I describe how my involvement with monopoles related to the multimonopole existence proof of Taubes, and how my later work on quaternionic quantum mechanics led to the classification theorem for generalized projective group representations of Tao and Millard.

1. Introduction

In this essay I discuss two unrelated subjects with a common theme: the influence of Yang–Mills theory on mathematics. In the first part I describe how my interest in monopole solutions in the period 1978–1980 led indirectly to the completion by Clifford Taubes of his multimonopole existence theorem during a visit to the Institute for Advanced Study in the spring of 1980. In the second part I shift my focus to a generalized definition of projective group representations that I proposed in the context of quaternionic quantum mechanics. This inspired a classification theorem proved by Terry Tao and Andrew Millard in 1996, which has useful implications for projective representations in standard, complex quantum mechanics.

2. Multimonopole Solutions

During the years 1978–1980 I became interested in trying to find a simple, semi-classical model for quark confinement. My first attempt, which did not succeed but which had useful by-products that I shall describe here, was based on the idea of considering the potential between classical quark sources in the background of a non-Abelian ’t Hooft–Polyakov [1] Prasad–Sommerfield [2] monopole or its generalizations, which I conjectured [3] might act as a quark-confining “bag”. To pursue (and ultimately rule out) this conjecture, I did a number of calculations of properties of monopole solutions. The first was a calculation of the Green’s function for a single Prasad–Sommerfield monopole, by using the multi-instanton representation of the monopole and a formalism for calculating multi-instanton Green’s functions given by Brown et al [4]. This calculation was spread over two papers that I wrote; setting up contour integral expressions for the Green’s function was done in Appendix A of Adler [5], and the final result for the monopole propagator, after evaluation of the contour integrals and considerable algebraic simplification, was given in Appendix A of Adler [6]. From the propagator formula, I concluded that a single monopole background would not lead to confinement.

Not yet ready to give up on the monopole background idea, I then wrote two papers speculating that the Prasad–Sommerfield monopole might be a member of
a larger class of solutions, in which the point at which the monopole Higgs field vanishes is extended to a higher-dimensional region, and in particular to a “string”-like configuration with a line segment as a zero set. In the first of these papers \cite{6}, I studied small deformations around the Prasad–Sommerfield monopole and found several series of such deformations. For normalized deformations I recovered the monopole zero modes that had already been obtained by Mottola \cite{7}, but I found that “if an axially symmetric extension exists, it cannot be reached by integration out along a tangent vector defined by a nonvanishing, non-singular small-perturbation mode”. This work was later extended into a complete calculation of the perturbations around the Prasad–Sommerfield solution by Akhoury, Jun, and Goldhaber \cite{8}, who also found “no acceptable nontrivial zero energy modes.” In my second paper \cite{9}, I employed nonperturbative methods and suggested that despite the negative perturbative results, there might still be interesting extensions of the Prasad–Sommerfield solution with extended Higgs field zero sets.

At just around the same time, Erick Weinberg wrote a paper \cite{10} extending an index theorem of Callias \cite{11} to give a parameter counting theorem for multi-monopole solutions. Weinberg concluded that “any solution with $n$ units of magnetic charge belongs to a $(4n - 1)$-parameter family of solutions. It is conjectured that these parameters correspond to the positions and relative $U(1)$ orientations of $n$ noninteracting unit monopoles”. For $n = 1$, his results agreed with the zero-mode counting implied by Mottola’s explicit calculation. Weinberg and I were aware of each other’s work, as evidenced by correspondence in my file dating from March to June of 1979, and references relating to this correspondence in our papers \cite{9} and \cite{10}.

My contact with Clifford Taubes was initiated by an April, 1979 letter from Arthur Jaffe, after I gave a talk at Harvard while Jaffe, as it happened, was visiting Princeton! In his letter, Jaffe noted that I was working on problems similar to those on which his students were working, and enclosed a copy of a paper by Clifford Taubes. (This preprint was not filed with Jaffe’s letter, so I am not sure which of the early Taubes papers listed on the SLAC Spires archive that it was.) Jaffe’s letter initiated telephone contacts with Taubes and some correspondence from him. On Jan. 6, 1980 Taubes wrote to me that he was making progress in proving the existence of multi-monopole Prasad–Sommerfield solutions, and in this letter and a second one dated on January 18, 1980 he reported results that were relevant to my conjectures on the possibility of deformed monopoles. His results placed significant restrictions on my conjectures; in a letter dated Feb. 1, 1980 I wrote to Lochlainn O’Raifeartaigh, who had also been interested in axially symmetric monopoles, saying that “On thinking some more about your paper (O’Raifeartaigh’s preprint was unfortunately not retained in my files) I realized that the enclosed argument by Cliff Taubes is strong evidence against $n = 2$ monopoles involving a line zero. What Taubes shows is that a finite action solution of the Yang–Mills–Higgs Lagrangian cannot have a line zero of arbitrarily great length; hence if $n = 2$ monopoles con-
tained a line zero joining the monopole centers, the monopole separation would be bounded from above. But this seems unlikely....” This correspondence and the result of Taubes was mentioned at the end of the published version, Houston and O’Raifeartaigh [12].

As a result of our overlapping interests, I arranged for Taubes to make an informal visit, of two or three months, to the Institute for Advanced Study during the spring of 1980. Clifford had expressed interest in this, he noted in a recent email, in part because Raoul Bott had suggested that he visit the Institute to get acquainted with Karen Uhlenbeck, who was visiting the IAS that year. In the course of his visit he met and interacted with Uhlenbeck, who, along with Bott, had a major impact on his development as a mathematician.

Taubes began the visit by looking at my conjecture of extended zero sets, but after a while told me that he could not find an argument for them. Partly as a result of his work, I was getting disillusioned with my own conjecture, so I asked him what was happening with his attempted proof of multi-monopole solutions. Taubes replied that he was stuck on that, and not sure whether they existed. I then mentioned to him Erick Weinberg’s parameter counting result, which strongly suggested a space of moduli much like that in the instanton case, where looking at deformations correctly suggests the existence and structure of the multi-instanton solutions. To my surprise, Taubes was not aware of Erick’s result, and knowing it impelled him into action on his multi-monopole proof. Within a week or two he had completed a proof, and wrote it up on his return to Harvard. In his paper he showed that “for every integer \( N \neq 0 \) there is at least a countably infinite set of solutions to the static \( SU(2) \) Yang-Mills-Higgs equations in the Prasad–Sommerfield limit with monopole number \( N \). The solutions are partially parameterized by an infinite sublattice in \( S_N(\mathbb{R}^3) \), the \( N \)-fold symmetric product of \( \mathbb{R}^3 \) and correspond to noninteracting, distinct monopoles.” This quote is taken from the Abstract of his preprint “The Existence of Multi-Monopole Solutions to the Static, \( SU(2) \) Yang-Mills-Higgs Equations in the Prasad-Sommerfield Limit”, which was received on the SLAC Spires data base in June, 1980, and which carried an acknowledgement on the title page noting that “This work was completed while the author was a guest at the Institute for Advanced Studies, Princeton, NJ 08540”. His preprint also ended with an Acknowledgment section noting his conversations with me, with Arthur Jaffe, and with Karen Uhlenbeck, as well as the Institute’s hospitality. The proof was not published in this form, however, but instead appeared (with acknowledgments edited out at some stage) as Chapter IV of the book by Arthur Jaffe and Clifford Taubes, “Vortices and Monopoles” (Birkhäuser Boston, 1980), that was completed soon after.”

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<sup>4</sup>Thus, there was a parallel to what happened a year before with respect to solutions of the first order Ginzburg–Landau equations. In that case Taubes had heard a lecture at Harvard by E. Weinberg on parameter counting for multi-vortex solutions (written up as Weinberg [13]) and then went home and came up with his existence proof for multi-vortices (Taubes [14]). The vortex work provided the initial impetus for Taubes’ turning to the monopole problem.
afterwards, in August of 1980. The multimonopole existence proof was a milestone in Taubes’ career; in a recent exchange of emails relating to this essay, Taubes commented on his visit “to hang out at the IAS during the spring of 1980. It profoundly affected my subsequent career...” He went on to further investigations of monopole solutions, that lead him to studies of 4-manifold theory which have had a great impact on mathematics.

O’Raifeartaigh, who had been following the monopole work at a distance, invited me during the spring of 1980 to come to Dublin that summer to lecture on my papers. However, since Taubes had much more interesting results I suggested to Lochlainn that he ask Clifford instead, and Taubes did go to Dublin to lecture. After Clifford’s visit, I redirected my search for semiclassical confinement models to a study of nonlinear dielectric models by analytic and numerical methods, in collaboration with Tsvi Piran; these models do give an interesting class of confining theories. Based on the observation that the Yang–Mills action is multiquadratic, Piran and I also applied the same numerical relaxation methods to give an efficient method for the computation of axially symmetric multimonopole solutions. This was described in our Reviews of Modern Physics article [15] that marked the completion of the research program on confining models, and as a by-product, on monopoles.\(^\text{b}\)

3. Projective Group Representations

Given two group elements \(b, a\) with product \(ba\), a unitary operator representation \(U_b\) in a Hilbert space is defined by \(U_b U_a = U_{ba}\). A more general type of representation, called a ray or projective representation, is relevant to describing the symmetries of quantum mechanical systems. In his famous paper on unitary ray representations of Lie groups, Bargmann [17] defines a projective representation as one obeying

\[
U_b U_a = U_{ba} \omega(b, a) , \tag{1}
\]

with \(\omega(b, a)\) a complex phase.

This definition is familiar, and seems obvious, until one asks the following question: Eq. (1) is assumed to hold as an operator identity when acting on all states in Hilbert space. However, we know that it suffices to specify the action of an operator on one complete set of states in Hilbert space to specify the operator completely. Hence why does one not start instead from the definition

\[
U_b U_a |f\rangle = U_{ba} |f\rangle \omega(f; b, a) , \tag{2}
\]

with \(\{|f\rangle\}\) one complete set of states, as defining a projective representation in Hilbert space? Let us call Bargmann’s definition of Eq. (1) a “strong” projective representation, and the definition of Eq. (2) a “weak” projective representation.

\(^{\text{b}}\)Our numerical calculations in the 2-monopole case served mainly to illustrate the computer methods, since by then exact analytic 2-monopole solutions had appeared; see Forgacs, Horvath, and Palla [16] and Ward [15].
Then the question becomes that of finding the relation between weak and strong projective representations.

Although I have formulated this question here in complex Hilbert space, it arose and was solved in the context of quaternionic Hilbert space, where the phases $\omega(f; b, a)$ are quaternions, which obey a non-Abelian or Yang–Mills group multiplication law isomorphic to $SO(3) \simeq SU(2)$. The “strong” definition of Eq. (1) was adopted for the quaternionic case by Emch [18], but in my book on quaternionic quantum mechanics [19] I introduced the “weak” definition of Eq. (2) in order for quaternionic projective representations to include embeddings of nontrivial complex projective representations into quaternionic Hilbert space; the state dependence of the phase is necessary because even a complex phase $\omega$ does not commute with general quaternionic rephasings of the state vector $|f\rangle$. I noted in my book that Eq. (2) can be extended to an operator relation by defining

$$\Omega(b, a) = \sum_f |f\rangle \omega(f; b, a) \langle f|,$$

so that Eq. (2) takes the form

$$U_b U_a = U_{ba} \Omega(b, a),$$

which gives the general operator form taken by projective representations in quaternionic quantum mechanics. I also introduced two specializations of this definition, motivated by the commutativity properties of the phase factor in complex projective representations. I defined [19] a \textit{multicentral} projective representation as one for which

$$[\Omega(b, a), U_a] = [\Omega(b, a), U_b] = 0$$

for all pairs $b, a$, and I defined a \textit{central} projective representation as one for which

$$[\Omega(b, a), U_c] = 0$$

for all triples $a, b, c$.

Subsequent to the completion of my book, I read Weinberg’s first volume on quantum field theory [20] and realized, from his discussion in Sec. 2.7 of the associativity condition for complex projective representations, that there must be an analogous associativity condition for weak quaternionic projective representations. I worked this out [21], and showed that it takes the operator form

$$U^{-1}_a \Omega(c, b) U_a = \Omega(cb, a)^{-1} \Omega(c, ba) \Omega(b, a),$$

which by the definition of Eq. (3) shows that $U^{-1}_a \Omega(c, b) U_a$ is diagonal in the basis $\{|f\rangle\}$, with the spectral representation

$$U^{-1}_a \Omega(c, b) U_a = \sum_f |f\rangle \omega(f; cb, a) \omega(f; c, ba) \omega(f; b, a) \langle f|.$$

\footnote{In Eq. (4.51a) of [19], $U_{ab}$ should read $U_{ba}$, that is, the only conditions are those already implied by Eq. (5).}
On the basis of some further identities, I also raised the question \cite{21} of whether one can construct a multicentral representation that is not central, or whether a multicentral representation is always central.

Subsequently, I discussed the issues of quaternionic projective representations with Andrew Millard, who was my thesis student in the mid-1990’s. He explained them to his roommate Terry Tao, a mathematics graduate student working for Elias Stein, and at my next conference with Andrew, Tao came along and presented the outline of a beautiful theorem that resolved the issues. This was written up as a paper of Tao and Millard \cite{22}, and consists of two parts. The first part is an algebraic analysis based on Eq. (8), which leads to the following theorem

**Structure Theorem:** Let $U$ be an irreducible projective representation of a connected Lie group $G$. There then exists a reraying of the basis $|f\rangle$ under which one of the following three possibilities must hold.

1. $U$ is a real projective representation. That is, $\omega(f; b, a) = \omega(b, a)$ is independent of $|f\rangle$ and is equal to $\pm 1$ for each $b$ and $a$.

2. $U$ is the extension of a complex projective representation. That is, the matrix elements $\langle f'|U_a|f\rangle$ are complex and $\omega(f; b, a) = \omega(b, a)$ is independent of $|f\rangle$ and is a complex phase.

3. $U$ is the tensor product of a real projective representation and a quaternionic phase. That is, there exists a decomposition $U_a = U^B_a \sum_f |f\rangle \sigma_a(f)|$, where the unitary operator $U^B_a$ has real matrix elements, $\sigma_a$ is a quaternionic phase, and $U^B_{ba} = \pm U^B_b U^B_a$ for all $b$ and $a$.

From the point of view of the Structure Theorem, case (1) corresponds to the only possibility allowed by the strong definition of quaternionic projective representations, as demonstrated earlier by Emch \cite{18}, while case (2) corresponds to an embedding of a complex projective representation in quaternionic Hilbert space, the consideration of which was my motivation for proposing the weak definition. Specializing the Structure Theorem to a complex Hilbert space, where case (3) cannot be realized, we see that in complex Hilbert space the weak projective representation defined in Eq. (2) implies the strong projective representation defined in Eq. (1), hence no generality is lost by starting from the strong definition, as in Bargmann’s paper.

The second part of the Tao–Millard paper was a proof, by real analysis methods, of a Corollary to the structure theorem, stating

**Corollary 1:** Any multicentral projective representation of a connected Lie group is central.

This thus solved the question of the relation of centrality to multicentrality that I raised in my paper \cite{21}.
Subsequent to this work, I had an exchange with Gerard Emch in the Journal of Mathematical Physics\cite{Emch1975, Emch1976} debating the merits of the strong and weak definitions. After a visit to Gainesville where we reconciled differing notations, we wrote a joint paper\cite{Taubes1977} clarifying the situation, and reexpressing the strong and weak definitions of Eqs. (1) and (2) in the language and notation often employed in mathematical discussions of projective group representations.

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