Fractional optical vortices in a uniaxial crystal

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Abstract
We have analyzed the solutions to the vector paraxial wave equation in the unbounded uniaxial crystal in the form of the transverse electric (TE) and transverse magnetic (TM) mode beams transporting the fractional optical vortices in the circularly polarized components. We revealed that the TE and TM beams have an asymmetric structure in distributing the local elliptic polarization over the beam cross-section and form two sets of singular beams depending on the real or imaginary value of the free $K$ parameter. We found that the fractional optical vortex born in the left-handed circularly polarized component of the beam with the real $K$ parameter can exist in the form of a holistic structure within small crystal lengths much smaller than the Rayleigh length while the beams with the imaginary $K$ parameter cannot maintain fractional optical vortices in the free state. However, such beam types can generate singly charged optical vortices in a far field. We also revealed that the energy efficiency and spin–orbit coupling are defined by the angular spectrum of the beam. The beam with the real $K$ parameter is characterized by a broad spectrum of plane waves propagating at small angles to the crystal optical axis. The beams with the imaginary $K$ parameter are shaped by two conical fans of plane waves. It is this circumstance that defines a very high value of the energy efficiency.

Keywords: optical vortices, uniaxial crystals, fractional charge

(Some figures may appear in colour only in the online journal)

1. Introduction

The problem of vortex beams bearing optical vortices with fractional topological charges is one of the most intriguing parts of advanced singular optics. The first experiments with fractional vortex beams on the base of computer-generated holograms considered by Soskin et al [1, 2] as far back as the beginning of the 1990s showed the principal instability of such wave constructions when propagating. Later the theoretical analysis of the Gaussian beam diffracted by the spiral plate with a fractional phase step presented by Berry [3] have shown that the diffracted beam splits at once into an infinite series of standard vortex beams with integer-order topological charge evolving in space in a complex way. The detailed experimental preparation of the phase structure of the fractional vortex beams of different kinds, with the help of the holographic technique researched by Padgett et al [4], have corroborated such a theoretical proposition. Subsequent study of the conical beams [5, 6] has also shown that fractional optical vortices cannot exist in the free state even in the initial plane but are a complex array of singly charged optical vortices. The vector features of the fractional vortex beams in free space and homogeneous isotropic media were considered in the paper [7].

At the same time, there are a number of mechanical and quantum mechanical analogues of the phase defects with a fractional phase step, e.g. in the Aharonov–Bohm effect and in the surface waves in liquid [8], indicating the possibility of observing the fractional vortex beam in the free state in optics. Recently we have shown [9] that there is at least one type of vortex beam with the 1/2-topological charge (the so-called erf-G beams) that can contain the fractional optical vortex in the free state near the initial plane. However, such a vortex state is a very unstable one splitting into the integer-order...
vortex array when propagating within a Rayleigh length. Also a reliable technique for generating such wave constructions has not yet been developed. However, there is the well-known technique for creating the integer-order vortices of highest orders on the base of uniaxial [10–16], biaxial [17–20], and biaxially induced crystals [21–24]. Thus, it is of great theoretical and practical interest to bring to light the basic physical processes responsible for the nucleation and propagation of the fractional optical vortices in light beams traveling through uniaxial crystals as the potential media for creating and manipulating such wave constructions.

The aim of our paper is to analyze the diffractive processes of the paraxial optical beams in a uniaxial crystal, resulting in the generation of fractional optical vortices, and to estimate their energy efficiency.

2. Asymmetric structure of TE and TM mode beams

We consider the propagation of the monochromatic paraxial singular beams with fractional optical vortices along the optical axis of the unbounded uniaxial crystal with the permeability tensor $\varepsilon = \text{diag} (\varepsilon_1, \varepsilon_1, \varepsilon_2, \varepsilon_3)$, where $\varepsilon_1 = \varepsilon_2 = n_0^2$, and $\varepsilon_3$ is the refractive index along the tensor principle axes, respectively. The electric field of the paraxial beam can be treated as $E(x, y, z) = \mathbf{E}(x, y, z) \exp(ik_0 z)$, where $\mathbf{E}(x, y, z)$ stands for the complex amplitude of the field, $k_0 = n_0 k$ and $k$ is the wavenumber. Then the paraxial wave equation for the transverse field components $E_\parallel = e_x E_x + e_y E_y$ is written in the form [12]:

$$\nabla^2 E_\parallel + 2ik_0 \partial_z E_\parallel = \alpha \nabla_\perp \cdot (\nabla_\perp E_\parallel), \quad \text{(1)}$$

whereas the longitudinal component can be found as

$$E_z \approx \frac{i n_0}{k n_2} \nabla_\perp E_\parallel, \quad \text{(2)}$$

where $\nabla_\perp = e_x \partial_x + e_y \partial_y$, $\alpha = \Delta \varepsilon / \varepsilon_{11}$, $\Delta \varepsilon = \varepsilon_{11} - \varepsilon_{33}$.

When dealing with the vertex beams it is convenient to use the complex variables [12, 15]

$$u = x + i y, \quad v = x - i y, \quad \text{(3)}$$

so that

$$\partial_u = \partial_x - i \partial_y = e^{-i \varphi} \left( \partial_x - \frac{i}{r} \partial_\varphi \right), \quad \text{(4)}$$

$$\partial_v = \partial_x + i \partial_y = e^{i \varphi} \left( \partial_x + \frac{i}{r} \partial_\varphi \right), \quad \text{(5)}$$

where $(r, \varphi)$ are the polar coordinates. The operators $\partial_u$ and $\partial_v$ can be treated as the operators of the annihilation and birth of optical vortices, respectively, while

$$\nabla^2 u = 4 \partial_{uu}, \quad \nabla_\perp E_\parallel = \partial_v E_+ + \partial_u E_- \quad \text{(6)}$$

In the last part of equation (6), the transverse electric field is represented in the circularly polarized basis

$$E_+ = E_x - i E_y, \quad E_- = E_x + i E_y. \quad \text{(7)}$$

Now the vector wave equation can be rewritten for each field component as

$$(\nabla^2 + 2ik_0 \partial_z) E_+ = 2\alpha \partial_u (\partial_u E_+ + \partial_\varphi E_-), \quad \text{(8)}$$

$$(\nabla^2 + 2ik_0 \partial_z) E_- = 2\alpha \partial_v (\partial_v E_+ + \partial_\varphi E_-). \quad \text{(9)}$$

The simplest solutions of the equation systems (8) and (9) have the form of transverse electric (TE) and transverse magnetic (TM) mode beams:

the TE mode beams, $E_z = 0$

$$E^{(\alpha)}_+ = \partial_u \Psi_\alpha, \quad E^{(\alpha)}_- = -\partial_\varphi \Psi_\alpha, \quad \text{(10)}$$

while the generatrix function $\Psi_\alpha$ obeys the paraxial wave equation

$$(\nabla^2 + 2ik_0 \partial_z) \Psi_\alpha = 0, \quad \text{(11)}$$

the TM mode beams, $H_z = 0$

$$E^{(\alpha)}_+ = \partial_\varphi \Psi_\alpha, \quad E^{(\alpha)}_- = \partial_v \Psi_\alpha, \quad \text{(12)}$$

while the generatrix function $\Psi_\alpha$ obeys the paraxial wave equation

$$\left( \nabla^2 + 2i \frac{k_0}{1 - \alpha} \partial_\varphi \right) \Psi_\alpha = 0. \quad \text{(13)}$$

From whence the TE mode beams are characterized by the ordinary refractive index $n_0$ and the wavenumber $k_0$ while the TM mode beams are associated with the extraordinary refractive index $n_\sigma = \frac{n_0}{\sqrt{n_0^2 - 1}} = \frac{n_0}{\sqrt{1 - \alpha}}$ and the wavenumber $k_\sigma = n_\sigma k$.

Thus, the solutions of the vector equation (1) are held back by the solutions of the scalar equations (11) and (13). We will find the wave functions $\Psi_\alpha$ and $\Psi_\sigma$ in the form

$$\Psi^{(q)}_\alpha = N_\alpha(z) F^{(q)}_\alpha(U_\alpha, V_\alpha) G_\alpha(u, v, z), \quad \text{(14)}$$

$$\Psi^{(q)}_\sigma = N_\alpha(z) F^{(q)}_\alpha(U_\sigma, V_\sigma) G_\sigma(u, v, z), \quad \text{(15)}$$

where $U_\alpha,v = \frac{u}{\sqrt{u^2 + v^2}}, \quad V_\alpha,v = \frac{v}{\sqrt{u^2 + v^2}}, \quad N_\alpha,\sigma(z)$ are the normalizing coefficients, $\sigma_{\alpha,\sigma} = 1 + iz/\sigma_{\alpha,\sigma}, \quad \alpha_{\alpha,\sigma} = k_\alpha w_0^2 / 2$, $w_0$ is the radius of the beam waist at $z = 0$, $G_{\alpha,\sigma} = \exp(-u^2 v^2 / w_0^2 \sigma_{\alpha,\sigma}) / \sigma_{\alpha,\sigma}$ are the Gaussian envelopes of the ordinary and extraordinary beams, while the modulation functions $F^{(q)}_{\alpha,\sigma}$ obey the Helmholtz–Kiselev equation [25]:

$$\left( \frac{4}{\partial U_{\alpha,\sigma}} \partial U_{\alpha,\sigma} + K^2_{\alpha,\sigma} \right) F^{(q)}_{\alpha,\sigma} = 0, \quad \text{(16)}$$

where the parameter $K_{\alpha,\sigma}$ can be an arbitrary complex value and $q$ is some number.

We choose the solutions of equation (16) in the form of the erf-$G$ beams (error function Gaussian beams) with $q = 1/2$:

$$F_{1/2}^{(q)} = \int_0^{2\pi} e^{i \varphi} \exp \left\{ - \frac{K_{\alpha}^2 r}{\sigma_0} \cos(\phi - \varphi) \right\} d\phi$$

$$= \int_0^{2\pi} e^{i \varphi} \exp \left\{ - \frac{K_{\alpha}^2 w_0^2}{2} \left[ U_\alpha e^{-i \varphi} + V_\alpha e^{i \varphi} \right] \right\} d\phi$$

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The solutions of the vector wave equation (1) can be obtained for the TE mode beams in the form

$$E_{\pm}^0 = \partial_{\phi} \psi_{1/2}^0 = -\frac{N_0 G_0}{\sigma_0} \left\{ F_{0}^{1/2} + \frac{e^{-i\phi}}{r_0^{1/2}} F_{0}^{1/2} \right\}, \quad (19)$$

and for the TM mode beams in the form

$$E_{\pm}^0 = -\partial_{\phi} \psi_{1/2}^0 = \frac{N_0 G_0}{\sigma_0} \left\{ F_{0}^{1/2} - \frac{e^{-i\phi}}{r_0^{1/2}} F_{0}^{1/2} \right\}, \quad (20)$$

where $r_0^{1/2} = K_0^{(o,e)} w_0^2/2$ and we made use of the relations

$$\partial_{\phi} U_{\phi} F_{0}^{(o,e)} = - (K_0^{(o,e)} w_0^2) F_{0}^{(o,e)}, \quad (23)$$

$$\partial_{\phi} V_{\phi} F_{0}^{(o,e)} = - (K_0^{(o,e)} w_0^2) F_{0}^{(o,e)}, \quad (23)$$

$$F_{0}^{(o,e)} = \int_0^{2\pi} e^{i\phi} \exp \left\{ -\frac{K_0^{(o,e)}}{\sigma_0} \cos(\phi - \varphi) \right\} d\phi. \quad (24)$$

Besides, the functions $F_{0}^{(o,e)}$ in equations (19)–(22) are expressed in the explicit form as

$$F_{-1/2}^{(o,e)} = -\frac{2\sqrt{\pi} e^{-i\phi/2}}{\sigma_0} \left\{ e^{-\sigma_0^2/2} \text{erf} \left( i\sigma_0 sin \frac{\varphi}{2} \right) \right\}, \quad (25)$$

$$F_{3/2}^{(o,e)} = -\frac{e^{i\phi/2}}{\sigma_0} \left\{ 2i\sqrt{\pi} \left\{ \frac{1}{2} \right\} \right\} \times \text{erf} \left( i\sigma_0 sin \frac{\varphi}{2} \right) + \frac{8}{\sigma_0} \text{erf} \left( \frac{\sigma_0^2 \sin^2 \frac{\varphi}{2}}{2} \right). \quad (26)$$

For the more general case of the highest order TE and TM beams we obtain from equations (19)–(22)

$$(E_{+}^{(m,n)})_{o,e} = \varphi_{e}^{m+1} \psi_{1/2}^{o,e}, \quad (27)$$

$$(E_{-}^{(m,n)})_{o,e} = s_0 e^{i\psi_{1/2}^{o,e}}, \quad (27)$$

where $s_0 = 1, \sigma_0 = -1$.

It should be noted that the field equations (19)–(22) of the TE and TM modes have asymmetric structure and are not linearly polarized in each point of the cross-section [9] in contrast to the standard TE and TM mode beams [12]. Besides, their field structure is transformed when propagating. Nevertheless, the corresponding longitudinal components of the electric or magnetic fields are definitely zero.

We will consider the case when only the $E_+$ component exists at the initial plane $z = 0$, i.e., $E_-(z = 0) = 0$. Then the field is described by the expression:

$$E(r, \varphi, z) = E_0(r, \varphi, z) + E_0(r, \varphi, z), \quad (28)$$

so that $K_{0}^{(o,e)} = K_{0}^{(o,e)} = K_{0}^{(o,e)}$, $r_0^{(o,e)} = r_0^{(o,e)}$, and $E_0(r, \varphi, z = 0) = e_0^{(o,e)} E_0(r, \varphi, z = 0)$. The corresponding intensity and phase distributions for the case $K_{0}^{(o,e)}$ being the real value are shown in figures 1 and 2, respectively. The characteristic feature of the phase distribution is the holostic helix-like surface with the step $\Delta \Psi = \pi$ (figure 2) in contrast to that for the case where $K_{0}^{(o,e)}$ is the imaginary value (see figures 3(a) and (b)) with the standard representation of the fractional vortex in the form of the infinite series of the integer-order optical vortices (see e.g. [2, 3]).

Our undivided attention should be drawn to the black line in the form of the broken circle cutting the intensity spike in figure 1. In order to analyze such a wave structure let us write the condition of the zeros in the field (19) $E_0(r, \varphi, z = 0) = 0$ far from the axis $r = 0$ when $z = 0$:

$$\left( 1 + \frac{r}{r_0} \right) e^{-K_{0}^{(o,e)}} \left( i\sigma_0 sin \frac{\varphi}{2} \right) + \left( 1 - \frac{r}{r_0} \right) e^{K_{0}^{(o,e)}} \left( i\sigma_0 sin \frac{\varphi}{2} \right) = 0. \quad (29)$$

First of all, the second term in equation (29) is much greater than the first one with the exception of the region near $\varphi = \pi$ because of $\exp(K_{0}^{(o,e)} r_0) \approx 7.21 \times 10^{48}$ and $\exp(-K_{0}^{(o,e)} r_0) \approx 1.4 \times 10^{48}$ for $K_{0}^{(o,e)} = 5 \times 10^{5} m^{-1}, w_0 = 30 \mu m$ but it vanishes at the circle

$$r = r_0^{(o,e)} = \frac{K_{0}^{(o,e)} w_0^2}{2}. \quad (30)$$

On the other hand, the first term in equation (29) is not the exact zero along the circle (30) excepting for the point $\varphi = 0$. 
Figure 1. Intensity distribution $E_z$ component of the TE mode at: $z = 0$, $w_0 = 30 \mu m$, $K_\perp = 3 \times 10^5 m^{-1}$.

Figure 2. Phase distribution $\Phi(x, y)$ near the axis $r = 0$ in the $E_z$ component at the initial plane $z = 0$.

Near the point $\varphi = \pi$ and $K_\perp r \gg 1$ its asymptotic value can be described by the expression:

$$
\left(1 + \frac{r}{r_\perp}\right) e^{-K_\perp r} \text{erf} \left( i \sqrt{2} \frac{K_\perp r \sin \frac{\varphi}{2}}{2} \right) \sim i \left(1 + \frac{r}{r_\perp}\right)
$$

$$
\times \frac{e^{-K_\perp r}}{\sqrt{\pi} \sqrt{2 K_\perp r}} \exp \left(2 K_\perp r \sin^2 \frac{\varphi}{2} \right),
$$

forming the bright spot at the end of the black ring $r = r_\perp$ in figure 1 while near the point $\varphi = 0$ it can be approximated as

$$
\left(1 + r \frac{r}{r_\perp}\right) e^{-K_\perp r} \text{erf} \left( i \sqrt{2} \frac{K_\perp r \sin \frac{\varphi}{2}}{2} \right) \sim i \left(1 + r \frac{r}{r_\perp}\right)
$$

$$
\times \frac{2 e^{-K_\perp r}}{\sqrt{\pi} \sqrt{2 K_\perp r}} \sqrt{2 K_\perp r \sin \frac{\varphi}{2}}.
$$

This means that the first term is an imaginary one over all the region $\varphi = (0, 2\pi)$ while the second term is real. Thus, the real and imaginary parts in equation (29) vanish simultaneously at the point $\varphi = 0$, $r = r_\perp$. Together with the factor $\exp(-i\varphi/2)$ they form a very intricate picture of the phase singularity whose phase portrait in the vicinity of the circle (30) is illustrated in figure 4.

3. The evolution of field states along the crystal

The inner structure of the centered and ring phase singularities described above is broken down when propagating the beam. We can for convenience outline two counter processes: (1) the beam diffraction and (2) spin–orbit coupling. The spin–orbit coupling results in nucleating something like a holistic screw dislocation (an optical vortex) with a fractional phase step proportional to $q = 2 - 1/2$ [15, 26] in the vector beam with $K_\perp$ real while the diffraction process splinters it into a set of integer-order optical vortices [9]. However, the process of the fractional vortex destruction is not an instantaneous one at the
Figure 3. Intensity ((a), (c)) and phase ((b), (d)) distributions in the TE mode with \( w_0 = 30 \mu m \), for ((a), (b)) the \( E_+ \) component, \( K_\perp = 5.5 \times 10^5 \) m\(^{-1}\) at the initial plane \( z = 0 \) and for ((c), (d)) the \( E_- \) component, \( K_\perp = 5.5 \times 10^5 \) m\(^{-1}\) at the initial plane \( z = 0.8 \) m.

initial plane as in the case shown in figure 3 but runs gradually along the beam length (see figure 5).

The non-disturbed state of the fractional vortex with \( q = -1/2 \) in the \( E_+ \) component and with \( q = 3/2 \) in the \( E_- \) component remains near the axis up to the \( z = 1 \) mm crystal length. At the same time there is nucleation of integer-order optical vortices at the periphery, in particular near the black broken ring in both components. At the length of about \( z = 1 \) cm we observe at last the nucleation of the vortex dipole near the axis in the \( E_+ \) component and the ensemble of three vortices (two with \( q = +1 \) and one with \( q = -1 \)) in the \( E_- \) component. At far field the vortex patterns in both components are very complicated, coming to be like that shown in figures 3(a) and (b) for the \( E_+ \) component and in figures 3(c) and (d) for the \( E_- \) component. It is interesting to note that there is not the double-charged vortex at the axis as could be expected [12]. Instead, we observe two separated singly charged optical vortices with the same sign of the topological charges (see figure 3(d)). Moreover, the energy exchange between circularly polarized components runs very slowly. The field transformations caused by the diffraction processes are more intense than those associated with the spin–orbit coupling. The vector structure of the beam changes very slowly.

An absolutely different situation occurs in the vortex beam with the imaginary \( K_\perp \) parameter. At the initial plane \( z = 0 \) we observe a complex array of integer-order optical vortices (figures 3(a) and (b)). The characteristic feature of each TE and TM mode with fractional optical vortices in their circularly polarized components is of the inherent asymmetry of the field distribution [9] that defines the further evolution of the field. The diffraction processes very quickly rebuild the field structure: the speckle-like intensity distribution turns into the regular \( C \)-like field without the black broken ring at a crystal distance of about 1–2 mm (see figure 6). At the same time, the intense energy exchange between circularly polarized components results in the periodical conversion of the right- and left-handed circular polarizations located at the intensity maximum of the \( C \)-like pattern. The uniform circular polarized field changes into the space variant polarization field with local linear polarization at the distance \( z = 2.5 \) mm. Then the field transforms into one with the dominant left-handed circular polarization, etc.

It is interesting to compare the vortex structures near the optical axis in the left-handed circularly polarized components in the beams with the real and imaginary \( K_\perp \) parameters (see figure 7) at far diffraction field. The set of equi-phase lines in figure 7(a) for the real \( K_\perp \) parameter has a smooth shape without any phase steps. This means that the left-handed field component does not carry over any centered optical vortices. At the same time the \( E_- \) component in the beam with the imaginary \( K_\perp \) parameter has a single optical
vortex with a positive topological charge \( l = 1 \) (figure 7(b)) in contrast to that of the initial standard beams \([12]\) bearing double-charged optical vortices near the crystal optical axis in the \( E_\perp \) component. Moreover, there are no integer-charged vortices at the periphery too safe for a weak phase perturbation in both circularly polarized components.

In order to clarify the physical picture of the above processes let us consider the angular spectrum of the field (21) and (22) at the initial plane \( z = 0 \) (\( N_o = N_e = N, G_o = G_e = G \)). First of all, it should be noted that the field \( F_q \) in equation (24) can be presented as the infinite superposition of the integer-order vortices \([3]\)

\[
\Psi_q = GN \int_0^{2\pi} e^{i q \phi} \exp \{-K_\perp r \cos(\phi - \varphi)\} d\phi
\]

\[
= A G N \sum_{m=-\infty}^{\infty} \int_0^{2\pi} \frac{e^{i m \phi}}{q - m} \exp \{-K_\perp r \cos(\phi - \varphi)\} d\phi
\]

\[
= 2\pi A G N \sum_{m=-\infty}^{\infty} \frac{e^{i m \phi} I_m(K_\perp r)}{q - m},
\]

where \( A = \sin(q \pi) e^{i q \pi}/\pi, I_m(x) \) is the modified Bessel function. The spectral function of the field (19) can be written as

\[
U_q(K_p) = \frac{k_o}{2\pi} \int_\infty E_+(r)e^{-i(k_x x + k_y y)} d^2r
\]

\[
= \frac{k_o}{2\pi} \int_0^\infty r dr \int_0^{2\pi} d\varphi E_+
\]

\[
\times (r, \varphi) e^{-i K_p r \cos(\varphi - \phi)},
\]

(34)

where \( E_+ = -\{\Psi_q - \frac{e^{-i \varphi}}{r_\perp} \Psi_{q-1}\}, K_p = \sqrt{k_x^2 + k_y^2}, r = (x, y), K_p = (k_x, k_y) \).

After the fourth integration (see 2.12.39.2-3 in [27]) of equation (34) we come to the expression for the imaginary \( K_\perp \) parameter

\[
U_{-1/2}(K_p) = -NA \frac{kw_0^2}{2}
\]

\[
\times \left\{ \sum_{m=-\infty}^{\infty} \frac{(-i)^m e((m-1)\varphi)}{1/2 - m} \left[ e^{i P_m + \frac{w_0}{r_\perp} Q_m}\right] \right\},
\]

(35)
Figure 5. Intensity $I$ and phase $\Phi$ distributions in the $E_+$ and $E_-$ components of TE mode beams of the lowest order along the length $z$ of the LiNbO$_3$ crystal $n_1 = 2.3$, $n_2 = 2.2$, $w_0 = 30 \mu$m, $K_\perp = 3 \times 10^5$ m$^{-1}$.

$$P_m = \frac{1}{2} \exp \left\{ -\frac{(K_p^2 + K_\perp^2)w_0^2}{4} \right\} I_m \left( -\frac{K_pK_\perp w_0^2}{2} \right), \quad (36)$$

$$Q_m = \frac{(K_\perp K_p w_0)^{2|m|}}{2^{2|m|+1}((|m|!)^2} \sum_{n=0}^{\infty} (-1)^n \times \frac{\Gamma(n + |m| + 3/2)}{n!} \left( \frac{K_p^2 w_0^2}{4} \right)^n \times 2F_1 \left( -n, -|m| - n; |m| + 1; -\frac{K_p^2}{K_\perp^2} \right), \quad (37)$$

where $\Gamma(v)$ is the gamma function, $2F_1$ stands for the hypergeometric function.

The energy efficiency of the vortex generation in the $E_-$ field component can be estimated by means of the spin–orbit coefficient [28]

$$\eta = \frac{1}{2} \left\{ 1 - 4 \text{Re} \left[ \iint_S E_+^* (E_-)^* \text{d}S \right] / I_0 \right\}, \quad (38)$$

where $I_0$ is the total intensity of the initial beam at $z = 0$, $S$ is the square of the beam cross-section at arbitrary crystal length.
Figure 6. Evolution of the polarization states in the vortex beam with $K_\perp = i5.5 \times 10^5$ m$^{-1}$, $w_0 = 30 \mu$m.

Figure 7. Phase distributions near the beam axis at $z = 10$ mm for (a) $E_+$ and (b) $E_-$ components, $K_\perp = i5.5 \times 10^5$ m$^{-1}$, $w_0 = 30 \mu$m.

$\eta(z)$ with the $K_\perp$ parameter corresponding to the spectral extremes $K_\perp = i5.5 \times 10^5$ m$^{-1}$ and $K_\perp = i4.5 \times 10^5$ m$^{-1}$ shown in figure 9 have quasi-periodical forms. Their energy efficiency can reach the very high value $\eta = 0.95$. This means that the singly charged optical vortex in the $E_-$ component is generated with efficiency 95% at the crystal lengths $z = 4$ mm and $z = 7.5$ mm, respectively. (The polarization distributions in figure 6 were obtained for crystal lengths corresponding to the extreme value $\eta$ and $\eta = 0.5$ of the curve in figure 9.) However, the energy efficiency in the second type of beam with the real $K_\perp = 5.5 \times 10^5$ m$^{-1}$ does not exceed 52% at a distance of about $z = 30$ mm. Moreover, this is the maximum value for any arbitrary large crystal length.

4. Conclusions

We have analyzed the solutions to the vector paraxial wave equation in the unbounded uniaxial crystal in the form of the
Figure 8. Angular spectra for the beams with $w_0 = 30 \, \mu m$ and (a) $K_\perp = 15.5 \times 10^4 \, m^{-1}$, (b) $K_\perp = 5.5 \times 10^4 \, m^{-1}$.

Figure 9. Energy efficiency $\eta$ of the spin–orbit coupling for the two types of vortex beams.

TE and TM mode beams transporting the fractional optical vortices in the circularly polarized components. We have found that both wave types make up two sets of singular beams with the real and imaginary free $K$ parameter. These beam types have different diffraction properties manifesting themselves in processes of the nucleating and breaking down of the fractional optical vortices. We revealed that the TE and TM beams have asymmetric structure in distributing the local elliptic polarization over the beam cross-section. The vector structure of the beams evolves along the beam length. The evolution of the vector structure and the generation of fractional optical vortices are controlled by two processes: the anisotropic diffraction of the beams and the spin–orbit coupling. The fractional optical vortex born in the left-handed circularly polarized component can exist in the form of a holistic structure within small crystal lengths much smaller the Rayleigh length. Then it splits into a great number of integer-order optical vortices. At far field, a pair of singly charged optical vortices with the same signs are created near the beam axis. The beams with the real $K$ parameter cannot form the holistic fractional vortices in both circularly polarized components. They always spilled into a set of integer-order optical vortices. However, there is only one singly charged vortex in the circularly polarized component with opposite handedness to the initial one at far field.

We estimated the energy efficiency of generating the fractional optical vortices and revealed that it can reach 95% in beams with the imaginary $K$ parameter while the energy efficiency cannot exceed 52% in beams with the real $K$ parameter. We found that the spin–orbit coupling and, consequently, the energy efficiency is defined by the angular spectrum of the beams. The beam with the real $K$ parameter is characterized by a broad spectrum of plane waves propagating at small angles to the crystal optical axis. The beams with the imaginary $K$ parameter are shaped by two conical fans of plane waves. It is this circumstance that defines a very high value of the energy efficiency.

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References

[1] Basisty I, Soskin M and Vasnetsov M 1995 Optical wavefront dislocations and their properties Opt. Commun. 119 604–12
[2] Basisty I, Pas’ko V, Slyusar V, Soskin M and Vasnetsov M 2004 Synthesis and analysis of optical vortexes with fractional topological charges J. Opt. A: Pure Appl. Opt. 6 S166–9
[3] Berry M V 2004 Optical vortices evolving from helicoidal integer and fractional phase steps J. Opt. A: Pure Appl. Opt. 6 S259–69
[4] Leach J, Yao E and Padgett M J 2004 Observation of the angular momentum order dependence J. Opt. A: Pure Appl. Opt. 6 154–62
[5] Garcia-Gracia H and Gutiérrez-Vega J C 2009 Diffraction of plane waves by finite-radius spiral phase plates of integer and fractional topological charges J. Opt. Soc. Am. A 26 794–803
[6] Gutiérrez-Vega J C and Lázaro-Martíscar C 2008 Nondiffracting vortex beams with continuous orbital angular momentum order dependence J. Opt. A: Pure Appl. Opt. 10 015009
[7] Mitri F G 2011 Vector wave analysis of an electromagnetic high-order Bessel vortex beam of fractional type Opt. Lett. 36 606–8
[8] Berry M V, Chambers R G, Large M D, Upstill C and Walmsley I J 1980 Wavefront dislocations in the Aharonov–Bohm effect and its water wave analogue Eur. J. Phys. 1 54–62
[9] Fadeyeva T, Alexeyev C, Rubass A and Volyar A 2012 Vector erf–Gaussian beams: fractional optical vortices and asymmetric TE and TM modes Opt. Lett. 37 1397–9
[10] Volyar A and Fadeyeva T 2003 Generation of singular beams in uniaxial crystals Opt. Spectrosc. 94 264–74
[11] Ciattoni A, Cincotti G and Palma C 2003 Circular polarized beams and vortex generation in uniaxial media J. Opt. Soc. Am. A 20 163–71
[12] Volyar A and Fadeyeva T 2006 Laguerre–Gaussian beams with complex and real arguments in uniaxial crystals Opt. Spectrosc. 101 297–304
[13] Egorov Yu, Fadeyeva T and Volyar A 2004 The fine structure of singular beams in crystals: colours and polarization J. Opt. A: Pure Appl. Opt. 6 S217–28
[14] Fadeyeva T and Volyar A 2010 Nondiffracting vortex-beams in a birefringent chiral crystal J. Opt. Soc. Am. A 27 13–20
[15] Fadeyeva T, Rubass A and Volyar A 2009 Transverse shift of high-order paraxial vortex-beam induced by a homogeneous anisotropic medium Phys. Rev. A 79 053815
[16] Belyi V N, Khilo N A, Kazak N S, Ryzhevich A A and Forbes A 2011 Propagation of high-order circularly polarized Bessel beams and vortex generation in uniaxial crystals Opt. Eng. 50 059001
[17] Kazak N S, Khilo N A and Ryzhevich A A 1999 Generation of Bessel light beams under the condition of internal conical refraction Quantum Electron. 29 1020–4
[18] Bel’skii A M and Stepanov M A 1999 Internal conical refraction of coherent light beams Opt. Commun. 167 1–5
[19] Vlolkh R, Volyar A, Mys O and Krupych O 2003 Appearance of optical vortex at conical refraction. Examples of NANO2 and YFeO3 crystals Ukr. J. Phys. Opt. 4 90–3
[20] Belyi V, King T, Kazak N, Khio N, Katranji E and Ryzhevich A 2001 Methods of formation and nonlinear conversion of Bessel optical vortices Proc. SPIE 4403 229–40
[21] Vasylykiv Yu, Skab I and Vlokh R 2011 Measurements of piezooptic coefficients π14 and π25 in Pb2Ge3O11 crystals using torsion induced optical vortex Ukr. J. Phys. Opt. 12 101–8
[22] Skab I, Vasylykiv Yu, Zapeka B, Savaryn V and Vlok R 2011 Appearance of singularities of optical fields under torsion of crystals containing threefold symmetry axes J. Opt. Soc. Am. A 28 1331–40
[23] Skab I, Vasylykiv Yu, Savaryn V and Vlok R 2010 Relations for optical indicatrix parameters in the condition of crystal torsion Ukr. J. Phys. Opt. 11 193–240
[24] Fadeyeva T, Alexeyev C, Anischenko P and Volyar A 2012 Engineering of the space-variant linear polarization of vortex-beams in biaxially induced crystals Appl. Opt. 51 C224–30
[25] Kiselev A P 2007 Localized light waves: paraxial and exact solutions of the wave equation (a review) Opt. Spectrosc. 102 603–22
[26] Ciattoni A, Cincotti G and Palma C 2003 Angular momentum dynamics of a paraxial beam in a uniaxial crystal Phys. Rev. E 67 036618
[27] Prudnikov A P, Brychkov Yu A and Marichev O I 1990 Special Functions: Integrals and Series vol 3 (New York: Gordon and Breach)
[28] Fadeyeva T and Volyar A 2010 Extreme spin–orbit coupling in the crystal traveling paraxial beams J. Opt. Soc. Am. A 27 381–9