MINIMAL ELEMENTARY END EXTENSIONS

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Abstract. Suppose that $\mathcal{M} \models \text{PA}$ and $\mathcal{X} \subseteq \mathcal{P}(\mathcal{M})$. If $\mathcal{M}$ has a finitely generated elementary end extension $\mathcal{N} \succ_{\text{end}} \mathcal{M}$ such that 
\[ \{X \cap M : X \in \text{Def}(\mathcal{N})\} = \mathcal{X}, \]
then there is such an $\mathcal{N}$ that is, in addition, a minimal extension of $\mathcal{M}$ iff every subset of $\mathcal{M}$ that is $\Pi^1_1$-definable in $(\mathcal{M}, \mathcal{X})$ is the countable union of $\Sigma^0_1$-definable sets.

The fundamental theorem of MacDowell & Specker [MS61] states that every model of Peano Arithmetic ($\text{PA}$) has an elementary end extension. Since it first appeared more than half a century ago, this theorem has been frequently refined, generalized, modified and applied. This note is in that spirit.

For a model $\mathcal{M} \models \text{PA}$, we let $\text{Def}(\mathcal{M})$ be the set of all parametrically definable subsets of $\mathcal{M}$. If $\mathcal{M} \prec \mathcal{N}$, then the set of subsets of $\mathcal{M}$ coded in $\mathcal{N}$ is $\text{Cod}(\mathcal{N}/\mathcal{M}) = \{X \cap M : X \in \text{Def}(\mathcal{N})\}$. An elementary extension $\mathcal{N} \succ \mathcal{M}$ is conservative iff $\text{Cod}(\mathcal{N}/\mathcal{M}) = \text{Def}(\mathcal{M})$. All conservative extensions are end extensions. Phillips [P74a], introducing the notion of a conservative extension, improved the MacDowell-Specker Theorem by observing that every model of $\text{PA}$ has a conservative extension. This was recently generalized by the following theorem.

Theorem 1: ([Sc14, Theorem 4]) If $\mathcal{M} \models \text{PA}$ and $\mathcal{X} \subseteq \mathcal{P}(\mathcal{M})$, then the following are equivalent:

(a) $\mathcal{X}$ is countably generated, $\text{Def}(\mathcal{M}) \subseteq \mathcal{X}$ and $(\mathcal{M}, \mathcal{X}) \models \text{WKL}^*_0$.

(b) There is a countably generated extension $\mathcal{N} \succ_{\text{end}} \mathcal{M}$ such that $\mathcal{X} = \text{Cod}(\mathcal{N}/\mathcal{M})$.

(c) There is a finitely generated extension $\mathcal{N} \succ_{\text{end}} \mathcal{M}$ such that $\mathcal{X} = \text{Cod}(\mathcal{N}/\mathcal{M})$.

The theory $\text{WKL}^*_0$ of second-order arithmetic was introduced by Simpson & Smith [SS86] and is, roughly speaking, $\text{WKL}_0$ with $\Sigma^0_1\text{-IND}$ replaced by $\Sigma^0_0\text{-IND}$. We say that $\mathcal{X}$ is countably generated if there is a countable $\mathcal{X}_0 \subseteq \mathcal{X}$ such that every set in $\mathcal{X}$ is $\Delta^0_1$-definable in $(\mathcal{M}, \mathcal{X}_0)$.

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It should be noted that an immediate consequence of [SS86, Theorem 4.8] is that \((b)\) implies that \((M, \mathcal{X}) \models \text{WKL}_0^*\).

Improving the MacDowell-Specker Theorem in another direction, Gaifman [Ga70] proved that every model of PA has a minimal elementary end extension. Phillips [P74a] observed that this result could also be improved: every model of PA has a conservative minimal extension. He also proved [P74b] that, on the other hand, the standard model has a nonconservative minimal elementary extension. These results were improved for countable models by the following theorem.

Theorem 2: ([Sc14, Theorem 5]) If \(M \models \text{PA}\) and \(\mathcal{X} \subseteq \mathcal{P}(M)\), then the following are equivalent:

1. \(M\) and \(\mathcal{X}\) are countable, \(\text{Def}(M) \subseteq \mathcal{X}\) and \((M, \mathcal{X}) \models \text{WKL}_0^*\).
2. There is a countable \(N \succ \text{end} M\) such that \(\mathcal{X} = \text{Cod}(N/M)\).
3. There is a superminimal extension \(N \succ \text{end} M\) such that \(\mathcal{X} = \text{Cod}(N/M)\).

Recall that the extension \(N \succ M\) is minimal iff there is no \(K\) such that \(M \prec K \prec N\), and it is superminimal iff there is no \(K\) such that \(M \nprec K \prec N\). Knight [Kn76] proved that every countable model of PA has a superminimal extension, and then this result was extended to conservative superminimal extensions ([KS06, Cor. 2.2.12]).

Every superminimal extension is minimal, and every minimal extension is finitely generated. If we weaken the implications \((c) \implies (a)\) of Theorem 1 by replacing “finitely generated” in \((c)\) by “minimal” and also weaken the implication \((a) \implies (c)\) of Theorem 2 by replacing “superminimal” in \((c)\) by “minimal”, then there is a wide gap between these two weakened implications. Question 6 of [Sc14] asked about the possibility of narrowing this gap. Theorem 3, the principal result of this paper, answers this question by completely characterizing such \((M, \mathcal{X})\).

Theorem 3: If \(M \models \text{PA}\) and \(\mathcal{X} \subseteq \mathcal{P}(M)\), then the following are equivalent:

1. There is a countably generated extension \(N \succ \text{end} M\) such that \(\mathcal{X} = \text{Cod}(N/M)\), and every set that is \(\Pi^0_1\)-definable in \((M, \mathcal{X})\) is the union of countably many \(\Sigma^0_1\)-definable sets.
2. There is a minimal extension \(N \succ \text{end} M\) such that \(\mathcal{X} = \text{Cod}(N/M)\).
The additional condition in (a) holds whenever \( \mathfrak{X} \) is countable. Also, it is implied by \( \text{ACA}_0 \), which is equivalent to: every \( \Pi^0_1 \)-definable set is \( \Sigma^0_1 \)-definable. Thus, the following corollary ensues.

**Corollary 4:** If \( (\mathcal{M}, \mathfrak{X}) \models \text{ACA}_0 \) and there is a countably generated extension \( \mathcal{N} \succ \text{end} \mathcal{M} \) such that \( \text{Cod}(\mathcal{N}/\mathcal{M}) = \mathfrak{X} \), then there is a minimal extension \( \mathcal{N} \succ \text{end} \mathcal{M} \) such that \( \text{Cod}(\mathcal{N}/\mathcal{M}) = \mathfrak{X} \).

\( \Sigma^0_1 \)-IND is equivalent to: every bounded \( \Pi^0_1 \)-definable set is \( \Sigma^0_1 \)-definable. If \( (\mathcal{M}, \mathfrak{X}) \models \text{\text{\text{\Sigma}}}^0_1 \)-IND and \( \mathcal{M} \) has countable cofinality, then every \( \Pi^0_1 \)-definable set is the union of countably many \( \Sigma^0_1 \)-definable sets. Thus, Corollary 4 can be improved when restricted to models of countable cofinality.

**Corollary 5:** If \( (\mathcal{M}, \mathfrak{X}) \models \text{WKL}_0 \), \( \text{cf}(\mathcal{M}) = \aleph_0 \) and there is a countably generated extension \( \mathcal{N} \succ \text{end} \mathcal{M} \) such that \( \text{Cod}(\mathcal{N}/\mathcal{M}) = \mathfrak{X} \), then there is a minimal extension \( \mathcal{N} \succ \text{end} \mathcal{M} \) such that \( \text{Cod}(\mathcal{N}/\mathcal{M}) = \mathfrak{X} \).

There are five numbered sections following this introduction. The purpose of §1 is to repeat some of the definitions from [Sc14] that are needed here. In particular, all definitions needed to understand Theorems 1–3 are given. The proof of a theorem that is stronger than the \((b) \implies (a)\) half of Theorem 3 is proved in §2. Since the proof of the other half of Theorem 3 depends on the proof of Theorem 1 as presented in [Sc14], an outline of that proof is given in §3. Theorem 3 is proved in §4. Finally, in §5, we show with examples that the additional condition in (a) of Theorem 3 is needed.

**§1. Definitions.** Let \( \mathcal{L}_{\text{PA}} = \{+, \times, 0, 1, \leq\} \) be the usual language appropriate for \( \text{PA} \). Models of \( \text{PA} \) will be denoted exclusively by script letters such as \( \mathcal{M}, \mathcal{N} \), etc., and their respective universes by the corresponding roman letters \( M, N \), etc. A subset \( R \subseteq M^n \) is \( \mathcal{M} \)-definable if it is definable in \( \mathcal{M} \) possibly with parameters. Thus, \( \text{Def}(\mathcal{M}) \) is the set of all \( \mathcal{M} \)-definable subsets of \( M \). If \( \mathcal{M} \models \text{PA} \) and \( X \subseteq M \), then \( \mathcal{L}_{\text{PA}}(X) = \mathcal{L}_{\text{PA}} \cup X \).

Suppose that \( \mathcal{M} \preceq \mathcal{N} \). The definitions of when \( \mathcal{N} \) is a conservative, minimal or superminimal extension of \( \mathcal{M} \) is given in the introduction. A subset \( A \subseteq N \) generates \( \mathcal{N} \) over \( \mathcal{M} \) if whenever \( \mathcal{M} \preceq K \preceq \mathcal{N} \) and \( A \subseteq K \), then \( K = \mathcal{N} \). We say that \( \mathcal{N} \) is a countably (respectively, finitely) generated extension of \( \mathcal{M} \) if there is a countable (respectively, finite) set \( A \subseteq N \) that generates \( \mathcal{N} \) over \( \mathcal{M} \).

Suppose \( \mathcal{M} \models \text{PA} \). Each \( s \in M \) codes the \( \mathcal{M} \)-finite sequence \( \langle (s)_i : i < \ell(s) \rangle \), where \( \ell(s) \), the length of \( s \), is the largest \( y \) such that \( 2^y \leq s+1 \).
and whenever \( i < \ell(s) \), then \( (s)_i \) is the \( i \)-th digit in the binary expansion \( s + 1 \). Define \(<\) on \( M \) so that if \( s, t \in M \), then \( s < t \) iff the sequence coded by \( s \) is a proper initial segment of the one coded by \( t \). In this way, we get the full binary \( \mathcal{M} \)-tree \((\mathcal{M}, \leq)\), which we suggestively denote by \( 2^{\leq M} \).

An \( \mathcal{M} \)-tree \( T \) is a subset of \( 2^{\leq M} \) such that whenever \( s < t \in T \), then \( s \in T \). A subset \( B \subseteq T \) is a branch (of \( T \)) if it is an unbounded maximal linearly ordered subset. We will say that the branch \( B \) indicates \( X \) if

\[
X = \{ k \in M : \text{ for some } s \in B, \, \mathcal{M} \models \ell(s) > k \land (s)_k = 0 \}
\]

or, equivalently,

\[
X = \{ k \in M : \text{ for all } s \in B, \, \mathcal{M} \models \ell(s) > k \land (s)_k = 0 \}.
\]

All models of second-order arithmetic encountered here will have the form \((\mathcal{M}, \mathfrak{X})\), where \( \mathcal{M} \models \text{PA} \) and \( \mathfrak{X} \subseteq \mathcal{P}(M) \). Consult Simpson’s book \([\text{Si99}]\) as a reference. Some schemes of second-order arithmetic will be needed. We let \( \text{WKL}^*_0 = \Delta^0_1 \text{-CA}_0 + \Sigma^0_1 \text{-IND} + \text{WKL} \), where \( \text{WKL} \) is Weak König’s Lemma, which asserts that every unbounded \( \mathcal{M} \)-tree in \( \mathfrak{X} \) has a branch in \( \mathfrak{X} \). As usual, \( \text{WKL}_0 = \text{WKL}^*_0 + \Sigma^0_1 \text{-IND} \) and \( \text{ACA}_0 = \text{WKL}_0 + \Sigma^0_1 \text{-CA} \).

If \( \mathfrak{X}_0 \subseteq \mathfrak{X} \subseteq \mathcal{P}(M) \), then we say that \( \mathfrak{X}_0 \) generates \( \mathfrak{X} \) if whenever \( \mathfrak{X}_0 \subseteq \mathfrak{X}_1 \subseteq \mathfrak{X} \), and \((\mathcal{M}, \mathfrak{X}_1) \models \Delta^0_1 \text{-CA} \), then \( \mathfrak{X}_1 = \mathfrak{X} \). We say that \( \mathfrak{X} \subseteq \mathcal{P}(M) \) is countably generated if there is a countable \( \mathfrak{X}_0 \subseteq \mathfrak{X} \) that generates \( \mathfrak{X} \). For example, \( \text{Def}(\mathcal{M}) \) is countably generated since it is generated by the set of all those \( X \in \text{Def}(\mathcal{M}) \) definable without parameters.

§2. The Proof, I. The purpose of this section is to prove Corollary 2.3, which, in the presence of Theorem 1, is the \((b) \Rightarrow (a)\) half of Theorem 3. This is exactly what was proved in an earlier version of this paper, but here we will prove the even stronger Theorem 2.1 in response to a question raised by Roman Kossak.

Suppose that \( \mathcal{M} \prec \mathcal{N} \). Recall that \( \text{Lt}(\mathcal{N}/\mathcal{M}) \) is the lattice of all \( \mathcal{K} \) such that \( \mathcal{M} \trianglelefteq \mathcal{K} \trianglelefteq \mathcal{N} \). We say that \( \mathcal{D} \) is dense (for this extension) if \( \mathcal{D} \subseteq \text{Lt}(\mathcal{N}/\mathcal{M})\setminus\{\mathcal{M}\} \) such that whenever \( \mathcal{M} \prec \mathcal{K}_0 \trianglelefteq \mathcal{N} \), then there is \( \mathcal{K}_1 \in \mathcal{D} \) such that \( \mathcal{K}_1 \trianglelefteq \mathcal{K}_0 \). Let \( \text{den}(\mathcal{N}/\mathcal{M}) \) be the least cardinal \( \delta \) such that \( \delta = |\mathcal{D}| \) for some dense \( \mathcal{D} \).

**Theorem 2.1**: Suppose that \( \mathcal{M} \prec_{\text{end}} \mathcal{N} \) is countably generated and that \( \delta = \text{den}(\mathcal{N}/\mathcal{M}) \). Then every set that is \( \Pi^0_1 \)-definable in \((\mathcal{M}, \mathfrak{X})\) is the union of \( \delta + \aleph_0 \) sets each of which is \( \Sigma^0_1 \)-definable in \((\mathcal{M}, \mathfrak{X})\).
Proof. Since $\mathcal{N}$ is a countably generated extension of $\mathcal{M}$, there is $\mathcal{N}' \succ_{\text{end}} \mathcal{N}$ and $\mathcal{N}'$ is a minimal elementary extension of $\mathcal{M}$. (The proof of this is a straightforward modification of the proof of [KS06 Theorem 2.1.12] concerning superminimal extensions.) Then $\text{Cod}(\mathcal{N}'/\mathcal{M}) = \text{Cod}(\mathcal{N}/\mathcal{M})$ and $\text{den}(\mathcal{N}'/\mathcal{M}) = \text{den}(\mathcal{N}/\mathcal{M})$. Thus, by replacing $\mathcal{N}'$ by $\mathcal{N}$, if necessary, we can assume that $\mathcal{N}$ is a finitely generated extension of $\mathcal{M}$. Let $c$ generate $\mathcal{N}$ over $\mathcal{M}$.

Let $\mathbf{D}$ be a dense set such that $|\mathbf{D}| = \delta$. For every $\mathcal{K}_0$ such that $\mathcal{M} \prec \mathcal{K}_0 \preceq \mathcal{N}$, there is $\mathcal{K}_1$ that is a finitely generated extension of $\mathcal{M}$ such that $\mathcal{M} \prec \mathcal{K}_1 \preceq \mathcal{K}_0$. Thus, we can safely assume that $\mathbf{D}$ consists only of finitely generated extensions of $\mathcal{M}$. Let $\{c_\alpha : \alpha < \delta\} \subseteq \mathcal{N}\setminus\mathcal{M}$ be such that each $\mathcal{K} \in \mathbf{D}$ is generated over $\mathcal{M}$ by some $c_\alpha$.

For each $\alpha < \delta$, let $g_\alpha : M \rightarrow M$ be an $\mathcal{M}$-definable function such that $g_\alpha(c) = c_\alpha$.

If $f, g : M \rightarrow M$ are $\mathcal{M}$-definable functions and $A \in \text{Def}(\mathcal{M})$, then we say that $f$ refines $g$ on $A$ if whenever $x, y \in A$ and $f(x) = f(y)$, then $g(x) = g(y)$. The following claim is the purpose for introducing this definition.

Claim: Suppose that $f : M \rightarrow M$ is an $\mathcal{M}$-definable function. Then, $f^N(c) > M$ iff there are $A \in \text{Def}(\mathcal{M})$ and $\alpha < \delta$ such that $c \in A^N$ and $f$ refines $g_\alpha$ on $A$.

We prove the Claim. Let $a = f^N(c)$.

First, suppose that $a > M$. Let $\alpha < \delta$ be such that $c_\alpha$ is in the elementary extension of $\mathcal{M}$ generated by $a$. Thus, there is an $\mathcal{M}$-definable function $h : M \rightarrow M$ such that $h^N(a) = c_\alpha$. Let $A \in \text{Def}(\mathcal{M})$ be the set of those $x \in M$ such that $hA(x) = g_\alpha(x)$. Then $f$ refines $g_\alpha$ on $A$, and since $hAf^N(c) = c_\alpha = g_\alpha^N(c)$, then $c \in A^N$.

Next, suppose that $A \in \text{Def}(\mathcal{M})$ and $\alpha < \delta$ are such that $c \in A^N$ and $f$ refines $g_\alpha$ on $A$. Define $h : f[A] \rightarrow g_\alpha[A]$ so that if $x \in A$, then $hf(x) = g_\alpha(x)$. This defines $h$ uniquely, and $h$ is $\mathcal{M}$-definable. Then, $hAf^N(c) = g_\alpha^N(c)$ so that $hA(a) = c_\alpha$. Since $c_\alpha > M$, then $f^N(c) = a > M$.

This completes the proof of the Claim.

Now let $D \subseteq M$ be $\Pi^0_1$-definable in $(\mathcal{M}, \mathcal{X})$. We will prove that there are $\Sigma^0_1$-definable sets $D_{\alpha,j}$ ($\alpha < \delta$, $j < \omega$) whose union is $D$. Since $D$ is $\Pi^0_1$-definable, we let $A \in \mathcal{X}$ and the $\mathcal{L}_{\text{PA}}$-formula $\theta(x, y)$ be such that

$$D = \{d \in M : (\mathcal{M}, \mathcal{X}) \models \forall y \in A \theta(d, y)\}.$$ 

To see this is possible, first observe that there is $B \in \mathcal{X}$ such that $D = \{d \in M : (\mathcal{M}, \mathcal{X}) \models \forall z (d, z) \in B\}$, and then let $A = M\setminus B$ and let $\theta(x, y)$ be $\forall z (y \neq (x, z))$. Let $S$ be the branch of $2^{<M}$ that indicates
A. Then $S \in \mathfrak{X} = \text{Cod}(\mathcal{N}/\mathcal{M})$, and $D$ is $\Pi_1^0$-definable from $S$ by the formula
\[
\forall s \in S \forall k < \ell(s) [(s)_k = 0 \rightarrow \theta(x, (s)_k)].
\]
Since $S \in \text{Cod}(\mathcal{N}/\mathcal{M})$, we assume that $c \in N \setminus M$ is such that
\[
S = \{s \in M : N \models s < c \}.
\]
For each $d \in M$, define the function $f_d : 2^{<M} \rightarrow M$ so that
\[
f_d(s) = \mu k < \ell(s) [ (s)_k = 0 \land \theta(d, k)].
\]
Clearly, each $f_d$ is $\mathcal{M}$-definable; in fact $\langle f_d : d \in M \rangle$ is an $\mathcal{M}$-definable family of functions. It is easily seen that for each $d \in M$,
\[(2.1.1) \quad d \in D \iff f_d^N(c) > M.
\]
From the Claim, we have that $d \in D$ iff there are $A \in \text{Def}(\mathcal{M})$ and $\alpha < \delta$ such that $c \in A^\mathcal{N}$ and $f_d$ refines $g_\alpha$ on $A$.

Let $\theta_0(x, y), \theta_1(x, y), \theta_2(x, y), \ldots$ be an enumeration of all the 2-ary $L_{PA}$-formulas. If $j < \omega$ and $b \in M$, let $A_j(b) \in \text{Def}(\mathcal{M})$ be the set defined by $\theta_j(x,b)$. Then let
\[
B_j = \{ b \in M : c \in A_j^\mathcal{N}(b) \},
\]
so that $B_j \in \text{Cod}(\mathcal{N}/\mathcal{M}) = \mathfrak{X}$. For $\alpha < \delta$ and $j < \omega$, let
\[
D_{\alpha,j} = \{ d \in M : \text{there is } b \in B_j \text{ such that } f_d \text{ refines } g_\alpha \text{ on } A_j(b) \}.
\]
Thus, each $D_{\alpha,j}$ is $\Sigma_1^0$-definable in $(\mathcal{M}, \mathfrak{X})$. There are at most $\delta + \aleph_0$ such $D_{\alpha,j}$.

To finish the proof, we will show that $D = \bigcup \{ D_{\alpha,j} : \alpha < \delta, \ j < \omega \}$.

$D \subseteq \bigcup \{ D_{\alpha,j} : \alpha < \delta, \ j < \omega \}$: Let $d \in D$. Then $f_d^N(c) > M$ by (2.1.1). By the Claim, we let $\alpha < \delta$ and $A \in \text{Def}(\mathcal{M})$ be such that $c \in A^\mathcal{N}$ and $f_d$ refines $g_\alpha$ on $A$. Let $j < \omega$ and $b \in B_j$ be such that $A = A_j(b)$. Then, $d \in D_{\alpha,j}$.

$\bigcup \{ D_{\alpha,j} : \alpha < \delta, \ j < \omega \} \subseteq D$: Suppose that $\alpha < \delta$, $j < \omega$ and $d \in D_{\alpha,j}$. Then there is $b \in B_j$ such that $f_d$ refines $g_i$ on $A_j(b)$. Since $A_j(b) \in \text{Def}(\mathcal{M})$ and $c \in A_j(b)^\mathcal{N}$, then $d \in D$, thereby completing the proof of the theorem. \hfill \Box

**Corollary 2.2**: Suppose that $\mathcal{M} \prec_{\text{end}} \mathcal{N}$ and $\text{Lt}(\mathcal{N}/\mathcal{M})$ is countable. Then every set that is $\Pi_1^0$-definable in $(\mathcal{M}, \mathfrak{X})$ is the union of countably many sets that are $\Sigma_1^0$-definable in $(\mathcal{M}, \mathfrak{X})$. \hfill \Box

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1This definition uses the minimalization operator $\mu$. Thus, if there is $k < \ell(s)$ such that $\mathcal{M} \models (s)_k = 0 \land \neg \theta(d,k)$, then $f_d(s)$ is the least such $k$; otherwise, $f_d(k) = \ell(s)$. 

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The special case of the previous corollary when $|\text{Lt}(N/M)| = 2$ should be mentioned.

**Corollary 2.3:** Suppose that $M \prec \text{end} N$ and $M$ is a minimal extension of $N$. Then every set that is $\Pi^0_1$-definable in $(M, \mathcal{X})$ is the union of countably many sets that are $\Sigma^0_1$-definable in $(M, \mathcal{X})$.

§3. Outlines. In this section we give an outline of the proof of the implication $(a) \implies (c)$ of Theorem 1, and also of the implication that $(a)$ of Theorem 2 implies the weakening of $(c)$ in Theorem 2 obtained by replacing “superminimal” with “minimal”. We first consider Theorem 1.

Suppose that $(M, \mathcal{X})$ satisfies $(a)$ of Theorem 1; specifically, $\mathcal{X}$ is countably generated, $\text{Def}(M) \subseteq \mathcal{X}$ and $(M, \mathcal{X}) \models \text{WKL}^*_0$. We describe how $N$ as in $(c)$ was obtained in [Sc14]. A set $\Phi(x)$ of 1-ary $\mathcal{L}_{PA}$-formulas is allowable if, for some $n < \omega$, there are $\mathcal{L}_{PA}$-formulas $\theta_0(x, y), \theta_1(x, y), \ldots, \theta_{n-1}(x, y)$ and $A_0, A_1, \ldots, A_{n-1} \in \mathcal{X}$ such that

$$\Phi(x) = \bigcup_{j < n} \{ \theta_j(x, b) : b \in A_j \}$$

and whenever $\Psi(x) \subseteq \Phi(x)$ is $M$-finite, then $\bigwedge \Psi(x)$ defines an unbounded subset of $M$. Since $\emptyset \in \mathcal{X}$, the set $\emptyset$ is allowable.

Our goal is to obtain an increasing sequence

$$\Phi_0(x) \subseteq \Phi_1(x) \subseteq \Phi_2(x) \subseteq \cdots$$

of allowable sets that has the following two properties:

(A1) ([Sc14] (4.1) and (5.1)]) Whenever $\theta(x, y)$ is an $\mathcal{L}_{PA}$-formula, then there are $m < \omega$ and $B \in \mathcal{X}$ such that

$$\{ \theta(x, b) : b \in B \} \cup \{ \neg \theta(x, b) : b \in M \setminus B \} \subseteq \Phi_m(x).$$

(A2) ([Sc14] (4.2) and (5.2)]) Whenever $B \in \mathcal{X}$, then there are $m < \omega$ and an $\mathcal{L}_{PA}$-formula $\theta(x, y)$ such that

$$\{ \theta(x, b) : b \in B \} \cup \{ \neg \theta(x, b) : b \in M \setminus B \} \subseteq \Phi_m(x).$$

If we have such a sequence, then $\bigcup_{m < \omega} \Phi_m(x)$ generates a complete type over $M$, thanks to (A1). Let $N$ be an extension of $M$ generated by an element realizing this complete type. Then $N \succ \text{end} M$. It follows from (A1) that $\text{Cod}(N/M) \subseteq \mathcal{X}$ and from (A2) that $\mathcal{X} \subseteq \text{Cod}(N/M)$.

The sequence is constructed by recursion, starting with $\Phi_0(x) = \emptyset$. Associated with (A1) is a lemma whose repeated application (once for each $\theta(x, y)$) assures that the constructed sequence of allowable sets satisfies (A1). There is also such a lemma associated with (A2), which,
when applied repeatedly (once for each $B$ in some countable set that generates $\mathcal{X}$), assures that (A2) is satisfied. In each of these lemmas, we start with an allowable set $\Phi(x)$ which we then enlarge it to an allowable set $\Phi(x) \cup \{\theta(x, b) : b \in B\} \cup \{-\theta(x, b) : b \in M \setminus B\}$.

The lemma for (A1) is the following.

**Lemma 3.1:** Suppose that $\Phi(x)$ is allowable and that $\theta(x, y)$ is an $\mathcal{L}_{PA}$-formula. Then there is $B \in \mathcal{X}$ such that

$$\Phi(x) \cup \{\theta(x, b) : b \in B\} \cup \{-\theta(x, b) : b \in M \setminus B\}$$

is allowable.

The lemma for (A2) is the following.

**Lemma 3.2:** Suppose that $\Phi(x)$ is allowable and $B \in \mathcal{X}$. Then there is an $\mathcal{L}_{PA}$-formula $\theta(x, y)$ such that

$$\Phi(x) \cup \{\theta(x, b) : b \in B\} \cup \{-\theta(x, b) : b \in M \setminus B\}$$

is allowable.

With Lemmas 3.1 and 3.2, we are able to prove $(a) \implies (c)$ of Theorem 1 as in [Sc14].

Next, we consider Theorem 2. In order to get $\mathcal{N} \succ_{\text{end}} \mathcal{M}$ to be minimal, we require that the sequence of allowable sets satisfies the following additional condition.

(A3) ([Sc14 (5.3)]) Whenever $f : M \rightarrow M$ is an $\mathcal{M}$-definable function, then there are $m < \omega$ and $\varphi(x) \in \Phi_m(x)$ such that $f$ is either bounded or one-to-one on the set defined by $\varphi(x)$.

It is easily seen (or consult the proof of [Sc14 Theorem 5]) that (A3) suffices to guarantee that $\mathcal{N}$ is a minimal extension of $\mathcal{M}$.

The next lemma is used to obtain (A3). By repeated applications of this lemma (once for each $\mathcal{M}$-definable $f : M \rightarrow M$), we can get the sequence of allowable sets to satisfy (A3) as long as $\mathcal{M}$ is countable. This will prove the weakening of $(a) \implies (c)$ of Theorem 2, with “minimal” replacing “superminimal”.

**Lemma 3.3:** Suppose that $\Phi(x)$ is allowable and $f : M \rightarrow M$ is $\mathcal{M}$-definable. Then there is a formula $\varphi(x)$ such that $\Phi(x) \cup \{\varphi(x)\}$ is allowable and $f$ is either bounded or one-to-one on the set defined by $\varphi(x)$. 
We remark that the proofs of Lemmas 3.1, 3.2 and 3.3 do not require that $\mathcal{X}$ be countably generated. Furthermore, the proof of Lemma 3.3 does not require that $\mathcal{M}$ be countable.

§4. The Proof, II. This section is devoted to completing the proof Theorem 3, the (b) $\implies$ (a) half following from Corollary 2.3.

Let $\mathcal{M} \models \text{PA}$ and $\mathcal{X} \subseteq \mathcal{P}(\mathcal{M})$. Suppose that (a) of Theorem 3 holds. By Theorem 1, we have that $\mathcal{X}$ is countably generated, Def($\mathcal{M}$) $\subseteq$ $\mathcal{X}$ and $(\mathcal{M}, \mathcal{X}) \models \text{WKL}^*_0$. In addition, every $\Pi^0_1$-definable set is the countable union $\Sigma^0_1$-definable sets. We will construct an increasing sequence $\Phi_0(x) \subseteq \Phi_1(x) \subseteq \Phi_2(x) \subseteq \cdots$ of allowable sets satisfying the three conditions (A1), (A2) and (A3) from §3. The challenge will be to satisfy (A3).

Following are two corollaries of Lemma 3.1.

**Corollary 4.1:** Suppose that $\Phi(x)$ is allowable and that $\varphi(x,y)$ is an $\mathcal{L}_{\text{PA}}$-formula. Then there are $B \in \mathcal{X}$, an $\mathcal{L}_{\text{PA}}$-formula $\theta(x,y)$ such that
\[
\Phi(x) \cup \{\theta(x,b) : b \in B\}
\]
is allowable and for each $k \in M$ there is $b \in B$ such that
\[
\mathcal{M} \models \forall y < k [ (\forall x (\theta(x,b) \implies \varphi(x,y)) \lor (\forall x (\theta(x,b) \implies \neg \varphi(x,y))].
\]

**Proof:** Apply Lemma 3.1 to get $A \in \mathcal{X}$ such that
\[
\Phi(x) \cup \{\varphi(x,a) : a \in A\} \cup \{\neg \varphi(x,a) : a \in M \setminus A\}
\]
is allowable. Let $B$ be the branch of $2^{<\mathcal{M}}$ that indicates $A$. Let $\varphi(x,y)$ be the formula
\[
\forall k < \ell(y) [(y)_k = 0 \iff \varphi(x,k)],
\]
and then let
\[
\Phi'(x) = \Phi(x) \cup \{\theta(x,b) : b \in B\}.
\]
It is clear that $\Phi'(x)$ is allowable. Moreover, for each $k \in M$, if we let $b \in B$ be such that $\ell(b) = k$, then $b$ is as required. □

**Corollary 4.2:** Suppose that $\Phi(x)$ is an allowable set and $t(u,x)$ is a Skolem $\mathcal{L}_{\text{PA}}$-term. Then there are $B \in \mathcal{X}$ and an $\mathcal{L}_{\text{PA}}$-formula $\theta(x,y)$ such that $\Phi(x) \cup \{\theta(x,b) \in B\}$ is allowable and for each $k \in M$, there is $b \in B$ such that for $i, j \leq k$, either
\[
(4.2.1) \quad \mathcal{M} \models \forall x [\theta(x,b) \implies t(i,x) \leq t(j,x)]
\]
or
\[
(4.2.2) \quad \mathcal{M} \models \forall x [\theta(x,b) \implies t(i,x) > t(j,x)].
\]
Proof. Let $\varphi(x, y)$ be the formula $\exists i, j [y = \langle i, j \rangle \land t(i, x) \leq t(j, x)]$. Apply Corollary 4.1. □

The next lemma is a generalization of Lemma 3.3.

**Lemma 4.3:** Suppose that $\Phi(x)$ is allowable, $t(u, x)$ is a Skolem $\mathcal{L}_{PA}$-term and $D \subseteq M$ is $\Sigma_1^0$-definable in $(\mathcal{M}, \mathfrak{X})$. Suppose further that whenever $i \in D$ and $a \in M$, then there is $\varphi(x) \in \Phi(x)$ such that

$$\mathcal{M} \models \forall x [\varphi(x) \rightarrow t(i, x) > a].$$

Then there is an allowable $\Phi'(x) \supseteq \Phi(x)$ such that whenever $i \in D$, there is $\varphi(x) \in \Phi'(x)$ such that

$$\mathcal{M} \models \forall x, y [\varphi(x) \land \varphi(y) \land x < y \rightarrow t(i, x) < t(i, y)].$$

**Proof.** Recall a definition from [Sc14]. An allowable set $\Psi(x)$ is tree-based if there is an $\mathcal{L}_{PA}$-formula $\psi(x, y)$ and a branch $B \in \mathfrak{X}$ of the full binary tree $(\mathcal{M}, \triangleleft)$ such that $\Psi(x) = \{ \theta(x, b) : b \in B \}$ and

$$\mathcal{M} \models \forall s, t [s \triangleleft t \rightarrow \forall x (\theta(x, t) \rightarrow \theta(x, s))].$$

Following [Sc14, p. 576], we suppose, without loss of generality, that $\Phi(x)$ is tree-based. Let $\Phi(x) = \{ \theta(x, s) : s \in B \}$, where $B$ is a branch. Let $T$ be the set of those $s \in 2^{<M}$ such that $\mathcal{M} \models \forall w \exists x [x > w \land \theta(x, s)]$. Then $T$ is $\mathcal{M}$-definable without parameters. Also, $T$ is an $\mathcal{M}$-tree and $B$ is a branch of $T$.

For $i \in M$, let $f_i : M \rightarrow M$ be the function defined by $t(i, x)$. (It will be easier, notationally, to work with $f_i$ rather than $t(i, x).$)

By Corollary 4.2, we can assume that whenever $k \in M$, then there is $s \in B$ such that whenever $i, j \leq k$, then either (4.2.1) or (4.2.2). Let $\gamma \subseteq M^2$ be such that whenever $i, j \in M$, then $\langle i, j \rangle \in \gamma$ iff

$$\mathcal{M} \models \exists s \in B \forall x [\theta(x, s) \rightarrow f_i(x) \leq f_j(x)].$$

Equivalently, $\langle i, j \rangle \in \gamma$ iff

$$\mathcal{M} \models \forall s \in B \exists x [\theta(x, s) \land f_i(x) \leq f_j(x)].$$

Hence, $\gamma$ is $\Delta^0_1$-definable from $B$ and a set in $\text{Def}(\mathcal{M})$, so that $\gamma \in \mathfrak{X}$.

It follows from (4.3.1) that $\gamma$ is transitive, and from (4.3.2) that $\gamma$ is connected. Being both a transitive and connected binary relation on $M$, $\gamma$ is, by definition, a linear quasi-order of $M$. Notice that there
is the possibility that \( \langle i, j \rangle, \langle j, i \rangle \in \gamma \) for distinct \( i, j \in M \), although \( \langle i, i \rangle \in \gamma \) for all \( i \in M \).

Since \( D \) is \( \Sigma_1^0 \)-definable in \((\mathcal{M}, \mathfrak{X})\), we let \( A \in \mathfrak{X} \) and the \( \mathcal{L}_{PA} \)-formula \( \psi(x, y) \) be such that the formula \( \exists y \in A \psi(x, y) \) defines \( D \) in \((\mathcal{M}, \mathfrak{X})\).

Let \( P = T \otimes 2^{<M} \) be the set of all pairs \( \langle s, t \rangle \), where \( s \in T, t \in 2^{<M} \) and \( \ell(s) = \ell(t) \). We consider \( P \) to be ordered componentwise by \( \leq_P \); that is, if \( p = \langle s_0, t_0 \rangle \) and \( q = \langle s_1, t_1 \rangle \) are in \( P \), then \( p \leq_P q \) iff \( s_0 \leq s_1 \) and \( t_0 \leq t_1 \). Then \((P, \leq_P)\) is (isomorphic to) an \( \mathcal{M} \)-tree that is definable in \( \mathcal{M} \) without parameters. Let

\[
C = \{ \langle s, t \rangle \in P : s \in B \text{ and } \mathcal{M} \models \forall i < \ell(t)[(t)_i = 0 \leftrightarrow i \in A] \}.
\]

Then, \( C \in \mathfrak{X} \) and \( C \) is a branch of \( P \).

**Convention:** Even though \( P \subseteq M^2 \), by the usual coding of pairs, we also assume that \( P \subseteq M \). We then have that \( < \) linearly orders \( P \). We assume the coding is done so that whenever \( q <_P p \in P \), then \( q < p \).

We will define \( P_0 \subseteq P \) and a function \( g : P_0 \rightarrow M \) by recursion in \( \mathcal{M} \). Suppose that \( p \in P \) and that, for all \( q < p \), we have already decided whether or not \( q \) is in \( P_0 \) and, if it is, what \( g(q) \) is. Then, \( p \in P_0 \) iff there is \( u \in M \) such that both:

\begin{align*}
(4.3.3) \quad & \mathcal{M} \models \theta(u, s) \land \forall q [(q \in P_0 \land q < p) \rightarrow u > g(q)]; \\
(4.3.4) \quad & \mathcal{M} \models \forall i, q \left[ (q \in P_0 \land q < p \land i < \ell(s) \land \exists k < \ell(t)[(t)_k = 0 \land \psi(i, (t)_k)] \right] \rightarrow f_i(u) > f_i(g(q)).
\end{align*}

If \( p \in P_0 \), let \( g(p) \) be the least \( u \) such that \((4.3.3)\) and \((4.3.4)\) hold.

Both \( P_0 \) and \( g \) are \( \mathcal{M} \)-definable without parameters.

If \( q <_P p \in P_0 \), then \( q \in P_0 \). For if \( u \in M \) is a witness demonstrating that \( p \in P_0 \) (that is, \( u \) satisfies both \((4.3.3)\) and \((4.3.4)\)), then it already had demonstrated that \( q \in P_0 \). Thus, we have that \( P_0 \) is a subtree of \( P \).

We claim that \( C \subseteq P_0 \). Consider \( p = \langle s, t \rangle \in C \), intending to show that \( p \in P_0 \). Thus, we want \( u \in M \) that satisfies \((4.3.3)\) and \((4.3.4)\). Since \( s \in B \), then \( \theta(x, s) \in \Phi(x) \) and, therefore, \( \theta(x, s) \) defines an unbounded subset of \( M \). Hence, there is an unbounded set of \( u \)'s satisfying \((4.3.3)\).

For \((4.3.4)\), the \( f_i \)'s that need to be considered are those for which

\[
\mathcal{M} \models i < \ell(s) \land \exists k [k < \ell(t) \land \psi(i, (t)_k)].
\]

Let \( I \) be the set of such \( i \). Clearly, \( I \subseteq D \). Let \( j \in I \) be a \( \gamma \)-minimal element of \( I \); that is, \( \langle j, i \rangle \in \gamma \) for all \( i \in I \). (Since \( I, \gamma \in \mathfrak{X} \) and \( I \) is bounded, then such a \( j \) does exist.)

Let

\[
a = \max\{f_i(g(q)) : p > q \in P_0, i < \ell(s)\}.
\]
Let $s' \in B$ be large enough so that $s' \succ s$,

(4.3.5) \[ \mathcal{M} \models \forall x [\theta(x, s') \rightarrow f_j(x) > a] \]

and

(4.3.6) \[ \mathcal{M} \models \forall i, j \in I \forall x [\theta(x, s') \rightarrow (f_i(x) \leq f_j(x) \leftrightarrow \langle i, j \rangle \in \gamma)] \]

For (4.3.5), $s'$ exists by the hypothesis of the lemma since $I \subseteq D$. For (4.3.6), invoke Corollary 4.1.

Then, for any $i \in I$, we have that

$$f_i(u) \geq f_j(u) > a \geq f_i(g(q))$$

whenever $u$ and $q$ are such that $\mathcal{M} \models \theta(u, s')$ and $p > q \in P_0$. Therefore, $p \in P_0$.

Let $\theta'(x, p)$ be the $L_{PA}$-formula

$$\exists q (p \leq_P q \in P_0 \land g(q) = x)$$

and let

$$\Phi'(x) = \Phi(x) \cup \{\theta'(x, p) : p \in C\}.$$ 

It is seen that $\Phi'(x)$ is allowable. We claim that $\Phi'(x)$ has the property required by the lemma.

To see that it does, consider $i \in D$. Let $p_0 = \langle s_0, t_0 \rangle \in C$ be large such that $\ell(s_0) = i + 1$. Let $\varphi(x) = \theta'(x, p_0) \in \Phi'(x)$, and let $x, y \in M$ be such that $\mathcal{M} \models \varphi(x) \land \varphi(y) \land x < y$. Then there are $q, r \geq_P p_0$ such that $q, r \in P_0$ and $g(q) = x$ and $g(r) = y$. Then (4.3.3) implies that $q < r$. Let $r = \langle s, t \rangle$. Since $t = t_0|(i+1)$, then $i$ satisfies the left side of the implication in (4.3.4), and therefore, $f_i(y) > f_i(z)$. This completes the proof of the lemma. \qed

We remark that the proofs of Corollary 4.1 and 4.2 and Lemma 4.3 do not require that $X$ be countably generated nor that every $\Pi^0_1$-definable set is the union of countably many $\Sigma^0_1$-definable sets.

We now construct the sequence $\Phi_0(x) \subseteq \Phi_1(x) \subseteq \Phi_2(x) \subseteq \cdots$. By repeated applications of Lemma 3.1 (once for each $\varphi(x, y)$), we will get (A1) to hold. By repeated applications of Lemma 3.2 (once for each $B$ in some countable set of generators for $X$), we will get (A2) to hold. We now describe how we get (A3) to hold.

Consider a Skolem $L_{PA}$-term $t(u, x)$. Let $\varphi(x, y)$ be the formula $\forall u, v [y = \langle u, v \rangle \land t(u, x) \leq v]$. At some point in the construction, we will have applied Lemma 3.1 to this formula, obtaining $\Phi_m(x)$. Therefore, there are $B \in X$ and an $L_{PA}$-formula $\theta(x, y)$ such that
\{\theta(x, b) : b \in B\} \subseteq \Phi_m(x) \text{ and for every } i, a \in M, \text{ there is } b \in B \text{ such either}

(4.1) \quad \mathcal{M} \models \forall x[\theta(x, b) \rightarrow t(x, i) \leq a] 

or

(4.2) \quad \mathcal{M} \models \forall x[\theta(x, b) \rightarrow t(x, i) > a].

Let \( D \) be the set of all those \( i \in M \) for which (4.1) holds for every \( a \in M \). Thus, the set \( D \) is \( \Pi^0_1 \) in \( B \) and some set in \( \text{Def}(\mathcal{M}) \). By the condition on \((M, X)\), there are countably many \( \Sigma^0_1 \)-definable sets \( D_0, D_1, D_2, \ldots \) whose union is \( D \). At any future stage of the construction, where we have \( \Phi_n(x) \), we will have, for any \( j < \omega \), that the hypothesis of Corollary 4.1 holds (with \( \Phi(x) = \Phi_n(x) \) and \( D = D_j \)). Applying Lemma 4.3, we get \( \Phi_{n+1}(x) = \Phi'(x) \) so that whenever \( i \in D_j \), there is \( \varphi(x) \in \Phi_{n+1}(x) \) such that

\[ \mathcal{M} \models \forall x,y[\varphi(x) \land \varphi(y) \land x < y \rightarrow t(i, x) < t(i, y)] \]

Thus by repeated applications of Lemma 4.3 (once for each \( D_j \)), we will get that for every \( i \in M \), there are \( n < \omega \) and \( \varphi(x) \in \Phi_n(x) \) such that either

\[ \mathcal{M} \models \exists w \forall x[\varphi(x) \rightarrow t(i, x) \leq w] \]

or

\[ \mathcal{M} \models \forall x,y[\varphi(x) \land \varphi(y) \land x < y \rightarrow t(i, x) < t(i, y)] \]

Since there are only countably many such \( t(u, x) \) we can do the construction by dovetailing the steps necessary for each possible \( t(u, x) \).

Thus, we get the sequence \( \Phi_0(x) \subseteq \Phi_1(x) \subseteq \Phi_2(x) \subseteq \cdots \) to satisfy (A3), thereby completing the proof of the \((a) \Rightarrow (b)\) half of Theorem 3.

With Theorem 1 and Corollary 2.3, the proof of Theorem 3 is complete.

\section*{§5. Appendix.} It may not be apparent that the added condition in (a) of Theorem 3 is actually needed. In other words, perhaps whenever \( \mathcal{M}, \mathcal{N} \) are such that \( \mathcal{N} >_{\text{end}} \mathcal{M} \) is a finitely generated extension, then \( \mathcal{M} \) has a minimal extension \( \mathcal{N}_0 >_{\text{end}} \mathcal{M} \) such that \( \text{Cod}(\mathcal{N}_0/\mathcal{M}) = \text{Cod}(\mathcal{N}/\mathcal{M}) \). The next theorem shows that that is not so.

\textbf{Theorem 5.1:} For every \( \mathcal{M}_0 \models \text{PA} \), there is \((\mathcal{M}, \mathfrak{X}) \models \text{WKL}_0\) such that \( \mathcal{M} \equiv \mathcal{M}_0 \), \( \mathfrak{X} \) is countably generated, \( \mathfrak{X} \supseteq \text{Def}(\mathcal{M}) \) and there is a \( \Pi^0_1 \)-definable set that is not the union of countably many \( \Sigma^0_1 \)-definable sets.

\textbf{Proof.} Suppose that \( \mathcal{M}_0 \models \text{PA} \). Viewing \((\mathcal{M}_0, \text{Def}(\mathcal{M}_0))\) as a 2-sorted first-order structure, we let \((\mathcal{M}, \mathfrak{X}_0) \equiv (\mathcal{M}_0, \text{Def}(\mathcal{M}_0))\) be
$\aleph_1$-saturated. By $\aleph_1$-saturation (although only recursive saturation is needed), there is $X \in \mathcal{X}_0$ such that $(\mathcal{M}, \mathcal{X}_0) \models (X)_n = \varnothing^{(n)}$ for every $n < \omega$. (That is, $(X)_n$ is the complete $\Sigma^0_n$-definable subset of $\mathcal{M}$.) Obtain by recursion, using the Low Basis Theorem, a sequence $X_0, X_1, X_2, \ldots$ such that $X_0 = X$ and, for each $n < \omega$, $X_{n+1} \in \mathcal{X}_0$ is low relative to $X_n$ (i.e., that $(\mathcal{M}, \mathcal{X}_0) \models (X_{n+1})' \equiv_T (X_n)'$) and is such that for any unbounded $\mathcal{M}_0$-tree $S$ that is $\Delta^0_1$-definable from $X_0 \oplus X_1 \oplus \cdots \oplus X_n$, there is $k \in M$ is such that $(X)_k$ is a branch of $S$. Let $\mathfrak{X}$ be the set of those subsets of $M$ that are $\Delta^0_1$-definable in $\text{Def}((\mathcal{M}, X_0, X_1, X_2, \ldots))$. We easily see that $(\mathcal{M}, \mathfrak{X})$ is as required.

Obviously, $\mathcal{M} \equiv \mathcal{M}_0$. The countable set $\{X_0, X_1, X_2, \ldots\}$ generates $\mathfrak{X}$. Since each $X_{n+1}$ is low relative to $X_n$, then each $X_n$ is low relative to $X$, so that $X' \not\in \mathfrak{X}$. Thus, $M \setminus X'$ is a $\Pi^0_1$-definable set that is not $\Sigma^0_1$-definable, nor is it the union of finitely many $\Sigma^0_1$-definable sets. By $\aleph_1$-saturation, it is not the union of countably many $\Sigma^0_1$-definable sets.

Remark: In the proof of the previous theorem, if $(\mathcal{M}, \mathcal{X}_0)$ happened to be $\kappa^+$-saturated, where $\kappa \geq \aleph_0$, then $M \setminus X'$ would be a $\Pi^0_1$-definable set that is not the union of $\kappa$ $\Sigma^0_1$-definable sets. □

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