ADAPTIVE NETWORKS BASED ON SERIAL-INSPIRED DIFFUSION

Cornelius T. Healy and Rodrigo C. de Lamare

CETUC - Pontifical Catholic University of Rio de Janeiro - PUC-RJ
Rio de Janeiro, Brazil
Email: cthealy@cetuc.puc-rio.br, delamare@cetuc.puc-rio.br

ABSTRACT
Distributed estimation and processing in networks modeled by graphs have received a great deal of interest recently, due to the benefits of decentralised processing in terms of performance and robustness to communications link failure between nodes of the network. Diffusion-based algorithms have been demonstrated to be among the most effective for distributed signal processing problems, through the combination of local node estimate updates and sharing of information with neighbour nodes through diffusion. In this work, we develop a serial-inspired approach based on message-passing strategies that provides a significant improvement in performance over prior art. The concept of serial processing in the graph has been successfully applied in sum-product based algorithms and here provides inspiration for an algorithm which makes use of the most up-to-date information in the graph in combination with the diffusion approach to offer improved performance.

Index Terms— Diffusion networks, wireless sensor networks, distributed processing.

1. INTRODUCTION
Distributed signal processing is an important tool for problems which may be modeled by a graph of nodes working to estimate a parameter of interest as it allows computations to be carried locally at the individual nodes, avoiding the need for a centralised processing unit and thus offering robustness to scenarios where the communication links to that central node are subject to channel effects. To achieve this each node makes use of its local observations in combination with the estimates produced at neighbour nodes to produce an improved estimate of the parameters of interest. This improved estimate is then shared with all neighbours of the node, leading to propagation of the information through the network.

The distributed estimation problem has been considered in terms of incremental [1], consensus [2] and diffusion [3] [4] [5] strategies. The incremental strategy in general demands a computationally costly operation for identifying a path through the graph nodes upon which to operate, while the diffusion based strategies have been demonstrated to be superior to those based on consensus in terms of convergence, performance and stability [6].

Recent work on the diffusion strategies has included sparsity-aware approaches of [7] [8] [9] [10] [11] [12] [13]., which exploit the knowledge that the parameter vector to be estimated may be sparse. Related work on improving the combiners in the information diffusion stage of the algorithm has been reported in [14] [15]. The effect of the network topology on the diffusion strategies and how it may be exploited has been considered in [16] [17] [18] [19] [20]. A number of studies have considered imperfect communications links between nodes in the network [21] [22]. Significant work has also been carried out on analysis of the diffusion strategies [23] [24].

In this paper, the schedule of node estimate updates is considered as a source of performance improvement. This is motivated by the observation that such an approach in the case of the sum-product algorithm operating on a bipartite graph, as for the decoding of LDPC codes, offers significantly faster convergence at almost no cost in terms of additional complexity. This approach was termed serial, shuffled or layered scheduling in the literature [25] [26]. Further improvements were found through more advanced update schedules [27] [28] [29] [30] [31]. Effectively, after each individual node update, the newly updated messages are made available to the neighbour nodes of the updated node, ensuring those neighbours compute their own updates with more up-to-date and accurate information. As the diffusion approach is based on the sharing of information with neighbour nodes in the graph of the network, this concept of serialisation translates well. In particular, we develop a serial-inspired (SI) least-mean square (LMS), which we denote SI-LMS and can exploit the schedule of node updates to obtain improved performance. In the proposed SI-LMS algorithm, serialisation is introduced through the inclusion of an additional diffusion combination which has access to the most recently updated estimates in the graph. The proposed SI-LMS algorithm offers significant improvements in convergence speed, as is demonstrated by the simulation study provided in this paper.

This paper is organized as follows. Section 2 provides the problem statement and introduces the diffusion strategy for system identification using the LMS algorithm. In Section 3, the proposed algorithm is developed and described in detail, along with pseudocode representation. Section 4 provides the numerical simulation results, and Section 5 concludes the paper.

Notation: Throughout this paper, lowercase letters such as $x$ indicate scalars, lowercase boldface letters such as $x$ denote column vectors and uppercase boldface letters such as $A$ denote matrices. The superscript $i$ denotes that $A^{(i)}$ is the realization of $A$ at the time index $i$, likewise for scalars as in $d^{(i)}$. Subscripts are used to identify the node or nodes in the graph with which a value is associated.

2. PROBLEM STATEMENT AND THE DIFFUSION STRATEGY
Consider a network modeled by a graph with $N$ nodes as depicted in Fig. 1. The value $d^{(i)}_k$ is the scalar observation at time instant $i$ for the node $k$ in the graph, and the observation is related to the input
signal \(x_k^{(i)}\) by
\[
d_k^{(i)} = \omega_0^H x_k^{(i)} + n_k^{(i)}, \quad i = 1, 2, \ldots, N, \tag{1}
\]
where the input signal vector \(x_k^{(i)}\) for node \(k\) at time index \(i\) is an \(M \times 1\) vector. The value \(n_k^{(i)}\) is the noise sample at node \(k\) and time index \(i\) and has zero mean and variance \(\sigma_{n,k}^2\). The goal of the distributed estimation problem is to estimate the value of \(\omega_0\) based on the knowledge at the nodes in the network of the observations \(d_k^{(i)}\), the input signal vectors \(x_k^{(i)}\) and the relation in (1) through use of that local knowledge and the ability to share information with neighbours in the network graph.

The diffusion strategy for distributed estimation involves a process of local adaptation with the information available using for example the LMS estimate update, followed by information sharing with neighbour nodes involving a weighted sum of the estimates across the neighbourhood of each node. This process leads to diffusion of information through the fully connected graph. The adaptation and combination steps of the diffusion strategy can be performed in either order, leading to one of the adapt—then—combine (ATC) diffusion strategy and in the other the combine—then—adapt (CTA) diffusion strategy [3]. The two steps of ATC diffusion are described by
\[
\begin{align*}
\psi_k^{(i)} &= \omega_k^{(i-1)} + \mu_k x_k^{(i)} (d_k^{(i)} - \omega_k^{(i-1)} H x_k^{(i)}), \\
\omega_k^{(i)} &= \sum_{l \in N_k} c_{kl} \psi_l^{(i)},
\end{align*}
\tag{2}
\]
where the values \(c_{kl}\) are known as the combination coefficients and provide the weighting in the combination step of the respective algorithms and \(\omega_k\) represents the parameter estimator. They are related to the topology of the graph, being nonzero only if node \(k\) and node \(l\) are neighbours and additionally must satisfy the constraint:
\[
\sum_l c_{kl} = 1, \quad l \in N_k \forall k. \tag{3}
\]
There are a number of rules specifying the combination coefficients to be found in the literature, including the uniform, Metropolis, relative degree and Laplacian rules. The Metropolis combiner is given by:
\[
\begin{align*}
c_{kl} &= \frac{1}{\max(1, N_k/N_l)}, & \text{if } l \in N_k, \ l \neq k \\
c_{kl} &= 1 - \frac{1}{\sum_{l \in N_k/k} c_{kl}}, & \text{if } l \in N_k, \ l = k \\
c_{kl} &= 0, & \text{if } l \notin N_k
\end{align*}
\tag{4}
\]
Fig. 2 provides the block diagram for the ATC diffusion algorithm which implements (2) - (4).

In this section, the proposed SI-LMS algorithm is introduced. In the case of sum-product type algorithms operating on bipartite graphs, the standard message update schedule is to activate all nodes of one type and then to activate all nodes of the other type. It was demonstrated that improvements in error rate convergence of the algorithms may be found at no increased computational complexity if the nodes in the graph are updated in a serial fashion, also termed shuffled or layered schedule [25] [26]. This improvement in convergence behaviour is derived from the fact that the most recently updated messages in the graph may be used to improve the next updates within an iteration. This observation has motivated the investigation of the diffusion LMS algorithm and resulted in the development of the algorithm proposed in this paper. Essentially, the process of information diffusion is used to improve the LMS adaptation through the use of new estimates at neighbour nodes as soon as they are available.

The MSE cost function at node \(k\) takes the form
\[
J_k(\omega) = E[|d_k^{(i)} - \omega^H x_k^{(i)}|^2], \tag{6}
\]
which through expansion and rearrangement results in
\[
J_k(\omega) = J_k_{\min} + ||\omega - \omega_0||_{R_{k,\omega}}^2, \tag{7}
\]
where \(J_{k,\min}\) is the value of \(J_k(\omega)\) evaluated at \(\omega = \omega_0\). The local cost function at node \(k\) when information sharing with neighbours is allowed becomes:
\[
J_k^{local}(\omega) = \sum_{l \in N_k} a_{l,k} J_l(\omega). \tag{8}
\]

The global cost function is simply the sum across all nodes \(l\) of this local cost function. Rewritten in terms of the node of interest and its neighbours, we have the global function from the perspective of node \(k\):
\[
J_k^{global}(\omega) = J_k^{local}(\omega) + \sum_{l \in N_k \setminus \{k\}} J_l^{local}(\omega). \tag{9}
\]
Applying the steepest-descent method we arrive at the recursion for updating the parameters of the estimator:
\[
\omega_k^{(i)} = \omega_k^{(i-1)} - \mu_k \nabla \omega J_k^{global}(\omega_k^{(i-1)}), \tag{10}
\]
where \(\nabla \omega J_k^{global}(\omega_k^{(i-1)})\) is the gradient of \(J_k^{global}(\omega)\) with respect to \(\omega\) evaluated at \(\omega_k^{(i-1)}\). Through the use of the expanded versions of (6) and through a number of approximations detailed in [4] [32],
the update recursion may be reformulated as:
\[
\omega_k^{(i)} = \omega_k^{(i-1)} + \mu_k \sum_{l \in N_k} a_{i,l,k} \left( r_{d,x,l} - R_{d,x,l} \omega_k^{(i-1)} \right) \\
+ \mu_k \sum_{l \in N_k \setminus \{k\}} b_{i,l,k} (\omega_l - \omega_k^{(i-1)}),
\]
where \( R_{d,x,l} = \mathbb{E}[x_k^{(i)} x_k^{(i)H}] \) and \( r_{d,x,l} = \mathbb{E}[d_k^{(i)} x_k^{(i)}] \). In (11) the previous estimate is corrected by a filter adaptation term and an information diffusion term. In the development of the ATC and CTA algorithms, the two correction terms are applied successively. In the proposed algorithm, the information diffusion term will be applied both before and after the adaptation of the estimator, and as in the development of those algorithms the best available estimate will be used to substitute for both \( \omega_0 \) and \( \omega_k^{(i-1)} \) in (11). In the proposed algorithm these best estimates are improved upon the previously presented works through the observation that the estimates which have been updated at neighbour nodes are available for use immediately, and so the first diffusion correction term is applied in a serial fashion. In particular, we employ instantaneous estimates of \( R_{d,x,l} \) and \( r_{d,x,l} \) to obtain the recursion for the proposed SI-LMS algorithm:
\[
\omega_k^{(i)} = \omega_k^{(i-1)} + \mu_k \sum_{l \in N_k} a_{i,l,k} x_k^{(i)} \left[ d_k^{(i)} - \omega_k^{(i-1)H} x_k^{(i)} \right] \\
+ \mu_k \sum_{l \in N_k \setminus \{k\}} b_{i,l,k} (\omega_l - \omega_k^{(i-1)}),
\]
Given the previous development, the steps of the proposed SI-LMS algorithm are as follows:
1. Combine prior and new estimates available from neighbour nodes, including the node of interest, by weighted sum.
2. Adapt the parameters according to the chosen rule, using local observations at the node and combined estimate from the first step. Make new estimate available to neighbour nodes.
3. Combine estimates available from neighbourhood.

The SI-LMS algorithm is presented as
\[
\psi_k^{(i)} = c_{k,i,k} \omega_k^{(i-1)} + \sum_{l \in N_k, l > k} c_{k,l} \omega_l^{(i-1)} + \sum_{m \in N_k, m < k} c_{k,m} \phi_m^{(i)},
\]
\[
\phi_k^{(i)} = \psi_k^{(i)} + \mu_k x_k^{(i)} \left[ d_k^{(i)} - \psi_k^{(i)} x_k^{(i)} \right]^H, \tag{13}
\]
\[
\omega_p^{(i)} = \sum_{q \in N_p} a_{p,q} \phi_q^{(i)}, \tag{15}
\]

Fig. 3 provides a block diagram of the proposed SI-LMS algorithm, with blocks for the initial serial information diffusion, the adaptation for the estimator and the final diffusion combination, respectively. In Alg. ?? the pseudocode for the proposed SI-LMS algorithm is provided. This gives the details of the algorithm.

3.1. Computational Cost and Bandwidth Requirements of the Proposed SI-LMS Algorithm

The proposed SI-LMS algorithm provides improvements in convergence speed through the use of an additional combination of new and old estimates prior to the LMS estimate adaptation. This additional weighted sum prior to the adaptation step (line 4 in Algorithm ??) comprises the only additional cost of the proposed algorithm. Thus, in terms of complexity per node, the proposed algorithm costs

\[\bullet |N_k| \text{ extra multiplications} \]
\[\bullet |N_k| \text{ extra additions} \]

With the complexity cost across the network being \( N \) times each of these. Note that in addition to the cost in terms of increased complexity, the proposed SI-LMS algorithm requires that the nodes in the graph share their updated estimates with their neighbours as soon as they are produced by the LMS adaptation. Thus the SI-LMS algorithm also incurs an extra communications cost of

\[\bullet N(|N_k| - 1) \text{ transmissions of a vector of dimension } M \times 1\]
when compared to the ATC-LMS diffusion algorithm \[3\].

4. SIMULATION RESULTS

In this section, the simulation study for the proposed algorithm is presented. Its performance, in terms of mean-square error (MSE), is compared to the diffusion ATC algorithm \[3\]. Fig. 4 provides the network graph topology, showing the network considered has \( N = 20 \) nodes. We adopted the Metropolis combining rule \[12\]. The unknown parameter vector to be estimated has length \( M = 5 \). Two cases are considered for the input signal, one with the signal variances equal at all nodes in the network, another with varying signal variances. The noise of \( i \) is modeled by white circular Gaussian random variables with zero mean, with signal variances that are arbitrary. Two cases are considered, the first in which the signal variances are the same across all nodes in the network, and another where they are allowed to vary. The variances are provided in Fig. 5. The variances of Fig. 5(a) and 5(b) correspond to the simulation environment for the results of Fig. 6 while the variances of Fig. 5(c) and 5(d) correspond to the simulation environment for the results of Fig. 7. The step size at all nodes in the network is \( \mu_k = 0.01 \) for both Fig. 6 and Fig. 7. Additional results for the scenario with different variances at the nodes in the network are provided for a larger step size of \( \mu_k = 0.05 \) in Fig. 8. The results provided are averaged over 100 independent runs.

Figs. 6 and 7 demonstrate that the proposed SI-LMS algorithm outperforms the standard-form ACT-LMS algorithm in speed of convergence, with approximately a 40% reduction in the number of iterations required to converge. Fig. 8 shows that the performance improvements of the proposed SI-LMS algorithm are consistent.

5. CONCLUSION

In this paper, a diffusion-based SI-LMS algorithm has been presented, which exploited the most recent estimates available in the network graph to improve the convergence of the estimates throughout the network. This was achieved through the inclusion of an additional information diffusion step, which is carried out in a serial
Fig. 4. The topology of the network for the results of Figs. 5 to 7.

Fig. 5. The details of signal and noise power at the nodes in the network of Fig. 4 considered in the results of Figs. 6 and 7 respectively.

Fig. 6. The network MSE of the network with topology of Fig. 4 and the signal and noise variance parameters provided in of Fig. 5(a) and 5(b).

Fig. 7. The network MSE of the network with topology of Fig. 4 and the signal and noise variance parameters provided in of Fig. 5(c) and 5(d).

Fig. 8. The network MSE of the network with topology of Fig. 4 and the variance parameters provided in of Fig. 5(c) and 5(d) for the case when a larger step size is used in the adaptive algorithms.

A discussion of the costs of the proposed SI-LMS algorithm in terms of increased computation required at the nodes in the manner. A discussion of the costs of the proposed SI-LMS algorithm in terms of increased computation required at the nodes in the graph and additional necessary communication of estimates required for information diffusion was provided, demonstrating that the proposed algorithm is not prohibitively costly considering the benefits offered. Numerical results justify the proposed SI-LMS algorithm and illustrate its performance advantages.

6. REFERENCES

[1] D. P. Bertsekas, “A new class of incremental gradient methods for least squares problems,” SIAM J. on Optimization, vol. 7, no. 4, pp. 913–926, Apr. 1997.

[2] J.N. Tsitsiklis and M. Athans, “Convergence and asymptotic agreement in distributed decision problems,” IEEE Trans. Autom. Control, vol. 29, no. 1, pp. 42–50, Jan 1984.

[3] C. G. Lopes and A. H. Sayed, “Diffusion least-mean squares over adaptive networks: Formulation and performance analysis,” IEEE Trans. Signal Process., vol. 56, no. 7, pp. 3122–3136, July 2008.

[4] F. S. Cattivelli and A. H. Sayed, “Diffusion lms strategies for
distributed estimation,” *IEEE Trans. Signal Process.*, vol. 58, no. 3, pp. 1035–1048, Mar. 2010.

[5] J. Chen and A. H. Sayed, “Diffusion adaptation strategies for distributed optimization and learning over networks,” *IEEE Trans. Signal Process.*, vol. 60, no. 8, pp. 4289–4305, Aug 2012.

[6] SY Tu and A. H. Sayed, “Diffusion strategies outperform consensus strategies for distributed estimation over adaptive networks,” *Signal Processing, IEEE Transactions on*, vol. 60, no. 12, pp. 6217–6234, Dec 2012.

[7] Y. Liu, C. Li, and Z. Zhang, “Diffusion sparse least-mean squares over networks,” *IEEE Trans. Signal Process.*, vol. 60, no. 8, pp. 4480–4485, Aug 2012.

[8] S. Chouvardas, K. Slavakis, Y. Kopsinis, and S. Theodoridis, “A sparsity promoting adaptive algorithm for distributed learning,” *IEEE Trans. Signal Process.*, vol. 60, no. 10, pp. 5412–5425, Oct 2012.

[9] R. C. de Lamare and R. Sampaio-Neto, “Adaptive reduced-rank processing based on joint and iterative interpolation, decimation, and filtering.” *IEEE Transactions on Signal Processing*, vol. 57, no. 7, pp. 2503–2514, 2009.

[10] Rui Fa, R. C. de Lamare, and Lei Wang, “Reduced-rank stap schemes for airborne radar based on switched joint interpolation, decimation and filtering algorithm,” *IEEE Transactions on Signal Processing*, vol. 58, no. 8, pp. 4182–4194, 2010.

[11] Yang Zhaochong, RC De Lamare, and Li Xiang, “L1 regul- larized stap algorithm with a generalized sidelobe canceler ar- chitecture for airborne radar,” *IEEE Transactions on Signal Processing*, vol. 60, no. 2, pp. 674–686, 2012.

[12] S. Xu, R.C. de Lamare, and H.V. Poor, “Distributed com- pressed estimation based on compressive sensing,” *IEEE Signal Processing Letters*, vol. 22, no. 9, pp. 1311–1315, Sept 2015.

[13] T. G. Miller, S. Xu, R. C. de Lamare, and H. V. Poor, “Distributed spectrum estimation based on alternating mixed discrete-continuous adaptation,” *IEEE Signal Processing Letters*, vol. 23, no. 4, pp. 551–555, 2016.

[14] N. Takahashi, I. Yamada, and A.H. Sayed, “Diffusion least-mean squares with adaptive combiners: Formulation and performance analysis,” *IEEE Trans. Signal Process.*, vol. 58, no. 9, pp. 4795–4810, Sept 2010.

[15] S-Y Tu and A.H. Sayed, “Optimal combination rules for adaptation and learning over networks,” in *IEEE Workshop on Comp. Adv. in Multi-Sensor Adaptive Process.*, CAMSAP, Dec 2011, pp. 317–320.

[16] C.G. Lopes and A.H. Sayed, “Diffusion adaptive networks with changing topologies,” in *Proc. IEEE ICASSP*, March 2008, pp. 3285–3288.

[17] P. Clarke and R. C. de Lamare, “Transmit diversity and relay selection algorithms for multirelay cooperative mimo systems,” *IEEE Transactions on Vehicular Technology*, vol. 61, no. 3, pp. 1084–1098, 2012.

[18] Tong Peng and Rodrigo C de Lamare, “Adaptive buffer-aided distributed space-time coding for cooperative wireless networks,” *IEEE Transactions on Communications*, vol. 64, no. 5, pp. 1888–1900, 2016.

[19] S. Xu, R.C. de Lamare, and H.V. Poor, “Adaptive link selection strategies for distributed estimation in diffusion wireless networks,” in *Proc. IEEE ICASSP*, May 2013, pp. 5402–5405.

[20] S. Xu, R. C. de Lamare, and H. V. Poor, “Adaptive link selection strategies for distributed estimation in wireless sensor networks,” *EURASIP Journal on Advances in Signal Processing*, 2015.

[21] X. Zhao, S-Y Tu, and A.H. Sayed, “Diffusion adaptation over networks under imperfect information exchange and non-stationary data,” *IEEE Trans. Signal Process.*, vol. 60, no. 7, pp. 3460–3475, July 2012.

[22] A. Rastegarnia, W.M. Bazzi, A. Khalili, and J.A. Chambers, “Diffusion adaptive networks with imperfect communications: link failure and channel noise,” *IET Signal Processing*, vol. 8, no. 1, pp. 59–66, Feb 2014.

[23] X. Zhao and A.H. Sayed, “Performance limits for distributed estimation over lms adaptive networks,” *IEEE Trans. Signal Process.*, vol. 60, no. 10, pp. 5107–5124, Oct 2012.

[24] J. Chen and A.H. Sayed, “On the learning behavior of adaptive networks part i: Transient analysis,” *IEEE Trans. Inform. Theory*, vol. 61, no. 6, pp. 3487–3517, June 2015.

[25] J. Zhang and M. Fossorier, “Shuffled belief propagation decoding,” in *Asilomar Conf. Signals, Systems and Computers*, Nov 2002, vol. 1, pp. 8–15 vol.1.

[26] D.E. Hocevar, “A reduced complexity decoder architecture via layered decoding of LDPC codes,” in *IEEE Workshop on Sig- nal Processing Systems, SIPS*, Oct 2004, pp. 107–112.

[27] C.T. Healy and R.C. de Lamare, “Knowledge-aided informed dynamic scheduling for LDPC decoding,” in *IEEE Conf. on Communication Workshop, ICCW*, June 2015, pp. 2212–2217.

[28] Cornelius T Healy and Rodrigo C de Lamare, “Design of ldpc codes based on multipath emd strategies for progressive edge growth,” *IEEE Transactions on Communications*, vol. 64, no. 8, pp. 3208–3219, 2016.

[29] A.G.D. Uchoa, C.T. Healy, and R.C. de Lamare, “Iterative de- tection and decoding algorithms for mimo systems in block- fading channels using ldpc codes,” *IEEE Trans. Vehicular Tech.*, 2016.

[30] R. C. de Lamare and R. Sampaio-Neto, “Minimum mean- squared error iterative successive parallel arbitrated decision feedback detectors for ds-cdma systems,” *IEEE Transactions on Communications*, vol. 56, no. 5, pp. 778–789, 2008.

[31] R. C. de Lamare, “Adaptive and iterative multi-branch mnse decision feedback detection algorithms for multi-antenna sys- tems,” *IEEE Transactions on Wireless Communications*, vol. 12, no. 10, pp. 5294–5308, 2013.

[32] A.H. Sayed, “Diffusion adaptation over networks,” in *E-Reference Signal Processing*, R. Chellapa and S. Theodoridis, Eds. Elsevier, 2013.