Massless excitations at $\theta = \pi$ in the $CP^{N-1}$ model with large values of $N$

A. M. M. Pruisken$^a$,* I. S. Burmistrov$^{a,b,c}$ and R. Shankar$^d$

$^a$Institute for Theoretical Physics, University of Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands

$^b$L.D. Landau Institute for Theoretical Physics RAS, 117940 Moscow, Russia

$^c$Department of Theoretical Physics, Moscow Institute of Physics and Technology, 141700 Moscow, Russia

$^d$The Institute for Mathematical Sciences, CIT Campus, Chennai 600 113, India

Abstract

We study the instanton vacuum of the $CP^{N-1}$ model with large values of $N$ in 1+1 space-time dimensions. Unlike the longstanding claims which state that the theory always has a mass gap, we for the first time establish a complete critical theory for the transition at $\theta = \pi$ obtained from a mapping onto the low temperature phase of the 1D Ising model. We derive a simple effective field theory in terms of 1D massless chiral fermions. Our results include, besides a diverging correlation length with an exponent $\nu = 1/2$, exact expressions for the $\beta$ functions. These expressions unequivocally demonstrate that the large $N$ expansion with varying $\theta$ displays all the fundamental features of the quantum Hall effect.

Key words: instanton vacuum, quantum Hall effect, quantum criticality
PACS: 73.43-f, 11.10.Hi, 11.15.Pg

1 Introduction

The quantum Hall effect [1] has remained one of the most important and outstanding laboratory systems where the instanton angle $\theta$ in asymptotically free field theory can be explored and investigated in great detail. As is well

* Corresponding author. Fax: +31 20 525 57 88 (A. M. M. Pruisken)

Email addresses: pruisken@science.uva.nl (A. M. M. Pruisken), burmi@itp.ac.ru (I. S. Burmistrov), shankar@imsc.res.in (R. Shankar).
known, these topological ideas [2] originally arose in the attempt to reconcile von Klitzing’s experimental discovery [3] with the scaling theory of Anderson localization [4]. From the experimental side, this theory set the stage for the observation of quantum criticality of the quantum Hall plateau transitions [5] and this subject subsequently became an important research objective both in the laboratory [6] as well as on the computer [7]. From the theoretical side however, major gaps remained in our understanding of the conceptual structure of the problem. For example, it has only recently been demonstrated that the topological concept of an instanton vacuum quite generally displays robust topological quantum numbers that explain the precision and stability of the quantum Hall plateaus [8]. This fundamental strong coupling feature of the theory had remained concealed for many years. The reason being that the very general aspect of the $\theta$ vacuum concept, the existence of massless chiral edge excitations, had historically not been recognized.

The novel insight of massless edge excitations [9] has important consequences for longstanding controversies such as the quantization of topological charge, the existence of discrete topological sectors in the theory as well as the meaning of instantons and instanton gases [10]. A detailed understanding of these issues has shown, for example, that the instanton vacuum of the Grassmannian $U(m+n)/U(m) \times U(n)$ non-linear $\sigma$ model in two dimensions quite generally displays the fundamental features of the quantum Hall effect and not merely in the limit of zero number of field components $m = n = 0$ (replica limit) alone. These fundamental or super universal features include

(i) massless edge excitations that are otherwise well known to exist in quantum Hall systems;
(ii) robust quantization of the Hall conductance;
(iii) gapless bulk excitations at $\theta = \pi$ that generally facilitate a transition to take place between adjacent quantum Hall plateaus.

This concept of super universality has in many ways been foreshadowed by the original papers on the subject. It however, clearly invalidates the many conflicting expectations and ideas on the $\theta$ parameter and in particular the historical papers [11,12] on the large $N$ expansion of the $CP^{N-1}$ model [13] which is a special case of the Grassmannian theory obtained by putting $m = 1$ and $n = N$. We can now say that these historical papers have merely promoted the wrong physical ideas in the literature and served incorrect mathematical objectives.

In this paper we address a particularly delicate and fundamental issue, item (iii) above, about which there has been a great deal of confusion over many years. The root of this confusion are long standing claims which say that within the large $N$ expansion of the $CP^{N-1}$ model “the mass gap at $\theta = \pi$ remains finite” [14], “no critical exponents can be defined” [15] etc. Upon assuming
that these claims are true, the statement of *super universality* of the quantum Hall effect would obviously be incorrect and the theory would also not make much sense. For example, the fundamental lesson that quantum Hall physics has taught us is that one generally cannot disentwine the problem of *quantum criticality* at $\theta = \pi$ from the *existence* of the quantum Hall plateaus. This simply means that one cannot make any random statements about the *transition* at $\theta = \pi$ without having a detailed understanding of the *quantization phenomenon* itself. This lack of knowledge is one of the reasons why it is always assumed incorrectly in the literature that the physics of the quantum Hall effect is merely a feature of the theory in the replica limit alone [14,15] or, for that matter, the super symmetric extensions of the disordered free electron gas [16].

It has now been discovered [8] that the large $N$ expansion of the $CP^{N-1}$ model displays all the *super universal* features of the quantum Hall effect as listed under (i) - (iii) above, and also provides a lucid and exactly solvable example of the quantum Hall *plateau transition*. For example, for the first time explicit *finite size scaling* results for the physical observables ("conductances") have been obtained and the expressions are found to be very similar to those observed experimentally. Advances such as these are extremely important, especially since we are dealing with a theory where the information that can be extracted is very limited otherwise.

Even though the recent results [8] on finite size scaling have clearly demonstrated that a divergent correlation length exists at $\theta = \pi$ with an exponent equal to 1/2, they do not elucidate the *nature* of the massless excitations. For this purpose we develop, in this paper, a detailed *critical theory* of the large $N$ expansion, obtained by identifying the critical operators and their correlation functions. By presenting a complete theory that has freed itself from any historical controversies or preconceived mathematical biases, the authors essentially reestablish the original ideas on the subject, especially where it says that *gapless* excitations at $\theta = \pi$ are a generic topological feature of the instanton vacuum concept with the replica method only playing a role of secondary importance. [1,2]

We start out, in Section 2, from the Coulomb gas representation of the large $N$ theory which corresponds to the geometry of a finite cylinder with radius $\beta/2\pi$ and length $L$. The appropriate objects to consider are the local charges of the Coulomb gas that are controlled by a fugacity $\sigma/\beta$ which is exponentially small in the linear dimension $\beta$. In the language of the $U(1)$ gauge theory, these correspond to Polyakov lines which wind around the cylinder. By considering the limit of *infinite* cylinders (i.e. $L \to \infty$ and finite $\beta$) we recognize these local charges in terms of the bosonic "quarks" and "anti-quarks" that appear in Coleman’s argument for periodicity in $\theta$ [17]. These particles have a vanishing string tension as $\theta$ passes through $\pi$ at which point it becomes energetically
favorable for the system to materialize a quark anti-quark pair that moves in opposite directions toward “edges” at spatial infinity. We identify the Coulomb charges/Coleman’s quarks/Polyakov lines in terms of critical operators of a one dimensional critical theory. Based on an explicit knowledge of all multi-point correlation functions we are able to establish a one-to-one correspondence (Section 4) between the series expansion of the Coulomb gas in powers of the fugacity $\sigma/\beta$ on the one hand, and the low temperature series expansion of the 1D Ising model on the other. This correspondence immediately suggests that the transition at $\theta = \pi$ can be mapped onto none other than the 1D Ising model at low temperatures. This mapping is extremely helpful since the Ising model, as is well known, is a prototypical example of a first order phase transition with a divergent correlation length $\xi$ [18].

The remaining of this paper is largely devoted to the details of the mapping of the large $N$ expansion onto exactly solvable models in one dimension. We shall benefit in particular from our introduction of a simple effective field theory in terms of massless chiral fermions that elegantly displays the complete operator structure of the theory as well as an underlying orthogonal symmetry (Section 6). This theory has previously emerged as the theory of massless chiral edge excitations in quantum Hall systems [9].

Having identified a complete critical theory for the transition at $\theta = \pi$ we next wish to use our results in order to obtain exact expressions for the physical observables ($\sigma_{xx}$ and $\sigma_{xy}$) which define the scaling behavior of the theory in two dimensions with varying linear dimensions $\beta \approx L$. By the phrase “exact” we mean that our mapping procedure facilitates a resummation to all orders in the fugacity $\sigma/\beta$ of the Coulomb gas. For this purpose we investigate finite size Ising spin chains and chiral fermions with a linear dimension $L$. In Section 8 we obtain expressions for $\sigma_{xx}$ and $\sigma_{xy}$ in terms of Ising model and chiral fermion correlations which are some of the most important results of this paper. In Section 9, finally, we study the consequences of our results in terms of the renormalization group. We end this paper with a conclusion in Section 10.

2 $CP^{N-1}$ model with large values of $N$

The action of the $CP^{N-1}$ model on a finite cylinder is

$$S = \int_{-L/2}^{L/2} dx \int_0^\beta d\tau \left( \frac{1}{g} \sum_{\alpha=1}^N |(\partial_\mu - iA_\mu)z_\alpha|^2 + i \frac{\theta}{2\pi} \epsilon_{\mu\nu} \partial_\mu A_\nu \right)$$

where $\sum_{\alpha=1}^N z_\alpha^* z_\alpha = 1$. We define the theory with the fields satisfying periodic boundary conditions in the $\tau$ direction and free boundary conditions at $x = \pm L/2$. 

4
To motivate this choice of boundary conditions, we note that the theory on a finite cylinder can be thought of in different physical situations. Firstly, in the context of the low energy dynamics of disordered electron gas in strong magnetic fields with $\theta/2\pi$ denoting the mean field value of the Hall conductance or, equivalently, the filling fraction of the Landau bands. It also describes the long wavelength behaviour of a dimerised $SU(N)$ quantum spin chain at temperature $\beta^{-1}$ where $\theta/2\pi$ is related to the degree of dimerisation. Finally, it can be thought of as the finite temperature theory of $N$ charged relativistic scalar particles in one spatial dimension, strongly interacting with $U(1)$ gauge fields and in the presence of a background electric field proportional to $\theta/2\pi$.

In the case of the electronic system, it is well known that edge currents exist and that they are crucial to the phenomenon of the quantum Hall effect. The spin chain has dangling spins at the edges which are the low energy degrees of freedom in the strongly dimerised limit. In the context of scalar electrodynamics, Coleman’s picture [17] leads us to expect charged degrees of freedom (“quarks” and “anti quarks”) at the edges. Thus in all the three cases, we have reason to expect that the fluctuations of the fields at the boundary play an important role in the physics.

In what follows we shall hardly distinguish between these three different physical interpretations of the theory and frequently make use of any one of them, wherever it is convenient.

2.1 Sine-Gordon model

In previous papers we have introduced 1D sine-Gordon model or Coulomb gas representation for the large $N$ expansion of the $CP^{N-1}$ model that effectively describes the $\theta$ dependence for a finite two dimensional cylindrical geometry with edges [8]. Even though ideas very similar to ours have been proposed a long time ago [12], the most fundamental aspects of the problem have nevertheless been overlooked and the exact meaning of the Coulomb gas representation has therefore not been understood until recently. We shall proceed by recalling the main features of the theory as it now stands.

The $z_\alpha$ fields can be integrated out in the standard large $N$ saddle point approximation and the partition function is written as

$$Z = \int \mathcal{D}[A_\mu] \exp \left\{ -\int_0^\beta d\tau \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \left[ \frac{1}{2g} F_{\mu\nu} F_{\mu\nu} + \frac{2\sigma}{\beta^2} (1 - \cos \beta \tilde{A}_0) + i \frac{\theta}{4\pi} \epsilon_{\mu\nu} F_{\mu\nu} \right] \right\}. \quad (2)$$

Here, the coupling constant $g$ and fugacity $\sigma$ are expressed in terms of the
mass $M$ of the large $N$ expansion according to

$$g = \frac{24\pi M^2}{N}, \quad \sigma = N\sqrt{\frac{\beta M}{2\pi}} e^{-\beta M} \ll 1.$$  \hspace{1cm} (3)

The $\tilde{A}_0(x)$ is the zero frequency component of $A_0(x, \tau)$ defined as

$$\tilde{A}_0(x) \equiv \frac{1}{\beta} \int_0^\beta d\tau \ A_0(x, \tau).$$  \hspace{1cm} (4)

It follows that $\exp(i\beta \tilde{A}_0(x))$ (the Polyakov line) is a gauge invariant quantity and hence so is $\cos(\beta \tilde{A}_0(x))$. Eq. (2) shows that the different frequency components of $A_\mu$ are decoupled and only the zero frequency sector depends on $\sigma$ and $\theta$. Therefore, hereafter we concentrate solely on the zero frequency sector. This sector is independent of $A_x$ and depends only on $\tilde{A}_0$. We will also change notation and henceforth refer to $\tilde{A}_0(x)$ as $A_0(x)$.

As in previous work, we separate the edge and bulk degrees of freedom. Since the action is periodic under $A_0 \to A_0 + \frac{2\pi}{\beta}$, we introduce the following resolution of identity in the path integral,

$$1 = \int_{-\pi/\beta}^{\pi/\beta} da^l_0 da^r_0 \sum_{m_l, m_r} e^{im_l(\beta A_0(L/2) - a^l_0)} \ e^{im_r(\beta A_0(-L/2) - a^r_0)}.$$  \hspace{1cm} (5)

The partition function then gets written as,

$$Z = \int_{-\pi/\beta}^{\pi/\beta} da^l_0 da^r_0 \ Z[a^l_0, a^r_0],$$

$$Z[a^l_0, a^r_0] = \sum_{m_l, m_r} Z(m_l, m_r) \exp(-i\beta a^l_0 m_l + i\beta a^r_0 m_r)$$  \hspace{1cm} (6)

with the following meaning of the symbols. The $Z(m_l, m_r)$ is defined as an (unconstrained) integral over static scalar potential field $A_0(x)$ according to

$$Z(m_l, m_r) = \int D[A_0] \exp\left\{-\beta \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \left[\frac{1}{g} (\partial_x A_0)^2 + \frac{2\sigma}{\beta^2} (1 - \cos \beta A_0) + iA_0 \rho_m \right] \right\}.$$  \hspace{1cm} (7)

Here, $\rho_m$ represents the integral charges $m_l, m_r$ as well as external fractional charges $\theta/2\pi$ located at the opposite edges $x = \pm L/2$ of a system of linear spatial dimension $L$. It is given by

$$\rho_m(x) = \left( m_l + \frac{\theta}{2\pi} \right) \delta(x + L/2) + \left( m_r - \frac{\theta}{2\pi} \right) \delta(x - L/2).$$  \hspace{1cm} (8)

One of the most significant quantities are phase factors $e^{-i\beta a^l_0 m_l}$ and $e^{i\beta a^r_0 m_r}$ in Eq. (6) which can be interpreted in terms of the fluctuations in the topological
charge of the theory about its integral values. In terms of the original $CP^{N-1}$ variables, these phase factors are the Berry phases that appear in the time evolution of $s = |m_{l,r}|/2$ $SU(N)$ spin degrees of freedom located at the edge ($x = \mp L/2$) [19]. More specifically, $\exp(-i\beta a_l^0 m_l)$ and $\exp(i\beta a_r^0 m_r)$ are given by the following,

$$e^{-i\beta a_l^0 m_l} = \exp S_l[z^*,z], \quad e^{i\beta a_r^0 m_r} = \exp S_r[z^*,z].$$  \hspace{1cm} (9)

The action $S_{l,r}$ for spin dynamics is given in terms of the original $CP^{N-1}$ complex vector field variables $z_\alpha$ according to

$$S_{l,r}[z^*,z] = |m_{l,r}| \int_0^\beta d\tau \left[ \mp z^*_\alpha \partial_0 z_\alpha + b z^*_1 z_1 \right].$$  \hspace{1cm} (10)

Here, the quantity $b$ denotes an external magnetic field which defines the theory in the infrared.

2.2 Coulomb gas representation

By series expanding Eq. (6) in powers of $\sigma$ one immediately sees that the $\cos \beta A_0$ term generates local charges with $\sigma/\beta$ playing the role of fugacity. We can write

$$\exp \frac{2\sigma}{\beta} \int_{-L/2}^{L/2} dx \cos \beta A_0 = \sum_{n_+} \frac{(\sigma/\beta)^{n_+}}{n_+!} \prod_{k=1}^{n_+} \int_{-L/2}^{L/2} dx^+_k \prod_{j=1}^{n_-} \int_{-L/2}^{L/2} dx^-_j \exp i\beta \int_{-L/2}^{L/2} dx A_0(x) \rho_n(x)$$  \hspace{1cm} (11)

where $\rho_n$ denotes the charge density in the bulk of the system

$$\rho_n(x) = \sum_{k=1}^{n_+} \delta(x-x^+_k) - \sum_{j=1}^{n_-} \delta(x-x^-_j).$$  \hspace{1cm} (12)

The free field $A_0$ can now be eliminated and the final result can be written as

$$Z(m_l, m_r) = \sum_{n_+=0}^{\infty} \frac{\delta_{m_l-m_r,n_-}}{n_+!n_-!} \left( \frac{\sigma}{\beta} \right)^{n_++n_-} \prod_{k=1}^{n_+} \prod_{j=1}^{n_-} \int_{-L/2}^{L/2} dx^+_k \int_{-L/2}^{L/2} dx^-_j e^{-\beta \mathcal{H}_coul}.$$  \hspace{1cm} (13)

The hamiltonian

$$\mathcal{H}_coul = -\frac{g}{8} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} dxdy \rho_{mn}(x)|x-y|\rho_{mn}(y).$$  \hspace{1cm} (14)
describes a system of interacting charges in one dimension. The total charge
density is given by the sum of the edge parts \( \rho_m \) and bulk parts \( \rho_n \)
\[
\rho_{mn}(x) = \rho_n(x) + \rho_m(x).
\] (15)

Finally, the symbol \( \delta_{m_1 - m, n_1 - n} \) in Eq. (13) ensures the charge neutrality of
the Coulomb gas.

3 Quantum Hall effect

3.1 Introduction

The results of the previous Sections are unchanged when considered as a model
of the quantum Hall effect. The only difference is that the imaginary time \( \tau \)
now plays the role of the \( y \)-coordinate such that \( L \) and \( \beta \) represent the linear
dimensions of the system in the \( x \) and \( y \) directions respectively. The action \( S_{l,r} \)
in Eq. (10) now describes the massless chiral edge excitations in the problem
rather than spin dynamics, and the quantity \( b \) stands for external frequency
rather than magnetic field. Furthermore, to retain the notation of previous
work we shall from now onward use the symbol \( \kappa \) rather than \( g \), i.e.,
\[
\kappa = \frac{1}{2\beta g} = \frac{N}{48\pi M^2 \beta}.
\] (16)

The remarkable thing about the massless edge excitations is that they are
identically the same for all members of the \( U(m + n)/U(m) \times U(n) \) non-
linear \( \sigma \) model. This previously unrecognized aspect of the problem is the sole
reason why the “conductances” or “response parameters” \( \sigma_{xx} \) and \( \sigma_{xy} \), to be
discussed further below, are in fact the most important physical quantities in
the problem that are uniquely defined for all values of \( m \) and \( n \). This is in
spite of the fact that their true significance as transport coefficients is retained
in the theory with \( m = n = 0 \) only.

As has been explained at many place elsewhere, the correct expressions for the
“response parameters” emerge from the definition of the effective action for
the edge field variables \( z_\alpha \) and \( \bar{z}_\alpha \) and can quite generally be considered as a
measure for the response of the interior of the system to infinitesimal changes
in the boundary conditions. For this purpose it is convenient to employ the
quantities \( \beta a_0^l \) and \( \beta a_0^r \) in Eq. (6). By expanding the free energy in small
fluctuations \( a_0^l \) and \( a_0^r \) it can be shown that
\[
\ln Z[a_0^l, a_0^r] = -F(\theta) + i\langle \sigma_{xy}' \rangle \beta (a_0^l - a_0^r) - \langle \sigma_{xx}' \rangle \left( \frac{\beta a_0^l + a_0^r}{2} \right)^2 + \ldots
\] (17)
where $\langle \sigma'_{xx} \rangle$ and $\langle \sigma'_{xy} \rangle$ are in all respects analogous to the dimensionless longitu-
dinal and Hall conductance respectively of the disordered free electron gas of size $\beta \times L$. Similarly, the coefficients of the higher order terms in the series of Eq. (17) can be interpreted in terms of conductance fluctuations or conductance distributions. An interesting feature of the large $N$ expansion is that these conductance distributions can quite generally be expressed in terms of the ensemble averaged quantities $\langle \sigma'_{xx} \rangle$ and $\langle \sigma'_{xy} \rangle$ alone. We will come back to this point at the end of Section 3.3.

3.2 The quantum Hall state, $\theta \approx 2\pi k$

To understand how the theory generates the physics of the quantum Hall effect let us first evaluate the partition function of the Coulomb gas, Eq. (13), to lowest orders in the fugacity $\sigma/\beta$. By considering the terms with $(m_l, m_r) = (m, m)$ and $(m \pm 1, m)$ for an arbitrary integer $m$ we obtain the following expression [8]

$$Z[a^l_0, a^r_0] = \sum_{m \in \mathbb{Z}} \zeta(m) e^{-\frac{L}{4\kappa} \left(m + \frac{\theta}{2\pi}\right)^2 + im\beta(a^r_0 - a^l_0)}$$

where

$$\zeta(m) = 1 - \frac{4k\sigma}{\beta} e^{i\beta a^l_0 + e^{-i\beta a^r_0}} + \frac{4k\sigma}{\beta} e^{-i\beta a^l_0 + e^{i\beta a^r_0}}.$$

Next, we take the quantity $\theta/2\pi$, representing the filling fraction of the Landau bands, to lie in the interval $k - 1/2 < \theta/2\pi < k + 1/2$ for an arbitrary integer $k$. It is easy to see that the partition function of Eq. (18) is then dominated by the single term with $m = -k$ and for large system sizes $\beta L$ we therefore have

$$Z[a^l_0, a^r_0] = \exp \left[ -\frac{L}{4\kappa} \left( \frac{\theta}{2\pi} - k \right)^2 + ik\beta(a^r_0 - a^l_0) \right] + \ldots$$

All the other terms represented by $\ldots$ are smaller by factors that are exponential in $\beta$ or $\beta L$. By comparing the result with Eq. (17) we now recognize the integer $k$ as the robustly quantized Hall conductance. More precisely, we can say that for all filling fractions in the range $k - 1/2 < \theta/2\pi < k + 1/2$ we have $\langle \sigma'_{xy} \rangle = k$ and $\langle \sigma'_{xx} \rangle = 0$ except for corrections that are exponentially small in the system size. The results are therefore precisely in accordance with the experimental observations of the quantum Hall effect.

Notice that in the language of the Coulomb gas the quantum Hall state, labelled by the integer $k$, is synonymous for having “quarks” and “anti quarks” at the edges of the system such as to maximally shield the fractional charges $\pm \theta/2\pi$. However, besides the quantized charges $\pm k$, the “quarks” and “anti
quarks" at the edges also carry spin degrees of freedom. Following the discussion in Section 2.1 we conclude that the complete expression for the partition function for the quantum Hall state reads, instead of Eq. (20),

$$ Z[a^l_0, a^r_0] \rightarrow e^{-F(\theta)} Z_l Z_r $$

(21)

where

$$ F(\theta) = \frac{L}{4\kappa} \left( \frac{\theta}{2\pi} - k \right)^2, \quad Z_{l,r} = \int D[z^*, z] \ e^{S_{l,r}[z^*, z]} $$

(22)

denote the bulk free energy and the one dimensional partition functions for the edge spins respectively. Here, the action for the edge $S_{l,r}$ is the same as in Eq. (10) but with the integer $k$ now replacing the spin quantum numbers $m_l$ and $m_r$.

From the results of this Section it is clear that the identification of Eqs (21) and (22) with the quantum Hall state is solely based on the edge parts of the theory $Z_{l,r}$ that describe the well known edge currents in the problem. These subtleties of the edges have historically not been recognized, however.

3.3 Plateau transitions, $\theta \approx 2\pi(k + 1/2)$

A transition takes place between the adjacent quantum Hall states, labelled by the integers $k$ and $k + 1$, at the exact values $\theta/2\pi = (k + 1/2)$ which in the language of the electron gas corresponds to half-integer filling fractions. Notice that in the limit where $\beta L$ tends to infinity this plateau transition is infinitely sharp. Similarly, from Eq (20) we conclude that, in the same limit, the free energy of the bulk $F(\theta)$ develops a cusp at $\theta = (2k + 1)\pi$ according to

$$ F(\theta) \approx - \frac{|2k + 1 - \theta/\pi|}{8\kappa} L $$

(23)

indicating that the transition is a first order one. Next, to develop a better understanding of the nature of the plateau transitions we evaluate the expression for the partition function Eqs (18) and (19) for $\theta$ close to an odd multiple of $\pi$. Taking $k = 0$ for simplicity then the sum in Eq. (18) is dominated by the terms with $m = 0, -1$ and the result can be written as

$$ Z[q, a_0] = e^{-F(\theta)} \left( \mathcal{P}_0 e^{6\pi i q} + \mathcal{P}_\pi e^{\pi i q} \cos \beta a_0 + \mathcal{P}_{2\pi} e^{2\pi i q} \right). $$

(24)

We have introduced the symbols

$$ q = \frac{\beta}{2\pi}(a^l_0 - a^r_0), \quad a_0 = \frac{1}{2}(a^l_0 + a^r_0). $$

(25)

The quantities $\mathcal{P}$ sum up to unity, $\mathcal{P}_0 + \mathcal{P}_\pi + \mathcal{P}_{2\pi} = 1$, and can be expressed as follows
\[ P_0 = 1 - \langle \sigma'_{xy} \rangle - \langle \sigma'_{xx} \rangle \]  
(26)  
\[ P_\pi = 2\langle \sigma'_{xx} \rangle \]  
(27)  
\[ P_{2\pi} = \langle \sigma'_{xy} \rangle - \langle \sigma'_{xx} \rangle. \]  
(28)  
The following expressions for the “ensemble averaged” conductances \( \langle \sigma'_{xx} \rangle \), \( \langle \sigma'_{xy} \rangle \) have been obtained \[8\]

\[ \langle \sigma'_{xx} \rangle = \frac{\eta}{e^x + e^{-x} + 2\eta}, \]  
(29)  
\[ \langle \sigma'_{xy} \rangle = \frac{e^{-x} + \eta}{e^x + e^{-x} + 2\eta}. \]  
(30)  
The two different scaling variables \( X \) and \( \eta \) are given as

\[ X = \frac{L}{8\kappa} \left( 1 - \frac{\theta}{\pi} \right) \]  
(31)  
\[ \eta = \frac{L}{\beta \sigma} \sinh X. \]  
(32)  
These explicit expressions which are defined for a system with linear dimensions \( \beta \) and \( L \) play a central role in the remainder of this paper. The most important features of these finite size scaling results are the symmetry about \( \theta = \pi \) (“particle-hole” symmetry) and the fact that they display all the characteristics of a continuous transition with a diverging correlation length \( \xi \). To see this we put \( \beta = L \) which is the most natural geometry to consider (see, however, the discussion at the end of this Section). The scaling variable \( X \) in Eq. (31) can then be written in the following manner

\[ X = \pm \frac{L^2}{\xi^2}, \quad \xi \propto \left| 1 - \frac{\theta}{\pi} \right|^{-1/2}. \]  
(33)  
The results clearly show that the theory at \( \theta = \pi \) must have gapless excitations. Moreover, by expressing Eqs (29) - (30) in differential form we obtain the following general results for the \( \beta \) functions

\[ \frac{d \langle \sigma'_{xx} \rangle}{d \ln L} = \beta_{xx} \left( \langle \sigma'_{xx} \rangle, \langle \sigma'_{xy} \rangle \right) \]  
(34)  
\[ \frac{d \langle \sigma'_{xy} \rangle}{d \ln L} = \beta_{xy} \left( \langle \sigma'_{xx} \rangle, \langle \sigma'_{xy} \rangle \right). \]  
(35)  
The renormalization group flow lines in the \( \langle \sigma'_{xx} \rangle, \langle \sigma'_{xy} \rangle \) conductance plane are illustrated in Fig. 1. The general result of Eqs (34) - (35) has previously been established in the weak coupling regime \( \langle \sigma'_{xx} \rangle \gg 1 \) based on instantons \[20\]. Therefore, by combining the known weak coupling form for the \( \beta \) functions
with the strong coupling results based on Eqs (29) - (30) we obtain complete information on the general phase structure of the large $N$ expansion. The super universal features of this theory are the symmetry about the line $\langle \sigma'_{xy} \rangle$ equal to half integer values, the infrared stable fixed points located at $\langle \sigma'_{xy} \rangle = k$ and the unstable fixed points located at $\langle \sigma'_{xy} \rangle = k + 1/2$.

Several remarks are in order. First of all, the result of Eq. (24) indicates that the plateau transition is described in terms of an admixture of three distinctly different phases with a well defined probability $\mathcal{P}$ each. These phases are labelled by an exponential factor $e^{i\theta' q}$ with $\theta'$ taking on the values $0, \pi$ and $2\pi$ respectively. The quantity $\mathcal{P}_\pi$ in Eq. (24) can be interpreted as the probability of finding the system in the dissipative or critical phase labelled by $\theta' = \pi$. In the same way, $\mathcal{P}_{2\pi}$ denotes the probability of finding the system in the $\theta' = 2\pi$ vacuum or $k = 1$ quantum Hall phase. Similarly, $\mathcal{P}_0$ is the probability of finding the system in the $\theta' = 0$ vacuum or $k = 0$ quantum Hall phase. As we shall see in the analysis that follows, correlation functions of the theory display exactly the same general structure as that of Eqs (24) - (28).

Secondly, it should be mentioned that the result of Eq. (24) determines not only the “ensemble averaged” conductances $\langle \sigma'_{xx} \rangle$ and $\langle \sigma'_{xy} \rangle$ but also the complete statistics of conductance distributions [21]. For example, expressing Eq. (24) in the exponential form of Eq. (17) then we directly see that besides the lowest order terms in $q$ and $\beta a_0$ we actually have an infinite series of higher order moments. These higher order moments can all be expressed in terms of the “ensemble averaged” quantities $\langle \sigma'_{xx} \rangle$ and $\langle \sigma'_{xy} \rangle$, however, and it suffices to limit the analysis to the “ensemble averaged” quantities alone.

Thirdly, in spite of the rich structure that emerges, it is important to keep in mind that we have considered the Coulomb gas to lowest order in an expansion in powers of the fugacity $\sigma/\beta$ only. Even though the expressions in Eqs (29) - (30) are generally well defined in the limit $\beta \approx L \to \infty$, they nevertheless show that the expansion in the fugacity $\sigma/\beta$ actually diverges at $\theta = \pi$ in the limit where $L$ is taken to infinity first. This complication in defining the thermodynamic limit clearly indicates that the higher order terms in the expansion are important. This will be the main subject of Section 8 where we make use of the mapping onto the Ising model and chiral fermion theory in order to be able to re-sum the series in the fugacity $\sigma/\beta$ to infinite order.
Fig. 1. Renormalization group flow in the \(\langle \sigma'_{xx} \rangle\) and \(\langle \sigma'_{xy} \rangle\) conductance plane according to Eqs (29) - (32), see text.

4 Correlation functions

4.1 Introduction

To develop a theory for gapless excitations at \(\theta = \pi\) we next embark on the subject of correlation functions. We are specifically interested in the finite temperature correlation function of the Coulomb gas

\[
G_2(x, y) = \langle e^{i\beta A_0(x)} e^{-i\beta A_0(y)} \rangle = \left\langle \exp \left( -i\beta \int_x^y dx \partial_x A_0 \right) \right\rangle
\]

which corresponds to a pair of static charges or a quark anti-quark pair at positions \(x\) and \(y\) respectively. In the limit of zero fugacity \((\sigma = 0)\) we immediately find from Eq. (13) (putting, for simplicity, the spin degrees of freedom at the edges equal to zero, \(q = \beta a_0 = 0\))

\[
G_2(x, y) = \frac{\sum_{m \in \mathbb{Z}} e^{-\frac{L\beta}{4}(\theta/2\pi + m)^2} e^{-\frac{2\beta}{4}U_2(x, y; m)}}{\sum_{m \in \mathbb{Z}} e^{-\frac{L\beta}{4}(\theta/2\pi + m)^2}}
\]

(37)

where

\[
U_2(x, y; m) = (\theta/\pi + 2m)(x - y) + |x - y|.
\]

(38)
We are interested from now onward in the limit \( L \to \infty \) while keeping \( \beta \) fixed which, as we shall see below, enables us to make contact with Coleman’s argument for having periodicity in \( \theta \). As discussed in the previous Section, this limit precisely corresponds to the situation where the naive expansion in powers of the fugacity \( \sigma/\beta \) gets complicated. To deal with these complications we shall proceed and obtain, in the remaining parts of this Section, expressions for arbitrary multi point correlation functions of the Coulomb gas. These expressions then serve as a starting point in Section 5 for the mapping of the Coulomb gas problem onto exactly solvable models in one dimension.

### 4.2 Coleman’s picture

First, let \( \theta \) approach \( \pi \) from below. The sum in Eq. (37) is then dominated by the term with \( m = 0 \) and we can write

\[
G_2(x, y) = \vartheta(y - x)e^{-(1-\theta/\pi)|y-x|/4\kappa} + \vartheta(x - y)e^{-((1+\theta/\pi)|y-x|/4\kappa}}, \quad (\theta \to \pi^-). \tag{39}
\]

Here, \( \vartheta \) denotes the Heaviside step function. On the other hand, when \( \theta \) approaches \( \pi \) from above the sum is dominated by the \( m = 1 \) term and the result is

\[
G_2(x, y) = \vartheta(y - x)e^{-(3-\theta/\pi)|y-x|/4\kappa} + \vartheta(x - y)e^{-(1-\theta/\pi)|y-x|/4\kappa}}, \quad (\theta \to \pi^+). \tag{40}
\]

In the limit of large separations one may replace the correlation function \( G_2(x, y) \) by its long ranged part which will be denoted by \( g_2(x, y) \)

\[
G_2(x, y) \to g_2(x, y) = \vartheta(y - x)e^{-(1-\theta/\pi)|y-x|/4\kappa}}, \quad (\theta \to \pi^-) \tag{41}
\]
\[
G_2(x, y) \to g_2(x, y) = \vartheta(x - y)e^{+(1-\theta/\pi)|y-x|/4\kappa}}, \quad (\theta \to \pi^+). \tag{42}
\]

Eqs (41) and (42) describe the mechanism that, following Coleman, is responsible for adding/removing charges at the edges at infinity when \( \theta \) passes through \( \pi \). Eq. (41) tells us, for example, that when \( \theta \) passes through \( \pi \) from below, it is energetically favorable for the system to materialize a quark and an anti-quark that “move” in opposite directions to the edges such as to maximally shield the background fractional charges \( \pm \theta/2\pi \). This picture is consistent with the result of Eq. (42) which says that when \( \theta \) passes through \( \pi \) from above, the pair correlation is dominated by the \( m = 1 \) vacuum which means that a quark and an anti-quark are present at the edges at infinity.

Coleman’s picture is strictly valid for the Coulomb gas with a vanishing fugacity (or \( \beta \to \infty \)) only and the physical processes associated with finite values of \( \sigma/\beta \) or \( \beta \) are by no means obvious. Nevertheless, it is extremely important to recognize that Coleman’s picture is in fact synonymous for having massless
excitations at $\theta = \pi$. More specifically, the statements made by Eqs (41) and (42) are the simplest possible example of a critical theory and the scaling dimension of the critical operators (quarks or Polyakov lines) is equal to unity. Moreover, exponential factors such as $\exp[\pm(1 - \theta/\pi)|x - y|/4\kappa]$ have a quite general significance in critical phenomena theory and for the problem at hand they demonstrate that when $\theta$ approaches $\pi$ there is a diverging correlation length according to $\xi = 4\kappa|1 - \theta/\pi|^{-1}$.

When looked upon as a critical theory in two dimensions rather than one, then the critical operators are nonlocal objects and the exponential factors are more appropriately understood in terms of an area law $\exp[\beta|x - y|/\xi^2]$ where $\xi$ is now given by Eq. (33) and $\beta|x - y|$ denotes the area enclosed by the Polyakov lines. The same area law appears in the scaling results for the conductances, Eqs (29) - (32). This clearly indicates that both Coleman's mechanism and the quantum Hall plateau transition should generally be regarded as distinctly different consequences of the same fundamental principle, notably the existence of massless excitations at $\theta = \pi$.

It is important to keep in mind that the considerations of this Section primarily apply to the Coulomb gas with zero fugacity. To deal with the complications of the theory with finite values of $\sigma/\beta$ we next proceed and embark on the subject of multi-point correlation functions.

4.3 Four point correlations ($\theta \to \pi^-$)

Write

$$G_4(x_1y_1x_2y_2) = \langle e^{i\beta A_0(x_1)} e^{-i\beta A_0(y_1)} e^{i\beta A_0(x_2)} e^{-i\beta A_0(y_2)} \rangle$$

then we immediately obtain from Eq. (13)

$$G_4(x_1y_1x_2y_2) = \sum_{m \in \mathbb{Z}} e^{-\frac{4\kappa}{2\pi}(\theta/2\pi + m)^2} e^{-\frac{1}{4\kappa}U_4(x_1y_1x_2y_2;m)}$$

where $U_4$ is given by

$$U_4(x_1y_1x_2y_2;m) = (\theta/\pi + 2m)(y_1 - x_1 + y_2 - x_2) + |y_1 - x_1| + |y_2 - x_2|$$
$$+ |y_2 - x_1| + |x_2 - y_1| - |y_2 - y_1| - |x_2 - x_1|.$$  

For simplicity we consider only the case where $\theta$ approaches $\pi$ from below. Like $G_2$ the sum in Eq. (44) is then dominated by the $m = 0$ term only. Next, assume the following arrangement of the coordinates (cf. Ref. [22])

$$x_1 < y_1 < x_2 < y_2$$

15
then Eq. (45) simplifies and we can write

\[ U_4(x_1 y_1 x_2 y_2; m) = (1 - \theta/\pi)(y_1 - x_1 + y_2 - x_2). \]  

(47)

This result is invariant under the interchange \( y_1 \leftrightarrow y_2 \) and \( x_1 \leftrightarrow x_2 \). From Eqs (46) and (47) one immediately concludes that the long ranged parts of \( G_4 \) can be expressed according to

\[ G_4(x_1 y_1 x_2 y_2) \to g_4(x_1 y_1 x_2 y_2) \]

\[ = [\vartheta_3(x_1 y_1 x_2 y_2) + \vartheta_3(x_2 y_1 x_1 y_2) + \vartheta_3(x_1 y_2 x_2 y_1) + \vartheta_3(x_2 y_2 x_1 y_1)]e^{-2\omega(y_1 - x_1 + y_2 - x_2)} \]  

(48)

where we have introduced

\[ \vartheta_3(x_1 y_1 x_2 y_2) = \vartheta(y_1 - x_1)\vartheta(x_2 - y_1)\vartheta(y_2 - x_2) \]  

(49)

and

\[ \omega = \frac{1}{8\kappa} \left( 1 - \frac{\theta}{\pi} \right) > 0. \]  

(50)

4.4 Multi point correlations \((\theta \to \pi^-)\)

A generalization of the results of the previous Section is straight forward. Let the \(2n\)-point correlation function be denoted by

\[ G_{2n}(x_1, y_1, \ldots, x_n, y_n) = \langle \prod_{k=1}^{n} e^{iA_0(x_k)} e^{-iA_0(y_k)} \rangle \]  

(51)

then by using Eq. (7) we immediately find

\[ G_{2n}(x_1, y_1, \ldots, x_n, y_n) = \frac{\sum_{m \in \mathbb{Z}} e^{-\frac{\kappa}{4}(\theta/2\pi + m)^2} e^{-\frac{\kappa}{2}U_{2n}(x_1, y_1, \ldots, x_n, y_n; m)}}{\sum_{m \in \mathbb{Z}} e^{-\frac{\kappa}{4}(\theta/2\pi + m)^2}} \]  

(52)

where

\[ U_{2n} = (\theta/\pi + 2m) \sum_{j=1}^{n} (x_j - y_j) + \sum_{j=1}^{n} |x_j - y_j| + \sum_{j>i} (|y_j - x_i| + |x_j - y_i| - |x_j - x_i| - |y_j - y_i|). \]  

(53)

Consider the case where \( \theta \) approaches \( \pi \) from below then the sum in Eq. (52) is again dominated by the \( m = 0 \) term only. Next, assume the following
arrangement of the coordinates

\[ x_1 < y_1 < x_2 < y_2 < \cdots < y_k < x_{k+1} < \cdots < y_n. \] (54)

It can then easily be shown that

\[ U_{2n}(x_1, y_1, \ldots, x_n, y_n) = \left( 1 - \frac{\theta}{\pi} \right) \sum_{i=1}^{n} (y_i - x_i). \] (55)

Since Eq. (55) is invariant under all permutations \( P(x_1 \ldots x_n) \) and \( P(y_1 \ldots y_n) \) we finally obtain the following total result for the long ranged parts of \( G_{2n} \)

\[ G_{2n} \to g_{2n} = \sum_{P(x_1 \ldots x_n)} \sum_{P(y_1 \ldots y_n)} \vartheta_{2n-1}(x_1, y_1, \ldots, x_n, y_n) \exp \left[ -2\omega \sum_{i=1}^{n} (y_i - x_i) \right] \] (56)

where the first two sums are over all permutations and

\[ \vartheta_{2n-1}(x_1, y_1, \ldots, x_n, y_n) = \vartheta(y_1 - x_1) \prod_{j=2}^{n} \vartheta(x_j - y_{j-1}) \vartheta(y_j - x_j). \] (57)

5 1D Ising model

5.1 Introduction

Given the explicit form of the correlation functions of the Coulomb gas with fugacity \( \sigma/\beta \) equal to zero, we next wish to exponentiate the operators \( e^{\pm i\beta A_0} \) in order to solve the problem with finite values of the fugacity. More specifically, we are interested in the partition function of the Coulomb gas which can be written as an operator statement according to (see also Eq. (11))

\[ Z_{CG} = \left\langle \exp \left[ \frac{2\sigma}{\beta} \int dx \cos \beta A_0(x) \right] \right\rangle_{\theta \to \pi}. \] (58)

By writing Eq. (58) as a series expansion in powers of \( \sigma/\beta \) we can formally express the result in terms of the multi-point correlation functions of the \( m = 0 \) sector as considered in the previous Section. At this stage of the analysis it may not be entirely obvious that such a procedure provides indeed the correct answer in the limit \( L \to \infty \) and for varying values of \( \theta \approx \pi \). In the Sections below we shall show, however, that the series expansion of Eq. (58) is in one-to-one correspondence with the well known low temperature series expansion of the one dimensional Ising model. Once the correspondence with the Ising model is established we can proceed in a variety of different ways and solve the Coulomb gas problem of Eq. (58) based on the results obtained from certain exactly solvable models.
5.2 Domain wall operators

Let the 1D Ising model be defined by

\[ S = \sum_i \left[ K s_i s_{i+1} + \frac{H}{2} (s_i + s_{i+1}) \right] \]  

(59)

where the temperature factor is absorbed in the symbols \( K \) and \( H \). At low temperatures \( (K \to \infty) \) the free energy for the system with \( L \) spins becomes simply

\[ F = -\ln Z = -(K + |H|)L \]  

(60)

indicating, like Eq. (23), that a first order transition occurs at \( H = 0 \). The excitations of lowest energy are described by the domain walls separating an array of up-spins from an array of down-spins. Consider first \( H > 0 \) such that the ground state has all the spins upward. Let furthermore \( a_{+} (j) \) denote the operator that creates a domain wall at lattice site \( j \), i.e. \( s_i = +1 \) for \( i \leq j \) and \( s_i = -1 \) for \( i > j \). Similarly \( a_{-} (j) \) denotes the domain wall operator with the + and − spins interchanged. In short hand notation one can write

\[ \langle a_{+} (i) a_{-} (j) \rangle = e^{-4K} \vartheta (j - i) e^{-2H(j - i)} \quad (H > 0). \]  

(61)

In the same way one finds

\[ \langle a_{+} (i) a_{-} (j) \rangle = e^{-4K} \vartheta (i - j) e^{2H(j - 1)} \quad (H < 0). \]  

(62)

By using an appropriate definition of the step function \( \vartheta \) on the lattice then these expressions precisely correspond to those of the Coulomb gas, Eqs (41) and (42), obtained from the \( m = 0 \) and \( m = -1 \) sector respectively. Next, one can easily check that all the terms of the low temperature expansion of the Ising model are in one-to-one correspondence with those of the series expansion of the Coulomb gas in powers of \( \sigma / \beta \). Considering \( H \gtrsim 0 \) from now onward then, under the appropriate identification of parameters, one can express the four point correlation function in terms of Eq. (48)

\[ \langle a_{+} (i_1) a_{-} (j_1) a_{+} (i_2) a_{-} (j_2) \rangle = e^{-8K} g_4 (i_1, j_1, i_2, j_2). \]  

(63)

In the same way one can express the 2n point function in terms of \( g_{2n} \), Eq. (56),

\[ \langle a_{+} (i_1) a_{-} (j_1) \ldots a_{+} (i_n) a_{-} (j_n) \rangle = e^{-4nK} g_{2n} (i_1, j_1, \ldots, i_n, j_n). \]  

(64)
### Table 1

Mapping of 1D Coulomb gas onto the 1D Ising model

| Quantity            | Coulomb gas          | Ising               |
|---------------------|----------------------|---------------------|
| Symmetry breaking   | $\omega > 0$         | $H > 0$             |
| Fugacity            | $e^{-2K}$            | $e^{-2K}$           |
| Operator            | $\sigma e^{i\beta A_0}$ | $a_{+-}$            |
| Operator            | $\sigma e^{-i\beta A_0}$ | $a_{-+}$            |
| Partition function  | $\left\langle e^{\frac{2i\omega}{\beta} \int dx \cos \beta A_0(x)} \right\rangle_{\theta \to \pi^-}$ | $\left\langle e^{\sum (a_{+-}(i)+a_{-+}(i))} \right\rangle_{H \to 0^+}$ |

On the basis of the low temperature series one readily concludes that the partition function of the Ising model can be written as an operator statement

$$Z_{\text{Ising}} = \left\langle \exp \sum_i (a_{+-}(i) + a_{-+}(i)) \right\rangle_{H \to 0^+} \quad (65)$$

which is completely analogous to the statement of Eq. (58). We identify the Ising model quantity $e^{-2K}$ with the fugacity $\sigma/\beta$ of the Coulomb gas and the role played by $H$ is the same as $\omega$, see Table 1.

Notice that in making the comparison between Eqs (58) and (65) we have only taken the long ranged parts ($g_{2n}$) of the Coulomb gas correlations ($G_{2n}$) into account. Although this point may at first instance be discarded, the presence of short ranged contributions in $G_{2n}$ will nevertheless result in slightly different expressions for Eqs (58) and (65). These differences will be investigated in a systematic manner in Section 7.

## 6 1D Chiral fermions

The simplest formalism that most effectively deals with operator statements like Eqs (58) and (65) is provided by none other than the theory of 1D chiral fermions. This theory has previously emerged as the theory of *massless* chiral edge excitations in quantum Hall systems and is otherwise known from studies on quantum spin chains. The chiral fermion action is given in terms of fermion fields as follows

$$S = \int dx \bar{\Psi}(x)(i\partial_x + iH\tau_z)\Psi(x) \quad (66)$$

where $\bar{\Psi} = \{\bar{\Psi}_+, \bar{\Psi}_-\}$, $\Psi = \{\Psi_+, \Psi_-\}^T$ and $\tau_a$ with $a = x, y, z$ denote the Pauli matrices. The free energy $F = -\ln Z = -\ln \int \mathcal{D}[\bar{\Psi}, \Psi] \exp S$ with varying $H$
is readily obtained as

$$F = -\int dx \, \text{Tr} \ln(i\partial_x + iH\tau_z) = -|H|L. \quad (67)$$

This result indicates that chiral fermion theory, like the Coulomb gas (Eq. (23)) and the Ising model (Eq. (60)), describes a \textit{first order} transition as $H$ passes through zero. Next we consider the expectations ($H > 0$)

$$\langle \Psi_\pm(x)\Psi_\pm(y) \rangle = \int \frac{dk}{2\pi} e^{ik(x-y)} = \pm i\partial(\pm(y-x))e^{-H|y-x|}. \quad (68)$$

To make contact with the results of the Coulomb gas as well as the Ising model we next introduce the following quantities

$$a(x) = i\bar{\Psi}(x)\frac{\tau_x + i\tau_y}{2}\Psi(x) \quad (69)$$

$$\bar{a}(x) = i\bar{\Psi}(x)\frac{\tau_x - i\tau_y}{2}\Psi(x). \quad (70)$$

Notice that under the transformation $H \to -H$ the operators $a$ and $\bar{a}$ are interchanged. Assuming $H > 0$ from now onward then the expression for the multi point correlation function becomes

$$\langle \bar{a}(x_1)a(y_1)\ldots\bar{a}(x_n)a(y_n) \rangle = g_{2n}(x_1, y_1, \ldots, x_n, y_n) \quad (71)$$

which coincides \textit{exactly} with the results found for the Coulomb gas, Eq. (56).

The operator algebra of the chiral fermion theory involves one more operator denoted by $Q$

$$Q = -\frac{i}{\sqrt{2}}\bar{\Psi}\tau_z\Psi. \quad (72)$$

Some examples of non-vanishing correlation functions containing the $Q$ are given by

$$\langle Q(x) \rangle = \frac{1}{\sqrt{2}} \quad (73)$$

$$\langle Q(x)\bar{a}(y)a(z) \rangle = \frac{1}{\sqrt{2}}\partial(z-y)e^{-2H(z-y)} \quad (74)$$

$$+ \frac{1}{\sqrt{2}}\partial(x-z)\partial(y-z)\partial(y-x)e^{-2H(y-x)}.$$  

6.1 \textit{Tentative solution of the Coulomb gas problem}

To see how the chiral fermion action elucidates the complete singularity structure of the theory we proceed and map the operator statement of Eq. (65)
directly onto the theory of chiral fermions. The relation between the Ising model and chiral fermion theory is given by

$$\sum_i (a_+ (i) + a_- (i)) \leftrightarrow e^{-2K} \int dx (\bar{a}(x) + a(x)) = -ie^{-2K} \int dx \bar{\Psi}(x) \tau_z \Psi(x).$$

(75)

The effective action for the Ising model at finite but low temperatures can therefore be written as follows

$$S_{\text{Ising}} = \int dx \bar{\Psi}(x) (i\partial_x + iH \tau_z + ie^{-2K} \tau_x) \Psi(x).$$

(76)

This theory is solved in a trivial manner. Introducing an orthogonal rotation

$$U = \exp(i\phi \tau_y)$$ on the \(\bar{\Psi}, \Psi\) fields

$$\chi = e^{i\phi \tau_y} \Psi, \quad \phi = \frac{1}{2} \arcsin \frac{e^{-2K}}{\sqrt{H^2 + e^{-4K}}}$$

(77)

then the action is diagonal

$$S_{\text{Ising}} = \int dx \bar{\chi}(x) (i\partial_x + i\tilde{H} \tau_z) \chi(x).$$

(78)

Here, the \(\tilde{H}\) is defined by

$$\tilde{H} = \sqrt{H^2 + e^{-4K}}$$

(79)

indicating, as is well known, that the Ising system at finite temperatures displays a finite mass gap of size \(e^{-2K}\). As a final remark, it should be mentioned that the chiral fermion action of Eq. (76) does not yet provide the complete answer to the Coulomb gas problem. The main reason is that the correlation functions \(G_{2n}\) of the Coulomb gas have short distance contributions that are generally different from the asymptotic scaling form that we have denoted by \(g_{2n}\). We will embark on the subtle differences between the Ising model and the Coulomb gas in Section 7.

6.2 Comparison with Ising model at finite temperatures

6.2.1 Magnetization

Let us next compare the predictions based on the chiral fermion action of Eq. (78) with the exact solutions of the Ising model. For example, the magnetization per spin is obtained as follows

$$\mathcal{M} = -\frac{1}{L} \frac{\partial F}{\partial H} = \frac{H}{\sqrt{H^2 + e^{-4K}}}$$

(80)
which for small values of $H$ precisely corresponds to the exact result [18]

$$\mathcal{M} = \frac{\sinh H}{\sqrt{\sinh^2 H + e^{-4K}}}.$$  \tag{81}

Notice that the spontaneous magnetization vanishes everywhere except at zero temperature ($K \to \infty$) as it should be.

**6.2.2 Domain wall operators**

Of interest next are the Ising model two point correlations at finite temperatures. For this purpose we make use of the orthogonal rotation $U$ discussed in the previous Section and express the local operators $\bar{a}$, $a$ and $Q$ in terms of the quantities

$$\tilde{a}(x) = i\bar{\chi}(x)\frac{\tau_x + i\tau_y}{2}\chi(x)$$  \tag{82}

$$\tilde{\bar{a}}(x) = i\bar{\chi}(x)\frac{\tau_x - i\tau_y}{2}\chi(x)$$  \tag{83}

$$\tilde{Q} = -\frac{i}{\sqrt{2}}\bar{\chi}\tau_z\chi.$$  \tag{84}

for which the correlation functions are simple. The relation between the two different sets of operators can be written as follows

$$\begin{pmatrix} \tilde{\bar{a}} \\ \bar{a} \\ \tilde{Q} \end{pmatrix} = \begin{pmatrix} \cos^2 \phi & -\sin^2 \phi & -\sqrt{2}\sin \phi \cos \phi \\ -\sin^2 \phi & \cos^2 \phi & -\sqrt{2}\sin \phi \cos \phi \\ \sqrt{2}\sin \phi \cos \phi & \sqrt{2}\sin \phi \cos \phi & \cos^2 \phi - \sin^2 \phi \end{pmatrix} \begin{pmatrix} \tilde{a} \\ \bar{a} \\ \tilde{Q} \end{pmatrix}. \tag{85}$$

On the basis of Eq. (85) we obtain the following expression for the pair correlation

$$g_2(x, y) = \langle \tilde{a}(x)a(y) \rangle = m_0 + m_+ \vartheta(y - x)e^{-\tilde{H}|y-x|} + m_- \vartheta(x - y)e^{-\tilde{H}|y-x|} \tag{86}$$

where

$$m_0 = \frac{1}{4}(1 - \mathcal{M}^2) \tag{87}$$

$$m_\pm = \frac{1}{4}(1 \pm \mathcal{M})^2 \tag{88}$$

$$\tilde{H} = \frac{H}{\mathcal{M}}. \tag{89}$$
It is a matter of simple algebra to show that the results of this Section, which include the orthogonal rotation $U$, are all in one-to-one correspondence with those obtained from the standard transfer matrix approach to the Ising model (see Appendix).

6.3 Ising model mass gap

Before proceeding with the details of the mapping, it is helpful to first digress on the meaning of the Ising model mass gap $\tilde{H} = e^{-2K}$ at $H = 0$ (see Eq. 79). Even though mass generation may generally be regarded as one of most interesting aspects of the 1D Ising model, from the Coulomb gas or $\theta$ vacuum point of view the meaning of this phenomenon is very different, however, and this so because of the difference in dimensionality. Notice that in the language of the Coulomb gas (Table 1) the mass gap at $\theta = \pi$ is solely induced by the fugacity $\sigma/\beta$ in the problem and both quantities vanish in the limit where the linear dimension $\beta$ goes to infinity which is the limit of physical interest. Therefore, the large $N$ expansion at $\theta = \pi$ is fundamentally gapless and the results so far indicate that the sought after critical theory is the same as that of the 1D Ising model at low temperatures or, equivalently, the theory of chiral fermions.

To further elucidate meaning of the correlation functions of Eqs (86) - (89) for the Coulomb gas at $\theta = \pi$ we next consider the critical correlations for which the exponential factor $e^{-\tilde{H}|y-x|}$ is close to unity. This means that we take the $H$ field to be close to zero (or $\theta \approx \pi$) and, at the same time, the distance $|x - y|$ to be much smaller than the Ising model correlation length $e^{2K}$ (or $\beta/\sigma$). After simple algebra it follows directly that Eq. (86) can be written in the general form

$$\langle \bar{a}(x)a(y) \rangle = e^{-2H|y-x|/M} \left\{ \hat{P}_x + \hat{P}_0 \vartheta(y-x) + \hat{P}_{2\pi} \vartheta(x-y) \right\}$$

where

$$\hat{P}_0 = 1 - \tilde{\sigma}_{xy} - \tilde{\sigma}_{xx}$$

$$\hat{P}_x = 2\tilde{\sigma}_{xx}$$

$$\hat{P}_{2\pi} = \tilde{\sigma}_{xy} - \tilde{\sigma}_{xx}$$

and

$$\tilde{\sigma}_{xy} = (m_- + m_0) = \frac{1}{2}(1 - M)$$

$$\tilde{\sigma}_{xx} = m_0 \left( e^{2H|y-x|/M} - 1 \right) \approx |x - y| \frac{e^{-4K}}{\sqrt{H^2 + e^{-4K}}} \ll 1.$$
These expressions are completely analogous to those of the free energy of the Coulomb gas with finite values of $\beta$ and $L$ and varying boundary conditions, see Eq. (24). The quantities $\tilde{\sigma}_{xx}$ and $\tilde{\sigma}_{xy}$ in Eqs (94) and (95) have the same meaning as the conductance parameters $\langle \sigma'_{xx} \rangle$ and $\langle \sigma'_{xy} \rangle$ in Eq. (24). Their detailed dependence on $H$ or $\theta$ is very different, however, and these differences reflect the distinctly different ways in which the infrared of the system is being regulated in each case.

In summary, based on Eqs (90) - (95) we can say that the “conductance” parameters $\langle \sigma'_{xx} \rangle$ and $\langle \sigma'_{xy} \rangle$ quite generally reveal themselves as the most important physical observables of the large $N$ expansion. Notice that these quantities naturally emerge from the Coulomb gas provided the infrared of the system is properly defined, e.g. either by taking the linear dimension $L$ to be finite as in Eq. (24), or by working with finite values of the fugacity $\sigma/\beta$ as in Eqs (90) - (95). This means that $\langle \sigma'_{xx} \rangle$ and $\langle \sigma'_{xy} \rangle$ are the fundamental objects of the theory in which the super universal strong coupling features of the $\theta$ vacuum can generally be expressed. Finally, the results of Eqs (90) - (95) explain, at the same time, why Coleman’s picture of the transition at $\theta = \pi$, which is based on the zero temperature expressions of Eqs (41) and (42) alone, is in many ways too simple. This picture all by itself does not facilitate a correct analysis of the topological features of an “edge”, in particular the appearance of an edge spin or edge currents, nor does it recognize the existence of physical quantities like $\langle \sigma'_{xx} \rangle$ and $\langle \sigma'_{xy} \rangle$ that generally provide the most important information on the low energy dynamics of the $\theta$ vacuum. Given this lack of insight in the infrared properties that one generally can associate with the topological issue of an instanton vacuum, it may no longer be any surprise to know that the concept of super universality has historically been overlooked completely [14,15].

7 Mapping of 1D Coulomb gas onto 1D chiral fermions

7.1 Finite renormalizations

In this Section we complete the mapping of the Coulomb gas onto chiral fermions and address the short distance parts of the correlations $G_{2n}$ which so far have been ignored. These short distance parts can in general be easily separated from the long distance contributions that we have denoted by $g_{2n}$. The basic idea therefore is to proceed by eliminating the short distance correlations from the Coulomb gas problem of Eq. (58) while retaining all the long distance parts $g_{2n}$. This procedure can in principle be carried out order by order in a series expansion of Eq. (58) in powers of the fugacity $\sigma/\beta$. The aim of this procedure is to eventually express the Coulomb gas problem in terms
of the pure scaling operators $\bar{a}$ and $a$ of the chiral fermion theory, rather than the original charge operators $e^{\pm i\beta A_0}$. Since this elimination procedure generally involves lengthy but elementary computations we shall proceed by first quoting the final results. Then, instead of embarking on the details of the computations we shall, in Section 7.2, present a more elegant and effective computational scheme based on the Hamiltonian approach.

Assuming as in Eq. (58) that $\theta$ approaches $\pi$ from below then the mass term $2\sigma^2 \cos \beta A_0$ of the Coulomb gas problem can be expressed in terms of the pure scaling operators $\bar{a}$ and $a$ according to

$$\left\langle e^{\frac{2i}{\pi} \int dx \cos \beta A_0} \right\rangle_{\theta \to \pi^-} \equiv \left\langle e^{\frac{2i}{\pi} \int dx \cos \beta A_0} \right\rangle_{\omega} = \left\langle e^{i \theta/\pi} \int dx (Z\bar{a} + Za + Z_0) \right\rangle_{Z_\omega \omega}. \quad (96)$$

Here, the expectations are defined for the theory with fugacity zero and the limit $L \to \infty$ is understood. Eq. (96) tells us that the aforementioned elimination process generally involves three distinct renormalization coefficients, i.e., one for the $\omega$ field $\omega \to Z_\omega \omega$, a second one for the $\sigma$ variable $\sigma \to Z\sigma$ and a third one, $Z_0$, which is a constant. The quantities $Z_\omega$, $Z$ and $Z_0$ are regular functions for small values of $\omega$ and can be expressed in terms of a regular series expansion in powers of $\sigma/\beta$. To illustrate the procedure we expand the left-hand side of Eq. (96) in powers of $\sigma/\beta$. To lowest non-trivial order we obtain the following terms

$$\left(\frac{\sigma}{\beta}\right)^2 \int dx_1 \int dx_2 G_2(x_1, x_2) = \left(\frac{\sigma}{\beta}\right)^2 \int dx_1 \left[ \int dx_2 g_2(x_1, x_2) + \frac{4\kappa}{1 + \theta/\pi} \right]. \quad (97)$$

Here, the short-ranged term in $G_2(x_1, x_2)$, Eq. (39), has been integrated out explicitly. After re-exponentiation of Eq. (97) we immediately obtain the right-hand side of Eq. (96) with $Z = Z_\omega = 1$ and

$$Z_0 = \left(\frac{\sigma \kappa}{\beta}\right) \frac{4}{1 + \theta/\pi}. \quad (98)$$

By continuing along the same lines but now taking the higher order terms in $\sigma/\beta$ into account one finds

$$Z_\omega = 1 + \left(\frac{\sigma \kappa}{\beta}\right)^2 \frac{32}{(1 + \theta/\pi)(3 - \theta/\pi)}, \quad (99)$$

$$Z = 1 - \left(\frac{\sigma \kappa}{\beta}\right)^2 \left[ \frac{32}{(1 + \theta/\pi)(3 - \theta/\pi)} + \frac{8}{(1 + \theta/\pi)^2} + \frac{8}{(3 - \theta/\pi)^2} \right]. \quad (100)$$

By comparing Eq. (96) with the operator statements of chiral fermion theory, Section 6, one can say that the charged particle operators $e^{\pm i\beta A_0}$ of the Coulomb gas define an operator algebra that generally involves three critical
operators only, namely the quantities $\bar{a}$ and $a$ that are associated with the fugacity $\sigma/\beta$ and a distinctly different operator denoted by $Q$ that is associated with $\omega$.

We next extend the result of Eq. (96) to include the expectations of the Coulomb gas operators $e^{\pm i \beta A_0}$ but now for the theory with finite values of the fugacity. Using the same notation as before we can write

$$\langle e^{\frac{2\pi}{\beta} \int dx \cos \beta A_0} e^{i \beta A_0(x)} \rangle_{\omega} = \langle e^{\frac{2\pi}{\beta} \int dx (Z\bar{a} + Z_0)} (Z\bar{a}(x) + Z_0) \rangle_{Z_\omega \omega}$$

and a similar result for $e^{-i \beta A_0}$. For the pair correlation one finds

$$\langle e^{\frac{2\pi}{\beta} \int dx \cos \beta A_0} e^{i \beta A_0(x) - i \beta A_0(y)} \rangle_{\omega} = \langle e^{\frac{2\pi}{\beta} \int dx (Z\bar{a} + Z_0)} (Z\bar{a}(x) + Z_0) (Za(y) + Z_0) \rangle_{Z_\omega \omega}.$$  

Eqs (101) and (102) involve the same coefficients $Z$, $Z_\omega$ and $Z_0$ as those obtained before, i.e., Eqs (98),(99) and (100). We can therefore replace the charge operators $e^{\pm i \beta A_0}$ of the Coulomb gas by the pure scaling operators $\bar{a}$ and $a$ following the general rule (see Table 2)

$$e^{i \beta A_0(x)} \rightarrow Z\bar{a}(x) + Z_0$$
$$e^{-i \beta A_0(x)} \rightarrow Za(x) + Z_0.$$  

Evaluation of Eq. (102) yields the following result for pair correlation function

$$\langle e^{i \beta A_0(x) - i \beta A_0(y)} \rangle_{CG} = M_0 + M_+ \vartheta(y - x)e^{-2\tilde{\omega}|y - x|} + M_- \vartheta(x - y)e^{-2\tilde{\omega}|y - x|}$$

where

$$M_0 = \left( \frac{1}{2} Z \sqrt{1 - \tilde{M}^2} + Z_0 \right)^2$$

$$M_\pm = \frac{Z^2}{4} \left( 1 \pm \tilde{M} \right)^2$$

$$\tilde{M} = \frac{Z_\omega \omega}{\sqrt{Z_\omega^2 \omega^2 + Z^2 \sigma^2 / \beta^2}}.$$
Table 2
Critical operators in Coulomb gas representation, chiral fermion theory and the 1D Ising model

| Operator | Coulomb gas | Chiral fermions | 1D Ising |
|----------|-------------|----------------|----------|
| $a$      | $e^{i\beta A_0} = Z\bar{a} + Z_0$ | $i\bar{\Psi}^{\tau_x+i\tau_y}\Psi$ | $a_{+-} = \frac{\tau_x+i\tau_y}{2}$ |
| $\bar{a}$ | $e^{-i\beta A_0} = Za + Z_0$ | $i\bar{\Psi}^{\tau_x-i\tau_y}\Psi$ | $a_{++} = \frac{\tau_x-i\tau_y}{2}$ |
| $Q$      | $Z_\omega Q$ | $-\frac{i}{\sqrt{2}}\Psi\tau_z\Psi$ | $s = \frac{1}{\sqrt{2}}\tau_z$ |

7.2 Hamiltonian approach

In this Section we show how ordinary quantum mechanics can be used very effectively to compute the various different numerical aspects of Coulomb gas problem, in particular the coefficients $Z$, $Z_\omega$ and $Z_0$. For this purpose we consider the hamiltonian of the 1D Coulomb gas action, Eq. (7), which for infinite systems ($L \to \infty$) can be written as

$$\mathcal{H} = \frac{1}{4\kappa} \left(-i \frac{\partial}{\partial (\beta A_0)} - \frac{\theta}{2\pi}\right)^2 - 2\frac{\sigma}{\beta} \cos(\beta A_0).$$  \hspace{1cm} (108)

This hamiltonian acts on wave functions with periodic boundary conditions $\psi(\beta A_0 + 2\pi) = \psi(\beta A_0)$. In the limit of zero fugacity $\sigma/\beta = 0$ the eigenvalues and eigenfunctions of $\mathcal{H}$ are easily found

$$E_m^{(0)} = \frac{1}{4\kappa} \left(m + \frac{\theta}{2\pi}\right)^2, \quad \psi_m^{(0)} = \frac{1}{\sqrt{2\pi}} e^{-im\beta A_0}, \quad m \in \mathbb{Z}. \hspace{1cm} (109)$$

Notice that the energy levels of the $m = 0$ and $m = -1$ sectors cross one another at $\theta = \pi$. The $(2\sigma/\beta) \cos \beta A_0$ term in Eq. (108) produces a band splitting which can be dealt with using standard perturbation theory.

We consider the two-point function $G_2(x, y)$

$$G_2(x, y) = \begin{cases} \sum_{J=0}^{\infty} |\langle 0 | e^{-i\beta A_0} | J \rangle|^2 e^{-(E_J-E_0)|y-x|} & x \leq y, \\ \sum_{J=0}^{\infty} |\langle J | e^{-i\beta A_0} | 0 \rangle|^2 e^{-(E_J-E_0)|y-x|} & x > y. \end{cases} \hspace{1cm} (110)$$

Here $E_J$ and $|J\rangle$ denote the exact eigenvalues and eigenstates respectively. The long range part of $G_2(x, y)$ is determined by the terms in Eq. (110) with $J = 0, 1$. 

27
\[ g_2(x, y) = |\langle 0 | e^{-i\beta A_0} | 0 \rangle|^2 + |\langle 0 | e^{-i\beta A_0} | 1 \rangle|^2 \vartheta(y - x) e^{-(E_1 - E_0)|y - x|} + |\langle 1 | e^{-i\beta A_0} | 0 \rangle|^2 \vartheta(x - y) e^{-(E_1 - E_0)|y - x|}. \] (111)

To study the limit \( \theta \to \pi^- \) we proceed by first projecting the Hamiltonian onto the subspace of eigenfunctions \( \psi_0^{(0)} \) and \( \psi_{-1}^{(0)} \). The following estimates for two lowest energies are obtained

\[ E_{0,1}^{(1)} = \frac{1}{4\kappa} \left( \frac{\theta}{2\pi} \right)^2 + \omega \mp \sqrt{\omega^2 + (\sigma/\beta)^2}. \] (112)

The corresponding eigenfunctions are

\[ \begin{pmatrix} \psi_0^{(1)} \\ \psi_{-1}^{(1)} \end{pmatrix} = e^{i\phi \tau_y} \begin{pmatrix} \psi_0^{(0)} \\ \psi_{-1}^{(0)} \end{pmatrix}. \] (113)

Here, \( \omega \) is defined by Eq. (50) and the angle \( \phi \) by Eq. (77). Using these results we obtain the same expression for \( g_2 \) as in Eq. (86), i.e.,

\[ g_2(x, y) = m_0 + m_+ \vartheta(y - x) e^{-2\tilde{\omega}|y - x|} + m_- \vartheta(x - y) e^{-2\tilde{\omega}|y - x|}. \] (114)

with \( \tilde{\omega} = \sqrt{\omega^2 + (\sigma/\beta)^2} \). The quantities \( m_0 \) and \( m_\pm \), under the appropriate substitution of parameters (see Table 1), are given by Eqs (87)-(88).

From this point onward we use perturbation theory in the fugacity \( \sigma/\beta \), taking into account the presence of the other levels. The results when compared to the general expression of Eq. (104) can be used to extract the coefficients \( Z_Z \), \( Z_0 \) and \( Z_\omega \). For example, for the energy gap between the first excited state and the ground state we obtain to next leading order in \( \sigma/\beta \)

\[ E_1^{(1)} - E_0^{(1)} = 2\sqrt{\omega^2 Z_\omega^2 + (\sigma/\beta)^2} \] (115)

where \( Z_\omega \) is given by Eq. (99). Similarly, the eigenfunctions to next leading order are given by

\[ \psi_0^{(2)} = \begin{bmatrix} 1 - \left( \frac{4\kappa \sigma}{\beta} \right)^2 \frac{1}{2(1 + \theta/\pi)^2} \end{bmatrix} \psi_0^{(1)} + \frac{4\kappa \sigma}{\beta} \frac{\sin \phi}{2(2 - \theta/\pi)} \psi_{-2}^{(0)} \] (116)

\[ + \frac{4\kappa \sigma}{\beta} \frac{\cos \phi}{1 + \theta/\pi} \psi_1^{(0)} + \frac{4\kappa \sigma}{\beta} \left( \frac{4\kappa \sigma}{\beta} \right)^2 \frac{1}{2(1 + \theta/\pi)(2 + \theta/\pi)} \psi_2^{(0)} \] (117)

\[ \psi_{-1}^{(2)} = \begin{bmatrix} 1 - \left( \frac{4\kappa \sigma}{\beta} \right)^2 \frac{1}{2(3 - \theta/\pi)^2} \end{bmatrix} \psi_{-1}^{(1)} + \frac{4\kappa \sigma}{\beta} \frac{\cos \phi}{3 - \theta/\pi} \psi_{-2}^{(0)} \] (118)

\[ - \frac{4\kappa \sigma}{\beta} \sin \phi \frac{1}{2\theta/\pi} \psi_1^{(0)} + \frac{4\kappa \sigma}{\beta} \left( \frac{4\kappa \sigma}{\beta} \right)^2 \frac{1}{2(3 - \theta/\pi)(4 - \theta/\pi)} \psi_3^{(0)} \] (119)
Using these expressions we find for the matrix elements in Eq. (111)

\[ |\langle 0| e^{-i\beta A_0} |0\rangle|^2 = \left( \frac{Z_2}{2} \sin 2\phi + Z_0 \right)^2, \quad |\langle 0| e^{-i\beta A_0} |1\rangle|^2 = Z^2 \cos^4 \phi \]  

(120)

where the quantities \( Z_0 \) and \( Z \) are given by Eqs. (98) and (100) respectively.

8 Finite size systems

8.1 Introduction

We have now completed one of the main objectives of this paper which is to lay the bridge between the large \( N \) expansion or Coulomb gas near \( \theta = \pi \) and exactly solvable models in one dimension. We next wish to extend this mapping to include the Coulomb gas with varying linear dimensions \( \beta \) and \( L \) which defines the conductances \( \langle \sigma'_{xx} \rangle \) and \( \langle \sigma'_{xy} \rangle \). For this purpose we will study spin chains and chiral fermion theory with finite values of \( L \). By expressing the conductances in terms of both Ising model and chiral fermion correlations we are able to extend the previously obtained scaling results of Eqs (29) and (30) to include infinite orders in the fugacity \( \sigma/\beta \). The final expressions that we obtain have the appropriate behavior in the thermodynamic limit \( L, \beta \to \infty \) and serve as a starting point in Section 9 where we discuss the renormalization behavior of the Coulomb gas.

8.2 1D Ising model (I)

To start we consider the partition function of an Ising spin chain of length \( L \). In terms of the transfer matrix

\[ T = e^K \left( \cosh H + \tau_z \sinh H + \tau_x e^{-2K} \right) \]  

(121)

we write

\[ Z = \text{Tr} T^L B. \]  

(122)

Here, the \( 2 \times 2 \) matrix \( B \) defines the boundary conditions on the spin chain. For example, \( B = 1 \) corresponds to periodic boundary conditions, \( B = \tau_x \) describes twisted boundary conditions and \( B = \cosh H + \tau_z \sinh H + \tau_x \) corresponds to free (or no) boundary conditions.

The idea next is to find the explicit form for \( B \) such that Eq. (122) can be identified with the partition function of the Coulomb gas, Eqs (24). After some
investigation it is not difficult to see that the correct expression for the matrix $B$ is given by

$$B(q, \beta a_0) = \begin{pmatrix} 1 & e^{i\pi q + i\beta a_0} \\ e^{i\pi q - i\beta a_0} & e^{2i\pi q} \end{pmatrix} \quad (123)$$

and the partition function of the Coulomb gas, Eqs (24), can be obtained as

$$Z[q, \beta a_0] = \frac{\text{Tr} \, T^L \, B(q, \beta a_0)}{\text{Tr} \, T^L \, B(0, 0)}. \quad (124)$$

To show this we consider the low temperature limit $K \to \infty$. To lowest non-trivial order in an expansion in powers of $e^{-2K}$ we can write Eq. (124) in the general form

$$Z[q, \beta a_0] = \left(1 - \langle \sigma'_{xy} \rangle - \langle \sigma'_{xx} \rangle\right) e^{0i q} + 2\langle \sigma'_{xx} \rangle e^{\pi i q} \cos \beta a_0 + \left(\langle \sigma'_{xy} \rangle - \langle \sigma'_{xx} \rangle\right) e^{2\pi i q}. \quad (125)$$

Here, $\langle \sigma'_{xx} \rangle$ and $\langle \sigma'_{xy} \rangle$ are given by the same expressions as in Eqs (29) and (30), i.e.,

$$\langle \sigma'_{xx} \rangle = \eta e^{X} + e^{-X} + 2\eta \quad (126)$$

$$\langle \sigma'_{xy} \rangle = \frac{1}{2} \left[1 - \frac{e^{X} - e^{-X}}{e^{X} + e^{-X} + 2\eta}\right]. \quad (127)$$

The quantities $X$ and $\eta$ now defined as

$$X = LH \quad (128)$$

$$\eta = L e^{-2K} \frac{\sinh LH}{L \sinh H} \to L e^{-2K} \frac{\sinh X}{X} \quad (129)$$

where in the last step we have taken the limit of small $H$. By comparing Eqs (128) and (129) with Eqs (31) and (32) we conclude that under the appropriate substitution of variables (see Table 1) the mapping of the Coulomb gas problem onto the 1D Ising model is retained also for finite size $L$.

8.3 1D Ising model (II)

We now can proceed and perform the summation of the series in powers of $e^{-2K}$ to infinite order by employing the orthogonal rotation $U$ introduced in
Section 6.1. More specifically, in the limit of small $e^{-2K}$ and $H$ we can write the transfer matrix $T$ as follows

$$T = e^{2K}U^{-1}e^{H\tau_z}U.$$  \hfill (130)

The quantity $Z[q, \beta a_0]$ can therefore be expressed as

$$Z[q, \beta a_0] = \frac{\text{Tr} e^{LH\tau_z} U B(q, \beta a_0) U^{-1}}{\text{Tr} e^{LH\tau_z} U B(0, 0) U^{-1}}.$$ \hfill (131)

The result is of the same general form as Eq. (125) but with the conductances $\langle \sigma'_{xx} \rangle$ and $\langle \sigma'_{xy} \rangle$ now given as

$$\langle \sigma'_{xx} \rangle = \frac{\eta}{e^X + e^{-X} + 2\eta}$$ \hfill (132)

$$\langle \sigma'_{xy} \rangle = \frac{1}{2} \left[ 1 - \frac{\sqrt{(e^X - e^{-X})^2 - 4\eta^2}}{e^X + e^{-X} + 2\eta} \right] = \frac{1}{2} \left[ 1 - \mathcal{M} \frac{e^X - e^{-X}}{e^X + e^{-X} + 2\eta} \right]$$ \hfill (133)

where

$$X = L\bar{H} = \sqrt{H^2 + e^{-4K}}$$ \hfill (134)

$$\eta = \sqrt{1 - \mathcal{M}^2} \sinh X = L e^{-2K} \sinh X.$$ \hfill (135)

Notice that these expressions cannot be obtained from the lowest order results of Eqs (126) - (129) by considering the series expansion to any finite order in powers of $e^{-2K}$. In the language of the Coulomb gas we can say that the higher order contributions in the fugacity $\sigma/\beta$ actually diverge as $\theta$ approaches $\pi$ and the infinite order results of Eqs (132) - (135) are therefore qualitatively different from the lowest order ones. For example, unlike Eqs (126) - (129) we can now consider the limit $L \to \infty$ keeping $e^{-2K}$ or $\sigma/\beta$ finite and the result is

$$\langle \sigma'_{xx} \rangle = \frac{1}{2} \frac{\sqrt{1 - \mathcal{M}^2}}{1 + \sqrt{1 - \mathcal{M}^2}}$$ \hfill (136)

$$\langle \sigma'_{xy} \rangle = \frac{1}{2} \left[ 1 - \frac{\mathcal{M}}{1 + \sqrt{1 - \mathcal{M}^2}} \right].$$ \hfill (137)

These expressions are well behaved at $H = 0$ (or $\theta = \pi$) and very similar to those obtained from the correlation functions, Eqs (94) and (95). Moreover, by taking $e^{-2K}$ or $\sigma/\beta$ to zero we obtain a sharp transition between the quantum Hall plateaus, i.e. $\langle \sigma'_{xx} \rangle = \langle \sigma'_{xy} \rangle = 0$ for $H > 0$ ($\theta < \pi$) and $\langle \sigma'_{xx} \rangle = 0$, $\langle \sigma'_{xy} \rangle = 1$ for $H < 0$ ($\theta > \pi$) as it should be.
From Eqs (136) and (137) we furthermore conclude that the role of the fugacity \( \sigma/\beta \) in the Coulomb gas problem or large \( N \) expansion is very similar to that of the temperature or external frequencies in the theory of localization and interaction effects. For example, the results of Eqs (136) and (137) are independent of the linear dimension \( L \) of the system and effectively describe the conductances of a “finite” sample with linear dimensions \( \beta \) and \( L = \xi = \beta/\sigma \) respectively. The consequences of Eqs (132) - (135) for the Coulomb gas problem will be discussed further in Section 8.6.

8.4 1D Ising model (III)

Although the one-to-one correspondence between the Coulomb gas and the 1D Ising model is limited to the low temperature regime \( e^{-2K} \to 0 \) of the latter only, it is nevertheless instructive to express the definition of \( Z[q, \beta a_0] \), Eq. (124), quite generally in terms of the Ising model parameters \( K \) and \( H \). For this purpose, write \( T^L \) in the form

\[
(T^L)_{\sigma\sigma'} = \exp \left[ -F + K'\sigma\sigma' + \frac{H'}{2}(\sigma + \sigma') \right].
\]

Here

\[
K' = -\ln \frac{\eta}{2}
\]
\[
H' = \ln \frac{\eta}{2} + \ln \left( \frac{1}{2} + \sqrt{1 - \mathcal{M}^2} \frac{\eta}{\mathcal{M}} \right)
\]

can be taken as the effective Ising model parameters of a chain of length \( L \). The results can again be written in the general form of Eq. (125) but with the parameters \( \langle \sigma'_{xx} \rangle \) and \( \langle \sigma'_{xy} \rangle \) now given by

\[
\langle \sigma'_{xx} \rangle = \frac{e^{-2K'}}{e^{H'} + e^{-H'} + 2e^{-2K'}}
\]
\[
\langle \sigma'_{xy} \rangle = \frac{e^{-H'} + e^{-2K'}}{e^{H'} + e^{-H'} + 2e^{-2K'}} = \frac{1}{2} \left( 1 - \frac{e^{H'} - e^{-H'}}{e^{H'} + e^{-H'} + 2e^{-2K'}} \right)
\]

It is easy to see that these general expressions reduce to the results of Eqs (132) and (133) in the limit of low temperatures and small \( H \).
8.5 Chiral fermions

We next introduce the idea of finite system sizes in the theory of chiral fermions. For this purpose we write the action of Eq. (66) as an integral over the finite interval $0 \leq x \leq L$

$$S_{\text{eff}} = \int_{0}^{L} dx \bar{\Psi} (i \partial_x + i H \tau_z + i e^{-2K} \tau_x) \Psi. \quad (143)$$

Assuming anti-periodic boundary conditions on the fermion fields for simplicity then we can express the derivative $i \partial_x$ in terms of a discrete set of frequencies $\omega_n = L^{-1} \pi (2n + 1)$. Next, by interpreting $x$ as the imaginary time and $L$ as the inverse temperature then the action of Eq. (143) corresponds to the following spin hamiltonian

$$\mathcal{H} = -H \tau_z - e^{-2K} \tau_x. \quad (144)$$

From the analysis on boundary conditions in the previous Sections we infer that Eq. (24) can be expressed in the hamiltonian formalism according to

$$Z[q, \beta a_0] = \frac{\text{Tr} B(q, \beta a_0) e^{-L \mathcal{H}}}{\text{Tr} B(0, 0) e^{-L \mathcal{H}}}. \quad (145)$$

The equivalence of Eqs (145) and (124) is readily established once it is recognized that in the limit of small $e^{-2K}$ and $H$ we can write

$$T^L = e^{2LK} e^{-L \mathcal{H}}. \quad (146)$$

The results obtained from chiral fermion theory are therefore identically the same as those obtained from the Ising model, Eq. (125), with the conductances given as in Section 8.3.

8.6 Coulomb gas

8.6.1 Conductances

As an important check on the results derived in Sections 8.3 and 8.5 we next present the expressions for the conductance parameters $\langle \sigma'_{xx} \rangle$ and $\langle \sigma'_{xy} \rangle$ as obtained directly from the Coulomb gas representation in a computation to second order in the fugacity $\sigma / \beta$. To start, we first list the results for the partition function which can be written as in Eq. (18), i.e.

$$Z[q, \beta a_0, \theta] = \sum_{m \in \mathbb{Z}} \zeta(m) e^{-\frac{1}{4\beta} (m+\theta/2\pi)^2 - i2\pi mq}. \quad (147)$$
The complete expression for $\zeta(m)$ to second order in $\sigma/\beta$ is as follows

$$
\zeta(m) = 1 - \frac{8\kappa\sigma}{\beta} \cos \beta a_0 \left[ \frac{e^{i\pi q}}{2m - 1 + \frac{\theta}{\pi}} - \frac{e^{-i\pi q}}{2m + 1 + \frac{\theta}{\pi}} \right] (148)
$$

$$
+ \left( \frac{4\kappa\sigma}{\beta} \right)^2 \cos 2\beta a_0 \left[ -\frac{2}{(2m + 1 + \frac{\theta}{\pi})(2m - 1 + \frac{\theta}{\pi})} \right]

+ \left( \frac{4\kappa\sigma}{\beta} \right)^2 \left[ \frac{L}{4\kappa} \left( \frac{1}{2m + 1 + \frac{\theta}{\pi}} - \frac{1}{2m - 1 + \frac{\theta}{\pi}} \right) \right]

+ \frac{1 - e^{i2\pi q}}{(2m - 1 + \frac{\theta}{\pi})^2} - \frac{1 - e^{-i2\pi q}}{(2m + 1 + \frac{\theta}{\pi})^2} \right]. (149)
$$

The contributions proportional to $\cos 2\beta a_0$ arise from the terms with charges $n_+ = 0, 2$ and $n_- = 2, 0$ in the interior of the system, see Eq. (13). Similarly, the other contributions of the order $(\sigma/\beta)^2$ originate from the terms with $n_+ = n_- = 1$. Notice that Eq. (147) has the following symmetry

$$
Z[q, \beta a_0, 2\pi - \theta] = e^{i2\pi q} Z[-q, \beta a_0, \theta]. (150)
$$

This result ensures that the “particle-hole symmetry” is retained by both the free energy and the conductances (see Eq. (17))

$$
F(2\pi - \theta) = F(\theta) (151)
$$

$$
\langle \sigma'_{xx}(2\pi - \theta) \rangle = \langle \sigma'_{xx}(\theta) \rangle (152)
$$

$$
\langle \sigma'_{xy}(2\pi - \theta) \rangle = 1 - \langle \sigma'_{xy}(\theta) \rangle. (153)
$$

Next, considering the limit $\theta \to \pi^-$ then the sum in Eq. (147) is dominated by the terms with $m = 0, -1$ only. The results for the conductances can be written as follows

$$
\langle \sigma'_{xx} \rangle = \frac{\eta_+(\omega)}{f(\omega) e^{i\omega L} + f(-\omega) e^{-i\omega L} + 2\eta_+(\omega)} (154)
$$

$$
\langle \sigma'_{xy} \rangle = \frac{1}{2} \left[ 1 - \frac{f(\omega) e^{i\omega L} - f(-\omega) e^{-i\omega L} + 2\eta_-(\omega)}{f(\omega) e^{i\omega L} + f(-\omega) e^{-i\omega L} + 2\eta_+(\omega)} \right] (155)
$$

with the following meaning of the symbols
\[ \eta_{\pm} = g_{\pm}(\omega)e^{\pm L} \pm g_{\pm}(-\omega)e^{-\omega L} \]

\[ f(\omega) = 1 + \left( \frac{\sigma}{\beta \omega} \right)^2 \frac{L\omega}{1 - 4\kappa \omega} \left[ 1 - \frac{6\kappa}{L} \left( 1 + \frac{32\kappa \omega(1 - 4\kappa \omega)}{(1 + 8\kappa \omega)(3 - 8\kappa \omega)} \right) \right] \]

\[ g_+(\omega) = \left( \frac{\sigma}{\beta \omega} \right) \frac{1}{2(1 - 4\kappa \omega)} + \left( \frac{\sigma}{\beta \omega} \right)^2 \frac{2(8\kappa \omega)^2}{(1 + 8\kappa \omega)(3 - 8\kappa \omega)} \]

\[ g_-(\omega) = \left( \frac{\sigma}{\beta \omega} \right) \frac{4\kappa \omega}{1 - 4\kappa \omega} - \frac{1}{4} \left( \frac{\sigma}{\beta \omega} \right)^2 \left[ 1 + \frac{8\kappa \omega}{1 + 8\kappa \omega} - \frac{(4\kappa \omega)^2}{(1 - 4\kappa \omega)(3 - 8\kappa \omega)} \right] \]

Notice that the functions \( f \) and \( g_{\pm} \) are given in terms of a regular series expansion in powers of \( \sigma / (\beta \omega) \), \( \omega L \) as well as \( \kappa \omega \). To understand these results let us first write Eqs (154) and (155) in a slightly more transparent manner according to

\[ \langle \sigma'_{xx} \rangle = \frac{\tilde{\eta}_+(\omega)}{e^X + e^{-X} + 2\tilde{\eta}_+(\omega)} \]

\[ \langle \sigma'_{xy} \rangle = \frac{1}{2} \left[ 1 - \frac{e^X - e^{-X} + 2\tilde{\eta}_-(\omega)}{e^X + e^{-X} + 2\tilde{\eta}_+(\omega)} \right] \]

where

\[ X = L\omega + \frac{1}{2} \ln \frac{f(\omega)}{f(-\omega)} \]

\[ \tilde{\eta}_{\pm} = \frac{\eta_{\pm}}{\sqrt{f(\omega)f(-\omega)}} \]

Next, it is not difficult to see that Eqs (160) - (163) precisely correspond to the results listed in Section 8.3 provided we drop all the terms with \( \kappa \omega \) and \( \kappa / L \) in the expressions for \( f \) and \( g_{\pm} \). The correct way of expressing this is by saying that we are interested in the thermodynamic limit \( L \approx \beta \to \infty \) while keeping the quantities \( \omega L \) and \( \sigma / (\beta \omega) \) fixed. Under these circumstances we have \( \kappa \omega = \mathcal{O}(L^{-2}) \) and \( \kappa / L = \mathcal{O}(L^{-2}) \) both of which therefore vanish. Keeping only the surviving terms we can write
\[ X \rightarrow \omega L \left( 1 + \frac{1}{2} \left( \frac{\sigma}{\beta \omega} \right)^2 \right) \]  
(164)

\[ \tilde{\eta}_+ \rightarrow \eta = \frac{\sigma}{\beta \omega} \sinh \omega L \]  
(165)

\[ \tilde{\eta}_- \rightarrow -\frac{\sigma^2}{\beta^2 \omega^2} \sinh \omega L \]  
(166)

These results can be written precisely in the form of Eqs (132) and (133) with \( X \) given in terms of a series expansion in powers of \( \sigma/(\beta \omega) \)

\[ X = L \sqrt{\omega^2 + \left( \frac{\sigma}{\beta} \right)^2} = \omega L \left( 1 + \frac{1}{2} \left( \frac{\sigma}{\beta \omega} \right)^2 + \ldots \right). \]  
(167)

Similarly, the square root in Eq. (133) is expanded according to

\[ \sqrt{(e^X - e^{-X})^2 - 4\eta^2} = (e^X - e^{-X}) \left( 1 - \frac{1}{2} \left( \frac{\sigma}{\beta \omega} \right)^2 + \ldots \right). \]  
(168)

In summary, the expressions for the conductances as derived in Sections 8.3 and 8.5 are entirely consistent with those obtained from the Coulomb gas representation to order \((\sigma/\beta)^2\). One may in principle proceed and introduce several different renormalization constants \(Z\) that absorb all or parts of the corrections of order \(\kappa \omega\) and \(\kappa/L\) in Eqs (157)-(159). For instance, the complete expression for \(X\) in Eq. (163) can be written in the form

\[ X = L \sqrt{Z_\omega^2 \omega^2 + \left( \frac{\sigma}{\beta} \right)^2} \]  
(169)

where to the appropriate order in \(\sigma/\beta\) the coefficient \(Z_\omega\) is equal to

\[ Z_\omega = 1 + \left( \frac{4\kappa \sigma}{\beta} \right)^2 \left[ \frac{1 - 6\kappa/L}{1 - (4\kappa \omega)^2} + \frac{96\kappa/L}{(1 - (8\kappa \omega)^2)(9 - (8\kappa \omega)^2)} \right]. \]  
(170)

Similarly, one can discuss the corrections in Eq. (168) but the expressions are somewhat cumbersome and generally different from those entering the correlation functions.

### 8.6.2 Distribution of quantum Hall states

For completeness we next present the final total result for the partition function of Eq. (147) which is generally more complex than the expressions en-
countered so far. Eq. (147) can most conveniently be written as follows

\[ Z[q, \beta a_0, \theta] = e^{-F(\theta)} \sum_{j=-2}^{4} K_j(\beta a_0) e^{i\pi q} \]  

(171)

where

\[
\begin{align*}
K_{-2} &= p_{-2,0} + p_{-2,2} \cos 2\beta a_0 \\
K_{-1} &= p_{-1,1} \cos \beta a_0 \\
K_0 &= \mathcal{P}_0 + p_{0,2} \cos 2\beta a_0 \\
K_1 &= \mathcal{P}_\pi \cos \beta a_0 \\
K_2 &= \mathcal{P}_{2\pi} + p_{2,2} \cos 2\beta a_0 \\
K_3 &= p_{3,1} \cos \beta a_0 \\
K_4 &= p_{4,0} + p_{4,2} \cos 2\beta a_0. 
\end{align*}
\]  

(172)

Here, the quantities \( K_j(\beta a_0) \) obey the general constraints

\[
\begin{align*}
\sum_{j=-2}^{4} K_j(0) &= 1, \\
\sum_{j=-2}^{4} jK_j(0) &= 2\langle \sigma'_{xy} \rangle, \\
\sum_{j=-2}^{4} K'_j(0) &= 2\langle \sigma'_{xx} \rangle.
\end{align*}
\]  

(173)

Based on these constraints one can express the \( \mathcal{P}_0, \mathcal{P}_\pi \) and \( \mathcal{P}_{2\pi} \) in Eqs (172) in terms of the quantities \( p_{i,j} \) as well as the conductances \( \langle \sigma'_{xy} \rangle \) and \( \langle \sigma'_{xx} \rangle \) as given by Eqs (154) and (155). The result is

\[
\begin{align*}
\mathcal{P}_0 &= 1 - \langle \sigma'_{xy} \rangle - \langle \sigma'_{xx} \rangle + \mathcal{P}'_0 \\
\mathcal{P}_\pi &= 2\langle \sigma'_{xx} \rangle + \mathcal{P}'_\pi \\
\mathcal{P}_{2\pi} &= \langle \sigma'_{xy} \rangle - \langle \sigma'_{xx} \rangle + \mathcal{P}'_{2\pi}
\end{align*}
\]  

(174)

where

\[
\begin{align*}
\mathcal{P}'_0 &= -2p_{4,0} + p_{-2,0} + p_{2,2} + 2p_{0,2} - p_{3,1} - p_{-1,1} + 3p_{-2,2}, \\
\mathcal{P}'_\pi &= -4p_{2,2} - 4p_{0,2} - p_{3,1} - p_{-1,1} - 4p_{4,2} - 4p_{-2,2}, \\
\mathcal{P}'_{2\pi} &= p_{4,0} - 2p_{-2,0} + 2p_{2,2} + p_{0,2} + p_{3,1} - p_{-1,1} + 3p_{4,2}.
\end{align*}
\]  

(175)

The quantities \( p_{j,k} \) in Eqs (172) indicate that the transition between the \( \theta = 0 \) and \( \theta = 2\pi \) vacuum generally involves a range of different \( \theta \) vacua or quantum Hall states. In total 8 different quantities \( p_{j,k} \) are given by
\[ p_{-2,0} = \left( \frac{2\kappa\sigma}{\beta} \right)^2 \frac{e^{\omega L}}{(1 - 4\kappa\omega)^2} D^{-1} \]
\[ p_{-2,2} = 2 \left( \frac{2\kappa\sigma}{\beta} \right)^2 \frac{e^{\omega L}}{(1 - 4\kappa\omega)(3 - 8\kappa\omega)} D^{-1} \]
\[ p_{-1,1} = \left( \frac{4\kappa\sigma}{\beta} \right) \frac{e^{\omega L}}{1 - 4\kappa\omega} D^{-1} \]
\[ p_{0,2} = \left( \frac{2\kappa\sigma}{\beta} \right) \left( \frac{\sigma}{\beta\omega} \right) \left( \frac{e^{\omega L}}{1 + 4\kappa\omega} - \frac{e^{-\omega L}}{1 - 8\kappa\omega} \right) D^{-1} \]
\[ p_{2,2} = \left( \frac{2\kappa\sigma}{\beta} \right) \left( \frac{\sigma}{\beta\omega} \right) \left( \frac{e^{\omega L}}{1 + 8\kappa\omega} - \frac{e^{-\omega L}}{1 - 4\kappa\omega} \right) D^{-1} \]
\[ p_{3,1} = \left( \frac{4\kappa\sigma}{\beta} \right) \frac{e^{-\omega L}}{1 + 4\kappa\omega} D^{-1} \]
\[ p_{4,0} = \left( \frac{2\kappa\sigma}{\beta} \right)^2 \frac{e^{-\omega L}}{(1 + 4\kappa\omega)^2} D^{-1} \]
\[ p_{4,2} = 2 \left( \frac{2\kappa\sigma}{\beta} \right)^2 \frac{e^{-\omega L}}{(1 + 4\kappa\omega)(3 + 8\kappa\omega)} D^{-1} \]

where

\[ D = f(\omega)e^{\omega L} + f(-\omega)e^{-\omega L} + 2\eta_{+}(\omega). \]

To lowest order in an series expansion in powers of \(\sigma/\beta\) and \(\kappa\omega\) we can write these results as follows

\[ p_{-2,0} \approx \left( \frac{2\kappa\sigma}{\beta} \right)^2 (1 - \langle \sigma'_{xy} \rangle) \]
\[ p_{-2,2} \approx \frac{2}{3} \left( \frac{2\kappa\sigma}{\beta} \right)^2 (1 - \langle \sigma'_{xy} \rangle) \]
\[ p_{-1,1} \approx \left( \frac{4\kappa\sigma}{\beta} \right) (1 - \langle \sigma'_{xy} \rangle) \]
\[ p_{0,2} \approx -\frac{3}{2} \left( \frac{4\kappa\sigma}{\beta} \right)^2 (1 - \langle \sigma'_{xy} \rangle) + \left( \frac{4\kappa\sigma}{\beta} \right) \langle \sigma'_{xx} \rangle \]
\[ p_{2,2} \approx -\frac{3}{2} \left( \frac{4\kappa\sigma}{\beta} \right)^2 \langle \sigma'_{xy} \rangle + \left( \frac{4\kappa\sigma}{\beta} \right) \langle \sigma'_{xx} \rangle \]
\[ p_{3,1} \approx \left( \frac{4\kappa\sigma}{\beta} \right) \langle \sigma'_{xy} \rangle \]
\[ p_{4,0} \approx \left( \frac{2\kappa\sigma}{\beta} \right)^2 \langle \sigma'_{xy} \rangle \]

38
The factors $\kappa \sigma / \beta$ indicate that the quantities $p_{i,j}$ are exponentially small corrections terms that can be ignored relative to the leading order terms contained in $P_0$, $P_\pi$ and $P_{2\pi}$, Eqs (174). Perhaps the most important conclusion that one can draw from the results of this Section is that both the robustly quantized quantum Hall plateau and the quantum critical behavior of the plateau transitions simultaneously emerge from the existence of discrete topological sectors in the theory. These topological sectors, labelled by the integer $j$ in Eq. (171), have not been recognized in the historical papers on the large $N$ expansion. They are nevertheless one of the most fundamental features of the instanton vacuum concept in scale invariant theories.

9 Scaling diagram for the large N expansion

Discarding the corrections to scaling discussed in the previous Section we next return to the expressions for the conductances $\langle \sigma'_{xx} \rangle$ and $\langle \sigma'_{xy} \rangle$, Eqs (132) - (135), which are some of the most important results of this paper. We have already mentioned several times earlier that these expressions have an entirely different significance depending on the physical context in which they are being used. These differences are clearly reflected in the renormalization behavior of the theory which is the main topic of the present Section.

9.1 Renormalization. Ising model

Let us first discuss the 1D Ising model which is in many ways standard. Eqs (132) - (135) involve two different scaling variables, namely

$$H' = LH, \quad e^{-2K'} = Le^{-2K}. \quad (179)$$

Here, $H'$ and $K'$ are the low temperature and small $H$ versions of the more general expressions given by Eqs (139) and (140). The renormalization group equations for small values of $H$ can generally be expressed as a series expansion in powers of $K^{-1}$

$$\frac{dH'}{d\ln L} = H'(1 + O(1/K')) \quad \text{and} \quad \frac{dK'}{d\ln L} = -\frac{1}{2} + O(1/K'). \quad (180)$$

These results show that the 1D Ising model is a prototypical example of an asymptotically free field theory with interesting features such dynamic mass
Next we turn to the Coulomb gas problem which has a different dimensionality. Substituting the parameters of the large $N$ expansion for the Ising model variables $H$ and $e^{-2K}$ we obtain

$$LH = L\omega = \left(1 - \frac{\theta}{\pi}\right)\frac{6\pi M^2 L^2}{N}, \quad Le^{-2K} = L\frac{\sigma}{\beta} = NML \frac{e^{-M\beta}}{\sqrt{2\pi M\beta}}. \quad (181)$$

To discuss the role of the parameter $N$ we introduce an arbitrary scale factor $b$ according to

$$b^2 = \frac{M^2 L^2}{N} \gg 1. \quad (182)$$

Equation (181) can then be written as

$$LH = 6\pi b^2 \left(1 - \frac{\theta}{\pi}\right), \quad Le^{-2K} = \alpha^2 n^2 \sqrt{\frac{nb}{2\pi}} e^{-nb} \quad (183)$$

where $\alpha$ and $n$ are defined as follows

$$\alpha = L/\beta, \quad n = \sqrt{N/\alpha}. \quad (184)$$

Since the quantities $b$ and $n$ in Eq. (183) are independent variables we conclude that the large $N$ limit of the theory is generally well defined and obtained by taking $n \to \infty$ first while keeping scale factor $b$ fixed. This definition furthermore ensures that the large $N$ expansion and the strong coupling expansion of the Coulomb gas in powers of the fugacity are mutually consistent. Next we combine Eqs (183) and (179) and express the renormalization of the Coulomb gas in terms of the Ising model quantities $H'$ and $K'$. Keeping in mind that we now have $K' = \mathcal{O}(n)$ then the renormalization group equations for large values of $n$ are obtained as follows

$$\frac{dH'}{d\ln b} = 2H', \quad \frac{dK'}{d\ln b} = K' \left[1 + \mathcal{O}\left(\frac{\ln K'}{K'}\right)\right]. \quad (185)$$

Unlike the Ising model, the results now indicate that the theory along the line $H' = 0$ or $\theta = \pi$ generally displays gapless excitations, rather than a mass gap. Moreover, the result for $H'$ is in accordance with the fact that the transition at $\theta = \pi$ is a first order one and, at the same time, displays a divergent correlation length with an exponent $1/2$.

Since the leading order results of Eq. (185) do not contain the parameter $N$ or $n$ explicitly, we can make use of Eqs (179) and (132) - (135) and project the
Fig. 2. Renormalization group flow in the $\langle \sigma'_xx \rangle$ and $\langle \sigma'_xy \rangle$ conductance plane according to Eqs (132) - (135) and (183), see text.

The renormalization group flow lines of the large $N$ theory directly onto the $\langle \sigma'_xx \rangle$, $\langle \sigma'_xy \rangle$ conductance plane. For illustration we have plotted in Fig. 2 the results obtained from numerical simulation. It is interesting to notice that the flow diagrams of Figs 1 and 2 are numerically very similar. This is in spite of the fact that the previously obtained scaling results (Eqs (29) -(32)) and those of the present paper (Eqs (132) - (135)) are qualitatively very different.

9.3 Regularity condition

As a final step in this paper we next make sure that the square root singularities appearing in Eqs (132) - (135) do not spoil the regularity conditions of the renormalization group equations. To investigate this point further we compute the $\beta$ functions

\begin{align*}
\frac{d\langle \sigma'_xx \rangle}{d \ln b} &= \beta_{xx}(\langle \sigma'_xx \rangle, \langle \sigma'_xy \rangle) \\
\frac{d\langle \sigma'_xy \rangle}{d \ln b} &= \beta_{xy}(\langle \sigma'_xx \rangle, \langle \sigma'_xy \rangle)
\end{align*}
based on the leading order results of Eq. (185). We obtain the following expressions

\[
\beta_{xx}(k, h) \approx k \left\{ 1 - 2k \frac{(1 - 2h)^2}{(1 - 2h)^2 + 4k^2} \ln \frac{1 - 4k + \sqrt{h^2 + 4k^2}}{1 - 2k - \sqrt{h^2 + 4k^2}} \right\}
\]

(188)

\[
\beta_{xy}(k, h) \approx k \ln k, \quad \beta_{xy}(0, h) = 2h - 1.
\]

(193)

Notice that these results are consistent with those of Eq. (185) under the identification \( k = e^{-2K'} \) and \( h = \frac{1}{2} - H' \).

10 Conclusion

Starting from the Coulomb gas representation of the \( CP^{N-1} \) model with large values of \( N \) in two dimensions we have identified the exact critical theory for the transition at \( \theta = \pi \). This theory is one dimensional and none other than the theory of massless chiral fermions that has previously emerged in the theory of the quantum Hall effect, in particular the Luttinger liquid theory of edge excitations [9].
We have benefitted from the various alternative approaches that we have introduced, in particular the hamiltonian approach as well as the mapping of the Coulomb gas onto the 1D Ising model. Besides computational advantages, these different mappings also elucidate the role played by the geometry of the system in general, and the meaning of topics such as mass generation at $\theta = \pi$ in particular.

Perhaps the most interesting conclusion of this paper is that the divergent correlation length $\xi \propto |\theta - \pi|^{-1/2}$ emerges not only from the physical objectives of the quantum Hall effect, but also from Coleman’s original ideas on the transition at $\theta = \pi$ [17]. Based on an explicit knowledge of the multi-point correlation functions we have shown that the mechanism responsible for changing the total number of charged particles at the edges of the system is in fact synonymous for the existence of gapless bulk excitations at $\theta = \pi$. Remarkably, neither the existence of these excitations nor the significance of Coleman’s mechanism in terms of quantum Hall physics has previously been recognized [14,15].

The results of this paper provide the complete conceptual structure that one in general can associate with the topological concept of an instanton vacuum. Besides an exactly solvable critical theory for $\theta \approx \pi$ this structure furthermore includes finite size scaling results for the “conductances”, robust topological quantum numbers that explain the precision and stability of the quantum Hall plateaus and also the massless chiral edge excitations that facilitate the flow of Hall currents. The fundamental features of the quantum Hall effect are therefore not merely a topic of replica field theory or disordered free electron systems alone. Rather than that, they are a super universal phenomenon that teaches us something fundamental about instanton vacuum in asymptotically free field theory in general.

As a final remark, it should be mentioned that the statement of super universality has recently been investigated and studied, with great success, in several completely different physical systems such as the theory in the presence of electron-electron interactions, [23] quantum spin chains [19] as well as the Ambegaokar-Eckern-Schön theory of the Coulomb blockade problem [24].

11 Acknowledgments

This research was initiated during the Amsterdam Summer Workshop “Low-D Quantum Condensed Matter”. The authors are grateful to the participants, in particular A. G. Abanov, for discussions. The work was funded in part by the Dutch National Science Foundations NWO and FOM. One of us (ISB) is indebted to P. M. Ostrovsky for fruitful discussions and to Russian Science Sup-
A Two-point correlation function $g_2$ in the Ising model

In this appendix we present a brief derivation of the result (86) in the Ising model representation. As well-known, the standard approach to exact solution the Ising model is via transfer matrix technique. For 1D Ising model the transfer matrix is given by Eq. (121). It can be diagonalized by an orthogonal rotation

$$U = e^{i\phi y}, \quad \phi = \frac{1}{2} \arcsin \left( \frac{e^{-2K}}{\sqrt{\sinh^2 H + e^{-4K}}} \right). \quad (A.1)$$

The eigenvalues of the $T$ are

$$\lambda_{\pm} = e^K \left( \cosh H \pm \sqrt{\sinh^2 H + e^{-4K}} \right). \quad (A.2)$$

The domain wall operators $a_{+-}$ and $a_{-+}$ can be expressed in the space in which the transfer matrix acts as follows

$$a_{+-} = \frac{\tau_x + i\tau_y}{2}, \quad a_{-+} = \frac{\tau_x - i\tau_y}{2}. \quad (A.3)$$

Defining the two-point correlation function $g_2(i, j)$ as

$$g_2(i, j) = \langle a_{+-}(i)a_{-+}(j) \rangle = \lim_{M \to \infty} \frac{\text{tr} T^i a_{+-} T^{-i} a_{-+} T^{M-i}}{\text{tr} T^M} \quad (A.4)$$

we easily find

$$g_2(i, j) = m_0 + m_+ \vartheta(j-i) e^{-|j-i|/\xi} + m_- \vartheta(i-j) e^{-|j-i|/\xi}. \quad (A.5)$$

Here the coefficients are given by Eqs (87)-(88) with the magnetization $\mathcal{M}$ defined in Eq. (81). The result (A.5) involves the correlation length $\xi$ of 1D Ising model

$$\xi = \frac{1}{\ln \lambda_+/\lambda_-} = \tanh^{-1} \frac{\tanh H}{\mathcal{M}}. \quad (A.6)$$

In the limit of weak magnetic field $H \ll 1$ the result (A.5) coincides with the result (86) for $g_2$ obtained in the framework of 1D chiral fermion theory.

References

[1] For an early review, see A. M. M. Pruisken in The Quantum Hall effect, edited by R. E. Prange and S. M. Girvin, (Springer-Verlag, Berlin, 1987).
2. H. Levine, S. B. Libby and A. M. M. Pruisken, Phys. Rev. Lett. 51 (1983) 1915,  
A. M. M. Pruisken, Nucl. Phys. B 235, 277 (1984).

3. K. von Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. 45 (1980) 494.

4. E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan,  
Phys. Rev. Lett. 42 (1979) 673.

5. A. M. M. Pruisken, Phys. Rev. Lett. 61, 1297 (1988)

6. H. P. Wei, D. C. Tsui, M. Palaanen, and A. M. M. Pruisken, Phys. Rev. Lett.  
61, 1294 (1988); L. A. Ponomarenko, D. T. N. de Lang, A. de Visser, D. Maude,  
B. N. Zvonkov, R. A. Lunin and A. M. M. Pruisken, Physica E 22 (2004) 236;  
W. Li, G. A. Csáthy, D. C. Tsui, L. N. Pfeiffer, and K. W. West, Phys. Rev. Lett.  
94 (2005) 206807.

7. For a review, see A. D. Mirlin, Phys. Rep. 326 (2000) 259 and references therein.

8. A. M. M. Pruisken, M. A. Baranov, and M. Voropaev, arXiv:cond-mat/0206011  
(unpublished);  
A. M. M. Pruïsken, M. A. Baranov, and M. Voropaev, arXiv:cond-mat/0101003  
(unpublished).

9. A. M. M. Pruïsken, B. Škorić, and M. A. Baranov, Phys. Rev. B 60 (1999) 16838,  
B. Škorić and A. M. M. Pruïsken, Nucl. Phys. B 599 (1999) 637.

10. A. M. M. Pruïsken and I. S. Burmistrov, Ann. of Phys. 316 (2005) 285.

11. A. D’Adda, M. Lüscher, P. Di Vecchia, Nucl. Phys. B 146 (1978) 63;  
E. Witten, Nucl. Phys. B 149 (1979) 285;

12. I. Affleck, Nucl. Phys. B 162 (1980) 461;  
I. Affleck, Nucl. Phys. B 171 (1980) 420.

13. H. Eichenherr, Nucl. Phys. B 146 (1978) 215;  
V. L. Golo and A. M. Perelomov, Phys. Lett. B 79 (1978) 112.

14. I. Affleck, Nucl. Phys. B 257 (1985) 397.

15. I. Affleck, Nucl. Phys. B 305 (1988) 582.

16. see e.g. K. B. Efetov, *Supersymmetry in disorder and chaos*, Cambridge  
University Press, New York (1997). The traditional discussions on the replica  
method versus supersymmetry are poorly motivated, however, and the matter  
has very much led an existence of its own, see e.g. K. Splittorff and J. J. M.  
Verbaarschot, Phys.Rev.Lett. 90 (2003) 041601 and references therein.

17. S. Coleman, Ann. of Phys. 101 (1976) 239.

18. see e.g. R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic  
Press, 1982.

19. A. M. M. Pruïsken, R. Shankar and N. Surendran, Phys. Rev. B 72 (2005)  
o35329.
[20] A. M. M. Pruisken, Nucl. Phys. B 285 (1987) 719;
    A. M. M. Pruisken, Nucl. Phys. B 290 (1987) 61.

[21] see e.g. B. L. Altshuler, V. E. Kravtsov, I. V. Lerner, in: B. L. Altshuler,
    P. A. Lee, R. A. Webb (Eds.), Mesoscopic Phenomena in Solids, North-Holland,
    Amsterdam, 1991, p. 449.

[22] K. D. Schotte and T. T. Truong, Phys. Rev. A 22 (1980) 2183.

[23] A. M. M. Pruisken and I. S. Burmistrov, cond-mat/0502488 (unpublished).

[24] I. S. Burmistrov and A. M. M. Pruisken, in preparation.