Quasinormal modes in Schwarzschild black holes due to arbitrary spin fields

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Abstract

The Newman-Penrose formalism is used to deal with the massless scalar, neutrino, electromagnetic, gravitino and gravitational quasinormal modes (QNMs) in Schwarzschild black holes in a united form. The quasinormal mode frequencies evaluated by using the 3rd-order WKB potential approximation show that the boson perturbations and the fermion perturbations behave in a contrary way for the variation of the oscillation frequencies with spin, while this is no longer true for the damping’s, which variate with $s$ in a same way both for boson and fermion perturbations. PACS: number(s): 04.70.Dy, 04.70.Bw, 97.60.Lf

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Ever since Chandrasekhar\cite{1} and Vishveshwara\cite{2} discovered the quasinormal modes of black holes, much effort has been devoted to investigating the QNMs of various black hole cases\cite{3,4,5,6,7}. By obtaining quasinormal mode(QN) frequencies, we can not only test the stability of the spacetime against small perturbations, but also probe the parameters of black hole, such as its mass, charge, and angular momentum, and thus, help uniquely identify a black hole.

QNMs are described as the \textit{pure tones} of black hole. They are defined as solutions of the perturbation equations belonging to certain complex characteristic frequencies which satisfy the boundary conditions appropriate for purely

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ingoing waves at the event horizon and purely outgoing waves at infinity[8]. QNMs are excited by the external perturbations (may be induced, for example, by the falling matter). They appear as damped oscillations described by the complex characteristic frequencies which are entirely fixed by the properties of background geometry, and independent of the initial perturbation. These frequencies can be detected by observing the gravitational wave signal[9]: this makes QNMs be of particular relevance in gravitational wave astronomy.

QNMs were firstly used to study the stability of a black hole. Detweiler and Leaver found the relations between the parameters of the black hole and QNMs. Latest studies show that QNMs play an important role in the quantum theory of gravity. This is related to the quantization of black hole area[10]. For example, there exit some possible relations between the classical vibrations of black holes and various quantum aspects, such as the relation between the real part of the quasinormal mode frequencies and the Barbero-Immirzi parameter, a factor introduced by hand in order that loop quantum gravity reproduces correctly entropy of the black hole[11,12,13]. All these works deal with asymptotically flat spacetimes. The recently proposed AdS/CFT correspondence makes the QNMs more appealing, due to its argument that string theory in anti-de Sitter (AdS) space is equivalent to conformal field theory (CFT) in one less dimension[14]. In addition to this context, many studies also have been done on QNMs of various spin-fields[3,7,8]. Chandrasekhar[8] investigated the QNMs of fields with spin \( s = 1/2, 1, 2 \) in Kerr black hole, and \( s = 2 \) in Schwarzschild and Reissner-Nordström black hole. Cardoso and Lemos[3] studied the QNMs of Schwarzschild-anti-de Sitter for fields with spin \( s = 1, 2 \). Cho[7] calculated the massive Dirac quasinormal mode frequencies of Schwarzschild black hole. However, problem on how to deal with QNMs due to arbitrary spin field (\( s = 0, 1/2, 1, 3/2, 2 \)) in a united form has never been discussed in these previous works. The main purpose of this article is to give a possible way to deal with QNMs in Schwarzschild black hole of arbitrary spin field in a united form.

We start with the line element in standard coordinates for the Schwarzschild space-time

\[
d s^2 = -e^{2U} dt^2 + e^{-2U} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2),
\]

(1)

with

\[
e^{2U} = 1 - \frac{2M}{r},
\]

(2)

where \( M \) is the mass of the black hole.

The Teukolsky’s master equations[15,16] for massless arbitrary spin fields \( s = 0, 1/2, 1, 3/2, 2 \) in Newman-Penrose formalism can be written as[17]

\[
\left\{ [D - (2s - 1)\epsilon + \epsilon^* - 2s\rho - \rho^*](\tilde{\Delta} - 2s\gamma + \mu) - [\delta - (2s - 1)\beta - \alpha^* - 2s\tau + \pi^*](\tilde{\delta} - 2s\alpha + \pi) - (s - 1)(2s - 1)\Psi_2 \right\} \Phi_s = 0,
\]

(3)
Assume that the wave-functions in Eqs. (3) and (4) have a $t$- and a $\varphi$-dependence specified in the form $e^{i(\omega t+\mathbf{m}\varphi)}$, i.e.,

$$
\Phi_+ = R_+(r)A_+(\theta)e^{i(\omega t+\mathbf{m}\varphi)}, \quad \Phi_- = R_-(r)A_-(\theta)e^{-i(\omega t+\mathbf{m}\varphi)}.
$$

and define

$$
P_+ = \Delta^s R_+, \quad P_- = r^{2s}R_-, \quad (6)
$$

(where $R_{\pm s}$ and $A_{\pm s}$ are, respectively, functions of $r$ and $\theta$ only, and $\Delta = r^2 - 2M$) one can decouple equations (3) and (4) as two pairs of equations,

$$
[\Delta \mathcal{D}_{1-s}\mathcal{D}_0 - 2(2s-1)i\omega r]P_+ = \lambda P_+, \quad (7)
$$

$$
\mathcal{L}_1 A_+ = -\lambda A_+, \quad (8)
$$

and

$$
[\Delta \mathcal{D}_{1-s}\mathcal{D}_0 + 2(2s-1)i\omega r]P_- = \lambda P_-, \quad (9)
$$

$$
\mathcal{L}_{1-s} A_- = -\lambda A_-, \quad (10)
$$

where $\lambda$ is a separation constant. The reason we have not distinguished the separation constants which derived from Eqs. (3) and (4) is that $\lambda$ is a parameter that is to be determined by the fact that $A_{\pm s}$ should be regular at $\theta = 0$ and $\theta = \pi$, and thus the operator acting on $A_{-s}$ on the left-hand side of Eq.(10) is the same as the one on $A_{+s}$ in Eq.(8) if we replace $\theta$ by $\pi - \theta$.

In Schwarzschild black hole, the separation constant can be determined analytically[16,18] for boson

$$
\lambda = (l + |s|)(l - |s| + 1), \quad l = |s|, |s| + 1, \cdots \quad (11)
$$

for fermion

$$
\lambda = (j + |s|)(j - |s| + 1), \quad j = |s|, |s| + 1, \cdots, \text{ and } j = l \pm |s| \quad (12)
$$

where $l$ and $j$ represent angular quantum number and total quantum number, respectively. Since $P_+\psi$ and $P_-\psi$ satisfy complex-conjugate equations (7) and (9), it will suffice to consider the equation (7) only.

By introducing a tortoise coordinate transformation $dr_\ast = \frac{r^2}{\Delta}dr$, one can
rewrite the operators $\mathcal{D}_0$ and $\mathcal{D}_0^\dagger$ as

$$\mathcal{D}_0 = \frac{r^2}{\Delta} \Lambda_+, \quad \text{and} \quad \mathcal{D}_0^\dagger = \frac{r^2}{\Delta} \Lambda_-.$$  \hspace{1cm} (13)

where we have defined $\Lambda_{\pm} = \frac{d}{dr} \pm i\omega$.

With the definition $Y = r^{1-2s}P_+^s$, equation (7) can be written as

$$\frac{r^{2s+3}}{\Delta} \left\{ \Lambda^2 Y + \left[ \frac{d}{d r^*} \ln \frac{r^{4s}}{\Delta_s} \right] \Lambda_- Y \right\} + \left[ \Delta^s \frac{d}{dr} \left( \frac{1}{\Delta_{s-1}} \frac{d}{dr} r^{2s-1} \right) - \lambda r^{2s-1} \right] Y = 0,$$  \hspace{1cm} (14)

where $\Lambda^2 = \frac{d^2}{dr^*^2} + \omega^2$. On further simplification, Eq.(14) can be brought to the form

$$\Lambda^2 Y + P \Lambda_- Y - QY = 0,$$  \hspace{1cm} (15)

where

$$P = \frac{d}{dr^*} \ln \frac{r^{4s}}{\Delta_s},$$  \hspace{1cm} (16)

and

$$Q = \frac{\Delta}{r^4} \left[ \lambda - (2s - 1)(s - 1) \left( \frac{2\Delta}{r^2} - \frac{\Delta'}{r} \right) \right].$$  \hspace{1cm} (17)

Considering the purpose of this article, we should seek to transform Eq.(15) to a one-dimensional wave-equation of the form

$$\Lambda^2 Z = V Z,$$  \hspace{1cm} (18)

where $V$ represents potential.

The transformation theory introduced in Ref.[8] is applicable to solve this problem. We assume that $Y$ is related to $Z$ in the manner

$$Y = \xi \Lambda_+ Z + W \Lambda_+ Z,$$  \hspace{1cm} (19)

where $\xi$ and $W$ are certain functions of $r^*$. One can then deduce the following equations[8]

$$\chi = \xi V + \frac{dT}{dr^*},$$  \hspace{1cm} (20)

$$\frac{d}{dr^*} \left( \frac{r^{4s}}{\Delta_s} \chi \right) = \frac{r^{4s}}{\Delta_s} (QT - 2i\omega \chi) + \beta,$$  \hspace{1cm} (21)

$$\chi \left( \chi - \frac{dT}{dr^*} \right) + \frac{\Delta^s}{r^{4s}} \beta T = \frac{\Delta^s}{r^{4s}} K,$$  \hspace{1cm} (22)

$$\chi V - Q \xi V = \frac{\Delta^s}{r^{4s}} \frac{d\beta}{dr^*},$$  \hspace{1cm} (23)

where $K$ is a constant, and $\chi$, $T$ are certain functions of $r_*$. The following work is to look for solutions of equations (20)-(23). These
equations provide four equations for five functions $\xi$, $\beta$, $\chi$, $T$, and $V$. As a result, there is considerable difficulty in seeking useful solutions of these equations. An obvious fact is that $\chi$ and $V$ are independent of $\omega$ (i.e., they do not contain any term linear in $i\omega$). Under these considerations, we can, without loss of generality, suppose that $T$, $\beta$, $K$ are of the forms

\begin{align*}
T &= T_1(r_*) + 2i\omega f(s), \\
\beta &= \beta_1(r_*) + 2i\omega \beta_2, \\
K &= \kappa_1 + 2i\omega \kappa_2,
\end{align*}

(24)

where $\beta_2$, $\kappa_1$, $\kappa_2$ are constants and $f(s)$ is function of $s$. In this article, we take the choice

\begin{equation}
f(s) = \frac{1}{6}s(2s-1)(6s^2 - 23s + 23) \quad \text{for} \quad s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2.
\end{equation}

(25)

Making use of the fact that, for a equation contains real and imaginary parts, the real parts and imaginary parts in two sides of the equation are equal, respectively, we can separate each of the Eqs.(21)-(22) into two equations by substituting Eq.(24), i.e.,

\begin{align*}
\chi &= fQ + \frac{\Delta^s}{r_{4s}} \beta_2, \\
\frac{d}{dr_*} \left( r_{4s}^4 \frac{\Delta^s}{\Delta^s} \chi \right) &= r_{4s}^4 \frac{\Delta^s}{\Delta^s} QT_1 + \beta_1,
\end{align*}

(26)

and

\begin{align*}
\beta_1 f + \beta_2 T_1 &= \kappa_2, \\
\chi^2 - \chi \frac{dT}{dr_*} + \frac{\Delta^s}{r_{4s}} \beta_1 T_1 &= \frac{\Delta^s}{r_{4s}} \kappa,
\end{align*}

(27)

(28)

(29)

where $\kappa = \kappa_1 + 4\omega^2 f \beta_2$.

Substituting Eqs.(26) and (28) into Eq.(27), we obtain

\begin{equation}
T_1 = \frac{f^2 F_{r_*} - \kappa_2}{f F - \beta_2}.
\end{equation}

(30)

Here we have defined $F = \frac{r_{4s}^4 Q}{\Delta^s}$, and ‘$r_*$’ denotes the differential with respect to $r_*$. Eq.(29) can then be written as

\begin{equation}
\frac{\Delta^s}{r_{4s}} (f F + \beta_2)^2 - f^2 \frac{(f F + \beta_2) F_{r_* r_*}}{f F - \beta_2} + \frac{(f^4 F_{r_*}^2 - \kappa_2^2) F}{(f F - \beta_2)^2} = \kappa,
\end{equation}

(31)

It’s obvious that Eq.(31) is a condition on $F$ if solutions of the chosen form are to exist, and hence the work to seek available $\beta_2$, $\kappa$, and $\kappa_2$ whose values satisfy Eq.(31) is a key step to obtain the function of potential $V$. Since $\kappa_2$ occurs as $\kappa_2^2$ in Eq.(31), two choices which associated with $+\kappa_2$ and $-\kappa_2$ are
possible to satisfy the equation. Further study finds, an available choice is the following

\[
\beta_2 = -\frac{1}{3}(2s - 3)(s - 2)(4s - 1)\lambda,
\]
\[
\kappa = \frac{1}{6}(2s - 1)(s - 1)(2s - 3)(5s - 8)\lambda(\lambda + s),
\]
\[
\kappa_2 = s(s - 1) \left[ \frac{8}{3}(2 - s)\lambda \sqrt{s - \frac{1}{2} + \lambda + (2s - 1)(2s - 3)M} \right].
\] (32)

The solution for \( V \) can then be obtained by substituting the expressions of \( \beta_2 \), \( \kappa \), and \( \kappa_2 \) into Eq.(23), i.e.,

\[
V^{(\pm)} = \frac{\Delta^s}{r^{4s}} F - \frac{(fF - \beta_2)fF_{r*}r* - f^2F_{r*}^2}{(fF - \beta_2)^2} \pm \frac{\kappa_2 F_{r*}}{(fF - \beta_2)^2},
\] (33)

where we have distinguished the transformations associated with \(+\kappa_2\) and \(-\kappa_2\) by superscripts \((\pm)\).

It is obvious that \( V^{(+)} \) equals to \( V^{(-)} \) for \( s = 0, 1 \) from the equation (33). For the case of \( s = \frac{1}{2}, \frac{3}{2}, 2 \), one can obtain a simpler form of \( V^{(\pm)} \) by defining a new function \( \tilde{F} \)

\[
V^{(\pm)} = \pm\kappa_2 \frac{d\tilde{F}}{dr_*} + \kappa_2^2 \tilde{F}^2 + \kappa \tilde{F}.
\] (34)

The expression of this new function is

\[
\tilde{F} = \frac{\Delta^{s-1}}{r^{4s-1}} \left[ \lambda + 2(2s - 1)(s - 1)\frac{M}{r} \right].
\] (35)

References[8,19] has shown that potentials \( V^{(+)} \) and \( V^{(-)} \) related in the way Eq.(34) shows are equivalent and hence possess the same spectra of quasinormal mode frequencies. We shall therefore concentrate on \( V^{(+)} \) only in evaluating the quasinormal mode frequencies.

So we can simplify the radial equation (7) to a one-dimensional wave-equation of the form

\[
\frac{d^2Z}{dr_*^2} + \omega^2Z = VZ,
\] (36)

with smooth real potentials, independent of \( \omega \), i.e.,

\[
V = \frac{\Delta^s}{r^{4s}} F - \frac{(fF - \beta_2)fF_{r*}r* - f^2F_{r*}^2 + \kappa_2 F_{r*}}{(fF - \beta_2)^2},
\] (37)

where \( F \) is a known function of \( r_* \). Note that we have written \( V^{(+)} \) as \( V \) because we shall not work with \( V^{(-)} \), which will give the same quasinormal mode frequencies.

Figure.1 demonstrates the variation of the effective potential \( V \) with spin
s. From this we can see that peak values of the effective potential $V$ decrease with $s$ for boson perturbations, while they increase with $s$ for fermion perturbations. This phenomena is closely related to the value of the separation constant $\lambda$.

Analytic expressions for the quasinormal mode frequencies are usually very difficult to obtain. One hence should appeal to approximation schemes to evaluate these frequencies. Many methods are available for our purpose. One often used is WKB approximation, a numerical method first proposed by Mashhoon[20], devised by Schutz and Will[21], and was subsequently extended to higher orders in[22,23]. An obvious character of this scheme is that it is very accurate for low-lying modes ($n < l$)[24]. Under these considerations, we may use WKB approximation to evaluate the quasinormal mode frequencies within low-lying modes. The values for $n < l$ are listed in Tables 1 and 2, which also include a comparison with the results of Cho[7], and those of Chandrasekhar and Detweiler(CD)[1]. Some gravitational modes ($l = 3, n = 2$, and $l = 4, n = 3$) of CD’s results have no counterpart in our results. This anomalous values were presumably results of numerical instabilities in their computational method that appeared when $\text{Im}(\omega) \sim \text{Re}(\omega)$ (see Ref.[1] for details). From Table 1, we see that our results for boson perturbations are almost the same as Iyer’s[24]. Our values for Dirac modes agree well also with the results of Cho from Table 2. Notice that during our evaluating procedures, we have let the mass $M$ of the black hole as a unit of mass so as to simplify the calculation. For negative $n$, the values are related to these with positive $n$ by reflection of the imaginary axis. Figure 2 and figure 3 depict the variation of the QN frequencies with spin $s$. Some significant results can be found from these figures. (i) For boson perturbations, the real parts of the QN frequencies decrease with the spin value for the same $l$ and $n$, while they behave in a contrary way for the fermion perturbations. (ii) The imaginary parts of the frequencies decrease slowly with spin value, regardless of boson or fermion perturbations, as showed in Fig.3. This means that the variation of the oscillation frequencies with spin for boson perturbations is totally different from fermion perturbations, while the variation of the damping’s with
spin has a same law for both boson and fermion perturbations. The method we used to deal with QNMs of arbitrary spin fields in this article is also available for discussion on other black hole cases, such as Reissner-Nordström and Kerr black hole, but a united form to deal with arbitrary spin perturbations in these black holes is still an unsolved problem.

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Table 1
Quasinormal mode frequencies for boson perturbations. $\omega_i (i = 0, 1, 2)$ represent scalar, electromagnetic and gravitational perturbations, respectively. $\omega_{CD}$ represent Chandrasekhar’s results for gravitational perturbations.

| $l$ | $n$ | $\omega_0$         | $\omega_1$         | $\omega_2$         | $\omega_{CD}$         |
|-----|-----|---------------------|---------------------|---------------------|------------------------|
| 0   | 0   | 0.1046+0.1152i       |                     |                     |                        |
| 1   | 0   | 0.2911+0.0980i       | 0.2459+0.0931i      |                     |                        |
| 2   | 0   | 0.4832+0.0968i       | 0.4571+0.0951i      | 0.3730+0.0891i      | 0.3737+0.0890i         |
|     | 1   | 0.4632+0.2958i       | 0.4358+0.2910i      | 0.3452+0.2746i      | 0.3484+0.2747i         |
| 3   | 0   | 0.6752+0.0965i       | 0.6567+0.0956i      | 0.5993+0.0927i      | 0.5994+0.0927i         |
|     | 1   | 0.6604+0.2923i       | 0.6415+0.2898i      | 0.5824+0.2814i      | 0.5820+0.2812i         |
|     | 2   | 0.6348+0.4941i       | 0.6151+0.4901i      | 0.5532+0.4767i      | 0.4263+0.3727i         |
| 4   | 0   | 0.8673+0.0964i       | 0.8530+0.0959i      | 0.8091+0.0942i      | 0.8092+0.0941i         |
|     | 1   | 0.8557+0.2909i       | 0.8411+0.2893i      | 0.7965+0.2844i      | 0.7965+0.2844i         |
|     | 2   | 0.8345+0.4895i       | 0.8196+0.4870i      | 0.7736+0.4790i      | 0.5061+0.4232i         |
|     | 3   | 0.8064+0.6926i       | 0.7909+0.6892i      | 0.7433+0.6783i      |                        |

Table 2
Quasinormal mode frequencies for fermion perturbations. $\omega_i (i = 1/2, 3/2)$ represent Dirac, Rarita-Schwinger perturbations, respectively. $\omega_{Cho}$ represent Cho’s results for Dirac perturbations. Notice that $\kappa = l + 1$ for $j = l + 1/2$, according to Cho’s definition.

| $l$ | $\kappa$ | $n$ | $\omega_{1/2}$          | $\omega_{Cho}$         | $\omega_{3/2}$         |
|-----|---------|-----|-------------------------|------------------------|------------------------|
| 1   | 2       | 0   | 0.3786+0.0965i          | 0.379+0.097i           |                        |
|     |         | 1   | 0.5562+0.2930i          | 0.556+0.293i           | 0.7206+0.2870i         |
| 3   | 4       | 0   | 0.7672+0.0963i          | 0.767+0.096i           | 0.9343+0.0954i         |
|     |         | 1   | 0.7540+0.2910i          | 0.754+0.291i           | 0.9233+0.2876i         |
|     |         | 2   | 0.7304+0.4909i          | 0.730+0.491i           | 0.9031+0.4835i         |
| 4   | 5       | 0   | 0.9602+0.0963i          | 0.960+0.096i           | 1.1315+0.0956i         |
|     |         | 1   | 0.9496+0.2902i          | 0.950+0.290i           | 1.1224+0.2879i         |
|     |         | 2   | 0.9300+0.4876i          | 0.930+0.488i           | 1.1053+0.4828i         |
|     |         | 3   | 0.9036+0.6892i          | 0.904+0.689i           | 1.0817+0.6812i         |
Fig. 2. Variation of the QN frequencies with spin $s$, the left for boson perturbations, the right for fermion perturbations.

Fig. 3: Variation of the imaginary parts of the QN frequencies with spin for $l = 4$. 