I. INTRODUCTION

Historically, the investigation of SU(2) quantum spin chains has been a great source of inspiration for theoretical physics. Besides Bethe’s solution of the $S = 1/2$ spin chain, one of the most notable achievements in this context was Haldane’s discovery that the physical properties of the SU(2) Heisenberg model in one dimension depend crucially on the type of spin that is used. For half-integer spins the system can easily be shown to be gapless, while for integer spins it has been conjectured to develop a gap, the so-called Haldane gap, while preserving translation invariance. This conjecture triggered a tremendous amount of work before it could be confirmed for various values of the spin using analytical and numerical approaches, or even in experiment.

The gapped phase of SU(2) spin chains with unique translation invariant ground state which arises for integer spin representations is nowadays commonly called the Haldane phase. As has been understood only recently, it provides a paradigmatic example of a symmetry protected topological phase. This means that its ground state cannot be transformed into a trivial product state except if either the relevant symmetry is broken or the system undergoes a phase transition. The symmetries that qualify for the protection of the Haldane phase are SO(3) = SU(2)/$\mathbb{Z}_2$, its dihedral subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$, time-reversal or space inversion. The prototypical example for a Hamiltonian and a ground state which reside in the Haldane phase is provided by the AKLT construction, named after its protagonists Affleck, Kennedy, Lieb and Tasaki. It provides the first realization of what these days is commonly referred to as a matrix product state.

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The study of Haldane phases reaches a new level of complexity for spin systems with higher rank symmetries such as SU(N). In contrast to SU(2) there are now up to $N − 1$ Haldane phases in each of them the ground state, which is assumed to be unique, is characterized by a distinct $\mathbb{Z}_N$-valued non-zero topological quantum number. The latter can be measured in terms of a non-local string order parameter and it is also reflected in the nature of emergent fractionalized boundary spins which arise if the system is studied with open boundary conditions. Information about the topological class of the system can also be inferred by looking at certain Berry phases or at the behavior of specific $\mathbb{Z}_N \times \mathbb{Z}_N$ subgroups of SU(N) = SU(N)/$\mathbb{Z}_N$.

Since the classification of SU(N) Haldane phases and the proof of existence in Ref. were based on abstract principles, the article did not give a detailed prescription for how to realize them in the phase diagram of concrete spin models with specific physical SU(N) spins. The first systematic attempt to overcome this deficiency took place in Ref. where a one-dimensional cut through the phase diagram of a SU(3)-invariant system with spins transforming in the eight-dimensional adjoint representation was studied. Along this line of interpolation a topological phase transition was observed which connects the two distinct SU(3) Haldane phases with topological quantum number $\pm[1] \in \mathbb{Z}_3$. The phase diagram of this model was then further explored in Ref.

Another concrete proposal, now with SU(N) spins transforming in the so-called “self-conjugate” representation, was provided in Ref. The definition of this representation is bound to even values of $N$ and the associated Haldane phase is characterized by the topological quantum number $[N/2] < N$. Further investigations illuminated various complementary aspects of this model. However, what is still lacking to date is a systematic construction of SU(N) Haldane phases which is valid for general values of $N$.

The main goal of our current paper is to close this gap and to provide explicit realizations of the four elementary non-trivial Haldane phases with topological quantum numbers $\pm[1]$ and $\pm[2]$ (regarded as elements of $\mathbb{Z}_N$). Following the ideas of Ref. we choose to work with physical SU(N) spins transforming in the adjoint representation. We then construct AKLT states whose auxiliary spins transform in either the (anti-)fundamental or the rank-2 anti-symmetric representation and we determine the associated parent Hamiltonians. Moreover, we also confirm the existence of a mass gap. For the special case SU(4), our construction leads to a realization of the complete set of three Haldane phases in a single phase

Gapped quantum spin chains with symmetry PSU(N) = SU(N)/$\mathbb{Z}_N$ are known to possess $N$ distinct symmetry protected topological phases. Besides the trivial phase, there are $N − 1$ Haldane phases which are distinguished by the occurrence of massless boundary spins. Motivated by the potential realization in alkaline-earth atomic Fermi gases, we explicitly construct previously unknown Hamiltonians for two classes of chiral AKLT states and we discuss their physical properties. We also point out a deep connection between symmetry protection in gapped and gapless 1D quantum spin systems and its implications for a potential multicritical nature of topological phase transitions.
AKLT models for SU(N) attracted some attention \cite{21,23}, even before realizing the intimate connection to the developing field of symmetry protected topological phases.\cite{23} Also our current setup with an AKLT state based on the adjoint representation and with auxiliary spins in the (anti-)fundamental representation already received a thorough treatment in the past. However, the authors of Ref. \cite{21,23} and later also those of Ref. \cite{16} have been satisfied with writing down a Hamiltonian which exhibits spontaneous inversion symmetry breaking. In this case, there are two ground states which both possess a simple AKLT form. The two states are related by an exchange of auxiliary spins and correspondingly space inversion maps one into the other.

From the perspective of the general classification\cite{11} these systems, however, do not reside in a well-defined pure Haldane phase but rather in a superposition of two different ones which are distinguished by their topological quantum numbers. Our current analysis resolves this degeneracy by incorporating suitable terms in the Hamiltonian which explicitly break inversion symmetry while preserving SU(N). The precise form of the desired Hamiltonian is derived using a powerful diagrammatic method called “birdtracks”\cite{25} The same technique also gives straightforward access to the eigenvalues of the ground states’ transfer matrix which, in turn, determine entanglement spectrum, spin-spin correlation functions and correlation length.

Another important outcome of our diagrammatic approach is a new type of universal parent Hamiltonian. While not a projector in general, it has the great advantage of having an extremely simple graphical (tensor) form.

Our analysis paves the way to a more detailed study of the phase diagram of SU(N) spin chains, including topological phase transitions. According to a recent proposal by Furuya and Oshikawa, the notion of symmetry protection is not only restricted to gapped phases but can also be used to characterize critical points.\cite{25} Specifically it leads to selection rules which impose restrictions to the existence of potential renormalization group flows between conformal field theories. While initially analyzed for SU(2), this statement was recently extended to SU(N) by Lecheminant \cite{27} We will add a further layer of insight to this analysis by showing that topological phase transitions between symmetry protected topological phases may generally be forced to be described by multicritical points, just by symmetry considerations.

Deeply intertwined with the study of Haldane phases is the investigation of a potential generalization of Haldane’s Conjecture to SU(N) spin systems, see Ref. \cite{3} for the original conjecture, Ref. \cite{5} for a review and Refs. \cite{11,22,28} for more recent studies and proposals. As suggested in Ref. \cite{11} the SU(N) Heisenberg model should generally lead to a gapped phase if and only if the physical spin permits the construction of a symmetry protected topological phase. Otherwise it should be gapless. Our current analysis leads us to our own version of Haldane’s Conjecture and to an educated guess about the nature of the phase realized by the SU(4) Heisenberg Hamiltonian for adjoint representation.

Last but not least, our analysis should also be seen in connection with recent experimental progress on the realization of SU(N) magnetism in ultra-cold gases of fermionic alkaline-earth atoms.\cite{22,23} The adjoint representation we are using to realize our Haldane phases is considerably smaller than the self-conjugate representation suggested in Ref. \cite{17} It is therefore conceivable that it enjoys a better stability when realized in terms of elementary fermionic degrees of freedom and better experimental addressability. For a general discussion of one-dimensional SU(N) spin systems and their realization in the strong coupling limit of Fermi-Hubbard models we refer to a recent review.\cite{33}

Our paper is organized as follows. In Section \textbf{II} we introduce the physical setup, i.e. we specify the physical SU(N) spin we would like to study and we classify the terms which may enter a general SU(N)-invariant Hamiltonian with nearest neighbor interactions. The technical foundations of the paper are presented in Section \textbf{III} After outlining the general philosophy of the AKLT construction we discuss in detail which auxiliary spins are permitted if the physical spin is to transform in the adjoint representation. We then provide an exhaustive discussion of SU(N)-invariant two-site operators, with a particular focus on higher order spin-spin couplings. In order to discuss the properties of these operators we then introduce birdtracks, a convenient graphical calculus, which provides a simple way to determine their eigenvalues. Our main results are contained in Section \textbf{IV} Here we explicitly state the AKLT Hamiltonians which give rise to chiral Haldane phases with topological quantum number $\pm |1\rangle$ and $\pm |2\rangle$. This is followed by a brief discussion of the special cases SU(3) and SU(4).

In Section \textbf{V} we show how the graphical calculus may be used to determine the eigenvalues of the transfer matrix, resulting in a proof of the gapfulness of our AKLT states. A more detailed exploration of the phase diagram is provided in Section \textbf{VI} with a particular focus on critical phases. In this context we also discuss the nature of potential topological phase transitions and the implications of symmetries for the occurrence of multicritical points. In Section \textbf{VII} we present our idea of a universal parent Hamiltonian. This is then adapted to compute the precise form of the AKLT Hamiltonians under consideration. Our approach leads to some thoughts on Haldane’s Conjecture for SU(N) quantum spin models which are presented in Section \textbf{VIII} We conclude our paper with a brief summary and an outlook to future research.

\section{II. Physical Setup}

In this article, we will solely be concerned with spin chains whose spins transform in the adjoint representa-
The AKLT construction and the emergence of boundary spins. The ellipses denote the projection from $\mathcal{V} \otimes \mathcal{V}^*$ to $\mathcal{P}$, the links between neighboring physical spins refer to singlet bonds.

FIG. 1. (Color online) The AKLT construction and the emergence of boundary spins. The ellipses denote the projection from $\mathcal{V} \otimes \mathcal{V}^*$ to $\mathcal{P}$, the links between neighboring physical spins refer to singlet bonds.

\[ \theta = \begin{bmatrix} \mathbf{1} \end{bmatrix}^{N-1} . \]  

Let us first discuss the most general form of translation invariant nearest-neighbor Hamiltonians for this particular choice of spin. Restricting our attention to two physical sites for a moment, the Hilbert space $\mathcal{H} \otimes \mathcal{H}$ decomposes according to (sketch for $N = 5$)

\[ \mathcal{H} \otimes \mathcal{H} = \mathbf{1} \oplus 2 \mathbf{2} \oplus \mathbf{3} \oplus \mathbf{4} \oplus \mathbf{5} \oplus \mathbf{6} \oplus \mathbf{7} \oplus \mathbf{8} \oplus \mathbf{9} . \]  

The essential difference and technical complication of the present case as compared to previous investigations in the literature is the non-trivial multiplicity of the adjoint representation in the decomposition. As we will shortly review, the AKLT recipe for the construction of a parent Hamiltonian requires one to embed the tensor product of two auxiliary spins into the tensor product of two physical spins. Since the former includes the adjoint representation once, we encounter the mathematical problem of identifying the proper one-dimensional subspace of this two-dimensional space and of constructing the corresponding projection operator (or rather its orthogonal complement). If we choose the wrong embedding, the corresponding AKLT state fails to have the desired properties, e.g. it might be degenerate.

\[ \mathbf{X}^2 = \mathcal{P}_S + \mathcal{P}_A \quad \text{and} \quad \{ \mathbf{X}, \Pi \} = 0 . \]  

These equations state that $\mathbf{X}$ is acting non-trivially only in the subspace of the two adjoint representations where it exchanges the symmetric and the anti-symmetric part. In Section III C we construct explicit expressions for all the invariant operators $\mathcal{P}_i$ and $\mathbf{X}$ and express them in terms of local spin operators.

The attentive reader may have realized that we only talked about eight operators so far. The desired nine invariant operators are obtained if $\mathbf{X}$ is split into two nilpotent contributions according to $\mathbf{X} = \mathbf{X}_{SA} + \mathbf{X}_{AS}$ where the subscripts indicate which irreducible representation space is mapped into which. The separate parts $\mathbf{X}_{SA}$ and $\mathbf{X}_{AS}$, however, will not play a role in what follows.

Our group theoretical analysis implies that the most general SU(N) and translation invariant Hamiltonian with nearest-neighbor interactions has nine parameters. Rescaling and shifting the energy leaving us with effectively seven non-trivial coupling constants which could be parametrized by the coordinates of the sphere $S^7$. All our considerations will take place in the phase diagram described by this sphere. In particular, we ignore the possibility of staggered couplings or long-range interactions.

### III. REALIZATION OF HALDANE PHASES

#### A. General considerations

We shall now investigate whether the phase diagram under consideration permits the construction of non-trivial Haldane phases. With a Haldane phase we mean a gapped spin liquid phase of a 1D quantum spin system which exhibits non-trivial symmetry protected topological order. As was shown in Ref. 11 SU(N) spin systems can exhibit up to $N - 1$ distinct Haldane phases. In principle, all of them can be realized by using the ideas of the original AKLT construction. Based on this philosophy, examples of generalized AKLT Hamiltonians for SU(N) have been worked out in Refs. 10, 12, 16, 17, 20, 22, and 28.

The essential difference and technical complication of the present case as compared to previous investigations in the literature is the non-trivial multiplicity of the adjoint representation in the decomposition. As we will shortly review, the AKLT recipe for the construction of a parent Hamiltonian requires one to embed the tensor product of two auxiliary spins into the tensor product of two physical spins. Since the former includes the adjoint representation once, we encounter the mathematical problem of identifying the proper one-dimensional subspace of this two-dimensional space and of constructing the corresponding projection operator (or rather its orthogonal complement). If we choose the wrong embedding, the corresponding AKLT state fails to have the desired properties, e.g. it might be degenerate.

AKLT states in the adjoint representation with the correct embedding have first been discussed in Ref. 12 for the particular case of an SU(3) spin chain. In what follows, we will generalize the construction of Ref. 12.
from SU(3) to SU(N), and we will also construct AKLT states based on the auxiliary spin $\mathbb{I}$ and its dual.

Before entering the technical details let us briefly outline the main ingredients of a general AKLT construction for a symmetry group $G$.

1. The physical spins transform in a representation $\mathcal{P}$ which is assumed to be self-dual, $\mathcal{P}^\ast = \mathcal{P}$. Self-duality of the physical spin $\mathcal{P}$ is required for the construction of a translation invariant AKLT state.

2. Each physical spin $\mathcal{P}$ is thought to be made up from two auxiliary spins $\mathcal{V}$ and $\mathcal{V}^\ast$. In other words, $\mathcal{P} \subset \mathcal{V} \otimes \mathcal{V}^\ast$ arises from a suitable projection on the tensor product of the two auxiliary spins.

3. For technical reasons we also demand that $\mathcal{V} \otimes \mathcal{V}^\ast \subset \mathcal{P} \otimes \mathcal{P}$. This should be thought of as a kind of non-degeneracy condition.

These structures allow to write down an AKLT state $|\text{AKLT}\rangle$ in matrix product form which is constructed from the Clebsch-Gordan coefficients for the embedding $\mathcal{P} \subset \mathcal{V} \otimes \mathcal{V}^\ast$, see Figure 1 for an illustration. The state will be unique for an infinite system or if periodic boundary conditions are imposed. In contrast, with open boundary conditions the AKLT state will generally lead to two different AKLT states $|\text{AKLT}\rangle_1$ and $|\text{AKLT}\rangle_2$ as its ground state, then its image under inversion will also automatically be a ground state. In other words: There is a non-trivial ground state degeneracy and spontaneous inversion symmetry breaking. This mechanism provides a general explanation for the observations in the papers 16, 21–23.

### B. Auxiliary spins in general anti-symmetric representations of SU(N)

From now on we will assume the physical spin $\mathcal{P} = \theta$ to transform in the adjoint representation of SU(N). On the other hand, we will explore the possibility to employ different types of auxiliary spins $\mathcal{V}$ for the AKLT construction, notably those of the form

$$\mathcal{V} = \Sigma_l = \{l\} \quad \text{with} \quad \mathcal{V}^\ast = \Sigma_{N-l} = \{N-l\} ,$$

The representation $\Sigma_l$ corresponds to the anti-symmetric part of the $l$-fold tensor product of the fundamental representation with itself. For this type of auxiliary spin, the dual representation is given by $\Sigma^\ast_l = \Sigma_{N-l}$, i.e. it belongs to the same family of representations. For symmetry reasons we can thus restrict our attention to cases with $l \leq N/2$. As explained in Section III A, the remaining cases can be obtained by space inversion.

For the construction of the associated AKLT Hamiltonian we need to know the tensor product decomposition of auxiliary spins which, in this case, reads (for $1 \leq l \leq N/2$)

$$\Sigma_l \otimes \Sigma_{N-l} = 0 \oplus \theta \oplus \bigoplus_{r=2}^{l} \Lambda_r = \bullet \oplus \theta \oplus \bigoplus_{r=2}^{l} \{r\} ,$$

where the symbols $0$ and $\bullet$ both refer to the trivial one-dimensional representation and the representations $\Lambda_r$ are described by the Young tableau

$$\Lambda_r = N-r \{r\}.$$ 

Formally, we can identify $\Lambda_0 = 0$ and $\Lambda_1 = \theta$. The representations we just encountered are summarized in Table I together with their most important properties.
We exclude the case $l = 0$ since it violates condition 2, i.e. it fails to feature the physical representation $\theta$ in the tensor product $[6]$. We note that the tensor product decomposition $[6]$ has $l + 1$ contributions. In general it will thus not be possible to satisfy requirement 3, i.e. $\mathcal{V} \otimes \mathcal{V}^* \subset \mathcal{P} \otimes \mathcal{P}$, since the right hand side only features up to seven representations.

A closer inspection reveals that condition 3 is satisfied if and only if $l = 1$ or $l = 2$. We will thus restrict our attention to these two special cases in what follows which will be referred to as the fundamental and the rank-2 anti-symmetric representation, respectively. All other cases presumably suffer from unwanted ground state degeneracies. However, a more detailed analysis of this issue is beyond the scope of the present paper.

The case where $\mathcal{V} = \Sigma_2$ (the rank-2 anti-symmetric representation) is very similar. Here one simply needs to add one more projector $\mathbb{P}_{S_2}$, resulting in

$$\mathbb{P} = \mathbb{P}_\bullet + \cos^2(\theta) \mathbb{P}_S + \sin^2(\theta) \mathbb{P}_A + \frac{1}{2} \sin(2\theta) \mathbb{X} + \mathbb{P}_{S_2}.$$  

(10)

Of course, for this case the correct angle $\theta$ will generally be different than in the previous setup.

While giving the correct and complete answer for the Hamiltonian and while even being suitable for an implementation on the computer (see Appendix A), the invariant operators $\mathbb{P}_i$ and $\mathbb{X}$ arose from rather abstract considerations and are thus not very explicit. Another, physically more intuitive, basis of invariant operators is provided by so-called Casimir operators which are associated with invariant tensors.

The Hamiltonian expressed in terms of invariant operators $\mathbb{P}_i$ and $\mathbb{X}$, is quite simple and suitable for an implementation on the computer (see Appendix A). However it is physically more intuitive to use a basis of the so-called Casimir operators which are written in terms of the spins. In order to define the relevant tensors we fix an arbitrary basis $\mathbb{S}^a$ of anti-hermitean $su(N)$ generators, i.e. of traceless $N \times N$ matrices. These generators satisfy the commutation relations

$$[\mathbb{S}^a, \mathbb{S}^b] = f^{abc} \mathbb{S}^c.$$  

(11)

The first tensor we consider is the Killing form

$$\kappa^{ab} = \text{tr}(\mathbb{S}^a \mathbb{S}^b)$$  

(12)

and its inverse $\kappa_{ab}$. The tensors $\kappa^{ab}$ and $\kappa_{ab}$ can be used to raise and lower indices. We also introduce two distinct rank-3 tensors, namely

$$f^{abc} = \text{tr}(\mathbb{S}^a \{\mathbb{S}^b, \mathbb{S}^c\})$$  

(13)

and

$$d^{abc} = \text{tr}(\mathbb{S}^a \{\mathbb{S}^b, \mathbb{S}^c\}).$$  

(14)

By construction $d^{abc}$ is completely symmetric while $f^{abc}$ is completely anti-symmetric.

The two main examples of Casimir operators, and also the only ones that will be needed in our paper, are the quadratic and the cubic Casimirs

$$\mathbb{Q} = \mathcal{S}^2 = \kappa_{ab} \mathbb{S}^a \mathbb{S}^b \quad \text{and} \quad \mathbb{C} = d_{abc} \mathbb{S}^a \mathbb{S}^b \mathbb{S}^c.$$  

(15)

While $\mathbb{C}$ is identically zero for SU(2), it is a non-trivial invariant operator for all SU(N) with $N \geq 3$.

Let us now focus our attention on the two-site Hilbert space $\mathcal{P} \otimes \mathcal{P}$ for which the spin $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$ is the sum of the two individual spins. On the tensor product we define the invariant operators

$$\mathcal{C}_S = d_{abc} (\mathbb{S}_1^a \mathbb{S}_1^b \mathbb{S}_2^c + \mathbb{S}_1^a \mathbb{S}_2^b \mathbb{S}_1^c)$$  

$$\mathcal{C}_A = d_{abc} (\mathbb{S}_1^a \mathbb{S}_2^b \mathbb{S}_2^c - \mathbb{S}_1^a \mathbb{S}_1^b \mathbb{S}_2^c).$$  

(16)

| Table I. Some representations of SU(N) and their data. |  |
|---|---|---|---|
| Dimension | $\Sigma_l, \Sigma_r$ | $\mathcal{V}$ | $\mathcal{P}$ |
| $N$ | $N^1$ | $N^2 - 1$ | $\frac{(N-2l+1)N(N+1)}{(l(N+l-1))^2}$ |
| Casimir | $\mathcal{Q}$ | $\theta$ | $\Lambda_l$ |
| Tableau | $l$ | $\mathbb{P}$ | $\mathbb{X}$ |
| $l$ | $\begin{cases} \mathbb{P} & \mathbf{S}_1 \mathbf{S}_2 \\ \mathbb{X} & \mathbf{S}_1 \mathbf{S}_2 \end{cases}$ | $\begin{cases} \mathbb{P} & \mathbf{S}_1 \mathbf{S}_2 \\ \mathbb{X} & \mathbf{S}_1 \mathbf{S}_2 \end{cases}$ | $\begin{cases} \mathbb{P} & \mathbf{S}_1 \mathbf{S}_2 \\ \mathbb{X} & \mathbf{S}_1 \mathbf{S}_2 \end{cases}$ |

---

C. Invariant operators on two physical sites

By construction, the two-site AKLT Hamiltonian $[1]$ is an SU(N) invariant operator. We will express it in terms of the seven projectors $\mathbb{P}$, and the ‘permutation operator’ $\mathbb{X}$, see Table I for our labeling conventions. In terms of these operators and choosing $\mathcal{V} = \Sigma_1$ (the fundamental representation), the most general candidate for the projection onto the subspace $\mathcal{V} \otimes \mathcal{V}^* \subset \mathcal{P} \otimes \mathcal{P}$ is given by

$$\mathbb{P} = \mathbb{P}_\bullet + \cos^2(\theta) \mathbb{P}_S + \sin^2(\theta) \mathbb{P}_A + \frac{1}{2} \sin(2\theta) \mathbb{X}.$$  

(8)

Choosing a convenient basis in the subspace of adjoints, the last four terms can simply be written as the matrix $[32]$

$$\mathbb{P}_{ad} = \begin{pmatrix} \cos^2(\theta) & \frac{1}{2} \sin(2\theta) \\ \frac{1}{2} \sin(2\theta) & \sin^2(\theta) \end{pmatrix},$$  

(9)

thereby making explicit its nature as a rank-1 projector. In other words, the angle $\theta$ parametrizes possible one-dimensional subspaces in the two-dimensional space of adjoints. For the construction of the AKLT Hamiltonian we need to identify the proper value of the angle $\theta$. This problem will be solved in Section VII below.
We note that the last operator is anti-symmetric under the exchange of the two tensor factors. We also define

\[ K = \varepsilon_{abc} d_{def} S^a_1 S^b_1 S^c_2 S^d_2 S^e_1 S^f_1 . \]  

(17)

One may easily convince oneself of the relations

\[ Q = Q_1 + Q_2 + 2 \vec{S}_1 \cdot \vec{S}_2 = 4N + 2 \vec{S}_1 \cdot \vec{S}_2 , \]

\[ C = C_1 + C_2 + 3C_S = 3C_S . \]  

(18)

Here we used that \( Q_1 = Q_2 = 2N \) and \( C_1 = C_2 = 0 \) for the adjoint representation. The vanishing of \( C_1 \) and \( C_2 \) is simply a consequence of the self-duality of the adjoint representation.

A complete basis in the nine-dimensional space of invariant operators on the tensor product \( \mathcal{P} \otimes \mathcal{P} \) is provided by

\[ 1, Q, Q^2, Q^3, Q^4, C_S, C_A, K, \{K, C_A\} . \]  

(19)

The eigenvalues of these operators can be calculated using a graphical calculus (see Section III D) and they are summarized in Table II. This table explicitly states how these operators may be expressed in terms of the projectors \( \mathcal{P}_i \) and \( \mathcal{X} \). Implicitly, it thus also contains information about all the algebraic relations between these operators.

Let us briefly outline why the expressions in (19) indeed provide a linearly complete independent set of invariant operators. An inspection of Table II for instance implies the relation

\[ Q(Q - 2N)(Q - 4N)(Q - 4N - 4)(Q - 4N + 4) = 0 . \]  

(20)

This explains the possibility to restrict one’s attention to polynomials in \( Q \) of order less or equal to four. Other relations allow for the simplification of mixed products such as \( KQ \) as a sum over a multiple of \( K \) and a polynomial in \( Q \). In particular, one easily confirms that \( K C_S = C_S C_A = 0 \).

From Table II we can also read off the important relation

\[ K = 2N^2(N^2 - 4)P_1 - 8N^2P_S \]

\[ + 2N^2(N - 2)P_{S_1} - 2N^2(N + 2)P_{S_2} , \]

(21)

which may be solved for the projector \( P_S \), thereby giving rise to

\[ P_S = -\frac{1}{8N^2} K + \frac{N^2 - 4}{4} P_1 + \frac{N - 2}{4} P_{S_1} - \frac{N + 2}{4} P_{S_2} . \]  

(22)

We emphasize that \( Q, C_S \) and \( K \) are all symmetric under the exchange of \( S_1 \) and \( S_2 \). These operators hence preserve the symmetric and the anti-symmetric part of the tensor product \( \mathcal{P} \otimes \mathcal{P} \). It is thus clear that we will have to use the anti-symmetric operator \( C_A \) for the construction of the “permutation operator” \( \mathcal{X} \). Their precise relation turns out to be

\[ \mathcal{X} = \pm \frac{1}{2N\sqrt{N^2 - 4}} C_A . \]  

(23)

To prove this relation we observe that both sides of the equation anti-commute with the permutation operator \( \Pi \), and compute the square of each side. More details may be found at the end of the appendix.

### D. Birdtracks

The projectors \( [5] \) and \( [10] \) can easily be constructed once one knows the eigenvalues of the invariant operators \( S_1 \cdot S_2, C_A \) and \( K \) (see Table II). To compute the eigenvalues of these invariant operators, we use a diagrammatic method called birdtracks, which was (re)discovered and refined by Cvitanovic. His book contains the diagrammatic form of the projectors \( \mathcal{P}_i \) for the various irreducible representations. We convert \( K, C_A \) etc. into diagrams, and find the answer for general values of \( N \) after some manipulation.

Throughout the paper we draw the fundamental and the adjoint representation with straight and wavy lines, respectively. We will therefore think of the \( su(N) \) matrices \( T^a \) in the fundamental representation as a three-index object represented by the diagram

\[ (T^a)_{\alpha\beta} = \begin{array}{ccc} \alpha & \beta \end{array} \]  

(24)

The fundamental and the anti-fundamental representation are distinguished by the orientation of their arrow. Since the adjoint representation is self-conjugate, there is no arrow on the wavy lines. In the rest of this paper, we usually omit the indices – after all, this is the point of using diagrams in the first place.

As an illustration we wish to depict the defining property \( \text{tr}(T^a) = 0 \) and the normalization convention \( \text{tr}(T^aT^b) = \delta^{ab} \) in terms of diagrams. They simply read

\[ = 0 \quad \text{and} \quad = . \]  

(25)

Similarly, the definition \( [13] \) of the structure constants \( f^{abc} \) and the completely symmetric tensor \( a^{abc} \) translates into the pictures

\[ = \]  

(26)

\[ = + \]  

(27)

Here we represented the tensors \( f^{abc} \) and \( a^{abc} \) by a filled and an open circle, respectively.
Now imagine a complicated diagram composed of straight and wavy lines (the former with arrows). These so-called birdtracks can be simplified and evaluated by means of two elementary identities. The first of these identities is

\[ \frac{1}{N} \begin{array}{c} \downarrow \\ \circ \end{array} + \begin{array}{c} \downarrow \\ \circ \end{array} = 1 \]

(28)

It simply expresses the familiar fact that \( V \otimes V^* = 0 \oplus \theta \) and decomposes the identity operator on \( V \otimes V^* \) into the corresponding projectors onto the trivial (the trace part) and the adjoint representation. From the left relation in Eq. (25) we see that the prefactor \( N \) in Eq. (28) is required in order to ensure the correct trace of the identity operator on \( V \). This trace \( \text{tr}_V(1) = N \) has a graphical representation as a solid loop. The second of the fundamental identities

\[ - \begin{array}{c} \downarrow \\ \circ \end{array} = \begin{array}{c} \downarrow \\ \circ \end{array} \]

(29)

simply implements the Lie algebra commutation relations. Note that there is no equivalent of this property for \( d^{abc} \) (empty circle).

Let us illustrate the previous discussion with an example. A diagrammatic equation that will be used in later sections is

\[ \begin{array}{c} \downarrow \\ \circ \end{array} = \frac{N^2 - 1}{N} \begin{array}{c} \downarrow \\ \circ \end{array} = N^2 - 1 = \text{tr} \left( \begin{array}{c} \downarrow \\ \circ \end{array} \right) \]

(30)

It has two interpretations. The right hand side simply determines the trace in the adjoint representation while on the left hand side we calculate the trace of the quadratic Casimir operator in the fundamental representation.

As pointed out in Section II the tensor product \( P \otimes P \) admits a nine-dimensional space of invariant operators when \( P \) is the adjoint representation and \( N \geq 4 \). This space is spanned by the invariant operators associated with the diagrams

\[ \begin{array}{c} \downarrow \\ \circ \end{array} \quad \begin{array}{c} \downarrow \\ \circ \end{array} \quad \begin{array}{c} \downarrow \\ \circ \end{array} \]

(31)

On the top row we recognize the first three diagrams to be the identity, the permutation and the (unnormalized) projection onto the singlet. It can be shown that any birdtrack with four external wavy lines and any combination of internal wavy and straight lines can be reduced to a linear superposition of the previous diagrams. In particular, there are expressions for the projectors \( P \) and \( X \) and we list them in Appendix A for later convenience.

Since any SU(\( N \)) representation arises in a multiple tensor product of the fundamental representation with itself, the graphical rules considered so far are, in principle, sufficient to deal with arbitrary physical representations. In order to describe the rank-2 anti-symmetric representation we only need to be able to project into the anti-symmetric part of two fundamental representations. Graphically, the (anti-)symmetrization will be indicated as follows,

\[ \begin{array}{c} \downarrow \\ \circ \end{array} = \begin{array}{c} \downarrow \\ \circ \end{array} - \begin{array}{c} \downarrow \\ \circ \end{array}, \quad \begin{array}{c} \downarrow \\ \circ \end{array} = \begin{array}{c} \downarrow \\ \circ \end{array} + \begin{array}{c} \downarrow \\ \circ \end{array} \]

(33)

These two operations will project onto the representations \( \begin{array}{c} \downarrow \\ \circ \end{array} \) and \( \begin{array}{c} \downarrow \\ \circ \end{array} \) respectively. Any Young tableau can be represented by projectors in this way, but these two are all we need in this paper.

Inversion and Conjugation (equivalent to time-reversal) also have a diagrammatic interpretation. The former is reflection about a vertical line and the latter corresponds to changing the direction of every arrow. The two are not equivalent in general. For example, the horizontal ‘figure-8’ is invariant under one but not the other.

\[ \begin{array}{c} \downarrow \\ \circ \end{array} \quad \text{Inversion} \quad \begin{array}{c} \downarrow \\ \circ \end{array} \quad \text{Conjugation} \]

(34)

E. Eigenvalues of invariant two-site operators

Physical Hamiltonians are generally written in terms of spin operators. The spin operator in the adjoint representation looks like

\[ S_i^a = \begin{array}{c} \downarrow \\ \circ \end{array} ^a = - \begin{array}{c} \downarrow \\ \circ \end{array} ^a \]

(35)

where the sign comes from the anti-symmetry in the structure constants \( f^{abc} \), see Eq. (26). The invariant operators \( S_1 \cdot S_2, C_A, C_S \) and \( K \) have been introduced in Section III C. In diagrammatic form, some of the relevant expressions read
\[ -S_1 \cdot S_2 = \begin{array}{c}
\end{array} \]
\[ \frac{C_S - C_A}{2} = d_{abc}S_1^a S_2^b S_2^c = - \]
\[ \frac{C_S + C_A}{2} = d_{abc}S_1^a S_2^b S_2^c = \]
\[ \mathcal{K} = d_{abcde} S_1^a S_1^b S_2^c S_2^d S_2^e = \]

These diagrams can be expressed in the standard basis (31) using the fundamental identities (28) and (29). After simplification, we get sums over the terms listed in (32). The final expressions read

\[ C_S = 2N \left\{ \begin{array}{c}
\end{array} \right\} , \quad -S_1 \cdot S_2 = \left\{ \begin{array}{c}
\end{array} \right\} + \left\{ \begin{array}{c}
\end{array} \right\} \]
\[ C_A = 2N \left\{ \begin{array}{c}
\end{array} \right\} , \quad \mathcal{K} = -N^2 \left\{ 2 + 2 \right\} \]

\section{IV. AKLT Hamiltonians for the Adjoint Representation}

This section will be used to state the SU(N) AKLT Hamiltonians for physical spins transforming in the adjoint representation. We will distinguish two distinct cases with auxiliary spins in the fundamental and in the rank-2 anti-symmetric representation, respectively. Two additional setups arise by dualizing these representations which results in a simple change of sign in the original AKLT Hamiltonians. We finally comment on the special case SU(4) which is of particular interest since our construction gives access to the complete set of three Haldane phases in a single phase diagram.

\subsection{A. Haldane phases based on the fundamental representation}

As discussed in Section III A, the two-site interaction featuring the AKLT Hamiltonian is the projection onto the orthogonal complement of \( V \otimes V^* \) in \( \mathcal{P} \otimes \mathcal{P} \). For auxiliary spins in the fundamental representation (i.e. \( V = \bigotimes \)) the relevant tensor product decomposition reads

\[ V \otimes V^* = \bullet \oplus \theta . \]

This needs to be compared with the tensor product decomposition of \( \mathcal{P} \otimes \mathcal{P} \) which can be found in Table I.

Our goal is to express the projector \( \bigotimes \) in terms of spin operators and to determine the proper value of the angle \( \theta \). The latter problem is conveniently solved by means of the birdtrack calculus which may be used to determine the correct embedding of \( V \otimes V^* \) into the physical two-site Hilbert space \( \mathcal{P} \otimes \mathcal{P} \). A detailed analysis (to be found in Section VII) fixes the corresponding projection matrix \( \theta \) to be

\[ \mathcal{P}_{\text{ad}} = \frac{1}{2N^2 - 4} \left( \frac{N^2 - 4}{\pm N\sqrt{N^2 - 4}} \right) . \]

By resolving the matrix structure, this expression can be written explicitly as a combination of the projectors \( \mathcal{P}_S, \mathcal{P}_A \) and the “permutation operator” \( \chi \). We then immediately obtain the two-site Hamiltonian

\[ h = 1 - \left( \mathcal{P}_S + \frac{N^2 - 4}{2(N^2 - 2)} \mathcal{P}_A + \frac{N^2 - 4}{2(N^2 - 2)} \chi \right) . \]

Finding an expression for this Hamiltonian which purely involves the operators \( S_1 \cdot S_2, C_A \) and \( \mathcal{K} \) is a straightforward but lengthy exercise. Instead of giving details of the calculation we restrict ourselves to pointing out the general procedure. We begin with writing the identity operator \( \mathcal{I} \) as a sum over projectors and by splitting the contributions for \( \mathcal{P}_S \) and \( \mathcal{P}_A \) into those of \( \mathcal{P}_S \) and \( \mathcal{P}_S + \mathcal{P}_A \). This is advantageous since \( \mathcal{P}_S + \mathcal{P}_A \) has a simple expression in terms of \( \chi = 4N + 2S_1 \cdot S_2 \) (see Appendix A) while \( \mathcal{P}_S \) can be rewritten in terms of \( \chi \) using Eq. (22). After also replacing \( \chi \) by \( C_A \) using Eq. (23) one then simply needs to expand the polynomials in \( S_1 \cdot S_2 \) in order to arrive at

\[ h = 1 - \left( \mathcal{P}_S + \frac{N^2 - 4}{4(N^2 - 2)} \mathcal{K} + \frac{1}{4(N^2 - 2)} C_A + \alpha_1 S_1 \cdot S_2 \right) + \alpha_2 (S_1 \cdot S_2)^2 + \alpha_3 (S_1 \cdot S_2)^3 + \alpha_4 (S_1 \cdot S_2)^4 \]

with coefficients

\[ \alpha_1 = \frac{N(N^2 - 3)}{4(N^2 - 1)(N^2 - 2)} \quad \alpha_2 = - \frac{N^2 + 1}{4(N^2 - 1)(N^2 - 2)} \]

\[ \alpha_3 = \frac{N}{8(N^2 - 1)(N^2 - 2)} \quad \alpha_4 = \frac{1}{8(N^2 - 1)(N^2 - 2)} . \]
The AKLT Hamiltonian for the configuration with swapped auxiliary spaces, \( V \leftrightarrow V^* \), is identical to (43) except for a change of sign in front of \( C_A \). This change of sign indeed just reflects the effect of inverting the order of the sites along the chain since \( C_A \) is anti-symmetric under the exchange of \( S_1 \) and \( S_2 \).

### B. Haldane phases based on the anti-symmetric representation

The analysis for the case of auxiliary spins transforming in the anti-symmetric rank-2 tensor completely parallels the previous calculation. Superficially, the only difference is the presence of an additional projector \( P_{S_2} \) in Eq. (5). However, apart from the need to incorporate \( P_{S_2} \), the adjoint representation in \( V \otimes V^* \) is also embedded differently into \( \Omega \otimes \Omega \). A detailed analysis (see again Section VII) results in the projector

\[
P_{\text{ad}} = \xi \left(\begin{array}{cc}
(N+2)(N-4)^2 & \pm N(N-4)\sqrt{N^2-4} \\
\pm N(N-4)\sqrt{N^2-4} & N^2(N-2)
\end{array}\right)
\]

with \( \xi = 1/[2(N^2(N-4)+16)] \). The associated two-site Hamiltonian is given by the expression

\[
h = 1 - P_{\text{ad}} - \left(\frac{(N+2)(N-4)^2}{2(16+N^2(N-4))} \right) P_{S_2}
\]

As before, the Hamiltonian can be rewritten in terms of spin operators. After some lengthy calculations this leads to the final form

\[
h = 1 + \beta_1 K + \beta_2 C_A + \alpha_1 S_1 \cdot S_2 + \alpha_2 (S_1 \cdot S_2)^2 + \alpha_3 (S_1 \cdot S_2)^3 + \alpha_4 (S_1 \cdot S_2)^4
\]

with coefficients

\[
\beta_1 = -\frac{N^2-8}{4N^2(16+N^2(N-4))} \quad \beta_2 = -\frac{N-4}{4(16+N^2(N-4))} \\
\alpha_1 = \frac{N(2N^3+N^2-2N-12)}{8(NN+1)(16+N^2(N-4))} \quad \alpha_2 = -\frac{N^4-N^3-4N^2-2N+16}{8(NN+1)(16+N^2(N-4))} \\
\alpha_3 = \frac{3N^3+10N^2-20N-16}{16(NN+1)(16+N^2(N-4))} \quad \alpha_4 = -\frac{N^2+4N-8}{16(NN+1)(16+N^2(N-4))}
\]

Just as in the previous case, the exchange of auxiliary spaces, \( V \leftrightarrow V^* \), leads to the opposite sign in the contribution involving \( C_A \).

### C. Discussion of special cases

In this section we will discuss some peculiarities arising in the special cases SU(3) and SU(4). For SU(3) it will be demonstrated that we recover the Hamiltonian of Ref. 12. We will also explicitly analyze the case SU(4) which is the only instance where the system with auxiliary spins in the rank-2 anti-symmetric representation is invariant under space inversion.

#### 1. The case of SU(3)

For SU(3), the tensor product of the adjoint with itself reads

\[
P \otimes P = \mathbb{1} \oplus 2P_{A_1} \oplus P_{A_2} \oplus P_{S_1} \oplus P_{S_2}
\]
This decomposition is slightly different than the generic one for $N \geq 4$ stated in Eq. (2) since one of the seven representations, the representation $P_{S_{+}}$, is missing here. For the remaining six projectors and for the AKLT Hamiltonians we may nevertheless use the expressions that we derived for general values of $N$. In this case one may actually convince oneself that the two Hamiltonians (42) and (47) reduce to the same expression up to a sign in front of the operator $\mathcal{K}$ (or $C_{A}$), just as it should be.

A straightforward calculation yields the concrete form

$$h = 1 - \left( P_{\cdot} + \frac{5}{14} P_{S} + \frac{9}{14} P_{A} \pm \frac{3\sqrt{5}}{14} \mathcal{K} \right)$$

$$= 1 - \frac{1}{28} \mathcal{K} + \frac{a}{20} (\vec{S}_{1} \cdot \vec{S}_{2} - \frac{17}{420} (\vec{S}_{1} \cdot \vec{S}_{2})^2$$

$$- \frac{1}{172} (\vec{S}_{1} \cdot \vec{S}_{2})^3 \mp \frac{1}{28} C_{A}$$

for the two-site AKLT Hamiltonian. In contrast to general values of $N$, this Hamiltonian only features spin-spin couplings up to third order in $\vec{S}_{1} \cdot \vec{S}_{2}$. This is due to an additional fourth order relation

$$\vec{S}_{1} \cdot \vec{S}_{2} (\vec{S}_{1} \cdot \vec{S}_{2} - 2)(\vec{S}_{1} \cdot \vec{S}_{2} + 3)(\vec{S}_{1} \cdot \vec{S}_{2} + 6) = 0$$

which arises from the absence of the seventh representation in the tensor product (see Table I). Needless to say, our Hamiltonian (51) perfectly matches the result obtained earlier in Ref. [12]. In order to compare with the results of Ref. [12] one needs the identifications $C_{A} = 8C_{a}$ and $K = 16C_{a}$. All which follow from the respective tables for the eigenvalues.

2. The case of $SU(4)$

For $SU(4)$, the physical two-site Hilbert space decomposes as

$$P \otimes P = \cdot \oplus 2 \cdot \oplus \hat{O} \oplus \hat{O} \oplus \hat{O} \oplus \hat{O} \oplus \hat{O} \oplus \hat{O}.$$  \hfill (53)

On the level of the auxiliary representations one can only reach

$$\hat{O} \otimes \hat{O} = \cdot \oplus \hat{O}$$

or

$$\hat{O} \otimes \hat{O} = \cdot \oplus \hat{O} \oplus \hat{O} \oplus \hat{O} \oplus \hat{O} \oplus \hat{O} \oplus \hat{O}. \hfill (54)$$

We note that the second option gives rise to an inversion symmetric AKLT state. In both cases, the contributions are contained in the two-site Hilbert space $P \otimes P$.

It remains to simplify the general expressions (43) and (48) to the cases at hand. The simplified two-site Hamiltonians read

$$h = 1 - \frac{C_{A}}{56} - \frac{\mathcal{K}}{896} + \frac{13}{210} (\vec{S}_{1} \cdot \vec{S}_{2} - \frac{17}{420} (\vec{S}_{1} \cdot \vec{S}_{2})^2$$

$$+ \frac{1}{420} (\vec{S}_{1} \cdot \vec{S}_{2})^3 \mp \frac{1}{1680} (\vec{S}_{1} \cdot \vec{S}_{2})^4$$

for an auxiliary spin in the anti-symmetric representation and

$$h = 1 - \frac{1}{128} \mathcal{K} + \frac{11}{20} (\vec{S}_{1} \cdot \vec{S}_{2} - \frac{7}{20} (\vec{S}_{1} \cdot \vec{S}_{2})^2$$

$$- \frac{1}{3} (\vec{S}_{1} \cdot \vec{S}_{2})^3 \mp \frac{1}{100} (\vec{S}_{1} \cdot \vec{S}_{2})^4$$

for an auxiliary spin in the anti-symmetric representation.

We recognize that the second Hamiltonian does not feature the term $C_{A}$ which is anti-symmetric under the exchange of the two spins $\vec{S}_{1}$ and $\vec{S}_{2}$. As a consequence, the resulting AKLT Hamiltonian is inversion symmetric. This property is related to the self-duality of the representation $\mathcal{V}$ which holds if and only if $N = 4$. We note that the Hamiltonians (55) and (56) realize the full set of three Haldane phases for $SU(4)$ if one takes into account the possibility to reverse the sign in front of $C_{A}$ in Eq. (55) which exchanges the roles of $\mathcal{V}$ and $\mathcal{V}^*$. 

V. Correlation Lengths

One of the characteristics of a physical system is the existence or absence of a gap. We will now show that both models under investigation are gapped by proving the existence of exponentially decaying correlations. This will be done by diagonalizing the associated transfer matrix and finding a gap between the largest (by absolute value) two eigenvalues.

A. Fundamental representation

The transfer matrix $E$ associated with our first AKLT state has the following simple graphical representation

$$E = \begin{array}{c} \hline \hline \hline \hline \hline \hline \hline \hline \end{array} \begin{array}{c} \hline \hline \hline \hline \hline \hline \hline \hline \end{array} \begin{array}{c} \hline \bigotimes \hline \end{array} \begin{array}{c} \hline \hline \hline \hline \hline \hline \hline \hline \end{array} \begin{array}{c} \hline \hline \hline \hline \hline \hline \hline \hline \end{array} \begin{array}{c} \hline \hline \hline \hline \hline \hline \hline \hline \end{array} \begin{array}{c} \hline \hline \hline \hline \hline \hline \hline \hline \end{array} \begin{array}{c} \hline \hline \hline \hline \hline \hline \hline \hline \end{array} \end{array}.$$  \hfill (57)

The matrix $E$ is a tensor with four indices and can, in principle, be interpreted in a variety of ways. In the context of calculating the norm of a state or correlation functions it is, however, convenient to read this diagram from left to right, i.e. to interpret $E$ as an operator on the space $\mathcal{V} \otimes \mathcal{V}$.

In the case of $\mathcal{V} = \hat{O}$ being the fundamental representation, this tensor product decomposes as $\mathcal{V} \otimes \mathcal{V}^* = 0 \oplus \theta$ and hence the transfer matrix possesses a decomposition in terms of projectors as $E = c_{1}P_{1} + c_{2}P_{2}$. The numbers $c_{1}$ and $c_{2}$ are the eigenvalues of the transfer matrix which determine the correlation length. In terms of our graphical calculus, this equation simply translates into

$$= c_{1} \frac{1}{N} \bigotimes P_{1} = c_{2} \bigotimes \bigotimes.$$

where the (normalized) diagrams correspond to the projectors $P_{0}$ and $P_{\theta}$ (compare with Eq. (28)). The eigenvalue $c_{1}$ can be determined by sandwiching both sides of the equation between two right/left cups. According to Eq. (25) this renders the rightmost diagram trivial while
Eq. (30) implies the value $c_1 = N - 1/N$. Similarly, one finds a relation between $c_1$ and $c_2$ by sandwiching the original equation with two cups from the top/bottom. We refer to these two operations as the horizontal and vertical trace, respectively. Eventually we find the eigenvalues $c_1 = N - 1/N$ and $c_2 = -1/N$ with multiplicities 1 and $N^2 - 1$ respectively. We note the strict inequality $|c_1| > |c_2|$ which proves the desired spectral gap. From this simple and intuitive calculation we can infer the correlation length as

$$\xi = 1/\ln(|c_1/c_2|) = 1/\ln(N^2 - 1). \quad (59)$$

This result confirms earlier computations by various groups[10,23,35,39].

### B. Anti-symmetric representation

While a neat exercise for the fundamental representation, the full power of the graphical calculus becomes visible when considering more complicated auxiliary spins such as the anti-symmetric representation $V^\otimes V$. In this case, the tensor product $V \otimes V^\star = 0 \oplus \otimes \oplus P_{S_2}$ decomposes into three irreducible representations and the associated transfer matrix reads $E = c_1 P_1 + c_2 P_2 + c_3 P_3$. When interpreted with the help of birdtracks, this equation reads

\[
\begin{align*}
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{birdtrack}\nonumber
\end{array}
\end{align*}
\]

\[
\begin{align*}
&= c_1 \frac{2}{N(N-1)} \begin{array}{c}
\includegraphics[width=0.5\textwidth]{birdtrack}\nonumber
\end{array} + c_2 \frac{1}{N-2} \begin{array}{c}
\includegraphics[width=0.5\textwidth]{birdtrack}\nonumber
\end{array} + c_3 \alpha(N) \begin{array}{c}
\includegraphics[width=0.5\textwidth]{birdtrack}\nonumber
\end{array}.
\end{align*}
\]

Here, we introduced the dotted double line as a new notation for the representation $P_{S_2}$ and $\alpha(N)$ is a normalization constant whose precise value will not be of any concern. Again we need to compute the eigenvalues $c_i$. Using the fact that the (horizontal) trace is zero, we may do so without needing the exact form of the last vertex or its normalization constant $\alpha(n)$. The eigenvalues are

$$c_1 = \frac{2(N+1)}{N}, \quad c_2 = \frac{N^2 - 2N - 4}{N(N-2)} \quad \text{and} \quad c_3 = \frac{4}{N(N-2)}. \quad (61)$$

We can now extract the correlation length which is the ratio of the next highest eigenvalue to the highest. We find

$$\xi = 1/\ln\left[\frac{N^2 - 2N - 4}{2(N+1)(N-2)}\right]. \quad (62)$$

We note that this correlation length (in contrast to $\xi$) has a non-trivial finite value in the limit $N \to \infty$.  

### VI. PHASE DIAGRAM

The two AKLT Hamiltonians we discussed in Sections [VI.A] and [VI.B] arise from fine-tuning several coupling constants such as to clearly exhibit the topological properties of the respective Haldane phases. It is a natural question which other Haldane phases can possibly be realized in the full space of SU(N) invariant Hamiltonians and where the phase boundaries are located. In particular, it appears natural that the topological phase transitions are described in terms of a critical conformal field theory (CFT).

#### A. Critical point with enhanced symmetry

While a general discussion of phase transitions is beyond the scope of the present paper we wish to point out that the phase diagram features a critical point with enhanced SU(N) symmetry where we introduced the abbreviation $\mathcal{N} = N^2 - 1$. This point is associated with a two-site Hamiltonian which degenerates into the permutation operator or, equivalently, to the projection onto the symmetric part of the two-site Hilbert space. This Hamiltonian can effectively be thought of as an integrable Heisenberg Hamiltonian for the fundamental representation of SU(N) whose dimension coincides with the dimension of the adjoint representation of SU(N). If $\Pi$ denotes the permutation operator on two physical sites and $\bar{T}$ the spin operator of SU($\mathcal{N}$), then the corresponding two-site Hamiltonian can be written as

$$h = \frac{1}{2} \left[ \mathbb{1} + \Pi \right] = \frac{1}{2} \left[ \bar{T}_1 \cdot \bar{T}_2 + \frac{N+1}{N} \mathbb{1} \right]. \quad (63)$$

As is well known, the associated universality class is described by a SU(N) WZW model at level $k = 1$. In terms of SU(N) projectors, the Hamiltonian has the form $h = P_\bullet + P_S + P_{S_1} + P_{S_2}$ which can easily be converted into the simple expression

$$h = -\frac{1}{8N^2} \mathbb{1} + \frac{N}{8} \bar{S}_1 \cdot \bar{S}_2 + \frac{1}{8} (\bar{S}_1 \cdot \bar{S}_2)^2 \quad (64)$$

using the spin operators $\bar{S}$ of SU(N).

#### B. Other integrable points

It is conceivable that there are other Bethe ansatz solvable Hamiltonians in the phase diagram of our model. A systematic discussion of integrable SU(N) Hamiltonians based on completely symmetric representations with $m$ boxes has been provided by Andrei and Johansson[41,42,43,44]. From the thermodynamic Bethe ansatz it could be inferred that these systems are critical and in the universality class of the SU(N) WZW model at level $k = m$. In principle, the approach of fusing the R-matrix of the fundamental representation to obtain integrable models for higher spins can be generalized to the adjoint representation. However, since the procedure is quite technical we will leave this analysis to future work. We believe that the corresponding theory flows to the critical SU(N) WZW model at level $k = N$. 
C. Multicritical points from symmetry protection

The universality classes of critical SU(N) quantum spin systems in one dimension are provided by SU(N) WZW models. These WZW models are parametrized by a number $k = 1, 2, \ldots$, commonly known as the level. The level can be thought of as a measure for the degree of multicriticality. In general, WZW models with level $k$ can be thought of as a measure for the degree of multicriticality. Specifically, these WZW models have a $Z_2$ symmetry which may be gauged for even values of the level (turning the SU(2) WZW model into an SO(3) WZW model) but which is anomalous for all odd values of $k$. As a consequence, the level is preserved modulo 2 if a renormalization group flow is triggered by relevant operators respecting this $Z_2$-symmetry. This means that there are in fact two stable fixed-points, namely $k = 1$ and $k = 2$. Here, the stability of the SU(2)$_2$ WZW model requires an additional $Z_2$ symmetry to be preserved.

The arguments of Ref.~26 can easily be generalized to other symmetry groups using the general classification of orbifold WZW theories. For the group of interest for us, namely SU(N), the corresponding analysis has been carried out in Ref.~27. In that case, the discrete symmetry which needs to be gauged is the group $Z_N$, the center of SU(N). This effectively turns the group SU(N) into the group PSU(N) = SU(N)/$Z_N$.

Let us now try a change of perspective. So far, the discussion in the literature was solely concerned with the stability of gapless phases and with selection rules on renormalization group flows between conformal field theories. Here, we would like to point out that the same arguments also have strong implications for the nature of topological phase transitions. More precisely, we will argue that the phase transitions between symmetry protected topological phases of SU(N) lattice models may correspond to multicritical points, i.e. to higher level SU(N) WZW models, under specific circumstances.

To explain the basic philosophy underlying our claim we first look at the familiar example of the spin-1 Haldane phase for SU(2). Since the spins transform in the spin-1 representation, the subgroup $Z_2 \subset$ SU(2) is acting trivially, turning the actual symmetry into SO(3) = SU(2)/$Z_2$. In other words, the model has an additional $Z_2$ symmetry (or rather redundancy) which needs to be present everywhere in the phase diagram as long as no extra degrees of freedom with half-integer spin are added. This $Z_2$-invariance then of course should also exist in the CFT describing the topological phase transition from the Haldane phase to the dimerized phase. And indeed, this transition is coinciding with the integrable Babujan-Takhtajan model which is well-known to be described by an SU(2)$_2$ WZW model (see Ref.~45 and references therein). In the previous argument we silently skipped over a subtle point. In fact, as explained in Ref.~45 the $Z_2$ symmetry is only present in the lattice model if the Hamiltonian is translation invariant. If it is broken, the transition from the Haldane phase to the dimerized phase is actually described by means of a SU(2)$_1$ WZW model.

In our present paper we deal with a translation invariant SU(N) spin chain with physical spins in the adjoint representation. Since the associated Young tableau has $N$ boxes, the central subgroup $Z_N \subset$ SU(N) is actually acting trivially and the actual symmetry group is PSU(N) = SU(N)/$Z_N$ (see 11). This means that the group $Z_N$ should be anomaly free in the SU(N) WZW model describing potential phase transitions. A careful inspection of these models shows that this is the case for all $k$ if $N$ is odd while $k$ is required to be even if $N$ is even. We are immediately led to the conjecture that the 2nd order phase transitions in the phase diagram of our model are generically described by SU($N$)$_1$ (for odd $N$) and SU($N$)$_2$ (for even $N$), respectively. Larger values of $k$ can be imagined if the microscopic system by chance does not feature certain relevant operators. The latter can not be excluded solely based on symmetry considerations. The same kind of argument also predicts a value of $k = 2$ for topological phase transitions in translation invariant SU(N) spin systems which are based on the self-conjugate representation of Ref.~17,20 since the definition of the latter requires $N$ to be even.

Of course, the idea just presented may be generalized to other continuous symmetry groups $G$ beyond SU(N). In order to fully appreciate the generality of our claims it is important to note that the existence of symmetry protected topological phases crucially relies on (part of) the central subgroup of $G$ acting trivially on the physical representation. It is remarkable that the mathematical structures governing potential anomalies in critical theories with continuous symmetry group and the classification of symmetry protected topological phases are related in such a deep fashion.

Let us finally reiterate that considerations very similar to the ones above have already appeared before in the literature. The focus of these works, however, was the stability of gapless phases. To our knowledge the connection to gapped symmetry protected topological phases and (topological) phase transitions has not been stressed so far.

D. Spontaneous inversion symmetry breaking

In order to obtain unique ground states we deliberately chose to add inversion symmetry breaking terms to our AKLT Hamiltonians. Of course one could also simply follow the alternative recipe of Refs.~10,22,23, where inversion symmetry in the Hamiltonian is restored at the
cost of having a two-fold degenerate ground state. The corresponding Hamiltonians should be of the form

\[ h = 1 - (P_{\Phi} + P_{S} + P_{A}) \]

\[ = 1 - 3\gamma N(5N^2 - 4)[4\vec{S}_1 \cdot \vec{S}_2 - (\vec{S}_1 \cdot \vec{S}_2)^3] \]

\[ - \gamma(7N^2 - 4)[4(\vec{S}_1 \cdot \vec{S}_2)^2 - (\vec{S}_1 \cdot \vec{S}_2)^4] \]

for an auxiliary spin in the (anti-)fundamental representation and, similarly,

\[ h = 1 - (P_{\Phi} + P_{S} + P_{A} + P_{S_2}) \]

\[ = 1 + 4N\delta(N^3 + 6N^2 + 18N + 12)\vec{S}_1 \cdot \vec{S}_2 \]

\[ - 2\delta(N^4 + 3N^3 - 12N - 8)(\vec{S}_1 \cdot \vec{S}_2)^2 \]

\[ - \delta N(3N^2 + 16N + 12)(\vec{S}_1 \cdot \vec{S}_2)^3 \]

\[ - \delta(N^2 + 6N + 4)(\vec{S}_1 \cdot \vec{S}_2)^4 \]

for an auxiliary spin in the anti-symmetric representation. Here, the constants \( \gamma \) and \( \delta \) are defined by

\[ \gamma = 1/[8N^2(N^2 - 1)(N^2 - 4)] \quad \text{and} \]

\[ \delta = 1/[16N^2(N + 1)(N + 2)]. \]

Hamiltonians with lower order in \( \vec{S}_1 \cdot \vec{S}_2 \) can be obtained by simply projecting out the unwanted contributions without paying attention to the relative normalization of the remaining projectors. While the lower order variant of the first expression has been known for a quite a while, the second Hamiltonian (and also the associated lower order variant) is new, at least to the best of our knowledge.

VII. A UNIVERSAL PARENT HAMILTONIAN

We now present a general strategy for the construction of a parent Hamiltonian for an AKLT state. The auxiliary spins \( \mathcal{V} \) and \( \mathcal{V}^* \) as well as the physical spin \( \mathcal{P} \) are assumed to be arbitrary until further notice as long as the conditions of Section VI A are satisfied. After outlining the general case we specialize to the case of interest for us.

A. General strategy

Our basic idea is to start with a natural Hamiltonian on the auxiliary level which is then projected onto the physical level. More precisely, the fundamental object is the projector

\[ h_{\text{aux}} = 1 - \frac{1}{d(\mathcal{V})} \]

\[ \mathcal{V} \quad \mathcal{V}^* \]

\[ \mathcal{V} \quad \mathcal{V}^* \]

\[ \mathcal{V} \quad \mathcal{V}^* \]

\[ \mathcal{V} \quad \mathcal{V}^* \]

\[ \mathcal{V} \quad \mathcal{V}^* \]

on \((\mathcal{V} \otimes \mathcal{V}^*) \otimes (\mathcal{V} \otimes \mathcal{V}^*)\), where \(d(\mathcal{V})\) denotes the dimension of the space \(\mathcal{V}\). Looking at Figure 1, the AKLT state on four auxiliary (two physical) sites has the spins in the middle joined into a singlet, while the boundary spins are free. Hence it is clear that it is annihilated by the above operator.

The auxiliary Hamiltonian needs to be projected onto the physical subspace space \(\mathcal{P} \otimes \mathcal{P} \subset (\mathcal{V} \otimes \mathcal{V}^*) \otimes (\mathcal{V} \otimes \mathcal{V}^*)\) using

\[ \mathcal{P}_{\text{phys}} = \]

\[ \mathcal{V} \quad \mathcal{V}^* \]

\[ \mathcal{V} \quad \mathcal{V}^* \]

\[ \mathcal{V} \quad \mathcal{V}^* \]

\[ \mathcal{V} \quad \mathcal{V}^* \]

\[ \mathcal{V} \quad \mathcal{V}^* \]

This projection results in the operator

\[ \mathcal{P}_{\text{phys}} h_{\text{aux}} \mathcal{P}_{\text{phys}} = 1_{\text{phys}} - \]

\[ \mathcal{V} \quad \mathcal{V}^* \]

\[ \mathcal{V} \quad \mathcal{V}^* \]

\[ \mathcal{V} \quad \mathcal{V}^* \]

\[ \mathcal{V} \quad \mathcal{V}^* \]

\[ \mathcal{V} \quad \mathcal{V}^* \]

Of course, what we really need is an operator which acts on the physical two-site Hilbert space \(\mathcal{P} \otimes \mathcal{P}\). Simply removing the fringes, gives

\[ \tilde{h}_{\text{AKLT}} = 1 - \frac{1}{d(\mathcal{V})} \]

\[ \mathcal{V} \quad \mathcal{V}^* \]

\[ \mathcal{V} \quad \mathcal{V}^* \]

\[ \mathcal{V} \quad \mathcal{V}^* \]

\[ \mathcal{V} \quad \mathcal{V}^* \]

\[ \mathcal{V} \quad \mathcal{V}^* \]

Our central claim is that the above construction provides a general form of the AKLT Hamiltonian on two physical sites. Note that \(71\) only requires knowledge of the ‘gluon vertex’ , which encodes how the physical spin can be written in terms of the boundary spins.

Let us discuss the properties of the operator \(71\). First, we note that \(\tilde{h}_{\text{AKLT}}\) as defined in \(71\) is not necessarily a projector. This property, usually deemed necessary, of AKLT Hamiltonians may indeed be lost during the embedding of the physical space \(\mathcal{P}\) into \(\mathcal{V} \otimes \mathcal{V}^*\). To be precise, \(\tilde{h}_{\text{AKLT}}\) is a sum of projectors into the constituent representations of \(\mathcal{P} \otimes \mathcal{P}\), with non-negative but (generally) distinct coefficients. Nevertheless, taken on a finite chain we find the same ground state which along with a gap, indicates that we are in the same topological phase. A way of ‘renormalizing’ the projectors’ coefficients back to their standard value 1 will be outlined after the next paragraph.

Furthermore, the Hamiltonian \(71\) is generally not self-conjugate. Conjugation amounts to reversing each arrow and physically leads to a parent Hamiltonian for another AKLT state which can be thought of as the image under inversion or time-reversal. Only if the auxiliary representation \(\mathcal{V} = \mathcal{V}^*\) is self-dual, will the associated AKLT state be invariant under these transformations.
This is the case for example, in the original AKLT state for SU(2) and for the AKLT states suggested in Ref. [17]. In contrast, the Hamiltonians considered in the present paper are all chiral, with the sole exception of SU(4), as discussed below.

As the bulk of the paper deals with the Hamiltonians written in terms of projectors we do the same for (71). As before, the Hilbert space for two physical sites as

\[ \mathcal{P} \otimes \mathcal{P} = (\mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots) \oplus \text{rest}, \]  

(72)

where we distinguish the irreducible representation that are also in \( \mathcal{V} \otimes \mathcal{V}^* \) from the ‘rest’. If the \( \mathcal{P}_i \) were distinct then we could write

\[ \hat{h}_{\text{AKLT}} = 1 - (c_1 \mathcal{P}_1 + c_2 \mathcal{P}_2 + \cdots) \]  

(73)

and normalizing each of the \( c_i \) to 1 gives a true projector if needed. Indeed, this is nothing but the usual prescription of projecting out the boundary spins from the space of two physical spins.

However, for the cases that are of chief interest in our paper, i.e. when \( \mathcal{P} \) is the adjoint representation, we have a non-trivial multiplicity and the right hand side of (73) is a block matrix where the diagonal contains projectors and the off-diagonal entries permute different copies of the degenerate representation. These coefficients may be computed using (71) and some diagrams, to which we now turn.

B. Computation for the fundamental representation

For the rest of this subsection \( \mathcal{P} = \) will always be the adjoint and \( \mathcal{V} = \) the fundamental representation, with dimensions \( N^2 - 1 \) and \( N \) respectively. We have

\[ \mathcal{V} \otimes \mathcal{V} = 0 \oplus \mathcal{P} \]  

\[ \mathcal{P} \otimes \mathcal{P} = \mathcal{P}_* \oplus \mathcal{P}_S \oplus \mathcal{P}_A + \text{rest}. \]  

(74)

Here \( \mathcal{P}_S \) and \( \mathcal{P}_A \) are two copies of the adjoint which are distinguished by their symmetry under permutation. Diagrammatically,

\[ \mathcal{V} \otimes \mathcal{V} = \begin{array}{c}
\mathcal{P}_* \\
\mathcal{P}_A \\
\mathcal{P}_S
\end{array} + \frac{1}{2N} + \frac{N}{2(N^2 - 4)} + \text{rest}, \]  

(75)

where we use the expressions listed in Appendix A. The Hamiltonian from (71) may be decomposed as,

\[ \hat{h}_{\text{AKLT}} = 1 - (c_\bullet \mathcal{P}_* + c_S \mathcal{P}_S + c_A \mathcal{P}_A + c_{\text{AS}} \mathcal{X}). \]  

(77)

It is convenient to write things in terms of a block matrix in the \( \mathcal{P}_S \oplus \mathcal{P}_A \) space. Then \( \Pi = (\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}) \) is the parity operator, and \( \mathcal{X} \) is defined (up to sign) by the requirements that \( \mathcal{X} \Pi = -\Pi \mathcal{X} \) and \( \Pi^2 = (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) \). We may then collect the last three terms of (77) into the symmetric matrix

\[ M = \begin{pmatrix}
\mathcal{P}_S & c_{\text{AS}} \\
c_{\text{AS}} & \mathcal{P}_A
\end{pmatrix} \]  

(78)

In other words, \( M \) is proportional to a projector. Its diagonal entries can be extracted by sandwiching \( \hat{h}_{\text{AKLT}} \) within the appropriate projectors. To illustrate this procedure, we show the computation of \( c_A \). From (71), \( \mathcal{P}_A (1 - \hat{h}_{\text{AKLT}}) \mathcal{P}_A = c_A \mathcal{P}_A \) and from (77) and (76)

\[ (2N \mathcal{P}_A) \left[ N(1 - \hat{h}_{\text{AKLT}}) \right] (2N \mathcal{P}_A) = \]  

\[ \begin{array}{c}
\mathcal{P}_A \\
\mathcal{P}_A \\
\mathcal{P}_A
\end{array} = \begin{array}{c}
n^2 \\
= N^2 \mathcal{P}_A,
\end{array} \]  

(79)

where we have repeatedly used the Lie algebra identity [29]. So \( \mathcal{P}_A \hat{h}_{\text{AKLT}} \mathcal{P}_A = 1/2 \mathcal{P}_A \) and \( c_A = \frac{1}{2} \).

A similar calculation gives \( c_S \) (which can also be read off from Table 11) and the off-diagonal entry \( c_{\text{AS}} \) is worked out from the condition in (78). Finally we divide by \( \alpha \) and set \( c_\bullet = 1 \) to get the normalized Hamiltonians displayed in Eqs. (42).

C. Computation for the antisymmetric representation

We now consider the case \( \mathcal{V} = \) with dimension \( N(N-1)/2 \). Recall that the dark bars in the diagram stand for anti-symmetrization. From

\[ \mathcal{V} \otimes \mathcal{V}^* = 0 \oplus \mathcal{P} \oplus \mathcal{P}_{S_2} \]  

\[ \mathcal{P} \otimes \mathcal{P} = \mathcal{P}_* \oplus \mathcal{P}_A \oplus \mathcal{P}_S \oplus \mathcal{P}_{S_2} \oplus \text{rest} \]  

(81)
we see that we have to deal with an additional projector $P_S$. From the universal form \( \tilde{h}_{AKLT} = 1 - \frac{2}{N(N-1)} P_S \), we find after some bookkeeping that

\[
\tilde{h}_{AKLT} = 1 - \left( N - 4 \right) \quad + \quad + \quad + \quad - \left\{ \begin{array}{l} \circ \quad + \quad \circ \\ \circ \quad + \quad \circ \end{array} \right\} - \left\{ \begin{array}{l} \circ \quad + \quad \circ \\ \circ \quad + \quad \circ \end{array} \right\}. \tag{83}
\]

\[
\hat{h}_{AKLT} = 1 - \left( N - 4 \right) + (N-1) \left( N-1 \right) P_S. \tag{85}
\]

Note that for $N = 4$ the first term on the right disappears and the rest is clearly invariant both under inversion, as well as conjugation (arrow reversal). This is the inversion symmetric topological phase for SU(4) that has been considered previously in Section IV C 2. In terms of projectors,

\[
\hat{h}_{AKLT} = 1 - \left( c_A P_A + c_S P_S + c_{AS} X + P_S \right) . \tag{85}
\]

c_A and c_S may be worked out from Table III and applying the normalization as shown before, gives the results in Eq. (47).

**VIII. COMMENTS ON HALDANE’S CONJECTURE**

The famous Haldane Conjecture for SU(2) states that anti-ferromagnetic Heisenberg spin chains behave differently, depending on whether the spin $s$ is an integer or half-integer. In the former case, assuming isotropy and in the limit of infinite length, we have a gapped phase with a unique ground state while the latter is gapless. This conjecture is well-supported by numerical evidence and may be arguably considered proven, in the large $s$ limit.\[46\]

For SU(N), there are two obvious directions in which to generalize. First, we may ask which representations realize a gapless phase. One result by Affleck and Lieb\[23\] states that rectangular Young tableaux where the total number of boxes has no common factors with $N$, are gapless. Greiter and Rachel suggested to extend this to non-rectangular tableaux with the same divisibility condition.\[22\] Here we would like to point out that this non-divisibility condition exactly rules out a non-trivial topological phase which is protected by any of the groups SU(N)/$Z_m$ where $m$ divides $N$.\[11\] In other words, from symmetry considerations alone, the phase must either be gapless or topologically trivial.

The second direction concerns the gapped phases – for simplicity we restrict ourselves to the case where $N$ divides the number of boxes in the tableau. Assuming the Heisenberg model is indeed gapped, what is the topological class (the protected boundary spins if any)? For SU(2) it is well known that the spin 1 chain has protected spin 1/2 modes at the boundary. Since the Heisenberg Hamiltonian is inversion symmetric, a unique ground state implies the boundary spin to be self-conjugate too. This is indeed possible for SU(4) if the boundary states are in a six-dimensional $V = \left\{ \begin{array}{l} \end{array} \right\}$ multiplet.

**Conjecture**: The Heisenberg chain for SU(4) in the adjoint representation is in the same phase as the AKLT Hamiltonian of Eq. (48).

When the physical spin is in the representation $\left\{ \begin{array}{l} \end{array} \right\}$ the analogous statement has been convincingly shown in Refs. \[18\] 20.

Finally we turn to the chiral phases which break inversion symmetry, and occupy most of this paper. A natural question is whether the Heisenberg model flows to such a chiral gapped phase when perturbed by a small inversion breaking term e.g. $C_A$. Both this question and the above conjecture are within the scope of present DMRG numerical methods.
IX. CONCLUSIONS AND OUTLOOK

In this paper we have provided a systematic analysis of Haldane phases in SU(N) spin chains with physical spins transforming in the adjoint representation. With the help of birdtracks we succeeded in constructing AKLT Hamiltonians for auxiliary spins transforming in the fundamental or the rank-2 anti-symmetric representation. It should be emphasized that our Hamiltonians have unique ground states which, by construction, reside in topological sectors characterized by the $\mathbb{Z}_N$ quantum numbers $\pm [1]$ and $\pm [2]$. This is in stark contrast to earlier investigations where only Hamiltonians leading to a two-fold ground state degeneracy and an associated spontaneous inversion symmetry breaking have been considered. For SU(4) our analysis gives the first complete account of all existing Haldane phases.

On the way we had to overcome a number of technical complications. The main problem arose from the fact that the decomposition of the physical two-site Hilbert space into irreducible representations of SU(N) features a non-trivial multiplicity. In this space we had to identify a specific one-dimensional subspace and we showed how this can be achieved transparently using birdtracks. Besides helping us with the construction of the AKLT states and their associated Hamiltonians this method also gave access to the eigenvalues of the ground states’ transfer matrix. As a consequence, we have been able to confirm the existence of a spectral gap, partially reproducing earlier results in the literature. It should be clear that our method can easily be extended to the determination of spin-spin correlation functions (see Ref. 16 for an algebraic derivation for the first of our two cases).

In a first and rather sketchy attempt to characterize the phase diagram of our spin chains we focused on the potential presence of special integrable and/or critical points. As one of the remarkable features the phase diagram shows the existence of an integrable point with enhanced SU($N^2 - 1$) symmetry and an effective description in terms of a SU($N^2 - 1$) WZW model at level $k = 1$. At least one other integrable point is likely to exist but the explicit construction of its Hamiltonian requires the fusion procedure for integrable models and has not been attempted here.

Let us stress that our paper also contains two results which are of rather general importance beyond the specific setup we have been investigating. The first one concerns a proposal for a universal parent Hamiltonian in Section VII which may function as a convenient replacement for the standard AKLT construction, also in the case of other symmetry groups and/or representations.

Secondly, we found an intimate relation between the classification of symmetry protected topological phases with continuous symmetry groups and a potential multicroticity of topological phase transitions. Specifically, we have been led to conjecture that the critical points in the phase diagram of our model for even values of $N$ should all be described by SU($N$) WZW models with level $k = 2$ (except at fine-tuned points where $k$ may be even larger). This, as well as our conjecture about the nature of the phase realized in the SU(4) Heisenberg chain, certainly deserve future study.

Of course there are also a few important issues that we did not touch upon in the present article. One concerns the nature of the excitations above the ground state which may lead to a similar interpretation as in Ref. 22. It is expected that our precise knowledge of the Hamiltonians for general values of $N$ will facilitate future large-N analyses. Another direction of research would be to enlarge the phase diagram by admitting terms in the Hamiltonian which break the SU($N$) symmetry but still preserve the subgroup $\mathbb{Z}_N \times \mathbb{Z}_N \subset $ PSU($N$). As was discussed in Ref. 14 (see also Ref. 15) this symmetry protects the same type of Haldane phases which are hence stable against such deformations. Some studies in this direction have been performed in Refs. 16, 18, and 19, albeit mostly for $N = 3$ or a different physical representation. Finally, in view of the experimental progress in the realization of systems exhibiting SU($N$) magnetism it is a pressing question to which extent the phases we have been discussing can actually be realized in a variant of the Fermi-Hubbard model or even in experiment.

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TABLE III. Eigenvalues of some tensors, represented as diagrams, including all the results used in the text.

### Appendix A: Diagrammatic form of the projectors

For convenience we include the diagrammatic form of the projectors into the irreducible representations inside product of two adjoint representations,

\[
\begin{align*}
\mathbb{P}_{\bullet} &= \frac{1}{N^2 - 1} \\
\mathbb{P}_A &= \frac{1}{2N} \\
\mathbb{P}_S &= \frac{N}{2(N^2 - 4)} \\
\mathbb{P}_{A_1} &= \frac{1}{2} \left( \frac{1}{2(N - 2)} + \frac{1}{2N} \right) \\
\mathbb{P}_{A_2} &= \frac{1}{2} \left( \frac{1}{2(N + 2)} + \frac{1}{2N} \right) \\
\mathbb{P}_{S_1} &= \frac{1}{2} \left( \frac{1}{2(N - 1) + 1} \right) \\
\mathbb{P}_{S_2} &= \frac{1}{2} \left( \frac{1}{2(N + 1) - 1} \right)
\end{align*}
\]
The projectors can also be expressed in terms of spin operators. This results in

\[
P_{\bullet} = (Q - 2N)(Q - 4N)(Q - 4N + 4) / \left[ 128N^2(N + 1)(N - 1) \right]
\]

\[
P_S = KQ(Q - 4N)(Q - 4N + 4) / \left[ 128N^4(N + 2)(N - 2) \right]
\]

\[
P_A = -(K + 8N^2)Q(Q - 4N)(Q - 4N + 4) / \left[ 128N^4(N + 2)(N - 2) \right]
\]

\[
P_{A_1/A_2} = -(C_S + 4N)Q(Q - 2N)(Q - 4N + 4) / \left[ 1024N^3 \right]
\]

\[
P_{A_2/A_1} = (C_S - 4N)Q(Q - 2N)(Q - 4N + 4) / \left[ 1024N^3 \right]
\]

\[
P_{S_1} = Q(Q - 2N)(Q - 4N)(Q - 4N + 4) / \left[ 256(N + 1)(N + 2) \right]
\]

\[
P_{S_2} = Q(Q - 2N)(Q - 4N)(Q - 4N - 4) / \left[ 256(N - 1)(N - 2) \right]
\]

We outline the proof of the relation (\ref{eq:relation}) obeyed by the operator $C_A$ since it is quite important. First, we note that $C_A$ is odd under inversion and hence must take an irreducible representation to another of opposite parity. Next we compute,

\[
\left( \frac{C_A}{(2N)} \right)^2 = \left( \begin{array}{c} \includegraphics[height=1cm]{spin_operator}\end{array} \right)^2 = \frac{N}{2} + (N^2 - 4) = (N^2 - 4)(P_S + P_A)
\]

which proves (\ref{eq:relation}) (the second equality follows from a long diagrammatic manipulation).

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30. For $N = 2$ the tensor product has only three parts while for $N = 3$ there are six.

31. The sign is basis dependent therefore cannot be determined unambiguously.

32. Our methods only allow us to fix this relation up to a sign. As will be explained in Section IV, changing this sign exchanges the roles played by the auxiliary spins $\nu$ and $\lambda$.

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