A NOTE ON THE MEAN VALUES OF THE DERIVATIVES OF $\zeta'/\zeta$

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Abstract. Assuming the Riemann hypothesis, we obtain a formula for the mean value of the $k$-derivative of $\zeta'$, depending on the pair correlation of zeros of the Riemann zeta-function. This formula allows us to obtain new equivalences to Montgomery’s pair correlation conjecture. This extends a result of Goldston, Gonek, and Montgomery where the mean value of $\zeta'/\zeta$ was considered.

1. Introduction

Let $\zeta(s)$ denote the Riemann zeta-function. The Riemann hypothesis (RH) states that the non-trivial zeros $\rho$ of $\zeta(s)$ have the form $\rho = \frac{1}{2} + i\gamma$ with $\gamma \in \mathbb{R}$. We will assume RH throughout this paper.

1.1. Montgomery’s pair correlation conjecture. In 1973, Montgomery [17] defined the pair correlation function

$$N(\beta, T) := \sum_{0 < \gamma, \gamma' \leq T} \frac{1}{0 < \gamma - \gamma' \leq \frac{2\pi}{\log T}} \sin 2\pi \beta \log T,$$

where the double sum runs over the ordinates $\gamma, \gamma'$ of two sets of non-trivial zeros of $\zeta(s)$, counted with multiplicity. Since there are $\sim T\log T/(2\pi)$ non-trivial zeros of $\zeta(s)$ with ordinates in the interval $(0, T]$ as $T \to \infty$, the function $N(\beta, T)$ counts the number of pairs of zeros within $\beta$ times the average spacing between zeros. The pair correlation conjecture of Montgomery asserts that

$$N(\beta, T) \sim \frac{T \log T}{2\pi} \int_0^\beta \left(1 - \frac{\sin \pi u}{\pi u}\right)^2 \, du, \quad \text{as } T \to \infty \quad \text{for any fixed } \beta > 0. \quad (1.1)$$

Assuming RH, there are several known equivalences to this conjecture. Define the function

$$F(\alpha, T) := \frac{2\pi}{T \log T} \sum_{0 < \gamma, \gamma' \leq T} T^{\alpha(\gamma - \gamma')} w(\gamma - \gamma'),$$

introduced by Montgomery [17], where $\alpha \in \mathbb{R}, T \geq 2,$ and $w(u) = 4/(4 + u^2)$. Using this function, Goldston [12] showed that the pair correlation conjecture (1.1) is equivalent to

$$\int_b^{b+\ell} F(\alpha, T) \, d\alpha \sim \ell, \quad \text{as } T \to \infty \quad \text{for any fixed } b \geq 1 \text{ and } \ell > 0. \quad (1.2)$$

Another equivalence for the pair correlation conjecture is related to the second moment of $\zeta'/\zeta$. In fact, Goldston, Gonek, and Montgomery [13, Theorem 3] established that the pair correlation conjecture is equivalent

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1 For an equivalence of the pair correlation conjecture related to the asymptotic formula for an integral of Selberg connected with primes in short intervals, see [12,13,14].
to the asymptotic
\[ I(a, T) := \int_1^T \left| \frac{\zeta'}{\zeta} \left( \frac{1}{2} + \frac{a}{\log T} + it \right) \right|^2 dt \sim \left( \frac{1 - e^{-2a}}{4a^2} \right) T \log^2 T, \text{ as } T \to \infty \text{ for any fixed } a > 0. \quad (1.3) \]

Since Montgomery’s pair correlation conjecture remains a difficult open problem, the efforts have thus been concentrated in obtaining upper and lower bounds for the functions \( N(\beta, T), \int_{b}^{b+\ell} F(\alpha, T) \, d\alpha, \) and \( I(a, T) \) in place of asymptotic formulae (see for instance [3, 4, 9, 11, 13, 14]).

1.2. **Mean values of the \( k \)-derivative of \( \zeta'/\zeta \).** The main goal in this paper is to extend the technique developed by Goldston, Gonek, and Montgomery in [14] to get new equivalences of the pair correlation conjecture, related to the mean values of the derivatives of \( \zeta'/\zeta \). Let \( k \geq 0 \) be an integer. For \( a > 0 \) and \( T \geq 2 \), define the second moment of the \( k \)-derivative of \( \zeta'/\zeta \) as

\[ I_k(a, T) = \int_1^T \left| \left( \frac{\zeta'}{\zeta} \right)^{(k)} \left( \frac{1}{2} + \frac{a}{\log T} + it \right) \right|^2 dt. \]

With this notation, we have \( I_0(a, T) = I(a, T) \).

**Theorem 1.** Assume RH and let \( k \geq 0 \) be an integer. The following statements are equivalent:

1. \( \int_{b}^{b+\ell} F(\alpha, T) \, d\alpha \sim \ell, \text{ as } T \to \infty \text{ for any fixed } b \geq 1 \text{ and } \ell > 0; \)
2. \( I_k(a, T) \sim \left( \frac{(2k + 1)!}{(2a)^{2k+2}} \sum_{m=1}^{\ell} \frac{m(2k)!}{(2k+1-m)!(2a)^{m+1}} e^{-2a} T(\log T)^{2k+2} \right), \text{ as } T \to \infty \text{ for any fixed } a > 0. \)

Note that Theorem 1 gives new equivalences for the pair correlation conjecture. When \( k = 0 \), it recovers the equivalence for the asymptotic formula (1.3). Moreover, our result shows the dependence of the asymptotic formulae for \( I_k(a, T) \) for all values of \( k \geq 0 \).

**Corollary 2.** Assume RH. Then, the asymptotic formula (II) holds for some \( k \geq 0 \) if and only if it holds for all \( k \geq 0 \).

One can estimate the right order of magnitude for \( I_k(a, T) \), as \( T \to \infty \), for a fixed \( a > 0 \). In fact, using Proposition 5 and the uniform estimate (see, for instance [13])

\[ \int_1^{\beta} F(\alpha, T) \, d\alpha \ll \beta, \quad (1.4) \]

it follows that for fixed \( k \geq 0 \) and \( a > 0 \), we have \( I_k(a, T) \sim_{k, a} T(\log T)^{2k+2} \).

On the other hand, Farmer proved a relation between \( I(a, T) \) and a certain discrete mean value of \( \zeta'/\zeta \). For \( k \geq 0 \) an integer, define

\[ D_k(a, T) = \sum_{0 < \gamma < T} \left( \frac{\zeta'}{\zeta} \right)^{(2k)} \left( \frac{1}{2} + \frac{a}{\log T} + i\gamma \right). \]

Then, Farmer [7, Lemma 3b] established that, for a fixed \( a > 0 \),

\[ D_0(a, T) = \frac{1}{2\pi} I_0 \left( \frac{a}{2}, T \right) + O(T^\varepsilon), \quad \text{for } T \geq 2 \text{ and } \varepsilon > 0 \text{ sufficiently small}. \quad (1.5) \]

In particular, using [13] we obtain that the pair correlation conjecture is equivalent to

\[ D_0(a, T) \sim \left( 1 - e^{-a} \right) \left( 2\pi a^2 \right) T \log^2 T, \text{ as } T \to \infty \text{ for any fixed } a > 0. \]
Extending \([1, 5]\) for \(D_k(a, T)\) and using Theorem \([4]\) we arrive at the following corollary.

**Corollary 3.** Assume RH and let \(k \geq 0\) be an integer. The following statements are equivalent:

(I) \(\int_b^{b+\ell} F(\alpha, T) \, d\alpha \sim \ell, \) as \(T \to \infty\) for any fixed \(b \geq 1\) and \(\ell > 0;\)

(II) \(D_k(a, T) \sim \frac{1}{2\pi} \left( \frac{(2k + 1)!}{a^{2k+2}} - \sum_{m=1}^{\infty} \frac{m (2k)!}{(2k+1-m)!} \frac{e^{-a}}{a^{m+1}} \right) T (\log T)^{2k+2}, \) as \(T \to \infty\) for any fixed \(a > 0;\)

\[\text{1.3. Related results.}\]

We would like to point out that for some objects related to the Riemann zeta-function, there are results where one relates the asymptotic formula of their second moments to suitably weighted integrals of \(F(\alpha, T).\) For instance, Goldston \([11, \text{Theorem 1}]\) showed, under RH, that

\[
\int_0^T |S(t)|^2 \, dt = \frac{T}{2\pi^2} \log \log T + \frac{T}{2\pi^2} \left[ \frac{\infty}{1} \frac{F(\alpha, T)}{\alpha^2} \, d\alpha + \gamma_0 - \sum_{m=2}^{\infty} \sum_p \left( \frac{1}{m} - \frac{1}{m^2} \right) \frac{1}{p^m} \right] + o(T), \quad \text{as} \quad T \to \infty,
\]

where \(\pi S(t)\) denote the argument of the Riemann zeta-function at the point \(\frac{1}{2} + it,\) and \(\gamma_0\) is Euler’s constant. Recently, this has been extended to the iterates of the function \(S(t)\) (see \([6, \text{Theorem 1}]\)). Note that, assuming \([1, 2],\) by integration by parts and \([1, 3]\) we get

\[
\int_0^T |S(t)|^2 \, dt = \frac{T}{2\pi^2} \log \log T + \frac{T}{2\pi^2} \left[ 1 + \gamma_0 - \sum_{m=2}^{\infty} \sum_p \left( \frac{1}{m} - \frac{1}{m^2} \right) \frac{1}{p^m} \right] + o(T), \quad \text{as} \quad T \to \infty.
\]

We refer the reader to Farmer \([4, 8]\) for other results related to pair correlation and certain asymptotic formulæ.

\[\text{2. The representation formula for} \ I_k(a, T)\]

In this section, we establish a representation formula for the second moment of the \(k\)-derivative of \(\zeta'/\zeta,\) related to the function \(F(\alpha, T).\) It can be seen as an extension of \([1, 4, \text{Theorem 1}]\). The Poisson kernel plays an important role in our formula. For \(b > 0,\) let \(h_b : \mathbb{R} \to \mathbb{R}\) be the Poisson kernel defined as

\[
h_b(x) = \frac{b}{b^2 + x^2}, \quad (2.1)
\]

and let \(\ell_b : \mathbb{R} \to \mathbb{R}\) be an auxiliary function \([5]\) defined as

\[
\ell_b(x) = \frac{b^2 - x^2}{(b^2 + x^2)^2}, \quad (2.2)
\]

The following technical lemma about the derivatives of \(h_b\) and \(\ell_b\) will be useful for us.

**Lemma 4.** Let \(k \geq 0\) be an even integer. Then, for all \(x \in \mathbb{R}\) we have

\[
|\langle h_b \rangle^{(k)}(x)| \ll_k \frac{1}{b^{k-1}(b^2 + x^2)}, \quad \text{and} \quad |\langle \ell_b \rangle^{(k)}(x)| \ll_k \frac{1}{b^k(b^2 + x^2)}.
\]

**Proof.** Let us prove the first estimate for \(b = 1.\) For any \(k \geq 0,\) it is easy to see by induction that

\[
\langle h_1 \rangle^{(k)}(x) = \frac{P(x)}{(1 + x^2)^{k+2}},
\]

where \(P\) is a polynomial of degree at most \(2^{k+1} - k - 2.\) In particular, when \(k = 2m\) with \(m \in \mathbb{Z}\) we have

\[
|\langle h_1 \rangle^{(2m)}(x)| \ll_m \frac{1}{(1 + x^2)^{m+1}}.
\]

The function \(\ell_b\) has previously been used to bound the real part of the derivative of \(\zeta'/\zeta\) (see \([5, \text{Theorem 3}]\)).
In the general case, since \( h_b(x) = h_1(x/b)/b \), it follows that
\[
|\langle (h_b)^{(2m)} \rangle(x)| = \frac{1}{b^{2m+1}} |\langle (h_1)^{(2m)} \rangle \left( \frac{x}{b} \right) | \ll_m \frac{b}{(b^2 + x^2)^{m+1}} \leq \frac{1}{b^{2m-1}(b^2 + x^2)}.
\]
We conclude the first estimate. The proof of the second estimate is similar. □

**Proposition 5.** Assume RH and let \( k \geq 1 \) be a fixed integer. Then, for \( 0 < a < 1 \) and \( T \geq 3 \) we have
\[
I_k(a, T) = \frac{(-1)^k}{2^{k+1}} \pi \int_0^T \alpha^{2k+1} e^{-2a\alpha} \, d\alpha + \int_1^T \alpha^{2k} e^{-2a\alpha} F(\alpha, T) \, d\alpha + o(1) \, T(\log T)^{2k+2}, \quad \text{as } T \to \infty. \tag{2.3}
\]

Proof. We start obtaining a bound for \( \left( \frac{\zeta'}{\zeta} \right)^{(k)} \). Let \( s = \sigma + it \), with \( \frac{1}{2} < \sigma < \frac{3}{2} \) and \( t \geq 2 \). From the partial fraction decomposition for \( \zeta'/\zeta \) [13, Eq. 2.12.7]
\[
\left( \frac{\zeta'}{\zeta} \right)(s) = B - \frac{1}{s-1} + \frac{1}{2} \log \pi - \frac{1}{2} \Gamma' \left( \frac{s}{2} + 1 \right) + \sum_{\rho} \left( \frac{1}{s-\rho} + \frac{1}{\rho} \right),
\]
where the sum runs over the non-trivial zeros \( \rho = \frac{1}{2} + i\gamma \) of \( \zeta(s) \) and \( B = -\text{Re} \sum_{\rho} \rho^{-1} \). Taking \( k \) derivatives in (2.3) and using the estimate
\[
\left( \frac{\Gamma'(1)}{\Gamma} \right)(w) = O \left( \frac{1}{|w|^k} \right), \quad \text{for Re } w \geq \sigma_0 > 0,
\]
it follows that
\[
\left( \frac{\zeta'}{\zeta} \right)^{(k)}(s) = (-1)^k k! \sum_{\rho} \frac{1}{(s-\rho)^{k+1}} + O \left( \frac{1}{|t|^k} \right). \tag{2.5}
\]
Since \( \sum_{|\gamma-t| \leq 1} 1 = O(\log t) \), we have
\[
\left| \sum_{\gamma-t \leq 1} \frac{1}{(s-\rho)^{k+1}} \right| \leq \sum_{|\gamma-t| \leq 1} \left\{ \sum_{t+n<\gamma \leq t+n+1} \frac{1}{|t-\gamma|^{k+1}} \right\} \leq \sum_{n \geq 1} \left\{ \sum_{t+n<\gamma \leq t+n+1} \frac{1}{n^{k+1}} \right\} \ll \sum_{n \geq 1} \frac{\log(t+n)}{n^{k+1}} \ll \log t.
\]
Similarly, we can prove the same estimate when the sum runs over \( \gamma < t \) instead. Therefore, in (2.3) we obtain\(^3\) for \( \frac{1}{2} < \sigma < \frac{3}{2} \) and \( t \geq 2 \),
\[
\left| \left( \frac{\zeta'}{\zeta} \right)^{(k)}(\sigma + it) \right| \ll \frac{\log t}{(\sigma - \frac{1}{2})^{k+1}}. \tag{2.6}
\]

Now, let us prove Proposition 5. Using the elementary identity \(|w|^2 = 2(\text{Re } \{w\})^2 - \text{Re } \{w^2\}\) for all \( w \in \mathbb{C} \), we write
\[
\int_1^T \left| \left( \frac{\zeta'}{\zeta} \right)^{(k)}(\sigma + it) \right|^2 dt = 2 \int_1^T \left( \text{Re } \left( \frac{\zeta'}{\zeta} \right)^{(k)}(\sigma + it) \right)^2 dt - \text{Re } \int_1^T \left( \left( \frac{\zeta'}{\zeta} \right)^{(k)}(\sigma + it) \right)^2 dt. \tag{2.7}
\]
We estimate the second integral on the right-hand side of (2.7) by pulling the contour to the right, up to the line \( \text{Re } s = \frac{3}{2} \) (see [14, p. 111]). In fact, to estimate the vertical edge at \( \text{Re } s = \frac{3}{2} \) we use the representation
\[^3\] It can be proved as the proof of Stirling’s formula, but starting after taking \( k \) derivatives in [14, Eq. (34) in p. 202].
\[^4\] The estimate (2.6) also holds when \( k = 0 \).
as a Dirichlet series of $(\zeta'/\zeta)^{(k)}(s)$, and for the upper horizontal edge we use the estimate (2.6). Therefore, in (2.7) we get
\[
\int_1^T \left| \frac{(\zeta'/\zeta)^{(k)}}{\zeta + it} \right|^2 dt = 2 \int_1^T \left( \Re \left( \frac{\zeta'}{\zeta} \right)(\sigma + it) \right)^2 dt + O\left( \log^2 T \frac{T}{(\sigma - \frac{1}{2})^{2k+1}} \right).
\]
(2.8)

On the other hand, note that
\[
(-1)^k! \Re \left\{ \sum \frac{1}{(s - \rho)^{k+1}} \right\}
= \sum \Re \left\{ (-i)^k \frac{d^k}{dx^k} \left( \frac{1}{(\sigma - \frac{1}{2}) + ix} \right) \right\} \bigg|_{x=t-\gamma}
= \sum \Re \left\{ (-i)^k \frac{d^k}{dx^k} \left( \frac{\sigma - \frac{1}{2}}{(\sigma - \frac{1}{2})^2 + x^2} \right) \right\} \bigg|_{x=t-\gamma}
= \sum \left\{ \Re \left\{ (-i)^k \right\} \frac{d^k}{dx^k} \left( \frac{\sigma - \frac{1}{2}}{(\sigma - \frac{1}{2})^2 + x^2} \right) \bigg|_{x=t-\gamma} + \Re \left\{ (-i)^{k+1} \right\} \frac{d^{k+1}}{dx^{k+1}} \left( \frac{(\sigma - \frac{1}{2})^2 - x^2}{((\sigma - \frac{1}{2})^2 + x^2)^2} \right) \bigg|_{x=t-\gamma} \right\},
\]

Therefore, taking the real part of (2.5) we arrive at
\[
\Re \left( \frac{(\zeta'/\zeta)^{(k)}}{\zeta + it} \right) + \frac{1}{|t|^k} = \sum \Re f_{k,\sigma}(t - \gamma),
\]
(2.9)

where $f_{k,\sigma}(x) = \Re \left\{ (-i)^k \right\} (h_{\sigma - 1/2}^{(k)})(x) + \Re \left\{ (-i)^{k+1} \right\} (\ell_{\sigma - 1/2}^{(k-1)})(x)$, and the functions $h_{\sigma - 1/2}$ and $\ell_{\sigma - 1/2}$ are defined in (2.11) and (2.2) respectively. Using the Fourier transforms:
\[
\widehat{h}_b(y) = \pi e^{-2\pi b|y|} \quad \text{and} \quad \widehat{\ell}_b(y) = 2\pi^2 |y| e^{-2\pi b|y|},
\]
the Fourier transform of $f_{k,\sigma}$ is given by
\[
\widehat{f}_{k,\sigma}(y) = \left( \Re \left\{ i^k \right\} y^k + \Re \left\{ i^{k+1} \right\} y^{k-1} \right) \left( -1 \right)^k 2^k \pi^{k+1} e^{-2\pi(\sigma - 1/2)|y|}.
\]
(2.10)

Now, we square (2.9), integrate from 1 to $T$, and use (2.7) to get
\[
\int_1^T \left( \Re \left( \frac{(\zeta'/\zeta)^{(k)}}{\zeta + it} \right) \right)^2 dt + O\left( \frac{\log^2 T}{(\sigma - \frac{1}{2})^{k+1}} \right) = \int_1^T \left( \sum \Re f_{k,\sigma}(t - \gamma) \right)^2 dt.
\]
(2.11)

We proceed to analyze the right-hand side of (2.11). From Lemma 44 it follows that \footnote{For a function $f \in L^1(\mathbb{R})$, we define its Fourier transform as $\hat{f}(y) = \int_{-\infty}^{\infty} e^{-2\pi i y x} f(x) \, dx$, and the convolution of $f$ and $g$ is defined as $(f * g)(y) = \int_{-\infty}^{\infty} f(x) g(y - x) \, dx$.}
\[
|f_{k,\sigma}(x)| \ll h_{\sigma - 1/2}(x) \left( \frac{1}{(\sigma - 1/2)^k} \right),
\]
and using Montgomery’s argument [17] we can restrict the inner sum over the zeros of $\zeta(s)$ such that $0 < \gamma \leq T$ and extend the integral to all $t \in \mathbb{R}$, with a final error at most \ll $(\sigma - 1/2)^{-2k} \log^3 T + (\sigma - 1/2)^{-2k-2} \log^2 T$ (see [14] p. 113]). Therefore, from (2.11) and using the fact that $f_{k,\sigma}$ is even,
\[
\int_{-\infty}^{\infty} \left( \sum_{0 < \gamma \leq T} f_{k,\sigma}(t - \gamma) \right)^2 dt = \sum_{0 < \gamma, \gamma' \leq T} \left( f_{k,\sigma} * f_{k,\sigma} \right)(\gamma - \gamma') = \sum_{0 < \gamma, \gamma' \leq T} \left( \widehat{f}_{k,\sigma} \right)^2 (\gamma - \gamma'),
\]

\footnote{We highlight that depending on the parity of $k$, only one of the terms of $f_{k,\sigma}$ appears.}
\[ = \pi(-1)^k \sum_{0 < \gamma, \gamma' \leq T} (h_{2\sigma-1})^{(2k)}(\gamma - \gamma'). \]

We want to add the weight \( w(\gamma - \gamma') \) to the last sum. In fact, Lemma 3 gives the bound
\[
\left| \sum_{0 < \gamma, \gamma' \leq T} (h_{2\sigma-1})^{(2k)}(\gamma - \gamma')(1 - w(\gamma - \gamma')) \right| \ll \frac{4}{(2\sigma - 1)^{2k-1}} \sum_{0 < \gamma, \gamma' \leq T} \frac{4}{4 + (\gamma - \gamma')^2} \\
\ll \frac{T \log TF(0,T)}{(2\sigma - 1)^{2k-1}} \ll \frac{T \log^2 T}{(2\sigma - 1)^{2k-1}},
\]
where in the last estimate we have used \((2.14)\). Thus,
\[
\int_1^T \left( \sum_{\gamma} f_{k,\sigma}(t - \gamma) \right)^2 dt = \pi(-1)^k \sum_{0 < \gamma, \gamma' \leq T} (h_{2\sigma-1})^{(2k)}(\gamma - \gamma') w(\gamma - \gamma') \\
+ O\left( \frac{T \log^2 T}{(2\sigma - 1)^{2k-1}} + \frac{\log^3 T}{(2\sigma - 1)^2k} + \frac{\log^2 T}{(2\sigma - 1)^{2k+2}} \right).
\]
Now, considering that \( \sigma = \frac{1}{2} + \frac{a}{\log T} \) for \( 0 < a \ll 1 \) and using the fact that \( h_{2\sigma-1}(x) = h_{a/\pi}(x \log T/2\pi) \log T/2\pi \) for \( x \in \mathbb{R} \), we obtain in \((2.11)\)
\[
\int_1^T \left( \operatorname{Re} \left( \frac{\zeta'}{\zeta} \right)^{(k)}(\sigma + it) \right)^2 dt = \frac{(-1)^k}{2^{2k+1} \pi 2k} \log T^{2k+1} \sum_{0 < \gamma, \gamma' \leq T} (h_{a/\pi})^{(2k)}(\gamma - \gamma') \left( \frac{\log T}{2\pi} \right) w(\gamma - \gamma') \\
+ O\left( \frac{T \log^2 T}{a^{2k-1}} + \frac{(\log T)^{2k+4}}{a^{2k+2}} \right).
\]
Inserting it in \((2.8)\) we conclude that
\[
\int_1^T \left( \frac{\zeta'}{\zeta} \right)^{(k)} \left( \frac{1}{2} + \frac{a}{\log T} + it \right)^2 dt = \frac{(-1)^k}{2^{2k+1} \pi 2k} \log T^{2k+1} \sum_{0 < \gamma, \gamma' \leq T} (h_{a/\pi})^{(2k)}(\gamma - \gamma') \left( \frac{\log T}{2\pi} \right) w(\gamma - \gamma') \\
+ O\left( \frac{T \log^2 T}{a^{2k-1}} + \frac{(\log T)^{2k+4}}{a^{2k+2}} \right).
\]
From Fourier inversion, it is known that for any function \( R \in L^1(\mathbb{R}) \) such that \( \hat{R} \in L^1(\mathbb{R}) \) we have the formula (see \(17\) Eq. (3))
\[
\sum_{0 < \gamma, \gamma' \leq T} R\left( \frac{\gamma - \gamma'}{2\pi} \right) w(\gamma - \gamma') = \frac{T \log T}{2\pi} \int_{-\infty}^\infty \hat{R}(\alpha) F(\alpha, T) d\alpha.
\]
Applying this formula to the function \( (h_{a/\pi})^{(2k)} \) and using the fact that \( \hat{(h_{a/\pi})^{(2k)}}(y) = (-1)^k 2^{2k+1} \pi 2k e^{-2\pi b |y|} \), we get in \((2.12)\) that, for a fixed \( a > 0 \),
\[
\int_1^T \left( \frac{\zeta'}{\zeta} \right)^{(k)} \left( \frac{1}{2} + \frac{a}{\log T} + it \right)^2 dt = \frac{T \log^2 T}{2} \int_{-\infty}^\infty \alpha^{2k} e^{-2a|\alpha|} F(\alpha, T) d\alpha + O(T \log^2 T)^{2k+1}.
\]
Refining the original work of Montgomery \((17)\), Goldston and Montgomery \((15)\) Lemma 8] proved that, under RH,
\[
F(\alpha, T) = (T^{-2|\alpha|} \log T + |\alpha|)(1 + o(1)), \quad \text{as } T \to \infty,
\]
uniformly for \( 0 \leq |\alpha| \leq 1 \). Using \((2.14)\) and the fact that \( F(\alpha, T) = F(-\alpha, T) \) for all \( \alpha \in \mathbb{R} \), we have
\[
\int_{-\infty}^\infty \alpha^{2k} e^{-2a|\alpha|} F(\alpha, T) d\alpha = 2 \int_0^1 \alpha^{2k+1} e^{-2a\alpha} d\alpha + 2 \int_1^\infty \alpha^{2k} e^{-2a\alpha} F(\alpha, T) d\alpha + o(1).
\]
3. A Tauberian lemma and the Proof of Theorem

3.1. A Tauberian lemma. The following lemma can be seen as a generalization of [14, Lemma 2], where the case $G \equiv 1$ was considered. The proof uses Karamata’s method and some examples of these Tauberian lemmas are given in [18, Section 7.12].

**Lemma 6.** Let $f(\alpha, T) \geq 0$ be a function such that the function $\alpha \mapsto f(\alpha, T)$ is continuous for each $T \geq 2$ fixed, and for $\beta > 0$ and $T \geq 2$,
\[
\int_0^\beta f(\alpha, T) \, d\alpha \ll \beta + 1. \tag{3.1}
\]

Let $G$ be a polynomial such that $G(\alpha) > 0$ for $\alpha \in [0, \infty)$. The following statements are equivalent:

(A) $\int_0^\infty f(\alpha, T) G(\alpha) e^{-b\alpha} \, d\alpha \sim \int_0^\infty G(\alpha) e^{-b\alpha} \, d\alpha$, as $T \to \infty$ for any fixed $b > 0$.

(B) $\frac{1}{d-c} \int_c^d f(\alpha, T) \, d\alpha \sim 1$, as $T \to \infty$ for any fixed $0 \leq c < d$.

**Proof.** Let us start assuming (A). Let $0 \leq c < d$ be fixed, and define the function $h : [0, 1] \to \mathbb{R}$ by
\[
h(u) = \begin{cases} 
0, & \text{if } 0 \leq u < e^{-d} \\
\frac{1}{u G(-\log u)}, & \text{if } e^{-d} \leq u \leq e^{-c} \\
0, & \text{if } e^{-c} < u \leq 1.
\end{cases}
\]

By the Weierstrass approximation theorem, for any $\varepsilon > 0$ sufficiently small we can construct a polynomial $P(u) = \sum_{n=0}^N a_n u^n$ (depending on $\varepsilon$) such that
\[
h(u) \leq P(u) \text{ for all } u \in [0, 1], \quad \text{and} \quad \int_0^1 (P(u) - h(u))^2 \, du = O(\varepsilon). \tag{3.2}
\]

Defining the function $Q(\alpha) = e^{-\alpha} P(e^{-\alpha})$, it follows that $\frac{\chi_{[c,d]}(\alpha)}{G(\alpha)} \leq Q(\alpha)$ for all $\alpha \geq 0$. Recalling that $G(\alpha) > 0$ we have
\[
\int_c^d f(\alpha, T) \, d\alpha \leq \int_0^\infty f(\alpha, T) G(\alpha) Q(\alpha) \, d\alpha = \int_0^\infty f(\alpha, T) G(\alpha) \sum_{n=0}^N a_n e^{-(n+1)\alpha} \, d\alpha
\]
\[
= \sum_{n=0}^N a_n \int_0^\infty f(\alpha, T) G(\alpha) e^{-(n+1)\alpha} \, d\alpha.
\]

Taking lim sup as $T \to \infty$ and using (A) we arrive at
\[
\limsup_{T \to \infty} \int_c^d f(\alpha, T) \, d\alpha \leq \sum_{n=0}^N a_n \int_0^\infty G(\alpha) e^{-(n+1)\alpha} \, d\alpha. \tag{3.3}
\]

\[\text{See [2] for another extension of [13, Lemma 2] depending of certain measures.}\]
\[\text{Here } \chi_{[c,d]}(\alpha) \text{ denotes the characteristic function of the interval } [c,d].\]
By a change of variables, the definition of $h$, the Cauchy-Schwarz inequality and \(3.2\), one can see that

$$\sum_{n=0}^{N} a_n \int_{0}^{\infty} G(\alpha) e^{-(\alpha+1)\alpha} \, d\alpha = \int_{0}^{1} G(-\log u) \, P(u) \, du$$

$$= \int_{0}^{1} G(-\log u) \, h(u) \, du + \int_{0}^{1} G(-\log u) \, (P(u) - h(u)) \, du$$

$$= \int_{e^{-\varepsilon}}^{1} \frac{1}{u} \, du + O \left( \left( \int_{0}^{1} G^2(-\log u) \, du \right)^{1/2} \left( \int_{0}^{1} (P(u) - h(u))^2 \, du \right)^{1/2} \right)$$

$$= d - c + O(\varepsilon^{1/2}).$$

Letting $\varepsilon \to 0$ and combining this with \(3.3\), we conclude that

$$\limsup_{T \to \infty} \int_{c}^{d} f(\alpha, T) \, d\alpha \leq d - c.$$

Similarly, we can proceed to prove that

$$d - c \leq \liminf_{T \to \infty} \int_{c}^{d} f(\alpha, T) \, d\alpha.$$

Therefore we obtain (B). Let us prove that (B) implies (A). Using integration by parts and \(3.1\), we see that

$$\int_{0}^{\infty} f(\alpha, T) \, G(\alpha) e^{-\alpha a} \, d\alpha = -\int_{0}^{\infty} \left( \int_{0}^{\alpha} f(\beta, T) \, d\beta \right) \left( G(\alpha) e^{-\alpha a} \right)' \, d\alpha. \quad (3.4)$$

Finally, using (B), the dominated convergence theorem, and integration by parts one more time, we conclude.

\[ \square \]

3.2. **Proof of Theorem 1**

Since the case $k = 0$ was considered in the work of Goldston, Gonek and Montgomery (see \[14\] Theorem 3), assume $k \geq 1$. Using the identity \[4.1\]

$$\int_{0}^{1} \alpha^{2k+1} e^{-2\alpha a} \, d\alpha + \int_{1}^{\infty} \alpha^{2k} e^{-2\alpha a} \, d\alpha = \frac{(2k+1)!}{(2a)^{2k+2}} - \frac{2k+1}{m} \frac{(2k)!}{(2k+1-m)! (2a)^{m+1}}, \quad \text{for any } a > 0,$$

and \[2.3\] we have that (II) is equivalent to

$$\int_{1}^{\infty} \alpha^{2k} e^{-2\alpha a} F(\alpha, T) \, d\alpha \sim \int_{1}^{\infty} \alpha^{2k} e^{-2\alpha a} \, d\alpha.$$

A translation gives that (II) is equivalent to

$$\int_{0}^{\infty} (\alpha + 1)^{2k} e^{-2\alpha a} F(\alpha + 1, T) \, d\alpha \sim \int_{0}^{\infty} (\alpha + 1)^{2k} e^{-2\alpha a} \, d\alpha.$$

Using Lemma \[6\] with the function $f(\alpha, T) = F(\alpha + 1, T)$, $G(\alpha) = (\alpha + 1)^{2k}$, and $b = 2a$ we conclude the proof. We remark that the additional constraint \[5.1\] follows from \[1.1\].

4. **Proof of Corollary 3**

Assume RH. From \[18\] p. 340, for each $n \in \mathbb{N}$ there is $T_n \in (n, n + 1)$ such that for $-1 < \sigma < 2$,

$$\left| \zeta' \zeta (\sigma + iT_n) \right| \ll (\log T_n)^2. \quad (4.1)$$

9 See \[16\] Eq. 3.351-1 and 3.351-2.
Now, let \( k \geq 1 \) be an integer, \( 0 < a < 1 \) and \( T \geq 4 \), \( T \neq N \). Choose \( n \in \mathbb{N} \) such that \( T, T_n \in (n, n + 1) \) and \( T_n \) satisfies (4.1). Note that \( \log T_n = \log T \). Using integration by parts \( k \) times and the bound (2.6), we have

\[
\int_1^{T_n} \left| \left( \frac{\zeta'(k)}{\zeta} \right) \left( \frac{1}{2} + \frac{a}{\log T} + it \right) \right|^2 dt = \int_1^{T_n} \left( \frac{\zeta'(k)}{\zeta} \right) \left( \frac{1}{2} + \frac{a}{\log T} + it \right) \left( \frac{\zeta'(k)}{\zeta} \right) \left( \frac{1}{2} + \frac{a}{\log T} - it \right) dt
\]

\[
= \frac{1}{i} \int \left[ \frac{1}{2} \frac{1}{\log T} + iT_n \zeta'(k) \left( s + \frac{2a}{\log T} \right) \left( \frac{\zeta'(k)}{\zeta} \right) (1 - s) ds
\]

\[
= \frac{1}{i} \int \left[ \frac{1}{2} \frac{1}{\log T} + iT_n \zeta'(k) \left( s + \frac{2a}{\log T} \right) \frac{\zeta'}{\zeta} (1 - s) ds + O \left( \frac{\log T}{a^{2k+1}} \right) .
\]

We use the residue theorem on the rectangle with vertices \( \frac{1}{2} - \frac{a}{\log T} + i, 2 + i, 2 + iT_n \) and \( \frac{1}{2} - \frac{a}{\log T} + iT_n \) (since RH holds, the function \( \left( \zeta'/\zeta \right)(2k) \left( s + \frac{2a}{\log T} \right) \) is analytic in this rectangle) and the bounds (2.6) and (4.1) to deduce that

\[
\int_1^{T_n} \left| \left( \frac{\zeta'(k)}{\zeta} \right) \left( \frac{1}{2} + \frac{a}{\log T} + it \right) \right|^2 dt = 2\pi \sum_{0 < \gamma < T_n} \left( \frac{\zeta'(k)}{\zeta} \right) \left( \rho + \frac{2a}{\log T} \right)
\]

\[
+ \frac{1}{i} \int_{2+i}^{2+iT_n} \left( \frac{\zeta'(k)}{\zeta} \right) \left( s + \frac{2a}{\log T} \right) \frac{\zeta'}{\zeta} (1 - s) ds + O \left( \frac{\log T}{a^{2k+1}} \right) .
\]

It is known that \( \zeta(s) \) satisfies the functional equation \( \zeta(s) = \chi(s)\zeta(1 - s) \), where

\[
\chi(s) = \frac{\pi^{s-\frac{1}{2}} \Gamma \left( \frac{1}{2} - \frac{s}{2} \right)}{\Gamma \left( \frac{s}{2} \right)}.
\]

Then, we write

\[
\frac{1}{i} \int_{2+i}^{2+iT_n} \left( \frac{\zeta'(k)}{\zeta} \right) \left( s + \frac{2a}{\log T} \right) \frac{\zeta'}{\zeta} (1 - s) ds = \frac{1}{i} \int_{2+i}^{2+iT_n} \left( \frac{\zeta'(k)}{\zeta} \right) \left( s + \frac{2a}{\log T} \right) \left( \frac{\chi'}{\chi} (s) - \frac{\zeta'}{\zeta} (s) \right) ds.
\]

Using the estimate

\[
\frac{\chi'}{\chi} (\sigma + it) = - \log \left| \frac{t}{2\pi} \right| + O \left( \frac{1}{|t|} \right) , \text{ for } |t| \geq 1 \text{ and } |\sigma| \ll 1,
\]

and the representation as a Dirichlet series of \( \left( \zeta'/\zeta \right)(2k) \) in the right-hand side of (4.2), we integrate term by term the right-hand side of (4.2) to obtain \( O(\log T) \). Therefore, we arrive at

\[
\int_1^{T_n} \left| \left( \frac{\zeta'(k)}{\zeta} \right) \left( \frac{1}{2} + \frac{a}{\log T} + it \right) \right|^2 dt = 2\pi \sum_{0 < \gamma < T_n} \left( \frac{\zeta'(k)}{\zeta} \right) \left( \rho + \frac{2a}{\log T} \right) + O \left( \frac{\log T}{a^{2k+1}} \right) .
\]

We can replace \( T_n \) by \( T \) using (2.6) and \( \sum_{|t - \gamma| \leq 1} = O(\log t) \) with an error at most \( \ll (\log T)^{2k+4}/a^{2k+2} \). Therefore, we conclude for \( 0 < a \ll 1 \) and \( T \) sufficiently large, that

\[
I_k(a, T) = 2\pi D_k(2a, T) + O \left( \frac{(\log T)^{2k+4}}{a^{2k+2}} \right) .
\]

Finally, we use Theorem 11 to conclude.

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