Nonlinear evolution of the $m = 1$ internal kink mode in the presence of magnetohydrodynamic turbulence

Andreas Bierwage, Sadruddin Benkadda, Satoshi Hamaguchi, and Masahiro Wakatani

Graduate School of Energy Science, Kyoto University, Gokasho, Uji, Kyoto 611-0011, Japan
Équipe Dynamique des Systèmes Complexes, UMR 6633 CNRS-Université de Provence, 13397 Marseille, France
Center for Atomic and Molecular Technologies, Osaka University, 2-1 Yamadaoka, Suita, Osaka 565-0871, Japan

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The nonlinear evolution of the $m = 1$ internal kink mode is studied numerically in a setting where the tokamak core plasma is surrounded by a turbulent region with low magnetic shear. As a starting point we choose configurations with three nearby $q = 1$ surfaces where triple tearing modes (TTMs) with high poloidal mode numbers $m$ are unstable. While the amplitudes are still small, the fast growing high-$m$ TTMs enhance the growth of the $m = 1$ instability. This is interpreted as a fast sawtooth trigger mechanism. The TTMs lead to a partial collapse, leaving behind a turbulent belt with $q \approx 1$ around the unconnected core plasma. Although, full reconnection can occur if the core displacement grows large enough, it is shown that the turbulence may actively prevent further reconnection. This is qualitatively similar to experimentally observed partial sawtooth crashes due to a saturated internal kink.

I. INTRODUCTION

Abrupt ejection of thermal energy and particles from a magnetized high-temperature plasma is frequently observed in astrophysical and laboratory plasmas. In the case of tokamak experiments, internal disruption events known as sawteeth bring about a sudden collapse of the core temperature through an internal kink instability. A thorough understanding of the underlying physical processes is important for the efficient operation of a tokamak device and the prospective application of the tokamak concept for thermonuclear fusion reactors.

The aim of this paper is to demonstrate the effect of magnetohydrodynamic (MHD) turbulence on the evolution of the $m = 1$ internal kink mode. Motivated by recent results on the instability of current-driven high-$m$ multiple tearing modes \[ 1, 2 \] we choose configurations with three $q = 1$ resonant surfaces located a small distance apart. Here, $q$ is the tokamak safety factor (measuring the field line pitch) and its central value is taken to be well below unity ($q_0 < 1$). The plasma is taken to have a finite resistivity to enable magnetic reconnection. This system is, in addition to the resistive $m = 1$ internal kink mode, unstable to a broad spectrum of triple tearing modes (TTMs) with helicity $q_{\text{res}} = m/n = 1$. At present, we neglect two-fluid and kinetic effects and also ignore the roles of finite pressure and toroidal curvature.

In this setting we address some open questions with regard to internal disruptions in tokamaks \[ 3, 4, 5, 6 \]. These include the issue of the rapid onset of a sawtooth crash, known as the trigger problem, and the possibility of partial reconnection and compound sawtooth crashes (e.g., \[ 7, 8, 9 \]). The fast sawtooth trigger is defined as a sudden transition from slow growth or stability to rapid growth of the $m = 1$ mode (e.g., \[ 10, 11 \]). The partial collapse is defined as a sawtooth crash during which the central core region remains intact, so that $q_0$ remains below unity (e.g., \[ 12, 13 \]). It is clear that explanations for the sawtooth trigger and partial reconnection events require nonlinear effects. Several possible mechanisms have been proposed in the past (see, e.g., Ref. \[ 14 \] for a review). Of particular interest here are scenarios that consider dynamics related to plasma turbulence.

In regimes where the $m = 1$ mode amplitude is still small, high-$m$ modes were previously shown to affect the growth of the $m = 1$ instability. Micro-turbulence \[ 14, 15 \] (oscillating modes) and TTMs \[ 11 \] (purely growing modes) were found to enhance the $m = 1$ growth rate. In other related studies it was shown that the $m = 1$ mode can be stabilized by local oscillations at the resonant surface \[ 16, 17 \] and micro-turbulence in the region of the thin current layer may lead to enhanced effective resistivity and thus faster reconnection \[ 18, 19, 20, 21 \]. Conversely, viscosity may be increased, which reduces the reconnection rate \[ 22 \]. In a broader sense, magnetic braiding and field line stochasticity may also be viewed as "turbulent" dynamics, and these were also shown to yield enhanced growth rates for reconnecting modes \[ 23, 24 \]. With regard to the long-time nonlinear evolution, pressure-driven instabilities (e.g., ballooning) were shown to be able to lead to a saturation of the internal kink at finite amplitude (partial collapse) \[ 25, 26 \].

The scenario considered here is comparatively simple and includes only a minimum of physical effects. In the first part of this paper the case of a TTM-driven internal kink mode \[ 4 \] is examined and two new results are presented: establishment of a fast growing $m = 1$ mode structure at low amplitudes and, in regimes with high viscosity ($Pr \gtrsim 10$), explosive growth during the transition.
The transition to turbulence occurs via a partial (an-
ular) collapse, whereupon a turbulent belt forms around
the central core. Indeed, in Ref. [28] the conjecture was
made that partial sawtooth crashes may be associated
with wide-spread MHD turbulence in the reconnected re-
gion. In Ref. [1] the possibility for the \( m = 1 \) mode to
saturate in such a state was demonstrated using a nu-
erical simulation where the core was constrained to a
linear motion in the poloidal plane.

The second part of this paper deals with the long-term
evolution of the internal kink, while surrounded by a low-
shear region with \( q \approx 1 \), governed by MHD tur-
bulence. The scenario considered here is more generic than
that in Ref. [1] since we apply a fully random pertur-
bation and allow each Fourier mode to alter its poloidal phase
angle through interaction with other modes. The resulting
changes in the kink flow lead to an irregular or “mea-
dering” motion of the core. One case is presented where
full reconnection occurs eventually. In another case, the
kink is found to saturate and decay, which shows that
the MHD turbulence may prevent full reconnection.

A reduced model is used for the simulations in order to
obtain first insights on a fundamental level and to lay
the foundations for further investigations with more realistic
models. Physical effects ignored in this work, such as
two-fluid, curvature, and finite-beta effects, may play a
significant role. In future work it would be interesting to
see under which conditions and in which way these effects
alter certain quantitative and qualitative features of the
results presented here, including the linear and nonlinear
instability growth, the dominant mode numbers in the
inter-resonance region, and the evolution of the internal
kink.

This paper is organized as follows. The physical model
is introduced in Sec. II and the numerical method is de-
scribed in Sec. III. The equilibrium used, its linear insta-
bility characteristics and the initial perturbation applied
are given in Sec. IV. In Sec. V we present results on
the early evolution of the \( m = 1 \) mode in the presence
of fast growing high-\( m \) TTM. The transition to tur-
bulence is treated in Sec. VI and the long-term evolu-
tion is described in Sec. VII. A summary, further discus-
sions and conclusions are given in Section VIII.

II. MODEL

We use the reduced set of magnetohydrodynamic
(RMHD) equations in a cylindrical geometry in the limit
of zero beta \( \beta \). This model has proven to be use-
ful in studies of MHD instabilities when the focus is on
a qualitative description of fundamental aspects of the
magnetized plasma system, as is the case here. The
RMHD model governs the evolution of the magnetic flux
function \( \psi \) and the electrostatic potential \( \phi \), as described
in Ref. [2]. The normalized RMHD equations are

\[
\begin{align*}
\partial_t \phi &= [\psi, \phi] - \partial_r \phi - S_{\psi,\phi}^{-1} (\hat{\eta} j - E_0) \\
\partial_t u &= [u, \phi] + [j, \psi] + \partial_r \phi + R_{\psi,\phi}^{-1} \nabla^2 u.
\end{align*}
\]  

(1)

(2)

The time is measured in units of the poloidal Alfvén time
\( \tau_{\psi,\phi} = \sqrt{\mu_0 \rho_{m} a / B_0} \) and the radial coordinate is normalized
by the minor radius \( a \) of the plasma. Here, \( \rho_{m} \)
is the mass density and \( B_0 \) the strong axial magnetic field.

The current density \( j \) and the vorticity \( u \) are related to
\( \psi \) and \( \phi \) through \( j = -\nabla^2 \psi \) and \( u = \nabla^2 \phi \), respectively.

In order to provide a simple mechanism for mag-
etic reconnection, a resistive diffusion term is included
in Eq. (1). Its strength is measured by the magnetic
Reynolds number \( S_{\psi,\phi}^{-1} \), with \( \tau_0 = a^2 \mu_0 / \rho_{\phi} \) be-
ing the resistive diffusion time and \( \eta_0 = \eta(r = 0) \) the
electrical resistivity in the plasma core. We use \( S_{\psi,\phi} = 10^6 \),
which is numerically efficient and physically reasonable
in the framework of the model used. Flow damping is
provided by an ion viscosity term in Eq. (2). Viscous dis-
sipation is measured by the kinematic Reynolds number
\( \nu_{\psi,\phi} = a^2 / \nu_\psi \), where \( \nu \) is the ion viscosity. Long-time calculations are performed for \( \nu_{\psi,\phi} = 10^6 \) and \( 10^8 \), as
will be specified case by case.

In order to ensure that, in the absence of magnetic
reconnection, the equilibrium remains unchanged, the
source term \( S_{\psi,\phi}^{-1} E_0 \) is included in Eq. (1). With \( E_0 = \hat{\eta} f \)
it compensates the resistive dissipation of the equilib-
rium current. The loop voltage measured by \( E_0 \) is
taken to be constant, so the resistivity profile is given
in terms of the equilibrium current density distribution
as \( \hat{\eta}(r) = \bar{J}(r = 0) / \bar{J}(r) \). For simplicity, the temporal
variation of the resistivity profile \( \hat{\eta} \) is neglected.

As in Ref. [2], each field variable \( f \) is decomposed into
an equilibrium part \( \bar{f} \) and a perturbation \( \delta f \) as

\[
f(r, \vartheta, \zeta, t) = \bar{f}(r, \vartheta, \zeta, t) + \delta f(r, \vartheta, \zeta, t).
\]  

(3)

The system is described in terms of the Fourier modes,
\( \psi_{m,n} \) and \( \phi_{m,n} \), obtained from the expansion

\[
f(r, \vartheta, \zeta, t) = \frac{1}{2} \sum_{m,n} f_{m,n}(r, t) e^{i[m\vartheta - n\zeta]} + c.c.,
\]  

(4)

with \( m \) being the poloidal mode number and \( n \) the
toroidal mode number. In the following, the \( (m, n) \)
subscripts will often be omitted for convenience. We con-
sider only the dynamics within a given helicity \( h = m/n \)
const, so the problem is reduced to two dimensions.

For the description of the dynamics in this system,
it is useful to define the helical flux function \( \psi_\ast \) with
corresponding current density \( j_\ast \) as

\[
\psi_\ast = \psi + \frac{n}{2m} r^2 \quad \text{and} \quad j_\ast = -\nabla^2 \psi_\ast = j - \frac{2n}{m},
\]  

(5)

The evolution of the individual Fourier modes is de-
scribed in terms of their kinetic and magnetic energies,

\[
E_{m,n}^{\text{kin}} = |\nabla \phi_{m,n}|^2 \quad \text{and} \quad E_{m,n}^{\text{mag}} = |\nabla \psi_{m,n}|^2.
\]  

(6)
and the corresponding nonlinear growth rates,
\[ \gamma_{m,n}^{\text{kin}}(t) = \frac{d \ln E_{m,n}^{\text{kin}}}{2dt} \quad \text{and} \quad \gamma_{m,n}^{\text{mag}}(t) = \frac{d \ln E_{m,n}^{\text{mag}}}{2dt} \]  
(7)
(these are amplitude growth rates, hence the factor 1/2).

In Eq. 6, \[ |f_{m,n}|^2 \equiv \int_0^1 dr r C_m |f_{m,n}(r)|^2, \]
with \( C_m = 4\pi \) and \( C_{m\neq0} = 2\pi \).

III. NUMERICAL METHOD

For the numerical solution of the model equations (1) and (2), a two-step predictor-corrector method is applied. In the first time step the dissipation terms are treated implicitly, all others explicitly, and the field variables are estimated at an intermediate time step \( t + \Delta t/2 \). The second is a full time step, \( t \to t + \Delta t \), with the right-hand sides of Eqs. (1) and (2) evaluated at the intermediate time step \( t + \Delta t/2 \) estimated before. In the nonlinear regime the time step size is of the order \( \Delta t \sim 10^{-3} \).

128 Fourier modes (including \( m = 0 \)) are carried. The Poisson brackets \[ [f, g] = \frac{1}{r} \partial_r f \partial_\rho g - \partial_r g \partial_\rho f \]
are evaluated in real space. This pseudo-spectral method has been applied together with an appropriate dealiasing technique. The outcomes of the long-term evolution in both cases studied, full reconnection and kink saturation, were also confirmed using 256 Fourier modes.

The radial coordinate is discretized using a non-uniformly spaced grid, with a grid density of up to \( N_r^{-1} \approx 1/2000 \) [1/5000 for numerical checks] in regions where sharp current density peaks occur. A fourth-order centered-finite-difference method is applied for the \( \partial_r \)-terms in the Poisson brackets. The Laplacians \( \nabla^2_{(m,n)} = \frac{1}{r} \partial_r r \partial_r - m^2/r^2 \) are evaluated at second-order accuracy (tridiagonal matrix equations).

Periodic boundary conditions are applied in the azimuthal and axial directions. At \( r = 1 \) an ideally conducting wall is assumed, requiring all perturbations to be identical to zero at that location: \( f(r = 1) = 0 \) (fixed boundary, no vacuum region). At \( r = 0 \), extraneous boundary conditions are applied to ensure smoothness: \( \partial_r \tilde{f}_{m=0}(r = 0) = 0 \) and \( \tilde{f}_{m\neq0}(r = 0) = 0 \).

IV. EQUILIBRIUM, LINEAR INSTABILITY AND INITIAL PERTURBATION

The equilibrium state is taken to be axisymmetric (only \( m = n = 0 \) components) and free of flows, i.e.,
\[ \bar{\phi} = \Pi = 0. \]
(8)

The equilibrium magnetic configuration is uniquely defined in terms of the safety factor \( q(r) \), and the magnetic

| Case   | \( q_0 \) | \( r_A \) | \( \mu_0 \) | \( \mu_1 \) | \( n \) | \( f_1 \) | \( \tau_{11} \) | \( \tau_{12} \) |
|--------|----------|----------|----------|----------|------|--------|----------|----------|
| (T-1)  | 0.73     | 0.455    | 0.93     | 1.45     | 1    | -0.09  | 0.5406   | 0.039    |
| (T-2)  | 0.73     | 0.455    | 0.93     | 1.45     | 1    | -0.098 | 0.5666   | 0.064    |

TABLE I: Parameter values for the \( q \) profiles shown in Fig. 1 using model formula (11) in Ref. 2.
faces with \( q_s \equiv q(r_{s1}) = m/n = 1 \), which are located at radii \( r_{s1} \) \((i = 1, 2, 3)\), are considered. The distances between the resonances, \( D_{ij} = |r_{s1} - r_{s2}| \), are chosen sufficiently small, so that the spectra of unstable TTMs are broad and the dominant modes have \( m \sim \mathcal{O}(10) \). The equilibrium \( q(r) \) profiles used are shown in Fig. 1. They are obtained using the model formula given by Eq. (11) in Ref. [2] with the parameters in Table I. The two cases studied are labeled (T-1) and (T-2).

The dispersion relations (spectra of linear growth rates) \( \gamma_{\text{lin}}(m) \) for all unstable eigenmodes are given in Fig. 2 for Case (T-1) and in Fig. 3 for Case (T-2). The linear eigenmode structures for Case (T-1) were shown in Ref. [1] and are similar for Case (T-2). Let us recall that eigenmode \( M^{(1)} \) (with growth rate \( \gamma_{\text{lin}}^{(1)} \)) is associated with the resonant surface \( r = r_{s1} \) and is an ordinary (single) \( m = 1 \) internal kink-tearing mode (stable for \( m > 1 \)). \( M^{(2)} \) (with \( \gamma_{\text{lin}}^{(2)} \)) is associated with \( r_{s2} \) in the sense that it is active in the region \( 0 < r < r_{s2} \) for \( m = 1 \) and \( r_{s1} < r < r_{s2} \) for \( m > 1 \). Thus, it may be regarded as a double tearing mode (DTM). Finally, \( M^{(3)} \) (with \( \gamma_{\text{lin}}^{(3)} \)) is associated with \( r_{s3} \) and may be regarded as the actual TTM eigenmode. In Figs. 2 and 3 the dominant eigenmodes and the \( m = 1 \) mode are indicated by arrows labeled with the corresponding mode numbers.

The characteristics of the equilibrium configurations are summarized in Table II including the values of the resistivity profile \( \eta \) at the resonances and the poloidal mode number \( m_{\text{peak}} \) of the dominant TTM eigenmode obtained from Figs. 2 and 3. We will see that the value of \( m_{\text{peak}} \) is an important clue for the interpretation of the nonlinear dynamics since it dictates a structure size that is most likely to be encountered in the poloidal direction.

Starting from an unstable equilibrium, the instability is excited by applying an initial perturbation of the form

\[
\tilde{\psi}(t = 0) = \frac{1}{2} \sum_{m=1}^{31} \Psi_{0,m} r(r-1)e^{im(\vartheta + \vartheta_{0,m})} + \text{c.c.,} \quad (10)
\]

where \( \Psi_{0,m} = 10^{-11} \) is the perturbation amplitude and \( \vartheta_s = \vartheta - q_s^{-1} \zeta \) is a helical angle coordinate. Each mode has an initial poloidal phase shift \( \vartheta_{0,m} \) with a randomly assigned value in the range \( \vartheta_{0,m} \in [0, \pi] \).

**Fig. 3:** (Color online). Spectra \( \gamma_{\text{lin}}(m) \) of unstable eigenmodes in Case (T-2) for \( S_{Hp} = 10^6 \) and \( Re_{Hp} = 10^6 \). The growth rates \( \gamma_{\text{lin}}^{(1)}, \gamma_{\text{lin}}^{(2)} \) and \( \gamma_{\text{lin}}^{(3)} \) of the three eigenmodes \( M^{(1)} \), \( M^{(2)} \) and \( M^{(3)} \) (cf. Fig. 4 in Ref. [2]) are shown.

**Fig. 4:** (Color online). Early evolution in Case (T-1). (a): Kinetic energies \( E_{m,n}^{lin} \) [Eq. (8)] of the \( m = 1 \) mode and the two fastest growing modes \( m = 13 \) and \( m = 14 \). (b) and (c): Magnetic and kinetic growth rates \( \gamma_{m,n}^{\text{mag}}, \gamma_{m,n}^{\text{kin}} \) [Eq. (9)]. The main stages to be distinguished are: (i) linear growth, (ii) nonlinearly driven growth, (iii) transition to the fully nonlinear (turbulent) regime. The value of the driven growth rate expected during stage (ii) is \( \gamma_{\text{drive}} \approx 2\gamma_{\text{lin}}(m_{\text{peak}}) \approx 16 \times 10^{-3} \), as indicated in (b). \( S_{Hp} = 10^6 \), \( Re_{Hp} = 10^6 \).

**Table II:** Properties of the \( q \) profiles shown in Fig. 1. The mode number of the fastest growing mode, \( m_{\text{peak}} \) (cf. Figs. 2 and 3), is valid for \( S_{Hp} = 10^6 \) and \( Re_{Hp} = 10^6 \) [Case (T-1)], \( Re_{Hp} = 10^6 \) [Case (T-2)].
V. EARLY DYNAMICS: ESTABLISHMENT OF NONLINEAR MODE STRUCTURE AND FAST GROWTH OF THE $m = 1$ MODE

Due to the disparate growth rates between the $m = 1$ mode and the fastest growing TTM’s [here, $\gamma_{\text{peak}} \sim 5 \times \gamma_{\text{lin}}(m = 1)$; cf. Figs. 2 and 3] nonlinear interactions begin already in regimes where mode amplitudes are still small. In this section we focus on these early stages of evolution where turbulence has not yet developed. They are most conveniently studied by considering the dynamics of individual Fourier modes. Results are presented only for Case (T-1), using $S_{\text{H}} = 10^6$ and $Re_{\text{H}} = 10^6$. Similar behavior is observed in Case (T-2).

Figure 2(a) shows the evolution of the kinetic energies of the $m = 1$ mode and the two fastest growing modes $m_{\text{peak}} = 13$ and $m_{\text{peak}} + 1 = 14$ (cf. Fig. 3). The corresponding growth rates are shown in Fig. 2(b) and (c). It can be seen that the $m = 13$ and $m = 14$ modes grow at their linear growth rates until $t \approx 170$ and saturate during $170 \leq t < 200$. The $m = 1$ mode grows linearly until $t \approx 100$ [phase (i) in Fig. 2]. Subsequently, its growth rate increases and exponential growth continues at an enhanced rate during $100 \leq t \leq 170$ [phase (ii)]. Upon entering the nonlinear regime, the $m = 1$ growth rate drops [phase (iii)].

The enhanced growth during phase (ii) is due to nonlinear driving by the fastest growing modes, predominantly $m_{\text{peak}} = 13$ and $m_{\text{peak}} = 14$. Since $\gamma_{\text{lin}}(m_{\text{peak}}) \approx 170 \approx 2 \gamma_{\text{lin}}(m_{\text{peak}} + 1)$ the driven growth rate of the $m = 1$ mode is $\gamma_{\text{drive}} \approx 2 \gamma_{\text{lin}}(m_{\text{peak}}) \approx 16 \times 10^{-2}$. The purely nonlinearly driven $m = 0$ mode grows at the same rate, as can be seen in Fig. 2(b) [phases (i) and (ii)]. Note that in the present case $\gamma_{\text{drive}}$ this is almost one order of magnitude higher than the linear growth rate of the $m = 1$ mode, $\gamma_{\text{lin}}(m = 1) = 1.7 \times 10^{-2}$.

The effect of the nonlinear driving on the $m = 1$ mode structure can be seen in Fig. 4. In Fig. 4(a) the linear mode structure of the TTM-type eigenmode is plotted, specifically the flux function $\psi_{(3)}(m = 1)$. The shapes of the driving terms arising from the convective nonlinearity [Fig. 1] are plotted in Fig. 4(b) and (c). In Fig. 4(d) it can be seen how this radially localized driving appears in the nonlinear $m = 1$ mode structure $\psi_{\text{drive}}(m = 1)$ in the form of “spikes” located near the resonant surfaces. It is particularly remarkable that, after a certain ratio between the amplitudes of the driving component (around $r_{s3}$) and the component remaining from the linear eigenmode structure (in the region $0 < r < r_{s1}$) is reached, the mode structure does not change further [Fig. 4(d), $t = 140–160$] and grows as a whole at the enhanced rate $\gamma_{\text{drive}}$. Thus, a new fast growing global $m = 1$ mode is created through radially localized driving. Comparisons made with other simulation runs indicate that the relative size of the driving component and the peak in the region $0 < r < r_{s1}$ in the established nonlinear mode structure depends on the ratio between $\gamma_{\text{drive}}$ and $\gamma_{\text{lin}}(m = 1)$.

VI. BEHAVIOR OF THE $m = 1$ MODE DURING THE TRANSITION TO TURBULENCE

Let us now consider the transition to the fully nonlinear, turbulent regime. In Fig. it this transition takes place via a gradual decrease in the growth rates during phase (iii). However, it turns out that this is only the case in regimes where the effect of viscosity is negligible. In Fig. 3 the evolution of the nonlinear growth rates $\gamma_{1,1,1}^{\text{mag}}$, $\gamma_{1,1,1}^{\text{kin}}$ and $\gamma_{0,0,0}^{\text{mag}}$ is shown for magnetic Reynolds numbers $S_{\text{H}} = 10^6$, $10^7$, $10^8$ (from left to right) and the kinematic Reynolds numbers $Re_{\text{H}} = 10^5, 10^6, 10^7$ (from top to bottom). Thus, the Prandtl number $Pr$, which measures the relative strengths of viscosity and (resistive) diffusion ($Pr = S_{\text{H}} / Re_{\text{H}} \approx \nu / \eta_0$) varies over the range $10^{-2} \leq Pr \leq 10^3$ (bottom left to top right). The growth rate of the $m = 0$ mode, $\gamma_{0,0,0}^{\text{mag}}$, is shown since it clearly reflects the level of nonlinear driving at all times. Qualitatively, the results in Fig. 4 may be summarized as follows:

1. $Pr \sim 0.1$ [Fig. 5(g)]: During the transition to the nonlinear regime, $\gamma_{1,1,1}^{\text{kin}}$ first decreases slowly ($t \approx 150–180$) and then drops rapidly. Similar behavior
is observed for \( Pr = 10^{-2} \) and is also expected for \( Pr < 10^{-2} \).

2. \( Pr \sim 1 \) [Fig. 6(d,h)]: Compared to the results obtained with lower \( Pr \), \( \gamma_{\text{drive}} \) is now reduced due to the stabilizing effect of the viscosity on \( \gamma_{\text{lin}}(m) \) (cf. Fig. 5 in Ref. [1]). During the transition to the fully nonlinear regime \( \gamma_{\text{lin}}^{m=1} \) remains high for a certain period of time and may exhibit oscillatory behavior as in Fig. 6(h) before dropping.

3. \( Pr \sim 10, 10^2, 10^3 \) [Fig. 6(a,b,c,e,f,i)]: The growth rate \( \gamma_{\text{drive}} \) is further decreased due to higher viscosity. However, at the end of the driving phase a significant increase in the growth rate from \( \gamma_{\text{drive}} \approx 2\gamma_{\text{peak}} \) to a value \( \gamma_{\text{max}} \) [indicated in Fig. 6(b)] is observed. This effect is most pronounced in the kinetic growth rate \( \gamma_{\text{kin}}^{m=1} \). Note in particular that \( \gamma_{\text{max}} \) can get close to the value of \( \gamma_{\text{drive}} \) obtained for \( Pr \sim 1 \) [compare, e.g., Fig. 6(b), (e) and (h)]. It is likely that this behavior can also be observed for \( Pr > 10^3 \).

Let us note that during this rapid growth the nonlinear interactions already include many Fourier modes. We suspect that the explosive growth phase observed here for \( Pr \gtrsim 10 \) is due to these nonlinear interactions becoming more important than viscous damping. The latter had reduced \( \gamma_{\text{drive}} \) through a reduction of the linear growth rates \( \gamma_{\text{lin}}(m) \).

VII. ANNULAR COLLAPSE AND EFFECT OF MHD TURBULENCE ON THE INTERNAL KINK

In this section we investigate the long-term evolution in Cases (T-1) and (T-2) (Fig. 4). We describe the magnetic and \( E \times B \) flow structures generated through TTM reconnection and analyze the evolution of the \( m = 1 \) internal kink mode while it is surrounded by MHD turbulence. The values for the dissipation parameters in Case (T-1) are \( S_{\text{Hp}} = 10^6 \) and \( R_{\text{cHp}} = 10^8 \), as in Sec. VI. For Case (T-2) we choose \( S_{\text{Hp}} = 10^8 \) and \( R_{\text{cHp}} = 10^6 \).

A. Case (T-1): Annular collapse and full reconnection

We begin with a discussion of snapshots taken in Case (T-1), which are shown in Figs. 7 and 8 and labeled (A)–(F). The initial perturbation is sufficiently random so that reconnection occurs all around the core [Fig. 7(A)].
FIG. 7: (Color online). Annular collapse and long-term evolution in Case (T-1). The first three snapshots taken at (A) $t = 200$, (B) $t = 240$, and (C) $t = 280$ are shown (continued in Fig. 8). Each snapshot consists of contour plots of the helical flux $\psi$, (left) and the electrostatic potential $\phi$, (right). Arrows indicate the flow directions. The dashed circles indicate the outermost $\psi$ contour of the core and have been superimposed on the $\phi$ contours for clarity. The small diagrams in the middle show the instantaneous profiles $q(r, t)$ and $\langle j_\ast \rangle \equiv \| j_\ast (r, t) \|_{0,0}$. $Si_{Hp} = 10^6$, $Re_{Hp} = 10^6$.

However, it is not isotropic, so that some magnetic islands grow faster than others. The sizes of the islands reflect the shape of the spectrum of linear growth rates $\gamma_{lin}(m)$ with dominant modes having $m \sim m_{peak} = 13$ [cf. Fig. 2]. During this annular collapse the amplitude of the $m = 1$ mode is still small and the $q(r)$ profile is flattened annularly [Fig. 7(A)]. The magnetic islands in the inter-resonance region exhibit complicated coalescence dynamics and a turbulent annular region is created around the core [Fig. 7(B) and (C)]. In the meantime, the core displacement becomes observable. The core traverses the turbulent belt [Fig. 8(D) and (E)] and upon
making contact with the flux surfaces beyond the outermost resonant radius ($r = r_{s3}$) core reconnection begins [Fig. 8(E) and (F)]. In the present case, (T-1), core reconnection is likely to proceed to completion, i.e., full reconnection is expected. For completeness, the evolution of the $m = 1$ energies and growth rates is shown in Fig. 9(a) and (b), respectively.

An important observation is the following. While the $m = 1$ mode was originally perturbed with $\vartheta_{0,m=1} = 0$ (i.e., core motion in the positive $x$ direction), the kink flow continuously changes its direction, as is obvious from the arrows drawn along with the $\phi$ contours in Figs. 7 and 8. Note that the core does not rotate; merely the direction of its translational motion alters (in RMHD, $\phi_{0,0} = 0$ at all times if it is zero initially). The core’s motion is quantified and shown in more detail in Fig. 9(c),
motion becomes rather complicated. The kink flow varies frequently, so that the core’s location and kink flow direction are shown. Note that the early evolution is very similar to that of Case (T-1) described in Section VI. It consists of (i) linear growth, (ii) nonlinearly driven fast growth, and (iii) transition to the turbulent regime with gradually decreasing growth rates (here $Pr = 0.01$). We may thus omit the further discussion of these stages, referring to Sec. VI above.

Similarly to Case (T-1), the first macroscopically observable event is an annular collapse due to high-$m$ TTMIs without significant displacement of the core [Fig. 10(A)]. Again, the $q(r)$ profile is flattened in the inter-resonance region. Subsequently the $m = 1$ mode grows inside the turbulent belt [Fig. 10(B)]. However, in contrast to Case (T-1), here the $m = 1$ mode saturates after reaching a relatively large amplitude [Fig. 10(C)]. This occurs at $t = 962$, as can clearly be seen in the $m = 1$ magnetic energy, $E_{1,1}^{\text{mag}}$, and the associated growth rate, $\gamma_{1,1}^{\text{mag}}$ [Fig. 12(a) and (b)]. The direction of the kink flow reverses, as is obvious from Snapshots (C)-(E) (Figs. 11 and 12) and from the 180-degree jump in $\dot{\psi}_{\text{kink}}$ in Fig. 12(c). Afterwards, the kink amplitude decays [Fig. 11(E)], overshoots, and grows again in a different direction [Fig. 11(F)].

The saturation of the $m = 1$ mode observed here seems to be due to an island-like structure developing “in front” of the displaced core, when the latter approaches the periphery. The island remains trapped, i.e., it is not expelled in the poloidal direction. Consequently core reconnection takes place at two separate points which in the present case are located an angle $\Delta \theta = 116^\circ$ apart, as indicated in Fig. 10(C). Since each reconnected flux surface adds to the island’s width, this structure can counter the internal kink, induce a rebound and send the core back into the center. This scenario is realized in the present case.

### B. Case (T-2): Kink saturation and partial reconnection

Here we discuss the long-term evolution in Case (T-2). Snapshots are presented in Figs. 10 and 11. In Fig. 12 the evolution of (a) the $m = 1$ energies, (b) corresponding growth rates, and (c) the core’s location and kink flow direction are shown. Using the angles $\dot{\psi}_{\text{mag}}^{\text{kink}}$ and $\dot{\psi}_{\text{kin}}^{\text{kink}}$. These are defined as

$$\dot{\psi}_{\text{mag}}^{\text{kink}} = \tan^{-1} \left[ \frac{\int dr \, r \, \text{Im}(\psi_{1,1}(r))}{\int dr \, r \, \text{Re}(\psi_{1,1}(r))} \right] \quad (11a)$$

$$\dot{\psi}_{\text{kin}}^{\text{kink}} = -\tan^{-1} \left[ \frac{\int dr \, r \, \text{Re}(\phi_{1,1}(r))}{\int dr \, r \, \text{Im}(\phi_{1,1}(r))} \right] \quad (11b)$$

(integration interval: $0 \leq r \leq 0.07$) and give an approximate image of the core’s motion when the kink amplitude is not too large. During the linear phase (i) both angles are zero, in agreement with the initial perturbation. During the driving phase (ii) the angles switch to $\dot{\psi}_{\text{mag}}^{\text{kink}} \approx \dot{\psi}_{\text{kin}}^{\text{kink}} \approx 3^\circ$ (determined by the driving modes). After the transition to turbulence in phase (iii), the direction of the kink flow varies frequently, so that the core’s motion becomes rather complicated.

### C. Modulation of the kink flow

In addition to the kink flow changing its direction, at certain times we observe an $m = 1$ modulation of the $\phi$ contours in the core’s interior. In Case (T-1) this occurs around $t = 400$, Snapshot (D) [Fig. 8]. This modulation of the $E \times B$ drift velocity is not strong enough to visibly alter the $\psi$ contours in the core, which therefore remain circular (not shown). Further details can be seen in Fig. 13, where the profile of the radial velocity $v_r \propto \phi/r$ is shown at several times in the interval $360 \leq t \leq 510$. Some peaks in $v_r$, like those near the magnetic axis, perform oscillations in the manner of a standing wave [left-hand side in Fig. 13(b)]. Other peaks, like those in the region $0.2 \lesssim r \lesssim 0.35$, do not change their signs until
about $t \approx 500$ [Fig. 13(c)]. The radial wavelength of the modulation is observed to change relatively slowly. Typically, it measures between 1/2 and 1/5 of the core’s radius. Realizations of this $m = 1$ modulation can also be observed in Case (T-2) (Figs. 10 and 11).

VIII. DISCUSSION AND CONCLUSIONS

We have studied the nonlinear evolution of the $m = 1$ internal kink mode in a configuration with three $q = 1$ resonant surfaces where high-$m$ TTM s are strongly unstable and lead to an annular collapse. The latter leaves behind a turbulent belt around the unreconnected core plasma. The simulation results show that high-$m$ TTM s
and MHD turbulence, which are localized in an annular region, are able to strongly affect the evolution of the $m = 1$ internal kink mode, which is a global instability. We conclude that multiple tearing modes (here, TTMs) and MHD turbulence may play a significant role during partial, compound or full sawtooth crashes in tokamak plasmas, as will be discussed in the following.

In the beginning, a fast sawtooth trigger, defined as a sudden transition from slow to rapid growth, was realized: after a phase of slow linear growth, rapidly growing $q = 1$ TTMs give rise to a new fast growing nonlinear $m = 1$ mode. This instability reaches an observable amplitude within a time much shorter than expected from the linear growth rate. Moreover, the transition to the fully nonlinear (turbulent) regime occurs via a phase of explosive growth when the Prandtl number is large ($Pr \gtrsim 10$). As proposed in Ref. [1], this enhancement of the $m = 1$ mode due to high-$m$ TTMs is a possible
mechanism for the fast sawtooth trigger, provided that multiple \( q = 1 \) resonant surfaces are formed during the sawtooth ramp.

During the further evolution, the turbulence in the collapsed annular region was seen to alter the direction of the kink flow responsible for the core displacement. Both continuing growth [Case (T-1)] and saturation of the kink [Case (T-2)] were observed, which shows that full as well as partial reconnection may occur in this setting. It was also found that the effect of the turbulence was not limited to the collapsed annular region and the overall motion of the core inside this turbulent belt: Perturbations of the electrostatic potential were even found to penetrate into the central core region in the form of an \( m = 1 \) modulation on top of the kink flow.

The converse effect, i.e., the influence of the core displacement on the surrounding turbulence has not been addressed and is left for future study. This is expected to be important since (a) the core displacement changes the geometry of the turbulent region and (b) the return flows of the internal kink are likely to interact with the turbulence. One particular question to be addressed is whether and how the core contributes to the formation of the trapped island that prevents further reconnection in Case (T-2) [cf. Figs. 10(C)].

Our results agree with some aspects of the partial sawtooth crash scenario suggested in earlier studies: A “shoulder” on the \( q(r) \) profile forms where \( q \approx 1 \), and this region is governed by electromagnetic turbulence (e.g., [28, 31, 32]). Indeed, the conjectures made by these authors imply that continued growth of the \( m = 1 \) mode [as in Case (T-1)] must be prevented by some means, so that the partial collapse remains partial. We have demonstrated that MHD turbulence is one possible mechanism leading to a saturation of the internal kink [as in Case (T-2)]. The residual core displacement may then account for the post-cursor oscillation observed experimentally after...
partial reconnection events (e.g., Ref. [33, 34]).

The fact that the long-term calculations in the two cases considered here have different outcomes, namely full reconnection in Case (T-1) and partial reconnection in Case (T-2), requires commenting on. Note that the initial conditions do not differ largely, except for the linear growth rates in Case (T-1) being twice as high as in Case (T-2) due to a higher magnetic shear (cf. Figs. 2 and 3). While we were able to demonstrate that MHD turbulence provides prerequisites for a saturation of the internal kink, the ultimate goal of determining quantitative criteria for partial reconnection requires further investigations using more realistic physical models. As mentioned in the introduction, potentially important effects to be considered include two-fluid, curvature, and finite-beta effects. In particular, if some of these would be found to have a stabilizing influence on the high-$m$ modes, which are essential in the present work, the results may be altered. To our knowledge, such studies have not been conducted for a comparable scenario, i.e., strongly coupled multiple tearing modes such as DTMs or TTM.

In this study, MHD turbulence was generated through current-driven resistive TTM, requiring multiple $q = 1$ resonant surfaces. The inclusion of a collisionless reconnection mechanism is expected to yield higher kinetic energies and thus stronger turbulence interacting with the internal kink. Furthermore, we conjecture that our principal result, namely that the internal kink mode can be strongly affected by tearing-mode-driven MHD turbulence, will also apply when the latter is generated by some other means, such as pressure-driven MHD or micro-instabilities [36].

Let us note that there are several other mechanisms which were proposed as possible explanations for the rapid collapse and partial crash phenomena, which were discovered assuming different pre-crash conditions (e.g., Ref. 3 and references therein). However, the currently available experimental data is not yet conclusive enough to rule out one proposed sawtooth crash scenario or another. Moreover, the detailed evolution may vary between different machines, shots and even between sawtooth crashes of a single discharge.

Through recent progress in plasma diagnostics it has become possible to detect high-$m$ magnetic islands associated with low-order $q = m/n$ resonant surfaces [37]. Thus, experimental checks of our simulation results seem feasible in the near future.

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