Linear stability estimates for Serrin’s problem via a modified implicit function theorem

Alexandra Gilsbach\textsuperscript{1,2} · Michiaki Onodera\textsuperscript{1}  

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Abstract
We examine Serrin’s classical overdetermined problem under a perturbation of the Neumann boundary condition. The solution of the problem for a constant Neumann boundary condition exists provided that the underlying domain is a ball. The question arises whether for a perturbation of the constant there still are domains admitting solutions to the problem. Furthermore, one may ask whether a domain that admits a solution for the perturbed problem is unique up to translation and whether it is close to the ball. We develop a new implicit function theorem for a pair of Banach triplets that is applicable to nonlinear problems with loss of derivatives except at the point under consideration. Combined with a detailed analysis of the linearized operator, we prove the existence and local uniqueness of a domain admitting a solution to the perturbed overdetermined problem. Moreover, an optimal linear stability estimate for the shape of a domain is established.

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\textsuperscript{1} Michiaki Onodera  
onodera@math.titech.ac.jp

\textsuperscript{1} Department of Mathematics, School of Science, Tokyo Institute of Technology, 2-12-1 Ookayama, Meguro-ku 152-8551, Tokyo, Japan

\textsuperscript{2} Present Address: Aachen, Germany
1 Introduction

We study the shape of a bounded domain $\Omega \subset \mathbb{R}^n, n \geq 2$, in which a solution $u$ to the Dirichlet problem

$$\begin{align*}
-\Delta u &= 1 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \Gamma = \partial \Omega
\end{align*} \quad (1.1a)$$

satisfies the overdetermined boundary condition

$$-rac{\partial u}{\partial v} = f \quad \text{on } \Gamma, \quad (1.1b)$$

where $v$ is the outer unit normal vector to $\Gamma$ and $f$ is a prescribed positive function defined on $\mathbb{R}^n$.

The overdetermined problem (1.1) arises in a shape optimization problem called the Saint-Venant problem, in which one maximizes the torsional rigidity

$$P(\Omega) = \sup_{u \in H^1_0(\Omega) \setminus \{0\}} \frac{\left(\int_{\Omega} u \, dx\right)^2}{\int_{\Omega} |\nabla u|^2 \, dx}$$

of a bar with cross section $\Omega$, among all sets $\Omega$ of equal weighted volume

$$V(\Omega) = \int_{\Omega} f^2 \, dx.$$ 

The Euler-Lagrange equation, after multiplying a normalizing constant, consists in (1.1). In the case where $f$ is a constant, Pólya [17] proved that the maximizer $\Omega$ of $P$ must be a ball with the prescribed volume $V$, using the symmetric rearrangement of a function. This applies to a more general situation, where $f$ is radially symmetric and non-decreasing in the radial direction.

In fact, the same symmetry result also holds for all critical points, namely, if $f$ is a constant, then (1.1) has a solution $u$ if and only if $\Omega$ is a ball. In particular, for the normalized constant $f = \frac{1}{n}$, $\Omega$ is a ball of radius one and $u(x) = \frac{1}{2^n} (1 - |x - c|^2)$ with $c$ being the center of the ball. This well-known symmetry result is due to Serrin [18]. The proof introduces the method of moving planes motivated by Alexandrov’s reflection principle [2] originally used to establish the soap bubble theorem. This symmetry result can be alternatively proven by an ingenious combination of the Rellich-Pohozaev integral identity and elementary inequalities (see Weinberger [19], and Brandolini, Nitsch, Salani, and Trombetti [6]), or by a continuous version of the Steiner symmetrization (see Brock and Henrot [7]).

The objective of this paper is the stability of a domain $\Omega$ under a perturbation of the Neumann boundary condition (1.1b), which naturally arises if one considers the torsional rigidity in anisotropic media. Namely, setting $\Omega_0 = B$, the $n$-dimensional unit ball centered at the origin, and

$$f(x) = \frac{1}{n} + g \left(\frac{x}{|x|}\right) \quad (x \in \mathbb{R}^n \setminus \{0\}) \quad (1.2)$$

with a prescribed function $g$ defined on $S = \partial B$, we prove the existence and local uniqueness of $\Omega$ admitting a solution $u$ to (1.1), and establish a quantitative estimate of the deviation of $\Omega$ from $\Omega_0$ in terms of the perturbation $g$.

The domain deviation is measured by a function $\rho = \rho(\zeta) \in (-1, \infty)$ which defines the star-shaped bounded domain $\Omega_\rho$ enclosed by the set

$$\Gamma_\rho = \{\zeta + \rho(\zeta) \zeta \mid \zeta \in S\}. \quad (1.3)$$
A domain $\Omega$ admitting a solution to (1.1) will also be referred to as a solution of the problem. In what follows, $h^{k+\alpha}(\overline{\Omega})$ denotes the little Hölder space defined as the closure of $C^{\infty}(\overline{\Omega})$ in $C^{k+\alpha}(\overline{\Omega})$, and similarly $h^{k+\alpha}(\Gamma)$ for a hypersurface $\Gamma$ (see Lunardi [13]).

In order to motivate our study, let us mention several related results concerning existence and stability of solutions to (1.1). The existence of $\Omega$ for non-constant $f$ is known (see Bianchini, Henrot and Salani [4]) in the case where $f$ is positively homogeneous, i.e.,

$$f(tx) = t^\gamma f(x) \quad (t > 0, \ x \in \mathbb{R}^n) \quad (1.4)$$

for $\gamma > 0$ with $\gamma \neq 1$ with $f$ being Hölder continuous on $\mathbb{R}^n \setminus \{0\}$. This condition ensures the existence of a maximizer $\Omega$ of the Saint-Venant problem, and a solution $u$ to (1.1a) then satisfies $-\partial_\nu u = \lambda f$ on $\Gamma$ with a Lagrange multiplier $\lambda > 0$. The $\gamma$-homogeneity of $f$ allows us to control $\lambda$ by considering the set $t\Omega = \{tx \mid x \in \Omega\}$, and indeed $t = \lambda^{1/(1-\gamma)}$ gives a desired domain. However, the 0-homogeneous case (1.2) cannot be treated by this variational approach, since the dichotomy of a maximizing sequence cannot be excluded in the concentration-compactness alternative.

Most of the existing stability results in the literature for (1.1), fitted into our context by translation and dilation, take inequalities of the form

$$\|\rho\|_{L^{\infty}(\mathcal{S}^{n-1})} \leq C \left[ \frac{\partial u_\Omega}{\partial \nu} + R \right]^\tau_X, \quad (1.5)$$

where $u_\Omega$ is a solution to (1.1a) in $\Omega = \Omega_\rho$ with $C^{2+\alpha}$-boundary, $0 < \tau \leq 1$ and $[\cdot]_X$ denotes a norm or seminorm which measures the deviation of $-\partial_\nu u_\Omega$ from a constant $R > 0$. Aftalion, Busca and Reichel [1] adopted a quantitative version of the method of moving planes and obtained a logarithmic version of (1.5) with $X = C^1(\Gamma)$. The method was further developed by Ciraolo, Magnanini and Vespri [8], and they obtained (1.5) for some $0 < \tau < 1$ in terms of the Lipschitz seminorm of $X = \operatorname{Lip}(\Gamma)$. In fact, these results also hold for semilinear equations $-\Delta u = f(u)$ with $u > 0$. On the other hand, Brandolini, Nitsch, Salani and Trombetti [5] made use of integral identities and proved (1.5) for $X = L^\infty(\Gamma)$ for some $0 < \tau < 1$. Moreover, they obtained an estimate of the volume of the symmetric difference of $\Omega$ and a union of balls by a weaker norm, i.e., $X = L^1(\Gamma)$. Note that the problem (1.1) admits a domain $\Omega$ composed of a finite number of balls joined by tiny tentacles if we only control the extra boundary condition (1.1b) by the $L^1$-norm. Following this approach, Feldman [9] obtained the sharp estimate

$$|\Omega \Delta B| \leq C \left\| \frac{\partial u_\Omega}{\partial \nu} + R \right\|_{L^2(\Gamma)},$$

where $|\Omega \Delta B|$ is the volume of the symmetric difference of $\Omega$ and $B$ and is considered as $\|\rho\|_{L^1(\Omega)}$ for star-shaped $\Omega = \Omega_\rho$. The linear (i.e., $\tau = 1$) stability estimate has also been expected in (1.5). Recently, Magnanini and Poggesi proved (1.5) with $X = L^2(\Gamma)$ and $\tau = 1$ for $n = 2$, $\tau$ arbitrarily close to 1 for $n = 3$, and $\tau = \frac{2}{n-1}$ for $n \geq 4$ [14].

In general, for overdetermined problems, the super-subsolution method based on the maximum principle provides an existence criterion. In our setting, a bounded domain $\Omega$ is called a supersolution to (1.1) if the unique solution $u = u_\Omega$ to (1.1a) satisfies

$$-\frac{\partial u_\Omega}{\partial \nu} \leq f \quad \text{on } \Gamma,$$

and a subsolution is defined analogously with the opposite inequality. The existence of a solution, i.e., a bounded domain $\Omega$ in which $u_\Omega$ satisfies (1.1b), is guaranteed provided there
are a supersolution $\Omega_{\text{sup}}$ and a subsolution $\Omega_{\text{sub}}$ satisfying $\Omega_{\text{sub}} \subset \Omega_{\text{sup}}$. Typically, balls $B_r$ with large or small radii $r > 0$ give super- or subsolutions. Indeed, for $\Omega = B_r$,  
$$u_{B_r}(x) = \frac{r^2 - |x|^2}{2n}$$
with $-\partial_v u_{B_r} = \frac{x}{n}$ on $\partial B_r$ solves (1.1), and we see that, in the $\gamma$-homogeneous setting (1.4) with $\gamma > 1$, $B_r$ with large (resp. small) $r > 0$ is a supersolution (resp. subsolution); while for $0 \leq \gamma < 1$, $B_r$ with large (resp. small) $r > 0$ is a subsolution (resp. supersolution). Hence these balls provide an appropriate pair of super- and subsolutions only if $\gamma > 1$.

We therefore take another approach in this paper based on an implicit function theorem, yielding linear stability estimates with Hölder norms on both sides of the estimate, as well as the existence and local uniqueness of $\Omega$ for a given perturbation $g$ in (1.2). We will need to exploit detailed properties of the linearized equation  
$$\begin{align*}
-\Delta p &= 0 \quad \text{in } \Omega_{\rho_0}, \\
\left( H - \frac{1}{f} \right) p + \frac{\partial p}{\partial v} &= -\varphi \quad \text{on } \Gamma_{\rho_0}, \\
p &= f \tilde{\rho} \quad \text{on } \Gamma_{\rho_0},
\end{align*}$$

where $H = H_{\Gamma_{\rho_0}}$ is the mean curvature of $\Gamma_{\rho_0}$ normalized such that $H = n - 1$ for $\Omega = B$. The linearized problem (1.6) is derived by substituting a solution pair $(\Omega_{\rho_0+\varepsilon\tilde{\rho}}, u_{\rho_0+\varepsilon\tilde{\rho}})$ with formal expansions  
$$\begin{align*}
\Gamma_{\rho_0+\varepsilon\tilde{\rho}} &= \{ \zeta + (\rho_0(\zeta) + \varepsilon \tilde{\rho}(\zeta)\nu(\zeta)) + o(\varepsilon) \mid \zeta \in S \}, \\
u_{\rho_0+\varepsilon\tilde{\rho}} &= u_{\rho_0} + \varepsilon p + o(\varepsilon)
\end{align*}$$
into (1.1) for a right hand side $f + \varepsilon \varphi$, and equating functions of order $\varepsilon$. Note that (1.6) is a decoupled system for $p$ and $\tilde{\rho}$, and we may consider only (1.6a) for the solvability of (1.6). Then (1.6b) with known $p$ yields a solution $\tilde{\rho}$.

Recall that the implicit function theorem states that, for each $g$ near $g_0$, the nonlinear equation $F(\rho, g) = 0$ has a unique solution $\rho$ near $\rho_0$ with $F(\rho_0, g_0) = 0$, if

(i) the mapping $F : X \times Y \to Z$ is $C^1$ in a neighbourhood of $(\rho_0, g_0)$ and if
(ii) the partial derivative $\partial_\rho F(\rho_0, g_0) \in \mathcal{L}(X, Z)$ is bijective.

Here, $X, Y$ and $Z$ are Banach spaces with $X \subset Z$. In addition to the solution $\rho(g)$ being locally unique, the mapping $g \mapsto \rho(g) \in X$ is in $C^1$. In the current setting, the Neumann boundary condition (1.1b) yields such a mapping $F$, and the linearized equation $\partial_\rho F(\rho_0, g_0)(\tilde{\rho}) = \varphi$ is reflected by (1.6). However, the linearized problem (1.6) has a regularity defect called loss of derivatives, i.e. $\partial_\rho F(\rho_0, g_0)^{-1} \notin \mathcal{L}(Z, X)$. Since solutions $\tilde{\rho}$ to (1.6) are less regular than $\rho_0$, and hence the typical iterative scheme in the classical implicit function theorem fails.

One method to overcome this regularity issue is the Nash-Moser theorem, a generalization of the classical implicit function theorem introduced by Nash in [16] and generalized by Moser in [15]. The introduction of a smoothing operator combined with Newton’s method for improved convergence was there shown to be a mean to overcome the regularity deficit. For the Nash-Moser theorem to work,

(i) regularity properties are required for $F : X_i \times Y \to Z_i$, where $(X_i, Z_i)$ is a family of pairs of Banach spaces such that $X_i \subset X_{i-1}$, $Z_i \subset Z_{i-1}$. Furthermore,  
(ii) a (right) inverse of $\partial_\rho F(\rho, g)$ has to exist for $(\rho, g)$ in a neighbourhood of $(\rho_0, g_0)$. 

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In this setting, for every $g$ in a neighbourhood of $g_0$, the existence of $\rho(g)$ in $X_0$ is then given. Note that there are various versions of the Nash-Moser theorem, also referred to as Nash-Moser-Hörmander theorem. We refer as an example to the work of Baldi and Haus [3] and the references therein.

Instead of applying the Nash-Moser theorem, we introduce a new modified version of the classical implicit function theorem, which has the constraint that a loss of derivatives may take place except at the point $(\rho_0, g_0)$. We require for a pair of Banach triplets $X_2 \subset X_1 \subset X_0$ and $Z_2 \subset Z_1 \subset Z_0$ that

(i) for $j = 1, 2$, $F$ is continuous in a neighbourhood of $(\rho_0, g_0)$ from $X_{j-1} \times Y$ to $Z_{j-1}$, and that it is in $C^1$ in a neighbourhood of $(\rho_0, g_0)$ from $X_j \times Y$ to $Z_{j-1}$. For $(\rho, g)$ in a neighbourhood of $(\rho_0, g_0) \in X_j \times Y$, we have that $\partial_{\rho_j} F(\rho, g) \in \mathcal{L}(Z_{j-1}, X_{j-1})$.

Further,

(ii) $F: X_j \times Y \to Z_j$ is Fréchet-differentiable at $(\rho_0, g_0)$ for $j = 1, 2$ and $\partial_{\rho_j} F(\rho_0, g_0) \in \mathcal{L}(X_j, Z_j)$ is invertible for $j = 0, 1, 2$.

The first point reflects the loss of regularity, the second point reflects that it does not occur at the point under consideration. Under these assumptions, we derive a modified implicit function theorem that yields local uniqueness of a solution $\rho(g) \in X_1$ for all $g$ in a neighbourhood of $g_0$, and the mapping $g \mapsto \rho(g) \in X_0$ is in $C^1$. Note that in the setting of (1.1) and (1.6), the loss of derivatives does indeed not occur in the case of the solution of (1.1) for constant $f$, as then $\Gamma$ and $u$ are smooth.

However, a second obstacle apart from the loss of derivatives arises. Due to the translational invariance of (1.1), the linearized equation (1.6a) for $g = 0$ and $\Omega_{\rho_0} = \mathbb{B}$ is not solvable for arbitrary $\varphi \in h^{2+\alpha}(\mathbb{S})$, and for $\varphi = 0$ it has an $n$-dimensional space of solutions. This implies that the partial derivative of $F$ at $(0,0)$ is not invertible, which is necessary also for the modified implicit function theorem. We will remove this degeneracy by imposing an additional condition

$$\int_{\Omega} x_j \, dx = 0 \quad (j = 1, \ldots, n), \tag{1.7}$$

so that the barycenter of $\Omega$ is fixed to be the origin, and by decomposing the space $h^{k+\alpha}(\mathbb{S})$ into $h^{k+\alpha}(\mathbb{S}) = X_1 \oplus K$, where we define

$$X_1 = \left\{ \rho \in h^{k+\alpha}(\mathbb{S}) \mid \langle \rho, x_j \rangle_{L^2(\mathbb{S})} = 0, \quad j = 1, \ldots, n \right\},$$

$$K = \text{span} \{x_1, \ldots, x_n\}. \tag{1.8}$$

This allows for a decomposition of a domain perturbation $\rho \in h^{k+\alpha}(\mathbb{S})$ into $\rho_1 \in X_1$, $\rho_2 \in K$, as well as a decomposition of the function $g$ in (1.2) likewise, and we examine $F(\rho_1 + \rho_2, g_1 + g_2) = 0$.

With these preparations, we present the main result of this work.

**Theorem 1.1** There exist neighbourhoods of zeros

$$V \subset X_2, \ U_2 \subset h^{2+\alpha}(\mathbb{S}) \times K \text{ and } U_3 \subset h^{3+\alpha}(\mathbb{S}) \times K,$$

such that for all $g_1 \in V$ there is a unique pair $(\rho, g_2) = (\rho(g_1), g_2(g_1)) \in U_3$ such that the following holds.

(i) $\Omega_{\rho(g_1)}$ defined by (1.3) admits a solution $u \in h^{3+\alpha}(\Omega_{\rho(g_1)})$ to (1.1) for (1.2) with $g = g_1 + g_2(g_1)$, and satisfies (1.7).
(ii) $\Omega_\rho(g_1)$ is locally unique up to translations in the sense that, if $\Omega_\rho$ with $(\rho, g_2(g_1)) \in U_3$ admits a solution $u$ to (1.1) for (1.2) with $g = g_2(g_1) + g_1$ and satisfies (1.7), then $\rho = \rho(g_1)$.

(iii) For the mapping $(\rho, g_2) : V \to U_3$, we have $(\rho, g_2) \in C^1(V, U_2)$ and the stability estimates

$$
\|\rho(g_1)\|_{h^{2+\alpha}(S)} \leq C \|g_1\|_{h^{2+\alpha}(S)},
$$

$$
\|g_2(g_1)\|_{h^{2+\alpha}(S)} \leq C \|g_1\|_{h^{2+\alpha}(S)}
$$

hold.

While the existence of $\rho$ is guaranteed in $h^{3+\alpha}(S)$, only the weaker norm, i.e. the $h^{2+\alpha}(S)$ norm, of $\rho$ is estimated in (1.9). This is due to the fact that the linear stability estimate requires $C^1$-regularity of the mapping $g_1 \mapsto \rho$, and this regularity is expected only when the image space is $h^{2+\alpha}(S)$ due to the loss of derivatives.

**Remark 1.1** The translational invariance of (1.1) is mirrored in that theorem by using the decomposition into the translational part and its orthogonal complement. In that regard, it also becomes clear why the setting of the little Hölder spaces $h^{l+\alpha}$ instead of the Hölder spaces $C^{l+\alpha}$ is necessary: The decomposition into subspaces is induced by the so-called spherical harmonics on $S$. They are dense in $h^{l+\alpha}(S)$, but not in $C^{l+\alpha}(S)$. This will be further discussed in Sect. 3.

The paper is organised as follows. In Sect. 2, we introduce the perturbed problem as well as derive in detail the formulation via the linearized equation (1.6). We motivate the application of an implicit function theorem to a mapping $F$ that is derived from the Neumann boundary condition. This application is obstructed by the degeneracy of the derivative of $F$ as well as the loss of derivatives. The degeneracy of the derivative of $F$ stemming from the inherent symmetry of (1.1) will be addressed in Sect. 3. There, also the decomposition for the little Hölder spaces is motivated as well as the necessity of using the setting of the little Hölder spaces. In Sect. 4, we will revisit the implicit function theorem and establish a modified version fitting our setting. This is then applied to the perturbed problem in Sect. 5 to prove Theorem 1.1.

## 2 Preliminaries

We formally set up the perturbed problem defined in (1.1). We want to know whether for a perturbation $g$ there exists an open bounded domain $\Omega = \Omega(g)$ admitting a solution $u_\Omega$ to (1.1) with (1.2), i.e. $f = \frac{1}{n} + g$.

We restrict the domain $\Omega$ to be in such a way that it may be modeled as a deviation of $\mathbb{B}$, the domain admitting a solution to (1.1) with $f = \frac{1}{n}$. For this reference domain $\Omega_0 = \mathbb{B}$, with $\partial \Omega_0 = \Gamma_0 = S$, we define the perturbed domain $\Omega_\rho$ by its $h^{m+\alpha}$-boundary $\Gamma_\rho = \partial \Omega_\rho$ in the following way. We define for $m \in \mathbb{N}$

$$
U_{\gamma, m} = \{ v \in h^{m+\alpha}(S) \mid \|v\|_{h^{m+\alpha}(S)} < \gamma \},
$$

with $\gamma \leq 1$ sufficiently small. Next, we define

$$
\theta : S \times (-1, \infty) \to \theta(S \times (-1, \infty)), \quad \theta(\zeta, r) = \zeta + r\nu_0(\zeta) = \zeta + r\zeta.
$$
In general, $v_\rho$ denotes the outer unit normal vector of $\Gamma_\rho$; for $\rho = 0$ we have $v_0(x) = x$ and $\Gamma_0 = S$. Then we set

$$\Gamma_\rho = \{ \zeta + \rho(\xi)v_0(\xi) \in \mathbb{R}^n \mid \xi \in \Gamma_0 \} = \{ \zeta + \rho(\xi)v_0(\xi) \in \mathbb{R}^n \mid \xi \in S \},$$

and $\rho \in U_{\gamma,m}$. The function $\rho$ models the radial perturbation of the boundary, and will be used to measure how much $\Gamma_\rho$ deviates from $\Gamma_0$. Using this, we define the diffeomorphism

$$\theta_\rho(x) = \begin{cases} x + \phi(|x|-1) \rho \left( \frac{x}{|x|} \right) x, & \text{for } x \neq 0, \\ 0 & \text{for } x = 0, \end{cases}$$

from $\Omega_0 =\mathbb{R}$ to $\Omega_\rho$, where $\phi : \mathbb{R} \to \mathbb{R}$ is a smooth cut-off function with $0 \leq \phi(r) \leq 1$, $\phi(r) = 1$ for $|r| \leq \frac{1}{3}$ and $\phi(r) = 0$ for $|r| \geq \frac{1}{3}$, as well as $|\frac{d\phi}{dr}(r)| \leq 4$. The diffeomorphism $\theta_\rho$ induces pullback and pushforward operators

$$\begin{align*}
\theta_\rho^* u &= u \circ \theta_\rho, \\
\theta_\rho^* v &= v \circ \theta_\rho^{-1},
\end{align*}$$

with $k \in \mathbb{N} \cup \{0\}$.

Our problem now becomes the following:

**Problem 2.1** For $g \in h^{1+\alpha}(S)$, is there a $\rho = \rho(g) \in U_{\gamma,2}$ such that $\Omega_\rho$ as defined above admits a solution $u_\rho$ to (2.1)?

$$\begin{align*}
-\Delta u_\rho &= 1 \quad \text{in } \Omega_\rho, \\
u_\rho &= 0 \quad \text{on } \Gamma_\rho, \\
\frac{\partial u_\rho}{\partial v_\rho}(x) &= \frac{1}{n} + g \left( \frac{x}{|x|} \right) \quad \text{on } \Gamma_\rho.
\end{align*}$$

(2.1a) (2.1b)

By elliptic regularity theory, we have, for given $\rho \in U_{\gamma,2}$, the existence and uniqueness of a solution $u_\rho \in h^{2+\alpha}(\Omega_\rho)$ when only considering (2.1a). Therefore, for the examination of this problem, it is sufficient to focus on the perturbation, i.e., (2.1b).

We define $F \in C(U_{\gamma,2} \times h^{1+\alpha}(S), h^{1+\alpha}(S))$ by

$$F(\rho, g) = \theta_\rho^* \left( \frac{\partial u_\rho}{\partial v_\rho} \right) + \frac{1}{n} + g,$$

(2.2)

where $u_\rho$ is the unique solution of (2.1a). Then $\Omega_\rho$ admits a solution to (2.1) for given $g \in h^{1+\alpha}(S)$ if and only if $F(\rho, g) = 0$.

This structure tempts to use the implicit function theorem to arrive at solutions in a neighbourhood of $(0, 0)$. However, we shall arrive at two obstacles. The first is the derivative $\partial_\rho F(0, 0)$ not being bijective due to the inherent translational invariance, an observation that will be treated in Sect. 3. The second is the loss of derivatives, a regularity issue of the $\rho$-derivative of $F$ that will be discussed in the following.

In view of this, note that for $g \in h^{2+\alpha}(S)$ and for $m = 2, 3$, we have

$$F \in C(U_{\gamma,m} \times h^{2+\alpha}(S), h^{m-1+\alpha}(S)).$$

(2.3)

We turn to the $\rho$-differentiability of $F$ at a point $(\rho_0, g)$. Due to the loss of derivatives, we need to assume $\rho_0 \in U_{\gamma,3}$. We consider

$$F(\rho_0 + \varepsilon \tilde{\rho}, g) - F(\rho_0, g) = A(\rho_0, g)[\varepsilon \tilde{\rho}] + o(\varepsilon)$$

where $A(\rho_0, g)$ is a linear mapping depending on $\rho_0$ and $g$.
for $\tilde{\rho} \in U_{\gamma,3}$ and $\varepsilon \to 0$. Since $u_{\rho}$ in $F(\rho, g)$ lives on $\Omega_{\rho}$, which varies for $\rho$, we consider the following approach.

In the following, $x$ is a point in $\Omega_{0}$ and we set $y = \theta_{\rho_{0}}(x) \in \partial \Omega_{\rho_{0}}$, as well as $z = \theta_{\rho_{0}+\varepsilon}\tilde{\rho}(x) \in \partial \Omega_{\rho_{0}+\varepsilon}\tilde{\rho}$. We consider the function $u(\rho, x) = u_{\rho}(\theta_{\rho}(x))$ defined for $\rho \in U_{\gamma,3}$ and $x \in \partial \Omega_{0}$. One may show that $u(\rho, \theta_{\rho}(x))$ is differentiable with respect to $\rho$, for the procedure see e.g. [12, Sect. 5.6]. Therefore, the following calculations are well-defined. Using the Taylor expansion, we write

$$u_{\rho_{0}+\varepsilon}\tilde{\rho}(z) = u(\rho_{0}, y) + p(\varepsilon)|\tilde{\rho}(\rho_{0})| + o(\varepsilon),$$

where $p$ and $V_{\rho_{0}}$ are defined by

$p(\varepsilon) = \partial_{\rho}u(\rho_{0}, y)\tilde{\rho}(y) + \varepsilon \frac{1}{|\nabla N_{\rho_{0}}|}$. Therefore, the following calculations are well-defined. Using the Taylor expansions, we write

$$u_{\rho_{0}+\varepsilon}\tilde{\rho}(z) = u(\rho_{0}, y) + p(\varepsilon)|\tilde{\rho}(\rho_{0})| + o(\varepsilon),$$

where $p$ and $V_{\rho_{0}}$ are defined by

$$p(\varepsilon) = \partial_{\rho}u(\rho_{0}, y)\tilde{\rho}(y) + \varepsilon \frac{1}{|\nabla N_{\rho_{0}}|}.$$
Next, we calculate
\[
\partial_{\zeta} u_{\rho_0 + \varepsilon \tilde{\rho}}(z) = \partial_{\zeta} u(\rho_0 + \varepsilon \tilde{\rho}, \theta_{\rho_0 + \varepsilon \tilde{\rho}}(x)) \\
= \frac{\partial}{\partial y_i} u(\rho_0, y) + \varepsilon \frac{\partial}{\partial y_i} p(y) + \varepsilon \frac{\partial}{\partial y_j} \frac{\partial}{\partial y_k} u(\rho_0, y) \theta_{\rho_0}^*(v_0 \tilde{\rho}) (y) + o(\varepsilon).
\]
This implies
\[
\nabla \varepsilon u_{\rho_0 + \varepsilon \tilde{\rho}}(z) \cdot v_{\rho_0 + \varepsilon \tilde{\rho}}(z) = \partial_{\zeta} u_{\rho_0 + \varepsilon \tilde{\rho}}(z) v^j_{\rho_0 + \varepsilon \tilde{\rho}}(z) \\
= \left[ \frac{\partial}{\partial y_i} u(\rho_0, y) + \varepsilon \frac{\partial}{\partial y_i} p(y) + \varepsilon \frac{\partial}{\partial y_j} \frac{\partial}{\partial y_k} u(\rho_0, y) \theta_{\rho_0}^*(v_0 \tilde{\rho}) (y) \right] v^j_{\rho_0}(y) + o(\varepsilon)
\]
\[
= \frac{\partial}{\partial y_i} u(\rho_0, y) + \varepsilon \frac{\partial}{\partial y_i} p(y) + \varepsilon \frac{\partial}{\partial y_j} \frac{\partial}{\partial y_k} u(\rho_0, y) \theta_{\rho_0}^*(\tilde{\rho}) (y) \left[ \frac{1}{|\nabla \rho_0|} v^j_{\rho_0} + \tau_{\rho_0} \right] v^i_{\rho_0}(y) + o(\varepsilon)
\]
\[
= \frac{\partial}{\partial \tau_{\rho_0}} u(\rho_0, y) + \varepsilon \frac{\partial}{\partial \tau_{\rho_0}} p(y) + \varepsilon \frac{\partial}{\partial \tau_{\rho_0}} \frac{\partial}{\partial \tau_{\rho_0}} u(\rho_0, y) \theta_{\rho_0}^*(\tilde{\rho}) (y) \frac{1}{|\nabla \rho_0|}
\]
\[
+ \varepsilon \frac{\partial}{\partial \tau_{\rho_0}} \frac{\partial}{\partial \tau_{\rho_0}} u(\rho_0, y) \theta_{\rho_0}^*(\tilde{\rho}) (y) + o(\varepsilon),
\]
where in the last step we used the identity \( \Delta u_{\rho_0} = \Delta \tau_{\rho_0} u_{\rho_0} + \frac{\partial^2}{\partial \tau_{\rho_0}} u_{\rho_0} + H_{\rho_0} \frac{\partial}{\partial \tau_{\rho_0}} u_{\rho_0} \), with \( \Delta \tau_{\rho_0} \) being the Laplace-Beltrami operator and \( H_{\rho_0} \) the mean curvature on \( \tau_{\rho_0} \). Using the Dirichlet boundary condition for \( p \) in (2.4), we arrive at
\[
F(\rho_0 + \varepsilon \tilde{\rho}, g) = F(\rho_0, g) + \varepsilon \theta_{\rho_0}^* \left[ \frac{\partial}{\partial \tau_{\rho_0}} p + H_{\rho_0} p - \frac{\theta_{\rho_0}^* \tilde{\rho}}{|\nabla \rho_0|} \right] + \frac{\partial}{\partial \tau_{\rho_0}} u(\rho_0, y) \theta_{\rho_0}^* \tilde{\rho} + o(\varepsilon).
\]
We see that the term
\[
\theta_{\rho_0}^* \left[ \frac{\partial}{\partial \tau_{\rho_0}} p + H_{\rho_0} p - \frac{\theta_{\rho_0}^* \tilde{\rho}}{|\nabla \rho_0|} \right] + \frac{\partial}{\partial \tau_{\rho_0}} u(\rho_0, y) \theta_{\rho_0}^* \tilde{\rho}
\]
is well-defined and lies in \( h^{1+\alpha}(\mathbb{S}) \) even when only assuming \( \tilde{\rho} \in h^{2+\alpha}(\mathbb{S}) \). Note that to verify this, one also needs to take into account the impact of the regularity assumption of \( \tilde{\rho} \) on the regularity of the solution \( p \) of (2.4) as mentioned in Remark 2.5. This gives us the following lemma.

**Lemma 2.1** The mapping \( F \) as defined in (2.2) satisfies
\[
F \in C(U_{\gamma,2} \times h^{1+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S})) \cap C^1(U_{\gamma,3} \times h^{1+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S})).
\]
Furthermore, we have the following.
(i) The \( \rho \)-Fréchet-derivative of \( F \) at \((\rho_0, g)\) \( \in U_{\gamma,3} \times h^{1+\alpha}(S) \) is
\[
\partial_\rho F(\rho_0, g) \left[ \hat{\rho} \right] = \theta^s \left[ \frac{\partial p}{\partial \nu_{\rho_0}} + H_{\rho_0} p - \frac{\theta^s_{\rho_0}}{\|\nabla N_{\rho_0}\|} \frac{\partial}{\partial \tau_{\rho_0}} \left( \frac{\partial u_{\rho_0}}{\partial \nu_{\rho_0}} \right) \theta^s_{\rho_0} \hat{\rho} \right] \in h^{1+\alpha}(S),
\]
where \( p \) is a unique solution to (2.4), i.e.
\[
\Delta p = 0 \quad \text{in } \Omega_{\rho_0},
\]
\[
p = -\frac{\partial u_{\rho_0}}{\partial \nu_{\rho_0}} \theta^s_{\rho_0} \frac{1}{\|\nabla N_{\rho_0}\|} \quad \text{on } \Gamma_{\rho_0}.
\]

(ii) We have
\[
\partial_\rho F(0, 0) \in L(h^{1+\alpha}(S), h^{2+\alpha}(S)) \quad \text{and}
\]
\[
\partial_\rho F(\rho, g) \in L(h^{1+\alpha}(S), h^{1+\alpha}(S)).
\]

(iii) The linear operator \( \partial_\rho F(\rho, g) \) has the extension
\[
\partial_\rho F(\rho, g) \in L(h^{2+\alpha}(S), h^{1+\alpha}(S)).
\]
In view of (2.3), one may verify that for \( m = 2, 3 \) and \( g \in h^{2+\alpha}(S) \), one has
\[
F \in C(U_{\gamma,m} \times h^{2+\alpha}(S), h^{m-1+\alpha}(S)) \cap C^1(U_{\gamma,m+1} \times h^{2+\alpha}(S), h^{m-1+\alpha}(S)).
\]

**Remark 2.2** We have the following characterisation of bijectivity of the \( \rho \)-Fréchet-derivative of \( F \) at a point \((\rho, g)\) with \( F(\rho, g) = 0 \), i.e. when \( \Omega_\rho \) is a solution to (2.1) for \( g \in h^{1+\alpha}(S) \):

The extended operator \( \partial_\rho F(\rho, g) \), with
\[
\partial_\rho F \in C \left( U_{\gamma,3} \times h^{1+\alpha}(S), L \left( h^{2+\alpha}(S), h^{1+\alpha}(S) \right) \right),
\]
has the bounded inverse
\[
\partial_\rho F(\rho, g)^{-1} \in L \left( h^{1+\alpha}(S), h^{2+\alpha}(S) \right)
\]
if and only if the boundary problem
\[
-\Delta p = 0 \quad \text{in } \Omega_\rho
\]
\[
\left( H_\rho - \frac{1}{n + \theta^s g} \left( 1 - \frac{\partial \theta^s g}{\partial \nu_\rho} \right) \right) p + \frac{\partial p}{\partial \nu_\rho} = \varphi \quad \text{on } \Gamma_\rho.
\]

is uniquely solvable for any \( \varphi \in h^{1+\alpha}(\Gamma_\rho) \). Unique solvability of (2.6) is given provided that
\[
\left( H_\rho - \frac{1}{n + \theta^s g} \right) > 0 \quad \text{and } \|p\|_{h^{2+\alpha}(\Omega_\rho)} \leq C \|\varphi\|_{h^{1+\alpha}(\Gamma_\rho)}.\]
This does not hold in the current setting. Thus, we have to examine the bijectivity in a different manner.

### 3 Degeneracy of \( \partial_\rho F \)

#### 3.1 Non-Bijecitivity of the partial derivative of \( F \)

We examine the \( \rho \)-derivative of \( F \) at \((\rho, g) = (0, 0)\), as we merely require the existence of an inverse of \( \partial_\rho F(0, 0) \) to use the modified implicit function theorem, Theorem 4.2. To this aim, we introduce the Dirichlet-to-Neumann operator.
Definition 3.1 Let $\varphi \in h^{l+\alpha}(\mathbb{S})$, $l \geq 2$ be arbitrary. The Dirichlet-to-Neumann operator on the sphere $\mathcal{N} : h^{l+\alpha}(\mathbb{S}) \to h^{l-1+\alpha}(\mathbb{S})$ is defined as

$$\mathcal{N} \varphi = \partial_{\nu} u, \text{ with } u \text{ being the unique solution of } \left\{ \begin{array}{l} \Delta u = 0 \quad \text{in } \mathbb{B}, \\ u = \varphi \quad \text{on } \mathbb{S}. \end{array} \right.$$ 

We see that

$$\mathcal{N} \in \mathcal{L} \left( h^{l+\alpha}(\mathbb{S}), h^{l-1+\alpha}(\mathbb{S}) \right), \quad l \in \mathbb{N}. $$

Using that notation, we have

$$\partial_{\rho} F(0, 0)[\tilde{\rho}] = -\frac{1}{n} \tilde{\rho} + \frac{1}{n} \mathcal{N} \tilde{\rho}, \quad \text{for } \tilde{\rho} \in h^{3+\alpha}(\mathbb{S}).$$

It suffices to test bijectivity of $\partial_{\rho} F(0, 0)$ for $\tilde{\rho} \in H_k$, $k \in \mathbb{N} \cup \{0\}$, where

$$H_k = \text{span} \left\{ h_{k,j} \mid j = 1, \ldots, d_k^n \right\}, \quad d_k^n < \infty,$$

is the set of harmonic homogeneous polynomials on the unit sphere of degree $k$. Indeed, the $h_{k,j}, k \in \mathbb{N} \cup \{0\}, \quad j = 1, \ldots, d_k^n$, form an orthonormal basis of $L^2(\mathbb{S})$, see e.g. [10, Thm. 2.53]. One may show that the set $\mathcal{H}$ defined by

$$\mathcal{H} = \text{span} \left\{ h_{k,j} \mid k \in \mathbb{N} \cup \{0\}, \quad j = 1, \ldots, d_k^n \right\}$$

is dense in $h^{l+\alpha}(\mathbb{S})$, which is why it is sufficient to consider $\tilde{\rho} \in \mathcal{H}$.

Therefore, let $\tilde{\rho} = h_{k,j}$ be a harmonic homogeneous polynomial on the unit sphere of order $k \in \mathbb{N} \cup \{0\}$, with $j \in \{1, \ldots, d_k^n\}$. We get

$$\partial_{\rho} F(0, 0)[h_{k,j}] = -\frac{1}{n} h_{k,j} + \frac{k}{n} h_{k,j} = \frac{k-1}{n} h_{k,j} \begin{cases} = 0 & \text{if } k = 1, \\ \neq 0 & \text{else}. \end{cases}$$

This shows that $\partial_{\rho} F(0, 0)$ is not bijective and that its kernel is

$$\ker (\partial_{\rho} F(0, 0)) = \text{span} \left\{ h_{1,j} \mid j = 1, \ldots, n \right\}, \quad (3.1)$$

with $\dim(\ker(\partial_{\rho} F(0, 0))) = n < \infty$. Furthermore, we see that the range of $\partial_{\rho} F(0, 0)$ is

$$\text{range} (\partial_{\rho} F(0, 0)) = \text{span} \left\{ h_{k,j} \mid k \in \mathbb{N}_{\geq 2} \cup \{0\}, \quad j = 1, \ldots, d_k^n \right\} \overset{\| \cdot \|_{h^{l+\alpha}(\mathbb{S})}}{\sim} h^{l+\alpha}(\mathbb{S}).$$

Notation 3.1 In view of the calculations to come, we define

$$X_l = \text{span} \left\{ h_{k,j} \mid k \in \mathbb{N}_{\geq 2} \cup \{0\}, \quad j = 1, \ldots, d_k^n \right\} \overset{\| \cdot \|_{h^{l+\alpha}(\mathbb{S})}}{\sim} h^{l+\alpha}(\mathbb{S}), \quad \text{for } l \in \mathbb{N}.$$ 

Note that this is equivalent to defining

$$X_l = \left\{ \rho \in h^{l+\alpha}(\mathbb{S}) \mid \langle \rho, h_{1,j} \rangle_{L^2(\mathbb{S})} = 0, \quad j = 1, \ldots, n \right\},$$

as in (1.8). To confirm this, also note that $h_{1,j}(x) = \omega_n^{-\frac{1}{n}} x_{j+1}, \quad j = 1, \ldots, n$. Since $X_l$ is a finite-codimensional subspace of $h^{l+\alpha}(\mathbb{S})$, we have $X_l \oplus X_l^\perp = h^{l+\alpha}(\mathbb{S})$, where $X_l^\perp$ denotes the orthogonal complement of $X_l$. Because $\dim(X_l^\perp) = n < \infty$ for all $l \in \mathbb{N}$, we do not need to differentiate between those spaces depending on $l$, as we have $X_l^\perp \cong \mathbb{R}^n$. Therefore, we may define

$$K = \text{span} \left\{ h_{1,j} \mid j = 1, \ldots, n \right\}.$$
and get \( X_l \oplus K = h^{l+\alpha}(\mathbb{S}) \). Finally, we denote 

\[
U_l = \left\{ \rho \in U_{\gamma,l} \mid \langle \rho, h_{1,j} \rangle_{L^2(\mathbb{S})} = 0, \ j = 1, \ldots, n \right\},
\]
as well as \( U_l^\perp \) for the subset of \( K \) such that \( U_{\gamma,l} = U_l \oplus U_l^\perp \).

**Remark 3.1** That the kernel of \( \partial_\rho F(0,0) \) is non-trivial is not surprising, in fact it is an obvious property resulting from the translational invariance of (2.1) with \( g = 0 \). The overdetermined problem is in this case solvable for any translated ball denoted by \( \mathbb{B} + c = B_1(c) \), with \( c \in \mathbb{R}^n \), and with solution \( u_c(x) = \frac{1}{2\pi} \left( 1 - |x - c|^2 \right) \).

To show the connection to the kernel of \( \partial_\rho F(0,0) \), we find \( \rho \in U_{\gamma,3} \) such that \( \Gamma_\rho = \partial (\mathbb{B} + c) \). With the ansatz \( x + \rho(x)x = y + c, \ x, y \in \mathbb{S} \), we arrive at \( \rho(x) = \rho(c, x) = x.c - 1 \pm \sqrt{1 - |c|^2} + (x.c)^2 \) for any \( |c| \leq 1 \). For \( c = te_j \), we then arrive at

\[
\frac{d}{dt} \rho(te_j, x)|_{t=0} = \left[ x_j + \frac{1}{2\sqrt{1 - t^2 + t^2x_j^2}} \left( -2t + 2tx_j^2 \right) \right]_{t=0} = x_j.
\]

Now, because of the translational invariance of (2.1), we have \( F(\rho(c, x), 0) = F(0, 0) = 0 \), which implies

\[
\partial_\rho F(0, 0) \left[ \frac{d}{dt} \left( \rho(te_j, \cdot) \right) \right]_{t=0} = 0
\]

and we arrive at \( \partial_\rho F(0, 0)[x_j] = 0 \) for \( j \in \{1, \ldots, n\} \). This coincides with (3.1).

**3.2 Re-formulation to eliminate degeneracy**

To eliminate the problem of non-bijectivity, i.e. the degeneracy of the problem (2.6), we need to eliminate the translation invariance in the original problem (2.1). Therefore, we replace \( F \) as defined in (2.2) by a mapping

\[
G \in C(U_2 \times U_2^\perp \times X_1 \times K, X_1 \times \mathbb{R}^n \times K),
\]

defined by

\[
G(\rho_1, \rho_2, g_1, g_2) = \begin{pmatrix}
G_0 \\
G_1 \\
\vdots \\
G_n \\
G_{n+1}
\end{pmatrix} (\rho_1, \rho_2, g_1, g_2) = \begin{pmatrix}
P_2 F(\rho_1 + \rho_2, g_1 + g_2) \\
\int_{\Omega_{\rho_1+\rho_2}} x_1 \, dx \\
\vdots \\
\int_{\Omega_{\rho_1+\rho_2}} x_n \, dx \\
(Id - P_2) F(\rho_1 + \rho_2, g_1 + g_2)
\end{pmatrix}.
\]

\( P_l \colon h^{l+\alpha}(\mathbb{S}) \rightarrow X_l \) denotes the projection onto \( X_l \). If it is clear which \( l \in \mathbb{N} \) is to be used, we write \( P \) for \( P_l \).

Note that for \( m = 2, 3 \), and \( g_1 \in X_2, G \) is also well-defined and we have

\[
G \in C \left( U_m \times U_m^\perp \times X_2 \times K, X_{m-1} \times \mathbb{R}^n \times K \right).
\]

By the condition \( \int_{\Omega_\rho} x_i \, dx = 0 \) for \( i = 1, \ldots, n \), we achieve that the center of mass of \( \Omega_\rho \) is in the origin and thus eliminate the possibility of translations, and thus, admissible \( \rho \) will
Lemma 3.1 \( \Omega_\rho \) with barycenter zero admits a solution to (2.1) for given \( g \in h^{1+\alpha}(\mathbb{S}) \) if and only if \( G(\rho_1, \rho_2, g_1, g_2) = 0 \), with \( \rho = \rho_1 + \rho_2 \) and \( g = g_1 + g_2 \).

### 3.3 Bijectivity of the partial derivative of \( G \)

The mapping \( G \) has the following regularity properties.

**Lemma 3.2** \( G \) is Fréchet-differentiable as a map from \( U_3 \times U_3^+ \times K \) to \( X_1 \times \mathbb{R}^n \times K \). We have

\[
\partial_{\rho_1, \rho_2, g_2} G(\rho_1, \rho_2, g_1, g_2) \in \mathcal{L}(X_3 \times K \times K, X_1 \times \mathbb{R}^n \times K),
\]

for \( (\rho_1 + \rho_2, g_1 + g_2) \in U_{y,3} \times h^{1+\alpha}(\mathbb{S}) \), which can be extended to

\[
\partial_{\rho_1, \rho_2, g_2} G(\rho_1, \rho_2, g_1, g_2) \in \mathcal{L}(X_2 \times K \times K, X_1 \times \mathbb{R}^n \times K).
\]

Furthermore,

\[
G \in C(U_2 \times U_2^+ \times X_1 \times K) \times X_1 \times \mathbb{R}^n \times K).
\]

In view of the application of the modified function theorem introduced in Sect. 4, we also need the following observation concerning the regularity of \( G \). For \( m = 2, 3 \), we have

\[
G \in C \left( U_m \times U_m^+ \times X_2 \times K, X_{m-1} \times \mathbb{R}^n \times K \right)
\]

\( \cap C^1(U_{m+1} \times U_{m+1}^+ \times X_2 \times K, X_{m-1} \times \mathbb{R}^n \times K) \).

**Proof** We have for \( i = 1, 2 \) and \( \tilde{\rho}_i \in X_3 \subset h^{3+\alpha}(\mathbb{S}) \), \( \tilde{\rho}_2 \in K \subset h^{3+\alpha}(\mathbb{S}) \)

\[
\partial_{\tilde{\rho}_i} G_0(\rho_1, \rho_2, g_1, g_2)(\tilde{\rho}_i) = \partial_{\rho_1} P F(\rho_1 + \rho_2, g_1 + g_2)(\tilde{\rho}_i) \quad \in X_1,
\]

\[
\partial_{\tilde{\rho}_i} G_n(\rho_1, \rho_2, g_1, g_2)(\tilde{\rho}_i) = \partial_{\rho_1} (\text{Id} - P) F(\rho_1 + \rho_2, g_1 + g_2)(\tilde{\rho}_i) \quad \in K
\]

and further for \( j = 1, \ldots, n \)

\[
\partial_{\tilde{\rho}_i} G_j(\rho_1, \rho_2, g_1, g_2)(\tilde{\rho}_i) = \int_{\Gamma_{\rho_1+\rho_2}} \frac{1}{|\nabla N_\rho|} \partial^2 \tilde{\rho}_i \cdot \sigma_j \text{d}\sigma, \quad \text{for } j = 1, \ldots, n.
\]

These expressions are still well-defined and of the same regularity for \( \tilde{\rho} \in h^{2+\alpha}(\mathbb{S}) \), implying the existence of an extension of \( \partial_{\rho_1, \rho_2} G(\rho_1, \rho_2, g_1, g_2) \) onto \( X_2 \times K \) and thus

\[
\partial_{\rho_1, \rho_2} G(\rho_1, \rho_2, g_1, g_2) \in \mathcal{L}(X_3 \times K, X_1 \times \mathbb{R}^n \times K) \cap \mathcal{L}(X_2 \times K, X_1 \times \mathbb{R}^n \times K).
\]
For the $g_2$-partial derivative and $\tilde{g}_2 \in K$, we get
\[
\partial_{g_2} G_0(\rho_1, \rho_2, g_1, g_2)[\tilde{g}_2] = \partial_{g_2} P F(\rho_1 + \rho_2, g_1 + g_2)[\tilde{g}_2],
\]
\[
\partial_{g_2} G_j(\rho_1, \rho_2, g_1, g_2)[\tilde{g}_2] = 0 \quad \text{and}
\]
\[
\partial_{g_2} G_{n+1}(\rho_1, \rho_2, g_1, g_2)[\tilde{g}_2] = \partial_{g_2} (\text{Id} - P) F(\rho_1 + \rho_2, g_1 + g_2)[\tilde{g}_2].
\]
This implies $\partial_{g_2} G(\rho_1, \rho_2, g_1, g_2) \in \mathcal{L}(K, X_1 \times \mathbb{R}^n \times K)$.

At zero, the partial derivative $\partial_{\rho_1, \rho_2, g_2} G$ is bijective. We abbreviate $(0, 0, 0, 0)$ by $(0)$.

**Lemma 3.3** $G$ is Fréchet-differentiable at $(\rho_1, \rho_2, g_1, g_2) = (0)$, and we have
\[
\partial_{\rho_1, \rho_2, g_2} G(0) \in \mathcal{L}(X_3 \times K \times K, X_2 \times \mathbb{R}^n \times K)
\]
with
\[
\partial_{\rho_1, \rho_2, g_2} G(0)[\tilde{\rho}_1, \tilde{\rho}_2, \tilde{g}_2] = \left( \frac{1}{n} (\mathcal{N} - \text{Id}) \tilde{\rho}_1 \right)
= \left( \frac{1}{n} \omega_n^{1/2} \frac{\rho_2}{\tilde{g}_2} \right).
\]
(3.4)

Indeed, for arbitrary $m \in \mathbb{N}$, we have
\[
\partial_{\rho_1, \rho_2, g_2} G(0) \in \mathcal{L}(X_{m+1} \times K \times K, X_m \times \mathbb{R}^n \times K).
\]

Further, $\partial_{\rho_1, \rho_2, g_2} G(0)$ is invertible with
\[
\partial_{\rho_1, \rho_2, g_2} G(0)^{-1} \in \mathcal{L}(X_m \times \mathbb{R}^n \times K, X_{m+1} \times K \times K)
\]
for $m \in \mathbb{N}$, and
\[
\partial_{\rho_1, \rho_2, g_2} G(0)^{-1}[\phi, \alpha_1, \ldots, \alpha_n, \psi] = \left( \frac{n}{\omega_n^{1/2}} \sum_{j=1}^{n} \alpha_j h_{1,j} \right) \in X_{m+1} \times K \times K
\]
(3.5)

where $\phi \in X_m$, $\alpha_j \in \mathbb{R}$ for $j = 1, \ldots, n$ and $\psi \in K$.

**Proof** Let $m \in \mathbb{N}$. Let $(\tilde{\rho}_1, \tilde{\rho}_2, \tilde{g}_2) \in X_{m+1} \times K \times K$ and $j = 1, \ldots, n$. We calculate
\[
\partial_{\rho_1} G_0(0)[\tilde{\rho}_1] = \partial_{\rho_1} P F(0)[\tilde{\rho}_1] = \frac{1}{n} (\mathcal{N} - \text{Id}) \tilde{\rho}_1,
\]
\[
\partial_{\rho_2} G_0(0)[\tilde{\rho}_2] = \partial_{\rho_2} P F(0)[\tilde{\rho}_2] = 0,
\]
\[
\partial_{\rho_1} G_j(0)[\tilde{\rho}_1] = \int_S \tilde{\rho}_1 \sigma_j \, d\sigma = \omega_n^{1/2} \int_S \tilde{\rho}_1 h_{1,j} \, d\sigma = 0,
\]
\[
\partial_{\rho_2} G_j(0)[\tilde{\rho}_2] = \int_S \tilde{\rho}_2 \sigma_j \, d\sigma = \omega_n^{1/2} \int_S \tilde{\rho}_2 h_{1,j} \, d\sigma = \omega_n^{1/2} (\tilde{\rho}_2)_j,
\]
\[
\partial_{\rho_1} G_{n+1}(0)[\tilde{\rho}_1] = \partial_{\rho_1} (\text{Id} - P) F(0)[\tilde{\rho}_1] = 0, \quad \text{for } i = 1, 2.
\]

$\mathcal{N}$ again denotes the Dirichlet-to-Neumann operator. Recall that the linear operator $(\mathcal{N} - \text{Id})$ is bijective as an operator in $\mathcal{L}(X_{m+1}, X_m)$. For the $g_2$-partial derivative, we find
\[
\partial_{g_2} G_0(0)[\tilde{g}_2] = \partial_{g_2} P F(0)[\tilde{g}_2] = 0,
\]
\[
\partial_{g_2} G_j(0)[\tilde{g}_2] = 0 \quad \text{and}
\]
\[
\partial_{g_2} G_{n+1}(0)[\tilde{g}_2] = \partial_{g_2} (\text{Id} - P) F(0)[\tilde{g}_2] = \tilde{g}_2.
\]
This implies (3.4). We directly arrive at (3.5) and also at the regularity properties of $\partial_{\rho_1, \rho_2, g_2} G(0)^{-1}$.
4 A modified implicit function theorem

To arrive at the existence, uniqueness and at a stability result for Problem 2.1, we introduce a modified version of the implicit function theorem, Theorem 4.2. Because of the regularity issues stated in Remark 2.5, we are not able to apply the classical implicit function theorem. In preparation, we need

**Theorem 4.1** Assume the following.

(I) Let \( X_0, X_1, \mathcal{Y}, Z_0, Z_1 \) be Banach spaces with \( X_1 \hookrightarrow X_0 \) and \( Z_1 \hookrightarrow Z_0 \). Let \( D_1 \subset D_0 \) be open sets such that \( (0,0) \in D_j \subset X_j \times \mathcal{Y} \) for \( j = 0, 1 \).

(II) Let \( F \in C^1(D_1, Z_0) \cap C(D_0, Z_0) \) with \( F(0,0) = 0 \) and \( \partial_x F \in C(D_1, \mathcal{L}(X_0, Z_0)) \), which is to be understood such that for \( (x, y) \in D_1 \), the partial derivative \( \partial_x F(x, y) \in \mathcal{L}(X_1, Z_0) \) can be extended to \( \overline{\partial_x F}(x, y) \in \mathcal{L}(X_0, Z_0) \) and \( \overline{\partial_x F} \in C(D_1, \mathcal{L}(X_0, Z_0)) \).

(III) We have \( F: D_1 \rightarrow Z_1 \) and \( F \) is Fréchet-differentiable at \( (0,0) \), hence \( \partial_x F(0,0) \in \mathcal{L}(X_1, Z_1) \) and \( \partial_y F(0,0) \in \mathcal{L}(\mathcal{Y}, Z_1) \).

(IV) The inverse \( \partial_x F(0,0)^{-1} \in \mathcal{L}(Z_1, X_1) \cap \mathcal{L}(Z_0, X_0) \) exists.

Then there exist neighbourhoods of zero \( 0 \in U_0 \subset X_0, 0 \in U_1 \subset X_1 \) and \( 0 \in V \subset \mathcal{Y} \), as well as a function \( u: V \rightarrow U_0 \) such that

(i) \( F(u(y), y) = 0 \) for all \( y \in V, u(0) = 0 \), and

(ii) for \( x_1, x_2 \in U_1, y \in V \) such that \( F(x_i, y) = 0 \) for \( i = 1, 2 \), we have \( x_1 = x_2 \).

**Proof** Let \( \varepsilon, \delta > 0 \) – we will redefine both later – and define

\[
U_1 = \{ x \in X_1 \mid \|x\|_{X_1} \leq \varepsilon \},
\]

\[
U_0 = \{ x \in X_0 \mid \|x\|_{X_0} \leq C \varepsilon \},
\]

\[
V = \{ y \in \mathcal{Y} \mid \|y\|_{\mathcal{Y}} < \delta \},
\]

with \( C > 0 \) a constant satisfying \( \|x\|_{X_0} \leq C \|x\|_{X_1} \), thus \( U_1 \subset U_0 \).

Step 1: Show that for all \( y \in V \), the function

\[
\Phi_y(x) = x - \partial_x F(0,0)^{-1} F(x, y) = \partial_x F(0,0)^{-1} (\partial_x F(0,0)x - F(x, y))
\]

is a contraction mapping from \( (U_1, \|\cdot\|_{X_0}) \) to itself, provided that \( \varepsilon, \delta > 0 \) are sufficiently small.

As the fundamental theorem of calculus holds on Banach spaces as well, we have for \( F \in C^1(D_1, Z_0) \) and for all \( x_1, x_2 \in D_1 \)

\[
F(x_1, y) - F(x_2, y) = \int_0^1 \partial_x F(x_2 + t(x_1 - x_2), y)(x_1 - x_2) \, dt. \tag{4.1}
\]

Note that \( \partial_x F(x_2 + t(x_1 - x_2), y) \in \mathcal{L}(X_1, Z_0) \) with extension in \( \mathcal{L}(X_0, Z_0) \).

Now let \( x_1, x_2 \in U_1, y \in V \). Then \( (x_j, y) \in D_1 \) for \( j = 1, 2 \), and we use (4.1) to arrive at

\[
\Phi_y(x_1) - \Phi_y(x_2) = \partial_x F(0,0)^{-1} \left[ \int_0^1 (\partial_x F(0,0) - \partial_x F(x_2 + t(x_1 - x_2), y)) \, dt(x_1 - x_2) \right].
\]
By choosing $\varepsilon, \delta > 0$ smaller, if necessary, we get
\[
\|\Phi_y(x_1) - \Phi_y(x_2)\|_{X_0} \leq \|\partial_x F(0, 0)^{-1}\|_{L^\infty(\mathbb{L}(X_0, Z_0))} \|x_1 - x_2\|_{X_0}
\]
\[
\cdot \sup_{0 \leq t \leq 1} \|\partial_x F(0, 0) - \partial_x F(x_2 + t(x_1 - x_2), y)\|_{L^\infty(X_0, Z_0)}
\]
\[
\leq \frac{1}{2} \|x_1 - x_2\|_{X_0},
\]
where the second inequality holds because of the condition $\partial_x F \in C(D_1, L^\infty(X_0, Z_0))$ in assumption (II), which for sufficiently small $\varepsilon, \delta > 0$ implies
\[
\sup_{0 \leq t \leq 1} \|\partial_x F(0, 0) - \partial_x F(x_2 + t(x_1 - x_2), y)\|_{L^\infty(X_0, Z_0)} < 1.
\]

Next, we show that $\Phi_y(x) \in U_1$ for $(x, y) \in D_1$. We estimate $\|\Phi_y(x)\|_{X_1}$ for $(x, y) \in D_1$: Choosing $\delta = \delta(\varepsilon) > 0$ smaller, if necessary, we obtain
\[
\|\Phi_y(x)\|_{X_1} \leq \|\partial_x F(0, 0)^{-1}\|_{L^\infty(Z_1, X_1)} \|\partial_x F(0, 0) x - F(x, y)\|_{Z_1}
\]
\[
\leq \|\partial_x F(0, 0)^{-1}\|_{L^\infty(Z_1, X_1)} \|\partial_x F(0, 0)\|_{L^\infty(Y, Z_1)} \delta + o(\varepsilon + \delta(\varepsilon))
\]
\[
\leq \varepsilon.
\]

Step 2: Construct a mapping $u: U_0 \to V$.

Let $y \in V$ be arbitrary but fixed. The inductively defined sequence $(x_j)_{j \in \mathbb{N}}$ with $x_0 = 0$, $x_{j+1} = \Phi_y(x_j) \in U_1 \subset U_0$ for $j \in \mathbb{N}$ is a Cauchy sequence and thus converges in $\|\cdot\|_{X_0}$ to some $x_\infty \in U_0$. Because $F \in C(D_0, Z_0)$, this implies
\[
\|F(x_\infty, y)\|_{Z_0} = \lim_{j \to \infty} \|F(x_j, y)\|_{Z_0} \leq \lim_{j \to \infty} \|\partial_x F(0, 0)\|_{L^\infty(X_0, Z_0)} \|x_j - x_{j+1}\|_{X_0} = 0,
\]
where we used $F(x_j, y) = \partial_x F(0, 0)(x_j - x_{j+1})$ for $j \in \mathbb{N}$ by definition of $\Phi_y(x)$. We set $u(y) = x_\infty \in U_0$ for $y \in V$.

Step 3: Show (ii) of the theorem.

If $x_1, x_2 \in U_1$ and $y \in V$ with $F(x_j, y) = 0$ for $j = 1, 2$, then
\[
\|x_1 - x_2\|_{X_0} = \|\Phi_y(x_1) - \Phi_y(x_2)\|_{X_0} \overset{(4.2)}{\leq} \frac{1}{2} \|x_1 - x_2\|_{X_0},
\]
and therefore $x_1 = x_2$.

**Theorem 4.2** Assume the following.

(I) Consider Banach spaces $X_2 \leftrightarrow X_1 \leftrightarrow X_0$, $Z_2 \leftrightarrow Z_1 \leftrightarrow Z_0$ and $\mathcal{Y}$. Let $D_2 \subset D_1 \subset D_0$ be open sets such that $(0, 0) \in D_j \subset X_j \times \mathcal{Y}$ for $j = 0, 1, 2$.

(II) For $j = 1, 2$, let $F \in C^1(D_j, Z_{j-1}) \cap C(D_{j-1}, Z_{j-1})$ with $F(0, 0) = 0$ and further $\partial_x F \in C(D_j, L^\infty(X_1, Z_{j-1}))$. This is to be understood such that for $(x, y) \in D_j$, the partial derivative $\partial_x F(x, y) \in L^\infty(X_j, Z_{j-1})$ can be extended to $\partial_x F(x, y) \in L^\infty(X_j, Z_{j-1})$ and $\partial_x F \in C(D_j, L^\infty(X_1, Z_{j-1}))$.

(III) For $j = 1, 2$, the mapping $F: D_j \to Z_{j}$ is Fréchet-differentiable at $(0, 0)$.

(IV) For $j = 0, 1, 2$, the inverse $\partial_x F(0, 0)^{-1} \in L^\infty(Z_1, X_1)$ exists.

Then there exist neighbourhoods of zero $0 \in U_0 \subset X_0$, $0 \in U_1 \subset X_1$ and $0 \in V \subset \mathcal{Y}$ such that there is a function $u: V \to U_1$ satisfying

(i) $F(u(y), y) = 0$ for all $y \in V$, $u(0) = 0$,

(ii) if $x \in U_1$, $y \in V$ such that $F(x, y) = 0$, then $x = u(y)$, and

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(iii) it holds \( u \in C^1(V, \mathcal{X}_0) \), and

\[
u'(y) = -\partial_x F(u(y), y)^{-1} \partial_y F(u(y), y),
\]

with \( \partial_x F(u(y), y)^{-1} \in \mathcal{L}(\mathcal{Z}_0, \mathcal{X}_0) \) and \( \partial_y F(u(y), y) \in \mathcal{L}(\mathcal{Y}, \mathcal{Z}_0) \).

**Proof**

Step 1: Existence and uniqueness of \( u \).

Applying Theorem 4.1 twice, we arrive at the existence of neighbourhoods \( 0 \in V \subset \mathcal{Y}, 0 \in U_j \subset \mathcal{X}_j, j = 0, 1 \), and at the existence of a mapping \( u : V \to U_1 \) such that

- \( F(u(y), y) = 0 \) for \( y \in V, u(0) = 0 \), and
- for \( x_1, x_2 \in U_1, y \in V \) with \( F(x_i, y) = 0, i = 1, 2 \), we have \( x_1 = x_2 \).

Thus, for \( x_1 \in U_1 \) and \( y \in V \) such that \( F(x, y) = 0 \), we have \( x = u(y) \). This shows (i) and (ii) of Theorem 4.2.

Step 2: Show Lipschitz-continuity of \( u \), i.e. \( u \in C^{0,1}(V, U_0) \).

Consider \( y_1, y_2 \in V \). Then with \( u \) as above, i.e. \( u(y_i) \in U_1 \) for \( i = 1, 2 \), we have

\[
\|u(y_1) - u(y_2)\|_{\mathcal{X}_0} = \left\| \Phi_{y_1}(u(y_1)) - \Phi_{y_2}(u(y_2)) \right\|_{\mathcal{X}_0}
\leq \left\| \Phi_{y_1}(u(y_1)) - \Phi_{y_1}(u(y_2)) \right\|_{\mathcal{X}_0} + \left\| \Phi_{y_1}(u(y_2)) - \Phi_{y_2}(u(y_2)) \right\|_{\mathcal{X}_0}
\leq \frac{1}{2} \|u(y_1) - u(y_2)\|_{\mathcal{X}_0} + \left\| \partial_x F(0, 0)^{-1} \right\|_{\mathcal{L}(\mathcal{Z}_0, \mathcal{X}_0)} \|F(u(y_2), y_1) - F(u(y_2), y_2)\|_{\mathcal{Z}_0}.
\]

This implies

\[
\|u(y_1) - u(y_2)\|_{\mathcal{X}_0} \leq 2 \left\| \partial_x F(0, 0)^{-1} \right\|_{\mathcal{L}(\mathcal{Z}_0, \mathcal{X}_0)} + \sup_{0 \leq t \leq 1} \left\| \partial_y F(u(y_2), y_2 + t(y_1 - y_2)) \right\|_{\mathcal{L}(\mathcal{Y}, \mathcal{Z}_0)} \|y_1 - y_2\|_{\mathcal{Y}}
\leq L \|y_1 - y_2\|_{\mathcal{Y}},
\]

because the sup-term is uniformly bounded in \( V \) for sufficiently small \( \varepsilon, \delta > 0 \) defining the sets as in the proof of Theorem 4.1, since \( \partial_y F \) is continuous around \( 0, 0 \).

Step 3: Show \( u \in C^1(V, \mathcal{X}_0) \).

We have for \( y, h \in V \) s.th. \( y + h \in V \)

\[
0 = F(u(y + h), y + h) - F(u(y), y)
= F(u(y + h), y + h) - F(u(y + h), y) + F(u(y + h), y) - F(u(y), y)
= \partial_y F(u(y + h), y)[h] + o_{\mathcal{Z}_0}(\|h\|_{\mathcal{Y}}) + F(u(y + h), y) - F(u(y), y)
= \partial_y F(u(y), y)[h] + (\partial_y F(u(y + h), y) + \partial_y F(u(y), y)) [h] + o_{\mathcal{Z}_0}(\|h\|_{\mathcal{Y}})
+ F(u(y + h), y) - F(u(y), y)
= \partial_y F(u(y), y)[h] + o_{\mathcal{Z}_0}(\|h\|_{\mathcal{Y}})
+ \int_0^1 \partial_x F(u(y) + t(u(y + h) - u(y)), y) \, dt |u(y + h) - u(y)|,
\]

and further

\[
\int_0^1 \partial_x F(u(y) + t(u(y + h) - u(y)), y) \, dt |u(y + h) - u(y)|
= \partial_x F(u(y), y)[u(y + h) - u(y)] + o_{\mathcal{Z}_0}(\|h\|_{\mathcal{Y}}).
\]
Therefore,
\[ u(y + h) - u(y) = \partial_x F(u(y), y)^{-1} \partial_y F(u(y), y)[h] + o(|h|), \]
yielding \( u \in C^1(V, X_0) \) and (iii).

Note that \( \partial_x F(u(y), y) \in \mathcal{L}(X_0, \mathbb{R}) \) is invertible for \( y \in V \), since \( \partial_x F(0, 0) \) is invertible and \( \|u(y)\|_{X_1} + \|y\|_{\mathcal{Y}} << 1 \).

## 5 Proof of theorem 1.1

With the tool of the modified implicit function theorem, Theorem 4.2, at hand, we are now able to prove Theorem 1.1, that is, the existence and uniqueness of admissible sets \( \Omega_\rho \) with barycenter zero that solve the perturbed overdetermined problem (2.1), as well as a stability estimate.

**Remark 5.1** Considering the somewhat unintuitive partial derivative \( \partial_{\rho_1, \rho_2, g_2} \) in Lemma 3.3 was necessary to arrive at bijectivity and to be able to apply Theorem 4.2. The partial derivative \( \partial_{\rho} G(0) \) is not bijective.

In addition to that, keeping in mind the nature of the problem discussed in Sect. 3, the set \( \Omega_\rho \) will only depend on the perturbations that do not induce a mere translation of the problem. \( \rho \) depending on \( g_1 \) (instead of \( g \)) is a consequence of that setting.

**Proof** Theorem 1.1 We confirm the requirements for Theorem 4.2. For 4.2, we set
\[
\mathcal{X}_j = h^{j+2+\alpha}(S) \times X_2^j = h^{j+2+\alpha}(S) \times K
\]
and
\[
\mathcal{Z}_j = X_{j+1} \times \mathbb{R}^n \times X_2^j = X_{j+1} \times \mathbb{R}^n \times K
\]
for \( j = 0, 1, 2, \mathcal{Y} = X_2 \), and \( D_j \) accordingly. By Lemma 3.2, (3.2) and (3.3), 4.2 is satisfied. Lemma 3.3 implies 4.2 and 4.2.

Thus, Theorem 4.2 implies the existence of a neighbourhood \( 0 \in V \subset X_2 \) as well as neighbourhoods \( 0 \in U_j \subset h^{j+2+\alpha}(S) \times X_2^j = h^{j+2+\alpha}(S) \times K, j = 2, 3 \), such that there is a function \( (\rho, g_2) : V \to U_3 \) with \( G(\rho(g_1), g_1 + g_2(g_1)) = 0 \) for all \( g_1 \in V \) and \( (\rho, g_2)(0) = 0 \).

Further, \( (\rho, g_2) \) is unique in \( U_3 \) and we have \( (\rho, g_2) \in C^1(V, U_2) \). Differentiating \( G(\rho(g_1), g_1 + g_2(g_1)) = 0 \) with respect to \( g_1 \) and evaluating it at \( g_1 = 0 \) in direction \( \tilde{g}_1 \), we get
\[
0 = D_{g_1} G(\rho_1(g_1), \rho_2(g_1), g_1 + g_2(g_1)) \big|_{(0)}[\tilde{g}_1]
= D_{\rho_{1,2, g_2}} G(0) \left[ \partial_{g_1} \rho_1(0)[\tilde{g}_1], \partial_{g_1} \rho_2(0)[\tilde{g}_1], \partial_{g_1} g_2(0)[\tilde{g}_1] \right] + \partial_{g_1} G(0)[\tilde{g}_1]
= \begin{pmatrix}
\int_{S} \sigma_1 \partial_{g_1} \rho_1(0)[\tilde{g}_1] d\sigma \\
\vdots \\
\int_{S} \sigma_n \partial_{g_1} \rho_2(0)[\tilde{g}_1] d\sigma \\
\partial_{g_1} g_2(0)[\tilde{g}_1]
\end{pmatrix}
+ \begin{pmatrix}
\tilde{g}_1 \\
0 \\
0
\end{pmatrix}.
\]
This yields
\[
\partial_{g_1} \rho_1(0)[\tilde{g}_1] = \partial_{\rho_1} F(0)^{-1}[\tilde{g}_1],
\partial_{g_1} \rho_2(0)[\tilde{g}_1] = 0, \quad \text{and}
\partial_{g_1} g_2(0)[\tilde{g}_1] = 0.
\]
where the last equation results from \( \partial \bar{g}_1 \rho_2(0) [\tilde{g}_1] \in X_2^1 = K \), and we arrive at the stability estimates in (1.9).

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