On the distribution of gaps between consecutive primes

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1 Introduction

The recent dramatic new developments in the study of bounded gaps between primes, reached by Zhang [Zha], Maynard [May1] and Tao [Pol8B] made other old conjectures about the distribution of primegaps accessible. One of the most interesting such conjectures was formulated in 1954 by Erdős [Erd2] as follows. Let $J$ denote the set of limit points of $d_n/\log n$, i.e.

\begin{equation}
J = \left\{ \frac{d_n}{\log n} \right\}'.
\end{equation}

Then $J = [0, \infty]$. While Westzynthius [Wes] proved already in 1931 the relation

\begin{equation}
\limsup_{n \to \infty} \frac{d_n}{\log n} = \infty \quad \text{i.e.} \quad \infty \in J,
\end{equation}

no finite limit point was known until 2005, when in a joint work of Goldston, Yildirim and the author [GPY1] it was shown that

\begin{equation}
\liminf_{n \to \infty} \frac{d_n}{\log n} = 0 \quad \text{i.e.} \quad 0 \in J.
\end{equation}

On the other hand, Erdős [Erd2] and Ricci [Ric] proved simultaneously and independently about 60 years ago that $J$ has positive Lebesgue measure.
In a recent work D. Banks, T. Freiberg and J. Maynard showed \[BFM\] that more than 2% of all nonnegative real numbers belong to \(J\).

The author has shown \[Pin3\] that for any \(f(n) \leq \log n, f(n) \nearrow \infty\) (i.e. \(f(n) \to \infty, f(n)\) is monotonically increasing) satisfying for any \(\varepsilon\)

\[
(1.5) \quad (1 - \varepsilon)f(N) \leq f(n) \leq (1 + \varepsilon)f(N) \text{ if } n \in [N, 2N], N > N_0(\varepsilon)
\]

we have an ineffective constant \(c_f\) such that

\[
(1.6) \quad [0, c_f] \subset J_f := \left\{\frac{d_n}{f(n)}\right\}'.
\]

Although \(\log n\) is the average value of \(d_n\), improving the result \((1.3)\) of Westzynthius, Erdős \[Erd1\] in 1935 and three years later Rankin \[Ran1\] proved stronger results about large gaps between consecutive primes. The 76-year-old result of Rankin, the estimate (\(\log_\nu n\) denotes the \(\nu\)-fold iterated logarithmic function)

\[
(1.7) \quad \limsup_{n \to \infty} \frac{d_n/\log n}{g(n)} \geq C_0, \quad g(x) = \frac{\log_2 x \log_4 x}{(\log_3 x)^2}
\]

was apart from the value of the constant \(C_0\) still until August 2014 the best known lower estimate for large values of \(d_n\). (The original value \(C_0 = 1/3\) of Rankin was improved in four steps, finally to \(C_0 = 2e^\gamma\) by the author \[Pin2\].) Then in two days two different new proofs appeared by Ford–Green–Konjagin–Tao \[FGKT\] and Maynard \[May2\] in the arXiv, proving Erdős’s famous USD 10,000 conjecture according to which \((1.7)\) holds with an arbitrarily large constant \(C_0\).

This raises the question whether the relation \((1.6)\) can be improved to functions of type \(f(x) = \omega(x) \log x\) with \(\omega(x) \to \infty\) and whether perhaps even \(\omega(x) = c_1 g(x)\) can be reached with some absolute constant \(c_1\), or, following the mentioned new developments, with an arbitrarily large \(c_1\) as well.

Another question is whether for some function \(f(n)\) we can reach

\[
(1.8) \quad [0, \infty] = J_f,
\]

i.e. the original conjecture of Erdős with the function \(f(n)\) in place of \(\log n\).

Using our notation \((1.1), (1.5)-(1.7)\), we will show the following results, which, although do not show the original conjecture \(J_{\log n} = [0, \infty]\) of Erdős,
but in several aspects approximate it and in other aspects they go even further.

Since the first version of this paper was written before the groundbreaking works \cite{FGKT} and \cite{May2} we will present the formulation and proofs of the original version of our results in the Introduction and Sections 3–5, while the formulation of the improved stronger versions appear in Section 2 and the needed changes in the proofs in Section 6, in this case the changes refer to the mentioned work of Maynard \cite{May2}.

**Theorem 1.** There exists an absolute constant $c_0$ such that for any function $f(x) \not> \infty$, satisfying (1.5) and

\begin{equation}
(1.9) \quad f(x) \leq c_0 g(x) \log x
\end{equation}

we have with a suitable (ineffective) constant $c_f$

\begin{equation}
(1.10) \quad [0, c_f] \subset J_f := \{ \frac{d_n}{f(n)} \}^\prime.
\end{equation}

**Theorem 2.** Let us consider a sequence of functions $\{f_i(x)\}_{i=1}^{\infty}$ satisfying (1.5), (1.9), $f(x) \not> \infty$ and

\begin{equation}
(1.11) \quad \frac{f_{i+1}(x)}{f_i(x)} \to \infty \text{ as } x \to \infty \text{ for every } i.
\end{equation}

Then apart from at most 98 functions $f_i(x)$ we have

\begin{equation}
(1.12) \quad [0, \infty] = J_f = \left\{ \frac{d_n}{f_i(n)} \right\}^\prime.
\end{equation}

Answering a question raised in a recent work of Banks, Freiberg and Maynard \cite{BFM} we show that the method of \cite{BFM} works also if we normalize the primegaps in place of $\log n$ with any function not exceeding the Erdős–Rankin function.

**Theorem 3.** Suppose $f(x) \not> \infty$ and satisfies (1.5) and (1.9). Then for any sequence of $k \geq 50$ non-negative real numbers $\beta_1 < \beta_2 < \cdots < \beta_k$ at least one of the numbers $\{\beta_j - \beta_i; 1 \leq i < j \leq k\}$ belongs to $J_f$. Consequently more than 2% of all non-negative real numbers belong to $J_f$. 

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As a by-product the method also gives a different new proof for the following result of Helmut Maier \[\text{[Mai1]}\] proved in 1981 by his famous matrix method:

**Theorem 4.** For any natural number \(m\) we have with the notation (1.7)

\[
\lim_{h \to \infty} \sup \frac{\min(d_{n+1}, \ldots, d_{n+m})}{g(n) \log n} > 0.
\]

An immediate corollary of Theorem 2 is the following

**Corollary 1.** Let \(\eta(x) \to 0\) be an arbitrary function. If \(\eta(x)F(x) \not\to \infty,\)
\(F(x) \not\to \infty,\) both functions \(F(x)\) and \(\eta(x)F(x)\) satisfy (1.5) and (1.9), then we have a function \(f(x) \not\to \infty,\)

\[
\eta(x)F(x) \leq f(x) \leq F(x),
\]

for which

\[
[0, \infty] = J_f := \left\{ \frac{d_n}{f(n)} \right\}.
\]

This means that although we can not show Erdős’s conjecture for the natural normalizing function \(\log n\), changing it a little bit, it will be already true for some function \(\xi(n) \log n\), where \(\xi(n)\) tends to 0 (or alternatively we can require \(\xi(n) \to \infty\)) arbitrarily slowly (even if this is not a natural normalization).

### 2 Stronger forms of Theorems [1–4] and Corollary [1]

We will use in the formulation and proof of our results the work of J. Maynard [May2] which implicitly defines an unspecified but actually explicitly calculable \(\omega_0(x)\) function with the property

\[
\lim_{x \to \infty} \omega_0(x) = \infty,
\]

such that defining (cf. (1.7))

\[
g_0(x) = \omega_0(x)g(x) = \omega_0(x)\frac{\log x \log_3 x}{(\log_3 x)^2},
\]
the result (1.7) holds with $g_0(n)$ in place of $g(n)$.

J. Maynard further mentions in the Remark at the end of his paper that he hopes to obtain his result with $\omega_0(x) = (\log_2 x)^{1+o(1)}$ which would be the limit of the Erdős–Rankin method.

We mention that such an improvement would almost surely lead to an improvement of our results too, since the mentioned idea (to show the same results with a uniformity in the variable $k$ of [May2] for $k$ as large as $k \asymp (\log x)^\alpha$) would leave the structure of the proof unchanged.

We will denote by Theorems 1’–4’ and Corollary 1’ the stronger versions of Theorems 1–4 and Corollary 1. They are the following ($g_0(x)$ is defined in (2.2)).

Theorem 1’. Theorem 1 holds with $g(x)$ replaced by $g_0(x)$ in (1.9).
Theorem 2’. Theorem 2 holds with $g(x)$ replaced by $g_0(x)$ in (1.9).
Theorem 3’. Theorem 3 holds with $g(x)$ replaced by $g_0(x)$ in (1.9).
Theorem 4’. Theorem 4 holds with $g(x)$ replaced by $g_0(x)$ in (1.9).
Corollary 1’. Corollary 1 holds with $g(x)$ replaced by $g_0(x)$ in (1.9).

## 3 The Maynard–Tao theorem

We call $H_m = \{h_1, \ldots, h_m\}$ an admissible $m$-tuple if $0 \leq h_1 < \cdots < h_m$ and $H_m$ does not occupy all residue classes mod $p$ for any prime $p$.

Further, we recall the Theorem of Landau–Page (see [Dav, p. 95]).

**Theorem.** If $c_1$ is a suitable positive constant, $N$ arbitrary, there is at most one primitive character $\chi$ to a modulus $r \leq N$: for which $L(s, \chi)$ has a real zero $\beta$ satisfying

\[
\beta > 1 - \frac{c_1}{\log N}.
\]

Such an exceptional character $\chi$ must be real, which means also that its conductor $r$ is squarefree, apart from the possibility that the prime 2 appears in the factorization of $r$ with an exponent 2 or 3. We have also

\[
\beta < 1 - \frac{c_2}{\sqrt{\log^2 r}} \quad [\text{Dav p. 96}], \quad \beta < 1 - \frac{c_3}{\sqrt{r}} \quad [\text{Pin1, GS}]
\]

with effective absolute constants $c_2, c_3 > 0$. 


We remark that (3.1) and the second inequality of (3.2) imply

\[(3.3) \quad r \geq c_4 \log^2 N \]

and for the greatest prime factor \(q_0\) of \(r\)

\[(3.4) \quad q_0 \geq 2 \log_2 N - c_5 > \log_2 N \quad \text{if} \quad N > N_0,\]

with effective absolute constants \(c_4, c_5 > 0\).

We will slightly reformulate Theorem 4.2 of [BFM] which itself is an improved reformulation of the original Maynard–Tao theorem. We remark that in order to obtain Theorems 1–3 (with a constant \(C\) larger than 50, respectively with a proportion less than \(1/50 = 2\%)\ one could also use the more complicated method of Zhang [Zha]. However, to obtain a new proof of Theorem 4 we need the Maynard–Tao method. Also the result stated below uses clearly the Maynard–Tao method.

Let \(P^+(n)\) denote the largest prime factor of \(n\).

**Theorem** (Maynard–Tao). Let \(k = k_m\) be an integer, \(\varepsilon = \varepsilon(k, n) > 0\) be sufficiently small, \(N > N_0(\varepsilon, k, m)\). Further, let

\[(3.5) \quad k + 1 < C_6(\varepsilon) < h_1 < h_2 < \cdots < h_k \leq N, \quad \mathcal{H} = \mathcal{H}_k = \{h_i\}_{i=1}^k \text{ admissible},\]

\[(3.6) \quad \Delta(\mathcal{H}) := \prod_{1 \leq i < j \leq k} (h_j - h_i), \quad \left( q_0 \prod_{i=1}^k h_i, \Delta(\mathcal{H}) \right) = 1, \quad \Delta(\mathcal{H}) < N^\varepsilon.\]

\[(3.7) \quad \text{For } m = 2 \text{ let } k = 50\]

and generally let

\[(3.8) \quad k_m = C_7 e^{5m}\]

with suitably chosen constants \(C_7\) and \(C_6(\varepsilon)\), depending on \(\varepsilon\). Then we have at least \(m\) primes among \(n + \mathcal{H}_k = \{n + h_i\}_{i=1}^k\) for some \(n \in (N, 2N]\).

**Remark 1.** In the proof of our Theorems 1–4 we will have

\[(3.9) \quad h_k \leq g(N) \log N < \log^2 N,\]

thus the second condition of (3.6) will be trivially fulfilled.
Remark 2. In the mentioned applications we will choose the values \( h_i \) as primes, so the first condition of (3.6) will be equivalent to

\[
q_0 \nmid h_j - h_i, \quad h_t \nmid h_j - h_i \quad \text{for any} \quad t \in [1, k], \quad 1 \leq i < j \leq k.
\]

Since in the applications the only other condition will be with some functions \( \xi_i(N) \) to have

\[
h_i = (1 + o(1))\xi_i(N), \quad \xi_i(N) \ll g(N) \log N
\]

it will make no problem to choose step by step primes \( h_i \) satisfying (3.9)–(3.10). Also \( h_i \in \mathcal{P}, \quad h_i > k \) assures that \( H_k \) is admissible.

The Maynard–Tao method assures the existence of at least \( m \) primes among numbers of the form

\[
n + h_i \ (1 \leq i \leq k) \quad \text{with} \quad n \equiv z \pmod{W}
\]

with any \( z \in [1, W] \) and for some \( n \in (N, 2N] \), \( N \) sufficiently large, if

\[
\prod_{1 \leq i < j \leq k} (z + h_i), W = 1.
\]

The pure existence of such a \( z \) follows from the admissibility of \( H_m \) but its actual choice is crucial in the applications.

In order for the method of Maynard–Tao and Banks–Freiberg–Maynard [BFM] to work we must assure still (see [BFM]) with a sufficiently large \( C_8(\varepsilon) \)

\[
\Delta(H) = \prod_{1 \leq i < j \leq k} (h_j - h_i) \mid W; \quad \prod_{p \leq C_8(\varepsilon)} p \mid W
\]

and for the possibly existing greatest prime factor \( q_0 \) of the possibly existing exceptional modulus \( r \),

\[
q_0 \nmid W \quad \text{(if such a modulus \( r \), and so \( q_0 \) exists)};
\]

further \( (P^+(n) \) will denote the greatest prime factor of \( n \)

\[
P^+(W) < N^{\varepsilon/\log_2 N}, \quad W < N^{2\varepsilon}.
\]

In the applications (3.9) will assure \( q_0 \nmid \Delta(H) \prod_{p \leq C_8(\varepsilon)} p \).
If we succeed to show the existence of a pair \((z, W)\) with (3.13)–(3.16) and the crucial additional property that with a suitable \(c_9(\varepsilon)\)
\[
(z + s, W) > 1 \quad \text{if} \quad s \not\in \mathcal{H}, \quad 1 < s \leq c_9(\varepsilon)g(N)\log N,
\]
then we can assure that all numbers \(z + s\) with (3.17) have a prime divisor \(p \mid W\). Consequently all \(n + s, s \neq h_i, s \in (1, c_9(\varepsilon)g(N)\log N]\) will be composite if \(n \in (N, 2N]\).

In order to achieve this, we will use the Erdős–Rankin method. After this we can show Theorems 1–4 with suitable choices of \(\mathcal{H}_k\).

We will choose the following parameters (\(p\) will always denote primes), \(\mathcal{H} = \mathcal{H}_i, c_{10}(\varepsilon) = 2c_9(\varepsilon)/\varepsilon,\)
\[
\mathcal{L} = \varepsilon \log N, \quad v = \log^3 \mathcal{L}, \quad U = c_{10}(\varepsilon)g(e^\mathcal{L})\mathcal{L} > c_9(\varepsilon)g(N)\log N,
\]
\[
y = \exp \left( \frac{1}{k + 5} \log \mathcal{L} \log_3 \mathcal{L} / \log_2 \mathcal{L} \right),
\]
\[
P_1 = \prod_{p \leq v}^* p,
\]
\[
P_2 = \prod_{v < p \leq y}^* p,
\]
\[
P_3 = \prod_{y < p \leq \mathcal{L}/2}^* p,
\]
\[
P_4 = \prod_{\mathcal{L}/2 < p \leq \mathcal{L}}^* p,
\]
where \(\prod_p^*\) means
\[
p \not\in \mathcal{H}' := \mathcal{H} \cup \{q_0\}.\]
Further, let

\[ (3.25) \quad W_0 = P_1 P_2 P_3 P_4, \quad W = \prod_{p \leq L}^* p, \Delta_0(\mathcal{H}) = [P_1 P_2 P_3 P_4, \Delta_0(\mathcal{H})] \]

where \( \Delta_0(\mathcal{H}) \) denotes the squarefree part of \( \Delta(\mathcal{H}) \). This choice of \( W \) clearly satisfies both conditions of (3.16) by the Prime Number Theorem if we additionally require the condition valid in all applications:

\[ (3.26) \quad h_k \leq \log^2 N. \]

4 The application of the Erdős–Rankin method

We will choose the congruence class \( z \) modulo any prime divisor of \( W \), which finally determines \( z \mod W \). Let \( p \) denote always primes; further let us choose

\[ (4.1) \quad z \equiv 0 \pmod{P_1 P_3}. \]

This implies by \( v \mathcal{L}/2 > U \) that

\[ (4.2) \quad (z + s, P_1 P_3) = 1 \quad (1 < s \leq U) \]

if and only if \( (s, P_1 P_3) = 1 \), that is, if and only if either

\[ (4.3) \quad s = pq_0^\alpha \prod_{i=1}^k h_i^{\alpha_i} \quad (\alpha \geq 0, \alpha_1 \geq 0, \ldots, \alpha_m \geq 0) \quad \text{and} \quad p > \mathcal{L}/2 \]

or

\[ (4.4) \quad s \text{ is composed only of primes } p \mid P_2 q_0 \prod_{i=1}^k h_i. \]

The first step is to estimate the number \( A_0 \) of numbers \( s \) satisfying (4.4). This will be relatively easy since

(i) we have an upper estimate for \( y \)-smooth numbers by the results of Dickman and (in a refined form) of de Bruijn \( [\text{Bru}] \). We quote a suitable result of \( [\text{Bru}] \) as our Lemma \( [\text{Mai}] \) in a simpler form as given in \( [\text{Mai}] \).
(ii) the additional factor $q_0^\alpha \prod_{i=1}^k h_i^{\alpha_i}$ leaves the asymptotic for numbers of the form (4.3) and (4.4) below $U$ nearly unchanged.

**Lemma 1.** Let $\Psi(x, y)$ denote the number of positive integers $n \leq x$ which are composed only of primes $\leq y$. For $y \leq x$, $y \to \infty$, $x \to \infty$ we have

\[
\Psi(x, y) \leq x \exp \left[ -\log x \frac{\log_3 y}{\log y} + (1 + o(1)) \log_2 y \right].
\]

This is a slightly simplified form of Lemma 5 of [Mai1].

Applying (4.5) and taking into account that $\alpha, \alpha_i \ll \log U \sim \log \mathcal{L}$, we obtain by the choice of $y$ in (3.19) for the number of $s \leq U$ with (4.4) the upper estimate

\[
(1 + o(1))(\log \mathcal{L})^{k+1}U \exp \left[ -\frac{(k+5+o(1))\log \mathcal{L} \log_3 \mathcal{L}}{\log_3 \mathcal{L}/\log_2 \mathcal{L}} + (1 + o(1)) \log_2 \mathcal{L} \right] \ll \frac{U}{\log^2 U} \leq \frac{C_{11}(\pi(\mathcal{L}) - \pi(\mathcal{L}/2))}{\log \mathcal{L}}
\]

which will be negligible compared with the numbers of $s \leq U$ with (4.3). This means that integers with (4.4) can be later sieved out by a tiny portion of primes dividing $P_4$ in (3.23). On the other hand, the number of integers with (4.3) is much larger than $\pi(\mathcal{L})$, although just slightly larger than $\pi(U) - \pi(\mathcal{L}/2)$, which corresponds to the case $\alpha = \alpha_1 = \cdots = \alpha_k = 0$ in (4.3). We have, namely, by the Prime Number Theorem,

\[
A_0 = \sum_{\alpha, \alpha_1, \ldots, \alpha_k \geq 0} \pi \left( \frac{U}{q_0^\alpha \prod_{i=1}^k h_i^{\alpha_i}} \right) - \pi \left( \frac{\mathcal{L}/2}{q_0^\alpha \prod_{i=1}^k h_i^{\alpha_i}} \right)
\]

\[
\ll \frac{(1 + o(1))U}{\log U} \left( 1 + \frac{1}{q_0} + 1 + \frac{1}{q_0^2} + \cdots \right) \prod_{i=1}^k \left( 1 + \frac{1}{h_i} + 1 + \frac{1}{h_i^2} + \cdots \right)
\]

\[
\sim \frac{U}{\log U} \left( 1 - \frac{1}{q_0} \right)^{-1} \prod_{i=1}^{m} \left( 1 - \frac{1}{h_i} \right)^{-1} \leq \frac{2U}{\log U}
\]

if $C_6(\varepsilon)$ was chosen sufficiently large depending on $k$. 

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We will choose the residue class \( z_{p_j} \pmod{p_j} \) for all \( p_j \mid P_2 \) consecutively for all primes. We have to take care in the \( j \)th step that

\[(4.8) \quad z_{p_j} + h_i \not\equiv 0 \pmod{p_j} \]

should hold; further, the additional property that at the \( j \)th step we choose the residue class \( z_p \mod p \) so that it should sieve out the maximal number of remaining elements from the remaining set of \( s \)'s of cardinality \( A_{j-1}' \). We distinguish two cases:

(i) if before the \( j \)th step the number of \( s \)'s satisfies

\[(4.9) \quad A_{j-1}' \leq \frac{\mathcal{L}}{5 \log \mathcal{L}} \left( < \frac{\pi(\mathcal{L}) - \pi(\mathcal{L}/2)}{2} \right), \]

then we stop the choice of new \( z_p \)'s.

Otherwise, if (4.9) is false, then we have in total at most \( (\log_2 \mathcal{L})^{k+1} \) possibilities for \( \alpha, \{\alpha_i\}_{i=1}^k \), since \( U_{pj} = o(\log \mathcal{L}) \) and even neglecting the primality of \( p \) we have in total at most

\[(4.10) \quad k \left[ \frac{U}{y} \right] (\log_2 \mathcal{L})^{k+1} < \frac{\mathcal{L}}{\log^{10} \mathcal{L}} < \frac{A_{j-1}'}{2(\log \mathcal{L})^8} \]

numbers \( s \) in forbidden residue classes (4.8). This means that choosing the residue class \( z_{p_j} \) so that we avoid the \( k \) forbidden residue classes \( h_1, \ldots, h_k \) but sieve out afterwards as many elements as possible, we obtain after the next step

\[(4.11) \quad A_j' < A_{j-1}' - \frac{A_{j-1}' \left( 1 - \frac{1}{2(\log \mathcal{L})^8} \right)}{p_j - k} \]

\[< A_{j-1}' \left( 1 - \frac{1}{2(\log \mathcal{L})^8} \right) \]

\[< A_j \left( 1 - \frac{1}{p_j} \right)^{1 - (\log \mathcal{L})^{-8}}. \]

By Mertens' theorem, (3.18)–(3.19) and (4.7), we obtain a final residual set
(after at most $\pi(y) - \pi(v)$ steps of $s$’s) of size at most

\begin{equation}
A^* < A'_0 \prod_{\nu < p \leq y} \left(1 - \frac{1}{p}\right)^{1-(\log \mathcal{L})^{-8}} \sim A'_0 \left(\frac{\log v}{\log y}\right)^{1-(\log \mathcal{L})^{-8}}
\end{equation}

\begin{align*}
&\sim A'_0 \left(\frac{3(k + 5) \log^2 \mathcal{L}}{\log \mathcal{L} \log_3 \mathcal{L}}\right) < \frac{7kU}{\log U g(e^2)} < \frac{7k_{10}(\varepsilon) \mathcal{L}}{\log \mathcal{L}} < \frac{\pi(\mathcal{L}) - \pi(\mathcal{L}/2)}{3}.
\end{align*}

This means that taking into account that the total number of $s$’s with \ref{eq:4.4} is by \ref{eq:4.6} a negligible portion of the above remaining quantity (even without sieving them out by the above procedure), we obtain finally that using the prime factors of $P_2$, with a suitable choice of $z_p$ for these primes we can already reach for the numbers $s$ with $1 < s \leq U$, $(z + s, P_1 P_3) = 1$ apart from an exceptional set $S$ of size at most $(\pi(\mathcal{L}) - \pi(\mathcal{L}/2))/2$ the crucial relations

\begin{equation}
(z + s, P_2) > 1 \text{ if } s \notin \mathcal{H}, \ s \notin S
\end{equation}

and

\begin{equation}
(z + s, P_2) = 1 \text{ if } s \in \mathcal{H}, \ s \notin S.
\end{equation}

Let

\begin{equation}
S' = S \setminus \mathcal{H}.
\end{equation}

Then by $|S| < (\pi(\mathcal{L}) - \pi(\mathcal{L}/2))/2$ we can easily find for any $s \in S$ a suitable prime $p \mid P_4 = \{p \in (\mathcal{L}/2, \mathcal{L}], p \notin \mathcal{H}\}$ with $p \nmid \prod_{i=1}^{k}(s - h_i)$ and consequently a $z_p(\mod p)$ with

\begin{equation}
z_p + s \equiv 0 (\mod p) \text{ for } s \in S', \ z_p + t \not\equiv 0 (\mod p) \text{ for } t \in \mathcal{H}.
\end{equation}

Thus we need still to determine $z \mod p$ for those primes which were not used before. These primes belong to one of the following categories (cf. \ref{eq:3.5}):

- (i) $p \in \mathcal{H} \rightarrow p > C_6(\varepsilon)$,
(ii) \( p = q_0 \rightarrow p \geq \log_2 N \) (if \( q_0 \leq W \)),

(iii) the remaining parts of unused \( p | P_4 \rightarrow p > L/2 \),

(iv) \( p | \frac{W}{W_0} \rightarrow p > L \).

Since we have for \( 1 < s \leq U, s \notin \mathcal{H} \) already by the earlier choices

\[
(z + s, P_1P_2P_3P_4) > 1
\]

this property will be valid independently from the further choices of \( z_p \) and so the condition \( z + s \) composite for \( 1 < s \leq U, s \notin \mathcal{H} \) will be true at the end as well.

So we have only to assure that for the primes in (i)–(iv) we should have

\[
z_p \not\equiv -t \pmod{p} \text{ if } t \in \mathcal{H}.
\]

But this makes no problem since \( |\mathcal{H}| = k < \min\{C_6(\epsilon), \log_2 N, L/2\} \) by (3.5).

So we finally determined a \( z \pmod{W} \) with the property that for \( s \in (1, U] \) we have

\[
(z + s, W) = 1 \text{ if and only if } s \in \mathcal{H}.
\]

Consequently if \( n \equiv z \pmod{W} \), then

\[
(n + s, W) = 1 \text{ if and only if } s \in \mathcal{H}.
\]

5 Proofs of Theorems 1–4

We summarize the results of Sections 3 and 4 with the aim of applications in Theorems 1–4.

Let \( k, m, \epsilon \) be chosen satisfying (5.1), let \( \epsilon = \epsilon(k, m) > 0 \) be a sufficiently small constant, \( N > N_0(\epsilon, k, m) \):

\[
k = 50, \quad m = 2 \quad \text{or} \quad k_m = C_7^5 m.
\]

Let \( \mathcal{H} = \{h_i\}_{i=1}^k \subset \mathcal{P} \) satisfying

\[
[h_t, q_0] \mid \prod_{1 \leq i < j \leq k} (h_j - h_i) \quad \text{for any } t \in [1, m], \quad q_0 \text{ defined in (3.1)–(3.4),}
\]

\[
|\mathcal{H}| = k < \min\{C_6(\epsilon), \log_2 N, L/2\} \text{ by (3.5).}
\]

So we finally determined a \( z \pmod{W} \) with the property that for \( s \in (1, U] \) we have

\[
(z + s, W) = 1 \text{ if and only if } s \in \mathcal{H}.
\]

Consequently if \( n \equiv z \pmod{W} \), then

\[
(n + s, W) = 1 \text{ if and only if } s \in \mathcal{H}.
\]
(5.3) \[ k + 1 < C_6(\varepsilon) < h_1 < \cdots < h_k \leq \log^2 N, \]

(5.4) \[ U = c_9(\varepsilon)g(N)\log N, \quad g(N) = \frac{\log_2 N \log_4 N}{\log_3 N}, \quad N' = \pi(N). \]

Then we can find suitable values of

(5.5) \[ W < N^\varepsilon, \quad z(\mod W) \]

and an \( n \in [N', 2N') \) such that we have at least \( m \) primes among \( n + h_i \) and all numbers of the form \( n + s \) are composite if \( s \in (1, U \setminus H) \).

**Remark.** This implies that all (at least \( m \)) primes in the interval \((n+1, n+U)\) are of the form \( n + h_j \), \( h_j \in H \).

**Remark.** We used the introduction of the variable \( \varepsilon \) since it was formulated in this way in \([BFM]\). However, since there is an exact connection \((5.1)\) between \( k \) and \( m \) and \( \varepsilon \) depends just on \( k \) and \( m \), in the applications we can write in \((5.3)\) \( C_6'(k) \) instead of \( C_6(\varepsilon) \), \( C_8'(k) \) in place of \( C_8(\varepsilon) \) in \((3.14)\) and \( c_9'(k) \) instead of \( c_9(\varepsilon) \) in \((5.4)\), \( c_{10}'(k) \) instead of \( c_{10}(\varepsilon) \) before \((3.18)\), further \( c_1'(k) \) in place of \( \varepsilon \) in \((5.5)\). Similarly we can choose \( \mathcal{L} = c_{12}(k) \log N \) with a small \( c_{12}(k) \) in \((3.18)\). Additionally, if \( m = 2, k = 50 \), these are just absolute constants (which is the case in Theorems \(1, 2\) and \(3\)). The elimination of \( \varepsilon \) in this part of the proof will also increase clarity since the condition \((1.5)\) for the function \( f(n) \) contains a parameter \( \varepsilon \) too.

In order to prove Theorem \(1\), suppose, in contrary to its assertion, that we have a sequence of 50 positive numbers \( c_\nu^*, \delta_\nu \) \((1 \leq \nu \leq 50)\) satisfying with two constants \( c^*, N^* > 0 \)

(5.6) \[ J_\nu := [c_\nu^*, c_\nu^* + \delta_\nu], \quad c_\nu^* > 4\delta_\nu > 20c_{\nu+1}, \quad c_1^* < c^*, \]

(5.7) \[ \left\{ \frac{d_n}{f(n)} \right\}_{n=N^*}^{\infty} \cap \left( \bigcup_{\nu=1}^{50} J_\nu \right) = \emptyset. \]

Let

(5.8) \[ I_\nu(n) := \left[ c_\nu^* f(n), (c_\nu^* + \delta_\nu) f(n) \right] \quad \text{for} \, \nu = 1, 2, \ldots, 50. \]
Then
\[ d_n \not\in \bigcup_{\nu=1}^{50} I_\nu(n) \quad \text{for} \quad \nu = 1, 2, \ldots, 50, \quad n \in [N', 2N'), \quad N' > N^*. \]  

We will choose now the primes \( h_1 < h_2 < \cdots < h_{50} \) consecutively, satisfying (5.2)–(5.3) and a sufficiently small \( \varepsilon > 0, \quad N' > \max(N_0(\varepsilon), N^*) \)

\[ h_\nu \in I'_{51-\nu}(n) := \left[ \left( c^*_{51-\nu} + \frac{\delta^*_{51-\nu}}{2} \right) (1 + \varepsilon)f(N'), \left( c^*_{51-\nu} + \delta_{51-\nu} \right)(1 - \varepsilon)f(N') \right] \]

This choice (for \( \nu = 1, \ldots, 50 \)) is easily assured by the Prime Number Theorem if \( \varepsilon \) was chosen sufficiently small, \( N_0(\varepsilon) \) sufficiently large depending on all \( c^*_\nu, \delta_\nu \) \((1 \leq \nu \leq 50)\) and \( \varepsilon \). So we have for \( 1 \leq \nu < \mu \leq 50 \) for large enough \( N' \)

\[ h_\mu - h_\nu \in \left[ \left( c^*_{51-\nu} + \frac{\delta^*_{51-\nu}}{2} - 2c^*_{51-\mu} \right) (1 + \varepsilon)f(N'), \left( c^*_{51-\nu} + \delta_{51-\nu} (1 - \varepsilon)f(N') \right) \right] := I^*_{51-\mu}(n) \subset I_{51-\mu}(n). \]

This contradicts to (5.9) since we have for at least one pair of consecutive primes

\[ d_n = h_\mu - h_\nu, \quad n \in [N', 2N') \quad \text{q.e.d.} \]

Now we turn to the proof of Theorem 2. Let us suppose that we have 50 functions with \( f_i(x) \not\to \infty \), satisfying (1.5), (1.9), (1.11) and 50 intervals

\[ J_\nu := [c^*_\nu, c^*_\nu + \delta_\nu], \quad I_\nu(n) := [c^*_\nu f_\nu(n), (c^*_\nu + \delta)f_\nu(n)] \]

such that with a sufficiently large \( N^* \) we have

\[ d_n \not\in \bigcup_{\nu=1}^{50} I_\nu(n) \quad \text{for} \quad \nu = 1, \ldots, 50, \quad n \in [N', 2N'), \quad N' > N^*. \]

Then, analogously to (5.10)–(5.11) we can choose the primes \( h_1 < h_2 < h_{50} \) with (5.2)–(5.3), \( \varepsilon > 0 \) and \( N > \max(N_0(\varepsilon), N^*) \) so that

\[ h_\nu \in I'_\nu(n) := \left[ \left( c^*_\nu + \delta^*_\nu \right)(1 + \varepsilon)f_\nu(N'), (c^*_\nu + \delta_\nu)(1 - \varepsilon)f_\nu(N') \right] \].

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This implies for sufficiently large $N'$ by (1.11) for any $1 \leq \nu < \mu \leq 50$

$$h_\mu - h_\nu \in \left[ (c_\mu^*(1 + \varepsilon)f_\mu(N), (c_\mu^* + \delta_\mu)(1 - \varepsilon)f_\mu(N) \right] \subset I_\mu(n)$$

and we obtain again a contradiction to (5.14). q.e.d.

The first assertion of Theorem 3 follows immediately from our summary in (5.1)–(5.5) if we choose simply $h_1 < h_2 < \cdots < h_{50}$ satisfying (5.2)–(5.3) and for $n \in [N', 2N')$ with

$$h_i = \beta_i f(N') \left( 1 + O\left( \frac{1}{\log_2 N'} \right) \right).$$

The consequence about the at least 2% density of $J_f$ follows in the same way as in the proof of Corollary 1.2 of [BFM].

Finally, Theorem 4 is also an obvious corollary of our summary (5.1)–(5.5). Namely, if for $n \in [N', 2N')$ we choose for $i \in [1, k_m]$

$$h_i = i \cdot \frac{U}{k_m + 1} \left( 1 + O\left( \frac{1}{\log U} \right) \right) \quad (k_m = C_7 e^{5m})$$

with (5.2)–(5.5) then we have in the interval $(n + 1, n + U)$ at least $m$ and at most $k_m$ primes, all among $n + h_i (1 \leq i \leq k_m)$. Consequently we get at least $m$ consecutive primegaps each of size at least

$$\frac{U}{2(k_m + 1)} \geq c_{12}(k)g(n)\log n \quad \text{with} \quad g(n) = \frac{\log_2 n \log_4 n}{\log_3^2 n}. \quad \text{q.e.d.}$$

6 Proofs of Theorems 1’–4’ and Corollary 1’

We first remark that apart from Theorem 2 we worked in all proofs (cf. our present Section 5) within a given interval $[N', 2N']$ where $N'$ was any sufficiently large constant and we worked with an $\mathcal{H}_k$ tuple satisfying

$$\mathcal{H}_k = \{h_i\}_{i=1}^k \quad \text{with} \quad h_i \asymp f(N').$$

Thus we will consider first the proofs of Theorems 1’, 3’, 4’. We will distinguish two cases as follows.

Case 1.

$$f(N') < \log N' (\log_2 N')^{1/2}.$$
In this case the assertions of Theorems 1’, 3’, 4’ follow directly from Theorems 1, 3, 4 for the specific interval \([N', 2N']\).

**Case 2.**

\[ f(N') \geq \log N' (\log_2 N')^{1/2}. \]

In this case we will use the method of [May2], and will describe the needed changes compared to [May2]. We will use (6.1) which in this case implies

\[ h_i \gg (\log N') (\log_2 N')^{1/2} \quad (i = 1, 2, \ldots, k). \]

In order to follow [May2] we will change our notation and choose with a given small \(\varepsilon_0\)

\[
\begin{align*}
  z &= \varepsilon_0 \log N', \quad x = \mathcal{L}, \quad P_y = \prod_{p \leq y} p, \\
  y &= \exp\left(\frac{(1 - \varepsilon_0) \log x \log_3 x}{\log_2 x}\right), \quad z = \frac{x}{\log_2 x}, \quad U = C_U \frac{x \log y}{\log_2 x},
\end{align*}
\]

where \(C_0\) is an arbitrarily large constant as in [May2], independent of \(\varepsilon_0\).

In contrast to Section 2 and in accordance with [May2] we will choose the residue classes \(a_p \pmod{p}\), in the first step for \(p \leq z, p \neq q_0\) (the greatest prime factor of the eventually existing single exceptional modulus, as in Sections 3–4)

\[
\begin{align*}
  a_p &= 0 \quad \text{for every prime } p \in (y, z], \quad p \neq q_0, \\
  a_p &= 1 \quad \text{for every prime } p \leq y, \quad p \neq q_0.
\end{align*}
\]

After removing elements of \([1, U]\) in these residue classes we obtain the set \(\mathcal{R} \cup \mathcal{R}' \cup \tilde{\mathcal{R}} \cup \tilde{\mathcal{R}}'\), where

\[
\begin{align*}
  \mathcal{R} &= \{mp \leq U : p > z, \ m \text{ is } y\text{-smooth}, \ (mp - 1, P_y) = 1\}, \\
  \tilde{\mathcal{R}} &= \{mpq_0 \leq U : p > z, \ m \text{ is } y\text{-smooth}, \ (mpq_0 - 1, P_y) = 1\}, \\
  \mathcal{R}' &= \{m \leq U : m \text{ is } y\text{-smooth}, \ (m - 1, P_y) = 1\}, \\
  \tilde{\mathcal{R}}' &= \{mq_0 \leq U : m \text{ is } y\text{-smooth}, \ (mq_0 - 1, P_y) = 1\}.
\end{align*}
\]

We obtain by Lemma 1 of Section 3 similarly to Lemma 2 of [May2]

\[ |\mathcal{R}' \cup \tilde{\mathcal{R}}'| \ll \frac{x}{(\log x)^{1+\varepsilon}}. \]
Again similarly to Lemma 3 of [May2] we have now for $V \in [z + z/ \log x, x(\log x)^2]$ (6.8) 
\[
\# \{z < p \leq V : (mp - 1, P_y) = 1\} = \frac{V - z}{\log x} \prod_{p \leq y \atop p \mid m} \frac{p - 2}{p - 1} (1 + o(1))
\]

and in particular for even $m \leq U(1 - 1/ \log x)/z$ (6.9) 
\[
|R_m| = \frac{2e^{-\gamma}U(1 + o(1))}{m(\log x)(\log y)} \left( \prod_{p > 2} \frac{p(p - 2)}{(p - 1)^2} \right) \left( \prod_{p \mid m, p > 2} \frac{p - 1}{p - 2} \right).
\]

Further the same methods show that by $q_0 \geq \log_2 N'$ (6.10) 
\[
|R_m| \ll \frac{U}{q_0 m \log x \log y} \ll \frac{U}{m \log^2 x \log y},
\]

which is negligible compared to (6.9). By $R_m$ and $\tilde{R}_m$, resp., we denoted the terms of $R$ and $\tilde{R}$, resp., which contain the specific parameter $m$ in (6.6) as in (2.8) of [May2].

After this initial choice of $a_p$ for $p \leq z$ we try to choose $a_p$ modulo $p$ for all $p \leq x$, $p \neq q_0$ in such a way that for the common solution $a \mod W$ of the congruences $a \equiv a_p \mod p$ for (6.11) 
\[
p \mid W := \frac{P_x}{q_0}
\]

we should have, similarly to (3.18) (6.12) 
\[
(a + s, W) > 1 \text{ if } s \notin \mathcal{H}, \quad 1 < s \leq U.
\]

The choice of $a_p$ for $z < p \leq x$ will follow closely that of [May2] with the following changes. We will choose in the applications (see Section 5) our $\mathcal{H}$ with (6.13) 
\[
h_i \in \mathcal{P}, \quad (h_i - 1, q_0 P_y) = 1
\]

and use (6.2) additionally. This means that $h_i \equiv 0 \mod p$ for some $p \in (y, z]$ will not occur and we will have (6.14) 
\[
h_i \in \mathcal{R}_1 \quad (i = 1, 2, \ldots, k).
\]
We note that $\mathcal{R}_m \cap \mathcal{R}_{m'} = \emptyset$ if $m \neq m'$, so $h_i \notin \mathcal{R}_m$ for $m > 1$.

Since any other essential requirements for $h_i$ in Sections 3–5 are concerning only the size of $h_i$ requiring

$$
(6.15) \quad \xi_i(N) \leq h_i \leq \xi_i(N)(1 + \eta_i)
$$

for some sufficiently small $\eta_i$ independent of $N$, with some functions $\xi_i(N)$, we can always fulfill these conditions with proper choice of the primes $h_i$ satisfying $(6.13)$, due to the relations $(6.8)$–$(6.9)$, which mean that we have sufficiently large sets to choose $\{h_i\}_{i=1}^k$.

After choosing our set $\mathcal{H} = \{h_i\}_{i=1}^k$ satisfying the requirements of Sections 3–5 for a given value of $N$ we will denote it by $\mathcal{H}_* = \{h_i^*\}_{i=1}^{k^*}$ and consider it fixed. After this we will choose $a_p$ for $p | W$, $p > z$ in a somewhat different way from that of [May2]. The difference affects only the case $m = 1$ and can be described as follows. We will choose the set $\mathcal{H} = \{h_1, \ldots, h_k\}$ in [May2] disjoint to our $\mathcal{H}_*$ (and also $k$ will be sufficiently large compared to $k^*$ while our $k^*$ will be equal to 50, 99 or $k = k(m)$ in Theorem 4'). The change in the choice of the probabilities of choosing $a \mod q \in I_1 \subseteq [x/2, x]$ will be that in contrast to (4.1) of [May2] we will set $(\mu_{1,q}^*(a) \text{ will denote the new probabilities, } \alpha_{1,q}^* \text{ the new normalizing number to have } \sum_{a(q)} \mu_{1,q}^*(a) = 1)$

$$
(6.16) \quad \mu_{1,q}^*(a) := 0 \text{ if } \exists i \in [1, k^*], \ a + h_i^* \equiv 0 \pmod{q},
$$

$$
(6.17) \quad \mu_{1,q}^*(a) := \mu_{1,q}(a) \cdot \frac{\alpha_{1,q}^*}{\alpha_{1,q}} \ \text{otherwise}.
$$

In this way we can avoid that by the random choice of $a_q$ modulo $q$ we should have $q | n + h_i^*$ for $n \equiv a \pmod{W}$ for some $i \in [1, k^*]$, since we give 0 probability to those $a_q$ in (6.16). Naturally we have to rescale the remaining probabilities as done in (6.17), which actually slightly increases all remaining probabilities. This means that none of the $h_i^* \in \mathcal{R}_1$ will be sieved out (with probability 1) by the above random sieve procedure. On the other hand we have to show that, similarly to the end of Section 6 on p. 13 of [May2], for all but $o_k (|\mathcal{R}_1|)$ primes $p_0 \in \mathcal{R}_1$ the expected number $\sum_q \mu_{1,q}^*(p_0)$ of times $p_0 \in \mathcal{R}_1$ is chosen will remain $\gg \delta \log k$ if $p_0 \notin \mathcal{H}^*$ as in [May2] in case of the original choice of $\mu_{1,q}(p_0)$.

If for a given $q$

$$
(6.18) \quad p_0 \not\equiv h_i^* \pmod{q} \text{ for every } i = 1, \ldots, k^*,
$$
then we have by (6.16)–(6.17)

\[(6.19)\]

which increases the corresponding term in our crucial sum. Let \( p_0 \not\in \mathcal{H}^* \) be given, and let us fix \( i \in [1, k^*] \). How many different \( q \)'s do we have at most in \( I_1 \subseteq [x/2, x] \) with

\[(6.20)\]

The answer is simple: at most one. If we had, namely, (6.20) for \( q = q_1, q_2 \in I_1 \ (q_1 \neq q_2) \), then this would imply by \( q_1q_2 \geq x^2/4 > U > \max(p_0, h_i^*) \)

\[(6.21)\]

consequently

\[(6.22)\]

So the remaining question is reduced to show that if we delete at most \( k^* = O(1) \) terms from the original sum \( \sum_q \mu_{1,q}(p_0) \) the sum will be still \( \gg \delta \log k \). But this follows already from the trivial relation \( \mu_{m,q}(a) \leq 1 \), although it is easy to see that even \( \mu_{m,q}(a) \ll \frac{x}{q} \ll \frac{1}{x^{1-\varepsilon}} \) holds for \( q \in [x/2, x] \). (\( k \) can be chosen sufficiently large compared with \( k^* \).)

This completes the proof of Theorems 1', 3', 4' (and thereby Corollary 1').

In case of Theorem 2' we consider a given \( N' \) and distinguish the following two cases. Let us consider a sequence of 99 exceptional functions.

**Case 1*.**

\( f_{50}(N') < \log N'(\log_2 N')^{1/2} \).

In this case the proof of the original Theorem 2 can be applied to the increasing subset \( \{f_i(x)\}_{i=1}^{50} \).

**Case 2*.**

\( f_{50}(N') \geq \log N'(\log_2 N')^{1/2} \).

In this case the new method of \[\text{May2}\] together with the changes of the present section yields the result for the increasing subset \( \{f_i(x)\}_{i=50}^{99} \).

Thus, in both cases we obtain a contradiction if we suppose for an increasing sequence of at least 99 functions that the relation (1.12) fails.

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Remark 3. With some extra effort it would also be possible to show Theorem 2’ with 49 instead of 98.

Remark 4. If we define additionally

\[(6.23) \quad \lambda_{d_1,d_k,e_1,e_k} = 0 \quad \text{if} \quad q_0 \mid \prod_{i=1}^{k} (d_i e_i),\]

then the whole result can be made effective. (We remark that actually we have a loss of size $1 + O\left(\frac{1}{q_0}\right) = 1 + o(1)$ due to (6.23) but this does not affect the validity of the argument.)

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