Complex Finsler metrics

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0. Introduction

A complex Finsler metric is an upper semicontinuous function $F: T^{1,0}M \to \mathbb{R}^+$ defined on the holomorphic tangent bundle of a complex Finsler manifold $M$, with the property that $F(p; \zeta v) = |\zeta|F(p; v)$ for any $(p; v) \in T^{1,0}M$ and $\zeta \in \mathbb{C}$.

Complex Finsler metrics do occur naturally in function theory of several variables. The Kobayashi metric introduced in 1967 ([K1]) and its companion the Carathéodory metric are remarkable examples which have become standard tools for anybody working in complex analysis; we refer the reader to [K2, 4], [L], [A] and [JP] to get an idea of the amazing developments in this area achieved in the past 25 years.

In general, the Kobayashi metric is not at all regular; it may even not be continuous. But in 1981 Lempert [Le] proved that the Kobayashi metric of a bounded strongly convex domain $D$ in $\mathbb{C}^n$ is smooth (outside the zero section of $T^{1,0}D$), thus allowing in principle the use of differential geometric techniques in the study of function theory over strongly convex domains (see also Pang [P2] for other examples of domains with smooth Kobayashi metric).

We started dealing with this kind of problems in [AP1]. In particular, [AP2] was devoted to the search of differential geometric conditions ensuring the existence in a complex Finsler manifold of a foliation in holomorphic disks like the one found by Lempert in strongly convex domains, where the disks were isometric embeddings of the unit disk $\Delta \subset \mathbb{C}$ endowed with the Poincaré metric. And indeed (see also [AP3]) we found necessary and sufficient conditions (see also Pang [P1] for closely related results). In that case, because the nature of the problem required the solution of certain P.D.E.’s, the conditions were mainly expressed in local coordinates somewhat hiding their geometric meaning.

The aim of this paper is to present an introduction to complex Finsler geometry in a way suitable to deal with global questions. Roughly speaking, the idea is to isometrically embed a complex Finsler manifold into a hermitian vector bundle, and then apply standard hermitian differential geometry techniques, in the spirit of [K3]. Here we provide just a coarse outline of the procedure. Let $\hat{M}$ be the complement of the zero section in $T^{1,0}M$. We assume that the complex Finsler metric $F$ is smooth on $\hat{M}$, and that $F$ is strongly pseudoconvex, that is that the Levi form of $G = F^2$ is positive definite. Now let $\mathcal{V} \subset T^{1,0}\hat{M}$ be the vertical bundle, that is the kernel of the differential of the canonical projection $\pi: T^{1,0}M \to M$. Using the Levi form of $G$, it is easy to define a hermitian metric on $\mathcal{V}$; moreover, there exists a canonical section $\iota$ of $\mathcal{V}$ giving an isometric embedding of $\hat{M}$ into $\mathcal{V}$.
— that is for any $v \in \tilde{M}$ the norm of $\iota(v)$ with respect to the given hermitian metric on $V$ is equal to $F(v)$. Let $D$ be the Chern connection on $V$ associated to the metric, and denote by $\mathcal{H}$ the kernel of the bundle map $X \mapsto \nabla_X \iota$. Then it turns out that $\mathcal{H}$ is a "horizontal bundle", that is $T^{1,0}\tilde{M} = \mathcal{H} \oplus \bar{V}$; furthermore, there is a canonically defined global bundle isomorphism $\Theta: V \to \mathcal{H}$. Using $\Theta$, we can transfer both the metric and the connection on $\mathcal{H}$, obtaining a canonical hermitian structure on $T^{1,0}\tilde{M}$, and the associated Chern connection preserves the splitting. Finally, the "horizontal radial vector field" $\chi = \Theta \circ \iota$ is a canonical isometric embedding of $\tilde{M}$ into $\mathcal{H}$. Then our idea is that the complex Finsler geometry of $\tilde{M}$ should be described by using the differential geometry of the Chern connection $D$ restricted to $\mathcal{H}$, using $\chi$ as a means of transferring informations from the tangent bundle to the horizontal bundle and back. For instance, the Kähler condition introduced in [AP2] becomes the vanishing of a suitable contraction of the horizontal part of the torsion of $D$ (here we say that the metric is weakly Kähler); and the necessary and sufficient condition for the existence of complex geodesic curves (see [AP2, 3]) are expressed by constant holomorphic curvature and a symmetry property of the horizontal part of the curvature of $D$; cf. Lemma 8.3.

This approach is in the spirit of the one developed by E. Cartan [C] for real Finsler metrics; see [Ru1], [M], [Ch], [BC], [Be] and the forthcoming monograph [AP4] for an account in modern language. On the other hand, to our surprise we were unable to find in the literature a comparable approach in the complex case. Rund, in [Ru2], described the Chern connection on the horizontal bundle, but only in local coordinates. Fukui in [Fu] studied the Cartan connection on a complex Finsler manifold, which is in general different from the Chern connection (see [AP4] for a comparison). Faran [F] studied the local equivalence problem, without dealing with global questions. Only Kobayashi [K3] explicitly used the Chern connection, but he seemed unaware of the relevance of the horizontal component. It should be mentioned that we choose to work on $\tilde{M}$ instead of the projectivized tangent bundle mainly for keeping more transparent the relationships between global objects and local computations (which are often simplified by consistently using the homogeneity of the function $G$ and its derivatives). However the two approach are completely equivalent. In fact, the role of the canonical sections $\iota$ and $\chi$ in our context is analogous to the role of the tautological line bundle in [K3]. We hope that our work will clarify the subject of complex Finsler geometry, opening the way to new research in the field.

The content of this paper is the following. In sections 1 and 2 we describe in detail the construction outlined above of the Chern-Finsler connection. In sections 3 and 4 we define the $(2,0)$-torsion, the $(1,1)$-torsion, the curvature of the Chern-Finsler connection on the horizontal bundle, we derive the Bianchi identities and we discuss Kähler Finsler metrics. In section 5 we introduce the notion of holomorphic curvature.

In sections 6 and 7 we derive the first and second variation formulas for a strongly pseudoconvex Kähler Finsler metric, giving a good example of global computations made using the tools introduced before. As a corollary, we prove the local existence and uniqueness of geodesics for a strongly pseudoconvex weakly Kähler metric, without assuming the strong convexity of the metric.

Finally, in section 8 we deal with strongly pseudoconvex Finsler metrics of constant
holomorphic curvature, providing a first step toward their classification. As a consequence of results of this section and of [AP2] we get for example the following:

**Theorem 0.1:** Let $F: T^{1,0}M \to \mathbb{R}^+$ be a complete strongly pseudoconvex Finsler metric on a simply connected complex manifold $M$. Assume that

(i) $F$ is Kähler;
(ii) $F$ has constant holomorphic curvature $-4$;
(iii) $R(H, K, \chi, \bar{\chi}) = R(\chi, K, H, \bar{\chi})$ for all $H, K \in \mathcal{H}$, where $R$ is the curvature operator of the Chern connection;
(iv) the indicatrices $I_F(p) = \{v \in T^{1,0}_p M \mid F(v) < 1\}$ of $F$ are strongly convex for all $p \in M$.

Then the exponential map $\exp_p: T^{1,0}_p M \to M$ is a homeomorphism, and a smooth diffeomorphism outside the origin, for any $p \in M$. Furthermore, a suitable reparametrization of $\exp_p$ induces a foliation of $M$ by isometric totally geodesic holomorphic embeddings of the unit disk $\Delta \subset \mathbb{C}$ endowed with the Poincaré metric. In particular, $F$ is the Kobayashi metric of $M$.

A version of this result also holds when the holomorphic curvature is identically zero; the precise statement can be found in Theorem 8.10

1. Definitions and preliminaries

Let $M$ be a complex manifold of complex dimension $n$. We shall denote by $T^{1,0}M$ the holomorphic tangent bundle of $M$, and by $\tilde{M}$ the complement in $T^{1,0}M$ of the zero section. The real tangent bundle of $M$ will be denoted by $T^R M$, and we set as usual $T^C M = T^R M \otimes \mathbb{C}$.

A complex Finsler metric $F$ on $M$ is an upper semicontinuous function $F: T^{1,0}M \to \mathbb{R}^+$ satisfying

(i) $G = F^2$ is smooth on $\tilde{M}$;
(ii) $F(p; v) > 0$ for all $p \in M$ and $v \in \tilde{M}_p$;
(iii) $F(p; \zeta v) = |\zeta| F(p; v)$ for all $p \in M$, $v \in T^{1,0}_p M$ and $\zeta \in \mathbb{C}$.

We shall systematically denote by $G$ the function $G = F^2$. Note that it is important to ask for the smoothness of $G$ only on $\tilde{M}$: in fact, it is easy to see that $G$ is smooth on the whole of $T^{1,0}M$ iff $F$ is the norm associated to a hermitian metric. In this case, we shall say that $F$ comes from a hermitian metric.

To start, we need a few notations and general formulas. In local coordinates, a vector $v \in T^{1,0}_p M$ is written as

$$v = v^\alpha \frac{\partial}{\partial z^\alpha} \bigg|_p,$$

where we adopt the Einstein convention. In particular, the function $G$ is locally expressed in terms of the coordinates $\{z^1, \ldots, z^n, v^1, \ldots, v^n\}$. We shall denote by indices like $\alpha$, $\beta$ and so on the derivatives with respect to the $v$-coordinates; for instance,

$$G_{\alpha\bar{\beta}} = \frac{\partial^2 G}{\partial v^\alpha \partial \bar{v}^\beta}.$$
On the other hand, the derivatives with respect to the \( z \)-coordinates will be denoted by indices after a semicolon; for instance,

\[
G_{,\mu
u} = \frac{\partial^2 G}{\partial z^\mu \partial z^\nu} \quad \text{or} \quad G_{\alpha;\bar{\nu}} = \frac{\partial^2 G}{\partial z^\nu \partial \bar{v}^\alpha}.
\]

For our aims, we ought to focus on a smaller class of Finsler metrics. A complex Finsler metric \( F \) will be said \textit{strongly pseudoconvex} if \( \text{(iv) the Levi matrix} \ (G_{\alpha\bar{\beta}}) \ \text{is positive definite on} \ \tilde{M} \).

This is equivalent to requiring that all the \( F \)-indicatrices

\[
I_F(p) = \{ v \in T^{1,0}_p M \mid F(v) < 1 \}
\]

are strongly pseudoconvexes. As we shall see in section 2, this hypothesis will allow us to define a hermitian metric on a suitable vector bundle.

The main (actually, almost the unique) property of the function \( G \) is its (1,1)-homogeneity: we have

\[
G(p; \zeta v) = \zeta \bar{\zeta} G(p; v)
\]

for all \( (p; v) \in T^{1,0} M \) and \( \zeta \in \mathbb{C} \). We now collect a number of formulas we shall use later on which are consequences of (1.1). First of all, differentiating with respect to \( v^\alpha \) and \( \bar{v}^\beta \) we get

\[
G_\alpha(p; \zeta v) = \bar{\zeta} G_\alpha(p; v), \\
G_{\alpha\bar{\beta}}(p; \zeta v) = G_{\alpha\bar{\beta}}(p; v), \\
G_{\alpha\beta}(p; \zeta v) = (\bar{\zeta}/\zeta) G_{\alpha\beta}(p; v).
\]

Thus differentiating with respect to \( \zeta \) or \( \bar{\zeta} \) and then setting \( \zeta = 1 \) we get

\[
G_{\alpha\beta} \bar{v}^\beta = G_\alpha, \quad G_{\alpha\beta} v^\beta = 0,
\]

and

\[
G_{\alpha\beta\gamma} v^\gamma = -G_{\alpha\beta}, \quad G_{\alpha\beta\gamma} \bar{v}^\gamma = G_{\alpha\beta}, \quad G_{\alpha\beta\gamma} v^\gamma = 0,
\]

where everything is evaluated at \( (p; v) \).

On the other hand, differentiating directly (1.1) with respect to \( \zeta \) or \( \bar{\zeta} \) and putting eventually \( \zeta = 1 \) we get

\[
G_\alpha v^\alpha = G, \quad G_{\alpha\beta} v^\alpha v^\beta = 0, \quad G_{\alpha\beta} \bar{v}^\alpha \bar{v}^\beta = G.
\]

It is clear that we may get other formulas applying any differential operator acting only on the \( z \)-coordinates, or just by conjugation. For instance, we get

\[
G_{\bar{\alpha};\mu} \bar{v}^\alpha = G_{,\mu},
\]

and so on.
Assuming from now on (unless explicitly noted otherwise) \( F \) strongly pseudoconvex, we get another bunch of formulas. As usual in hermitian geometry, we shall denote by \( (G^\beta_\alpha) \) the inverse matrix of \( (G_\alpha^\beta) \), and we shall use it to raise indices.

First of all, applying \( G^\beta_\alpha \) to the first equation in (1.3) we get

\[
G^\beta_\alpha G_\alpha = \bar{v}^\beta,
\]

and thus, applying (1.6),

\[
G^\beta_\mu G^\beta_\alpha G_\alpha = G^\beta_\mu.
\]

Recalling that \( (G^\beta_\alpha) \) is the inverse matrix of \( (G_\alpha^\beta) \), we may also compute derivatives of \( G^\beta_\alpha \):

\[
DG^\beta_\alpha = -G^\nu_\alpha G^\beta_\mu (DG^\mu_\nu),
\]

(1.9)

where \( D \) denotes any first order linear differential operator. As a consequence of (1.4) and (1.9) we get

\[
G^\beta_\alpha \bar{v}^\nu = -G^\nu_\alpha G^\beta_\mu G^\mu_\nu \bar{v}^\nu = 0,
\]

(1.10)

and recalling also (1.7) we obtain

\[
G^\beta_\gamma G^\beta_\alpha = -G^\beta_\gamma G^\beta_\mu G^\nu_\alpha G^\mu_\nu \gamma v^\mu = 0.
\]

(1.11)

2. The Chern-Finsler connection

To any hermitian metric is associated a unique (1,0)-connection such that the metric tensor is parallel: the Chern connection. The main goal of this section is to define the analogue for strongly pseudoconvex Finsler metrics.

Let \( \pi: \tilde{M} \to M \) denote the restriction of the canonical projection of \( T^{1,0}M \) onto \( M \). The vertical bundle \( \mathcal{V} \subset T^{1,0}M \) is, by definition, the kernel of the differential \( d\pi: T^{1,0}\tilde{M} \to T^{1,0}M \).

It is easy to check that \( \mathcal{V} \) is a complex vector bundle of rank \( n \) over \( \tilde{M} \); a local frame for \( \mathcal{V} \) is given by \( \{\dot{\partial}_1, \ldots, \dot{\partial}_n\} \), where we set

\[
\dot{\partial}_\alpha = \frac{\partial}{\partial v^\alpha} \quad \text{and} \quad \partial_\mu = \frac{\partial}{\partial z^\mu},
\]

for \( \alpha, \mu = 1, \ldots, n \). We shall denote by \( \mathcal{X}(\mathcal{V}) \) the space of smooth sections of \( \mathcal{V} \); more generally, \( \mathcal{X}(E) \) will denote the space of smooth sections of any vector bundle \( p: E \to B \).

Let \( j_p: T^{1,0}_pM \to T^{1,0}M \) be the inclusion and, for \( v \in \tilde{M}_p \), let \( k_v: T^{1,0}_pM \to T^{1,0}_v(T^{1,0}_pM) \) denote the usual identification. Then we get a natural isomorphism

\[
\iota_v = d(j_{\pi(v)})_v \circ k_v: T^{1,0}_\pi(v)M \to \mathcal{V}_v,
\]

and, by restriction, the all-important natural section \( \iota: \tilde{M} \to \mathcal{V} \) given by

\[
\iota(v) = \iota_v(v) \in \mathcal{V}_v.
\]
In local coordinates,
\[ \iota_v \left( \frac{\partial}{\partial z^\alpha} \bigg|_{\pi(v)} \right) = \dot{\partial}_\alpha | v; \]
in particular, if \( v = v^\alpha (\partial / \partial z^\alpha) \) then
\[ \iota(v) = v^\alpha \dot{\partial}_\alpha | v. \]
\( \iota \) is called the radial vertical vector field.

The first observation is that a strongly pseudoconvex Finsler metric \( F \) defines a hermitian metric on the vertical bundle \( \mathcal{V} \). Indeed, if \( v \in \tilde{M} \) and \( W_1, W_2 \in \mathcal{V}_v \), with \( W_j = W_j^\alpha \dot{\partial}_\alpha \), we set
\[ \langle W_1, W_2 \rangle_v = G^\alpha_{\bar{\beta}}(v) W_1^\alpha \overline{W_2^\beta}. \]
Being \( F \) strongly pseudoconvex, \( \langle , \rangle \) is a hermitian metric. Note that the third equation in (1.5) says that \( G = \langle \iota, \iota \rangle \); so \( \iota \) is an isometric embedding of \( \tilde{M} \) into \( \mathcal{V} \).

Following Kobayashi [K3], we now consider the Chern connection \( D \) on the vector bundle \( \mathcal{V} \): it is the unique (1,0)-connection on \( \mathcal{V} \) such that the hermitian structure previously defined is parallel. In other words, \( D : \mathcal{X}(\mathcal{V}) \to \mathcal{X}(T^*_C \tilde{M} \otimes \mathcal{V}) \) is such that
\[ X \langle V, W \rangle = \langle \nabla_X V, W \rangle + \langle V, \nabla_X W \rangle, \]
for any \( X \in T^{1,0} \tilde{M} \) and \( V, W \in \mathcal{X}(\mathcal{V}) \).

In local coordinates, the connection matrix \( (\omega^\alpha_\beta) \) is given by
\[ \omega^\alpha_\beta = G^{\bar{\tau} \alpha} \partial G^\beta_{\bar{\tau}} = \tilde{\Gamma}^\alpha_{\beta;\mu} d z^\mu + \tilde{\Gamma}^\alpha_{\beta;\gamma} d v^\gamma, \]
where
\[ \tilde{\Gamma}^\alpha_{\beta;\gamma} = G^{\bar{\tau} \alpha} G^\beta_{\bar{\tau};\gamma} \quad \text{and} \quad \tilde{\Gamma}^\alpha_{\beta;\mu} = G^{\bar{\tau} \alpha} G^\beta_{\bar{\tau};\mu}. \]

This is only part of the connection we are looking for: our next goal is to canonically extend \( D \) to a (1,0)-connection on \( T^{1,0} \tilde{M} \). Let us consider the bundle map \( \Lambda : T^{1,0} \tilde{M} \to \mathcal{V} \) defined by
\[ \Lambda(X) = \nabla_X \iota, \]
and set \( \mathcal{H} = \ker \Lambda \subset T^{1,0} \tilde{M} \). We claim that \( \mathcal{H} \) is a horizontal bundle, that is \( T^{1,0} \tilde{M} = \mathcal{H} \oplus \mathcal{V} \). Indeed, in local coordinates
\[ \Lambda(X) = [\dot{X}^\alpha + \omega^\alpha_\beta(X) v^\beta] \dot{\partial}_\alpha, \]
where \( X = X^\mu \partial_\mu + \dot{X}^\alpha \dot{\partial}_\alpha \). Then a local frame for \( \mathcal{H} \) is given by \( \{ \delta_1, \ldots, \delta_n \} \), where
\[ \delta_\mu = \partial_\mu - \tilde{\Gamma}^\alpha_{\beta;\mu} v^\beta \dot{\partial}_\alpha. \]
— note that $\tilde{\Gamma}^\alpha_{\beta\gamma} v^\beta \equiv 0$ — and the claim is proved.

It is not difficult to check (see [AP4] for a coordinate-free proof) that setting

$$\Theta(\hat{\partial}_\alpha) = \delta_\alpha$$

for $\alpha = 1, \ldots, n$ we get a well-defined global bundle isomorphism $\Theta: V \rightarrow H$; then we can define a $(1,0)$-connection $D$ on $H$ just by setting

$$\nabla_X H = \Theta[\nabla_X (\Theta^{-1} H)]$$

for any $X \in T_{\tilde{\mathcal{C}}} \tilde{M}$ and $H \in \mathcal{X}(H)$. By linearity, this yields a $(1,0)$-connection on $T^{1,0} \tilde{M}$, still denoted by $D$: the Chern-Finsler connection.

Using the bundle isomorphism $\Theta: V \rightarrow H$ we can also transfer the hermitian structure $\langle \cdot, \cdot \rangle$ on $H$ just by setting

$$\forall H, K \in H \quad \langle H, K \rangle_v = \langle \Theta^{-1}(H), \Theta^{-1}(K) \rangle_v,$$

and then we can define a hermitian structure on $T^{1,0} \tilde{M}$ by requiring $H$ be orthogonal to $V$. It is easy to check then that $D$ is the Chern connection associated to this hermitian structure, that is

$$\langle \chi \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$$

for any $X \in T^{1,0} \tilde{M}$ and $Y, Z \in \mathcal{X}(T^{1,0} \tilde{M})$.

¿From now on we shall work only with the frame $\{\delta_1, \ldots, \delta_n\}$ and its dual co-frame $\{dz^\mu, \psi^\alpha\}$ given by

$$\psi^\alpha = dv^\alpha + \Gamma^\alpha_{\beta\mu} dz^\mu = dv^\alpha + G^\tau^\alpha G^{\tau\mu} \tilde{\tau} \mu d\tilde{\tau},$$

where we have set

$$\Gamma^\alpha_{\beta\mu} = \tilde{\Gamma}^\alpha_{\beta\mu} v^\beta = G^\tau^\alpha G^{\tau\mu} \tilde{\tau} \mu.$$

Writing

$$\omega^\alpha_\beta = \Gamma^\alpha_{\beta\mu} dz^\mu + \Gamma^\alpha_{\beta\gamma} \psi^\gamma,$$

we get

$$\begin{align*}
\Gamma^\alpha_{\beta\gamma} &= G^\tau^\alpha G^{\beta\tau} \tilde{\tau} \gamma = \Gamma^\alpha_{\gamma\beta},
\Gamma^\alpha_{\beta\mu} &= G^\tau^\alpha \delta_\mu (G^{\beta\tau}) = G^\tau^\alpha (G^{\beta\tau} \mu - G^{\beta\tau} \gamma \mu).
\end{align*}$$

(2.1)

Note that

$$\begin{align*}
\Gamma^\alpha_{\beta\mu} &= \hat{\partial}_\beta (\Gamma^\alpha_{\mu\beta}) \quad \text{and} \quad \Gamma^\alpha_{\beta\mu} = \Gamma^\alpha_{\beta\mu} v^\beta.
\end{align*}$$

(2.2)

in particular, this is exactly the connection introduced by Rund [Ru2].

So we have described a canonical splitting of the holomorphic tangent bundle of $\tilde{M}$ in a vertical and a horizontal bundle, and defined a canonical connection on it, preserving this splitting. In the following subsections we shall begin the study of this connection, introducing torsions and curvatures; here we first describe a few properties of the splitting.

First of all, the next lemma shows that the local frames $\{\delta_1, \ldots, \delta_n\}$ enjoy some nice and convenient properties:
Lemma 2.1: Let $D$ be the Chern-Finsler connection associated to a strongly pseudoconvex Finsler metric $F$, and let $\{\delta_1, \ldots, \delta_n\}$ be the corresponding local horizontal frame. Then

(i) $[\delta_\mu, \delta_\nu] = 0$ for all $1 \leq \mu, \nu \leq n$;
(ii) $[\delta_\mu, \hat{\delta}_\alpha] = \Gamma^\sigma_{\alpha; \mu} \hat{\delta}_\sigma$ for all $1 \leq \alpha, \mu \leq n$;
(iii) $\delta_\mu(G) = \delta_{\mu}(G) = 0$ for all $1 \leq \mu \leq n$;
(iv) $\delta_{\mu}(G_\alpha) = 0$ for all $1 \leq \alpha, \mu \leq n$.

Proof: (i) If suffices to compute. First of all,

$$[\delta_\mu, \delta_\nu] = (\Gamma^\alpha_{\mu; \nu} - \Gamma^\alpha_{\nu; \mu} + \Gamma^\alpha_{\sigma; \mu} \Gamma^\sigma_{\nu; \mu} - \Gamma^\alpha_{\sigma; \mu} \Gamma^\sigma_{\nu; \mu}) \hat{\delta}_\alpha,$$

where $\Gamma^\alpha_{\mu; \nu} = \partial_\nu (\Gamma^\alpha_{\mu})$ and so on. Now,

$$\Gamma^\alpha_{\mu; \nu} = G^{\bar{\tau}}(G_{\bar{\tau}; \mu} - G_{\sigma \bar{\tau}; \mu} \Gamma^\sigma_{\mu}), \quad \Gamma^\alpha_{\sigma; \mu} \Gamma^\sigma_{\mu} = G^{\bar{\tau}}(G_{\sigma \bar{\tau}; \mu} \Gamma^\sigma_{\mu} - G_{\sigma \bar{\tau}; \mu} \Gamma^\sigma_{\mu});$$

and the assertion follows. Note that we have actually proved that

$$\delta_\nu (\Gamma^\alpha_{\mu}) = \delta_\mu (\Gamma^\alpha_{\nu}). \quad (2.3)$$

(ii) Indeed,

$$[\delta_\mu, \hat{\delta}_\alpha] = [\partial_\mu - \Gamma^\sigma_{\mu} \hat{\delta}_\sigma, \hat{\delta}_\alpha] = \hat{\delta}_\alpha (\Gamma^\sigma_{\mu}) \hat{\delta}_\sigma = \Gamma^\sigma_{\alpha; \mu} \hat{\delta}_\sigma.$$

(iii) In fact, using (1.8) we get

$$\delta_\mu(G) = G_{\mu} - \Gamma^\sigma_{\mu} G_\sigma = G_{\mu} - G^{\bar{\tau}} G_{\bar{\tau}; \mu} G_\sigma = G_{\mu} - G_{\mu} = 0.$$

(iv) Finally,

$$\delta_{\mu}(G_\alpha) = G_{\alpha; \mu} - \Gamma^{\bar{\tau}}_{\mu} G_{\alpha \bar{\tau}} = G_{\alpha; \mu} - G_{\alpha; \mu} = 0,$$

where $\Gamma^{\bar{\tau}}_{\mu} = \Gamma_{\bar{\tau}; \mu}.$

The philosophical idea behind our work is that to study the geometry of a complex Finsler metric one should transfer everything (or most of it) in the horizontal bundle, and then apply the usual techniques of hermitian geometry there. We shall better substantiate this idea later, for instance in sections 6 and 7 discussing variation formulas; here we begin to show how to lift objects (e.g., vector fields) from the tangent bundle up to $\mathcal{H}$.

The main tool is provided by the horizontal analogues of the isomorphisms $\iota_\nu$. If $\nu \in \tilde{M}$, we set

$$\chi_\nu = \Theta_\nu \circ \iota_\nu: T_{\pi(\nu)} M \rightarrow \mathcal{H}_\nu.$$

The horizontal radial vector field $\chi \in \mathcal{X}(\mathcal{H})$ is then defined by

$$\chi = \Theta \circ \iota;$$
in local coordinates, if \( v = v^\alpha (\partial/\partial z^\alpha)|_p \) we have
\[
\chi(v) = v^\alpha \delta_\alpha|_v.
\]

Using the isomorphisms \( \chi_v \) we can induce an embedding of \( \tilde{M} \) into \( \mathcal{H} \) which respects the Lie algebra structure. To be precise, a vector field \( \xi \in \mathcal{X}(T^{1,0}M) \) may be lifted in two different ways to vector fields in \( T^{1,0}\tilde{M} \): via the horizontal lift
\[
\xi^H (v) = \chi_v \left( \xi(\pi(v)) \right),
\]
and via the vertical lift
\[
\xi^V (v) = \iota_v \left( \xi(\pi(v)) \right).
\]

A consequence of Lemma 2.1 is that the horizontal lift is a Lie algebra homomorphism:

**Proposition 2.2:** Let \( D \) be the Chern-Finsler connection associated to a strongly pseudoconvex Finsler metric \( F \) on a complex manifold \( M \). Then:

(i) \( [\mathcal{X}(\mathcal{H}), \mathcal{X}(\mathcal{H})] \subset \mathcal{X}(\mathcal{H}) \) and \( [\mathcal{X}(\mathcal{V}), \mathcal{X}(\mathcal{V})] \subset \mathcal{X}(\mathcal{V}) \);

(ii) if \( \xi_1, \xi_2 \in \mathcal{X}(\tilde{M}) \) then \( [\xi^H_1, \xi^H_2] = [\xi_1, \xi_2]^H, [\xi^V_1, \xi^V_2] = 0 \) and \( [\xi^H_1, \xi^V_2] \in \mathcal{X}(\mathcal{V}) \).

**Proof:** (i) Take \( H_1, H_2 \in \mathcal{X}(\mathcal{H}) \). Locally, \( H_j = H^\mu_j \delta_\mu \); hence
\[
[H_1, H_2] = \left( H^\mu_1 \delta_\nu \left( H^\mu_2 \right) - H^\mu_2 \delta_\nu \left( H^\mu_1 \right) \right) \delta_\mu \tag{2.4}
\]
(where we used Lemma 2.1) is horizontal. Analogously, if \( V_1, V_2 \in \mathcal{X}(\mathcal{V}) \) with \( V_j = V^\alpha_j \dot{\delta}_\alpha \), we get
\[
[V_1, V_2] = \left( V^\beta_j \dot{\delta}_\beta \left( V^\alpha_2 \right) - V^\beta_2 \dot{\delta}_\beta \left( V^\alpha_1 \right) \right) \dot{\delta}_\alpha, \tag{2.5}
\]
which is vertical.

(ii) Locally, \( \xi_j = \xi_j^\mu (\partial/\partial z^\mu) \) and \( \xi^H_j = (\xi_j^\mu \circ \pi) \delta_\mu \); so (2.4) yields
\[
[\xi^H_1, \xi^H_2] = \left( \left( \xi^\nu_1 \circ \pi \right) \delta_\nu \left( \xi^\mu_2 \circ \pi \right) - \left( \xi^\nu_2 \circ \pi \right) \delta_\nu \left( \xi^\mu_1 \circ \pi \right) \right) \delta_\mu.
\]
Now \( \delta_\nu (\xi_j^\mu \circ \pi) = (\partial \xi^\mu_j / \partial z^\nu) \circ \pi \); therefore
\[
[\xi^H_1, \xi^H_2] = \left[ \left( \xi^\nu_1 \frac{\partial \xi^\mu_2}{\partial z^\nu} - \xi^\nu_2 \frac{\partial \xi^\mu_1}{\partial z^\nu} \right) \circ \pi \right] \delta_\mu = [\xi_1, \xi_2]^H.
\]
On the other hand, \( \xi_j^V = (\xi_j^\alpha \circ \pi) \dot{\delta}_\alpha \) and \( \dot{\delta}_\beta (\xi_j^\alpha \circ \pi) = 0 \) yield
\[
[\xi^V_1, \xi^V_2] = 0.
\]
Finally,
\[
[\xi^H_1, \xi^V_2] = \left[ \left( \xi^\mu_1 \frac{\partial \xi^\alpha_2}{\partial z^\mu} \right) \circ \pi + \left( \left( \xi^\mu_1 \xi^\beta_2 \circ \pi \right) \Gamma^\alpha_{\beta \mu} \right) \dot{\delta}_\alpha \right],
\]
again by Lemma 2.1.

Note that, as a consequence of (ii), the obvious map of \( \mathcal{X}(\mathcal{V}) \) into \( \mathcal{X}(\mathcal{H}) \) induced by the complex horizontal map \( \Theta : \mathcal{V} \to \mathcal{X} \) is not an isomorphism of Lie algebras; it suffices to remark that \( \Theta(\xi^V) = \xi^H \) for all \( \xi \in \mathcal{X}(\mathcal{M}) \).
3. Torsions and kählerianity

As it may be expected, the next step is the study of the Chern-Finsler connection is to describe its torsion(s) and clarify their geometrical meaning.

The tangent bundle $T^{1,0}M$ (and hence $\tilde{M}$ too) is naturally equipped with a $T^{1,0}\tilde{M}$-valued global $(1,0)$-form, the canonical form

$$\eta = dz^\mu \otimes \delta_\mu + dv^\alpha \otimes \hat{\partial}_\alpha \in \mathcal{X}(\bigwedge^{1,0} \tilde{M} \otimes T^{1,0}\tilde{M}).$$

It is easy to see that as soon as we have a strongly pseudoconvex Finsler metric — and hence the canonical splitting $T^{1,0}M = \mathcal{H} \oplus \mathcal{V}$ — one has

$$\eta = dz^\mu \otimes \delta_\mu + \psi^\alpha \otimes \hat{\partial}_\alpha.$$

Extending as usual the Chern-Finsler connection $D$ to an exterior differential (still denoted by $D$) on $T^{1,0}\tilde{M}$-valued differential forms, it is very natural to consider the torsion $D\eta$ of the connection. Since $\eta$ is a $(1,0)$-form, $D\eta$ splits in the sum of a $(2,0)$-form $\theta$ and a $(1,1)$-form $\tau$. We shall call $\theta$ the $(2,0)$-torsion of the Chern-Finsler connection, and $\tau$ the $(1,1)$-torsion of the Chern-Finsler connection.

Locally, we may write

$$\theta = \theta^\mu \otimes \delta_\mu + \hat{\theta}^\alpha \otimes \hat{\partial}_\alpha \quad \text{and} \quad \tau = \tau^\alpha \otimes \hat{\partial}_\alpha,$$

where, setting $\Gamma^\alpha_{\beta;\mu} = \hat{\partial}_\beta (\Gamma^\alpha_{\mu;\beta})$,

$$\tau^\alpha = \hat{\partial} \psi^\alpha = - \delta_\nu (\Gamma^\alpha_{\mu;\nu}) dz^\mu \wedge d\bar{z}^\nu - \Gamma^\alpha_{\beta;\mu} dz^\mu \wedge \bar{\psi}^\beta;$$

$$\theta^\mu = - dz^\nu \wedge \omega^\mu_\nu = \frac{1}{2} [\Gamma^\mu_{\nu;\sigma} - \Gamma^\mu_{\sigma;\nu}] dz^\sigma \wedge dz^\nu + \Gamma^\mu_{\nu;\sigma} \psi^\gamma \wedge d\bar{z}^\nu; \quad (3.1)$$

and

$$\hat{\theta}^\alpha = \partial \psi^\alpha - \psi^\beta \wedge \omega^\alpha_\beta$$

$$= \frac{1}{2} [\delta_\nu (\Gamma^\alpha_{\mu;\nu}) - \delta_\nu (\Gamma^\alpha_{\nu;\mu})] dz^\mu \wedge d\bar{z}^\nu + [\hat{\partial}_\beta (\Gamma^\alpha_{\mu;\nu}) - \Gamma^\alpha_{\beta;\mu}] \psi^\gamma \wedge d\bar{z}^\nu + \frac{1}{2} [\Gamma^\alpha_{\beta;\gamma} - \Gamma^\alpha_{\gamma;\beta}] \psi^\beta \wedge \bar{\psi}^\gamma = 0, \quad (3.2)$$

by (2.3), (2.1) and (2.2).

One may wonder whether these torsions are the right generalizations of the usual torsion in the hermitian case. The answer is a double yes. First of all, a standard argument using the definitions shows that torsions and covariant derivative are related as usual:

$$\nabla_X Y - \nabla_Y X = [X,Y] + \theta(X,Y),$$

$$\nabla_X \tilde{Y} - \nabla_{\tilde{X}} X = [X,\tilde{Y}] + \tau(X,\tilde{Y}) + \tilde{\tau}(X,\tilde{Y}), \quad (3.3)$$

for any $X, Y \in \mathcal{X}(T^{1,0}\tilde{M})$, where, by definition,

$$\nabla_X \tilde{Y} = \nabla_{\nabla_X Y}.$$
Furthermore, the vanishing of (part of) the (2,0)-torsion can be again interpreted as a Kähler condition — but with some care, because $\theta$ is composed by a horizontal part and a mixed part. To be precise, we shall say that a differential form $\gamma$ on $\tilde{M}$ is horizontal if it vanishes contracted with any $V \in \mathcal{X}(\mathcal{V})$. The decomposition $T^{1,0}\tilde{M} = \mathcal{H} \oplus \mathcal{V}$ induces a projection $p^*_H$ of the differential forms onto the horizontal forms; the horizontal part of a form $\gamma$ is then $p^*_H(\gamma)$.

There is a corresponding projection on the vertical forms, of course, but we shall not need it now because the vertical part of both torsions $\theta$ and $\tau$ is zero. For this reason, the form $\theta - p^*_H(\theta)$ will be called the mixed part of $\theta$. In local coordinates,

$$p^*_H(\theta) = (\Gamma^\sigma_{\nu,\mu} dz^\mu \wedge dz^{\nu'}) \otimes \delta_\sigma$$

and

$$\theta - p^*_H(\theta) = (\Gamma^\sigma_{\nu,\gamma} \psi^\gamma \wedge dz^{\nu'}) \otimes \delta_\sigma.$$ 

The next proposition discusses the meaning of the vanishing of the (2,0)-torsion $\theta$ or of one of its parts.

**Proposition 3.1:** Let $F$ be a strongly pseudoconvex Finsler metric on a complex manifold $M$. Then:

(i) the mixed part of the (2,0)-torsion vanishes iff $F$ comes from a hermitian metric;

(ii) $\theta$ vanishes iff $F$ comes from a hermitian Kähler metric.

**Proof:** (i) The mixed part of the torsion vanishes iff $G^\beta_{\bar{\mu}\gamma} = 0$ for all $\beta$, $\mu$ and $\gamma$. Conjugating, this is equivalent to having $\hat{\partial}_\gamma(G^\beta_{\bar{\mu}}) = \hat{\partial}_{\bar{\gamma}}(G^\beta_{\bar{\mu}}) = 0$, that is $G^\beta_{\bar{\mu}}(v)$ depends only on $\pi(v)$ — and this happens iff $F$ comes from a hermitian metric.

(ii) It follows from (i) and the fact that when $F$ comes from a hermitian metric $g = (g_{\alpha\bar{\beta}})$ one has

$$\Gamma^\alpha_{\beta,\mu} = g^{\tau\alpha} \frac{\partial g_{\beta\tau}}{\partial z^\mu}.$$ 

For this reason we say that a strongly pseudoconvex Finsler metric $F$ is strongly Kähler if the horizontal part of the (2,0)-torsion vanishes, that is iff

$$\forall H, K \in \mathcal{H} \quad \theta(H, K) = 0.$$ 

This is exactly the notion of kählerianity introduced by Rund [Ru2]. However, as we shall see later on (see sections 6 and 7), studying the geometry of a strongly pseudoconvex Finsler metric it turns out that this assumption is too strong and not quite natural. So it is appropriate to introduce two more notions of kählerianity. We shall say that $F$ is Kähler if

$$\forall H \in \mathcal{H} \quad \theta(H, \chi) = 0,$$

and that $F$ is weakly Kähler if

$$\forall H \in \mathcal{H} \quad \langle \theta(H, \chi), \chi \rangle = 0.$$
In local coordinates, $F$ is strongly Kähler iff
\[ \Gamma_{\mu;\nu}^\alpha = \Gamma_{\nu;\mu}^\alpha; \]
it is Kähler iff
\[ \Gamma_{\mu;\nu}^\alpha v^\mu = \Gamma_{\nu;\mu}^\alpha v^\mu; \]
it is weakly Kähler iff
\[ G_{\alpha} [\Gamma_{\mu;\nu}^\alpha - \Gamma_{\nu;\mu}^\alpha] v^\mu = 0, \]
that is iff
\[ 0 = [G_{\mu;\nu} - G_{\nu;\mu} + G_{\nu\sigma} \Gamma_{;\mu}^\sigma] v^\mu = [G_{\mu;\nu} - G_{\nu;\mu} + G_{\nu\sigma} \Gamma_{;\mu}^\sigma] v^\mu \bar{v}^\tau. \]

In particular, if $F$ comes from a hermitian metric then these three conditions are all equivalent to the usual Kähler condition, because $G_{\nu\sigma} \bar{\tau} \equiv 0$ for a Finsler metric coming from a hermitian metric.

There are other characterizations of strongly Kähler Finsler metrics. To $F$ we may associate the fundamental form
\[ \Phi = iG_{\alpha\bar{\beta}} \, dz^\alpha \wedge \bar{dz}^\beta, \]
which is a well-defined real $(1,1)$-form on $\tilde{M}$. Then the strong Kähler condition is equivalent to the vanishing of the horizontal part of $d\Phi$. To express it more clearly, set
\[ d_H = p_H^* \circ d, \quad \partial_H = p_H^* \circ \partial \quad \text{and} \quad \bar{\partial}_H = p_H^* \circ \bar{\partial}, \]
so that again $d_H = \partial_H + \bar{\partial}_H$.

**Theorem 3.2:** Let $F$ be a strongly pseudoconvex Finsler metric on a complex manifold $M$. Then the following assertions are equivalent:

(i) $F$ is a strongly Kähler Finsler metric;
(ii) $\nabla_H K - \nabla_K H = [H, K]$ for all $H, K \in \mathcal{X}(\mathcal{H})$;
(iii) $d_H \Phi = 0$;
(iv) $\partial_H \Phi = 0$;
(v) for any $p_0 \in M$ there is a neighbourhood $U$ of $p_0$ in $M$ and a real-valued function $\phi \in C^\infty(\pi^{-1}(U))$ such that $\Phi = i\partial_H \bar{\partial}_H \phi$ on $\pi^{-1}(U)$.

**Proof:**
(i) $\iff$ (ii) follows from (3.3).

(iii) $\iff$ (iv) holds simply because $\Phi$ is a real $(1,1)$-form.

(iv) $\iff$ (i). Indeed, (2.1) yields
\[ \partial \Phi(X, Y, Z) = i\langle \theta(X, Y), Z \rangle \]
for all $X, Y, Z \in T^{1,0} \tilde{M}$; hence $\partial_H \Phi$ vanishes iff $p_H^* \circ \theta$ vanishes, that is iff $F$ is strongly Kähler.

(v) $\implies$ (iv) follows from Lemma 2.1.(i).
(iii) $\implies$ (v). Let $\gamma$ be any horizontal form. In local coordinates, defined on a coordinate neighbourhood of the form $\pi^{-1}(U)$, one has

$$\gamma|_{(p,v)} = \gamma_{AB}(p; v)\,dz^A \wedge d\overline{z}^B,$$

for suitable multi-indices $A$ and $\overline{B}$. On $U$ we may then consider the family of forms

$$\gamma_v|_p = \gamma_{AB}(p; v)\,dz^A \wedge d\overline{z}^B,$$

where here $\{dz^j\}$ is the dual frame of $\{\partial/\partial z^j\}$; in other words, we are considering the $v$-coordinates just as parameters.

The gist is that the following formula holds:

$$(d_H \gamma)_v = d(\gamma_v).$$

Then we may now apply the Dolbeault and Serre theorems (with parameters) to $\Phi_v$ in a possibly smaller neighbourhood of $p_0$ — still denoted by $U$ — to get a function $\phi_v \in C^\infty(U, \mathbb{R})$ depending smoothly on $v$ such that $\Phi_v = i\partial\bar{\partial}\phi_v$. Then setting

$$\phi(p; v) = \phi_v(p)$$

we get $\Phi = i\partial_H \bar{\partial} \phi$, as required.

¿From this point of view, a strongly pseudoconvex Finsler metric is Kähler iff

$$d_H \Phi(\cdot, \chi, \cdot) \equiv 0,$$

and it is weakly Kähler iff

$$d_H \Phi(\cdot, \chi, \bar{\chi}) \equiv 0.$$

We end this section pointing out that also the vanishing of the $(1,1)$-torsion $\tau$ has a nice geometric meaning:

**Proposition 3.3:** The $(1,1)$-torsion $\tau$ vanishes iff the frame $\{\delta_\mu, \dot{\alpha}\}$ is holomorphic.

**Proof:** Indeed the frame $\{\delta_\mu, \dot{\alpha}\}$ is holomorphic iff its dual coframe $\{dz^\mu, \psi^\alpha\}$ is, which happens iff the forms $\psi^\alpha$ are holomorphic, that is iff $\tau^\alpha = \partial\bar{\partial}\psi^\alpha = 0$ for $\alpha = 1, \ldots, n$. □

4. The curvature tensor

The curvature tensor $R: \mathcal{X}(T^{1,0}\tilde{M}) \to \mathcal{X}(\Lambda^2(T^*_c\tilde{M}) \otimes T^{1,0}\tilde{M})$ of the Chern-Finsler connection is given by $R = D \circ D$, that is

$$\forall X \in \mathcal{X}(T^{1,0}\tilde{M}) \quad R_X = D(DX).$$

Analogously we have the curvature operator $\Omega \in \mathcal{X}(\Lambda^2(T^*_c\tilde{M}) \otimes \Lambda^{1,0}\tilde{M} \otimes T^{1,0}\tilde{M})$ defined by (cf. also [K3])

$$\Omega(X,Y)Z = R_Z(X,Y).$$
Locally, $\Omega$ is given by
\[
\Omega = \Omega^\alpha_\beta \otimes \left[ dz^\beta \otimes \delta_\alpha + \psi^\beta \otimes \dot{\theta}_\alpha \right],
\]
where
\[
\Omega^\alpha_\beta = d\omega^\alpha_\beta - \omega^\gamma_\beta \wedge \omega^\alpha_\gamma.
\]

Decomposing $\Omega$ into types, we get
\[
\Omega = \Omega' + \Omega'',
\]
where $\Omega'$ is a (2,0)-form and $\Omega''$ a (1,1)-form. Locally,
\[
(\Omega')^\alpha_\beta = \partial_\alpha \omega^\beta - \omega^\gamma_\beta \wedge \omega^\alpha_\gamma,
\]
\[
(\Omega'')^\alpha_\beta = \overline{\partial} \omega^\alpha_\beta.
\]
\[\Omega\] has no (0,2)-components because the connection forms are (1,0)-forms. Actually, even $\Omega'$ vanishes: indeed, by definition
\[
\omega^\beta_\alpha = G^{\tau \beta} \partial G^\alpha_{\bar{\tau}}.
\]
So
\[
\partial \omega^\beta_\alpha = \partial G^{\tau \beta} \wedge \partial G^\alpha_{\bar{\tau}} = -G^{\tau \mu} G^{\bar{\sigma} \beta} \partial G^\mu_{\bar{\sigma}} \wedge \partial G^\alpha_{\bar{\tau}}
\]
\[
= (G^{\tau \mu} \partial G^\alpha_{\bar{\tau}}) \wedge (G^{\bar{\sigma} \beta} \partial G^\mu_{\bar{\sigma}}) = \omega^\mu_\alpha \wedge \omega^\beta_{\mu}.
\]
So $\Omega = \Omega''$ and
\[
\Omega^\alpha_\beta = \overline{\partial} \omega^\alpha_\beta,
\]
exactly as in the hermitian case.

The relation between curvature and covariant derivatives is the usual one:
\[
\nabla_X \nabla_Y - \nabla_Y \nabla_X = \nabla_{[X,Y]}
\]
\[
\nabla_X \nabla_{\bar{Y}} - \nabla_{\bar{Y}} \nabla_X = \nabla_{[X,\bar{Y}]} + \Omega(X,\bar{Y});
\]
\[
\nabla_{\bar{X}} \nabla_{\bar{Y}} - \nabla_{\bar{Y}} \nabla_{\bar{X}} = \nabla_{[\bar{X},\bar{Y}]};
\]
for any $X, Y \in \mathcal{X}(T^{1,0}\bar{M})$.

We can also recover the Bianchi identities in this setting:

**Proposition 4.1:** Let $D: \mathcal{X}(T^{1,0}\bar{M}) \rightarrow \mathcal{X}(T^{*}_c\bar{M} \otimes T^{1,0}\bar{M})$ be the complex linear connection on $\bar{M}$ induced by a good complex vertical connection. Then
\[
D\theta = \eta^H \wedge \Omega,
\]
\[
D\tau = \eta^V \wedge \Omega,
\]
\[
D\Omega = 0,
\]
where $\eta^H = dz^\mu \otimes \delta_\mu$ and $\eta^V = \psi^\alpha \otimes \dot{\theta}_\alpha$.

**Proof:** It suffices to compute. First of all,
\[
\overline{\partial} \theta^\mu = dz^\nu \wedge \overline{\partial} \omega^\mu_\nu = dz^\nu \wedge \Omega^\mu_\nu,
\]
\[
\partial \theta^\mu + \theta^\mu \wedge \omega^\mu_\nu = dz^\nu \wedge \partial \omega^\mu_\nu - dz^\nu \wedge \omega^\mu_\nu = 0,
\]
by (4.1), and so $D\theta = \eta^H \wedge \Omega$. Next
\[
\bar{\partial}_\tau \alpha = 0,
\]
\[
\partial \tau^\alpha + \tau^\beta \wedge \omega^\alpha_\beta = \partial \bar{\partial} \psi^\alpha + \bar{\partial} \psi^\beta \wedge \omega^\alpha_\beta = -\bar{\partial} \partial \psi^\alpha + \bar{\partial} \psi^\beta \wedge \omega^\alpha_\beta = \psi^\beta \wedge \bar{\partial} \omega^\alpha_\beta,
\]
by (3.2), and so $D\tau = \eta^V \wedge \Omega$. Finally, $\bar{\partial} \Omega^\alpha_\beta = 0$ and
\[
\partial \Omega^\alpha_\beta - \omega^\gamma_\beta \wedge \Omega^\alpha_\gamma + \Omega^\alpha_\beta \wedge \omega^\alpha_\gamma = \partial \bar{\partial} \omega^\alpha_\beta - \omega^\gamma_\beta \wedge \bar{\partial} \omega^\alpha_\gamma + \bar{\partial} \omega^\gamma_\beta \wedge \omega^\alpha_\gamma = -\bar{\partial}(\partial \omega^\alpha_\beta - \omega^\gamma_\beta \wedge \omega^\alpha_\gamma) = 0,
\]
by (4.1).

In local coordinates, the curvature operator is given by
\[
\Omega^\alpha_\beta = R^\alpha_{\beta;\mu\nu} dz^\mu \wedge dz^\nu + R^\alpha_{\beta;\delta;\nu} \psi^\delta \wedge dz^\nu + R^\alpha_{\beta;\gamma;\mu} dz^\mu \wedge \bar{\psi}^\gamma + R^\alpha_{\beta;\delta;\gamma} \psi^\delta \wedge \bar{\psi}^\gamma,
\]
where
\[
\begin{align*}
R^\alpha_{\beta;\mu\nu} &= -\delta^\alpha_\nu (\Gamma^\alpha_{\beta;\mu}) - \Gamma^\alpha_{\beta\sigma} \delta^\sigma_\nu (\Gamma^\sigma_{\beta;\mu}), \\
R^\alpha_{\beta;\delta;\nu} &= -\delta^\alpha_\nu (\Gamma^\alpha_{\beta\delta}) = R^\alpha_{\beta\delta;\nu}, \\
R^\alpha_{\beta;\gamma;\mu} &= -\delta^\alpha_\gamma (\Gamma^\alpha_{\beta;\mu}) - \Gamma^\alpha_{\beta\sigma} \Gamma^\sigma_{\beta;\mu}, \\
R^\alpha_{\beta;\delta;\gamma} &= -\delta^\alpha_\gamma (\Gamma^\alpha_{\beta\delta}) = R^\alpha_{\beta\delta;\gamma}.
\end{align*}
\]
(4.2)

In particular, since
\[
(D\tau)^\alpha = (\eta^V \wedge \Omega)^\alpha = \psi^\sigma \wedge \Omega^\sigma_\alpha = R^\alpha_{\sigma;\mu\nu} \psi^\sigma \wedge dz^\mu \wedge dz^\nu + R^\alpha_{\sigma;\delta;\nu} \psi^\sigma \wedge \psi^\delta \wedge dz^\nu
+ R^\alpha_{\sigma;\gamma;\mu} \psi^\sigma \wedge dz^\mu \wedge \bar{\psi}^\gamma + R^\alpha_{\sigma;\delta;\gamma} \psi^\sigma \wedge \psi^\delta \wedge \bar{\psi}^\gamma
= -R^\alpha_{\sigma;\mu\nu} dz^\mu \wedge \psi^\sigma \wedge dz^\nu - R^\alpha_{\sigma;\gamma;\mu} dz^\mu \wedge \psi^\sigma \wedge \bar{\psi}^\gamma,
\]
the vanishing of $\tau$ implies the vanishing of most of the curvature.

Another consequence of (4.2) is an unexpected relation between $\Omega$ and $\tau$:

**Lemma 4.2:** Let $D$ be the Chern-Finsler connection associated to a strongly pseudoconvex Finsler metric $F$ on a complex manifold $M$. Then
\[
\tau = \Omega(\cdot, \cdot)_t.
\]

**Proof:** Recalling (4.2), (1.3), (1.4) and
\[
\Gamma^\alpha_{\beta;\mu} v^\beta = \Gamma^\alpha_{\mu}, \quad \Gamma^\alpha_{\beta;\gamma} v^\beta = 0,
\]
we have
\[
\begin{align*}
R^\alpha_{\beta;\mu\nu} v^\beta &= -\delta^\alpha_\nu (\Gamma^\alpha_{\beta;\mu}), \\
R^\alpha_{\beta\delta;\nu} v^\beta &= 0, \\
R^\alpha_{\beta;\gamma;\mu} v^\beta &= -\Gamma^\alpha_{\beta;\mu}, \\
R^\alpha_{\beta;\delta;\gamma} v^\beta &= 0,
\end{align*}
\]
and the assertion follows from (3.1).
5. Holomorphic curvature

One of the most useful concepts in hermitian geometry is the notion of holomorphic sectional curvature. To find the correct analogue in our setting, we first need a closer look to the horizontal part of the curvature operator. We define the horizontal curvature tensor $R$ by

$$R_v(H, K, L, M) = \langle \Omega(H, K)L, M \rangle_v$$

for all $H, K, L, M \in \mathcal{H}_v$ and $v \in \hat{M}$. In local coordinates,

$$R(H, K, L, M) = G_{\sigma\overline{\beta}} R_{\alpha;\mu\nu}^\sigma H^\mu K^\nu L^\alpha M^\beta.$$ 

The symmetries of $R$ are easily described:

**Proposition 5.1:** Take $v \in \hat{M}$ and $H, K, L, M \in \mathcal{H}_v$. Then

$$R(K, H, L, M) = -R(H, K, L, M);$$

$$R(K, L, M, \overline{L}) = R(H, \overline{K}, L, \overline{M}).$$

Furthermore, if $\bar{\partial}_H \theta = 0$ we also have

$$R(L, K, H, \overline{M}) = R(H, K, L, \overline{M}) = R(H, \overline{M}, L, \overline{K}).$$

**Proof:** (5.1) follows immediately from the observation $\Omega_\beta^\alpha(K, H) = -\Omega_\beta^\alpha(H, K)$. To prove (5.2), we start from

$$\Omega_\beta^\alpha = \bar{\partial} \omega_\beta^\alpha = \bar{\partial}(G^{\bar{\tau}\alpha} \partial G_{\beta\bar{\tau}}) = -G^{\bar{\tau}\mu} G^{\bar{\nu}\alpha} \bar{\partial} G_{\mu\bar{\nu}} \wedge \partial G_{\beta\bar{\tau}} + G^{\bar{\tau}\alpha} \bar{\partial} \partial G_{\beta\bar{\tau}};$$
in particular,

$$G_{\alpha\gamma} \Omega_\beta^\alpha = -G^{\bar{\tau}\mu} \bar{\partial} G_{\mu\bar{\gamma}} \wedge \partial G_{\beta\bar{\tau}} + \bar{\partial} \partial G_{\bar{\beta}\bar{\gamma}}.$$

On the other hand,

$$\Omega_\gamma^\beta = G^{\bar{\tau}\gamma} G^{\bar{\nu}\mu} \bar{\partial} G_{\mu\bar{\gamma}} \wedge \partial G_{\nu\bar{\tau}} - G^{\bar{\gamma}\tau} \bar{\partial} \partial G_{\nu\bar{\tau}};$$
hence

$$G_{\alpha\gamma} \Omega_\beta^\alpha = -G_{\beta\bar{\alpha}} \Omega_\gamma^\beta.$$ 

In our case, this means that

$$R(K, L, M, \overline{L}) = G_{\alpha\gamma} \Omega_\beta^\alpha(K, L) M^\beta \overline{L}^\gamma = -G_{\beta\gamma} \Omega_\alpha^\beta(K, L) M^\beta \overline{L}^\gamma$$

$$= G_{\alpha\gamma} \Omega_\beta^\alpha(H, \overline{K}) L^\gamma \overline{M}^\beta$$

$$= R(H, K, L, \overline{M}),$$

and (5.2) is proved.

Now, (5.3). First of all, $\bar{\partial}_H \theta = p^*_H(D\theta)$, because we saw that the (2,0)-part of $D\theta$ vanishes. Proposition 4.1 says that $D\theta = \eta^H \wedge \Omega$; in local coordinates,

$$(\eta^H \wedge \Omega)^\alpha = dz^\alpha \wedge \Omega^\alpha = R_{\sigma;\mu\nu}^\alpha dz^\sigma \wedge dz^\mu \wedge dz^\nu + R_{\alpha;\delta\bar{\nu}}^\alpha dz^\sigma \wedge \psi^\delta \wedge dz^\bar{\nu}$$

$$+ R_{\alpha;\gamma;\mu}^\alpha dz^\sigma \wedge dz^\mu \wedge \overline{\psi}^\gamma + R_{\alpha;\delta\bar{\gamma}}^\alpha dz^\sigma \wedge \psi^\delta \wedge \overline{\psi}^\bar{\gamma};$$

in particular, $\bar{\partial}_H \theta = 0$ iff $R_{\sigma;\mu\nu}^\alpha = R_{\mu;\sigma\nu}^\alpha$. Then

$$R(L, K, H, \overline{M}) = G_{\alpha\gamma} R_{\sigma;\mu\nu}^\alpha L^\mu K^\nu H^\sigma \overline{M}^\tau = G_{\alpha\gamma} R_{\mu;\sigma\nu}^\alpha L^\mu K^\nu H^\sigma \overline{M}^\tau = R(H, K, L, \overline{M}).$$

Finally,

$$R(H, \overline{M}, L, K) = R(M, H, K, \overline{L}) = R(K, \overline{H}, M, \overline{L}) = R(H, K, L, \overline{M}).$$

$\square$
We remark that (5.2) is equivalent to
\[
\langle \Omega(H, K) L, M \rangle = \langle L, \Omega(K, H) M \rangle
\]
for all \(H, K, L, M \in \mathcal{H}\).

Now, one possible approach to the holomorphic sectional curvature is to consider the (horizontal) holomorphic flag curvature \(\tilde{K}_F(H)\) of \(F\) along a horizontal vector \(H \in \mathcal{H}_v\):
\[
\tilde{K}_F(H) = \frac{2}{\langle H, H \rangle_v^2} R(H, \overline{H}, H, \overline{H}).
\]

Exactly as in the hermitian case, if \(\bar{\partial}_H \theta = 0\) then the holomorphic flag curvature completely determines the horizontal curvature tensor:

**Proposition 5.2:** Let \(R, S : \mathcal{H}_v \times \mathcal{H}_v \times \mathcal{H}_v \times \mathcal{H}_v \rightarrow \mathbb{C}\) be two quadrilinear maps satisfying (5.2) and (5.3). Assume that
\[
\forall H \in \mathcal{H}_v \quad R(H, \overline{H}, H, \overline{H}) = S(H, \overline{H}, H, \overline{H}).
\]
Then \(R \equiv S\).

The proof is very similar to the traditional one for hermitian metrics; see [KN] and [AP4] for the details. We do not discuss it here because, from a certain point of view, the holomorphic flag curvature is not the right generalization of the holomorphic sectional curvature. In fact, roughly speaking, it contains too many informations. Requiring, for instance, that the holomorphic flag curvature is constant means imposing very strong constraints on the behavior of the complex Finsler metric, constraints that are somewhat beyond the geometry of the metric which lives naturally on the tangent bundle of the manifold. Of course, one may study such requirements, but in this case the theory seems to be a standard consequence of the hermitian geometry of vector bundles without significant application to the function theory of the manifold.

A different notion appears to be a more appropriate tool for the applications in complex geometry (see [K3], [AP1], [AP2], [AP3], and sections 6 and 7 where we discuss variational formulas; cf. also [Ch] and [BC] for similar arguments in the real case). Namely, let \(F : T^{1,0} M \rightarrow \mathbb{R}^+\) be a strongly pseudoconvex Finsler metric on a complex manifold \(M\), and take \(v \in \tilde{M}\). Then the holomorphic curvature \(K_F(v)\) of \(F\) along \(v\) is given by
\[
K_F(v) = \tilde{K}_F(\chi(v)) = \frac{2}{G(v)^2} R(\chi(v), \overline{\chi}(v), \chi(v), \overline{\chi}(v)).
\]

Clearly,
\[
K_F(\zeta v) = K_F(v)
\]
for all \(\zeta \in \mathbb{C}^\ast\); so this is the holomorphic curvature discussed by Kobayashi [K3]. Note that, by Proposition 5.1, the holomorphic curvature is necessarily real-valued.

In local coordinates we get
\[
K_F = -\frac{2}{G^2} G_\alpha \delta_\nu (\Gamma^\alpha_{\mu \nu} v^\mu v^\nu).
\]
If $F$ comes from a hermitian metric, (5.6) gives exactly the classical holomorphic sectional curvature. Furthermore, our definition recovers another important geometrical characterization of the holomorphic sectional curvature, and provides a firm link with the theory of invariant metrics on complex manifolds (cf. [AP2]). Wu [Wu] has shown that for a hermitian metric $g$ on a complex manifold $M$, the holomorphic sectional curvature of $g$ along $v \in T^1,0_p M$ is the maximum value attained by the Gaussian curvature at the origin of the pull-back metric $\varphi^*g$ when $\varphi$ varies among the holomorphic maps from the unit disk $\Delta \subset \mathbb{C}$ into $M$ with $\varphi(0) = p$ and $\varphi'(0) = \lambda v$ for some $\lambda \in \mathbb{C}^*$. Well, this is true in our case too:

**Theorem 5.3:** Let $F: T^{1,0}M \to \mathbb{R}^+$ be a strongly pseudoconvex Finsler metric on a complex manifold $M$, and take $p \in M$ and $v \in \tilde{M}_p$. Then

$$K_F(v) = \sup \{K(\varphi^*G)(0)\},$$

where $K(\varphi^*G)(0)$ is the Gaussian curvature at the origin of the pull-back metric $\varphi^*G$, and the supremum is taken with respect to the family of all holomorphic maps $\varphi: \Delta \to M$ with $\varphi(0) = p$ and $\varphi'(0) = \lambda v$ for some $\lambda \in \mathbb{C}^*$.

For the proof, see [AP2]. We also recall that this variational interpretation of the holomorphic curvature makes sense for upper semicontinuous Finsler metrics, and has been previously investigated in geometric function theory (see [W], [R] and [S]).

This ends the general presentation of the setting we suggest for studying complex Finsler geometry. To substantiate this suggestion, in the next sections we give a few applications: the variation formulas and a close look to manifold with constant holomorphic curvature.

### 6. First variation of the length integral and geodesics

Let $F: T^{1,0}M \to \mathbb{R}^+$ be a strongly pseudoconvex Finsler metric on a complex manifold $M$. To $F$ we may associate a function $F^\circ: T_{\mathbb{R}}M \to \mathbb{R}^+$ just by setting

$$\forall u \in T_{\mathbb{R}}M \quad F^\circ(u) = F(u_\circ),$$

where $u \mapsto u_\circ = (u - iJu)/2$ is the standard isomorphism between $T_{\mathbb{R}}M$ and $T^{1,0}M$ ($J$ is the complex structure on $T_{\mathbb{R}}M$). $F^\circ$ satisfies all the properties defining a real Finsler metric, but perhaps the indicatrices are not necessarily strongly convex. Nevertheless, we may use it to measure the length of curves, and so to define geodesics; and one of the main results of this section is a theorem ensuring the local existence and uniqueness of geodesics for weakly Kähler Finsler metrics only under the strong pseudoconvexity hypothesis — a striking by-product of the complex structure.

Let us fix the notations needed to study variations of the length integral in this setting. The idea is, as usual, to pull back the connection along a curve; but since our connection lives on the tangent-tangent bundle, the details are a bit delicate.

A *regular curve* $\sigma: [a, b] \to M$ is a $C^1$ curve with never vanishing tangent vector. Here, we mean the tangent vector in $T^{1,0}M$, obtained via the canonical isomorphism with $T_{\mathbb{R}}M$: so we set

$$\dot{\sigma}(t) = d^{1,0}\sigma_t \left( \frac{d}{dt} \right) = \frac{d\sigma^\alpha}{dt}(t) \frac{\partial}{\partial z^\alpha}|_{\sigma(t)},$$
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where \( d^{1,0} \) is the composition of the differential with the projection of \( T_C M \) onto \( T^{1,0} M \) associated to the splitting \( T_C M = T^{1,0} M \oplus T^{0,1} M \).

The length of a regular curve \( \sigma \) with respect to the strongly pseudoconvex Finsler metric \( F \) is given by

\[
L(\sigma) = \int_a^b F(\dot{\sigma}(t)) \, dt,
\]

exactly as in the hermitian case.

A geodesic for \( F \) is a curve which is a critical point of the length functional. To be more precise, let \( \sigma_0: [a, b] \to M \) be a regular curve with \( F(\dot{\sigma}_0) \equiv c_0 > 0 \). A regular variation of \( \sigma_0 \) is a \( C^1 \) map \( \Sigma: (-\varepsilon, \varepsilon) \times [a, b] \to M \) such that

(a) \( \sigma_0(t) = \Sigma(0, t) \) for all \( t \in [a, b] \);
(b) for every \( s \in (-\varepsilon, \varepsilon) \) the curve \( \sigma_s(t) = \Sigma(s, t) \) is a regular curve in \( M \);
(c) \( F(\dot{\sigma}_s) \equiv c_s > 0 \) for every \( s \in (-\varepsilon, \varepsilon) \).

A regular variation \( \Sigma \) is fixed if it moreover satisfies

(d) \( \sigma_s(a) = \sigma_0(a) \) and \( \sigma_s(b) = \sigma_0(b) \) for every \( s \in (-\varepsilon, \varepsilon) \).

If \( \Sigma \) is a regular variation of the curve \( \sigma_0 \), we define the function \( \ell_\Sigma: (-\varepsilon, \varepsilon) \to \mathbb{R}^+ \) by

\[
\ell_\Sigma(s) = L(\sigma_s).
\]

We shall say that a regular curve \( \sigma_0 \) is a geodesic for \( F \) iff

\[
\frac{d\ell_\Sigma}{ds}(0) = 0
\]

for all fixed regular variations \( \Sigma \) of \( \sigma_0 \).

Our first goal is to write the first variation of the length functional; we shall then find the differential equation satisfied by the geodesics (see also [AP1]).

Let \( \Sigma: (-\varepsilon, \varepsilon) \times [a, b] \to M \) be a regular variation of a regular curve \( \sigma_0: [a, b] \to M \). Let \( p: \Sigma^*(T^{1,0} M) \to (-\varepsilon, \varepsilon) \times [a, b] \) be the pull-back bundle, and \( \gamma: \Sigma^*(T^{1,0} M) \to T^{1,0} M \) be the bundle map such that the diagram

\[
\begin{array}{ccc}
\Sigma^*(T^{1,0} M) & \xrightarrow{\gamma} & T^{1,0} M \\
p \downarrow & & \downarrow \pi \\
(-\varepsilon, \varepsilon) \times [a, b] & \xrightarrow{\Sigma} & M
\end{array}
\]

commutes.

Two particularly important sections of \( \Sigma^*(T^{1,0} M) \) are

\[
T = \gamma^{-1} \left( d^{1,0} \Sigma \left( \frac{\partial}{\partial t} \right) \right) = \frac{\partial \Sigma^\alpha}{\partial t} \frac{\partial}{\partial z^\alpha},
\]

and

\[
U = \gamma^{-1} \left( d^{1,0} \Sigma \left( \frac{\partial}{\partial s} \right) \right) = \frac{\partial \Sigma^\alpha}{\partial s} \frac{\partial}{\partial z^\alpha}.
\]
the restriction of $U$ to $s = 0$ is the variation vector of the variation $\Sigma$. Note that setting $\Sigma^* \tilde{M} = \gamma^{-1}(\tilde{M})$, we have $T \in \mathcal{X}(\Sigma^* \tilde{M})$ and

$$T(s, t) = \gamma^{-1}(\dot{\sigma}_s(t)).$$

Now we pull-back $T^{1,0} \tilde{M}$ over $\Sigma^* \tilde{M}$ by using $\gamma$, obtaining the commutative diagram

\[
\begin{array}{ccc}
\gamma^*(T^{1,0} \tilde{M}) & \xrightarrow{\tilde{\gamma}} & T^{1,0} \tilde{M} \\
\downarrow & & \downarrow \\
\Sigma^* \tilde{M} & \xrightarrow{\gamma} & \tilde{M} \\
\downarrow & & \downarrow \\
(-\varepsilon, \varepsilon) \times [a, b] & \xrightarrow{\Sigma} & M
\end{array}
\]

note that $\gamma^*(T^{1,0} \tilde{M})$ is a complex vector bundle over a real manifold. The bundle map $\tilde{\gamma}$ induces a hermitian structure on $\gamma^*(T^{1,0} \tilde{M})$ by

$$\forall X, Y \in \gamma^*(T^{1,0} \tilde{M})_v \quad \langle X, Y \rangle_v = \langle \tilde{\gamma}(X), \tilde{\gamma}(Y) \rangle_{\gamma(v)}.$$ 

Analogously, the Chern connection $D$ gives rise to a $(1,0)$-connection

$$D^* : \mathcal{X}(\gamma^*(T^{1,0} \tilde{M})) \to \mathcal{X}(T_{\mathbb{C}}^*(\Sigma^* \tilde{M}) \otimes \gamma^*(T^{1,0} \tilde{M})),
$$

where $T_{\mathbb{C}}^*(\Sigma^* \tilde{M}) = T_{\mathbb{R}}^*(\Sigma^* \tilde{M}) \otimes \mathbb{C}$, by setting

\[
\begin{align*}
\nabla^*_{X,Y} &= \tilde{\gamma}^{-1}\left(\nabla_{d^{1,0}\gamma(X)} \tilde{\gamma}(Y)\right), \\
\nabla^*_{X} &= \tilde{\gamma}^{-1}\left(\nabla_{d^{1,0}\gamma(X)} \tilde{\gamma}(Y)\right),
\end{align*}
\]

for all $X \in T_{\mathbb{R}}(\Sigma^* \tilde{M})$ and $Y \in \mathcal{X}(\gamma^*(T^{1,0} \tilde{M}))$. In particular we have

\[
X \langle Y, Z \rangle = X(\langle \tilde{\gamma}(Y), \tilde{\gamma}(Z) \rangle) = d\gamma(X)(\langle \tilde{\gamma}(Y), \tilde{\gamma}(Z) \rangle)
\]

\[
= (d^{1,0}\gamma(X) + \overline{d^{1,0}\gamma(X)})(\langle \tilde{\gamma}(Y), \tilde{\gamma}(Z) \rangle)
\]

\[
= \langle \nabla^*_X Y, Z \rangle + \langle Y, \nabla^*_X Z \rangle + \overline{\langle \nabla^*_X Y, Z \rangle} + \langle Y, \nabla^*_X Z \rangle,
\]

for all $X \in T_{\mathbb{R}}(\Sigma^* \tilde{M})$ and $Y, Z \in \mathcal{X}(\gamma^*(T^{1,0} \tilde{M}))$.

We may also decompose $T_{\mathbb{R}}(\Sigma^* \tilde{M}) = \mathcal{H}^* \oplus \mathcal{V}^*$, where as usual a local real frame for $\mathcal{V}^*$ is given by $\{\partial_{\alpha}, i\partial_{\alpha}\}$, and a local frame for $\mathcal{H}^*$ is given by

$$\delta_t = \partial_t - (\Gamma^\mu_{\alpha} \circ \gamma) \frac{\partial \Sigma^\alpha}{\partial t} \partial_{\mu}, \quad \delta_s = \partial_s - (\Gamma^\mu_{\alpha} \circ \gamma) \frac{\partial \Sigma^\alpha}{\partial s} \partial_{\mu},$$
where \( \partial_t = \partial/\partial t \) and \( \partial_s = \partial/\partial s \). Therefore, setting \( T^H = d^{1,0} \gamma (\partial_t) \) and \( U^H = d^{1,0} \gamma (\partial_s) \), we have

\[
T^H (v) = \frac{\partial \Sigma}{\partial t} (s, t) \delta_\mu |_{\gamma (v)} = \chi_{\gamma (v)} (\dot{\sigma}_s (t)) \in \mathcal{H}_{\gamma (v)}
\]

and

\[
U^H (v) = \frac{\partial \Sigma}{\partial s} (s, t) \delta_\mu |_{\gamma (v)} = \chi_{\gamma (v)} (\gamma (U(s, t))) \in \mathcal{H}_{\gamma (v)}.
\]

for all \( v \in \Sigma \cdot \tilde{M}(s, t) \); they are the horizontal lifts of \( \gamma (T) \) and \( \gamma (U) \) respectively. In particular,

\[
T^H (\gamma^{-1} (\dot{\sigma}_s)) = \chi (\dot{\sigma}_s).
\]

If we take \( v \in (\Sigma \cdot \tilde{M})(s, t) \), then

\[
d^{1,0} \gamma (T_{\mathbb{R}} (\Sigma \cdot M)) \subset T^{1,0} \tilde{M} \quad \text{and} \quad \tilde{\gamma} (\gamma^*(T^{1,0} \tilde{M})_v) = T^{1,0} \tilde{M}.
\]

Therefore we also have a bundle map \( \Xi : T_{\mathbb{R}} (\Sigma \cdot \tilde{M}) \to \gamma^*(T^{1,0} \tilde{M}) \) such that the diagram

\[
\begin{array}{ccc}
T_{\mathbb{R}} (\Sigma \cdot \tilde{M}) & \xrightarrow{\Xi} & \gamma^*(T^{1,0} \tilde{M}) \\
\downarrow \cong & & \downarrow \tilde{\gamma} \\
T^{1,0} \tilde{M}
\end{array}
\]

commutes. Using \( \Xi \) we may prove three final formulas:

\[
\tilde{\gamma} (\nabla^*_X \Xi (Y) - \nabla^*_Y \Xi (X)) = \nabla^{d^{1,0} \gamma (X)} (d^{1,0} \gamma (Y) - \nabla^{d^{1,0} \gamma (Y)} d^{1,0} \gamma (X))
\]

\[
= [d^{1,0} \gamma (X), d^{1,0} \gamma (Y)] + \theta (d^{1,0} \gamma (X), d^{1,0} \gamma (Y)),
\]

for all \( X, Y \in \mathcal{X} (T_{\mathbb{R}} (\Sigma \cdot \tilde{M}) \); \)

\[
\tilde{\gamma} \circ (\nabla^*_X \nabla^*_Y - \nabla^*_Y \nabla^*_X) = (\nabla^{d^{1,0} \gamma (X)} \nabla^{d^{1,0} \gamma (Y)} - \nabla^{d^{1,0} \gamma (Y)} \nabla^{d^{1,0} \gamma (X)}) \circ \tilde{\gamma}
\]

\[
= \nabla^{[d^{1,0} \gamma (X), d^{1,0} \gamma (Y)]} \circ \tilde{\gamma},
\]

and

\[
\tilde{\gamma} \circ (\nabla^*_X \nabla^*_Y - \nabla^*_Y \nabla^*_X) = (\nabla^*_{[d^{1,0} \gamma (X), d^{1,0} \gamma (Y)]} + \Omega (d^{1,0} \gamma (X), d^{1,0} \gamma (Y)) \circ \tilde{\gamma},
\]

for all \( X, Y \in T_{\mathbb{R}} (\Sigma \cdot \tilde{M}) \).

We are now able to prove the first variation formula for weakly Kähler Finsler metrics:

**Theorem 6.1:** Let \( F : T^{1,0} M \to \mathbb{R}^+ \) be a weakly Kähler Finsler metric on a complex manifold \( M \). Take a regular curve \( \sigma_0 : [a, b] \to M \) with \( F(\dot{\sigma}_0) = c_0 > 0 \), and a regular variation \( \Sigma : (-\varepsilon, \varepsilon) \times [a, b] \to M \) of \( \sigma_0 \). Then

\[
\frac{d}{ds} (0) = \frac{1}{c_0} \left\{ \Re \langle U^H, T^H \rangle_{\dot{\sigma}_0} \bigg|_a^b - \Re \int_a^b \langle U^H, \nabla_{T^H + \overline{T^H}} T^H \rangle_{\dot{\sigma}_0} dt \right\}.
\]
In particular, if $\Sigma$ is a fixed variation, that is $\Sigma(\cdot, a) \equiv \sigma_0(a)$ and $\Sigma(\cdot, b) \equiv \sigma_0(b)$, we have

$$\frac{d\ell_\Sigma}{ds}(0) = -\frac{1}{c_0} \Re \int_a^b \langle U^H, \nabla_{TH+T^H}T^H \rangle_{\delta_\sigma} dt. \quad (6.6)$$

Proof: By definition,

$$\ell_\Sigma(s) = \int_a^b (G(\delta_s))^1/2 dt;$$

therefore

$$\frac{d\ell_\Sigma}{ds} = \frac{1}{2c_s} \int_a^b \frac{\partial}{\partial s}[G(\delta_s)] dt = \frac{1}{2c_s} \int_a^b \frac{\partial}{\partial s} \langle \Xi(\delta_t), \Xi(\delta_t) \rangle_T dt,$$

where $c_s \equiv F(\delta_s)$ and we used

$$G(\delta_s) = \langle \chi(\delta_s), \chi(\delta_s) \rangle_{\delta_\sigma} = \langle \Xi(\delta_t), \Xi(\delta_t) \rangle_T,$$

by (6.2). Now, using (6.1) and (6.3), we get

$$\frac{1}{2} \frac{\partial}{\partial s} \langle \Xi(\delta_t), \Xi(\delta_t) \rangle_T = \frac{1}{2} \delta_s \langle \Xi(\delta_t), \Xi(\delta_t) \rangle_T$$

$$= \frac{1}{2} \left\{ \langle \nabla^*_{\delta_s} \Xi(\delta_t), \Xi(\delta_t) \rangle_T + \langle \Xi(\delta_t), \nabla^*_{\delta_s} \Xi(\delta_t) \rangle_T \right\}$$

$$+ \langle \nabla^*_{\delta_s} \Xi(\delta_t), \Xi(\delta_t) \rangle_T + \langle \Xi(\delta_t), \nabla^*_{\delta_s} \Xi(\delta_t) \rangle_T \right\}$$

$$= \Re \left\{ \langle \nabla^*_{\delta_s} \Xi(\delta_t), \Xi(\delta_t) \rangle_T + \langle \nabla^*_{\delta_s} \Xi(\delta_t), \Xi(\delta_t) \rangle_T \right\}$$

$$= \Re \left\{ \langle \nabla^*_{\delta_t} \Xi(\delta_s), \Xi(\delta_t) \rangle_T + \langle [U^H, T^H] + \nabla_{U^H}T^H, T^H \rangle_{\delta_s} + \langle \theta(U^H, T^H), T^H \rangle_{\delta_s} \right\}.$$
Then
\[ \frac{1}{2} \frac{\partial}{\partial s} \langle \Xi(\delta_t), \Xi(\delta_t) \rangle_T = \text{Re} \left\{ \langle \nabla_{\delta_t}^* \Xi(\delta_s), \Xi(\delta_t) \rangle_T + \langle \nabla_{\delta_t}^* \Xi(\delta_s), \Xi(\delta_t) \rangle_T \right\} \]
\[ = \text{Re} \left\{ \delta_t \langle \Xi(\delta_s), \Xi(\delta_t) \rangle_T - \langle \Xi(\delta_s), \nabla_{\delta_t}^* \Xi(\delta_t) \rangle_T \right\} \]
\[ = \text{Re} \left\{ \frac{\partial}{\partial t} \langle U^H, T^H \rangle_{\delta_s} - \langle U^H, \nabla_{T^H + T^H_T} T^H \rangle_{\delta_s} \right\}, \]
and the assertion follows.

As a corollary we get the equation of geodesics:

**Corollary 6.2:** Let \( F : T^{1,0} M \to \mathbb{R}^+ \) be a weakly Kähler Finsler metric on a complex manifold \( M \), and let \( \sigma : [a, b] \to M \) be a regular curve with \( F(\dot{\sigma}) \equiv c_0 > 0 \). Then \( \sigma \) is a geodesic for \( F \) iff
\[ \nabla_{T^H + T^H_T} T^H \equiv 0, \] (6.10)
where \( T^H(v) = \chi_v(\dot{\sigma}(t)) \in \mathcal{H}_v \) for all \( v \in M_{\sigma(t)} \).

**Proof:** It follows immediately from (6.6).

**Corollary 6.3:** Let \( F : T^{1,0} M \to \mathbb{R}^+ \) be a weakly Kähler Finsler metric on a complex manifold \( M \). Then for any \( p \in M \) and \( v \in \tilde{M}_p \) with \( F(v) = 1 \) there exists a unique geodesic \( \sigma : (-\varepsilon, \varepsilon) \to M \) such that \( \sigma(0) = p \) and \( \dot{\sigma}(0) = v \).

**Proof:** In local coordinates we have
\[ \nabla_{T^H + T^H_T} T^H = \left[ (\dot{\sigma}^\mu \delta_\mu + \overline{\dot{\sigma}}^\mu \delta_\mu)(\dot{\sigma}^\alpha) + \Gamma^\alpha_{\nu,\mu}(\dot{\sigma})\dot{\sigma}^\mu \dot{\sigma}^\nu \right] \delta_\alpha = [\overline{\dot{\sigma}}^\alpha + \Gamma^\alpha_{\nu,\mu}(\dot{\sigma})\dot{\sigma}^\mu] \delta_\alpha. \]
So (6.10) is a quasi-linear O.D.E. system, and the assertion follows.

Thus the standard O.D.E. arguments apply in this case too, and we may recover for weakly Kähler Finsler metrics the usual theory of geodesics. In particular, if the metric \( F \) is complete we can define the exponential map \( \text{exp}_p : T^0_{p,0} M \to M \) for any \( p \in M \). See [AP4] for details.

**7. Second variation of the length integral**

Our next goal is the second variation formula, which holds for Kähler Finsler metrics. To express it correctly, we need two further ingredients. The first one is the horizontal \((1,1)\)-torsion \( \tau^H \), simply defined by
\[ \tau^H(X, Y) = \Theta(\tau(X, Y)) = \Omega(X, Y) \chi. \]
The second one is the symmetric product \( \langle \langle , \rangle \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C} \) locally given by
\[ \forall H, K \in \mathcal{H}_v \quad \langle \langle H, K \rangle \rangle_v = G_{\alpha\beta}(v) H^\alpha K^\beta. \]
It is clearly globally well-defined, and it satisfies
\[ \forall H \in \mathcal{H} \quad \langle \langle H, \chi \rangle \rangle = 0. \]
Theorem 7.1: Let \( F: T^{1,0}M \to \mathbb{R}^+ \) be a Kähler Finsler metric on a complex manifold \( M \). Take a geodesic \( \sigma_0: [a, b] \to M \) with \( F(\dot{\sigma}_0) \equiv 1 \), and let \( \Sigma: (-\varepsilon, \varepsilon) \times [a, b] \to M \) be a regular variation of \( \sigma_0 \). Then

\[
\frac{d^2f_\Sigma}{ds^2}(0) = \text{Re} \left( \nabla_{U^H + \overline{U^H}} U^H, T^H \right)_{\dot{\sigma}_0}^b_a \\
+ \int_a^b \left\{ \left\| \nabla_{T^H + \overline{T^H}} U^H \right\|_\sigma^2 - \left| \frac{\partial}{\partial t} \text{Re} \left( U^H, T^H \right)_{\dot{\sigma}_0} \right|^2 \\
- \text{Re} \left[ \left( \Omega(T^H, \overline{U^H}) U^H, T^H \right)_{\dot{\sigma}_0} - \left\langle \Omega(U^H, \overline{T^H}) U^H, T^H \right\rangle_{\dot{\sigma}_0} \\
+ \left\langle \tau^H(U^H, \overline{T^H}), U^H \right\rangle_{\dot{\sigma}_0} - \left\langle \tau^H(T^H, \overline{U^H}), U^H \right\rangle_{\dot{\sigma}_0} \right] \right\} dt.
\]

In particular, if \( \Sigma \) is a fixed variation such that \( \text{Re} \left( U^H, T^H \right)_{\dot{\sigma}_0} \) is constant we have

\[
\frac{d^2f_\Sigma}{ds^2}(0) = \int_a^b \left\{ \left\| \nabla_{T^H + \overline{T^H}} U^H \right\|_\sigma^2 \\
- \text{Re} \left[ \left( \Omega(T^H, \overline{U^H}) U^H, T^H \right)_{\dot{\sigma}_0} - \left\langle \Omega(U^H, \overline{T^H}) U^H, T^H \right\rangle_{\dot{\sigma}_0} \\
+ \left\langle \tau^H(U^H, \overline{T^H}), U^H \right\rangle_{\dot{\sigma}_0} - \left\langle \tau^H(T^H, \overline{U^H}), U^H \right\rangle_{\dot{\sigma}_0} \right] \right\} dt.
\]

Proof: During the proof of the first variation formula — in (6.9) — we saw that

\[
\frac{dl_\Sigma}{ds}(s) = \text{Re} \left( \int_a^b \left( \nabla_{\delta_t} \Xi(\delta_s), \Xi(\delta_t) \right)_T + \left( \nabla_{\delta_t}^* \Xi(\delta_s), \Xi(\delta_t) \right)_T \right) \left( \left\langle \Xi(\delta_t), \Xi(\delta_t) \right\rangle_T \right)^{1/2} dt.
\]

So we need to compute

\[
\frac{\partial}{\partial s} \left[ \left( \nabla_{\delta_t} \Xi(\delta_s), \Xi(\delta_t) \right)_T + \left( \nabla_{\delta_t}^* \Xi(\delta_s), \Xi(\delta_t) \right)_T \right] = \frac{\delta_s \left( \nabla_{\delta_t} \Xi(\delta_s), \Xi(\delta_t) \right)_T + \delta_s \left( \nabla_{\delta_t}^* \Xi(\delta_s), \Xi(\delta_t) \right)_T}{\left( \left\langle \Xi(\delta_t), \Xi(\delta_t) \right\rangle_T \right)^{1/2}} \\
- \frac{1}{2} \left( \nabla_{\delta_t} \Xi(\delta_s), \Xi(\delta_t) \right)_T + \left( \nabla_{\delta_t}^* \Xi(\delta_s), \Xi(\delta_t) \right)_T \right) \frac{3/2}{\left( \left\langle \Xi(\delta_t), \Xi(\delta_t) \right\rangle_T \right)^{3/2}} \delta_s \left( \Xi(\delta_t), \Xi(\delta_t) \right)_T.
\]

Since, when \( s = 0 \), the denominator of the first term is equal to 1, and the denominator of the second term is equal to 2, we may forget them. Let us call (I) the numerator of the first term, and (II) the numerator of the second term. First of all, (6.9) yields

\[
\frac{1}{2} \text{Re} (\text{II}) = \left| \text{Re} \left[ \frac{\partial}{\partial t} \left( U^H, T^H \right)_{\dot{\sigma}_s} - \left( U^H, \nabla_{T^H + \overline{T^H}} T^H \right)_{\dot{\sigma}_s} \right] \right|^2;
\]
in particular, for $s = 0$ we get

$$
\frac{1}{2} \text{Re} (\mathcal{I}(0)) = \left| \frac{\partial}{\partial t} \text{Re}(U^H, T^H)_{\sigma_0} \right|^2,
$$

(7.2)

because $\sigma_0$ is a geodesic.

The computation of (I) is quite longer. First of all, using (6.3), (6.4) and (6.5) we get

(I) = $\langle \nabla^*_s \nabla^*_t \Xi(\delta_s), \Xi(\delta_t) \rangle_T + \langle \nabla^*_s \nabla^*_t \Xi(\delta_s), \Xi(\delta_t) \rangle_T$

+ $\langle \nabla^*_s \Xi(\delta_s), \nabla^*_t \Xi(\delta_t) \rangle_T + \langle \nabla^*_s \Xi(\delta_s), \nabla^*_t \Xi(\delta_t) \rangle_T$

+ $\langle \nabla^*_s \nabla^*_t \Xi(\delta_s), \Xi(\delta_t) \rangle_T + \langle \nabla^*_s \nabla^*_t \Xi(\delta_s), \Xi(\delta_t) \rangle_T$

+ $\langle \nabla^*_s \Xi(\delta_s), \nabla^*_t \Xi(\delta_t) \rangle_T + \langle \nabla^*_s \Xi(\delta_s), \nabla^*_t \Xi(\delta_t) \rangle_T$

$= \langle \nabla^*_s \nabla^*_t \Xi(\delta_s), \Xi(\delta_t) \rangle_T - \langle \nabla_{[TH, U^H]} U^H, T^H \rangle_{\sigma_s}$

+ $\langle \nabla^*_s \nabla^*_t \Xi(\delta_s), \Xi(\delta_t) \rangle_T - \langle \nabla_{[TH, U^H]} U^H, T^H \rangle_{\sigma_s} - \langle \Omega(T^H, U^H) U^H, T^H \rangle_{\sigma_s}$

+ $\langle \nabla^*_s \Xi(\delta_s), \nabla^*_t \Xi(\delta_t) \rangle_T + \langle \nabla_{TH} U^H, [U^H, T^H] + \nabla_{U^H T^H} \rangle_{\sigma_s}$

+ $\langle \nabla_{TH} U^H, \theta(U^H, T^H) \rangle_{\sigma_s}$

+ $\langle \nabla^*_s \nabla^*_t \Xi(\delta_s), \Xi(\delta_t) \rangle_T - \langle \nabla_{[TH, U^H]} U^H, T^H \rangle_{\sigma_s} + \langle \nabla_{TH} U^H, [U^H, T^H] + \nabla_{U^H T^H} \rangle_{\sigma_s}$

+ $\langle \nabla_{TH} U^H, \theta(U^H, T^H) \rangle_{\sigma_s}$

+ $\langle \nabla^*_s \Xi(\delta_s), \nabla^*_t \Xi(\delta_t) \rangle_T + \langle \nabla_{TH} U^H, [U^H, T^H] + \nabla_{U^H T^H} \rangle_{\sigma_s}$

+ $\langle \nabla_{TH} U^H, \theta(U^H, T^H) \rangle_{\sigma_s}$

Recalling (6.8), (6.2) and that $F$ is Kähler we get

(I) = $\langle \nabla^*_s \nabla^*_t \Xi(\delta_s), \Xi(\delta_t) \rangle_T + \langle \nabla^*_s \nabla^*_t \Xi(\delta_s), \Xi(\delta_t) \rangle_T$

+ $\langle \nabla^*_s \nabla^*_t \Xi(\delta_s), \Xi(\delta_t) \rangle_T + \langle \nabla^*_s \nabla^*_t \Xi(\delta_s), \Xi(\delta_t) \rangle_T$

+ $\langle \nabla^*_s \nabla^*_t \Xi(\delta_s), \Xi(\delta_t) \rangle_T + \langle \nabla^*_s \nabla^*_t \Xi(\delta_s), \Xi(\delta_t) \rangle_T$

+ $\langle \nabla^*_s \nabla^*_t \Xi(\delta_s), \Xi(\delta_t) \rangle_T + \langle \nabla^*_s \nabla^*_t \Xi(\delta_s), \Xi(\delta_t) \rangle_T$

Now

$$
[T^H, U^H] = \left[ \frac{\partial \Sigma^\mu}{\partial t} \right] \delta_\mu \left( \frac{\partial \Sigma^\nu}{\partial s} \right) - \frac{\partial \Sigma^\mu}{\partial s} \delta_\mu \left( \frac{\partial \Sigma^\nu}{\partial t} \right) \delta_\nu,
$$

$$
[T^H, U^H] = \frac{\partial \Sigma^\mu}{\partial t} \frac{\partial \Sigma^\nu}{\partial s} \left[ \delta_\nu (\Gamma^\alpha_{\mu \nu}) \delta_{\alpha} - \delta_\mu (\Gamma^\alpha_{\nu \alpha}) \delta_{\nu} \right] + \frac{\partial \Sigma^\mu}{\partial t} \delta_\mu \left( \frac{\partial \Sigma^\nu}{\partial s} \right) \delta_\nu - \frac{\partial \Sigma^\nu}{\partial s} \delta_\nu \left( \frac{\partial \Sigma^\mu}{\partial t} \right) \delta_\mu,
$$

$$
\overline{[T^H, U^H]} = \frac{\partial \Sigma^\mu}{\partial t} \frac{\partial \Sigma^\nu}{\partial s} \left[ \delta_\nu (\Gamma^\beta_{\mu \nu}) \delta_{\beta} - \delta_\mu (\Gamma^\beta_{\nu \beta}) \delta_{\nu} \right] + \frac{\partial \Sigma^\mu}{\partial t} \delta_\mu \left( \frac{\partial \Sigma^\nu}{\partial s} \right) \delta_\nu - \frac{\partial \Sigma^\nu}{\partial s} \delta_\nu \left( \frac{\partial \Sigma^\mu}{\partial t} \right) \delta_\mu,
$$

$$
\overline{[T^H, U^H]} = \left[ \frac{\partial \Sigma^\mu}{\partial t} \delta_\mu \left( \frac{\partial \Sigma^\nu}{\partial s} \right) - \frac{\partial \Sigma^\mu}{\partial s} \delta_\mu \left( \frac{\partial \Sigma^\nu}{\partial t} \right) \right] \delta_\nu,
$$
and so (6.7) and (3.1) yield

\[ [T^H, U^H] + [T^H, U^H] + [T^H, U^H] + [T^H, U^H] = \tau(U^H, TH) - \tau(T^H, U^H) + \tau(U^H, TH) - \tau(T^H, U^H). \]

Furthermore, if \( V \in \mathcal{V} \) we have

\[
\langle \nabla_V U^H, T^H \rangle_{\dot{s}} = G_{\alpha\beta}(\dot{s}) V^\gamma \left[ \dot{\gamma} \left( \frac{\partial \Sigma^\alpha}{\partial s} + \Gamma^\alpha_{\gamma\delta}(\dot{s}) \left( \frac{\partial \Sigma^\delta}{\partial s} \right) \right) \right]_{\dot{s}}^\beta
\]

\[
= G_{\alpha}(\dot{s}) \Gamma^\alpha_{\delta\gamma}(\dot{s}) \left( \frac{\partial \Sigma^\delta}{\partial s} \right) V^\gamma = G_{\delta\gamma}(\dot{s}) \left( \frac{\partial \Sigma^\delta}{\partial s} \right) V^\gamma
\]

\[
= \langle \Theta(V), U^H \rangle_{\dot{s}},
\]

and

\[
\langle \nabla \nabla U^H, T^H \rangle_{\dot{s}} = G_{\alpha\beta}(\dot{s}) V^\gamma \left[ \dot{\gamma} \left( \frac{\partial \Sigma^\alpha}{\partial s} \right) \right]_{\dot{s}}^\beta
\]

\[
= 0.
\]

Therefore

\[
\langle \nabla_{[T^H, U^H] + [T^H, U^H] + [T^H, U^H] + [T^H, U^H]} U^H, T^H \rangle_{\dot{s}} = \langle \tau(U^H, TH), U^H \rangle_{\dot{s}} - \langle \tau(T^H, U^H), U^H \rangle_{\dot{s}},
\]

and thus

\[
(I) = \delta_t \langle \nabla^*_{\delta_t + \dot{s}} \Xi(\delta_t), \Xi(\delta_t) \rangle_T - \langle \nabla^*_{\dot{s} + \delta_t} \Xi(\delta_t), \nabla^*_{\dot{s} + \delta_t} \Xi(\delta_t) \rangle_T
\]

\[
- \langle \Omega(T^H, U^H) U^H, T^H \rangle_{\dot{s}} + \langle \Omega(U^H, TH) U^H, U^H \rangle_{\dot{s}}
\]

\[
- \langle \langle T^H(U^H, TH), U^H \rangle_{\dot{s}} + \langle T^H(U^H, TH), U^H \rangle_{\dot{s}} + \| \nabla^*_{\delta_t + \dot{s}} \Xi(\delta_t) \|_T^2.
\]

Recalling that for \( s = 0 \) we have \( \nabla^*_{\delta_t + \dot{s}} \Xi(\delta_t) \equiv 0 \) because \( \sigma_0 \) is a geodesic, (7.1), (7.2) and (7.3) evaluated at \( s = 0 \) yield the assertion. \( \square \)

So we have obtained the second variation formula for strongly pseudoconvex Kähler Finsler metrics. Besides its own intrinsic interest, we need it to compare the curvature of the real Finsler metric \( F^o \) and our original complex Finsler metric \( F \). The idea is that both measuring the length of curves using \( F^o \) and using \( F \) we end up with the same function \( \ell_\Sigma \); therefore the second variation formula should be the same written in terms of \( F \) or in terms of \( F^o \) — assuming the convexity of the latter, of course.

The second variation formula for real Finsler metrics has been computed by Auslander [Au1] (see also Chern [Ch], Bao and Chern [BC] and [AP4]), in a setting similar to ours and in terms of the so-called horizontal flag curvature of the Cartan connection. So comparing the two formulas we get an expression for the horizontal flag curvature of the Cartan connection for convex Kähler Finsler metrics:
Corollary 7.2: Let $F: T^{1,0}M \to \mathbb{R}^+$ be a convex (i.e., with strongly convex indicatrices) Kähler Finsler metric on a complex manifold $M$. Then the horizontal flag curvature of the Cartan connection associated to $F^o$ is given by

$$R_v(H, H) = \text{Re} \left[ \langle \Omega(\chi, \overline{\Omega})H, \chi \rangle_v - \langle \Omega(H, \overline{\chi})H, \chi \rangle_v \right.$$

$$+ \langle \langle \tau^H(\chi), H \rangle_v - \langle \langle \tau^H(\overline{\chi}), H \rangle_v \right]$$

for all $H \in \mathcal{H}$.

We shall need this result to apply Auslander’s version [Au2] of the classical Cartan-Hadamard theorem. By the way, it turns out that a direct computation of the Cartan connection (and its curvature) in terms of the Chern-Finsler connection (and its curvature) is unexpectedly difficult; see [AP4] for details.

8. Manifolds with constant holomorphic curvature

A very natural problem now is the classification of Kähler Finsler manifolds of constant holomorphic curvature. In this respect, the Finsler situation is much richer than the hermitian one; for instance, Lempert’s work [Le] and [AP2] imply that all strongly convex domains of $\mathbb{C}^n$ endowed with the Kobayashi metric are weakly Kähler Finsler manifolds with constant holomorphic curvature $-4$.

The last theorem of this paper is a step toward this classification; roughly speaking, we shall prove that a simply connected Kähler Finsler manifold of nonpositive constant holomorphic curvature is diffeomorphic to an euclidean space. Furthermore, in the case of constant negative holomorphic curvature our results show that the Finsler geometry of the manifold is pretty much the same of the one of strongly convex domains endowed with the Kobayashi metric.

The idea is to apply the Cartan-Hadamard theorem; to do so, we need to estimate the curvature terms appearing in the second variation formula.

Let $F: T^{1,0}M \to \mathbb{R}^+$ be a strongly pseudoconvex Finsler metric on a complex manifold $M$. We say that $F$ has constant holomorphic curvature $2c \in \mathbb{R}$ if

$$\langle \Omega(\chi, \overline{\chi})\chi, \chi \rangle \equiv cG^2,$$  \hspace{1cm} (8.1)

that is iff $K_F \equiv 2c$. The idea is to differentiate (8.1) in such a smart way to get all the informations we need.

We start with a couple of computational lemmas.

Lemma 8.1: Let $F: T^{1,0}M \to \mathbb{R}^+$ be a strongly pseudoconvex Finsler metric on a complex manifold $M$. Then

$$\langle (\nabla_W \Omega)(H, \overline{K})\chi, \chi \rangle = \langle \tau^H(\chi), H \rangle_v - \langle \langle \tau^H(\overline{\chi}), H \rangle_v$$

for all $W \in \mathcal{V}$ and $H, K \in \mathcal{H}$. In particular,

$$\langle (\nabla_W \Omega)(H, \overline{\chi})\chi, \chi \rangle = 0$$
for all $W \in \mathcal{V}$ and $H \in \mathcal{H}$.

**Proof:** Since we are interested only in the horizontal part, we may replace $\Omega$ by

$$\Omega^H = \Omega^\alpha_{\beta} \otimes dz^\beta \otimes \delta_\alpha.$$  

Since $\nabla_W dz^\beta = 0$ and $\nabla_W \delta_\alpha = 0$, we have

$$\nabla_W \Omega^H = (\nabla_W \Omega^\alpha_{\beta}) \otimes dz^\beta \otimes \delta_\alpha.$$  

Again, we only need the horizontal part, that is

$$p^*_H(\nabla_W \Omega^\alpha_{\beta}) = \nabla(W R^\alpha_{\beta;\mu\nu}) dz^\mu \wedge dz^\nu - R^\alpha_{\beta;\mu\rho} \bar{\omega}_\rho(W) dz^\mu \wedge dz^\nu.$$  

Recalling (4.2), taking $H$, $K$, $L \in \mathcal{H}$ we get

$$\langle (\nabla_W \Omega^H)(H, K) L, \chi \rangle = G_\alpha [W (R^\alpha_{\beta;\mu\nu}) - R^\alpha_{\beta;\mu\rho} \bar{\omega}_\rho(W)] H^\mu K^\nu L^\beta$$

$$= - G_\alpha [\partial_\gamma \partial_\beta (\Gamma^\alpha_{\beta;\mu}) - \partial_\beta (\Gamma^\alpha_{\gamma;\mu}) \Gamma^\gamma_{\beta;\mu} + \partial_\gamma (\Gamma^\alpha_{\beta;\mu}) - \Gamma^\alpha_{\beta;\mu} \Gamma^\beta_{\gamma;\mu}] H^\mu K^\nu L^\beta$$

$$= - G_\alpha [\partial_\gamma \partial_\beta (\Gamma^\alpha_{\beta;\mu}) - \Gamma^\alpha_{\beta;\mu} \Gamma^\beta_{\gamma;\mu}] H^\mu K^\nu L^\beta$$

(where $\Gamma^\alpha_{\gamma;\mu} = \partial_\gamma (\Gamma^\alpha_{\beta;\mu})$ and we used Lemma 2.1.(ii)),

$$= - [\delta_\nu (G_\alpha \partial_\beta (\Gamma^\alpha_{\nu;\gamma})) + G_\alpha (G_\alpha \partial_\beta (\Gamma^\alpha_{\nu;\gamma})) - G_\alpha (\Gamma^\alpha_{\beta;\mu}) - \Gamma^\alpha_{\beta;\mu} \Gamma^\beta_{\gamma;\mu}] H^\mu K^\nu L^\beta$$

(where we used $G_\alpha \Gamma^\alpha_{\beta;\gamma} = G_\beta \gamma$, $\delta_\nu (G_\alpha) = 0$ and $G_\alpha \partial_\gamma (\Gamma^\alpha_{\beta;\gamma}) = G_\beta \gamma - G_\alpha \gamma \beta_\gamma = 0$),

$$= - [\delta_\nu (G_\alpha \beta_\gamma (\Gamma^\alpha_{\gamma;\nu} + \Gamma^\beta_{\gamma;\nu} \Gamma^\alpha_{\nu;\gamma})) - G_\alpha (\Gamma^\alpha_{\beta;\mu}) - \Gamma^\alpha_{\beta;\mu} \Gamma^\beta_{\gamma;\mu}] H^\mu K^\nu L^\beta$$

(where we used $G_\alpha \Gamma^\alpha_{\gamma;\nu} = 0$),

$$= - [\delta_\nu (G_\alpha \beta_\gamma (\Gamma^\alpha_{\nu;\gamma})) - G_\alpha \beta_\gamma (\Gamma^\alpha_{\nu;\gamma} + \Gamma^\beta_{\gamma;\nu} \Gamma^\alpha_{\nu;\gamma})) - G_\alpha (\Gamma^\alpha_{\beta;\mu}) - \Gamma^\alpha_{\beta;\mu} \Gamma^\beta_{\gamma;\mu}] H^\mu K^\nu L^\beta$$

Hence

$$\langle (\nabla_W \Omega)(H, K) \chi, \chi \rangle$$

$$= - [\delta_\nu (G_\alpha \beta_\gamma (\Gamma^\alpha_{\nu;\gamma})) - G_\alpha \beta_\gamma (\Gamma^\alpha_{\nu;\gamma} + \Gamma^\beta_{\gamma;\nu} \Gamma^\alpha_{\nu;\gamma})) - G_\alpha (\Gamma^\alpha_{\beta;\mu}) - \Gamma^\alpha_{\beta;\mu} \Gamma^\beta_{\gamma;\mu}] H^\mu K^\nu L^\beta,$$

(where we used (1.3) and $\nu^\beta \Gamma^\alpha_{\beta;\mu} = \Gamma^\alpha_{\gamma;\mu}$),

$$= G_\alpha \beta_\gamma (\Gamma^\alpha_{\nu;\gamma}) H^\mu K^\nu L^\beta$$

$$= \langle \tau^H (H, \theta (K, W)), \chi \rangle,$$

because $\theta (K, W) = - \Gamma^\nu_{\nu;\gamma} K^\nu W^\gamma \delta_\mu$, by (3.1).

Finally,

$$\langle (\nabla_W \Omega)(H, \chi), \chi \rangle = G_\alpha \delta_\nu (\Gamma^\alpha_{\gamma;\nu}) \bar{\omega}_\nu H^\mu K^\nu L^\beta$$

$$= 0,$$

because $\Gamma^\alpha_{\gamma;\nu} \bar{\omega}_\nu = 0$. 

Lemma 8.2: Let $F : T^{1,0}M \to \mathbb{R}^+$ be a strongly pseudoconvex Finsler metric on a complex manifold $M$. Then
\[
\langle (\nabla_V \Omega)(H, \overline{K}) \chi, \chi \rangle = \langle \nu^H(\theta(H, V), \overline{K}), \chi \rangle
\]
for all $V \in \mathcal{V}$ and $H, \ K \in \mathcal{H}$. In particular,\[
\langle (\nabla_V \Omega)(\chi, \overline{K}) \chi, \chi \rangle = 0
\]
for all $V \in \mathcal{V}$ and $K \in \mathcal{H}$.

Proof: Again it suffices to consider $\Omega^H = \Omega^\beta_\gamma \otimes dz^\beta \otimes \delta_\alpha$; so
\[
\nabla_V \Omega^H = (\nabla_V \Omega^\beta) \otimes dz^\beta \otimes \delta_\alpha - \Omega^\gamma_\beta \otimes \omega^\gamma_\beta(V) dz^\beta \otimes \delta_\alpha + H^\gamma \circ d^\beta \otimes \omega^\gamma_\beta (V) \delta_\alpha.
\]

We are interested only in the horizontal part. Taking $H, \ K \in \mathcal{H}$ we get
\[
G_{\alpha}(\nabla_V \Omega^\beta_\gamma)(H, \overline{K}) = G_{\alpha}[V(R^\gamma_{\beta, \mu}) - R^\gamma_{\beta, \mu \nu} \omega^\mu_\nu(V)] H^\mu \overline{K}^\nu
\]
\[
= -G_{\alpha}[d_{\nu} \delta_{\nu}(\Gamma^\alpha_{\beta, \mu}) - \delta_{\nu}(\Gamma^\alpha_{\beta, \nu}) \Gamma_{\mu \lambda} + \delta_{\nu}(\Gamma^\alpha_{\beta, \lambda}) \Gamma_{\mu \nu} - \Gamma^\alpha_{\beta, \delta} \delta_{\nu}(\Gamma^\gamma_{\mu \lambda})] \nu^\delta H^\nu \overline{K}^\nu
\]
\[
= -G_{\alpha}[\delta_{\nu} \delta_{\nu}(\Gamma^\alpha_{\beta, \mu}) - \delta_{\nu}(\Gamma^\alpha_{\beta, \nu}) \Gamma_{\mu \lambda} + \delta_{\nu}(\Gamma^\alpha_{\beta, \lambda}) \Gamma_{\mu \nu} - \Gamma^\alpha_{\beta, \delta} \delta_{\nu}(\Gamma^\gamma_{\mu \lambda})] \nu^\delta H^\nu \overline{K}^\nu
\]
\[
(\text{where we used } [\delta_{\nu}, \delta_{\nu}] = \Lambda_{\nu \nu} \hat{\sigma}_{\nu}),
\]
\[
= -[\delta_{\nu}(G_{\alpha} \Gamma^\alpha_{\beta, \mu}) + \delta_{\nu}(\Gamma^\alpha_{\beta, \nu}) \Gamma_{\mu \lambda} + \Gamma_{\nu \lambda} \delta_{\nu}(\Gamma^\alpha_{\beta, \lambda}) - \delta_{\nu}(\Gamma^\alpha_{\beta, \lambda}) \Gamma_{\mu \nu} - \Gamma^\alpha_{\beta, \delta} \delta_{\nu}(\Gamma^\gamma_{\mu \lambda})] \nu^\delta H^\nu \overline{K}^\nu
\]
\[
(\text{where we used } \delta_{\nu}(G_{\alpha}) = 0 \text{ and } G_{\alpha} \Gamma^\alpha_{\beta, \sigma} = \Gamma^\alpha_{\beta, \sigma}),
\]
\[
= -[-\delta_{\nu}(G_{\alpha} \Gamma^\alpha_{\beta, \mu}) + \delta_{\nu}(\Gamma^\alpha_{\beta, \nu}) \Gamma_{\mu \lambda} + \Gamma_{\nu \lambda} \delta_{\nu}(\Gamma^\alpha_{\beta, \lambda}) - \delta_{\nu}(\Gamma^\alpha_{\beta, \lambda}) \Gamma_{\mu \nu} - \Gamma^\alpha_{\beta, \delta} \delta_{\nu}(\Gamma^\gamma_{\mu \lambda})] \nu^\delta H^\nu \overline{K}^\nu
\]
\[
(\text{where we used } G_{\alpha} \Gamma^\alpha_{\beta, \mu} = 0 \text{ and } G_{\alpha} \Gamma^\alpha_{\beta, \lambda} = \delta_{\nu}(G_{\alpha})),
\]
\[
= -[-\delta_{\nu}(G_{\alpha} \Gamma^\alpha_{\beta, \mu}) + \delta_{\nu}(\Gamma^\alpha_{\beta, \nu}) \Gamma_{\mu \lambda} + \Gamma_{\nu \lambda} \delta_{\nu}(\Gamma^\alpha_{\beta, \lambda}) - \delta_{\nu}(\Gamma^\alpha_{\beta, \lambda}) \Gamma_{\mu \nu} - \Gamma^\alpha_{\beta, \delta} \delta_{\nu}(\Gamma^\gamma_{\mu \lambda})] \nu^\delta H^\nu \overline{K}^\nu.
\]

Furthermore,
\[
G_{\alpha} \Omega^\gamma_\beta(H, \overline{K}) \omega^\gamma_\beta(V) = -G_{\alpha}[\delta_{\nu}(\Gamma^\gamma_{\beta, \mu}) + \Gamma^\gamma_{\beta, \delta} \delta_{\nu}(\Gamma^\gamma_{\mu \lambda})] \nu^\delta H^\nu \overline{K}^\nu
\]
\[
= -[\delta_{\nu}(\Gamma^\gamma_{\beta, \mu}) + \Gamma^\gamma_{\beta, \delta} \delta_{\nu}(\Gamma^\gamma_{\mu \lambda})] \nu^\delta H^\nu \overline{K}^\nu;
\]
\[
G_{\alpha} \omega^\gamma_\beta(V) \Omega^\gamma_\beta(H, \overline{K}) = -G_{\alpha}[\delta_{\nu}(\Gamma^\gamma_{\beta, \mu}) + \Gamma^\gamma_{\beta, \delta} \delta_{\nu}(\Gamma^\gamma_{\mu \lambda})] \nu^\delta H^\nu \overline{K}^\nu
\]
\[
= -G_{\gamma \nu}[\delta_{\nu}(\Gamma^\gamma_{\beta, \mu}) + \Gamma^\gamma_{\beta, \delta} \delta_{\nu}(\Gamma^\gamma_{\mu \lambda})] \nu^\delta H^\nu \overline{K}^\nu.
\]
Summing up we find
\[
\langle (\nabla_V \Omega)(H, K) \chi, \chi \rangle = -[\delta_V (G_{\alpha\beta} \Gamma^\alpha_{\chi, \mu}) + \delta_V (\hat{\partial}_\beta \delta_\mu (G_\lambda)) - \delta_V (G_{\alpha\gamma} \Gamma^\alpha_{\beta; \rho}) \Gamma^\rho_{\mu \chi} + G_{\alpha} \hat{\partial}_\lambda (\Gamma^\alpha_{\beta; \rho}) \delta_V (\Gamma^\gamma_{\mu \chi}) + G_{\beta \sigma} \delta_V (\Gamma^\gamma_{\alpha; \rho}) \Gamma^\rho_{\mu \chi} + \delta_V (\delta_\mu (G_\gamma)) \Gamma^\gamma_{\beta \chi} + G_{\gamma \sigma} \delta_V (\Gamma^\gamma_{\beta; \rho}) \delta_V (\Gamma^\gamma_{\mu \chi})]
\]
\[
\quad + G_{\gamma \mu} \delta_V (\Gamma^\gamma_{\beta; \rho}) [v^\beta V^\lambda H^\mu \bar{K}^\nu]
\]
\[
= -[\delta_V (G_{\alpha\beta} \Gamma^\alpha_{\chi, \mu}) + \delta_V (\hat{\partial}_\beta \delta_\mu (G_\lambda)) - G_{\lambda \gamma} \delta_V (\Gamma^\gamma_{\beta; \rho}) \Gamma^\rho_{\mu \chi} + G_{\lambda} \hat{\partial}_\gamma (\Gamma^\alpha_{\beta; \rho}) \delta_V (\Gamma^\gamma_{\mu \chi})]
\]
\[
\quad + G_{\lambda \gamma} \delta_V (\Gamma^\gamma_{\beta; \rho}) [v^\beta V^\lambda H^\mu \bar{K}^\nu]
\]

(8.2)

(8.3)

The final assertion follows from \( \theta(\chi, V) = 0 \).

In the following computations we shall need some symmetries of the curvature operator, summarized in

**Lemma 8.3:** Let \( F: T^{1,0} M \to \mathbb{R}^+ \) be a strongly pseudoconvex Finsler metric on a complex manifold \( M \). Then

(i) \( \langle \Omega(H, \bar{\chi}) \chi, \chi \rangle = \langle \Omega(\chi, \bar{\chi}) H, \chi \rangle \) for all \( H \in \mathcal{H} \) iff

\[
\langle \hat{\partial}_H \theta(H, \chi, \bar{\chi}), \chi \rangle = 0
\]

for all \( H \in \mathcal{H} \);

(ii) \( \langle \Omega(H, K) \chi, \chi \rangle = \langle \Omega(\chi, K) H, \chi \rangle \) for all \( H, K \in \mathcal{H} \) iff

\[
\langle \hat{\partial}_H \theta(H, \chi, K), \chi \rangle = 0
\]

for all \( H, K \in \mathcal{H} \).

**Proof:** It follows immediately from (5.4) and Proposition 4.1.

Now we can start. The first step is:

**Proposition 8.4:** Let \( F: T^{1,0} M \to \mathbb{R}^+ \) be a strongly pseudoconvex Finsler metric on a complex manifold \( M \), with constant holomorphic curvature \( 2c \in \mathbb{R} \). Then

\[
\langle \hat{\partial}_H \theta(H, \chi, \bar{\chi}), \chi \rangle = 0
\]

for all \( H \in \mathcal{H} \) iff

\[
\tau^H (\chi, \bar{\chi}) = cG \chi.
\]
Furthermore, they both imply
\[ \langle \Omega(\chi, \overline{K}) \chi, \chi \rangle = cG(\chi, K) \quad (8.6) \]
for all \( K \in \mathcal{H} \).

**Proof:** Take \( W \in \mathcal{V} \) and let \( K = \Theta(W) \in \mathcal{H} \); note that \( \nabla_W \chi = 0 \) and \( \nabla_W \chi = \Theta(W) = K \). Then
\[
\nabla_W (cG^2) = 2cG \langle \chi, K \rangle; \\
\nabla_W \langle \Omega(\chi, \overline{\chi}) \chi, \chi \rangle = \langle (\nabla_W \Omega)(\chi, \overline{\chi}) \chi, \chi \rangle + \langle \Omega(\chi, \overline{K}) \chi, \chi \rangle + \langle \Omega(\chi, \overline{\chi}) \chi, K \rangle \quad (8.7)
\]
where we used Lemmas 8.1 and 4.2. Since \( F \) has constant holomorphic curvature \( 2c \), we have
\[ \langle \Omega(\chi, \overline{\chi}) \chi, \chi \rangle = cG^2 \]
and hence (8.7) yields
\[ \langle \Omega(\chi, \overline{K}) \chi, \chi \rangle = 2cG \langle \chi, K \rangle - \langle \tau^H(\chi, \overline{\chi}), K \rangle. \quad (8.8) \]
Subtracting \( \langle \tau^H(\chi, \overline{\chi}), K \rangle = \langle \Omega(\chi, \overline{\chi}) \chi, K \rangle \) to both sides, we find that (8.5) holds if and only if
\[ \langle \Omega^H(\chi, \overline{K}) \chi, \chi \rangle = \langle \Omega^H(\chi, \overline{\chi}) \chi, K \rangle \]
for all \( K \in \mathcal{H} \), that is, recalling (5.5), iff
\[ \langle \Omega(K, \overline{\chi}) \chi, \chi \rangle = \langle \Omega(\chi, \overline{\chi}) K, \chi \rangle, \]
and thus, by Lemma 8.3, iff (8.4) holds.
Finally, if (8.5) holds, (8.8) yields (8.6).

The second step requires (8.3):

**Proposition 8.5:** Let \( F: T^{1,0}M \to \mathbb{R}^+ \) be a strongly pseudoconvex Finsler metric on a complex manifold \( M \) with constant holomorphic curvature \( 2c \in \mathbb{R} \). Assume that (8.2) holds. Then
\[ \langle \Omega(H, \overline{K}) \chi, \chi \rangle + \langle \Omega(\chi, \overline{K}) H, \chi \rangle = c \{ \langle H, \chi \rangle \langle \chi, K \rangle + \langle \chi, \chi \rangle \langle H, K \rangle \}, \quad (8.9) \]
for all \( H, K \in \mathcal{H} \). In particular, if (8.3) holds then
\[ \langle \Omega(\chi, \overline{K}) H, \chi \rangle = \frac{c}{2} \{ \langle H, \chi \rangle \langle \chi, K \rangle + \langle \chi, \chi \rangle \langle H, K \rangle \} \quad (8.10) \]
for all \( H, K \in \mathcal{H} \).

**Proof:** Take \( V, W \in \mathcal{V} \) such that \( \Theta(V) = H \) and \( \Theta(W) = K \) and extend them in any way to sections of \( \mathcal{V} \) (and thus extend \( H \) and \( K \) as sections of \( \mathcal{H} \) via \( \Theta \)). We have
\[ V(cG(\chi, K)) = c[\langle H, \chi \rangle \langle \chi, K \rangle + G(H, K) + G(\chi, \nabla_{\overline{\chi}} K)], \]
and
\[
\langle V(\Omega(\chi, K) \chi, \chi) \rangle
= \langle (\nabla_V \Omega)(\chi, K) \chi, \chi \rangle + \langle \Omega(H, K) \chi, \chi \rangle + \langle \Omega(\chi, \nabla_V K) \chi, \chi \rangle + \langle \Omega(\chi, K) H, \chi \rangle,
\]
thanks to Lemma 8.2. Since (8.2) holds, we can use Lemma 8.4 (that is, (8.6) applied both to $K$ and to $\nabla_V K$) to get exactly (8.9).

Finally, (8.10) follows from Lemma 8.3.

So we have obtained one of the hermitian product terms. This immediately yields one of the symmetric product terms:

**Proposition 8.6:** Let $F: T^{1,0} M \to \mathbb{R}^+$ be a strongly pseudoconvex Finsler metric on a complex manifold $M$ with constant holomorphic curvature $2c \in \mathbb{R}$. Assume that (8.3) holds. Then
\[
\tau^H(K, \bar{\chi}) = \frac{c}{2} \{ \langle K, \chi \rangle \chi + \langle \chi, \chi \rangle K \} \tag{8.11}
\]
for all $K \in \mathcal{H}$. In particular,
\[
\langle \langle H, \tau^H(K, \bar{\chi}) \rangle \rangle = \frac{c}{2} \langle \chi, \chi \rangle \langle \langle H, K \rangle \rangle \tag{8.12}
\]
for all $H, K \in \mathcal{H}$.

**Proof:** We get
\[
\langle H, \tau^H(K, \bar{\chi}) \rangle = \langle H, \Omega(K, \bar{\chi}) \chi \rangle = \langle \Omega(\chi, \bar{K}) H, \chi \rangle
\]
for all $H, K \in \mathcal{H}$, thanks to Lemma 4.2 and (5.5). Then (8.10) yields (8.11), and (8.12) follows immediately.

For the other symmetric product term we need the weak Kähler condition:

**Proposition 8.7:** Let $F: T^{1,0} M \to \mathbb{R}^+$ be a weakly Kähler-Finsler metric on a complex manifold $M$ such that (8.3) holds. Then
\[
\langle \langle H, \tau^H(K, \bar{\chi}) \rangle \rangle = 0
\]
for all $H, K \in \mathcal{H}$.

**Proof:** The weak Kähler condition $\langle \theta(H, \chi), \chi \rangle = 0$ for all $H \in \mathcal{H}$ implies
\[
\forall H, K \in \mathcal{H} \quad \langle (\nabla_R^\chi \theta)(H, \chi), \chi \rangle = 0 \tag{8.13}
\]
because $\nabla_R^\chi \chi = 0 = \nabla_K \chi$. Now, writing $\theta = \theta^\alpha \otimes \delta_\alpha$, we have $\nabla_R^\chi \theta = (\nabla_R^\chi \theta^\alpha) \otimes \delta_\alpha$ and
\[
\nabla_R^\chi \theta^\alpha = K^\tau \delta_\tau (\Gamma^\alpha_{\nu,\mu}) dz^\mu \wedge dz^\nu + K^\tau \delta_\tau (\Gamma^\alpha_{\mu,\nu}) \psi^\gamma \wedge dz^\nu.
\]
Therefore (8.13) implies
\[
G_\alpha \left[ \delta_\tau (\Gamma^\alpha_{\nu,\mu}) - \delta_\tau (\Gamma^\alpha_{\mu,\nu}) \right] H^\mu K^\tau v^\nu = 0 \tag{8.14}
\]
for all $H, K \in \mathcal{H}$.

Writing the curvature in local coordinates we find
\[
\langle \Omega(H, K)H, \chi \rangle = -G_\alpha [\delta_\tau (\Gamma^\alpha_{\mu\nu}) + \Gamma^\alpha_{\mu\sigma} \delta_\tau (\Gamma^\sigma_{\nu\mu})] H^\mu \overline{K^\tau} v^\nu,
\]
\[
\langle \Omega(H, K) \chi, \chi \rangle = -G_\alpha [\delta_\tau (\Gamma^\alpha_{\nu\mu}) + \Gamma^\alpha_{\nu\sigma} \delta_\tau (\Gamma^\sigma_{\mu\nu})] H^\mu \overline{K^\tau} v^\nu.
\]

So (8.14) yields
\[
\langle \Omega(H, K)H, \chi \rangle - \langle \Omega(H, K) \chi, \chi \rangle = -G_\alpha [\delta_\tau (\Gamma^\alpha_{\nu\mu}) + \Gamma^\alpha_{\nu\sigma} \delta_\tau (\Gamma^\sigma_{\mu\nu})] H^\mu \overline{K^\tau} v^\nu
\]
\[
= \langle \langle H, K \rangle \rangle,
\]
and the assertion follows from (8.3).

We are left with the last term:

**Proposition 8.8**: Let $F: T^{1,0} M \to \mathbb{R}^+$ be a strongly pseudoconvex Finsler metric on a complex manifold $M$ with constant holomorphic curvature $2c \in \mathbb{R}$. Assume that (8.2) holds. Then
\[
\langle \Omega(H, \bar{\chi}) H, \chi \rangle = c \{ \langle H, \chi \rangle \langle K, \chi \rangle + \langle \chi, \chi \rangle \langle \langle H, K \rangle \rangle \}
\]
for all $H, K \in \mathcal{H}$.

**Proof**: First of all, we have
\[
\langle \Omega(H, \bar{\chi}) \chi, \chi \rangle = \langle \chi, \Omega(H, \overline{\chi}) \chi \rangle = \langle \Omega(H, \overline{\chi}) \chi, \chi \rangle = cG \langle H, \chi \rangle,
\]
by (5.5) and (8.6). Now take $W \in \mathcal{V}$ such that $\Theta(W) = K$; then
\[
W(cG \langle H, \chi \rangle) = c \{ \langle K, \chi \rangle \langle H, \chi \rangle + G \langle \nabla_W H, \chi \rangle \},
\]
\[
W \langle \Omega(H, \bar{\chi}) \chi, \chi \rangle = \langle \langle \nabla_W \Omega \rangle (H, \bar{\chi}) \chi, \chi \rangle + \langle \Omega(\nabla_W H, \bar{\chi}) \chi, \chi \rangle + \langle \Omega(H, \bar{\chi}) K, \chi \rangle,
\]
and so (8.15) yields
\[
\langle \Omega(H, \bar{\chi}) K, \chi \rangle = c \langle K, \chi \rangle \langle H, \chi \rangle - \langle \langle \nabla_W \Omega \rangle (H, \bar{\chi}) \chi, \chi \rangle.
\]

Now Lemma 8.2 gives
\[
\langle \langle \nabla_W \Omega \rangle (H, \bar{\chi}) \chi, \chi \rangle = \langle \tau^H(\theta(H, W), \bar{\chi}), \chi \rangle = \langle \Omega(\theta(H, W), \bar{\chi}) \chi, \chi \rangle
\]
\[
= cG \langle \theta(H, W), \chi \rangle,
\]
again by (8.15). But
\[
\langle \theta(H, W), \chi \rangle = -G_\alpha \Gamma^\alpha_{\nu\beta} K^\beta H^\nu = -\langle \langle H, K \rangle \rangle,
\]
and we are done.

We can finally collect all our computations in
Corollary 8.9: Let \( F: T^{1,0}M \to \mathbb{R}^+ \) be a weakly Kähler-Finsler metric on a complex manifold \( M \). Assume \( F \) has constant holomorphic curvature \( 2c \in \mathbb{R} \) and that (8.3) holds. Then
\[
\operatorname{Re} \left[ \left( \Omega(\chi, \bar{\Omega}H, \chi) - \langle \Omega(\chi, K), \chi \rangle + \langle \langle H, \tau^H(K, \chi) \rangle \rangle - \langle \langle H, \tau^H(\chi, K) \rangle \rangle \right) \right] = \frac{c}{2} \operatorname{Re} \left[ G \{ \langle H, K \rangle - \langle \langle H, K \rangle \rangle \} + \langle H, \chi \rangle \{ \langle \chi, K \rangle - 2\langle K, \chi \rangle \} \right]
\]
for all \( H, K \in \mathcal{H} \).

Proof: It follows from Propositions 8.5, 8.6, 8.7, 8.8 and Corollary 7.2.

We are then able to prove the announced

Theorem 8.10: Let \( F: T^{1,0}M \to \mathbb{R}^+ \) be a complete Finsler metric on a simply connected complex manifold \( M \). Assume that:

(i) \( F \) is Kähler;
(ii) \( F \) has nonpositive constant holomorphic curvature \( 2c \leq 0 \);
(iii) \( \langle \partial_H \theta(H, \chi), \chi \rangle = 0 \) for all \( H, K \in \mathcal{H} \);
(iv) the indicatrices of \( F \) are strongly convex.

Then \( \exp_p: T^{1,0}_p \to M \) is a homeomorphism, and a smooth diffeomorphism outside the origin, for any \( p \in M \). Furthermore, \( M \) is foliated by isometric totally geodesic holomorphic embeddings of the unit disk \( \Delta \) endowed with a suitable multiple of the Poincaré metric if \( c < 0 \), or by isometric totally geodesic holomorphic embeddings of \( \mathbb{C} \) endowed with the euclidean metric if \( c = 0 \). In particular, if \( 2c = -4 \) then \( F \) is the Kobayashi metric of \( M \), and if \( c = 0 \) then the Kobayashi metric of \( M \) vanishes identically.

Proof: Let \( F^\circ: T_R M \to \mathbb{R}^+ \) be the real Finsler metric associated to \( F \) as at the beginning of section 6. Then Corollary 7.2 and Corollary 8.9 show that the horizontal flag curvature of \( F^\circ \) is given by
\[
R(H, H) = \frac{c}{2} \operatorname{Re} \left\{ G \{ \langle H, H \rangle - \langle \langle H, H \rangle \rangle \} + \langle H, \chi \rangle \{ \langle \chi, H \rangle - 2\langle H, \chi \rangle \} \right\}.
\]
In particular, if \( H = \chi \) we get
\[
R(\chi, \chi) = 0,
\]
and if \( \langle H, \chi \rangle = 0 \) we get
\[
R(H, H) = \frac{cG}{2} \operatorname{Re} \left\{ \langle H, H \rangle - \langle \langle H, H \rangle \rangle \right\} = \frac{cG}{2} \operatorname{Re} \left\{ \langle iH, iH \rangle + \langle \langle iH, iH \rangle \rangle \right\}.
\]
Now, in local coordinates the quadratic form
\[
H \mapsto \operatorname{Re} \left\{ \langle H, H \rangle + \langle \langle H, H \rangle \rangle \right\}
\]
is represented by the Hessian of \( G \); by (iv), it is positive definite. So \( \langle H, \chi \rangle \) implies
\[
R(H, H) \leq 0.
\]
Now, take \( K \in \mathcal{H} \) and write \( K = \zeta \chi + H \), with \( \langle H, \chi \rangle = 0 \). Then Corollary 8.9, (8.16) and (8.17) yield
\[
R(K, K) = R(H, H) \leq 0.
\]
In conclusion, the horizontal flag curvature is negative semi-definite, and the first assertion follows from Auslander’s version of the Cartan-Hadamard theorem [Au2]. Finally, the last assertion has been proved under weaker assumptions in [AP2, 3].
We remark that, contrarily to what happens in the hermitian case, condition (iii) does not seem to be a consequence of the Kähler condition. For instance, the proof of Proposition 8.7 shows that if $F$ is weakly Kähler (but even Kähler does not help much) then condition (iii) holds iff

$$
\forall H, K \in \mathcal{H} \quad \langle H, \tau^H(\chi, K) \rangle = 0.
$$

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