On the Convergence of Adam and Adagrad

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Abstract

We provide a simple proof of the convergence of the optimization algorithms Adam and Adagrad with the assumptions of smooth gradients and almost sure uniform bound on the $\ell_\infty$ norm of the gradients. This work builds on the techniques introduced by Ward et al. (2019) and extends them to the Adam optimizer. We show that in expectation, the squared norm of the objective gradient averaged over the trajectory has an upper-bound which is explicit in the constants of the problem, parameters of the optimizer and the total number of iterations $N$. This bound can be made arbitrarily small. In particular, Adam with a learning rate $\alpha = 1/\sqrt{N}$ and a momentum parameter on squared gradients $\beta_2 = 1 - 1/N$ achieves the same rate of convergence $O(\ln(N)/\sqrt{N})$ as Adagrad. Thus, it is possible to use Adam as a finite horizon version of Adagrad, much like constant step size SGD can be used instead of its asymptotically converging decaying step size version.

1. Introduction

First order methods with adaptive step sizes have proved useful in many fields of machine learning, be it for sparse optimization (Duchi et al., 2013), tensor factorization (Lacroix et al., 2018) or deep learning (Goodfellow et al., 2016).

Adagrad (Duchi et al., 2011) rescales each coordinate by a sum of squared past gradient values. While Adagrad proved effective for sparse optimization (Duchi et al., 2013), experiments showed that it under-performed when applied to deep learning (Wilson et al., 2017). The large impact of past gradients prevents it from adapting to local changes in the smoothness of the function. With RMSProp, Tieleman & Hinton (2012) proposed an exponential moving average instead of a cumulative sum to forget past gradients. Adam (Kingma & Ba, 2014), currently one of the most popular adaptive algorithms in deep learning, built upon RMSProp by introduced a finite horizon version of Adagrad, much like constant step size SGD can be used instead of its asymptotically converging decaying step size version.

and added corrective term to the step sizes at the beginning of training, together with heavy-ball style momentum.

We provide a simple proof of the convergence of the optimization algorithms Adam and Adagrad with the assumptions of smooth gradients and almost sure uniform bound on the $\ell_\infty$ norm of the gradients. This work builds on the techniques introduced by Ward et al. (2019) and extends them to the Adam optimizer. We show that in expectation, the squared norm of the objective gradient averaged over the trajectory has an upper-bound which is explicit in the constants of the problem, parameters of the optimizer and the total number of iterations $N$. This bound can be made arbitrarily small. In particular, Adam with a learning rate $\alpha = 1/\sqrt{N}$ and a momentum parameter on squared gradients $\beta_2 = 1 - 1/N$ achieves the same rate of convergence $O(\ln(N)/\sqrt{N})$ as Adagrad. Thus, it is possible to use Adam as a finite horizon version of Adagrad, much like constant step size SGD can be used instead of its asymptotically converging decaying step size version.

In this paper, we present a new proof of convergence to a critical point for Adagrad and Adam for stochastic non-convex smooth optimization, under the assumptions that the stochastic gradients of the iterates are almost surely bounded. These assumptions are weaker and more realistic than those of prior work on these algorithms. In particular, we show for a fully connected feed forward neural networks with sigmoid activation trained with $\ell_2$ regularization, the iterates of Adam or Adagrad almost surely stay bounded, which in turn implies a bound on the stochastic gradient as long as the training input data is also bounded. We recover the standard $O(\ln(N)/\sqrt{N})$ convergence rate for Adagrad for all step sizes, and the same rate with Adam with an appropriate rescaling of the step sizes and decay parameters. Compared to previous work, our bound significantly improves the dependency on the momentum parameter $\beta_1$. The best known bounds for Adagrad and Adam are respectively in $O((1 - \beta_1)^{-3})$ and $O((1 - \beta_1)^{-5})$ (see Section 3), while our result is in $O((1 - \beta_1)^{-1})$ for both algorithms.

Another important contribution of this work is a significantly simpler proof than previous ones. The reason is that...
On the Convergence of Adam and Adagrad

in our approach, the main technical steps are carried out jointly for Adagrad and Adam with constant parameters, while previous attempts at unified proofs required varying parameters through the iterations (Chen et al., 2018; Zou et al., 2019b;a).

The precise setting and assumptions are stated in the next section, and previous work is then described. Next, we discuss the relevance of our assumptions in the context of deep learning using containment arguments inspired by Bottou (1999). The main theorems are presented in Section 5, followed by a full proof for the case without momentum in Section 6. The full proof of the convergence with momentum is deferred to the supplementary material.

2. Setup

2.1. Notation

Let $d \in \mathbb{N}$ be the dimension of the problem and take $[d] = \{1, 2, \ldots, d\}$. Given a function $h : \mathbb{R}^d \to \mathbb{R}$, we note $\nabla h$ its gradient and $\nabla_i h$ the $i$-th component of the gradient. In the entire paper, $\epsilon$ represents a small constant, e.g., $10^{-8}$, used for numerical stability. Given a sequence $(u_n)_{n \in \mathbb{N}}$ with $\forall n \in \mathbb{N}, u_n \in \mathbb{R}^d$, we note $u_{n,i}$ for $n \in \mathbb{N}$ and $i \in [d]$ the $i$-th component of the $n$-th element of the sequence.

We want to optimize a function $F : \mathbb{R}^d \to \mathbb{R}$. We assume there exists a random function $f : \mathbb{R}^d \to \mathbb{R}$ such that $\mathbb{E}[\nabla f(x)] = \nabla F(x)$ and that we have access to an oracle providing i.i.d. samples $(f_n)_{n \in \mathbb{N}}$. In machine learning, $x \in \mathbb{R}^d$ typically represents the weights of a linear or deep model, $f$ represents the loss from individual training examples or minibatches, and $F$ is the full training objective function. The goal is to find a critical point of $F$.

2.2. Adaptive methods

We study a family of algorithms that covers both Adagrad (Duchi et al., 2011) and Adam (Kingma & Ba, 2014). We assume we have an infinite stream $(f_n)_{n \in \mathbb{N}^*}$ of i.i.d. copies of $f$, $0 \leq \beta_2 \leq 1$ and $0 \leq \beta_1 < \beta_2$, and a non negative sequence $(\alpha_n)_{n \in \mathbb{N}^*}$.

Given $x_0 \in \mathbb{R}^d$ our starting point and $m_0 = 0$, $v_0 = 0$, we iterate, for every $n \in \mathbb{N}^*$,

\[
m_{n,i} = \beta_1m_{n-1,i} + \nabla_i f_n(x_{n-1}), \quad v_{n,i} = \beta_2v_{n-1,i} + (\nabla_i f_n(x_{n-1}))^2, \quad x_{n,i} = x_{n-1,i} - \alpha_n \frac{m_{n,i}}{\sqrt{\epsilon + v_{n,i}}}. \tag{2.3}
\]

The real number $\beta_1$ is a heavy-ball style momentum parameter (Polyak, 1964), while $\beta_2$ controls the rate at which the scale of past gradients is forgotten.

Taking $\beta_1 = 0$, $\beta_2 = 1$ and $\alpha_n = \alpha$ gives Adagrad. While the original Adagrad algorithm (Duchi et al., 2011) did not include a heavy-ball-like momentum, our analysis also applies to the case $\beta_1 > 0$. On the other hand, when $0 < \beta_2 < 1$, $0 \leq \beta_1 < \beta_2$, taking

\[
\alpha_n = \alpha(1 - \beta_1) \sqrt{\frac{1}{\beta_2}}, \tag{2.4}
\]

leads to an algorithm close to Adam. Indeed, the step size in (2.4) is rescaled based on the number of past gradients that were accumulated. This is equivalent to the correction performed by Adam, which compensates for the possible smaller scale of $v_n$ when only few gradients have been accumulated.\footnote{Adam updates are usually written $\alpha_n = \alpha(1 - \beta_1) \sqrt{1 - \beta_2^2}$ and $v_{n,i} = \beta_2v_{n-1,i} + (1 - \beta_1)(\nabla_i f_n(x_{n-1}))^2$. These are equivalent to ours because the factor $(1 - \beta_1)$ is transferred to a multiplication of $\alpha_n$ by $1/\sqrt{1 - \beta_2}$. The same apply to $m_n.$}

When there is no momentum ($\beta_1 = 0$) the only difference with Adam is that $\epsilon$ in (2.3) is outside the square root in the original algorithm. When $\beta_1 > 0$, an additional difference is that we do not compensate for $m_n$ being smaller during the first few iterations.

The slight difference in step size when $\beta_2 > 0$ simplifies the proof at a minimum practical cost: the first few iterations of Adam are usually noisy, in particular due to $v_n$ having seen few samples, and (2.4) is equivalent to taking a smaller step size during the first $1/\beta_2$ iterations. Since Kingma & Ba (2014) suggested a default value of $\beta_1 = 0.9$, our update rule differs significantly from the original Adam only during the first few tens of iterations.

2.3. Assumptions

We make four assumptions. We first assume $F$ is bounded below by $F_\ast$, that is,

\[
\forall x \in \mathbb{R}^d, \quad F(x) \geq F_\ast. \tag{2.5}
\]

We assume the iterates are contained within an $\ell_\infty$ ball almost surely,

\[
\forall n \in \mathbb{N}, \quad \|x_n\|_{\infty} \leq B \quad \text{a.s..} \tag{2.6}
\]

We then assume the $\ell_\infty$ norm of the stochastic gradients is almost surely bounded over this ball: for all $x \in \mathbb{R}^d$ such that $\|x\|_{\infty} \leq B$,

\[
\|\nabla f(x)\|_{\infty} \leq R - \sqrt{\epsilon} \quad \text{a.s.,} \tag{2.7}
\]

and finally, the smoothness of the objective function over this ball, e.g., its gradient is $L$-Liptchitz-continuous with respect to the $\ell_2$-norm: for all $x, y \in \mathbb{R}^d$ such that $\|x\|_{\infty} \leq B$ and $\|y\|_{\infty} \leq B$,

\[
\|\nabla F(x) - \nabla F(y)\|_2 \leq L \|x - y\|_2. \tag{2.8}
\]
On the Convergence of Adam and Adagrad

Note that, if $F$ is $L$-smooth over $\mathbb{R}^d$ and the stochastic gradients are uniformly almost surely bounded over $\mathbb{R}^d$, then one can take $B = \infty$, and (2.6) is then verified. This case matches more usual assumptions, but it is rarely met in practice, as explained in Section 3. However, note that (2.6) is verified with $B < \infty$ for some cases of deep neural network training, as proven in Section 4.

3. Related work

Work on adaptive optimization methods started with the seminal papers of McMahan & Streeter (2010) and Duchi et al. (2011). They showed that adaptive methods like Adagrad achieve an optimal rate of convergence of $O(1/\sqrt{N})$ for convex optimization (Agarwal et al., 2009). Practical experiences with training deep neural networks led to the development of adaptive methods using an exponential moving average of past squared gradients like RMSProp (Tieleman & Hinton, 2012) or Adam (Kingma & Ba, 2014).

Kingma & Ba (2014) claimed that Adam with decreasing step sizes converges to an optimal solution for convex objectives. However, the proof contained a mistake spotted by Reddi et al. (2019), who also gave examples of convex problems where Adam does not converge to an optimal solution. They proposed AMSGrad as a convergent variant of Adam, which consisted in retaining the maximum value of the exponential moving average. The examples given by Reddi et al. (2019) illustrate a behavior of Adam that is coherent with our results and previous work (Zou et al., 2019b), because they use a small exponential decay parameter $\beta_2 < 1/5$. Under our assumptions, Adam with constant $\beta_2$ is guaranteed to not diverge, but it is not guaranteed to converge to a stationary point.

Regarding the non-convex setting, Li & Orabona (2019) showed the convergence of Adagrad for the non-convex case but under unpractical conditions, in particular the step size $\alpha$ should verify $\alpha \leq \sqrt{\epsilon}/L$. Ward et al. (2019) showed the convergence of a variant of Adagrad (in the sense of the expected squared norm at a random iterate) for any value of $\alpha$, but only for the “scalar” version of Adagrad, with a rate of $O(\ln(N)/\sqrt{N})$. While our approach builds on this work, we significantly extend it to apply to both Adagrad and Adam, in their coordinate-wise version used in practice, while also supporting heavy-batch momentum.

Zou et al. (2019a) showed the convergence of Adagrad with either heavy-ball or Nesterov style momentum. We recover a similar result for Adagrad with heavy-ball momentum, under different but interchangeable hypotheses, as explained in Section 5.2. Their proof technique worked with a variety of averaging scheme for the past squared gradients, including Adagrad. In that case, we obtain the same rate as them as a function of $N$ (i.e., $O(\ln(N)/\sqrt{N})$), but we improve the dependence on the momentum parameter $\beta_1$ from $O((1 - \beta_1)^{-3})$ to $O((1 - \beta_1)^{-1})$. Chen et al. (2019) also present bounds for Adagrad and Adam, without convergence guarantees for Adam. The dependence of their bounds in $\beta_1$ is worse than that of Zou et al. (2019a).

Zou et al. (2019b) propose unified convergence bounds for Adagrad and Adam. We recover the same scaling of the bound with respect to $\alpha$ and $\beta_2$. However their bound has a dependency in $O((1 - \beta_1)^{-5})$ with respect to $\beta_1$, while we prove $O((1 - \beta_1)^{-1})$, a significant reduction.

In previous work (Zou et al., 2019b,a), the assumption given by (2.7) is replaced by

$$\forall x \in \mathbb{R}^d, \mathbb{E} \left[ \|\nabla f(x)\|^2_2 \right] \leq R^2. \quad (3.1)$$

First, notice that we assume an almost sure bound instead of a bound on the expectation of the squared stochastic gradients. However this lead to a weaker convergence result, e.g., a bound on the expected norm of the full gradient at the itertes taken to the power 4/3 instead of 2, as explained in Section 5.2. The proof remains mostly identical whether we assume an almost sure bound or bound in expectation of the squared stochastic gradients. Given that for a fixed $x \in \mathbb{R}^d$, the variance of the stochastic gradients for machine learning models comes from the variance of the training data, going from a bound in expectation of the squared gradients to an almost sure bound is easily accomplished by the removal of outliers in the training set.

Second, assumption (3.1) rarely hold in practice as it assume boundness of the gradient over $\mathbb{R}^d$. It is not verified by any deep learning network with more than one layer, linear regression, nor logistic regression with $\ell_2$ regularization. In fact, a deep learning network with two layers is not even $L$-smooth over $\mathbb{R}^d$, as the norm of the gradient for the first layer is multiplied by the norm of the gradient for the second layer. We show in the next section that for deep neural networks with sigmoid activations and $\ell_2$ regularization, (2.6) is verified, as long as the data in the training set is bounded, which implies both (2.7) and (2.8).

4. Containment of the iterates

Following Bottou (1999) we show in this section that (2.6) is verified for a fully connected feed forward neural network with sigmoid activations and $\ell_2$ regularization. The goal of this section is to show that there is an upper-bound on the weights of this neural network when trained with Adam or Adagrad even though the bound we obtain grows super exponentially with the depth.

We assume that $\beta_1 = 0$ for simplicity, so that for any iteration $n \in \mathbb{N}^*$ and coordinate $i \in [d], m_{n,i} = \nabla_i f_n(x_{n-1}).$ We assume $x \in \mathbb{R}^d$ is the concatenation of $[w_1 w_2 \ldots w_l],$
where \( l \) is the number of layers and for all \( s \in [l], w_s \in \mathbb{R}^{c_{s-1} \times c_s} \) is the weight of the \( s \)-th layer, \( c_0 \) being the dimension of the input data. For clarity, we assume \( c_l = 1 \), i.e. the neural network has a single output. The fully connected network is represented by the function,

\[
\forall z \in \mathbb{R}^{c_0}, h(x, z) = \sigma(w_0 \sigma(w_{l-1} \ldots \sigma(w_1 z))).
\]

Then, the stochastic objective function is given by,

\[
f(x) = D(h(x, Z), Y) + \frac{\lambda}{2} \|x\|_2^2,
\]

where \( Z \) is a random variable over \( \mathbb{R}^{c_0} \) representing the input training data, \( Y \) is the label over a set \( \mathcal{Y} \), \( D \) is the loss function, and \( \lambda \) the \( \ell_2 \) regularization parameter. We assume that the \( \ell_\infty \) norm of \( Z \) is almost surely bounded by 1 and that for any label \( y \in \mathcal{Y}, |D'(\cdot, y)| \leq M' \). This is verified for the Huber loss, or the cross entropy loss. When writing \( D' \), we always mean its derivative with respect to its first argument. Finally, we note \( o_s(x, z) \) the output of the \( s \)-th layer, i.e.

\[
\forall s \in [l], o_s(x, z) = \sigma(w_s \sigma(w_{s-1} \ldots \sigma(w_1 z)));
\]

and \( o_0(x, z) = z \). In particular, \( \|o_s(x, z)\|_\infty \leq 1 \).

We will prove the bound on the iterates through induction, starting the output layer and going backward up to the input layer. We assume all the weights are initialized with a size much smaller than the bound we will derive.

### 4.1. Containment of the last layer

In the following, \( \nabla_w \) is the Jacobian operator with respect to the weights of a specific layer \( w \). Taking the derivative of \( f(x) \) with respect to \( w_l \), we get,

\[
\nabla_{w_l} D(h(x, Z), Y) = D'(h(x, Z), Y) \nabla_{w_l} h(x, Z)
\]

\[
= D'(h(x, Z), Y) \sigma'(w_s o_{l-1}) o_{l-1}.
\]

Given that \( \sigma' \leq 1/4 \), we have,

\[
\|\nabla_{w_l} D(h(x, Z), Y)\|_\infty \leq \frac{M'}{4}.
\]  

#### Updates of Adam or Adagrad are bounded

For any iteration \( n \in \mathbb{N}^* \), we have for Adam \( \alpha_n \leq \frac{\alpha}{\sqrt{n}} \) and for Adagrad \( \alpha_n \leq \alpha \). We note \( A = \max_{n \in \mathbb{N}^*} \alpha_n \). Besides, for any coordinate \( i \in [d] \), we have \( |x_{n-1,i} - x_{n,i}| \leq A \). Given (4.1), we have,

\[
\nabla_{i} f(x_{n0}) \geq -\frac{M'}{4} + \lambda \frac{M'}{4\lambda} \geq 0.
\]

Thus \( m_{n0,i} \geq 0 \) and using (4.2),

\[
0 \leq w_{n1,i} - A \leq w_{n0,i} \leq w_{n-1,i}.
\]

so that \( |x_{n0,i} - x_{n-1,i}| \leq A \). Thus, if at any point \( x_{n-1,i} \) goes over \( \frac{M'}{4\lambda} + A \), the next iterates decrease until they go back below \( \frac{M'}{4\lambda} + A \). Given that the maximum increase between two update is \( A \), it means we have for any iteration \( n \in \mathbb{N}^* \), and for any coordinate \( i \) corresponding to a weight of the last layer,

\[
x_{n,i} \leq \frac{M'}{4\lambda} + 2A.
\]

Applying the same technique we can show that \( x_{n,i} \geq -\frac{M'}{4\lambda} - 2A \) and finally,

\[
|x_{n,i}| \leq \frac{M'}{4\lambda} + 2A.
\]

In particular, this implies that the Frobenius norm of the weight of the last layer \( w_l \) stays bounded for all the iterates.

#### 4.2. Containment of the previous layers

Now taking a layer \( s \in [l-1] \), we have,

\[
\nabla_{w_s} D(h(x, Z), Y) = \frac{D'(h(x, Z), Y)}{D'(h(x, Z), Y)} \left( \prod_{k=s}^{s+1} \sigma'(o_{k-1}) w_k \right) o_{s-1}.
\]

Let us assume we have shown that for layers \( k > s \), \( \|w_k\|_F \leq M_k \), then we can immediately derive that the above gradient is bounded in \( \ell_\infty \) norm. Applying the same method as in 4.1, we can then show that the weights \( w_s \) stay bounded as well, with respect to the \( \ell_\infty \) norm, by \( \frac{M'}{4\lambda} \prod_{k=s}^{s+1} M_k + 2A \). Thus, by induction, we can show that the weights of all layers stay bounded for all iterations, albeit with a bound growing more than exponentially with depth.

### 5. Main results

For any total number of iterations \( N \in \mathbb{N}^* \), we define \( \tau_N \) a random index with value in \( \{0, \ldots, N-1\} \), verifying

\[
\forall j \in \mathbb{N}, \ P[\tau = j] \propto 1 - \beta_1^{N-j}.
\]

If \( \beta_1 = 0 \), this is equivalent to sampling \( \tau \) uniformly in \( \{0, \ldots, N-1\} \). If \( \beta_1 > 0 \), the last few \( \frac{1}{1-\beta_1} \) iterations are sampled rarely, and all iterations older than a few times that number are sampled almost uniformly. All our results bound the expected squared norm of the total gradient at iteration \( \tau \), which is standard for non convex stochastic optimization (Ghadimi & Lan, 2013).
5.1. Convergence bounds

For simplicity, we first give convergence results for $\beta_1 = 0$, along with a complete proof in Section 6.1 and Section 6.2. We show convergence for any $\beta_1 < \beta_2$, however the theoretical bound is always worse than for $\beta_1 = 0$, while the proof becomes significantly more complex. Therefore, we delay the complete proof with momentum to the Appendix, Section A.5. We still provide the results with momentum in the second part of this section. Note that the disadvantageous dependency of the bound on $\beta_1$ is not specific to our proof but can be observed in previous adaptive methods bounds (Chen et al., 2019; Zou et al., 2019a).

**Theorem 1** (Convergence of Adam without momentum). Given the assumptions introduced in Section 2.3, the iterates $x_n$ defined in Section 2.2 with hyper-parameters verifying $0 < \beta_2 < 1$, $\alpha_n = \sqrt{\sum_{j=0}^{n-1} \beta_2^j} \alpha$ with $\alpha > 0$ and $\beta_1 = 0$, we have for any $N \in \mathbb{N}^*$, taking $\tau$ defined by (5.1),

$$
\mathbb{E} \left[ \left\| \nabla F(x_{\tau}) \right\|^2 \right] \leq 2 R^2 \left( \frac{F(x_0) - F_*}{\alpha N} + C \left( \frac{1}{N} \ln \left( 1 + \frac{R^2}{(1 - \beta_2)\epsilon} \right) - \ln(\beta_2) \right) \right),
$$

(5.2)

with

$$
C = \frac{4dR^2}{\sqrt{1 - \beta_2}} + \frac{\alpha dRL}{1 - \beta_2}.
$$

**Theorem 2** (Convergence of Adagrad without momentum). Given the assumptions introduced in Section 2.3, the iterates $x_n$ defined in Section 2.2 with hyper-parameters verifying $\beta_2 = 1$, $\alpha_n = \alpha$ with $\alpha > 0$ and $\beta_1 = 0$, we have for any $N \in \mathbb{N}^*$, taking $\tau$ defined by (5.1),

$$
\mathbb{E} \left[ \left\| \nabla F(x_{\tau}) \right\|^2 \right] \leq 2 R^2 \left( \frac{F(x_0) - F_*}{\alpha \sqrt{N}} + \frac{1}{\sqrt{N}} \left( 4dR^2 + \alpha dRL \right) \ln \left( 1 + \frac{NR^2}{\epsilon} \right) \right),
$$

(5.3)

**Theorem 3** (Convergence of Adam with momentum). Given the assumptions introduced in Section 2.3, the iterates $x_n$ defined in Section 2.2 with hyper-parameters verifying $0 < \beta_2 < 1$, $\alpha_n = (1 - \beta_1) \sqrt{\sum_{j=0}^{n-1} \beta_2^j} \alpha$ with $\alpha > 0$ and $0 \leq \beta_1 < \beta_2$, we have for any $N \in \mathbb{N}^*$ such that $N > \frac{1}{1 - \beta_1}$, taking $\tau$ defined by (5.1),

$$
\mathbb{E} \left[ \left\| \nabla F(x_{\tau}) \right\|^2 \right] \leq 2 R^2 \left( \frac{F(x_0) - F_*}{\alpha N} + C \left( \frac{1}{N} \ln \left( 1 + \frac{R^2}{(1 - \beta_2)\epsilon} \right) - \frac{N}{N} \ln(\beta_2) \right) \right),
$$

(5.4)

with

$$
\tilde{N} = N - \frac{\beta_1}{1 - \beta_1},
$$

and,

$$
C = \frac{\alpha dRL (1 - \beta_1)}{(1 - \beta_1/\beta_2)(1 - \beta_2)} + \frac{12dR^2}{\sqrt{1 - \beta_2}} \frac{\sqrt{1 - \beta_2}}{(1 - \beta_1/\beta_2)^{3/2} + \frac{2\alpha^2dL^2\beta_1}{(1 - \beta_1/\beta_2)(1 - \beta_2)^{3/2}}},
$$

(5.5)

**Theorem 4** (Convergence of Adam with momentum). Given the assumptions introduced in Section 2.3, the iterates $x_n$ defined in Section 2.2 with hyper-parameters verifying $\beta_2 = 1$, $\alpha_n = \alpha$ with $\alpha > 0$ and $0 \leq \beta_1 < 1$, we have for any $N \in \mathbb{N}^*$ such that $N > \frac{1}{1 - \beta_1}$, taking $\tau$ as defined by (5.1),

$$
\mathbb{E} \left[ \left\| \nabla F(x_{\tau}) \right\|^2 \right] \leq 2 R^2 \left( \frac{F(x_0) - F_*}{\alpha N} + \frac{\sqrt{N} \tilde{C} \ln \left( 1 + \frac{NR^2}{\epsilon} \right)}{\sqrt{N} \sqrt{C}} \right),
$$

(5.6)

with

$$
\tilde{N} = N - \frac{\beta_1}{1 - \beta_1},
$$

(5.7)

and,

$$
C = \left( \frac{\alpha dRL + 12dR^2}{1 - \beta_1} + \frac{2\alpha^2dL^2\beta_1}{1 - \beta_1} \right).
$$

(5.8)

5.2. Analysis of the bounds

**Dependence in $d$.** Looking at bounds introduced in the previous section, one can notice the presence of two terms: the forgetting of the initial condition, proportional to $F(x_0) - F_*$, and a second term that scales as $d$. The scaling as $d$ is inevitable given our hypothesis, in particular the use of a bound on the $\ell_\infty$-norm of the objectives. Indeed, for any bound valid for a function $F_1$ with $d = 1$, then we can build a new function $F_d = \sum_{i \in [d]} F_i(x_i)$, i.e., we replicate $d$ times the same optimization problem. The Hessian of $F_d$ is diagonal with each diagonal element being the same as the Hessian of $F_1$, thus the smoothness constant is unchanged, nor is the $\ell_\infty$ bound on the stochastic gradients. Each dimension is independent from the other and equivalent to the single dimension problem given by $F_1$, thus

$$
\mathbb{E} \left[ \left\| \nabla F_d(x_{\tau}) \right\|^2 \right] \text{ scales as } d.
$$

**Almost sure bound on the gradient.** We chose to assume the existence of an almost sure $\ell_\infty$-bound on the gradients given by (2.7). We use it only in (6.16) and (6.18). It is possible instead to use the Hlder inequality, which is the choice made by Ward et al. (2019) and Zou et al. (2019a). This however deteriorate the bound, instead of a bound on $\mathbb{E} \left[ \left\| \nabla F(x_{\tau}) \right\|^2 \right]$, this would give a bound on
Adam and Adagrad are twins. We discovered an important fact from the bounds we introduced in Section 5.1: Adam is to Adagrad like constant step size SGD is to decaying step size SGD. While Adagrad is asymptotically optimal, it has a slower forgetting of the initial condition than Adam, as $1/\sqrt{N}$ instead of $1/N$ for Adam. Furthermore, Adam adapts to local change of the smoothness faster than Adagrad as it eventually forgets about past gradients. This fast forgetting of the initial condition and improved adaptivity comes at a cost as Adam does not converge. It is however possible to chose parameters $\alpha$ and $\beta_2$ as to achieve an $\epsilon$ critical point for $\epsilon$ arbitrarily small and in particular, for a known time horizon, they can be chosen to obtain the exact same bound as Adagrad.

### 6. Proofs for $\beta_1 = 0$ (no momentum)

We assume here for simplicity that $\beta_1 = 0$, i.e., there is no heavy-ball style momentum. The recursions introduced in Section 2.2 can be simplified into

\[ v_{n,i} = \beta_2 v_{n-1,i} + (\nabla_i f_n(x_{n-1}))^2 \]

\[ x_{n,i} = x_{n-1,i} - \alpha_n \nabla_i f_n(x_{n-1}) \sqrt{\epsilon + \bar{v}_{n,i}} \]  

Throughout the proof we note by $E_{n-1}$ [·] the conditional expectation with respect to $f_1, \ldots, f_{n-1}$. In particular, $x_{n-1}, v_{n-1}$ is deterministic knowing $f_1, \ldots, f_{n-1}$. For all $n \in \mathbb{N}^+$, we also define $\bar{v}_n \in \mathbb{R}^d$ so that for all $i \in [d]$, 

\[ \bar{v}_{n,i} = \beta_2 v_{n-1,i} + E_{n-1} [(\nabla_i f_n(x_n))^2] \]

i.e., $\bar{v}_n$ is obtained from $v_n$ by replacing the last gradient contribution by its expected value knowing $f_1, \ldots, f_{n-1}$.

### 6.1. Technical lemmas

A problem posed by the update in (6.2) is the correlation between the numerator and denominator. This prevents us from easily computing the conditional expectation and as noted by Reddi et al. (2019), the expected direction of update can have a positive dot product with the objective gradient. It is however possible to control the deviation from the descent direction, following Ward et al. (2019) with this first lemma.

**Lemma 6.1 (adaptive update approximately follow a descent direction).** For all $n \in \mathbb{N}^+$ and $i \in [d]$, we have:

\[ E_{n-1} \left[ \nabla_i F(x_{n-1}) \nabla_i f_n(x_{n-1}) \right] \geq \frac{(\nabla_i F(x_{n-1}))^2}{2\sqrt{\epsilon + \bar{v}_{n,i}}} - 2R E_{n-1} \left[ \frac{(\nabla_i f_n(x_{n-1}))^2}{\epsilon + \bar{v}_{n,i}} \right] . \]
Proof. We take \( i \in [d] \) and note \( G = \nabla_i F(x_{n-1}) \), \( g = \nabla_i f_n(x_{n-1}) \), \( \nu = \nu_{n,i} \) and \( \tilde{v} = \tilde{v}_{n,i} \).

\[
\mathbb{E}_{n-1} \left[ \frac{Gg}{\sqrt{\epsilon + \nu}} \right] = \mathbb{E}_{n-1} \left[ \frac{Gg}{\sqrt{\epsilon + \tilde{v}}} \right] + \mathbb{E}_{n-1} \left[ Gg \left( \frac{1}{\sqrt{\epsilon + \nu}} - \frac{1}{\sqrt{\epsilon + \tilde{v}}} \right) \right]. \tag{6.5}
\]

Given that \( g \) and \( \tilde{v} \) are independent given \( f_1, \ldots, f_{n-1} \), we immediately have

\[
\mathbb{E}_{n-1} \left[ \frac{Gg}{\sqrt{\epsilon + \tilde{v}}} \right] = \frac{G^2}{\sqrt{\epsilon + \tilde{v}}}. \tag{6.6}
\]

Now we need to control the size of \( A \),

\[
A = \frac{Gg}{\sqrt{\epsilon + \nu}} \sqrt{\epsilon + \nu} \left( \sqrt{\epsilon + \nu} + \sqrt{\epsilon + \tilde{v}} \right)
\]  
\[
= \frac{Gg}{\sqrt{\epsilon + \nu}} \sqrt{\epsilon + \nu} \left( \frac{\mathbb{E}_{n-1} [g^2] - g^2}{\sqrt{\epsilon + \nu}} \right)
\]

\[
|A| \leq \left( \frac{|G|}{\sqrt{\epsilon + \nu}} \right) \mathbb{E}_{n-1} [g^2] + \left( \frac{|G|}{\epsilon + \nu} \right) \mathbb{E}_{n-1} [g^2].
\]

the last inequality coming from the fact that \( \sqrt{\epsilon + \nu} + \sqrt{\epsilon + \tilde{v}} \geq \max(\sqrt{\epsilon + \nu}, \sqrt{\epsilon + \tilde{v}}) \) and \( \mathbb{E}_{n-1} [g^2] - g^2 \leq \mathbb{E}_{n-1} [g^2] + g^2 \).

Following Ward et al. (2019), we can use the following inequality to bound \( \kappa \) and \( \rho \),

\[
\forall \lambda > 0, \ x, y \in \mathbb{R}, xy \leq \frac{\lambda}{4} x^2 + \frac{y^2}{2 \lambda}. \tag{6.7}
\]

First applying (6.7) to \( \kappa \) with

\[
\lambda = \frac{\sqrt{\epsilon + \tilde{v}}}{\sqrt{\epsilon + \nu}}, \ x = \frac{|G|}{\sqrt{\epsilon + \nu}}, \ y = \frac{|G| \mathbb{E}_{n-1} [g^2]}{\sqrt{\epsilon + \tilde{v}}} \sqrt{\epsilon + \nu}
\]

we obtain

\[
\kappa \leq \frac{G^2}{4 \sqrt{\epsilon + \tilde{v}}} + \frac{g^2 \mathbb{E}_{n-1} [g^2]}{(\epsilon + \nu)^{3/2}}. \tag{6.8}
\]

Given that \( \epsilon + \tilde{v} \geq \mathbb{E}_{n-1} [g^2] \) and taking the conditional expectation, we can simplify as

\[
\mathbb{E}_{n-1} [\kappa] \leq \frac{G^2}{4 \sqrt{\epsilon + \tilde{v}}} + \frac{\mathbb{E}_{n-1} [g^2]}{\epsilon + \nu}. \tag{6.9}
\]

Given that \( \sqrt{\mathbb{E}_{n-1} [g^2]} \leq \sqrt{\epsilon + \tilde{v}} \) and \( \sqrt{\mathbb{E}_{n-1} [g^2]} \leq R \), we can simplify (6.9) as

\[
\mathbb{E}_{n-1} [\kappa] \leq \frac{G^2}{4 \sqrt{\epsilon + \tilde{v}}} + R \mathbb{E}_{n-1} [g^2]. \tag{6.10}
\]

Now turning to \( \rho \), we use (6.7) with

\[
\lambda = \frac{\sqrt{\epsilon + \nu}}{\sqrt{\epsilon + \nu}}, \ x = \frac{|G|}{\sqrt{\epsilon + \nu}}, \ y = \frac{g^2}{\epsilon + \nu} \tag{6.11}
\]

we obtain

\[
\rho \leq \frac{G^2}{4 \sqrt{\epsilon + \nu}} \mathbb{E}_{n-1} [g^2] + \frac{\mathbb{E}_{n-1} [g^2]}{\epsilon + \nu} g^2. \tag{6.12}
\]

Given that \( \epsilon + \nu \geq g^2 \) and taking the conditional expectation we obtain

\[
\mathbb{E}_{n-1} [\rho] \leq \frac{G^2}{4 \sqrt{\epsilon + \nu}} + \frac{\mathbb{E}_{n-1} [g^2]}{\epsilon + \nu}, \tag{6.13}
\]

which we simplify using the same argument as for (6.9) into

\[
\mathbb{E}_{n-1} [\rho] \leq \frac{G^2}{4 \epsilon + \nu} + R \mathbb{E}_{n-1} [g^2]. \tag{6.14}
\]

Notice that in (6.10), we possibly divide by zero. It suffice to notice that if \( \mathbb{E}_{n-1} [g^2] = 0 \) then \( g^2 = 0 \) a.s. so that \( \rho = 0 \) and (6.12) is still verified.

Summing (6.9) and (6.12) we can bound

\[
\mathbb{E}_{n-1} [||A||] \leq \frac{G^2}{4 \epsilon + \nu} + 2 R \mathbb{E}_{n-1} [g^2]. \tag{6.15}
\]

Injecting (6.13) and (6.16) into (6.5) finishes the proof. \qed

Anticipating on Section 6.2, we can see that for a coordinate \( i \in [d] \) and iteration \( n \in \mathbb{N}^{*} \), the deviation from a descent direction is at most

\[
2 R \mathbb{E}_{n-1} \left[ \left( \nabla_i f_n(x_{n-1}) \right)^2 \right].
\]

While for any specific iteration, this deviation can take us away from a descent direction, the next lemma tells us that when we sum those deviations over all iterations, it cannot grow larger than a logarithmic term. This key insight introduced by Ward et al. (2019) is what makes the proof work.

**Lemma 6.2** (sum of ratios with the denominator increasing as the numerator). We assume we have \( 0 < \beta_2 \leq 1 \) and a non-negative sequence \( (a_n)_{n \in \mathbb{N}} \). We define \( b_n = \sum_{j=1}^{n} \beta_{n-j} a_j \) with the convention \( b_0 = 0 \). Then we have

\[
\sum_{j=1}^{N} \frac{a_j}{\epsilon + b_j} \leq \ln \left( 1 + \frac{b_N}{\epsilon} \right) - N \ln(\beta_2). \tag{6.16}
\]

**Proof.** Given that concavity of ln, and the fact that \( b_j > a_j \), we have for all \( j \in \mathbb{N}^{*} \),

\[
\frac{a_j}{\epsilon + b_j} \leq \ln(\epsilon + b_j) - \ln(\epsilon + b_j - a_j)
\]

\[
= \ln(\epsilon + b_j) - \ln(\epsilon + \beta_2 b_j - 1)
\]

\[
= \ln \left( \frac{\epsilon + b_j}{\epsilon + b_j - 1} \right) + \ln \left( \frac{\epsilon + b_j}{\epsilon + \beta_2 b_j - 1} \right).
\]
The first term on the right-hand side forms a telescoping series, while the last term is bounded by \( -\ln(\beta) \) as \( \epsilon \geq \beta \epsilon \). Increasing with \( b_{j-1} \) and thus is bounded by \( -\ln(\beta_2) \). Summing over all \( j \in [N] \) gives the desired result. \( \Box \)

### 6.2. Proof of Adam and Adagrad without momentum

For all iterations \( n \in \mathbb{N}^* \), we define the update \( u_n \in \mathbb{R}^d \),

\[
\forall i \in [d], u_{n,i} = \frac{\nabla_i f_n(x_{n-1})}{\sqrt{\epsilon + v_{n,i}}}.
\]

(6.15)

**Adam** Let us take an iteration \( n \in \mathbb{N}^* \). We note \( \alpha_n = \alpha \sqrt{\sum_{j=0}^{n-1} \beta_j^2} \) (see (2.4)) in Section 2.2. Using the smoothness of \( F \) defined in (2.8), we have

\[
F(x_n) \leq F(x_{n-1}) - \alpha_n \nabla F(x_{n-1})^T u_n + \frac{\alpha_n^2 L}{2} \| u_n \|_2^2.
\]

Notice that due to the a.s. \( \ell_\infty \) bound on the gradients (2.7), we have for any \( i \in [d] \), \( \sqrt{\epsilon + v_{n,i}} \leq R \sqrt{\sum_{j=0}^{n-1} \beta_j^2} \), so that,

\[
\alpha_n \frac{(\nabla_i F(x_{n-1}))^2}{2 \sqrt{\epsilon + v_{n,i}}} \geq \frac{\alpha ((\nabla_i F(x_{n-1}))^2)}{R}.
\]

(6.16)

Taking the conditional expectation with respect to \( f_0, \ldots, f_{n-1} \) we can apply the descent Lemma 6.1 and use (6.16) to obtain,

\[
\mathbb{E}_{n-1} [F(x_n)] \leq F(x_{n-1}) - \frac{\alpha}{2R} \| \nabla F(x_{n-1}) \|_2^2 + \left( 2\alpha_n R + \frac{\alpha_n^2 L}{2} \right) \mathbb{E}_{n-1} \left[ \| u_n \|_2^2 \right].
\]

Given that \( \beta_2 < 1 \), we have \( \alpha_n \leq \frac{\alpha}{\sqrt{1 - \beta_2}} \). Summing the previous inequality for all \( n \in [N] \) and taking the complete expectation yields

\[
\mathbb{E} [F(x_N)] \leq F(x_0) - \frac{\alpha}{2R} \sum_{n=0}^{N-1} \mathbb{E} \left[ \| \nabla F(x_n) \|_2^2 \right] + \left( 2\alpha R + \frac{\alpha^2 L}{2(1 - \beta_2)} \right) \sum_{n=0}^{N-1} \mathbb{E} \left[ \| u_n \|_2^2 \right].
\]

(6.17)

The application of Lemma 6.2 immediately gives for all \( i \in [d] \),

\[
\mathbb{E} \left[ \sum_{n=0}^{N-1} u_{n,i}^2 \right] \leq \ln \left( 1 + \frac{R^2}{(1 - \beta_2)\epsilon} \right) - N \ln(\beta).
\]

Injecting into (6.17) and rearranging the terms, the result of Theorem 1 follows immediately.

**Adagrad.** Let us now take \( \alpha_n = \alpha \) and \( \beta_2 \) to recover Adagrad. Using again the smoothness of \( F \) defined in (2.8), we have

\[
F(x_{n+1}) \leq F(x_n) - \alpha \nabla F(x_n)^T u_n + \frac{\alpha^2 L}{2} \| u_n \|_2^2.
\]

Notice that due to the a.s. \( \ell_\infty \) bound on the gradients (2.7), we have for any \( i \in [d] \), \( \sqrt{\epsilon + v_{n,i}} \leq R \sqrt{n} \), so that,

\[
\alpha \frac{(\nabla_i F(x_{n-1}))^2}{2 \sqrt{\epsilon + v_{n,i}}} \geq \frac{\alpha ((\nabla_i F(x_{n-1}))^2)}{2R \sqrt{n}}.
\]

(6.18)

Taking the conditional expectation with respect to \( f_0, \ldots, f_{n-1} \) we can apply the descent Lemma 6.1 and use (6.18) to obtain,

\[
\mathbb{E}_{n-1} [F(x_n)] \leq F(x_{n-1}) - \frac{\alpha}{2R} \| \nabla F(x_{n-1}) \|_2^2 + \left( 2\alpha R + \frac{\alpha^2 L}{2} \right) \mathbb{E}_{n-1} \left[ \| u_n \|_2^2 \right].
\]

Summing the previous inequality for all \( n \in [N] \), taking the complete expectation, and using that \( \sqrt{n} \leq \sqrt{N} \) gives us,

\[
\mathbb{E} [F(x_N)] \leq F(x_0) - \frac{\alpha}{2R \sqrt{N}} \sum_{n=0}^{N-1} \mathbb{E} \left[ \| \nabla F(x_n) \|_2^2 \right] + \left( 2\alpha R + \frac{\alpha^2 L}{2} \right) \sum_{n=0}^{N-1} \mathbb{E} \left[ \| u_n \|_2^2 \right].
\]

(6.19)

The application of Lemma 6.2 immediately gives for all \( i \in [d] \),

\[
\mathbb{E} \left[ \sum_{n=0}^{N-1} u_{n,i}^2 \right] \leq \ln \left( 1 + \frac{R^2 N}{\epsilon} \right).
\]

Injecting into (6.19) and rearranging the terms, the result of Theorem 2 follows immediately.

### 7. Conclusion

We provided a simple proof on the convergence of Adam and Adagrad without heavy-ball style momentum. The extension to include heavy-ball momentum is slightly more complex, but our approach leads to simpler proofs than previous ones, while significantly improving the dependence on the momentum parameter. The bounds clarify the important parameters for the convergence of Adam. A main practical takeaway, increasing the exponential decay factor is as critical as decreasing the learning rate for converging to a critical point. Our analysis also highlights a link between Adam and a finite-horizon version of Adam: for fixed \( N \), taking \( \alpha = 1/\sqrt{N} \) and \( \beta_2 = 1 - 1/N \) for Adam gives the same convergence bound as Adagrad.
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Appendix

A. Convergence of adaptive methods with heavy-ball momentum

A.1. Setup and notations

We recall the dynamic system introduced in Section 2.3. In the rest of this section, we take an iteration \( n \in \mathbb{N}^* \), and when needed, \( i \in [d] \) refers to a specific coordinate. Given \( x_0 \in \mathbb{R}^d \) our starting point, \( m_0 = 0 \), and \( v_0 = 0 \), we define

\[
\begin{align*}
    m_{n,i} &= \beta_1 m_{n-1,i} + \nabla_i f_n(x_{n-1}), \\
    v_{n,i} &= \beta_2 v_{n-1,i} + (\nabla_i f_n(x_{n-1}))^2, \\
    x_{n,i} &= x_{n-1,i} - \alpha_n \frac{m_{n,i}}{\sqrt{v_{n,i} + \epsilon}}.
\end{align*}
\]

(A.1)

For Adam, the step size is given by

\[
\alpha_n = \alpha (1 - \beta_1) \sqrt{\frac{n-1}{\sum_{j=0}^{n-1} \beta_j^2}}. \tag{A.2}
\]

For Adagrad (potentially extended with heavy-ball momentum), we have \( \beta_2 = 1 \) and

\[
\alpha_n = \alpha (1 - \beta_1). \tag{A.3}
\]

Notice we include the factor \( 1 - \beta_1 \) in the step size rather than in (A.1), as this allows for a more elegant proof. The original Adam algorithm included compensation factors for both \( \beta_1 \) and \( \beta_2 \) (Kingma & Ba, 2014) to correct the initial scale of \( m \) and \( v \) which are initialized at 0. Adam would be exactly recovered by replacing (A.2) with

\[
\alpha_n = \alpha \sqrt{\frac{\sum_{j=0}^{n-1} \beta_j^2}{\sum_{j=0}^{n-1} \beta_j^2}}. \tag{A.4}
\]

However, the denominator \( \sum_{j=0}^{n-1} \beta_j^2 \) potentially makes \((\alpha_n)_{n \in \mathbb{N}^*} \) non monotonic, which complicates the proof. Thus, we instead replace the denominator by its limit value for \( n \to \infty \). This has little practical impact as (i) early iterates are noisy because \( v \) is averaged over a small number of gradients, so making smaller step can be more stable, (ii) for \( \beta_1 = 0.9 \) (Kingma & Ba, 2014), (A.2) differs from (A.4) only for the first few tens of iterations. We could have replaced the numerator by its limit value but chose not to as: (i) \( \beta_2 \) is typically much closer to 1 than \( \beta_1 \), thus our update rule would have differed from Adam for longer (ii) without this correction, the step size is either too large early on or too small at the end.

Throughout the proof we note \( \mathbb{E}_{n-1} [\cdot] \) the conditional expectation with respect to \( f_1, \ldots, f_{n-1} \). In particular, \( x_{n-1}, v_{n-1} \) is deterministic knowing \( f_1, \ldots, f_{n-1} \). We introduce

\[
G_n = \nabla F(x_{n-1}) \quad \text{and} \quad g_n = \nabla f_n(x_{n-1}). \tag{A.5}
\]

Like in Section 6.2, we introduce the update \( u_n \in \mathbb{R}^d \), as well as the update without heavy-ball momentum \( U_n \in \mathbb{R}^d \):

\[
u_{n,i} = \frac{m_{n,i}}{\sqrt{v_{n,i} + \epsilon}} \quad \text{and} \quad U_{n,i} = \frac{g_{n,i}}{\sqrt{v_{n,i} + \epsilon}}. \tag{A.6}
\]

For any \( k \in \mathbb{N} \) with \( k < n \), we define \( \tilde{v}_{n,k} \in \mathbb{R}^d \) by

\[
\tilde{v}_{n,k,i} = \beta_2^{n-k} v_{n-k,i} + \mathbb{E}_{n-k-1} \left[ \sum_{j=n-k+1}^{n-1} \beta_2^{n-j} g_{j,i}^2 \right]. \tag{A.7}
\]

i.e. the contribution from the \( k \) last gradients are replaced by their expected value for know values of \( f_1, \ldots, f_{n-k-1} \). For \( k = 1 \), we recover the same definition as in (6.3).
A.2. Results

For any total number of iterations \( N \in \mathbb{N}^* \), we define \( \tau_N \) a random index with value in \( \{0, \ldots, N-1\} \), verifying

\[
\forall j \in \mathbb{N}, j < N, \mathbb{P} [ \tau = j ] \propto 1 - \beta_1^{N-j}.
\]

(A.8)

If \( \beta_1 = 0 \), this is equivalent to sampling \( \tau \) uniformly in \( \{0, \ldots, N-1\} \). If \( \beta_1 > 0 \), the last few \( \frac{1}{\beta_1} \) iterations are sampled rarely, and all iterations older than a few times that number are sampled almost uniformly. We bound the expected squared norm of the total gradient at iteration \( \tau \), which is standard for non convex stochastic optimization (Ghadimi & Lan, 2013).

Note that we have improved the results compared with the one stated in Section 5, in particular the dependency in \( (1 - \beta_1) \) which is now \( (1 - \beta_1)^{-1} \). This is a significant improvement over the existing bound for Adagrad with heavy-ball momentum, which scale as \( (1 - \beta_1)^{-3} \) (Zou et al., 2019a), or the best known bound for Adam which scale as \( (1 - \beta_1)^{-5} \) (Zou et al., 2019b).

Technical lemmas to prove the following theorems are introduced in Section A.4, while the proof of Theorems A.1 and A.2 are provided in Section A.5.

**Theorem A.1** (Convergence of Adam with momentum). Given the hypothesis introduced in Section 2.3, the iterates \( x_n \) defined in Section 2.2 with hyper-parameters verifying \( 0 < \beta_2 < 1 \), \( \alpha_n = \alpha (1 - \beta_1) \sqrt{\sum_{j=0}^{n-1} \beta_2^j} \) with \( \alpha > 0 \) and \( 0 < \beta_1 < \beta_2 \), we have for any \( N \in \mathbb{N}^* \) such that \( N > \frac{\beta_1}{1 - \beta_1} \), taking \( \tau \) defined by (A.8),

\[
\mathbb{E} \left[ \| \nabla F(x_\tau) \|^2 \right] \leq 2R \frac{F(x_0) - F_*}{\alpha N} + \frac{E}{N} \left( \ln \left( 1 + \frac{R^2}{\epsilon (1 - \beta_2)} \right) - N \ln(\beta_2) \right),
\]

(A.9)

with

\[
\tilde{N} = N - \frac{\beta_1}{1 - \beta_1}.
\]

and

\[
E = \frac{\alpha dR (1 - \beta_1)}{(1 - \beta_1/\beta_2)(1 - \beta_2)} + \frac{12dR^2 \sqrt{1 - \beta_1}}{(1 - \beta_1/\beta_2)^{3/2} \sqrt{1 - \beta_2}} + \frac{2\alpha^2 dL^2 \beta_1}{(1 - \beta_1/\beta_2)(1 - \beta_2)^{3/2}}.
\]

(A.10)

**Theorem A.2** (Convergence of Adagrad with momentum). Given the hypothesis introduced in Section 2.3, the iterates \( x_n \) defined in Section 2.2 with hyper-parameters verifying \( \beta_2 = 1 \), \( \alpha_n = (1 - \beta_1) \alpha \) with \( \alpha > 0 \) and \( 0 < \beta_1 < 1 \), we have for any \( N \in \mathbb{N}^* \) such that \( N > \frac{\beta_1}{1 - \beta_1} \), taking \( \tau \) as defined by (A.8),

\[
\mathbb{E} \left[ \| \nabla F(x_\tau) \|^2 \right] \leq 2R \sqrt{N} \frac{F(x_0) - F_*}{\alpha N} + \frac{\sqrt{N}}{N} \left( \alpha dR L + \frac{12dR^2}{1 - \beta_1} + \frac{2\alpha^2 dL^2 \beta_1}{1 - \beta_1} \right) \ln \left( 1 + \frac{NR^2}{\epsilon} \right),
\]

(A.11)

with

\[
\tilde{N} = N - \frac{\beta_1}{1 - \beta_1}.
\]

(A.12)

A.3. Analysis of the results with momentum

First notice that taking \( \beta_1 \to 0 \) in Theorems A.1 and A.2, we almost recover the same result as stated in 1 and 2, only losing on the term \( 4dR^2 \) which becomes \( 12dR^2 \).

**Simplified expressions with momentum**  Assuming \( N \gg \frac{\beta_1}{1 - \beta_1} \) and \( \beta_1/\beta_2 \approx \beta_1 \), which is verified for typical values of \( \beta_1 \) and \( \beta_2 \) (Kingma & Ba, 2014), it is possible to simplify the bound for Adam (A.9) as

\[
\mathbb{E} \left[ \| \nabla F(x_\tau) \|^2 \right] \approx 2R \frac{F(x_0) - F_*}{\alpha N} + \left( \alpha dR L + \frac{12dR^2}{1 - \beta_1} + \frac{2\alpha^2 dL^2 \beta_1}{1 - \beta_1} \right) \left( \frac{1}{N} \ln \left( 1 + \frac{R^2}{\epsilon (1 - \beta_2)} \right) - \ln(\beta_2) \right).
\]

(A.13)

Similarly, if we assume \( N \gg \frac{\beta_1}{1 - \beta_1} \), we can simplify the bound for Adagrad (A.12) as

\[
\mathbb{E} \left[ \| \nabla F(x_\tau) \|^2 \right] \approx 2R \frac{F(x_0) - F_*}{\alpha \sqrt{N}} + \frac{1}{\sqrt{N}} \left( \alpha dR L + \frac{12dR^2}{1 - \beta_1} + \frac{2\alpha^2 dL^2 \beta_1}{1 - \beta_1} \right) \ln \left( 1 + \frac{NR^2}{\epsilon} \right),
\]

(A.14)
Optimal finite horizon Adam is still Adagrad  We can perform the same finite horizon analysis as in Section 5.3. If we take \( \alpha = \frac{\alpha}{\sqrt{N}} \) and \( \beta_2 = 1 - 1/N \), then (A.14) simplifies to

\[
\mathbb{E} \left[ \left\| \nabla F(x_t) \right\|^2 \right] \leq 2R \frac{F(x_0) - F_*}{\alpha \sqrt{N}} + \frac{1}{\sqrt{N}} \left( \hat{\sigma} d R L + \frac{12dR^2}{1 - \beta_1} + \frac{2\hat{\sigma}^2 d L^2 \beta_1}{1 - \beta_1} \right) \left( \ln \left( 1 + \frac{NR^2}{\epsilon} \right) + 1 \right).
\]

(A.16)

The term \((1 - \beta_2)^{3/2}\) in the denominator in (A.14) is indeed compensated by the \(\alpha^2\) in the numerator and we again recover the proper \(\ln(N)/\sqrt{N}\) convergence rate, which matches (A.15) up to a \(+1\) term next to the log.

A.4. Technical lemmas

We first need an updated version of 6.1 that includes momentum.

Lemma A.1 (Adaptive update with momentum approximately follows a descent direction). Given \(x_0 \in \mathbb{R}^d\), the iterates defined by the system (A.1) for \((\alpha_j) \in \mathbb{N}^*\) that is non-decreasing, and under the conditions (2.5), (2.7), and (2.8), as well as \(0 \leq \beta_1 < \beta_2 \leq 1\), we have for all iterations \(n \in \mathbb{N}^*\),

\[
\mathbb{E} \left[ \sum_{i \in [d]} G_{n,i} \frac{m_{n,i}}{\epsilon + v_{n,i}} \right] \geq \frac{1}{2} \left( \sum_{i \in [d]} \sum_{k=0}^{n-1} \beta_k^2 \mathbb{E} \left[ \frac{G_{n-k,i}^2}{\epsilon + v_{n,k+1,i}} \right] \right) - \alpha_n^2 \frac{L^2}{4R} \sqrt{1 - \beta_1} \sum_{k=0}^{n-1} \left( \frac{1}{\epsilon + v_{n-k,i}} \right) - \frac{3R}{\sqrt{1 - \beta_1}} \left( \sum_{k=0}^{n-1} \frac{\beta_k^2}{\beta_2} \right) \sqrt{k + 1} \left( \frac{\epsilon + v_{n-k,i}}{\epsilon + v_{n,i}} \right)^2.
\]

(A.17)

Proof.  We use multiple times (6.7) in this proof, which we repeat here for convenience,

\[
\forall \lambda > 0, x, y \in \mathbb{R}, xy \leq \frac{\lambda}{2} x^2 + \frac{y^2}{2\lambda}.
\]

(A.18)

Let us take an iteration \(n \in \mathbb{N}^*\) for the duration of the proof. We have

\[
\sum_{i \in [d]} G_{n,i} \frac{m_{n,i}}{\epsilon + v_{n,i}} = \sum_{i \in [d]} \sum_{k=0}^{n-1} \beta_k^2 G_{n,i} \frac{g_{n-k,i}}{\epsilon + v_{n,i}} = \sum_{i \in [d]} \sum_{k=0}^{n-1} \beta_k^2 \frac{G_{n-k,i}}{\epsilon + v_{n,i}} + \sum_{i \in [d]} \sum_{k=0}^{n-1} \beta_k^2 \frac{(G_{n,k,i} - G_{n-k,i})}{\epsilon + v_{n,i}}.
\]

(A.19)

Let us now take an index \(0 \leq k \leq n - 1\). We show that the contribution of past gradients \(G_{n-k}\) and \(g_{n-k}\) due to the heavy-ball momentum can be controlled thanks to the decay term \(\beta_k^2\). Let us first have a look at \(B\). Using (A.18) with

\[
\lambda = \frac{\sqrt{1 - \beta_1}}{2R \sqrt{k + 1}}, x = \left| G_{n,i} - G_{n-k,i} \right|, y = \frac{|g_{n-k,i}|}{\epsilon + v_{n,i}},
\]

we have

\[
|B| \leq \sum_{i \in [d]} \sum_{k=0}^{n-1} \beta_k^2 \left( \frac{\sqrt{1 - \beta_1}}{4R \sqrt{k + 1}} \left( G_{n,i} - G_{n-k,i} \right)^2 + \frac{R}{\sqrt{1 - \beta_1}} \sqrt{k + 1} \frac{g_{n-k,i}^2}{\epsilon + v_{n,i}} \right).
\]

(A.20)

Notice first that for any dimension \(i \in [d]\), \(\epsilon + v_{n,i} \geq \epsilon + \beta_k^2 v_{n-k,i} \geq \beta_k^2 (\epsilon + v_{n-k,i})\), so that

\[
\frac{g_{n-k,i}^2}{\epsilon + v_{n,i}} \leq \frac{1}{\beta_k^2} \frac{g_{n-k,i}^2}{v_{n-k,i}}
\]

(A.21)
Besides, using the L-smoothness of $F$ given by (2.8), we have
\[
\|G_n - G_{n-k}\|_2^2 \leq L^2 \|x_{n-1} - x_{n-k-1}\|_2^2
\]
\[
= L^2 \left\| \sum_{l=1}^{k} \alpha_{n-l} u_{n-l} \right\|_2^2
\]
\[
\leq \alpha_n^2 L^2 k \sum_{l=1}^{k} \|u_{n-l}\|_2^2,
\]
(A.22)
using Jensen inequality and the fact that $\alpha_n$ is non-decreasing. Injecting (A.21) and (A.23) into (A.20), we obtain
\[
|B| \leq \left( \sum_{k=0}^{n-1} \frac{\alpha_n^2 L^2}{4R} \sqrt{1 - \beta_1 \beta_2^k} \sqrt{k} \sum_{l=1}^{k} \|u_{n-l}\|_2^2 \right) + \left( \sum_{k=0}^{n-1} \frac{R}{\sqrt{1 - \beta_1}} \left( \beta_1 \right)^{k} \sqrt{k + 1} \|U_{n-k}\|_2^2 \right)
\]
\[
= \frac{\alpha_n^2 L^2}{4R} \sqrt{1 - \beta_1} \left( \sum_{l=0}^{n-1} \|u_{n-l}\|_2^2 \sum_{k=0}^{n-1} \beta_2^l \sqrt{l} \right) + \frac{R}{\sqrt{1 - \beta_1}} \left( \sum_{k=0}^{n-1} \left( \beta_1 \right)^{k} \sqrt{k + 1} \|U_{n-k}\|_2^2 \right).
\]
(A.24)
Now going back to the $A$ term in (A.19), we will study the main term of the summation, i.e. for $i \in [d]$ and $k < n$
\[
\mathbb{E} \left[ G_{n-k,i} g_{n-k,i} \right] = \mathbb{E} \left[ \nabla_i F(x_{n-1}) \nabla_i f_{n-k}(x_{n-1}) \right].
\]
(A.25)
Notice that we could almost apply Lemma 6.1 to it, except that we have $v_{n,i}$ in the denominator instead of $v_{n,k,i}$. Thus we will need to extend the proof to decorrelate more terms. We will further drop indices in the rest of the proof, noting $G = G_{n-k,i}$, $g = g_{n-k,i}$, $\tilde{v} = \tilde{v}_{n,k+1,i}$ and $v = v_{n,i}$. Finally, let us note
\[
\delta^2 = \sum_{j=n-k}^{n} \beta_2^{n-j} g_{j,i} \quad \text{and} \quad r^2 = \mathbb{E}_{n-k} \left[ \delta^2 \right].
\]
(A.26)
In particular we have $\tilde{v} - v = r^2 - \delta^2$. With our new notations, we can rewrite (A.25) as
\[
\mathbb{E} \left[ G g \sqrt{\frac{1}{\epsilon + \tilde{v}}} \right] = \mathbb{E} \left[ G g \sqrt{\frac{1}{\epsilon + v}} + G g \left( \frac{1}{\sqrt{\epsilon + \tilde{v}}} - \frac{1}{\sqrt{\epsilon + v}} \right) \right]
\]
\[
= \mathbb{E} \left[ G g \sqrt{\frac{1}{\epsilon + v}} \right] + G g \frac{r^2 - \delta^2}{\sqrt{\epsilon + v} \sqrt{\epsilon + \tilde{v}} \sqrt{\epsilon + v} \sqrt{\epsilon + \tilde{v}}}
\]
\[
= \mathbb{E} \left[ G g \sqrt{\frac{1}{\epsilon + v}} \right] + G g \frac{r^2 - \delta^2}{\epsilon + v} \sqrt{\epsilon + v} \sqrt{\epsilon + \tilde{v}} \sqrt{\epsilon + v} \sqrt{\epsilon + \tilde{v}}
\]
\[
\leq \frac{G^2}{\sqrt{\epsilon + v}} + \frac{r^2 - \delta^2}{\epsilon + v} \sqrt{\epsilon + v} \sqrt{\epsilon + \tilde{v}} \sqrt{\epsilon + v} \sqrt{\epsilon + \tilde{v}}
\]
(A.27)
We first focus on $C$:
\[
|C| \leq |G g| \frac{r^2}{\sqrt{\epsilon + v} \sqrt{\epsilon + \tilde{v}}} + |G g| \frac{\delta^2}{(\epsilon + v) \sqrt{\epsilon + \tilde{v}}}
\]
due to the fact that $\sqrt{\epsilon + v} \sqrt{\epsilon + \tilde{v}} \geq \max(\sqrt{\epsilon + v}, \sqrt{\epsilon + \tilde{v}})$ and $|r^2 - \delta^2| \leq r^2 + g^2$.
Applying (A.18) to $\kappa$ with
\[
\lambda = \frac{\sqrt{1 - \beta_1} \sqrt{\epsilon + \tilde{v}}}{2}, \quad x = \frac{|G|}{\sqrt{\epsilon + v}}, \quad y = \frac{|g| r^2}{\sqrt{\epsilon + \tilde{v} \sqrt{\epsilon + v}}}
\]
we obtain
\[
\kappa \leq \frac{G^2}{4 \sqrt{\epsilon + v}} + \frac{1}{\sqrt{1 - \beta_1}} \frac{g^2 r^4}{(\epsilon + \tilde{v})^{3/2} (\epsilon + v)}.
\]
Given that $\epsilon + \tilde{v} \geq r^2$ and taking the conditional expectation, we can simplify as

$$
\mathbb{E}_{n-k-1}[\rho] \leq \frac{G^2}{4\sqrt{\epsilon + \tilde{v}}} + \frac{1}{\sqrt{1 - \beta_1}} \frac{r^2}{\sqrt{\epsilon + \tilde{v}}} \mathbb{E}_{n-k-1} \left[ \frac{g^2}{\epsilon + \tilde{v}} \right].
$$

(A.28)

Now turning to $\rho$, we use (A.18) with

$$
\lambda = \frac{\sqrt{1 - \beta_1} \sqrt{\epsilon + \tilde{v}}}{2r^2},
$$

we obtain

$$
\rho \leq \frac{G^2}{4\sqrt{\epsilon + \tilde{v}}} + \frac{1}{\sqrt{1 - \beta_1}} \frac{r^2}{\sqrt{\epsilon + \tilde{v}}} \frac{g^2\delta^2}{(\epsilon + \tilde{v})^2}.
$$

(A.29)

Given that $\epsilon + v \geq \delta^2$, and $\mathbb{E}_{n-k-1} \left[ \frac{g^2}{\epsilon + \tilde{v}} \right] = 1$, we obtain after taking the conditional expectation,

$$
\mathbb{E}_{n-k-1}[\rho] \leq \frac{G^2}{2\sqrt{\epsilon + \tilde{v}}} + \frac{2}{\sqrt{1 - \beta_1}} \frac{r^2}{\sqrt{\epsilon + \tilde{v}}} \mathbb{E}_{n-k-1} \left[ \frac{g^2}{\epsilon + \tilde{v}} \right].
$$

(A.30)

Notice that in (A.29), we possibly divide by zero. It suffice to notice that if $r^2 = 0$ then $\delta^2 = 0$ a.s. so that $\rho = 0$ and (A.30) is still verified. Summing (A.28) and (A.30), we get

$$
\mathbb{E}_{n-k-1} [\| C \|] \leq \frac{G^2}{2\sqrt{\epsilon + \tilde{v}}} + \frac{2}{\sqrt{1 - \beta_1}} \frac{r^2}{\sqrt{\epsilon + \tilde{v}}} \mathbb{E}_{n-k-1} \left[ \frac{g^2}{\epsilon + \tilde{v}} \right].
$$

(A.31)

Given that $r \leq \sqrt{\epsilon + \tilde{v}}$ by definition of $\tilde{v}$, and that using (2.7), $r \leq \sqrt{k + 1} R$, we have, reintroducing the indices we had dropped

$$
\mathbb{E}_{n-k-1} [\| C \|] \leq \frac{G^2_{n-k,i}}{2\sqrt{\epsilon + \tilde{v}_{n,k,i}}} + \frac{2R}{\sqrt{1 - \beta_1}} \frac{\sqrt{k + 1} \mathbb{E}_{n-k-i} \left[ \frac{g^2_{n-k,i}}{\epsilon + \tilde{v}_{n,k,i}} \right]}{\sqrt{\epsilon + \tilde{v}_{n,k,i}}}.
$$

(A.32)

Taking the complete expectation and using that by definition $\epsilon + v_{n,i} \geq \epsilon + \beta_2^k v_{n-k,i} \geq \beta_2^k (\epsilon + v_{n-k,i})$ we get

$$
\mathbb{E} \left[ \| C \| \right] \leq \frac{1}{2} \mathbb{E} \left[ \frac{G^2_{n-k,i}}{\sqrt{\epsilon + \tilde{v}_{n,k,i}}} \right] + \frac{2R}{\sqrt{1 - \beta_1}} \frac{\sqrt{k + 1} \mathbb{E} \left[ \frac{g^2_{n-k,i}}{\epsilon + \tilde{v}_{n,k,i}} \right]}{\sqrt{\epsilon + \tilde{v}_{n,k,i}}}.
$$

(A.33)

Injecting (A.33) into (A.27) gives us

$$
\mathbb{E} \left[ A \right] \geq \sum_{i \in [d]} \sum_{k=0}^{n-1} \beta_1^k \left( \mathbb{E} \left[ \frac{G^2_{n-k,i}}{\sqrt{\epsilon + \tilde{v}_{n,k,i}}} \right] - \frac{1}{2} \mathbb{E} \left[ \frac{G^2_{n-k,i}}{\sqrt{\epsilon + \tilde{v}_{n,k,i}}} \right] + \frac{2R}{\sqrt{1 - \beta_1}} \frac{\sqrt{k + 1} \mathbb{E} \left[ \frac{g^2_{n-k,i}}{\epsilon + \tilde{v}_{n,k,i}} \right]}{\sqrt{\epsilon + \tilde{v}_{n,k,i}}} \right)
$$

$$
= \frac{1}{2} \left( \sum_{i \in [d]} \sum_{k=0}^{n-1} \beta_1^k \mathbb{E} \left[ \frac{G^2_{n-k,i}}{\sqrt{\epsilon + \tilde{v}_{n,k,i}}} \right] - \frac{2R}{\sqrt{1 - \beta_1}} \left( \sum_{i \in [d]} \sum_{k=0}^{n-1} \beta_1^k \mathbb{E} \left[ \frac{g^2_{n-k,i}}{\epsilon + \tilde{v}_{n,k,i}} \right] \right) \right). \quad (A.34)
$$

Injecting (A.34) and (A.24) into (A.19) finishes the proof.

\square

Similarly, we will need an updated version of 6.2.

**Lemma A.2** (sum of ratios of the square of a decayed sum and a decayed sum of square). We assume we have $0 < \beta_2 \leq 1$ and $0 < \beta_1 < \beta_2$, and a sequence of real numbers $(a_n)_{n \in \mathbb{N}}$. We define $b_n = \sum_{j=1}^{n} \beta_2^{n-j} a_j$ and $c_n = \sum_{j=1}^{n} \beta_1^{n-j} a_j$. Then we have

$$
\sum_{j=1}^{n} \frac{c_j^2}{\epsilon + b_j} \leq \frac{1}{(1 - \beta_1)(1 - \beta_1/\beta_2)} \left( \ln \left( 1 + \frac{b_n}{\epsilon} \right) - n \ln(\beta_2) \right). \quad (A.35)
$$
Proof. Now let us take \( j \in \mathbb{N}^*, j \leq n \), we have using Jensen inequality

\[
c_j^2 \leq \frac{1}{1 - \beta_1} \sum_{l=1}^{j} \beta_1^{j-l} a_l^2,
\]

so that

\[
\frac{c_j^2}{\epsilon + b_j} \leq \frac{1}{1 - \beta_1} \sum_{l=1}^{j} \beta_1^{j-l} \frac{a_l^2}{\epsilon + b_j}.
\]

Given that for \( l \in [j] \), we have by definition \( \epsilon + b_j \geq \epsilon + \beta_2^{j-l} b_l \geq \beta_2^{j-l} (\epsilon + b_j) \), we get

\[
\frac{c_j^2}{\epsilon + b_j} \leq \frac{1}{1 - \beta_1} \sum_{l=1}^{j} \left( \beta_1 \beta_2 \right)^{j-l} \frac{a_l^2}{\epsilon + b_l}.
\]

(A.36)

Thus, when summing over all \( j \in [n] \), we get

\[
\sum_{j=1}^{n} \frac{c_j^2}{\epsilon + b_j} \leq \frac{1}{1 - \beta_1} \sum_{j=1}^{n} \sum_{l=1}^{j} \left( \beta_1 \beta_2 \right)^{j-l} \frac{a_l^2}{\epsilon + b_l}
\]

\[
= \frac{1}{1 - \beta_1} \sum_{l=1}^{n} \frac{a_l^2}{\epsilon + b_l} \sum_{j=l}^{n} \left( \beta_1 \beta_2 \right)^{j-l}
\]

\[
\leq \frac{1}{(1 - \beta_1)(1 - \beta_1/\beta_2)} \sum_{l=1}^{n} \frac{a_l^2}{\epsilon + b_l}.
\]

(A.37)

Applying Lemma 6.2, we obtain (A.35).

We also need two technical lemmas on the sum of series.

**Lemma A.3** (sum of a geometric term times a square root). *Given* \( 0 < a < 1 \) and \( Q \in \mathbb{N} \), *we have,*

\[
\sum_{q=0}^{Q-1} a^q \sqrt{q + 1} \leq \frac{1}{1 - a} \left( 1 + \frac{\sqrt{\pi}}{2 \sqrt{-\ln(a)}} \right) \leq \frac{2}{(1 - a)^{3/2}}.
\]

(A.38)

**Proof.** We first need to study the following integral:

\[
\int_0^\infty \frac{a^x}{2 \sqrt{x}} \, dx = \int_0^\infty \frac{e^{\ln(a) x}}{2 \sqrt{x}} \, dx, \quad \text{then introducing} \quad y = \sqrt{x},
\]

\[
= \int_0^\infty e^{\ln(a) y^2} \, dy, \quad \text{then introducing} \quad u = \sqrt{-2 \ln(a)} y,
\]

\[
= \frac{1}{\sqrt{-2 \ln(a)}} \int_0^\infty e^{-u^2/2} \, du
\]

\[
\int_0^\infty \frac{a^x}{2 \sqrt{x}} \, dx = \frac{\sqrt{\pi}}{2 \sqrt{-\ln(a)}},
\]

(A.39)

where we used the classical integral of the standard Gaussian density function.

Let us now introduce \( A_Q \):

\[
A_Q = \sum_{q=0}^{Q-1} a^q \sqrt{q + 1},
\]
then we have

\[
A_Q - aA_Q = \sum_{q=0}^{Q-1} a^q \sqrt{q} + 1 - \sum_{q=1}^{Q} a^q \sqrt{q}, \quad \text{then using the concavity of } \sqrt{\cdot},
\]

\[
\leq 1 - aQ \sqrt{Q} + \sum_{q=1}^{Q-1} a^q \sqrt{\frac{q}{q+1}}
\]

\[
\leq 1 + \int_{0}^{\infty} \frac{a^x}{2\sqrt{x}} \, dx
\]

\[
(1 - a)A_Q \leq 1 + \frac{\sqrt{\pi}}{2\sqrt{-\ln(a)}},
\]

where we used (A.39). Given that \(\sqrt{-\ln(a)} \geq \sqrt{1 - a}\) we obtain (A.38).

\[\square\]

**Lemma A.4** (sum of a geometric term times roughly a power 3/2). Given \(0 < a < 1\) and \(Q \in \mathbb{N}\), we have,

\[
\sum_{q=0}^{Q-1} a^q \sqrt{q}(q + 1) \leq \frac{4a}{(1 - a)^{3/2}}.
\]  
\[\text{(A.40)}\]

**Proof.** Let us introduce \(A_Q\):

\[
A_Q = \sum_{q=0}^{Q-1} a^q \sqrt{q}(q + 1),
\]

then we have

\[
A_Q - aA_Q = \sum_{q=0}^{Q-1} a^q \sqrt{q}(q + 1) - \sum_{q=1}^{Q} a^q \sqrt{q-1}
\]

\[
\leq \sum_{q=1}^{Q-1} a^q \sqrt{q} \left( (q + 1) - \sqrt{q} \sqrt{q-1} \right)
\]

\[
\leq \sum_{q=1}^{Q-1} a^q \sqrt{q} \left( (q + 1) - (q - 1) \right)
\]

\[
\leq 2a \sum_{q=1}^{Q-1} a^q \sqrt{q}
\]

\[
= 2a \sum_{q=0}^{Q-2} a^q \sqrt{q+1}, \quad \text{then using Lemma A.3},
\]

\[
(1 - a)A_Q \leq \frac{4a}{(1 - a)^{3/2}}.
\]  
\[\square\]

**A.5. Proof of Adam and Adagrad with momentum**

**Common part of the proof** Let us a take an iteration \(n \in \mathbb{N}^*\). Using the smoothness of \(F\) defined in (2.8), we have

\[
F(x_n) \leq F(x_{n-1}) - \alpha_n G_n^T u_n + \frac{\alpha_n^2 L}{2} \|u_n\|^2.
\]
Taking the full expectation and using Lemma A.1,

\[
\mathbb{E}[F(x_n)] \leq \mathbb{E}[F(x_{n-1})] - \frac{\alpha_n}{2} \left( \sum_{i \in [d]} \sum_{k=0}^{n-1} \beta_1^k \mathbb{E} \left[ \frac{G_{n-k,i}^2}{2\sqrt{\epsilon + v_{n,k+1,i}}} \right] \right) + \frac{\alpha_n^2 L^2}{2} \mathbb{E} \left[ \|u_n\|^2 \right] \\
+ \frac{\alpha_n^2 L^2}{4R} \sqrt{1 - \beta_1} \left( \sum_{l=0}^{n-1} \|u_{n-l}\|^2 \sum_{k=1}^{n-1} \beta_1^k \sqrt{l} \right) + \frac{3\alpha_n R}{\sqrt{1 - \beta_1}} \left( \sum_{k=0}^{n-1} \left( \frac{\beta_1}{\beta_2} \right)^k \right) \sqrt{k + 1} \|U_{n-k}\|^2 \right).
\]

(A.41)

Notice that because of the bound on the \( \ell_{\infty} \) norm of the stochastic gradients at the iterates (2.7), we have for any \( k \in \mathbb{N}, k < n \), and any coordinate \( i \in [d], \sqrt{\epsilon + v_{n,k,i}} \leq R \sum_{j=0}^{n-1} \beta_2^j \). Introducing \( \Omega_n = \sqrt{\sum_{j=0}^{n-1} \beta_2^j} \), we have

\[
\mathbb{E}[F(x_n)] \leq \mathbb{E}[F(x_{n-1})] - \frac{\alpha_n}{2R \Omega_n} \sum_{k=0}^{n-1} \beta_1^k \mathbb{E} \left[ \|G_{n-k}\|^2 \right] + \frac{\alpha_n^2 L^2}{2} \mathbb{E} \left[ \|u_n\|^2 \right] \\
+ \frac{\alpha_n^2 L^2}{4R} \sqrt{1 - \beta_1} \left( \sum_{l=0}^{n-1} \|u_{n-l}\|^2 \sum_{k=1}^{n-1} \beta_1^k \sqrt{l} \right) + \frac{3\alpha_n R}{\sqrt{1 - \beta_1}} \left( \sum_{k=0}^{n-1} \left( \frac{\beta_1}{\beta_2} \right)^k \right) \sqrt{k + 1} \|U_{n-k}\|^2 \right).
\]

(A.42)

Now summing over all iterations \( n \in [N] \) for \( N \in \mathbb{N}^* \), and using that for both Adam (A.2) and Adagrad (A.3), \( \alpha_n \) is non-decreasing, as well the fact that \( F \) is bounded below by \( F_* \) from (2.5), we get

\[
\frac{1}{2R} \sum_{n=1}^{N} \Omega_n \sum_{k=0}^{n-1} \beta_1^k \mathbb{E} \left[ \|G_{n-k}\|^2 \right] \leq F(x_0) - F_* + \frac{\alpha_n^2 L^2}{2} \sum_{n=1}^{N} \mathbb{E} \left[ \|u_n\|^2 \right] \\
+ \frac{\alpha_n^2 L^2}{4R} \sqrt{1 - \beta_1} \sum_{n=1}^{N} \sum_{k=0}^{n-1} \mathbb{E} \left[ \|u_{n-k}\|^2 \right] \sum_{l=k}^{n-1} \beta_1^l \sqrt{l} + \frac{3\alpha_n R}{\sqrt{1 - \beta_1}} \sum_{n=1}^{N} \sum_{k=0}^{n-1} \left( \frac{\beta_1}{\beta_2} \right)^k \sqrt{k + 1} \mathbb{E} \left[ \|U_{n-k}\|^2 \right].
\]

(A.43)

First looking at \( B \), we have using Lemma A.2,

\[
B \leq \frac{\alpha_n^2 L}{2(1 - \beta_1)(1 - \beta_1 / \beta_2)} \sum_{i \in [d]} \left( \ln \left( 1 + \frac{v_{N,i}}{\epsilon} \right) - N \log(\beta_2) \right).
\]

(A.44)

Then looking at \( C \) and introducing the change of index \( j = n - k \),

\[
C = \frac{\alpha_n^3 L^2}{4R} \sqrt{1 - \beta_1} \sum_{n=1}^{N} \sum_{j=1}^{n} \mathbb{E} \left[ \|u_j\|^2 \right] \sum_{l=n-j}^{n-1} \beta_1^l \sqrt{l}
\]

\[
= \frac{\alpha_n^3 L^2}{4R} \sqrt{1 - \beta_1} \sum_{j=1}^{N} \mathbb{E} \left[ \|u_j\|^2 \right] \sum_{n-j=l}^{n-1} \sum_{l=1}^{n-j} \beta_1^l \sqrt{l}
\]

\[
= \frac{\alpha_n^3 L^2}{4R} \sqrt{1 - \beta_1} \sum_{j=1}^{N} \mathbb{E} \left[ \|u_j\|^2 \right] \sum_{l=0}^{N-1} \sqrt{l} \sum_{j=l}^{N-1} \beta_1^l \sqrt{l}
\]

\[
= \frac{\alpha_n^3 L^2}{4R} \sqrt{1 - \beta_1} \sum_{j=1}^{N} \mathbb{E} \left[ \|u_j\|^2 \right] \sum_{l=0}^{N-1} \beta_1^l \sqrt{l} \sum_{j=0}^{N-1} \beta_1^l \sqrt{l}
\]

\[
\leq \frac{\alpha_n^3 L^2}{R} \sum_{j=1}^{N} \mathbb{E} \left[ \|u_j\|^2 \right] \frac{\beta_1}{(1 - \beta_1)^2},
\]

(A.45)
On the Convergence of Adam and Adagrad

using Lemma A.4. Finally, using Lemma A.2, we get

\[ C \leq \frac{\alpha_n^3 dL^2 \beta_1}{R(1 - \beta_1)\beta(1 - \beta_1/\beta_2)} \sum_{i \in [d]} \left( \ln \left(1 + \frac{v_{N,i}}{\epsilon}\right) - N \log(\beta_2) \right). \]  \hfill (A.46)

Finally, introducing the same change of index \( j = n - k \) for \( D \), we get

\[ D = \frac{3\alpha N R}{\sqrt{1 - \beta_1}} \sum_{n=1}^{N} \sum_{j=1}^{n} \left( \frac{\beta_1}{\beta_2} \right)^{n-j} \sqrt{1 + n - j} \mathbb{E} [\|U_j\|_2^2] \]

\[ = \frac{3\alpha N R}{\sqrt{1 - \beta_1}} \sum_{j=1}^{N} \mathbb{E} [\|U_j\|_2^2] \sum_{n=j}^{N} \left( \frac{\beta_1}{\beta_2} \right)^{n-j} \sqrt{1 + n - j} \]

\[ \leq \frac{3\alpha N R}{\sqrt{1 - \beta_1}} \sum_{j=1}^{N} \mathbb{E} [\|U_j\|_2^2] \frac{1}{(1 - \beta_1/\beta_2)^{3/2}}, \]  \hfill (A.47)

using Lemma A.3. Finally, using Lemma 6.2 or equivalently Lemma A.2 with \( \beta_1 = 0 \), we get

\[ D \leq \frac{6\alpha N dR}{\sqrt{1 - \beta_1}} \sum_{i \in [d]} \left( \ln \left(1 + \frac{v_{N,i}}{\epsilon}\right) - N \log(\beta_2) \right). \]  \hfill (A.48)

This is as far as we can get without having to use the specific form of \( \alpha_n \) given by either (A.2) for Adam or (A.3) for Adagrad. We will now split the proof for either algorithm.

**Adam** For Adam, using (A.2), we have \( \alpha_n = (1 - \beta_1)\Omega_n \alpha \). Thus, we can simplify the \( A \) term from (A.43), also using the usual change of index \( j = n - k \), to get

\[ A = \frac{1}{2R} \sum_{n=1}^{N} \frac{\alpha_n}{\Omega_n} \sum_{j=1}^{n} \beta_1^{n-j} \mathbb{E} [\|G_j\|_2^2] \]

\[ = \frac{\alpha (1 - \beta_1)}{2R} \sum_{j=1}^{N} \mathbb{E} [\|G_j\|_2^2] \sum_{n=j}^{N} \beta_1^{n-j} \]

\[ = \frac{\alpha}{2R} \sum_{j=1}^{N} (1 - \beta_1^{N-j+1}) \mathbb{E} [\|G_j\|_2^2] \]

\[ = \frac{\alpha}{2R} \sum_{j=1}^{N} (1 - \beta_1^{N-j+1}) \mathbb{E} [\|\nabla F(x_{j-1})\|_2^2] \]

\[ = \frac{\alpha}{2R} \sum_{j=0}^{N-1} (1 - \beta_1^{N-j}) \mathbb{E} [\|\nabla F(x_j)\|_2^2]. \]  \hfill (A.49)

If we now introduce \( \tau \) as in (A.8), we can first notice that

\[ \sum_{j=0}^{N-1} (1 - \beta_1^{N-j}) = N - \beta_1 \frac{1 - \beta_1^N}{1 - \beta_1} \geq N - \frac{\beta_1}{1 - \beta_1}. \]  \hfill (A.50)

Introducing

\[ \tilde{N} = N - \frac{\beta_1}{1 - \beta_1}, \]  \hfill (A.51)

we then have

\[ A \geq \frac{\alpha \tilde{N}}{2R} \mathbb{E} [\|\nabla F(x_{\tau})\|_2^2]. \]  \hfill (A.52)
On the Convergence of Adam and Adagrad

Further notice that for any coordinate $i \in [d]$, we have $v_{N,i} \leq \frac{R^2}{1-\beta_2}$, besides $\alpha_N \leq \alpha \frac{1-\beta_1}{\sqrt{1-\beta_2}}$, so that putting together (A.43), (A.52), (A.44), (A.46) and (A.48) we get

$$
E \left[ \left\| \nabla F(x_{\tau}) \right\|_2^2 \right] \leq 2RF_0 \frac{F_0 - F_*}{\alpha N} + \frac{E}{N} \left( \ln \left( 1 + \frac{R^2}{\epsilon(1-\beta_2)} \right) - N \log(\beta_2) \right),
$$

(A.53)

with

$$
E = \frac{\alpha d RL(1-\beta_1)}{(1-\beta_1/\beta_2)(1-\beta_2)} + \frac{2\alpha^2 d L^2 \beta_1}{(1-\beta_1/\beta_2)^{3/2}} + \frac{12d R^2 \sqrt{1-\beta_1}}{(1-\beta_1/\beta_2)^{3/2} \sqrt{1-\beta_2}}.
$$

(A.54)

This conclude the proof of theorem A.1.

**Adagrad** For Adagrad, we have $\alpha_n = (1-\beta_1)\alpha$, $\beta_2 = 1$ and $\Omega_n \leq \sqrt{N}$ so that,

$$
\begin{align*}
A &= \frac{1}{2R} \sum_{n=1}^{N} \frac{\alpha_n}{\Omega_n} \sum_{j=1}^{n} \beta_1^{n-j} \mathbb{E} \left[ \left\| G_j \right\|_2^2 \right] \\
&\geq \frac{\alpha(1-\beta_1)}{2R\sqrt{N}} \sum_{j=1}^{N} \mathbb{E} \left[ \left\| G_j \right\|_2^2 \right] \sum_{n=j}^{N} \beta_1^{n-j} \\
&= \frac{\alpha}{2R\sqrt{N}} \sum_{j=1}^{N} (1-\beta_1^{N-j+1}) \mathbb{E} \left[ \left\| G_j \right\|_2^2 \right] \\
&= \frac{\alpha}{2R\sqrt{N}} \sum_{j=1}^{N} (1-\beta_1^{N-j+1}) \mathbb{E} \left[ \left\| \nabla F(x_{j-1}) \right\|_2^2 \right] \\
&= \frac{\alpha}{2R\sqrt{N}} \sum_{j=0}^{N-1} (1-\beta_1^{N-j}) \mathbb{E} \left[ \left\| \nabla F(x_j) \right\|_2^2 \right].
\end{align*}
$$

(A.55)

(A.56)

Reusing (A.50) and (A.51) from the Adam proof, and introducing $\tau$ as in (5.1), we immediately have

$$
A \geq \frac{\alpha \tilde{N}}{2R \sqrt{N}} \mathbb{E} \left[ \left\| \nabla F(x_{\tau}) \right\|_2^2 \right].
$$

(A.57)

Further notice that for any coordinate $i \in [d]$, we have $v_{N,i} \leq NR^2$, besides $\alpha_N = (1-\beta_1)\alpha$, so that putting together (A.43), (A.57), (A.44), (A.46) and (A.48) with $\beta_2 = 1$, we get

$$
\mathbb{E} \left[ \left\| \nabla F(x_{\tau}) \right\|_2^2 \right] \leq 2R \sqrt{N} \frac{F_0 - F_*}{\alpha N} + \frac{E}{N} \ln \left( 1 + \frac{NR^2}{\epsilon} \right),
$$

(A.58)

with

$$
E = \alpha d RL + \frac{2\alpha^2 d L^2 \beta_1}{1-\beta_1} + \frac{12d R^2}{1-\beta_1}.
$$

(A.59)

This conclude the proof of theorem A.2.