Local topology via the invariants of the velocity gradient tensor within vortex clusters and intense Reynolds stress structures in turbulent channel flow

Abel-John Buchner¹, Adrián Lozano-Durán², Vassili Kitsios¹, Calum Atkinson¹, Julio Soria¹,³

¹Laboratory for Turbulence Research in Aerospace and Combustion (LTRAC), Department of Mechanical and Aerospace Engineering, Monash University, Clayton, Australia
²School of Aeronautics, Universidad Politécnica de Madrid, Madrid, Spain
³Department of Aeronautical Engineering, King Abdulaziz University, Jeddah, Kingdom of Saudi Arabia

E-mail: abel-john.buchner@monash.edu

Abstract.

Previous works have shown that momentum transfer in the wall-normal direction within turbulent wall-bounded flows occurs primarily within coherent structures defined by regions of intense Reynolds stress [1]. Such structures may be classified into wall-attached and wall-detached structures with the latter being typically weak, small-scale, and isotropically oriented, while the former are larger and carry most of the Reynolds stresses. The mean velocity fluctuation within each structure may also be used to separate structures by their dynamic properties. This study aims to extract information regarding the scales, kinematics and dynamics of these structures within the topological framework of the invariants of the velocity gradient tensor (VGT). The local topological characteristics of these intense Reynolds stress structures are compared to the topological characteristics of vortex clusters defined by the discriminant of the velocity gradient tensor. The alignment of vorticity with the principal strain directions within these structures is also determined, and the implications of these findings are discussed.

1. Introduction

Turbulent wall-bounded flows contain features at a range of scales and with various topological properties and dynamic behaviours. In studying the motion of fluid within a turbulent wall-bounded flow, it is common to separate the flow into distinct regions of spatially and/or temporally coherent velocity or vorticity. These are often labelled “coherent structures” [2, 3]. This has the advantage of defining a tangible entity whose topology and evolution may be quantitatively studied. There is however significant ambiguity and confusion about how to formally define these coherent structures, and the physical relevance of each proposed method. Commonly, coherent vortical structures may be defined by the second invariant of the velocity gradient tensor (VGT), \( Q_A \) [4], or by contiguous positive regions of its discriminant, \( D \) [5, 6]. Lagrangian methods such as the Finite Time Lyapunov Exponents (FTLE) have also been used [7, 8] and, more recently, Lozano–Durán et al. [1] studied the dynamics of structures defined
based on regions of intense Reynolds stress. Cucitore et al. [9] and Chakraborty et al. [10] present comprehensive descriptions and analyses of popular coherent structure identification criteria.

This paper focuses on the study of the intense Reynolds stress structures of Lozano–Durán et al. [1] in a turbulent channel flow, and their comparison with structures defined based on the VGT discriminant, as in del Alamo et al. [6]. The term “vortex cluster” is used here to identify structures defined based on the VGT discriminant.

The overall topology and dynamics of intense Reynolds stress structures and vortex clusters has been characterised previously [1, 6], but questions remain regarding the topology and alignment of the flow within these structures on the local level. It is unknown what the most likely topological states are within these structures, whether this differs from the whole flow, and how structures delineated based on different criteria differ from one another locally.

To address these unknowns, intense Reynolds stress structures and vortex clusters are compared here in the topological framework of the invariants of the VGT [5], which may be used to elicit information regarding the local topology within a velocity field. A comparison is also made regarding the likely local alignment of the strain and vorticity fields within each type of structure. Chong et al. [5], Soria et al. [11], Perry and Chong [12], Blackburn et al. [13], and Ooi et al. [14] present comprehensive discussion on the physical interpretation of the VGT invariants.

2. Intense Reynolds stress structures
Lozano–Durán et al. [1] investigated the three-dimensional structure of momentum transfer in turbulent channels by studying the structure of the intense Reynolds stresses that carry most of the wall-normal flux of momentum. Particular emphasis was placed on the logarithmic and outer layers. The flow in this study was separated into coherent structures based on contiguous regions of Reynolds stress values above a threshold, \( H \),

\[
|\tau(x)| / (u'(y) v'(y)) > H
\]  

where the Reynolds stress, \( \tau \), is normalised by \( u' \) and \( v' \), the root–mean–square (RMS) of each velocity component at each wall-normal location, \( y \). An apostrophe is used throughout this paper to denote the RMS of a quantity. A percolation analysis was used to set a normalised Reynolds stress threshold of \( H_1 = 1.75 \). Individual structures are defined by spatially connecting neighbouring points satisfying this criterion.

Sub-classification of these coherent structures is possible, using the quadrant analysis of Wallace et al. [15], Willmarth and Lu [16], and Lu and Willmarth [17]. This analysis categorizes the intense Reynolds stress structures as sweeps, ejections, and inward and outward interactions based on the sign of the fluctuating streamwise and spanwise velocity components, \( u \) and \( v \), averaged within each structure as

\[
u_m = \frac{\int_{\Omega} u \cdot dV}{\int_{\Omega} dV} \quad (2)\]

and

\[
v_m = \frac{\int_{\Omega} v \cdot dV}{\int_{\Omega} dV} \quad (3)\]

where integration is performed over the points, \( \Omega \), belonging to each individual structure’s volume. Lozano-Durán et al. [1] label these as \( Q_1 \), \( Q_2 \), \( Q_3 \), and \( Q_4 \) structures. Note that \( Q_i \) refers to quadrant, in contrast to the \( Q_A \) used before to denote the second invariant of the VGT. This terminology, illustrated in figure 1, is used here to separate intense Reynolds stress
structures into families which travel towards the wall, or away from the wall, and those which move faster or slower than the local mean velocity in the streamwise direction.

\[ u_m \text{ and } v_m, \text{ as defined in equations 2 and 3, are the average fluctuating streamwise and wall–normal velocity within individual structures.} \]

It has also been previously shown that a natural separation exists in the classification of structures based on the wall-normal location of the structures’ minimum extent. Those structures whose minimum bound lies within twenty wall units of the wall may be considered “wall–attached” structures, while all others may be considered “wall–detached”. This separation is based on a probability density function (PDF) of structure heights and occurs for both vortex clusters [6] and intense Reynolds stress structures [1] at the same wall-normal location, \( y^+ = 20 \). Each of these families of structures display different properties; wall–detached structures appear isotropically oriented, with their size depending on the distance to the wall indirectly through the Kolmogorov length–scale \( \eta \), although this dependence is very weak, varying as \( \eta \sim y^{1/4} \). Wall–attached structures on the other hand tend to be much taller, in some cases extending from near the wall, well beyond the centreplane of the channel. Lozano-Durán et al. [1] showed that the predominant structure within the logarithmic layer of a turbulent channel flow is a pair of wall–attached intense Reynolds stress structures (one sweep and one ejection), with an associated vortex cluster embedded in the ejection. These tall–attached sweeps (\( Q_4 \)'s) and ejections (\( Q_2 \)'s) account for a disproportionately large volume fraction of the flow, when compared to the inward and outward interactions (\( Q_1 \) and \( Q_3 \) structures), and are the focus of the topological analysis presented here.

3. Topological analysis via the velocity gradient tensor
The invariants \( P_A, Q_A, \) and \( R_A \) of the VGT may be computed from

\[ P_A = -A_{ii} \quad (4) \]

\[ Q_A = -\frac{1}{2} A_{ij}A_{ji} \quad (5) \]

\[ R_A = -\frac{1}{3} A_{ij}A_{jk}A_{ki} \quad (6) \]

where \( A_{ij} \) represents the VGT [5]. The \( P_S, Q_S, R_S, P_W, Q_W, \) and \( R_W \) invariants are similarly defined from the rate of strain \( S_{ij} \) and rate of rotation \( W_{ij} \) tensors, respectively. The rate of strain and rate of rotation tensors are the symmetric, and skew–symmetric, parts of the velocity gradient tensor, such that
\[ A_{ij} = S_{ij} + W_{ij}. \]  

(7)

The relative dominance of each of the invariants indicate the local topology of the flow [5], as classified in figure 2. The first invariant, \( P_A \), is identically zero due to incompressibility. The second invariant, \( Q_A \), relates to the rotational nature of the flow, with values greater than zero indicating dominance of enstrophy over strain, while negative values imply the opposite.

The topological effect of the third invariant, \( R_A \), depends strongly on the value of \( Q_A \). For \( Q_A > 0 \) and large, \( R_A < 0 \) represents vortex stretching and \( R_A > 0 \) vortex compression. In the limit of large negative values of \( Q_A \) however, \( R_A \) is dominated by strain self–amplification, \( R_A \approx R_S \), and positive and negative values of \( R_A \) are associated with sheet-like and tube-like structures respectively. The tent–like curve in figure 2(a) is the curve along which the discriminant,

\[ D_A = \frac{27}{4} R_A^2 + Q_A^3 \]  

(8)

of the velocity gradient tensor is equal to zero. This curve separates focal from non–focal local topologies, as illustrated on the figure.

In addition to the \( Q_A – R_A \) plane, we examine in this paper the probabilistic topological distribution of the flow in the \( Q_S – Q_W \) and \( Q_W – \Sigma \) planes, where \( \Sigma \) is the normalised vortex stretching rate given by

\[ \Sigma = \frac{R_S - R_A}{Q_W}. \]  

(9)

The second invariant of the rate of strain tensor, \( Q_S \), is exclusively negative, and when its magnitude is large compared to \( Q_W \), the flow topology is dominated by irrotational dissipative topology. On the other hand, when \( Q_W \) is large and \( Q_S \) is close to zero the flow takes on highly enstrophic topology with little dissipation, such as in the solid body rotation at the centre of vortex–tubes. When \( Q_S \) and \( Q_W \) are in balance, \( -Q_S = Q_W \), the value of \( Q_A \) is low and vortex sheets dominate.

\[ \text{Figure 2.} \]  

Local flow topology classification based on invariants of the velocity gradient tensor, in (a) the \( Q_A – R_A \) invariant plane, (b) the \( Q_S – Q_W \) invariant plane, and (c) the \( Q_W – \Sigma \) invariant plane. The tent–like curve in (a) is the zero–contour of the VGT discriminant, \( D_A \). Cartoons are taken from Soria et al. [11].
4. Numerical simulation and statistics

The direct numerical simulation (DNS) of a turbulent channel flow, performed by del Álamo et al. [18] provides us with a database for the calculation of intense Reynolds stress structures and vortex clusters and the study of the local topology therein. The Navier–Stokes equations are solved in the manner of Kim, Moin, and Moser [19] for the wall–normal vorticity, $\omega_y$, and the wall-normal velocity Laplacian, $\nabla^2 v$. Dealiased Fourier expansions are used in the streamwise and spanwise directions, and spatial discretisation is implemented by Chebyshev polynomials in the wall-normal coordinate. Integration in time is via the third order semi-implicit Runge-Kutta method of Moser, Kim, and Mansour [20].

The friction Reynolds number of the simulation is $h^+ = 934$ and the simulation domain extends to $L_x \times L_z = 8\pi \times 3\pi$ channel half heights, $h$, in the streamwise, $x$, and spanwise, $z$, directions. $N_x = 3072$, $N_z = 2304$, and $N_y = 385$ grid points are used in the streamwise, spanwise, and wall-normal directions respectively. The viscous–scaled grid resolution, $\Delta x^+ = 7.6$, after dealiasing in the streamwise direction is equal to the wall–normal spacing, $\Delta y_c^+$, at the channel centre–plane. In the spanwise direction, it is $\Delta z^+ = 3.8$. The statistics presented here are accumulated from six statistically independent velocity fields containing a total of $8.8 \times 10^5$ vortex clusters and $4.0 \times 10^5$ intense Reynolds stress structures, of which 28% are $Q_2$’s and 32% are $Q_4$’s. The basic numerical parameters of the simulation are given in table 1.

| Parameter | Value |
|-----------|-------|
| $h^+$ | 934 |
| $L_x/h$ | $8\pi$ |
| $L_z/h$ | $3\pi$ |
| $\Delta x^+$, $\Delta z^+$, $\Delta y_c^+$ | (7.6, 3.8, 7.6) |
| $N_x$, $N_z$, $N_y$ | (3072, 2304, 385) |

Table 1. Relevant simulation parameters. In the table, $h$ is the channel half–height, $L_x$ and $L_z$ are the streamwise and spanwise lengths of the simulation domain. $\Delta x$ and $\Delta z$ are the dealiased grid resolutions in the streamwise and spanwise directions respectively. At the channel centreplane, the wall–normal resolution is $\Delta y_c$. In number of grid points in the streamwise and spanwise directions are represented by $N_x$ and $N_z$, while $N_y$ is the number of Chebyshev collocation points in the wall–normal direction. The + exponent indicates viscous scaling.

Figure 3. Examples structures, coloured by wall–normal location. Taken from Lozano–Durán et al. [1] (a) Attached $Q_2$ (ejection) structure, and (b) attached vortex cluster. Axes are labelled in wall units.

Illustrative examples taken from Lozano–Durán et al. [1] of a wall–attached $Q_2$ intense Reynolds stress structure, and a wall attached vortex cluster taken from the simulation are shown in figure 3. The surfaces shown are the threshold isosurfaces of the normalised Reynolds stress and the VGT discriminant, respectively. The axis scales are labelled in wall units, and
colour represents wall–normal location. Qualitatively, it may be observed that the structures display a complex, fractal geometry, containing detail at a range of scales. As stated by Lozano–Durán et al. [1], the immediate impression is that vortex clusters are composed of “sponges of strings” while intense Reynolds stress structures are composed of “sponges of flakes”.

Figure 4. (a) Mean streamwise velocity profile, and (b) RMS of velocity components as a function of wall–normal location. Inner scaling applied.

Figure 5. (a) Mean, and (b) RMS of VGT invariants and vortex stretching rate as a function of wall–normal location.
Mean and RMS profiles of each velocity component are shown in figure 4 as a function of wall-normal location, $y$. Averaging is performed in each wall-parallel $x-z$ plane, and fluctuating velocity components are defined as the departure from the mean in each $x-z$ plane also. The mean streamwise velocity profile displays a logarithmic behaviour in the approximate range $30^+ < y < 0.2h$, taking $\kappa = 0.42$ and with an additive constant of $B = 5.1$. This logarithmic relation and also the linear law of the wall in the viscous sublayer are represented by dashed curves on the figure. The vertical dashed lines, repeated in each pair of axes, delineate the viscous, buffer, and logarithmic layers at $y^+ = 5$ and $y^+ = 30$. The RMS profile of the streamwise component peaks at $\sqrt{\langle u^+ u^+ \rangle} = 2.8$ at $y^+ \approx 10.5$, while the spanwise component fluctuations are stronger than those in the wall-normal component.

The statistics of each tensor invariant are also plotted, in figure 5. Near the wall both $Q_A$ and $R_A$ are on average strongly negative, implying predominance of stable flow topologies as described in figure 2(a). Above some wall-normal location within the buffer layer both become positive. The mean vortex stretching rate, $\langle \Sigma \rangle$, is positive at all wall-normal locations, peaking in magnitude at the lower bound of the logarithmic layer. The RMS of both $Q_A$ and $R_A$ peaks within the buffer layer, with the $R_A$ fluctuations being stronger than the fluctuations in $Q_A$. The RMS of $\Sigma$ is much smaller than the RMS of both $Q_A$ and $R_A$, despite its mean magnitude being of the same order. The curves of $Q_S$ and $Q_W$ appear to nearly mirror one another about zero, since by definition $Q_A = Q_S + Q_W$ and the mean magnitude of $Q_A$ is much smaller than that of $Q_S$ and $Q_W$. This observation also accounts for the close match between the RMS curves of $Q_S$ and $Q_W$. The magnitude of both the mean and fluctuating values of $Q_S$ and $Q_W$ is large near the wall and monotonically decreases with $y$, while the mean and RMS values of $R_S$ peak within the viscous layer before also decreasing with distance from the wall.

5. Joint probability density functions

Joint probability density functions (JPDFs) in the $Q_A - R_A$ invariant plane are given in figure 6. The joint PDFs are separately calculated and plotted for each type of structure, and for the whole channel flow domain. The joint PDFs are normalised such that their integral equals 1, and contour lines are logarithmically spaced in increments of a decade, with the outermost contour shown indicating a normalised JPDF level of $10^{-3}$.

The $Q_A - R_A$ joint PDF for the whole channel exhibits the classic “teardrop” shape documented in Soria et al. [11] for a plane mixing layer, and also by Blackburn et al. [13] for turbulent flow in a channel, Ooi et al. [14] for isotropic turbulence, and Atkinson et al. [21] in a turbulent boundary layer. The teardrop shaped joint PDF in all of these cases has a strong preferential alignment towards positive $R_A$ values along the zero-discriminant line. Compared with the whole channel, the conditional joint PDF for vortex clusters displays a greater likelihood for values of the invariants $Q_A$ and $R_A$ in the upper left quadrant, indicating strong vortex stretching. This is especially the case for wall-detached vortex clusters. By definition the topology within vortex clusters does not include negative discriminant values, and this is reflected in the joint PDFs. Additionally, it appears that small positive values of discriminant are more likely to occur within wall-attached vortex clusters than in the wall-detached variety, where the joint PDF is shifted upwards away from the zero discriminant curve. This suggests stronger gradients associated with the local topology in the detached case.

The joint PDFs of $Q_A - R_A$ conditional on topology within high Reynolds stress structures are a less radical departure from the joint PDFs calculated from whole channel domain. Each displays a teardrop shape and it is clear that, unlike vortex clusters, the high Reynolds stress structures include topology with both positive and negative discriminant. The joint PDFs for the $Q_2$ structures display proportionally larger $Q_A$ and $R_A$ discriminants than the channel as a whole, but not to the same extent as the vortex clusters. The asymmetric teardrop tail in the lower right quadrant is especially pronounced and this is especially the case for wall-attached $Q_2$. 

7
structures. The joint PDFs for $Q_4$ structures exhibit significantly less asymmetry and a greater tendency towards small magnitudes of the $Q_A$ and $R_A$ invariants. In general, wall–attached $Q_4$ structures tend to contain smaller invariant values than their wall-detached equivalents. It is well-known that $Q_2$’s are associated with stronger $uv$ values than $Q_4$’s [1], but the present results indicate that this is also the case for gradients. The small tail or bump along the zero discriminant curve in the lower left quadrant is also more pronounced in the $Q_4$ distributions than in the other joint PDFs presented here.

Figure 6. Joint PDFs of $Q_A$ vs $R_A$ for each structure type. Decade–spaced contour intervals, with outermost contour at $10^{-3}$.

Figure 7 illustrates the probabilistic topological behaviour, conditional on structure type in the $Q_S – Q_W$ plane. Physically, $Q_W$ correlates with enstrophy density and high values indicate strongly rotational topology, while values of $–Q_S \gg Q_W$ imply irrotational dissipation of kinetic energy. The joint PDF calculated from the whole channel domain shows a broad distribution across both invariants, but with a bias towards high $Q_W$ values. This is consistent with previous numerical studies [22, 23] suggesting that turbulent flows tend to arrange as worm–like or sheet–like vortices at the small scales, choosing such topology preferentially over irrotational dissipation.

The topology within vortex clusters is significantly more rotational than the flow as a whole, with the entire joint PDF shifted to high $Q_W$ values. The $Q_S – Q_W$ distribution peaks at a non–negligible value of $Q_W$, and the probability density drops to very small values near the $Q_S$ axis. Wall–detached vortex clusters appear to be more rotational than the wall-attached counterparts. This behaviour is consistent with previous observations in a turbulent boundary layer of greater sheet–like structure in the near–wall region [24]. Unlike the joint PDF for the whole channel (and for the high Reynolds stress structures), the joint PDFs for the vortex clusters display an increasing probability of high dissipation ($Q_S$) with increasing $Q_W$.

The shape of the $Q_S – Q_W$ joint PDFs for intense Reynolds stress structures more closely resembles that for the whole channel. A preference for rotational topology (high $Q_W$) over high negative $Q_S$ magnitudes may be observed in a broadly distributed PDF. As was the case in the $Q_A – R_A$ plane, $Q_4$ structures display a greater tendency towards small invariant magnitudes. The most striking feature here is shape of the $Q_S – Q_W$ joint PDF for wall-attached $Q_4$’s. Although overall there is a broad probability distribution, there is a significant proportion
of these structures within which $Q_S$ and $Q_W$ balance one another manifesting in the highly correlated JPDF along the 45° line. This behaviour is associated with the existence of vortex sheet–like topology near the wall [24] within the lower parts of the attached $Q_4$’s and, since $Q_A = Q_W + Q_S$, explains the low invariant magnitudes within the $Q_4$ structures in the $Q_A - R_A$ plane.

![Figure 7. Joint PDFs of $Q_S$ vs $Q_W$ for each structure type. Decade–spaced contour intervals, with outermost contour at $10^{-3}$.](image)

When the joint PDF of the invariant $Q_W$ and the vortex stretching, $\Sigma$, is plotted (figure 8) it becomes apparent that low values of $Q_W$ are associated with a wide range of vortex stretching rates, while the spread of vortex stretching rates diminishes with increasingly enstrophic flow. The probabilistic topology distribution is slightly biased towards positive vortex stretching rates. Similarly shaped distributions occur when the joint PDF is calculated conditional on structure type. For all structure types, the spread of vortex stretching diminishes with increasing $Q_W$, and in all cases there is a bias towards positive $\Sigma$. Additionally, vortex clusters have a greater bias towards positive vortex stretching, and this bias increases with increasing $Q_W$, indicating strong self–stretching behaviour of these structures. Consistent with figures 7, $Q_W$ values are generally higher in vortex clusters than in intense Reynolds stress structures, and $Q_4$ structures show less enstrophic behaviour than do the $Q_2$ structures.

6. Alignment of principal strains

The equation governing the evolution of enstrophy in the flow is

$$\frac{1}{2} \frac{D}{Dt} (\omega^2) = \omega_i S_{ij} \omega_j + \nu \omega_i \nabla^2 \omega_i,$$

where two terms on the right hand side of this equation relate to the production and dissipation of enstrophy. It is clear from the first term on the right hand side that the production of enstrophy is dependent on the strain field. This term may be rewritten in terms of the principal strains, $\alpha_i$, and their directions, $\hat{e}_i$, as

$$\omega_i S_{ij} \omega_j = \omega^2 \alpha_i (\hat{e}_i \cdot \hat{\omega})^2.$$
The angle of alignment $\theta_i = \cos^{-1}(\hat{e}_i \cdot \hat{\omega})$ between the local vorticity vector, $\omega$, and the eigenvectors $e_i$ of the rate-of-strain tensor thus directly drives the enstrophy production rate.

The cosine of the angle between $e_i$ and $\omega$ is calculated locally for each point in the flow as $\cos(\theta_i) = \hat{e}_i \cdot \hat{\omega}$, and probability density functions conditional on structure type are given in figure 9. $\Phi$ represents the value of the PDF. In each set of axes the black lines represent the PDF taken over the entire computational domain, and PDFs conditional on each classification of turbulent structure are compared directly. 120 evenly spaced bins are used in the approximation of each PDF. Only every second symbol is shown, for clarity.

It is clear that the second, or intermediate, strain eigenvector aligns strongly with the the local vorticity direction, while the compressive eigenvector, $\hat{e}_3$, predominantly aligns orthogonally with the vorticity vector. The first, most extensional eigenvector, $\hat{e}_1$, appears mostly randomly oriented. This alignment has been noted previously by Ashurst et al. [25], Sondergaard et al. [26], and Tsinober et al. [27], and appears to be a property universal to turbulent flows.

| Table 2. Ratios of probability of $\hat{e}_2 - \omega$ alignment to orthogonality. |
|-----------------------------------------------|---------------|
| Attached | Detached |
| Whole channel | 10.3:1 | 21.6:1 |
| Vortex clusters | 17.4:1 | 21.6:1 |
| $Q_2$ | 10.3:1 | 7.5:1 |
| $Q_4$ | 7.2:1 | 5.1:1 |

Figure 9(b) shows that these alignment preferences persist within both wall-attached and wall-detached $Q_2$ structures, where all three eigenvectors appear to have very similar alignment to the alignment over the entire domain. In comparison, within vortex clusters the intermediate eigenvector is more strongly aligned with the vorticity field. The alignment is weaker within...
$Q_4$ structures. Ashurst et al. [25] reports a ratio of the probability of perfect alignment of $\hat{e}_2$ and $\omega$ and the probability of these vectors orienting orthogonally as 15:1 in a free shear flow. The value here for the whole channel is approximately 10.3:1, indicating a less strongly aligned nature. The alignment ratios for each classification of coherent structure are listed in table 2.

![Figure 9](image)

**Figure 9.** PDF of the alignment of the principal strain directions with the local vorticity vector. (a) Within vortex clusters, (b) $Q_2$, and (c) $Q_4$ structures, each compared with the distribution for the whole computational domain.

Within vortex clusters, the first eigenvector is no longer randomly oriented, but aligns transversely to the vorticity field. No such variation from the behaviour of the whole flowfield is observed in intense Reynolds stress structures of both the $Q_2$ and $Q_4$ type. It appears that the strong increase in $\hat{e}_1 - \omega$ alignment close to $\cos(\theta) = 0$ is due primarily to the behaviour of the flow within wall-attached $Q_4$ structures. This is also the case for the similar feature seen in the alignment PDF relating to the $\hat{e}_3$ eigenvector. For most of the structures for which alignment distributions are plotted, the third eigenvector is equivalently, or less strongly, transversally aligned with the vorticity field. The exception is the alignment distribution calculated within detached vortex clusters wherein the transversal alignment is stronger than that exhibited by the flow overall.

The Euclidean norm, $\|\alpha\|$, of the principal strain eigenvalues is used to normalise the three eigenvalues of the rate-of-strain tensor, which are ordered by magnitude as

$$\frac{\alpha_1}{\|\alpha\|} > \frac{\alpha_2}{\|\alpha\|} > \frac{\alpha_3}{\|\alpha\|}$$

Due to the incompressible nature of this flow, the largest strain rate $\alpha_1$ must be positive definite and the smallest, $\alpha_3$, must be always negative. The intermediate eigenvalue however may take either sign and, since the second eigenvector, $\hat{e}_2$, is preferentially aligned with the vorticity within the vortex clusters and the intense Reynolds stress structures, its relative extensional or compressive nature strongly affects the behaviour of these structures [28]. Incompressibility also dictates that the maximum value that any one strain rate may take is $\alpha_i/\|\alpha\| = \pm 2/\sqrt{6}$. Normalised probability density functions of the principal strain rates are computed and shown in figure 10. PDFs of $\alpha_i/\|\alpha\|$ calculated over the entire computational domain are compared with conditional PDFs computed from vortex clusters, $Q_2$, and $Q_4$ structures in figure 10(a), (b), and (c), respectively.

The intermediate strain rate distribution is clearly skewed towards positive values in all cases, indicating a predominantly vortex stretching behaviour and thus the amplification of enstrophy. The distribution of the strain eigenvalues within $Q_2$ structures varies negligibly from the distribution calculated over the whole domain. Within $Q_4$ structures the first eigenvalue
tends to be more extensional, while the distribution of the second eigenvalue is less strongly skewed towards positive values. Very slight, raised bumps in the strain rate distributions are observed near the origin and at $\alpha_1/\|\alpha\| = 1/\sqrt{2}$. These appear only in the whole domain PDFs and, more clearly, in the attached $Q_4$ structure PDFs, and are presumably driven by two–dimensional flow near the wall within the $Q_4$ structures. The distribution of principal strain rates within the vortex clusters displays similarly increased likelihood of highly extensional first principal strain, but to a greater extent than the $Q_4$ structures. This is especially true for vortex clusters originating within 20 wall–units of the wall.

It is notable that the PDFs of $\alpha_1$, $\alpha_2$, and $\alpha_3$ do not overlap. This implies that the most extensional strain rate eigenvalue at each point in the flow is larger than the intermediate eigenvalue at every other point in the flow, and that the same follows for the relationship between the intermediate and most compressional strain rate eigenvalues also. This is the case since, due to incompressibility, the values of neighbouring eigenvalues coincide when they are equal to $\alpha_i/\|\alpha\| = \pm 1/\sqrt{6}$, and this represents a bound on each normalised principal strain rate.

7. Remarks

In comparing the topological distribution within intense Reynolds stress structures to that within vortex clusters, it was found that joint PDFs of the VGT invariants conditionally calculated within intense Reynolds stress structures show similar overall topology due to their non–mutually exclusive nature, yet still exhibit multiple distinct differences both with each other and with the overall channel flow itself. Wall–attached $Q_4$ structures originating at the lower bound within twenty wall units of the wall ($y^+ \leq 20$) show the greatest difference in topology from the other structures, with greater correlation between $Q_W$ (enstrophy) and $Q_S$ (dissipation) exhibited within these structures. This is likely due to attached $Q_4$ structures’ closer proximity to the wall, which is reflected also in the distribution of principal strain rates, where the presence of two–dimensional flow is evident. Vortex clusters by definition include topology with only positive discriminant, and this is reflected in the shape of joint PDFs of the invariants $Q_A$ and $R_A$ of the velocity gradient tensor within these vortex clusters. They also show a different joint PDF structure to the intense Reynolds stress structures in the $Q_S – Q_W$ plane $Q_W – \Sigma$ plane, with stronger vortex stretching, and greater likelihood of strong rotational motion.

Alignment of the principal strain directions with the vorticity field is qualitatively similar to that observed in other turbulent flows, with strong alignment of the intermediate strain direction within all coherent structures examined here. The intermediate principal strain rate is skewed towards positive values, implying predominance of vorticity amplification within these
structures, but both the likelihood of close alignment and the relative extensional nature of the intermediate strain rate vary depending on which type of coherent structure is considered. As with the joint PDFs of the VGT invariants, the differences between the $Q_2$ structures and the whole flow domain are less distinct when compared to the flow within $Q_4$ structures.

Acknowledgements
This research was funded in part by the Multiflow program of the European Research Council (ERC) and was undertaken with the assistance of resources from the National Computational Infrastructure (NCI), which is supported by the Australian Government. Further computational resources were provided by the Partnership for Advance Computing in Europe (PRACE). The support of the Australian Research Council (ARC) is also gratefully acknowledged. Julio Soria gratefully acknowledges the support of an Australian Research Council Discovery Outstanding Researcher Award fellowship.

References
[1] Lozano-Durán A, Flores O and Jiménez J 2012 The three-dimensional structure of momentum transfer in turbulent channels J. Fluid Mech. 694 100–130
[2] Robinson S K 1991 Coherent structures in the turbulent boundary layer Ann. Rev. Fluid Mech. 23 601–639
[3] Holmes P, Lumley J L and Berkooz G 1996 Turbulence, coherent structures, dynamical systems and symmetry. Cambridge Monographs in Mechanics. 1st ed (Cambridge Univ. Press)
[4] Hunt J C R, Wray A A and Moin P 1988 Eddies, streams, and convergence zones in turbulent flows Proc. CTR Summer Prog., Center Turb. Res., NASA Ames/Stanford University
[5] Chong M S, Perry A E and Cantwell B J 1990 A general classification of three-dimensional flow fields Phys. Fluids 2 765–777
[6] del Álamo J C, Jiménez J, Zandonade P and Moser R D 2006 Self–similar vortex clusters in the turbulent logarithmic region J. Fluid Mech. 561 329–358
[7] Haller G 2002 Lagrangian coherent structures from approximate velocity data Phys. Fluids 14 1851–1861
[8] Shadden S C, Lekien F and Marsden J E 2005 Definition and properties of Lagrangian coherent structures from finite-time Lyapunov exponents in two-dimensional aperiodic flows Physica D 212 271–304
[9] Cucitore R, Quadrio M and Baron A 1999 On the effectiveness and limitations of local criteria for the identification of a vortex Europ. J. Mech. B: Fluids 18 261–282
[10] Chakraborty P, Balachandar S and Adrian R J 2005 Relationships between local vortex identification schemes J. Fluid Mech. 535 189–214
[11] Soria J, Sondergard R, Cantwell B, Chong M and Perry A 1994 A study of the fine–scale motions of incompressible time–developing mixing layers Phys. Fluids 6 871–884
[12] Perry A and Chong M 1994 Topology of flow patterns in vortex motions and turbulence App. Sci. Res. 53 357–374
[13] Blackburn H, Mansour N and Cantwell B 1996 Topology of fine–scale motions in turbulent channel flow J. Fluid Mech. 310 269–292
[14] Ooi A, Martin J, Soria J and Chong M 1999 A study of the evolution and characteristics of the invariants of the velocity–gradient tensor in isotropic turbulence J. Fluid Mech. 381 141–174
[15] Wallace J M, Eckelman H and Brodkey R S 1972 The wall region in turbulent shear flow J. Fluid Mech. 54 39–48
[16] Willmarth W W and Lu S S 1972 Structure of the Reynolds stress near the wall J. Fluid Mech. 55 65–92

2nd Multiflow Summer School on Turbulence IOP Publishing
Journal of Physics: Conference Series 708 (2016) 012005 doi:10.1088/1742-6596/708/1/012005
[17] Lu S S and Willmarth W W 1973 Measurements of the structure of the Reynolds stress in a turbulent boundary layer J. Fluid Mech. 60 481–511
[18] del Álamo J C, Jiménez J, Zandonade P and Moser R D 2004 Scaling of the energy spectra of turbulent channels J. Fluid Mech. 500 135–144
[19] Kim J, Moin P and Moser R D 1987 Turbulence statistics in fully developed channel flow at low Reynolds number J. Fluid Mech. 177 133–166
[20] Moser R D, Kim J and Mansour N N 1999 Direct numerical simulation of turbulent channel flow up to $Re_f = 590$ Phys. Fluids 11 943–945
[21] Atkinson C, Chumakov S, Bermejo–Moreno I and Soria J 2012 Lagrangian evolution of the invariants of the velocity gradient tensor in a turbulent boundary layer Phys. Fluids 24 105104
[22] Siggia E D 1981 Numerical study of small-scale intermittency in three-dimensional turbulence J. Fluid Mech. 107 375–406
[23] Jiménez J, Wray A A, Saffman P G and Rogallo R S 1993 The structure of intense vorticity in isotropic turbulence J. Fluid Mech. 255 65–90
[24] Bermejo-Moreno I, Atkinson C, Chumakov S, Soria J and Wu X 2010 Flow topology and non–local geometry of structures in a flat–plate turbulent boundary layer Proc. CTR Summer Prog., Center Turb. Res., NASA Ames/Stanford University
[25] Ashurst W T, Kerstein A R, Kerr R M and Gibson C H 1987 Alignment of vorticity and scalar gradient with strain rate in simulated Navier–Stokes turbulence Phys. Fluids 30 2343–2353
[26] Sondergaard R, Chen J H, Soria J and Cantwell B J 1991 Local topology of small scale motions in turbulent shear flows 8th Symp. Turbulent Shear Flows, Munich Germany
[27] Tsinober A, Kit E and Dracos T 1992 Experimental investigation of the field of velocity gradients in turbulent flows J. Fluid Mech. 242 169–192
[28] Vincent A and Meneguzzi M 1994 The dynamics of vorticity tubes in homogeneous turbulence J. Fluid Mech. 258 245–254