COMPUTING THE DEGREE OF A LATTICE IDEAL OF DIMENSION ONE

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Abstract. We show that the degree of a graded lattice ideal of dimension 1 is the order of the torsion subgroup of the quotient group of the lattice. This gives an efficient method to compute the degree of this type of lattice ideals.

1. Introduction

Let $S = K[t_1, \ldots, t_s]$ be a graded polynomial ring over a field $K$, where each $t_i$ is homogeneous of degree one, and let $S_d$ denote the set of homogeneous polynomials of total degree $d$ in $S$, together with the zero polynomial. The set $S_d$ is a $K$-vector space of dimension $\binom{d+s-1}{s-1}$. If $I \subset S$ is a graded ideal, i.e., $I$ is generated by homogeneous polynomials, we let $I_d = I \cap S_d$, denote the set of homogeneous polynomials in $I$ of total degree $d$, together with the zero polynomial. Note that $I_d$ is a vector subspace of $S_d$. Then the Hilbert function of the quotient ring $S/I$, denoted by $H_I(d)$, is defined by

$$H_I(d) = \dim_K(S_d/I_d).$$

According to a classical result of Hilbert [5, Theorem 4.1.3], there is a unique polynomial

$$h_I(t) = c_k t^k + \text{(terms of lower degree)}$$

of degree $k$, with rational coefficients, such that $h_I(d) = H_I(d)$ for $d \gg 0$. By convention the zero polynomial has degree $-1$, that is, $h_I(t) = 0$ if and only if $k = -1$. The integer $k + 1$ is the Krull dimension of $S/I$ and $h_I(t)$ is the Hilbert polynomial of $S/I$. If $k \geq 0$, the positive integer $c_k(k!)$ is called the degree of $S/I$. The degree of $S/I$ is defined as $\dim_K(S/I)$ if $k = -1$. The index of regularity of $S/I$, denoted by $\text{reg}(S/I)$, is the least integer $r \geq 0$ such that $h_I(d) = H_I(d)$ for $d \geq r$. The degree and the Krull dimension are denoted by $\deg(S/I)$ and $\dim(S/I)$, respectively. As usual, by the dimension of $I$ we mean the Krull dimension of $S/I$.

The notion of degree plays an important role in algebraic geometry [8, 9, 21] and commutative algebra [5, 12]. Consider a projective space $\mathbb{P}^{s-1}$ over the field $K$. The degree and the dimension of a projective variety $X \subset \mathbb{P}^{s-1}$ can be read off from the Hilbert polynomial $h_I(t)$, where $I = I(X)$ is the vanishing ideal of $X$ generated by the homogeneous polynomials of $S$ that vanish at all points of $X$. For the geometric interpretation of the degree of $S/I(X)$ see the references above. If $X$ is a finite set of points, the Hilbert polynomial of $S/I(X)$ is a non-zero constant, the degree of $S/I(X)$ is equal to $|X|$ (the number of points in $X$), and the dimension of $S/I(X)$ is equal to 1 [21, p. 164]. In view of its applications to coding theory, we are interested in the case when $K$ is a finite field and $X$ is parameterized by monomials (see Section 5).

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Let \( \mathcal{L} \subset \mathbb{Z}^s \) be a lattice, i.e., \( \mathcal{L} \) is a subgroup of \( \mathbb{Z}^s \). The lattice ideal of \( \mathcal{L} \), denoted by \( \mathcal{I}(\mathcal{L}) \), is the ideal of \( S \) generated by the set of all binomials \( t^a + t^{-a} \) such that \( a \in \mathcal{L} \), where \( a^+ \) and \( a^- \) are the positive and negative part of \( a \) (see Section 3). A first hint of the rich interaction between the group theory of \( \mathcal{L} \) and the algebra of \( \mathcal{I}(\mathcal{L}) \) is that the rank of \( \mathcal{L} \), as a free abelian group, is equal to \( s - \dim(S/\mathcal{I}(\mathcal{L})) \) [27, Proposition 7.5]. This number is the height of the ideal \( \mathcal{I}(\mathcal{L}) \) in the sense of [5], and is usually denoted by \( \text{ht}(\mathcal{I}(\mathcal{L})) \). Another useful result is that \( \mathbb{Z}^s/\mathcal{L} \) is a torsion-free group if and only if \( \mathcal{I}(\mathcal{L}) \) is a prime ideal [27, Theorem 7.4]. In the same vein, for a certain family of lattice ideals, we will relate the structure of the finitely generated abelian group \( \mathbb{Z}^s/\mathcal{L} \) and the degree of \( S/\mathcal{I}(\mathcal{L}) \).

The set of nonnegative integers (resp. positive integers) is denoted by \( \mathbb{N} \) (resp. \( \mathbb{N}_+ \)). The structure of \( \mathbb{Z}^s/\mathcal{L} \) can easily be determined, as we now explain. Let \( A \) be an integral matrix of order \( m \times s \) whose rows generate \( \mathcal{L} \). There are unimodular integral matrices \( U \) and \( V \) such that \( UAV = D \), where \( D = \text{diag}(d_1, d_2, \ldots, d_r, 0, \ldots, 0) \) is a diagonal matrix, with \( d_i \in \mathbb{N}_+ \) and \( d_i \) divides \( d_j \) if \( i \leq j \). The matrix \( D \) is the Smith normal form of \( A \) and the integers \( d_1, \ldots, d_r \) are the invariant factors of \( A \). Recall that the torsion subgroup of \( \mathbb{Z}^s/\mathcal{L} \), denoted by \( T(\mathbb{Z}^s/\mathcal{L}) \), consists of all \( \mathbf{r} \in \mathbb{Z}^s/\mathcal{L} \) such that \( \ell \mathbf{r} = \mathbf{0} \) for some \( \ell \in \mathbb{N}_+ \). From the fundamental structure theorem for finitely generated abelian groups [22] one has:

\[
\mathbb{Z}^s/\mathcal{L} \cong \mathbb{Z}/(d_1) \oplus \mathbb{Z}/(d_2) \oplus \cdots \oplus \mathbb{Z}/(d_r) \oplus \mathbb{Z}^{s-r},
\]
\[
T(\mathbb{Z}^s/\mathcal{L}) \cong \mathbb{Z}/(d_1) \oplus \mathbb{Z}/(d_2) \oplus \cdots \oplus \mathbb{Z}/(d_r),
\]

where \( r \) is the rank of \( \mathcal{L} \). In particular the order of \( T(\mathbb{Z}^s/\mathcal{L}) \) is \( d_1 \cdots d_r \). Thus, using any algebraic system that compute Smith normal forms of integral matrices, Maple [7] for instance, one can determine the order of \( T(\mathbb{Z}^s/\mathcal{L}) \).

Let \( \mathcal{I}(\mathcal{L}) \subset S \) be a graded lattice ideal of dimension one. Note that the corresponding lattice \( \mathcal{L} \) is homogeneous and has rank \( s-1 \) (see Definition 3.5 and Remark 3.6). The aim of this paper is to give a new method, using integer linear algebra, to compute the degree of \( S/\mathcal{I}(\mathcal{L}) \).

The contents of this paper are as follows. In Section 2 we present some well known results about the behavior of Hilbert functions of graded ideals. In particular, we recall a standard method, using Hilbert series, to compute the degree and the index of regularity.

In Section 3, we use linear algebra and Gröbner bases methods to describe the torsion subgroup of \( \mathbb{Z}^s/\mathcal{L} \) (see Lemmas 3.8 and 3.10). Then, using standard Hilbert functions techniques, we give an upper bound for the index of regularity (see Proposition 3.12).

The main result of this paper is the following formula for the degree:

\[
\deg S/\mathcal{I}(\mathcal{L}) = |T(\mathbb{Z}^s/\mathcal{L})|,
\]

where \( |T(\mathbb{Z}^s/\mathcal{L})| \) is the cardinality of the torsion subgroup (see Theorem 3.13). As a consequence, if \( \mathcal{L} \) is generated as a \( \mathbb{Z} \)-module by the rows of an integral matrix \( A \), then

\[
\deg S/\mathcal{I}(\mathcal{L}) = d_1 \cdots d_{s-1},
\]

where \( d_1, \ldots, d_{s-1} \) are the invariant factors of \( A \) (see Corollary 3.14). This gives a method to compute the degree directly from a set of generators of the lattice using the Smith normal form from linear algebra. It would be interesting to compute the index of regularity in terms of the lattice using linear algebra methods. Some other problems for future works are included in Section 4.

If \( \mathcal{B} \) is a \( \mathbb{Z} \)-basis for \( \mathcal{L} \) and \( \mathcal{P} \) is the convex hull of \( \mathcal{B} \cup \{0\} \) in \( \mathbb{R}^s \), we obtain the following expression for the degree:

\[
\deg S/\mathcal{I}(\mathcal{L}) = (s-1)!\text{vol}(\mathcal{P}),
\]
where “vol” is the relative volume in the sense of \[15, 36\] (see Corollary 3.18). If an integral polytope in \(\mathbb{R}^s\) has dimension \(s\), then its relative volume agrees with its usual volume \[36, p. 239\]. Note that in our situation the usual volume of \(\mathcal{P}\) is 0 because \(\mathcal{P}\) has dimension \(s - 1\). This is why we express the degree in terms of the relative volume.

There are standard methods to compute the degree of any grade \(d\) lattice ideal, using Gröbner bases and Hilbert series (see Sections 2 and 4), but our method is far more efficient, especially with large examples (see Examples 4.2 and 4.4). Our main result cannot be generalized to arbitrary lattice ideals (see Example 4.6).

In Section 5, we consider the case when \(K\) is a finite field and \(X\) is a subset of the projective space \(\mathbb{P}^{s-1}\) over the field \(K\). If \(X\) is parameterized by monomials, by Theorem 5.2, our results can be applied to the vanishing ideal \(I(X)\). The degree of a vanishing ideal is relevant from the viewpoint of coding theory because it occurs as one of the main parameters of evaluation codes \[32\]. We show an instance where the algebraic structure of the lattice is reflected in the algebraic structure of the vanishing ideal (see Corollary 5.3).

For all unexplained terminology and additional information, we refer to \[13, 27, 38\] (for the theory of binomial and lattice ideals) and \[8, 35\] (for Gröbner bases and Hilbert functions).

2. Hilbert functions and the degree

In this section we recall some well known results about the behavior of Hilbert functions of graded ideals and recall a standard method, using Hilbert series, to compute the degree.

We continue to use the notation and definitions used in Section 1. Let

\[ S = K[t_1, \ldots, t_s] = \bigoplus_{d=0}^{\infty} S_d \]

be a graded polynomial ring over a field \(K\), with the grading induced by setting \(\text{deg}(t_i) = 1\) for \(i = 1, \ldots, s\), and let \(I \subset S\) be a graded ideal. The ring \(S/I\) inherits a graded structure

\[ S/I = \bigoplus_{d=0}^{\infty} (S/I)_d \]

whose \(d\)-th component is given by \((S/I)_d = S_d/I_d\), where \(I_d = I \cap S_d\). Recall that the Hilbert function of \(S/I\) is given by

\[ H_I(d) = \dim_K S_d/I_d. \]

The degree and the regularity of \(S/I\) can be computed using Hilbert series, as we now explain. By the Hilbert Serre theorem, there is a unique polynomial \(g(t) \in \mathbb{Z}[t]\) with \(g(1) \neq 0\) such that the Hilbert series \(F_I(t)\) of \(S/I\) can be written as

\[ F_I(t) := \sum_{d=0}^{\infty} H_I(d)t^d = \frac{g(t)}{(1 - t)^\lambda}, \]

where \(\lambda\) is the Krull dimension of \(S/I\). The degree of \(S/I\) is equal to \(g(1)\) and the index of regularity of \(S/I\) is equal to 0 if \(\text{deg}(g(t)) - \lambda < 0\) and is equal to \(\text{deg}(g(t)) - \lambda + 1\) otherwise. Thus, the computation of the degree is reduced to the computation of the Hilbert series of \(S/I\). There are a number of computer algebra systems (\textit{Macaulay2 19}, \textit{CoCoA}, \textit{Singular}) that compute the Hilbert series and the degree of \(S/I\) using Gröbner bases. Two excellent references for computing Hilbert series, using elimination of variables, are \[2, 9\]. For toric ideals there
are methods, implemented in Normaliz [3], to compute its Hilbert series and its degree using polyhedral geometry.

We could not find a reference for the next simple lemma.

Lemma 2.1. (a) If \( S_i = I_i \) for some \( i \geq 1 \), then \( S_d = I_d \) for all \( d \geq i \).

(b) If \( \dim S/I \geq 1 \), then \( H_I(i) > 0 \) for \( i \geq 0 \).

Proof. (a) It suffices to prove the case \( d = i + 1 \). As \( I_{i+1} \subset S_{i+1} \), we need only show \( S_{i+1} \subset I_{i+1} \). Take a monomial \( f \) in \( S_{i+1} \). Then, \( f = t_1^{a_1} \cdots t_s^{a_s} \) with \( \sum_{i=1}^{s} a_i = i + 1 \) and \( a_j > 0 \) for some \( j \). Thus, \( f \in S_1 S_i \). As \( S_1 S_i = S_1 I_i \subset I_{i+1} \), we get \( f \in I_{i+1} \).

(b) The Hilbert polynomial \( h_I \) of \( S/I \) has degree \( \dim(S/I) - 1 \geq 0 \). Hence, \( h_I \) is a non-zero polynomial. If \( H_I(i) = \dim_K(S/I) \) \( = 0 \) for some \( i \), then \( S_i = I_i \). Thus, by (a), \( H_I(d) \) vanishes for \( d \geq 1 \), a contradiction because the Hilbert polynomial of \( S/I \) is non-zero.

Next, we recall and prove a general fact about 1-dimensional Cohen-Macaulay graded ideals: the Hilbert function is increasing until it reaches a constant value. This behaviour was pointed out in [11, p. 456] (resp. [17, Remark 1.1, p. 166]) for finite (resp. infinite) fields, see also [10]. No proof was given in neither of these places, likely because the result is not hard to show.

Proposition 2.2. (i) If \( \dim S/I \geq 2 \) and depth \( S/I > 0 \), then \( H_I(i) < H_I(i+1) \) for \( i \geq 0 \).

(ii) If depth \( S/I = \dim S/I = 1 \), then there is an integer \( r \geq 0 \) such that

\[ 1 = H_I(0) < H_I(1) < \cdots < H_I(r-1) < H_I(i) = \deg(S/I) \quad \text{for} \ i \geq r. \]

Proof. Consider the algebraic closure \( \overline{K} \) of the field \( K \). We set

\[ \overline{S} = S \otimes_K \overline{K} \quad \text{and} \quad \overline{I} = I \overline{S}. \]

By [35] Lemma 1.1, \( S/I \) and \( \overline{S}/\overline{I} \) have the same Krull dimension, the same depth, and the same Hilbert function. This follows by considering the minimal free resolution \( F_* \) of \( S/I \) and observing that \( F_* \otimes_K \overline{K} \) is the minimal free resolution of \( \overline{S}/\overline{I} \) and has the same numerical data (Betti numbers and shifts) as \( F_* \). Hence, replacing \( K \) by \( \overline{K} \), we may assume that \( K = \overline{K} \). As \( S/I \) has positive depth, there is \( h \in S_1 \) which is a non zero-divisor of \( S/I \). Applying the function \( \dim_K(\cdot) \) to the exact sequence

\[ 0 \rightarrow (S/I)[-1] \rightarrow S/I \rightarrow S/(h,I) \rightarrow 0, \]

we get \( H_I(i+1) - H_I(i) = H(i + 1) \geq 0 \) for \( i \geq 0 \), where \( H(i) = \dim_K(S/(h,I))_i \). We set \( S' = S/(h,I) \). Notice that \( \dim(S') = \dim(S/I) - 1 \).

(i) If \( H(i+1) = 0 \) for some \( i \geq 0 \), then, by Lemma 2.1(a), \( \dim_K(S') < \infty \). Hence \( S' \) is Artinian, i.e., \( \dim(S') = 0 \), a contradiction. Thus, \( H_I(i+1) > H_I(i) \) for \( i \geq 0 \).

(ii) Since \( \dim(S/I) = 1 \), the Hilbert polynomial of \( S/I \) is a non-zero constant equal to \( \deg(S/I) \). Let \( r \geq 0 \) be the first integer such that \( H_I(r) = H_I(r+1) \), thus \( S'_r + 1 = (0) \), i.e., \( S_{r+1} = (h,I)_{r+1} \). Then, by Lemma 2.1(a), \( S_i = (0) \) for \( i \geq r + 1 \). Hence, the Hilbert function of \( S/I \) is constant for \( i \geq r \) and strictly increasing on \( [0, r-1] \). \( \square \)

3. The degree of a lattice ring

We continue to use the notation and definitions used in Sections 1 and 2. Given a homogeneous lattice \( L \subset \mathbb{Z}^s \) of rank \( s - 1 \), in this section we describe the torsion subgroup of \( \mathbb{Z}^s/L \) using linear algebra and Gröbner bases techniques. Then, we show that the degree of the lattice ring \( S/I(L) \) is the order of the torsion subgroup of \( \mathbb{Z}^s/L \).
Recall that a binomial in $S$ is a polynomial of the form $t^a - t^b$, where $a, b \in \mathbb{N}^s$ and where, if $a = (a_1, \ldots, a_s) \in \mathbb{N}^s$, we set 

$$t^a = t_1^{a_1} \cdots t_s^{a_s} \in S.$$ 

A polynomial of the form $t^a - t^b$ is usually referred to as a pure binomial [13], although here we are dropping the adjective “pure”. A binomial ideal is an ideal generated by binomials.

Given $a = (a_i) \in \mathbb{Z}^s$, the set $\text{supp}(a) = \{i | a_i \neq 0\}$ is called the support of $a$. The vector $a$ can be written uniquely as $a = a^+ - a^-$, where $a^+$ and $a^-$ are two nonnegative vectors with disjoint support, the positive and the negative part of $a$ respectively.

**Definition 3.1.** Let $\mathcal{L} \subset \mathbb{Z}^s$ be a lattice, that is, $\mathcal{L}$ is a subgroup of $\mathbb{Z}^s$. The lattice ideal of $\mathcal{L}$ is the binomial ideal

$$I(\mathcal{L}) := \langle \{t^a - t^b | a \in \mathcal{L}\} \rangle \subset S.$$ 

The lattice ring of $\mathcal{L}$ is the quotient ring $S/I(\mathcal{L})$.

Lattice ideals have been studied extensively, see [13, 16, 24, 27, 28] and the references therein. The concept of a lattice ideal is a natural generalization of a toric ideal [38, Corollary 7.1.4]. A lattice ideal $I(\mathcal{L})$ is a toric ideal if and only if $\mathbb{Z}^s/\mathcal{L}$ is torsion-free [27, Theorem 7.4].

The next lemma gives the general form of the binomials in a lattice ideal.

**Lemma 3.2.** Let $\mathcal{L} \subset \mathbb{Z}^s$ be a lattice and let $t^a - t^b$ be a binomial in $S$. Then, $a - b$ is in $\mathcal{L}$ if and only if $t^a - t^b$ is in $I(\mathcal{L})$.

**Proof.** If $a - b \in \mathcal{L}$, then by definition of a lattice ideal one has $t^a - t^b \in I(\mathcal{L})$. To show the converse assume that $t^a - t^b$ is in $I(\mathcal{L})$. We define an equivalence relation $\sim_{\mathcal{L}}$ on the set of monomials of $S$ by $t^c \sim_{\mathcal{L}} t^d$ if and only if $c - d \in \mathcal{L}$. Since $t^a$ and $t^b$ are not in $I(\mathcal{L})$, by [14, Lemma 2.2], we get that $t^a \sim_{\mathcal{L}} t^b$. Thus, $a - b \in \mathcal{L}$. \hfill $\square$

Given a binomial $g = t^a - t^b$, we set $\tilde{g} = a - b$. If $B$ is a subset of $\mathbb{Z}^s$, $(B)$ denotes the subgroup of $\mathbb{Z}^s$ generated by $B$.

A lattice ideal is defined by a unique lattice.

**Lemma 3.3.** [24] Let $\mathcal{L} \subset \mathbb{Z}^s$ be a lattice and let $I(\mathcal{L})$ be its lattice ideal. If $g_1, \ldots, g_r$ is a set of binomials that generate $I(\mathcal{L})$, then $\mathcal{L} = \langle \tilde{g}_1, \ldots, \tilde{g}_r \rangle$. In particular if $L$ is a lattice ideal, there is a unique lattice $\mathcal{L}$ such that $L = I(\mathcal{L})$.

The following is a well known description of lattice ideals that follows from [13, Corollary 2.5].

**Theorem 3.4.** [13] If $L$ is a binomial ideal of $S$, then $L$ is a lattice ideal if and only if $t_i$ is a non-zero divisor of $S/L$ for all $i$.

**Proof.** Let $g_1, \ldots, g_r$ be a set of generators of $L$ consisting of binomials. Since $L$ does not contain any monomials, by [13, Corollary 2.5], there is a unique lattice $\mathcal{D} \subset \mathbb{Z}^s$ such that

$$(\dagger) \quad (L; h^\infty) = I(\mathcal{D}),$$ 

where $h = t_1 \cdots t_s$ and $(L; h^\infty) := \{f \in S | fh^m \in L \text{ for some } m \geq 1\}$. It is not hard to see that $\mathcal{D}$ is generated by $\tilde{g}_1, \ldots, \tilde{g}_r$ (cf. Proposition [4.1]).

$\Rightarrow$ If $L = I(\mathcal{L})$ for some lattice $\mathcal{L}$, by Lemma 3.3 $\mathcal{L} = \mathcal{D}$. Thus, $(L; h^\infty) = L$. To show that $t_i$ is a non zero-divisor of $S/L$ assume that there is $\overline{f} \in S/L$ such that $t_i\overline{f} = \overline{0}$. Thus, $t_i f \in L$ and $hf \in L$. Hence $f \in L$, i.e., $\overline{f} = \overline{0}$, as required.
Hence, taking inner products with \( 1 \), we need only show the equality \((L; h^\infty) = L\). The inclusion \((L; h^\infty) \supset L\) is clear. To show the reverse inclusion take \( f \in (L; h^\infty) \), i.e., \( h^m f \in L \). Since \( t_i \) is a nonzero divisor for all \( i \), we get that \( f \in L \) as required.

The unique lattice that defines a graded lattice ideal is homogeneous in the following sense.

**Definition 3.5.** If \( a = (a_1, \ldots, a_s) \in \mathbb{Z}^s \), we set \(|a| = \sum_{i=1}^s a_i\). A lattice \( L \) is called *homogeneous* if \(|a| = 0\) for all \( a \in L \).

**Remark 3.6.** (i) A lattice is homogeneous if and only if its lattice ideal is graded. This follows from Lemma 3.3.

(ii) If \( L \) is a homogeneous lattice in \( \mathbb{Z}^s \) of rank \( s - 1 \), then \( S/I(L) \) is a Cohen-Macaulay ring of dimension 1. This follows from Theorem 3.4 and using the fact that the height of \( I(L) \) is the rank of \( L \) [27, Proposition 7.5].

**Definition 3.7.** The *torsion subgroup* of an abelian group \((M, +)\), denoted by \( T(M) \), is the set of all \( x \) in \( M \) such that \( px = 0 \) for some \( 0 \neq p \in \mathbb{N} \).

Next, we determine a generating set for the torsion subgroup of \( \mathbb{Z}^s/L \).

**Lemma 3.8.** Let \( L \subset \mathbb{Z}^s \) be a homogeneous lattice of rank \( s - 1 \) and let \( \mathbb{Q}L \) be the \( \mathbb{Q} \)-linear space spanned by \( L \). Then

(a) \( \mathbb{Q}L \cap \mathbb{Z}^s = \mathbb{Z}(e_1 - e_s) \oplus \cdots \oplus \mathbb{Z}(e_{s-1} - e_s) \), where \( e_i \) is the \( i \)th unit vector in \( \mathbb{Q}^s \).

(b) \( T(\mathbb{Z}^s/L) = \mathbb{Z}(e_1 - e_s) \oplus \cdots \oplus \mathbb{Z}(e_{s-1} - e_s)/\mathbb{L} \).

**Proof.** (a) “⊆”: Take \( a = (a_1, \ldots, a_s) \) in \( \mathbb{Q}L \cap \mathbb{Z}^s \). Then, \( a_s = -a_1 - \cdots - a_{s-1} \) and we can write

\[
\begin{align*}
a &= a_1(e_1 - e_s) + \cdots + a_{s-1}(e_{s-1} - e_s) \in \mathbb{Q}L \cap \mathbb{Z}^s.
\end{align*}
\]

Thus, \( a \) is a \( \mathbb{Z} \)-linear combination of \( e_1 - e_s, \ldots, e_{s-1} - e_s \).

“⊇”:
It suffices to show that \( e_k - e_s \) is in \( \mathbb{Q}L \) for all \( k \). The dimension of \( \mathbb{Q}L \) is equal to \( \text{rank}(L) = s - 1 \). Notice that \( e_s \notin \mathbb{Q}L \) because \( L \) is homogeneous. Hence, \( \mathbb{Q}e_s + \mathbb{Q}L = \mathbb{Q}^s \).

Therefore, we can write

\[
\begin{align*}
e_k &= \mu_{ks} e_s + \lambda_{k1} \alpha_1 + \cdots + \lambda_{km} \alpha_m \quad (\mu_{ks} \in \mathbb{Q}; \lambda_{ki} \in \mathbb{Q}; \alpha_j \in L \text{ for all } i, j).
\end{align*}
\]

Hence, taking inner products with \( 1 = (1, \ldots, 1) \) and using that \( (1, \alpha_i) = 0 \) for all \( i \), we get \( \mu_{ks} = 1 \). Thus, \( e_k - e_s \in \mathbb{Q}L \).

(b): By [15] Lemma 2.3], the torsion subgroup of \( \mathbb{Z}^s/L \) is \( \mathbb{Q}L \cap \mathbb{Z}^s/L \). Hence, the expression for the torsion follows from (a).

In what follows we shall assume that \( > \) is the *reverse lexicographical order* (revlex order for short) on the monomials of \( S \). This order is given by \( t^b > t^a \) if and only if the last non-zero entry of \( b - a \) is negative. As usual, if \( g \) is a polynomial of \( S \), we denote the leading term of \( g \) by \( \text{in}(g) \). If \( L \) is an ideal of \( S \), the initial ideal of \( L \), denoted by \( \text{in}(L) \), is generated by the leading terms of the polynomials of \( L \).

**Remark 3.9.** By Buchberger’s algorithm [8, Theorem 2, p. 89] and [8, Proposition 6, p. 91], a graded lattice ideal \( I(L) \) has a unique reduced Gröbner basis \( G \) consisting of homogeneous binomials and, by Theorem 3.4, each binomial \( t^a - t^b \in G \) satisfies that \( \text{supp}(a) \cap \text{supp}(b) = \emptyset \).

**Lemma 3.10.** Let \( L \subset \mathbb{Z}^s \) be a homogeneous lattice of rank \( s - 1 \). Then, given \( \bar{\alpha} = \alpha + L \) in the torsion subgroup \( T(\mathbb{Z}^s/L) \) there exists a unique \( a = (a_1, \ldots, a_{s-1}, a_s) \in \mathbb{Z}^s \) such that
Definition 3.11. An ideal \( I \subset S \) is called a complete intersection if there exists \( f_1, \ldots, f_r \) in \( S \) such that \( I = (f_1, \ldots, f_r) \), where \( r \) is the height of \( I \).

A graded ideal \( I \) is a complete intersection if and only if \( I \) is generated by a homogeneous regular sequence with \( \text{ht}(I) \) elements (see [38 Proposition 1.3.17, Lemma 1.3.18]).

Proposition 3.12. If \( L \subset S \) is a graded lattice ideal of dimension 1, then there are positive integers \( n_1, \ldots, n_s-1 \) such that

(a) \( L' = (t_{i_1}^{n_1} - t_{s-1}^{n_s}, \ldots, t_{i_{s-1}}^{n_{s-1}} - t_{s-1}^{n_s}) \subset L \),

(b) \( \text{reg}(S/(t_s, L)) = \text{reg}(S/(t_s, L')) = 1 + \sum_{i=1}^{s-1}(n_i - 1) \), and
Then, the monomial $a_i$ from Lemma 3.10, the map $n_i$ by noticing the equality $I_i$ such that

If $t_i, L \mapsto H_{(t_i, L)}(d) = \deg S/L$ for $d \geq \sum_{i=1}^{s-1}(n_i - 1) + 1$.

Proof. (a): Let $L \subset \mathbb{Z}^s$ be the lattice that defines $L$, i.e., $L = I(L)$. By Lemma 3.8, there are positive integers $n_1, \ldots, n_{s-1}$ such that $n_i(e_i - e_s) \in L$ for all $i$. Thus, $t_i^{n_i} - t_s^{n_s} \in L$ for all $i$.

(b): Since $\dim(S/(t_s, L)) = \dim(S/(t_s, L))$, the Hilbert polynomials of $S/(t_s, L')$ and $S/(t_s, L)$ are equal to zero. Using the epimorphism

$S/(t_s, L') \twoheadrightarrow S/(t_s, L) \rightarrow 0$,

we get that $H_{(t_s, L')}(d) = H_{(t_s, L)}(d)$ for $d \geq 0$. Hence, the index of regularity of $S/(t_s, L)$ is bounded from above by the index of regularity of $S/(t_s, L')$. The ideal $I = (t_1^{n_1}, \ldots, t_{s-1}^{n_{s-1}})$ is a complete intersection of the polynomial ring $R = K[t_1, \ldots, t_{s-1}]$, hence the Hilbert series of $R/I$ is equal to the polynomial

$F_I(t) = (1 + t + \cdots + t^{n-1}) \cdots (1 + t + \cdots + t^{n-1-1}),$

see [38, p. 104]. Thus, the index of regularity of $R/I$ is $\sum_{i=1}^{s-1}(n_i - 1) + 1$ (see Section 2). As $S/(t_s, L') \simeq R/I$, we get $\deg(S/(t_s, L')) = \sum_{i=1}^{s-1}(n_i - 1) + 1$.

(c): Assume that $d \geq \sum_{i=1}^{s-1}(n_i - 1) + 1$. There is an exact sequence of graded rings

$0 \rightarrow (S/L)[-1] \rightarrow S/L \rightarrow S/(t_s, L) \rightarrow 0$.

Hence, $H_L(d) = H_L(d-1) = \dim_K(S/(t_s, L))_d = H_{(t_s, L)}(d)$. Therefore, using (b), we obtain $H_{(t_s, L)}(d) = 0$ and $H_L(d) = \deg(S/L)$ (cf. Proposition 2.2(ii)).

We come to the main result of this paper.

**Theorem 3.13.** If $I(L) \subset S$ is a graded lattice ideal of dimension 1, then

$\deg S/I(L) = |T(\mathbb{Z}^s/L)|$.

Proof. Let $\succ$ be the revlex order on the monomials of $S$ and let $\text{in}(I(L))$ be the initial ideal of $I(L)$. We set $d = \sum_{i=1}^{s-1}(n_i - 1) + 1$. By Proposition 3.12, there are positive integers $n_1, \ldots, n_{s-1}$ such that $t_i^{n_i} - t_s^{n_s} \in L(L)$ for all $i$ and $H_L(L)(d) = \deg S/I(L)$. There is an injective map

$B_d = \{t^c \mid t^c \notin \text{in}(I(L))\} \cap S_d \rightarrow (S/I(L))_d, \ t^c \mapsto t^c + I(L)$.

By a classical result in Gröbner bases theory [31 Proposition 1, p. 228], the image of this map is a basis for the $K$-vector space $(S/I(L))_d$. Thus, $|B_d| = H_L(L)(d)$. Consider the map

$\phi: B_d \rightarrow T(\mathbb{Z}^s/L), \ t^c = t_1^{c_1} \cdots t_s^{c_s} \mapsto \phi(c_1, \ldots, c_{s-1}, c_s - d) + L$.

The map $\phi$ is well defined, i.e., $\phi(t^c)$ is in $T(\mathbb{Z}^s/L)$ for all $t^c$ in $B_d$. This follows directly from Lemma 3.8(b) by noticing the equality

$(c_1, \ldots, c_{s-1}, c_s - d) = c_1(e_1 - e_s) + \cdots + c_{s-1}(e_{s-1} - e_s)$.

Altogether, we need only show that $\phi$ is bijective. Notice that $t_s^{d}$ maps to $\tilde{0}$ under $\phi$. By Lemma 3.10, the map $\phi$ is injective. To show that $\phi$ is onto, take $\tilde{a} \in T(\mathbb{Z}^s/L)$. By Lemma 3.10, we may assume that $a_i \geq 0$ for $i = 1, \ldots, s-1$ and $t_1^{a_1} \cdots t_{s-1}^{a_{s-1}} \notin \text{in}(I(L))$. Notice that $0 \leq a_i \leq n_i - 1$ for $i = 1, \ldots, s-1$ because $t_i^{n_i} - t_s^{n_s} \in L(L)$ for all $i$. Thus, $\sum_{i=1}^{s-1} a_i \leq \sum_{i=1}^{s-1}(n_i - 1) < d$.

Consider the vector $c = (c_1, \ldots, c_s)$ given by $c_i = a_i$ for $i = 1, \ldots, s-1$ and $c_s = d - \sum_{i=1}^{s-1} a_i$. Then, the monomial $t^c$ is in $B_d$ and maps to $\tilde{a}$ under the map $\phi$. □
Corollary 3.14. Let $\mathcal{L} \subset \mathbb{Z}^s$ be a homogeneous lattice of rank $s - 1$ generated as a $\mathbb{Z}$-module by the rows of an integral matrix $A$. Then

$$\deg S/I(\mathcal{L}) = d_1 \cdots d_{s-1},$$

where $d_1, \ldots, d_{s-1}$ are the invariant factors of $A$.

Proof. It is well known [30, Theorem II.9, pp. 26-27] that there are unimodular integral matrices $U$ and $V$ such that

$$UAV = D = \text{diag}\{d_1, \ldots, d_{s-1}, 0, \ldots, 0\},$$

di $> 0$ for $1 \leq i \leq s - 1$ and $d_i$ divides $d_{i+1}$ for all $i$. In matrix theory terminology, this means that $D = \text{diag}\{d_1, \ldots, d_{s-1}, 0, \ldots, 0\}$ is the Smith normal form of $A$ and $d_1, \ldots, d_{s-1}$ are the invariant factors of $A$. Hence, by the fundamental structure theorem for finitely generated abelian groups $[22]$ pp. 187-188], we get

$$\mathbb{Z}^s/\mathcal{L} \simeq \mathbb{Z}/(d_1) \oplus \cdots \oplus \mathbb{Z}/(d_{s-1}) \oplus \mathbb{Z} \quad \text{and} \quad T(\mathbb{Z}^s/\mathcal{L}) \simeq \mathbb{Z}/(d_1) \oplus \cdots \oplus \mathbb{Z}/(d_{s-1}).$$

Thus, the result follows from Theorem 3.13.

□

Corollary 3.15. Let $L \subset S$ be a graded lattice ideal of dimension 1. If $L$ is generated by the binomials $t^{a_1} - t^{a_i}, \ldots, t^{a_m} - t^{a_m}$. Then

$$\deg S/L = d_1 \cdots d_{s-1},$$

where $d_1, \ldots, d_{s-1}$ are the invariant factors of the matrix $A$ whose rows are $\alpha_1, \ldots, \alpha_m$.

Proof. Let $\mathcal{L}$ be the homogeneous lattice that defines the lattice ideal $L$. By Lemma 3.13 one has the equality $\mathcal{L} = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_m$. Thus, the result follows at once from Corollary 3.14.

□

Lemma 3.16. [37] pp. 32-33) If $H \subset G$ are free abelian groups of the same rank $r$ with $\mathbb{Z}$-bases $\delta_1, \ldots, \delta_r$ and $\gamma_1, \ldots, \gamma_r$ related by $\delta_i = \sum_j g_{ij} \gamma_j$, where $g_{ij} \in \mathbb{Z}$ for all $i, j$, then $|G/H| = |\det(g_{ij})|.

Definition 3.17. Let $\Delta$ be a lattice $n$-simplex in $\mathbb{R}^s$, i.e., $\Delta$ is the convex hull of a set of $n + 1$ affinely independent points in $\mathbb{Z}^s$. The normalized volume of $\Delta$ is defined as $n!\text{vol}(\Delta)$.

The next result shows that the degree is the normalized volume of any $(s - 1)$-simplex arising from a $\mathbb{Z}$-basis of $\mathcal{L}$.

Corollary 3.18. If $\mathcal{L} \subset \mathbb{Z}^s$ is a homogeneous lattice and $\alpha_1, \ldots, \alpha_{s-1}$ is a $\mathbb{Z}$-basis of $\mathcal{L}$, then

$$\deg S/I(\mathcal{L}) = (s - 1)!\text{vol}(\text{conv}(0, \alpha_1, \ldots, \alpha_{s-1})),$$

where $\text{vol}$ is the relative volume and $\text{conv}$ is the convex hull.

Proof. By hypothesis, $\mathcal{L} = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_{s-1}$. Hence, using Lemma 3.8(b), we get the equality

$$T(\mathbb{Z}^s/\mathcal{L}) = \mathbb{Z}(e_1 - e_s) \oplus \cdots \oplus \mathbb{Z}(e_{s-1} - e_s)/\mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_{s-1}.$$ 

For $1 \leq i \leq s - 1$, we can write $\alpha_i = \alpha_{i,1}(e_1 - e_s) + \cdots + \alpha_{i,s-1}(e_{s-1} - e_s)$, where $\alpha_{i,j}$ is the $j$th entry of $\alpha_i$. Applying Theorem 3.13 and Lemma 3.16 gives

$$\deg S/I(\mathcal{L}) = |T(\mathbb{Z}^s/\mathcal{L})| = \left| \begin{array}{ccc} \alpha_{1,1} & \cdots & \alpha_{1,s-1} \\ \vdots & \ddots & \vdots \\ \alpha_{s-1,1} & \cdots & \alpha_{s-1,s-1} \end{array} \right| = (s - 1)!\text{vol}(\Delta),$$

where $\Delta = \text{conv}(0, (\alpha_{1,1}, \ldots, \alpha_{1,s-1}), \ldots, (\alpha_{s-1,1}, \ldots, \alpha_{s-1,s-1}))$ is a simplex in $\mathbb{R}^{s-1}$. To finish the proof we need only show that $\text{vol}(\Delta) = \text{vol}(\text{conv}(0, \alpha_1, \ldots, \alpha_{s-1}))$. This follows from the very definition of the notion of a relative volume (see [15] Section 2 and [36] p. 238]).

□
Corollary 3.19. Let $I(\mathcal{L}) \subset S$ be a graded lattice ideal of dimension 1. If $I(\mathcal{L})$ is a complete intersection generated by $t^{\alpha_1} - t^{\alpha_1 - 1}, \ldots, t^{\alpha_s} - t^{\alpha_s - 1}$, then
\[
\deg S/I(\mathcal{L}) = (s - 1)!\vol(\conv(0, \alpha_1, \ldots, \alpha_{s-1})).
\]
Proof. By Lemma 3.3, one has the equality $\mathcal{L} = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_{s-1}$. Thus, the formula for the degree follows from Corollary 3.18. □

Corollary 3.20. If $I(\mathcal{L}) \subset S$ is a graded lattice ideal of dimension 1, then $\mathbb{Z}^s/\mathcal{L}$ is torsion-free if and only if $I(\mathcal{L}) = (t_1 - t_s, \ldots, t_{s-1} - t_s)$.

Proof. Assume that $\mathbb{Z}^s/\mathcal{L}$ is torsion-free. Then, by Lemma 3.8(b), one has the equality.
\[
\mathcal{L} = \mathbb{Z}(e_1 - e_s) \oplus \cdots \oplus \mathbb{Z}(e_{s-1} - e_s).
\]
Hence, $I(\mathcal{L}) = (t_1 - t_s, \ldots, t_{s-1} - t_s)$. The converse is clear because the $(s - 1) \times s$ matrix with rows $e_1 - e_s, \ldots, e_{s-1} - e_s$ diagonalizes over the integers to an identity matrix. □

4. Computing some examples

Given a set of generators of a homogeneous lattice $\mathcal{L} \subset \mathbb{Z}^s$, a standard method to compute the degree of the lattice ring $S/I(\mathcal{L})$ consists of two steps. First, one computes a generating set for $I(\mathcal{L})$ using the following result:

Proposition 4.1. [27, Lemma 7.6] If $\mathcal{L} \subset \mathbb{Z}^s$ is a lattice generated by $\alpha_1, \ldots, \alpha_m$ and $Q$ is the ideal generated by $t^{\alpha_1} - t^{\alpha_1 - 1}, \ldots, t^{\alpha_m} - t^{\alpha_m - 1}$, then
\[
(Q: (t_1 \cdots t_s)^\infty) = I(\mathcal{L}),
\]
where $(Q: h^\infty) := \{ f \in S \mid fh^p \in Q \text{ for some } p \geq 1 \}$ is the saturation of $Q$ and $h = t_1 \cdots t_s$.

Second, one uses Hilbert functions (as described in Section 2) to compute the degree of $S/I(\mathcal{L})$. The handy command “degree” of Macaulay2 [19] computes the degree.

This standard method works for any homogeneous lattice. For homogeneous lattices of rank $s - 1$, our method is far more efficient, especially with large examples.

Example 4.2. Let $\mathcal{L} \subset \mathbb{Z}^5$ be the homogeneous lattice of rank 4 generated by the rows of the matrix
\[
A = \begin{pmatrix}
1001 & -500 & -501 & 0 & 0 \\
0 & 3500 & -3500 & 0 & 0 \\
0 & 0 & 3200 & -200 & -3000 \\
5000 & -1000 & -1000 & -1001 & -1999
\end{pmatrix}.
\]
The following procedure for Maple [7]
\begin{verbatim}
with(LinearAlgebra):
A:=<1001,-500,-501,0,0; 0,3500,-3500,0,0;
0,0,3200,-200,-3000; 5000,-1000,-1000,-1001,-1999>:
SmithForm(A);
\end{verbatim}
computes the Smith normal form of $A$:
\[
D = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 100 & 0 & 0 \\
0 & 0 & 0 & 91203112000 & 0 \\
\end{pmatrix}.
\]
Thus, by Theorem 3.13 we obtain $\deg S/I(L) = (2^8)(5^5)(7^2)(11)(13)(1627)$. The standard procedure for computing the degree of $S/I(L)$ fails for this example. Indeed, Macaulay2 does not even compute the saturation $(Q : h^\infty)$ of the ideal

$$Q = (t_1^{1001} - t_2^{500}t_3^{501}, t_2^{3500} - t_3^{3200}t_4^{500}, t_1^{5000} - t_2^{1000}t_3^{1001}t_4^{1999})$$

with respect to $h = t_1t_2t_3t_4t_5$. Notice that $Q$ is a complete intersection and accordingly $\deg(S/Q) = (1001)(3500)(3200)(5000) = (2^{12})(5^9)(7^2)(11)(13)$.

**Remark 4.3.** Given an integral matrix $A$, the Macaulay2 function “smithNormalForm” produces a diagonal matrix $D$, and unimodular matrices $U$ and $V$ such that $D = UAV$. Warning: even though this function is called the Smith normal form, it doesn’t necessarily satisfy the more stringent condition that the diagonal entries $d_1, d_2, \ldots, d_m$ of $D$ satisfy: $d_1 \mid d_2 \mid \cdots \mid d_m$. For this reason we prefer to use Maple\[7\] to compute the Smith normal form of $A$.

**Example 4.4.** Let $L \subset \mathbb{Z}^3$ be the homogeneous lattice of rank 2 generated by the rows of the matrix

$$A = \begin{pmatrix} 18 & -18 & 0 \\ 45 & 0 & -45 \\ 0 & 10 & -10 \end{pmatrix}.$$ 

The following procedure for Maple\[7\]

with(LinearAlgebra):
A:=<18,-18,0; 45,0,-45; 0,10,-10>:
SmithForm(A);

computes the Smith normal form of $A$:

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 90 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

Thus, by Theorem 3.13 we obtain $\deg S/I(L) = 90$. The standard procedure for computing the degree of $S/I(L)$ works fine in this “small” example. Indeed, using the following procedure for Macaulay2

S=QQ[t1,t2,t3]
Q=ideal(t1^18-t2^18,t1^45-t3^45,t2^10-t3^10)
saturate(Q,t1^2*t2*t3)
degree saturate(Q,t1^2*t2*t3)

we obtain

$$I(L) = (Q : (t_1^2t_2t_3)^\infty) = (t_1^8 - t_2^5t_3^2, t_2^{10} - t_3^{10})$$
and
$$\deg(S/I(L)) = 90.$$ 

**Remark 4.5.** The program Normaliz\[6\] computes the normalized volume of lattice polytopes. Hence, by Corollary 3.18 we can use this program with the handy option -v to compute the degree. This of course requires computing a $\mathbb{Z}$-basis of the lattice first. We computed the degree of Example 4.4 without any problem using “normbig.exe”.

Our main result does not extends to graded lattice ideals of dimension $\geq 2$.

**Example 4.6.** Consider the homogeneous lattice $L = \mathbb{Z}(-1, 2, -1) \subset \mathbb{Z}^3$. Then,

$$I(L) = (t_2^2 - t_1t_3)$$
and
$$\deg \mathbb{Q}[t_1,t_2,t_3]/I(L) = 2 \neq 1 = |T(\mathbb{Z}^3/L)|.$$
5. Vanishing ideals over finite fields

In this section, we link our results to vanishing ideals over finite fields and present an application. Vanishing ideals are connected to coding theory as is seen below.

**Definition 5.1.** The projective space of dimension \( s - 1 \) over a field \( K \), denoted by \( \mathbb{P}^{s-1} \), is the quotient space

\[
(K^s \setminus \{0\}) / \sim
\]

where two points \( \alpha, \beta \) in \( K^s \setminus \{0\} \) are equivalent, \( \alpha \sim \beta \), if \( \alpha = \lambda \beta \) for some \( \lambda \in K \). We denote the equivalence class of \( \alpha \) by \([\alpha]\).

Let \( \mathbb{F}_q \) be a finite field with \( q \) elements and let \( v_1, \ldots, v_s \) be a sequence of vectors in \( \mathbb{N}^n \) with \( v_i = (v_{i1}, \ldots, v_{in}) \) for \( 1 \leq i \leq s \). Consider the projective algebraic toric set

\[
X := \{[(x_1^{v_{11}} \cdots x_n^{v_{1n}}), \ldots, x_1^{v_{sn}} \cdots x_n^{v_{sn}})] \mid x_i \in \mathbb{F}_q^* \text{ for all } i \} \subset \mathbb{P}^{s-1}
\]

parameterized by the monomials \( x^{v_1}, \ldots, x^{v_s} \), where \( \mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\} \) and \( \mathbb{P}^{s-1} \) is the projective space of dimension \( s - 1 \) over the field \( \mathbb{F}_q \). The set \( X \) is a multiplicative group under componentwise multiplication.

Let \( S = \mathbb{F}_q[t_1, \ldots, t_s] = S_0 \oplus S_1 \oplus \cdots \oplus S_q \oplus \cdots \) be a polynomial ring over the field \( \mathbb{F}_q \) with the standard grading. Recall that the vanishing ideal of \( X \), denoted by \( I(X) \), is the ideal of \( S \) generated by the homogeneous polynomials that vanish at all points of \( X \).

According to the next theorem, our results can be applied to this family of vanishing ideals.

**Theorem 5.2.** If \( \mathbb{F}_q \) is a finite field, then

(a) \[ I(X) \text{ is a radical 1-dimensional Cohen-Macaulay ideal.} \]
(b) \[ \text{There is a unique homogeneous lattice } \mathcal{L} \text{ such that } I(X) = I(\mathcal{L}). \]
(c) \[ \text{Lecture 13} \quad H_{I(X)}(d) = |X| \text{ for } d \geq |X| - 1. \]

Hence, by (c), the degree of \( S/I(X) \) is equal to \( |X| \). Thus, our results can be used to compute \( |X| \), especially in cases where the homogeneous lattice that defines the ideal \( I(X) \) is known (see for instance \[ 32 \] Theorem 2.5) for such cases).

The degree of \( S/I(X) \) is relevant from the viewpoint of algebraic coding theory as we now briefly explain. Roughly speaking, an evaluation code over \( X \) of degree \( d \) is a linear space obtained by evaluating all homogeneous \( d \)-forms of \( S \) on the set of points \( X \subset \mathbb{P}^{s-1} \) (see \[ 11 \] \[ 18 \]). An evaluation code over \( X \) of degree \( d \) has length \( |X| \) and dimension \( H_{I(X)}(d) \). The main parameters (length, dimension, minimum distance) of evaluation codes of this type have been studied in \[ 11 \] \[ 18 \] \[ 20 \] \[ 23 \] \[ 33 \]. The index of regularity is useful in coding theory because potentially good evaluation codes can occur only if \( 1 \leq d < \text{reg}(S/I(X)) \).

The complete intersection property of \( I(X) \) was recently characterized in \[ 24 \] in algebraic and geometric terms (see also \[ 33 \]). If \( X \) is parameterized by the edges of a clutter, then \( I(X) \) is a complete intersection if and only if \( X \) is a projective torus \[ 33 \].

Let \( \mathcal{L} \) be the homogeneous lattice that defines \( I(X) \). The next result shows how the algebraic structure of \( \mathbb{Z}^s/\mathcal{L} \) is reflected in the algebraic structure of \( I(X) \).

**Corollary 5.3.** If \( q - 1 \) is a prime number such that \( v_i \not\equiv v_j \mod (q - 1) \) for \( i \neq j \) and \( T(\mathbb{Z}^s/\mathcal{L}) \cong (\mathbb{Z}_{q-1})^{s-1} \), then \( I(X) \) is a complete intersection if and only if

\[
I(X) = (t_1^{q-1} - t_s^{q-1}, \ldots, t_{s-1}^{q-1} - t_s^{q-1}).
\]
Thus, our main result gives a combinatorial formula for the order of graphs (see [1, 25] and the references therein). By the Kirchhoff’s matrix tree theorem, the ideal \( S/I \) of \( G \), \( I \) of \( G \) is called the sandpile group of \( G \). Its group structure is only known for a few families of graphs (see [1, 25] and the references therein). By the Kirchhoff’s matrix tree theorem, the order of \( T(\mathbb{Z}^s/\mathcal{L}) \) is the number of spanning trees of \( G \) (see [4, Theorem 6.3, p. 39] and [26, Theorem 1.1]). Thus, by our main result, one obtains a nice combinatorial formula for the degree of \( S/I \). On the other hand in the setting of commutative algebra and coding theory [29, 32], the degree of \( S/I(X) \) has been computed in terms of the combinatorics of \( G \) [29, Theorem 3.2]. Thus, our main result gives a combinatorial formula for the order of \( T(\mathbb{Z}^s/\mathcal{L}) \).

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