Second Order Approximations for Slightly Trimmed Sums

N.V. Gribkova∗, R. Helmers†

Abstract

We investigate the second order asymptotic behavior of trimmed sums \( T_n = \frac{1}{n} \sum_{i=k_n+1}^{n-m_n} X_{i:n} \), where \( k_n, m_n \) are sequences of integers, \( 0 \leq k_n < n - m_n \leq n \), such that \( \min(k_n, m_n) \to \infty \), as \( n \to \infty \), the \( X_{i:n} \)'s denote the order statistics corresponding to a sample \( X_1, \ldots, X_n \) of \( n \) i.i.d. random variables. In particular, we focus on the case of slightly trimmed sums with vanishing trimming percentages, i.e. we assume that \( \max(k_n, m_n)/n \to 0 \), as \( n \to \infty \), and heavy tailed distribution \( F \), i.e. the common distribution of the observations \( F \) is supposed to have an infinite variance.

We derive optimal bounds of Berry – Esseen type of the order \( O\left(r_n^{-1/2}\right) \), \( r_n = \min(k_n, m_n) \), for the normal approximation to \( T_n \) and, in addition, establish one-term expansions of the Edgeworth type for slightly trimmed sums and their studentized versions.

Our results supplement previous work on first order approximations for slightly trimmed sums by Csorgo, Haeusler & Mason (1988) and on second order approximations for (Studentized) trimmed sums with fixed trimming percentages by Gribkova & Helmers (2006, 2007).

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1 Introduction and main results

Let \( X_1, X_2, \ldots \) be a sequence of independent identically distributed (i.i.d.) real-valued nondegenerate random variables (r.v.) with common distribution function \( (df) \) \( F \), and for each integer \( n \geq 1 \) let \( X_{1:n} \leq \cdots \leq X_{n:n} \) denote the order statistics based on the sample \( X_1, \ldots, X_n \). Introduce the left-continuous inverse function \( F^{-1} \) defined as \( F^{-1}(u) = \inf\{x : F(x) \geq u\}, \ 0 < u \leq 1, \ F^{-1}(0) = F^{-1}(0^+) \), and let \( F_n \) and \( F_n^{-1} \) denote the empirical \( df \) and its inverse respectively.

∗St.Petersburg State University, Mathematics and Mechanics Faculty, 198504, St.Petersburg, Stary Peterhof, Universitetsky pr. 28, Russia; E-mail: nv.gribkova@gmail.com
†Centre for Mathematics and Computer Science, P.O.Box 94079, 1090 GB Amsterdam, The Netherlands; E-mail: R.Helmers@cwi.nl
Define the population truncated mean and variance functions

\[ \mu(u, 1 - v) = \int_u^{1-v} F^{-1}(s) \, ds, \]
\[ \sigma^2(u, 1 - v) = \int_u^{1-v} \int_u^{1-v} (s \land t - st) \, dF^{-1}(s) \, dF^{-1}(t), \]  

(1.1)

where \( 0 \leq u < 1 - v \leq 1 \), and \( s \land t = \min(s, t) \). Note that \( \sigma^2(0, 1) \) equals the variance of \( X_1 \) whenever \( EX_1^2 \) is finite.

Let \( k_n \) and \( m_n \) be sequences of integers such that \( 0 \leq k_n < n - m_n \leq n \), and \( k_n \land m_n \to \infty \), as \( n \to \infty \). Put \( \alpha_n = k_n/n \), \( \beta_n = m_n/n \).

Consider the trimmed sum given by

\[ T_n = \frac{1}{n} \sum_{i=k_n+1}^{n-m_n} X_{i:n} = \int_{\alpha_n}^{1-\beta_n} F_n^{-1}(u) \, du. \]  

(1.2)

The first order asymptotic properties of trimmed sums and slightly trimmed sums (i.e. \( \alpha_n \lor \beta_n \to 0 \)) were investigated by many authors (cf. [32], [9], [10], [19] and references therein). In particular in Csörge et al. [9] a necessary and sufficient condition for the existence of \( \{a_n\}, \{b_n\} \) such that the distribution of the properly normalized slightly trimmed sum \( a_n^{-1}(T_n - b_n) \) tends to the standard normal law was obtained, and (using a different approach than in [9]) Griffin and Pruitt [19] derived an equivalent \( \text{iff} \) condition for asymptotic normality of \( T_n \). In Griffin and Pruitt [19] the class of all subsequential limit laws for the sequences of slightly trimmed sums \( a_n^{-1}(T_n - b_n) \) was characterized and sufficient conditions were given for \( F \) to be in the domain of partial attraction of a given law from this class. The members of this class are of the form

\[ \tau N_1 + f(N_2) - g(N_3), \]

where \( N_1, N_2, N_3 \) are independent \( N(0, 1) \), \( \tau \geq 0 \) and \( f \) and \( g \) are arbitrary nondecreasing convex functions. Both in [9] and in [19] a classical result by Stigler [32] for the trimmed mean with fixed trimming percentages was extended to the case that the fraction of trimming data is vanishing when \( n \) gets large.

The second order asymptotic properties of trimmed sums with fixed trimmed percentages were investigated by Gribkova and Helmers [15]-[16]: the validity of the one-term Edgeworth expansion (EE) for a (Studentized) trimmed mean and bootstrapped trimmed mean were established and simple explicit formulas of the first leading terms of these expansions were found. (We note in passing that in Helmers et al. [22] a saddle-point approximation – a completely different way of approximating \( df \) of the trimmed mean with fixed trimming percentages accurately – was obtained.)

Here we extend the result in [15] and establish second order asymptotic properties to slightly trimmed means and their Studentized versions. Except for a non optimal Berry – Esseen type bound for slightly trimmed means in [11], to the best of our knowledge, no such second order results are available in the literature. In this article we focus on the case of heavy-tailed distributions, i.e. we assume that \( \sigma^2(0, 1) = \infty \) (cf. (1.1)). We refer to Remark 1.1 for a detailed discussion of the different behavior of our second order approximations in the case of a heavy tailed respectively light tailed distribution \( F \).

To begin with we shall obtain bounds of Berry – Esseen type for the normal approximation to \( T_n \) under a weak condition on the density, assuming its existence in the tails of the distribution of the observations, we also show that the bounds we give in this paper, namely \( O((k_n \land m_n)^{-1/2}) \), in absence of any moment assumptions are of the best possible order.
Secondly we will supplement our results on the rates of convergence towards normality by deriving the one-term expansions of the Edgeworth type for slightly trimmed sums and Studentized slightly trimmed sums and obtain simple explicit formulas for these expansions. In a way we in particular refine the first order limit results of Csörgő et al. [9] by establishing more accurate second order approximation of Edgeworth type for slightly trimmed means with vanishing trimming percentages.

We show that the first leading term of our one-term expansion (in absence of symmetry) has the exact order \((k_n \wedge m_n)^{-1/2}\) (cf. (1.27), Remark 1.1), when the density is regular varying in the tails with index \(\rho = -(1 + \gamma)\), \(0 < \gamma < 2\), (cf. Bingham et.al [6] on the topic regular variation and Borovkov and Mogulskii [7] for assumptions similar to our condition \([R]\) on p. [8]), which directly imply optimality of our Berry – Esseen type bounds. (When the underlying distribution has finite second moment, the order of the bound can be improved to \(O(n^{-3(1/2-1/\gamma)})\) when \(2 < \gamma < 3\), and to \(O(n^{-1/2})\) when \(3 \leq \gamma; -(1 + \gamma)\) is the index of regularity of the density in the tails (cf. condition \([R]\) on p. [8]). We will pursue this topic elsewhere).

Similarly as in [15]–[16], our method of proof is based on a stochastic approximation of a (slightly) trimmed sum by a \(U\)-statistic of degree two with a kernel depending on \(n\). We use also a Bahadur–Kiefer type approximation (cf. section 4).

We conclude this introduction by noting that the case of heavy-tailed distribution we focus on in this article is interesting in particular due to the following statistical motivation: suppose that \(E|X_1| < \infty\) and that we are interested in estimating of \(EX_1\), whereas \(\sigma^2(0,1) = \infty\). The trimmed mean with fixed trimming percentages (robust estimate) is not consistent in absence of symmetry of the underlying distribution, but the slightly trimmed mean \(T_n\) tends to \(EX_1\) a.s., so it is a consistent estimator. The next issue one may want to consider is interval estimation. Fortunately, the suitably normalized (or studentized) \(T_n\) has a standard normal asymptotic distribution. The rate of convergence in case of the heavy tailed \(F\) can be rather slow (cf. Theorem 1.2 and Corollary 1.1 below). However, if we know a second order asymptotic approximation to a \(df\) of the normalized \(T_n\) (cf. Theorem 1.5) and of the studentized \(T_n\) (cf. Theorem 1.7) correcting the bias and skewness, we can improve the standard normal approximation to an approximation having smaller remainder.

The paper is organized as follows: in Section 1, we formulate sets of conditions and state our main results on Berry – Esseen type bounds and the Edgeworth type expansions for a normalized (slightly) trimmed sums and for its Studentized versions. In Section 2, we state and prove the auxiliary results on the \(U\)-statistic approximation for \(T_n\) and for the plug-in estimate of its asymptotic variance. The proofs of the main results are relegated to Section 3. In Section 4 we state and prove two Bahadur–Kiefer type lemmas, which we use in our proofs, in particular, lemma 4.2 provides a representation for a sum of order statistics lying between the \(\alpha_n\)-th population quantile and the corresponding empirical quantile. A lemma used in the proofs of the Bahadur–Kiefer type results is relegated to the Appendix.

Define the \(\nu\)-th quantile of \(F\) by \(\xi_\nu = F^{-1}(\nu)\), \(0 < \nu < 1\), and let \(W_i(n), i = 1, \ldots, n\), denote \(X_i\) Winsorized outside of \((\xi_{\alpha_n}, \xi_{1-\beta_n}]\), that is

\[
W_i(n) = \xi_{\alpha_n} \lor (X_i \land \xi_{1-\beta_n}),
\]

(1.3)

where \(s \lor t = \max(s, t)\). Define the quantile function

\[
Q_n(u) = \xi_{\alpha_n} \lor (F^{-1}(u) \land \xi_{1-\beta_n}),
\]

(1.4)
and the first two cumulants of $W_i(n)$:
\[
\mu_{W(n)} = \int_0^1 Q_n(u) du, \quad \sigma_{W(n)}^2 = \int_0^1 (Q_n(u) - \mu_{W(n)})^2 du.
\] (1.5)

Note that $\sigma_{W(n)}^2 = \sigma^2(\alpha_n, 1-\beta_n)$ (cf. [11]), and its square root is a suitable scale parameter for $T_n$ when establishing its asymptotic normality (cf. Csörgő et al. [9], see also Griffin and Pruitt [19]). We will suppose throughout this article that $\lim \inf_{n \to \infty} \sigma_{W(n)} > 0$ (i.e. $\xi_{\alpha_n} \neq \xi_{1-\beta_n}$ for all sufficiently large $n$).

Define four numbers
\[
a_1 = \liminf_{n \to \infty} \alpha_n, \quad a_2 = \limsup_{n \to \infty} \alpha_n, \\
b_1 = \liminf_{n \to \infty} (1-\beta_n), \quad b_2 = \limsup_{n \to \infty} (1-\beta_n),
\] (1.6)

where $0 \leq a_1 \leq a_2$, $b_1 \leq b_2 \leq 1$, and suppose that $a_2 < b_1$.

We will assume throughout this article that the following smoothness condition is satisfied.

[A1]. There exist two open sets $U_a, U_b \subset (0,1)$ such that $F^{-1}$ is differentiable in $U = U_a \cup U_b$, and
\[
U_a \ni (0,\varepsilon), \quad \text{if } 0 = a_1 = a_2, \quad U_a \ni (0,a_2), \quad \text{if } 0 < a_1 < a_2, \\
[a_1,a_2], \quad \text{if } 0 < a_1 \leq a_2, \quad U_b \ni (1-\varepsilon,1), \quad \text{if } b_1 = b_2 = 1, \\
[b_1,1), \quad \text{if } b_1 < b_2 = 1, \\
[b_1,b_2], \quad \text{if } b_1 \leq b_2 < 1,
\] (1.7)

(with some $0 < \varepsilon < 1$ in cases given in the first lines of (1.7)), i.e. the density $f = F'$ exists and is positive in $F^{-1}(U)$.

Define two sequences:
\[
q_{\alpha_n} = \frac{1}{\sqrt{n} \sigma_{W(n)}} \frac{\alpha_n}{f(\xi_{\alpha_n})}, \quad q_{\beta_n} = \frac{1}{\sqrt{n} \sigma_{W(n)}} \frac{\beta_n}{f(\xi_{1-\beta_n})}.
\] (1.8)

We note that it is a simple consequence of [A1] that these quantities are well defined for all sufficiently large $n$. The same remark also applies to some other quantities we introduce below.

Our second assumption is:

[A2].\quad $q_{\alpha_n} \vee q_{\beta_n} \to 0$, as $n \to \infty$.

Note that [A2] holds true if $\sigma^2 = \sigma^2(0,1) < \infty$ and $\frac{\alpha_n}{f(\xi_{\alpha_n})} \vee \frac{\beta_n}{f(\xi_{1-\beta_n})} = o(\sqrt{n})$.

Moreover, [A2] is also satisfied if $\sigma^2 = \infty$ and $\frac{\sqrt{\alpha_n}}{f(\xi_{\alpha_n})} \vee \frac{\sqrt{\beta_n}}{f(\xi_{1-\beta_n})} = o(\sqrt{n})$, because, due to Lemma 2.1 of Csörgő et al. [9], for any quantile function $F^{-1}$:
\[
\lim \sup_{u,v \in (0,1)} \frac{u (F^{-1}(u))^2 + v (F^{-1}(1-v))^2}{\sigma^2(u, 1-v)} < \infty.
\] (1.9)

In a way relation (1.9) will be crucial for our purposes. Note first of all that in the special case that the second moment of $F$ is assumed to be finite (cf. Theorem 1.3) the upper limit in (1.9) is not only bounded but is in fact equal to zero. This simple fact
is at the basis of the slightly better rates obtained in Theorem 1.3 in comparison with the rate established in the more general Theorem 1.2.

Let $h$ be a real-valued function defined on the set $F^{-1}(U)$ (cf. (1.17)). Take an arbitrary $0 < B < \infty$ and for all sufficiently large $n$ define

$$
\Psi_{\alpha_n, h}(B) = \sup_{|t| \leq B} \left| h \circ F^{-1} \left( \alpha_n + t \sqrt{\alpha_n \ln \frac{k_n}{n}} \right) - h \circ F^{-1} \left( \alpha_n \right) \right|,
$$

$$
\Psi_{1-\beta_n, h}(B) = \sup_{|t| \leq B} \left| h \circ F^{-1} \left( 1 - \beta_n + t \sqrt{\beta_n \ln \frac{m_n}{n}} \right) - h \circ F^{-1} \left( 1 - \beta_n \right) \right|, \quad (1.10)
$$

where $h \circ F^{-1}(u) = h (F^{-1}(u))$. Note that $\alpha_n + t \sqrt{\alpha_n \ln \frac{k_n}{n}} = \alpha_n \left( 1 + o(1) \right)$, and $1 - \beta_n + t \sqrt{\beta_n \ln \frac{m_n}{n}} = 1 - \beta_n \left( 1 + o(1) \right)$, as $n \to \infty$. In particular, this implies that the two functions introduced in (1.10) are well-defined for all sufficiently large $n$.

We will use in what follows the auxiliary functions: $\Psi_{\nu, \frac{1}{f(x)}}(B)$, $\Psi_{\nu, \frac{1}{f(x)}}(B)$, $\Psi_{\nu, \frac{1}{f(x)}}(B)$, $\Psi_{\nu, \frac{1}{f(x)}}(B)$, $\Psi_{\nu, \frac{1}{f(x)}}(B)$, $\Psi_{\nu, \frac{1}{f(x)}}(B)$, corresponding to $h(x) = x$, $1/f(x)$ and $x/f(x)$ in (1.10).

It is easy to see in any case that the following inequalities are valid:

$$
\Psi_{\alpha_n, \frac{1}{f(x)}}(B) \leq \alpha_n B \left( \ln \frac{k_n}{n} \right)^{1/2} \left( \frac{1}{f(\xi_{\alpha_n})} + \Psi_{\alpha_n, \frac{1}{f(x)}}(B) \right)^2 + |\xi_{\alpha_n}| \Psi_{\alpha_n, \frac{1}{f(x)}}(B),
$$

$$
\Psi_{1-\beta_n, \frac{1}{f(x)}}(B) \leq \beta_n B \left( \ln \frac{m_n}{n} \right)^{1/2} \left( \frac{1}{f(\xi_{1-\beta_n})} + \Psi_{1-\beta_n, \frac{1}{f(x)}}(B) \right)^2 + |\xi_{1-\beta_n}| \Psi_{1-\beta_n, \frac{1}{f(x)}}(B). \quad (1.11)
$$

These inequalities will be especially useful in the proof of Lemma 2.2 in Section 2.

Our third assumption is

$[A_3] \quad$ For every $0 < B < \infty$

$$
\frac{\alpha_n}{\sqrt{n} \sigma W(n)} \Psi_{\alpha_n, \frac{1}{f(x)}}(B) \to 0, \quad \frac{\beta_n}{\sqrt{n} \sigma W(n)} \Psi_{1-\beta_n, \frac{1}{f(x)}}(B) \to 0,
$$

as $n \to \infty$.

Define the df of the normalized $T_n$ by

$$
F_{T_n}(x) = P \left( \sigma_{W(n)}^{-1} n^{1/2} \left( T_n - \mu(\alpha_n, 1 - \beta_n) \right) \leq x \right). \quad (1.12)
$$

First we show that the conditions $[A_1] - [A_3]$ together yields

$$
\sup_{x \in R} |F_{T_n}(x) - \Phi(x)| = o(1), \quad \text{as} \quad n \to \infty, \quad (1.13)
$$

where $\Phi$ is standard normal df. To check this we verify that the iff conditions of asymptotic normality of the trimmed sum $T_n$ (cf. Csörgő et al. [9], Theorem 4, p. 677) are automatically satisfied whenever our conditions $[A_1] - [A_3]$ hold true. Consider the first auxiliary function defined on page 674 of Csörgő et al. [9], which corresponds to the trimming of the $k_n$ smallest observations on our sample of size $n$; the treatment of the second auxiliary function on page 674 of the same paper, which deals with the trimming
of the $m_n$ largest observations, is similar and therefore omitted. For (1.13) to hold we must verify that for every $c \in \mathbb{R}$

$$Q_{1,n}(c) \to 0, \quad n \to \infty,$$

(1.14)

where

$$Q_{1,n}(c) = \begin{cases} 
\left(\frac{\alpha_n}{\sigma_W(n)}\right)^{1/2} \left\{ F^{-1}\left(\alpha_n + c\sqrt{\frac{m_n}{n}}\right) - F^{-1}(\alpha_n) \right\}, & |c| \leq \frac{1}{2} \sqrt{\alpha_n n}, \\
Q_{1,n}\left(-\frac{1}{2} \sqrt{\alpha_n n}\right), & -\infty < c < -\frac{1}{2} \sqrt{\alpha_n n}, \\
Q_{1,n}\left(\frac{1}{2} \sqrt{\alpha_n n}\right), & \frac{1}{2} \sqrt{\alpha_n n} < c < \infty,
\end{cases}$$

(cf. Csörgő et al. [9]). Note that $\alpha_n n = k_n \to \infty$, and for each $c \in \mathbb{R}$ and all sufficiently large $n$ we have $|c| < \frac{1}{2} \sqrt{\alpha_n n}$, and $\alpha_n + c\sqrt{\frac{m_n}{n}} = \alpha_n(1 + c\sqrt{\frac{1}{k_n}})$ belongs to the set $U_a$ (cf. (1.7)). So we have

$$Q_{1,n}(c) = \left(\frac{\alpha_n}{\sigma_W(n)}\right)^{1/2} c \sqrt{\frac{\alpha_n}{n}} \frac{1}{f\left(F^{-1}\left(\alpha_n + \theta c\sqrt{\frac{m_n}{n}}\right)\right)}$$

(1.15)

for some $0 < \theta < 1$, and the quantity (1.15) in absolute value is less than

$$|c| \frac{\alpha_n}{\sqrt{n} \sigma_W(n)} \left(\frac{1}{f(\xi_{\alpha_n})} + \Psi_{\alpha_n,1/f(x)}(\theta |c|)\right),$$

which tends to zero by $[A_2]$ and $[A_3]$, and (1.14) follows.

Our conditions $[A_1] - [A_3]$ are slightly stronger than if $f$ conditions of asymptotic normality of Csörgő et al. [9], but these conditions enable us to establish a bound for the error in the normal approximation for the df of $T_n$.

We note in passing that Peng [27] has shown that it is impossible in general to replace the truncated mean $\mu(\alpha_n, 1 - \beta_n)$ employed in (1.12), by the ordinary mean of the trimmed sum $ET_n$ (which is always finite of course when $F$ and $1 - F$ are regular varying at minus and plus infinity respectively); centering by a truncated mean is really needed to obtain a standard normal limit in (1.12).

Here is our general result on the rate of convergence of the distribution of a properly normalized trimmed sum $T_n$ to the standard normal law.

**Theorem 1.1** Assume that the conditions $[A_1]$ and $[A_2]$ are satisfied. Then

$$\sup_{x \in \mathbb{R}} |F_{T_n}(x) - \Phi(x)| \leq \frac{A}{\sqrt{n}}\left(\delta_{1,n} + \delta_{2,n} + \delta_{3,n} + \delta_{4,n}\right) + C\left(k_n^- + m_n^-\right),$$

(1.16)

$$\delta_{1,n} = \frac{E\left|W_1(n)\right|^3}{\sigma_W(n)}; \quad \delta_{2,n} = \frac{1}{\sigma_W(n)} \left(\frac{\alpha_n}{f(\xi_{\alpha_n})} + \frac{\beta_n}{f(\xi_{1-\beta_n})}\right),$$

$$\delta_{3,n} = \left(\alpha_n\right)^{1/3} \left(\frac{\alpha_n}{f(\xi_{\alpha_n})\sigma_W(n)}\right)^{5/3} + \left(\beta_n\right)^{1/3} \left(\frac{\beta_n}{f(\xi_{1-\beta_n})\sigma_W(n)}\right)^{5/3},$$

$$\delta_{4,n} = \frac{1}{\sigma_W(n)} \left(\alpha_n \ln k_n \Psi_{\alpha_n,1/f(x)}(B) + \beta_n \ln m_n \Psi_{1-\beta_n,1/f(x)}(B)\right),$$

for every $c > 0$, where $A, B, C > 0$ are some constants, depending only on $c$. 
Note that at the r.h.s. of (1.16) we have: $\frac{1}{\sqrt{n}} \delta_{1,n} = O\left(\frac{1}{\sqrt{k_n}} + \frac{1}{\sqrt{m_n}}\right)$ in view of (1.9) (cf. Proof of Theorem 1.2, Section 3), $\frac{1}{\sqrt{n}} \delta_{2,n} = o(1)$ in view of $[A_2]$, $\frac{1}{\sqrt{n}} \delta_{3,n} = o(1)$ if we additionally assume that $q_{\alpha_n} = o(k_n^{-1/5})$, $q_{\beta_n} = o(m_n^{-1/5})$ and $\frac{1}{\sqrt{n}} \delta_{4,n} = o(1)$ if $\frac{\alpha_n}{\sqrt{n} \sigma W(n)} \Psi_{\alpha_n f, \xi_1}(B) = o\left(\frac{1}{\ln k_n}\right)$, $\frac{\beta_n}{\sqrt{n} \sigma W(n)} \Psi_{1-\beta_n f, \xi_1}(B) = o\left(\frac{1}{\ln m_n}\right)$.

**Example 1.1.** Let us consider an example, where the underlying distribution $F$ has super-heavy tails. Let $F$ is such that $F(x) = 1 - F(x) = \frac{1}{2} \left(\ln |x|\right)^\rho$, $\rho > 0$, for all $x$: $|x| > x_0 > 0$ (cf. [19]). Simple computations on the quantities $\delta_{i,n}$ (it turns out that the term, corresponding $i = 3$ is the largest one in this case) show that at the r.h.s. of (1.16) we have a bound of the order $O(k_n^{-d} + m_n^{-d})$ with some $0 < d \leq 1/2$ when $\limsup_{n \to \infty} (k_n^{-d} + m_n^{-d}) n^{5+3\rho/2-\alpha} < \infty$, and the bound of the order $O(k_n^{-1/2} + m_n^{-1/2})$ is possible if and only if $k_n \sim n$ and $m_n \sim n$. We can obtain a bound of the order $O(k_n^{-1/3} + m_n^{-1/3})$ (say) if we take $k_n$, $m_n \sim n^{10/13}$.

To obtain more explicit bounds than the bound given in (1.16) we need some more restrictive conditions. The following assumption is somewhat stronger than $[A_2]$:

$[A'_2]$. **Suppose that**

\[
\limsup_{n \to \infty} \frac{\alpha_n^{3/2}}{\sigma W(n) f(\xi_1) \xi_1} < \infty, \quad \limsup_{n \to \infty} \frac{\beta_n^{3/2}}{\sigma W(n) f(\xi_1-\beta_n)} < \infty.
\]

The latter condition implies

\[
q_{\alpha_n} = O\left(\frac{1}{\sqrt{k_n}}\right), \quad q_{\beta_n} = O\left(\frac{1}{\sqrt{m_n}}\right), \quad \text{as} \quad n \to \infty.
\]

Note that in view of (1.9) condition $[A'_2]$ holds true if the following slightly stronger condition is satisfied:

$[A''_2]$. **Suppose that**

\[
\limsup_{n \to \infty} \frac{\alpha_n}{\xi_1} \xi_1 f(\xi_1) < \infty, \quad \limsup_{n \to \infty} \frac{\beta_n}{\xi_1-\beta_n} f(\xi_1-\beta_n) < \infty.
\]

Note that in the case of a slightly trimmed sum (i.e. when $\alpha_n \vee \beta_n \to 0$) condition $[A''_2]$ is certainly satisfied when $\limsup_{x \to -\infty} \frac{F(x)}{f(x)} \rho(x, f(x)) < \infty$ and $\limsup_{x \to +\infty} \frac{1-F(x)}{f(x)} \rho(x, f(x)) < \infty$, and that the latter requirement is true when the df $F$ has a density for all sufficiently large $|x|$, and $f = F'$ is regularly varying at the infinity with index $\rho < -1$ (cf. condition [R], Corollary 1.1, Theorems 1.5, 1.7).

The following condition is stronger than smoothness condition $[A_3]$:

$[A'_3]$. **For every** $B > 0$

\[
\Psi_{\alpha_n f, \xi_1}(B) = O\left(\frac{1}{f(\xi_1 n \ln k_n)}\right), \quad \Psi_{1-\beta_n f, \xi_1}(B) = O\left(\frac{1}{f(\xi_1-\beta_n n \ln m_n)}\right),
\]

Now we are in a position to state our second result of Berry–Esseen type, which yields an explicit upper bound of a much simpler form:
Theorem 1.2 Suppose that conditions $[A_1]$, $[A'_2]$ and $[A'_3]$ hold. Then

$$\sup_{x \in \mathbb{R}} |F_{T_n}(x) - \Phi(x)| \leq C \left( \frac{1}{\sqrt{k_n}} + \frac{1}{\sqrt{m_n}} \right),$$

(1.18)

where $C$ is a positive constant not depending on $n$.

This result can be compared with an earlier result by Egorov & Nevzorov [11], where a non optimal bound of the order $O\left( \frac{\ln k_n}{\sqrt{k_n}} + \frac{\ln m_n}{\sqrt{m_n}} \right)$ under stronger conditions was obtained. In contrast, our bound (1.18) is sharp and yields an optimal order bound of Berry–Esseen type for slightly trimmed means when $F$ is, for instance, the Cauchy distribution. The optimality of the bound in (1.18) follows directly from our results on the Edgeworth type expansions and computations given in Remark 1.1 (cf. (1.27)).

Our next assertion concerns the case of a slightly trimmed mean for the special case when $EX_1^2 < \infty$, i.e. the case of a light tailed distribution.

Theorem 1.3 Suppose that $\sigma^2 = \sigma^2(0, 1) < \infty$, $\alpha_n \vee \beta_n \to 0$ as $n \to \infty$, and that the conditions $[A_1]$, $[A'_2]$ hold true. In addition assume that for every $B > 0$:

$$\Psi_{\alpha_n, \frac{1}{(1-\epsilon)}}(B) = o\left( (f(\xi_{\alpha_n}) \ln k_n)^{-1} \right), \quad \Psi_{1-\beta_n, \frac{1}{(1-\epsilon)}}(B) = o\left( (f(\xi_{1-\beta_n}) \ln m_n)^{-1} \right),$$

(1.19)

as $n \to \infty$. Then

$$\sup_{x \in \mathbb{R}} |F_{T_n}(x) - \Phi(x)| = o\left( \frac{1}{\sqrt{k_n}} + \frac{1}{\sqrt{m_n}} \right),$$

(1.20)

as $n \to \infty$.

This result — i.e. the order bound (1.20) — applies for instance to a df $F$ with a regular varying density $f$ which behaves like $|x|^{-(3+\epsilon)}$ (with some $\epsilon > 0$) in the tails, so that the variance of $F$ is indeed finite. If we take in addition — by way of an example — $k_n = m_n = \lfloor n^{1/2} \rfloor$ then we obtain a sharper bound of order $o(n^{-1/4})$ instead of $O(n^{-1/4})$ which would follow from the previous Theorem 1.2. Moreover, if moments of higher order than 2 are assumed to be finite, it appears possible to establish the exact order of the normal approximation error in (1.20), rather than asserting only that the order of magnitude of the normal error in (1.20) is smaller than the one in (1.18). A detailed study of these exact rates, however, is outside the scope of the present paper. The authors hope to pursue this matter elsewhere.

Next we obtain some consequences of Theorems 1.1, 1.2 and 1.3. Our first corollary concerns the case of slightly trimmed sum when the df $F$ belongs to a domain of attraction of a stable law. Let $RV_\rho^\infty$ be a class of regularly varying in the infinity functions: $g \in RV_\rho^\infty \iff g(x) = |x|^{\rho} L(x)$, for $|x| > x_0$, with some $x_0 > 0$, $\rho \in \mathbb{R}$, and $L(x)$ is a positive slowly varying function at infinity. We will need the following regularity condition on the tails for the density $f$:

$[R]$. Suppose that $f \in RV_\rho^\infty$, where $\rho = -(1 + \gamma)$, $\gamma > 0$, and assume that

$$|f(x + \Delta x) - f(x)| = O\left( f(x) \left| \frac{\Delta x}{x} \right| \right),$$

(1.21)

when $\Delta x = o(|x|)$, as $|x| \to \infty$. 

\[ 8 \]
Note that (1.21) holds true for $f$ if $|\frac{L(x+\Delta x)}{L(x)}-1|=O\left(\frac{\Delta x}{x}\right)$, as $|x| \to +\infty$, where $L$ is the corresponding slowly varying function, and it is satisfied if $L$ is continuously differentiable for sufficiently large $|x|$ and $|L'(x)|=O\left(\frac{L(x)}{|x|}\right)$, as $|x| \to +\infty$, which is valid for instance when $L$ is some power of the logarithm. We refer to Borovkov and Mogulskii [7], p. 568 for some conditions closely related to ours.

The following corollary holds true for a slightly trimmed mean in case of a regular varying density:

Corollary 1.1 Suppose that $\alpha_n \vee \beta_n \to 0$, as $n \to \infty$, condition [A1] holds true for some $\varepsilon > 0$, and the density $f$ satisfies [R] with $0 < \gamma \leq 2$ on the set $F^{-1}(U)$. Then:
(i) the bound (1.18) is valid;
(ii) in addition if $\gamma = 2$ and $\sigma^2 < \infty$ then also the sharper order bound (1.20) holds true.

It is clear from our proofs (cf Section 3) that the latter assertion is valid as well if the density $f$ has different indices of regularity near $-\infty$ respectively to $+\infty$ (in particular, at least one of them can be greater than 2), we keep these two indices equal to each other for simplicity. Moreover this situation corresponds to the important special case when $F$ belongs to a domain of attraction of a stable law.

Our second corollary concerns the classical case when trimming occur on the levels of the central order statistics. Let $a_1, b_2, U_a$ and $U_b$ are as in (1.6)-(1.7).

Corollary 1.2 Suppose that $0 < a_1 < b_2 < 1$, and assume that the condition [A1] is satisfied. In the addition suppose that the density $f$ satisfies a Hölder condition of degree $d$ (for some $d > 0$) on the sets $F^{-1}(U_c), c = a, b$. Then
\[
\sup_{x \in \mathbb{R}} |F_{T_n}(x) - \Phi(x)| \leq \frac{C}{\sqrt{n}},
\]
where $C > 0$ is a constant, not depending on $n$.

Note that the smoothness assumptions imposed in Corollary 1.2 are especially well suited for obtaining our results on the Edgeworth type expansions, which we will state and prove below. So the smoothness assumption in corollary 1.2 is slightly excessive for obtaining of the Berry – Esseen type bound (1.22) (cf. for instance, [14], where the optimal bound was obtained under a somewhat weaker smoothness assumption that $F^{-1}$ satisfies a Lipschitz condition on the sets $U_a$ and $U_b$, by an application of Theorem 1.1 of van Zwet [33] for symmetric statistics).

Next we will go one step further and establish one-term Edgeworth type expansions for $df$ of a normalized and of a Studentized slightly trimmed sum.

Define
\[
\gamma_{3,W(n)} = \int_0^1 (Q_n(u) - \mu_{W(n)})^3 \, du,
\]
where $Q_n(u), \mu_{W(n)}$ as in (1.4)-(1.5), and put
\[
\delta_{2,W(n)} = -\alpha_n^2 \frac{(\mu_{W(n)} - \xi_{\alpha n})^2}{f(\xi_{\alpha n})} + \beta_n^2 \frac{(\mu_{W(n)} - \xi_{1-\beta n})^2}{f(\xi_{1-\beta n})}.
\]
Define two sequences of the real numbers
\[ \lambda_{1(n)} = \frac{\gamma 3 W(n)}{\sigma^3 W(n)}, \quad \lambda_{2(n)} = \frac{\delta 2 W(n)}{\sigma^3 W(n)} \] (1.24)

We establish the validity of the Edgeworth type expansion for the df \( F_{T_n} \) under conditions \([A_1]-[A_3]\). This expansion is given by
\[ G_n(x) = \Phi(x) - \frac{\phi(x)}{6\sqrt{n}} \left( (\lambda_{1(n)} + 3\lambda_{2(n)}) (x^2 - 1) + 6\sqrt{n} \frac{b_n}{\sigma W(n)} \right), \] (1.25)
where \( \phi = \Phi' \), and \( b_n = \frac{1}{2\sqrt{n}} (-\alpha (1-\alpha) + \beta (1-\beta)), \) \( b_n \) is a bias term which is present in the expansion despite of the absence of any moment assumptions (cf. [15]).

Note that if \( \alpha_n = \beta_n \) and the underlying distribution is symmetric, we have \( G_n(x) \equiv \Phi(x) \) because the second term of the expansion is equal to zero in this case.

Similarly as when proving of Theorem 1.2 it is easy to check that if conditions \([A_1] \) and \([A'_2] \) are satisfied the second term of \( G_n(x) \) at the r.h.s. of (1.25) (for each fixed \( x \)) is a magnitude of the order \( O \left( \frac{1}{\sqrt{k_n}} + \frac{1}{\sqrt{m_n}} \right) \) as \( n \to \infty \). And under some proper conditions (cf. Theorem 1.5 and Corollary 1.3) the remainder in approximating of the df of the normalized slightly trimmed sum by its expansion \( G_n(x) \) is of the Bahadur type order \( O \left( \frac{(\ln k_n)^{3/4}}{k_n^2} + \frac{(\ln m_n)^{3/4}}{m_n} \right) \), as \( n \to \infty \).

Some simple computations show that in case of underlying distribution \( F \) considered in Example 1.1 (\( F \) has no finite moments) \( |G_n(x) - \Phi(x)| \propto \frac{1}{\sqrt{k_n}} (\frac{1}{k_n})^{\gamma} + \frac{1}{\sqrt{m_n}} (\frac{1}{m_n})^{\gamma} \), \( x \in R \). So, \( |G_n(x) - \Phi(x)| \propto \frac{1}{\sqrt{k_n}} + \frac{1}{\sqrt{m_n}} \) if and only if \( k_n \approx n \) and \( m_n \approx n \).

**Remark 1.1** Let us investigate the order of magnitude of the various terms appearing in the Edgeworth type correction (1.25). Two of these terms are correcting for skewness, let us denote them by \( t_{j,n} = \frac{1}{\sqrt{n}} \lambda_{j(n)}, \) \( j = 1, 2, \) respectively, while a third term is correcting for the bias present, which we denote by \( t_{3,n} = b_n/\sigma W(n) \). Suppose now that \( \max(\alpha, \beta) \to 0 \), condition \([A_1] \) is satisfied and the density \( f = F' \) is regularly varying at the infinity with index \( \rho = -1(\gamma) \), where \( \gamma > 0 \), moreover, we will suppose that there is no symmetry, i.e. that we are not in a situation where \( f(x)/f(-x) \to 1, \) \( |x| \to \infty \) and simultaneously \( \alpha_n/\beta_n \to 1 \). (If \( f(x) = f(-x) \) for all sufficiently large \( |x| \) and \( \alpha_n = \beta_n \), then \( t_{j,n} = 0, \) \( j = 1, 2, 3 \).

Note first of all that by \([A_1] \) we have \( \alpha_n = F(\xi_{\alpha_n}), \beta_n = 1 - F(\xi_{1-\beta_n}) \) and that the regularity condition implies
\[ \lim_{n \to \infty} \frac{\alpha_n}{|\xi_{\alpha_n}|f(\xi_{\alpha_n})} = \lim_{n \to \infty} \frac{\beta_n}{\xi_{1-\beta_n}f(\xi_{1-\beta_n})} = \frac{1}{\gamma}. \] (1.26)

Let \( h(n) \sim g(n) \) denotes that \( \lim_{n \to \infty} h(n)/g(n) = c, \) where \( 0 < c < \infty \) is some constant. We will now distinguish three cases:

1. \( 0 < \gamma < 2 \). In this case \( \sigma^2(0, 1) = \infty \), i.e. we are dealing with a heavy tailed distribution \( F \). Using Karamata type property (cf. Feller [12], Vol. II, Chp. VIII, paragraph 9, Theorem 2), we find that \( t_{1,n} \sim \frac{1}{\sqrt{n}} \left( \frac{\alpha_n \xi_{\alpha_n}^3 + \beta_n \xi_{1-\beta_n}^3}{(\alpha_n \xi_{\alpha_n}^3 + \beta_n \xi_{1-\beta_n}^3)^{3/2}} \right), \) the latter in absolute value is less than \( \frac{1}{\sqrt{n}} \left( \frac{\alpha_n \xi_{\alpha_n}^3 + \beta_n \xi_{1-\beta_n}^3}{(\alpha_n \xi_{\alpha_n}^3 + \beta_n \xi_{1-\beta_n}^3)^{3/2}} \right) \). Similarly we easy check
that both \( t_{2,n} \) and \( t_{3,n} \) are of the same order \( O\left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{m_n}}\right) \). Thus, in case (1) (in absence of symmetry) we obtain

\[
|G_n(x) - \Phi(x)| \asymp \frac{1}{\sqrt{k_n}} + \frac{1}{\sqrt{m_n}}
\]

(2) \( 2 < \gamma < 3 \), the second moment of \( F \) is finite, but the third moment is infinite. In this case \( \sigma_n^2 \to \sigma^2(0, 1) < \infty \), and again using Karamata type properties of truncated moments we obtain that \( t_{1,n} \sim \frac{\alpha_n^3 + \beta_n^3}{\sqrt{n}} \), and the latter quantity in absolute value is \( \sim \frac{1}{\sqrt{n}} \left(1 - \frac{2}{7}\right) L(n) \), where \( L(n) \) is some positive slowly varying function.

The latter quantity is the same as \( n^{-3(\frac{2}{7} - \frac{1}{4})} \left(k_n^{-\frac{2}{7}} + m_n^{-\frac{2}{7}}\right) L(n) \). Note that \( 1 - \frac{3}{7} < 0 \) since \( \gamma < 3 \). For \( t_{2,n} \) we easily find that it is of the same order as \( t_{1,n} \). Note that when \( \gamma \) gets close to 3 the order of \( t_{1,n} + t_{2,n} \) becomes close to \( n^{-1/2} \). However, for \( t_{3,n} \) we obtain a slower rate of convergence to zero than for \( t_{j,n} \), \( j = 1, 2 \). Simple computations using regularity condition show that \( t_{3,n} \) in absolute value is of the order \( n^{-\frac{1}{2} - \frac{1}{7}} \left(k_n^{-\frac{2}{7}} + m_n^{-\frac{2}{7}}\right) L(n) \). Since \( -\frac{1}{7} < 1 - \frac{3}{7} \) when \( \gamma > 2 \), we see that the bias term \( t_{3,n} \) is of bigger order than \( t_{1,n} + t_{2,n} \).

(3) \( \gamma \geq 3 \). In this case the third absolute moment of \( F \) is finite and obviously \( |t_{1,n} + t_{2,n}| = O\left(\frac{1}{\sqrt{n}}\right) \). However for the bias term \( t_{3,n} \) as before we have (in absence of symmetry) the exact order \( n^{-\frac{1}{2} - \frac{1}{7}} \left(k_n^{-\frac{2}{7}} + m_n^{-\frac{2}{7}}\right) L(n) \), the latter quantity is close to \( n^{-1/6} \) when \( \gamma > 3 \) is close to 3. So, in the case of a light tailed distribution \( F \) with finite third absolute moment the bias part of the Edgeworth type expansion \( (1.25) \) is of the order close to \( n^{-1/6} \).

We conclude this remark by noting that in the cases (2) and (3) centering by \( ET_n \) in fact to be preferred. However in the 'heavy tailed' case (1) this is not possible as already was shown by Peng \[27\]. In a way all this tell us that the expansion \( (1.25) \) in its present form is only really suitable for 'heavy tailed' distribution \( F \), i.e. in case (1). Otherwise one should center \( T_n \) by its exact expectation \( ET_n \) and consequently delete the bias term \( -\phi(x)t_{3,n} \) presented in \( (1.25) \).

Here is our general result on the validity of one-term Edgeworth type expansion for the \( df \)'s of a normalized slightly trimmed sum.

**Theorem 1.4** Suppose that the conditions \([A_1]-[A_3]\) hold. Then

\[
\sup_{x \in \mathbb{R}} |F_{T_n}(x) - G_n(x)| \leq C_1 \frac{1}{n} \left(\delta_{1,n} + \delta_{2,n} + \delta_{3,n}\right) + \frac{C_2}{n^{3/4}} \delta_{4,n} + \frac{C_3}{n^{1/2}} \left(\delta_{5,n} + \delta_{6,n}\right) + C_4 \left(k_n^{-\epsilon} + m_n^{-\epsilon}\right),
\]

for every \( c > 0 \) and some constants \( C_i > 0, i = 1, \ldots, 4 \), not depending on \( n \), whereas \( \delta_{1,n} = \frac{E(W_i(n))}{\sigma^2 W(n)} \), \( \delta_{2,n} = \alpha^2 \left(\frac{1}{\sigma^2 W(n)}\right)^{2+\epsilon} + \beta^2 \left(\frac{1}{\sigma^2 W(n)}\right)^{2+\epsilon} \), for every \( \epsilon > 0 \), while \( \delta_{3,n} \) is same as \( \delta_{2,n} \), but with \( \epsilon = 0 \), in addition \( \delta_{4,n} = \left(\frac{\ln k_n}{\sigma^2 W(n)}\right)^{3/4} \), for every \( \epsilon > 0 \).
Suppose that
\[ \frac{(\ln m_n)^{3/4}(\beta_n)^{3/4}}{f(\xi_1-\beta_n)\sigma_W(n)} \cdot \delta_{5,n} = \alpha_n \frac{\ln k_n}{\sigma_W(n)} \psi_{\gamma_n,1} \left( \frac{B}{\sigma_W(n)} \right) + \beta_n \frac{\ln m_n}{\sigma_W(n)} \psi_{1-\beta_n,1} \left( \frac{B}{\sigma_W(n)} \right), \]
where \( B > 0 \) is some constant depending only on \( c \). Finally \( \delta_{6,n} = \frac{(\lambda_1(n) + 3\lambda_2(n))b_n}{\sigma_W(n)} \). The constants \( C_i \) on the r.h.s. of (1.28) depend on \( c \) and \( \varepsilon \).

Note that in any case we have \( \frac{1}{n} \delta_{1,n} = O \left( \frac{1}{k_n} + \frac{1}{m_n} \right) \) at the r.h.s. of (1.28) in view of (1.9) (cf. proof of Theorem 1.5, section 3).

The next corollary provides an explicit upper bound of a much simpler form. To state it we will need the following assumption:

\[ [L]. \] There exists \( 0 < s \leq 1 \) such that
\[ \limsup_{n \to \infty} \frac{n^s}{k_n} \wedge m_n < \infty. \]

**Corollary 1.3** Suppose that the conditions \([A_1], [A_2]'\), and \([L]\) hold true, in addition, assume that for every \( 0 < B < \infty \)
\[ \psi_{\alpha_n,1} \left( \frac{B}{\sigma_W(n)} \right) = O \left( \frac{1}{f(\xi_1-\beta_n)} \left( \frac{\ln m_n}{m_n} \right)^{1/4} \right) \]
and
\[ \psi_{1-\beta_n,1} \left( \frac{B}{\sigma_W(n)} \right) = O \left( \frac{1}{f(\xi_1-\beta_n)} \left( \frac{\ln m_n}{m_n} \right)^{1/4} \right), \]
as \( n \to \infty \). Then the bound on the r.h.s. in (1.28) is of the order
\[ O \left( \frac{(\ln k_n)^{5/4}}{k_n^{3/4}} + \frac{(\ln m_n)^{5/4}}{m_n^{3/4}} \right). \]

The following theorem ensures an expansion of the Edgeworth type for \( df \) of a normalized slightly trimmed mean.

**Theorem 1.5** Suppose that \( \alpha_n \vee \beta_n \to 0 \), as \( n \to \infty \), conditions \([A_1]\) and \([L]\) hold true, and the density satisfies condition \([R]\) with \( 0 < \gamma < 2 \). Then
\[ \sup_{x \in \mathbb{R}} |F_{T_n}(x) - G_n(x)| = O \left( \frac{(\ln k_n)^{5/4}}{k_n^{3/4}} + \frac{(\ln m_n)^{5/4}}{m_n^{3/4}} \right), \quad n \to \infty. \]  

(1.29)

The following corollary of Theorem 1.4 can be viewed as a version of the result by Gribkova and Helmers [16] under slightly weaker conditions.

**Corollary 1.4** Suppose that \( 0 < a_1 < b_2 < 1 \), where \( a_1, b_2 \) as in (1.6), condition \([A_1]\) holds true, and the density \( f \) satisfies a Hölder condition of degree \( d > 0 \) on the sets \( F^{-1}(U_a) \) and \( F^{-1}(U_b) \), \( U_a \) and \( U_b \) as in (1.7). Then
\[ \sup_{x \in \mathbb{R}} |F_{T_n}(x) - G_n(x)| = o \left( n^{-1/2-p} \right), \quad n \to \infty, \]  

(1.30)

for every \( p < \min(1/4, d/2) \).

To proceed we state our results on the Edgeworth type expansion for a Studentized slightly trimmed sum.

Define an empirical quantile function \( \hat{Q}_n(u) = X_{k_n,n} \vee F_n^{-1}(u) \wedge X_{n-m,n,n} \), and the plug-in estimates of \( \mu_W(n) \) and \( \sigma^2_W(n) \):
\[ \hat{\mu}_W(n) = \int_0^1 \hat{Q}_n(u) \, du = \frac{k_n}{n} X_{k_n,n} + \frac{1}{n} \sum_{i=k_n+1}^{n-m_n} X_{i,n} + \frac{m_n}{n} X_{n-m_n,n}, \]  

(1.31)
\[ \hat{\sigma}_{W(n)}^2 = \int_0^1 (\hat{Q}(u) - \hat{\mu}_{W(n)})^2 \, du = \frac{k_n}{n} X_{k_n:n}^2 + \frac{1}{n} \sum_{i=k_n+1}^{n-m_n} X_{i:n}^2 + \frac{m_n}{n} X_{n-m_n:n}^2 - \hat{\mu}_{W(n)}^2. \]

Define the df of a Studentized trimmed sum by 
\[ F_{T_n,S}(x) = P \left( \frac{\sqrt{n}(T_n - \mu_{a_n,1-\beta_n})}{\sigma_{W(n)}} \leq x \right). \]

We prove that the one-term expansion for \( F_{T_n,S}(x) \) is given by
\[ H_n(x) = \Phi(x) + \frac{\phi(x)}{\sigma_{W(n)} f(\xi_{\alpha_n})} \left( 2x^2 + 1 + \ln \frac{k_n}{m_n} + 3(x^2 + 1) \lambda_{1(n)} + 3(\ln \frac{k_n}{m_n} + 3) \lambda_{2(n)} - 6 \sqrt{n} \frac{b_n}{\sigma_{W(n)}} \right), \]  
(1.32)

where \( b_n \) is as in (1.25). Define a quantity
\[ \Delta_{n,S} = \sum_{i=1}^{5} \delta_i(n), \]  
(1.33)

with
\[ \begin{align*}
\delta_1(n) &= \frac{\alpha_n^{3/2}}{\sigma_{W(n)} f(\xi_{\alpha_n})} \left( \ln \frac{k_n}{m_n} \right)^{3/4} + \frac{\beta_n^{3/2}}{\sigma_{W(n)} f(\xi_{1-\beta_n})} \left( \ln \frac{m_n}{k_n} \right)^{3/4}, \\
\delta_2(n) &= B \left[ (\ln \frac{k_n}{m_n}) \left( \frac{1}{\alpha_n} \right) (B) + \ln m_n \left( \frac{1}{\beta_n} \right) (B) \right] + \ln m_n \left( \frac{1}{\beta_n} \right) (B), \\
\delta_3(n) &= \frac{\alpha_n^{3/2}}{\sigma_{W(n)} f(\xi_{\alpha_n})} \left( \ln \frac{k_n}{m_n} \right)^{1/2} + \frac{\beta_n^{3/2}}{\sigma_{W(n)} f(\xi_{1-\beta_n})} \left( \ln \frac{m_n}{k_n} \right)^{1/2}, \\
\delta_4(n) &= \frac{1}{\sqrt{n}} \left( \frac{1}{k_n} + \frac{1}{m_n} \right) \left[ \frac{\alpha_n^{3/2} \ln k_n}{\sigma_{W(n)} f(\xi_{\alpha_n})} + \frac{\beta_n^{3/2} \ln m_n}{\sigma_{W(n)} f(\xi_{1-\beta_n})} \right], \\
\delta_5(n) &= \frac{\ln \frac{k_n}{m_n} \xi_{\alpha_n}^2 + \ln m_n \xi_{1-\beta_n}^2}{n \sigma_{W(n)}^2} = O \left( \ln \frac{k_n}{m_n} + \ln \frac{m_n}{k_n} \right).
\end{align*} \]

The quantity \( \Delta_{n,S} \) will determine the order of the remainder term in the stochastic approximation for the difference \( \frac{\hat{\sigma}_{W(n)}^2}{\sigma_{W(n)}^2} - 1 \) by a sum of i.i.d. r.v.’s (cf Lemma 2.2, Section 2):

Here is our result for a Studentized slightly trimmed sum.

**Theorem 1.6** Suppose that the conditions \([A_1], [A_2] \) and \([A_3] \) hold true. Then
\[ \sup_{x \in \mathbb{R}} \left| F_{T_n,S}(x) - H_n(x) \right| \leq C(\delta_n + \delta_{n,S}), \]  
(1.34)

where \( C > 0 \) is some constant not depending on \( n \), \( \delta_n \) is the bound on the r.h.s. of (1.28) (cf. Theorem 1.4) and \( \delta_{n,S} = \Delta_{n,S} + \sum_{i=1}^{5} \delta_i(n) \), where \( \Delta_{n,S} \) is as in (1.33); \( \delta_1,S(n) = \ln k_n (\sqrt{k_n} q_{\alpha_n} + \sqrt{m_n} q_{\beta_n}) + \ln m_n (\sqrt{m_n} q_{\beta_n} + \sqrt{k_n} q_{\alpha_n}); \) \( \delta_2,S(n) = (\ln k_n)^2 q_{\alpha_n}^2 + (\ln m_n)^2 q_{\beta_n}^2 + \ln k_n \ln m_n q_{\alpha_n} q_{\beta_n} (q_{\alpha_n} + q_{\beta_n}). \)

The next corollary is analogous to Corollary 1.3, now for a Studentized \( T_n \).
Corollary 1.5 Suppose that the conditions of Corollary 1.3 hold true. Then the bound on the r.h.s. in (1.34) is of the order \( O\left(\frac{(\ln k)^{5/4}}{k^{3/4}} + \frac{(\ln m)^{5/4}}{m^{3/4}}\right)\), \( n \to \infty \).

Finally, we state our Edgeworth type result for a Studentized \( T_n \) parallel to Theorem 1.5.

Theorem 1.7 Suppose that \( \alpha_n \vee \beta_n \to 0 \), as \( n \to \infty \), conditions \([A_1]\) and \([L]\) hold true, and the density satisfies condition \([R]\) with \( 0 < \gamma < 2 \). Then

\[
\sup_{x \in \mathbb{R}} \left| F_{T_n, S}(x) - H_n(x) \right| = O\left(\frac{(\ln k)^{5/4}}{k^{3/4}} + \frac{(\ln m)^{5/4}}{m^{3/4}}\right), \quad n \to \infty.
\]  

(1.35)

Remark 1.2 We conjecture that both (1.29) and (1.35) are also valid without condition \([L]\). The latter condition is only used the formula \( \delta_{2,n} \) on the r.h.s. of (1.28) and a similar term in the studentized case. The \( \varepsilon > 0 \) appearing in the expression of \( \delta_{2,n} \) and in its counterpart for the studentized case is due to the presence of a similar error term involving \( \varepsilon \) in Bentkus et al. [3]. These authors, however, also conjecture in their Remark 1.3 that assuming the existence of such positive \( \varepsilon \) is in fact superfluous, i.e. taking \( \varepsilon = 0 \) will also work. Without condition \([L]\) an extra term of order \( O\left(\frac{(\ln k)^{3/2}}{k^{3/2}} + \frac{(\ln m)^{3/2}}{m^{3/2}}\right)\) shows up; this term can be absorbed in the r.h.s.’s of (1.29) and (1.35) in case condition \([L]\) is satisfied.

To conclude this section we want to mention a by now classical paper by van Zwet [33] on Berry Esseen bounds for general symmetric statistics. We also refer to recent work by Chen & Shao [8], using a method due originally to C.Stein, and also to Bentkus, Jing and Zhou [3] who obtained optimal results on rates of convergence for \( U \)-statistics of general degree \( k \).

For interesting recent probabilistic work on slightly trimmed sums when data are long range dependent linear processes rather than i.i.d. observations we refer to Kulik (cf. [26]).

2 A \( U \)-statistic approximation

In this section we will approximate \( T_n \) by a suitable \( U \)-statistic of degree 2. This will enable us to establish second order approximations – Berry – Esseen bounds and Edgeworth type expansions – for \( T_n \) and its studentized version by applying known results of this type for \( U \)-statistics of degree 2 (cf. Friedrich [13] and Bentkus et.al [4]). This method of proof is well known in the literature; we refer to Bentkus et al [3] for recent work on this topic. However, our remainder term – i.e. the difference between \( T_n \) and the approximating \( U \)-statistic – has a different structure compared with the error terms appearing in previous work on ’smooth statistics’ (cf., for instance, Putter & van Zwet [29]): no terms of higher order in the Hoeffding decomposition, but instead a remainder term of Bahadur type.

Set \( 1_\nu(X_i) = 1\{X_i \leq \xi_\nu\} \), where \( \xi_\nu = F^{-1}(\nu) \), \( 0 < \nu < 1 \), and \( 1_A \) is the indicator of the event \( A \).

Define a \( U \)-statistic of degree 2 with kernel, depending on \( n \), by

\[
L_n + U_n = \sum_{i=1}^n L_{n,i} + \sum_{1 \leq i < j \leq n} U_{n,(i,j)} ,
\]  

(2.1)
where

\[ L_{n,i} = \frac{1}{\sqrt{n}} \left( W_i(n) - \mu W(n) \right) = \frac{1}{\sqrt{n}} \left[ X_i \mathbf{1}_{1-\beta_n}(X_i)(1 - \mathbf{1}_{\lambda_n}(X_i)) + \xi_{\lambda_n} \mathbf{1}_{\alpha_n}(X_i) + \xi_{1-\beta_n}(1 - \mathbf{1}_{1-\beta_n}(X_i)) - \mu W(n) \right], \tag{2.2} \]

with \( W_i(n) \) and \( \mu W(n) \) as in (1.3) and (1.5) respectively, and

\[ U_{n,(i,j)} = \frac{1}{n \sqrt{n}} \left[ - \frac{1}{f(\xi_{\alpha_n})} \left( \mathbf{1}_{\alpha_n}(X_i) - \alpha_n \right) \left( \mathbf{1}_{\alpha_n}(X_j) - \alpha_n \right) \right.
\]

\[ + \frac{1}{f(\xi_{1-\beta_n})} \left( \mathbf{1}_{1-\beta_n}(X_i) - (1 - \beta_n) \right) \left( \mathbf{1}_{1-\beta_n}(X_j) - (1 - \beta_n) \right). \tag{2.3} \]

Note that

\[ E L_{n,i} = 0, \quad i = 1, \ldots, n, \tag{2.4} \]

and

\[ E U_{n,(i,j)} = 0, \quad E \left( L_{n,3} U_{n,(i,j)} \right) = 0, \quad i, j = 1, \ldots, n \quad (i \neq j). \tag{2.5} \]

Using (2.1)–(2.5), we easily check that \( E \left( L_n + U_n \right)^2 = \sigma_{W(n)}^2 + E \left( U_n^2 \right) \), where \( \sigma_{W(n)}^2 \) is given as in (1.5) and \( E \left( U_n^2 \right) = \frac{n-1}{2n} E \left( n^{3/2} U_{n,1,2} \right)^2 \leq \frac{1}{n} \left( \frac{\alpha_n^2}{f(\xi_{\alpha_n})} + \frac{\beta_n^2}{f(\xi_{1-\beta_n})} \right) \). So we obtain:

\[ E \left( \frac{L_n + U_n}{\sigma_{W(n)}} \right)^2 = 1 + \varepsilon_n, \tag{2.6} \]

where \( 0 < \varepsilon_n \leq q_{\alpha_n}^2 + q_{\beta_n}^2 \) (cf. (1.8)), and \( \varepsilon_n \to 0 \), as \( n \to \infty \), provided condition [A2] is satisfied.

For the third moment we have

\[ E \left( \frac{L_n + U_n}{\sigma_{W(n)}} \right)^3 = \frac{1}{\sqrt{n}} \lambda_{1,n} + 3 \sigma_{W(n)}^{-3} E \left( L_n^2 U_n \right) + 3 \sigma_{W(n)}^{-3} E \left( L_n U_n^2 \right) + \sigma_{W(n)}^{-3} E \left( U_n^3 \right), \tag{2.7} \]

where \( \lambda_{1,n} \) is as in (1.24). For the second term on the r.h.s. of (2.7) we obtain

\[ 3 \sigma_{W(n)}^{-3} E \left( L_n^2 U_n \right) = 3n(n-1) \sigma_{W(n)}^{-3} E \left( L_{n,1} L_{n,2} U_{n,1,2} \right), \]

which is equal to

\[ \frac{3}{\sqrt{n}} \frac{1}{\lambda_{2,n}} \left( 1 - \frac{1}{n} \right) = 3 \frac{1}{\sqrt{n}} \lambda_{2,n} + o \left( \frac{1}{n} \right), \tag{2.8} \]

where \( \lambda_{2,n} \) is as in (1.24). The last equality on the r.h.s. of (2.8) is valid because by (1.9) there exists a constant \( C > 0 \) such that

\[ \frac{1}{\sqrt{n}} |\lambda_{2,n}| \leq C \frac{\max(\alpha_n, \beta_n)}{\sqrt{n} \sigma_{W(n)} f(\xi_{\alpha_n}) + \sqrt{n} \sigma_{W(n)} f(\xi_{1-\beta_n})} = O \left( \frac{1}{n} q_{\alpha_n} + q_{\beta_n} \right) = o \left( \frac{1}{n} \right). \]

Using relations (2.2–2.5) we find that

\[ \frac{3}{\sigma_{W(n)}^3} E \left( L_n U_n^2 \right) = \frac{3(n-1)}{\sigma_{W(n)}^{3/2}} \left[ \frac{\alpha_n^2 (\xi_{\alpha_n} - \mu W(n)) (1-\alpha_n) (1-2\alpha_n)}{f^2(\xi_{\alpha_n})} - \frac{2\alpha_n \beta_n}{f(\xi_{\alpha_n}) f(\xi_{1-\beta_n})} \right] \]

\[ + \frac{3}{\sigma_{W(n)}^{3/2}} \left[ \frac{\beta_n (\xi_{1-\beta_n} - \mu W(n)) (1-\beta_n)}{f^2(\xi_{1-\beta_n})} - \frac{2\alpha_n \beta_n}{f(\xi_{\alpha_n}) f(\xi_{1-\beta_n})} \right]. \]
where \( \sigma \) term at the r.h.s. of (2.7) we have \( \sigma > \) for every \( a \) bias term 

then \( \square \)

- \( o = \) 

\( \text{Lemma 2.1} \)

Suppose that the conditions \( [A_1] \) and \( [A_2] \) hold true. Then

\[
P\left( \left| n^{1/2} \left( T_n - \mu(\alpha_n, 1 - \beta_n) \right) - (L_n + U_n + b_n) \right| > \Delta_n \right) = O\left( (k_n \wedge m_n)^{-c} \right), \tag{2.10}
\]

for every \( c > 0 \), where \( b_n \) is as in \( (1.25) \). \( \Delta_n = A(\Delta_{\alpha,n} + \Delta_{\beta,n}) \),

\[
\Delta_{\alpha,n} = \alpha_n \frac{\ln k_n}{\sqrt{n}} \left[ \frac{1}{f(\xi_{\alpha,n})} \left( \frac{\ln k_n}{k_n} \right)^{1/4} + \Psi_{1 - \beta_n, 1/(\gamma)}(B) \right],
\]

\[
\Delta_{\beta,n} = \beta_n \frac{\ln m_n}{\sqrt{n}} \left[ \frac{1}{f(\xi_{1 - \beta_n})} \left( \frac{\ln m_n}{m_n} \right)^{1/4} + \Psi_{1 - \beta_n, 1/(\gamma)}(B) \right], \tag{2.11}
\]

and where the constants \( A, B > 0 \) depend only on \( c \).

**Proof.** Define a binomial r.v. \( \xi_{\nu} = \#(i : X_i \leq \xi_{\nu}) \), \( 0 < \nu < 1 \), and note that

\[
W_n = \frac{1}{n} \sum_{i=1}^{n} W_i(n) = \frac{N_{\alpha,n}}{n} \xi_{\alpha,n} + \frac{1}{n} \sum_{i=N_{\alpha,n}+1}^{N_1-\beta_n} X_i + \frac{n - N_1-\beta_n}{n} \xi_{1-\beta_n}. \tag{2.12}
\]

Then

\[
T_n = \mu(\alpha_n, 1 - \beta_n) - [W_n - EW_n]
\]

\[
= \frac{1}{n} \left[ \text{sgn}(N_{\alpha,n} - k_n) \sum_{i=(k_n \wedge N_{\alpha,n})+1}^{N_{\alpha,n} \vee k_n} (X_i - \xi_{\alpha,n}) 
- \text{sgn}(N_1-\beta_n - (n - m_n)) \sum_{i=(n-m_n) \vee N_1-\beta_n+1}^{N_1-\beta_n} (X_i - \xi_{1-\beta_n}) \right]. \tag{2.13}
\]
where \( sgn(s) = s/|s|, sgn(0) = 0 \), and by lemma 4.2 (cf. Section 4) the latter is equal to
\[
- \frac{(N_{\alpha_n} - \alpha_n n)^2}{2n^2} \frac{1}{f(\xi_{\alpha_n})} + \frac{(N_{1-\beta_n} - (1 - \beta_n) n)^2}{2n^2} \frac{1}{f(\xi_{1-\beta_n})} + R_n, \tag{2.14}
\]
where \( P(|R_n| > \frac{A_{\alpha_n}}{\sqrt{n}} (\Delta_{\alpha,n} + \Delta_{\beta,n})) = O((k_n \land m_n)^{-c}) \), and \( \Delta_{\alpha,n}, \Delta_{\beta,n} \) are given as in (2.11). Relations (2.1), (2.3), and (2.13) – (2.14) yield
\[
n^{1/2}(T_n - \mu(\alpha_n, 1 - \beta_n)) = L_n + U_n - \frac{1}{2n\sqrt{n}} \left[ \frac{1}{f(\xi_{\alpha_n})} \sum_{i=1}^{n} (1_{\alpha_n}(X_i) - \alpha_n)^2 \right] + \frac{1}{n^{1/2}} R_n
+ \frac{1}{f(\xi_{1-\beta_n})} \sum_{i=1}^{n} (1_{1-\beta_n}(X_i) - (1 - \beta_n))^2 \right] + n^{1/2} R_n
= L_n + U_n + b_n + \frac{1}{2\sqrt{n}} \tau_n + n^{1/2} R_n, \tag{2.15}
\]
where \( b_n \) is as in (2.25) and \( \tau_n = -\frac{1}{f(\xi_{\alpha_n})} \tau_{n,1} + \frac{1}{f(\xi_{1-\beta_n})} \tau_{n,2} \).

\( \tau_{n,1} = \frac{1}{n} \sum_{i=1}^{n} \left[ (1_{\alpha_n}(X_i) - \alpha_n)^2 - \alpha_n (1 - \alpha_n) \right], \)
\( \tau_{n,2} = \frac{1}{n} \sum_{i=1}^{n} \left[ (1_{1-\beta_n}(X_i) - (1 - \beta_n))^2 - \beta_n (1 - \beta_n) \right]. \)

We consider only \( \tau_{n,1} \), the treatment for \( \tau_{n,2} \) is similar. Note that \( \tau_{n,1} \) is an average of i.i.d. centered r.v.'s, \( \tau_{n,1} = \frac{1}{n} S_{n,1} \), where \( S_{n,1} = \sum_{k=1}^{n} Y_k \), \( EY_k = 0 \), and \( B_n = D(S_{n,1}) = n\sigma_1^2 \) with \( \sigma_1^2 = EY_1^2 = \alpha_n(1 - \alpha_n)[1 - 2\alpha_n]^2 \). Moreover, for each integer \( m \geq 2 \) we have
\[
EY_1^m = \sigma_1^2 [1 - 2\alpha_n]^{m-2} [(1 - \alpha_n)^{m-1} + (-1)^m (\alpha_n)^{m-1}],
\]
and hence, \( |EY_1^m| \leq \sigma_1^2 \). Then by applying an exponential bound (cf. Petrov [25], chapter 3, Theorem 17, with \( H = 1 \)) we obtain
\[
P\left( |S_{n,1}| \geq x \right) \leq \exp \left( -\frac{x^2}{4B_n} \right) \tag{2.16}
\]
for every \( 0 \leq x \leq B_n \). Take \( x = A(n \ln k_n \alpha_n(1 - \alpha_n))^{1/2} |1 - 2\alpha_n| \). If \( \alpha = 1/2 \) is not a partial limit point of the sequence \( \alpha_n \), we can easily see that \( 0 \leq x \leq B_n \) for all sufficiently large \( n \). Otherwise note that we can consider only cases when \( \delta_n = |\alpha_n - \frac{1}{2}| > A_1 \left( \frac{\ln k_n}{k_n} \right)^{1/2} \) with some \( A_1 > 0 \) which we will choose later. Indeed, if it is not so, we can write: \( \tau_{n,1} = \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{1}{2} + \delta_{n,i} \right]^2 - (\frac{1}{2} - \delta_{n,i}) \left( \frac{1}{2} + \delta_{n,i} \right) \], where \( \delta_{n,i} = (-1)^{1_{\alpha_n}(X_i)} (\alpha_n - \frac{1}{2}) \), \(|\delta_{n,i}| = \delta_n \), and \(|\tau_{n,1}| \leq \delta_n (1 + 2 \delta_n) = O \left( \frac{\ln k_n}{k_n} \right)^{1/2} \). Since \( \xi_{1/2} \in U_n \cup U_b \) under condition \([A_1]\), we have \( f(x) \geq f_0 > 0 \) in some neighbourhood of \( \xi_{1/2} \), and we obtain \( \tau_{n,1} \frac{1}{f(\xi_{\alpha_n})} = O \left( \frac{\ln k_n}{k_n} \right)^{1/2} \), and hence \( \frac{1}{2\sqrt{n}} |\tau_{n,1}| = o(\Delta_n) \) (cf. (2.15)).

For the case \( \delta_n > A_1 \left( \frac{\ln k_n}{k_n} \right)^{1/2} \) we easily check that \( x < B_n \) for all sufficiently large \( n \) when we choose \( A_1 \) such that \( A_1 > A(1 - a_2)^{-1/2} \), and by (2.16) we obtain
\[ P\left( |\overline{r}_{n,1}| \geq A \frac{1}{\sqrt{n}} |1 - 2\alpha_n| (\ln k_n \alpha_n (1 - \alpha_n))^{1/2} \right) \leq \exp \left( -\frac{1}{4} A^2 \ln k_n \right), \] the latter is of the order \( O(k_n^{-c}) \) when \( A^2 > 4c \). So \( \frac{1}{j(\xi_{\alpha_n})} |\overline{r}_{n,1}| \leq A \frac{\ln k_n^{1/2} \alpha_n}{\sqrt{n} f(\xi_{\alpha_n}) (\alpha_n)^{1/2}} = A \alpha_n \left( \frac{\ln k_n}{k_n} \right)^{1/2}, \) and hence \( \frac{1}{2 \sqrt{n}} \frac{1}{j(\xi_{\alpha_n})} |\overline{r}_{n,1}| = o(\Delta_n) \) (cf. (2.11) and (2.15)). The lemma is proved. \( \square \)

We complete this section by a lemma which can be viewed as an extension of Lemma 5.1 from Gribkova and Helmers [15] to slightly trimmed means, i.e. to the case corresponding to the first two lines of (1.7).

**Lemma 2.2** Suppose that the conditions \([A_1]\) and \([A_2]\) hold true. Then for every \( c > 0 \)

\[
P\left( \left| \frac{\hat{\sigma}_{W(n)}^2}{\sigma_{W(n)}^2} - 1 - \frac{V_n}{\sigma_{W(n)}^2} \right| > A \Delta_n, S \right) = O\left( k_n^{-c} + m_n^{-c} \right), \tag{2.17}
\]

where \( \Delta_n, S \) is as in (1.33),

\[ V_n = V_{n,1} + V_{n,2}, \tag{2.18} \]

with

\[ V_{n,1} = 2 \alpha_n \frac{N_{\alpha_n} - \alpha_n n}{n} (\mu_{W(n)} - \xi_{\alpha_n}) \]
\[ + 2 \frac{\beta_n}{f(\xi_{1 - \beta_n})} \frac{N_{1 - \beta_n} - (1 - \beta_n) n}{n} (\mu_{W(n)} - \xi_{1 - \beta_n}) \]

and

\[ V_{n,2} = \frac{1}{n} \sum_{i=1}^{n} \left[ (W_i(n) - \mu_{W(n)})^2 - \sigma_{W(n)}^2 \right]. \]

\( A, B > 0 \) (\( B \) a constant appearing in \( \delta_i(n), i = 2, 3 \)) are some constants not depending on \( n \). Moreover,

\[ EV_n = 0, \quad E\left( \frac{V_n}{\sigma_{W(n)}^2} \right)^2 = O\left( \frac{E(W_1(n))^4}{n \sigma_{W(n)}^4} + q_{\alpha_n}^2 + q_{\beta_n}^2 \right). \tag{2.19} \]

**Proof.** First we note that relations (2.19) follow directly by definition (2.18) of \( V_{n,i}, \) \( i = 1, 2, \) and (1.9).

To prove (2.17) fix an arbitrary \( c > 0 \) and define the auxiliary quantity \( \tilde{S}_{W(n)}^2 = \frac{1}{n} \sum_{i=1}^{n} W_i^2(n) - \overline{W}^2(n) \), where \( \overline{W}(n) = \frac{1}{n} \sum W_i(n) \). First we prove that

\[ \tilde{\sigma}_{W(n)}^2 = S_{W(n)}^2 + V_{n,1} + R_{n,1}, \tag{2.20} \]

where

\[ P\left( \left| R_{n,1} \right| > A_1 \Delta_n, S \right) = O(k_n^{-c} + m_n^{-c}). \tag{2.21} \]
Here and elsewhere $A, A_i > 0$, $i = 1, 2, \ldots$, denote the constants, independent of $n$. We have
\[
\hat{\sigma}_{W(n)}^2 - S_{W(n)}^2 = \left[\frac{k_n}{n} X_{k_n:n}^2 + \frac{1}{n} \sum_{i=k_n+1}^{n-m_n} X_{i:n}^2 + \frac{m_n}{n} \sum_{i=k_n+1}^{n-n-m_n} - \frac{N_{\alpha_n} \xi_{\alpha_n}}{n} - \frac{1}{n} \sum_{i=N_{\alpha_n}+1}^{N_1-\beta_n} X_{i:n}^2 \right] - \frac{n - N_1-\beta_n \xi_{1-\beta_n}^2}{n}
\]
\[
+ \left(\frac{N_{\alpha_n} \xi_{\alpha_n} + 1}{n} \sum_{i=N_{\alpha_n}+1}^{N_1-\beta_n} X_{i:n} + \frac{n - N_1-\beta_n \xi_{1-\beta_n}}{n}\right)^2
\]
\[
- \left(\frac{k_n}{n} X_{k_n:n} + \frac{1}{n} \sum_{i=k_n+1}^{n-m_n} X_{i:n} + \frac{m_n}{n} X_{n-m_n:n}\right)^2.
\]

(2.22)

Rewrite the term within the first square brackets on the r.h.s. of (2.22) as
\[
\frac{k_n}{n} (X_{k_n:n}^2 - \xi_{\alpha_n}^2) + \frac{1}{n} sgn(N_{\alpha_n} - k_n) \sum_{i=(k_n \land N_{\alpha_n})+1}^{k_n \lor N_{\alpha_n}} (X_{i:n}^2 - \xi_{\alpha_n}^2) + \frac{m_n}{n} (X_{n-m_n:n}^2 - \xi_{1-\beta_n}^2)
\]
\[
- \frac{1}{n} sgn(N_{1-\beta_n} - (n-m_n)) \sum_{i=(N_{1-\beta_n} \land (n-m_n)) + 1}^{(n-m_n) \lor N_{1-\beta_n}} (X_{i:n}^2 - \xi_{1-\beta_n}^2),
\]

(2.23)

then by Lemmas 4.1 and 4.2 where $G(x) = x^2$ (cf. Section 4), the latter quantity is equal to
\[
- 2\alpha_n \frac{N_{\alpha_n} - \alpha_n}{n} \frac{\xi_{\alpha_n}}{f(\xi_{\alpha_n})} - \frac{(N_{\alpha_n} - \alpha_n)^2}{n^2} \frac{\xi_{\alpha_n}}{f(\xi_{\alpha_n})} - 2\beta_n \frac{N_{1-\beta_n} - (1 - \beta_n)}{n} \frac{\xi_{1-\beta_n}}{f(\xi_{1-\beta_n})}
\]
\[
+ \frac{(N_{1-\beta_n} - (1 - \beta_n))^2}{n^2} \frac{\xi_{1-\beta_n}}{f(\xi_{1-\beta_n})} + R^{(1)}_{n,1},
\]

(2.24)

where $R^{(1)}_{n,1}$ is a remainder term appearing as result of application of Lemma 4.1 two times: in the first and third terms of (2.23). Using (4.1), (4.9) and inequalities (1.11) we obtain that $|R^{(1)}_{n,1}|/\sigma_{W(n)}^2 = O(\delta_1(n) + \delta_2(n) + \delta_3(n))$ with probability $1 - O(k_n^{-c} + m_n^{-c})$, where $\delta_2(n)$, $\delta_3(n)$ involve $B > 0$, which depends on $c$ and does not depend on $n$. Note that the remainder term appearing as result of application of Lemma 4.2 in (2.23) is of the negligible order and contribute to $R^{(1)}_{n,1}$. Moreover, the Bernstein's inequality and (1.9) together imply that $\frac{(N_{\alpha_n} - \alpha_n)^2}{n^2 \sigma_{W(n)}^2} |\frac{1}{f(\xi_{\alpha_n})}| = O\left(\frac{2^{1/2} \ln k_n}{\alpha_n \sigma_{W(n)} f(\xi_{\alpha_n})}\right)$ with probability $1 - O(k_n^{-c})$, and it is $o(\delta_1(n))$. The same is valid for the second quadratic term in (2.24). Thus, we obtain that (2.24) is equal
\[
- 2\alpha_n \frac{N_{\alpha_n} - \alpha_n}{n} \frac{\xi_{\alpha_n}}{f(\xi_{\alpha_n})} - 2\beta_n \frac{N_{1-\beta_n} - (1 - \beta_n)}{n} \frac{\xi_{1-\beta_n}}{f(\xi_{1-\beta_n})} + R^{(1)}_{n,1},
\]

(2.25)
Now consider the term within the second square brackets on the r.h.s. of (2.22).

Arguing as before, we can rewrite it as

\[
\left( \frac{\alpha_n}{f(\xi_{\alpha_n})} \frac{N_{\alpha_n} - \alpha_n n}{n} + \frac{\beta_n}{f(\xi_{1-\beta_n})} \frac{N_{1-\beta_n} - (1 - \beta_n)n}{n} + R_{n,1}^{(2)} \right) \times \left( \frac{1}{n} \sum_{i=1}^{n} W_i(n) - \frac{\alpha_n}{f(\xi_{\alpha_n})} \frac{N_{\alpha_n} - \alpha_n n}{n} - \frac{\beta_n}{f(\xi_{1-\beta_n})} \frac{N_{1-\beta_n} - (1 - \beta_n)n}{n} - R_{n,1}^{(2)} \right),
\]

where by Lemma 4.1 \(|R_{n,1}^{(2)}| \leq A_1 \left\{ \alpha_n \left[ \frac{1}{f(\xi_{\alpha_n})} \left( \frac{\ln n}{k_n} \right)^{3/4} + \Psi_{\alpha_n, \frac{1}{f(\xi_{\alpha_n})}}(B) \left( \frac{\ln n}{k_n} \right)^{1/2} \right] \right\}

+ \beta_n \left[ \frac{1}{f(\xi_{1-\beta_n})} \left( \frac{\ln m_n}{m_n} \right)^{3/4} + \Psi_{\beta_n, \frac{1}{f(\xi_{1-\beta_n})}}(B) \left( \frac{\ln m_n}{m_n} \right)^{1/2} \right] \right\}

with probability \(1 - O(k_n^{-c} + m_n^{-c})\).

The quadratic and remainder terms caused by application of Lemma 4.2 are of the negligible order and contribute to \(R_{n,1}^{(2)}\) again. Simple computations using (2.22), (2.25) – (2.29), Bernstein’s inequality and (1.9) lead to the following relation

\[
\frac{\hat{\sigma}_{W(n)}^2}{\sigma_{W(n)}^2} - S_{W(n)}^2 = V_{n,1} + O(\delta_1(n) + \delta_2(n) + \delta_3(n)) + R_{n,1}^{(3)}
\]

where \(R_{n,1}^{(3)} = \frac{2}{n} \sum_{i=1}^{n} \left( W_i(n) - \mu_{W(n)} \right) \left[ \frac{\alpha_n}{f(\xi_{\alpha_n})} \frac{N_{\alpha_n} - \alpha_n n}{n} + \frac{\beta_n}{f(\xi_{1-\beta_n})} \frac{N_{1-\beta_n} - (1 - \beta_n)n}{n} \right]. \)

Since by Hoeffding’s inequality for sum of i.i.d. centered bounded r.v.’s (cf. Hoeffding [23]) \(\left| \sum_{i=1}^{n} \left( W_i(n) - \mu_{W(n)} \right) \right| \leq A_2 \ln^{1/2}(k_n \wedge m_n)(\xi_{1-\beta_n} - \xi_{\alpha_n}) n^{1/2} \) with probability \(1 - O(k_n^{-c} + m_n^{-c})\), where \(A_2 > 0\) is some constant, depending on \(c\) and not depending on \(n\), using the latter bound and Bernstein’s inequality and (1.9) after the simple computations we obtain

\[
\left| \frac{R_{n,1}^{(3)}}{\sigma_{W(n)}^2} \right| \leq A_2 \frac{\left| \xi_{\alpha_n} \right| + \left| \xi_{1-\beta_n} \right|}{n \sigma_{W(n)}^2} \left[ \frac{\alpha_n^{3/2}}{f(\xi_{\alpha_n})} + \frac{\beta_n^{3/2}}{f(\xi_{1-\beta_n})} \right] = O(\delta_4(n))
\]

with probability \(1 - O(k_n^{-c} + m_n^{-c})\), and (2.20) – (2.21) follow.

Finally we prove that

\[
S_{W(n)}^2 = \sigma_{W(n)}^2 + V_{n,2} + R_{n,2},
\]

where \(R_{n,2}\) satisfies (2.21). We have

\[
S_{W(n)}^2 - \sigma_{W(n)}^2 - V_{n,2} = S_{W(n)}^2 - \frac{1}{n} \sum_{i=1}^{n} (W_i(n) - \mu_{W(n)})^2 = -(\hat{W}(n) - \mu_{W(n)})^2,
\]

and applying Hoeffding’s inequality once more, we obtain that the quantity at the r.h.p. of (2.30) divided by \(\sigma_{W(n)}^2\) in absolute value is of the order \(\frac{\ln k_n \wedge \ln m_n}{n \sigma_{W(n)}^2} \left( \xi_{1-\beta_n} - \xi_{\alpha_n} \right)^2 = O(\delta_5(n))\) with probability \(1 - O(k_n^{-c} + m_n^{-c})\), and (2.29) follows. The lemma is proved. \(\square\)
3 Proofs

In this section we prove Theorems 1.1.-1.7 and their corollaries stated in Section 1.

**Proof of Theorem 1.1.** By Lemma 2.1 we can write \( n^{1/2}(T_n - \mu(\alpha_n, \beta_n)) = L_n + U_n + b_n + R_n \), where \( L_n + U_n \) is \( U \)-statistic of degree 2, (cf. (2.1)), \( b_n \) is as in (1.25), and \( R_n \) is a remainder term (cf. (2.10)). Define the df of a normalized \( U \)-statistic:

\[
F_{U,n}(x) = \Phi \left( \frac{L_n + U_n}{\sigma_W(n)} \right).
\]

Since \( \frac{b_n}{\sigma_W(n)} \leq \frac{1}{2}(q_{\alpha_n} + q_{\beta_n}) = \frac{1}{2}\sqrt{n}\delta_{2,n} \) (cf. (1.16)), the following inequalities are valid:

\[
F_{U,n}(x - \delta_n) - P(\|R_n\| > \Delta_n) \leq F_{T_n}(x) \leq F_{U,n}(x + \delta_n) + P(\|R_n\| > \Delta_n),
\]

where \( \delta_n = \frac{\Delta_n}{\sigma_W(n)} + \frac{1}{2}\sqrt{n}\delta_{2,n} \), \( \Delta_n \) is as in (2.10), and by Lemma 2.1 \( P(\|R_n\| > \Delta_n) = O(k_n^{-c} + m_n^{-c}) \) for every \( c > 0 \).

For \( F_{U,n}(x \pm \delta_n) \) we can write

\[
\sup_{x \in \mathbb{R}} |F_{U,n}(x \pm \delta_n) - \Phi(x)| \leq \Delta_{n,1} + \Delta_{n,2},
\]

where

\[
\Delta_{n,1} = \sup_{x \in \mathbb{R}} |F_{U,n}(x) - \Phi(x)|, \quad \Delta_{n,2} = \sup_{x \in \mathbb{R}} |\Phi(x \pm \delta_n) - \Phi(x)|.
\]

To estimate \( \Delta_{n,1} \) we apply the Berry – Esseen bound for \( U \)-statistics (cf. Friedrich [13]):

\[
\Delta_{n,1} \leq \frac{C}{\sqrt{n}} \left[ \frac{E|W_1(n)|^3}{\sigma_W(n)^3} + \frac{E|\sqrt{n}U_{n,(1,2)}|^{5/3}}{\sigma_W(n)^{5/3}} \right],
\]

where \( C > 0 \) is an absolute constant. Using formula (2.3), we easily check that

\[
E|\sqrt{n}U_{n,(1,2)}|^{5/3} \leq 2^{2/3} \left[ \left( \frac{1}{f(\xi_{\alpha_n})} \right)^{5/3} \left( E|1_{\alpha_n}(X_1) - \alpha_n|^{5/3} \right)^2 + \left( \frac{1}{f(\xi_{1-\beta_n})} \right)^{5/3} \left( E|1_{1-\beta_n}(X_1) - (1 - \beta_n)|^{5/3} \right)^2 \right] \leq 2^{4/3} \left[ \frac{(\alpha_n(1 - \alpha_n))^2}{f^{5/3}(\xi_{\alpha_n})} + \frac{(\beta_n(1 - \beta_n))^2}{f^{5/3}(\xi_{1-\beta_n})} \right].
\]

Relations (3.4)-(3.5) together imply that

\[
\Delta_{n,1} \leq \frac{C_1}{\sqrt{n}} (\delta_{1,n} + \delta_{3,n}),
\]

where \( C_1 > 0 \) is some absolute constant.

Finally, consider \( \Delta_{n,2} \). Note that \( \frac{\Delta_n}{\sigma_W(n)} = \frac{1}{\sqrt{n}}(o(\delta_{2,n}) + \delta_{4,n}) \) (cf. (1.10) and (2.10)), therefore \( \delta_n = \frac{1}{\sqrt{n}}(\delta_{2,n}(1/2 + o(1)) + \delta_{4,n}) \), and we obtain

\[
\Delta_{n,2} \leq \frac{C_2}{\sqrt{n}} (\delta_{2,n} + \delta_{4,n}),
\]

(3.7)
where $C_2 > 0$ is some constant, depending only on $c$ (cf. Lemma 2.1). Relations (3.1)–(3.2), and (3.6)–(3.7) imply (1.16). The theorem is proved. □

**Proof of Theorem 1.2.** We obtain this theorem as a consequence of Theorem 1.1. First choose $c = 1/2$ in (1.16). To prove (1.18) we must verify that under conditions $[A_1], [A'_2]$ and $[A'_3]$

$$
\frac{1}{\sqrt{n}} \delta_{i,n} = O\left(\frac{1}{\sqrt{k_n}} + \frac{1}{\sqrt{m_n}}\right), \quad i = 1, \ldots, 4.
$$

For $i = 1$ we have

$$
\frac{1}{\sqrt{n}} \delta_{1,n} = \frac{\alpha_n^2}{\sqrt{n} \sigma^3_{W(n)}} \left(\frac{\left(\alpha_n \right)^{1/2}}{\sigma_{W(n)}}\right)^3,
$$

We consider the three terms in the nominator on the r.h.s. of (3.8). For the first one we have $\frac{\alpha_n^2}{\sqrt{n} \sigma^3_{W(n)}} = \frac{1}{\sqrt{n}} \left(\frac{\left(\alpha_n \right)^{1/2}}{\sigma_{W(n)}}\right)^3$ and by (1.9) it is a magnitude of the exact order $O\left(\frac{1}{\sqrt{n}}\right)$, because $\lim \inf_{n \to \infty} \sigma^3_{W(n)} > 0$ under the condition $[A_1]$ when $a_2 < b_1$. Similarly for the third term we obtain the bound of the order $O\left(\frac{1}{\sqrt{n}}\right)$, whereas for the second term we have

$$
\frac{1}{\sqrt{n} \sigma^3_{W(n)}} \int_{\alpha_n}^{1-\beta_n} |F^{-1}(u)|^3 \, du = \frac{1}{\sqrt{n}} \frac{|\xi_{\alpha_n}| \sqrt{\sigma_{W(n)}}}{\sigma_{W(n)}} = O\left(\frac{1}{\sqrt{k_n}} + \frac{1}{\sqrt{m_n}}\right),
$$

hence, for $\delta_{1,n}$ the desired estimate is valid. For $\delta_{2,n}$ it follows directly from (1.17). For the first term of $\delta_{3,n}$ by (1.17) we obtain

$$
\frac{1}{\sqrt{n}} \left(\frac{\left(\alpha_n \right)^{1/2}}{\sigma_{W(n)}}\right)^{5/3} \leq C \frac{\alpha_n^{1/3}}{\sqrt{n}} \left(\frac{\left(\alpha_n \right)^{-1/2}}{\sigma_{W(n)}}\right)^{5/3} \leq C \frac{\gamma}{\sqrt{n}},
$$

where $C > 0$ is some constant, independent of $n$ (cf. (1.17)), and for the second term of $\delta_{3,n}$ we similarly obtain the bound $O\left(\frac{1}{\sqrt{m_n}}\right)$. Finally, for $\delta_{4,n}$ conditions $[A'_2]$ and $[A'_3]$ directly yield $\frac{1}{\sqrt{n}} \delta_{4,n} = O\left(\frac{1}{\sqrt{k_n}} + \frac{1}{\sqrt{m_n}}\right)$. The theorem is proved. □

**Proof of Theorem 1.3.** Also the validity of this theorem is a simple consequence of Theorem 1.1. Take an arbitrary $c > 1/2$ and $A, B$ on the r.h.s. of (1.16), corresponding to the value of $c$. Now to prove (1.20) it suffices to repeat the proof of Theorem 1.2, taking into account that $\alpha_n^2 \sigma_{\alpha_n}^2 \beta_n^2 \sigma_{1-\beta_n}^2 \to 0$, as $n \to \infty$ when $\sigma^2 < \infty$. This gives us the desired bound for $\frac{1}{\sqrt{n}} \left(\delta_{1,n} + \delta_{2,n} + \delta_{3,n}\right)$ at the r.h.s. of (1.16). Finally, an application of the condition: $\Psi_{\alpha_n, f(x)}(B) = o\left((f(\xi_{\alpha_n}) \ln k_n)^{-1}\right)$ and $\Psi_{1-\beta_n, f(x)}(B) = o\left((f(\xi_{1-\beta_n}) \ln m_n)^{-1}\right)$ for every $B > 0$, as $n \to \infty$, directly provides the bound $\frac{1}{\sqrt{n}} \delta_{4,n} = o\left(\frac{1}{\sqrt{k_n}} + \frac{1}{\sqrt{m_n}}\right)$. The theorem is proved. □

**Proof of Corollary 1.1.** To prove this corollary we apply (1.16) with constants $A$ and $B$, corresponding to $c = 1/2$ in the general case, and to some value $c > 1/2$ in the special case where $\gamma = 2$ and $\sigma^2 < \infty$ to obtain the bound (1.20). We must verify that the conditions $[A'_2]$ and $[A'_3]$ are satisfied in our case. Note that $F(x)$ and
imply that both \( \liminf_{n \to \infty} \frac{\alpha_n}{f(\xi_{\alpha_n})} = \lim_{x \to -\infty} \frac{F(x)}{xf(x)} = \frac{1}{\gamma} \) and \( \lim_{n \to \infty} \frac{\beta_n}{x^{1 - \beta_n} f(\xi_{1 - \beta_n})} = \lim_{x \to -\infty} \frac{1 - F(x)}{xf(x)} = \frac{1}{\gamma} \) (cf. Bingham et al. [8]), and the conditions \([A']_2\) (and hence, \([A'_2]\)) is satisfied. This implies that the quantity \( \frac{1}{\sqrt{n}}(\delta_{1,n} + \delta_{2,n} + \delta_{3,n}) \) has a magnitude of the order \( O \left( \frac{1}{\sqrt{k_n}} + \frac{1}{\sqrt{m_n}} \right) \). Moreover, in special case that \( \gamma = 2 \) and \( \sigma^2 < \infty \) the same quantity is of the smaller order, i.e. \( o \left( \frac{1}{\sqrt{k_n}} + \frac{1}{\sqrt{m_n}} \right) \) (cf. proof of Theorems 1.2 and 1.3).

It remains to check that \([A'_3]\) holds true in our case. We will verify that the first inequality in \([A'_3]\) is satisfied, for the second one we can apply similar argument. Set \( x_n = F^{-1}(\alpha_n) \) and \( x_n + \triangle x_n = F^{-1}(\alpha_n + t \sqrt{\frac{\ln k_n}{k_n}}) = F^{-1}(\alpha_n(1 + t \sqrt{\frac{\ln k_n}{k_n}})) \), where \( |t| \leq B \). Then \( \Psi_{\alpha_n, \frac{1}{f(x)}} \) holds true. Moreover, since \( \alpha_n \to 0 \), and \( |x_n| = (\alpha_n)^{-1/\gamma} L_1(\alpha_n) \), where \( L_1 \) is a slowly varying function when its argument tends to zero. Therefore \( \triangle x_n = F^{-1}(\alpha_n(1 + t \sqrt{\frac{\ln k_n}{k_n}})) - F^{-1}(\alpha_n) = |x_n| \left[ (1 + t \sqrt{\frac{\ln k_n}{k_n}})^{-1/\gamma} L_1(\alpha_n(1 + t \sqrt{\frac{\ln k_n}{k_n}})) - 1 \right] \) with \( L_1 \) is as before and satisfying the requirement that it is in absolute value of order \( o(|x_n|) \). Then by the condition \([R]\) for every fixed \( t \) such that \( |t| \leq B \) we can write

\[
\frac{|f(x_n) - f(x_n + \triangle x_n)|}{f(x_n + \triangle x_n)} = O \left( \frac{f(x_n)}{|\triangle x_n|} \right) = O \left( \frac{f(x_n)}{|x_n|} \right),
\]

as \( n \to \infty \). Next we note that \( \triangle x_n = \frac{1}{f(F^{-1}(\alpha_n + \theta \alpha_n \sqrt{\frac{\ln k_n}{k_n}}))} t \alpha_n \sqrt{\frac{\ln k_n}{k_n}}, \) where \( 0 < \theta < 1 \), and by \([R]\) the latter quantity is equal to

\[
\frac{1}{f(x_n) + O \left( \frac{f(x_n)}{|x_n|} |\triangle x_n| \right)} t \alpha_n \sqrt{\frac{\ln k_n}{k_n}}.
\]

Then at the r.h.s. of (3.3) we have a quantity of the order \( O \left( \frac{t \alpha_n \sqrt{\ln k_n}}{|x_n| f(x_n) (1 + o(1))} \right) \), as \( n \to \infty \). Since \( |x_n| f(x_n) \sim F(x_n)^{\frac{1}{\gamma}} \) due to the regularly varying property, and because \( F(x_n) = F(F^{-1}(\alpha_n)) = \alpha_n \), we obtain that the quantity at the r.h.s. of (3.9) is of the order \( O \left( \sqrt{\frac{\ln k_n}{k_n}} \right) \) uniformly in all \( |t| \leq B \). This implies that \( \Psi_{\alpha_n, \frac{1}{f(x)}}(B) = o \left( f(\xi_{\alpha_n}) \ln k_n \right)^{-1} \), and similarly we obtain that \( \Psi_{1 - \beta_n, \frac{1}{f(x)}}(B) = o \left( f(\xi_{1 - \beta_n}) \ln m_n \right)^{-1} \). We can conclude that under our conditions we have \( \frac{1}{\sqrt{k_n}}(\delta_{4,n}) = o \left( \frac{1}{\sqrt{k_n}} + \frac{1}{\sqrt{m_n}} \right) \), what completes the proof of the corollary. □

**Proof of Corollary 1.2.** It suffices to check that the conditions of Theorem 1.2 are satisfied. Condition \([A'_1]\), inequalities \( 0 < a_1 < b_2 < 1 \) and continuity of \( f \) together imply that both \( \liminf_{n \to \infty} (f(\xi_{a_1}) \wedge f(\xi_{b_2})) > 0 \) and \( \liminf_{n \to \infty} \sigma W_n > 0 \) are automatically satisfied. Hence, \([A'_1]\) holds true. Moreover, by the Hölder condition of degree \( d > 0 \) we have: \( \Psi_{\alpha_n, \frac{1}{f(x)}}(B) = O \left( \frac{\alpha_n \ln k_n}{n} \right)^{d/2} = O \left( \frac{\ln n}{n} \right)^{d/2} = o \left( \frac{1}{m_n} \right) \), as \( n \to \infty \).
The same argument is valid for \( \Psi_{1-\beta_n, \mathbb{P}_{(x)}}(B) \). Thus, \([A'_3]\) also holds true. The corollary is proved. \( \square \)

**Proof of Theorem 1.4.** Similarly as in the proof of Theorem 1.1 (cf. (3.1)) we write

\[
F_{U,n}(x - \frac{\Delta_n}{\sigma_{W(n)}}) - P(\{R_n > \Delta_n\}) \leq F_{U,n}(x + \frac{b_n}{\sigma_{W(n)}}) \\
\leq F_{U,n}(x + \frac{\Delta_n}{\sigma_{W(n)}}) + P(\{|R_n| > \Delta_n\}), \tag{3.10}
\]

where \( \Delta_n \) is as in Lemma 2.1, \( b_n \) is as in (1.25), and \( F_{U,n} \) is the df of \( (L_n + U_n)/\sigma_{W(n)} \) (cf. (3.1)) — the \( U \)-statistic of degree 2 with a kernel which depends on \( n \). Our smoothness condition \([A_1]\) implies that the df of the r.v. \( W_1(n) \) has a positive density on a Borel set \( D_n \subset \mathbb{R} \) with nonzero Lebesgue measure and that there exists an integer \( n_0 \) such that \( \bigcap_{n=n_0}^{\infty} D_n \supset D \neq \emptyset \), and the set \( D \) has nonzero Lebesgue measure. Hence, the members of the consequence of the df’s of r.v.’s \( W_1(n) \) have a common nontrivial absolutely continuous component independent of \( n \) for all sufficiently large \( n \). This yields

\[
\limsup_{n \to \infty} \limsup_{|t| \to \infty} |E \exp(it\sqrt{n} L_{n,1})| = \limsup_{n \to \infty} \limsup_{|t| \to \infty} |E \exp(it W_1(n))| < 1, \tag{3.11}
\]

hence the sequence of the first canonical functions of the \( U \)-statistic satisfies the Cramer condition, and we can apply a result by Bentkus et al. \([4]\). The one term Edgeworth expansion of the df \( F_{U,n}(x) = P\left((L_n + U_n)/\sigma_{W(n)} \leq x\right) \) is equal to \( G_{U,n}(x) = \Phi(x) - \frac{\phi(x)}{\sigma_{W(n)}}(\lambda_{1(n)} + 3\lambda_{2(n)})(x^2 - 1) \) (cf. Section 2, cf. also Bentkus et al. \([4]\), page 855). Write

\[
\sup_{x \in \mathbb{R}} \left| F_{U,n}(x + \frac{\Delta_n}{\sigma_{W(n)}}) - G_{U,n}(x) \right| \leq \Delta_{n,1} + \Delta_{n,2}, \tag{3.12}
\]

where \( \Delta_{n,1} = \sup_{x \in \mathbb{R}} |F_{U,n}(x) - G_{U,n}(x)| \), \( \Delta_{n,2} = \sup_{x \in \mathbb{R}} \left| G_{U,n}(x + \frac{\Delta_n}{\sigma_{W(n)}}) - G_{U,n}(x) \right| \).

To estimate \( \Delta_{n,1} \) we apply Theorem 1.2 of Bentkus et al. \([4]\), taking into account the Remark 1.3, given on page 856 in cited paper. Then we obtain

\[
\Delta_{n,1} \leq \frac{C}{n} \left( \frac{E(W_1(n))^4}{\sigma_{W(n)}^4} + \frac{\gamma_{2+\varepsilon}}{\sigma_{W(n)}^{2+\varepsilon}} \right), \tag{3.13}
\]

where \( \varepsilon > 0 \) is an arbitrary constant, the constant \( C > 0 \) depends on \( \varepsilon \) and does not depend on \( n \) (Note that the quantity \( \Delta_3^2 \) appearing in Theorem 1.2 is zero in case of a \( U \)-statistic of degree 2 (cf. Bentkus et al. \([4]\), page 858)), and

\[
\gamma_{2+\varepsilon} = E \left| n^{3/2} U_{n,(1,2)} \right|^{2+\varepsilon} \leq 2^{1+\varepsilon} \left[ \frac{(E |I_{\alpha_n}(X_1) - \alpha_n|^{2+\varepsilon})^2}{f^{2+\varepsilon}(\xi_{\alpha_n})} + \frac{(E |I_{1-\beta_n}(X_1) - (1 - \beta_n)|^{2+\varepsilon})^2}{f^{2+\varepsilon}(\xi_{1-\beta_n})} \right] \leq 2^{1+\varepsilon} \left[ \frac{\alpha_n^2}{f^{2+\varepsilon}(\xi_{\alpha_n})} + \frac{\beta_n}{f^{2+\varepsilon}(\xi_{1-\beta_n})} \right]. \tag{3.14}
\]
Relations (3.13) - (3.14) imply that $\Delta_{n,1} \leq \frac{C_1}{n} (\delta_{1,n} + \delta_{2,n})$, and since $G'_{U,n}(x)$ is bounded uniformly in $x$, we obtain $\Delta_{n,2} \leq C_2 \frac{\delta_{1,n}}{\sigma_W(n)} + \frac{C_2}{n^{3/4}} \delta_{4,n} + \frac{C_2}{n^{1/2}} \delta_{5,n} + C_3 (k_{n}^{-c} + m_{n}^{-c})$. Thus, the corollary follows directly from (1.28).

It remains to note that since $G'_{U,n}(x)$ and $G''_{U,n}(x)$ are bounded uniformly in $x$, we have

$$\sup_{x \in \mathbb{R}} \left| F_{T,n}(x) - G_{U,n}(x) - \frac{b_n}{\sigma_W(n)} \right| \leq C \left( \frac{\lambda_{1(n)} + 3\lambda_{2(n)}}{\sqrt{n} \sigma_W(n)} + \frac{b_n^2}{\sigma_W(n)^2} \right) \leq C \left( \frac{\delta_{6,n}}{n^{1/2}} + \frac{\delta_{3,n}}{n} \right),$$

where $C$ is some constant not depending on $n$. Relations (3.15) - (3.16) imply (1.28). The proof is complete.

**Proof of Corollary 1.3.** To prove this corollary we apply Theorem 1.4 with $c = 3/4$, and check that the quantity at the r.h.s. of (1.28) is of the order $O \left( \frac{(\ln k_n)^{5/4}}{k_n^{3/4}} + \frac{m_n}{k_n^{3/4}} \right)$ under conditions of Corollary 1.3. Similarly as in proof of Theorem 1.2 using condition $[A'_2]$ and (1.9) we easily verify that $\frac{1}{n} \left( \delta_{1,n} + \frac{\delta_{3,n}}{m_n} \right) = O \left( \frac{1}{k_n} + \frac{1}{m_n} \right)$. By condition $[A'_2]$ for $\frac{1}{n} \delta_{2,n}$ we have the bound $O \left( \frac{1}{n} \left[ \alpha_1^{-1-3\varepsilon/2} + \beta_1^{-1-3\varepsilon/2} \right] \right) = O \left( \frac{1}{k_n} + \frac{m_n}{n} \right)^{3/2} + \frac{m_n}{n} \left( \frac{n}{m_n} \right)^{3/2}$, and by condition $[L]$ the latter quantity is of the order $O \left( \frac{1}{k_n^{1/4}} + \frac{1}{m_n^{3/4}} \right)$ if $s \geq 6\varepsilon/(1+6\varepsilon)$. For $\frac{1}{n} \delta_{2,n}$ we have the desired bound directly by $[A'_2]$, for $\frac{1}{n^{1/2}} \delta_{5,n}$ we get the same bound directly by the conditions $[A'_2]$ and the condition on $\Psi_{\alpha_n, \frac{1}{\sigma_W(n)}}(B)$ and $\Psi_{1-\beta_n, \frac{1}{\sigma_W(n)}}(B)$. To treat $\frac{1}{n^{1/2}} \delta_{6,n}$ we use the same argument as before based on (1.9) and condition $[A'_2]$, which leads to a bound of order $O \left( \frac{1}{k_n} + \frac{1}{m_n} \right)$ for this term. The corollary is proved.

**Proof of Theorem 1.5.** Similarly as in proof of Corollary 1.1 we check that the condition $[A'_2]$ (and hence $[A'_2]$) is satisfied. Moreover, conditions for $\Psi_{\alpha_n, \frac{1}{\sigma_W(n)}}(B)$ and $\Psi_{1-\beta_n, \frac{1}{\sigma_W(n)}}(B)$ (cf. corollary 1.3) are satisfied if the conditions $[A_1]$, $[R]$ hold true (cf. proof of the Corollary 1.1). Thus, we obtain the validity of (1.29) as a consequence of Corollary 1.3. The proof is complete.

**Proof of Corollary 1.4.** This corollary follows directly from (1.28). Indeed, in our conditions we have $\frac{1}{n} \left( \delta_{1,n} + \delta_{2,n} + \delta_{3,n} \right) = O \left( \frac{1}{n} \right)$, $n^{-3/4} \delta_{4,n} = O \left( \frac{(\ln k_n)^{5/4}}{n^{3/4}} + \frac{m_n}{n^{3/4}} \right)$, $n^{-1/2} \delta_{6,n} = O \left( \frac{1}{n} \right)$, and by Hölder condition $\ln n(\Psi_{\alpha_n, \frac{1}{\sigma_W(n)}}(B) + \Psi_{1-\beta_n, \frac{1}{\sigma_W(n)}}(B)) = O \left( \ln n \left( \ln \frac{n}{m_n} \right)^{d/2} \right) = o(n^{-d/2+\varepsilon})$ for every $\varepsilon > 0$. These bounds imply that $n^{-3/4} \delta_{4,n} + n^{-1/2} \delta_{6,n} = o(n^{-1/2-p})$ for every $p < \min(1/4, d/2)$. The corollary is proved.

**Proof of Theorem 1.6.** First we write $F_{T,n,S}(x) = P \left( \frac{L_n + U_n + b_n}{\sigma_W(n)} + \frac{R_n}{\sigma_W(n)} \leq x \right)$, where by Lemma 2.1: $p_{n,1} := P(\lvert R_n \rvert > \Delta_n) = O(k_{n}^{-c} + m_{n}^{-c})$, for every $c > 0$, and $\Delta_n$ is as in (2.10). By Lemma 2.2 the main term of the quantity $\frac{\sigma_W(n)}{\sigma_W(n)} - 1$ is $V_n \sigma_W(n).$
for which by Chebyshev’s inequality for every \( t > 0 \) we have \( p_{n,2} = P\left( \frac{|V_n|}{\sigma_{\tilde{W}(n)}} > 2t \right) \leq \frac{EV_n^2}{4t^2\sigma_{\tilde{W}(n)}^4} \leq C \left( \frac{E(W_n^4)}{n \sigma_{\tilde{W}(n)}^4} + \frac{\beta_2^2}{\sigma_{\tilde{W}(n)}^4} \right) \), where \( C > 0 \) is some constant independent of \( n \) and \( t \) (cf. (2.19)), and because (1.9) the latter quantity is of the order \( O\left( \frac{1}{n} + \frac{1}{m \sigma_{\tilde{W}(n)}^4} \right) = o(\delta_1, S(n) + \delta_2(n)) \), where \( \delta_1, S(n), \delta_2(n) \) as in (1.31) and (1.33) respectively. This implies that \( \left| \frac{\bar{W}(n)}{\tilde{W}(n)} - 1 \right| \leq t \) with probability of the order \( p_{n,2} \). Put \( P_n = p_{n,1} + p_{n,2} \). Then we obtain

\[
\tilde{F}_{U_n,S}(x - \frac{\Delta_n(1+t)}{\sigma_{W(n)}}) - P_n \leq F_{T_n,S}(x) \leq \tilde{F}_{U_n,S}(x + \frac{\Delta_n(1+t)}{\sigma_{W(n)}}) + P_n, \tag{3.17}
\]

where

\[
\tilde{F}_{U_n,S}(x) = P\left( \frac{L_n + U_n + b_n}{\sigma_{W(n)}} \leq x \left( 1 + \frac{V_n}{\sigma_{\tilde{W}(n)}^2} - R_n,S \right)^{1/2} \right), \tag{3.18}
\]

where \( R_{n,S} \) is the remainder term from Lemma 2.2. Note that \( 1 + \frac{V_n}{\sigma_{\tilde{W}(n)}} > 0 \) for all sufficiently large \( n \) with probability of the order \( P_n \), and \( |R_{n,S}| = O(\Delta_{n,S}) \) with probability \( 1 - O(\tilde{k}^{-c} n^{-c}) \), where \( \Delta_{n,S} \) is as in (1.33).

Since \( H_n'(x) \) is bounded from above uniformly in \( x \), it is enough to prove that \( H_n(x) \) is the expansion for the r.h.s. of (3.18) without \( R_{n,S} \), because omitting of it gives a remainder term of the order \( O(\Delta_{n,S}) \), which presents at the r.h.s. of (3.18). Write

\[
P\left( \frac{L_n + U_n + b_n}{\sigma_{W(n)}} \leq x \left( 1 + \frac{V_n}{\sigma_{\tilde{W}(n)}^2} \right)^{1/2} \right) = P\left( \frac{L_n + U_n + b_n}{\sigma_{W(n)}} - x \left( 1 + \frac{V_n}{\sigma_{\tilde{W}(n)}^2} \right)^{1/2} - 1 \right) \leq x. \tag{3.19}
\]

Since \( \frac{|V_n|}{\sigma_{\tilde{W}(n)}} \leq t \) with probability \( 1 - P_{n,1} \) for every \( t > 0 \), we can apply as in Putter and van Zanten (cf. also (15)–(16)) the following inequality: \( 1 + \frac{z}{t^2} - t^2 \leq (1 + z)^{1/2} \leq 1 + \frac{z}{t} \), \( |z| \leq 4/t \). Note that \( \frac{V_n^2}{4\sigma_{W(n)}^4} \leq \frac{1}{2} \left( \frac{V_{n,1}^2}{\sigma_{W(n)}^4} + \frac{V_{n,2}^2}{\sigma_{W(n)}^4} \right) \), where \( V_{n,i}, i = 1, 2 \), are as in (2.18), and note that \( \frac{V_{n,1}^2}{\sigma_{W(n)}^4} = O(\ln k_n q_{\alpha,n}^2 + \ln m_n q_{\beta,n}^2) \) (cf. (2.19)), the latter quantity contributes to to \( \delta_{n,S} \) on the r.h.s. of (1.34) (because it is a term of \( \Delta_{n,S} \)). It follows that we have to show that

\[
\sup_{x \in \mathbb{R}} \left| P\left( \frac{L_n + U_n + b_n}{\sigma_{W(n)}} - \frac{x V_n}{2\sigma_{\tilde{W}(n)}^2} \leq x \right) - H_n(x) \right| \leq C(\delta_n + \delta_{n,S}), \tag{3.20}
\]

\[
\sup_{x \in \mathbb{R}} \left| P\left( \frac{L_n + U_n + b_n}{\sigma_{W(n)}} - \frac{x V_n}{2\sigma_{\tilde{W}(n)}^2} + \frac{x V_{n,2}^2}{2\sigma_{\tilde{W}(n)}^4} \leq x \right) - H_n(x) \right| \leq C(\delta_n + \delta_{n,S}), \tag{3.21}
\]

where \( C > 0 \) is some constant independent of \( n \) and \( \delta_n \) is as in (1.34). Define \( \tilde{H}_n(x) = H_n(x) + \frac{b_n\sigma_{\tilde{W}(n)}}{\sigma_{W(n)}} \) (i.e. \( \tilde{H}_n(x) \) is \( H_n(x) \) without bias term). Note that \( \frac{b_n}{\sigma_{W(n)}} \leq \frac{1}{2}(q_{\alpha,n} + q_{\beta,n}) \). Since \( x \tilde{H}_n'(x) \) and \( x^2 \tilde{H}_n'(x) \) are bounded, we obtain: \( H_n(x + \frac{b_n}{\sigma_{\tilde{W}(n)}}) = H_n(x) + \)
\[ \phi(x) \frac{b_n}{\sigma W(n)} + O\left(\frac{1}{\sqrt{n}} \left| \frac{b_n}{\sigma W(n)} \right| \right) = H_n(x) \frac{b_n}{\sigma W(n)} + O\left(\frac{1}{\sqrt{n}} \delta_n \sigma W(n) \right) = H_n(x) + \phi(x) \frac{b_n}{\sigma W(n)} + O(\delta_n + \sigma W(n)). \]

It follows that we should prove that

\[ \sup_{x \in \mathbb{R}} \left| P\left( \frac{L_n + U_n}{\sigma W(n)} - \frac{x V_n}{2 \sigma W(n)} \right) \leq x \right| - \tilde{H}(x) \leq C(\delta_n + \sigma W(n)), \] (3.22)

\[ \sup_{x \in \mathbb{R}} \left| P\left( \frac{L_n + U_n}{\sigma W(n)} - \frac{x V_n}{2 \sigma W(n)} + \frac{1}{2} x V_n^2 \right) \right| - \tilde{H}(x) \leq C(\delta_n + \sigma W(n)). \] (3.23)

First we prove (3.22). Since \( V_n \) is a sum of centered i.i.d. r.v.’s, we obtain that \( U_x = \frac{L_n + U_n}{\sigma W(n)} - \frac{x V_n}{2 \sigma W(n)} \) is a centered U-statistic of degree two, and as in proof of Theorem 1.4 we find that in view of our smoothness assumption \( A_1 \) the Cramer condition is satisfied. Put \( \nu_n = \ln(k_n \land m_n) \). First we prove that (3.22) holds true uniformly in \( x \): \( |x| < \nu_n \). By Theorem 1.1 of Bentkus et al. [4] (taking into account the Remark 1.3 given on page 856 in cited paper) after simple computation of the fourth moment of \( U_x \) we obtain

\[ \sup_{|x| < \nu_n} \left| \frac{E(W_1(n))}{2 \sigma W(n)} + \frac{\| \beta \|^4}{n^2} \left( \frac{\Phi(x)}{\sigma W(n)} \right)^4 \right| \leq C \left( \frac{k_3 x}{\sigma W(n)} \right)^2 + \left( \frac{e}{\sigma W(n)} \right)^2, \] (3.24)

where \( \sigma W(n) = EU_x^2, k_3 x = EU_x^3 \). Relation [1.9] implies that

\[ \frac{\| \beta \|^4}{n^2} \left( \frac{\Phi(x)}{\sigma W(n)} \right)^4 = O \left( \frac{\nu_n^4}{n^4} \right) + \frac{1}{n} \left( \frac{1}{\nu_n^4} \right) + \frac{\| \beta \|^4}{n^2} \left( \frac{1}{\nu_n^4} \right)^2 \]

\[ = o(\delta_1 S(n) + \delta_2 S(n)), \]

where \( \delta_i S(n) \) is as in (1.34). Moreover, as in proof of Theorem 1.4 we obtain that \( \gamma_2^{\alpha + \beta} = O(\delta_2 n) \). Thus, at the r.h.s. of (3.21) we have desired bound \( C \left( \frac{k_3 x}{\nu_n} + \delta_2 S(n) \right) \leq C(\delta_n + \sigma W(n)) \) (here \( \delta_i n, i = 1, 2, \) are two terms of \( \delta_n, \) cf. (1.23)).

Next consider \( \tilde{H}_n(x) \). We have \( \sigma_x^2 = EU_x^2 = E \left( \frac{L_n + U_n}{\sigma W(n)} \right)^2 - \frac{x}{\sigma W(n)} E \left( \frac{L_n + U_n}{\sigma W(n)} \right) \). Since \( U_n \) and \( V_n \) are uncorrelated, after simple computations using formulas (2.1)-(2.5) and (2.18), we obtain \( E(L_n + U_n) V_n = E(L_n V_n) = \frac{1}{n} \left( \frac{1}{n} \right) \left( 3 W(n) + 2 \delta_2 W(n) \right), \) and hence (cf. (2.6), (2.19)),

\[ \sigma_x^2 = 1 - \frac{x(\lambda_1(n) + 2 \lambda_2(n))}{\sqrt{n}} + O\left( \nu_n^2 \left[ \frac{E(W_1(n))^4}{\sigma W(n)^4} + \delta_1 n + \delta_2 n \right] \right). \] (3.26)

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Moreover, relations (2.14)-(2.5), (2.18) and (1.9) after simple computations yield

\[ k_{3,x} = EU_x^3 = \frac{\lambda_1(n) + 3\lambda_2(n)}{\sqrt{n}} + O(\delta_{n,S}). \]  

(3.27)

Note that estimating of the remainder term at the r.h.s. of (3.27) is essentially based on relation (1.9), which we use to bound the moments \( EW_1^2(n)/\sigma W_1(n) \) (cf. proof of Theorem 1.2), where the largest power appearing here is \( r = 6 \). The relations (3.25) – (3.27) together imply that

\[ G_n(x) = \Phi\left(\frac{x}{\sigma_x}\right) - \frac{\lambda_1(n) + 3\lambda_2(n)}{6\sqrt{n}}(x^2 - 1)\phi(x) + O(\delta_{n,S}), \]

(3.28)

for \( |x| \leq \nu_n \), that is \( \sigma_x \) influences the first term of the expansion only through the term \( \Phi\left(\frac{x}{\sigma_x}\right) \) (cf. proof of Theorem 1.2 in Putter & van Zwet [29]). Then using (3.26) and (1.9), we obtain:

\[ \Phi\left(\frac{x}{\sigma_x}\right) = \Phi(x) + \phi(x) \frac{x^2(\lambda_1(n) + 2\lambda_2(n))}{2\sqrt{n}} + O(\delta_{n,S}). \]

Thus, \( G_n(x) = \tilde{H}_n(x) + O(\delta_{n,S}) \) for \( |x| \leq \nu_n \). Then we argue as Putter & van Zwet [29] (cf. also [15]-[16]): for \( x < -\nu_n \) \( \tilde{H}_n(x) = O(k_n^{-c} + m_n^{-c}) \), and for \( x > \nu_n \) \( 1 - \tilde{H}_n(x) = O(k_n^{-c} + m_n^{-c}) \), where \( c > 0 \) is an arbitrary constant. So, monotonicity of a distribution function implies (3.22).

It remains to prove (3.23). As before we see that it is enough to prove it taking supremum in \( x : |x| < \nu_n \). We must prove that the presence of \( xV_{n,2}^2/\sigma W_{n}(n) \) does not influence on the expansion and the order of the bound at the r.h.s. of (3.23). Note that \( xV_{n,2}^2 = \frac{\nu_n}{n}\left(\sum_{i=1}^{n}[(W_i(n) - \mu W_{n}(n))^2 - \sigma^2 W_{n}(n)]\right)^2 \) is a U-statistic of degree two, and its Hoeffding’s decomposition is

\[ \frac{x}{n} E[(W_1(n) - \mu W_{n}(n))^2 - \sigma^2 W_{n}(n)]^2 + \frac{x}{n^2} \sum_{i=1}^{n} \left( [(W_i(n) - \mu W_{n}(n))^2 - \sigma^2 W_{n}(n)]^2 - E[(W_i(n) - \mu W_{n}(n))^2 - \sigma^2 W_{n}(n)] \right)^2 \]

\[ + \frac{2x}{n^2} \sum_{1 \leq i < j \leq n} [(W_i(n) - \mu W_{n}(n))^2 - \sigma^2 W_{n}(n)] [(W_j(n) - \mu W_{n}(n))^2 - \sigma^2 W_{n}(n)]. \]

Since \( \frac{x}{n\sigma^2 W_{n}(n)} E[(W_1(n) - \mu W_{n}(n))^2 - \sigma^2 W_{n}(n)]^2 = O\left(\frac{\nu_n}{n} \frac{EW_1^4(n)}{\sigma W_{n}(n)}\right) \) and as \( \tilde{H}'(x) \) is bounded uniformly in \( x \in \mathbb{R} \), the constant term of Hoeffding’s decomposition contributes to a remainder term and can be omitted. Since \( \tilde{U}_x = \frac{L_n + U_n}{\sigma W_{n}(n)} - \frac{x V_{n,2}}{2\sigma W_{n}(n)} \)

\[ - \frac{x}{2n\sigma^2 W_{n}(n)} E[(W_1(n) - \mu W_{n}(n))^2 - \sigma^2 W_{n}(n)]^2 \]

is a centered U-statistic of degree two, and as \( \frac{x V_{n,2}^2}{2\sigma^2 W_{n}(n)} \) is even less than \( \frac{x V_{n,2}}{2\sigma^2 W_{n}(n)} \) for all sufficiently large \( n \), applying Theorem 1.1 of Bentkus et al. [4] we come to the same estimate as in (3.24) with \( \tilde{G}_n(x) = \Phi\left(\frac{x}{\sigma_x}\right) - \frac{\lambda_1(n) + 3\lambda_2(n)}{6\sqrt{n}}(x^2 - 1)\phi(x) + O(\delta_{n,S}) \), where for \( \tilde{\sigma}^2_x = EU_x^2 \) after some but rather tedious computations we obtain:

\[ \tilde{\sigma}^2_x = 1 - \frac{x(\lambda_1(n) + 2\lambda_2(n))}{\sqrt{n}} + O\left(\nu_n^2 \frac{EW_1^4(n)}{n\sigma W_{n}(n)} + q_{\alpha n}^2 + q_{\beta n}^2\right) + o(\delta_{n,S}). \]

Thus, as well as before \( \Phi\left(\frac{x}{\sigma_x}\right) = \Phi(x) + \phi(x) \frac{x^2(\lambda_1(n) + 2\lambda_2(n))}{2\sqrt{n}} + O(\delta_{n,S}), \)
and \( \tilde{G}_n(x) = \tilde{H}_n(x) + O(\delta_{n,S}) \). This implies that (3.23) is valid when supremum is taken all over \( x : |x| < \nu_n \), then as before by a monotonicity argument we obtain the validity of it on the whole real line. The theorem is proved.

**Proof of Corollary 1.5.** To prove this corollary we apply Theorem 1.6 with \( c = 3/4 \), and check that \( \delta_n + \delta_{n,S} = O\left( \frac{(\ln k_n)^{3/4}}{k_n^{3/4}} + \frac{(\ln m_n)^{3/4}}{m_n^{3/4}} \right) \), as \( n \to \infty \). We obtain the desired bound for \( \delta_n \) similarly as in the proof of Corollary 1.3. The check for \( \delta_{n,S} \) is similar. The corollary is proved.

**Proof of Theorem 1.7.** The proof is similar to the proof of Theorem 1.5.

\[ \square \]

4 Some Bahadur – Kiefer type representations

In this section we state and prove two lemmas used in our proofs. In essence this lemmas extend corresponding auxiliary results obtained in [15], [16] for a special case of the central sample quantiles \( \xi_{\alpha,n} \) (\( 0 < \alpha < 1 \) is fixed) to the case that \( \alpha_n \) is a sequence, which in particular can tend to 0 or to 1 (i.e. to the case of intermediate sample quantiles).

Let \( k_n \) be a sequence of positive integers such that \( k_n \to \infty \), recall that \( \alpha_n = k_n/n \), \( 0 \leq \lim \inf \alpha_n \leq \lim \sup \alpha_n < 1 \), \( \xi_{\alpha,n} = F^{-1}(\alpha_n) \) denote the corresponding sample quantiles, and let \( U_a \) be the set defined in (1.7). Let \( G(x), x \in \mathbb{R} \), be a real-valued function, \( g = G' \) – its derivative when it exists, and let \( (g/f)(x) \) denote the ratio \( g(x)/f(x) \), \( (|g|/f)(x) \) — the ratio \( |g(x)|/f(x) \).

**Lemma 4.1** Suppose that \( F^{-1} \) and \( G \) are differentiable on the sets \( U_a \) and \( F^{-1}(U_a) \) respectively. Then

\[
G(\xi_{\alpha,n}) - G(\xi_{\alpha}) = -\left[ F_n(\xi_{\alpha,n}) - F(\xi_{\alpha,n}) \right] \frac{g}{f}(\xi_{\alpha,n}) + R_n, 
\]

where \( P(|R_n| > \Delta_n) = O\left( k_n^{-c} \right) \) for each \( c > 0 \), and

\[
\Delta_n = A \alpha_n \left[ \left| \frac{g}{f}(\xi_{\alpha}) \right| \left( \frac{\ln k_n}{k_n} \right)^{3/4} + \Psi_{\alpha,n}\alpha \left( \frac{\ln k_n}{k_n} \right)^{1/2} \right],
\]

where \( A \) and \( B \) are some positive constants, which depend only on \( c \).

Lemma 4.1 is a Bahadur-Kiefer type result. For a special case when \( 0 < \alpha < 1 \) is fixed it is stated in lemmas 3.1 [15] (cf. also lemmas 4.1, [16] and Reiss [30]). We prove this lemma below in this section.

Note that we prove our results assuming that \( \alpha_n < 1 \) for all sufficiently large \( n \), where \( \alpha_n \) can tend to 0. Certainly, the same results can be obtained in case when \( \alpha_n > 0 \) for all sufficiently large \( n \), where \( \alpha_n \), in particular, can tend to 1, i.e. on the right tail of the sample. Some new results on the Bahadur – Kiefer representations for intermediate sample quantiles can be found in our recent paper [18].

Lemma 4.2 extends lemma 4.3 from [16] (cf. also lemma 3.2, [15]), where it was proved for a fixed \( \alpha \) to the case that \( \alpha_n \) is a sequence.
Lemma 4.2 Suppose that the conditions of Lemma 4.1 hold true. Then
\[
\int_{\xi_{\alpha,n,n}} (G(x) - G(\xi_{\alpha,n})) dF_n(x) = -\frac{1}{2} [F_n(\xi_{\alpha,n}) - F(\xi_{\alpha,n})]^2 \frac{g}{f}(\xi_{\alpha,n}) + R_n, \tag{4.2}
\]
where \( P(|R_n| > \Delta_n) = O(k_n^{-c}) \) for each \( c > 0 \), and
\[
\Delta_n = A \alpha_n \frac{\ln k_n}{n} \left[ \frac{|g|}{f}(\xi_{\alpha,n}) \left( \frac{\ln k_n}{k_n} \right)^{1/4} + \Psi_{\alpha_n,\frac{q}{2}}(B) \right],
\]
where \( A, B \) are some positive constants, which depend only on \( c \).

Remark 4.1 Suppose that \( k_n^{-1} \ln n \to 0 \), as \( n \to \infty \), and replace \( \ln k_n \) by \( \ln n \) in definition of function \( \Psi_{\alpha,n,B}(B) \) (cf. (1.10)). Then lemmas 4.1, 4.2 remain valid if we replace \( \ln k_n \) by \( \ln n \) in formula for \( \Delta_n \) in (4.1)–(4.2). Furthermore, \( P(|R_n| > \Delta_n) = O(n^{-c}) \) for each \( c > 0 \) in (4.1)–(4.2). To see the validity of this remark, it is enough to replace \( \ln k_n \) by \( \ln n \) in the proof of lemmas 4.1 and 4.2 and use the assumption \( k_n^{-1} \ln n \to 0 \), no more changes in the proofs are needed. This remark is useful for obtaining of some results similar to Theorems 1.1, 1.4 and 1.6 in the case of light tails (\( F \) has a finite variance), it allows us to get the bounds of the order \( O(n^{-r}) \), \( 0 < r \leq 1/2 \), which are as one would expect in this case.

Let \( U_1, \ldots, U_n \) denote a sample of independent uniform \((0,1)\) distributed r.v.'s, and \( U_{1:n} \leq \cdots \leq U_{n:n} \) – the corresponding order statistics. Put
\[
N_{\alpha}^x = \#\{i : X_i \leq \xi_{\alpha,n}\}, \quad N_{\alpha} = \#\{i : U_i \leq \alpha\}, \tag{4.3}
\]
and note that \( \xi_{\alpha,n,n} = X_{k_n:n} \) (because \( \alpha_n = k_n/n \)).

Proof of lemma 4.1 We must prove that \( P(|R_n| > \Delta_n) = O(k_n^{-c}) \) for each \( c > 0 \) (cf. (4.1)), and since the joint distribution of \( X_{k_n:n}, N_{\alpha,n} \) coincide with joint distribution of \( F^{-1}(U_{k_n:n}), N_{\alpha,n} \) it is suffices to verify it for a remainder given by
\[
R_n = G(F^{-1}(U_{k_n:n})) - G(F^{-1}(\alpha_n)) + \frac{N_{\alpha,n} - \alpha_n n g}{n} \frac{g}{f}(\xi_{\alpha,n}).
\]
Since \( P(U_{k_n:n} \notin U_0) = O(exp(-\delta n)) \) for some \( \delta > 0 \) not depending on \( n \), we can rewrite \( R_n \) for all sufficiently large \( n \) as
\[
\frac{g}{f}(\xi_{\alpha,n}) R_{n,1} + R_{n,2}, \tag{4.4}
\]
where \( R_{n,1} = U_{k_n:n} - \alpha_n + \frac{N_{\alpha,n} - \alpha_n n}{n} \), and \( R_{n,2} = \left( \frac{g}{f}(F^{-1}(\alpha_n + \theta(U_{k_n:n} - \alpha_n))) - \frac{g}{f}(F^{-1}(\alpha_n)) \right) (U_{k_n:n} - \alpha_n), \) \( 0 < \theta < 1 \). Fix an arbitrary \( c > 0 \) and note that we can estimate \( R_{n,j}, j = 1, 2, \) on the set \( E = \{\omega : |N_{\alpha,n} - \alpha_n n| < A_0 (\alpha_n n \ln k_n)^{1/2}\} \), where \( A_0 \) is a positive constant, depending only on \( c \), because by Bernstein inequality \( P(\Omega \setminus E) = O(k_n^{-c}) \) (in fact we can take every \( A_0; A_0^2 > 2c \)). We will prove that
\[
P(|R_{n,1}| > A_1 (\alpha_n)^{1/4} (\ln k_n/n)^{3/4}) = O(k_n^{-c}) \tag{4.5}
\]
and that
\[ P(\{R_{n,2} > A_2 \alpha_n \Psi_{\alpha_n} g(B)(\ln k_n/k_n)^{1/2} \}) = O(k_n^{-c}). \]  
Here and elsewhere \( A_i, i = 1, 2, \ldots, \) and \( B \) denote some positive constants, depending only on \( c \). Relations (4.4)–(4.6) imply (4.1).

First we prove (4.3) using a similar conditioning on \( N_{\alpha_n} \) argument as in proof of lemmas 4.1, 4.3 in [16]. First let \( k_n \leq N_{\alpha_n} \), then conditionally on \( N_{\alpha_n} \), the order statistic \( U_{k_n:n} \) is distributed as \( k_n \)-th order statistic \( U'_{k_n:N_{\alpha_n}} \) of the sample \( U'_1, \ldots, U'_{N_{\alpha_n}} \) independent \((0, \alpha_n)\) uniformly distributed r.v.’s. Its expectation \( E(U_{k_n:n} \mid N_{\alpha_n}, k_n \leq N_{\alpha_n}) = \alpha_n \frac{k_n}{N_{\alpha_n}+1} \), and the conditional variance \( V^2_{k_n} = \sigma^2_n/\left(\alpha_n + 1\right) \) and on the set \( E \) we have an estimate \( V^2_{k_n} \leq A_0(\alpha_n)^{1/2} n^{-3/2} \ln^{1/2} k_n \). Then rewrite \( R_{n,1} \) (at the event \( k_n \leq N_{\alpha_n} \)) as
\[ U_{k_n:n} - \alpha_n \frac{k_n}{N_{\alpha_n} + 1} + R'_{n,1}, \]  
where \( R'_{n,1} = \alpha_n \frac{k_n}{N_{\alpha_n} + 1} - \alpha_n + \frac{N_{\alpha_n} - k_n}{n} \) and on the set \( E \) the latter quantity is of the order \( O(k_n^{-1/n}) \), and since \( \ln k_n = o((\alpha_n)^{1/4} (\ln k_n)^{3/4}) \), the remainder term \( R'_{n,1} \) is of negligible order for our purposes. For the first two terms in (4.7) we have
\[ P \left( \left| U_{k_n:n} - \alpha_n \frac{k_n}{N_{\alpha_n} + 1} \right| > A_1(\alpha_n)^{1/4} \left( \frac{\ln k_n}{n} \right)^{3/4} \left| N_{\alpha_n} : k_n \leq N_{\alpha_n} \right) \right) = P_1 + P_2, \]  
where \( N_{\alpha_n} \) is fixed, \( k_n \leq N_{\alpha_n} \), \( A_1 \) is a constant which we will choose later, 
\[ P_1 = P \left( U'_{k_n:N_{\alpha_n}} > \alpha_n \frac{k_n}{N_{\alpha_n} + 1} + A_1(\alpha_n)^{1/4} \left( \ln k_n/n \right)^{3/4} \left| N_{\alpha_n} \right) \right) = \text{constant} \]  
and 
\[ P_2 = P \left( U'_{k_n:N_{\alpha_n}} < \alpha_n \frac{k_n}{N_{\alpha_n} + 1} - A_1(\alpha_n)^{1/4} \left( \ln k_n/n \right)^{3/4} \right). \]  
We evaluate \( P_1 \), the treatment for \( P_2 \) is similar. Consider a binomial r.v. \( S'_n = \sum_{i=1}^{N_{\alpha_n}} 1 \{ U'_i \leq \alpha_n \frac{k_n}{N_{\alpha_n} + 1} + A_1(\alpha_n)^{1/4} \left( \ln k_n/n \right)^{3/4} \} \) with parameter \((p'_n, N_{\alpha_n})\), where \( p'_n = \min(1, \frac{k_n}{N_{\alpha_n} + 1} + t_n) \), where \( t_n = A_1 \left( \frac{\ln k_n}{k_n} \right)^{3/4} \). If \( p'_n = 1 \), then \( P_1 = 0 \) and the inequality we need is valid trivial. Let \( p'_n < 1 \) and let \( S'_n \) denote the average \( S'_n/N_{\alpha_n} \), then the probability \( P_1 \) is equal to
\[ P(S'_n < k_n) = P \left( \left( \frac{S'_n}{n} - p'_n \right) < \frac{k_n}{N_{\alpha_n}} - \frac{k_n}{N_{\alpha_n} + 1} - t_n \right). \]  
Note that \( \frac{k_n}{N_{\alpha_n}} - \frac{k_n}{N_{\alpha_n} + 1} = \frac{k_n}{N_{\alpha_n}(N_{\alpha_n} + 1)} < \frac{1}{N_{\alpha_n}} \), and since the latter quantity is \( o\left( t_n k_n^{-1/4} \right) = o(t_n) \) on the set \( E \), this term can be omitted at the r.h.s. of (4.9) in our estimating. To evaluate \( P(S'_n - p'_n < -t_n) \) we note that \( p'_n - t_n = \frac{k_n}{N_{\alpha_n} + 1} \in (0, 1) \), and that \( p'_n > 1/2 \) for all sufficiently large \( n \) (and hence \( k_n \) and \( N_{\alpha_n} \)) on the set \( E \). So, we may apply an inequality (2.2) of Hoeffding [23] with \( \mu = p'_n \) and with \( g(\mu) = 1/(2\mu(1-\mu)) \). Then we obtain
\[ P(S'_n < k_n) \leq \exp \left( -N_{\alpha_n} t_n^2 g(p'_n)) = \exp \left( -\frac{N_{\alpha_n} A^2 \left( \log k_n/k_n \right)^{3/2}}{2p'_n(1-p'_n)} \right). \]  
(4.10)

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Finally we note that $1 - p_n' = 1 - \frac{k_n}{N_{\alpha_n} + 1} - A_1 \left( \frac{\log k_n}{N_{\alpha_n} + 1} \right)^{3/4} \leq \frac{N_{\alpha_n} + 1 - k_n}{N_{\alpha_n} + 1}$, and on the set $E$ the latter quantity is not greater than $\frac{A_0(\log k_n)^{3/2}}{N_{\alpha_n}}$. Then we can get a low bound for the ratio at the r.h.s. in (4.10): $\frac{N_{\alpha_n} A_n^2 (\log k_n/k_n)^{3/2}}{2p_n'(1-p_n') \geq A_n^2 N_{\alpha_n}^2 (\log k_n/k_n)^{3/2}} = \frac{A_n^2}{2A_n} \log k_n \left( \frac{N_{\alpha_n}}{k_n} \right)^2 = \frac{A_n^2}{2A_n} \log k_n (1 + o(1)).$ This bound and (4.10) together yield that when $\frac{A_n^2}{2A_n} \geq c$ the desired relation $P_1 = O(k_n^{-c})$ hold true. The same estimate is valid for $P_2$.

In case $N_{\alpha_n} < k_n$ we use the fact that $U_{k_n:n}$ conditioned on $N_{\alpha_n}$ is distributed as $(k_n - N_{\alpha_n})$-th order statistic $U''_{k_n-n_{\alpha_n}-N_{\alpha_n}}$ of the sample $U_1', \ldots, U''_{N_{\alpha_n}}$ from $(1 - \alpha_n, 1)$ uniform distribution, its expectation is $\alpha_n + \frac{k_n}{N_{\alpha_n} - 1}$, and for the conditional variance we have the estimate $V_{k_n-n_{\alpha_n}}^2 \leq A_0(\log k_n\alpha_n)^{1/2} n^{-3/2}$. In this case we use a representation for $R_{n,1} = R_{n,1}'' + R_{n,2}''$, where $R_{n,1}'' = U_{k_n:n} - \alpha_n - \frac{k_n-n_{\alpha_n}}{N_{\alpha_n}+1} (1 - \alpha_n)$, and $R_{n,2}'' = \frac{N_{\alpha_n} - N_{\alpha_n} n}{n} + \frac{k_n-n_{\alpha_n}}{N_{\alpha_n}+1} (1 - \alpha_n)$. Similarly as in first case we obtain that $R_{n,2}'' = O\left( \frac{\log k_n}{n} \right)$ with probability $1 - O(k_n^{-c})$, and this term is of the negligible order in our estimating. Using Hoeffding’s inequality we obtain for $R_{n,1}$ same estimate as for $R_{n,1}''$. So (4.5) is proved.

It remains to prove (4.6). First note that by (4.5) on the set $E$ with probability $1 - O(k_n^{-c})$ we have $|U_{k_n:n} - \alpha_n| \leq A_0 \left( \frac{\log k_n}{k_n} \right)^{1/2} + A_1 \alpha_n \left( \frac{\log k_n}{k_n} \right)^{3/4} = (\alpha_n + \frac{k_n}{n})^{1/2} (1 + o(1))$. Thus, there exists $A_2$, depending only on $c$, such that $|R_{n,2}''| \leq A_2 \left( \frac{\log k_n}{n} \right)^{1/2} \Psi_{\alpha_n, \frac{\log k_n}{k_n}}(A_2)$ with probability $1 - O(k_n^{-c})$. This implies (4.6). The lemma is proved. □

**Proof of lemma 4.2** Let $N_{\alpha_n}^2$ and $N_{\alpha_n}$ are given as in (4.3), then we can rewrite integral on the l.h.s. of (4.2) as $\frac{\text{sgn}(N_{\alpha_n}^2)}{\text{sgn}(N_{\alpha_n})} \sum_{i = (k_n,N_{\alpha_n})+1} G(X_{i:n}) - G(\xi_{\alpha_n}))$, where $\text{sgn}(x) = x/|x|$, $\text{sgn}(0) = 0$. Let us adopt the following notation: for any integer $k$ and $m$ define a set $I_{(k,m)} := \{ i : (k \land m) + 1 \leq i \leq k \lor m \}$ and let $\sum_{i \in I_{(k,m)}}(.) = \text{sgn}(m - k) \sum_{i = (k,m) + 1} (.)$. Then we must estimate $R_n = \frac{1}{n} \sum_{i \in I_{(k_n,n)}} G(X_{i:n}) - G(\xi_{\alpha_n})) + \frac{N_{\alpha_n} - \alpha_n}{2n^2} \frac{g(\xi_{\alpha_n})}{f(\alpha_n)}$, and similarly as in proof of lemma 4.1 we note that $R_n$ is distributed as

$$\frac{1}{n} \sum_{i \in I_{(k_n,n)}} \left( G \circ F^{-1}(U_{i:n}) - G \circ F^{-1}(\alpha_n) \right) + \frac{(N_{\alpha_n} - \alpha_n)^2}{2n^2} \frac{g(\xi_{\alpha_n})}{f(\alpha_n)}$$

$$= \frac{g}{f}(\xi_{\alpha_n}) R_{n,1} + R_{n,2},$$

(4.11)

where $R_{n,1} = \frac{1}{n} \sum_{i \in I_{(k_n,n)}} (U_{i:n} - \alpha_n) + \frac{(N_{\alpha_n} - \alpha_n)^2}{2n^2}$, $R_{n,2} = \frac{1}{n} \sum_{i \in I_{(k_n,n)}} \left[ \frac{g}{f} \circ F^{-1}(\alpha_n + \theta_i (U_{i:n} - \alpha_n)) - \frac{g}{f} \circ F^{-1}(\alpha_n) ^{1/2} \right] (U_{i:n} - \alpha_n)$, where $0 < \theta_i < 1$, $i \in I_{(k_n,n)}$.

Fix an arbitrary $c > 0$ and prove that

$$P\left( |R_{n,1}| > A_1 \left( \alpha_n \right)^{3/4} (\log k_n/n)^{5/4} \right) = O(k_n^{-c}),$$

(4.12)
\[
P\left( |R_{n,2}| > A_2 \alpha_n \frac{\ln k_n}{n} \Psi_{\alpha_n, \frac{1}{2}}(A_2) \right) = O(k_n^{-c}). \quad (4.13)
\]

Since \((\alpha_n)^{3/4}(\ln k_n / n)^{5/4} = \alpha_n \ln k_n / (\ln k_n / n)^{1/4}\), relations (4.12)–(4.13) imply (4.2).

Note that as in proof of lemma 4.1, it is enough to estimate \(R_{n,j}\), \(j = 1, 2\), on the set \(E = \{ \omega : |N_{\alpha_n} - \alpha_n n| < A_0 (\alpha_n n \ln k_n)^{1/2} \}\), where \(A_0 > 0\) is a constant, depending only on \(c\), such that \(P(\Omega \setminus E) = O(k_n^{-c})\).

First we treat \(R_{n,2}\). Note that

\[
\max_{i \in I(k_n, N_{\alpha_n})} |U_{i:n} - \alpha_n| = |U_{k_n:n} - \alpha_n| \lor |U_{N_{\alpha_n}+1:n} - \alpha_n| \lor |U_{N_{\alpha_n}+1:n} - \alpha_n|,
\]

\[
P\left( |U_{k_n:n} - \alpha_n| > A_0 (\alpha_n \ln k_n / n)^{1/2} \right) = O(k_n^{-c}) \quad (\text{cf. proof of lemma 4.1}), and for \(j = N_{\alpha_n}, N_{\alpha_n}+1\) simultaneously we have \(P\left( |U_{j:n} - \alpha_n| > A_1 \ln k_n / n \right) \leq P\left( U_{N_{\alpha_n}+1:n} > A_1 \ln k_n / n \right) = P\left( U_{1:n} > A_1 \ln k_n / n \right) = \left( 1 - A_1 \ln k_n / n \right)^n = O(k_n^{-c})\) for \(A_1 > c\). Since \(\ln k_n / n = o(\alpha_n \ln k_n / n)^{1/2}\), on the set \(E\) we obtain

\[
|R_{n,2}| \leq \frac{1}{n} \Psi_{\alpha_n, \frac{1}{2}}(A_0) \frac{A_2^2}{2} (\alpha_n n \ln k_n)^{1/2} \left( \frac{\ln k_n}{n} \right)^{1/2} = A_2 \alpha_n \ln k_n \Psi_{\alpha_n, \frac{1}{2}}(A_0)
\]

with probability \(1 - O(k_n^{-c})\), and (4.13) is proved.

Finally, consider \(R_{n,1}\). Note that conditionally on \(N_{\alpha_n}\), \(k_n \leq N_{\alpha_n}\), the order statistics \(U_{i:n}, k_n \leq i \leq N_{\alpha_n}\), are distributed as the order statistics \(U'_{i:n_{\alpha_n}}\) from the uniform \((0, \alpha_n)\) distribution (cf. lemma 5.1, Section 5), their conditional expectations are equal to \(\alpha_n / N_{\alpha_n}\).

Then in the case \(k_n \leq N_{\alpha_n}\) (the proof for the case \(N_{\alpha_n} < k_n\) is similar (cf. proof of lemma 4.1) with respect to interval \((1 - \alpha_n, 1)\), and we omit the details) we rewrite \(R_{n,1}\) as

\[
R_{n,1} = \frac{1}{n} \sum_{i = k_n+1}^{N_{\alpha_n}} \left( U_{i:n} - \alpha_n \frac{i}{N_{\alpha_n}+1} \right) + R_{n,1}'\quad (4.14)
\]

where

\[
R_{n,1}' = \frac{1}{n} \sum_{i = k_n+1}^{N_{\alpha_n}} \alpha_n \left( \frac{i}{N_{\alpha_n}+1} - 1 \right) + \frac{(N_{\alpha_n} - \alpha_n n)^2}{2n^2} = \frac{-k_n (N_{\alpha_n} - \alpha_n n)}{2(N_{\alpha_n}+1)n^2}, \quad \text{and on the set } E \text{ the latter quantity is of the order } O\left( k_n^2 / (\ln k_n)^{3/2} \right) = o\left( (\alpha_n)^{3/4} (\ln k_n)^{5/4} \right), \text{i.e. } R_{n,1}' \text{ is of negligible order (cf. (4.12)) for our purposes.}
\]

It remains to evaluate the dominant first term on the r.h.s. in (4.14). Fix an arbitrary \(c_1 > c + 1/2\), and note that conditional on \(N_{\alpha_n}\) the variance of \(U_{i:n}\) in \((k_n + 1, i \leq N_{\alpha_n})\) is equal to \(V_i^2 = (\alpha_n)^{2} \frac{1}{N_{\alpha_n}+2} \frac{i}{N_{\alpha_n}+1} (1 - \frac{i}{N_{\alpha_n}+1})\), and on the set \(E\) it is less than \((\alpha_n)^{2} \frac{A_0 k_n^{1/2} (\ln k_n)^{1/2}}{N_{\alpha_n}}\), and \(V_i \leq \alpha_n A_0^{1/2} k_n^{1/4} (\ln k_n)^{1/4} / N_{\alpha_n} \leq A_0^{1/2} \alpha_n k_n^{-3/4} (\ln k_n)^{1/4} \leq A_0^{1/2} (\alpha_n)^{1/4} n^{-3/4} (\ln k_n)^{1/4}\).

Using Hoeffding’s inequality (similarly as in proof of lemma 4.1), we find that

\[
P\left( \left| U_{i:n} - \alpha_n \frac{i}{N_{\alpha_n}+1} \right| > A_0 (\alpha_n)^{1/4} (\ln k_n)^{3/4} \right| N_{\alpha_n} : k_n \leq N_{\alpha_n} = O(k_n^{-c})
\]


where \( A_1 \) depends only on \( c_1 \) (in fact it is true for every \( A_1 \) such that \( A_1^2 > 2A_0 c \)). Thus,

\[
P\left( \frac{1}{n} \sum_{i=k_n}^{N_n} (U_{i:n} - \alpha_n \frac{i}{N_{\alpha_n} + 1}) \right) > A_0 A_1 (\alpha_n)^{3/4} \left( \ln k_n / n \right)^{5/4} \left| N_{\alpha_n} : k_n \leq N_{\alpha_n} \right|
\]

\[
\leq A_0 (k_n \ln k_n)^{1/2} O(k_n^{-c_1}) = O(k_n^{-c_1}). \tag{4.15}
\]

Combining (4.14)–(4.15) and similar estimates for the case \( N_{\alpha_n} < k_n \), we come to (4.12).

The lemma is proved. \( \square \)

5 Appendix

Let as before, \( N_\alpha = \sharp \{ i : X_i \leq \xi_\alpha, i = 1, \ldots, n \} \), where \( 0 < \alpha < 1 \) is fixed. In this appendix we prove that conditionally on \( N_\alpha \) the order statistics \( X_{1:n}, \ldots, X_{N_{\alpha:n}} \) are distributed as order statistics corresponding to a sample of \( N_\alpha \) i.i.d. r.v.’s with distribution function \( F(x) / \alpha, x \leq \xi_\alpha \). Though this fact is known (cf. [21], [32]), we give a brief proof of it. Let \( U_1, \ldots, U_n \) be independent r.v.’s uniformly distributed on \((0, 1)\) and let \( U_{1:n}, \ldots, U_{n:n} \) denote the corresponding order statistics. Put \( N_{\alpha,u} = \sharp \{ i : U_i \leq \alpha \} \).

Since \( X_{i:n} \overset{d}{=} F^{-1}(U_{i:n}) \) and \( N_{\alpha} \overset{d}{=} N_{\alpha,u} \), it is enough to prove the assertion for the uniform distribution.

**Lemma 5.1** Conditionally given \( N_{\alpha,u} \), the order statistics \( U_{1:n}, \ldots, U_{N_{\alpha,u}:n} \) are distributed as order statistics corresponding to a sample of \( N_{\alpha,u} \) independent \((0, \alpha)\)-uniform distributed r.v.’s.

**Proof.** a). First consider the case \( N_{\alpha,u} = n \). Take arbitrary \( 0 < u_1 \leq \cdots \leq u_n < \alpha \) and write

\[
P(U_{1:n} \leq u_1, \ldots, U_{N_{\alpha,u}:n} \leq u_n \mid N_{\alpha,u} = n) = \frac{P(U_{1:n} \leq u_1, \ldots, U_{n:n} \leq u_n)}{\alpha^n} = \frac{n!}{\alpha^n} \int_{u_1}^{u_2} \cdots \int_{u_{n-1}}^{u_n} d x_1 d x_2 \cdots d x_n,
\]

and the latter is d.f. of the order statistics corresponding to the sample of \( n \) independent \((0, \alpha)\)-uniform distributed r.v.’s.  

b). Consider the case \( N_{\alpha,u} = k < n \). Let \( F_{i,n}(u) = P(U_{i:n} \leq u) \) be a df of \( i \)-th order statistic, put \( P_n(k) = P(N_{\alpha,u} = k) = \binom{n}{k} \alpha^k (1-\alpha)^{n-k} \). Then we have

\[
P(U_{1:n} \leq u_1, \ldots, U_{N_{\alpha,u}:n} \leq u_k \mid N_{\alpha,u} = k) = \frac{P(U_{1:n} \leq u_1, \ldots, U_{k:n} \leq u_k, U_{k+1:n} > \alpha)}{P_n(k)}. \tag{5.1}
\]

The probability in the nominator on the r.h.s. of (5.1) is equal to

\[
\int_{u_1}^{u_k} \cdots \int_{u_{k-1}}^{u_k} d x_1 d x_2 \cdots d x_k \left( \int_{u_k}^{1} d x \right),
\]

and by the Markov property of order statistics the latter quantity equals

\[
\int_{u_1}^{u_k} \cdots \int_{u_{k-1}}^{u_k} d x_1 d x_2 \cdots d x_k \left( \int_{u_k}^{1} d x \right) = \alpha^k \int_{u_1}^{u_k} \cdots \int_{u_{k-1}}^{u_k} d x_1 d x_2 \cdots d x_k \times \alpha^k \int_{u_k}^{1} d F_{k+1,n}(v),
\]

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and since $\alpha^k \int_0^1 \frac{1}{x^k} dF_{k+1,n}(v) = \alpha^k \int_0^1 \frac{(1-v)^{n-k-1}}{B(k+1,n-k)} dv = \binom{n}{k} \alpha^k (1 - \alpha)^{n-k} = P_n(k)$, where $B(k+1,n-k) = k!(n-k-1)!/n!$, we obtain that conditional probability in [5.1] is equal
\[
\frac{k!}{\alpha^k} \int_0^{u_1} \int_{u_1}^{u_2} \cdots \int_{u_{k-1}}^{u_k} dx_1 dx_2 \cdots dx_k,
\]
which corresponds to the $(0, \alpha)$-uniform distribution. The lemma is proved. □

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