Notes on general SIC-POVMs

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An unavoidable task in quantum information processing is how to obtain data about the state of an individual system by suitable measurements. From this viewpoint, informationally complete measurements are relevant in quantum state tomography, quantum cryptography, quantum cloning, and other issues. Symmetric informationally complete measurements (SIC-POVMs) form an especially important class of such measurements. We formulate some novel properties and relations for general SIC-POVMs in a finite-dimensional Hilbert space. It is known that general SIC-POVMs exist in all dimensions. For a given density matrix and any general SIC-POVM, the so-called index of coincidence of generated probability distribution is exactly calculated. Using this result, we obtain state-dependent entropic bounds for a single general SIC-POVM. Lower entropic bounds are derived in terms of the Rényi $\alpha$-entropies for $\alpha \in [2; \infty)$ and the Tsallis $\alpha$-entropies for $\alpha \in (0; 2]$. A lower bound on the min-entropy for a SIC-POVM is separately examined. For a pair of general SIC-POVMs, entropic uncertainty relations of the Maassen–Uffink type are considered.

Keywords: Rényi entropy, Tsallis entropy, general SIC-POVM, index of coincidence

I. INTRODUCTION

The quantum information theory treats quantum states and effects as tools for information processing [1]. At the final stage of any protocol, one or another measurement is required. Hence, we ask how to obtain information about the state of a quantum system. Informationally complete measurements have found to be useful in many issues. Among numerous methods for retrieving information on the state, the informationally complete measurements [2, 3] seem to be the most versatile. Especially interesting cases are when the measurement is symmetric [4] or covariant with respect to a group of physical transformations [5, 6]. Symmetric informationally complete (SIC) measurements are the subject of active research. Although the formulation itself is enough simple, SIC-POVMs are difficult to construct. In particular, they are intimately related to problem of building a complete set of mutually unbiased bases (MUBs) [7]. Weyl–Heisenberg (WH) covariant SIC-sets of states and their connection with standard constructions for MUBs were examined in [8]. Tight informationally complete measurements were introduced in [9].

In their original version, SIC-POVMs are assumed to be constructed of only rank-one elements. Some concrete examples in low dimensions are discussed in [9]. For quantum tomography, rank-one SIC-POVMs are maximally efficient at estimating the quantum state [6]. The seamy side is that such a measurement erases the original state of the system being measured. These reasons pertain to situation, when some unknown state is the subject of tomography and also post-tomography processing. It is typical in such a case that only a part of the total system is measured through the tomography process. As a rule, there is a trade-off between efficiency of used measurement and disturbance, which will influence on further stages. In this regard, other versions of SIC-POVMs are of interest. An approximate version of rank-one SIC-POVMs were examined in [10]. General SIC-POVMs with elements of any rank are considered in [11, 12]. It has recently been shown that general SIC-POVMs exist in all dimensions [13]. Explicit constructions for such POVMs and their dual bases have been presented in [13].

The aim of the present work is to study some generic properties of general symmetric informationally complete measurements. The preliminary material is reviewed in Section II. In particular, dual bases informationally complete POVMs are briefly considered. In Section III we exactly calculate the so-called index of coincidence of probability distribution obtained with a general SIC-POVM and arbitrary state. Uncertainty bounds for a single general SIC-POVM are considered in Sections IV and V. For these purposes, we respectively use the Tsallis and Rényi entropies, including the so-called min-entropy. In Section VI, uncertainty relations of the Maassen–Uffink type are obtained for a pair of general SIC-POVMs. In Section VII we conclude the paper with a summary of results.

II. NOTATION

Let $\mathcal{L}(\mathcal{H})$ be the space of linear operators on $d$-dimensional Hilbert space $\mathcal{H}$. By $\mathcal{L}_{\text{sa}}(\mathcal{H})$ and $\mathcal{L}_+(\mathcal{H})$, we respectively mean the space of Hermitian operators on $\mathcal{H}$ and the set of positive ones. For two operators $X, Y \in \mathcal{L}(\mathcal{H})$, their Hilbert–Schmidt inner product is defined by [14]

$$\langle X, Y \rangle_{\text{hs}} := \text{tr}(X^\dagger Y) .$$

(1)
The inner product \( \langle X,Y \rangle \) induces the Frobenius norm, also called the Hilbert–Schmidt norm:
\[
\|X\|_2 := \langle X, X \rangle_{\text{hs}}^{1/2} = \text{tr}(X^\dagger X)^{1/2}.
\] (2)

For each \( X \in \mathcal{L}(\mathcal{H}) \), we will use positive operator \( |X| := \sqrt{X^\dagger X} \). The singular values \( \sigma_i(X) \) of arbitrary \( X \in \mathcal{L}(\mathcal{H}) \) are defined as the eigenvalues of \( |X| \in \mathcal{L}_+(\mathcal{H}) \). Then the Schatten \( q \)-norm is introduced for all \( q \in [1; \infty] \) as
\[
\|X\|_q := \left( \sum_{i=1}^d \sigma_i(X)^q \right)^{1/q}.
\] (3)

The family \( \{|X|\} \) gives the trace norm \( \|X\|_1 = \text{tr}|X| \) for \( q = 1 \), the Frobenius norm \( \|X\|_2 \) for \( q = 2 \), and the spectral norm \( \|X\|_\infty = \max \{ \sigma_i(X) : 1 \leq i \leq d \} \) for \( q = \infty \). These norms and relations between them have found to be useful in various questions of quantum information \[13,14,15,16,17\]. For each \( q \in [1; \infty] \) and \( X,Y, \in \mathcal{L}(\mathcal{H}) \), we have \[14\]
\[
\|XY\|_q \leq \|X\|_\infty \|Y\|_q.
\] (4)

This inequality will be used in section \[VI\]. For all \( q > p \geq 1 \) and arbitrary \( X \in \mathcal{L}(\mathcal{H}) \), we also have
\[
\|X\|_q \leq \|X\|_p.
\] (5)

This relation is actually a consequence of theorem 19 of the classical book \[18\].

A density matrix \( \rho \in \mathcal{L}_+(\mathcal{H}) \) has unit trace, i.e., \( \text{tr}(\rho) = 1 \). Generalized quantum measurements are commonly described within the POVM formalism \[19\]. Let \( \mathcal{N} = \{N_j\} \) be a set of elements \( N_j \in \mathcal{L}_+(\mathcal{H}) \), satisfying the completeness relation
\[
\sum_j N_j = 1.
\] (6)

Here, the \( I \) denotes the identity operator on \( \mathcal{H} \). This set \( \mathcal{N} = \{N_j\} \) is a positive operator-valued measure (POVM). Consider some POVM with \( d^2 \) elements \( N_j \), which satisfy the following two conditions. First, for all \( j = 1, \ldots, d^2 \) we have
\[
\langle N_j, N_j \rangle_{\text{hs}} = a.
\] (7)

This condition can be rewritten as \( \|N_j\|_2 = \sqrt{a} \). Second, the pairwise inner products are all symmetrical, namely
\[
\langle N_j, N_k \rangle_{\text{hs}} = b \quad (j \neq k).
\] (8)

Then the POVM \( \mathcal{N} = \{N_j\} \) is called general SIC-POVM. Combining the formula \( \langle I, I \rangle_{\text{hs}} = d \) with the above properties finally leads to the relation \[13\]
\[
b = \frac{1 - ad}{d(d^2 - 1)}.
\] (9)

Further, we obtain \( \text{tr}(N_j) = d^{-1} \) for all \( j = 1, \ldots, d^2 \). Therefore, the value \( a \) is the only parameter characterizing the type of a general SIC-POVM. This parameter is restricted as \[13\]
\[
d^{-3} \leq a \leq d^{-2}.
\] (10)

One reaches the lower bound \( a = d^{-3} \) in the only case \( N_j = d^{-2} I \). The upper bound \( a = d^{-2} \) is achieved, if and only if the POVM elements are all rank-one. The latter is actually the case of usual SIC-POVMs, when each element is represented in terms of the corresponding unit vector as
\[
N_j = d^{-1} |\phi_j \rangle \langle \phi_j|.
\] (11)

The formulas \( \text{8} \) and \( \text{9} \) then result in \( \langle \phi_j | \phi_k \rangle = (d + 1)^{-1/2} \). In the further discussion, we will usually exclude both the least values of the relation \[10\]. That is, we do not focus on the least case \( N_j = d^{-2} I \) as well as rank-one SIC-POVMs.

In the next sections, we will use the following properties of general SIC-POVMs. First, the elements \( N_j \) of a general SIC-POVM form a basis in the space \( \mathcal{L}_{s.a.}(\mathcal{H}) \) \[13\]. Second, each operator \( X \in \mathcal{L}_{s.a.}(\mathcal{H}) \) can be represented in terms of elements of the dual basis as
\[
X = \sum_{j=1}^{d^2} \langle N_j, X \rangle_{\text{hs}} \bar{N}_j.
\] (12)
Here, the dual basis \( \{ \tilde{N}_j \} \) is a basis in \( L_{s.a.}(H) \) such that
\[
\langle N_j, \tilde{N}_k \rangle_{hs} = \delta_{jk} \quad \forall \ j, k \in \{1, \ldots, d^2\} .
\] (13)

When some basis in \( L_{s.a.}(H) \) is formed by positive operators, its dual basis cannot consist of only positive elements \[20, 21\]. For a general SIC-POVM \( \{ N_j \} \) with \( a \neq d^{-3} \), the dual basis is comprised by operators \[13\]
\[
\tilde{N}_j = \frac{d}{ad^3 - 1} \left( (d^2 - 1)N_j - (1-ad)1 \right) .
\] (14)

For a usual SIC-POVM with elements \[11\], the formula \[14\] is reduced to \( \tilde{N}_j = (d + 1)|\phi_j\rangle\langle \phi_j| - 1 \). Explicit constructions and related properties of general SIC-POVMs are presented in \[13\].

III. INDEX OF COINCIDENCE

In this section, we calculate the so-called index of coincidence of probability distribution generated by a general SIC-POVM on any mixed state. Using this notion, the writers of \[22\] have derived some entropic bounds for a set of several mutually unbiased bases. For usual SIC-POVMs, this issue was considered in \[23\]. The index of coincidence of probability distribution \( \{ p_j \} \) is defined as \[24\]
\[
C(p) := \sum_j p_j^2 .
\] (15)

If the pre-measurement state is described by density matrix \( \rho \), \( j \)-th outcome occurs with the probability
\[
p_j(N|\rho) = \text{tr}(N_j\rho) .
\] (16)

For the given SIC-POVM \( N \) and state \( \rho \), the quantity \( C(N|\rho) \) is obtained by substitution of probabilities \[16\] into \[15\]. The following statement takes place.

**Proposition 1** Let general SIC-POVM \( N \) be characterized by the parameter \( a \) in the sense of \[7\]. For arbitrary \( \rho \), the index of coincidence of generated probability distribution is equal to
\[
C(N|\rho) = \frac{(ad^3 - 1) \text{tr}(\rho^2) + d(1-ad)}{d(d^2 - 1)} .
\] (17)

**Proof.** We first suppose that \( a \neq d^3 \). Using \[12\] and \[16\], we represent the density matrix in the dual basis as
\[
\rho = \sum_{j=1}^{d^2} p_j \tilde{N}_j .
\] (18)

For brevity, we put the quantities \( \tilde{a} = \langle \tilde{N}_j, \tilde{N}_j \rangle_{hs} \) and \( \tilde{b} = \langle \tilde{N}_j, \tilde{N}_k \rangle_{hs} \) for \( j \neq k \). They are calculated by substitution of \( \tilde{N}_j \) and \( \tilde{N}_k \) according to \[14\]. The resulting expressions are then written as
\[
\tilde{a} - \tilde{b} = \frac{d(d^2 - 1)}{ad^3 - 1} ,
\] (19)
\[
\tilde{b} = -\frac{d(1-ad)}{ad^3 - 1} .
\] (20)

Here, we used \( \text{tr}(N_j) = d^{-1} \), the definitions \[7\] and \[8\], and the condition \[11\]. Substituting the right-hand side of \[18\] into \( \text{tr}(\rho^2) \) leads to the formula
\[
\text{tr}(\rho^2) = \tilde{a} C(p) + \tilde{b} \sum_{j \neq k} p_j p_k = (\tilde{a} - \tilde{b}) C(p) + \tilde{b} ,
\] (21)

where the normalization condition was used at the last step. Combining \[21\] with \[13\] and \[20\] finally leads to the claim \[17\]. We also note that the result \[17\] holds for \( a = d^{-3} \), when \( N_j = d^{-2}1 \) for all \( j = 1, \ldots, d^2 \). In this case, we merely have \( p_j(N|\rho) = d^{-2} \) and, herewith, \( C(N|\rho) = d^{-2} \). The latter coincides with \[17\].
The statement of Proposition [H] gives the expression for the index of coincidence in terms of the parameter $a$ and dimensionality $d$. For the completely mixed state $\rho_*=\mathbb{I}/d$ with $\text{tr}(\rho_*)^2=d^{-1}$, the formula (17) gives

$$C(\mathcal{N}|\rho_*) = d^{-2},$$

irrespectively to $a$. The right-hand side of (22) is valid, since $p_j(\mathcal{N}|\rho_*) = d^{-2}$ for any general SIC-POVM. For a usual SIC-POVMs with only rank-one elements (11), we substitute $a = d^{-2}$ into (17) and obtain

$$C(\mathcal{N}|\rho) = \frac{\text{tr}(\rho^2)}{d(d+1)}. \tag{23}$$

The result (23) has been derived in [23] by other method. For pure states, the numerator in the right-hand side of (22) is equal to 2. This pure-state case of (23) was previously presented in [25]. The method of the paper [25] is based on the fact that the unit vectors $|φ_j⟩$ form a spherical 2-design.

In the case $d=2$, the right-hand side of (17) can be represented with use of the Bloch vector. Here, we denote the identity $2 \times 2$-matrix by $\mathbb{I}$ and the usual Pauli matrices by $\sigma_x$, $\sigma_y$, and $\sigma_z$. Arbitrary density matrix is written as

$$\rho = \frac{1}{2} (\mathbb{I} + \vec{r} \cdot \vec{σ}). \tag{24}$$

where $\vec{r} = (r_x, r_y, r_z)$ is the Bloch vector. Positivity of this matrix implies $r = |r| \leq 1$. Calculating $\text{tr}(\rho^2) = (1+r^2)/2$, the index of coincidence is equal to

$$C(\mathcal{N}|\rho) = \sum_{j=1}^{d^2-1} p_j(\mathcal{N}|\rho)^2 = \frac{3 + (8a - 1)r^2}{12}. \tag{25}$$

This result shows a dependance of $C(\mathcal{N}|\rho)$ on the parameter $a$ and the Bloch vector $\vec{r}$. For $a = 1/4$, the formula (26) gives the fraction $(3+r^2)/12$, which was already noted in [23]. The Bloch-vector representation for finite-level systems is one of important state representations [26]. Similarly to (25), the formula (17) could be rewritten in terms of the generalized Bloch vector of a $d$-level system. By $\lambda_n \in \mathcal{L}_{s,a}(\mathcal{H})$, with $n = 1, \ldots, d^2-1$, we denote the generators of $\text{SU}(d)$ which satisfy $\text{tr}(\lambda_n) = 0$ and

$$\text{tr}(\lambda_m \lambda_n) = 2δ_{mn}. \tag{26}$$

The factor 2 in (26) is traditionally used. Arbitrary density operator can be represented in the form [26, 27]

$$\rho = \frac{1}{d} \left( \mathbb{I} + \sum_{n=1}^{d^2-1} r_n \lambda_n \right), \tag{27}$$

where $r_n = (d/2) \text{tr}(\rho \lambda_n)$. These components form a $(d^2-1)$-dimensional real vector, which represents the density matrix $\rho$. Although the definition of the Bloch vector is simple, the space of the Bloch vectors for $d$-level system is difficult to determine. Some general properties of the Bloch-vector space are studied in [28, 30]. By calculations, we obtain

$$\text{tr}(\rho^2) = \frac{1}{d^2} + \frac{2}{d^2} \|r\|_2^2, \tag{28}$$

where $\|r\|_2$ denotes the vector 2-norm. The formula (17) is then represented as

$$C(\mathcal{N}|\rho) = \frac{1}{d^2} + \frac{2(ad^3-1)}{d^3(d^2-1)} \|r\|_2^2. \tag{29}$$

For $d=2$, this result is reduced to (25). For the completely mixed state $\rho_* = \mathbb{I}/d$, components of the generalized Bloch vectors are all zero. Hence, the formula (29) directly leads to (22). Thus, we have useful expressions in terms of the generalized Bloch vector.

IV. TSALLIS-ENTROPY BOUNDS FOR A SINGLE SIC-POVM

In this section, we obtain entropic uncertainty relations for a single general SIC-POVM in terms of its Tsallis entropy. Entropic functions are natural and flexible tools to measure an uncertainty in quantum measurements
The Rényi and Tsallis entropies form an especially important family of one-parametric generalizations of the Shannon entropy. For $\alpha > 0 \neq 1$, the Tsallis $\alpha$-entropy of probability distribution $\{p_j\}$ is defined by

$$H_\alpha(p) := \frac{1}{1-\alpha} \left( \sum_j p_j^\alpha - 1 \right). \quad (30)$$

The right-hand side of (30) is usually rewritten in terms of the $\alpha$-logarithm

$$\ln_\alpha(x) := \frac{x^{1-\alpha} - 1}{1-\alpha}, \quad (31)$$

where $\alpha > 0 \neq 1$ and $x > 0$. The Tsallis $\alpha$-entropy reads

$$H_\alpha(p) = -\sum_j p_j^\alpha \ln_\alpha(p_j) = \sum_j p_j \ln_\alpha \left( \frac{1}{p_j} \right). \quad (32)$$

In statistical physics, the entropy (30) was originally introduced in [33]. Taking $\alpha \to 1$, the $\alpha$-logarithm is reduced to the standard logarithm $\ln x$. Then the entropy (30) leads to the Shannon entropy $H_1(p) = -\sum_j p_j \ln p_j$. Functional properties of the entropy (30) and its conditional versions are considered in [34, 35].

For given SIC-POVM $N = \{N_j\}$, the entropy $H_\alpha(N|\rho)$ is obtained by substituting the probabilities (19) into the formula (30). It turns out that these entropies are bounded from below. We will derive lower bounds on the Tsallis $\alpha$-entropy for $\alpha \in (0;2]$.

**Proposition 2.** Let general SIC-POVM $N$ be characterized by the parameter $\alpha$ in the sense of [7]. For $\alpha \in (0;2]$ and arbitrary density matrix $\rho$, the Tsallis $\alpha$-entropy satisfies the state-dependent bound

$$H_\alpha(N|\rho) \geq \ln_\alpha \left( \frac{d(d^2 - 1)}{(ad^3 - 1)\tr(\rho^2) + d(1-ad)} \right). \quad (33)$$

**Proof.** The following point was noted in [23]. For $\alpha \in (0;2]$ and arbitrary probability distribution, the Tsallis $\alpha$-entropy obeys

$$H_\alpha(p) \geq \ln_\alpha \left( \frac{1}{C(p)} \right). \quad (34)$$

This formula is a direct consequence of Jensen’s inequality for the function $x \mapsto \ln_\alpha(1/x)$. Indeed, this function is convex for $\alpha \in (0;2]$. Combining (17) with (34) immediately gives the claim (33). ■

For all $\alpha \in (0;2]$, the result (33) provides a state-dependent lower bound on the Tsallis $\alpha$-entropy of probability distribution generated by a general SIC-POVM. Using (29), we rewrite the bound (33) in terms of the generalized Bloch vector, namely

$$H_\alpha(N|\rho) \geq \ln_\alpha \left( \frac{d^3(d^2 - 1)}{2(ad^3 - 1)||\rho||^2_2 + d(d^2 - 1)} \right), \quad (35)$$

where $\alpha \in (0;2]$. For $\alpha = 1$, we obtain the lower bound on the Shannon entropy, namely

$$H_1(N|\rho) \geq \ln \left( \frac{d(d^2 - 1)}{(ad^3 - 1)\tr(\rho^2) + d(1-ad)} \right) \quad (36)$$

With a pure state $\rho = |\psi\rangle\langle\psi|$, the entropic bound (33) is reduced to the inequality

$$H_\alpha(N|\psi) \geq \ln_\alpha \left( \frac{d(d + 1)}{ad^2 + 1} \right). \quad (37)$$

For impure states, we have a stronger lower bound (33). The latter follows from increasing of the $\alpha$-logarithm and the fact that $\tr(\rho^2) < 1$ for an impure state. Here, we see a natural dependence on the measured state. The right-hand side of (33) reaches its maximum with the completely mixed state $\rho_* = \mathbb{I}/d$. Using $\tr(\rho_*^2) = d^{-1}$, the formula (33) becomes

$$H_\alpha(N|\rho_*) \geq \ln_\alpha \left( \frac{d(d + 1)}{ad^2 + 1} \right). \quad (38)$$

The bound (38) is just saturated. Indeed, for all $j = 1, \ldots, d^2$ we have $\tr(N_j) = d^{-1}$ and, herewith, $p_j(N|\rho_*) = d^{-2}$. Substituting this probability into right-hand side of (32), we actually obtain (38) with the sign of equality. With the completely mixed state, the equality takes place for all $\alpha > 0$ irrespectively to the parameter $a$. In the mentioned sense, the state-dependent bound (33) is tight. At the same time, we proved (33) only for $\alpha \in (0;2]$. Formulating lower bounds on the entropy $H_\alpha(N|\rho)$ for $\alpha > 2$ is an open question.
V. RÉNYI-ENTROPY BOUNDS FOR A SINGLE SIC-POVM

In this section, we obtain entropic bounds for a single general SIC-POVM in terms of the Rényi entropy. For \( \alpha > 0 \neq 1 \), the Rényi \( \alpha \)-entropy of the probability distribution \( \{ p_j \} \) is defined as

\[
R_\alpha(p) := \frac{1}{1-\alpha} \ln \left( \sum_j p_j^\alpha \right) .
\] (39)

In the limit \( \alpha \to 1 \), this expression is reduced to the standard Shannon entropy. The entropy (39) is a non-increasing function of order \( \alpha \) \([36]\). Taking \( \alpha = 2 \), the expression (39) gives the collision entropy

\[
R_2(p) = -\ln \left( \sum_j p_j^2 \right) .
\] (40)

In the limit \( \alpha \to \infty \), we have the so-called min-entropy

\[
R_\infty(p) = -\ln(\max_j p_j) .
\] (41)

The min-entropy is of particular interest in cryptography \([37]\). This entropy is also related to the extrema of the discrete Wigner function \([38]\). Rényi-entropies uncertainty relations are significant in studying the connection between complementarity and uncertainty principles \([39]\). Using the Rényi entropy, the writers of \([40]\) formulated trade-off relations for a trace-preserving quantum operation. An extension of such trade-off relations in terms of the so-called unified entropies was discussed in \([41]\).

For a SIC-POVM \( N = \{ N_j \} \), the entropy \( R_\alpha(N | \rho) \) is obtained by substituting the probabilities (16) into (39). We now consider lower bounds on this entropy.

**Proposition 3** Let general SIC-POVM \( N \) be characterized by the parameter \( a \) in the sense of (7). For \( \alpha \in [2; \infty) \) and arbitrary density matrix \( \rho \), the Rényi \( \alpha \)-entropy satisfies the state-dependent bound

\[
R_\alpha(N | \rho) \geq \frac{\alpha}{2(1-\alpha)} \ln \left( \frac{d(d^2 - 1)}{(ad^3 - 1) \text{tr}(\rho^2) + d(1-ad)} \right) .
\] (42)

**Proof.** For \( \alpha \geq 2 \) and arbitrary probability distribution, we write the inequality

\[
\left( \sum_j p_j^\alpha \right)^{1/\alpha} \leq \left( \sum_j p_j^2 \right)^{1/2} = C(p)^{1/2} .
\] (43)

This inequality follows from theorem 19 of the book \([18]\). The function \( x \mapsto (1-\alpha)^{-1} \ln x \) decreases for \( \alpha > 1 \). Combining this with (39) and (43) further gives

\[
R_\alpha(p) \geq \frac{\alpha}{2(1-\alpha)} \ln C(p) .
\] (44)

The formulas (17) and (44) completes the proof. \( \blacksquare \)

The formula (42) provides a state-dependent lower bound on the Rényi \( \alpha \)-entropy of probability distribution generated by a general SIC-POVM. Due to (29), we can rewrite the bound (33) in the form

\[
R_\alpha(N | \rho) \geq \frac{\alpha}{2(1-\alpha)} \ln \left( \frac{1}{d^2} + \frac{2(ad^3 - 1)}{d^3(d^2 - 1)} \| r \|_2^2 \right) .
\] (45)

For \( \alpha = 2 \), the inequality (42) gives a bound on the collision entropy written as

\[
R_2(N | \rho) \geq \ln \left( \frac{d(d^2 - 1)}{(ad^3 - 1) \text{tr}(\rho^2) + d(1-ad)} \right)
\] (46)

\[
= -\ln \left( \frac{1}{d^2} + \frac{2(ad^3 - 1)}{d^3(d^2 - 1)} \| r \|_2^2 \right) .
\] (47)

As the Rényi \( \alpha \)-entropy does not increase with \( \alpha \), the bound (46) is valid for all Rényi’s entropies of order \( \alpha \in (0; 2] \), including the Shannon-entropy case (36). For \( \rho = | \psi \rangle \langle \psi | \), the entropic bound (42) is reduced to its pure-state form

\[
R_\alpha(N | \psi) \geq \frac{\alpha}{2(\alpha - 1)} \ln \left( \frac{d(d+1)}{ad^2 + 1} \right) ,
\] (48)
where \(\alpha \in [2; \infty)\). Due to \(\text{tr}(\rho^2) < 1\) and increase of the logarithm, the lower bound (48) is weaker than (42). It is natural that the right-hand side of (42) reaches its maximum with \(\rho_c = \mathbb{1}/d\). It is easy to see that \(p_j(N|\rho_c) = d^{-2}\) and, therefore, \(R_\alpha(N|\rho_c) = 2 \ln d\) for all \(\alpha > 0\). With the completely mixed state, the right-hand side of (17) actually gives the bound \(2 \ln d\) for all \(\alpha \in (0; 2]\). In this sense, the derived Rényi-entropy bound (42) is tight for such values of \(\alpha\). For \(\alpha > 2\), the result (42) is always approximate. For instance, it gives gives the lower bound \(R_\alpha(N|\rho_c) \geq 2 \ln d\), which is only a half of the exact value \(R_\infty(N|\rho_c) = 2 \ln d\). In the paper [23], we have mentioned a way to improve the min-entropy relation for usual SIC-POVMs. We now extend this treatment to general SIC-POVMs.

**Proposition 4** Let general SIC-POVM \(N\) be characterized by the parameter \(a\) in the sense of (7). For arbitrary density matrix \(\rho\), the min-entropy satisfies the state-dependent bound

\[
R_\infty(N|\rho) \geq 2 \ln d - \ln 1 + \sqrt{ad^3 - 1} \sqrt{\text{tr}(\rho^2) d - 1} .
\]

**Proof.** In appendix A of the paper [23], we proved the following statement. If the \(n\) positive numbers \(x_j\) satisfy the two conditions \(\sum_{j=1}^n x_j = 1\) and \(\sum_{j=1}^n x_j^2 = b^2\), then

\[
\max\{x_j : 1 \leq j \leq n\} \leq \frac{1}{n} \left(1 + \sqrt{n - 1} \sqrt{n b^2 - 1}\right) .
\]

Substituting (17) instead of \(b^2\) and \(d^2\) instead of \(n\) into (50), we finally obtain

\[
\max\{p_j(N|\rho) : 1 \leq j \leq d^2\} \leq \frac{1}{d^2} \left(1 + \sqrt{ad^3 - 1} \sqrt{\text{tr}(\rho^2) d - 1}\right) .
\]

Combining (11) with (51) directly leads to (49), since the function \(x \mapsto -\ln x\) decreases.

The lower bound (49) is clearly stronger than the limiting case \(\alpha \to \infty\) of the right-hand side of (42). With the completely mixed state, the result (49) gives the tight bound \(2 \ln d\) due to \(\text{tr}(\rho^2) = d^{-1}\). The right-hand side of (49) increases as the quantity \(\text{tr}(\rho^2)\) decreases. In other words, the more a state is mixed, the more the bound (49). Replacing \(\text{tr}(\rho^2)\) with 1, the right-hand side of (49) becomes the lower bound for the pure-state case. For \(a = d^{-2}\), the upper bound (51) on the maximal probability leads to the analogous bound for a usual SIC-POVM. This particular case was already given in [23].

**VI. UNCERTAINTY RELATIONS OF THE MAASEN–UFFINK TYPE**

In this section, we will discuss entropic uncertainty relations for a pair of general SIC-POVMs. Since the celebrated Heisenberg’s result was published [42], much many approaches to incompatibilities in quantum measurements have been proposed. Entropic uncertainty relations were studied in many important cases [31, 52]. Results of such a kind are mainly based on the Maasen–Uffink approach [43]. This approach has been developed with use of various entropic functions. Entropic bounds in terms of generalized entropies entropic bounds were utilized in studying many topics such as the case of conjugate observables [44, 45], quantifying number-phase uncertainties [46, 47], incompatibilities of anti-commuting observables [48] and reformulations in quasi-Hermitian models [49]. In the context of simultaneous measurements of complementary observables, uncertainty relations in terms of both the Tsallis and Rényi entropies are examined in [50]. Since the Maasen–Uffink uses the Riesz theorem, it leads to lower bound on the sum of two entropies, whose orders obey a certain condition [44, 46]. Recently, new universal approach to entropic uncertainty relations has been proposed [51, 52]. Apparently, this approach will play a significant role in future research. We will formulate uncertainty relations for two general SIC-POVMs in terms of the Rényi and Tsallis entropies as well as their symmetrized versions. We have the following statement.

**Proposition 5** Let \(\mathcal{M} = \{M_i\}\) and \(\mathcal{N} = \{N_j\}\) be general SIC-POVMs. To any density matrix \(\rho\), we assign the quantity

\[
g(\mathcal{M},\mathcal{N}|\rho) := \max\left\{p_i(M|\rho)^{-1/2} p_j(N|\rho)^{-1/2} |\text{tr}(M_i N_j \rho)| : i, j = 1, \ldots, d^2\right\} .
\]

Let positive orders \(\alpha\) and \(\beta\) obey \(1/\alpha + 1/\beta = 2\), and let \(\mu = \max\{\alpha, \beta\}\). Then the corresponding Tsallis entropies satisfy the inequality

\[
H_\alpha(\mathcal{M}|\rho) + H_\beta(\mathcal{N}|\rho) \geq \ln_\mu \left(g(\mathcal{M},\mathcal{N}|\rho)^{-2}\right) .
\]

Under the same preconditions, the corresponding Rényi entropies satisfy the inequality

\[
R_\alpha(\mathcal{M}|\rho) + R_\beta(\mathcal{N}|\rho) \geq -2 \ln g(\mathcal{M},\mathcal{N}|\rho) .
\]
The presented formulations \ref{eq:58} and \ref{eq:55} immediately follows from the results of section 3 of \ref{63}. The quantity \ref{eq:52} explicitly depends on the pre-measurement density matrix $\rho$. It is of certain interest to obtain the state-independent form of entropic bounds \ref{54}. For the entropic relations \ref{55} and \ref{54}, a way to obtain such forms was considered in \ref{49} \ref{53}. Using the Cauchy-Schwarz inequality for the Hilbert–Schmidt inner product and the inequality \ref{41}, we finally obtain

$$g(M, N|\rho) \leq f(M, N) = \max \left\{ \|M_i\|_2 \|N_j\|_2 : i, j = 1, \ldots, d^2 \right\}.$$ \hfill (55)

Here, we used $p_i(M|\rho) = \|M_i\|_2 \sqrt{\rho}|_{i}^{2}$, $p_j(N|\rho) = \|N_j\|_2 \sqrt{\rho}|_{j}^{2}$, and $\text{tr}(M_i N_j \rho) = \langle M_i \sqrt{\rho}, N_j \sqrt{\rho} \rangle_{hs}$. Due to \ref{55}, we obtain the state-independent entropic bounds

$$H_\alpha(M|\rho) + H_\beta(N|\rho) \geq \ln \left( f(M, N)^{-2} \right),$$ \hfill (56)

$$R_\alpha(M|\rho) + R_\beta(N|\rho) \geq -2 \ln f(M, N),$$ \hfill (57)

in which the parameters $\alpha$, $\beta$, and $\mu$ are defined as in Proposition \ref{5}. We can also reformulate these uncertainty relations in terms of the parameters $a_M$ and $a_N$ defined according to \ref{6}. Using the inequality \ref{5}, we write

$$\|M_i\|_2 \|N_j\|_2 \leq \|M_i\|_2 \|N_j\|_2 = \sqrt{a_M a_N}.$$ \hfill (58)

This inequality is always saturated with the usual SIC-POVMs, when $a_M a_N = d^{-4}$. Due to positivity, we have $f(M, N)^2 = \max \{ \|M_i\|_2 \|N_j\|_2 : i, j = 1, \ldots, d^2 \}$. Combining this with \ref{55} finally gives

$$g(M, N|\rho)^{-2} \geq f(M, N)^{-2} \geq a_M^{-1/2} a_N^{-1/2}.$$ \hfill (59)

Since the function $x \mapsto \ln(x)$ is increasing, the inequality \ref{59} leads to entropic uncertainty relations

$$H_\alpha(M|\rho) + H_\beta(N|\rho) \geq \ln \left( a_M^{-1/2} a_N^{-1/2} \right),$$ \hfill (60)

$$R_\alpha(M|\rho) + R_\beta(N|\rho) \geq -\frac{1}{2} \left( \ln a_M + \ln a_N \right),$$ \hfill (61)

which follow from \ref{56} and \ref{57}, respectively. The parameters $a_M$ and $a_N$ range according to \ref{51}. For two usual SIC-POVMs, the right-hand sides of \ref{60} and \ref{61} take values $\ln(d^2)$ and $2 \ln d$ due to $a_M = a_N = d^{-2}$. For other general SIC-POVMs, these bounds are strictly stronger. In principle, they may increase up to $\ln(d)$ and $3 \ln d$, when $a_M = a_N = d^{-3}$. As a general SIC-POVM has $d^2$ different outcomes, its Tsallis and Renyi $\alpha$-entropies are respectively bounded from above by $\ln_{\mu}(d^2)$ and $2 \ln d$. For two general SIC-POVMs, the formulas \ref{60} and \ref{61} show that the sum of the two corresponding entropies is not less than the maximal possible value for one of them. Thus, we have obtained non-trivial entropic bound of the Maassen–Uffink type.

Finally, we discuss reformulations in terms of the so-called symmetrized entropies. For a pair of observables, uncertainty relations in terms of symmetrized entropies were given in both the Rényi \ref{44} and Tsallis formulations \ref{55}. In \ref{23}, lower bounds on the sum of symmetrized entropies have been derived for several mutually unbiased bases. Let us assume that the entropic orders obey $1/\alpha + 1/\beta = 2$. Using $s \in [0, 1)$, we parametrize these orders as

$$\max\{\alpha, \beta\} = \frac{1}{1-s}, \quad \min\{\alpha, \beta\} = \frac{1}{1+s}.$$ \hfill (62)

The symmetrized Tsallis and Rényi entropies are respectively defined by

$$\tilde{H}_s(N|\rho) := \frac{1}{2} \left( H_\alpha(N|\rho) + H_\beta(N|\rho) \right),$$ \hfill (63)

$$\tilde{R}_s(N|\rho) := \frac{1}{2} \left( R_\alpha(N|\rho) + R_\beta(N|\rho) \right).$$ \hfill (64)

Taking $\mu = (1-s)^{-1}$, for the sum $\tilde{H}_s(M|\rho) + \tilde{H}_s(N|\rho)$ we have the three lower bounds \ref{53}, \ref{56}, and \ref{60}. Similarly, the lower bounds \ref{54}, \ref{57}, and \ref{61} are all valid for the sum $\tilde{R}_s(M|\rho) + \tilde{R}_s(N|\rho)$. The use of symmetrized entropies allow to extend bounds of the Maassen–Uffink type to more than two measurements. An example with several mutually unbiased bases has been analyzed in \ref{23}. In principle, this idea could be applied to general SIC-POVMs.
VII. CONCLUSIONS

We have reported some properties of general symmetric informationally complete POVMs. SIC-POVMs are of interest in various topics such as quantum state tomography and quantum cryptography. Thus, the presented results may be useful within quantum technologies. For a general SIC-POVM and arbitrary measured state, the index of coincidence of generated probability distribution is exactly calculated. This result is a generalization of the previous calculation for a rank-one SIC-POVM. The obtained index of coincidence is expressed in terms of dimensionality, the trace of squared density matrix, and one natural parameter characterizing the given SIC-POVM. The trace of squared density matrix is one of measures quantifying a degree of state impurity. The calculation of the index of coincidence leads to entropic uncertainty relations for a single general SIC-POVM. We have expressed state-dependent formulations in terms of both the Rényi and Tsallis entropies. These formulations are an immediate extension of entropic relations previously given in [23]. The min-entropy uncertainty relation is separately considered. For a pair of general SIC-POVMs, we discussed uncertainty relations of the Maassen-Uffink type. Reformulations in terms of the symmetrized entropies are briefly considered. A new important approach to obtaining entropic uncertainty bounds with the use of majorization technique has recently been proposed in the papers [51, 52]. It may be interesting to study uncertainty relations for general SIC-POVMs on the base of the majorization approach.

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