The Gelfand-Tsetlin-Zhelobenko base vectors for the series $B$

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Using the method of $Z$-invariants of Zhelobenko we construct base vectors of Gelfand-Tsetlin type in the space of $\mathfrak{o}_{2n-1}$-highest vectors in a representation of $\mathfrak{o}_{2n+1}$. The construction is based on a relation between restriction problems $\mathfrak{o}_{2n+1} \downarrow \mathfrak{o}_{2n-1}$ and $\mathfrak{gl}_{n+1} \downarrow \mathfrak{gl}_{n-1}$.

1 Introduction

In the paper [1] Gelfand and Tsetlin constructed a base in a representation of the Lie algebra $\mathfrak{o}_N$. The construction is based on an investigation of a branching of an irrep of $\mathfrak{o}_N$ under the restriction of Lie algebras $\mathfrak{o}_N \downarrow \mathfrak{o}_{N-1}$. The restriction problem $\mathfrak{g} \downarrow \mathfrak{k}$, where $\mathfrak{k}$ is a subalgebra in a Lie algebra $\mathfrak{g}$, is a problem of an explicit description of $\mathfrak{k}$-highest vectors in a representation of $\mathfrak{g}$.

Later it turned out that it is natural to have a construction of a base of Gelfand-Tsetlin type for a representation of $\mathfrak{o}_{2n+1}$, based on restrictions $\mathfrak{o}_{2n+1} \downarrow \mathfrak{o}_{2n-1}$ inside the series $B$. Thus in physical literature there were attempts to obtain such a base for $\mathfrak{o}_5$. Such a base is needed in the problem of classification of states of a five-dimensional quasi-spin in a shell models of nuclear kernels [3], [4], [5].

Zhelobenko in [6] constructed base vectors of Gelfand-Tsetlin type for $\mathfrak{sp}_{2n}$. He used a simpler technique of $Z$-invariants. This technique allowed him to find a relation between the restriction problems $\mathfrak{sp}_{2n} \downarrow \mathfrak{sp}_{2n-2}$ and $\mathfrak{gl}_{n+1} \downarrow \mathfrak{gl}_{n-1}$.

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1 But he did not manage to obtain formulas for the action of generators of the algebra in this base
Later V.V. Shtepin investigated the problem of restriction \( o_{2n+1} \downarrow o_{2n-1} \) in [8] using the technique of \( Z \)-invariants of Zhelobenko, but he did not find a relation with the problem of restriction \( gl_{n+1} \downarrow gl_{n-1} \).

Finally the problem of construction of a Gelfand-Tsetlin type base for the series \( B, D, \) and \( C \) was solved completely by Molev (see [2]). To obtain such a construction a solution of restriction problems were obtained. Molev constructed base vectors and obtained formulas for the action of generators of the algebras in this base. But he used a much more difficult technique. The key step in the Molev’s approach is a construction of an action of a Yangian on the space of \( o_{2n-1} \)-highest vectors with a fixed highest weight. Also he did not point out a relation of restriction problems \( o_{2n+1} \downarrow o_{2n-1} \) and \( gl_{n+1} \downarrow gl_{n-1} \).

In the present paper in Section 5 using the technique of \( Z \)-invariants of Zhelobenko we find a relation between the problems of restriction \( o_{2n+1} \downarrow o_{2n-1} \) and \( gl_{n+1} \downarrow gl_{n-1} \). Using it in Section 6 we construct in the same manner as in the case \( sp_{2n} \) in [6] base vectors of Gelfand-Tsetlin type for the algebra \( o_{2n+1} \) (Theorem 1). Unfortunately we have not managed to find the formulas for the action of generators of the algebra. The structure of our Gelfand-Tsetlin tableaux for \( o_{2n+1} \) is the same as the structure of Gelfand-Tsetlin tableaux constructed by Molev (see [2]).

2 The algebra \( o_{2n+1} \), the method of \( Z \)-invariants

The algebra \( o_{2n+1} \) is generated by \((2n + 1) \times (2n + 1)\) matrices, whose rows and columns are indexed by \(-n, \ldots, -1, 0, 1, \ldots, n\), of type

\[ F_{i,j} = E_{i,j} - E_{-j, -i}, \quad i, j = -n, \ldots, -1, 0, 1, \ldots, n, \]

where \( E_{i,j} \) is a matrix unit.

The subalgebra \( o_{2n-1} \) is generated by \( F_{i,j} \) for \( i, j \in -n, \ldots, -2, 0, 2, \ldots, n \).

3 Zhelobenko’s realization

On the open dense subset \( O^0_{2n+1} \subset O_{2n+1} \) the Gauss decomposition takes place

\[
O^0_{2n+1} = Z^{-1}DZ, \quad X = \zeta \delta z, \\
X \in O_{2n+1}, \quad \zeta \in Z^-, \quad \delta \in D, \quad z \in Z, 
\]
where $Z$ is a subgroup of upper-triangular unipotent matrices form $O_{2n+1}$, $D$ is a subgroup of diagonal matrices in $O_{2n+1}$, and $Z$ is a subgroup of lower-triangular unipotent matrices. On the space of polynomial functions on $Z$ there exists an action of $O_{2n+1}$ by the following ruler. Let us be given a function on $Z$ of type $f(z) = f(z_{i,j}), i < j$. For $X \in O_{2n+1}$ put

$$(Xf)(z) = \alpha(\delta)f(\tilde{z}), \quad zX = \tilde{\delta}\tilde{z}, \quad \alpha(\delta) = \delta_{-n}...\delta_{-1},$$

where $\delta = \text{diag}(\delta_{-n}, \delta_{-n+1}, ..., \delta_{n})$

Thus the space of all such functions form a representation of $O_{2n+1}$. A finite-dimensional representation with the highest weight $[m_{-n}, ..., m_{-1}]$, where numbers $m_{-i}$ are simultaneously integers or half-integers, is formed by functions that satisfy a system of PDE called the indicator system:

$$L_{r_{-n}-n+1}f = 0, ..., L_{r_{-1}+1}f = 0,$$

where $r_{-i}$ are defined as follows

$$r_{-n} = m_{-n} - m_{-n+1}, ..., r_{-2} = m_{-2} - m_{-1}, \quad r_{-1} = 2m_{-1}. \quad (2)$$

Here $L_{i,j}$ are operator that do left infinitesimal shifts of a function $f(z)$ by $F_{i,j}$.

The procedure of a construction of the Gelfand-Tsetlin type base is based on an investigation of a branching of an irrep under the restriction of the algebra. The method of $Z$-invariants gives us a description of functions that are $\mathfrak{o}_{2n-1}$-highest vectors. As in the case $\mathfrak{sp}_{2n}$ (see [6]) one can easily show that a function $f$ is a $\mathfrak{o}_{2n+1}$-highest vector $f$ if and only if the following conditions hold.

1. The function $f$ depends on the following variables

$$f = f(z_{-n,-1}, ..., z_{-2,-1}, z_{-n,1}, ..., z_{-2,1}, z_{0,1}). \quad (3)$$

We used the relation $z_{-1,1} = -\frac{z_{0,1}}{2}$, that holds for the matrix elements of the group $Z$.

2. The function $f$ satisfies the indicator system.

4 An explicit form of the indicator system and it’s solutions

Let us write the explicit form of the indicator system being restricted to the functions of type [3]. The indicator system looks as follows
\( L_{-n,-n+1}^{r-n+1} f = (z_{-n+1,-1} \frac{\partial}{\partial z_{-n,-1}} + z_{n+1,1} \frac{\partial}{\partial z_{n,1}}) f = 0, \)

\[
\begin{align*}
L_{-3,-2}^{r-3+1} f &= (z_{-2,-1} \frac{\partial}{\partial z_{-3,-1}} + z_{-2,1} \frac{\partial}{\partial z_{-3,1}})^{r-3+1} f = 0 \\
L_{-2,-1}^{r-2+1} f &= \left( \frac{\partial}{\partial z_{-2,-1}} + \frac{z_{0,1}}{2} \frac{\partial}{\partial z_{-2,1}} \right)^{r-2+1} f = 0 \\
L_{-1,0}^{r-1+1} f &= \left( \frac{\partial}{\partial z_{0,1}} \right)^{r-1+1} f = 0.
\end{align*}
\] (4)

To solve it let us introduce new variables

\[
u_{-k} = z_{-k,1} + \frac{z_{0,1}}{2} z_{-k,-1}, \quad v_{k} = z_{-k,1} - \frac{z_{0,1}}{2} z_{-k,-1}, \quad k = 2, ..., n.
\]

\[
u_{-1} = z_{0,1}, \quad v_{-1} = 0.
\]

The variables \( z_{-k,-1}, z_{-k,1} \) and be reconstructed as follows \( u_{-k}, v_{-k} \):

\[
z_{-k,1} = \frac{u_{-k} + v_{-k}}{2}, \quad z_{-k,-1} = \frac{u_{-k} - v_{-k}}{z_{0,1}/2}.
\]

In the space of polynomials in variables \( z_{-k,-1}, z_{-k,1}, ..., z_{-2,-1}, z_{-2,1}, z_{0,1} \) there exists a base

\[
u_{-1}^{p-1} \prod_{k=2}^{n} (u_{-k} + v_{-k})^{p-k} (u_{-k} - v_{-k})^{q-k},
\]

\[
p-k, q-k \geq 0, \quad k = 2, ..., n, \quad p_{-1} + 2 \sum_{k=2}^{n} p_{-k} \geq 0.
\] (5)

Let us find a condition under which (12) is a solution.

Consider first the equations \( L_{-k,-k+1}^{r-k+1} f = 0 \) for \( k = n, ..., 2 \). One has

\[
L_{-k,-k+1} u_{-k} = u_{-k+1}, \quad L_{-k,-k+1} v_{-k} = v_{-k+1}, \\
L_{-k,-k+1} u_{-l} = L_{-k,-k+1} v_{-l} = 0, \quad k \neq l, \quad k = 2, ..., n,
\]

Thus the operator \( L_{-k,-k+1}^{r-k+1} \) maps a polynomial in variables \( u_{-i}, v_{-i} \) into zero if and only if in each monomial the sum of degrees of \( u_{-k} \) and \( v_{-k} \) is not greater than \( r_{-k} \). That is if
\[ p_{-k} + q_{-k} \leq r_{-k}, \quad k = 2, ..., n. \]

Consider the equation \( L_{r_{-1},0}^{r_{-1}+1} f = 0 \). One has

\[ L_{-1,0} = \frac{\partial}{\partial z_{0,1}}, \]

the operator \( L_{r_{-1},0}^{r_{-1}+1} \) maps a polynomial in variables \( u_{-i}, v_{-i} \) into zero if and only if in variables \( z \) we have a polynomial in variable \( z_{0,1} \) of degree not greater than \( r_{-1} \). The polynomial (12) being rewritten in variables \( z \) has a degree in variable \( z_{0,1} \) equal to \( p'_{-1} + 2 \sum_{k=2}^{n} (p_{-k} + q_{-k}) \). That is the following condition must hold

\[ p_{-1} + 2 \sum_{k=2}^{n} (p_{-k} + q_{-k}) \leq r_{-1} \]

Thus we obtain

**Proposition 1.** In the space of solutions of (1) there exists a base

\[ f = u_{-1}^{p_{-1}} \prod_{k=2}^{n} (u_{-k} + v_{-k})^{p_{-k}} (u_{-k} - v_{-k})^{q_{-k}}, \tag{6} \]

where

\begin{align*}
    p_{-k}, q_{-k} & \geq 0, \quad k = 2, ..., n, \quad p_{-1} + 2 \sum_{k=2}^{n} p_{-k} \geq 0, \\
    p_{-k} + q_{-k} & \leq r_{-k}, \quad k = 2, ..., n, \quad p_{-1} + 2 \sum_{k=2}^{n} (p_{-k} + q_{-k}) \leq r_{-1}. \tag{7}
\end{align*}

Let us find the action of \( F_{-i,-i} \) onto these functions. The matrix \( \z \) from (11) for \( X = e^{tF_{-i,-i}} \) can be obtained from \( z \) by multiplication of the row \(-i\) onto \( e^{-t} \), of the row \( i \) onto \( e^{t} \), of the column \(-i\) onto \( e^{t} \), of the column \( i \) onto \( e^{-t} \). The matrix \( \delta \) equals \( e^{tF_{-i,-i}} \). Thus for the infinitesimal action one has

\begin{align*}
    F_{-i,-i} f &= -z_{-i,-1} \frac{\partial}{\partial z_{-i,-1}} f - z_{i,1} \frac{\partial}{\partial z_{i,1}} f + m_{-i} f, \quad i = 2, ..., n, \\
    F_{-1,-1} f &= \sum_{i=2}^{n} (z_{-i,-1} \frac{\partial}{\partial z_{-i,-1}} - z_{i,1} \frac{\partial}{\partial z_{i,1}}) f - z_{0,1} \frac{\partial}{\partial z_{0,1}} f + m_{-1} f. \tag{8}
\end{align*}
4.1 The restriction problem $\mathfrak{gl}_{n+1} \downarrow \mathfrak{gl}_{n-1}$

4.1.1 The indicator system and it’s solutions

Consider the algebra of all matrices $\mathfrak{gl}_{n+1}$ acting in the space with coordinates indexed by $-n, ..., -1, 1$. Representations of this Lie algebra can be realized in the space of functions on upper-triangular unipotent matrices. An irreducible representation is selected by an indicator system (see [6]).

Consider the subalgebra $\mathfrak{gl}_{n-1}$ generated by $E_{i,j}, i, j \in \{-n, ..., -2\}$. Let us be given an irreducible representation of $\mathfrak{gl}_{n+1}$ with the highest weight $[m_{-n}, ..., m_{-1}, m_1 = 0]$. Consider the problem of restriction $\mathfrak{gl}_{n+1} \downarrow \mathfrak{gl}_{n-1}$.

It turns out that (see [6]) $\mathfrak{gl}_{n-1}$-highest vectors are functions of type $f = f(z_{-n,-1}, ..., z_{-2,-1}, z_{-n,1}, ..., z_{-1,1})$, satisfying the indicator system.

Being restricted to these functions the indicator system takes the following explicit form

$$L_{r_{-n-1}+1}^{r_{-n}} f = (z_{-n+1,-1} \frac{\partial}{\partial z_{-n,-1}} + z_{-n+1,1} \frac{\partial}{\partial z_{-n,1}})^{r_{-n+1}} f = 0,$$

$$...$$

$$L_{r_{-3}-2}^{r_{-2}+1} f = (z_{-2,-1} \frac{\partial}{\partial z_{-3,-1}} + z_{-2,1} \frac{\partial}{\partial z_{-3,1}})^{r_{-3+1}} f = 0$$

(9)

$$L_{r_{-2}-1}^{r_{-1}+1} f = (\frac{\partial}{\partial z_{-2,-1}} + z_{-1,1} \frac{\partial}{\partial z_{-2,1}})^{r_{-2+1}} f = 0$$

$$L_{r_{-1}+1}^{r_{-1}} f = (\frac{\partial}{\partial z_{-1,1}})^{r_{-1+1}} f = 0,$$

where

$$r_{-n} = m_{-n} - m_{n+1}, ..., r_{-2} = m_{-2} - m_{-1}, r_{-1} = m_{-1}. \quad (10)$$

To solve this system let us introduce new variables

$$x_{-k} = z_{-k,1} + z_{-1,1}z_{-k,-1}, \quad y_{-k} = z_{-k,1} - z_{-1,1}z_{-k,-1}, \quad k = 2, ..., n.$$

$$x_{-1} = z_{-1,1}, \quad y_{-1} = 0.$$

The variables $z_{-k,-1}, z_{-k,1}$ can be reconstructed from $x_{-k}, y_{-k}$:

$$z_{-k,1} = \frac{x_{-k} + y_{-k}}{2}, \quad z_{-k,-1} = \frac{x_{-k} - y_{-k}}{2z_{-1,1}}. \quad (11)$$
Thus the space of polynomials $z_{-k,-1}, z_{-k,1}, \ldots, z_{-2,-1}, z_{-2,1}, z_{-1,1}$ there exists a base

$$x_{-1}^{p-1} \prod_{k=2}^{n} (x_{-k} + y_{-k})^{p-k}(x_{-k} - y_{-k})^{q-k},$$

(12)

$$p_{-k}, q_{-k} \geq 0, \ k = 2, \ldots, n, \ p_{-1} + \sum_{k=2}^{n} p_{-k} \geq 0.$$

Let us write conditions under which this polynomial is a solution. One has

$$L_{-k,-k+1}x_{-k} = x_{-k+1}, \quad L_{-k}y_{-k} = y_{-k+1},$$

$$L_{-k,-k+1}x_{-l} = L_{-k,-k+1}y_{-l} = 0, \ k \neq l, k = 2, \ldots, n.$$

As in the previous Section we obtain the following statement.

**Proposition 2.** In the space of polynomial solutions of the system (9) there exists a base of type

$$f = x_{-1}^{p-1} \prod_{k=2}^{n} (x_{-k} + y_{-k})^{p-k}(x_{-k} - y_{-k})^{q-k},$$

(13)

where

$$p_{-k}, q_{-k} \geq 0, \ k = 2, \ldots, n, \ p_{-1} + \sum_{k=2}^{n} p_{-k} \geq 0,$$

$$p_{-k} + q_{-k} \leq r_{-k}, \ k = 2, \ldots, n, \ p_{-1} + \sum_{k=2}^{n} (p_{-k} + q_{-k}) \leq r_{-1}.\ (14)$$

Let us find the action of $E_{-i,-i}$ onto these functions. The matrix $\bar{z}$ from (1) for $X = e^{tE_{-i,-i}}$ can be obtained from the row $-i$ by multiplication of the row $-i$ onto $e^{-t}$ and by multiplication of the column $-i$ onto $e^t$. The matrix $\delta$ equals $e^{tE_{-i,-i}}$. Thus for an infinitesimal action one has

$$E_{-i,-i}f = -z_{-i,-1} \frac{\partial}{\partial z_{-i,-1}} f - z_{-i,1} \frac{\partial}{\partial z_{-i,1}} f + m_{-i} f, \ i = 2, \ldots, n,$$

$$E_{-1,-1}f = \sum_{i=2}^{n} z_{-i,-1} \frac{\partial}{\partial z_{-i,-1}} f - z_{-1,1} \frac{\partial}{\partial z_{-1,1}} f + m_{-1} f,$$

$$E_{1,1}f = \sum_{i=1}^{n} z_{-i,1} \frac{\partial}{\partial z_{-i,1}} f.$$

(15)
4.1.2 The Gelfand-Tsetlin base.

In the space of $\mathfrak{gl}_{n-1}$-highest vectors there exists the Gelfand-Tsetlin base encoded by tableaux in which the betweeness conditions hold.

$$
\begin{array}{cccc}
m_{-n} & m_{-n+1} & \cdots & m_{-1} \\
m'_{-n,n} & & \cdots & m'_{-n,1} \\
m_{-n,n-1} & \cdots & m_{-2,n-1} & 0
\end{array}
$$

To prove the main statement below we need a realization of a representation in the functions on the whole group. Onto a function $f(g)$ an element $X \in GL_n$ acts by the ruler

$$(Xf)(g) = f(gX).$$

Let $a_{ij}$ be a function of a matrix element, where $j$ is a row index and $i$ is a column index. Put

$$a_{i_1,\ldots,i_k} := \det(a_{ij})_{i=i_1,\ldots,i_k}^{j=-n,\ldots,-n+k-1}.$$

One can easily check that the function

$$v_0 = \prod_{k=-n}^{-1} (a_{-n,\ldots,-k})^{r-k},$$

is a highest vector for $\mathfrak{gl}_{n+1}$ with the weight $[m_{-n}, \ldots, m_{-1}, 0]$. Indeed the operator $E_{i,j}$ acts onto $a_{i_1,\ldots,i_k}$ by the ruler

$$a_{i_1,\ldots,i_k} \mapsto a_{i_1,\ldots,i_k}\vert_{j \rightarrow i},$$

where $j \rightarrow i$ is an operation of substitution of the index $i$ instead of $j$, if $j \notin \{i_1, \ldots, i_k\}$ then the determinant is mapped to zero. Onto a product of determinant the operator $E_{i,j}$ act by the Leibnitz ruler.

To write the formulas for a vector corresponding to a tableau let us introduce operators $e_{i,-i}$, $i = n, \ldots, 1$ acting onto determinants by the ruler

$$a_{-n,\ldots,-1,-i} \mapsto a_{-n,\ldots,-1,1},$$
other determinants \( \mapsto 0 \),
and acting onto a product of determinants by the Leibnitz ruler. Also let us introduce operators \( e_{-1,-i} \), \( i = n, \ldots, 2 \) acting onto determinant by the ruler

\[
\begin{align*}
a_{-n,\ldots,-i-1,-1} & \mapsto a_{-n,\ldots,-i-1,-1}, \\
-1, \ldots, 1 \quad 1, \ldots, 1 & \mapsto a_{-n,\ldots,-i-1,1}, \\
\end{align*}
\]

and acting onto a product of determinants by the Leibnitz ruler.

Then for a \( \mathfrak{gl}_{n-1} \)-highest vector \( v \), defined by a Gelfand-Tsetlin tableau one has a formula

\[
v = \text{const} \cdot \prod_{i=-n}^{-2} e_{-1,-i}^{m'_{i,-n},-m_{i,-n}} \prod_{i=-n}^{-1} e_{1,-i}^{m_{i,-n},-m'_{i,-n}} v_0. \tag{17}
\]

See for example [7]. Our operators \( e_{\pm 1,-1} \) correspond to operators \( pE_{\pm 1,-1} \).

Indeed the extremal projector \( p \) maps a vector \( v \) to zero in the case \( v = E_- w \), where \( E_- \) is an element of \( \mathfrak{gl}_{n-1} \), corresponding to a negative root. The coincidence of actions of \( pE_{\pm 1,-i} \) and \( e_{\pm 1,-i} \) onto determinant can be easily checked. One has to prove that their action onto products of determinants coincide. Let us write it as follows

\[
\prod_{i=\pm n,-k} a_{-n,\ldots,-k,-1}^{\alpha_k} a_{-n,\ldots,-k,1}^{\beta_k} a_{n,\ldots,-k,1}^{\gamma_k} a_{n,\ldots,-k,-1}. \tag{18}
\]

The operator \( E_{-1,-2} \) act only on factors with \( k = 2 \), the application of \( p \) changes nothing. The resulting action of \( pE_{-1,-2} \) coincides with \( e_{-1,-2} \). Now consider \( E_{-1,-3} \), this operator can be represented as \([E_{-1,-2}, E_{-2,-3}] = E_{-1,-2} E_{-2,-3} - E_{-2,-3} E_{-1,-2}\). After application of \( p \) we obtain \( pE_{-1,-2} E_{-2,-3} \). The operator \( E_{-2,-3} \) act onto determinants in \( (18) \) with \( k = 3 \). Under the action of \( E_{-1,-2} E_{-2,-3} \) one obtains a product of determinant of type \( (18) \), which consists of determinants that are highest with respect to \( \mathfrak{gl}_{n-1} \). Thus \( pE_{-1,-3} = E_{-1,-2} E_{-2,-3} \). One can easily prove that \( E_{-1,-2} E_{-2,-3} \) equals to \( e_{-1,-3} \). Thus finally \( pE_{-1,-3} = e_{-1,-3} \). For the rest operators \( e_{\pm 1,-i} \) the proof is the same.

In the formula analogous to \( (17) \) not \( pE_{\pm 1,-i} \) but the operators denoted in \[7\] as \( z_{\pm 1,-i} \) occur. However under the action onto weight vectors the operators \( pE_{\pm 1,-i} \) and \( z_{\pm 1,-i} \) are proportional. Thus the formula \( (17) \) follows from the results of [7].

We need the following statement.

\footnote{Note that in [7] and in the present paper the indexation of coordinates is different}
Proposition 3. If one decomposes a vector corresponding to a Gelfand-Tsetlin tableau by the base (13), then for the summands one has the equality $p_{-1} + \sum_{k=-2}^{n}(p_{-k} + q_{-k}) \leq m_{-1,n} - m'_{-1,n}$. For at least one of the summand the equality takes place.

Proof. Take a realization of a representation on the functions on the whole group. From explicit formulas for the action of $e_{\pm 1,-i}$ one obtains that $v$ defined by (17) is a linear combination of products of determinants of type $a_{-n,-i}, a_{-n,-i,1}, a_{-n,-i,-1}, a_{-n,-i,-1,1}$. In which such a product the sum of degrees of $a_{-n,-i,-2,1}$ and $a_{-n,-i,-3,1,1}$ equals to $m_{-1,n} - m'_{-1,n}$.

Indeed these determinants appear as a result of action of $e_{1,-1}$ and $e_{-1,-2}$. The sum of degrees of these operators in (17) equals $m_{-1,n} - m'_{-1,n}$. Thus the sum of degrees of $a_{-n,-i,-2,1}$ and $a_{-n,-i,-3,1,1}$ in (17) equals $m_{-1,n} - m'_{-1,n}$.

Consider the value of these determinants on the subgroup $Z$. This is a polynomial in variables $z_{-n,1}, z_{-n,-1}$, and $z_{-1,1}$ can appear only from the determinants $a_{-n,-i,-2,1}$ and $a_{-n,-i,-3,1,1}$. More precisely $a_{-n,-i,-2,1}|z = z_{-1,1}, a_{-n,-i,-3,1,1}|z = z_{-2,1}z_{-1,1} - z_{-2,1}$. Thus its degree in the variable $z_{-1,1}$ equals to $m_{-1,n} - m'_{-1,n}$.

Now let us pass to the variables $x_{-i}, y_{-i}$. Let us be given a base polynomial (13) in these variables, rewrite it in variables $z_{-n,1}, z_{-n,-1}$, then it’s degree in $z_{-1,1}$ equals to $p_{-1} + \sum_{k=-2}^{n}(p_{-k} + q_{-k})$.

Thus when we pass from $z_{-n,1}, z_{-n,-1}$ to $x_{-i}, y_{-i}$ only monomials with $p_{-1} + \sum_{k=-2}^{n}(p_{-k} + q_{-k}) \leq m_{-1,n} - m'_{-1,n}$ appear and at least for one of them the equality takes place.

\[\square\]

Corollary 1. The span of vectors (13) with $p_{-1} + \sum_{k=-2}^{n}(p_{-k} + q_{-k}) \leq m_{-1,n} - 1$ contains a subspace the span of vectors corresponding to tableaux with $m'_{-1,n} > 0$.

5 A relation between restriction problems

Let us establish a relation between restriction problems $\mathfrak{o}_{2n+1} \downarrow \mathfrak{o}_{2n-1}$ with exponents (2) and $\mathfrak{gl}_{n+1} \downarrow \mathfrak{gl}_{n-1}$ with exponents (10).

Consider the cases when $p_{-1}$ is odd and even separately.

5.1 The case of even $p_{-1}$

Let us write $p_{-1} = 2p'_{-1}$. Take the solution (6) with exponents $r_{-n}, ..., r_{-2}, r_{-1}$ and relate to it a solution (13) with exponents $r_{-n}, ..., r_{-2}, \left[\frac{n-1}{2}\right]$, where $[\cdot]$ is an integer part, according to the ruler
\[ (p_1, p_k, q_k), \quad k = -2, ..., -n \mapsto p'_{-1} = \frac{p_1 - 1}{2}, p_k, q_k, \quad k = -2, ..., -n \quad (19) \]

The inequalities (7) for \( p_{-1}, p_k, q_k \), and \( r_{-n}, ..., r_{-2}, r_{-1} \) are equivalent to inequalities (14) with \( p'_{-1}, p_k, q_k \) and \( r_{-n}, ..., r_{-2}, \left\lceil \frac{r_{-1}}{2} \right\rceil \). Thus we obtain the following statement.

**Proposition 4.** The correspondence (19) is a bijection between the solution space (4) with even \( p_{-1} \) and the space of all solutions (9).

### 5.2 The case of odd \( p_{-1} \)

Let us write \( p_{-1} = 2p'_{-1} + 1 \). Take a solution (6) of the system with exponents \( r_{-n}, ..., r_{-2}, r_{-1} \) and let us relate to it a solution (13) with exponents \( r_{-n}, ..., r_{-2}, \left\lceil \frac{r_{-1}}{2} \right\rceil \) according to the ruler

\[ (p_{-1}, p_k, q_k), \quad k = -2, ..., -n \mapsto (p'_{-1} = \frac{p_{-1} - 1}{2}, p_k, q_k), \quad k = -2, ..., -n \quad (20) \]

Let us first prove that this correspondence is well defined. It is necessary to check that for the image of (20) the inequalities (14) hold. Inequalities (7) for \( p_{-1}, p_k, q_k \) and \( r_{-n}, ..., r_{-2}, r_{-1} \) give us inequalities (14) for \( p'_{-1}, p_k, q_k \) and \( r_{-n}, ..., r_{-2}, \left\lceil \frac{r_{-1}}{2} \right\rceil \). That is (20) is a well-defined embedding of the space of solutions with even \( p_{-1} \) of the problem of restriction \( \mathfrak{o}_{2n+1} \downarrow \mathfrak{o}_{2n-1} \) into the space of solutions of the problem of restriction \( \mathfrak{gl}_{n+1} \downarrow \mathfrak{gl}_{n-1} \).

Let us describe the image of this embedding.

**Proposition 5.** If the highest weight of the representation of \( \mathfrak{o}_{2n+1} \) is half-integer then (20) is an isomorphism.

If the highest weight of the representation of \( \mathfrak{o}_{2n+1} \) is integer then the image of (20) is a span of tableaux with \( m'_{-1,n} > 0 \).

**Proof.** Take the inequality \( p_{-1} + 2 \sum_{k=-2}^{n} (p_k + q_k) \leq r_{-1} \), and divide it by two, one obtains

\[ 0 \leq p'_{-1} + \frac{1}{2} + \sum_{k=-2}^{n} (p_k + q_k) \leq \frac{r_{-1}}{2}. \quad (21) \]

Suggest that the highest weight is half-integer, that is \( r_{-1} \) is odd. Because of the fact that \( p'_{-1}, p_{-i}, q_{-i} \) are integer, the inequality (21) is equivalent to the following one: \( p'_{-1} + \sum_{k=-2}^{n} (p_k + q_k) \leq \frac{r_{-1}}{2} = \left\lceil \frac{r_{-1}}{2} \right\rceil \). Thus from the equality (14) the equality (7) follows. That is (20) is an isomorphism.
Suggest that the highest weight is integer, that is \( r_{-1} \) is even.

Let us first prove that the image of the correspondence contains the linear span of tableaux with \( m'_{-1,n} > 0 \). We obtain the equality

\[
p'_{-1} + \frac{1}{2} + \sum_{k=-2}^{-n} (p_k + q_k) \leq \frac{r-1}{2}.
\]

(22)

Since \( p'_{-1}, p_{-i}, q_{-i} \) are integer we obtain that \( p'_{-1} + \sum_{k=-2}^{-n} (p_k + q_k) \) takes the maximal value not \( \frac{r}{2} \) but \( \frac{r-1}{2} - 1 \).

By Proposition 3 if one decomposes a vector corresponding to a tableau with \( m'_{-1,n} > 0 \) by the base (6) then only monomials with \( p'_{-1} + \sum_{k=-2}^{-n} (p_k + q_k) \leq m_{-1,n} - 1 = \frac{r-1}{2} - 1 \) appear. Thus the image of (20) contains the span of vectors corresponding to tableaux with \( m'_{-1,n} > 0 \).

To prove the inverse embedding let us calculate the dimension of the image and of the span.

We obtained that the inequality (13) for \( p_{-1}, p_k, q_k \) and \( r_{-n}, \ldots, r_{-2}, r_{-1} \) are equivalent to inequalities (13) for \( p'_{-1}, p_k, q_k \) and \( r_{-n}, \ldots, r_{-2}, \frac{r_{-1}}{2} - 1 \).

The corresponding monomials (13), given by \( p'_{-1}, p_k, q_k \) and \( r_{-n}, \ldots, r_{-2}, \frac{r_{-1}}{2} - 1 \), define a base in the space of the problem of restriction \( \mathfrak{gl}_{n+1} \downarrow \mathfrak{gl}_{n-1} \) for the highest weight \([m_{-n} - 1, \ldots, m_{-1} - 1, 0]\). In the same space there exists another base indexed by tableau, which elements are integers

\[
\begin{align*}
m_{-n} - 1 & \quad m_{-n+1} - 1 & \quad \ldots & \quad m_{-1} - 1 & \quad 0 \\
\quad m'_{-n,n} & \quad \ldots & \quad m'_{-1,n} \\
\quad m_{-n,n-1} & \quad \ldots & \quad m_{-2,n-1}
\end{align*}
\]

Hence the dimension of the image of (20) is the number of such tableaux. To each such a tableau there corresponds a tableau composed of integers

\[
\begin{align*}
m_{-n} & \quad m_{-n+1} & \quad \ldots & \quad m_{-1} & \quad 0 \\
\quad m'_{-n,n} + 1 & \quad \ldots & \quad m'_{-1,n} + 1 \\
\quad m_{-n,n-1} + 1 & \quad \ldots & \quad m_{-2,n-1} + 1
\end{align*}
\]
In this tableaux $m'_{-1,n} = m'_{-1,n} + 1 > 0$ and each tableau $n$ with $m'_{-1,n} > 0$ can be written in this manner. Hence the dimension of linear span of tableaux with $m'_{-1,n} > 0$ equals to the dimension of the image of the correspondence (20). Thus they are equal.

6 The Gelfand-Tsetlin-Zhelobenko base in the space of $\mathfrak{o}_{2n-1}$-highest vectors

In the previous Section we investigated the mapping

$$(p_{-1}, p_k, q_k) \mapsto (p'_{-1} = \frac{p_{-1}}{2}, p_k, q_k, \sigma = 0, 1), \ k = -2, ..., -n,$$ (23)

where $\sigma$ is a residue of the division of $p_{-1}$ by 2, with relates to the solutions (6) corresponding to $r_{-n}, ..., r_{-2}, r_{-1}$ the solution (13) corresponding to $r_{-n}, ..., r_{-2}, \lfloor \frac{r_{-1}}{2} \rfloor$.

Compare formulas (6) and (13). We obtain the following statement.

Proposition 6. The mapping inverse to (23) (non everywhere-defined), can be written as follows

$$f\left(z_{-n,-1}, z_{-n,1}, ..., z_{-2,-1}, z_{-2,1}, z_{-1,1}\right) \mapsto z_{0,1}^p f\left(z_{-n,-1}, z_{-n,1}, ..., z_{-2,-1}, z_{-2,1}, \frac{1}{2} z_{0,1}^p\right)$$ (24)

Let us construct Gelfand-Tsetlin tableaux and Gelfand-Tselin base for the restriction problem $\mathfrak{o}_{2n+1} \downarrow \mathfrak{o}_{2n-1}$. For this purpose we construct a special base in the solution space of the restriction problem $\mathfrak{gl}_{n+1} \downarrow \mathfrak{gl}_{n-1}$ that occur on the side in (24).

Suppose that the highest weight as integer. Then in the space of solutions of the restriction problem $\mathfrak{gl}_{n+1} \downarrow \mathfrak{gl}_{n-1}$ c $r_{-n}, ..., r_{-2}, \lfloor \frac{r_{-1}}{2} \rfloor = m_{-1}$ there exist a base encoded by tableaux

$$
\begin{array}{cccc}
m_{-n} & m_{-n+1} & ... & m_{-1} & 0 \\
m'_{-n,n} & ... & m'_{-1,n} \\
m_{n,n-1} & ... & m_{2,n-1}
\end{array}
$$

13
Thus in this case in the solution space of the restriction problem $\mathfrak{o}_{2n+1} \downarrow \mathfrak{o}_{2n-1}$ there exists a base encoded by such tableaux and a number $\sigma = 0, 1$.

Suppose that the highest weight is half integer. In the solution space of the restriction problem $\mathfrak{g}^r_{n+1} \downarrow \mathfrak{g}^r_{n-1}$ c $r_n, ..., r_{-2}, \left\lfloor \frac{r_{-1}}{2} \right\rfloor = m_{-1} - 1/2$ there exists a base encoded by such tableaux

\[
\begin{array}{cccc}
m_{-n} - 1/2 & m_{-n+1} - 1/2 & \ldots & m_{-1} - 1/2 \\
\bar{m}'_{-n,n} & \ldots & \bar{m}'_{-1,n} \\
\bar{m}_{-n,n-1} & \ldots & \bar{m}_{-2,n-1}
\end{array}
\]

Chose another indexation of this base. All element of this tableau is integer. Let us add to each element of this tableau 1/2 and obtain a tableau with half-integer elements. In particular the lower row in a collection of eigenvalues of $E_{-i, -i} + \frac{1}{2}id, \ i = 2, ..., n$.

Thus in this case in the solution space of the restriction problem $\mathfrak{o}_{2n+1} \downarrow \mathfrak{o}_{2n-1}$ there exist a base encoded by tableau of the same type as in the case of integer highest weight but with half-integer elements and a $\sigma = 0, 1$. Thus we have proved the Theorem.

Proposition ?? we described the domain of definition of (24). Thus we come to the Theorem.

**Theorem 1.** Let $m_{-n,n} := m_{-n}, ..., m_{-1,n} := m_{-1}$. Then in the space of $\mathfrak{o}_{2n-1}$-highest vectors in a $\mathfrak{o}_{2n+1}$-representation $V$ there exists a base indexed by tableaux

\[
m_{-n,n} \geq m'_{-n,n} \geq m_{-n+1,n} \geq m'_{-n+1,n} \geq \ldots \geq m_{-1,n} \geq m'_{-1,n} \geq 0 \\
\sigma \ m'_{-n,n} \geq m_{-n,n-1} \geq m'_{-n+1,n} \geq m_{-n+1,n-1} \geq \ldots \geq m_{-2,n-1} \geq m'_{-1,n}
\]

Here $\sigma$ takes only values 1 and 0, and other numbers are simultaneously integers or half-integers. If the highest weight is integer and $m'_{-1,n} = 0$ then $\sigma = 0$.

As in the case $\mathfrak{sp}_{2n}$ the following statement takes place.

**Proposition 7.** The lower row of the tableau (25) is a $\mathfrak{o}_{2n-1}$-weight of the corresponding $\mathfrak{o}_{2n-1}$-highest vector.
Proof. In the case of integer highest weight the correspondence \((24)\) conjugates the actions of \(E_{-i,-1}\) and \(F_{-i,-1}\) for \(i = n, \ldots, 2\). This follows from formulas \((8)\) and \((15)\). The lower row of \((25)\) is a collection of eigenvalues of \(E_{-i,-1}\) for \(i = n, \ldots, 2\). Hence after application of \((24)\) to this tableau we obtain a vector encoded by \((25)\) and the lower row of \((25)\) is a collection of eigenvalues of \(F_{-i,-1}\) for \(i = n, \ldots, 2\).

In the case of half-integer highest weight the formulas \((8)\) and \((15)\) show that the correspondence \((24)\) conjugates the actions \(F_{-i,-1} f\) and \(E_{-i,-1} f + \frac{1}{2} f\). But in this case the tableau \((25)\) is constructed in such way that \(m_{-i,n-1}\) is an eigenvalue of \(E_{-i,-1} f + \frac{1}{2} f\).

Using Theorem \([1]\) and Proposition \([7]\) one can construct the Gelfand-Tsetlin-Zhelobenko base vectors in a representation of \(\mathfrak{so}_{2n+1}\).

Let us write the formula for the \((-1)\)-component of the weight of the vector defined by a tableau

**Proposition 8.** The \((-1)\)-component of the weight equals

\[
\sigma + 2 \sum_{k=1}^{n} m'_{-k,n} - \sum_{k=1}^{n} m_{-k,n} + \sum_{k=2}^{n} m_{-k,n-1}.
\]

**Proof.** Let the highest weight be integer. By \((8)\) and \((15)\) one has

\[
F_{-1,-1} f = \sum_{i=2}^{n} (z_{-i,-1} \frac{\partial}{\partial z_{-i,-1}} - z_{-i,1} \frac{\partial}{\partial z_{-i,1}}) f - z_{0,1} \frac{\partial}{\partial z_{0,1}} f - m_{-1} f,
\]

\[
(E_{-1,-1} - E_{1,1}) f = \sum_{i=2}^{n} (z_{-i,-1} \frac{\partial}{\partial z_{-i,-1}} - z_{-i,1} \frac{\partial}{\partial z_{-i,1}}) f - 2 z_{-1,1} \frac{\partial}{\partial z_{-1,1}} f - m_{-1} f.
\]

Let \(\sigma = 0\). The under the correspondence \((24)\) the change of variables

\[
z_{-1,1} \mapsto \frac{z_{-1,1}^2}{2}
\]

is performed. Thus the actions \(2 z_{-1,1} \frac{\partial}{\partial z_{-1,1}}\) and \(z_{0,1} \frac{\partial}{\partial z_{0,1}}\) are conjugated. Hence the actions of \(F_{-1,-1}\) and \(E_{-1,-1} - E_{1,1}\) are conjugated. Thus the eigenvalue of \(E_{-1,-1}\) on the vector \((25)\) is a difference of eigenvalues of \(E_{-1,-1}\) and \(E_{1,1}\). These eigenvalues are equal to \(\sum_{k=1}^{n} m'_{-k,n} - \sum_{k=2}^{n} m_{-k,n-1}\) and \(\sum_{k=1}^{n} m_{-k,n} - \sum_{k=1}^{n} m'_{-k,n}\). The difference of these expressions is \((26)\).

In the case \(\sigma = 1\) при соответствии \((24)\) происходит еще умножение на \(z_{0,1}\). Поэтому к разности собственных значений \(E_{-1,-1}\) и \(E_{1,1}\) добавляется 1.

Suppose that the weight is half-integer. Let \(\sigma = 0\). Then by formulas \((8)\) and \((15)\) under the action of \((24)\) the actions of \(F_{-1,-1}\) and \(E_{-1,-1} - E_{1,1} + \frac{1}{2} id\)
are conjugated. In term of the tableau (25) the eigenvalues of $E_{-1,-1}$ and $E_{1,1}$ are equal $\sum_{k=1}^{m} m_{k,n} - \sum_{k=2}^{n} m_{k,n-1} - \frac{1}{2}$ and $\sum_{k=1}^{m} m_{-k,n} - \sum_{k=1}^{n} m_{-k,n}$. Their difference plus $\frac{1}{2}$ equals to (26).

In the case $\sigma = 1$ to this expression 1 must be added.

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