Long-time behaviour and propagation of chaos for mean field kinetic particles

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Abstract

The trend to equilibrium in large time is studied for a large particle system associated to a Vlasov-Fokker-Planck equation in the presence of a convex external potential, without smallness restriction on the interaction. From this are derived uniform in time propagation of chaos estimates, which themselves yield in turn an exponentially fast convergence for the semi-linear equation itself. The approach is quantitative.

1 Settings and main results

The Vlasov-Fokker-Planck equation reads

$$\begin{align*}
\frac{\partial}{\partial t} m_t(x, y) + y \cdot \nabla_x m_t &= \nabla_y \cdot \left( \frac{\sigma^2}{2} \nabla_y m_t + \left( \int \nabla_x U(x, u) m_t(u, v) du dv + \gamma y \right) m_t \right) \\
\end{align*}$$

(1)

where $m_t(x, y)$ is a density at time $t$ of particles at point $x \in \mathbb{R}^d$ with velocity $y \in \mathbb{R}^d$, $\sigma, \gamma > 0$, $\nabla$ and $\nabla \cdot$ stand for the gradient and divergence operators and the potential $U$ is a function from $\mathbb{R}^{2d}$ to $\mathbb{R}$. This equation is naturally linked to the stochastic system of $N$ kinetic particles $(X_i, Y_i)_{i \in [1,N]}$ that solves

$$\begin{align*}
\forall i \in [1, N] \quad \begin{cases}
\mathrm{d}X_i = Y_i \mathrm{d}t \\
\mathrm{d}Y_i = -\gamma Y_i \mathrm{d}t - \left( \frac{1}{N} \sum_{j=1}^{N} \nabla_x U(X_i, X_j) \right) \mathrm{d}t + \sigma \mathrm{d}B_i
\end{cases}
\end{align*}$$

(2)

with the initial conditions $(X_i(0), Y_i(0))$ being i.i.d. random variables of law $m_0$, independent from the standard Brownian motion $B$. Here, naturally linked means, depending on one’s point of view, that for large $N$ the semi-linear PDE is an approximation of the particle system, or that the particle system is an approximation of the semi-linear PDE: for large $N$, the random empirical law $M^N_t = \frac{1}{N} \sum_{i \in [1,N]} \delta_{X_i,Y_i}$ (where $\delta_{x,y}$ is the Dirac mass at point $(x,y)$) should behave like the deterministic solution $m_t$ of (1).

Equation (2) describes a mean field dynamics in the sense each particle is equally influenced by all the others at once. As $N$ goes to infinity, two given particles should be less and less correlated, and a given particle $(X_i, Y_i)$ should behave like $(\overline{X}_i, \overline{Y}_i)$ which solves

$$\begin{align*}
\begin{cases}
\mathrm{d}\overline{X}_i = \overline{Y}_i \mathrm{d}t \\
\mathrm{d}\overline{Y}_i = -\gamma \overline{Y}_i \mathrm{d}t - \left( \int \nabla_x U(\overline{X}_i, u) m_t(u, v) du dv \right) \mathrm{d}t + \sigma \mathrm{d}B_i
\end{cases}
\end{align*}$$

(3)

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with \((\overline{X}_i(0), \overline{Y}_i(0)) = (X_i(0), Y_i(0))\). This is the so-called propagation of chaos phenomenon, and it is equivalent for interchangeable particles to the convergence of the empirical measure of the particle system to \(m_t\) (see [18]). Note the \((\overline{X}_i, \overline{Y}_i)\)'s are i.i.d. random variables with law \(m_t\).

Aside from the propagation of chaos \((N \to \infty)\) another natural question is the long time behaviour \((t \to \infty)\) of both \(m_t\) and \((X_i, Y_i)_{i \in [1, N]}\). There is a large literature for the corresponding space homogeneous model (the Mc Kean-Vlasov equation, which can also be seen as an overdamped version of \([11]\) when the particles have no mass), or when there is no interaction (i.e. when \(U(x, c)\) only depends on \(x\), which is the kinetic Fokker-Planck or Langevin equation); see the introductions of \([1, 3]\) for references.

However obtaining explicit speed of convergence toward equilibrium for the Vlasov-Fokker-Planck equation (at least when there is a unique equilibrium, which is false in general, see [8]), coping with both the kinetic dynamics and the interaction, is more difficult. There are several results for small interactions \([12, 11, 20]\) or when the forces are close to be linear \([4]\). See also [9] for a case with possibly several equilibria.

In the recent work of Hérau and Thomann [12], since the interaction is small, exponential convergence to equilibrium is established from the results on the linear Fokker-Planck equation (at least when there is a unique equilibrium, which is false in general, see [18]). Note the \((\overline{X}_i, \overline{Y}_i)\) in the definition of \(\chi^t\) only depends on \(x\).

Theorem 1. Under Assumption 1, \(m_0\) admits a smooth density in \(L \log L\) with respect to the Lebesgue measure and a finite second moment.

Note that the term \(V(u)\) in the definition of \(U(x, u)\) does not alter the force \(\nabla_u U(x, u)\). It is added only for the sake of symmetry and of the definition of the global potential \(U_N\) in Section 2.

These conditions discard Coulomb interaction forces, but it treats mollified approximations of those, as in [11]. Under Assumption 1 as we will see, the solution \(m_t^{(N)}\) of \([11]\) is well defined, along with the processes \(Z_N = ((X_i, Y_i))_{i \in [1, N]}\) and \(\overline{Z}_N = ((\overline{X}_i, \overline{Y}_i))_{i \in [1, N]}\). We call \(m_t^{(N)}\) the law of \(Z_N\) when \(m_0^{(N)} = m_0^{\otimes N}\), while the law of \(\overline{Z}_N\) is \(m_t^{\otimes N}\). The process \(Z_N\) admits a unique invariant law, denoted by \(m_{\infty}^{(N)}\). For two probability laws \(\nu\) and \(\mu\) we write

\[
\mathcal{H}(\nu \| \mu) = \begin{cases} 
\int \ln \left(\frac{d\nu}{d\mu}\right) \, d\nu & \text{if } \nu \ll \mu \\
+\infty & \text{else}
\end{cases}
\]

the relative entropy of \(\nu\) with respect to \(\mu\). Under Assumption 1 at initial time, \(\mathcal{H}\left( m_0^{(N)} \| m_{\infty}^{(N)} \right) < \infty\) (more precisely, see Lemma 9 below).

Theorem 1. Under Assumption 1, there exist \(C, \chi > 0\) that depends only on \(U, \gamma, \sigma\) such that for all \(N\) and \(t \geq 0\),

\[
\mathcal{H}\left( m_t^{(N)} \| m_{\infty}^{(N)} \right) \leq C e^{-\chi t} \mathcal{H}\left( m_0^{(N)} \| m_{\infty}^{(N)} \right),
\]
Note that $m_{it}^{(N)}$ is a solution of a (large dimensional) linear Fokker-Planck equation, for which the exponential decay of the entropy is already known (see e.g. [20]). The key point in Theorem 1 is that $C$ and $\chi$ do not depend on $N$. This enables us to prove the following:

**Corollary 2.** Under Assumption 1, Equation (1) admits a unique equilibrium $m_\infty$. Moreover, for any $\beta > d + 2$ there exist $K, N_0 > 0$ that depends only on $U, \gamma, \sigma, \beta$ and $m_0$ such that for all $\epsilon, t$ and $N \geq N_0 (1 \wedge \epsilon)^{-\beta},$

$$\mathbb{P} \left( \mathcal{W}_2 \left( M_t^{(N)}, m_\infty \right) \geq \epsilon \right) \leq \frac{K}{\epsilon^2} \left( e^{-\chi t} + \frac{1}{N} \right)$$

where $\chi$ is given by Theorem 1 and $\mathcal{W}_2$ is the Wasserstein distance

$$\mathcal{W}_2^2 (\nu_1, \nu_2) = \inf \left\{ \mathbb{E} \left( |A_1 - A_2|^2 \right), \ Law(A_i) = \nu_i \text{ for } i = 1, 2 \right\}.$$

(We emphasize that throughout this work $|\cdot|$ always stands for the usual Euclidean distance, and not a renormalization of it, as in some other works on propagation of chaos.)

As shown in [5, Proposition 2.1], such a result yields confidence intervals with respect to the uniform metric for a numerical approximation of $m_\infty$ by $M_t^{(N)} \ast \xi$ where $\xi$ is a smooth kernel. Since $m_\infty$ turns to be the tensor product of an explicit Gaussian distribution in velocity and of the invariant measure of an order one McKean-Vlasov equation (such as considered in [5, 13]) in position, there are thus two possibilities for its approximation, and we may wonder if one is better than the other. This question already arises to sample a Gibbs law $e^{-U}$ even when there is no interaction and $U$ is explicit: this can be done either with a Fokker-Planck reversible process, or with a kinetic Langevin non-reversible one. The reversible dynamics is simpler but the kinetic Langevin sampler may converge to equilibrium faster. This has been observed numerically in [17], proven for some toy models in [10] and there are some philosophical reasons to believe it, but it is hard to state in general since the rates of convergence obtained for so-called hypocoercive processes are usually not sharp.

From Theorem 1 one can also expect to recover a rate of convergence for the deterministic equation (1). This requires explicit estimates for the propagation of chaos that behave nicely with respect to time, which are not so easy to establish. In the close to linear case (and with a small interaction), Bolley, Guillin and Malrieu in [4] are able to use the exterior coercive force to prove uniform in time estimates directly from parallel coupling, namely by considering $\overline{Z}_N$ and $\underline{Z}_N$ which respectively solve (2) and (3) driven by the same Brownian motion. In our case, this does not work. However, using the parallel coupling for small times and the coupling of the equilibria for large times, an intertwining between results of propagation of chaos and of large time convergence ultimately yields the two following consequences:

**Theorem 3.** Under Assumptions 1, there exist $\alpha > 0$ (depending only on $U, \gamma$ and $\sigma$) and $K > 0$ (depending only on $U, \gamma, \sigma$ and $m_0$) such that for all $N \geq 1, t > 0,$

$$\mathcal{W}_2 \left( m_{it}^{(1,N)}, m_t \right) \leq \frac{K}{N^\alpha}$$

where $m_{it}^{(1,N)}$ stands for the first 2d-dimensional marginal of $m_{it}^{(N)}$, namely for the law of $(X_1, Y_1)$.

(And in fact we also get a uniform in time propagation of chaos estimate for the total variation distance, see the last remark of the paper.)

**Theorem 4.** Under Assumptions 1, there exist $\chi' > 0$ (depending only on $U, \gamma$ and $\sigma$) and $K > 0$ (depending only on $U, \gamma, \sigma$ and $m_0$) such that for all $t \geq 0,$

$$\|m_t - m_\infty\|_1 \leq K e^{-\chi' t}.$$  \(\text{(5)}\)
Note that uniform in time propagation of chaos cannot hold when the non-linear dynamics admits several equilibria, since \( m_t \) will converge in large time to one of those while the invariant law of \( m_t^{(N)} \) is unique and should converge as \( N \to \infty \) to a combination of all of these equilibria. Nevertheless we may hope the same kind of arguments to work when \( m_t^{(N)} \) is replaced by a quasi-stationary distribution (QSD) of the particle system, and the convergence of \( t \) and \( N \) to infinity is more intricate, namely for a given \( N \) we let \( t \) be large enough so that \( m_t^{(N)} \) converges to its QSD but not so large so that it doesn’t leave the catchment area of a particular equilibrium (to which the QSD should converge as \( N \to \infty \)). This is the subject of ongoing research.

The rest of the paper is organized as follow: Theorem 1, Corollary 2, Theorem 3 and Theorem 4 are respectively proven in Sections 2, 3, 4 and 5.

**Notation:** throughout all the paper, from lines to lines, we keep denoting by \( K \) different constants as long as they depend only on \( U, \gamma, \sigma \) and \( m_0 \).

## 2 The particle system

Let us recall some known facts whose proofs may be found in [19, 15]. The SDE (2) admits a unique strong solution defined for all times. Denoting by \( P_t^N \) its associated semi-group, defined on (say bounded) functions on \( \mathbb{R}^{2dN} \) by \( P_t^N f(z) = \mathbb{E}(f(Z_N(t)) \mid Z_N(0) = z) \) (recall we write \( Z_N = ((X_i, Y_i))_{i \in [1,N]} \), then \( P_t^N \) is strongly Feller and fixes the set of smooth functions whose all derivatives grow at most polynomially. The process admits a unique invariant probability measure whose density with respect to the Lebesgue measure is

\[
m^{(N)}_\infty(x, y) \propto \exp\left(-\frac{2\gamma}{\sigma^2} \left(U_N(x) + \frac{|y|^2}{2}\right)\right)
\]

where the full potential

\[
U_N(x) := \frac{1}{2N} \sum_{i,j=1}^{N} U(x_i, x_j)
\]

is such that the term that depends on \( X \) in (2) in \( dY_t \) is exactly

\[
\nabla_x U_N(x) = \frac{1}{N} \sum_{j=1}^{N} \nabla U(x_i, x_j).
\]

**Lemma 5.** Under Assumption \( \square \) for all \( N \in \mathbb{N} \) and all \( x, u \in \mathbb{R}^{dN} \),

\[
(c_1 - 2c_2) |u|^2 \leq u \cdot \nabla^2 U_N(x) u \leq (\|\nabla^2 V\|_\infty + 2\|\nabla^2 W\|_\infty) |u|^2.
\]

**Proof.** This is a direct consequence of

\[
 u \cdot \nabla^2 U_N(x) u = \sum_{i=1}^{N} u_i \cdot \nabla^2 V(x_i) u_i + \frac{1}{2N} \sum_{i,j=1}^{N} (u_i - u_j) \cdot \nabla^2 W(x_i - x_j)(u_i - u_j).
\]
Recall we say a measure $\mu$ satisfies a log-Sobolev inequality with constant $\eta > 0$ if
\[ \forall f > 0 \text{ s.t. } \int f \, d\mu = 1, \quad \int f \ln f \, d\mu \leq \eta \int \frac{\|\nabla f\|^2}{f} \, d\mu. \] (6)

**Lemma 6.** Under Assumption 1, the measure $m_{\infty}^{(N)}$ satisfies a log-Sobolev inequality with a constant $\eta$ that does not depend on $N$.

**Proof.** For $z \in \mathbb{R}^{2dN}$, let $F(z) = \frac{2\sigma^2}{\gamma^2} \left( U_N(x) + \frac{|w|^2}{2} \right)$. For all $z, u \in \mathbb{R}^{2dN}$ with $|u| = 1$,
\[ u \cdot \nabla^2 F(z) u \geq \frac{2\gamma}{\sigma^2} \min (c_1 - 2c_2, 1) := \frac{1}{2\eta} > 0. \]
Since $m_{\infty}^{(N)} \propto e^{-F}$ is the invariant measure of the semi-group with generator $-\nabla F \cdot \nabla + \Delta$, the Bakry-Emery curvature criterion (see [2]) concludes. □

Consider the generator
\[ L_N = -y \cdot \nabla_x + (\nabla U_N(x) - \gamma y) \cdot \nabla_y + \frac{\sigma^2}{2} \Delta_y. \] (7)

Then $h_i^{(N)} = \frac{m_i^{(N)}}{m_{\infty}^{(N)}}$, the density of the law of the particle system [2] with respect to its equilibrium, solves $\partial_t h_i^{(N)} = L_N h_i^{(N)}$. This is a linear kinetic Fokker-Planck equation, for which convergence to equilibrium has been proven by many ways. All we need to check is that the explicit estimates we obtain do not depend on $N$. For instance we can use the following

**Theorem 7** (from Theorem 10 of [16]). Consider a diffusion generator $L$ on Hörmander form
\[ L = B_0 + \sum_{i=1}^d B_i^2 \]
where the $B_j$'s are derivation operators. Suppose there exist $N_c \in \mathbb{N}$ and $\lambda, \Lambda, m, \rho, K > 0$ such that for $i \in [0, N_c + 1]$ there exist smooth derivation operators $C_i$ and $R_i$ and a scalar field $Z_i$ satisfying:

(i) $C_{N_c+1} = 0$, and $[B_0, C_i] = Z_{i+1} C_{i+1} + R_{i+1}$ for all $i \in [0, N_c]$, where $[A, B] = AB - BA$ stands for the Poisson bracket of two operators,

(ii) $[B_j, C_i] = 0$ for all $i \in [0, N_c], j \in [1, d],$

(iii) $\lambda \leq Z_i \leq \Lambda$ for all $i \in [0, N_c]$,

(iv) $|C_0 f|^2 \leq m \sum_{j \geq 1} |B_j f|^2$ and $|R_i f|^2 \leq m \sum_{j < i} |C_j f|^2$ for all $i \in [0, N_c + 1]$ and smooth Lipschitz $f$.

(v) $\sum_{i \geq 0} |C_i f|^2 \geq \rho |\nabla f|^2$.

Suppose moreover that there exists a probability measure $\mu$ which is invariant for $e^{tL}$ and satisfies a log-Sobolev inequality with constant $\eta$.

Then for all $t > 0$ and for all $f > 0$ with $\int f \, d\mu = 1$,
\[ \int (e^{tL} f) \ln (e^{tL} f) \, d\mu \leq e^{-\eta (1-e^{-1})^{2N_c}} \int f \ln f \, d\mu \] (8)
with
\[ \kappa = \frac{\rho}{\eta} \left( \frac{100}{\lambda} \left( N_c^2 + \frac{\Lambda^2}{\lambda} + m \right) \right)^{-20N_c^2}. \]

**Proof of Theorem 1.** The generator \( B \) is on Hörmander form
\[ B_0 + \sum_{i=1}^{N} \sum_{j=1}^{d} B_{i,j} \]
with, writing \( y_i = (y_i^{(1)}, \ldots, y_i^{(d)}) \in \mathbb{R}^d \),
\[ B_0 = -y \cdot \nabla_x + (\nabla U_N(x) - \gamma y) \cdot \nabla_y \]
\[ B_{i,j} = \frac{\sigma}{\sqrt{2}} \partial_{y_{ij}}. \]
Since
\[ [B_0, \nabla_y] = [L_N, \nabla_y] = \nabla_x + \gamma \nabla_y, \quad [B_0, \nabla_x] = [L_N, \nabla_x] = -\nabla_x^2 U_N \nabla_y, \]
Theorem 7 applies with
\[ C_0 = \nabla_y, \quad C_1 = \nabla_x, \]
\[ R_1 = \gamma \nabla y, \quad R_2 = -\nabla_x^2 U_N \nabla_y, \]
\[ Z_1 = Z_2 = N_c = \lambda = \Lambda = \rho = 1, \]
\[ m = \frac{2}{\sigma^2} + \gamma^2 + (\|\nabla^2 V\|_{\infty} + 2\|\nabla^2 W\|_{\infty})^2 \]
and \( \eta \) given by Lemma 6.

Remark that instead of [16, Theorem 10], we could have referred to the work of Villani, which is anterior, to get the same result. Indeed, combining the Theorems 28 and A.15 of [20] yields a similar hypocoercive convergence in the relative entropy sense. Nevertheless, this would have required a thorough examination of the somewhat tedious computations in the proofs of these theorems, in order to explicit the constants \( C \) and \( \chi \) and check that they do not depend on \( N \). Besides, since we did not conduct such an examination, it is not clear whether the computations in [20], which are known not to be sharp, can indeed give constants \( C \) and \( \chi \) which are uniform. Yet, this is the crucial argument on which relies all the rest of the present work.

Remark also that, on the contrary, we could not use the method of Dolbeault, Mouhot and Schmeiser (see the seminal work [7]) since, rather than in the relative entropy sense, the latter states a convergence in the \( L^2 \) distance sense, which behaves badly with respect to tensorization. More precisely, \( \mathcal{H}(m_1 | m_2) \) would be replaced by
\[ \int \left( \frac{m_1}{m_2} - 1 \right)^2 m_2 = \int \left( \frac{m_1}{m_2} \right)^2 m_2 - 1 \]
and thus, for independent variables, \( \mathcal{H} (m_1^{\otimes N} | m_2^{\otimes N}) \) would be replaced by
\[ \left( \int \left( \frac{m_1}{m_2} \right)^2 m_2 \right)^N - 1. \]
In our case, the explicit bound on the rate given by Theorem 7 reads
\[
\chi = \frac{2 \min (c_1 - 2c_2, 1)}{\sigma^2 \left( 100 \left( 2 + \frac{2}{\sigma^2} + \gamma^2 + (\|\nabla^2 V\|_\infty + 2\|\nabla^2 W\|_\infty)^2 \right) \right)^{20}},
\]
which is quite rough. When the potentials are quadratic, we can compare this to the real rate, which is known. Suppose \(V(x) = \frac{a}{2} |x|^2\) and \(W(x) = \frac{b}{2} |x|^2\) so that \(Z = (X,Y)\) is a generalized Ornstein-Uhlenbeck process that solves
\[
dZ = -AZ + \sigma \left( \begin{array}{c} 0 \\ I \end{array} \right) dB
\]
with, denoting by \(\pi = \frac{1}{N} \left( \begin{array}{ccccc} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{array} \right)\) the orthogonal projector on \(\left( \begin{array}{c} 1 \\ \vdots \\ 1 \end{array} \right)\).
\[
A = \left( \begin{array}{cc} 0 & -I \\ aI + b(I - \pi) & \gamma I \end{array} \right).
\]
As proved in [16, Corollary 12], the rate of convergence in the entropic sense of \(Z\) to its equilibrium is exactly the spectral gap of \(A\). Note that \(aI + b(I - \pi)\) is diagonalizable in an orthonormal basis \((u_k)_{k \in \{1, dN\}}\), with the corresponding eigenvalues \(\lambda_k\) being either \(a + b\) or \(a\). Then a vector of the form \((u_k, ru_k)\) for some \(r \in \mathbb{C}\) is an eigenvector of \(A\) if only if
\[
r = -\frac{\gamma}{2} \pm \sqrt{\left(\frac{\gamma}{2}\right)^2 - \lambda_k},
\]
in which case the corresponding eigenvalue is \(-r\). It means that, in this quadratic case, Theorem 4 holds in fact with
\[
\chi = \begin{cases} \frac{\gamma}{2} & \text{if } \left(\frac{\gamma}{2}\right)^2 < \min (a, a + b) \\ \frac{\gamma}{2} - \sqrt{\left(\frac{\gamma}{2}\right)^2 - \min (a, a + b)} & \text{if } \left(\frac{\gamma}{2}\right)^2 > \min (a, a + b) \end{cases}
\]
(in the case \(\gamma^2 = 4 \min (a, a + b)\), an additional polynomial factor should be added, see [16]).
On the other hand, the bound (9) here only reads
\[
\chi \geq \frac{2 \min (a + b, 1)}{\sigma^2 \left( 100 \left( 2 + \frac{2}{\sigma^2} + \gamma^2 + (a + 2|b|)^2 \right) \right)^{20}}.
\]
For instance, if \(\sigma = a = b = \gamma = 1\), we should have \(\chi = \frac{1}{2}\), while from (9) we only get \(\chi \geq 2 \times 10^{-63}\) (again, the computations that lead to (9) were rough but, should we carefully refine them step by step, we would still miss the target by many orders of magnitude).

3 Confidence intervals

From the work of Malrieu on the McKean-Vlasov equation, we obtain the two following Lemmas:

Lemma 8. Under Assumption 4, the Vlasov-Fokker-Planck equation (1) admits a unique equilibrium \(m_\infty\) (with normalized mass) which satisfies a log-Sobolev inequality with constant
η (given by Lemma 9), and there exists $K > 0$ that depends only on $U, \gamma, \sigma$ and $m_0$ such that for all $N$

\[
W^2_2 (m^{(N)}_\infty, m^{(N)}_\infty) \leq K
\]

\[
\|m^{(1,N)}_\infty - m_\infty\|_1 \leq \frac{K}{\sqrt{N}}.
\]

**Proof.** According to [1], a measure $\mu$ is an equilibrium of (1) if and only if $\mu(dx, dy) = \nu(x)dx \otimes G(dy)$ where $G$ is the Gaussian distribution with variance $\sigma^2 \gamma$ and $\nu$ is an equilibrium of the McKean-Vlasov equation

\[
\partial_t \rho_t = \nabla \cdot \left( \nabla \rho + \rho_t \int \nabla U(x, u)\rho(u)du \right).
\]

Under the convexity condition of Assumption 1 according to [13] such an equilibrium $\nu$ is unique and satisfies the same log-Sobolev inequality as does the invariant measure of the corresponding particle system uniformly in $N \geq 1$. Thus by tensorization $m_\infty$ satisfies a log-Sobolev inequality.

Since the second $d\Sigma^N$-dimensional marginals of $m^{(N)}_\infty$ and $m^{(N)}_\infty$ are equal, equal to $G^{\otimes N}$, the $W_2$ and total variation bounds only concern the first marginals. The $W_2$ (resp. total variation) bound is obtained by letting $t$ go to infinity in [13] Theorem 1.2 (resp. [13] Proposition 3.13) when the initial law is $m_\infty$, which states that $W_2^2 (m^{(N)}_t, m^{(N)}_\infty)$ is bounded uniformly in $N$ and $t$ (and similarly for the total variation distance). The convergence of $m^{(N)}_t$ to its equilibrium, from Theorem 1 together with the Talagrand and Pinsker’s Inequalities, concludes.

\[\square\]

**Lemma 9.** Under Assumption 1, there exists $K$ depending only on $U, \gamma, \sigma$ and $m_0$ such that

\[
\mathcal{H} (m^{(N)}_\infty | m^{(N)}_\infty) \leq KN.
\]

**Proof.** Without loss of generality, up to some translations, we suppose the potentials $V$ and $W$ are positive and vanish at the origin. Writing $\Psi_N(x, y) = \frac{2\gamma}{\sigma^2} (U_N(x) + \frac{|y|^2}{2})$, note that from Lemma 5 $\Psi_N \leq K (|x|^2 + |y|^2)$ for some $K$ that do not depend on $N$. Then,

\[
\mathcal{H} (m^{(N)}_\infty | m^{(N)}_\infty) = \int m^{(N)}_\infty \ln m^{(N)}_\infty + \int m^{(N)}_\infty \Psi_N + \ln \int e^{-\Psi_N} \\
\leq N \left( \int m_0 \ln m_0 + K \int m_0 (|x|^2 + |y|^2) + \ln \int e^{-\frac{2\gamma}{\sigma^2} \left(V(x) + \frac{|y|^2}{2}\right)} \right).
\]

\[\square\]

**Lemma 10.** Let $\nu_1$ and $\nu_2$ be probability laws on $\mathbb{R}^{dN} = (\mathbb{R}^d)^N$ which are fixed by any permutation of the $d$-dimensional coordinates (in other words, if $(A_i)_{i \in [1,N]}$ is of law $\nu$, the $A_i$’s are interchangeable). Let $(A, B) = (A_i, B_i)_{i \in [1,N]}$ be a coupling of $\nu_1$ and $\nu_2$ such that

\[
\mathbb{E} (|A - B|^2) = W^2_2 (\nu_1, \nu_2).
\]

Then

\[
\mathbb{E} \left( W^2_2 \left( \frac{1}{N} \sum_{i=1}^N \delta_{A_i}, \frac{1}{N} \sum_{i=1}^N \delta_{B_i} \right) \right) \leq \frac{1}{N} W^2_2 (\nu_1, \nu_2).
\]
Proof. Let $I$ be uniformly distributed on $[1, N]$. Then $(A_I, B_I)$ is a coupling of \( \frac{1}{N} \sum \delta_{A_i} \) and \( \frac{1}{N} \sum \delta_{B_i} \), hence
\[
\mathbb{E} \left( \mathcal{W}_2^2 \left( \frac{1}{N} \sum_{i=1}^{N} \delta_{A_i}, \frac{1}{N} \sum_{i=1}^{N} \delta_{B_i} \right) \right) \leq \mathbb{E} \left( |A_I - B_I|^2 \right) = \frac{1}{N} \mathbb{E} \left( |A - B|^2 \right).
\]

Remark: consider the \( \mathcal{W}_2 \) metric on \( \mathcal{P}(\mathcal{P}(\mathbb{R}^d)) \) when \( \mathcal{P}(\mathbb{R}^d) \) is itself endowed with the Euclidean \( \mathcal{W}_2 \) metric. Let \( \Pi_N \) be the application from \( \mathcal{P}(\mathbb{R}^d) \) to \( \mathcal{P}(\mathcal{P}(\mathbb{R}^d)) \) defined by \( \Pi_N(\nu) = \mathcal{L} \left( \frac{1}{N} \sum \delta_{X_i}, X \sim \nu \right) \). Lemma 10 implies that for laws of interchangeable particles,
\[
\mathcal{W}_2^2 (\Pi_N(\nu_1), \Pi_N(\nu_2)) \leq \frac{1}{N} \mathcal{W}_2^2 (\nu_1, \nu_2).
\]

Proof of Corollary 2. Consider \( (Z, \overline{Z}) = (Z_i, \overline{Z}_i)_{i \in [1, N]} \) an optimal coupling of \( m_t^{(N)} \) and \( m_\infty^{\otimes N} \), in the sense
\[
\mathbb{E} \left( |Z - \overline{Z}|^2 \right) = \mathcal{W}_2^2 \left( m_t^{(N)}, m_\infty^{\otimes N} \right).
\]
Let \( M_t^N = \frac{1}{N} \sum \delta_{Z_i} \) and \( \overline{M}_N = \frac{1}{N} \sum \delta_{\overline{Z}_i} \), so that
\[
\mathbb{P} \left( \mathcal{W}_2 \left( M_t^N, m_\infty \right) \geq \varepsilon \right) \leq \mathbb{P} \left( \mathcal{W}_2 \left( M_t^N, \overline{M}_N \right) \geq \frac{\varepsilon}{2} \right) + \mathbb{P} \left( \mathcal{W}_2 \left( \overline{M}_N, m_\infty \right) \geq \frac{\varepsilon}{2} \right).
\]
Since the \( \overline{Z}_i \)'s are independent, the second term falls within the scope of Theorem 1.1] (the log-Sobolev inequality satisfied by \( m_\infty \) implies a \( T_2 \) Talagrand one), which gives an exponential bound for \( N \) large enough. As far as the first term is concerned, from Lemma 10 and the Markov inequality,
\[
\mathbb{P} \left( \mathcal{W}_2 \left( M_t^N, \overline{M}_N \right) \geq \frac{\varepsilon}{2} \right) \leq \frac{8}{N \varepsilon^2} \left( \mathcal{W}_2^2 (m_t^{(N)}, m_\infty^{(N)}_t) + \mathcal{W}_2^2 (m_\infty^{(N)}, m_\infty^{\otimes N}) \right).
\]
The last term is bounded by \( K \) from Lemma 8. On the other hand the law \( m_\infty^{(N)} \) satisfies a log-Sobolev (hence \( T_2 \)) inequality with a constant that does not depend on \( N \), which together with Theorem 1 and Lemma 9 yields
\[
\mathcal{W}_2^2 \left( m_t^{(N)}, m_\infty^{(N)} \right) \leq K e^{-\chi t} \mathcal{H} \left( m_0^{(N)} \mid m_\infty^{(N)} \right) \leq KN e^{-\chi t}.
\]
Altogether we have obtained
\[
\mathbb{P} \left( \mathcal{W}_2 \left( M_t^N, m_\infty \right) \geq \varepsilon \right) \leq \frac{K e^{-\chi t}}{\varepsilon^2} + \frac{K}{N \varepsilon^2} + e^{-K^{-1} N \varepsilon^2}.
\]
\]
4 Propagation of chaos

The existence and uniqueness of solutions for Equations (1) and (3) under Assumption 1 are ensured by [14]. Let $Z_N = (X, Y)$ and $Y_N = (X, Y)$ respectively solve Equations (2) and (3) with the same initial data, of law $m_0$. We first prove a uniform bound on the second moments of both processes:

Lemma 11. Under Assumptions 1 there exists $K > 0$ depending only on $U, \gamma, \sigma$ and $m_0$ such that for all $N \geq 1$, $t > 0$,

$$
\mathbb{E} \left( |X(t)|^2 + |Y(t)|^2 \right) \leq K.
$$

Proof. Let $(X, Y)$ solves

$$
\begin{align*}
\frac{dX}{dt} &= Ydt \\
\frac{dY}{dt} &= -\gamma Y dt - U_N(X) dt + \sigma dB,
\end{align*}
$$

where $B$ is a $dN$-dimensional Brownian motion. Talay proved in [19] uniform in time moment bounds for this diffusion, but we need to check the dependency on $N$.

Recall that, as a symmetric matrix, $\nabla^2 U_N \geq \eta$ where $\eta$ is given by Lemma 6 and in particular depends neither on $t$ nor $N$. Up to a translation we can suppose the unique minimum of $U_N$ is 0 and attained at $x = 0$. Consider the generator

$$
L_N = y \cdot \nabla_x - (\nabla U_N(x) + \gamma y) \cdot \nabla_y + \frac{\sigma^2}{2} \Delta_y
$$

(which is the dual in $L^2(m_\infty)$ of $L_N$ given by (12)) and

$$
H(x, y) = U_N(x) + \frac{1}{2} |y|^2 + \varepsilon x \cdot y
$$

for some $\varepsilon > 0$. Then

$$
L_N H(x, y) = - (\gamma - \varepsilon) |y|^2 + \frac{\sigma^2}{2} dN - \varepsilon \gamma x \cdot y - \varepsilon x \cdot \nabla U_N(x)
$$

$$
\leq - (\gamma - \varepsilon - \gamma^2 \varepsilon) |y|^2 - \varepsilon (\eta - \sqrt{\varepsilon}) |x|^2 + \frac{\sigma^2}{2} dN.
$$

Note that on the other hand $U_N(x) \leq (\|\nabla^2 V\|_\infty + 2\|\nabla^2 W\|_\infty) |x|^2$. For $\varepsilon$ small enough (and independent from $t$ and $N$) we thus have

$$
L_N H \leq -c_1 H + c_2 N
$$

where $c_1$ and $c_2$ are independent from $t$ and $N$. The Grönwall Lemma yields

$$
\mathbb{E}(H(X(t), Y(t))) \leq \mathbb{E}(H(X(0), Y(0))) + \frac{c_2 N}{c_1} \leq K N
$$

where we used interchangeability together with $U_N(x) \leq K |x|^2$. We conclude the case of $(X, Y)$ with interchangeability again and $U_N(x) \geq \eta |x|^2$.

Now let $(\overline{X}, \overline{Y})$ solve (3) and write

$$
\overline{U}_N(x) = \sum_{i=1}^N \int U(x_i, v)m_t(v, w)dvdw,
$$

(11)
so that
\[ \partial_t \mathbb{E} (H \left( \overline{X}(t), \overline{Y}(t) \right)) = \]
\[ \mathbb{E} \left( \left( \mathcal{L}_N + \nabla (U_N - \overline{U}_N) \nabla_y \right) H \left( \overline{X}(t), \overline{Y}(t) \right) \right) \]
\[ \leq -\frac{1}{2} c_1 \mathbb{E} \left( H \left( \overline{X}(t), \overline{Y}(t) \right) \right) + c_2 \mathbb{E} \left( \left| \nabla (U_N - \overline{U}_N) \right|^2 \left( \overline{X}(t), \overline{Y}(t) \right) \right) \]

for some \( c_3 \) independent from \( t \) and \( N \). The \((\overline{X}_i, \overline{Y}_i)\)'s being independent with law \( m_t \),
\[ \mathbb{E} \left( \left| \nabla (U_N - \overline{U}_N) \right|^2 \left( \overline{X}(t), \overline{Y}(t) \right) \right) \]
\[ = \frac{1}{N} \mathbb{E} \left( \left| \sum_{j=1}^N \nabla W(\overline{X}_1 - \overline{X}_j) - \int \nabla W(\overline{X}_1 - u) m_t(u, v) \right|^2 \right) \]
\[ = \frac{1}{N} \mathbb{E} \left( \left| \sum_{j=1}^N \nabla W(\overline{X}_1 - \overline{X}_j) - \int \nabla W(\overline{X}_1 - u) m_t(u, v) \right|^2 \right) \]
\[ \leq \frac{\| \nabla^2 W \|_\infty^2}{N} \mathbb{E} \left( \left| \overline{X}(t) \right|^2 \right). \] (12)

As a consequence for \( N \) large enough
\[ \partial_t \mathbb{E} \left( H \left( \overline{X}(t), \overline{Y}(t) \right) \right) \leq -\frac{c_1}{4} \mathbb{E} \left( H \left( \overline{X}(t), \overline{Y}(t) \right) \right) + c_2 N \]
and the conclusion is similar to the case of \((X, Y)\). \( \square \)

In a first instance, from a classical strategy (the parallel coupling of \( Z_N \) and \( \overline{Z}_N \)), we prove a propagation of chaos estimate which badly behaves in time:

**Proposition 12.** Under Assumptions \( \text{[1]} \) there exist \( b > 0 \) (depending only on \( U, \gamma \) and \( \sigma \)) and \( K > 0 \) (depending only on \( U, \gamma, \sigma \) and \( m_0 \)) such that for all \( N \geq 1 \), \( t > 0 \), if \( Z_N \) and \( \overline{Z}_N \) respectively solve Equations \( \text{[2]} \) and \( \text{[3]} \) driven by the same Brownian motion, then
\[ \mathbb{E} \left( \left| Z_N(t) - \overline{Z}_N(t) \right|^2 \right) \leq Ke^{bt}. \]

Note that by interchangeability, this reads
\[ \mathbb{E} \left( \left| X_1(t) - \overline{X}_1(t) \right|^2 + \left| Y_1(t) - \overline{Y}_1(t) \right|^2 \right) \leq \frac{Ke^{bt}}{N} \]
and, recalling that \( m_t^{(1,N)} \) stands for the law of \((X_1, Y_1)\), it implies
\[ \mathcal{W}_2 \left( m_t^{(1,N)}, m_t \right) \leq \frac{Ke^{bt}}{N}. \]

**Proof.** Let \((x, y) = (X - \overline{X}, Y - \overline{Y})\). The potential \( U \) being Lipschitz, it is clear there exists \( b' > 0 \) such that
\[ \partial_t \left( |x|^2 + |y|^2 \right) \leq b' \left( |x|^2 + |y|^2 \right) \]
\[ - \frac{2}{N} \sum_{i,j=1}^N y_i \left( \nabla W(X_i - X_j) - \int W(X_i - u) m_t(u, v) \right). \]
Decomposing the last factor in
\[
(\nabla W(X_i - X_j) - \nabla W(X_i - X_j))
+ \left( \nabla W(X_i - X_j) - \int W(X_i - u)m_t(u, v) \right),
\]
and using \( W \) is Lipschitz, the bound \eqref{prop12} and the moment estimate of Lemma \ref{lem11}, we obtain
\[
\partial_t \mathbb{E} (|x|^2 + |y|^2) \leq b \mathbb{E} (|x|^2 + |y|^2) + K
\]
for some \( b > 0 \), which concludes. \( \square \)

From this first rough estimate, together with the convergence in large time of the particle system given by Theorem \ref{thm1}, we obtain a first result in large time for the non-linear process:

**Proposition 13.** Under Assumptions \ref{ass1}, there exists \( K > 0 \) depending only on \( U, \gamma, \sigma \) and \( m_0 \) such that for all \( t > 0 \),
\[
\mathcal{W}_2^2 (m_t, m_\infty) \leq Ke^{-\chi t}.
\]
where \( \chi \) is given by Theorem \ref{thm1}.

**Proof.** From the bound \eqref{prop12}, together with Proposition \ref{prop12}, Lemma \ref{lem8} and interchangeability, it comes
\[
\mathcal{W}_2^2 (m_t, m_\infty) = \frac{1}{N} \mathcal{W}_2^2 (m_t^{\otimes N}, m_\infty^{\otimes N})
\leq \frac{1}{N} \left( \mathcal{W}_2 (m_t^{\otimes N}, m_t^{(N)}) + \mathcal{W}_2 (m_t^{(N)}, m_\infty^{(N)}) + \mathcal{W}_2 (m_\infty^{(N)}, m_\infty^{\otimes N}) \right)^2
\leq \frac{K}{N} \left( e^{bt} + Ne^{-\chi t} \right)
\]
and we can let \( N \) go to infinity. \( \square \)

Combining our previous propagation of chaos results (at equilibrium, or with an exponential prefactor) together with the convergence in large time ones (for the particle system and for the non-linear process) we can now prove the claimed uniform in time propagation of chaos result:

**Proof of Theorem 3.** According to Proposition \ref{prop12} for \( t \leq \varepsilon \ln N \) for some \( \varepsilon > 0 \),
\[
\mathcal{W}_2^2 (m_t^{(1,N)}, m_t) \leq \frac{K}{N^{1-b\varepsilon}}.
\]
For \( t \geq \varepsilon \ln N \), according to Proposition \ref{prop13} Theorem \ref{thm1} and Lemma \ref{lem8}
\[
\mathcal{W}_2^2 (m_t^{(1,N)}, m_t) \leq \frac{1}{N} \left( \mathcal{W}_2 (m_t^{(N)}, m_t^{(N)}) + \mathcal{W}_2 (m_t^{(N)}, m_\infty^{(N)}) + \mathcal{W}_2 (m_\infty^{(N)}, m_t^{\otimes N}) \right)^2
\leq K \left( \frac{1}{N^{\varepsilon \chi}} + \frac{1}{N} \right).
\]
Taking \( \varepsilon = (\chi + b)^{-1} \), we obtain the result with \( \alpha = (1 + b/\chi)^{-1} \). \( \square \)

One could expect the result to hold with \( \alpha = \frac{1}{2} \), which is the case in the space-homogeneous McKean-Vlasov equation. Nevertheless, since the \( b \) obtained in the proof of Proposition \ref{prop12} is clearly greater than 1, the best lower bound we could get on \( \alpha \) would be less than \( \chi \) given in \eqref{prop12}, which is pretty bad. Note that in the quadratic case, the results of \cite{3} applies, so that Theorem 3 holds with \( \alpha = \frac{1}{2} \).
5 The Vlasov-Fokker-Planck equation

Lemma 14. Under Assumptions 7 there exist $K$ depending only on $U, \gamma, \sigma$ and $m_0$ such that for all $N \geq 1, t > 0$,

$$\|m^{(1,N)}_t - m_t\|_1 \leq \frac{K \sqrt{t}}{N^\alpha}$$

where $\alpha$ is given by Theorem 13.

**Proof.** We follow the idea of [13, Lemma 3.15], namely we compute the derivative of

$$F(t) = \mathcal{H}(m^{(N)}_t | m^{\otimes N}_t).$$

To do so, let $u_1 = m^{(N)}_t, u_2 = m^{\otimes N}_t$,

$$b_1(x,y) = \left(-\gamma y - \frac{y}{\nabla_x U_N(x)}\right), \quad b_2(x,y) = \left(-\gamma y - \frac{y}{\nabla_x U_N(x)}\right)$$

(where $U_N$ is given by (11)) and $L_i f = -\nabla \cdot (b_i f) + \frac{\sigma_i^2}{2} \Delta_y f$ for $i = 1, 2$. With these notations, $\partial_t(u_i) = L_i u_i$, and the dual in the Lebesgue sense of $L_i$ is $L'_i = b_i \cdot \nabla + \frac{\sigma_i^2}{2} \Delta_y$. From the conservation of the mass of $u_1$, we get

$$0 = \partial_t \left( \int \frac{u_1}{u_2} u_2 \right) = \int \left( L_1 u_1 - \frac{u_1}{u_2} L_2 u_2 + L'_2 \left( \frac{u_1}{u_2} \right) u_2 \right).$$

Since $L'_1$ is a diffusion operator with carré du champ operator $\Gamma f = \frac{\sigma_i^2}{2} |\nabla_y f|^2$ (see [11] p.20 & 42 for the definitions),

$$u_1 L'_1 \ln \left( \frac{u_1}{u_2} \right) = u_1 \frac{L'_1 \left( \frac{u_1}{u_2} \right)}{u_2} - u_1 \frac{\Gamma \left( \frac{u_1}{u_2} \right)}{u_2} = u_2 L'_1 \left( \frac{u_1}{u_2} \right) - u_1 \Gamma \left( \ln \frac{u_1}{u_2} \right).$$

Using both these relations,

$$\partial_t \left( \int \ln \left( \frac{u_1}{u_2} \right) u_1 \right) = \int \left( \frac{L_1 u_1}{u_2} - \frac{L_2 u_2}{u_2} + L'_1 \ln \left( \frac{u_1}{u_2} \right) \right) u_1$$

$$= \int -\Gamma \left( \frac{u_1}{u_2} \right) u_1 + u_2 L'_1 \left( \frac{u_1}{u_2} \right) - u_2 L_2' \left( \frac{u_1}{u_2} \right)$$

$$= \int -\Gamma \left( \frac{u_1}{u_2} \right) u_1 + (b_1 - b_2) \cdot \nabla \ln \left( \frac{u_1}{u_2} \right) u_1.$$

Applying Young’s Inequality, we get

$$F'(t) \leq \frac{1}{2\sigma^2} \int \left| \nabla U_N(x) - \nabla U_N(x) \right|^2 m^{(N)}_t$$

$$= \frac{N}{2\sigma^2} \mathbb{E} \left( \frac{1}{N} \sum_{j=1}^{N} \nabla W(X_1 - X_j) - \int \nabla W(X_1 - v)m_t(v,w) \right)^2$$

by interchangeability. Developing the square of the sum, the diagonal terms are bounded by

$$\|\nabla^2 W\|_\infty^2 \left( \mathbb{E} (|X_j|^2) + \int |v|^2 m_t(v,w) \right) \leq K$$
where we used Lemma 11. For the extra-diagonal terms, we consider an optimal coupling $Z_N = (X, Y)$ and $Z_N = (X, Y)$ in the sense the law of $Z_N$ is $m_t^{\otimes N}$ and
\[
\mathbb{E} \left( \left| Z_N(t) - Z_N(t) \right|^2 \right) = W_2^2 \left( m_t^{\otimes N}, m_t^{(N)} \right)
\]
and write
\[
\left( \nabla W(X_1 - X_j) - \int \nabla W(X_1 - v)m_t \right) \left( \nabla W(X_1 - X_k) - \int \nabla W(X_1 - v)m_t \right)
\]
\[
= \left( \nabla W(X_1 - X_j) - \nabla W(X_1 - X_j) \right) \left( \nabla W(X_1 - X_k) - \int \nabla W(X_1 - v)m_t \right)
\]
\[
+ \left( \nabla W(X_1 - X_j) - \int \nabla W(X_1 - v)m_t \right) \left( \nabla W(X_1 - X_k) - \int \nabla W(X_1 - X_k) \right)
\]
\[
+ \left( \nabla W(X_1 - X_j) - \int \nabla W(X_1 - X_k) \right) \left( \nabla W(X_1 - X_k) - \int \nabla W(X_1 - v)m_t \right).
\]
The $X_i$'s being independent with law the first marginal of $m_t$, the expectation of the third term vanishes, while the expectations of the other terms is bounded by the Cauchy-Schwarz inequality, interchangeability and Theorem 3 by
\[
\left\| \nabla^2 W \right\|_\infty \sqrt{\mathbb{E} \left( |X_1 - \overline{X}_1|^2 \right) \left( \mathbb{E} \left( |X_1|^2 \right) + \mathbb{E} \left( |\overline{X}_1|^2 \right) \right)} \leq \frac{K}{N^{\frac{2}{3}}},
\]
Hence $F'(t) \leq KN^{1-\frac{2}{3}}$ and moreover $F(0) = 0$, so that we conclude by the Csiszár’s inequality which reads
\[
\mathcal{H} \left( \mu^{(1,N)} | \nu \right) \leq \frac{2}{N} \mathcal{H} \left( \mu^{(N)} | \nu^{\otimes N} \right)
\]
when $\mu^{(N)}$ is an interchangeable probability law (see [1]) and the Pinsker’s one.

From this last propagation of chaos estimate (with a not-so-bad behaviour in time) together with the convergence in large time of the particle system, we are finally able to recover the convergence of the total variation distance for the Vlasov-Fokker-Planck equation:

**Proof of Theorem 4.** From the Pinsker’s inequality, Theorem 1 and Lemma 9
\[
\|m_t^{(1,N)} - m_t^{(1,N)}\|_1 = \sup_{f \in L^\infty(\mathbb{R}^d)} \int f \left( m_t^{(1,N)} - m_t^{(1,N)} \right)
\]
\[
\leq \sup_{f \in L^\infty(\mathbb{R}^{dN})} \int f \left( m_t^{(N)} - m_t^{(N)} \right)
\]
\[
= \|m_t^{(N)} - m_t^{(N)}\|_1
\]
\[
\leq 2\mathcal{H} \left( m_t^{(N)} | m_t^{(N)} \right)
\]
\[
\leq KN e^{-\chi t}.
\]
By Lemmas 8 and 14
\[
\|m_t - m_\infty\|_1 \leq \|m_t - m_t^{(1,N)}\|_1 + \|m_t^{(1,N)} - m_t^{(1,N)}\|_1 + \|m_t^{(1,N)} - m_\infty\|_1
\]
\[
\leq \frac{K \sqrt{t}}{N^{\frac{1}{2}}} + \sqrt{K N e^{-\frac{t}{2}}} + \frac{K}{\sqrt{N}}
\]
and we take $N$ of order $e^{\frac{2t}{\alpha+2}}$.  

\[\square\]
Remarks: in turn this leads to a self-improvement of Lemma 14 by writing
\[
\|m_t^{(1,N)} - m_t\|_1 \leq \|m_t^{(1,N)} - m_\infty^{(1,N)}\|_1 + \|m_\infty - m^{(1,N)}\|_1 + \|m_t - m_\infty\|_1 \\
\leq K \left( e^{-\frac{1}{2} \chi t} + \frac{1}{N} \right)
\]
which, used for, say, \( t \geq N^{\frac{2}{\chi}} \) while Lemma 14 is used for small times, reads
\[
\|m_t^{(1,N)} - m_t\|_1 \leq \frac{K}{N^{\frac{2}{\chi}}}
\]
for all \( N \geq 1, \ t > 0. \)

In the quadratic case, we obtain a rate of convergence equal to \( \frac{1}{t} \chi \), where \( \chi \) is given at the end of Section 2.

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References

[1] D. Bakry, I. Gentil, M. Ledoux, Analysis and geometry of Markov diffusion operators, Vol. 348 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer, Cham, 2014. doi:10.1007/978-3-319-00227-9, URL http://dx.doi.org/10.1007/978-3-319-00227-9

[2] F. Bolley, I. Gentil, Phi-entropy inequalities for diffusion semigroups, J. Math. Pures Appl. (9) 93 (5) (2010) 449–473. doi:10.1016/j.matpur.2010.02.004, URL http://dx.doi.org/10.1016/j.matpur.2010.02.004

[3] F. Bolley, I. Gentil, A. Guillin, Uniform convergence to equilibrium for granular media, Arch. Ration. Mech. Anal. 208 (2) (2013) 429–445. doi:10.1007/s00205-012-0599-z, URL http://dx.doi.org/10.1007/s00205-012-0599-z

[4] F. Bolley, A. Guillin, F. Malrieu, Trend to equilibrium and particle approximation for a weakly selfconsistent Vlasov-Fokker-Planck equation, M2AN Math. Model. Numer. Anal. 44 (5) (2010) 867–884. doi:10.1051/m2an/2010045, URL http://dx.doi.org/10.1051/m2an/2010045

[5] F. Bolley, A. Guillin, C. Villani, Quantitative concentration inequalities for empirical measures on non-compact spaces, Probab. Theory Related Fields 137 (3-4) (2007) 541–593. doi:10.1007/s00440-006-0004-7, URL http://dx.doi.org/10.1007/s00440-006-0004-7

[6] I. Csiszár, Sanov property, generalized I-projection and a conditional limit theorem, Ann. Probab. 12 (3) (1984) 768–793. URL http://links.jstor.org/sici?sici=0091-1798(198408)12:3<768:SPGAAC>2.0.CO;2-Y

[7] J. Dolbeault, C. Mouhot, C. Schmeiser, Hypocoercivity for kinetic equations with linear relaxation terms, C. R. Math. Acad. Sci. Paris 347 (9-10) (2009) 511–516. doi:10.1016/j.crma.2009.02.025, URL http://dx.doi.org/10.1016/j.crma.2009.02.025
[8] M. H. Duong, J. Tugaut, Stationary solutions of the Vlasov-Fokker-Planck equation: existence, characterization and phase transition, Appl. Math. Lett. 52 (2016) 38–45. doi:10.1016/j.aml.2015.08.003. URL http://dx.doi.org/10.1016/j.aml.2015.08.003

[9] R. Esposito, Y. Guo, R. Marra, Stability of the front under a Vlasov-Fokker-Planck dynamics, Arch. Ration. Mech. Anal. 195 (1) (2010) 75–116. doi:10.1007/s00205-008-0184-7. URL http://dx.doi.org/10.1007/s00205-008-0184-7

[10] S. Gadat, L. Miclo, Spectral decompositions and $L^2$-operator norms of toy hypocoercive semi-groups, Kinet. Relat. Models 6 (2) (2013) 317–372.

[11] F. Hérau, Short and long time behavior of the Fokker-Planck equation in a confining potential and applications, J. Funct. Anal. 244 (1) (2007) 95–118. doi:10.1016/j.jfa.2006.11.013. URL http://dx.doi.org/10.1016/j.jfa.2006.11.013

[12] F. Hérau, L. Thomann, On global existence and trend to the equilibrium for the Vlasov-Poisson-Fokker-Planck system with exterior confining potential, ArXiv e-print arXiv:1505.01698.

[13] F. Malrieu, Logarithmic Sobolev inequalities for some nonlinear PDE’S, Stochastic Process. Appl. 95 (1) (2001) 109–132. doi:10.1016/S0304-4149(01)00095-3. URL http://dx.doi.org/10.1016/S0304-4149(01)00095-3

[14] S. Méléard, Asymptotic behaviour of some interacting particle systems; McKean-Vlasov and Boltzmann models, in: Probabilistic models for nonlinear partial differential equations (Montecatini Terme, 1995), Vol. 1627 of Lecture Notes in Math., Springer, Berlin, 1996, pp. 42–95. doi:10.1007/BFb0093177. URL http://dx.doi.org/10.1007/BFb0093177

[15] P. Monmarché, Hypocoercivity in metastable settings and kinetic simulated annealing, ArXiv e-print arXiv:1502.07263.

[16] P. Monmarché, Generalized Γ calculus and application to interacting particles on a graph, ArXiv e-print arXiv:1510.05936.

[17] A. Scemama, T. Lelièvre, G. Stoltz, M. Caffarel, An efficient sampling algorithm for variational monte carlo, Journal of Chemical Physics 125.

[18] A.-S. Sznitman, Topics in propagation of chaos, in: École d’Été de Probabilités de Saint-Flour XIX—1989, Vol. 1464 of Lecture Notes in Math., Springer, Berlin, 1991, pp. 165–251. doi:10.1007/BFb0085169. URL http://dx.doi.org/10.1007/BFb0085169

[19] D. Talay, Stochastic Hamiltonian systems: exponential convergence to the invariant measure, and discretization by the implicit Euler scheme, Markov Process. Related Fields 8 (2) (2002) 163–198, inhomogeneous random systems (Cergy-Pontoise, 2001).

[20] C. Villani, Hypocoercivity, Mem. Amer. Math. Soc. 202 (950) (2009) iv+141. doi:10.1090/S0065-9266-09-00567-5. URL http://dx.doi.org/10.1090/S0065-9266-09-00567-5