Running-mass models of inflation, and their observational constraints

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Abstract

If the inflaton sector is described by softly broken supersymmetry, and the inflaton has unsuppressed couplings, the inflaton mass will run strongly with scale. Four types of model are possible. The prediction for the spectral index involves two parameters, while the COBE normalization involves a third, all of them calculable functions of the relevant masses and couplings. A crude estimate is made of the region of parameter space allowed by present observation.

1 Introduction

It is generally supposed that inflation sets the initial conditions for the subsequent hot big bang. In particular the primordial density perturbation, that is thought to be the origin of structure in the Universe, is supposed to come from the vacuum fluctuation of the inflaton field. The spectrum $\delta_H^2$ of the primordial density perturbation involves only the potential $V(\phi)$ of the inflaton field $\phi$, evaluated while cosmological scales are leaving the horizon. It is given by

$$\delta_H^2(k) = \frac{1}{75\pi^2 M_P^3} \frac{V^3}{V'} ,$$

where $k/a$ is the comoving wavenumber, and the right hand side is evaluated at the epoch of horizon exit $k = aH$. The scale-dependence of the spectrum is conveniently specified

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1The Planck scale is $M_P = (8\pi G)^{-1/2} = 2.4 \times 10^{18}$ GeV. The scale factor of the Universe is $a$, normalized to $a = 1$ at the present epoch, and $H \equiv \dot{a}/a$ defines the Hubble parameter, with a dot indicating the time-derivative.
by a spectral index $n = 1 + d \ln \delta_H^2/d \ln k$, for which the prediction is

$$\frac{n - 1}{2} = \eta \equiv M_P^2 V''/V.$$ (2)

On a scale $k \simeq 10 H_0$ ($H_0$ being the present value of $H$), the COBE observation of the cosmic microwave background anisotropy gives

$$\delta_H = 1.91 \times 10^{-5}.$$ (3)

A variety of observations indicate that $\delta_H(k)$ is approximately scale-independent over the cosmological range of scales $H_0 \lesssim k \lesssim 10^4 H_0$. The constraint assuming roughly constant $n$ is [2]

$$|n - 1| \lesssim 0.2.$$ (4)

Observations of the cosmic microwave background anisotropy and of galaxies, plus accurate determinations of the cosmological parameters, will strongly discriminate between models of inflation within the next decade or so. In particular, the Planck satellite [3] will eventually measure $n$ with an accuracy $\Delta n \sim 0.01$. Partly for this reason, the level of activity in inflation model-building is quite high at present, and likely to become higher over the next few years. In this paper we propose a strategy for comparing with observation a whole class of models. These are the models with a running inflaton mass [4, 5, 6].

Let us begin by placing this class of models in perspective. For many years, the standard paradigm was the tree-level potential

$$V(\phi) = V_0 + \frac{1}{2} m^2 \phi^2 + \frac{1}{4} \lambda \phi^4 + \cdots.$$ (5)

The dots represent non-renormalizable terms, and the constants $V_0$, $m^2$ and $\lambda$ are supposed to have negligible dependence on the renormalization scale (no running). For the underlying field theory to be under control one needs $\phi \lesssim M_P$, and we focus on this case. Then inflation requires a very flat potential, with $V_0$ dominating. Observation requires [4]

$$M_P^2 m^2/V_0 \lesssim 0.1,$$ (6)

and

$$\lambda \lesssim 10^{-8}.$$ (7)

Because of the last constraint, $\lambda$ is usually supposed to be completely negligible during inflation, along with the non-renormalizable terms. This leaves the mass term, giving the

[2] The precise slow-roll result is $\frac{1}{2}(n - 1) = \eta - \frac{3}{2} M_P^2 (V''/V)^2$, but the second term is negligible in most models including the running mass ones. The reader is referred to reference [4] for a comprehensive review of inflation models, including many details and references omitted in the present brief account.
distinctive tree-level prediction\footnote{In writing the tree-level potential, we ignored odd terms on the assumption that they are forbidden by some symmetry. We also ignored terms with negative or fractional powers, that arise in mutated hybrid inflation models, where the non-inflaton field has a $\phi$-dependent value. If any such term dominates, it gives a spectral index fairly close to 1, with the mild scale-dependence $n - 1 \propto 1/N$.} that $n$ is scale-independent and possibly indistinguishable from 1. We note for future reference that a generic supergravity theory makes all scalar field masses, and in particular the inflaton mass, of order

$$ |m| \sim V_0^{1/2}/M_P, \quad (8) $$

in contradiction with the bound Eq. (6).

Because the tree-level potential is so flat, the one-loop correction may in fact be significant. In the regime where $\phi$ is much bigger than all of the relevant masses, it typically has the form $\Delta V = (a_0 + a_2\phi^2 + a_4\phi^4)\ln(\phi/Q)$. Here $Q$ is the renormalization scale, which should be fixed at some value within the relevant range of $\phi$ so that the loop correction is small (and therefore believable).

In a non-supersymmetric theory, the quartic term dominates and each loop gives a contribution

$$ \Delta V = g^2\phi^4\ln(\phi/Q), \quad (9) $$

where $g \ll 1$ is a typical coupling (times a loop-suppression factor like $(8\pi)^{-1/2}$). This is roughly equivalent to a change $\Delta \lambda \sim g^2$ in the tree-level potential, so the constraint Eq. (7) requires very suppressed couplings ($g^2$ many orders of magnitude below 1). In particular the inflaton is presumably a gauge singlet, since gauge couplings as opposed to Yukawa couplings are not supposed to be suppressed.

With the favoured paradigm of supersymmetry, things are quite different. During inflation, the sector of the theory occupied by the inflaton field is usually described by global supersymmetry as opposed to supergravity\footnote{By ‘sector’ we mean the set of fields that couple to the inflaton with more than gravitational strength.}. The quartic term then vanishes by virtue of non-renormalization theorems. If supersymmetry is broken only spontaneously, and the relevant masses-squared have zero supertrace, the quadratic term vanishes as well leaving a contribution of the form

$$ \Delta V = g^2V_0\ln(\phi/Q). \quad (10) $$

Again, $g$ is a typical coupling. On the assumption that the slope of the tree-level potential is negligible, this paradigm has been widely studied, particularly in the manifestation known as ‘$D$-term inflation’. It makes the distinctive prediction $n \simeq 0.97$ which can eventually be tested. Unfortunately, it also (through Eq. (3)) requires $\phi_{\text{COBE}} \gtrsim M_P$ if $g$ is unsuppressed, which is typically the case in this type of model. (A subscript COBE will indicate the epoch when the scale explored by COBE leaves the horizon.) As a result the non-renormalizable terms of the tree-level potential will generically spoil inflation.

We are here concerned with the opposite possibility, that supersymmetry during inflation is broken explicitly as opposed to spontaneously. As usual the breaking is supposed
to be soft, and then the dominant loop correction is quadratic,

$$\Delta V = g^2 \tilde{m}^2 \phi^2 \ln(\phi/Q).$$

(11)

Here $\tilde{m}$ is a typical soft mass appearing in the loop, and $g$ is still a typical coupling. Again, the loop correction is significant if $g$ is unsuppressed, assuming that $|\tilde{m}|$ has the typical value $V_0^{1/2}/M_P$. We assume that these conditions are satisfied in what follows.

With this paradigm, $\phi$ typically varies by many orders of magnitude during the era of interest (starting when COBE scales leave the horizon, and ending when slow-roll inflation ends). As a result, no single choice of $Q$ will make the loop correction valid during the entire era. To handle this situation, one can drop the loop correction in favour of the renormalization-group-improved tree-level potential, in which $m^2(Q)$ is evaluated at $Q = \phi$;

$$V = V_0 + \frac{1}{2} m^2(\phi) \phi^2 + \cdots.$$

(12)

In the approximation that $m^2$ is linear in $\ln \phi$, one recovers the prescription ‘unimproved tree-level plus loop correction’ by fixing the renormalization scale.

Following Stewart [4, 5], our central strategy is to assume the linear approximation while cosmological scales are leaving the horizon, but not afterwards. The predictions then involve only three parameters, which we call $c$, $\sigma$ and $\tau$. The first two give the prediction for the spectral index, while the COBE prediction involves all three.

The plan of the paper is as follows. In Section 2 we evaluate the predictions, and the four possible types of model to which they apply. In Section 3 we make a crude estimate of the observational constraints on the three parameters, for each of the four possible models. In Section 4 we see how the present discussion applies to the simplest possible model [4, 5], in which only a single gauge coupling is significant. In Section 5 we summarize the results, and point to future directions for comparison with observation.

2 The running-mass models and their predictions

The generic estimate Eq. (8) applies to all of the scalar masses, when the renormalization scale is $M_P$. As Stewart pointed out, it can be avoided for the inflaton provided that $m^2(\phi)$ decreases in magnitude, as $\phi$ decreases from $M_P$. We assume that this happens, and focus on the case [4, 5] where $m^2(\phi)$ changes sign at some point. (The opposite case will be mentioned briefly.) Because the couplings are small compared with unity, $V'$ then vanishes at some relatively nearby point, which we denote by $\phi_\ast$.

2.1 The linear approximation

It is useful to write Eq. (12) in the form

$$V(\phi) = V_0 \left( 1 - \frac{1}{2} M_P^{-2} \mu^2(\phi) \phi^2 \right),$$

(13)
where
\[ \mu^2(\phi) \equiv -M_P^2 m^2(\phi)/V_0. \] (14)

We are supposing that \( V_0 \) dominates, since this is necessary for inflation in the regime \( \phi \lesssim M_P \) where the field theory is under control. Then
\[
M_P \frac{V'}{V_0} = -\phi \left[ \mu^2 + \frac{1}{2} \frac{d\mu^2}{dt} \right],
\] (15)
\[
\eta \equiv M_P^2 \frac{V''}{V_0} = -\left[ \mu^2 + \frac{3}{2} \frac{d\mu^2}{dt} + \frac{1}{2} \frac{d^2\mu^2}{dt^2} \right],
\] (16)

where \( t \equiv \ln(\phi/M_P) \).

We assume that while observable scales are leaving the horizon one can make a linear expansion in \( \ln \phi \),
\[
\mu^2 \simeq \mu^2_* + c \ln(\phi/\phi_*),
\] (17)
where \( |c| \ll 1 \) is related to the couplings involved. This gives
\[
M_P \frac{V'}{V_0} = c\phi \ln(\phi_*/\phi)
\] (18)
\[
\eta \equiv M_P^2 \frac{V''}{V_0} = c[\ln(\phi_*/\phi) - 1].
\] (19)

Note that \( \mu^2_* = -\frac{1}{2}c \), and that \( \mu^2 = 0 \) at \( \ln(\phi_*/\phi) = -\frac{1}{2} \) while \( V'' = 0 \) at \( \ln(\phi_*/\phi) = 1 \).

The number \( N(\phi) \) of e-folds to the end of slow-roll inflation is given by
\[
N(\phi) = M_P^{-2} \int_{\phi_{\text{end}}}^{\phi} \frac{V}{V'} d\phi.
\] (20)

Using the linear approximation near \( \phi_* \), this gives
\[
N(\phi) = -\frac{1}{c} \ln \left( \frac{c}{\sigma} \ln \frac{\phi_*}{\phi} \right),
\] (21)
or
\[
(\sigma/c)e^{-cN} = \ln(\phi_*/\phi).
\] (22)

Knowing the functional form of \( m^2(\phi) \), and the value of \( \phi_{\text{end}} \), the constant \( \sigma \) can be evaluated by taking the limit \( \phi \to \phi_* \) in the full expression Eq. (20). We shall see in the next section that in most cases one expects
\[ |c| \lesssim |\sigma| \lesssim 1. \] (23)

Using Eq. (2), the spectral index is given in terms of \( c \) and \( \sigma \) by
\[
\frac{n(k) - 1}{2} = \sigma e^{-cN} - c.
\] (24)
As usual, the right hand side is evaluated at the epoch of horizon exit \( k = aH \). Since \( H \) is slowly varying, \( d \ln k \simeq -dN \), where \( dN \equiv -d \ln a \) is the change in the number \( N(\phi) \) of e-folds to the end of slow-roll inflation.

This prediction is relevant while cosmological scales are leaving the horizon. As we noted earlier, cosmological scales span roughly the four decades of \( H_0 \lesssim k \lesssim 10^4 H_0 \). Taking round figures, this corresponds to \( \Delta N = 10 \), with the COBE measurement probing more or less the top of the range. Cosmological scales therefore leave the horizon while

\[
N_{\text{COBE}} - 10 \lesssim N \lesssim N_{\text{COBE}}
\]

(25)

To determine \( N_{\text{COBE}} \) one needs to know what happens after slow-roll inflation ends, but an upper bound is obtained by assuming instant reheating which lasts until the present matter-dominated era. This gives\(^5\)

\[
N_{\text{COBE}} < 48 + \ln(V_0^{1/4}/10^{10} \text{GeV}) .
\]

(26)

From Eq. (11), the COBE measurement gives

\[
\frac{V_0^{1/2}}{M_P^2} = 5.3 \times 10^{-4} M_P \frac{|V'|}{V_0} ,
\]

(27)

In our case it is convenient to define a constant \( \tau \) by

\[
\ln(M_P/\phi_*) \equiv \tau/|c| .
\]

(28)

Assuming that \( |m^2| \) has the typical value \( V_0/M_P^2 \) at the Planck scale, the linear approximation Eq. (17) applied at that scale would give \( \tau \simeq 1 \). Will the linear approximation apply at that scale? If all relevant masses at the Planck scale are of order \( V_0/M_P^2 \), one expects on dimensional grounds that the linear approximation will be valid in the regime \( |c \ln(\phi/\phi^*)| \ll 1 \). Then the approximation will be just beginning to fail at the Planck scale. At least in this case, one expects \( \tau \) to be very roughly of order 1.

Using the definition of \( \tau \), Eqs. (18) and (22) give

\[
\frac{V_0^{1/2}}{M_P^2} = e^{-\tau/|c|} \exp \left( -\frac{\sigma}{c} e^{-cN_{\text{COBE}}} \right) |\sigma| e^{-cN_{\text{COBE}}} \times 5.3 \times 10^{-4} .
\]

(29)

In these models, the spectral index may be strongly scale-dependent. In fact, using \( d \ln k = -dN \) one finds

\[
\frac{dn}{d \ln k} = 2c \sigma e^{-cN} .
\]

(30)

The observational constraint \( |n - 1| < 0.2 \) applies only if \( n \) has negligible variation, \( \Delta n \ll 0.4 \). Taking cosmological scales to span \( \Delta \ln k \sim 10 \), and considering the centre of the range, the predicted variation is in fact negligible only if

\[
|c \sigma| e^{-c(N_{\text{COBE}} - 5)} < 0.02 .
\]

(31)

\(^5\)This expression takes \( k = H_0 \) to be the scale probed by COBE. The more correct \( k \simeq 10H_0 \) would reduce the right hand side by \( \ln 10 \simeq 2 \).
Figure 1: A possible form for the renormalization-group-improved inflaton potential. In units of $M_P$, the maximum is located roughly at $\phi \sim e^{-1/c}$. In the case illustrated, the minimum corresponds to the vacuum, where $V$ vanishes. This and the following figure are taken from [1].

In a large region of parameter space, Eq. (31) is violated and $n$ has significant variation. In that situation, a detailed comparison with cosmological observations is probably best done by directly considering the scale-dependence of $\delta_H(k)$. Integrating $\frac{1}{2}(n-1) = d\ln \delta_H/d\ln k$, it is given by

$$\ln \left[ \frac{\delta_H(k)}{\delta_H(k_{COBE})} \right] = +\frac{\sigma}{c} \left( e^{-cN} - e^{-cN_{COBE}} \right) + c(N - N_{COBE}).$$

(32)

The Planck satellite will explore a range $\Delta \ln k \simeq 6$ of scales, and will measure $n$ with an accuracy of around .01. This means that a scale-dependence $|dn/d\ln k| \gtrsim 2 \times 10^{-3}$ should be observable by Planck.

2.2 The four models

Four types of inflation model are possible, corresponding to whether $\phi_*$ is a maximum or a minimum, and whether $\phi$ during inflation is smaller or bigger than $\phi_*$. 

If $\phi_*$ is a maximum, one expects the potential to have the form shown in Figures 1 and 2. There is a minimum at $\phi = 0$, and the non-renormalizable terms will ensure that there is a minimum also at some value $\phi_{\text{min}} > \phi_*$. The latter will generally be lower than the one at the origin, and we assume that this is the case. This lowest minimum represents the true vacuum if $V$ vanishes there as in Figure 1. If instead $V$ is positive as in Figure 2, the vacuum lies in some other field direction, ‘out of the paper’.

In the case that $\phi_*$ is a minimum, one expects that the origin will be a maximum, and that $V$ will increase monotonically to the right of the minimum. The unique minimum represented by $\phi_* = 0$ is the vacuum if $V$ vanishes there, otherwise the vacuum lies in some other field direction.
Figure 2: Alternatively, the true vacuum may lie in another field direction, ‘out of the paper’.

2.2.1 Model (i); $\phi_*$ a maximum with $\phi < \phi_*$

This model \cite{5, 6} corresponds to $m^2(M_P) < 0$, $c > 0$ and $\sigma > 0$, with $\phi$ decreasing during inflation. The spectral index increases as the scale $k^{-1}$ decreases, and can be either bigger or less than 1.

For inflation to end, the form Eq. (12) of $V(\phi)$ must be modified when $\phi$ falls below some critical value $\phi_c$, presumably through a hybrid inflation mechanism. On the other hand, if the inflaton mass continues to run until $m^2 \approx V_0/M_P^2$, slow-roll inflation will end then. Let us suppose first that this is the case, and define $\phi_{\text{fast}}$ by

$$m^2(\phi_{\text{fast}}) = \frac{V_0}{M_P^2}. \quad (33)$$

This is equivalent to defining $\eta(\phi_{\text{fast}}) = 1$, up to corrections of order $c$ which presumably should not be included in a one-loop calculation. The end of slow-roll inflation corresponds to $\phi_{\text{end}} = \phi_{\text{fast}}$, and the linear approximation Eq. (17) gives the rough estimate $|\ln(\phi_{\text{end}}/\phi_*)| \sim 1/c$, making $\sigma \sim 1$.

Now consider the case where inflation ends at some value $\phi_c$, with $|m^2(\phi_c)| < V_0/M_P^2$. If the mass is still running at that point, the linear estimate Eq. (21) gives $\sigma \sim c\ln(\phi_*/\phi_c) < 1$. Values $\sigma \ll c$ can be achieved only with $\phi_c$ very close to $\phi_*$ which would represent fine-tuning. Therefore we expect in this case $c \lesssim \sigma \lesssim 1$.

If the mass stops running before $\phi_c$ is reached, at some point $\phi_{\text{low}}$, then $m^2$ has a constant value $m^2_{\text{low}} = m^2(\phi_{\text{low}})$ in the regime $\phi_c < \phi < \phi_{\text{low}}$. In this regime, some number $\Delta N$ of $e$-folds of slow-roll inflation occur. We are assuming that cosmological scales leave the horizon while the mass is still running, which requires

$$\Delta N < N_{\text{COBE}} - 10 \quad (34)$$

$$< 38 + \ln\left(\frac{V_0^{1/4}}{10^{10} \text{GeV}}\right). \quad (35)$$

Retaining the estimate of the previous paragraph for the $e$-folds of inflation before the
mass stops running, the constant $\sigma$ to be used in Eq. (22) will be in the range

$$c \lesssim \sigma \lesssim e^{c\Delta N}. \quad (36)$$

According to the estimates of the next section, $e^{c\Delta N}$ will not be more than one or two orders of magnitude above unity.

2.2.2 Model (ii); $\phi_*$ a maximum with $\phi > \phi_*$

Like the previous model, this one corresponds to $m^2(M_P) < 0$ and $c > 0$, but now $\sigma < 0$ and $\phi$ increases during inflation. The spectral index is less than 1, and decreases as the scale decreases.

In contrast with the previous case, inflation can end without any need for a hybrid inflation mechanism, or a change in the form of the potential Eq. (12), if the minimum at $\phi > \phi_*$ is the true vacuum. If the form Eq. (12) holds until $\phi$ reaches the value $\phi_{\text{fast}}$ defined by $\eta(\phi_{\text{fast}}) = -1$, slow roll inflation will end there. To leading order in $c$ this corresponds to

$$m^2(\phi_{\text{fast}}) = -V_0/M_P^2. \quad (37)$$

Setting $\phi_{\text{end}} = \phi_{\text{fast}}$, and using the crude linear approximation one finds $\phi_{\text{end}} \sim e^{1/|c|}\phi_* \sim M_P$, and $\sigma \sim -1$.

On the other hand, slow-roll inflation might end at some point earlier $\phi_c$. In the true-vacuum case illustrated in Figure 1, this may happen through a steepening in the form of $V(\phi)$. Otherwise it may happen through an inverted hybrid inflation mechanism. In both cases, we expect $c \lesssim |\sigma| \lesssim 1$.

In contrast with the previous model, this one also makes sense if $m^2$ stops running (as $\phi$ decreases) before it changes sign; in other words, if it stops running at $\phi_{\text{low}}$ with $m^2(\phi_{\text{low}}) < 0$, but very small. In this case the maximum of the potential is at the origin and $\eta$ is small and constant up to $\phi = 0$. The above treatment remains valid if $m^2$ has started to run before cosmological scales leave the horizon (remember that in this model, $\phi$ increases during inflation). Otherwise, one has a different model that we shall not consider.

2.2.3 Model (iii); $\phi_*$ a minimum with $\phi < \phi_*$

This corresponds to $m^2(M_P) > 0$, $c < 0$ and $\sigma < 0$, and $\phi$ increases during inflation. The spectral index can be either above or below 1, and it increases as the scale decreases.

Now $|m^2|$ decreases during inflation, and slow-roll inflation ends only when the potential Eq. (12) ceases to hold at some value $\phi_{\text{end}} = \phi_c$. In a single-field model, corresponding to $V$ vanishing at the minimum, this can occur through a steepening of the form of the potential.

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6This estimate of $\phi_{\text{fast}}$ assumes that quartic and higher terms in the tree-level potential Eq. (5) are negligible at $\phi_{\text{fast}}$. Assuming that only one such term is significant, one easily checks that the estimate is roughly correct, unless the dimension of the term is not extremely large. We do not consider that case, or the case where more than one term is significant.
tree-level potential, as higher powers of $\phi$ become important. Alternatively, if $V$ is positive at the minimum it can occur through a hybrid inflation mechanism (inverted hybrid inflation).

To estimate $\sigma$ in this case, suppose first that (as $\phi$ decreases) the mass continues to run until $m^2 = -V_0/M_P^2$, and denote the point where this happens by $\phi_{\text{fast}}$. Slow roll inflation can then only occur in the regime $\phi > \phi_{\text{fast}}$. It follows that

$$\phi_{\text{end}} \gtrsim \phi_{\text{fast}},$$

and the linear approximation $\phi_{\text{fast}} \sim e^{-1/|c|}\phi_*$ then gives $|\sigma| \lesssim 1$. As before $|\sigma| \gtrsim |c|$ is required to avoid the fine-tuning $\ln(\phi_*/\phi_c) \ll 1$.

If the mass stops running at some point $\phi_{\text{low}}$, with $|m^2(\phi_{\text{low}})| \ll 1$, inflation can begin at arbitrarily small field values. If cosmological scales start to leave the horizon only after the mass has started to run, Eq. (38) still applies and the estimate for $\sigma$ is unchanged. We do not consider the opposite case.

**Model (iv); $\phi_*$ a minimum with $\phi > \phi_*$**

Like the previous case this one corresponds to $m^2(M_\text{P}) > 0$ and $c < 0$, but now $\sigma > 0$ and $\phi$ decreases during inflation. The spectral index is bigger than 1, and it decreases as the scale decreases.

Everything is the same as in the previous case, except that a hybrid inflation mechanism will definitely be needed to end inflation, since higher-order terms in $\phi$ can hardly become more important as $\phi$ decreases. We again expect $|c| \lesssim \sigma \lesssim 1$, with the lower limit needed to avoid the fine-tuning $\ln(\phi_c/\phi_*) \ll 1$. As a result we expect $|c| \lesssim \sigma \lesssim 1$.

Like Model (iii), this one can still make sense if the mass stops running before $\phi_*$ is reached. The above treatment applies if cosmological scales leave the horizon while the mass is still running. We do not consider the opposite case.

**2.3 When will the mass stop running?**

The expression for the one-loop correction given in eq. (11) is valid when $\phi$ is larger than any other mass scale and only in this case is the choice $Q = \phi$ reasonable. In the other case $Q$ will be given by some other relevant scale and the mass would no more depend on the value of $\phi$. We can therefore assume that running stops at the value $\phi_{\text{low}}$, defined by $\phi_{\text{low}} = |\tilde{m}(\phi_{\text{low}})|$, where $\tilde{m}$ is the biggest relevant mass.

At the Planck scale, the estimate Eq. (8) applies to all masses. As the renormalization scale decreases, some will increase in magnitude but as a rough estimate we can take

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\textsuperscript{7}Stewart \textsuperscript{1} took the view that models (iii) and (iv) require a fine-tuning of $\phi_c$ over the whole range of parameter space. As with all views on fine-tuning, this is a matter of taste.

\textsuperscript{8}At least it applies to the scalar masses. Gaugino masses can be smaller, depending on the gauge kinetic function.
$|\tilde{m}(\phi)|$ to have the roughly scale-independent value $V_0^{1/2}/M_P$. According to the COBE normalization Eq. (43) given below, this gives

$$|	ilde{m}| \sim (10^{-4} \text{ to } 10^{-5}) e^{-\tau/|c|} M_P$$

(39) \equiv (10^{-4} \text{ to } 10^{-5}) \phi_* .

According to this crude estimate, the running continues until well below $\phi_*$, where $m^2 \simeq 0$. In the specific model [6] that we consider later, it continues all the way down to $\phi_{\text{fast}}$, where $|m^2| = V_0^{1/2}/M_P$.

### 3 Observational constraints

The range of observable scales corresponds to

$$N_{\text{COBE}} - 10 \ll N \ll N_{\text{COBE}} .$$

(41)

In the case that $n$ is roughly constant over this range, it is constrained by observation to $|n - 1|/2 < 0.1$. As a rough estimate we have applied this constraint at the two values $N_{\text{COBE}}$ and $N = N_{\text{COBE}} - 10$. The results are shown in Figures 3 to 6. An alternative view is provided by Figure 7, in which the lines $n - 1 = 0.1, 0.0$ and $-0.1$ are shown, for different values of $N$. Each quadrant corresponds to one of the models; top right is model (i), bottom right is model (ii), bottom left is model (iii) and top left is model (iv).

We have yet to consider the COBE normalization Eq. (29). According to the above bounds, it is roughly

$$\frac{V_0^{1/2}}{M_P^2} \sim (10^{-4} \text{ to } 10^{-5}) e^{-\tau/|c|}$$

(42) \equiv (10^{-4} \text{ to } 10^{-5}) (\phi_*/M_P) .

(43)

Imposing only the requirement that inflation ends before nucleosynthesis, one requires $V_0^{1/4} > 10 \text{ MeV}$ corresponding to

$$0.012 \lesssim |c|/\tau \lesssim 1 .$$

(44)

But if the supersymmetry breaking scale during inflation is the same as in the true vacuum one will have

$$10^5 \text{ GeV} \lesssim V_0^{1/4} \lesssim 10^{10} \text{ GeV} .$$

(45)

The upper limit corresponds to gravity-mediated susy breaking in the true vacuum, while the rest of the range corresponding to gauge-mediated susy breaking in the true vacuum.$^9$ This corresponds to

$$0.019 \lesssim |c|/\tau \lesssim 0.033 .$$

(46)

$^9$The upper limit corresponds to the condition $M_P |V'/V| \ll 1$ for slow-roll inflation, which implies $V_0^{1/4} \lesssim 10^{16} \text{ GeV}$.

$^{10}$In the latter case, the inflaton sector can have only gravitational strength couplings with the visible sector.
Figure 3: This and the following three figures show a crude observational constraint on the parameter space, obtained by requiring \(|n - 1| < 0.2\) at both \(N = N_{\text{COBE}}\) and \(N = N_{\text{COBE}} - 10\). These values of \(N\) correspond respectively to the biggest and smallest scales on which \(n\) can be observed. The first panel takes \(N_{\text{COBE}} = 50\), while the second takes \(N_{\text{COBE}} = 25\). In all cases, a full line corresponds to \(n = 0.8\) while a long–dashed line corresponds to \(n = 1.2\). The dotted straight line \(|\sigma| = |c|\) is also shown in all cases; since the theoretically expected parameters satisfy \(|\sigma| \geq |c|\), the allowed parameter space is bounded by these three lines. If \(|\sigma|\) is bigger than the value indicated by the short–dashed line (labelled P-LB), the Planck satellite will measure the scale dependence of \(n\).

Figure 4: Model (ii), in which \(n\) is less than 1, and decreases as the scale decreases. The full line corresponds to \(n = 0.8\) evaluated at \(N = N_{\text{COBE}} - 10\). The allowed region lies above this line and below the dotted one. For parameters lying below the short–dashed line (P–LB) the spectral index scale dependence will be detected by the Planck satellite.
Figure 5: Model (iii), in which \( n - 1 \) increases as the scale decreases. The full line corresponds to \( n = 0.8 \) evaluated at \( N = N_{\text{COBE}} \). The long–dashed line (which is invisible in the first panel) corresponds to \( n = 1.2 \) evaluated at \( N_{\text{COBE}} - 10 \). The allowed region lies between the two lines and the axes and below the dotted line. For parameters lying below the short–dashed line (P–LB) the spectral index scale dependence will be detected by the Planck satellite.

Figure 6: Model (iv), in which \( n - 1 \) is positive, and decreases as the scale decreases. The long–dashed line corresponds to \( n = 1.2 \), evaluated at \( N = N_{\text{COBE}} \). The allowed region lies below this line and above the dotted line \( \sigma = -c \). For parameters lying above the short–dashed line (P–LB) the spectral index scale dependence will be detected by the Planck satellite.
Figure 7: The lines $n = 0.8$, $n = 1.0$ and $n = 1.2$ are shown, for $N = 50$ and $N = 25$. Each quadrant corresponds to one of the four models of inflation. In the regime $c < 0$, the line $n = 1.0$ falls marginally below the line $\sigma = 0$.

Finally, we should mention that in some cases, a strong constraint is imposed by the requirement that excessive black hole formation does not occur soon after inflation ends. With some assumptions, the requirement for this [7] is that the $rms$ perturbation smoothed over a Hubble scale does not exceed .04 at the end of inflation. With further assumptions, this is equivalent [8] to

$$\delta_H < 0.01,$$

(47)

at the end of slow-roll inflation.

As the linear approximation ceases to be accurate before the end of inflation, the black hole constraint cannot in general be evaluated in terms of the parameters $c$, $\sigma$ and $\tau$. We examined the constraint in a case where it is likely to be particularly strong, namely the case of model (i) with inflation ending at $\phi_{\text{fast}}$. We considered the case where only a single, asymptotically free gauge group is relevant, using the formulas given in the next section.

What we found was that $\delta_H$ increases very quickly as $\phi_{\text{fast}}$ is approached. If Eq. (17) is imposed precisely at the point $\phi_{\text{fast}}$ defined by $\eta(\phi_{\text{fast}}) = 1$, it is a strong constraint. But if we make the equally reasonable estimate $\eta(\phi_{\text{fast}}) = \frac{1}{2}$ for the point where slow-roll inflation ends, and impose Eq. (17) there instead, it ceases to be a significant constraint. The conclusion, at least in this case, is that a more careful calculation of black hole formation is required, than any appearing in the literature.
4 The case of a single gauge coupling

The Renormalization Group Equations can easily be solved in the case where only a single gauge coupling is relevant. The result is [5, 6]

\[ m^2(\phi) = m_0^2 + \frac{2c_b}{b} \tilde{m}_0^2 \left[ 1 - \frac{1}{\ln(\phi/M_P)} \right] \text{.} \]  

(48)

Here \( m_0 \) is the inflaton mass, \( \tilde{m}_0 \) is the gaugino mass, \( \alpha_0 \) is the gauge coupling, all evaluated at the Planck scale, and \( b \) and \( c > 0 \) are numerical factors depending on the gauge group and its representations.

We want the magnitude of \( m^2 \) to decrease as one goes down from the Planck scale. This requires \( m_0^2 < 0 \), corresponding to model (i) or model (ii). Only model (i) in the case of asymptotic freedom (\( b < 0 \)) has been studied so far [5, 6].

We evaluate \( c, \sigma \) and \( \tau \) to leading order in \( \alpha \), which is presumably all that is justified in a one-loop calculation. It is convenient to use the following definitions, taken from [3] but now applied to models (i) and (ii).

\[ \mu^2 \equiv -m^2 M_P^2 / V_0 \text{,} \]  

(49)

\[ A \equiv -\frac{2c}{b} \tilde{m}^2 M_P^2 / V_0 \text{,} \]  

(50)

\[ \tilde{\alpha} \equiv -\frac{b\alpha}{2\pi} \text{,} \]  

(51)

\[ y \equiv \left[ 1 + \tilde{\alpha}_0 \ln(\phi/M_P) \right]^{-1} \text{,} \]  

(52)

\[ y_* \equiv \sqrt{1 + \frac{\mu^2}{A_0}} \text{,} \]  

(53)

where the subscript 0 for a running quantity denotes the value of that quantity at the Planck scale. Applying the linear approximation one finds

\[ c = 2y_*^3 A_0 \tilde{\alpha}_0 \]  

(54)

\[ \tau = 2A_0 y_*^2 (y_* - 1) \text{.} \]  

(55)

We shall investigate models (i) and (ii) for arbitrary \( b \), and in both cases assume that inflation continues until \( |\mu^2| = 1 \).

4.1 Model (i)

Assuming that the mass continues to run until the end of inflation, the condition \( N(\phi_{\text{end}}) = 0 \) gives

\[ \ln \sigma = 2y_*^2 \left( y_* - y_{\text{end}}^{-1} \right) + \ln \left[ \frac{4A_0 y_*^2 (y_{\text{end}} - y_*)}{y_{\text{end}} + y_*} \right] \text{.} \]  

(56)
Figure 8: Lines of constant $n$ and of constant $V_0/M_4^4$, in the plane $A_0-\mu_2^2$, for $N_{\text{Cobe}} = 45$ and $\tilde{\alpha}_0 = 0.01$. The right figure has been obtained using the linear approximation and equations (54), (55) and (56), while the left figure shows the exact result given in [3]. For this value of $N_{\text{Cobe}}$, Eq. (26) requires $V_0/M_4^4 \gtrsim 10^{-36}$, so the regime above that line is forbidden.

with

$$y_{\text{end}} \equiv \left(y_{*+}^2 + A_0^{-1}\right)^{1/2}.$$  \hfill (57)

This expressions are valid both in the case of negative $b$ (i.e. positive $A_0$ and $\tilde{\alpha}_0$ ) and of positive $b$ (i.e. negative $A_0$ and $\tilde{\alpha}_0$ ).

In Figure 8 we make a comparison between the present formalism, and the exact results of [3] for the case of asymptotic freedom. Lines of constant $n$ and of constant $V_0$ are shown, and the agreement is seen to be very good. In order to make a direct comparison with [3], the lines of constant $n$ have all been evaluated at $N = N_{\text{Cobe}}$. The allowed region, as defined in the previous Section, lies between the line $n = 0.8$ evaluated at $N = N_{\text{Cobe}}$ and the line $n = 1.2$ evaluated at $N = N_{\text{Cobe}} - 10$; the latter actually lies a little below the line $n = 1.0$ shown in the Figure.

In Figure 8 we give a similar plot for the case $N_{\text{Cobe}} = 25$, but this time plotting the line $n = 1.2$ at $N = N_{\text{Cobe}} - 10$ (curve marked LB). The allowed region is between this one and the line $n = 0.8$ evaluated at $N = N_{\text{Cobe}}$ (marked UB). We can see that the general behaviour is similar to Figure 8.

Let us now consider the other case $b > 0$, i.e. $A_0, \tilde{\alpha}_0 < 0$. As we can see from eq. (23), the parameter space is limited to the region where $|A_0| > \mu_0^2$; this is due to the fact that the running is weaker in this case and a large gaugino mass is necessary to change the sign of the inflaton mass. The limiting case where $|A_0| = \mu_0^2$ correspond to a vanishing mass in the asymptotic limit $\phi = 0$, but such result is not reliable since the running would surely stop before that point. Since we are actually setting $y_{\text{end}}$ by eq. (24), our parameter space will be even more restricted by the condition $|A_0| > \mu_0^2 + 1$. In Figure 10 we show the allowed region for negative $A_0$ both for $N_{\text{Cobe}} = 50$ and $N_{\text{Cobe}} = 25$ and for $\tilde{\alpha}_0 = -0.01$. It is bounded by the two lines UB, corresponding to
Figure 9: The contour lines in the linear approximation for the spectral index \( n(N_{\text{COBE}}) = 0.8 \) and \( n(N_{\text{COBE}} - 10) = 1.2 \) and \( V_0 \) for two different values of \( \tilde{\alpha}_0 = 0.01, 0.001 \) in the \( \mu_0^2 - A_0 \) plane for \( N_{\text{COBE}} = 25 \). For this value of \( N_{\text{COBE}} \), Eq. (26) requires \( V_0/M_4^4 \gtrsim 10^{-72} \), so the regime above that line is forbidden.

\( n = 1.2 \) at \( N = N_{\text{COBE}} - 10 \), and LB, corresponding to \( n = 0.8 \) at \( N = N_{\text{COBE}} \). Notice in this case a long narrow strip satisfies the experimental constraints, but the larger values of \( \mu_0^2 \) are excluded because of the low value of \( V_0 \).

### 4.2 Model (ii)

Finally, we consider model (ii) in the same setting, i.e., considering that the inflaton rolls on the right of the maximum (\( \phi > \phi_* \)) and inflation ends when \( |\eta| \simeq 1 \). In this case \( \sigma < 0 \) and is given by:

\[
\ln |\sigma| = 2y_{**}^2 \left( y_{**}^{-1} - y_{\text{end}}^{-1} \right) + \ln \left[ \frac{4A_0y_{**}^2(y_{**} - y_{\text{end}})}{y_{\text{end}} + y_{**}} \right], \tag{58}
\]

with

\[
y_{\text{end}} \equiv \left( y_{**}^2 - A_0^{-1} \right)^{1/2}. \tag{59}
\]

We see from the expression of \( y_{\text{end}} \) that our parameter space will in this case be restricted to the region where \( A_0 + \mu_0^2 \geq 1 \); otherwise \( |\eta| \simeq 1 \) will never be reached and some other mechanism should be responsible for the end of inflation. Since we naturally expect \( \mu_0^2 \) to be of order 1, this condition is not very restrictive, apart for negative \( A_0 \).

We see from Figure 11 that in this model too, there is a region of parameter space where \( \mu_0^2 \) and \( A_0 \) can have their expected values of order 1, with the reasonable potential \( 10^5 \text{GeV} \lesssim V_0^{1/4} \lesssim 10^{10} \text{GeV} \) (\( 10^{-72} \lesssim V_0/M_4^4 \lesssim 10^{-32} \)).

\[^1\text{This reverses the more pessimistic conclusion of Stewart} \ [4], \text{which was made on the basis of rough estimates only.}\]
Figure 10: The contour lines in the linear approximation for the spectral index \( n(N_{\text{COBE}}) = 0.8 \) and \( n(N_{\text{COBE}} - 10) = 1.2 \) and \( V_0 \) for \( \tilde{\alpha}_0 = -0.01 \) in the \( \mu_0^2 - A_0 \) plane for \( N_{\text{COBE}} = 50, 25 \). The line marked RB is the upper bound to the parameter space given by requiring \( |A_0| > \mu_0^2 + 1 \). From Eq. (26), the regime \( V_0/M_p^4 \lesssim 10^{-29} \) is forbidden if \( N_{\text{COBE}} = 50 \), and the regime \( V_0/M_p^4 \lesssim 10^{-72} \) is forbidden if \( N_{\text{COBE}} = 25 \).

Figure 11: Contour lines for the spectral index at \( N = N_{\text{COBE}} - 10 \) (line UB) and \( V_0 \) at \( N_{\text{COBE}} \) for two different values of \( N_{\text{COBE}} = 50, 25 \) in the \( \mu_0^2 - A_0 \) plane for \( \tilde{\alpha}_0 = 0.01 \). This is a model of type (ii) and therefore \( n - 1 < 0 \) in all the parameter space; the experimentally allowed region is enclosed between the line UB and the line \( A_0 + \mu_0^2 = 1 \) (LB). From Eq. (26), the regime \( V_0/M_p^4 \lesssim 10^{-29} \) is forbidden if \( N_{\text{COBE}} = 50 \), and the regime \( V_0/M_p^4 \lesssim 10^{-72} \) is forbidden if \( N_{\text{COBE}} = 25 \).
We considered also the case of negative \( A_0 \) and \( \tilde{\alpha}_0 \), but no region of the parameter space gives in this case an acceptable spectral index, a part the line \(|A_0| = \mu_0^2\), corresponding to \( \phi_* = 0 \). Since, as we already discussed, we cannot extrapolate the running mass up to such low values of the field, this kind of model is ruled out for the non asymptotically free case. Notice moreover that the large negative \( n \) is due to the fact that almost all the parameter space correspond to large \( |\sigma| \) because of the positive exponential \( \exp(1 - y_{**}/y_{\text{end}}) \) (negative for \( b < 0 \)). Therefore it seems improbable that considering another mechanism for the end of inflation could change this result.

5 Conclusion

A model of inflation with a running inflaton mass may contain many parameters, corresponding to the masses and couplings of the particles responsible for the loop correction to the inflaton potential. We have seen how to calculate the observable predictions in terms of just three parameters, which in a given model can be calculated in terms of the masses and couplings.

Observation presently allows a wide range of parameter space, which we have partially delineated. In the fairly near future, the allowed region will become much smaller, because these models typically predict a spectral index \( n \) with the distinctive form \( \frac{1}{2}(n(k) - 1) = \sigma e^{-cN} - c \). In this formula, the variation in wavenumber \( k \) is given by \( dk = -dN \), and the parameters typically satisfy \( |c| \lesssim |\sigma| \lesssim 1 \). If this model is correct, the scale dependence of \( n(k) \) will be detected by Planck in a large region of the parameter space.

On the theoretical side, the exploration of running-mass models is only just beginning, and many questions remain unanswered regarding their possible relation with other aspects of physics beyond the Standard Model. In particular, one would like to know if the inflaton can belong to the visible sector, or whether it must belong to a hidden sector consisting of particles that have only gravitational-strength coupling with particles possessing the gauge interactions of the Standard Model.

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