On a (2 + 1)-dimensional generalization of the Ablowitz–Ladik lattice and a discrete Davey–Stewartson system

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Abstract

We propose a natural (2 + 1)-dimensional generalization of the Ablowitz–Ladik lattice that is an integrable space discretization of the cubic nonlinear Schrödinger system in 1 + 1 dimensions. By further requiring rotational symmetry of order 2 in the two-dimensional lattice, we identify an appropriate change of dependent variables, which translates the (2 + 1)-dimensional Ablowitz–Ladik lattice into a suitable space discretization of the Davey–Stewartson system. The space-discrete Davey–Stewartson system has a Lax pair and allows the complex conjugation reduction between two dependent variables as in the continuous case. Moreover, it is ideally symmetric with respect to space reflections. Using the Hirota bilinear method, we construct some exact solutions such as multidromion solutions.

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1. Introduction

More than 40 years have passed since the Korteweg–de Vries (KdV) equation was solved by Gardner et al [1] using the inverse scattering method based on its Lax pair [2]. The number of known integrable systems following the KdV equation, particularly partial differential equations (PDEs) in 1 + 1 spacetime dimensions, has increased enormously, and various techniques to study them have been developed. Recently, the center of researchers’ interest has shifted from continuous PDEs to differential-difference or partial difference equations wherein at least one of the independent variables takes discrete values. A major problem in this trend is how to find a suitable difference analog of a given differential equation. The
suitable discretization of an integrable continuous system is generally required to retain the integrability [3], but that is not sufficient if the original continuous system has some essential internal symmetries. This becomes conspicuous if we consider integrable discretizations of nonlinear Schrödinger (NLS)-type systems, i.e. two-component systems of second order that allow the complex conjugation reduction between the two dependent variables. As a prototypical example, we discuss the cubic NLS system in 1 + 1 dimensions [4]:

\[
\begin{align*}
    i q_t + q_{xx} - 2q^2 r &= 0, \\    ir_t - r_{xx} + 2r^2 q &= 0.
\end{align*}
\]

(1.1a) (1.1b)

Note that the reduction \( r = \sigma q^* \) with a real constant \( \sigma \) simplifies the two-component system (1.1) to the scalar NLS equation [5, 6]. In addition, (1.1) is invariant under the space reflection \( x \to -x \) as well as the time reflection \( t \to -t \) with \( q \leftrightarrow r \). The suitable and elegant space discretization of the NLS system (1.1) was proposed by Ablowitz and Ladik [7] in the form

\[
\begin{align*}
    i q_{n,t} + (q_{n+1} + q_{n-1} - 2q_n) - q_n r_n (q_{n+1} + q_{n-1}) &= 0, \\    i r_{n,t} - (r_{n+1} + r_{n-1} - 2r_n) + r_n q_n (r_{n+1} + r_{n-1}) &= 0.
\end{align*}
\]

(1.2a) (1.2b)

Indeed, the system (1.2) is integrable and, with a rescaling of variables, reduces to (1.1) in the continuous space limit. Moreover, (1.2) allows the complex conjugation reduction between \( q_n \) and \( r_n \) and possesses the same invariance properties with respect to the space/time reflection as the continuous system (1.1). Although 35 years have already passed since their work, the Ablowitz–Ladik discretization (1.2) is still a rare example of success. Indeed, even now, only a small number of suitable space discretizations of integrable NLS-type systems are known (see, e.g., [8]); they are all \((1 + 1)\)-dimensional systems with only one discrete spatial variable. The problem of how to discretize the continuous time variable in such systems is an interesting topic [9], but we do not discuss it in this paper.

The main objective of this paper is to provide the first example of a suitable discretization of an NLS-type system in 2 + 1 dimensions. In particular, we consider the discretization of both spatial variables in a \((2 + 1)\)-dimensional NLS system known as the Davey–Stewartson system [10] (also see [11]). Note that the Davey–Stewartson system is integrable [12–16] and appears to be the only genuinely \((2 + 1)\)-dimensional generalization of the NLS system (1.1) (cf the Calogero–Degasperis system [17]). Moreover, even if we include other types of integrable systems, the list of known systems with two discrete and one continuous independent variables is still very short. Thus, it is a highly nontrivial and challenging task to obtain the suitable space discretization of the Davey–Stewartson system. To solve this problem, we first propose a natural \((2 + 1)\)-dimensional generalization of the Ablowitz–Ladik lattice (cf (1.2)) by constructing its Lax pair. This \((2 + 1)\)-dimensional Ablowitz–Ladik lattice certainly reduces to the Davey–Stewartson system in the continuous space limit. A relevant Lax pair as well as the resulting system was previously studied by other authors [18] (also see [19]), but the time part of our Lax pair is essentially more general than the previously known one [18]. As a result, the time evolution of our system is a linear combination of four elementary time evolutions, two of which were previously unknown. Moreover, it can be shown that the four time evolutions are mutually commutative. Thus, the \((2 + 1)\)-dimensional Ablowitz–Ladik lattice is general enough and appears to be promising. However, it does not allow the complex conjugation reduction directly and thus is not a suitable space discretization of the Davey–Stewartson system in its present form. To fix this shortcoming, we only have to consider a certain nonlocal transformation of dependent variables, which symmetrizes the equations of
motion. Thus, we obtain the suitable space discretization of the Davey–Stewartson system that indeed allows the complex conjugation reduction between the new variables after the transformation. In addition, the invariance properties of the continuous Davey–Stewartson system with respect to space/time reflections turn out to be properly incorporated in our space-discrete Davey–Stewartson system.

This paper is organized as follows. In section 2, we propose a $(2+1)$-dimensional version of the Ablowitz–Ladik lattice by considering an appropriate generalization of the Lax pair for the original Ablowitz–Ladik lattice. To uncover how the complex conjugation reduction can be imposed as an NLS-type system, we consider a nonlocal change of dependent variables; it can turn the $(2+1)$-dimensional Ablowitz–Ladik lattice into the suitable space discretization of the Davey–Stewartson system that indeed allows the complex conjugation reduction between the new dependent variables. In section 3, we elucidate how the general time evolution considered can be decomposed into four elementary time evolutions corresponding to the four directions on the two-dimensional lattice. On the basis of this decomposition and using the Hirota bilinear method [20], we construct some exact solutions of the $(2+1)$-dimensional Ablowitz–Ladik lattice and the space-discrete Davey–Stewartson system. In particular, multiodromion solutions are presented explicitly. The last section, section 4, is devoted to concluding remarks.

2. Derivation based on Lax pairs

2.1. $(2+1)$-dimensional Ablowitz–Ladik lattice

As a generalization of the Lax pair introduced by Ablowitz and Ladik [7], we consider the following linear system on the two-dimensional lattice:

\[ \Delta^+_n \psi_{n,m} := \psi_{n+1,m} - \psi_{n,m} = q_n,m \phi_{n,m}, \]  
\[ \Delta^+_m \phi_{n,m} := \phi_{n,m+1} - \phi_{n,m} = r_n,m \psi_{n,m}, \]  
\[ \frac{\partial \psi_{n,m}}{\partial t} = a \psi_{n,m+1} + A_{n,m} \psi_{n,m} + C_{n,m} \psi_{n,m-1} + b q_{n-1,m} \phi_{n,m} - q_{n,m} D_{n,m} \phi_{n-1,m}, \]  
\[ \frac{\partial \phi_{n,m}}{\partial t} = b \phi_{n,m+1} + B_{n,m} \phi_{n,m} + D_{n,m} \phi_{n-1,m} + a r_{n-1,m} \psi_{n,m} - r_{n,m} C_{n,m} \psi_{n,m-1}. \]

Here, $\Delta^+_n$ and $\Delta^+_m$ denote the forward difference operators in each spatial direction, and the parameters $a$ and $b$ are arbitrary constants. The time dependence of the functions is usually suppressed. The compatibility conditions $\partial \Delta^+_n \psi_{n,m} = \Delta^+_n \partial \psi_{n,m}$ and $\partial \Delta^+_m \phi_{n,m} = \Delta^+_m \partial \phi_{n,m}$ for the linear system (2.1) provide the $(2+1)$-dimensional Ablowitz–Ladik lattice:

\[ \frac{\partial q_{n,m}}{\partial t} + q_{n+1,m} D_{n+1,m} + b q_{n-1,m} - a q_{n,m+1} - C_{n+1,m} q_{n,m-1} + q_{n,m} B_{n,m} - A_{n+1,m} q_{n,m} = 0, \]  
\[ \frac{\partial r_{n,m}}{\partial t} + r_{n,m+1} C_{n+1,m} + a r_{n,m-1} - b r_{n+1,m} - D_{n+1,m} r_{n,m-1} + r_{n,m} A_{n,m} - B_{n,m+1} r_{n,m} = 0, \]  
\[ A_{n+1,m} - A_{n,m} = -a(q_{n,m+1} r_{n,m} - q_{n,m} r_{n,m-1}), \]  
\[ B_{n,m+1} - B_{n,m} = -b(r_{n+1,m} q_{n,m} - r_{n,m} q_{n-1,m}), \]  
\[ (1 - q_{n,m} r_{n,m}) C_{n,m} = C_{n+1,m} (1 - q_{n,m-1} r_{n,m-1}), \]  
\[ (1 - r_{n,m} q_{n,m}) D_{n,m} = D_{n,m+1} (1 - r_{n-1,m} q_{n-1,m}). \]
Indeed, if all the functions depend on \( n \) and \( m \) only through \( n + m \), (2.2) reduces to the original Ablowitz–Ladik lattice [7]; the latter contains the integrable discrete NLS system (1.2) as a special case. To restore the (1 + 1)-dimensional Lax pair involving the spectral parameter \( \gamma \), we set \( \psi_{n,m} = z^{n-m} \psi'_{n,m} \) and \( \phi_{n,m} = z^{n-m+1} \phi'_{n,m} \), rewrite the linear problem (2.1) in terms of \( \psi'_{n,m} \) and \( \phi'_{n,m} \) and then consider the dimensional reduction.

Under appropriate boundary conditions at spatial infinity, we can use (2.2c)–(2.2f) recursively to express the auxiliary fields \( A_{n,m}, B_{n,m}, C_{n,m} \) and \( D_{n,m} \) globally in terms of \( q_{n,m} \) and \( r_{n,m} \). Thus, they can be considered as the defining relations for the auxiliary fields, and the (2 + 1)-dimensional Ablowitz–Ladik lattice (2.2) has intrinsically nonlocal nonlinearity. Note that relations (2.2c)–(2.2f) already appeared in the literature on an integrable time discretization of the (1 + 1)-dimensional Ablowitz–Ladik lattice [3, 21, 22]. Incidentally, in the stationary case of \( \partial_q q_{n,m} = \partial_r r_{n,m} = 0 \), (2.2) reduces to a nontrivial system of partial difference equations in 1 + 1 dimensions.

Using the simple transformation

\[
q_{n,m} = q'_{n,m} e^{\gamma t}, \quad r_{n,m} = r'_{n,m} e^{-\gamma t},
\]

where \( \gamma \) is a constant, and omitting the prime, we can introduce the terms \( +\gamma q_{n,m} \) and \( -\gamma r_{n,m} \) in (2.2a) and (2.2b), respectively. Clearly, these terms can be absorbed by constant shifts of \( A_{n,m} \) and \( B_{n,m} \).

### 2.2. Continuum limit

By choosing the parameters appropriately and taking the continuous space limit, we can reduce the system (2.2) to the continuous Davey–Stewartson system. To see this, we first shift the auxiliary fields as

\[
A_{n,m} = -a + A_{n,m}, \quad B_{n,m} = a - b + B_{n,m}, \quad C_{n,m} = a + C_{n,m}, \quad D_{n,m} = b + D_{n,m},
\]

and rewrite (2.2) as

\[
\frac{\partial q_{n,m}}{\partial t} + b(q_{n+1,m} + q_{n-1,m} - 2q_{n,m}) - a(q_{n,m+1} + q_{n,m-1} - 2q_{n,m}) + q_{n+1,m} \tilde{D}_{n+1,m} - \hat{C}_{n+1,m} q_{n,m+1} + q_{n,m} \hat{B}_{n,m} - \tilde{A}_{n+1,m} q_{n,m} = 0, \quad (2.3a)
\]

\[
\frac{\partial r_{n,m}}{\partial t} + a(r_{n+1,m} + r_{n-1,m} - 2r_{n,m}) - b(r_{n+1,m} + r_{n-1,m} - 2r_{n,m}) + r_{n+1,m} \hat{C}_{n+1,m} + \hat{D}_{n+1,m} r_{n+1,m} + r_{n,m} \hat{A}_{n,m} - \hat{B}_{n+1,m} r_{n,m} = 0, \quad (2.3b)
\]

\[
\tilde{A}_{n+1,m} - \hat{A}_{n,m} = -a(q_{n+1,m}r_{n,m} - q_{n,m}r_{n+1,m}), \quad (2.3c)
\]

\[
\hat{B}_{n,m+1} - \hat{B}_{n,m} = -b(r_{n,m+1}q_{n,m} - r_{n,m}q_{n,m+1}), \quad (2.3d)
\]

\[
\hat{C}_{n+1,m} - \hat{C}_{n,m} = -a(q_{n+1,m}r_{n,m} - q_{n,m}r_{n+1,m}) + \hat{C}_{n+1,m} q_{n,m+1} - q_{n,m} \hat{C}_{n+1,m} - q_{n,m+1} r_{n,m} \hat{C}_{n,m}, \quad (2.3e)
\]

\[
\hat{D}_{n,m+1} - \hat{D}_{n,m} = -b(r_{n,m+1}q_{n,m} - r_{n,m}q_{n,m+1}) + \hat{D}_{n,m+1} r_{n,m+1} q_{n,m} - r_{n,m} q_{n,m+1} \hat{D}_{n,m}, \quad (2.3f)
\]

Subsequently, we rescale the variables and parameters as

\[
q_{n,m} = \Delta x \cdot q(x, y), \quad r_{n,m} = \Delta y \cdot r(x, y), \quad x := n \Delta x, \quad y := m \Delta y,
\]

\[
A_{n,m} = \alpha(x, y), \quad B_{n,m} = \beta(x, y), \quad C_{n,m} = C(x, y), \quad D_{n,m} = D(x, y),
\]

wherein the time dependence is suppressed and

\[
a = \frac{\alpha}{(\Delta y)^2}, \quad b = \frac{\beta}{(\Delta x)^2}.
\]
Thus, in the continuum limit $\Delta x, \Delta y \to 0$, (2.3) reduces to the continuous Davey–Stewartson system [13, 14, 16, 23, 24]:

\begin{align}
q_t + \beta q_{xx} - \alpha q_{yy} - (A + C)q + q(B + D) &= 0, \\
\alpha r_{xx} + r(A + C) - (B + D)r &= 0,
\end{align}

(2.4a, 2.4b)

\begin{align}
A_x &= C_x = -\alpha(qr)_y, \\
B_y &= D_y = -\beta(rq)_x.
\end{align}

(2.4c, 2.4d)

Note that the Davey–Stewartson system (2.4) is a linear combination of the two commuting flows corresponding to $\alpha = 0$, $\beta \neq 0$ and $\alpha \neq 0$, $\beta = 0$ [25, 26] (also see [24]). In subsection 3.1, we present its discrete analog; that is, the $(2+1)$-dimensional Ablowitz–Ladik lattice (2.2) is a linear combination of four commuting flows. This is a quite natural result because (i) each of the two Davey–Stewartson flows provides an asymmetric $(2+1)$-dimensional generalization of the NLS system and (ii) the Ablowitz–Ladik discretization of the NLS system is actually a sum of two elementary flows (and one trivial flow) in the same hierarchy [3, 7, 27–29].

2.3. Noncommutative extension

Actually, the $(2+1)$-dimensional Ablowitz–Ladik lattice (2.2) is integrable in the general case where the dependent variables take their values in matrices, as long as the operations such as addition and multiplication make sense. In that case, ‘l’ in (2.2e) and (2.2f) should be interpreted as the identity matrix.

We can further generalize it to a variable-coefficient system wherein the parameters $a$ and $b$ become arbitrary matrix-valued functions of one spatial variable as $a_m := a(m)$ and $b_n := b(n)$. To obtain such an extension, we consider the following generalization of the linear system (2.1):

\begin{align}
\psi_{n+1,m} &= z\psi_{n,m} + q_{n,m}\phi_{n,m}, \\
\phi_{n,m+1} &= z^{-1}\phi_{n,m} + r_{n,m}\psi_{n,m},
\end{align}

(2.5a, 2.5b)

\begin{align}
\frac{\partial \psi_{n,m}}{\partial t} &= zq_{n,m}\psi_{n,m+1} + q_{n,m}\phi_{n,m} + z^{-1}q_{n-1,m}b_{n-1,m}\phi_{n,m} - q_{n,m}D_{n,m}\phi_{n-1,m},
\end{align}

(2.5c)

\begin{align}
\frac{\partial \phi_{n,m}}{\partial t} &= z^{-1}b_{n,m}\phi_{n+1,m} + B_{n,m}\phi_{n,m} + zD_{n,m}\phi_{n-1,m} + zr_{n,m-1}a_{n-1,m}\psi_{n,m} - r_{n,m}C_{n,m}\psi_{n-1,m}.
\end{align}

(2.5d)

Note that the ‘spectral parameter’ $z$ is nonessential in the $(2+1)$-dimensional case and can be fixed at 1 as described in subsection 2.1. The compatibility conditions for the linear system (2.5) indeed provide the noncommutative system with site-dependent coefficients:

\begin{align}
\frac{\partial q_{n,m}}{\partial t} + q_{n+1,m}D_{n+1,m} + q_{n-1,m}b_{n-1} - a_{n,m}q_{n,m+1} - C_{n+1,m}q_{n,m-1} + q_{n,m}B_{n,m} - A_{n+1,m}q_{n,m} &= 0,
\end{align}

(2.6a)

\begin{align}
\frac{\partial r_{n,m}}{\partial t} + r_{n+1,m}C_{n+1,m} + r_{n-1,m}a_{n-1} - b_{n}r_{n+1,m} - D_{n+1,m}r_{n-1,m} + r_{n,m}A_{n,m} - B_{n+1,m}r_{n,m} &= 0,
\end{align}

(2.6b)
\[ A_{n+1,m} - A_{n,m} = -a_m q_{n,m+1} r_{n,m} + q_{n,m} r_{n,m-1} a_{n-1}, \]
\[ B_{n+1,m} - B_{n,m} = -b_m r_{n+1,m} q_{n,m} + r_{n,m} q_{n-1,m} b_{n-1}, \]
\[ (I - q_{n,m} r_{n,m}) C_{n,m} = C_{n+1,m} (I - q_{n,m-1} r_{n,m-1}), \]
\[ (I - r_{n,m} q_{n,m}) D_{n,m} = D_{n+1,m} (I - r_{n-1,m} q_{n-1,m}). \]

If \( q_{n,m} \) and \( r_{n,m} \) are rectangular matrices, the identity matrix \( I \) in (2.6e) and that in (2.6f) have unequal sizes. In the commutative case of the parameters, the site-dependent nature of (2.6) is nonessential if both \( \prod_{m=-\infty}^{\infty} a_m \) and \( \prod_{m=-\infty}^{\infty} b_n \) take nonzero finite values. Indeed, if we change the variables as

\[ q_{n,m} = \left( \prod_{j=-\infty}^{m-1} a_j \right)^{-1} \left( \prod_{k=-\infty}^{n-1} b_k \right) \tilde{q}_{n,m}, \quad r_{n,m} = \left( \prod_{j=-\infty}^{m-1} a_j \right) \left( \prod_{k=-\infty}^{n-1} b_k \right)^{-1} \tilde{r}_{n,m}, \]

\[ C_{n,m} = a_m^{-1} \tilde{C}_{n,m}, \quad D_{n,m} = b_n^{-1} \tilde{D}_{n,m}, \]

the site-dependent parameters can be normalized to 1. We can also obtain a similar result in the noncommutative case.

### 2.4. Appropriate change of dependent variables

For simplicity, in the following discussion, we consider only the commutative and constant-coefficient case wherein the parameters \( a \) and \( b \) are constants and all the quantities are scalar. Thus, the lowest-order conservation law for (2.2) is given by

\[
\frac{\partial \log(1 - q_{n,m} r_{n,m})}{\partial t} = \Delta_a^* \left[ -b q_{n,m+1} r_{n,m} + D_{n,m} (1 - q_{n,m} r_{n,m})^{-1} q_{n,m} r_{n,m+1} \right] \\
+ \Delta_b^* \left[ -a q_{n,m} r_{n,m+1} + C_{n,m} (1 - q_{n,m} r_{n,m}^{-1})^{-1} q_{n,m} r_{n,m+1} \right].
\]

The existence of an ultralocal conserved density \( \log(1 - q_{n,m} r_{n,m}) \) implies that a nonlocal transformation involving infinite products of \( (1 - q_{n,m} r_{n,m})^\delta \) with \( \delta \neq 0 \) could be applied (cf [30]); this is indeed the case as we will see below.

The \((2+1)\)-dimensional Ablowitz–Ladik lattice (2.2) is invariant under a space reflection \((n, m) \rightarrow (-m, -n)\) with a minor redefinition of the parameters and the auxiliary fields. However, (2.2) does not allow the complex conjugation reduction between \( q_{n,m} \) and \( r_{n,m} \) in the local form. Therefore, we need to identify new ‘conjugate’ variables instead of \( q_{n,m} \) and \( r_{n,m} \) and rewrite (2.2) in a more symmetric form using the new variables. For this purpose, we consider a gauge transformation so that the spatial part of the Lax representation obtains invariance with respect to the combined space reflection \((n, m) \rightarrow (-n, -m)\) or, equivalently, a 180° rotation around the origin. Thus, we apply the gauge transformation

\[ \psi_{n,m} = X_{n,m} \Psi_{n,m}, \quad \phi_{n,m} = Y_{n,m} \Phi_{n,m} \]

(2.8) to (2.1a) and (2.1b) and change the dependent variables as

\[ u_{n,m} = \frac{Y_{n,m}}{X_{n,m}} q_{n,m}, \quad v_{n,m} = \frac{X_{n,m}}{Y_{n,m}} r_{n,m}. \]

(2.9)

Here, \( X_{n,m} \) and \( Y_{n,m} \) are defined as

\[
X_{n,m} := \frac{1}{h_m} \prod_{j=-\infty}^{m-1} \sqrt{1 - q_{j,m} r_{j,m}}, \quad Y_{n,m} := \frac{1}{l_n} \prod_{k=-\infty}^{n-1} \sqrt{1 - q_{n,k} r_{n,k}}.
\]

(2.10)

The norming functions \( h_m(t) \) and \( l_n(t) \) are introduced to realize the complex conjugation reduction between \( u_{n,m} \) and \( v_{n,m} \); they will be determined later. One can also use
Before applying the transformation described in subsection 2.4, we fix the boundary conditions 2.5. Space-discrete Davey–Stewartson system equations of motion.

\[
\left( \prod_{k=-\infty}^{\infty} \sqrt{1-q_{n,k}r_{n,k}} \right)^{-1} \text{ instead of } \prod_{k=-\infty}^{m-1} \sqrt{1-q_{n,k}r_{n,k}} \text{ to maintain the invariance under the space reflection } (n,m) \rightarrow (-m,-n). \]

This modification causes no essential difference in the following discussion, so the transformed system can become ideally symmetric with respect to space reflections. Here and hereafter, we assume that \( |q_{n,m}r_{n,m}| \ll 1 \) so that \( \sqrt{1-q_{n,m}r_{n,m}} \) and its inverse as well as their infinite products as considered above are uniquely and well defined. For example, we consider that \( q_{n,m} = O(\Delta x) \) and \( r_{n,m} = O(\Delta y) \) (cf. (2.1a) and (2.1b)), and \( \sqrt{1-q_{n,m}r_{n,m}} \) is defined as the Maclaurin series in \( q_{n,m}r_{n,m} = O(\Delta x \Delta y) \). Note that \( u_{n,m}v_{n,m} = q_{n,m}r_{n,m} \), so that the inverse transformation of (2.9) can be obtained immediately. Thus, the spatial part of the Lax representation acquires the form

\[
\begin{bmatrix}
\tilde{\Psi}_{n+1,m} \\
\Phi_{n,m+1}
\end{bmatrix} = \frac{1}{\sqrt{1-u_{n,m}v_{n,m}}} \begin{bmatrix}
u_{n,m} & 1 \\
u_{n,m} & 1
\end{bmatrix} \begin{bmatrix}
\tilde{\Psi}_{n,m} \\
\Phi_{n,m}
\end{bmatrix},
\]

(2.11)

Very recently, Zakharov has considered essentially the same scattering problem in [31, 32]. However, this is an accidental coincidence because the first author arrived at this Lax representation as well as its generalization implied in subsection 2.5 independently before the papers [31, 32] appeared. The invariance under the combined space reflection \((n,m) \rightarrow (-n,-m)\) can be easily seen if we shift the indices of the linear wavefunction by 1/2, i.e.

\[
\begin{bmatrix}
\tilde{\Psi}_{n+\frac{1}{2},m} \\
\Phi_{n,m+\frac{1}{2}}
\end{bmatrix} = \frac{1}{\sqrt{1-u_{n,m}v_{n,m}}} \begin{bmatrix}
u_{n,m} & 1 \\
u_{n,m} & 1
\end{bmatrix} \begin{bmatrix}
\tilde{\Psi}_{n-\frac{1}{2},m} \\
\Phi_{n,m-\frac{1}{2}}
\end{bmatrix},
\]

where \( \tilde{\Psi}_{n,m} = \tilde{\Psi}_{n-\frac{1}{2},m} \) and \( \Phi_{n,m} = \Phi_{n,m-\frac{1}{2}} \). Indeed, because the determinant of the spatial Lax matrix above is unity, we obtain

\[
\begin{bmatrix}
\tilde{\Psi}_{n-\frac{1}{2},m} \\
\Phi_{n,m-\frac{1}{2}}
\end{bmatrix} = \frac{1}{\sqrt{1-u_{n,m}v_{n,m}}} \begin{bmatrix}
u_{n,m} & 1 \\
u_{n,m} & 1
\end{bmatrix} \begin{bmatrix}
\tilde{\Psi}_{n+\frac{1}{2},m} \\
\Phi_{n,m+\frac{1}{2}}
\end{bmatrix}.
\]

It should be possible to apply the transformation (2.9) with (2.10) directly to the \((2+1)\)-dimensional Ablowitz–Ladik lattice (2.2) and derive the transformed equations of motion with the aid of the conservation law (2.7). However, the nonlocal nature of (2.2) makes such a computation rather complicated and difficult. Thus, as an alternative, we apply the gauge transformation (2.8) with (2.9) and (2.10) to the time part of the Lax representation, (2.1c) and (2.1d), and determine the time evolution of the gauge-transformed wavefunction, \( \delta_t \tilde{\Psi}_{n,m} \) and \( \delta_t \tilde{\Phi}_{n,m} \). Its compatibility with the scattering problem (2.11) can provide the transformed equations of motion.

2.5. Space-discrete Davey–Stewartson system

Before applying the transformation described in subsection 2.4, we fix the boundary conditions for the \((2+1)\)-dimensional Ablowitz–Ladik lattice (2.2) as

\[
\lim_{m \to -\infty} (q_{n,m}, r_{n,m}) = \lim_{m \to -\infty} (q_{n,-m}, r_{n,-m}) = 0,
\]

(2.12a)

\[
\lim_{m \to -\infty} C_{n,m} = e^{\left( \frac{L_{m-1}}{L_m} \right)^2}, \quad \lim_{m \to -\infty} D_{n,m} = d \left( \frac{L_{m-1}}{L_m} \right)^2.
\]

(2.12b)

However, we do not fix \( \lim_{m \to -\infty} A_{n,m} \) and \( \lim_{m \to -\infty} B_{n,m} \) in order to obtain interesting solutions such as dromion solutions; they can also depend on the remaining spatial variable and time \( t \). In (2.12a), the dynamical variables \( q_{n,m} \) and \( r_{n,m} \) are assumed to approach zero.
sufficiently rapidly. In (2.12b), $c$ and $d$ are constants. The defining relations (2.2e) and (2.2f) enable $C_{n,m}$ and $D_{n,m}$ to be expressed globally as

$$ C_{n,m} = c \left( \frac{X_{n,m}}{X_{n,m-1}} \right)^2, \quad D_{n,m} = d \left( \frac{Y_{n,m}}{Y_{n-1,m}} \right)^2. $$

Thus, the gauge transformation (2.8) with (2.9) and (2.10) changes (2.1c) and (2.1d) to

$$ \frac{\partial \Psi_{n,m}}{\partial t} = a \frac{X_{n,m+1}}{X_{n,m}} \psi_{n,m+1} + \tilde{A}_{n,m} \psi_{n,m} + c \frac{X_{n,m}}{X_{n,m-1}} \psi_{n,m-1} $$

$$ + b \frac{Y_{n,m}}{Y_{n-1,m+1}} \psi_{n-1,m} + d \frac{Y_{n,m}}{Y_{n-1,m}} \psi_{n-1,m} $$

and

$$ \frac{\partial \Phi_{n,m}}{\partial t} = b \frac{Y_{n+1,m}}{Y_{n,m}} \Phi_{n+1,m} + \tilde{B}_{n,m} \Phi_{n,m} + d \frac{Y_{n,m}}{Y_{n-1,m}} \Phi_{n-1,m} $$

$$ + a \frac{X_{n,m}}{X_{n+1,m-1}} \psi_{n,m-1} \psi_{n,m} - c \frac{X_{n,m}}{X_{n-1,m}} \psi_{n,m-1} \psi_{n,m-1}, $$

where

$$ \tilde{A}_{n,m} := A_{n,m} = \frac{\partial X_{n,m}}{\partial n}, \quad \tilde{B}_{n,m} := B_{n,m} = \frac{\partial Y_{n,m}}{\partial m}. $$

Recalling that $q_{n,m} r_{n,m} = u_{n,m} v_{n,m}$, the above relations combined with (2.11) comprise the Lax representation for the transformed system. To express it in a concise form, we introduce the quantities

$$ u_{n,m} := \frac{1}{\sqrt{1 - q_{n,m} r_{n,m}}} = \frac{1}{\sqrt{1 - u_{n,m} v_{n,m}}}, $$

(2.14a)

$$ f_{n,m} := \frac{X_{n,m+1}}{X_{n,m}} = \frac{h_{m} \prod_{j=-\infty}^{n-1} \sqrt{1 - u_{j,m+1} v_{j,m+1}}}{h_{m+1} \prod_{j=-\infty}^{n} \sqrt{1 - u_{j,m} v_{j,m}}}, $$

(2.14b)

$$ g_{n,m} := \frac{Y_{n+1,m}}{Y_{n,m}} = \frac{l_{n} \prod_{k=-\infty}^{m-1} \sqrt{1 - u_{n+1,k} v_{n+1,k}}}{l_{n+1} \prod_{k=-\infty}^{m} \sqrt{1 - u_{n,k} v_{n,k}}}. $$

(2.14c)

Thus, we obtain the Lax representation in the form

$$ \Psi_{n+1,m} = \psi_{n,m} \psi_{n,m} + w_{n,m} u_{n,m} \Phi_{n,m}, $$

(2.15a)

$$ \Phi_{n+1,m} = \psi_{n,m} \Phi_{n,m} + w_{n,m} v_{n,m} \Phi_{n,m}, $$

(2.15b)

$$ \frac{\partial \Psi_{n,m}}{\partial t} = a f_{n,m} \psi_{n,m+1} + \tilde{A}_{n,m} \psi_{n,m} + c f_{n,m-1} \psi_{n,m-1} $$

$$ + b w_{n-1,m} g_{n-1,m} u_{n-1,m} \Phi_{n,m-1} - d g_{n-1,m} u_{n,m} \Phi_{n-1,m}, $$

(2.15c)

$$ \frac{\partial \Phi_{n,m}}{\partial t} = b g_{n,m} \Phi_{n+1,m} + \tilde{B}_{n,m} \Phi_{n,m} + d g_{n-1,m} \Phi_{n-1,m} $$

$$ + a w_{n+1,m} f_{n+1,m} \psi_{n+1,m-1} \psi_{n+1,m-1} \psi_{n,m-1} - c f_{n,m-1} \psi_{n,m-1} \psi_{n,m-1}. $$

(2.15d)

The corresponding boundary conditions are given by

$$ \lim_{n \to -\infty} (u_{n,m}, v_{n,m}) = \lim_{m \to -\infty} (u_{n,m}, v_{n,m}) = 0, $$

$$ \lim_{n \to -\infty} f_{n,m} = \frac{h_{m}}{h_{m+1}}, \quad \lim_{m \to -\infty} g_{n,m} = \frac{l_{n}}{l_{n+1}}. $$
Actually, we can generalize (2.15) to a more general form wherein the spatial part is given by
\[
\Psi_{n+1,m} = w_{n,m} \Psi_{n,m} + q_{n,m} \Phi_{n,m},
\]
\[
\Phi_{n+1,m} = s_{n,m} \Phi_{n,m} + r_{n,m} \Psi_{n,m},
\]
with four independent functions \(w_{n,m}, s_{n,m}, q_{n,m}\) and \(r_{n,m}\). Thus, it is possible to start with this general Lax representation and then consider the reduction. However, we skip such a discussion to maintain an easy-to-read flow of the paper.

The compatibility conditions for the linear system (2.15) with \(w_{n,m} = (1 - u_{n,m} v_{n,m})^{-\frac{1}{2}}\) provide the time evolution equations for \(u_{n,m}\) and \(v_{n,m}\). They can be written in a natural compact form using new auxiliary fields \(\alpha_{n,m}\) and \(\beta_{n,m}\) defined as
\[
\tilde{A}_{n,m} = \frac{1}{2} w_{n-1,m} g_{n-1,m} (bu_{n-1,m} v_{n,m} - du_{n,m} v_{n-1,m}) - \frac{1}{2} \alpha_{n,m}, \tag{2.16a}
\]
\[
\tilde{B}_{n,m} = \frac{1}{2} w_{n,m-1} f_{n,m-1} (au_{n,m} v_{n-1,m} - cu_{n,m-1} v_{n,m}) - \frac{1}{2} \beta_{n,m}. \tag{2.16b}
\]

Thus, we finally arrive at the desired system:
\[
\frac{\partial u_{n,m}}{\partial t} + (1 - u_{n,m} v_{n,m}) (d w_{n,m} g_{n,m} u_{n+1,m} + b w_{n-1,m} g_{n-1,m} u_{n-1,m}) - au_{n,m} f_{n,m} u_{n,m+1} - cu_{n,m-1} f_{n,m-1} u_{n,m-1} + \frac{1}{2} u_{n,m} [u_{n,m} - w_{n,m-1} f_{n,m-1} (au_{n,m} v_{n-1,m} + cu_{n,m-1} v_{n,m}) - \beta_{n,m} + w_{n-1,m} g_{n-1,m} (bu_{n-1,m} v_{n,m} + du_{n,m} v_{n-1,m})] = 0, \tag{2.17a}
\]
\[
\frac{\partial v_{n,m}}{\partial t} + (1 - u_{n,m} v_{n,m}) (c w_{n,m} f_{n,m} v_{n,m+1} + a w_{n-1,m} f_{n,m-1} v_{n,m-1}) - bw_{n,m} g_{n,m} v_{n+1,m} - dw_{n-1,m} g_{n-1,m} v_{n-1,m} - \frac{1}{2} v_{n,m} [\alpha_{n,m} - w_{n,m-1} f_{n,m-1} (au_{n,m} v_{n-1,m} + cu_{n,m-1} v_{n,m}) - \beta_{n,m} + w_{n-1,m} g_{n-1,m} (bu_{n-1,m} v_{n,m} + du_{n,m} v_{n-1,m})] = 0, \tag{2.17b}
\]
\[
w_{n,m} f_{n,m} = w_{n+1,m} f_{n+1,m} \quad \text{if } (a, c) \neq 0, \tag{2.17c}
\]
\[
w_{n,m} g_{n,m} = w_{n+1,m} g_{n+1,m} \quad \text{if } (b, d) \neq 0, \tag{2.17d}
\]
\[
\Delta_{n}^{w} \alpha_{n,m} = \Delta_{n}^{w} w_{n-1,m} f_{n,m-1} (au_{n,m} v_{n-1,m} + cu_{n,m-1} v_{n,m}), \tag{2.17e}
\]
\[
\Delta_{n}^{w} \beta_{n,m} = \Delta_{n}^{w} w_{n-1,m} g_{n-1,m} (bu_{n-1,m} v_{n,m} + du_{n,m} v_{n-1,m}). \tag{2.17f}
\]

Here, \(a, b, c\) and \(d\) are constants and \(w_{n,m} = (1 - u_{n,m} v_{n,m})^{-\frac{1}{2}}\). In the same way as (2.2), (2.17) also admits a dimensional reduction to the Ablowitz–Ladik lattice \([7]\). Using (2.17) \textit{a}–(2.17 \textit{f}), we can rewrite (2.17a) and (2.17b) in a more symmetric form with respect to space reflections. When \(c = -a^{*}\) and \(d = -b^{*}\), the (2+1)-dimensional system (2.17) allows the complex conjugation reduction \(v_{n,m} = \sigma u_{n,m}^{*}\) with a real constant \(\sigma\); in this reduction, the auxiliary fields \(f_{n,m}\) and \(g_{n,m}\) become real-valued, while the auxiliary fields \(\alpha_{n,m}\) and \(\beta_{n,m}\) become purely imaginary. In particular, (2.17) with purely imaginary \(a, b, c (= -a^{*})\) and \(d (= -b^{*})\) provides the suitable space discretization of the Davey–Stewartson system (cf (2.4)).

Similar to the continuous case (cf [33–35]), when \(c = -a\) and \(d = -b\), we can consider the reduction of \(v_{n,m} = \sigma u_{n,m}\) and \(\alpha_{n,m} = \beta_{n,m} = 0\) to obtain a (2+1)-dimensional analog of
the modified Volterra lattice [36]:

\[
\frac{\partial u_{n,m}}{\partial t} = (1 - \sigma_{u_{n,m}}^2) \left( bw_{n,m} g_{n,m} u_{n+1,m} - bw_{n-1,m} g_{n-1,m} u_{n-1,m} \right. \\
\left. + a w_{n,m} f_{n,m} u_{n,m+1} - a w_{n,m-1} f_{n,m-1} u_{n,m-1} \right),
\]

(2.18a)

\[w_{n,m} f_{n,m} = w_{n,m+1} f_{n+1,m} \quad \text{if} \quad a \neq 0,\]  
(2.18b)

\[w_{n,m} g_{n,m} = w_{n+1,m} g_{n,m+1} \quad \text{if} \quad b \neq 0.\]  
(2.18c)

Here, \(w_{n,m} = \left(1 - \sigma u_{n,m}^2\right)^{-\frac{1}{2}}\). It would be interesting to look for a relationship between (2.18) and the discrete modified Nizhnik–Veselov–Novikov hierarchy in [31] (also see [37–39]).

3. Solutions by the Hirota method

In this section, we discuss how to construct exact solutions of the discrete Davey–Stewartson system (2.17) using the Hirota bilinear method [20]. Because of the complexity and irrationality of the equations of motion, it would be too hard to solve (2.17) directly, so we take an alternative approach. First, we bilinearize the \((2 + 1)\)-dimensional Ablowitz–Ladik lattice (2.2). Subsequently, we consider the effect of the nonlocal transformation (2.10), (2.13), (2.14) and (2.16) in the bilinear formalism. The infinite products appearing in the nonlocal transformation can essentially be expressed locally in terms of a ‘tau function’. Thus, we can obtain exact solutions of (2.2) and (2.17) concurrently from the same set of bilinear equations.

3.1. Decomposition into four commutative flows

Before applying the Hirota bilinear method, we demonstrate that the \((2 + 1)\)-dimensional Ablowitz–Ladik lattice (2.2) can be decomposed into the four elementary flows. In view of (2.2c), (2.2d) and (2.12b), we rescale the auxiliary fields as

\[A_{n,m} = a A_{n,m}^{(0)}, \quad B_{n,m} = b B_{n,m}^{(0)}, \quad C_{n,m} = c C_{n,m}^{(0)}, \quad D_{n,m} = d D_{n,m}^{(0)}.\]  
(3.1)

The corresponding boundary conditions are

\[\lim_{n \to -\infty} (q_{n,m}, r_{n,m}) = \lim_{m \to -\infty} (q_{n,m}, r_{n,m}) = 0,\]

\[\lim_{n \to -\infty} C_{n,m}^{(0)} = \left(\frac{h_{m-1}}{h_m}\right)^2, \quad \lim_{m \to -\infty} D_{n,m}^{(0)} = \left(\frac{l_{n-1}}{l_n}\right)^2.\]

Thus, considering the simplest cases where only one of the parameters \(a, b, c\) and \(d\) does not vanish, we obtain the four elementary systems:

- \(a\)-system

\[\partial_n q_{n,m} = q_{n,m+1} + A_{n+1,m}^{(0)} q_{n,m},\]  
(3.2a)

\[\partial_n r_{n,m} = -r_{n,m-1} - r_{n,m} A_{n,m}^{(0)},\]  
(3.2b)

\[A_{n+1,m}^{(0)} = A_{n,m}^{(0)} = -(q_{n,m+1} r_{n,m} - q_{n,m} r_{n,m-1}).\]  
(3.2c)
which indeed implies the relation

\[ q_{n,m} = q_{n-1,m} - q_{n,m}B_{n,m}^{(0)}; \]
\[ \partial_t q_{n,m} = -q_{n-1,m} - q_{n,m}B_{n,m}^{(0)}; \] (3.3a)
\[ \partial_t r_{n,m} = r_{n+1,m} + B_{n,m+1}^{(0)}; \] (3.3b)
\[ B_{n,m+1}^{(0)} - B_{n,m}^{(0)} = -(r_{n+1,m}q_{n,m} - r_{n,m}q_{n-1,m}); \] (3.3c)

\[ \partial_t \alpha = \alpha, \quad \partial_t \beta = \beta \]

Because the four systems (3.2)–(3.5) are compatible, in the sense that their flows mutually check the commutativity conditions

\[ \{\partial_t A^{(0)}_{n,m}, \partial_t B^{(0)}_{n,m+1}\} = -\{\partial_t B^{(0)}_{n,m}, \partial_t C^{(0)}_{n+1,m}\}; \]

Using (3.2)–(3.5), we can obtain all necessary expressions in the local forms, e.g.

\[ (1 - q_{n,m}r_{n,m})C_{n,m}^{(0)} = C_{n+1,m}^{(0)}(1 - q_{n,m-1}r_{n,m-1}); \] (3.4c)

\[ \partial_t q_{n,m} = C_{n+1,m}^{(0)} - q_{n,m-1}r_{n,m-1}, \]
\[ \partial_t r_{n,m} = C_{n,m+1}^{(0)} - C_{n,m}^{(0)}; \] (3.4b)
\[ (1 - q_{n,m+1}r_{n,m})D_{n,m}^{(0)} = D_{n,m+1}^{(0)}(1 - r_{n-1,m}q_{n-1,m}). \] (3.5c)

\[ \partial_t q_{n,m} = -q_{n+1,m}D_{n+1,m}^{(0)}; \] (3.5a)
\[ \partial_t r_{n,m} = D_{n,m+1}^{(0)}r_{n-1,m} - q_{n,m}D_{n,m}^{(0)}r_{n-1,m-1}; \] (3.5b)

Clearly, the time evolution in (2.2) is a linear combination of these four time evolutions, that is \( \partial_t = a\partial_a + b\partial_b + c\partial_c + d\partial_d \). In fact, they are mutually commutative, so the above four systems belong to the same integrable hierarchy as the original system (2.2). To check the commutativity conditions \( \partial_t \partial_x q_{n,m} = \partial_x \partial_t q_{n,m} \) and \( \partial_t \partial_x r_{n,m} = \partial_x \partial_t r_{n,m} \) for \( [\alpha, \beta, c] \subset [a, b, c, d] \), we need to know how to express time derivatives of the auxiliary fields. Using (3.2)–(3.5), we can obtain all necessary expressions in the local forms, e.g.

\[ \partial_t A_{n,m}^{(0)} = -(q_{n-1,m+1}r_{n,m} - q_{n-1,m}r_{n,m-1}); \]
\[ \partial_t B_{n,m}^{(0)} = -(C_{n+1,m}^{(0)} - C_{n,m}^{(0)}); \]
\[ \partial_t C_{n,m}^{(0)} = q_{n,m+1}D_{n,m+1}^{(0)}r_{n-1,m} - q_{n,m}D_{n,m}^{(0)}r_{n-1,m-1}; \] etc.

Here, we assumed that all ‘integration constants’ etc can be set equal to zero. With these local expressions, we can check the commutativity of the four flows by direct computations. Note that \( \lim_{t \to -\infty} \partial_t A_{n,m}^{(0)} \) \( \lim_{t \to -\infty} \partial_t B_{n,m}^{(0)} \), \( \lim_{t \to -\infty} \partial_t C_{n,m}^{(0)} \) and \( \lim_{t \to -\infty} \partial_t D_{n,m}^{(0)} \) do not vanish in general. Thus, the \( t_r \)-flow can change the boundary value of the auxiliary field in the \( t_c \)-flow and vice versa; the same applies for the \( t_b \) and \( t_d \)-flow.

### 3.2. Bilinearization

Because the four systems (3.2)–(3.5) are compatible, in the sense that their flows mutually commute, we will consider here their common solution denoted as \( q_{n,m}(t_a, t_b, t_c, t_d), r_{n,m}(t_a, t_b, t_c, t_d) \), etc. Here, \( t_a, t_b, t_c \) and \( t_d \) are independent arguments. Thus, the solution of the original system (2.2) is obtained by setting

\[ t_a = at, \quad t_b = bt, \quad t_c = ct, \quad t_d = dt; \] (3.6)

which indeed implies the relation \( \partial_t = a\partial_a + b\partial_b + c\partial_c + d\partial_d \).

We assume a solution expressible in the form

\[ q_{n,m} = G_{n,m}/F_{n+1,m}, \quad r_{n,m} = H_{n,m}/F_{n,m+1}; \] (3.7a)
Once a solution of the bilinear equations (3.8)–(3.11) is obtained, formula (3.7) with (3.10) and (3.13) provides the solution of the (2 + 1)-dimensional Ablowitz–Ladik lattice (2.2). We assume that it satisfies the boundary conditions (2.12). Thus, by applying the nonlocal transformation (2.9) with (2.10), (2.13), (2.14) and (2.16), we can also obtain the solution of the discrete Davey–Stewartson system (2.17). To evaluate the effect of this nonlocal transformation, we use (3.7a) and (3.10c) (or (3.11c)) to rewrite the infinite products as

\[
\prod_{j=-\infty}^{n-1} \sqrt{1 - q_{j,m} r_{j,m}} = \prod_{j=-\infty}^{n-1} \frac{F_{j+1,m+1} F_{j,m}}{F_{j+1,m} F_{j,m+1}} = \lim_{j \to -\infty} \frac{F_{j,m+1}}{F_{j,m}} F_{j,m+1},
\]

\[
\prod_{k=-\infty}^{m-1} \sqrt{1 - q_{k,n} r_{k,n}} = \prod_{k=-\infty}^{m-1} \frac{F_{k+1,n+1} F_{k,n}}{F_{k+1,n} F_{k,n+1}} = \lim_{k \to -\infty} \frac{F_{k,n+1}}{F_{k,n}} F_{k,n+1}.
\]

and bilinearize the four systems (3.2)–(3.5) in terms of the ‘tau functions’ $F_{n,m}$, $G_{n,m}$ and $H_{n,m}$ as follows:

- **a-system**
  
  \[
  F_{n+1,m+1} \partial_y G_{n,m} - G_{n,m} \partial_y F_{n+1,m+1} = F_{n+1,m} G_{n,m+1},
  \]
  \[
  F_{n,m} \partial_y H_{n,m} - H_{n,m} \partial_y F_{n,m} = -F_{n+1,m} H_{n,m-1},
  \]
  \[
  F_{n,m} \partial_y F_{n+1,m} - F_{n+1,m} \partial_y F_{n,m} = -G_{n,m} H_{n,m-1}.
  \]

- **b-system**

\[
F_{n,m} \partial_y G_{n,m} - G_{n,m} \partial_y F_{n,m} = -F_{n+1,m} G_{n-1,m},
\]
\[
F_{n+1,m} \partial_y H_{n,m} - H_{n,m} \partial_y F_{n+1,m+1} = F_{n+1,m} H_{n+1,m},
\]
\[
F_{n,m} \partial_y F_{n+1,m} - F_{n+1,m} \partial_y F_{n,m} = -G_{n-1,m} H_{n+1,m}.
\]

- **c-system**

\[
F_{n+1,m} \partial_y G_{n,m} - G_{n,m} \partial_y F_{n+1,m} = F_{n+1,m+1} G_{n,m-1},
\]
\[
F_{n,m+1} \partial_y H_{n,m} - H_{n,m} \partial_y F_{n,m+1} = -F_{n,m} H_{n+1,m},
\]
\[
F_{n+1,m} F_{n,m+1} = F_{n+1,m+1} F_{n,m} = G_{n,m} H_{n,m}.
\]

- **d-system**

\[
F_{n+1,m} \partial_y G_{n,m} - G_{n,m} \partial_y F_{n+1,m} = -F_{n,m} G_{n+1,m},
\]
\[
F_{n,m+1} \partial_y H_{n,m} - H_{n,m} \partial_y F_{n,m+1} = F_{n+1,m} H_{n-1,m},
\]
\[
F_{n+1,m} F_{n+1,m+1} = F_{n+1,m+1} F_{n,m} = G_{n,m} H_{n,m}.
\]

To be precise, each triplet of bilinear equations gives a sufficient condition for the corresponding original system. Note that for the **c-** and **d-systems**, the bilinear forms as well as some exact solutions were studied in [18].

### 3.3. General solution formulas

Once a solution of the bilinear equations (3.8)–(3.11) is obtained, formula (3.7) with (3.1) and (3.6) provides the solution of the (2 + 1)-dimensional Ablowitz–Ladik lattice (2.2). We assume that it satisfies the boundary conditions (2.12). Thus, by applying the nonlocal transformation (2.9) with (2.10), (2.13), (2.14) and (2.16), we can also obtain the solution of the discrete Davey–Stewartson system (2.17). To evaluate the effect of this nonlocal transformation, we use (3.7a) and (3.10c) (or (3.11c)) to rewrite the infinite products as
Because we assumed $|q_{n,m}r_{n,m}| \ll 1$, the value of $(F_{n+1,m+1}F_{n,m})/(F_{n+1,m}F_{n,m+1})$ is always restricted to the neighborhood of 1. For simplicity, in considering the solution of (2.17), we also assume that $F_{n,m}$ is positive (or, at least, $|\arg F_{n,m}|$ is sufficiently small); the positivity condition $F_{n,m} > 0$ can fully justify the use of the formulas for the square root, such as $\sqrt{X^2} = X$ and $\sqrt{X/Y} = \sqrt{X}/\sqrt{Y}$. We set the norming functions $h_m$ and $l_n$ in (2.10) as

$$h_m = \lim_{j \to -\infty} \frac{F_{j,m}}{F_{j+1,m}}, \quad l_n = \lim_{k \to -\infty} \frac{F_{n,k}}{F_{n+1,k}}.$$ 

Thus, we obtain

$$X_{n,m} = \sqrt{F_{n,m}} + 1 F_{n,m}, \quad Y_{n,m} = \sqrt{F_{n+1,m}} F_{n,m} + 1.$$

After all, we can express the transformation from (2.2) to (2.17) locally in terms of the ‘tau function’ $F_{n,m}$. Combining (2.9), (2.13), (2.14), (3.1), (3.7) and (3.12), we arrive at general solution formulas for the discrete Davey–Stewartson system (2.17) in the form

$$u_{n,m} = \frac{G_{n,m}}{\sqrt{F_{n+1,m}} F_{n,m+1}}, \quad v_{n,m} = \frac{H_{n,m}}{\sqrt{F_{n+1,m}} F_{n,m+1}},$$

(3.13a)

$$\tilde{A}_{n,m} = \frac{1}{2} (a \partial_x - b \partial_y - c \partial_x - d \partial_y) \log \left( \frac{F_{n,m+1}}{F_{n,m}} \right),$$

(3.13b)

$$\tilde{B}_{n,m} = \frac{1}{2} (b \partial_x - a \partial_y - c \partial_x - d \partial_y) \log \left( \frac{F_{n+1,m}}{F_{n,m}} \right),$$

(3.13c)

$$f_{n,m} = \frac{\sqrt{F_{n,m+2}F_{n,m}}}{F_{n+1,m}}, \quad g_{n,m} = \frac{\sqrt{F_{n+2,m}F_{n,m}}}{F_{n+1,m}}.$$ (3.13d)

Here, the time variables are set as in (3.6) and the auxiliary fields $\alpha_{n,m}$ and $\beta_{n,m}$ are determined from $\tilde{A}_{n,m}$ and $\tilde{B}_{n,m}$ through (2.16). Using the bilinear equations (3.8e) and (3.9e) and noting that (2.17e) and (2.17f) are identities in $a$, $b$, $c$ and $d$, we obtain compact expressions for $\alpha_{n,m}$ and $\beta_{n,m}$:

$$\alpha_{n,m} = (-a \partial_x + c \partial_y) \log \left( \frac{F_{n,m+1}}{F_{n,m}} \right),$$

(3.14a)

$$\beta_{n,m} = (-b \partial_x + d \partial_y) \log \left( \frac{F_{n+1,m}}{F_{n,m}} \right),$$

(3.14b)

and new bilinear equations:

$$F_{n,m} \partial_x F_{n+1,m} - F_{n+1,m} \partial_x F_{n,m} = G_{n,m+1} H_{n,m},$$

(3.15)

$$F_{n,m} \partial_y F_{n+1,m} - F_{n+1,m} \partial_y F_{n,m} = G_{n,m+1} H_{n,m}.$$ (3.16)

Note that (3.15) and (3.16) fill in the piece missing in (3.8)–(3.11). In the next subsection, we construct common solutions to all these bilinear equations. As described below (2.17), when $c = -a^*$ and $d = -b^*$, we can impose the reduction $v_{n,m} = \sigma u_{n,m}^*$ with a real constant $\sigma$. This reduction can be realized by requiring that $F_{n,m} > 0$ and $H_{n,m} = \sigma G_{n,m}^*$. 

13
3.4. Solitons and dromions

The set of bilinear equations (3.8)–(3.11) together with (3.15) and (3.16) is not ideally symmetric in its present form. In particular, it is not clear why reductions such as \( F_{n,m}^* = F_{n,m} \) and \( H_{n,m} = \sigma G_{n,m}^* \) are allowed. To restore the symmetry, we need only to rewrite (3.8a), (3.8b), (3.9a) and (3.9b) using (3.8c), (3.9c) and (3.10c) (or (3.11c)). For example, using (3.8c) and then (3.10c), (3.8a) can be rewritten as

\[
F_{n,m+1} + \partial_t H_{n,m} = G_{n,m} \partial_t F_{n,m+1} = F_{n,m} G_{n,m+1}.
\]

Thus, the full set of bilinear equations can be reformulated in the symmetric form:

\[
F_{n+1,m} + F_{n+1,m} G_{n,m} + G_{n,m} F_{n+1,m} = F_{n,m} G_{n,m+1} = F_{n,m} G_{n,m+1},
\]

(3.17)

\[
F_{n+1,m} \partial_t G_{n,m} - G_{n,m} \partial_t F_{n+1,m} = F_{n+1,m} G_{n,m+1},
\]

(3.18a)

\[
F_{n+1,m} \partial_t F_{n+1,m} - H_{n,m} \partial_t F_{n+1,m} = -F_{n+1,m} H_{n+1,m-1},
\]

(3.18b)

\[
F_{n,m} \partial_t F_{n+1,m} - F_{n+1,m} \partial_t F_{n,m} = -G_{n+1,m} H_{n+1,m},
\]

(3.18c)

\[
F_{n+1,m} \partial_t G_{n,m} - G_{n,m} \partial_t F_{n+1,m} = -F_{n+1,m} G_{n-1,m},
\]

(3.19a)

\[
F_{n+1,m} \partial_t H_{n,m} - H_{n,m} \partial_t F_{n+1,m} = F_{n+1,m} H_{n+1,m},
\]

(3.19b)

\[
F_{n+1,m} \partial_t F_{n+1,m} - F_{n+1,m} \partial_t F_{n,m} = -G_{n+1,m} H_{n+1,m},
\]

(3.19c)

\[
F_{n+1,m} \partial_t G_{n,m} - G_{n,m} \partial_t F_{n+1,m} = F_{n+1,m} G_{n-1,m},
\]

(3.20a)

\[
F_{n+1,m} \partial_t H_{n,m} - H_{n,m} \partial_t F_{n+1,m} = -F_{n+1,m} H_{n+1,m},
\]

(3.20b)

\[
F_{n+1,m} \partial_t F_{n+1,m} - F_{n+1,m} \partial_t F_{n,m} = G_{n+1,m} H_{n+1,m},
\]

(3.20c)

\[
F_{n+1,m} \partial_t G_{n,m} - G_{n,m} \partial_t F_{n+1,m} = -F_{n+1,m} G_{n+1,m},
\]

(3.21a)

\[
F_{n+1,m} \partial_t H_{n,m} - H_{n,m} \partial_t F_{n+1,m} = F_{n+1,m} H_{n+1,m},
\]

(3.21b)

\[
F_{n+1,m} \partial_t F_{n+1,m} - F_{n+1,m} \partial_t F_{n,m} = G_{n+1,m} H_{n+1,m},
\]

(3.21c)

It is now clear that the \( t_r \)-flow and \( t_r' \)-flow can be identified with the \( t_r \)-flow and \( t_r' \)-flow, respectively, through the complex conjugation reduction. Moreover, (3.18) and (3.19) correspond to each other by the interchange of \( n \) and \( m \), up to a redefinition of the variables. With these symmetries in mind, we can considerably reduce the task of constructing explicit solutions to the above 13 bilinear equations.

In the same way as in the continuous case (see, e.g., [40]), we can construct the one-soliton solution and a two-soliton solution straightforwardly. The one-soliton solution is given by

\[
F_{n,m} = 1 - \frac{g^G}{(1 - p^G)(1 - q^G)} (p^G)^n (q^G)^m e^{\omega_G},
\]

\[
G_{n,m} = g p^n q^m e^\omega, \quad H_{n,m} = \mp \bar{p}^n \bar{q}^m e^{\bar{\omega}},
\]

where \( \omega := q t_a - p^{-1} t_b + q^{-1} t_c - p t_d, \bar{\omega} := -\bar{q}^{-1} t_a + \bar{p} t_b - \bar{q} t_c + \bar{p}^{-1} t_d \), and \( g, p, q, \) etc are nonzero constants. The constant \( q \) should not be confused with \( q_{t_a,m} \). Actually, we can shift \( t_a, t_b, t_c \) and \( t_d \) in (3.6) by arbitrary constants, but this freedom can be absorbed by rescaling \( g \) and \( \bar{g} \). When \( c = -a^* \) and \( d = -b^* \) (cf (3.6)), we set \( \bar{p} = p^*, \bar{q} = q^* \), and \( \bar{g} = \sigma g^* \) with
\[ \sigma (1 - |p|^2)(1 - |q|^2) < 0. \] Thus, \( F_{n,m} > 0 \) and \( H_{n,m} = \sigma G_{n,m}^* \), so the complex conjugation reduction is realized.

To save space, we omit a rather lengthy expression for a two-soliton solution. These solitons are direct \((2 + 1)\)-dimensional analogs of the soliton solutions of the \((1 + 1)\)-dimensional systems such as (1.2). With an appropriate choice of the parameters, they represent straight line solitons in the physical variables (cf (3.7a) or (3.13a)) and thus are not localized.

In the following we obtain more interesting solutions, that is, dromion solutions; dromions [41] are spatially localized ‘solitons’ that decay exponentially in all directions [35] and can exhibit nontrivial interaction properties [40, 42]. More details as well as an extensive list of references can be found in the review article [43]. In analogy with the continuous case [40], the one-dromion solution is obtained as

\[
F_{n,m} = 1 + \frac{\alpha_{11}}{1 - \overline{p} p} (\overline{p} p) e^{\omega_{11} t_{1} + \overline{\omega}_{11} t_{2}} + \frac{\alpha_{22}}{1 - \overline{q} q} (\overline{q} q) e^{\omega_{22} t_{1} + \overline{\omega}_{22} t_{2}},
\]

\[
G_{n,m} = \alpha_{12} p^n q^m e^{\omega_{12} t_{1} + \overline{\omega}_{12} t_{2}},
\]

\[
H_{n,m} = \alpha_{21} \overline{p} n \overline{q} m e^{\overline{\omega}_{12} t_{1} + \omega_{22} t_{2}},
\]

where

\[
\omega_{1} := -p^{-1} t_{1} - p t_{2}, \quad \overline{\omega}_{1} := \overline{p} t_{1} + \overline{p}^{-1} t_{2},
\]

\[
\omega_{2} := q t_{1} + q^{-1} t_{2}, \quad \overline{\omega}_{2} := -\overline{q}^{-1} t_{1} - \overline{q} t_{2}.
\]

When \( c = -a^* \) and \( d = -b^* \) (cf (3.6)), we set \( \overline{p} = p^* \), \( \overline{q} = q^* \) and \( \alpha_{12} = \sigma \alpha_{21}^* \) so that \( H_{n,m} = \sigma G_{n,m}^* \). Moreover, if the constant coefficients of the three terms in \( F_{n,m} \) are positive, then \( F_{n,m} > 0 \), so the complex conjugation reduction is realized. Note that the above \( F_{n,m} \) can be written in a \( 2 \times 2 \) determinant form:

\[
F_{n,m} = \det \left( I + \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} p^n e^{\omega_{11}} & 0 \\ 0 & q^m e^{\overline{\omega}_{22}} \end{pmatrix} \begin{pmatrix} 1 - p \overline{p} & 0 \\ 0 & 1 - q \overline{q} \end{pmatrix} \begin{pmatrix} p^n e^{\overline{\omega}_{12}} & 0 \\ 0 & q^m e^{\omega_{22}} \end{pmatrix} \right).
\]

Following the Gilson–Nimmo approach [42] in the continuous case, we construct the multidromion solution called the \((M, N)\)-dromion solution. The one-dromion solution corresponds to the simplest case of \( M = N = 1 \). Hereinafter, we suppress the subscripts of the functions representing their dependence on the spatial variables \( n \) and \( m \). When they are shifted, we express it using the shift operators

\[
(S_n Z)_{n,m} := Z_{n+1,m}, \quad (S_m Z)_{n,m} := Z_{n,m+1}.
\]

We will consider various \((M + N) \times (M + N)\) matrices; they all have the same shape as \( 2 \times 2 \) block matrices, so operations can be performed blockwise. For simplicity, off-diagonal zeros in the block diagonal matrices are omitted. We introduce two diagonal matrices as

\[
\Xi := \begin{pmatrix} \Xi_1 \\ \Xi_2 \end{pmatrix}, \quad \Xi_1 := \text{diag}(\psi_1, \ldots, \psi_M), \quad \Xi_2 := \text{diag}(\chi_1, \ldots, \chi_N),
\]

\[
\overline{\Xi} := \begin{pmatrix} \overline{\Xi}_1 \\ \overline{\Xi}_2 \end{pmatrix}, \quad \overline{\Xi}_1 := \text{diag}(\overline{\psi}_1, \ldots, \overline{\psi}_M), \quad \overline{\Xi}_2 := \text{diag}(\overline{\chi}_1, \ldots, \overline{\chi}_N),
\]

where

\[
\psi_i (n) := p_i^n e^{-p_i^{-1} t_1 - p_i t_2}, \quad \overline{\psi}_i (n) := \overline{p}_i^n e^{\overline{p}_i^{-1} \overline{t}_1 + \overline{p}_i \overline{t}_2}, \quad \chi_i (m) := q_i^m e^{-q_i^{-1} t_2 - q_i t_1}, \quad \overline{\chi}_i (m) := \overline{q}_i^m e^{\overline{q}_i^{-1} \overline{t}_2 + \overline{q}_i \overline{t}_1}.
\]
We also introduce
\[
R := \begin{pmatrix} P & Q \\ \overline{P} & \overline{Q} \end{pmatrix}, \quad P := \text{diag}(p_1, \ldots, p_M), \quad Q := \text{diag}(q_1, \ldots, q_N).
\]
\[
\overline{R} := \begin{pmatrix} \overline{P} & \overline{Q} \\ P & Q \end{pmatrix}, \quad \overline{P} := \text{diag}(\overline{p}_1, \ldots, \overline{p}_M), \quad \overline{Q} := \text{diag}(\overline{q}_1, \ldots, \overline{q}_N).
\]

Then, the following relations hold:
\[
S_n \Xi_1 = \Xi_1 (P I), \quad S_m \Xi_1 = \Xi_1 (I Q),
\]
\[
S_n \overline{\Xi}_1 = \overline{\Xi}_1 (\overline{P} I), \quad S_m \overline{\Xi}_1 = \overline{\Xi}_1 (I \overline{Q}).
\]
\[
\partial_t a /\Xi_1 = -\Xi_1 (\overline{O} Q - 1), \quad \partial_t b /\Xi_1 = -\Xi_1 (P - 1 \overline{O}),
\]
\[
\partial_t c /\Xi_1 = -\Xi_1 (O \overline{Q}), \quad \partial_t d /\Xi_1 = -\Xi_1 (P \overline{O}),
\]
\[
\partial_t a /\overline{\Xi}_1 = \overline{\Xi}_1 (\overline{O} \overline{Q} - 1), \quad \partial_t b /\overline{\Xi}_1 = \overline{\Xi}_1 (P - 1 \overline{O}),
\]
\[
\partial_t c /\overline{\Xi}_1 = \overline{\Xi}_1 (O \overline{Q}), \quad \partial_t d /\overline{\Xi}_1 = \overline{\Xi}_1 (P \overline{O}).
\]

Note that the order of two diagonal matrices on the right-hand side can be changed because they commute. Moreover, we introduce \(M \times M\) matrices \(K\) and \(E_M\) as
\[
(K)_{ij} := \frac{1}{1 - p_i \overline{p}_j}, \quad (E_M)_{ij} := 1, \quad 1 \leq i, j \leq M,
\]
and \(N \times N\) matrices \(L\) and \(E_N\) as
\[
(L)_{ij} := \frac{1}{1 - q_k \overline{q}_l}, \quad (E_N)_{kl} := 1, \quad 1 \leq k, l \leq N.
\]

They satisfy the relations
\[
K - PK \overline{P} = E_M, \quad L - QL \overline{Q} = E_N.
\]

We define \((M + N)\)-component column vectors as
\[
e_M := (1, \ldots, 1, 0, \ldots, 0)^T, \quad e_N := (0, \ldots, 0, 1, \ldots, 1)^T,
\]
and
\[
l := \Xi e_M = (\varphi_1, \ldots, \varphi_M, 0, \ldots, 0)^T, \quad m := \Xi e_N = (0, \ldots, 0, \chi_1, \ldots, \chi_N)^T,
\]
\[
\overline{l} := \overline{\Xi} e_M = (\overline{\varphi}_1, \ldots, \overline{\varphi}_M, 0, \ldots, 0)^T, \quad \overline{m} := \overline{\Xi} e_N = (0, \ldots, 0, \overline{\chi}_1, \ldots, \overline{\chi}_N)^T.
\]

Then, we can easily show the following relations:
\[
S_n l = R l, \quad S_m l = l, \quad S_n m = m, \quad S_m m = R m,
\]
\[
S_n \overline{l} = \overline{R} \overline{l}, \quad S_m \overline{l} = \overline{l}, \quad S_n \overline{m} = \overline{m}, \quad S_m \overline{m} = \overline{R} \overline{m}.
\]

We set the ‘tau function’ \(F_{n,m}\) as
\[
F = \det F.
\]

Here, the \((M + N) \times (M + N)\) matrix \(F\) is defined as
\[
F := I + \Xi \Xi \begin{pmatrix} K & L \\ L & K \end{pmatrix} \Xi.
\]
where $A$ is a constant $(M + N) \times (M + N)$ matrix. We also set the other ‘tau functions’ $G_{n,m}$ and $H_{n,m}$ as

$$G = (\overline{m}^T \mathcal{F}^{-1} A I) F, \quad H = (\overline{I}^T \mathcal{F}^{-1} A m) F.$$  

Note that the one-dromion solution is reproduced by setting $M = N = 1$, up to a minor redefinition of the parameters. Let us check that these ‘tau functions’ indeed satisfy the only bilinear equation without time derivatives (3.17). With the aid of the previous relations, we have

$$S_{n,F} = \det(\mathcal{F} - A I \overline{I^T})$$
$$= \det(\mathcal{I} - \mathcal{F}^{-1} A I \overline{I^T}) F$$
$$= (1 - \overline{I^T} \mathcal{F}^{-1} A I) F.$$

Similarly, we obtain

$$S_{m,F} = (1 - \overline{m}^T \mathcal{F}^{-1} A m) F.$$

To compute $S_{n} S_{m} F$, we still need to know $S_{n} \mathcal{F}^{-1}$. This can be expressed as

$$S_{n,\mathcal{F}^{-1}} = (S_{n,\mathcal{F}})^{-1}$$
$$= (\mathcal{F} - A I \overline{I^T})^{-1}$$
$$= \mathcal{F}^{-1} + \mathcal{F}^{-1} A I \overline{I^T} \mathcal{F}^{-1} \frac{1}{1 - \overline{I^T} \mathcal{F}^{-1} A I}.$$

Here, we used the so-called Sherman–Morrison formula; recall that $A I$ is a column vector and $\mathcal{F}^T$ is a row vector. Combining the above results, we obtain

$$(S_{n} S_{m} F) F = [1 - \overline{m}^T (S_{n,\mathcal{F}^{-1}}) A m] (S_{n} F) F$$
$$= (S_{n} F) [1 - \overline{m}^T \mathcal{F}^{-1} A m] F - (\overline{m}^T \mathcal{F}^{-1} A I) F (\overline{I^T} \mathcal{F}^{-1} A m) F$$
$$= (S_{n} F) (S_{m} F) - GH.$$

This completes the proof of (3.17).

It is a direct but lengthy calculation to check the remaining 12 equations (3.18a)–(3.21c) involving time derivatives. For example, in order to check (3.18b), we need the following intermediate formulas:

$$\partial_{t} F = - (\overline{m}^T \mathcal{F}^{-1} A R^{-1} m) F,$$
$$\partial_{t} H = (\overline{I^T} \mathcal{F}^{-1} A R^{-1} m)(\overline{m}^T \mathcal{F}^{-1} A m - 1) F - (\overline{I^T} \mathcal{F}^{-1} A m)(\overline{m}^T \mathcal{F}^{-1} A R^{-1} m) F,$$
$$S_{m}^{-1} H = (\overline{I^T} \mathcal{F}^{-1} A R^{-1} m) F.$$
To obtain the first formula, we first compute \( \partial_t F \) as

\[
\partial_t F = -A \Xi \begin{pmatrix} O \\ Q^{-1} \end{pmatrix} \begin{pmatrix} K \\ L \end{pmatrix} \Xi + A \Xi \begin{pmatrix} K \\ L \end{pmatrix} \begin{pmatrix} O \\ Q^\dagger \end{pmatrix} \Xi \\
= -A \Xi \begin{pmatrix} O \\ Q^{-1} E_N \end{pmatrix} \Xi \\
= -A R^{-1} e_N e_N^\dagger \Xi \\
= -A R^{-1} m \bar{m}^T.
\]

Then, we multiply it by \( F^{-1} \) and take the trace.

4. Concluding remarks

In this paper, we have studied a suitable space discretization of the Davey–Stewartson system. The Davey–Stewartson system is an integrable NLS system in 2+1 dimensions, which involves two spatial variables on an equal footing and allows the complex conjugation reduction between the dependent variables. We started with a natural (2 + 1)-dimensional generalization of the Ablowitz–Ladik lattice and then considered a nonlocal change of dependent variables to symmetrize the equations of motion. Consequently, we obtained the space-discrete Davey–Stewartson system inheriting most of the important properties of the continuous system; in particular, it is integrable and allows the complex conjugation reduction. The price to pay is the irrationality of the equations of motion and their high degree of nonlocality, which are not seen in the continuous case. Through a simple reduction, we reduced the degree of nonlocality and obtained a discrete modified KdV-type system in two spatial dimensions, namely the (2 + 1)-dimensional modified Volterra lattice (2.18).

The (2 + 1)-dimensional Ablowitz–Ladik lattice, as well as the space-discrete Davey–Stewartson system, is a superposition of four elementary flows that are mutually commutative. Naturally, the number of elementary flows is equal to the number of directions on the square lattice. Note also that both the (1 + 1)-dimensional Ablowitz–Ladik lattice and the continuous Davey–Stewartson system can be written as a sum of two commuting flows. We conjecture that the (2 + 1)-dimensional Ablowitz–Ladik lattice and the space-discrete Davey–Stewartson system possess four infinite sets of higher symmetries. As in the original Ablowitz–Ladik lattice [44, 45], each set of symmetries could be generated from a single discrete-time system using the Maclaurin expansion in the step-size parameter. It would be interesting to provide a more precise description within the framework of the Sato theory, e.g. the discrete two-component KP hierarchy (cf [19]).

Using the Hirota bilinear method, we have constructed exact solutions such as the multidromion solutions of the (2 + 1)-dimensional Ablowitz–Ladik lattice and the space-discrete Davey–Stewartson system concurrently. Their solutions can be obtained from the same set of bilinear equations, although their bilinearizing transformations are rather different (cf (3.7a) and (3.13a)). Note that (3.13a) reflects the irrationality of the space-discrete Davey–Stewartson system that can, however, allow the complex conjugation reduction. The solutions were derived as the common solutions of the four elementary flows. On the level of the bilinear equations, the four flows look fully symmetric and stand on an equal footing. Thus, it is relatively easy to construct their common solutions despite the high number of bilinear equations.
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