Tangent-like Spaces to Local Monoids

Keqin Liu
Department of Mathematics
The University of British Columbia
Vancouver, BC
Canada, V6T 1Z2

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Abstract

The main new notions are the notions of tangent-like spaces and local monoids. The main result is the passage from a local monoid to its tangent-like space which is a local Leibniz algebra.

Based on my belief that Leibniz algebras are too general to establish a fair counterpart of Lie theory in the context of Leibniz algebras, I introduced the notion of a local Leibniz algebra in Section 1.6 of [3]. The purpose of this paper is to construct the analogue of the passage from a linear Lie group to its Lie algebra in the context of local Leibniz algebras. The group-like objects I need in constructing the analogue are local monoids, which are obtained by adding more algebraic structures to monoids with diconjugations introduced in Section 4.2 of [2]. One of the difficulties I experienced in constructing the analogue is to find a suitable definition of a tangent space. The notion of a tangent-like space introduced in this paper is good enough for the purpose of this paper, but some changes may be needed in order to use it to develop the counterpart of differential geometry in a more general context.

The paper consists of three sections. Section 1 discusses trimonoids, which give the algebraic foundation of this paper. Section 2 introduces the notion of a local monoid. Section 3 constructs the passage from a local monoid to its tangent-like space which is a local Leibniz algebra.

Thoughout this paper, we will use Chapter 3 and Chapter 4 of [2].

1 Trimonoids

We begin this section by introducing the notion of a trisemigroup.
Definition 1.1 Let \( G \) be a nonempty set together with three binary operations: \( \sharp, \cdot, \rightarrow \). The set \( G \) is called a trisemigroup if the following two properties hold:

(i) For each \( \ast \in \{ \sharp, \cdot, \rightarrow \} \), \( (G, \ast) \) is a semigroup.

(ii) The three binary operations \( \sharp, \cdot, \rightarrow \) satisfy the Hu-Liu triassociative law:

\[
\begin{align*}
(x \cdot y) \sharp z &= x \sharp (y \cdot z) \\
(x \sharp y) \cdot z &= x \sharp (y \cdot z) \\
x \cdot (y \sharp z) &= (x \cdot y) \sharp z \\
x \cdot (y \cdot z) &= x \cdot y \cdot z \\
(x \cdot y) \rightarrow z &= x \rightarrow y \rightarrow z
\end{align*}
\]

for all \( x, y, z \in G \).

A trisemigroup \( G \) is also denoted by \( (G, \sharp, \cdot, \rightarrow) \), and the three binary operations \( \sharp, \cdot, \rightarrow \) are called the product, the left product and the right product, respectively.

The Hu-Liu triassociative law was introduced in Definition 3.4 of [2]. If we drop (1) in Definition 1.1 then we obtain the notion of a quasitrisemigroup. The Hu-Liu quasitriassociative law means the law consisting of (2), (3), (4) and (5).

Definition 1.2 Let \( (G, \sharp, \cdot, \rightarrow) \) be a trisemigroup (quasitrisemigroup). \( G \) is called the trimonoid (quasitrimonoid) with a triunit \( e \) if \( e \in G \) and \( e \) satisfies

\[
\begin{align*}
x \sharp e &= x = e \sharp x \quad \text{for all } x \in G. \quad (6) \\
x \cdot e &= x = e \cdot x \quad \text{for all } x \in G. \quad (7)
\end{align*}
\]

Note that if \( e \) is a triunit of a trimonoid \( G \), then \( e \) satisfies:

\[
e \rightarrow x = x \rightarrow e \quad \text{for all } x \in G. \quad (8)
\]

A triunit of a trimonoid \( G \) is also called an identity of \( G \). By Proposition 3.1 in [2], the left product \( \rightarrow \) and the right product \( \leftarrow \) of a trimonoid with a triunit satisfy the diassociative law.
2 Local Monoids

This section depends on the notion of a 7-tuple, which was introduced in Definition 4.1 of [2]. Let \((A, +, \#, \vdash, \cdot)\) be a 7-tuple with an identity \(1^\times\). The set
\[ h^\times(A) := \{ e \in A \mid e \cdot x = x = x \cdot e \text{ for all } x \in A \} \]
is called the **halo** of \(A\). An element of \(h^\times(A)\) is called a **bar-unit** of \(A\). If \(e\) is a bar-unit of \(A\), then the set
\[ h^+(A) := \{ \alpha \in A \mid e \cdot \alpha = 0 \} \]
is called the **additive halo** of \(A\).

By Section 1.2 of [3], every bar-unit \(e\) of a 7-tuple \(A\) produces three more binary operations \(\vdash, \cdot, \bullet\) on \(A\) in the following way:
\[
\begin{align*}
x \vdash y & : = x + e \cdot y, \\
x \cdot y & : = x \cdot e + y, \\
x \bullet y & : = x \cdot y + x \cdot y - x \cdot e \cdot y, \\
\end{align*}
\]
where \(x, y \in R\). The binary operations \(\uplus, \maplus\) are called the **left addition** and the **right addition** induced by \(e\), respectively. The binary operation \(\bullet\) is called the **Hu-Liu product** induced by \(e\). Hence, a 7-tuple \((A, +, \#, \vdash, \cdot, \bullet)\) always carries the following seven binary operations:
\[
+, \maplus, \vdash, \cdot, \maprime, \uplus, \maplus, \bullet.
\]

This is our reason of using 7-tuple to name the algebraic object introduced in Definition 4.1 of [2]. If it is necessary to indicate explicitly that the left addition, the right addition and the Hu-Liu product are induced by a bar-unit \(e\), then we use \(\maplus, \mapl, \mapr\) to denote \(\vdash, \cdot, \bullet\), respectively.

**Proposition 2.1** Let \((A, +, \#, \vdash, \cdot, \bullet)\) be a 7-tuple. If \(e\) is a bar-unit of \(A\), then \((A, \mapr, \cdot, \mapr)\) is a quasitrimonoid with the triunit \(e\).

**Proof** By Proposition 1.3 in [3], \((A, \bullet, e)\) is a monoid with the unit \(e\). A direct computation shows that the three binary operations \(\mapr, \cdot, \mapr\) satisfy the Hu-Liu quasitriassociative law. Hence, Proposition 2.1 is true.

The next definition is based on the equation (6.1) in [2].

**Definition 2.1** An element \(x\) of a 7-tuple \((A, +, \#, \vdash, \cdot, \mapr)\) is said to be **one-sided invertible** if there exist two elements \(x_e^l\) and \(x_e^r\) of \(A\) such that
\[
x_e^l \cdot x = e = x \cdot x_e^r \quad \text{for some } e \in h^\times(A).
\]
The two elements \( x_l^{-1} \) and \( x_r^{-1} \) are called the \textbf{left inverse} and the \textbf{right inverse} of \( x \) with respect to the bar-unit \( e \), respectively. By Proposition 6.2 in [2], \( x_l^{-1} = x_r^{-1} \) if and only if \( e \) satisfies \( \mathbf{8} \). The set of all one-sided invertible elements of a 7-tuple \( A \) is denoted by

\[ A^{-1} := \{ a \in A | \ y \vee a = e = a \vee z \ \text{for some} \ y, z \in A \}, \]

where \( e \) is a fixed bar-unit of \( A \). The definition of \( A^{-1} \) does not depend on the choice of the bar-unit \( e \).

For an one-sided invertible element \( a \) of a 7-tuple \( A \), we define a map \( \Psi_a : A \to A \) by

\[ \Psi_a(x) := a_l^{-1} \vee x = a_r^{-1} \vee x = a_l^{-1} \vee x = a_r^{-1} \vee x \] for \( x \in A, \]

(14)

where \( e \) is a bar-unit of \( A \). The map \( \Psi_a \) is called the \textbf{diconjugation} of \( A \) determined by \( a \). Since the definition of \( \Psi_a \) is independent of the choice of the bar-unit \( e \), (14) is also written as

\[ \Psi_a(x) := a_l^{-1} \vee x = a_r^{-1} \vee x = a_l^{-1} \vee x = a_r^{-1} \vee x \] for \( x \in A, \]

where \( a_l^{-1} \) and \( a_r^{-1} \) denote the left inverse and the right inverse of \( a \) with respect to any bar-unit of \( A \), respectively.

**Proposition 2.2** Let \( (A, +, \ #, \ \vee, \ \leftarrow, \ \rightleftarrows) \) be a 7-tuple with an identity \( 1^\times \).

(i) \( (h^\times(A), \ #) \) is a monoid with the unit \( 1^\times \).

(ii) \( (h^+(A), +, \ #) \) is a rng.

(iii) For any \( e \in h^\times(A) \), \( (h^\times(A), \ #_e) \) is a monoid with the unit \( e \).

(iv) For any \( a \in A^{-1} \), we have

\[ \Psi_a\left(h^\times(A)\right) = h^\times(A) \quad \text{and} \quad \Psi_a\left(h^+(A)\right) = h^+(A). \]

(v) If \( a \in A^{-1} \), then the diconjugation \( \Psi_a \) preserves each of the first four binary operations in the list (12) and

\[ \Psi_a(x \#_e y) = \Psi_a(x) \#_e \Psi_a(y), \]

\[ \Psi_a(x \leftarrow_e y) = \Psi_a(x) \leftarrow_e \Psi_a(y), \]

\[ \Psi_a(x \circ_e y) = \Psi_a(x) \circ_e \Psi_a(y), \]

where \( x, y \in A \) and \( e \in h^\times(A) \).
Proof They are the direct consequences of the Hu-Liu triassociative law.

An element $1^\sharp$ of a 7-tuple $(A, +, \#, \vec{\cdot}, \vec{\cdot})$ is called the local identity if $(\bar{h}^+(A), +, \#)$ is a ring with the identity $1^2$. The notion of a local identity is of importance to rewriting commutative ring theory in a more general context (see Chapter 4 or Chapter 5 in [3] for the application of local identity).

Let $(A, +, \# , \vec{\cdot}, \vec{\cdot})$ be a 7-tuple with an identity $1\times$. An element $a$ of $A$ is said to be invertible if there exists an element $b$ of $A$ such that

$$a \# b = 1\times = b \# a. \quad (15)$$

The element $b$ satisfying (15) is called the inverse of $a$ and is denoted by $a^{-1}$. We use $A^{-1}$ and $h\times(A)^{-1}$ to denote the set of all invertible elements in $A$ and $h\times(A)$, respectively.

We now define local monoids, which are the group-like object we need to construct the analogue of the passage from a Lie group to its Lie algebra in the context of local Leibniz algebras.

Definition 2.2 Let $(A, +, \# , \vec{\cdot}, \vec{\cdot})$ be a 7-tuple with an identity $1\times$. A subset $G$ of $A^{-1}$ is called a $(\Delta, \Omega)$-local monoid of $A$ if the following five properties hold:

(i) $\Psi_a(G) \subseteq G$ for each $a \in G$.

(ii) $\Delta \subseteq \{ \#, \bullet_e | e \in G \cap h\times(A) \text{ and } \bullet_e \text{ is the Hu-Liu product induced by } e \}$.

(iii) $\bullet_e \in \Delta \implies \bullet_{\Psi_a(e)} \in \Delta$ for each $a \in G$.

(iv) $(G, *)$ is a monoid with a unit for each $* \in \Delta$.

(v) $\Omega \subseteq G \cap h\times(A)^{-1}$, $\Psi_a(\Omega) \subseteq \Omega$ for each $a \in G$ and $(\Omega, \#)$ is a group with the identity $1\times$.

It is clear that if $G$ is a $(\Delta, \Omega)$-local monoid of a 7-tuple $A$, then $G$ is also a $(\Delta, \{1\times\})$-local monoid of $A$. We will see in the next section that every $(\Delta, \Omega)$-local monoid $G$ of a finite dimensional complete 7-tuple $A$ produces a local Leibniz algebra, where the notion of a finite dimensional complete 7-tuple was introduced in Definition 4.4 of [2].

3 Tangent-like Spaces

The notion of a local Leibniz algebra was introduced in Definition 1.13 of [3] by using the Leibniz identity, the Jacobi identity and the Hu-Liu identity. Before presenting the definition of a local Leibniz algebra, we explain where the Hu-Liu identity comes from.
In a 7-tuple \( (A, +, \sharp, \cdot, \cdot, \leftarrow, \rightarrow) \), we can define an angle bracket \( \langle \cdot, \cdot \rangle \) and a square bracket \([\cdot, \cdot]\) by:

\[
\langle x, y \rangle := x \cdot y - y \cdot x \quad \text{for } x, y \in A
\]

and

\[
[x, y] := x \sharp y - y \sharp x \quad \text{for } x, y \in A.
\]

By Proposition 4.8 in [2], the angle bracket \( \langle \cdot, \cdot \rangle \) defined by (16) and the square bracket \([\cdot, \cdot]\) defined by (17) satisfy the following identity:

\[
\begin{align*}
&\quad [x, \langle y, z \rangle] + [y, \langle z, x \rangle] + [z, \langle x, y \rangle] \\
&= [x, \langle z, y \rangle] + [z, \langle y, x \rangle] + [y, \langle x, z \rangle] + \langle [x, y], z \rangle + \langle [y, z], x \rangle + \langle [z, x], y \rangle,
\end{align*}
\]

where \( x, y, z \in A \). If \( x := \alpha \) and \( y := \beta \) are two elements of the additive halo \( \bar{h}^+(A) \), then (18) becomes

\[
\langle [\alpha, \beta], z \rangle + \langle [\beta, z], \alpha \rangle + \langle \beta, \langle \alpha, z \rangle \rangle = 0
\]

for all \( \alpha, \beta \in \bar{h}^+(A) \) and \( z \in A \). The identity (19) is called the **Hu-Liu identity**.

**Definition 3.1** A vector space \( L \) over a field \( k \) is called a **local Leibniz Algebra** if there exists a subspace \( L_1 \) of \( L \) such that the following three properties hold:

(i) There is a bilinear map \( \langle \cdot, \cdot \rangle : L \times L \to L \) satisfying the Leibniz identity and

\[
\langle L, L_1 \rangle = 0, \quad \langle L_1, L \rangle \subseteq L_1.
\]

(ii) There is a bilinear map \( [\cdot, \cdot] : L_1 \times L_1 \to L_1 \) satisfying the Jacobi identity and

\[
[\alpha, \beta] = -[\beta, \alpha],
\]

where \( \alpha, \beta, \gamma \in L_1 \).

(iii) The angle bracket \( \langle \cdot, \cdot \rangle \) and the square bracket \([\cdot, \cdot]\) satisfy the Hu-Liu identity (19) for all \( \alpha, \beta \in L_1 \) and \( z \in L \).

The subspace \( L_1 \) in Definition 3.1 is called the **local part** of the local Leibniz algebra \( L \).

Clearly, a 7-tuple \( (A, +, \sharp, \cdot, \cdot, \leftarrow, \rightarrow) \) is a trialgebra introduced by Definition 1.12 in [3]. By Proposition 1.11 in [3], \( (A, +, \langle \cdot, \cdot \rangle, [\cdot, \cdot]) \) is a local Leibniz Algebra with the local part \( \bar{h}^+(A) \), where the angle bracket \( \langle \cdot, \cdot \rangle \) and the square bracket \([\cdot, \cdot]\) are defined by (16) and (17), respectively. This fact gives the passage from a 7-tuple to a local Leibniz Algebra.

In the remaining part of this paper, \( k \) will denote the field \( \mathbb{R} \) of real numbers or the field \( \mathbb{C} \) of complex numbers.
Definition 3.2 Let \((A, +, \zeta, \tilde{\cdot}, \tilde{\cdot})\) be a finite dimensional complete 7-tuple with an identity \(1^\times\). If \(G\) is a \((\Delta, \Omega)\)-local monoid of \(A\). The tangent-like space \(T_{\Delta,\Omega}(G)\) to \(G\) is defined by

\[
T_{\Delta,\Omega}(G) := T_{\Omega}(G) + \sum_{* \in \Delta} T_*(G),
\]

where

\[
T_{\Omega}(G) := \{ A'(0) \mid A(t) \text{ is a differentiable curve in } \Omega \text{ with } A(0) = 1^\times \},
\]

\[
T_*(G) := \{ a'(0) \mid a(t) \text{ is a differentiable curve in } G \text{ with } a(0) = e \}
\]

and

\[
T_{\tilde{\cdot}}(G) := \{ a'(0) \mid a(t) \text{ is a differentiable curve in } G \text{ with } a(0) = 1^\times \}.
\]

The next proposition gives the main result of this paper.

Proposition 3.1 Let \((A, +, \zeta, \tilde{\cdot}, \tilde{\cdot})\) be a finite dimensional complete 7-tuple with an identity \(1^\times\). If \(G\) is a \((\Delta, \Omega)\)-local monoid of \(A\), then the tangent-like space \((T_{\Delta,\Omega}(G), +, \langle, \rangle, \lbrack, \rbrack)\) to \(G\) is a real local Leibniz algebra with the local part \(T_{\Omega}(G)\), where the angle bracket \(\langle, \rangle\) and the square bracket \(\lbrack, \rbrack\) are defined by (16) and (17), respectively. Moreover, the local part \(T_{\Omega}(G)\) is a subset of the additive halo \(\bar{\mathfrak{h}}(A)\) of \(A\).

Proof First, using the standard argument in linear Lie groups, we have

\[
(T_{\Omega}(G), +, \lbrack, \rbrack) \text{ is a real Lie algebra}
\]

and

\[
(T_*(G), +, \cdot) \text{ is a real vector space for each } * \in \Delta.
\]

Next, using the argument in the proof of Proposition 4.4 of [2], we have

\[
\langle T_{\Omega}(G), T_{\Omega}(G) \rangle = 0,
\]

\[
\langle T_*(G), T_{\Omega}(G) \rangle = 0 \quad \text{for } * \in \Delta,
\]

\[
(T_{\Omega}(G), T_*(G)) \subseteq T_{\Omega}(G) \quad \text{for } * \in \Delta,
\]

\[
\langle T_{*1}(G), T_{*2}(G) \rangle \leq \sum_{* \in \Delta} T_*(G) \quad \text{for } *_1, *_2 \in \Delta.
\]

By [28] and [24], \(T_{\Delta,\Omega}(G)\) is a real vector space, \(T_{\Omega}(G)\) is a subspace of \(T_{\Delta,\Omega}(G)\), and \(T_{\Omega}(G)\) is closed under the square bracket \(\lbrack, \rbrack\). By [26], [26], [27] and [28], \(T_{\Delta,\Omega}(G)\) is closed under the angle bracket \(\langle, \rangle\) and [20] holds for \(L_1 := T_{\Omega}(G)\). This proves that \(T_{\Delta,\Omega}(G)\) is a real local Leibniz algebra with the local part \(T_{\Omega}(G)\).
Finally, since \( T_{\Omega}(G) \) is a closed real subspace of \( A \) and

\[
\hat{h}^x(A) - \hat{h}^x(A) \subseteq \hat{h}^+(A),
\]
we have that \( T_{\Omega}(G) \) is a real subspace of \( \hat{h}^+(A) \).

Note that if \( G \) is a \((\Delta, \{1^x\})\)-local monoid, then the tangent-like space \( T_{\Delta,\{1^x\}}(G) \) to \( G \) is a Leibniz algebra. Moreover, if \((G, \#)\) is a monoid with a unit \( 1^x \), then \( G \) is a \((\{\#\}, \{1^x\})\)-local monoid and the tangent-like space \( T_{\{\#\},\{1^x\}}(G) = T_{\#}(G) \) is a Leibniz algebra, which is Proposition 4.4 of [2]. Hence, the passage established in Proposition 4.4 of [2] is contained in Proposition 3.1.

The main idea of this paper can be used to construct the analogue of the passage from a linear Lie group to its Lie algebra in the context of Hu-Liu Leibniz algebras, where Hu-Liu Leibniz algebras are more general than local Leibniz algebras and were introduced in Definition 1.15 of [3].

References

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