Method of regularized sources for Stokes flow problems with improved calculation of velocity derivatives at the boundary

Božidar Šarler1,2,3, Shiting Wen1 and Ming Li1
1 Institute of Metals and Technology, Lepi pot 11, SI-1000 Ljubljana, Slovenia
2 University of Nova Gorica, Vipavska 13, SI-5000, Nova Gorica, Slovenia
3 College of Mathematics, Taiyuan University of Technology, Yingze West Street 79, 30024 Taiyuan, Shanxi Province, China

E-mail: bozidar.sarler@ung.si, bozidar.sarler@imt.si

Abstract. The solution of Stokes flow problems with Dirichlet and Neumann boundary conditions is performed by a non-singular Method of Fundamental Solutions which does not require artificial boundary, i.e. source points of fundamental solution coincide with the collocation points on the boundary. Instead of Dirac delta force, an exponential function, called blob, with a free parameter epsilon is employed, which limits to Dirac delta function when epsilon limits to zero. The solution of the problem is sought as a linear combination of the fields due to the regularized sources that coincide with the boundary and with their intensities chosen in such a way that the solution complies with the boundary conditions. A two-dimensional flow between parallel plates is chosen to assess the properties of the method. The results of the method are accurate except for the derivatives at the boundary. A correction of the method is proposed which can be used to properly assess also the derivatives at the boundary.

1. Introduction

Stokes or creeping flow is a type of fluid flow [1] where the inertial forces are small compared with the viscous forces. This typically occurs in situations where the fluid velocities are very slow, the viscosities are very large, or the length-scales of the flow are very small. The present research was initiated by the simulation needs in microfluidics, specifically gas focused micro-jets [2]. The Method of Fundamental Solutions (MFS) [3] has been widely applied in recent years for the computational analysis of fluid flows. The MFS belongs to the boundary meshless methods. The solution in MFS is represented by trial functions defined by fundamental solutions of the governing equation. Their expansion coefficients match, in collocation or least squares sense, the boundary conditions. Since the fundamental solution is usually singular, the source points are in general not allowed to be put on the boundary. Their position forms so called artificial boundary. This represents the main disadvantage of the classical MFS, particularly pronounced in geometrically complex and multi-region situations. The first ideas regarding how to overcome the artificial boundary issue came from the group of Young et al. [4], where the desingularization has been performed through the properties of double layer potential. Since 2006, several publications appear on the subject of non-singular MFS, treating a spectra of different partial differential equations. Young et al. [5] developed classical MFS for 2D and 3D Stokes flows, based on

1 To whom any correspondence should be addressed.
the Stokes fundamental solution. Curteanu et al. [6] developed classical MFS solution of Stokes flow by using Laplace decomposition and Laplace fundamental solution. The pioneering developments of the non-singular MFS for potential flow, Darcy flow and Stokes flow appear in [7,8,9], respectively. The desingularization was achieved by integration of the fundamental solution over a small vicinity of the singularity. In case of Neumann boundary conditions, reference solutions need to be constructed for resolving the diagonal coefficients. Cortez desingularized the Stokes flow fundamental solution by introducing smoothed sources instead of delta sources, which appear in the classical fundamental solution, and applied the method to two [10] and three [11] dimensional flow problems. The present group recently upgraded the related Method of Regularized Sources (MRS) for axisymmetric Stokes flow problems [12] and re-derived this approach for different types of boundary conditions and external as well internal Stokes flow problems. Additionally, we made a systematic sensitivity study [13] on the accuracy of the method as a function of the rational and exponential desingularization function types and desingularization parameter on example of a driven cavity problem and flow between parallel plates. A further development of the method for accurate calculation of the derivatives at the boundary is presented in the present paper.

2. Governing equations
Consider a fixed domain $\Omega$ with boundary $\Gamma$ filled with fluid that exhibits steady incompressible Stokes flow with constant fluid viscosity. The boundary value problem is governed by the following set of mass and momentum conservation equations

\[ \nabla \cdot \mathbf{v}(\mathbf{p}) = 0, \]  
\[ -\nabla P(\mathbf{p}) + \mu \nabla^2 \mathbf{v}(\mathbf{p}) + \mathbf{f}(\mathbf{p}) = 0, \]  
and boundary conditions of the Dirichlet

\[ v_\zeta(\mathbf{p}) = v_\zeta^D(\mathbf{p}); \mathbf{p} \in \Gamma^D, \]  
and Neumann type

\[ \partial v_\zeta(\mathbf{p}) / \partial \nu_\zeta = v_\zeta^N(\mathbf{p}); \mathbf{p} \in \Gamma^N, \]  
where $\mathbf{p}$ represents position vector, $\mathbf{v}$ velocity, $P$ pressure, $\mu$ viscosity, $\mathbf{f}$ the body force, $\Gamma^D$ Dirichlet part of the boundary for coordinate $\zeta$, $\Gamma^N$ Neumann part of the boundary for the coordinate $\zeta$ and derivative over coordinate $\xi$, respectively. $v_\zeta^D$ and $v_\zeta^N$ stand for known boundary conditions forcing functions. We seek the solution of the pressure and the velocity field in $\Omega$ and unknown parts of $\Gamma$.

3. Solution procedure
In case the Dirac delta function $\delta(\mathbf{p} - \mathbf{s})$ is selected for function $\mathbf{f}(\mathbf{p})$, and the equations (1-2) are solved in infinite media, the solution results in well known Stokes fundamental pressure and velocity. Consider a two-dimensional situation with $\mathbf{p} = \mathbf{p}_\zeta \mathbf{i}_\zeta + \mathbf{p}_\eta \mathbf{i}_\eta$ ($\mathbf{p}_\zeta$ and $\mathbf{i}_\zeta$ stand for Cartesian coordinates and base vectors, respectively). Instead of selecting Dirac delta function for the source shape, the exponential blob is selected.
$\phi(p-s) = \frac{1}{\pi \varepsilon^2} \exp \left( \frac{|p-s|^2}{\varepsilon^2} \right)$,

$|p-s|^2 = r^2 = (p_x-s_x)^2 + (p_y-s_y)^2$ \hspace{1cm} (5)

It has the same strength like the Dirac delta function, $\int_\Omega \delta d\Omega = 1$, i.e. $\int_\Omega \phi(r(p-s))2\pi rd\Omega = 1$; $r \in 2D$, and approaches Dirac delta function when $\varepsilon \to 0$. The boundary of the domain is discretized with collocation points $p_n$, $n=1,2,...,N$, where the desingularized sources with $s_n = p_n$ and $\varepsilon = \varepsilon(s_n)$ are put. The unknown forces $f_{in}$ and $f_{in}$ in source points are determined from $2N \times 2N$ system of linear equations in such a way that the boundary conditions are satisfied. The system of equations has the following form

$\begin{align*}
\mathbf{A} \mathbf{x} &= \mathbf{b}, \quad A_{in} x_n = b_j; \quad j = 1,2,...,2N, \quad n = 1,2,...,2N, \\
A_{2j-1,j-1} &= \mathcal{Y}^{(p)}(p) c_{\chi_{\chi\chi}}(p_j) + \mathcal{Y}^{(p)}(p) c_{\chi_{\chi\chi}}(p_j) + \mathcal{Y}^{(p)}(p) c_{\chi_{\chi\chi}}(p_j), \\
A_{2j-1,j-1} &= \mathcal{Y}^{(p)}(p) c_{\chi_{\chi\chi}}(p_j) + \mathcal{Y}^{(p)}(p) c_{\chi_{\chi\chi}}(p_j) + \mathcal{Y}^{(p)}(p) c_{\chi_{\chi\chi}}(p_j), \\
A_{2j,2j} &= \mathcal{Y}^{(p)}(p) c_{\chi_{\chi\chi}}(p_j) + \mathcal{Y}^{(p)}(p) c_{\chi_{\chi\chi}}(p_j) + \mathcal{Y}^{(p)}(p) c_{\chi_{\chi\chi}}(p_j), \\
x_{2j-1} = f_{in}, \quad x_{2n} = f_{in}, \quad (6)
\end{align*}$

The following boundary conditions indicators have been introduced in order to make the notation compact

$\begin{align*}
\mathcal{Y}^{(p)}(p) &= \begin{cases} 1; p \in \Gamma^D \cap \zeta = x, y & \text{,} \\
0; p \in \Gamma^{N \cap \zeta = x, y} & \text{.}
\end{cases} \\
\mathcal{Y}^{(p)}(p) &= \begin{cases} 1; p \in \Gamma^N \cap \zeta, \xi = x, y & \text{,} \\
0; p \in \Gamma^{N \cap \zeta, \xi = x, y} & \text{.}
\end{cases}
\end{align*}$ \hspace{1cm} (14)

as well as the following coefficients

$\begin{align*}
c_{\chi_{\chi\chi}}(p) &= \frac{1}{\mu} \frac{\partial^2}{\partial p_{\zeta}^2} \delta(p-s), \quad c_{\chi_{\chi\chi}}(p) = \frac{1}{\mu} \frac{\partial^2}{\partial p_{\zeta}^2} \delta(p-s), \\
c_{\chi_{\chi\chi}}(p) &= \frac{1}{\mu} \frac{\partial^2}{\partial p_{\zeta}^2} \delta(p-s), \quad c_{\chi_{\chi\chi}}(p) = \frac{1}{\mu} \frac{\partial^2}{\partial p_{\zeta}^2} \delta(p-s), \\
c_{\chi_{\chi\chi}}(p) &= \frac{\partial}{\partial p_{\zeta}}, \quad c_{\chi_{\chi\chi}}(p) = \frac{\partial}{\partial p_{\zeta}} \delta(p-s), \quad c_{\chi_{\chi\chi}}(p) = \frac{\partial}{\partial p_{\zeta}} \delta(p-s), \\
\end{align*}$ \hspace{1cm} (15,16)

with

$\tilde{\delta} = -\frac{1}{4\pi} \mathrm{Ei} \left( -\frac{r^2}{\varepsilon^2} \right) + \frac{1}{2\pi} \log(r), \quad \tilde{\phi} = -\frac{1}{16\pi} \left[ 2r^2 + e^{-r/\varepsilon} \varepsilon^2 + \left( r^2 + \varepsilon^2 \right) \mathrm{Ei} \left( -\frac{r^2}{\varepsilon^2} \right) - 2 \log(r) \right]$, \hspace{1cm} (23,24)
\[
v_x(p_x, p_y) = \sum_{m=1}^{N} \left[ c_{x,m}(p_x, p_y) f_m + c_{y,m}(p_x, p_y) f_m \right], \quad \xi = x, y,
\]
(25)

\[
P(p_x, p_y) = \sum_{m=1}^{N} \left[ c_{x,m}(p_x, p_y) f_m + c_{y,m}(p_x, p_y) f_m \right].
\]
(26)

4. Numerical example
Consider a flow between two parallel plates with dimensions \( p_{x-} \leq p_x \leq p_{x+}, \quad p_{y-} \leq p_y \leq p_{y+}, \quad p_{x-} = -5.0\text{m} , \quad p_{y-} = -0.5\text{m} \) with \( p_{x+} = +5.0\text{m} , \quad p_{y+} = +0.5\text{m} \). The boundary conditions are of the Dirichlet type on the four boundaries. The velocity is set to \( v_x = v_y = 0\text{m/s} \) at South and North boundaries, and to \( v_x = -0.5(\Delta P/\mu)(y_{-} - y_{+})(y_{-} - y) \text{m/s}, v_y = 0\text{m/s} \) at the West and East boundaries. The boundary conditions define a developed flow between parallel plates with fluid viscosity \( \mu \) and pressure drop \( \Delta P = -1\text{N/m}^2 \), set to \( \mu = 1\text{kg/(m/s)} \) and \( \Delta P = -1\text{N/m}^2 \). The numerical solution is calculated in uniformly distributed nodes on the boundary. The root mean square (RMS) error \( L_2 \) between the numerical and analytical solutions is calculated as a function of the shape parameter. The results are given in Figure 1. The analytical solution of the velocity derivative is \( \frac{\partial v_x}{\partial p_x} = (\Delta P/\mu) y \text{m/s} \). The velocity and derivative across the line \( p_{y-} \leq p_y \leq p_{y+} \) at \( p_x = (p_{y+} + p_{y-})/2 \) is evaluated at 101 equidistant points and the results are given in Fig. 2.

**Figure 1.** \( L_2 \) error of MRS solution with exponential blob of \( v_x \) as a function of \( \varepsilon \) (left) and as a function of the factor \( f_x \) (right) between epsilon and nodal distance (\( f_x = \varepsilon / \Delta p_x \)). The RMS error between the solutions is calculated on uniform 101 node arrangement across the line \( p_{y-} \leq p_y \leq p_{y+} \) at \( p_x = (p_{y+} + p_{y-})/2 \).
Figure 2. Profile of \( v_x \) (left) and \( \frac{\partial v_x}{\partial p_y} \) (right) calculated by MRS with exponential blobs and analytical solution, evaluated at \( N = 101, \varepsilon = 0.007 \) across the line \( p_y \leq p_y \leq p_y \), at \( p_x = \left( p_{x^+} + p_{x^-} \right) / 2 \).

5. The correction algorithm

One can see from the numerical results (See Figs.1 and 2), that the function values are extremely well represented, but the behaviour of the method for derivatives is not appropriate in the boundary layer. This can be easily explained by the fact that we solve Stokes equation with a finite source instead of Stokes equation without a source in the boundary layer. The derivatives on the boundary are respectively corrected by an ad-hoc correction in the form

\[
\frac{\partial}{\partial p_y} v_x(p_x, p_y) = \sum_{n=1}^{N} \left[ c_{m_n, z} \left( p_y, p_y \right) f_{m_n} + c_{z_m, z} \left( p_y, p_y \right) f_{z_m} \right] \left[ 1 + \lambda \frac{\phi(p_x, p_y)}{\varepsilon \phi(p_x, p_x)} \right],
\]

(27)

where \( p_x \) stands for the closest discretisation point on the boundary to \( p_x \), and \( \lambda \) for a heuristic constant, taken as 1.2. The correction is effective at the boundary, where the finite sources are positioned, and decays away from it (see Fig. 3 left). The correction has to be applied also in assembling of the system of equations (6) in case the Neumann boundary conditions are present.

Figure 3. Correction \( 1 + \lambda \frac{\phi(p_x, p_y)}{\varepsilon \phi(p_x, p_x)} \) with \( \lambda = 1.2 \) at the boundary (left) and profile of \( \frac{\partial v_x}{\partial p_y} \) calculated by MRS with correction and analytical solution, evaluated at \( N = 101, \varepsilon = 0.007 \) across the line \( p_y \leq p_y \leq p_y \), at \( p_x = \left( p_{x^+} + p_{x^-} \right) / 2 \).
6. Conclusions
The represented novel MRS is a very simple and efficient boundary meshless method without artificial boundary. The solution is constructed through the superposition of the exact solutions due to regularized sources placed on the physical boundary of the system. The coefficients of the trial functions, obtained in this way, are requiring the solution to comply with the boundary conditions. The calculated derivatives of the method have been in the present paper corrected, since they represent the derivatives of the solution that exhibits finite sources in the boundary layer, which is not in accordance with the original governing equation. A study of the performance of the method when using exponential blobs with and without boundary derivative correction has been performed. Accurate results in terms of velocity and velocity derivatives can be obtained when the regularization parameter scales with the typical nodal distance and when the represented correction is applied. In the future, the method can be connected with a robust algorithm for an estimation of the specific optimum regularization parameter for each of the boundary nodes separately. If needed, this would make it possible to obtain even more accurate results. Such an algorithm might be of the leave-one-out-cross-validation-type [15], as recently demonstrated for a determination of artificial boundary position in classic MFS.

Acknowledgement
The present paper is sponsored by Slovenian Grant Agency under program group P2-0379, project J2-7384 and project: Innovative methods for imaging with the X-ray Free Electron Laser (XFEL) and synchrotron sources, sponsored by Desy, Germany. The project is also supported by the National Natural Science Foundation of China (Grant No. 11472184) and the National Youth Science Foundation of China (Grant No. 11401423).

References
[1] Kaviani M S 1995 Principles of Heat Transfer in Porous Media (Berlin: Springer)
[2] Beyerlein K R, Adriano L, Heymann M, Kirian R, Knoška J, Wilde F, Chapman H N and Bajt S 2015 Review of Scientific Instruments 86 125104, 2015; http://dx.doi.org/10.1063/1.4936843
[3] Chen W, Fu Z J and Chen C S 2014 Recent Advances in Radial Basis Function Collocation Methods (Heidelberg: Springer)
[4] Young D L, Chen K H and Lee C W 2006 Journal of Computational Physics 209 290
[5] Young D L, Jane S J, Fan C M, Murgesan K and Tsai C C 2006 Journal of Computational Physics, 211 1
[6] Curteanu A E, Elliot L, Ingham D B, Lesnic D 2007 Engineering Analysis with Boundary Elements 31 501
[7] Šarler B 2009 Engineering Analysis with Boundary Elements 33 1374
[8] Perne M, Šarler B and Gabrovšek F 2012 Engineering Analysis with Boundary Elements 36 1649
[9] Sincich E, Šarler B 2014 Computer Modeling in Engineering & Sciences 99 393
[10] Cortez R 2005 SIAM Journal of Scientific Computing 23 1204
[11] Cortez R, Fauci L and Medovikov A 2005 Physics of Fluids 17 031504-1
[12] Wang K, Wen S T, Zahoor R, Li M and Šarler B 2016 International Journal of Numerical Methods in Heat & Fluid Flow (in print)
[13] Wen S T, Wang K, Zahoor R, Li M and Šarler B 2016 Computer Assisted Methods in Engineering and Science (in print)
[14] Lamichhane A R, Chen C S 2015 Applied Mathematics Letters 46 50
[15] Chen C S, Karageorghis A and Li Y 2016 Numerical Algorithms DOI 10.1007/s11075-015-0036-0