ASYMMETRIC BURKHOLDER INEQUALITIES IN NONCOMMUTATIVE SYMMETRIC SPACES

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Abstract. In this paper, we establish noncommutative Burkholder inequalities with asymmetric diagonals in symmetric operator spaces. Our proof mainly relies on a noncommutative good-\(\lambda\) approach and a new complex interpolation result on asymmetric vector valued spaces. We include as well the asymmetric versions of noncommutative Johnson-Schechtman inequalities.

1. Introduction

Based on duality arguments, an asymmetric version of Doob’s maximal inequalities for noncommutative martingales was established by Junge [30]. This result answered a question posed by Pisier and initiated the study of asymmetric martingale inequalities in noncommutative setting. Via algebraic atomic decompositions and Davis decompositions, Hong et al. [21] further discussed asymmetric maximal inequalities in noncommutative \(L_p\) spaces. Recently, their results were extended to the context of noncommutative symmetric spaces in [54].

Following [30, 21, 54], the purpose of this paper is to discuss asymmetric Burkholder inequalities for noncommutative martingales in symmetric operator spaces.

To put our results in an appropriate context, let us present some development of Burkholder’s inequalities in the classical case. Inspired by searching for Banach spaces linearly isomorphic to a complemented subspace of an \(L_p\) space, Rosenthal [55] established the following statement: for \(2 \leq p < \infty\) and for every sequence of independent mean zero random variables \((g_n)_{n \geq 1}\) in \(L_p\), we have

\[
\left\| \sum_{n \geq 1} g_n \right\|_p \precsim_p \left( \sum_{n \geq 1} \|g_n\|_2^2 \right)^{\frac{1}{2}} + \left( \sum_{n \geq 1} \|g_n\|_p^p \right)^{1/p},
\]

where the notation \(A \precsim_p B\) means that there exists a constant \(c_p\) depending only on \(p\) such that \(c_p^{-1}B \leq A \leq c_pB\). In 1973, Burkholder [7] extended the equivalence (1.1) to the martingale setting. It was shown in [7] that if \(2 \leq p < \infty\) and \((f_n)_{n \geq 1}\) is an \(L_p\) bounded martingale (adapted to some discrete-time filtration \((\Sigma_n)_{n \geq 1}\)), then

\[
\|f\|_p \precsim_p \|s(f)\|_p + \left( \sum_{n \geq 1} \|df_n\|_p^p \right)^{1/p},
\]

where \(df_n\) is the martingale difference of \(f\) and \(s(f)\) is the conditioned square function of \(f\). Moreover, as proved in [7], the diagonal term \(\left( \sum_{n} \|df_n\|_p^p \right)^{1/p}\) can be replaced by the maximal function of martingale difference sequence; namely, if \(2 \leq p < \infty\) and if \(f \in L_p\),
then
\[
\|f\|_p \simeq_p \|s(f)\|_p + \sup_n \|df_n\|_p.
\]

The above inequalities (1.1), (1.2) and (1.3) have turned out to be very useful probabilistic tools and have been found many interesting generalizations and applications; see for instance [1, 2, 3, 9, 20, 24, 29, 36].

In 1997, Pisier and Xu [45] formulated a noncommutative analogue of Burkholder-Gundy inequalities. This pioneering work stimulates the development of noncommutative martingale theory. A lot of classical martingale inequalities have been gradually generalized to the noncommutative setting; see [6, 11, 14, 30, 31, 34, 35, 47, 48, 49, 50] and many others. In particular, the work due to Junge and Xu [34, 35] provided a comprehensive study of (1.1), (1.2) and (1.3) in the context of noncommutative \( L_p \) spaces. One of the main results in [34] can be summarized as follows: if \( 2 \leq p < \infty \) and if \( x \in L_p(M) \), then
\[
\|x\|_p \simeq_p \max \left\{ \|s_c(x)\|_{p'}, \|s_r(x)\|_{p'}, \left( \sum_{n \geq 1} \|dx_n\|_p^{1/p} \right)^{1/p} \right\},
\]

where \( s_c(x) \) and \( s_r(x) \) denote the column and the row versions of conditioned square functions for which we refer to the next section for concrete definitions. Here and what follows, \((M, \tau)\) will always denote a finite von Neumann algebra equipped with a faithful normal finite trace \( \tau \) with \( \tau(1) = 1 \). Moreover, by using an interpolation method, Junge and Xu [35] proved that, similar to the classical case, the diagonal term in (1.4) can be replaced by a maximal function version: if \( 2 < p < \infty \) and if \( x \in L_p(M) \), then
\[
\|x\|_p \simeq_p \max \left\{ \|s_c(x)\|_{p'}, \|s_r(x)\|_{p'}, \left( \sum_{n \geq 1} \|dx_n\|_p^{1/p} \right)^{1/p} \right\},
\]

The above results (1.4), (1.5) can be regarded as noncommutative extensions of (1.2), (1.3), respectively.

Recently, the equivalence (1.4) was investigated in more general symmetric operator spaces by some authors (c.f. [10, 14, 51, 53]). In particular, the following is proved in [53]: if the symmetric Banach function space \( E \) with Fatou property lies in \( \text{Int}(\mathcal{L}_2, \mathcal{L}_q) \) for some \( 2 < q < \infty \) and if \( x \in E(M) \), then
\[
\|x\|_E \simeq_E \max \left\{ \|s_c(x)\|_{E'}, \|s_r(x)\|_{E'}, \left( \sum_{n \geq 1} \|dx_n \otimes e_n\|_{E(M, \ell_q^\infty)} \right)^{1/q} \right\},
\]

where \((e_n)_{n \geq 1}\) are the standard unit vectors in \( \ell_q \). Applying a noncommutative good-\( \lambda \) approach (we refer to [23] for more information about this method), Jiao, Zani and Zhou [28] generalized (1.5) to the symmetric operator spaces: if the symmetric Banach function space \( E \) lies in \( \text{Int}(\mathcal{L}_p, \mathcal{L}_q) \) for \( 2 < p \leq q < \infty \) and if \( x \in E(M) \), then
\[
\|x\|_E \simeq_E \max \left\{ \|s_c(x)\|_{E'}, \|s_r(x)\|_{E'}, \left( \sum_{n \geq 1} \|dx_n\|_{E(M, \ell_q^\infty)} \right)^{1/q} \right\},
\]

The purpose of this paper is to explore asymmetric versions of (1.7), in the sense that the maximal diagonal part is replaced with an “asymmetric” maximal diagonal term. As mentioned in the beginning, this is motivated by recent progress on asymmetric maximal inequalities of noncommutative martingales (\[30, 21, 54\]).

Our first version of asymmetric Burkholder inequalities reads as follows. We should mention that a special case (i.e., \( \theta=1 \) or 0) of the following result was achieved in [54] by using Davis decompositions.

**Theorem 1.1.** Let \( 0 \leq \theta \leq 1 \). Assume that \( E \) is a symmetric Banach function space which is an interpolation of the couple \((\mathcal{L}_p, \mathcal{L}_q)\) for \( 2 < p \leq q < \infty \). If \( x \in E(M) \), then
\[
\|x\|_E \simeq_E \max \left\{ \|s_c(x)\|_{E'}, \|s_r(x)\|_{E'}, \left( \sum_{n \geq 1} \|dx_n\|_{E(M, \ell_q^\infty)} \right)^{1/q} \right\}.
\]
We might also prove a stronger version of asymmetric Burkholder inequalities under a more strict assumption on the symmetric function space.

**Theorem 1.2.** Let \( 0 \leq \theta \leq 1 \). Assume that \( E \) is a symmetric Banach function space which is an interpolation of the couple \((L_p, L_q)\) for \( 2 < p \leq q < \infty \). If \( E \) has Fatou norm and is \( s \)-concave, then for \( x \in E(\mathcal{M}) \), we have

\[
\|x\|_E \simeq_{E} \max \left \{ \|s_e(x)\|_E, \|s_r(x)\|_E, \|(dx_n)_{n \geq 1}\|_E(\mathcal{M}; \ell^q_n) \right \}.
\]

Once Theorem 1.1 and Theorem 1.2 have been established, a natural question is to consider the dual versions. Our third result deals with this problem.

**Theorem 1.3.** Let \( 0 \leq \theta \leq 1 \). Assume that \( E \) is a symmetric Banach function space which is an interpolation of the couple \((L_p, L_q)\) for \( 1 < p \leq q < 2 \). If \( E \) is \( s \)-concave and \( x \in E(\mathcal{M}) \), then

\[
\|x\|_E \simeq_{E} \inf \left \{ \|s_e(y)\|_E + \|s_r(z)\|_E + \|(dw_n)_{n \geq 1}\|_E(\mathcal{M}; \ell^q_n) \right \},
\]

where the infimum is taken over all decompositions \( x = y + z + w \) with \( y, z \) and \( w \) being martingales.

As one may notice, Theorem 1.1, Theorem 1.2, and Theorem 1.3 are new even for \( E = L_p \). They strengthen the results from [14, 28, 51, 53], and in particular, Theorem 1.1 improves [54, Corollary 4.11] greatly.

The proof of Theorem 1.1 is mainly based on a noncommutative good-\( \lambda \) approach. Our strategy for the proof of Theorem 1.2 and Theorem 1.3 are as follows. To prove Theorem 1.2, we establish a new complex interpolation result for asymmetric vector valued spaces. Then, using a duality approach, we may deduce Theorem 1.3 from Theorem 1.2.

Our arguments for Theorem 1.2 can also be applied to get an asymmetric version of noncommutative Johnson-Schechtman inequalities. Recall that Johnson and Schechtman [29] extended the Rosenthal inequalities (1.1) to the symmetric Banach function spaces. This result has been extensively investigated in the noncommutative setting (see [57, 25, 26, 27]). Recently, a version of noncommutative Johnson-Schechtman inequalities involving maximal diagonal term was given in [28, Theorem 1.5]. In this paper, by exploiting our new complex interpolation, we further extend their result to the asymmetric case (see Theorem 7.2 below).

The paper is organized as follows. In the next preliminary section, we review the basics of noncommutative spaces and noncommutative martingales. This section also includes some background on the pointwise product of functions spaces and complex interpolation method. Section 3 is devoted to studying several fundamental properties of the asymmetric vector valued spaces \( E(\mathcal{M}; \ell^q_\infty) \) and \( E(\mathcal{M}; \ell^q_1) \). In Section 4, we will establish a complex interpolation result for \( E(\mathcal{M}; \ell^q_\infty) \), which constitutes one of the key ingredients in the proof of asymmetric inequalities. This result, together with a complementary argument, yields the corresponding interpolation result for the martingale Hardy spaces \( h^{\infty_a}_E \). Section 5 contains the proofs of asymmetric Burkholder inequalities (namely, Theorem 1.1 and Theorem 1.2). A dual theorem between \( E(\mathcal{M}; \ell^q_\infty) \) and \( E(\mathcal{M}; \ell^q_1) \) is established in Section 6. Combining the duality with a complementary result, we also prove a similar duality for the martingale Hardy spaces \( h^{\infty_a}_E \) and \( h^1_E \). Using this duality, we deduce Theorem 1.3 from Theorem 1.2. The last section focus on asymmetric form of Johnson-Schechtman inequalities.

We close the introduction with an interesting question.

**Question 1.4.** Let \( 0 \leq \theta \leq 1 \). Assume that \( E \) is a symmetric Banach function space satisfies \( E \in \text{Int}(L_p, L_q) \) with \( \max\{2(1-\theta), 2\theta\} < p \leq q < \infty \). For \( x \in E(\mathcal{M}) \), do we have

\[
\inf \left \{ \|s_e(y)\|_E + \|s_r(z)\|_E + \|(dw_n)_{n \geq 1}\|_E(\mathcal{M}; \ell^q_n) \right \} \lesssim_{E} \|x\|_E,
\]
where the infimum is taken over all decompositions \( x = y + z + w \) with \( y, z, w \) being martingales, and
\[
\|x\|_E \lesssim E \max \left\{ \|s_c(x)\|_E, \|s_r(x)\|_E, \|(dx_n)_{n \geq 1}\|_{E(M, \mathcal{M}_n^{\alpha})} \right\}.
\]

2. Preliminaries

Throughout this paper, we write \( A \lesssim B \) if there is a constant \( C_\alpha \) depending only on the parameter \( \alpha \) such that the inequality \( A \leq C_\alpha B \) is satisfied, and write \( A \simeq B \) if both \( A \lesssim B \) and \( B \lesssim A \) hold.

2.1. Noncommutative symmetric spaces. Throughout, \((\mathcal{M}, \tau)\) will always denote a finite von Neumann algebra equipped with a faithful normal finite trace \( \tau \). Without loss of generality, we assume that \( \tau(1) = 1 \). Let \( L_0(\mathcal{M}, \tau) \) be the associated topological \( * \)-algebra of measurable operators in the sense of \([42]\); see also \([58]\) for more details. Since \( \tau \) is finite, \( L_0(\mathcal{M}, \tau) \) consists of all the operators affiliated to \( \mathcal{M} \). For \( x \in L_0(\mathcal{M}, \tau) \), we define its generalized singular number \( \mu(x) \) by
\[
\mu(x) = \inf\{s > 0 : \tau(\chi_{(s, \infty)}(|x|)) \leq t\}, \quad 0 < t \leq 1,
\]
where \( \chi_{(s, \infty)}(|x|) \) is the spectral projection of \( |x| \). If \( \mathcal{M} \) is the abelian von Neumann algebra \( L_\infty(0, 1) \) with the trace given by integration with respect to Lebesgue measure, then \( L_0(\mathcal{M}, \tau) \) is the space of measurable complex valued functions on \((0, 1] \). For \( f \in L_0(\mathcal{M}, \tau) \), \( \mu(f) \) is the usual decreasing rearrangement of \( |f| \). We refer to \([46]\) for more information on noncommutative integration.

A Banach (or, quasi Banach) function space \((E, \| \cdot \|_E)\) of measurable functions on the interval \((0, 1] \) is called symmetric if for every \( g \in E \) and every measurable function \( f \) with \( \mu(f) \leq \mu(g) \), we have \( f \in E \) and \( \|f\|_E \leq \|g\|_E \). Moreover, \( E \) is called fully symmetric if for every \( g \in E \) and every measurable function \( f \) with \( \mu(f) \ll \mu(g) \) (which means \( \int_0^t \mu(s, f) ds \leq \int_0^t \mu(s, g) ds \) for \( 0 < t \leq 1 \)), we have \( f \in E \) and \( \|f\|_E \leq \|g\|_E \).

A symmetric Banach function space \( E \) is said to have Fatou norm if for every net \( (f_\beta) \subset E \) and \( f \in E \) satisfying \( 0 \leq f_\beta \uparrow f \), we have \( \|f_\beta\|_E \uparrow \|f\|_E \). A symmetric Banach space \( E \) is said to have Fatou property if for every net \( (f_\beta) \subset E \) and measurable function \( f \) satisfying \( 0 \leq f_\beta \uparrow f \) and \( \sup_\beta \|f_\beta\|_E < \infty \), we have \( f \in E \) and \( \|f_\beta\|_E \uparrow \|f\|_E \). We say that \( E \) has order continuous norm if for every net \((f_\beta)\) in \( E \) such that \( f_\beta \downarrow 0 \) we have \( \|f_\beta\|_E \downarrow 0 \).

For a symmetric Banach space \( E \), we define the corresponding noncommutative space by
\[
E(\mathcal{M}, \tau) = \left\{ x \in L_0(\mathcal{M}, \tau) : \mu(x) \in E \right\}.
\]
Endowed with the norm \( \|x\|_{E(\mathcal{M}, \tau)} := \|\mu(x)\|_E \), the linear space \( E(\mathcal{M}, \tau) \) becomes a Banach space \((\mathcal{M}, \tau)\) and is usually referred to as the noncommutative symmetric space associated with \((\mathcal{M}, \tau)\) and \( E \). An extensive discussion of the various properties of such spaces can be found in \([17, 46, 59]\). We remark that in particular if \( E = L_p(0, 1) \) with \( 1 \leq p < \infty \), then \( E(\mathcal{M}, \tau) = L_p(\mathcal{M}, \tau) \) where \( L_p(\mathcal{M}, \tau) \) is the noncommutative \( L_p \)-space associated with \((\mathcal{M}, \tau)\). In the sequel, \( E(\mathcal{M}, \tau) \) will be abbreviated to \( E(\mathcal{M}) \) and the norm will be denoted by \( \| \cdot \|_E \); \( E(\mathcal{M})^+ \) is denoted by the set of all positive elements in \( E(\mathcal{M}) \).

In this paper, we consider symmetric spaces that are interpolations of the couple \((L_p, L_q)\) for \( 1 \leq p < q < \infty \). For a given compatible Banach couple \((X, Y)\), we recall that a Banach space \( Z \) is called an interpolation space if \( X \cap Y \subseteq Z \subseteq X + Y \) and whenever a bounded linear operator \( T : X + Y \to X + Y \) is such that \( T(X) \subseteq X \) and \( T(Y) \subseteq Y \), we have \( T(Z) \subseteq Z \) and \( \|T : Z \to Z\| \leq C \max\{\|T : X \to X\|, \|T : Y \to Y\|\} \) for some constant \( C \). In this case, we write \( Z \in \text{Int}(X, Y) \).
Lemma 2.1 ([38, Theorem II.3.4]). A symmetric quasi Banach function space $E$ lies in $\text{Int}(L_p, L_\infty)$ if and only if $E$ is fully symmetric.

Given a symmetric space $E$, the Köthe dual of a symmetric space $E$ is the function space defined by setting

$$E^\times = \left\{ f \in L_0(0, 1) : \int_0^1 |f(t)g(t)| \, dt < \infty, \forall g \in E \right\};$$

$$\|f\|_{E^\times} := \sup \left\{ \int_0^1 |f(t)g(t)| \, dt : \|g\|_E \leq 1 \right\}, \quad f \in E^\times.$$

The space $E^\times$ is fully symmetric and has the Fatou property.

A symmetric Banach function space $E$ on $(0, 1]$ has a Fatou norm if and only if $E$ embeds isometrically into its second Köthe dual $E^{\times \times}$. It has the Fatou property if and only if $E = E^{\times \times}$ isometrically. It has order continuous norm if and only if it is separable, which is also equivalent to the statement $E^\times = E$. It is well-known that if $E \in \text{Int}(L_p, L_q)$ for some $1 \leq p < q \leq \infty$, then $E^\times \in \text{Int}(L_{q'}, L_{p'})$ where $p'$ and $q'$ denote the conjugate indices of $p$ and $q$ respectively.

For $0 < r < \infty$, the $r$-convexification of a Banach function space $E$ is defined by

$$E^{(r)} := \left\{ f \in L_0(0, 1) : |f|^r \in E \right\}$$

equipped with the norm (or, quasi-norm)

$$\|f\|_{E^{(r)}} = \| |f|^r \|_E^{1/r}.$$

It is easy to verify that if $E$ is symmetric then so is $E^{(r)}$. Moreover, according to [15, Proposition 3.5], if $E \in \text{Int}(L_p, L_q)$ for some $1 \leq p < q \leq \infty$, then we have

$$E^{(r)} \in \text{Int}(L_{pr}, L_{q'r}).$$

These facts will be used repeatedly throughout.

Let $0 < p, q < \infty$. A symmetric quasi Banach function space $E$ is said to be $p$-convex if there is a constant $c > 0$ such that for any finite sequence $(f_i)_{i=1}^n$ in $E$ we have

$$\left\| \left( \sum_{i=1}^n |f_i|^p \right)^{1/p} \right\|_E \leq c \left( \sum_{i=1}^n \|f_i\|_E^p \right)^{1/p}.$$

The least constant satisfying the above inequality is called the $p$-convexity constant of $E$ and will be denoted by $M^{(p)}(E)$. A symmetric quasi Banach function space $E$ is said to be $q$-concave if there is a constant $c > 0$ such that for any finite sequence $(f_i)_{i=1}^n$ in $E$ we have

$$\left( \sum_{i=1}^n \|f_i\|_E^q \right)^{1/q} \leq c \left\| \left( \sum_{i=1}^n |f_i|^q \right)^{1/q} \right\|_E.$$

The least constant satisfying the above inequality is called the $q$-concavity constant of $E$ and will be denoted by $M^{(q)}(E)$. From the definitions, one can easily show that if $E$ is $p$-convex and $q$-concave for $0 < p \leq q < \infty$, then $E^{(r)}$ is $pr$-convex and $qr$-concave. Moreover, if $E$ is $p$-convex for some $0 < p < \infty$, then $E$ is $r$-convex for any $0 < r \leq p$ and $M^{(r)} \leq M^{(p)}$. Similarly, if $E$ is $q$-concave for some $0 < q < \infty$, then $E$ is $s$-concave for any $q \leq s \leq \infty$ and $M^{(s)} \leq M^{(q)}$. It is also well known that if $E$ is $q$-concave for some $0 < q < \infty$, then $E$ has order continuous norm, which is equivalent to saying that $E$ is separable (see [13, Lemma 4.12]).

The following result is taken from [18, Proposition 3.4].
Proposition 2.2. If $E$ is a symmetric Banach function space with Fatou norm, and if $E$ is $p$-convex for some $0 < p < \infty$, then there exists a symmetric norm $\| \cdot \|$ on $E^{(1/p)}$ such that
\[
\|x\|_E \leq \|x\|_{E^{(1/p)}} \leq M^{(p)}(E) \|x\|_E.
\]

We now discuss the pointwise product of function spaces that we will need in the sequel. Let $E_i$ ($i = 1, 2$) be symmetric Banach function spaces on $(0, 1]$. We define the pointwise product of $E_1$ and $E_2$ by setting:
\[
E_1 \odot E_2 = \{ x : x = x_1 x_2, x_1 \in E_1, x_2 \in E_2 \}.
\]
For $f \in E_1 \odot E_2$, set
\[
\|f\|_{E_1 \odot E_2} := \inf \{ \|f_1\|_{E_1} \|f_2\|_{E_2} : f_1 \in E_1, f_2 \in E_2, f = f_1 f_2 \}.
\]
According to [39, Theorem 2], $E_1 \odot E_2$ is a symmetric quasi Banach function space when equipped with the quasi norm $\| \cdot \|_{E_1 \odot E_2}$. Similarly, one can also define the product of symmetric operator spaces as follows:
\[
E_1(M) \odot E_2(M) = \{ x : x = x_1 x_2, x_1 \in E_1(M), x_2 \in E_2(M) \}.
\]
For $x \in E_1(M) \odot E_2(M)$, we define
\[
\|x\|_{E_1(M) \odot E_2(M)} := \inf \{ \|x_1\|_{E_1} \|x_2\|_{E_2} : x_1 \in E_1(M), x_2 \in E_2(M), x = x_1 x_2 \}.
\]
From [56, Theorems 3, 4] (see also [4, Theorem 2.5]), we know that if $E = E_1 \odot E_2$, then $E(M) = E_1(M) \odot E_2(M)$.

We record some facts about pointwise products of symmetric spaces for further use.

Lemma 2.3 ([39, Corollary 2, Theorem 1]). Let $X$, $Y$ be symmetric Banach function spaces. Then the following hold:
(i) if $1 < p < \infty$, then $[X^{(p)}]^\times = [X^\times]^{(p)} \odot L_p$, where $p'$ is the conjugate index of $p$;
(ii) if $0 < p < \infty$, then $(X \odot Y)^{(p)} = X^{(p)} \odot Y^{(p)}$.

Lemma 2.4 ([5, Lemma 3.1]). Let $E$, $E_1$ and $E_2$ be symmetric Banach function spaces on $(0, 1]$ such that $E = E_1 \odot E_2$. If $x \in E(M)^+$, then for any $\varepsilon > 0$, there exist $a \in E_1(M)^+$ and $b \in E_2(M)^+$ such that $x = ab$,
\[
\|a\|_{E_1} \|b\|_{E_2} \leq (1 + \varepsilon) \|x\|_E
\]
and $a$ is invertible with bounded inverse.

2.2. Martingales and Hardy spaces. We now describe the general setup for martingales in noncommutative symmetric spaces. Denote by $(M_n)_{n \geq 1}$ an increasing sequence of von Neumann subalgebras of $M$ whose union is weak*-dense in $M$. For $n \geq 1$, we assume that there exists a trace preserving conditional expectation $\mathcal{E}_n$ from $M$ onto $M_n$. It is well-known that if $\tau_n$ denotes the restriction of $\tau$ on $M_n$, then $\mathcal{E}_n$ extends to a contractive projection from $L_p(M, \tau)$ onto $L_p(M_n, \tau_n)$ for all $1 \leq p \leq \infty$. More generally, if $E \in \text{Int}(L_1, L_\infty)$, then for every $n \geq 1$, $\mathcal{E}_n$ is bounded from $E(M, \tau)$ onto $E(M_n, \tau_n)$.

Definition 2.5. A sequence $x = (x_n)_{n \geq 1}$ in $L_1(M)$ is called a noncommutative martingale with respect to $(M_n)_{n \geq 1}$ if $\mathcal{E}_n(x_{n+1}) = x_n$ for every $n \geq 1$.

If, in addition, all $x_n$’s belong to $E(M)$, then $x$ is called an $E(M)$-martingale. In this case, we set
\[
\|x\|_E = \sup_{n \geq 1} \|x_n\|_E.
\]
If $\|x\|_E < \infty$, then $x$ is called a bounded $E(M)$-martingale.

Let $x = (x_n)_{n \geq 1}$ be a noncommutative martingale with respect to $(M_n)_{n \geq 1}$. Define $dx_n = x_n - x_{n-1}$ for $n \geq 2$ and $dx_1 = x_1$. The sequence $dx = (dx_n)_{n \geq 1}$ is called the martingale difference sequence of $x$. A martingale $x$ is called a finite martingale if there
exists $N$ such that $dx_n = 0$ for all $n \geq N$. In the sequel, for any operator $x \in L_1(\mathcal{M})$, we denote $x_n = \mathcal{E}_n(x)$ for $n \geq 1$.

Let us now review the definitions of the conditioned square functions and martingale conditioned Hardy spaces for the case of noncommutative symmetric spaces. Let $x = (x_n)_{n \geq 1}$ be a martingale in $L_2(\mathcal{M})$. We set (with the convention that $\mathcal{E}_0 = \mathcal{E}_1$):

$$s_{c,n}(x) = \left(\sum_{k=1}^{n} \mathcal{E}_{k-1}|dx_k|^2\right)^{\frac{1}{2}}, \quad s_c(x) = \left(\sum_{k=1}^{\infty} \mathcal{E}_{k-1}|dx_k|^2\right)^{\frac{1}{2}}.$$

The operator $s_{c}(x)$ is called the column conditioned square function of $x$. In order to define the conditioned Hardy spaces, we need to use Junge’s representation of conditioned spaces ([30]). The corresponding extension to noncommutative symmetric spaces can be found in [51, 52, 53], but since we will need some of the finer details in our proofs, we include a brief description.

For every $n \geq 1$ and $1 \leq p < \infty$, define the space $L_p^\mathcal{E}(\mathcal{M}, \mathcal{E}_n)$ to be the completion of $\mathcal{M}$ with respect to the quasi-norm

$$\|x\|_{L_p^\mathcal{E}(\mathcal{M}, \mathcal{E}_n)} = \|\mathcal{E}_n(x^*)x\|_{p/2}.$$

According to [30, Proposition 2.8], there exists an isometric right $\mathcal{M}_n$-module map

$$u_{n,p} : L_p^\mathcal{E}(\mathcal{M}, \mathcal{E}_n) \to L_p(\mathcal{M}_n; \ell_2^c)$$

such that

$$(2.3) \quad u_{n,p}(x^*)u_{n,q}(y) = \mathcal{E}_n(x^*y) \otimes e_{1,1}$$

for all $x \in L_p^\mathcal{E}(\mathcal{M}, \mathcal{E}_n)$ and $y \in L_q^\mathcal{E}(\mathcal{M}, \mathcal{E}_n)$ with $1/p + 1/q \leq 1$.

Denote by $\mathcal{F}$ the collection of all finite sequences $(a_n)_{n \geq 1}$ in $\mathcal{M}$. For $1 \leq p \leq \infty$, define the space $L_p^{\mathcal{E}}(\mathcal{M}; \ell_2^c)$ to be the completion of $\mathcal{F}$ with respect to the norm

$$\| (a_n)_{n \geq 1} \|_{L_p^{\mathcal{E}}(\mathcal{M}; \ell_2^c)} = \left\| \left( \sum_{n \geq 1} \mathcal{E}_{n-1}|a_n|^2 \right)^{\frac{1}{2}} \right\|_p.$$

The space $L_p^{\mathcal{E}}(\mathcal{M}; \ell_2^c)$ can be isometrically embedded into an $L_p$-space associated to a semifinite von Neumann algebra by means of the following map:

$$U_p : L_p^{\mathcal{E}}(\mathcal{M}; \ell_2^c) \to L_p(\mathcal{M} \otimes B(\ell_2(\mathbb{N}^2)))$$

defined by setting

$$U_p((a_n)_{n \geq 1}) = \sum_{n \geq 1} u_{n-1,p}(a_n) \otimes e_{n,1}.$$

From (2.3), it follows that if $(a_n)_{n \geq 1} \in L_p^{\mathcal{E}}(\mathcal{M}; \ell_2^c)$ and $(b_n)_{n \geq 1} \in L_q^{\mathcal{E}}(\mathcal{M}; \ell_2^c)$ for $1/p + 1/q \leq 1$, then

$$U_p((a_n)_{n \geq 1})^*U_q((b_n)_{n \geq 1}) = \left( \sum_{n \geq 1} \mathcal{E}_{n-1}(a_n^*b_n) \right) \otimes e_{1,1} \otimes e_{1,1}.$$

In particular, $\| (a_n)_{n \geq 1} \|_{L_p^{\mathcal{E}}(\mathcal{M}; \ell_2^c)} = \| U_p((a_n)_{n \geq 1}) \|_p$ and hence $U_p$ is indeed an isometry. An important fact here is that $U_p$ is independent of $p$ and hence we will simply write $U$ for $U_p$.

Now, we generalize the notion of conditioned spaces to the setting of symmetric spaces. Fix a symmetric Banach function space $E$. We consider the algebraic linear map $U$ restricted to the linear space $\mathcal{F}$ that takes its values in $L_1(\mathcal{M} \otimes B(\ell_2(\mathbb{N}^2))) \cap (\mathcal{M} \otimes B(\ell_2(\mathbb{N}^2)))$. For a given sequence $(a_n)_{n \geq 1} \in \mathcal{F}$, we set:
This is well-defined and induces a norm on the linear space $\mathcal{F}$. We define the Banach space $E^{cond}(\mathcal{M}; E^2)$ to be the completion of $\mathcal{F}$ with respect to the above norm. Then $U$ extends to an isometry from $E^{cond}(\mathcal{M}; E^2)$ into $E(\mathcal{M}, X)$ which we still denote by $U$.

Let $x = (x_n)_{n \geq 1}$ be a finite martingale in $E(\mathcal{M})$. Define
\[
\|x\|_{E(\mathcal{M})} = \|s_c(x)\|_{E(\mathcal{M})}.
\]
From the definition of $E^{cond}(\mathcal{M}; E^2)$, one can easily see that $\|\cdot\|_{E(\mathcal{M})}$ is a norm on the set of all finite martingales in $\mathcal{M}$. The conditioned Hardy space $h_E^c(\mathcal{M})$ is defined as the completion of the set all finite martingales under the norm $\|\cdot\|_{E(\mathcal{M})}$.

If we denote by $\mathcal{D}_c : h_E^c(\mathcal{M}) \to E^{cond}(\mathcal{M}; E^2)$ the “extension” of the natural map $\mathcal{D}_c : \mathcal{M} \to h_E^c(\mathcal{M})$, then its composition with $U$ induces the isometric embedding:
\[
UD_c : h_E^c(\mathcal{M}) \to E(\mathcal{M}, X)
\]
with the property that if $x \in h_E^c(\mathcal{M})$ and $y \in h_E^c(\mathcal{M})$ then
\[
(UD_c(x))^* UD_c(y) = \left( \sum_{n \geq 1} e_n - 1 (dx_n^* dy_n) \right) \otimes e_{1,1} \otimes e_{1,1}.
\]
In particular, whenever $s_c(x)$ is a well-defined operator, we have
\[
\|UD_c(x)\|^2 = (s_c(x))^2 \otimes e_{1,1} \otimes e_{1,1}.
\]

Similarly, we may define the row conditioned square functions $s_r(x)$ and row conditioned Hardy spaces $h_E^r(\mathcal{M})$.

2.3. Complex interpolation. Let $(X_0, X_1)$ be an interpolation couple of Banach spaces; namely $X_j, j = 0, 1$ are continuously embedded into a Hausdorff topological vector space $Y$. Let $\mathcal{S}$ (respectively, $\overline{\mathcal{S}}$) denote the open strip $\{z : 0 < \text{Re} z < 1\}$ (respectively, the closed strip $\{z : 0 \leq \text{Re} z \leq 1\}$) in the complex plane $\mathbb{C}$. Denote by $\partial_0 = \{z \in \overline{\mathcal{S}} : \text{Re} z = 0\}$, $\partial_1 = \{z \in \overline{\mathcal{S}} : \text{Re} z = 1\}$ the boundaries of $\overline{\mathcal{S}}$. Let $\mathcal{F}(X_0, X_1)$ be the space of bounded analytic functions $f : \mathcal{S} \to X_0 + X_1$ which extend continuously to $\overline{\mathcal{S}}$ such that the functions $t \mapsto f(j + it)$ are bounded and continuous from $\mathbb{R}$ into $X_j, j = 0, 1$.

We equip $\mathcal{F}(X_0, X_1)$ with the norm
\[
\|f\|_{\mathcal{F}(X_0, X_1)} = \max \left\{ \sup_{z \in \partial_0} \|f(z)\|_{X_0}, \sup_{z \in \partial_1} \|f(z)\|_{X_1} \right\}.
\]
Then $\mathcal{F}(X_0, X_1)$ is a Banach space. For $0 \leq \theta \leq 1$, we define the complex interpolation space $[X_0, X_1]_\theta$ as the set of all $x \in X_0 + X_1$ satisfying that $x = f(\theta)$ for some $f \in \mathcal{F}(X_0, X_1)$. The norm on $[X_0, X_1]_\theta$ is defined by setting
\[
\|x\|_{[X_0, X_1]_\theta} = \inf \left\{ \|f\|_{\mathcal{F}(X_0, X_1)} : f \in \mathcal{F}(X_0, X_1), f(\theta) = x \right\}.
\]
It then follows that $[X_0, X_1]_\theta$ is a Banach space for $0 \leq \theta \leq 1$ (see [8, Theorem 4.1.2]).

The following complex interpolation result might be well known to experts. We still include a short proof for the convenience of the reader.

Lemma 2.6. Let $0 < \theta < 1$. Assume that $E_1, E_2$ are fully symmetric Banach function spaces. We have
\[
\ell(\mathcal{M})_\theta = E_1^{\ell(\mathcal{M})} \otimes E_2^{(\theta)}(\mathcal{M}).
\]
Proof. The desired result follows from [36, Theorem 4.6] and [16, Theorem 3.2]. In fact, by [36, Theorem 4.6], we have

\[ |E_1, E_2|_a = E_1^{\left(\frac{1}{1-p}\right)} \oplus E_2^\theta. \]

Now, applying [16, Theorem 3.2], the preceding interpolation automatically lifts to the noncommutative setting. \( \square \)

3. Asymmetric vector valued spaces

In this section, we introduce two asymmetric vector valued operator spaces associated with symmetric Banach function spaces \( E \): \( E(\mathcal{M}; \ell^\theta_\infty) \) and \( E(\mathcal{M}; \ell^1_\infty) \). Some elementary results about these spaces are presented for further use.

Let us begin with the definition of \( E(\mathcal{M}; \ell^\theta_\infty) \). Let \( 0 \le \theta \le 1 \). Suppose that \( E \) is a symmetric Banach function space such that \( E \in \text{Int}(L_p, L_q) \) with \( 1 \le p \le q \le \infty \). We define \( E(\mathcal{M}; \ell^\theta_\infty) \) to be the space of all sequences \( x = (x_n)_{n \ge 1} \) in \( E(\mathcal{M}) \) for which there exist \( a \in E^{\left(\frac{1}{1-p}\right)}(\mathcal{M}) \), \( b \in E^\theta(\mathcal{M}) \) and \( y = (y_n)_{n \ge 1} \in L_\infty(\mathcal{M}) \) such that

(3.1) \[ x_n = aby_n, \quad n \ge 1. \]

For \( x \in E(\mathcal{M}; \ell^\theta_\infty) \), we define

\[ \|x\|_{E(\mathcal{M}; \ell^\theta_\infty)} = \inf \left\{ \|a\|_E : A \ge 0, |x_n|^2 \le A^2, \forall n \ge 1 \right\}, \]

where the infimum is taken over all factorizations as above. We should mention that the case \( \theta = 1 \) reduces to the symmetric space \( E(\mathcal{M}; \ell_\infty) \). In the case \( \theta = 0 \) or 1, the spaces \( E(\mathcal{M}; \ell^\theta_\infty) \) and \( E(\mathcal{M}; \ell^1_\infty) \) will be denoted by \( E(\mathcal{M}; \ell^\theta_\infty) \) and \( E(\mathcal{M}; \ell^1_\infty) \), respectively. It can be verified that

\[ \|x\|_{E(\mathcal{M}; \ell^\theta_\infty)} = \inf \left\{ \|A\|_E : A \ge 0, |x_n|^2 \le A^2, \forall n \ge 1 \right\}. \]

Similarly,

\[ \|x\|_{E(\mathcal{M}; \ell^1_\infty)} = \inf \left\{ \|A\|_E : A \ge 0, |x_n|^2 \le A^2, \forall n \ge 1 \right\}. \]

The following fact will be frequently used. We leave the proof as an exercise for the interested reader.

**Fact 3.1.** Observe that if \( \|x\|_{E(\mathcal{M}; \ell^\theta_\infty)} < 1 \), then there exist \( a \in E^{\left(\frac{1}{1-p}\right)}(\mathcal{M}) \), \( b \in E^\theta(\mathcal{M}) \) and \( y = (y_n)_{n \ge 1} \in L_\infty(\mathcal{M}) \) such that \( x = aby \) and

\[ \max \left\{ \|a\|_E^{\left(\frac{1}{1-p}\right)}, \sup_{n \ge 1} \|y_n\|_\infty, \|b\|_E^\theta \right\} < 1. \]

It is easy to check that \( \|\cdot\|_{E(\mathcal{M}; \ell^\theta_\infty)} \) satisfies the positive definiteness and the homogeneity. The lemma below shows that \( \|\cdot\|_{E(\mathcal{M}; \ell^\theta_\infty)} \) is a quasi norm.

**Lemma 3.2** ([54, Page 59]). Let \( 0 \le \theta \le 1 \) and let \( E \) be a symmetric Banach function space. For every \( x = (x_n)_{n \ge 1} \) and \( y = (y_n)_{n \ge 1} \in E(\mathcal{M}; \ell^\theta_\infty) \), we have

\[ \|x + y\|_{E(\mathcal{M}; \ell^\theta_\infty)} \le 2(\|x\|_{E(\mathcal{M}; \ell^\theta_\infty)} + \|y\|_{E(\mathcal{M}; \ell^\theta_\infty)}). \]

Though \( \|\cdot\|_{E(\mathcal{M}; \ell^\theta_\infty)} \) is just a quasi norm in general, the next lemma shows that it is equivalent to a norm under certain assumption. In the case \( \theta = 0,1 \) and \( E = L_p \), the below result is just [41, Lemma 3.5].

**Lemma 3.3.** Let \( 0 \le \theta \le 1 \) and let \( E \) be a symmetric Banach function space with Fatou norm. If \( E = \max\{2\theta, 2(1 - \theta)\} \)-convex, then \( \|\cdot\|_{E(\mathcal{M}; \ell^\theta_\infty)} \) is equivalent to a norm. In other words, \( E(\mathcal{M}; \ell^\theta_\infty) \) can be renormed.
Proof. Note that $E$ has Fatou norm, and is $\max\{2\theta, 2(1 - \theta)\}$-convex. According to Proposition 2.2, $\|\cdot\|_{E^{(\frac{1}{1 - \theta})}}$ is a equivalent to a norm which we denote by $\|\cdot\|_{E^{(\frac{1}{1 - \theta})}}$. Similarly, $\|\cdot\|_{E^{(\frac{1}{\vartheta})}}$ is equivalent to a norm $\|\cdot\|_{E^{(\frac{1}{\vartheta})}}$. Therefore, for each $x = (x_n)_{n \geq 1} \in E(M; \ell_\infty^\theta)$, $\|x\|_{E(M; C_\infty^\theta)}$ is equivalent to

$$
\inf \left\{ \|a\|_{E^{(\frac{1}{1 - \theta})}} \sup_{n \geq 1} \|y_n\|_{\infty} \|b\|_{E^{(\frac{1}{\vartheta})}} \right\},
$$

where the infimum is taken over all factorizations $x = aby$ with $a \in E^{(\frac{1}{1 - \theta})}(M)$, $b \in E^{(\frac{1}{\vartheta})(M)}$ and $y = (y_n)_{n \geq 1} \in L_\infty(M)$.

In the following, we show that $\|\cdot\|_{E(M; C_\infty^\theta)}$ is a norm. We verify only the triangle inequality here. Take $x^{(1)}, x^{(2)} \in E(M; \ell_\infty^\theta)$. According to the homogeneity of symmetric spaces, for any $\varepsilon > 0$, there exist factorizations $x^{(1)} = a_1 y^{(1)} b_1$ and $x^{(2)} = a_2 y^{(2)} b_2$ such that $\sup_{n \geq 1} \|y^{(1)}_n\|_{\infty} \leq 1$, $\sup_{n \geq 1} \|y^{(2)}_n\|_{\infty} \leq 1$, and

$$
\begin{align*}
\max \left\{ \|a_1\|_{E^{(\frac{1}{1 - \theta})}}, \|b_1\|_{E^{(\frac{1}{\vartheta})}} \right\} &\leq \left( \|x^{(1)}\|_{E(M; C_\infty^\theta)} + \varepsilon \right)^{\frac{1}{2}}, \\
\max \left\{ \|a_2\|_{E^{(\frac{1}{1 - \theta})}}, \|b_2\|_{E^{(\frac{1}{\vartheta})}} \right\} &\leq \left( \|x^{(2)}\|_{E(M; C_\infty^\theta)} + \varepsilon \right)^{\frac{1}{2}}.
\end{align*}
$$

(3.2)

Then we define

$$
\alpha = (a_1 a_1^* + a_2 a_2^* + \varepsilon 1)^{\frac{1}{2}}, \quad \beta = (b_1^* b_1 + b_2^* b_2 + \varepsilon 1)^{\frac{1}{2}}.
$$

Since for $k \in \{1, 2\}$ we have $a_k a_k^* \leq \alpha^2$ and $b_k^* b_k \leq \beta^2$, there exist contractions $u_1, u_2, w_1$ and $w_2$ such that $a_1 = \alpha u_1, a_2 = \alpha u_2, b_1 = w_1 \beta$ and $b_2 = w_2 \beta$. It is clear that

$$
x^{(1)} + x^{(2)} = \alpha (u_1 y^{(1)} w_1 + u_2 y^{(2)} w_2) \beta.
$$

Note that $\|\cdot\|_{E^{(\frac{1}{2(1 - \theta)})}}$ is a norm. It follows that

$$
\|\alpha\|_{E^{(\frac{1}{1 - \theta})}} = \|a_1 a_1^* + a_2 a_2^* + \varepsilon 1\|_{E^{(\frac{1}{1 - \theta})}}^{\frac{1}{2}} \leq \left( \|a_1 a_1^*\|_{E^{(\frac{1}{2(1 - \theta)})}} + \|a_2 a_2^*\|_{E^{(\frac{1}{2(1 - \theta)})}} + \varepsilon \right)^{\frac{1}{2}} \leq \left( \|x^{(1)}\|_{E(M; C_\infty^\theta)} + \|x^{(2)}\|_{E(M; C_\infty^\theta)} + 2\varepsilon \right)^{\frac{1}{2}}.
$$

Similarly, we have

$$
\|\beta\|_{E^{(\frac{1}{\vartheta})}} \leq \left( \|x^{(1)}\|_{E(M; C_\infty^\theta)} + \|x^{(2)}\|_{E(M; C_\infty^\theta)} + 2\varepsilon \right)^{\frac{1}{2}}.
$$

Also, it is clear that

$$
(u_1 y^{(1)} w_1 + u_2 y^{(2)} w_2) \otimes e_{1, 1} = \left[ \begin{array}{cc} u_1 & u_2 \\ 0 & 0 \end{array} \right] \left[ \begin{array}{cc} y^{(1)} & 0 \\ 0 & y^{(2)} \end{array} \right] \left[ \begin{array}{cc} w_1 & 0 \\ w_2 & 0 \end{array} \right].
$$

Note that $\alpha, \beta$ are invertible and

$$
u_1 u_1^* + u_2 u_2^* = \alpha^{-1} (a_1 a_1^* + a_2 a_2^*) \alpha^{-1} \leq 1,
$$

$$
w_1^* w_1 + w_2^* w_2 = \beta^{-1} (b_1^* b_1 + b_2^* b_2) \beta^{-1} \leq 1.
$$

Moreover, since $\sup_{n \geq 1} \|y^{(1)}_n\|_{\infty} \leq 1$, $\sup_{n \geq 1} \|y^{(2)}_n\|_{\infty} \leq 1$, we have

$$
\|u_1 y^{(1)} w_1 + u_2 y^{(2)} w_2\|_{L_\infty(M; C_\infty^\theta)} \leq 1.
$$

Therefore, we have found a factorization

$$
x^{(1)} + x^{(2)} = \alpha (u_1 y^{(1)} w_1 + u_2 y^{(2)} w_2) \beta.
$$
with
\[ \|\alpha\|_{E'} \leq \left\| u_1 y^{(1)} w_1 + u_2 y^{(2)} w_2 \right\|_{L_\infty(M \otimes \ell_\infty)} \|\beta\|_{E'} \]
\[ \leq \|x^{(1)}\|_{E(M; \ell_\infty^2)} + \|x^{(2)}\|_{E(M; \ell_\infty^2)} + 2\varepsilon. \]

Letting \( \varepsilon \to 0 \), the desired assertion follows. \( \square \)

**Proposition 3.4.** Let \( 0 \leq \theta \leq 1 \) and let \( E \) be a symmetric Banach function space with Fatou norm. If \( E \) is max\{2\theta, 2(1 - \theta)\}-convex, then \( E(M; \ell_\infty^2) \) can be renormed to a Banach space.

**Proof.** In fact, we will prove that \( E(M; \ell_\infty^2) \) is a Banach space with respect to the norm \( \| \cdot \|_{E(M; \ell_\infty^2)} \). Without causing any confusion, we replace \( \| \cdot \| \) by \( \| \cdot \|_{E} \) for simplicity below.

Let \( x^{(1)}, x^{(2)}, \ldots, x^{(k)}, \ldots \) be a Cauchy sequence in \( E(M; \ell_\infty^2) \). We may find a subsequence \( x^{(n_1)}, x^{(n_2)}, \ldots, x^{(n_k)}, \ldots \) such that
\[ \|x^{(n_1)}\|_{E(M; \ell_\infty^2)} < 4^{-1} \]
and
\[ \|x^{(n_k)} - x^{(n_{k-1})}\|_{E(M; \ell_\infty^2)} < 4^{-k}, \quad k \geq 2. \]
It is well known that, by the triangle inequality proved in Lemma 3.3, to obtain the completeness, it suffices to show that \( \sum_k x^{(n_{k+1})} - x^{(n_k)} \) belongs to \( E(M; \ell_\infty^2) \). Set
\[ x^{(1)} = x^{(n_1)} \]
and
\[ x^{(k)} = x^{(n_k)} - x^{(n_{k-1})}, \quad k \geq 2. \]
Then, by Fact 3.1 there exist \( a_k \in E^{(\frac{1}{1 - \theta})}(M) \), \( b_k \in E^{(\frac{1}{\theta})}(M) \) and \( y^{(k)} \in L_\infty(M \otimes \ell_\infty) \) such that \( x^{(k)} = 4^{-k} a_k y^{(k)} b_k \) with
\[ \max \left\{ \|a_k\|_{E^{(\frac{1}{1 - \theta})}}, \|y^{(k)}\|_{L_\infty(M \otimes \ell_\infty)}, \|b_k\|_{E^{(\frac{1}{\theta})}} \right\} < 1. \]
We define for \( \varepsilon > 0 \)
\[ S_r(a) = \left( \sum_{k=1}^{\infty} 2^{-k} a_k a_k^* + \varepsilon 1 \right)^{\frac{1}{2}}, \quad S_c(b) = \left( \sum_{k=1}^{\infty} 2^{-k} b_k b_k^* + \varepsilon 1 \right)^{\frac{1}{2}}, \]
where the series converge in \( E^{(\frac{1}{1 - \theta})}(M) \) and \( E^{(\frac{1}{\theta})}(M) \), respectively. In fact, it follows from Lemma 3.3 that
\[ \|S_r(a)\|_{E^{(\frac{1}{1 - \theta})}} \leq \left( \varepsilon + \sum_{k=1}^{\infty} 2^{-k} \|a_k\|^2_{E^{(\frac{1}{1 - \theta})}} \right)^{\frac{1}{2}} \leq (1 + \varepsilon)^{\frac{1}{2}}. \]
A similar estimate applies to \( S_c(b) \):
\[ \|S_c(b)\|_{E^{(\frac{1}{\theta})}} \leq (1 + \varepsilon)^{\frac{1}{2}}. \]
Clearly, there exist contractions \( \alpha_k, \beta_k \) such that
\[ 2^{-k/2} a_k = S_r(a) \alpha_k, \quad 2^{-k/2} b_k = \beta_k S_c(b). \]
We may write
\[ \sum_{k=1}^{\infty} x^{(k)} = S_r(a) \left( \sum_{k=1}^{\infty} 2^{-k} \alpha_k y^{(k)} \beta_k \right) S_c(b). \]
For the middle term of the above expression, we have
\[ \left\| \sum_{k=1}^{\infty} 2^{-k} \alpha_k y^{(k)} \beta_k \right\|_{L_\infty(M \otimes \ell_\infty)} \leq \sum_{k=1}^{\infty} \left\| 2^{-k} \alpha_k y^{(k)} \beta_k \right\|_{L_\infty(M \otimes \ell_\infty)} \leq 1. \]
Therefore,
\[
\left\| \sum_{k=1}^{\infty} x^{(k)} \right\|_{E(\mathcal{M}; E_{\infty})} \leq \| S_{\theta} (a) \|_{E(\mathcal{M}; F)} \left\| \sum_{k=1}^{\infty} 2^{-k} a_k y^{(k)} \right\|_{L_{\infty}(\mathcal{M}; E_{\infty})} \| S_{\theta} (b) \|_{E(\mathcal{M}; F)} \leq 1 + \varepsilon.
\]
The proof is complete. \( \square \)

We now turn our attention to the space \( E(\mathcal{M}; E_{\theta_1}) \). Let \( 0 \leq \theta \leq 1 \) and let \( E \) be a symmetric Banach function space such that \( E^\times \) is \( \max \{ 2\theta, 2(1-\theta) \} \)-convex. We define
\[
E_{1-\theta} := \left( ([E^\times]^{(1-\theta)/2}) \times \right)^{(2)}, \quad E_\theta := \left( ([E^\times]^{\theta/2}) \times \right)^{(2)}.
\]
Then the space \( E(\mathcal{M}; E_{\theta_1}) \) is defined to be the set of all sequences \( x = (x_n)_{n \geq 1} \) in \( E(\mathcal{M}) \) which can be decomposed as
\[
x_n = \sum_{k \geq 1} v_{n,k} w_{n,k}, \quad n \geq 1
\]
for two families \( v_{n,k} \in E_{1-\theta} (\mathcal{M}) \) and \( w_{n,k} \in E_\theta (\mathcal{M}) \) satisfying
\[
\left( \sum_{n,k \geq 1} v_{n,k} v_{n,k}^* \right)^{1/2} \in E_{1-\theta} (\mathcal{M}) \quad \text{and} \quad \left( \sum_{n,k \geq 1} w_{n,k} w_{n,k}^* \right)^{1/2} \in E_\theta (\mathcal{M})
\]
where the series converge in norm. For \( x \in E(\mathcal{M}; E_{\theta_1}) \), define
\[
\| x \|_{E(\mathcal{M}; E_{\theta_1})} = \inf \left\{ \left\| \left( \sum_{n,k \geq 1} v_{n,k} v_{n,k}^* \right)^{1/2} \right\|_{E_{1-\theta}} \cdot \left\| \left( \sum_{n,k \geq 1} w_{n,k} w_{n,k}^* \right)^{1/2} \right\|_{E_\theta} \right\}
\]
where the infimum is taken over all factorizations as above. Obviously, the case \( \theta = \frac{1}{2} \) reduces to the symmetric space \( E(\mathcal{M}; E_{\frac{1}{2}}) \). In the case \( \theta = 0 \) or 1, the spaces \( E(\mathcal{M}; E_{\theta_1}) \) and \( E(\mathcal{M}; E_{\theta_1}) \) will be denoted by \( E(\mathcal{M}; \ell_1) \) and \( E(\mathcal{M}; \ell_\infty) \), respectively.

**Fact 3.5.** If \( \| x \|_{E(\mathcal{M}; E_{\theta_1})} < 1 \), then there exist two families \( v_{n,k} \in E_{1-\theta} (\mathcal{M}) \) and \( w_{n,k} \in E_\theta (\mathcal{M}) \) such that \( x_n = \sum_{k \geq 1} v_{n,k} w_{n,k} \) for \( n \geq 1 \) and
\[
\max \left\{ \left\| \left( \sum_{n,k \geq 1} v_{n,k} v_{n,k}^* \right)^{1/2} \right\|_{E_{1-\theta}}, \left\| \left( \sum_{n,k \geq 1} w_{n,k} w_{n,k}^* \right)^{1/2} \right\|_{E_\theta} \right\} < 1.
\]

**Lemma 3.6.** Let \( 0 \leq \theta \leq 1 \) and let \( E \) be a symmetric Banach function space such that \( E^\times \) is \( \max \{ 2\theta, 2(1-\theta) \} \)-convex. Then \( \| \cdot \|_{E(\mathcal{M}; E_{\theta_1})} \) is equivalent to a norm.

**Proof.** Note that \( E^\times \) has Fatou property, and is \( \max \{ 2\theta, 2(1-\theta) \} \)-convex. According to Proposition 2.2, \( (E^\times)^{\frac{1}{\theta(1-\theta)}} \) and \( (E^\times)^{\frac{1}{\theta \theta}} \) can be renormed to be Banach spaces. Similar to Lemma 3.3, \( \| x \|_{E(\mathcal{M}; E_{\theta_1})} \) is equivalent to
\[
\| x \|_{E(\mathcal{M}; E_{\theta_1})} = \inf \left\{ \left\| \left( \sum_{n,k \geq 1} v_{n,k} v_{n,k}^* \right)^{1/2} \right\|_{E_{1-\theta}} \cdot \left\| \left( \sum_{n,k \geq 1} w_{n,k} w_{n,k}^* \right)^{1/2} \right\|_{E_\theta} \right\},
\]
where the infimum is taken over all factorizations as in (3.4). \( \| \cdot \|_{E_{1-\theta}}, \| \cdot \|_{E_\theta} \) denote the norms that are equivalent to \( \| \cdot \|_{E_{1-\theta}}, \| \cdot \|_{E_\theta} \) (these notions are referred to (3.3)), respectively.

We now show that \( \| x \|_{E(\mathcal{M}; E_{\theta_1})} \) is norm. For simplicity, in what follows, we write \( \| \cdot \| \) instead of \( \| \cdot \|_{E_{1-\theta}}, \| \cdot \|_{E_\theta} \). The positive definiteness and the homogeneity can be checked easily. We only provide the proof of the triangle inequality. Take \( x = (x_n)_{n \geq 1} \) and \( y = (y_n)_{n \geq 1} \) in \( E(\mathcal{M}; E_{\theta_1}) \). By Fact 3.5 and the homogeneity of symmetric spaces, for any \( \varepsilon > 0 \),
there exist two families \( v_{n,k} \in E_{1-\theta}(\mathcal{M}) \) and \( w_{n,k} \in E_{\theta}(\mathcal{M}) \) such that for every \( n \geq 1 \),

\[
x_n = \sum_{k \geq 1} v_{n,k} w_{n,k},
\]

and

\[
\max \left\{ \left\| \sum_{n,k \geq 1} v_{n,k}^* v_{n,k}^* \right\|_{E_{1-\theta}}, \left\| \sum_{n,k \geq 1} w_{n,k}^* w_{n,k}^* \right\|_{E_{\theta}} \right\} \leq \left( \|x\|_{E(\mathcal{M}; \ell_1^n)} + \varepsilon \right)^{1/2}.
\]

Similarly, there exist two families \( a_{n,k} \in E_{1-\theta}(\mathcal{M}) \) and \( b_{n,k} \in E_{\theta}(\mathcal{M}) \) such that for every \( n \geq 1 \),

\[
y_n = \sum_{k \geq 1} a_{n,k} b_{n,k}
\]

and

\[
\max \left\{ \left\| \sum_{n,k \geq 1} a_{n,k}^* a_{n,k}^* \right\|_{E_{1-\theta}}, \left\| \sum_{n,k \geq 1} b_{n,k}^* b_{n,k}^* \right\|_{E_{\theta}} \right\} \leq \left( \|y\|_{E(\mathcal{M}; \ell_1^n)} + \varepsilon \right)^{1/2}.
\]

Obviously, for every \( n \geq 1 \), we have

\[
x_n + y_n = \sum_{k \geq 1} v_{n,k} w_{n,k} + \sum_{k \geq 1} a_{n,k} b_{n,k}.
\]

Note that \( \| \cdot \|_{E_{1-\theta}(\mathcal{M})} \) is a norm. Therefore,

\[
\left\| \left( \sum_{n,k \geq 1} v_{n,k} v_{n,k}^* + \sum_{n,k \geq 1} a_{n,k} a_{n,k}^* \right)^{1/2} \right\|_{E_{1-\theta}} = \left\| \sum_{n,k \geq 1} v_{n,k} v_{n,k}^* + \sum_{n,k \geq 1} a_{n,k} a_{n,k}^* \right\|_{E_{1-\theta}}^{1/2} \leq \left( \left\| \sum_{n,k \geq 1} v_{n,k} v_{n,k}^* \right\|_{E_{1-\theta}}^{1/2} + \left\| \sum_{n,k \geq 1} a_{n,k} a_{n,k}^* \right\|_{E_{1-\theta}}^{1/2} \right)^{1/2} \leq \left( \|x\|_{E(\mathcal{M}; \ell_1^n)} + \|y\|_{E(\mathcal{M}; \ell_1^n)} + 2\varepsilon \right)^{1/2}.
\]

Similarly, one can show that

\[
\left\| \left( \sum_{n,k \geq 1} w_{n,k} w_{n,k}^* + \sum_{n,k \geq 1} b_{n,k} b_{n,k}^* \right)^{1/2} \right\|_{E_{1-\theta}} \leq \left( \|x\|_{E(\mathcal{M}; \ell_1^n)} + \|y\|_{E(\mathcal{M}; \ell_1^n)} + 2\varepsilon \right)^{1/2}.
\]

Combining the last two estimates and letting \( \varepsilon \to 0 \), the desired assertion follows. \( \square \)

The proof of the following result is similar to that of Proposition 3.4. We include details for the convenience of the reader.

**Proposition 3.7.** Let \( 0 \leq \theta \leq 1 \) and let \( E \) be a symmetric Banach function space such that \( E^* \) is \( \max\{2\theta, 2(1-\theta)\} \)-convex. Then \( E(\mathcal{M}; \ell_1^n) \) can be renormed to be a Banach space.

**Proof.** In fact, we will prove that \( E(\mathcal{M}; \ell_1^n) \) is a Banach space with respect to the norm \( \| \cdot \|_{E(\mathcal{M}; \ell_1^n)} \) coming from Lemma 3.6. Without causing any confusion, we replace \( \| \cdot \|_{E(\mathcal{M}; \ell_1^n)} \) by \( \| \cdot \| \) for simplicity below.

Let \( x^{(1)}, x^{(2)}, \ldots, x^{(k)}, \ldots \) be a Cauchy sequence in \( E(\mathcal{M}; \ell_1^n) \). We may find a subsequence \( x^{(n_1)}, x^{(n_2)}, \ldots, x^{(n_k)}, \ldots \) such that

\[
\|x^{(n_1)}\|_{E(\mathcal{M}; \ell_1^n)} < 4^{-1}
\]

and

\[
\|x^{(n_k)} - x^{(n_{k-1})}\|_{E(\mathcal{M}; \ell_1^n)} < 4^{-k}, \quad k \geq 2.
\]
It is well known that, by the triangle inequality proved in Lemma 3.6, to obtain the completeness, it suffices to show that $\sum_k x^{(nk+1)} - x^{(nk)}$ belongs to $E(M; \ell^q_1)$. Set

$$x^{(1)} = x^{(n_1)}$$

and

$$x^{(k)} = x^{(nk)} - x^{(nk-1)}, \quad k \geq 2.$$  

Then by Fact 3.1, there exist two families $v_{n,j}^{(k)} \in E_{1-\theta}(M)$ and $w_{n,j}^{(k)} \in E_\theta(M)$ such that $x_n^{(k)} = 4^{-k} \sum_j v_{n,j}^{(k)} w_{n,j}^{(k)}$ with

$$\max \left\{ \left\| \left( \sum_{n,j \geq 1} v_{n,j}^{(k)} v_{n,j}^{(k)*} \right)^{1/2} \right\|_{E_1-\theta}, \left\| \left( \sum_{n,k \geq 1} w_{n,j}^{(k)} w_{n,j}^{(k)*} \right)^{1/2} \right\|_{E_\theta} \right\} < 1.$$  

Using a similar factorization trick as in Proposition 3.4, we define for $\varepsilon > 0$

$$V_{n,j} = \left( \sum_{k=1}^\infty 2^{-k} v_{n,j}^{(k)} v_{n,j}^{(k)*} + \varepsilon 1 \right)^{1/2}, \quad W_{n,j} = \left( \sum_{k=1}^\infty 2^{-k} w_{n,j}^{(k)} w_{n,j}^{(k)*} + \varepsilon 1 \right)^{1/2}.$$  

We claim that

$$\left( V_{n,j} \right)_{n,j} \in E_{1-\theta}(M; \ell^q_2(\mathbb{N}^2)), \quad \left( W_{n,j} \right)_{n,j} \in E_\theta(M; \ell^q_2(\mathbb{N}^2)).$$

Indeed, the triangle inequality holds true in $E^{(1/2)}_{1-\theta}(M)$. Thus,

$$\left\| \left( \sum_{n,j=1}^{\infty} V_{n,j} V_{n,j}^* \right)^{1/2} \right\|_{E_1-\theta} = \left\| \sum_{k=1}^{\infty} 2^{-k} \sum_{n,j=1}^{\infty} v_{n,j}^{(k)} v_{n,j}^{(k)*} + \varepsilon 1 \right\|^{1/2}_{E_1-\theta} \leq \left( \sum_{k=1}^{\infty} 2^{-k} \sum_{n,j=1}^{\infty} v_{n,j}^{(k)} v_{n,j}^{(k)*} \right)^{1/2}_{E_1-\theta} + \varepsilon \varepsilon^{1/2}.$$

According to (3.5), it follows that

$$\left\| \left( \sum_{n,j=1}^{\infty} V_{n,j} V_{n,j}^* \right)^{1/2} \right\|_{E_1-\theta} \leq (1 + \varepsilon) \varepsilon^{1/2}.  $$

A similar argument can be applied to get that

$$\left\| \left( \sum_{n,j=1}^{\infty} W_{n,j}^* W_{n,j} \right)^{1/2} \right\|_{E_\theta} \leq (1 + \varepsilon) \varepsilon^{1/2}.  $$

This shows our claim. Obviously, there exist contractions $\alpha_{n,j}^{(k)}, \beta_{n,j}^{(k)}$ such that

$$2^{-k/2} v_{n,j}^{(k)} = V_{n,j} \alpha_{n,j}^{(k)}, \quad 2^{-k/2} w_{n,j}^{(k)} = W_{n,j} \beta_{n,j}^{(k)}.$$  

It is clear that, for every $n \geq 1$, the $n$-th term of $\sum_{k=1}^\infty x_n^{(k)}$ equals to

$$\sum_{k=1}^{\infty} x_n^{(k)} = \sum_{j=1}^{\infty} V_{n,j} \left( \sum_{k=1}^{\infty} 2^{-k} \alpha_{n,j}^{(k)} \beta_{n,j}^{(k)} \right) W_{n,j}.$$  

For fixed $n, j$, the middle term of the above expression is a contraction:

$$\left\| \sum_{k=1}^{\infty} 2^{-k} \alpha_{n,j}^{(k)} \beta_{n,j}^{(k)} \right\|_\infty \leq \sum_{k=1}^{\infty} 2^{-k} \| \alpha_{n,j}^{(k)} \beta_{n,j}^{(k)} \|_\infty \leq 1.$$  

Therefore,

$$\left\| \sum_{k=1}^{\infty} x_n^{(k)} \right\|_{E(M; \ell^q_1)} \leq \left\| \left( \sum_{n,j=1}^{\infty} V_{n,j} V_{n,j}^* \right)^{1/2} \right\|_{E_1-\theta} \left\| \left( \sum_{n,j=1}^{\infty} W_{n,j}^* W_{n,j} \right)^{1/2} \right\|_{E_\theta} \leq 1 + \varepsilon.$$
The proof is complete. □

**Remark 3.8.** Let $\mathcal{F}$ be the set of elements $x = (x_n)_{n \geq 1}$ with

$$x_n = \sum_{k \geq 1} v_{n,k} w_{n,k}, \quad n \geq 1$$

and

$$\text{Card}\{ (n,k) : v_{n,k} \neq 0 \text{ or } w_{n,k} \neq 0 \} < \infty.$$ 

Then $\mathcal{F}$ is dense in $E(\mathcal{M}; \ell^\theta_1)$. This can be seen from the fact that the set of finite sequences is dense in $E_{1-\theta}(\mathcal{M}; \ell^\theta_2)$ and also in $E_{\theta}(\mathcal{M}; \ell^\theta_2)$.

Let $h_E^{1\theta}$ (resp. $h_E^{\infty\theta}$) be the subspace of $E(\mathcal{M}; \ell^\theta_1)$ (resp. $E(\mathcal{M}; \ell^\theta_\infty)$) consisting of martingale difference sequences. We have the following complemented result.

**Proposition 3.9.** Let $0 \leq \theta \leq 1$. Assume that $E$ is a symmetric Banach function space with $E \in \text{Int}(L_p, L_q)$ for $1 < p \leq q < \infty$.

(i) If $E^* \text{ is max}\{2\theta, 2(1-\theta)\}$-convex and $q' > \text{max}\{2\theta, 2(1-\theta)\}$, then $h_E^{1\theta}$ is complemented in $E(\mathcal{M}; \ell^\theta_1)$.

(ii) If $E$ is max\{2\theta, 2(1-\theta)\}-convex with Fatou norm, and $p > \text{max}\{2\theta, 2(1-\theta)\}$, then $h_E^{\infty\theta}$ is complemented in $E(\mathcal{M}; \ell^\theta_\infty)$.

**Proof.** Let us first show (i). By Proposition 3.7, $E(\mathcal{M}; \ell^\theta_1)$ can be renormed to be a Banach space. It therefore suffices to prove that the Stein projection

$$\mathcal{D}(x_n)_{n \geq 1} = (dx_n)_{n \geq 1}$$

is bounded on $E(\mathcal{M}; \ell^\theta_1)$. Let $x_n = \sum_k v_{n,k} w_{n,k}$ be the decomposition of $x_n$ as in (3.4). Then for each $n$ we may write

$$\mathcal{E}_n(x_n) \otimes e_{1,1} = \sum_k \mathcal{E}_n(v_{n,k} w_{n,k}) \otimes e_{1,1} = \sum_{k,j} u_n(v_{n,k}^* (j)^* u_n(w_{n,k})(j)).$$

Note that $E^{(1/p)}_{1-\theta} \in \text{Int}(L_r, L_s)$ for some $1 < r \leq s < \infty$ when $q' > \text{max}\{2\theta, 2(1-\theta)\}$. Applying the dual Doob inequality for symmetric spaces ([14, Corollary 4.13]), we have

$$\left\| \left( \sum_{n,k,j} |u_n(v_{n,k}^* (j)^* u_n(w_{n,k}) \right) \right\|_{E_{1-\theta}} \leq \| \left( \sum_{n,k} \mathcal{E}_n(v_{n,k}^* w_{n,k}) \right) \|_{E_{1-\theta}}$$

Similarly,

$$\left\| \left( \sum_{n,k} |u_n(w_{n,k}) (j)| \right) \right\|_{E_{\theta}} \leq \| \left( \sum_{n,k} \mathcal{E}_n(w_{n,k}^* w_{n,k}) \right) \|_{E_{\theta}}.$$

Therefore, $(\mathcal{E}_n(x_n))_{n \geq 1} \in E(\mathcal{M}; \ell^\theta_1)$, which shows that the space $h_E^{1\theta}$ is complemented in $E(\mathcal{M}; \ell^\theta_1)$.

Now we turn to prove (ii). Let $x_n = ay_n b$ be the decomposition of $x_n$ as in (3.1). It suffices to show that $(\mathcal{E}_n(ay_n b))_{n \geq 1} \in E(\mathcal{M}; \ell^\theta_\infty)$. Without loss of generality, we may assume that $\sup_n \|y_n\|_{\infty} \leq 1$. From (2.3), we immediately have

$$\mathcal{E}_n(ay_n b) \otimes e_{1,1} = u_n(y_n^* a^*) u_n(b).$$
We claim that \( (u_n(y_n a^*)^*)_{n \geq 1} \in E^{(1/p)}(M; \ell_r^\infty) \) and \( (u_n(b))_{n \geq 1} \in E^{(1/q)}(M; \ell_r^\infty) \). Indeed, we have
\[
\|u_n(y_n a^*)^* u_n(y_n a^*)\| = \mathcal{E}_n(a) |y_n^*|^2 a^* \otimes e_{1,1} \leq \mathcal{E}_n(aa^*) \otimes e_{1,1}.
\]
Note that \( E^{(1/p)}(M; \ell_r^\infty) \in \text{Int}(L_m, L_n) \) for some \( 1 < m \leq n < \infty \) when \( p > \max\{2\theta, 2(1 - \theta)\} \).

Combining the above inequality with the Doob inequality ([14, Theorem 5.7]), we obtain that \( (u_n(y_n a^*)^*)_{n \geq 1} \in E^{(1/p)}(M; \ell_r^\infty) \). Similar arguments can be applied to prove that \( (u_n(b))_{n \geq 1} \in E^{(1/q)}(M; \ell_r^\infty) \). It is obvious that
\[
E(M; \ell_r^\theta) = E^{(1/p)}(M; \ell_r^\infty) \otimes E^{(1/q)}(M; \ell_r^\infty).
\]
Therefore, we have \( (\mathcal{E}_n(ay_n b))_{n \geq 1} \in E(M; \ell_r^\theta) \). The assertion is verified. \( \square \)

4. AN INTERPOLATION RESULT

Our main result of this section is a new complex interpolation result on asymmetric vector valued spaces \( E(M; \ell_r^\theta) \), which plays an essential role in the proofs of asymmetric Burkholder and Johnson-Schechtman inequalities.

**Theorem 4.1.** Let \( \theta_0, \theta_1 \) be such that \( 0 \leq \theta_0 < \theta_1 \leq 1 \). Let \( E \) be a fully symmetric Banach function. Suppose that \( E \) is \( 2 \max\{1 - \theta_0, 1 - \theta_1\} \)-convex with Fatou norm. Let \( 0 \leq \theta, \eta \leq 1 \) be such that
\[
\eta = \frac{\theta - \theta_0}{\theta_1 - \theta_0}.
\]

Then
\[
E(M; \ell_\theta^\theta) = \left[ E(M; \ell_{\theta_0}^0), E(M; \ell_{\theta_1}^0) \right]_\eta.
\]

**Proof.** From Proposition 3.4, by the assumptions that \( E \) has Fatou norm and \( E \) is \( 2 \max\{2(1 - \theta_0), 2\theta_0, 2(1 - \theta_1), 2\theta_1\} \)-convex, we know that \( E(M; \ell_{\theta_0}^0) \) and \( E(M; \ell_{\theta_1}^0) \) can be renormed to be Banach spaces. Therefore,
\[
\left[ E(M; \ell_{\theta_0}^0), E(M; \ell_{\theta_1}^0) \right]_\eta
\]
is also a Banach space. For the sake of clarity, we split the proof into several steps.

**Step 1.** The lower estimate is easy. Indeed, let
\[
T_1 : E^{(1/p)}(M) \times L_\infty(M; \ell_r^\infty) \times E^{(1/q)}(M) \rightarrow E(M; \ell_{\theta_0}^0),
\]
\[
T_2 : E^{(1/p)}(M) \times L_\infty(M; \ell_r^\infty) \times E^{(1/q)}(M) \rightarrow E(M; \ell_{\theta_1}^0)
\]
be the maps given by
\[
T_1(a, (y_n)_{n \geq 1}, b) = (ay_n b)_{n \geq 1} = T_2(a, (y_n)_{n \geq 1}, b).
\]

Clearly, both \( T_1, T_2 \) are contractive and multilinear. Therefore, from Lemma 2.6, we have the following continuous embedding:
\[
E(M; \ell_\theta^\theta) \subset \left[ E(M; \ell_{\theta_0}^0), E(M; \ell_{\theta_1}^0) \right]_\eta.
\]

**Step 2.** We claim that it suffices to show that
\[
(4.1) \quad \|x\|_{E(M; \ell_\theta^\theta)} \leq \|x\|_{\left[ E(M; \ell_{\theta_0}^0), E(M; \ell_{\theta_1}^0) \right]_\eta}, \quad \forall x = (x_n)_{n \geq 1} \in L_\infty(M; \ell_r^\infty).
\]
In fact, assume that the inequality (4.1) holds true. Then, combining inequality (4.1) with the estimate established in Step 1, we deduce that
\[
\|x\|_{E(M; \ell_\theta^\theta)} = \|x\|_{\left[ E(M; \ell_{\theta_0}^0), E(M; \ell_{\theta_1}^0) \right]_\eta}, \quad \forall x = (x_n)_{n \geq 1} \in L_\infty(M; \ell_r^\infty).
\]
It is obvious that $L_\infty(M,\ell_\infty)$ continuously embeds into $E(M;\ell_\infty^0)$ as a dense subspace. According to Proposition 3.4, we know that the space $E(M;\ell_\infty^0)$ is a Banach space. Therefore, it follows that

$$
\|x\|_{E(M;\ell_\infty^0)} = \|x\|_{[E(M;\ell_\infty^0), E(M;\ell_\infty^0)]_\eta}, \quad \forall x = (x_n)_{n \geq 1} \in E(M;\ell_\infty^0).
$$

This implies that $E(M;\ell_\infty^0)$ isometrically embeds into $[E(M;\ell_\infty^0), E(M;\ell_\infty^0)]_\eta$. Since $L_\infty(M,\ell_\infty)$ is dense in the space $E(M;\ell_\infty^0)$ and therefore dense in the space $[E(M;\ell_\infty^0), E(M;\ell_\infty^0)]_\eta$, we further deduce that the space $E(M;\ell_\infty^0)$ is norm dense in $[E(M;\ell_\infty^0), E(M;\ell_\infty^0)]_\eta$. Hence, $E(M;\ell_\infty^0)$ and $[E(M;\ell_\infty^0), E(M;\ell_\infty^0)]_\eta$ coincide, which proves the claim.

**Step 3.** We now verify (4.1). Let $x = (x_n)_{n \geq 1} \in L_\infty(M,\ell_\infty)$ such that

$$
\|x\|_{[E(M;\ell_\infty^0), E(M;\ell_\infty^0)]_\eta} < 1.
$$

Then, there exists a bounded analytic function (here, $S$ is referred to Section 2.3)

$$
f : S \rightarrow E(M;\ell_\infty^0) + E(M;\ell_\infty^0)
$$

such that $f(\theta) = x$ and

$$
\max \left\{ \sup_{z \in \partial S} \|f(z)\|_{E(M;\ell_\infty^0)}, \sup_{z \in \partial S} \|f(z)\|_{E(M;\ell_\infty^0)} \right\} < 1. \tag{4.2}
$$

From the boundary condition (4.2) and the fact that $L_\infty(M,\ell_\infty)$ can be seen as a dense subspace of $E(M;\ell_\infty^0)$ for any $0 \leq \theta \leq 1$, we deduce that $f|_{\partial S}$ can be written as follows:

$$
f(z) = a(z)y(z)b(z), \quad z \in \partial S,
$$

where $a : \partial \varphi \rightarrow M$, $b : \partial \varphi \rightarrow M$ and $y : \partial \varphi \rightarrow L_\infty(M,\ell_\infty)$ with $j = 0, 1$ satisfying the following estimates

$$
\max_{z \in \partial S} \left\{ \|a(z)\|_{E(\frac{1}{1-\xi_0})}, \|y(z)\|_{L_\infty(M,\ell_\infty)}, \|b(z)\|_{E(\frac{1}{1-\xi_0})} \right\} < 1, \tag{4.3}
$$

$$
\max_{z \in \partial S} \left\{ \|a(z)\|_{E(\frac{1}{1-\xi_1})}, \|y(z)\|_{L_\infty(M,\ell_\infty)}, \|b(z)\|_{E(\frac{1}{1-\xi_1})} \right\} < 1.
$$

Fix $\varepsilon > 0$. Define $\varphi_\varepsilon : \partial S \rightarrow M$ as $\varphi_\varepsilon(z) = a(z)a(z)^* + \varepsilon 1$. By Devinatz’s factorization theorem (see [44, Theorem 2.2], [12]) using a conformal mapping from $S$ onto the unit disc $D$, there exists an analytic function $\alpha : S \rightarrow M$ satisfying that $\alpha^{-1}$ exists and it is also bounded analytic on $S$, and such that $\varphi_\varepsilon = \alpha^*\alpha$ on $\partial S$, i.e., we have the following identity on the boundary of $S$:

$$
\alpha(z)^*\alpha(z) = a(z)a(z)^* + \varepsilon 1. \tag{4.4}
$$

Similarly, we may find a bounded analytic function $\beta : S \rightarrow M$ with bounded analytic inverse $\beta^{-1}$ satisfying the following identities on the boundary:

$$
\beta(z)^*\beta(z) = b(z)b(z)^* + \varepsilon 1. \tag{4.5}
$$

Now we may write

$$
f(z) = a(z)\tilde{y}(z)\beta(z), \quad z \in S
$$

with

$$
\tilde{y}(z) = \alpha^{-1}(z)f(z)\beta^{-1}(z).
$$
Note that $\tilde{g}: S \to E(\mathcal{M}; \ell^p_\infty) + E(\mathcal{M}; \ell^q_\infty)$ is bounded analytic. From (4.4) and (4.5), we conclude the following estimates:

$$\sup_{z \in \partial_0} \|a(z)\|_{E(\mathcal{M}; \ell^p_\infty)}^2 \leq \sup_{z \in \partial_0} \|a(z)\|_{E(\mathcal{M}; \ell^p_\infty)}^2 + \varepsilon < 1 + \varepsilon,$$

$$\sup_{z \in \partial_1} \|a(z)\|_{E(\mathcal{M}; \ell^q_\infty)}^2 \leq \sup_{z \in \partial_1} \|a(z)\|_{E(\mathcal{M}; \ell^q_\infty)}^2 + \varepsilon < 1 + \varepsilon,$$

$$\sup_{z \in \partial_0} \|b(z)\|_{E(\mathcal{M}; \ell^p_\infty)}^2 \leq \sup_{z \in \partial_0} \|b(z)\|_{E(\mathcal{M}; \ell^p_\infty)}^2 + \varepsilon < 1 + \varepsilon,$$

$$\sup_{z \in \partial_1} \|b(z)\|_{E(\mathcal{M}; \ell^q_\infty)}^2 \leq \sup_{z \in \partial_1} \|b(z)\|_{E(\mathcal{M}; \ell^q_\infty)}^2 + \varepsilon < 1 + \varepsilon.$$

According to Lemma 2.6 (note that Lemma 2.1 assures that the related symmetric Banach spaces are fully symmetric), the above estimates imply that

$$\|a(\theta)\|_{E(\mathcal{M}; \ell^p_\infty)}^2 < (1 + \varepsilon)\frac{1}{2}, \quad \|b(\theta)\|_{E(\mathcal{M}; \ell^q_\infty)}^2 < (1 + \varepsilon)^\frac{1}{2}.$$

At the same time, it is clear that

$$\|a^{-1}(z)a(z)\|_{\mathcal{M}}^3 < 1, \quad \|b(z)\|_{\mathcal{M}}^2 < 1, \quad \forall z \in \partial S.$$

Therefore,

$$\sup_{z \in \partial_0} \|\tilde{g}(z)\|_{L_\infty(\mathcal{M}; \ell^p_\infty)} = \sup_{z \in \partial_0} \|a^{-1}(z)a(z)y(z)b^{-1}(z)\|_{L_\infty(\mathcal{M}; \ell^q_\infty)}$$

$$\leq \sup_{z \in \partial_0} \|a^{-1}(z)a(z)\|_{\mathcal{M}} \|y(z)\|_{L_\infty(\mathcal{M}; \ell^p_\infty)} \|b(z)b^{-1}(z)\|_{\mathcal{M}} < 1$$

and similarly,

$$\sup_{z \in \partial_1} \|\tilde{g}(z)\|_{L_\infty(\mathcal{M}; \ell^q_\infty)} < 1.$$

Then it is easy to see that

$$\|\tilde{g}(\theta)\|_{L_\infty(\mathcal{M}; \ell^p_\infty)} < 1.$$

In conclusion, we have the following factorization of $x$:

$$x = f(\theta) = a(\theta)\tilde{g}(\theta)b(\theta)$$

with

$$\|a(\theta)\|_{E(\mathcal{M}; \ell^p_\infty)}^2 < (1 + \varepsilon)\frac{1}{2}, \quad \|\tilde{g}(\theta)\|_{L_\infty(\mathcal{M}; \ell^p_\infty)} < 1, \quad \|b(\theta)\|_{E(\mathcal{M}; \ell^q_\infty)}^2 < (1 + \varepsilon)^\frac{1}{2}.$$

Hence, we get

$$\|x\|_{E(\mathcal{M}; \ell^p_\infty)} < 1 + \varepsilon.$$

Letting $\varepsilon \to 0$ in the above inequality, we obtain the desired result. The proof is complete.

**Remark 4.2.** We should point out that, if $E = L_p$, then the above theorem goes back to a particular case of [32, Theorem A]. Moreover, if $\theta = 0, 1$ and $E = L_p$, the above theorem goes back to [41, Proposition 3.7].

5. **Asymmetric Burkholder inequalities**

This section is devoted to proving Theorem 1.1 and Theorem 1.2.
5.1. **Proof of Theorem 1.1.** Our proof depends on the Cuculescu projections for self-adjoint martingales; see e.g. [43, Proposition 1.4]. Let \( y \in L_1(\mathcal{M}) \) be a self-adjoint operator. Let \( R_{n-1}^\lambda = 1 \) for \( \lambda \in \mathbb{R} \) and define by induction
\[
R_n^\lambda = R_{n-1}^\lambda (\lambda (-\infty, \lambda))(R_{n-1}^\lambda E_n(y) R_{n-1}^\lambda), \quad n \geq 0.
\]
It is obvious that \( (R_n^\lambda)_{n \geq 0} \) is decreasing. Let
\[
\lambda = \bigwedge_{n \geq 0} R_n^\lambda,
\]
and let \( Q_n^\lambda = R_{n-1}^\lambda - R_n^\lambda, \quad n \geq 1. \) Then
\[
\sum_{n \geq 1} Q_n^\lambda = 1 - \lambda.
\]

The following result can be obtained by the combination of [28, Proposition 3.4] and [28, (3.8)].

**Lemma 5.1.** For every \( y = y^* \in L_2(\mathcal{M}) \) and \( \lambda > 0 \), we have
\[
\tau(1 - R^\lambda y) \leq 12 \sum_{n \geq 1} \|Q_n^\lambda y_n Q_n^\lambda y_n\|^2_2 + 12 \sum_{n \geq 1} \|Q_n^\lambda (y - E_n(y))\|^2_2.
\]

Before going further, we present two basic estimates.

**Lemma 5.2.** For every \( y = y^* \in L_2(\mathcal{M}) \) and \( \lambda > 0 \), we have
\[
\sum_{n \geq 0} \|Q_n^\lambda (y - E_n(y))\|^2_2 \leq \tau((1 - R^\lambda)s(y)^2),
\]
where \( s(y) := s_c(y) \).

**Proof.** Note that
\[
E_n(y - E_n(y))^2 = E_n \left((\sum_{k > n} dy_k)^2\right) = E_n \left(\sum_{k_1, k_2 > n} dy_{k_1} dy_{k_2}\right) = \sum_{k > n} \sum_{k_1, k_2 > n} \|dy_{k_1} dy_{k_2}\|^2.
\]
Since \( Q_n^\lambda \in \mathcal{M}_n \), it follows that
\[
\|Q_n^\lambda (y - E_n(y))\|^2_2 = \tau(Q_n^\lambda \cdot E_n((y - E_n(y))^2) \cdot Q_n^\lambda)
= \sum_{k > n} \tau(Q_n^\lambda \cdot E_n(dy_k^2) \cdot Q_n^\lambda)
= \sum_{k > n} \tau(Q_n^\lambda \cdot E_{k-1}(dy_k^2) \cdot Q_n^\lambda)
\]
which, combining with (5.3), further implies the desired inequality. \( \square \)

**Lemma 5.3.** Let \( \theta \in [0, 1] \), and \( y = y^* \in L_2(\mathcal{M}) \). Assume that the martingale difference sequence \((dy_n)_{n \geq 1}\) has a factorization \( dy_n = av_n b, \) \( n \geq 1, \) with \( \sup_{n \geq 1} \|v_n\|_\infty \leq 1. \) Then
\[
\sum_{n \geq 1} \|Q_n^\lambda dy_n Q_n^\lambda y_n\|^2_2 \leq \theta \tau((1 - R^\lambda)|a|^{2/\theta}) + (1 - \theta)\tau((1 - R^\lambda)|b|^{2/(1-\theta)}).
\]

**Proof.** Note that
\[
\frac{1}{2} = \frac{\theta}{2} + \frac{1 - \theta}{2}.
\]
By the Hölder inequality, we have, for each \( n \geq 1, \)
\[
\|Q_n^\lambda dy_n Q_n^\lambda y_n\|^2_2 = \|Q_n^\lambda av_n b Q_n^\lambda y_n\|^2_2 \leq \|Q_n^\lambda a\|^2_2 \|b Q_n^\lambda y_n\|^2_2 \frac{1}{\theta} \leq \|Q_n^\lambda a\|^2_2 \|b Q_n^\lambda y_n\|^2_2 \frac{1}{\theta}.
\]
Recall that the Hansen inequality (see [19, Page 249]) states that for bounded operator \( B \) and positive operator \( A \),
\[
B^* A B \leq (B^* A^p B)^{\frac{1}{p}}, \quad \forall p \geq 1.
\]
From this, we have
\[
Q_n^\lambda |a|^2 Q_n^\lambda \leq (Q_n^\lambda |a|^{2/\theta} Q_n^\lambda)^\theta, \quad Q_n^\lambda |b|^2 Q_n^\lambda \leq (Q_n^\lambda |b|^{2/(1-\theta)} Q_n^\lambda)^{1-\theta}.
\]
Therefore,
\[
\|Q_n^\lambda d_{y_n} Q_n^\lambda\|_2^2 \leq \|Q_n^\lambda |a|^{2/\theta} Q_n^\lambda\|_1 \|Q_n^\lambda |b|^{2/(1-\theta)} Q_n^\lambda\|_1^{1-\theta} \\
\leq \theta \|Q_n^\lambda |a|^{2/\theta} Q_n^\lambda\|_1 + (1-\theta) \|Q_n^\lambda |b|^{2/(1-\theta)} Q_n^\lambda\|_1 \\
= \theta \tau(Q_n^\lambda |a|^{2/\theta}) + (1-\theta) \tau(Q_n^\lambda |b|^{2/(1-\theta)}),
\]
where the second inequality is due to the Young inequality. Then, the desired result follows from (5.3).

The following result can be deduced from the above last three lemmas.

**Corollary 5.4.** Let \( \theta \in [0, 1] \), and \( y = y^* \in L_2(\mathcal{M}) \). Assume that the martingale difference sequence \( (d_{y_n})_{n \geq 1} \) has a factorization \( d_{y_n} = av_n b_n, n \geq 1, \) with \( \sup_{n \geq 1} \|v_n\|_\infty \leq 1 \). Then, for every \( y = y^* \in L_2(\mathcal{M}) \) and \( \lambda > 0 \), we have
\[
\tau(1 - R^{2\lambda}) \leq 12 \tau((1 - R^\lambda)A^2_{\theta}),
\]
where
\[
A_\theta = [\theta |a|^{2/\theta} + (1-\theta)|b|^{2/(1-\theta)} + s(y)^2]^{\frac{1}{2}}.
\]

**Lemma 5.5.** Let \( \theta \in (0, 1) \) and \( x = (x_n)_{n \geq 1} \in E(\mathcal{M}, \ell_\infty^\theta) \). For any \( \varepsilon > 0 \), there exists a factorization
\[
x_n = av_n b_n, \quad n \geq 1
\]
such that \( \sup_{n \geq 1} \|v_n\|_\infty \leq 1 \),
\[
\|a\|_{E(\frac{1}{\theta})} \|b\|_{E(\frac{1}{1-\theta})} \leq \|x\|_{E(\mathcal{M}, \ell_\infty^\theta)} + \varepsilon
\]
and
\[
\|a\|_{E(\frac{1}{\theta})}^\frac{1}{\theta} = \|b\|_{E(\frac{1}{1-\theta})}^{1-\theta}.
\]

**Proof.** Take \( \varepsilon > 0 \). According to the definition of \( \|\cdot\|_{E(\mathcal{M}, \ell_\infty^\theta)} \), there exists a factorization
\[
x_n = cv_n d_n, \quad n \geq 1, \text{ such that}
\]
\[
\|c\|_{E(\frac{1}{\theta})} \|d\|_{E(\frac{1}{1-\theta})} =: AB = Y \leq \|x\|_{E(\mathcal{M}, \ell_\infty^\theta)} + \varepsilon.
\]
Setting
\[
a := \frac{Y^\theta}{A} c, \quad b := \frac{Y^{1-\theta}}{B} d,
\]
the desired assertion follows.

Now we are ready to provide the proof of Theorem 1.1. We also need a result proved in [28, Theorem 3.8]. Let \( \Phi : \mathbb{R} \to \mathbb{R}_+ \) be an Orlicz function, that is, \( \Phi \) is an even convex function such that \( \Phi(0) = 0 \) and \( \Phi(\infty) = \infty \). Then \( L_\Phi(\mathcal{M}) \) can be defined according to (2.1). Given \( 1 \leq p \leq q \leq \infty \), an Orlicz function \( \Phi \) is said to be \( p \)-convex if the function \( t \mapsto \Phi(t^{1/p}) \), \( t > 0 \), is convex; and \( \Phi \) is said to be \( q \)-concave if the function \( t \mapsto \Phi(t^{1/q}) \), \( t > 0 \), is concave. Suppose that \( \Phi \) is a \( p \)-convex and \( q \)-concave Orlicz function with \( 2 < p \leq q < \infty \). Let \( y \in L_\Phi(\mathcal{M}) \) be self-adjoint and \( 0 \leq A \in L_\Phi(\mathcal{M}) \) satisfy
\[
\tau(1 - R^{2\lambda}) \leq \lambda^{-2} \tau((1 - R^\lambda)A^2), \quad \lambda > 0.
\]
Then
\[
\tau(\Phi(\|y\|)) \leq c_{p,q} \tau(\Phi(A)).
\]

(5.4)
Proof of Theorem 1.1. By the definitions of $E(M; \ell^p_{\infty})$, $E(M; \ell^p_\infty)$ and the Burkholder-Gundy inequality (see e.g. [27, Theorem 1.3] or [15]), we have

$$\| (dx_k)_{k \geq 1} \|_{E(M; \ell^p_{\infty})} \leq \left( \sum_{k \geq 1} |dx_k|^2 \right)^{\frac{1}{2}} \|_E \lesssim_E \| x \|_E$$

and

$$\| (dx_k)_{k \geq 1} \|_{E(M; \ell^p_\infty)} \leq \left( \sum_{k \geq 1} |dx_k|^2 \right)^{\frac{1}{2}} \|_E \lesssim_E \| x \|_E.$$  

Note that, for any $\theta \in [0, 1]$, $E(M, \ell^p_{\infty}) \cap E(M, \ell^p_\infty) \subset E(M, \ell^p_\infty)$. Then

$$\| (dx_k)_{k \geq 1} \|_{E(M, \ell^p_{\infty})} \leq \max\{ \| (dx_k)_{k \geq 1} \|_{E(M, \ell^p_{\infty})}, \| (dx_k)_{k \geq 1} \|_{E(M, \ell^p_\infty)} \} \lesssim_E \| x \|_E.$$  

The symmetric Burkholder inequality (1.7) gives that

$$\max\{ \| s_c(x) \|_E, \| s_r(x) \|_E \} \lesssim_E \| x \|_E.$$  

Hence, we have

$$\max\{ \| (dx_k)_{k \geq 1} \|_{E(M, \ell^p_{\infty})}, \| (dx_k)_{k \geq 1} \|_{E(M, \ell^p_\infty)}, \| s_c(x) \|_E, \| s_r(x) \|_E \} \lesssim_E \| x \|_E.$$  

Now we show the inverse inequality, i.e.,

$$\| x \|_E \lesssim_E \| (dx_k)_{k \geq 1} \|_{E(M, \ell^p_{\infty})} + \| (dx_k)_{k \geq 1} \|_{E(M, \ell^p_\infty)} + \| s_c(x) \|_E + \| s_r(x) \|_E.$$  

We only prove this inequality for the case $\theta \in (0, 1)$ since the case $\theta = 0, 1$ can be easily obtained with slight modification. To this end, we first write

$$\| x \|_E \leq \| y \|_E + \| \tau \|_E,$$  

where $y = \text{Re}(x) = (x + x^*)/2$ and $z = (x - x^*)/2i$. According to Lemma 5.5, for any $\varepsilon > 0$, we can find a factorization

$$dy_n = av_kb, \quad n \geq 1$$  

such that $\sup_{k \geq 1} \| v_k \|_\infty \leq 1$,

$$\| a \|_{E(1, \frac{1}{2})} \leq \| (dy_k)_{k \geq 1} \|_{E(M, \ell^p_{\infty})} + \varepsilon,$$  

then, combining Corollary 5.4 and (5.4), for any $p$-convex and $q$-concave Orlicz function $\Phi$, we have,

$$\tau(\Phi(|y|)) \leq c_{p, q} \tau(\Phi(A_0)),$$

which, together with Theorem 7.1 in [36], gives that

$$\| y \|_E \lesssim_E \| A_0 \|_{E(\frac{1}{2}, \frac{1}{p})}^2 \lesssim_E \| \theta |a|^p \|_{E(\frac{1}{2}, \frac{1}{p})}^2 + \| (1 - \theta) |b|^{2(\frac{1}{1 - \theta})} \|_{E(\frac{1}{2}, \frac{1}{q})}^2 + \| s(y) \|_E^2$$

$$= \| a \|_{E(\frac{1}{2}, \frac{1}{p})}^2 \| b \|_{E(\frac{1}{1 - \theta}, \frac{1}{q})}^2 + \| s(y) \|_E^2$$

$$\leq \| (dy_k)_{k \geq 1} \|_{E(M, \ell^p_{\infty})} + \| s(y) \|_E^2 + \varepsilon,$$  

where the last equality we used the Young inequality. Note that $\| \cdot \|_{E(M, \ell^p_{\infty})}$ is at least a quasi-norm; see Lemma 3.2. Then,

$$\| y \|_E \lesssim_E \| (dx_k)_{k \geq 1} \|_{E(M, \ell^p_{\infty})} + \| (dx_k)_{k \geq 1} \|_{E(M, \ell^p_\infty)} + \| s_c(x) \|_E + \| s_r(x) \|_E$$

$$= \| (dx_k)_{k \geq 1} \|_{E(M, \ell^p_{\infty})} + \| (dx_k)_{k \geq 1} \|_{E(M, \ell^p_\infty)} + \| s_c(x) \|_E + \| s_r(x) \|_E.$$  

Similarly, we have

$$\| z \|_E \lesssim_E \| (dx_k)_{k \geq 1} \|_{E(M, \ell^p_{\infty})} + \| (dx_k)_{k \geq 1} \|_{E(M, \ell^p_\infty)} + \| s_c(x) \|_E + \| s_r(x) \|_E.$$  

We finish the proof by combining the estimates of $\| y \|_E$ and $\| z \|_E$. 

\[ \square \]
5.2. **Proof of Theorem 1.2.** In this subsection, we apply the interpolation theorem to prove Theorem 1.2.

**Proof of Theorem 1.2.** According to (5.6), we only need to show, for any \( \theta \in [0, 1] \),
\[
\|x\|_E \lesssim_E \max \left\{ \|x\|_{h^E}^\theta, \|x\|^\theta_{h^E}, \|x\|_{h^\infty}^\theta \right\}.
\]

By (1.7), we already have
\[
(5.7) \quad \|x\|_E \lesssim_E \max \left\{ \|x\|_{h^E}^\theta, \|x\|^\theta_{h^E}, \|x\|^\theta_{h^\infty} \right\}.
\]

For the case \( \|x\|^\theta_{h^\infty} < \max \left\{ \|x\|_{h^E}^\theta, \|x\|^\theta_{h^E} \right\} \), inequality (5.7) implies that
\[
\|x\|_E \lesssim_E \max \left\{ \|x\|_{h^E}^\theta, \|x\|^\theta_{h^E} \right\}.
\]

Therefore,
\[
\|x\| \lesssim \max \left\{ \|x\|_{h^E}^\theta, \|x\|^\theta_{h^E} \right\} \leq \|x\|_{h^\infty}^\theta.
\]

It follows from (5.7) that
\[
(5.8) \quad \|x\|_E \lesssim \|x\|_{h^\infty}^\theta \leq \|(dx_k)_{k \geq 1}\|_{E(M; \ell_\infty)}.
\]

For \( \frac{1}{2} < \theta \leq 1 \), Theorem 4.1 gives that
\[
E(M; \ell_\infty) = [E(M; \ell^\theta_\infty), E(M; \ell^\theta_\infty)]_{\frac{1}{2\theta}}.
\]

Therefore,
\[
\|(dx_k)_{k \geq 1}\|_{E(M; \ell_\infty)} \leq \|(dx_k)_{k \geq 1}\|_{E(M; \ell^\theta_\infty)}^{1 - \frac{1}{2\theta}} \|E(M; \ell^\theta_\infty)\|_{E(M; \ell^\theta_\infty)}^{\frac{1}{2\theta}},
\]

which, together with (5.8) and (5.5), implies
\[
\|x\|_E \lesssim \|(dx_k)_{k \geq 1}\|_{E(M; \ell^\theta_\infty)} = \|x\|_{h^\infty}^\theta, \quad \frac{1}{2} < \theta \leq 1.
\]

As for \( 0 \leq \theta \leq \frac{1}{2} \), by Theorem 4.1, we have
\[
E(M; \ell_\infty) = [E(M; \ell^{1-\theta}_\infty), E(M; \ell^{\eta}_\infty)]_\eta, \quad \eta = (1 - 2\theta)/[2(1 - \theta)].
\]

Using the same argument as above, we get
\[
\|x\|_E \lesssim \|(dx_k)_{k \geq 1}\|_{E(M; \ell^\theta_\infty)} = \|x\|_{h^\infty}^\theta, \quad 0 \leq \theta < \frac{1}{2}.
\]

Therefore, we conclude that
\[
\|x\|_E \lesssim \max \left\{ \|x\|_{h^E}^\theta, \|x\|^\theta_{h^E}, \|x\|^\theta_{h^\infty} \right\}, \quad 0 \leq \theta \leq 1.
\]

The proof is complete. \(\square\)
6. Duality: proof of Theorem 1.3

In this section, we establish the following duality and then use the duality to prove Theorem 1.3. In particular, this duality is a common generalization of the symmetric duality established by Junge in [30] (see also [14]) as well as the column version proved by Junge and Perrin in [33].

**Theorem 6.1.** Let $0 \leq \theta \leq 1$ and let $E$ be a separable symmetric Banach function space with Fatou property. Suppose that $E^\infty$ is max $\{2(1-\theta), 2\theta\}$-convex. Then

$$(E(M; \ell_1^0))^* = E^\infty(M; \ell_\infty^0)$$

isometrically.

In order to prove this theorem, we need some preparation. Recall that

$$E_{1-\theta} := \left(\left([E^\infty(\frac{1}{2(1-\theta)})]^x\right)^{(2)}\right), \quad E_\theta := \left(\left([E^\infty(\frac{1}{2\theta})]^x\right)^{(2)}\right).$$

The next lemma discusses the connection between $E_{1-\theta}$ (respectively, $E_\theta$) and $(E^\infty(\frac{1}{2\theta}))$ (respectively, $(E^\infty(\frac{1}{2\theta}))$).

**Lemma 6.2.** Let $0 \leq \theta \leq 1$ and let $E$ be a symmetric Banach function space with Fatou property. Suppose that $E^\infty$ is max $\{2(1-\theta), 2\theta\}$-convex. We have

(i) $E = E_{1-\theta} \circ E_\theta$.

(ii) $L_2 = E_{1-\theta} \circ (E^\infty(\frac{1}{2\theta})) = E_\theta \circ (E^\infty(\frac{1}{\theta}))$.

**Proof.** (i): The case $\theta = \frac{1}{2}$ is trivial. It suffices for us to verify the result for $\frac{1}{2} < \theta \leq 1$ since the remaining case can be treated in a similar way. Note that $E$ has the Fatou property. Then $E = E^{xx}$, and hence, it follows from Lemma 2.3(i) that

$$E = E^{xx} = \left(\left([E^\infty(\frac{1}{2\theta})]^x\right)^{(2)}\right) \circ L_{\frac{2}{2\theta}}.$$ 

Now using Lemma 2.3(ii), we further get

$$E^{(\frac{1}{2\theta})} = \left(\left([E^\infty(\frac{1}{2\theta})]^x\right)^{(2)}\right) \circ L_{\frac{2}{2\theta}} = E_\theta \circ L_{\frac{2}{2\theta}}.$$

At the same time, for $\theta > \frac{1}{2}$, we have $\frac{1}{2(1-\theta)} > 1$. Then, applying Lemma 2.3 again,

$$(E_{1-\theta}) = \left(\left([E^\infty(\frac{1}{2(1-\theta)})]^x\right)^{(2)}\right) = (E^{xx}) \circ L_{\frac{2}{2(1-\theta)}} = E^{\frac{1}{1-\theta}} \circ L_{\frac{2}{2(1-\theta)}}.$$ 

With the above argument, we conclude that

$$E_{1-\theta} \circ E_\theta = E^{\frac{1}{1-\theta}} \circ L_{\frac{2}{2(1-\theta)}} \circ E_\theta = E^{\frac{1}{1-\theta}} \circ E^{\frac{1}{\theta}} = E.$$ 

We now turn to verify (ii). If $\frac{1}{2} < \theta \leq 1$, then it follows from (6.2), Lemma 2.3(ii) and the well-known Lozanovskii factorization theorem ($L_1 = E \circ E^\infty$) that

$$E_{1-\theta} \circ (E^\infty(\frac{1}{2\theta})) = E^{\frac{1}{1-\theta}} \circ L_{\frac{2}{2\theta}} \circ (E^\infty(\frac{1}{2\theta})) = L_{\frac{1}{1-\theta}} \circ L_{\frac{2}{2\theta}} = L_2.$$ 

Assume now $0 \leq \theta < \frac{1}{2}$. Note that in this case $\frac{1}{2} < 1 - \theta \leq 1$. Similar to (6.1), we have

$$E^{\frac{1}{1-\theta}} = E_{1-\theta} \circ L_{\frac{2}{2(1-\theta)}}.$$ 

By (6.1) and the Lozanovskii factorization theorem, we have

$$L_{\frac{1}{1-\theta}} = E^{\frac{1}{1-\theta}} \circ (E^\infty(\frac{1}{1-\theta})) = E_{1-\theta} \circ L_{\frac{2}{2(1-\theta)}} \circ (E^\infty(\frac{1}{1-\theta})).$$
At the same time, it is obvious that
\[
L_{\frac{1}{\theta}} = L_2 \odot L_{\frac{2}{1-\theta}}.
\]
Comparing the last two equations, we conclude that
\[
L_2 = E_{1-\theta} \odot (E^\times)^{\left(\frac{1}{1-\theta}\right)},
\]
which is the desired assertion. The proof of \(L_2 = E_\theta \odot (E^\times)^{\left(\frac{1}{\theta}\right)}\) is similar. \(\square\)

**Lemma 6.3.** Let \(0 \leq \theta \leq 1\) and let \(E\) be a symmetric Banach function space with Fatou property. Suppose that \(E^\times\) is \(\max\{2(1-\theta), 2\theta\}\)-convex. We have
\[
\|a\|_{E_{1-\theta}} \|b\|_{E_\theta} \leq \|a\|_{E} \|b\|_{E}.
\]

**Proof.** For a finite sequence \(x = (x_n)_{n=1}^N\) with \(x_n \in E(M)\), we may write
\[
x = \sum_{n=1}^N x^{(n)}
\]
where
\[
x^{(n)} = (x^{(n)}_k)_{k \geq 1} := (\delta_{n,k} x_k)_{k \geq 1}
\]
with \(\delta_{n,k} = 1\) if \(k = n\) and \(\delta_{n,k} = 0\) if \(n \neq k\). From Lemma 2.4 and Lemma 6.2(i), we see that for any \(\varepsilon > 0\) and \(k \geq 1\), there exist \(a_k \in E_{1-\theta}(M)^+\) and \(b_k \in E_{\theta}(M)^+\) such that \(|x_k| = a_k b_k\) and
\[
\|a_k\|_{E_{1-\theta}} \|b_k\|_{E_\theta} \leq \|a\|_{E_{1-\theta}} \|b\|_{E_\theta} \leq (1 + \varepsilon)\|x_k\|_{E}.
\]
Then we write
\[
v^{(n)}_k = \delta_{n,k} u_k a_k \quad \text{and} \quad w^{(n)}_k = \delta_{n,k} b_k
\]
where \(x_k = u_k \|x_k\|\) is the polar decomposition of \(x_k\). It is obvious that
\[
x^{(n)}_k = v^{(n)}_k w^{(n)}_k, \quad k \geq 1.
\]
Consider
\[
v^{(n)}_{k,j} = \begin{cases} 
v^{(n)}_k & \text{if } j = k, \\
0 & \text{if } j \neq k,
\end{cases} \quad w^{(n)}_{k,j} = \begin{cases} 
w^{(n)}_k & \text{if } j = k, \\
0 & \text{if } j \neq k.
\end{cases}
\]
Then we have
\[
x^{(n)}_k = \sum_{j \geq 1} v^{(n)}_{k,j} w^{(n)}_{k,j}, \quad k \geq 1.
\]
Therefore,
\[
\|x^{(n)}\|_{E(M; \ell^\theta_1)} \leq \left( \left\| \sum_{k,j} v^{(n)}_{k,j} v^{(n)*}_{k,j} \right\|_{E_{1-\theta}} \cdot \left\| \sum_{k,j} w^{(n)*}_{k,j} w^{(n)}_{k,j} \right\|_{E_{\theta}} \right)^{\frac{1}{2}}
\]
\[
= \left( \left\| \sum_{k,j} v^{(n)}_k v^{(n)*}_k \right\|_{E_{1-\theta}} \cdot \left\| \sum_{k} w^{(n)*}_k w^{(n)}_k \right\|_{E_{\theta}} \right)^{\frac{1}{2}}
\]
\[
= \|a_n u_n\|_{E_{1-\theta}} \cdot \|b_n\|_{E_\theta}
\]
\[
\leq (1 + \varepsilon)\|x_n\|_E,
\]
which yields that
\[
\|x\|_{E(M; \ell^\theta_1)} \leq \sum_{n=1}^N \|x^{(n)}\|_{E(M; \ell^\theta_1)} \leq (1 + \varepsilon) \sum_{n=1}^N \|x_n\|_E = (1 + \varepsilon)\|x\|_{\ell_1(E(M))}.
\]
We finish the proof by letting \(\varepsilon \to 0\). \(\square\)

**Lemma 6.4.** Let \(1 < q < \infty\). Suppose that \(E, F\) are symmetric Banach function spaces which satisfy \(L_q = E \odot F\). The following are true:
(i) if $a$ is a positive operator in $E(\mathcal{M})$ with supp $a = 1$, then $aF(\mathcal{M})$ is norm dense in $L_q(\mathcal{M})$;
(ii) if $b$ is a positive operator in $E(\mathcal{M})$ with supp $b = e$, then $bF(\mathcal{M})$ is norm dense in $eL_q(\mathcal{M})$.

Proof. For item (i), consider the operator $L_a : F(\mathcal{M}) \to L_q(\mathcal{M})$ defined by $x \mapsto ax$. One can easily see that the adjoint $L^*_a : L^q(\mathcal{M}) \to F^\times(\mathcal{M})$ is given by $y \mapsto ay$. Since supp $a = 1$, $L^*_a$ is one to one. This implies that the range of $L_a$ is weak*-dense in $L_q(\mathcal{M})$. Since $L_q(\mathcal{M})$ is reflexive, $aF(\mathcal{M}) = \text{Im}(L_a)$ is norm dense in $L_q(\mathcal{M})$.

Now we check item (ii). Fix $c \in E(\mathcal{M})^\perp$ with supp $c = 1$. Consider the operator
$$a' = b + (1 - e)c(1 - e).$$
Then $a'$ is positive with supp $a' = 1$. If $z \in eL_q(\mathcal{M})$, then by (i), there is a sequence of operators $(x_n)_{n \geq 1}$ in $F(\mathcal{M})$ such that $a'x_n \to z$.

Note that $z$ is left-supported in $e$. Therefore,
$$ea'x_n = bx_n \to z.$$This shows that $bF(\mathcal{M})$ is norm dense in $eL_q(\mathcal{M})$.

Lemma 6.5. Let $E$ be a symmetric Banach function space with Fatou property. Suppose that $E^\times$ is max $\{2(1 - \theta), 2\theta\}$-convex. Suppose that $(z_n)_{n \geq 1}$ is a sequence of operators in $E^\times(\mathcal{M})$. Let $\alpha \in (E^\times)\left(\frac{1}{1 - \theta}\right)(\mathcal{M})$, $\beta \in (E^\times)\left(\frac{1}{\theta}\right)(\mathcal{M})$ be positive. If for any $v \in E_{1 - \theta}(\mathcal{M})$, $w \in E_\theta(\mathcal{M})$,
$$|\tau(z_n^*vw)| \leq \|\alpha v\|_2\|w\beta\|_2, \quad n \geq 1,$$then $(z_n)_{n \geq 1} \in E^\times(\mathcal{M}; \ell^\theta_\infty)$. Moreover,
$$\|(z_n)_{n \geq 1}\|_{E^\times(\mathcal{M}; \ell^\theta_\infty)} \leq \|\alpha\|_{(E^\times)\left(\frac{1}{1 - \theta}\right)}\|\beta\|_{(E^\times)\left(\frac{1}{\theta}\right)}.$$

Proof. Denote by $q_\alpha$, $q_\beta$ the support projections of $\alpha$, $\beta$, respectively. We claim that
$$z_n^* = q_\beta z_n^* q_\alpha, \quad n \geq 1.$$In fact, for any $v \in E_{1 - \theta}(\mathcal{M})$ and $w \in E_\theta(\mathcal{M})$, the assumption (6.3) implies that
$$|\tau(z_n^*(1 - q_\alpha)vw)| \leq \|\alpha^2(1 - q_\alpha)vw^*\| \tau(w^*w\beta^2) \leq 0$$and
$$|\tau((1 - q_\beta)z_n^*vw)| \leq \|\alpha^2 vv^*\| \tau(w^*w(1 - q_\beta)\beta^2) \leq 0.$$By Lemma 6.2(i), we get $z_n^*(1 - q_\alpha) = 0$ and $(1 - q_\beta)z_n^* = 0$, which implies the claim. Therefore, we may write
$$z_n^* = q_\beta z_n^* q_\alpha = q_\beta \beta^{-1} z_n^* \alpha^{-1} q_\alpha q_\alpha = (\alpha y_n z_n)^*, \quad n \geq 1.$$where $y_n = q_\alpha \alpha^{-1} z_n \beta^{-1} q_\beta$ for $n \geq 1$.

Now it remains to check:
$$\|y_n\|_{\infty} = \|q_\alpha \alpha^{-1} z_n \beta^{-1} q_\beta\|_{\infty} \leq 1, \quad n \geq 1.$$Fix $n \geq 1$ and $x \in L_1(\mathcal{M})$. We may write $x = x_1x_2$ with $x_1 \in L_2(\mathcal{M})$, $x_2 \in L_2(\mathcal{M})$ and $\|x\|_1 = \|x_1\|_2\|x_2\|_2$. By Lemma 6.4(ii) and Lemma 6.2(ii), we may assume that $x_1 = \alpha v$ and $x_2 = w\beta$ for some $v \in E_{1 - \theta}(\mathcal{M})$ and some $w \in E_\theta(\mathcal{M})$. Therefore,
$$|\tau(y_n^*x)| = |\tau(q_\beta \beta^{-1} z_n^* \alpha^{-1} q_\alpha x)| = |\tau(q_\beta \beta^{-1} z_n^* \alpha^{-1} q_\alpha x_1 x_2)| = |\tau(z_n^*vw)| \leq \|\alpha v\|_2\|w\beta\|_2 = \|x_1\|_2\|x_2\|_2 = \|x\|_1,$$which implies (6.4) by duality. The proof is complete.
We now prove the duality.

**Proof of Theorem 6.1.** We first show that

\[ E^\times(\mathcal{M}; \ell^0_1) \subset (E(\mathcal{M}; \ell^0_1))^*. \]

Let \( z = (z_n)_{n \geq 1} \) be in \( E^\times(\mathcal{M}; \ell^0_\infty) \). For any \( \varepsilon > 0 \), there exist \( a \in (E^\times(\mathcal{M}; \ell^0_\infty))^\perp \), \( b \in (E^\times(\mathcal{M}; \ell^0_\infty))^\perp \) and \( y = (y_n)_{n \geq 1} \subset L_\infty(\mathcal{M}) \) such that \( z_n = ay_n b \) for all \( n \geq 1 \), and

\[
\|a\|_{(E^\times(\mathcal{M}; \ell^0_\infty))^\perp} \sup_{n \geq 1} \|y_n\| \|b\|_{(E^\times(\mathcal{M}; \ell^0_\infty))^\perp} \leq (1 + \varepsilon)\|z\|_{E^\times(\mathcal{M}; \ell^0_\infty)}. 
\]

Define the mapping \( \phi_z : E(\mathcal{M}; \ell^0_1) \to \mathbb{C} \) as follows:

\[ \phi_z(x) = \sum_{n \geq 1} \tau(x_n z_n^*), \quad \forall x = (x_n)_{n \geq 1} \in E(\mathcal{M}; \ell^0_1). \]

Suppose that \( x = (x_n)_{n \geq 1} \in E(\mathcal{M}; \ell^0_1) \). By the definition of \( E(\mathcal{M}; \ell^0_1) \), there are families \( v_{n,k} \in E_{1-\theta}(\mathcal{M}) \) and \( w_{n,k} \in E_{\theta}(\mathcal{M}) \) such that

\[ x_n = \sum_{k \geq 1} v_{n,k} w_{n,k}, \quad \forall n \geq 1, \]

and

\[
\left\| \left( \sum_{n,k \geq 1} v_{n,k}^* v_{n,k} \right)^{\frac{1}{2}} \right\|_{E_{1-\theta}} \cdot \left\| \left( \sum_{n,k \geq 1} w_{n,k}^* w_{n,k} \right)^{\frac{1}{2}} \right\|_{E_{\theta}} \leq (1 + \varepsilon)\|x\|_{E(\mathcal{M}; \ell^0_1)}. 
\]

Therefore, we deduce from Hölder’s inequality that

\[
|\phi_z(x)| = \left| \sum_{n \geq 1} \tau(x_n z_n^*) \right| = \left| \sum_{n \geq 1} \tau\left( \sum_{k \geq 1} v_{n,k} w_{n,k} z_n^* \right) \right| 
\]

\[ = \left| \sum_{n \geq 1} \tau\left( v_{n,k} w_{n,k} b^* y_n a^* \right) \right| = \left| \sum_{n \geq 1} \tau\left( a^* v_{n,k} w_{n,k} b^* y_n^* \right) \right| 
\]

\[ \leq \sup_{n \geq 1} \|y_n^*\| \sum_{n \geq 1} \|a^* v_{n,k} w_{n,k} b^*\| \|1\|_1 
\]

\[ \leq \sup_{n \geq 1} \|y_n\| \left( \sum_{n \geq 1} \|a^* v_{n,k}\|^{\frac{2}{2}} \right)^{\frac{1}{2}} \cdot \left( \sum_{n \geq 1} \|w_{n,k} b^*\|^{\frac{1}{2}} \right)^{\frac{1}{2}} 
\]

\[ = \sup_{n \geq 1} \|y_n\| \left( \sum_{n \geq 1} \|v_{n,k}^* v_{n,k}\| \right)^{\frac{1}{2}} \cdot \left( \sum_{n \geq 1} \|w_{n,k} w_{n,k}\|^2 \right)^{\frac{1}{2}} 
\]

Using Lemma 6.2(ii), we get

\[
|\phi_z(x)| \leq \sup_{n \geq 1} \|y_n\| \|a\|_{(E^\times(\mathcal{M}; \ell^0_1))^\perp} \left( \sum_{n,k \geq 1} v_{n,k}^* v_{n,k} \right)^{\frac{1}{2}} \|E_{1-\theta}\| \left( \sum_{n,k \geq 1} w_{n,k}^* w_{n,k} \right)^{\frac{1}{2}} \|E_{\theta}\| 
\]

\[ \leq (1 + \varepsilon)^2 \|x\|_{E(\mathcal{M}; \ell^0_1)} \|z\|_{E^\times(\mathcal{M}; \ell^0_\infty)}, \quad \forall \varepsilon > 0, \]

which implies that

\[ \|\phi_z\| \leq \|z\|_{E^\times(\mathcal{M}; \ell^0_\infty)}. \]

Hence, \( E^\times(\mathcal{M}; \ell^0_1) \subset (E(\mathcal{M}; \ell^0_1))^* \).

Now we show that all the continuous functionals on \( E(\mathcal{M}; \ell^0_1) \) are in \( E^\times(\mathcal{M}; \ell^0_\infty) \). This direction is more involved. Let \( \phi : E(\mathcal{M}; \ell^0_1) \to \mathbb{C} \) be a norm one functional. By Lemma 6.3,

\[ \ell_1(E(\mathcal{M})) \subset E(\mathcal{M}; \ell^0_1). \]
Since \( E \) is separable, we may assume that there exists \( z = (z_n)_{n \geq 1} \in \ell_\infty(E^*(\mathcal{M})) \) such that
\[
\phi(x) = \sum_{n \geq 1} \tau(x_n z_n^*), \quad \forall x = (x_n)_{n \geq 1} \in \ell_1(E(\mathcal{M})).
\]

Define
\[
B_{1-\theta} = \{ c \in (E^*)^\prime(\mathcal{M})^+ : \|c\|_{(E^*)^\prime(\mathcal{M})^+} \leq 1 \}
\]
and
\[
B_{\theta} = \{ d \in (E^*)^\prime(\mathcal{M})^+ : \|d\|_{(E^*)^\prime(\mathcal{M})^+} \leq 1 \}.
\]

According to Lemma 6.2(ii) and Lemma 2.3(ii), we have
\[
L_1 = (E_{1-\theta})(\frac{1}{2}) \odot (E^*)^\prime(\mathcal{M})^+ = (E_{\theta})(\frac{1}{2}) \odot (E^*)^\prime(\mathcal{M})^+.
\]

Therefore \( B_{1-\theta} \) is compact with respect to the \( \sigma((E^*)^\prime(\mathcal{M}), (E_{1-\theta})(\frac{1}{2})(\mathcal{M})) \)-topology. Similarly, \( B_{\theta} \) is compact when equipped with the \( \sigma((E^*)^\prime(\mathcal{M}), (E_{\theta})(\frac{1}{2})(\mathcal{M})) \)-topology.

According to the definition of \( E(\mathcal{M}; \ell^d_\theta) \), we have
\[
\left| \sum_{n,k \geq 1} \tau(z_n^* v_n k w_n k) \right| = \left| \phi\left( \left( \sum_{j \geq 1} v_{n,j} w_{n,j} \right)_{n \geq 1} \right) \right| 
\leq \left\| \sum_{n,k \geq 1} v_{n,k} v_n^* k \right\|_{E_{1-\theta}^\prime(\mathcal{M})} \cdot \left\| \sum_{n,k \geq 1} w_n^* k w_{n,k} \right\|_{E_{\theta}^\prime(\mathcal{M})} 
\leq \frac{1}{2} \sup_{c,d} \left\{ \sum_{n,j \geq 1} \tau(v_{n,j} v_n^* c) + \sum_{n,j \geq 1} \tau(w_n^* k w_{n,j} d) : c \in B_{1-\theta}, d \in B_{\theta} \right\}.
\]

The right hand side remains unchanged under multiplication with signs \( \varepsilon_{n,j} \). Therefore,
\[
\sum_{n,k \geq 1} \left| \tau(z_n^* v_n k w_n k) \right| 
\leq \frac{1}{2} \sup_{c,d} \left\{ \sum_{n,j \geq 1} \tau(v_{n,j} v_n^* c) + \sum_{n,j \geq 1} \tau(w_n^* k w_{n,j} d) : c \in B_{1-\theta}, d \in B_{\theta} \right\}.
\] (6.5)

For any finite sequences \( v = (v_n)_{n \geq 1} \) in \( E_{1-\theta}(\mathcal{M}) \) and \( w = (w_n)_{n \geq 1} \) in \( E_{\theta}(\mathcal{M}) \), we define the function
\[
f_{v,w}(c,d) := \sum_{n \geq 1} \tau(v_n v_n^* c) + \tau(w_n^* w_n d) - 2 |\tau(z_n^* v_n w_n)|, \quad c \in B_{1-\theta}, \ d \in B_{\theta}.
\]

It is clear that \( f_{v,w} \) is a real valued continuous function on \( B_{1-\theta} \times B_{\theta} \). Moreover, from (6.5), it follows that
\[
\sup_{c,d} \left\{ f_{v,w}(c,d) : c \in B_{1-\theta}, \ d \in B_{\theta} \right\} \geq 0.
\]

Let \( C \) be the set of all functions \( f_{v,w} \) as above. Then \( C \) is a cone. Indeed, if \( \lambda \geq 0 \) and \( f_{v,w} \in C \), then \( \lambda f_{v,w} = f_{\lambda v, \lambda w} \in C \). Moreover, if \( f_{v,w}, f_{\tilde{v}, \tilde{w}} \in C \), then \( f_{v,w} + f_{\tilde{v}, \tilde{w}} \) can be realized as \( f_{v+\tilde{v}, w+\tilde{w}} \), where \( v + \tilde{v} \) denotes the sequence starting with \( v \) and followed by \( \tilde{v} \). It is obvious that \( C \) is disjoint from the cone
\[
C_- = \left\{ g \in C(B_{1-\theta} \times B_{\theta}) : \sup g < 0 \right\},
\]
where \( C(B_{1-\theta} \times B_{\theta}) \) denotes all real valued continuous functions on \( B_{1-\theta} \times B_{\theta} \). By the Hahn-Banach separation theorem, there exists a measure \( \mu \) on \( B_{1-\theta} \times B_{\theta} \) and a scalar \( t \) such that for \( f \in C \) and \( g \in C_- \),
\[
\int_{B_{1-\theta} \times B_{\theta}} g d\mu < t \leq \int_{B_{1-\theta} \times B_{\theta}} f d\mu.
\] (6.6)
Since \( C \) and \( C_- \) are cones, it follows that \( t = 0 \) and \( \mu \) is positive. By normalization, we may assume that \( \mu \) is a probability measure. Now define positive operators \( a, b \) by

\[
a = \int_{B_{1-\theta} \times B_\theta} c \, d\mu, \quad b = \int_{B_{1-\theta} \times B_\theta} d \, d\mu.
\]

By convexity of \( B_{1-\theta} \) and \( B_\theta \), we deduce that \( a \in B_{1-\theta} \) and \( b \in B_\theta \). Let \( (v_n)_{n \geq 1} \) in \( E_{1-\theta}(\mathcal{M}) \) and \( (w_n)_{n \geq 1} \) in \( E_\theta(\mathcal{M}) \) be finite sequences. From (6.6), we have

\[
0 \leq \int_{B_{1-\theta} \times B_\theta} f \, v_n \, d\mu = \sum_{n \geq 1} \left[ \tau(v_n v_n^* c) + \tau(w_n^* w_n d) - 2|\tau(z_n^* v_n w_n)| \right] \, d\mu
\]

\[
= \sum_{n \geq 1} \int_{B_{1-\theta} \times B_\theta} \left[ \tau(v_n v_n^* c) + \tau(w_n^* w_n d) \right] \, d\mu - 2 \sum_{n \geq 1} |\tau(z_n^* v_n w_n)|
\]

\[
= \sum_{n \geq 1} \left[ \tau(v_n v_n^* a) + \tau(w_n^* w_n b) - 2|\tau(z_n^* v_n w_n)| \right].
\]

Therefore,

\[
\sum_{n \geq 1} 2|\tau(z_n^* v_n w_n)| \leq \sum_{n \geq 1} \left[ \tau(v_n v_n^* a) + \tau(w_n^* w_n b) \right].
\]

Moreover, note that for any \( r > 0 \),

\[
\sum_{n \geq 1} 2|\tau(z_n^* v_n w_n)| = \sum_{n \geq 1} 2|\tau(z_n^* (r^\theta v_n)(r^{-\theta} w_n))|
\]

Repeating the arguments above, we conclude that

\[
\sum_{n \geq 1} 2|\tau(z_n^* v_n w_n)| \leq r \sum_{n \geq 1} \tau(v_n v_n^* a) + r^{-1} \sum_{n \geq 1} \tau(w_n^* w_n b).
\]

Taking the infimum over \( r \) and using the fact \( 2st = \inf_{r > 0} \{ rs^2 + r^{-1}t^2 \} \), we get

\[
\sum_{n \geq 1} |\tau(z_n^* v_n w_n)| \leq \left( \sum_{n \geq 1} \tau(v_n v_n^* a) \right)^{\frac{1}{2}} \left( \sum_{n \geq 1} \tau(w_n^* w_n b) \right)^{\frac{1}{2}}
\]

\[
= \left( \sum_{n \geq 1} \|a^{\frac{1}{2}} v_n\|_2^2 \right)^{\frac{1}{2}} \left( \sum_{n \geq 1} \|b^{\frac{1}{2}} w_n\|_2^2 \right)^{\frac{1}{2}}.
\]

In particular, the above implies that for any \( v \in E_{1-\theta}(\mathcal{M}) \) and \( w \in E_\theta(\mathcal{M}) \),

\[
|\tau(z_n^* v w)| \leq \|a^{\frac{1}{2}} v\|_2 \|w b^{\frac{1}{2}}\|_2, \quad n \geq 1.
\]

Applying Lemma 6.5, we conclude that \( z = (z_n)_{n \geq 1} \in E^\times(\mathcal{M}; \ell^0_\infty) \) and

\[
\|z\|_{E^\times(\mathcal{M}; \ell^0_\infty)} \leq 1.
\]

According to Remark 3.8, \( \mathfrak{S} \) is dense in \( E(\mathcal{M}; \ell^0_\infty) \). Therefore, the functional \( \phi \) is uniquely determined by the sequence \( z = (z_n)_{n \geq 1} \) and the duality is proved.

We also have the duality between \( h^1_E \) and \( h^\infty_\mathcal{S}_E \).

**Corollary 6.6.** Let \( 0 \leq \theta \leq 1 \) and let \( E \) be a symmetric Banach function space satisfying \( E \in \text{Int}(L_p, L_q) \) with \( 1 < p \leq q < 2 \). If \( E \) is \( 2 \)-concave, then

\[
(h^1_E(\mathcal{M}))^* = h^\infty_\mathcal{S}_E(\mathcal{M})
\]

with equivalent norms.
Proof. Since \( E \) is 2-concave, it follows that \( E \) is separable and \( E^\times \) is 2-convex (see e.g. [13, Lemma 4.12, Theorem 4.13]). On the other hand, according to [40, Page 52], if \( E \) is \( q \)-concave with \( q < \infty \), then \( E \) does not have a subspace which is isomorphic to \( c_0 \). Combining this fact with the separability of \( E \), we conclude that \( E \) has Fatou property (see e.g. [40, Page 119]). Therefore, Theorem 6.1 and Proposition 3.9 are applicable here. A combination of Proposition 3.9 and Theorem 6.1 yields the desired duality. \( \square \)

Now we provide the proof of Theorem 1.3.

Proof of Theorem 1.3. Since \( E \in \text{Int}(L_p, L_q) \) with \( 1 < p \leq q < 2 \) and \( E \) is 2-concave, it follows from [5, Theorem 3.2] that

\[
(h^*_E)^* = h^*_{E^\times}, \quad (h^*_E)^* = h^*_{E^\times}.
\]

Combining this with Theorem 1.2 and Corollary 6.6, we complete the proof. \( \square \)

Remark 6.7. Indeed, it can be seen from above that the assumption of \( E \) in Theorem 1.3 can be relaxed to the following: \( E \) is a separable symmetric Banach function space with Fatou property and \( E^\times \) is 2-convex.

7. Comments for asymmetric Johnson-Schechtman inequalities

We now turn to the asymmetric versions of noncommutative Johnson-Schechtman inequalities. Let \( L^h_0(\mathcal{M}) \) denote the set of all self-adjoint elements in \( L_0(\mathcal{M}) \). We say that \( x \in L_1(\mathcal{M}) \cap L^h_0(\mathcal{M}) \) is mean zero if \( \tau(x) = 0 \). We introduce the definition of noncommutative independence in the sense of Junge and Xu [35].

Definition 7.1. Let \( \mathcal{M} \) be a finite von Neumann algebra equipped with a finite faithful trace \( \tau \). Assume that \( (\mathcal{M}_k)_{k \geq 1} \) are von Neumann subalgebras of \( \mathcal{M} \).

(i) We say that \( (\mathcal{M}_k)_{k \geq 1} \) are independent with respect to \( \tau \) if \( \tau(xy) = \tau(x)\tau(y) \) holds true for every \( x \in \mathcal{M}_k \) and for every \( y \) in the von Neumann algebra generated by \( (\mathcal{M}_j)_{j \neq k} \).

(ii) A sequence \( (x_k)_{k \geq 1} \subset L^h_0(\mathcal{M}) \) is said to be independent with respect to \( \tau \) if the unital von Neumann subalgebras \( \mathcal{M}_k, k \geq 1 \), generated by \( x_k \) are independent.

Similar to Theorem 1.2, we also have asymmetric form of Johnson-Schechtman inequalities for noncommutative independent random variables.

Theorem 7.2. Let \( 0 \leq \theta \leq 1 \). Assume that \( E \) is a symmetric Banach function space which is an interpolation of the couple \( (L_p, L_q) \) for \( 1 < p \leq q < \infty \). If \( E \) is 2-convex with Fatou norm, then for any sequence \( (x_k)_{k \geq 1} \) of mean zero independent random variables,

\[
\left\| \sum_{k \geq 1} x_k \right\|_E \leq E \left( \| (x_k)_{k \geq 1} \|_{E(\mathcal{M}; \ell^0_1)} + \left\| \sum_{k \geq 1} x_k \otimes e_k \right\|_{(L_1 + L_2)(\mathcal{M}; \ell^0_1)} \right).
\]

Proof. The proof is similar to that of Theorem 1.2. We include details for the convenience of the reader. It follows from [28, Theorem 1.5] that

\[
\left\| \sum_{k \geq 1} x_k \right\|_E \leq E \left( \| (x_k)_{k \geq 1} \|_{E(\mathcal{M}; \ell_1)} + \left\| \sum_{k \geq 1} x_k \otimes e_k \right\|_{(L_1 + L_2)(\mathcal{M}; \ell_1)} \right).
\]

Clearly, if \( \| (x_k)_{k \geq 1} \|_{E(\mathcal{M}; \ell_1)} \leq \| \sum_{k \geq 1} x_k \otimes e_k \|_{(L_1 + L_2)(\mathcal{M}; \ell_1)} \), then we get the desired inequality immediately. Now suppose that

\[
\left\| \sum_{k \geq 1} x_k \otimes e_k \right\|_{(L_1 + L_2)(\mathcal{M}; \ell_1)} \leq \| (x_k)_{k \geq 1} \|_{E(\mathcal{M}; \ell_1)}.
\]

Then we have

\[
\left\| \sum_{k \geq 1} x_k \right\|_E \leq E \| (x_k)_{k \geq 1} \|_{E(\mathcal{M}; \ell_1)}.
\]
For $\frac{1}{2} < \theta \leq 1$, by Theorem 4.1, we get
\[
\| (x_k)_{k \geq 1} \|_{E(M; \ell_\infty)} \leq \left( x_{\| E(\mathcal{M}; \ell_\infty) \| E(\mathcal{M}; \ell_\infty) k \geq 1}^\eta k \geq 1 \right)^{1-\eta} \| (x_k)_{k \geq 1} \|_{E(\mathcal{M}; \ell_\infty)} \|, \quad \eta = \frac{\theta - 1}{2\theta}.
\]
By the definition of $E(\mathcal{M}; \ell_\infty)$ and [25, Theorem 1.4], we have
\[
\| (x_k)_{k \geq 1} \|_{E(\mathcal{M}; \ell_\infty)} \leq \left( \left( \sum_{k \geq 1} \| x_k \|_{2}^2 \right)^{\frac{1}{2}} \right)_{E} \simeq \| \sum_{k \geq 1} x_k \|_{E},
\]
Combining the last three estimates, we conclude that
\[
\left\| \sum_{k \geq 1} x_k \right\|_{E} \lesssim \| (x_k)_{k \geq 1} \|_{E(\mathcal{M}; \ell_\infty)}.
\]
The case $0 \leq \theta < \frac{1}{2}$ can be dealt with in a similar manner. Therefore, we obtain
\[
\left\| \sum_{k \geq 1} x_k \right\|_{E} \lesssim \| (x_k)_{k \geq 1} \|_{E(\mathcal{M}; \ell_\infty)} + \| \sum_{k \geq 1} x_k \otimes e_k \|_{(L_1(L_2))(M \otimes \ell_\infty)}.
\]
For the converse inequality, the proof is similar to Theorem 1.2. We leave details to the reader. \qed

As a consequence of Theorem 7.2, if we consider noncommutative positive independent random variables, we will have the following corollary. The proof is routine, and therefore we omit the details.

**Corollary 7.3.** Let $0 \leq \theta \leq 1$. Assume that $E$ is a symmetric Banach function space which is an interpolation of the couple $(L_p, L_q)$ for $1 < p \leq q < \infty$. If $E$ is 2-convex with Fatou norm, then for any sequence $(x_k)_{k \geq 1}$ of positive independent random variables,
\[
\left\| \sum_{k \geq 1} x_k \right\|_{E} \simeq \| (x_k)_{k \geq 1} \|_{E(\mathcal{M}; \ell_\infty)} + \| \sum_{k \geq 1} x_k \otimes e_k \|_{L_1(M \otimes \ell_\infty)}.
\]

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