Optimal Hierarchical Signaling for Quadratic Cost Measures and General Distributions: A Copositive Program Characterization

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Abstract—In this paper, we address the problem of optimal hierarchical signaling between a sender and a receiver for a general class of square integrable multivariate distributions. The receiver seeks to learn a certain information of interest that is known to the sender while the sender seeks to induce the receiver to perceive that information as a certain private information. For the setting where the players have quadratic cost measures, we analyze the Stackelberg equilibrium, where the sender leads the game by committing his/her strategies beforehand. We show that when the underlying state space is “finite”, the optimal signaling strategies can be computed through an equivalent linear optimization problem over the cone of completely positive matrices. The equivalence established enables us to use the existing computational tools to solve this class of cone programs approximately with any error rate. For continuous distributions, we also analyze the error of approximation, if the optimal signaling strategy is computed for a discretized version obtained through a quantization scheme, and we provide an upper bound in terms of the quantization error.

Index Terms—Stackelberg games, Signaling, Bayesian persuasion, Deception, Privacy, Copositive Programming.

I. INTRODUCTION

To induce intelligent decision makers to take certain actions, we can create incentives via external means, e.g., explicit payments, but we can also persuade them to take the actions by their own will without the need for any external means if we can craft the information available to them [1]. In informed (rational) decisions, understanding of how the information is generated plays an essential role. Without such an understanding, rationality necessitates the consideration of who generates the information. For example, if the objectives of the information provider and the decision maker are known to be misaligned, then the rational decision maker should take into account that the (selfish) information provider must have generated the information in a way that manipulates the decision. Such a problem of interest was originally introduced and analyzed by Crawford and Sobel in their inaugural paper [2]. They have considered the scenarios where the information of interest is drawn from a compact state space according to a commonly known distribution. When the (quite general) cost measures of the players are misaligned by a commonly known distribution. When the (quite general) cost measures of the players are misaligned by a commonly known distribution, the authors have drawn the conclusion that in any incentive compatible, i.e., Nash, equilibrium, the information provider partitions the state space in a certain way and then signals the partition of the information realized.

On the other hand, if how the information/signal generated is transparent to the decision maker, the decision maker would just follow the (non-strategic) machinery of the Bayesian reaction. Correspondingly, transparency of the signals sent gives more power to the information provider seeking to control the rational reaction of the decision maker. This brings in the possibility of persuading intelligent decision makers to take certain actions through transparent information transmissions. One way to ensure transparency is the commitment power of the information provider. Relatively recently, such a problem of interest has been examined by Kamenica and Gentzkow in their seminal paper [1]. They have considered the scenarios where the decision maker is aware of the content of the messages received due to the committed transparency/honesty even though the information provider may not reveal the underlying information completely. They have provided a geometrical interpretation of the optimal persuasive signaling strategy and examined the persuasion capability of the information provider. A detailed justification of the information provider’s commitment power can be found in [1] and a recent survey of literature on Bayesian persuasion could be found in [3].

In this paper, we are also interested in the signaling models with policy commitment. As a special subclass of the Bayesian persuasion framework, here the information provider is interested in the perception of the decision maker about the underlying information of interest with respect to some quadratic cost measure rather than any decision made (possibly) based on that perception. Without loss of generality, we incorporate the commitment power of the information provider via a hierarchical structure where the information provider leads the interaction from a hierarchically higher position by announcing his/her signaling strategies publicly. This hierarchical viewpoint enables us to examine the interaction between the information provider and the decision maker under the solution concept of Stackelberg equilibrium [4], where the information provider is the leader.

We note that hierarchical signaling could have important applications in multi-agent noncooperative environments since asymmetry of information that agents have access to is preva-
lent due to the diversity of the agents’ perspectives. For example, distributed signal processing over sensor networks seeks to exploit this diversity through information exchanges among cooperative sensors in order to broaden their horizons (e.g., see [5]). However, in a noncooperative environment, selfish agents could have incentive to share the information private to them strategically with the other agents in order to gain advantage with respect to their own distinct objective. Correspondingly, as argued in [6], commitment/ transparency could play an important role for the credibility of the agents on the long run. Furthermore, in adversarial environments, commitment of the defender could be viewed as the defender avoiding the vulnerability of obscurity based defense against the possibility that the attacker could have learned the defense strategy, e.g., via regression analysis, once it is deployed widely.

A. Prior Literature

Originating in the fields of economics, recently, strategic signaling has also attracted significant attention in the fields of communication and control due to its compelling applications in noncooperative multi-agent environments. Due to the versatility of Gaussian distribution in engineering applications, these studies have mainly focused on Gaussian information models, different from the literature in the economics.

Within the framework of [2], in [7], the authors have identified the condition under which a non-partition equilibrium can exist in the scenarios where the underlying information is Gaussian, there exists an additive Gaussian noise channel, the players have quadratic cost measures, and the information provider has a soft power constraint on the signal sent. Under the solution concept of Stackelberg equilibrium, in [8], the authors have studied the problem setting of [2] when the misalignment factor between the objectives of the information provider and the decision maker, i.e., the bias term, is private to the information provider. They have shown that the optimal signaling strategy is linear in the scenarios where the underlying information is (multivariate) Gaussian, the players have quadratic cost measures, and the decision maker has bounded rationality by using linear estimates only. Indeed, for the same settings of [8], in [9, 10], the author had shown that the optimal signaling strategy is linear even when the decision maker is completely rational by selecting any measurable decision policy and provided an analytical formulation of the optimal signaling strategy. Under the same settings with [8] yet for a completely rational receiver and scalar Gaussian information, [10] has shown the optimality of linear signaling strategies when there is an additive Gaussian noise channel and the information provider has a hard power constraint.

In our previous papers [11]–[13], we have addressed hierarchical Gaussian signaling in non-cooperative dynamic communication and control systems over a finite horizon. For discrete-time (multivariate) Gauss Markov processes, in [11], we have shown the optimality of linear signaling strategies within the general class of measurable policies and formulated an equivalent (in optimality) semi-definite program (SDP), which enables computation of the optimal signaling strategies numerically through existing SDP solvers efficiently. In [12], we have shown that the equivalence to an SDP is not limited to equivalence in optimality and would still hold when we include certain additional constraints in the optimization. In that way, for non-cooperative linear quadratic Gaussian control problems, we have addressed in [12] optimal linear signaling strategies of a sensor who seeks to deceive a private-type controller in settings where the distribution over the types of the controller is not known. And reference [13] provides an overview of the results of [11] and [12].

For Gaussian information, we can obtain well structured results, e.g., linear signaling strategies, also in dynamic and noisy environments, as shown in the studies reviewed above. However, for distributions other than Gaussian, we still have significant but yet not completely explored problems. Notably, for general distributions with compact support, [11] brings in a geometrical interpretation into the problem, which requires the computation of a convex envelope of a function, which can be prominently challenging even for finite yet relatively large state spaces [14]. We also note that, as studied in [15]–[17], a relatively simpler characterization of the solution is possible for a special class of Bayesian persuasion problems, which is different from our problem settings. There have also been computational approaches for the Bayesian persuasion problem, e.g., [18], [19]. Particularly, the non-strategic nature of the decision maker makes it possible to formulate the problem as a single optimization problem faced by the information provider. Based on the revelation principle, the authors formulate a linear program to compute the optimal signaling strategy, which, however, turns out to be impractical to solve numerically unless the finite state space is fairly small [18], [19]. Therefore the authors consider the scenarios where the players’ cost measures are independently (and identically) drawn from a known distribution for each state, and examine connections with auction theory. We also note that for such an LP formulation, the action space of the receiver is considered to be finite, however, in our setting, the receiver’s decision is his/her belief and correspondingly his/her action space is a continuum even when the underlying information is discrete.

B. Contributions of This Paper

In this paper, our goal is to address hierarchical signaling for a general class of square integrable multivariate distributions. We again consider the scenario where there are two decision makers: a sender and a receiver. The sender has access to the realizations of two random vectors (with commonly known statistical profiles): information of interest and some private information. The receiver seeks to estimate the information of interest in the mean-square-error sense based on the signal he/she receives. To this end, the receiver seeks to compute the optimal Bayesian (possibly nonlinear) estimate of the information of interest using the signal he/she receives. However, the sender constructs the signal strategically in a stochastic way in order to deceive the receiver to perceive the information of interest as that private information with respect to another quadratic cost measure. The sender selects the signaling strategy from the general class of stochastic kernels.
Furthermore, by turning the problem around, the proposed setting could also be applicable to preserve privacy. While sharing the information of interest, the sender could seek to minimize the informational leakage of the private information when the information of interest and the private information are not independent of each other. Although we mainly focus on the former, i.e., deceptive signaling, problem, we will also show how the results would be extended to the latter, i.e., persuasive privacy, problem.

Under the solution concept of Stackelberg equilibrium, where the sender is the leader, we seek to compute the least possible cost for the sender at an equilibrium. Note that a Stackelberg game admits a unique value for the leader at all the equilibria if they exist. Particularly, the follower reacts in a non-strategic way given the leader’s strategy. Correspondingly, we can formulate the problem faced by the leader as a “single” optimization problem by characterizing the follower’s non-strategic reaction. In other words, the commitment power of the sender brings the signaling problem into the domain of optimization instead of fixed-point analysis as in the incentive compatible models, e.g., [2].

For any signaling strategy, the optimal reaction of the receiver is given by the conditional expectation of the information of interest, where the conditioning is on the signal sent by the sender. We note that in general, it is challenging to obtain the conditional expectation in an analytical form for arbitrary joint statistical profiles of the signal and the information of interest. Therefore, we seek to examine the problem faced by the sender further in order to transform it into a structured, exploitable, form. As shown in [9], the problem faced by the sender turns out to be a linear function of the correlation matrix of the posterior estimate. We note that the relationship between the correlation matrix of the posterior estimate and the signaling strategies is highly nonlinear. However, by characterizing tractable necessary and sufficient conditions on the correlation matrix of the posterior estimate enables us to compute the minimum cost for the sender. In [13], we have shown that for Gaussian distributions, the necessary and sufficient condition on the correlation matrix of the posterior can be expressed via certain linear matrix inequalities while these linear matrix inequalities are not sufficient for the general class of distributions. This is another intriguing feature of Gaussian distribution which provides the sender with utmost flexibility to deceive the receiver within the general class of square integrable distributions with fixed mean and covariance.

As an initial step toward addressing the problem for arbitrary distributions, we first consider discrete distributions. Although, now, the problem is a finite-dimensional optimization problem, obtaining its solution is still challenging due to its highly nonlinear and non-convex nature. One of our contributions in this paper is to transform this challenging problem into a structured, exploitable, form. To this end, we formulate an equivalent linear optimization problem over the convex cone of completely positive matrices. We say that a symmetric matrix $A \in S^n$ is completely positive if there exists $B \in \mathbb{R}^{n \times k}$ such that $\Xi = BB'$ [20]. Even though it is a linear optimization problem, the cone of completely positive matrices is still not tractable [20]. Furthermore, the proof of equivalence is constructive yet requires the factorization of a completely positive matrix (see [21], [22]). Notably this cone turns out to be tractable for sizes up to 4 [20], [23].

On the other hand, the dual of the equivalent problem is a linear optimization problem over the cone of copositive matrices. We say that a symmetric matrix $A \in S^n$ is copositive provided that $b'Ab \geq 0$ for all $b \in \mathbb{R}^n$ [20]. We can show that the strong duality is attained for our problem setting. We note that some classes of NP-hard problems could be transformed into copositive programs [24]. It is an active research area to exploit the compact structure of the cone of copositive matrices to approximate the underlying constraint space with sequential polyhedral or semi-definite cones for any accuracy level at the expense of computational complexity, e.g., [24]–[29]. Therefore, by transforming the hierarchical signaling problem into this class of problems, we can benefit from these powerful computational tools developed (or to be developed) in the optimization community over the course of time. We examine the approximation power of the proposed solution concept given a discretization of a continuous distribution via, e.g., a quantization scheme [30]. We also formulate an upper bound on the approximation error in terms of the quantization error. Finally, we examine the performance of the proposed solution concept over various numerical examples.

We can list the main contributions of this paper as follows:

- We address the problem of optimal hierarchical signaling to deceive a receiver to perceive the underlying information of interest as some private information for a general class of distributions.
- We show that the proposed solution concept also works for privacy applications where a sender seeks to minimize the informational leakage on his/her private information while sharing the information of interest with the receiver.
- For discrete distributions, we formulate an equivalent linear optimization problem over the convex cone of completely positive matrices and show its strong duality with a linear optimization problem over the convex cone of copositive matrices. This equivalence enables us to use the existing computational tools [24]–[26] to solve this class of problems at any level of accuracy.
- We formulate an upper bound on the error of approximation if we use the proposed solution concept for a discretized version of the underlying continuous distribution, e.g., via a quantization scheme.

The paper is organized as follows: In Section [I] we formulate the deceptive signaling game for the general class of continuous distributions. In Section [II] we compute the optimal signaling strategy for finite state spaces by formulating an equivalent linear optimization problem over the cone of completely positive matrices, and examine the loss induced if we use the proposed solution concept for a discretized version of a continuous state space. In Section [III] we discuss the computational approaches to solve the equivalent optimization problem. In Section [IV] we provide illustrative numerical examples. Section [V] concludes the paper with several remarks and possible research directions. An Appendix provides the
Information of Interest

Fig. 1: Hierarchical signaling model, where a sender has access to some information of interest as well as private information, and sends a signal composed of them to a receiver while the receiver seeks to learn the information of interest. The receiver will be learning both of them while the sender seeks to induce the receiver to perceive the information of interest as the private information.

proofs of three technical results.

Notation. We denote random variables by bold lower case letters, e.g., \( \mathbf{x} \). The correlation matrix of a random vector \( \mathbf{x} \) is given by \( \mathbb{E} \{ \mathbf{x} \mathbf{x}' \} \). Sets are denoted by calligraphic letters, e.g., \( \mathcal{Z} \). We denote the set of all probability distributions on a set \( \mathcal{Z} \) by \( \Delta(\mathcal{Z}) \). For a vector \( \mathbf{x} \) and a matrix \( \mathbf{A} \) and \( \mathbf{A}' \) denote their transposes; further \( ||\mathbf{x}|| \) and \( ||\mathbf{A}||_2 \) denote the Euclidean (\( L^2 \)) norms of the vector \( \mathbf{x} \) and the matrix \( \mathbf{A} \), respectively. For a matrix \( \mathbf{A} \), \( \text{tr} \{ \mathbf{A} \} \) denotes its trace. We denote the identity and zero matrices with the associated dimensions by \( \mathbf{I} \) and \( \mathbf{O} \), respectively. For positive semi-definite matrices \( \mathbf{A} \) and \( \mathbf{B} \), \( \mathbf{A} \succeq \mathbf{B} \) means that \( \mathbf{A} - \mathbf{B} \) is also a positive semi-definite matrix. \( \mathcal{S}^m \) (or \( \mathcal{S}^n_m \)) denotes the set of symmetric (or positive semi-definite) matrices of dimensions \( m \times m \) while \( \mathbb{R}_+^{n \times m} \) (or \( \mathbb{R}_+^{n \times m \times m} \)) denotes the set of \( n \)-by-\( m \) matrices with non-negative (or positive) entries.

II. Problem Formulation

Consider two non-cooperating decision makers: a sender (\( \mathcal{P}_S \)) and a receiver (\( \mathcal{P}_R \)), as seen in Fig. 1. \( \mathcal{P}_S \) has access to an underlying information of interest and also to some private information, which are realizations of \( m \)-dimensional random vectors \( \mathbf{x} \) and \( \mathbf{y} \) with supports \( \mathcal{X} \subset \mathbb{R}^m \) and \( \mathcal{Y} \subset \mathbb{R}^m \), respectively. The supports are not necessarily compact and can be as large as the entire \( \mathbb{R}^m \), e.g., the support of a multivariate Gaussian distribution. The random variables are defined on a common probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), where \( \Omega \) is the outcome space, \( \mathcal{F} \) is a proper \( \sigma \)-algebra, and \( \mathbb{P} \) is the probability measure, i.e.,

\[
(\Omega, \mathcal{F}, \mathbb{P}) \xrightarrow{\mathcal{X}} (\mathcal{X}, \mathcal{B}^m, \mathbb{P}_x),
\]

\[
(\Omega, \mathcal{F}, \mathbb{P}) \xrightarrow{\mathcal{Y}} (\mathcal{Y}, \mathcal{B}^m, \mathbb{P}_y),
\]

where \( \mathcal{B}^m \) denotes the Borel \( \sigma \)-algebra on \( \mathbb{R}^m \). We note that explicit definition of the probability space does not play a role in the analysis. We consider the scenarios where \( \mathbf{x} \) and \( \mathbf{y} \) belong to the normed space of all square integrable \( m \)-dimensional random vectors with the norm: \( ||\mathbf{x}|| = \mathbb{E} \{ \mathbf{x} \mathbf{x}' \}^{1/2} \).

\( \mathcal{P}_S \) selects his strategy \( \pi(\cdot) \) as a “stochastic kernel” from \( \mathcal{X} \times \mathcal{Y} \) to \( \mathcal{S} \subset \mathbb{R}^m \) such that the signal sent is given by

\[
\mathbf{s} = \pi(\mathbf{x}, \mathbf{y}), \text{ a.e. over } \mathcal{S}.
\]

Let us denote the set of all such signaling rules by \( \Pi \), i.e., \( \pi \in \Pi \). On the other side, after a realization of the signal sent is received, \( \mathcal{P}_R \) selects her strategy \( \gamma(\cdot) \) that is a Borel measurable function from \( \mathcal{S} \) to \( \mathcal{X} \) such that her estimate of the underlying information is given by

\[
\hat{x} = \gamma(\mathbf{s}), \text{ a.e. over } \mathcal{X}.
\]

The strategy space of \( \mathcal{P}_R \) is denoted by \( \Gamma \), which is the set of all measurable functions from \( \mathcal{S} \) to \( \mathcal{X} \).

The decision makers select their strategies according to distinct cost measures. \( \mathcal{P}_R \) has the cost measure

\[
c_R(\pi, \gamma) = ||\mathbf{x} - \hat{x}||^2
\]

to be minimized via \( \gamma \in \Gamma \), i.e., seeks to estimate the underlying information of interest. On the other side, \( \mathcal{P}_S \) has the cost measure

\[
c_S(\pi, \gamma) = ||\mathbf{y} - \hat{x}||^2
\]

to be minimized via \( \pi \in \Pi \), i.e., seeks to induce \( \mathcal{P}_R \) to perceive the information of interest as the private information instead of true itself.

We consider a hierarchical setting where \( \mathcal{P}_R \) makes a Bayes estimate of the information of interest based on the signal received by knowing the content of the signal, i.e., the signaling rule \( \pi \in \Pi \). Correspondingly, we model the interaction between the decision makers under the solution concept of Stackelberg equilibrium \[4\], where \( \mathcal{P}_S \) is the leader, who commits and announces to play his strategy beforehand, while \( \mathcal{P}_R \) is the follower, who selects her strategy knowing \( \mathcal{P}_S \)'s strategy. In particular, we can view the sequence of moves as follows:

1) \( \mathcal{P}_S \) selects \( \pi \in \Pi \) according to (6) and by anticipating the optimal response of \( \mathcal{P}_R \).
2) \( \mathcal{P}_R \) observes the signaling rule chosen by \( \mathcal{P}_S \) and selects \( \gamma \in \Gamma \) according to (5).
3) Nature chooses the outcome \( \omega \in \Omega \), and correspondingly \( \mathbf{x} = \mathbf{x}(\omega), \mathbf{y} = \mathbf{y}(\omega) \), and \( \mathbf{s} = \mathbf{s}(\omega) \), which is \( \mathbf{s}(\omega) = \pi(\mathbf{x}(\omega), \mathbf{y}(\omega)) \).
4) \( \mathcal{P}_R \) observes the realization \( \mathbf{s} \in \mathcal{S} \).
5) \( \mathcal{P}_R \) estimates \( \mathbf{x} \in \mathcal{X} \) given \( \mathbf{s} \in \mathcal{S} \) via \( \gamma \in \Gamma \).

Note that we can view the estimate of \( \mathbf{x} \in \mathcal{X} \) as a realization of the random vector \( \hat{x} \), as described in [4]. In the following, we provide a formal description of this game.

Definition (Deceptive Signaling Game). The deceptive signaling game

\[
\mathcal{G} := (\Pi, \Gamma, \mathbf{x}, \mathbf{y}, c_S(\cdot), c_R(\cdot))
\]

is a Stackelberg game between the leader \( \mathcal{P}_S \) and the follower \( \mathcal{P}_R \). We let \( B(\pi) \subset \Gamma \) be the optimum reaction set of the follower \( \mathcal{P}_R \) for a given strategy \( \pi \in \Pi \) of \( \mathcal{P}_S \). Then, the

\[2\]We use the pronouns “he” and “she” while referring to \( \mathcal{P}_S \) and \( \mathcal{P}_R \), respectively, only for clear referral.
pair of the strategy and the optimum reaction set \((\pi^*, B(\pi^*))\) attains the Stackelberg equilibrium provided that

\[
\begin{align*}
\pi^* &\in \operatorname{argmin}_{\pi \in \Pi} \max_{\gamma \in B(\pi^*)} c_S(\pi, \gamma) \quad (8a) \\
B(\pi) &= \operatorname{argmin}_{\gamma \in \Gamma^p} c_R(\pi, \gamma). \quad (8b)
\end{align*}
\]

Note that different from the previous literature on Bayesian persuasion, here \(P_S\) considers the scenarios where \(P_R\) breaks the tie in the reverse direction of \(P_S\)'s favor, which would be more justifiable given the deceptive intention of \(P_S\). However, in our case, due to the strictly convex cost measure of \(P_R\), the best reaction set of \(P_R\) for any given signaling strategy \(\pi \in \Pi\) will already turn out to be a singleton with probability 1.

We also note that we can turn the problem around by considering different quadratic cost measures so that the proposed solution concept could also have applications for, e.g., privacy, as long as the cost measures are quadratic and \(P_R\)'s optimum reaction turns out to be the mean of the posterior estimate. In that respect, we introduce the “persuasive privacy” problem, where \(P_S\) and \(P_R\) have a contract dictating \(P_S\) to disclose the information of interest \(x\), but following the contract, \(P_S\) also seeks to control the perception of \(P_R\) about the private information \(y\) since \(P_R\) could infer the private information based on the signal sent. Correspondingly, if \(x\) and \(y\) are not completely independent of each other, direct disclosure of \(x\) could reveal \(y\) to a certain extent. Therefore, \(P_S\) might seek to preserve such leakage of private information by signaling to \(P_R\) strategically. For example, we might view \(P_R\) seeking to estimate both \(x\) and \(y\) based on the signal \(s = \pi^p(x, y)\). To this end, \(P_R\) selects decision rules \(\gamma^p_x : \mathcal{X} \rightarrow \mathcal{X}\) and \(\gamma^p_y : \mathcal{Y} \rightarrow \mathcal{Y}\) that minimize

\[
t_R^P(\pi^p, \gamma^p) = ||x - \gamma^p_x(\pi^p(x, y))||^2 + ||y - \gamma^p_y(\pi^p(x, y))||^2,
\]

where \(\gamma^p := (\gamma^p_x, \gamma^p_y) \in \Gamma^p\) and \(\Gamma^p\) denotes the space of such decision rule pairs. On the other side, \(P_S\) constructs the signal \(s\) via a signaling rule \(\pi^p \in \Pi\) to minimize

\[
c_S^P(\pi^p, \gamma^p) = ||x - \gamma^p_x(\pi^p(x, y))||^2 - ||y - \gamma^p_y(\pi^p(x, y))||^2.
\]

We note that the sign of the second term in the cost measure of \(P_R\) is positive since \(P_R\) seeks to estimate the private information \(y\), whereas the sign of the second term in the cost measure of \(P_S\) is negative since \(P_S\) seeks to maximize the estimation error of \(P_R\) for the private information.

Similar to the description of the deceptive signaling game \(G\), in the following, we provide a formal description of this “persuasive privacy” game.

**Definition (Persuasive Privacy Game).** The persuasive privacy game \(G^p := (\Pi, \Gamma^p, x, y, c_S^p(\cdot), c_R^p(\cdot))\) (9)

is a Stackelberg game between the leader \(P_S\) and the follower \(P_R\). The pair of the strategy and the optimum reaction set \((\pi^p, B^p(\pi^p))\) attains the Stackelberg equilibrium provided that

\[
\begin{align*}
\pi^p &\in \operatorname{argmin}_{\pi \in \Pi} \max_{\gamma \in B^p(\pi)} c_S^p(\pi, \gamma) \quad (10a) \\
B^p(\pi) &= \operatorname{argmin}_{\gamma \in \Gamma^p} c_R^p(\pi, \gamma). \quad (10b)
\end{align*}
\]

In the following, we mainly address the deceptive signaling game \(G\) while remarking how the results would also be applicable for the persuasive privacy game \(G^p\).

### III. MAIN RESULT

Before delving into the technical details, in the following we first provide an overview of the main result of this paper and the steps we follow toward its solution. We first focus on identifying the underlying optimization problem faced by \(P_S\). Note that \(P_R\) follows the machinery of Bayesian estimation for any signaling rule selected/committed by \(P_S\). By incorporating this (non-strategic) machinery into \(P_S\)'s cost measure (6), we can write the optimal hierarchical signaling problem as a single optimization problem faced by \(P_S\).

Before addressing the problem for the general class of distributions, we seek to address the problem for finite state spaces. This will also enable us to formulate the best achievable performance for \(P_S\) approximately for continuous distributions if we discretize them via, e.g., a quantization scheme. Although the problem faced by \(P_S\) is no longer an infinite-dimensional optimization problem when the underlying state space is finite, the problem turns out to be a highly nonlinear and non-convex optimization problem. To mitigate this issue, we formulate an equivalent linear optimization problem over the cone of completely positive matrices. Although this problem is a linear optimization problem, which has compact structure, because of the underlying constraint space, i.e., the cone of completely positive matrices, it is not tractable and indeed some classes of NP-hard problems could be transformed into this form [20].

The difficulty of this equivalent problem is also a formal indication of the difficulty level for the original optimization problem. In other words, a polynomial-time solution for the hierarchical signaling problem would have implied a polynomial-time solution for a large class of completely-positive programs.

Furthermore, the dual of the equivalent problem is a linear optimization problem over the cone of copositive matrices, and we can show that the strong duality is attained for our problem setting. However, this cone is also not tractable, but its compact structure enables us to approximate the underlying constraint space with sequential polyhedral or semi-definite cones for any level of accuracy at the expense of computational complexity [24]–[26]. Therefore, by transforming the hierarchical signaling problem into this form, we can use these existing computational tools. In order to extend the results to continuous distributions, we examine the approximation error on the cost metric of \(P_S\) when the underlying distribution is discretized via a given quantization scheme. In the following, we provide the technical details.

#### A. Optimization Problem Faced by \(P_S\)

Since \(P_R\)'s objective is a mean-square-error minimization problem, as described in [5], it is well known that the best reaction of \(P_R\), for any given signaling rule \(\pi \in \Pi\), is uniquely given by \(\pi = \mathbb{E}[x|s]\) almost everywhere over \(\mathcal{X}\). Consider the augmented vector \(z := [x' \ y']'\) composed of the information of interest \(x\) and the private information \(y\). We denote the
support of $z$ by $Z := \mathcal{X} \times \mathcal{Y}$ and define $\hat{z} := \mathbb{E}\{z|s\}$ almost everywhere over $Z$. Henceforth we will call $\hat{z}$ “posterior” instead of posterior estimate of the augmented vector.

Through this new auxiliary parameter, we can write the problem faced by $P_S$ in a compact form. To this end, let the optimum reaction of $P_R$ for a committed signaling strategy $\pi \in \Pi$ be denoted by $\gamma^*(\pi)$ so that we can show its dependence on $\pi$ explicitly. Then the problem faced by $P_S$, as defined in (6), can be written as

$$c_S(\pi, \gamma^*(\pi)) = \text{tr} \{\mathbb{E}\{y'\}\} + \text{tr} \{V \mathbb{E}\{\hat{z}'\}\}, \quad (11)$$

where

$$V := \begin{bmatrix} I & -I \\ -I & O \end{bmatrix} \quad (12)$$

since we have $\mathbb{E}\{z'\} = \mathbb{E}\{\mathbb{E}\{z'|s\}\} = \mathbb{E}\{\hat{z}'\}$.

Remark. In the persuasive privacy game $\mathcal{G}_p$, the optimum reaction of $P_R$ would be $\gamma^*_p(s) = \mathbb{E}\{z|s\}$, almost everywhere over $X$, and $\gamma^*_p(s) = \mathbb{E}\{y|s\}$, almost everywhere over $\mathcal{Y}$. Correspondingly, the problem faced by $P_S$ could be written in that case as

$$c_S^p(\pi^p, \gamma^p(\pi^p)) = \text{tr} \{V^p \mathbb{E}\{\hat{z}'\}\} - \text{tr} \{V^p \mathbb{E}\{z'\}\}, \quad (13)$$

where

$$V^p := \begin{bmatrix} -I & O \\ O & I \end{bmatrix}. \quad (14)$$

The following results are also applicable to $\mathcal{G}_p$ if we simply replace $V \in \mathbb{R}^{2m}$ by $V^p \in \mathbb{R}^{2m}$.

Therefore, the infinite-dimensional optimization problem faced by $P_S$ can be written as

$$\text{tr} \{\mathbb{E}\{y'\}\} + \min_{\pi \in \Pi} \text{tr} \{V \mathbb{E}\{\hat{z}'\}\}, \quad (15)$$

where we have taken the first term at the right-hand-side of (11) out of the optimization objective since $\text{tr} \{\mathbb{E}\{y'\}\}$ has a fixed value, i.e., does not depend on the optimization argument $\pi \in \Pi$. Furthermore, the problem faced by $P_S$ depends on the optimization argument $\pi \in \Pi$ only via $\mathbb{E}\{\hat{z}'\}$ while the optimization objective is linear in $\mathbb{E}\{\hat{z}'\}$. This motivates us to examine the relation between the correlation matrix of the posterior $\mathbb{E}\{\hat{z}'\}$ and the signaling rule $\pi \in \Pi$ further.

Instead of attempting the problem as direct optimization on the strategy space of $P_S$, we seek to formulate the necessary and sufficient conditions on the correlation matrix of the posterior estimate. Note that the optimization objective (15) depends on the (infinite-dimensional) optimization argument $\pi \in \Pi$ only through a (finite-dimensional) matrix corresponding to the correlation matrix of the posterior estimate of the underlying information. By exploiting the relation between the signaling strategies and the correlation matrix of the posterior estimate, we seek to obtain a tractable finite-dimensional problem equivalent to the original infinite-dimensional optimization problem. This has already been shown to be possible via a linear optimization problem with tractable constraints, e.g., linear matrix inequalities, for the special class of Gaussian distributions [9], [13]. In [13], however, we have also shown that this finite-dimensional problem can only be viewed as a lower bound (indeed not a tight one) for the general class of distributions.

B. An Equivalent Problem Over the Cone of Completely Positive Matrices

In order to address (8) in the most general form, let us first focus on the special case where $n := |Z| < \infty$. Recall that the geometrical approach developed in [1] can be used effectively for finite state spaces with fairly small sizes, while the geometrical approach developed in [15] can be effectively used for relatively larger size problems if the problem faced by $P_S$ only depends on the mean of the posterior. Therefore the solution for (8) has remained open even for finite state spaces since here the problem faced by $P_S$ depends on the correlation matrix of the posterior. Further, by addressing (8) for finite state spaces at large scales brings in the possibility of addressing (8) approximately for continuum state spaces.

Taking the state space to be finite, we can restrict ourselves to the scenarios where $n \leq k := |S| < \infty$ without loss of generality. In the web appendix of [1], the authors have shown that the size of the signal space can be set the same with the size of the state space, i.e., $k = n$, without loss of generality. However, we do not impose such an upper bound on $k$. We can view the signaling strategy selected by $P_S$ as if $P_S$ selects a mixed strategy over $S$ for each $z \in Z$, i.e., determines the probabilities of sending the signals in $S$. With a slight abuse of notation, we denote the probability that $P_S$ sends $s \in S$ for $z \in Z$ by $\pi(s|z) \in [0,1]$. Note that for each $z \in Z$, $P_S$ will be sending a signal with probability 1, which yields that

$$\sum_{s \in S} \pi(s|z) = 1. \quad (16)$$

Note that null signaling, i.e., sending no signal, does not lead to a contradiction since it can practically be viewed as sending the same signal for all the states. Suppose that the prior distribution has complete support on $Z$ and let $p_o(z) \in (0,1]$ denote the probability that the state $z \in Z$ is realized. Then Bayes rule yields that the probability of state $z \in Z$ being realized given that signal $s \in S$ is received is given by

$$p_s(z) := \frac{\pi(s|z)p_o(z)}{p(s)}, \quad (17)$$

where $p(s)$, denoting the probability that the signal $s \in S$ is sent, is given by

$$p(s) = \sum_{z \in Z} \pi(s|z)p_o(z). \quad (18)$$

Therefore, for given $s \in S$, we have

$$\hat{z} := \mathbb{E}\{z|\mathbb{E}\{s\} = s\} = \sum_{z \in Z} p_s(z)z, \quad (19)$$

which yields that

$$\mathbb{E}\{\hat{z}'\} = \sum_{s \in S} p(s) \left( \sum_{z \in Z} p_s(z)z \right) \left( \sum_{z \in Z} p_s(z)z' \right). \quad (20)$$

3Information drawn from square integrable distributions ensures that $\mathbb{E}\{\hat{z}'\}$ is well defined.

4A signal $s \in S$ would be received if the associated probability is positive, i.e., $p(s) > 0$. 
We can write the correlation matrix of the posterior \((20)\) in a compact form as
\[
\mathbb{E}\{\hat{z}z'\} = Z\Xi_\pi Z',
\]
where \(Z := [z_1 \ldots z_n] \in \mathbb{R}^{2m \times n}\) and we introduce \(\Xi_\pi \in \mathbb{S}^n\) whose \(i\)th row and \(j\)th column entry is given by
\[
\Xi_\pi[i, j] = \sum_{s \in \mathcal{S}} p(s)p_s(z_i)p_s(z_j).
\]
Then, the problem faced by \(\mathcal{P}_S\) can be written as
\[
\min_{\pi \in \mathcal{H}} \text{tr}\{\mathbb{E}\{\hat{z}z'\} V\} = \min_{\pi \in \mathcal{H}} \text{tr}\{\Xi_\pi \tilde{V}\},
\]
where \(\tilde{V} := Z'VZ\).

Remark. We can address the problem according to the solution concept developed in [1] as follows. The cost for \(\mathcal{P}_S\) if \(\mathcal{P}_R\) has the posterior belief \(\tilde{p} \in \Delta(Z)\) is given by
\[
\tilde{c}(\tilde{p}) = \text{tr}\{\tilde{V} \tilde{p}p'\}
\]
which corresponds to
\[
\mathbb{E}\{\hat{z}z'\} \preceq Z\Xi_\pi Z' \succeq \mathbb{E}\{z\} \mathbb{E}\{z'\}'.
\]
However, given the necessary and sufficient condition that \(\Xi_\pi \in \mathbb{C}P^n\) and \(\Xi_\pi 1 = \bar{p}_o\), this condition \((28)\) turns out to be redundant.

Based on Proposition [1] the following corollary shows that the highly-nonlinear non-convex optimization problem faced by \(\mathcal{P}_S\) could be written, "equivalently", as a linear optimization problem over the convex cone of completely positive matrices. We emphasize that there is no relaxation on the optimization problem and the equivalence is not limited to optimality.

Corollary 1. The problem faced by \(\mathcal{P}_S\) can be written in an equivalent form as
\[
\min_{\pi \in \mathcal{H}} \text{tr}\{\mathbb{E}\{\hat{z}z'\} V\} = \min_{\Xi \in \mathbb{C}P^n} \text{tr}\{\Xi \tilde{V}\},
\]
s.t. \(\Xi 1 = p_o\),

where \(\tilde{V} := Z'VZ\).

Interestingly, the following proposition shows that the equivalent problem, as described by the right-hand-side of \((29)\) turns out to be a semi-definite program (SDP), which can be solved with the existing numerical tools/solvers efficiently if \(n \leq 4\).

Proposition 2. Let \(n \leq 4\). Then, we have
\[
\min_{\Xi \in \mathbb{C}P^n} \text{tr}\{\Xi \tilde{V}\} = \min_{\Xi \in \mathbb{C}P^n} \text{tr}\{\Xi \tilde{V}\},
\]
s.t. \(\Xi 1 = p_o\), s.t. \(\Xi 1 = p_o\), \(\Xi \in \mathbb{R}_{++}^{n \times n}\),

Furthermore, an SDP relaxation of the problem for \(n > 4\) is given by
\[
\min_{\Xi \in \mathbb{C}P^n} \text{tr}\{\Xi \tilde{V}\} \geq \min_{\Xi \in \mathbb{C}P^n} \text{tr}\{\Xi \tilde{V}\},
\]
s.t. \(\Xi 1 = p_o\), s.t. \(\Xi 1 = p_o\), \(\Xi \in \mathbb{R}_{++}^{n \times n}\).

Proof. This follows since \(\mathbb{C}P^n \subseteq \mathbb{S}_+^n \cap \mathbb{R}_{++}^{n \times n}\) for all \(n\), while \(\mathbb{C}P^n \subseteq \mathbb{S}_+^n \cap \mathbb{R}_{++}^{n \times n}\) if, and only if, \(n \leq 4\) [20] [23].

With respect to the trace inner product, given the primal problem:
\[
\min_{\Xi \in \mathbb{C}P^n} \text{tr}\{\Xi \tilde{V}\}, \ s.t. \ \Xi 1 = p_o,
\]
the dual problem is given by
\[
\max_{\gamma \in \mathbb{R}_+^n, s \in \mathbb{C}P^n} \gamma^T y, \ s.t. \ 1^T y - S = \tilde{V}.
\]
Furthermore, the following proposition shows the strong duality between \((32)\) and \((33)\), which enables us to solve only one of them while obtaining the value of both of them.

Proposition 3. The primal problem \((32)\) is feasible, has finite value, and has an interior point. Therefore, there exists a strong duality between the primal problem \((32)\) and its dual \((33)\), i.e., we have
\[
\min_{\Xi \in \mathbb{C}P^n} \text{tr}\{\Xi \tilde{V}\} = \max_{\gamma \in \mathbb{R}_+^n, s \in \mathbb{C}P^n} \gamma^T y, \ \ s.t. \ 1^T y - S = \tilde{V}.
\]
Proof. We can show that there exists an interior point in the feasible set of the optimization problem based on the characterization of the interior of $\mathcal{CP}^n$ provided in \cite{27}, \cite{28}. Particularly, a mixture of full and null signaling leads to $\Xi = \{\mathcal{CP}^n\}$ and $\Xi_{\pi}^1 = p$. The technical details of the proof is provided in Appendix \[3\].

\section{C. Theoretical Approximation Guarantees}

We have formulated the minimum cost of $\mathcal{P}_S$ for the scenarios where the state space is finite. We can, however, adopt the solution concept to the general class of distributions by discretizing a continuous state space through a quantization scheme. However, such discretization would lead to loss of information and correspondingly the computed minimum cost could deviate from the true one. Therefore, in this subsection, we seek to restrain this deviation. To this end, by turning the problem around, we can view the problem setting as $\mathcal{P}_S$ selecting a random vector within the general class of square integrable distributions and sending a realization of that signal rather than selecting a signaling strategy within the general class of stochastic kernels. Note that $\mathcal{P}_S$ should take into account the joint distribution of the underlying distribution and the signal sent, which would normally have been determined by the signaling strategy.

Based on this observation, the following corollary provides theoretical guarantees on the approximation capability of the solution concept for a given quantization scheme.

\begin{corollary}
Consider a quantization of the continuous random variable $z \in \mathbb{Z}$, denoted by $z_q \in \mathbb{Z}$, i.e., $z_q$ attains the same value within any bin of the quantization. Let $e = z - z_q$, almost everywhere over $\mathbb{Z}$, denote the quantization error. Then, we have
\begin{equation}
\min_{z} \text{tr} \{E \{\hat{z}z\} V\} - \min_{z_q} \text{tr} \{E \{\hat{z}_q\hat{z}_q\} V\} \leq \epsilon,
\end{equation}
where $\hat{z} = E \{z|s\}$, $\hat{z}_q = E \{z_q|s\}$, and
\begin{equation}
\epsilon = (2\|z\| + \|e\|)||V||2\|e\|,
\end{equation}
which yields that $\epsilon \rightarrow 0$ when $\|e\| \rightarrow 0$.
\end{corollary}

\begin{proof}
The proof is provided in Appendix \[4\].
\end{proof}

\section{IV. Computational Approaches}

Although the equivalent problems \cite{22}, \cite{28} are linear optimization problems over convex constraint sets, they are difficult to solve numerically since the cones of completely positive and copositive matrices are not tractable if $n > 4$. However, for $n > 4$, we can approximate the solution with any desired error rate at the expense of computational complexity by using existing computational tools developed in the optimization community over the course of time, e.g., \cite{24}--\cite{29}. Furthermore, each new development in this active research area (due to its broad applications) will bring in new computational tools and insights to address the problem. Indeed, as remarked below, the signaling framework could also bring in new insights to copositive programs.

\subsection{A. Polyhedral Inner and Outer Approximations}

Note that the extreme rays of $\mathcal{CP}^n$ have rank 1, i.e., they can be written as $b_i b_i^T$, where $b_i \in \mathbb{R}^n_+$. Correspondingly, $\mathcal{CP}^n$ can be viewed as the conic hull of vectors from the unit simplex $\Delta_{n-1}$ in $\mathbb{R}^n$. Consider a family of simplices $\mathcal{P} = \{\Delta^1, \ldots, \Delta^t\}$ satisfying
\begin{equation}
\Delta_{n-1} = \bigcup_{i=1}^t \Delta^i \text{ and } \text{int } \{\Delta^i\} \cap \text{int } \{\Delta^j\} = \emptyset \text{ if } i \neq j.
\end{equation}
Given the family of simplices $\mathcal{P}$, we define the polyhedral cones $I_{\mathcal{P}}$ and $O_{\mathcal{P}}$ as follows:

$$I_{\mathcal{P}} := \left\{ \sum_{b \in V_{\mathcal{P}}} \lambda_b b^l : \lambda_b \geq 0 \right\},$$

$$O_{\mathcal{P}} := \left\{ \sum_{b \in V_{\mathcal{P}}} \lambda_{b,c} (be^l + ce^r) : \lambda_{b,c} \geq 0 \right\},$$

where $V_{\mathcal{P}}$ denotes the set of vertices in $\mathcal{P}$. In [24], the authors have shown that $I_{\mathcal{P}} \subseteq CP^n \subseteq O_{\mathcal{P}}$ for any $\mathcal{P}$. Let $CP(\mathcal{K})$ denote the solution of the following $\mathcal{K}$-cone program:

$$CP(\mathcal{K}) = \min \tau \left\{ \sum_{b \in V_{\mathcal{P}}} \lambda_b b^l \right\} \quad \text{s.t.} \quad \sum_{b \in V_{\mathcal{P}}} \lambda_b b^l = p_o, \quad \mathcal{K} \subseteq \mathbb{R}^n.$$  

Since $I_{\mathcal{P}} \subseteq CP^n \subseteq O_{\mathcal{P}}$, we have

$$CP(I_{\mathcal{P}}) \geq CP(CP^n) \geq CP(O_{\mathcal{P}}).$$

Correspondingly, through a sequence of simplical partitions, we can construct a sequence of nested polyhedral cones $I_1 \subseteq I_2 \subseteq \ldots$ and $O_1 \supseteq O_2 \supseteq \ldots$ that converge to $CP^n$, i.e.,

$$CP^n = \bigcup_{i \in \mathbb{N}} I_i \quad \text{and} \quad CP^n = \bigcap_{i \in \mathbb{N}} O_i,$$

from below and above, respectively [24].

B. Resemblance to the Equivalent Problem Formulated in [1]

We can view a polyhedral inner approximation of $CP^n$, e.g., via the conic hull of the vertices of the simplical partition of the unit simplex, as a discretization of the feasible set of the optimization problem through a discretization of the unit simplex [28]. Correspondingly, based on (37), the upper bound in (40) can be written as

$$\min_{\Delta \in \mathbb{R}^n_{\mathcal{P}}} \tau \left\{ \sum_{b \in V_{\mathcal{P}}} \lambda_b b^l \right\} \quad \text{s.t.} \quad \sum_{b \in V_{\mathcal{P}}} \lambda_b b^l = p_o,$$

which is equivalent to

$$\min_{\Delta \in \mathbb{R}^n_{\mathcal{P}}} \sum_{b \in V_{\mathcal{P}}} c_b \lambda_b \quad \text{s.t.} \quad \sum_{b \in V_{\mathcal{P}}} \lambda_b b^l = p_o,$$

where $c_b := \tau \left\{ b^l b^r \right\}$.

Note that, in [1, Corollary 1], the authors have shown that the problem faced by $\mathcal{P}_S$ is equivalent to the following optimization problem:

$$\min_{\tau \in \Delta(\Delta(Z))} \int_{\Delta(Z)} \hat{c}_S(p) \tau(dp) \quad \text{s.t.} \quad \int_{\Delta(Z)} \tau(dp) = p_o,$$

where, with a slight abuse of notation, $\hat{c}_S(p)$ denotes $\mathcal{P}_S$'s cost for a given common belief $p \in \Delta(Z)$, the constraint $\int_{\Delta(Z)} \tau(dp) = p_o$ is called the Bayes plausibility, and $\tau$ could be viewed as a probability measure over the posterior distributions. The resemblance between (44) and (43) is notable. Particularly, a discretization of the simplex $\Delta(Z)$ in (44) would have lead to (43).

C. Relaxed Specifications and Sender-Favorite Discrete Distributions

Recall that the Gaussian distribution leads to the least possible cost for the sender when the correlation matrix of the underlying random vector is fixed. Similarly, within the proposed framework, we can relax certain specifications on the prior distribution to find out the corresponding most favorable distribution for the sender. For example, an interesting relaxation would be to consider the scenarios where only the mean of the underlying prior distribution is fixed, i.e., $Z \Xi = \mu$ $\in \mathbb{R}^n$. Corresponding equivalent linear optimization problem is given by

$$\min_{\Xi \in \mathbb{R}^n} \tau \left\{ \Xi \right\} \quad \text{s.t.} \quad \left[ \begin{array}{c} \Xi \\ 1 \end{array} \right] \Xi = \left[ \begin{array}{c} p_o \\ 1 \end{array} \right],$$

where we have $2n + 1$ linear constraints different from the primal problem, where there are $n$ linear constraints. We can enrich the class of problems that can be formulated within the proposed framework by considering partial specifications of the underlying distribution or its mean.

V. ILLUSTRATIVE EXAMPLES

Similar to the example introduced in [1], let us consider the interaction between a prosecutor ($\mathcal{P}_S$) and a judge ($\mathcal{P}_J$) during the trial of a defendant. Particularly, now the prosecutor has access to

- the information of interest $x \in \mathcal{X}$ corresponding to the status of a defendant based on some evidence
- the private information $y \in \mathcal{Y}$ corresponding to the prosecutor’s intuition about the status of the defendant.

Irrespective of the evidence, our self-confident and righteous prosecutor seeks to induce the judge to perceive the status of
the defendant in line with his intuition. On the other side, the judge is only interested in what the evidence says about the status of the defendant. We consider three scenarios where the underlying joint distributions over the prosecutor’s intuition and what the evidence suggests about the defendant’s status are as tabulated in Tables I, II, and III, where (NG), (I), and (G) correspond to the status of ‘not guilty’, ‘innocent’, and ‘guilty’, respectively.

Note that in Scenario I, the prosecutor’s intuition always says that the defendant is guilty, which can also be viewed as the prosecutor always seeks for conviction similar to the example studied in [1]. However, this differs from the example in [1] in terms of the cost measures, and therefore also the result is different. The cost of $P_S$ when $P_R$ has the belief $[1-\mu \mu]^T$, where $\mu := P \{ z = [-1 1] \}$, is given by

$$\hat{c}(\mu) = 4\mu^2 - 8\mu + 3,$$

(46)

which is a convex function. Note that even though $tr \{ Vp^-p^- \}$ is a non-convex function of $p$, it turns out to be a convex function over the unit simplex under the specific settings of this example. However, as observed in the following examples, this is not always the case. Furthermore, as characterized in [1], since $\hat{c}(\cdot)$ is a convex function, null signaling is the optimal one in Scenario I. Alternatively, we have its convex envelope $C(\mu) = \hat{c}(\mu)$ and the minimum cost for $P_S$ is given by $C(\mu) = \hat{c}(\mu_o) = 0.96$, where $\mu_o$ is the prior probability of $[-1 1]^T$, i.e., 0.3.

On the other hand, in Scenarios II and III, the associated cost of $P_S$ for a given posterior belief of $P_R$ does not end up to be a convex function. The dimensions of freedom to select the posterior are 3 and 8, respectively, in Scenarios II and III. Therefore, in order to apply the same solution concept, we need to compute the convex envelope of some non-convex functions defined over $\mathbb{R}^3$ and $\mathbb{R}^8$, instead of $\mathbb{R}$ as we do in Scenario I. However, through the proposed framework, we can efficiently compute the minimum cost for $P_S$ and the associated optimal signaling strategies in those scenarios. In Table [V] we tabulate the results. Proposition 2 says that the SDP-relaxation and the primal problem are equivalent for $n \leq 4$, as also seen in Table [V]. We also note that the SDP-relaxation has turned out to be a tight lower bound in Scenario III, where $n = 9$. In Fig. 2 we illustrate the convergence behavior of the solutions of the inner and outer polyhedral approximations across the iterations. Note that with the increase of the size of the state space, the corresponding number of iterations necessary for convergence increase significantly while the computational complexity of each iteration increases further with an increase in the size of the state space.

We have also analyzed the cost of $P_S$ in the persuasive privacy game over Scenarios I-III. We have tabulated the results in Table [V]. Similar to the case in the deceptive signaling game, we can apply the solution process introduced in [1] to this problem for Scenario I. Correspondingly, the cost of $P_S$ when $P_R$ has the belief $[1-\mu \mu]^T$, where $\mu := P \{ z = [-1 1] \}$, is now given by

$$\hat{c}(\mu) = -4\mu^2 + 4\mu,$$

(47)
which is a concave function. As characterized in \( \Pi \), since \( c(\cdot) \) is a concave function, we can conclude that full signaling is the optimal one in Scenario I. Alternatively, a geometrical inspection yields that the convex envelope of \( \hat{c}(\mu) \) is given by \( C(\mu) = 0 \), which coincides with the cost for the full signaling case as expected. We also note that the number of iterations for the convergence of inner and outer approximations are observed to be similar with the cases in the deceptive signaling game.

VI. CONCLUDING REMARKS

We have addressed the problem of optimal hierarchical signaling between a sender and a receiver for a general class of square integrable multivariate distributions, which has applications to deception and privacy. We have considered a noncooperative communication setting where the sender and the receiver have different cost measures. This leads to the possibility of strategic crafting of the messages sent by the sender. The sender has had access to the information of interest and some private information, both of them being realizations of random vectors with arbitrary distributions. We have shown that the proposed solution concept can compute the minimum cost for the sender in the deception and privacy applications. For the former one, the sender seeks to induce the receiver to perceive the information of interest as the private information. In the latter one, the sender seeks to minimize the informational leakage of the private information while sharing the information of interest with the receiver.

For discrete distributions, we have formulated an equivalent linear optimization problem over the cone of completely positive matrices. This has brought the hierarchical signaling problem into the framework of copositive programs so that we could use the existing computational tools that can solve it to any level of accuracy. For continuous distributions, we have also addressed the approximation error of the minimum cost for the sender if we have employed the solution concept for its discretized version obtained via a quantization scheme. Finally we have analyzed the performance of the proposed solution concept over various numerical examples.

In addition to developing efficient computational tools to address the equivalent cone program, some other future directions of research include: formulation of optimal hierarchical signaling in dynamic and/or noisy environments with multiple senders and/or multiple receivers, and applications of the setting in other noncooperative communication and control scenarios.

APPENDIX A

PROOF OF PROPOSITION \( \Pi \)

Claim 1. For any signaling rule \( \pi \in \Pi \), \( \Xi_\pi \in S^n \) satisfies

- \( \Xi_\pi \in CP^n \),
- \( \Xi_\pi 1 = p_o \),

where \( p_o := [p_o(z_1) \ldots p_o(z_n)]' \).

Proof. We can decompose \( \Xi_\pi \in S^n \), as described in (22), as \( \Xi_\pi = AA' \), where

\[
A := \begin{bmatrix}
p_{s_1}(z_1)p(s_1)^{1/2} & \cdots & p_{s_k}(z_1)p(s_k)^{1/2} \\
\vdots & \ddots & \vdots \\
p_{s_1}(z_n)p(s_1)^{1/2} & \cdots & p_{s_k}(z_n)p(s_k)^{1/2}
\end{bmatrix}, \tag{48}
\]

which is clearly a nonnegative matrix since all the entries are products of (nonnegative) probability measures. This yields that \( \Xi_\pi \in CP^n \).

For a given signaling rule \( \pi \in \Pi \), let \( S_o \subseteq S \) denote the set of signals that \( P_\pi \) sends with positive probability, i.e., \( p(s) > 0 \) for all \( s \in S_o \). Then, the sum of entries of \( \Xi_\pi \) at the \( i \)th row is given by

\[
\sum_{j=1}^{n} \Xi_\pi[i,j] = \sum_{s \in S_o} p(s)p_o(z_i) \sum_{j=1}^{n} p_s(z_j), \tag{49}
\]

\[
= \sum_{s \in S_o} p(s)\pi(s|z_i)p_o(z_i), \tag{50}
\]

\[
= p_o(z_i) \sum_{s \in S_o} \pi(s|z_i), \tag{51}
\]

\[
= p_o(z_i), \tag{52}
\]

where (a) follows from (17) and (18), (b) follows from (17), and (c) follows from (16). By (52), we have \( \Xi_\pi 1 = p_o \), which completes the proof of the claim.

Claim 2. For any completely positive matrix \( \Xi \in CP^n \) that satisfies \( \Xi 1 = p_o \), there exists a signaling rule \( \pi \in \Pi \) such that \( \Xi_\pi = \Xi \), where \( \Xi_\pi \in CP^n \) is as described in (22).

Proof. Consider any completely positive matrix \( \Xi \in CP^n \) that satisfies \( \Xi 1 = p_o \). By the definition of completely positive matrices, we can decompose \( \Xi \in CP^n \) into \( \Xi = BB' \), where

\[
B = [b_1 \ldots b_k] \in \mathbb{R}_+^{n \times k} \tag{53}
\]

is some nonnegative matrix. We note that the decomposition is not necessarily a unique one [20]. For example, by padding zero columns into \( B \), we can generate infinitely many decompositions. Correspondingly, we assume, without loss of generality, that in the decomposition of \( \Xi \in CP^n \), the nonnegative matrix \( B \in \mathbb{R}^{n \times k} \) does not have an all zero column, i.e., \( b_i \neq 0 \) for all \( i = 1, \ldots, k \).

Recall that for any given signaling rule \( \pi \in \Pi \), we can decompose \( \Xi_\pi = AA' \), where the matrix \( A \in \mathbb{R}_+^{n \times |S|} \) is as described in (48). Correspondingly, if we can show that there exists a signaling rule \( \pi \in \Pi \) such that \( A = B \), then this would imply that \( \Xi = \Xi_\pi \) for that signaling rule. To this end, we let \( |S| = k \), and introduce auxiliary vectors

\[
\Xi_o := [\pi(s_1|z_1) \ldots \pi(s_k|z_1)]', \tag{54}
\]

which, by (18), yields that \( p(s_i) = \Xi_o p_o \). Therefore, by substituting (17) in (48), we can write the matrix \( A \) as

\[
A = P_o \left[ \frac{\Xi_o}{\sqrt{2|S|p_o}} \cdots \frac{\Xi_o}{\sqrt{2|S|p_o}} \right], \tag{55}
\]
where \( P_o := \text{diag}\{p_i\} \), which is nonsingular since the prior distribution \( p_o \) has complete support on \( Z \). Correspondingly, \( A \) would be equal to \( B \) provided that
\[
\frac{P_o \pi_i}{\sqrt{\pi' o}} = b_i, \quad \forall i = 1, \ldots, k, \tag{56}
\]
and it can be verified that we have \( \pi_i = b_i' P_o^{-1} b_i \). \( \pi_i \)

However, we also need to inspect the validity of \( \Xi \) as a signaling rule. Particularly, we have the constraints on the signaling strategies that \( \pi(s|z) \geq 0 \) for all \( s \in S \) and \( z \in Z \), and \( \Xi(B) \) satisfies
\[
\Xi(B) \geq \Xi(B') \quad \forall B, B' \in \mathbb{R}^{n \times k}. \tag{57}
\]
The former constraint is satisfied by the definition, since \( \Xi(B) \) is a column of the nonnegative matrix \( B \in \mathbb{R}^{n \times k} \).

Verification of the latter constraint \( \Xi(B) \geq \Xi(B') \) is relatively more involved. To this end, let us introduce
\[
\Pi := [\pi_1 \ldots \pi_n]. \tag{58}
\]
Then, the latter constraint is equivalent to \( \Pi \Pi' \geq 1 \), i.e., the sum of columns of \( \Pi \) is an all 1s vector. If we set \( \pi_i \) as in \( \Xi \), then we obtain
\[
\Pi = [(b_1' A) P_o^{-1} b_1 \ldots (b_n' A) P_o^{-1} b_n], \tag{59}
\]
and correspondingly the sum of columns of \( \Pi \) is given by
\[
\Pi^T = P_o^{-1} \sum_{i=1}^k b_i (b_i' A). \tag{60}
\]

Note that \((b_i' A) P_o^{-1} b_i = P_o^{-1} b_i (b_i' A) \) since \( b_i' A \) is just a scalar.

Recall that the completely positive matrix \( \Xi \in \mathcal{C}^n \) satisfies \( \Xi(B) \geq p_o \), which can also be written as
\[
BB' 1 = P_o \Leftrightarrow \sum_{i=1}^k (b_i' A) = p_o. \tag{61}
\]
Therefore, by \( \Xi \) and \( \Pi \), we obtain \( \Pi \Pi' \geq 1 \) and correspondingly \( \Xi \) is a valid signaling strategy, which completes the proof of the claim. \( \square \)

Based on \( \Xi \), Claims \( 1 \) and \( 2 \) yield \( \Pi \), and \( \Pi' \) follows from \( \Xi \), which completes the proof.

### Appendix B

#### Proof of Proposition \( 3 \)

The primal problem is feasible since the constraint set is not empty based on Claims \( 1 \) and \( 2 \).

The fact that the primal problem has finite value follows by the extreme value theorem since the optimization objective is linear in the optimization argument and the constraint set
\[
\{ \pi \in \mathcal{C}^n | \Xi = p_o \} \tag{62}
\]
is a closed and bounded subset of \( \mathcal{C}^n \).

Note that if both of these constraints hold, this would also imply that \( \pi(s|z) \in [0, 1] \) for all \( s \in S \) and \( z \in Z \).

It is relatively more involved to show that the primal problem entails an interior point. Particularly, a characterization of the interior of \( \mathcal{C}^n \) is given by \( \Xi \) \[31\] Theorem 3.3]
\[
\int \{ \mathcal{C}^n \} = \{ AA' \mid \text{rank} \{ A \} = n, A = [g \ A] \geq 0, \bar{A} \geq 0 \}. \tag{63}
\]

Therefore, the question is whether there exists a \( \Xi \in \int \{ \mathcal{C}^n \} \) such that \( \Xi = p_o \).

Based on Claims \( 1 \) and \( 2 \), let us consider the associated signaling problem where we set the signal space as \( S = \{ s_o = \emptyset, s_1 = z_1, \ldots, s_n = z_n \} \) and the prior distribution over \( Z \) to have full support, without loss of generality. Consider the two extreme cases: full disclosure and null disclosure, respectively, given and denoted by \( \pi(z_i) = z_i \), and \( \pi(z_i) = \emptyset \) for all \( i = 1, \ldots, n \). Note that for a given signaling strategy \( \pi \in \Pi \), an entry of the associated completely positive matrix \( \Xi(\pi) \) is described in \( \Xi \). Furthermore, a component of a decomposition of \( \Xi(\pi) \) is described in \( \Xi \).

Correspondingly, we obtain \( \Xi \)
\[
\Xi(\pi) = \sum_{i=1}^n \frac{1}{s} \Xi(s) \tag{64}
\]

This yields that if \( P_S \) selects a signaling strategy \( \pi \in \Pi \) that discloses \( z \in Z \) truthfully, i.e., \( s_i = z_i \), with probability \( \lambda \in (0,1) \) and discloses \( s_o \) otherwise, then we obtain
\[
\Xi(\pi) = AA' \Leftrightarrow A := \left[ (1 - \lambda) p_o \ A \right], \tag{65}
\]
in which the first column is a positive vector and rank \( \{ A \} = n \) since \( p_o \) is an all-positive vector, i.e., the prior distribution has full support over \( Z \) by the formulation. This yields that \( \Xi(\pi) \in \int \{ \mathcal{C}^n \} \) and \( \Xi(\pi) = p_o \).

Since the conditions for the strong duality theorem \( [34] \) Theorem 4.7.1 hold, we have strong duality between the primal and dual problems, which concludes the proof.

### Appendix C

#### Proof of Corollary \( 2 \)

We first note that the following inequality always holds
\[
\min \{ \text{tr} \{ \mathbb{E} \{ \hat{z} \hat{z}' \} V \} \} \leq \min \{ \text{tr} \{ \mathbb{E} \{ \hat{z} \hat{z}' \} V \} \tag{67}
\]
since any quantization would restrict \( P_S \)'s strategy space for continuous distributions.

Let \( \hat{e} = \hat{z} - \hat{z}_q \) for a given signal \( s \), i.e., \( \hat{e} = \mathbb{E} \{ e|s \} \). Then, for any signal \( s \sim Z \), we have
\[
\mathbb{E} \{ \hat{z} \hat{z}' \} = \mathbb{E} \{ \hat{z}_q \hat{z}_q' \} + \mathbb{E} \{ \hat{z}_q \hat{z}' \} + \mathbb{E} \{ e \hat{e}' \}. \tag{68}
\]

Correspondingly, we obtain
\[
\min \{ \text{tr} \{ \mathbb{E} \{ \hat{z} \hat{z}' \} V \} \} \geq \min \{ \text{tr} \{ \mathbb{E} \{ \hat{z}_q \hat{z}_q' \} V \} \}
+ \min \{ \text{tr} \{ \mathbb{E} \{ \hat{z}_q \hat{z}' \} + \mathbb{E} \{ e \hat{e}' \} \} \} \tag{68}
\]

Let us take a closer look at the second term on the right-hand-side, which can also be written as
\[
- \max_{\hat{e}} \{ \text{tr} \{ \mathbb{E} \{ \hat{z}_q \hat{z}' \} + \mathbb{E} \{ e \hat{e}' \} \} \} \tag{69}
\]
Then, the Cauchy-Schwarz inequality for random vectors yields that (69) is bounded from above by
\[ -2E \{ \bar{z}'qV \bar{e} \} - E \{ \bar{e}'V \bar{e} \} \leq (2||\bar{z}_q|| + ||\bar{e}||)||V||_2||\bar{e}||, \]
since \( || - V ||_2 = ||V||_2 \). Note that the right-hand-side depends on the signal \( s \). However, we also have
\[
\begin{align*}
||z_q - \hat{z}_q||^2 &= ||z_q||^2 - 2\bar{z}_q\bar{z}'_q + ||\bar{z}_q||^2 \geq 0, \\
||e - \hat{e}||^2 &= ||e||^2 - 2\bar{e}\bar{e}' + ||\bar{e}||^2 \geq 0.
\end{align*}
\]
(70)
(71)
Therefore, by (70) and (71), we obtain
\[
-2E \{ \bar{z}'qV \bar{e} \} - E \{ \bar{e}'V \bar{e} \} \leq (2||\bar{z}_q|| + ||\bar{e}||)||V||_2||\bar{e}||, \quad (72)
\]
where, now, the right-hand-side does not depend on the signal \( s \). Therefore, we obtain
\[
\epsilon \geq \min \{ E \{ \bar{z}'qV \} - \min \{ E \{ \bar{e}'V \} \} \},
\]
where \( \epsilon \) is as described in (36), which completes the proof.

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