PARTIAL TRACE OF A FULL SYMMETRIZER

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Abstract. A formula for the partial trace of a full symmetrizer is obtained. The formula is used to provide an inductive proof of the well-known formula for the dimension of a full symmetry class of tensors.

0. Introduction

Let $A$ and $B$ be vector spaces over the field $\mathbb{C}$ of complex numbers. The “partial trace relative to $B$” of a linear operator $T$ on the tensor product $A \otimes B$ is a certain linear operator $\text{tr}_B T$ on the space $A$ (see Section 1).

In quantum physics, the partial trace is used in the definition of the density operator for a state on a subsystem. This operator captures all of the information about the state that can be gained from measurements on the subsystem alone [RP11, p. 207].

The partial trace has uses in pure mathematics as well. In this paper, we provide a formula for the partial trace relative to $V$ of a full symmetrizer of the tensor power $V^\otimes m = V^\otimes (m-1) \otimes V$ of a vector space $V$ (see Corollary 2.6).

As an application, we use the formula to give a proof by induction of the well-known formula for the dimension of a full symmetry class of tensors (see Theorem 3.1).

1. Partial trace

Let $V$ be a vector space over $\mathbb{C}$ with basis $\{v_1, \ldots, v_n\}$. For each $i$ the dual vector corresponding to $v_i$ is the linear map $v_i^* : V \to \mathbb{C}$ given by $v_i^*(v_j) = \delta_{ij}$ (Kronecker delta). Denote by $L(V)$ the space of linear operators on $V$. For $T \in L(V)$ denote by $\text{tr} T$ the trace of $T$, so $\text{tr} T = \sum_i v_i^* (T(v_i))$.

Let $A$ and $B$ be vector spaces over $\mathbb{C}$ with bases $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_n\}$, respectively. Let $T \in L(A \otimes B)$. Define $\text{tr}_B T \in L(A)$ by

$$\text{tr}_B T(a_k) = \sum_{i=1}^m \sum_{j=1}^n (a_i \otimes b_j)^* (T(a_k \otimes b_j)) a_i.$$
The linear operator $\text{tr}_B T$ on $A$ is the partial trace of $T$ relative to $B$ [RP11, pp. 210–211]. If $A = C$, the space $A \otimes B$ identifies with $B$ and $\text{tr}_B T = \text{tr} T$. At the other extreme, if $B = C$, the space $A \otimes B$ identifies with $A$ and $\text{tr}_B T = T$.

1.1 Lemma. For every $T \in L(A \otimes B)$, we have $\text{tr} T = \text{tr} (\text{tr}_B T)$.

Proof. Let $T \in L(A \otimes B)$ and let the notation be as above. We have

$$\text{tr}(\text{tr}_B T) = \sum_{k=1}^{m} a_k^* (\text{tr}_B T(a_k))$$

$$= \sum_{k=1}^{m} \sum_{j=1}^{n} (a_k \otimes b_j)^* (T(a_k \otimes b_j))$$

$$= \text{tr} T.$$  

□

1.2 Lemma. For every $S \in L(A)$ and $R \in L(B)$, we have $\text{tr}_B (S \otimes R) = (\text{tr} R) S$.

Proof. Let $S \in L(A)$ and $R \in L(B)$ and let the notation be as above. For each $1 \leq k \leq m$, we have

$$\text{tr}_B (S \otimes R)(a_k) = \sum_{i=1}^{m} \sum_{j=1}^{n} (a_i \otimes b_j)^* (S(a_k) \otimes R(b_j)) a_i$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} a_i^* (S(a_k)) b_j^* (R(b_j)) a_i$$

$$= \sum_{j=1}^{n} b_j^* (R(b_j)) \sum_{i=1}^{m} a_i^* (S(a_k)) a_i$$

$$= (\text{tr} R) S(a_k),$$

and the claim follows. □

2. Symmetrizer

For $l \in \mathbb{N} := \{0, 1, 2, \ldots \}$, put

$$\Gamma^+_l = \{ \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_l, 0, 0, \ldots) \mid \gamma_i \in \mathbb{N} \text{ for each } i \}$$

and put $\Gamma^+ = \cup_l \Gamma^+_l$.

Let $\gamma \in \Gamma^+$ and let $m$ be a positive integer. We write $\gamma \models m$ and say that $\gamma$ is an improper partition of $m$ if $\sum_i \gamma_i = m$. We write $\gamma \vdash m$ and say that $\gamma$ is a partition of $m$ if $\gamma \models m$ and $\gamma_i \geq \gamma_{i+1}$ for each $i$. 
Let $m$ be a nonnegative integer. Denote by $S_m$ the symmetric group of degree $m$. The cycle partition of a permutation $\sigma \in S_m$ is the partition of $m$ obtained by writing the lengths of the cycles in a disjoint cycle decomposition of $\sigma$ in nonincreasing order (followed by zeros). Two elements of $S_m$ are conjugate if and only if they have the same cycle partition (see [JK81, p. 9] or [Mer97, Theorem 3.18]).

The number of (ordinary) irreducible characters of $S_m$ is the same as the number of conjugacy classes of $S_m$, so these irreducible characters are indexed by the partitions of $m$. For $\alpha \vdash m$, denote by $\chi_\alpha$ the irreducible character of $S_m$ corresponding to $\alpha$ as in [JK81, p. 36] and [Mer97, p. 99].

Let $n$ be a positive integer. Put

$$\Gamma_{m,n} = \{ \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in \mathbb{Z}^m \mid 1 \leq \gamma_i \leq n \text{ for each } i \}$$

(which equals the set consisting of the empty tuple $(\ )$ if $m = 0$). There is a right action of the group $S_m$ on the set $\Gamma_{m,n}$ given by place permutation:

$$\gamma \sigma := (\gamma_{\sigma(1)}, \ldots, \gamma_{\sigma(m)}) \ (\gamma \in \Gamma_{m,n}, \ \sigma \in S_m).$$

Let $V$ be a vector space over $\mathbb{C}$ of dimension $n$ and let $\{e_1, e_2, \ldots, e_n\}$ be a basis of $V$. The $n$th tensor power $V^{\otimes m}$ of $V$ has basis $\{e_\gamma \mid \gamma \in \Gamma_{m,n}\}$, where $e_\gamma := e_{\gamma_1} \otimes \cdots \otimes e_{\gamma_m}$ (with $V^{\otimes 0} = \mathbb{C}$ and $e_{()} = 1$). For $\sigma \in S_m$ denote by $P_m(\sigma)$ the linear operator on $V^{\otimes m}$ given by $P_m(\sigma)(e_\gamma) = e_{\gamma \sigma^{-1}}$ ($\gamma \in \Gamma_{m,n}$).

Let $\alpha \vdash m$. The linear operator $T_\alpha$ on $V^{\otimes m}$ given by

$$T_\alpha = \frac{\chi_\alpha(1)}{m!} \sum_{\sigma \in S_m} \chi_\alpha(\sigma) P_m(\sigma)$$

is the symmetrizer of $V^{\otimes m}$ corresponding to $\alpha$ (or to $\chi_\alpha$). This operator is often referred to as a full symmetrizer to distinguish it from a symmetrizer corresponding to an irreducible character of a proper subgroup of $S_m$, which is defined similarly.

From now on, we assume that $m > 0$. Put $A = V^{\otimes (m-1)}$ and $B = V$. We have $V^{\otimes m} = A \otimes B$ so the partial trace $\text{tr}_B T_\alpha$ is defined. The main result of the paper (see Corollary 2.6) expresses this partial trace as a linear combination of symmetrizers $T_{\beta} : V^{\otimes (m-1)} \to V^{\otimes (m-1)}$ for various $\beta \vdash m-1$.

The proof of this result requires several preliminaries, to which we now turn. For $1 \leq i \leq m$, put

$$\kappa_i = \begin{cases} n, & i = m, \\ 1, & i \neq m. \end{cases}$$

2.1 Lemma. Let $\tau \in S_{m-1}$ and $1 \leq i \leq m$, and put $\sigma = \tau(i, m) \in S_m$. We have

$$\text{tr}_B P_m(\sigma) = \kappa_i P_{m-1}(\tau).$$
2.2 Lemma. The set $S_m$ is the disjoint union of the cosets $S_{m-1}(i,m)$, $1 \leq i \leq m$.

Proof. Let $\sigma \in S_m$ and put $i = \sigma^{-1}(m)$. We have $\tau := \sigma(i,m) \in S_{m-1}$ and $\sigma = \tau(i,m) \in S_{m-1}(i,m)$, so $S_m$ is the union of the indicated cosets.

Let $1 \leq i, j \leq m$ and assume that the cosets $S_{m-1}(i,m)$ and $S_{m-1}(j,m)$ intersect. Then the cosets are equal, implying $(i,m)(j,m) = (i,m)(j,m)^{-1} \in S_{m-1}$, that is, $(i,m)(j,m)$ fixes $m$. Therefore, $i = j$ and we conclude that the indicated cosets are disjoint. \[\square\]
For $\beta \in \Gamma^+$, denote by $l(\beta)$ (length of $\beta$) the least $l \in \mathbb{N}$ for which $\beta_l \in \Gamma^+$. If $\beta$ is nonzero, $l(\beta)$ is the least index $l$ for which $\beta_l \neq 0$; otherwise, $l(\beta) = 0$.

Any $\beta \models m$ corresponds, by an arrangement of its entries in nonincreasing order, to a uniquely determined partition of $m$ and hence to a uniquely determined conjugacy class of $S_m$, which we refer to as the conjugacy class corresponding to $\beta$. For $\alpha \vdash m$ and $\beta \models m$ we write $\chi_\alpha(\beta)$ for the value of the character $\chi_\alpha$ on the conjugacy class corresponding to $\beta$.

For each positive integer $i$ we write $\varepsilon_i$ for the element $(0, \ldots, 0, 1_i, 0, \ldots)$ of $\Gamma^+$.

**2.3 Theorem** [Hol18, 4.5]. For every $\alpha \vdash m$ and every $\beta \models m - 1$, we have

$$
\sum_{\substack{i=1 \\alpha_i > \alpha_{i+1}}}^{l(\alpha)} (\alpha_i - i) \chi_{\alpha - \varepsilon_i}(\beta) = \sum_{j=1}^{l(\beta)} \beta_j \chi_\alpha(\beta + \varepsilon_j).
$$

□

**2.4 Theorem** (Branching theorem) [JK81, 2.4.3]. For every $\alpha \vdash m$ and every $\tau \in S_{m-1}$ we have

$$
\chi_\alpha(\tau) = \sum_{\substack{i=1 \\alpha_i > \alpha_{i+1}}}^{l(\alpha)} \chi_{\alpha - \varepsilon_i}(\tau).
$$

□

For $\alpha \vdash m$, put

$$
T'_\alpha = \sum_{\sigma \in S_m} \chi_\alpha(\sigma) P_m(\sigma).
$$

**2.5 Theorem.** For every $\alpha \vdash m$, we have

$$
\text{tr}_B T'_\alpha = \sum_{\substack{i=1 \\alpha_i > \alpha_{i+1}}}^{l(\alpha)} (n + \alpha_i - i) T'_{\alpha - \varepsilon_i}.
$$
Proof. Let $\alpha \vdash m$. Using linearity of the partial trace, and then Lemmas 2.2 and 2.1, we get

$$\tr_B T'_\alpha = \sum_{\sigma \in S_m} \chi_\alpha(\sigma) \tr_B(P_m(\sigma))$$

$$= \sum_{\tau \in S_{m-1}} \sum_{i=1}^{m} \chi_\alpha(\tau(i, m)) \tr_B(P_m(\tau(i, m)))$$

$$= \sum_{\tau \in S_{m-1}} \sum_{i=1}^{m-1} \chi_\alpha(\tau(i, m)) P_{m-1}(\tau) + n \sum_{\tau \in S_{m-1}} \chi_\alpha(\tau) P_{m-1}(\tau).$$

Fix $\tau \in S_{m-1}$ and let $\tau = \tau_1 \cdots \tau_t$ be a complete factorization of $\tau$ as a product of disjoint cycles (including 1-cycles, so that each $1 \leq i \leq m - 1$ actually appears), and assume that the factors are in order of nonincreasing lengths. Fix $1 \leq i < m$. We have $i \in \tau_j$ (meaning $i$ appears in $\tau_j$) for a uniquely determined $j$. If $\tau_j$ has length $l$, then $\tau_j(i, m)$ is a cycle of length $l + 1$ since it is the same as $\tau_j$ except with $m$ inserted directly after $i$. In particular, if $\beta \vdash (m - 1)$ is the cycle partition of $\tau$, it follows that $\tau(i, m)$ is in the conjugacy class of $S_m$ corresponding to $\beta + \epsilon_j$. Therefore,

$$\sum_{i=1}^{m-1} \chi_\alpha(\tau(i, m)) = \sum_{j=1}^{t} \sum_{i \in \tau_j} \chi_\alpha(\beta + \epsilon_j)$$

$$= \sum_{j=1}^{t} \beta_j \chi_\alpha(\beta + \epsilon_j)$$

$$= \sum_{i=1}^{t} (\alpha_i - i) \chi_{\alpha - \epsilon_i}(\tau),$$
where the last step uses Theorem 2.3. Substituting this into the earlier equation and using Theorem 2.4 in the second sum, we get

$$\text{tr}_B T'_\alpha = \sum_{\tau \in S_{m-1}} \sum_{i=1}^{l(\alpha)} (\alpha_i - i) \chi_{\alpha_i - \varepsilon_i}(\tau) P_{m-1}(\tau)$$

$$+ n \sum_{\tau \in S_{m-1}} \sum_{i=1}^{l(\alpha)} \chi_{\alpha_i - \varepsilon_i}(\tau) P_{m-1}(\tau)$$

$$= \sum_{i=1}^{l(\alpha)} (n + \alpha_i - i) \chi_{\alpha_i - \varepsilon_i}(\tau) P_{m-1}(\tau)$$

$$= \sum_{i=1}^{l(\alpha)} (n + \alpha_i - i) T'_{\alpha - \varepsilon_i}.$$

Since $T_\alpha = (\chi_\alpha(1)/m!) T'_\alpha$ for $\alpha \vdash m$, we get an immediate consequence.

2.6 Corollary. For every $\alpha \vdash m$, we have

$$\text{tr}_B T_\alpha = \sum_{i=1}^{l(\alpha)} \frac{\chi_\alpha(1)(n + \alpha_i - i)}{m \chi_\alpha - \varepsilon_i(1)} T'_{\alpha - \varepsilon_i}.$$

3. Dimension of symmetry class

As an application of Theorem 2.5 we give a proof by induction of the well-known formula for the dimension of the space $V^{\chi_\alpha} := T_\alpha(V^\otimes m)$ ($\alpha \vdash m$), which is known as the symmetry class of tensors corresponding to the irreducible character $\chi_\alpha$ of $S_m$ and the vector space $V$.

3.1 Theorem [Mer97]. For every $\alpha \vdash m$, we have

$$\dim V^{\chi_\alpha} = \frac{(\chi_\alpha(1))^2}{m!} \prod_{i=1}^{l(\alpha)} \prod_{j=1}^{\alpha_i} (n + j - i),$$

where $n = \dim V$.

Proof. Let $\alpha \vdash m$. Since $T_\alpha^2 = T_\alpha$ [Mer97, Theorem 6.3], every eigenvalue of $T_\alpha$ is either 0 or 1, so $\dim V^{\chi_\alpha}$ is the rank of $T_\alpha$, which equals the trace of $T'_\alpha$:

$$\dim V^{\chi_\alpha} = \text{tr} T_\alpha = \frac{\chi_\alpha(1)}{m!} \text{tr} T'_\alpha.$$
Therefore, it suffices to show that
\[ \text{tr} T'_{\alpha} = \chi_{\alpha}(1) \prod_{i=1}^{l(\alpha)} \prod_{j=1}^{\alpha_i}(n+j-i). \]

We proceed by induction on \( m \). If \( m = 1 \), then \( \alpha = (1,0,0,\ldots) \) and \( \chi_{\alpha} \) is the trivial character, implying that \( T'_{\alpha} \) is the identity map on \( V \) and both sides of the equation equal \( n \). Assume that \( m > 1 \). Using Lemma 1.1, Theorem 2.5, and the induction hypothesis in turn, we get
\[ \text{tr} T'_{\alpha} = \text{tr}(\text{tr}_B T'_{\alpha}) \]
\[ = \sum_{k=1}^{l(\alpha)} \sum_{\alpha_k > \alpha_{k+1}} (n + \alpha_k - k) \text{tr}(T'_{\alpha - \varepsilon_k}) \]
\[ = \sum_{k=1}^{l(\alpha)} (n + \alpha_k - k) \chi_{\alpha - \varepsilon_k}(1) \prod_{i=1}^{l(\alpha - \varepsilon_k)} \prod_{j=1}^{\alpha_i}(n+j-i). \]

Now, for each \( 1 \leq k \leq l(\alpha) \) with \( \alpha_k > \alpha_{k+1} \),
\[ (n + \alpha_k - k) \prod_{i=1}^{l(\alpha - \varepsilon_k)} \prod_{j=1}^{\alpha_i}(n+j-i) \]
\[ = (n + \alpha_k - k) \prod_{i=1}^{l(\alpha)} \prod_{j=1}^{\alpha_i}(n+j-i) \prod_{j=1}^{\alpha_k-1} (n+j-k) \]
\[ = \prod_{i=1}^{l(\alpha)} \prod_{j=1}^{\alpha_i} (n+j-i), \]
so substituting into the earlier formula we get
\[ \text{tr} T'_{\alpha} = \sum_{k=1}^{l(\alpha)} \chi_{\alpha - \varepsilon_k}(1) \prod_{i=1}^{l(\alpha)} \prod_{j=1}^{\alpha_i}(n+j-i) \]
\[ = \chi_{\alpha}(1) \prod_{i=1}^{l(\alpha)} \prod_{j=1}^{\alpha_i}(n+j-i), \]
where the last step uses Theorem 2.4. This completes the proof. \( \square \)

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