The Lyapunov function for Schrödinger operators with a periodic $2 \times 2$ matrix potential

Andrei Badanin * Jochen Brüning † Evgeny Korotyaev ‡

January 25, 2018

Abstract

We consider the Schrödinger operator on the real line with a $2 \times 2$ matrix valued 1-periodic potential. The spectrum of this operator is absolutely continuous and consists of intervals separated by gaps. We define a Lyapunov function which is analytic on a two-sheeted Riemann surface. On each sheet, the Lyapunov function has the same properties as in the scalar case, but it has branch points, which we call resonances. We prove the existence of real as well as non-real resonances for specific potentials. We determine the asymptotics of the periodic and anti-periodic spectrum and of the resonances at high energy. We show that there exist two type of gaps: 1) stable gaps, where the endpoints are periodic and anti-periodic eigenvalues, 2) unstable (resonance) gaps, where the endpoints are resonances (i.e., real branch points of the Lyapunov function). We also show that periodic and anti-periodic spectrum together determine the spectrum of the matrix Hill operator.

1 Introduction and main results

We consider the self-adjoint operator $Ty = -y'' + V(x)y$, acting in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, where $V$ is a symmetric 1-periodic $2 \times 2$ matrix potential which belongs to the real space $\mathcal{H}^p, p = 1, 2$, given by

$$\mathcal{H}^p = \{ V = V^* = V^T = \left( \begin{array}{cc} V_1 & V_3 \\ V_3 & V_2 \end{array} \right) : \int_0^1 V_3(x) dx = 0 \},$$

equipped with the norm $\|V\|_p^p = \int_0^1 (|V_1(x)|^p + |V_2(x)|^p + 2|V_3(x)|^p) dx < \infty$. Without loss of generality we assume

$$V(0) = \int_0^1 V(t) dt = \text{diag}\{V_{10}, V_{20}\}, \quad V_{10} \leq V_{20}, \quad V_{m0} = \int_0^1 V_m(x) dx, \quad m = 1, 2.$$
Introduce the self-adjoint operator $T^0 = -\frac{\partial^2}{\partial x^2}$, with the domain $\text{Dom}(T^0) = W^2_2(\mathbb{R}) \oplus W^2_2(\mathbb{R})$. In order to get self-adjointness of $T$ we use symmetric quadratic forms. We shortly recall a well known argument (see [RS]). We define the quadratic form $(V\psi,\psi_1) = -\int_{\mathbb{R}} V\psi\bar{\psi}_1dx$, $\psi,\psi_1 \in \text{Dom}(T^0)$. Using the estimate (see [K1]) $|(q',f)| < \varepsilon(f', f) + b_\varepsilon(f, f)$ for any small $\varepsilon > 0$ and some $b_\varepsilon > 0$ and any $f \in W^2_2(\mathbb{R}), q \in L^2(\mathbb{R}/\mathbb{Z})$ we deduce that

$$|(V\psi, \psi_1)| < (1/2)(\psi', \psi')+b(\psi, \psi), \quad \psi \in W^2_2(\mathbb{R}) \oplus W^2_2(\mathbb{R}).$$

Thus we can apply the KLMN Theorem (see [RS]) to define $T = -\frac{\partial^2}{\partial x^2} + V$. There exists a unique self-adjoint operator $T$ with form domain $\mathcal{Q}(T) = W^2_1(\mathbb{R}) \oplus W^2_1(\mathbb{R})$ and

$$(T\psi, \psi_1) = (-\psi'', \psi_1) + (V\psi, \psi_1), \quad \text{all} \quad \psi, \psi_1 \in \mathcal{Q}(T^0) = W^2_1(\mathbb{R}).$$

Any domain of essential self-adjointness for $T^0$ is a form core for $T$.

It is well known (see [DS] p.1486-1494, [Ge]) that the spectrum $\sigma(T)$ of $T$ is absolutely continuous and consists of non-degenerated intervals $S_n, n = 1, 2, \ldots$. These intervals are separated by the gaps $G_n$ with lengths $|G_n| > 0, n = 1, 2, \ldots, N_G \leq \infty$. Introduce the fundamental $2 \times 2$-matrix solutions $\varphi(x, \lambda), \vartheta(x, \lambda)$ of the equation

$$-y'' + V(x)y = \lambda y, \quad \lambda \in \mathbb{C}, \quad (1.1)$$

with initial conditions $\varphi(0, \lambda) = \vartheta'(0, \lambda) = 0, \varphi'(0, \lambda) = \vartheta(0, \lambda) = I_2$, where $I_m, m \geq 1$ is the identity $m \times m$ matrix. Here and below we use the notation $' = \partial/\partial x$. We define the $4 \times 4$-monodromy matrix by

$$M(\lambda) = \mathcal{M}(1, \lambda), \quad \mathcal{M}(x, \lambda) = \left( \begin{array}{cc} \vartheta(x, \lambda) & \varphi(x, \lambda) \\ \vartheta'(x, \lambda) & \varphi'(x, \lambda) \end{array} \right). \quad (1.2)$$

The matrix valued function $M$ is entire. An eigenvalue of $M(\lambda)$ is called a multiplier, i.e., it is a root of the algebraic equation $D(\tau, \lambda) = 0$, where

$$D(\tau, \lambda) := \det(M(\lambda) - \tau I_4), \quad \tau, \lambda \in \mathbb{C}. \quad (1.3)$$

There is an enormous literature on the scalar Hill operator including the inverse spectral theory [M], [GT], [KK]. In the recent paper [K1] one of the authors solved the inverse problem (including characterization) for the operator $-y'' + \nu y', \nu$ a function $\nu \in L^2(\mathbb{T})$ and $\nu'$ is the distribution and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. However, in spite of the importance of extending these studies to vector differential equations, apart from the information given by Lyapunov and Krein (see [YS]), until recently nothing essential has been done. The matrix potential poses interesting new problems: 1) to construct the Lyapunov function, 2) to define the quasimomentum as a conformal mapping, 3) to derive appropriate trace formulae (e.g. analog to the trace formulas in [K2]), 4) to obtain a priori estimates of potentials in terms of gap lengths, 5) to define and to study the integrated density of states. In fact this is the motivation of our paper.

The basic results in the direct spectral theory for the matrix case were obtained by Lyapunov [Ly] and Krein [Kr] (see also Gel’fand and Lidskii [GL]). Below we need the
following well-known results of Lyapunov [YS], which we formulate only for the case $2 \times 2$ matrices.

**Theorem (Lyapunov)** Let $V \in \mathcal{H}^1$.

i) If $\tau(\lambda)$ is a multiplier for some $\lambda \in \mathbb{C}$ (or $\lambda \in \mathbb{R}$), then $\tau^{-1}(\lambda)$ (or $\overline{\tau}(\lambda)$) is a multiplier too.

ii) $M(\lambda)$ has exactly four multipliers $\tau_1(\lambda), \tau_1^{-1}(\lambda), \tau_2(\lambda), \tau_2^{-1}(\lambda)$ for all $\lambda \in \mathbb{C}$. Moreover, $\lambda \in \sigma(T)$ iff $|\tau_m(\lambda)| = 1$ for some $m = 1, 2$.

iii) If $\tau(\lambda)$ is a simple multiplier and $|\tau(\lambda)| = 1$, then $\tau'(\lambda) \neq 0$.

We mention the papers relevant in our context, for various problems periodic systems we refer the reader to [CGHL]. In the paper [Ca1,2] Carlson obtained the trace formulas. In [Ca3] he proved the compactness of Floquet isospectral sets for the matrix Hill operator.

By the Lyapunov Theorem, each $M(\lambda), \lambda \in \mathbb{C}$ has exactly four multipliers $\tau_m(\lambda), \tau_m^{-1}(\lambda), m = 1, 2$, which are the roots of the characteristic polynomial $\det(M(\lambda) - \tau I_4) = 0$. If $c_0 = \frac{V_{20} - V_{10}}{2} > 0$, then the multipliers have the following asymptotics

$$\tau_m(\lambda) = e^{i(\sqrt{\lambda} - \frac{V_{10}}{2\sqrt{\lambda}} + O(1/\lambda))}, \quad |\lambda| \to \infty, \quad |\sqrt{\lambda} - \pi n| > \frac{\pi}{4}, \quad m = 1, 2,$$

(see Lemma 3.2). Next we define the functions

$$\mu_m(\lambda) = \frac{\text{Tr}}{4} M^m(\lambda), \quad m = 1, 2, \quad \rho = \frac{\mu_2 + 1}{2} - \mu_1^2, \quad \rho_0(\lambda) = c_0 \sin \sqrt{\lambda} \frac{2}{2\sqrt{\lambda}}.$$  

(1.5)

Note that $\varphi(x, \lambda), \vartheta(x, \lambda), \mu_1(\lambda), \mu_2(\lambda), \rho(\lambda)$ are real for all $x, \lambda \in \mathbb{R}$ and entire. If $c_0 = \frac{V_{20} - V_{10}}{2} > 0$, then Lemma 3.1 yields $\rho(\lambda) = \rho_0(\lambda)(1 + o(1))$, as $|\lambda| \to \infty$ in $D_1$, where

$$D_r := \left\{ \lambda \in \mathbb{C} : |\lambda| > r^2, |\sqrt{\lambda} - \pi n| > \frac{\pi}{4}, n \in \mathbb{N} \right\}, \quad r > 0.$$  

(1.6)

Thus we may define the analytic function $\sqrt{\rho(\lambda)}, \lambda \in D_r$, for large $r > 0$, by the requirement that $\sqrt{\rho(t)} = \rho_0(\lambda)(1 + o(1))$ as $|\lambda| \to \infty$ in $D_r$. Then there exists a unique analytic continuation of $\sqrt{\rho}$ from $D_r$ into the two sheeted Riemann surface $\Lambda$ (in general, of infinite genus) defined by $\sqrt{\rho}$. We now introduce our Lyapunov function $\Delta(\lambda)$ by

$$\Delta(\lambda) = \mu_1(\lambda) + \sqrt{\rho(\lambda)}, \quad \lambda \in \Lambda.$$  

(1.7)

Let $\Delta_1(\lambda) = \mu_1(\lambda) + \sqrt{\rho(\lambda)}$ on the first sheet $\Lambda_1$ and let $\Delta_2(\lambda) = \mu_1(\lambda) - \sqrt{\rho(\lambda)}$ on the second sheet $\Lambda_2$ of $\Lambda$. Now we formulate our main result concerning the function $\Delta$.

**Theorem 1.1.** Let $V \in \mathcal{H}^1$ with $V_{10} < V_{20}$. Then the function $\Delta = \mu_1 + \sqrt{\rho}$ is analytic on the two sheeted Riemann surface $\Lambda$ and has the following properties:

i) $\Delta_m(\lambda) = \frac{\tau_m(\lambda) + \tau_m^{-1}(\lambda)}{2}, \quad \lambda \in \Lambda_m, \quad m = 1, 2,$

(1.8)

$$\Delta_m(\lambda) = \cos \sqrt{\lambda} + V_{m0} \sin \sqrt{\lambda} \frac{\sin \sqrt{\lambda}}{2\sqrt{\lambda}} + O\left(\frac{1}{\sqrt{\lambda}}\right), \quad m = 1, 2, \quad \lambda \in \mathcal{D}_1.$$  

(1.9)
\( \lambda \in \mathbb{C} \) belongs to \( \sigma(T) \) iff \( \Delta_m(\lambda) \in [-1, 1] \) for some \( m = 1, 2 \).

ii) If \( \lambda \in \sigma(T) \), then \( \rho(\lambda) \geq 0 \).

iv) (The monotonicity property.) Let \( \Delta_m \) be real analytic on some interval \( I = (\alpha_1, \alpha_2) \subset \mathbb{R} \) and \( -1 < \Delta_m(\lambda) < 1 \), for any \( \lambda \in I \) for some \( m \in \{1, 2\} \). Then \( \Delta'_m(\lambda) \neq 0 \) for each \( \lambda \in I \).

Remark. i) For the scalar Hill operator the monodromy matrix has exactly two eigenvalues \( \tau \) and \( \tau^{-1} \). The Lyapunov function \( \frac{1}{\tau}(\tau + \tau^{-1}) \) is an entire function of the spectral parameter and it defines the band-gap structure of the spectrum. By Theorem 1.1, our Lyapunov function for the matrix Hill operator also defines the band-gap structure of the spectrum, but it is the sheeted analytic function.

ii) Consider the case of a diagonal potential, i.e. \( V_3 = 0 \). Then the Riemann surface degenerates into two components, and we get

\[
\mu_1 = \frac{1}{2}(\Delta(1) + \Delta(2)), \quad \rho = \frac{1}{4}(\Delta(1) - \Delta(2))^2, \quad \sqrt{\rho} = \frac{1}{2}(\Delta(1) - \Delta(2)),
\]

where \( \Delta_{(m)} \) is the Lyapunov function for the scalar Hill operator \( -y'' + V_m y \), \( m = 1, 2 \). Thus \( \Delta_1 = \mu_1 + \sqrt{\rho} = \Delta(1) \) and \( \Delta_2 = \mu_1 - \sqrt{\rho} = \Delta(2) \).

iii) A lot of papers are devoted to the resonances for the Schrödinger operator with compactly supported potentials on the real line, see [BKW],[K3],[S],[Z]. Assume that we have the case of a diagonal potential, i.e. \( V_3 = 0 \). Then the Riemann surface degenerates into two components, and we get

\[
\mu_1 = \frac{1}{2}(\Delta(1) + \Delta(2)), \quad \rho = \frac{1}{4}(\Delta(1) - \Delta(2)^2), \quad \sqrt{\rho} = \frac{1}{2}(\Delta(1) - \Delta(2)),
\]

where \( \Delta_{(m)} \) is the Lyapunov function for the scalar Hill operator \( -y'' + V_m y \), \( m = 1, 2 \). Thus \( \Delta_1 = \mu_1 + \sqrt{\rho} = \Delta(1) \) and \( \Delta_2 = \mu_1 - \sqrt{\rho} = \Delta(2) \).

Denote by \( \{r_n\}_n^\infty \) the sequence of zeros of \( \rho \) in \( \mathbb{C} \) (counted with multiplicity) such that \( 0 \leq |r_1| \leq |r_2| \leq |r_3| \leq \ldots \). We call these zeros of \( \rho \) the resonances of \( T \). We formulate the theorem about the recovering of the spectrum of \( T \) and the asymptotics of the periodic and anti-periodic eigenvalues and resonances at high energy. Furthermore, we write \( a_n = b_n + \ell^2(n) \) iff the sequence \( \{a_n - b_n\}_{n \geq 1} \in \ell^2 \). Recall that \( V_0, V_m, m = 1, 2 \) are defined by (1.15).

**Theorem 1.2.** Let \( V \in \mathcal{H}^2 \) with \( V_{20} - V_{10} > 0 \). Then the following asymptotics are fulfilled:

\[
\lambda_{n,m+k} = (\pi n)^2 + V_{m0} + \ell^2(n), \quad m, k = 1, 2, \quad n \to \infty,
\]

\[
r_{2n-m} = (\pi n)^2 + \ell^2(n), \quad m = 0, 1, \quad n \to \infty.
\]

ii) Let \( V \in \mathcal{H}^2, V_{20} - V_{10} > 0 \). Then the following statements hold

a) The periodic spectrum and the anti-periodic spectrum recover the resonances and the spectrum of the operator \( T \).

b) The periodic (anti-periodic) spectrum is recovered by the anti-periodic (periodic) spectrum and resonances.
Let a potential $q_{\gamma, \nu} = a J + \gamma v_\nu(x) J_1$ is a potential for some constants $a, \gamma \in \mathbb{R}$, $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $J_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $q_{\gamma, \nu}$ satisfies

**Condition A.** \( \frac{a}{2\pi^2} - n_a \in (0, 1) \) for some integer \( n_a \geq 0 \). Each function \( v_\nu \in L^1(\mathbb{T}), \nu = 1, \frac{1}{2}, \frac{1}{3}, \ldots \), is such that \( v_\nu(x) = v_\nu(1 - x), \ x \in (0, 1), \) and \( \int_0^1 v_\nu(x)dx = 1 \) and for any \( f \in C(0,k), k \in \mathbb{N} \), the following convergence holds:

$$
\int_0^k v_\nu(x)f(x)dx \to \int_0^k \delta_{\text{per}}(x)f(x)dx \quad \text{as} \quad \nu \to 0, \quad \delta_{\text{per}} = \sum_{-\infty}^{\infty} \delta(x - n - \frac{1}{2}). \quad (1.13)
$$

If \( \gamma = 0 \), then we have the operator \( T^0 = -y'' + q^0 \) with a constant potential \( q^0 = aJ \). In this case there are no gaps in the spectrum. The fundamental solutions of Eq.\((1.1)\) with \( q^0 = aJ \) have the form \( \vartheta^0 = \text{diag}(c_+, c_-) \), \( \varphi^0 = \text{diag}(s_+, s_-) \), where

$$
c_{\pm}(x, \lambda) = \cos \eta_{\pm} x, \quad s_{\pm}(x, \lambda) = \frac{\sin \eta_{\pm} x}{\eta_{\pm}}, \quad \eta_{\pm} = \sqrt{\lambda \mp a}, \quad \text{diag}(b, c) = \begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix}
$$

and the branch of \( \sqrt{\lambda} \) is given by \( \sqrt{\lambda} = 1 \). Below we sometimes will write \( \rho(\lambda, V), M(\lambda, V), \ldots \), instead of \( \rho(\lambda), M(\lambda), \ldots \), when several potentials are being dealt with. Then the functions \( \rho^0(\cdot) = \rho(\cdot, q^0), \mu^0_m(\cdot) = \mu_m(\cdot, q^0), \ldots \) corresponding to \( q^0 \) have the forms

$$
\mu^0_m(\lambda) = \frac{c_+(m, \lambda) + c_-(m, \lambda)}{2}, \quad m = 1, 2, \quad \rho^0(\lambda) = \frac{(c_+(1, \lambda) - c_-(1, \lambda))^2}{4}, \quad (1.14)
$$

Thus all roots of \( \rho^0 = \rho(\cdot, q^0) \) have multiplicity 2 and are given by \( \Delta^0_{(m)}(\lambda) = c_+(1, \lambda), \quad \Delta^0_{(2)}(\lambda) = c_-(1, \lambda) \). Here \( \Delta^0_{(m)} \) is the Lyapunov function for the Eq. \(-y'' - (-1)^m ay = \lambda y, m = 1, 2 \). The entire function \( c_+(1, \lambda) - c_-(1, \lambda) \) has only the simple roots \( z^0_n \) given by

$$
z^0_n = (\pi n)^2 + \frac{a^2}{(2\pi n)^2} = \pm a + (\pi n \pm \frac{a}{2\pi n})^2, \quad n \geq 1. \quad (1.16)
$$

Note that \( z^0_1 > z^0_2 > \ldots > z^0_n \) and \( z^0_{n+1} < z^0_{n+2} < \ldots \). Thus all roots of \( \rho^0 = \rho(\cdot, q^0) \) have multiplicity 2 and are given by \( \lambda^0_{m,n} = a - (-1)^m (\pi n)^2, \quad \lambda^0_{m,n} \neq \lambda^0_{2,s}, \quad m = 1, 2, \quad n, s \geq 0 \). \( (1.17) \)

Note that \( \lambda^0_{m,n}, m = 1, 2, n \geq 0 \) are the roots of \( D^0_{\pm}(\lambda) \). If \( \gamma, \nu \neq 0 \) are small enough, then there exist gaps in the spectrum of \( T_{\gamma, \nu} \). Define the disk \( \mathbb{D}_r = \{ \lambda : |\lambda| < \pi^2 r^2 \}, r > 0 \). We show that there exist the non-degenerated resonance gaps for some \( V \). In this example some resonances \( r_n \in \mathbb{R} \) and some \( r_n \notin \mathbb{R} \).

**Proposition 1.3.** Let a potential \( q_{\gamma, \nu} = aJ + \gamma v_\nu(x) J_1, a > 0, \gamma \in \mathbb{R}, \) satisfy Condition \( A \), and let \( \frac{4a}{\pi^2} \notin \mathbb{N} \). Then for any \( N \geq 1 + a \) there exist small \( \varepsilon, \varepsilon_1, \varepsilon_2 > 0 \) such that
in general, they are not the periodic (or anti-periodic) eigenvalues. These endpoints are the branch points of the Lyapunov function, and, therefore, we show the existence of new gaps (so-called resonance gaps).

Remark. i) If $0 < a < 2\pi^2$, then $\rho(\lambda, q_{\gamma,\nu})$ has only real roots $z^\pm_n(q_{\gamma,\nu})$ in the disk $\mathbb{D}_{N+\frac{1}{2}}$ for small $\gamma, \nu$. If $a > 2\pi^2$, then $\rho(\lambda, q_{\gamma,\nu})$ has at least two non-real roots $z^\pm_n(q_{\gamma,\nu})$ for small $\gamma, \nu$.

ii) We show that the operator $T_{\gamma,\nu}$ has the gaps associated with the periodic or anti-periodic spectrum. Moreover, we show the existence of new gaps (so-called resonance gaps). The endpoints of the resonance gap are the branch points of the Lyapunov function, and, in general, they are not the periodic (or anti-periodic) eigenvalues. These endpoints are not stable. If they are real ($n = n_a + 1, \ldots, N$), then we have a gap. If they are complex ($1 \leq n \leq n_a$), then we have not a gap, we have only the branch points of the Lyapunov function in the complex plane.

2 Fundamental solutions

In this section we study $\vartheta, \varphi$. We begin with some notational convention. A vector $h = \{h_n\}_1^N \in \mathbb{C}^N$ has the Euclidean norm $|h|^2 = \sum_1^N |h_n|^2$, while a $N \times N$ matrix $A$ has the operator norm given by $|A| = \sup_{|h|=1} |Ah|$. The function $\varphi$ satisfies the following integral equations

$$\varphi(x, \lambda) = \varphi_0(x, \lambda) + \int_0^x \frac{\sin \sqrt{\lambda}(x-t)}{\sqrt{\lambda}} V(t) \varphi(t, \lambda) dt, \quad \varphi_0(x, \lambda) = \frac{\sin \sqrt{\lambda}x}{\sqrt{\lambda}} I_2,$$

where $(x, \lambda) \in \mathbb{R} \times \mathbb{C}$. The standard iterations in \cite{24} yields

$$\varphi(x, \lambda) = \sum_{n \geq 0} \varphi_n(x, \lambda), \quad \varphi_{n+1}(x, \lambda) = \int_0^x \frac{\sin \sqrt{\lambda}(x-t)}{\sqrt{\lambda}} V(t) \varphi_n(t, \lambda) dt. \tag{2.2}$$

The similar expansion $\vartheta = \sum_{n \geq 0} \vartheta_n$ with $\vartheta_0(x, \lambda) = (\cos \sqrt{\lambda}x) I_2$ holds. We introduce the functions

$$I^0_m(\lambda) = \int_0^m dt \int_0^t \cos \sqrt{\lambda}(m-2t+2s) F(t, s) ds, \quad F(t, s) = \text{Tr} V(t) V(s), \tag{2.3}$$

$m = 1, 2$. In Lemma 2.1 we shall show the simple identity

$$I^0_2(\lambda) = 4 I^1_1(\lambda) \cos \sqrt{\lambda}. \tag{2.4}$$

We define $|\lambda|_1 \equiv \max\{1, |\lambda|\}$ and

$$V(0) = \int_0^1 V(x) dx, \quad V(1) = \text{Tr} V(0), \quad V(2) = \text{Tr} V_0^2, \quad A = e^{\text{Im} \sqrt{\lambda} x}, \quad x = \frac{||V||_1}{\sqrt{|\lambda|_1}}. \tag{2.5}$$
We prove

**Lemma 2.1.** For each \((x, V) \in \mathbb{R}_+ \times \mathcal{H}^1\) the functions \(\varphi(x, \cdot), \vartheta(x, \cdot)\) are entire and for any \(N \geq -1\) the following estimates are fulfilled:

\[
\max \left\{ \left| \vartheta(x, \lambda) - \sum_{n=0}^{N} \vartheta_n(x, \lambda) \right|, \left| \sqrt{\lambda} \left( \varphi(x, \lambda) - \sum_{n=0}^{N} \varphi_n(x, \lambda) \right) \right|, \left| \frac{1}{\sqrt{\lambda}} \left( \vartheta'(x, \lambda) - \sum_{n=0}^{N} \vartheta'_n(x, \lambda) \right) \right| \right\} \leq \frac{(x \lambda)^{N+1}}{(N+1)!} A^x. \tag{2.6}
\]

Moreover, each \(\mu_n(\lambda), m = 1, 2\) is real for \(\lambda \in \mathbb{R}\) and entire and the following estimates are fulfilled:

\[
|\mu_m(\lambda)| \leq A^m, \quad |\mu_m(\lambda) - \cos mz| \leq m \lambda A^m, \tag{2.7}
\]

\[
|\mu_m(\lambda) - \cos mz - \frac{\sin mz}{4z} mV(1)| \leq \frac{(m \lambda)^2}{2} A^m, \tag{2.8}
\]

\[
|\mu_m(\lambda) - \cos mz - \frac{\sin mz}{4z} mV(1) - \frac{1}{8z^2} \left( I_m(\lambda) - \frac{m^2}{2} \cos mz V(2) \right)| \leq \frac{(m \lambda)^3}{3!} A^m, \tag{2.9}
\]

where \(I_0, I_2\) satisfy (2.4) and \(z = \sqrt{\lambda}\).

**Proof.** We prove the estimates of \(\varphi\), the proof for \(\varphi', \vartheta, \vartheta'\) is similar. \(2.2\) gives

\[
\varphi_n(x, \lambda) = \int_{0 < x_1 < x_2 < \ldots < x_{n+1}} \varphi_0(x_1, \lambda) \left( \prod_{1 \leq k \leq n} \frac{\sin(\sqrt{\lambda}(x_{k+1} - x_k))}{\sqrt{\lambda}} V(x_k) \right) dx_1 dx_2 \ldots dx_n, \tag{2.10}
\]

where for matrices \(a_1, a_2, \ldots, a_n\) we denote \(\prod_{1 \leq k \leq n} a_k = a_n a_{n-1} \ldots a_1\). Substituting the estimate

\[
|\sqrt{\lambda} \varphi_0(x, \lambda)| \leq e^{\text{Im} \sqrt{\lambda} x}
\]

into (2.10) we obtain

\[
|\sqrt{\lambda} \varphi_0(x, \lambda)| \leq \frac{(x \lambda)^n}{n!} e^{\text{Im} z m}
\]

which shows that for each \(x \geq 0\) the series (2.2) converges uniformly on bounded subsets of \(\mathbb{C}\). Each term of this series is an entire function. Hence the sum is an entire function. Summing the majorants we obtain estimates (2.6).

We have

\[
4 \mu_m = \text{Tr } M^m(\lambda) = \text{Tr } M(m, \lambda) = \text{Tr } \sum_{n \geq 0} M_n(m, \lambda), \quad \text{where } m = 1, 2
\]

\[
\text{Tr } M_0(m, \lambda) = 4 \cos mz, \quad \text{Tr } M_n(m, \lambda) = \text{Tr } \vartheta_n(m, \lambda) + \text{Tr } \vartheta'_n(m, \lambda), \quad n \geq 1. \tag{2.11}
\]

The estimates

\[
|\vartheta'_n(m, \lambda)| \leq \frac{(x \lambda)^n}{m!} e^{\text{Im} z m} \quad \text{and} \quad |\vartheta_n(m, \lambda)| \leq \frac{(x \lambda)^n}{m!} e^{\text{Im} z m}
\]

yield

\[
|\text{Tr } M_n(m, \lambda)| \leq 4 \frac{(x \lambda)^n}{m!} e^{m |\text{Im} \sqrt{\lambda}|}, \quad n \geq 0. \tag{2.12}
\]

Using (2.11) we obtain

\[
\text{Tr } M_1(m, \lambda) = \frac{1}{z} \int_0^m (\sin z(m-t) \cos z t + \cos z(m-t) \sin z t) \text{Tr } V(t) dt = \frac{\sin mz}{z} mV(1). \tag{2.13}
\]
and
\[
\text{Tr } M_2(m, \lambda) = \frac{1}{z^2} \int_0^m \int_0^t \sin(z(t-s))z(m-t-s)F(t,s)dtds.
\]
\[
= \frac{1}{2z^2} \int_0^m \int_0^t (\cos(z(m-2t+2s)) - \cos(zm))F(t,s)dtds = \frac{1}{2z^2} (I_0^0(\lambda) - \frac{m^2}{2} \cos mz V(2)),
\]
since
\[
\int_0^m \int_0^t F(t,s)dtds = \frac{1}{2} \text{Tr} \left( \int_0^m V(t)dt \right) = \frac{m^2}{2} V(2).
\]
We have that \(\mu_1, \mu_2\) are entire. Moreover, the trace of the monodromy matrix is a sum of its eigenvalues. By Lyapunov Theorem (see Sect.1), the set of these eigenvalues is symmetric with respect to the real axis, as \(\lambda \in \mathbb{R}\). Hence \(\mu_1, \mu_2\) are real on \(\mathbb{R}\).

We show (2.3). Let \(I_0^0 = I_0^0(\lambda)\). We have 2 cos \(z I_1^0 = Y_0 + Y_1\), where
\[
Y_0 = \int_0^1 \int_0^t \cos 2z(t-s) F(t,s)dtds, \quad Y_1 = \int_0^1 \int_0^t \cos 2z(1-t+s) F(t,s)dtds.
\]
We get \(I_2^0 = Y_1 + Y_2 + Y_3\), where
\[
Y_2 = \int_1^2 \int_1^t \cos 2z(1-t+s) F(t,s)dtds, \quad Y_3 = \int_1^2 \int_1^t \cos 2z(1-t+s) F(t,s)dtds
\]
and using the new variable \(\tau = t-1\) we get \(Y_2 = \int_1^1 \int_0^\tau \cos 2z(\tau-s) F(\tau,s)d\tau ds = 2Y_0\). We use the new variables \(\tau = t-1, \sigma = s-1\) and obtain \(Y_3 = \int_0^1 \int_0^\tau \cos 2z(1-\tau+\sigma) F(\tau,\sigma)d\tau d\sigma = Y_1\), which yields \(I_2^0 = 2Y_1 + 2Y_0\). Thus we have (2.4). \(\blacksquare\)

We need the basic properties of the monodromy matrix.

**Lemma 2.2** Let \(V \in \mathcal{H}^1\). Then the function \(D(\tau, \lambda) = \det(M(\lambda) - \tau I_4)\), \(\lambda, \tau \in \mathbb{C}^2\) is entire on \(\mathbb{C}^2\) and the following identities are fulfilled:
\[
D'_\tau(\tau, \lambda) = -D(\tau, \lambda) \text{Tr} \left( M(\lambda) - \tau I_4 \right)^{-1}, \quad (2.14)
\]
\[
D(\tau, \cdot) = \tau^4 - 4\mu_1 \tau^3 + 2(4\mu_1^2 - \mu_2) \tau^2 - 4\mu_1 \tau + 1
\]
\[
= \left( \tau^2 - 2(\mu_1 - \sqrt{\rho}) \tau + 1 \right) \left( \tau^2 - 2(\mu_1 + \sqrt{\rho}) \tau + 1 \right). \quad (2.15)
\]

**Proof.** Let \(D(\tau) \equiv D(\tau, \lambda)\). The standard identity \(D'(\tau) = D(\tau) \text{Tr} ((M-\tau)\frac{d(M-\tau)}{d\tau})\) yields (2.14). We prove (2.15). Since \(\det M = D(0) = 1\), we have: \(D(\tau) = 1 + a\tau + b\tau^2 + c\tau^3 + d\tau^4, a, b, c, d \in \mathbb{C}\). Then \(D(\tau) = (\tau - \tau_1)(\tau - \tau_1^{-1})(\tau - \tau_2)(\tau - \tau_2^{-1})\), where \(\tau_1, \tau_1^{-1}, \tau_2, \tau_2^{-1}\) are the multipliers. Therefore \(d = 1, a = c\). Then we have
\[
D(\tau) = 1 + D'(0)\tau + \frac{1}{2}D''(0)\tau^2 + D'(0)\tau^3 + \tau^4. \quad (2.16)
\]
Identity (2.14) yields \(D'(0) = -\text{Tr } M^{-1} = -\text{Tr } M = -4\mu_1\). Differentiating (2.14) we obtain \(D''(0) = -D'(0) \text{Tr}(M-\tau)^{-1} - D(\tau) \text{Tr}(M-\tau)^{-2} \big|_{\tau = 0} = 4(4\mu_1^2 - \mu_2)\). Substituting these identities into (2.16) we obtain the first identity in (2.15). The second identity is proved by direct calculation. \(\blacksquare\)
3 The Lyapunov function

We need some results about the functions $\rho$, $\rho_0 = c_0 \frac{\sin \sqrt{\lambda}}{2\sqrt{\lambda}}$, where $c_0 = \frac{1}{2}(V_{20} - V_{10})$.

**Lemma 3.1.** i) For each $V \in \mathcal{H}^1$ the function $\rho = \frac{1}{2}(\mu_2 + 1) - \mu_1^2$ is entire and real on the real line. Moreover, the following estimate is fulfilled:

$$|\rho(\lambda) - \rho_0^2(\lambda)| \leq 2 \mu^3 e^{\text{Im}\sqrt{\lambda}} + 2 \mu, \quad \lambda \in \mathbb{C}, \quad \mu = \|V\|_1/\sqrt{\|\lambda\|_1}. \quad (3.1)$$

ii) Let $V \in \mathcal{H}^1$ with $c_0 = \frac{1}{2}(V_{20} - V_{10}) > 0$. Then for each integer $N > \frac{27\|V\|_1^2}{c_0^3}$ the function $\rho(\lambda)$ has exactly $2N$ roots, counted with multiplicity, in the disk $\{\lambda : |\lambda| < \pi^2(N + \frac{1}{2})^2\}$ and for each $n > N$, exactly two roots, counted with multiplicity, in the domain $\{\lambda : |\sqrt{\lambda} - \pi n| < 1\}$. There are no other roots.

iii) Let $V \in \mathcal{H}^1, c_0 > 0$. Then the function $\sqrt{\rho}$ is an analytic function in the domain $D_r, r = \frac{2\|V\|_1^3}{c_0^5}$ given by (1.6) and the following estimate is fulfilled:

$$|\sqrt{\rho(\lambda)} - \rho_0(\lambda)| \leq \frac{3C_0 |\rho_0(\lambda)|}{5 \sqrt{|\lambda|}}, \quad C_0 = 4^4 \|V\|_1^3 c_0^5 < \frac{\sqrt{|\lambda|}}{2}, \quad \lambda \in D_r. \quad (3.2)$$

**Proof.** i) By Lemma 2.2, $\rho$ is entire and real on the real line. Let $\mu_m \equiv \mu_{m0} + \mu_{m1} + \mu_{m2} + \tilde{\mu}_{m3}$, $m = 1, 2$, where

$$\mu_{m0} = \cos mz, \quad \mu_{m1} = \frac{\sin mz}{4z} mV_{(1)}, \quad \mu_{m2} = \frac{1}{8z^2} (I_m^0(\lambda) - \frac{m^2}{2} \cos mzV_{(2)}), \quad (3.3)$$

and $z = \sqrt{\lambda}$, where $V_{(m)}$ is defined by (2.5). We obtain $\mu_1^2 = B_1 + B_2$, where

$$B_1 = (\mu_{10} + \mu_{11})^2 + 2\mu_{10}\mu_{12}, \quad B_2 = 2\mu_{10}\tilde{\mu}_{13} + (\mu_{1} - \mu_{10} + \mu_{11})(\mu_{1} - \mu_{10} - \mu_{11}).$$

Then (3.3) yields

$$B_1 = \cos^2 z + \frac{\sin 2z}{4z} V_{(1)} + \frac{4 \cos z I_1^0(\lambda) + \sin^2 zV_{(1)}^2 - 2 \cos^2 zV_{(2)}}{16z^2}.$$ 

Thus we get

$$\rho = \frac{\mu_2 + 1}{2} - \mu_1^2 = G_1 + G_2, \quad G_1 = \frac{1 + \mu_{20} + \mu_{21} + \mu_{22}}{2} - B_1, \quad G_2 = \frac{\tilde{\mu}_{23}}{2} - B_2,$$

$$G_1 = \frac{I_0^0(\lambda) - 2 \cos 2zV_{(2)} - 4 \cos z I_1^0(\lambda) + 2 \cos^2 zV_{(2)} - \sin^2 zV_{(1)}^2}{16z^2} = \frac{\sin^2 z}{16z^2} (2V_{(2)} - V_{(1)}^2),$$

which yields $G_1 = \rho_0^2$. Using (2.7)–(2.9) we obtain

$$|B_2| \leq 2 \frac{\mu^3}{3!} A^2 + (\mu A + \mu A) \frac{\mu^2}{2!} A^2 = \frac{4\mu^3}{3} A^2,$$
\[ |G_2| \leq \frac{|\bar{\rho}_{m3}|}{2} + |B_2| \leq \frac{4(\pi)^3}{3!} A^2 + \frac{4\pi^3}{3} A^2 = 2\pi^3 A^2, \quad (3.4) \]

which yields (3.1).

ii) Let \( N' > N \) be another integer and \( r = \pi(N + \frac{1}{2}) \). Note that \( \pi \leq \frac{V_0^2}{2r_0\pi^2} \leq \frac{1}{2\pi} \) for \( \lambda \in \mathcal{D}_r \). Using the estimate \( e^{\Im \sqrt{\lambda}} < 4 \sin \sqrt{\lambda}, \lambda \in \mathcal{D}_r \) and (3.1) we obtain (on all contours)

\[ |\rho(\lambda) - \rho_0^2(\lambda)| \leq 2 \frac{\|V\|^2}{|z|^3} e^{2(\Im z + \pi\lambda)} \leq \frac{C_0}{\sqrt{|\lambda|}} |\rho_0^2(\lambda)|, \quad C_0 \equiv 4 \frac{\|V\|^3}{c_0^2} < \frac{\sqrt{|\lambda|}}{2}, \lambda \in \mathcal{D}_r. \quad (3.5) \]

Hence, by the Rouché theorem, \( \rho(\lambda) \) has as many roots, counted with multiplicity, as \( \frac{\sin^2 \sqrt{\lambda}}{\lambda} \) in the bounded domain \( \mathcal{D}_r \cap \{ \lambda : |\lambda| \leq \pi(N' + \frac{1}{2}) \} \). Since \( \frac{\sin^2 \sqrt{\lambda}}{\lambda} \) has exactly one double root at \((\pi n)^2, n \geq 1\), and since \( N' > N \) can be chosen arbitrarily large, ii) follows.

iii) Let \( \rho \equiv \rho_0^2 + \rho_1, \rho_0 = c_0 \frac{\sin \sqrt{\lambda}}{2\sqrt{\lambda}} \). Estimates (3.5) imply

\[ \sqrt{\rho(\lambda)} = \rho_0(\lambda) \sqrt{1 + b(\lambda)}, \quad b = \frac{\rho_1}{\rho_0}, \quad |b(\lambda)| \leq \frac{C_0}{\sqrt{|\lambda|}} < \frac{1}{2}, \lambda \in \mathcal{D}_r. \quad (3.6) \]

Using the estimate \( |\sqrt{1 + b(\lambda)} - 1| \leq \frac{3}{2} |b(\lambda)| \) for \( \lambda \in \mathcal{D}_r \), we obtain

\[ |\sqrt{\rho(\lambda)} - \rho_0(\lambda)| = |\rho_0(\lambda)(\sqrt{1 + b(\lambda)} - 1)| \leq \frac{3C_0 |\rho_0(\lambda)|}{5 \sqrt{|\lambda|}}. \]

We need asymptotics of the eigenvalues of the monodromy matrix.

**Lemma 3.2.** Let \( V \in \mathcal{H}_1, V_0 > 0 \). Then the monodromy matrix \( M(\lambda) \) has two multipliers \( \tau_m(\lambda), m = 1, 2 \) with asymptotics \((1.4)\). Moreover, asymptotics \((1.9)\) is fulfilled.

**Proof.** Using estimates (2.3), (3.2) we obtain asymptotics \((1.9)\). Asymptotics \((1.9)\) yield

\[ \Delta_m^2(\lambda) - 1 = \left( \cos \sqrt{\lambda} + V_{m0} \frac{\sin \sqrt{\lambda}}{2\sqrt{\lambda}} \right)^2 - 1 + O(E(\lambda)) \]

\[ = -\sin^2 \sqrt{\lambda} + V_{m0} \frac{\sin \sqrt{\lambda} \cos \sqrt{\lambda}}{\sqrt{\lambda}} + O(E(\lambda)) = -\sin^2 \sqrt{\lambda} \left( 1 - V_{m0} \frac{\cos \sqrt{\lambda}}{\sqrt{\lambda} \sin \sqrt{\lambda}} \right) + O(E(\lambda)) \]

as \( \lambda \in \mathcal{D}_1, |\lambda| \to \infty \), where \( E(\lambda) = \frac{e^{i\Im \sqrt{\lambda}}}{\lambda} \), which implies

\[ \sqrt{\Delta_m^2(\lambda) - 1} = i \sin \sqrt{\lambda} - i V_{m0} \frac{\cos \sqrt{\lambda}}{2\sqrt{\lambda}} + O(E(\lambda)). \quad (3.7) \]

By (2.15), the matrix \( M(\lambda), \lambda \in \mathcal{D}_r \), has the eigenvalues \( \tau_m(\lambda) \) satisfying the identities \( \tau_m(\lambda) + \tau_m(\lambda)^{-1} = 2\Delta_m(\lambda) \). Then \( \tau_m(\lambda) \) has the form \( \tau_m(\lambda) = \Delta_m(\lambda) + \sqrt{\Delta_m^2(\lambda) - 1} \) and the asymptotics give

\[ \tau_m(\lambda) = e^{iv\sqrt{\lambda}} + V_{m0} \frac{\sin \sqrt{\lambda}}{2\sqrt{\lambda}} - i V_{m0} \frac{\cos \sqrt{\lambda}}{2\sqrt{\lambda}} + O(E(\lambda)) = e^{iv\sqrt{\lambda}} \left( 1 - \frac{i V_{m0}}{2\sqrt{\lambda}} \right) + O(E(\lambda)) \]
which yields (1.3).

Now we prove our first result about the Lyapunov function \( \Delta = \mu_1 + \sqrt{\rho} \).

**Proof of Theorem 1.1.** From Lemma 3.1 we obtain the analytic function \( \Delta \) on the Riemann surface of the function \( \sqrt{\rho} \).

i) Identity (2.13) shows that \( \Delta_m = \frac{\tau + \tau^{-1}}{2} \) for some multiplier \( \tau \). Lemma 3.2 gives the asymptotics of \( \Delta_m \) and \( \tau_m, m = 1, 2 \).

ii) The result follows from the statement i) and the Lyapunov Theorem (see Sect.1).

iii) If \( \lambda \in \sigma(T) \), then \( \mu_1(\lambda) \) is real. By ii), \( \Delta(\lambda) \) is also real. Hence by (1.7), \( \sqrt{\rho(\lambda)} \) is real and \( \rho(\lambda) \geq 0 \).

iv) Assume that \( \Delta_m' (\lambda_0) = 0 \) for some \( \lambda_0 \in I \subset \sigma(T) \) and \( m \in \{1, 2\} \). Then we have

\[
\Delta_m(\lambda) = \Delta_m(\lambda_0) + (\lambda - \lambda_0)p^{(\Delta_m(\lambda_0))} + O(|\lambda - \lambda_0|^{p+1}), \quad \text{as} \quad \lambda - \lambda_0 \to 0,
\]

where \( \Delta_m^{(p)}(\lambda_0) \neq 0 \) for some \( p \geq 1 \). By the Implicit Function Theorem, there exists some curve \( \Gamma \subseteq \{ \lambda : \lambda - \lambda_0 < \varepsilon \} \cap \mathbb{C}_+ \), \( \Gamma \neq \emptyset \), for some \( \varepsilon > 0 \) such that \( \Delta_m(\lambda) \in (-1, 1) \) for any \( \lambda \in \Gamma \). Thus we have a contradiction with the Lyapunov Theorem in Sect. 1. ■

Recall that \( D_{\pm}(\lambda) = \frac{1}{4} \text{det}(M(\lambda) \mp I_4) \), the set \( \{ \lambda : D_{\pm}(\lambda) = 0 \} \) is a periodic spectrum and the set \( \{ \lambda : D_{-}(\lambda) = 0 \} \) is an anti-periodic spectrum. Now we prove a lemma about the number of periodic and anti-periodic eigenvalues in a large disc.

**Lemma 3.3.** For each \( V \in H^1 \) the functions \( \Delta_1 + \Delta_2, \Delta_1\Delta_2, D_{\pm} \) are entire and satisfy the following identities:

\[
\Delta_1^2 + \Delta_2^2 = 1 + \mu_2, \quad \Delta_1\Delta_2 = 2\mu_1^2 - \frac{\mu_2 + 1}{2},
\]

\[
D_{\pm} = (\mu_1 \mp 1)^2 - \rho = \frac{(2\mu_1 \mp 1)^2 - \mu_2}{2}, \quad D_{+} - D_{-} = -4\mu_1.
\]

Let in addition \( \text{Tr} \int_0^1 V(t)dt = 0 \). Then the following estimates and properties are fulfilled:

\[
\max \left\{ \left| D_{\pm}(\lambda) - 4\sin^4 \frac{\sqrt{\lambda}}{2} \right|, \left| D_{-}(\lambda) - 4\cos^4 \frac{\sqrt{\lambda}}{2} \right| \right\} \leq \frac{\kappa^2}{2}(2+\kappa)^2 e^{2|\text{Im} \sqrt{\lambda}|} e^{\frac{1}{2}} \kappa \quad \text{as} \quad \lambda \in \mathbb{C}.
\]

i) For each integer \( N > 8\|V\|_1 \) the function \( D_{\pm} \) has exactly \( 4N + 2 \) roots, counted with multiplicity, in the disc \( \{ |\lambda| < 4\pi^2(N + \frac{1}{2})^2 \} \) and for each \( n > N \), exactly four roots, counted with multiplicity, in the domain \( \{ |\sqrt{\lambda} - 2\pi n| < \frac{\pi}{2} \} \). There are no other roots.

ii) For each integer \( N > 8\|V\|_1 \) the function \( D_{\pm} \) has exactly \( 4N \) roots, counted with multiplicity, in the disc \( \{ |\lambda| < 4\pi^2N^2 \} \) and for each \( n > N \), exactly four roots, counted with multiplicity, in the domain \( \{ |\sqrt{\lambda} - \pi(2n + 1)| < \frac{\pi}{2} \} \). There are no other roots.

**Proof.** By Lemma 2.1, 2.2 the functions \( \Delta_1 + \Delta_2, \Delta_1\Delta_2, D_{\pm} \) are entire and identities (3.9), (3.10) are fulfilled. Using Lemmas 2.1 and 3.1 we obtain

\[
\mu_1(\lambda) = \cos z + \mu_{12}(\lambda), \quad |\mu_{12}(\lambda)| \leq \frac{\kappa^2}{2} A^2, \quad |\rho(\lambda)| \leq (1 + 2\kappa)\frac{\kappa^2}{2} A^2.
\]
Substituting these relations into $D_{\pm} = (\mu_1 \mp 1)^2 - \rho$ we get (3.11).
i) Let $N' > N$ be another integer. Let $\lambda$ belong to the contours $C_0(2N + 1), C_0(2N' + \frac{1}{2}), C_2n(\frac{1}{2}), n > N$, where $C_n(r) = \{ \sqrt{\lambda} : |\sqrt{\lambda} - \pi n| = \pi r \}, r > 0$. Note that $\pi \leq \frac{1}{16\pi}$ on all contours. Then (3.11) and the estimate $e^{\frac{1}{2} \Im \sqrt{\lambda}} < 4 |\sin \frac{\sqrt{\lambda}}{2}|$ on all contours yield

$$
\left| D_{+}(\lambda) - 4 \sin^{4} \frac{\sqrt{\lambda}}{2} \right| \leq \frac{e^{2\pi r} (2 + \rho)^2}{4^{4} \pi n} e^{2|\Im \sqrt{\lambda}|} < \frac{1}{4} \left| 4 \sin^{4} \frac{\sqrt{\lambda}}{2} \right|.
$$

Hence, by Rouché’s theorem, $D_{+}(\lambda)$ has as many roots, counted with multiplicities, as $\sin^{4} \frac{\sqrt{\lambda}}{2}$ in each of the bounded domains and the remaining unbounded domain. Since $\sin^{4} \frac{\sqrt{\lambda}}{2}$ has exactly one root of the multiplicity four at $(2\pi n)^{2}$, and since $N' > N$ can be chosen arbitrarily large, i) follows. The proof for $D_{-}$ is similar. 

We are ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** i) It is enough to consider the case $V_{10} = c_0 = -V_{20}$. We prove asymptotics (11) for $\lambda_{2n,m}, 1 \leq m \leq 4$. The proof for $\lambda_{2n+1,m}$ is similar. Firstly, we prove the rough asymptotics of $\lambda_{2n,m}, r_{n}$. Lemma 3.3.i yields $\sqrt{\lambda_{2n,m}} = 2\pi n + \varepsilon_{n}, |\varepsilon_{n}| < 1$, as $n \to \infty$. By Lemma 3.3, $D_{+}(\lambda) = 4 \sin^{4} \frac{\sqrt{\lambda}}{2} + O(n^{-2})$ as $|\sqrt{\lambda} - \pi n| \leq 1, n \to \infty$. Then identity $D_{+}(\lambda_{2n,m}) = 0$ implies

$$
\sqrt{\lambda_{2n,m}} = 2\pi n + \varepsilon_{n}, \quad \varepsilon_{n} = O(n^{-1/2}), \quad 1 \leq m \leq 4.
$$

(3.12)

Lemma 3.1.i implies $\sqrt{r_{2n-m}} = \pi n + \delta_{n}, |\delta_{n}| < 1$ for $n \to \infty, m = 0, 1$. Moreover, Lemma 3.1.i gives $\rho(\lambda) = c_0^{2} \sin^{2} \sqrt{\lambda} + O(n^{-3}), |\sqrt{\lambda} - \pi n| \leq 1$ as $n \to \infty$. Since $\rho(r_{n}) = 0$, we have

$$
\sqrt{r_{2n-m}} = \pi n + \delta_{n}, \quad \delta_{n} = O(n^{-1/2}), \quad m = 0, 1.
$$

(3.13)

Secondly, in order to improve these asymptotics of $\lambda_{2n,m}, \lambda_{n}$ we need asymptotics of the multipliers in a neighborhood of the points $\pi n$. We introduce the matrix $\tilde{M} = U^{-1}MU$ with the same eigenvalues, where $U = \begin{pmatrix} I_2 & 0 \\ 0 & \sqrt{\lambda} I_2 \end{pmatrix}$. We shall show the asymptotics

$$
\tilde{M}(\lambda) = \tilde{M}_0(\lambda) + \ell_1^2(n), \quad \tilde{M}_0(\lambda) = \begin{pmatrix} C(\lambda) & S(\lambda) \\ -S(\lambda) & C(\lambda) \end{pmatrix}, \quad \sqrt{\lambda} = \pi n + O(n^{-1/2}),
$$

(3.14)

where

$$
C(\lambda) = \text{diag}(\cos \eta_{+}, \cos \eta_{-}), \quad S(\lambda) = \text{diag}(\sin \eta_{+}, \sin \eta_{-}), \quad \eta_{\pm} = \sqrt{\lambda \mp c_0}.
$$

(3.15)

Estimate (2.6) gives

$$
\vartheta(1, \lambda) = \cos \sqrt{\lambda} I_2 + \frac{1}{\sqrt{\lambda}} \int_{0}^{1} \sin \sqrt{\lambda}(1 - t) \cos \sqrt{\lambda} t V(t) dt + O(n^{-2})
$$

$$
= \cos \sqrt{\lambda} I_2 + \frac{1}{2\sqrt{\lambda}} \int_{0}^{1} \left( \sin \sqrt{\lambda} + \sin \sqrt{\lambda}(1 - 2t) \right) V(t) dt + O(n^{-2}), \quad |\sqrt{\lambda} - \pi n| \leq 1,
$$

(3.16)
as \( n \to +\infty \). Let \( \sqrt{\lambda} = \pi n + u_n, u_n = O(n^{-1/2}) \). The Taylor formula gives \( \sin 2t(\pi n + u_n) = \sin 2\pi nt + 2tu_n \cos 2\pi nt + O(n^{-1}) \) and the similar formula for \( \cos 2t(\pi n + u_n) \). Substituting these asymptotics into (3.18) we obtain

\[
\vartheta(1, \lambda) = \cos \sqrt{\lambda} I_2 + \frac{\sin \sqrt{\lambda}}{2\sqrt{\lambda}} \varepsilon_0 J + \ell_1^2(n) = C(\lambda) + \ell_1^2(n), \quad \sqrt{\lambda} = \pi n + O(n^{-1/2}).
\]

Similar arguments for \( \varphi(1, \lambda), \vartheta'(1, \lambda), \varphi'(1, \lambda) \) yield

\[
\varphi(1, \lambda) = \frac{1}{\sqrt{\lambda}} S(\lambda) + \ell_2(n), \quad \vartheta'(1, \lambda) = -\sqrt{\lambda} S(\lambda) + \ell^2(n), \quad \varphi'(1, \lambda) = C(\lambda) + \ell_1^2(n),
\]
as \( \sqrt{\lambda} = \pi n + O(n^{-1/2}) \). Substituting the obtained asymptotics into the definition (1.12) of \( M \) and using the identity \( \tilde{M} = U^{-1} MU \) we get (3.14).

We will use standard arguments from the perturbation theory (see [Ka,p.291]). Let \( A, B \) be bounded operators, \( A \) be a normal operator and \( \sigma(A), \sigma(B) \) be spectra of \( A, B \). Then \( \text{dist}\{\sigma(A), \sigma(B)\} \leq \|A - B\| \). Note that \( \tilde{M}_0(\lambda) \) is a normal operator having eigenvalues \( e^{i\eta \pm} \). Hence \( \tilde{M} \) has eigenvalues \( \tau_\pm \) satisfying the estimates \( |\tau_\pm - e^{i\eta \pm}| < |\tilde{M} - \tilde{M}_0| \). Then (3.14) implies

\[
\tau_\pm(\lambda) = e^{i\eta \pm} + \ell_1^2(n), \quad \text{as} \quad \sqrt{\lambda} = \pi n + O(n^{-1/2}). \tag{3.17}
\]

Now we improve asymptotics (3.12) for \( \lambda_{2n,m} \). Note that \( \lambda = \lambda_{2n,m} \) iff \( \tau_+(\lambda) = 1 \) or \( \tau_-(\lambda) = 1 \). Then (3.17) and (3.12) yield

\[
e^{i\eta \pm}(\lambda_{2n,m}) = 1 + \ell_1^2(n), \quad \lambda = \lambda_{2n,m}. \tag{3.18}
\]

Substituting asymptotics (3.12) into (3.18) we have

\[
\sqrt{(2\pi n + \varepsilon_n)^2 - \varepsilon_0^2} = 2\pi n + \ell_1^2(n), \quad n \to +\infty.
\]

and therefore \( \varepsilon_n = \frac{\varepsilon_0}{4\pi n} + \ell_1^2(n) \). Substituting this asymptotics into (3.12) we obtain (1.11) for \( \lambda_{2n,m} \).

We improve asymptotics (3.13) of \( r_n \). Note that \( \lambda = r_n \) iff the following condition is fulfilled: \( \tau_+(\lambda) = \tau_+^{-1}(\lambda) \) or \( \tau_+(\lambda) = \tau_-(\lambda) \). Asymptotics (3.17) imply that the second equation has no solutions for large \( n \). We rewrite the first equation in the form:

\[
e^{i\eta \pm} = e^{-i\eta \pm} + \ell_1^2(n), \quad \text{as} \quad \sqrt{\lambda} = \pi n + O(n^{-1/2}). \tag{3.19}
\]

Substituting asymptotics (3.13) into (3.19) we obtain

\[
\sqrt{(\pi n + \delta_n)^2 - \varepsilon_0^2} = 2\pi n + \ell_1^2(n),
\]

which gives \( \delta_n = \ell_1^2(n) \). Substituting this asymptotics into (3.13) we obtain (1.12).

ii) a) Assume that we have the periodic spectrum \( \lambda_{0,m}, m = 1, 2, 3, 4, n \geq 1 \). Using the asymptotics (1.11) and repeating the standard arguments (see [PT,pp.39-40]) we obtain the Hadamard factorization for the function \( D_1 \):

\[
D_+ (\lambda) = \prod_{m=1,2,3,4, n \geq 1} \frac{\lambda - \lambda_{2n,m}}{(2\pi n)^2}.
\]
In a similar way, we determine $D_-$ by the anti-periodic spectrum. Using (3.10) we get $\rho$. Thus, we recover the resonances.

b) Suppose, that we have the periodic spectrum and the set of the resonances. Then we determine the functions $D_+$ by the periodic spectrum and $\rho$ by the resonances. Using (3.10) we get $\mu_1$, $\mu_2$ and then $D_-$. Thus, we recover the anti-periodic spectrum. The proof of another case is similar. ■

4 Example

1. Periodic $\delta$-potentials. Consider the operator $T^\gamma = -\frac{d^2}{dx^2} + q^\gamma, \gamma \in \mathbb{C}$, where $q^\gamma = aJ + \gamma \delta_{per}J_1$ and the potential $\delta_{per} = \sum_{-\infty}^{\infty} \delta(x - n - \frac{1}{2})$. Let $\mu_1 = \mu_1(\cdot, q^\gamma), \rho^\gamma = \rho(\cdot, q^\gamma), ...$

Lemma 4.1. For the operator $T^\gamma = -\frac{d^2}{dx^2} + aJ + \gamma \delta_{per}J_1$ the following identities are fulfilled:

$$\mu_1^\gamma = \mu_1^0, \quad \mu_2^\gamma = \mu_2^0 + 2h, \quad \rho^\gamma = \rho^0 + h, \quad D_\pm^\gamma = D_\pm^0 - h, \quad h = \frac{\gamma^2}{4}s_+s_-, \quad \gamma \in \mathbb{C}. \quad (4.1)$$

Proof. The solution $y(x), x \in \mathbb{R}$ of the system $-y'' + q^\gamma y = \lambda y$ is continuous and $y'(x_n + 0) - y'(x_n - 0) = \gamma J_1 y(x_n)$ at the points $x = x_n = n + \frac{1}{2}$. Then the fundamental solutions have the form: $\vartheta^\gamma(x) = \vartheta^0(x), \quad \varphi^\gamma(x) = \varphi^0(x), \quad 0 \leq x < \frac{1}{2}$,

$$\vartheta^\gamma(x) = \left( \begin{array}{c} c_+(x) \\ \gamma c_+_1(x) - \frac{1}{2} s_-(x - \frac{1}{2}) \\ \gamma c_-(x) s_-(x - \frac{1}{2}) \end{array} \right), \quad \frac{1}{2} \leq x < \frac{3}{2},$$

$$\varphi^\gamma(x) = \left( \begin{array}{c} s_+(x) \\ \gamma s_+(x) - \frac{1}{2} s_-(x - \frac{1}{2}) \\ \gamma s_-(x) s_-(x - \frac{1}{2}) \end{array} \right), \quad \frac{1}{2} \leq x < \frac{3}{2},$$

and

$$\vartheta^\gamma(x) = \left( \begin{array}{c} c_+(x) + \gamma^2 c_+ \frac{1}{2} s_-(1) s_+(x - \frac{3}{2}) \\ \gamma c_-(x) + \gamma^2 c_- \frac{1}{2} s_-(1) s_-(x - \frac{3}{2}) \end{array} \right),$$

$$\varphi^\gamma(x) = \left( \begin{array}{c} s_+(x) + \gamma^2 s_+ \frac{1}{2} s_-(1) s_+(x - \frac{3}{2}) \\ \gamma s_-(x) + \gamma^2 s_- \frac{1}{2} s_+(1) s_-(x - \frac{3}{2}) \end{array} \right),$$

for $\frac{3}{2} \leq x < \frac{5}{2}$, where $*$ is some term. These relations yield

$$\vartheta(1) = \left( \begin{array}{c} c_+ \frac{1}{1} \\ c_-(1) \end{array} \right), \quad \varphi(1) = \left( \begin{array}{c} c_+(1) \frac{1}{1} \\ c_-(1) \end{array} \right),$$

$$\vartheta(2) = \left( \begin{array}{c} c_+(2) + 2h \\ c_-(2) + 2h \end{array} \right), \quad \varphi(2) = \left( \begin{array}{c} c_+(2) + 2h \\ c_-(2) + 2h \end{array} \right).$$

The last identities and (1.14) imply

$$\mu_1^\gamma = \frac{\text{Tr}(\vartheta^\gamma(1) + (\varphi^\gamma)'(1))}{4} = \mu_1^0,$$

$$\mu_2^\gamma = \frac{\text{Tr}(\vartheta^\gamma(2) + (\varphi^\gamma)'(2))}{4} = \mu_2^0 + 2h.$$
which give $\rho^\gamma = \frac{1}{2}(1 + \mu_2^\gamma) - (\mu_1^\gamma)^2 = \rho^0 + h$ and $D^1_\pm = (\mu_1^\gamma \mp 1)^2 - \mu_2^\gamma = D^0_\pm - h$.  

We describe the spectrum of the operator $T^\gamma$.

**Lemma 4.2.** Let the operator $T^\gamma = -\frac{d^2}{dx^2} + q^\gamma$, where $q^\gamma = aJ + \gamma \delta_{\perp} J_1$, $a > 0$ and $\gamma \in \mathbb{R}$.  

i) Let $0 < \frac{a}{\pi^2} - n_a < 1$, for some integer $n_a \geq 0$. Then for each $n \in \mathbb{N}$ there exist analytic functions $z_{n}^\pm(\gamma), |\gamma| < \gamma_n$ for some $\gamma_n > 0$ such that $z_{n}^\pm(\gamma)$ is the zero of $\rho^\gamma(\lambda)$ and

$$
z_{n}^\pm(\gamma) = z_{n}^0 \pm (-1)^n \gamma \alpha_0 (c_n + O(\gamma)), \quad c_n > 0, \quad k_n = \begin{cases} 1 & \text{if } 1 \leq n \leq n_a, \\ 0 & \text{if } n > n_a \end{cases}, \quad \gamma \to 0. \quad (4.2)
$$

Moreover, each spectral interval $(z_{n}^-(\gamma), z_{n}^+(\gamma)) \subset \mathbb{R}, n > n_a$ is a gap in the spectrum of $T^\gamma$.

ii) If $\frac{a}{\pi^2} \notin \mathbb{N}$, then for each $n \geq 0, m = 1, 2$ there exist real analytic functions $\lambda_{n,m}^\pm(\gamma), \gamma \in (-\gamma_n, \gamma_n)$ for some $\gamma_n > 0$ such that $\lambda_{n,m}^\pm(\gamma)$ is the zero of the function $\Delta_{n,m}^2(\lambda) - 1$ and

$$
\lambda_{n,m}^-(\gamma) < \lambda_{n,m}^+(\gamma), \quad \lambda_{n,m}^+(0) = \lambda_{n,m}^0. \quad (4.3)
$$

Moreover, each spectral interval $(\lambda_{n,m}^-(\gamma), \lambda_{n,m}^+(\gamma)) \neq \emptyset, n \geq 1$ has multiplicity 2.

**Proof.** i) Recall that $\eta_\pm = \sqrt{\lambda \mp a}$. The zero of $\rho^\gamma(\lambda) = 0$ satisfies the equation

$$
0 = \rho^\gamma(\lambda) = f^2(\lambda) + \frac{\gamma^2}{4} s_+^2(\lambda) s_-(\lambda), \quad f \equiv 2 \sin \frac{\eta_+ - \eta_-}{2} \sin \frac{\eta_+ + \eta_-}{2}. \quad (4.4)
$$

The zeros $z_n^0$ of $\rho^0 = \rho(\cdot, q^0)$ have the form (1.16) and satisfy the following identities

$$
\sqrt{z_n^0} + a + \sqrt{z_n^0} - a = \frac{a}{\pi n}, \quad \sqrt{z_n^0} + a - \sqrt{z_n^0} - a = 2\pi n, \quad 1 \leq n \leq n_a, \quad (4.5)
$$

$$
\sqrt{z_n^0} + a + \sqrt{z_n^0} - a = 2\pi n, \quad \sqrt{z_n^0} + a - \sqrt{z_n^0} - a = \frac{a}{\pi n}, \quad n > n_a \geq 0. \quad (4.6)
$$

Using (4.5), (4.6) we have the identity

$$
\sin \eta_+(\lambda) \sin \eta_-(\lambda)|_{\lambda=z_n^0} = \begin{cases} 1 - \cos \frac{a}{\pi n} > 0 & \text{if } 1 \leq n \leq n_a, \\ \cos \frac{a}{\pi n} - 1 < 0 & \text{if } n > n_a \end{cases}. \quad (4.7)
$$

Recall that the function $f$ has only simple zeros $\lambda = z_n^0, n \geq 1$. Consider the case $n > n_a$, the proof for $1 \leq n \leq n_a$ is similar. We rewrite $\rho^\gamma(\lambda) = 0$ in the form

$$(f(\lambda) - \gamma F(\lambda))(f(\lambda) + \gamma F(\lambda)) = 0, \quad F(\lambda) \equiv \sqrt{-s_+^2(\lambda)s_-(\lambda)/2}, \quad F(z_n^0) > 0.
$$

Applying the Implicit Function Theorem to $\Phi(\lambda, \gamma) = 0$, where $\Phi(\lambda, \gamma) = f(\lambda) \pm \gamma F(\lambda)$ and $\frac{\partial}{\partial \lambda} \Phi(z_n^0, 0) \neq 0$ we obtain a unique solution $z_{n}^\pm = u = u(\gamma), \gamma \in (-\gamma_0, \gamma_0), u(0) = z_n^0$ of the equation $\Phi(\lambda, \gamma) = 0$, such that $\Phi(u(\gamma), \gamma) = 0, \gamma \in (-\gamma_0, \gamma_0)$ for some $\gamma_0 > 0$.

ii) Consider the equation $0 = D^1_\pm(\lambda) = D^0_\pm(\lambda) - \frac{\gamma^2}{4} s_+^2(\lambda) s_-(\lambda)$. Using

$$
D^0_\pm(\lambda) = 4 \sin^2 \frac{\eta_+}{2} \sin^2 \frac{\eta_-}{2}, \quad s_\pm(\lambda) = \frac{2}{\eta_\pm} \sin \frac{\eta_+}{2} \cos \frac{\eta_-}{2}, \quad \cos \eta_-(\lambda, 2n_1) \neq 1 \quad (4.8)
$$

15
the Implicit Function Theorem to $\Phi(\lambda, \gamma) = \eta$. Applying
the fundamental solutions

$$\varphi_{\nu, \gamma}(x, \lambda) = \varphi_{0}(x, \lambda) + \int_{0}^{x} \varphi_{0}(x - t, \lambda)q_{\gamma, \nu}(t)\varphi_{\nu, \gamma}(t, \lambda)dt.$$  (4.11)

The standard iterations in (4.11) yield

$$\varphi_{n, \nu, \gamma}(x, \lambda) = \int_{0}^{x} \varphi_{0}(x - t, \lambda)q_{\gamma, \nu}(t)\varphi_{n-1, \nu, \gamma}(t, \lambda)dt.$$  (4.12)

The last identity gives

$$\varphi_{n, \nu, \gamma}(x, \lambda) = \int_{0 < x_{1} < x_{2} < \ldots < x_{n+1} = x} \varphi_{0}(x_{1}, \lambda) \left( \prod_{1 \leq k \leq n} \varphi_{0}(x_{k+1} - x_{k}, \lambda)q_{\gamma, \nu}(x_{k}) \right) dx_{1}dx_{2}...dx_{n},$$  (4.13)

where for matrices $a_{1}, a_{2}, ..., a_{n}$ we define $\prod_{1 \leq k \leq n} a_{k} = a_{n}a_{n-1}...a_{1}$. Substituting the estimate

$$||\sqrt{\lambda}\varphi_{0}(x, \lambda)|| \leq e^{1\text{Im} \sqrt{\lambda}x} \text{ into (4.13)}$$

we obtain $||\sqrt{\lambda}\varphi_{n, \nu, \gamma}(x, \lambda)|| \leq \frac{(2x(\lambda + |\gamma|))^{n}}{n!} e^{1\text{Im} \lambda x}$, which shows that for each $x \geq 0$ the formal series (4.12) converges uniformly on bounded subsets of $\mathbb{C}$. Each term of this series is an entire function. Hence the sum is an entire function. Since $\nu_{\nu} \rightarrow p$ in the sense of distributions we obtain $\varphi_{n, \nu, \gamma}(x, \lambda) \rightarrow \varphi_{n, 0, \gamma}(x, \lambda)$ as $\nu \rightarrow 0$ uniformly on bounded subset of $\mathbb{R} \times \mathbb{C}^{2}$, which yields (4.10). $\blacksquare$

**Proof of Proposition 1.3.** Lemma 4.3 yields $\rho(\lambda, q_{\gamma, \nu}) \rightarrow \rho(\lambda, q^{\gamma})$ and $\mu_{m}(\lambda, q_{\gamma, \nu}) \rightarrow \mu_{m}(\lambda, q^{\gamma})$, $m = 1, 2$, uniformly on any compact set in $\mathbb{C}^{2}$ as $\nu \rightarrow 0$. Then their zeros converge
to the corresponding zeros at $\nu = 0$, uniformly on any compact in $\mathbb{C}^2$. Due to Lemma 3.3 we have convergence of $D_\pm(\lambda, q_{\gamma,\nu})$ and of the Lyapunov function $\Delta(\lambda, q_{\gamma,\nu})$. Thus using Lemmas 4.1-4.2 we obtain the proof of Proposition 1.3. ■

References

[BKW] Brown, B., Knowles, I., Weikard, R.: On the inverse resonance problem. J. London Math. Soc. (2) 68 (2003), no. 2, 383-401

[Ca1] Carlson, R.: Eigenvalue estimates and trace formulas for the matrix Hill’s equation. J. Differential Equations 167 (2000), no. 1, 211–244.

[Ca2] R. Carlson. Large eigenvalues and trace formulas for the matrix Sturm-Liouville problems. SIAM J. Math. Anal.30 (1999), 949-962

[Ca3] Carlson, R.: Compactness of Floquet isospectral sets for the matrix Hill’s equation. Proc. Amer. Math. Soc. 128 (2000), no. 10, 2933–2941.

[CHGL] Clark S., Holden H., Gesztesy, F., Levitan, B.: Borg-type theorem for matrix-valued Schrödinger and Dirac operators, J. Diff. Eqs. 167(2000), 181-210

[DS] Dunford, N. and Schwartz, J.: Linear Operators Part II: Spectral Theory, Interscience, New York, 1988.

[GL] Gel’fand I., Lidskii, V.: On the structure of the regions of stability of linear canonical systems of differential equations with periodic coefficients. (Russian) Uspehi Mat. Nauk (N.S.) 10 (1955), no. 1(63), 3–40.

[Ge] Gel’fand, I.: Expansion in characteristic functions of an equation with periodic coefficients. (Russian) Doklady Akad. Nauk SSSR (N.S.) 73, (1950). 1117–1120.

[KK] Kargaev P., Korotyaev E.: Inverse problem for the Hill operator, the direct approach. Invent. Math., 129(1997), no. 3, 567-593

[Ka] Kato, T.: Perturbation theory for linear operators. Springer-Verlag, Berlin, 1995.

[K1] Korotyaev, E.: Characterization of the spectrum of Schrödinger operators with periodic distributions. Int. Math. Res. Not. (2003) no. 37, 2019–2031

[K2] Korotyaev, E.: The estimates of periodic potentials in terms of effective masses. Comm. Math. Phys. 183 (1997), no. 2, 383–400.

[K3] Korotyaev, E.: Stability for inverse resonance problem, Int. Math. Res. Not. 73(2004), 3927-3936

[Kr] Krein, M.: The basic propositions of the theory of $\lambda$-zones of stability of a canonical system of linear differential equations with periodic coefficients. In memory of A. A. Andronov, pp. 413–498. Izdat. Akad. Nauk SSSR, Moscow, 1955.

[Ly] Lyapunov, A.: The general problem of stability of motion, 2 nd ed. Gl. Red. Obshchetekh. Lit., Leningrad, Moscow, 1935; reprint Ann. Math. Studies, no. 17, Prinston Univ. Press, Prinston, N.J. 1947

[PT] Pöshel,J., Trubowitz, E.: Inverse spectral theory. Pure and Applied Mathematics, 130. Academic Press, Inc., Boston, MA, 1987. 192 pp.

[RS] Reed, M., Simon, B.: Methods of Modern Mathematical Physics, Vol.II, Fourier Analysis, Self-Adjointness, Academic Press, New York, 1975

[S] Simon B.: Resonances in one dimension and Fredholm determinants, J. Funct. Anal. 178 (2000), no. 2, 396-420.
[YS] Yakubovich, V., Starzhinskii, V.: Linear differential equations with periodic coefficients. 1, 2. Halsted Press [John Wiley & Sons] New York-Toronto, 1975. Vol. 1, Vol. 2

[Z] Zworski M.: Distribution of poles for scattering on the real line, J. Funct. Anal. 73, 277-296, 1987