Violation of local realism vs detection efficiency

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We put bounds on the minimum detection efficiency necessary to violate local realism in Bell experiments. These bounds depends of simple parameters like the number of measurement settings or the dimensionality of the entangled quantum state. We derive them by constructing explicit local-hidden variable models which reproduce the quantum correlations for sufficiently small detectors efficiency.

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I. INTRODUCTION

Since the work of Bell [1] it is well known that “non-local” correlations can be extracted from entangled states by performing certain measurements on spatially separated regions. More precisely by “non-local” correlations, one means correlations that cannot be reproduced by local realistic theories. Non-locality and entanglement are closely connected and they form a most remarkable features of quantum mechanics. But the relation between these two concepts is still not perfectly clear. One difference between them is that while entanglement is a characteristic per se of a quantum system, non-locality depends on the specific experiment carried on the quantum system, in particular it depends on the measurements performed and on practical details such as the efficiency and the background of the detectors, the amount of noise present, etc. It is therefore much more difficult to compare non-locality exhibited by different experiments and to find measures of non-locality than it is for entanglement.

Possible ways to quantify the non-local character of quantum correlations exploit their dependence to experimental imperfections like the maximum amount of noise or the minimum detection efficiency still allowing a violation of local realism. The amount of communication needed to reproduce the quantum correlations in a classical scenario can also serve to gauge their non-local nature.

In the present paper we concentrate on the resistance to inefficient detectors and try to put bounds on how much increase in non-locality can be expected from that point of view. We suppose that each detector has a probability \( \eta \) of giving a result and a probability \( 1 - \eta \) of not giving a result. If \( \eta \) is sufficiently small, the quantum correlations produced in a Bell experiment can be explained by a local hidden variables (LHV) model. We denote by \( \eta^* \) the maximum detection efficiency for which a LHV model exists. Thus if \( \eta > \eta^* \) the correlations are indeed non-local. To put bounds on \( \eta^* \), we construct several LHV models that take advantage of the inefficiency of the detectors to reproduce the quantum correlations. LHV models exploiting the detection loophole have already been constructed to reproduce the result of specific experiments [2, 3]. There have also been attempts to build more general LHV models that can for example reproduce measurements performed on the singlet state [4, 5] or experiments performed using parametric-down conversion sources [6]. In this paper we try to be more general than that. Indeed, our purpose is to understand how \( \eta^* \) is constrained by simple parameters such as the number of measurements settings or the dimensionality of the quantum system. We therefore introduce a first LHV model in section II which depend only on the number of measurements settings at each site (it is a generalization of a model first discussed in [2, 3]). We describe it both in the case of two parties and in the case of many parties. In the case of two measurements per site, the bound on \( \eta^* \) our LHV model implies is saturated by Eberhard’s [7] and Larsson and Semitecolos’s [8] schemes. In section II we introduce a second model for maximally entangled states that depend only of the dimension \( d \) of the Hilbert space and which reproduce the quantum correlations up to small errors. This LHV model will be analyzed in the case of two parties, although it could probably be generalized to more parties. These two LHV models work for arbitrary measurements (POVM’s) carried out by the parties. Before presenting them, let us briefly recall the principle of Bell experiments and the content of LHV theories.

II. BELL EXPERIMENTS AND LHV THEORIES

In a typical Bell experiment, two parties Alice and Bob (the generalization to \( N \) parties is straightforward) share an entangled state \( \rho_{AB} \). Alice selects one of \( M_A \) measurements on his sub-system and Bob one of \( M_B \). We will consider the most general type of measurements, namely Positive Operator Valued Measurements (POVM). Let \( X \) be Alice’s measurement and \( Y \) be Bob’s measurements and \( a \) and \( b \) Alice and Bob’s outcomes. The POVM \( X \) thus consists of the positive operators \( x_a \) with the

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property that $\sum_a x_a = I_A$. Similarly the POVM $Y$ consists of the positive operators $y_b$ with the property that $\sum_b y_b = I_B$. Here $I_A$ and $I_B$ are the identity operators. Quantum mechanics predicts the probabilities

$$
P^{QM}(a, b|X, Y) = \text{Tr}(x_a \otimes y_b \rho_{AB}),$$

$$
P^{QM}(a|X) = \text{Tr}(x_a \otimes \mathbf{I} \rho_{AB}),$$

$$
P^{QM}(b|Y) = \text{Tr}(\mathbf{I} \otimes y_b \rho_{AB}).$$

If the detectors aren’t perfect, i.e. $\eta < 1$, a supplementary outcome is possible, corresponding to the case where the detector don’t fire. We denote this outcome by the symbol $\emptyset$. We then have the probabilities:

$$
P^{QM}_\emptyset(a, b|X, Y) = \eta^2 P^{QM}(a, b|X, Y) \quad a, b \neq \emptyset,$$

$$
P^{QM}_\emptyset(\emptyset, b|X, Y) = \eta(1-\eta) P^{QM}(b|Y) \quad b \neq \emptyset,$$

$$
P^{QM}_\emptyset(a, \emptyset|X, Y) = \eta(1-\eta) P^{QM}(a|X) \quad a \neq \emptyset,$$

$$
P^{QM}_\emptyset(\emptyset, \emptyset|X, Y) = (1-\eta)^2.$$  

(1)

In a local hidden variable theory, the quantum correlations (1) or (2) are reproduced with the help of a random variable $\lambda$ shared by both parties. Moreover the outcomes of measurements performed by one of the parties are determined by the settings of the measurement apparatus of that party only. Correlations predicted by these theories are thus of the form:

$$
P^{LHV}(a, b|X, Y) = \int d\lambda \ p(\lambda) P(a|X, \lambda) P(b|Y, \lambda)$$  

(3)

where $\lambda$ is the shared randomness. In the case of inefficient detectors, $a$ and $b$ can take either a value different from $\emptyset$, or the value $\emptyset$. In the latter case the LHV model just instructs the detectors not to fire.

III. A LHV MODEL THAT DEPEND ONLY ON THE NUMBER OF SETTINGS

A classical theory can reproduce all the results of quantum mechanics if information on which measurement has been selected can flow from one side to the other. It is to guarantee that such mechanism cannot account of the observed data that measurements in Bell tests must be carried out at spatially separated regions. A LHV model can nevertheless exploit the limited detection efficiency by guessing a priori which measurement will be performed on one side. If the actual measurement and the guessed one coincide, the model will output results in agreement with quantum mechanics. If they don’t, it simply tells the detectors not to fire. Building a LHV model out of this idea will enable us to prove the following bound:

**Theorem 1:** In experiments where Alice can choose between $M_A$ measurements and Bob $M_B$, the maximum detection efficiency $\eta^*$ for which a LHV model exist is at least

$$
\eta^* \geq \frac{M_A + M_B - 2}{M_A M_B - 1}.
$$

(4)

**Proof:** The proof consist of constructing a LHV model that reproduce the correlations (2) with $\eta$ given by the bound. In this model, the local hidden variable $\lambda$ consist of the pair $\lambda = (a', X')$ where $X'$ correspond to one of the $M_A$ possible measurements of Alice and $a'$ to one of the possible outcomes. $X'$ is chosen with probability $1/M_A$ and $a'$ with probability $P^{QM}(a'|X')$, so that $p(\lambda) = P^{QM}(a'|X')/M_A$. If Alice’s actual measurement $X$ coincides with $X'$ (this occurs with probability $1/M_A$), Alice outputs $a'$, otherwise she outputs $\emptyset$. We thus have $P(a|X, \lambda) = \delta_{a a'} \delta_{X X'}$ if $a \neq \emptyset$ and $P(a|X, \lambda) = 1 - \delta_{X X'}$ if $a = \emptyset$. On the other hand, Bob always gives an output different from $\emptyset$. He randomly chooses a result $b$ using the probability distribution $P(b|Y, \lambda) = P^{QM}(a', b|X', Y)/P^{QM}(a'|X')$.

So far, Alice’s efficiency $\eta_A$ is equal to $1/M_A$ and Bob’s efficiency $\eta_B = 1$. To make the protocol symmetric, Alice and Bob must exchange their role part of the time. This is done with the help of a supplementary hidden variable which tells both parties to run the protocol as above with probability $p$ and the permuted one with probability $1-p$. There is then one problem left with the model, it never happens that both detector don’t fire. This can be corrected by adding yet another supplementary LHV that instruct Alice’s and Bob’s detectors to both produce the result $\emptyset$ with probability $(1-q)$ and to proceed as above with probability $q$. Using (3), it is then not difficult to check that our model produces the following correlations:

$$
P^{LHV}(a, b|X, Y) = q \left( \frac{p}{M_A} + \frac{1-p}{M_B} \right) P^{QM}(a, b|X, Y),$$

$$
P^{LHV}(\emptyset, b|X, Y) = q \frac{M_A-1}{M_A} P^{QM}(b|Y),$$

$$
P^{LHV}(a, \emptyset|X, Y) = q (1-p) \left( \frac{M_B-1}{M_B} \right) P^{QM}(a|X),$$

$$
P^{LHV}(\emptyset, \emptyset|X, Y) = 1-q.$$  

(5)

These correlations are similar to the quantum ones (2), modulo the detection probabilities, i.e. the probability that Alice’s and Bob’s, Alice’s only, Bob’s only or neither detector fire. The two distributions will be identical if these detection probabilities coincide:

$$
\eta^2 = p \left( \frac{q}{M_A} + \frac{1-q}{M_B} \right),
$$

$$
\eta(1-\eta) = q \frac{M_A-1}{M_A},
$$

$$
\eta(1-\eta) = p (1-q) \left( \frac{M_B-1}{M_B} \right),
$$

$$
(1-\eta)^2 = 1-p.$$  

(6)

Solving for $\eta$ gives the right-hand side of (4). □

When $M_A = M_B = 2$, the simplest non-trivial case, our bound predicts $\eta^* \geq 2/3$. It follows from Eberhard’s result that this value is optimal. Indeed Eberhard has shown that there exists a 2-settings Bell experiments performed on a non-maximally entangled state of two qubits.
that violate local realism for value of $\eta$ arbitrarily close to $2/3$. For larger values of $M_A$ and $M_B$, $\eta^*$ as given by (5) decreases and tends to zero when both $M_A$ and $M_B$ tends to infinity. It is not known whether our bound can be attained by quantum mechanics in these situations. However note that there are quantum correlations produced by experiments with exponentially many measurement settings, and for which $\eta^*$ is exponentially small [4]. It is thus at least possible to approach the bound (5) for large $M_A$, $M_B$.

We have attempted to generalise this result to the case of many parties. For simplicity we have considered the case where each party can choose between the same number $M$ of measurements.

We have only been able to prove our strongest result for less than 500 parties because we had to resort to numerical computations to finish the proof. We state it as a conjecture:

**Conjecture 2** (proven for $N \leq 500$): In a Bell experiment with $N$ parties, each of whose measuring apparatus can have $M$ settings,

$$\eta^* \geq \frac{N}{(N-1)M + 1}.$$  \hspace{1cm} (7)

When the number of measurements on each site is $M = 2$, the bound (5) reduces to

$$\eta^* \geq \frac{N}{2N - 1}.$$  \hspace{1cm} (8)

For two parties, we recover Eberhard threshold $\eta^* \geq 2/3$ and as we have already mentioned this bound can be saturated by quantum mechanics. However, the threshold (8) can be saturated by quantum mechanics for the other values of $N$ as well. Indeed Larsson and Semitecolos [4] have generalized Eberhard’s result to the case of many parties and have shown that $N$ qubits in a non-maximally entangled state can lead to violation of local realism for detection efficiencies $\eta$ arbitrarily close to (5) for any $N$.

For number of measurements settings $M > 2$, it is not known whether the bound (5) can be saturated. However one can come close to saturating it when the number of parties is large. Indeed for large $N$, fixed $M$, eq. (7) becomes $\eta^* \geq 1/M + O(1/N)$. And in (10) it is shown that there exists a measurement scenario for $M = 2^l$ $(l = 1, 2, \ldots)$ settings performed on $N$ qubits that exhibit non-locality for value of $\eta$ approaching $1/M$ as $N \to \infty$ for fixed $l$.

As a final remark, note that our conjecture seems quite constraining as regards the possible decrease of $\eta^*$ by increasing the number of parties. Indeed, for fixed $M$, replacing $N = 2$ by $N \to \infty$ one can expect at best a decrease of $\eta^*$ by a factor of $2M/(M + 1) \leq 2$. From the resistance to detection inefficiency point of view, it seems thus more advantageous to consider experiments with many settings than with many parties.

As mentioned above we have not been able to prove eq. (5) for all numbers of parties. However we have been able to prove a weaker result valid for any number $N$ of parties. In this weaker result we do not ask the LHV model to reproduce all the quantum correlations. Rather we only ask that if all the detectors click, then the correlations exactly coincide with the quantum correlations.

On the other hand we do not put any constraint on the correlations when one or more of the detectors do not click. This type of model has been considered previously in [5, 10].

**Theorem 3**: Consider Bell experiments with $N$ parties and $M$ measurements settings per site. We require that if all detectors click, the correlations should coincide with the quantum correlations, but we do not put any condition on the correlations when one or more of the detectors do not click. Then the maximum detection efficiency $\eta^*$ for which a LHV model exists satisfies

$$\eta^* \geq \frac{1}{M(N-1)/N}$$  \hspace{1cm} (9)

We begin by proving Theorem 3. We then turn to the arguments behind Conjecture 2.

**Proof of Theorem 3**: As in Theorem 1, we can build a LHV model to reproduce the correlations based on the remark that it is possible to predict outcomes for all measurements performed at one site if measurements are guessed at the other sites. A LHV will thus predetermine particular measurements and corresponding outcomes for $N - 1$ of the parties. If the guessed and the actual measurements coincide which happens with probability $1/M$, these parties output the selected result, if not, which happens with probability $(M - 1)/M$ their detectors keep quiet. Assuming that the measurements performed by the other parties are the ones specified by the hidden variable, the last party always output a result different from 0. Since each party has the choice between the same number $M$ of measurements there is no privileged site and each party has the same probability $1/N$ to be selected as the special one for which the detector always fire.

Thus when all detectors click, which occurs with probability $1/M(N-1)$, the results obtained will agree with those of quantum mechanics. This probability should be identified with $\eta^N$, the probability that all detectors click. This proves Theorem 3. □

We now turn to Conjecture 2.

**Proof of Conjecture 2 for $N \leq 500$**: The basic idea is to try to use the LHV model introduced in the proof of Theorem 3 to reproduce all the correlations, and not only the restricted one obtained when all detectors clicks.

Note that in the model introduced in the proof of Theorem 3, a detector clicks only if we are sure that it will output an answer that agrees with quantum mechanics. The only way for the LHV model and quantum mechanics to differ is thus in the probabilities that the detectors click, not in the correlations of outputs conditional on the firing of the detector. Similarly to (5), predictions
of quantum mechanics and the LHV model will therefore be identical provided they give the same detection probabilities \( q(k) \) that \( k \) given detectors don’t fire and the remaining \( N - k \) do. For quantum mechanics these probabilities are given by

\[
q^{QM}(k) = \eta^{N-k}(1-\eta)^k
\]

(10)

In particular this implies that the ratios

\[
\frac{q^{QM}(k)}{q^{QM}(k+1)} = \frac{\eta}{1-\eta}
\]

(11)

are independent of \( k \).

The LHV model introduced in Theorem 3 predicts the probabilities

\[
q^{LHV}(k) = \frac{N-k(M-1)^k}{N \cdot M^{N-1}}
\]

(12)

(see eq. (15) with \( i = 0 \) and the explanation in the paragraph following eq. (15)). It has thus the property that

\[
\frac{q^{LHV}(0)}{q^{LHV}(1)} = \frac{N}{(N-1)(M-1)}
\]

(13)

Using eq. (11) and solving for \( \eta \) yields eq. (7). This is the basis for Conjecture 2.

But from (12) we also deduce

\[
\frac{q^{LHV}(1)}{q^{LHV}(2)} > \frac{q^{LHV}(0)}{q^{LHV}(1)}
\]

(14)

in contradiction with (11). Furthermore the model introduced in Theorem 3 never instructs the \( N \) detector to keep quiet simultaneously.

We can try to correct the model so as to recover eq. (11) while leaving (13) unchanged, by increasing the probability \( q^{LHV}(k), k \geq 2 \) that more than one party does not fire.

A natural way to extend our protocol so that it can reproduce the whole set of correlations is to introduce the possibility for it to constrain \( i (i = 2, \ldots, N) \) of the parties to output \( \emptyset \), similarly to the proof of Theorem 1 where part of the time Alice and Bob had both to produce result \( \emptyset \).

The new LHV model will therefore be built out of a family of \( N \) protocols \( \mathcal{P}_i \) \((i = 0, 2, \ldots, N)\). In protocol \( \mathcal{P}_i \), a subset of \( i \) of the \( N \) parties is forced to output \( \emptyset \) independently of the measurement performed at these \( i \) sites. Since they are \( \binom{N}{i} \) possible choices of \( i \) parties among the \( N \), the probability that one particular subset is chosen is \( 1/\binom{N}{i} \). The protocol then works as before with \( N \) replaced by \( N - i \). The probabilities \( q^i(k) \) that \( k \) given detectors don’t fire and the remaining \( N - k \) do for protocol \( \mathcal{P}_i \) are given by

\[
q^i(k) = \begin{cases} 
0 & \text{if } k < i, \\
\binom{N}{i}^{-1} \frac{N-k(M-1)^{k-i}}{M^{N-i-1}} & \text{if } k \geq i \\
1 & \text{if } k \text{ and } i = N.
\end{cases}
\]

(15)

The first and the last case of (15) are trivial. Indeed, in our protocols at least \( i \) parties produce the result \( \emptyset \) so that their contribution to events where \( k < i \) parties don’t fire is null. On the other hand, the protocol \( \mathcal{P}_N \) always output \( \emptyset \) for the \( N \) parties. For the remaining case when \( k \geq i \) detectors don’t click, the subset of \( i \) parties that are forced to output \( \emptyset \) must certainly be included in the subset of the \( k \) parties that don’t click. Since they are \( \binom{k}{i} \) subset of the \( \binom{N}{k} \) possible that satisfy this condition we have the term \( \binom{k}{i}/\binom{N}{k} \). Secondly, the special party for which the detector always fire can not be one of the \( k \) not clicking. There thus remain only \( N - k \) possibilities over the \( N - i \) original one, hence the term \( (N-k)/(N-i) \). Finally, in the remaining \( N - i - 1 \) parties \( k - i \) of them must output \( \emptyset \), which happens with probability \( (M-1)^{k-i}/M^{N-i-1} \).

If the LHV model instructs to use protocol \( \mathcal{P}_i \) \((i = 0, 2, \ldots, N)\) with probability \( p_i \) we find

\[
q^{LHV}(k) = p_0 q^0(k) + \sum_{i=2}^{N} p_i q^i(k)
\]

(16)

since \( q^i(k) = 0 \) for \( i > k \).

As already stated above our model predict the correct probabilities conditional on the firing of the detectors. It will thus properly reproduce the quantum probabilities obtained in an experiment provided the detection probabilities satisfy \( q^{LHV}(k) = q^{QM}(k) \) or

\[
\frac{\eta^{N-k}(1-\eta)^k}{\eta^{N-k}(1-\eta)^k} = p_0 q^0(k) + \sum_{i=2}^{k} p_i q^i(k) \quad \text{for all } k.
\]

(17)

This will be the case if this set of equations for the \( p_i \) admits a solution such that the \( p_i \) are positive and sum to one, i.e. they form an actual probability distribution.

The fact that they sum to one is already implied by the structure of (17). Indeed summing both sides of (17) over all possible subset of parties for which the detector fire and don’t fire, we deduce that \( p_0 + \sum_{i=2}^{N} p_i = 1 \), since

\[
\sum_{k} \binom{N}{k} \eta^{N-k}(1-\eta)^k = \sum_{k} \binom{N}{k} q^k(k) = 1.
\]

To check whether the \( p_i \) are positive we use

\[
\frac{\eta}{1-\eta} = \frac{\eta^{0}(0)}{\eta^{0}(1)} = \frac{\sum_{i=0}^{k-1} p_i q^i(k-1)}{\sum_{i=0}^{k} p_i q^i(k)}
\]

(18)

to write

\[
p_k = \frac{1}{q^k(k)} \sum_{i=0}^{k-1} p_i \frac{q^i(1)}{q^i(0)} q^i(k-1) - q^i(k)
\]

(19)

This define recursively the \( p_i \) starting from \( p_0 = \text{cst} > 0 \) and \( p_1 = 0 \). Note that the \( p_i \) depends of \( N \) and \( M \). If we define \( r_k = M^k/(M-1)^k \) \( p_k \) we obtain for the \( r_k \) the...
recursive definition
\[ r_k = \frac{1}{q^k(k)} \sum_{i=0}^{k-1} r_i \left( \frac{q^i(1)}{q^i(0)} q^i(k - 1) - q^i(k) \right) \]

where \( q^i(k) = M^{N-i-1}/(M-1)^{k-i} \). Since the \( q^i(k) \) are independent of \( M \) so are the \( r_k \). If all the \( r_i \) are positive for given \( N \) it thus follows that all the \( p_i \) are also positive for that given \( N \) and for all values of \( M \). We checked this positivity condition for the \( r_i \) for \( N \leq 500 \) using a symbolic mathematics software (Mathematica) that performs exact computations (indeed, non-linear recursive equations as \( (20) \) are sensitive to small numerical perturbations and we didn’t find any stable method of solving \( (20) \) using finite precision arithmetics). This concludes the proof of Conjecture 2 for \( N \leq 500 \). □

IV. A LHV MODEL THAT APPROXIMATELY REPRODUCES THE QUANTUM CORRELATIONS FOR GIVEN DIMENSIONALITY

We now present a LHV model inspired by the communication protocol described in [11]. Though this model is probably not optimal, it shows that it is in principle possible to build LHV models that depend only on the dimension of the quantum system. In this model \( \eta \) decreases exponentially with \( d \). This behavior of \( \eta \) must be shared by all models that depends only on the dimension since in [7] it is shown that there are quantum correlations which are non local even when the detector efficiency is exponentially small in \( d \). Note however that the quantum correlations in [7] require an almost complete absence of noise to exhibit non-locality, whereas the model described below reproduces noisy correlations (although the amount of noise decreases with the dimension for fixed \( \eta \)).

Note: for simplicity of notation, in this section all the probabilities \( P^{LHV} \) or \( P^{QM} \) we compute or refer to are probabilities conditional on the firing of both the detectors.

**Theorem 4:** For measurements performed on the maximally entangled state \( |\Phi\rangle = \sum_{i=0}^{d-1} \frac{1}{\sqrt{d}} |ii\rangle \) and for given \( \epsilon < 2d \), there exists a LHV model that produces a probability distribution \( P^{LHV}(a,b|X,Y) \) such that for all \( X,Y \), \( P^{LHV}(a|X) = P^{QM}(a|X) \), \( P^{LHV}(b|Y) = P^{QM}(b|Y) \) and \( |P^{LHV}(a,b|X,Y) - P^{QM}(a,b|X,Y)| \leq \epsilon P^{QM}(a|X)P^{QM}(b|Y) \) when the efficiency of the detectors is

\[ \eta = (\frac{\epsilon}{4d})^{2(d-1)} \]  

**Proof:** We recall that Alice and Bob carry out the POVM’s \( X \) and \( Y \) with elements \( x_a \) and \( y_b \). Without loss of generality we can suppose that \( x_a \) and \( y_b \) are rank one [12]. We rewrite them as

\[ x_a = |x_a\rangle \langle x_a| \quad , \quad y_b = |y_b\rangle \langle y_b| \]

where \( |x_a\rangle, |y_b\rangle \) are normalized states. In the case of the maximally entangled state, the marginals and the joint outcome probability are

\[
P^{QM}(a|X) = \frac{|x_a|}{d} \quad , \quad P^{QM}(b|Y) = \frac{|y_b|}{d}
\]

\[
P^{QM}(a,b|X,Y) = \frac{1}{d} |x_a| |y_b| \langle x_a^* | y_b \rangle^2
\]

where \( |x_a^* \rangle = \sum_i x_i^a |i\rangle \) with \( x_i^a \) the components of \( |x_a\rangle \) in the basis where \( |\Phi\rangle = \frac{1}{\sqrt{d}} \sum_i |ii\rangle \).

The local hidden variable consists of the classical description of a pure quantum state \( |\phi\rangle \). This state is uniformly chosen in the Hilbert space using the invariant measure over \( SU(d) \). Alice’s strategy is the following: she first chooses a probability \( |x_a|/d \), in agreement with the marginal probability \( P^{QM}(a|X) \). Having fixed \( a \) she then computes \( s = |\langle \phi | x_a \rangle|^2 \). If \( s < \cos^2 \delta \), she outputs “no result”. If \( s \geq \cos^2 \delta \), she outputs \( a \) (where \( \delta > 0 \) will be fixed below). The probability \( Q \) for Alice to give an outcome is

\[
Q = \int_{SU(d)} d\phi \ \Theta( |\langle \phi | x_a \rangle|^2 - \cos^2 \delta)
\]

To compute this expression we write \( |\phi\rangle = \cos \theta |x_a \rangle + e^{i\phi} \sin \theta |\phi_{d-1}\rangle \) where \( |\phi_{d-1}\rangle \) lies in the subspace orthogonal to \( |x_a\rangle \). Since \( d\phi = \frac{4}{\pi} \cos \theta \sin \theta \cos^2 \theta - \cos \delta \ d\theta \ d\phi \ d\phi_{d-1} \) we find

\[
Q = 2(d-1) \int_{0}^{\pi/2} d\theta \cos \theta \sin \theta \cos^2 \theta - \cos \delta \ = (\sin \delta)^{2(d-1)}.
\]

As expected, the probability to give an outcome is independent of Alice’s particular result \( a \).

Bob’s strategy is as follows: he gives output \( b \) with probability

\[
P(b|Y, \phi) = |y_b| |\langle \phi^* | y_b \rangle|^2.
\]

This results in the marginal probability

\[
P^{LHV}(b|Y) = \int_{SU(d)} d\phi P(b|Y, \phi) = 2(d-1) |y_b| \int_{0}^{\pi/2} d\theta \cos^2 \theta \sin \theta \cos^2 \theta - \cos \delta
\]

\[
= \frac{|y_b|^2}{d}
\]

where we have taken \( |\phi\rangle = \cos \theta |y_b^* \rangle + e^{i\phi} \sin \theta |\phi_{d-1}\rangle \) and \( |\phi_{d-1}\rangle \) orthogonal to \( |y_b\rangle \) to pass from the first line to the second one.

Let us now compute the joint probability of outcomes \( a \) and \( b \) given that an outcome has been produced. This
is

\[
P^{LHV}(a,b|X,Y) = \frac{1}{Q} \int_{SU(d)} d\phi \; P(a|X,\phi)P(b|Y,\phi)
\]

\[
= \frac{1}{Q} \int_{SU(d)} d\phi \; \frac{|x_a|}{d} \Theta(|\langle x_a |^2 - \cos^2 \delta||y_b||\langle \phi^*|y_b \rangle|^2
\]

(27)

To compute how much this differs from the true probability, let us evaluate

\[
D = |\langle \phi^*|y_b \rangle|^2 - |\langle x_a^*|y_b \rangle|^2.
\]

(28)

Writing \(|\phi\rangle = \cos \theta |x_a\rangle + e^{i\varphi} \sin \theta |\phi_{d-1}\rangle\) where \(\langle x_a|\phi_{d-1}\rangle = 0\) we find

\[
|D| = | - \sin^2 \theta |\langle x_a^*|y_b \rangle|^2 + \sin^2 \theta |\langle \phi_{d-1}^*|y_b \rangle|^2 + (\sin \theta \cos \theta |\langle x_a|y_b \rangle \langle \phi_{d-1}^*|y_b \rangle + c.c. |)
\]

\[
\leq \sin^2 \theta + 2 \sin \theta .
\]

(29)

From which we deduce

\[
|P^{LHV}(a,b|X,Y) - P^{QM}(a,b|X,Y)|
\]

\[
\leq \frac{1}{Q} \frac{|x_a|}{d} |y_b|2(d-1) \int_0^{\pi/2} d\theta \cos \theta (\sin^2 \theta)^{2d-3}
\]

\[
\times \Theta(\cos^2 \theta - \cos^2 \delta) |(\sin^2 \theta + 2 \sin \theta)
\]

\[
\leq \frac{1}{Q} \frac{|x_a|}{d} |y_b|2(d-1)(\sin^2 \delta + 2 \sin \delta)
\]

\[
\times \int_0^\delta d\theta \cos \theta (\sin^2 \theta)^{2d-3}
\]

\[
= \frac{1}{d} |x_a| |y_b| (\sin^2 \delta + 2 \sin \delta) = \epsilon P(a|X)P(b|Y)
\]

where we have taken \(\epsilon = d(\sin^2 \delta + 2 \sin \delta)\).

In the above protocol the roles of Alice and Bob are not symmetric and it never happens that both detectors don’t fire. Upon letting them take randomly one of the two roles above and forcing both detectors to stay quiet part of the times, as in the previous LHV models, one sees that the model we have constructed has detector efficiency \(\eta/(1-\eta) = 2Q/(1-Q)\) or

\[
\eta = \frac{2(\sin \delta)^{2d-1}}{1 + (\sin \delta)^{2d-1}} \geq \sin \delta^{2(d-1)} \geq \left(\frac{\epsilon}{4d}\right)^{2(d-1)}
\]

(30)

since \(\sin \delta \geq 1/2 - \epsilon^2/8d^2 \geq \epsilon/4d\) when \((\epsilon < 2d)\). \(\Box\)

V. CONCLUSION

We have exhibited LHV models that depend only on the dimensionality of the quantum system or only on the number of settings of each party’s measurement apparatus. These models show that there exist general constraints on the violation of local realism independently of the particular settings of Bell experiments. They help point out which parameters are important when trying to find quantum experiments that exhibit strong non locality. For instance the existence of these LHV models served as a guiding principle for a recent numerical search that yielded several Bell inequalities resistant to detection inefficiency [13].

Our models can also have implications in the design of loophole-free tests of Bell inequalities. Loophole-free tests of Bell inequalities are important both from a fundamental point of view and for the security of some quantum cryptographic protocols [14]. In experiments involving photons the detection loophole remain the last serious loophole to be closed. It would therefore be interesting to find Bell scenarios that violate local realism for efficiencies of the detectors close to the actual value of our current photo-detectors. Our result shows that to go beyond Eberhard’s threshold of \(\eta' \geq 2/3\) (or to go beyond Larsson and Semitecolos’s threshold for many parties) it is necessary to consider Bell experiments with more than two measurements per site. This strengthens the recent interest in Bell inequalities involving many measurements settings which have been shown to exhibit more strongly the non-locality of quantum mechanics than usual Bell inequalities based on two settings [13,14,16].

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