Extracting the geometry of an obstacle and a zeroth-order coefficient of a boundary condition via the enclosure method using a single reflected wave over a finite time interval

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Abstract
This paper considers an inverse problem for the classical wave equation in an exterior domain. It is a mathematical interpretation of an inverse obstacle problem which employs the dynamical scattering data of an acoustic wave over a finite time interval. It is assumed that the wave satisfies a Robin-type boundary condition with an unknown variable coefficient. The wave is generated from the initial data localized outside the obstacle and observed over a finite time interval at the same place as the support of the initial data. It is already known that, using the enclosure method, one can extract the maximum sphere whose exterior encloses the obstacle, from the data. In this paper, it is shown that the enclosure method enables us to extract also: (i) a quantity which indicates the deviation of the geometry between the maximum sphere and the boundary of the obstacle at the first-reflection points of the wave; (ii) the value of the coefficient of the boundary condition at an arbitrary first-reflection point of the wave provided, for example, that the surface of the obstacle is known in a neighbourhood of the point. Further new knowledge is obtained as follows: the enclosure method can cover the case where the data are taken over a sphere whose centre coincides with that of the support of an initial datum, and yields results corresponding to (i) and (ii).

Keywords: enclosure method, acoustic wave, inverse obstacle scattering problem, back-scattering data, wave equation
1. Introduction

Produce a single wave with compact support at the initial state outside an unknown obstacle and observe the reflected wave at some place not far away from the obstacle over a finite time interval. The problem of extracting information about the geometry and properties of the surface of the obstacle from this observed wave is a prototype of so-called inverse obstacle problems and the solution may have many applications in, e.g., sonar and radar imaging. In this paper, we consider an inverse problem for the classical wave equation in an exterior domain which is a mathematical interpretation of this inverse obstacle problem.

Let us describe a mathematical formulation of the problem. Let $D$ be a non-empty bounded open subset of $\mathbb{R}^3$ with $C^2$-boundary such that $\mathbb{R}^3 \setminus \overline{D}$ is connected. Let $0 < T < \infty$. Let $f \in L^2(\mathbb{R}^3)$ satisfy $\text{supp} \ f \cap \overline{D} = \emptyset$. Let $u = u_f(x, t)$ denote the weak solution of the following initial–boundary value problem for the classical wave equation:

$$
\begin{align*}
\partial_t^2 u - \Delta u &= 0 \quad \text{in} \quad (\mathbb{R}^3 \setminus \overline{D}) \times [0, T], \\
u(x, 0) &= 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{D}, \\
\partial_t u(x, 0) &= f(x) \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{D}, \\
\frac{\partial u}{\partial \nu} - \gamma(x)\partial_t u - \beta(x)u &= 0 \quad \text{on} \quad \partial D \times [0, T].
\end{align*}
$$

Here $\nu$ denotes the unit outward normal to $D$ on $\partial D$. The coefficients $\gamma (\geq 0)$ and $\beta$ in the boundary condition in (1.1) are essentially bounded functions on $\partial D$. The weak solution for the wave equation over a finite time interval is the same as that used in [17, 19] which comes from [6].

Given $p \in \mathbb{R}^3$, define $d_{\partial D}(p) = \inf_{q \in \partial D} |x - p|$. We denote by $\Lambda_{\partial D}(p)$ the set of all points $q \in \partial D$ such that $|q - p| = d_{\partial D}(p)$. We call this the first reflector from $p$ to $\partial D$ and the points in the first reflector are called the first-reflection points, going from $p$ to $\partial D$.

In this paper, first we consider the following inverse problem.

**Problem 1.** Let $B$ be a known open ball centred at $p \in \mathbb{R}^3$ and with radius $\eta$ such that $\overline{B} \cap \overline{D} = \emptyset$. Let $\chi_B$ denote the characteristic function of $B$ and set $f = \chi_B$.

(i) Assume that $D$ is unknown and both $\gamma$ and $\beta$ are unknown. Extract information about the location and shape of $D$ from the data $u_f(x, t)$ given at all $x \in B$ and $t \in [0, T]$.

(ii) Assume that a point $q \in \Lambda_{\partial D}(p)$ is known and there exists an open ball $U$ centred at $q$ such that $U \cap \partial D$ is known. Extract the values of $\gamma$ and $\beta$ at $q$ from the data $u_f(x, t)$ given at all $x \in B$ and $t \in [0, T]$.

The aim of this paper is to give some solutions to (i) and (ii) by employing the enclosure method as the guiding principle. The enclosure method was introduced in [10, 11] and aims at extracting a domain that encloses an unknown discontinuity, such as an inclusion, cavity, crack, obstacle, etc, embedded in a known background medium. It was based, originally, on the decaying and growing property of the complex exponential solutions or the complex geometrical optics solutions of the governing equation of the signal used, which propagates in the background medium. The idea of the enclosure method used here goes back to [10]. It is a single-measurement version of the enclosure method. Therein the governing equation is given by the Laplace equation and the idea yielded an extraction formula for the support function for a polygonal cavity from a single set of the Cauchy data. The formula has been tested numerically in [25] and the idea of this enclosure method has been realized also for the inverse conductivity problem in two dimensions [12, 13], inverse obstacle scattering problems at a fixed wavenumber in two dimensions [14, 15, 18, 24] and, in an extension of [10], to elastic bodies in two dimensions ([22] and references therein).
Recently the single-measurement version of the enclosure method was developed also in [16] for the heat and wave equations in one space dimension. This paper opened the door to the possibility of using several exponential solutions of the time-dependent governing equation in the framework of that method. Now we already have some results using this time domain single-measurement version of the enclosure method in three space dimensions for the wave equation in [17, 19–21] and the heat equation in [23].

This paper is closely related to some results in [19, 20]. For (i) we have already that from the data \( u_f(x, t) \) given at all \( x \in B \) and \( t \in [0, T] \), one can extract \( \text{dist}(D, B) \) via the formula

\[
\lim_{\tau \to \infty} \frac{1}{2\tau} \log \left| \int_B (w_f - v_f) \, dx \right| = -\text{dist}(D, B),
\]

where

\[
w_f(x, t) = \int_0^T e^{-\tau t} u_f(x, t) \, dt, \quad x \in \mathbb{R}^3 \setminus \overline{D}, \quad \tau > 0
\]

and \( v_f \in H^1(\mathbb{R}^3) \) is the unique weak solution of the modified Helmholtz equation \( (\Delta - \tau^2) v + f = 0 \) in \( \mathbb{R}^3 \) which is given by

\[
v_f(x, \tau) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{-\tau |x-y|}}{|x-y|} f(y) \, dy = \frac{1}{4\pi} \int_B \frac{e^{-\tau |x-y|}}{|x-y|} \omega \, dy.
\]

Note that the function \( v(x, \tau) = e^{-\tau} v_f(x, \tau) \) satisfies the inhomogeneous wave equation \( (\partial_t^2 - \Delta) v + e^{-\tau} f = 0 \) in \( \mathbb{R}^3 \times \mathbb{R} \) and decays everywhere as \( \tau \to \infty \) unlike previous complex exponential solutions of, e.g., the Laplace equation.

Therein it is assumed that \( \gamma \) satisfies \( \gamma(x) \leq 1 - C \) a.e. \( x \in \partial D \) or \( \gamma(x) \geq 1 + C \) a.e. \( x \in \partial D \) for a positive constant \( C \) and the following reasonable constraint on \( T \):

\[
T > 2 \text{dist}(D, B).
\]

See theorem 1.2 in [19] for the details. Since \( \text{dist}(D, B) = d_{\partial D}(p) - \eta \), the formula above yields \( d_{\partial D}(p) \) which is the radius of the largest sphere whose exterior encloses \( D \).

One gets, as a corollary of (1.2), a criterion for whether, given a direction \( \omega \in S^2 \), the point \( p + d_{\partial D}(p) \omega \) belongs to \( \Lambda_{\partial D}(p) \), by using the back-scattering data \( u_f \) on \( B_{p+\eta \omega} \times [0, T] \) for \( f \equiv \chi_{B_{p+\eta \omega}}(p+s \omega) \) with a fixed \( s \in [0, \eta] \), that is, \( p + d_{\partial D}(p) \omega \in \Lambda_{\partial D}(p) \) if and only if the quantity \( d_{\partial D}(p+s \omega) \) computed by using the data \( u_f \) via the formula (1.2) with \( f \) above coincides with \( d_{\partial D}(p) - s \) (see proposition 5.1 in [20] for this type of statement in the interior problem).

How about the shape of \( D \)? This is one of two questions considered in this paper. Before describing a first result concerning the question, we give some remarks.

- If \( q \in \Lambda_{\partial D}(p) \), then \( q \in \partial B_{d_{\partial D}(p)}(p) \) and the two tangent planes at \( q \) of \( \partial D \) and \( \partial B_{d_{\partial D}(p)}(p) \) coincide.
- We denote by \( S_q(\partial D) \) and \( S_q(\partial B_{d_{\partial D}(p)}(p)) \) the shape operators (or the Weingarten maps) at \( q \) with respect to \( v_q \), which is the unit outward normal on \( \partial D \) and the inward one on \( \partial B_{d_{\partial D}(p)}(p) \). Those are symmetric linear operators on the common tangent space at \( q \) of \( \partial D \) and \( \partial B_{d_{\partial D}(p)}(p) \).
- \( S_q(\partial B_{d_{\partial D}(p)}(p)) - S_q(\partial D) \geq 0 \) as the quadratic form on the same tangent space at \( q \), since \( q \) attains \( \min_{q \in \partial D} |x-p| \).

Now we can describe the following result which is the core of an answer to the question for the case where \( \gamma \equiv 0 \).

**Theorem 1.1.** Let \( \gamma \equiv 0 \). Assume that \( \partial D \) is \( C^3 \) and \( \beta \in C^2(\partial D) \); \( \Lambda_{\partial D}(p) \) is finite and satisfies

\[
\det (S_q(\partial B_{d_{\partial D}(p)}(p)) - S_q(\partial D)) > 0, \quad \forall q \in \Lambda_{\partial D}(p).
\]
If $T$ satisfies (1.3), then we have
\[ \lim_{\tau \to -\infty} \tau^4 e^{2\tau \text{dist}(D,B)} \int_B (w_f - v_f) \, dx = \frac{\pi}{2} \left( \frac{\eta}{d_{\partial D}(p)} \right)^2 A_{\partial D}(p), \tag{1.4} \]
where
\[ A_{\partial D}(p) = \sum_{q \in \Lambda_{\partial D}(p)} \frac{1}{\sqrt{\det(S_q(\partial B_{d_{\partial D}(p)}(p)))}}. \]

**Remark 1.1.** Theorem 1.1 tells us that formula (1.4) is invariant with respect to the zeroth-order perturbation $\partial u/\partial v - \beta(x)u = 0$ of the Neumann boundary condition $\partial u/\partial v = 0$ on $\partial D$. It seems that the proof of theorem 1.1 cannot cover the case where $\gamma \not\equiv 0$. The study for this case belongs to our next project. See also [28] for some results using the scattering amplitude in the Lax–Phillips scattering theory when $\gamma \not\equiv 0$ and $\beta \equiv 0$.

**Remark 1.2.** Let $k_1(q) \leq k_2(q)$ denote the eigenvalues of $S_q(\partial D)$. They are called the principal curvatures of $\partial D$ at $q$ with respect to $v_q$. Since $S_q(\partial B_{d_{\partial D}(p)}(p)) = (1/d_{\partial D}(p))I$, we have
\[ \det(S_q(\partial B_{d_{\partial D}(p)}(p)) - S_q(\partial D)) = (\lambda - k_1(q))(\lambda - k_2(q)), \]
where $\lambda = 1/d_{\partial D}(p)$. Recall that the Gauss curvature $K_{\partial D}(q)$ of $\partial D$ at $q$ and mean curvature $H_{\partial D}(q)$ with respect to $v_q$ are given by
\[ K_{\partial D}(q) = k_1(q)k_2(q), \quad H_{\partial D}(q) = (k_1(q) + k_2(q))/2. \]
This yields another expression:
\[ \det(S_q(\partial B_{d_{\partial D}(p)}(p)) - S_q(\partial D)) = \lambda^2 - 2H_{\partial D}(q)\lambda + K_{\partial D}(q), \tag{1.5} \]
where $\lambda$ is the same as above.

We have as a corollary of theorem 1.1 the following result which enables us to extract information about the shape of $\partial D$ at a known $q \in \Lambda_{\partial D}(p)$ from two back-scattering data corresponding to two suitably chosen initial data.

**Corollary 1.1.** Let $\gamma \equiv 0$. Assume that $\partial D$ is $C^3$ and $\beta \in C^2(\partial D)$. Let $p \in \mathbb{R}^3 \setminus \overline{D}$ and assume that $q \in \Lambda_{\partial D}(p)$ is known. Let $B_1$ and $B_2$ denote two open balls centred at $p - s_jv_q$, $j = 1, 2$, respectively, with $0 < s_1 < s_2 < d_{\partial D}(p)$, and satisfy $\overline{B_1} \cup \overline{B_2} \subset \mathbb{R}^3 \setminus \overline{D}$.

Then, one can extract $K_{\partial D}(q)$ and $H_{\partial D}(q)$ from $u_f$ on $B_j \times [0, T]$ with $f = \chi_{B_j}$ for $j = 1, 2$ provided $T$ satisfies $T > 2 \max_{k_1, k_2} \text{dist}(D, B_j)$.

Note that the centre of $B_2$ lies on the segment connecting the centre of $B_1$ with $q$; $d_{\partial D}(p) = |p - q|$ and $v_q = (p - q)/(q - p)$.

The points are:
- $\Lambda_{\partial D}(p - s_jv_q) = \{q\}$ and $\det(S_q(\partial B_{d_{\partial D}(p-s_jv_q)}(p - s_jv_q)) - S_q(\partial D)) > 0$;
- $d_{\partial D}(p - s_jv_q) = d_{\partial D}(p) - s_j$.

These enable us to apply theorem 1.1 to the case where $B = B_j$ and $f = \chi_{B_j}$ with $j = 1, 2$. Then with the help of (1.5), from (1.4) with $j = 1, 2$ we have the following $2 \times 2$ linear system for two unknowns $K_{\partial D}(q)$ and $H_{\partial D}(q)$ via formula (1.4):
\[ Q(s) = (d_{\partial D}(p) - s)^{-2} - 2H_{\partial D}(q)(d_{\partial D}(p) - s)^{-1} + K_{\partial D}(q), \]
where $Q(s), s = s_j$, are known quantities coming from (1.4).

By solving this system we obtain both the Gauss and the mean curvatures of $\partial D$ at $q$. Note that this idea goes back to the proof of theorem 5.1 in [20] where the interior problem for the case where $\gamma = \beta = 0$ has been considered.
And how about the question (ii)? This is a new question in the framework of the enclosure method and the complete answer remains open. Clearly (1.3) does not help us since it does not contain any information about the coefficient \( \beta \). One possible direction is to clarify the remainder term of (1.4) as \( \tau \to \infty \), that is,

\[
\tau^4 e^{2\text{dist}(D,B)} \int_B (w_f - v_f) \, dx - \frac{\pi}{2} \left( \frac{\eta}{d_{BD}(p)} \right)^2 A_{BD}(p).
\]

The following theorem is closely related to this question and the main result of this paper.

**Theorem 1.2.** Let \( \gamma \equiv 0 \). Assume that \( \partial D \) is \( C^5 \) and \( \beta \in C^2(\partial D) \); \( \Lambda_{\partial D}(p) \) is finite and satisfies

\[
det \left( S_q(\partial B_{d_{BD}(p)}(p)) - S_q(\partial D) \right) > 0, \forall q \in \Lambda_{\partial D}(p).
\]

For each \( q \in \Lambda_{\partial D}(p) \) let \( e_j, j = 1, 2, \) be an orthonormal basis of the tangent space at \( q \) of \( \partial D \) with \( e_1 \times e_2 = v_q \). Choose an open ball \( U \) centred at \( q \) with radius \( r_q \) in such a way that there exists an \( h \in C^0_\infty(\mathbb{R}^2) \) with \( h(0,0) = 0 \) and \( \nabla h(0,0) = 0 \) such that \( U \cap \partial D = \{ q + \sigma_1 e_1 + \sigma_2 e_2 + h(\sigma_1, \sigma_2) v_q | \sigma_1^2 + \sigma_2^2 + h(\sigma_1, \sigma_2)^2 < r_q^2 \} \).

If \( T \) satisfies (1.3), then we have

\[
\lim_{\tau \to \infty} \tau^5 \left \{ e^{2\text{dist}(D,B)} \int_B (w_f - v_f) \, dx - \frac{1}{\tau^2} \left( \frac{\eta}{d_{BD}(p)} \right)^2 A_{BD}(p) \right \} = -\frac{\pi \eta}{d_{BD}(p)^2} A_{BD}(p) + \frac{\pi}{2} \eta^2 B_{BD}(p), \tag{1.6}
\]

where

\[
B_{BD}(p) = \sum_{q \in \Lambda_{\partial D}(p)} \frac{C_{BD}(q)}{\sqrt{\det \left( S_q(\partial B_{d_{BD}(p)}(p)) - S_q(\partial D) \right)}},
\]

\[
C_{BD}(q) = -\frac{1}{d_{BD}(p)^3} + \frac{11 - 12d_{BD}(p) \beta_{BD}(q)}{8d_{BD}(p)^2 \det \left( S_q(\partial B_{d_{BD}(p)}(p)) - S_q(\partial D) \right)} - \frac{1}{4d_{BD}(p)^2} h_{\sigma_1, \sigma_2}(0) h_{\sigma_1, \sigma_2}(0) \left( \frac{1}{4} B_{qB} B_{qB} + \frac{1}{6} B_{qB} B_{qB} \right) + \frac{1}{16d_{BD}(p)^2} h_{\sigma_1, \sigma_2}(0) B_{qB} B_{qB} - \frac{\beta(q)}{d_{BD}(p)^2},
\]

and

\[
B = (B_{pq}) = -\left( \frac{1}{d_{BD}(p)} I_2 - \nabla^2 h(0) \right)^{-1}.
\]

Note that we have used the summation convention where repeated indices are to be summed from 1 to 2.

From theorem 1.2 we have immediately the following corollary.

**Corollary 1.2.** Let \( \gamma \equiv 0 \). Assume that: \( \partial D \) is \( C^5 \) and \( \beta \in C^2(\partial D) \); \( q \in \Lambda_{\partial D}(p) \) is known; and there exists an open ball \( U \) centred at \( q \) with radius \( r_q \) and an orthonormal basis \( e_1 \) and \( e_2 \) of the tangent space at \( q \) of \( \partial D \) with \( e_1 \times e_2 = v_q \) such that \( U \cap \partial D = \{ q + \sigma_1 e_1 + \sigma_2 e_2 + h(\sigma_1, \sigma_2) v_q | \sigma_1^2 + \sigma_2^2 + h(\sigma_1, \sigma_2)^2 < r_q^2 \} \) with an \( h \in C^0_\infty(\mathbb{R}^2) \) satisfying \( h(0,0) = 0 \), \( \nabla h(0,0) = 0 \) and that all the second-, third- and fourth-order derivatives of \( h \) at \( \sigma = (0,0) \) are known. Let \( 0 < s < d_{BD}(p) \). Let \( B' \) denote the open ball centred at \( p - sv_q \) and satisfy \( B' \cap \bar{D} = \emptyset \). Then, one can extract \( \beta(q) \) from \( u_f \) on \( B' \) for \( f = \chi_B \) provided \( T \) satisfies \( T > 2\text{dist}(D,B') \).
This corollary says, in short, the following. Assume that we know in advance a point \( q \in \Lambda(D) \) and thus \( d_{\Lambda(D)}(p) = |p - q| \) and \( v_q = (p - q)/|p - q| \), too. Thus the tangent plane \( (x - q) \cdot v_q = 0 \) at \( q \) of \( \partial D \) is known. Moreover, assume that we know that \( \partial D \) near \( q \) is given by making a rotation around the normal at \( q \) of a graph of a function \( h \) defined on the tangent plane and that, in appropriate orthogonal coordinates on the tangent plane, say \( \sigma = (\sigma_1, \sigma_2) \), the Taylor expansion of the function at \( \sigma = 0 \) has the form \( h(\sigma_1, \sigma_2) = \sum_2|\sigma|<4 h_{\sigma} \sigma^\nu + \cdots \) with known coefficients \( h_{\sigma} \) for \( 2 \leq |\sigma| \leq 4 \). Then, produce the wave \( u_f \) with \( f = \chi_B \) for a small \( s > 0 \) and measure the wave on \( B \). Since \( \Lambda(D)(p - sv_q) = \{ q \} \), computing (1.6) in which \( B \) is replaced with \( B' \) and \( p \) with \( p - sv_q \), we obtain \( C_{\Lambda(D)}(q) \) which is a linear equation with unknown \( \beta(q) \) and thus solving this, one obtains \( \beta(q) \).

By the way, it is also interesting to consider the following inverse problem.

**Problem II.** Let \( B \) and \( f \) be the same as those of problem I.

Let \( B_R(p) \) denote the open ball centred at \( p \) with radius \( R \) and satisfy \( B_R(p) \subset \mathbb{R}^3 \setminus \overline{D} \) and \( B \subset B_R(p) \).

(i) Assume that \( D \) is unknown and both \( \gamma \) and \( \beta \) are unknown. Extract information about the location and shape of \( D \) from the data \( u_f(x, t) \) given at all \( x \in \partial B_R(p) \) and \( t \in ]0, T[ \).

(ii) Assume that a point \( q \in \Lambda(D) \) is known and there exists an open ball \( U \) centred at \( q \) such that \( U \cap \partial D \) is known. Extract the values of \( \gamma \) and \( \beta \) at \( q \) from the data \( u_f(x, t) \) given at all \( x \in \partial B_R(p) \) and \( t \in ]0, T[ \).

The difference between problems I and II lies in the place where the wave is observed. The latter case is clearly desirable since the dimension of the observation place is lower than that for the former case. The sphere \( \partial B_R(p) \) is a model of the place where many receivers are placed.

For this problem we show that the enclosure method does not make an issue of this difference at all. The point is the following asymptotic formula, between two data.

**Proposition 1.1.** Let \( B \) and \( f \) be the same as those of problem I. Let \( B_R(p) \) denote the open ball centred at \( p \) with radius \( R \) and satisfy \( B_R(p) \subset \mathbb{R}^3 \setminus \overline{D} \) and \( B \subset B_R(p) \).

We have, as \( \tau \to \infty \),
\[
\int_{\partial B_R(p)} (w_f - v_f) \, dS = \frac{R}{\eta} e^{\nu(R-\eta)} (1 + O(\tau^{-1} e^{-\nu\tau})) \int_B (w_f - v_f) \, dx + O(\tau e^{-\tau(R-(R-\eta))}).
\]

(1.7)

Note that \( w_f \) on \( \partial B_R(p) \) can be computed from the trace of \( u_f(\cdot, t) \) on \( \partial B_R(p) \) given at almost all \( t \in ]0, T[ \) via the formula
\[
w_f = \int_0^T e^{-\tau t} u_f(\cdot, t)_{|\partial B_R(p)} \, dt \in H^{1/2}(\partial B_R(p)).
\]

We can say that throughout formula (1.7), all the results mentioned above are transplanted in this case. For the detailed description of the transplanted results, see section 4.

A brief outline of the rest of the paper is as follows. In section 2 we derive an asymptotic representation formula for the indicator function
\[
\tau \mapsto \int_B (w_f - v_f) \, dx
\]
as \( \tau \to \infty \). It clarifies the principal term in terms of integrals of \( v_f \) over \( D \) and \( \partial D \). The key to the proof is an argument developed in [27] which we call the Lax–Phillips reflection
argument. Previously, we applied the argument for the case where $\gamma \equiv 0$ and $\beta \equiv 0$ in [20]. In this paper, we found that some modification of the argument still works in our problem setting.

Having the formula established in section 2, we prove theorems 1.1 and 1.2 in section 3. They are an application of the Laplace method; however, we need the second term of the asymptotic expansion of a double integral of Laplace type. As can be seen in [2], the coefficient is quite complicated. To make the relation between the obstacle and the coefficient as concise as possible, we did some additional calculations and they are summarized as lemma 3.1. Although the calculation is simple, it is tedious. Thus, we put the proof of lemma 3.1 in the appendix. In section 4 we give a proof of proposition 1.1 and its implications for problem II.

Before closing the introduction, we give comments on some of the other approaches to inverse obstacle scattering problems whose governing equation is given by the wave equation.

In the Lax–Phillips scattering theory, because the purpose differs from ours, the wave is observed far away from the obstacle and, thus, infinite observation time is needed. In the case where $\gamma \not\equiv 0$ and $\beta \equiv 0$, there is a classical result due to Majda [28] using the high frequency asymptotics of the scattering amplitude, which is the Fourier transform of the scattering kernel in the Lax–Phillips scattering theory. The method of the proof used therein is based on the idea of geometrical optics and, thus, different from ours. And our result yields not only the Gauss curvature but also the mean curvature. This is also another difference from that result.

The main tool in the proof for realizing the enclosure method presented here exactly is just an integration by parts and the proof is free from the unique continuation property of the governing equations unlike for the previous applications to elliptic equations; see [10, 12, 14]. This elementary feature is a point of decisive difference from the time domain linear sampling method [9], which is functional analytic or operator theoretic. It makes use of reflected waves corresponding to infinitely many incident waves over an infinite time interval and aims at recovering unknown obstacles itself, and thus the purpose is different from ours.

The procedure of extraction of the geometry of unknown obstacles based on theorems 1.1 and 1.2 does not make use of the knowledge of the boundary condition, in contrast to the continuation procedure for the solutions of the governing equation close to an obstacle such as that of [26], which aims at recovering an unknown surface by using the boundary condition in the procedure. See also [3], which is also based on a continuation method close to an obstacle, using a combination of a continuation method in the frequency domain and the Fourier transform. To present another continuation approach, one can cite also [5] which gives a combination of the time-reversal technique [7] and an optimization method.

Finally we mention an approach in [29] which is based on the boundary control method [1]. That work considers an inverse initial-boundary value problem for the wave equation in a bounded domain and employs the local hyperbolic Neumann-to-Dirichlet operator as the observation data. Its approach enables us to compute the volume of a set which is closely related to an unknown obstacle embedded in the domain and the computed volume yields some information about the location of the obstacle. Note that in the crucial step of the proof it makes use of the unique continuation property of the governing equation.

2. Extracting the principal term of the indicator function

In this section we set $c = c(x, \tau) = \gamma(x)\tau + \beta(x)$. Let $f \in \mathcal{L}^2(\mathbb{R}^3)$ satisfy supp $f \subset B$.

We give a proof of the following asymptotic formula.

Proposition 2.1. Let $\gamma \equiv 0$. Assume that $\partial D$ is $C^3$ and $\beta \in C^2(\partial D)$. We have

$$\int_B f(w_f - v_f) \, dx = 2J(\tau)(1 + O(\tau^{-1/2})) + O(\tau^{-1} e^{-\tau T}),$$

(2.1)
where
\[ J(\tau) = \int_D (|\nabla v_f|^2 + \tau^2 |v_f|^2) \, dx - \int_{\partial D} c|v_f|^2 \, dS. \]

Let \( \epsilon^0 \in H^1(\mathbb{R}^3 \setminus \overline{D}) \) solve
\[ (\Delta - \tau^2)\epsilon^0 = 0 \text{ in } \mathbb{R}^3 \setminus \overline{D}, \]
\[ \frac{\partial \epsilon^0}{\partial v} - c\epsilon^0 = -\left( \frac{\partial v_f}{\partial v} - cv_f \right) \quad \text{on } \partial D \quad (2.2) \]
and define
\[ E(\tau) = \int_{\mathbb{R}^3 \setminus D} (|\nabla \epsilon^0|^2 + \tau^2 |\epsilon^0|^2) \, dx + \int_{\partial D} c|\epsilon^0|^2 \, dS. \]

Clearly proposition 2.1 is a consequence of two asymptotic formulae (2.3) and (2.8) described in the following two lemmas. The core of this section is the proof of (2.8) which forms a subsection independently and can be considered as one of the essential parts of this paper.

**Lemma 2.1.** As \( \tau \rightarrow \infty \) we have
\[ \int_B f(w_f - v_f) \, dx = J(\tau) + E(\tau) + O(\tau^2 e^{-T \text{dist}(D,B)}) + O(\tau^{-1} e^{-\tau T}). \quad (2.3) \]

**Proof.** It is easy to see that \( w_f \) satisfies
\[ (\Delta - \tau^2)w_f + f = e^{-\tau T} F \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \]
where
\[ F = F(x, \tau) = \partial u(x, T) + \tau u(x, T). \]
This together with the governing equation of \( v_f \) yields
\[ (\Delta - \tau^2)(w_f - v_f) = e^{-\tau T} F \quad \text{in } \mathbb{R}^3 \setminus \overline{D}. \]
Multiplying both sides of this equation with \( v_f \), integrating over \( \mathbb{R}^3 \setminus \overline{D} \) and applying integration by parts, we obtain
\[ \int_{\partial D} \left( (w_f - v_f) \frac{\partial v_f}{\partial v} - \frac{\partial}{\partial v} (w_f - v_f) v_f \right) \, dS - \int_{\mathbb{R}^3 \setminus D} f(w_f - v_f) \, dx = e^{-\tau T} \int_{\mathbb{R}^3 \setminus D} F v_f \, dx. \]
Since \( w_f \) satisfies
\[ \frac{\partial}{\partial v} w_f = cw_f + e^{-\tau T} \gamma(x) u(x, T) \quad \text{on } \partial D, \]
finally we obtain
\[ \int_B f(w_f - v_f) \, dx = \int_{\partial D} \left( \frac{\partial v_f}{\partial v} - cv_f \right) w \, dS - e^{-\tau T} \left( \int_{\mathbb{R}^3 \setminus D} F v_f \, dx + \int_{\partial D} G v_f \, dS \right), \quad (2.4) \]
where
\[ G = G(x) = \gamma(x) u(x, T). \]
Note that the definition of the weak solution taken from [6] ensures that \( \|F\|_{L^2(\mathbb{R}^3 \setminus D)} = O(\tau) \) and \( \|G\|_{L^2(\partial D)} < \infty \) (see [19]). Since \( v_f \) has the bounds \( \|v_f\|_{L^2(\mathbb{R}^3)} = O(\tau^{-2}) \) and \( \|v_f\|_{L^\infty(\partial D)} = O(e^{-\tau \text{dist}(D,B)}) \), we see that the second term of (2.4) has the bound \( O(\tau^{-1} e^{-\tau T}) + O(e^{-T \text{dist}(D,B)}) \).
One can write \( w_f = v_f + \epsilon_f^0 + Z \), where \( Z \) solves
\[
(\Delta - \tau^2)Z = e^{-\tau T} F \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{D},
\]
\[
\frac{\partial Z}{\partial v} - cZ = e^{-\tau T} G \quad \text{on} \quad \partial D.
\]
Using the same argument as was used in the proof of lemma 2.1 in [19], one has
\[
\int_{\mathbb{R}^3 \setminus \overline{D}} (|\nabla Z|^2 + \tau^2 |Z|^2) \, dx = O(\tau^2 e^{-2\tau T}). \tag{2.5}
\]
From a combination of this, the trace theorem on \( \partial D \) and the trivial estimate \( \|\partial v_f/\partial v - cv_f\|_{L^2(\partial D)} = O(\tau e^{-\tau \text{dist}(D,B)} \), we obtain
\[
\int_{\partial D} \left( \frac{\partial v_f}{\partial v} - cv_f \right) Z \, dS = O(\tau^2 e^{-\tau (T+\text{dist}(D,B)})).
\]
It is easy to see that integration by parts yields
\[
J(\tau) = \int_{\partial D} \left( \frac{\partial v_f}{\partial v} - cv_f \right) v_f \, dS \tag{2.6}
\]
and
\[
E(\tau) = \int_{\partial D} \left( \frac{\partial v_f}{\partial v} - cv_f \right) \epsilon_f^0 \, dS. \tag{2.7}
\]
Now (2.3) is clear. \( \square \)

**Lemma 2.2.** Let \( \gamma \equiv 0 \) and \( \beta \in C^2(\partial D) \). There exists a \( \tau_0 > 0 \) such that for all \( \tau \geq \tau_0 \), we have \( J(\tau) > 0 \), and as \( \tau \to \infty \),
\[
E(\tau) = J(\tau)(1 + O(\tau^{-1/2})). \tag{2.8}
\]

### 2.1. Proof of lemma 2.2

The proof is a modification of the Lax–Phillips reflection argument [27] (see also [20] for the case where \( c \equiv 0 \)). In this subsection we always assume that \( \partial D \) is \( C^1 \) and \( c \in C^2(\partial D) \). And for simplicity we denote \( v_f \) and \( \epsilon_f^0 \) by \( v \) and \( \epsilon_0 \), respectively.

#### 2.1.1. A representation formula for \( E(\tau) - J(\tau) \) via a reflection.

There exists a positive constant \( \delta_0 \) such that given \( x \in \mathbb{R}^3 \setminus D \) with \( d_{\partial D}(x) < 2\delta_0 \), there exists a unique \( q = q(x) \) that is a boundary point on \( \partial D \) such that \( x = q \pm d_{\partial D}(x) v_q \) ([8]). Both \( d_{\partial D}(x) \) and \( q(x) \) are \( C^2 \) for \( x \in \mathbb{R}^3 \setminus D \) with \( d_{\partial D}(x) < 2\delta_0 \) (lemma 1 of the appendix in [8]). Note that \( v_q \) is the unit normal vector to \( \partial D \) at \( q \).

For \( x \) with \( d_{\partial D}(x) < 2\delta_0 \) define
\[
x' = 2q(x) - x.
\]
Let \( 0 < \delta < \delta_0 \). Let \( \phi = \phi_0 \) be a smooth cut-off function, with \( 0 \leq \phi(x) \leq 1 \), and such that \( \phi(x) = 1 \) if \( d_{\partial D}(x) < \delta \) and \( \phi(x) = 0 \) if \( d_{\partial D}(x) > 2\delta \); \( |\nabla \phi(x)| \leq C\delta^{-1} \); \( |\nabla^2 \phi(x)| \leq C\delta^{-2} \).

Define
\[
v'(x) = \phi(x) v(x'), x \in \mathbb{R}^3 \setminus \overline{D}.
\]
Let \( \eta = \eta(x) \) be a \( C^2 \)-function of \( x \in \mathbb{R}^3 \setminus D \) with \( d_{\partial D}(x) < 2\delta_0 \).
**Proposition 2.2.** Assume that $\partial D$ is $C^3$ and that $\gamma \in C^2(\partial D)$ and $\beta \in C^2(\partial D)$. Let $\eta$ satisfy $\eta = 1$ on $\partial D$ and

\[
\frac{\partial \eta}{\partial v} = 2(\gamma(x)\tau + \beta(x)) \quad \text{on} \quad \partial D.
\]

Then, we have

\[
E(\tau) - J(\tau) = \int_{\mathbb{R}^3 \setminus D} \epsilon_0(\Delta - \tau^2)(\eta v') \, dx.
\] (2.9)

**Proof.** Integration by parts (or the weak formulation of (2.2)) yields

\[
\int_{\mathbb{R}^3 \setminus D} \epsilon_0(\Delta - \tau^2)(\eta v') \, dx = \int_{\partial D} \left\{ \frac{\partial \epsilon_0}{\partial v} (\eta v') - \epsilon_0 \frac{\partial}{\partial v} (\eta v') \right\} \, dS. \tag{2.10}
\]

Since

\[
\frac{\partial v'}{\partial v} = -\frac{\partial v}{\partial v}, \quad v' = v \quad \text{on} \quad \partial D,
\]
we have

\[
\frac{\partial}{\partial v} (\eta v') = \frac{\partial \eta}{\partial v} v - \eta \frac{\partial v}{\partial v} \quad \text{on} \quad \partial D.
\]

Substituting this together with

\[
\frac{\partial \epsilon_0}{\partial v} = c\epsilon_0 - \left( \frac{\partial v}{\partial v} - cv \right) \quad \text{on} \quad \partial D
\]
into (2.10), we obtain

\[
\int_{\mathbb{R}^3 \setminus D} \epsilon_0(\Delta - \tau^2)(\eta v') \, dx = \int_{\partial D} \left\{ \left( c\epsilon_0 - \left( \frac{\partial v}{\partial v} - cv \right) \right) \eta v - \epsilon_0 \left( \frac{\partial \eta}{\partial v} v - \eta \frac{\partial v}{\partial v} \right) \right\} \, dS
\]

\[
= \int_{\partial D} \left\{ \left( c\eta - \frac{\partial \eta}{\partial v} \right) v + \eta \frac{\partial v}{\partial v} \right\} \epsilon_0 \, dS - \int_{\partial D} \left( \frac{\partial v}{\partial v} - cv \right) \eta v \, dS
\]

\[
= \int_{\partial D} \left( \frac{\partial v}{\partial v} - \frac{1}{\eta} \left( \frac{\partial \eta}{\partial v} - cv \right) \right) v \eta \epsilon_0 \, dS - \int_{\partial D} \left( \frac{\partial v}{\partial v} - cv \right) \eta v \, dS.
\]

This, together with (2.6) and (2.7), yields (2.9). \qed

One possible choice of $\eta$ in (2.9) is

\[
\eta(x) = 1 + 2c(q(x), \tau) d_{\partial D}(x), \quad x \in \mathbb{R}^3 \setminus D, \quad d_{\partial D}(x) < 2\delta_0 \tag{2.11}
\]

since we have

\[
\nabla (d_{\partial D}(x)) = v(q(x))
\]
and

\[
\nabla q^j(x) \cdot v(q(x)) = 0, \quad j = 1, 2, 3.
\]

In what follows we always make use of $\eta$ given by (2.11) and thus we have (2.9) for this $\eta$.  

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2.1.2. The Lax–Phillips reflection argument. We assume that \( \gamma \equiv 0 \) and \( \beta \in C^2(\partial D) \). Thus (2.11) becomes \( \eta(x) = \beta(q(x))d_{\partial D}(x) \) with \( d_{\partial D}(x) < 2\delta_0 \).

In this part we give the following upper bound for the integral in the right-hand side of (2.9): there exists a \( \tau_0 > 0 \) such that, for all \( \tau \geq \tau_0, J(\tau) > 0 \) and \( E(\tau) > 0 \) and, as \( \tau \to \infty \),

\[
\left| \int_{\mathbb{R}^1 \setminus D} \epsilon_0(\Delta - \tau^2)(\eta\nu') \, dx \right| \leq O(\tau^{-1/2}(E(\tau)J(\tau))^{1/2}).
\]  

(2.12)

Once we have this, from (2.9) we have

\[
|E(\tau) - J(\tau)| \leq O(\tau^{-1/2}(E(\tau)J(\tau))^{1/2}).
\]  

(2.13)

Then, taking the square on both sides in (2.13), we obtain

\[
E^2(\tau) \leq (2 + O(\tau^{-1}))E(\tau)J(\tau)
\]

and thus \( E(\tau) \leq (2 + O(\tau^{-1}))J(\tau) \). A combination of this and (2.13) yields

\[
|E(\tau) - J(\tau)| \leq O(\tau^{-1/2})J(\tau).
\]

Therefore we obtain (2.8). This completes the proof of lemma 2.2.

Thus the point is the derivation of (2.12). Here we apply the Lax–Phillips reflection argument developed in [27] (see also appendix A in [20]) to our situation. We focus only on the essential part of the derivation.

The key point is a differential identity for \((\Delta - \tau^2)(\nu')\) which is a consequence of (4.15) in [27]. It takes the form

\[
(\Delta - \tau^2)(\nu')(x) = \phi(x) \sum_{i,j} d_{\partial D}(x)a_{ij}(x)(\partial_i \partial_j \nu)(x')
\]

\[\quad + \sum_{i,j} \left( \sum_k b_{jk}(x) \frac{\partial \phi}{\partial x_k}(x) + d_{ij}(x)\phi(x) \right) \frac{\partial v}{\partial x_j}(x') + (\Delta \phi)(x)\nu(x'),
\]

where \( a_{ij}(x), b_{jk}(x) \) and \( d_{ij}(x) \) with \( i,j,k = 1,2,3 \) are independent of \( \phi \) and \( \nu \); \( a_{ij}(x) \) and \( b_{jk}(x) \) are \( C^1 \) and \( d_{ij}(x) \) is \( C^0 \) for \( x \in \mathbb{R}^1 \setminus D \) with \( d_{\partial D}(x) < 2\delta_0 \).

A change of variables \( x = y' \) gives

\[
\int_{\mathbb{R}^1 \setminus D} \epsilon_0(x)\eta(x)(\Delta - \tau^2)\nu'(x) \, dx = \int_D \epsilon_0(y')\eta(y')
\]

\[\times \left\{ \phi(y') \sum_{i,j} d_{\partial D}(y')a_{ij}(y')(\partial_i \partial_j \nu)(y) + \text{lower order terms} \right\} J(y) \, dy,
\]

(2.14)

where \( J(y) \) denotes the Jacobian. The point is the bound on the first term involving the second-order derivatives of \( \nu \). Since \( d_{\partial D}(x) = 0 \) on \( \partial D \), applying integration by parts to the term, one can rewrite as follows:

\[
\int_D \epsilon_0(y')\eta(y')\phi(y') \sum_{i,j} d_{\partial D}(y')a_{ij}(y')(\partial_i \partial_j \nu)(y)J(y) \, dy
\]

\[= -\sum_{i,j} \int_D \partial_i [\epsilon_0(y')\eta(y')\phi(y')d_{\partial D}(y)a_{ij}(y')J(y)] \partial_j \nu(y) \, dy.
\]

Note that \( d_{\partial D}(y') = d_{\partial D}(y) \). This yields

\[
\int_D \epsilon_0(y')\eta(y')\phi(y') \sum_{i,j} d_{\partial D}(y')a_{ij}(y')(\partial_i \partial_j \nu)(y)J(y) \, dy
\]

\[= \left\{ O(\delta) \| \nabla \epsilon_0 \|_{L^2(\partial D)} + O(1) \| \epsilon_0 \|_{L^2(\partial D)} \right\} \| \nabla \nu \|_{L^2(\partial D)}.
\]  

(2.15)
where \( \epsilon'(y) = \epsilon_0(y') \) and \( D_3 = \{ y \in D \mid d_{AD}(y) < 28 \} \). Note that \( O(\delta) \) comes from \( d_{AD}(y) \).

Just simply estimating the integrals involving the lower order terms in the right-hand side of (2.14) and combining the results with (2.15), we obtain

\[
\left| \int_{R^n \backslash D} \epsilon_0(x) \eta(x) (\Delta - \tau^2) v'(x) \, dx \right| = O(\delta) \| \nabla \epsilon_0 \|_{L^2(D_0)} \| \nabla v \|_{L^2(D)} + O(\delta^{-1}) \| \epsilon_0 \|_{L^2(D_0)} \| v \|_{L^2(D)}. \tag{2.16}
\]

It is easy to see that

\[
\| \epsilon_0 \|_{L^2(R^n \backslash D)} \leq C \| \epsilon_0 \|_{L^2(R^n \backslash D)}, \quad \| \nabla \epsilon_0 \|_{L^2(R^n \backslash D)} \leq C \| \nabla \epsilon_0 \|_{L^2(R^n \backslash D)}. \tag{2.17}
\]

Moreover, using a trace theorem, one can easily obtain the following inequalities: there exist constants \( \tau_0 > 0 \) and \( C > 0 \) such that, for all \( \tau \geq \tau_0 \),

\[
\| \epsilon_0 \|_{L^2(R^n \backslash D)} \leq C \tau^{-2} E(\tau), \tag{2.18}
\]

\[
\| \nabla \epsilon_0 \|_{L^2(R^n \backslash D)} \leq C E(\tau), \tag{2.19}
\]

\[
\| v \|_{L^2(D)} \leq C \tau^{-2} J(\tau), \tag{2.20}
\]

\[
\| \nabla v \|_{L^2(D)} \leq CJ(\tau). \tag{2.21}
\]

Note that these include also \( E(\tau) > 0 \) and \( J(\tau) > 0 \) for all \( \tau \geq \tau_0 \) provided \( \gamma = 0 \). Applying (2.18) and (2.19) to (2.17), we obtain

\[
\| \epsilon_0 \|_{L^2(D_0)} \leq C \tau^{-1} E(\tau)^{1/2}, \quad \| \nabla \epsilon_0 \|_{L^2(D_0)} \leq C E(\tau)^{1/2}. \tag{2.22}
\]

Now applying (2.20)–(2.22) to the right-hand side of (2.16), we obtain

\[
\left| \int_{R^n \backslash D} \epsilon_0(x) \eta(x) (\Delta - \tau^2) v'(x) \, dx \right| \leq \{ O(\delta) + O(\delta^{-1} \tau^{-1}) + O(\delta^{-2} \tau^{-2}) \} (E(\tau)J(\tau))^{1/2}. \tag{2.23}
\]

On the other hand, a direct computation yields

\[
(\Delta - \tau^2)(\eta v') - \eta(\Delta - \tau^2)(v') = (\Delta \eta)(\phi(x) v(x')) + 2(\nabla \eta \phi(x) \cdot (\nabla v)(x')).
\]

From this and the regularity of \( q(x) \) and \( d_{AD}(x) \) for \( x \in R^3 \backslash D \) with \( d_{AD}(x) < 2\delta_0 \) in [8], one obtains that there exists a positive constant \( C \) such that, for all \( x \in R^3 \backslash D \),

\[
|(\Delta - \tau^2)(\eta v') - \eta(\Delta - \tau^2)(v')| \leq C|v(x')(\| \phi(x) \| + \| \nabla \phi(x) \|) + |(\nabla v)(x')\| \phi(x)|. \tag{2.24}
\]

Using the change of variables \( x = y' \), we have also from (2.20)–(2.22)

\[
\left| \int_{R^n \backslash D} \epsilon_0(x)(|v(x')(\| \phi(x) \| + \| \nabla \phi(x) \|) + |(\nabla v)(x')\| \phi(x)|) \, dx \right| \leq C(\tau^{-2}(1 + O(\delta^{-1})) + \tau^{-1})(E(\tau)J(\tau))^{1/2}. \tag{2.25}
\]

From this, (2.23) and (2.24) we obtain

\[
\left| \int_{R^n \backslash D} \epsilon_0(\Delta - \tau^2)(\eta v') \, dx \right| \leq \{ O(\delta) + O(\delta^{-1} \tau^{-1}) + O(\delta^{-2} \tau^{-2}) + C(\tau^{-2}(1 + O(\delta^{-1})) + \tau^{-1})(E(\tau)J(\tau))^{1/2} \}
\]

Now choosing \( \delta = \tau^{-1/2} \), we obtain (2.12).

**Remark 2.1.** If \( \gamma \neq 0 \), then the argument above does not work because of the presence of the growing factor \( \gamma(q(x)) \tau \) as \( \tau \to \infty \). Note also that \( J(\tau) < 0 \) for \( \tau \gg 1 \) when \( \inf_{x \in dAD}(\gamma(x) < 1) > 0 \) (see [19]).
3. Proof of theorems 1.1 and 1.2

In this section, we set \( f = \chi_B \) and \( v = v_f \). By (2.1), it suffices to study the asymptotic behaviour of

\[
J(\tau) = \int_{\partial D} \left( \frac{\partial v}{\partial v} - cv \right) v \, dS = \int_{\partial D} \frac{\partial v}{\partial v} v \, dS - \int_{\partial D} c|v|^2 \, dS. \tag{3.1}
\]

We will see that it is the second-order term, not the top term, that contains information about \( c = \beta(x) \) at \( x \in \Lambda_{\partial D}(p) \).

By (3.26) in [20], we have, for a positive constant \( C_1 \),

\[
\int_{\partial D} \frac{\partial v}{\partial v} v \, dS = \frac{1}{4\tau^3} \left( \eta - \frac{1}{\tau} \right)^2 \int_{\partial D} \left( \frac{\beta(x)}{|x-p|^2} \right) \frac{p-x}{|p-x|} v \, dS + O(\tau^{-1} e^{-2\tau \text{dist} (B, (1+C_1))}). \tag{3.2}
\]

Note that \( v_k \) is outward with respect to \( D \).

On the other hand, by (3.22) and (3.24) in [20], we have, for a positive constant \( C_2 \),

\[
\int_{\partial D} c|v|^2 \, dS = \frac{1}{4\tau^3} \left( \eta - \frac{1}{\tau} \right)^2 \int_{\partial D} \frac{\beta(x)}{|x-p|^2} \frac{\beta(x) e^{-2\tau \text{dist} (B, (1+C_1))}}{|x-p|^2} \, dS + O(\tau^{-5} e^{-2\tau \text{dist} (B, (1+C_1))}). \tag{3.3}
\]

Substituting (3.2) and (3.3) into (3.1), we obtain

\[
J(\tau) = \frac{1}{4\tau^3} \left( \eta - \frac{1}{\tau} \right)^2 \left( I_1(\tau) + \frac{1}{\tau} I_2(\tau) \right) + O(\tau^{-1} e^{-2\tau \text{dist} (B, (1+C_1))}), \tag{3.4}
\]

where

\[
I_1(\tau) = \int_{\partial D} \frac{1}{|x-p|^2} \frac{p-x}{|p-x|} v \, e^{-2\tau \text{dist} (B, (1+C_1))} \, dS,
\]

and

\[
I_2(\tau) = \int_{\partial D} \left( \frac{1}{|x-p|^3} \frac{p-x}{|p-x|} v \right) - \frac{\beta(x)}{|x-p|^2} e^{-2\tau \text{dist} (B, (1+C_1))} \, dS.
\]

A combination of (2.1) and (3.4) gives

\[
\int_B (w_f - v_f) \, dx = \frac{1}{2\tau^2} \left( \eta - \frac{1}{\tau} \right)^2 \left( I_1(\tau) + \frac{1}{\tau} I_2(\tau) \right) + O(\tau^{-1} e^{-2\tau \text{dist} (B, (1+C_1))}) \times (1 + O(\tau^{-1/2})) + O(\tau^{-1} e^{-\tau^2}). \tag{3.5}
\]

Remark 3.1. Note that, to make the derivation of (3.4) self-contained, an alternative direct proof is also given in the appendix.

Now we study the asymptotic behaviour of \( I_1(\tau) \) and \( I_2(\tau) \) as \( \tau \to \infty \).

For each \( q \in \Lambda_{\partial D}(p) \) let \( e_j, j = 1, 2 \), be the orthonormal basis of the tangent space at \( q \) of \( \partial D \) with \( e_1 \times e_2 = v_q \). Choose an open ball \( U \) centred at \( q \) with radius \( r_q \) in such a way that there exists an \( h \in C^2_0(\mathbb{R}^3) \) with \( h(0,0) = 0 \) and \( \nabla h(0,0) = 0 \) such that \( U \cap \partial D = \{ q + \sigma_1 e_1 + \sigma_2 e_2 + h(\sigma_1, \sigma_2) r_q \} \), \( \sigma_1^2 + \sigma_2^2 + h(\sigma_1, \sigma_2) < r_q^2 \). In \( U \cap \partial D \), we have

\[
x - p = \sigma e_1 + \sigma_2 e_2 + (h(\sigma) - d_{\partial D}(p)) v_q
\]

and

\[
v_q \, dS = (-h_{\sigma_1}(\sigma) e_1 - h_{\sigma_2}(\sigma) e_2 + v_q) \, d\sigma.
\]
Thus we have
\[
\frac{p-x}{|x-p|} \cdot v_x \, dS_x = \Psi_q(\sigma)^{-1} (\nabla h(\sigma) \cdot \sigma + d_{\partial D}(p) - h(\sigma)) \, d\sigma,
\]
where
\[
\Psi_q(\sigma) = |x - p| = \sqrt{|\sigma|^2 + (d_{\partial D}(p) - h(\sigma))^2}.
\]
Define
\[
\phi(\sigma) = \Psi_q(0) - \Psi_q(\sigma).
\]
We have
\[
\phi_{\sigma,i}(0) = -\left( \frac{1}{d_{\partial D}(p)} I_2 - \nabla^2 h(0) \right)
\]
and thus
\[
\det (\phi_{\sigma,i}(0)) = \det (S_q(\partial B(d_{\partial D}(p))) - S_q(\partial D)). \tag{3.6}
\]
Using the finiteness of \(\Lambda_{\partial D}(p)\) and the partition of the unity, we see that for the study of the asymptotic expansion of \(I_1(\tau)\) and \(I_2(\tau)\) it suffices to compute the asymptotic expansion of the following two integrals, respectively, if necessary choosing a smaller \(r_q\):
\[
\tilde{I}_1(\tau) = e^{-2\tau \text{dist} (D,B)} \int_{|\sigma| < r_q} g_0(\sigma) \, e^{2\tau \phi(\sigma)} \, d\sigma
\]
and
\[
\tilde{I}_2(\tau) = e^{-2\tau \text{dist} (D,B)} \int_{|\sigma| < r_q} g_1(\sigma) \, e^{2\tau \phi(\sigma)} \, d\sigma,
\]
where
\[
g_0(\sigma) = \Psi_q(\sigma)^{-3} (\nabla h(\sigma) \cdot \sigma + d_{\partial D}(p) - h(\sigma)) \tag{3.7}
\]
and
\[
g_1(\sigma) = \frac{g_0(\sigma)}{\Psi_q(\sigma)} - \frac{\beta(\sigma)}{\Psi_q(\sigma)^2} \sqrt{1 + |\nabla h(\sigma)|^2}.
\]
Since we have (3.6), by the Laplace method ([2]) we obtain
\[
\sqrt{\det (\phi_{\sigma,i}(0))} e^{2\tau \text{dist} (D,B)} \tilde{I}_1(\tau) = \frac{\pi}{\tau d_{\partial D}(p)^2} + O(\tau^{-2}), \tag{3.8}
\]
and
\[
\sqrt{\det (\phi_{\sigma,i}(0))} e^{2\tau \text{dist} (D,B)} \tilde{I}_2(\tau) = \frac{\pi}{\tau} \left( \frac{1}{d_{\partial D}(p)^3} - \frac{\beta(q)}{d_{\partial D}(p)^2} \right) + O(\tau^{-2}). \tag{3.9}
\]
Thus theorem 1.1 directly follows from (3.5), (3.8) and (3.9).

For the proof of theorem 1.2 we have to compute the second term of the asymptotic expansion of \(\tilde{I}_1(\tau)\) as \(\tau \to \infty\) since the term will make a contribution to the second-order term in the right-hand side of (3.5). This can be done, in principle; see (8.3.50) on p 338 in [2].

However, as pointed out there (see (8.3.53) therein), the coefficient is quite complicated except for the leading term:
\[
\sqrt{\det (\phi_{\sigma,i}(0))} e^{2\tau \text{dist} (D,B)} \tilde{I}_1(\tau) = \frac{\pi}{\tau d_{\partial D}(p)^2} = \frac{\pi}{4\tau^2} \Delta \chi G_0|_{\xi=0} \sqrt{\det (\phi_{\sigma,i}(0))} + O(\tau^{-3}), \tag{3.10}
\]
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where $G_0(\xi)$ is given by (8.3.26) on p 334 in [2] and $\triangle G_0|_{\xi=0}$ has the following form in our notation:

$$
\triangle G_0|_{\xi=0}\sqrt{\det (\phi_{\sigma,\sigma}(0))} = \phi_{\sigma,\sigma,\sigma}(0)B_{pr}B_{qs}(g_0)_{\sigma}(0) - \text{Trace} (CB)
$$

$$
\left\{ -g_0(0) \phi_{\sigma,\sigma,\sigma}(0) \left( \frac{1}{4} B_{pr}B_{qs} + \frac{1}{6} B_{pr}B_{qs} \right) - \frac{1}{4} \phi_{\sigma,\sigma,\sigma}(0)B_{pr}B_{qs} \right\},
$$

(3.11)

where

$$
(B_{pq}) = (\phi_{\sigma,\sigma}(0))^{-1},
$$

$$
C = ((g_0)_{\sigma,\sigma}(0)).
$$

Note that we have used the summation convention where repeated indices are to be summed from 1 to 2.

Thus, from (3.9) and (3.10) we obtain

$$
\sqrt{\det (\phi_{\sigma,\sigma}(0))} e^{2r\text{dist} (D,B)} \left( \tilde{I}_1(\tau) + \frac{1}{\tau} \tilde{I}_2(\tau) \right)
$$

$$
= \frac{\pi}{d_3} \frac{1}{\frac{1}{\tau^2} + \frac{1}{\tau^2} \left( \frac{1}{2} \triangle G_0|_{\xi=0} \sqrt{\det (\phi_{\sigma,\sigma}(0))} - \frac{1}{d_3} \frac{1}{d_3} \phi_{\sigma,\sigma}(0) \right) \left( \frac{1}{d_3} \frac{1}{d_3} \right) \frac{1}{\tau^2}}
$$

$$
+ O \left( \frac{1}{\tau^2} \right) = \frac{A}{\tau} + B + O \left( \frac{1}{\tau^2} \right),
$$

where

$$
A = \frac{\pi}{d_3} \frac{1}{\frac{1}{\tau^2} + \frac{1}{\tau^2} \left( \frac{1}{2} \triangle G_0|_{\xi=0} \sqrt{\det (\phi_{\sigma,\sigma}(0))} - \frac{1}{d_3} \frac{1}{d_3} \phi_{\sigma,\sigma}(0) \right) \left( \frac{1}{d_3} \frac{1}{d_3} \right) \frac{1}{\tau^2}}
$$

and

$$
B = \pi \left( \frac{1}{2} \triangle G_0|_{\xi=0} \sqrt{\det (\phi_{\sigma,\sigma}(0))} - \frac{1}{d_3} \frac{1}{d_3} \phi_{\sigma,\sigma}(0) \right) \left( \frac{1}{d_3} \frac{1}{d_3} \right) \frac{1}{\tau^2}.
$$

Thus, we have

$$
\sqrt{\det (\phi_{\sigma,\sigma}(0))} e^{2r\text{dist} (D,B)} \left( \eta - \frac{1}{\tau} \right) \left( \tilde{I}_1(\tau) + \frac{1}{\tau} \tilde{I}_2(\tau) \right)
$$

$$
= \eta^2 A \frac{\tau}{\eta B - 2A} + O \left( \frac{1}{\tau^2} \right).
$$

(3.12)

Here we prepare the following formulae whose proofs are given in the appendix.

Lemma 3.1. We have

$$
(g_0)_{\sigma}(0) = 0,
$$

(3.13)

$$
\phi_{\sigma,\sigma,\sigma}(0) = h_{\sigma,\sigma,\sigma}(0),
$$

(3.14)

$$
\phi_{\sigma,\sigma,\sigma,\sigma}(0)B_{pr}B_{qs} = h_{\sigma,\sigma,\sigma,\sigma}(0)B_{pr}B_{qs} + \frac{14}{d_3} - 16 \frac{dH_{3D}(q)}{d \sqrt{\det \phi_{\sigma,\sigma}(0)}}
$$

(3.15)

and

$$
\text{Trace} (CB) = \frac{8}{d_3} - \frac{2(1 - dH_{3D}(q))}{d \sqrt{\det \phi_{\sigma,\sigma}(0)}}.
$$

(3.16)
From this lemma we obtain

\[ \frac{\triangle G_0}{|\xi|_0} \psi_0 = \sqrt{\det (\phi_{\sigma_1 \sigma_1}(0))} \left( -\frac{8}{d^3} + \frac{11 - 12 d H_d^q(q)}{2d^2 \det (\phi_{\sigma_1 \sigma_1}(0))} \right) \]

\[ - d^{-2} h_{\sigma_1 \sigma_1 \sigma_1 \sigma_1}(0) h_{\sigma_1 \sigma_1 \sigma_1 \sigma_1}(0) \left( \frac{1}{4} B_{\sigma_1 \sigma_1 \sigma_1 \sigma_1} B_{\eta_1 \eta_1 \eta_1 \eta_1} + \frac{1}{6} B_{\sigma_1 \sigma_1 \sigma_1 \sigma_1} B_{\eta_1 \eta_1 \eta_1 \eta_1} \right) \]

\[ + \frac{1}{4d^2} h_{\sigma_1 \sigma_1 \sigma_1 \sigma_1}(0) B_{\eta_1 \eta_1 \eta_1 \eta_1}, \]

(3.17)

where \( d = d_{\partial D}(p) \). Thus if one knows the values of all the second-, third- and fourth-order derivatives of \( h \) at \( \sigma = 0 \), then the right-hand side of (3.17) is known. Finally summing (3.12) over all \( q \in \Lambda_{\partial D}(p) \), from (3.5) we see that (1.6) in theorem 1.2 is true. The proof of lemma 3.1 is given in the appendix.

4. A data reduction and its implication

4.1. Proof of proposition 1.1.

Let \( U \) be a non-empty open set of \( \mathbb{R}^3 \). The mean value theorem [4] states that if \( \psi \in C^2(U) \) is a solution of the modified Helmholtz equation

\[ (\triangle - \tau^2) \psi = 0 \quad \text{in} \quad U \]

and \( B_r(x) \subset U \), then

\[ \psi(x) = \frac{\tau}{4\pi r \sinh(\tau r)} \int_{\partial B_r(x)} \psi(y) \, dS(y). \]  (4.1)

Substituting this into the identity

\[ \int_{B_r(x)} \psi(z) \, dz = \int_0^r \left( \int_{\partial B_r(x)} \psi(y) \, dS(y) \right) \, ds, \]

we obtain

\[ \int_{B_r(x)} \psi(z) \, dz = \frac{4\pi}{\tau^3} (\tau r \cosh(\tau r) - \sinh(\tau r)) \psi(x). \]  (4.2)

Lemma 4.1. Let \( B \) and \( f \) be the same as those of problem I. Let \( B_R(p) \) denote the open ball centred at \( p \) with radius \( R \) and satisfy \( B_R(p) \subset \mathbb{R}^3 \setminus \overline{D} \) and \( B \subset B_R(p) \). We have

\[ \int_{\partial B_R(p)} (w_f - v_f) \, dS = \frac{\tau^2 R \sinh(\tau R)}{\tau \eta \cosh(\tau \eta) - \sinh(\tau \eta)} \int_B (w_f - v_f) \, dx + O(e^{-\tau(T - (R - \eta))}). \]  (4.3)

Proof. We have

\[ \int_B (w_f - v_f) \, dx = \int_B e_f^0 \, dx + \int_B Z \, dx. \]

Since we have (2.5), it holds that

\[ \int_B Z \, dx = O(e^{-\tau T}) \]

and thus

\[ \int_B (w_f - v_f) \, dx = \int_B e_f^0 \, dx + O(e^{-\tau T}). \]  (4.4)
Applying (4.1) and (4.2) to \( \epsilon^0_f \) and putting \( \varphi(\xi) = \xi \cosh \xi - \sinh \xi \), we have
\[
\int_B \epsilon^0_f \, dx = \frac{4\pi}{\tau} \varphi(\tau \eta) \epsilon^0_f(p) = \frac{4\pi}{\tau^3} \varphi(\tau \eta) \times \frac{\tau}{4\pi R \sinh(\tau R)} \int_{\partial B_0(p)} \epsilon^0_f \, dS
\]
\[
= \frac{\varphi(\tau \eta)}{\tau^3 R \sinh(\tau R)} \int_{\partial B_0(p)} (w_f - v_f - Z) \, dS
\]
\[
= \frac{\varphi(\tau \eta)}{\tau^3 R \sinh(\tau R)} \left( \int_{\partial B_0(p)} (w_f - v_f) \, dS - \int_{\partial B_0(p)} Z \, dS \right).
\]

Just simply applying the trace theorem to \( \partial B_0(p) \), from (2.5) we have
\[
\int_{\partial B_0(p)} Z \, dS = O(\tau e^{-\tau T}).
\]
Thus, we obtain
\[
\int_B \epsilon^0_f \, dx = \frac{\varphi(\tau \eta)}{\tau^3 R \sinh(\tau R)} \left( \int_{\partial B_0(p)} (w_f - v_f) \, dS + O(\tau e^{-\tau T}) \right).
\]
From this and (4.4) we obtain
\[
\int_B (w_f - v_f) \, dx = \frac{\varphi(\tau \eta)}{\tau^3 R \sinh(\tau R)} \left( \int_{\partial B_0(p)} (w_f - v_f) \, dS + O(\tau e^{-\tau T}) \right) + O(e^{-\tau T}). \tag{4.5}
\]
Since \( R \gtrsim \eta \) and
\[
O\left( e^{-\tau T} \frac{\tau^2 R \sinh(\tau R)}{\varphi(\tau \eta)} \right) = O\left( e^{-\tau T} \frac{\tau^2 R e^{\tau R}}{\tau \eta e^{\tau \eta}} \right) = O(\tau e^{-\tau T} e^{R-\eta}),
\]
from (4.5) one gets (4.3). \( \Box \)

Since we have
\[
\frac{\tau^2 R \sinh(\tau R)}{\tau \eta \cosh(\tau \eta) - \sinh(\tau \eta)} = \frac{\tau^2 R (e^{\tau R} - e^{-\tau R})}{\tau \eta (e^{\tau \eta} + e^{-\tau \eta}) - (e^{\tau \eta} - e^{-\tau \eta})}
\]
\[
= \frac{\tau^2 R e^{\tau R}(1 - e^{-2\tau R})}{\tau \eta e^{\tau \eta}(1 + e^{-2\tau R}) - e^{\tau \eta}(1 - e^{-2\tau R})}
\]
\[
= \frac{\tau^2 R e^{\tau R}(1 - e^{-2\tau R})}{\tau \eta e^{\tau \eta}(1 + O(\tau^{-1} e^{-\tau \eta}))}
\]
\[
= \frac{R}{\eta} e^{\tau(R-\eta)} \frac{1}{\tau^{-1} e^{-\tau \eta} + O(e^{-2\tau R})}
\]
\[
= \frac{R}{\eta} e^{\tau(R-\eta)} (1 + O(\tau^{-1} e^{-\tau \eta} + O(e^{-2\tau R}))
\]
\[
= \frac{R}{\eta} e^{\tau(R-\eta)} (1 + O(\tau^{-1} e^{-\tau \eta})).
\]

(4.3) yields (1.7). This completes the proof of proposition 1.1.

### 4.2. Some transplanted results

Let \( B \) and \( f \) be the same as those of problem I. Let \( B_R(p) \) denote the open ball centred at \( p \) with radius \( R \) and satisfy \( B_R(p) \subset \mathbb{R}^3 \setminus D \) and \( B \subset B_R(p) \).
Let \( T > 2 \text{ dist}(D, B) \). From (1.7) we have

\[
\frac{\tau^3 \eta}{R} e^{t(2 \text{ dist}(D, B) - (R - \eta))} \int_{\partial B_{\eta}(p)} (w_f - v_f) dS = \tau^4 e^{2 \text{ dist}(D, B)} (1 + O(\tau^{-1} e^{-\tau\eta}))
\]

\[
\times \int_B (w_f - v_f) dx + O(\tau^4 e^{-\tau (T - 2 \text{ dist}(D, B))}),
\]

(4.6)

This enables us to transplant several results with data \( u_f \) on \( B \times [0, T] \) provided \( T > 2 \text{ dist}(D, B) \) not \( T > 2 \text{ dist}(D, B) - (R - \eta) \). We think that this is reasonable since the quantity \( 2 \text{ dist}(D, B) - (R - \eta) \) coincides with the time of flight of a wave with propagation speed 1 that starts at points on \( \partial B \) and hits points on \( \partial D \) and returns to \( \partial B_{\eta}(p) \). However, the surface is thin, so to make the received signal on \( \partial B_{\eta}(p) \) strong enough some redundant observation time may be needed.

Using (4.6), from theorem 1.2 in [19] one can deduce the following result.

**Corollary 4.1.** Assume that \( \partial D \) is \( C^2 \). Let \( T \) satisfy \( T > 2 \text{ dist}(D, B) \). Let \( C \) be a positive constant.

We have the following:

if \( \gamma(x) \leq 1 - C \) a.e. \( x \in \partial D \), then there exists \( \tau_0 > 0 \) such that, for all \( \tau \geq \tau_0 \),

\[
\int_{\partial B_{\eta}(p)} (w_f - v_f) dS > 0;
\]

if \( \gamma(x) \geq 1 + C \) a.e. \( x \in \partial D \), then there exists \( \tau_0 > 0 \) such that, for all \( \tau \geq \tau_0 \),

\[
\int_{\partial B_{\eta}(p)} (w_f - v_f) dS < 0.
\]

In both cases, the formula

\[
\lim_{\tau \to \infty} \frac{1}{\tau} \log \left| \int_{\partial B_{\eta}(p)} (w_f - v_f) dS \right| = -(2 \text{ dist}(D, B) - (R - \eta))
\]

is valid.

The following result is a transplanted version of theorem 1.1 via (4.6).

**Corollary 4.2.** Let \( \gamma \equiv 0 \). Assume that \( \partial D \) is \( C^3 \) and \( \beta \in C^2(\partial D) \); \( \Lambda_{3D}(p) \) is finite and satisfies

\[
\det (S_q(\partial B_{\eta}(p))) - S_q(\partial D)) > 0, \forall q \in \Lambda_{3D}(p).
\]

If \( T \) satisfies \( T > 2 \text{ dist}(D, B) \), then we have

\[
\lim_{\tau \to \infty} \frac{\tau^3 \eta}{R} e^{t(2 \text{ dist}(D, B) - (R - \eta))} \int_{\partial B_{\eta}(p)} (w_f - v_f) dS = \frac{\pi}{2} \left( \frac{\eta}{d_{3D}(p)} \right)^2 \Lambda_{3D}(p).
\]

From (4.6) we have

\[
\tau^4 \left\{ \frac{\eta}{R} e^{t(2 \text{ dist}(D, B) - (R - \eta))} \int_{\partial B_{\eta}(p)} (w_f - v_f) dS - \frac{1}{\tau^3} \left( \frac{\eta}{d_{3D}(p)} \right)^2 \Lambda_{3D}(p) \right\}
\]

\[
= (1 + O(\tau^{-1} e^{-\tau\eta})) \tau^5 \left\{ e^{2 \text{ dist}(D, B)} \int_B (w_f - v_f) dx \right.
\]

\[
- \frac{1}{\tau^3} \left( \frac{\eta}{d_{3D}(p)} \right)^2 \Lambda_{3D}(p) \right\} + O(e^{-\tau\eta}) + O(\tau^5 e^{-\tau (T - 2 \text{ dist}(D, B)))}.
\]

This enables us to deduce the following result from theorem 1.2.
Corollary 4.3. Let $\gamma \equiv 0$. Assume that $\partial D$ is $C^5$ and $\beta \in C^2(\partial D)$; $\Lambda_{3D}(p)$ is finite and satisfies

$$\det (S_q(\partial B_{d_{3D}}(p)) - S_q(\partial D)) > 0, \forall q \in \Lambda_{3D}(p).$$

For each $q \in \Lambda_{3D}(p)$ let $e_j, j = 1, 2$, be the orthonormal basis of the tangent space at $q$ of $\partial D$ with $e_1 \times e_2 = v_q$. Choose an open ball $U$ centred at $q$ with radius $r_q$ in such a way that there exists an $h \in C^5_0(\mathbb{R}^2)$ with $h(0,0) = 0$ and $\nabla h(0,0) = 0$ such that $U \cap \partial D = \{q + \sigma_1 e_1 + \sigma_2 e_2 + h(\sigma_1, \sigma_2) v_q \mid \sigma_1^2 + \sigma_2^2 + h(\sigma_1, \sigma_2)^2 < r_q^2 \}$.

If $T$ satisfies $T \geq 2 \text{dist}(D, B)$, then we have

$$\lim_{\tau \to \infty} \tau^4 \left\{ e^{\text{t}(2 \text{dist}(D, B) - (R - \eta))} \frac{R}{4} \int_{\partial B_{\eta}(p)} (w_f - v_f) \, dS - \frac{1}{\tau^3} \left( \frac{\eta}{d_{3D}(p)} \right)^2 A_{3D}(p) \right\}
= -\frac{\pi \eta}{d_{3D}(p)^2} A_{3D}(p) + \frac{\pi}{2} \eta^2 B_{3D}(p).$$

Some trivial modifications of corollaries 1.1 and 1.2 are also valid; however, we omit their description.

5. Concluding remarks

In this paper we considered only the case where $\gamma \equiv 0$. Thus, the next problem is to consider general $\gamma \geq 0$. The author thinks that proposition 2.1 should be changed because of the dissipation of the energy of the wave on the obstacle.

In [21] we have considered the case where so-called bistatic data are available. This means that one observes the wave at a different place, for example, even possibly far from the centre of the support of the initial data. We found that a spheroid plays a role similar to that of a sphere. However, the technique used in [21] is based on the maximum principle and depends heavily on the homogeneous Dirichlet boundary condition on the obstacle which corresponds to the case where $\gamma \equiv 0$ and $\beta = \infty$. Thus, one cannot readily apply the method to the present case.

For further open problems see section 6 of [21] and also section 4 of [20].

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Appendix

A.1. A direct derivation of (3.4)

Applying the mean value theorem (4.2) to $v_f(x)$ for $x \in \mathbb{R}^3 \setminus B$ with $B = B_{\eta}(p)$, we obtain

$$v_f(x) = \frac{\varphi(\tau \eta) e^{-\tau|x-p|}}{\tau^3} \frac{1}{|x-p|}$$

and thus

$$\nabla v_f(x) = \frac{\varphi(\tau \eta)}{\tau^2} \left( 1 + \frac{1}{\tau|x-p|} \right) \frac{(p-x) e^{-\tau|x-p|}}{|x-p| \frac{x-p}{|x-p|}}.$$
where
\[ \varphi(\xi) = \xi \cosh \xi - \sinh \xi. \]
Substituting these into \( J(\tau) \), we obtain
\[ J(\tau) = \frac{\varphi(\tau \eta)^2}{\tau^5} \left( \tilde{I}_1(\tau) + \frac{1}{\tau} I_2(\tau) \right), \]
where
\[ \tilde{I}_1(\tau) = \int_{\partial D} \frac{1}{|x - p|^2} \cdot \nu_x e^{-2\tau|x - p|} \, dS_x \]
and
\[ I_2(\tau) = \int_{\partial D} \frac{1}{|x - p|^3} \cdot \nu_x - \frac{\beta(x)}{|x - p|^2} e^{-2\tau|x - p|} \, dS_x. \]
Since we have
\[ \varphi(\xi) e^{-\xi} = \xi - 1 + \frac{\xi + 1}{2} e^{-2\xi}, \]
one gets
\[ \frac{\varphi(\tau \eta)^2 e^{-2\tau \eta}}{\tau^5} = \frac{1}{4\tau^3} \left( \eta - \frac{1}{\tau} \right)^2 + O(\tau^{-3} e^{-2\tau \eta}). \]
Note also that \( d_\partial(x) = |x - p| - \eta \) for \( x \in \partial D \) and, thus, \( I_j(\tau) = e^{2\tau \eta} \tilde{I}_j(\tau), j = 1, 2. \)
From these one obtains (3.4).

A.2. Proof of lemma 3.1
In this section we set \( d = d_{\partial D}(p) \).

A.2.1. Proof of (3.13). From (3.7) we have
\[
\frac{\partial}{\partial \sigma} g_0 = -3\Psi^{-4} \frac{\partial \Psi}{\partial \sigma_p} (h_{\sigma_p} \sigma_q + d - h) + \Psi^{-3} \left( h_{\sigma_p} \sigma_q + h_{\sigma_q} - h_{\sigma_p} \right)
\]
\[
= -3\Psi^{-4} \frac{\partial \Psi}{\partial \sigma_p} g_0 + \Psi^{-3} h_{\sigma_p} \sigma_q. \tag{A.1}
\]
Thus we obtain (3.13).

A.2.2. Proof of (3.14). We have
\[
\frac{\partial \Psi_q}{\partial \sigma_p}(\sigma) = \Psi_q(\sigma)^{-1} \{ \sigma_p + h_{\sigma_p}(\sigma)(h(\sigma) - d) \}. \tag{A.2}
\]
This gives
\[
\phi_{\sigma_p}(\sigma) = -\Psi_q(\sigma)^{-1} \{ \sigma_p + h_{\sigma_p}(\sigma)(h(\sigma) - d) \} \tag{A.3}
\]
and
\[
\phi_{\sigma_p}(0) = \frac{\partial \Psi_q}{\partial \sigma_p}(0) = 0. \tag{A.4}
\]
It follows from (A.3) that
\[
\phi_{\sigma_p, \sigma_q}(\sigma) = (\Psi_q(\sigma)^{-1})^{-1} \frac{\partial \Psi_q}{\partial \sigma_q}(\sigma) \{ \sigma_q + h_{\sigma_q}(\sigma)(h(\sigma) - d) \}
\]
\[
- \Psi_q(\sigma)^{-1} \{ \delta_{pq} + h_{\sigma_p, \sigma_q}(\sigma)(h(\sigma) - d) + h_{\sigma_q}(\sigma) h_{\sigma_q}(\sigma) \}. \]
Applying (A.2) to the first term in the right-hand side, we obtain

\[ \Psi_q(\sigma)\phi_{\sigma q}(\sigma) = \phi_{\sigma q}(\sigma)\phi_{\sigma q}(\sigma) - \{\delta_{qp} + h_{\sigma q}(\sigma)(h(\sigma) - d) + h_{\sigma q}(\sigma)h_{\sigma q}(\sigma)\}. \]  

Differentiating both sides of (A.5) with respect to \( \sigma \), and using (A.2) and (A.3), we obtain

\[ -\phi_{\sigma q}(\sigma)\phi_{\sigma q}(\sigma) + \Psi_q(\sigma)\phi_{\sigma q}(\sigma) = \phi_{\sigma q}(\sigma)\phi_{\sigma q}(\sigma) + \phi_{\sigma q}(\sigma)\phi_{\sigma q}(\sigma) \]

\[ -[h_{\sigma q}(\sigma)(h(\sigma) - d) + h_{\sigma q}(\sigma)h_{\sigma q}(\sigma) + h_{\sigma q}(\sigma)h_{\sigma q}(\sigma)] \]

Noting that \( h(0) = h_{\sigma q}(0) = 0 \), from this together with (A.4) we obtain (3.14).

A.2.3. Proof of (3.15). From (A.4) we have

\[ \frac{\partial}{\partial \sigma_s}(\phi_{\sigma q}(\sigma)\phi_{\sigma q}(\sigma) + \Psi_q(\sigma)\phi_{\sigma q}(\sigma))|_{\sigma = 0} = -\phi_{\sigma q}(0)\phi_{\sigma q}(0) + d\phi_{\sigma q}(\sigma)(\sigma)|_{\sigma = 0} \]  

and also

\[ \frac{\partial}{\partial \sigma_s}(\phi_{\sigma q}(\sigma)\phi_{\sigma q}(\sigma) + \phi_{\sigma q}(\sigma)\phi_{\sigma q}(\sigma))|_{\sigma = 0} = \phi_{\sigma q}(0)\phi_{\sigma q}(0) + \phi_{\sigma q}(0)\phi_{\sigma q}(0). \]

Moreover we have

\[ \frac{\partial}{\partial \sigma_s}[h_{\sigma q}(\sigma)(h(\sigma) - d) + h_{\sigma q}(\sigma)h_{\sigma q}(\sigma) + h_{\sigma q}(\sigma)h_{\sigma q}(\sigma)]|_{\sigma = 0} \]

\[ = -d[h_{\sigma q}(\sigma)(0) + h_{\sigma q}(\sigma)(0) + h_{\sigma q}(\sigma)(0) + h_{\sigma q}(\sigma)(0)]. \]

Differentiating both sides of (A.6) with respect to \( \sigma \), and using (A.7)-(A.9), we obtain

\[ \phi_{\sigma q}(\sigma)(\sigma) = \frac{1}{d}(\phi_{\sigma q}(0)\phi_{\sigma q}(0) - h_{\sigma q}(0)h_{\sigma q}(0)) \]

\[ + \frac{1}{d}(\phi_{\sigma q}(0)\phi_{\sigma q}(0) - h_{\sigma q}(0)h_{\sigma q}(0)) \]

\[ + \frac{1}{d}(\phi_{\sigma q}(0)\phi_{\sigma q}(0) - h_{\sigma q}(0)h_{\sigma q}(0) + h_{\sigma q}(0)h_{\sigma q}(0)). \]

A combination of this and (A.5) yields

\[ \phi_{\sigma q}(\sigma)(\sigma) = \frac{1}{d}(\delta_{pq}\delta_{rs} + \delta_{qr}\delta_{ps} + \delta_{ps}\delta_{qr}) - \frac{1}{d^2}(\delta_{pq}h_{\sigma q}(0) + \delta_{rs}h_{\sigma q}(0) + \delta_{qr}h_{\sigma q}(0) + \delta_{ps}h_{\sigma q}(0) + \delta_{qr}h_{\sigma q}(0) + h_{\sigma q}(0)). \]

We have also

\[ \delta_{pq}\delta_{rs}B_{pq}\delta_{rs} = B_{pq}B_{qp}, \quad \delta_{qr}\delta_{ps}B_{qr}\delta_{ps} = B_{pq}B_{qp}, \quad \delta_{ps}B_{pq}\delta_{ps} = B_{pq}B_{qp}. \]

Since \( B_{pq} = B_{qp} \) and \( B_{pq} = \text{Trace } B \), from these we obtain

\[ (\delta_{pq}\delta_{rs} + \delta_{qr}\delta_{ps} + \delta_{ps}\delta_{qr})B_{pq}B_{pq} = 2|B|^2 + (\text{Trace } B)^2. \]

Moreover, we have

\[ \delta_{pq}h_{\sigma q}(0)B_{pq} = \text{Trace } (B\nabla^2 h(0)B); \]

\[ \delta_{rs}h_{\sigma q}(0)B_{pq} = \text{Trace } (\nabla^2 h(0)B^2); \]

\[ \delta_{qr}h_{\sigma q}(0)B_{pq} = \text{Trace } (\nabla^2 h(0)B^2); \]

\[ \delta_{ps}h_{\sigma q}(0)B_{pq} = \text{Trace } (\nabla^2 h(0)B^2); \]

\[ \delta_{qr}h_{\sigma q}(0)B_{pq} = \text{Trace } B \text{ Trace } (\nabla^2 h(0)B); \]

\[ \delta_{ps}h_{\sigma q}(0)B_{pq} = \text{Trace } B \text{ Trace } (\nabla^2 h(0)B). \]
Now it follows from these, (A.10) and (A.11) that
\[
\phi_{\sigma,\sigma,\sigma,\sigma}(0)B_{pr}B_{qs} = \frac{1}{d^4} \left( 2|B|^2 + \text{Trace}(B) \right)^2 - \frac{1}{d^2} \left( \text{Trace}(B) \nabla^2 h(0)B + 3\text{Trace}(\nabla^2 h(0)B^2) \right) \\
+ 2\text{Trace} B \text{ Trace}(\nabla^2 h(0)B) + h_{\sigma,\sigma,\sigma,\sigma}(0)B_{pr}B_{qs}. \tag{A.12}
\]

Note that
\[
\nabla^2 h(0) = B^{-1} + \frac{1}{d} I;
\]
\[
B\nabla^2 h(0)B = \nabla^2 h(0)B^2 = B + \frac{1}{d} B^2;
\]
\[
\text{Trace}(\nabla^2 h(0)B) = 2 + \frac{1}{d} \text{Trace} B;
\]
\[
\text{Trace}(B\nabla^2 h(0)B) = \text{Trace}(\nabla^2 h(0)B^2) = \text{Trace} B + \frac{1}{d}|B|^2. \tag{A.13}
\]

Substituting these into the right-hand side of (A.12), we obtain
\[
d^4(\phi_{\sigma,\sigma,\sigma,\sigma}(0)B_{pr}B_{qs} - h_{\sigma,\sigma,\sigma,\sigma}(0)B_{pr}B_{qs}) = |B|^2 - (\text{Trace} B)^2 - 8d \text{ Trace} B. \tag{A.14}
\]

A direct computation yields
\[
|B|^2 = \left( \frac{1}{d} - h_{\sigma,\sigma}(0) \right)^2 + \left( \frac{1}{d} - h_{\sigma,\sigma}(0) \right)^2 + 2h_{\sigma,\sigma}(0)^2, \tag{A.15}
\]
and thus
\[
(\det \phi_{\sigma,\sigma}(0))^2 |B|^2 - (\text{Trace} B)^2 = \left( \frac{1}{d} - h_{\sigma,\sigma}(0) \right)^2 + \left( \frac{1}{d} - h_{\sigma,\sigma}(0) \right)^2 \\
+ 2h_{\sigma,\sigma}(0)^2 - \left\{ \left( \frac{1}{d} - h_{\sigma,\sigma}(0) \right) + \left( \frac{1}{d} - h_{\sigma,\sigma}(0) \right) \right\}^2 \\
= 2 \left\{ h_{\sigma,\sigma}(0)^2 - \left( \frac{1}{d} - h_{\sigma,\sigma}(0) \right) \left( \frac{1}{d} - h_{\sigma,\sigma}(0) \right) \right\} \\
= -2 \left| \begin{array}{ccc}
\frac{1}{d} - h_{\sigma,\sigma}(0) & -h_{\sigma,\sigma}(0) \\
-h_{\sigma,\sigma}(0) & \frac{1}{d} - h_{\sigma,\sigma}(0)
\end{array} \right| = -2 \det \phi_{\sigma,\sigma}(0).
\]

This yields
\[
\det \phi_{\sigma,\sigma}(0)|(B|^2 - (\text{Trace} B)^2) = -2.
\]

Then we have
\[
\det \phi_{\sigma,\sigma}(0)|(B|^2 - (\text{Trace} B)^2 - 8d \text{ Trace} B) = 14 - 16d H_{\sigma\sigma}(q).
\]

Now from (A.14) we obtain (3.15).

### A.2.4. Proof of (3.16).

From (A.1) we have
\[
\frac{\partial^2}{\partial \sigma_q \partial \sigma_p} g_0 = \frac{\partial}{\partial \sigma_q} \left( -3 \Psi^{-1} \frac{\partial \Psi}{\partial \sigma_p} g_0 + \Psi^{-3} h_{\sigma,\sigma,\sigma,\sigma} \right) \\
= 3 \Psi^{-2} \frac{\partial \Psi}{\partial \sigma_q} \frac{\partial \Psi}{\partial \sigma_p} g_0 - 3 \Psi^{-1} \frac{\partial^2 \Psi}{\partial \sigma_q \partial \sigma_p} g_0 - 3 \Psi^{-1} \frac{\partial \Psi}{\partial \sigma_p} \frac{\partial g_0}{\partial \sigma_q} \\
- 3 \Psi^{-1} \frac{\partial \Psi}{\partial \sigma_q} h_{\sigma,\sigma,\sigma,\sigma} + \Psi^{-3} h_{\sigma,\sigma,\sigma,\sigma} + \Psi^{-3} h_{\sigma,\sigma,\sigma,\sigma}.
\]
This together with (A.4) and (3.13) yields
\[
(g_0)_{\sigma,\sigma}(0) = -3d^{-3} \frac{\partial^2 \psi}{\partial \sigma_q \partial \sigma_p}(0) + d^{-3} h_{\sigma,\sigma_q}(0) = d^{-3} \left( h_{\sigma,\sigma_q}(0) - 3 \frac{\partial^2 \psi}{\partial \sigma_q \partial \sigma_p}(0) \right).
\]
(A.16)

Since we have
\[
\frac{\partial^2 \psi}{\partial \sigma_p \partial \sigma_q}(0) = \frac{1}{d} \delta_{pq} - h_{\sigma,\sigma_q}(0),
\]
from (A.16), we obtain
\[
(g_0)_{\sigma,\sigma}(0) = d^{-3} \left( 4h_{\sigma,\sigma_q}(0) - \frac{3}{d} \delta_{pq} \right).
\]

This together with (A.13) yields
\[
\text{Trace}(CB) = (g_0)_{\sigma,\sigma_q} B_{\sigma p} = d^{-3} \left( 4h_{\sigma,\sigma_q}(0) B_{\sigma p} - \frac{3}{d} \delta_{pq} B_{\sigma p} \right)
= d^{-3} \left( 4 \text{Trace}(\nabla^2 h(0) B) - \frac{3}{d} \text{Trace} B \right)
= d^{-3} \left( 4 \left( 2 + \frac{1}{d} \text{Trace} B \right) - \frac{3}{d} \text{Trace} B \right)
= d^{-3} \left( 8 + \frac{1}{d} \text{Trace} B \right).
\]

Now from this and the second formula of (A.15) we obtain (3.16).

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