EFFECTIVE BOUNDS ON MULTIPLICATIVELY DEPENDENT ORBITS OF INTEGER POLYNOMIALS MODULO S-INTEGERS

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Abstract. We obtain effective bounds on the heights of algebraic integers whose orbits contain multiplicatively dependent values modulo S-integers. Our method is based on a new upper bound on the so-called S-height of polynomial values over the ring of integers of \( K \). Our results provide an effective variant of a recent result of A. Bérczes, A. Ostafe, I. E. Shparlinski and J. H. Silverman (2019) on multiplicative dependence modulo a finitely generated subgroup by eliminating the use of non-effective results by K. F. Roth and G. Faltings.

1. Introduction

1.1. Background. For a polynomial \( f(X) \in K[X] \) and \( n \geq 0 \), we write \( f^{(n)}(X) \) for the \( n \)th iterate of \( f \), that is, \( f^{(0)}(X) = X \) and
\[
f^{(n)}(X) = f \circ f \circ \cdots \circ f(X), \quad n \geq 1.
\]
The orbit of \( \alpha \in K \) is the set \( \{ \alpha, f(\alpha), f^{(2)}(\alpha), \ldots \} \). In case the set is finite we say that \( \alpha \) is preperiodic and we use PrePer(\( f \)) to denote the set of preperiodic points \( \alpha \in K \).

A famous theorem of Northcott [18] says that for any number field \( K \), for any nontrivial polynomial \( f(X) \in K[X] \) the set PrePer(\( f \)) is finite. Namely there are only finitely many \( \alpha \in K \) such that
\[
(1.1) \quad f^{(m)}(\alpha) = f^{(n)}(\alpha)
\]
for two distinct iterations of \( f \) (that is, for \( m \neq n \)).

Coupled with modern counting results on the number of algebraic numbers of bounded height and degree, see [3, 4, 23, 24], one can obtain various effective and rather explicit versions of this result.

Several generalisations of the finiteness result of Northcott [18] have recently been considered in [6, 19, 20], where Equation (1.1) has been
replaced by various restrictions of multiplicative type on the ratios $f(m)(\alpha)/f(n)(\alpha)$ or even the ratios of the powers $f^{(m)}(\alpha)^r/f^{(n)}(\alpha)^s$.

For example, it is shown in [6, Theorems 1.3 and 1.4] that if $f(X) \in \mathbb{K}[X]$ of degree $d \geq 2$ is not of the form $f(X) = aX(X - b)^{d-1}$ with $a \in \mathbb{K}^*$ and $b \in \mathbb{K}$, then for any finitely generated multiplicative subgroup $\Gamma \subset \mathbb{K}^*$, there are only finitely many $\alpha \in \mathbb{K}$ for which
\begin{equation}
(1.2) \quad f(n)(\alpha)/f(m)(\alpha) \in \Gamma
\end{equation}
for some integers $m > n \geq 0$.

1.2. New results. Unfortunately the method of [6] relies on the results of Faltings [10,11] and thus is not effective. We consider the more general equation
\begin{equation}
(1.3) \quad f(n)(\alpha) = af(m)(\alpha)
\end{equation}
for some integers $m > n \geq 0$ and an $S$-integer $a \in \mathfrak{o}_S$ (see Equation (1.6) for a definition). Note that Equation (1.2) is a special case of Equation (1.3). However, Equation (1.3) is no longer symmetric in $m$ and $n$.

In Theorem 2.2 (see also Theorem 2.4) we show that in the case where $f(X) \in \mathfrak{o}[X]$, where $\mathfrak{o}$ is the ring of integers of $\mathbb{K}$, and $\alpha \in \mathfrak{o}$, one can obtain an effective bound on the size of $\alpha$. In fact, we trace the explicit dependence on $S$. This is an effective version of [6, Theorem 1.4].

Furthermore, we also provide an effective variant of [6, Theorem 1.7] which states that, under mild additional constraints, there are only finitely many $\alpha \in \mathbb{K}$ that satisfy the following relation of multiplicative dependence modulo $S$-units among values in an orbit
\[ f(n+k)(\alpha)^r \cdot f(k)(\alpha)^s \in \mathfrak{o}_S^* \]
for some $n, k \geq 1$ and $(r, s) \neq (0, 0)$. That is, we give an effective upper bound on the height of $\alpha \in \mathfrak{o}$ that satisfy
\begin{equation}
(1.4) \quad (f^{(m)}(\alpha))^r = u (f^{(n)}(\alpha))^s
\end{equation}
for some integers $m > n \geq 1$, $(r, s) \neq (0, 0)$ and an $S$-unit $u \in \mathfrak{o}_S^*$, see Equation (1.5) for a definition.

As in [6], the key to proving Theorem 2.2 is an upper bound on the $S$-height of polynomial values, see Equation (1.7) for a precise definition, which we believe is of independent interest and may find other applications. Recall that in [6] this upper bound is provided by [15, Theorem 11(c)] which is unfortunately not effective. Here we modify the argument of [6] to use an effective variant of [15, Theorem 11(c)] which we provide by extending [8, Theorem 2.2] to number fields.
We note that obtaining such extensions in terms of the norm in \( \mathfrak{o} \) can be done by following the arguments in [8]. However, such a generalisation is not sufficient for our purpose, so we add some additional ideas and ingredients in order to obtain a bound in terms of the \( S \)-height (see Equation (1.7)).

1.3. **General notation and conventions.** We set the following notation which we use for the rest of this paper. We refer to [7] for a background on valuations, height and other notions we introduce below.

Throughout this paper, we assume that \( K \) is a number field of degree \( d \), with class number \( h \), regulator \( R \) and ring of integers \( \mathfrak{o} \).

We use \( \mathcal{M} \) to denote the set of places of \( K \) and write

\[
\mathcal{M} = \mathcal{M}^\infty \cup \mathcal{M}^0,
\]

where \( \mathcal{M}^\infty \) and \( \mathcal{M}^0 \) are the set of archimedean (infinite) and non-archimedean (finite) places of \( K \) respectively.

We always assume that \( S \) is a finite set of places containing \( \mathcal{M}^\infty \) and use \( S^0 = S \cap \mathcal{M}^0 \) to denote the set of finite places in \( S \). We also define

\[
s = \#S \quad \text{and} \quad t = \#S^0.
\]

As usual, \( \mathfrak{o}_S^\ast \) denotes the group of \( S \)-units, that is

\[
(1.5) \quad \mathfrak{o}_S^\ast = \{ u \in K^\ast : |u|_v = 1 \ \forall v \in \mathcal{M} \setminus S \}.
\]

In particular, \( \mathfrak{o}^\ast = \mathfrak{o}_{\mathcal{M}^\infty}^\ast \) is the group of units which, by the Dirichlet Unit Theorem, is a finitely generated group of rank \( \#\mathcal{M}^\infty - 1 \).

Similarly, \( \mathfrak{o}_S \) denotes the ring of \( S \)-integers, that is

\[
(1.6) \quad \mathfrak{o}_S = \{ a \in K : |a|_v \leq 1 \ \forall v \in \mathcal{M} \setminus S \}.
\]

We use \( N(a) \) for the norm of the ideal \( a \), we also write \( N(\alpha) \) to mean \( N([\alpha]) \), where \([\alpha] \) is the principal ideal in \( \mathfrak{o} \) generated by \( \alpha \in \mathfrak{o} \). In particular, \( N(\alpha) > 0 \) for \( \alpha \neq 0 \).

For \( x \geq 0 \) it is convenient to introduce the functions

\[
\log^+ x = \max \{ \log x, 0 \} \quad \text{and} \quad \log^* x = \max \{ \log x, 1 \},
\]

with \( \log^+ 0 = 0 \), \( \log^* 0 = 1 \). We are now able to define the *logarithmic height* of \( \alpha \in K \) as

\[
h(\alpha) = \sum_{v \in \mathcal{M}} \frac{\ell_v}{d} \log^+ |\alpha|_v,
\]

where
• $|\alpha|_v$ is the absolute value extending the valuation on $\mathbb{Q}$. That is, for a finite place $v$ corresponding to a prime ideal $p \mid p$

$$|\alpha|_v = p^{-\text{ord}_p \alpha/e_v},$$

where $\text{ord}_p \alpha$ is the $p$-adic order of $\alpha$.

• $\ell_v$ denotes the local degree of the valuation $v$, that is

$$\ell_v = [K_v : \mathbb{Q}_v],$$

where $K_v$ and $\mathbb{Q}_v$ are the completions at $v$.

Finally, for a set $T \subseteq M$ we use

$$h_T(\alpha) = \sum_{v \in T} \frac{\ell_v}{d} \log^+ |\alpha|_v$$

to denote the $T$-height of $\alpha \in K$.

We also recall the identity,

$$\sum_{v_i \in M^\infty} \frac{\ell_{v_i}}{d} = 1,$$

which is a special case of [7, Corollary 1.3.2] applied to the archimedean valuation of $\mathbb{Q}$.

Let $\mathcal{P}_K$ denote the set of all prime ideals of $\mathfrak{a}$. For $\alpha \in \mathbb{K}^*$ define

$$\text{supp}(\alpha) = \{p \in \mathcal{P}_K \mid \text{ord}_p(\alpha) > 0\}$$

and

$$\mathfrak{P}(\alpha) = \max_{p \in \text{supp}(\alpha)} N(p)$$

with the convention that $\mathfrak{P}(\alpha) = 1$ if $\text{supp}(\alpha) = \emptyset$.

We use $A$ with or without subscripts or arguments for fully explicit constants, while $c$ and $C$ are used for not explicit but effective constants depending on their arguments.

2. Main Results

2.1. Height of $S$-parts of polynomials in number fields. We start with an effective version of [6, Theorem 1.4] where we also make the dependence on $S$ completely explicit. This result is proven by extending [8, Theorem 2.2] to number fields. We note that it is also indicated in [8] that such an extension to number fields should be possible. However, if one follows closely the argument of the proof of [8, Theorem 2.2] this leads to such an extension in terms of the norm, while for our purpose we need it in terms of the height, which requires bringing in additional tools.

First we need to define some notation stemming from the use of [13].
Suppose we are working with a finite set of places $\mathcal{S}$ and the prime ideals $\{p_1, \cdots, p_t\}$ correspond to the places in $\mathcal{S}_0$. Then define

$$P = \max_{i \in [1,t]} N(p_i), \quad Q = N(p_1 \cdots p_t), \quad \mathcal{L} = \sum_{i=1}^{t} \log^* \log N(p_i)$$

with the convention that for $\mathcal{S} = M^\infty$ we set $P = Q = 1$, $\mathcal{L} = 0$.

Also define the functions

$$A_1(u, v) = v^{2u+3.5}2^7v \log(2v) u^{2v}$$

and

$$A_2(u, v) = (2048u)^v v^{3.5}$$

which stem from [13] which underlies our argument.

We recall the definition of $h_\mathcal{S}$ in Section 1.3 and also that $d = [K : \mathbb{Q}]$.

**Theorem 2.1.** Let $f(X) \in \mathfrak{o}[X]$ be a polynomial with at least 3 distinct roots. Let $\mathbb{L}$ be a splitting field of $f$ over $K$, let $D = [\mathbb{L} : K]$ and let $h_\mathbb{L}$ denote the class number of $\mathbb{L}$. Let $\mathcal{S}$ be a finite set of $s$ places of $K$ containing all infinite places and let $t = \#\mathcal{S}_0$. Then for all $\alpha \in \mathfrak{o}$, $f(\alpha) \neq 0$ we have

$$h_\mathcal{S}(f(\alpha)^{-1}) < (1 - \eta_1([K, f, \mathcal{S}]) (h(f(\alpha)) + 1),$$

where

$$\eta_1([K, f, \mathcal{S}])^{-1} = c_1([K, f]) A_1(dD, sD) \max\{1, t\}$$

$$\times P^D(\log^* P + \mathcal{L}) \prod_{i=1}^{t} \log(N(p_i))^D,$$

and, for $t > 0$, we have

$$h_\mathcal{S}(f(\alpha)^{-1}) < (1 - \eta_2([K, f, \mathcal{S}]) (h(f(\alpha)) + 1),$$

where

$$\eta_2([K, f, \mathcal{S}])^{-1} = c_1([K, f]) A_2(dDh_\mathbb{L}, tD) t P^D \prod_{i=1}^{t} \log(N(p_i))^D,$$

where $c_1([K, f]) > 0$ is effectively computable.

Note that Equation (2.5) omits the $v^v$ term in Equation (2.2). This is necessary for the proofs of Theorems 2.4 and 2.5.

We also note that, using the recent improvement [12, Corollary 4] in place of Lemma 3.1, we can replace the main dependence on $P$ by a dependence on the third largest value of $N(p_i)$, $i = 1, \ldots, t$. 
2.2. Effective bounds on points with multiplicatively dependent orbits. We recall that $d = [\mathbb{K} : \mathbb{Q}]$.

**Theorem 2.2.** Let $f(X) \in \mathfrak{o}[X]$ be a polynomial with at least 3 distinct roots and for which 0 is not periodic. Let $\mathcal{S}$ be a finite set of places of $\mathbb{K}$ containing all infinite places and let $t = \#\mathcal{S}_0$. Then for any $\alpha \in \mathfrak{o}$ such that Equation (1.3) holds for some non-negative integers $m > n$ and $a \in \mathfrak{o}_\mathcal{S}$ we have

\begin{equation}
(2.6) \quad h(\alpha) < c_2(\mathbb{K}, f) \eta_1(\mathbb{K}, f, \mathcal{S})^{-1},
\end{equation}

and, for $t > 0$,

\begin{equation}
(2.7) \quad h(\alpha) < c_2(\mathbb{K}, f) \eta_2(\mathbb{K}, f, \mathcal{S})^{-1},
\end{equation}

where $\eta_1(\mathbb{K}, f, \mathcal{S})$, $\eta_2(\mathbb{K}, f, \mathcal{S})$ are as in Theorem 2.1 and $c_2(\mathbb{K}, f)$ is an effectively computable constant.

With Theorem 2.2 we can also prove the following effective variant of [6, Theorem 1.7].

**Theorem 2.3.** Let $f(X) \in \mathfrak{o}[X]$ be a polynomial of degree at least 3 without multiple roots and for which 0 is not periodic. Let $\mathcal{S}$ be a finite set of places of $\mathbb{K}$ containing all infinite places. Then for any tuple $(m, n, \alpha, r, s)$ for which Equation (1.4) holds with $n \geq 1$ we have

\[ h(\alpha) < c_3(\mathbb{K}, f, \mathcal{S}) \]

for some effectively computable $c_3(\mathbb{K}, f, \mathcal{S})$.

Note that we have assumed $m, n \neq 0$, otherwise there are trivially infinitely many solutions of the form $(f^{(m)}(u))^0 = u^{-1}(f^{(0)}(u))$.

Theorem 2.3 almost directly follows from the proof of [6, Theorem 1.7] but instead using Theorem 2.2 in place of [6, Theorem 1.3].

2.3. Applications to the existence of large prime ideals in factorisations. For $\alpha \in \mathbb{K}$, define the function

\[ \lambda(\alpha) = \log^* h(\alpha). \]

We obtain an effective lower bound on the largest norm of a prime ideal appearing with a higher order in $f^{(m)}(\alpha)$ than in $f^{(n)}(\alpha)$.

**Theorem 2.4.** Let $f(X) \in \mathfrak{o}[X]$ be a polynomial with at least 3 distinct roots and for which 0 is not periodic. Let $\alpha \in \mathfrak{o}$, $m, n \in \mathbb{Z}$, $m > n \geq 0$ such that $f^{(m)}(\alpha)$, $f^{(n)}(\alpha) \neq 0$. Then

\[ \mathfrak{p} \left( \frac{f^{(m)}(\alpha)}{f^{(n)}(\alpha)} \right) > c_4(\mathbb{K}, f) \frac{\lambda(f^{(m)}(\alpha)) \log^* \lambda(f^{(m)}(\alpha))}{\log^* \log^* \lambda(f^{(m)}(\alpha))}, \]

where $c_4(\mathbb{K}, f) > 0$ is an effectively computable constant.
Using standard properties of height, such as Equation (4.1) below, we see that Theorem 2.4 implies that if, in addition, $\alpha$ is not preperiodic, then
\[
\mathfrak{H} \left( \frac{f^{(m)}(\alpha)}{f^{(n)}(\alpha)} \right) > c_5(\mathbb{K}, f) \frac{m \log^* m}{\log^* \log^* m},
\]
where $c_5(\mathbb{K}, f) > 0$ is an effectively computable constant.

Finally, we obtain a result on the existence of primitive divisors within small sets of iterates.

**Theorem 2.5.** Let $f(X) \in \mathfrak{o}[X]$ be a polynomial with at least 3 distinct roots and for which 0 is not periodic. Then there exists an effectively computable constant $c_6(\mathbb{K}, f) > 0$ such that, letting
\[
k(m, \alpha) = \lfloor c_6(\mathbb{K}, f) \log \lambda(f^{(m)}(\alpha)) \rfloor,
\]
for every $m \in \mathbb{Z}$, $m > 0$, and every $\alpha \in \mathfrak{o}$, $f^{(m)}(\alpha)$ not a unit, there exists a prime ideal $\mathfrak{p}$ that divides $f^{(m)}(\alpha)$ but does not divide any element in the set
\[
\{ f^{(\max(0, m-k(m, \alpha)))}(\alpha), f^{(\max(0, m-k(m, \alpha))+1)}(\alpha), \ldots, f^{(m-1)}(\alpha) \}.
\]

If, in addition, $\alpha$ is not preperiodic, then, using Equation (4.1), Theorem 2.5 also holds for
\[
k(m, \alpha) = \lfloor c_7(\mathbb{K}, f) \log m \rfloor
\]
for some effectively computable $c_7(\mathbb{K}, f) > 0$.

3. Proof of Theorem 2.1

3.1. Preliminaries. As in the proof of [8, Theorem 2.10], the main tool is [13, Theorem 3]. We state the special case for 2 variables where it is easy to state a sufficient condition for $F$ to be triangularly connected. We maintain the dependence on $S$; however, we omit the explicit dependence on $\mathbb{K}$ and $F$.

Let $R$ be the regulator of $\mathbb{K}$.

In Lemma 3.1 below, which is a simplified version of [13, Theorem 3], we have made use of the inequality (see [9])
\[
R_S \leq h_R \prod_{i=1}^{t} \log N(\mathfrak{p}_i),
\]
where $h$ is the class number of $\mathbb{K}$ and $R_S$ is the $S$-regulator of $\mathbb{K}$ (see [9] for a definition, it is the natural generalisation of the regulator to $S$-units). In particular, we absorb $h, R$ into the constant $C_1(\mathbb{K}, F)$. 

We recall that a binary form (that is, a homogeneous polynomial) \( F \in \mathbb{K}[X, Y] \) is called decomposable over \( \mathbb{K} \), if \( F \) factors into linear factors over \( \mathbb{K} \).

**Lemma 3.1.** Let \( F \in \mathbb{K}[X, Y] \) be a decomposable form over \( \mathbb{K} \) which has at least 3 pairwise non-proportional linear factors. Let \( S \) be a finite set of \( s \) places of \( \mathbb{K} \) containing all infinite places and let \( t = \#S_0 \). Let \( P, Q, \mathcal{L} \) be as defined in Equation (2.1). Let \( \beta \in \mathbb{K} \setminus \{0\} \). Then all solutions \( (x_1, x_2) \in \mathbb{O}_S \) of
\[
F(x_1, x_2) = \beta
\]
satisfy
\[
h(x_1), h(x_2) < C_1(\mathbb{K}, F)A_1(d, s) (\log^* Q + h(\beta)) \times P(1 + \mathcal{L} / \log^* P) \prod_{i=1}^{t} \log N(p_i),
\]
and, for \( t > 0 \),
\[
h(x_1), h(x_2) < C_2(\mathbb{K}, F)A_2(d, h, t) (\log^* Q + h(\beta)) \times (P/\log^* P) \prod_{i=1}^{t} \log N(p_i),
\]
where \( A_1 \) is defined as in Equation (2.2), \( A_2 \) is defined as in Equation (2.3), \( d = \left[ \mathbb{K} : \mathbb{Q} \right] \), \( h \) is the class number of \( \mathbb{K} \) and \( C_1(\mathbb{K}, F) \), \( C_2(\mathbb{K}, F) \) are effectively computable constants.

We refer to [13] for a fully explicit statement. For the case \( t > 0 \) see also the recent improvement [12, Corollary 4].

To adapt the proof of [8, Theorem 2.10] to number fields, we need a well known fact on the approximation of archimedean valuations by units. To obtain explicit bounds, we first need [9, Lemma 1] in the case where \( S = \mathcal{M}^\infty \) (see also [13, Lemma 2] for an alternative bound when the unit rank is at least 2).

**Lemma 3.2.** Let \( \mathbb{K} \) be a number field with unit rank at least 1. Then there exists a fundamental system of units \( \varepsilon_1, \ldots, \varepsilon_r \) such that
\[
\max_{1 \leq i \leq r} h(\varepsilon_i) \leq A_3(\mathbb{K})R,
\]
where
\[
A_3(\mathbb{K}) = \frac{(r!)^2}{2^{r-1}d^r} \left( \frac{\delta_\mathbb{K}}{d} \right)^{1-r},
\]
where \( d = \left[ \mathbb{K} : \mathbb{Q} \right] \) and \( \delta_\mathbb{K} \) is any positive constant such that every non-zero algebraic number \( \alpha \in \mathbb{K} \) which is not a root of unity satisfies \( h(\alpha) \geq \delta_\mathbb{K} / d \).
A result of Voutier [22] states that we can take
\[ \delta_K = \begin{cases} \frac{\log 2}{d} & \text{if } d = 1, 2, \\ \frac{1}{4} \left( \frac{\log \log d}{\log d} \right)^3 & \text{if } d \geq 3, \end{cases} \]
in Lemma 3.2.

In the following result, little effort has been made to optimise the right hand side as it suffices for our results that it is effectively computable in terms of \( \mathbb{K} \). In fact, it is essentially established in the proof of [9, Lemma 2] (see also [13, Lemma 3]); however, for the sake of completeness, we give a short proof.

**Lemma 3.3.** For every \( \alpha \in \mathfrak{a} \setminus \{0\} \) and for every integer \( n \geq 1 \) there exists an \( \varepsilon \in \mathfrak{a}^\ast \) such that
\[ |\log |\varepsilon^n \alpha|_v - \frac{1}{d} \log (N(\alpha))| \leq \frac{1}{2} A_3(\mathbb{K})nd^2 R \]
for all \( v \in \mathcal{M}^\infty \) with \( A_3(\mathbb{K}) \) as in Lemma 3.2 and \( d = [\mathbb{K} : \mathbb{Q}] \).

**Proof.** For this proof let \( \mathcal{M}^\infty = \{v_1, \ldots, v_{r+1}\} \). Since the case \( r = 0 \) is trivial, we henceforth assume that \( r \geq 1 \). Let \( \varepsilon_1, \ldots, \varepsilon_r \) be a fundamental system of units satisfying the inequalities of Lemma 3.2.

We note that by the Dirichlet Unit Theorem (see, for example, [17, Theorem I.7.3]), the columns of the \((r + 1) \times r\) matrix \( M \)
\[ M_{i,j} = \ell_v \log |\varepsilon_j|_{v_i} \]
(where as before \( \ell_v = [\mathbb{K}_v : \mathbb{Q}_v] \) for \( v \in \mathcal{M} \)) form a basis for the hyperplane in \( \mathbb{R}^{r+1} \) of vectors whose coordinates sum to 0.

Let \( \mathbf{v} \) be the column vector of dimension \( r + 1 \), where
\[ (\mathbf{v})_i = \ell_{v_i} \log (N(\alpha))^{-1/d} |\alpha|_{v_i}, \quad i = 1, \ldots, r + 1. \]

Then there exists a unique vector \( \mathbf{x} = (x_1, \ldots, x_r)^T \) such that
\[ M\mathbf{x} = \mathbf{v}. \]

For each \( i = 1, \ldots, r \), we write
\[ x_i = ny_i + z_i, \quad y_i, z_i \in \mathbb{Z}, \quad z_i \in \left(-\frac{n}{2}, \frac{n}{2}\right). \]

Let \( \varepsilon = \varepsilon_1^{-y_1} \cdots \varepsilon_r^{-y_r} \). Then, for all \( v_i \),
\[ \log |\varepsilon_1^{z_1} \cdots \varepsilon_r^{z_r}|_{v_i} = \log |\varepsilon^n \alpha|_{v_i} - \frac{1}{d} \log (N(\alpha)). \]

For each \( j = 1, \ldots, r \), by Lemma 3.2, we have
\[ |z_j \log |\varepsilon_j|_{v_i}| \leq \frac{n}{2} dh(\varepsilon_j) \leq \frac{n}{2} dA_3(\mathbb{K}) R. \]
and summing over $j = 1, \ldots, r$ yields the desired statement. \hfill \Box

We now have all the tools we need for the proof of Theorem 2.1.

3.2. **Concluding the proof.** We will first prove Equation (2.4) holds assuming that $f$ splits in $K$.

Let $F(X, Y)$ be the homogenisation of $f$, that is,

$$F(X, Y) = Y^\varnothing f(X/Y),$$

where $\varnothing = \text{deg } f$. Then $F$ is a decomposable form in $K$ with $F(x, 1) = f(x)$.

Suppose $x \in \mathfrak{o}$ and $f(x) \neq 0$. Let $b = F(x, 1) = f(x)$. We can write $[b]$ uniquely in the form

$$[b] = p_1^{b_1} \ldots p_t^{b_t} a,$$

where $a$ is an ideal coprime to $p_1, \ldots, p_t$ and $b_i = \text{ord}_{p_i} b$, $i = 1, \ldots, t$.

Decompose each $b_i$ (uniquely) as

$$b_i = d_i q_i + r_i,$$

where $d_i$ is the class number of $K$ and $q_i, r_i \in \mathbb{Z}_{\geq 0}$, $r_i < d_i$.

For each $i \in [1, t]$, define $p_i \in \mathfrak{o}$ to be any generator of $p_i^{d_i}$ (which is a principal ideal).

Now let

$$c = F \left( \frac{x}{p_1^{q_1} \ldots p_t^{q_t}}, \frac{1}{p_1^{q_1} \ldots p_t^{q_t}} \right) = \frac{b}{p_1^{q_1} \ldots p_t^{q_t}},$$

so that

$$[c] = p_1^{r_1} \ldots p_t^{r_t} a.$$

We now apply Lemma 3.3 to $c$ and let $\varepsilon \in \mathfrak{o}^*$ be any unit satisfying Equation (3.3) where $\alpha$ is replaced by $c$ and $n$ by $\varnothing$.

Multiplying the arguments of $F$ by $\varepsilon$ we get

$$F \left( \frac{\varepsilon x}{p_1^{q_1} \ldots p_t^{q_t}}, \frac{\varepsilon}{p_1^{q_1} \ldots p_t^{q_t}} \right) = \varepsilon^\varnothing c$$

and applying Lemma 3.1 we obtain the inequality

$$h(\varepsilon/p_1^{q_1} \ldots p_t^{q_t}) < C_1(K, f)A_1(d, s) \left( \log^* Q + h(\varepsilon^\varnothing c) \right)$$

$$\times P(1 + \mathcal{L}/\log^* P) \prod_{i=1}^{t} \log N(p_i).$$

(3.7)

We separately lower bound $h(\varepsilon/p_1^{q_1} \ldots p_t^{q_t})$ and upper bound $h(\varepsilon^\varnothing c)$. 
— Lower bound on $h(\varepsilon/p_1^{q_1} \ldots p_t^{q_t})$: Since $\varepsilon/p_1^{q_1} \ldots p_t^{q_t}$ is an $S$-integer

$$h(\varepsilon/p_1^{q_1} \ldots p_t^{q_t}) = \sum_{v_i \in M^0 \cap S} \frac{\ell_{v_i}}{d} \log^+ |\varepsilon/p_1^{q_1} \ldots p_t^{q_t}|_{v_i}$$

$$+ \sum_{v_i \in M^\infty} \frac{\ell_{v_i}}{d} \log^+ |\varepsilon/p_1^{q_1} \ldots p_t^{q_t}|_{v_i}. \tag{3.8}$$

From Equation (3.5), we have that for all $v_i$

$$\varepsilon \log |\varepsilon/p_1^{q_1} \ldots p_t^{q_t}|_{v_i} - \log (|\varepsilon^b |_{v_i}) = \log (|b|_{v_i}^{-1}).$$

If $v_i \in M^0 \cap S$, by direct calculation we get

$$\log^+ |\varepsilon/p_1^{q_1} \ldots p_t^{q_t}|_{v_i} > \frac{1}{d} \log^+ (|b|_{v_i}^{-1}) - \frac{h}{e_{v_i}} \log \rho_i \tag{3.9}$$

(where $\rho_i$ is the prime in $\mathbb{Z}$ that $p_i$ lies over and $e_{v_i}$ is the ramification index of $v_i$, $i = 1, \ldots, t$).

If $v_i \in M^\infty$, using the bound of Lemma 3.3 and dividing by $\varepsilon$ we get

$$\log |\varepsilon/p_1^{q_1} \ldots p_t^{q_t}|_{v_i} \geq \frac{1}{d} \left( \log (|b|_{v_i}^{-1}) + \frac{1}{d} \log (N(c)) \right) - \frac{1}{2} A_3(\mathbb{K}) d^2 R. \tag{3.10}$$

Hence

$$\log^+ |\varepsilon/p_1^{q_1} \ldots p_t^{q_t}|_{v_i} \geq \frac{1}{d} \log^+ (|b|_{v_i}^{-1}) - \frac{1}{2} A_3(\mathbb{K}) d^2 R. \tag{3.11}$$

Substituting Equation (3.9) and Equation (3.10) into Equation (3.8) and using the trivial bound $e_{v_i} \geq 1$ and Equation (1.8) we obtain

$$h(\varepsilon/p_1^{q_1} \ldots p_t^{q_t}) \geq \frac{1}{d} h_S(b^{-1}) \log Q - \frac{1}{2} A_3(\mathbb{K}) d^2 R,$$

where again we let $Q = N(p_1 \cdots p_t) \geq \rho_1 \ldots \rho_t$.

— Upper bound on $h(\varepsilon^b c)$: Since $\varepsilon^b c \in \mathfrak{o}$ we have

$$h(\varepsilon^b c) = \sum_{v_i \in M^\infty} \frac{\ell_{v_i}}{d} \log^+ |\varepsilon^b c|_{v_i}.$$ 

From Equation (1.8) and Equation (3.3) we obtain

$$h(\varepsilon^b c) \leq \frac{1}{d} \log (N(c)) + \frac{1}{2} A_3(\mathbb{K}) d^2 R.$$

Since $r_i < \vartheta h$, from Equation (3.6) we get that

$$h(\varepsilon^b c) \leq \frac{1}{d} \log (N(a)) + \frac{\vartheta h}{d} \log Q + \frac{1}{2} A_3(\mathbb{K}) \vartheta d^2 R, \tag{3.12}$$

where again we let $Q = N(p_1 \cdots p_t)$.
By definition ord_p a = 0 for all finite valuations in S. Substituting Equation (3.4) into Equation (3.12) we obtain the upper bound
\[ h(\varepsilon^0 c) \leq h_{\mathcal{M}\setminus S}(b^{-1}) + \frac{dh}{d} \log Q + \frac{1}{2} A_3(\mathbb{K}) d^2 R. \]

--- Combining the bounds: --- Substituting Equation (3.11) and Equation (3.13) into Equation (3.7) we obtain
\[ h_S(b^{-1}) < C_3(\mathbb{K}, f) A_1(d, s) \left( \log^* Q + h_{\mathcal{M}\setminus S}(b^{-1}) \right) \times P(1 + \mathcal{L} / \log^* P) \prod_{i=1}^t \log N(p_i), \]
where \( C_3(\mathbb{K}, f) \) is an effectively computable constant.

Noting that \( \log Q \leq t \log P \) we can simplify to get
\[ h_S(b^{-1}) < C_3(\mathbb{K}, f) A_4(d, S) \left( 1 + h_{\mathcal{M}\setminus S}(b^{-1}) \right), \]
where
\[ A_4(d, S) = A_1(d, s) \max\{1, t\} P(\log^* P + \mathcal{L}) \prod_{i=1}^t \log N(p_i). \]
Using
\[ h(b) = h(b^{-1}) = h_{\mathcal{M}\setminus S}(b^{-1}) + h_S(b^{-1}) \]
we now arrive to
\[ h_S(b^{-1}) < \frac{C_3(\mathbb{K}, f) A_4(d, S)}{1 + C_3(\mathbb{K}, f) A_4(d, S)} (1 + h(b)), \]
concluding the proof of Equation (2.4) in the case where \( f \) splits in \( \mathbb{K} \).

--- Proving Equation (2.4): --- Now, suppose that \( f \) does not split in \( \mathbb{K} \).
Let \( \mathbb{L} \) be the splitting field of \( f \) over \( \mathbb{K} \) and let \( \mathcal{T} \) be the set of places in \( \mathbb{L} \) lying over \( S \).

Then Equation (3.14) holds in \( \mathbb{L} \) where we replace \( S \) by \( \mathcal{T} \). For ease
of notation, we introduce the subscript \( \mathbb{L} \) when talking about constants defined in terms of \( \mathbb{L} \) (some of them also depend on \( S \)). In particular,
\[ d_\mathbb{L} = [\mathbb{L} : \mathbb{Q}], \quad s_\mathbb{L} = \#\mathcal{T} \] and so on. Let \( D = [\mathbb{L} : \mathbb{K}] \).

We note that
\[ d_\mathbb{L} = D d, \quad t_\mathbb{L} \leq D t, \quad s_\mathbb{L} \leq D s, \]
\[ P_\mathbb{L} \leq P^D, \quad \mathcal{L}_\mathbb{L} \leq D \mathcal{L} + C_4(\mathbb{K}, f), \]
\[ \prod_{q \in \pi_0} \log N(q) < C_5(\mathbb{K}, f) \prod_{p \in S_0} (\log N(p))^D, \]
where \( C_4(\mathbb{K}, f), C_5(\mathbb{K}, f) \) are effective constants that depend on \( D \) and
the number of prime ideals of \( \mathbb{K} \) with norm less than \( e^e \).
We also note that heights are independent of extension in the sense that if $b \in \mathbb{K}$, then
\[ h(b) = h_L(b), \quad \text{and} \quad h_S(b^{-1}) = h_T(b^{-1}). \]

Using Equation (3.14) with $\mathbb{K}$ replaced by $L$ and other parameters replaced by the upper bounds in Equation (3.15) we conclude the proof of Equation (2.4).

— Proving Equation (2.5): This is the same proof as above, except using Equation (3.2) in place of Equation (3.1) in the derivation of Equation (3.7).

4. Proofs of Theorems 2.2 and 2.3

4.1. Dynamical canonical height function. We introduce the dynamical canonical height function which is useful in the proofs of Theorems 2.2 and 2.3.

The following result is standard and proofs of its statements can be found in [21, Section 3.4]; see also [14, Remark B.2.7] and [25, Proposition 3.2] regarding the effectiveness of the result.

**Lemma 4.1.** For a fixed $f \in \mathbb{K}(X)$ with $\mathfrak{d} = \deg f \geq 2$ there exists a function $\hat{h}_f : \mathbb{K} \to [0, \infty)$ such that:

(a) There is an effectively computable constant $C_6(\mathbb{K}, f)$ such that
\[ |\hat{h}_f(\alpha) - h(\alpha)| < C_6(\mathbb{K}, f), \]
for all $\alpha \in \mathbb{K}$.

(b) For all $\alpha \in \mathbb{K}$ we have
\[ \hat{h}_f(f(\alpha)) = \mathfrak{d}\hat{h}_f(\alpha). \]

(c) For all $\alpha \in \mathbb{K}$ we have
\[ \hat{h}_f(\alpha) = 0 \iff \alpha \in \text{PrePer}(f). \]

As a consequence, there exists an effectively computable constant $C_7(\mathbb{K}, f)$ such that for all $\ell \in \mathbb{Z}$, $\ell \geq 0$ and $\alpha \in \mathbb{K}$, $h(\alpha) > C_7(\mathbb{K}, f)$,
\[ \mathfrak{d}^\ell C_7(\mathbb{K}, f) h(\alpha) > h(f^{(\ell)}(\alpha)) > \mathfrak{d}^\ell C_7(\mathbb{K}, f)^{-1} h(\alpha). \]  

4.2. Proof of Theorem 2.2. We first prove Equation (2.6). Suppose $\alpha$ satisfies Equation (1.3). Let $\mathfrak{d} = \deg f$. We assume that
\[ h(\alpha) > \max \left\{ \frac{\mathfrak{d} + 1}{\mathfrak{d} - 1} C_6(\mathbb{K}, f), \ 2\eta_1(\mathbb{K}, f, \mathcal{S})^{-1} \right\} \]
with $C_6(\mathbb{K}, f)$ as in Lemma 4.1 (a) and $\eta_1(\mathbb{K}, f, \mathcal{S})$ as in Theorem 2.1.
The first term in the maximum on the right hand side of Equation (4.2), along with Lemma 4.1 (a) and (b), ensures that

\[ h(f(\alpha)) > \hat{h}_f(f(\alpha)) - C_6(\mathbb{K}, f) = \hat{\delta} h_f(\alpha) - C_6(\mathbb{K}, f) \]
\[ > \delta h(\alpha) - (\delta + 1)C_6(\mathbb{K}, f) > h(\alpha). \]

The second term in the maximum in Equation (4.2) and Theorem 2.1, along with Equation (4.3), implies that

\[ h_{M\backslash S}(f(l)(\alpha) - 1) > \eta_1(\mathbb{K}, f, S) h(f(l)(\alpha)) - 1 \]
\[ > \frac{\eta_1(\mathbb{K}, f, S)}{2} h(f(l)(\alpha)), \]

for any iterate \( f(l)(\alpha) \) with \( l \geq 1 \).

For any \( \alpha \in \sigma \), we write \( J_S(\alpha) \) to mean the \( S \)-free part of \([\alpha]\), that is, the ideal

\[ J_S(\alpha) = \frac{[\alpha]}{\prod_{p \in S_0} p^{\text{ord}_p \alpha}}. \]

We now write \([f(m)(\alpha)] = a \cdot b\) where

\[ a = J_S(f(m)(\alpha)) \quad \text{and} \quad b = \prod_{p \in S_0} p^{\text{ord}_p f(m)(\alpha)}. \]

Observe that Equation (1.3) implies that \( a \mid f(n)(\alpha) \). Setting

\[ k = m - n > 0, \]

we write

\[ f(k)(f(n)(\alpha)) = f(m)(\alpha) \]

which, with the above observation, implies that \( a \mid f(k)(0) \).

Since 0 is not a periodic point, we have \( f(k)(0) \neq 0 \) and combining the above observation with the notation in Lemma 4.1 we obtain

\[ h_{M\backslash S}(f(m)(\alpha)^{-1}) \leq h_{M\backslash S}(f(k)(0)^{-1}) \]
\[ \leq h(f(k)(0)^{-1}) = h(f(k)(0)) \]
\[ < \delta^k \hat{h}_f(0) + C_6(\mathbb{K}, f). \]

On the other hand, Equation (4.4) along with Lemma 4.1 (a) implies that

\[ h_{M\backslash S}(f(m)(\alpha)^{-1}) > \frac{\eta_1(\mathbb{K}, f, S)}{2}(\delta^m \hat{h}_f(\alpha) - C_6(\mathbb{K}, f)). \]

Comparing Equation (4.5) and Equation (4.6), since \( k \leq m \), we obtain

\[ \hat{h}_f(\alpha) < 2\eta_1(\mathbb{K}, f, S)^{-1} \hat{h}_f(0) + \frac{2\eta_1(\mathbb{K}, f, S)^{-1} + 1}{\delta^m} C_6(\mathbb{K}, f). \]
As $\hat{h}_f(0) < C_6(\mathbb{K}, f)$ we obtain the upper bound

$$h(\alpha) < (2\eta_1(\mathbb{K}, f, S)^{-1} + 1) \left(1 + \frac{1}{d_m}\right) C_6(\mathbb{K}, f),$$

as required.

The proof for Equation (2.7) is the same, except with $\eta_2$ instead of $\eta_1$ throughout.

4.3. Proof of Theorem 2.3. First we establish the result when one of $r$ or $s$ is equal to 0. We obtain an explicit dependence on $S$ which may be interesting in its own right. In particular, this gives a somewhat explicit version of [16, Proposition 1.5(a)].

More generally, an explicit version of Lemma 4.2 for $f(z) \in \mathbb{K}(z)$ can be derived from the proof of [16, Proposition 1.5(a)]. The key ingredient of the proof is Siegel’s Theorem for curves of genus 0 which can be made fully explicit using Baker’s method (see, for example, the end of [2, Theorem 4.3]).

**Lemma 4.2.** Let $f(X) \in \mathfrak{o}[X]$ be a polynomial with at least 3 distinct roots. Suppose that $\alpha \in \mathfrak{o}$ satisfies

$$f(\alpha) \in \mathfrak{o}^*_S.$$

Then

$$h(\alpha) < \frac{\eta_1(\mathbb{K}, f, S)^{-1}}{\mathfrak{d}} + \left(1 + \frac{1}{\mathfrak{d}}\right) C_6(\mathbb{K}, f),$$

with $\eta_1(\mathbb{K}, f, S)$ as in Theorem 2.1, $\mathfrak{d} = \deg f$ and $C_6(\mathbb{K}, f)$ as in Lemma 4.1.

**Proof.** If $f(\alpha) \in \mathfrak{o}^*_S$, then $h_S(f(\alpha)^{-1}) = h(f(\alpha)^{-1}) = h(f(\alpha))$. Substituting into Theorem 2.1 we obtain

(4.7) $h(f(\alpha)) < \eta_1(\mathbb{K}, f, S)^{-1}$.

By Lemma 4.1 we have the inequality

(4.8) $h(f(\alpha)) > \hat{h}_f(f(\alpha)) - C_6(\mathbb{K}, f) = \mathfrak{d}\hat{h}_f(\alpha) - C_6(\mathbb{K}, f)$

$$> \mathfrak{d}h(\alpha) - (\mathfrak{d} + 1)C_6(\mathbb{K}, f).$$

The result now follows from substituting Equation (4.8) into Equation (4.7). \[\square\]

We now prove Theorem 2.3. Theorem 2.3 essentially follows from the proof of [6, Theorem 1.7] (in the case where $\alpha \in R_{S,f,r}$) upon replacing the use of [6, Theorem 1.2] with Lemma 4.2 and the use of [6, Theorem 1.3] with Theorem 2.2. We use the same cases as in the proof of [6, Theorem 1.7] and just indicate the changes necessary.
As in [6], we can effectively bound the height of elements of $\text{PrePer}(f)$ (see Lemma 4.1 (a) and (c)), hence we assume $\alpha \notin \text{PrePer}(f)$ from now on.

First, if $r = 0$ or $s = 0$, we then have that $f^{(n)}(\alpha) \in \mathfrak{o}_S^*$ for some $n \geq 1$. Lemma 4.2 bounds the height of $f^{(n-1)}(\alpha)$. From this, Lemma 4.1 provides an effective upper bound on $h(\alpha)$ as required.

By replacing $(r, s)$ by $(-r, -s)$ we may assume that $r > 0$.

If, in addition, $s < 0$, then, as in [6], we can conclude that $f^{(m)}(\alpha) \in \mathfrak{o}_S^*$ and bound $h(\alpha)$ as above.

If either $s \geq 2$ or $r \geq 2$, then the argument in [6] applies directly (noting that as $\deg f \geq 3$, we can always apply one of [5, Theorem 2.1] or [5, Theorem 2.2] to obtain effective results).

Finally, the case $r = s = 1$ is just a consequence of Theorem 2.2, which concludes the proof.

5. Proof of Theorem 2.4

5.1. The case where $m$ is much larger than $n$.

Lemma 5.1. Let $f(X) \in \mathfrak{o}[X]$ be a polynomial with at least 3 distinct roots. Let $\alpha \in \mathfrak{o}, m, n \in \mathbb{Z}, m > n \geq 0$ such that $f^{(m)}(\alpha), f^{(n)}(\alpha) \neq 0$. Let

$$L = \log^* \left( \frac{h(f^{(m)}(\alpha))}{h(f^{(n)}(\alpha)) + 1} \right).$$

Then

$$\mathfrak{P} \left( \frac{f^{(m)}(\alpha)}{f^{(n)}(\alpha)} \right) > C_8(\mathbb{K}, f) \frac{L \log^* L}{\log^* \log^* L},$$

where $C_8(\mathbb{K}, f) > 0$ is an effectively computable constant.

Proof. For any $X > 0$, let $S^X = \mathcal{M}^\infty \cup \mathcal{M}^{<X}$, where

$$\mathcal{M}^{<X} = \{ |\cdot|_v | N(p) \leq X \}.$$

If $X > 1$, then, since at most $d$ prime ideals lie over each prime $p \in \mathbb{Z}$, using an explicit bound on the prime counting function [1, Theorem 4.6] we derive

$$t_X = \# (S^X \cap \mathcal{M}^0) \leq 6dX / \log X,$$

$$s_X = \# S^X \leq d + 6dX / \log X.$$

Let

$$X = \mathfrak{P} \left( \frac{f^{(m)}(\alpha)/f^{(n)}(\alpha)}{f^{(m)}(\alpha)} \right).$$
Then \((f^{(n)}(\alpha)/f^{(m)}(\alpha)) \in o_{sX}\). Therefore,

\[
(5.2) \quad h_{M \setminus S X}(f^{(m)}(\alpha)^{-1}) \leq h_{M \setminus S X}(f^{(n)}(\alpha)^{-1}) \leq h(f^{(n)}(\alpha)^{-1}) = h(f^{(n)}(\alpha)).
\]

Suppose that \(X > 1\), hence \(t_X > 0\). Then Theorem 2.1 implies that

\[
(5.3) \quad h_{M \setminus S X}(f^{(m)}(\alpha)^{-1}) > \eta_2(\mathbb{K}, f, S^X) \cdot h(f^{(m)}(\alpha)) - 1.
\]

Combining Equation (5.2) and Equation (5.3) we obtain

\[
(5.4) \quad \frac{h(f^{(m)}(\alpha))}{h(f^{(n)}(\alpha))} < \eta_2(\mathbb{K}, f, S^X)^{-1}.
\]

We note that

\[
(5.5) \quad A_2(dDhL, t_X D) \leq C_9(\mathbb{K}, f)^{X/\log X}
\]

for some effectively computable constant \(C_9(\mathbb{K}, f)\).

Substituting Equation (5.1) and Equation (5.5) into Equation (5.4) we obtain

\[
\frac{h(f^{(m)}(\alpha))}{h(f^{(n)}(\alpha))} + 1 < (C_{10}(\mathbb{K}, f) \log^* X)^{C_{10}(\mathbb{K}, f) X/\log^* X},
\]

where \(C_{10}(\mathbb{K}, f) > 0\) is effectively computable. Taking logs, we obtain

\[
\log^* \left( \frac{h(f^{(m)}(\alpha))}{h(f^{(n)}(\alpha))} + 1 \right) < C_{11}(\mathbb{K}, f) X \frac{\log^* \log^* X}{\log^* X},
\]

where \(C_{11}(\mathbb{K}, f) > 0\) is effectively computable. The desired result follows after some simple calculation.

In the case where \(X = 1\), the same procedure but using Equation (2.4) instead of Equation (2.5) shows that

\[
\frac{h(f^{(m)}(\alpha))}{h(f^{(n)}(\alpha))} + 1 < C_{12}(\mathbb{K}, f),
\]

where \(C_{12}(\mathbb{K}, f)\) is an effectively computable constant, as required. \(\square\)

5.2. The case where \(m\) and \(n\) are of comparable sizes.

Lemma 5.2. Let \(f(X) \in o[X]\) be a polynomial with at least 3 distinct roots and for which 0 is not periodic. Let \(\alpha \in o, m, n \in \mathbb{Z}, m > n \geq 0\) such that \(f^{(m)}(\alpha), f^{(n)}(\alpha) \neq 0\). Then

\[
(5.6) \quad \mathfrak{B} \left( \frac{f^{(m)}(\alpha)}{f^{(n)}(\alpha)} \right) > C_{13}(\mathbb{K}, f) \frac{\lambda (f^{(m)}(\alpha)) \log^* \lambda (f^{(n)}(\alpha))}{\log^* \log^* \lambda (f^{(n)}(\alpha))},
\]

where \(C_{13}(\mathbb{K}, f) > 0\) is an effectively computable constant.
Proof. Define $S^X$ and $X$ as in the proof of Lemma 5.1.

Then $(f^{(n)}(\alpha)/f^{(m)}(\alpha)) \in \mathfrak{G}_{\infty}$. If $X = 1$, hence $S^X = \mathcal{M}^\infty$, applying Theorem 2.2 with $S = \mathcal{M}^\infty$ and $\alpha = f^{(n)}(\alpha)$ yields an effective upper bound on $h(f^{(n)}(\alpha))$ in terms of $\mathbb{K}$ and $f$, as required.

Otherwise, $\#(S^X \cap \mathcal{M}^0) > 0$. Hence, applying Theorem 2.2 with $S = S^X$ and $\alpha = f^{(n)}(\alpha)$, we obtain

$$h(f^{(n)}(\alpha)) \leq c_2(\mathbb{K}, f, S^X) \cdot \log d.$$

(5.6)

We can now proceed as in the proof of Lemma 5.1, except with Equation (5.6) in place of Equation (5.4), to obtain the desired result. □

5.3. Concluding the proof. We now prove Theorem 2.4.

If $h(f^{(n)}(\alpha)) > C_7(\mathbb{K}, f)$, with $C_7(\mathbb{K}, f)$ as in Equation (4.1), then a combination of Lemma 5.1, used for

$$m - n \geq \frac{\lambda(f^{(n)}(\alpha))}{\log \delta},$$

and of Lemma 5.2 otherwise implies the result.

Otherwise, $h(f^{(n)}(\alpha)) \leq C_7(\mathbb{K}, f)$. The result now follows from Lemma 5.1.

6. Proof of Theorem 2.5

If $p \mid f^{(m)}(\alpha)$ and $p \mid f^{(n)}(\alpha)$ with $m > n$, then, writing

$$f^{(m)}(\alpha) = f^{(m-n)}(f^{(n)}(\alpha)),$$

we see that $p \mid f^{(m-n)}(0)$.

Fix a $k \in \mathbb{Z}$, $k > 0$ and let $S_k$ be the finite set of places containing $\mathcal{M}^\infty$ and all finite places corresponding to a prime dividing a value in the set

$$\{ f^{(1)}(0), f^{(2)}(0), \ldots, f^{(k)}(0) \}$$

($S_k$ is finite since 0 is not periodic).

With the above observation, to prove the desired statement for $k(m, \alpha) = k$,

it suffices to show that

$$h_{S_k}(f^{(m)}(\alpha)^{-1}) < h(f^{(m)}(\alpha)).$$

The case where $S_k$ contains no finite places is trivial. Hence, we assume that $S_k$ contains at least one finite place. Then, by Theorem 2.1,
we have that
\begin{equation}
    h(f^{(m)}(\alpha)) - h_{S_k}(f^{(m)}(\alpha)^{-1}) > \eta_2(\mathbb{K}, f, S_k)h(f^{(m)}(\alpha)) - 1.
\end{equation}

We note the following inequalities for $S_k$ which are consequences of Lemma 4.1 and simple calculation (for the last inequality note that \[ \sum_{i=1}^{t} \log(N(p_i)) < C_{14}(\mathbb{K}, f) d_k \]):
\[ t < C_{14}(\mathbb{K}, f)d_k, \quad P < e^{C_{14}(\mathbb{K}, f) d_k}, \quad \prod_{i=1}^{t} \log(N(p_i)) < e^{C_{14}(\mathbb{K}, f) d_k} \]
for an effectively computable $C_{14}(\mathbb{K}, f)$. Hence,
\begin{equation}
    \eta_2(\mathbb{K}, f, S_k)^{-1} < e^{C_{15}(\mathbb{K}, f) d_k}.
\end{equation}
Substituting Equation (6.2) into Equation (6.1), the required statement holds for any $k$ such that
\[ \log h(f^{(m)}(\alpha)) > C_{15}(\mathbb{K}, f) d_k. \]

If $h(f^{(m)}(\alpha))$ is sufficiently large, then Theorem 2.5 follows immediately. Otherwise, $h(f^{(m)}(\alpha))$ is bounded and we may pick $c_6(\mathbb{K}, f)$ small enough such that $k(m, \alpha) = 0$.

Acknowledgement

The authors are grateful to Attila Bérczes for supplying a proof of a version of Theorem 2.1 in terms of the norm of the $S$-part of $f(\alpha)$ and Alina Ostafe for her encouragement and comments on an initial draft of the paper.

This work was supported, in part, by the Australian Research Council Grant DP180100201.

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MULTIPlicatively DEPENDENT ORBITS MODULO $S$-INTEGERS

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