Spectral - integral representation of the photon polarization operator in a constant uniform magnetic field

V.M. Katkov
Budker Institute of Nuclear Physics,
Novosibirsk, 630090, Russia
e-mail: katkov@inp.nsk.su

Abstract

The polarization operator in a constant and homogeneous magnetic field of arbitrary strength is investigated on mass shell. The calculations are carried out at all photon energies higher the pair creation threshold as well as lower this threshold. The general formula for the effective mass of the photon with given polarization has been obtained being useful for an analysis of the problem under consideration as well as at a numerical work. Approximate expressions for strong and weak fields $H$, compared with the critical field $H_0 = 4.41 \cdot 10^{13}$ G, have been found. Depending on $H/H_0$ we consider the pure quantum region of photon energy, where particles are created on lower Landau levels or created not at all. Also the energy region of large level numbers is considered where the quasiclassical approximation is valid.

1. The study of QED processes in a strong magnetic field close to and exceeding the critical field strength $H_0 = m^2/e = 4.41 \cdot 10^{13}$ G (the system of units $\hbar = c = 1$ is used ) is stimulated essentially by the existence of very strong magnetic fields in nature. It is universally recognized the magnetic field of neutron stars (pulsars) run up $H \sim 10^{11} \div 10^{13}$ G [1]. These values of field strength gives the rotating magnetic dipole model, in which the pulsar loses rotational energy through the magnetic dipole radiation. The prediction of this model is in quite good agreement with the observed radiation from pulsars in the radio frequency region. There are around some thousand radio pulsars. Another class of neutron stars, now referred to as magnetars [2], was discovered on examination of the observed radiation at x-ray and $\gamma$-ray energies and may possess even stronger surface magnetic fields $H \sim 10^{14} \div 10^{15}$ G. The photon propagation in these fields and the dispersive properties of the space region with magnetic is of very much interest. This propagation accompanied by the photon conversion into a pair of charged particles when the transverse photon momentum is larger than the process threshold value $k_\perp > 2m$. When the field change is small on
the characteristic length of process formation (for example, when this length is smaller then the scale of heterogeneity of the neutron star magnetic field), the consideration can be realized in the constant field approximation. In 1971 Adler [3] had calculated the photon polarization operator in a magnetic field using the proper-time technique developed by Schwinger [4]. In the same year Batalin and Shabad [5] had calculated this operator in an electromagnetic field using the Green function found by Schwinger [4]. In 1975 the contribution of charged-particles loop in an electromagnetic field with \( n \) external photon lines had been calculated in [6]. For \( n = 2 \) the explicit expressions for the contribution of spinor particles to the polarization operator of photon were given in this work. For the contribution of spinor particles obtained expressions coincide with the result of [5], but another form is used.

The polarization operator in a constant magnetic field has been investigated well enough in the energy region lower and near the pair creation threshold (see, for example, the papers [7, 8, 9] and the bibliography cited there. In the present paper we consider in detail the polarization operator on mass shell (\( k^2 = 0 \), the metric \( ab = a^0b^0 - ab \) is used ) at arbitrary value of the photon energy and magnetic field strength. The restriction of our consideration is only the applicability of the perturbation theory over the electromagnetic interaction constant \( \alpha \) [10].

2. Our analysis is based on the general expression for the contribution of spinor particles to the polarization operator obtained in a diagonal form in [6] (see Eqs. (3.19), (3.33)). For the case of pure magnetic field we have in a covariant form the following expression

\[
\Pi^{\mu\nu} = - \sum_{i=2,3} \kappa_i \beta_i^\mu \beta_i^\nu, \quad \beta_i \delta_{ij} = \beta_i k = 0; \quad (1)
\]

\[
\beta_2^\mu = (F^*k)^\mu / \sqrt{-(F^*k)^2}, \quad \beta_3^\mu = (Fk)^\mu / \sqrt{-(Fk)^2},
\]

\[
FF^* = 0, \quad F^2 = F^{\mu\nu}F_{\mu\nu} = 2(H^2 - E^2) > 0, \quad (2)
\]

where \( F^{\mu\nu} \) - the electromagnetic field tensor , \( F^{*\mu\nu} \) - dual tensor, \( k^\mu \) - the photon momentum, \( (Fk)^\mu = F^{\mu\nu}k_\nu \),

\[
\kappa_i = \frac{\alpha}{\pi} m^2 r \int_{-1}^{\infty} \int_0 f_i(v, x) \exp[i\psi(v, x)] dx. \quad (3)
\]

Here

\[
f_2(v, x) = 2 \frac{\cos(vx) - \cos x}{\sin^3 x} - \frac{\cos(vx)}{\sin x} + \frac{v \cos x \sin(vx)}{\sin^2 x},
\]

\[
f_3(v, x) = \frac{\cos(vx)}{\sin x} - \frac{v \cos x \sin(vx)}{\sin^2 x} - (1 - v^2) \cot x,
\]

\[
\psi(v, x) = \frac{1}{\mu} \left\{ 2r \frac{\cos x - \cos(vx)}{\sin x} + [r(1 - v^2) - 1]x \right\}; \quad (4)
\]

\[
r = -(F^*k)^2 / 2m^2 F^2, \quad \mu^2 = F^2 / 2H_0^2. \quad (5)
\]
The real part of $\kappa_i$ determines the refractive index $n_i$ of the photon with polarization $e_i = \beta_i$:

$$n_i = 1 - \frac{\text{Re}\kappa_i}{2\omega^2}.$$  

At $r > 1$ the proper value of polarization operator $\kappa_i$ includes the imaginary part which determines the probability per unit length of pair production by photon with the polarization $\beta_i$:

$$W_i = -\frac{1}{\omega} \text{Im}\kappa_i$$

For $r < 1$ the integration counter over $x$ in Eq. (3) may be turn to the lower semiaxis ($x \to -ix$), then the value $\kappa_i$ becomes real in the explicit form.

3. As well as in our work [11] (see Appendix A) we present the effective mass in the form of

$$\kappa_i = \alpha m^2 \frac{r}{\pi} T_i; \quad T_i = \int dv \int_{-1}^{0} f_i(v, x) \exp[i\psi(v, x)] dx,$$

$$T_i = \sum_{n=0}^{\infty} \left(1 - \frac{\delta_{n0}}{2}\right) T_i^{(n)}; \quad T_i^{(n)} = \int dv \int_{-1}^{0} F_i^{(n)}(v, x) \exp[i a_n(v) x] dx,$$

where

$$F_i^{(n)} = (-i)^n \exp(i z \cot x) \left[ \frac{i}{\sin x} (J_{n+1}(t) - J_{n-1}(t)) - \frac{2\mu n}{z} \cot x J_n(t) \right],$$

$$F_2^{(n)} = (-i)^n \exp(i z \cot x) \frac{4}{z} \left( b \cot x - \frac{i}{\sin^2 x} \right) J_n(t) - F_1^{(n)},$$

$$F_3^{(n)} = F_1^{(n)} - 2(-i)^n \exp(i z \cot x) (1 - v^2) \cot x J_n(t);$$

$$a_n(v) = nv - b, \quad b = \frac{1}{\mu} (1 - r(1 - v^2)), \quad z = \frac{2r}{\mu}, \quad t = \frac{z}{\sin x}.$$

Let’s note that at $x \to -i\infty$ the asymptotic of the Bessel function $J_n(t)$ is

$$J_n(t) \simeq J_n(2iz e^{-|x|}) \simeq \frac{(iz)^n}{n!} e^{-n|x|},$$

and under the condition $a_n(v) < n$, the integration counter over $x$ in Eq. (3) can be unrolled to the lower semiaxis. Then $T_i^{(n)}$ becomes real in the explicit form.

The functions $F_i^{(n)}(v, x)$ are periodical over $x$. So one can present $T_i^{(n)}$ as
\[
T_i^{(n)} = \int_{-1}^{1} dv \int_{0}^{2\pi} F_i^{(n)}(v, x) \exp[ian(v)x]dx \sum_{k=0}^{\infty} \exp[2\pi ika_n(v)]
\]
\[
= \int_{-1}^{1} \frac{dv}{1 - \exp[2\pi ian(v)] + i0} \int_{0}^{2\pi} F_i^{(n)}(v, x) \exp[ian(v)x]dx. \quad (13)
\]

Using the expression
\[
\frac{1}{1 - \exp[2\pi ian(v)] + i0} = \mathcal{P} \frac{1}{1 - \exp[2\pi ian(v)]} - i\pi \delta(1 - \exp[2\pi ian(v)]), \quad (14)
\]
taking into account the above notation
\[
- i\pi \delta(1 - \exp[2\pi ian(v)]) = - i\pi \sum_m \delta(1 - \exp[2\pi i(a_n(v) - m)]) \rightarrow \frac{1}{2} \sum_{m \geq n} \delta(a_n(v) - m). \quad (15)
\]
and allowing for \(F_i^{(n)}(v, x + \pi) = (-1)^n F_i^{(n)}(v, x)\), we have
\[
T_i^{(n)} = (-1)^n \frac{1}{2} \mathcal{P} \int_{-1}^{1} \frac{dv}{\sin(\pi a_n(v))} \int_{-\pi}^{\pi} F_i^{(n)}(v, x) \exp[ian(v)x]dx
\]
\[
+ \sum_{m=0}^{\min(n, \frac{\pi}{2\mu})} \left(1 + (-1)^{m+n} \frac{1}{2a'_n(v)}\right) \vartheta(g(n, m, r)) \int_{-\pi}^{\pi} F_i^{(n)}(v_{1,2}, x) \exp[imx]dx, \quad (16)
\]
where
\[
g(n, m, r) = r^2 - (1 + m\mu)r + n^2 \mu^2 / 4,
\]
\[
v_{1,2} = \frac{n\mu}{2r} \pm \frac{1}{r} \sqrt{g}, \quad a'_n(v) = \frac{2}{\mu} \sqrt{g}; \quad (17)
\]
\[
n_{\text{max}} = [d(r)], \quad d(r) = \frac{2(r - \sqrt{7})}{\mu}. \quad (18)
\]

Here \([d]\) is the integer part of \(d\).

Bringing out the distinction in the explicit form we present \(T_i^{(n)}\) as
\[ T_i^{(nr)} = T_i^{(nr)} + T_i^{(ns)}; \quad (19) \]
\[ T_i^{(nr)} = (-1)^n \frac{i}{2} P \int_{-1}^{1} dv \int_{-\pi}^{\pi} \left[ F_i^{(n)}(v, x) \frac{\exp[ia_n(v)x]}{\sin(\pi a_n(v))} \right] \]
\[ - \sum_{m=n}^{m_{\text{max}}} \sum_{v=1,2} (-1)^m P_i^{(n)}(v, x) \frac{\exp[imx]}{a_n(v) - im} \frac{dv}{dx}, \quad (20) \]
\[ T_i^{(ns)} = \sum_{m=n}^{m_{\text{max}}} \sum_{v=1,2} \frac{\mu \pi}{2\sqrt{g}} \left[ 1 - \frac{1}{\pi} \left( \arctan \frac{2\sqrt{-g}}{2r - \mu n} + \arctan \frac{2\sqrt{-g}}{2r + \mu n} \right) \right] \]
\[ \times \int_{0}^{\pi} F_i^{(n)}(v, x) \exp[imx] dx. \quad (21) \]

Here the regularized function \( T_i^{(nr)} \) is singularity-free, and for \( n > n_{\text{max}} \) the integration counter in \( T_i^{(n)} \) can be unrolled to the lower semiaxis. After that we present \( T_i \) in the form

\[ T_i = \sum_{n>n_{\text{max}}}^{\infty} T_i^{(n)} + \sum_{n=0}^{n_{\text{max}}} T_i^{(n)} = \left( T_i - \sum_{n=0}^{n_{\text{max}}} T_i^{(n)} \right) + \sum_{n=0}^{n_{\text{max}}} T_i^{(n)} \quad (22) \]
\[ = \int_{-1}^{1} dv \int_{0}^{\infty} \left\{ F_i(v, x) \exp[-\chi(v, x)] \right\} \]
\[ + i \sum_{n=0}^{n_{\text{max}}} F_i^{(n)}(v, -ix) \exp[a_n(v)x] \frac{dx}{dx} + \sum_{n=0}^{n_{\text{max}}} T_i^{(n)}. \quad (23) \]

Here the functions \( F_i(v, x), \chi(v, x), \chi_00(v, x) \) have a form

\[ F_2(v, x) = \frac{1}{\sinh x} \left( \frac{2 \cosh x - \cosh(vx)}{\sinh^2 x} - \cosh(vx) + v \sinh(vx) \coth x \right), \quad (24) \]
\[ F_3(v, x) = \frac{\cosh(vx)}{\sinh x} - v \frac{\cosh x \sinh(vx)}{\sinh^2 x} - (1 - v^2) \coth x; \quad (25) \]
\[ \chi(v, x) = \frac{1}{\mu} \left[ 2r \frac{\cosh x - \cosh(vx)}{\sinh x} + (rv^2 - r + 1)x \right], \quad (26) \]
\[ \chi_00(v, x) = \frac{1}{\mu} \left[ 2r + (rv^2 - r + 1)x \right]. \quad (27) \]

and \( a_n(v) \) is given by \[11\]

The integrals over \( x \) in the expression for \( T_i^{(ns)} \) (only there the imaginary terms are contained) have been calculated in Appendix A \[11\]. Along with
integers $m$ and $n$ we use also $l = (m + n)/2$ and $k = (m - n)/2$ which are straight the level numbers

$$r_{lk} = (\varepsilon(l) + \varepsilon(k))^2/4m^2, \quad \varepsilon(l) = \sqrt{m^2 + 2eHl} = m\sqrt{1 + 2\mu l}. \quad (28)$$

We have

$$\kappa^3_i = \alpha m^2 r \sum_{n=0}^{\mu_{\text{max}}} \left(1 - \delta_{n0}\frac{\varepsilon_n}{2}\right) \frac{T_i^{(ns)}}{n^2} = -i\alpha m^2 \mu e^{-\zeta} \sum_{n,m} (2 - \delta_{n0}) \frac{\zeta^n k!}{\sqrt{g!}}$$

$$\times \left[1 - \frac{1}{\pi} \left(\arctan\frac{2\sqrt{-g}}{2r - \mu n} + \arctan\frac{2\sqrt{-g}}{2r + \mu n}\right)\right] D_i; \quad (29)$$

$$D_2 = \left(\frac{m\mu}{2} - \frac{n^2\mu^2}{4r}\right) F + 2\mu l \vartheta(k - 1) \left[2L_{k-1}^{n+1}(\zeta)L_k^{n-1}(\zeta) - L_k^n(\zeta) L_{k-1}^n(\zeta)\right],$$

$$D_3 = \left(1 + \frac{m\mu}{2} - \frac{n^2\mu^2}{4r}\right) F + 2\mu l \vartheta(k - 1) L_k^n(\zeta) L_{k-1}^n(\zeta),$$

$$F = [L_k^n(\zeta)]^2 + \vartheta(k - 1) \frac{l}{k} [L_{k-1}^n(\zeta)]^2, \quad \zeta = \frac{2r}{\mu}. \quad (30)$$

where $L_k^n(\zeta)$ is the generalized Laguerre polynomial.

At $\mu << 1$, $(r - 1)/\mu \lesssim 1$, $g/\mu \simeq \left|(r - 1)/\mu - m\right| << 1$ the main terms of sum in Eq. (29) have a form:

$$\kappa^3_i \simeq -i\alpha m^2 \mu e^{-\zeta} \zeta^m g^{-1/2} \sum_{k+l=m} \frac{1}{k! l!} \quad (31)$$

$$= -i\alpha m^2 \mu e^{-\zeta} \zeta^m g^{-1/2} \frac{2m}{m!}, \quad \kappa_2^s \simeq \frac{1}{2} m\mu \kappa_3^s.$$
References

[1] M. Ruderman, in The Electromagnetic Spectrum of Nutron Stars, NATO ANSI Proceedings (Springer, New York, 2004).
[2] R.C. Duncan and C. Tompson, Astrophys. J. 392, 19 (1992)
[3] S.L. Adler, Ann. Phys. (N.Y.), 67, 599 (1971).
[4] J. Schwinger, Phys. Rev., 82, 664 (1951).
[5] I.A. Batalin and A.E.Shabad, Sov. Phis. JETP 33, 483 (1971).
[6] V.N. Baier, V.M. Katkov and V.M. Strakhovenko, Sov. Phis. JETP 41, 198 (1975).
[7] V.N. Baier, A.I. Milstein and R.Zh. Shaisultanov, Zh. Eksp. Teor. Fiz. 111, 52 (1997).
[8] A.C. Harding, M.G. Baring and P.L. Conthier, Astrophys. J. 476, 246 (1997).
[9] A.E.Shabad, Zh. Eksp. Teor. Fiz. 125, 210, (2004).
[10] V.M. Katkov, arXiv:1403.3983 [hep-th] (2014).
[11] V.N. Baier and V.M. Katkov, Phys. Rev., D 75, 073009 (2007).
[12] V.M. Katkov, arXiv:1311.6206 [hep-ph] (2013).