The anomalous behavior of the Hall constant $R_{\text{H}}$ in the normal state of cuprates remains the challenge for theorists for over a decade. Two aspects, possibly interrelated, are evident and should be understood: a) the d.c. $R_{\text{H}}^0$ at low temperatures $T \to 0$ is clearly doping dependent. In the prototype material La$_2$−$_x$Sr$_x$CuO$_4$ (LSCO) it changes from positive $R_{\text{H}}^0 \propto 1/x$ at low doping $x < x^* \approx 0.3$, consistent with the picture of hole-doped (Mott-Hubbard) insulator, to the electron-like $R_{\text{H}}^0 < 0$ at $x > x^*$ in agreement with the usual band picture. b) $R_{\text{H}}^0$ is also strongly temperature dependent, both at low doping and optimum doping. At optimum doping, the attention has been devoted to the anomalous variation of the Hall angle $\theta_{\text{H}} \propto T^2$ in YBa$_2$Cu$_3$O$_y$ [3]. On the other hand, at low hole concentration $c_h < 0.15$, $R_{\text{H}}(T)$ in LSCO has been shown to follow an universally behaved decrease with $T$ in which $R_{\text{H}}^0(T \to 0)$ and the characteristic temperature $T^*$ of vanishing $R_{\text{H}}^0(T^*) \to 0$ both scale with $c_h$. In underdoped cuprates, the same $T^*(c_h)$ has been in fact associated with the (large) pseudogap crossover scale in universal susceptibility $\chi_0(T)$, in-plane resistivity $\rho(T)$, specific heat $c_v(T)$, and some other quantities [3].

A number of theoretical investigations have addressed the first question, i.e. the doping dependence of $R_{\text{H}}$ in models of strongly correlated electrons, in particular within the $t$-$J$ model and the Hubbard model on a planar lattice. The advantage is that one can study the dynamical Hall response and the d.c. Hall constant as a ground state ($T = 0$) property, in particular in systems with finite transverse dimension and in the ladder geometry [3]. It has been also shown that within the $t$-$J$ model the change from a hole-like to an electron-like Hall response can be qualitatively reproduced by studying the high-frequency $R_{\text{H}}^\infty = R_{\text{H}}(\omega \to \infty)$ [10, 11], analytically tractable at $T \to \infty$. Recently, a connection of the reactive $R_{\text{H}}^\infty(T=0)$ to the charge stiffness has also been found [9].

The anomalous temperature dependence of $R_{\text{H}}(T)$, being the main subject of this work, has been much less clarified in the literature. The Hall mobility $\mu_{\text{H}}(T)$ of a single charge carrier in the Mott-Hubbard insulator has been first evaluated within the generalized retraceable path approximation [10]. The high-frequency $R_{\text{H}}^\infty(T)$ has been calculated using the high-$T$ expansion [11]. At low doping, $c_h < 0.15$, it has been observed that on decreasing temperature $R_{\text{H}}^\infty$ is also decreasing instead of approaching presumed (larger) semiclassical and experimentally observed d.c. result $R_{\text{H}}^0 = 1/c_h e_0 \approx 4 R_{\text{H}}^\infty(T = \infty)$. Related are the conclusions of the quantum Monte-Carlo study of the planar Hubbard model [12], where close to the half-filling electron-like $R_{\text{H}}^0 < 0$ has been found at low $T$. The same has been claimed generally for $R_{\text{H}}^\infty(\omega)$ even for low $\omega$ [12]. Quite controversial are also results for $R_{\text{H}}^\infty(\omega)$ on ladders [13]. In regard to that, we should also mention the questionable relation of the off-diagonal $\sigma_{xy}$ to the orbital susceptibility $\chi_{\text{d}}$ [13], potentially useful as an alternative route to the understanding of $R_{\text{H}}^\infty(T)$ [14].

In the following we present numerical results for the dynamical $R_{\text{H}}(\omega)$, as obtained within the low doping regime of the $t$-$J$ model using the finite-temperature Lanczos method (FTLM) [14]. The aim of this letter is to approach the low-$\omega$ and low-$T$ limit as much as possible and to investigate the relation between $R_{\text{H}}^\infty(T)$ and $R_{\text{H}}(T)$. We find these two quantities essentially different for $T \to T^*$, establishing the pseudogap scale $T^* < J$ both in the ladder and planar systems.

We study the $t$-$J$ model in an external homogeneous magnetic field $\mathbf{B} = \text{curl} \mathbf{A}$,

$$H(\mathbf{A}) = - t \sum_{(ij)s} (e^{i\theta_{ij}} \hat{c}^\dagger_{is} \hat{c}_{js} + \text{H.c.}) + J \sum_{(ij)} (\mathbf{S}_i \cdot \mathbf{S}_j - \frac{1}{2} n_i n_j),$$

where the (inhomogeneous) vector potential enters the phases $\theta_{ij} = e\mathbf{A}(\mathbf{r}_i) \cdot \mathbf{r}_{ij}$. The hopping is only between the nearest neighbors (ij). Projected fermionic operators $\hat{c}^\dagger_{is}, \hat{c}_{is}$ do not allow for the double occupancy of sites.

In order to calculate the dynamical Hall coefficient

$$R_{\text{H}}(\omega) = \frac{\partial \sigma_{xy}(\omega)}{\partial B} \bigg|_{B \to 0} = \frac{\sigma_{xy}(\omega)}{B \sigma_{xx}(\omega) \sigma_{yy}(\omega)} \bigg|_{B \to 0},$$

the conductivity tensor is evaluated within the linear response

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operators are given by
\[ T = \int_0^\infty dt e^{i\omega t} \langle [j_\alpha(t), j_\beta] \rangle, \] 
where in the presence of \( B \neq 0 \) the particle current \( j \) and the stress tensor \( \tau \) operators are given by
\[ j = t \sum_{(ij)s} \mathbf{r}_{ij} (e^{i\theta_{ij} \phi^\dagger_{is} \phi_{js}} + \text{H.c.}), \]
\[ \tau = t \sum_{(ij)s} \mathbf{r}_{ij} \otimes \mathbf{r}_{ij} (e^{i\theta_{ij} \phi^\dagger_{is} \phi_{js}} + \text{H.c.}). \] 

On a square lattice with \( N \) sites and periodic boundary conditions \( \text{(b.c.)} \) one cannot apply arbitrary magnetic field \( B \) since only quantized \( B = B_m = m\Phi_0 a^2/N \) can be made compatible with the periodic b.c. [13]. Therefore the smallest but finite \( B = B_1 \) is used in calculations. The square lattices used are in general Euclidean (tilted) \( N = l^2 + n^2 \), in particular we investigate systems \( N = 10, 16, 18 \). On the other hand, the ladder geometry of \( N = L \times M \) sites with the periodic b.c. in the \( L \) direction and open b.c. in the perpendicular \( M \) direction allows for any finite \( B \neq 0 \), the fact already used in several \( T = 0 \) calculations [3, 4, 8]. The advantage of ladder systems is also the existence of the reference ground-state results \( R_{\text{H}}^0(T=0) \) which seem to be better understood [3, 4]. Furthermore, at low doping they reproduce the simple semiclassical behavior \( R_{\text{H}}^0(T=0) \sim R^\infty_{\text{H}} = 1/\cosh \omega_0 \).

Dynamical components \( \sigma_{\alpha\beta}(\omega) \) are evaluated using the FTLM [13], employed so far for various dynamic and static quantities within the \( t-J \) model [14], among them also the \( B = 0 \) optical conductivity \( \sigma(\omega) = \sigma_{\alpha\alpha}(\omega) \) on a square lattice. Comparing to the diagonal \( \sigma_{\alpha\alpha} \), the evaluation of the off-diagonal \( \sigma_{\alpha\beta}(\omega) \) is more demanding for several reasons: a) the introduction of \( B > 0 \) in the model [9] breaks the translational invariance and prevents the reduction of the basis states in the Lanczos procedure, hence available finite-size systems are somewhat smaller, b) we expect \( \sigma_{\alpha\beta}(\omega) \propto B \) while \( \sigma_{\alpha\alpha}(B=0, \omega) \) does not vanish identically within the FTLM: consequently larger sampling over initial wavefunctions [13, 14] are needed to reduce the statistical error, c) on a finite square lattice the reference result \( R_{\text{H}}^0(T=0) \) is not meaningful for \( B_m > 0 \), while in ladder systems it is quite sensitive to the introduction of an additional flux [3]. Nevertheless, in general, restrictions for the validity of the FTLM results are similar to other quantities. Through the thermodynamic partition function \( Z(T_{\text{H}}) = Z^* \), we can define the marginal finite-size \( T_{\text{H}} \) below which too few levels contribute to the average and results loose the thermodynamic validity [14]. In the following, we analyze results for \( J = 0.4t \) at low hole doping \( c_0 = N_{\text{h}}/N \) \((N_{\text{h}} = 1, 2)\). In this regime we can estimate \( T_{\text{H}}/t \sim 0.15 - 0.2 \lesssim 0.5J/t \).

Let us first present results for the dynamical \( R_{\text{H}}(\omega) \). In Fig. 1 we show the normalized real part \( r_{\text{H}} = e_{0\text{cH}}\cosh R_{\text{H}} \) for systems with a single hole \( N_{\text{h}} = 1 \). In the evaluation of \( R_{\text{H}}(\omega) \) from Eq. [9] we insert complex \( \sigma_{\alpha\alpha} \) at \( B = 0 \) and the most sensitive quantity remains \( \sigma_{xy}(\omega) \) calculated at \( B = B_1 \) on a square lattice and \( B \sim 0.3B_1 \) on ladders. In the presentation of results an additional frequency smoothening \( \delta = 0.2t \) is used. The normalization of \( R_{\text{H}} \) is chosen such that at low doping \( r_{\text{H}} \sim 1 \) would show up in the case of the semiclassical result.

In Fig. 1 several common features of \( R_{\text{H}} \) in the ladder geometry and in the 2D systems are recognized: a) \( r_{\text{H}}(\omega) \) is quite smoothly varying function of \( \omega \), at least in contrast to strongly \( \omega \)-dependent \( \text{Re} \sigma(\omega) \) on a 2D system, which is found [14] to decay with an anomalous relaxation rate \( 1/\tau(\omega) \propto \omega + \xi T \). b) At high temperatures \( T > t \) we get a hole-like \( r_{\text{H}} > 0 \) for all systems. In this regime \( r_{\text{H}}(\omega) \) is very smooth, in particular for the \( M = 4 \) ladder and the 2D lattice. c) For low temperatures \( T < t \), \( r_{\text{H}}(\omega) \) is less smooth and the dependence is more pronounced for the 2-leg ladder. On the other hand, \( M = 4 \) ladder clearly approaches the behavior of the 2D system, whereby both of the latter show quite a modest variation of \( r_{\text{H}}(\omega) \). In all systems the resonances (and the variation) visible in \( r_{\text{H}}(\omega) \) at high \( \omega > t \) reflect the predominantly local physics of the hole motion and are thus not related to a current relaxation rate deduced from \( \sigma(\omega) \).

Results for \( r_{\text{H}}(\omega) \) are the basis for the calculation of high-frequency \( r_{\text{H}}(\omega = \infty) \) as well as the d.c. limit \( r_{\text{H}}^\infty = r_{\text{H}}(\omega \rightarrow 0) \). The latter is more sensitive since in a finite system (even at \( T > 0 \)) \( \sigma_{\alpha\beta}(\omega \rightarrow 0) \) can be singular due to the coherent charge transport in a system with periodic b.c.. The coherent transport shows up in a finite (but small) charge stiffness [14].
which should be omitted in the evaluation of Eq. (3). In any case, one should take into account proper $\omega \to 0$ behavior of dissipative systems at $T > 0$ which is different in ladders and in 2D lattices, respectively: a) On a ladder we get in the leading order of $\omega \to 0$ a normal conductance along the $x$-direction, i.e. $\sigma_{xx}(\omega \to 0) \sim \sigma_0$, but a finite polarizability along the $y$-direction, $\sigma_{yy}(\omega \to 0) \propto \omega \chi_{yy}^0$. Hence, we expect $\sigma_{xy} \propto \omega$ and finite $r_H^0$. b) For a macroscopic isotropic 2D system we get $\sigma_{xx}(\omega \to 0) \sim \sigma_0$ and we expect as well $\sigma_{xy} \to \sigma_{xy}^0$, leading to finite $r_H^0$.

In Fig. 2 we present results for $r_H^0(T)$ and $r_H^0(T)$ for the ladder systems with $N_h = 1$. Results are shown for 2-leg ladders with various lengths $L = 9, 10, 11$ and for $M = 4$ ladder with $L = 4$. Since $r_H^0$ and $r_H^*$ are properly scaled, for given $M$ curves are expected to approach a well defined macroscopic limit at $L \to \infty$. In fact, $r_H^*$ are nearly independent of $L$ (as well as of $M$) down to $T \sim T_s$. A crossover at $T_s \sim 0.6t$ from a hole-like $r_H^* > 0$ into an electron-like $r_H^* < 0$ can be explicitly observed. $r_H^0$ results are more size ($L$) dependent, nevertheless they reveal a crossover nearly at the same $T \sim T_s$. In contrast to $r_H^*$ which remains negative for the whole regime $T < T_s$, $r_H^*$ changes sign again at $T = T^* \sim 0.2t$. Although our data for $T^*$ are more scattered the crossover into the hole-like $r_H^0(T < T^*) > 0$ is expected. Namely, from the ground state calculations in same systems [8] we know that $r_H^0(T = 0) \sim 1.5$ and $r_H^0(T = 0) \sim 1.2$ for $M = 2$ and $M = 4$ ladders, respectively. Therefore, it is not surprising that the observed dependence $r_H^0(T < T^*)$ is very steep.

Corresponding results for the planar lattice in Fig. 3 are both qualitatively and quantitatively similar. Note, that at low doping the limiting value $r_H^0(T \to \infty) = 1/4$ agrees with the analytical result [1], while obtained $r_H^0(T \to \infty) \sim 0.3$ is also quite close. Again, the crossover into an electron-like regime appears at $T_s \sim 0.6t$. For larger sizes $N \geq 16$ the lower crossover $T^* \sim 0.2t$ is visible as well. In finite 2D systems a reference numerical result at $T = 0$ does not exist, however, the analytical theory [7] indicates that in a macroscopic limit with a single hole ($N_h = 1$) in an ordered antiferromagnet one should get $r_H^0 = 1$.

In numerically available systems, $N_h = 2$ represents already a substantial doping. Therefore, results for $r_H^*$ and $r_H^*$
shown in Fig. 3 should be interpreted in relation with the corresponding finite doping $c_h$. Main message of Fig. 3 is that upper crossover $T_s$, still nearly the same in both $\frac{1}{H}(T)$ and $R^0_{\text{HI}}(T)$, shifts down quite systematically with increasing $c_h$, i.e. with decreasing $N$ at given $N_h$. At least in ladder systems at $c_h < 0.3$, we still find $R^0_{\text{HI}}(T=0) > 0$ in the ground state [8], therefore also the lower crossover $T^* < T_s$ is expected. However, we cannot detect such a crossover in $R^0_{\text{HI}}(T)$ down to $T_s \sim 0.15t$, not surprisingly also since the experimental value, e.g. in LSCO at $c_h > 0.1$, is $T^* > 600K \sim 0.15t$ (assuming $t \sim 0.4eV$).

Let us finally comment on the relation of the d.c. $\sigma^0_{xy}$ to the orbital susceptibility $\chi_d$ in a macroscopic 2D system. Namely, $\sigma^0_{xy} = eB\partial \chi_d / \partial \mu = eB(\partial \chi_d / \partial c_h)(\partial c_h / \partial \mu)$, (where $\mu$ denotes chemical potential) was derived using seemingly quite general thermodynamic relations [6,3], but at the same time put under question [6]. Since the d.c. $\sigma^0_{xy}(T) > 0$ is quite a smooth function the above relation seems to yield also a qualitative connection between $\chi_d(T)$ and $R^0_{\text{HI}}(T)$. The situation should be particularly simple at low doping (but not too low $T$), where $\partial c_h / \partial \mu \sim c_h/T$ and $\chi_d \propto c_h$ is expected, and consequently $\sigma^0_{xy} \propto -B\chi_d/T$. Indeed, results for $N_h = 1$ indicate [14] that both crossovers $T_s$ and $T^*$ appear also as a change of sign in $\chi_d(T)$ nearly at the same values. Here, the intermediate regime $T^* < T < T_s$ corresponds to an anomalous paramagnetic response $\chi_d > 0$. On the other hand, it is quite evident from our results that the relation is not valid at high $T \gg t$. Namely, in this regime $\sigma^0_{xy} \propto 1/T$ and $\sigma^0_{xy} \propto B/T^2$ [15] is obtained, leading to $R^0_{\text{HI}}(T \to \infty) \sim \text{const}$. On the other hand, from the high-$T$ expansion $\chi_d \propto 1/T^3$ is acquired [4], so that the assumed relation would demand $\sigma^0_{xy} \propto B/T^2$, in conflict with previous $\sigma^0_{xy} \propto B/T^2$.

In conclusion, we have presented results for both dynamical and d.c. Hall constant within the $t$-$J$ model on ladders and on square lattices. The main novel point is the observation of two crossover temperatures $T_s$ and $T^*$ which are at low doping generally present in all systems. Both $R^0_{\text{HI}}$ and $R^0_{\text{HI}}$ are positive at $T > T_s$ and change sign at $T_s$. While $R^0_{\text{HI}}(T<T_s)$ stays negative, $R^0_{\text{HI}}$ reveals a sign change into a hole-like behavior at $T=T^* < T_s$ as well as steep variation of $R^0_{\text{HI}}(T<T^*)$. This reconciles some seemingly controversial theoretical results [11,12]. Our results are in agreement with high-$T$ expansion results for $R^0_{\text{HI}}(T)$ which at low $c_h$ also show decreasing positive values with decreasing $T$. Quantum Monte Carlo results within the Hubbard model for $R^0_{\text{HI}}(t\omega)$ correspond effectively to high (imaginary) frequencies and low $T$, and being negative they are in agreement with our findings for $R^*_{\text{HI}}$.

How should we understand the above numerical results? At high $T \gg t$ and low doping $c_h \ll 1$, $R^0_{\text{HI}}$ as well as $R^0_{\text{HI}}$ are governed by a loop motion (that is where the dependence on $B \neq 0$ comes from) of a hole within a single plaquette [10,11]. One expects $R^0_{\text{HI}} > 0$, but $R^0_{\text{HI}} = 1/4$ is a non-universal value which e.g. depends on the lattice coordination [14]. The electron-like $R^0_{\text{HI}}(T=0) < 0$ represents an instantaneous Hall response within the ground state near half filling is harder to explain, but is clearly the signature of strong correlations. On the other hand at low $T$, $R^0_{\text{HI}}$ tests the (low energy) quasiparticle properties. Evidently, at low doping and $T < T^*$ at least a single hole in an antiferromagnetic spin background behaves as a well defined hole-like quasiparticle leading to $R^0_{\text{HI}}(T \to 0) \sim 1$ both in 2D [17] and in ladders [8]. Our results for vanishing $R^0_{\text{HI}}(T^*) \sim 0$ indicate that the quasiparticle character is essentially lost at quite low $T \sim T^* < J$, with the pseudogap scale $T^*(c_h)$ decreasing with doping. Such phenomenon is possibly consistent with the scenario of electrons being effectively composite particles (spinons and holons) in strongly correlated systems [3,9], at least at $T > T^*(c_h)$, whereby $T^*(c_h)$ vanishes at optimum doping.

Finally, let us note that our results for $R^0_{\text{HI}}$ are in several aspects consistent with experiments on cuprates, and with LSCO in particular. At low doping $c_h < 0.1$ we find $T^* \sim J/2$, close to the observed $T^* \sim 800 K$. At the same time, we find a very steep dependence in $R^0_{\text{HI}}(T<T^*)$. With increasing $c_h$, $T^*(c_h)$ seems to have desired decreasing tendency, although to establish this beyond a reasonable doubt more work is needed.

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