An intriguing application of telescoping sums

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Abstract. This note offers a simple, yet very intriguing application of telescoping sums. Particularly, the cancellation technique which is known as the method of differences is employed to establish analytically the closed-form solutions of some systems of nonlinear difference equations. The results delivered here, in addition, generalizes several results found in recent literatures.

1. Introduction
There has been an enormous amount of published work dealing with solution forms of some systems of nonlinear difference equations that are solvable in closed-forms. This type of equations appear very simple in form, however, in some situations, the structure and behavior of their solutions are quite difficult to formulate and completely understand. Nevertheless, several studies have dealt with the solution forms of several solvable systems of nonlinear difference equations, see, for instance, a recent work of Elsayed [3], Elsayed and El-Metwally [4], Elsayed and Ibrahim [5], Tollu et al. [12, 13], Touafek [14], Yazlik [16] and the papers cited therein. In these aforementioned works, we noticed that the results exhibited, particularly the solution forms of the systems being studied, were established through a mere application of induction principle. (in fact, at some point, the proofs of some of the statements presented are insufficient and are even completely omitted). We stress that this method can only validate analytically the solution form of a particular system of difference equation. However, it does not give much information on how these formulas are obtained. In an effort to explain theoretically some intriguing results in this research line, the author have revisited and re-examined in [8, 9, 10] and also in [11], with Bacani, several systems of nonlinear difference equations which have been studied previously in [1, 6, 14, 15, 16].

In this work, we turn our attention to a two-dimensional system of nonlinear difference equation which could be seen as a generalization of recent investigations. In particular, we shall show how the solution form of the system

\[
\begin{align*}
x_{n+1} &= \frac{x_n y_{n-k}}{y_{n-k+1}(\pm 1 - x_n y_{n-k})}, \\
y_{n+1} &= \frac{y_n x_{n-k}}{x_{n-k+1}(\pm 1 \pm y_n x_{n-k})},
\end{align*}
\]

(1.1)

with real initial conditions \(\{x_n\}_{n=0}^{\infty} := \{x_n\}_{n-k}^{0} \) and \(\{y_n\}_{n-k}^{0}\), can be established analytically. Our technique uses appropriate transformation on the phase variables \(x_n\) and \(y_n\), reducing the system into a linear type and employs the “method of differences” or, in particular, the idea of
“telescoping sums”. This method has been used effectively by the author, along with Bacani [11], in dealing with the solution forms of several solvable systems of nonlinear difference equations, covering also the equations considered in [14]. The solution form for the case when $k = 1$ has been established by Elsayed in [5] through the principle of induction. For this reason, we have intended to re-examine and approach the problem in a different perspective which will perhaps provide interested readers some new ideas on how to attack similar types of problems.

The paper is organized as follows. In the next section (Section 2), we discuss shortly the idea behind the method of differences and prove a lemma which will be essential for our main result. Our main contribution is presented in Section 3 where we will exhibit the solution forms of system (1.1) through the use of telescoping. We end our paper with a short summary in Section 4.

2. Preliminaries

Let $\{a_n\}_{n=1}^{\infty}$ be a number sequence. A telescoping sum is a sum in which subsequent terms cancel each other, leaving only its initial and final terms. For instance,

$$\sum_{n=1}^{N}(a_n - a_{n-1}) = a_N - a_0,$$

is a telescoping sum. This idea, obviously, can be extended into products. For example, we have the following relation

$$\prod_{n=1}^{N}\left\{ \frac{b_n}{b_{n-1}} \right\} = \frac{b_N}{b_0},$$

where $\{b_n\}_{n=1}^{\infty}$ is some number sequence with nonzero terms. As alluded in the Introduction, we are interested in the application of these identities especially in obtaining closed-form solutions of some solvable systems on nonlinear difference equations. With this set objective, the following results are essential.

Lemma 2.1 ([7]). Let $n \in \mathbb{N}_0$. Then, we have the following results:

(i) the equation $z_{n+1} - z_n = \pm 1$ has the solution $z_n = z_0 \pm n$,

(ii) the equation $z_{n+1} + z_n = \pm 1$ has the solution $z_n = (-1)^n[z_0 \mp \xi(n)]$, where $\xi(n) = 0$ when $n$ is even and $\xi(n) = 1$ when $n$ is odd.

Proof. The proof follows directly from telescoping. Indeed, for (i), we have $z_n - z_0 = \sum_{j=0}^{n-1}(z_{j+1} - z_j) = \pm \sum_{j=0}^{n-1}1 = \pm n$. Rearranging the equation, we get the desired equation. Meanwhile, for (ii), we have $(-1)^n z_n + z_0 = \sum_{j=0}^{n-1}(-1)^j(z_{j+1} + z_j) = \pm \sum_{j=0}^{n-1}(-1)^j$. The right hand side (RHS) equates to 0 (resp. 1) when $n$ is even (resp. odd). Thus, $z_n = (-1)^n[z_0 \mp \xi(n)]$, where $\xi(n) = 0$ when $n$ is even and $\xi(n) = 1$ when $n$ is odd.

Proposition 2.2. Let $n$ be a non-negative integer and

$$\xi(n) = \begin{cases} 0, & \text{if } n \text{ is even}, \\ 1, & \text{if } n \text{ is odd}. \end{cases}$$

Then, $\xi(a + b) = \xi(\xi(a) + \xi(b))$ for all non-negative integers $a$ and $b$. Moreover, if $a > 0$ is an odd integer, then we have $\xi(an) = \xi(n)$. 

Proof. To see the first result, we consider two possibilities: (i) \( a \) and \( b \) are of the same parity and (ii) \( a \) and \( b \) are of different parity. For the first possibility, we note that \( \xi(a) = \xi(b) \) which implies that \( \xi(a) + \xi(b) = 2\xi(b) \). Hence, \( \xi(\xi(a) + \xi(b)) = \xi(2\xi(b)) = 0 \). However, since \( a \) and \( b \) are of the same parity, then \( a + b \) is always even. Thus, \( \xi(a + b) \) and so \( \xi(a + b) = \xi(\xi(a) + \xi(b)) \).

Meanwhile, if \( a \) and \( b \) are of different parity, then we can assume without-loss-of-generality that \( a \) is even while \( b \) is odd. Hence, \( \xi(a) = 0 \) and \( \xi(b) = 1 \). It follows that \( \xi(a) + 1 = \xi(b) \) and \( \xi(b) - 1 = 0 \). Adding \( \xi(b) - 1 \) on both sides of the former equation, we get \( \xi(a) + \xi(b) = 2\xi(b) - 1 \). Therefore, \( \xi(\xi(a) + \xi(b)) = \xi(2\xi(b) - 1) = 1 \). But \( a \) and \( b \) are of different parity, then \( a + b \) is always odd. Thus, we have \( \xi(a + b) = 1 \) which in turn implies that \( \xi(a + b) = \xi(\xi(a) + \xi(b)) \).

Now, for the second result, we note that the parity of the product \( a \cdot n \) for any odd integer \( a > 0 \) depends on the parity of \( n \). Thus, \( \xi(an) = \xi(n) \) for all non-negative integers \( a \) and \( n \) where \( a \) is odd. This proves the proposition. 

3. Main Results

In the following, we derive the solution form of the system (1.1), taking into account the following substitutions on the phase variables:

\[
\begin{align*}
  w_n &= \frac{1}{x_n y_{n-k}} \quad \text{and} \quad z_n = \frac{1}{y_n x_{n-k}}, \quad \text{for all } n \in \mathbb{N}_0, \quad (T)
\end{align*}
\]

Our discussion proceeds according to the following particular cases of system (1.1):

\[
\begin{align*}
  x_{n+1} &= \frac{x_n y_{n-k}}{y_{n-k+1}(1 - x_n y_{n-k})}, & y_{n+1} &= \frac{y_n x_{n-k}}{x_{n-k+1}(1 \pm y_n x_{n-k})}, \quad n \in \mathbb{N}_0; \quad (S.1)
  x_{n+1} &= \frac{x_n y_{n-k}}{y_{n-k+1}(-1 - x_n y_{n-k})}, & y_{n+1} &= \frac{y_n x_{n-k}}{x_{n-k+1}(1 \pm y_n x_{n-k})}, \quad n \in \mathbb{N}_0; \quad (S.2)
  x_{n+1} &= \frac{x_n y_{n-k}}{y_{n-k+1}(-1 - x_n y_{n-k})}, & y_{n+1} &= \frac{y_n x_{n-k}}{x_{n-k+1}(-1 \pm y_n x_{n-k})}, \quad n \in \mathbb{N}_0. \quad (S.3)
\end{align*}
\]

3.1. Solution form of system (S.1)

Using the substitution (T), system (S.1) can be transformed into the following equations:

\[
\begin{align*}
  w_{n+1} &= w_n - 1, & z_{n+1} &= z_n \pm 1, \quad n \in \mathbb{N}_0.
\end{align*}
\]

In view of (i) of Lemma 2.1, we have

\[
\forall n \in \mathbb{N}_0 : \quad w_n = w_0 - n \quad \text{and} \quad z_n = z_0 \pm n,
\]

or equivalently, in reference to equation (T),

\[
\forall n \in \mathbb{N}_0 : \quad \begin{cases} x_n y_{n-k} = \frac{x_0 y_{-k}}{1 - n y_0 y_{-k}} \\ y_n x_{n-k} = \frac{y_0 x_{-k}}{1 \pm n y_0 x_{-k}}. \end{cases} \quad (3.1)
\]

With these equations at hand, we could easily compute for \( \{x_{2kn+k-1}\}_{k=1}^0 \) and \( \{y_{2kn+k-1}\}_{k=1}^0 \). For instance, to find for \( x_{2kn+(2k-1)} \) we can proceed as follows. First, replace \( n \) by \( 2kn + (2k - 1) \) (resp. by \( 2kn + k - 1 \)) in the first (resp. second) equation in (3.1) to obtain

\[
x_{2kn+2k-1} y_{2kn+k-1} = \frac{x_0 y_{-k}}{1 - (2kn + 2k - 1) x_0 y_{-k}}.
\]
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and
\[ y_{2kn+k-1} x_{2kn-1} = \frac{y_0 x_{-k}}{1 \pm (2kn + k - 1)y_0 x_{-k}}. \]

Then, we can eliminate the second phase variable \( y_n \) by taking the ratio of these two equations; that is, we have
\[
\frac{x_{2kn+2k-1}}{x_{2kn-1}} = \frac{x_{2kn+2k-1} y_{2kn+k-1}}{y_{2kn+k-1} x_{2kn-1}} = \frac{x_0 y_{-k}(1 \pm (2kn + k - 1)y_0 x_{-k})}{y_0 x_{-k}(1 - (2kn + 2k - 1)x_0 y_{-k})}.
\]

After that, we replace \( n \) by \( j \) and then take the product of the resulting equation from \( j = 0 \) to \( j = n - 1 \), to obtain
\[
\frac{x_{2kn+2k-1}}{x_{-1}} = \prod_{j=0}^{n-1} \left\{ \frac{x_{2kj+(2k-1)}}{x_{2kj-1}} \right\} = \prod_{j=0}^{n-1} \left\{ \frac{x_0 y_{-k}(1 \pm (2kj + k - 1)y_0 x_{-k})}{y_0 x_{-k}(1 - (2kj + 2k - 1)x_0 y_{-k})} \right\}
\]

Finally, by rearranging this equation, with an adjustment on the index on the RHS of the equation, we get
\[
\forall n \in \mathbb{N}_0 : \ x_{2kn-1} = x_{-1} \left( \frac{x_0 y_{-k}}{y_0 x_{-k}} \right)^n \prod_{j=0}^{n-1} \left\{ \frac{1 \pm (2kj + k - 1)y_0 x_{-k}}{1 - (2kj + 2k - 1)x_0 y_{-k}} \right\}.
\]

Following the same inductive lines as above, the solution forms for the other phase variables can easily be established. Thus, we have the following theorem.

**Theorem 3.1.** Every solution \( \{(x_n, y_n)\}_i^{\infty} \) of system (S.1) takes, for all \( n \in \mathbb{N}_0 \), the form
\[
x_{2kn+i} = x_1 \left( \frac{x_0 y_{-k}}{y_0 x_{-k}} \right)^n \prod_{j=0}^{n-1} \left\{ \frac{1 \pm (2kj + k + i)y_0 x_{-k}}{1 - (2kj + 2k + i)x_0 y_{-k}} \right\},
\]

for all \( i = -2(k-1), 0 \) and
\[
x_{2kn+1} = \frac{x_0 y_{-k}}{y_{-1} (1 - x_0 y_{-k})} \left( \frac{x_0 y_{-k}}{y_0 x_{-k}} \right)^n \times \prod_{j=0}^{n-1} \left\{ \frac{1 \pm (2kj + k + 1)y_0 x_{-k}}{1 - (2kj + 2k + 1)x_0 y_{-k}} \right\},
\]

and
\[
y_{2kn+i} = y_i \left( \frac{y_0 x_{-k}}{x_0 y_{-k}} \right)^n \prod_{j=0}^{n-1} \left\{ \frac{1 \pm (2kj + k + i)x_0 y_{-k}}{1 - (2kj + 2k + i)y_0 x_{-k}} \right\},
\]

for all \( i = -2(k-1), 0 \) and
\[
y_{2kn+1} = \frac{y_0 x_{-k}}{x_{-1} (1 - y_0 x_{-k})} \left( \frac{y_0 x_{-k}}{x_0 y_{-k}} \right)^n \times \prod_{j=0}^{n-1} \left\{ \frac{1 \pm (2kj + k + 1)x_0 y_{-k}}{1 - (2kj + 2k + 1)y_0 x_{-k}} \right\}.
\]

**3.2. Solution form of system (S.2)**

In deriving the solution form for system (S.2), we apply the same argument used in the previous section. Through the substitution (T), system (S.2) can be transformed into the following equations:
\[
w_{n+1} = -w_n - 1, \quad z_{n+1} = z_n \pm 1, \quad n \in \mathbb{N}_0.
\]
In reference to Lemma 2.1, the solution to these equations can readily be found and are given by
\[ \forall n \in \mathbb{N}_0 : \quad w_n = (-1)^n[w_0 + \xi(n)] \quad \text{and} \quad z_n = z_0 \pm n, \]
respectively. Using the substitution defined in (T), we then have
\[ \forall n \in \mathbb{N}_0 : \quad \begin{align*}
x_n y_{n-k} &= \frac{(-1)^n x_0 y_{-k}}{1 + \xi(n)x_0 y_{-k}}, \\
y_n x_{n-k} &= \frac{y_0 x_{-k}}{1 \pm n y_0 x_{-k}}.
\end{align*} \tag{3.2} \]

Again, with these relations at hand, we can easily derive the solution form for \( \{x_{2kn+i}\}_{i=-(2k-1)} \) and \( \{y_{2kn+i}\}_{i=-(2k-1)} \). This time we only show how to compute for \( y_{2kn-k} \). The formulas for the other phase variables, however, can be established in a similar fashion. Now, to proceed, we replace \( n \) by \( 2kn \) (resp. \( 2kn + k \)) in the first (resp. second) equation in (3.2) to obtain
\[ \forall n \in \mathbb{N}_0 : \quad \begin{align*}
x_{2kn} y_{2kn-k} &= \frac{x_0 y_{-k}}{1 + \xi(2kn)x_0 y_{-k}}, \\
y_{2kn+k} x_{2kn} &= \frac{y_0 x_{-k}}{1 \pm (2kn + k)y_0 x_{-k}},
\end{align*} \]
and so we have, using the definition of \( \xi(n) \) in Lemma 2.1,
\[ \frac{y_{2kn+k} x_{2kn}}{x_{2kn} y_{2kn-k}} = \frac{y_0 x_{-k}}{x_0 y_{-k}(1 \pm (2kn + k)y_0 x_{-k})}. \]
Replacing \( n \) by \( j \) and then taking the product of the resulting expressions from \( j = 0 \) to \( j = n-1 \) yields
\[ \frac{y_{2kn-k}}{y_{-k}} = \prod_{j=0}^{n-1} \left\{ \frac{y_0 x_{-k}}{x_0 y_{-k}(1 \pm (2kj + k)y_0 x_{-k})} \right\}. \]

Rearranging this equation and after some algebra, we get
\[ y_{2kn-k} = \frac{y_0 x_{-k}}{x_0 y_{-k}} \prod_{j=0}^{n-1} \{1 \pm (2kj + k)y_0 x_{-k}\}. \]

In concluding, we have proved analytically the following theorem.

**Theorem 3.2.** Every solution \( \{(x_n, y_n)\}_{n=0}^{\infty} \) of system (S.2) takes, for all \( n \in \mathbb{N}_0 \), the form
\[ x_{2kn+i} = \begin{cases} 
\frac{x_i}{(\pm \pm 1 \pm x_0 y_{-k})^n} \left( \frac{x_0 y_{-k}}{y_0 x_{-k}} \right)^n \prod_{j=0}^{n-1} \{1 - (2kj + k + i)y_0 x_{-k}\}, & \text{for odd } i, \\
\frac{x_i}{\prod_{j=0}^{n-1} \{1 \pm (2kj + k + i)y_0 x_{-k}\}} & \text{for even } i,
\end{cases} \]
for all \( i = -2(k-1), 0 \) and
\[ x_{2kn+1} = \frac{x_0 y_{-k}(1 \pm x_0 y_{-k})^{-1}}{x_0 y_{-k}(1 \pm x_0 y_{-k})^n} \left( \frac{x_0 y_{-k}}{y_0 x_{-k}} \right)^n \prod_{j=0}^{n-1} \{1 - (2kj + k + 1)y_0 x_{-k}\}, \]
and

\[
y_{2kn+i} = \begin{cases} 
  y_i \left( \frac{y_0x_{-k}}{x_0y_{-k}} \right)^n \prod_{j=0}^{n-1} \left\{ \mp 1 - (2kj + 2k + i)y_0x_{-k} \right\}, & \text{for odd } i, \\
  y_i \left( \frac{y_0x_{-k}}{x_0y_{-k}} \right)^n \prod_{j=0}^{n-1} \left\{ 1 + (2kj + 2k + i)y_0x_{-k} \right\}^{-1}, & \text{for even } i,
\end{cases}
\]

for all \( i = -2(k-1), 0 \) and

\[
y_{2kn+1} = \frac{y_0x_{-k}}{x_{-k+1}(1 + y_0x_{-k})} \left( \frac{y_0x_{-k}}{x_0y_{-k}} \right)^n \prod_{j=0}^{n-1} \left\{ \mp 1 - (2kj + 2k + 1)y_0x_{-k} \right\}^{-1}.
\]

### 3.3. Solution form of system (S.3)

Now we turn our attention to system (S.3) with initial conditions \( x_{-k}y_0 \neq 1 \) and \( x_0y_{-k} \neq 1 \). Again, our approach follows the same argument as in the previous two cases. Using the substitution (T), system (S.3) can be expressed into the following equivalent forms:

\[
w_{n+1} = -w_n - 1, \quad z_{n+1} = -z_n + 1, \quad n \in \mathbb{N}_0.
\]

In view of Lemma 2.1, the solution to these equations are given by

\[
\forall n \in \mathbb{N}_0 : \quad w_n = (-1)^n[w_0 + \xi(n)] \quad \text{and} \quad z_n = (-1)^n[z_0 + \xi(n)],
\]

respectively. Once again, using the substitution defined in (T), we then have

\[
\forall n \in \mathbb{N}_0 : \quad \begin{cases} 
  x_ny_{n-k} = \frac{(-1)^n x_0y_{-k}}{1 + \xi(n)y_0y_{-k}}, \\
  y_nx_{n-k} = \frac{(-1)^n y_0x_{-k}}{1 + \xi(n)y_0x_{-k}}.
\end{cases} \tag{3.3}
\]

Through these equations, we can easily find for the form of the phase variables \( \{x_{2kn+i}\}_{i=-(2k-1)}^{0} \) and \( \{y_{2kn+i}\}_{i=-(2k-1)}^{0} \) in terms of \( n \) and the initial conditions \( \{x_n\}_{k}^{0} \) and \( \{y_n\}_{n=-k}^{0} \). In the sequel, we shall give complete details in deriving the solution form for \( x_{2kn} \) and \( x_{2kn-1} \). The other formulas, however, can be achieved using similar arguments. Now, for \( x_{2kn} \), we replace \( n \) by \( 2kn \) (resp. \( 2kn-1 \)) in the first (resp. second) equation in (3.3) to obtain

\[
\forall n \in \mathbb{N}_0 : \quad \begin{cases} 
  x_{2kn}y_{2kn-k} = \frac{x_0y_{-k}}{1 + \xi(2kn)y_0y_{-k}}, \\
  y_{2kn-k}x_{2kn-2k} = \frac{(-1)^k y_0x_{-k}}{1 + \xi(2k-1)y_0x_{-k}}.
\end{cases}
\]

Hence, using the definition of \( \xi(n) \) from Lemma 2.1, we have

\[
\frac{x_{2kn}}{x_{2(kn-1)}} = \frac{x_{2kn}y_{2kn-2k}}{y_{2kn-2k}x_{2kn-2k}} = \frac{x_0y_{-k}(-1)^k(1 + \xi(k)y_0x_{-k})}{y_0x_{-k}}.
\]
Replacing \( n \) by \( j \) and then taking the product of the resulting expressions from \( j = 1 \) to \( j = n \) yields the relation

\[
x_{2kn}/x_0 = \prod_{j=1}^{n} \left\{ \frac{x_0y_{-k}(-1)^k(1 + \xi(k)y_0x_{-k})}{y_0x_{-k}} \right\} = \left\{ \frac{x_0y_{-k}(-1)^k(1 + \xi(k)y_0x_{-k})}{y_0x_{-k}} \right\}^n
\]

or equivalently,

\[
x_{2kn} = x_0(-1)^kn(1 + \xi(k)y_0x_{-k})^n \left( \frac{x_0y_{-k}}{y_0x_{-k}} \right)^n.
\]

On the other hand, to get the form of \( x_{2kn-1} \), we replace \( n \) by \( 2kn + 2k - 1 \) (resp. by \( 2kn + k - 1 \)) in the first (resp. second) equation in (3.3) to obtain, \( \forall n \in \mathbb{N}_0 \) :

\[
\begin{align*}
x_{2kn+2k-1}y_{2kn+k-1} &= -\frac{x_0y_{-k}}{1 + \xi(2kn + 2k - 1)x_0y_{-k}}, \\
y_{2kn+k-1}^2 &= -\frac{(-1)^kx_0y_{-k}}{1 + \xi(2kn + k - 1)y_0x_{-k}}.
\end{align*}
\]

Using the definition of \( \xi(n) \) from Lemma 2.1 and utilizing Proposition 2.2, we get

\[
\frac{x_{2kn+2k-1}}{x_{2kn-1}} = \frac{x_{2kn+2k-1}y_{2kn+k-1}}{y_{2kn+k-1}x_{2kn-1}} = \frac{(-1)^kx_0y_{-k}(1 + \xi(k - 1)y_0x_{-k})}{y_0x_{-k}(1 + x_0y_{-k})}.
\]

Replacing \( n \) by \( j \) and then taking the product of the resulting expressions from \( j = 0 \) to \( j = n - 1 \) yields the relation

\[
x_{2kn-1}/x_{-1} = \prod_{j=1}^{n} \left\{ \frac{(-1)^kx_0y_{-k}(1 + \xi(k - 1)y_0x_{-k})}{y_0x_{-k}(1 + x_0y_{-k})} \right\} = \left\{ \frac{(-1)^kx_0y_{-k}(1 + \xi(k - 1)y_0x_{-k})}{y_0x_{-k}(1 + x_0y_{-k})} \right\}^n
\]

or equivalently,

\[
x_{2kn-1} = x_{-1}(-1)^kn \left( 1 + \xi(k - 1)y_0x_{-k} \right)^n \left( \frac{x_0y_{-k}}{y_0x_{-k}} \right)^n.
\]

After computing for the other formulas through the same inductive lines used above, we’ll finally arrive at the following theorem.

**Theorem 3.3.** If \( x_0y_{-k} \neq -2 \) and \( y_0x_{-k} \neq \pm 2 \), then every solution \( \{ (x_n, y_n) \}_{n=0}^{\infty} \) of system (S.3) takes the form

\[
x_{2kn+i} = \begin{cases} (-1)^knx_{i} \left( \frac{1 + \xi(k - 1)y_0x_{-k}}{1 + x_0y_{-k}} \right)^n \left( \frac{x_0y_{-k}}{y_0x_{-k}} \right)^n, & \text{for odd } i, \\ (-1)^knx_{i}(1 + \xi(k)y_0x_{-k})^n \left( \frac{x_0y_{-k}}{y_0x_{-k}} \right)^n, & \text{for even } i, \end{cases}
\]

for all \( i = -2(k - 1), 0 \) and

\[
x_{2kn+1} = \frac{(-1)^kn}{y_{-k+1}} \left( \frac{x_0y_{-k}}{-1 - x_0y_{-k}} \right)^{n+1} \times \left( \frac{-1 + \xi(k - 1)y_0x_{-k}}{y_0x_{-k}} \right)^n.
\]
and
\[ y_{2kn+i} = \begin{cases} \frac{(-1)^{kn}y_i(1 + \xi(k - 1)x_{0}y_{-k})^n}{1 + y_{0}x_{-k}} \left(\frac{y_{0}x_{-k}}{x_{0}y_{-k}}\right)^n, & \text{for odd } i, \\ \frac{(-1)^{kn}y_i}{(1 + \xi(k)x_{0}y_{-k})^n} \left(\frac{y_{0}x_{-k}}{x_{0}y_{-k}}\right)^n, & \text{for even } i, \end{cases} \]
for all \( i = -2(k - 1), 0 \) and
\[ y_{2kn+1} = \frac{(-1)^{kn}}{x_{-k+1}} \left(\frac{y_{0}x_{-k}}{-1 \pm y_{0}x_{-k}}\right)^{n+1} \times \left(\frac{1 - \xi(k - 1)x_{0}y_{-k}}{x_{0}y_{-k}}\right)^n. \]

It can be observed easily that the solutions of system (S.3) are all unbounded, except for the case when \( x_{0}y_{-k} = -2 \) and \( y_{0}x_{-k} = \pm 2 \). If, instead, we have the case \( x_{0}y_{-k} = -2 \) and \( y_{0}x_{-k} = \pm 2 \), then the above formulas suggest that system (S.3) has a periodic solution. More precisely, we have the following results.

**Corollary 3.4.** Every solution \( \{(x_n, y_n)\}^{\infty}_{1} \) of the system
\[ x_{n+1} = \frac{x_{n}y_{n-k}}{y_{n-k+1}(-1 - x_{n}y_{-k})}, \quad y_{n+1} = \frac{y_{n}x_{n-k}}{y_{n-k+1}(-1 + y_{n}x_{n-k})}, \]
for all \( n \in \mathbb{N}_{0} \), is periodic with period \( 2k \) if and only if \( x_{0}y_{-k} = -2 \) and \( y_{0}x_{-k} = 2 \).

**Corollary 3.5.** Every solution \( \{(x_n, y_n)\}^{\infty}_{1} \) of the system
\[ x_{n+1} = \frac{x_{n}y_{n-k}}{y_{n-k+1}(-1 - x_{n}y_{-k})}, \quad y_{n+1} = \frac{y_{n}x_{n-k}}{y_{n-k+1}(-1 - y_{n}x_{n-k})}, \]
for all \( n \in \mathbb{N}_{0} \), is periodic with period \( 4k \) if and only if \( x_{0}y_{-k} = y_{0}x_{-k} = -2 \).

The above results can be verified using the solution forms of the systems given in Theorem 3.3 from which their structure can also be obtained.

**Remark 3.6.** Some immediate consequences of our main results can be found in earlier papers such as in [4] and [5]. In these previous works, however, the results were established merely through induction principle and no further explanations at all were given by the authors to explain how the solution forms exhibited appear in such structures.

**4. Summary**
It was shown that the method of differences provides an elegant approach in dealing with the solution form of some particular systems of nonlinear difference equations. This work, in turn, has provided interested readers a new way to attack similar type of problems. It is possible that the technique used here can be applied to other classes of difference equations whose closed-form solutions are, in structure, similar to the ones obtained here – this provides considerable interest to continue investigating these type of equations.

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References

[1] Bayram M, Das S E 2010 On the positive solutions of the difference equation system $x_{n+1} = 1/y_{n-k}, y_{n+1} = x_{n-k}/y_{n-k}$ Appl. Math. Sci. (Hikari) 4(7) pp 817–821
[2] Elsayed E M 2015 The expressions of solutions and periodicity for some nonlinear systems of rational difference equations Adv. Stud. in Contemp. Math. 25(3) pp 341–367
[3] Elsayed E M 2014 Solution for systems of difference equations of rational form of order two Comp. Appl. Math. 33(3) pp 751–765
[4] Elsayed E M and El-Metwally H 2013 On the solutions of some nonlinear systems of difference equations Adv. Diff. Equ. 161 14 pages
[5] Elsayed E M and Ibrahim T F 2015 Periodicity and solutions for some systems of nonlinear rational difference equations Hacet. J. Math. Stat. 44(6) pp 1361–1390
[6] Elsayed E M 2010 On the solutions of a rational system of difference equations Fasc. Math. 45 pp 25–36
[7] Rabago J F T 2016 Note on a paper of E. M. Elsayed (submitted for publication)
[8] Rabago J F T 2017 Effective methods on determining the periodicity and form of solutions of some systems of nonlinear difference equations Int. J. Dyn. Syst. Differ. Equ. 7(2) pp 112–135
[9] Rabago J F T 2016 On an open question concerning a product type difference equation (submitted for publication)
[10] Rabago J F T 2017 On the closed-form solution of a nonlinear difference equation and another proof to Sroysang’s conjecture Iran. J. Math. Sci. Inform. to appear
[11] Rabago J F T and Bacani J B 2016 New techniques on solving some solvable systems of nonlinear difference equations (preprint)
[12] Tollu D T, Yazlik Y and Taskara N 2014 On fourteen solvable systems of difference equations Appl. Math. Comp. 233 pp.310–319
[13] Tollu D T, Yazlik Y and Taskara N 2013 On the solutions of two special types of Riccati difference equation via Fibonacci numbers Adv. Differ. Equ. 174 7 pages
[14] Touafek N 2014 On some fractional systems of difference equations Iran. J. Math. Sci. Inform. 9(2) pp 303–305
[15] Yang Y, Chen L and Shi Y G 2011 On solutions of a system of rational difference equations Acta Math. Univ. Comenian. (N.S.) 80(1) pp 63–70
[16] Yazlik Y, Tollu D T and Taskara N 2013 On the solutions of difference equation systems with Padovan numbers Applied Mathematics, 4(12A) pp 15–20