Log-concavity of the Excedance Enumerators in positive elements of Type A and Type B Coxeter Groups

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October 21, 2020

Abstract

The classical Eulerian Numbers $A_{n,k}$ are known to be log-concave. Let $P_{n,k}$ and $Q_{n,k}$ be the number of even and odd permutations with $k$ excedances. In this paper, we show that $P_{n,k}$ and $Q_{n,k}$ are log-concave. For this, we introduce the notion of strong synchronisation and ratio-alternating which are motivated by the notion of synchronisation and ratio-dominance, introduced by Gross, Mansour, Tucker and Wang in 2014.

We show similar results for Type B Coxeter Groups. We finish with some conjectures to emphasize the following: though strong synchronisation is stronger than log-concavity, many pairs of interesting combinatorial families of sequences seem to satisfy this property.

1 Introduction

Log-concavity and unimodality are well-studied properties of combinatorial sequences. They often appear in various areas of mathematics such as combinatorics, probability and algebra. The papers of Brändén [4], Brenti ([5], [6]) and Stanley [24] contain a wealth of information about various results on log-concavity.

Definition 1 A sequence $(a_k)_{k=0}^n$ is said to be log-concave if for all $i = 1, 2, \ldots, n - 1$, we have $a_i^2 \geq a_{i-1}a_{i+1}$.

Definition 2 A sequence $(a_k)_{k=0}^n$ is said to be unimodal if there exists an index $0 \leq r \leq n$ such that $a_0 \leq a_1 \leq \ldots \leq a_{r-1} \leq a_r \geq a_{r+1} \geq \ldots \geq a_n$.

In this work, we will only deal with finite and non-negative sequences. Define a polynomial to be log-concave (and unimodal respectively), if the sequence of its coefficients is log-concave (and
unimodal respectively). If a non-negative sequence \((a_k)_{k=0}^n\) is log-concave and does not have any internal zero, then there cannot be any \(j\) such that \(a_{j-1} > a_j < a_{j+1}\) and so the sequence \((a_k)_{k=0}^n\) must be unimodal. Many methods have been incorporated to establish the log-concavity of various combinatorial sequences. If the sequence satisfies some ‘nice’ formula or recurrence, then by direct manipulation one can show log-concavity. Another approach towards proving the log-concavity of a sequence is showing the real-rootedness of the associated polynomial. Combinatorial polynomials are often real-rooted and Newton showed that real-rooted polynomials are log-concave. Thus this criterion directly solves many log-concavity related problems (see Petersen [19, Chapter 4]). Another interesting way of attacking a log-concavity problem is by directly giving a combinatorial proof. If \(a_0, a_1, \ldots, a_n\) is any sequence of non-negative integers for which a combinatorial meaning is known (that is, we have sets \(S_0, S_1, \ldots, S_n\) such that \(|S_i| = a_i\)), then constructing an injection \(\phi_k : S_{k-1} \times S_{k+1} \to S_k \times S_k\) yields a combinatorial proof of \(a_k^2 \geq a_k a_{k-1}\). One can take a look at [21] where Sagan gave combinatorial proof of log-concavity of some combinatorial sequences. In this work, we are interested in the following question:

Suppose we have two sequences \(A = (a_k)_{k=0}^n\) and \(B = (b_k)_{k=0}^n\). Let us define \(S(A, B)\) to be the set of all sequences \(C = (c_k)_{k=0}^n\) such that for each \(k\), \(c_k \in \{a_k, b_k\}\). \(S(A, B)\) is actually the set of all \(2^{n+1}\) sequences, which can be cooked up by using the two given sequences \(A = (a_k)_{k=0}^n\) and \(B = (b_k)_{k=0}^n\). The natural question, that comes to mind, is whether all the sequences in \(S(A, B)\) are log-concave or not. In this work, we investigate the above question for some interesting combinatorial pair of sequences.

For a positive integer \(n\), let \([n] = \{1, 2, \ldots, n\}\) and let \(\mathcal{S}_n\) be the set of permutations on \([n]\). For \(\pi = \pi_1, \pi_2, \ldots, \pi_n \in \mathcal{S}_n\), define its excedance set as \(\text{EXC}(\pi) = \{i \in [n] : \pi_i > i\}\) and its number of excedances as \(\text{exc}(\pi) = |\text{EXC}(\pi)|\). Define its number of antieexcences as \(\text{nexc}(\pi) = |\{i \in [n] : \pi_i \leq i\}|\) and its number of inversions as \(\text{inv}(\pi) = |\{1 \leq i < j \leq n : \pi_i > \pi_j\}|\). Let \(\text{DES}(\pi) = \{i \in [n - 1] : \pi_i > \pi_{i+1}\}\) and \(\text{ASC}(\pi) = \{i \in [n - 1] : \pi_i < \pi_{i+1}\}\) be its set of descents and ascents respectively. Let \(\text{des}(\pi) = |\text{DES}(\pi)|\) be its number of descents and \(\text{asc}(\pi) = |\text{ASC}(\pi)|\) be its number of ascents. Let \(\mathcal{A}_n \subseteq \mathcal{S}_n\) be the subset of even permutations. Let \(E_{n,k}, P_{n,k}\) and \(Q_{n,k}\) be the number of permutations with \(k\) excedges in \(\mathcal{S}_n, \mathcal{A}_n\) and \(\mathcal{S}_n - \mathcal{A}_n\) respectively. Define

\[
A_n(t) = \sum_{\pi \in \mathcal{S}_n} t^{\text{des}(\pi)} = \sum_{k=0}^{n-1} A_{n,k} t^k \quad \text{and} \quad \text{AExc}_n(t) = \sum_{\pi \in \mathcal{S}_n} t^{\text{exc}(\pi)} = \sum_{k=0}^{n-1} E_{n,k} t^k, \tag{1}
\]

\[
\text{AExc}_n^+(t) = \sum_{\pi \in \mathcal{A}_n} t^{\text{exc}(\pi)} = \sum_{k=0}^{n-1} P_{n,k} t^k \quad \text{and} \quad \text{AExc}_n^-(t) = \sum_{\pi \in \mathcal{S}_n - \mathcal{A}_n} t^{\text{exc}(\pi)} = \sum_{k=0}^{n-1} Q_{n,k} t^k. \tag{2}
\]

It is a well known result of MacMahon [16] that both descents and antieexcences are equidistributed over \(\mathcal{S}_n\). That is, for all positive integers \(n\) and \(0 \leq k \leq n - 1\), \(A_{n,k} = E_{n,k}\). \(A_n(t)\) is known to be real-rooted for all \(n\) and hence the \(A_{n,k}s\) are log-concave. But the excedance enumerating polynomial over \(\mathcal{A}_n\) and \(\mathcal{S}_n - \mathcal{A}_n\) are not always real-rooted and hence log-concavity of \(P_{n,k}\) and \(Q_{n,k}\) are not immediate. Moreover, we ask whether all the sequences that can be cooked up by using \(P_{n,k}\) and \(Q_{n,k}\) are log-concave. To answer this question, we introduce a notion of strong synchronisation which is influenced by the notion of synchronisation as defined in the paper by Gross, Mansour, Tucker and Wang [14]. They defined the following.
Definition 3. Two non-negative sequences \( A = (a_k)_{k=0}^n \) and \( B = (b_k)_{k=0}^n \) are said to be synchronised, denoted as \( A \sim B \) if both are log-concave and they satisfy \( a_{k-1}b_{k+1} \leq a_kb_k \) and \( a_{k+1}b_{k-1} \leq a_kb_k \) for all \( 1 \leq k \leq n-1 \).

Here we generalise this further and define the following notion of strong synchronisation of two sequences.

Definition 4. Two non-negative sequences \( A = (a_k)_{k=0}^n \) and \( B = (b_k)_{k=0}^n \) are said to be strongly synchronised, denoted as \( A \approx B \) if the following holds for all \( 1 \leq k \leq n-1 \):

\[
(\min\{a_k, b_k\})^2 \geq \max\{a_{k+1}, b_{k+1}\} \cdot \max\{a_{k-1}, b_{k-1}\}.
\]

Clearly strong synchronisation implies log-concavity of both sequences \( A \) and \( B \).

For \( n = 5 \), consider the following sequences \( (P_5 \circ k)_{k=0}^4 \) and \( (Q_5 \circ k)_{k=0}^4 \):

\[
P_5 = (1, 11, 36, 11, 1),
\]

\[
Q_5 = (0, 15, 30, 15, 0).
\]

It is easy to check that the sequences \( (P_5 \circ k)_{k=0}^4 \) and \( (Q_5 \circ k)_{k=0}^4 \) satisfy (3) and hence they are strongly synchronised.

Clearly, strong synchronisation is a much stronger property than synchronisation. Recall the sequences \( P_{n,k} \) and \( Q_{n,k} \) from (2). One of our main results in this paper is the following:

Theorem 5. For positive integers \( n \), the sequences \( P_n = (P_{n,k})_{k=0}^{n-1} \) and \( Q_n = (Q_{n,k})_{k=0}^{n-1} \) are strongly synchronised and hence log-concave.

The sum of two log-concave sequences need not be log-concave. But Gross et al. in [14, Theorem 2.3] showed that the sum of two synchronised sequences is log-concave. Hence Theorem 5 refines the log-concavity of \( A_{n,k} \). We generalize our results to the case when excedances are summed over the elements with positive sign in Type B Coxeter Groups. Let \( \mathcal{B}_n \) be the set of permutations \( \pi \) of \( \{-n, -(n-1), \ldots, -1, 1, 2, \ldots n\} \) satisfying \( \pi(-i) = -\pi(i) \). \( \mathcal{B}_n \) is referred to as the hyperoctahedral group or the group of signed permutations on \( [n] \) and \( |\mathcal{B}_n| = 2^n n! \). We use Brenti’s [7] definition for Type B excedance and define the excedance polynomials of Type B. There is a natural notion of length in these groups and we get results when excedance enumeration is restricted to elements with even length. For Type B Coxeter Groups, our main result is Theorem 6.

Theorem 6. For positive integers \( n \), the sequences \( P^B_n = (P^B_{n,k})_{k=0}^n \) and \( Q^B_n = (Q^B_{n,k})_{k=0}^n \) are strongly synchronised.

We organize this paper as follows. In Section 2 we state and prove some basic properties of strongly synchronised sequences. In Section 3 we prove Theorem 5. In Section 4 we prove Theorem 6. In Section 5 we modify a Theorem of Sagan to prove log-concavity of some combinatorial sequences directly.
2 Properties of strong synchronisation

Recall the sequence \( A_{n,k} \) from Section 1. Let \( \mathcal{S}\mathcal{D}_n \) be the set of derangements in \( \mathcal{S}_n \) and let \( D_{n,k} = |\{ \pi \in \mathcal{S}\mathcal{D}_n : \text{des}(\pi) = k \}|. \) Consider the following sequences:

\[
(A_{6,0}, A_{6,1}, A_{6,2}, A_{6,3}, A_{6,4}, A_{6,5}) = (1, 57, 302, 302, 57, 1)
\]
\[
(D_{6,0}, D_{6,1}, D_{6,2}, D_{6,3}, D_{6,4}, D_{6,5}) = (0, 16, 104, 120, 24, 1)
\]

Then it can be checked that these two sequences are synchronised in \( k \) but not strongly synchronised in \( k \) as \( D_{6,1}^2 = 16^2 \leq 302 = A_{6,0}A_{6,2}. \)

By definition, clearly the strong synchronisation relation is symmetric but neither reflexive nor transitive. Consider the following example:

\[
A = (1, 4, 5), \quad B = (1, 5, 10), \quad C = (1, 6, 25).
\]

Here, \( A \approx B \) and \( B \approx C \) but \( A \) and \( C \) are not strongly synchronised. Moreover, note that \( A \) and \( C \) are not even synchronised. We also note that for any log-concave sequence \( A \), we have \( A \approx A \), but for 2 different scalars \( \lambda \) and \( \mu \), \( \lambda A \) and \( \mu A \) may, or may not be strongly synchronised. The following is a useful result which gives a nice connection between the strong synchronisation of two sequences \( A \) and \( B \) and log-concavity of all the sequences in \( S(A, B) \).

**Theorem 7** Two non-negative sequences \( A \) and \( B \) are strongly synchronised if and only if for all \( C \in S(A, B) \), \( C \) is log-concave.

**Proof:** Let \( A \) and \( B \) be strongly synchronised and \( C \in S(A, B) \). Then

\[
c_k^2 \geq \left( \min\{a_k, b_k\} \right)^2 \geq \max\{a_{k+1}, b_{k+1}\}. \max\{a_{k-1}, b_{k-1}\} \geq c_{k+1}^{k+1}.
\]

Hence, \( C \) is log-concave.

Conversely, let \( A \) and \( B \) be two non-negative sequences such that for all \( C \in S(A, B) \), \( C \) is log-concave. We fix \( k \) and we need to show

\[
(\min\{a_k, b_k\})^2 \geq \max\{a_{k+1}, b_{k+1}\}. \max\{a_{k-1}, b_{k-1}\}.
\]

Let us consider the following sequence \( C \) with

\[
c_r = \begin{cases} 
\max\{a_r, b_r\} & \text{if } r = k - 1 \text{ and } k + 1 . \\
\min\{a_r, b_r\} & \text{if } r = k . \\
a_r & \text{elsewhere} .
\end{cases}
\]

Then log-concavity of \( C \) ensures (4). For each \( k \), we can construct such a sequence \( C \) whose log-concavity will ensure (4) and hence, we are done. \( \blacksquare \)

Next we consider \( l \) non-negative sequences \( T^1 = (T^1_k)_{k=0}^n, T^2 = (T^2_k)_{k=0}^n, \ldots, T^l = (T^l_k)_{k=0}^n \). Let \( S(T^1, T^2, \ldots, T^l) \) be the set of sequences \( C = (c_k)_{k=0}^n \) such that for each \( k, c_k \in \{T^1_k, \ldots, T^l_k\} \). \( S(T^1, T^2, \ldots, T^l) \) is essentially the set of all \( n+1 \) sequences, which can be made up from the given sequences \( T^1, T^2, \ldots, T^l \).
Corollary 8 Let $T^1, T^2, \ldots, T^l$ be $l$ non-negative sequences. Suppose $C$ is log-concave for all $C \in S(T^1, T^2, \ldots, T^l)$. Then for all $1 \leq i < j \leq l$, the sequences $T^i$ and $T^j$ are strongly synchronised.

As $C$ is log-concave for all $C \in S(T^1, T^2, \ldots, T^l)$, $C$ is log-concave for all $C \in S(T^i, T^j)$. Hence, Corollary 8 follows. Surprisingly, the converse of the above statement is not true. Consider the following three sequences:

$$T^1 = (1, 5, 3), \quad T^2 = (7, 6, 3), \quad T^3 = (6, 6, 4).$$

Here, $T^1 \approx T^2, T^2 \approx T^3$ and $T^1 \approx T^3$. Consider the sequence $(7, 5, 4) \in S(T^1, T^2, T^3)$ which is not log-concave. It is easy to see that log-concavity of a sequence $A = (a_k)_{k=0}^n$ with all intermediate terms positive is equivalent to saying $a_j a_l \geq a_{j-i} a_{l+i}$ for all positive integers $j \leq l$ and $i \leq j$. Thus, Theorem 7 gives the following corollary which we will need later in Section 3.

Corollary 9 Let, $A = (a_k)_{k=0}^n$ and $B = (b_k)_{k=0}^n$ be two sequences. The following are equivalent:

1. $A = (a_k)_{k=0}^n$ and $B = (b_k)_{k=0}^n$ are strongly synchronised.

2. For all $j \leq l$ and for all positive integers $i$, we have

$$\min\{a_j, b_j\}, \min\{a_i, b_i\} \geq \max\{a_{j-i}, b_{j-i}\}, \max\{a_{l+i}, b_{l+i}\}. \quad (5)$$

Proof: We prove the forward implication at first. Let $A = (a_k)_{k=0}^n$ and $B = (b_k)_{k=0}^n$ be strongly synchronised. Thus, by Theorem 7 any sequence $C \in S(A, B)$ is log-concave and hence, we are done. The other direction follows by setting $j = l$ and $i = 1$ in (5).

2.1 Ratio-Alternating Sequences

Gross et al. introduced the ratio-dominance relation between two sequences in [14]. Then they gave several results connecting ratio-dominance and synchronisation. Motivated by those, we introduce a similar but different notion of ratio-alternating defined as follows:

Definition 10 Two non-negative sequences $A = (a_k)_{k=0}^n$ and $B = (b_k)_{k=0}^n$ are said to be ratio-alternating if they satisfy either

$$a_{2i} \leq b_{2i} \quad \forall \ 0 \leq 2i \leq n \quad \text{and} \quad a_{2i+1} \geq b_{2i+1} \quad \forall \ 0 \leq 2i+1 \leq n, \quad (6)$$

or

$$a_{2i} \geq b_{2i} \quad \forall \ 0 \leq 2i \leq n \quad \text{and} \quad a_{2i+1} \leq b_{2i+1} \quad \forall \ 0 \leq 2i+1 \leq n. \quad (7)$$

The relation ratio-alternating is reflexive and symmetric but not transitive. Consider the following example:

$$A = (1, 5, 7), \quad B = (3, 4, 10), \quad C = (2, 6, 8).$$

Here, $A$ and $B$ are ratio-alternating, $B$ and $C$ are also ratio-alternating but $A$ and $C$ are not ratio-alternating. We need the following two definitions for the next Theorem.
Definition 11 A sequence \( A = (a_k)_{k=0}^n \) is said to be even log-concave (respectively, odd log-concave) if we have \( a_i^2 \geq a_{i-1}a_{i+1} \) for \( i \) even (respectively, \( i \) odd) and \( 1 \leq i \leq n-1 \).

Theorem 12 Let \( A = (a_k)_{k=0}^n \) and \( B = (b_k)_{k=0}^n \) be two non-negative sequences which satisfy (6). Then the following statements are equivalent:

1. \( A \) is even log-concave and \( B \) is odd log-concave.
2. \( A \) and \( B \) are strongly synchronised.

In a similar manner, if \( A \) and \( B \) be two non-negative sequences which satisfy (7), then the following are equivalent:

1. \( A \) is odd log-concave and \( B \) is even log-concave.
2. \( A \) and \( B \) are strongly synchronised.

Proof: We consider only the first case, that is, when \( A \) and \( B \) satisfy (6). It is easy to see that [2] implies [1]. We prove [1] implies [2]. Assume that \( A \) is even log-concave and \( B \) is odd log-concave. Then for even \( k \), we have

\[
(\min\{a_k, b_k\})^2 = a_k^2 \geq a_{k+1}a_{k-1} = \max\{a_{k+1}, b_{k+1}\} \cdot \max\{a_{k-1}, b_{k-1}\}. \tag{8}
\]

The first equality follows as \( A \) and \( B \) satisfy (6). The inequality follows as \( A \) is even log-concave. The last inequality also follows from (6). Similarly for \( k \) odd, we have

\[
(\min\{a_k, b_k\})^2 = b_k^2 \geq b_{k+1}b_{k-1} = \max\{a_{k+1}, b_{k+1}\} \cdot \max\{a_{k-1}, b_{k-1}\}. \tag{9}
\]

Hence, \( A \) and \( B \) are strongly synchronised. The proof for the case when \( A \) and \( B \) satisfy (7) is identical and hence omitted.

3 Proof of Theorem 5

At first, we mention the following well-known recurrence, satisfied by the Eulerian numbers \( A_{n,k} \). For reference, one can see [19, Theorem 1.3].

Theorem 13 For positive integers \( n, k \) with \( n \geq 2 \) and \( 0 \leq k \leq n-1 \), the numbers \( A_{n,k} \) satisfy the following recurrence:

\[
A_{n,k} = (k+1)A_{n-1,k} + (n-k)A_{n-1,k-1}, \tag{10}
\]

where \( A_{1,0} = 1 \) and \( A_{1,1} = 0 \).

Remark 14 Setting \( k = 1 \) in Equation (10), we have \( A_{n,1} = 2A_{n-1,1} + n - 1 \). Thus, when \( n \geq 3 \), we have \( A_{n,1} \geq n + 1 \).
Next, we recall the numbers $P_{n,k}$ and $Q_{n,k}$ from (2). The following identity involving $P_{n,k}$ and $Q_{n,k}$ was shown by Mantaci (see [17], [18]).

**Theorem 15 (Mantaci)** For positive integers $n$ and $0 \leq k \leq n - 1$, $P_{n,k}$ and $Q_{n,k}$ satisfy the following:

\[
P_{n,k} - Q_{n,k} = (-1)^k \binom{n-1}{k}.
\] (11)

Later Sivasubramanian in [22] gave a proof of Theorem 15 using determinant enumeration of suitably defined matrices. In [17] and [18], Mantaci showed the following recurrences involving $P_{n,k}$ and $Q_{n,k}$. Though Mantaci did with anti-excedance enumerator, in an identical manner we can get the same recurrences for excedance enumerator. The recurrences are also shown by Dey and Sivasubramanian in [10].

**Lemma 16 (Mantaci)** For positive integers $n$ and $0 \leq k \leq n - 1$, the coefficients $P_{n,k}$ and $Q_{n,k}$ satisfy the following:

\[
P_{n,k} = kQ_{n-1,k} + (n-k)Q_{n-1,k-1} + P_{n-1,k},
\] (12)

\[
Q_{n,k} = kP_{n-1,k} + (n-k)P_{n-1,k-1} + Q_{n-1,k}.
\] (13)

where $P_{1,0} = 1$ and $Q_{1,0} = 0$.

**Remark 17** From Lemma 16 it is easy to show that for any positive integer $n$, we have $P_{n,0} = 1$ and $Q_{n,0} = 0$.

From (11), we get the following corollary.

**Corollary 18** For positive integers $n$, the sequences $P_n = (P_{n,k})_{k=0}^{n-1}$ and $Q_n = (Q_{n,k})_{k=0}^{n-1}$ are ratio-alternating. Moreover, they satisfy (7).

**Proof:** By Equation (11), we have $P_{n,k} \geq Q_{n,k}$ for $k$ being even and $P_{n,k} \leq Q_{n,k}$ for $k$ being odd, completing the proof. $lacksquare$

**Lemma 19** For positive integers $n$ and $1 \leq k \leq n - 2$, the coefficients $P_{n,k}$ satisfy following relation:

\[
P_{n,k}^2 - P_{n,k+1}P_{n,k-1} = \sum_{i=1}^{9} T_i(n, k),
\] (14)

where
\[ T_1(n, k) = (k^2 - 1)(Q_{n-1,k}^2 - Q_{n-1,k+1}Q_{n-1,k-1}), \]
\[ T_2(n, k) = Q_{n-1,k}^2 + Q_{n-1,k-1}^2 - 2Q_{n-1,k-1}Q_{n-1,k}, \]
\[ T_3(n, k) = ((n - k)^2 - 1)(Q_{n-1,k-1}^2 - Q_{n-1,k}Q_{n-1,k-2}), \]
\[ T_4(n, k) = P_{n-1,k}^2 - P_{n-1,k+1}P_{n-1,k-1}, \]
\[ T_5(n, k) = (k + 1)(n - k + 1)(Q_{n-1,k}Q_{n-1,k-1} - Q_{n-1,k+1}Q_{n-1,k-2}), \]
\[ T_6(n, k) = (k - 1)(n - k - 1)(Q_{n-1,k-1}Q_{n-1,k} - Q_{n-1,k}Q_{n-1,k+1}), \]
\[ T_7(n, k) = (n - k - 1)(Q_{n-1,k-1}P_{n-1,k} - Q_{n-1,k}P_{n-1,k-1}), \]
\[ T_8(n, k) = (n - k + 1)(Q_{n-1,k}P_{n-1,k} - Q_{n-1,k-1}P_{n-1,k+1}), \]
\[ T_9(n, k) = 2kQ_{n-1,k}P_{n-1,k} - (k + 1)Q_{n-1,k+1}P_{n-1,k-1} - (k - 1)Q_{n-1,k-1}P_{n-1,k+1}. \]

Similar identity holds for \( Q_{n,k} \).

**Proof:** By (12).

\[ P_{n,k}^2 = k^2Q_{n-1,k}^2 + (n - k)^2Q_{n-1,k-1}^2 + P_{n-1,k}^2 + 2k(n - k)Q_{n-1,k}Q_{n-1,k-1} + 2kQ_{n-1,k}P_{n-1,k} + 2(n - k)Q_{n-1,k-1}P_{n-1,k}, \]  
(15)

and

\[ P_{n,k+1}P_{n,k-1} = (k^2 - 1)Q_{n-1,k+1}Q_{n-1,k-1} + [(n - k)^2 - 1]Q_{n-1,k}Q_{n-1,k-2} + P_{n-1,k+1}P_{n-1,k-1} + (k + 1)(n - k + 1)Q_{n-1,k+1}Q_{n-1,k-2} + (k - 1)(n - k - 1)Q_{n-1,k-1}Q_{n-1,k} + (k + 1)Q_{n-1,k+1}P_{n-1,k-1} + (k - 1)Q_{n-1,k}P_{n-1,k-1} + (n - k + 1)Q_{n-1,k-1}P_{n-1,k+1}. \]  
(16)

Subtracting (16) from (15) and rearranging suitably, we get (14).

**Proof of Theorem 5:** We use induction on \( n \). The base cases when \( n = 2 \) and \( n = 3 \) are easy to be verified. Assume that \( P_{n-1} \) and \( Q_{n-1} \) are strongly synchronised. Hence, for all \( k \in \mathbb{N} \) we have \( P_{n-1,k}^2 \geq P_{n-1,k-1}P_{n-1,k+1} \) and \( Q_{n-1,k}^2 \geq Q_{n-1,k-1}Q_{n-1,k+1} \). In particular, for odd \( k \) with \( 1 \leq k \leq n - 2 \), we have \( P_{n-1,k}^2 \geq P_{n-1,k-1}P_{n-1,k+1} \) and for even \( k \) with \( 1 \leq k \leq n - 2 \), we have \( Q_{n-1,k}^2 \geq Q_{n-1,k-1}Q_{n-1,k+1} \).

We show that for odd \( k \) with \( 1 \leq k \leq n - 1 \), \( P_{n,k}^2 \geq P_{n,k-1}P_{n,k+1} \).

We first observe that

1. \( T_2(n, k) = (Q_{n-1,k} - Q_{n-1,k-1})^2 \geq 0 \)

2. Next, we show that \( T_1(n, k), T_3(n, k), T_4(n, k), T_5(n, k), T_6(n, k), T_8(n, k) \) and \( T_9(n, k) \) are non-negative using our inductive hypothesis. Non-negativity of \( T_1, T_3 \) and \( T_5 \) follows from the log-concavity of \( Q_{n-1} \) while non-negativity of \( T_4 \) follows from log-concavity of \( P_{n-1} \). As \( P_{n-1} \) and \( Q_{n-1} \) are strongly synchronised, we have \( Q_{n-1,k}P_{n-1,k} \geq Q_{n-1,k+1}P_{n-1,k-1} \) and also \( Q_{n-1,k}P_{n-1,k} \geq Q_{n-1,k-1}P_{n-1,k+1} \). Thus, \( T_9 \) is non-negative. Non-negativity of
We need three further cases: (i) when $k = 1$, (ii) when $k = n - 1$, (iii) when $3 \leq k \leq n - 2$.

**Case 1: When $k = 1$**

Setting $k = 1$ and $k = 2$ in Equation (12), we get

$$P_{n,1}^2 - P_{n,2}P_{n,0} = (Q_{n-1,1} + P_{n-1,1})^2 - [2Q_{n-1,2} + (n - 2)Q_{n-1,1} + P_{n-1,2}]$$

$$= [Q_{n-1,1}^2 - Q_{n-1,2}] + [Q_{n-1,1}P_{n-1,1} - Q_{n-1,2}] + [Q_{n-1,1}(P_{n-1,1} - (n - 2))]$$

$$+ [P_{n-1,1}^2 - P_{n-1,2}]$$

By induction, the sequences $P_{n-1}$ and $Q_{n-1}$ are strongly synchronised. Hence, $Q_{n-1,1}^2 - Q_{n-1,2} = Q_{n-1,1}^2 - Q_{n-1,2} + P_{n-1,0} = 0$. Similarly, we can show that $Q_{n-1,1}P_{n-1,1} - Q_{n-1,2} = 0$ and $P_{n-1,1}^2 - P_{n-1,2} \geq 0$. Further, by Lemma 16 and Remark 14, we have $P_{n-1} = Q_{n-2,1} + P_{n-2,1} = A_{n-2,1} \geq n$. Thus, we have $P_{n,1}^2 - P_{n,2}P_{n,0} \geq 0$.

**Case 2: When $k = n - 1$**

$$P_{n,k} \geq P_{n,k-1}P_{n,k+1} = 0.$$  

(18)

**Case 3: When $3 \leq k \leq n - 2$ and $k$ odd**

$$T_1(n, k) = (k^2 - 1)Q_{n-1,k}^2 - (k^2 - 1)Q_{n-1,k+1}Q_{n-1,k-1}$$

$$\geq (k^2 - 1)Q_{n-1,k}^2 - (k^2 - 1)P_{n-1,k}^2$$

$$= (k^2 - 1)A_{n-1,k}(Q_{n-1,k} - P_{n-1,k})$$

$$= (k^2 - 1)\binom{n-2}{k}A_{n-1,k}$$

$$= \frac{(k^2 - 1)(n - 1 - k)}{k} \binom{n-2}{k-1}A_{n-1,k}$$

$$\geq (n - k - 1)\binom{n-2}{k-1}Q_{n-1,k}.$$  

(19)

The second line follows by induction. The third line uses $A_{n-1,k} = P_{n-1,k} + Q_{n-1,k}$. The fourth line uses (11). The last line follows from the fact that $k^2 - 1 \geq k^2$ when $k \geq 2$. We observe that

$$Q_{n-1,k+1}Q_{n-1,k-2} = \frac{Q_{n-1,k+1}Q_{n-1,k}Q_{n-1,k-1}Q_{n-1,k-2}}{Q_{n-1,k}Q_{n-1,k-1}}$$

$$\leq \frac{Q_{n-1,k}P_{n-1,k}Q_{n-1,k-1}}{Q_{n-1,k}Q_{n-1,k-1}} = P_{n-1,k}Q_{n-1,k-1}.$$  

(20)

9
The first step is legitimate since \( Q_{n-1,k} \) and \( Q_{n-1,k-1} \) are positive for \( 3 \leq k \leq n-2 \). The second step follows using strong synchronisation of the sequences \( P_{n-1,k} \) and \( Q_{n-1,k} \).

\[
T_5(n,k) = (k+1)(n-k+1)(Q_{n-1,k}Q_{n-1,k-1} - Q_{n-1,k+1}Q_{n-1,k-2}) \\
\geq (k+1)(n-k+1)(Q_{n-1,k}Q_{n-1,k-1} - Q_{n-1,k+1}P_{n-1,k}) \\
= (k+1)(n-k+1)Q_{n-1,k-1}\binom{n-2}{k} \tag{21}
\]

Here the second line uses (20) and the third line uses line uses (11).

\[
-T_7(n,k) = (n-k-1)(Q_{n-1,k}P_{n-1,k-1} - P_{n-1,k}Q_{n-1,k-1}) \\
= (n-k-1)\left[ (P_{n-1,k} + \binom{n-2}{k})Q_{n-1,k-1} + \binom{n-2}{k-1}Q_{n-1,k-1} \right] \\
= (n-k-1)\left[ \binom{n-2}{k-1}Q_{n-1,k} + \binom{n-2}{k}Q_{n-1,k-1} \right] \\
= (n-k-1)\left( \binom{n-2}{k-1}Q_{n-1,k} + (n-k-1)\binom{n-2}{k}Q_{n-1,k-1} \right) \\
\leq T_1(n,k) + T_5(n,k). \tag{22}
\]

Here, both the second and the third line uses Theorem 13. The fifth line follows from (19) and (21). Hence, when \( 3 \leq k \leq n-2 \) and \( k \) odd, \( T_1(n,k) + T_5(n,k) + T_7(n,k) \geq 0. \)

Thus for odd \( k \) with \( 1 \leq k \leq n-1 \), we have \( P_{n,k}^2 \geq P_{n,k-1}P_{n,k+1} \). In an identical manner we can get \( Q_{n,k}^2 \geq Q_{n,k-1}Q_{n,k+1} \) for even \( k \) with \( 1 \leq k \leq n-1 \). So we proved that \( P_n \) is odd log-concave and \( Q_n \) is even log-concave. By Corollary 18 the sequences \( P_n \) and \( Q_n \) are ratio-alternating, hence by Theorem 12 the sequences \( P_n \) and \( Q_n \) are strongly synchronised.

**Corollary 20** For positive integers \( n \), all the sequences in \( S(P_n, Q_n) \) are log-concave.

**Proof:** By Theorem 5 the sequences \( P_n = (P_{n,k})_{k=0}^{n-1} \) and \( Q_n = (Q_{n,k})_{k=0}^{n-1} \) are strongly synchronised. Hence, by Theorem 7 we get that all the sequences in \( S(P_n, Q_n) \) are log-concave.

---

### 4 Type B Coxeter groups

Let \( \mathcal{B}_n \) be the set of permutations \( \pi \) of \( \{-n, -(n-1), \ldots, -1, 1, 2, \ldots n\} \) that satisfy \( \pi(-i) = -\pi(i) \). For \( \pi \in \mathcal{B}_n \), for \( 1 \leq i \leq n \), we alternatively denote \( \pi(i) \) as \( \pi_i \). For \( \pi \in \mathcal{B}_n \), define \( \text{Negs}(\pi) = \{i : i > 0, \pi_i < 0\} \) be the set of elements which occur with a negative sign. Define \( \text{inv}_B(\pi) = \{|1 \leq i < j \leq n : \pi_i > \pi_j\} + \{|1 \leq i < j \leq n : -\pi_i > \pi_j\} + |\text{Negs}(\pi)| \). Let \( \mathcal{B}^+_n \subseteq \mathcal{B}_n \) denote the subset of elements having even \( \text{inv}_B(\pi) \) value and let \( \mathcal{B}^-_n = \mathcal{B}_n \setminus \mathcal{B}^+_n \). Following Brenti’s definition of excedance from [7], define \( \text{exc}_B(\pi) = \{|i \in [n] : \pi_{|\pi(i)|} > \pi_i\}| + \{|i \in [n] : \pi_i = -i\}| \) and define \( \text{wkexc}_B(\pi) = \{|i \in [n] : \pi_{|\pi(i)|} > \pi_i\}| + \{|i \in [n] : \pi_i = i\}\). For \( \pi \in \mathcal{B}_n \), let \( \pi_0 = 0 \). We refer the reader to Petersen’s book [19] Chapter 13 for the following
of signed permutations with $k$ denote the number of signed permutations with $k$ descents in $\mathcal{B}_n$, $\mathcal{B}_n^+$ and $\mathcal{B}_n^-$ respectively. Let $E^B_{n,k}$, $P^B_{n,k}$ and $Q^B_{n,k}$ denote the number of signed permutations with $k$ excedances in $\mathcal{B}_n$, $\mathcal{B}_n^+$ and $\mathcal{B}_n^-$ respectively.

Brenti in [7, Theorem 3.15] proved the type B counterpart of MacMahon’s theorem and showed that $B_{n,k} = E^B_{n,k}$. Brenti proved the result by showing the following.

Theorem 21 (Brenti) For positive integers $n$, there exists a bijection $h_n : \mathcal{B}_n \mapsto \mathcal{B}_n$ such that $\text{asc}_B(h_n(\pi)) = \text{wkexc}_B(\pi)$ and $|\text{Negs}(h_n(\pi))| = |\text{Negs}(\pi)|$.

Reiner in [20, Theorem 3.2] proved the following:

Theorem 22 (Reiner) For positive integers $n$ and $0 \leq k \leq n$, $B^+_{n,k}$ and $B^-_{n,k}$ satisfy the following recurrence relations:

$$B^+_{n,k} - B^-_{n,k} = (-1)^k \binom{n}{k}. \quad (23)$$

In [9, Lemma 34], Dey and Sivasubramanian proved the following recurrence between the coefficients $B^+_{n,k}$ and $B^-_{n,k}$.

Lemma 23 For positive integers $n$ and $1 \leq k \leq n$, $B^+_{n,k}$ and $B^-_{n,k}$ satisfy the following recurrence relations:

1. $B^+_{n,k} = 2kB^-_{n-1,k} + (2n - 2k + 1)B^-_{n-1,k-1} + B^+_{n-1,k}$,
2. $B^-_{n,k} = 2kB^+_{n-1,k} + (2n - 2k + 1)B^+_{n-1,k-1} + B^-_{n-1,k}$,

where $B^+_{1,0} = 1$, $B^-_{1,0} = 0$, $B^+_{1,1} = 0$ and $B^-_{1,1} = 1$.

Sivasubramanian in [23, Theorem 8] enumerated the signed excedance polynomial over $\mathcal{B}_n$ and showed the following:

Theorem 24 (Sivasubramanian) For positive integers $n$ and $0 \leq k \leq n$,

$$P^B_{n,k} - Q^B_{n,k} = (-1)^k \binom{n}{k}. \quad (24)$$

By Theorem 21 we have $E^B_{n,k} = B_{n,k}$. From Theorem 22 and Theorem 24 we have $P^B_{n,k} - Q^B_{n,k} = B^+_n - B^-_n$. The coefficients $P^B_{n,k}$ and $Q^B_{n,k}$ also satisfy same initial conditions, hence we have $P^B_{n,k} = B^+_{n,k}$ and $Q^B_{n,k} = B^-_{n,k}$. Thus, we get the following lemma.

Lemma 25 For positive integers $n$ and $1 \leq k \leq n$, $P^B_{n,k}$ and $Q^B_{n,k}$ satisfy the following recurrence relations:

1. $P^B_{n,k} = 2kQ^B_{n-1,k} + (2n - 2k + 1)Q^B_{n-1,k-1} + P^B_{n-1,k}$,
2. \( Q_{n,k}^B = 2kP_{n-1,k}^B + (2n - 2k + 1)P_{n-1,k-1}^B + Q_{n-1,k}^B \).

where \( P_{1,0}^B = 1, \ P_{1,1}^B = 0, \ Q_{1,0}^B = 0 \text{ and } Q_{1,1}^B = 1. \)

From Theorem 24 we immediately have the following corollary.

**Corollary 26** For positive integers \( n \), the sequences \( P_{n,k}^B \) and \( Q_{n,k}^B \) are ratio-alternating. Moreover, they satisfy (7).

We first prove Lemma 27 which is analogous to Lemma 19.

**Lemma 27** For positive integers \( n \) and \( 1 \leq k \leq n - 1 \), the coefficients \( P_{n,k}^B \) satisfy the following:

\[
(P_{n,k}^B)^2 - P_{n,k+1}^B P_{n,k-1}^B = \sum_{i=1}^{9} T_i^B(n, k)
\]

where

\[
\begin{align*}
T_1^B(n, k) &= 4(k^2 - 1)[(Q_{n-1,k}^B)^2 - Q_{n-1,k+1}^B Q_{n-1,k-1}^B], \\
T_2^B(n, k) &= 4(Q_{n-1,k}^B)^2 + 4(Q_{n-1,k-1}^B)^2 - 8Q_{n-1,k-1}^B Q_{n-1,k}^B, \\
T_3^B(n, k) &= ((2n - 2k + 1)^2 - 4)[(Q_{n-1,k-1}^B)^2 - Q_{n-1,k}^B Q_{n-1,k-2}^B], \\
T_4^B(n, k) &= (P_{n-1,k}^B)^2 - P_{n-1,k+1}^B P_{n-1,k-1}^B, \\
T_5^B(n, k) &= 2(k + 1)(2n - 2k + 3)(Q_{n-1,k}^B Q_{n-1,k-1}^B - Q_{n-1,k+1}^B Q_{n-1,k-2}^B), \\
T_6^B(n, k) &= 2(k - 1)(2n - 2k - 1)(Q_{n-1,k-1}^B Q_{n-1,k}^B - Q_{n-1,k-2}^B Q_{n-1,k}^B), \\
T_7^B(n, k) &= (2n - 2k - 1)(Q_{n-1,k-1}^B P_{n-1,k}^B - Q_{n-1,k}^B P_{n-1,k-1}^B), \\
T_8^B(n, k) &= (2n - 2k + 3)(Q_{n-1,k-1}^B P_{n-1,k}^B - Q_{n-1,k-2}^B P_{n-1,k}^B), \\
T_9^B(n, k) &= 4kQ_{n-1,k}^B P_{n-1,k}^B - 2(k + 1)Q_{n-1,k+1}^B P_{n-1,k-1}^B - 2(k - 1)Q_{n-1,k-1}^B P_{n-1,k+1}^B.
\end{align*}
\]

**Proof:** This proof follows by calculating \( (P_{n,k}^B)^2 \) and \( P_{n,k+1}^B P_{n,k-1}^B \) using the recurrences in Lemma 23 as was done in the proof of Lemma 27.

Now we are in a position to prove our main result of this section.

**Proof of Theorem 5** We prove this by induction along the same lines as in the proof of Theorem 5. By induction assume that \( P_{n-1}^B \) and \( Q_{n-1}^B \) are strongly synchronised in \( k \). Hence, for \( 1 \leq k \leq n - 1 \) with \( k \) odd, \( (P_{n-1,k}^B)^2 \geq P_{n-1,k-1}^B P_{n-1,k+1}^B \) and for \( 1 \leq k \leq n - 1 \) with \( k \) even, \( (Q_{n-1,k}^B)^2 \geq Q_{n-1,k-1}^B Q_{n-1,k+1}^B \). Proceeding along the same line as in proof of Theorem 5 we get that for odd \( k \) with \( 0 \leq k \leq n \), we have \( (P_{n,k}^B)^2 \geq P_{n,k-1}^B P_{n,k+1}^B \). In an identical manner we can get \( (Q_{n,k}^B)^2 \geq Q_{n,k-1}^B Q_{n,k+1}^B \) for even \( k \) with \( 0 \leq k \leq n \). Thus, the sequence \( P_n^B \) is odd log-concave and \( Q_n^B \) is even log-concave. By Corollary 18, \( P_n^B \) and \( Q_n^B \) are ratio-alternating, hence by Theorem 12, the sequences \( P_n^B \) and \( Q_n^B \) are strongly synchronised.

As the sequences \( P_{n,k}, Q_{n,k}, P_{n,k}^B \) and \( Q_{n,k}^B \) don’t have any internal zeroes, hence log-concavity of those polynomials directly gives the following result.

**Corollary 28** For positive integers \( n \), the sequences \( P_{n,k}, Q_{n,k}, P_{n,k}^B \) and \( Q_{n,k}^B \) are unimodal.
5 Modification of Sagan’s Theorem

Given a triangular array of non-negative integers, Sagan in [21 Theorem 1] gave the following condition which ensures that every row of the array is log-concave.

**Theorem 29 (Sagan)** Suppose that for \( n \geq 1 \) and \( 0 \leq k \leq n \), a non-negative integral sequence \( t_{n,k} \) satisfies the following triangular recurrence relation: \( t_{n,k} = c_{n,k}t_{n-1,k-1} + d_{n,k}t_{n-1,k} \) where the multiplicative coefficients \( c_{n,k}, d_{n,k} \) are all non-negative integers and \( t_{a,b} = 0 \) whenever \( a < b \). Suppose the following conditions hold: (i) For each positive integer \( n \), \( c_{n,k} \) and \( d_{n,k} \) are log-concave in \( k \). (ii) \( c_{n,k-1}d_{n,k+1} + c_{n,k+1}d_{n,k-1} \leq 2c_{n,k}d_{n,k} \) for all \( n \geq 1 \) and \( 0 \leq k \leq n \). Then, for each positive integer \( n \), the sequence \( t_{n,k} \) is log-concave in \( k \).

Looking at Theorem 29 one may not consider this to be much of a labor-saving method, but working with the coefficient arrays in most of the cases is much simpler than working with the original ones. This theorem has some nice applications. This directly gives the log-concavity of binomial coefficients and Stirling Number of both kinds. But this does not directly prove the log-concavity of Eulerian Numbers as (ii) is not satisfied. Hence we modify Sagan’s theorem. The proof of Theorem 30 goes along the same line as the original proof of Theorem 29 but we give it for completeness.

**Theorem 30** Suppose that for \( n \geq 1 \) and \( 0 \leq k \leq n \), a non-negative integral sequence \( t_{n,k} \) satisfies the following triangular recurrence relation: \( t_{n,k} = c_{n,k}t_{n-1,k-1} + d_{n,k}t_{n-1,k} \) where \( c_{n,k}, d_{n,k} \) are all non-negative integers and \( t_{a,b} = 0 \) whenever \( a < b \). Suppose the following conditions hold: (i) For each positive integers \( n \), \( c_{n,k} \) and \( d_{n,k} \) are log-concave in \( k \).

(ii) \( 2\sqrt{(c_{n,k}^2 - c_{n,k+1}c_{n,k-1})(d_{n,k}^2 - d_{n,k+1}d_{n,k-1})} \geq c_{n,k-1}d_{n,k+1} + c_{n,k+1}d_{n,k-1} - 2c_{n,k}d_{n,k} \) for all \( n \geq 1 \) and all \( 0 \leq k \leq n \).

Then, for each positive integers \( n \), the sequence \( t_{n,k} \) is log-concave in \( k \).

**Proof:** We prove this by induction. Assume \( t_{n-1,k} \) to be log-concave in \( k \).

\[
t_{n,k}^2 - t_{n,k+1}t_{n,k-1} = c_{n,k}^2t_{n-1,k-1}^2 - c_{n,k+1}c_{n,k-1}t_{n-1,k-1}t_{n-1,k-2} + d_{n,k}^2t_{n-1,k}^2 - d_{n,k+1}d_{n,k-1}t_{n-1,k-1} + 2c_{n,k}d_{n,k}t_{n-1,k-1}t_{n-1,k} - c_{n,k+1}d_{n,k-1}t_{n-1,k-1}t_{n-1,k} - c_{n,k-1}d_{n,k+1}t_{n-1,k-2}t_{n-1,k+1} + (c_{n,k}^2 - c_{n,k+1}c_{n,k-1})t_{n-1,k-1}^2 + (d_{n,k}^2 - d_{n,k+1}d_{n,k-1})t_{n-1,k}^2 + (2c_{n,k}d_{n,k} - c_{n,k-1}d_{n,k+1} - c_{n,k+1}d_{n,k-1})t_{n-1,k-1}t_{n-1,k}
\]

By A.M-G.M inequality \( Ax^2 + By^2 \geq 2\sqrt{AB}xy \) whenever \( A \) and \( B \) are non-negative. Let, \( A = c_{n,k}^2 - c_{n,k+1}c_{n,k-1} \) and \( B = d_{n,k}^2 - d_{n,k+1}d_{n,k-1} \). Then \( A \) and \( B \) are non-negative due to log-concavity of \( c_{n,k} \) and \( d_{n,k} \) respectively. Here, \( 2\sqrt{AB} \geq \frac{4c_{n,k}^2 - c_{n,k+1}c_{n,k-1})(d_{n,k}^2 - d_{n,k+1}d_{n,k-1})}{(c_{n,k-1}d_{n,k+1} + c_{n,k+1}d_{n,k-1} - 2c_{n,k}d_{n,k})} \).

Hence, \( t_{n,k}^2 - t_{n,k+1}t_{n,k-1} \) is non-negative and so we are done.
Remark 31 Note that, Theorem 30 also gives a necessary condition to ensure that every row of a triangular array satisfying that condition will be log-concave. But Theorem 30 is more general because, if the multiplicative coefficients \( c_{n,k} \) and \( d_{n,k} \) satisfy (ii) of Theorem 29 then they certainly satisfy (ii) of Theorem 30. But, there are examples (all the examples in the next subsection) for which \( c_{n,k} \) and \( d_{n,k} \) satisfy (ii) of Theorem 30 but don’t satisfy (ii) of Theorem 29. Here also we work with the coefficient arrays instead of the original ones.

5.1 Direct Applications of Modified Sagan’s Theorem

Here, we give some applications of Theorem 30. At first, we consider some combinatorial sequences, whose log-concavity is already known using real-rootedness or some other tools. But here, we give direct proofs of log-concavity of those sequences.

1. Log-concavity of Eulerian Numbers: Frobenius showed that the Eulerian polynomials \( A_n(t) \) are real-rooted (for reference, one can see [19, Chapter 4]) and hence log-concave. Theorem 30 immediately provides us an alternate proof of the log-concavity of Eulerian Numbers. We know from [19, Theorem 1.3] that \( A_{n,k} \) satisfy the following recurrence:

\[
A_{n,k} = (k + 1)A_{n-1,k} + (n - k)A_{n-1,k-1}.
\]

It is easy to see that both \( c_{n,k} = k + 1 \) and \( d_{n,k} = (n - k) \) are log-concave in \( k \). Further,

\[
2 \sqrt{(c_{n,k}^2 - c_{n,k+1})d_{n,k+1} - d_{n,k+1}d_{n,k-1}} = 2 \geq 2 = (c_{n,k-1}d_{n,k+1} + c_{n,k+1}d_{n,k-1} - 2c_{n,k}d_{n,k}).
\]

Thus, \( c_{n,k} \) and \( d_{n,k} \) satisfy the conditions of Theorem 30. Hence \( A_{n,k} \) is log-concave.

2. Log-concavity of Type B Eulerian Numbers: Let us consider the Eulerian polynomials of Type B: \( B_n(t) = \sum_{\pi \in S_n} t^{\text{des}(\pi)} = \sum_{k=0}^n B_{n,k}t^k \). Brenti [7] showed that these polynomials are real-rooted and hence \( B_{n,k} \)s are log-concave. Theorem 30 gives another proof of log-concavity of the sequence \( B_{n,k} \). From [7], we get that they satisfy the following recurrence:

\[
B_{n,k} = (2k + 1)B_{n-1,k} + [2(n - k) + 1]B_{n-1,k-1}.
\]

Taking \( c_{n,k} = 2k + 1 \) and \( d_{n,k} = 2(n - k) + 1 \) works here as both of them are log-concave and

\[
2 \sqrt{(c_{n,k}^2 - c_{n,k+1})d_{n,k+1} - d_{n,k+1}d_{n,k-1}} = 8 \geq 8(c_{n,k-1}d_{n,k+1} + c_{n,k+1}d_{n,k-1} - 2c_{n,k}d_{n,k}).
\]

Thus, by Theorem 30 they are also log-concave.

3. Log-concavity of Second order Eulerian Numbers: Let \( Q_n \) be the set of permutations of \( \{1, 1, 2, 2, \ldots, n, n\} \) such that for all \( i \), entries between two occurrences of \( i \) are larger than \( i \).
For a permutation $\pi \in Q_n$, let $\text{DES}(\pi) = \{i \in [2n-1] : \pi_i > \pi_{i+1}\}$ be its number of descents. Let $H_{n,k} = |\{\pi \in Q_n : \text{des}(\pi) = k\}|$. These numbers $H_{n,k}$ are called as the second-order Eulerian Numbers. Bona in [2] proved that the associated polynomials $H_{n}(t) = \sum_{k=0}^{n} H_{n,k} t^k$ are real-rooted which immediately gives log-concavity of $H_{n,k}$. Here we provide another proof of log-concavity of $H_{n,k}$. From [15], we get that these coefficients $H_{n,k}$ satisfy the following recurrence:

$$H_{n,k} = kH_{n-1,k} + (2n-k)H_{n-1,k-1}.$$ 

Taking $c_{n,k} = k$ and $d_{n,k} = 2n-k$ and applying Theorem [30] we immediately get an alternate proof of the log-concavity of $H_{n,k}$ for all positive integers $n$.

We now turn our attention to palindromic polynomials. A polynomial $f(t) = \sum_{i=0}^{n} a_i t^i$ is said to be palindromic if $a_i = a_{n-i}$ for all $0 \leq i \leq \lfloor n/2 \rfloor$. A palindromic polynomial $f(t) = \sum_{i=0}^{n} a_i t^i$ is said to be gamma-positive if $f(t) = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_{n,i} t^i (1+t)^{n-2i}$ with $\gamma_{n,i} \geq 0$ for all $0 \leq i \leq \lfloor n/2 \rfloor$. One can see the survey paper of Athanasiadis [1] for a good reference on various gamma-positivity results.

1. **Log-concavity of the gamma-coefficients of Type A Eulerian polynomials:** Foata and Schützenberger in [11] showed that the Eulerian polynomials of Type A are gamma positive. Let $T_{n,k}$ be the coefficient of $t^{2k}(1+t)^{n-1-2k}$ in $A_n(t)$. Foata and Strehl in [12] gave a combinatorial interpretation of $T_{n,k}$. They proved that $T_{n,k}$ is actually the number of elements in $S_n$ with $k$ descents and no double descents. From [9, Theorem 8] we get that these coefficients satisfy the following recurrence:

$$T_{n,k} = (k+1)T_{n-1,k} + (2n-4k)T_{n-1,k-1}.$$ 

Let $c_{n,k} = (k+1)$ and $d_{n,k} = 2n-4k$. Then, $c_{n,k}$ and $d_{n,k}$ satisfy the conditions of Theorem [30]. Thus, by Theorem [30] the sequence $T_{n,k}$ is log-concave for any $n$.

2. **Log-concavity of gamma-coefficients of Type B Eulerian polynomials:** Chow in [8, Theorem 4.7] proved that the Type B Eulerian polynomials $B_n(t) = \sum_{k=0}^{\lfloor n/2 \rfloor} R_{n,k} t^k (1+t)^{n-2k}$ where $R_{n,k}$ satisfies the following recurrence: $R_{n,k} = (2k+1)R_{n-1,k} + 4(n+1-2k)R_{n-1,k-1}$. We can take $c_{n,k} = (2k+1)$ and $d_{n,k} = 4(n+1-2k)$ to get the log-concavity of $R_{n,k}$.

6 **Open Problems**

In this Section, we raise some questions and make some interesting conjectures. Define $A_{n,k}^+$ and $A_{n,k}^-$ to be the number of permutations with $k$ descents in $A_n$ and $\tilde{S}_n - A_n$ respectively. Based on data, we make the following conjecture about the sequences $A_{n,k}^+$ and $A_{n,k}^-$. 

**Conjecture 32** For positive integers $n$, the sequences $(A_{n,k}^+)^{n-1}_{k=0}$ and $(A_{n,k}^-)^{n-1}_{k=0}$ are strongly synchronised.
Conjecture 32 will show log-concavity of $A_{n,k}^+$ and $A_{n,k}^-$. Real-rootedness of the polynomials $A_n^+(t) = \sum_{\pi \in A_n} t^{\text{des}(\pi)}$ and $A_n^-(t) = \sum_{\pi \in S_n - A_n} t^{\text{des}(\pi)}$ was conjectured by Dey and Sivasubramanian [9, Conjecture 48] when $n \equiv 0,1 \mod 4$ and extended by Fulman, Kim, Lee and Petersen [13, Conjecture 1.3] for all $n$.

Though descents and excedances are not equidistributed over $A_n$, they seem to be strongly synchronised over $A_n$ for all $n$, that is,

**Conjecture 33** For positive integers $n$, the sequences $(A_{n,k}^+)^{n-1}_{k=0}$ and $(P_{n,k})^{n-1}_{k=0}$ are strongly synchronised. Similarly, the sequences $(A_{n,k}^-)^{n-1}_{k=0}$ and $(Q_{n,k})^{n-1}_{k=0}$ are strongly synchronised.

**Problem 34** It would be very interesting to find combinatorial proofs of Theorem 5, Theorem 6, Conjecture 32 and Conjecture 33. Bóna and Ehrenborg in [3] have given a combinatorial proof of log-concavity of $A_{n,k}$ but the coefficients $A_{n,k}^+$ and $A_{n,k}^-$ are not even ratio-alternating. Hence, this argument directly does not prove strong synchronisation of $A_{n,k}^+$ and $A_{n,k}^-$. 

**Acknowledgements**

The author would like to thank his advisor Sivaramakrishnan Sivasubramanian for all the insightful discussions and comments during the preparation of the paper. The author also thanks Subhajit Ghosh, Venkitesh Iyer and Brahadeesh Sankarnarayanan for some suggestions during the later phase of this work. The author also acknowledges funding from CSIR-SPM fellowship.

**References**

[1] Athanasiadis, C. A. Gamma-positivity in combinatorics and geometry. Available at users.uoa.gr/~caath/gp.pdf (2017).

[2] Bóna, M. Real zeros and normal distribution for statistics on Stirling permutations defined by Gessel and Stanley. *SIAM J. Discrete Math.* 23 (2008/09), 401–406.

[3] Bóna, M., and Ehrenborg, R. A Combinatorial Proof of the Log-Concavity of the Numbers of Permutations with $k$ Runs. *Journal of Combinatorial Theory, Series A* 90 (2000), 293–303.

[4] Brändén, P. Unimodality, Log-concavity, Real-rootedness and Beyond. In *Handbook of Enumerative Combinatorics*, M. Bona, Ed. Chapman & Hall CRC Press, 2015, ch. 7.

[5] Brenti, F. Unimodal, log-concave and Pólya frequency sequences in combinatorics,. *Mem. Amer. Math. Soc.* 413 (1989).

[6] Brenti, F. Log-concave and unimodal sequences in algebra, combinatorics, and geometry: An update,. *Contemp. Math.* 178 (1994), 71–89.
[7] BRENTI, F. $q$-Eulerian Polynomials Arising from Coxeter Groups. *European Journal of Combinatorics* 15 (1994), 417–441.

[8] CHOW, C.-O. On certain combinatorial expansions of the Eulerian polynomials. *Advances in Applied Math* 41 (2008), 133–157.

[9] DEY, H. K., AND SIVASUBRAMANIAN, S. Gamma Positivity of the Descent based Eulerian polynomial in positive elements of Classical Weyl Groups. *Electronic Journal of Combinatorics* 27(3) (2020).

[10] DEY, H. K., AND SIVASUBRAMANIAN, S. Gamma Positivity of the Excedance-Based Eulerian Polynomial in Positive Elements of Classical Weyl Groups. *Annals of Combinatorics*, https://doi.org/10.1007/s00026-020-00511-6 (2020).

[11] FOATA, D., AND SCHÜTZENBERGER, M.-P. Théorie géométrique des polynômes Eulériens, available at http://www.mat.univie.ac.at/~slc/books/ ed. Lecture Notes in Mathematics, 138, Berlin, Springer-Verlag, 1970.

[12] FOATA, D., AND STREHL, V. Euler numbers and variations of permutations. *Colloquio Internazionale sulle Teorie Combinatoire (Roma 1973) Tomo I Atti dei Convegni Lincei, No 17, Accad. Naz. Lincei, Rome* (1976), 119–131.

[13] FULMAN, J., KIM, G., LEE, S., AND PETERSEN, T. K. On the joint distribution of descents and signs of permutations. available at https://arxiv.org/abs/1910.04258 (2019), 16 pages.

[14] GROSS, J. L., MANSOUR, T., TUCKER, W., AND WANG. LOG-CONCAVITY OF COMBINATIONS OF SEQUENCES AND APPLICATIONS TO GENUS DISTRIBUTIONS. *SIAM J. Discrete Math.* (2014).

[15] HAGLUND, J., AND VISONTAI, M. Stable multivariate Eulerian polynomials and generalized Stirling permutations. *European Journal of Combinatorics* 33 (2012), 477–487.

[16] MACMAHON, P. A. *Combinatory Analysis*. Cambridge University Press, 1915-1916 (Reprinted by AMS Chelsea, 2000).

[17] MANTACI, R. *Statistiques Eulériennes sur les Groupes de Permutation*. PhD thesis, Université Paris, 1991.

[18] MANTACI, R. Binomial Coefficients and Anti-excedances of Even Permutations: A Combinatorial Proof. *Journal of Combinatorial Theory, Ser A* 63 (1993), 330–337.

[19] PETERSEN, T. K. *Eulerian Numbers*, 1st ed. Birkhäuser, 2015.

[20] REINER, V. Descents and one-dimensional characters for classical Weyl groups. *Discrete Mathematics* 140 (1995), 129–140.

[21] SAGAN, B. E. Inductive and injective proofs of log concavity results. *Discrete Mathematics* 68 (1988), 281–292.
[22] SIVASUBRAMANIAN, S. Signed excedance enumeration via determinants. *Advances in Applied Math* 47 (2011), 783–794.

[23] SIVASUBRAMANIAN, S. Signed Excedance Enumeration in the Hyperoctahedral group. *Electronic Journal of Combinatorics* 21(2) (2014), P2.10.

[24] STANLEY, R. Log-concave and unimodal sequences in algebra, combinatorics and geometry. *Ann. New York Acad. Sci.* 576 (1989), 500–534.