ON THE NUMBER OF ERGODIC PHYSICAL/SRB MEASURES OF
SINGULAR-HYPERBOLIC ATTRACTING SETS

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ABSTRACT. It is known that sectional-hyperbolic attracting sets, for a $C^2$ flow on a finite
dimensional compact manifold, have at most finitely many ergodic physical invariant
probability measures. We prove an upper bound for the number of distinct ergodic phys-
ical measures supported on a connected singular-hyperbolic attracting set for a 3-flow.
This bound depends only on the number of Lorenz-like equilibria contained in the at-
tracting set. Examples of singular-hyperbolic attracting sets are provided showing that
the bound is sharp.

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1. INTRODUCTION

In 1963, the meteorologist Edward Lorenz published in the Journal of Atmospheric Sciences [25] an example of a parametrized polynomial system of differential equations

\[
\begin{align*}
\dot{X} &= a(Y - X) & a &= 10 \\
\dot{Y} &= rX - Y - XZ & \text{where} & b &= 8/3 \\
\dot{Z} &= XY - bZ & r &= 28
\end{align*}
\]  

(1.1)

as a very simplified model for thermal fluid convection, motivated by an attempt to understand the foundations of weather forecast. Numerical simulations performed by Lorenz for an open neighborhood of the chosen parameters suggested that almost all points in phase space tend to a chaotic attractor, whose well known “butterfly” picture can be easily found in the literature.

The mathematical study of these equations began with the geometric Lorenz flows, introduced independently by Afraimovich-Bykov-Shil’nikov [1] and Guckenheimer-Williams [21, 47] as an abstraction of the numerically observed features of solutions to (1.1). Tucker [43] showed that the maximal invariant subset of the classical Lorenz equations (1.1) is in fact a geometric Lorenz attractor through a computer assisted proof. For more on the rich history of the study of this system of equations, the reader can consult [46, 8].

The geometric Lorenz attractor is the most representative example of the class of singular-hyperbolic flows, an extension of the notion of (uniform) hyperbolicity encompassing invariant sets with equilibria accumulated by regular orbits inside the set [29, 30]. Singular-hyperbolic attracting sets were shown to be sensitive to initial conditions, to posses finitely many ergodic physical measures with full basin and with strong statistical properties; see e.g. [9, 7, 5, 6].

Arguably one of the most important concepts in Dynamical Systems theory is the notion of physical (or SRB) measure. We say that an invariant probability measure \( \mu \) for a flow \( \phi_t \) induced by a vector field \( G \) is physical if the set

\[
B(\mu) = \left\{ z \in M : \lim_{t \to \infty} \frac{1}{t} \int_0^t \psi(\phi_s(z)) \, ds = \int \psi \, d\mu, \forall \psi \in C^0(M, \mathbb{R}) \right\}
\]

has non-zero volume, with respect to any volume form on the ambient compact manifold \( M \). The set \( B(\mu) \) is by definition the (ergodic) basin of \( \mu \). It is assumed that time averages of these orbits be observable if the flow models a physical phenomenon.

The study of the existence of these special measures and their statistical properties for uniformly hyperbolic diffeomorphisms and flows has a long and rich history, starting with the works of Sinai, Ruelle and Bowen [15, 16, 38, 39, 41] which were inspired in the work of Anosov [2]. Hyperbolic attractors (transitive basic pieces of the Smale “Spectral Decomposition”) admit a unique physical measure, as well as transitive Anosov flows, as long as the smoothness of the underlying vector field is at least \( C^2 \). For non-transitive Anosov flows in 3-manifolds, constructed by Franks and Williams [20], there are finitely many possible ergodic physical measures after Brunella [17].
It is well-known that singular-hyperbolic attractors (that is, containing a dense forward regular orbit of the flow) supports a unique ergodic physical/SRB measure whose basin covers an open neighborhood the attractor, except for a subset of Lebesgue measure (volume) zero; see \cite{9}. Analogously singular-hyperbolic attracting sets (possibly without a transitive trajectory) support finitely many ergodic physical/SRB measures whose basins together cover an open neighborhood the attractor, again with the exception of a zero volume subset; see e.g. \cite{3} and references therein.

In this work, we study the number of distinct ergodic physical measures supported on a singular-hyperbolic attracting sets and provide an upper bound for this number, depending only on the number of Lorenz-like equilibria contained in the attracting set. Examples of singular-hyperbolic attracting sets are provided showing that the bound is sharp.

### 1.1. Definitions and statements of results.

Let $M$ be a compact connected Riemannian manifold with dimension $\dim M = m$, induced distance $d$ and volume form $\text{Leb}$. Let $\mathcal{X}^r(M), r \geq 1$, be the set of $C^r$ vector fields on $M$ and denote by $\phi_t$ the flow generated by $G \in \mathcal{X}^r(M)$. For any given subset $S \subset M$ we denote by $\overline{S}$ the topological closure of $S$.

#### 1.1.1. Sectional-hyperbolic attracting sets.

An invariant set $\Lambda$ for the flow $\phi_t$ is a subset of $M$ which satisfies $\phi_t(\Lambda) = \Lambda$ for all $t \in \mathbb{R}$. Given a compact invariant set $\Lambda$ for $G \in \mathcal{X}^r(M)$, we say that $\Lambda$ is isolated if there exists an open set $U \supset \Lambda$ such that $\Lambda = \bigcap_{t \in \mathbb{R}} \overline{\text{clos} \phi_t(U)}$.

If $U$ can be chosen so that $\overline{\text{clos} \phi_t(U)} \subset U$ for all $t > 0$, then we say that $\Lambda$ is an attracting set and $U$ a trapping region (or isolated neighborhood) for $\Lambda = \Lambda_G(U) = \bigcap_{t>0} \overline{\text{clos} \phi_t(U)}$.

For a compact invariant set $\Lambda$, we say that $\Lambda$ is partially hyperbolic if the tangent bundle over $\Lambda$ can be written as a continuous $D\phi_t$-invariant sum $T\Lambda M = E^s \oplus E^{cu}$, where $d_s = \dim E^s_x \geq 1$ and $d_{cu} = \dim E^{cu}_x \geq 2$ for $x \in \Lambda$, and there exist constants $C > 0, \lambda \in (0,1)$ such that for all $x \in \Lambda, t \geq 0$, we have

- uniform contraction along $E^s$: $\|D\phi_t|_{E^s_x}\| \leq C\lambda^t$; and
- domination of the splitting: $\|D\phi_t|_{E^s_x}\| \cdot \|D\phi_{-t}|_{E^{cu}_{\phi_t x}}\| \leq C\lambda^t$.

We say that $E^s$ is the stable bundle and $E^{cu}$ the center-unstable bundle. A partially hyperbolic attracting set is a partially hyperbolic set that is also an attracting set.

We say that the center-unstable bundle $E^{cu}$ is volume expanding if there exists $K, \theta > 0$ such that $|\det(D\phi_t|_{E^{cu}_x})| \geq Ke^{\theta t}$ for all $x \in \Lambda, t \geq 0$. More generally, $E^{cu}$ is sectional expanding if for every two-dimensional subspace $P_x \subset E^{cu}_x$,

\begin{equation}
|\det(D\phi_t(x) \mid P_x)| \geq Ke^{\theta t} \quad \text{for all } x \in \Lambda, t \geq 0.
\end{equation}

If $\sigma \in M$ and $G(\sigma) = 0$, then $\sigma$ is called an equilibrium or singularity in what follows and we denote by $\text{Sing}(G)$ the family of all such points. We say that a singularity $\sigma \in \text{Sing}(G)$ is hyperbolic if all the eigenvalues of $DG(\sigma)$ have non-zero real part.

A point $p \in M$ is periodic for the flow $\phi_t$ generated by $G$ if $G(p) \neq 0$ and there exists $\tau > 0$ so that $\phi_{\tau}(p) = p$; its orbit $O_G(p) = \phi_R(p) = \phi_{[0,\tau]}(p)$ is a periodic orbit, an
invariant simple closed curve for the flow. The family of periodic orbits of $G$ is written $\text{Per}(G)$.

The critical elements $\text{Crit}(G)$ of a vector field $G$ are its equilibria and periodic orbits, that is, $\text{Crit}(G) = \text{Sing}(G) \cup \text{Per}(G)$. An invariant set is nontrivial if it is not a critical element of the vector field.

We say that a compact invariant set $\Lambda$ is a sectional hyperbolic set if $\Lambda$ is partially hyperbolic with sectional expanding center-unstable bundle and all equilibria in $\Lambda$ are hyperbolic. A sectional hyperbolic set which is also an attracting set is called a sectional hyperbolic attracting set.

A singular hyperbolic set is a compact invariant set $\Lambda$ which is partially hyperbolic with volume expanding center-unstable subbundle and all equilibria within the set are hyperbolic. A sectional hyperbolic set is singular hyperbolic and both notions coincide if, and only if, $d_{cu} = 2$. In what follows, singular-hyperbolicity implicitly means that $d_{cu} = 2$.

**Lemma 1.1** (Hyperbolic Lemma). [29, Lemma 3] (or [8, Proposition 6.2]) A sectional hyperbolic set with no equilibria is a (uniformly) hyperbolic set.

This lemma precisely means that, in our setting, the central unstable subbundle admits a splitting $E_{cu}^{\Lambda} = \mathbb{R}\{G(x)\} \oplus E^{\prime}_{\Lambda}$ for all $x \in \Lambda$ (so that $d_{cu} = 1 + d_{u}$ with $d_{u} = \dim E^{u}$) where $E^{u}_{\Lambda}$ is uniformly contracting under the time reversed flow; see e.g. [8].

That is, $\Lambda$ is a (uniformly) hyperbolic set if $T_{\Lambda} M = E^{s} \oplus \mathbb{R}\{G\} \oplus E^{u}$, where $E^{s}$ is uniformly contracting, $\mathbb{R}\{G\}$ is the one-dimensional subspace along the direction of the vector field $G$, and we have for the same $C, \lambda$ used in the uniform contraction of $E^{s}$

- uniform expansion (backward contraction) along $E^{u}$: $\|D\phi_{-t}|E^{u}_{\Lambda}\| \leq C\lambda^{t}$ for all $x \in \Lambda$ and $t \geq 0$.

A periodic orbit $O_{G}(p)$ is hyperbolic if $O_{G}(p)$ is a hyperbolic subset for $G$. If moreover $E^{u}$ is trivial (i.e. $E^{u}_{q} = \{0\}, q \in O_{G}(p)$), then the periodic orbit is a periodic sink.

**Remark 1.2.** A singular hyperbolic attracting set cannot contain isolated periodic orbits. For otherwise such orbit must be a periodic sink, contradicting volume expansion.

We recall that a subset $\Lambda \subset M$ is transitive if it has a full dense orbit, that is, there exists $x \in \Lambda$ such that $\text{clo}_{\Lambda} \{ \phi_{t} x : t \geq 0 \} = \Lambda = \text{clo}_{\Lambda} \{ \phi_{t} x : t \leq 0 \}$.

A nontrivial transitive sectional hyperbolic attracting set is a sectional hyperbolic attractor.

The prototype of a sectional-hyperbolic attractor for 3-flows is the Lorenz attractor; see e.g. [25, 44, 8]. For higher dimensional flows we have the multidimensional Lorenz attractor; see [14]. More examples are provided in Section 2 and many more in [28].

1.1.2. **Invariant manifolds.** An embedded disk $\gamma \subset M$ is a (local) strong-unstable manifold, or a strong-unstable disk, if $\text{dist}(\phi_{+t}(x), \phi_{+t}(y))$ tends to zero exponentially fast as $t \to +\infty$, for every $x, y \in \gamma$. In the same way $\gamma$ is called a (local) strong-stable manifold, or a strong-stable disk, if $\text{dist}(\phi_{-t}(x), \phi_{-t}(y)) \to 0$ exponentially fast as $n \to +\infty$, for every $x, y \in \gamma$. 


It is well-known that there exists $\varepsilon_0 > 0$ so that every point in a hyperbolic set possesses a local strong-stable manifold $W_{ss}^{loc}(x)$ and a local strong-unstable manifold $W_{uu}^{loc}(x)$ which are disks tangent to $E^s_x$ and $E^u_x$ at $x$ respectively with topological dimensions $d_s = \dim(E^s)$ and $d_u = \dim(E^u)$ and inner radius $\varepsilon_0$; see e.g. [19, Chap. 6]. It is common to write $W^\ast_\varepsilon(x), \ast = ss, uu$ for the corresponding local manifolds with inner radius $\varepsilon$.

Considering the action of the flow we get the (global) strong-stable manifold

$$W^{ss}(x) = \bigcup_{t > 0} \phi_{-t}\left(W_{loc}^{ss}(\phi_t(x))\right)$$

and the (global) strong-unstable manifold

$$W^{uu}(x) = \bigcup_{t > 0} \phi_t\left(W_{loc}^{uu}(\phi_{-t}(x))\right)$$

for every point $x$ of a uniformly hyperbolic set. Similar notions are defined in a straightforward way for diffeomorphisms. These are immersed submanifolds with the same differentiability of the flow or the diffeomorphism.

In the case of a flow we also consider the stable manifold $W^s(x) = \bigcup_{t \in \mathbb{R}} \phi_t(W^{ss}(x))$ and unstable manifold $W^u(x) = \bigcup_{t \in \mathbb{R}} \phi_t(W^{uu}(x))$ for $x$ in a uniformly hyperbolic set, which are flow invariant.

We note that these notions are well defined for a hyperbolic periodic orbit, since this compact set is itself a hyperbolic set. Since all periodic orbits in a singular-hyperbolic set are hyperbolic, then these manifolds also exist in this setting.

In general, (local) stable manifolds exist for every point of a singular-hyperbolic set due to partial hyperbolicity; see Subsection 3.3.1 and [5].

1.1.3. Singularities in singular-hyperbolic attracting sets.

**Proposition 1.3.** [3, Proposition 2.1] Let $\Lambda$ be a sectional hyperbolic attracting set and let $\sigma \in \Lambda$ be an equilibrium. If there exists $x \in \Lambda \setminus \{\sigma\}$ so that $\sigma \in \omega(x) \cup \alpha(x)$, then $\sigma$ is generalized Lorenz-like: that is, $DG(\sigma)|E^c_{ss}^{\sigma}$ has a real eigenvalue $\lambda^s$ and $\lambda^u = \inf\{\Re(\lambda) : \lambda \in \text{Spec}(DG(\sigma)), \Re(\lambda) \geq 0\}$ satisfies $-\lambda^u < \lambda^s < 0 < \lambda^u$ and so the index of $\sigma$ is $\dim E^s_\sigma = d_s + 1$.

**Remark 1.4.**  
1. Partially hyperbolicity of $\Lambda$ ensures that the direction $G(x)$ of the flow is contained in the central-unstable subbundle $G(x) \subset E^c\times_{x}$ for all $x \in \Lambda$; see [4, Lemma 5.1] 
2. If $\sigma \in \text{Sing}(G) \cap \Lambda$ is a generalized Lorenz-like singularity and $\gamma^s_\sigma$ is its local stable manifold, then at $w \in \gamma^s_\sigma \setminus \{\sigma\}$ we have $T_w\gamma^s_\sigma = E^c_w = E^s_w \oplus \mathbb{R} \cdot \{G(w)\}$ since $T\gamma^s_\sigma$ is $D\phi_t$-invariant and contains $G(w)$ (because $\gamma^s_\sigma$ is $\phi_t$-invariant) and the dimensions coincide.
(3) If \( \sigma \in \text{Sing}(G) \cap \Lambda \) is a generalized Lorenz-like singularity, then the strong-stable manifold of \( \sigma \) (with dimension \( d_s = \dim E_s \)), that is
\[
W_{ss}^\sigma = \left\{ x \in M : \text{dist}(\phi_t(x), \sigma) e^{-\lambda_s t} \xrightarrow{t \to +\infty} 0 \right\}
\]
does not intersect any other point of \( \Lambda \): \( W_{ss}^\sigma \cap \Lambda = \{ \sigma \} \); see e.g. [8, Lemma 5.30 & Remark 5.31].

(4) If an equilibrium \( \sigma \in \text{Sing}(G) \cap \Lambda \) is not generalized Lorenz-like, then \( \sigma \) is not in the limit set of \( \Lambda \ \{ \sigma \} \), i.e. there is no \( x \in \Lambda \ \{ \sigma \} \) so that \( \sigma \in \alpha(x) \cup \omega(x) \). An example is provided by the pair of equilibria of the Lorenz system of equations away from the origin: these are saddles with an expanding complex eigenvalue which belong to the attracting set of the trapping ellipsoid already known to E. Lorenz; see e.g. [8, Section 3.3] and references therein and also Subsections 2.2 and 2.3.

1.1.4. Statements of the results. Singular-hyperbolic (and sectional-hyperbolic) attracting sets for \( C^2 \) smooth flows admit finitely many ergodic physical measures; see e.g. [10] and [3]. Here we provide a bound for the number of such ergodic physical measures. One of the motivations for our statement comes from the following result of Morales [27].

**Theorem 1.5.** Let \( \Lambda \) be a singular-hyperbolic attractor of a 3-flow of a \( C^r \) vector field \( X \), \( r \geq 1 \). Then, there is a neighborhood \( U \) of \( \Lambda \) such that every attractor in \( U \) of a \( C^r \) vector field \( C^r \) close to \( X \) is singular.

Therefore, in this setting, it is natural to consider the number of ergodic physical measures whose support contains a singularity, since there are no hyperbolic attractors. In addition, from Proposition 1.3, any singularity contained in the support of these physical measures are necessarily (generalized) Lorenz-like, since the support of an ergodic measure for a continuous invertible map on a metric space admits a dense forward and backward orbit; see e.g. [26].

We recall that all hyperbolic sets for flows are singular-hyperbolic in particular. Moreover, for \( C^2 \) smooth flows, each hyperbolic attractor admits a unique physical measure (see e.g. Bowen-Ruelle [16] or [19, Theorem 7.4.10]), then we can find flows with any number of physical measures within hyperbolic attracting sets; see below and Subsection 2.1.

**Theorem A.** Let \( G \) be a 3-vector field of class \( C^2 \), \( \Lambda \) be a connected singular-hyperbolic attracting set of \( G \), and \( s_L \) be the number of Lorenz-like singularities of \( \Lambda \). Then the number \( s \) of ergodic physical measures supported in \( \Lambda \) whose support contains a singularity satisfies \( s \leq 2 \cdot s_L \).

Let \( V \) be the \( C^r \) neighborhood of a 3-vector field \( X \) which admits a singular-hyperbolic attractor \( \Lambda \) with trapping region \( U \), for some \( r \geq 2 \), according to the previous Theorem 1.5. Then the previous bound applies to all physical measures of the attracting set within \( U \) for all vector fields in \( V \). More precisely, we have the following.
Corollary B. Let $G \in V$ be given. Let $s_0$ be the number of singularities of $G$ in $U$. Then the number $s$ of ergodic physical measures supported in $\Lambda = \Lambda_G(U)$ satisfies $s \leq 2 \cdot s_0$ (and all of them contain some singularity in their support).

About the inequality above, we observe the following.

(1) If $\Lambda$ contains no equilibria of $G$, that is, $s_L = 0$, then we have a hyperbolic attracting set, so we obtain equality in the bound given by Theorem A in this particular case: $s = 0$ when $s_L = 0$.

In Subsection 2.1, we provide a construction of a connected singular-hyperbolic attracting set with no singularities (and thus, uniformly hyperbolic) and any given finite number of ergodic physical measures (none of which contain equilibria).

(2) Moreover, Morales [28, Theorem A] describes the construction of a singular-hyperbolic attracting set whose unique equilibrium is non-Lorenz-like, and admits a non-singular transitive component which supports an ergodic physical measure. We again have $s = 0 = s_L$ but with a non-Lorenz-like equilibrium.

(3) In addition, the (geometric) Lorenz attractor, provides an example where $s_L = s = 1$; see Subsection 2.2. Hence, a strict inequality is obtained in the statement of Theorem A.

(4) It is easy to increase the number of Lorenz-like and non-Lorenz like equilibria while keeping the number of ergodic physical measures: see Remarks 2.2, 2.3 and 2.6. Moreover, Morales [28, Theorem B] presents an example of a singular-hyperbolic attractor with several Lorenz-like equilibria.

We also describe examples showing that the inequality is sharp.

Theorem C. Given an integer $s_L > 0$ there exists a $C^2$ vector field, on a bounded and open subset of $\mathbb{R}^3$, having a connected singular-hyperbolic attracting set $\Lambda$ supporting exactly $2s_L$ ergodic physical measures and $s_L$ equilibria, all of them Lorenz-like: i.e. $s = 2 \cdot s_L$.

The constructions can be easily adapted to provide examples where the number of (hyperbolic non-Lorenz-like) equilibria in the singular-hyperbolic attracting set is larger than the number of Lorenz-like equilibria; see Remark 1.4 after the proof of Theorem C.

1.2. Possible extension of the results. The above results should be extended, either for higher dimensional manifolds, or for less smooth vector fields. Moreover, the examples presented in the proof of Theorem C are not robust, so the inequality $s < s_L$ should be generic, and the variation of the number of physical measures with the vector field should be considered.

1.2.1. Semicontinuity of the number of physical measures. Since we have obtained an upper bound for the number of singular physical measures (the ones containing some equilibria in their support) on singular-hyperbolic attracting sets; these attracting sets are robust (for each $C^1$ close vector field the trapping region still contains a singular-hyperbolic attracting set); and the number of Lorenz-like equilibria is locally constant in a $C^1$ neighborhood of the vector field, it is natural to conjecture the following.
Moreover, the number of ergodic physical measures containing singularities in their support varies upper semicontinuously with the vector field, but a pair of physical ergodic measures may fuse under arbitrarily small perturbations.

**Conjecture 1.** Let $\mathcal{V}$ be the open family of $C^2$ vector fields admitting a singular-hyperbolic attracting set $\Lambda$ and $s: \mathcal{V} \to \mathbb{Z}_0^+$ the function associating to each $G \in \mathcal{V}$ the number $s(G)$ of ergodic physical measures supported in $\Lambda$ and containing some equilibria. Then $s$ is upper semicontinuous.

1.2.2. **Genericity of the inequality $s < s_L$.** The existence of periodic orbits at the boundary of the transversal section in the examples presented in the proof of Theorem C (see Section 2) is not a robust situation under small perturbations of the vector field. This, it is natural to state the following.

**Conjecture 2.** Among the family of 3-vector fields $G$ of class $C^2$ exhibiting singular-hyperbolic attracting set containing some Lorenz-like singularity, there exists an open and dense subset of vector fields for which the bound $s < s_L$ holds, for each singular-hyperbolic attracting set.

1.2.3. **Extension to higher dimensional vector fields.** The results stated above hold for 3-vector fields of class $C^2$ having singular-hyperbolic attracting sets. But the existence of finitely many ergodic physical measures also holds for higher dimensional singular-hyperbolic attracting sets with any stable dimension, that is, $d_s \geq 1$ and $d_{cu} = 2$. Recently this was extended in [3] to sectional-hyperbolic attracting sets with any combination of $d_s \geq 1$ and $d_{cu} \geq 2$ values. Moreover, the existence of homoclinic classes in sectional-hyperbolic attracting sets was obtained in[11]. So it is natural to state the following.

**Conjecture 3.** The results of Theorem A and Corollary B also hold for any sectional-hyperbolic attracting sets, replacing “Lorenz-like” singularities by “generalized Lorenz-like” singularities.

Since part of the statement of the results is inspired on Theorem 1.5 it is only natural to propose the following.

**Conjecture 4.** The statement of Theorem 1.5 holds also for sectional-hyperbolic attracting sets $C^1$ close to sectional-hyperbolic attractors, without dimensional restrictions (i.e. admitting any combination of $d_s \geq 1$ and $d_{cu} \geq 2$).

1.2.4. **Extension to $C^1$ smooth vector field.** Recently existence and uniqueness of physical/SRB measures was obtained for any uniformly hyperbolic attractor of a $C^1$ generic diffeomorphism on any compact manifold by Qiu [36].

More recently, Crovisier-Yang-Zhang [18, Corollary B.1 & Theorem J] obtained existence of a SRB invariant probability measure for any sectional-hyperbolic attracting set for a $C^1$ vector field and, assuming that this SRB measure is unique, they are able to deduce that it is also the unique physical measure. This points to the following possibility.
Conjecture 5. For generic $C^1$ vector fields having a sectional-hyperbolic attracting set the number of ergodic physical/SRB measures satisfies the same bounds as given by Theorem A and Corollary B.

For more on SRB versus physical measures, see Section 3.1.

1.3. Organization of the text. In the following, Section 2 contains descriptions of the construction of the examples mentioned after the statement of Corollary B including Theorem C.

In Section 3, we present properties of singular-hyperbolic attracting sets for 3-flows in Subsections 3.1 and 3.2, which will be necessary for the proof of the bounds in Theorem A and Corollary B in Subsection 3.3.

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2. Description of Classes of Examples

Here we describe the construction of the examples mentioned in Theorem C.

2.1. Connected attracting set containing any finite number of hyperbolic attractors.

2.1.1. Criteria for connectedness of attracting sets. We start with a simple criteria to obtain a connected attracting set.

Lemma 2.1. If $U$ is a connected open subset of $M$ which is a trapping region, then the attracting set $\Lambda = \bigcap_{t>0} \text{clos} \phi_t(U)$ is a compact and connected subset.

Proof. Indeed, the assumption on $U$ ensures that $\phi_t(U)$ is connected and its closure is contained in $U$ for all $t \geq T$. Since the attracting set can be written

$$\Lambda = \bigcap_{t>0} \text{clos} \phi_t(U) = \lim_{t \to +\infty} \text{clos} \phi_t(U)$$

where the limit is in the Hausdorff topology of compact subsets of Euclidean 3-space, then $\Lambda$ is compact and connected. More precisely, for each $\varepsilon > 0$ there exists $T_0 > 0$ so that for all $T > T_0$ we get

$$\Lambda \subset \bigcap_{0 < t \leq T} \text{clos} \phi_t(B) = \text{clos} \phi_T(B) \subset B(\Lambda, \varepsilon) = \bigcup_{x \in \Lambda} B(x, \varepsilon).$$

Arguing by contradiction, if we assume that $\Lambda$ is not connected, then there would exist two disjoint non-empty closed subsets $\Lambda_1 \cup \Lambda_2 = \Lambda$. Hence there would also exist disjoint open neighborhoods $W_i$ of $\Lambda_i$, $i = 1, 2$ with $W = W_1 \cup W_2$ an open neighborhood of $\Lambda$. Thus we can find $T_0 > 0$ so that $\text{clos} \phi_t(U) \subset W$ for all $t > T_0$ and, by connectedness of $\phi_t(U)$, this subset is contained in either $W_1$ or $W_2$. Moreover, by continuity of the flow, $\phi_t(U)$ is contained in the same $W_i$, let us say $W_1$.

But then $\Lambda = \bigcap_{t>0} \text{clos} \phi_t(U) \subset \Lambda_1$ contradicting the existence of nonempty disjoint closed subsets $\Lambda_1, \Lambda_2$ whose union is $\Lambda$. \hfill \square
2.1.2. A Morse-Smale of the disk with an attracting set containing finitely many attractors. We consider the plane flow $\psi_t$ defined by the ODE $\dot{x} = g'(x), \dot{y} = -y$ with $g(x) = \sin(\pi x)$ on the rectangle $R_k = [-1, 2k] \times [-1, 1]$ for some fixed integer $k > 0$.

Then $R_k$ contains $k + 1$ sinks at $S_0 = \{2i - 1/2 : i = 0, \ldots, k\} \times \{0\}$ and $k$ saddles $S_1 = \{2i + 1/2 : i = 0, \ldots, k - 1\} \times \{0\}$ and $\psi_t(R_k) \subset \text{int} R_k$ for all $t > 0$; see the left hand side of Figure 1.

![Figure 1](image)

**Figure 1.** The phase portrait of the plane flow $\psi_t$, on the left. On the right: suspension of a Plykin attractor for the Poincaré first return map to the disk $D$ of a flow inside a solid torus $D \times S^1$.

We note that $R_k$ is diffeomorphic to the disk and the maximal invariant set $\Lambda = \bigcap_{t>0} \phi_t(R_k)$ given by $[0, 2k - 1] \times \{0\}$ is an attracting set.

We also consider the vector field $X$ in the solid torus $R_k \times S^1$ given by $X(x, y, z) = (g'(x), -y, 1)$ in what follows.

2.1.3. The suspension of the Plykin attractor on the disk. For the construction of the Plykin attractor see Plykin [35], Robinson [37] and/or Fisher-Hasselblatt [19]. It is easy to see from the construction presented by Kuznetsov [24, 23], that the diffeomorphism of the disk proposed by Plykin is diffeotopic to the identity; see also [31] for an animated graphics presentation.

This allows us to define a smooth $C^\infty$ vector field $Y = (Y_0, Y_1)$ of the solid torus $D \times S^1$ which points inward at the boundary and admitting a cross-sectional disk whose Poincaré first return map is the diffeomorphism $f$ described by Plykin, having an expanding period three periodic orbit $\{p_0, p_1, p_2\}$; see the right hand side of Figure 1.

The maximal invariant subset $A = \bigcap_{t\in\mathbb{R}} \mathcal{Y}_t(D \times S^1)$ of the flow generated by $Y$ can be written as the union of two connected components $P \cup O$, where $P$ is the Plykin (expanding) attractor and $O$ is a periodic source.

2.1.4. Attaching the Plykin suspension to the Morse-Smale example. We now “attach” this vector field in a neighborhood $V_s(\varepsilon) = B(s, \varepsilon) \times S^1$ of each sink $s \in S_0$ as follows, for $\varepsilon \in (0, 1/100)$ small enough. We find a $C^\infty$ partition of unity $(\mathcal{Y}_s)_{s \in S_0 \cup O}$ subordinated to the open cover $\{V_s(\varepsilon) : s \in S_0\} \cup R_k \times S^1 \setminus \cup_{s \in S_0} V_s(\varepsilon/2)$ so that $\text{supp } \mathcal{Y}_s \subset V_s(\varepsilon)$ and $\text{supp } \mathcal{Y}_0 \subset R_k \times S^1 \setminus \cup_{s \in S_0} V_s(\varepsilon/2)$. Then we replace $X$ by the $C^\infty$ vector field

$$G = \mathcal{Y}_0 \cdot X + \sum_{s \in S_0} \mathcal{Y}_s \cdot \left(s + \frac{\varepsilon}{2} Y_0, Y_1\right)$$
which coincides with a rescaled and translated version of $Y$ in the interior of each $V_s(\varepsilon)$. Now the limit set in $R_k \times S^1$ with respect to the flow $\phi_t$ induced by $G$ is given by

$$L = \text{clos}\{ \alpha(x) \cup \omega(x) : x \in R_k \times S^1 \} = \bigcup_{u \in S_1} u \times S^1 \cup \bigcup_{s \in S_0} (P_s \cup O_s),$$

where $u \times S^1$ is a periodic hyperbolic saddle for each $u \in S_1$ and $P_s, O_s$ are the Plykin attractor and periodic source within $V_s(\varepsilon)$ for each $s \in S_0$; see Figure 2.

![Figure 2](image)

**Figure 2.** A phase portrait of the flow $\phi_t$ of the vector field $G$, with a suspension of a Plykin attractor in the place of each periodic sink of the flow of $X$.

If we remove small neighborhoods $U_s$ around each $O_s$, we obtain that in the connected trapping region $W = R_k \times S^1 \setminus \bigcup_{s \in S_0} U_s$ we have the attracting set

$$\Lambda = \bigcap_{t > 0} \phi_t(W) = \bigcup_{u \in S_1} W^u(u \times S^1) \cup \bigcup_{s \in S_0} P_s,$$

which is connected by Lemma 2.1. Moreover, $\Lambda$ contains $k$ hyperbolic attractors which are suspensions of the Plykin example.

Since $k > 0$ was arbitrarily fixed, we have constructed an attracting set with no equilibria and any given finite number $(k + 1)$ of hyperbolic attractors in a $C^\infty$ smooth 3-flow.

2.2. **Lorenz-like attracting sets: inequality $s < 2s_L$.** From the original work of Lorenz [25] and Sparrow [42, Appendix C] it is well known that there exists a trapping region bounded by an ellipsoid $E$ for an attracting set and a smaller trapping region bounded by a bitorus $U \subset E$ for the Lorenz attractor; this was proved later by Tucker using a computer assisted proof [43, 44].

This is depicted in Figure 3: the maximal invariant subset inside $E$ contains, besides the Lorenz attractor, the two equilibria $\sigma_1, \sigma_2$ with expanding complex eigenvalues around which the trajectories in the “lobes” of the attractor rotate; see the left hand side of Figure 4 for a picture of the geometric Lorenz attractor [1, 47].

The geometric Lorenz attractor contains a Lorenz-like equilibrium at the origin and a unique physical measure which is ergodic; see e.g. [45, Section 6.3] or [9, Section 6]. Hence we obtain an example where the number $s$ of ergodic physical measures and $s_L$ of Lorenz-like singularities are both equal to one: $s = 1 = s_L$. 
Figure 3. Local stable and unstable manifolds near $\sigma_0$, $\sigma_1$ and $\sigma_2$, and the ellipsoid $E$, on the left; and the trapping bitorus $U$ on the right.

Figure 4. The geometric Lorenz attractor on the left, with one Lorenz-like equilibrium at the origin and a unique (ergodic) physical measure; and an adaptation of the construction on the right, providing a singular-hyperbolic attracting set with two Lorenz-like equilibria and still a unique physical measure.

It is easy to increase the number of Lorenz-like equilibria while keeping the number of ergodic physical measures, as depicted in the right hand side of Figure 4, where we have a unique physical measure and a pair of Lorenz-like singularities.

The “doubling” of the construction of the geometric Lorenz attractor, depicted in the left hand side of Figure 5, provides an attracting set with two transitive geometric Lorenz components, $H_1$ and $H_2$, above and below the horizontal plane through the origin. Then the maximal invariant set is singular-hyperbolic and supports two ergodic physical measures: $s = 2$. Moreover, the set contains three Lorenz-like singularities marked in the picture: $s_L = 3$.

This is a consequence of the existence of a pair of Poincaré first return transformations which have the same properties of the geometric Lorenz first return map, one for each of the cross-sections presented in the left hand side of Figure 5, and so generate distinct physical measures; see e.g. [8, Chap. 7, §3].

Remark 2.2. We can easily increase the number of Lorenz-like equilibria to any number we like by a simple adaptation of this construction, as shown in the right hand side of Figure 5 where we have $s_L = 4$ and still only two ergodic physical measures in the attracting set. The same trick can be applied to the geometric Lorenz construction to
obtain any number of Lorenz-like equilibria in a singular-hyperbolic attractor, and so with a unique physical invariant probability measure.

**Remark 2.3.** We can include in the above construction non-Lorenz-like equilibria by extending the trapping region to contain the “lobes” around which the unstable manifolds of the Lorenz-like equilibria wind. This means replacing the solid bitorus by an ellipsoid as the trapping region and include the two hyperbolic saddle-focus equilibria $\sigma_1, \sigma_2$ in the corresponding attracting set; see Figure 3.

2.3. **Example with sharp equality** $s = 2s_L = 2$. Next we again adapt the geometric Lorenz construction to obtain the equality $s = 2 \cdot s_L$ with $s_L = 1$.

2.3.1. **Overview of the construction.** We start with a one-dimensional Lorenz-like transformation with two expanding fixed repellers at the boundary of the interval; see the left hand side of Figure 6. Then we perform the geometric Lorenz construction in such a way to obtain this map as the quotient over the stable leaves of the Poincaré first return map to the global cross-section of a vector field $G_0$; see the right hand side of Figure 6.

As usual in the geometric Lorenz construction, we assume that in the cube $[-1,1] \times [-1,1] \times [-1,1]$, between the two cross-sections $S = [-1,1]^2 \times \{1\}$ and $S_- = [-1,1]^2 \times \{-1\}$, the flow is linear $G_0 = A \cdot G_0$ with $A = \text{diag}\{\lambda_1, \lambda_2, \lambda_3\}$ and a Lorenz-like singularity at the origin $o_0$ satisfying $\lambda_1 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$; see e.g. the left hand side of Figure 4.

Then we modify the flow to send a neighborhood $D_+$ of the interior of a fundamental domain $J$ of the local strong-unstable manifold $W^{uu}_t(p_+)$ of a point $p_+$ of the periodic orbit $O(p_+)$ to the interior of the cross-section $S_-$; see the left hand side of Figure 7 and the upper half of the right hand side of the same figure.

The boundary of $J$ is a pair of points of a trajectory $\gamma$ of the unstable manifold $W^u(p_+)$. We also send a neighborhood $D_0$ of a point $q_+$ in the $\omega$-limit of $\gamma$ to
Figure 6. Lorenz one-dimensional transformation with repelling fixed points at the extremes of the interval on the left; and the geometric Lorenz construction with this map as the quotient over the contracting invariant foliation on the cross-section $S$, with two corresponding periodic saddle-type periodic orbits $O(p_{\pm})$.

(i) either the interior of the cross-section $S$, so that the future trajectories of the points of $D_0$ will accumulate the transitive set on $\{z \geq 0\}$, as depicted in Figure 7;

(ii) or to the interior of another cross-section of a similar distinct attracting set away from $\sigma_0$ (to be able to replicate a finite number of similar attracting sets forming a connected subset).

Note that in this way the cusp section $\Sigma_-$, coming from one half of $S_-$ in Figure 7, is eventually sent by the flow back to the interior of $S_-$. To complete the construction of the attracting set:

• we modify the flow to send an open neighborhood of a fundamental domain of the local strong-unstable manifold $W_{uu}^{\epsilon}(p_-)$ to the interior of $S_-$ in a symmetrical way to what as done around $O(p_+)$ with respect to the origin, by the transformation $T : (x, y, z) \mapsto (-x, -y, z)$ reminiscent of the symmetry of the original Lorenz system of differential equations;

• analogously we connect a neighborhood of a point of a trajectory inside $W_u^{\epsilon}(O(p_-))$ to

  (i) either the interior of the cross-section $S$;

  (ii) or to the interior of another cross-section of a similar attracting set away from $\sigma_0$, see below.

We observe that, independently of the choices (i) or (ii), we obtain an attracting set accumulating $\sigma_0$ from both “sides” $\{z > 0\}$ and $\{z < 0\}$, and having two transitive components. The return map to $S_-$, after quotienting over the contracting leaves, is sketched at the lower right hand side of Figure 7. This attracting set is singular-hyperbolic since

• the Poincaré first return map to $S$ has the same properties as the geometric Lorenz return map; and
the first return map to $S_-$ has a quotient over the contracting leaves (whose existence is guaranteed by construction similarly to the geometric Lorenz construction) which is piecewise expanding with long branches.

This also provides a pair of ergodic physical measures: $\mu_+$ whose support is contained in $\{z \geq 0\}$; and $\mu_-$ whose support intersects $\{z \leq 0\}$ in a neighborhood of $\sigma_0$. There are now the following possibilities to complete the vector field.

1. If we choose $(i)$ above we obtain an attracting set with $s = 1$ and $s_L = 2$; see below for more details.
2. If we choose $(ii)$ above to connect the unstable manifold of either $\mathcal{O}(p_+)$ or $\mathcal{O}(p_-)$ (or both) to another similar attracting set, then we can obtain examples of connected attracting sets with any given number $s_L$ of Lorenz-like singularities and precisely $2 \cdot s_L$ ergodic physical measures.
2.3.2. Modifying the geometric Lorenz vector field. More precisely, around the periodic orbits \( \mathcal{O}(p_\pm) \) we extend the vector field to specify the trapping region as follows; we refer to Figure 8.

![Figure 8. The potential defining the vector field to be attached to a neighborhood of \( \mathcal{O}(p_+) \) in the upper right hand side; a sketch of the perturbation of the vector field in \( D_0 \) and \( D_+ \) in the left hand side; and a sketch of the vector field inside the cylinders in the lower right hand side.](image)

We consider the \( C^\infty \) potential \( \varphi \) depicted in the upper right hand side of Figure 8 so that \( \varphi < 0 \) on \((0,1)\) and \( \varphi(0) = \varphi(1) = 0 \). We set \( Z_0 \) to be the vector field on the unit disk given by the gradient of \( \varphi(x^2 + y^2) \) on the \( xy \)-plane.

Then we attach this vector field to the periodic orbit \( \mathcal{O}(p_+) \) so that this orbit corresponds to the repelling periodic orbit of the gradient vector field; see the plane vector field at the left hand side of Figure 8. We note that there exists an attracting periodic orbit (periodic sink) for \( Z_0 \), corresponding to the minimum at 1/2 of \( \varphi \).

We extend the planar field \( Z_0 \) to a neighborhood in \( \mathbb{R}^3 \) through a contraction on a perpendicular direction, to obtain the field \( Z_1 \) – we locally identify the stable manifold of \( \mathcal{O}(p_+) \) with \( W^s_{\varepsilon_0}(p_+) \) of the initial vector field \( G_0 \). Repeating the construction symmetrically through the transformation \( T \), this provides a smooth \( C^\infty \) vector field \( G_1 \) defined on a ball around the origin.

At this point, we have a pair of extra saddle-focus equilibria \( \sigma_+ \) and \( \sigma_- = T(\sigma_+) \), together with a pair of periodic sinks – incompatible with singular-hyperbolicity – and still no recurrence on \( \{ z < 0 \} \) for the vector field \( G_1 \). We modify \( G_1 \) by

1. fixing a point \( p_+ \) in the periodic hyperbolic saddle and choosing
   a fundamental domain \( J \) of its local strong-unstable manifold \( W^u_{\varepsilon}(p_+) \): \( J \) is an interval between two points of a trajectory \( \gamma \) in the local unstable manifold \( W^u_{\varepsilon}(p_+) \);
   an open ball \( D_0 \) in \( \mathbb{R}^3 \) containing the interior of \( J \) in \( W^u_{\varepsilon}(p_+) \) and disjoint from \( \gamma \);
(c) a point \( q_+ \) in the periodic sink which is also in the closure of \( W_{e}^{uu}(p_+) \) and in the \( \omega \)-limit of \( \gamma_+ \), together with
(d) a small open ball \( D_0 \) containing \( q_+ \) and disjoint from \( D_+ \).

(2) We consider \( C^{\infty} \) smooth regular curves \( \xi_+, \xi_0 : I = [0, 1] \to \mathbb{R}^3 \) as follows:
- \( \xi_+ \) starting at a point \( q_+ \in J \) and ending in the interior of \( S_- \); and
- \( \xi_0 \) starting at \( q \) and ending in the interior of \( S \);
so that they start and end tangent to \( G_1 \) and move either transversely or in the same general direction of \( G_1 \): for \( * \in \{+, 0\} \)
  (a) \( \dot{\xi}_*(t) = G_1(\xi_*(t)) \) for \( t \in \{0, 1\} \); and
  (b) either \( \xi_* \wedge G_1 \circ \xi_* \neq 0 \) or \( \langle \xi_*, G_1 \circ \xi_* \rangle > 0 \).

(3) We take open neighborhoods \( U_* \supset V_* \) of \( \xi_*(I) \) so that \( U_* \cap D_+ = D_+ \) and \( U_* \supset \text{clos} \ V_* \), for \( * \in \{+, 0\} \). We also assume that \( U_* \) contain tubular neighborhoods of \( \xi_*(I) \) and that we can parameterize \( U_* \) as a cylinder \( D \times (0, 1) \subset \mathbb{R}^3 \) (where \( D \) is the open unit disk in \( \mathbb{R}^2 \)). We introduce the “inward pointing tubular vector field” \( H_* \) in \( U_* \) for \( * \in \{+, 0\} \), given by the following expression in the coordinates \( D \times (0, 1) \):
\[
H_* = \left( \nabla \varphi(40(x^2 + y^2) - 1/2), 1 \right),
\]
whose phase portrait is depicted in the lower right hand side of Figure 8.

(4) We let \( \psi, \psi_0, \psi_+ \) be the partition of unity subordinated to the open cover
\[
\mathbb{R}^3 \setminus (U_0 \cup U_+), U_0, U_+
\]
respectively, so that \( \text{supp} \psi_* = \text{clos} \ U_* \) and \( \psi_* \mid V_* \equiv 1 \), for \( * \in \{+, 0\} \). We define the perturbation
\[
G = \psi_0 \cdot H_0 + \psi_+ \cdot H_+ + \psi \cdot G_1. \tag{2.1}
\]
We note that points in \( J \cap D_+ \) and in a neighborhood of \( q \) are eventually sent to the other end of the respective cylinder by the action of the modified flow, so that, as depicted in the left hand side of Figure 7:
(a) the points of \( D_+ \cap J \) are taken to \( S_- \);
(b) points in a neighborhood of \( q \) are sent as in (i) to the cross-section \( S \) as described in the overview.

(5) Finally, we perform the same construction symmetrically using the transformation \( T \) around \( \partial(p_-) \) to obtain a flow on a ball around the origin with the phase portrait depicted in the left hand side of Figure 7.

We note that the choice of the curves \( \xi_+ \) and \( \xi_0 \) ensures that the definition of \( G \) through (2.1) avoids the introduction of any new equilibrium point, since \( G \) is everywhere a linear convex combination of two vector fields which either make an acute angle or are transverse.

The modified vector field \( G \) has no periodic sink and has a Lorenz-like attracting set containing \( \sigma_0 \), which accumulates this equilibrium only on a neighborhood of \( \sigma_0 \) in
\{z \geq 0\}, and whose Poincaré return map to the cross-section \(S_\pm\), after quotienting over the contracting stable leaves, is sketched in the lower right hand side of Figure 7.

The infinitely many branches of this map are due to the fact that the cuspidal sections \(\Sigma_\pm\) are sent by the flow close to the stable manifold of the periodic saddles \(O(p_\pm)\), which winds around the orbit. The Inclination Lemma ensures that each fundamental domain of the local unstable manifold of \(O(p_\pm)\) eventually accumulates in \(D_+\) (and its symmetric \(T(D_+)\) corresponding to \(O(p_-)\)). These pieces of \(\Sigma_\pm\) are then sent back to the interior of \(S_\pm\) as in the left hand side of Figure 7.

To complete the example, we specify the trapping region: take a large ellipsoid like \(E\) in Figure 3 centered around \(\sigma_0\) encompassing the recurrence region with negative \(z\); and remove a neighborhood of the stable manifold of the saddle-foci \(\sigma_\pm\) obtaining a bitorus like \(U\) of Figure 3, which is a trapping region.

The maximal invariant subset \(\Lambda_G(U)\) is a connected singular-hyperbolic attracting set with one Lorenz-like singularity \(\sigma_0\) and, due to the existence of two independent Poincaré return maps, we obtain a pair \(\mu_\pm\) of ergodic physical measures for the flow. The singular-hyperbolicity of the trajectories through the cylinders around \(\xi_+, \xi_0\) is a consequence of the following properties:

- the \(\alpha\)-limit is one periodic hyperbolic saddle, or the singular-hyperbolic transitive set with cross-section \(S_\pm\), or is outside of the trapping region; and
- the \(\omega\)-limit is the singular-hyperbolic transitive set with cross-section \(S_\); and moreover
- in dimension three, the dimensions of the singular-hyperbolic splitting, at the \(\alpha\)-limit and at the \(\omega\)-limit, are compatible.

For future use, we can assume without loss of generality that \(U\) is contained in the cube \(Q_0 = [-4, 4]^3\) and \(G\) is defined in \(Q_0\).

**Remark 2.4.** The boundary of the trapping region of this attracting set is a bitorus with two handles around the arcs \(\xi_-\) and \(T(\xi_-)\), thus a 4-torus.

### 2.4. Examples with sharp equality \(s = 2s_L\) with any given \(s_L > 1\).

To build an attracting set with any given finite number \(s_L > 1\) of Lorenz-like singularities and exactly \(2s_L\) ergodic physical measures, we “copy and paste” the previous construction \(s_L\) times translated along the \(x\)-axis and connect the attracting sets around each equilibria through the perturbation \((ii)\) loosely described in the previous Subsection 2.3.1; see Figure 9. For that we choose the regular curve \(\xi_0\) as in the previous section, except that \(\xi_0\) now ends at a cross-section above a different Lorenz-like singularity.

More precisely, similarly to the construction presented in Subsection 2.1, we start with a Morse-Smale vector field \(X_0\) defined by the ODE \(\dot{x} = g'(x/20), \dot{y} = y, \dot{z} = -z\) on the box \(R_k = [-10, 40k] \times [-5, 5] \times [-5, 5]\) for some fixed integer \(k = s_L - 1 > 0\), where \(g : \mathbb{R} \to \mathbb{R}\) is the same function defined in Subsection 2.2.
We obtain $k + 1 = s_L$ Lorenz-like singularities at $S_0 = \{ \sigma_i = (40i - 5, 0, 0) : i = 0, \ldots, k \}$ and $k$ non-Lorenz-like singularities at $S_1 = \{ \varrho_i = (40i + 5, 0, 0) : i = 0, \ldots, k - 1 \}$; see Figure 10.

We attach the vector field constructed in Subsection 2.3 at every box $Q_i = [\sigma_i - 4, \sigma_i + 4] \times [-4, 4]^2$ to the Morse-Smale vector field $X_0$, as follows. We consider the open cover $W_i = (\sigma_i - 5, \sigma_i + 5) \times (-4, 4)^2$ with $i = 0, \ldots, k$, and $R_k \setminus \sum_i W_i$; and let the corresponding $C^\infty$ partition of unity be $\psi_i, i = 0, \ldots, k$ and $\psi_-$, subordinated to this open cover. We define the $C^\infty$ vector field

$$X_1(w) = \psi_-(w) \cdot X_0(w) + \sum_{i=0}^{k} \psi_i(w) \cdot G(w + \sigma_i)$$

on $R_k$. This vector field has $k + 1 = s_L$ copies of the attracting set constructed in Subsection 2.3, together with non-Lorenz-like singularities between these attracting sets.

Next we modify $X_1$ into a field $X_2$ so that we obtain a connected attracting set containing the union of the transitive pieces of $X_1$.

The attracting set around the “last” Lorenz-like equilibrium $\sigma_{s_L}$ will keep the connection from $D_0$ to the cross-section “above” the same equilibrium. The attracting sets around the singularity $\sigma_i$ will be modified according to option (ii) (from the overview
Figure 10. The phase portrait of the Morse-Smale 3-dimensional vector field at the base of the construction of the example, with Lorenz-like $\sigma_i$, $i = 0, \ldots, k$ and non-Lorenz-like $\varrho_i$, $i = 0, \ldots, k - 1$ singularities.

Subsection 2.3.1) to have a connection from $D_0$ to the cross-section “above” the “next” equilibrium $\sigma_{i+1}$, for $i = 0, \ldots, s_L - 1$; see Figure 9.

Figure 11. The trapping region $U$ for the attracting set with $2 \cdot s_L = s > 2$ ergodic physical measures, given by the union of the 4-tori $B_i$ and tubular neighborhoods $T_i$. The repelling equilibria are not contained in $U$.

The trapping neighborhood $U$ will be the union of bitori $B_i$ shown in Figure 11 around each $\sigma_i$, where a neighborhood around all the saddle-foci has been removed, together with the “tubular” neighborhoods $T_i$ connecting a trajectory of the unstable set of $O(p_i^-)$ to the cross-section above $\sigma_{i+1}$ for $i = 0, \ldots, s_L - 1$.

This provides exactly a pair of ergodic physical measures containing each Lorenz-like equilibrium in their support. The connections between the unstable manifolds of periodic orbits in the support of one ergodic measure and the cross-section of the flow around a different equilibrium, ensure that the maximal invariant subset $\Lambda$ within $U$ will be connected.

This completes the proof of Theorem C.
Remark 2.5. We note that the boundary of the topological basin of attraction of this attracting set is diffeomorphic to a $4 \cdot s_L$-tori (a topologically connected sum of $4s_L$ tori; or a “torus with $4s_L$ holes”).

Remark 2.6. If we do not remove the neighborhoods around the saddle-foci in Figure 11, that is, if we replace each 4-torus by an ellipsoid containing the maximal invariant set around $\sigma_i$ and the nearby cross-sections, then we include in the new attracting set $\tilde{\Lambda}$ all the saddle-focus equilibria plus some trajectories connecting these singularities with the transitive set accumulating on $\sigma_i$ in $\{z < 0\}$.

This $\tilde{\Lambda}$ is a singular-hyperbolic attracting set with the same number of Lorenz-like singularities and ergodic physical measures as before, but containing a pair of non-Lorenz-like hyperbolic equilibria for each Lorenz-like equilibrium.

3. The Bound on the Number of Ergodic Physical Measures

Here we start the proof of Theorem A and Corollary B. After some preliminaries, the proof is presented in Subsection 3.3.

The definition of sectional-hyperbolicity ensures that every invariant probability measure supported in a sectional-hyperbolic set is a hyperbolic measure. Moreover, if the vector field is smooth (at least of class $C^2$) from the proof [3, Theorem A] – or of [9, Theorem B, §4] or explicitly from [40, Theorem 1.5] for $d_{cu} = 2$ – we get the first part of the following statement.

**Theorem 3.1.** Every sectional-hyperbolic attracting set for a $C^2$ smooth flow admits finitely many $\mu_0, \ldots, \mu_k$ ergodic physical invariant measures which are SRB measures for the system. Moreover, the union of the ergodic basins of these measures covers a full Lebesgue measure subset of the topological basin of attraction of $\Lambda$.

In addition, the support of an ergodic physical/SRB measure without singularities contained in $\Lambda$ is a hyperbolic attractor.

In fact, the same result is true in higher dimensions; see [3]. The last statement of Theorem 3.1 is not contained in [9] and for completeness we provide a proof in the following Subsection 3.1 together with the definition of SRB measure.

3.1. The support of a non-singular physical measure is a hyperbolic attractor. Similarly to the uniformly hyperbolic setting (see e.g. [19, Theorems 7.4.8 & 7.4.10]), in the sectional-hyperbolic setting we have that physical and SRB measures coincide.

**Theorem 3.2.** Let $\Lambda$ be a sectional-hyperbolic attracting set for a $C^2$ vector field $X$ with the open subset $U$ as trapping region. Then

(1) there are finitely many ergodic physical measures $\mu_1, \ldots, \mu_k$ supported in $\Lambda$ such that the union of their ergodic basins covers $U$ Lebesgue almost everywhere:

$$\text{Leb} \left( U \setminus \left( \bigcup_{i=1}^{k} B(\mu_i) \right) \right) = 0.$$
Moreover, for each $X$-invariant ergodic probability measure $\mu$ supported in $\Lambda$ the following are equivalent:

(a) $h_\mu(X_1) = \int \log \left| \det D_{\mathcal{E}_{cu}} X_1 \right| d\mu > 0$;

(b) $\mu$ is a SRB measure, that is, there is a positive Lyapunov exponent at $\mu$-a.e. point and $\mu$ admits an absolutely continuous disintegration along the corresponding unstable manifolds;

(c) $\mu$ is a physical measure, i.e., its basin $B(\mu)$ has positive Lebesgue measure.

In addition, the family $E$ of all $X$-invariant probability measures which satisfy item (2) above is the convex hull $E = \sum_{i=1}^k t_i \mu_i : \sum_{i=1}^k t_i = 1; 0 \leq t_i \leq 1, i = 1, \ldots, k$.

Proof. This is [10, Theorem 1.7] proved for a sectional-hyperbolic attracting set whose center-unstable dimension is two: $\dim E_{cu} = 2$. The proof is the same in any dimension; see [3].

The following is a consequence of the Hyperbolic Lemma 1.1 together with the absolutely continuous disintegration of physical/SRB measures.

**Proposition 3.3.** Let $\mu$ be a physical/SRB ergodic probability measure whose support $\text{supp} \mu$ is contained in a sectional-hyperbolic attracting set $\Lambda = \Lambda_G(U)$. If $\text{supp} \mu$ does not contain Lorenz-like singularities, then $\text{supp} \mu$ is a hyperbolic attractor.

Proof. Let $\mu$ be a physical ergodic probability measure supported in $\Lambda$, and let us assume that $A = \text{supp} \mu$ contains no Lorenz-like singularities. Since $A$ is transitive, then by Remark 1.4 there can be no other (hyperbolic) singularities in $A$. Then the compact invariant subset $A$ is uniformly hyperbolic, by the Hyperbolic Lemma 1.1.

The SRB property can be geometrically described as follows; see e.g [34]. For $\mu$-a.e. $x$ there exists a neighborhood $V_x$ where $\mu$ admits a disintegration $\{\mu_\gamma\}_{\gamma \in \mathcal{F}(V_x)}$ over the strong-unstable leaves $\mathcal{F}(V_x)$ that cross $V_x$ which is absolutely continuous with respect to the induced volume measure on the leaves. More precisely, we have

- $\mu(\varphi) = \int \mu_\gamma(\varphi) d\hat{\mu}(\gamma)$ for each bounded measurable observable $\varphi : M \to \mathbb{R}$, where $\hat{\mu}$ is the quotient measure on the leaf space induced by $\mu$ and, in addition
- $\mu_\gamma = \psi_\gamma \text{Leb}_\gamma$ for $\mu$-a.e. $\gamma$, where $\text{Leb}_\gamma$ denotes the measure induced on the leaf $\gamma \in \mathcal{F}(V_x)$ (which is a submanifold of $M$) by the Lebesgue volume measure; and $\psi_\gamma : \gamma \to [0, +\infty)$ is a strictly positive measurable and Leb$_\gamma$-integrable density function.

In particular, Leb$_\gamma$-a.e. point of $\mu$-a.e. leaf in $\mathcal{F}(V_x)$ belongs to $A$. Indeed, given any full measure subset $A_1$ of $A$, we have that $\mu_\gamma(A_1) = 1$ for $\mu$-a.e. $\gamma$ and hence Leb$_\gamma(A_1) = 1$ also. Since a full Lebesgue measure subset is dense and $A$ is closed, we see that the unstable leaf $\gamma$ is contained in $A$ for a $\mu$-positive measure subset of $V_x$.

In addition, unstable leaves of inner radius $\epsilon > 0$ are defined on all points of $A$ by uniform hyperbolicity and the map $A \ni x \mapsto W^u_\epsilon(x)$ is continuous in the $C^1$ topology of disk embeddings; see e.g. [19, Chapter 6].

Then, since the previous property holds for a full $\mu$-measure subset of points $x$, which is dense in $A$, we see that $A$ contains the unstable leaves through a dense subset of its...
points. By continuity of the unstable foliation in a hyperbolic set, we conclude that $A$ contains the unstable manifold through each of its points. This ensures that $A$ is a hyperbolic attractor:

- the support of the ergodic measure $\mu$ is topologically transitive;
- the union the center stable leaves $W^s(y)$, through each point $y$ of the local strongly unstable leaf $W^u_i(x)$, is an open subset contained in the topological basin of $A$;
- the flow is at least $C^2$, hence we can apply [16, Proposition 5.4 & Theorem 5.6].

The proof is complete. □

3.2. Non-existence of hyperbolic attracting sets near singular-hyperbolic attractors.

Here we state and prove a small extension of the statement of Theorem 1.5, showing that not only attractors but also attracting sets must be singular near a singular-hyperbolic attractor.

Corollary 3.4. Let $\Lambda$ be a singular-hyperbolic attractor of a $C^r$ vector field $X$, $r \geq 1$. Then, there is a neighborhood $U$ of $\Lambda$ such that every attracting set in $U$ of a $C^r$ vector field $G$ close to $X$ is singular.

Proof. Let $\Lambda = \Lambda_X(U)$ be a singular-hyperbolic attractor as in the statement of the corollary (and Theorem 1.5) and $V$ be a $C^r$ neighborhood of $X$ satisfying the conclusion of Theorem 1.5: for every $G \in V$ every attractor $\Gamma \subset U$ with respect to $G$ contains some (Lorenz-like) singularity – since these are the only equilibria allowed in singular-hyperbolic attractors; see e.g. [30, 8].

Arguing by contradiction, let $A = \Lambda_G(W)$ be an attracting set: the maximal invariant subset within $W \subset U$; and let us assume that $A$ is non-singular.

Hence, by the Hyperbolic Lemma 1.1, $A$ becomes a hyperbolic locally maximal attracting set for the flow generated by $G$. Thus, from the standard uniform hyperbolic theory for flows [19, Corollaries 5.3.21 & 5.3.22] the set $A$ satisfies shadowing and periodic orbits in $W$ are dense in $A$. Therefore, by “Spectral Decomposition” [19, Proposition 5.3.33], $A$ is the disjoint union of finitely many basic sets: $A = \sum_i A_i$ where each $A_i$ is compact, hyperbolic, transitive and locally maximal.

In particular, we can find pairwise disjoint open neighborhoods such that $A_i \subset W_i \subset W$ and $A_i = \bigcap_{t \in \mathbb{R}} \text{clos} \phi^G_t(W_i)$. But if $x \in \Lambda_G(W_i)$, then $x \in W_i \subset W$ and $\phi^G_{-t}(x) \in W_i \subset W$ for all $t \geq 0$. Hence $x \in \Lambda_G(W) = A$ and $x \in W_i$; thus $x \cap A \cap W_i = A_i$. This shows that $\Lambda_G(W_i) \subset A_i$. Since clearly $\Lambda_G(W_i) \subset A_i$ by invariance of $A_i$, we see that $A_i$ is a hyperbolic attractor.

However, such attractors cannot exist by Theorem 1.5. This contradiction shows that every attracting set within $U$ must contain some (Lorenz-like) singularity for all $G \in V$, as we wanted to prove. □

3.3. Number of singular ergodic physical measures. Here we prove Theorem A and Corollary B.

We present some auxiliary results: the existence of an invariant stable foliation covering the trapping region of any partially hyperbolic attracting set; the denseness of
the stable manifold of singularities in the support of an ergodic physical measure; and finally use these results, considering the ergodic physical measures whose support contains some equilibrium, to complete the proofs of the main Theorem A and Corollary B.

3.3.1. Existence of stable foliation covering \( U \). We recall the following useful property of partially hyperbolic attracting sets.

**Proposition 3.5.** [5, Proposition 3.2] Let \( \Lambda \) be a partially hyperbolic attracting set. The stable bundle \( E^s \) over \( \Lambda \) extends to a continuous uniformly contracting \( D\phi_t \)-invariant bundle \( E^s \) over an open neighborhood of \( \Lambda \).

Let \( D^k \) denote the \( k \)-dimensional open unit disk and let \( \text{Emb}'(D^k, M) \) denote the set of \( C^r \) embeddings \( \gamma : D^k \to M \) endowed with the \( C^r \) distance. In the present setting, we have \( d = \dim E^s = 1 \) in what follows.

**Proposition 3.6.** [5, Theorem 4.2 and Lemma 4.8] Let \( \Lambda \) be a partially hyperbolic attracting set. There exists a positively invariant neighborhood \( U_0 \) of \( \Lambda \), and constants \( C > 0, \lambda \in (0, 1) \), such that the following are true:

1. For every point \( x \in U_0 \) there is a \( C^r \) embedded \( d \)-dimensional disk \( W^s_x \subset M \), with \( x \in W^s_x \), such that
   - \( T_x W^s_x = E^s_x \).
   - \( \phi_t(W^s_x) \subset W^s_{\phi_t x} \) for all \( t \geq 0 \).
   - \( d(\phi_t x, \phi_t y) \leq C \lambda^t d(x, y) \) for all \( y \in W^s_x, t \geq 0 \).

2. The disks \( W^s_x \) depend continuously on \( x \) in the \( C^0 \) topology: there is a continuous map \( \gamma : U_0 \to \text{Emb}^0(D^d, M) \) such that \( \gamma(x)(0) = x \) and \( \gamma(x)(D^d_x) = W^s_x \). Moreover, there exists \( L > 0 \) such that the Lipschitz constant of \( \gamma(x) \) is bounded above \( \text{Lip} \gamma(x) \leq L \) for all \( x \in U_0 \).

3. The family of disks \( \{ W^s_x : x \in U_0 \} \) defines a topological foliation of \( U_0 \).

We can naturally assume that \( U = U_0 \) in what follows (using the flow invariance of the foliation).

3.3.2. Density of stable leaves of equilibria contained in the support of an ergodic physical measure. We state and prove the following useful property of singular-hyperbolic attracting sets of smooth 3-flows.

**Proposition 3.7.** Let \( \Lambda \) be a connected singular-hyperbolic attracting set for a \( C^2 \)-smooth 3-vector field \( G \). Let \( \mu \) be an ergodic physical measure supported on \( \Lambda \) and whose support contains a singularity. Then the stable manifold of the singularity transversely intersects the unstable manifold of every periodic orbit in the support of \( \mu \).

**Proof.** Let \( \mu \) be an ergodic physical measure supported in \( \Lambda \), in the setting of the statement of the proposition.

It is well-known from the Non-Uniform Hyperbolic Theory (Pesin’s Theory) that the support of a non-atomic hyperbolic ergodic probability measure \( \mu \) is contained in a
homoclinic class of a hyperbolic periodic orbit \( O(p) \); see e.g. \([22, \text{Appendix}]\) or \([12, \text{Theorem 15.4.3}]\).

In particular, the set of periodic points \( \text{Per}(G) \cap \text{supp} \mu \) is dense in \( \text{supp} \mu \) and all of them are homoclinically related – this conclusion also follows from \([40, \text{Theorem 1.5(8)}]\). This means, more precisely, that given any \( p, q \in \text{Per}(G) \cap \text{supp} \mu \) there exists a transversal intersection between the stable and unstable manifolds of \( p, q \):

\[
W^s(p) \cap W^u(q) \neq \emptyset \neq W^u(p) \cap W^s(q).
\]

**Lemma 3.8.** Fix \( p_0 \in \text{Per}(G) \cap \text{supp} \mu \) and let \( J = [a, b] \) be an arc on a connected component of \( W^u_G(p_0) \setminus \{p_0\} \) with \( a \neq b \). Then \( H = \text{clos} \cup_{t>0} \phi_t(J) \) contains a singularity of \( \Lambda \).

*Proof.* Since \( \mu \) is a SRB measure, for \( \mu \text{-a.e.} \ x \) we have \( W^u_x \subset \text{supp} \mu \) and \( W^u_x \cap W^s(\emptyset(p)) \neq \emptyset \). Thus by the Inclination Lemma (see \([32]\)) we have \( W^u(p) \subset W^u(x) \subset \text{supp} \mu \).

Because every periodic point \( p_0 \in \text{supp} \mu \) is homoclinically related to \( p \), then we also have \( W^u(p_0) \subset W^u(p_0) \subset \text{supp} \mu \).

Note that \( H \subset \text{clos} W^u_0(p_0) \subset \text{supp} \mu \) and \( H \) is a compact invariant set by construction, where \( W^u_0(p_0) \) is the connected component of \( W^u(p_0) \setminus \emptyset(p_0) \) containing \( J \). In addition, \( H \) is clearly connected, since \( H \) is also the closure of the orbit of the connected set \( J \) under a continuous flow.

If \( H \) has no singularities, then \( H \) is a compact connected hyperbolic set of saddle-type. Moreover, \( H \) contains the strong-unstable manifolds through any of its points, since every point in \( H \) is accumulated by forward iterates of the arc \( J \).

This means that \( H \) is a hyperbolic attracting set and so \( H = \text{supp} \mu \) by the existence of a dense regular orbit in \( \text{supp} \mu \). Consequently, \( H \) contains all singularities of \( \text{supp} \mu \). This contradiction proves that \( H \) must contain a singularity of \( \text{supp} \mu \). \( \square \)

Fix \( p_0 \) and \( \sigma \in \text{Sing}(G) \cap H \cap \text{supp} \mu \) as in the statement of Lemma 3.8. We have shown that \( \text{clos} W^u(p_0) \cap \text{Sing}(X) \neq \emptyset \). Moreover, the flow of \( G \) is inwardly transverse to the boundary of the manifold \( \overline{U} \) and has \( \Lambda \) as its maximal invariant subset, which is singular-hyperbolic. Hence, \( G \) in \( \overline{U} \) is a sectional-Anosov flow and we are in the setting of the next result from Bautista and Morales \([13]\).

**Theorem 3.9.** \([13, \text{Corollary 1.4}]\) If \( \emptyset \) is a periodic orbit of a sectional-Anosov flow \( G \) on a compact manifold satisfying \( \text{clos} W^u(p_0) \cap \text{Sing}(G) \neq \emptyset \), then there exists \( \sigma \in \text{Sing}(G) \) such that \( W^u(\emptyset) \cap W^s(\sigma) \neq \emptyset \).

Now we take \( \emptyset = \emptyset_G(p_0) \) and since \( G \) is a sectional-Anosov flow in the trapping region of \( \Lambda \), we apply Theorem 3.9 to obtain the existence of \( \sigma \in \text{Sing}(G) \) satisfying \( W^u(\emptyset) \cap W^s(\sigma) \neq \emptyset \). By definition of singular-hyperbolicity, this implies that \( W^u(p_0) \cap W^s(\sigma) \neq \emptyset \).

This is enough to conclude the proof of Proposition 3.7. Indeed, since all periodic orbits in \( \text{supp} \mu \) are homoclinically related, it is enough to obtain \( W^u(p_0) \cap W^s(\sigma) \neq \emptyset \) for one periodic point \( p_0 \in \text{supp} \mu \). \( \square \)
3.3.3. Proof of the main theorem and corollary. Here we show that a Lorenz-like equilibrium can be accumulated by at most two supports of ergodic physical measures inside a singular-hyperbolic attracting set.

**Lemma 3.10.** Given a Lorenz-like singularity $\sigma_0$ in a singular-hyperbolic attracting set $\Lambda = \Lambda_G(U)$ in a trapping region $U$ for a 3-dimensional $C^2$ vector field $G$, then there are at most two distinct ergodic physical measures whose support contains $\sigma_0$.

Using this we are ready to prove Theorem A.

**Proof of Theorem A.** Let then $\mathcal{P}$ be the family of ergodic physical measures supported in $\Lambda$ whose support contains some (Lorenz-like) singularity, and $\mathcal{L}$ be the set of Lorenz-like singularities in $\Lambda$. Proposition 1.3 ensures that there is a relation $R$ between $\mathcal{P}$ and $\mathcal{L}$ associating to each element of $\mathcal{P}$ the singularities it contains from $\mathcal{L}$:

$$\mu R \sigma \iff \sigma \in \text{supp } \mu.$$ 

Lemma 3.10 implies that $R^{-1}\sigma := \{\eta \in \mathcal{P} : \eta R \sigma\}$ satisfies $n(\sigma) := \#(R^{-1}\sigma) \in \{0, 1, 2\}$. Hence we obtain $\mathcal{P} = \bigcup_{\sigma \in \mathcal{L}} R^{-1}\sigma$ and since the union is not necessarily disjoint, we conclude

$$s = \# \mathcal{P} \leq \sum_{\sigma \in \mathcal{L}} n(\sigma) \leq 2 \# \mathcal{L} = 2 \cdot s_L.$$

This completes the proof, depending only on Lemma 3.10. $\square$

The following is straightforward.

**Proof of Corollary B.** In a trapping neighborhood $U$ of a singular-hyperbolic attractor $\Lambda_X(U)$ of a $C^1$-smooth 3-vector field $X$, there exists a $C^1$ neighborhood $\mathcal{U}$ of $X$ so that for all $G \in \mathcal{U}$ there are no hyperbolic attracting sets with respect to $G$, after Corollary 3.4. Hence, all attracting sets for $G \in \mathcal{U}$ are singular-hyperbolic and contain the continuation of some equilibrium from the original attractor $\Lambda$, which is necessarily Lorenz-like, by Proposition 1.3.

Therefore, the number of singularities in $\Lambda_G(U)$ is the same as the number of Lorenz-like singularities in the set. Hence, for a $C^2$ vector field $G \in \mathcal{U}$ (these form a $C^1$ dense subset of $\mathcal{U}$; see e.g. [32, Chapter 0]) we can apply Theorem A to obtain that the number of ergodic physical measures supported in $\Lambda = \Lambda_G(U)$ is bounded by twice the number of singularities in $\Lambda$. $\square$

To finish, we present a proof of Lemma 3.10.

**Proof of Lemma 3.10.** Given an ergodic physical probability measure $\mu_1$ so that supp $\mu_1 \subset \Lambda$ and $\sigma \in \text{supp } \mu_1 \cap \text{Sing}(G)$, then Proposition 3.7 implies that the unstable manifold of $\mu_1$-a.e. point crosses the stable manifold of $\sigma$ and so accumulates $\sigma$.

Indeed, ergodic physical measures are (non-uniformly) hyperbolic measures and also non-atomic, because they are also SRB measures. Hence their support is contained in
a homoclinic class of a hyperbolic periodic orbit $\mathcal{O}(p)$; see [12, Theorem 15.4.3]. In our singular-hyperbolic setting these periodic orbits are hyperbolic saddles with index $1^1$.

This ensures that for $\mu$-a.e. $x$ there exists $p \in \text{Per}(G) \cap \Lambda$ so that $W^s(p) \cap W^u(x) \neq \emptyset$. Since $W^u(p) \cap W^s(\sigma_0) \neq \emptyset$, the Inclination Lemma now ensures that we also have $W^u(x) \cap W^s(\sigma) \neq \emptyset$.

Thus we find a center unstable leaf $W_1$ through a $\mu_1$ generic point which transversely crosses the local stable manifold of $\sigma$ and so, by the Inclination Lemma, become arbitrarily close to $\sigma$.

According to Remark 1.4, singular-hyperbolic attracting sets satisfy $W^{ss}(\sigma) \cap \Lambda = \{\sigma\}$ for all Lorenz-like singularities $\sigma \in \text{Sing}(G)$. Then the accumulation of $\sigma$ by the trajectory of $W_1$ is performed along the one-dimensional weak-stable direction of $\sigma_0$; see the left hand side of Figure 12.

We can repeat the above arguments for any other ergodic physical measure $\mu_2$ obtaining a center unstable leaf $W_2$ through a $\mu_2$ generic point which also transversely crosses the stable manifold of some Lorenz-like equilibria $\sigma'$.

If $\sigma = \sigma'$, then the future trajectories of $W_i$, $i = 1, 2$ both accumulate $\sigma$ and so accumulate each other; see Figure 12.

![Figure 12](https://example.com/figure12.png)

**Figure 12.** On the left hand side: the unstable manifolds $W_1, W_2$ associated to generic points of $\mu_1, \mu_2$ on opposite sides of $W^s(\sigma_0)$. On the right hand side: the unstable manifolds $W_1, W_2$ associated to generic points of $\mu_1, \mu_2$ on the same side of $W^s(\sigma_0)$ and a cross-section to the flow very close to $\sigma_0$.

We note that the local strong-stable manifold $W^{ss}_{\text{loc}}(\sigma_0)$ divides the local stable manifold $W^s_{\text{loc}}(\sigma_0)$ in two “sides”. More precisely, let $h : B(\sigma_0, r) \to T_{\sigma_0}M$ for some $r > 0$ be the homeomorphism conjugating $h \circ \phi_t = e^{tDG_{\sigma_0}} \cdot h$ the flow $\phi_t$ of $G$ with the linear flow given by $DG_{\sigma_0}$ according to the Hartman-Grobman Theorem; see e.g. [32]. By a linear change of coordinates on $T_{\sigma_0}M$, we may assume without loss of generality that

---

$^1$The index of a hyperbolic periodic orbit is the dimension of the stable direction.
the eigenspaces associated to the eigenvalues $\lambda_1 \leq \lambda_2 < 0 < \lambda_3$ are respectively the $x$, $z$ and $y$ axis, as depicted in Figure 12. Then we set, for a small enough $\rho > 0$

$$B(\sigma_0)^+ = h^{-1}(B(0, \rho) \cap \{z > 0\}) \quad \text{and} \quad B(\sigma_0)^- = h^{-1}(B(0, \rho) \cap \{z < 0\})$$

the “top” and “down” sides of a neighborhood of the singularity $\sigma_0$.

Now we observe that if $\sigma_0$ is accumulated by $W_1$ and $W_2$ along the same side of the weak-stable direction, as in the right hand side of Figure 12 then, since

- both unstable leaves belong to $\Lambda$ (because it is an attracting set), and
- the stable leaves through points of $\Lambda$ have uniform size (by Proposition 3.6),

we conclude that the stable foliation $\mathcal{F}_{ss}$ of $\Lambda$ transversely crosses both leaves near $\sigma_0$.

Moreover, it is well-known that the stable foliation for $C^2$ partially hyperbolic flows is absolutely continuous (see e.g. [34] and [5])), and so a positive Lebesgue measure subset of $W_1$ will be sent through the stable holonomy to a positive Lebesgue measure subset of $W_2$. Since $\mu_i$ is an ergodic SRB measure, a full measure subset of $W_i$ is formed by $\mu_i$-generic points, $i = 1, 2$. Then we can find a $\mu_1$-generic point $x_1$ and a $\mu_2$-generic point $x_2$ in the same stable leaf. This implies that $\mu_1 = \mu_2$.

The only possible way to avoid this phenomenon is for $W_1, W_2$ to accumulate $\sigma_0$ along different sides of the local stable manifold of $\sigma_0$, as in the left hand side of Figure 12.

However, if there existed still another ergodic physical measure $\mu_3$ whose support contains the same equilibrium $\sigma_0$, then some pair of the measures $\mu_i, i = 1, 2, 3$ would accumulate $\sigma_0$ through central-unstable leaves along the same side of $\sigma_0$, and so this pair of measures would coincide.

This shows that at most a pair of distinct ergodic physical measures can share a singularity on their respective supports. The proof of the lemma is complete. 

\[\square\]

**Figure 13.** Singular-hyperbolic attractor whose singularities are accumulated by both sides by the support of the unique physical measure.
Remark 3.11. There are examples of singular-hyperbolic attractors with Lorenz-like singularities accumulated on both sides; see e.g. the proof of [28, Theorem B, pg. 345-346] and Figure 13.

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[44] URL: [https://sites.google.com/site/vdaraujo99/](https://sites.google.com/site/vdaraujo99/)
NUMBER OF ERGODIC PHYSICAL MEASURES OF SINGULAR-HYPERBOLIC ATTRACTING SETS

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