BOREL AND VOLUME CLASSES FOR DENSE REPRESENTATIONS OF DISCRETE GROUPS

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ABSTRACT. We show that the bounded Borel class of any dense representation $\rho : G \to \text{PSL}_n \mathbb{C}$ is non-zero in the degree three bounded cohomology and has maximal Gromov semi-norm, for any countable discrete group $G$. For $n = 2$, the Borel class is equal to the 3-dimensional hyperbolic volume class. We show that the volume classes of dense representations $\rho : G \to \text{PSL}_2 \mathbb{C}$ are uniformly separated in semi-norm from any other representation $\rho' : G \to \text{PSL}_2 \mathbb{C}$ for which there is a subgroup $H \leq G$ on which $\rho$ is still dense but $\rho'$ is discrete and faithful or indiscrete but not Zariski dense. We show that the Banach subspace of reduced bounded cohomology that is the closure of volume classes of dense representations has dimension equal to the cardinality of the continuum.

1. INTRODUCTION

The bounded cohomology of discrete groups admits an almost entirely algebraic description, but many bounded classes are most naturally understood in the context of the geometry of non-positively curved metric spaces. Isometric actions of a discrete group on quasi-trees and hyperbolic spaces are responsible for producing an abundance non-trivial bounded classes. Conversely, bounded cohomology can be used as a tool to understand the space of isometric actions that a discrete group admits on, say, a non-compact symmetric space $X$. Broadly, we aim to understand to what extent bounded cohomology parameterizes such actions, including those that are not covering actions or which factor through some other group. We narrow our focus and consider the setting that $\text{Isom}^+(X) = \text{PSL}_n \mathbb{C}$, $n \geq 2$. Intriguingly, from the point of view of bounded cohomology, the behavior of isometric actions of a countable discrete group $G$ with a dense orbit in $X$ is controlled in a certain sense by a single discrete and faithful representation $\rho_2 : \text{PSL}_2 \mathbb{C} \to \text{Isom}^+(\mathbb{H}^3) = \text{PSL}_2 \mathbb{C}$, i.e. the geometry of a complete hyperbolic 3-manifold homeomorphic to a genus 2 handlebody. This is the phenomenon that we will explain in this note.

Immersed locally geodesic tetrahedra in a complete hyperbolic 3-manifold $M$ lift to embedded geodesic tetrahedra in the universal cover $\mathbb{H}^3$. We can measure the (signed) hyperbolic volume of such a tetrahedron, which is bounded above by $v_3$, defining a class in the bounded cohomology of the manifold. The bounded cohomology ring is an invariant of the fundamental group $\pi_1(M) = \Gamma$ [Gro82], and any other action $\rho : \Gamma \to \text{Isom}^+(\mathbb{H}^3)$ yields a bounded class $[\rho^* \text{vol}_3] \in H^3_3(\Gamma; \mathbb{R})$, by measuring volumes of the geodesic tetrahedra with vertex set contained in the orbit $\rho(\Gamma).x$, for some (any) fixed $x \in \mathbb{H}^3 \cup \partial \mathbb{H}^3$. We say that a representation is elementary if its image has a global fixed point in $\mathbb{H}^3 \cup \partial \mathbb{H}^3$ or it preserves a totally geodesic plane.

**Theorem 1.1.** If $G$ is a countable discrete group and $\rho : G \to \text{PSL}_2 \mathbb{C}$ is a dense representation, i.e. $\rho(G) = \text{PSL}_2 \mathbb{C}$, then

$$||[\rho^* \text{vol}_3]||_\infty = v_3,$$

which is the largest possible value. Moreover, if $\rho_0 : G \to \text{PSL}_2 \mathbb{C}$ is any other representation and there is a subgroup $H \leq G$ such that $\rho(H) = \text{PSL}_2 \mathbb{C}$, but $\rho_0$ is either discrete and faithful or elementary restricted to $H$, then

$$||[\rho^* \text{vol}_3] - [\rho_0^* \text{vol}_3]||_\infty \geq v_3.$$

Suppose $M$ is hyperbolic 3-manifold of finite volume and $\rho : \pi_1(M) \to \text{PSL}_2 \mathbb{C}$; the volume of $\rho$ is a numerical invariant that can be obtained by pairing the bounded fundamental class or volume class $[\rho^* \text{vol}_3] \in H^3_3(\pi_1(M); \mathbb{R})$ of the representation with a (relative) fundamental cycle of $M$. This numerical invariant has been studied from this perspective by [BBI13], where Bucher, Burger, and Iozzi show that when the maximal volume for a representation is achieved, the representation must be conjugate to the lattice embedding $\pi_1(M) \hookrightarrow \mathbb{H}^3$. We refer to this kind of result informally as a volume rigidity result.
We are most interested in hyperbolic manifolds with infinite volume, and instead of assigning a numerical invariant to a representation, we work directly with its volume class in bounded cohomology. For example, let $S$ be a hyperbolic surface of finite type and $\Gamma = \pi_1(S)$. The quasi-isometry class of an injective Kleinian surface group representation $\rho : \Gamma \to \text{PSL}_2 \mathbb{C}$ without parabolic elements is characterized by the volume class $[\rho^* \text{vol}_3] \in H^3_\beta(\Gamma; \mathbb{R})$ [Far18a, Far18b]. The semi-norm in bounded cohomology can be used to detect faithfulness of arbitrary representations [Far18a, Theorem 7.8], and Theorem 1.1 says that the semi-norm on bounded cohomology detects discreteness for Zariski dense representations. We have the following rigidity property, which is a consequence of the work here and in [Far18a, Far18b]. We think of this as a kind of volume rigidity result.

**Corollary 1.2.** Let $S$ be an orientable surface of finite type. There exists a constant $\varepsilon = \varepsilon(S)$ such that the following holds. Suppose that $\rho_0 : \pi_1(S) \to \text{PSL}_2 \mathbb{C}$ is a discrete and faithful representation without parabolic elements, and $M_{\rho_0}$ has a degenerate end. If $\rho : \pi_1(S) \to \text{PSL}_2 \mathbb{C}$ is any other representation without parabolics satisfying

$$\| [\rho_0^* \text{vol}_3] - [\rho^* \text{vol}_3] \|_\infty < \varepsilon,$$

then $\rho$ and $\rho_0$ are in the same quasi-isometry class. If $\rho_0$ is totally degenerate, then $\rho_0$ and $\rho$ are conjugate.

We explain on the role of parabolic cusps and elaborate on the quasi-isometric equivalence relation in Section 2.6. In the setting of surface groups, quasi-isometric equivalence of discrete and faithful representations is equivalent to the existence of a volume preserving bi-Lipschitz homeomorphism of quotient manifolds $M_\rho \to M_{\rho_0}$ inducing $\rho_0 \circ \rho^{-1}$ on $\pi_1$.

Recently, Bucher, Burger, and Iozzi proved a volume rigidity result for representations of finite volume hyperbolic 3-manifold groups into $\text{PSL}_n \mathbb{C}$ with respect to the so-called Borel invariant of a representation, defined in [BB18]. To do so, they computed the continuous bounded cohomology $H^3_\beta(\text{PSL}_n \mathbb{C}; \mathbb{R})$, which is generated by a single class $\beta_n$, called the bounded Borel class. They also computed the semi-norm of the bounded Borel class and how it behaves under various natural inclusions $\text{PSL}_k \mathbb{C} \hookrightarrow \text{PSL}_n \mathbb{C}$, $k \leq n$ (see Section 2.5). The bounded Borel class is a generalization of the hyperbolic volume, which is the setting that $n = 2$. The argument that proves Theorem 1.1 generalizes almost immediately to dense representations into $\text{PSL}_n \mathbb{C}$. Recall that there is a unique (up to conjugacy) irreducible representation $\iota_n : \text{PSL}_2 \mathbb{C} \to \text{PSL}_n \mathbb{C}$.

**Theorem 1.3.** Let $G$ be a countable discrete group and $\rho : G \to \text{PSL}_n \mathbb{C}$ be dense. Then

$$\| \rho^* \beta_n \|_\infty = v_3 \frac{n(n^2 - 1)}{6},$$

which is the largest possible value. Suppose that $\rho_0 : G \to \text{PSL}_2 \mathbb{C}$ is such that there exists a finitely generated subgroup $H \leq G$ such that $\rho(H) = \text{PSL}_n \mathbb{C}$ and $\rho_0$ is discrete, faithful, and geometrically finite or elementary restricted to $H$. Then

$$\| \rho^* \beta_n - (\iota_k \circ \rho_0)^* \beta_k \|_\infty \geq v_3 \frac{n(n^2 - 1)}{6},$$

for all $k \geq 2$.

Degree three bounded cohomology is a very big space. We will consider the reduced bounded cohomology $\overline{H}^3_\beta(F_2; \mathbb{R}) = H^3_\beta(F_2; \mathbb{R}) / Z$, where $Z$ is the subspace of zero-norm bounded cohomology. The reduced space $\overline{H}^3_\beta(F_2; \mathbb{R})$ is a Banach space with respect to the quotient norm. The volume class defines an $\text{Out}(G)$-equivariant function on the character variety

$$\text{Hom}(G, \text{PSL}_2 \mathbb{C}) / \text{PSL}_2 \mathbb{C} \to \overline{H}^3_\beta(G; \mathbb{R})$$

with respect to the natural actions of $\text{Out}(G)$ on each space. Call the closure of the linear span of the image of the discrete and faithful locus $G \subset \overline{H}^3_\beta(G; \mathbb{R})$, the geometric subspace. Denote the closure of the linear span of the image of conjugacy classes of dense representations $\mathcal{D} \subset \overline{H}^3_\beta(G; \mathbb{R})$.

**Theorem 1.4.** The $\mathbb{R}$-dimension of the Banach subspace $\mathcal{D}$ of reduced bounded cohomology generated by the volume classes of dense representations of a free group has cardinality equal to $\# \mathbb{R}$. The $v_3$-norm ball in $\mathcal{D}$ has $\| \cdot \|_\infty$ distance at least $v_3$ from the $v_3$-norm ball in $G$. 

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The following elementary observation about Banach spaces is proved using the Baire Category Theorem: if a Banach space $V$ has $\dim V \geq \#\mathbb{N}$, then $\dim V \geq \#\mathbb{R}$. Theorem 1.4 is obtained by constructing (in Section 6.1) collections of dense representations that satisfy the hypotheses of

**Theorem 1.5.** Suppose $\{\rho_i : G \to \text{PSL}_2 \mathbb{C}\}_{i=1}^\infty$ are dense and $(a_i) \in \ell^1(\mathbb{N})$. If there are subgroups $H_i \leq G$ such that $\rho_i(H_i) = \text{PSL}_2 \mathbb{C}$ but $\rho_i|_{H_j}$ is discrete and faithful or elementary for $i \neq j$, then

$$\left\| \sum_{i=1}^\infty a_i[\rho_i^* \text{vol}_3]\right\|_\infty \geq \max\{|a_i|\} \cdot v_3.$$ 

Consequently, $\{[\rho_i^* \text{vol}_3] : i = 1, 2, \ldots\} \subset \prod_i^\infty(G; \mathbb{R})$ is a linearly independent set.

**Remark 1.6.** The main argument we record in this note is very generally applicable in the sense that it can be adapted to work in all dimensions, contingent on the existence of a sequence of hyperbolic $n$-manifolds with prescribed topological and geometric properties. So, if one can show that there is a sequence $M_k$ of hyperbolic $n$-manifolds that have efficient fundamental cycles that are ‘well approximated by a free group,’ then the degree $n$ bounded cohomology of a free group $F_2$ on two letters has dimension equal to the cardinality of the continuum, at least when $n \geq 4$ is even. Compare this remark to the following sketch.

The main idea behind the proofs of our theorems is that a subgroup of a dense countable group can approximate the geometry of *any* finitely generated discrete group, up to a certain scale in the universal cover. More precisely, given a finitely generated Kleinian group $\Gamma = \langle \gamma_1, \ldots, \gamma_k \rangle$ and a dense representation $\rho : G \to \text{PSL}_2 \mathbb{C}$, positive integer $N$, and $\varepsilon > 0$, there are $g_1, \ldots, g_k \in G$ such that all words of length at most $N$ in the $\rho(g_i)$ are in the $\varepsilon$-neighborhood of corresponding words in the $\gamma_i$. If $\mathbb{H}^n/\Gamma$ has a submanifold $K \subset M$ with large volume and small area, then we can lift this submanifold to the universal cover. Assume that $K$ is equipped with a straight triangulation by geodesic tetrahedra, and that the triangulation does not have too many triangles on the boundary. Then we can homotope the triangulation so that there is only one vertex, and such that we do not loose too much volume during the homotopy, which is a small miracle of hyperbolic geometry. The edges of the tetrahedra are now labeled by elements of $\pi_1(M)$, because they are closed, based loops. The (finite) triangulation lifts to the universal cover. The idea is now to use our approximation of words of length at most $N$ in the $\gamma_i$ by words of length at most $N$ in $\rho(g_i)$ to build a chain on $G$ that has almost the same shape as our lifted chain, via its $\rho$-action. In this way, we use the geometry of discrete groups to build chains on our abstract group $G$ that have large volume and small boundary, which is enough to show that $[\rho^* \text{vol}_3] \neq 0$; if in addition, the chains on $\Gamma$ are $\varepsilon$-efficient, we can show that $\|[\rho^* \text{vol}_3]\| \geq \varepsilon$.

There are some technical issues to consider, such as how the algebraic structure of $G$ and that of $\Gamma$ interact. Indeed, this is a fairly important point, since our approximation of $\Gamma$ by words in $G$ will typically not respect the relations in $\Gamma$. In dimension 3, we are able to work exclusively with Kleinian free groups $F_k \cong \Gamma \leq \text{PSL}_2 \mathbb{C}$, and so we can avoid this issue, because it is easy to construct homomorphisms $F_k \to G$. It would be interesting to try to apply the techniques presented here to higher dimensional hyperbolic manifolds. The idea would be to try to take the algebraic structure of $G \leq \text{Isom}^+(\mathbb{H}^n)$ into account in the approximation of $\Gamma$ by a free subgroup of $\rho(G)$, where $\rho : G \to \text{Isom}^+(\mathbb{H}^n)$ is dense. The issue is that the boundary of a chain in $G$ that approximates the geometry of a chain on $\Gamma$ may have boundary which is much larger than the boundary of a chain on $\Gamma$.

The structure of the paper is as follows. We provide some background on bounded cohomology, marked Kleinian groups, and the bounded Borel class. In Section 3 we explain how to approximate discrete groups by dense groups up to finitely many multiplications and provide a very general framework to show that dense representations have volume or Borel classes with positive semi-norm. In Section 4 we prove Theorem 1.4 and explain how to handle $\ell^1$ linear combinations of dense representations. In Section 6.1 for each $N$, we construct $N$ representations that satisfy the hypotheses of Theorem 1.5 to show that the dimension of bounded cohomology spanned by the volume classes for dense representations is at least $\#\mathbb{N}$. The number of $\mathbb{R}$-co-chains on a discrete countable group is at most $\#\mathbb{N}$, and this establishes Corollary 1.4. Finally, in Section 5 we consider a higher rank formulation of volume, known as the bounded Borel class of a representation $\rho : \Gamma \to \text{PSL}_n \mathbb{C}$; the argument we present for $n = 2$ generalizes to the higher rank setting due to the work of [BB18].
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2. BACKGROUND AND NOTATION

2.1. Continuous bounded cohomology of groups. Let $G$ be a topological group. We define a co-chain complex for $G$ by considering the collection of continuous, $G$-invariant functions

$$C^n(G; \mathbb{R}) = \{ f : G^{n+1} \to \mathbb{R} : g.f = f, \forall g \in G \}.$$ 

The homogeneous co-boundary operator $\delta$ for the trivial $G$ action on $\mathbb{R}$ is, for $f \in C^n(G; \mathbb{R})$,

$$\delta f(g_0, ..., g_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(g_0, ..., \hat{g}_i, ..., g_{n+1}),$$

where $\hat{g}_i$ means to omit that element, as usual. The co-boundary operator gives the collection $C^\bullet(G; \mathbb{R})$ the structure of a co-chain complex. An $n$-co-chain $f$ is bounded if

$$\|f\|_{\infty} = \text{sup} |f(g_0, ..., g_n)| < \infty,$$

where the supremum is taken over all $n+1$ tuples $(g_0, ..., g_n) \in G^{n+1}$.

The operator $\delta : C^0_b(G; \mathbb{R}) \to C^1_b(G; \mathbb{R})$ is a bounded linear (continuous) operator of Banach spaces with operator norm at most $n+2$, so the collection of bounded co-chains $C^\bullet_b(G; \mathbb{R})$ forms a subcomplex of the ordinary co-chain complex. The cohomology of $(C^\bullet_b(G; \mathbb{R}), \delta)$ is called the continuous bounded cohomology of $G$, and we denote it $H^\bullet_b(G; \mathbb{R})$. When $G$ is a discrete group, the continuity assumption is vacuous, and we write $H^\bullet_b(G; \mathbb{R})$ to denote the bounded cohomology of $G$ in the case that it is discrete. The $\infty$-norm $\| \cdot \|_{\infty}$ descends to a semi-norm on bounded cohomology, so that if $\alpha \in H^\bullet_{cb}(G; \mathbb{R})$,

$$\|\alpha\|_{\infty} = \text{inf}_{\|\alpha\|_{\infty}} \|\alpha\|_{\infty}.$$

A continuous group homomorphism $\varphi : H \to G$ induces a map $\varphi^* : H^\bullet_{cb}(G; \mathbb{R}) \to H^\bullet_{cb}(H; \mathbb{R})$ that is norm non-increasing. In [Som98], Soma shows that the pseudo-norm is in general not a norm in degree $\geq 3$. We will consider the quotient $\overline{H}^n_{cb}(G; \mathbb{R}) = H_{cb}^n(G; \mathbb{R})/Z$ where $Z \subset H^3_{cb}(G; \mathbb{R})$ is the subspace of zero-semi-norm elements. Then $\overline{H}^\bullet_{cb}(G; \mathbb{R})$ is a Banach space with the quotient norm $\| \cdot \|_{\infty}$.

2.2. Norms on chain complexes. Given a connected countable CW-complex $X$, we define a norm on the singular chain complex of $X$ as follows. Let $\Sigma_n = \{ \sigma : \Delta_n \to X \}$ be the collection of singular $n$-simplices. Write a simplicial chain $A \in C_n(X; \mathbb{R})$ as an $\mathbb{R}$-linear combination

$$A = \sum \alpha_\sigma \sigma,$$

where each $\sigma \in \Sigma_n$. The 1-norm or Gromov-norm of $A$ is defined as

$$\|A\|_1 = \sum |\alpha_\sigma|.$$

This norm promotes the algebraic chain complex $C^\bullet_n(X; \mathbb{R})$ to a chain complex of normed linear spaces; the boundary operator is a bounded linear operator. Keeping track of this additional structure, we can take the topological dual chain complex

$$(C^\bullet_n(X; \mathbb{R}), \partial, \| \cdot \|_1)^* = (C_b^\bullet_n(X; \mathbb{R}), \delta, \| \cdot \|_{\infty}).$$

The $\infty$-norm is naturally dual to the 1-norm, so the dual chain complex consists of bounded co-chains. Define the bounded cohomology $H^\bullet_n(X; \mathbb{R})$ as the co-homology of this complex. Gromov [Gro82] proved the remarkable fact that for any countable connected CW-complex $M$, the homotopy class of a classifying map $K(\pi_1(M), 1) \to M$ induces an isometric isomorphism $H^\bullet_b(M; \mathbb{R}) \to H^\bullet_n(\pi_1(M); \mathbb{R})$. We therefore identify the two spaces $H^\bullet_b(\pi_1(M); \mathbb{R}) = H^\bullet_n(\pi_1(M); \mathbb{R})$.

For a countable discrete group $G$, we will also consider the normed chain complex $(C^\bullet_n(G; \mathbb{R}), \partial, \| \cdot \|_1)$ that defines the homology of $G$. The collection of $n$-co-invariants $C_n(G; \mathbb{R})$ of $G$ is the $\mathbb{R}$-linear span of $\Sigma_n(G) =$
\{(g_0, ..., g_n) : g_i \in G\} / \sim$, where $\sim$ is the equivalence relation generated by $(g_0, ..., g_n) \sim (gg_0, ..., gg_n)$; we denote an equivalence class $[g_0, ..., g_n] \in \Sigma_n(G)$, and we think of $[g_0, ..., g_n]$ as an $n$-simplex in the universal cover of a $K(G, 1)$ for $G$, defined only up to covering transformation, thus defining a simplex in the quotient $K(G, 1)$. The ordered vertex set is $(g_0, ..., g_n)$, up to deck transformation. A group chain or $n$-co-invariant $Z \in \mathcal{C}_n(G; R)$ is then a sum

$$Z = \sum_{i=1}^{k} a_i(g_{i_1}^1, ..., g_{i_n}^1),$$

where $[g_{i_1}^1, ..., g_{i_n}^1] \neq [g_{j_1}^2, ..., g_{j_n}^2]$ for $i \neq j$. The 1-norm is then $\|Z\|_1 = \sum_{i=1}^{k} |a_i|$, by definition. The boundary operator $\partial : \mathcal{C}_n(G) \to \mathcal{C}_{n-1}(G)$ is the dual of the co-boundary operator. One thinks of $\partial$ as the alternating sum of face maps on $n$-simplices. If $f \in \mathcal{C}_n^0(G; R)$ and $Z \in \mathcal{C}_n(G; R)$, then we have a trivial inequality $|f(Z)| \leq \|f\|_\infty \|Z\|_1$.

### 2.3. Isometric chain maps.

We will be interested in free marked Kleinian groups $\rho : F_d \to \text{PSL}_2 \mathbb{C}$, i.e. $F_d$ is a free group of rank $d$ and $\rho$ is a discrete and faithful representation. Thus $\text{im} \rho = \Gamma$ acts properly discontinuously by orientation preserving isometries on $\mathbb{H}^3$, and the orbit space $M_\rho = \mathbb{H}^3/\Gamma$ is a complete hyperbolic 3-manifold of infinite volume. Call the orbit projection $\pi : \mathbb{H}^3 \to M_\rho$. By the Tameness Theorem [CG06, Ago04], there is a homeomorphism $\mathcal{H}_d \to M_\rho$ inducing $\rho$ on $\pi_1$, where $\mathcal{H}_d$ is a genus $d$ handlebody. There is a natural subspace of the singular chain complex $\mathcal{C}_*(M_\rho)$ obtained by straightening. We have a 1-norm non-increasing chain map $\mathcal{H}_\rho$

$$\text{str} : \mathcal{C}_*(M_\rho) \to \mathcal{C}_*(M_\rho),$$

defined by homotoping a singular $n$-simplex $\sigma : \Delta_n \to M_\rho$, relative to its vertex set, to the unique totally geodesic hyperbolic tetrahedron $\text{str} \sigma$, where we ignore issues of parameterization, though Thurston provides the relevant details in [Thu82, Chapter 6.1]. Then $\mathcal{C}_*^{\text{str}}(M_\rho; \mathbb{R})$ denotes the image of $\text{str}$, and if $\bar{x} \in M_\rho$, let $\mathcal{C}_*^{\text{str}}(M_\rho, \{\bar{x}\}; \mathbb{R})$ be the subcomplex of straight simplices spanning by the straight simplices whose vertices all map to $\bar{x}$. We will now construct a map $\text{str}_\bar{x} : \mathcal{C}_*(M_\rho; \mathbb{R}) \to \mathcal{C}_*^{\text{str}}(M_\rho, \{\bar{x}\}; \mathbb{R})$.

Fix $x \in \pi^{-1}(\bar{x})$, and let $D = \{y \in \mathbb{H}^3 : d(x, y) \leq d(\gamma(x), y) \text{ for all } \gamma \in \Gamma\}$ be the Dirichlet fundamental polyhedron for $\Gamma$ centered at $x$; delete a face of $D$ in each face-pair $(F, \gamma F)$ to obtain a fundamental domain for $\Gamma$, which we still call $D$. Let $\sigma : \Delta_n \to M_\rho$ and choose a lift $\bar{\sigma} : \Delta_n \to \mathbb{H}^3$. The vertices $v_0, ..., v_n$ of $\bar{\sigma}$ are uniquely labeled by group elements $v_i = \gamma_i y_i$ where $\gamma_i \in \Gamma$, $y_i \in D$. Define $\text{str}_\bar{x} \sigma = \pi(\sigma_x(\gamma_0, ..., \gamma_n))$, where $\sigma_x(\gamma_0, ..., \gamma_n)$ is the straightening of any simplex whose ordered vertex set is $(\gamma_0 x, ..., \gamma_n x)$. The definition is independent of the choice of lift, because any two lifts of $\sigma$ have vertex set equal to $(\gamma_0 y_0, ..., \gamma_n y_n)$ for some $\gamma \in \Gamma$. All maps are chain maps and the operator norm $\|\text{str}_\bar{x}\| \leq 1$. This is just because some simplices in a chain may collapse and cancel after applying $\text{str}_\bar{x}$. If $Z \in \mathcal{C}_n^{\text{str}}(M_\rho, \{\bar{x}\}; \mathbb{R})$, then $Z$ defines a chain in $\mathcal{C}_n(\Gamma; \mathbb{R})$ by linear extension of the rule

$$\pi(\sigma_x(\gamma_0, ..., \gamma_n)) \mapsto [\gamma_0, ..., \gamma_n],$$

where again, this map is well defined for exactly the reason that $\pi(\sigma_x(\gamma_0, ..., \gamma_n)) = \pi(\sigma_x(\gamma_0 y_0, ..., \gamma_n y_n))$, for any $\gamma \in \Gamma$.

One checks that this is an isometric isomorphism of normed chain complexes $\ell_x : \mathcal{C}_*^{\text{str}}(M_\rho, \{\bar{x}\}; \mathbb{R}) \to \mathcal{C}_*(\Gamma; \mathbb{R})$ with their 1-norms. Thus if $Z \in \mathcal{C}_n(M, \{x\}; \mathbb{R})$, then we have $\|Z\|_1 = \|\ell_x(Z)\|_1$ and $\|\partial_x(Z)\|_1 = \|\partial_x(Z)\|_1$. Conversely, if we have a chain $Z \in \mathcal{C}_n(\Gamma; \mathbb{R})$, one sees that $\pi_*(Z) = \ell^{-1}_x(Z) \in \mathcal{C}_n^{\text{str}}(M_\rho, \{\bar{x}\}; \mathbb{R})$.

### 2.4. The volume class.

Let $x \in \mathbb{H}^3$ and consider the function $\text{vol}_x^3 : (\text{PSL}_2 \mathbb{C})^4 \to \mathbb{R}$ which assigns to $(g_0, ..., g_3)$ the signed hyperbolic volume of the convex hull of the points $g_0 x, ..., g_3 x$. Any geodesic tetrahedron in $\mathbb{H}^3$ is contained in an ideal geodesic tetrahedron, and there is an upper bound $v_3$ on volume that is maximized by a regular ideal geodesic tetrahedron. That is, $\|\text{vol}_x^3\|_\infty = v_3$. One checks that $\delta \text{vol}_x^3 = 0$, so that $[\text{vol}_x^3] \in H_3^3(\text{PSL}_2 \mathbb{C}; \mathbb{R})$. Moreover, for any $x, y \in \mathbb{H}^3 \cup \partial \mathbb{H}^3$, we have $[\text{vol}_x^3] = [\text{vol}_y^3]$. This is because the straight line homotopy between geodesic triangles can be triangulated by 3 (ideal) tetrahedra using the prism operator, and so

$$\text{vol}_x^3 - \text{vol}_y^3 = \delta H_{x,y},$$
Proof. Lemma 2.2. and Proposition 3.3 for most of the proof. We record here an observation.

Theorem 2.1 characterizes when a finitely generated Kleinian representation has non-trivial volume class.

Lemma 2.2. Let G be a countable discrete group. If \( \rho : G \to \text{PSL}_2 \mathbb{C} \) is indiscrete but not dense, then \( [\rho^* \text{vol}_3] = 0 \in H^3_c(G; \mathbb{R}) \).

Proof. Since \( \rho \) is indiscrete and not dense, \( \rho(G) \leq \text{PSL}_2 \mathbb{C} \) is a proper, closed Lie subgroup strictly containing \( \rho(G) \), and is therefore conjugate to one of the following: the stabilizer of \( \infty \in \partial \mathbb{H}^3 \) which is the group \( P \) of upper triangular matrices, \( \text{PSL}_2 \mathbb{R} \), \( \mathbb{R}, \mathbb{R}^2 \), or the circ group \( \text{PSU}(1) \). We claim that \( [\rho^* \text{vol}_3] = 0 \) for some \( e \in H^3 \mathbb{R} \). We just need to choose \( e \) to be the global fixed point of one of these groups or contained in the geodesic hyperbolic plane or line that is preserved, so that every tetrahedron is degenerate and has zero volume. Since \( [\rho^* \text{vol}_3] = 0 \in [\rho^* \text{vol}] \), it follows that \( [\rho^* \text{vol}] = 0 \) as well.

If \( \rho : G \to \text{PSL}_2 \mathbb{C} \) is indiscrete but not dense, we say that \( \rho \) is elementary.

2.5. The bounded Borel class. Let \( \mathcal{F}(\mathbb{C}^n) \) be the space of complete flags. That is, \( F \in \mathcal{F}(\mathbb{C}^n) \) is a sequence of vector subspaces \( \{0\} \leq F^1 \leq \ldots \leq F^n = \mathbb{C}^n \) such that \( \dim \mathbb{C}(F^j) = j \). The group \( \text{PSL}_n \mathbb{C} \) acts transitively on \( \mathcal{F}(\mathbb{C}^n) \); the stabilizer of the standard flag \( \{0\} \leq (e_1) \leq (e_1, e_2) \leq \ldots \leq \mathbb{C}^n \) is the group \( P \) of upper triangular matrices. Since \( P \) is a minimal parabolic subgroup, we can identify \( \text{PSL}_n \mathbb{C}/P = \mathcal{F}(\mathbb{C}^n) \) with the Furstenberg boundary of \( \text{PSL}_n \mathbb{C} \). There is a unique irreducible representation \( \iota_n : \text{PSL}_2 \mathbb{C} \to \text{PSL}_n \mathbb{C} \), up to conjugation, and it induces an equivariant map of Furstenburg boundaries \( \iota_n : \partial \mathbb{H}^3 \to \mathcal{F}(\mathbb{C}^n) \) called the Veronese embedding.

We now describe a class \( \beta_n \in H^3_c(\text{PSL}_n \mathbb{C}; \mathbb{R}) \) called the bounded Borel class, that has a representative \( B^F_n : (\text{PSL}_n \mathbb{C})^4 \to \mathbb{R} \), which measures the projective placement of 4-tuples of flags \( (g_0 F_0, \ldots, g_3 F_3) \) in the following sense. For \( n = 2 \), the flag space is \( \mathcal{F}(\mathbb{C}^2) = \mathbb{P}(\mathbb{C}^2) = \partial \mathbb{H}^3 \), and the volume co-cycle can be computed by evaluating the Bloch-Wigner function (a variant of the di-logarithm) on the cross ratio of a four tuple \( (g_0 z, g_1 z, g_2 z, g_3 z) \). To evaluate the continuous co-cycle \( B^F_n \) we calculate linear projections of \( g_i F^j \) onto vector spaces \( \mathbb{C}^2 \) that arise as the spans and quotients of various combinations of \( g_k F^j \), for different values of \( i, j, k, \) and \( \ell \). For each such \( \mathbb{C}^2 \), there is a projection coming from each flag \( g_i F^j, i = 0, \ldots, 3 \). After projectivization, we have a 4-tuple in \( \mathbb{P}(\mathbb{C}^2) \), and one then evaluates the Bloch-Wigner function on the corresponding cross ratios. There are always at most \( \frac{n(n^2-1)}{6} \) different ways of combining the \( g_i F^j \)'s to form \( \mathbb{P}(\mathbb{C}^2) \)'s on which the projections are non-degenerate, i.e. on which all of the projections have the correct dimension. The definition of the cocycle is rather technical, so we refer to [BB18 Section 2], Bucher, Burger, and Iozzi give a detailed description in the setting \( n = 3 \) of a (boundedly cohomologous) co-cycle, which is illustrative of the general situation.

The flags in the image of the Veronese embedding or one of its translates maximize the co-cycle representing \( \beta_n \). Let \( x \in \partial \mathbb{H}^3 \), then \( B^F_n(x)(\iota_n(g_0), \ldots, \iota_n(g_3)) = \frac{n(n^2-1)}{6} \text{vol}_3(g_0, \ldots, g_3) \) by [BB18 Proposition 21]; which is
to say that for flags in the image of the Veronese embedding, all of their projections in the above sketch of the definition of the co-cycle are projectively equivalent to \((g_0 x, ..., g_3 x)\), and so all of their cross ratios are equal to that of \((g_0 x, ..., g_3 x)\). An interesting fact that is quite important in the work of [BBI18, Theorem 1] (but which we will not use here) is that if \(B^n(g_0, ..., g_3) = v_3\frac{n(n^2 - 1)}{6}\), then there exists \(g \in \text{PSL}_n \mathbb{C}\), \(h_i \in \text{PSL}_2 \mathbb{C}\), and \(x \in \partial \mathbb{H}^3\) such that \(F = g \iota_n(x), g_i = g \iota_n(h_i)\), and \(\text{vol}_3(h_0, ..., h_3) = v_3\) [BBI18, Theorem 19]. In other words, Borel-maximal 4-tuples of flags come exactly from the images of regular, ideal tetrahedra in \(\mathbb{H}^3\). One can think about the the co-cycle as ‘picking out’ the hyperbolic directions, or the boundaries of geodesically embedded copies of \(\mathbb{H}^3\) in the symmetric space for \(\text{PSL}_n \mathbb{C}\). The Borel class generalizes hyperbolic volume in the sense that \(\beta_2 = \text{vol}_3\).

**Theorem 2.3** ([BBI18, Theorem 2]). For each \(n \geq 2\), the Borel class \(\beta_n\) generates \(H^3_{cb}(\text{PSL}_n \mathbb{C}; \mathbb{R})\), and its Gromov norm is

\[
\|\beta_n\|_{\infty} = v_3\frac{n(n^2 - 1)}{6}.
\]

For the irreducible representation \(\iota_n : \text{PSL}_2 \mathbb{C} \to \text{PSL}_n \mathbb{C}\), the pullback satisfies

\[
\iota_n^* \beta_n = \frac{n(n^2 - 1)}{6}\text{vol}_3 \in H^3_{cb}(\text{PSL}_2 \mathbb{C}; \mathbb{R}).
\]

For a countable discrete group \(G\) and representation \(\rho : G \to \text{PSL}_n \mathbb{C}\), the bounded Borel class of \(\rho\) or the Borel class of \(\rho\) is \(\rho^* \beta_n \in H^3_{cb}(G; \mathbb{R})\). Note that by Theorem 2.3, if \(\rho : F_2 \to \text{PSL}_2 \mathbb{C}\) is discrete, faithful, and geometrically infinite, then \(\|(\iota_n \circ \rho)^* \beta_n\|_{\infty} = \frac{n(n^2 - 1)}{6}\|\rho^* \text{vol}_3\|_{\infty} = v_3\frac{n(n^2 - 1)}{6}\), where the last equality was by Theorem 2.4.

2.6. Geometrically infinite ends of hyperbolic 3-manifolds and quasi-isometric equivalence. We say that two discrete and faithful representations \(\rho_1, \rho_2 : \Gamma \to \text{PSL}_2 \mathbb{C}\) of a finitely generated group \(\Gamma\) are quasi-isometric, and write \(\rho_1 \sim_{q.i.} \rho_2\), if there is a \((\rho_1, \rho_2)\) equivariant quasi-isometry \(\mathbb{H}^3 \to \mathbb{H}^3\). Note that conjugate representations are (quasi-)isometric. For \(\Gamma = \pi_1(S)\) where \(S\) is an oriented hyperbolic surface of finite type, by the quasi-conformal deformation theory of Kleinian groups and the Ending Lamination Theorem, to say that \(\rho_1 \sim_{q.i.} \rho_2\) is equivalent to saying that there is a volume preserving bi-Lipschitz diffeomorphism \(M_{\rho_1} \to M_{\rho_2}\) inducing \(\rho_2 \circ \rho_1^{-1}\) on \(\pi_1\). In particular, every convex co-compact representation of a closed surface group is quasi-isometric to a single Fuchsian representation.

**Theorem 2.4** ([Far18, Far18b]). Let \(S\) be an orientable hyperbolic surface of finite type. There is a constant \(\epsilon > 0\) such that if \(\rho_1, \rho_2 : \pi_1(S) \to \text{PSL}_2 \mathbb{C}\) are discrete and faithful with no parabolic elements, then the following are equivalent.

- \(\|\rho_1^* \text{vol}_3 - \rho_2^* \text{vol}_3\|_{\infty} < \epsilon\)
- \(\rho_1^* \text{vol}_3 = \rho_2^* \text{vol}_3\)
- \(\rho_1 \sim_{q.i.} \rho_2\).

Moreover, if \(M_{\rho_1}\) has a geometrically infinite end and \(\rho_3 : \pi_1(S) \to \text{PSL}_2 \mathbb{C}\) is an arbitrary representation satisfying \(\|\rho_1^* \text{vol}_3 - \rho_3^* \text{vol}_3\|_{\infty} < \epsilon\), then \(\rho_3\) is faithful.

There are geometrically finite representations that are not quasi-isometric to each other. For example, if \(\mathcal{P}_1\) and \(\mathcal{P}_2\) are pants decompositions of a closed surface \(S\) with no common curves, then there is a unique conjugacy class of representation \(\rho : \Gamma \to \text{PSL}_2 \mathbb{C}\) such that the quotient manifold has \(|\mathcal{P}_1| + |\mathcal{P}_2|\) rank-one cusps. The curves in \(\mathcal{P}_1\) correspond to curves on the ‘top’ of the convex core that have length 0, and the components of \(\mathcal{P}_2\) are the cusps on the ‘bottom’ of the convex core. The convex core has finite volume, i.e. \(\rho\) is geometrically finite. The volume class of this example vanishes by Theorem 2.4, but \(\rho\) is not quasi-isometric to any Fuchsian representation. We see that the assumption that representations have no parabolic elements cannot be dropped in the above theorem (to have the same conclusion). From Theorem 2.4, once we establish Theorem 1.1, Theorem 1.2 follows.

By the Ending Lamination Theorem [Min10, BCM12], the quasi-isometry classes of discrete and faithful geometrically infinite representations of surface and free groups without parabolics are parameterized by spaces of ending laminations. We briefly describe the situation when \(S\) is a closed surface of genus \(g \geq 2\). Bonahon [Bon89] proved that if an incompressible end \(E\) of a hyperbolic 3-manifold \(M\) has no neighborhood that is disjoint from its convex core, then \(E\) is simply degenerate. That is, \(E \cong S \times (-1, \infty)\) and there is
a sequence of simple closed curves $\gamma_n \subset S \times \{0\}$ such that the geodesic representatives $\gamma^*_n \subset E$ exit every compact subset of $E$. Such an end is called simply degenerate or just degenerate. Thurston [Thu82] proved that there is a geodesic lamination $\lambda_E \subset S$, which is minimal and filling in the absence of parabolic cusps of $M_\rho$, such that $\gamma_n$ accumulate onto it in $\mathcal{PML}$ and the topological support of the simplex is constant and equal to $\lambda_E$. We say that $\lambda_E$ is the ending lamination for $E$. The feature of simply degenerate ends that is interesting for our purposes is that from the geodesic curves $\gamma^*_n$, one can ‘hang’ hyperbolic surfaces $\gamma^*_n \times X_n \subset E$ off of the $\gamma^*_n$ (one can think of the $X_n$ as simplicial hyperbolic surfaces or pleated surfaces). All of the hyperbolic surfaces $X_n$ are mutually homotopic, and while the $X_n$ may not be embedded, there are close by embedded surfaces. One can find collections $X_i$ and $X_j$ that nearly bound submanifolds $W_{i,j} \subset E$ with volume that grows like $|i - j|$ and such that the area of $\partial W_{i,j}$ is uniformly bounded from above. Canary showed that compressible geometrically infinite ends of topologically tame hyperbolic 3-manifolds are degenerate (a slight strengthening of the definition is needed) and that the ending laminations are well defined, up to a natural equivalence [Can92]. See [Far18a] for a more precise discussion of how degenerate ends are related to bounded fundamental classes.

3. An approximation scheme

Suppose we have a dense representation $\rho : G \rightarrow \text{Isom}^+(\mathbb{H}^n)$ where $G$ is countable and discrete. We would like to use the geometry and topology of quotients $\mathbb{H}^n/\Gamma$ for certain finitely generated discrete, torsion free subgroups $\Gamma \leq \text{Isom}^+(\mathbb{H}^n)$ to find chains on $G$ that have large volume (via their $\rho$-action) and small boundary. Indeed, this is exactly our strategy, and we elaborate on the setting that $n = 3$ and $\Gamma$ is a geometrically infinite free group of rank 2 with no parabolic elements.

The manifold $\mathbb{H}^3/\Gamma$ is homeomorphic to a genus 2 handlebody with one end which is homeomorphically the product of a closed surface of genus 2 with an open interval. Geometrically, $\mathbb{H}^3/\Gamma$ is simply degenerate, i.e. there is a sequence of exiting hyperbolic surfaces, all homotopic to inclusion of a level surface in a product neighborhood of the end. Pairs of these surfaces bound regions of large volume, whereas the boundary always has the same topological complexity. In the next section, we use this feature of the simply degenerate ends of hyperbolic 3-manifolds to find straight 3-chains in $\mathbb{H}^3/\Gamma$ with only one vertex, small boundary, and large volume. Since they have only one vertex, the chains are actually defined on $\Gamma = \langle a, b \rangle$. By density, we can approximate $a$ and $b$ by sequences $a_n = \rho(x_n)$ and $b_n = \rho(y_n)$ for suitably chosen $x_n, y_n \in G$. The shape of the chain on $\Gamma$ with large volume and small boundary can be approximated by chains on the groups $\langle a_n, b_n \rangle$. The chains in the approximates thus have essentially the same volume as the chains which came from the manifold $\mathbb{H}^3/\Gamma$.

We now give a criterion for the $n$-dimensional volume class of a representation to be non-zero and have positive seminorm in bounded cohomology.

**Lemma 3.1.** Let $G$ be a countable, discrete group and $\rho : G \rightarrow \text{Isom}^+(\mathbb{H}^n)$ a homomorphism. Choose $x \in \mathbb{H}^n \cup \partial \mathbb{H}^n$, and suppose there exist $\epsilon > 0$ and chains $Z_k \in C_n(G; \mathbb{R})$ for $k = 1, 2, \ldots$ such that

(i) $\frac{\rho^* \text{vol}_n^\epsilon(Z_k)}{\|Z_k\|_1} > \epsilon$ for all $k$,

(ii) $\liminf_{k \rightarrow \infty} \frac{\|\partial Z_k\|_1}{\|Z_k\|_1} = 0$.

Then $[\rho^* \text{vol}_n] \neq 0 \in \mathbb{H}_b^n(G; \mathbb{R})$ and $\|[\rho^* \text{vol}_n]\|_\infty \geq \epsilon$.

**Proof.** Given $b \in C_b^{n-1}(G; \mathbb{R})$, we need to show that $\|\rho^* \text{vol}_n^\epsilon + \partial b\|_\infty > \epsilon$. We have the trivial inequality

$|(\rho^* \text{vol}_n^\epsilon + \partial b)(Z_k)| \leq \|\rho^* \text{vol}_n^\epsilon + \partial b\|_\infty \|Z_k\|_1$.

By the triangle inequality, we have

$|(\rho^* \text{vol}_n^\epsilon + \partial b)(Z_k)| \geq |\rho^* \text{vol}_n^\epsilon(Z_k)| - |\partial b(Z_k)|$.

Another application of the trivial inequality yields

$|\partial b(Z_k)| = |b(\partial Z_k)| \leq \|b\|_\infty \|\partial Z_k\|_1$. 

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Stringing together the inequalities and dividing through by \( \|Z_k\|_1 \), we obtain
\[
\|\rho^* \text{vol}_n^\epsilon + \delta b\|_\infty \geq \frac{\|\rho^* \text{vol}_n^\epsilon(Z_k)\|}{\|Z_k\|_1} - \|b\|_\infty \frac{\|\partial Z_k\|_1}{\|Z_k\|_1}.
\]

By passing to a subsequence, we assume that
\[
\lim_{k \to \infty} \frac{\|\partial Z_k\|_1}{\|Z_k\|_1} = 0,
\]
and we see that for any \( \epsilon > 0 \), there is a \( K \) such that for \( k \geq K \), we have \( \|\rho^* \text{vol}_n^\epsilon + \delta b\|_\infty > \epsilon - \epsilon \). Since \( b \) was arbitrary, this implies that \( \|\rho^* \text{vol}_n\| \geq \epsilon \).

\[ \blacksquare \]

**Remark 3.2.** The arguments in the rest of this section apply to all dimensions \( n \geq 2 \) of hyperbolic space. In fact, all we need is that \( \text{PSL}_2 \mathbb{C} \) is a locally compact topological group, and that \( [\text{vol}] \in H^A_{cb}(\text{PSL}_2 \mathbb{C}; \mathbb{R}) \) has positive semi-norm; we just use continuity over and over. For concreteness, we supply arguments in dimension 3, though we will apply techniques in another setting, as well. See Corollary 3.4, below.

Recall that \( \text{PSL}_2 \mathbb{C} \) has a left invariant metric which is invariant by right multiplication by a maximal compact \( K \leq \text{PSL}_2 \mathbb{C} \) and which descends to the hyperbolic metric on \( \text{PSL}_2 \mathbb{C}/K = \mathbb{H}^3 \). By \( \mathcal{N}_\epsilon(\gamma) \subset \text{PSL}_2 \mathbb{C} \), we mean the \( \epsilon \)-neighborhood of \( \gamma \) with respect to this metric. Suppose \( \rho_0 : F_2 \to \text{PSL}_2 \mathbb{C} \) is discrete and faithful, and let \( W \subset F_2 \) be any finite set. The following is an easy consequence of continuity, properness, and invertibility of multiplication in \( \text{PSL}_2 \mathbb{C} \).

**Lemma 3.3.** Suppose \( \rho_0 : F_2 \to \text{PSL}_2 \mathbb{C} \) is discrete and faithful, and let \( W \subset F_2 \) be any finite set. Let \( \{z^1, ..., z^d\} \) be a free basis for \( F_2 \). Given \( \epsilon > 0 \), there is a \( \delta > 0 \) such that if \( \rho : F_2 \to \text{PSL}_2 \mathbb{C} \) is any representation such that \( \rho(z^i) \in \mathcal{N}_\delta(\rho_0(z^i)) \) for each \( i = 1, ..., d \), then for each \( w \in W \), \( \rho(w) \in \mathcal{N}_\epsilon(\rho_0(w)) \).

Note that in the above lemma, we do not require \( \rho \) to be faithful. For example, given a representation \( \rho' : G \to \text{PSL}_2 \mathbb{C} \), and a set \( \{g_1, ..., g_d\} \subset G \), the rule \( z^i \mapsto \rho'(g_i) \) defines a representation \( \rho : F_2 \to \text{PSL}_2 \mathbb{C} \). We will approximate a Kleinian group \( \Gamma \leq \text{PSL}_2 \mathbb{C} \) by a dense representation \( \rho : G \to \text{PSL}_2 \mathbb{C} \) and approximate the volumes of certain chains on \( G \).

**Proposition 3.4.** Let \( \rho_0 : F_2 \to \text{PSL}_2 \mathbb{C} \) be discrete and faithful and \( \rho : G \to \text{PSL}_2 \mathbb{C} \) be dense. For any \( Z \in C_3(F_2; \mathbb{R}) \) and for any \( \epsilon > 0 \), there exists \( Z(\epsilon) \in C_3(G; \mathbb{R}) \) such that
\[
\|\rho(\text{vol}_n^\epsilon(Z(\epsilon)) - \rho_0 \text{vol}_n(Z))\| < \epsilon.
\]
Moreover, \( \|Z(\epsilon)\|_1 \leq \|Z\|_1 \) and \( \|\partial Z(\epsilon)\|_1 \leq \|\partial Z\|_1 \).

**Proof.** Recall that the volume function for straight hyperbolic 3-simplices is continuous in the vertices. That is, if \( \sigma(x_0, ..., x_3) \) is the oriented, straight hyperbolic 3-simplex with ordered vertex set \( (x_0, ..., x_3) \in \mathbb{H}^3 \times ... \times \mathbb{H}^3 \), then the map
\[
(x_0, ..., x_3) \mapsto \text{vol}_3 \sigma(x_0, ..., x_3)
\]
is continuous. Thus, given a finite collection of straight simplices, \( \{\sigma(x_0^i, ..., x_3^i)\}_{i=1}^M \) and \( \epsilon > 0 \), there is an \( \epsilon' > 0 \) such that if \( y_j^i \in \mathcal{N}_\epsilon(\rho^{-1}(x_j^i)) \) for each \( i = 1, ..., M \) and \( j = 0, ..., 3 \), then we have
\[
\|\text{vol}_3 \sigma(x_0^i, ..., x_3^i) - \text{vol}_3 \sigma(y_0^i, ..., y_3^i)\| < \epsilon, \quad \forall i = 1, ..., M.
\]
Take \( \epsilon = \frac{\epsilon}{\|Z\|_1} \) and find the appropriate \( \epsilon' > 0 \) so that inequality \( \blacksquare \) holds for the collection of isometry classes of simplices in \( \rho_0 \ast \rho(Z) \).

Write \( Z = \sum_{j=1}^M a_j [w_0^j, ..., w_3^j] \), and choose the representative \( (id, w_0^j, w_2^j, w_3^j) \in [w_0^j, ..., w_3^j] \) where \( w_i^j = (w_0^j)^{-1} w_i^j \) for each \( i = 1, 2, 3 \). Take \( W = \{w_i^j : i = 1, 2, 3, j = 1, ..., M\} \). Let \( \{z_1^1, ..., z_d^3\} \) be a free basis for \( F_2 \). By Lemma 3.3, there exists \( \delta > 0 \) such that if \( z_i^j \in G \) satisfy \( \rho(z_i^j) \in \mathcal{N}_\delta(\rho_0(z_i^j)) \) for each \( i = 1, ..., d \), then \( \rho(w) \in \mathcal{N}_\epsilon(\rho_0(w)) \) for all \( w \in W \). Such \( z_i^j \) exist by density of \( \rho \). We consider the map \( i_e : F_2 \to G \) defined by \( z_i^j \mapsto z_i^j \) and denote \( Z(\epsilon) = i_e \ast Z \in C_3(G; \mathbb{R}) \) the chain corresponding to \( Z \) under this identification. Then
\[
\|\rho(\text{vol}_n^\epsilon(Z(\epsilon)) - \rho_0 \text{vol}_n^\epsilon(Z))\| \leq \epsilon \|Z\|_1,
\]
because each isometry class of simplex of \( \rho(Z) \) has volume that is at most \( \epsilon \) different from the volume of the corresponding isometry class of simplex in \( \rho_0(Z) \) by our choice of \( z_i^j \). The map \( i_e \) is 1-form non-increasing chain map, so \( \|Z(\epsilon)\|_1 \leq \|Z\|_1 \) and \( \|\partial Z(\epsilon)\|_1 \leq \|\partial Z\|_1 \). Finally, \( \epsilon \|Z\|_1 = \epsilon \) by our choice of \( e \).
Remark 3.5. We will work with free Kleinian subgroups \( \Gamma \leq \text{PSL}_2 \mathbb{C} \); there is an analogue of Proposition 3.4 for groups that are not free, but the topology of the quotient hyperbolic \( n \)-manifold \( \mathbb{H}^n / \Gamma' \) for arbitrary discrete and torsion free \( \Gamma' \leq \text{Isom}^+(\mathbb{H}^n) \) must be taken into account when constructing chain maps between groups. That is, we are exploiting the fact that homomorphisms from free groups are easy to construct.

Remark 3.6. For even \( n \geq 4 \), it is known that there is an \( \epsilon_n > 0 \) such that the Cheeger constant of \( \mathbb{H}^n / \rho(H) \) is at least \( \epsilon_n \) when \( H \) is a free group of finite rank, a closed surface group of genus at least 2, or a finite volume hyperbolic 3-manifold group and \( \rho \) is discrete and faithful [Bow13]. Vanishing of the Cheeger constant is equivalent to the non-vanishing of \( n \) \( \epsilon_0 \). Moreover, Remark 3.6.

Lemma 3.8. Proposition 4.1.

Remark 3.5. We will work with free Kleinian subgroups \( \Gamma \leq \text{PSL}_2 \mathbb{C} \); there is an analogue of Proposition 3.4 for groups that are not free, but the topology of the quotient hyperbolic \( n \)-manifold \( \mathbb{H}^n / \Gamma' \) for arbitrary discrete and torsion free \( \Gamma' \leq \text{Isom}^+(\mathbb{H}^n) \) must be taken into account when constructing chain maps between groups. That is, we are exploiting the fact that homomorphisms from free groups are easy to construct.

Corollary 3.7. Let \( \rho_0 : F_d \to \text{PSL}_n \mathbb{C} \) be discrete and faithful and \( \rho : G \to \text{PSL}_n \mathbb{C} \) be dense. For any \( Z \in C_3(F_d; \mathbb{R}) \) and for any \( \epsilon > 0 \), there exists \( Z(\epsilon) \in C_3(G; \mathbb{R}) \) such that

\[
\rho^* B_n^\epsilon(Z(\epsilon)) - \rho_0^* B_n^\epsilon(Z) < \epsilon.
\]

Moreover, \( \|Z(\epsilon)\|_1 \leq \|Z\|_1 \) and \( \|\partial Z(\epsilon)\|_1 \leq \|\partial Z\|_1 \).

We record here also a direct analogue of Lemma 3.1.

Lemma 3.8. Let \( G \) be a countable, discrete group and \( \rho : G \to \text{PSL}_n \mathbb{C} \) a homomorphism. Choose \( x \in \mathcal{F}(C^n) \), and suppose there exist \( \epsilon > 0 \) and chains \( Z_k \in C_3(G; \mathbb{R}) \) for \( k = 1, 2, ... \) such that

i. \( \|\rho^* B_n^\epsilon(Z_k)\|_1 > \epsilon \) for all \( k \),

\( \lim_{k \to \infty} \|\partial Z_k\|_1 = 0. \)

Then \( \rho^* \beta_n \neq 0 \in H^3_n(G; \mathbb{R}) \) and \( \|\rho^* \beta_n\|_\infty \geq \epsilon \).

4. Volume classes of dense representations

We call a 3-chain \( Z \) \( \epsilon \)-efficient or just \( \epsilon \)-efficient if \( |\text{vol}_3(Z)| > \epsilon \). There are several chain complexes in which \( Z \) could live, e.g. in a group \( \Gamma \) or a quotient manifold \( \mathbb{H}^3 / \Gamma \), and \( \text{vol}_3 \) should be interpreted in whichever context it makes sense. In what follows, we show that for chains \( Z \) with \( \|\partial Z\|_1 \ll K \), in the context of hyperbolic geometry it is not so important where we compute the volume. This is essentially the content of Proposition 4.1. below.

In [Far15], the author constructed sequences of 3-chains on geometrically infinite genus \( g \) handlebodies with large volume and small boundary that were \( \epsilon_g > 0 \) efficient and for which the boundary surfaces of the chains were well understood. Soma [Som97a, Som97b] constructed \( \nu \) \( (v_3 - \epsilon) \)-efficient chains, for any \( \epsilon > 0 \), but the boundary of his chains are not well controlled, and grow wilder as the efficiency constant gets closer to \( v_3 \). In what follows, we will use Soma’s chains, because we do not need to control the boundary as was necessary in [Far15]. Since handlebodies have free fundamental group, we can approximate these chains with dense representations as in Section 5.

Proposition 4.1. For every positive integer \( d \geq 2 \), there is a \( K_d > 0 \) with the following properties. Let \( \rho_0 : F_d \to \text{PSL}_2 \mathbb{C} \) be discrete, faithful, and geometrically infinite with no parabolic elements. For every \( \epsilon > 0 \) there exists a sequence of chains \( Z_k \in C_3(F_d; \mathbb{R}) \) such that for any \( x \in \mathbb{H}^3 \):

i. \( \|\rho_0^* \text{vol}_3^\epsilon(Z_k)\|_1 > v_3 - \epsilon \), for all \( k \)

\( \|\partial Z_k\|_1 \leq K_d \), for all \( k \)

\( \lim_{k \to \infty} \|\partial Z_k\|_1 = 0. \)
Proof. Let $\Gamma = \rho_0(F_d)$; by the Tameness Theorem [CG06, Aro04], the quotient manifold $M_{\rho_0} = \mathbb{H}^3/\Gamma$ is homeomorphic to a handlebody $H_d$ of genus $d$. Let $f : H_d \to M_{\rho_0}$ be a homeomorphism inducing $\rho_0$ on $\pi_1$. We have a map $f_* : \text{C}_*(H_d; \mathbb{R}) \to \text{C}_*(M_{\rho_0}; \mathbb{R})$ induced at the level of chains, and since $f$ is a homeomorphism, $f_*$ is an isometry with respect to 1-norms. From [Sor97a, Lemma 3.2 and Proposition 3.3] (see also the proof of [Sor97a, Theorem 1]), there exists a $K_d > 0$ such that for every $\epsilon > 0$, there is a sequence $V_k \in \text{C}_3(H_d; \mathbb{R})$ such that the following properties hold:

(a) $\frac{|\text{vol}_3(\text{str}_x V_k)|}{\|V_k\|_1} > v_3 - \epsilon$ for all $k$

(b) $\|\partial V_k\|_1 \leq K_d$, for all $k$

(c) $|\text{vol}_3(\text{str}_x V_k)| \to \infty$.

Choose a point $\bar{x} \in M_{\rho_0}$, and construct the chain map $\text{str}_x : \text{C}_*(M_{\rho_0}; \mathbb{R}) \to \text{C}_*(\Gamma; \mathbb{R})$ from Section 2.3; recall that the operator norm satisfies $\|\text{str}_x\| \leq 1$. By [Par18b, Lemma 3.8],

$$|\text{vol}_3(\text{str}_x f_*, V_k) - \text{str}_x f_* V_k)| \leq 3v_3\|f_*\partial V_k\|_1 \leq 3v_3K_d,$$

where the rightmost inequality is by property (b). We note that $\|\text{str}_x f_* V_k\|_1 \leq \|V_k\|_1$ and $\|\partial \text{str}_x f_* V_k\|_1 \leq K_d$.

Apply the map $\iota_x : \text{C}^{\text{str}}_*(M_{\rho_0}; \mathbb{R}) \to \text{C}_*(\Gamma; \mathbb{R})$ to obtain a sequence $W_k = \iota_x(\text{str}_x f_* V_k)$ so that $\text{vol}_3(W_k) = \text{vol}_3(\text{str}_x f_* V_k)$. Now apply $\rho_0^{-1} : \text{C}_*(\Gamma; \mathbb{R}) \to \text{C}_*(F_d; \mathbb{R})$ to obtain $Z_k = \rho_0^{-1}W_k$ with $\rho_0^* \text{vol}_3(Z_k) = \text{vol}_3(W_k)$. Since $\rho_0$ and $\iota_x$ are both isometries, we have $\|Z_k\|_1 \leq \|V_k\|_1$. Collecting inequalities and notation, we see that

$$|\rho_0^* \text{vol}_3(Z_k) - \text{vol}_3(\text{str}_x f_* V_k)| \leq 3v_3K_d.$$

Since $|\text{vol}_3(\text{str}_x f_* V_k)| \to \infty$ as $k \to \infty$, it follows that $|\rho_0^* \text{vol}_3(Z_k)| \to \infty$, as well. Then $3v_3K/\|V_k\|_1 \to 0$ as $k \to \infty$, and so for $k$ large enough, we have

$$v_3 - \epsilon < \frac{|\text{vol}_3(\text{str}_x f_* V_k)|}{\|V_k\|_1} \leq \frac{|\rho_0^* \text{vol}_3(Z_k)|}{\|Z_k\|_1}.$$

This establishes property (i) after reindexing, and (iii) is immediate because $\|\partial Z_k\|_1 \leq K_d$. \hfill $\square$

Lemma 4.2. In the statement of Proposition 4.7, one may take $x \in \partial \mathbb{H}^3$ so that (ii) holds.

Proof. From Section 2.3 for any point $y \in \mathbb{H}^3 \cup \partial \mathbb{H}^3$ and for any $Z \in \text{C}_3(F_d; \mathbb{R})$,

$$|\rho_0^* \text{vol}_3^y(Z) - \rho_0^* \text{vol}_3^y(Z)| \leq \|\partial Z\|_1 3v_3.$$

So, after reindexing, we can choose $x \in \partial \mathbb{H}^3$. \hfill $\square$

We would like to use our approximation scheme from Section 3 to transfer this information to our dense representation as follows:

Proposition 4.3. Let $G$ be a countable, discrete group and fix $x \in \mathbb{H}^3 \cup \partial \mathbb{H}^3$. If $\rho : G \to \text{PSL}_2 \mathbb{C}$ is dense, then there is a $K > 0$ such that for every $\epsilon > 0$, there is a sequence of chains $D_k \in \text{C}_3(G; \mathbb{R})$ satisfying

(I) $|\rho^* \text{vol}_3^y(D_k)| > v_3 - \epsilon$, for all $k$

(II) $\lim_{k \to \infty} \frac{\|\partial D_k\|_1}{\|D_k\|_1} = 0$

Proof. Fix a geometrically infinite discrete and faithful representation $\rho_0 : F_d \to \text{PSL}_2 \mathbb{C}$ without parabolic cusps, take $\epsilon > 0$, and apply Proposition 4.1 to obtain $Z_k \in \text{C}_3(F_2)$ that satisfy the conclusions (i), (ii), and (iii). For each $k$, we can now apply Proposition 4.4 to obtain $Z_k(1) \in \text{C}_3(G)$ such that

$$|\rho^* \text{vol}_3^y(Z_k(1)) - \rho_0^* \text{vol}_3^y(Z_k)| < 1,$$

and $\|Z_k(1)\|_1 \leq \|Z_k\|_1, \|\partial Z_k(1)\|_1 \leq \|\partial Z_k\|_1$. By property (iii) and the above approximation, we have

$$|\rho^* \text{vol}_3^y(C_k(1))| > v_3 - \epsilon,$$

for large enough $k$. By (iii), $\lim_{k \to \infty} \frac{\|\partial Z_k(1)\|_1}{\|Z_k(1)\|_1} = 0$. Take $D_k = Z_k(1)$. \hfill $\square$
We can now prove the first part of Theorem 1.1.

**Theorem 4.4.** If $G$ is a countable, discrete group and $\rho : G \to \text{PSL}_2 \mathbb{C}$ is dense, then $[\rho^* \text{vol}_3] \neq 0 \in H^3_!(G; \mathbb{R})$ and $\| [\rho^* \text{vol}_3] \|_\infty = v_3$.

**Proof.** The chains $D_k$ from Proposition 4.3 satisfy the hypotheses of Lemma 5.1 so that $\| [\rho^* \text{vol}_3] \|_\infty \geq v_3 - \epsilon$. But $\epsilon > 0$ was arbitrary, so $\| [\rho^* \text{vol}_3] \|_\infty \geq v_3$. On the other hand, $\| [\rho^* \text{vol}_3] \|_\infty \leq \| \text{vol}_3 \|_\infty = v_3$. \(\square\)

We now want to explain how to get definite distance between the volume class for a dense representation and a discrete and faithful representation. The point is that for a finitely generated Kleinian group, *most* infinite index subgroups are geometrically finite. We know that the volume classes for geometrically finite classes are trivial [Som96, Ch 3]. The following technical lemma makes repeated use of the Ending Lamination Theorem [Min10, BCM12, BCM, Bow16] and the Covering Theorem [Can96]. The boundary of a handlebody $H_2$ of genus 2 is a closed surface of genus 2; we consider the proper free rank 2 subgroups of $\pi_1(H_2) = F_2$ case by case.

**Lemma 4.5.** Suppose $\rho_0 : F_2 \to \text{PSL}_2 \mathbb{C}$ is discrete and faithful. Suppose $H_\ell < \cdots < H_1 < H_0 = F_2$ is any chain of proper subgroups (each inclusion is strict) and $H_j$ is free of rank 2 for each $j = 1, \ldots, \ell$. Then for every $\ell \geq j \geq 2$, $\rho_0 \circ i_j : H_j \to \text{PSL}_2 \mathbb{C}$ is geometrically finite with infinite co-volume, where $i_j : H_j \to F_2$ denotes inclusion.

**Proof.** First, we note that by covering space theory of graphs, $H_{\ell+1}$ has infinite index in $H_\ell$, since both groups have the same rank. By Tameness, $M_{\rho_0}$ is homeomorphic to a handlebody $H_2$ of genus 2, so it has one end and at most 2 relative ends. More specifically, the end invariant of $M_{\rho_0}$ breaks $\partial H_2$ into a union of subsurfaces. There is a collection of disjoint, simple closed curves $P$ which are mutually non-isotopic in $H_2$ corresponding to the *parabolic locus* of $M_{\rho_0}$, a collection of *ending laminations* on components of $\partial H_2 \setminus P$ corresponding to the geometrically infinite relative ends of $M_{\rho_0}$, and points in the Teichmüller space for the remaining complementary components (see [Min10] and [Bow16] for details and discussion of the history of the Ending Lamination Theorem and end invariants). By the Covering Theorem [Can96], if $M_{\rho_0}$ was already geometrically finite, then so is $\rho_0 \circ i_j$, for each $j \geq 0$. Indeed, if $M_{\rho_0}$ is geometrically finite, then we are done, so we assume that $M_{\rho_0}$ is geometrically infinite.

Applying Tameness again, $M_{\rho_0 \circ i_1}$ is also a genus 2 handlebody. By the Covering Theorem, if $M_{\rho_0 \circ i_1}$ is geometrically infinite, then the covering projection $\pi : M_{\rho_0 \circ i_1} \to M_{\rho_0}$ is finite to one on a neighborhood $\hat{E} \subset M_{\rho_0 \circ i_1}$ of a relative end of $M_{\rho_0 \circ i_1}$. Moreover, $\hat{E}$ is homeomorphically a product $\hat{E} \cong \hat{Y} \times (0, \infty)$, where $\hat{Y} \subset \partial H_2$ is a homotopically essential subsurface supporting an ending lamination. The locally isometric image $E = \pi(\hat{E})$ also admits a product structure $E \cong Y \times (0, \infty)$, $Y$ finitely covers $Y \subset \partial H_2$, and $Y$ supports an ending lamination. Let $S_{g,n}$ be the orientable surface of genus $g$ and $n$ punctures. The only subsurfaces $Y, \hat{Y} \subset \partial H_2$ that can support ending laminations are topologically $S_{2,0} = \partial H_2, S_{1,2}, S_{1,1}$, and $S_{0,4}$. Before we consider the cases, we claim

**Claim 4.6.** If $Y \cong S_{1,1}$, then $Y$ is incompressible.

**Proof of claim 4.6.** Note that $\partial Y \subset P$ corresponds to a parabolic cusp in $M_{\rho_0}$, so $\partial Y$ is homotopically non-trivial. If $Y$ is compressible, then by the Loop Theorem, there is an essential simple closed curve $\alpha \subset Y$ that bounds a properly embedded disk $D \subset H_2$. Then since $\alpha$ is non-separating, $H_2 \setminus D$ is a solid torus $T$ with two marked points on its boundary. Then $\partial Y$ bounds a disk in $T$ containing the two marked points, because $\partial Y$ separates. The interior of this disk may be homotoped away from $\partial T$, making it a compressing disk with boundary $\partial Y$, which is a contradiction. \(\square\)

Now we begin analyzing cases. In the first case $\hat{Y} = \partial H_2$, $\pi$ is one-to-one on $\hat{E}$. The end invariants of $M_{\rho_0}$ are $M_{\rho_0 \circ i_1}$ are the same, and so $\pi$ is an isometry, by the Ending Lamination Theorem. Thus $H_1 < F_2$ has index 1, which contradicts the assumption that $H_1 < F_2$ is a proper subgroup, proving that $M_{\rho_0 \circ i_1}$ is geometrically finite in this case.

If $Y \cong S_{1,2}$, then by Euler characteristic considerations $Y \cong S_{1,2}$ or $Y \cong S_{1,1}$. In the first case, again $\pi$ is one-to-one, hence an isometry on $\hat{E}$. The complements $\partial H_2 \setminus \hat{Y}$ and $\partial H_2 \setminus Y$ consist of a single annulus in the same homotopy class. The end invariants of $\rho_0 \circ i_1$ and $\rho_0$ are hence the same, which implies that $\pi$ is an isometry and again contradicts the assumption that $H_1 < F_2$ is proper. If $Y \cong S_{1,1}$, then $\hat{Y}$ covers an
incompressible surface, and so is itself incompressible in \( \mathcal{H}_2 \), which is impossible because the rank of \( \pi_1(\hat{Y}) \) is 3 and \( \pi_1(\mathcal{H}_2) = \mathcal{H}_2 \) has rank 2.

Suppose \( \hat{Y} \cong S_{1,1} \). Then \( Y \cong S_{1,1} \), \( \pi \) is an isometry on \( \hat{E} \), and \( Y \to \mathcal{H}_2 \) is a homotopy equivalence by Claim 4.1. We claim that \( M_{\rho_{0i1}} \) has only one geometrically infinite relative end. The complement \( \hat{Y}' = \partial \mathcal{H}_2 \setminus \hat{Y} \) is topologically \( S_{1,1} \). If the end invariant of \( \hat{Y}' \) is an ending lamination, then the end invariants of the two manifolds \( M_{\rho_0} \) and \( M_{\rho_{0i1}} \) coincide, which is impossible. So \( M_{\rho_0} \) has only geometrically infinite relative end. If \( M_{\rho_{0i2}} \) is geometrically infinite, then another application of Tameness and the Covering Theorem yields a geometrically infinite relative end \( \hat{E} \subset M_{\rho_{0i2}} \) with a product structure \( \hat{Y} \times (0, \infty) \), where \( \hat{Y} \subset \partial \mathcal{H}_2 \) finitely covers \( \hat{Y} \cong S_{1,1} \). But \( \hat{Y} \) must be \( S_{1,1} \), because \( \hat{Y} \) is incompressible and the rank of \( \pi_1(S_{1,2}) \) is too large, as before. Moreover \( \hat{Y} \to \hat{Y} \) has degree 1 and \( \hat{Y} \) is incompressible, which implies that \( H_2 = H_1 \).

This is a contradiction, and so \( \pi \) is an isometry, a contradiction. This completes the proof of the lemma, because in all cases \( M_{\rho_{0i2}} \) is geometrically finite.

\[ \text{Theorem 4.7.} \quad \text{Let } G \text{ be a countable discrete group and } \rho : G \to \text{PSL}_2 \mathbb{C} \text{ be dense. If } \rho_0 : G \to \text{PSL}_2 \mathbb{C} \text{ is any other representation and if there is a subgroup } H \leq G \text{ on which } \rho_0 \text{ is faithful or elementary such that } \rho(H) = \text{PSL}_2 \mathbb{C} \text{ but } \rho_0(H) \neq \text{PSL}_2 \mathbb{C}, \text{ then} \]

\[ \| [\rho^* \text{vol}_3] - [\rho_0^* \text{vol}_3] \|_\infty \geq v_3. \]

\[ \text{Proof.} \quad \text{Pass the the subgroup } H \text{ where } \rho_0 \text{ is dense. If } \rho_0(H) \text{ is indiscrete, then it is elementary. If } i : H \to G \text{ denotes inclusion, then } [(\rho_0 \circ i)^* \text{vol}] = 0 \text{ by Lemma 2.2.} \]

In this case, the proof of Theorem 4.4 applies without changes, and we are done, using the fact that \( i^* : H_0^*(G; \mathbb{R}) \to H_0^*(H; \mathbb{R}) \) is norm nonincreasing, i.e.

\[
(2) \quad v_3 = \| [\rho^* \text{vol}_3] \|_\infty \\
(3) \quad = \| [\rho_0^* \text{vol}_3] - [\rho_0^* \text{vol}_3] \|_\infty \\
(4) \quad = \| [i^* (\rho^* \text{vol}_3) - \rho_0^* \text{vol}_3] \|_\infty \\
(5) \quad \leq \| [\rho_0^* \text{vol}_3] - [\rho_0^* \text{vol}_3] \|_\infty.
\]

Since \( \rho \circ i : H \to \text{PSL}_2 \mathbb{C} \) is dense, by [BG03], there is a free rank 2 subgroup \( F_2 \leq H \) on which \( \rho \) is faithful; let \( i_0 : F_2 \to G \) be inclusion. By hypothesis, \( \rho_0 \circ i_0 : F_2 \to \text{PSL}_2 \mathbb{C} \) is faithful, and we may assume that it is not elementary, which means that \( \rho_0 \circ i_0 \) is dense and faithful. We claim that we can find \( H' \leq F_2 \), a proper free subgroup of rank 2 such that \( \rho(H') = \text{PSL}_2 \mathbb{C} \). This is because we can break the Margulis Lemma in the following sense: \( \rho(F_2) \) contains twoloxodromic elements \( a \) and \( b \) that have nearly intersecting axes, very short real translation length (shorter than the 3-dimensional Margulis constant \( \epsilon_3 \)), and are not conjugate into \( \text{PSL}_2 \mathbb{R} \). A point close to their common intersection will be moved very little by both elements. By the Margulis Lemma, \( (a, b) \) is either indiscrete group or virtually abelian. Since large enough powers generate a classical Schottky group, \( (a, b) \) is indiscrete and dense because it is clearly not elementary. Take \( H' = (a, b) \).

By Lemma 4.3, we have only to repeat this process once more to find a subgroup \( H'' \leq H' \) so that \( \rho(H'') \) is dense in \( \text{PSL}_2 \mathbb{C} \), whereas \( \rho_0(H'') \) is geometrically finite. Let \( i'' : H'' \to G \) be the inclusion. By the Tameness Theorem [CG00, Ago04], we can apply [Som97c, Theorem 1] to obtain \( [(\rho_0 \circ i'')^* \text{vol}_3] = 0 \). We can now apply Theorem 4.4 to the class \( [\rho \circ i''^* \text{vol}_3] - [(\rho_0 \circ i'')^* \text{vol}_3] = [\rho \circ i''^* \text{vol}_3] \) because the pullback is norm nonincreasing, as in \( (2) - (5) \).

If we are more careful, we can obtain the following without too much extra work.

\[ \text{Theorem 4.8.} \quad \text{Suppose } \{ \rho_i : G \to \text{PSL}_2 \mathbb{C} \}_{i=1}^\infty \text{ is a collection of dense representations of a countable discrete group } G \text{ such that for every } i = 1, 2, \ldots \text{ there is a subgroup } \iota_i : H_i \to G \text{ such that } \rho_0(H_i) = \text{PSL}_2 \mathbb{C}, \text{ and } \rho_i \circ \iota_j : H_j \to \text{PSL}_2 \mathbb{C} \text{ is either elementary or discrete and faithful for } i \neq j. \text{ Then for any } (a_i) \in \ell^1(\mathbb{N}), \]
we have
\[ \left\| \sum_{i=1}^{\infty} a_i(\rho_i^* \vol) \right\|_\infty \geq \max\{|a_i|\} \cdot v_3. \]
Consequently, \( \{\rho_i^* \vol\} \subset H^3_0(G; \mathbb{R}) \) is a linearly independent set.

**Proof.** The strategy is to prove the inequality for finite sums, and pass to the limit; to this end, let \( N \geq 1 \) for convenience, we assume that \( |a_1| = \max\{|a_i|\} \). First of all, by [BG05] we may assume that \( H_1 \cong F_2 \).

Find \( H_1' \subset H_1 \) a proper rank 2 free subgroup for which \( \rho_1(H_1') = \text{PSL}_2 \mathbb{C} \), which can always be done by ‘breaking the Margulis Lemma’ as in the proof of Theorem 4.7 Repeat this process once more to obtain \( H_1'' \subset H_1' \). By Lemma 4.3 if \( j \neq 1 \) and \( \rho_j(H_1) \) was discrete, then \( \rho_j(H_1'') \) is geometrically finite. Otherwise, \( \rho_j(H_1'') \) is elementary. Abusing notation, assume that \( H_1 = H_1'' \), so that by Theorem 2.1 and Lemma 2.2 we have \( \{\rho_j \circ \iota_1\}^* \vol_3 = 0 \), for \( j > 1 \).

That is, there is a \( \beta_j \in C^2_0(H_1; \mathbb{R}) \) such that \( (\rho_j \circ \iota_1)^* \vol_3 = \delta \beta_j \). Thus
\[ \sum_{j=2}^{N} a_j(\rho_j \circ \iota_1)^* \vol_3 = \sum_{j=2}^{N} a_j \delta \beta_j = \delta \left( \sum_{j=2}^{N} a_j \beta_j \right) = \delta \beta \in C^2_0(H_1; \mathbb{R}), \]
and
\[ \iota_1^* \left( \sum_{i=1}^{N} a_i \rho_i^* \vol_3 \right) = a_1(\rho_1 \circ \iota_1)^* \vol_3 + \delta \beta. \]

Let \( \epsilon > 0 \) be given and apply Proposition 4.3 to \( \rho_1 \circ \iota_1 : H_1 \to \text{PSL}_2 \mathbb{C} \) to obtain chains \( \iota_1^* (D_k) \in C_3(G; \mathbb{R}) \) satisfying properties (1) and (11). Lemma 4.3 tells us that
\[ \left\| \iota_1^* \left( \sum_{i=1}^{N} a_i \rho_i^* \vol_3 \right) \right\|_\infty = |a_1| \cdot \|((\rho_1 \circ \iota_1)^* \vol_3 + \delta \beta)\|_\infty \geq |a_1| \cdot (v_3 - \epsilon). \]

Since \( \iota_1^* : H^3_0(H_1; \mathbb{R}) \to H^3_0(G; \mathbb{R}) \) is norm non-increasing, \( \epsilon \) was arbitrary, and the inequality holds for all \( N \), the first result follows. If there is a sequence \( (a_i) \in \ell^1(\mathbb{N}) \) such that \( \sum_{i=1}^{N} a_i \rho_i^* \vol_3 = 0 \), then \( \max\{|a_i|\} = 0 \) which means that \( a_i = 0 \) for all \( i \), establishing linear independence.

5. **Borel classes of dense representations**

In this section, we show that pullbacks of the Borel class under dense representations have maximal semi-norm. Since the structure of discrete subgroups \( \Gamma \leq \text{PSL}_n \mathbb{C} \) is not well understood, we cannot give a simple criterion for the differences of the Borel classes to be separated in semi-norm. For example, we used the Covering Theorem and the Ending Lamination Theorem in the proof of Lemma 4.3 and geometrical infiniteness is not well understood in the higher rank setting. It would be interesting to investigate whether maximality of the semi-norm of the pullback of the Borel class under discrete and faithful representations \( \rho : \tau_1(S) \to \text{PSL}_n \mathbb{C} \) characterizes geometrical infiniteness, where \( S \) is an orientable surface of finite type.

The proof of the main result in this section is nearly identical to what we do in the previous section. In fact, much of the work of the previous section can be thought of as a corollary of the results in this section. We prefer the organization of the paper as it is, because the main construction is really one about hyperbolic 3-manifolds.

**Theorem 5.1.** Let \( G \) be a countable discrete group and \( \rho : G \to \text{PSL}_n \mathbb{C} \) be dense. Then
\[ \|\rho^* \beta_n\|_\infty = v_3 \frac{n(n^2-1)}{6}. \]
Recall that there is a unique (up to conjugacy) irreducible representation \( \iota_k : \text{PSL}_2 \mathbb{C} \to \text{PSL}_k \mathbb{C} \). The following will be immediate from the proof of Theorem 5.1 and Theorem 4.7

**Corollary 5.2.** Let \( G \) be a countable discrete group and \( \rho : G \to \text{PSL}_n \mathbb{C} \) be dense. Suppose also that \( \rho_0 : G \to \text{PSL}_2 \mathbb{C} \) is such that there exists a finitely generated \( H \leq G \) such that \( \rho(H) = \text{PSL}_n \mathbb{C} \) and \( \rho_0 \) is discrete, faithful, and geometrically finite or elementary restricted to \( H \). Then
\[ \|\rho^* \beta_n - (\iota_k \circ \rho_0)^* \beta_k\|_\infty \geq v_3 \frac{n(n^2-1)}{6} \]
for all \( k \geq 2 \).

The point is that \((\rho_0 \circ i)^*[\text{vol}_3] = 0\) and so \((\iota_k \circ \rho_0 \circ i)^* \beta_k = 0\), as well by Theorem 2.3 where \( i : H \to G \) is inclusion.

**Remark 5.3.** Another easy consequence of Theorem [2.4] and the triangle inequality is that if \( \rho_1 : G \to \text{PSL}_n \mathbb{C} \) and \( \rho_2 : G \to \text{PSL}_k \mathbb{C} \) are both dense, then

\[
\|\rho_1^* \beta_n - \rho_2^* \beta_k\| \geq v_3 \frac{n(n^2 - 1)}{6} - \frac{k(k^2 - 1)}{6}.
\]

We begin by finding a sequence of efficient chains on which the Borel class evaluates to a large number with controlled boundary. These chains come from hyperbolic geometry, together with the explicit description of the continuous co-cycle given by [BB18] and the explicit formula for the semi-norm of the pull-back under the irreducible representation \( \iota_n : \text{PSL}_2 \mathbb{C} \to \text{PSL}_n \mathbb{C} \).

**Lemma 5.4.** If \( G \) is a countable discrete group and \( \rho : G \to \text{PSL}_n \mathbb{C} \) is dense, then for every \( \epsilon > 0 \), there is a \( y \in \mathcal{F}(\mathbb{C}^n) \) and a sequence of chains \( D_k \in C_3(G; \mathbb{R}) \) such that

\[
(A) \quad \frac{|\rho^* B_n^y(D_k)|}{\|D_k\|[1]} > (v_3 - \epsilon) \frac{n(n^2 - 1)}{6}, \quad \text{for all } k
\]

\[
(B) \quad \lim_{k \to +\infty} \frac{1}{\|\partial D_k\|[1]} = 0
\]

**Proof.** As in the proof of Proposition 4.3 fix a geometrically infinite discrete and faithful representation \( \rho_0 : F_2 \to \text{PSL}_2 \mathbb{C} \), take \( \epsilon > 0 \) and apply Proposition 4.1. We now have \( K_2 > 0 \) and \( Z_k \in C_3(F_2; \mathbb{R}) \) that satisfy conditions (I), (II), and (III) of the conclusion of Proposition 4.1 where we may assume that \( x \in \partial \mathbb{H}^3 \) for the conclusion (I), by Lemma 4.2. Recall that the irreducible representation \( \iota_n \) induces an equivariant continuous map of boundaries \( \iota_n \circ \rho_0 \circ \iota_n : \partial \mathbb{H}^3 \to \mathcal{F}(\mathbb{C}^n) \). Take \( y = \iota_n(x) \), so that by equivariance of \( \iota_n \), we have \( \iota_n(\rho_0(\gamma_k(x)) = (\iota_n \circ \rho_0)_\gamma(\gamma_k)^n(y) \). By the explicit description of \( B_n^y \in \beta_n \) [BB18, Proposition 21], we have

\[
\iota_n^* B_n^y = \frac{n(n^2 - 1)}{6} \text{vol}_x.
\]

Thus,

\[
(t_n \circ \rho_0)^* B_n^y(Z_k) = \frac{n(n^2 - 1)}{6} \rho_0^* \text{vol}_x^n(Z_k).
\]

For each \( k \), apply Corollary 5.7 to obtain \( Z_k(1) \in C_3(G; \mathbb{R}) \) such that

\[
|\rho^* B_n^y(Z_k(1)) - (t_n \circ \rho_0)^* B_n^y(Z_k)| < 1.
\]

As in the proof of Proposition 4.3 the two conclusions now follow with \( D_k = Z_k(1) \).

**Proof of Theorem 5.7**. The chains from Lemma 5.4 satisfy the hypotheses of Lemma 5.8. Thus \( \|\rho^* \beta_n\| \geq (v_3 - \epsilon) \frac{n^2 - 1}{6}, \) for all \( \epsilon > 0 \). On the other hand, by [BB18], \( \|\rho^* \beta_n\| \leq v_3 \frac{n^2 - 1}{6}, \) which yields the desired equality.

6. **On the cardinality of dimension of dense volume classes**

6.1. **Constructing incompatible representations.** Given any natural number \( N \) we will construct \( N \) dense representations \( \{\rho_i : F_2 \to \text{PSL}_2 \mathbb{C}\}_{i=1}^N \) and free-rank 2 subgroups \( H_i \subset F_2 \) such that \( \rho_i(H_i) \subset \text{PSL}_2 \mathbb{C} \) is dense, but \( \rho_i(H_j) \) is a classical Schottky group for \( i \neq j \). We start with a loxodromic element \( a \in \text{PSL}_2 \mathbb{C} \) such that the complex translation length of \( a \) has norm \( r \) and non-zero rotational part, i.e. \( a \) is not conjugate into \( \text{PSL}_2 \mathbb{R} \), and the translation length of \( a \) is \( re^{it} \), where \( t \in (0, 2\pi) \). Find an elliptic element \( b(\theta) \) with irrational rotation angle \( \theta \in \mathbb{R}/\mathbb{Z} \) and with fixed line contained in an \( \mathbb{H}^2 \) containing the axis of \( a \), meeting it orthogonally. For infinitely many values \( n \), the axis of \( b(\theta)^n ab(\theta)^{-n} \) is nearly parallel to the axis of \( a \).

**Lemma 6.1.** Let \( r > 0 \) be given. There is a threshold threshold \( \tau_0 \in (0, 1) \) such that if \( n \theta \in (-\tau_0, \tau_0) \) mod 1, then \( \langle a, b(\theta)^n ab(\theta)^{-n} \rangle \) is dense in \( \text{PSL}_2 \mathbb{C} \).

**Proof.** Let \( \gamma = \text{axis}(a) \). Then for any \( \tau \in (0, 1) \), \( \text{axis}(b(\tau)ab(\tau)^{-1}) = b(\tau)(\gamma) \), and \( \gamma \cap b(\tau)(\gamma) = \{x\} \) since the fixed line of \( b(\tau) \) meets \( \gamma \) in the point \( x \in \mathbb{H}^2 \). By hyperbolic trigonometry, there is a \( \tau_0 = \tau_0(r) \) such that if \( -\tau_0 < \tau < \tau_0 \) mod 1, then in an \( (r + 1) \)-neighborhood of \( x \), the Hausdorff distance between
γ and b(ρ)(γ) is at most ε3, the 3-dimensional Margulis constant. Specifically, take τ₀ ∈ (0, 1) such that 2 sinh⁻¹(sin(2πτ₀) sinh(r + 1)) < ε₃. Suppose −τ₀ < τ < τ₀ mod 1, and for convenience, write b for b(ρ).

Then d(ax, bab⁻¹x) < ε₃, because d(x, ax) = d(x, bab⁻¹x) = r < r + 1. This means that d(ba⁻¹b⁻¹ax, x) < ε₃. Similarly, d(bab⁻¹a⁻¹x, x) < ε₃. By the Margulis Lemma, the group ⟨a, bab⁻¹⟩ is therefore indiscrete if it is not virtually abelian. For large enough k, ⟨aᵏ, bab⁻¹k⟩ is free, by the Ping-Pong Lemma, so ⟨a, bab⁻¹⟩ is indiscrete. Moreover, ⟨a, bab⁻¹⟩ is dense because it is clearly non-elementary, hence Zariski dense. The lemma follows. □

There are also infinitely many values of n such that γ is nearly orthogonal to b(θ)ⁿγ. The following is immediate from the Ping-Pong Lemma and a direct computation in □

Lemma 6.2. For r > 1 + cos(π/8)/1 - cos(π/8), if the norm of the complex translation length of a is at least r and nθ ∈ (1/8, 3/8) mod 1, then ⟨a, b(θ)ⁿab(θ)⁻ⁿ⟩ is a classical Schottky group of rank 2.

Now fix a ∈ PSL₂ C with translation length re⁻ᵗ, with r > 1 + cos(π/8)/1 - cos(π/8) and t ∈ (0, 2π).

Lemma 6.3. Let {θ₁, ..., θₙ} ⊂ (0, 1) be a rationally independent set, F₂ = ⟨z₁, z₂⟩, and let ρᵢ : F₂ → PSL₂ C be defined by ρᵢ(z₁) = a and ρᵢ(z₂) = b(θᵢ). There are numbers n₁, ..., nₙ such that Hᵢ := ⟨z₁, z₂, z₃⁻ⁿ_i⟩ satisfies ρᵢ(Hᵢ) ≤ PSL₂ C is dense but ρᵢ|Hᵢ is discrete, faithful, and geometrically finite.

Proof. Since {θᵢ} is a rationally independent set, the obvious self mapping of the N-torus Θ : (R/Z)ᴺ → (R/Z)ᴺ defined by {θᵢ} is minimal. Let τ₀ be the threshold form Lemma 6.4 and let D = (−τ₀, τ₀) ⊂ R/Z and F = (1/8, 3/8) ⊂ R/Z. For each i, let pᵢ : (R/Z)ᴺ → R/Z be projection onto the i-th factor and take Uᵢ ⊂ (R/Z)ᴺ to be the product of D’s and F’s where pᵢ(Uᵢ) = D and p_j(Uᵢ) = F, if i ≠ j. By minimality of Θ, there is an nᵢ such that Θᴺ(0) ∈ Uᵢ. By Lemma 6.1, ρᵢ(Hᵢ) = PSL₂ C, and ρᵢ|Hᵢ : Hᵢ → PSL₂ C is discrete, faithful, and convex cocompact for i ≠ j, by Lemma 6.2. □

Corollary 6.4. The cardinality of the dimension of the Banach subspace of reduced bounded cohomology D ⊂ H₀ b(F₂; R) that is the closure of the subspace spanned by the volume classes of dense representations is #R.

Proof. Let N ∈ N and construct the representations {ρᵢ : F₂ → PSL₂ C : i = 1, ..., N} from Lemma 6.3. By Theorem 4.3, {ρᵢ | vol₃} ≠ 0 ∈ H₀ b(F₂; R) for each i = 1, ..., N. The collection {ρᵢ}ᴺᵢ₌₁ satisfies the hypotheses of Theorem 4.3, so {ρᵢ | vol₃} ⊂ H₀ b(F₂; R) is a linearly independent set. Since N was arbitrary, dimᵣ D ≥ #N. Then D is a Banach subspace of H₀ b(F₂; R), so dimᵣ D ≥ #R. On the other hand, dimᵣ C₃(F₂; R) = #R, so dimᵣ D = #R. □

6.2. On the collection of ρ-dense subgroups. Let G be a countable discrete group and suppose ρ : G → PSL₂ R is dense. Define the set DS(ρ) of ρ-dense subgroups by DS(ρ) = {H ⊆ G : ρ(H) = PSL₂ R}, where H ⊆ G. G means that H is a finitely generated subgroup of G. The following fact may seem surprising, at first. An outline of the proof was communicated to the author by Yair Minsky.

Fact 6.5. Let ρ₁, ρ₂ : F₂ → PSL₂ R be dense and faithful. If DS(ρ₁) = DS(ρ₂), then ρ₁ is conjugate to ρ₂ in PSL₂ R.

The proof of fact 6.5 uses the fact that dense representations contain elliptic elements, since (0, 2) is an open in the image of the absolute value of the trace function. Generically, dense representations F₂ → PSL₂ C do not contain any elliptics. Due to the fractal nature of the boundary of Schottky space, one might expect an answer to the following to be more involved.

Question 6.6. Let ρ₁, ρ₂ : F₂ → PSL₂ C be dense and faithful. If DS(ρ₁) = DS(ρ₂), then is ρ₁ necessarily conjugate to ρ₂ in PSL₂ C?

If Question 6.6 has an affirmative answer, then the volume class of every conjugacy class of dense representation is distinguished and separated in semi-norm from every other such class.
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