A NEW PINCHING THEOREM FOR COMPLETE SELF-SHRINKERS AND ITS GENERALIZATION

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Abstract. In this paper, we firstly verify that if \( M \) is a complete self-shrinker with polynomial volume growth in \( \mathbb{R}^{n+1} \), and if the squared norm of the second fundamental form of \( M \) satisfies \( 0 \leq |A|^2 - 1 \leq \frac{1}{18} \), then \( |A|^2 \equiv 1 \) and \( M \) is a round sphere or a cylinder. More generally, let \( M \) be a complete \( \lambda \)-hypersurface with polynomial volume growth in \( \mathbb{R}^{n+1} \) with \( \lambda \neq 0 \). Then we prove that there exists an positive constant \( \gamma \), such that if \( |\lambda| \leq \gamma \) and the squared norm of the second fundamental form of \( M \) satisfies \( 0 \leq |A|^2 - \beta \lambda \leq \frac{1}{18} \), then \( |A|^2 \equiv \beta \lambda \), \( \lambda > 0 \) and \( M \) is a cylinder. Here \( \beta \lambda = \left( \frac{1}{2} + \lambda^2 + |\lambda|\sqrt{\lambda^2 + 4} \right) \).

1. Introduction

Suppose \( X : M \to \mathbb{R}^{n+1} \) is an isometric immersion. If the position vector \( X \) evolves in the direction of the mean curvature vector \( \vec{H} \), this yields a solution of mean curvature flow:

\[
\begin{align*}
\frac{\partial}{\partial t} X(x, t) &= \vec{H}(x, t), \quad x \in M, \\
X(x, 0) &= X(x).
\end{align*}
\]

An important class of solutions to the above mean curvature flow equations are self-shrinkers [17], which satisfy

\[ H = -X^N, \]

where \( X^N \) is the projection of \( X \) on the unit normal vector \( \xi \), i.e., \( X^N = \langle X, \xi \rangle \).

We remark that some authors have a factor \( \frac{1}{2} \) on the right-hand side of the defining equation for self-shrinkers.

Rigidity problems of self-shrinkers have been studied extensively. As is known, there are close relations between self-shrinkers and minimal submanifolds. But they are quite different on many aspects. We refer the readers to [14] for the rigidity problems of minimal submanifolds. In [11], Abresch–Langer classified all smooth closed self-shrinker curves in \( \mathbb{R}^2 \). In 1990, Huisken [17] proved that the only smooth closed self-shrinkers with nonnegative mean curvature in \( \mathbb{R}^{n+1} \) are round spheres for \( n \geq 2 \). Based on the work due to Huisken [17, 18], Colding–Minicozzi [11] proved that if \( M \) is an \( n \)-dimensional complete self-shrinker with nonnegative mean curvature and polynomial volume growth in \( \mathbb{R}^{n+1} \), then \( M \) is isometric to either

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the hyperplane $\mathbb{R}^n$, a round sphere or a cylinder. In [2], Brendle verified that the round sphere is the only compact embedded self-shrinker in $\mathbb{R}^3$ of genus zero.

In 2011, Le–Sesum [20] proved that any $n$-dimensional complete self-shrinker with polynomial volume growth in $\mathbb{R}^{n+1}$ whose squared norm of the second fundamental form satisfies $|A|^2 < 1$ must be a hyperplane. Afterwards, Cao–Li [3] generalized this rigidity result to arbitrary codimension and proved that if $M$ is an $n$-dimensional complete self-shrinker with polynomial volume growth in $\mathbb{R}^{n+1}$, and if $|A|^2 \leq 1$, then $M$ must be one of the generalized cylinders. In 2014, Ding–Xin [13] proved the following rigidity theorem for self-shrinkers in the Euclidean space.

**Theorem A.** Let $M$ be an $n$-dimensional complete self-shrinker with polynomial volume growth in $\mathbb{R}^{n+1}$. If the squared norm of the second fundamental form satisfies $0 \leq |A|^2 - 1 \leq \frac{1}{|\lambda|}$, then $|A|^2 \equiv 1$ and $M$ is a round sphere or a cylinder.

In [8], Cheng–Wei proved that if $M$ is an $n$-dimensional complete self-shrinker with polynomial volume growth in $\mathbb{R}^{n+1}$, and if $0 \leq |A|^2 - 1 \leq \frac{2}{|\lambda|}$, where $|A|$ is constant, then $|A|^2 = 1$.

Recently, Xu–Xu [37] improve Theorem A and proved the following rigidity theorem.

**Theorem B.** Let $M$ be an $n$-dimensional complete self-shrinker with polynomial volume growth in $\mathbb{R}^{n+1}$. If the squared norm of the second fundamental form satisfies $0 \leq |A|^2 - 1 \leq \frac{1}{|\lambda|}$, then $|A|^2 \equiv 1$ and $M$ is a round sphere or a cylinder.

In this paper, we firstly prove the following rigidity theorem for self-shrinkers in the Euclidean space.

**Theorem 1.1.** Let $M$ be an $n$-dimensional complete self-shrinker with polynomial volume growth in $\mathbb{R}^{n+1}$. If the squared norm of the second fundamental form satisfies $0 \leq |A|^2 - 1 \leq \frac{1}{|\lambda|}$, then $|A|^2 \equiv 1$ and $M$ is one of the following cases:

(i) the round sphere $S^n(\sqrt{n})$;

(ii) the cylinder $S^k(\sqrt{k}) \times \mathbb{R}^{n-k}$, $1 \leq k \leq n-1$.

More generally, we consider the rigidity of $\lambda$-hypersurfaces. The concept of $\lambda$-hypersurfaces was introduced independently by Cheng–Wei [7] via the weighted volume-preserving mean curvature flow and McGonagle–Ross [25] via isoperimetric type problem in a Gaussian weighted Euclidean space. Precisely, the hypersurfaces of Euclidean space satisfying the following equation are called $\lambda$-hypersurfaces:

\[
H = -X^N + \lambda,
\]

where $X^N$ is the projection of $X$ on the unit normal vector $\xi$ and $\lambda$ is a constant.

In recent years, the rigidity of $\lambda$-hypersurfaces has been investigated by several authors [2, 3, 7, 13, 33]. In [13], Guang showed that if $M$ is a $\lambda$-hypersurface with polynomial volume growth in $\mathbb{R}^{n+1}$, and if $|A|^2 \leq \alpha_\lambda$, then $M$ must be one of the generalized cylinders, where $\alpha_\lambda = \frac{1}{2}(2 + \lambda^2 - |\lambda|\sqrt{\lambda^2 + 4})$. In the second part of this paper, we prove the following second pinching theorem for $\lambda$-hypersurfaces in the Euclidean space.

**Theorem 1.2.** Let $M$ be an $n$-dimensional complete $\lambda$-hypersurface with polynomial volume growth in $\mathbb{R}^{n+1}$ with $\lambda \neq 0$. There exists an positive constant $\gamma$, such that if $|\lambda| \leq \gamma$ and the squared norm of the second fundamental form satisfies $0 \leq |A|^2 - \beta_\lambda \leq \frac{1}{|\lambda|}$, then $|A|^2 \equiv \beta_\lambda$, $\lambda > 0$ and $M$ must be the cylinder $S^{(\sqrt{n+1}-|A|)}(\frac{\sqrt{n+1}-|A|}{2}) \times \mathbb{R}^{n-1}$. Here $\beta_\lambda = \frac{1}{2}(2 + \lambda^2 + |\lambda|\sqrt{\lambda^2 + 4})$. 
2. Rigidity of self-shrinkers

Let $M$ be an $n$-dimensional complete hypersurface in $\mathbb{R}^{n+1}$. We shall make use of the following convention on the range of indices:

$$1 \leq i, j, k, \ldots \leq n.$$

We choose a local orthonormal frame field $\{e_1, e_2, \ldots, e_{n+1}\}$ near a fixed point $x \in M$ over $\mathbb{R}^{n+1}$ such that $\{e_i\}_{i=1}^n$ are tangent to $M$ and $e_{n+1}$ equals to the unit normal vector $\xi$. Let $\{\omega_1, \omega_2, \ldots, \omega_{n+1}\}$ be the dual frame fields of $\{e_1, e_2, \ldots, e_{n+1}\}$. Denote by $R_{ijkl}$, $A := \sum_{ij} h_{ij} \omega_i \otimes \omega_j$, $H := \text{Trace } A$ and $S := \text{Trace } A^2$ the Riemann curvature tensor, the second fundamental form, the mean curvature and the squared norm of the second fundamental form of $M$, respectively. We denote the first, the second and the third covariant derivatives of the second fundamental form of $M$ by

$$\nabla A = \sum_{i,j,k} h_{ijk} \omega_i \otimes \omega_j \otimes \omega_k,$$

$$\nabla^2 A = \sum_{i,j,k,l} h_{ijkl} \omega_i \otimes \omega_j \otimes \omega_k \otimes \omega_l,$$

$$\nabla^3 A = \sum_{i,j,k,l,m} h_{ijklm} \omega_i \otimes \omega_j \otimes \omega_k \otimes \omega_l \otimes \omega_m.$$

The Gauss and Codazzi equations are given by

(2.1)

$$R_{ijkl} = h_{ik} h_{jl} - h_{il} h_{jk},$$

(2.2)

$$h_{ijk} = h_{ikj}.$$

We have the Ricci identities on $M$

(2.3)

$$h_{ijkl} - h_{ijk l} = \sum_m h_{im} R_{mkjl} + \sum_m h_{mj} R_{mikl},$$

(2.4)

$$h_{ijklm} - h_{ijkm} = \sum_r h_{rjk} R_{ritl} + \sum_r h_{irk} R_{rjlm} + \sum_r h_{ijr} R_{rklm}.$$}

We choose a local orthonormal frame $\{e_i\}$ such that $h_{ij} = \mu_i \delta_{ij}$ at $x$. By the Gauss equation (2.1) and the Ricci identity (2.3), we have

(2.5)

$$t_{ij} := h_{ijij} - h_{jiji} = \mu_i \mu_j (\mu_i - \mu_j).$$

Set $u_{ijkl} = \frac{1}{4} (h_{ijkl} + h_{ijlk} + h_{ikjl} + h_{kijl})$. Then we have

$$\sum_{i,j,k,l} (h_{ijkl}^2 - u_{ijkl}^2) \geq \frac{6}{16} \sum_{i \neq j} [(h_{ijij} - h_{jiji})^2 + (h_{jiji} - h_{ijij})^2]$$

(2.6)

$$= \frac{3}{4} G,$$

i.e.,

(2.7)

$$|\nabla^2 A|^2 \geq \frac{3}{4} G,$$

where $G = \sum_{i,j} t_{ij}^2 = 2(S f_4 - f_j^2)$ and $f_k = \text{Trace } A^k = \sum_i \mu_i^k$. In [11], Colding-Minicozzi introduced the linear operator

$$\mathcal{L} = \Delta - \langle X, \nabla (\cdot) \rangle = e^{\frac{3x^2}{2}} \text{Div} \left( e^{-\frac{3x^2}{2}} \nabla (\cdot) \right).$$
They showed that $L$ is self-adjoint respect to the measure $\rho \, d\mu$, where $\rho = e^{-\frac{|X|^2}{2}}$.

Let $M$ be a self-shrinker with polynomial volume growth. By a computation (see [13, 37]), we have following equalities

(2.8) $L|A|^2 = 2|\nabla A|^2 - 2|A|^4 + 2|\nabla A|^2$,

(2.9) $|\nabla S|^2 = \frac{1}{2}LS^2 + 2S^2(S - 1) - 2S|\nabla A|^2$,

(2.10) $|\nabla^2 A|^2 = \frac{1}{2}L|\nabla A|^2 + (|A|^2 - 2)|\nabla A|^2 + 3(B_1 - 2B_2) + \frac{3}{2}|\nabla S|^2$,

(2.11) $\int_M (B_1 - 2B_2) \rho \, d\mu = \int_M \left( \frac{1}{2}G - \frac{1}{4}|\nabla S|^2 \right) \rho \, d\mu$,

where $B_1 = \sum_{i,j,k,l,m} h_{ij}h_{ijkl}h_{klm}$ and $B_2 = \sum_{i,j,k,l,m} h_{ijkl}h_{ijkl}h_{ijkl}$.

Now we are in a position to prove our rigidity theorem for self-shrinkers in the Euclidean space.

**Proof of Theorem 1.1.** From (2.7), (2.10) and (2.11), we have

(2.12) $\int_M (B_1 - 2B_2) \rho \, d\mu = \int_M \left( \frac{1}{2}G - \frac{1}{4}|\nabla S|^2 \right) \rho \, d\mu$

This implies that

(2.13) $\int_M (B_1 - 2B_2) \rho \, d\mu \geq \int_M \left[ \frac{2}{3}(2 - S)|\nabla A|^2 - \frac{3}{4}|\nabla S|^2 \right] \rho \, d\mu$.

By Lemma 4.2 in [13] and Young’s inequality, for $\sigma > 0$, we have

(2.14) $3(B_1 - 2B_2) \leq (S + C_1 G^{1/3})|\nabla A|^2 \leq S|\nabla A|^2 + \frac{1}{3}C_1 \sigma^2 G + \frac{2}{3}C_1 \sigma^{-1}|\nabla A|^3$,

where $C_1 = \frac{2\sqrt{2} + 3}{\sqrt{21\sqrt{6} + 103/2}}$. Notice that

(2.15) $-\int_M (\nabla |\nabla A|, \nabla S) \rho \, d\mu = \int_M |\nabla A|L\nabla S \rho \, d\mu$.

This together with (2.8) implies

(2.16) $\int_M |\nabla A|^3 \rho \, d\mu = \int_M \left( \frac{1}{2}LS - S^2 \right) |\nabla A| \rho \, d\mu$

$\leq \int_M \left[ (S^2 - S)|\nabla A| - \frac{1}{2} (|\nabla A|, \nabla S) \right] \rho \, d\mu$

$\leq \int_M \left[ (S^2 - S)|\nabla A| + \epsilon |\nabla^2 A|^2 + \frac{1}{16\epsilon}|\nabla S|^2 \right] \rho \, d\mu.$
From (2.10), (2.11), (2.14), (2.16), we have

\[
3 \int_M (B_1 - 2B_2) \rho \, d\mu \\
\leq \int_M \left( S|\nabla A|^2 + \frac{2}{3} C_1 \sigma^2 G + \frac{2}{3} C_1 \sigma^{-1} |\nabla A|^3 \right) \rho \, d\mu \\
\leq \int_M \left( S|\nabla A|^2 + \frac{1}{3} C_1 \sigma^2 G \right) \rho \, d\mu \\
+ \frac{2}{3} C_1 \sigma^{-1} \int_M \left[ (S^2 - S)|\nabla A| + \epsilon|\nabla^2 A|^2 + \frac{1}{16\epsilon} |\nabla S|^2 \right] \rho \, d\mu \\
= \int_M \left( S|\nabla A|^2 + \frac{2}{3} C_1 \sigma^2 \left( B_1 - 2B_2 + \frac{1}{4} |\nabla S|^2 \right) \right) \rho \, d\mu \\
+ \frac{2}{3} C_1 \sigma^{-1} \int_M \left[ (S^2 - S)|\nabla A| + \frac{1}{16\epsilon} |\nabla S|^2 \right] \rho \, d\mu \\
(2.17)
\]

Thus, we obtain

\[
3 \theta \int_M (B_1 - 2B_2) \rho \, d\mu \\
\leq \int_M \left[ S + \frac{2}{3} C_1 \sigma^{-1} \epsilon (S - 2) \right] |\nabla A|^2 \rho \, d\mu \\
+ \left( \frac{1}{6} C_1 \sigma^2 + C_1 \sigma^{-1} \epsilon + \frac{1}{24\epsilon} C_1 \sigma^{-1} \right) \int_M |\nabla S|^2 \rho \, d\mu \\
+ \frac{2}{3} C_1 \sigma^{-1} \int_M (S^2 - S)|\nabla A| \rho \, d\mu,
\]

where \( \theta = 1 - \left( \frac{2}{3} C_1 \sigma^2 + \frac{2}{3} C_1 \sigma^{-1} \epsilon \right) \). We restrict \( \sigma \) and \( \epsilon \) such that \( \theta \geq 0 \).

Combining (2.13) and (2.18), we have

\[
0 \leq \int_M \left[ S + \frac{2}{3} C_1 \sigma^{-1} \epsilon + 2\theta (S - 2) \right] |\nabla A|^2 \rho \, d\mu \\
+ \left( \frac{1}{6} C_1 \sigma^2 + C_1 \sigma^{-1} \epsilon + \frac{1}{24\epsilon} C_1 \sigma^{-1} + \frac{9}{4} \theta \right) \int_M |\nabla S|^2 \rho \, d\mu \\
+ \frac{2}{3} C_1 \sigma^{-1} \int_M (S^2 - S)|\nabla A| \rho \, d\mu.
\]

(2.19)

To simplify the notation, we put

\[
L_1 := \frac{2}{3} C_1 \sigma^{-1} \epsilon + 2\theta,
\]

\[
L_2 := \frac{1}{6} C_1 \sigma^2 + C_1 \sigma^{-1} \epsilon + \frac{1}{24\epsilon} C_1 \sigma^{-1} + \frac{9}{4} \theta.
\]

Then (2.19) is reduced to

\[
0 \leq \int_M \left[ S + L_1 (S - 2) \right] |\nabla A|^2 \rho \, d\mu \\
+ L_2 \int_M |\nabla S|^2 \rho \, d\mu + \frac{2}{3} C_1 \sigma^{-1} \int_M (S^2 - S)|\nabla A| \rho \, d\mu.
\]

(2.20)
When \(0 \leq S - 1 \leq \delta\), we have
\[
\frac{1}{2} \int_M |\nabla S|^2 \rho \, d\mu = \int_M S(S - 1)^2 \rho \, dM - \int_M (S - 1)|\nabla A|^2 \rho \, d\mu \\
\leq \int_M (1 - S + \delta)|\nabla A|^2 \rho \, d\mu.
\] (2.21)

For \(\kappa > 0\), we have
\[
\int_M S(S - 1)|\nabla A| \rho \, dM \leq 2(1 + \delta)\kappa \int_M S(S - 1) \rho \, dM \\
+ \frac{1}{8(1 + \delta)\kappa} \int_M S(S - 1)|\nabla A|^2 \rho \, d\mu \\
\leq 2(1 + \delta)\kappa \int_M |\nabla A|^2 \rho \, d\mu \\
+ \frac{1}{8\kappa} \int_M (S - 1)|\nabla A|^2 \rho \, dM.
\] (2.22)

Substituting (2.21) and (2.22) into (2.20), we obtain
\[
0 \leq \int_M [(1 + L_1)(S - 1) + 1 - L_1]|\nabla A|^2 \rho \, d\mu \\
+ 2L_2 \int_M (1 - S + \delta)|\nabla A|^2 \rho \, d\mu \\
+ \frac{4}{3}C_1\sigma^{-1}(1 + \delta)\kappa \int_M |\nabla A|^2 \rho \, d\mu \\
+ \frac{1}{12\kappa}C_1\sigma^{-1} \int_M (S - 1)|\nabla A|^2 \rho \, dM \\
= \int_M \left( 1 + L_1 - 2L_2 + \frac{1}{12\kappa}C_1\sigma^{-1} \right)(S - 1)|\nabla A|^2 \rho \, d\mu \\
+ \int_M \left[ 1 - L_1 + \frac{4}{3}C_1\sigma^{-1}\kappa + \left( \frac{4}{3}C_1\sigma^{-1}\kappa + 2L_2 \right)\delta \right]|\nabla A|^2 \rho \, d\mu.
\] (2.23)

Let \(\sigma = 0.616, \epsilon = 0.0577\) and \(\kappa = 0.0434\). By a computation, we have
\[
\theta > 0, \quad 1 + L_1 - 2L_2 + \frac{1}{12\kappa}C_1\sigma^{-1} < 0, \\
1 - L_1 + \frac{4}{3}C_1\sigma^{-1}\kappa < -0.452, \\
\frac{4}{3}C_1\sigma^{-1}\kappa + 2L_2 < 8.03.
\]

We take \(\delta = 1/18\). Then the coefficients of the integrals in (2.23) are both negative.

Therefore, we have \(|\nabla A| \equiv 0\) and \(S \equiv 1\), i.e., \(M\) either the round sphere \(S^n(\sqrt{n})\),
or the cylinder \(S^k(\sqrt{k}) \times \mathbb{R}^{n-k}, 1 \leq k \leq n - 1\). \(\square\)

3. RIGIDITY OF \(\lambda\)-HYPERSURFACES

Let \(M\) be an \(n\)-dimensional complete \(\lambda\)-hypersurface with polynomial volume growth in \(\mathbb{R}^{n+1}\).
We adopt the same notations as in Section 2. To simplify the computation, we choose local frame \(\{e_i\}\), such that \(\nabla e_i e_j = 0\) at \(p \in M\), i.e.,
\[
\nabla e_i e_j = h_{ij} \xi, \text{ and } h_{ij} = \mu_i \delta_{ij}.
\]
Then we have
\[
\nabla e_i H = -\nabla e_i \langle X, \xi \rangle = h_{ik} \langle X, e_k \rangle,
\] (3.1)
Lemma 3.1. If \( M \) is a \( \lambda \)-hypersurface of \( \mathbb{R}^{n+1} \), then we have:

\( \nabla^2 A \) is complete self-shrinkers and its generalization.

and

\[
\text{Hess} H(e_i, e_j) = -\nabla e_i \nabla e_j (X, \xi)
\]

(3.2) \[ = h_{ijk}(X, e_k) + h_{ij} - (H - \lambda) h_{i} h_{kj}. \]

Taking \( f_k = \text{Trace} A_k = \sum_i \mu_i \), we obtain

\[
\mathcal{L} |A|^2 = \Delta |A|^2 - \langle X, \nabla |A|^2 \rangle
\]

(3.3) \[ = 2 \sum_{i,j} h_{ij} \Delta h_{ij} + 2 |\nabla A|^2 - 2 \sum_{i,j,k} h_{ij} h_{ijk}(X, e_k) \]

\[ = 2 \sum_{i,j} h_{ij} \nabla e_i \nabla e_j H + \sum_{i,j,k} h_{ij} h_{ijk} h_{ki} - |A|^4 \]

\[ + 2 |\nabla A|^2 - 2 \sum_{i,j,k} h_{ij} h_{ijk}(X, e_k) \]

(3.4) \[ = 2 |A|^2 - 2 |A|^4 + 2 \lambda f_3 + 2 |\nabla A|^2. \]

Proof of Theorem. Putting \( F_\lambda = |A|^4 - |A|^2 - \lambda f_3 \), we have

(3.5) \[ \int_M F_\lambda \rho dM = \int_M |\nabla A|^2 \rho dM. \]

We also have

\[ |\nabla S|^2 = \frac{1}{2} \mathcal{L} S^2 + 2SF_\lambda - 2S|\nabla A|^2. \]

Notice that

(3.6) \[ F_\lambda \geq |A|^2(|A|^2 - 1 - |\lambda| \cdot |A|). \]

If \( |A|^2 \geq \beta_\lambda = \frac{1}{2}(2 + \lambda^2 + |\lambda|\sqrt{\lambda^2 + 4}) \), then \( F_\lambda \geq 0 \). Moreover, if \( F_\lambda = 0 \), then \( |A|^2 = \beta_\lambda \). Denote by \( \alpha_\lambda = \frac{1}{2}(2 + \lambda^2 - |\lambda|\sqrt{\lambda^2 + 4}) \). When \( \beta \leq |A|^2 \leq \beta_\lambda + \delta \), we have the following upper bound for \( F_\lambda \).

\[
F_\lambda \leq |A|^2(|A|^2 - 1 + |\lambda| \cdot |A|)
\]

(3.7) \[ = |A|^2(|A| + \sqrt{\beta_\lambda})(|A| + \sqrt{\alpha_\lambda})^{-1}(|A|^2 - \beta_\lambda + q_\lambda)
\]

\[ \leq |A|^2(|A|^2 - \beta_\lambda + q_\lambda) \left( 1 + \frac{\sqrt{\beta_\lambda} - \sqrt{\alpha_\lambda}}{\sqrt{\beta_\lambda} + \sqrt{\alpha_\lambda}} \right)
\]

where \( q_\lambda = |\lambda|\sqrt{\lambda^2 + 4} \), \( r_\lambda = q_\lambda + \lambda^2 + \frac{|\lambda|\delta}{\lambda^2 + 4} \).

For \( |\nabla^2 A| \) and the integral of \( B_1 - 2B_2 \), we obtain the following lemma.

Lemma 3.1. If \( M \) is a \( \lambda \)-hypersurface of \( \mathbb{R}^{n+1} \), then we have:

(i) \( |\nabla^2 A|^2 = \frac{1}{2} \mathcal{L} |A|^2 + (|A|^2 - 2) |\nabla A|^2 + 3(B_1 - 2B_2) + \frac{3}{2} |\nabla S|^2 - 3\lambda C \),

(ii) \( \int_M (B_1 - 2B_2) \rho dM = \int_M \left( \frac{1}{2} G - \frac{3}{4} |\nabla S|^2 \right) \rho dM \),

where \( B_1 = \sum_{i,j,k,m} h_{ijk} h_{ijk} h_{km}, B_2 = \sum_{i,j,k,l,m} h_{ijk} h_{klm} h_{im}, C = \sum_{i,j,k,l} h_{ijk} h_{ijkl} \),

\[ G = \sum_{i,j} t_{ij}^2 = 2(Sf_4 - f_3^2) \]

and \( t_{ij} = h_{ij} - h_{i} h_{jj} = \mu_i \mu_j (\mu_i - \mu_j) \).
Proof. (i) Applying Ricci identities (2.3) and (2.4), we have

\[ \Delta h_{ijkl} = (h_{ijlk} + h_{ir} R_{rjkl} + h_{rj} R_{rkl})_l \]

\[ = h_{ijlk} + h_{rj} R_{rkl} + h_{ir} R_{rjlk} + h_{ir} R_{rkl} + (h_{ir} R_{rjkl} + h_{rj} R_{rkl})_l \]

\[ = (h_{ijlk} + h_{ir} R_{rjkl} + h_{rj} R_{rkl})_k + h_{rj} R_{rkl} + h_{ir} R_{rjkl} + h_{ir} R_{rkl} \]

\[ + h_{ir} R_{rjkl} + h_{rj} R_{rkl} - h_{ij} R_{rijkl} \]

\[ + h_{ir} (R_{rjkl})_l + h_{rj} (R_{rkl})_l + h_{ir} (R_{rjkl})_l + h_{rj} (R_{rkl})_l \]

(3.8)

It follows from (3.2) that

\[ H_{ji} = h_{jli} \langle X, e_l \rangle + h_{ij} + (\lambda - H)h_{ij}h_{ji}. \]

Since \( \langle X, \xi \rangle = \lambda - H \), we compute the covariant derivative of \( H_{ji} \)

\[ H_{jik} = h_{jik} \langle X, e_l \rangle + h_{jli} \langle e_k, e_l \rangle + h_{jil} (X, \nabla e_i e_l) + h_{ijk} \]

\[ - H_k h_{il} h_{ji} + (\lambda - H) (h_{kl} h_{ji} + h_{ij} h_{jkl}) \]

\[ = h_{jik} \langle X, e_l \rangle + 2h_{ijk} - H_k h_{il} h_{ji} \]

\[ + (\lambda - H) (h_{li} h_{jkl} + h_{ji} h_{ikl} + h_{kl} h_{jli}). \]

Combining (3.8) and (3.10), we have

\[ \Delta h_{ijk} = h_{jik} \langle X, e_l \rangle + 2h_{ijk} - h_{ij} h_{ji} H_k \]

\[ + (\lambda - H) (h_{li} h_{jkl} + h_{ji} h_{ikl} + h_{kl} h_{jli}) \]

\[ + h_{rkl} R_{rjkl} + h_{rjkl} R_{rjkl} + 2h_{rj} R_{rkl} + 2h_{rj} R_{rkl} + h_{rj} R_{rkl} \]

\[ + h_{ir} (R_{rjkl})_l + h_{rj} (R_{rkl})_l + h_{ir} (R_{rjkl})_l + h_{rj} (R_{rkl})_l \]

(3.11)

The Gauss equation (2.15) implies

\[ h_{ijk}(h_{rkl} R_{rjkl} + h_{rj} R_{rkl} + 2h_{rj} R_{rkl} + 2h_{rj} R_{rkl} + h_{rj} R_{rkl}) \]

\[ + h_{ir} (R_{rjkl})_l + h_{rj} (R_{rkl})_l + h_{ir} (R_{rjkl})_l + h_{rj} (R_{rkl})_l \]

\[ = h_{ijk} (6h_{rkl} h_{rj} h_{ji} - 6h_{rkl} h_{rj} h_{ji} + 3h_{rj} h_{rkl} h_{ij} + 3h_{rj} h_{rkl} h_{ij} - 3h_{rj} h_{rkl} h_{ij} \]

\[ + 3h_{ir} h_{rj} h_{ij} - 2h_{ir} h_{rj} h_{ij} - h_{ijk} h_{rkl}^2). \]

(3.12)

From (2.3) and (3.1), we have

\[ h_{ijk}(h_{ijlk} - h_{ijk}) \langle X, e_l \rangle = h_{ijk} (h_{ir} R_{rjlk} + h_{rj} R_{rkl}) \langle X, e_l \rangle \]

\[ = 2h_{ijk} h_{ir} h_{jkl} H + 2h_{ijk} h_{ir} h_{jkl} H. \]

(3.13)
Substituting (3.12) and (3.13) into (3.11), we obtain

\[
\frac{1}{2}(\Delta - \langle X, \nabla \rangle)h^2_{ijk} = h_{ijk}(\Delta h_{ijk} - h_{ijkl}(X, e_i)) + h^2_{ijkl}
\]

Therefore, we have

\[
\frac{1}{2}(\Delta - \langle X, \nabla \rangle)h^2_{ijk} = h_{ijk}(\Delta h_{ijk} - h_{ijkl}(X, e_i)) + h^2_{ijkl}
\]

This together with the divergence theorem implies

\[
\frac{1}{2}(\Delta - \langle X, \nabla \rangle)h^2_{ijk} = h_{ijk}(\Delta h_{ijk} - h_{ijkl}(X, e_i)) + h^2_{ijkl}
\]

Substituting (3.12) and (3.13) into (3.11), we obtain

\[
\frac{1}{2}(\Delta - \langle X, \nabla \rangle)h^2_{ijk} = h_{ijk}(\Delta h_{ijk} - h_{ijkl}(X, e_i)) + h^2_{ijkl}
\]

Applying Ricci identity (2.3), we get

\[
\frac{1}{2}(\Delta - \langle X, \nabla \rangle)h^2_{ijk} = h_{ijk}(\Delta h_{ijk} - h_{ijkl}(X, e_i)) + h^2_{ijkl}
\]

(ii) It follows from the divergence theorem that

\[
\int_M \sum_{i,j} (f_3)_{ij} h_{ij} \rho dM = - \int_M \sum_{i,j} (f_3)_{ij} (h_{ij} \rho)_j dM.
\]

By the condition \( H = -X^N + \lambda \), we have

\[
\sum_{i,j} (h_{ij} \rho)_j = \sum_{i,j} h_{jjij} \rho - \sum_{i,j} h_{ij} \rho (e_i, X)
\]

\[
= -\epsilon_i(\langle \xi, X \rangle) \rho + \sum_{i,j} (\nabla e_i \xi, X) \rho
\]

(3.16)

\[
= 0.
\]

This together with the divergence theorem implies

\[
\int_M \sum_{i,j,k} h_{ik} h_{kj} S_{ij} \rho dM = - \int_M \sum_{i,j,k} h_{ikj} h_{kj} S_{ij} \rho dM
\]

(3.17)

\[
= - \frac{1}{2} \int_M |\nabla S|^2 \rho dM.
\]

Applying Ricci identity (2.3), we get

\[
h_{ijij} - h_{ijji} = \mu_i \mu_j (\mu_i - \mu_j).
\]

Thus, we have

\[
\frac{1}{3} \sum_{i,j} (f_3)_{ij} = \sum_{i,k} h_{iikk} \mu_k \mu_i^2 + 2 \sum_{i,j,k} h^2_{ijk} \mu_i \mu_k
\]

\[
= \sum_{i,k} [h_{kii} + (\mu_i - \mu_k) \mu_i \mu_k] \mu_k \mu_i^2 + 2B_2
\]

\[
= \sum_i (\frac{S_{ii}}{2} - \sum_{j,k} h^2_{ijk}) \mu_i^2 + \sum_{i,k} \mu_i^3 \mu_k (\mu_i - \mu_k) + 2B_2
\]

(3.19)

\[
= \sum_{i,j,k} \frac{h_{ik} h_{kj}}{2} S_{ij} + S f_4 - f_3^2 - (B_1 - 2B_2).
\]
Substituting (3.16), (3.17) and (3.19) into (3.15), we obtain
\begin{equation}
\int_M (B_1 - 2B_2) \rho dM = \int_M \left[ S f_4 - f_3^2 - \frac{1}{4} |\nabla S|^2 \right] \rho dM.
\end{equation}
\(\square\)

Combining (2.7) and Lemma 3.1, we derive the following inequality.
\begin{equation}
\int_M (B_1 - 2B_2) \rho dM = \int_M \left[ \frac{1}{2} G - \frac{1}{4} |\nabla S|^2 \right] \rho dM \\
\leq \frac{2}{3} \int_M |\nabla A|^2 \rho dM - \frac{1}{4} \int_M |\nabla S|^2 \rho dM \\
= \frac{2}{3} \int_M (S - 2) |\nabla A|^2 \rho dM + 2 \int_M (B_1 - 2B_2) \rho dM \\
+ \frac{3}{4} \int_M |\nabla S|^2 \rho dM - 2\lambda \int_M C \rho dM.
\end{equation}
This implies
\begin{equation}
\int_M (B_1 - 2B_2) \rho dM \geq - \frac{2}{3} \int_M (S - 2) |\nabla A|^2 \rho dM \\
- \frac{3}{4} \int_M |\nabla S|^2 \rho dM + 2\lambda \int_M C \rho dM.
\end{equation}

For any \(\sigma > 0\), using Lemma 4.2 in [13] and Young’s inequality, we have
\begin{equation}
3(B_1 - 2B_2) \leq (S + C_1 G^{1/3}) |\nabla A|^2 \leq S |\nabla A|^2 + \frac{1}{3} C_1 \sigma^2 G + \frac{2}{3} C_1 \sigma^{-1} |\nabla A|^3,
\end{equation}
where \(C_1 = \frac{2\sqrt{5} + 3}{\sqrt{21\sqrt{6+103}/2}}\). Notice that
\begin{equation}
- \int_M \nabla |\nabla A| \cdot \nabla S \rho dM = \int_M |\nabla A| \nabla S \rho dM.
\end{equation}
This together with (3.18) implies
\begin{equation}
\int_M |\nabla A|^3 \rho dM \\
= \int_M (F_\lambda + \frac{1}{2} \mathcal{L} |A|^2) |\nabla A| \rho dM \\
= \int_M F_\lambda |\nabla A| \rho dM - \frac{1}{2} \int_M |\nabla A| \cdot \nabla S \rho dM \\
\leq \int_M F_\lambda |\nabla A| \rho dM + \epsilon \int_M |\nabla^2 A|^2 \rho dM + \frac{1}{10\epsilon} \int_M |\nabla S|^2 \rho dM,
\end{equation}
for arbitrary \(\epsilon > 0\). We assume that \(S\) satisfies the pinching condition \(\beta_\lambda \leq S \leq \beta_\lambda + \delta\). From (3.4) and (3.5), we have
\begin{equation}
\frac{1}{2} \int_M |\nabla S|^2 \rho dM = \int_M (S - \beta_\lambda) F_\lambda \rho dM - \int_M (S - \beta_\lambda) |\nabla A|^2 \rho dM \\
\leq \int_M (-S + \beta_\lambda + \delta) |\nabla A|^2 \rho dM.
\end{equation}
For any $\kappa > 0$, (3.7) implies
\[
\int_M F_\lambda |\nabla A| \rho dM \leq 2(\beta_\lambda + \delta)\kappa \int_M F_\lambda \rho dM \\
+ \frac{1}{8(\beta_\lambda + \delta)\kappa} \int_M F_\lambda |\nabla A|^2 \rho dM \\
\leq 2(\beta_\lambda + \delta)\kappa \int_M F_\lambda \rho dM \\
+ \frac{1}{8(\beta_\lambda + \delta)\kappa} \int_M (|A|^2 - \beta_\lambda + r_\lambda)|A|^2 |\nabla A|^2 \rho dM \\
\leq 2(\beta_\lambda + \delta)\kappa \int_M |\nabla A|^2 \rho dM \\
+ \frac{1}{8\kappa} \int_M (|A|^2 - \beta_\lambda + r_\lambda)|\nabla A|^2 \rho dM.
\]
(3.27)

For $C$, we have the estimate
\[
|C| = \left| \sum_{i,j,k} \mu_i h_{ijk}^2 \right| \leq |A||\nabla A|^2.
\]
(3.28)

Combining (3.23), (3.25) and Lemma 3.1, we obtain
\[
3 \int_M (B_1 - 2B_2) \rho dM \\
\leq \int_M \left( S|\nabla A|^2 + \frac{1}{3} C_1 \sigma^2 G + \frac{2}{3} C_1 \sigma^{-1} |\nabla A|^3 \right) \rho dM \\
\leq \int_M S|\nabla A|^2 \rho dM + \frac{1}{3} C_1 \sigma^2 \int_M G \rho dM \\
+ \frac{2}{3} C_1 \sigma^{-1} \int_M F_\lambda |\nabla A| \rho dM + \frac{C_1}{24\sigma \epsilon} \int_M |\nabla S|^2 \rho dM \\
+ \frac{2}{3} C_1 \sigma^{-1} \epsilon \int_M |\nabla^2 A|^2 \rho dM \\
= \int_M S|\nabla A|^2 \rho dM + \frac{2}{3} C_1 \sigma^2 \int_M \left( B_1 - 2B_2 + \frac{1}{4}|\nabla S|^2 \right) \rho dM \\
+ \frac{2}{3} C_1 \sigma^{-1} \int_M F_\lambda |\nabla A| \rho dM + \frac{C_1}{24\sigma \epsilon} \int_M |\nabla S|^2 \rho dM \\
+ \frac{2}{3} C_1 \sigma^{-1} \epsilon \int_M \left( (S - 2)|\nabla A|^2 + 3(B_1 - 2B_2) + \frac{3}{2}|\nabla S|^2 - 3\lambda C \right) \rho dM.
\]
(3.29)

Hence
\[
3\theta \int_M (B_1 - 2B_2) \rho dM \\
\leq \int_M \left[ S + \frac{2}{3} C_1 \sigma^{-1} \epsilon (S - 2) \right] |\nabla A|^2 \rho dM \\
+ \left( \frac{1}{6} C_1 \sigma^2 + \frac{C_1}{24\sigma \epsilon} + C_1 \sigma^{-1} \epsilon \right) \int_M |\nabla S|^2 \rho dM \\
+ \frac{2}{3} C_1 \sigma^{-1} \int_M F_\lambda |\nabla A| \rho dM - 2C_1 \sigma^{-1} \epsilon \lambda \int_M C \rho dM.
\]
(3.30)
where $\theta = 1 - \left(\frac{2}{3} C_1 \sigma^2 + \frac{2}{3} C_1^{-1} \epsilon \right)$. When $\theta > 0$, this together with (3.22) implies

\[
0 \leq \int_M \left[ (S + L_1(S - 2)) \left| \nabla A \right|^2 \rho dM + L_1 \int_M \left| \nabla S \right|^2 \rho dM \right] + \frac{2}{3} C_1^{-1} \int_M F_\lambda \left| \nabla A \right| \rho dM - 2\lambda (C_1^{-1} \epsilon + 3\theta) \int_M C \rho dM,
\]

(3.31)

where $L_1 = \frac{2}{3} C_1^{-1} \epsilon + 2\theta$. Substituting (3.26), (3.27) and (3.28) into (3.31), we obtain

\[
0 \leq \int_M \left[ (S + L_1(S - 2)) \left| \nabla A \right|^2 \rho dM + L_1 \int_M \left| \nabla S \right|^2 \rho dM \right] + \frac{2}{3} C_1^{-1} \int_M F_\lambda \left| \nabla A \right| \rho dM - 2\lambda (C_1^{-1} \epsilon + 3\theta) \int_M C \rho dM,
\]

(3.32)

where $L_1 = \frac{2}{3} C_1^{-1} \epsilon + 2\theta$. Substituting (3.26), (3.27) and (3.28) into (3.31), we obtain

\[
0 \leq \int_M \left[ (S + L_1(S - 2)) \left| \nabla A \right|^2 \rho dM + L_1 \int_M \left| \nabla S \right|^2 \rho dM \right] + \frac{2}{3} C_1^{-1} \int_M F_\lambda \left| \nabla A \right| \rho dM - 2\lambda (C_1^{-1} \epsilon + 3\theta) \int_M C \rho dM,
\]

(3.33)

By a computation, we have

\[
\theta > 0, \quad 1 + L_1 - 2L_2 + \frac{C_1}{12\sigma \kappa} < 0,
\]

(3.34)

\[
1 - L_1 + \frac{4}{3\sigma} C_1 \kappa < -0.452, \quad 2L_2 + \frac{4}{3\sigma} C_1 \kappa < 8.03.
\]

Take $\delta = 1/18$. There exists an positive constant $\gamma$, such that $\eta_\lambda \leq 0.005$ when $|\lambda| \leq \gamma$. Then the coefficients of the integral in (3.33) are both negative. Therefore, $|\nabla A| \equiv 0$. By a classification theorem due to Lawson [19], $M$ must be $\mathcal{S}^k(r) \times \mathbb{R}^{n-k}$, $1 \leq k \leq n$. For $\lambda \neq 0$, the radius $r$ satisfies $|\lambda| = \frac{r}{\tau} - r$. Hence,

\[
r = \frac{\sqrt{\lambda^2 + 4k} - \lambda}{2},
\]

(3.35)

\[
\mu_1 = \ldots = \mu_k = \frac{1}{2k} (\sqrt{\lambda^2 + 4k} + \lambda),
\]

where $\mu_k$ is the $k$-th principal curvature of $M$.

We consider the following two cases:

(i) for $\lambda > 0$, the squared norm of the second fundamental form $M$ satisfies

\[
S_k = \sum_{i=1}^k \mu_i^2 = \frac{1}{2k} (\lambda^2 + 2k + |\lambda|\sqrt{\lambda^2 + 4k}).
\]
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Hence, \( S_1 = \beta \lambda \). When \( k \geq 2 \), \( S_k < \beta \lambda \).

(ii) for \( \lambda < 0 \), by a computation, we have

\[
S_k = \sum_{i=1}^{k} \mu_i^2 = \frac{1}{2k} (\lambda^2 + 2k - |\lambda| \sqrt{\lambda^2 + 4k}).
\]

When \( 1 \leq k \leq n \), \( S_k < \beta \lambda \).

Therefore, \( \lambda > 0 \) and \( M \) must be \( \mathbb{S}\left(\frac{\sqrt{\lambda^2 + 4k} - |\lambda|}{2}\right) \times \mathbb{R}^{n-1} \).

\[\square\]

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