A double bounded key identity for Göllnitz’s (BIG) partition theorem

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Abstract

Given integers \(i, j, k, L, M\), we establish a new double bounded \(q\)-series identity from which the three parameter \((i, j, k)\) key identity of Alladi-Andrews-Gordon for Göllnitz’s (big) theorem follows if \(L, M \to \infty\). When \(L = M\), the identity yields a strong refinement of Göllnitz’s theorem with a bound on the parts given by \(L\). This is the first time a bounded version of Göllnitz’s (big) theorem has been proved. This leads to new bounded versions of Jacobi’s triple product identity for theta functions and other fundamental identities.

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1 Introduction

Our goal here is to prove the following double bounded key identity for Göllnitz’s (big) partition theorem: If $i, j, k, L, M$, are integers, then

$$
\sum_{i,j,k \text{ constraints}} q^{T_i+T_{ab}+T_{ac}+T_{bc}} \left[ \begin{array}{ccc} L - t + a & L - t + b & M - t + c \\ a & b & c \end{array} \right] \times \left[ \begin{array}{ccc} L - t & M - t & M - t \\ ab & ac & bc \end{array} \right] + \\
\sum_{i,j,k \text{ constraints}} q^{T_i+T_{ab}+T_{ac}+T_{bc}-1} \left[ \begin{array}{ccc} L - t + a - 1 & L - t + b & M - t + c \\ a - 1 & b & c \end{array} \right] \times \\
\left[ \begin{array}{ccc} L - t & M - t & M - t \\ ab & ac & bc - 1 \end{array} \right]
$$

$$
= \sum_{s \geq 0} q^{s(M+2)-T_i+T_{i-s}+T_{j-s}+T_{k-s}} \left[ \begin{array}{ccc} L - s & M - i - j \\ s, i - s, j - s & k - s \end{array} \right],
$$

(1.1)

where $t = a + b + c + ab + ac + bc$ and the summations on the left in (1.1) are over parameters $a, b, c, ab, ac$ and $bc$, satisfying the $i, j, k$ constraints

$$
i = a + ab + ac, \\
j = b + ab + bc, \\
k = c + ac + bc.
$$

We emphasize that in (1.1) and everywhere, $ab$ is a variable, and is not equal to $a$ times $b$, with similar interpretation for $ac$ and $bc$. The role of the variables $a, b, \ldots, bc$ will become clear in the sequel.

In (1.1) and in what follows, $T_n = n(n+1)/2$, and the $q$-binomial and $q$-multinomial coefficients are defined by

$$
\left[ \begin{array}{c} \mbox{ } \\ n + m \\ \mbox{ } \end{array} \right] = \left[ \begin{array}{c} \mbox{ } \\ n + m \\ \mbox{ } \end{array} \right]_q = \begin{cases} \frac{(q^{m+1})_n}{(q)_n}, & \text{if } n \geq 0, \\
0, & \text{otherwise}, \end{cases}
$$

(1.3)

and

$$
\left[ \begin{array}{c} L \\ a, b, c, \ldots \end{array} \right] = \left[ \begin{array}{c} L \\ a \end{array} \right] \left[ \begin{array}{c} L - a \\ b \end{array} \right] \left[ \begin{array}{c} L - a - b \\ c \end{array} \right] \ldots \\
= \begin{cases} \frac{(q^{L-a-b-c-\ldots})_{a+b+c-\ldots}}{(q)_a(q)_b(q)_c}, & \text{if } a \geq 0, b \geq 0, c \geq 0, \ldots, \\
0, & \text{otherwise}, \end{cases}
$$

(1.4)
where the symbols \((a)_n\) are given by

\[
(a; q)_n = (a)_n = \begin{cases} 
\prod_{j=0}^{n-1} (1 - aq^j), & \text{if } n > 0, \\
1, & \text{if } n = 0, \\
\prod_{j=1}^{-n} (1 - aq^{-j})^{-1}, & \text{if } n < 0.
\end{cases}
\]  

(1.5)

The connections between (1.1) and the partition theorem of Göllnitz [13] will be explained subsequently. Note that when \(L, M \to \infty\), only the term corresponding to \(s = 0\) on the right hand side of (1.1) survives, and so (1.1) reduces to

\[
\sum_{\text{constraints}} q^{T_i + T_{ab} + T_{ac} + T_{bc} - 1} (1 - q^a + q^{a+bc}) = \frac{q^{T_i + T_j + T_k}}{(q)_i(q)_j(q)_k},
\]

(1.6)

which is the three parameter key identity for Göllnitz’s theorem due to Alladi-Andrews-Gordon [2]. If any one of the parameters \(i, j, k\) is set equal to 0, then (1.1) reduces to the double bounded key identity for Schur’s theorem we have recently established [3]. For instance, with \(i = 0\), (1.1) becomes

\[
\sum_{\text{constraints}} q^{T_b + T_{bc} + T_{bc}} \left[ \begin{array}{c} L - k \\ j - bc \\ k - bc \end{array} \right] \left[ \begin{array}{c} M - j \\ M - b - c - bc \\ bc \end{array} \right] = q^{T_j + T_k} \left[ \begin{array}{c} L \\ j \\ k \end{array} \right].
\]

(1.7)

Our proof of (1.1) has two parts. Denoting the left hand side of (1.1) by \(g_{i,j,k}(L, M)\) and the right hand side of (1.1) by \(p_{i,j,k}(L, M)\), we first show in §2 that the functions \(g_{i,j,k}(L, M)\) and \(p_{i,j,k}(L, M)\) satisfy identical second order recurrences in \(L\). To complete the proof of the equality

\[
g_{i,j,k}(L, M) = p_{i,j,k}(L, M)
\]

(1.8)

we show in §3 that both functions satisfy the same initial conditions

\[
g_{i,j,k}(i + j, M) = p_{i,j,k}(i + j, M).
\]

(1.9)

This is not as easy as it sounds; the proof of (1.9) in §3 requires the use of Jackson’s \(q\)–analog of Dougall’s summation. When \(L = M\), the right hand side of (1.1) can be evaluated elegantly in terms of a product of \(q\)–binomial coefficients with cyclic dependence on \(i, j,\) and \(k\) (see §4). This has a nice partition interpretation yielding a strong refinement of Göllnitz’s theorem with a bound on the size of the parts. To the best of our knowledge, this is the first time a bounded version of Göllnitz’s theorem has been found. There are a number of important consequences of this theorem one of which is a new finite version of Jacobi’s triple product identity which is stated as identity (5.2) in §5 (also see (5.3), (5.5)); the proof of (5.2) and finite versions of many other fundamental results in the theory of partitions and \(q\)-series will be given elsewhere [3], [7]. In §5
some problems for further investigation motivated by this work are briefly indicated as well. Finally, certain technical details pertaining to recurrences for \( q \)-multinomial coefficients and to partition theoretical interpretation of (1.1) are relegated to Appendix A and B, respectively.

## 2 Recurrences

Define for integers \( i, j, k, \delta, L, M \), the sum

\[
X_{i,j,k}(L, M) = \sum_{i,j,k \text{ constraints}} q^{Ti+T_{ab}} \left[ \begin{array}{c} L - t + a - \delta \\ a - \delta \\ L - t + b \end{array} \right] \left[ \begin{array}{c} L - t \\ b \\ L - t \\ ab \end{array} \right] f(M - t; c, ac, bc),
\]

where \( t = a + b + c + ab + ac + bc \) as before, and the explicit form of the function \( f(M; c, ac, bc) \) will not be required for the recurrences. However, it is important that \( f(M; c, ac, bc) \) does not depend on \( L; a, b, ab \). We wish to show that \( X_{i,j,k}(L, M) \) satisfies the following second order recurrence in \( L \):

\[
X_{i,j,k}(L, M) = X_{i,j,k}(L - 1, M) + q^{L} X_{i-1,j,k}(L - 1, M - 1) + q^{L} X_{i,j-1,k}(L - 1, M - 1)
\]

\[
+ q^{L} X_{i-1,j-1,k}(L - 2, M - 1) - q^{2L-1} X_{i-1,j-1,k}(L - 2, M - 2).
\]

To this end we will use repeatedly the \( q \)-binomial recurrence

\[
\binom{n + m}{n} = \binom{n + m - 1}{n} + q^{m} \binom{n - 1 + m}{n - 1}
\]

which holds for all integers \( m, n \), to expand the right hand side of (2.1) in a telescopic fashion as follows:

\[
\sum_{i,j,k \text{ constraints}} q^{Ti+T_{ab}} \left[ \begin{array}{c} L - 1 - t + a - \delta \\ a - \delta \\ L - 1 - t + b \\ b \\ L - 1 - t \\ ab \end{array} \right] f(M - t; c, ac, bc)
\]

\[
+ \sum_{i,j,k \text{ constraints}} q^{Ti-1+T_{ab}-1+L} \left[ \begin{array}{c} L - 1 - t + a - \delta \\ a - \delta \\ L - 1 - t + b \\ b \\ L - 1 - t \\ ab - 1 \end{array} \right] f(M - t; c, ac, bc)
\]

\[
+ \sum_{i,j,k \text{ constraints}} q^{Ti-1+T_{ab}+L} \left[ \begin{array}{c} L - 1 - t + a - \delta \\ a - \delta \\ L - t + b - 1 \\ b - 1 \\ L - t \\ ab \end{array} \right] f(M - t; c, ac, bc)
\]

\[
+ \sum_{i,j,k \text{ constraints}} q^{Ti-1+T_{ab}+L} \left[ \begin{array}{c} L - t + a - 1 - \delta \\ a - 1 - \delta \\ L - t + b \\ b \\ L - t \\ ab \end{array} \right] f(M - t; c, ac, bc).
\]
Let us denote each of the four sums in (2.4) by $\Sigma_1, \Sigma_2, \Sigma_3,$ and $\Sigma_4,$ respectively. To see that (2.4) is an expansion of (2.1), we merge $\Sigma_1$ and $\Sigma_2$ in (2.4) into a single sum with the aid of (2.3). This single sum can in turn be merged with $\Sigma_3$ in (2.4) using (2.3), and this finally can be merged with $\Sigma_4$ in (2.4) to yield (2.1). (This telescopic expansion technique was introduced in [9] and later was used extensively by Berkovich-McCoy-Schilling [10] and by Schilling-Warnaar [15].) From the definition of $X_{i,j,k}(L, M)$ it is clear that

$$\Sigma_1 = X_{i,j,k}(L - 1, M).$$  

If we perform the change $ab \mapsto ab + 1$ in $\Sigma_2,$ then $t \mapsto t + 1, i \mapsto i - 1, j \mapsto j - 1,$ and so

$$\Sigma_2 = \sum_{i-1,j-1,k} q^{T_t + T_{ab} + L} \left[ \begin{array}{ccc} L - 2 - t + a - \delta \\ -a - \delta \end{array} \right] \left[ \begin{array}{ccc} L - 2 - t + b \\ b \end{array} \right] \left[ \begin{array}{ccc} L - 2 - t \\ ab \end{array} \right] \times f(M - 1 - t; c, ac, bc) = q^L X_{i-1,j-1,k}(L - 2, M - 1).$$

Similarly, replacing $a$ by $a + 1$ in $\Sigma_4,$ we obtain

$$\Sigma_4 = q^L X_{i-1,j,k}(L - 1, M - 1).$$

With regard to $\Sigma_3,$ we write it as a difference to recognize it as

$$\Sigma_3 = \sum_{i,j,k} q^{T_{t-1} + T_{ab} + L} \left[ \begin{array}{ccc} L - t + a - \delta \\ -a - \delta \end{array} \right] \left[ \begin{array}{ccc} L - t + b - 1 \\ b - 1 \end{array} \right] \left[ \begin{array}{ccc} L - t \\ ab \end{array} \right] f(M - t; c, ac, bc)$$

$$- \sum_{i,j,k} q^{T_{t-2} + T_{ab} + 2L - 1} \left[ \begin{array}{ccc} L - t + a - 1 - \delta \\ -a - 1 - \delta \end{array} \right] \left[ \begin{array}{ccc} L - t + b - 1 \\ b - 1 \end{array} \right] \left[ \begin{array}{ccc} L - t \\ ab \end{array} \right] f(M - t; c, ac, bc)$$

$$= q^L X_{i,j-1,k}(L - 1, M - 1) - q^{2L-1} X_{i-1,j-1,k}(L - 2, M - 2).$$

Observe that $g_{i,j,k}(L, M),$ the left hand side of (1.4), is a sum of two functions $X_{i,j,k}(L, M),$ one with $\delta = 0,$ and the other with $\delta = 1,$ and with $f(M - t; c, ac, bc)$ suitably identified. So it follows that

$$g_{i,j,k}(L, M) = g_{i,j,k}(L - 1, M) + q^L g_{i-1,j,k}(L - 1, M - 1) + q^L g_{i,j-1,k}(L - 1, M - 1)$$

$$+ q^L g_{i-1,j-1,k}(L - 2, M - 1) - q^{2L-1} g_{i-1,j-1,k}(L - 2, M - 2).$$
In [3], Andrews had derived a fourth order recursion relation in $L$ for $g_{i,j,k}(L, L)$. His recurrence can be generalized as

$$g_{i,j,k}(L, M) = g_{i,j,k}(L-1, M-1)+ (q^L g_{i-1,j,k}(L-1, M-1)+q^L g_{i,j-1,k}(L-1, M-1)+q^M g_{i,j,k-1}(L-1, M-1)) + (1 - q^{L-1}) (q^L g_{i-1,j-1,k}(L-2, M-2) + q^M g_{i,j-1,k-1}(L-2, M-2) + q^M g_{i,j,k-1}(L-2, M-2)) + q^{2L+M-3} g_{i-1,j-1,k-1}(L-3, M-3) + (q^{L+M-1} g_{i-2,j-1,k-1}(L-3, M-3) + q^{L+M-1} g_{i-1,j-2,k-1}(L-3, M-3) + q^{2M-1} g_{i-1,j-1,k-2}(L-3, M-3)) + q^{L+2M-3} g_{i-2,j-2,k-2}(L-4, M-4).$$

The introduction of an extra parameter $M$ has enabled us to bring down the order of the recursion relation in $L$ for $g_{i,j,k}(L, M)$ to just two.

Next, we claim that $p_{i,j,k}(L, M)$, the right hand side of (1.1), satisfies the same recurrence, namely,

$$p_{i,j,k}(L, M) = p_{i,j,k}(L-1, M) + q^L p_{i-1,j,k}(L-1, M-1) + q^L p_{i,j-1,k}(L-1, M-1) + q^L p_{i-1,j-1,k}(L-2, M-1) - q^{2L-1} p_{i-1,j-1,k}(L-2, M-2).$$

(2.10)

For this purpose we employ the following recursion relation for the $q$–multinomial coefficients (see Appendix A for a proof):

$$q^{L-j} \left[ \begin{array}{c} L-1-s \\ s, i-s, j-1-s \end{array} \right] + q^{L-i-j} \left[ \begin{array}{c} L-1-s \\ s-1, i-s, j \end{array} \right] + q^{L+s-i-j} \left[ \begin{array}{c} L-2-s \\ s, i-1-s, j-1-s \end{array} \right] - q^{2L-1-i-j} \left[ \begin{array}{c} L-2-s \\ s, i-1-s, j-1-s \end{array} \right].$$

(2.11)

Substituting (2.11) into the right hand side of (1.1), we see that

$$p_{i,j,k}(L, M) = p_{i,j,k}(L-1, M) + q^L p_{i-1,j,k}(L-1, M-1) + q^L p_{i,j-1,k}(L-1, M-1)$$

$$\sum_{s \geq 0} q^{L-i-j+s(M+2)-T_s+T_{i-s}+T_{j-s}+T_{k-s}} \left[ \begin{array}{c} L-1-s \\ s-1, i-s, j-s \end{array} \right] \left[ \begin{array}{c} M-i-j \\ k-s \end{array} \right] +$$

$$\sum_{s \geq 0} q^{L+s-i-j+s(M+2)-T_s+T_{i-s}+T_{j-s}+T_{k-s}} \left[ \begin{array}{c} L-2-s \\ s, i-1-s, j-1-s \end{array} \right] \left[ \begin{array}{c} M-i-j \\ k-s \end{array} \right].$$
Now we replace \( s \) by \( s + 1 \) in the first sum in (2.12). This enables us to merge this sum with the second sum in (2.12) to obtain

\[
q^L \sum_{s \geq 0} q^{s(M-1+2)-Ts_0+T_{i-1-s}+T_{j-1-s}+T_{k-s}} \left[ \begin{array}{c} L - 2 - s \\ s, i - 1 - s, j - 1 - s \\ k - s \end{array} \right] = q^L p_{i-1,j-1,k}(L - 2, M - 1). \tag{2.13}
\]

The recurrence (2.10) follows from (2.12) and (2.13).

### 3 The boundary identity

Having established that \( g_{i,j,k}(L, M) \) and \( p_{i,j,k}(L, M) \) satisfy identical recurrences (2.9) and (2.10), we note now that

\[
g_{i,j,k}(L, M) = p_{i,j,k}(L, M) = 0 \tag{3.1}
\]

if any one of the parameters \( i, j, k \) is negative. Thus if we show that the boundary identity (1.9) is true, then we can conclude that

\[
g_{i,j,k}(L, M) = p_{i,j,k}(L, M), \ \forall (i, j, k, L, M) \in \mathbb{Z}^5. \tag{3.2}
\]

A few comments are in order concerning the nonstandard choice of the diagonal boundary \( L = i + j - 1 \). The conventional choice \( L = 0, 1 \) leads to difficulties because the terms of (1.1) do not collapse in these cases. Moreover, the truth of (1.1) for \( L = 0, 1 \) leads us to conclude its validity only for \( L \geq 0 \). Consequently, the case \( L < 0 \) of (1.1) which is highly nontrivial would not be covered. On the other hand the choice \( L = i + j - 1 \) enables us to prove (1.1) for all \( L \in \mathbb{Z} \). To provide additional motivation for the choice \( L = i + j - 1 \), we now show that \( p_{i,j,k}(L, M) \) collapses radically in this case. Indeed,

\[
p_{i,j,k}(i + j - 1, M) = \sum_{s \geq 0} q^{s(M+2)-Ts_0+T_{i-s}+T_{j-s}+T_{k-s}} \left[ \begin{array}{c} i + j - s - 1 \\ s \\ i - s \\ j - s \end{array} \right] = \delta_{i,0} \delta_{j,0} q^T_k \left[ \begin{array}{c} M - i - j \\ k - s \end{array} \right]. \tag{3.3}
\]
by noticing that
\[
\begin{bmatrix}
  j - s - 1 \\
  j - s 
\end{bmatrix} = \delta_{j,s},
\begin{bmatrix}
  i - j - 1 \\
  i - j 
\end{bmatrix} = \delta_{i,j},
\begin{bmatrix}
  i - 1 \\
  i 
\end{bmatrix} = \delta_{i,0} \tag{3.4}
\]

where
\[
\delta_{i,j} = \begin{cases} 
1, & \text{if } i = j, \\
0, & \text{otherwise.} 
\end{cases}
\]

Thus the boundary identity (1.9) can be stated as
\[
g_{i,j,k}(i + j - 1, M) = \delta_{i,0} \delta_{j,0} q^T_k \begin{bmatrix} \Delta \\ k \end{bmatrix}, \tag{3.5}
\]

where \(\Delta = M - i - j\). Next, by repeated use of the \(q\)-binomial formula
\[
\begin{bmatrix} -\alpha \\ k \end{bmatrix} = (-1)^k \begin{bmatrix} k + \alpha - 1 \\ k \end{bmatrix} q^{-\alpha k - T_{k-1}} \tag{3.6}
\]

(see Gasper and Rahman \[12\], formula (I.44)), we may rewrite (3.5) as
\[
\sum_{i,j,k \text{ constraints}} (-1)^{a+b+ab} q^{T_{\tau}+T_{\Gamma}+T_{\Delta}+T_{\Delta-\Gamma}+T_{\Delta-\Gamma}} \times
\begin{bmatrix} \Gamma \\ a \end{bmatrix} \begin{bmatrix} \Gamma \\ b \end{bmatrix} \begin{bmatrix} \Delta \\ \Gamma \end{bmatrix} \begin{bmatrix} ab + \Delta \\ ab \end{bmatrix} \begin{bmatrix} \Delta - \Gamma \\ ac \end{bmatrix} \begin{bmatrix} \Delta - \Gamma \\ bc \end{bmatrix}
\]

\[
\sum_{i,j,k \text{ constraints}} (-1)^{a+1+b+ab} q^{T_{\tau}+T_{\Gamma}+T_{\Delta}+T_{\Delta-\Gamma}+T_{\Delta-\Gamma}} \times
\begin{bmatrix} \Gamma \\ a-1 \end{bmatrix} \begin{bmatrix} \Gamma \\ b \end{bmatrix} \begin{bmatrix} \Delta \\ \Gamma \end{bmatrix} \begin{bmatrix} ab + \Delta \\ ab \end{bmatrix} \begin{bmatrix} \Delta - \Gamma \\ ac \end{bmatrix} \begin{bmatrix} \Delta - \Gamma \\ bc - 1 \end{bmatrix}
\]

\[
= \delta_{i,0} \delta_{j,0} q^T_k \begin{bmatrix} \Delta \\ k \end{bmatrix}, \tag{3.7}
\]

where
\[
\Gamma = c - ab \quad \text{and} \quad \tau = a + b + 2ab + ac + bc. \tag{3.8}
\]

It is convenient to treat \(\Delta\) as an independent parameter and \(\Gamma\) as an independent summation variable. In this case the constraints in (3.7) become
\[
\begin{align*}
i &= a + ab + ac, \\
j &= b + ab + ac, \\
k &= \Gamma + ab + ac + bc.
\end{align*} \tag{3.9}
\]
Next, multiply both sides of (3.7) by \( A^i B^j C^k \) and sum over \( i, j, k \). For the right hand side we get immediately

\[
\sum_{i,j,k \geq 0} \delta_{i,0} \delta_{j,0} \delta_{k,0} q^{T_k} \begin{bmatrix} \Delta \\ k \end{bmatrix} A^i B^j C^k = (-Cq)\Delta.
\]  

(3.10)

To treat the left hand side of (3.7), we get rid of the condition on \( \tau \) in (3.8) and rewrite it as

\[
\omega^0 \left\{ \theta(\omega, q) \sum_{a,b,\Gamma,ab,ac,bc} q^{T_{ac}+T_{bc}-1} \left( -\frac{A}{\omega} \right)^a \left( -\frac{B}{\omega} \right)^b \left( -\frac{ABCq^\Gamma}{\omega^2} \right)^{ab} C^\Gamma \left( \frac{ACq^\Gamma}{\omega} \right)^{ac} \left( \frac{BCq^\Gamma}{\omega} \right)^{bc} \right. 
\]

\[
\times \begin{bmatrix} \Gamma \\ b \end{bmatrix} \begin{bmatrix} \Delta \\ \Gamma \end{bmatrix} \begin{bmatrix} \Delta + ab \\ ab \end{bmatrix} \begin{bmatrix} \Delta - \Gamma \\ ac \end{bmatrix} \left. \right. 
\]

\[
\left( q^{T_{bc}+T_{a-1}} \begin{bmatrix} \Gamma \\ a \end{bmatrix} \begin{bmatrix} \Delta - \Gamma \\ bc \end{bmatrix} - q^{T_{bc}-1+T_{a-2}} \begin{bmatrix} \Gamma \\ a-1 \end{bmatrix} \begin{bmatrix} \Delta - \Gamma \\ bc-1 \end{bmatrix} \right) \right\},
\]  

(3.11)

where

\[
\theta(\omega, q) = \sum_{\tau = -\infty}^{\infty} \omega^\tau q^{T_{\tau}},
\]

and \([\omega^m]f(\omega)\) is the coefficient of \( \omega^m \) in the Laurent expansion of \( f(\omega) \). Thanks to the two \( q \)-binomial theorems

\[
\sum_{n \geq 0} z^n q^{T_n} \begin{bmatrix} \Delta \\ n \end{bmatrix} = (-zq)\Delta,
\]  

(3.12)

and

\[
\sum_{n \geq 0} z^n \begin{bmatrix} \Delta + n \\ n \end{bmatrix} = \frac{1}{(z)^{\Delta+1}},
\]  

(3.13)

we can evaluate the summations in (3.11) over the variables \( a, b, ab, ac, \) and \( bc \), to cast the left hand side of (3.7) as

\[
[\omega^0] \left\{ \theta(\omega, q) \sum_{\Gamma \geq 0} \frac{(1 + \frac{ABCq^{2\Gamma}}{\omega^2}q^{2\Gamma})}{(\frac{ABCq^{2\Gamma}}{\omega^2}q^{2\Gamma})^{\Delta+1}} \begin{bmatrix} \Delta \\ \Gamma \end{bmatrix} \begin{bmatrix} A \omega \Gamma \\ B \omega \Gamma \end{bmatrix} \frac{ACq^{\Gamma+1}}{\omega} \begin{bmatrix} \Delta - \Gamma \\ -\frac{BCq^{\Gamma+1}}{\omega} \end{bmatrix} \right\}.
\]  

(3.14)

We would like to write (3.14) in \( q \)-hypergeometric form. This can be done with the aid of the
following formulas:

$$(i) \quad \left[ \begin{array}{c}
\Delta \\
\Gamma
\end{array} \right] = \frac{(q-\Delta)^\Gamma (q-\Delta)^\Gamma}{(q)^\Gamma} q^{-T\Gamma-1},$$

$$(ii) \quad (xq^\Gamma)\Delta = \frac{(x)^\Delta}{(x)^\Gamma},$$

$$(iii) \quad \frac{1}{(xq^\Gamma)\Delta} = \frac{(x)^\Gamma}{(x)(xq^\Delta)^\Gamma},$$

$$(iv) \quad 1 + \frac{ABC}{\omega^2} q^{2\Delta} = \left(1 + \frac{ABC}{\omega^2}\right) \left(\frac{q\sqrt[-ABC\omega^2]}{\omega^2}, -q\sqrt[-ABC\omega^2]}{\omega^2}\right)^\Gamma, \quad (3.15)$$

where

$$(a_1, a_2, \ldots, a_r; q)_m = (a_1, a_2, \ldots, a_r)_m = (a_1)_m(a_2)_m \cdots (a_r)_m. \quad (3.16)$$

Thus the expression in (3.14) is

$$[\omega^0] \left\{ \theta(\omega, q) \frac{-ACq}{\omega} \frac{-BCq}{\omega} \Delta \right\} 6\phi_5 \left( \frac{y, q\sqrt{y}, -q\sqrt{y}, A}{\omega^2}, \frac{B}{\omega^2}, q^-\Delta \sqrt{y}, -\sqrt{y}, \frac{-AC}{\omega} q, \frac{-BC}{\omega} q, yq^{\Delta+1} ; q, -cq^{\Delta+1} \right), \quad (3.17)$$

where $y = -\frac{ABC}{\omega^2}$, and we have made use of standard notation

$$r+1\phi_r \left( \frac{a_1, a_2, \ldots, a_{r+1}}{b_1, b_2, \ldots, b_r} ; q, z \right) = \sum_{n \geq 0} \frac{(a_1, a_2, \ldots, a_{r+1})_n}{(q, b_1, b_2, \ldots, b_r)_n} z^n. \quad (3.18)$$

Actually the $6\phi_5$ in (3.17) can be evaluated by Jackson’s $q$–analog of Dougall’s summation (see [12], formula (II.21)) to be

$$\frac{(-\frac{ABC}{\omega^2} q, -Cq)\Delta}{(-\frac{AC}{\omega} q, -\frac{BC}{\omega} q)\Delta} \quad (3.19)$$

Finally, combining (3.7), (3.10), (3.11), (3.14), (3.17) and (3.19), we can rewrite (3.5) as

$$[\omega^0] (\theta(\omega, q) \cdot (-Cq)\Delta) = (-Cq)\Delta,$$

which is obviously true because

$$[\omega^0]\theta(\omega, q) = 1.$$

Thus, we have completed the proof of the boundary identity (1.9) and consequently the truth of (1.8) (and (1.7)) is established.
4 A bounded version of Göllnitz’s partition theorem

In this section we assume that \( L = M \), and for this case we first establish the representation

\[
p_{i,j,k}(L, L) = q^{T_i + T_j + T_k} \begin{bmatrix} L - k \\ i \end{bmatrix} \begin{bmatrix} L - i \\ j \end{bmatrix} \begin{bmatrix} L - j \\ k \end{bmatrix}.
\] (4.1)

We then discuss the partition interpretation of the identity

\[
g_{i,j,k}(L, L) = p_{i,j,k}(L, L).
\] (4.2)

First note that

\[
\begin{bmatrix} L - s \\ s, i - s, j - s \end{bmatrix} \begin{bmatrix} L - i - j \\ k - s \end{bmatrix} = \begin{bmatrix} L - s \\ s, i - s, j - s, k - s \end{bmatrix}.
\] (4.3)

With (4.3) in mind, we rewrite \( p_{i,j,k}(L, L) \) as

\[
p_{i,j,k}(L, L) = \lim_{\ell \to L} \sum_{s \geq 0} q^{s(\ell + 2) - T_s + T_{i-s} + T_{j-s} + T_{k-s}} \begin{bmatrix} \ell - s \\ s, i - s, j - s, k - s \end{bmatrix} = \lim_{\ell \to L} \frac{q^{T_i + T_j + T_k}}{(q)_i(q)_j(q)_k} (q^{1+\ell-i-j-k})_{i+j+k} \times 3\phi_2 \left( \begin{array}{c} q^{-i}, q^{-j}, q^{-k} \\ q^{1+\ell-i-j-k}, q^{-\ell} \end{array} ; q, q \right),
\] (4.4)

where we have used the limit definition to make sure that all objects in (4.4) are well defined. It turns out that by the use of the \( q \)-Pfaff-Saalschütz summation (see Gasper and Rahman [12], eqn.(II.12))

\[
3\phi_2 \left( \begin{array}{c} q^{-i}, q^{-j}, q^{-k} \\ q^{1+\ell-i-j-k}, q^{-\ell} \end{array} ; q, q \right) = \frac{(q^{1+\ell-j-k}, q^{1+\ell-i-k})_k}{(q^{1+\ell-i-j-k}, q^{1+\ell-k})_k},
\] (4.5)

and so (4.4) becomes

\[
p_{i,j,k}(L, L) = \lim_{\ell \to L} \frac{q^{T_i + T_j + T_k}}{(q)_i(q)_j} \frac{(q^{1+\ell-i-j-k})_{i+j+k} (q^{1+\ell-i-k})_k}{(q^{1+\ell-i-j-k})_k (q^{1+\ell-k})_k} \begin{bmatrix} \ell - j \\ k \end{bmatrix}.
\] (4.6)

Finally it can be shown by repeated use of the formula (ii) of (3.15) that (4.6) yields (we omit the lengthy details of this calculation)

\[
p_{i,j,k}(L, L) = \lim_{\ell \to L} \frac{q^{T_i + T_j + T_k}}{(q)_i(q)_j} \begin{bmatrix} \ell - k \\ i \end{bmatrix} \begin{bmatrix} \ell - i \\ j \end{bmatrix} \begin{bmatrix} \ell - j \\ k \end{bmatrix},
\] which is (4.1), thus completing the proof.
Remark 1: Zeilberger [16] has given a combinatorial proof of the q-Pfaff-Saalschütz summation.

His parameters translate to ours in (4.3) by suitable change of variables.

Now when

\[ L \geq \max(i + j, j + k, k + i), \] (4.7)

\( p_{i,j,k}(L, L) \) given by (4.1) can be interpreted as the generating function of partitions \( \pi \) whose parts occur in three (primary) colors \( A, B, C \) ordered as

\[ A < B < C \] (4.8)

such that parts in the same color are distinct and

\[
\begin{align*}
\nu(A; \pi) &= \nu(A) = i, & \lambda(A; \pi) &= \lambda(A) \leq L - k, \\
\nu(B) &= j, & \lambda(B) &\leq L - i, \\
\nu(C) &= k, & \lambda(C) &\leq L - j,
\end{align*}
\] (4.9)

where \( \nu(A; \pi) = \nu(A) \) is the number of parts of \( \pi \) in color \( A \) and \( \lambda(A; \pi) = \lambda(A) \) is the largest part of \( \pi \) in color \( A \), and the other notations in (4.9) have similar interpretation. Now consider partitions \( \tilde{\pi} \) such that part 1 may occur in three primary colors \( A, B, C \), but parts \( \geq 2 \) could occur in the three primary colors as well as in three secondary colors \( AB, AC, BC \) ordered as

\[ AB < AC < A < BC < B < C \] (4.10)

for any given part occurring in these colors and such that the gap between the parts is \( \geq 1 \) where gap=1 only if both parts are either of the same primary color or if the larger part is in a color of higher order (as given by (4.10)). We call such a partition \( \tilde{\pi} \) a Type-1 partition as in [2]. It is at this point the interpretation of the parameters \( a, b, c, ab, ac, bc \) becomes clear. Indeed, we denote \( \nu(A; \tilde{\pi}) \) by \( a \), \( \nu(B; \tilde{\pi}) \) by \( b \), \ldots, \( \nu(BC; \tilde{\pi}) \) by \( bc \). With this interpretation, we will now show that

\[ g_{i,j,k}(L, L) \] for \( L \geq \max(i + j, j + k, k + i) \) is the generating function for Type-1 partitions \( \tilde{\pi} \) such that

\[ \lambda(\tilde{\pi}) \leq L, \] (4.11)

and the constraints on the frequencies \( a, b, \ldots, bc \), are as in (1.4). To this end subtract 1 from the smallest part of \( \tilde{\pi} \), 2 from the second smallest part, \ldots, \( t \) from the largest part \( \lambda(\tilde{\pi}) \) of \( \tilde{\pi} \) so that \( T_t \) is the total amount subtracted. (Note that \( t = a + b + c + ab + ac + bc \) is the number of parts of \( \tilde{\pi} \).) Clearly, this subtraction procedure is reversible. Let the resulting partitions after subtraction be denoted by \( \pi' \). The colors of the parts of \( \pi' \) are those of the parts of \( \tilde{\pi} \) from which they were
derived. We decompose \( \pi' \) into monochromatic partitions in colors \( \mathbf{A}, \mathbf{B}, \mathbf{C} \), \( \mathbf{AB}, \mathbf{AC}, \mathbf{BC} \), denoted as \( \pi'_A, \pi'_B, \pi'_C, \pi'_{AB}, \pi'_{AC}, \pi'_{BC} \). The monochromatic partitions satisfy the following conditions:

\[
P_{\mathbf{A}} \quad \lambda(\pi'_A) = L - t, \quad \nu(\pi'_A) = a,
0 \text{ could be part of } \pi'_A.
\]

\[
P_{\mathbf{B}} \quad \lambda(\pi'_B) = L - t, \quad \nu(\pi'_B) = b,
0 \text{ could be part of } \pi'_B.
\]

\[
P_{\mathbf{C}} \quad \lambda(\pi'_C) = L - t, \quad \nu(\pi'_C) = c,
0 \text{ could be part of } \pi'_C.
\]

The monochromatic partitions satisfy the following conditions:

\[
\pi'_{AB} \text{ has distinct parts }, \quad \lambda(\pi'_{AB}) \leq L - t, \quad \nu(\pi'_{AB}) = ab,
\]

\[
\pi'_{AC} \text{ has distinct parts }, \quad \lambda(\pi'_{AC}) \leq L - t, \quad \nu(\pi'_{AC}) = ac,
\]

\[
\pi'_{BC} \text{ has distinct parts }, \quad \lambda(\pi'_{BC}) \leq L - t, \quad \nu(\pi'_{BC}) = bc,
\]

and

\[
s(\pi'_{BC}) \geq 0, \quad \text{if } s(\pi'_A) = 0,
s(\pi'_{BC}) \geq 1, \quad \text{otherwise},
\]

where \( s(\pi'_{BC}) \) is the smallest part in \( \pi'_{BC} \) with similar interpretation for \( s(\pi'_A) \). The first summation on the left in (1.1) is the generating function of Type–1 partitions \( \bar{\pi} \) such that \( \pi' \) satisfies (4.12), (4.13) and \( s(\pi'_{BC}) \geq 1, s(\pi'_A) \geq 0 \). The second summation on the left in (1.1) is the generating function of Type–1 partitions \( \bar{\pi} \) such that \( \pi' \) satisfies (4.12), (4.13) and \( s(\pi'_{BC}) = 0, s(\pi'_A) = 0 \). Hence, \( g_{i,j,k}(L, L) \) is the generating function of all Type–1 partitions satisfying (4.11). Thus, we have the following new bounded version of the Alladi-Andrews-Gordon [2] refinement of Göllnitz’s theorem [13].

**Theorem 1** Let \( G_L(n; a, b, c, ab, ac, bc) \) denote the number of Type–1 partitions \( \bar{\pi} \) of \( n \) such that \( \lambda(\bar{\pi}) \leq L, \nu(\mathbf{A}; \bar{\pi}) = a, \ldots, \nu(\mathbf{BC}; \bar{\pi}) = bc \). Let \( P_L(n; i, j, k) \) denote the number of partitions \( \pi \) of \( n \) into parts occurring in three colors \( \mathbf{A} < \mathbf{B} < \mathbf{C} \) such that parts of the same color are distinct and conditions (1.9) are satisfied. Then for \( L \geq \max(i + j, j + k, k + i) \) we have

\[
\sum_{i,j,k \text{ constraints}} G_L(n; a, b, c, ab, ac, bc) = P_L(n; i, j, k),
\]

where the \( i, j, k \) constraints on the summation variables \( a, b, c, ab, ac, bc \) are as in (1.2).

**Remark 2**: We would like to stress that the condition (4.7) is crucial for our partition theoretical interpretation of \( g_{i,j,k}(L, L) \). Indeed, this condition along with \( M = L \) guarantees that whenever the summands in (1.1) are non zero, then \( L - t \geq 0 \). Thus the largest parts \( \lambda(\pi'_A), \ldots, \lambda(\pi'_{BC}) \) in (4.12), (4.13) will never take negative values (see Appendix B for details).

**Remark 3**: To see the connection between Theorem 1 and Göllnitz’s theorem [13], proceed as follows. First, denote part \( n \) of color \( \mathbf{A} \) by \( A_n \), part \( n \) of color \( \mathbf{B} \) by \( B_n \), etc. Next, replace \( A_n \) by
6n − 4, \( B_n \) by 6n − 2, \( C_n \) by 6n − 1, \( AB_n \) by 6n − 6, \( AC_n \) by 6n − 5, \( BC_n \) by 6n − 3, let \( L \to \infty \), and sum over \( i, j, k \). This yields Göllnitz’s (Big) partition theorem [13]:

**Theorem 2** Let \( B(n) \) denote the number of partitions of \( n \) into distinct parts \( \equiv 2, 4, 5(\text{mod } 6) \).
Let \( C(n) \) denote the number of partitions of \( n \) in the form \( m_1 + m_2 + \ldots + m_s \), no part equals 1 or 3, and such that \( m_s - m_{s+1} \geq 6 \) with strict inequality if \( m_s \equiv 6, 7 \) or \( 9(\text{mod } 6) \).

Then, \( B(n) = C(n) \).



## 5 Prospects

In [2], Alladi, Andrews and Gordon discuss companions to Göllnitz’s theorem generated by different orderings of the colored integers. They show that these companion partition functions are bijectively equivalent to \( G(n; a, b, c, ab, ac, bc) \), and therefore the left hand side of their key identity (1.6) is the generating function for all these companion partition functions. It turns out that when bounds are imposed on the parts, these bijections can fail at the boundary. Thus the finite key identity (1.1) in the case \( L = M \) corresponds to the bounded Göllnitz partition function in §4 only with the ordering in (4.10). If a different ordering were considered as in [2], then this might lead to a bounded key identity different from (1.1), but one which still reduces to (1.6) when \( L, M \to \infty \).

In a recent paper [4], by studying a reformulation of Göllnitz’s theorem as a weighted identity involving partitions into parts differing by \( \geq 2 \), we deduced several well known results as special cases, including Jacobi’s triple product identity in the form

\[
\sum_{n=-\infty}^{\infty} A^n q^n^2 = \prod_{m=1}^{\infty} (1 + Aq^{2m-1})(1 + A^{-1}q^{2m-1})(1 - q^{2m}). \tag{5.1}
\]

Motivated by the method in [4] and our Theorem 1 in §4, we have now obtained the following new bounded version of (5.1)

\[
\sum_{\ell=0}^{L} (-1)^{L+\ell} q^{2(T_L-2T_\ell)} \sum_{n=-\ell} A^n q^n^2 = \sum_{L \geq \max(i+j+i+k+j+k)} (-1)^k A^{i-j} q^{2T_i+2T_j+2T_k-i-j} \begin{bmatrix} L-k \cr i \cr \end{bmatrix} \begin{bmatrix} L-i \cr j \cr \end{bmatrix} \begin{bmatrix} L-j \cr k \cr \end{bmatrix} q^2. \tag{5.2}
\]

When \( L \to \infty \), (5.2) reduces to (5.1). The proof and discussion of (5.2) will be presented elsewhere [5]. Also in [5] we will show that (5.2) implies the new false theta function identity

\[
\sum_{\ell \geq 0} (-1)^{\ell} q^{T_\ell} = \sum_{i, k \geq 0} (-1)^{i+k} \frac{q^{T_i+T_k-ik}}{(q)_i(q)_k} \begin{bmatrix} k+i \cr k \cr \end{bmatrix}. \tag{5.3}
\]
Closely related to (5.1) is another Jacobi’s formula
\[ \sum_{\ell \geq 0} (-1)^\ell (2\ell + 1)q^{T\ell} = (q)_\infty^3. \] (5.4)

Using Theorem 1, we found in [6] a new polynomial analog of (5.4)
\[ \sum_{\ell=0}^{L} (-1)^\ell (2\ell + 1)q^{T\ell} = \sum_{i,j,k \geq 0} (-1)^{i+j+k} q^{T_i + T_j + T_k} \begin{bmatrix} L - k \\ i \\ \end{bmatrix} \begin{bmatrix} L - i \\ j \\ \end{bmatrix} \begin{bmatrix} L - j \\ k \\ \end{bmatrix}, \] (5.5)

which is unexpectedly elegant and is very different from the polynomial identity proven by Hirschhorn [14]. Clearly, as \( L \to \infty \) (5.5) reduces to (5.4). More generally, we derived
\[ \sum_{\ell=0}^{L} a^{-\ell} + a^{2\ell+1} q^{T\ell} = \sum_{i,j,k \geq 0} a^{i-j} (-1)^k q^{T_i + T_j + T_k} \begin{bmatrix} L - k \\ i \\ \end{bmatrix} \begin{bmatrix} L - i \\ j \\ \end{bmatrix} \begin{bmatrix} L - j \\ k \\ \end{bmatrix}. \] (5.6)

If we set \( q = 1 \) in (5.6) it becomes
\[ \frac{a^{L+1} - a^{-L-1}}{a - a^{-1}} = \sum_{i,j,k \geq 0} a^{i-j} (-1)^k \begin{bmatrix} L - k \\ i \\ \end{bmatrix} \begin{bmatrix} L - i \\ j \\ \end{bmatrix} \begin{bmatrix} L - j \\ k \\ \end{bmatrix}, \] (5.7)

which is, essentially, a special case of Carlitz’s formula (3.7) in [11] with \( n = L \) and \( x = y^{-1} = a \). Actually, from Theorem 1, many new finite versions of other fundamental results can be deduced, and these will be presented in [5] and [6]. Interestingly, all these new finite identities can be interpreted as \( q \)–analogs of Carlitz’s formulas for the binomial cycles of length 3.

It was the appearance of the \( q \)–binomial cycles in (4.1) in a special form that led to Theorem 1. In collaboration with Andrews we intend to conduct a systematic study of \( q \)–binomial cycles; in particular we will show that the generating function of the \( q \)–binomial cycles of length 3 can be evaluated in terms of infinite products.

Now that we have succeeded in obtaining a partition interpretation of (1.1) when \( L = M \), it would be worthwhile to see what combinatorial interpretation (1.1) has when \( L \neq M \). In the case of Schur’s partition theorem with two bounds \( L \neq M \) such interpretation turned out to be quite delicate [3].

Recently, in collaboration with Andrews [1], we have obtained the following remarkable four parameter key identity:
\[ \sum_{i,j,k,l \text{\ constraints}} q^{T_i + T_{ab} + T_{bc} + T_{ad} + T_{bd} + T_{cd} - bc - bd - cd + 4T_{Q-1} + Q(3+2t)} \frac{(q)_{a} (q)_{b} (q)_{c} (q)_{d} (q)_{ab} (q)_{ac} (q)_{ad} (q)_{bd} (q)_{cd} (q)_{Q} \times}{(q_{a} (q)_{b} (q)_{c} (q)_{d} (q)_{ab} (q)_{ac} (q)_{ad} (q)_{bd} (q)_{cd} (q)_{Q}}} \]
\[
\left\{(1 - q^a) + q^{a+bc+bd+Q}(1 - q^b) + q^{a+bc+bd+Q+b+cd}\right\} = \frac{q^{T_i+T_j+T_k+T_\ell}}{(q)_i(q)_j(q)_k(q)_\ell},
\]

(5.8)

where

\[
t = a + b + c + d + ab + ac + ad + bc + bd + cd,
\]

(5.9)

and the \(i, j, k, \ell\) constraints on the summation variables \(a, b, \ldots, cd, Q\) are

\[
\begin{aligned}
i &= a + ab + ac + ac + Q, \\
j &= b + ab + bc + bd + Q, \\
k &= c + ac + bc + cd + Q, \\
\ell &= d + ad + bd + cd + Q.
\end{aligned}
\]

(5.10)

Identity (5.8) reduces to (1.6) when any one of the parameters \(i, j, k, \ell\) is set equal to 0. The combinatorial interpretation of (5.8) yields a four parameter generalization of Göllnitz’s theorem. The discovery and proof of (5.8) settles a thirty year old problem of Andrews [8] who asked whether there exists a partition theorem that lies “beyond” the (big) theorem of Göllnitz. It would be worthwhile to seek a bounded identity that reduces to (5.8) when the bounds go to infinity, just as (1.1) reduces to (1.6) when \(L, M \to \infty\).

**Note Added** Axel Riese informed us that he significantly improved WZ algorithm and, as a result, was able to obtain a computer proof of the identities (1.1) and (1.6).

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Here we will prove that

\[
\begin{bmatrix}
L \\
_{s,i,j}
\end{bmatrix} = \begin{bmatrix}
L - 1 \\
_{s,i,j}
\end{bmatrix} + q^{L-i} \begin{bmatrix}
L - 1 \\
_{s,i-1,j}
\end{bmatrix} + q^{L-j} \begin{bmatrix}
L - 1 \\
_{s,i,j-1}
\end{bmatrix} + q^{L-s-i-j} \begin{bmatrix}
L - 1 \\
_{s-1,i,j}
\end{bmatrix} + q^{L-i-j} (1 - q^{L-1}) \begin{bmatrix}
L - 2 \\
_{s,i-1,j-1}
\end{bmatrix}.
\]

(6.1)

First we note that

\[
\begin{bmatrix}
L \\
_{s,i,j}
\end{bmatrix} = \begin{bmatrix}
L \\
_{i,j,s}
\end{bmatrix} = \begin{bmatrix}
L \\
_{i,j}
\end{bmatrix} \begin{bmatrix}
L - i - j \\
_{s}
\end{bmatrix}.
\]

(6.2)

Next, we recall the symmetric recursion relation

\[
\begin{bmatrix}
L \\
_{i,j}
\end{bmatrix} = \begin{bmatrix}
L - 1 \\
_{i,j}
\end{bmatrix} + q^{L-i} \begin{bmatrix}
L - 1 \\
_{i-1,j}
\end{bmatrix} + q^{L-j} \begin{bmatrix}
L - 1 \\
_{i,j-1}
\end{bmatrix} + q^{L-i-j} (1 - q^{L-1}) \begin{bmatrix}
L - 2 \\
_{i-1,j-1}
\end{bmatrix},
\]

(6.3)

proven in [3]. Combining (6.2) and (6.3), we obtain

\[
\begin{bmatrix}
L \\
_{s,i,j}
\end{bmatrix} = \begin{bmatrix}
L - 1 \\
_{i,j}
\end{bmatrix} + q^{L-i} \begin{bmatrix}
L - 1 \\
_{i-1,j,s}
\end{bmatrix} + q^{L-j} \begin{bmatrix}
L - 1 \\
_{i,j-1,s}
\end{bmatrix} + q^{L-i-j} (1 - q^{L-1}) \begin{bmatrix}
L - 2 \\
_{i-1,j-1,s}
\end{bmatrix}.
\]

(6.4)

Finally, using the \(q\)-binomial recurrence (2.3) with \(m = L - s - i - j\) and \(n = s\), we obtain from (6.4)

\[
\begin{bmatrix}
L \\
_{s,i,j}
\end{bmatrix} = \begin{bmatrix}
L - 1 \\
_{i,j,s}
\end{bmatrix} + q^{L-s-i-j} \begin{bmatrix}
L - 1 \\
_{i,j,s-1}
\end{bmatrix} + q^{L-i} \begin{bmatrix}
L - 1 \\
_{i-1,j,s}
\end{bmatrix} + q^{L-j} \begin{bmatrix}
L - 1 \\
_{i,j-1,s}
\end{bmatrix} + q^{L-i-j} (1 - q^{L-1}) \begin{bmatrix}
L - 2 \\
_{i-1,j-1,s}
\end{bmatrix},
\]

(6.5)

which is essentially (1.1). Formula (2.11) follows from (6.1) with the substitutions

\[
\begin{align*}
L & \mapsto L - s, \\
i & \mapsto i - s, \\
j & \mapsto j - s.
\end{align*}
\]

(6.6)
Throughout this appendix we assume that $M = L$ in (1.1). We need to show that when (4.7) holds, $L - t \geq 0$ in all cases where the summands in (1.1) are non zero in value. To this end, it is important to observe that definition (1.3) implies that

$$\begin{bmatrix} n \\ m \end{bmatrix} \neq 0, \text{ iff } m \geq 0 \text{ and either } n < 0 \text{ or } n \geq m. \quad (7.1)$$

So, if $n, m \geq 0$, then

$$\begin{bmatrix} n \\ m \end{bmatrix} \neq 0, \text{ iff } n \geq m. \quad (7.2)$$

Let us now assume that $k \leq \min(i, j)$ and consider

$$\begin{bmatrix} L - t + c \\ c \end{bmatrix} = \begin{bmatrix} L - i - j + ab \\ c \end{bmatrix}, \quad (7.3)$$

which appears in the lhs of (1.1). Since $L - i - j \geq 0$ (by (4.7)) and $ab \geq 0$, then $L - t + c \geq 0$ and, as a result, $L - t \geq 0$ (by (7.2)). Obviously, the case $j \leq \min(i, k)$ can be treated in the analogous fashion. If $i \leq \min(j, k)$, one needs to consider two $q$-binomials

$$\begin{bmatrix} L - t + a - 1 \\ a - 1 \end{bmatrix} \begin{bmatrix} L - t \\ bc - 1 \end{bmatrix} = \begin{bmatrix} L - j - k + bc - 1 \\ a - 1 \end{bmatrix} \begin{bmatrix} L - t \\ bc - 1 \end{bmatrix}, \quad (7.4)$$

which appear in the second product in (1.1). Since $L - j - k \geq 0$ (by (4.7)) and $bc - 1 \geq 0$, we conclude that $L - t + a - 1 \geq 0$, which again implies that $L - t \geq 0$. 

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