COMMUTING CATEGORIES FOR BLOCKS AND FUSION SYSTEMS

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Abstract. We extend the notion of a commuting poset for a finite group to \( p \)-blocks and fusion systems, and we generalize a result, due originally to Alperin and proved independently by Aschbacher and Segev, to commuting graphs of blocks, with a very short proof based on the \( G \)-equivariant version, due to Thévenaz and Webb, of a result of Quillen.

Let \( k \) be a field of prime characteristic \( p \). A block of a finite group \( G \) is a primitive idempotent \( b \) in \( \mathbb{Z}(kG) \). A \( b \)-Brauer pair is a pair \((Q, e)\) consisting of a \( p \)-subgroup \( Q \) of \( G \) and a block \( e \) of \( C_G(Q) \) satisfying \( \text{Br}_Q(b)e \neq 0 \), where \( \text{Br}_Q : (kG)^Q \to kC_G(Q) \) is the Brauer homomorphism; the set of \( b \)-Brauer pairs is a \( G \)-poset with respect to the conjugation action of \( G \) (see [10] for more details and background material on block theory). We denote by \( \mathcal{A}(b) \) the \( G \)-poset containing all \( b \)-Brauer pairs \((Q, e)\) such that \( Q \) is nontrivial and elementary abelian.

Two subgroups \( R, R' \) of \( G \) are said to commute if they commute elementwise; that is, if \([R, R'] = 1\). For any nonempty set \( \kappa \) of pairwise commuting subgroups of \( G \) we denote by \( \Pi \kappa \) the product in \( G \) of all subgroups belonging to \( \kappa \); this is clearly a subgroup of \( G \). If all elements of \( \kappa \) are \( p \)-subgroups (respectively, abelian subgroups) of \( G \), then \( \Pi \kappa \) is a \( p \)-subgroup (respectively, abelian subgroup) of \( G \). For any abelian subgroup \( Q \) of \( G \) we denote by \( c(Q) \) the set of subgroups of order \( p \) of \( Q \).

Definition 1. Let \( G \) be a finite group and \( b \) a block of \( G \). The commuting poset of \( b \) is the \( G \)-poset \( \mathcal{K}(b) \) whose elements are pairs \((\kappa, e)\), where \( \kappa \) is a nonempty set of pairwise commuting subgroups of order \( p \) of \( G \) and where \( e \) is a block of \( C_G(\Pi \kappa) \) such that \((\Pi \kappa, e)\) is a \( b \)-Brauer pair for \( \Pi \kappa \).
pair, with partial order given by

\[(\lambda, f) \leq (\kappa, e), \text{ if } \begin{cases} 
\lambda \subseteq \kappa, \\
(\Pi \lambda, f) \leq (\Pi \kappa, e)
\end{cases}\]

for \((\kappa, e), (\lambda, f) \in \mathcal{K}(b)\).

If \(b\) is the principal block of \(G\) then \(\mathcal{K}(b)\) is the clique complex \(\mathcal{K}_p(G)\) of the commuting graph \(\Lambda_p(G)\), where the notation is as in [3]. For nonprincipal blocks, however, \(\mathcal{K}(b)\) need not be the clique complex of a graph (e.g., see Example 5).

Given a \(G\)-poset \(\mathcal{X}\) we denote by \(\Delta \mathcal{X}\) the \(G\)-simplicial complex whose set of \(n\)-simplices consists of all chains of \(n\) proper inclusions in \(X\), where \(n \geq 0\). For any simplicial complex \(Y\), we denote the geometric realization of \(Y\) by \(|Y|\). Two \(G\)-spaces \(X\) and \(Y\) are called \(G\)-homotopically equivalent if there are \(G\)-equivariant maps \(f : X \to Y, g : Y \to X\) and \(G\)-equivariant homotopies \(h : I \times X \to X, h' : I \times Y \to Y\) such that \(h(0, -) = \text{Id}_A, h(1, -) = f, h'(0, -) = \text{Id}_Y,\) and \(h'(1, -) = g\), where the unit interval \(I = [0, 1]\) is viewed as a \(G\)-space with the trivial \(G\)-action. Two \(G\)-posets \(\mathcal{X}\) and \(\mathcal{Y}\) are called \(G\)-homotopically equivalent if the \(G\)-spaces \(|\Delta \mathcal{X}|\) and \(|\Delta \mathcal{Y}|\) are \(G\)-homotopically equivalent. By the \(G\)-equivariant version [11, (1.1)] of [9, 1.3], in order to show that \(\mathcal{X}\) and \(\mathcal{Y}\) are \(G\)-homotopically equivalent, it suffices to find \(G\)-equivariant functors \(\Phi : \mathcal{X} \to \mathcal{Y}\) and \(\Psi : \mathcal{Y} \to \mathcal{X}\) such that there is a natural transformation between \(\text{Id}_{\mathcal{X}}\) and \(\Psi \circ \Phi\) (in either direction) and a natural transformation between \(\text{Id}_{\mathcal{Y}}\) and \(\Phi \circ \Psi\).

**Theorem 2.** Let \(b\) be a block of a finite group \(G\). The maps:

\[\Phi : \begin{cases} 
\mathcal{A}(b) \to \mathcal{K}(b) \\
(Q, e) \mapsto (c(Q), e)
\end{cases}\]

\[\text{and} \quad \Psi : \begin{cases} 
\mathcal{K}(b) \to \mathcal{A}(b) \\
(\kappa, e) \mapsto (\Pi \kappa, e)
\end{cases}\]

are inverse \(G\)-homotopy equivalences.

**Proof.** The maps \(\Phi, \Psi\) are obviously order preserving and \(G\)-equivariant. We have \(\Psi \circ \Phi = \text{Id}_{\mathcal{A}(b)}\). There is a natural transformation \(\text{Id}_{\mathcal{K}(b)} \to \Phi \circ \Psi\) given by \((\kappa, e) \leq (c(\Pi \kappa), e)\), which shows that \(\Psi\) is a \(G\)-homotopy inverse of \(\Phi\). \(\square\)

Applied to principal blocks, this theorem yields, in particular, a proof of the fact, due independently to Alperin [1] Theorem 3] and to Aschbacher and Segev [4, 9.7], that \(\mathcal{K}_p(G)\) and \(\mathcal{A}_p(G)\) have the same homotopy type (see also [3, 5.2]). The \(G\)-orbit space of \(\mathcal{K}(b)\) admits a generalization to fusion systems and, in fact, to arbitrary categories on finite \(p\)-groups (cf. [7, 2.1]).
Definition 3. Let $\mathcal{F}$ be a category on a finite $p$-group $P$. The commuting category of $\mathcal{F}$ is the category $\mathcal{K}(\mathcal{F})$ whose objects are the nonempty sets of pairwise commuting subgroups of $P$ of order $p$, and for objects $\kappa, \lambda \in \mathcal{K}(\mathcal{F})$,

$$\text{Hom}_{\mathcal{K}(\mathcal{F})}(\kappa, \lambda) = \{ \psi \in \text{Hom}_F(\Pi\kappa, \Pi\lambda) \mid \text{if } Q \in \kappa, \text{ then } \psi(Q) \in \lambda. \}$$

The composition of morphisms in $\mathcal{K}(\mathcal{F})$ is induced by the usual composition of group homomorphisms. We denote by $[\mathcal{K}(\mathcal{F})]$ the poset consisting of the isomorphism classes $[\kappa]$ of objects $\kappa$ of $\mathcal{K}(\mathcal{F})$ with partial order given by

$$[\kappa] \leq [\lambda], \text{ if } \text{Hom}_{\mathcal{K}(\mathcal{F})}(\kappa, \lambda) \neq \emptyset$$

for $\kappa, \lambda \in \mathcal{K}(\mathcal{F})$.

Clearly $\mathcal{K}(\mathcal{F})$ is an $EI$-category. As a consequence of results in [2], any choice of a maximal $b$-Brauer pair $(P, e)$ of a block $b$ of a finite group $G$ determines a category $\mathcal{F}(P, e)(G, b)$ on $P$ that, if $k$ is large enough, is a saturated fusion system (see e.g., [6, §3.3] for details and further references).

Theorem 4. Let $b$ be a block of a finite group $G$, let $(P, e)$ be a maximal $b$-Brauer pair and let $\mathcal{F} = \mathcal{F}(P, e)(G, b)$. We have an isomorphism of posets

$$[\mathcal{K}(\mathcal{F})] \cong \mathcal{K}(b)/G$$

mapping the isomorphism class of an object $\kappa \in \mathcal{K}(\mathcal{F})$ to the $G$-conjugacy class of the unique Brauer pair $(P\kappa, e)$ contained in $(P, e)$.

Proof. For $(\kappa, e) \in \mathcal{K}(b)$, let $[(\kappa, e)]$ denote its $G$-conjugacy class. For elements $(\kappa, e), (\lambda, f) \in \mathcal{K}(b)$, one has $[(\kappa, e)] = [(\lambda, f)]$ if and only if there exists $g \in G$ such that $\kappa^g = \lambda$ and $e^g = f$. Define a poset map $\eta : \mathcal{K}(b)/G \to [\mathcal{K}(\mathcal{F})]$ by setting $\eta([(\kappa, e)]) = [\kappa^g]$, where $g \in G$ such that $(P\kappa, e)^g \leq (P, e)$. One verifies that this map is the inverse of the given map in the statement. \qed

Example 5. The following example was communicated to the authors by R. Kessar. Suppose $p = 2$. Set $G = S_n$, where $n \geq 6$ is an integer such that $kG$ has a block $b$ with a dihedral defect group $P \cong D_8$ of order 8. By results in [3], $b$ is of principal type; that is, for any 2-subgroup $Q$ of $G$ either $\text{Br}_Q(b) = 0$ or $\text{Br}_Q(b)$ is a block of $kC_G(Q)$. Moreover, $P$ may be chosen as a Sylow 2-subgroup of $S_4$, canonically embedded into $G$ and such that $P$ contains the involutions $x = (1 \ 2), y = (3 \ 4)$. Setting $z = (5 \ 6)$, we have $x, z \in P^{(3 \ 5)(4 \ 6)}$ and $y, z \in P^{(1 \ 5)(2 \ 6)}$. Since $b$ is of principal type, there are unique blocks $e_x, e_y, e_z$ of $kC_G(x), kC_G(y), kC_G(z)$, respectively, and unique blocks
$e_{xy}, e_{xz}, e_{yz}$ of $kC_G(\langle x, y \rangle)$, $kC_G(\langle x, z \rangle)$, $kC_G(\langle y, z \rangle)$, respectively, giving the following inclusions of $b$-Brauer pairs:

\[
\begin{array}{ccc}
(\langle x, y \rangle, e_{xy}) & (\langle x, z \rangle, e_{xz}) & (\langle y, z \rangle, e_{yz}) \\
(\langle x \rangle, e_x) & (\langle y \rangle, e_y) & (\langle z \rangle, e_z) \\
(1, b)
\end{array}
\]

Suppose that $\Gamma$ is a graph whose clique complex is $K(b)$. The $b$-Brauer pairs $(\langle x \rangle, e_x)$, $(\langle y \rangle, e_y)$, and $(\langle z \rangle, e_z)$ are minimal in the poset $K(b)$ and are pairwise contained in a common $b$-Brauer pair, implying that the graph $\Gamma$ has a clique of the form:

\[
\begin{array}{ccc}
(\langle x \rangle, e_x) & (\langle y \rangle, e_y) & (\langle z \rangle, e_z)
\end{array}
\]

However, the corresponding clique is not an element of the poset $K(b)$ because the group $\langle x, y, z \rangle$ is not contained in a defect group of $b$. This contradiction shows that there is no graph whose clique complex yields $K(b)$ and explains why we have refrained from defining a commuting graph of $b$ in this way.

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