Smooth Loops and Thomas Precession

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Abstract

Fundamentals of the local smooth loops due to Sabinin are concisely outlined together with the corresponding infinitesimal objects, so-called $\nu$-hyperalgebras, and the analogue of the Lie groups theory. We apply here this theory to formulation of a new concept of loop of boosts. A quaternionic model of the three-parametric loop of boosts is obtained and a remarkable connection with geodesic loops of Lobachevskii space is found. A description of Thomas precession in the light of general theory of smooth loops is given.

Key words: Quasigroups, Loops, Thomas precession

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1 Introduction

In this article we discuss the problem of special Lorentz transformations (hyperbolic rotations or boosts) and Thomas precession in the light of general theory of smooth loop, see Sabinin (1972a,b, 1977, 1981, 1985, 1988, 1990, 1991, 1994). This theory is a direct generalization of the Lie groups theory and it proved to be effective in applications to Mathematical Physics. There is a lot of areas where this theory has good promises, among of them Special and General Relativity, Quantum Theory and Renormgroup Theory, see Nesterov (1989,1990), Kuusk et al (1994).

The paper is organized ed as following.

In Sec. 2 fundamentals of the local smooth loops theory are given, the corresponding infinitesimal objects (so-called $\nu$-hyperalgebras) introduced and an analogue of Lie groups theory is outlined.

In Sec. 3 the three-parametric loop of boosts is introduced, the theory of smooth loops is used for its description and remarkable connections with geodesic loops of Lobachevskii space are traced.

In Sec. 4 we are treating the Thomas precession from the point of view of smooth loops theory. Relevant loop $QH(2)$ and its properties are considered.

In Sec. 5 concluding remarks are given.

2 Smooth loops

General results on the subject can be extract from Belousov (1967), Sabinin (1972a,b, 1977, 1981, 1985, 1988, 1990, 1991, 1994). A quasigroup is a groupoid $\langle Q, \ast \rangle$ in which equations $a \ast x = b$, $y \ast a = b$ have the unique solutions: $x = a \setminus b$, $y = b/a$. A loop is a quasigroup with a two-sided identity $a \ast e = e \ast a = a$, $\forall a \in Q$. A loop $\langle Q, \ast, e \rangle$ with a smooth functions $\varphi(a, b) := a \ast b$ is called the smooth loop. Let $\langle Q, \ast, e \rangle$ be a smooth local loop with a neutral element $e$. $Q$ being a manifold, dim $Q = n$. We define

$$L_a b = a \ast b = R_b a,$$
$$l_{(a,b)} = L_{a^{-1}} b \circ L_a b,$$  (1)

where $L_a$ is a left translation, $R_b$ is a right translation and $l_{(a,b)}$ is an associator.

It is known that the infinitesimal theory of Lie groups arises from the
associativity of operation $\ast$

$$a \ast (b \ast c) = (a \ast b) \ast c.$$  

For the smooth quasigroups we have the modified law of associativity (quasi-associativity)

$$a \ast (b \ast c) = (a \ast b) \ast l_{(a,b)}c. \tag{2}$$

We will show below that the identity (2) leads to the infinitesimal theory of smooth loops (Sabinin 1988, 1991).

**Definition 2.1:** The vector fields $A_\alpha \ (\alpha = 1, \ldots, n)$ defined on $Q$ are called the **left fundamental vector fields** of a local loop $\langle Q, \ast, e \rangle$ if

$$A_\alpha(x) = A_\alpha^\beta(x) \frac{\partial}{\partial x^\beta}, \tag{3}$$

where

$$A_\alpha^\beta(x) = [(L_x)_e]_\alpha^\beta = \left[ \frac{\partial(x \ast y)^\beta}{\partial y^\alpha} \right]_{y=e},$$

$\{x^\alpha\}$ being coordinates on $Q$.

Analogously one can introduce the **right fundamental vector fields**.

Obviously, $A_1, \ldots, A_n$ are linearly independent at any point.

**Definition 2.2:** The differential 1-forms $\omega^\alpha \ (\alpha = 1, \ldots, n)$ of the base dual to $A_\alpha \ (\alpha = 1, \ldots, n)$ are called the **left fundamental 1-forms** of a loop $Q$.

**Definition 2.3:** We define a **quasialgebra** $q$ on $Q$ as the vector space spanned by left fundamental vector fields of $Q$ with the Lie commutator as operation.

Evidently,

$$[A_\alpha, A_\beta](x) = C_{\alpha\beta}^\gamma(x)A_\gamma(x). \tag{4}$$

**Theorem 2.1:** (Sabinin (1988, 1991)). Let $\langle Q, \ast, e \rangle$ be a local loop. Then $\varphi^\alpha = (a \ast x)^\alpha$ and $\bar{l}_\tau^\sigma = [\bar{l}_{(a,x)}^\tau]_e^\sigma = \bar{l}_\tau^\sigma(a, x)$ (see (1)) are the solutions of the system of differential equations

$$\frac{\partial \varphi^\alpha}{\partial x^\mu} = A^\alpha_\gamma(\varphi)\bar{l}_\sigma^\gamma B^\sigma_\mu(x),$$

$$A_\nu^\sigma(x) \frac{\partial \bar{l}_\mu^\kappa}{\partial x^\sigma} - A_\mu^\sigma(x) \frac{\partial \bar{l}_\nu^\kappa}{\partial x^\sigma} = C^\sigma_{\nu\mu}(x)\bar{l}_\sigma^\kappa - C^\kappa_{\tau\lambda}(\varphi)\bar{l}_\nu^\tau \bar{l}_\mu^\lambda \tag{5}$$
with initial conditions $\varphi^\alpha|_e = a^\alpha$, $\tilde{l}_\kappa^\alpha|_e = \delta_\kappa^\alpha$ $\varphi^\alpha(a,e) = a^\alpha$, $\tilde{l}_\kappa^\alpha(a,e) = \delta_\kappa^\alpha$). Functions $A^\alpha_\beta(x)$ are supposed to be given and satisfy the conditions $A^\alpha_\beta(x)(e) = \delta^\alpha_\beta$; $B^\alpha_\beta(x)$ is the inverse matrix for $A^\alpha_\beta(x)$ and $C^\nu_\mu_\gamma(x)$ are expressed through $A^\alpha_\beta(x)$ by formula (4).

Remark 2.1: One might exclude $\tilde{l}_\kappa^\alpha$ from (5) and obtain the system of second order in unknowns $\varphi^\alpha$.

Remark 2.2: A solution of (5) is nonunique, there are some arbitrariness, which will be seen further.

Let us introduce $\text{Exp} : T_eQ \to Q$ by $\xi \to \varphi(1,\xi)$, where $\varphi(t,\xi)$ is the solution of the differential equation

$$\frac{d\varphi^\alpha(t,\xi)}{dt} = A^\alpha_\beta(\varphi(t,\xi))\xi^\beta, \quad \varphi^\alpha(0,\xi) = e^\alpha, \quad A^\alpha_\beta(e) = \delta^\alpha_\beta \quad (6)$$

Here $A^\alpha_\beta(x)$ is supposed to be given. Due to the theorem of existence and uniqueness of solution it is easily verified that $\varphi(t,\xi) = \varphi(1,t\xi)$ def $\text{Exp}(t\xi)$. It is easy to see that $(\text{Exp})_{*,0} = \text{Id}$, what means that $\text{Exp}^{-1}$ locally exists.

Remark 2.3: Taking into account that our construction depends on the fields $[A_\alpha(a)]^\beta = A^\alpha_\beta(a)$, we should write $\text{Exp}(t\xi, A_1, \ldots, A_n)$ instead of $\text{Exp}(t\xi)$, but we shall do it in the case of misunderstanding only.

Now let us define in a neighbourhood of $e$ operations of a vector space, Sabinin (1988, 1991)

$$x + y = \text{Exp} (\text{Exp}^{-1}x + \text{Exp}^{-1}y), \quad tx = \text{Exp}(t\text{Exp}^{-1}x) \quad (t \in \mathbb{R}; x, y \in Q). \quad (7)$$

It is obvious that

$$\frac{d(tb)^\alpha}{dt} = A^\alpha_\beta(tb)(\text{Exp}^{-1}b)^\beta. \quad (8)$$

The operations $\omega_t : x \to tx$ can be added to an initial loop $\langle Q, *, e \rangle$ and we obtain the structure of a canonical preodule of a given smooth loop. All operations defined in (7) can be added to an initial loop and we get the canonical prediodule of a given smooth loop.

Remark 2.4: The structure of any smooth odule (diodule), Sabinin (1981, 1990, 1991), coincides with the canonical structure of its preodule (prediodule). Indeed, differentiating the identity of monoassociativity of an odule: $tb * ub = (t+u)b$, by $u$ at $u = 0$, we see that $tb$ satisfies (13) with $\xi = \text{Exp}^{-1}b$.

Analogously, $t(x + y) = tx + ty$ implies that after differentiating by $t$

$$A^\alpha_\beta(t(x + y))(\text{Exp}^{-1}(x + y))^\beta = A^\alpha_\beta(tx)(\text{Exp}^{-1}x)^\beta + A^\alpha_\beta(ty)(\text{Exp}^{-1}y)^\beta. \quad (9)$$
And at $t = 0$ we have $\text{Exp}^{-1}(x + y) = \text{Exp}^{-1}x + \text{Exp}^{-1}y$. Thus, $x + y = \text{Exp}(\text{Exp}^{-1}x + \text{Exp}^{-1}y)$.

One may now introduce (locally) for a smooth loop so called normal coordinates $b \to ((\text{Exp}^{-1})^1, \ldots, (\text{Exp}^{-1})^n)$. In such a coordinates $b^\alpha = \text{Exp}^{-1}b^\alpha$ ($\alpha = 1, \ldots, n$).

**Definition 2.4:** (Sabinin (1988, 1991). Let $V$ be a vector space over $\mathbb{R}$ (dim $V = n$), $d(\xi, \eta)$ be a continous binary operation on $V$, $d(\xi, \xi) = 0$, admitting the representation $d(\xi, \eta) = [d^\alpha_{\alpha\beta}(\xi)\eta^\beta]e_\gamma$ in an arbitrary base $e_1, \ldots, e_n$. We say in the case, that $V$ is an hyperalgebra. If on $V$, additionally, a continuous binary operation $\nu(\eta, \xi)$, admitting representation $\nu(\eta, \xi) = \nu^\beta(\eta, \xi)\xi^\alpha$ in an arbitrary base $e_1, \ldots, e_n$ with the properties $\nu(0, \xi) = \xi$, $\nu^\beta(\eta, 0)\xi^\beta = \xi^\alpha$ is given, then we say that $V$ is a $\nu$-hyperalgebra (a hyperalgebra with a multioperator $\nu$).

Evidently that any smooth loop $Q$ with the neutral $e$ generates a $\nu$-hyperalgebra on $T_e(Q)$ with the operations $d(\xi, \eta) = C^\alpha_{\alpha\beta}(\text{Exp}\xi)\xi^\alpha\eta^\beta e_\gamma$, $d(\xi, \xi) = \vec{t}\beta(\text{Exp}\eta, \text{Exp}\xi)\xi^\beta e_\alpha$ (where $e_\alpha = (\partial\alpha)/\partial e$). Such a $\nu$-hyperalgebra is called the tangent $\nu$-hyperalgebra of a loop $\langle Q, *, e \rangle$ (see Sabinin (1988, 1991)).

Let us consider the system

$$\frac{d\varphi^\alpha}{dt} = A^{\alpha\beta}_\beta(\varphi)\nu^\beta(\text{Exp}^{-1}a, t\text{Exp}^{-1}b)t^{-1},$$

where

$$A^{\alpha\beta}_\beta(e) = \delta^\alpha_\beta,$$  \quad $\nu^\beta(\xi, \eta) = \nu^\beta(\xi, \eta)\eta^\gamma$, 

$\nu^\beta(0, \xi) = \eta$, \quad $\nu^\beta(\xi, 0) = \delta^\beta_\gamma$, \quad $\xi, \eta \in T_e(Q)$

with the initial conditions $\varphi^\alpha|_{t=0} = a^\alpha$. Here $\text{Exp}\eta = \text{Exp}(\eta, A_1, \ldots, A_n)$, $tb = \text{Exp}(t\text{Exp}^{-1}b)$ are defined by given $A^\alpha_\beta(x)$. The functions $\nu^\beta$ are also given. Note, that representation $\nu^\beta(\xi, \eta)$ through $\nu^\beta_\alpha(\xi, \eta)$ is nonunique (it is possible that $\nu^\beta(\xi, \eta) = \nu^\beta_\alpha(\xi, \eta)\eta^\gamma = \nu^\beta_\gamma(\xi, \eta)\eta^\gamma$).

**Theorem 2.2:** (Sabinin (1988, 1991)). Let a smooth local loop $\langle Q, *, e \rangle$ together with its canonical operations be given. Then $\varphi(a, b, t) = a \ast tb$ is a solution of the equation (10), where

$$A^{\alpha\beta}_\beta(a) = \left[(L_a)_\ast e\right]^{\alpha\beta}_\beta, \quad \nu^\beta(\xi, \eta) = \vec{L}\beta^\beta_\gamma(\xi, \eta)\eta^\gamma$$

**Proof:** Differentiating $(a \ast tb)^\alpha$ by $t$ as the composition of functions and using (5) from Theorem 2.1 and (10) we get our assertion.
Theorem 2.3: (Sabinin (1988, 1991)). A solution \( \varphi(a, b, t) \) of the equation (10) defines a local loop \( a \ast b = \varphi(a, b, 1) \) with the neutral \( e \) and its canonical unary operations \( tb = \varphi(e, b, t) \). Moreover, \( \varphi(a, b, t) = a \ast tb \).

Let us introduce, Sabinin (1988, 1991)

\[
\tilde{A}_\alpha^\sigma(x) = A^\lambda_\alpha(x) \frac{\partial (\text{Exp}^{-1} x)^\sigma}{\partial x^\lambda}.
\]

Using (4) we have

\[
\tilde{A}_\alpha^\sigma(x) \frac{\partial \tilde{A}_\beta^\gamma(x)}{\partial (\text{Exp}^{-1} x)^\nu} - \tilde{A}_\beta^\gamma(x) \frac{\partial \tilde{A}_\alpha^\sigma(x)}{\partial (\text{Exp}^{-1} b)^\nu} = C_{\alpha\beta}^\sigma(x) \tilde{A}_\gamma^\nu(x).
\]

Setting here \( x = tb \) and contracting it with \((\text{Exp}^{-1} b)^\alpha\) we get, taking into account (6),

\[
\frac{d\tilde{A}_\gamma^\nu(tb)}{dt} - \tilde{A}_\gamma^\nu(tb) t^{-1} [\delta^\nu_\gamma - \tilde{A}_\sigma^\nu(tb)] = [C_{\alpha\beta}^\sigma(tb)(\text{Exp}^{-1} b)^\alpha] \tilde{A}_\gamma^\nu(tb).
\]

At the right hand side this equation has a singularity at \( t = 0 \), which does not allow us to use the theorem of existence and uniqueness. Therefore, equivalently, we write an equation with matrix inverse to \( \tilde{A}_\beta^\gamma(x) \) :

\[
\tilde{B}_\mu^\lambda(x) = B^\lambda_\nu(x) \frac{\partial x^\nu}{\partial (\text{Exp}^{-1} x)^\mu}.
\]

After evident calculations such an equation has the form

\[
\frac{d[t \tilde{B}_\mu^\lambda(tb)]}{dt} = \delta^\lambda_\mu - [C_{\alpha\mu}^\sigma(tb)(\text{Exp}^{-1} b)^\alpha][t \tilde{B}_\sigma^\lambda(tb)].
\]

Thus, having solved the equation

\[
\frac{d\psi_\mu^\lambda}{dt} = \delta^\lambda_\mu - C_{\alpha\mu}^\sigma(tb)(\text{Exp}^{-1} b)^\alpha \psi_\sigma^\lambda, \quad \psi_\sigma^\lambda|_{t=0} = 0,
\]

we get

\[
\tilde{B}_\mu^\lambda(b) = \psi_\mu^\lambda(1, b).
\]

Note, that \( \lambda \psi_\sigma^\alpha(t, \lambda b) = \psi_\sigma^\alpha(\lambda t, b) \), being solutions of the same equation co-inciding at \( t = 0 \). Thus, at \( t = 0 \) we receive \( \psi_\sigma^\alpha(\lambda, b) = \lambda \psi_\sigma^\alpha(1, \lambda b) \) and the solution takes the required form.
In such a way we can reconstruct $\tilde{B}_{\mu}^{\alpha}(x)$ and, consequently, $\tilde{A}_{\mu}^{\alpha}(x)$ and $A_{\mu}^{\alpha}(x)$ by means of

$$d_{\mu}^{\alpha}(\xi) = C_{\alpha\mu}^{\alpha}(\text{Exp}\xi)\xi^{\alpha}. \quad (19)$$

In virtue of $C_{\alpha\mu}^{\alpha} = -C_{\mu\alpha}^{\alpha}$ it is evident, that

$$d_{\mu}^{\alpha}(\xi)\xi^{\mu} = 0. \quad (20)$$

As for the rest, $d_{\mu}^{\alpha}(\xi)$ is an arbitrary function of the form

$$d_{\mu}^{\alpha}(\xi) = d_{\alpha\mu}(\xi)\xi^{\alpha}. \quad (21)$$

(The representation of $d_{\mu}^{\alpha}(\xi)$ in such a view is not unique, generally speaking.)

Having given an arbitrary $d_{\mu}^{\alpha}$ satisfying (20) and (21), let us take invertible $\text{Exp}$ arbitrarily. Let us introduce $\bar{tb} = \text{Exp}(t\text{Exp}^{-1}b)$ and solve the equation

$$\frac{d\psi_{\lambda}^{\mu}}{dt} = \delta_{\lambda}^{\mu} - t^{-1}d_{\mu}^{\sigma}(\text{Exp}^{-1}\bar{tb})\psi_{\sigma}^{\lambda}, \quad \psi_{\sigma}^{\lambda}|_{t=0} = 0. \quad (22)$$

We get $\tilde{B}_{\alpha}^{\lambda}(\bar{tb}) = t^{-1}\psi_{\alpha}^{\lambda}(t, b)$ and $\tilde{A}_{\beta}^{\alpha}(\bar{tb})$ (as an inverse matrix). Further, using (13)–(17) we get

$$C_{\alpha\mu}^{\alpha}(b)(\text{Exp}^{-1}b)^{\alpha} = d_{\mu}^{\alpha}(\text{Exp}^{-1}b). \quad (23)$$

And $d_{\mu}^{\alpha}$ has a prescribed meaning, that is by means of arbitrary $d_{\mu}^{\alpha}$ we have reconstructed in the unique manner $\tilde{A}_{\beta}^{\alpha}$ and, consequently, $A_{\beta}^{\alpha}(x)$ for which (23) is valid. The property (21) implies $\bar{tb} = tb$. Indeed, contracting (22) with $(\text{Exp}^{-1}b)^{\mu}$ we get $d[t\tilde{B}_{\mu}^{\lambda}(\bar{tb})(\text{Exp}^{-1}b)^{\mu}]/dt = (\text{Exp}^{-1}b)^{\lambda}$ whence $\tilde{B}_{\mu}^{\lambda}(\bar{tb})(\text{Exp}^{-1}b)^{\mu} = (\text{Exp}^{-1}b)^{\lambda}$, or

$$B_{\nu}^{\lambda}(\bar{tb}) \frac{\partial(\bar{tb})^{\nu}}{\partial(\text{Exp}^{-1}b)^{\mu}}(\text{Exp}^{-1}b)^{\mu} = (\text{Exp}^{-1}b)^{\lambda},$$

or

$$B_{\nu}^{\lambda}(\bar{tb}) \frac{\partial(\bar{tb})^{\nu}}{\partial(\text{Exp}^{-1}b)^{\mu}} \frac{d(\text{Exp}^{-1}b)^{\mu}}{dt} = (\text{Exp}^{-1}b)^{\lambda} \quad (24)$$

(we have used $\bar{t} = \text{Exp}(\text{Exp}^{-1}$, or $\text{Exp}^{-1}\bar{tb} = t\text{Exp}^{-1}b$). Finally, $d(\bar{tb})^{\nu}/dt = A_{\mu}^{\sigma}(\bar{tb})(\text{Exp}^{-1}b)^{\lambda}$ $\bar{tb}|_{t=0} = e$, which means $\bar{tb} = tb$. Since $d_{\mu}^{\sigma}(\xi, \eta) = d_{\mu}^{\sigma}(\xi)\eta^{\mu}$ we have
Theorem 2.4: (Sabinin (1988, 1991)). Any \( \nu \)-hyperalgebra \( V \) with an operation \( d \) uniquely defines a smooth loop \( \langle Q, *, e \rangle \) with the tangent \( \nu \)-hyperalgebra isomorphic to initially given.

It is easy, also, to prove the following.

Theorem 2.5: (Sabinin (1988, 1991)). Any morphism of smooth loops induces a morphism of corresponding \( \nu \)-hyperalgebras and vice versa.

Remark 2.5: If operations \( \nu, d \) are analytic, then expanding them into series one can introduce a countable system of multilinear operations (with identities), which is equivalent to initial \( \nu \)-hyperalgebra (of course, some conditions of convergence are needed).

Theorem 2.6: (Sabinin (1988, 1991)). A smooth loop is rightmonoalternative (that is \( (x * ty) * uy = x * (t + u)y \) if and only if for its tangent \( \nu \)-hyperalgebra \( \nu(\eta, \xi) = \xi \) is valid (equivalently, \( \nu(\eta, \xi) \) is linear in \( \xi \)).

Theorem 2.7: (Sabinin (1988, 1991). If the tangent \( \nu \)-hyperalgebra of a smooth loop is a Lie algebra, (that is an operation \( \xi * \eta = d(\xi, \eta) \) are bilinear, \( \xi * \xi = 0, \xi * (\eta * \zeta) + \zeta * (\xi * \eta) + \eta * (\zeta * \xi) = 0 \) and this loop is rightmonoalternative (equivalently, \( \nu(\eta, \xi) = \xi \) then our loop is a Lie group. Converse is also true, since any Lie group is rightmonoalternative.

Note, that by definition a loop is monoalternative if it is right and left monoalternative, that is \( (x * ty) * uy = x * (t + u)y, \, tx * (ux*)y = (t + u)x \).

In particular this is the case of smooth Moufang loops, Belousov (1967), Sabinin (1985, 1990).

The following result is valid.

Theorem 2.8: (Sabinin (1988, 1991)). A local smooth monoalternative loop is defined in the unique manner by its tangent bilinear algebra.

3 Quaternions and smooth loops

A quaternionic algebra over a field \( \mathbb{F} \) is a set
\[
H = \{ \alpha + \beta i + \gamma j + \delta k \mid \alpha, \beta, \gamma, \delta \in \mathbb{F} \}
\]
with multiplication operation defined by the property of bilinearity and following rules for \( i, j, k \)
\[
i^2 = j^2 = k^2 = -1, \quad jk = -kj = i, \quad ki = -ik = j, \quad ij = -ji = k.
\] (25)
We consider a quaternionic algebra over complex field $\mathbb{C}(1, i)$

$$H_C = \{ \alpha + \beta i + \gamma j + \delta k \mid \alpha, \beta, \gamma, \delta \in \mathbb{C} \}.$$  

The quaternionic conjugation (denoted by $^+$) is defined by

$$q^+ = \alpha - \beta i - \gamma j - \delta k.$$  

(26)

for $q = \alpha + \beta i + \gamma j + \delta k$. This definition implies

$$(qp)^+ = p^+q^+, \quad p, q \in H_C.$$  

(27)

For real quaternions $q \in H_R$:

$$q = \alpha + \beta i + \gamma j + \delta k; \quad \alpha, \beta, \gamma, \delta \in \mathbb{R},$$  

(28)

we define the norm $\|q\|^2$ setting

$$\|q\|^2 = qq^+ = \alpha^2 + \beta^2 + \gamma^2 + \delta^2.$$  

(29)

Let us consider a set of unit quaternions $H_I$:

$$H_I = \{ q = \alpha + i(\beta i + \gamma j + \delta k) : \|q\|^2 = 1, \; i^2 = -1, \; i \in \mathbb{C}; \; \alpha, \beta, \gamma, \delta \in \mathbb{R} \},$$  

(30)

where the norm $\|q\|^2$ is given by

$$\|q\|^2 = qq^+ = \alpha^2 - \beta^2 - \gamma^2 - \delta^2.$$  

(31)

It is easy to see that for $p, q \in H_I$ the product $pq \not\in H_I$. It means that the set $H_I$ is not closed with respect to the multiplication of quaternions. We introduce a new operation $\oplus$ such one that $p \oplus q \in H_I$. This new operation is defined by

$$p \oplus q = pql^+, \quad (32)$$

where $p, q \in H_I$, \quad $l \in H_R : \|p\|^2 = \|q\|^2 = \|l\|^2 = 1$. It leads to $\|p \oplus q\|^2 = 1$ as well. Requiring $p \oplus q \in H_I$, we shall find a quaternion $l$.

In order to simplify following considerations let us set

$$p = \zeta_0(1 + i\zeta), \quad q = \eta_0(1 + i\eta), \quad l = \alpha_0(1 + \alpha), \quad (33)$$

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where

\[ \zeta = \zeta^1 i + \zeta^2 j + \zeta^3 k, \quad \eta = \eta^1 i + \eta^2 j + \eta^3 k, \quad \alpha = \alpha^1 i + \alpha^2 j + \alpha^3 k, \ (\alpha^1, \eta^1, \zeta^k \in \mathbb{R}), \]

\[ \zeta_0 = \frac{1}{\sqrt{1 - \zeta \zeta}}, \quad \eta_0 = \frac{1}{\sqrt{1 - \eta \eta}}, \quad \alpha_0 = \frac{1}{\sqrt{1 + \alpha \alpha}} \]

and \( \zeta = (\zeta^1, \zeta^2, \zeta^3) \), \( \zeta \zeta = (\zeta^1)^2 + (\zeta^2)^2 + (\zeta^3)^2 \), etc.

Requiring \( p \odot q \in \mathbb{H}_I \), we find from (32)

\[ l = \frac{1 + \zeta^+ \eta}{\|1 + \zeta^+ \eta\|}. \tag{34} \]

Finally, using (32) and (34) we obtain

\[ p \odot q = \frac{\|1 + \zeta^+ \eta\|}{\sqrt{(1 - \zeta \zeta)(1 - \eta \eta)}} \left( 1 + i \frac{(\zeta + \eta)(1 + \eta^+ \zeta)}{\|1 + \zeta^+ \eta\|^2} \right), \tag{35} \]

and obviously \( p \odot q \in \mathbb{H}_I \).

It easy to see that the set of unit quaternions \( \mathbb{H}_I \) with the nonassociative operation (35) is a loop. We call this loop \( QH_\mathbb{I} \). The loop \( QH_\mathbb{I} \) forms so-called non-associative quaternionic representation and evidently it is isomorphic to the loop \( QH(3) \), which we define in open ball \( D_3 = \{ \zeta : |\zeta| < 1 \} \) by

\[ \zeta \star \eta = (\zeta + \eta)/(1 + \zeta^+ \eta), \tag{36} \]

where \( \zeta = \alpha i + \beta j + \gamma k \leftrightarrow \zeta = (\alpha, \beta, \gamma) \) and \( / \) denotes the right division.

The associator (see (1)) is given by

\[ l_{(\zeta, \eta)} \xi = (1 + \eta^+ \zeta) \backslash (1 + \zeta^+ \eta) \xi, \tag{37} \]

where \( \backslash \) is the left division. This expression one can write as

\[ l_{(\zeta, \eta)} \xi = \frac{1 + \zeta \eta - |\zeta \times \eta|^2}{1 + \zeta \eta + |\zeta \times \eta|^2} \xi. \tag{38} \]

In the vector form the operation (36) is written as

\[ \zeta \star \eta = \frac{1 + 2 \zeta \eta + \eta^2}{1 + 2 \zeta \eta + \zeta^2 \eta^2} \zeta + \frac{1 - \zeta^2}{1 + 2 \zeta \eta + \zeta^2 \eta^2} \eta \tag{39} \]
Remark 3.1: Using the identity
\[(\zeta + \eta)(1 + \eta^+\zeta) = (1 + \zeta^+\eta)(\zeta + \eta),\]
where \(\zeta = \alpha i + \beta j + \gamma k\), \(\eta = \lambda i + \mu j + \nu k\), one can write (36) as
\[\zeta \ast \eta = (1 + \eta^+\zeta) \setminus (\zeta + \eta), \quad (40)\]

Remark 3.2: The loop \(QH(3)\) is isomorphic to geodesic loops of three-dimensional Lobachevskii space realized as the upper part of two-sheeted unit hyperboloid \(H^3\). The isomorphism is established by exponential mapping (see Eq. (47))
\[\zeta = \text{Exp}\tau = \frac{\tau}{\tau} \tanh \tau, \quad \tau = \text{Exp}^{-1}\zeta = \frac{\zeta}{|\zeta|} \tanh^{-1} |\zeta|,\]
where we introduced \(\tau \equiv |\tau| = \tanh^{-1} |\zeta|\).

Note also, that in this case it is valid so-called identity of pseudolinearity, Sabinin et al (1986), Sabinin and Miheev (1993):
\[\zeta \ast \eta = \text{Exp}(\alpha(\zeta, \eta)\text{Exp}^{-1}\zeta + \beta(\zeta, \eta)\text{Exp}^{-1}\eta), \quad (41)\]
where
\[\alpha(\zeta, \eta) = \left(\frac{1 + 2\zeta\eta + \eta^2}{1 + 2\zeta\eta + \zeta^2\eta^2}\right) \left(\frac{|\zeta| \tanh^{-1} |\zeta \ast \eta|}{|\zeta \ast \eta| \tanh^{-1} |\zeta|}\right),\]
\[\beta(\zeta, \eta) = \left(\frac{1 - \zeta^2}{1 + 2\zeta\eta + \zeta^2\eta^2}\right) \left(\frac{|\eta| \tanh^{-1} |\zeta \ast \eta|}{|\zeta \ast \eta| \tanh^{-1} |\eta|}\right).\]

It is easy to verify (see Belousov (1967), Sabinin (1990, 1991), Sabinin and Miheev (1993) that
\[\xi \ast (\eta \ast (\xi \ast \zeta)) = (\xi \ast (\eta \ast \xi)) \ast \zeta \quad \text{(left Bol identity)}, \quad (42)\]
and
\[\xi \ast (\eta \ast (\xi \ast \zeta)) = (\zeta \ast \eta) \ast (\xi \ast \eta) \quad \text{(left Bruck identity)}. \quad (43)\]

Let us consider the infinitesimal theory of the loop \(QH(3)\). From (39) we obtain the left fundamental vector fields of the loop \(QH(3)\):
\[A_i = (1 - \zeta \zeta) \frac{\partial}{\partial \zeta^i}. \quad (44)\]
These vector fields obey the commutation relations of quasialgebra

$$[A_i, A_j] = C^k_{ij}(\zeta) A_k,$$  \hspace{1cm} (45)$$

where

$$C^k_{ij}(\zeta) = 2(\delta^k_i \zeta_j - \delta^k_j \zeta_i)$$

are the structure functions and $\zeta^i = \zeta_i$. Dual base of 1-forms are determined by

$$\omega^i = \frac{d\zeta^i}{1 - \zeta \zeta}.$$  \hspace{1cm} (46)$$

Now let us find the exponential mapping $T_{t} Q \xrightarrow{\text{Exp}} Q$ by solving the Eq. (6). Using (44), in normal coordinates $\tau^i = \tau n^i$ ($|n| = 1$), one can write (6) as

$$\frac{d\varphi^i}{dt} = (1 - \Sigma_{j=1}^{3} \varphi^j \varphi^j) n^i.$$  \hspace{1cm} (47)$$

The solution of this system is $\varphi^i = n^i \tanh t$. Thus the exponential mapping is given by

$$\zeta = \text{Exp} \tau = \frac{\tau}{\tanh \tau}, \quad \tau = \text{Exp}^{-1} \zeta = \frac{\zeta}{|\zeta|} \tanh^{-1} |\zeta|.$$  \hspace{1cm} (48)$$

Let us consider the two-parametric subloop $QH(2) \subset QH(3)$ which we define in the open disk $D_2 = \{\zeta : |\zeta| < 1\}$ by

$$\zeta \ast \eta = (\zeta + \eta)/(1 + \zeta^* \eta),$$

where $\zeta = \alpha i + \beta j \iff \zeta = (\alpha, \beta)$. The one of the remarkable peculiarities of this loop is that it can realized on the complex numbers. Let $\mathbb{C}$ be the complex plane, and let $D_2 \subset \mathbb{C}$ be the open unit disk, $D_2 = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$. Inside the disk $D_2$ we define the binary operation

$$L_{\zeta} \eta \equiv \zeta \ast \eta = \frac{\zeta + \eta}{1 + \zeta^* \eta},$$  \hspace{1cm} (49)$$

where $\zeta, \eta \in D$ and $\zeta^*$ is the complex conjugate number ($\zeta = \zeta^1 + i\zeta^2, \quad \zeta^* = \zeta^1 - i\zeta^2$).
The set of complex numbers with the operation $\ast$ on $D_2$ forms two-sided loop $QH(2)$, Nesterov and Stepanenko (1986), Nesterov (1989, 1990). The associator $l_{(\zeta,\eta)} = L_{\zeta}^{-1} \circ L_{\eta} \circ L_{\zeta}$ on $QH(2)$ is determined by

$$l_{(\zeta,\eta)}\xi = \frac{1 + \eta \zeta^*}{1 + \eta^* \zeta} \xi \equiv e^{-i\phi} \xi, \quad \phi = 2\text{arg}(1 + \eta^* \zeta). \quad (49)$$

This loop is isomorphic to the geodesic loop of two-dimensional Lobachevskii space realized as the upper part of two-sheeted unit hyperboloid $H^2$. The isomorphism is established by exponential mapping

$$\zeta = e^{i\varphi} \tanh \frac{\theta}{2}, \quad (50)$$

where $(\theta, \varphi)$ are inner coordinates of unit $H^2$.

4 Loop of boosts and Thomas precession

The set of matrices

$$U_\theta = \cosh (\theta/2) + (n \cdot \sigma) \sinh (\theta/2), \quad |n| = 1. \quad (51)$$

determines in the Minkowski space hyperbolic rotations (boosts) (see, e.g., Misner et al (1973)) by

$$X' = UXU^*, \quad (52)$$

where the matrix $X$ is connected with four-vector $x^\mu$ by

$$X = x^0 + x^i \sigma_i.$$

The angle $\theta$ is related with the velocity of the reference system by $\beta = \tanh \theta$, where $\beta = v/c$ and $c$ is the speed of light. The unit vector $n$ determines the direction of boost. It is well known that the set of hyperbolic rotations does not form the group. It forms the loop with the following nonassociative operation, Nesterov (1989, 1990)

$$U_{\theta_1} \ast U_{\theta_2} = U_{\theta_1} U_{\theta_2} \Lambda_{(\theta_1,\theta_2)}, \quad (53)$$
where $\theta = \theta \mathbf{n}$, $|\mathbf{n}| = 1$ and
\begin{align*}
    U_{\theta_1} &= \cosh(\theta_1/2) + (\mathbf{n}_1 \cdot \boldsymbol{\sigma}) \sinh(\theta_1/2), \\
    U_{\theta_2} &= \cosh(\theta_2/2) + (\mathbf{n}_2 \cdot \boldsymbol{\sigma}) \sinh(\theta_2/2), \\
    \Lambda_{(\theta_1, \theta_2)} &= \cos(\alpha/2) + i(\mathbf{n}_\alpha \cdot \boldsymbol{\sigma}) \sin(\alpha/2), \\
    \cot(\alpha/2) &= \frac{\mathbf{n}_1 \cdot \mathbf{n}_2 + \cot(\theta_1/2) \cot(\theta_2/2)}{\sqrt{1 - (\mathbf{n}_1 \cdot \mathbf{n}_2)^2}}, \\
    \mathbf{n}_\alpha &= \frac{\mathbf{n}_1 \times \mathbf{n}_2}{\sqrt{1 - (\mathbf{n}_1 \cdot \mathbf{n}_2)^2}}.
\end{align*}

Now let us consider the quaternionic formulation of boosts. The space-time points can be represented by quaternions as follows

\begin{equation}
    X = t + i(x_1 + y_2 + z_3).
\end{equation}

and the Lorentz invariant norm is given by

\begin{equation}
    XX^+ = t^2 - x^2 - y^2 - z^2.
\end{equation}

Then the special Lorentz transformations (boosts), which are determined by (53), can be represented by the action of the linear quaternions

\begin{equation}
    Q = \zeta_0 (1 - i\zeta), \quad Q \in QH_x
\end{equation}

on the quaternions

\begin{equation}
    X = t - i(x_1 + y_2 + z_3),
\end{equation}

namely,

\begin{equation}
    X' = QXQ^+.
\end{equation}

The three-parametric loop of boosts is isomorphic to the loop $QH_I$. This isomorphism is established by

\begin{align*}
    \zeta &= \mathbf{n} \tanh \frac{\theta}{2}, \quad |\mathbf{n}| = 1, \\
    i &\rightarrow i\sigma_z, \quad j \rightarrow i\sigma_y, \quad k \rightarrow i\sigma_x,
\end{align*}

where $\sigma_x, \sigma_y, \sigma_z$, are Pauli matrices. Hence, we have

\begin{equation}
    Q \rightarrow U = \cosh (\theta/2) + (\mathbf{n} \cdot \boldsymbol{\sigma}) \sinh (\theta/2).
\end{equation}
Setting $\beta = \zeta$ (or $\beta = \zeta = n \tanh \frac{\theta}{2}$, where $\beta$ is the three-velocity of the observer, the speed of light $c = 1$), we obtain from (36) the quaternionic formula for the addition of relativistic three-velocities

$$\beta_1 \star \beta_2 = (\beta_1 + \beta_2)/(1 + \beta_1^+ \beta_2),$$

or equivalently

$$\beta_1 \star \beta_2 = (1 + \beta_2^+ \beta_1) \setminus (\beta_1 + \beta_2).$$

Now let us consider the loop $QH(2)$ with nonassociative operation $*$ (see (48))

$$L_\zeta \eta \equiv \zeta * \eta = \frac{\zeta + \eta}{1 + \zeta * \eta}.$$  

We assign to each element $\zeta \in QH(2)$ the matrix $U_\zeta \in SU(1, 1)$:

$$\zeta \longrightarrow U_\zeta = \begin{pmatrix} a & b \\ b^* & a \end{pmatrix}, \quad a^2 - |b|^2 = 1,$$

where

$$a = \frac{1}{\sqrt{1 - |\zeta|^2}}, \quad b = \frac{\zeta}{\sqrt{1 - |\zeta|^2}}$$

and define the nonassociative operation $\odot$ on the set $K$ of matrices (67) as

$$U_\eta \odot U_\zeta = U_{\eta \ast \zeta}.$$  

Note, that

$$U_{\eta \ast \zeta} = U_\eta U_\zeta \Lambda^{-1}(\eta, \zeta),$$

where

$$\Lambda = \begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix}, \quad \varphi = 2 \text{arg}(1 + \eta^* \zeta).$$

The set $K$ with the composition law $\odot$ forms the so-called the matrix representation of the loop $QH(2)$.

Let $\eta = \nu \tanh (\theta/2)$, where $|\nu| = 1$. Then

$$a = \frac{1}{\sqrt{1 - |\zeta|^2}} = \cosh (\theta/2), \quad b = \frac{\zeta}{\sqrt{1 - |\zeta|^2}} = \nu \sinh (\theta/2),$$
and hence

$$\zeta \rightarrow U_\theta = \cosh (\theta/2) + (\nu \cdot \sigma) \sinh (\theta/2) = \exp\left(\frac{1}{2} \theta \cdot \sigma\right),$$

where $\theta = \theta \nu$, $\nu = (\Re \nu, \Im \nu)$ and $\sigma_i$ are Pauli matrices.

Let us consider the combination of two boosts in different directions but with the same angle $\theta$:

$$\eta = \nu \tanh \frac{\theta}{2}, \quad \zeta = (\nu + \delta \nu) \tanh \frac{\theta}{2}. \quad (71)$$

Setting $\nu = \exp(i\alpha)$ one can obtain for the infinitesimal $\delta \varphi$ (see (49), (70)) the following expression:

$$\delta \varphi = 2 \delta \alpha \tanh^2 \frac{\theta}{2} \frac{\tanh \frac{\theta}{2}}{1 + \tanh^2 \frac{\theta}{2}}. \quad (72)$$

Let $\delta \varphi = \omega dt$, then (72) leads to

$$\omega = 2 \frac{\dot{\alpha} \tanh^2 \frac{\theta}{2}}{1 + \tanh^2 \frac{\theta}{2}}. \quad (73)$$

For the slow motion ($\theta \ll 1$) one can easy obtain

$$\omega = \frac{1}{2} \theta^2 \dot{\alpha} = \frac{1}{2} \alpha v, \quad (74)$$

where $a = v^2/r$ is the acceleration, $v$ being the velocity of the reference system. In an arbitrary coordinate system (73) takes the following form

$$\omega = \frac{1}{2} a \times v. \quad (75)$$

The last one is exactly the expression for Thomas precession (see, e.g. Misner et al (1973)). We see that associator $l(\zeta, \eta)$ completely determines the Thomas precession.
5 Concluding remarks

For the first time the complex model of relativistic addition of velocities was introduced by Nesterov (Nesterov and Stepanenko (1986), Nesterov (1989, 1990)) and later by Ungar (1991a,b, 1992, 1994). The other models of addition of three-velocities was considered by Sabinin and Miheev (1993). So-called “gyrogroup”, see Ungar (1991a), is exactly left Bol loop with left Bruck identity, Sabinin (1995). Some vectorlike properties of the complex disk, Ungar (1994), are contained in the vector space operations (7). In particular, Eq.(8.4b) in Ungar (1991a) then expresses obvious properties of the exponential mapping (50), and some of axioms of gyrogroups are superfluous, Sabinin (1995). The so-called gyrosemidirect product, Ungar (1991a), proved to be a subcase of the well known semidirect product of a loop by its transassociant, Sabinin (1972a,b). The fact that several authors independently rediscovered certain results of the theory of smooth loops in connection with some physical problems, is remarkable and valuable, showing the vitality and importance of the smooth loops theory.

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