ON THE FORMALITY OF NEARLY KÄHLER MANIFOLDS
AND OF JOYCE’S EXAMPLES IN G₂-HOLONOMY

MANUEL AMANN AND ISKANDER A. TAIMANOV

Abstract. It is a prominent conjecture (relating Riemannian geometry and algebraic topology) that all simply-connected compact manifolds of special holonomy should be formal spaces, i.e. their rational homotopy type should be derivable from their rational cohomology algebra already—an as prominent as particular property in rational homotopy theory. Special interest now lies on exceptional holonomy $G₂$ and $\text{Spin}(7)$.

In this article we provide a method of how to confirm that the famous Joyce examples of holonomy $G₂$ indeed are formal spaces; we concretely exert this computation for one example which may serve as a blueprint for the remaining Joyce examples (potentially also of holonomy $\text{Spin}(7)$).

These considerations are preceded by another result identifying the formality of manifolds admitting special structures: we prove the formality of nearly Kähler manifolds.

A connection between these two results can be found in the fact that both “special holonomy” and “nearly Kähler” naturally generalize compact Kähler manifolds, whose formality is a classical and celebrated theorem by Deligne–Griffiths–Morgan–Sullivan.

Introduction

In topology we find a beautiful synthesis of two both deeply linked yet “competing” theories: (co)homology and homotopy. If one is willing to forget torsion information, a similar picture is preserved, yet it is enriched by the beauty of rational homotopy theory which provides a common framework to observe the interplay of homotopy and homology “in wildlife”. The “rational homotopy type” thus is elegantly encoded in so-called Sullivan models, or, respectively, in free Lie models, and contains both homotopical as well as (co)homological invariants. This immediately prompts one of the most considered meta-questions in the field: when is the homotopy type no more complicated than the cohomology type, when is all homotopical information already contained in the cohomology algebra, i.e. when is the space formal?

It is one of the mysteries of the manifold interlinks between topology and (Riemannian) geometry that this abstract concept of formality suddenly seeks attention in the form of analytically defined properties of Riemannian structures. More concretely, formality holds for symmetric spaces (and their generalizations), it is satisfied by Kähler manifolds, and it is conjectured (and
partly confirmed) to hold true for Riemannian manifolds of special holonomy (see Sections 1, 3).

With this article we want to cast more light on the depicted “opaque interweavings” of geometry and topology with our two main theorems.

**Theorem A.** Closed simply-connected nearly Kähler manifolds are formal.

Nearly Kähler manifolds are one way to generalize Kähler manifolds and we point the reader to Section 2 for their definition. In order to prove this result, it suffices to bring together known classification results for nearly Kähler manifolds on the one hand, and, on the other hand, formality results for the respective “building blocks” in these classifications.

It is a prominent conjecture that all simply-connected manifolds of special holonomy are formal. With the formality of Kähler manifolds (including hyperkähler ones) and of positive quaternion Kähler manifolds established the attention focuses on $G_2$- and $\text{Spin}(7)$-manifolds. In both cases there are several simply-connected closed examples, the first and most prominent ones established by Joyce. Until recently (see [35]) not even their cohomology rings were known. We extend the analysis of the geometric construction of the example in this article via means from differential topology and rational homotopy. More precisely, we use the intersection homology of concrete submanifolds in order to build a Sullivan model over the reals up to a certain degree. For a typical example of a Joyce manifold we consider the simplest one, which we denote simply by $M$ (see [20, p. 349, Example 1]). Note that this example, however, already carries the main features of the construction principles and can serve as a model case (see Remark 3.2). Consequently, the concept of $s$-formality allows us to derive

**Theorem B.** The Joyce example $M$ ([20, Example 1]) of exceptional holonomy $G_2$ is formal.

The manifold in question is described in detail in [35] and in Section 4. This approach and reasoning can be considered a blueprint result in order to analyze and establish the formality of the remaining Joyce examples (in $\text{Spin}(7)$ holonomy as well) (cf. Remark 3.2).

**Structure of the article.** In Section 1 we provide necessary concepts from rational homotopy theory. In particular, we provide a certain background on formality. Section 2 is devoted to compiling necessary results and to the proof of Theorem A. In Section 3 we comment on the importance of formality within special holonomy, we review the (generalized) Kummer construction underlying the examples by Joyce, and we recall the concepts from intersection homology which we use on the side of differential topology. Next, in Section 4 we provide the proof of Theorem B. This illustrates the general method of deriving formality of such examples.

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1. FORMALITY IN RATIONAL HOMOTOPY THEORY

We cannot and do not intend to provide an introduction to rational homotopy theory here. For this we point the reader to the textbooks [9], and [10]. Let us just briefly recall some concepts necessary for this article.

A most important tool in the theory are so-called (minimal) Sullivan models of commutative differential graded algebras, respectively of nilpotent or simply-connected topological spaces \( X \), which are encoded via the algebras of polynomial differential forms \( (A_{\text{PL}}(X), d) \) (see [9, Chapters 10 and 12]). Indeed, the rational homotopy type of any such space is captured via a quasi-isomorphism (i.e. a morphism of differential graded algebras inducing an isomorphism on cohomology)

\[
(\Lambda V, d) \xrightarrow{\sim} (A_{\text{PL}}(X), d)
\]

from a minimal Sullivan algebra \((\Lambda V, d)\). For path-connected spaces such a model can always be constructed and the algebra \((\Lambda V, d)\) is unique up to isomorphism. For example, the fact that it encodes the rational homotopy type of \( X \) is reflected by the fact that up to duality the underlying graded vector space \( V \) over \( \mathbb{Q} \) is gradedly isomorphic to \( \text{Hom}(\pi_*(X), \mathbb{Q}) \), the rational homotopy groups of \( X \), provided that \( X \) is a nilpotent space.

It is one of the central questions in rational homotopy theory to understand the nature, behaviour and examples of formality.

**Definition 1.1.** A path-connected topological space \( X \) is called formal if already its rational cohomology algebra \( H^*(X; \mathbb{Q}) \) (endowed with the trivial differential) encodes its rational homotopy type, i.e. if there is a chain of quasi-isomorphisms

\[
(A_{\text{DR}}, d) \xrightarrow{\sim} (A_1, d) \xrightarrow{\sim} \cdots \xrightarrow{\sim} (A_k, d) \xrightarrow{\sim} (H^*(X), 0)
\]

Equivalently, a minimal Sullivan model of \((H^*(X), 0)\) is one of \((A_{\text{DR}}, d)\).

So far, the algebras we consider were over the rational numbers. Clearly, one obtains the analog results passing to algebras over the reals, thus dealing with the real homotopy type. On smooth manifolds one may observe that the commutative differential graded algebra of smooth differential forms \((A_{\text{DR}}(M), d)\) together with the exterior derivative encodes the real homotopy type. Usually, several rational homotopy types may fall into one real homotopy type. However, from [18, Corollary 6.9, p. 265] we recall that a space is formal over \( \mathbb{Q} \) if and only if it is formal over any field extension, i.e. in particular over \( \mathbb{R} \). This result will allow us to draw on computations with smooth differential forms in order to establish formality.

There are several prominent examples of formal spaces. Since products of harmonic forms on compact symmetric spaces are harmonic again, these spaces are “geometrically formal” and formal, in particular, due to the Hodge decomposition. We remark that this is one of the few cases (cf. Theorem 1.2, see Remark 2.2) in which formality can be derived by an analytic property.
There is the larger class of so-called generalized symmetric spaces. These are Riemannian manifolds equipped with a (regular) s-structure, i.e. at each point \( x \in M \) there exists an isometry \( s_x : M \to M \) with isolated fixed point \( x \) and satisfying that
\[
 s_x s_y = s_{s_x(y)} s_x
\]
In case there exists a \( k \in \mathbb{N} \) such that for all \( x \in M \) it holds that \( s_x^k = \text{id} \) and \( s_x^l \neq \text{id} \) for \( l < k \), then \( k \) is the order of the s-structure. The smallest order \( k \) of any s-structure on \( (M,g) \) is the order of the generalized symmetric space. We then call \( (M,g) \) a \( k \)-symmetric space. We remark that the given definition is not the most general one used in the literature, but suffices for our purposes.

From the main results in [34], [25], [15] we cite that \( k \)-symmetric spaces are formal (but not necessarily geometrically formal).

This article is particularly motivated by another highly important class of formal spaces, namely compact Kähler manifolds.

**Theorem 1.2** ([8]). Compact Kähler manifolds are formal.

Moreover, since, due to the Künneth formula, the cohomology of a finite product space is the tensor product of the respective cohomology algebras, and since a Sullivan model of such a tensor product is the tensor product of the respective models, it is immediate that a product of formal spaces is again formal—actually, it is formal if and only if so are both spaces.

One-point unions of formal topological spaces and connected sums of formal manifolds are formal again.

### 2. Formality of nearly Kähler manifolds

There are two natural generalizations of Kähler manifolds: almost Kähler manifolds and nearly Kähler manifolds. These notions are orthogonal in the sense that if a manifold is almost Kähler and nearly Kähler at the same time, then it is Kähler (see [17]).

We recall that a manifold is called *almost Hermitian* if it is endowed with an almost complex structure \( J \) and Hermitian metrics in the fibers of the tangent bundle. It is assumed that \( J \) and the metric are smooth.

An *almost Kähler* manifold is an almost Hermitian manifold for which the smooth 2-form on the fibers of the tangent bundles
\[
(1) \quad \omega(X,Y) = \langle JX,Y \rangle
\]
is closed. Here \( \langle \cdot, \cdot \rangle \) is the Hermitian metric.

It is easy to notice that these are exactly symplectic manifolds for which \( \omega \) is a symplectic form. There was a conjecture that simply-connected compact symplectic manifolds are formal. For symplectic nilmanifolds it was known to be wrong: for instance, the Kodaira–Thurston manifold \( M_{KT} = M_H \times S^1 \), where \( M_H \) is a nilpotent Heisenberg 3-manifold, is not formal due to a nontrivial triple Massey product in \( M_H \). In [2, 3] the formality conjecture for simply-connected symplectic manifolds of dimension \( \geq 10 \) was disproved as follows. \( M_{KT} \) is symplectically embedded into \( \mathbb{C}P^n, n \geq 5 \), and after the
symplectic blow up of \( \mathbb{CP}^n \) along \( M_{KT} \) gives rise to a nontrivial Massey product in the resulting manifold. For eight-dimensional manifolds this conjecture was disproved later by other means (see [13]).

A nearly Kähler manifold is an almost Hermitian manifold such that the (2,1)-tensor \( \nabla J \) is skew symmetric:

\[
(\nabla_X J)(X) = 0 \quad \text{for all } X \in TM.
\]

This notion was introduced in [16] where it was mentioned that there are plenty of examples of nearly Kähler manifolds which are not Kähler. Among these are the 6-sphere with the canonical almost complex structure and the round metric, and the quotients \( G/K \) where \( G \) is a compact semisimple Lie group, and \( K \) is the fixed point set of an automorphism \( \sigma : G \to G \) of order three: \( \sigma^3 = 1 \). These are 3-symmetric spaces in the terminology of Section 1.

We remark that it is a prominent problem if the 6-sphere admits a complex structure; clearly, it is certainly not Kähler.

Contrast the definition of nearly Kähler manifolds with the definition of a Kähler manifold wherein, in particular, it is required that \( J \) is a genuine complex structure, i.e. \( \nabla J = 0 \), and \( J \) is parallel. Thus, clearly, nearly Kähler manifolds are obvious generalizations.

It appears that, in contrast to (1), Condition (2) is very rigid.

Let us recall that a nearly Kähler manifold is called strict if

\[
\nabla_X J \neq 0 \quad \text{for } X \neq 0.
\]

It was proved in [24] (see also [30]) that a complete and simply-connected nearly Kähler manifold \( M \) is represented as a Riemannian product

\[
M = M_1 \times M_2
\]

of a Kähler manifold \( M_1 \) and a strict nearly Kähler manifold \( M_2 \) (one of the factors may be a point).

The classification of complete simply-connected strict nearly Kähler manifolds is given by Nagy’s Theorem in [29]: every such manifold is a Riemannian product whose factors belong to one of three classes, namely

1. 6-dimensional nearly Kähler manifolds,
2. homogeneous nearly Kähler manifolds of certain types,
3. twistor spaces over quaternionic Kähler manifolds with positive scalar curvature, endowed with the canonical nearly Kähler metric.

Moreover, Nagy derived from this classification the proof of the Wolf–Gray conjecture for manifolds of dimension \( \geq 8 \). This conjecture states that every homogeneous nearly Kähler manifold is a 3-symmetric space with the canonical almost complex structure, and its proof was completed by Butruille who confirmed it for 6-dimensional manifolds in [5].

In order to prove the formality of nearly Kähler manifolds, we recall the following observations from Section 1. First of all, from Theorem (1.2) we recall the formality of compact Kähler manifolds. Next, a product is formal if (and only if) so are its factors. Hence, in view of Decomposition (3) it
remains to prove the formality of strict nearly Kähler manifolds, i.e. we have to discuss formality for the three possible types of factors given above.

(1) all simply-connected closed 6-dimensional manifolds are formal for purely algebraic reasons (see [31]);
(2) 3-symmetric manifolds and moreover all $k$-symmetric spaces $G/H$ with $G$, $H$ compact connected are formal by [15, 25, 34];
(3) positive quaternion Kähler manifolds and twistor spaces over them are formal (see [8], [1]).

Let us quickly reflect upon and justify the third item. Quaternion Kähler manifolds are Einstein. If their scalar curvature is positive, they are called positive quaternion Kähler manifolds. In particular, they are simply-connected then (see [33, Theorem 6.6, p. 163]).

Quaternion Kähler manifolds admit so-called twistor fibrations $\mathbb{CP}^1 \hookrightarrow E \rightarrow M$ with complex contact Einstein twistor spaces $E$. The complex structure is constructed using the quaternionic structure of the base and the complex structure of $\mathbb{CP}^1$—see [33, Theorem 4.1, p. 152]. (The simplest example of this is given by $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^{2n+1} \to \mathbb{HP}^n$.) Due to [33, Theorem 6.1, p. 158] twistor spaces $E$ over positive quaternion Kähler manifolds actually admit a Kähler structure. Hence they are formal. (Note that in [1] this formality was used to derive the formality of positive quaternion Kähler manifolds as well.) We should remark that this Kähler structure on $E$ is a priori independent of the nearly Kähler structure, which justifies that, endowed with the nearly Kähler structures, the manifolds $E$ are actually strict nearly Kähler manifolds.

Since a product of formal manifolds is formal, this then yields

Theorem 2.1. Simply-connected closed nearly Kähler manifolds are formal.

Remark 2.2. The formality of Kähler manifolds is derived from the so-called $dd^c$-lemma. It would be interesting to derive the formality of nearly Kähler manifolds straightforwardly from the analytical condition (2) without checking one by one special cases as done above.

3. Formality and special holonomy

3.1. The formality problem for manifolds with special holonomy.

There is another way to generalize Theorem 1.2 by Deligne–Griffiths–Morgan–Sullivan on the formality of Kähler manifolds. This is based on the observation that Kähler manifolds are exactly those Riemannian manifolds which have $U(n)$-holonomy.

The following prominent conjecture (see for example [1, Conjecture 2, p. 2049]) guides our work with respect to special holonomy.

Conjecture 3.1. A simply-connected compact Riemannian manifold of special holonomy is a formal space.

By Berger’s Theorem, the holonomy group of an $n$-dimensional oriented manifold which is neither locally reducible (i.e. not locally a product) nor locally symmetric either coincides with the entire group $SO(n)$ or has special holonomy from the following list: $U\left(\frac{n}{2}\right)$ (Kähler), $SU\left(\frac{n}{2}\right)$ (Calabi–Yau),
Sp\left(\frac{n}{4}\right) (hyperkähler), Sp\left(\frac{n}{2}\right) Sp(1) (quaternion Kähler), G_2 (n = 7), and Spin(7) (n = 8).

Since Sp\left(\frac{n}{4}\right) \subset U\left(\frac{n}{2}\right) and SU\left(\frac{n}{2}\right) \subset U\left(\frac{n}{2}\right), manifolds whose holonomy belongs to the first three special types are Kähler. By Theorem 1.2, they are formal.

As we recalled in Section 2 quaternion Kähler manifolds are Einstein. If they are simply-connected and have vanishing scalar curvature, they are hyperkähler, Kähler and formal, in particular. Positive quaternion Kähler manifolds (those with positive scalar curvature) are simply-connected due to [33, Theorem 6.6, p. 163] and formal due to [1]. If they have negative Einstein constant, they are called negative quaternion Kähler manifolds (of which we do not know many interesting examples).

Hence we are left with three open cases:

- negative quaternion Kähler manifolds,
- G_2-manifolds,
- Spin(7)-manifolds.

Let us recall that the first examples of closed manifolds with holonomy G_2 and Spin(7) were found in the 1990s by Joyce (see [19, 20, 21]). He used for that the generalized Kummer construction. Since then other methods of constructing such examples were introduced (see [22, 26, 27, 6, 23, 32]).

We would like to consider the formality problem for manifolds constructed with this method by Joyce.

3.2. The Kummer construction and its generalizations. The original Kummer method deals with the involution \(\sigma: T^4 \rightarrow T^4\) which in the linear coordinates \(x = (x_1, x_2, x_3, x_4)\) on the four-torus \(T^4 = \mathbb{R}^4/\mathbb{Z}^4\) takes the form \(\sigma(x) = -x\). We recall that these coordinates are defined modulo integers and that the involution has exactly 16 fixed points, i.e. the half-periods, which are defined by the condition \(2x \in \mathbb{Z}^4\). The singular set of \(T^4/\langle\sigma\rangle\) consists of 16 conic points whose neighborhoods have the form \(B^4/\{\pm 1\}\) where \(B^4\) is the open 4-dimensional disc. Each singularity is resolved via the mapping \(T^*\mathbb{CP}^1 \rightarrow \mathbb{C}^2/\{\pm 1\} \supset B^4/\{\pm 1\}\), i.e., every copy of \(B^4/\{\pm 1\}\) is replaced by an open neighborhood \(U\) of the zero section of the fiber bundle \(T^*\mathbb{CP}^1 \rightarrow \mathbb{CP}^1\). The space \(T^*\mathbb{CP}^1\) admits the Ricci-flat asymptotically locally Euclidean (ALE) metric called the Eguchi–Hanson metric. The open manifold \(T^4/\langle\sigma\rangle\) is flat. By perturbations these metrics are made compatible and form a Ricci-flat metric on the resulting four-manifold, which is homeomorphic to a K3 surface. Since the preimage of zero under the mapping \(U \rightarrow B^4/\{\pm 1\}\) is a two-sphere, the resolution of each singularity increases \(b_2(T^4/\langle\sigma\rangle)\), the second Betti number of \(T^4/\langle\sigma\rangle\), by one.

In [19, 20, 21] this construction is generalized as follows. Joyce considers an action of a finite group \(\Gamma\) on a flat torus \(T^7\) or \(T^8\) such that the singular set \(T^7/\Gamma\) consists of finitely many orbifolds. In fact, in these articles finitely many examples for which every orbifold singularity is resolved by an analogue of the Kummer construction are considered.

The simplest case is the orbifold \(T^3\) with the neighborhood \(T^3 \times B^4/\{\pm 1\}\). Such a singularity is resolved by applying the Kummer construction to every fiber of the fibration \(T^3 \times B^4/\{\pm 1\} \rightarrow T^3\). Therewith, \(T^3 \times B^4/\{\pm 1\}\) is
replaced by $T^3 \times U$ where $U$ is a neighborhood of the zero section of the bundle $T^* \mathbb{C}P^1 \to \mathbb{C}P^1$, the resolution map $\pi: U \to B^4/\{\pm 1\}$ is a diffeomorphism on the preimage of the punctured disc $(B^4 \setminus \{0\})/\{\pm 1\}$, and $\pi^{-1}(0)$ is the zero section of the bundle $T^* \mathbb{C}P^1 \to \mathbb{C}P^1$. Therefore, the preimage of $T^3 \times \{0\}$ under the resolution map is a product $T^3 \times \mathbb{C}P^1$, and we can easily describe the contribution of the resolution to the Betti numbers and the intersections of the appearing cycles.

For seven-dimensional manifolds the other possible singularities are

- $T^3/\langle \tau \rangle$, where in linear coordinates the involution $\tau$ is given by $\tau(x_1, x_2, x_3) = (x_1 + \frac{1}{2}, -x_2, -x_3)$, and the neighborhoods look like $(T^3 \times B^4/\{\pm 1\})/\mathbb{Z}_2$.
- or $(T^3 \times B^4/\mathbb{Z}_3)/\mathbb{Z}_2$, and $S^1$ for which there are three possible types of neighborhoods: $S^1 \times B^2/\mathbb{Z}_3, (S^1 \times B^2/\mathbb{Z}_3)/\mathbb{Z}_2$, and $S^1 \times B^6/\mathbb{Z}_7$. In the last three cases the Eguchi–Hanson metric is replaced by other explicitly described Ricci-flat ALE metrics.

For eight-dimensional manifolds five model types of singularities are explicitly described in [21, Section 3]. In the seven-dimensional case all model singular orbifolds are pairwise-nonintersecting. In the eight-dimensional case one of the model singularities appears as the intersection of singular orbifolds of two other types.

3.3. The intersection homology ring. As a general reference for this section we point the reader to [4, §6, p. 53].

The homology groups $H_*(M; \mathbb{R})$ of manifolds obtained by the generalized Kummer construction are generated by cycles from $H_*(T^7 (or 8); \mathbb{R})$ invariant under $\Gamma$ and by the explicitly described cycles induced by the resolutions.

Let us assume that the manifold $M^n$ is oriented. It is known that if two homology cycles $u \in H_k(M^n; \mathbb{R})$ and $v \in H_l(M^n; \mathbb{R})$ are realized by oriented submanifolds $X^k$ and $Y^l$ and intersect transversally, then their intersection $X \cap Y$ realizes the cycle $w$ such that

$$ Dw = Du \cup Dv $$

where

$$ D : H_*(M^n; \mathbb{R}) \to H^*(M^n; \mathbb{R}) $$

is Poincare duality (see, for instance, [4, §6]). Here the orientation of $X \cap Y$ is defined as follows. Let $(e_1, \ldots, e_{k+l-n})$ be a basis in the tangent space to $Y \cap Z$ at some point $x$, and let $(e_1, \ldots, e_{k+l-n}, e'_1, \ldots, e'_{n-l})$ and $(e_1, \ldots, e_{k+l-n}, e''_1, \ldots, e''_{n-k})$ be positively oriented bases in the tangent spaces to $Y$ and $Z$ at the same point. Now we are left to define the orientation of $Y \cap Z$. If the basis $(e_1, \ldots, e_{k+l-n}, e'_1, \ldots, e'_{n-l}, e''_1, \ldots, e''_{n-k})$ at the tangent space to $M^n$ is positively oriented, we assume that $(e_1, \ldots, e_{k+l-n})$ is positively oriented in the tangent space to $Y \cap Z$. Otherwise, we assume that the orientation of $Y \cap Z$ is determined by the basis $(-e_1, e_2, \ldots, e_{k+l-n})$.

Moreover, for every such cycle $u$ its Poincare dual can be realized by a closed form $\omega$ supported on an open neighborhood of the corresponding submanifold $X$ (see [4]). Such a neighborhood can be taken arbitrarily small.

This way we obtain the homology intersection ring which is dual to the cohomology ring. It was originally defined by Lefschetz for algebraic varieties
and later his sketch of the definitions for all oriented manifolds was rigorously realized in [14]. However, if all generators are realized by submanifolds this ring is quite clear.

From the generalized Kummer construction it is possible to write down explicitly all (additive) generators of the homology, to realize them by submanifolds which are pairwise transversally intersecting, and to explicitly describe the ring structure. For the simplest example, the Joyce manifold with all singularities of the form $T^3 \times B^4/\{\pm 1\}$ this was done in [35]. In fact, in [35] there was presented an algorithm for computing case by case the (real) cohomology rings for all of Joyce’s manifolds. For another example, this was realized in the diploma work of I.V. Fedorov, a student of the second named author (I.A.T.) (see [11]).

The example from [35] is used in the next section. The reader should understand and interpret this as a blueprint of how to prove the formality of Joyce’s $G_2$-manifolds. We suggest to do this following the same method.

**Remark 3.2.** We think that after some technical modifications—keeping also in mind that the orbifold singularities in this case may intersect—the same method can be applied to prove the formality of not only the Joyce examples of holonomy $G_2$, but also of Joyce’s $\text{Spin}(7)$-manifolds.

## 4. Formality of the Joyce Examples

### 4.1. Strategy and Preliminaries

Denote by $M$ the Joyce example from [35] with holonomy $G_2$. We want to prove that $M$ is formal in the sense of Rational Homotopy Theory. Since formality does not depend on the field extension of $Q$ (see [18, Corollary 6.9, p. 265]), we work with real coefficients, i.e. with usual deRham differential forms $A_{\text{DR}}(M)$ and with minimal Sullivan models and cohomology over the reals.

We recall [12, Theorem 3.1, p. 157], namely

**Theorem 4.1.** Let $M$ be a connected and orientable compact differentiable manifold of dimension $2n$ or $2n - 1$. Then $M$ is formal if and only if it is $(n - 1)$-formal.

Hence, in order to prove the formality of $M$ it suffices to show that $M$ possesses some 3-formal minimal model $(\Lambda V, d)$. That is we shall construct a minimal model $(\Lambda V, d)$ of $M$ with the property that

- there is a homogeneous splitting $V = C \oplus N$ with $dC = 0$,
- $d: N \to \Lambda V$ is injective,
- and any closed form in the ideal $I(N^{\leq 3}) = N^{\leq 3} \cdot \Lambda V^{\leq 3}$ is exact in $(\Lambda V, d)$.

This minimal model will be derived from a special model $(A, d)$ of $M$ which we shall construct in the following.

From [35, Theorem 1, p. 9] we cite the structure of the rational cohomology ring. It is the quotient of

$$Q[c_{\delta i}, c'_{\delta ij}, t_i, t'_i, 1 \leq i \leq 4, 1 \leq j \leq 3, 1 \leq \delta \leq 3] \otimes \Lambda(t_{\delta i}, t_i, c_{\delta ij}, c'_{\delta i})1 \leq i \leq 4, 1 \leq j \leq 3, 1 \leq \delta \leq 3$$
with degrees
\[ \deg c_{\delta i} = 2, \deg c_{\delta ij} = \deg t_\delta = \deg t_i = 3, \]
\[ \deg c'_{\delta i} = \deg t'_\delta = \deg t'_i = 4, \deg c'_{\delta i} = 5 \]
by the relations
\[ c_{\delta i} c'_{\delta i} = -2, c_{\delta ij} c'_{\delta ij} = -2, t_\delta t'_\delta = 8, t_i t'_i = 8, c^2_{\delta i} = -2t'_\delta \]
and with all other products of generators vanishing.

4.2. Construction of the minimal model. As was done in [35], up to duality, we now choose representing embedded submanifolds of complementary dimensions. The manifold \( M \) is obtained from \( T^7 \) by resolving singularities of \( T^7/\Gamma \), where the group \( \Gamma = \mathbb{Z}_2^3 \) is generated by three involutions. The singularity set in \( T^7/\Gamma \) consists of 12 three-dimensional tori \( T_{\delta i}, 1 \leq \delta \leq 3, 1 \leq i \leq 4 \), with neighborhoods of the form \( T^3 \times B^4/\{\pm1\} \).

These tori split into three families corresponding to fixed point sets of three generating involutions numerated by \( \delta \). The resolution gives rise to the submanifolds \( (C'_{\delta i})^5 = T_{\delta i} \times C_{\delta i} \) with \( [(C'_{\delta i})'] \in H_5(M) \) which are dual to \( c_{\delta i} \in H^2(M) \) and embedded into \( M \) via \( \iota_{\delta i}: C'_{\delta i} \hookrightarrow M \). The \( C_{\delta i} \) are pairwise disjoint \( \mathbb{CP}^1s \). The products of one- and two-dimensional cycles in \( T_{\delta i} \) with two-spheres \( C_{\delta i} \) realize the cycles \( C_{\delta ij} \) and \( C'_{\delta ij} \). There are also cycles realized by tori invariant under \( \Gamma \): these are three-dimensional cycles \( T_{\delta} \) and \( T_i \) and four-dimensional cycles \( T'_{\delta} \) and \( T'_i \). These split into two classes, as the tori \( T_{\delta i} \) are homologous to \( T_{\delta} \) for all \( \delta \) and \( i \).

In this notation the cycles \([Z]\) and \([Z']\) are Poincaré dual to the cocycles \( z' \) and \( z \).

From [4, Proposition 6.24, p. 67] we recall that the homology class represented by a closed submanifold \( \iota: S^k \hookrightarrow M^n \) in cohomology corresponds to the Thom class of its normal bundle, which we usually identify with a tubular neighborhood. Since the Thom class is compactly supported we may extend it to all of \( M \).

Moreover, this Thom class identifies with the uniquely determined class \( [\nu_S] \in H^{n-k}(M) \) satisfying
\[
\int_S \iota^* \omega = \int_M \omega \wedge \nu_S
\]
for all \( \omega \) with \( [\omega] \in H^k(M) \) (see [4, p. 67, (5.13), p. 51, (5.21), p. 65]). Transversal intersection of the representing submanifolds corresponds to the cup product of their duals in cohomology (see [4, p. 69]). By the “localization principle” (see [4, Proposition 6.25, p. 67], without restriction, a cohomology class dual to \([S]\) can be represented by a form with support contained in an arbitrary tubular neighborhood of \( S \). This will play an important role in the arguments to come.

We now define the following commutative connected cochain algebra \((A, d)\) over \( \mathbb{R} \). We take \((A, d)\) to be the commutative differential graded subalgebra of \((A, d) \xrightarrow{\iota} A_{DR}(M)\) (together with the restriction of the usual exterior
differential d) generated by the following graded subspaces $A^i$ of smooth differential forms on $M$.

$$
\begin{align*}
A^0 &:= \mathbb{R} \\
A^1 &:= 0 \\
A^2 &:= \langle c_{\delta i} \rangle_{\delta,i} \\
A^3 &:= \langle n_{\delta k} \rangle_{\delta, 2 \leq k \leq 4} \oplus \langle n_{\delta} \rangle \oplus \langle t_{\delta, t_1, c_{\delta ij}} \rangle_{\delta, i, j} \\
A^{\geq 4} &:= A_{\text{DR}}^{\geq 4}.
\end{align*}
$$

It sounds reasonable to take for $c_{\delta i}$ representatives of the Thom classes which are supported in certain tubular neighborhoods $\nu_{\delta i}$ of the $C'_{\delta i}$ (hence representing the respective cohomology classes in $H^*(M)$ we cited above). Thereupon one has to choose these tubular neighborhoods $\nu_{\delta i}$ small enough to guarantee that all pairwise intersections $\nu_{\delta_1 i_1} \cap \nu_{\delta_2 i_2} = \emptyset$ are disjoint whenever $\delta_1 \neq \delta_2$ (which is possible, since the $C'_{\delta i}$ are pairwise disjoint). However, in order to prove that 7-dimensional Massey products vanish we construct the following representatives of these elements.

**The construction of $c_{\delta i}$ and $n_{\delta k}$.**

Let us construct the elements $c_{\delta i}$ and $n_{\delta k}$. We recall that the manifold $M$ is obtained by blow-up of singularities of type $T^3 \times (\mathbb{C}^2 / \pm 1)$ of the quotient space $T^7 / \Gamma$.

The singularities correspond to 48 tori which are fixed by different involutions from $\Gamma$. These tori are disjoint and split into three families which correspond to the involutions $\alpha, \beta, \text{and } \gamma$ that generate $\Gamma$. On every such family the group $\Gamma$ acts by translations, every orbit of $\Gamma$ consists of 4 tori and different orbits are mapped into each other by translations by half-periods of $T^7 = \mathbb{R}^7 / \mathbb{Z}^7$.

Since the group $\Gamma$ consists of involutions of the form

$$
(x_1, x_2, \ldots, x_7) \to (\pm x_1 + c_1, \pm x_2 + c_2, \ldots, \pm x_7 + c_7)
$$

with constants $c_1, \ldots, c_7$ all translations by half-periods commute with elements of $\Gamma$.

Every such torus $T^3$ gives rise to a singularity in $T^7 / \Gamma$ due to a certain element $g \in \Gamma$ which, in convenient linear coordinates $(y_1, \ldots, y_7)$ on $T^3$, acts near $T^3$ as follows

$$
(y_1, \ldots, y_3, y_4, \ldots, y_7) \to (y_1, \ldots, y_3, -y_4, \ldots, -y_7),
$$

where $T^3$ is distinguished by the equation $y_4 = \cdots = y_7 = 0$. To every such torus $T^3$ there exist neighborhoods $T^3 \subset T^3 \times U_0 \subset T^3 \times U \subset T^7 \subset T^3 \times U_1$ such that

- for different tori these neighborhoods are disjoint;
- if tori from two orbits of $\Gamma$ are mapped into each other by a translation by a half-period then the corresponding neighborhoods are also mapped into each other by the same translation;
- in the local coordinates $\{y_k\}$ the sets $U_0, U$, and $U_1$ are given by neighborhoods of the origin in $\mathbb{R}^4$ with the coordinates $y_4, \ldots, y_7$ and we assume that $U_0, U$ and $U_1$ are invariant under the reflection $(y_4, \ldots, y_7) \to (-y_4, \ldots, -y_7)$.
Now let us construct the generators $c_{di}$ and $u_{d\delta}$ of the minimal model.

Without loss of generality we assume that $\delta = \alpha$. The involution $\alpha$ acts on the torus $T^7 = \mathbb{R}^7/\mathbb{Z}^7$ as
\[
(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \rightarrow (x_1, x_2, x_3, -x_4, -x_5, -x_6, -x_7)
\]
and has $16$ fixed tori of the form $(x_1, x_2, x_3, a_4, a_5, a_6, a_7)$ with $a_k \in \{0, 1/2\}$, $k = 4, \ldots, 7$. The group $\Gamma/\mathbb{Z}_2(\alpha) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ acts on these tori effectively and splits them into four orbits. Every such orbit induces a singularity in $T^7/\Gamma$, and another four singularities similarly correspond to the involutions $\beta$ and $\gamma$ each. By construction, \[20\], (see the more detailed description in \[35\]) the orbits of different involutions are disjoint.

Let us take a representative $T_{A1}^3$ of one of these orbits.

The torus $T^7$ splits into a product $T^3 \times T^4$ such that all tori $T_{A1}^3$ have the form $T^3 \times \{\text{const}\}$. We denote the projection of $T^7$ onto the $T^4$ component by $q$, i.e. $q(x_1, \ldots, x_7) = (x_4, x_5, x_6, x_7)$.

Suppose that $q(T_{A1}^3) = (a_4, \ldots, a_7) =: a$. We recall that to every fixed torus and, in particular, to $T_{A1}^3$, we attribute a pair of neighborhoods $T^3 \times U \subset T_{A1}^3 \times U$ in $T^7$ where $U_0$ and $U$ are sufficiently small and are invariant under the involution $(x - a) \rightarrow -(x - a)$. In $U$ we take the complex coordinates $z_1 = (x_4 - a_4) + i(x_5 - a_5)$, $z_2 = (x_6 - a_6) + i(x_7 - a_7)$.

Let us take a non-negative $C^\infty$-function $f_{a1}$ on $T^4$ (with the coordinates $x_4, \ldots, x_7$) such that

1. $0 \leq f_{a1} \leq 1$,
2. $f_{a1}$ is supported on $U$, and positive inside $U \setminus U_0$,
3. $f_{a1}$ vanishes identically on $U_0$,
4. $f_{a1}$ is invariant under the involution $(x - a) \rightarrow -(x - a)$.

Subsequently, we want to define the form $c_{a1}$ on $T^4$. For this we begin with
\[
\tilde{s}_{a1} = f_{a1}(x_4, \ldots, x_7)(dx_4 \wedge dx_5 + dx_6 \wedge dx_7).
\]
Now we define a form $\tilde{c}_{a1}$ on $T^7$ by the formula
\[
\tilde{c}_{a1} = \sum_{g \in \Gamma} g^* q^* \tilde{s}_{a1}.
\]
This form is $\Gamma$-invariant. By \(5\), it has the form
\[
f(x_4, \ldots, x_7)(dx_4 \wedge dx_5 + dx_6 \wedge dx_7).
\]
Its integral over the torus $T^4 = T^4(x_4, \ldots, x_7)$ is positive and, hence, in $H^*(T^7; \mathbb{R})$ it represents the class dual to $[T_{A1}^3]$.

By construction of $f_{a1}$, the pullback of $\tilde{s}_{a1}$ onto the blow-up is smooth.

Indeed, locally after the blow up of $C^2/\mathbb{Z}_2$ we obtain a smooth variety which is covered by two charts $U_+$ and $U_-$ with coordinates $(u_1, u_2)$ and $(v_1, v_2)$. \[1\]

\[1\]
For completeness, let us describe this construction. We realize the singular variety $C^2/\mathbb{Z}_2$ as a singular curve in $\mathbb{C}^3$ as follows. The set of $\mathbb{Z}_2$-invariant polynomials on $\mathbb{C}^2$ is generated by $z_1^2$, $z_2^2$, and $z_1 z_2$. To very point in $\mathbb{C}^2/\mathbb{Z}_2$ there corresponds a point in $\mathbb{C}^3$ with the coordinates $x_1 = z_1^2$, $x_2 = z_2^2$, $x_3 = z_1 z_2$. Therewith the variety $\mathbb{C}^2/\mathbb{Z}_2$ is embedded into $\mathbb{C}^3$ as a singular curve $X$ given by
\[
x_1 x_2 = x_3^2.
\]
These charts are projected onto \((U \setminus \{z_1 = 0\})/\mathbb{Z}_2\) and \((U \setminus \{z_2 = 0\})/\mathbb{Z}_2\) as follows:
\[
q : U_+ \setminus \{u_1 = 0\} \to (U \setminus \{z_1 = 0\})/\mathbb{Z}_2, \quad u_1 = z_1^2, u_2 = \frac{z_2}{z_1},
\]
\[
q : U_- \setminus \{v_1 = 0\} \to (U \setminus \{z_2 = 0\})/\mathbb{Z}_2, \quad v_1 = z_2^2, v_2 = \frac{z_1}{z_2}.
\]
The pullbacks of the forms \(dz_1 \wedge dz_2\) and \(d\tilde{z}_1 \wedge d\tilde{z}_2\) are extended regular on the whole blown-up space. However, for instance,
\[
q^*(dz_1 \wedge d\tilde{z}_1) = \frac{1}{|u_1|} du_1 \wedge d\tilde{u}_1,
\]
\[
q^*(dz_1 \wedge d\tilde{z}_2) = \frac{1}{4|u_1|} du_1 \wedge d\tilde{u}_1 + \frac{1}{2} \sqrt{\frac{u_1}{u_1}} du_1 \wedge d\tilde{u}_2.
\]
Hence, we multiply the forms on \(\mathbb{C}^2\) by \(f_{\alpha_1}\) in order to pull them back to regular forms on \(M\).

To define the forms \(\tilde{c}_{\alpha k}\) for \(k = 2, 3, 4\), i.e., for other orbits, we notice that all orbits are obtained from each other by translations by half-periods. We recall that by (5) all translations by half-periods commute with elements of \(\Gamma\). Therefore, we obtain the forms \(\tilde{c}_{\alpha k}\) for \(k = 2, 3, 4\) by acting on \(\tilde{c}_{\alpha 1}\) by convenient translations. Therewith, if \(k \neq 1\), the form
\[
\tilde{c}_{\alpha k}^2 - \tilde{c}_{\alpha 1}^2
\]
is \(\Gamma\)-invariant. By (6), it is a pullback of a 4-form \(d_{\alpha, k_1}\) on \(T^4\) such that its integral over \(T^4\) vanishes. Therefore there exists a form \(\eta_{\alpha k}\) such that
\[
d\eta_{\alpha k} = \tilde{c}_{\alpha k}^2 - \tilde{c}_{\alpha 1}^2
\]
and \(\eta_{\alpha k}\) is a pullback of a 3-form \(m_{\alpha k}\) on \(T^4\) such that \(dm_{\alpha k} = d_{\alpha, k_1}\).

We notice that both forms \(\tilde{c}_{\alpha k}^2\) and \(\tilde{c}_{\alpha 1}^2\) are \(\Gamma\)-invariant. By (5), the form \(\eta_{\alpha k}\) can also be chosen to be \(\Gamma\)-invariant.

Near all fixed tori of \(\Gamma\) the form \(\eta_{\alpha k}\) is closed, i.e. \(d\eta_{\alpha k} = 0\). In particular, by construction of the forms \(\tilde{c}_{\alpha k}\), this holds in the neighborhoods \(T^3 \times U_0\). In every such neighborhood the form \(\eta_{\alpha k}\) is cohomologically trivial because its integral over a torus fixed by every involution vanishes. This follows from the fact that \(\eta_{\alpha k}\) is a wedge product of forms \(dx_4, dx_5, dx_6,\) and \(dx_7\) with certain coefficients and from the particular description of the fixed tori (see, for instance, the part of the description of the topology of the \(G_2\)-manifold \(M^7\) in [35]) which implies that the integrals of three-forms which are wedge products of these one-forms over all fixed tori vanish).

Now we distinguish two cases:

The blowup \(\tilde{X}\) of \(X\) at a singular point \(x = 0\) lies in the blowup of \(\mathbb{C}^3\) at the same point: \(\tilde{X} \subset \mathbb{C}^3\). In \(\mathbb{C}^3 \times \mathbb{CP}^2\) the subvariety \(\mathbb{C}^3\) is distinguished by the equations
\[
(8) \quad x_j y_k - x_k y_j = 0, \quad 1 \leq j, k \leq 3,
\]
where \((y_1 : y_2 : y_3)\) are the homogeneous coordinates in \(\mathbb{CP}^2\). Let us consider a chart on \(\mathbb{CP}^2\) defined by the condition \(y_1 = 1\). By (8), \(x_k = x_1 y_k\), and it follows from (7) that \(y_2 = y_3^2\). Therefore to this chart there corresponds a chart \(U_1\) on \(\tilde{X}\) with the coordinates \(u_1 = x_1 = z_1^2\) and \(u_2 = y_3\) such that if \(u_1 \neq 0\) then \(u_2 = \frac{z_2}{z_1}\) and \(v_2 = y_3\) such that if \(v_1 \neq 0\) then \(v_2 = \frac{z_1}{z_2}\). Analogously, to the chart \(\{y_2 = 1\}\) there corresponds a chart \(U_2\) on \(\tilde{X}\) with the coordinates \(u_1 = x_2 = z_2^2\) and \(u_2 = y_3\) such that if \(u_1 \neq 0\) then \(u_2 = \frac{z_1}{z_2}\). These two charts cover \(\tilde{X}\).
(1) the torus $T^3$ is a fixed torus of the involution $\alpha$,
(2) the torus $T^3$ is a fixed torus of $\beta$ or of $\gamma$.

Case 1. For any such torus let us take small neighborhoods $T^3 \times U_{-1}$ and $T^3 \times U_0$ ($U_0$ as above) such that $U_{-1} \subset U_0$, and such that they are invariant under reflections $z \to -z$ in the convenient coordinates $z$. By construction $\eta_{ak}$ is closed in $T^3 \times U_0$.

For such neighborhoods we take a $C^\infty$-function $f$ such that $f$ is supported in $U_0$, $0 \leq f \leq 1$, $f = 1$ in $U_{-1}$, and such that $f$ is invariant under the reflection. We also take a form $\omega$ in $T^3 \times U_0$ such that $d\omega = \eta_{ak}$ and $\omega$ is a restriction onto $T^3 \times U_0$ of a $\Gamma$-invariant form (the latter is easily achieved by averaging over the action of $\Gamma$). Let us now replace in $T^3 \times U_0$ the form $\eta_{ak}$ by $\eta'_{ak} = \eta_{ak} - d(f\omega)$.

By construction, $\Gamma$ acts near $T^3$ by the reflection, and other involutions generate the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action which consists in translations of the domain $T^3 \times U_0$ by certain half-periods. The action of $\Gamma$ on $\eta'_{ak}$ extends it onto $\Gamma(T^3 \times U_0)$, the $\Gamma$-orbit of $T^3 \times U_0$, and, moreover, its properties near other fixed tori are similar. By the choice of $f$, this form is glued with the restriction of $\eta_{ak}$ onto $T^3 \setminus \Gamma(T^3 \times U_0)$ to a smooth form which we denote again by $\eta_{ak}$.

After performing this procedure to all orbits of fixed tori we obtain a form $\tilde{\eta}_{ak}$ such that

1. $\tilde{\eta}_{ak}$ is $\Gamma$-invariant,
2. $\tilde{\eta}_{ak}$ vanishes near the fixed torus $T^3$ and all tori from its $\Gamma$-orbit;
3. $d\tilde{\eta}_{ak} = \tilde{c}_{ak}^2 - \tilde{c}_{a1}^2$.

Case 2. Let us take for any such torus small neighborhoods $T^3 \times U$ and $T^3 \times U_1$ (both as above). By construction, $\eta_{ak}$ is closed in $T^3 \times U$. Now we repeat the same construction as above replacing $U_{-1}$ by $U$ and $U_0$ by $U_1$. The resulting form $\tilde{\eta}_{ak}$ vanishes in $T^3 \times U$. This finishes Case 2.

Now we have $\Gamma$-invariant forms $\tilde{c}_{ai}$ and $\tilde{\eta}_{ak}$, $1 \leq i \leq 4$, $k \leq 2 \leq 4$, such that

$$d\tilde{\eta}_{ak} = \tilde{c}_{ak}^2 - \tilde{c}_{a1}^2$$

and all these forms vanish near all 48 fixed tori of $\Gamma$. Therefore these forms can be pushed forward to forms on $T^7/\Gamma$ which vanish at singularities and hence are pulled back to the resolution

$$M \to T^7/\Gamma.$$

Therewith $\tilde{c}_{ai}$ give rise to $c_{ai}$, and the $\tilde{\eta}_{ai}$ induce $n_{ai}$. By construction we have

$$dn_{ak} = c_{ak}^2 - c_{a1}^2.$$  

We are left to show that

$$\int n_{ai}c_{ak}^2 = 0 \text{ for all } i \text{ and } k.$$  

There are two possibilities: 1) $\alpha \neq \delta$, and 2) $\alpha = \delta$.

In the first case $\tilde{\eta}_{ai}$ vanishes on $T^3 \times U$ where $T^3$ are fixed tori of $\delta$, and the $\tilde{c}_{ak}$ are supported on the same domains. Therefore, $\tilde{\eta}_{ai}\tilde{c}_{ak} = 0$, which implies (9).

In the second case whenever $\tilde{\eta}_{ai}$ and $\tilde{c}_{ak}$ do not vanish they both are wedge products of the forms $dx_4, dx_5, dx_6$, and $dx_7$ with certain coefficients.
Therefore, the 7-form $\tilde{n}_{\alpha i} c_{\alpha k}^2$ vanishes everywhere for dimension reasons and this implies (9).

Let us summarize in view of the previous constructions. Degree 0 is generated by the constant functions. In degree 3, the subspace $\langle t_\delta, t_i, c_{\delta ij} \rangle_{\delta, i, j}$ just consists of representatives of the respective cohomology classes. The forms $n_{\delta k}$ and $n_{\delta}$ are chosen to satisfy
\begin{align}
\text{(10)} & \quad dn_{\delta k} = c_{\delta k} \wedge c_{\delta k} - c_{\delta 1} \wedge c_{\delta 1} \\
\text{(11)} & \quad dn_{\delta} = c_{\delta i} \wedge c_{\delta i} - t_{\delta}'
\end{align}
for $2 \leq k \leq 4$ representing the cohomological equalities $[c_{\delta i}]^2 = [t_{\delta}']$ (and the thereby induced mutual equalities amongst the squares) for all $1 \leq i \leq 4$. Note that by our choice of representatives in degree 2 it holds that
\begin{equation}
\text{(12)} \quad c_{\delta_1 i_1} \wedge c_{\delta_2 i_2} = 0
\end{equation}
for $\delta_1 \neq \delta_2$ or $i_1 \neq i_2$.

As a consequence, by construction, we obtain that the inclusion of cochain algebras $\iota: (A, d) \rightarrow A_{\text{DR}}(M)$ is a quasi-isomorphism, i.e. $H(\iota)$ yields the isomorphism

$$H(A, d) \cong H(A_{\text{DR}}(M)) \cong H^*(M)$$

**Passing to the minimal model.**

We now construct a minimal Sullivan model $\langle \Lambda V, d \rangle$ (in normal form) for $(A, d)$ and thereby, by uniqueness of minimal models up to isomorphism, also doing so for $M$, i.e. we construct a quasi-isomorphism

$$m: \langle \Lambda V, d \rangle \xrightarrow{\cong} (A, d) \simeq A_{\text{DR}}(M)$$

inductively over degree. That is, we construct the models and the morphisms $m^l$: $(\Lambda V^{\leq l}, d) \rightarrow (A, d) \simeq A_{\text{DR}}(M)$ by constructing $V^{\leq l}$ together with the differential $d$ and the morphisms $m^l$ such that differentials commute with the morphism, and such that $m$ is a quasi-isomorphism up to degree $l$.

Set $V^1 = 0$, and obtain again by a little abuse of notation (and $m^2$, $m^3$ reflecting it exactly) that

$$V^2 = \langle c_{\delta i} \rangle_{\delta, i}$$
$$V^3 = \langle t_\delta, t_i, c_{\delta ij} \rangle_{\delta, i, j} \oplus \langle n_{\delta k} \rangle_{\delta, k} \oplus \langle n_s \rangle_{s}$$

with the induced differentials as in (10). This procedure then may be continued over degree to yield $(\Lambda V, d)$ and $m$ in the limit as described in Algorithm [9, Page 144]. That is, we have constructed the quasi-isomorphisms

$$m: \langle \Lambda V, d \rangle \xrightarrow{\cong} (A, d) \xrightarrow{\cong} A_{\text{DR}}(M) \cong H^*(M)$$

We need to comment on degree 3 and, especially, on the additional summand $\langle n_s \rangle_{s}$: Since, in contrast to $(A, d)$, the algebra $(\Lambda V, d)$ is a free commutative differential graded algebra, it needs to encode the additional relations (12) via differentials. Hence we choose the $n_s$ in bijection with these relations.
and set their differentials to equal each respective one. It is clear that \( m \) can be set to restrict to an isomorphism on \( \langle t_δ, t_i, c_{δij} \rangle_{δ,i,j} \oplus \langle n_{δk} \rangle_{δ,k} \) and can be set to vanish on \( \langle n_s \rangle_s \).

Note further that these \( n_s \) have nothing to do with the \( n_δ \) from Equation (11). The latter actually do not have to be represented in the minimal model, since the \( t_δ' \) just do not appear as new spherical cohomology; i.e. in the minimal model—exactly due to minimality—they directly appear in the form \( c_{δi}^2 \) (for any \( i \)).

4.3. 3-formality of the model. In view of Theorem 4.1 it remains to show that this model \((ΛV, d)\) is 3-formal. For this we split \( V = C \oplus N \) as indicated above in such a way that

\[
C^2 = \langle c_{δi} \rangle_{δ,i} \\
N^2 = 0 \\
C^3 = \langle t_δ, t_i, c_{δij} \rangle_{δ,i,j} \\
N^3 = \langle n_{δk} \rangle_{δ,k} \oplus \langle n_s \rangle_s
\]

(13)

It remains to show that every closed form in \(\langle n_{δk} \rangle_{δ,k} \oplus \langle n_s \rangle_s \cdot (ΛV \leq^3)\) is exact in \((ΛV, d)\). By degree and dimension this is only necessary in degrees 5 to 7, and by Poincaré duality this is clear in degree 6. Hence we need to verify this in degrees 5 and 7.

4.3.1. Degree 5. Suppose that there is a closed non-exact form from (13) in degree 5. By Poincaré duality, i.e. the non-degeneracy of the intersection form, there exist a cohomology class in degree 2 which multiplies with the class in degree 5 to a volume form. Consequently, this becomes a closed non-exact form from (13) now in degree 7. This implies that once we have proved that there are now such classes in degree 7 the case of degree 5 is settled as well.

We point the reader to the appendix where we have a closer look at dimension 5, in particular illustrating the used techniques once again.

4.3.2. Degree 7. Now we show that there are no closed non-exact forms from (13) in degree 7 (thereby, as depicted, proving the analogue for degree 5).

So we need to show that any closed form in \(N \leq^3 \cdot (ΛV \leq^3)\) of degree 7 is exact. We will do so by actually showing that any form in there has an exact image in the quasi-isomorphic algebra \((A, d)\).

By the structure of \(V\) and the decomposition \( V = C \oplus N \), it follows that any form in \(N \leq^3 \cdot (ΛV \leq^3)\) of degree 7 is necessarily in

\[
C^2 \cdot C^2 \cdot N^3
\]

(14)

Note that by slight abuse of notation we will suppress the morphism \( m \) in the following, etc., and consider forms to lie in the minimal model, the algebra \( A \) and the differential forms \( A_{DR}(M) \) depending on context.

Next, in the algebra \((A, d)\) it holds that \( c_{δ,i} \cdot c_{δ,j} = 0 \) unless \( δ_1 = δ_2 \) and \( i = j \), since the forms are concentrated around disjoint submanifolds. Hence
any non-exact element in \( m(C^2 \cdot C^2 \cdot N^3) \subseteq (\mathcal{A}, d) \) (which is closed by degree) is necessarily in the vector space

\[
\langle c^2_{\delta_1} \cdot n_{\delta_2} k \rangle \delta_1, \delta_2, i, k \subseteq \mathcal{A}^7
\]

By (9) we now show that

\[
0 = \int_{C_{\delta_1}'} (c_{\delta_1} \wedge n_{\delta_2} k) = \int_M (c_{\delta_1} \wedge n_{\delta_2} k) \wedge c_{\delta_1}
\]

for the embeddings \( \iota_{\delta_1}: C'_{\delta_1} \to M \). This proves that all the forms in (15) above are exact. Hence \( M \) is 3-formal and formal consequently.

\[
\int_M c_{\delta_1} \wedge c_{\delta_1} \wedge t_{\delta}, \quad \int_M c_{\delta_1} \wedge c_{\delta_1} \wedge c_{\delta k j}, \quad \int_M c_{\delta_1} \wedge c_{\delta_1} \wedge c_{\delta 1 l}
\]

\( U \subseteq T^4 \) containing \( x \) and \( y \) and satisfying the additional property that the image of \( T^3 \times U \)

**Remark 4.2.** On the one hand in [8] it is stated that formality is equivalent to “uniform vanishing of Massey products”. Since then a lot of work has been dedicated to making this statement precise in the form of several interpretations. This has led to various different characterizations of formality ranging from the “original” one in [8, Theorem 4.1, p. 261] over \( A^\infty \)- or \( L^\infty \)-algebras, etc., to the depiction cited in Remark 4.5 below. On the other hand intersection theory was used before to represent Massey products via submanifolds. Hence in view of Theorem 4.1 from [12] which generalises the characterization in [8] and which is our main algebraic tool we make this interplay of Massey products, formality, and submanifolds precise and geometric.

**Remark 4.3.** The reader may notice that our arguments, the blow-up construction, etc., do not really depend on the concrete structure of the group \( \Gamma \). Hence this group certainly can be replaced by more general ones. Simply-connected manifolds are known to be formal below dimension 7, whence also here a more general situation involving more complicated submanifolds should not lead to many additional difficulties. Hence, in summary, we are confident that our approach should pave the way to analyzing the formality of the remaining Joyce examples respectively of further instances like examples of \( \text{Spin}(7) \)-manifolds, etc.

**Remark 4.4.** We note that although the general theory (see for example [10, Theorem 8.28, p. 337] and [28]) for blow-ups allows for explicit rational models only in stable ranges (which in our case of a blow-up of a 3-dimensional submanifold begins in dimension 9) we managed to construct a rational model of \( M \), which, due to formality clearly is given by its cohomology algebra. Hence, for example, it is a triviality to compute its rational homotopy groups (which clearly have exponential growth) in a certain range.

**Remark 4.5.** We remark that in the case of a simply-connected 7-dimensional manifold, we may equivalently consider the “Bianchi–Massey tensor” introduced in [7]. Our arguments transcribe to show its vanishing, and according to [7, Theorem 1.3, p. 3] this yields formality.
Appendix A. Forms in degree 5 in the Joyce example

Here we have a closer look at the forms in degree 5. By the arguments presented above, this discussion is not necessary for the actual proof, it does provide however better insight into the structure of the manifolds and provides similar but slightly alternative arguments for the vanishing of Massey products in degree 5.

Using SageMath 9.0 we compute a basis of the cohomology $H^5(N^{\leq 3}, \Lambda V^{\leq 5}, d)$. This space is rather large. We may take the quotient by those forms in the kernel of $m_5$, i.e. those lying in the ideal generated by $\langle n_s \rangle_s$, as by construction of the minimal model as a quasi-isomorphism they are also exact in $(\Lambda V, d)$. This quotient still has dimension 76. A careful check, however, shows that they come in “two types”; we discuss two prototypical examples of each, namely

$$c_{a1} \cdot n_1 - c_{a4} \cdot n_2 - c_{a2} \cdot n_{a4}$$

and

$$c_{a2} \cdot n_1 - c_{a4} \cdot n_3 + c_{a1} \cdot n_{a2} - c_{a1} \cdot n_{a4}$$

with

$$dn_1 = c_{a1} \cdot c_{a2}$$
$$dn_2 = c_{a1} \cdot c_{a3}$$
$$dn_3 = c_{a1} \cdot c_{a4}$$

Clearly, (10) implies that $dn_{\delta k} = c_{\delta k} \cdot c_{\delta k} - c_{\delta 1} \cdot c_{\delta 1}$.

In view of the quasi-isomorphism $m$ it suffices to show that the images under $m$ of these classes

$$m(c_{a1} \cdot n_1 - c_{a4} \cdot n_2 - c_{a2} \cdot n_{a4}) = -c_{a2} \cdot n_{a4}$$

and

$$m(c_{a2} \cdot n_1 - c_{a4} \cdot n_3 + c_{a1} \cdot n_{a2} - c_{a1} \cdot n_{a4}) = c_{a1} \cdot n_{a2} - c_{a1} \cdot n_{a4}$$

are exact in $(A, d)$. (By construction, this is equivalent to the exactness of the forms themselves in $(AV, d)$.)

As for $-c_{a2} \cdot n_{a4}$ this can be seen as follows. Recall that $dn_{a4} = c_{a1}^2 - c_{a4}^2$. We have that up to duality $c_{a1}$ is represented by $C_{a1} = C_{a1} \times T_{a1}$ and $c_{a4}$ by $C_{a4} = C_{a4} \times T_{a4}$. Hence $c_{a1}^2 - c_{a4}^2$ is supported in a tubular neighborhood of $C_{a1} \times T_{a1} \cup C_{a4} \times T_{a4}$.

Hence, since $dn_{a4}$ encodes the relation $c_{a1}^2 - c_{a4}^2$, we have that $dn_{a4}$ restricted to a tubular neighborhood of $C_{a2} \times T_{a2}$ vanishes. Thus, the restriction of $n_{a4}$ is a closed form and defines a cohomology class supported around $C_{a2} \times T_{a2}$. By construction, all the cohomology classes concentrated in a tubular neighborhood of $C_{a2} \times T_{a2}$ are actually represented by Thom classes of respective further submanifolds. Hence, they are compactly supported, vanish outside of a small tubular neighborhood and exist globally on $M$. Thus, we may add such a suitable closed Thom class $\omega$—globally defined, but vanishing outside of a small tubular neighborhood—to $n_{a4}$ such that $n_{a4} - \omega$ restricted to the given tubular neighborhood of $C_{a2} \times T_{a2}$ is exact on this tubular neighborhood, i.e. on there there exists some form $\zeta$ with
dζ = n_{a4} − ω. That is, without restriction, replacing n_{a4} by n_{a4} − ω, we may assume this to hold for n_{a4} in the first place, i.e. its restriction is exact from the very beginning. Since c_{a2} is concentrated in a tubular neighborhood of C_{a2} × T_{a2}, the form c_{a2} : ζ (now with dζ = n_{a4} without restriction) is globally defined and satisfies d(c_{a2} : ζ) = c_{a2} : n_{a4}.

Assume now that the second form from (18) is not exact. Due to the non-degeneracy of the duality pairing, for its induced cohomology class there is a dual class in \( H^2(M) \) such that their product is non-zero. By construction the chosen form is localised in a neighborhood around \( C'_{a1} \). Since \( H^2(M) \cong \langle c_{\delta i} \rangle_{\delta, i} \) and due to this localization (also being expressed in the relations \([c_{\delta i}]^2 = [c_{\delta j}]^2\) and \([c_{\delta i} c_{\delta j}] \cdot [c_{\delta j} c_{\delta k}] = 0\) for \( \delta_1 \neq \delta_2 \) in cohomology — although we need to carefully set these two properties apart) we deduce that up to (non-trivial) real multiples the only form which multiplies cohomologically non-trivially in the depicted cases can and must be \( c_{a1} \), i.e.

\[
[c_{a1}^2 \cdot (n_{a2} - n_{a4})] \neq 0
\]

In \((\mathcal{A}, d)\) we compute, however, that

\[
d(-n_{a2} \cdot n_{a4}) = -(c_{a1}^2 - c_{a2}^2) \cdot n_{a4} + n_{a2} \cdot (c_{a1}^2 - c_{a4}^2)
\]

\[
= -c_{a1}^2 \cdot (n_{a4} - n_{a2}) + (c_{a2}^2 \cdot n_{a4} - n_{a2} \cdot c_{a4}^2)
\]

As above, again we argue that, without restriction, the second summand \((c_{a2}^2 \cdot n_{a4} - n_{a2} \cdot c_{a4}^2)\) is exact. For this we argue again that \( d\alpha_{a4} \) differentiates to 0 on a tubular neighborhood of \( C_{a2} \times T_{a2} \), and can be assumed to be exact there, which then globalizes again to the exactness of \( c_{a2}^2 n_{a4} \). The analog holds for \( c_{a4}^2 n_{a2} \). This then shows the exactness of \(-c_{a1}^2 \cdot (n_{a4} - n_{a2})\).

This contradicts the original assumption that the class \([c_{a1} \cdot n_{a2} - c_{a1} \cdot n_{a4}] \in H(\mathcal{A}, d)\) was not exact. As a consequence all closed forms in \( N^{\leq 3} \cdot \Lambda V^{\leq 3} \) in degree 5 are exact. We made this more rigorous for all potential forms in Section 4.3.2.

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