L-THEORY OF GROUPS WITH UNSTABLE DERIVED SERIES

S. K. ROUSHON

Abstract. In this short note we prove that the Farrell-Jones Fibered Isomorphism Conjecture in $L$-theory, after inverting 2, is true for a group whose some derived subgroup is free.

1. Introduction

In [3] it was shown that the Fibered Isomorphism Conjecture in $L^{-\infty}$-theory (([1], 1.7)), after inverting 2 (that is for $L^{-\infty} = L^{-\infty} \otimes \mathbb{Z}[\frac{1}{2}]$), is true for a large class (denoted $\mathcal{FICWF}$) of groups including poly-free groups and one-relator groups.

Here we deduce the conjecture, using the results in [3], for groups whose some derived subgroup is free. See Remark 1.1 regarding the relevance of this class of groups.

Throughout the article, by ‘group’ we mean ‘countable group’. We prove the following.

Theorem 1.1. Let $\Gamma$ be a group. Then the Fibered Isomorphism Conjecture of Farrell and Jones for the $L^{-\infty}$-theory is true for the group $\Gamma \triangleright F$ if it is true for $\Gamma^{(n)} \triangleright F$ for some $n$, where $F$ is a finite group and $\Gamma^{(n)}$ denotes the $n$-th derived subgroup of $\Gamma$.

In other words, the $\text{FICwF}_{\mathcal{V}}^{G}$ (or the $\text{FICwF}_{\mathcal{F}_{\mathcal{I}}_{\mathcal{N}}}$) is true for $\Gamma$ if the $\text{FICwF}_{\mathcal{V}}^{G}$ (or the $\text{FICwF}_{\mathcal{F}_{\mathcal{I}}_{\mathcal{N}}}$) is true for $\Gamma^{(n)}$ for some $n$.

For notations and statement of the conjecture see [3], section 2.

Proof of Theorem 1.1. Consider the following exact sequence.

$$1 \rightarrow \Gamma^{(n)} \rightarrow \Gamma \rightarrow \Gamma/\Gamma^{(n)} \rightarrow 1.$$
Note that $\Gamma/\Gamma^{(n)}$ is a solvable group. Hence applying the hypothesis, Corollary 1.2 and [[3], (2) of lemma 2.13] we complete the proof of the Theorem.

**Corollary 1.1.** Let $G$ be a finite index subgroup of a group $\Gamma$. Assume that $G^{(n)}$ is a free group for some $n$. Then the $\text{FICwF}_{\text{VC}}^{\mathcal{H}^G}$ (or the $\text{FICwF}_{\text{FIN}}^{\mathcal{H}^G}$) is true for $\Gamma$.

**Proof.** Using Theorem 1.2 and Lemma 1.1 we can assume that $\Gamma = G$. Next we only need to recall that by [[3], main lemma] the $\text{FICwF}_{\text{VC}}^{\mathcal{H}^G}$ (or the $\text{FICwF}_{\text{FIN}}^{\mathcal{H}^G}$) is true for any free group and then apply Theorem 1.1.

**Remark 1.1.** Groups whose derived series does not stabilize (or some derived subgroup is free or surjects onto a free group) are of interest in group theory and topology. See [4]. In fact in [4] we predicted that these kind of groups appear more often than other groups.

Let us recall the following definition from [3].

**Definition 1.1.** ([[3], definition 1.1]) Let $\text{FICWF}$ be the smallest class of groups satisfying the following conditions

- The following groups belong to $\text{FICWF}$.
  1. Finite groups. 2. Finitely generated free groups. 3. Compact discrete subgroups of linear Lie groups with finitely many components.
- (Subgroup) If $H < G \in \text{FICWF}$ then $H \in \text{FICWF}$
- (Free product) If $G_1, G_2 \in \text{FICWF}$ then $G_1 * G_2 \in \text{FICWF}$.
- (Direct limit) If $\{G_i\}_{i \in I}$ is a directed sequence of groups with $G_i \in \text{FICWF}$. Then the limit $\lim_{i \in I} G_i \in \text{FICWF}$.
- (Extension) For an exact sequence of groups $1 \to K \to G \to N \to 1$, if $K, N \in \text{FICWF}$ then $G \in \text{FICWF}$.

Let $A$ and $B$ be two groups then by definition the wreath product $A \wr B$ is the semidirect product $A^B \rtimes B$, where the action of $B$ on $A^B$ is the regular action. Let $\mathcal{VC}$ and $\mathcal{FIN}$ denote the class of virtually cyclic groups and the class of finite groups respectively.

We proved the following theorem in [[3], theorem 1.1].

**Theorem 1.2.** ([[3], theorem 1.1]) Let $\Gamma \in \text{FICWF}$. Then the following assembly maps are isomorphisms for all $n$, for any group homomorphism $\phi : G \to \Gamma \wr F$ and for any finite group $F$.

$$
\mathcal{H}_n^G(p, L^{-\infty}) : \mathcal{H}_n^G(E_{\phi \cdot \mathcal{VC}(\Gamma \wr F)}(G), L^{-\infty}) \to \mathcal{H}_n^G(pt, L^{-\infty}) \simeq L_n^{-\infty}(ZG).
$$

$$
\mathcal{H}_n^G(p, L^{-\infty}) : \mathcal{H}_n^G(E_{\phi \cdot \mathcal{FIN}(\Gamma \wr F)}(G), L^{-\infty}) \to \mathcal{H}_n^G(pt, L^{-\infty}) \simeq L_n^{-\infty}(ZG).
$$
In other words the Fibered Isomorphism Conjecture of Farrell and Jones for the $L^{-\infty}$-theory is true for the group $\Gamma \backslash F$. Equivalently, the $\text{FICwF}_{\text{VC}}^{N_L}(\Gamma)$ and the $\text{FICwF}_{\text{FIN}}^{N_L}(\Gamma)$ are satisfied (see [3], definition 2.1 for notations).

In [3] we showed that $\text{FICwF}$ contains some well-known classes of groups. Here we see that $\text{FICwF}$ also contains any virtually solvable group.

**Theorem 1.3.** $\text{FICwF}$ contains the class of virtually solvable groups.

*Proof.* Let $\Gamma$ be a virtually solvable group. Using the ‘direct limit’ condition in the definition of $\text{FICwF}$ we can assume that $\Gamma$ is finitely generated, for any countable infinitely generated group is a direct limit of finitely generated subgroups. The following Lemma shows that we can also assume that the group $\Gamma$ is solvable.

**Lemma 1.1.** Let $G$ be a finitely generated group and contains a finite index subgroup $K$. If $K \in \text{FICwF}$ then $G \in \text{FICwF}$.

*Proof.* By taking the intersection of all conjugates of $K$ in $G$ we get a subgroup $K'$ of $G$ which is normal and of finite index in $G$. Therefore, we can use ‘subgroup’ and ‘extension’ conditions in the definition of $\text{FICwF}$ to conclude the proof of the Lemma. \qed

Hence we have $\Gamma$ a finitely generated solvable group. We say that $\Gamma$ is $n$-step solvable if $\Gamma^{(n+1)} = (1)$ and $\Gamma^{(n)} \neq (1)$. The proof is by induction on $n$. Since countable abelian groups belong to $\text{FICwF}$ (see [3], lemma 4.1), the induction starts.

So assume that a finitely generated $k$-step solvable group for $k \leq n-1$ belong to $\text{FICwF}$ and $\Gamma$ is $n$-step solvable.

We have the following exact sequence.

$$1 \rightarrow \Gamma^{(n)} \rightarrow \Gamma \rightarrow \Gamma/\Gamma^{(n)} \rightarrow 1.$$ 

Note that $\Gamma^{(n)}$ is abelian and $\Gamma/\Gamma^{(n)}$ is $(n-1)$-step solvable. Using the ‘extension’ condition and the induction hypothesis we complete the proof. \qed

Applying Theorems 1.2 and 1.3 we get the following.

**Corollary 1.2.** The $\text{FICwF}_{\text{VC}}^{N_L}$ (or the $\text{FICwF}_{\text{FIN}}^{N_L}$) is true for any virtually solvable group.
References

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School of Mathematics, Tata Institute, Homi Bhabha Road, Mumbai 
400005, India

E-mail address: roushon@math.tifr.res.in

URL: http://www.math.tifr.res.in/~roushon/