Generalized Pólya’s theorem on connected locally compact Abelian groups of dimension 1

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Abstract. According to the generalized Pólya theorem, the Gaussian distribution on the real line is characterized by the property of equidistribution of a monomial and a linear form of independent identically distributed random variables. We give a complete description of \( a \)-adic solenoids for which an analog of this theorem is true. The proof of the main theorem is reduced to solving some functional equation in the class of continuous positive definite functions on the character group of an \( a \)-adic solenoid.

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1. Introduction

According to the classical characterization theorems of mathematical statistics, the Gaussian distribution on the real line is characterized by the independence of two linear forms of independent random variables (the Skitovich–Darmois theorem), as well as by the symmetry of the conditional distribution of one linear form given another (the Heyde theorem). A number of papers have been devoted to generalizing of these theorems to various algebraic structures, in particular, to locally compact Abelian groups (see e.g. [5, 6, 8–14, 18], and also [7, Chapters IV–VI]). In so doing, the coefficients of linear forms are topological automorphisms of the group. In this note, we consider a group analog of the generalized Pólya theorem, in which the Gaussian distribution on the real line is characterized by the property of equidistribution of a monomial and a linear form of independent identically distributed random variables. We study the generalized Pólya theorem on \( a \)-adic solenoids. This is an important class of locally compact Abelian groups, since each connected locally compact Abelian group of dimension 1 is topologically isomorphic to either the additive group of real numbers, or the circle group, or an \( a \)-adic solenoid. Furthermore, the coefficients of a monomial and a linear form, as in the group analogs of the Skitovich–Darmois and Heyde theorems, are topological automorphisms of the \( a \)-adic solenoid. As far as we know, such problem for groups has not been considered earlier. Note also that in the list of unsolved problems in [17, Ch. 14] the problem of constructing a theory of equidistributed linear forms on algebraic structures was formulated. A step in this direction is this note.

Recall the generalized Pólya theorem.

**Theorem A** ([17 §13.7]). Let \( \xi_j, j = 1, 2, \ldots, n, n \geq 2 \), be independent identically distributed random variables. Let \( \alpha_j \) be nonzero real numbers such that

\[
\alpha_1^2 + \cdots + \alpha_n^2 = 1. \tag{1}
\]

If \( \xi_j \) and \( \alpha_1 \xi_1 + \cdots + \alpha_n \xi_n \) are identically distributed, then \( \xi_j \) are Gaussian.

If we denote by \( f(s) \) the characteristic function of \( \xi_j \) it is easy to see that the generalized Pólya theorem is equivalent to the statement that all solutions of the equation

\[
f(s) = f(\alpha_1 s) \cdots f(\alpha_n s), \quad s \in \mathbb{R},
\]

in the class of normed continuous positive definite functions are of the form \( f(s) = \exp\{-\sigma s^2 + i\beta s\} \), where \( \sigma \geq 0, \beta \in \mathbb{R} \).
The first theorem of this kind, namely when $n = 2$ and $\alpha_1 = \alpha_2 = \frac{1}{\sqrt{2}}$, was proved by G. Pólya in 1923. Subsequently, the characterization of the Gaussian distribution on the real line by the property of equidistribution of two linear forms of independent identically distributed random variables were studied by I. Marcinkiewicz, Yu. V. Linnik, A.A. Zinger and others (see [15, Chapter 2]).

Let $X$ be a second countable locally compact Abelian group. We will consider only such groups, without mentioning it specifically. Let $\text{Aut}(X)$ be the group of topological automorphisms of the group $X$, and $I$ be the identity automorphism of a group. Denote by $Y$ the character group of the group $X$, and by $(x, y)$ the value of a character $y \in Y$ at an element $x \in X$. For a closed subgroup $K$ of $X$, denote by $A(Y, K) = \{y \in Y : (x, y) = 1 \text{ for all } x \in K\}$ its annihilator. If $\alpha \in \text{Aut}(X)$, then define the adjoint automorphism $\tilde{\alpha} \in \text{Aut}(Y)$ by the formula $(x, \tilde{\alpha}y) = (\alpha x, y)$ for all $x \in X, y \in Y$. Denote by $\mathbb{R}$ the group of real numbers, by $\mathbb{Z}$ the group of integers and by $\mathbb{T}$ the circle group. Let $n$ be an integer, $n \neq 0$. Denote by $f_n : X \mapsto X$ an endomorphism of the group $X$ defined by the formula $f_n x = nx, x \in X$.

Let $M^1(X)$ be the convolution semigroup of probability distributions on the group $X$. For $\mu \in M^1(X)$ denote by

$$\hat{\mu}(y) = \int_X (x, y) d\mu(x), \quad y \in Y,$$

the characteristic function (Fourier transform) of the distribution $\mu$, and define the distribution $\tilde{\mu} \in M^1(X)$ by the rule $\tilde{\mu}(B) = \mu(-B)$ for all Borel subsets $B$ in $X$. We have $\hat{\mu}(y) = \tilde{\mu}(y)$.

A distribution $\gamma \in M^1(X)$ is called Gaussian (see [19, Chapter IV, §6]) if its characteristic function can be represented in the form

$$\hat{\gamma}(y) = (x, y) \exp\{-\varphi(y)\}, \quad y \in Y,$$

where $x \in X$, and $\varphi(y)$ is a continuous non-negative function on the group $Y$ satisfying the equation

$$\varphi(u + v) + \varphi(u - v) = 2[\varphi(u) + \varphi(v)], \quad u, v \in Y.$$

Note that, in particular, the degenerate distributions are Gaussian. Denote by $\Gamma(X)$ the set of Gaussian distributions on $X$.

Let $K$ be a compact subgroup of the group $X$. Denote by $m_K$ the Haar distribution on $K$. The characteristic function of the distribution $m_K$ is of the form

$$\hat{m_K}(y) = \begin{cases} 1, & \text{if } y \in A(Y, K), \\ 0, & \text{if } y \notin A(Y, K). \end{cases} \tag{2}$$

2. Main theorem

Let $a=(a_0, a_1, a_2, \ldots)$, where all $a_j \in \mathbb{Z}, a_j > 1$, and let $\Delta_a$ be the group of $a$-adic integers ([15, (10.2)]). As a set $\Delta_a$ coincides with the Cartesian product $\prod_{n=0}^{\infty}\{0, 1, \ldots, a_n - 1\}$. Put $u = (1, 0, \ldots, 0, \ldots) \in \Delta_a$. Consider the group $\mathbb{R} \times \Delta_a$ and denote by $B$ a subgroup of the form $B = \{(n, nu)\}_{n=-\infty}^{\infty}$. The factor-group $\Sigma_a = (\mathbb{R} \times \Delta_a)/B$ is called an $a$-adic solenoid. The group $\Sigma_a$ is compact, connected and has dimension 1 ([15, (10.12), (10.13), (24.28)]). The character group of the group $\Sigma_a$ is topologically isomorphic to a discrete additive group $H_a$ of the rational numbers of the form

$$H_a = \left\{ \frac{m}{a_0a_1 \cdots a_n} : n = 0, 1, \ldots; m \in \mathbb{Z} \right\} \tag{3}$$
It is convenient for us to assume that $H_a$ is the character group of the group $\Sigma_a$. Each topological automorphism $\alpha \in \text{Aut}(\Sigma_a)$ is of the following form $\alpha = f_p f_q^{-1}$ for some mutually prime $p$ and $q$, where $f_p, f_q \in \text{Aut}(\Sigma_a)$. We will identify $\alpha = f_p f_q^{-1}$ with the rational number $\frac{f_p}{f_q}$. Note that $f_n = f_n$. Hence, if $\alpha = f_p f_q^{-1} \in \text{Aut}(\Sigma_a)$, then $\tilde{\alpha} = f_p f_q^{-1} \in \text{Aut}(H_a)$.

The main result of this note is the following theorem.

**Theorem 2.1.** Let $X = \Sigma_a$ be an $\alpha$-adic solenoid satisfying the condition:

(i) There is a unique prime number $p$ such that the group $X$ contains no elements of order $p$.

Let $\alpha_j, j = 1, 2, \ldots, n, n \geq 2$, be topological automorphisms of the group $X$ such that

$$\alpha_1^2 + \cdots + \alpha_n^2 = I.$$  

(4)

Let $\xi_j$ be independent identically distributed random variables with values in $X$ and distribution $\mu$. If $\xi_j$ and $\alpha_1 \xi_1 + \cdots + \alpha_n \xi_n$ are identically distributed, then $\mu = \gamma \cdot m_K$, where $\gamma \in \Gamma(X)$, and $K$ is a compact subgroup of the group $X$. Moreover, $f_p(K) = K$.

We note that if $\mu = \gamma \cdot m_K$, where $\gamma \in \Gamma(X)$ and $K$ is a compact subgroup of the group $X$, then $\mu$ is invariant with respect to $K$, and $\mu$ induces on the factor-group $X/K$ under the natural mapping $X \to X/K$ a Gaussian distribution. The condition (4) for $\alpha$-adic solenoids is an analog of the condition (1) for the real line. Thus, Theorem 2.1 can be considered as an analog for $\alpha$-adic solenoids satisfying the condition (i) of the generalized Pólya theorem. Observe that group analogs of the generalized Pólya theorem for locally compact Abelian groups in the case of equidistribution of a monomial and a linear form with integer coefficients were studied in [3, §11].

**Corollary 2.2.** Let $X = \Sigma_a$ be an $\alpha$-adic solenoid satisfying the condition (i) of Theorem 2.1. Let $\alpha_j, j = 1, 2, \ldots, n, n \geq 2$, be topological automorphisms of the group $X$ satisfying the condition (4). Let $\xi_j$ be independent identically distributed random variables with values in $X$ and distribution $\mu$ with a non-vanishing characteristic function. If $\xi_j$ and $\alpha_1 \xi_1 + \cdots + \alpha_n \xi_n$ are identically distributed, then $\mu \in \Gamma(X)$.

**Remark 2.3.** Let $\alpha = (a_0, a_1, a_2, \ldots)$, and let $\Sigma_a$ be the corresponding $\alpha$-adic solenoid.

It is easy to verify that the following statements are equivalent:

(i) There is a unique prime number $p$ such that $\Sigma_a$ contains no elements of order $p$;

(ii) There is a unique prime number $p$ such that infinite number of the numbers $a_j$ are divided by $p$.

An example of an $\alpha$-adic solenoid $\Sigma_a$ that satisfies the condition (i) of Theorem 2.1 is an $\alpha$-adic solenoid $\Sigma_a$, where $\alpha = (p, p, p, \ldots)$. Its character group is topologically isomorphic to a discrete additive group $F$ of the rational numbers of the form

$$F = \left\{ \frac{m}{p^n} : n = 0, 1, \ldots; m \in \mathbb{Z} \right\}.$$  

(5)

It is easy to see that the following statements are equivalent:

(a) There are topological automorphisms $\alpha_j, j = 1, 2, \ldots, n, n \geq 2$, of $\Sigma_a$ satisfying the condition (4);

(b) There is a prime number $p$ such that $\Sigma_a$ contains no elements of order $p$.

(\gamma) There is a prime number $p$ such that infinite number of the numbers $a_j$ are divided by $p$.

To prove it, note that there is an alternative: either $f_p \in \text{Aut}(\Sigma_a)$ for a prime number $p$ or $\text{Aut}(\Sigma_a) = \{ \pm I \}$.

To prove Theorem 2.1 we need two lemmas.
Lemma 2.4. Let $X$ be a locally compact Abelian group, and let $\alpha_j$, $j = 1, 2, \ldots, n$, $n \geq 2$, be topological automorphisms of the group $X$. Let $\xi_j$ be independent identically distributed random variables with values in $X$ and distribution $\mu$. Then $\xi_j$ and $\alpha_1 \xi_1 + \cdots + \alpha_n \xi_n$ are identically distributed if and only if the characteristic function $\hat{\mu}(y)$ satisfies the equation

$$\hat{\mu}(y) = \hat{\mu}(\hat{\alpha}_1 y) \cdots \hat{\mu}(\hat{\alpha}_n y), \ y \in Y. \quad (6)$$

The proof of the lemma is the same as in the classical case and follows directly from the fact that $\hat{\mu}(y) = E[(\xi_j, y)]$ for all $y \in Y$, and the independence of the random variables $\xi_j$.

The following lemma can be considered as an analog for locally compact Abelian groups of the well-known Cramér theorem on decomposition of the Gaussian distribution on the real line ([1], see also [7, Theorem 4.6]).

**Lemma 2.5.** Let $X$ be a locally compact Abelian group containing no subgroup topologically isomorphic to the circle group $\mathbb{T}$. Let $\gamma \in \Gamma(X)$. If $\gamma = \lambda_1 * \lambda_2$, where $\lambda_j \in M^1(X)$, then $\lambda_j \in \Gamma(X)$, $j = 1, 2$.

**Proof of Theorem 2.1.** In order not to complicate the notation we will identify $Y$ with $H_a$ and consider $Y$ as a subset of the real line $\mathbb{R}$. It follows from the definition of the Gaussian distribution that $\gamma \in \Gamma(X)$ if and only if its characteristic function can be represented in the form

$$\hat{\gamma}(y) = (x, y) \exp\{-\sigma y^2\}, \ y \in Y,$$

where $x \in X$, $\sigma \geq 0$.

By Lemma 2.4, the characteristic function $\hat{\mu}(y)$ satisfies equation (6). First, we will prove the theorem assuming that $\hat{\mu}(y) \geq 0$ for all $y \in Y$. This implies that $\hat{\mu}(-y) = \hat{\mu}(y)$ for all $y \in Y$.

Since $X$ is a connected group, $f_n$ is an epimorphism for each nonzero integer $n$ ([15, (24.25)]). Since $X$ contains no elements of order $p$, we have that $f_p$ is a monomorphism. In view of the fact that $f_p$ is an epimorphism, $f_p \in \text{Aut}(X)$, and for this reason $f_p \in \text{Aut}(Y)$. It means that both $X$ and $Y$ are groups with unique division by $p$. Note that in view of the condition (i) of the theorem, if $q$ is a prime number and $q \neq p$, then $f_q \notin \text{Aut}(X)$. This implies that each topological automorphism of the group $X$ is of the form $\pm f_p^{\pm 1}$, where $m$ is a non-negative integer. Taking into account ([4]), we conclude that each topological automorphism $\alpha_j$ in the theorem is $\alpha_j = \pm f_p^{-m_j}$, where $m_j$ is a natural number, i.e. $\alpha_j$ is a multiplication by a number $\pm p^{-m_j}$. Put $f(y) = \hat{\mu}(y)$. If necessary, changing the numbering of the random variables $\xi_j$ and taking into account that $f(-y) = f(y)$, we can rewrite equation (6) in the form

$$f(y) = (f(p^{-1}y))^{k_1} \cdots (f(p^{-l}y))^{k_l}, \ y \in Y, \quad (7)$$

where $k_j \geq 0$, $j = 1, 2, \ldots, l$, $k_1 + \cdots + k_l = n$. It follows from (4) that

$$\sum_{j=1}^l \frac{k_j}{p^{2j}} = 1. \quad (8)$$

Assume that for each $j$ the inequality $k_j \leq p^j$ is valid. Then we have

$$\sum_{j=1}^l \frac{k_j}{p^{2j}} \leq \sum_{j=1}^l \frac{1}{p^j} < 1,$$

contrary to (8). Hence, there is $j_0$ such that

$$k_{j_0} > p^{j_0}. \quad (9)$$
Taking this into account and the fact that \(0 \leq f(y) \leq 1\), (7) implies the inequality
\[
f(y) \leq (f(p^{-j_0}y))^{k_{j_0}} \leq (f(p^{-j_0}y))^{p_{j_0}}, \quad y \in Y. \tag{10}
\]

Set \(m = p^{j_0}\) and rewrite (10) in the form
\[
f(my) \leq (f(y))^m, \quad y \in Y. \tag{11}
\]

Observe that we can assume that \(m > 2\) in (11). Indeed, if \(m = 2\), then \(p = 2\) and \(j_0 = 1\). In this case (9) implies that \(k_1 > 2\), and in view of (5), either \(k_1 = 4\) or \(k_1 = 3\). Let \(k_1 = 4\). Then (7) takes the form \(f(y) = (f(2^{-1}y))^4\). This implies that \(f(4y) = (f(y))^{16}\), and we may suppose that \(m = 4\) in (11).

Let \(k_1 = 3\). Then (7) can be written as \(f(y) = (f(2^{-1}y))^3\phi(y)\), where \(\phi(y)\) is a characteristic function. It follows from this that \(f(4y) = (f(y))^9(\phi(2y))^3\phi(4y)\), and we may also suppose that \(m = 4\) in (11).

It may be noted that if \(\mu = m_X\), then the theorem is proved. Therefore, suppose that \(\mu \neq m_X\).

Then there is an element \(y_0 \in Y\), \(y_0 \neq 0\), such that \(f(y_0) > 0\). Consider a group \(F\) of the form (5). Since \(Y\) is a group with unique division by \(p\), we can consider on the subgroup \(F\) the function \(g(r) = f(ry_0)\), \(r \in F\). Verify that (11) implies the uniform continuity of the function \(g(r)\) in the topology induced on \(F\) by the topology of \(\mathbb{R}\). Taking into account that \(g(r)\) is a positive definite function on \(F\), for this it is enough to show that the function \(g(r)\) is continuous at zero. We will follow the scheme of the proof of Lemma 10 in [2] and Lemma 8 in [1].

Obviously, it follows from (11) that \(f(m^ky_0) \leq (f(y_0))^{m^k}\) for all natural \(k\). Hence,
\[
f(m^{-k}y_0) \geq (f(y_0))^{1/m^k}.
\]

It implies that
\[
g \left( \frac{1}{m^k} \right) \leq (g(1))^{1/m^k}. \tag{12}
\]

Let \(a > 0\). Note that for all \(x > 0\) the inequality \(1 - e^{-ax} \leq ax\) holds. Taking this into account we get from (12) that
\[
1 - g \left( \frac{1}{m^k} \right) \leq \frac{C}{m^k}, \tag{13}
\]
where \(C = -\ln g(1)\).

It is obvious that for any arbitrary real-valued characteristic function \(f(y)\) the inequality
\[
1 - f(y_1 + y_2) \leq 2[(1 - f(y_1)) + (1 - f(y_2))], \quad y_1, y_2 \in Y, \tag{14}
\]
is fulfilled. By induction, this implies the inequality
\[
1 - f(y_1 + \cdots + y_q) \leq \sum_{j=1}^{q} 2^j (1 - f(y_j)), \quad y_j \in Y. \tag{15}
\]

Let \(i\) be an integer such that \(0 < i < m^k\). We have
\[
\frac{i}{m^k} = \frac{i_1}{m} + \cdots + \frac{i_k}{m^k}, \tag{16}
\]
where \(i_l\) are integers such that \(0 \leq i_l < m\), \(l = 1, \ldots, k\). Substituting \(q = i_l\), \(y_1 = \cdots = y_{i_l} = \frac{y}{m^l}\) in (15), we get
\[
1 - g \left( \frac{i}{m^l} \right) \leq \sum_{j=1}^{i_l} 2^j \left( 1 - g \left( \frac{1}{m^l} \right) \right) \leq 2^m \left( 1 - g \left( \frac{1}{m^l} \right) \right). \tag{17}
\]
Suppose that
\[ \frac{i}{m^k} < \frac{1}{m^s}. \] (18)

It follows from this that \( i_l = 0 \) for all \( l \leq s \) in (16). In this case, taking into account (13), (15) and (17), we find from (15)

\[
1 - g \left( \frac{i}{m^k} \right) = 1 - g \left( \frac{i_{s+1}}{m^{s+1}} + \cdots + \frac{i_k}{m^k} \right) \leq \sum_{j=1}^{k-s} 2^j \left( 1 - g \left( \frac{i_{s+j}}{m^{s+j}} \right) \right)
\]

\[
\leq \sum_{j=1}^{k-s} 2^j 2^{m} \left( 1 - g \left( \frac{1}{m^{s+j}} \right) \right) \leq \sum_{j=1}^{k-s} C_2^{m+j} \sum_{j=1}^{k-s} \left( \frac{2}{m} \right)^j.
\] (19)

In view of \( m > 2 \), we find from (22) that the inequality

\[ 1 - g \left( \frac{i}{m^k} \right) \leq C_2^{m+1} \frac{1}{m^s} \] (20)
is valid. Obviously, (18) and (20) imply the continuity at zero of the function \( g(r) \) in the topology induced on \( F \) by the topology of \( \mathbb{R} \), and hence, the uniform continuity of \( g(r) \) in this topology.

It follows from (17) that the function \( g(r) \) satisfies the equation

\[ g(r) = (g(p^{-1}r))^{k_1} \cdots (g(p^{-l}r))^{k_l}, \quad r \in F. \] (21)

We extend the function \( g(r) \) by continuity from \( F \) to a continuous positive definite function on \( \mathbb{R} \). We keep the notation \( g \) for the extended function. It follows from (21) that the function \( g(s) \) satisfies the equation

\[ g(s) = (g(p^{-1}s))^{k_1} \cdots (g(p^{-l}s))^{k_l}, \quad s \in \mathbb{R}. \] (22)

By the generalized Pólya theorem, it follows from (22) that \( g(s) \) is the characteristic function of a symmetric Gaussian distribution on the real line, and hence, we get the representation

\[ f(y) = \exp\{-\sigma(y_0)y^2\}, \quad y = ry_0, \quad r \in F, \] (23)

where \( \sigma(y_0) \geq 0 \).

When we obtained (23) we fixed an element \( y_0 \in Y, \ y_0 \neq 0 \), such that \( f(y_0) > 0 \). If we fix another element \( \tilde{y} \in Y, \ \tilde{y} \neq 0 \), such that \( f(\tilde{y}) > 0 \), we will obtain the representation

\[ f(y) = \exp\{-\sigma(\tilde{y})y^2\}, \quad y = r\tilde{y}, \quad r \in F, \]

where \( \sigma(\tilde{y}) \geq 0 \). Since the subgroups \( \{y = r y_0 : r \in F\} \) and \( \{y = r \tilde{y} : r \in F\} \) have a nonzero intersection, this implies that \( \sigma(y_0) = \sigma(\tilde{y}) = \sigma \). Put \( E = \{y \in Y : f(y) \neq 0\} \). Thus, we have the representation

\[ f(y) = \exp\{-\sigma y^2\}, \quad y \in E. \] (24)

We will verify that \( E \) is a subgroup of \( Y \). First, observe that when we got (23) we proved that if \( y \in E \), then \( ry \in E \) for all \( r \in F \). Take \( y_1, y_2 \in E \). It follows from (21) that \( f(p^{-k}y_j) \to 1, \ j = 1, 2, \) as \( k \to \infty \), and then (14) implies that \( p^{-k}(y_1 + y_2) \in E \) for sufficiently large natural numbers \( k \). Hence \( y_1 + y_2 \in E \).

Thus, we obtain the representation

\[ f(y) = \begin{cases} \exp\{-\sigma y^2\}, & \text{if } y \in E, \\ 0, & \text{if } y \notin E. \end{cases} \] (25)
Put $K = A(X, E)$. Let $\gamma$ be a Gaussian distribution on the group $X$ with the characteristic function $\hat{\gamma}(y) = \exp(-\sigma y^2)$, $y \in Y$. Since $E = A(Y, K)$, it follows from (2) and (25) that $f(y) = \hat{\mu}(y) = \hat{\gamma}(y)\hat{m}_K(y)$, and hence $\mu = \gamma \ast m_K$. Note that the subgroup $E$ has the property: if $py \in E$, then $y \in E$. This implies that $f_p(K) = K$.

Get rid of the restriction that $\hat{\mu}(y) \geq 0$. Put $\nu = \mu \ast \hat{\mu}$. Then $\hat{\nu}(y) = |\hat{\mu}(y)|^2 \geq 0$ for all $y \in Y$, and the characteristic function $\hat{\nu}(y)$ also satisfies equation (6). Then, as has been proved above, the function $\hat{\nu}(y)$ is represented in the form (25). On the one hand, the subgroup $E$ possesses the property: if $y \in E$, then $\frac{1}{p^j}y \in E$. Therefore $E$ is not isomorphic to the group $\mathbb{Z}$. Taking into account that if a subgroup of the group of rational numbers is not isomorphic to $\mathbb{Z}$, then it is isomorphic to a group of the form (3), this implies that the group $E$ is topologically isomorphic to a discrete additive group $H_b$ of the rational numbers of the form

$$H_b = \left\{ \frac{m}{b_0b_1\cdots b_n} : n = 0, 1, \ldots; m \in \mathbb{Z} \right\}$$

for some $b = (b_0, b_1, b_2, \ldots)$, where all $b_j \in \mathbb{Z}$, $b_j > 1$. On the other hand, since $E = A(Y, K)$, the group $E$ is topologically isomorphic to some character group of the group $X/K$. By the Pontryagin duality theorem, the factor-group $X/K$ is topologically isomorphic to the corresponding $b$-adic solenoid $\Sigma_b$, and hence the group $X/K$ contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$.

Applying Lemma 2.5 to the group $X/K$, and taking into account that each character of the subgroup $E$ can be written as $(x, y)$, $x \in X$, $y \in E$, we get the statement of the theorem in the general case from representation (25) for the function $\hat{\nu}(y)$.

**Remark 2.6.** Let $\Sigma_a$ be an $a$-adic solenoid. Let $\gamma$ be a Gaussian distribution on $\Sigma_a$ with the characteristic function of the form

$$\hat{\gamma}(y) = \exp(-\sigma y^2), \quad y \in H_a,$$

(26)

where $\sigma \geq 0$. Let $\alpha_j$, $j = 1, 2, \ldots, n$, $n \geq 2$, be topological automorphisms of the group $\Sigma_a$ satisfying the condition (1). It is obvious that the characteristic function $\hat{\gamma}(y)$ satisfies equation (6).

Let $\Sigma_a$ be an $a$-adic solenoid with the property that there is a prime number $p$ such that $\Sigma_a$ contains no elements of order $p$. This implies that $\Sigma_a$ is a group with unique division by $p$. Let $\alpha_j$, $j = 1, 2, \ldots, n$, $n \geq 2$, be topological automorphisms of the group $\Sigma_a$ of the form $\alpha_j = \pm f_p^{-1}$. Let $K$ be a compact subgroup of $\Sigma_a$ such that $f_p(K) = K$. Let $\xi_j$ be independent identically distributed random variables with values in $\Sigma_a$ and distribution $m_K$. Note that the condition $f_p(K) = K$ implies that if $py \in A(H_a, K)$, then $y \in A(H_a, K)$. Taking this into account and using (2) it is not difficult to verify that the characteristic function $\hat{m}_K(y)$ also satisfies equation (6). In view of Lemma 2.4 it follows from has been said the following statement.

Let $\Sigma_a$ be an $a$-adic solenoid with the property that there is a prime number $p$ such that $\Sigma_a$ contains no elements of order $p$. Let $\alpha_j$, $j = 1, 2, \ldots, n$, $n \geq 2$, be topological automorphisms of $\Sigma_a$ of the form $\alpha_j = \pm f_p^{-1}$. Let $\alpha_j$ satisfy the condition (1). Let $\gamma$ be a Gaussian distribution on $\Sigma_a$ with the characteristic function of the form (26), and $K$ be a compact subgroup of $\Sigma_a$ such that $f_p(K) = K$. Put $\mu = \gamma \ast m_K$. Let $\xi_j$ be independent identically distributed random variables with values in $\Sigma_a$ and distribution $m_K$. Then $\xi_j$ and $\alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ are identically distributed.

Hence, we cannot narrow down the class of distributions in Theorem 2.1 which is characterized by the property of equidistribution of $\xi_j$ and $\alpha_1\xi_1 + \cdots + \alpha_n\xi_n$.

It turns out that Theorem 2.1 is false if the condition (i) is not satisfied. Namely, the following statement holds.
Proposition 2.7. Let \( X = \Sigma_a \) be an \( a \)-adic solenoid satisfying the condition:

(i) There are two prime numbers \( p \) and \( q \) such that the group \( X \) contains no elements of order \( p \) and \( q \).

Then there are topological automorphisms \( \alpha_j, j = 1, 2, \ldots, n, n \geq 2 \), of the group \( X \) satisfying the condition \( \{1\} \), and independent identically distributed random variables \( \xi_j \) with values in \( X \) and distribution \( \mu \) such that \( \xi_j \) and \( \alpha_1\xi_1 + \cdots + \alpha_n\xi_n \) are identically distributed, whereas \( \mu \) can not be represented as a convolution \( \mu = \gamma \ast m_K \), where \( \gamma \in \Gamma(X) \), and \( K \) is a compact subgroup of the group \( X \).

Theorem 2.11 and Proposition 2.7 imply the following statement.

Corollary 2.8. Let \( X = \Sigma_a \) be an \( a \)-adic solenoid. Let \( \alpha_j, j = 1, 2, \ldots, n, n \geq 2 \), be topological automorphisms of the group \( X \) satisfying the condition \( \{1\} \). Let \( \xi_j \) be independent identically distributed random variables with values in \( X \) and distribution \( \mu \). The equidistribution of \( \xi_j \) and \( \alpha_1\xi_1 + \cdots + \alpha_n\xi_n \) implies that \( \mu = \gamma \ast m_K \), where \( \gamma \in \Gamma(X) \) and \( K \) is a compact subgroup of the group \( X \), if and only if the condition (i) of Theorem 2.11 is satisfied.

To prove Proposition 2.7 we need the following lemma ([16, (32.43)]).

Lemma 2.9. Let \( Y \) be an Abelian group, let \( L \) be a subgroup of \( Y \), and let \( g(y) \) be a positive definite function on \( L \). If a function \( f(y) \) on the group \( Y \) is represented in the form

\[
f(y) = \begin{cases} 
g(y), & \text{if } y \in L, \\
0, & \text{if } y \notin L,
\end{cases}
\]

then \( f(y) \) is a positive definite function.

Proof of Proposition 2.7. Suppose for definiteness that \( p < q \). In order not to complicate the notation, we will identify \( Y \) with \( H_a \). The condition (i) in Proposition 2.7 implies that \( f_p, f_q \in \text{Aut}(X) \), and hence, \( f_p, f_q \in \text{Aut}(Y) \). Therefore both \( X \) and \( Y \) are groups with unique division by \( p \) and by \( q \).

Denote by \( H \) a subgroup of \( Y \) of the form

\[
H = \left\{ \frac{m}{n} \in Y : n \text{ is not divided by } p; \ m \in \mathbb{Z} \right\}.
\]

Let \( k \) be a natural number. Set

\[
H_k = \left\{ \frac{m}{p^kn} \in Y : m, n \text{ are not divided by } p \right\}
\]

and

\[
L = H \cup H_1.
\]

It is obvious that \( Y = H \cup \bigcup_{k=1}^{\infty} H_k \) and \( L \) is a subgroup of \( Y \). Denote by \( G \) the character group of the group \( L \), and put \( F = \text{Aut}(G, H) \). It is easy to see that \( L/H \cong \mathbb{Z}(p) \), where \( \mathbb{Z}(p) \) is the group of residue classes modulo \( p \). Since the character group of the factor-group \( L/H \) is topologically isomorphic to the annihilator \( A(G, H) \), we have \( F \cong \mathbb{Z}(p) \). Take \( 0 < c < 1 \). Let \( \omega \) be a distribution on the group \( F \) of the form \( \omega = cE_0 + (1 - c)m_F \), where \( E_0 \) is the distribution concentrated at zero. Consider \( \omega \) as a distribution on the group \( G \). Then (2) implies that the characteristic function \( \hat{\omega}(l), l \in L \), is of the form

\[
\hat{\omega}(l) = \begin{cases} 
1, & \text{if } l \in H, \\
c, & \text{if } l \in H_1.
\end{cases}
\]
Obviously, $\hat{\omega}(l)$ is a positive definite function. Consider on the group $Y$ the function

$$
\hat{\omega}(y) = \begin{cases} 
\hat{\omega}(y), & \text{if } y \in L, \\
0, & \text{if } y \notin L.
\end{cases}
$$

By Lemma 2.9, $f(y)$ is a positive definite function. By the Bochner theorem, there is a probability distribution $\mu$ on the group $X$ such that $f(y) = \hat{\mu}(y)$. It is obvious that the distribution $\mu$ is not represented as a convolution $\mu = \gamma \ast m_K$, where $\gamma \in \Gamma(X)$, and $K$ is a compact subgroup of $X$.

Take natural numbers $l$ and $m$, where $l < m$, such that the remainders of the division of the numbers $q^{2l}$ and $q^{2m}$ by $p^2$ are equal. Then $q^{2m} - q^{2l}$ is divided by $p^2$. Hence, $q^{2m-2l} - 1$ is also divided by $p^2$, and the remainder of the division $q^{2m-2l}$ by $p^2$ is equal to 1. Put $a = m - l$. We have

$$
q^{2a} = p^2b + 1.
$$

Put $\alpha_1 = \cdots = \alpha_b = f pf^{q^{-1}}$, $\alpha_{b+1} = f q^{a-1}$, $n = b + 1$. In view of (27), the topological automorphisms $\alpha_j$, $j = 1, 2, \ldots, n$, of the group $X$ satisfy the condition (1).

Let $\xi_j$, $j = 1, \ldots, n$, be independent identically distributed random variables with values in $X$ and distribution $\mu$. Verify that the function $f(y)$ satisfies the equation

$$
f(y) = \left(f \left(\frac{p}{q^r}y\right) \right)^b f \left(\frac{1}{q}y\right), \quad y \in Y.
$$

Indeed, if $y \in H$, then $\frac{p}{q^r}y, \frac{1}{q}y \in H$, and both parts of equation (28) are equal to 1. If $y \in H_1$, then $\frac{p}{q^r}y \in H$, $\frac{1}{q}y \in H_1$, and both parts of equation (28) are equal to $c$. If $y \in H_k$, $k \geq 2$, then $\frac{1}{q}y \in H_k$, and both parts of equation (28) are equal to zero. Thus, the function $f(y)$ satisfies equation (28), and by Lemma 2.4 $\xi_1$ and $\alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ are identically distributed. ■

**Remark 2.10.** Let $X$ be a locally compact Abelian group and let $\nu \in M^1(X)$. Put $B = \{y \in Y : \hat{\nu}(y) = 1\}$. It is well-known that $B$ is a closed subgroup of $Y$ and the distribution $\nu$ is supported in the subgroup $A(X,B)$. Therefore, in the notation of Proposition 2.7 the distribution $\mu$ constructed in the proof of Proposition 2.7 is supported in the annihilator $A(X,H)$, i.e in a proper closed subgroup of the group $X$.

We can strengthen Proposition 2.7 as follows. Let $X = \Sigma_a$ be an $a$-adic solenoid satisfying the condition (i) of Proposition 2.7. Let $\gamma$ be a non-degenerate Gaussian distribution on $X$ with the characteristic function of the form (26). Set $\lambda = \gamma \ast \mu$, where the distribution $\mu$ constructed in the proof of Proposition 2.7. Then the characteristic function of the distribution $\lambda$ is of the form

$$
\hat{\lambda}(y) = \begin{cases} 
\exp\{-\sigma y^2\}, & \text{if } y \in H, \\
\exp\{-\sigma y^2\}, & \text{if } y \in H_1, \\
0, & \text{if } y \notin L.
\end{cases}
$$

It is obvious that the distribution $\lambda$ can not be represented as a convolution of a Gaussian distribution on $X$ and the Haar distribution of a compact subgroup of the group $X$. Let $\xi_j$ be independent identically distributed random variables with values in $X$ and distribution $\lambda$. Arguing as in the proof of Proposition 2.7 and using Lemma 2.4 we are convinced that $\xi_j$ and $\alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ are identically distributed. As is easily seen, the distribution $\lambda$ is not supported in any proper close subgroup of the group $X$.

**Remark 2.11.** On the real line, both the Skitovich–Darmin theorem and the generalized Pólya theorem characterize the Gaussian distribution. Corollary 2.8 gives a complete description of $a$-adic
solenoids on which the generalized Pólya theorem is valid. It is interesting to note that, as follows from [12], there are no $a$-adic solenoids on which the Skitovich–Darmois theorem holds. Namely, the following statement takes place.

Let $X = \Sigma_a$ be an $a$-adic solenoid. Then there are topological automorphisms $\alpha_j, \beta_j$ of the group $X$ and independent random variables $\xi_1$ and $\xi_2$ with values in $X$ and distributions $\mu_1$ and $\mu_2$ such that the linear forms $L_1 = \alpha_1 \xi_1 + \alpha_2 \xi_2$ and $L_2 = \beta_1 \xi_1 + \beta_2 \xi_2$ are independent, whereas $\mu_j$ can not be represented as convolutions $\mu_j = \gamma_j \ast m_{K_j}$, where $\gamma_j \in \Gamma(X)$, and $K_j$ are compact subgroups of the group $X$, $j = 1, 2$.

**Remark 2.12.** As noted earlier each connected locally compact Abelian group of dimension 1 is topologically isomorphic to either the additive group of real numbers, or the circle group, or an $a$-adic solenoid. The generalized Pólya theorem on the additive group of real numbers characterizes Gaussian distributions (Theorem A). The generalized Pólya theorem on $a$-adic solenoids $\Sigma_a$ satisfying the condition (i) of Theorem 2.1 characterizes convolutions of Gaussian distributions on $\Sigma_a$ and Haar distributions on compact subgroups of $\Sigma_a$ (Theorem 2.1). Discuss the generalized Pólya theorem on the circle group.

Let $\alpha_j, j = 1, 2, \ldots, n, n \geq 2$, be topological automorphisms of the circle group $\mathbb{T}$, and let $\xi_j$ be independent identically distributed random variables with values in $\mathbb{T}$ and distribution $\mu$. Consider the linear form $L = \alpha_1 \xi_1 + \cdots + \alpha_n \xi_n$. Taking into account that $I$ and $-I$ are the only topological automorphisms of $\mathbb{T}$, $L$ is of the form $L = \pm \xi_1 + \cdots + \pm \xi_n$. Assume for definiteness that $L = \xi_1 + \cdots + \xi_m - \xi_{m+1} - \cdots - \xi_n$. Set $f(y) = \hat{\mu}(y)$. If $\xi_j$ and $L$ are identically distributed, then by Lemma 2.4, the function $f(y)$ satisfies equation (6) which takes the form

$$f(y) = (f(y))^m (f(-y))^{n-m}, \quad y \in \mathbb{Z}. \quad (29)$$

Put $E = \{ y \in \mathbb{Z} : f(y) \neq 0 \}$. It follows from (29) that $|f(y)| = 1$ for all $y \in E$. This implies that $E$ is a subgroup of $\mathbb{Z}$ and there is an elements $x \in \mathbb{T}$ such that $f(y) = (x,y)$ for all $y \in E$. Thus, we have

$$f(y) = \begin{cases} (x,y), & \text{if } y \in E, \\ 0, & \text{if } y \notin E. \end{cases} \quad (30)$$

Put $K = A(\mathbb{T}, E)$. Then $E = A(\mathbb{Z}, K)$. Taking into account (2), it follows from (30) that $\mu$ is a shift by the element $x$ of the Haar distribution $m_K$. We see that the generalized Pólya theorem on the circle group $\mathbb{T}$ characterizes shifts of Haar distributions on compact subgroups of $\mathbb{T}$.

Thus, we have completely studied the generalized Pólya theorem on connected locally compact Abelian groups of dimension 1.
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