Local-to-Global-rigidity of graphs quasi-isometric to the 
Bruhat-Tits building of $\text{PSL}_n(\mathbb{Q}_p)$

And application to p-adic lattices

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Abstract

A vertex-transitive graph $\mathcal{G}$ is called Local-to-Global rigid if there exists $R > 0$ such that every other graph whose balls of radius $R$ are isometric to the balls of radius $R$ in $\mathcal{G}$ is covered by $\mathcal{G}$. An example of such a graph is given by the Bruhat-Tits building of $\text{PSL}_n(\mathbb{Q}_p)$ with $n \geq 4$. In this paper we extend this rigidity property to a class of graphs quasi-isometric to the building including torsion-free lattices of $\text{SL}_n(\mathbb{Q}_p)$. The proof is the occasion to prove a result on the local structure of the building. We show that if we fix a $\text{PSL}_n(\mathbb{Q}_p)$-orbit in it, then a vertex is uniquely determined by the neighbouring vertices in this orbit.

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Introduction

A well known problem in geometric group theory is the classification of groups according to their large scale geometry. A group or a class of group that is characterized by its large scale geometry is called QI-rigid. This is for example the case of the class of free group of finite rank.

Although this point of view did lead to powerful results such as the classification of irreducible lattices in semi-simple Lie groups [Far97], one can also adopt the opposite perspective. Indeed, instead of asking whether an object is determined by its coarse geometry, one can ask whether local properties of an object can have global implications for its geometry. A classical example of such implications is given by Lie groups and their locally defined Lie algebras. Another striking illustration is provided by the work of Tits [Tit81] who gave a local characterization of a particular family of graphs called “buildings of type $\tilde{A}_{d-1}$” (see Section 1.1 for a definition). Precisely, graphs and their local-to-global properties are the objects we focus on in this article. All graphs will be equipped with the usual metric, fixing the length of an edge to one.

A natural local condition to impose on a graph is to be $d$-regular for some $d \in \mathbb{N}$, which means that all the vertices must have degree $d$. A well-known result about such a graph is that the $d$-regular tree is its universal covering. This is a first example of a global information deduced only by a local knowledge of the graph.

One can now ask what happens if we impose a local condition which is stronger than $d$-regularity. We formalize this in the next definition.

**Definition 0.1**

Let $R > 0$ and let $X$ and $Y$ be two graphs.

We say that $Y$ is **$R$-locally** $X$ if every ball of radius $R$ in $Y$ is isometric to a ball of radius $R$ in $X$.

If $Y$ is $R$-locally $X$ and $X$ is $R$-locally $Y$ then we say that they are **$R$-locally the same**.

**Example 0.2.** In the following example, $B_X(x_0, 2)$ is isometric to $B_Y(y_0, 2)$.

The previous covering result on the $d$-regular tree is a first example of a more general notion called the Local-to-Global rigidity, also named LG-rigidity.

**Definition 0.3**

Let $R > 0$. We say that $X$ is **Local-to-Global-rigid** (or **LG-rigid** for short) at scale $R$ if every graph $Y$ which is $R$-locally $X$ is covered by $X$.

**Example 0.4.** Benjamini and Ellis [BE16] showed that for any $d \geq 2$ the Cayley graph of $\mathbb{Z}^d$ endowed with its usual generating set is $3$-LG-rigid. They also proved that $3$ was optimal showing that $\mathbb{Z}^3$ is not LG-rigid at scale $2$.
Example 0.5. De la Salle et Tessera [dlST19, Theorem C] proved that every graph quasi-isometric to a tree is LG-rigid.

Benjamini [Ben13] and Georgakopoulos [Geo17] conjectured that any Cayley graph of a finitely presented group is LG-rigid at some scale $R > 0$. That conjecture was proven to be false in [dlST19, Theorem B], where the authors built counter-examples using groups with torsion elements.

Counter-example 0.6. The groups $F_2 \times F_2 \times \mathbb{Z}/2\mathbb{Z}$ and $SL_4(\mathbb{Z})$ admit Cayley graphs that are not LG-rigid.

Remark here that we do not state that every Cayley graph of these groups is non-LG-rigid, but that each group admits a non-LG-rigid Cayley graph. Indeed, in [dlST19, Theorem J] the authors also showed that every finitely presented group with an element of infinite order has a Cayley graph which is LG-rigid. Hence, LG-rigidity for a Cayley graph depends on the generating set. In particular LG-rigidity is not invariant under quasi-isometries.

With a little bit more of material, we will be able to give a topological interpretation of Local-to-Global rigidity (see page 8).

That rigidity notion can be refined in what is called the Strong Local-to-Global rigidity, also named SLG-rigidity.

Definition 0.7

Let $r, R > 0$. We say that $X$ is **SLG-rigid** at scale $(r, R)$ if for all $Y$ which is $R$-locally $X$ and for all isometry $f$ from $B_X(x, r)$ to $B_Y(y, R)$, the restriction of $f$ to $B_X(x, r)$ extends to a covering of $Y$ by $X$.

Such a refinement is far more than just a subtlety: it actually proves necessary to obtain our main result (see page 19 for more details).

The following proposition gives us many examples of SLG-rigid graphs.

Proposition 0.8 (de la Salle, Tessera [dlST19, Proposition 3.8])

A graph with cocompact isometry group is LG-rigid if and only if it is SLG-rigid.

For example, any LG-rigid Cayley graph is actually SLG-rigid. In the same article, de la Salle and Tessera proved a powerful condition relating to the isometry group of a Cayley graph. We will refer to the isometry group of a Cayley graph $(\Gamma, S)$ as $Is(\Gamma, S)$.

Theorem 0.9 (de la Salle, Tessera [dlST19, Theorem E])

Let $\Gamma$ be a finitely presented group and $S$ be a symmetric generating set and denote by $(\Gamma, S)$ the corresponding Cayley graph. If $Is(\Gamma, S)$ is discrete, then $(\Gamma, S)$ is SLG-rigid.

As stated in [dlST19, Corollary F], we can deduce two new classes of examples from the above theorem.
Example 0.10. Torsion-free groups of polynomial growth are SLG-rigid.

Example 0.11. Torsion-free, non-virtually free lattices in connected simple real Lie groups are SLG-rigid.

So far, the graphs chosen as examples are mostly Cayley graphs, but these are not the only LG-rigid ones. Indeed, besides the case of quasi-trees seen above, another interesting example is given by Bruhat-Tits buildings (see Section 1.1 for a definition).

Theorem 0.12 (de la Salle, Tessera, [dlST16, Theorem 0.1])

Let \( p \) be a prime number and \( n \geq 4 \).

The Bruhat-Tits building of \( \text{PSL}_n(\mathbb{Q}_p) \) is SLG-rigid.

Keeping in mind the above theorem, consider the following question asked in [dlST19].

Question 0.13. Among lattices in semi-simple Lie groups, which ones are LG-rigid?

This question concerns real Lie groups but one can also wonder what happens for the \( p \)-adic case. Indeed, by a well known result of Svarc and Milnor, any lattice of \( \text{SL}_n(\mathbb{Q}_p) \) is quasi-isometric to this building (see Lemma 4.2 for more details). The fact that such a lattice is “almost” a building encouraged us to study the \( p \)-adic version of question 0.13.

Question 0.14. Among lattices in \( p \)-adic Lie groups, which ones are LG-rigid?

Our first result provides an element of response to that question in the case of lattices in \( \text{SL}_n(\mathbb{Q}_p) \).

Theorem 0.15

Let \( p \) be a prime number and \( n \geq 4 \).

The torsion-free lattices of \( \text{SL}_n(\mathbb{Q}_p) \) are SLG-rigid.

This result is actually a corollary of our main theorem below which goes beyond the lattices framework and gives a rigidity result in a more general case.

Theorem 0.16

Let \( p \) be a prime and \( n \geq 4 \). Let \( \mathcal{X} \) be the Bruhat-Tits building of \( \text{PSL}_n(\mathbb{Q}_p) \) and \( X \) be a transitive graph. If \( X \) verifies that

- There is an injective homomorphism \( \rho \) from \( \text{Is}(X) \) to \( \text{Is}(\mathcal{X}) \) such that \( \rho(\text{Is}(X)) \) is of finite index in \( \text{Is}(\mathcal{X}) \);
- There is a \( \text{Is}(X) \)-equivariant injective quasi-isometry \( q \) from \( X \) to \( \mathcal{X} \);

then \( X \) is SLG-rigid.

Before moving to the sketch of the proof let us discuss the hypothesis made on the torsion in Theorem 0.15. First, introducing torsion in a group is in some case a useful way to build non-LG-rigid graphs. Indeed the counter-example 0.6 is built this way. Second, in order to link \( (\Gamma, S) \) to \( \mathcal{X} \) we will need an injection of \( \text{Is}(\Gamma, S) \) into \( \text{Is}(\mathcal{X}) \). Using a famous result of Kleiner and Leeb we will show that \( \text{Is}(\Gamma, S) \) acts on the buildings by isometries. The injection into \( \text{Is}(\mathcal{X}) \) will then be allowed by the following proposition.

Proposition 0.17 (de la Salle, Tessera [dlST19, Proposition 6.2])

Let \( \Gamma \) be an infinite, torsion-free, finitely generated group and let \( S \) be a finite symmetric generating subset of \( \Gamma \). Then the isometry group of \( (\Gamma, S) \) has no non-trivial compact normal subgroup.

For more details on how we use this proposition, see the proof of Lemma 4.3.
Introduction

Sketch of the proof of Theorem 0.16 Let $p$ be a prime and $n \geq 4$ and denote by $X$ the Bruhat-Tits building of $\text{PSL}_n(\mathbb{Q}_p)$. Let $X$ be the studied graph and $Y$ be a graph $\mathbb{R}$-locally the same as $X$. The main idea of the proof is to use the rigidity of $X$ to build the wanted covering from $X$ to $Y$ (see Figure 2), thus we need to build a graph locally the same as $X$. We will denote such a graph $\mathcal{Y}$.

Moreover, for the rigidity of the building to induce a covering between $X$ and $Y$, we want $\mathcal{Y}$ to contain a copy of the vertices of $Y$. Hence the goal is to define the vertices of $\mathcal{Y}$ to be composed of the vertices of $Y$ and a copy of each vertex in $X \backslash \text{q}(X)$ and define the edges to correspond to edges in $X$. With such a description $\mathcal{Y}$ is a “hybrid” graph and to define its edges we might need to understand how to link a vertex coming from $Y$ to a vertex coming from $X$. Hence, to avoid such a hybridation we chose to define the vertices only with informations encoded in $Y$. That is why we introduce the notion of print in the building (see Section 2.1). It allows us to characterize a vertex in $X$ by a set of neighbouring vertices in $\text{im}(q)$ and, using a well chosen set of isometries from $Y$ to $X$, to transfer this print notion to $Y$. Each print in $Y$ corresponding to a vertex in $X \backslash \text{q}(X)$. The vertices of the wanted graph $\mathcal{Y}$ will be composed of the vertices of $Y$ and of prints in $Y$. It will now be easier to build edges between these vertices; the key argument to construct such edges is presented in Section 1.3.

Using the rigidity of the building we will obtain an isometry between $X$ and $\mathcal{Y}$. To conclude the proof we will show that this isometry induces the wanted covering between $Y$ and $X$.

Organization of the paper The first section is devoted to the definition of our framework. We recall some material about Bruhat-Tits buildings and large scale simple connectedness and present a fundamental result on isometries’ extension. The second and third sections are devoted to the proof of Theorem 0.16. In the second section we develop the necessary engineering to build a graph locally the same as the building —this is where we define and study prints— while in the third one we use the rigidity of the building to prove the rigidity of the studied graph. We prove Theorem 0.15 in the fourth section where we check that the lattice verifies the hypothesis of our main theorem.

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1 Framework

Let us start by setting up the framework of the next sections. We first recall some material about Bruhat-Tits buildings, and large scale simple connectedness. Then we present a useful tool concerning the extension of isometries. We conclude by a result one step further in to the proof of our main theorem, linking the \( \text{PSL}_n(Q_p) \)-orbits in the building and the image \( q(X) \) of the graph studied.

1.1 Bruhat-Tits building

Let \( n \geq 2 \). Since it is the object at the center of our proof, let us recall the description of the Bruhat-Tits building associated to \( \text{PSL}_n(Q_p) \) where \( n \geq 2 \), see \[AB18\] for more details.

A \( Z_p \)-lattice of \( Q^n \) is a \( Z_p \)-submodule which generates \( Q_n \) as a \( Q_p \) vector space. Such a lattice can be written as \( Z_p e_1 + \cdots + Z_p e_n \) for a basis \( (e_1, \ldots, e_n) \) of \( Q^n \). Since for any \( a \in Q_p^* \) and any lattice \( L \), the module \( aL \) is also a lattice, we can define the equivalence relation of \( lattices \ mod \ lattices \ homothety \). We denote by \([L]\) the class of a lattice \( L \).

The Bruhat-Tits building of \( \text{PSL}_n(Q_p) \) is a simplicial complex of dimension \( n-1 \) whose 1-skeleton (denoted by \( \mathcal{X} \)) is described as follows. The vertices are the classes of \( Z_p \)-lattices modulo homothety. Two vertices \( x_1 \) and \( x_2 \) are linked by an edge if there exists representatives \( L_1 \) of \( x_1 \) and \( L_2 \) of \( x_2 \) such that:

\[
pL_1 \subset L_2 \subset L_1.
\]

Example 1.1. One can show that the building of \( \text{PSL}_2(Q_p) \) is a \((p + 1)\)-regular tree. Figure 3a gives a representation of the building when \( p = 2 \).

(a) The building has two \( SL_2(Q_2) \)-orbits
(b) Representation of one apartment

Figure 3: The building of \( \text{PSL}_2(Q_2) \)

The usual action of \( GL_n(Q_p) \) on \( Q^n \) induces an action of \( \text{PGL}_n(Q_p) \) on \( \mathcal{X} \) by isometry. Since \( GL_n(Q_p) \) acts transitively on the bases, the action of \( \text{PGL}_n(Q_p) \) on the vertices of \( \mathcal{X} \) is also transitive.

Denote by \( v \) the \( p \)-adic valuation on \( Q_p \). If \( L = \bigoplus_i Z_p e_i \) is a lattice we define its type to be \( v(\det(e_1, \ldots, e_n)) \). Since:

\[
\forall a \in Q_p^* \quad v(\det(ae_1, \ldots, ae_n)) = v(\det(e_1, \ldots, e_n)) \mod n,
\]
one can define the type of a vertex \( x \) in \( \mathcal{X} \) to be the value modulo \( n \) of the type of one of its representatives. We denote by \( \tau(x) \) the type of \( x \).

If \( L' \) is a second lattice, we can choose our basis \( e_1, \ldots, e_n \) for \( L \) in such a way that \( L' \) admits a basis of the form \( a_1 e_1, \ldots, a_n e_n \) for some \( a_i \in \mathbb{Q}_p^* \). The scalars \( a_i \) can be taken to be powers of \( p \). The incidence relation defined above implies that if the classes of \( L \) and \( L' \) are linked by an edge in \( \mathcal{X} \), then they have different types. The action of \( \text{SL}_n(\mathbb{Q}_p) \) on \( \mathcal{X} \) preserves the determinant and is transitive on the pairs of vertices of the same type. So there are exactly \( n \) orbits under the action of \( \text{SL}_n(\mathbb{Q}_p) \) (see Figure 3a and Figure 4 for examples).

If \( e \) is a basis of \( \mathbb{Q}_p^n \) then the sub-complex \( \mathcal{A} \) induced by the set of vertices \( \{ \oplus_{i=1}^n \mathbb{Z}_p p^{k_i} e_i \mid k_i \in \mathbb{Z} \} \) is isometric to a \((n-1)\)-dimensional Euclidean space tiled by regular \((n-1)\)-simplices. We call such sub-complexes apartments. For example an apartment in the building of \( \text{PSL}_2(\mathbb{Q}_2) \) is isometric to \( \mathbb{R}^1 \) tiled with segments of length 1 (see Figure 3b), whereas for \( \text{PSL}_3(\mathbb{Q}_2) \) the apartment are isometric to \( \mathbb{R}^2 \) and tiled with triangles (see Figure 4).

![Figure 4: Apartment in the building of PSL_3(Q_2). The colors correspond to SL_3(Q_2)-orbits.](image)

### 1.2 Large scale simple connectedness

For a graph \( \mathcal{G} \) and \( k \in \mathbb{N} \), we define a 2-complex, noted \( P_k(\mathcal{G}) \), such that:

- Its 1-skeleton is given by \( \mathcal{G} \);
- Its 2-skeleton is composed of \( m \)-gons (for \( m \in [0,k] \)) defined by the simple loops of length \( m \) in \( \mathcal{G} \) (up to cyclic permutations).

**Definition 1.2**

We say that \( \mathcal{G} \) is \( k \)-simply connected or simply connected at scale \( k \) if \( P_k(\mathcal{G}) \) is simply connected.

**Example 1.3.** Let \( G \) be a finitely generated group and \( T \) a finite symmetric generating set. The Cayley graph \( (G,T) \) is simply connected at scale \( k \) if and only if \( G \) has a presentation \( \langle T, \mathcal{R} \rangle \) with relations of length at most \( k \).

Remark that if \( k \leq k' \), then every \( k \)-simply connected graph is \( k' \)-simply connected. The following proposition allows us to restrict the study of the LG-rigidity of a graph \( \mathcal{G} \) to some smaller class of graphs.

**Proposition 1.4 (de la Salle, Tessera, [dlST16, Proposition 1.5])**

Let \( k \in \mathbb{N} \) and \( \mathcal{G} \) be a \( k \)-simply connected graph, with cocompact isometry group. Then \( \mathcal{G} \) is LG-rigid if and only if there exists \( R \) such that every \( k \)-simply connected graph which is \( R \)-locally \( \mathcal{G} \) is isometric to \( \mathcal{G} \).

To apply this result to our proof we will need to show that the studied graph \( \mathcal{X} \) is simply connected. The following proposition shows that being simply connected is invariant under quasi-isometry.
Proposition 1.5 (de la Salle, Tessera, [dlST16, Theorem 2.2])

Let $k \in \mathbb{N}^*$ and let $\mathcal{G}$ be a $k$-simply connected graph. If $\mathcal{H}$ is quasi-isometric to $\mathcal{G}$, then there exists $k' \in \mathbb{N}^*$ such that $\mathcal{H}$ is simply connected at scale $k'$.

Before moving to the next section, let us mention a consequence of that last property. Indeed, this result allows us to look at the LG-rigidity notion with a topological point of view. Let’s denote $\mathfrak{G}_k$ the set of isometry classes of locally finite $k$-simply connected graphs. We can define a distance on this set by:

$$d_{\mathfrak{G}_k}(X,Y) := \inf \{2^{-r} : X \text{ and } Y \text{ are } \mathbb{R}\text{-close}\},$$

which endows $\mathfrak{G}_k$ with a topology. The above proposition implies that a graph is LG-rigid if and only if its isometry class in $\mathfrak{G}_k$ is isolated for this topology.

### 1.3 Extension of isometries

In order to build the “hybrid” graph mentioned above, we will need some result to extend globally our local definition of edges. We recall here the result of de la Salle and Tessera [dlST19, Lemma 4.1] that will serve our purpose.

Proposition 1.6 (de la Salle, Tessera)

Let $\mathcal{G}$ be a graph with cocompact discrete isometry group. Given some $r_1 \geq 0$, there exists $r_2 > 0$ such that, for every $g \in \mathcal{G}$, the restriction to $B_{\mathcal{G}}(g,r_1)$ of an isometry $f : B_{\mathcal{G}}(g,r_2) \rightarrow \mathcal{G}$ coincides with the restriction of an element of $\text{Is}(\mathcal{G})$.

It is however not necessarily true that $f$ coincides on the whole $B(g,r_2)$ with an isometry of $\mathcal{G}$. Indeed, truncating the entire graph to some ball might allow some kind of flexibility near the boundary of the ball. Hence, in order to coincide with a global isometry we need to restrict $f$ to a smaller ball which does not contain the flexible area.

### 1.4 Preliminary results on $X$

Lemma 1.7

If $X$ verifies the hypothesis of Theorem 0.16, then $\text{PSL}_n(\mathbb{Q}_p)$ is included in $\rho(\text{Is}(X))$. Moreover, if $q(X)$ contains a vertex of a certain type $i$, then $q(X)$ contains all the vertices of type $i$.

Proof. Since $\rho(\text{Is}(X))$ is of finite index in $\text{Is}(X)$, the same goes for its normal core $\cap_{g \in \text{Is}(X)} \rho(\text{Is}(X)) g^{-1}$. Then, by simplicity of $\text{PSL}_n(\mathbb{Q}_p)$, the normal core of $\rho(\text{Is}(X))$ contains $\text{PSL}_n(\mathbb{Q}_p)$. Hence the result.

Then, the second part of the lemma follows from the equivariance of $q$ and the transitivity of $\text{PSL}_n(\mathbb{Q}_p)$ on vertices of the same type.

Without loss of generality, we can assume that $\text{im}(q)$ contains type 0 vertices, that is to say $\tau^{-1}(0) \subset \text{im}(q)$. Moreover, using Proposition 1.5 we obtain that $X$ is simply connected at some scale $k > 0$.

\*\*

The aim of the next two sections is to prove Theorem 0.16. For the sake of clarity we recapitulate here the needed assumptions for the proof.
Hypothesis (H)

1. Let $X$ be a $k$-simply-connected transitive graph;
2. Let $Y$ be a graph $R$-locally $X$ and $k$-simply connected;
3. Let $p$ be a prime, let $n \geq 4$ and $X$ be the Bruhat-Tits building of $\text{PSL}_n(\mathbb{Q}_p)$;
4. Let $\rho : \text{Is}(X) \to \text{Is}(X)$ be an injective homomorphism and $q : X \to X$ an $\text{Is}(X)$-equivariant injective quasi-isometry;
5. Assume that $\rho(\text{Is}(X))$ is of finite index in $\text{Is}(X)$ and that $q(X)$ contains $\tau^{-1}(0)$.

2 Tracking vertices through their imprints

This section is dedicated to the definition of a graph locally the same as $X$ which we will call $Y$. Before moving to the detailed definition let us explain the idea of the construction. Recall that the vertices of $X$ are partitioned into different types (see Section 1.1) denoted by integers in $\{0, \ldots, n-1\}$. By Lemma 1.7, if $q(X)$ contains a vertex of a certain type then it contains all the vertices of that type. Denote by $T$ the set of types that are not contained in $q(X)$, namely $T = \{0, \ldots, n-1\} \setminus \tau(q(X))$. We have the following partition

$$X = q(X) \cup (\bigcup_{i \in T} \tau^{-1}(i)).$$

Example 2.1. Take $p = 2$ and assume that $\text{im}(q)$ is composed only of type zero vertices. When $n = 2$ we have $T = \{1\}$ and the building is represented in Figure 3a. The partition in eq. (1) corresponds to the partition of vertices in two different colors.

When $n = 3$, we get $T = \{1, 2\}$. An apartment of $X$ is represented in Figure 4 and the partition of this part of $X$ corresponds to the partition in three different colors.

Example 2.2. Let $n = 4$ and $p = 2$ and assume that $\text{im}(q)$ contains type zero and type 2 vertices. Then $T = \{1, 3\}$. We will not try to represent $X$ or an apartment but recall that it is tiled by tetrahedrons. The partition is illustrated on a tetrahedron in Figure 5, where $\text{im}(q)$ corresponds to the two blue vertices.

![Figure 5: Partition of a simplex](image)

The idea of the construction of $Y$ is to take the vertices of $Y$ and add to them vertices of the missing types, i.e. vertices with type in $T$ (see Figure 9 for an example). But we want to build this vertices only
with informations encoded in $V(Y)$. That is why we introduce the local characterization of a vertex in the building (see Section 2.1). Then, using a well chosen set of isometries from $Y$ to $X$, we transfer this print notion to $Y$, each print in $Y$ corresponding to a vertex of a missing type.

## 2.1 Prints in a building

In this section we show that a vertex in $X$ can be determined by a part of its 1-neighbourhood. More precisely, we prove that a vertex in the building is entirely determined by its type and the vertices in its 1-neighbourhood having type zero.

**Definition 2.3**

Let $x$ be a vertex of $X$. We define the print of $x$, denoted by $\mathcal{P}(x)$, to be the intersection of the 1-neighbourhood of $x$ with the vertices of type zero, viz. $\mathcal{P}(x) := B_X(x, 1) \cap \tau^{-1}(0)$.

**Example 2.4.** Figure 6 represents a ball of radius 1 when $n = 2 = p$ and when $p = 2$ and $n = 3$. In each case, the print of $x$ corresponds to the set of blue vertices.

![Figure 6: Prints and 1-neighbourhood of a vertex in $X$](image)

The following result proves that a vertex in $X$ is uniquely determined by its type and its print.

**Proposition 2.5**

Let $i \in \{0, \ldots, n\}$. If $x_1$ and $x_2$ are two vertices of type $i$ in $X$ such that $\mathcal{P}(x_1) = \mathcal{P}(x_2)$, then $x_1 = x_2$.

To prove this property, we show that if such $x_1$ and $x_2$ are different then, there exists an isometry of the building sending one to the other and acting like a translation on an apartment containing them. Noting that such an isometry belongs to the stabilizer of $\mathcal{P}(x_1)$, we arrive at a contradiction since this stabilizer has to be compact.

**Proof.** Let $x_1, x_2 \in X$ be such that $\tau(x_1) = i = \tau(x_2)$. Denote by $K$ the stabilizer of $\mathcal{P}(x_1)$ under the action of $\text{Is}(X)$. Since $\mathcal{P}(x_1)$ is finite, the stabilizer $K$ is compact. Moreover if $\alpha$ belongs to $\text{PSL}_n(\mathbb{Q}_p)$ and sends $x_1$ to $x_2$, then $\alpha$ belongs to $K$.

Let $\mathcal{A}$ be an apartment containing both $x_1$ and $x_2$. There exists a basis $e := (e_1, \ldots, e_n)$ of $\mathbb{Q}_p^n$ such that $A = \{ \oplus_{i=1}^n \mathbb{Z}_p k_i e_i \mid k_i \in \mathbb{Z} \}$. Now let $(c_1, \ldots, c_n)$ and $(d_1, \ldots, d_n)$ be two $n$-tuples of integers verifying

$$x_1 := \left[ \oplus_{i=1}^n \mathbb{Z}_p k_i e_i \right], \quad x_2 := \left[ \oplus_{i=1}^n \mathbb{Z}_p d_i e_i \right].$$
Let \( \alpha \) be the element of \( \text{GL}_n(\mathbb{Q}_p) \) having as matrix in \( \mathbf{e} \)

\[
\text{Mat}_\mathbf{e}(\alpha) = \begin{pmatrix}
p^{d_1 - c_1} & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
p^{d_n - c_n} & \cdots & \cdots & \cdots 
\end{pmatrix}.
\]

Then \( \alpha(x_1) = \left[ \oplus_{i=1}^n \mathbb{Z}_p p^{c_i} \alpha(e_i) \right] = \left[ \oplus_{i=1}^n \mathbb{Z}_p p^{c_i} p^{d_i - c_i} e_i \right] = x_2 \). Moreover since \( \tau(x_1) = \tau(x_2) \), the value of \( \sum_{i=1}^n c_i \) modulo \( n \) is the same as \( \sum_{i=1}^n d_i \).

Thus \( v(\det \text{Mat}_\mathbf{e}(\alpha)) = v \left( p^{d_1 - c_1} \cdots p^{d_n - c_n} \right) = \sum_{i=1}^n d_i - c_i = 0 \mod n. \)

Hence \( \alpha \) belongs to \( \text{PSL}_n(\mathbb{Q}_p) \) and thus belongs to \( \mathcal{K} \).

Assume that \( x_1 \neq x_2 \), then the difference \( d_i - c_i \) is non-constant and \( \alpha \) is not a homothety. Hence \( \alpha \) acts as a non-trivial translation on \( \mathcal{A} \) and generates a subgroup with non-compact closure. Recalling that \( \alpha \) belongs to the compact group \( \mathcal{K} \) we obtain the wanted contradiction. Thus \( \alpha \) is a homothety and \( x_1 = x_2 \).

This proves that a vertex in \( \mathcal{X} \) is uniquely determined by its print and its type. Thus, we can introduce the following definition without ambiguity.

**Definition 2.6**
Let \( x \) be a vertex of type \( i \) in \( \mathcal{X} \). We say that \( x \) is the source (of type \( i \)) of \( \mathcal{P}(x) \).

In order to prove Theorem 0.16, we will need to know how prints behave under the action of \( \text{PSL}_n(\mathbb{Q}_p) \). So, let \( x \in \mathcal{X} \) of type \( i \in \{0, \ldots, n\} \) and let \( \alpha \in \text{PSL}_n(\mathbb{Q}_p) \). Since \( \alpha \) is an isometry and is type-preserving, we get

\[
\alpha \left( \mathcal{P}(x) \right) = \alpha \left( B(x, 1) \cap \tau^{-1}(0) \right) = \alpha \left( B(x, 1) \right) \cap \alpha \tau^{-1}(0) = B \left( \alpha(x), 1 \right) \cap \tau^{-1}(0).
\]

We deduce the following lemma.

**Lemma 2.7**
Let \( x \in \mathcal{X} \) of type \( i \in \{0, \ldots, n\} \). If \( \alpha \) belongs to \( \text{PSL}_n(\mathbb{Q}_p) \), then \( \alpha \left( \mathcal{P}(x) \right) = \mathcal{P} \left( \alpha(x) \right) \).

### 2.2 Atlas of local isometries

To build our graph locally the same as \( \mathcal{X} \), we need to restrict ourselves to a particular set of local isometries from \( \mathcal{Y} \) to \( \mathcal{X} \). More precisely, if \( y_1 \) and \( y_2 \) are close in \( \mathcal{Y} \) and \( f_1 \) (resp. \( f_2 \)) is an isometry from \( B_\mathcal{Y}(y_1, R) \) (resp. \( B_\mathcal{Y}(y_1, R) \)) to \( \mathcal{X} \), we want the transition map \( f_2 f_1^{-1} \) to coincide with an element in \( \rho^{-1}\text{PSL}_n(\mathbb{Q}_p) \) on a small ball. This is what we formalize here and schematize in Figure 7.

**Definition 2.8**
Let \( \mathfrak{A} \) be a set of isometries from balls of radius \( R \) in \( \mathcal{Y} \) to \( \mathcal{X} \). We say that \( \mathfrak{A} \) is an atlas of local isometries from \( \mathcal{Y} \) to \( \mathcal{X} \) if the map that associates to each isometry in \( \mathfrak{A} \) the center of its ball of definition is a bijection from \( \mathfrak{A} \) to \( \mathcal{Y} \). That is to say, we can write

\[
\mathfrak{A} := \left\{ f_y : B_\mathcal{Y}(y, R) \to X \mid y \in \mathcal{Y} \right\},
\]

where the map that associates \( f_y \) to \( y \) is bijective.

We say that \( f_y \) is the isometry associated to \( y \) in \( \mathfrak{A} \).
Let $H_0 := \rho^{-1} \text{PSL}_n(\mathbb{Q}_p)$. Now, we show that we can construct an atlas of local isometries from $Y$ to $X$ such that the transition maps between two isometries defined on balls with neighbouring centers coincide with elements of $H_0$. We will note a path between two vertices $v_1$ and $v_2$ as a sequence $(v_1, \ldots, v_l)$ of adjacent vertices.

**Lemma 2.9**

Let $r_A > 0$ and let $H_0 := \rho^{-1} \text{PSL}_n(\mathbb{Q}_p)$. For $R$ large enough, if $Y$ is $R$-locally $X$, then there exists an atlas $\mathfrak{A}$ such that for any two neighbours $y$ and $z$ in $Y$

$$
\exists a_0 \in H_0 \quad f_y \cdot f_z^{-1}|_{B(f_z(z), r_A)} = a_{B(f_z(z), r_A)}.
$$

Before proving it, let us schematize the framework of this lemma. In Figure 7 we represent two isometries $f_y$ and $f_z$ with $z$ neighbour to $y$. The larger discs correspond to balls of radius $R$ and the smaller ones to balls of radius $r_A$. The map $f_y f_z^{-1}$ restricted to $B(f_z(z), r_A)$ takes $f_z(z)$ to $f_y(z)$ which is a neighbour of $f_y(y)$ and coincide on this ball with and element in $H_0$.

![Figure 7: Composition of isometries with neighbouring centers](image)

Let us discuss the idea of the proof. First, for two neighbours $y$ and $z$ we use Proposition 1.6 to prove that $f_y f_z^{-1}$ coincides on a small ball with an element $a$ in $\text{Is}(X)$. This isometry corresponds to the “default” of belonging to $H_0$ we want to correct. Hence, we consider in our atlas the new isometry defined on $B(z, R)$ by $af_z$. Finally, we extend this construction along paths in $Y$ and prove that the wanted property for $\mathfrak{A}$ does not depend on the choice of path.

**Proof.** Let $r_A > 0$ and let $H_0 := \rho^{-1} \text{PSL}_n(\mathbb{Q}_p)$. Now, let $y \in Y$ and $f_y$ be an isometry from $B(y, R)$ to $X$. Let $z$ be a neighbour of $y$ in $Y$ and $f_z$ be an isometry from $B(z, R)$ to $X$. Then the map

$$
f_y \cdot f_z^{-1} : B_X \left( f_z(z), R - 1 \right) \to B_X \left( f_y(z), R - 1 \right)
$$

is a well defined local-isometry of $X$. By Proposition 1.6 if $R$ is large enough, there exists $a$ in $\text{Is}(X)$ such that $f_y \cdot f_z^{-1}$ coincides with $a$ on $B_X \left( f_z(z), r_A + k \right)$, where we recall that $k$ refers to the scale at which $Y$ is simply connected. We will see below why we need to consider such a radius.
Now let \( f_z := a f_z \). By definition we have

\[
    f_z : \begin{cases} 
        B_Y(z, R) & \rightarrow B \left( f_y(z), R \right), \\
        z & \mapsto a f_z(z) = f_y(z), 
    \end{cases}
\]

thus the transition map \( f_y f_z^{-1} \) is well defined on \( B_X(f_z(z), R - 1) \). Moreover, by choice of \( f_z \) we get that \( f_y f_z^{-1} \) restricted to \( B(f_y(z), r_A + k) \) coincides with the identity and thus belongs to \( H_0 \).

Extending this construction along paths in \( Y \) we get an atlas \( \mathfrak{A} \) of local isometries from \( Y \) to \( X \).

Now if \( y \in Y \) and \( f_y \) is the associated isometry in \( \mathfrak{A} \), we want to show that (up to a multiplication by an element in \( \text{PSL}_n(Q_p) \)) this isometry does not depend on the choice of path. So let \( y \in Y \) and \( (y_0, y_1, \ldots, y_l = y) \) be a loop of length \( l \). Take \( f_0 \) to be an isometry from \( B_Y(y_0, R) \) to \( X \) and using the process detailed above, build a sequence of isometries \( f_1, \ldots, f_l \) such that \( f_i \) is defined on \( B_Y(y_i, R) \) and

\[
    \forall i \in \{1, \ldots, l\} \exists a_i \in H_0 \mid \left( f_{i-1} f_i^{-1} \right)_{|B(f_i(y_i), r_A + k)} = a_i \left| B(f_i(y_i), r_A + k) \right.
\]

We have to prove that the restrictions to \( B(y_0, r_A) \) of \( f_0 \) and \( f_m \) are equal up to a multiplication by an element in \( H_0 \). Since \( Y \) is simply connected at scale \( k \), we only have to prove this for loop of length smaller than \( k \). Hence, we assume that \( l \leq k \).

First, remark that for all \( i \in \{0, \ldots, l - 1\} \)

\[
    \begin{cases} 
        f_{i-1} f_i^{-1} : B_X(f_i(y_i), r_A + k) & \rightarrow B_X(f_{i-1}(y_i), r_A + k), \\
        f_i f_{i+1}^{-1} : B_X(f_{i+1}(y_{i+1}), r_A + k) & \rightarrow B_X(f_i(y_{i+1}), r_A + k). 
    \end{cases}
\]

Now since \( y_i \) and \( y_{i+1} \) are at distance \( 1 \), the ball \( B_X(f_i(y_i), r_A + k - 1) \) is included in \( B_X(f_{i+1}(y_{i+1}), r_A + k) \). Hence the map \( (f_{i-1} f_i^{-1}) f_i f_{i+1}^{-1} \) is well defined and coincides with \( a_i a_{i+1} \) on \( B_X(f_{i+1}(y_{i+1}), r_A + k - 1) \). By induction we get that for all \( x \) in \( B_X(f_{i+1}(y_{i+1}), r_A + k - 1) \)

\[
    f_0 f_1 \cdots f_{l-1} (x) = (f_0 f_1^{-1}) \cdots (f_{l-1} f_l^{-1}) (x) = a_1 \cdots a_l (x).
\]

Since \( \prod_{i=1}^l a_i \) belongs to \( H_0 \) and \( l \) is smaller than \( k \), it implies that \( f_0 \) is equal to \( f_1 \) on \( B_Y(y_0, r_A) \) up to multiplication by an element in \( H_0 \).

The atlas is defined such that a transition map between two isometries defined on balls with neighbouring centers belongs to \( H_0 \). But in fact, this property is also true when the centers are at a slightly bigger distance.

**Lemma 2.10**

Let \( r > 0 \) and \( \mathfrak{A} \) be an atlas verifying the conditions of Lemma 2.9 with \( r_A > 3r \). Let \( y \) and \( z \) in \( Y \) be at distance less than \( 2r \) and \( f_y, f_z \) the associated isometries in \( \mathfrak{A} \). Then

\[
    \exists a \in H_0 \quad \left( f_y f_z^{-1} \right)_{|B_Y(z, r)} = a \left| B_Y(z, r) \right.
\]

**Proof.** Let \( r > 0 \) and assume \( r_A > 3r \). Let \( y, z \in Y \) be at distance \( l \leq 2r \) and let \( f_y, f_z \) be two elements of \( \mathfrak{A} \) such that

\[
    f_y : B_Y(y, R) \rightarrow X \quad f_z : B_Y(z, R) \rightarrow X.
\]

Take \( (y_0 = y, y_1, \ldots, y_l = z) \) to be a geodesic between \( y \) and \( z \), and for all \( i \in \{0, \ldots, l\} \), let \( f_i \in \mathfrak{A} \) be the
2. Tracking vertices through their imprints

isometry associated to \( y_i \). Remark that by definition of an atlas, it implies \( f_0 = f_y \) and \( f_1 = f_z \) and

\[
\forall i \in \{0, \ldots, l-1\} \quad \exists a_i \in H_0 \quad (f_{i+1}^{-1} f_i)_{|B(0, r_A)} = a_i |_{B(f_i+1(y_i+1), r_A)}.
\]

Now, if \( r_A > 3r \) and \( l \leq 2r \), then \( B_{Y}(z, r) \) is contained in \( B_{Y}(y, r_A) \). Hence the composition of transition maps \( (f_0 f_1^{-1}) \cdots (f_{l-1} f_l^{-1}) \) is well defined on \( B_{Y}(t_i(y_i), r_A - 1) \) and verifies on that ball

\[
f_0 f_1^{-1} = (f_0 f_1^{-1}) \cdots (f_{l-1} f_l^{-1}) = a_0 \cdots a_{l-1}.
\]

Hence the result.

2.3 Prints in \( Y \)

Using the atlas built above, we can now transfer this print notion to the graph \( Y \). Let \( r_p > 0 \) and assume that \( Y \) is endowed with an atlas of isometries \( \mathfrak{A} \) as given by Lemma 2.9 with \( r_A > 3r_p \). Hence, we have

\[
R > r_A > 3r_p > r_p.
\]

**Definition 2.11**

Let \( P \) be a set of vertices in \( Y \) and \( i \in \{1, \ldots, n\} \). We say that \( P \) is a **print of type** \( i \) if there exists \( y \) in \( Y \) and \( f \in \mathfrak{A} \) an isometry from \( B_{Y}(y, R) \) to \( X \) such that

- The set \( P \) is contained in \( B_{Y}(y, r_p) \);
- There exists \( x \in X \) such that \( \tau(x) = i \) and \( \mathcal{P}(x) = qf(P) \).

**Example 2.12.** If \( n = 3 \) and \( p = 2 \) there are exactly 3 types of vertices, each represented in Figure 8 by a different color. The 1-neighbourhood of a vertex \( x \) in \( X \) is then composed of fourteen vertices, represented on the right side of the aforementioned figure. If \( x \) is of type “brown” then these fourteen vertices are composed of seven vertices of type “orange” and seven of type “blue”. On the left side of the figure is represented \( P \) (the black dots) inside \( B(y, r_p) \) (the darker disc). The set \( qf(P) \) is exactly the set of blue vertices. Hence \( P \) is a print of type “brown”.

![Figure 8: Definition of a print in \( Y \)](image)

For now, let's say that \( P \) verifying the definition above is a print of type \( i \) associated to \( y \) and \( f \). We are going to show that this definition depends neither on \( y \) nor \( f \).
Lemma 2.13

Let $i \in T$, let $y_1, y_2 \in Y$ and $f_1, f_2$ be the associated isometries in $\mathfrak{X}$. Let $P$ be a print of type $i$ associated to $y_1$ and $f_1$. If $P \subset B(y_2, r_P)$ then $P$ is a print of type $i$ associated to $y_2$ and $f_2$.

Proof. First, remark that since $P \subset B(y_2, r_P) \cap B(y_1, r_P)$, then taking any $y$ in $P$ we get

$$d_Y(y_1, y) \leq d_Y(y_1, y_2) + d_Y(y, y_2) \leq 2r_P.$$

Applying Lemma 2.10 with $r = r_P$, we get that there exists $a \in H_0$ such that $(f_1 f_2^{-1})|_{B_Y(f_2(y_2), r_P)} = a|_{B_Y(f_1(y_2), r_P)}$. Now let $x \in X$ be such that $\tau(x) = i$ and $P(x) = qf_1(P)$. Using the equivariance of $q$ and Lemma 2.7, we get

$$qf_2(P) = \rho(a)^{-1}qf_1(P) = \rho(a)^{-1}P(x) = P(\rho(a)^{-1}(x)).$$

Moreover $\rho(a)^{-1}(x)$ has type $i$ since $\rho(a)$ is type preserving.

Hence $P$ is a print of type $i$ associated to $y_2$ and $f_2$. \hfill $\Box$

This last lemma proves that being a print of type $i$ does not depend on the choice of local isometry.

2.4 Definition of $\mathfrak{Y}$: a building's replica

The following property defines the graph $\mathfrak{Y}$ we will demonstrate to be locally the same as $\mathfrak{X}$.

Proposition 2.14

Let $r_P > 0$ and $\mathfrak{A}$ be the atlas given by Lemma 2.9 for $r_A > 3r_P$. If $R$ is large enough, then the following graph is well defined.

Let $\mathfrak{Y}$ be the graph which vertices are given by

$$V(\mathfrak{Y}) := V(Y) \cup \{(i, P) : i \in T, P \text{ is a print of type } i\},$$

and edges are given by:

- If $y_1, y_2 \in V(\mathfrak{Y})$, then $(y_1, y_2)$ is an edge if there exists $z \in Y$ and $f \in \mathfrak{A}$ defined on $B_Y(z, R)$ such that $y_1, y_2 \in B(z, r_P)$ and $d_X(qf(y_1), qf(y_2)) = 1$.

- If $y \in V(\mathfrak{Y})$ and $P$ is a print of type $i$, then $(y, (i, P))$ is an edge if there exists $z \in Y$ and $f \in \mathfrak{A}$ defined on $B_Y(z, R)$ containing $y$ and $P$ and such that $qf(y)$ is at distance 1 from the source of $qf(P)$.

- If $P_1$ is a print of type $i$ and $P_2$ a print of type $j$, then $((i, P_1), (j, P_2))$ is an edge if there exists $z \in Y$ and $f \in \mathfrak{A}$ defined on $B_Y(z, R)$ such that $P_1, P_2 \subset B_Y(z, r_P)$ and such that the source of $qf(P_1)$ is at distance 1 from the source of $qf(P_2)$.

Before looking to the proof of this property, let us sketch some part of this graph.

Example 2.15. If $n = 4$ then $\mathfrak{X}$ is composed of vertices of type $0, 1, 2$ and $3$. Assume that $q(\mathfrak{X})$ is composed of vertices of type $0$ and $2$, then $T = \{1, 3\}$ and we saw the corresponding partition of $\mathfrak{X}$ in example 2.2 and Figure 5. The appearance of the corresponding $V(\mathfrak{Y})$ is represented in Figure 9.
Proof. Let \( Y \) be as in Proposition 2.14 and let us show that the definition of the edges does not depend on the choice of \( f \) in the atlas.

First, let \( y_1, y_2 \in Y \) and \( y, z \in Y \) such that \( y_1 \) and \( y_2 \) belongs to \( B(y, r_\mathcal{P}) \cap B(z, r_\mathcal{P}) \). Then, take two local maps \( f_y, f_z \) in \( \mathfrak{A} \) associated to \( y \) and \( z \) respectively. Then \( d(y, z) \leq 2r_\mathcal{P} \) and by Lemma 2.10 there exists \( a \in Is(X) \) verifying eq. (3). Hence, by \( Is(X) \)-equivariance of \( q \) we get

\[
\mathcal{X}(qf_y(y_1), qf_y(y_2)) = \mathcal{X}(\rho(a)qf_y(y_1), \rho(a)qf_y(y_2)) = \mathcal{X}(a f_\mathcal{P}(y_1), a f_\mathcal{P}(y_2)) = \mathcal{X}(qf_y(y_1), qf_y(y_2)).
\]

Thus \( \mathcal{X}(qf_y(y_1), qf_y(y_2)) = 1 \) if and only if \( \mathcal{X}(qf_y(y_1), qf_y(y_2)) = 1 \) and the definition of edges between two vertices of \( Y \) does not depend on the choice of local isometry.

Now take \( y \in Y \) and \( i \in T \) and let \( P \) be a print of type \( i \). Let \( z, z' \) such that \( y \) and \( P \) are contained in \( B(z, r_\mathcal{P}) \cap B(z', r_\mathcal{P}) \) and take \( f \) (resp. \( f' \)) in \( \mathfrak{A} \) defined on \( B(z, R) \) (resp. \( B(z', R) \)). Then \( d(z, z') \leq 2r_\mathcal{P} \) and by Lemma 2.10 there exists \( a \in Is(X) \) verifying eq. (3). Hence,

\[
\mathcal{X}(qf(y), x) = \mathcal{X}(\rho(a)qf(y), \rho(a)(x)) = \mathcal{X}(a f_\mathcal{P}(y), \rho(a)(x)) = \mathcal{X}(qf'(y), \rho(a)(x)).
\]

If \( x \) is the source of \( qf(P) \) then, by Lemma 2.7 we get

\[
\mathcal{P}(\rho(a)(x)) = \rho(a)(\mathcal{P}(x)) = \rho(a)qf(P) = qf'(P).
\]

Thus, the existence of an edge between \( y \) and \( (i, P) \) in \( \mathcal{Y} \) does not depend of the choice of map in \( \mathfrak{A} \).

Finally, take \( P_1 \) a print of type \( i \) and \( P_2 \) a print of type \( j \), and let \( z, z' \) in \( Y \) and \( f \) (resp. \( f' \)) defined on \( B_\mathcal{V}(z, R) \) (resp. \( B_\mathcal{V}(z', R) \)) such that \( P_1, P_2 \subset B_\mathcal{V}(z, r_\mathcal{P}) \cap B_\mathcal{V}(z', r_\mathcal{P}) \). Again \( d(z, z') \leq 2r_\mathcal{P} \) and by Lemma 2.10 there exists \( a \in Is(X) \) verifying eq. (3). Hence if \( x_1 \) is the source of \( qf(P_1) \) and \( x_2 \) the source of \( qf(P_2) \), then \( d(x_1, x_2) = 1 \) if and only if \( d(\rho(a)(x_1), \rho(a)(x_2)) = 1 \). Moreover, by Lemma 2.7

\[
\forall i = 1, 2 \quad \mathcal{P}(\rho(a)(x_i)) = \rho(a)(\mathcal{P}(x_i)) = \rho(a)qf(P_i) = qf'(P_i).
\]

Hence the existence of an edge between \( (i, P_1) \) and \( (j, P_2) \) in \( \mathcal{Y} \) does not depend of the choice of map in the atlas \( \mathfrak{A} \).

\[\square\]

3 From one graph to the other

In this section we prove the isometry between the graph \( \mathcal{Y} \) built and the Bruhat-Tits building and show that it induces an isometry between \( X \) and \( Y \).
3.1 Isometry with the building

We can now prove that $\mathcal{Y}$ is isometric the Bruhat-Tits building. Recall that $r_\alpha$ is the radius used to define our atlas $\mathfrak{A}$ (see Lemma 2.9) and $r_p$ is the radius used to define prints in $\mathcal{Y}$ (see Definition 2.11). These constants verify $R > r_\alpha > 3r_p > r_p$.

**Lemma 3.1**

Let $R_\mathcal{X} > 0$. If $r_p$ (and hence $R$) is large enough, then $\mathcal{Y}$ is $R_\mathcal{X}$-locally $\mathcal{X}$.

To prove this lemma, we define explicetly the local isometries on balls of radius $R_\mathcal{X}$ and prove that these maps are well defined injections. Then, we compute the minimal value of $r_p$ necessary for these applications to be surjective on balls of radius $R_\mathcal{X}$. We conclude by showing that these maps preserve the distance.

**Proof.** Let $v \in V(Y)$. If $v \in V(Y)$ let $f \in \mathfrak{A}$ be the isometry defined on $B_y(v, R)$. If $v = (i, P)$ let $y$ and $f \in \mathfrak{A}$ be such that $P$ is a print associated to $y$ and $f$. Our goal is to show that the map

$$
\Phi_f : \begin{cases}
B_y(v, R_\mathcal{X}) &\to \mathcal{X}, \\
X &\to qf(y), \\
(j, Q) &\to x \text{ where } \tau(x) = j, \text{ and } \mathcal{P}(x) = qf(Q),
\end{cases}
$$

is an isometry.

By Proposition 2.5, it is a well defined map. Moreover, using the injectivity of $q$ and Proposition 2.5 and eq. (i) we get that $\Phi_f$ is an injective map.

Now, recall that since $q$ is a quasi-isometry, two elements $q(x_1)$ and $q(x_2)$ joined by an edge in $\mathcal{X}$ might be at distance greater than 1 in $\mathcal{X}$. If we want to prove that $\Phi_f$ is surjective on $B_\mathcal{X}(\Phi_f(v), R_\mathcal{X})$ and preserves the distance, we have to show that there exists a radius $r_p$ allowing us to “reconstruct” all the edges of $B_\mathcal{X}(\Phi_f(v), R_\mathcal{X})$ in $B_y(v, R_\mathcal{X})$. Let $L, \varepsilon > 0$ be such that $q$ is a $(L, \varepsilon)$-quasi-isometry. We distinguish three cases, represented in Figure 10.

If $x_1, x_2 \in \text{im}(q)$, then let $x_1, x_2 \in \mathcal{X}$ such that $q(x_1) = x_1$. They verify $d_\mathcal{X}(x_1, x_2) \leq Ld_\mathcal{X}(x_1, x_2) + \varepsilon$. This case is represented in Figure 10a.

If $x_1 \in \text{im}(q)$ and $x_2 \notin \text{im}(q)$, let $x_1 = q^{-1}(x_1)$. For all $x_2 \in \mathcal{X}$ such that $q(x_2) \in \mathcal{P}(x_1)$, we have (see Figure 10b)

$$
d_\mathcal{X}(q(x_1), q(x_2)) \leq 1 + d_\mathcal{X}(x_1, x_2) \Rightarrow d_\mathcal{X}(x_1, x_2) \leq Ld_\mathcal{X}(x_1, x_2) + L + \varepsilon.
$$

If $x_1, x_2 \notin \text{im}(q)$, let $x_1 \in \mathcal{X}$ such that $q(x_1) \in \mathcal{P}(x_i)$ for $i = 1, 2$. Then (see Figure 10b)

$$
d_\mathcal{X}(q(x_1), q(x_2)) \leq 2 + d_\mathcal{X}(x_1, x_2) \Rightarrow d_\mathcal{X}(x_1, x_2) \leq Ld_\mathcal{X}(x_1, x_2) + 2L + \varepsilon.
$$

Hence, assume $r_p > LR_\mathcal{X} + 2L + \varepsilon$ and let us show that $\Phi_f$ is an isometry.

Let $x \in B_{\mathcal{X}}(\Phi_f(v), R_\mathcal{X})$, by choice of $r_p$ either $x \in \text{im}(q)$ and then there exists $z \in B_y(y, r_p)$ such that $qf(z) = x$ or $x \notin \text{im}(q)$ and then there exists $P \subset B_y(v, r_p)$ such that $qf(P) = \mathcal{P}(x)$. Hence, in both cases $x \in \text{im}(\Phi_f)$ and thus, $\Phi_f$ is a bijection from $B_y(v, R_\mathcal{X})$ to $B_{\mathcal{X}}(\Phi_f(v), R_\mathcal{X})$. Now take $v_1, v_2 \in B_y(v, R_\mathcal{X})$ at distance 1 in $\mathcal{Y}$ and let $(w_0 = v_1, w_1, ..., w_i = v_2)$ be a geodesic in $\mathcal{Y}$. By definition of $\mathcal{Y}$ and choice of $r_p$, for all $i \in \{0, ..., l - 1\}$ if there is an edge between $w_i$ and $w_{i+1}$, then $d(\Phi_f(w_i), \Phi_f(w_{i+1})) = 1$.

Hence $d_{\mathcal{X}}(\Phi_f(v_1), \Phi_f(v_2)) \leq l$. To get the reversed inequality, take $x_1, x_2 \in B_{\mathcal{X}}(\Phi_f(v), R_\mathcal{X})$. Since $\Phi_f$ is bijective there exists $v_1, v_2 \in \mathcal{Y}$ such that $(\Phi_f(v_1), ..., \Phi_f(v_{l+1}))$ is a geodesic between $x_1$ and $x_2$. Again, by definition of $\mathcal{Y}$ and choice of $r_p$, an edge between $\Phi_f(v_1)$ and $\Phi_f(v_{l+1})$ gives an edge between $v_1$ and $v_{l+1}$ in $\mathcal{Y}$ and thus $d_{\mathcal{X}}(v_1, v_2) \leq l$. 
Hence, if \( r_p > LR_X + 2L + \varepsilon \) then \( \phi_1 \) is an isometry. \( \square \)

Thanks to the previous lemma, we can now use the rigidity of the Bruhat-Tits building.

**Proposition 3.2**

If \( R_X \) (and hence \( R \)) is large enough, then \( Y \) is isometric to \( X \).

**Proof.** Recall that we have \( R > r_A > 3r_p > r_p > 3R_X + 2L + \varepsilon > R_X \).

By Theorem 0.12, the building \( X \) is \( LG \)-rigid. Moreover, since its isometry group is transitive Proposition 1.4 gives us the existence of some radius \( R_{sc} > 0 \) such that every graph which is 3-simply connected and \( R_{sc} \)-locally \( X \) is isometric to \( X \).

By definition of the edges on \( Y \), this graph is simply connected at scale 3. Taking \( r_p \) (and hence \( R \)) large enough so that \( R_X \geq R_{sc} \) the preceding paragraph combined with Lemma 3.1 give us the existence of an isometry between \( X \) and \( Y \). \( \square \)

### 3.2 Change of local map, change of global isometry

Let \( y \in Y \) and \( f_y \in \mathfrak{A} \) be the isometry defined on \( B(y, R) \). Let

\[
\phi_y : \begin{cases} 
B_y(y, R_X) & \to X \\
z & \mapsto qf_y(z) \\
(j, Q) & \mapsto x \quad \text{where } \tau(x) = j, \text{ and } \mathcal{P}(x) = qf_y(Q).
\end{cases}
\]

**Lemma 3.3**

Let \( y \) and \( z \) be neighbours in \( Y \) and \( a \in H_0 \) such that \( f_yf_z^{-1} \) coincide with \( a \) on \( B_X(f(z), r_A) \). If \( R_X \) is large enough, then \( \phi_y \phi_z^{-1} \) coincide with \( \rho(a) \) on \( B_X(\phi_z(z), 2) \).

**Proof.** Let \( y \) and \( z \) be neighbours in \( Y \) and \( a \in H_0 \) such that \( f_yf_z^{-1} \) coincide with \( a \) on \( B_X(f(z), r_A) \). If \( R_X \) (and hence \( R \)) is large enough, then \( B_y(z, 2) \) is contained in \( B_y(y, R_X) \). Thus, \( \phi_y \phi_z^{-1} \) is well defined on \( B_X(\phi_z(z), 2) \).
Let $v \in B_y(z, 2)$. If $v \in V(Y)$, then
\[ \phi_y(v) = qf_y(v) = qaf_z(v) = \rho(a)qf_z(v) = \rho(a)\phi_z(v). \]
If $v = (i, P)$ with $P$ a print of type $i$, then
\[ \phi_y(v) = qf_y(P) = qaf_z(P) = \rho(a)qf_z(P) = \rho(a)\phi_z(v), \]
and $\tau(\phi_y(v)) = \tau(\phi_z(v))$. Thus $\phi_y(v) = \rho(a)\phi_z(v)$. Hence the result. \hfill \Box

Now let $r_\mathcal{X} > 0$. If $r_\mathcal{X}$ is large enough then, by SLG-rigidity of $\mathcal{X}$ there exists an isometry $\iota_y$ from $\mathcal{Y}$ to $\mathcal{X}$ that coincides with $\phi_y$ on $B(y, r_\mathcal{X})$. Thus, recalling that $z = y$, we can work with a set of isometries from $\mathcal{Y}$ to $\mathcal{X}$ that differs only by a multiplication by an element of $\text{PSL}_n(Q_p)$.

**Lemma 3.4**

If $y$ and $z$ belong to $\mathcal{Y}$ and $r_\mathcal{X}$ is large enough, then $\iota_y\iota_z^{-1} \in \text{PSL}_n(Q_p)$. Hence for all $y \in \mathcal{Y}$, the isometry $\iota_y$ sends the copy of $V(Y)$ contained in $\mathcal{Y}$ to $\text{im}(q)$ and sends prints of type $i$ in $\mathcal{Y}$ to vertices of type $i$ in $\mathcal{X}$.

**Proof.** Let $y$ and $z$ be neighbours in $\mathcal{Y}$. Since $\iota_y\iota_z^{-1}$ is an isometry of $\mathcal{X}$ it permutes the $\text{PSL}_n(Q_p)$-orbits. Recall that $\iota_y$ coincides with $\phi_y$ on $B(y, r_\mathcal{X})$. Hence, if $r_\mathcal{X}$ (and hence $R$) is large enough, then $B_y(z, 2)$ is contained in $B_y(y, r_\mathcal{X})$, thus
\[ \left( \iota_y\iota_z^{-1} \right)_{|B_y(\iota_z(z), 2)} = \phi_y\phi_z^{-1}. \]
But $\phi_y\phi_z^{-1}$ coincides with an element of $\text{PSL}_n(Q_p)$ on $B_\mathcal{X}(\phi_z(z), 2)$, by Lemma 3.3. Hence $\iota_y\iota_z^{-1}$ restricted to a ball of radius 2 preserves the $\text{PSL}_n(Q_p)$-orbits. Since such a ball contains a vertex of each type, it implies that $\iota_y\iota_z^{-1}$ preserves the $\text{PSL}_n(Q_p)$-orbits and thus belongs to $\text{PSL}_n(Q_p)$.

Now take $y$ and $z$ in $\mathcal{Y}$ (not necessarily neighbours), denote by $(y_0 = y, y_1, ..., y_l = z)$ a geodesic in $\mathcal{Y}$. By the preceding paragraph, there exists a sequence $\alpha_1, ..., \alpha_l$ of elements in $\text{PSL}_n(Q_p)$ such that
\[ \forall i \in \{1, ..., l\} \quad \iota_{y_0}\iota_{y_{i-1}}^{-1} = \alpha_i. \]
Thus, recalling that $z = y_l$ and $y = y_0$, we get $\iota_z = \alpha_l ... \alpha_1 \iota_y$. Which proves the first assertion of the lemma.

Let us now prove the second part of the lemma. Let $y \in \mathcal{Y}$ and $v \in \mathcal{Y}$. There exists $z \in \mathcal{Y}$ such that $v \in B_y(z, 2)$, and using the paragraph above, there exists $\alpha \in \text{PSL}_n(Q_p)$ such that $\iota_y = \alpha\iota_z$. In particular, since $v$ belongs to $B_y(z, R_\mathcal{X})$,
\[ \iota_y(v) = \alpha\iota_z(v) = \alpha\phi_z(v). \]
By definition of $\phi_z$, if $v \in V(Y)$ then $\phi_z(v)$ belongs to $\text{im}(q)$ and if $v = (i, P)$ with $P$ print of type $i$, then $\phi_z(v)$ is a vertex of type $i$ in $\mathcal{X}$. This finish the proof of the lemma. \hfill \Box

Now we have all the tools we need to prove the isometry between $\mathcal{Y}$ and $\mathcal{X}$.

### 3.3 Isometry from $\mathcal{Y}$ to $\mathcal{X}$

Let $\kappa$ be the natural injection of $\mathcal{Y}$ in $\mathcal{Y}_Z$ and $\iota$ an isometry given by Proposition 3.2. With the objects constructed so far we get the diagram in Figure 11.

The aim of this section is to prove the following result.
Thus, using the equivariance of $\phi$, hence if $R > r_Y > 3r_p > R_X + 2L + \epsilon > R_X > r_X$. Let $y \in Y$ and recall that $L$ and $\epsilon$ are constants such that $q$ is a $(L, \epsilon)$-quasi-isometry. If $r_X \geq Lr_Y + \epsilon$ (and hence if $R$ is large enough) then $\kappa(B_Y(y, r_Y))$ is included in $B_Y(y, r_X)$. Indeed if $z \in B_Y(y, r_Y)$ then
\[
d_X(qf_y(y), qf_y(z)) \leq Ld_X(f_y(y), f_y(z)) + \epsilon = Ld_Y(y, z) + \epsilon \leq Lr_Y + \epsilon \leq r_X.
\]

Thus $\phi_y(\kappa(z)) = qf_y(z)$ and $d_Y(\kappa(y), \kappa(z)) = d_X(\phi_y(\kappa(y)), \phi_y(\kappa(z))) = d_X(qf_y(y), qf_y(z)) \leq r_X$.

Now, recall that $H_y = \rho^{-1}PSL_n(O_p)$. Then, by Lemma 3.4 there exists $a_y \in H_y$ such that $t_y \kappa^{-1} = \rho(a_y)$.

Hence, using the equivariance of $q$ we get that for all $z_1$ and $z_2$ in $B_Y(y, r_Y)$
\[
d_X(q^{-1}\kappa(z_1), q^{-1}\kappa(z_1)) = d_X(a_y q^{-1}\kappa(z_1), a_y q^{-1}\kappa(z_1))
\]
\[
= d_X(q^{-1}\rho(a_y)\kappa(z_1), q^{-1}\rho(a_y)\kappa(z_1)) = d_X(q^{-1}t_y\kappa(z_1), q^{-1}t_y\kappa(z_1)).
\]

But $z_1$ and $z_2$ belong to $B_Y(y, r_Y)$, hence for $i = 1, 2$ we have $t_y \kappa(z_i) = qf_y(z_i)$. Thus,
\[
d_X(q^{-1}\kappa(z_1), q^{-1}\kappa(z_1)) = d_X(q^{-1}f_y(z_1), q^{-1}f_y(z_2))
\]
\[
= d_X(f_y(z_1), f_y(z_2)) = d_Y(z_1, z_2).
\]

Thus $q^{-1}\kappa$ restricted to $B_Y(y, r_Y)$ preserves the distance. 

Let's show that the claim forces $q^{-1}\kappa$ to be an isometry from $Y$ to $X$. Take $r_Y \geq 2$ and let $y, y' \in Y$ and $(y_0 = y, y_1, \ldots, y_i = y')$ be a geodesic in $Y$. Since for all $i$ the vertices $y_i$ and $y_{i+1}$ are adjacent, then
claim 3.6 implies that \( d_X(q^{-1} \iota k(y_i), q^{-1} \iota k(y_{i+1})) = 1 \). Hence

\[
d_X(q^{-1} \iota k(y), q^{-1} \iota k(y')) \leq \sum_{i=0}^{l-1} d_X(q^{-1} \iota k(y_i), q^{-1} \iota k(y_{i+1})) = l.
\]

Moreover, if \((x_0 = q^{-1} \iota k(y), x_1, \ldots, x_m = q^{-1} \iota k(y'))\) is a geodesic in \(X\), then by bijectivity of \(q^{-1} \iota k\) there exists \(z_i \in Y\) such that \(q^{-1} \iota k(z_i) = x_i\) for all \(i\) in \(\{1, \ldots, m-1\}\). Denote \(z_0 = y\) and \(z_m = y'\). Since for all \(i\) the vertices \(x_i\) and \(x_{i+1}\) are adjacent, then claim 3.6 implies that \(d_X(z_i, z_{i+1}) = d_X(q^{-1} \iota k(z_i), q^{-1} \iota k(z_{i+1}))\).

Thus

\[
d_Y(y, y') \leq \sum_{i=0}^{m-1} d_X(z_i, z_{i+1}) = \sum_{i=0}^{m-1} d_X(q^{-1} \iota k(z_i), q^{-1} \iota k(z_{i+1})) = \sum_{i=0}^{m-1} d_X(x_i, x_{i+1}) = m.
\]

\[\square\]

### 4 Application to \(p\)-adic lattices

In this section we prove Theorem 0.15 which we recall below.

**Corollary 4.1**

Let \(p\) be a prime number and \(n \geq 4\).

The torsion-free lattices of \(\text{SL}_n(\mathbb{Q}_p)\) are SLG-rigid.

Let \(n \neq 3\) and \(\Gamma \leq \text{SL}_n(\mathbb{Q}_p)\) be a lattice without torsion. Denote by \((\Gamma, S)\) one of its Cayley graphs. Recall that any lattice in \(\text{SL}_n(\mathbb{Q}_p)\) is uniform (i.e. cocompact).

#### 4.1 Quasi-isometry between the lattice and the building

To show the corollary, we first check that the lattice is quasi-isometric to the building. Then, using a famous result of Kleiner and Leeb we show that the isometry group of the lattice acts on the building and that the quasi-isometry can be chosen to be equivariant under this action.

**Lemma 4.2**

Let \(\Lambda\) be a lattice of \(\text{SL}_n(\mathbb{Q}_p)\). Then \(\Lambda\) is quasi-isometric to \(X\).

**Proof.** First, recall that any lattice in \(\text{SL}_n(\mathbb{Q}_p)\) is uniform, viz. cocompact (see for example [BQ14]).

Since \(\Lambda\) is a lattice of \(\text{SL}_n(\mathbb{Q}_p)\), there is a natural action on the Bruhat-Tits building induced by the action of \(\text{PSL}_n(\mathbb{Q}_p)\). Moreover, since \(\Lambda\) is cocompact and the \(\text{PSL}_n(\mathbb{Q}_p)\) action has exactly \(n\) orbits, the \(\Lambda\) action is also cocompact. Hence by the Svarc-Milnor’s lemma \(\Lambda\) is quasi-isometric to \(X\). \[\square\]

By a result of Kleiner and Leeb [KL97] and Cornulier [Cor18, Theorem 3.B.1] applied to our lattice \(\Gamma\), this quasi-isometry implies the existence of a homomorphism from \(\text{Is}(\Gamma, S)\) to \(\text{Is}(X)\) and a quasi-isometry from \((\Gamma, S)\) to \(X\) which is \(\text{Is}(\Gamma, S)\)-equivariant. Since \(\Gamma\) is assumed to be torsion-free, we can refine the informations about these two applications.

**Lemma 4.3**

Let \(\Lambda\) be a lattice of \(\text{SL}_n(\mathbb{Q}_p)\) and \(T\) a symmetric generating set. If \(\Lambda\) is torsion-free, then there exists an injective homomorphism \(\rho : \text{Is}(\Lambda, T) \rightarrow \text{Is}(X)\),
and an injective quasi-isometry which is $\text{Is}(\Lambda, T)$-equivariant
\[ q : (\Lambda, T) \to X. \]

**Proof.** Since we assumed that $\Lambda$ has no torsion element, by Proposition 0.17 the isometry group of $(\Lambda, T)$ contains no non-trivial compact normal subgroup. Hence the morphism $\rho$ given by Kleiner-Leeb’s theorem is injective.

Assume that there exist $\lambda_1, \lambda_2 \in \Lambda$ such that $\lambda_1 \neq \lambda_2$ and $q(\lambda_1) = q(\lambda_2)$. Then, the equivariance of $q$ implies that $q(\{\lambda_1^n : n \in \mathbb{N}\}) = \{q(\lambda_2)\}$, which contradicts the fact that $q$ is a quasi-isometry. \qed

### 4.2 Relation between the isometry groups

To apply Theorem 0.16, we still need to check that $\text{Is}(\Gamma, S)$ is of finite index in $\text{Is}(X)$. As stated in the lemma below, this is not always the case: the lattice’s isometry group can also be discrete. But as we will see in Section 4.3 we will be able to prove the rigidity of the lattice in that case too.

**Lemma 4.4**

Using the previous notations,

- Either $\text{Is}(\Gamma, S)$ is discrete.
- Or $\text{Is}(\Gamma, S)$ is of finite index in $\text{Is}(X)$ and contains $\text{PSL}_n(\mathbb{Q}_p)$.

Before proving this lemma, let us recall a useful consequence of a theorem of Benoist and Quint. The original and more general statement can be found in [BQ14, Corollary 4.5].

**Proposition 4.5 (Benoist, Quint [BQ14])**

Let $G$ be $p$-adic Lie group and $H$ be a finite covolume closed subgroup of $G$, with Lie algebra $\mathfrak{h}$. If $G$ has no proper cocompact normal subgroup, then $G$ normalizes $\mathfrak{h}$.

**Proof of Lemma 4.4.** Let $G = \text{PSL}_n(\mathbb{Q}_p)$ and $H = \text{Is}(\Gamma, S) \cap G$ and note $\mathfrak{h} = \text{Lie}(H)$ and $\mathfrak{g} = \text{Lie}(G)$ their respective Lie algebras. Since $\Gamma$ is a lattice in $\text{SL}_n(\mathbb{Q}_p)$, we get that $\rho(\Gamma) \cap \text{PSL}_n(\mathbb{Q}_p)$ is a lattice in $\text{PSL}_n(\mathbb{Q}_p)$. Hence $H$ contains the uniform lattice $\rho(\Gamma) \cap G$ of $G$, thus $H$ has finite covolume in $\text{PSL}_n(\mathbb{Q}_p)$.

The above property applied to $G$ and $H$ implies that $G$ normalises $\mathfrak{h}$, in other words $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. Since $\mathfrak{g}$ is simple, we get that $\mathfrak{h}$ is either trivial or the full Lie algebra $\mathfrak{g}$. If $\text{Is}(\Gamma, S)$ isn’t discrete, then it is a closed subgroup of $\text{Is}(X)$. Hence $H$ is a closed subgroup of $G$ and its Lie algebra is non-trivial. By the previous point it can only be $\mathfrak{g}$. Hence, it implies that is an open subgroup of $G$. Since it is also cocompact, it is necessarily of finite index in $G$. Thus, we get that $\rho(\text{Is}(\Gamma, S))$ is of finite index in $\text{Is}(X)$.

Let’s show that $\text{PSL}_n(\mathbb{Q}_p) \leq \rho(\text{Is}(\Gamma, S))$. First assume that $\rho(\text{Is}(\Gamma, S))$ is strictly contained in $\text{PSL}_n(\mathbb{Q}_p)$. Since these two groups are of finite index in $\text{Is}(X)$, we get that $\rho(\text{Is}(\Gamma, S))$ is of finite index in $\text{PSL}_n(\mathbb{Q}_p)$. But then the core:

\[ \bigcap_{g \in \text{PSL}_n} g \cdot \rho(\text{Is}(\Gamma, S)) \cdot g^{-1} \]
of $ρ(\text{Is}(Γ,S))$ is itself of finite index in $\text{PSL}_n(ℚ_p)$ (and different from $\text{PSL}_n(ℚ_p)$), which contradicts the simplicity of $\text{PSL}_n(ℚ_p)$.

Now, let’s go back to the general case. Assume that $\text{PSL}_n(ℚ_p)$ isn’t included in $ρ(\text{Is}(Γ,S))$ and remark that:

$$\mathfrak{h} = \text{Lie } (\text{Is}(X)) = \text{Lie } \left(\text{PSL}_n(ℚ_p)\right).$$

In particular $ρ(\text{Is}(Γ,S))$ is “locally” $\text{PSL}_n(ℚ_p)$ so, up to apply what precedes to an open set centered on $e_r$ sufficiently small of $ρ(\text{Is}(Γ,S))$, we obtain a contradiction.

Hence $\text{PSL}_n(ℚ_p)$ is contained in $ρ(\text{Is}(Γ,S))$.

4.3 Rigidity of $p$-adic lattices

We conclude by the proof of Corollary 4.1.

Proof of Corollary 4.1. Let $n ≠ 3$ and $p$ be a prime. Let $Γ$ be a torsion-free lattice of $\text{PSL}_n(ℚ_p)$ and $S$ be a symmetric generating part.

If $n = 2$, then $X$ is the $(p + 1)$-regular tree. Since by Lemma 4.2, the graph $(Γ, S)$ is quasi-isometric to $X$, example 0.5 implies that $(Γ, S)$ is LG-rigid.

Assume now that $n > 3$. If Is$(Γ, S)$ is discrete the LG-rigidity of the lattice is given by Theorem 0.9. If Is$(Γ, S)$ is non-discrete, then by Lemma 4.4 it has finite index in Is$(X)$ and in this case the hypothesis of Theorem 0.16 are satisfied, hence the rigidity of the lattice.

Conclusion and open problems

Our main result is proved for graphs quasi-isometric to the Bruhat-Tits building of $\text{PSL}_n(ℚ_p)$ and the key idea of the proof is to use the rigidity of this building to “transfer it” to the graph quasi-isometric thereto. One can ask whether we can generalize this idea to other LG-rigid graphs.

Question 4.6. Let $\mathcal{Y}$ be quasi-isometric to a LG-rigid graph $\mathcal{X}$, both having cocompact isometry group. If the quasi-isometry is Is$(\mathcal{Y})$-equivariant, is $\mathcal{Y}$ LG-rigid?

Remark that if $\mathcal{X}$ and $\mathcal{Y}$ are two Cayley graphs of the same group, we can chose $\mathcal{X}$ to be LG-rigid and $\mathcal{Y}$ to be non-rigid (see the discussion below counter-example 0.6 for more details). In that case the hypothesis of the preceding question are satisfied without $\mathcal{Y}$ being LG-rigid. Thus, more restrictive hypothesis will be needed to get the rigidity of $\mathcal{Y}$.

Our result on $p$-adic lattices is proved for $n ≠ 3$; when $n = 3$ we don’t know (yet) the answer. Indeed, our proof is based on the rigidity of the Bruhat-Tits building of $\text{PSL}_n(ℚ_p)$, a result known to be true only for $n ≥ 4$. In the $n = 3$ case, a lot of flexibility seems to be allowed (see for example [BP07]) obstructing any local recognizability result. Hence the following question:

Question 4.7. Are torsion-free lattices of $\text{SL}_3(ℚ_p)$ LG-rigid?

Lattices in $p$-adic Lie groups can be viewed as particular cases of $S$-arithmetic lattices.

Definition 4.8

Let $S$ be a set of prime.

We say that $Γ$ an $S$-arithmetic lattice if it’s a lattice in a product of the form $\prod_i G_i$ where $G_i$ is either a real Lie group or a $p$-adic Lie group for $p ∈ S$. 

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Hence, one we can ask what happens in that more general case.

**Question 4.9.** Are torsion-free $S$-arithmetic lattices LG-rigid?

A result by Bader, Furman and Sauer [BFS20, Theorem B] can be used to deal with irreducible torsion-free $S$-arithmetic lattices. Indeed, if the product $\prod_i G_i$ contains at least a non-compact real factor, then the aforementioned theorem implies that the isometry group of a Cayley graph of $\Gamma$ is discrete. Thus, by Theorem 0.9 the lattice is LG-rigid. Now, if the product contains a compact real factor then the isometry group of the Cayley graph might not be discrete and in that case, the problem is still open.

When the lattice is reducible, we now know that the projection on the $p$-adic factors gives LG-rigid lattices. Moreover, if we suppose the real factors to be simple and connected, then a result by de la Salle and Tessera [dlST19] shows that the projection on these factors are also LG-rigid. Hence it remains to understand how to combine these results on the factors in order to get a result on the product.
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\textbf{Notations Index}

\( \mathfrak{A} \) Atlas of isometries from \( Y \) to \( X \).

\( (\Gamma, S) \) Cayley graph of \( \Gamma \) with respect to the generating part \( S \).

\( H_0 \) The group \( \rho^{-1}(\text{PSL}_n(\mathbb{Q}_p)) \).

\( \text{Is}(\mathfrak{g}) \) Isometry group of \( \mathfrak{g} \).

\( \kappa \) Natural injection of \( Y \) in \( \hat{Y} \) (see Section 3.3).

\( [L] \) The class modulo homothety of the lattice \( L \).

\( \mathfrak{P}(x) \) The print of the vertex \( x \) (see Definition 2.3).

\( P \) A print of a certain type \( i \) in \( Y \) (see Definition 2.11).

\( q \) The quasi-isometry between \( X \) and \( \hat{X} \).

\( R \) Radius such that \( Y \) is \( R \)-locally the same as \( X \).

\( r_A \) See Lemma 2.9.

\( r_p \) Radius considered to define prints (see Definition 2.11).

\( R_X \) Radius such that \( Y \) is \( R_X \)-locally \( X \).

\( r_x \) Radius such that \( \tau_y \) coincide with \( \phi_y \) on \( B_y(y, r_x) \) (see page 19).

\( r_Y \) See claim 3.6.

\( \tau(x) \) The type of the vertex \( x \), where \( x \) belongs to the Bruhat-Tits building of \( \text{PSL}_n(\mathbb{Q}_p) \).

\( \hat{X} \) The Bruhat-Tits building of \( \text{PSL}_n(\mathbb{Q}_p) \).

\( (y_1, \ldots, y_l) \) A path of adjacent vertices \( y_1, y_2, \ldots, y_l \).