RESIDUES FOR AKIZUKI’S ONE-DIMENSIONAL LOCAL DOMAIN

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Abstract. For a one-dimensional local domain \( C \) constructed by Akizuki, we find residue maps which give rise to a local duality. The completion of \( C \) is described using these residue maps.

Injective hulls of a given module are all isomorphic. For this reason, people often speak of the injective hull to indicate its “uniqueness”. However, isomorphisms between these injective hulls are not canonical. In fact, they are a part of the structure of the given module. For instance, a local duality for a power series ring \([\mathbb{Z}[^n]](5.9)) is interpreted as an isomorphism between two injective hulls - one given by local cohomology and another by continuous homomorphisms. This isomorphism is induced by a residue map, which was not observed from the viewpoint of “uniqueness” of injective hulls. In this article, our philosophy is taken up again by a Noetherian local ring \( C \) constructed by Akizuki \([1\). Although \( C \) behaves beyond geometric expectation, we can still define certain maps, which give rise to a local duality as an identification of local cohomology classes and continuous homomorphisms. These maps, also called residue maps, determine all endomorphisms of an injective hull of the residue field of \( C \). So we are able to describe the completion of \( C \).

We recall Akizuki’s construction. Let \( A \) be a discrete valuation ring with the maximal ideal \( m = tA \), let \( \hat{A} \) be its completion, and let \( K \) (resp. \( \hat{K} \)) be the quotient field of \( A \) (resp. \( \hat{A} \)). Assume that there is an element
\[
z = a_0 + a_1 t^{n_1} + a_2 t^{n_2} + \cdots \in \hat{A} \quad (a_i \in A \setminus m)
\]
transcendental over \( A \) with the condition
\[
n_r \geq 2n_{r-1} + 2 \quad (r \geq 1)
\]
on exponents, where \( n_0 = 0 \). Let
\[
z_r = a_r + a_{r+1} t^{n_{r+1} - n_r} + \cdots \quad (r \geq 0)
\]
and
\[
C = A[t(z_0 - a_0), \{(z_i - a_i)^2\}^\infty_{i=0}].
\]
\( C \) is defined to be the localization of \( C \) at the maximal ideal \( M \) generated by \( t \) and \( t(z_0 - a_0) \). Akizuki \([1\] showed that \( C \) is a one-dimensional Noetherian local domain, whose normalization is not a finite \( C \)-module.

The quotient field of \( C \) is \( K(z) \), which equals \( (C_M)_{t} \) as \( t \) is a system of parameter of \( C_M \). We can use the exact sequence
\[
(1) \quad 0 \rightarrow C_M \xrightarrow{\text{localization}} (C_M)_{t} \rightarrow H^1_{MC_M}(C_M) \rightarrow 0
\]
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to describe elements of the first local cohomology module $H^1_{MC_M}(C_M)$ of $C_M$ supported at the maximal ideal of $C_M$: For $f \in C_M$ and $n > 0$, the generalized fraction

$$\left[ \frac{f}{t^n} \right]_{C_M}$$

is defined as the image of $f/t^n$ in $H^1_{MC_M}(C_M)$ under the map in (1). The description of top local cohomology modules by generalized fractions is essentially used in the concrete realizations of Grothendieck duality [2, 4, 5]. It can be applied to combinatorial analysis [3, 6]. See also [9] for an alternate treatment to generalized fractions.

For $f \in \hat{A}$ and $n > 0$, we denote by

$$(2) \quad \left[ \frac{f}{t^n} \right]_{\hat{A}}$$

the image of $f/t^n$ in $\hat{K}/\hat{A}$ under the map in the exact sequence

$$0 \rightarrow \hat{A} \xrightarrow{\text{inclusion}} \hat{K} \rightarrow \hat{K}/\hat{A} \rightarrow 0.$$ We remark that there is a canonical isomorphism between $\hat{K}/\hat{A}$ and $H^1_{\hat{m}_A}(\hat{A})$, with which the representation (2) of elements in $\hat{K}/\hat{A}$ and the representation of elements in $H^1_{\hat{m}_A}(\hat{A})$ by generalized fractions agree.

Vanishing of elements in $\hat{K}/\hat{A}$ and $H^1_{MC_M}(C_M)$ can be described in terms of ideal membership:

$$\left[ \frac{f}{t^n} \right]_{\hat{A}} = 0 \quad \text{resp.} \quad \left[ \frac{f}{t^n} \right]_{C_M} = 0$$

if and only if $f$ is contained in the ideal of $\hat{A}$ (resp. $C_M$) generated by $t^n$. For instance,

$$\left[ \begin{array}{c} t(z_0 - a_0) \\ t \end{array} \right]_{\hat{A}} = 0$$

but

$$\left[ \begin{array}{c} t(z_0 - a_0) \\ t \end{array} \right]_{C_M} \neq 0.$$ The canonical map $K/A \rightarrow \hat{K}/\hat{A}$ is an isomorphism. We use the notation in (2) to represent elements in $K/A$ under this isomorphism.

**Lemma 1.** Every element of $H^1_{MC_M}(C_M)$ can be written as

$$\left[ \begin{array}{c} X + Yt(z_0 - a_0) \\ t^n \end{array} \right]_{C_M}$$

for some $X, Y \in A$, $n \geq 0$.

**Proof.** It suffices to show that, for any specified $n > 0$, any element $f \in C_M$ can be written as

$$f = X + Yt(z_0 - a_0) + t^nZ$$

with $X, Y \in A$ and $Z \in C_M$. Write $f$ as

$$f = \frac{f_1}{1 - f_2}.$$
for some $f_1 \in C$ and $f_2 \in M$. Then
\[ f - f_1(1 + f_2 + f_2^2 + \cdots + f_2^n) = \frac{f_1 f_2^{n+1}}{1 - f_2} \in M^{n+1}C_M. \]
Since $M^{n+1}C_M \subset t^nC_M$,
\[ f - f_1(1 + f_2 + f_2^2 + \cdots + f_2^n) = t^nZ_2 \]
for some $Z_2 \in C_M$. By [3, p. 140, Section 9.5, equation (5)], there exist $X, Y \in A$ and $Z_1 \in C$ such that
\[ f_1(1 + f_2 + f_2^2 + \cdots + f_2^n) = X + Y t(z_0 - a_0) + t^nZ_1. \]
The representation
\[ f = X + Y t(z_0 - a_0) + t^n(Z_1 + Z_2) \]
is of the required form. \hfill \square

**Lemma 2.** If $X, Y \in A$ and
\[ \left[ \begin{array}{c} X + Y t(z_0 - a_0) \\ t^n \end{array} \right]_{C_M} = 0, \]
then $X, Y \in t^nA$.

**Proof.**
\[ X + Y t(z_0 - a_0) = t^nZ \]
for some $Z \in C_M$. Write $X = t^\ell X_1$ and $Y = t^mY_1$ with invertible $X_1, Y_1 \in A$. If $\ell \leq m$, then $n \leq \ell$, otherwise $X_1 = -t^{m-\ell}Y_1 t(z_0 - a_0) + t^{n-\ell}Z \in tA$. If $\ell > m$, then $m \geq n$, otherwise $t(z_0 - a_0) = -t^{\ell-m}X_1 Y_1^{-1} + t^{n-m}Z Y_1^{-1} \in tC_M$. In either case, $X, Y \in t^nA$. \hfill \square

With these lemmas, we are able to define the following map for any $\sigma, \rho \in \hat{A}$.

**Definition 3.**
\[ \text{res}_{\sigma, \rho} : H^1_{MC,M}(C_M) \to K/A \]
is defined to be the $A$-linear map given by
\[ \left[ \begin{array}{c} X + Y t(z_0 - a_0) \\ t^n \end{array} \right]_{C_M} \mapsto \left[ \begin{array}{c} X \sigma + Y \rho \\ t^n \end{array} \right]_A. \]

Adopting the terminology of [2], we call $\text{res}_{\sigma, \rho}$ a residue map. Let
\[ \text{Hom}^c_A(C_M, K/A) = \{ \varphi \in \text{Hom}_A(C_M, K/A) \mid \varphi(M^nC_M) = 0 \text{ for some } n \} \]
be the $C_M$-module of continuous homomorphism. As a special case of J. Lipman’s result [2, Proposition 3.4], $\text{Hom}^c_A(C_M, K/A)$ is an injective hull of the residue field of $C_M$. Note that $t^{n+1}C_M \subset M^{n+1}C_M \subset t^nC_M$. Hence a $A$-linear map $\varphi : C_M \to K/A$ is continuous if and only if $\varphi(t^nC_M) = 0$ for some $n$. Using the representation (3) of elements of $C_M$, we see that a continuous homomorphism is determined by an integer $n$ with which $t^nC_M$ is in the kernel and by its values at 1 and $t(z_0 - a_0)$.

**Definition 4.**
\[ \Phi_{\sigma, \rho} : H^1_{MC,M}(C_M) \to \text{Hom}^c_A(C_M, K/A) \]
is defined to be the $C_M$-linear map given by
\[ \Phi_{\sigma, \rho}(\omega)(f) = \text{res}_{\sigma, \rho}(f \omega), \]
where $\omega \in H^1_{MC,M}(C_M)$ and $f \in C_M$. 

Local Duality. If $\rho$ is invertible, then $\Phi_{\sigma,\rho}$ is an isomorphism.

Proof. The inverse map of $\Phi_{\sigma,\rho}$ can be written down explicitly. Let

$$s_r := a_1 t^{n_1} + a_2 t^{n_2} + \cdots + a_r t^{n_r} \in A.$$ 

Then we have

$$t(z_0 - a_0) = t^{n_r+1}(z_r - a_r) + ts_r$$

and

$$t^2(z_0 - a_0)^2 + t^2 s_r^2 - 2ts_r t(z_0 - a_0) = t^{2n_r+2}(z_r - a_r)^2 \in t^r C_M.$$ 

Given $\varphi \in \text{Hom}_A^1(C_M, K/A)$ with $\varphi(t^r C_M) = 0$ and

$$\varphi(1) = \begin{bmatrix} \alpha \\ t^r \end{bmatrix}_A$$

$$\varphi(t(z_0 - a_0)) = \begin{bmatrix} \beta \\ t^r \end{bmatrix}_A,$$

the system of equations

$$\begin{cases}
X\sigma + Y\rho \\
X\rho - Yt^2 s_r^2 + 2Yts_r \rho
\end{cases} = \begin{bmatrix} \alpha \\ t^r \end{bmatrix}_A$$

$$\begin{bmatrix} X - \frac{\alpha ts_r (\sigma ts_r - 2\rho) + \beta \rho}{(\rho - ts_r \sigma)^2} \\
Y - \frac{\alpha \rho - \beta \sigma}{(\rho - ts_r \sigma)^2}
\end{bmatrix} \in t^r \hat{A}.$$ 

We define

$$\Phi^{-1}(\varphi) := \begin{bmatrix} X + Yt(z_0 - a_0) \\ t^r \end{bmatrix}_C \in C_M,$$

which is independent of the choices of $r$, $\alpha$, $\beta$, $X$ or $Y$. Then

$$\Phi_{\sigma,\rho}(\Phi^{-1}(\varphi))(1) = \begin{bmatrix} \alpha \\ t^r \end{bmatrix}_A$$

and

$$\Phi_{\sigma,\rho}(\Phi^{-1}(\varphi))(t(z_0 - a_0)) = \text{res}_{\sigma,\rho} \left[ X t(z_0 - a_0) - Yt^2 s_r^2 + 2Yts_r t(z_0 - a_0) \right]_{C_M} \text{ (by (4))}$$

$$= \begin{bmatrix} \beta \\ t^r \end{bmatrix}_A.$$ 

Hence $\Phi_{\sigma,\rho}(\Phi^{-1}(\varphi)) = \varphi$. For any $\omega \in H^1_{MC_M}(C_M)$, it is also straightforward to check that $\Phi^{-1}(\Phi_{\sigma,\rho}(\omega)) = \omega$. So $\Phi^{-1}$ is indeed the inverse of $\Phi_{\sigma,\rho}$. 

Corollary 5. $C_M$ is Gorenstein.

Proof. $H^1_{MC_M}(C_M)$ is injective. Hence (4) is a finite injective resolution of $C_M$. 

\[\square\]
We remark that a Noetherian local ring $R$ whose maximal ideal is generated by $1 + \text{depth } L$ elements is always Gorenstein \[\text{p. 163, Exercise 1}].

**Proposition 6.** Any $C_M$-linear map

\[\Phi: H^1_M(C_M) \to \text{Hom}_A^e(C_M, K/A)\]

equals $\Phi_{\sigma, \rho}$ for some $\sigma, \rho \in \hat{A}$.

**Proof.** For each $n$, there exist $\sigma_n, \rho_n \in A$ such that

\[
\begin{align*}
\left[ \begin{array}{c} \sigma_n \\ t^n \end{array} \right] & = \Phi \left( \left[ \begin{array}{c} 1 \\ t^n \end{array} \right] \right) \left(1\right) \\
\left[ \begin{array}{c} \rho_n \\ t^n \end{array} \right] & = \Phi \left( \left[ \begin{array}{c} 1 \\ t^n \end{array} \right] \right) \left(t(z_0 - a_0)\right).
\end{align*}
\]

Since $\sigma_n - \sigma_{n+1}$ and $\rho_n - \rho_{n+1}$ are contained in $t^nA$, the limits

\[
\begin{align*}
\sigma &= \lim_{n \to \infty} \sigma_n \\
\rho &= \lim_{n \to \infty} \rho_n
\end{align*}
\]

exist in $\hat{A}$. For any $X, Y \in A$,

\[
\Phi \left[ \begin{array}{c} X + Y(t(z_0 - a_0)) \\ t^n \end{array} \right]_{C_M} \left(1\right) = \left[ \begin{array}{c} X\sigma_n + Y\rho_n \\ t^n \end{array} \right]_A = \left[ \begin{array}{c} X\sigma + Y\rho \\ t^n \end{array} \right]_A.
\]

Hence

\[\left(\Phi(\omega)\right)(f) = \Phi(f\omega)(1) = \Phi_{\sigma, \rho}(f\omega)(1) = \left(\Phi_{\sigma, \rho}(\omega)\right)(f)\]

for any $\omega \in H^1_M(C_M)$ and $f \in C_M$. That is, $\Phi = \Phi_{\sigma, \rho}$. \qed

Now we fix a $\sigma_0$ and an invertible $\rho_0$. All endomorphisms of the $C_M$-module $H^1_M(C_M)$ are of the form $\Phi_{-1}^{-1} \circ \Phi_{\sigma, \rho}$. Since different pairs of $\sigma$ and $\rho$ determine different $C_M$-linear maps $\Phi_{\sigma, \rho}$, the completion $\hat{C}_M$ of $C_M$ can be described as the set \{\Phi_{\sigma, \rho}\}_{\sigma, \rho \in \hat{A}} with addition

\[\Phi_{\sigma_1, \rho_1} + \Phi_{\sigma_2, \rho_2} = \Phi_{\sigma_1 + \sigma_2, \rho_1 + \rho_2} ,\]

unit $\Phi_{\sigma_0, \rho_0}$, and multiplication

\[\Phi_{\sigma_1, \rho_1} \ast \Phi_{\sigma_2, \rho_2} = \Phi_{\sigma_1, \rho_1} \circ \Phi_{\sigma_0, \rho_0} \circ \Phi_{\sigma_2, \rho_2} \]

given by composition of endomorphisms. If $\sigma_0 = 0$ and $\rho_0 = 1$, then

\[\Phi_{\sigma_1, \rho_1} \ast \Phi_{\sigma_2, \rho_2} = \Phi_{\sigma_1 + \sigma_2, \rho_1 \rho_2 - 2\sigma_1\sigma_2(t(z_0 - a_0))^2} ,\]

Identify $\Phi_{\sigma, \rho}$ with $\rho + \sigma X$ in $\hat{A}/(X + t(z - a_0))^2$, compare their additions and multiplications, we get the following description of $\hat{C}_M$.

**Corollary 7.** $\hat{C}_M \simeq \hat{A}/(X + t(z - a_0))^2$

For $\rho \in C_M$, the endomorphism $\Phi_{0, \rho}^{-1} \circ \Phi_{0, \rho}$ of $H^1_M(C_M)$ is multiplication by $\rho$. Therefore, with respect to the isomorphism in Corollary \[\text{p. 163, Exercise 1}\] the embedding $C_M \to \hat{A}/(X + t(z - a_0))^2$ of completion is the composition

\[C_M \xrightarrow{\text{inclusion}} \hat{A} \xrightarrow{\text{canonical}} \hat{A}/(X + t(z - a_0))^2.\]
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