A SIMPLE PROOF OF NON-EXPLOSION FOR MEASURE SOLUTIONS OF
THE KELLER-SEGEL EQUATION

NICOLAS FOURNIER AND YOAN TARDY

Abstract. We give a simple proof, relying on a two-particles moment computation, that there
exists a global weak solution to the 2-dimensional parabolic-elliptic Keller-Segel equation when
starting from any initial measure \( f_0 \) such that \( f_0(\mathbb{R}^2) < 8\pi \).

1. Introduction

1.1. The model. We consider the classical parabolic-elliptic Keller-Segel model, also called Patlak-
Keller-Segel, of chemotaxis in \( \mathbb{R}^2 \), which writes
\[
\begin{align*}
\partial_t f + \nabla \cdot (f \nabla c) &= \Delta f \\
\Delta c + f &= 0.
\end{align*}
\]
(1)
The unknown \((f, c)\) is composed of two nonnegative functions \( f_t(x) \) and \( c_t(x) \) of \( t \geq 0 \) and \( x \in \mathbb{R}^2 \),
and the initial condition \( f_0 \) is given.

This equation models the collective motion of a population of bacteria which emit a chemical
substance that attracts them. The quantity \( f_t(x) \) represents the density of bacteria at position
\( x \in \mathbb{R}^2 \) at time \( t \geq 0 \), while \( c_t(x) \) represents the concentration of chemical substance at position
\( x \in \mathbb{R}^2 \) at time \( t \geq 0 \). Note that in this model, the speed of diffusion of the chemo-attractant
is supposed to be infinite. This equation has been introduced by Keller and Segel [10], see also
Patlak [12]. We refer to the recent book of Biler [3] and to the review paper of Arumugam and
Tyagi [1] for some complete descriptions of what is known about this model.

We classically observe, see e.g. Blanchet-Dolbeault-Perthame [6, Page 4], that necessarily \( \nabla c_t = K * f_t \) for each \( t \geq 0 \), where
\[
K(x) = - \frac{x}{2\pi \|x\|^2} \quad \text{for} \quad x \in \mathbb{R}^2 \setminus \{0\} \quad \text{and (arbitrarily)} \quad K(0) = 0.
\]
Hence (1) may be rewritten as
\[
\partial_t f + \nabla \cdot [f (K * f)] = \Delta f.
\]
(2)

1.2. Weak solutions. We will deal with weak measure solutions. For each \( M > 0 \), we set
\[
\mathcal{M}_M(\mathbb{R}^2) = \left\{ \mu \text{ nonnegative measure on } \mathbb{R}^2 \text{ such that } \mu(\mathbb{R}^2) = M \right\}
\]
and we endow \( \mathcal{M}_M(\mathbb{R}^2) \) with the weak convergence topology, i.e. taking \( C_0(\mathbb{R}^d) \), the set of
continuous and bounded functions, as set of test functions. We also denote by \( C^2_0(\mathbb{R}^d) \) the set of
\( C^2 \)-functions, bounded together with all their derivatives. The following notion of weak solutions
is classical, see e.g. Blanchet-Dolbeault-Perthame [6, Page 5].

2010 Mathematics Subject Classification. 35K57, 35D30, 92C17.
Key words and phrases. Keller-Segel equation, Chemotaxis, Existence of weak solutions.
Definition 1. Fix \( M > 0 \). We say that \( f \in C([0,\infty),\mathcal{M}(\mathbb{R}^2)) \) is a weak solution of (2) if for all \( \varphi \in C^0_b(\mathbb{R}^2) \), all \( t \geq 0 \),

\[
\int_{\mathbb{R}^2} \varphi(x)f_t(dx) = \int_{\mathbb{R}^2} \varphi(x)f_0(dx) + \int_0^t \int_{\mathbb{R}^2} \Delta \varphi(x)f_s(dx)ds \\
+ \frac{1}{2} \int_0^t \int_{\mathbb{R}^2} K(x-y) \cdot (\nabla \varphi(x)-\nabla \varphi(y)) f_s(dx)f_s(dy)ds.
\]

All the terms in this equality are well-defined. In particular concerning the last term, it holds that \( |K(x-y) \cdot (\nabla \varphi(x)-\nabla \varphi(y))| \leq \|\nabla^2 \varphi\|_\infty/2\pi \). However, \( K(x-y) \cdot (\nabla \varphi(x)-\nabla \varphi(y)) \), which equals 0 when \( x = y \) because we (arbitrarily) imposed that \( K(0) = 0 \), is not continuous near \( x = y \). Hence a good weak solution has to verify that \( \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathbb{1}_{\{x=y\}} f_s(dx)f_s(dy) = 0 \) for a.e. \( s \geq 0 \).

1.3. Main result. Our goal is to give a simple proof of the following global existence result.

Theorem 2. Fix \( M \in (0,8\pi) \) and assume that \( f_0 \in \mathcal{M}(\mathbb{R}^2) \). There exists a global weak solution \( f \) to (2) with initial condition \( f_0 \). Moreover, for all \( \gamma \in (M/(4\pi),2) \), there is a constant \( A_{M,\gamma} > 0 \) depending only on \( M \) and \( \gamma \) such that for all \( T > 0 \),

\[
\int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \|x-y\|^{\gamma-2} f_s(dx)f_s(dy)ds \leq A_{M,\gamma}(1+T).
\]

These solutions indeed satisfy that \( \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathbb{1}_{\{x=y\}} f_s(dx)f_s(dy) = 0 \) for a.e. \( s \geq 0 \).

1.4. References. Actually, a stronger result is already known: gathering the results of Bedrossian-Masmoudi \[2\] and Wei \[13\], there exists a global mild solution for any \( f_0 \in \mathcal{M}(\mathbb{R}^2) \) with \( M < 8\pi \). But the proof in \[2,13\] is long and complicated, and the goal of the present paper is to provide a simple and robust non explosion proof, even if the solution we build is weaker. Actually, a global solution is also built in \[2,13\] when \( f_0 \in \mathcal{M}_{8\pi}(\mathbb{R}^2) \) satisfies \( \max_{x \in \mathbb{R}^2} f_0(\{x\}) < 8\pi \), a case we could also treat with a little more work.

This model was first introduced by Patlak \[12\] and Keller-Segel \[11\], as a model for chemotaxis. For an exhaustive summary of the knowledge about this equation and related models, we refer the reader to the review paper Arumugam-Tyagi \[1\] and to the book of Biler \[3\]. The main difficulty of this model lies in the tight competition between diffusion and attraction. Therefore it is not clear that a solution exists because a blow-up could occur due to the emergence of a cluster, i.e. a Dirac mass. Thus, the whole problem is about determining if the solution ends-up by being concentrated in finite time or not.

As shown in Jäger-Luckhaus \[9\], this depends on the initial mass of the solution \( M = \int_{\mathbb{R}^2} f_0(dx) \), the solution globally exists if \( M \) is small enough and explodes in the other case. The fact that solutions must explode in finite time if \( M > 8\pi \) is rather easy to show. But the fact that \( 8\pi \) is indeed the correct threshold was much more difficult.

Biler-Karch-Laurençot-Nadzieja \[4,5\] proved the global existence of a weak solution in the subcritical case for every initial data which is a radially symmetric measure such that \( f_0(\{0\}) = 0 \) and \( f_0(\mathbb{R}^2) = M \leq 8\pi \), with a few other anodyne technical conditions.

At the same time, Blanchet-Dolbeault-Perthame \[6\] proved the existence of a global weak free energy solution for initial data \( f_0 \in L^1(\mathbb{R}^2) \) with mass \( M < 8\pi \), a finite moment of order 2 and a finite entropy. The core of the argument lies in the use of the logarithmic Hardy-Littlewood-Sobolev inequality applied on a well chosen free-energy quantity. Something noticeable is that the authors use this inequality with its optimal constant to get the correct threshold \( 8\pi \).
In Bedrossian-Masmoudi [2], it is proven that under the condition that \( \max_{x \in \mathbb{R}^2} f_0(\{x\}) < 8\pi \), one can build mild solutions even in the supercritical case, which are stronger solutions than weak solutions, but these solutions are local in time. Wei [13] built global mild solutions in the subcritical and critical cases and local mild solutions in the supercritical case for every initial data \( f_0 \in L^1(\mathbb{R}^2) \), without any other condition. And these two last papers can be put together to build global mild solutions as soon as \( f_0(\mathbb{R}^2) \leq 8\pi \) and \( \max_{x \in \mathbb{R}^2} f_0(\{x\}) < 8\pi \).

Let us finally mention [8], where global weak solutions were built for any measure initial condition \( f_0 \) such that \( f_0(\mathbb{R}^2) < 2\pi \), with a light additional moment condition. This work was inspired by the work of Osada [11] on vortices. The present paper consists in refining this approach, and surprisingly, this allows us to treat the whole subcritical case.

1.5. Motivation. Our main goal is to present a simple proof of non explosion. This proof relies on a two-particles moment computation: roughly, we show that for \( \gamma \in (0, 2) \) and for \((f_t)_{t \geq 0}\) a solution to (2), it \emph{a priori} holds that

\[
\frac{d}{dt} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} ||x-y||^{\gamma} f_t(dx)f_t(dy) \geq c_{\gamma,M} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} ||x-y||^{\gamma-2} I_{\{||x-y|| \leq 1\}} f_t(dx)f_t(dy),
\]

with \( c_{\gamma,M} > 0 \) as soon as \( \gamma \in (4M/\pi, 2) \). By integration, this implies [5] and such an \emph{a priori} estimate is sufficient to build a global solution.

This computation seems simple and robust. Although they build a more regular weak solution, Blanchet-Dolbeault-Perthame [6] use some optimal Hardy-Littlewood-Sobolev inequality. Moreover, they have some little restrictions on the initial conditions (finite entropy and moment of order 2). The proof of Bedrossian-Masmoudi [2] and Wei [13] is much longer and relies on a fine study of what happens near each possible atom of the initial condition. Let us say again that they build a much stronger solution.

In particular, due to its robustness, we hope to be able to apply such a method to study the convergence of the empirical measure of some stochastic particle system, as the number of particles tends to infinity, to the solution of (2). To establish such a convergence, one needs to show the non explosion of the particle system, uniformly in \( N \) in some sense. It seems that the present method works very well and we hope to be able to treat the whole subcritical case \( M \in (0, 8\pi) \) and even the critical case \( M = 8\pi \). To our knowledge, only the case where \( M \in (0, 4\pi) \) has been studied, by an entropy method, see Bresch-Jabin-Wang [7].

We do not treat the critical case \( M = 8\pi \) in the present paper for the sake of conciseness.

2. Proof

We fix \( M > 0 \) and \( f_0 \in \mathcal{M}_M(\mathbb{R}^2) \). For \( \varepsilon \in (0, 1] \), we introduce the following regularized versions

\[
K_\varepsilon(x) = \frac{x}{2\pi(||x||^2 + \varepsilon)} \quad \text{and} \quad f_0^\varepsilon(x) = \frac{1}{2\pi\varepsilon} \int_{\mathbb{R}^2} e^{-||x-y||^2/(2\varepsilon)} f_0(dy).
\]

of \( K \) and \( f_0 \). Since \( K_\varepsilon \) and \( f_0^\varepsilon \) are smooth, the equation

\[
\partial_t f^\varepsilon + \nabla \cdot [f^\varepsilon (K_\varepsilon * f^\varepsilon)] = \Delta f^\varepsilon
\]

starting from \( f_0^\varepsilon \) has a unique classical solution \((f_t^\varepsilon(x))_{t \geq 0, x \in \mathbb{R}^2}\). This solution preserves mass, i.e

\[
\int_{\mathbb{R}^2} f_t^\varepsilon(dx) = \int_{\mathbb{R}^2} f_0^\varepsilon(dx) = \int_{\mathbb{R}^2} f_0(dx) = M \quad \text{for all } t \geq 0,
\]
where we write $f_t^\varepsilon(dx) = f_t^\varepsilon(x)dx$. Multiplying (4) by $\varphi \in C^2_b(\mathbb{R}^2)$, integrating on $[0, t] \times \mathbb{R}^2$, proceeding to some integrations by parts and using a symmetry argument, we classically find that

(6) \[ \int_{\mathbb{R}^2} \varphi(x)f_t^\varepsilon(dx) = \int_{\mathbb{R}^2} \varphi(x)f_0^\varepsilon(dx) + \int_0^t \int_{\mathbb{R}^2} \Delta \varphi(x)f_s^\varepsilon(dx)ds \]

\[ + \frac{1}{2} \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} K_\varepsilon(x-y) \cdot [\nabla \varphi(x) - \nabla \varphi(y)]f_u^\varepsilon(dx)f_u^\varepsilon(dy)ds. \]

We now prove some compactness result.

**Proposition 3.** Fix $M > 0$, $f_0 \in \mathcal{M}_M(\mathbb{R}^2)$ and consider the corresponding family $(f^\varepsilon)_{\varepsilon \in (0, 1]}$. The family $(f^\varepsilon)_{\varepsilon \in (0, 1]}$ is relatively compact in $C([0, \infty), \mathcal{M}_M(\mathbb{R}^2))$, endowed with the uniform convergence on compact time intervals, $\mathcal{M}_M(\mathbb{R}^2)$ being endowed with the weak convergence topology.

**Proof.** We first prove that for each $t \geq 0$, the family $(f_t^\varepsilon)_{\varepsilon \in (0, 1]}$ is tight in $\mathcal{P}(\mathbb{R}^2)$. Since the family $(f_t^0)_{\varepsilon \in (0, 1]}$ is clearly tight, by the de la Vallée Poussin theorem, there exists $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ such that $\lim_{|z| \rightarrow \infty} \psi(z) = \infty$ and $A = \sup_{\varepsilon \in (0, 1]} \int_{\mathbb{R}^2} \psi(x)f_0^\varepsilon(dx) < \infty$. Moreover, we can choose $\psi$ smooth and such that $||\nabla^2 \psi||$ is bounded by some constant $C$. It then immediately follows from (6), since $||\psi||K_\varepsilon(z)|| \leq 1/(2\pi)$, that for all $\varepsilon \in (0, 1]$, all $t \geq 0$,

\[ \int_{\mathbb{R}^2} \psi(x)f_t^\varepsilon(dx) \leq \int_{\mathbb{R}^2} \psi(x)f_0^\varepsilon(dx) + C\left(M + \frac{M^2}{4\pi}\right)t \leq A + C\left(M + \frac{M^2}{4\pi}\right)t. \]

As $\lim_{|z| \rightarrow \infty} \psi(x) = \infty$, we conclude that indeed, $(f_t^\varepsilon)_{\varepsilon \in (0, 1]}$ is tight for each $t \geq 0$.

By the Arzela-Ascoli theorem, it is enough to prove that $f^\varepsilon$ is uniformly Lipschitz continuous in time, in that there exists a constant $C > 0$ such that for all $\varepsilon \in (0, 1]$, all $t \geq s \geq 0$, $\delta(f_t^\varepsilon, f_s^\varepsilon) \leq C|t - s|$, where $\delta$ metrizes the weak convergence topology on $\mathcal{M}_M(\mathbb{R}^2)$. As is well-known, we may find a family $(\varphi_n)_{n \geq 0}$ of elements of $C^2_b(\mathbb{R}^2)$ satisfying

\[ ||\varphi_n||_\infty + ||\nabla \varphi_n||_\infty + ||\nabla^2 \varphi_n||_\infty \leq 1 \quad \text{for all } n \geq 0 \]

and such that the distance $\delta$ on $\mathcal{M}_M(\mathbb{R}^2)$ defined through

\[ \delta(f, g) = \sum_{n \geq 0} 2^{-n} \left| \int_{\mathbb{R}^2} \varphi_n(x)f(dx) - \int_{\mathbb{R}^2} \varphi_n(x)g(dx) \right| \]

is suitable. But using (6), for all $n \geq 0$,

\[ \left| \int_{\mathbb{R}^2} \varphi_n(x)f_t^\varepsilon(dx) - \int_{\mathbb{R}^2} \varphi_n(x)f_s^\varepsilon(dx) \right| \]

\[ = \left| \int_s^t \int_{\mathbb{R}^2} \Delta \varphi_n(x)f_u^\varepsilon(dx)du + \frac{1}{2} \int_s^t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} K_\varepsilon(x-y) \cdot [\nabla \varphi_n(x) - \nabla \varphi_n(y)]f_u^\varepsilon(dx)f_u^\varepsilon(dy)du \right| \]

\[ \leq (M + M^2/(4\pi))(t-s), \]

by (5), since $||\nabla^2 \varphi_n||_\infty \leq 1$ and since $||z||K_\varepsilon(z)|| \leq 1/(2\pi)$. We conclude that

\[ \delta(f_t^\varepsilon, f_s^\varepsilon) \leq \sum_{n \geq 0} 2^{-n}(M + M^2/(4\pi))(t-s) = 2(M + M^2/(4\pi))(t-s) \]

as desired. \qed

The following simple geometrical observation is crucial for our purpose.
Lemma 4. For all pair of nonincreasing functions $\varphi, \psi: (0, \infty) \to (0, \infty)$, for all $X, Y, Z \in \mathbb{R}^2$ such that $X + Y + Z = 0$, we have

$$\Delta = [\varphi(||X||)X + \varphi(||Y||)Y + \varphi(||Z||)Z] \cdot [\psi(||X||)X + \psi(||Y||)Y + \psi(||Z||)Z] \geq 0.$$ 

Proof. We may study only the case where $||X|| \leq ||Y|| \leq ||Z||$. Since $Y = -X - Z$,

$$\varphi(||X||)X + \varphi(||Y||)Y + \varphi(||Z||)Z = \lambda X - \mu Z,$$

$$\psi(||X||)X + \psi(||Y||)Y + \psi(||Z||)Z = X' - \mu' Z,$$

where $\lambda = \varphi(||X||) - \varphi(||Y||) \geq 0$, $\mu = \varphi(||Y||) - \varphi(||Z||) \geq 0$, $X' = \psi(||X||) - \psi(||Y||) \geq 0$ and $\mu' = \psi(||Y||) - \psi(||Z||) \geq 0$. Therefore,

$$\Delta = \lambda \lambda' ||X||^2 + \mu \mu' ||Z||^2 - (\lambda \mu' + \lambda' \mu) X \cdot Z \geq 0$$

as desired, because $X \cdot Z \leq 0$. Indeed, if $X \cdot Z > 0$, then $||Y||^2 = ||Z + X||^2 = ||Z||^2 + ||X||^2 + 2X \cdot Z > ||Z||^2 \geq ||Y||^2$, which is absurd. \qed

The following computation is the core of the paper.

Proposition 5. Recall that $M \in (0, 8\pi)$, that $f_0 \in M_M(\mathbb{R}^2)$. For all $\gamma \in (M/(4\pi), 2)$, there is a constant $A_M, \gamma > 0$ depending only on $M$ and $\gamma$ such that for all $\varepsilon \in (0, 1)$, all $T > 0$,

$$\int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \|x - y\|^{-2} f_\varepsilon^r (dx) f_\varepsilon^r (dy) ds \leq A_M, \gamma (1 + T).$$

Proof. For any smooth $\psi: (\mathbb{R}^2)^2 \to \mathbb{R}$ such that $\psi(x, y) = \psi(y, x)$, it holds that

$$\frac{d}{dt} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \psi(x, y) f_\varepsilon^r (dx) f_\varepsilon^r (dy) = 2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \psi(x, y)[\Delta f_\varepsilon^r (x) - \nabla \cdot (f_\varepsilon^r (x)(K_x * f_\varepsilon^r))(x)] f_\varepsilon^r (y) dx dy$$

$$= 2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} [\Delta_x \psi(x, y) + (K_x * f_\varepsilon^r)(x) \cdot \nabla_x \psi(x, y)] f_\varepsilon^r (x) f_\varepsilon^r (y) dx dy.$$

We fix $\gamma \in (M/(4\pi), 2)$, introduce $\varphi(r) = r^{\gamma/2}/(1 + r^{\gamma/2})$, and set $\psi(x, y) = \varphi(||x - y||^2)$. We have

$$\varphi'(r) = \frac{\gamma}{2} \frac{r^{\gamma/2 - 1}}{(1 + r^{\gamma/2})^2}$$

and

$$\varphi''(r) = \frac{\gamma}{2} \frac{r^{\gamma/2 - 2}}{(1 + r^{\gamma/2})^2} \left( 2 \frac{\gamma - \gamma r^{\gamma/2}}{1 + r^{\gamma/2}} \right)$$

and

$$\nabla_x \psi(x, y) = 2 \varphi'(||x - y||^2) (x - y) = \gamma \frac{||x - y||^{-2}}{(1 + ||x - y||^2)} (x - y),$$

$$\Delta_x \psi(x, y) = 4 \varphi'(||x - y||^2) + 4 \varphi(2 \varphi'(||x - y||^2)) = \gamma^2 \frac{||x - y||^{-2}}{(1 + ||x - y||^2)^2} \left( 1 - 2 \frac{||x - y||^2}{1 + ||x - y||^2} \right).$$

Hence

$$\varphi(||x - y||^2) f_\varepsilon^r (dx) f_\varepsilon^r (dy) = J_\varepsilon^r + S_\varepsilon^r,$$

where

$$J_\varepsilon^r = 2 \gamma^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{||x - y||^{-2}}{(1 + ||x - y||^2)^2} \left( 1 - 2 \frac{||x - y||^2}{1 + ||x - y||^2} \right) f_\varepsilon^r (x) f_\varepsilon^r (y) dx dy,$$

$$S_\varepsilon^r = 2 \gamma \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{||x - y||^{-2}}{(1 + ||x - y||^2)^2} (x - y) \cdot K_x(x, y) f_\varepsilon^r (x) f_\varepsilon^r (y) dx dy dz.$$
First, we have

\( J_t^r \geq \gamma (\gamma + M/(4\pi)) \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{||x-y||^{\gamma - 2}}{(1 + ||x-y||^\gamma)^2} f_t^r(x)f_t^r(y) dx dy - M^2 C_{M,\gamma}, \)

where \( C_{M,\gamma} > 0 \) is a constant such that for all \( a > 0 \), recall that \( \gamma > M/(4\pi), \)

\[ 2\gamma^2 \frac{a^{\gamma - 2}}{(1 + a^\gamma)^2} \left( 1 - 2 \frac{a^\gamma}{1 + a^\gamma} \right) \geq 2\gamma \left( \frac{\gamma + M/(4\pi)}{2} \right) \frac{a^{\gamma - 2}}{(1 + a^\gamma)^2} - C_{M,\gamma}. \]

Next, by symmetrization, we have

\[ S_t^r = \gamma \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{||x-y||^{\gamma - 2}}{(1 + ||x-y||^\gamma)^2} (x-y) \cdot (K_\varepsilon(x-z) - K_\varepsilon(y-z)) f_t^r(x)f_t^r(y)f_t^r(z) dx dy dz, \]

where

\[ F_t(x, y, z) = [K_\varepsilon(x-z) - K_\varepsilon(y-z)] \cdot (x-y) \frac{||x-y||^{\gamma - 2}}{(1 + ||x-y||^\gamma)^2} \]

\[ + [K_\varepsilon(y-x) - K_\varepsilon(z-x)] \cdot (y-z) \frac{||y-z||^{\gamma - 2}}{(1 + ||y-z||^\gamma)^2} \]

\[ + [K_\varepsilon(z-y) - K_\varepsilon(x-y)] \cdot (z-x) \frac{||z-x||^{\gamma - 2}}{(1 + ||z-x||^\gamma)^2}. \]

Introducing \( X = x - y, Y = y - z \) and \( Z = z - x \) and recalling that \( K_\varepsilon(X) = \frac{-X}{2\pi(||X||^2 + \varepsilon)} \), we find

\[ 2\pi F_t(x, y, z) = \frac{Z}{||Z||^2 + \varepsilon} + \frac{Y}{||Y||^2 + \varepsilon} \cdot \frac{||X||^{\gamma - 2}}{(1 + ||X||^\gamma)^2} \]

\[ + \left[ \frac{X}{||X||^2 + \varepsilon} + \frac{Z}{||Z||^2 + \varepsilon} \right] \cdot \frac{Y}{(1 + ||Y||^\gamma)^2} \]

\[ + \left[ \frac{Y}{||Y||^2 + \varepsilon} + \frac{X}{||X||^2 + \varepsilon} \right] \cdot Z \frac{||Z||^{\gamma - 2}}{(1 + ||Z||^\gamma)^2}. \]

We now introduce

\[ G(x, y, z) = \frac{||X||^{\gamma - 2}}{(1 + ||X||^\gamma)^2} + \frac{||Y||^{\gamma - 2}}{(1 + ||Y||^\gamma)^2} + \frac{||Z||^{\gamma - 2}}{(1 + ||Z||^\gamma)^2} \]

\[ \geq X \cdot \frac{X}{||X||^2 + \varepsilon} \frac{||X||^{\gamma - 2}}{(1 + ||X||^\gamma)^2} + Y \cdot \frac{Y}{||Y||^2 + \varepsilon} \frac{||Y||^{\gamma - 2}}{(1 + ||Y||^\gamma)^2} + Z \cdot \frac{Z}{||Z||^2 + \varepsilon} \frac{||Z||^{\gamma - 2}}{(1 + ||Z||^\gamma)^2}. \]

Hence \( G(x, y, z) + 2\pi F_t(x, y, z) \) is larger than

\[ \left( \frac{X}{||X||^2 + \varepsilon} + \frac{Y}{||Y||^2 + \varepsilon} + \frac{Z}{||Z||^2 + \varepsilon} \right) \cdot \left( X \frac{||X||^{\gamma - 2}}{(1 + ||X||^\gamma)^2} + Y \frac{||Y||^{\gamma - 2}}{(1 + ||Y||^\gamma)^2} + Z \frac{||Z||^{\gamma - 2}}{(1 + ||Z||^\gamma)^2} \right), \]

which is nonnegative according to Lemma \( \text{H} \) since \( r \to 1/(r^2 + \varepsilon) \) and \( r \to r^{\gamma - 2}/(1 + r^\gamma)^2 \) are both nonincreasing on \((0, \infty)\) and since \( X + Y + Z = 0 \). Thus \( F_t(x, y, z) \geq -G(x, y, z)/(2\pi) \). Recalling
with the uniform convergence on compact time intervals, $M$-convergence topology. By definition of $\gamma$

We finally give the

Gathering (7)-(8)-(10), we find

Integrating on $[0,T]$, using that $\gamma > M/(4\pi)$ and that $\varphi$ is $[0,1]$-valued, we end with

One easily completes the proof, using that there is $D_\gamma > 0$ such that $a^{\gamma-2} \leq 2a^{\gamma-2}/(1+a^\gamma)^2 + D_\gamma$

for all $a > 0$

We finally give the

Proof of Theorem 2

Recall that $M \in (0,8\pi)$, that $f_0 \in \mathcal{M}_M(\mathbb{R}^2)$, and that $(f^\varepsilon)_{\varepsilon \in (0,1]}$ is the corresponding family of regularized solutions. By Proposition 3 we can find $(\varepsilon_k)_{\varepsilon > 0}$ and $f \in C([0,\infty),\mathcal{M}_M(\mathbb{R}^2))$ such that $\lim_{k} \varepsilon_k = 0$ and $\lim_k f^{\varepsilon_k} = f$ in $C([0,\infty),\mathcal{M}_M(\mathbb{R}^2))$, endowed with the uniform convergence on compact time intervals, $\mathcal{M}_M(\mathbb{R}^2)$ being endowed with the weak convergence topology. By definition of $f^{\varepsilon_0}$, we obviously have $f|_{t=0} = f_0$. By Proposition 5 and the Fatou lemma, for all $\gamma \in (M/(4\pi),2)$, we have

It only remains to check that $f$ is a weak solution to 2. We fix $\varphi \in C_0^\infty(\mathbb{R}^2)$ and use 4 to write

where

Since $\varphi$ and $\Delta \varphi$ are continuous and bounded, we immediately conclude that

and it only remains to check that for $J(t) = \frac{1}{2} \int_0^t \int_{\mathbb{R}^2} K(x-y) \cdot [\nabla \varphi(x) - \nabla \varphi(y)] f_s(dx)f_s(dy) ds$, we have $\lim_k J_k(t) = J(t)$. To this end, we write $J_k(t) = J_k^1(t) + J_k^2(t)$, where

$$J_k^1(t) = \frac{1}{2} \int_0^t \int_{\mathbb{R}^2} K(x-y) \cdot [\nabla \varphi(x) - \nabla \varphi(y)] f_s^{\varepsilon_k}(dx)f_s^{\varepsilon_k}(dy) ds,$$

$$J_k^2(t) = \frac{1}{2} \int_0^t \int_{\mathbb{R}^2} [K_{\varepsilon_k}(x-y) - K(x-y)] \cdot [\nabla \varphi(x) - \nabla \varphi(y)] f_s^{\varepsilon_k}(dx)f_s^{\varepsilon_k}(dy) ds.$$
Recalling the expression of $K$ and that $\varphi \in C^2_b(\mathbb{R}^2)$, we see that $g(x, y) = K(x-y) \cdot [\nabla \varphi(x) - \nabla \varphi(y)]$ is bounded and continuous on the set $\mathbb{R}^2 \setminus D$, where $D = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 : x = y\}$. Since $f_k^x \otimes f_k^y$ goes weakly to $f_x \otimes f_y$ for each $s \geq 0$ and since $(f_x \otimes f_y)(D) = 0$ for a.e. $s \geq 0$ by (11), 1, we conclude that $\lim_k \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} g(x, y) f_k^x (dx) f_k^y (dy) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} g(x, y) f_x (dx) f_y (dy)$ for a.e. $s \geq 0$, whence $\lim_k J_k^2(t) = J(t)$ by dominated convergence.

We finally have to verify that $\lim_k J_k^2(t) = 0$. We fix $\gamma \in (M/(4\pi), 2)$ and write
\[
\|z\| \|K(z) - K_\varepsilon(z)\| = \frac{\varepsilon}{2\pi(\varepsilon + \|z\|^2)} \leq \min(1, \varepsilon \|z\|^{-2}) \leq (\varepsilon \|z\|^{-2})^{1-\gamma/2} = \varepsilon^{1-\gamma/2} \|z\|^{\gamma-2}.
\]
Thus
\[
|J_k^2(t)| \leq \|\nabla^2 \varphi\| \|\varepsilon_k^{1-\gamma/2} \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \|x - y\|^{\gamma-2} f_{s_k}^x (dx) f_{s_k}^y (dy) ds,
\]
which tends to 0 as desired since $\sup_{x \in (0,1]} \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \|x - y\|^{\gamma-2} f_{s_k}^x (dx) f_{s_k}^y (dy) ds < \infty$ by Proposition 5.

\section*{References}

[1] G. Arumugam, J. Tyagi, Keller-Segel chemotaxis models: a review, Acta Appl. Math. 171, (2021), Paper 6, 82 pp.
[2] J. Bedrossian, N. Masmoudi, Existence, uniqueness and Lipschitz dependence for Patlak-Keller-Segel and Navier-Stokes in $\mathbb{R}^2$ with measure-valued initial data, Arch. Ration. Mech. Anal. 214, (2014), 717–801.
[3] P. Biler, Singularities of solutions to chemotaxis systems, De Gruyter Series in Mathematics and Life Sciences, De Gruyter, Berlin, 2020.
[4] P. Biler, G. Karch, P. Laurençot, T. Nadzieja, The $8\pi$-problem for radially symmetric solutions of a chemotaxis model in a disc, Topol. Methods Nonlinear Anal. 27, (2006), 133–147.
[5] P. Biler, G. Karch, P. Laurençot, T. Nadzieja, The $8\pi$-problem for radially symmetric solutions of a chemotaxis model in the plane, Math. Methods Appl. Sci. 29, (2006), 1563–1583.
[6] A. Blanchet, J. Dolbeault, B. Perthame, Two-dimensional Keller-Segel model: optimal critical mass and qualitative properties of the solutions, Electron. J. Differential Equations 44 (2006), 32 pp.
[7] D. Bresch, P.E. Jabin, Z. Wang, On mean-field limits and quantitative estimates with a large class of singular kernels: application to the Patlak-Keller-Segel model. C. R. Math. Acad. Sci. Paris 357 (2019), 708–720.
[8] N. Fournier, B. Jourdain, Stochastic particle approximation of the Keller-Segel equation and two-dimensional generalization of Bessel processes, Ann. Appl. Probab. 27, (2017), 2807–2861.
[9] W. Jäger, S. Luckhaus, On explosions of solutions to a system of partial differential equations modelling chemotaxis, Trans. Amer. Math. Soc. 329, (1992), 819–824.
[10] E. Keller, L. Segel, Initiation of slime mold aggregation viewed as an instability, J. Theoret. Biol. 26, (1970), 399–415.
[11] H. Osada, A stochastic differential equation arising from the vortex problem, Proc. Japan Acad. Ser. A Math. Sci. 61, (1985), 333–336.
[12] C.S. Patlak, Random walk with persistence and external bias, Bull. Math. Biophys. 15, (1953), 311–338.
[13] D. Wei, Global well-posedness and blow-up for the 2-D Patlak-Keller-Segel equation, J. Funct. Anal. 274, (2018), 388–401.

Sorbonne Université, LPSM-UMR 8001, Case courrier 158, 75252 Paris Cedex 05, France.
Email address: nicolas.fournier@sorbonne-universite.fr, yoan.tardy@sorbonne-universite.fr