Toward the Understanding of Quark Matter Formation

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Abstract

We present the evidence that crossover, fluctuation phenomena and possible Anderson transition are the precursors of quark matter formation.

1 Introduction

During the last decade, the investigation of quark matter at finite temperature and density has become one of the QCD focal points. It is expected that at densities which are 3-5 times larger than the normal nuclear density baryons are crushed into quarks. It is also expected that if the temperature of such quark matter is low enough (below few tens of MeV) the system is unstable with respect to the formation of quark-quark Cooper-pair condensate [1]-[3]. This phenomenon is called color superconductivity since diquarks belong to the 3 color channel. At present there is a fair understanding of color superconductivity physics in the regime of ultra-high density when $\alpha_s$ is small [4]. Most interesting is, however, the region of moderate densities (3-5 times larger than the normal nuclear density). This region is important for physics of neutron stars and may possibly be investigated in the laboratory, in particular, in future experiments at the GSI heavy-ion machine\(^1\).

\(^1\)In heavy-ion collisions high-density state is naturally formed with high temperature thus preventing diquark condensation.
In the moderate density/strong coupling regime the theory faces the well-known difficulties of the nonperturbative QCD. Lattice QCD calculations encounter serious obstacle at nonzero density since in this case the determinant of the Dirac operator is complex resulting in non-positive measure of the corresponding path integral. Still, several attempts to perform lattice calculations at nonzero density have been performed – see the review paper [5]. The region of moderate densities has been extensively studied within the framework of the Nambu-Jona-Lasinio (NJL) type models (see [6, 7] and references therein), using the instanton gas model [8], or chiral perturbation theory [9].

The main conclusion reached within the NJL-type models is that transition from the nuclear matter (NM) phase to the quark matter (QM) phase occurs very early, namely when the quark density reaches the value only three times larger than the density of quarks in normal nuclear matter. This corresponds to the value \( \mu \simeq 400 \text{ MeV} \) of the quark chemical potential. In the NJL-type models the \((NM) \rightarrow (QM)\) transition has two main signatures, namely:

(i) The gap equation for the diquark channel acquires a nontrivial solution.

(ii) The quark constituent mass tends to zero\(^2\) and the chiral symmetry is restored. At \( T = 0 \) the transition is believed to be of the first order.

Both conclusions should be taken with reservations due to oversimplifications inherent for the models and certain arbitrariness in the interpretation of the results.

The principal deficiency of the existing approaches to the \((NM) \rightarrow (QM)\) transition is the lack of understanding what happens to the gluon sector when the density increases. From thermodynamic arguments it follows that possible color diquark condensate and gluon condensate have the same energy scale which results in competition between them somewhat similar to Meissner effect [10]. It also has been demonstrated within the Ginzburg-Landau approach (see below) that fluctuations of the gluon field are important and may significantly influence the character of the phase transition and the critical temperature [11, 12]. From the fact that the would-be diquark condensate belongs to color anti-triplet it follows that 5 of 8 gluons become massive [3]. However, first principle derivation of the QCD string and gluon condensate

\(^2\)In the density region under consideration only \( u \)- and \( d \)-quarks participate in possible pairing
sate evolution with the increase of the density is not currently available. An investigation aimed at the resolution of these problems has been attempted very recently [13].

In the present paper we specify the set of the key parameters which characterize the \((N M) \rightarrow (Q M)\) transition and estimate the values of these parameters. In this way we obtain a model-independent though schematic picture which exhibits several nontrivial phenomena. The onset of the quark phase and its further evolution to higher densities may be viewed as a crossover from the strong coupling regime of composite nonoverlapping bosons (diquarks) to the weak coupling regime of macroscopic overlapping Cooper pair condensate\(^3\). Another feature of the transition region is the drastic increase of fluctuations. The \((N M) \rightarrow (Q M)\) evolution possesses also some features of the Anderson transition. All these three properties are interrelated.

2 BEC-BCS Crossover and the Ginzburg-Levanyuk Number

As already mentioned in the Introduction, the gap equation for the diquark channel derived within the framework of the NJL-type models acquires a nontrivial solution (a gap) at \(\mu \simeq 0.4\) GeV \([6, 7]\). To describe the crossover we will need the corresponding value of the quark number density. Relation between \(\mu\) and \(n\) is given by the well-known equation

\[
n = -\frac{\partial \Omega}{\partial \mu},
\]

where \(\Omega(T, \mu, \Delta)\) is the thermodynamic potential, \(\Delta\) - the gap parameter. For the NJL-type of models \(\Omega\) is easily calculated \([6, 7]\) (see below). According to \([6]\) for \(N_f = 2\) and \(T = 0\) transition to \(\Delta \neq 0\) phase occurs at \(\mu = 0.292\) GeV \(^4\). Then Eq. (1) yields \(n^{1/3} = 0.18\) GeV \([4]\). This number was obtained in the chiral limit, i.e. under the condition that the quark constituent mass goes to zero when \(\Delta \neq 0\). In our view the conclusion that \(m = 0\) when \(\Delta \neq 0\)

\(^3\)This is a particular case of the Bose-Einstein condensate to Bardeen-Cooper-Schrieffer (BEC-BCS) crossover – see below.

\(^4\)Strange quark starts to participate in pairing at much higher densities when \(\mu \gg m_s \simeq 150\) MeV.
may be a specific feature of the NJL model (or some versions of this model). The behavior of the quark constituent mass throughout the \((NM) \rightarrow (QM)\) transition is still a moot point.

The above result for \(n\) may be with a good accuracy reproduced using the equation for free degenerate quarks

\[
n = N_c N_f \frac{k_F^3}{3\pi^2} = \frac{2}{\pi^2} k_F^3 \simeq \frac{2}{\pi^2} \mu^3,
\]

where \(\mu = (k_F^2 + m^2)^{1/2}\), and \(\mu = k_F\) in the chiral limit. For \(\mu = 0.292\) GeV Eq.\(\text{[2]}\) yields \(n^{1/3} = 0.17\) GeV in a good agreement with Eq.\(\text{[1]}\). For \(\mu = 0.4\) GeV Eq.\(\text{[2]}\) gives \(n^{1/3} = 0.23\) GeV. Thus we conclude that \(n^{1/3} \simeq 0.2\) GeV \(\simeq 1\) fm\(^{-3}\) in the transition region.

We now turn our attention to the physics behind the fact that the gap equation derived within the framework of the NJL model and in the mean-field approximation acquires a nontrivial solution starting from \(\mu \simeq 0.4\) GeV. It took quite some time before it was realized \([14, 15]\) that a nonzero value of the gap does not mean the onset of the color superconductivity (the BCS regime). It is only a signal of the presence of fermion pairs. Depending on the dynamics of the system, on the fermion density, and on the temperature, such pairs may be either stable, or fluctuating in time, may form a BCS condensate, or a dilute Bose gas, or undergo a Bose-Einstein (BE) condensation. The fact that there is a continuous transition (crossover) from the strong coupling/low density regime of independent bound state formation to the weak coupling/high density cooperative Cooper pairing is well known\([16]\). In contrast to macroscopic Cooper pairs, the compact molecular – like states which are formed in strong coupling/low density regime are called Schaffroth pairs \([17, 18]\).

The dimensionless crossover parameter is \(n^{1/3} \xi\), where \(\xi\) is the characteristic length of pair correlation when the system is in the BCS regime and the root of the mean square radius of the bound state when the system is in the strong coupling regime. Some arbitrariness occurs in the definition of the crossover parameter. For example, in \([19]\) it is defined as \(k_F \xi \simeq 1.7 n^{1/3} \xi\) (see \([20]\)). Another definition of the crossover parameter is \(x_0 = \mu/\Delta\) \([20]\).

In the BCS theory \(\xi\) is given by \(\xi = v_F/\pi \Delta\), where \(v_F\) is the velocity at the Fermi surface. For a typical metal superconductor \(v_F \simeq c/137\), \(\Delta \simeq 5 K\), so that \(\xi \simeq 10^{-4}\) cm. The density of electrons is \(n \simeq 10^{22}\) cm\(^{-3}\). Therefore in the BCS regime \(n^{1/3} \xi \gtrsim 10^3\). In coordinate space the wave function of
the Cooper pair is proportional to \((\sin k_F r/k_F r) \exp(-r/\xi)\) and hence it has \(\sim 10^3\) nodes.

The crossover from the BCS to the strong coupling regime occurs at \([16, 19-22]\)
\[n^{1/3}\xi \sim 1.\] (3)

The width of the crossover region with respect to the above parameter is several units and is model-dependent \([19-22]\). It was first pointed out in \([14, 15]\) that at \(\mu \sim (0.3 - 0.5)\) GeV the quark system is in the crossover regime and not in the BCS regime as it was inferred from the fact that at such values of \(\mu\) the gap equation acquires a nontrivial solution.

Let us estimate the value of the crossover parameter at the onset of the phase with \(\Delta \neq 0\), i.e., at \(\mu \sim (0.3 - 0.5)\) GeV. We have seen that \(n^{1/3} \sim 1\) fm\(^{-1}\) in this region.\(^5\) The value of \(\xi\) in the strong coupling regime cannot be evaluated from the first principles. One may expect that it is of a typical hadronic scale \(\xi \sim (1 - 2)\) fm and that it grows with density asymptotically approaching the BCS value. Model calculations confirm this expectations \([23] - [25]\). At zero density the root-mean square radius of the diquark is \(\sim 1\) fm \([23]\). At low density the single-gluon exchange model leads to the result \(\xi \sim n^{-1/3}\) \([23]\), while at \(\mu \sim 1\) GeV \(\xi n^{1/3} \sim 10\) \([24]\). It should be noted that the calculation of the pair size in the crossover region is a complicated task even for the “simpler” system of ultracold fermionic atoms \([26]\). To sum up, we conclude that \(\xi n^{1/3} \sim 1\) at the onset of the \(\Delta \neq 0\) phase. The \((NM) \rightarrow (QM)\) transition brings the system to the quark matter state in the crossover regime, but not in color superconducting state.

Next we turn to the structure of the diquark wave function in the strong coupling regime at nonzero density. The corresponding relativistic equation has been derived in \([21]\). The pair wave function has the following asymptotic form
\[\psi \sim \frac{1}{r} \exp \left\{- \frac{(\mu + m) \varepsilon_b}{2} \right\},\] (4)
where the binding energy is given by \(\varepsilon_b/2 = m - \mu\). If we introduce the nonrelativistic chemical potential \(\tilde{\mu} = \mu - m\), we get \(\varepsilon_b = -2\tilde{\mu}\) in complete agreement with the result obtained by Nozieres and Schmitt-Rink \([16]\) (recall that \(\tilde{\mu} < 0\) in the strong coupling limit). In the nonrelativistic case we may
\(^5\)In NJL-type models transition to \(\Delta \neq 0\) phase is accompanied by a sharp increase of density towards the value indicated above. Fluctuation corrections should smooth this behavior.
also write \( \mu + m = \bar{\mu} + 2m \approx 2m \), and then (4) reduces to
\[
\psi \sim \frac{1}{r} \exp \left(-r/a_s\right), \tag{5}
\]
where \( a_s = (m\varepsilon_b)^{1/2} \) is the scattering length. This parametrization is very convenient for the system of cold fermionic atoms where the scattering length is a tunable parameter [27]. However, for quarks the scattering length is an ill-defined quantity.

Let us see, then, what happens to the wave function (4) in ultrarelativistic case if \( m \to 0 \) in line with chiral symmetry restoration. Then
\[
\psi \sim \frac{1}{r} \exp \left(-\frac{r}{2}(-\varepsilon_b^2)^{1/2}\right). \tag{6}
\]
This result shows that the system develops an instability due to the coalescence of the quark and antiquark branches [21]. The arguments presented here are based on the asymptotic form (4) of the wave function. To investigate the vicinity of the instability point one should consider the system of coupled equations for positive and negative energy components written down in [21]. This will be done in a forthcoming paper. Therefore the chiral symmetry restoration in \((NM) \to (QM)\) transition is a subtle point which deserves further investigation.

Next we turn from the crossover parameter to the Ginzburg-Levanyuk number \( G_i \) [28]. The two quantities are interrelated since both characterize the fluctuating pairs. Let us start with the expression for \( G_i \) in case of a clean three-dimensional superconductor [25]
\[
G_i = \frac{27\pi^4}{28\zeta(3)} \left( \frac{T_c}{E_F} \right)^4 \approx 80 \left( \frac{T_c}{E_F} \right)^4, \tag{7}
\]
where \( \zeta(3) \approx 1.2 \), \( T_c \) is the critical temperature and \( E_F \) – the Fermi energy. We note in passing that in [29] \( G_i \) was underestimated by two orders of magnitude. This means an extreme narrowness of the fluctuation region. If we apply (7) to the quark system, take \( \mu \) for \( E_F \) and put \( \mu \simeq (0.3 - 0.4) \) GeV, \( T_c \simeq (0.04 - 0.05) \) GeV [3, 6, 7], we obtain
\[
G_i \gtrsim 10^{-2} \tag{8}
\]
which is a huge number as compared to that for ordinary superconductors. It is by two orders of magnitude larger than the value of \( G_i \) for quarks presented in [29].
At this point one may ask to what extent is the equation (7) for $G_i$ applicable to the quark system. One of the ways to derive it [30] is to compare the radius of interaction between fluctuations with the correlation radius in the system far from the transition point. Then to arrive at (7) use is made of the Ginzburg-Landau functional. Therefore the estimate (9) relies on the applicability of the Ginzburg-Landau theory to the quark system at moderate density. We shall discuss this point at the end of the paper.

Next we wish to express $G_i$ in terms of the crossover parameter. We remind the expression for the coherence length [28]

$$\xi^2 = \frac{7\zeta(3)}{48\pi^2T_c^2}v_F^2,$$

where $v_F$ is the velocity at the Fermi surface. Using Eqs. (7) and (9) we obtain

$$G_i = \frac{21\zeta(3)}{64}(k_F\xi)^{-4} \approx \frac{5 \cdot 10^{-2}}{(n^{1/3}\xi)^4}.$$ (10)

We see that the fluctuation effects in a quark system at moderate density are very strong. Next we have to determine what is the dominant type of fluctuations – that of the order parameter, or of the gauge field potential. The fluctuation contribution to the free energy can be estimated [28] dividing the volume of a specimen by the cube of the fluctuation correlation length of the given type of fluctuation. The correlation length of the order parameter is, as we have seen, $\sim 1$ fm, while that of the gluon field is $\sim 0.2$ fm$^6$ [31]. Therefore fluctuation of the gauge field dominate.

3 The Ginzburg-Landau Functional

The standard framework to consider fluctuations is the Ginzburg-Landau functional (GLF). In this section we shall present the derivation of the GLF for quarks starting from the general form of the effective action common for the NJL, gluon exchange, or instanton models. Considerations presented below may be regarded as an extension of our paper [11].

We start with the QCD Euclidean partition function

$$Z = \int DAD\bar{\psi}D\psi \exp(-S),$$ (11)
where
\[ S = \int d^4x \bar{\psi}(-i\gamma_\mu D_\mu - im + i\mu\gamma_4)\psi + \frac{1}{4} \int d^4xF_\mu^aF_\mu^a. \] \hspace{1cm} (12)

In (12) color and flavor indices are suppressed, \( N_f = 2, N_c = 3, \) and the chemical potential \( \mu \) is introduced. Performing integration over the gauge fields one gets effective fermion action in terms of cluster expansion \[ Z = \int D\bar{\psi}D\psi\exp(-\int d^4xL_0 - S_{eff}), \] \hspace{1cm} (13)
with \( L_0 = \bar{\psi}(-i\gamma_\mu \partial_\mu - im + i\mu\gamma_4)\psi \) and effective action \( S_{eff} = \sum_{n=2}^\infty \frac{1}{n!} \ll \theta^n \gg, \) where \( \theta = \int d^4x\bar{\psi}(x)g\gamma_\mu A_\mu^a(x)t^a\psi(x) \) and the double brackets denote irreducible cumulants \[32\].

The derivation of the GLF from the effective action entering into (13) is a complicated task for further work. Only the first step in this direction has been done recently \[13]. Here we shall follow much simpler route which is used to derive the GLF for electronic superconductors \[33\].

We replace \( S_{eff} \) in (13) by an effective four-fermion interaction with symmetry properties of the two-flavor QCD. This might be either NJL, or instanton vertex, or contact interaction imitating one-gluon exchange. Symbolically such interaction looks like
\[ L_{int} = g(\bar{\psi} \hat{K}\psi)(\bar{\psi} \hat{K}\psi), \] \hspace{1cm} (14)
where the constant \( g \) has a dimension \( m^{-2} \).

Then we perform the Fierz transformation in the Lorentz, color and flavor spaces. As a result the interaction term takes the form
\[ L_{int} = g(\bar{\psi}^c \hat{R}\psi)(\bar{\psi}^c \hat{R}^+\psi^c). \] \hspace{1cm} (15)
Here \( (\bar{\psi}^c \hat{R}\psi) \) is a scalar diquark in color \( \bar{3} \) state
\[ \bar{\psi}^c \hat{R}\psi = \psi^T_{\alpha i}C\delta_{\gamma 3}\varepsilon_{\alpha\beta\gamma}(\tau_2)_{ij}\gamma_5\psi^c_{\beta j}, \] \hspace{1cm} (16)
where \( C = \gamma_2\gamma_4 \) and the presence of the color structure \( \delta_{\gamma 3}\varepsilon_{\alpha\beta\gamma} \) signals that color symmetry is broken by such diquarks.

Next step is to integrate the partition function over quark fields. The interaction is quartic in the fermion fields and therefore one has to apply the Hubbard-Strotonovich trick (bosonization)
\[ \exp\left\{ g \int_0^\beta d\tau \int d\mathbf{r}(\bar{\psi}^c \hat{R}\psi)(\bar{\psi}^c \hat{R}^+\psi^c) \right\} = \]
\[ = \int D\Delta^* D\Delta \exp \left\{ - \int_0^\beta d\tau \int d\mathbf{r} \left[ \frac{|\Delta|^2}{g} - \Delta (\bar{\psi}^c \hat{R}^+ \psi^c) - \Delta^* (\bar{\psi}^c \hat{R} \psi) \right] \right\}, \]  
\tag{17}

where \( \Delta \) is a complex scalar field. From the Lagrange equation of motion for \( \Delta^* \) we see that \( \Delta = g \langle \bar{\psi}^c \hat{R} \psi \rangle \), so that \( \Delta \) has a dimension of mass as it should be for the gap parameter. Now we can integrate the partition function over the quark fields and obtain

\[ Z = \int D\Delta^* D\Delta \exp \left\{ - \frac{1}{g} \int_0^\beta d\tau \int d\mathbf{r} |\Delta|^2 - \Omega_B(T, \mu, \Delta^*, \Delta) \right\}, \]  
\tag{18}

where \( \Omega_B \) is the Bogolubov functional

\[ \Omega_B = -\frac{1}{2} tr \ln \left( \begin{array}{cc} \Delta \hat{R} & i\partial_\mu \gamma_\mu - i\mu \gamma_4 \\ -i\partial_\mu \gamma_\mu T & \Delta^* \hat{R}^+ \end{array} \right). \]  
\tag{19}

We use the following representation

\[ \gamma = \begin{pmatrix} 0 & -i\sigma \\ i\sigma & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]  
\tag{20}

\[ C = \gamma_2 \gamma_4, \quad C^{-1} \gamma_\mu C = -\gamma_\mu T, \quad \partial_\mu T = \bar{\gamma}_\mu = -\bar{\partial}_\mu. \]  
\tag{21}

In (19) the quark mass is for simplicity set to zero. The important assertion we are making is that the gauge field may be introduced at the last step by replacing the gradient by the covariant derivative and adding the Yang-Mills part of the Lagrangian. We may also recall [6, 7, 11] that the Fierz transformation from (14) to (15) results in some extra terms in (15) including the chiral condensate \( \langle \bar{\psi}_\alpha \delta_{\alpha \beta} \delta_{ij} \psi_\beta \rangle \). It has been shown in [11] that in the leading approximation the thermodynamics of the system is determined by diquarks. Finally, we wish to note that according to (18) the partition function is obtained by averaging the Bogolubov functional \( \Omega_B \) with the Gaussian weight factor \( |\Delta|^2/g \).

Next we rearrange (19) using the identity

\[ tr \ln \left( \begin{array}{cc} B & A \\ -A^T & B^+ \end{array} \right) = tr \ln AA^T + tr \ln \{1 + A^{-1} B (A^T)^{-1} B^+ \}. \]  
\tag{22}
Before we apply (22) to $\Omega_B$ given by (19), we rewrite the operator $\hat{R}$ (see (16)) as $\hat{R} = \hat{R}_0 C$, where $C = \gamma_2 \gamma_4$ is the charge conjugation operator. Then the following formula is easily worked out to be

$$A^{-1} \Delta \hat{R}(A^T)^{-1} \Delta^* \hat{R}^+ = \hat{R}_0 \hat{R}_0^+ \sum_p \Delta(p) \Delta^*(p) \sum_k G(k) G(p - k),$$

(23)

where $A = i \partial_\mu \gamma_\mu - i \mu \gamma_4$, and $G$ is the thermal propagator

$$G(q) = \frac{i}{q \gamma + q_4 \gamma_4 + i \mu \gamma_4} \equiv \frac{i}{q + i \mu \gamma_4}$$

(24)

with $q_4 = -\pi (2n + 1)T, T = \beta^{-1}$. Next we return to formula (18) and write

$$Z Z_0^{-1} = \exp \{-(\Omega - \Omega_0)\} ,$$

(25)

with $(\Omega - \Omega_0)$ being the $\Delta \neq 0$ part of the thermodynamic potential. Now we can finally combine (18)-(19) and (23)-(25) and somewhat symbolically write

$$\Omega - \Omega_0 = \frac{1}{g} \int_0^\beta d\tau \int dr |\Delta|^2 - \frac{1}{2} tr \ln \left\{1 + \hat{R}_0 \hat{R}_0^+ \Delta^* GG\right\} .$$

(26)

Expanding the logarithm in (26) in powers of $\Delta^2$ we arrive at the GLF. Following the standard approach we take into account the spatial variation of $\Delta$ in the quadratic term and neglect such dependence in the quartic term. The diagrams corresponding to the terms proportional to $|\Delta|^2$ and $|\Delta|^4$ are shown in Fig.1.
We start with the quadratic term in (26). It reads

\[ \Omega^{(2)} = \frac{1}{g} \int_{\tau_0}^\beta d\tau \int d\mathbf{r} |\Delta|^2 - \frac{1}{2} tr \hat{R}_0 \hat{R}_0^+ \sum_p \sum_k \frac{-\Delta(p)\Delta^*(p)}{(k + i\mu\gamma_4)(\hat{p} - k + i\mu\gamma_4)}. \] (27)

The trace over flavor and color indices is

\[ tr_{f,c} \hat{R}_0 \hat{R}_0^+ = N_f(N_c - 1) = 4. \] (28)

After this trace is taken, formula (27) still contains \( tr' \) over the Lorentz indices (\( \gamma \)-matrices).

The GLF corresponds to the 'soft loop” approximation

\[ tr'\sum_k \frac{-1}{(k + i\mu\gamma_4)(\hat{p} - k + i\mu\gamma_4)} \simeq A + B \mathbf{p}^2. \] (29)

First we compute the static term \( A \). We have

\[ A = \frac{1}{\beta} tr' \sum_{\omega_n} \int \frac{d\mathbf{k}}{(2\pi)^2} \frac{1}{(k + i\mu\gamma_4)(k - i\mu\gamma_4)}. \] (30)

By simple algebra it is easily seen that

\[ \frac{1}{(k + i\mu\gamma_4)(k - i\mu\gamma_4)} = 2 \left\{ \frac{1}{k^2 + (k - \mu)^2} + \frac{1}{k^2 + (k + \mu)^2} \right\}. \] (31)

We assume that one can neglect the contribution of antiparticles. This means that only the first term should be kept in the representation (31). Then we perform the Matsubara summation in (30) and obtain

\[ A = \frac{\nu_F}{4} \int_{-\Lambda}^\Lambda d\xi \frac{\xi}{\xi th \frac{\xi}{2T}}. \] (32)

Here \( \nu_F = 2\mu^2/\pi^2 \) is the density of states at the Fermi surface. Due to color and flavor degrees of freedom it is four times larger than the corresponding BCS theory factor. Parameter \( \Lambda \) is the cut-off of the four fermion interaction. In BCS theory it is the Debye frequency, in the NJL model \( \Lambda \simeq 0.8 \text{ GeV} \). As we shall see, the value of \( \Lambda \) is irrelevant as soon as \( \Lambda/2T_c \gg 1 \). Integration in (32) is over \( \xi = k - \mu \).
To proceed further we make an important assumption. In the summation
over \( p \) in (27) we integrate over Fourier coefficients with \( \omega \neq 0 \). In BCS theory this leads to small corrections to the GLF coefficients \[33\]. This means that we keep in (27) only \( \Delta(p) \) and \( \Delta^*(p) \) having \( \omega = 0 \) frequency. Then

\[
\sum_p \Delta(p)\Delta^*(p) = \beta \int \frac{dp}{(2\pi)^3} \Delta(p)\Delta^*(p) = \beta \int d|\Delta(r)|^2. \tag{33}
\]

Under the same assumption we can integrate over \( \tau \) the first term in (27). Next we expand the integrand in (32) in powers of \((T_c - T)/T_c\)

\[
\frac{1}{\xi} \approx \frac{1}{\xi_0} + \frac{T_c - T}{2T_c^2} \left( \frac{ch \xi_0}{2T_c} \right)^2. \tag{34}
\]

Upon insertion into (27) and with the account of (28) the first term in the right-hand side of (34) cancels with the first term in the right-hand side of (27). This is because of the gap equation or equivalently from the definition of the critical temperature. Then we perform the integration in (32) keeping only the last term in (34) and setting the limits of integration to \( \pm \infty \) since \( \Lambda/2T_c \sim 10 \) and the integrand is a rapidly decreasing function. The result for \( A \) reads

\[
A = \frac{\nu_F}{2} \left( \frac{T_c - T}{T_c} \right). \tag{35}
\]

Now we turn to the coefficient \( B \) in (29). Keeping in mind the assumption formulated prior to Eq. (33), we write

\[
B = \frac{1}{2p^2\beta tr'} \sum_{\omega_n} \int \frac{dk}{(2\pi)^3} \frac{1}{(k + i\mu\gamma_4)} \left( P \frac{\partial}{\partial k} \right)^2 \frac{1}{(k - i\mu\gamma_4)} =
\]

\[
= -\frac{\nu_F}{48\beta t \rho} \gamma^2 \sum_{\omega_n} \int_{-\infty}^{+\infty} \frac{d\xi}{(\xi^2 + k_4^2)^2} = -\nu_F \frac{7\zeta(3)}{96\pi^2 T_c^2}. \tag{36}
\]

With the coefficients \( A \) and \( B \) at hand we can write down the quadratic part of the thermodynamic potential

\[
\Omega^{(2)} = \beta \nu_F \int d|t|\Delta(r)|^2 + \frac{7\zeta(3)}{48\pi^2 T_c^2} |\grad \Delta(r)|^2 \}, \tag{37}
\]

\[^7\text{In fact the above assumption was tacitly used already in Eq. (29).}\]
where \( t = (T - T_c)/T_c \).

The derivation of the quartic term essentially repeats that of the quadratic term presented above. Therefore we omit the technical details. The result reads

\[
\Omega^{(4)} = \beta \nu_F \frac{7 \zeta(3)}{16\pi^2 T_c^2} \int dr |\Delta(r)|^4.
\]  

(38)

Now we can assemble the pieces together. We remind the two assumptions formulated above: (i) non-zero Matsubara are decoupled, and (ii) the gluon field enters via the covariant derivative. The result reads

\[
\Omega = \beta \int dr F(T, \mu, \Delta^*, \Delta, A^l),
\]  

(39)

\[
F = \nu_F \left\{ t|\Delta|^2 + \frac{\beta}{2} |\Delta|^4 + \gamma |D\Delta|^2 \right\} - \frac{1}{2} A^l_k \nabla^2 A^l_k.
\]  

(40)

Here

\[
D = \nabla - ig\lambda^i 2 A^l,
\]  

(41)

\[
\beta = \frac{7 \zeta(3)}{8\pi^2 T_c^2}, \quad \gamma = \frac{7 \zeta(3)}{48\pi^2 T_c^2}.
\]  

(42)

Note that \( \gamma = \xi^2 \), where \( \xi^2 \) is the BCS theory correlation length (see Eq. (8)).

The last term in (40) is the contribution of the gluon field in the Coulomb gauge with cubic and quartic terms neglected.

One may argue that expression (40) contains nothing new. It coincides with the standard GLF with the coefficients (42) obtained by Gorkov almost half a century ago [34]. For the quark system the GLF was presented in [1, 35] and some later papers. What we have tried to do here is to expose several subtle points in the derivation of the GLF for quarks. It should be clear that even if one takes for granted that in some mysterious way the role of the gluon sector reduces to the generation of an effective four-quark interaction it is by far not enough to straightforwardly derive the GLF.

4 Fluctuation Induced Color Diamagnetism

We have seen that the \((NM) \rightarrow (QM)\) transition takes place in the strong fluctuation regime. Fluctuating diquarks are the precursors for the Cooper
pairs forming the BCS condensate. Below we consider fluctuations of the gluon field. This type of fluctuations is responsible for two effects [36]: (i) the lowering of the critical temperature (fluctuation diamagnetism), and (ii) modification of the order of the phase transition (first order instead of second). Both points with regard to quark system have been discussed in recent years [37, 38].

Our emphasis here will be on another aspect of the gluon field fluctuation phenomena, namely ”the emerging phenomenology of \(\langle (A^a)^2 \rangle\)” gluon condensate [39, 40]. In the last few years there has been a growing interest in condensates of dimension two [41]. It might be that there is some (indirect) connection between the phenomenology of \(\langle A^2 \rangle\) and gluon field fluctuation in dense quark matter. There is a further possibility that the problem of \(\langle A^2 \rangle\) is linked to the Anderson transition in quark matter. In order to clearly define what we mean by \(\langle A^2 \rangle\) in the context of dense quark matter we start with the derivation of the basic equations.

The partition function expressed in terms of the variables \(\Delta, \Delta^*\) and \(A^i\) reads

\[
Z = \int D\Delta^* D\Delta DA^i \exp \left\{ -\beta \int dr F(T, \mu, \Delta^*, \Delta, A^i) \right\} = \int D\Delta^* D\Delta Z_A, \tag{43}
\]

where \(F\) is the GLF [40] and

\[
Z_A = \int DA^i \exp \left\{ -\beta \int dr F \right\}. \tag{44}
\]

Integration over the gauge fields yields

\[
Z_A = \exp\{-\beta \nu_F \int dr (t|\Delta|^2 + \frac{\beta}{2}|\Delta|^4) + \frac{1}{\beta} \int \frac{dk}{(2\pi)^3} \ln(2\bar{\gamma} g^2 |\Delta|^2 + k^2)\}. \tag{45}
\]

Here \(\bar{\gamma} = \nu_F \gamma\). The coupling constant \(g^2\) in (45) includes the averaging of the kinetic term \(|D\Delta|^2\) over the generators \(\lambda^i\). The corresponding calculations
may be found in Ref. [12]. Taking the derivative of $\ln Z_A$ with respect to $|\Delta|$ we obtain

\[ -\frac{T}{2V} \frac{\partial}{\partial |\Delta|} \ln Z_A = \nu_F \{ t|\Delta| + \beta |\Delta|^3 \} + \]

\[ + T \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{2\gamma g^2 |\Delta|}{\mathbf{k}^2 + 2\gamma g^2 |\Delta|^2}. \]  

(46)

The same derivative calculated directly from (44) and (40) reads

\[ -\frac{T}{2V} \frac{\partial}{\partial |\Delta|} \ln Z_A = \nu_F \{ t|\Delta| + \beta |\Delta|^3 \} + \]

\[ + \gamma g^2 |\Delta| Z_A^{-1} \int DA^i(A^2) \exp(-\beta \int d\mathbf{r} F) \} = \]

\[ = \nu_F \{ t|\Delta| + \beta |\Delta|^3 + \gamma g^2 |\Delta| \langle A^2 \rangle \}. \]  

(47)

Comparing (46) and (47), one obtains

\[ \langle A^2 \rangle = Z_A^{-1} \int DA^i(A^2) \exp \{ -\beta \int d\mathbf{r} F \} = \]

\[ = 2T \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{1}{\mathbf{k}^2 + M^2}, \]  

(48)

where $M^2 = 2\nu_F \gamma g^2 |\Delta|^2$. We recognize in $M$ the London penetration depth. The average $\langle A^2 \rangle$ is the expectation value of $A^2$ in a fixed bosonic field $\Delta$. Next we integrate (47) back over $\Delta$ and obtain

\[ F = \nu_F \left\{ (t + g^2 \xi^2 \langle A^2 \rangle) |\Delta|^2 + \frac{\beta}{2} |\Delta|^4 \right\} . \]  

(49)

The critical temperature shifted due to gluon field fluctuations corresponds to zero of the quadratic term coefficient [36, 28, 43]

\[ T'_C = T_C (1 - g^2 \xi^2 \langle A^2 \rangle). \]  

(50)

We can estimate $g^2 \xi^2$ as $g^2 = 4\pi \alpha_S \simeq 4$, $\xi \simeq 2$ fm, $g^2 \xi^2 \simeq 400$ GeV$^{-2}$. Then the formal absolute upper bound on $\langle A^2 \rangle$ following from (50) would be $\langle A^2 \rangle < (50$ MeV)$^2$. Later we shall return to the estimate of $\langle A^2 \rangle$.  

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The second effect of the gauge field fluctuations is the replacement of the second order phase transition by the first order one. This point was thoroughly discussed in the literature [36, 1, 37, 38, 11]. A careful look at the expression (49) for $\langle A^2 \rangle$ shows that it induces the $|\Delta|^3$ term in the GLF. To see this consider equation (48) for $\langle A^2 \rangle$ and expand it in terms of $M/\Lambda$, where $\Lambda \simeq 0.8$ GeV is the cut-off parameter.

$$
\langle A^2 \rangle = \frac{T \Lambda}{\pi^2} \left( 1 - \frac{\pi M}{2 \Lambda} + \frac{M^2}{\lambda^2} - \ldots \right) \simeq \frac{T \Lambda}{\pi^2} - \frac{g}{\pi^2} T (\mu \xi) |\Delta|.
$$

Then instead of (49) we have

$$
F = \nu_F \left\{ \left( t + g^2 \xi^2 \frac{T \Lambda}{\pi^2} \right) |\Delta|^2 - \frac{g^3}{\pi^2} T \mu \xi^3 |\Delta|^3 + \frac{3}{2} |\Delta|^4 \right\}.
$$

Due to the term proportional to $|\Delta|^3$ the potential $F$ acquires a second minimum at a finite value of $|\Delta|$. This means that the phase transition is of the first order. For further discussion of this question see [37, 38].

From (51) we obtain another upper bound on $\langle A^2 \rangle$

$$
\langle A^2 \rangle < \frac{T_C \Lambda}{\pi^2} \simeq (60 \text{MeV})^2
$$

for $T_C = 50$ MeV, $\Lambda = 800$ MeV.

We see that the $(NM) \rightarrow (QM)$ transition is accompanied by the gluon field fluctuations with the typical magnitude $\langle A^2 \rangle \lesssim (50 \text{MeV})^2$. What might be the meaning of this result? Is this number large or small? The quantity $\langle A^2 \rangle$ is gauge-variant, but in [40] it was shown that it attains its minimum in the Coulomb gauge. The comparison of the above result with lattice calculations of $\langle A^2 \rangle$ at zero baryon density below and above $T_C$ [39] shows that our value is by more than an order of magnitude smaller than the lattice result. One may argue that the two quantities are defined somewhat differently (see [39, 40]). The physical reason for possible mismatch is that the increase of the baryon/quark density leads to the suppression of the gluon field [10, 13]. However the suppression by a factor of 10 - 20 seems to contradict the expected magnitude of the effect [10]. The possible way to resolve this difficulty (if it exists) is to return to the general expression for the GLF [40] and reanalyze the loop diagram which gives rise to the coefficient $\gamma$ in [40] and then to the coefficient $\xi^2$ in front of $\langle A^2 \rangle$ in [50].

8According to [37] $\Lambda = T_C$ (in the high density regime).
5 A Marginal Remark on Anderson Localization

Anderson localization (AL) [41] is one of the basic concepts of contemporary condensed matter physics. Recently it was realized [45, 28] that this phenomena is important for the description of superconductivity in strongly disordered systems.

The explanation of AL can be given in terms of the two-particle Green’s function (29) (the loop diagram in Fig.1). This diagram embedded into the disordered background determines the dynamical momentum dependent diffusion coefficient which turns zero at the localization edge. In the theory of superconductivity the disorder is created by impurities. According to Ioffe-Regel criterion [46] the phase transition to the AL regime takes place at

\[ k_F l \lesssim 1, \quad (54) \]

where \( l \) is the mean free path. Approaching the localization region the coefficient \( \gamma \) in the GLF (40) (and hence the coefficient \( \xi^2 \) in (50)) undergoes a drastic change [45]. In particular, close to mobility edge (condition (54)) the coefficient \( \gamma = \xi^2 \) in (40) is substituted by [45]

\[ \gamma = \left( \frac{D_0 l}{T_C} \right)^{2/3}, \quad (55) \]

where \( D_0 = \frac{1}{3}v_F l \) is the Drude diffusion coefficient. Formula (55) is one of several possible expressions for \( \gamma \). The concrete scenario of the evolution of \( \gamma \) as a function of the disorder strength depends upon the values of the critical exponent and parameters (54) – see Refs. [28, 45]. Using (55) we may illustrate the idea of possible presence of AL in the quark system in the transition region and estimate the renormalization of the coefficient \( \gamma \) in GLF.

The hypothesis that AL may play some role in quark systems was first formulated in [47]. The role of impurities was attributed to random components of the gluon fields. The behavior of the quark Green’s function was analyzed and it was shown that localization should not be confused with confinement. In the framework of this idea let us estimate the renormalization of the GLF coefficient \( \gamma \) according to (55). The characteristic scale of the quark mean free path is \( l \simeq 1 \text{ fm} \). This might be, e.g., the average distance between
instantons in the instanton vacuum picture of QCD. For $l = 1 \text{ fm}$, $v_F = 1, T_C = 50 \text{ MeV}$ equation (55) yields $\gamma \simeq 1.2 \text{ fm}^2$, while $\gamma = \xi^2$ with $\xi = 2 \text{ fm}$ gives $\gamma \simeq 4 \text{ fm}^2$. Approaching the edge of localization the expression (51) and the estimate (53) are not valid any more since the inverse penetration depth $M$ in (51) becomes energy dependent through the energy dependence of $\gamma$ (expression (55) is valid only in the immediate vicinity of the mobility edge). With the above estimate for $\gamma$ we may return to Eq. (50) and obtain a new upper bound on $\langle A^2 \rangle$ which is $\langle A^2 \rangle \lesssim (100 \text{ MeV})^2$. Hence the discrepancy between our estimate for $\langle A^2 \rangle$ and that given in Ref. [39] may be at least partly eliminated by the account of AL.

6 Summary

It is clear that further work is needed before the process of quark matter formation is finally understood. Three important features of this process have been examined in the present paper. They are: (i) crossover from strong coupling/low density to weak coupling/high density, (ii) strong fluctuation regime, (iii) possible Anderson localization. We have seen that these three points are interrelated.

It can be definitely stated that there is no direct transition from nuclear matter phase to quark superconducting BCS-like phase. In between the two phases there is a crossover region with strong fluctuations and possible Anderson localization. Despite the extensive investigations our understanding of the $(NM) \rightarrow (QM)$ transition is in no way being complete. In particular, the behavior of the string tension in this region is almost intractable (see, however, Ref. [13]).

We have presented a detailed derivation of the GLF within the mean-field three-dimensional effective theory. We have found that the GL approach is on a shaky ground in this region. Small correlation length of the gluon field makes the local approximation baseless, fluctuation corrections are large and the influence of antiquarks is not negligible. Therefore the investigation of the interior of neutron stars using GLF (see, e.g., [49]) can hardly bring reliable results.

We have found an interesting intersection between the QM physics and Anderson localization in highly disordered media. The behavior of the quark

\footnote{Because of quark spin-flip instantons may be similar to magnetic impurities.}
loop diagram shown in Fig.1 embedded into the gauge field background should be studied more deeply.

Finally we wish to mention that the location of the crossover region on the QCD phase diagram has been discussed in a recent publication [50].

7 Acknowledgments

We would like to thank N.O.Agasian, F.V.Gubarev, V.I.Shevchenko and Yu.A.Simonov for discussions and critical remarks. This work was partially supported by the RFBR grant 06-02-17012, Leading Scientific Schools grant # 843.2006.2 and State Contract 02.445.11.7424. 2006-112. Finally, we wish to thank Prof. H.Abuki for bringing our attention to Ref [51], which contains a comprehensive analysis of several questions discussed in the present paper. We are also grateful to Prof. N.Itoh, whose paper [52] was not known to us before.

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