Stochastic semiclassical gravity and fluctuations during inflation

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Stochastic semiclassical gravity is a theory for the interaction of gravity with quantum matter fields which goes beyond the semiclassical limit. The theory predicts stochastic fluctuations of the classical gravitational field induced by the quantum fluctuations of the stress energy tensor of the matter fields. Here we use an axiomatic approach to introduce the Einstein-Langevin equations as the consistent set of dynamical equations for a first order perturbative correction to semiclassical gravity and review their main features. We then describe the application of the theory in a simple chaotic inflationary model, where the fluctuations of the inflaton field induce stochastic fluctuations in the gravitational field. The correlation functions for these gravitational fluctuations lead to an almost Harrison-Zel’dovich scale invariant spectrum at large scales, in agreement with the standard theories for structure formation. A summary of recent results and other applications of the theory is also given.

I. INTRODUCTION

Let us briefly summarize the road to stochastic semiclassical gravity. It starts with quantum field theory in a curved spacetime which is now a well understood and well defined theory at least for free fields [1]. In this theory the gravitational field is the classical field of general relativity, that is the metric of the spacetime, and the quantum fields propagate in such a spacetime. Since the spacetime is now dynamical it is not always possible to define a physically meaningful vacuum state for the quantum field and when this is possible in some “initial” times it is usually unstable, in the sense that it may differ from the vacuum state at latter times, and spontaneous creation of particles occurs. Applications of this in cosmology, such as particle production in expanding Friedmann-Robertson-Walker models [2], and black hole physics, such as Hawking radiation [3], are well known. This is one aspect of the interaction of gravity with quantum matter fields.

Another aspect of this interaction is the back-reaction of the quantum fields on the spacetime. Since the gravitational field couples to the stress tensor of matter fields, the key object here is the expectation value in some quantum state of the stress energy tensor of the quantum field, which is a classical observable. However, since this object is quadratic in the field operator, which is only well defined as a distribution on spacetime, it involves ill defined quantities which translate into ultraviolet divergencies. To be able to define a physically meaningful quantity a regularization and a renormalization procedure is required. The ultraviolet divergencies associated to the expectation value of the stress energy tensor are also present in Minkowski spacetime, but in a curved background the renormalization procedure is more sophisticated as it needs to preserve general covariance. A regularization procedure which is specially adapted to the curved background is the so called point-splitting method [4]. The final expectation value of the stress energy tensor using point splitting or any other reasonable regularization technique is essentially unique, modulo some terms which depend on the spacetime curvature and which are independent of the quantum state. This uniqueness was essentially proved by Wald [5] who investigated the criteria that a physically meaningful expectation value of the stress energy tensor ought to satisfy.

Now the back-reaction problem may be formulated in terms of the so called semiclassical Einstein equations. These are Einstein equations which have the expectation value in some quantum state of the stress energy tensor as a matter source. The back-reaction problem was investigated in cosmology, in particular to see whether cosmological anisotropies could be damped by back-reaction [6]. This was an earlier attempt [7] previous to inflation to explain why the universe is so isotropic at present.

Another step forward in the back-reaction problem was the use of effective action methods [8] so familiar in quantum field theory. These methods were of great help in the study of cosmological anisotropies since they allowed the introduction of familiar perturbative treatments into the subject. The most common effective action method, however, led to equations of motion which were not real because they were taylored to compute transition elements of quantum operators rather than expectation values. Fortunately the appropriate technique had already been developed by Schwinger and Keldysh [9] in the so called Closed Time Path (CTP) or in-in effective action method, and was soon adapted to the gravitational context [10]. These techniques were then applied to different problems in the cosmological context including the effects of arbitrary perturbations on homogeneous backgrounds [11]. As a result one was now...
able to deduce the semiclassical Einstein equations by the CTP functional method: starting with an action for the interaction of gravity with matter fields, treating the matter fields as quantum fields and the gravitational field at tree level only.

The semiclassical Einstein equations have limitations, it is clear that even outside Planck scales if fluctuations on the expectation value of matter fields are large, and that depends on the quantum state, the semiclassical equations should break down [2]. One expects, in fact, that a better approximation would describe the gravitational field in a probabilistic way. In other words, that the semiclassical equations should be substituted by some Langevin-type equations with a stochastic source that describes the quantum fluctuations. A significant step in this direction was made by Hu [13] who proposed to view the back-reaction problem in the framework of an open quantum system, where the quantum fields are seen as the “environment” and the gravitational field is seen as the “system”. Following this proposal a systematic study of the connection between semiclassical gravity and open quantum systems resulted in the development of a framework where semiclassical Einstein-Langevin equations could be derived [16]. The key technical factor to most of these results was the use of the influence functional method of Feynman and Vernon [17] for the description of the system-environment interaction when only the state of the system is of interest. The CTP method for open systems involves, in fact, the influence functional.

However although several Einstein-Langevin equations were derived, the results were somewhat formal and some concern could be raised on the physical reality of the solutions of the stochastic equations for the gravitational field. This is related to the issue of the environment induced quantum to classical transition. In fact, for the existence of a semiclassical regime for the dynamics of the system one needs two requirments, in the language of the consistent histories formulation of quantum mechanics [18]. The first is decoherence, which guarantees that probabilities can be consistently assigned to histories describing the evolution of the system, and the second is that these probabilities should peak near histories which correspond to solutions of classical equations of motion. The effect of the environment is crucial on the one hand to provide decoherence [19] and on the other hand to produce both dissipation and noise to the system through back-reaction, thus inducing a semiclassical stochastic dynamics on the system. As shown by Gell-Mann and Hartle [20] in an open quantum system stochastic semiclassical equations are obtained after a coarse graining of the environmental degrees of freedom and a further coarse graining in the system variables. That this mechanism could also work for decoherence and classicalization of the metric field was not so clear lacking a quantum description of the gravitational field, and the analogy could be made only formaly [21].

Thus, an axiomatic approach to the Einstein-Langevin equations which was independent of the open system analogy was suggested: it was based on the formulation of consistent dynamical equations for a perturbative correction to semiclassical gravity able to account for the lowest order stress energy fluctuations of matter fields [22]. It was later shown that these same equations could be derived, in this general, case from the influence functional of Feynman and Vernon in which, the gravitational field is treated at tree level and and quantum fields are quantized, the first being, in fact, the “system” and the seconds the “environment” [23].

Here we review some of these developments in semiclassical stochastic gravity and also some of its applications. In section II a brief sketch of semiclassical gravity is given. In section III the axiomatic approach to the Einstein-Langevin equations is discussed. To illustrate the relation between the semiclassical, stochastic semiclassical and quantum theories, a simplified model of linear gravity is used. Throughout this section we use a simplified notation, avoiding tensorial indices when possible and emphasizing the conceptual aspects. In section IV an important application of stochastic gravity is discussed in some detail, it concerns the computation of the two-point correlations of the metric perturbations induced by the fluctuation in the stress energy tensor of the inflaton field during inflation. The results agree with the standard results but the present method in which the gravitational field and the matter fields are treated separately may have some advantages over the methods where both the metric and matter perturbations are treated on the same footing from the start. Finally, in section V we summarize our result and briefly discuss other applications. We should mention that an extensive and stimulating review of stochastic semiclassical gravity is given in ref. [24], it includes the suggestive idea of considering classical general relativity it terms of collective variables and of viewing semiclassical gravity as mesoscopic physics.

II. SEMICLASSICAL GRAVITY

Semiclassical gravity is a theory which describes the interaction of the gravitational field assumed to be a classical field with matter fields which are quantum. It is supposed to be some limit of the still unknown theory describing the interaction of quantum gravity with quantum fields. Due to the lack of the quantum theory, the semiclassical limit cannot be rigurously derived. However, it can be formally derived in several ways. One of them is the leading-order $1/N$ approximation of quantum gravity [25], where $N$ is the number of independent free quantum fields which interact with gravity only, and where one keeps finite the value of $NG$, where $G$ is Newton’s gravitational constant. In this
limit, after path integration one arrives at a theory in which formally the gravitational field can be treated as a c-number (i.e. is quantized at tree level) and the quantum fields are fully quantized. If we call \( g \) the metric tensor and \( \phi \) the scalar field (for simplicity we consider just one scalar field) one arrives at the semiclassical Einstein equation as the dynamical equation for the metric \( g \):

\[
G_g = 8\pi G (\bar{T}^R)_g,
\]

where \( \bar{T} = T[\phi^2] \) is the stress energy tensor in a simplified notation, which is quadratic in the field operator \( \phi \). This operator, being the product of distribution valued operators, is ill defined and needs to be regularized and renormalized, the \( R \) in \( \bar{T}^R \) means that the operator has been renormalized. The angle brackets on the right hand side mean that the expectation value of the stress tensor operator is computed in some quantum state, say \( |\psi\rangle \), compatible with the geometry described by the metric \( g \). On the left hand side \( G_g \) stands for the Einstein tensor for the metric \( g \) together with the cosmological constant term and other terms quadratic in the curvature which are generally needed to renormalize the stress energy tensor operator. The quantum field operator \( \phi \) propagates in the background defined by the metric \( g \), it thus satisfies a Klein-Gordon equation, which we write also schematically as

\[
\Box_g \phi = 0, \tag{2}
\]

where \( \Box_g \) stands for the D’Alambert operator in the background of \( g \). Equation (1) is the semiclassical Einstein equation, it is the dynamical equation for the metric tensor \( g \) and describes the back-reaction of the quantum matter fields on the geometry. A solution of semiclassical gravity consists of the set \((g, \phi, |\psi\rangle)\) where \( g \) is a solution of (1), \( \phi \) is a solution of (2) and \( |\psi\rangle \) is the quantum state in which the expectation value of the stress energy tensor in (1) is computed.

For a free quantum field this theory is robust in the sense that it is consistent and fairly well understood. Note that it in some sense unique as a theory where the gravitational field is classical. In fact, the (classical) gravitational field interacts with matter fields through the stress energy tensor, and the only reasonable c-number stress energy tensor that one may construct with the operator \( \bar{T} \) is just the right hand side of (1). However the scope and limits of the theory are not so well understood as a consequence of the lack of the full quantum theory. It is assumed that the semiclassical theory should break down at Planck scales, which is when simple order of magnitude estimates suggest that the quantum effects of gravity cannot be ignored: the gravitational energy of a quantum fluctuation of energy in a Planck size region, determined by the Heisenberg uncertainty principle, are of the same order of magnitude.

There is also another situation when the semiclassical theory should break down, namely, when the fluctuations of the stress energy tensor are large. This has been emphasized by Ford and collaborators. It is less clear how to quantify what a large fluctuation here means and some criteria have been proposed [13, 20]. Generally this depends on the quantum state and may be illustrated by the example used in ref. [13] as follows.

Let us assume a quantum state formed by an isolated system which consists of a superposition with equal amplitude of one configuration with mass \( M_1 \) and another with mass \( M_2 \). Semiclassical theory as described in (1) predicts that the gravitational field of this system is produced by the average mass \((M_1 + M_2)/2 \), that is a test particle will move on the background spacetime produced by such a source. However one would expect that if we send a succession of test particles to probe the gravitational field of the above system half of the time they would react to the field of a mass \( M_1 \) and the other half to the field of a mass \( M_2 \). If the two masses differ substantially the two predictions are clearly different, note that the fluctuations in mass of the quantum state is of the order of \((M_1 - M_2)^2 \). Although the previous example is suggestive a word of caution should be said in order not to take it too literally. In fact, if the previous masses are macroscopic the quantum system decoheres very quickly [19] and instead of a pure quantum state it is described by a density matrix which diagonalizes in a certain pointer basis. Thus for observables associated to this pointer basis the matrix density description is equivalent to that provided by a statistical ensemble. In any case, however, from the point of view of the test particles the predictions differ from that of the semiclassical theory.

**III. EINSTEIN-LANGEVIN EQUATION**

The purpose of semiclassical stochastic gravity is to be able to deal with the situation of the previous example in which the predictions of the semiclassical theory may be inaccurate. Consequently, our first point is to characterize the quantum fluctuations of the stress energy tensor.

The physical observable that measures these fluctuations is \( \langle \bar{T}^2 \rangle - \langle \bar{T} \rangle^2 \). To make this more precise let us introduce the tensor operator \( \hat{I} = \bar{T} - \langle \bar{T} \rangle \hat{I} \), where \( \hat{I} \) is the identity operator, then we introduce the noise kernel as the four index bi-tensor defined as the expectation value of the anticommutator of the operator \( \hat{I} \):

\[
\langle \bar{T}^2 \rangle - \langle \bar{T} \rangle^2 = \langle \hat{I} \rangle^2.
\]
\[ N(x, y) = \frac{1}{2} \langle \{ \hat{t}(x), \hat{t}(y) \} \rangle_g. \] (3)

The subindex \( g \) here means that this expectation value in taken in the background metric which is a solution of the semiclassical equation (\[ 3 \]). An important property of the symmetric bi-tensor \( N(x, y) \) is that it is finite because the tensor operator \( \hat{t} \) is finite since the ultraviolet divergencies of \( \hat{T} \) are cancelled by the substraction of \( \langle \hat{T} \rangle \). Since the operator \( \hat{T} \) is selfadjoint \( N(x, y) \), which is the expectation value of an anticommutator, is real and positive semi-definite. This last property allows for the introduction of a classical Gaussian stochastic tensor \( \xi \) defined by

\[ \langle \xi(x) \rangle_c = 0, \quad \langle \xi(x) \xi(y) \rangle_c = N(x, y). \] (4)

This stochastic tensor is symmetric \( \xi_{\mu\nu} = \xi_{\nu\mu} \) and divergenceless, \( \nabla^\mu \xi_{\mu\nu} = 0 \), as a consequence of the fact that the stress tensor operator is divergenceless. The subindex \( c \) means that the expectation value is just a classical average. Note that we assume that \( \xi \) is Gaussian just for simplicity in order to include the main effect. The idea now is simple we want to modify the semiclassical Einstein equation (\[ 3 \]) by introducing a linear correction to the metric tensor \( g \), such as \( g + h \), which accounts consistently for the fluctuations of the stress energy tensor. The simplest equation is,

\[ G_{g+h} = 8\pi G(\langle \hat{T}^R \rangle_{g+h} + \xi), \] (5)

where \( g \) is assumed to be a solution of equation (\[ 3 \]). This stochastic equation must be thought of as a linear equation for the metric perturbation \( h \) which will behave consequently as a stochastic field tensor. Note that the tensor \( \xi \) is not a dynamical source, since it has been defined in the background metric \( g \) which is solution of the semiclassical equation. Note also that this source is divergenceless with respect to the metric, and it is thus consistent to write it on the right hand side of the Einstein equation. This equation is gauge invariant with respect to diffeomorphisms defined by any field on the background spacetime (\[ 22 \]). If we take the statistical average equation (\[ 5 \]) becomes just the semiclassical equation for the metric \( g + h \) where now the expectation value of \( \hat{T} \) is taken in the perturbed spacetime.

The stochastic equation (\[ 3 \]) is known as the Einstein-Langevin equation. It is linear in \( h \), thus its solutions can be written as the sum of a solution of the homogeneous equation plus a stochastic part: \( h = h^b + h^s \). The equation predicts that the gravitational field has stochastic fluctuations over the background \( g \). The correlation function for the gravitational field is simply given by \( \langle h^s(x)h^s(y) \rangle_c \). This is the physically most relevant observable, to find it requires to solve the Einstein-Langevin equation and to know the noise kernel \( N(x, y) \). Note that the noise kernel should be thought of as a distribution function, the limit of coincidence points has meaning only in the sense of distributions. Explicit expressions of this kernel in terms of the two point Wightman functions is given in (\[ 22 \]), expression based on point-splitting methods have also been given in (\[ 23 \]).

This stochastic theory goes beyond semiclassical gravity in the following sense. The semiclassical theory, which is based on the expectation value of the stress energy tensor, carries information on the field two-point correlations only, since \( \langle \hat{T} \rangle \) is quadratic in the field operator \( \phi \). The stochastic semiclassical theory on the other hand, is based on the noise kernel (\[ 3 \]) which is quartic in the field. It is thus clear that it carries information beyond the semiclassical theory. In some sense the theory represents a middle road between the semiclassical and the quantum theory. It does not carry information, however, on the graviton-graviton interaction.

**A. A simplified model**

To clarify this point it is useful to introduce a toy model which would describe exactly a linear theory such as electromagnetism, and captures some essential features of linearized gravity. Let us assume that the gravitational equations are described by the linear equations for the field \( h \), with a source \( T[\phi^2] \). The semiclassical equations now correspond to

\[ \Box h = \langle \hat{T} \rangle, \] (6)

where now \( \hat{T} \) depends on the field operator \( \phi \). Note that this theory does not correspond to linearized semiclassical gravity around the Minkowski background since in that case \( \langle \hat{T} \rangle = 0 \) assuming that \( T \) does not depend on \( h \) (in the linearized semiclassical theory \( \langle \hat{T} \rangle \) is, in fact, linear \( h \) (\[ 24 \])). The model, however, can be extended to linearized gravity (\[ 30 \]).

The solutions of this equation may be written in terms of the retarded propagator \( G_{xy} \) as

\[ h_x = h_x^0 + \int G_{xx'} \langle \hat{T}_{x'} \rangle, \] (7)
where $h_0^0$ is the homogeneous solution which is determined by the initial conditions.

Let us now consider the quantum theory which, in Heisenberg representation, may be written as

$$\Box \hat{h} = \hat{T}.$$  

(8)

The solutions of this equation may be written as

$$\hat{h}_x = \hat{h}_0^0 + \int G_{xx'} \hat{T}_{x'} ,$$  

(9)

and one may compute the two point quantum correlation function as

$$\langle \hat{h}_x \hat{h}_y \rangle = \langle \hat{h}_x^0 \hat{h}_y^0 \rangle + \int \int G_{xx'} G_{yy'} \langle \hat{T}_{x'} \hat{T}_{y'} \rangle ,$$  

(10)

where the expectation value is taken in the quantum state in which both fields $\phi$ and $h$ have been quantized, and we have used that for the free field $\langle \hat{h}_0^0 \rangle = 0$.

At this point we may now compare with the stochastic theory as described by equation (8) which, in this simplified model, may be written as

$$\Box h = \langle \hat{T} \rangle + \xi ,$$  

(11)

where $\xi$ is a Gaussian stochastic source defined by $\langle \xi_x \rangle_c = 0$ and $\langle \xi_x \xi_y \rangle_c = \langle \hat{T}_x \hat{T}_y \rangle - \langle \hat{T}_x \rangle \langle \hat{T}_y \rangle$, where the expectation values on the right hand side are defined in a given state of the field $\phi$ and the subscript $c$ on the left hand sides means statistical average, see (3) and (4). Now the solution of this equation may be written in terms of the retarded propagator as,

$$h_x = h_0^0 + \int G_{xx'} \langle \langle \hat{T}_{x'} \rangle + \xi_x \rangle ,$$  

(12)

from where the two point correlation function for the classical field $h$, after using the definition of $\xi$ and that $\langle h_0^0 \rangle_c = 0$, is given by

$$\langle h_x h_y \rangle_c = \langle \hat{h}_x^0 \hat{h}_y^0 \rangle_c + \int \int G_{xx'} G_{yy'} \langle \hat{T}_{x'} \hat{T}_{y'} \rangle .$$  

(13)

Comparing (11) with (13) we see that the respective second terms on the right hand side are identical provided the expectation values are computed in the same quantum state for the field $\phi$, note that we have assumed that $\hat{T}$ does not depend on $h$. The fact that the field $h$ is also quantized in (10) does not change the previous statement. In the real theory of gravity $T$, in fact, depends also on $h$ and then the previous statement is only true approximately, i.e perturbatively in $h$. The nature of the first terms on the right hand sides of equations (10) and (13) is different: in the first case it is the two point expectation value of the free quantum field $\hat{h}_0$ whereas in the second case it is the average of the two point classical average of the homogeneous field $\hat{h}_0$, which depends on the initial conditions. Now we can still make these terms to be equal to each other if we assume for the homogeneous field $h$ a distribution of initial conditions such that $\langle h_x^0 h_y^0 \rangle_c = \langle \hat{h}_x^0 \hat{h}_y^0 \rangle$. Thus, under this assumption on initial conditions for the field $h$ the two point correlation function of (13) equal the quantum expectation value of (10) exactly. Thus in a linear theory as in the model just described one may just use the statistical description given by (11) to compute the quantum two point function of equation (10). Of course, the statistical description is not able to account for graviton-graviton effects which go beyond the linear approximation in $h$.

B. Functional approach

To end this section we should mention that the Einstein-Langevin equation (5) may also be formally derived using the CTP functional method (21). As remarked in the introduction the CTP functional was introduced by Schwinger to compute expectation values. One just considers the interaction of the gravitational field $g$ at tree level and of the quantum field $\phi$ fully quantum. Then the effective action for the gravitational field is derived after integrating out the degrees of freedom of the quantum field, and the CTP influence action reduces basically to the Feynman and Vernon influence functional (17) used in quantum open systems. Here the system is the gravitational field and the environment is the quantum field. The stochastic terms for the gravitational field are found by suitably interpreting some pure imaginary term which appear in the influence action. These terms are closely connected to Gell-Mann and Hartle decoherence functional (20) used to study decoherence and classicalization in open quantum systems. The net result of these analogies is that the interaction with the environment induces fluctuations in the system dynamics. It is precisely the noise kernel introduced in (3) that accounts for this effect.
IV. GRAVITATIONAL FLUCTUATIONS DURING INFLATION

An important application of stochastic semiclassical gravity is the derivation of the cosmological perturbations generated during inflation. Let us consider the Lagrangian density for the inflaton field \( \phi \) of mass \( m \)

\[
\mathcal{L}(\phi) = \frac{1}{2}(\partial \phi)^2 + \frac{1}{2}m^2\phi^2,
\]

which is the basis of the simplest chaotic inflationary model. The conditions for the existence of an inflationary period, which is characterized by an accelerated expansion of the spacetime, is that the value of the field averaged over a region with the typical size of the Hubble radius is higher than the Planck mass, \( m_P \). This is because in order to solve the cosmological horizon and flatness problem more than 60 e-folds of expansion are needed, to achieve this the scalar field should begin with a value higher than \( 3m_P \). Furthermore, as we will see, the large scale anisotropies measured restrict the inflaton mass to be of the order of \( 10^{-5}m_P \).

We want to study the metric perturbations produced by the stress tensor fluctuations of the inflaton field on the homogeneous background of a flat Friedmann-Robertson-Walker model, described by the cosmological scale factor \( a(\eta) \), where \( \eta \) is the conformal time, which is driven by the homogenous inflaton field \( \phi(\eta) = \langle \hat{\phi} \rangle \). Thus we write the inflaton field in the following form

\[
\hat{\phi} = \phi(\eta) + \hat{\varphi}(x),
\]

where \( \hat{\varphi}(x) \) corresponds to a free massive quantum scalar field with zero expectation value on the homogeneous background metric: \( \langle \hat{\varphi} \rangle_g = 0 \). Restricting ourselves to scalar-type perturbations the perturbed metric \( \tilde{g} = g + h \) can be written in the longitudinal gauge as,

\[
d\bar{s}^2 = a^2(\eta)[(1 + 2\Psi(x))d\eta^2 + (1 - 2\Psi(x))\delta_{ij}dx^idx^j],
\]

where the metric perturbations \( \Phi(x) \) and \( \Psi(x) \) correspond to Bardeen’s gauge invariant variables.

The Einstein-Langevin equation as described in the previous section is gauge invariant, and thus we can work in a desired gauge and then extract the gauge invariant quantities. The Einstein-Langevin equation (3) reads now:

\[
G^{(0)} - 8\pi G\langle \hat{T}^{(0)} \rangle_g + G^{(1)}(h) - 8\pi G\langle \hat{T}^{(1)} \rangle_g = 8\pi G\xi,
\]

where the two first terms cancel, that is \( G^{(0)} - 8\pi G\langle \hat{T}^{(0)} \rangle_g = 0 \), as the background metric satisfies the semiclassical Einstein equations. Here the subscripts \( (0) \) and \( (1) \) refer to functions in the background metric \( g \) and linear in the metric perturbation \( h \), respectively. The stress tensor operator \( \hat{T} \) for the minimally coupled inflaton field in the perturbed metric \( \tilde{g} = g + h \) is:

\[
\hat{T}_{\mu\nu} = \partial_\mu\hat{\phi}\partial_\nu\hat{\phi} + \frac{1}{2}\tilde{g}_{\mu\nu}(\partial_\rho\hat{\phi}\partial^\rho\hat{\phi} + m^2\hat{\phi}^2).
\]

Now using the decomposition of the scalar field into its homogeneous and inhomogeneous part, see (13), and the metric \( \tilde{g} \) into its homogenous background \( g \) and its perturbation \( h \), the renormalized expectation value for the stress tensor can be written as

\[
\langle \hat{T}^R(\tilde{g}) \rangle = \langle \hat{T}(\tilde{g}) \rangle_{\phi\phi} + \langle \hat{T}(\tilde{g}) \rangle_{\phi\varphi} + \langle \hat{T}^R(\tilde{g}) \rangle_{\varphi\varphi},
\]

where only the homogeneous solution for the scalar field contributes to the first term. The second term is proportional to \( \langle \hat{\varphi}^2 \rangle \) which is not zero because the field dynamics is considered on the perturbed spacetime, i.e. this term includes the coupling of the field with \( h \). The last term corresponds to the expectation value to the stress tensor for a free scalar field on the spacetime of the perturbed metric.

We can now compute the noise kernel \( N(x,y) \) defined in equation (3), which after using the previous decomposition can be written as

\[
\langle \{ \hat{t}, \hat{t} \} | g \rangle = \langle \{ \hat{t}, \hat{t} \} |_{\phi\phi} | g \rangle + \langle \{ \hat{t}, \hat{t} \} |_{\varphi\varphi} | g \rangle,
\]

where we have used the fact that \( \langle \hat{\varphi} \rangle_g = 0 = \langle \hat{\varphi}^2 \rangle_g \) for Gaussian states on the background geometry. We have considered the vacuum state to be the Euclidean vacuum which is preferred in the de Sitter background, and this state is Gaussian. In the above equation the first term is quadratic in \( \hat{\varphi} \) whereas the second one is quartic, both contributions to the noise kernel are separately conserved since both \( \phi(\eta) \) and \( \hat{\varphi} \) satisfy the Klein-Gordon field equations on the
background spacetime. Consequently, the two terms can be considered separately. On the other hand if one treats \( \dot{\varphi} \) as a small perturbation the second term in (22) is of lower order than the first and may be neglected consistently, this corresponds to neglecting the last term of (19). The stress tensor fluctuations due to a term of that kind were considered in ref. [33].

We can now write down the Einstein-Langevin equations (17). It is easy to check that the space-space components coming from the stress tensor expectation value terms and the stochastic tensor are diagonal, i.e. \( \langle T_{ij} \rangle = 0 = \xi_{ij} \) for \( i \neq j \). This, in turn, implies that the two functions characterizing the scalar metric perturbations are equal: \( \Phi = \Psi \) in agreement with ref. [32]. The equation for \( \Phi \) can be obtained from the 0\( i \)-component of the Einstein-Langevin equation, which in Fourier space reads

\[
2ik_i (H\Phi_k + \Phi_k') = \frac{8\pi}{m_P} \xi_{k0i},
\]

(21)

where \( k_i \) is the comoving momentum component associated to the comoving coordinate \( x^i \). Here primes denote derivatives with respect to the conformal time \( \eta \) and \( H = a'/a \). A non-local term of dissipative character which comes from the second term in (19) should also appear on the left hand side of equation (21), but since we are mainly interested in the fluctuating part we have ignored this term. To solve this equation, whose left hand side comes from the linearized Einstein tensor for the perturbed metric [32], we need the retarded propagator for the gravitational potential \( \Phi_k \),

\[
G_k(\eta, \eta') = -i\frac{4\pi}{k_im_P^2} \eta \frac{a(\eta')}{a(\eta)} + f(\eta, \eta')
\]

(22)

where \( f \) is a homogeneous solution of (21), related to the initial conditions chosen. For instance, if we take \( f(\eta, \eta') = -\theta(\eta_0 - \eta')a(\eta')/a(\eta) \) the solution would correspond to “turning on” the stochastic source at \( \eta_0 \).

The correlation function for the metric perturbations is now given by

\[
\langle \Phi_k(\eta) \Phi_{k'}(\eta') \rangle_c = (2\pi)^2 \delta(k + k') \int_0^\eta d\eta_1 \int_0^{\eta'} d\eta_2 G_k(\eta, \eta_1)G_{k'}(\eta', \eta_2)\langle \xi_k \xi_{k'} \rangle_c.
\]

(23)

The correlation function for the stochastic source , which is connected to the stress tensor fluctuations through the noise kernel is given by,

\[
\langle \xi_k \xi_{k'} \rangle_c = \frac{1}{2} \langle \{i_k^{\dagger} \xi_{\eta_1}, i_k^{-1} \xi_{\eta_2} \} \rangle_{\phi \phi} = \frac{1}{2} k_i k_i^* \langle \phi(\eta_1) \phi(\eta_2) \rangle_{\phi \phi}^{(1)}(\eta_1, \eta_2),
\]

(24)

where \( G_k^{(1)}(\eta_1, \eta_2) = \langle \{ \dot{\varphi}_k(\eta_1), \dot{\varphi}_k(\eta_2) \} \rangle \) is the k-mode Hadamard function for a free minimally coupled scalar field which is in the Euclidean vacuum on the de Sitter background.

It is useful to compute the Hadamard function for a massless field and consider a perturbative expansion in terms of the dimensionless parameter \( m/m_P \). Thus we consider \( G_k^{(1)}(\eta_1, \eta_2) = \{ \phi_k(\eta_1), \phi_k(\eta_2) \} | 0 \rangle = 2\mathcal{R} (u_k(\eta_1)u_k^*(\eta_2)) \) with \( \dot{y}_k(\eta) = a(\eta)\dot{\varphi}_k(\eta) = \dot{a}_k u_k(\eta) + \dot{a}_k^* u_{-k}^*(\eta) \) and where \( u_k = (2k)^{-1/2} e^{ik\eta} (1 - i/\eta) \) are the positive frequency k-mode for a massless minimally coupled scalar field on a de Sitter background, which define the Euclidean vacuum state: \( \tilde{a}_k | 0 \rangle = 0 \).

The background geometry, however, is not exactly that of de Sitter spacetime, for which \( a(\eta) = -H(\eta)^{-1} \) with \( -\infty < \eta < 0 \). One can expand in terms of the “slow-roll” parameters and assume that to first order \( \dot{\varphi}(t) \approx m_P^2(m/m_P) \), where \( t \) is the physical time. The correlation function for the metric perturbation (23) is the computed, see ref. [27] for details. The final result, however, is very weakly dependent on the initial conditions as one may understand from the fact that the accelerated expansion of de quasi-de Sitter spacetime during inflation erases the information about the initial conditions. Thus one may take the initial time to be \( \eta_0 = -\infty \) and obtain to lowest order in \( m/m_P \) the expression

\[
\langle \Phi_k(\eta) \Phi_{k'}(\eta') \rangle_c \approx 8\pi^2 \left( \frac{m}{m_P} \right)^2 k^{-3} (2\pi)^3 \delta(k + k') \cos k(\eta - \eta').
\]

(25)

From this result two main conclusions are derived. First, the prediction of an almost Harrison-Zel’dovich scale-invariant spectrum for large scales, i.e. small values of \( k \). Second, since the correlation function is of order of \( (m/m_P)^2 \) a severe bound to the mass \( m \) is imposed by the gravitational fluctuations derived from the small values of the Cosmic Microwave Background (CMB) anisotropies detected by COBE. This bound is of the order of \( (m/m_P) \sim 10^{-6} \) [33, 32].
One possible advantage of the Einstein-Langevin approach to the gravitational fluctuations in inflaton over the approach based on the quantization of the linear perturbations of both the metric and the inflaton field [32], is that an exact treatment of the inflaton quantum fluctuations is in principle possible, keeping the metric perturbations to linear order. On the other hand although the gravitational fluctuations are here assumed to be classical, the correlation functions obtained correspond to the quantum expectation values of the quantum metric perturbations [36,30], at least in the linear regime. This means that even in the absence of decoherence the fluctuations predicted by the Einstein-Langevin equation, whose solutions do not describe the actual dynamics of the gravitational field any longer, still give the correct quantum two-point functions.

V. SUMMARY AND OUTLOOK

We have reviewed the semiclassical theory of gravity as the theory of the interaction of classical gravity with quantum matter fields. The most important equations in this theory are the semiclassical Einstein equations (1) which describe the back-reaction of the gravitational fluctuations in its interaction with the quantum fields. We noticed that the theory may seriously fail when the fluctuations on the stress energy tensor of the quantum fields are important. We have then sought an axiomatic approach by which the semiclassical equations can be corrected in order to take into account those fluctuations. These equations turn out to be uniquely defined and are the Einstein-Langevin equations (5) which are linear in the metric perturbations $h$ over the semiclassical background. These equations predict stochastic fluctuations in the metric perturbations induced by the stress tensor fluctuations described by the noise kernel (3).

We have also noticed that the Einstein-Langevin equations can be formally derived from the open quantum system paradigm, in which the gravitational fluctuations $h$ and the quantum field interact when the interest is in the dynamics of the gravitational field. Thus treating the quantum field as the “environment” and the gravitational field as the “system”. The mathematical tools to carry out this approach are the CTP functional method and, in this context, its closely related Feynman and Vernon influence functional.

We have finally used the stochastic theory in the inflationary cosmological context. We have computed the two-point correlation functions of the metric fluctuations during a quasi-de Sitter expansion induced by the stress tensor fluctuations of the inflaton field. The results are in agreement with other approaches to the same problem [32], an approximate Harrison-Zel’dovich spectrum is predicted. We noticed that in our approach the quantum fields and the gravitational fields are treated separately, and this may have some advantages to go one step further and consider the quantum field fully, not just to linear order.

In is worth mentioning that other applications of the stochastic theory have been carried out and others are in progress. Thus, the fluctuations on the ground state of semiclassical gravity, which consists of the Minkowski metric and the quantum state in its vacuum state, have been considered [37]. The computation of the two-point correlations of the linearized Einstein tensor indicate that a typical correlation length is present.

Other applications of stochastic semiclassical gravity to semiclassical cosmology have been performed [38], some including thermal fields [39,23]. It has been shown that noise produced by a quantum field on the cosmological scale factor of an isotropic closed Friedmann-Robertson-Walker, in the presence of a cosmological constant, may take the scale factor from a region where it is nearly zero to a region where it describes a de Sitter inflationary era [40]. Thus jumping over the barrier by activation, this is the semiclassical analogue of the tunneling from nothing in quantum cosmology [41] and gives yet another mechanism to produce inflation.

An important application of stochastic gravity which is now beginning is in the physics of black holes [42,24]. In particular in black hole thermodynamics, the stress tensor fluctuations near the black hole horizon may induce fluctuations in the horizon area. The relevance of this back-reaction effect in Hawking radiation has not been yet explored, although preliminary investigations seem to indicate that Hawking result should not be substantially different [43]. The contribution of the horizon fluctuations to the black hole entropy [44] is another tantalizing issue that may deserve some attention in the present context.

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[1] N.D. Birrell and P.C.W. Davies, *Quantum fields in curved space* (Cambridge: Cambridge University Press, 1984).

[2] L. Parker, Phys. Rev. Lett. 21, 562 (1968); Phys. Rev. 183, 1057 (1969).

[3] S.W. Hawking, Nature 248, 30 (1974); Commun. Math. Phys. 43, 199 (1975).

[4] B.S. DeWitt in *Relativity, groups and topology* eds. B.S. DeWitt and C. DeWitt (New York: Gordon and Breach, 1975); S.M. Christensen, Phys. Rev. D 14, 2490 (1976); Phys. Rev. D 17, 946 (1978).

[5] R.M. Wald, Commun. Math. Phys. 54, 1 (1977); Phys. Rev. D 17, 1477 (1978); Ann. Phys. 110, 472 (1978).

[6] V.N. Lukash and A.A. Starobinsky, Sov. Phys. JETP 39, 742 (1974); B.L. Hu and L. Parker, Phys. Rev. D 17, 933 (1978).

[7] Y.B. Zel’dovich, JETP Lett. 12, 307 (1970); C.W. Misner, Phys. Rev. Lett. 28, 994 (1972).

[8] J.B. Hartle, Phys. rev. Lett. 39, 1373 (1977); M. Fischetti, J.B. Hartle and B.L. Hu, Phys. Rev. D 20, 1757 (1979); J.B. Hartle and B.L. Hu, Phys. Rev. D 20, 1772 (1979).

[9] J. Schwinger, J. Math. Phys. 2, 407 (1961); L.V. Keldysh, Zh. Eksp. Teor. Fiz 47, 1515 (1964) [Sov. Phys. JETP 20, 1018 (1965)]; K. Chou, Z. Su, B. Hao, and L. Yu, Phys. Rep. 118, 1 (1985).

[10] R.D. Jordan, Phys. Rev. D 33, 444 (1986); E. Calzetta and B.L. Hu, Phys. Rev. D 35, 495 (1987); R.D. Jordan, Phys. Rev. D 36, 3593 (1987); J.P. Paz, Phys. Rev. D 41, 1054 (1990).

[11] A. Campos and E. Verdaguer, Phys. Rev. D 49, 1861 (1994).

[12] R.M. Wald, Commun. Math. Phys. 54, 1 (1977).

[13] L.H. Ford, Ann. Phys. (N.Y.) 144, 238 (1982).

[14] C.-I. Kuo and L.H. Ford, Phys. Rev. D 47, 4510 (1993).

[15] B.L. Hu, Physica A 158, 399 (1989).

[16] E. Calzetta and B.L. Hu, Phys. Rev. D 49, 6636 (1994); B.L. Hu and S. Sinha, Phys. Rev. D 51, 1587 (1995); B.L. Hu and A. Matacz, Phys. Rev. D 51, 1577 (1995); A. Campos and E. Verdaguer, Phys. Rev. D 53, 1927 (1996).

[17] R.P. Feynman and F.L. Vernon, Ann. Phys. (N.Y.) 24, 118 (1963); R.P. Feynman and A.R. Hibbs, *Quantum Mechanics and Path Integrals* (New York: McGraw-Hill, 1965).

[18] R.B. Griffiths, J. Stat. Phys. 36, 219 (1984); R. Omnès, Rev. Mod. Phys. 64, 339 (1992).

[19] W.H. Zurek, Physics Today 44, 36 (1991).

[20] M. Gell-Mann and J.B. Hartle, Phys. Rev. D 47, 3345 (1993).

[21] R. Martín and E. Verdaguer, Int. J. Theor. Phys. 38, 3049 (1999).

[22] R. Martín and E. Verdaguer, Phys. Lett. B 465, 113 (1999).

[23] R. Martín and E. Verdaguer, Phys. Rev. D 60, 084008 (1999).

[24] B.L. Hu, Int. J. Theor. Phys. 38, 2987 (1999).

[25] J.B. Hartle and G.T. Horowitz, Phys. Rev. D 24, 257 (1981).

[26] N.G. Phillips and B.L. Hu, Phys. Rev. D 55, 6123 (1997); Phys. Rev. D 62, 084017 (2000); B.L. Hu and N.G. Phillips, gr-qc/0004006.

[27] A. Roura and E. Verdaguer, Int. J. Theor. Phys. 39, 1831 (2000).

[28] G.N. Phillips and B.L. Hu, gr-qc/0010019.

[29] G.T. Horowitz, Phys. Rev. D 21, 1445 (1980).

[30] A. Roura and E. Verdaguer, in preparation (2001).

[31] A. Linde, *Particle physics and inflationary cosmology*, Harwood Academic Publishers, 1990.

[32] V.F. Mukhanov, H.A. Feldman and R.H. Brandenberger, Phys. Rep. 215, 203 (1992).

[33] G.F. Smoot et al Astrophysical J. Lett. 396, L1 (1992).

[34] J.M. Bardeen, Phys. Rev. D 22, 1882 (1980).

[35] A. Roura and E. Verdaguer, Int. J. Theor. Phys. 38, 3123 (1999).

[36] E. Calzetta, A. Roura and E. Verdaguer, gr-qc/0011007.

[37] R. Martín and E. Verdaguer, Phys. Rev. D 61, 124024 (2000).

[38] E. Calzetta, A. Campos and E. Verdaguer, Phys. Rev. D 56, 2163 (1997).

[39] A. Campos and B.L. Hu, Phys. Rev. D 58, 125021 (1998).

[40] E. Calzetta and E. Verdaguer, Phys. Rev. D 59, 083513 (1999).

[41] A. Vilenkin, Phys. Lett. 117B, 25 (1982); Phys. Rev. D 27, 2848 (1983); Phys. Rev. D 30, 509 (1984).

[42] B.L. Hu and A. Raval, gr-qc/9901010; C. Barrabés, V. Frolov and R. Parentani, gr-qc/0001102.

[43] L.H. Ford and N.F. Svaiter, Phys. Rev. D 56, 2226 (1997); C.-H. Wu and L.H. Ford, Phys. Rev. D 60, 104013 (1999).

[44] R. Sorkin, gr-qc/9701056; R.D. Sorkin and D. Sudarsky, Class. Quantum Grav. 16, 3835 (1999).