When does a Measurement or Event Occur?

J. Oppenheim\(^{(a)}\), * B. Reznik\(^{(b)}\), † and W. G. Unruh\(^{(a)}\)‡

\(^{(a)}\) Department of Physics, University of British Columbia, 6224 Agricultural Rd. Vancouver, B.C., Canada V6T1Z1

\(^{(b)}\) Theoretical Division, T-6, MS B288, Los Alamos National Laboratory, Los Alamos, NM, 87545

Abstract

Within quantum mechanics it is possible to assign a probability to the chance that a measurement has been made at a specific time \(t\). However, the interpretation of such a probability is far from clear. We argue that a recent measuring scheme of Rovelli’s \(^{[\text{quant-ph/9802020}]}\) yields probabilities which do not correspond to the conventional probabilities usually assigned in quantum mechanics. The same arguments also apply to attempts to use the probability current to measure the time at which a particle arrives at a given location.

* jono@physics.ubc.ca
† reznik@t6-serv.lanl.gov
‡ unruh@physics.ubc.ca
The observable $A$ of a quantum system $S$ can be measured by coupling a macroscopic apparatus $O$ to it, via an interaction such as

$$H = g(t)PA$$

where $P$ is the conjugate momentum to the pointer $Q$ of the measuring device, and $g(t)$ is a function which is zero everywhere, except during a small interval of time. After the measurement is complete, the measuring apparatus will be correlated with the state of the system. If initially, $S$ is in a superposition of eigenstates $|\phi_i\rangle$ of the observable $A$, so that $|\psi_S\rangle = \sum_i c_i |\phi_i\rangle$, then we expect the initial state of the combined $S - O$ system to evolve into a correlated state.

$$\sum_i c_i |\phi_i\rangle \otimes |O\rangle \rightarrow \sum_i c_i |\phi_i\rangle \otimes |O_i\rangle$$

where $|O\rangle$ is the original state of the device and the $|O_i\rangle$ are orthogonal states of the measuring apparatus which are correlated with the system. If the coupling is small, then the duration of the measurement might need to be long in order to distinguish between the various eigenvalues of $A$. At any time during the measurement, it is possible to calculate the density matrix of the combined S-O system. One can imagine that a second apparatus $O'$ measures the state of the first apparatus $O$ to determine whether a measurement has occurred. This has been studied for the case when the measurement is gradual $[1] \ [2]$. Rovelli $[3]$ has recently proposed that the apparatus $O'$ might measure the operator

$$M = \sum_i |\phi_i\rangle \otimes |O_i\rangle \langle O_i| \otimes \langle \phi_i|.$$ 

This is a projector onto the space of states where a correlation exists between the measuring apparatus and the quantum system. In the measurement scheme proposed by Rovelli, the probability that a measurement has been made at time $t$ is given in the Heisenberg representation by

$$P_M(t) = \langle \psi_{SO}|M(t)|\psi_{SO}\rangle$$
where $|\psi_{SO}\rangle$ is the state of the combined S-O system. The operator which gives the probability that a measurement was made between times $t$ and $t + dt$ is

$$m(t) = \frac{dM(t)}{dt}. \quad (5)$$

It is interesting to compare this method for measuring the time of a measurement to the use of the probability current to measure the time-of-arrival [4]. One imagines that a particle is localized in the region $x < 0$ and travelling towards the origin. The projector $\Pi_+ = \int_0^\infty dx |x\rangle \langle x|$ is an operator which is equal to one when $x > 0$ and zero otherwise. The probability of detecting the particle in the positive x-axis is given by

$$P_+(t) = \langle \psi | \Pi_+(t) | \psi \rangle. \quad (6)$$

In the Schrödinger representation, this expression is just $P_+(t) = \int_0^\infty |\psi(x,t)|^2 dx$. It is then claimed that the current $J_+$, given by

$$\frac{\partial J_+}{\partial x} = \frac{d\Pi_+(t)}{dt} \quad (7)$$

will give the probability that the particle arrives between $t$ and $t + dt$.

The problem lies with interpreting these probabilities as *probabilities in time*. In conventional quantum mechanics, for each observable, we can assign an operator $A$. At each time $t$ there exists a Hilbert space and inner product which enable one to calculate the probability $P_a(t)$ that an observation yields the result $a$. The probability of finding the result $a$ at time $t$ is independent of the probability of finding the result $a'$ at the same time. ie. if $\Pi_a(t)$ is the projection operator onto the state with eigenvalue $a$, then

$$[\Pi_a(t), \Pi_{a'}(t)] = 0. \quad (8)$$

If $a \neq a'$ then the projection operators project onto orthogonal states.

If we interpret the probabilities $P_a(t) = \langle \psi | \Pi_a(t) | \psi \rangle$ as probabilities in time, then our conventional notions of what these probabilities mean, break down. In general,

$$[\Pi_a(t), \Pi_{a'}(t')] \neq 0. \quad (9)$$
Measurements made at earlier times influence measurements made at later times. One doesn’t expect any of the operators $\mathbf{M}(t)$ and $\mathbf{m}(t)$ (or for the time-of-arrival case, $\Pi_+(t)$ and $\mathbf{J}_+(t)$) to commute with themselves at different times $\text{[]}$. Furthermore, one no longer has the same notion of orthogonality as with conventional probabilities. At a given time $t$ a system can only take on one value $a$, but for a given value $a$, a system may attain this value at many different times $t$. Furthermore, the probabilities $P_a(t)$, while they are normalized at a given time

$$\sum_a P_a(t) = 1 \quad (10)$$

are not normalized as probabilities in time.

$$\int_{-\infty}^{\infty} dt P_a(t) \neq 1 \quad (11)$$

The norm can even be zero or infinite. One can try to normalize the probabilities, but the normalization is different for each state $\psi$, and one needs to know the state $\psi$ at all times before the normalization can be done.

Another problem is that probabilities derived from operators such as $\mathbf{m}(t)$ will be negative when $\mathbf{M}(t)$ is not a monotonically increasing function of time. In addition, one finds that an operator such as $\mathbf{J}_+(t)$ can attain negative values, even for particles which only contain modes of positive frequency $\text{[]}$. The possibility of negative values for these quantities makes it impossible to interpret them as probabilities. Furthermore, it can be shown $\text{[]}$ that formally, an operator which measures the time of the occurrence of an event cannot exist.

Instead of considering operators, a more physical meaningful method of measuring the occurrence of an event is to consider continuous measurement processes. For example, the operator $\Pi_a(t)$ can be measured continuously or at small time intervals. The probability of

$\text{[}A$ similar problem is encountered when one attempts to measure the dwell time for a particle tunnelling through a potential barrier $\text{]}$
finding that the system enters the state $\phi_a$ at time $t_a$ is given by the probability that it isn’t in the state $\phi_a$ before $t_a$, times the probability that it is in the state $\phi_a$ at $t_a$.

To see how such a scheme might work, let us see how one would measure the time of an occurrence of a measurement. A measurement of the operator $M(t)$ will tell us whether a measurement has occurred. We can then measure $M(t)$ at times $t_k = k\Delta$ for integral $k$ in order to determine when the measurement occurred. We will now work in the Schrödinger representation, simply because it is the most natural arena to talk about successive measurements on a system.

At time $t_1$, the probability that a measurement has occurred is given by

$$P(\uparrow_z, t_1) = \langle \psi_0(0)|U_\Delta^\dagger MU_\Delta|\psi_0(0)\rangle$$

and the probability that it hasn’t is

$$P(\downarrow_z, t_1) = \langle \psi_0(0)|U_\Delta^\dagger(1-M)U_\Delta|\psi_0(0)\rangle$$

where $\uparrow_z$ and $\downarrow_z$ correspond to detection or null, $\psi_0(0)$ is the initial state of the system and $U_\Delta$ is the evolution operator $e^{-iH\Delta}$. If the result is null, we collapse the wave function and evolve it to the next instant. The normalized state before the second measurement is:

$$|\psi_2(t_2)\rangle = \frac{U_\Delta(1-M)U_\Delta|\psi_0\rangle}{\langle \psi_0|U_\Delta^\dagger(1-M)U_\Delta|\psi_0\rangle^{1/2}}$$

The probability that a measurement has occurred at $t_2$ is given by the probability that a measurement didn’t occur at $t_1$ times the probability that $\psi_2$ is in one of the states $|\phi_i\rangle \otimes |O_i\rangle$

$$P(\uparrow_z, t_2) = \frac{\langle \psi_0|U_\Delta^\dagger(1-M)U_\Delta^\daggerMU_\Delta(1-M)U_\Delta|\psi_0\rangle}{\langle \psi_0|U_\Delta^\dagger(1-M)U_\Delta|\psi_0\rangle} \times \langle \psi_0|U_\Delta^\dagger(1-M)U_\Delta|\psi_0\rangle$$

The probability that a measurement didn’t occur is given by

$$P(\downarrow_z, t_2) = \langle \psi_0|U_\Delta^\dagger(1-M)U_\Delta^\dagger(1-M)U_\Delta(1-M)U_\Delta|\psi_0\rangle$$

By repeating this process, we find that at time $t_k$ the probability that a measurement has occurred is given by
\[ P(\uparrow_z, t_k) = \langle \psi_0 | A_k | \psi_0 \rangle \]  \hspace{1cm} (17) 

where

\[ A_k = U_\Delta \dagger (1 - M) U_\Delta \dagger (1 - M) \ldots U_\Delta \dagger M U_\Delta \ldots (1 - M) U_\Delta (1 - M) U_\Delta \]  \hspace{1cm} (18) 

and the probability that a measurement hasn’t occurred is

\[ P(\downarrow_z, t_k) = \langle \psi_0 | B_k | \psi_0 \rangle \]  \hspace{1cm} (19) 

with

\[ B_k = U_\Delta \dagger (1 - M) U_\Delta \dagger (1 - M) \ldots U_\Delta \dagger (1 - M) U_\Delta \ldots (1 - M) U_\Delta (1 - M) U_\Delta \]  \hspace{1cm} (20) 

By acting the unitary operators on the projection operators we can write the \( A_k \) or \( B_k \) in the "Heisenberg representation." For example

\[ A_k = (1 - M)(t_1) \ldots (1 - M)(t_{k-1})(1 - M)(t_k)(1 - M)(t_{k-1}) \ldots (1 - M)(t_1) \]  \hspace{1cm} (21) 

However, while the operators \( M(t) \) can be found by unitary time-evolution of \( M(0) \), the operators \( A_k \) and \( B_k \) are not related by a unitary transformation to \( A_0 \) and \( B_0 \). Nor are the \( A_k \) and \( B_k \) projection operators. By construction, the sum of the probabilities, \( \sum_{k=1}^{\infty} P(\uparrow_z, t_k) = 1 \), however, these probabilities are not universal. In this case, they apply only to the particular measurement scenario under discussion. As we will now show, the probability distribution is sensitive to the frequency at which \( M \) is measured, a phenomenon which is related to the Zeno paradox [4].

As an example, consider a spin 1/2 particle which is in a state given by

\[ |\psi_S\rangle = a |\uparrow_z\rangle + b |\downarrow_z\rangle \]  \hspace{1cm} (22) 

and a simple measuring device which is also a spin 1/2 particle initially in the state \( |O\rangle = |\uparrow_z\rangle \), which evolves according to the Hamiltonian

\[ H = g(t)\sigma_x'(1 - \sigma_z) \]  \hspace{1cm} (23)
where \( g(t) = \frac{\pi}{T} \) when \( 0 < t < T \) and the primed Pauli matrix acts on the measuring device, while the unprimed Pauli matrix acts on the system. After a time \( T \), the spin of the measuring device will be correlated with the system. Since this measurement is rather crude, (the initial state of the device is the same as one of the measurement states), the operator \( M \) at \( t = 0 \) is not zero. Let us simplify the problem further, by assuming that \( a = 0 \) and \( b = 1 \). In this case, the only relevant matrix element of \((1-M)\) is \( | \uparrow_z \rangle \otimes | \downarrow_z' \rangle \langle \downarrow_z' | \otimes \langle \uparrow_z | = | \psi_o \rangle \langle \psi_o | \). We then find the probability that the measuring apparatus has not responded at time \( t_k \) is

\[
P(\downarrow_z, t_k) = |\langle \psi_o | U_\Delta | \psi_o \rangle|^{2k}
\]

\[
\simeq |\langle \psi_o | 1 - i \Delta \mathbf{H} - \Delta^2 \mathbf{H}^2 | \psi_o \rangle|^{2k}
\]

\[
\simeq 1 - \Delta^{2k}(\langle \mathbf{H}^2 \rangle - \langle \mathbf{H} \rangle^2)^k
\]

(24)

If we fix a value of \( \tau = t_k \) and then make \( \Delta \) go to zero, we find

\[
P(\downarrow_z, \tau) \simeq e^{-(\Delta dE)^2/\Delta}
\]

\[
\simeq 1,
\]

(25)

which implies that the measuring apparatus becomes frozen and never records a measurement. In order not to freeze the apparatus, we need \( \Delta > 1/dE \). There is always an inherent inaccuracy when measuring the time that the event (of the measurement) occurred. This inaccuracy is similar to the one found when trying to measure the time-of-arrival or the traversal-time \( 3 8 10 9 \). Note that this inaccuracy is not related to the so-called "Heisenberg energy-time uncertainty relationship" as it applies to every single measurement and not to the width of measurements carried out on an ensemble.

We have seen that operators which classically might give the time of an event cannot be given a physical interpretation. Several authors \( 11 12 \) have maintained that the problems with defining an operator for the time of an event are technical, and can be circumvented by slightly modifying these operators. However, we have argued that probabilities in time are fundamentally different from traditional probabilities in Quantum Mechanics, and that there is a limitation on these measurements.
REFERENCES

[1] A Peres, WK Wootters: Phys Rev D32 (1985) 1968

[2] Y. Aharonov, B. Reznik, quant-ph/9704001

[3] C. Rovelli, quant-ph/902020

[4] R.S. Dumont, T.L. Marchioro II, Phys. Rev. A 47 85 (1993); C.R. Leavens, Phys. Lett. A 178, 27 (1993); W.R. MacKinnon, C.R. Leavens, Phys. Rev. A, 51 2748 (1995); J.G. Muga, S. Brouard, D. Macias, Ann. Phys. (N.Y.) 240, 27 (1995); Ph. Blanchard, A. Jadczyk, quant-ph/9602010; V. Delgado, J.G. Muga, Phys. Rev. A 56 3425 (1997) (quant-ph/9704010); V. Delgado, quant-ph/9709037

[5] Bracken A.J. and Melloy G.F., J. Phys. A: Math. Gen 27 (1994) 2197

[6] Y. Aharonov, J. Oppenheim, S. Popescu, B. Reznik, W.G. Unruh, to be published in Phys. Rev. A, (quant-ph/9709031).

[7] J. von Neumann, Mathematische Grundlagen der Quantenmechanik (Springer, Berlin, 1932) p. 195, [English translation: Mathematical Foundations of Quantum Mechanics, trans E.T. Beyer (Princeton University Press, Princeton, 1995) p.366]; B. Misra and E.C.G. Sudarshan, J. Math. Phys. 18, 756 (1977),

[8] J. Oppenheim, B. Reznik, W.G. Unruh, quant-ph/9801034

[9] G.R. Allcock, Ann Phys 53 (1996), 253

[10] A. Peres, Am. J. Phys., 48, 552 (1980).

[11] N. Grot, C. Rovelli, R.S. Tate, Phys. Rev. A 54 4676 (1996), (quant-ph/9603021)

[12] V. Delgado, J.G. Muga, Phys. Rev. A 56 3425 (1997), (quant-ph/9704010); For a recent review, see J.G. Muga, R. Sala, J.P. Palao, quant-ph/9801043.