Quantum Chains with $\mathcal{U}_q(sl(2))$ Symmetry and Unrestricted Representations

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Abstract

We consider two-state ($q^2 = -1$) and three-state ($q^3 = 1$) one-dimensional quantum spin chains with $\mathcal{U}_q(sl(2))$ symmetry. Taking unrestricted (type $B$) representations (periodic, semi-periodic and nilpotent), we show which are the necessary conditions to obtain a Hermitian Hamiltonian.
One-dimensional quantum chains appear in equilibrium statistical mechanics \cite{1,2} and in non-equilibrium master equations which describe annihilation-diffusion processes \cite{3} or critical dynamics \cite{4}. A detailed description of the last two topics is given in Ref. \cite{4}. In problems of equilibrium, one is often (but not always!) interested in Hermitian Hamiltonians, whereas in non-equilibrium problems one is interested in Hamiltonians where all the matrix elements are real since they are related to probabilities. In the present note we examine the possibility of obtaining (up to boundary terms) Hermitian or real Hamiltonians with two and three states using $U_q(sl(2))$ unrestricted representations. The case of restricted representations is already known \cite{6}. Note that at this point we do not ask any questions about the integrability of the Hamiltonian.

**Definitions**

The quantum algebra $U_q(sl(2))$ is defined by the generators $k$, $k^{-1}$, $e$, $f$, and the relations

\begin{align}
 kk^{-1} &= k^{-1}k = 1, & kkek^{-1} &= q^2e,
 [e, f] &= \frac{k - k^{-1}}{q - q^{-1}}, & kfk^{-1} &= q^{-2}f.
\end{align}

The coalgebra structure is given by the coproduct

\begin{align}
 \Delta(k) &= k \otimes k \\
 \Delta(e) &= e \otimes 1 + k \otimes e \\
 \Delta(f) &= f \otimes k^{-1} + 1 \otimes f,
\end{align}

**Centre of $U_q(sl(2))$ when $q$ is a root of unity**

Take $q$ such that $q^{m'} = 1$ and define $m$ such that $m = m'/2$ if $m'$ is even, $m = m'$ otherwise. Then the centre of $U_q(sl(2))$ is generated by $e^m$, $f^m$, $k^\pm m$ and $C$ with

\begin{align}
 C &= fe + (q - q^{-1})^{-2} (qk + q^{-1}k^{-1}) ,
\end{align}

altogether satisfying the polynomial relation

\begin{align}
 P_m(C) &= e^m f^m + q^m \frac{k^m + k^{-m}}{(q - q^{-1})^{2m}},
\end{align}

where

\begin{align}
 P_m(X) &= \frac{2}{(q - q^{-1})^{2m}} T_m \left( \frac{1}{2} (q - q^{-1})^2 X \right),
\end{align}
\( T_m \) being the \( m \)-th Chebychev polynomial of the first kind

\[
T_m(X) = \cos(m \arccos X).
\]  

(6)

**Unrestricted, or type B representations of \( \mathcal{U}_q(sl(2)) \)**

For the construction of our quantum spin chains, we will use the type of irreducible representations (irreps) that exists only when \( q \) is a root of unity \([7]\). We call these irreps type \( B \) irreps (these representations all have dimension \( m \) and depend on continuous complex parameters), as opposed to type \( A \) irreps corresponding to \( q \)-deformations of ordinary representations of \( \mathcal{U}(sl(2)) \) (which have dimension \( \leq m \)).

On these irreps, the central elements \( e^m, f^m, k^\pm 1 \) and \( C \) take the values \( x, y, z^\pm 1 \), and \( c \) lying on a 3-dimensional manifold because of relation (4).

\( \lambda \) being an \( m \)-th root of \( z \), the type \( B \) representation denoted in the following by \( B(x, y, z, c) \) is given, in the basis \( \{v_0, ..., v_{m-1}\} \), by

\[
\begin{align*}
kv_p &= \lambda q^{-2p} v_p & \text{for } 0 \leq p \leq m - 1 \\
f v_p &= \alpha_p v_{p+1} & \text{for } 0 \leq p \leq m - 2 \\
f v_{m-1} &= \alpha_{m-1} v_0 \\
ev_p &= \alpha_p^{-1} \left( c - \frac{1}{(q-q^{-1})^2} \left( \lambda q^{-2p+1} + \lambda^{-1} q^{2p-1} \right) \right) v_{p-1} & \text{for } 1 \leq p \leq m - 1 \\
ev_0 &= \alpha_{m-1}^{-1} \left( c - \frac{1}{(q-q^{-1})^2} \left( \lambda q + \lambda^{-1} q^{-1} \right) \right) v_{m-1}.
\end{align*}
\]

(7)

where \( \prod_{p=0}^{m-1} \alpha_p = y \).

The freedom left by the choice of the \( \alpha_p \) satisfying \( \prod \alpha_p = y \), corresponding to rescalings of the basis, will be used in the following to explore more easily equivalent Hamiltonians for the quantum chains.

We will distinguish three types of type \( B \) irreps:

a) Periodic representations, corresponding to injective action of \( e \) and \( f \), i.e. to \( xy \neq 0 \);

b) Semi-periodic representations, for which either \( x \) or \( y \) is 0;

c) Nilpotent representations, corresponding to \( x = y = 0 \) with only one complex parameter \( z \).

The nilpotent representations are also representations of the quantum algebra generated by \( e, f \) and \( h \), where \( k = q^h \). (Note that this is not true in the other cases since \( [h, e^m] = 0 \) on a representation implies \( e^m = 0 \) on it, and idem for \( f^m \).) The logarithm
of \( k \) on nilpotent representations is actually well defined, since the highest weight and the
lowest weight provide a cut in the values of \( k \).

**Quantum spin chains**

We define a quantum spin chain with \( \mathcal{U}_q(sl(2)) \) symmetry as follows: to each site
\( j = 1, ..., L \) of the chain, we assign a type \( \mathcal{B} \) representation \( \pi_j = \mathcal{B}(x_j, y_j, z_j, c_j) \). We write
the Hamiltonian

\[
H = \sum_{j=1}^{L-1} \text{Id} \otimes ... \otimes \text{Id} \otimes H_j \otimes ... \otimes \text{Id}
\]

with \( H_j \) acting on sites \( j \) and \( j + 1 \) as

\[
H_j = (\pi_j \otimes \pi_{j+1})(Q_j(\Delta(C)))
\]

where \( C \) is the quadratic Casimir \( [3] \) and \( Q_j \) is a polynomial of degree \( d < m \).

This Hamiltonian is by construction \( \mathcal{U}_q(sl(2)) \) invariant.

**On tensor products of type \( \mathcal{B} \) irreps**

On the tensor product of two type \( \mathcal{B} \) irreps characterized by the parameters
\( (x_1, y_1, z_1, c_1) \) and \( (x_2, y_2, z_2, c_2) \), the central elements \( \Delta(c^m) \), \( \Delta(f^m) \) and \( \Delta(k^m) \) take
the (scalar) values

\[
x = x_1 + z_1 x_2,
\]

\[
y = y_1 z_2^{-1} + y_2,
\]

\[
z = z_1 z_2.
\]

The possible values for \( \Delta(C) \) are given by the relation \( [4] \). More precisely, if we define \( \zeta \) by

\[
xy + q^m \frac{z + z^{-1}}{(q - q^{-1})^2m} = \frac{\zeta^m + \zeta^{-m}}{(q - q^{-1})^2m},
\]

then the possible values for \( \Delta(C) \) are

\[
c_l = \frac{\zeta q^{2l} + \zeta^{-1} q^{-2l}}{(q - q^{-1})^2}
\]

\( l = 0, ..., m - 1 \).

Then, if \( \zeta^{2m} \neq 1 \) we have the fusion rule

\[
\mathcal{B}(x_1, y_1, z_1, c_1) \otimes \mathcal{B}(x_2, y_2, z_2, c_2) = \bigoplus_{l=0}^{m-1} \mathcal{B}(x, y, z, c_l).
\]
With this, we know that the states of the whole chain lie in a sum of $m$-dimensional multiplets, at least for generic values of the parameters $\mathcal{E}$.

**Opposite coproduct and algebraic curve**

We see from the fusion rules that the tensor product $\pi_2 \otimes \pi_1$ is equivalent to $\pi_1 \otimes \pi_2$ iff

$$\frac{x_1}{1-z_1} = \frac{x_2}{1-z_2}, \quad \frac{y_1}{1-z_1^{-1}} = \frac{y_2}{1-z_2^{-1}}. \quad (14)$$

These conditions will be imposed on each pair of consecutive representations of the chain. In other words, the parameters of all the representations will lie on the same manifold defined by

$$x = A(1-z) \quad y = B(1-z^{-1}) \quad (15)$$

for fixed $A, B$ (possibly 0 or infinite. Note that $A$ or $B$ infinite means that $z$ is fixed to 1 and that $x$ or $y$ are free.)

**Some general results for the $m$-state quantum chains**

- A sufficient condition for the existence of a $U(1)$ invariance is obtained when $x_j = y_j = 0$ for all $j$‘s, i.e. $A = B = 0$, that is for nilpotent representations. As explained before, the generator $h$ is well-defined on each representation, and the Hamiltonian commutes with

$$\Delta^{(L)}(h) = \sum_{j=1}^{L} \text{Id} \otimes \ldots \otimes h \otimes \ldots \otimes \text{Id} \quad (16)$$

which generates a $U(1)$ symmetry.

- A sufficient condition for having a real spectrum is the following: if the parameters $x_j, y_j,$ and $z_j$ are such that

$$(x_j + z_j x_{j+1})(y_j z_{j+1}^{-1} + y_{j+1}) + q^m \frac{z_j z_{j+1}^{-1} + z_{j+1}^{-1} z_j^{-1}}{(q-q^{-1})^2 m} = \frac{\xi_j^m + \xi_j^{-m}}{(q-q^{-1})^2 m} \quad (17)$$

is a real number between $-2$ and 2, then $\Delta(C)$ on sites $j, j+1$ will have real eigenvalues (given by (12) with $\zeta = \xi_j$). If the polynomials $Q_j$ used to define the Hamiltonian have real coefficients, then the spectrum is real.

**Two-state quantum chain:** $q = i, m = 2$
Let us consider a Hamiltonian constructed as above, with first finite values for $A$ and $B$ in (13). Then $H_j$ is

$$H_j = F_j \left( \rho_j \sigma^z_j + \rho_j^{-1} \sigma^z_{j+1} + \eta_j \sigma^x_j \sigma^x_{j+1} + \eta_j^{-1} \sigma^y_j \sigma^y_{j+1} \right)$$  \hspace{1cm} (18)$$

with $\eta_j = 1$, $\rho_j = \left( \frac{z_{j+1} - 1}{1 - z_j} \right)^{1/2}$ and $F_j$ is an arbitrary constant (note that the freedom in the choice of $Q_j$ corresponds to a redefinition of $F_j$). For the special choice $\rho_j = \rho$ independent of $j$, we recover the $U_q(sl(1/1))$ supersymmetric Hamiltonian of Ref. [9] with $q = \rho$. It is interesting to notice that the symmetries of this chain can now be understood in a new way through type $B$ representations of $U_q(sl(2))$.

*Remark:* The case of nilpotent representations is included here: it corresponds to $A = B = 0$. It has nothing remarkable since $A$ and $B$ do not change Eq. (18).

Suppose now that (13) is satisfied with $A$ and $B$ infinite, i.e. $z_j = 1$, $\forall j$, and $x_j$, $y_j$ are free. Then we obtain the Hamiltonian (18) with $\rho_j = \left( \frac{x_{j+1} y_{j+1}}{x_j y_j} \right)^{1/4}$ and $\eta_j = \left( \frac{x_j y_{j+1}}{x_{j+1} y_j} \right)^{1/4}$. If we take $\rho_j = \rho$ and $\eta_j = \eta$ independent of $j$ we find the Hamiltonian discussed in Ref. [10]. This Hamiltonian is invariant under a two-parameter ($\alpha$ and $\beta$ with $\alpha = \rho/\eta$ and $\beta = \rho\eta$) deformation of a superalgebra with two odd generators. As for the $U_q(sl(1/1))$ case, here we understand the symmetries in a different way.

The crucial feature here is that the original parameters $\frac{x_{j+1}}{x_j}$ and $\frac{y_{j+1}}{y_j}$, which were parameters of representations (we had $q = i$ in $SU(2)_q$ so no continuous deformation of the Lie algebra) appear as deformation parameters of super Lie algebras. An analogous phenomenon was already observed in [11,12].

On the whole chain, $\Delta^{(L)}(e)^2$, $\Delta^{(L)}(f)^2$ and $\Delta^{(L)}(k)^2$ take the scalar values $x_1 \left( \frac{1 - \alpha^{2L}}{1 - \alpha^2} \right)$, $y_1 \left( \frac{1 - \beta^{2L}}{1 - \beta^2} \right)$ and 1. So for generic values of the parameters (of the representations) $\alpha$ and $\beta$, the total representation will be periodic. However, for $\alpha$ or $\beta$ a root of unity (say $\alpha^{2L_0} = 1$; $L_0$ having nothing to do with $m = 2l$), the total representation will lose its periodicity for chains of lengths $L$ which are multiples of $L_0$, and the total representation will contain indecomposable subrepresentations.

All the Hamiltonians obtained with type $B$ representations of $U_q(sl(2))$ with $q = i$ are hence integrable [10], and are Hermitian if $\rho$ and $\eta$ are real. If $\rho$ is on the unit circle, the Hamiltonian is Hermitian up to boundary terms.
Three-state quantum chain: $q^3 = 1, m = 3$

We consider $H_j = a_2^{(j)} \Delta(C)^2 + a_1^{(j)} \Delta(C) + a_0^{(j)}$ with the freedom for the choice of $a_0^{(j)}$, $a_1^{(j)}$, $a_2^{(j)}$ and for the parameters $(x_j, y_j, z_j, c_j)$ of the representations $\pi_j$, as well as for the $a_p^{(j)}$ (freedom in the bases of the representations). We then look for the conditions on these parameters for having a Hermitian Hamiltonian, or for having a Hamiltonian with real matrix elements. A tedious calculation proves that the following conditions are necessary conditions for both cases:

- All the representations are nilpotent, i.e. $x_j = y_j = 0 \ (\forall j)$. The representations are then completely characterized by their highest weights $\lambda_j$ (with $z_j = \lambda_j^3$ and $c_j = \frac{q\lambda+q^{-1}\lambda^{-1}}{(q-q^{-1})^2}$).
- All the representations $\pi_j$ are equivalent to $\pi_1 \equiv \pi$ (i.e. $\lambda_j = \lambda, \forall j$).
- $|\lambda| = 1$

As explained before, the Hamiltonian will have a $U(1)$ symmetry since we have to consider nilpotent representations only.

A necessary and sufficient condition for the Hamiltonian to be Hermitian (up to boundary terms) is

$$\text{Re} \left( (q\lambda)^2 \right) \geq -\frac{1}{2}. \quad (19)$$

The Hamiltonian density $H_j$ is then written

$$H_j = D \left( E^{01} \otimes E^{10} + E^{10} \otimes E^{01} \right) + H \left( E^{12} \otimes E^{21} + E^{21} \otimes E^{12} \right)$$

$$- \left( E^{02} \otimes E^{20} + E^{20} \otimes E^{02} \right)$$

$$+ F \left( E^{01} \otimes E^{21} + E^{21} \otimes E^{01} + E^{10} \otimes E^{12} + E^{12} \otimes E^{10} \right) \quad (20)$$

with

$$D = \frac{\lambda q^{-1} - \lambda^{-1} q}{q - q^{-1}} \quad H = -\frac{\lambda - \lambda^{-1}}{q - q^{-1}} \quad F = \frac{i}{q - q^{-1}} \left( 1 + \lambda^2 q^2 + \lambda^{-2} q^{-2} \right)^{1/2} \quad (21)$$

for the non-diagonal part,

$$A \otimes 1 + 1 \otimes A \quad \text{with} \quad A = \frac{1}{2} \text{Diag} \left( \frac{\lambda^2 q^{-1} - \lambda^{-2} q}{q - q^{-1}} + 1, 2, -\frac{\lambda^2 q^{-1} - \lambda^{-2} q}{q - q^{-1}} + 1 \right) \quad (22)$$

for the real diagonal part and

$$B \otimes 1 - 1 \otimes B \quad \text{with} \quad B = \frac{1}{2} \frac{(\lambda - \lambda^{-1})(\lambda q^{-1} - \lambda^{-1} q)}{q - q^{-1}} \text{Diag} (1, 0, -1) \quad (23)$$
for the imaginary diagonal part (boundary term). This Hamiltonian (up to the imaginary boundary term) coincides with the one discovered by Gomez and Sierra [13].

The limit $\lambda \to 1$ of this Hamiltonian is

$$H_j = - (E^{01} \otimes E^{10} + E^{10} \otimes E^{01}) - (E^{02} \otimes E^{20} + E^{20} \otimes E^{02})$$

$$+ (E^{00} \otimes E^{11} + E^{11} \otimes E^{00} + E^{10} \otimes E^{02} + E^{20} \otimes E^{00})$$

$$+ 2 (E^{11} \otimes E^{11} + E^{11} \otimes E^{22} + E^{22} \otimes E^{11} + E^{22} \otimes E^{22})$$

(24)

This last Hamiltonian, obtained with $\lambda = 1$, is actually the only Hamiltonian (up to equivalence transformations) which has only real matrix elements. This special case is discussed in detail in [5], where one considers a two-species model of diffusion-annihilation processes.

When this work was completed, we received Ref. [14], where the q-chains based on nilpotent representations are studied.

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