Photon-subtracted squeezed thermal state: nonclassicality and decoherence

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Abstract

We investigate nonclassical properties of the field states generated by subtracting any number photon from the squeezed thermal state (STS). It is found that the normalization factor of photon-subtracted STS (PSSTS) is a Legendre polynomial of squeezing parameter \( r \) and average photon number \( \bar{n} \) of thermal state. Expressions of several quasi-probability distributions of PSSTS are derived analytically. Furthermore, the nonclassicality is discussed in terms of the negativity of Wigner function (WF). It is shown that the WF of single PSSTS always has negative values if \( \bar{n} < \sinh^2 r \) at the phase space center. The decoherence effect on PSSTS is then included by analytically deriving the time evolution of WF. The results show that the WF of single PSSTS has negative value if \( 2\kappa t < \ln\{(1 - (2\bar{n} + 1)(\bar{n} - \sinh^2 r)) / (2(2\bar{n} + 1)(\bar{n}\cosh 2r + \sinh^2 r))\} \), which is dependent not only on average number \( \bar{N} \) of environment, but also on \( \bar{n} \) and \( r \).

Keywords: Nonclassicality, decoherence, Photon-subtraction, Squeezed thermal state

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1 Introduction

Nonclassical Gaussian states play an important role in quantum information processing with continuous variables, such as teleportation, dense coding, and quantum cloning. In a quantum optics laboratory, Gaussian states have been generated but there is some limitation in using them for various tasks of quantum information process. For example, when a two-mode squeezed vacuum state (a Gaussian state) with low squeezing is used as an entangled resource to realize quantum teleportation, the average fidelity is just more (8 ± 2) % than the classical limits. On the other hand, it is possible to generate and manipulate various nonclassical optical fields by subtracting or adding photons from/to traditional

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quantum states or Gaussian states, which are useful ways to conditionally manipulate nonclassical state of optical field \([2\text{--}11]\). Recently, subtracting or adding photon states have received more attention from both experimentalists and theoreticians \([12\text{--}17]\). One of reasons is that photon subtraction can be applied to improve entanglement between Gaussian states \([18\text{--}19]\), loophole-free tests of Bell’s inequality \([20\text{--}21]\), and quantum computing \([22]\). Thus the photon subtraction (a non-Gaussian operation) can satisfy the requirement of quantum information protocols for long-distance communication. Nevertheless, the photon addition and subtraction have been successfully demonstrated experimentally for probing quantum commutation rules by Parigi et al. \([23\text{--}24]\). In fact, they have implemented simple alternated sequences of photon creation (addition) and annihilation (subtraction) on a thermal field and observed the noncommutativity of the creation and annihilation operators.

In addition, Olivares et al. \([25]\) theoretically discussed the relation between the photon subtracted squeezed vacuum (PSSV), as an output state passing through a beamsplitter, and two parameters (the transmissivity of beamsplitter and the photodetection quantum efficiency). Then the case of two-mode photon-subtraction is also further discussed in the presence of noise \([26\text{--}27]\). Kitagawa et al. \([28]\) investigated the degree of entanglement for non-Gaussian mixed (pure) states generated by photon subtraction from two-mode squeezed vacuum states with on-off photon detectors. For the single PSSV, furthermore, its nonclassical properties and decoherence was investigated theoretically in two different decoherent channels (amplitude decay and phase damping) by Biswas and Agarwal \([5]\). They indicated that the WF losses its non-Gaussian nature and becomes Gaussian at long times in amplitude decay case. Recently, it is found that consecutive applications of photon subtraction (or subtracting a well-defined number of photons) from a squeezed vacuum state result in the generation of a squeezed superpositions of coherent state (SSCS) with nearly the perfect fidelity regardless of the number of photons subtracted \([6]\). The amplitude of the SSCS increases as the number of the subtracted photons gets larger.

It is interesting in noticing that single-mode displaced-squeezed thermal state can be considered as a generalized Gaussian state, which has received more attention \([29\text{--}33]\). For example, phase estimations for squeezed thermal states (STSs) and displaced thermal states are presented \([29]\), which shows that a larger temperature can enhance the estimation fidelity for the former. Another example is, for Gaussian squeezed states of light, that a scheme is also presented experimentally to measure its squeezing, purity and entanglement \([32\text{--}33]\). To our knowledge, however, the investigation of photon subtraction from STS (even for single photon subtraction case) has not been previously addressed (especially when this state interacts with its surrounding environment). In addition, the exact threshold value of the decay time has not been explicitly given. In this paper, we focus on any number photon-subtracted single-mode STS (PSSTS), which is optically produced single-mode non-Gaussian states, and explore theoretically its nonclassical properties and decoherence in a thermal channel by deriving analytically some expressions, such as normalized constant, photon-number distribution and Wigner function (WF). For single PSSTS, it is shown that the WF of single PSSTS always has the negative values under the condition of \(\bar{n} < \sinh^2 r\) at the phase space center (\(\bar{n}\) and \(r\) are an average number of thermal state and a squeezing parameter, respectively), and that the threshold value of the decay time is dependent not only on the average number of environment, but also on \(\bar{n}\) and \(r\).

In section II, we introduce the single-mode PSSTS, where the normalized factor turns out to be a Legendre polynomial with a remarkable result. In Sec. III, the nonclassical properties of the PSSTS, such as Mandel’s \(Q\)-parameter, and distribution of photon number
(related to a Legendre polynomial), are calculated analytically and then be discussed in details. In Sec. IV, the explicitly analytical expressions of quasi-probability distributions for PSSTS, such as P-distribution, Q-function and WF of the PSSTS, are derived by using the Weyl ordered operators’ invariance under a similar transformations. Then we derive an explicitly analytical expression of time evolution of WF for the arbitrary PSSTS in the thermal channel and discuss the loss of nonclassicality in reference of the negativity of WF in Sec. V. It is found that the threshold value of decay time corresponding to the transition of WF from partial negative to completely positive definite is obtained at the center of the phase space, which is not only dependent on the average number $\bar{n}$ of environment, but also on $\bar{n}$ and $r$. We show that the WF for single PSSTS has always negative value if the decay time $\kappa t < \frac{1}{2} \ln \{1 - (2\bar{n} + 1)(\bar{n} - \sinh^2 r) /[2(\bar{n} + 1)(\bar{n} \cosh 2r + \sinh^2 r)]\}$ (see Eq. (51) below), where $\kappa$ denotes a dissipative coefficient of interacting with the environment. Sec. VI is devoted to calculating the fidelity between the PSSTS and the STS. It is shown that the fidelity decreases monotonously with the increment of both photon-subtraction number $m$ and the squeezing parameter $r$. We end with the main conclusions of our work.

2 Photon-subtraction squeezed thermal state

At first, let’s introduce the photon-subtraction squeezed thermal state (PSSTS). For a squeezed thermal field, its density operator is

$$\rho_s = S(r) \rho_c S^\dagger(r),$$

where $S(r) = \exp[r(a^{12} - a^2)/2] = \exp[-ir(QP + PQ)/2]$ is the squeezing operator [34, 35] with squeezing parameter $r$, here the coordinate operators $Q = (a + a^\dagger)/\sqrt{2}$ and the momentum operators $P = (a - a^\dagger)/(\sqrt{2}i)$ ([a, a$^\dagger$] = 1) are introduced as functions of create and annihilation operators $a^\dagger$ and $a$, respectively, and $\rho_c$ is a density operator of thermal state,

$$\rho_c = (1 - e^\sigma)e^{\sigma a^\dagger a}, \quad \sigma = -\frac{\hbar \omega}{kT},$$

where $k$ is a Boltzmann constant, and the temperature $T$ is qualified to be a density operator of thermal (chaotic) field with $\text{tr} \rho_c = 1$. Using the operator identity [37, 38]

$$e^{\sigma a^\dagger a} =: \exp[(e^\sigma - 1)a^\dagger a] = \frac{2}{e^\sigma + 1} : \exp\left\{\frac{e^\sigma - 1}{e^\sigma + 1} (Q^2 + P^2)\right\} :,$$

where these two symbols $:\ :$ and $:\ :$ denote normal ordering and Weyl ordering, respectively, and using the Weyl ordering invariance under similarity transformations [37, 38], which means that

$$S^\dagger : (\circ \circ \circ) : S^{-1} = : S (\circ \circ \circ) S^{-1} :,$$

as if the “fence” $:\ :$ did not exist, so $S$ can pass through it, as well as the technique of integration within an ordered product of operators (IWOP), one can convert $\rho_s$ to its normally ordered Gaussian form [38] (see Appendix A), i.e.,

$$\rho_s = \frac{1}{\tau_1 \tau_2} : \exp \left\{ -\frac{Q^2}{2\tau_1^2} - \frac{P^2}{2\tau_2^2} \right\} :,$$

where

$$2\tau_1^2 = (2\bar{n} + 1)e^{2r} + 1, \quad 2\tau_2^2 = (2\bar{n} + 1)e^{-2r} + 1,$$
which leads to the following relations,
\begin{align}
\tau_2^2 - \tau_2^2 &= (2\bar{n} + 1) \sinh 2r, \\
\tau_2^2 + \tau_2^2 &= (2\bar{n} + 1) \cosh 2r + 1, \\
\tau_1^2 \tau_2 &= \bar{n}^2 + (2\bar{n} + 1) \cosh^2 r,
\end{align}
\tag{7}\tag{8}\tag{9}
and \(\bar{n} = \text{tr}(\rho_c a^\dagger a) = (e^{-\sigma} - 1)^{-1}\) \[\text{39}\] denotes the average photon number of thermal (chaotic) field \(\rho_c\) in Eq. (2). The form in Eq. (5) is similar to the bivariate normal distribution in statistics, which is useful for us to further derive the marginal distributions of \(\rho_s\).

Theoretically, the PSSTS can be obtained by repeatedly operating the photon annihilation operator \(a\) on a squeezed thermal state, so its density operator is given by
\[
\rho = C_m^{-1} a^m \rho_s a^\dagger m,
\tag{10}
\]
where \(m\) is the subtracted photon number (a non-negative integer), and \(C_m\) is a normalized constant with (see Appendix B)
\[
C_m = \text{Tr}(a^m \rho_s a^\dagger m) = m! D^m/2 P_m \left( B/\sqrt{D} \right),
\tag{11}
\]
which indicates that \(C_m\) is just related to Legendre polynomial \(P_m(x)\) (see Appendix B (B10)), and
\begin{align}
B &= \frac{1}{2} [(2\bar{n} + 1) \cosh 2r - 1], \\
D &= \bar{n}^2 - (2\bar{n} + 1) \sinh^2 r.
\tag{12}\tag{13}
\end{align}
It is noted that, for the case of no-photon-subtraction with \(m = 0\), \(C_0 = 1\) as expected. Under the case of \(m\)-photon-subtraction thermal state with \(B = \bar{n}, D = \bar{n}^2\), and \(P_m(1) = 1, C_m = m! \bar{n}^m\). The same result as Eq.(24) can be found in Ref. [40].

Here we should point out that, as Agarwal et al. introduced the excitations on a coherent state by repeated application of the photon creation operator on the coherent state \[\text{41}\], we introduce theoretically the PSSTS \[\text{11}\]. In realistic situations, one the other hand, the photon subtraction would be done by on/off detector and the tapping beam splitters with a non-unity transmittance, which leads to a generated mixed state. For various schemes for generating photon subtraction, one can refer to Refs. [1, 28, 42].

3 Nonclassical properties of PSSTS

3.1 Mandel’s \(Q\)-parameter

The analytical expression of \(C_m\) is of importance for further investigating the properties of PSSTS. For instance, one can easily calculate
\begin{align}
\langle a^\dagger a \rangle &= \text{Tr}(C_m^{-1} a^{m+1} \rho_s a^\dagger m+1) = \frac{C_{m+1}}{C_m}, \\
\langle a^2 a^2 \rangle &= \text{Tr}(C_m^{-1} a^{m+2} \rho_s a^\dagger m+2) = \frac{C_{m+2}}{C_m}.
\tag{14}\tag{15}
\end{align}
thus the Mandel’s $Q$-parameter is given by

$$Q_M = \frac{\langle a^\dagger a^2 \rangle}{\langle a^\dagger a \rangle} - \frac{\langle a^4 \rangle}{\langle a^\dagger a \rangle} = \frac{C_{m+2}}{C_{m+1}} - \frac{C_{m+1}}{C_m},$$  \hspace{1cm} (16)$$

which measures the deviation of the variance of the photon number distribution of the field state under consideration from the Poissonian distribution of the coherent state. If $Q_M = 0$ we say the field has Poissonian photon statistics, while for $Q_M > 0$ ($Q_M < 0$), the field has super-(sub-) Poissonian photon statistics. It is well-known that the negativity of the $Q_M$-parameter refers to sub-Poissonian statistics of the state. But a state may be nonclassical even though $Q_M$ is positive as pointed out in Ref. [43]. This case is true for the present state. In fact, if $Q_M$ is positive, it does not mean that the state is classical. In such cases, we have to use other parameters to test the non-classicality [43]. From Fig.1, one can see clearly that for odd number $m$, $Q_M$ becomes negative when the squeezing parameter $r$ is less than a certain threshold value which decreases as $m$ increases. Differently from the case of odd number $m$, $Q_M$ is always positive for even number $m$. It is necessary to emphasize that the Wigner function (WF) has negative region for all $r$, and thus the PSSTS is nonclassical. In addition, when the average photon number $\bar{n}$ is larger than a certain threshold value, $Q_M$ is also always positive. Without loss of generality, thus, we consider only the (ideal) PSSTS in a thermal channel in our following work.

### 3.2 Photon-number distribution of PSSTS

Next we discuss the photon-number distribution (PND) of PSSTS. Noticing $a^m |n\rangle = \sqrt{(m+n)!/n!} |n+m\rangle$ and using the un-normalized coherent state $|\alpha\rangle = \exp[\alpha a^\dagger] |0\rangle$, leading to $|n\rangle = \frac{1}{\sqrt{n!}} \frac{d^n}{d\alpha^n} |\alpha\rangle |_\alpha=0$, \hspace{1cm} (\langle \beta | = e^{\alpha\beta^*}),$ as well as the normal ordering form of $\rho_s$
in Eq. (5), the probability of finding \( n \) photons in the field is given by
\[
\mathcal{P}(n) = \langle n \mid \rho \mid n \rangle = C_m^{-1} \langle n \mid a^m \rho_s a^m \mid n \rangle 
\]
\[
= \frac{(m+n)!}{n!C_m} \langle m+n \mid \rho_s \mid m+n \rangle 
\]
\[
= C_m^{-1} \frac{d^{m+n}}{n!\tau_1 \tau_2 d\beta^{m+n} d\alpha^{m+n}} \langle \beta \mid : \exp \left\{ -\frac{(a + a^\dagger)^2}{4\tau_1^2} + \frac{(a - a^\dagger)^2}{4\tau_2^2} \right\} : \mid \alpha \rangle \bigg|_{\alpha = \beta^* = 0} 
\]
\[
= C_m^{-1} \frac{d^{m+n}}{n!\tau_1 \tau_2 d\beta^{m+n} d\alpha^{m+n}} \exp \left\{ -\frac{(\alpha + \beta^*)^2}{4\tau_1^2} + \frac{(\alpha - \beta^*)^2}{4\tau_2^2} + \alpha \beta^* \right\} \bigg|_{\alpha = \beta^* = 0} 
\]
\[
= C_m^{-1} \frac{d^{2m+2n}}{n!\tau_1 \tau_2 d\beta^{m+n} d\alpha^{m+n}} \exp \left\{ A_1 \alpha \beta^* + A_2 (\alpha^2 + \beta^* 2) \right\} \bigg|_{\alpha = \beta^* = 0}, 
\]
where \( A_1 \) and \( A_2 \) are defined by
\[
A_1 = 1 - \frac{1}{2\tau_1^2} - \frac{1}{2\tau_2^2} = \frac{\tilde{n} (\tilde{n} + 1)}{n^2 + (2\tilde{n} + 1) \cosh^2 r}, 
\]
\[
A_2 = \frac{1}{4\tau_1^2} - \frac{1}{4\tau_2^2} = \frac{1}{4} \frac{(2\tilde{n} + 1) \sinh 2r}{(\tilde{n} + 2) \cosh^2 r}. 
\]
In a similar way to deriving Eq.(B11), we finally obtain
\[
\mathcal{P}(n) = C_m^{-1} \frac{d^{2m+2n}}{n!\tau_1 \tau_2 d\beta^{m+n} d\alpha^{m+n}} \exp \left\{ -\left( i \sqrt{A_2} \alpha \right)^2 - \left( -i \sqrt{A_2} \beta^* \right)^2 + \frac{A_1 A_2}{2} \left( i \sqrt{A_2} \alpha \right) \left( -i \sqrt{A_2} \beta^* \right) \right\} \bigg|_{\alpha = \beta^* = 0} 
\]
\[
= \frac{(A_2)^{m+n}}{n!\tau_1 \tau_2} C_m^{-1} \frac{d^{2m+2n}}{d\beta^{m+n} d\alpha^{m+n}} \exp \left\{ -\alpha^2 - \beta^* 2 + \frac{A_1 A_2}{2} \alpha \beta^* \right\} \bigg|_{\alpha = \beta^* = 0} 
\]
\[
= \frac{(m+n)!}{n!\tau_1 \tau_2 C_m} E^{(m+n)/2} P_{m+n} \left( A_1 / \sqrt{E} \right), 
\]
where \( P_{m+n} (x) \) is Legendre polynomial in \( B(10) \), and
\[
E = A_1^2 - 4 A_2^2 = \frac{\tilde{n}^2 - (2\tilde{n} + 1) \sinh^2 r}{\tilde{n}^2 + (2\tilde{n} + 1) \cosh^2 r}. 
\]
In particular, when \( m = 0 \) \((C_0 = 1)\), Eq.(20) reduces to
\[
\mathcal{P}(n) = \frac{E^{n/2}}{\tau_1 \tau_2} P_n \left( A_1 / \sqrt{E} \right), 
\]
which is just the PND of STS which seems a new result; while for \( r = 0 \) \((\tau_1 \tau_2 = \tilde{n} + 1, A_1 = \tilde{n} / (\tilde{n} + 1), E = \tilde{n}^2 / (\tilde{n} + 1)^2, P_m (1) = 1, C_m = m! \tilde{n}^m \)\), Eq.(20) becomes
\[
\mathcal{P}(n) = \frac{(m+n)!}{m! n! \tilde{n}^m (\tilde{n} + 1)^{m+n+1}}, 
\]
which is the PND of \( m \)-photon-subtracted thermal state which also seems a new result, and the PND \((\tilde{n} / (\tilde{n} + 1)^{n+1})\) of thermal state without photon-subtraction [16][17].

In Fig.2, the PND is shown for different values \((\tilde{n}; r)\) and \( m \), from which we can see that by subtracting photons, we have been able to move the peak from zero photons to nonzero photons (see Fig.2 (a)-(c)). The position of peak depends on how many photons are annihilated and how much the state is squeezed initially. In addition, the PND mainly shifts to the bigger number states and becomes more “flat” and “wide” with the increasing parameter \( r \) and the average photon-number of thermal field \( \rho_s \) (see Fig.2 (b) and (d)).
In this section, several quasiprobability distributions of PSSTS are derived in order to provide a convenient way for studying the nonclassical properties of fields.

4.1 P distribution

We first calculate the Glauber-Sudarshan P distribution function \[48\] of PSSTS. For this purpose, we start from the anti-normal ordering form of \(\rho\) in Eq. (10). Recalling the integration formula converting an operator \(\hat{O}\) into its anti-normal ordering form \[49\] with anti-normal ordering \(\cdot\cdot\cdot\); i.e.,

\[
\hat{O} = \int \frac{d^2 \beta}{\pi} \langle -\beta | \hat{O} | \beta \rangle \exp \left( |\beta|^2 + \beta^* a - \beta a^\dagger + a^\dagger a \right); \tag{24}
\]

where \(|\beta\rangle\) is a coherent state, one can obtain the anti-normal ordering form of the squeezed thermal state \(\rho_s\) by substituting Eq. (5) into Eq. (24) and using the integration formula (B7). Such as

\[
\rho_s = \frac{1}{\tau_1 \tau_2} \int \frac{d^2 \beta}{\pi} \exp \left\{ -A_1 |\beta|^2 - \beta a^\dagger + \beta^* a + A_2 \left( \beta^2 + \beta^* 2 \right) + a^\dagger a \right\};
\]

\[
= \frac{1}{\sqrt{D}} \exp \left[ \left( 1 - \frac{A_1}{E} \right) a^\dagger a + \frac{A_2}{E} \left( a^2 + a^2 \right) \right];
\]

\[
= \frac{1}{\sqrt{D}} \exp \left[ \frac{2 - \tau_1^2 - \tau_2^2}{2D} a^\dagger a + \frac{\tau_1^2 - \tau_2^2}{4D} \left( a^2 + a^2 \right) \right]; \tag{25}
\]
thus the anti-normal ordering form of $\rho$ in Eq. (10) is
\[
\rho = C^{-1} \sqrt{D} a^m \exp \left[ \frac{2 - \tau_1^2 - \tau_2^2}{2D} a^\dagger a + \frac{\tau_1^2 - \tau_2^2}{4D} \left( a^\dagger a + a^2 \right) \right] a^\dagger m^:, \tag{26}
\]
which leads to the P-function $P(\alpha)$ of PSSTS,
\[
P(\alpha) = C^{-1} |\alpha|^{2m} P_0(\alpha), \tag{27}
\]
where $P_0(\alpha)$ is the P-function of STS,
\[
P_0(\alpha) = \frac{1}{\sqrt{D}} \exp \left[ \frac{2 - \tau_1^2 - \tau_2^2}{2D} |\alpha|^2 + \frac{\tau_1^2 - \tau_2^2}{4D} (\alpha^* \alpha + \alpha^2) \right]. \tag{28}
\]
It is interesting in noticing that when $r = 0$, Eq. (28) becomes
\[
P(\alpha) = \frac{|\alpha|^{2m} e^{-|\alpha|^2/\bar{n}}}{\bar{n} C_m},
\]
which is just the P-function of $m$-photon-subtraction thermal state which seems a new result. From Eq. (27) one can see that the P-representation of density operator $\rho$ can be expanded as
\[
\rho = C^{-1} \sqrt{D} \int \frac{d^2z}{\pi} |z|^{2m} P_0(z) |z \rangle \langle z|, \tag{29}
\]
which is a non-Gaussian function due to the presence of $|z|^{2m}$.

### 4.2 Q-function

The Q-function is the absolute magnitude squared of the projection of a state of the field onto a coherent state $\langle \alpha |$, defined by
\[
Q(\alpha, \alpha^*) = \frac{1}{\pi} \langle \alpha | \rho | \alpha \rangle. \tag{30}
\]
Substituting Eq. (29) into (30), we can obtain
\[
Q(\alpha, \alpha^*) = R_m(\alpha, \alpha^*) Q_0(\alpha, \alpha^*), \tag{31}
\]
where $Q_0(\alpha, \alpha^*)$ is the Q-function of STS,
\[
Q_0(\alpha, \alpha^*) = \frac{1}{\pi \tau_1 \tau_2} \exp \left[ -\frac{\tau_1^2 + \tau_2^2}{2\tau_1 \tau_2} |\alpha|^2 + \frac{\tau_1^2 - \tau_2^2}{4\tau_1 \tau_2} (\alpha^* \alpha + \alpha^2) \right], \tag{32}
\]
which seems a new result not reported before, and $R_m(\alpha, \alpha^*)$ is a factor generated from the photon-subtraction, i.e.,
\[
R_m(\alpha, \alpha^*) = C_m^{-1} \sum_{l=0}^{m} \frac{(m)!^2 M^m (2O)_{l}}{l! [(m - l)!]^2} \left| H_{m-l}(-i\sqrt{M}(O\alpha^* + \alpha)) \right|^2, \tag{33}
\]
where $M = [(2\bar{n} + 1) \sinh 2r]/[4 (\bar{n}^2 + (2\bar{n} + 1) \cosh^2 r)]$, and $O = 2\bar{n}(\bar{n} + 1)/[(2\bar{n} + 1) \sinh 2r]$. Eq. (31) indicates that the Q-function of PSSTS is also a non-Gaussian type due to the presence of $R_m(\alpha, \alpha^*)$ and always positive since $O > 0$. In particular, when $m = 0, R_m(\alpha, \alpha^*) = 1$, thus $Q(\alpha, \alpha^*) = Q_0(\alpha, \alpha^*)$, as expected.
4.3 Wigner function

Next, the P-function is applied to deduce the WF of PSSTS. The partial negativity of WF is indeed a good indication of the highly nonclassical character of the state. Therefore it is worth of obtaining the WF for any states. The WF $W(\alpha, \alpha^*)$ associated with a quantum state can be derived as follows [36]:

$$W(\alpha, \alpha^*) = \text{tr}[\rho \Delta(\alpha, \alpha^*)], \quad \alpha = (q + \sqrt{2}p) / \sqrt{2},$$

where $\Delta(\alpha, \alpha^*)$ is Wigner operator, whose coherent state representation is

$$\Delta(\alpha, \alpha^*) = e^{2|\alpha|^2} \int \frac{d^2\beta}{\pi^2} |\beta\rangle \langle -\beta| e^{2(\alpha\beta^* - \alpha^*\beta)},$$

where $|\beta\rangle = \exp(-|\beta|^2/2 + \beta a^1)|0\rangle$ is the coherent state. Using the vacuum projector $|0\rangle \langle 0| =: e^{-a^1a^1}$, and the IWOP technique [37] one can put Eq. (35) into its normal ordering form,

$$\Delta(\alpha, \alpha^*) = \frac{1}{\pi} \exp \left[ -2 \left( a^1 - \alpha^* \right) (a - \alpha) \right].$$

Thus substituting Eqs. (28), (36) and (29) into Eq. (34), we can finally obtain the WF of PSSTS (see Appendix C),

$$W(\alpha, \alpha^*) = F_m(\alpha, \alpha^*) W_0(\alpha, \alpha^*),$$

where $W_0(\alpha, \alpha^*)$ is the WF of STS,

$$W_0(\alpha, \alpha^*) = \frac{1}{\pi(2n+1)} \exp \left[ -\frac{2 \cosh 2r}{2n+1} |\alpha|^2 + \frac{\sinh 2r}{2n+1} (\alpha^2 + \alpha^*2) \right],$$

and

$$F_m(\alpha, \alpha^*) = \frac{(m!)^2 C_m^{-1} \sinh^{2r} m}{2^{2mn}(2n+1)^m} \sum_{l=0}^{m} \frac{2^{2l} (\bar{n} - \sinh^2 r)^l}{l! [(m-l)!]^2 \sinh^l 2r} \left| H_{m-l}(\bar{\beta}) \right|^2,$$

where $\bar{\beta} = [2\alpha^*(\bar{n} - \sinh^2 r) + \alpha \sinh 2r]/\{i[(2\bar{n}+1) \sinh 2r]^{1/2}\}$. Eq. (37) is the analytical expression of WF for PSSTS, related to single-variable Hermite polynomials. It is obvious that there does not exist negative region for WF in phase space when $\bar{n} > \sinh^2 r$ which is agreement with Eq. (28) in Ref. [50]. In particular, when $m = 0$, $F_0(\alpha, \alpha^*) = 1$, Eq. (37) becomes $W(\alpha, \alpha^*) = W_0(\alpha, \alpha^*)$; while for $r = 0$, note $C_m = m!\bar{n}^m$, $W_0(\alpha, \alpha^*) = e^{-2|\alpha|^2/(2\bar{n}+1)} / \pi (2\bar{n}+1)$ (Eq. (30) in Ref. [40]) and $F_m(\alpha, \alpha^*) = \frac{1}{(2\bar{n}+1)^m} L_m \left( -\frac{4\bar{n}}{2\bar{n}+1} |\alpha|^2 \right)$, Eq. (37) reduces to

$$W(\alpha, \alpha^*) = \frac{1}{\pi(2\bar{n}+1)^{m+1}} e^{-2|\alpha|^2/(2\bar{n}+1)} L_m \left( -\frac{4\bar{n}}{2\bar{n}+1} |\alpha|^2 \right),$$

which corresponds to the WF of m-photon subtracted thermal state [40], and can be checked directly from Eq. (C6). In addition, for $m = 1$, (single-photon-subtracted squeezed thermal state (SPSSTS)), $C_1 = B$ [12], the special WF of SPSSTS is

$$W_1(\alpha, \alpha^*) = F_1(\alpha, \alpha^*) W_0(\alpha, \alpha^*),$$
Figure 3: (Color online) Wigner function distributions $W(\alpha, \alpha^*)$ of PASSTS for different $(\bar{n}, r)$ and $m$ values (a) $\bar{n} = 0.1, r = 0.5, m = 1$; (b) $\bar{n} = 0.1, r = 0.5, m = 2$; (c) $\bar{n} = 0.2, r = 0.5, m = 1$; (d) $\bar{n} = 0.1, r = 0.8, m = 1$.

where

\[
F_1(\alpha, \alpha^*) = \frac{1}{(2\bar{n} + 1)B} \left( (2\bar{n} + 1) |\alpha|^2 + \bar{n} - \sinh^2 r \right) \\
= \frac{2\alpha^* (\bar{n} - \sinh^2 r) + \alpha \sinh 2r}{(2\bar{n} + 1)^2 B} + \frac{\bar{n} - \sinh^2 r}{(2\bar{n} + 1) B}.
\] (42)

Noting $B > 0$, thus from Eq.(41) one can see that when the factor $F_1(\alpha, \alpha^*) < 0$, the WF of SPSSTS has its negative distribution in phase space. This indicates that the WF of SPSSTS always has the negative values under the condition: $\bar{n} < \sinh^2 r$ at the phase space center $\alpha = 0$, which is similar to the case of single-photon-subtracted squeezed vacuum [5,7].

Using Eq.(37), the WFs of PSSTS are depicted in Fig.3 for several different values of $\bar{n}, r$ and $m$ in phase space. It is easy to see that the the WF is non-Gaussian in phase space. As an evidence of nonclassicality of the state, squeezing in one of the quadratures is clear in the plots (see Figs.3(a) and 3(d)). In addition, we can clearly see that there is some negative region of WF, which is another evidence of nonclassicality of the state, and that the negative region of WF gradually disappears as the $\bar{n}$ (or the temperature) increases for given $r$ and $m$ (see Fig.3(a) and (c)). Furthermore, for a larger squeezing, the WF shows a smaller minimum negative value at the center of phase space (Figs.3(a) and 3(d)). For two-photon subtracted case, the WF presents two positive peaks and two negative peaks, different from the case of single-photon subtracted case.

5 Decoherence of PSSTS in thermal environment

When the $m$-PSSTS evolves in the thermal channel, the evolution of the density matrix can be described by master equation [51]

\[
\frac{d\rho}{dt} = \kappa (N + 1) \left( 2a\rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a \right) + \kappa N \left( 2a^\dagger \rho a - aa^\dagger \rho - \rho aa^\dagger \right).
\] (43)
where \( \kappa \) represents the dissipative coefficient and \( \mathcal{N} \) denotes the average thermal photon number of the environment. When \( \mathcal{N} = 0 \), \( \text{Eq.} (43) \) reduces to the master equation describing the photon-loss channel. The evolution of the WF is governed by the following integration equation \( \text{[52]} \),

\[
W(\zeta, \zeta^*, t) = \frac{2}{(2\mathcal{N} + 1) \mathcal{T}} \int \frac{d^2 \alpha}{\pi} W(\alpha, \alpha^*, 0) e^{-2 |\zeta - \alpha e^{-\kappa t}|^2 / (2\mathcal{N} + 1)}, \tag{44}
\]

where \( W(\alpha, \alpha^*, 0) \) is the WF of the initial state, and \( \mathcal{T} = 1 - e^{-2\kappa t} \). \( \text{Eq.} (44) \) is just the evolution formula of WF in thermal channel. Thus the WF at evolving time may be obtained by performing the integration with an initial value.

Substituting Eqs. (37)-(39) into (44), and using Eq.(B7) we finally obtain the evolution of WF for PSSTS in thermal environment (see Appendix D, in a similar way to deriving Eq. (11)),

\[
W(\zeta, \zeta^*, t) = F_m(\zeta, \zeta^*, t) W_0(\zeta, \zeta^*, t), \tag{45}
\]

where \( W_0(\zeta, \zeta^*, t) \) is the WF of squeezed thermal state in thermal channel,

\[
W_0(\zeta, \zeta^*, t) = \frac{1}{(2\mathcal{N} + 1) \mathcal{T} \sqrt{G}} \exp \left[ -\Delta_2 |\zeta|^2 + \frac{g_2 g_3^2}{4G} (\zeta^2 + \zeta^{*2}) \right], \tag{46}
\]

\[
F_m(\zeta, \zeta^*, t) = C_m^{-1} \sum_{l=0}^{m} \frac{(m!)^2 \chi^l \Delta_1^{m-l}}{l! [(m-l)!]^2} \left| H_{m-l} \left[ \omega / (2i \sqrt{\Delta_1}) \right] \right|^2, \tag{47}
\]

and

\[
g_0 = \frac{\cosh 2r}{2n + 1}, \quad g_1 = \frac{n - \sinh^2 r}{2n + 1}, \quad g_2 = \frac{\sinh 2r}{2n + 1}, \quad g_3 = \frac{e^{-\kappa t}}{(2\mathcal{N} + 1) \mathcal{T}}, \tag{48}
\]

as well as

\[
G = (g_0 + g_3 e^{-\kappa t}/2)^2 - g_2^2, \quad \omega = \frac{2e^{-\kappa t}}{2\mathcal{N} \mathcal{T} + 1} (\chi \zeta + 2\Delta_1 \zeta^*),
\]

\[
\Delta_1 = \frac{g_2}{4G} \left( 1 + g_3 e^{-\kappa t}/2 \right)^2, \quad \Delta_2 = \frac{2}{(2\mathcal{N} + 1) \mathcal{T}} - \frac{g_2^2}{2G} (g_0 + g_3 e^{-\kappa t}/2), \tag{49}
\]

\[
\chi = \frac{1 + g_3 e^{-\kappa t}/2}{2G} \left[ g_0 + g_1 g_3 e^{-\kappa t} - 1 / (2n + 1)^2 \right].
\]

It is noted that, at the initial time \( t = 0 \), \((2\mathcal{N} + 1) \mathcal{T} \sqrt{G} \rightarrow 1, \Delta_2 \rightarrow 2g_0, \frac{g_2 g_3^2}{4G} \rightarrow g_2, \Delta_1 \rightarrow \frac{1}{4} g_2, \chi \rightarrow g_1, \omega \rightarrow 2g_1 \zeta + g_2 \zeta^* \), Eqs. (46) and (47) just reduce to Eqs. (38) and (39), respectively, i.e., the WF of PSSTS. In addition, for the case of \( m = 1 \), corresponding to the case of SPSSTS, \( \text{Eq.} (47) \) just becomes

\[
F_1(\zeta, \zeta^*, t) = C_1^{-1} \left( |\omega|^2 + \chi \right), \tag{50}
\]

from which one can see that when the factor \( F_1(\zeta, \zeta^*, t) < 0 \), the WF of SPSSTS in thermal channel has its negative distribution in phase space. At the phase space center \( \zeta = 0 \),
the W of SPSSTS always has the negative values when \( \chi < 0 \), leading to the following condition:

\[
\kappa t < \kappa t_c = \frac{1}{2} \ln \left[ 1 - \frac{2\bar{n} + 1}{2N + 1} \frac{\bar{n} - \sinh^2 r}{\bar{n} \cosh 2r + \sinh^2 r} \right],
\]

which implies that the threshold value \( \kappa t_c \) is dependent not only on the average number \( \mathcal{N} \) of environment, but also on the average number \( \bar{n} \) of thermal state and the squeezing parameter \( r \) (a result different from other discussions about the threshold value \( \kappa t_c \) in thermal channel \([5, 7]\)). The WF of SPSSTS is always positive in the whole phase space when \( \kappa t \) exceeds the threshold value \( \kappa t_c \). Actually, Eq.(51) is also true for the case with any number (\( m \)) photon-subtraction (see Eq.(47)). From Eq.(51) one can clarify how the thermal noise (\( \bar{n}, \mathcal{N} \)) shortens the threshold value of the decay time.

Using Eq. (45) we present the time-evolution of WF at different times scales in Fig.4. From Fig.4, one can see clearly that the partial negative region of WF gradually diminishes. At long times \( \kappa t \to \infty \), one has \( \omega \to 0, \chi \to \bar{n} \cosh 2r + \sinh^2 r, \Delta_1 \to \frac{1}{2} (2\bar{n} + 1) \sinh 2r \) and \( H_m(0) = (-1)^k m! \delta_{m,2k} / k! \). Thus

\[
W(\zeta, \zeta^*, \infty) = \frac{1}{\pi (2N + 1)} e^{-\frac{2|\zeta|^2}{2N+1}},
\]

which is independent of photon-subtraction number \( m \) and corresponds to thermal states with mean thermal photon number \( \mathcal{N} \). This implies that the system reduces to thermal state after a long time interaction with the environment. Eq.(52) denotes a Gaussian distribution. Thus the thermal noise causes the absence of the partial negative of the WF if the decay time \( \kappa t \) exceeds a threshold value. In addition, from Fig.4, it is found that the SPSSTS is similar to a Schrodinger cat state.

6 Fidelity as a non-Gaussianity measure for PSSTS

Recently, some quantitative measures to assess non-Gaussianity are proposed \([53, 54]\). A non-Gaussianity measure may serve as a guideline to quantify the non-Gaussian states.
Therefore, it is of interest to evaluate the degree of the resulting non-Gaussianity and assess this operation as a resource to obtain non-Gaussian states starting from Gaussian ones. Here, we examine the fidelity between the PSSTS $\rho_s$ and the STS $\rho$. Since the STS can be considered as a generalized Gaussian state, the fidelity may be seen as a non-Gaussianity measure able to quantify the non-Gaussian character of a quantum state. In order to quantify the non-Gaussian character of the PSSTS, we introduce the fidelity by defining

$$F = \frac{\text{tr} (\rho_s \rho)}{\text{tr} (\rho_s^2)},$$  \hspace{1cm} (53)$$

where $\rho_s$ and $\rho$ are the squeezed thermal state (a generalized Gaussian state) and the PSSTS, respectively. Obviously, when photon-subtraction number $m = 0$, leading to $\rho = \rho_s$, then $F = 1$ which means that $\rho$ is a Gaussian state described by $\rho_s$.

Using Eqs.(1) and (2), one has

$$\text{tr} (\rho_s^2) = \frac{1}{2\bar{n} + 1}.$$  \hspace{1cm} (54)$$

On the other hand, the fidelity ($\text{tr} (\rho_s \rho)$) can then be calculated as the overlap between the two WFs:

$$\text{tr} (\rho_s \rho) = 4\pi \int d^2\alpha W_0 (\alpha, \alpha^*) W_\rho (\alpha, \alpha^*),$$  \hspace{1cm} (55)$$

where $W_0 (\alpha, \alpha^*)$ is the WF of squeezed thermal state $\rho_s$. Using Eq.(37) we may express Eq.(55) as

$$\text{tr} (\rho_s \rho) = 4\pi \int F_m (\alpha, \alpha^*) W_0^2 (\alpha, \alpha^*) d^2\alpha.$$  \hspace{1cm} (56)$$

Then employing Eqs.(38) and (C6), similarly to Eq.(11), Eq. (56) may rewritten as (see Appendix E)

$$\text{tr} (\rho_s \rho) = \frac{m! B_m^{m/2}}{(2\bar{n} + 1) C_m} P_m \left(\frac{B_1}{\sqrt{B_2}}\right),$$  \hspace{1cm} (57)$$

where $P_m (x)$ is the Legendre polynomial with

$$B_1 = \frac{\bar{n} (\bar{n} + 1)}{2\bar{n} + 1} \cosh 2r, \quad B_2 = \frac{\bar{n}^2 (\bar{n} + 1)^2}{(2\bar{n} + 1)^2} - \sinh^2 r \cosh^2 r.$$  \hspace{1cm} (58)$$

Thus the fidelity (53) for the PSSTS is given by

$$\bar{F} = \frac{m! B_m^{m/2}}{C_m^2} \frac{P_m \left(\frac{B_1}{\sqrt{B_2}}\right)}{P_m \left(\frac{B}{\sqrt{D}}\right)} = \left(\frac{B_2}{D}\right)^{m/2} \frac{P_m \left(\frac{B_1}{\sqrt{B_2}}\right)}{P_m \left(\frac{B}{\sqrt{D}}\right)},$$  \hspace{1cm} (59)$$

which is an analytical expression for the fidelity between PSSTS and SSTS. We see that when $m = 0$ (the case of no photon-subtraction), $\bar{F} = 1$; while for $m = 1$ (the case of SPSSTS), Eq.(59) reduces to

$$\bar{F} = \frac{\bar{n} (\bar{n} + 1) \cosh 2r}{(2\bar{n} + 1) (\sinh^2 r + \bar{n} \cosh 2r)}.$$  \hspace{1cm} (60)$$

In Fig.5, we plot the fidelity between PSSTS and STS as the function of squeezing parameter $r$ for different photon-subtraction number $m$. From Fig. 5 one can see that the fidelity decreases monotonously with the increment of both photon-subtraction number $m$ and the squeezing parameter $r$, as expected.
Figure 5: (Color online) The fidelity between PSSTS and squeezed thermal state as the function of squeezing parameter $r$ for different photon-subtraction number $m = 0, 1, 2, 3, 4, 19, 20. (\bar{n} = 0.2)$. The cases of $m = 19$ and $20$ are not identical, but they are almost overlap each other, which cannot be seen clearly from figure due to the use of thick style for line.

7 Conclusions and Remarks

In summary, we investigate the nonclassicality photon-subtracted squeezed thermal state (PSSTS) and its decoherence in thermal channel with average thermal photon number $\mathfrak{N}$ and dissipative coefficient $\kappa$. For arbitrary number PSSTS, we have, for the first time, obtained an analytical express for the normalization factor, which turns out to be a Legendre polynomial of squeezing parameter $r$ and average photon number $\bar{n}$ of thermal state, a remarkable result. Based on Legendre polynomials’ behavior the nonclassical properties of the field, such as Mandel’s $Q$-parameter and photon number distribution, are also derived analytically. Furthermore, the nonclassicality of PSSTS is discussed in terms of the negativity of WF after deriving the explicit expression of WF, which implies the highly nonclassical properties of quantum states. It is shown that the WF of single PSSTS always has negative values if $\bar{n} < \sinh^2 r$ at the phase space center. Then the decoherence of PSSTS in thermal channel is also demonstrated according to the compact expression for the WF. It is found that the threshold value of the decay time corresponding to the transition of the WF from partial negative to completely positive definite is obtained at the center of the phase space, which is dependent not only on the average number $\mathfrak{N}$ of environment, but also on the average number $\bar{n}$ of thermal state and the squeezing parameter $r$. We show that the WF for single PSSTS has always negative value if the decay time $\kappa t < \frac{1}{2} \ln\{1 - (2\bar{n} + 1)(\bar{n} - \sinh^2 r)/[(2\mathfrak{N} + 1)\bar{n}\cosh 2r + \sinh^2 r]\}$. A non-Gaussianity measure may serve as a guideline to quantify them for the class of non-Gaussian states, where the fidelity decreases monotonously with the increment of both photon-subtraction number $m$ and the squeezing parameter $r$.

In addition, Mandel’s $Q$ parameter does not always indicate a negative value for nonclassical state. In fact, for the photon-subtracted squeezed states by even number, this parameter is positive. Thus the negativity of $Q$ parameter is a sufficient condition to distinguish non-classical state from classical one. While for photon subtracted squeezed states by odd number, the negativity of single photon subtracted case is noticeable. To
compare further non-classicality of quantum states for different number subtracted case, the measures based on the volume of the negative part of the Wigner function \[55\], on the nonclassical depth \[56\] and on the entanglement potential \[57\] may be other alternative methods. Non-classical state introduced in this work will maybe used in combination with other non-classical states such as entangled states.

On the other hand, we should mention that for a photon-subtracted squeezed state generated with some realistic probability, its non-classicality, in particular, its non-Gaussianity would not be always superior to the input Gaussian state. For example, photon-subtracted two-mode squeezed vacuum state has more entanglement than initial two-mode squeezed state in not so strong squeezing parameter; while for strong squeezing region, its superiority disappears \[28\]. Entanglement evaluation investigation for photon-subtracted two-mode squeezed thermal state is a future problem.

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APPENDIX A: Derivation of Eq.(5)

Using the operator identity \[4\] and noticing the single-mode squeezing operator yields the transformations,

\[ S(r) QS^\dagger (r) = e^{-r} Q, \quad S(r) PS^\dagger (r) = e^r P, \] (A1)

one has

\[ \rho_s = (1 - e^\sigma) S(r) e^{\sigma a^\dagger a} S^\dagger (r) = \frac{2(1 - e^\sigma)}{e^\sigma + 1} : \exp \left\{ \frac{e^\sigma - 1}{e^\sigma + 1} \left( e^{-2r} Q^2 + e^{2r} P^2 \right) \right\} : , \] (A2)

which is still in Weyl ordering, in deriving (A2) we have used the Weyl ordering invariance under similarity transformations \[4\]. According to the definition of Weyl correspondence rule \[58\], i.e., the classical Weyl function \( f(q,p) \) of operator \( \rho_s \) can be given by replacing the \( Q \) and by \( q \) and \( p \) in its Weyl ordered form, respectively,

\[ f(q,p) = \frac{2(1 - e^\sigma)}{e^\sigma + 1} \exp \left\{ \frac{e^\sigma - 1}{e^\sigma + 1} \left( e^{-2r} q^2 + e^{2r} p^2 \right) \right\} , \] (A3)

then using the relation between \( \rho_s \) and Wigner operator \( \Delta(q,p) \), i.e., operator \( \rho_s \) can be expanded in terms of \( \Delta(q,p) \),

\[ \rho_s = \int_{-\infty}^{\infty} dqdp f(q,p) \Delta(q,p) , \] (A4)

where the normal ordering form of \( \Delta(q,p) \) is given by

\[ \Delta(q,p) = \frac{1}{\pi} : \exp \left[ - (q - Q)^2 - (p - P)^2 \right] : \] (A5)

one can see that

\[ \rho_s = \frac{2(1 - e^\sigma)}{\pi (e^\sigma + 1)} \int_{-\infty}^{\infty} dqdp \exp \left\{ \frac{e^\sigma - 1}{e^\sigma + 1} \left( e^{-2r} q^2 + e^{2r} p^2 \right) \right\} \times \exp \left[ - (q - Q)^2 - (p - P)^2 \right] : = \text{Eq.(5)} . \] (A6)
thus we complete the proof of Eq. (5).

APPENDIX B: Deduction of Eq. (11)

Using the completeness relation and \(\rho_s\) normal ordering form in (5), as well as the overlap of coherent state,

\[
\langle \beta | \alpha \rangle = \exp \left[ -\frac{1}{2} |\alpha|^2 - \frac{1}{2} |\beta|^2 + \beta^* \alpha \right], \tag{B1}
\]

we have

\[
C_m = \frac{1}{\tau_1 \tau_2} \text{Tr} \left\{ a^m \int \frac{d^2\alpha d^2\beta}{\pi^2} |\alpha\rangle \langle \alpha| : \exp \left[ -\frac{Q^2}{2\tau_1^2} - \frac{P^2}{2\tau_2^2} \right] : |\beta\rangle \langle \beta| a^m \right\}
\]

\[
= \frac{1}{\tau_1 \tau_2} \int \frac{d^2\alpha d^2\beta}{\pi^2} \alpha^m \beta^m \exp \left[ -|\alpha|^2 - |\beta|^2 + \beta^* \alpha + A_1 \beta^* A_2 (\beta^2 + \alpha^2) \right] \tag{B2}
\]

\[
= \frac{1}{\tau_1 \tau_2} \int \frac{d^2\alpha d^2\beta}{\pi^2} \exp \left[ -|\alpha|^2 + (\beta^* + k) \alpha + A_1 \beta^* + A_2 \alpha^2 \right] \times \exp \left| -|\beta|^2 + \beta^* s + A_2 \beta^2 \right|_{s=0} \tag{B3}
\]

\[
= \frac{1}{\tau_1 \tau_2} \int \frac{d^2\alpha d^2\beta}{\pi^2} \exp \left[ (k^2 + s^2) A + B s \right]_{s=0},
\]

where

\[
A_1 = 1 - \frac{1}{2\tau_2^2} - \frac{1}{2\tau_1^2}, \quad A_2 = \frac{1}{4\tau_2^2} - \frac{1}{4\tau_1^2} > 0, \tag{B4}
\]

\[
A_3 = (1 - A_1)^2 - 4A_2^2 = \frac{1}{\tau_1 \tau_2^2}, \tag{B5}
\]

\[
A = \frac{A_2}{A_3} = \frac{1}{4} \left( \tau_1^2 - \tau_2^2 \right) = \frac{2\tau + 1}{4} \sinh 2\tau > 0, \tag{B6}
\]

\[
B = \frac{A_1 - A_2^2 + 4A_2^2}{A_3} = \frac{1}{2} \left( \tau_1^2 + \tau_2^2 \right) - 1,
\]

and using the integration formula [59]

\[
\int \frac{d^2z}{\pi} \exp \left\{ \zeta |z|^2 + \xi z + \eta z^* + f z^2 + g z^2 \right\} = \frac{1}{\sqrt{\xi^2 - 4fg}} \exp \left\{ -\zeta \eta + \xi^2 g + \eta^2 f \right\}, \tag{B7}
\]

whose convergent condition is \(\text{Re}(\zeta \pm f \pm g) < 0\) and \(\text{Re} \left( \frac{\xi^2 - 4fg}{\zeta^2 - 4fg} \right) < 0\) and noting that

\[
\frac{\partial^{2m}}{\partial t^m \partial \tau^m} \exp (-t^2 - \tau^2 + 2xt \tau) \bigg|_{t,\tau=0}
\]

\[
= \sum_{n,l,k=0}^{\infty} \frac{(-)^{n+l}}{n!l!k!} (2x)^k \frac{\partial^{2m}}{\partial t^m \partial \tau^m} \tau^{2n+k} l^{2l+k} \bigg|_{t,\tau=0}
\]

\[
= 2^m m! \sum_{n=0}^{\lfloor m/2 \rfloor} \frac{m!}{2^{2n} (n!)^2 (m - 2n)!} \xi^{m-2n}, \tag{B8}
\]
one rewritten Eq. (B2) as

\[ C_m = (-A)^m \frac{\partial^{2m}}{\partial k_m \partial s^m} \exp \left[ -k^2 - s^2 - \frac{B}{A} ks \right] \bigg|_{s=k=0} \]

\[ = (-A)^m 2^m m! \sum_{n=0}^{[m/2]} \frac{m! (\frac{B}{2A})^{m-2n}}{2^{2n} (n!)^2 (m-2n)!}. \]  

(B9)

Recalling the newly found expression of Legendre polynomial (its equivalence to the well-known Legendre polynomial’s \((P_m(x))\) expression is \([60]\)

\[ x^m \sum_{l=0}^{[m/2]} \frac{m!}{2^{2l} (l!)^2 (m-2l)!} \left( 1 - \frac{1}{x^2} \right)^l = P_m(x), \]

(B10)

we derive the compact form for \(C_m:\)

\[ C_m = m! B^m \sum_{n=0}^{[m/2]} \frac{m!}{2^{2n} (n!)^2 (m-2n)!} \left( \frac{4A^2}{B^2} \right)^n \]

\[ = m! D^{m/2} P_m \left( B/\sqrt{D} \right), \]  

(B11)

where

\[ D = B^2 - 4A^2 = \bar{n}^2 - (2\bar{n} + 1) \sinh^2 r. \]  

(B12)

Eq.(B11) indicates that the normalization factor \(C_m\) is just related to Legendre polynomial. Combining Eqs.(B8) and (B10), On the other hand, one can derive a new formula for Legendre polynomial, i.e.,

\[ \frac{\partial^{2m}}{\partial k^m \partial t^m} \exp \left( -t^2 - \tau^2 + \frac{2xt}{\sqrt{x^2 - 1}} \right) \bigg|_{t, \tau = 0} = \frac{2^m m!}{(x^2 - 1)^{m/2}} P_m(x). \]

(B13)

**APPENDIX C: Derivation of WF (11) for PSSTS**

According to Eqs.(34), (36) and (29), we have

\[ W(\alpha, \alpha^*) = C_m^{-1} \text{tr} \left[ \int \frac{d^2z}{\pi} |z|^{2m} P_0(z) \langle z | \Delta(\alpha, \alpha^*) \rangle \right] \]

\[ = C_m^{-1} \int \frac{d^2z}{\pi} |z|^{2m} P_0(z) \exp \left[ -2(z^* - \alpha^*)(z - \alpha) \right] \]

\[ = C_m^{-1} e^{-2|\alpha|^2} \int \frac{d^2z}{\pi} \frac{1}{|z|^{2m}} \exp \left[ -g |z|^2 + 2\alpha^* z + 2\alpha z^* + \frac{\tau - 4D}{4D} (z^{*2} + z^2) \right] \]

\[ = C_m^{-1} e^{-2|\alpha|^2} \frac{\partial^{2m}}{\partial k_m \partial t^m} \int \frac{d^2z}{\pi} \exp \left[ -g |z|^2 + (2\alpha + k) z^* + (2\alpha^* + t) z + \frac{\tau - 4D}{4D} (z^{*2} + z^2) \right] \bigg|_{k=t=0}, \]

(C1)

where

\[ g = \frac{\tau_+ - 2}{2D} + 2 = \frac{(2\bar{n} + 1)}{D} (\bar{n} - \sinh^2 r), \]

(C2)
which leads to
\[ g^2 - \frac{r^2}{4D^2} = \frac{(2\bar{n} + 1)^2}{D}. \] (C3)
Then using the integration formula (B7), we can write Eq.(C1) as following form,
\[ W(\alpha, \alpha^*) = \frac{C_m^{-1}e^{-2|\alpha|^2}}{\pi (2\bar{n} + 1)} \frac{\partial^{2m}}{\partial k^m \partial t^m} \exp \left[ g_1 (2\alpha + k) (2\alpha^* + t) + \frac{g_2}{4} \left( (2\alpha + k)^2 + (2\alpha^* + t)^2 \right) \right]_{k=t=0} = F_m(\alpha, \alpha^*) W_0(\alpha, \alpha^*), \] (C4)
where \( W_0(\alpha, \alpha^*) \) is the WF of squeezed thermal state defined in Eq.(38), and
\[ \bar{\alpha} = 2g_1\alpha^* + g_2\alpha, \quad g_1 = \frac{\bar{n} - \sinh^2 r}{2\bar{n} + 1}, \quad g_2 = \frac{\sinh 2r}{2\bar{n} + 1}, \] (C5)
as well as
\[ F_m(\alpha, \alpha^*) = \frac{C_m^{-1}}{2^{2m}g_2^m} \sum_{l=0}^{\infty} \frac{g_1^l}{l!} \frac{\partial^{2l}}{\partial \alpha^l \partial \alpha^{*l}} \exp \left[ \bar{\alpha} k + \bar{\alpha}^* t + \frac{g_2}{4} \left( k^2 + t^2 \right) \right]_{k=t=0}. \] (C6)
Further expanding the exponential term \( kt \) included in (C6) into sum series, and using the generating function of single-variable Hermite polynomials,
\[ H_n(x) = \left. \frac{\partial^n}{\partial t^n} \exp \left( 2tx - t^2 \right) \right|_{t=0}, \] (C7)
which leads to
\[ \left. \frac{\partial^n}{\partial t^n} \exp \left( At + Bt^2 \right) \right|_{t=0} = \left( i\sqrt{B} \right)^n H_n \left[ A/(2i\sqrt{B}) \right] = \left( -i\sqrt{B} \right)^n H_n \left[ A/(-2i\sqrt{B}) \right], \] (C8)
thus we can see
\[ F_m(\alpha, \alpha^*) = \frac{C_m^{-1}}{2^{2m}g_2^m} \sum_{l=0}^{\infty} \frac{g_1^l}{l!} \frac{\partial^{2l}}{\partial \alpha^l \partial \alpha^{*l}} H_m(\beta)H_m(\beta^*), \] (C9)
where
\[ \beta = \frac{\sqrt{2\bar{n} + 1}}{i\sqrt{\sinh 2r}} \bar{\alpha} = 2\alpha^* \left( \frac{\bar{n} - \sinh^2 r}{2\bar{n} + 1} \right) + \alpha \sinh 2r. \] (C10)
Then using the recurrence relation of \( H_n(x) \),
\[ \frac{d}{dx} H_n(x) = \frac{2^n n!}{(n - l)!} H_{n-l}(x), \] (C11)
Eq.(C9) becomes
\[ F_m(\alpha, \alpha^*) = \frac{(m!)^2 g_2^m}{2^{2m}C_m} \sum_{l=0}^{\infty} \frac{1}{l! \left( \frac{\bar{n} - \sinh^2 r}{\sinh 2r} \right)^l} \frac{\partial^{2l}}{\partial \beta^l \partial \beta^{*l}} H_m(\beta)H_m(\beta^*) = \text{Eq.} \text{(C9)}. \] (C12)
Thus we complete the derivation of WF Eq. (11) by combing Eqs. (C4) and (C12).

**APPENDIX D: Derivation of (45)**

Substituting Eqs. (37)-(39) into (44), we have

\[
W(\zeta, \zeta^*, t) = \frac{2C_m^{-1}/(2n+1)}{\pi(2n+1)T} \exp\left[\frac{-2|\zeta|^2}{(2n+1)T}\right] \frac{\partial^{2m}}{\partial k^m \partial \tau^m} \exp\left[g_1 k \tau + \frac{g_2}{4} (k^2 + \tau^2)\right]
\]
\[
\times \int \frac{d^2 \alpha}{\pi} \exp \left[- (2g_0 + g_3 e^{-\kappa t}) |\alpha|^2 + (2\kappa g_1 + k g_2 + g_3 \zeta^*) \alpha\right.
\]
\[
+ (2k g_1 + \kappa g_2 + g_3 \zeta^*) \alpha^* + g_2 (\alpha^2 + \alpha^*2)\right]_{k=\tau=0}
\]
\[
= \frac{C_m^{-1}/(2n+1)}{\pi(2n+1)T \sqrt{G}} \exp\left[- \Delta_2 |\zeta|^2 + \frac{g_2g_3^2}{4G} (\zeta^2 + \zeta^{*2})\right]
\]
\[
\times \frac{\partial^{2m}}{\partial k^m \partial \tau^m} \exp\left[\chi k \tau + \omega^*k + \omega \tau + \Delta_1 (k^2 + \tau^2)\right]_{k=\tau=0},
\]

where \(T = (1 - e^{-2\kappa t})\), \((g_0, g_1, g_2, g_3)\) and \((\chi, \omega, G, \Delta_1, \Delta_2)\) are defined in Eqs. (48) and (49), respectively. In a similar way to deriving Eq. (11), we can further put Eq. (D1) into Eqs. (45)-(47).

**APPENDIX E: Derivation of (57)**

Then employing Eqs. (38) and (C6) as well as the integration formula (B7), we can treat the integration in a similar way to deriving Eq. (11),

\[
tr(\rho_s \rho) = \frac{4C_m^{-1}/(2n+1)^2}{\partial^{2m}/\partial k^m \partial \tau^m} \exp\left[\frac{g_2}{4} (k^2 + t^2) + g_1 kt\right]
\]
\[
\times \int \frac{d^2 \alpha}{\pi} \exp \left[-4g_0 |\alpha|^2 + (k g_2 + 2t g_1) \alpha + (2k g_1 + t g_2) \alpha^* + 2g_2 (\alpha^2 + \alpha^*2)\right]_{k=t=0}
\]
\[
= \frac{C_m^{-1}/(2n+1)^2 \sqrt{g_0^2 - g_2^2}}{\partial^{2m}/\partial k^m \partial \tau^m} \exp\left[\frac{g_2}{4} (k^2 + t^2) + g_1 kt\right]
\]
\[
\times \exp\left[\frac{g_2 (4g_1^2 + 4g_0 g_1 + g_2^2)}{8 (g_0^2 - g_2^2)} (k^2 + t^2) + \frac{4g_0 g_1^2 + 4g_1 g_2^2 + g_0 g_2^2}{4 (g_0^2 - g_2^2)} kt\right]_{k=t=0}
\]
\[
= \frac{C_m^{-1}/(2n+1)^2 \sqrt{g_0^2 - g_2^2}}{\partial^{2m}/\partial k^m \partial \tau^m} \exp\left[B_2' (k^2 + t^2) + B_1' kt\right]_{k=t=0},
\]

where \(g_0^2 - g_2^2 = \frac{1}{(2n+1)^2}\) and

\[
B_1' = \frac{1}{4} \frac{g_0}{g_0^2 - g_2^2} (4g_1^2 + 4g_0 g_1 + g_2^2)
\]
\[
= \frac{\bar{n}}{2n+1} \cosh 2r = g_0 \bar{n} (\bar{n} + 1),
\]

\[
B_1' = \frac{1}{8} \frac{g_2 (2g_0^2 + 4g_0 g_1 + 4g_1^2 - g_2^2)}{g_0^2 - g_2^2}
\]
\[
= \frac{2n^2 + 2\bar{n} + 1}{4 (2n+1)} \sinh 2r = \frac{g_2}{4} (2n^2 + 2n + 1).
\]

Similarly to deriving Eq. (B11), we have

\[
\frac{\partial^{2m}}{\partial k^m \partial \tau^m} \exp\left[B_2' (k^2 + t^2) + B_1' kt\right]_{k=t=0} = m! B_2'^{m/2} P_m \left(B_1'/\sqrt{B_2'}\right),
\]
and $B_2 \equiv B_1^2 - 4B_2'^2$ given in Eq. (58), which leads to Eq. (57).

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