Exact Solution of Heisenberg-liquid models with long-range coupling

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We present the exact solution of two Heisenberg-liquid models of particles with arbitrary spin $S$ interacting via a hyperbolic long-range potential. In one model the spin-spin coupling has the simple antiferromagnetic Heisenberg exchange form, while for the other model the interaction is of the ferromagnetic Babujian-Takhatajan type. It is found that the Bethe ansatz equations of these models have a similar structure to that of the Babujian-Takhatajan spin chain. We also conjecture the integrability of a third new spin-lattice model with long-range interaction.

I. INTRODUCTION

Spin fluctuations are a common feature of strongly correlated electron systems such as high $T_c$ superconductors and heavy fermion compounds. Exchange interactions of the Heisenberg type, originally introduced to describe the properties of insulating magnets, have now been realized playing a central role in doped Mott insulators and heavy fermion systems. Two related models are the $t - J$ model and the Kondo lattice model. In these systems, the spin exchange among the itinerant electrons or between the conduction electrons and the local moments induce fascinating physical phenomena, such as very large mass enhancements and non-Fermi liquid behavior. For some low-carrier-density systems, especially in low dimensions, the small number of carriers and their reduced mobility do not provide an effective mechanism for screening and long-range interactions should be considered. A special class of integrable systems with long-range interactions are the Calogero-Sutherland model (CSM)\cite{1,2} and its $SU(N)$ generalizations\cite{3-5}. The lattice versions of these models, which generically are referred to as Haldane-Shastry models\cite{6}, can be obtained by freezing the orbital dynamics within a proper scheme. In this letter, we present two integrable models of particles with arbitrary spin interacting via a Calogero-Sutherland potential with a hyperbolic space dependence. In case I the spin-spin coupling is of the simple Heisenberg exchange form, i.e. $\vec{S}_i \cdot \vec{S}_j$, while in case II, the interaction is of the Babujian-Takhatajan model (BTM) type. Finally, based on the structure of the Bethe ansatz equations (BAE), an $SU(2)$-invariant high-spin chain model, which corresponds to the long-range-coupling generalization of the BTM\cite{8}, is conjectured to be integrable.

The structure of the present paper is the following: In the subsequent section, we construct the model Hamiltonians. The energy spectra and the BAE are derived via the so-called asymptotic Bethe ansatz (ABA)\cite{2}. In sect.III, we study the thermodynamics as well as the ground state properties. Concluding remarks are given in sect.IV.

II. MODELS AND BETHE ANSATZ

The general form of the model Hamiltonian we shall consider is

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2 \sum_{i<j}^{N} \frac{\gamma^2 g_{ij}}{\sinh^2 \gamma(x_i - x_j)},$$  \hspace{1cm} (1)

where $x_j$ is the position of the particle $j$, $\gamma > 0$ is a real constant, and $g_{ij}$ is a spin-dependent coupling, which for our two cases is:

$$g_{ij} = 2\vec{S}_i \cdot \vec{S}_j + 2S(S + 1),$$  \hspace{1cm} (2)

for model I and

$$g_{ij} = -4SQ_2S(\vec{S}_i \cdot \vec{S}_j) \equiv [2S - \hat{l}_{ij}][2S + 1 - \hat{l}_{ij}],$$  \hspace{1cm} (3)

for model II, where
\[ \hat{I}_{ij} = \sqrt{\frac{1}{4} + (\vec{S}_i + \vec{S}_j)^2 - \frac{1}{2}} \]

which has eigenvalues 0, \ldots, 2S and \( Q_{2S} \) is just the Babujian polynomial. Hence, in case I the spin coupling is a simple antiferromagnetic exchange, while in case II it is ferromagnetic and of the Babujian-Takhatajan form. It is interesting to compare model I to the \( t - J \) model. Their physics is quite similar, although in the present model the coordinate \( x_j \) is continuous while it is restricted to a lattice in the \( t - J \) model, and that here the interaction is of variable range. For some high \( T_c \) compounds, the next-nearest-neighbor spin exchange is believed to be relevant. In the present model the range of the interaction is governed by the free parameter \( \gamma \). For instance, for sufficiently large \( \gamma \) the interaction is short-ranged, while the long-range Calogero-Sutherland interaction is readily recovered in the limit \( \gamma \to 0 \). We note that in both cases, I and II, the total spin is conserved in the two-body scattering processes and the Hamiltonian has a basic \( SU(2) \)-invariance.

The asymptotic Bethe ansatz (ABA) is a powerful tool to derive the energy spectrum for models with nonlocal interactions, like e.g. the CSM. If the model is integrable it is sufficient to know the asymptotic long-distance behavior, i.e. the scattering phase shifts, which can be obtained without the full knowledge of the wave functions. The ABA also assumes that the particles are asymptotically free, i.e., that they do not form charge bound states. The ABA only needs the two-body phase shifts (or two-body scattering matrix), but requires an independent proof of the integrability of the model Hamiltonian. The integrability of the present model is related to the unitary matrix model of Minahan and Polychronakos who have shown that the CSM with a coupling strength \( g_{ij} = J_{ij}J_{ji} \) is exactly soluble. In Ref. [ J_{ij} \) was chosen to be the \( SU(q) \) angular momentum. For the 2S-fold \( SU(2) \) case, we can map \( J_{ij}J_{ji} \) to \((\vec{S}_i + \vec{S}_j)^2\), which is of more interest to condensed matter problems and corresponds to our case I. In fact, a potential \( \vec{J}^2/r^2 \) can be generated naturally from the dynamics of two free particles in three dimensions in spherical coordinates. Such a potential describes the angular dynamics of the center of mass of two free particles. If the angular momentum is considered a “spin”, the problem is reduced to a one-dimensional one and the scattering matrix must be factorizable, which already infers the integrability of many particles in one dimension with such a potential. Hence, for case I our choice of \( g_{ij} \) is quite natural. On the other hand, case II can be realized by a proper quantization of \( J_{ij} \) using projection operators.

To derive the two-body \( S \)-matrix, we note that the two-particle scattering occurs in \( 2S + 1 \) different channels of total spin \( l \) ranging from 0 to 2S. In a given channel of spin \( l \), the coupling strength is \( l(l+1) \) for case I and \((2S-l)(2S-l+1)\) for case II. For convenience we will express our results in terms of the operator \( \hat{I}_{ij} \) defined above. Below we study the two cases separately.

### A. Model I

The two-body transmission \( S \)-matrix can be derived following Sutherland’s method. Consider the two-particle case. The asymptotic wave function for \( |x_1 - x_2| \to \infty \) can be written as

\[
\Psi(x_1, x_2) \to A_{12}(12)e^{i k_1 x_1 + i k_2 x_2} + A_{12}(21)e^{i k_2 x_1 + i k_1 x_2},
\]

for \( x_1 < x_2 \) and

\[
\Psi(x_1, x_2) \to A_{21}(21)e^{i k_1 x_1 + i k_2 x_2} + A_{21}(12)e^{i k_2 x_1 + i k_1 x_2},
\]

for \( x_1 > x_2 \), where \( k_1, k_2 \) are the asymptotic momenta carried by the particles and the \( A \)'s are constant coefficients, which determine the \( S \)-matrices. For example, \( S_{1,2}(k_1 - k_2) = A_{21}(21)/A_{12}(12) \). By solving the Schrödinger equation for two particles we obtain

\[
S_{i,j}(k_1 - k_j) = -\frac{\Gamma(1 + i k_{ij}) \Gamma(1 + \hat{I}_{ij} - i k_{ij})}{\Gamma(1 - i k_{ij}) \Gamma(1 + \hat{I}_{ij} + i k_{ij})} P_{ij},
\]

where \( k_{ij} = (k_i - k_j)/2\gamma \), \( \Gamma(x) \) is the gamma function and \( P_{ij} \) is the spin permutation operator. In fact, \( S_{i,j} \) is a polynomial of \( \vec{S}_i \cdot \vec{S}_j \) of order \( 2S \) and is proportional to the Lax-operator of the BTM. Hence, it satisfies the Yang-Baxter relation. Hence, in case II we have

\[
S_{i,j} = \frac{k_{ij} + i(1/2 + 2\vec{S}_i \cdot \vec{S}_j)}{k_{ij} - i}.
\]
while for $S = 1$ it is

$$S_{i,j} = - \frac{2\vec{S}_i \cdot \vec{S}_j (1 - ik_{ij} + \vec{S}_i \cdot \vec{S}_j) - k_{ij}^2 + ik_{ij} - 2}{(ik_{ij} + 1)(ik_{ij} + 2)}. \quad (8)$$

Because the system is non-diffractive, the wave function must asymptotically be given by plane waves $\Psi(x|Q) \sim \sum_p \Psi(P|Q) \exp[i \sum_{j=1}^N x_{Q,j} k_{P,j}]$ when $|x_i - x_j| \to \infty$ for all pairs $i, j$, where $x_{Q,i} < \cdots < x_{Q,N}$, and $P$ and $Q$ are permutations of $(1, 2, \cdots, N)$. With the periodic boundary conditions of the wave function, the momenta $k_j$ are determined by a set of Yang eigenvalue equations.

$$S_{j,j-1} \cdots S_{j,1} S_{j,N} \cdots S_{j,j+1} \psi(P|Q) = e^{-ik_j L} \psi(P|Q). \quad (9)$$

For convenience, we introduce an auxiliary spin $\vec{S}_0$ and define $S_{0,j} = S_{0,j}(\lambda - k_j)$, so that $S_{0,j}(0) = P_{0j}$. Further we define the $(2S + 1) \times (2S + 1)$ monodromy matrix

$$T_0(\lambda) = S_{0,j-1} \cdots S_{0,1} S_{0,N} \cdots S_{0,j+1}. \quad (10)$$

By summing over the auxiliary spin of the matrix $T_0(\lambda)$, it is easy to show that the trace

$$tr_0 T_0(k_j) = S_{j,j-1} \cdots S_{j,1} S_{j,N} \cdots S_{j,j+1}$$

is just the transfer matrix of the BTM model. The eigenvalue problem Eq.(9) can be solved following the standard method of the algebraic Bethe ansatz. The first set of Eqs. (11) is determined by the eigenvalue of the transfer matrix $tr_0 T_0(k_j) = \exp(-ik_j L)$, while the second set arises as the condition that ensures that the Bethe states are eigenstates of the Hamiltonian. The energy eigenvalues are given by

$$E = \sum_{j=1}^N k_j^2, \quad (12)$$

by acting the Hamiltonian on the asymptotic wave function.

**B. Model II**

For our second model the two-body $S$-matrix is given by

$$S_{i,j} = \frac{\Gamma(1 + ik_{ij}) \Gamma(1 + 2S - \hat{\lambda}_{ij} - ik_{ij})}{\Gamma(1 - ik_{ij}) \Gamma(1 + 2S - \hat{\lambda}_{ij} + ik_{ij})} \rho_{ij}. \quad (13)$$

With the same procedure, we obtain the BAE as

$$e^{ik_j L} = \prod_{\alpha=1}^M \frac{k_j - \lambda_{\alpha} - 2iS\gamma}{k_j - \lambda_{\alpha} + 2iS\gamma}, \quad \prod_{j=1}^N \frac{\lambda_{\alpha} - k_j - 2iS\gamma}{\lambda_{\alpha} - k_j + 2iS\gamma} = \prod_{\beta, \beta \neq \alpha}^M \frac{\lambda_{\alpha} - \lambda_{\beta} - 2i\gamma}{\lambda_{\alpha} - \lambda_{\beta} + 2i\gamma}. \quad (14)$$

The energy eigenvalues are still given by Eq.(12).
III. THERMODYNAMICS

A. model I

Since $g_{ij} \geq 0$, the potential is repulsive and no charge bound state can exist. This can be seen clearly in the two-particle case. In fact, the two-particle wave function can be solved exactly in terms of two hypergeometric functions which has no bound state solution in the parameter region we considered. The solutions of the BAE, Eqs.(11), are therefore classified into real charge rapidities $k_i$ and strings of length $n$ of spin-rapidities $\eta_i$. In the thermodynamic limit $L \to \infty, N \to \infty, N/L \to n_c$, we denote with $\rho(k)$, $\sigma_n(\lambda)$ and $\rho_n(k)$, $\sigma^h_n(\lambda)$ the rapidity densities and the densities of their holes, respectively. By taking the thermodynamic limit of the BAE, we have

$$\frac{1}{2\pi} = \rho(k) + \rho_n(k) + \hat{B}' \rho(k) - \sum_n \hat{B}_{2S,n} \sigma_n(k),$$

$$\sigma^h_n(\lambda) + \sum_{m=1}^{\infty} \hat{A}_{m,n} \sigma_n(\lambda) = \hat{B}_{2S,n} \rho(\lambda),$$

where the integral operators are given by $\hat{B}' = \sum_{m=1}^{2S}[2m], \hat{B}_{2S,n} = \sum_{l=1}^{\min(2S,n)}[2S + n + 1 - 2l], \hat{A}_{m,n} = [m + n] + 2[m + n - 2] + \cdots + 2[m - n] + 2[n],$ and $[n]$ is the integral operator with the kernel

$$a_n(k) = \frac{n \gamma}{\pi(k^2 + n^2 \gamma^2)}.$$  

The free energy is expressed as

$$F/L = \int (k^2 - \mu - Sh) \rho(k) dk + \int n h \int \sigma_n(\lambda) d\lambda$$

$$- T \int [(\rho(k) + \rho_n(k)) \ln(\rho(k) + \rho_n(k)) - \rho(k) \ln \rho(k) - \rho_n(k) \ln \rho_n(k)] dk$$

$$- T \sum_n \int [(\sigma_n(\lambda) + \sigma^h_n(\lambda)) \ln(\sigma_n(\lambda) + \sigma^h_n(\lambda)) - \sigma_n(\lambda) \ln \sigma_n(\lambda) - \sigma^h_n(\lambda) \ln \sigma^h_n(\lambda)] d\lambda.$$  

where $\mu, T$ and $h$ are the chemical potential, the temperature and the magnetic field, respectively. It is convenient to introduce the quantities $\zeta(k) = \rho_n(k)/\rho(k)$ and $\eta_n(\lambda) = \sigma^h_n(\lambda)/\sigma_n(\lambda)$. By minimizing the free energy with respect to the particle densities, we obtain

$$\ln \zeta = \frac{k^2 - \mu - Sh}{T} + \hat{B}' \ln(1 + \zeta^{-1}) - \sum_{n=1}^{\infty} \hat{B}_{2S,n} \ln(1 + \eta_n^{-1}),$$

$$\ln(1 + \eta_n) = \frac{nh}{T} + \sum_{m=1}^{\infty} \hat{A}_{m,n} \ln(1 + \eta_n^{-1}) - \hat{B}_{2S,n} \ln(1 + \zeta^{-1}).$$

Substituting the above equations into Eq.(17), we obtain

$$F/L = - \frac{T}{2\pi} \int_{-\infty}^{\infty} \ln[1 + \zeta^{-1}(k)] dk.$$  

The thermodynamic BAE (18) can be further simplified. For convenience, we define $\hat{G} = [1]/([0] + [2])$. With the following operator identities

$$\hat{A}_{m,n} - \hat{G} [\hat{A}_{m,n+1} + \hat{A}_{m,n-1}] = \delta_{m,n},$$

$$\hat{A}_{1,m} - \hat{G} \hat{A}_{2,m} = \delta_{1,m},$$

$$\hat{B}_{m,n} - \hat{G} [\hat{B}_{m,n+1} + \hat{B}_{m,n-1}] = \delta_{m,n} \hat{G},$$

$$\hat{B}_{1,m} - \hat{G} \hat{B}_{2,m} = \delta_{1,m} \hat{G},$$

we can rewrite the second equation of Eqs.(18) as
\[
\ln \eta_n = -\delta_{n,2S} \hat{G} \ln(1 + \zeta^{-1}) + \hat{G} [\ln(1 + \eta_{n+1}) + \ln(1 + \eta_{n-1})],
\]

with the boundary conditions \(\eta_0 = 0\), and \(\lim_{n \to \infty} \ln \eta_n/n = h/T \equiv 2\pi_0\). From Eqs.\((20)\) obviously \(\hat{G} \hat{A}_{m,n} = \hat{B}_{m,n}\), so that from the second of Eqs.\((18)\) we have

\[
\hat{G} \ln(1 + \eta_{2S}) = \frac{Sh}{T} + \sum_m \hat{B}_{m,2S} \ln(1 + \eta_{m-1}^-) - \hat{G} \hat{B}_{2S,2S} \ln(1 + \zeta^{-1}).
\]

Substituting the above equation into the first equation of Eqs.\((18)\), we get

\[
\ln \zeta = \frac{k^2 - \mu}{T} + \hat{B}' \ln(1 + \zeta^{-1}) - \hat{G} [\hat{B}_{2S,2S} \ln(1 + \zeta^{-1}) + \ln(1 + \eta_{2S})]
\]

\[
= \frac{k^2 - \mu}{T} + \hat{B}'' \ln(1 + \zeta^{-1}) - \hat{G} \ln(1 + \eta_{2S}),
\]

where \(\hat{B}'' = \hat{G} \sum_{m=1}^{2S} [2m + 1]\).

For \(T \to 0\) and \(h \to 0\) we obtain from Eq.\((23)\) that \(\zeta \to 0\) for \(k < \sqrt{\mu}\) and \(\zeta \to \infty\) for \(k > \sqrt{\mu}\). This implies that the charges form a Fermi sea. In addition, in the same limit \(\eta_{2S} \to 0\) while all other \(\eta_n\) tend to some constant as indicated by Eq.\((21)\). Therefore, the spin quanta form a Fermi sea of \(2S\)-strings in the ground state, as in the case of the BTM. The ground state properties are then given in terms of \(\rho(k)\) and \(\sigma_{2S}(\lambda)\) (which describe the two Fermi seas). The rapidities are densely distributed in the intervals \([-K, K]\) and \([-\Lambda, \Lambda]\), respectively, where in zero-field \(\Lambda \to \infty\) for finite \(\gamma\), and the integral equations are

\[
\rho(k) = \frac{1}{2\pi} - \int_{-K}^{K} B'(k - k')\rho(k')dk' + \int_{-\Lambda}^{\Lambda} B_{2S,2S}(k - \lambda)\sigma_{2S}(\lambda)d\lambda,
\]

\[
\int_{-\Lambda}^{\Lambda} A_{2S,2S}(\lambda - \lambda')\sigma_{2S}(\lambda')d\lambda' = \int_{-K}^{K} B_{2S,2S}(k - \lambda)\rho(k)dk,
\]

where \(B'(k), B_{m,n}(k)\) and \(A_{m,n}(\lambda)\) are the kernels of \(\hat{B}', \hat{B}_{m,n}\) and \(\hat{A}_{m,n}\), respectively. The elementary excitations are uniquely determined by the dressed energies \(\epsilon_c(k)\) (of charges) and \(\epsilon_s(\lambda)\) (of spins), which for \(h \to 0\) satisfy

\[
\epsilon_c(k) = k^2 - \mu - \int_{-K}^{K} B'(k - k')\epsilon_c(k')dk' + \int_{-\Lambda}^{\Lambda} B_{2S,2S}(k - \lambda)\epsilon_s(\lambda)d\lambda,
\]

\[
\int_{-\Lambda}^{\Lambda} A_{2S,2S}(\lambda - \lambda')\epsilon_s(\lambda')d\lambda' = \int_{-K}^{K} B_{2S,2S}(k - \lambda)\epsilon_c(k)dk,
\]

The group velocities of the two bands are \(v_c = \epsilon'_c(K)/2\pi\rho(K)\) and \(v_s = \epsilon'_s(\Lambda)/2\pi\sigma_{2S}(\Lambda)\), where the prime denotes derivative with respect to the rapidity. By integrating the second of Eqs.\((25)\) for \(h = 0\) (\(\Lambda = \infty\)) we have \(N = 2M\), i.e. as expected the ground state is a spin singlet. At low \(T\) the system then is a two-component Luttinger liquid with charge dynamics of central charge 1 and spin dynamics with central charge 2S. The latter is very similar to the BTM.

In the Calogero-Sutherland limit, all the above quantities can be derived explicitly, since for \(\gamma \to 0\), the integral operator \([m]\) becomes a \(\delta\)-function,

\[
\rho(k) = 2\sigma_{2S}(k) = \frac{\theta(K - |k|)}{2\pi(S + 1)},
\]

\[
\epsilon_c(k) = 2\epsilon_s(k) = \frac{k^2 - K^2}{S + 1},
\]

\[
v_c = v_s = 2\pi n_c(S + 1),
\]

where \(\theta(x)\) is the step function and the Fermi momentum is \(K = \pi n_c(S + 1)\) with \(n_c = N/L\). Without magnetic field the particles occupy each spin state equally, so that the distribution function \(n_s(k)\) \((s = -S, \cdots, S)\) is that of an exclusion statistics with the statistical weight \(g = (S + 1)(2S + 1)\),

\[
n_s(k) = \frac{\theta(K - |k|)}{(S + 1)(2S + 1)}.
\]
B. Model II

The solutions of the BAE for model II are still given by real $k_j$ and $\lambda$-strings of length $n$. In the thermodynamic limit, the BAE, Eqs.(14), yield

\[
\frac{1}{2\pi} = \rho(k) + \rho_h(k) + \sum_{n=1}^{\infty} \hat{B}_{n,2S} \sigma_n(k),
\]

\[
\sum_{m=1}^{\infty} \hat{A}_{m,n} \sigma_m(\lambda) + \sigma_n^h(\lambda) = \hat{B}_{2S,n} \rho(\lambda).
\]

The free energy is still given by Eqs.(17) and (19). Using a similar procedure, i.e., by minimizing the the free energy functional with respect to $\rho(k)$ and $\sigma_n(\lambda)$, we readily obtain the thermal BAE of model II

\[
\ln \zeta = \frac{k^2 - \mu - Sh}{T} - \sum_{n=1}^{\infty} \hat{B}_{2S,n} \ln(1 + \eta_n^{-1}),
\]

\[
\ln(1 + \eta_n) = \frac{n h}{T} + \sum_{m=1}^{\infty} \hat{A}_{m,n} \ln(1 + \eta_m^{-1}) + \hat{B}_{2S,n} \ln(1 + \zeta^{-1}),
\]

or equivalently

\[
\ln \zeta = \frac{(k^2 - \mu)}{T} - \hat{G} \ln(1 + \eta_{2S}) - \hat{G} \hat{B}_{2S,2S} \ln(1 + \zeta^{-1}),
\]

\[
\ln \eta_n = \delta \hat{G} \ln(1 + \zeta^{-1}) + \hat{G} \ln([1 + \eta_{n+1}](1 + \eta_{n-1})],
\]

with the same boundary conditions as in Eq.(21). A strikingly different feature of this model is that when $T \to 0$, $h \to 0$, all $\eta_n \to \infty$, which indicates that there are no flipped spins in the ground state. Hence, the ground state is ferromagnetic, since all the electrons align along a spin direction. The rapidity distributions in the ground state are then

\[
\rho(k) = \frac{1}{2\pi} \theta(K - |k|),
\]

\[
\sigma_n(\lambda) = 0,
\]

where $K$ is the cutoff (integration limit or Fermi momentum) given by

\[
K = \pi n_e.
\]

The group velocity of the charge excitations at the Fermi surface is

\[
v_c = 2\pi n_e.
\]

The spin sector is no longer a Luttinger liquid, but has a quantum critical point. A single spin-wave excitation with momentum $p$, has an excitation energy $\epsilon_s(p) \sim p^2$ if $p$ is small. At low but finite $T$, the charge sector behaves almost like non-interacting spinless fermions, while for the spin sector we have a ferromagnetic Babujian-Takhatajan spin liquid, so that asymptotically as $T \to 0$ the specific heat and the spin susceptibility are

\[
C \sim T^\frac{1}{2}, \quad \chi \sim T^{-2} \ln^{-1} T.
\]

The low-$T$ specific heat can also be inferred from the spin-wave dispersion relation.

In the Calogero-Sutherland limit $\gamma \to 0$, both the charge and spin sectors, obey ideal fractional statistics. The thermal BAE can then be solved analytically, since in this limit the integral equations reduce to algebraic ones. Solving for $\eta_n$ separately for $n \leq 2S$ and $n \geq 2S$ we obtain

\[
\eta_n = \frac{\sin^2[(n + 1)\alpha]}{\sin^2 \alpha} - 1, \quad n \leq 2S,
\]

\[
\eta_n = \frac{\sinh^2[(n - 2S)x_0 + \beta]}{\sinh^2 x_0} - 1, \quad n \geq 2S,
\]
where α and β are functions of ζ(k) determined by
\[ \sin((2S + 1)\alpha)\sinh x_0 = \sinh \beta \sin \alpha, \]
\[ \sin(2S\alpha)\sqrt{1 + \zeta^{-1}} = \frac{\sin \alpha}{\sinh x_0} |\sinh(\beta - x_0)|. \] (36)

The solution of Eqs. (35) and (36) yields η_n as a function of ζ. Substituting the result into the first of Eqs. (30) we obtain a relation for ζ(k), which uniquely determines the free energy. The distribution function for the charges is
\[ n(k) = -\frac{\partial}{\partial \mu}\ln[1 + \zeta^{-1}(k)]. \] (37)

A careful calculation also shows that the logarithm in Eq. (34) disappears, i.e., \( \chi \sim T^{-2} \). This is due to the free nature of the particles in the limit \( \gamma \to 0 \).

### IV. CONCLUDING REMARKS

In conclusion, we constructed two integrable models with long-range spin couplings. In model I, the coupling between spins is of the antiferromagnetic exchange form, while in model II, the spin coupling is ferromagnetic of the Babujian-Takhtajan type. The BAE for the spin sector are the same as for the BTM (here the \( k_j \) introduce a lattice disorder). As we have learned from ordinary integrable models, almost all lattice models have a continuum counterpart in the sense that their BAE share a common structure. Examples are the spin-1/2 isotropic Heisenberg chain [15] with the δ-potential boson gas model [16] the Hubbard model [17] with the δ-potential Fermi gas model [10] and the \( SU(N) \)-invariant \( t-J \) model [18] with the \( SU(N) \)-invariant δ-potential Fermi gas model [19]. However, no continuum counterpart to the BTM with contact interaction was known so far. The present work fills this gap in the integrable model family.

Finally, we conjecture that the following lattice model
\[ H = \sinh^2 \gamma \sum_{m<n}^N \frac{Q(\vec{S}_m \cdot \vec{S}_n)}{\sinh^2[\gamma(m-n)]}, \] (38)
is integrable for the following reasons: (a) With the proper freezing process (eliminating the charge dynamics), the continuum models may be reduced to Eq. (38), since the BAE have the same form as for the BTM [20] (b) For \( \gamma \to \infty \), Eq. (38) flows to the BTM, which has been demonstrated to be integrable.

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