In the present paper the Ising model with competing binary \((J)\) and binary \((J_1)\) interactions with spin values \(±1\), on a Cayley tree of order 2 is considered. The structure of Gibbs measures for the model considered is studied. We completely describe the set of all periodic Gibbs measures for the model with respect to any normal subgroup of finite index of a group representation of the Cayley tree. Types of von Neumann algebras, generated by GNS-representation associated with diagonal states corresponding to the translation invariant Gibbs measures, are determined. It is proved that the factors associated with minimal and maximal Gibbs states are isomorphic, and if they are of type III\(_{λ}\) then the factor associated with the unordered phase of the model can be considered as a subfactors of these factors respectively. Some concrete examples of factors are given too.

**Keywords:** Cayley tree, Ising model, competing interactions, Gibbs measure, GNS-construction, Hamiltonian, von Neumann algebra.

1 **Introduction**

The present paper is a continuation of the paper \([MR]\). In that paper it has been given motivations to study of the Ising models with competing interactions and we have investigated the Ising model with ternary interactions on a Cayley tree.

Recall that the Cayley tree \(Γ^k\) of order \(k ≥ 1\) is an infinite tree, i.e., a graph without cycles, such that each vertex of which lies on \(k + 1\) edges. Let \(Γ^k = (V, Λ)\), where \(V\) is the set of vertices of \(Γ^k\), \(Λ\) is the set of edges of \(Γ^k\). The vertices \(x\) and \(y\) are called *nearest neighbors*, if there exists an edge connecting them, such vertices are denoted by \( < x, y > \). The distance \(d(x, y), x, y ∈ V\), on the Cayley tree, is the length of the shortest path from \(x\) to \(y\).

We set \(W_n = \{x ∈ V | d(x, x^0) = n\}\), \(V_n = \bigcup_{m=1}^{n} W_m\), for an arbitrary point \(x^0 ∈ V\).

Denote \(S(x) = \{y ∈ W_{n+1} : d(x, y) = 1\}, \ x ∈ W_n\)
this set is called a set of direct successors of \( x \).

Two vertices \( x, y \in V \) is called one level next-nearest-neighbor vertices if there is a vertex \( z \in V \) such that \( x, y \in S(z) \), and they are denoted by \( > x, y < \). In this case the vertices \( x, z, y \) was called ternary and denoted by \( < x, z, y > \).

In this paper we consider the Ising model with competing interactions, where the spin takes values in the set \( \Phi = \{-1, 1\} \), on the Cayley tree which is defined by the following Hamiltonian

\[
H(\sigma) = -J \sum_{>x,y<} \sigma(x)\sigma(y) - J_1 \sum_{<x,y>} \sigma(x)\sigma(y) \tag{1.1}
\]

where \( J, J_1 \in \mathbb{R} \) are coupling constants and \( \sigma \) a configuration on \( V \), i.e. \( \sigma \in \Omega = \Phi^V \).

The other parts of the paper is organized as follows. In section 2 using the similar argument as [MR] we reduce the problem of describing limit Gibbs measures to the problem of solving a nonlinear functional equation. By means of the obtained equation we construct periodic Gibbs measures and find ground states the model. In section 3, we determine the types of von Neumann algebras generated by GNS-representation associated with diagonal states corresponding to the translation invariant measures. In addition, we will demonstrate more concrete examples of factors. In section 4 we discuss the results.

2 On the set of Gibbs measures

In this section recall the construction of a special class of limiting Gibbs measures for the Ising model on a Cayley tree with competing interactions.

Let \( h : x \to \mathbb{R} \) be a real valued function of \( x \in V \). Given \( n = 1, 2, \ldots \) consider the probability measure \( \mu^{(n)} \) on \( \Phi^V_n \) defined by

\[
\mu^{(n)}(\sigma_n) = Z^{-1}_n \exp\{-\beta H(\sigma_n) + \sum_{x \in W_n} h_x \sigma(x)\} \tag{2.1}
\]

Here, \( \beta = \frac{1}{T} \) and \( \sigma_n : x \in V_n \to \sigma_n(x) \) and \( Z_n \) is the corresponding partition function:

\[
Z_n = \sum_{\sigma_n \in \Omega_{V_n}} \exp\{-\beta H(\bar{\sigma}_n) + \sum_{x \in W_n} h_x \bar{\sigma}(x)\},
\]

\[
H(\sigma_n) = -J \sum_{>x,y<; x,y \in V_n} \sigma_n(x)\sigma_n(y) - J_1 \sum_{<x,y>; x,y \in V_n} \sigma_n(x)\sigma_n(y) \tag{2.2}
\]

Recall that the consistency condition for \( \mu^{(n)}(\sigma_n), n \geq 1 \) is

\[
\sum_{\sigma^{(n)}} \mu^{(n)}(\sigma_{n-1}, \sigma^{(n)}) = \mu^{(n-1)}(\sigma_{n-1}) \tag{2.3}
\]

where \( \sigma^{(n)} = \{\sigma(x), x \in W_n\} \).

The following statement describes conditions on \( h_x \) guaranteeing the consistency condition of measures \( \mu^{(n)}(\sigma_n) \). In the sequel for the simplicity we will consider the case \( k = 2 \).
Theorem 2.1. The measures $\mu^{(n)}(\sigma_n)$, $n = 1, 2, \ldots$ satisfy the consistency condition (2.3) if and only if for any $x \in V$ the following equation holds:

$$h_x = \frac{1}{2} \log \left( \frac{\theta^2 \theta e^{2(h_y + h_z)} + \theta_1 (e^{2h_y} + e^{2h_z}) + \theta}{\theta e^{2(h_y + h_z)} + \theta_1 (e^{2h_y} + e^{2h_z}) + \theta_1^2 \theta} \right)$$  \hspace{1cm} (2.4)

here $\theta = e^{2\beta J}$, $\theta_1 = e^{2\beta J_1}$ and $y, x, z$ are ternary neighbors.

Proof. Using (2.1) it is easy to see that (2.3) and (2.4) are equivalent (cf. [MR]).

This theorem reduces the problem of describing of Gibbs measures to the description of solutions of the functional equation (2.4).

According to Proposition 2.1 [MR] that there exists a one-to-one correspondence between the set $V$ of vertices of the Cayley tree of order $k \geq 1$ and the group $G_k$ of the free products of $k + 1$ cyclic groups of the second order with generators $a_1, a_2, \ldots, a_{k+1}$.

Recall that $h = \{h_x : x \in G_k\}$ is $\hat{G}_k$-periodic if $h_yx = h_x$ for all $x \in G_k$ and $y \in \hat{G}_k$, here $\hat{G}_k$ is a normal subgroup of $G_k$ with finite index. A Gibbs measure is called $\hat{G}_k$-periodic if it corresponds to $\hat{G}_k$-periodic function $h$. If it is $G_k$-periodic, then this measure is translation-invariant.

As before in the sequel we will consider the group $G_2$.

As in [MR] we firstly find translation-invariant solutions of (2.4). This case recently has been investigated in Ref. [GPW]. For the sake of completeness and since throughout the paper we will use this result, we recall it.

In this setting (2.4) has the form

$$u = \frac{\theta^2 u^2 + 2\theta_1 u + \theta}{\theta u^2 + 2\theta_1 u + \theta_1^2 \theta}$$  \hspace{1cm} (2.5)

where $u = e^{2h}$.

Proposition 2.2. [GPW]. If $\theta_1 > \sqrt{3}$ and $\theta > \frac{2\theta_1}{\theta_1^2 - 3}$ then for all pairs $(\theta, \theta_1)$ the equation (2.5) has three positive solutions $u_1^* < u_2^* < u_3^*$, here $u_2^* = 1$. Otherwise (2.5) has a unique solution $u^*_1 = 1$.

Remark 2.1. The numbers $u_1^*$ and $u_3^*$ are the solutions of the following equation

$$u^2 + (1 + \alpha)u + 1 = 0,$$  \hspace{1cm} (2.6)

here $\alpha = \frac{2\theta_1}{\theta} - \theta_1^2$. Hence, if $\beta \to \infty$ then $u_3^* \to \infty$ and $u_1^* \to 0$.

By $\mu_1, \mu_2, \mu_3$ we denote Gibbs measures corresponding to these solutions.

Using the same argument as [MR] one can prove the following

Theorem 2.3. For the model (1.1) with parameters $J_1 > 0$ and $J \in \mathbb{R}$ on the Cayley tree $\Gamma^2$ the following assertions hold

(i) if $\theta_1 > \sqrt{3}$, $\theta > \frac{2\theta_1}{\theta_1^2 - 3}$ then the measures $\mu_1$ and $\mu_3$ are extreme;
Remark 2.2. This theorem specifies the result obtained in [GPW], as there it was proved that a phase transition occurs if and only if the above indicated conditions is satisfied, and the extremity was open. The formulated theorem answers that the found Gibbs measures are extreme. In spite of this, further we will show that a phase transition can be occur when the condition of theorem is not satisfied.

Remark 2.3. The measure $\mu_2$ corresponding to the solution $h_x = 0$, $x \in V$ is the unordered phase, i.e. the spin $\sigma(x)$ takes its values $\pm 1$ with respect to $\mu_2$ with probability $1/2$.

Now we turn to the constructions of periodic Gibbs measures. Let $H_0$ be a subgroup of index $r$ in $G_2$, and let $G_2|H_0 = \{H_0, H_1, ..., H_{r-1}\}$ be the quotient group. Let $g_i(x) = |S^*(x) \cap H_i|, i = 0, 1, ..., r-1; N(x) = |\{j : q_j(x) \neq 0\}|$, where $S^*(x)$ is the set of all nearest neighbors of $x \in G_2$. Denote $Q(x) = (q_0(x), q_1(x), ..., q_{r-1})$. We note (see [R]) that for every $x \in G_2$ there is a permutation $\pi_x$ of the coordinates of the vector $Q(e)$ (where $e$ is the identity of $G_2$) such that

$$\pi_x Q(e) = Q(x). \quad (2.7)$$

It follows from (2.7) that $N(x) = N(e)$ for all $x \in G_2$.

It is clear that each $H_0$-periodic function $h_x$ is given by

$$\{h_x = h_i \text{ for } x \in H_i, i = 0, 1, ..., r-1\}.$$

By $G_2^{(2)}$ we denoted in [MR] the subgroup of $G_2$ consisting of all words with even length. This $G_2^{(2)}$ has an index 2.

Theorem 2.4. Let $H_0$ be a subgroup of finite index in $G_2$. Then each $H_0$-periodic Gibbs measure for (1.1) model is either translation-invariant or $G_2^{(2)}$-periodic.

Proof. Let $f(x, y)$ be function defined as follows

$$f(x, y) = \frac{\theta_1^2 \theta xy + \theta_1 (x + y) + \theta}{\theta xy + \theta_1 (x + y) + \theta_1^2 \theta}, \quad x, y > 0. \quad (2.8)$$

For $\theta_1 \neq 1$ it is easy to see that $f(u_1, v) = f(u_2, v)$ if and only if $u_1 = u_2$. Also $f(u, v_1) = f(u, v_1)$ if and only if $v_1 = v_2$. Using this property of $f(u, v)$, by Theorem 2.1 and (2.7) we have

$$h_x = h_y = h_1, \quad \text{if } x, y \in S^*(z), z \in G_2^{(2)};$$

$$h_x = h_y = h_2, \quad \text{if } x, y \in S^*(z), z \in G_2 \setminus G_2^{(2)}.$$

Thus the measures are translational-invariant (if $h_1 = h_2$) or $G_2^{(2)}$-periodic (if $h_1 \neq h_2$). The theorem is proved.

If $H_0$ is a subgroup of finite index in $G_2$, then it natural to ask: what condition on $H_0$ guarantees that each $H_0$-periodic Gibbs measure to be translation-invariant? We put $I(H_0) = H_0 \cap \{a_1, a_2, a_3\}$, where $a_i, i = 1, 2, 3$ are generators of $G_2$.

Theorem 2.5. If $I(H_0) \neq \emptyset$, then each $H_0$-periodic Gibbs measure for (1.1) model is translation-invariant.

4
By Theorem 2.4 we have $h_I(u)$ where $h$ equation has the form (2.10). This proves the theorem.

Theorems 2.4, 2.5 reduce the problem of describing $H_0$-periodic Gibbs measures with $I(H_0) \neq \emptyset$ to describing the fixed points of the function $f(u,u)$ (see (2.8)) which describe the translation-invariant Gibbs measures. If $I(H_0) = \emptyset$, this problem is reduced to the describing solutions of the system:

$$
\begin{align*}
      u &= \frac{\theta_1^2 \theta v^2 + 2 \theta_1 v + \theta}{\theta v^2 + 2 \theta_1 v + \theta_1^2 \theta^2} \\
      v &= \frac{\theta_2^2 \theta u^2 + 2 \theta_1 u + \theta}{\theta u^2 + 2 \theta_1 u + \theta_1^2 \theta^2}
\end{align*}
$$

(2.9)

where $u = \exp\{2h_1\}, v = \exp\{2h_2\}$.

The analysis of the equation (2.9) is carried in the following

**Proposition 2.6.** The equation (2.9) has three positive solutions $(1,1),(u_*,v_*)$ and $(v_*,u_*)$ (here $u_* < v_*$) if and only if $\theta_1 < 1/\sqrt{3}$ and $\theta > \frac{2\theta_1}{1 - 3\theta_1^2}$. Here $u_*, v_*$ are the solutions of the equation:

$$
\theta_1^2 \theta (\theta_1^2 \theta + 2 \theta_1 + \theta)(x^2 + 1) + ((\theta_1^2 \theta)^2 + 4 \theta_1^3 \theta + 4 \theta_1^2 - \theta^2)x = 0.
$$

(2.10)

**Proof.** It is clear that $(1,1)$ is a solution of (2.9). The equation (2.9) can be written as $u = g(g(u))$, here $g(u) = f(u,u)$. Hence, the solutions of the equations $u = g(u)$ are the solution of (2.9), but they describe only the translation-invariant Gibbs measures. Now we should find solutions of $\frac{g(g(u)) - u}{g(u) - u} = 0$. After some calculations it can be shown that the last equation has the form (2.10).

Full analysis of the equation (2.10) shows that parameters $\theta, \theta_1$ must satisfy the condition of the proposition. This completes the proof.

Thus we can formulate the following

**Theorem 2.7.** For the model (1.1) with respect to any subgroup $H_0$ of finite index the following assertions hold:

(i) Let $\theta_1 > \sqrt{3}, \theta > \frac{2\theta_1}{\theta_1^2 - 3}$ be satisfied, then $H_0$-periodic Gibbs measures coincide with the translation-invariant Gibbs measures.

(ii) Let $\theta_1 < 1/\sqrt{3}, \theta > \frac{2\theta_1}{1 - 3\theta_1^2}$ be satisfied.

(a) If $I(H_0) \neq \emptyset$ then $H_0$-periodic Gibbs measures coincide with the translation-invariant Gibbs measures.
(b) If \( I(H_0) = \emptyset \) then there are three \( H_0 \)-periodic (= \( G_2^{(2)} \)-periodic) Gibbs measures \( \mu_{12}, \mu_{21} \) and \( \mu_\ast \). Here the measure \( \mu_\ast \) corresponds to the unique solution of equation (2.5).

**Proof.** Let the condition (i) be satisfied. Then Proposition 2.6 implies that there is no \( G_2^{(2)} \)-periodic Gibbs measures in this setting. Therefore, according to Theorem 2.4 we conclude that the \( H_0 \)-periodic Gibbs measures are translation-invariant. Now let (ii) hold. Then the assertions (a) and (b) immediately follow from Proposition 2.6, Theorems 2.4. and 2.5. This completes the proof.

**Remark 2.4.** In [MR] we have investigated only \( G_2^{(2)} \)-periodic Gibbs measures, the proved Theorem 2.7 completely describes all periodic Gibbs measures, associated with subgroups of \( G_2 \) with finite index, of the model.

Now comparing Theorems 2.3 and 2.7 we infer the following

**Corollary 2.8.** If \( 1/\sqrt{3} < \theta_1 < \sqrt{3} \) then for the model (1.1) there is no phase transition.

By using the similar argument as in [MR] we can prove the extremity of \( \mu_{12}, \mu_{21} \).

**Theorem 2.9.** Let \( \theta_1 < 1/\sqrt{3} \) and \( \theta > \frac{2\theta_1}{1 - 3\theta_1^2} \) be satisfied. Then the measures \( \mu_{12}, \mu_{21} \) and \( \mu_\ast \) are extreme.

Using the analogical way as in [BG], [GR] with the aid of measures \( \mu_1, \mu_2 \) and \( \mu_3 \) one can construct uncountable number of extreme Gibbs measures.

From the construction of the Gibbs measures we easily see that the measures \( \mu_1 \) and \( \mu_3 \) depend on parameter \( \beta \). Now we are interested on the behaviour of these measures when \( \beta \) goes to \( \infty \).

Put

\[
\sigma_+ = \{ \sigma(x) : \sigma(x) = 1, \forall x \in \Gamma^2 \}, \\
\sigma_- = \{ \sigma(x) : \sigma(x) = -1, \forall x \in \Gamma^2 \}.
\]

**Theorem 2.10.** Let \( \theta_1 > \sqrt{3} \) and \( \theta > \frac{2\theta_1}{\theta_1^2 - 3} \), then

\[
\mu_1 \rightarrow \delta_{\sigma_-}, \quad \mu_3 \rightarrow \delta_{\sigma_+} \quad \text{as} \quad \beta \rightarrow \infty,
\]

here \( \delta_\sigma \) is a delta-measure concentrated on \( \sigma \).

**Proof.** Consider the measure \( \mu_3 \). This measure corresponds to the function \( h_x = h_3, x \in V \), here \( h_3 > 0 \) (see Proposition 2.2). Let us first consider a case:

\[
\mu_3(\sigma(x) = 1) = \frac{e^{h_3}}{e^{h_3} + e^{-h_3}} = \frac{u_3^*}{u_3^* + 1} \rightarrow 1 \quad \text{as} \quad \beta \rightarrow \infty,
\]

since \( u_3^* \rightarrow \infty \) as \( \beta \rightarrow \infty \), here \( x \in V \). Let us turn to the general case. From the condition imposed in the Theorem we find that \( J_1 > 0 \). Now separately consider two cases.
First case. Let \( J > 0 \). Then from the form of Hamiltonian (1.1) it is easy to check that \( H(\sigma_n|V_n) \geq H(\sigma_+|V_n) \) for all \( \sigma \in \Omega \) and \( n > 0 \). It follows that
\[
\mu_3(\sigma_+|V_n) = \frac{\exp\{-\beta H(\sigma_+|V_n) + h_3 W_n\}}{\sum_{\tilde{\sigma}_n \in \Omega_{V_n}} \exp\{-\beta H(\tilde{\sigma}_n) + h_3 \sum_{x \in W_n} \tilde{\sigma}(x)\}} = \frac{1}{1 + \sum_{\tilde{\sigma}_n \in \Omega_{V_n}, \tilde{\sigma}_n \neq \sigma_+|V_n} \frac{\exp\{-\beta H(\tilde{\sigma}_n) + h_3 \sum_{x \in W_n} \tilde{\sigma}(x)\}}{\exp\{-\beta H(\sigma_+|V_n) + h_3 W_n\}}}
\]

The last inequality yields that \( \mu_3 \to \delta_{\sigma_+} \).

Second case. Let \( J < 0 \). Let us introduce some notations.
\[
A(\sigma_n) = \sum_{x,y:x,y \in V_n} \sigma(x)\sigma(y), \quad A = A(\sigma_+|V_n),
\]
\[
B(\sigma_n) = \sum_{<x,y:x,y \in V_n} \sigma(x)\sigma(y), \quad B = B(\sigma_+|V_n),
\]
\[
C(\sigma_n) = \sum_{x \in V_n} \sigma(x), \quad C = C(\sigma_+|V_n).
\]

Then it is easy to see that the following equality holds
\[
\mu_3(\sigma_+|V_n) = \frac{1}{1 + \sum_{\tilde{\sigma}_n \in \Omega_{V_n}, \tilde{\sigma}_n \neq \sigma_+|V_n} \frac{1}{e^{J\beta(A-A(\tilde{\sigma}_n))} e^{J_1\beta(B-B(\tilde{\sigma}_n))} e^{h_3(C-C(\tilde{\sigma}_n))}}}.
\]

We want to show that
\[
\sum_{\tilde{\sigma}_n \in \Omega_{V_n}, \tilde{\sigma}_n \neq \sigma_+|V_n} \frac{1}{e^{J\beta(A-A(\tilde{\sigma}_n))} e^{J_1\beta(B-B(\tilde{\sigma}_n))} e^{h_3(C-C(\tilde{\sigma}_n))}} \to 0 \quad \text{as } \beta \to \infty.
\]

It is enough to prove that
\[
\frac{1}{e^{J\beta(A-A(\tilde{\sigma}_n))} e^{J_1\beta(B-B(\tilde{\sigma}_n))} e^{h_3(C-C(\tilde{\sigma}_n))}} \to 0 \quad \text{as } \beta \to \infty
\]
for all \( \tilde{\sigma}_n \in \Omega_{V_n}, \tilde{\sigma}_n \neq \sigma_+|V_n \). We rewrite the last sentence as follows
\[
\frac{1}{e^{J\beta(A-A(\tilde{\sigma}_n))} e^{J_1\beta(B-B(\tilde{\sigma}_n))} e^{h_3(C-C(\tilde{\sigma}_n))}} \leq \frac{1}{\theta(\tilde{\sigma}_n)^2} \frac{1}{(u_3^*)^2} \leq \frac{(\theta_1^2 - 3)(A-A(\tilde{\sigma}_n))/2}{\theta_1(\tilde{\sigma}_n)^2 + h_3(C-C(\tilde{\sigma}_n))/2} \leq \frac{(\theta_1^2 - 3)(A-A(\tilde{\sigma}_n))/2}{\theta_1(A-A(\sigma_+)+B-B(\sigma_+))/2 u_3^*}.
\]
here we have used the inequality $\theta > \frac{2\theta_1}{\theta_1^2 - 3}$.

Obviously, if $\beta$ is large enough we have
\[
\frac{(\theta_1^2 - 3)(A - A(\sigma_n))}{\theta_1^{A - A(\sigma_n) + B - B(\sigma_n)}} \sim \frac{\theta_1^{A - A(\sigma_n) + B - B(\sigma_n)}}{\theta_1^{B - B(\sigma_n)}} = \frac{\theta_1^{B(\sigma_n) - A(\sigma_n)}}{\theta_1^{B - A}}.
\]

If $B(\sigma_n) - A(\sigma_n) \leq B - A$ then the last relation implies that $\frac{(\theta_1^2 - 3)(A - A(\sigma_n))}{\theta_1^{A - A(\sigma_n) + B - B(\sigma_n)}}$ is bounded, and hence from (2.11) we get the required relation.

Now it remains to prove the following

**Lemma 2.11.** For every $n > 0$ and $\sigma_n \in \mathcal{O}_n$, the following inequality holds
\[ B(\sigma_n) - A(\sigma_n) \leq B - A. \] (2.12)

**Proof.** Denote $\mathcal{C}(\sigma_n) = \{x \in V_n : \sigma(x) = -1\}$. Maximal connected components of $\mathcal{C}(\sigma_n)$ we will denote by $\mathcal{K}_1(\sigma_n), \cdots, \mathcal{K}_m(\sigma_n)$. For a connected subset $\mathcal{K}$ of $V_n$ put
\[
\partial \mathcal{K} = \{x \in V_n \setminus \mathcal{K} : < x, y > \text{ for some } y \in \mathcal{K}\},
\]
\[
\partial^2 \mathcal{K} = \{x \in V_n \setminus \mathcal{K} : > x, y < \text{ for some } y \in \mathcal{K}\}.
\]

From definition of $A(\sigma_n)$ and $B(\sigma_n)$ we get
\[ B(\sigma_n) = B - 2 \sum_j |\partial \mathcal{K}_j(\sigma_n)|, \]
\[ A(\sigma_n) = A - 2 \sum_j |\partial^2 \mathcal{K}_j(\sigma_n) \setminus \cup_{m \neq j} \mathcal{K}_m(\sigma_n)|, \]

here $|A|$ stands for a number of elements of a set $A$.

To prove (2.12) it enough to show that $|\partial^2 \mathcal{K}| \leq |\partial \mathcal{K}|$ for all connected subsets $\mathcal{K}$ of $V_n$. For each $x \in \partial^2 \mathcal{K}$ we can show a $y = y(x) \in \partial \mathcal{K}$. Indeed, if $> x, t <, t \in \mathcal{K}$ and $< x, z, t >$ then $y(x) = x$ if $z \in \mathcal{K}$ and $y(x) = z$ if $z \notin \mathcal{K}$. It is clear $y(x) \in \partial \mathcal{K}$. Now we will prove that if $x_1 \neq x_2 \in \partial^2 \mathcal{K}$ then $y(x_1) \neq y(x_2) \in \partial \mathcal{K}$. Let $< x_1, z_1, t_1 >,$ $< x_2, z_2, t_2 >$, where $t_1, t_2 \in \mathcal{K}$. By definition of $y(x)$ we have $y(x_i) \in \{x_i, z_i\}, i = 1, 2$. So to prove $y(x_1) \neq y(x_2)$ for $x_1 \neq x_2$ it is enough to show $z_1 \neq z_2$. Note that in our case (i.e. $k = 2$) if $x_1 \neq x_2 \in \partial^2 \mathcal{K}$ then $d(x_1, x_2) \geq 3$, since $< x_i, z_i >, i = 1, 2$ and hence we get $z_1 \neq z_2$. Thus $|\partial^2 \mathcal{K}| \leq |\partial \mathcal{K}|$. So Lemma is proved.

By similar argument the theorem can be proved for the measure $\mu_1$. Thus the theorem is proved.

From Theorem 2.10 we conclude that $\sigma_+$ and $\sigma_-$ are ground states of the considered model.

**Remark 2.6.** If in the condition of Theorem 2.10, we put $J = 0$ then the obtained result coincides with Theorem 2.3 of [BRZ2].
Now introduce two configurations as follows
\[ \sigma_{+-} = \{ \sigma_-(x), \ x \in V \}, \ \sigma_{-+} = \{ \sigma_-(x), \ x \in V \}, \]

where
\[ \sigma_+(x) = \begin{cases} 1, & \text{if } x \in G_2^{(2)}, \\ -1, & \text{if } x \in G_2 \setminus G_2^{(2)}, \end{cases} \]
\[ \sigma_-(x) = \begin{cases} -1, & \text{if } x \in G_2^{(2)}, \\ 1, & \text{if } x \in G_2 \setminus G_2^{(2)}. \end{cases} \]

We can formulate the following

**Theorem 2.12.** Let \( \theta_1 < 1/\sqrt{3} \) and \( \theta > \frac{2\theta_1}{1 - 3\theta_1^2} \) then
\[ \mu_{12} \to \delta_{\sigma_{-+}}, \ \mu_{21} \to \delta_{\sigma_{-+}} \ \text{as } \beta \to \infty. \]

The proof is similar to the proof of Theorem 2.10.

### 3 Diagonal states generated by Gibbs measures and corresponding von Neumann algebras

In this section we consider a case \( \theta_1 > \sqrt{3}, \theta > \frac{2\theta_1}{\theta_1^2 - 3} \) and determine types of von Neumann algebras generated by the GNS-representation associated with the diagonal states corresponding to the translation invariant measures.

As the paper [MR] we consider \( C^* \)-algebra \( A = \otimes_{\mathbb{T}^n} M_2(\mathbb{C}) \), where \( M_2(\mathbb{C}) \) is the algebra of 2x2 matrices over the field \( \mathbb{C} \) of complex numbers.

By \( \omega_i \ (i = 1, 2, 3) \) we denote the diagonal state generated by the translation invariant measures \( \mu_1, \mu_2, \mu_3 \) respectively. On the finite dimensional \( C^* \)-subalgebra \( A_{V_n} = \otimes_{V_n} M_2(\mathbb{C}) \subset A \) we rewrite the state \( \omega_i \) as follows
\[ \omega_i(x) = \frac{\text{tr}(e^{\hat{H}_i(V_n)x})}{\text{tr}(e^{\hat{H}_i(V_n)})}, \ x \in A_{V_n}, \quad (3.1) \]

where \( \text{tr} \) is a trace on \( A_{V_n} \). The term \( \sigma(x)\sigma(y), \ (> x, y <) \) in (2.2) we represent as a diagonal element of \( M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \) in the standard basis as follows
\[ \sigma(x)\sigma(y) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \quad (3.2) \]
Using (3.2), the form of Hamiltonian (1.1), (2.1) and (3.1) the Hamiltonian \( \tilde{H}_i(V_n) \) in the standard basis of \( A_{V_n} \) has the form

\[
\tilde{H}_i(V_n) = \sum_{x,y:x,y\in V_n} F_{x,y} + \sum_{<x,y>:x,y\in V_n} G_{<x,y>} + \sum_{x\in W_n} h_i \sigma_x^z,
\]

here and below

\[
F_{x,y} = \left(\begin{array}{cc}
A \otimes 1 & O \\
O & B \otimes 1
\end{array}\right),
\]

\[
A = \left(\begin{array}{cc}
\log p_1 & 0 \\
0 & \log p_2
\end{array}\right),
\]

\[
B = UAU,
\]

\[
U = \left(\begin{array}{cc}
0 & 1 \\
1 & 0
\end{array}\right),
\]

(3.3)

\[
p_1 = \frac{1}{e^{-2βJ} + 1},
\]

\[
p_2 = 1 - p_1 = \frac{e^{-2βJ}}{e^{-2βJ} + 1},
\]

(3.4)

\[
G_{<x,y>} = \left(\begin{array}{cc}
A_1 & O \\
O & UA_1 U
\end{array}\right),
\]

\[
A_1 = \left(\begin{array}{cc}
\log p_{11} & 0 \\
0 & \log p_{22}
\end{array}\right),
\]

(3.5)

\[
p_{11} = \frac{1}{e^{-2βJ_1} + 1},
\]

\[
p_{22} = 1 - p_{11} = \frac{e^{-2βJ_1}}{e^{-2βJ_1} + 1},
\]

\[
\sigma_x^z = \left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right),
\]

(3.6)

and \( h_i = \log u_i^*/2 \), where \( u_i^* \) is a solution of (2.5).

Hence the state \( \omega_i \) is an Gibbs state for quantized Hamiltonian

\[
\tilde{H} = \sum_{x,y} F_{x,y,z} + \sum_{<x,y>} G_{<x,y>}.
\]

Denote \( M_i = \pi_{\omega_i}(A)^{\nu_i} \), where \( \pi_{\omega_i} \) is a GNS - representation associated with \( \omega_i \) (see Ref. [BR1, definition 2.3.18]). Note \( M_i \) is a factor, since the measures \( \mu_i \) are translation invariant and satisfy mixing property, i.e.

\[
\lim_{|g|\to\infty} \omega_i(T_g(x)y) = \omega_i(x)\omega_i(y),
\]

here \( T_g \) is a left shift transformation of \( G_2 \). Our goal in the present section is to determine a type of \( M_i \).

We note that the modular group of \( M_i \) associated with \( \omega_i \) is defined by

\[
\sigma_i^{w_i}(x) = \lim_{V_n\to V} \exp\{it\tilde{H}_i(V_n)\}x \exp\{-it\tilde{H}_i(V_n)\}, \quad x \in M_i,
\]

(3.7)

here as before

\[
\tilde{H}(L) = \sum_{x,y:x,y\in V_n} F_{x,y} + \sum_{<x,y>:x,y\in V_n} G_{<x,y>} + \sum_{x\in W_n} h_i \sigma_x^z.
\]

The existence of the last limit easily can be checked by using Theorem 6.2.4 [BR2] (see [MR]).

**Lemma 3.1.** Let the following condition be satisfied: there exist integers \( k_i \) and \( m_j^{(i)} \), \( j \in \{1, 2, 3\} \) and the smallest number \( \delta_i \in (0, 1) \) such that

\[
\frac{p_1}{p_2} = \delta_i^{m_1^{(i)}}, \quad \frac{p_{11}}{p_{22}} = \delta_i^{m_2^{(i)}}, \quad \frac{p_1}{p_{11}} = \delta_i^{m_3^{(i)}}, \quad \exp\{h_i\} = \delta_i^{k_i},
\]

(3.8)
then for the modular group $\sigma_{t_0}$ and the number $t_0 = -2\pi / \log \delta_i$, the equality holds

$$\sigma_{t_0} = Id.$$ 

**Proof.** From (3.8) we have

$$\begin{align*}
p_1 &= \frac{\delta_{m_1}^{(i)}}{\delta_{m_1}^{(i)} + 1}, \quad p_2 = \frac{1}{\delta_{m_2}^{(i)} + 1}, \\
p_{11} &= \frac{\delta_{m_1}^{(i)-m_3}^{(i)}}{\delta_{m_1}^{(i)} + 1}, \quad p_{22} = \frac{\delta_{m_2}^{(i)-m_3}^{(i)}}{\delta_{m_1}^{(i)} + 1}.
\end{align*}$$

(3.9)

Hence from (3.3), (3.4) and (3.9) we can get that $\sigma_{t_0} = Id$. This completes the proof.

Now using Lemma 3.1, Proposition 5.2[MR] and the argument of [MR] we can prove the following

**Theorem 3.3.** Let $\theta_1 > \sqrt{3}$, $\theta > \frac{2\theta_1}{\theta_1^2 - 3}$ and the condition (3.8) be satisfied. Then von Neumann algebras $M_i$ corresponding to the translation invariant Gibbs states $\mu_i$ of the Ising model with competing interactions (1.1) on the Cayley tree $\Gamma^2$ are factors of type $\text{III}_{\delta_i}$.

Since $u_1^*$ and $u_3^*$ are the solution of the equation (2.6) then from (3.8) we find that $k_1 = -k_3$ and $\delta_1 = \delta_3$. This implies that the factors $M_1$ and $M_3$ have the same type. It is easy to see that $k_2 = 0$.

From Theorem 3.3 and the argument of [MR] we have the following

**Corollary 3.4.** Let $\theta_1 > \sqrt{3}$, $\theta > \frac{2\theta_1}{\theta_1^2 - 3}$ and the following condition be satisfied: there exist integers $m_i, i \in \{1, 2, 3\}$ and the smallest number $\delta \in (0, 1)$ such that

$$\begin{align*}
p_1 &= \delta_{m_1}^{(i)}, \quad p_{11} = \delta_{m_2}^{(i)}, \quad p_1 = \delta_{m_3}^{(i)},
\end{align*}$$

(3.10)

then a von Neumann algebras $M_2$ corresponding to the unordered phase of the Ising model with competing interactions (1.1) on the Cayley tree $\Gamma^2$ is a factor of type $\text{III}_{\delta}$. Otherwise $M_2$ is a factor of type $\text{III}_1$.

**Corollary 3.5.** Let $\theta_1 > \sqrt{3}$, $\theta > \frac{2\theta_1}{\theta_1^2 - 3}$ and the following conditions be satisfied: there exist integers $k$ and $n_i, i \in \{1, 2, 3\}$ and the smallest number $\delta_1 \in (0, 1)$ such that

$$\begin{align*}
p_1 &= \delta_{n_1}^{(i)}, \quad p_{11} = \delta_{n_2}^{(i)}, \quad p_1 = \delta_{n_3}^{(i)},
\end{align*}$$

(3.11)

and

$$\exp \{h_1\} = \delta_1^k, \quad h_1 > 0,$$

(3.12)

then von Neumann algebras $M_1$ and $M_3$ corresponding to the Gibbs states $\mu_1$ and $\mu_3$ respectively, of the Ising model with competing interactions (1.1) on the Cayley tree $\Gamma^2$ are factors of type $\text{III}_{\delta_1}$. Otherwise they are factors of type $\text{III}_1$. 

11
Remark 3.2. If we consider the case $\theta_1 < 1/\sqrt{3}$ and $\theta > \frac{2\theta_1}{\theta_1^2 - 3}$, then there are two strictly periodic (non-translation invariant) Gibbs measures. By similar arguments as above we can prove analogical theorems as Theorem 3.3 for these periodic measures.

Remark 3.3. Here it would be good to mention that there is an example of factor generated by Cayley tree, but it does not appear from a physical system (see [RR]).

It is clear that if (3.10) is not satisfied then (3.11) is too, consequently, the algebras $\mathcal{M}_i$ are factors of type $\text{III}_1$. Suppose (3.10) is valid then (3.11) is also satisfied with $\delta_1 \geq \delta$, more exactly, $\delta_1 = \delta^r$, where $r \in (0, 1] \cap \mathbb{Q}$. But it is interesting whether the equality $\delta_1 = \delta$ is satisfied. The following theorem answers to this question.

**Theorem 3.6.** Let $\theta > \sqrt{3}$, $\theta > \frac{2\theta_1}{\theta_1^2 - 3}$ be satisfied. Suppose the equalities (3.10)-(3.12) are satisfied. Then the factor $\mathcal{M}_1$ and $\mathcal{M}_3$ have types $\text{III}_{3r}$, $0 < r < 1, r \in \mathbb{Q}$, while the factor $\mathcal{M}_2$ has type $\text{III}_3$.

**Proof.** The conditions (3.10) and (3.12) imply that there exists a rational number $s \in \mathbb{Q}$ such that $\theta = \theta_1^s$. We note this is a necessary condition that $\mathcal{M}_2$ to be a factor of type $\text{III}_3$. We want to prove that $\delta_1 > \delta$. Let us assume that $\delta_1 = \delta$. Keeping in mind that the numbers $e^{2\delta_1}$ and $e^{-2\delta_1}$ are the solutions of the equality (2.6) from (3.12) we obtain

$$2 \cosh(2k \log \delta) = \theta_1^2 - 2\theta_1^{1-s} - 1$$

(3.13)

here we have used that $\alpha = 2\theta_1^{1-s} - \theta_1^2$. From (3.11) we find that $\log \delta = -2nJ_1\beta$, substituting it into (3.13) we have

$$2 \cosh(4nJ_1\beta) = \theta_1^2 - 2\theta_1^{1-s} - 1,$$

(3.14)

here without loss of generality we may assume that $n > 0$, $n \in \mathbb{Z}$, since $\cosh(x)$ is an even function. Defining $f(n) = 2 \cosh(4nJ_1\beta)$ from (3.14) it is easy to see that $f(1) > \theta_1^2 - 2\theta_1^{1-s} - 1$, since $\theta_1 > \sqrt{3}$. It is clear that $f(n)$ is an increasing function, so this implies that the equality (3.14) can not be satisfied for any positive integer $n$. This means $\delta_1 > \delta$. Consequently, the factors $\mathcal{M}_1$ and $\mathcal{M}_3$ can not have the same type with the factor $\mathcal{M}_2$. This completes the proof.

The proved Theorem means that the factor $\mathcal{M}_2$ can be considered as a subfactor of $\mathcal{M}_1$ and $\mathcal{M}_3$ respectively.

**Corollary 3.7.** Let $\theta > \sqrt{3}$, $\theta > \frac{2\theta_1}{\theta_1^2 - 3}$ be satisfied. If there is an irrational $\gamma$ such that $J = \gamma J_1$ then the factors $\mathcal{M}_i$ ($i = 1, 2, 3$) have type $\text{III}_1$.

Let us consider some more concrete examples of factors.

**Example 3.1.** Suppose that $J = 0$ and $\theta_1 > \sqrt{3}$. Then the condition $\theta > \frac{2\theta_1}{\theta_1^2 - 3}$ implies that $\theta_1 > 3$. In this case the equality (3.10) reduces to the following one

$$\frac{p_{11}}{p_{22}} = \delta^m,$$

here as before $\delta \in (0, 1)$ and $m \in \mathbb{Z}$, which is automatically satisfied with $\delta = \theta_1^{-1}$ and $m = -1$. So in this case $\mathcal{M}_2$ is a factor of type $\text{III}_3$. But it is interesting question is whether the factors
$\mathcal{M}_1$ and $\mathcal{M}_3$ can have type III$_{\delta_1}$, while the factor $\mathcal{M}_2$ has type III$_{\delta}$. Now we going to show that this can be occur.

Indeed, we firstly note that in the considered case the equation (2.4) can be written as follows

$$h = 2\text{arctanh}(\tilde{\theta}\tanh h)$$

(3.15)

here $\tilde{\theta} = \tanh(J_1\beta)$ (see, [BRZ1]). We recall that the condition $\theta_1 > 3$ is equivalent to $\tilde{\theta} > \frac{1}{2}$.

Now using the formula

$$\tanh(2x) = \frac{2\tanh x}{1 + \tanh^2 x}$$

from (3.15) we obtain

$$\tanh h_1 = \frac{2\tilde{\theta}\tanh h_1}{1 + (\tilde{\theta}\tanh h_1)^2},$$

it yields that

$$h_1 = \text{arctanh}\left(\frac{\sqrt{2\tilde{\theta} - 1}}{\tilde{\theta}}\right).$$

(3.16)

Let us turn to the conditions (3.11) and (3.12). In our case they can be reduced to the following ones

$$\frac{p_{11}}{p_{22}} = \delta_1^n, \quad \exp\{h_1\} = \delta_1^k, \quad n, k \in \mathbb{Z}.$$  

(3.17)

Choose the number $\tilde{\theta}$ such that which satisfies the following equation

$$\tilde{\theta}^3 + 5\tilde{\theta}^2 + 7\tilde{\theta} - 5 = 0.$$  

(3.18)

It is not hard to check that the required $\tilde{\theta}$ does exist, i.e. with the property $1/2 < \tilde{\theta} < 1$. Put $\delta_1 = \sqrt{3}$ or $\delta_1 = \theta_1^{-1/4}$. It easy to see that for such $\delta_1$ we have $n = -4$. From (3.17) we find

$$h_1 = -\frac{k}{2}J_1\beta,$$

which yields

$$h_1 = -\frac{k}{2}\text{arctanh}\tilde{\theta},$$

(3.19)

here we have used the $J_1\beta = \text{arctanh}\tilde{\theta}$.

We will show that this equality is satisfied when $k = -1$. Indeed, using (3.16) from (3.19) we have

$$\frac{\sqrt{2\tilde{\theta} - 1}}{\tilde{\theta}} = \tanh\left(\frac{\text{arctanh}(\tilde{\theta})}{2}\right).$$

(3.20)

Now according to the formula

$$\tanh\frac{x}{2} = \frac{1 - \sqrt{1 - \tanh^2 x}}{\tanh x}, \quad x > 0$$

from (3.20) we get

$$\sqrt{2\tilde{\theta} - 1} + \sqrt{1 - \tilde{\theta}^2} = 1.$$
The last equation equivalent to the following one

\[(\hat{\theta} - 1)(\hat{\theta}^3 + 5\hat{\theta}^2 + 7\hat{\theta} - 5) = 0.\]

The condition (3.18) yields that the last equality is satisfied, hence (3.20) is valid. Thus the factors \(M_1\) and \(M_3\) have type III\(\sqrt{\delta}\).

These results clarifies and specifies the results obtained in [M],[MR].

**Example 3.2.** Suppose that \(J = J_1\) and \(J \neq 0\), this means \(\theta = \theta_1\). Hence the equality (3.10) is satisfied with parameters: \(\delta = \theta^{-1}\), \(m_1 = -1\), \(m_2 = -1\), \(m_3 = 0\). So according to Corollary 3.4 we conclude that \(M_2\) is a factor of type III\(\delta\). Now assume that there is a phase transition, i.e. the condition \(\theta_1 > \sqrt{3}\), \(\theta > 2\theta_1 / \theta_1^2 - 3\) is satisfied, which implies in our case (i.e. \(J = J_1\)) that \(\theta > \sqrt{3}\). Now we are going to find another \(\delta_1\) for which the factors \(M_1\) and \(M_3\) have type III\(\delta_1\).

Put \(\theta = 1 + \sqrt{2}\) and \(\delta_1 = \sqrt{\delta}\). It is clear that (3.11) is satisfied. Now we should check (3.12). Keeping in mind that the numbers \(e^{2h_1}\) and \(e^{-2h_1}\) are the solutions of (2.6) from (3.12) we get

\[\delta^k + \delta^{-k} = \theta^2 - 3.\]  
\((3.21)\)

Put \(k = 1\). Let us show this equality is satisfied. The equality (3.21) can be written as follows

\[(\theta + 1)(\theta^2 - 2\theta - 1) = 0.\]

The chosen \(\theta\) satisfies this equation, hence (3.12) is valid. This is the required.

If \(J = 0\) then the phase transition does not occur and the factor \(M_2\) has type \(\Pi_1\). We note in the case \(\theta_1 > \sqrt{3}\) and \(\theta > 2\theta_1 / \theta_1^2 - 3\) the factor \(M_2\) can not have a type \(\Pi_1\).

4 Discussion of the results

It is known that to exact calculations in statistical mechanics are paid attention by many of researchers, because those are important not only for their own interest but also for some deeper understanding of the critical properties of spin systems which are not obtained form approximations. So, those are very useful for testing the credibility and efficiency of any new method or approximation before it is applied to more complicated spin systems. In the previous paper [MR] we have exactly solved an Ising model on a Cayley tree, the Hamiltonian of which contained ternary interactions. In addition, we found some conditions on parameters which enabled to determine exactly types of von Neumann algebras associated with periodic Gibbs states of that model. In the present paper we continue investigations of the Ising model, but now we consider a model with the next-nearest-neighbor binary interactions. Using the same way as [MR] we exactly solve a phase transition problem for the model, namely, we calculated critical curve such that there is a phase transitions above it, and a single Gibbs state is found elsewhere. Comparing with the results of [MR] in the present paper we describe all periodic
Gibbs states associated with subgroups of $G_2$ with finite index, while in the mentioned paper we only found $G_2^{(2)}$-periodic Gibbs states. Besides, we also find ground states of the considered model. Here (in the paper) as in [MR] we also find some conditions of parameters $J$ and $J_1$ which completely determine types of von Neumann algebras corresponding to the translation-invariant Gibbs states, but now we show how these algebras related with each other, more precisely speaking, we prove that the factor corresponding to the unordered phase is a subfactor of the factors associated with the minimum and maximum Gibbs states. We note that this kind of question was not considered in [MR]. Finally, we demonstrate some more concrete examples of such factors, which clarify the results obtained in [MR, MR].

We note that some computer simulations results of the model considered were studied in [MTA]. Some other phase transitions problems were considered in [L].

Acknowledgements The work was done within the scheme of Borsa di Studio CNR-NATO. One of authors (F.M.) thanks CNR for providing financial support and II Universita di Roma "Tor Vergata" for all facilities. Besides, he also thanks Prof.E.Presutti for kind hospitality and useful discussions. U.R. thanks Institute des Hautes Etudes Scientifiques (IHES) for supporting the visit to Bures-sur-Yvette (IHES, France) in September-December 2003. The work is also partially supported by Grant Φ-1.1.2 Rep. Uzb.

References

[BG] P.M. Bleher and N.N. Ganikhodjaev, On pure phases of the Ising model on the Bethe lattice, *Theor. Probab. Appl.* **35**:216-227 (1990).

[BRZ1] P.M. Bleher, J. Ruiz and V.A. Zagrebnov, On the purity of the limiting Gibbs state for the Ising model on the Bethe lattice, *Journ. Statist. Phys.* **79**: 473-482 (1995).

[BRZ2] P.M. Bleher, J. Ruiz and V.A. Zagrebnov, On the phase diagram of the random field Ising model on the Bethe lattice, *Journ. Statist. Phys.* **93**: 33-78 (1998).

[BR1] O. Bratteli and D. Robinson, *Operator algebras and Quantum Statistical Mechanics I* (Springer-Verlag, Berlin/New-York, 1979).

[BR2] O. Bratteli and D. Robinson, *Operator algebras and Quantum Statistical Mechanics,II* (Springer-Verlag, Berlin/New-York, 1981).

[GPW] N.N.Ganikhodjaev,C.H.Pah and M.R.B.Wahiddin, Exact solution of an Ising model with competing interactions on a Cayley tree, *J.Phys. A: Math. Gen.* **36**: 4283-4289 (2003).

[GR] N.N. Ganikhodjaev and U.A. Rozikov, A description of periodic extremal Gibbs measures of some lattice models on the Cayley tree, *Theor. Math. Phys.* **111**: 480-486 (1997).

[L] R.Lyons, Phase transitions on nonamenable graphs, *Jour. Math. Phys.* **41**, 3 : 1099-1126 (2000).

[MTA] M.Mariz, C.Tsalis and A.L.Albuquerque, Phase diagram of the Ising model on a Cayley tree in the presence of competing interactions and magnetic field, *Jour. of Stat. Phys.* **40**: 577-592 (1985).

[M] F.M. Mukhamedov, Von Neumann algebras corresponding translation - invariant Gibbs states of Ising model on the Bethe lattice, *Theor. Math. Phys.* **123**: 489-493 (2000).
F.M.Mukhamedov and U.A.Rozikov, On Gibbs measures of models with competing ternary and binary interactions and corresponding von Neumann algebras. *Jour. of Stat.Phys.* **114**(2004), 3/4, 825-848.

J.Ramagge and G.Robertson, Farctors from trees, *Proc. Amer. Math. Soc.*, **125**: 2051-2055 (1997).

U.A. Rozikov, Partition structures of the group representation of the Cayley tree into cosets by finite-index normal subgroups and their applications to the description of periodic Gibbs distributions, *Theor. Math. Phys.* **112**: 929-933 (1997).