Functor of extension in Hilbert cube and Hilbert space

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Abstract: It is shown that if $\Omega = \mathbb{Q}$ or $\Omega = \ell_2$, then there exists a functor of extension of maps between $Z$-sets in $\Omega$ to mappings of $\Omega$ into itself. This functor transforms homeomorphisms into homeomorphisms, thus giving a functorial setting to a well-known theorem of Anderson [Anderson R.D., On topological infinite deficiency, Michigan Math. J., 1967, 14, 365–383]. It also preserves convergence of sequences of mappings, both pointwise and uniform on compact sets, and supremum distances as well as uniform continuity, Lipschitz property, nonexpansiveness of maps in appropriate metrics.

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1. Introduction

Anderson in his celebrated paper [2] showed that if $\Omega = \mathbb{Q}$ or $\Omega = \ell_2$, then every homeomorphism between two $Z$-sets in $\Omega$ can be extended to an autohomeomorphism of $\Omega$ (see also [1] or [8]). The theorem on extending homeomorphisms between $Z$-sets was generalized [3, 9] and settled in any manifold modelled on an infinite-dimensional Fréchet space [9] (which is, in fact, homeomorphic to a Hilbert space, see [16, 17]), and is one of the deepest results in infinite-dimensional topology. (For more information on $Z$-sets consult e.g. [8, Chapter V].) The aim of this paper is to strengthen Anderson’s theorem in a functorial manner. To formulate our results, let us fix the notation.

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Notation and terminology

Below, \( \Omega \) continues to denote the Hilbert cube \( Q = [-1, 1]^\omega \) or the Hilbert space \( \ell_2 \), and for metrizable spaces \( X \) and \( Y \) we denote by \( C(X, Y) \) the set of all maps (that is, continuous functions) from \( X \) to \( Y \). The closure operation is marked by an overline; in particular, \( \overline{\text{im}}(\varphi) \) denotes the closure of the image of a function \( \varphi \). For a topological space \( X \), \( Z(X) \) stands for the collection of all \( Z \)-sets in \( X \); that is, \( K \in Z(X) \) if \( K \) is closed and \( C(Q, X \setminus K) \) is dense in \( C(Q, X) \) (in the uniform convergence topology). An embedding \( u \colon (X, d) \to (Y, q) \) between metric spaces is called uniform if both \( u \) and \( u^{-1} \) are uniformly continuous (with respect to the metrics \( d \) and \( q \)). By a compatible metric on a metrizable space we mean any metric which induces the topology of the space. The collection of all compatible bounded metrics on a metrizable space \( X \) is denoted by \( \text{Metr}(X) \). For \( d \in \text{Metr}(Y) \) the supremum metric on \( C(X, Y) \) induced by \( d \) is denoted by \( d_{\text{sup}} \). The category of continuous functions between topological spaces is denoted by \( \text{Top} \). Whenever \( K \) is a class of topological spaces, \( \text{Top}|_K \) denotes the category of (all) maps between members of \( K \) (thus \( \text{Top} \) is the class of all objects in \( \text{Top}|_K \)). The identity map on \( X \) is denoted by \( \text{id}_X \).

Let \( \mathfrak{X} = \text{Top}|_{Z(\Omega)} \) and \( \mathfrak{C} = \text{Top}|_{\epsilon(\Omega)} \). Notice that whenever \( \varepsilon \colon \mathfrak{X} \to \mathfrak{C} \) is a functor, then necessarily \( Z(K) = \Omega \) and \( \varepsilon(K) \) is a map of \( \Omega \) into itself for each \( K \in Z(\Omega) \) and every map \( \varphi \) between two \( Z \)-sets in \( \Omega \). Our main result is

**Theorem 1.1.**
There exists a functor \( \varepsilon \colon \mathfrak{X} \to \mathfrak{C} \) such that for any \( \varphi \in C(K, L) \) with \( K, L \in Z(\Omega) \),

\( \varepsilon(\varphi) \) extends \( \varphi \);

(\( \varepsilon_0 \)) \( \varepsilon(\varphi) \) is an injection (resp. a surjection or an embedding) iff \( \varphi \) is so;

(\( \varepsilon_1 \)) the image of \( \varepsilon(\varphi) \) is closed in \( \Omega \) iff the image of \( \varphi \) is closed in \( L \), and similarly with “dense” in place of “closed”;

(\( \varepsilon_2 \)) for an arbitrary sequence \( \varphi_1, \varphi_2, \ldots \in C(K, L) \), the maps \( \varepsilon(\varphi_n) \) converge to \( \varepsilon(\varphi) \) pointwise (resp. uniformly on compact sets) iff the maps \( \varphi_n \) converge so to \( \varphi \).

Condition (\( \varepsilon_0 \)) of the above result asserts that \( \varepsilon \) (being a functor) extends homeomorphisms to autohomeomorphisms of \( \Omega \). The functor \( \varepsilon \) has also additional properties listed in the following proposition.

**Proposition 1.2.**
Under the notation of Theorem 1.1,

(\( \varepsilon_4 \)) \( \overline{\text{im}}(\varepsilon(\varphi)) \) is homeomorphic to \( \Omega \); and \( \text{im}(\varphi) \) is completely metrizable iff so is \( \text{im}(\varepsilon(\varphi)) \), \( \text{im}(\varepsilon(\varphi)) \) is homeomorphic to \( \Omega \);

(\( \varepsilon_5 \)) \( \overline{\text{im}}(\varepsilon(\varphi)) \) either is a \( Z \)-set in \( \Omega \) or coincides with \( \Omega \);

(\( \varepsilon_6 \)) the image of \( \varepsilon(\varphi) \) and its closure in \( \Omega \) intersect \( L \) along \( \text{im}(\varphi) \) and \( \overline{\text{im}}(\varphi) \), respectively.

Our method also enables extending metrics in a way that the extensor for metrics harmonize with the functor \( \varepsilon \) discussed above, as shown by

**Theorem 1.3.**
Under the notation of Theorem 1.1, to each (complete) metric \( d \in \text{Metr}(L) \) one may assign a (complete) metric \( \varepsilon(d) \in \text{Metr}(\Omega) \) such that:

(\( \varepsilon_7 \)) \( \varepsilon(d) \) extends \( d \);

(\( \varepsilon_8 \)) for any \( K \in Z(\Omega) \) the map \( C(K, L), d_{\text{sup}} \rightarrow K \to \varepsilon(K) \in C(\Omega, \Omega), \varepsilon(d_{\text{sup}}) \) is isometric;

(\( \varepsilon_9 \)) if \( K \in Z(\Omega) \) and \( q \in \text{Metr}(K) \), then a map \( \varphi \in C(K, L) \) is uniformly continuous (resp. a uniform embedding, \( (\text{bi-})\text{Lipschitz}, \text{nonexpansive}, \text{isometric} \)) with respect to \( q \) and \( d \) iff \( \varepsilon(\varphi) \) is so with respect to \( \varepsilon(q) \) and \( \varepsilon(d) \).

Our proofs are surprisingly easy and use simple ideas. However, the main tools of this paper are Anderson’s result (mentioned at the beginning of the section) and the well-known theorem of Keller [11] (for \( \Omega = Q \)) and the result of Bessaga and Pelczyński [7] on spaces of measurable functions (for \( \Omega = \ell_2 \)). What is more, the main proof is
nonconstructive and — in contrast to homeomorphism extension theorems — the extensor will not be continuous in the limitation topologies (see [16] for the definition) in case $\Omega = \ell_2$.

In the next section we shall prove the above results in case of $\Omega = Q$, while Section 3 deals with the Hilbert space settings.

## 2. Hilbert cube

Throughout this section we assume that $\Omega = Q$. In this case, the main tool to build the functor $E$ will be the well-known theorem of Keller [11] (the proof may also be found in [8, Chapter III, §3]).

**Theorem 2.1.**
Every compact, metrizable, infinite-dimensional convex subset of a locally convex topological vector space is homeomorphic to $Q$.

We shall build the functor $E$ using two additional functors, denoted by $J$ and $M$. Below we will write $A \oplus B$ for the topological disjoint union (or the direct sum) of topological spaces $A$ and $B$. For simplicity, we shall write $A \cong B$ if $A$ and $B$ are homeomorphic, and we follow the convention that $A \subset A \oplus B$.

In order to define $J$, let us fix a homeomorphic copy $\Omega'$ of $Q$ with a metric $d' \in \operatorname{Metr}(\Omega')$ such that $\operatorname{diam}(\Omega', d') = 1$. For any $K, L \in \mathcal{Z}(Q)$, $d \in \operatorname{Metr}(L)$ and $\varphi \in C(K, L)$ let $J(K) = K \oplus \Omega'$, and let $J(d) \in \operatorname{Metr}(J(L))$ and $J(\varphi) \in C(J(K), J(L))$ be defined as follows: $J(d)$ coincides with $d$ on $K \times K$, with $d'$ on $\Omega' \times \Omega'$ and $J(d)(x, y) = \max(\operatorname{diam}(K, d), 1)$ if one of $x$ and $y$ belongs to $K$ and the other to $\Omega'$; $J(\varphi)(x) = \varphi(x)$ for $x \in K$ and $J(\varphi)(x) = x$ for $x \in \Omega'$. Notice that

$$J(K) \text{ is a compact metrizable space having infinitely many points} \quad (1)$$

(this fact shall be used later), $K$ is a closed subset of $J(K)$, and $J(\varphi)$ and $J(d)$ extend $\varphi$ and $d$, respectively. Now we turn to the definition of the functor $M$.

Fix a compact metrizable space $K$. Let $\mathcal{M}(K)$ be the set of all probabilistic Borel measures on $K$ equipped with the standard weak topology (inherited, thanks to the Riesz characterization theorem, from the weak-* topology of the dual Banach space of $C(K, \mathbb{R})$), i.e. the topology with the basis consisting of finite intersections of sets of the form

$$B(\mu, f, \varepsilon) = \left\{ \lambda \in \mathcal{M}(K) : \left| \int_K f \, d\mu - \int_K f \, d\lambda \right| < \varepsilon \right\},$$

where $\mu \in \mathcal{M}(K)$, $f \in C(K, \mathbb{R})$ and $\varepsilon > 0$. The space $\mathcal{M}(K)$ is compact, convex and metrizable. What is more, $\mathcal{M}(K)$ is infinite-dimensional provided $K$ is an infinite set, and hence, by Theorem 2.1,

$$\operatorname{card}(K) \geq \aleph_0 \quad \Rightarrow \quad \mathcal{M}(K) \cong Q. \quad (2)$$

For $a \in K$, let $\epsilon_a \in \mathcal{M}(K)$ denote the Dirac measure at $a$; that is, $\epsilon_a$ is the probabilistic measure on $K$ such that $\epsilon_a(\{a\}) = 1$. It is clear that the map $\gamma_K : K \ni a \mapsto \epsilon_a \in \operatorname{im}(\gamma_K) \subset \mathcal{M}(K)$ is a homeomorphism. What is more,

$$\operatorname{card}(K) > 1 \quad \Rightarrow \quad \operatorname{im}(\gamma_K) \in \mathcal{Z}(\mathcal{M}(K)). \quad (3)$$

Although (3) is elementary and simple, we will see in the sequel that it is a crucial property. Further, for a metric $d \in \operatorname{Metr}(K)$ let $M(d) : \mathcal{M}(K) \times \mathcal{M}(K) \to \mathbb{R}_+$ (where $\mathbb{R}_+ = [0, \infty)$) be defined by

$$M(d)(\mu, \nu) = \sup \left\{ \left| \int_K f \, d\mu - \int_K f \, d\nu \right| : f \in \operatorname{Contr}(K, \mathbb{R}) \right\},$$

where $\operatorname{Contr}(K, \mathbb{R})$ is the set of all continuous real-valued functions on $K$. What is more, $M(d)$ is a pseudo-metric on $\mathcal{M}(K)$, and $M(d)$ is a set of all continuous real-valued functions on $K$. What is more, $M(d)$ is a pseudo-metric on $\mathcal{M}(K)$, and $M(d)$ is a set of all continuous real-valued functions on $K$.
where \( \text{Contr}(K, \mathbb{R}) \) stands for the family of all \( d \)-nonexpansive maps of \( K \) into \( \mathbb{R} \). Then \( \mathcal{M}(d) \in \text{Metr}(\mathcal{M}(K)) \) (the metric \( \mathcal{M}(d) \) was rediscovered by many mathematicians, e.g., by Kantorovich, Monge, Rubinstein, Wasserstein; in Fractal Geometry it is known as the Hutchinson metric). Observe that

\[
\mathcal{M}(d)(\epsilon_a, \epsilon_b) = d(a, b), \quad a, b \in K. \tag{4}
\]

Finally, for a map \( \varphi: K \to L \) between compact metrizable spaces, we define \( \mathcal{M}(\varphi): \mathcal{M}(K) \to \mathcal{M}(L) \) by the formula

\[
(\mathcal{M}(\varphi)(\mu))(B) = \mu(\varphi^{-1}(B)), \quad \mu \in \mathcal{M}(K), \quad B \subset L \text{ is Borel}.
\]

Thus \( \mathcal{M}(\varphi)(\mu) \) is the transport of the measure \( \mu \) by the map \( \varphi \). Observe that if \( \lambda = \mathcal{M}(\varphi)(\mu) \), then for each \( g \in C(L, \mathbb{R}) \),

\[
\int_K g \, d\lambda = \int_K g \circ \varphi \, d\mu. \quad \text{This implies that } \mathcal{M}(\varphi) \in \mathcal{C}(\mathcal{M}(K), \mathcal{M}(L)).
\]

Moreover, \( \mathcal{M}(\varphi) \) is affine (so its image is a compact convex set) and

\[
\mathcal{M}(\varphi)(\gamma(\alpha)) = \gamma(\varphi(\alpha)), \quad \alpha \in K. \tag{5}
\]

It is easy to check that both \( J \) and \( \mathcal{M} \) are functors. Now define a functor \( \mathcal{L} \) as their composition; that is, \( \mathcal{L} = \mathcal{M} \circ J \). For transparency, for each \( K \in \mathcal{Z}(Q) \) let \( \delta_K: K \to \gamma_{J(K)}(K) \subset \mathcal{L}(K) \) be a map obtained by restricting \( \gamma_{J(K)} \). Below we collect most important properties of the functor \( \mathcal{L} \).

**Lemma 2.2.**

Under the notation introduced above, for arbitrary two \( Z \)-sets \( K \) and \( L \) in \( Q \), a map \( \varphi: K \to L \) and compatible metrics \( d \) and \( q \) on \( K \) and \( L \), respectively, the following conditions hold:

\( (L1) \) \( \mathcal{L}(K) \equiv Q \);

\( (L2) \) \( \delta_K \) is a homeomorphism and \( \text{im}(\delta_K) \subset \mathcal{Z}(\mathcal{L}(K)) \);

\( (L3) \) \( \mathcal{L}(\varphi)(\delta_K(x)) = \delta_L(\varphi(x)) \) for each \( x \in K \);

\( (L4) \) \( \mathcal{L}(d)(\delta_K(x), \delta_K(y)) = d_L(x, y) \) for any \( x, y \in K \);

\( (L5) \) the function \( C(K, L, \psi_{sup}) \ni \xi \mapsto \mathcal{L}(\xi) \in C(\mathcal{L}(K), \mathcal{L}(L)), \mathcal{L}(\psi_{sup}) \) is isometric;

\( (L6) \) the assignment \( C(K, L, \psi_{def}) \ni \xi \mapsto \mathcal{L}(\xi) \in C(\mathcal{L}(K), \mathcal{L}(L)) \) preserves pointwise convergence of sequences;

\( (L7) \) \( \mathcal{L}(\varphi) \) is \( (bi-) \)Lipschitz (resp. nonexpansive, isometric) with respect to \( \mathcal{L}(d) \) and \( \mathcal{L}(q) \) iff such is \( \varphi \) with respect to \( d \) and \( q \);

\( (L8) \) \( \text{im}(\mathcal{L}(\varphi)) = \{ \mu \in \mathcal{L}(L) : \mu(\text{im}(\varphi) \oplus \Omega') = 1 \} \);

\( (L9) \) \( \mathcal{L}(\varphi) \) is injective iff \( \varphi \) is such.

**Proof.** \( (L1) \) follows from (1) and (2), while \( (L2) \) is a consequence of (3). Further, \( (L3) \) and \( (L4) \) are implied by (5) and (4), respectively. Let us briefly show conditions \( (L5) – (L8) \) (point \( (L9) \) is left to the reader). \( (L6) \) follows from the definition of the topology of \( \mathcal{L}(L) \) and Lebesgue’s dominated convergence theorem, while \( (L8) \) is a consequence of the Krein–Milman theorem: \( \text{im}(\mathcal{L}(\varphi)) \) is a convex compact subset of \( \mathcal{L}(\text{im}(\varphi)) \) (if we naturally identify the latter set with the set of all measures on \( J(L) \) which are supported on \( J(\text{im}(\varphi)) \)) and contains all Dirac’s measures concentrated on points of \( J(\text{im}(\varphi)) \), which are precisely the extreme points of \( \mathcal{L}(\text{im}(\varphi)) \). In order to check \( (L5) \), take \( \varphi, \psi \in C(K, L) \) and \( \mu \in \mathcal{L}(K) \), and put \( \mu_\varphi = \mathcal{L}(\varphi)(\mu) \) and \( \mu_\psi = \mathcal{L}(\psi)(\mu) \). Note that

\[
\mathcal{L}(\psi)(\mathcal{L}(\varphi)(\mu), \mathcal{L}(\psi)(\mu)) = \sup \left\{ \left| \int_{J(K)} f \, d\mu_\psi - \int_{J(K)} f \, d\mu_\varphi \right| : f \in \text{Contr}(J(L), \mathbb{R}) \right\}
\]

\[
= \sup \left\{ \left| \int_{J(K)} f \circ \mathcal{L}(\varphi) \, d\mu - \int_{J(K)} f \circ \mathcal{L}(\psi) \, d\mu \right| : f \in \text{Contr}(J(L), \mathbb{R}) \right\}
\]

\[
\leq \sup \left\{ \int_{J(K)} |f \circ \mathcal{L}(\varphi) - f \circ \mathcal{L}(\psi)| \, d\mu : f \in \text{Contr}(J(L), \mathbb{R}) \right\}
\]

\[
\leq \int_{J(K)} q\left( \mathcal{L}(\varphi)(x), \mathcal{L}(\psi)(x) \right) \, d\mu(x) \leq q_{sup}(\varphi, \psi).
\]
This gives \( L(q)_{\sup}(\mathcal{L}(\varphi), \mathcal{L}(\psi)) \leq q_{\sup}(\varphi, \psi) \). Since the reverse inequality is immediate (thanks to (L3) and (L4), we see that (L5) is fulfilled.

It remains to show (L7). This property is actually well known in Fractal Geometry, but for the reader’s convenience, we shall prove it here. Assume that for some \( L \in [1, \infty] \) and any \( x, y \in K \), \( g(\varphi(x), \varphi(y)) \leq Ld(x, y) \). Then also \( \mathcal{I}(g)[\mathcal{J}(\varphi(x), \mathcal{J}(\varphi)(y))] \leq Ld(x, y) \) for any \( x, y \in \mathcal{J}(K) \). We conclude that \((1/L)f \circ \mathcal{J}(\varphi) \in \text{Contr}(\mathcal{J}(K), \mathbb{R})\) for \( f \in \text{Contr}(\mathcal{J}(L), \mathbb{R}) \). This simply yields that \( \mathcal{L}(\varphi) \) satisfies Lipschitz condition with constant \( L \). Similarly, if \( g(\varphi(x), \varphi(y)) \geq d(x, y)/L \) for any \( x, y \in K \) (where still \( L \geq 1 \)), then \( \mathcal{I}(g)[\mathcal{J}(\varphi)(x), \mathcal{J}(\varphi)(y)] \geq d(x, y)/L \) for all \( x, y \in \mathcal{J}(K) \) and therefore for any \( f \in \text{Contr}(\mathcal{J}(K), \mathbb{R}) \) the function

\[
\text{im}(\mathcal{J}(\varphi)) \ni y \mapsto \frac{1}{L} f(\mathcal{J}(\varphi)^{-1}(y)) \in \mathbb{R}
\]

(is well defined and) extends to a function \( g \in \text{Contr}(\mathcal{J}(L), \mathbb{R}) \). This implies that every \( f \in \text{Contr}(\mathcal{J}(K), \mathbb{R}) \) may be written in the form \( f = L \cdot g \circ \mathcal{J}(\varphi) \) with \( g \in \text{Contr}(\mathcal{J}(L), \mathbb{R}) \) chosen appropriately. Hence, for any \( \mu_1, \mu_2 \in \mathcal{L}(K) \), putting \( v_j = \mathcal{L}(g)(\mu_j), \quad j = 1, 2 \), we obtain

\[
\mathcal{L}(g)\{\mathcal{L}(\varphi)(\mu_1), \mathcal{L}(\varphi)(\mu_2)\} = \sup \left\{ \left| \int_{\mathcal{J}(K)} g \, dv_1 - \int_{\mathcal{J}(K)} g \, dv_2 \right| : g \in \text{Contr}(\mathcal{J}(L), \mathbb{R}) \right\}
\]

\[
= \sup \left\{ \left| \int_{\mathcal{J}(K)} g \circ \mathcal{J}(\varphi) \, d\mu_1 - \int_{\mathcal{J}(K)} g \circ \mathcal{J}(\varphi) \, d\mu_2 \right| : g \in \text{Contr}(\mathcal{J}(L), \mathbb{R}) \right\}
\]

\[
\geq \frac{1}{L} \sup \left\{ \left| \int_{\mathcal{J}(K)} f \, d\mu_1 - \int_{\mathcal{J}(K)} f \, d\mu_2 \right| : f \in \text{Contr}(\mathcal{J}(K), \mathbb{R}) \right\} = \frac{1}{L} \mathcal{L}(d)(\mu_1, \mu_2)
\]

and we are done.

Now we are ready to give

**Proof of Theorem 1.1 for \( \Omega = Q \).** We continue the notation of the section. By (L2) and Anderson’s theorem [2], for every \( K \in \mathcal{Z}(Q) \) there exists a homeomorphism \( H_K : Q \to \mathcal{L}(K) \) which extends \( \delta_K \). We define \( \mathcal{E} \) by: \( \mathcal{E}(K) = Q \) for \( K \in \mathcal{Z}(Q) \) and

\[
\mathcal{E}(\varphi) = H_k^{-1} \circ \mathcal{L}(\varphi) \circ H_k, \quad \varphi \in \mathcal{C}(K, L), \quad K, L \in \mathcal{Z}(Q).
\]

It is readily seen that \( \mathcal{E} \) is a functor (since \( \mathcal{L} \) is). Let us check (E0). If \( \varphi \in \mathcal{C}(K, L) \) and \( x \in K \), then \( H_k(x) = \delta_K(x) \) and thus \( \mathcal{E}(\varphi)(x) = \varphi(x) \) by (E3). Finally, observe that conditions (E1) and (E2) immediately follow from (L8)–(L9) and (L5)–(L6), respectively, while (E2) is trivial in the compact case.

**Proof of Proposition 1.2 for \( \Omega = Q \).** Point (E4) follows from (L8) and Keller’s theorem (Theorem 2.1) and both (E5) and (E6) are consequences of (L8). (Indeed: if \( X \) is a proper closed subset of a compact metrizable space \( Y \), then \( \{ \mu \in \mathcal{M}(Y) : \mu(X) = 1 \} \) is a \( Z \)-set in \( \mathcal{M}(Y) \); and if \( \mathcal{E}(\varphi)(x) = y \in L \) for some \( x \in Q \), then \( \mathcal{L}(\varphi)(\mu) = \epsilon_y \) for \( \mu = H_k(x) \), which yields \( \mu(\mathcal{E}(\varphi)^{-1}(y)) = 1 \) and therefore \( y \in \text{im}(\varphi) \)).

**Proof of Theorem 1.3 for \( \Omega = Q \).** For \( d \in \text{Metr}(K) \) where \( K \in \mathcal{Z}(Q) \) and \( x, y \in Q \) we put \( \mathcal{E}(d)(x, y) = \mathcal{L}(d)(H_k(x), H_k(y)) \). Since \( H_k \) is a homeomorphism between \( Q \) and \( \mathcal{L}(K) \) and

\[
H_k \quad \text{is an isometry of} \quad (Q, \mathcal{E}(d)) \quad \text{onto} \quad (\mathcal{L}(K), \mathcal{L}(d)).
\]

we conclude that \( \mathcal{E}(d) \in \text{Metr}(Q) \). Moreover, (L4) implies (L7). Finally, (E8) and (E9) follow from (6) and, respectively, (L5) and (L7).

Recall that \( \text{Auth}(X) \) is the group of all autohomeomorphisms of a topological space \( X \). Theorem 1.1 has the following consequence.
**Corollary 2.3.** Let $K$ be a $Z$-set in $Q$. Let $\text{Auth}(Q, K) = \{ h \in \text{Auth}(Q) : h(K) = K \}$ and $\text{Auth}_0(Q, K) = \{ h \in \text{Auth}(Q, K) : h(x) = x \text{ for } x \in K \}$ be equipped with the topology of uniform convergence. Then there is a closed subgroup $\mathcal{G}$ of $\text{Auth}(Q, K)$ such that the map

$$\Phi : \text{Auth}_0(Q, K) \times \mathcal{G} \ni (u, v) \mapsto u \circ v \in \text{Auth}(Q, K)$$

is a homeomorphism.

**Proof.** It is enough to put $\mathcal{G} = \{ \varepsilon(h) : h \in \text{Auth}(K) \}$. Since the map $\Psi : \text{Auth}(K) \ni h \mapsto \varepsilon(h) \in \text{Auth}(Q)$ is an embedding and a group homomorphism and both $\text{Auth}(K)$ and $\text{Auth}(Q)$ are completely metrizable, therefore $\mathcal{G}$ is closed (see e.g. [12]). Now it remains to notice that

$$\Phi^{-1}(h) = \{ h \circ [\Psi(h)_k]^{-1}, \Psi(h)_k \}.$$

It is worth mentioning that the functor $\mathcal{M}$ introduced above in its full generality was investigated by Banakh in [5, 6].

### 3. Hilbert space

In this section we assume that $\Omega = \ell_2$. The proof in that case goes similarly. The main difference is that we shall change the functor $\mathcal{M}$. (Actually, the same functor $\mathcal{M}$ as used in Section 2, combined with the functor $\mathcal{I}$ described below, leads us, in the same way as before, to the functor of extension. However, the author is unable to resolve whether in this way one obtains a functor with all desired properties.) Moreover, the lack of compactness of the space $\Omega$ makes the details more complicated. Instead of Keller’s theorem, which was used in the previous part, here we need a theorem of Bessaga and Pełczyński [7]. In order to state their result, we have to describe spaces of measurable functions.

Let $X$ be a separable nonempty metrizable space and let $\mathcal{M}(X)$ be the space of all Lebesgue measurable functions of $[0, 1]$ into $X$ up to almost everywhere equality. For $d \in \text{Metr}(X)$ the function

$$\mathcal{M}(d) : \mathcal{M}(X) \times \mathcal{M}(X) \ni (f, g) \mapsto \int_0^1 d(f(t), g(t)) \, dt \in \mathbb{R}_+$$

is a bounded metric on $\mathcal{M}(X)$, $(\mathcal{M}(X), \mathcal{M}(d))$ is separable; and $\mathcal{M}(d)$ is complete iff $d$ is so. The topology on $\mathcal{M}(X)$ induced by $\mathcal{M}(d)$ is independent of the choice of $d \in \text{Metr}(X)$, and functions $f_1, f_2, \ldots \in \mathcal{M}(X)$ converge to $f \in \mathcal{M}(X)$ iff they converge to $f$ in measure in the sense e.g. of Halmos (see [10, § 2]). Equivalently,

$$\lim_{n \to \infty} f_n = f \iff \text{every subsequence of } (f_n)_{n=1}^\infty \text{ has a subsequence converging to } f \text{ pointwise almost everywhere.} \quad (7)$$

For us the most important property of $\mathcal{M}(X)$ is the following theorem of Bessaga and Pełczyński [7] (see also [8, Theorem VI.7.1]; for generalizations consult [14]).

**Theorem 3.1.** If $X$ is a separable metrizable space, then the space $\mathcal{M}(X)$ is homeomorphic to $\ell_2$ iff $X$ is completely metrizable and has more than one point.

Fix for a moment a separable metrizable space $X$. For $x \in X$ let $\varepsilon_x = x$. Put $\gamma_X : X \ni x \mapsto \varepsilon_x \in \text{im}(\gamma_X) \subset \mathcal{M}(X)$. Clearly, $\gamma_X$ is a homeomorphism. What is more,

$$\text{card}(X) > 1 \implies \text{im}(\gamma_X) \in \mathcal{Z}(\mathcal{M}(X)). \quad (8)$$
As in Section 2, observe that
\[ M(d)(\epsilon, \epsilon) = d(x, y), \quad x, y \in X, \quad d \in \text{Metr}(X). \tag{9} \]

If \( A \) is a subset of \( X \), \( M(A) \) naturally embeds in \( M(X) \) and therefore we shall consider \( M(A) \) as a subset of \( M(X) \). Under such an agreement one has \( M(A) = M(A) \).

Now let \( Y \) be another separable metrizable space and \( f \in C(X, Y) \). We define \( M(f): M(X) \to M(Y) \) by the formula \( (M(f))(u) = f \circ u \). It is easy to verify that \( M(f) \in C(M(X), M(Y)) \) and \( M \) is a functor in the category of separable metrizable spaces. We collect further properties of the above functor \( M \) in the following lemma.

**Lemma 3.2.**

Let \( X \) and \( Y \) be separable metrizable spaces, \( f: X \to Y \) be a map and let \( d \) and \( q \) be any bounded compatible metrics on \( X \) and \( Y \), respectively.

(M1) If \( A \) is a closed *proper* subset of \( X \), then \( M(A) \) is a \( Z \)-set in \( M(X) \).

(M2) \( M(f)(y)(x) = y(f(x)) \) for any \( x \in X \).

(M3) \( M(f) \) is an injection or an embedding iff so is \( f \).

(M4) If \( f_1, f_2, f_3, \ldots \in C(X, Y) \) converge pointwise or uniformly on compact sets to \( f \), then \( M(f_n) \) converge so to \( M(f) \).

(M5) The map \( (C(X), q_{uw}) \ni u \to M(u) \in (C(M(X), M(Y)), M(q_{uw})) \) is isometric.

(M6) \( M(f) \) is uniformly continuous (resp. a uniform embedding, (bi-)Lipschitz, nonexpansive, isometric) with respect to \( M(d) \) and \( M(q) \) iff so is \( f \) with respect to \( d \) and \( q \).

**Proof.** To see (M1), take \( a \in X \setminus A \) and observe that the maps
\[ \Phi_a : M(X) \ni f \mapsto \epsilon_a \left| f_{[0,1/a]} \cup f_{[1/a,1]} \right| \in M(X) \]
converge uniformly on compact subsets of \( M(X) \) to \( id_{M(X)} \) and their images are disjoint from \( M(A) \) (and, of course, \( M(A) \) is closed in \( M(X) \)).

Items (M2), (M5) and first claims of (M3) and (M4) are quite easy and we leave them to the reader. Also the part of (M6) concerning (bi-)Lipschitz, nonexpansive and isometric maps is immediate. Let us show the second claim of (M4). Below we involve criterion (7). Assume \( f_1, f_2, f_3, \ldots \in C(X, Y) \) converge uniformly on compact sets to \( f \in C(X, Y) \). Let \( (u_n)_{n=1}^\infty \) be a sequence of elements of \( M(X) \) which is convergent to \( u \in M(X) \). We have to prove that \( (M(f_n))(u_n)_{n=1}^\infty \) converges to \( M(f)(u) \). For an arbitrary subsequence \( (u_{n_k})_{k=1}^\infty \) take its subsequence \( (u_{n_{k_l}})_{l=1}^\infty \) such that the set \( T = \{ t \in [0,1]: u_{n_k}(t) \to u(t) \} \) has Lebesgue measure equal to 1. Observe that \( f_{n_k}(u_{n_k}(t)) \to f(u(t)) \) for \( t \in T \). But this means that \( (M(f_n))(u_{n_k})_{k=1}^\infty \) tends to \( M(f)(u) \) in the topology of \( M(Y) \) and we are done. In a similar manner one checks that \( M(f) \) is an embedding provided \( f \) is so.

Now assume \( f \) is uniformly continuous with respect to \( d \) and \( q \). Since \( q \) is bounded, we conclude that there exists a bounded continuous monotone concave function \( \omega: \mathbb{R}_+ \to \mathbb{R}_+ \) vanishing at 0 such that
\[ q(f(x), f(y)) \leq \omega(d(x, y)), \quad x, y \in X \tag{10} \]
(see e.g. [4]). Now Jensen’s inequality (applied for a convex function \( t \mapsto M - \omega(t) \) where \( M \) is an upper bound of \( \omega \)) combined with (10) yields that, for any \( u, v \in M(X) \),
\[ M(q)(M(f)(u), M(f)(v)) = \int_0^1 q(f(u(t)), f(v(t))) dt \leq \int_0^1 \omega(d(u(t), v(t))) dt \leq \omega \left( \int_0^1 d(u(t), v(t)) dt \right) = \omega(M(d)(u, v)) \]
and thus \( M(f) \) is uniformly continuous. Conversely, if \( M(f) \) is uniformly continuous, then \( f \) is so, by (M2) and (9).

Finally, if \( f \) is a uniform embedding, we may repeat the above argument, starting from a bounded continuous monotone concave function \( \tau: \mathbb{R}_+ \to \mathbb{R}_+ \) vanishing at 0 such that \( d(x, y) \leq \tau(q(f(x), f(y))) \) for any \( x, y \in X \), and finishing with \( M(d)(u, v) \leq \tau(M(q)(M(f)(u), M(f)(v))) \) for all \( u, v \in M(X) \).
Our last step is to prove that \( \text{im}(\mathcal{M}(f)) = \mathcal{M}(\text{im}(f)) \). It is however not as simple as it looks. To show this, we shall apply two theorems of the descriptive set theory and we have to introduce the terminology.

A **Souslin space** is the empty topological space or a metrizable space which is a continuous image of the space \( \mathbb{R} \setminus \mathbb{Q} \). We shall need the following three properties of Souslin spaces (for proofs and more information see e.g. [13, Chapter XIII] or [15, Appendix]):

(S1) Every Souslin subspace of a Borel subset of a separable completely metrizable space is a Souslin space;

(S2) The inverse image of a Souslin space under a Borel function (between Borel subsets of separable completely metrizable spaces) is Souslin as well;

(S3) Every Souslin subspace of the interval \([0,1]\) is Lebesgue measurable ([13, Theorem XIII.4.1] or [15, Theorem A.13]).

The main tool used in the next result is the following theorem.

**Theorem 3.3.**

Let \( Y \neq \emptyset \) be a separable completely metrizable space; let \( X \neq \emptyset \) be any set and let \( \mathcal{R} \) be a \( \sigma \)-algebra of subsets of \( X \).

If a function \( F : X \to 2^Y \) satisfies the following two conditions:

(i) \( F(x) \) is a nonempty and closed subset of \( Y \) for any \( x \in X \),

(ii) \( \{ x \in X : F(x) \cap U \neq \emptyset \} \in \mathcal{R} \) for any open subset \( U \) of the space \( Y \),

then there exists a function \( f : X \to \mathcal{R} \) such that \( f(x) \in F(x) \) for every \( x \in X \) and \( f \) is \( \mathcal{R} \)-measurable, that is, \( f^{-1}(U) \in \mathcal{R} \) for all open sets \( U \subseteq Y \).

For a proof and a discussion, consult [13, Theorem XIV.1.1].

**Proposition 3.4.**

If \( X \) and \( Y \) are two separable metrizable spaces and \( f \in C(X, Y) \), then \( \text{im}(\mathcal{M}(f)) = \mathcal{M}(\text{im}(f)) \), provided \( X \) is completely metrizable.

**Proof.** The inclusion "\( \subseteq \)" easily follows from the relation \( \text{im}(\mathcal{M}(f)(u)) \subseteq \text{im}(f) \). To prove the reverse one, take a Borel function \( v : [0,1] \to Y \) such that \( v([0,1]) \subseteq \text{im}(f) \). Let \( \mathcal{L} \) denote the \( \sigma \)-algebra of all Lebesgue measurable subsets of \([0,1]\). Define \( F : [0,1] \to 2^Y \) by the formula \( F(t) = F^{-1}(\{v(t)\}) \). Clearly, \( F(t) \) is nonempty and closed in \( X \) for any \( t \in [0,1] \). What is more, if \( U \) is open in \( X \), then, by \((S1)\), \( I(U) \) is a Souslin space and hence, by \((S2)\), so is the set \( v^{-1}(f(U)) \) and therefore it is Lebesgue measurable (by \((S3)\)). But \( v^{-1}(f(U)) = \{ t \in [0,1] : F(t) \cap U \neq \emptyset \} \subseteq \mathcal{L} \). Now Theorem 3.3 gives us a Lebesgue measurable function \( u : [0,1] \to X \) such that \( u(t) \in F(t) \) for any \( t \in [0,1] \). This means that \( u \in M(X) \) and \((M(f))(u) = v \).  

**Proof of Theorem 1.1 for \( \Omega = \ell_2 \).** As in Section 2, we fix a homeomorphic copy \( \Omega' \) of \( \ell_2 \) and a complete metric \( d' \in \text{Metr}(\Omega') \) such that \( \text{diam}(\Omega', d') = 1 \). Now let \( J \) be a functor built in the same way as in the previous part of the paper:

- \( J \) assigns to each \( Z \)-set \( K \) in \( \ell_2 \) the space \( K \oplus J \ell_2 \);
- for \( K, L \in Z(\ell_2) \) and a map \( f : K \to L, fL \in C \mathcal{J}(K) \), \( fL \) coincides with \( f \) on \( K \) and with \( \text{id} \mathcal{J} \) on \( \Omega' \);
- for \( K \in Z(\ell_2) \) and \( d \in \text{Metr}(K) \), \( J(d) \in \text{Metr}(\mathcal{J}(K)) \) coincides with \( d \) on \( K \times K \), with \( d' \) on \( \Omega' \times \Omega' \) and \( J(d)(x, y) = \max \{ \text{diam}(K, d), 1 \} \) otherwise.

Now we mimic the proof of the theorem for \( \Omega = \mathbb{Q} \). We define a functor \( \mathcal{L} \) by \( \mathcal{L} = \mathcal{M} \circ J \), and for any \( K \in Z(\ell_2) \) denote by \( \delta_K : K \to \mathcal{Y}(K) \) the restriction of \( \mathcal{Y}(K) \) and take a homeomorphism \( H_K : \ell_2 \to \mathcal{L}(K) \) which extends \( \delta_K \), based on Theorem 3.1 and (8). Finally, for \( \varphi \in C(K, L) \) (with \( K, L \in Z(\ell_2) \)) we put \( \mathcal{E}(\varphi) = H_K^{-1} \circ \mathcal{L}(\varphi) \circ H_K \). Note that, by Proposition 3.4,

\[
\text{im}(\mathcal{M}(\varphi)) = \mathcal{M}(\text{im}(\varphi) \oplus \Omega').
\]

Now in the same way as in Section 2 one checks \((E0), (E1)-(E2)\) (use \((M3)\) and \((11)\)) and \((E3)\) (apply \((M4)\)). The details are left to the reader. 

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Proof of Proposition 1.2 for \( \Omega = \ell_2 \). It is readily seen that (E4) follows from Theorem 3.1 and (11), while (E5) from (M1) and (11). Finally, (E6) may briefly be deduced from (11) and the formula for \( \mathcal{E}(\varphi) \) (cf. the proof of the proposition for \( \Omega = Q \)).

Proof of Theorem 1.3 for \( \Omega = \ell_2 \). As in Section 2, for \( d \in \text{Met}(K) \) (where \( K \in \mathcal{Z}(\ell_2) \)) define \( \mathcal{E}(d) \in \text{Met}(\ell_2) \) by \( \mathcal{E}(d)(x, y) = L(d)(H_k(x), H_k(y)) \). Now it suffices to repeat the proof from the previous case, involving (M5) and (M6).

We end the paper with a note that we do not know if there exists an analogous functor of extension of mappings between \( Z \)-sets of \( \ell_2 \) which is continuous in the limitation topologies.

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