SCALAR CURVATURE RIGIDITY OF ALMOST HERMITIAN MANIFOLDS WHICH ARE ASYMPTOTIC TO $\mathbb{C}H^{2n}$

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Abstract. We show that an almost Hermitian manifold $(M,g)$ of real dimension $4n$ which is strongly asymptotic to $\mathbb{C}H^{2n}$ and satisfies a certain scalar curvature bound must be isometric to the complex hyperbolic space. Assuming Kähler instead of almost Hermitian this gives the already known rigidity result by H. Boualem and M. Herzlich proved in Ann. Scuola Norm. Sup Pisa (Ser. V), vol. 1(2).

1. Introduction

Scalar curvature rigidity of hyperbolic spaces is a frequently studied problem (cf. [11, 11, 11, 3, 10, 9]). M. Herzlich showed in [6] that a strongly asymptotically complex hyperbolic Kähler spin manifold $(M^{2m},g)$ of odd complex dimension $m$ and with scalar curvature $\text{scal} \geq -4m(m+1)$ must be isometric to the complex hyperbolic space $\mathbb{C}H^m$. In [3] H. Boualem and M. Herzlich gave the corresponding result in the even complex dimensional case, but because of a different representation theory, the spin assumption has to be replaced by another topological condition. In [9] we generalized the odd complex dimensional case in the way that we only assumed almost Hermitian instead of Kähler. In particular, we proved that a complete almost Hermitian spin manifold $(M,g,J)$ of odd complex dimension $m$ which is strongly asymptotically complex hyperbolic and satisfies the scalar curvature bound

$$\text{scal} \geq -4m(m+1) + 6\sqrt{2m}|\nabla J|$$

must be Kähler and isometric to the complex hyperbolic space. In this paper we consider the case of even complex dimension $m$. In order to do so, the spin assumption is replaced by the existence of an appropriate complex line bundle which is the associated line bundle of a chosen spin$^c$ structure. Since an almost Hermitian manifold $(M,g,J)$ is already spin$^c$ (cf. [3], App. D)), there is no need for additional topological assumptions on $M$.

Definition 1. $(\mathbb{C}H^m,g_0)$ denotes the complex hyperbolic space of complex dimension $m$ and holomorphic sectional curvature $-4$, i.e. $K \in [-4,-1]$.

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as well as $B_R(q) \subset M$ is the set of all $p \in M$ with geodesic distance to $q$ less than $R$. Let $(M^{2m}, g, J)$ be an almost Hermitian manifold, i.e. $g$ is a Riemannian metric and $J$ is a $g$–compatible almost complex structure. $(M, g, J)$ is said to be strongly asymptotically complex hyperbolic if there is a compact manifold $C \subset M$ and a diffeomorphism $f : E := M - C \to \mathbb{C}H^m - B_R(0)$ in such a way that the positive definite gauge transformation $A \in \Gamma(\text{End}(TM_E))$ given by

$$g(AX, AY) = (f^*g_0)(X, Y) \quad g(AX, Y) = g(X, AY)$$

satisfies:

1. $A$ is uniformly bounded.
2. Suppose $r$ is the $f^*g_0$–distance to a fixed point, $\nabla^0$ is the Levi–Civita connection for $f^*g_0$ and $J_0$ is the complex structure of $\mathbb{C}H^m$ pulled back to $E$, then (for some $\epsilon > 0$)

$$|\nabla^0 A| + |A - \text{Id}| + |J_0 - J| \in O(e^{-(m+1+\epsilon)r}).$$

**Theorem 1.** Let $(M, g, J)$ be a complete almost Hermitian manifold of even complex dimension $m = 2n$ which is strongly asymptotically complex hyperbolic. Suppose $\frac{\omega}{2\pi} \in \Gamma(\Lambda^1, M)$ is a closed 2–form representing the real Chern class of a complex line bundle $\lambda$ which defines a spin$^c$ structure on $M$. If

$$2\Omega + \omega \in O(e^{-(m+1+\epsilon)r})$$

$\{\Omega = g(., J.)\}$ and the scalar curvature satisfies

$$\text{scal} \geq -4m(m+1) + c_1|d^*\Omega| + c_2\left(|D^\prime\Omega| + |D''\Omega| + |4\Omega + 2\omega|\right),$$

then $(M, g, J)$ is Kähler and isometric to $\mathbb{C}H^m$.

In this case $c_1$ and $c_2$ are constants depending on the complex dimension:

$$c_1 := 2\sqrt{\frac{m+1}{m-1}}, \quad c_2 := 2\left(m + 1 - \sqrt{m^2 - 1} + \frac{2}{\sqrt{m^2 - 1}}\right),$$

in particular $c_1, c_2 \approx 2$ for large $m$. Throughout this paper $\Omega = g(., J.)$ always denotes the 2–form associated to $J$, $d^*$ is formal $L^2$–adjoint of the exterior derivative $d$ and $D' + D''$ is the Dolbeault decomposition of $D = d + d^*$ in $\Lambda^*(TM) \otimes \mathbb{C}$, i.e. if $e_1, \ldots, e_{2m}$ is an orthonormal base, we define $D' = \sum e_j^{1,0} \cdot \nabla e_j$ and $D'' = \sum e_j^{0,1} \cdot \nabla e_j$. Introduce $D^c := d^c + d^c*$ with $d^c := \sum J(e_k) \wedge \nabla e_k$ and $d^c* := -\sum J(e_k) \nabla e_k$, we obtain $D' = \frac{1}{2}(D + iD^c)$ as well as $D'' = \frac{1}{2}(D - iD^c)$. In particular, we can estimate

$$|D'\Omega| + |D''\Omega| \leq |D\Omega| + |D^c\Omega| = |d^*\Omega| + |d\Omega| + |d^c*\Omega| + |d^c\Omega|.$$ 

Moreover, using the facts $|D\eta| \leq \sum_j |\nabla e_j\eta| \leq \sqrt{2m}|\nabla \eta|$ as well as $|D^c\eta| \leq \sqrt{2m}|\nabla \eta|$ for all $\eta \in \Gamma(\Lambda^* M)$ we get

$$|D'\Omega| + |D''\Omega| \leq 2\sqrt{2m}|\nabla \Omega|.$$
In complex dimension \( m = 2 \), the constant \( c_2 \) in inequality (2) can be improved by setting \( c_2 := 2(3 - \sqrt{3}) \). Furthermore, condition (1) can be replaced (in all dimensions) by one of the following asymptotic assumptions on \( g \):

(i) \( \text{scal} + 4m(m + 1) \in O(e^{-2(m+1+\epsilon)r}) \)

(ii) \( |\nabla^0 \nabla^0 A| \in O(e^{-2(m+1+\epsilon)r}) \).

Note that (ii) together with the asymptotic assumptions imply (i), and (i) together with inequality (2) yield (1). Moreover, assuming that \( \Omega \) is a symplectic form (i.e. \( d\Omega = 0 \)), leads to a much simpler statement of the previous theorem.

**Corollary 1.** Let \((M, g, J)\) be a complete almost Hermitian manifold of even complex dimension \( m = 2n \) which is strongly asymptotically complex hyperbolic, and suppose one of the following conditions:

(i) \( d\Omega = 0 \) and \( M \) is diffeomorphic to \( \mathbb{C}H^{2n} \).

(ii) \( \Omega \) is exact and \( M \) is spin.

If the scalar curvature satisfies

\[
\text{scal} \geq -4m(m + 1) + (c_1 + 2c_2)\sqrt{2m}|
\nabla\Omega|,
\]

then \((M, g, J)\) is Kähler and isometric to \( \mathbb{C}H^{2n} \).

Combining the methods in this paper and in [9] yield a non-spin version of the result in [9].

**Proposition 1.** Let \((M, g, J)\) be a complete almost Hermitian manifold of odd complex dimension \( m = 2n+1 \) which is strongly asymptotically complex hyperbolic. Suppose \( \lambda \) is the associated complex line bundle of a chosen spin\(^c\) structure on \( M \), and \( \frac{\omega\lambda}{2\pi} \in \Gamma(\Lambda^{1,1}M) \) represents the real Chern class of \( \lambda \). If \( \omega \in O(e^{-2(m+1+\epsilon)r}) \) and the scalar curvature satisfies

\[
\text{scal} \geq -4m(m + 1) + 2\left[|d^*\Omega| + |D'\Omega| + |D''\Omega|\right] + 2|\omega|,
\]

then \((M, g, J)\) is Kähler and isometric to \( \mathbb{C}H^{2n} \).

The proof in the complex even dimensional case is much more technical than in the odd dimensional case. We have to modify the ordinary Killing structure in order to show a suitable Bochner–Weitzenböck formula which is necessary to use the non-compact Bochner technique.

2. Preliminaries

Let \((M, g, J)\) be an almost Hermitian manifold of complex dimension \( m \), then \( M \) is a spin\(^c\) manifold (cf. [3] App. D]). Suppose \( S^cM \) is a complex spinor bundle of \( M \) with associated complex line bundle \( \lambda \). We denote
by \( \gamma \) respectively · the Clifford multiplication on \( S^c M \). \( S^c M \) decomposes orthogonal into

\[
S^c M = S^c_0 \oplus \cdots \oplus S^c_m
\]

(cf. \[7, 8\]) where each \( S^c_j \) is an eigenspace of \( \gamma(\Omega) \) to the eigenvalue \( i(m-2j) \).
Let \( \pi_j \) be the orthogonal projections \( S^c M \to S^c_j \). The decomposition \( S^c \) is parallel (i.e. \( \nabla \pi_j = 0 \) for all \( j \)) if \( (g, J) \) is Kähler. As usual we introduce \( X^{1,0} := \frac{1}{2}(X - iJ(X)) \) and \( X^{0,1} := \frac{1}{2}(X + iJ(X)) \). We obtain \( \gamma(X^{1,0}) : S^c_j \to S^c_{j+1} \) as well as \( \gamma(X^{0,1}) : S^c_j \to S^c_{j-1} \), where \( S^c_0 = \{0\} \) if \( j \notin \{0, \ldots, m\} \). If \( \nabla^c \)

is a spin\(^c\) connection on \( S^c M \), \( \nabla^c \) denotes the corresponding Dirac operator. The 2–form \( \omega \) appearing in the Lichnerowicz formula (cf. \[8\] Thm. D12):

\[
(\nabla^c)^2 = \nabla^c \nabla^c + \frac{\text{scal}}{4} + \frac{i}{2} \gamma(\omega)
\]

will be called the the curvature 2–form associated to \( \nabla^c \) (for the moment \( \nabla^c \) is an arbitrary spin\(^c\) connection, but in order to show the main theorem we will consider the canonical spin\(^c\) connection induced by the choice of the connection on the complex line bundle \( \lambda \)). Let \( (g, J) \) be Kähler and consider the connection \( \tilde{\nabla}^c := \nabla^c + \mathfrak{A} \) with

\[
\mathfrak{A}_X := \kappa_1 \gamma(X^{1,0}) \pi_{n-1} + \kappa_2 \gamma(X^{0,1}) \pi_n
\]

\( n := \left\lfloor \frac{m+1}{2} \right\rfloor \). Since \( \mathfrak{A} \) is parallel w.r.t. \( \nabla^c \), we obtain

\[
\tilde{R}^c_{X,Y} = R^c_{X,Y} + \kappa_1 \kappa_2 (\gamma(X^{0,1}) \gamma(Y^{1,0}) - \gamma(Y^{0,1}) \gamma(X^{1,0})) \pi_{n-1}
\]

\[
+ \kappa_1 \kappa_2 (\gamma(X^{1,0}) \gamma(Y^{0,1}) - \gamma(Y^{1,0}) \gamma(X^{0,1})) \pi_n.
\]

Suppose \( \mathcal{R} : \Lambda^2 M \to \Lambda^2 M \) is the Riemannian curvature operator, then the curvature of \( \nabla^c \) satisfies

\[
R^c_{X,Y} = \frac{1}{2} \gamma(\mathcal{R}(X \wedge Y)) + \frac{i}{2} \omega(X, Y).
\]

Therefore,

\[
X^{1,0} \cdot Y^{0,1} - Y^{1,0} \cdot X^{0,1} = \frac{1}{2} \gamma(X \wedge Y + JX \wedge JY - 2i\Omega(X, Y))
\]

\[
X^{0,1} \cdot Y^{1,0} - Y^{0,1} \cdot X^{1,0} = \frac{1}{2} \gamma(X \wedge Y + JX \wedge JY + 2i\Omega(X, Y))
\]

leads to the following proposition:

**Proposition 2.** Suppose \( (M, g, J) \) is a simply connected Kähler manifold of constant holomorphic sectional curvature \( \kappa \) and complex dimension \( m \), then \( \tilde{\nabla}^c \) is a flat connection on the subbundle \( \mathcal{V} = S^c_{n-1} \oplus S^c_n \subset S^c M \), \( n = \left\lfloor \frac{m+1}{2} \right\rfloor \), if

1. \( m \) is odd, \( \kappa = 4\kappa_1\kappa_2 \) and \( \omega = 0 \) (i.e. in particular \( M \) is spin)
2. \( m \) is even, \( \kappa = 4\kappa_1\kappa_2 \) and \( \omega = 2\kappa_1\kappa_2 \Omega = \frac{1}{2} \kappa \Omega \).
Proof. Since $M$ is simply connected, $\hat{\nabla}^0$ is flat if and only if $\hat{R}^0 = 0$. $(M, g, J)$ is of constant holomorphic curvature $\kappa$ if and only if the Riemannian curvature operator satisfies

$$\mathcal{R}(X \wedge Y) = -\frac{\kappa}{4}(X \wedge Y + JX \wedge JY + 2\Omega(X, Y)\Omega).$$

Thus, we conclude the claim from $\gamma(\Omega) = i\sum_j (m - 2j)\pi_j$. \hfill \Box

3. Integrated Bochner–Weitzenböck formula

In order to show the main theorem, we need a suitable integrated Bochner–Weitzenböck formula. In the case of even complex dimension $m$ it is not possible to show a useful Bochner–Weitzenböck formula for the Killing structure $\mathfrak{A}$ introduced above. However, a minor modification of the Killing structure yields the correct formula. Suppose $(M, g, J)$ is almost Hermitian of even complex dimension $m = 2n$, $n \geq 1$. We define $\mathcal{V} := \mathfrak{S}^c_{n-1} \oplus \mathfrak{S}^c_n$, its projection $\text{pr}_\mathcal{V} := \pi_{n-1} + \pi_n$ and

$$\mathfrak{S}_X := i(\alpha_1 X^{1.0} \cdot \pi_{n-1} + \alpha_2 X^{0.1} \cdot \pi_n + \beta_1 X^{1.0} \cdot \pi_{n-2} + \beta_2 X^{0.1} \cdot \pi_{n+1})$$

with (note that $\pi_n = 0$ if $n < 2$)

$$\alpha_1 = \sqrt{\frac{m-1}{m+1}}, \quad \beta_1 = m + 1 - (m + 2)\alpha_1,$$

$$\alpha_2 = \sqrt{\frac{m+1}{m-1}}, \quad \beta_2 = m + 1 - m\alpha_2,$$

then $\mathfrak{S} \circ \text{pr}_\mathcal{V}$ equals $\mathfrak{A}$ if we set $\kappa := i\alpha_j$. Moreover, define

$$\mathcal{T} := -i(m + 2)\alpha_1 \pi_{n-1} - i m\alpha_2 \pi_n - i(m + 1) \sum_{j \neq n-1, n} \pi_j$$

$$= -i(m + 1) + i\beta_1 \pi_{n-1} + i\beta_2 \pi_n.$$

Lemma 1. $\mathfrak{S}_X + \gamma(X)\mathcal{T}$ is a selfadjoint endomorphism on the complex spinor bundle $\mathfrak{S}^c M$ (for every vector field $X$). Moreover, we have (for any orthonormal base $e_1, \ldots, e_{2m}$)

$$\mathcal{T}' := \sum_{j=1}^{2m} (e_j \cdot \mathfrak{S}_e_j) = \mathcal{T} \circ \text{pr}_\mathcal{V} - i\beta_1 (m + 4)\pi_{n-2} - i\beta_2 (m + 2)\pi_{n+1}.$$ \hfill (8)

Proof. Using the facts $(\gamma(X^{1.0})\pi_j)^* = -\gamma(X^{0.1})\pi_{j+1}$ and $(\gamma(X^{0.1})\pi_j)^* = -\gamma(X^{1.0})\pi_{j-1}$ we compute

$$(\mathfrak{S}_X)^* = i(\alpha_1 X^{0.1} \pi_n + \alpha_2 X^{1.0} \pi_{n-1} + \beta_1 X^{0.1} \pi_{n-1} + \beta_2 X^{1.0} \pi_n)$$

as well as

$$(\gamma(X)\mathcal{T})^* = -i(m + 1)\gamma(X) + i\beta_1 (X^{0.1} \pi_n + X^{1.0} \pi_{n-2}) +$$

$$+ i\beta_2 (X^{0.1} \pi_{n+1} + X^{1.0} \pi_{n-1}).$$
This leads to
\[ \mathcal{X}_X + X \cdot \mathcal{T} - (\mathcal{X}_X + X \cdot \mathcal{T})^* = i(\alpha_1 - \alpha_2 + \beta_1 - \beta_2)(X^{1.0}\pi_{n-1} - X^{0.1}\pi_n) = 0. \]

The second claim follows from the facts
\[ \sum_{k=1}^{2m} e_k \cdot e_k^{1.0} = -m + i\gamma(\Omega) \quad \text{and} \quad \sum_{k=1}^{2m} e_k \cdot e_k^{0.1} = -m - i\gamma(\Omega). \]

\[ \square \]

Proposition 3. Let \((M,g,J)\) be almost Hermitian of even complex dimension \(m = 2n\). Suppose \(\nabla^c\) is a spin connection on \(\mathcal{S}^c M\) and \(\mathcal{X}\) as well as \(\mathcal{T}\) are given as above. Define the connection \(\hat{\nabla} := \nabla^c + X\) and the operator \(\tilde{\mathcal{D}} := \mathcal{D}^c + \mathcal{T}\). Then the integrated Bochner–Weitzenböck formula
\[ \int_{\partial N} \langle \hat{\nabla}_\nu \varphi + \nu \cdot \tilde{\mathcal{D}} \varphi, \psi \rangle = \int_{N} \langle \hat{\nabla} \varphi, \hat{\nabla} \psi \rangle - \langle \tilde{\mathcal{D}} \varphi, \tilde{\mathcal{D}} \psi \rangle + \langle \hat{\mathcal{R}} \varphi, \psi \rangle \]
holds for any compact \(N \subset M\) and \(\varphi, \psi \in \Gamma(\mathcal{S}^c M)\). In this case \(\nu\) is the outward normal vector field on \(\partial N\) and \(\hat{\mathcal{R}}\) is given by
\[ \frac{\text{scal}}{4} + \frac{i}{2} \gamma(\omega) + (m+2)(m-1)\pi_{n-1} + m(m+1)\pi_n + (m+1)^2 \text{pr}_{\mathcal{V}^\perp} \]
\[ - (m+4)\beta_1^2\pi_{n-2} - (m+2)\beta_2^2\pi_{n+1} + \delta \mathcal{X} + \mathcal{D}^c \mathcal{T} \]
while \(\omega\) is the curvature 2–form associated to \(\nabla^c\) [cf. (4)], \(\text{pr}_{\mathcal{V}^\perp}\) is the projection to the orthogonal complement of \(\mathcal{V}\) in \(\mathcal{S}^c M\), \(\delta \mathcal{X}\) is the divergence of \(\mathcal{X}\), i.e. \(\delta \mathcal{X} = \sum (\nabla_{e_j} \mathcal{X}) e_j\) and \(\mathcal{D}^c \mathcal{T}\) is given by \(\sum e_j \cdot (\nabla_{e_j} \mathcal{T})\). Moreover, the boundary operator \(\hat{\nabla}_\nu + \nu \cdot \tilde{\mathcal{D}}\) is selfadjoint.

Proof. The selfadjointness of the boundary operator \(\hat{\nabla}_\nu + \nu \cdot \tilde{\mathcal{D}}\) follows immediately from the selfadjointness of \(\nabla^c_\nu + \nu \cdot \mathcal{D}^c\) and \(\mathcal{X}_\nu + \nu \cdot \mathcal{T}\) (since \(\nabla^c\) is a Hermitian connection). The formal \(L^2\)–adjoint of \(\tilde{\mathcal{D}}\) is given by \(\tilde{\mathcal{D}}^* = \mathcal{D}^c - \mathcal{T}\). Thus, we can easily verify
\[ \int_{N} \langle \tilde{\mathcal{D}}^* \varphi, \tilde{\mathcal{D}} \psi \rangle = - \int_{\partial N} \langle \nu \cdot \tilde{\mathcal{D}} \varphi, \psi \rangle + \int_{N} \langle \tilde{\mathcal{D}}^* \tilde{\mathcal{D}} \varphi, \psi \rangle \]
as well as
\[ \tilde{\mathcal{D}}^* \tilde{\mathcal{D}} = (\mathcal{D}^c)^2 + (m+1)^2 \text{pr}_{\mathcal{V}^\perp} + \alpha_1^2(m+2)^2\pi_{n-1} + \alpha_2^2m^2\pi_n + \mathcal{D}^c \mathcal{T} + \sum_{i=1}^{2m} (\gamma(e_i)\mathcal{T} - \mathcal{T} \gamma(e_i)) \nabla^c_{e_i}. \]
Moreover, a straightforward calculation shows

\[
\int_N \left\langle \nabla^\phi \varphi, \nabla^\psi \psi \right\rangle = \int_N \left\langle \nabla^c \varphi, \nabla^c \psi \right\rangle + \left\langle \nabla^c \varphi, \xi \psi \right\rangle + \left\langle \xi \varphi, \nabla^c \psi \right\rangle + \left\langle \xi \varphi, \xi \psi \right\rangle
\]

\[
= \int_\partial N \left\langle \nabla^c \varphi + \xi \varphi, \psi \right\rangle + \int_N \left\langle \nabla^c \varphi, \psi \right\rangle + \int_N \left\langle \nabla^c \varphi, \psi \right\rangle + \left\langle \sum_{i=1}^{2m} (\pi_{e_i} - \xi \pi_{e_i}) \nabla^c \varphi, \psi \right\rangle
\]

for all \(\varphi, \psi \in \Gamma(S^cM)\). Therefore, the Lichnerowicz formula (4) and the fact (previous lemma, \(T^* = -T\))

\[
\xi_X^* - \xi_X = \gamma(X) T - T \gamma(X)
\]

yields the claim with

\[
\widehat{\mathcal{R}} = \frac{1}{4} \scal + \frac{1}{2} \gamma(\omega) + (m+1)^2 \pr_{V^\perp} + (m+2) \alpha_1^2 \pi_{n-1} + m^2 \alpha_2^2 \pi_n + \mathcal{D} T + \delta \xi - \sum_{j=1}^{2m} \xi_{\pi_{e_j}} \circ \xi_{e_j}.
\]

Use \(\pi_j \gamma(X) \pi_{j-1} = \gamma(X^{1,0}) \pi_{j-1}\) and \(\pi_j \gamma(X) \pi_{j+1} = \gamma(X^{0,1}) \pi_{j+1}\) as well as (9) and \(X^{1,0} \cdot X^{1,0} = X^{0,1} \cdot X^{0,1} = 0\) to compute

\[
\sum_{j=1}^{2m} \xi_{\pi_{e_j}} \circ \xi_{e_j} = (m+2) \alpha_1^2 \pi_{n-1} + m \alpha_2^2 \pi_n + (m+4) \beta_1^2 \pi_{n-2} + (m+2) \beta_2^2 \pi_{n+1}.
\]

Therefore, we obtain \(\widehat{\mathcal{R}}\) from

\[
(m + 2)^2 \alpha_1^2 - (m + 2) \alpha_1^2 = (m + 2)(m - 1)
\]

\[
m^2 \alpha_2^2 - m \alpha_2^2 = m(m + 1).
\]

\[\square\]

Remark 1. Since \(\bar{D}\) is not the Dirac operator of \(\hat{\nabla}\), we made a difference in the notation. However, if \(\bar{D} = \sum e_j \cdot \hat{\nabla}_{e_j}\) denotes the Dirac operator of \(\hat{\nabla}\), equation (9) yields \(\bar{D} \circ \pr_Y = \bar{D} \circ \pr_Y\). Moreover, if \(\varphi\) is a section in \(S^cM\) with \(\hat{\nabla} \varphi = 0\) as well as \(\bar{D} \varphi = 0\), then \(\varphi\) is a section in \(\mathcal{V}\) [use the fact \(\bar{D} \varphi = 0\) and equation (9)].

Lemma 2. The endomorphism

\[
-\iota \gamma(\Omega) + (m + 2)(m - 1) \pi_{n-1} + m(m + 1) \pi_n + (m + 1)^2 \pr_{V^\perp}
\]

\[
- (m + 4) \beta_1^2 \pi_{n-2} - (m + 2) \beta_2^2 \pi_{n+1} - m(m + 1) \text{Id}
\]

is non-negative definite on \(S^cM\) \([\Omega = g(., J)]\).
Proof. Since $i_\gamma(\Omega) = -\sum_j (m - 2j)\pi_j$ we conclude the claim on $\mathcal{S}_j$ for $j$ different from $n - 2$ and $n + 1$. It remains to show that

\[ f_1(m) := 4 + (m + 1) - (m + 4)\beta_1^2 \geq 0 \]
\[ f_2(m) := -2 + (m + 1) - (m + 2)\beta_2^2 \geq 0 \]

Both functions are increasing and since $f_1(2) > 0$ and $f_2(2) > 0$, we get the claim (we only consider the case of complex dimension $m \geq 2$). \hfill \Box

**Lemma 3.** Suppose inequality (2) of the main theorem holds, then at each point of $M$, $\hat{\mathcal{R}}$ has no negative eigenvalues: $\hat{\mathcal{R}} \geq 0$.

Proof. If $\eta$ is a two form, the operator norm of $\gamma(\eta)$ on $\mathcal{S}^c M$ can be estimated by $|\eta|$

\[ |\gamma(\eta)| \leq |\eta|. \]

Using the last lemma we obtain

\[ \hat{\mathcal{R}} \geq \frac{\text{scal}}{4} + m(m + 1) - \left| \Omega + \frac{1}{2}\omega \right| - |\delta_{\Sigma} + D\mathcal{T}|. \]

Therefore, we have to find an estimate for $\delta_{\Sigma} + D\mathcal{T}$. A straightforward calculation shows

\[ \delta_{\Sigma} = \sum_{j=1}^{2m} (\nabla_{e_j} \Sigma)_{e_j} \]

\[ = \frac{1}{2} \gamma(\delta J)(\alpha_1 \pi_{n-1} - \alpha_2 \pi_n + \beta_1 \pi_{n-2} - \beta_2 \pi_{n+1}) + i \sum_{j=1}^{2m} (\alpha_1 e_j^{1,0} \nabla_{e_j}^c \pi_{n-1} \]

\[ + \alpha_2 e_j^{0,1} \nabla_{e_j}^c \pi_n + \beta_1 e_j^{1,0} \nabla_{e_j}^c \pi_{n-2} + \beta_2 e_j^{0,1} \nabla_{e_j}^c \pi_{n+1}) \]

as well as

\[ D\mathcal{T} = i \sum_{j=1}^{2m} (\beta_1 e_j \nabla_{e_j}^c \pi_{n-1} + \beta_2 e_j \nabla_{e_j}^c \pi_n). \]

Set $\alpha = \alpha_1 + \beta_1 = \alpha_2 + \beta_2$, then $\nabla^c = \nabla$ on $\text{Cl}_C(TM) = \text{End}(\mathcal{S}^c M)$ yields

\[ |\delta_{\Sigma} + D\mathcal{T}| \leq \frac{\alpha^2}{2} |\omega| + \alpha \sum \left( e_j^{1,0} \nabla_{e_j} \pi_{n-1} + e_j^{0,1} \nabla_{e_j} \pi_n \right) + \]

\[ + \beta_1 \left( e_j^{1,0} \nabla_{e_j} \pi_{n-2} + e_j^{0,1} \nabla_{e_j} \pi_{n-1} \right) + \]

\[ + \beta_2 \left( e_j^{1,0} \nabla_{e_j} \pi_{n+1} + e_j^{0,1} \nabla_{e_j} \pi_{n+1} \right) \]

Thus, we have to estimate $\nabla X \pi_r$. We conclude from $\pi_{n-k} \gamma(\Omega) = 2k \pi_{n-k}$

\[ (\nabla X \pi_{n-k})(2k \pi - \gamma(\Omega)) = \pi_{n-k} \gamma(\nabla X \Omega) \]

as well as from $\gamma(\Omega) \pi_{n-k} = 2k \pi_{n-k}$

\[ (2k \pi - \gamma(\Omega))(\nabla X \pi_{n-k}) = \gamma(\nabla X \Omega) \pi_{n-k}. \]
Using the facts $\pi_r(\nabla_X \pi_r)\pi_r = 0$ for all $r$ and 
$$2ki - \gamma(\Omega) = \sum_{j\neq n-k} c_j \pi_j$$
with $|c_j| \geq 2$, $|\nabla_X \pi_r|$ can be estimated by $\frac{1}{2} |\nabla_X \Omega|$. Moreover, 
$$\sum_{j=1}^{2m} e_j^{1,0} \cdot (\nabla e_j \pi_{n-k})(2ki - \gamma(\Omega)) = \pi_{n-k+1} \sum_{j=1}^{2m} \gamma(e_j^{1,0} \cdot \nabla e_j \Omega)$$
leads to 
$$\left| \sum_{j=1}^{2m} e_j^{1,0} (\nabla e_j \pi_{n-k}) \phi \right| \leq \frac{1}{2} |\gamma(D' \Omega) \phi|,$n

if $\pi_{n-k}(\phi) = 0$, and 
$$\sum_{j=1}^{2m} \gamma(e_j^{1,0} \cdot \nabla e_j \Omega) \pi_{n-k} = \sum_{j=1}^{2m} \gamma(e_j^{1,0} \cdot \nabla e_j \Omega)(2ki - \gamma(\Omega))(\nabla e_j \pi_{n-k})$$
shows 
$$\left| \sum_{j=1}^{2m} e_j^{1,0} (\nabla e_j \pi_{n-k}) \phi \right| \leq \frac{1}{2} |\gamma(D' \Omega) \phi|,$n

if $\phi \in \mathcal{F}_{n-k}$. In this case we used $\pi_{n-k+1}(e_j^{1,0} \cdot \nabla e_j \pi_{n-k}) \pi_{n-k} = 0$ and the fact that $(2k - 2)i - \gamma(\Omega)$ has absolute minimal eigenvalue 2 on $\mathcal{F}_{n-k+1}$. The same procedure yields 
$$\left| \sum_{j=1}^{2m} e_j^{0,1} (\nabla e_j \pi_r) \right| \leq \frac{1}{2} |\mathcal{D}' \Omega|,$n

for all $r$. Thus, we obtain 
$$|\delta \Sigma + \mathcal{D}' \mathcal{T}| \leq \frac{\alpha_2}{2} |d^* \Omega| + \frac{\alpha + |\beta_1| + |\beta_2|}{2} (|\mathcal{D}' \Omega| + |\mathcal{D}'' \Omega|),$$

and 
$$|\beta_1| + |\beta_2| = \beta_1 - \beta_2 = \frac{2}{\sqrt{m^2 - 1}}, \quad \alpha = m + 1 - \sqrt{m^2 - 1}$$

supplies the claim: $\mathcal{R} \geq 0$. In complex dimension $m = 2$ there is a better estimate of $\delta \Sigma + \mathcal{D}' \mathcal{T}$. Since the decomposition $\mathcal{F}^c M = (\mathcal{F}^c M)^+ \oplus (\mathcal{F}^c M)^-$ induced by the volume form is parallel and we have $(\mathcal{F}^c M)^- = \mathcal{F}_1^c$ as well as $(\mathcal{F}^c M)^+ = \mathcal{F}_0^c \oplus \mathcal{F}_2^c$, $\pi_1 = \pi_-$ is parallel and we obtain the improvement if $m = 2$ from (10) and the above considerations: 
$$|\delta \Sigma + \mathcal{D}' \mathcal{T}| \leq \frac{\alpha_2}{2} |d^* \Omega| + \frac{\alpha}{2} |\mathcal{D}' \Omega| + \frac{|\beta_1| + |\beta_2|}{2} |\mathcal{D}'' \Omega|. $$

□
Proposition 4. Suppose $(M, g, J)$ is a complete almost Hermitian manifold of complex dimension $m$, $\mathcal{S}^c M$ is a complex spinor bundle of $M$, $\nabla^c$ is a spin$^c$ connection and $\omega$ is the curvature two form associated to $\nabla^c$. If the scalar curvature is uniformly bounded with

\[
\frac{\text{scal}}{4} \geq -m(m + 1) + \left| \Omega + \frac{1}{2} \omega \right|,
\]

then the Dirac operator

\[
\tilde{D} = \mathcal{D} + T : W^{1,2}(M, \mathcal{S}^c M) \to L^2(M, \mathcal{S}^c M)
\]

is an isomorphism of Hilbert spaces.

Proof. First we show that

\[
\bar{D}_\pm := \mathcal{D} \pm i(m + 1) : W^{1,2}(M, \mathcal{S}^c M) \to L^2(M, \mathcal{S}^c M)
\]

are isomorphism of Hilbert spaces if the scalar curvature inequality (11) is satisfied. $\bar{D}_\pm$ is bounded on $W^{1,2}$, i.e. the symmetric bilinear form

\[
B_\pm(\varphi, \psi) := \int_M \left\langle \bar{D}_\pm \varphi, \bar{D}_\pm \psi \right\rangle
\]

is well defined and bounded on $W^{1,2}$. Let $\phi$ be a section in $\mathcal{S}^c M$ with compact support in $M$, then using the Lichnerowicz formula (4) and inequality (11) leads to

\[
B_\pm(\phi, \phi) = \int_M |\nabla^c \phi|^2 + (m + 1)|\phi|^2 + \left( m(m + 1) + \frac{\text{scal}}{4} + \frac{i}{2} \omega \cdot \phi, \phi \right) \geq \int_M |\nabla^c \phi|^2 + (m + 1)|\phi|^2 - \langle i\Omega \cdot \phi, \phi \rangle.
\]

Therefore, $|\gamma(\Omega)| \leq m$ on $\mathcal{S}^c M$ implies that $B_\pm$ is coercive, in particular $B_\pm$ is a scalar product on $W^{1,2}$. This proves the injectivity of $\bar{D}_\pm$. The surjectivity of $\bar{D}_\pm$ follows from the Riesz representation theorem and [5, Thm. 2.8] (cf. [1, 6, 11]). The Dirac operator $\tilde{D} = \mathcal{D} + T$ is bounded w.r.t. the $W^{1,2}$–norm, i.e. $\tilde{D}$ is well defined and the bilinear form

\[
B(\varphi, \psi) := \int_M \left\langle \tilde{D} \varphi, \tilde{D} \psi \right\rangle
\]
is well defined and bounded on $W^{1,2}(M, \mathcal{S}^e M)$. Using the definition of $\overline{\Psi}$ and the above estimate lead to

$$B(\phi, \phi) = \int_M |\overline{\Psi}_- \phi|^2 + i \left< \beta_1 \pi_{n-1}(\phi) + \beta_2 \pi_n(\phi), \overline{\Psi}_- \phi \right>$$

$$\geq \int_M \left( 1 - \frac{|\beta_1| + |\beta_2|}{2} \right) |\overline{\Psi}_- \phi|^2 - \frac{|\beta_1|}{2} |\pi_{n-1} \phi|^2 - \frac{|\beta_2|}{2} |\pi_n \phi|^2$$

$$\geq \int_M \left( 1 - \frac{|\beta_1| + |\beta_2|}{2} \right) (|\nabla^c \phi|^2 + |\phi|^2) - \frac{|\beta_1|}{2} |\pi_{n-1} \phi|^2 - \frac{|\beta_2|}{2} |\pi_n \phi|^2.$$ 

If $m \geq 2$ one can verify that $|\beta_1|, |\beta_2| < 1$ as well as $|\beta_1| + \frac{1}{2}|\beta_2| < 1$ and $|\beta_2| + \frac{1}{2}|\beta_1| < 1$ (use the fact that $|\beta_1|$ and $|\beta_1|$ are decreasing and the inequalities hold in case $m = 2$). Thus we conclude that $B$ is coercive on $W^{1,2}(M, \mathcal{S}^e M)$. In particular, $\overline{\Psi}$ has to be injective and the surjectivity remains to show. Suppose $\psi \in L^2(M, \mathcal{S}^e M)$, then

$$l(\phi) := \int_M \langle \psi, \overline{\Psi}_- \phi \rangle$$

is a bounded linear functional on $W^{1,2}(M, \mathcal{S}^e M)$. The Lax–Milgram theorem (cf. [3 Ch. 5.8]) yields a spinor $\xi \in W^{1,2}(M, \mathcal{S}^e M)$ with

$$B(\xi, \phi) = l(\phi)$$

for all $\phi \in W^{1,2}$. Set $\zeta := \overline{\Psi}_- \xi - \psi \in L^2$, then $\zeta$ is a weak solution of $\overline{\Psi}_- \zeta$, in this case we used $(\overline{\Psi}_-)^* = \overline{\Psi}_+$. Elliptic theory supplies that $\zeta$ is smooth and since $\overline{\Psi}_- \zeta = -i(m + 1) \zeta \in L^2$, theorem 2.8 in [3] yields $\zeta \in W^{1,2}$. But $\overline{\Psi}_+$ is injective on $W^{1,2}$, i.e. $\zeta = 0$ shows the surjectivity of $\overline{\Psi}$.

**Lemma 4.** Suppose $\theta$ is a closed two form on $\mathbb{C}H^m$ with $\theta \in O(e^{-\delta r})$, $\delta > 3$ (r is the complex hyperbolic distance to a fixed point). Then there is a 1-form $\eta \in O(e^{-\delta r})$ with $d\eta = \theta$.

**Proof.** Suppose $X$ is the unit radial vector field of some polar chart on $\mathbb{C}H^m - B_{r_0}(0)$, $r_0 > 0$. Let $\varphi_1$ be the flow of $X$ and define $\eta_1$ by

$$\eta_1 := - \int_0^\infty X \varphi_1^* \theta \, dt,$$

then $\varphi_1^* \in O(e^{3t})$ yields $d\eta_1 = \theta$ as well as $\eta \in O(e^{-\delta r})$ on $\mathbb{C}H^m - B_{r_0}(0)$. Suppose $f$ is a cut off function for $B_{r_0}(0)$, i.e. $f$ is smooth, $f = 0$ on $B_{r_0}(0)$ and $f = 1$ on $\mathbb{C}H^m - B_{r_1}(0)$ for some $r_1 > r_0$, then $\theta - d(f\eta_1)$ is a closed and compact supported two form on $\mathbb{C}H^m$. Thus, $\mathbb{C}H^m \approx \mathbb{R}^{2m}$ yields a compact supported 1-form $\eta_2$ with $d\eta_2 = \theta - d(f\eta_1)$, and $\eta := f\eta_1 + \eta_2$ satisfies $d\eta = \theta$ as well as $\eta \in O(e^{-\delta r})$. 

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1Private communication with M. Herzlich
Lemma 5. Suppose \((V,q)\) is a vector space of real dimension \(2m\) with a quadratic form \(q\) and a \(q\)-compatible complex structure \(J\). Denote by \(S = \oplus S_r\) the complex spinor space of \(V\) where \(S_r\) are induced by the action of the Kähler form \(\Omega\). Then Clifford multiplication

\[
\gamma|_{\mathfrak{su}} : \Lambda^1_{\mathfrak{1}} V = \mathfrak{su}(V) \subset \text{Cl}^c(V,q) \to \text{End}(S_r)
\]

is injective if \(0 < r < m\) and trivial if \(r = 0\) or \(r = m\). Moreover, if \(\text{Id}\) denotes the identity in \(\text{End}(S_r)\), \(i \cdot \text{Id}\) is not in the image of \(\gamma|_{\mathfrak{su}}\) for all \(r = 0 \ldots m\).

Proof. We follow a proof given in [6]. Clifford multiplication of two forms supplies a representation

\[
\lambda : \Lambda^1_{\mathfrak{1}}(V) \to \text{End}(S_r),
\]

since \(\theta : S_r \subset S_r\) for each \(\theta \in \Lambda^1(V)\). With the inclusion \(\mathfrak{su}(m) \subset \mathfrak{so}(2m)\) the representation \(\rho : \mathfrak{so}(2m) \to \text{End}(\text{Cl}^c_{2m})\) (cf. [6] Ch. II (3.1)) restricts to a representation

\[
\rho : \mathfrak{su}(m) \to \text{End}(\text{Cl}^c_{2m}) = \text{End}(\text{End}(S))
\]

[\(\rho\) is the Lie–algebra version of \(\text{Ad} : \text{SU}(m) \subset \text{Spin}(2m) \to \text{Aut}(\text{Cl}^c_{2m})\)]. Moreover, the adjoint representation

\[
\text{ad} : \mathfrak{su}(m) \to \text{End}(\Lambda^1_{\mathfrak{1}}(V))
\]

is irreducible (since \(\mathfrak{su}(m)\) is simple). With these definitions \(\lambda\) is \(\mathfrak{su}(m)\)–equivariant, that means

\[
\rho(B)\lambda(\theta_0) = \lambda(\text{ad}(B)\theta_0)
\]

holds on \(S_r\) for any \(B \in \mathfrak{su}(m)\) and \(\theta_0 \in \Lambda^1_{\mathfrak{1}}(V)\). Thus, if \(\theta_0 \in \Lambda^1_{\mathfrak{1}}(V)\) is in the kernel of \(\lambda\), \(\lambda(\text{ad}(B)\theta_0)\) vanishes for all \(B \in \mathfrak{su}(m)\), in particular \(\ker(\lambda)\) is \(\mathfrak{su}(m)\)–invariant (in the representation \(\text{ad}\)). In particular the irreducibility of \(\text{ad}\) supplies \(\ker(\lambda) = \{0\}\) or \(\ker(\lambda) = \Lambda^1_{\mathfrak{1}}(V)\). The second case appears if and only if \(r = 0\) or \(r = m\): let \(e \in V\) with \(q(e) = 1\), then Clifford multiplication on \(S_r\) with \(Je \wedge e - \frac{1}{m} \Omega \in \Lambda^1_{\mathfrak{1}} V\) is invertible:

\[
\gamma \left( e \wedge Je - \frac{1}{m} \Omega \right) \gamma \left( Je \wedge e - \frac{1}{m} \Omega \right) = \text{Id} + \frac{1}{m^2} \gamma(\Omega)^2 = \frac{4mr - 4r^2}{m^2} \text{Id}_{S_r}.
\]

This implies \(\ker(\lambda) = \{0\}\) if \(0 < r < m\). That \(\gamma|_{\mathfrak{su}}\) can not be injective on \(S_0\) or \(S_m\) follows immediately from \(\dim S_0 = \dim S_m = 1\). In order to see the second claim in the case \(r \neq 0, m\) we use again the irreducibility of \(\text{ad} : \mathfrak{su}(m) \to \text{End} (\mathfrak{su}(m))\). The image of \(\gamma|_{\mathfrak{su}(m)}\) is isomorphic (as Lie algebras) to \(\mathfrak{su}(m)\). Thus, considering the adjoint transformation

\[
\text{ad} : \text{Im}(\gamma|_{\mathfrak{su}}) \to \text{End}(\text{Im}(\gamma|_{\mathfrak{su}}))
\]

yields \(\text{ad}(f) \neq 0\) for all \(f \in \text{Im}(\gamma|_{\mathfrak{su}}) - \{0\}\) (otherwise \(\text{ad}\) is reducible). In particular, if \(i \cdot \text{Id}\) is contained in \(\text{Im}(\gamma|_{\mathfrak{su}})\), \(\text{ad}(i \cdot \text{Id}) = 0\) leads to a contradiction. \(\square\)
Corollary 2. Clifford multiplication with $\Lambda^{1,1}M$–forms on $\mathcal{V}$ is injective.

Proof. Suppose

$$\eta = \eta_0 - \frac{1}{m} \langle \eta, \Omega \rangle \Omega$$

is a $\Lambda^{1,1}$–form with $\eta_0 \in \Lambda^{1,1}_0$ and $\gamma(\eta) = 0$ on $\mathcal{V} = s_{n-1} \oplus s_n$. Since $\gamma(\Omega) = 0$ on $s_n$, the last lemma yields $\eta_0 = 0$ ($0 < n < m$). Thus, $\gamma(\Omega) = 2i$ on $s_{n-1}$ supplies $\langle \eta, \Omega \rangle = 0$ which shows $\eta = 0$. \qed

4. Proof of the main theorem

Let $(M,g,J)$ be an almost Hermitian manifold which is strongly asymptotically complex hyperbolic, where $E \subset M$ is supposed to be the Euclidean end of $M$. We denote by $s^cM$ the considered spin$^c$ bundle as well as by $\nabla^c$ the canonical spin$^c$ connection determined by the choice of the connection on the complex line bundle $\lambda$ which has curvature $\omega$ (cf. [5 Prop. D11]). The spin connection $\nabla$ is well defined and unique on $s^cM_E$ and differs from $\nabla^c$ by an imaginary valued 1–form: $\eta(X) := \nabla^c_X - \nabla_X$ which satisfies $2d\eta = \omega$ on $E$. Let $g_0$ be the complex hyperbolic metric on $E$ with Kähler structure $\Omega_0 := g_0(J_0, J_0)$ $[(g_0, J_0)$ is of constant holomorphic sectional curvature $-4]$. $\nabla^0$ denotes the Levi–Civita connection for $g_0$ on $TM_E$ as well as the canonical spin connection for $g_0$ on $s^cM_E$. We conclude from the asymptotic assumptions and condition $\{\}$: $\Omega_0 + \frac{1}{2} \omega \in O(e^{-\delta r})$ with $\delta = 2(m + 1 + \epsilon)$. Therefore, lemma $\{\}$ supplies a 1–form $\eta_0$ on $E$ (use again a cut off argument) with $d\eta_0 = \Omega_0 + \frac{1}{2} \omega$ and $\eta_0 \in O(e^{-\delta r})$. Thus, the connection

$$\nabla^{0,c} := \nabla^0 + i\eta(\cdot) - i\eta_0(\cdot)$$

is a spin$^c$ connection on $s^cM_E$ with associated curvature two form

$$2d(\eta - \eta_0) = \omega - 2d\eta_0 = -2\Omega_0.$$

Define $\widehat{\nabla}^0 := \nabla^{0,c} + \Theta^0$ on $s^cM_E$ with $\kappa_j := i\alpha_j$ ($j = 1, 2$), we conclude from proposition $2$ that $\widehat{\nabla}^0$ is a flat connection on the subbundle $\mathcal{V}^0$. We consider the connection $\widehat{\nabla} := \nabla^c + \mathcal{S}$ on $s^cM$ and show that the restriction of $\widehat{\nabla}$ to $\mathcal{V}$ is asymptotic to $\widehat{\nabla}^0$. The gauge transformation $A$ extends to a bundle isomorphism $A : s^cM_E \rightarrow s^cM_E$ with (cf. $\{\}$)

$$|\nabla \varphi - \nabla^c \varphi| \leq C |A^{-1}| |\nabla^0 A| |\varphi| + |\eta_0||\varphi|,$$

where $\nabla$ is a connection on $s^cM_E$ given by $A\nabla^{0,c}A^{-1}$. Let $\psi$ be a spinor on $E \subset M$ which is parallel with respect to $\widehat{\nabla}^0$. Let $\psi := h(A\psi_0)$ for some cut off function $h$, i.e. $h = 1$ at infinity, $h = 0$ in $M - E$ and supp$(dh)$ compact. We compute

$$\widehat{\nabla}_X \psi = (Xh)A\psi_0 + h(\nabla^c_X A\psi_0 + \mathcal{S}_X(A\psi_0))$$

$$= (Xh)A\psi_0 + h(\nabla^c_X - \nabla_X)A\psi_0 - hA\Omega^0_X \psi_0 + h\mathcal{S}_X A\psi_0$$
and thus, the asymptotic assumptions supply
\[ \hat{\nabla} \psi \in O(e^{(1-\delta)r}) \subset L^2(M, T^* M \otimes S^c M) \]
and
\[ (12) \quad \langle \hat{\nabla}_\nu \psi + \nu \cdot \hat{\Phi} \psi, \psi \rangle \in O(e^{(2-\delta)r}) \subset L^1(M) \]
\(|\psi_0|^2 \) can be estimated by \( cc^{2r} \), \(|\nu| = 1 \). Using proposition 4 gives a spinor \( \xi \in W^{1,2}(M, S^c M) \) with \( \hat{\Phi} \xi = \hat{\Phi} \psi \in L^2 \). In particular \( \varphi := \psi - \xi \) is \( \hat{\Phi} \)-harmonic and non–trivial \( (\psi \notin L^2) \). Moreover, the selfadjointness of the boundary operator \( \hat{\nabla}_\nu + \nu \cdot \hat{\Phi} \) together with (12) implies as usual
\[ \lim_{r \to \infty} \int_{\partial M_r} \langle \hat{\nabla}_\nu \varphi + \nu \cdot \hat{\Phi} \varphi, \varphi \rangle = 0 \]
for a non–degenerate exhaustion \( \{ M_r \} \) of \( M \) (cf. 11). Since inequality (2) gives \( \hat{R} \geq 0 \), we conclude from the integrated Bochner–Weitzenböck formula:
\[ \int_{\partial M_r} \langle \hat{\nabla}_\nu \varphi + \nu \cdot \hat{\Phi} \varphi, \varphi \rangle \geq \int_{M_r} \hat{\nabla} \varphi \|^2 \geq 0, \]
that \( \varphi \) is parallel w.r.t. \( \hat{\nabla} \). Furthermore, \( \varphi \) has to be a section of \( \mathcal{V} = S_{n-1} \oplus S_n \) (cf. remark 11), since \( 0 = \hat{\Phi} \varphi = \Phi^c \varphi + T^c \varphi \) and \( 0 = \hat{\Phi} \varphi = \Phi^c \varphi + T \varphi \).
Because \( \hat{\nabla}^0 \) is a flat connection in \( \mathcal{V}^0, \mathcal{V} \) is trivialized by spinors parallel w.r.t. \( \hat{\nabla} \). In particular, \( \hat{\nabla}^0 \varphi \) preserves sections of \( \mathcal{V} \) which implies that \( \pi_\varphi \) is parallel w.r.t. \( \nabla^c = \nabla \). The complex spinor bundle admits an orthogonal and parallel decomposition \( S^c M = (S^c M)^+ \oplus (S^c M)^- \) induced from the volume form. If \( \pi_+ \) as well as \( \pi_- \) denote the orthogonal projections of this decomposition, \( \pi_{n-1} \) is given by \( \pi_\varphi \circ \pi_+ \) if \( n \) is odd and given by \( \pi_\varphi \circ \pi_- \) if \( n \) is even. Therefore, \( \pi_{n-1} \) as well as \( \pi_n = \pi_\varphi - \pi_{n-1} \) have to be parallel and the Killing structure \( \mathfrak{X} = \mathfrak{X} \circ \pi_\varphi \) is also parallel w.r.t. \( \nabla^c \). Moreover, the integrated Bochner–Weitzenböck formula implies \( \hat{R}(\varphi) = 0 \) for all \( \varphi \in \Gamma(\mathcal{V}) \), in particular \( \delta \mathfrak{X} = \partial \mathfrak{T} = 0 \) on \( \mathcal{V} \) supply
\[ 0 = \frac{\text{scal}}{4} \varphi + \frac{i}{2} \omega \cdot \varphi + (m + 2)(m - 1)\pi_{n-1} \varphi + m(m + 1)\pi_n \varphi \]
for all \( \varphi \in \mathcal{V} \). Thus, lemma 5 and \( \omega \in \Lambda^{1,1} M \) yield scal = \(-4m(m + 1)\) and \( \omega = -2\Omega \). Therefore, \( \hat{R} = 0 \) on \( \mathcal{V}, \nabla^c \mathfrak{X} = 0 \) as well as equations (5), (6) and (7) imply
\[ 0 = R^c_{X,Y} + [\mathfrak{X}_X, \mathfrak{X}_Y] \]
\[ (13) \quad = \frac{1}{2} \gamma(\mathcal{R}(X \land Y) + \kappa_1 \kappa_2 (X \land Y + JX \land JY + 2\Omega(X,Y)\Omega)) \]
on \( \mathcal{V} \). From the fact (cf. 22)
\[ \gamma(\text{Ric}(X)) = 2 \sum_i e_i \cdot R^s_{e_i,X} = \sum_i e_i \cdot \gamma(\mathcal{R}(e_i \land X)), \]
we conclude $\text{Ric}(X) = -2(m+1)X$, i.e. $g$ is Einstein of scalar curvature $-4m(m+1)$. Inequality (2) yields $d^*\Omega = 0$ as well as $\mathcal{D}'\Omega = 0$ and $\mathcal{D}''\Omega = 0$. In particular, $\mathcal{D}' + \mathcal{D}'' = d + d^*$ supplies $d\Omega = 0$. Moreover, equation (13), $\kappa_1\kappa_2 = -1$ and corollary 2 show

\[ \text{pr}_{\Lambda^{1,1}M} \circ \mathcal{R}(X \wedge Y) = X \wedge Y + JX \wedge JY + 2\Omega(X,Y)\Omega. \]

Using this equation, the symmetry of the Riemannian curvature tensor and $\Omega \in \Gamma(\Lambda^{1,1}M)$ lead to

\[ \langle \mathcal{R}(\Omega), X \wedge Y \rangle = \langle \mathcal{R}(X \wedge Y), \Omega \rangle = 2(m+1)\Omega(X,Y). \]

Consider the Bochner–Weitzenböck formula on $\Lambda^2M$:

\[ \Delta = d^*d + dd^* = \nabla^*\nabla + \mathfrak{R}, \]

then $\mathfrak{R}$ is given by $\text{Ric} + 2\mathcal{R}$ (cf. [12, Ap. B]), where Ric acts as derivation on $\Lambda^2M$. We already know, that $g$ is Einstein, i.e. $\text{Ric} = -4(m+1)\text{Id}_{\Lambda^2M}$ supplies $\mathfrak{R}(\Omega) = 0$. Moreover, $d\Omega = 0$ and $d^*\Omega = 0$ imply that $\Omega$ is harmonic: $\Delta\Omega = 0$, i.e. we obtain $\nabla^*\nabla\Omega = 0$. Using the fact

\[ 0 = \Delta|\Omega|^2 = d^*d|\Omega|^2 = 2\langle \nabla^*\nabla\Omega, \Omega \rangle - 2\langle \nabla\Omega, \nabla\Omega \rangle \]

we conclude that $(g,J)$ is Kähler. Thus, $\mathcal{R} : \Lambda^2M \to \Lambda^{1,1}M$ together with (14) yield constant holomorphic sectional curvature $-4$ of $(M,g,J)$. Since the end of $M$ is diffeomorphic to $\mathbb{C}^m - \overline{B_R(0)}$, $M$ must be isometric to $\mathbb{C}H^m$.

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