AN INTEGRAL EXPRESSION FOR THE DUNKL KERNEL IN THE DIHEDRAL SETTING

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Abstract. In this paper, we establish an integral expression for the Dunkl kernel in the context of Dihedral group of an arbitrary order by using the results in [12] where a construction of the Dunkl intertwining operator for a large set of regular parameter functions is provided. We introduce a differential system that leads to the explicit expression of the Dunkl kernel whenever an appropriate solution of it is obtained. In particular, an explicit expression of the Dunkl kernel \( E_k(x, y) \) is given when one of its argument \( x \) or \( y \) is invariant under the action of a known reflection in the Dihedral group. We obtain also a generating series for the homogeneous components \( E_m(x, y), m \in \mathbb{Z}^+ \), of the Dunkl kernel from which we derive new sharp estimates for the Dunkl kernel when the parameter function \( k \) satisfies \( \text{Re}(k) > -\nu \), \( \nu \) an arbitrary nonnegative integer, which, up to our knowledge, is the largest context for such estimates so far.

1. Introduction and Preliminaries

In [3], Dunkl defined a family of first-order differential-difference operators associated to a Coxeter group. These operators generalize in a certain manner the usual differentiation and have gained during recent years considerable interest in various fields of mathematics and also in physical applications (see [14, 15, 9] and the references therein). Our paper gives some new results around the Dunkl kernel [14] which is a key tool in this theory and whose expression is unfortunately known explicitly only in few cases.

In this paper, we use the results in [12] to establish an integral expression for the Dunkl kernel in the context of Dihedral group of an arbitrary order. This allows us to obtain the explicit expression of the Dunkl kernel \( E_k(x, y) \) when one of its argument \( x \) or \( y \) is invariant under the action of a known reflection in the Dihedral group. We obtain also a generating series for the homogeneous components \( E_m(x, y), m \in \mathbb{Z}^+ \), of the Dunkl kernel from which we derive new sharp estimates for the Dunkl kernel when the parameter function \( k \) satisfies \( \text{Re}(k) > -\nu \), \( \nu \) an arbitrary nonnegative integer, which, up to our knowledge, is the largest context for such estimates so far.

The paper is organized as follows. The remaining of this section serves to introduce concepts and notations needed for the sequel. In section 2 we set up the Dihedral group setting and we establish an important relationship between the homogeneous components \( E_m, m \in \mathbb{Z}^+ \), of the Dunkl kernel, which turn to be crucial for our integral representation. In section 3 we introduce and study a fundamental differential system whose solutions are closely related to the explicit expression of the Dunkl kernel. Section 4 establishes the integral expression for the Dunkl kernel and finally section 5 is devoted to some applications.

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Fix a reduced root system $R$ (see [3]) and consider its associated Coxeter group $G$ generated by the reflections $\sigma_\alpha$ where $\alpha \in R$ and
\[
\sigma_\alpha(x) := x - 2 \langle x, \alpha \rangle \alpha / |\alpha|^2, \quad (x \in \mathbb{R}^d).
\]
Here $\langle \cdot, \cdot \rangle$ denotes the canonical inner product in the space $\mathbb{R}^d$ and $||\cdot||$ its euclidean norm.
We extend the form $\langle \cdot, \cdot \rangle$ to a bilinear form on $\mathbb{C}^d \times \mathbb{C}^d$ again denoted by $\langle \cdot, \cdot \rangle$.

Throughout the paper, if $B \in \mathbb{C}^{d \times d}$, $\| B \|$ stands for the supremum norm of $B$.

The action of $G$ on functions $f : \mathbb{R}^d \rightarrow \mathbb{C}$ is defined by
\[
L_g(f) := f \circ g^{-1}, \quad g \in G.
\]
Set $R_+ := \{ \alpha \in R : \langle \alpha, \beta \rangle > 0 \}$ for some $\beta \in \mathbb{R}^d$ such that $\langle \alpha, \beta \rangle \neq 0$ for all $\alpha \in R$.
Let $k$ be a parameter function on $R$, that is, $k : R \rightarrow \mathbb{C}$ and $G$-invariant.
For $\xi \in \mathbb{R}^n$, the Dunkl operator $T_\xi$ on $\mathbb{R}^d$ associated to the group $G$ and the parameter function $k$ is acting on functions $f \in C^1(\mathbb{R}^d)$ by
\[
T_\xi(k)(f)(x) = \partial_\xi f(x) + \sum_{\alpha \in R_+} k(\alpha) \langle \alpha, \xi \rangle \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle}, \quad (x \in \mathbb{R}^d).
\]
Let $M$ be the vector space of all parameter functions and set
\[
M^{reg} := \{ k \in M : \cap_{\xi \in \mathbb{R}^d} \ker(T_\xi(k)) = \mathbb{C} \cdot 1 \}.
\]
$M^{reg}$ is called the set of regular parameter functions.

We let $\mathcal{P}_m, m \in \mathbb{Z}^+$, denotes the space of all homogeneous polynomials of degree $m$ in the $\mathbb{C}$-algebra $\Pi^d := \mathbb{C}[\mathbb{R}^d]$.

It has been shown in [4] and [2] that for each $k \in M^{reg}$ there exists a unique isomorphism $V_k$ of $\Pi^d$, called the intertwining operator, satisfying
\[
V_k(\mathcal{P}_n) \subset \mathcal{P}_n, \quad V_k(1) = 1 \quad \text{and} \quad T_\xi V_k = V_k \partial_\xi, \quad \forall \xi \in \mathbb{R}^d.
\]

The Dunkl kernel or the generalized exponential $E_k(\cdot, \cdot)$, is a key tool in the Dunkl’s theory and it was introduced [5] as the unique function satisfying
\[
E_k(0, y) = 1, \quad T_\xi E_k(x, y) = \langle \xi, y \rangle E_k(x, y), \quad \xi, x \in \mathbb{R}^d, \quad y \in \mathbb{C}^d.
\]

Further details about this kernel may be found in [14][5].

The expression of the kernel $E_k$ is known explicitly only in few cases. Following [12] we have
\[
E_k(x, y) = \sum_{m=0}^{\infty} E_m(x, y), \quad x \in \mathbb{R}^d, \quad y \in \mathbb{C}^d,
\]
where
\[
E_m(x, y) := \frac{1}{m!} V_k(\langle \cdot, y \rangle^m)(x),
\]
for a large set $M^*$ of regular parameter functions $k$ presented later in this section.

We recall from [12] some notations used along this paper. We consider the operator $A$ acting on functions $f : \mathbb{R}^d \rightarrow \mathbb{C}$ by
\[
A := \sum_{\beta \in R_+} k(\beta)L_{\sigma_\beta},
\]
where $L_\sigma$ is defined by (1.1). It is clear that $A$ leaves invariant the space $\mathcal{P}_m$ for all nonnegative integer $m$, thus $A_m := A|_{\mathcal{P}_m}$ is an endomorphism of $\mathcal{P}_m$. 

Let $M^*$ denotes the set of all parameter functions for which the operator
\[ H_m := ((m + \gamma) - A_m)^{-1}, \quad \gamma := \sum_{\alpha \in R^+} k(\alpha) \]
is defined for all $m \geq 1$. When $k \in M^*$, define the operator $H$ by $H|P_m := H_m$.

Appealing to [12], $M^*$ includes the set of all parameter functions whose real part is nonnegative and $M^* \subset M^{reg}$. Moreover we have the crucial result:

**Theorem 1.1.** [12] Assume that $k \in M^*$. Let $m$ be a positive integer and $p \in P_m$. Then
\[ V_k(p)(x) = \sum_{j=0}^{d} x_j V_k(\partial_j (Hp))(x), \quad (x \in \mathbb{R}^d). \]

### 2. The Dihedral Group Setting

In this section, we suppose that the Coxeter group $G$ associated to the root system $R$ is the Dihedral group $D_n$ of order $2n$ where $n \geq 2$ is arbitrary.

$D_n$ is the symmetry group of a regular $n$-gons in the plane $\mathbb{R}^2$. Using the identification $\mathbb{R}^2 = \mathbb{C}$ and letting $w := e^{i\pi/n}$, then an associated positive root system is $R^+ = \{iw^j, j = 0, \ldots, n-1\}$ while the rotations and reflections in $D_n$ are given respectively by $r_j : z \mapsto zw^{2j}$ and $\sigma_j : z \mapsto zw^{2j}$, $j = 0, \ldots, n-1$.

Taking $r = r_1$ and $\sigma = \sigma_0$, we may then write $D_n = D_n^+ \cup D_n^-$ with
\[ D_n^+ := \{r^j, j = 0, \ldots, n-1\} \quad \text{and} \quad D_n^- := \{r^j \sigma, j = 0, \ldots, n-1\}. \]

It is worth to note that one has $\sigma r \sigma = r^{-1}$ and
\[ \sum_{j=0}^{n-1} r^j x = 0, \]
for all $x \in \mathbb{R}^2$. It is also important to keep in mind (see [6, Theorem 4.2.7]) that in our setting the reflections in $D_n$ are given exactly by the set $D_n^-$. In all the sequel, we fix an arbitrary integer $n \geq 2$ and consider $D_n$ as above. We will also suppose that the parameter function $k$ is constant, that is $k := k(\alpha)$ for all $\alpha \in R^+$. Although that in our setting we have $\gamma = nk$, we will always write $\gamma$ instead of $nk$ to conserve the standard notations for this theory.

With these settings in mind, the operator $A$ given by (1.4) rewrites
\[ A = k \sum_{i=0}^{n-1} L_{r^i \sigma}. \]

The next result pursues the developments in [12] and gives more elements in the set $M^*$. Further, it makes concrete the action of $H_m$ on polynomials.

**Proposition 2.1.** Consider $k \in \mathbb{C}$ such that
\[ \gamma \notin \left\{ -\frac{m}{2}, \quad m \in \mathbb{Z}^+ \right\}. \]
Then $k \in M^*$. Moreover, in this case the operator $H_m$ is given by
\[ H_m(p)(x) = \sum_{j=0}^{n-1} a_j(m)p(r^j x) + \sum_{j=0}^{n-1} b_j(m)p(r^j \sigma x), \]
for all positive integer $m$, where for $j = 0, \ldots, n - 1$, we set

$$a_j(m) = \delta_{j,0} \frac{1}{m + \gamma} + \frac{\gamma^2}{nm(m + \gamma)(m + 2\gamma)}$$

and

$$b_j(m) = \frac{\gamma}{nm(m + 2\gamma)},$$

with $\delta_{j,0} = 1$ if $j = 0$ and $\delta_{j,0} = 0$ otherwise.

**Proof.** Take $k \in \mathbb{C}$ satisfying (2.4) and fix any positive integer $m$. From (2.3), we see that $A = \gamma L_\sigma R$ where $R := \frac{1}{d} \sum_{j=0}^n L_{rj}$ is a projector.

Hence, each polynomial $p \in \mathcal{P}_m$ decomposes uniquely as $p = p_1 + p_2$ where $p_1, p_2 \in \mathcal{P}_m$ with $A(p_1) = 0$ and $A(p_2) = \gamma L_\sigma p_2$. Therefore we get,

$$(m + \gamma - A)p_1 = (m + \gamma)p_1 \quad \text{and} \quad (m + \gamma - A)p_2 = ((m + \gamma) - \gamma L_\sigma)p_2.$$

By view of $\gamma \notin \{-\frac{d}{m}, m \in \mathbb{Z}^+\}$, we may define the operator $\Lambda$ in $\mathcal{P}_m$ by

$$\Lambda(p_1) = \frac{1}{m + \gamma}p_1,$$

$$\Lambda(p_2) = \left(\frac{m + \gamma}{m(m + 2\gamma)} + \frac{\gamma}{m(m + 2\gamma)}L_\sigma\right)p_2.$$

A straightforward calculation shows that $W \circ ((m + \gamma) - A) = \text{Id}_{\mathcal{P}_m}$, thus $k \in M^*$ and $\Lambda = H_m$. Further, since $\gamma p_2 = L_\sigma A(p)$, we get

$$H_m(p) = \frac{1}{(m + \gamma)}p + \frac{\gamma^2}{m(m + 2\gamma)(m + 2\gamma)}p_2 + \frac{\gamma}{m(m + 2\gamma)}L_\sigma p_2$$

$$= \frac{1}{(m + \gamma)}p + \frac{\gamma}{m(m + \gamma)(m + 2\gamma)}L_\sigma A(p) + \frac{1}{m(m + 2\gamma)}A(p),$$

and the proof is complete. \qed

From now on, we assume that the parameter function $k$ satisfies the condition (2.4).

Based on the definition (1.3), Theorem 1.1, and a direct application of Proposition 2.1 the proposition below gives a useful relationship between $E_{m+1}$ and $E_m$.

**Proposition 2.2.** For all $m \in \mathbb{Z}^+$ and all $x, y \in \mathbb{R}^d$ we have

$$E_{m+1}(x, y) = \sum_{j=0}^{n-1} a_j(m+1) \langle r^j x, y \rangle E_m(r^j x, y) + \sum_{j=0}^{n-1} b_j(m+1) \langle r^j \sigma x, y \rangle E_m(r^j \sigma x, y),$$

where the constants $a_j, b_j, j = 0, \ldots, n - 1$ are given by Proposition 2.1.

As an important consequence of Proposition 2.2 with (2.2) in mind, one has

$$E_1(x, y) = \frac{1}{1 + \gamma} \langle x, y \rangle, \quad x, y \in \mathbb{R}^2.$$  

(2.5)

In the remaining of this section, we will proceed to construct a sequence that turn to be an essential ingredient for the expression of the Dunkl kernel. Before to go on, we need to introduce some additional notations.

For a positive integer $m$, we set $I_m := \text{diag}(1, \ldots, 1) \in \mathbb{C}^{m \times m}$.

For $X \in \mathbb{C}^{m \times 1}$, we define its rank one associated matrix $X \otimes X \in \mathbb{C}^{m \times m}$ by

$$X \otimes X := X \times^t X.$$

For $x, y \in \mathbb{R}^2$, we define the diagonal matrix $D := D(x, y)$ by

$$D(x, y) := \text{diag} \left( \langle r^j x, y \rangle, \quad j = 0, \ldots, n - 1 \right) \in \mathbb{C}^{n \times n}.$$  

(2.6)
Finally, we define the one column vectors $W, W_s \in \mathbb{C}^{2n \times 1}$ by

$W := [1, \ldots, 1]^T$ and $W_s := [1, \ldots, 1, -1, \ldots, -1]^T$.

In the sequel we will always identify $\mathbb{C}^{m \times 1}$ with $\mathbb{C}^m, m \geq 1$.

For $x, y \in \mathbb{R}^2$ consider the vector $X_m := X_m(x, y) \in \mathbb{C}^n$ whose components are given by:

$$X_{m,i}(x, y) = E_m(r^i x, y), \quad i = 0, \ldots, n-1.$$

For the sake of simplicity, we will omit the dependance on $y$ or on $x$ and $y$ together, when there is no need to reveal them. For instance, we will write, $X_{m,i}(x)$ in place of $X_{m,i}(x, y)$ and $D(x)$ or $D$ in place of $D(x, y)$ and so on.

By Proposition 2.2 and using notation modulo $n$ for the indices we get for $i = 0, \ldots, n-1$:

$$X_{m+1,i}(x) = E_{m+1}(r^i x, y)$$

$$= \sum_{j=0}^{n-1} a_j(m + 1) \langle r^{i+j} x, y \rangle E_m(r^{i+j} x, y) + \sum_{j=0}^{n-1} b_j(m + 1) \langle r^{i-j} x, y \rangle E_m(r^{i-j} x, y)$$

$$= \sum_{j=0}^{n-1} a_{j-i}(m + 1) \langle r^j x, y \rangle E_m(r^j x, y) + \sum_{j=0}^{n-1} b_{j+i}(m + 1) \langle r^j x, y \rangle E_m(r^j x, y)$$

$$= \sum_{j=0}^{n-1} a_{j-i}(m + 1) \langle r^j x, y \rangle X_{m,j}(x) + \sum_{j=0}^{n-1} b_{j+i}(m + 1) \langle r^j x, y \rangle X_{m,j}(x).$$

The $n$ latter equations for $i = 0, \ldots, n-1$, can be gathered in a matrix form as follows:

$$X_{m+1}(x) = A_m D(x) X_m(x) + B_m D(x) X_m(x)$$

where the matrices $A_m$ and $B_m$ are defined by

$$(2.9) \quad A_m := \frac{1}{m + 1 + \gamma} I_n + \frac{\gamma^2}{n(m+1)(m + 1 + \gamma)(m + 1 + 2\gamma)} O_n,$$

$$(2.10) \quad B_m := \frac{\gamma}{n(m+1)(m + 1 + 2\gamma)} O_n,$$

where $O_n$ is the square matrix of order $n$ whose entries are all equal to 1. This leads to

$$\begin{bmatrix} X_{m+1}(x, y) \\ X_{m+1}(x, y) \end{bmatrix} = W_m D(x, y) \begin{bmatrix} X_m(x, y) \\ X_m(x, y) \end{bmatrix}$$

where $W_m$ is defined by

$$W_m := \frac{1}{m + 1 + \gamma} I_{2n} + \frac{\gamma}{n(m+1)(m + 1 + 2\gamma)} \begin{bmatrix} \frac{\gamma}{m+1+\gamma} O_n & O_n \\ O_n & \frac{\gamma}{m+1+\gamma} O_n \end{bmatrix}.$$  

Noting that

$$\begin{bmatrix} \frac{\gamma}{m+1+\gamma} O_n & O_n \\ O_n & \frac{\gamma}{m+1+\gamma} O_n \end{bmatrix} = \frac{m+1+2\gamma}{2(m+1+\gamma)} W \otimes W - \frac{m+1}{2(m+1+\gamma)} W_s \otimes W_s,$$ 

and setting

$$(2.11) \quad Y_m := Y_m(x, y) = (1 + \gamma)_m \begin{bmatrix} X_m(x, y) \\ X_m(x, y) \end{bmatrix},$$

where $(1 + \gamma)_m$ is the Pochhammer symbol, we get the starting useful relationship:

$$(2.12) \quad Y_{m+1} = D Y_m + \frac{\gamma}{2n(m+1)} (DY_m, W) W - \frac{\gamma}{2n(m+1+2\gamma)} (DY_m, W_s) W_s,$$
for all $m \in \mathbb{Z}^+$. Next, an easy induction gives

\[
Y_m = D^{m-1}Y_1 + \frac{\gamma}{2n} \sum_{j=1}^m \frac{1}{j} \langle DY_{j-1}, W \rangle D^{m-j}W \\
- \frac{\gamma}{2n} \sum_{j=1}^m \frac{1}{j + 2\gamma} \langle DY_{j-1}, W_s \rangle D^{m-j}W_s,
\]

for all positive integer $m$.

From (2.5) and (2.12) we get $Y_1 = DW$ and the equation (2.13) rewrites

\[
Y_m = D^mW + \frac{\gamma}{2n} \sum_{j=1}^m \frac{1}{j} \langle DY_{j-1}, W \rangle D^{m-j}W \\
- \frac{\gamma}{2n} \sum_{j=1}^m \frac{1}{j + 2\gamma} \langle DY_{j-1}, W_s \rangle D^{m-j}W_s,
\]

for all positive integer $m$. Taking the scalar products of both members in (2.14) with $DW$ and $DW_s$ respectively, we get

\[
\langle Y_m, DW \rangle = \langle D^{m+1}W, W \rangle + \frac{\gamma}{2n} \sum_{j=1}^m \frac{1}{j} \langle Y_{j-1}, DW \rangle \langle D^{m-j+1}W, W \rangle \\
- \frac{\gamma}{2n} \sum_{j=2}^m \frac{1}{j + 2\gamma} \langle DY_{j-1}, W_s \rangle \langle D^{m-j+1}W_s, W \rangle,
\]

and

\[
\langle Y_m, DW_s \rangle = \langle D^{m+1}W_s, W \rangle + \frac{\gamma}{2n} \sum_{j=1}^m \frac{1}{j} \langle Y_{j-1}, DW \rangle \langle D^{m-j+1}W_s, W \rangle \\
- \frac{\gamma}{2n} \sum_{j=2}^m \frac{1}{j + 2\gamma} \langle DY_{j-1}, W_s \rangle \langle D^{m-j+1}W_s, W \rangle,
\]

for all $m \geq 1$. The equations (2.15) and (2.16) are the fundamental pieces of our main result.

3. The Group Associated Differential System

In this section we present and study a differential system related to the Dihedral group $D_n$, which we will refer to in this paper, simply, as the group differential system since. It will be shown in section 2 and 3 that the explicit expression of the Dunkl kernel as well as its integral representation are closely related to this differential system. More precisely, we get the explicit expression of the Dunkl kernel whenever we get an appropriate solution of the homogeneous differential system associated to the group differential system.

Before to pursue our developments, we need to introduce some specific additional notations here. Recall that for $B \in \mathbb{C}^{2 \times 2}$, $\|B\|$ denotes its supremum norm, while for $X \in \mathbb{C}^2$, $\|X\|$ stands for its euclidean norm.

In all the sequel we fix $x, y \in \mathbb{R}^2$ and we define

\[
a(x, y) := \max \{ |\langle gx, y \rangle|, \quad g \in D_n \}.
\]

In order to include the case $a(x, y) = 0$ into our developments (especially when $x = 0$ or $y = 0$), we will allow $\frac{1}{a(x, y)} = +\infty$ when $a(x, y) = 0$. 

For $0 < \rho \leq +\infty$, $\mathbb{D}_\rho$ is the open disk in the complex plane $\mathbb{C}$ centered at 0 with radius $\rho$ if $\rho < +\infty$ and the hole plane $\mathbb{C}$ otherwise. We let also $\mathbb{D}^*_\rho := \mathbb{D}_\rho - \{0\}$ denotes its associated punctured disk.

Define the functions $g$ and $g_s$ by
\begin{align}
g(z) := g(z, x, y) &= \sum_{i=0}^{n-1} \frac{(r^i x, y)}{1 - z (r^i x, y)} + \sum_{i=0}^{n-1} \frac{(r^i \sigma x, y)}{1 - z (r^i \sigma x, y)}, \\
g_s(z) := g_s(z, x, y) &= \sum_{i=0}^{n-1} \frac{(r^i x, y)}{1 - z (r^i x, y)} - \sum_{i=0}^{n-1} \frac{(r^i \sigma x, y)}{1 - z (r^i \sigma x, y)}.
\end{align}

Finally, consider the $\mathbb{C}^2$-valued function $\Lambda$ given by
\begin{equation}
\Lambda(z) := \Lambda(z, x, y, z) = \frac{\gamma}{2n} \begin{bmatrix} g(z) & -g_s(z) \\ g_s(z) & -g(z) \end{bmatrix}.
\end{equation}

$\Lambda$ is a $\mathbb{C}^2$-holomorphic function in $\mathbb{D}^*_R$ with $h(z) = \sum_{p=0}^{\infty} B_p(x, y)z^p$ and
\begin{equation}
B_p(x, y) := \frac{\gamma}{2n} \sum_{i=0}^{n-1} \left[ (r^i x, y)^{p+1} + (r^i \sigma x, y)^{p+1} - \left( (r^i x, y)^{p+1} - (r^i \sigma x, y)^{p+1} \right) \right].
\end{equation}

From now on, for the sake of simplicity, we will always omit the dependance on $x$ or $y$ both when there is no need to reveal them.

Consider the group differential system
\begin{equation}
X'(z) = \begin{bmatrix} g(z) \\ g_s(z) \end{bmatrix} + \begin{bmatrix} \frac{2\gamma}{n} g(z) \\ \frac{2\gamma}{n} g_s(z) \end{bmatrix} - \frac{2\gamma}{n} g_s(z) - \frac{2\gamma}{n} g(z),
\end{equation}
and its associated homogeneous system
\begin{equation}
X'(z) = \begin{bmatrix} \frac{2\gamma}{n} g(z) \\ \frac{2\gamma}{n} g_s(z) \end{bmatrix} - \frac{2\gamma}{n} g_s(z) - \frac{2\gamma}{n} g(z).
\end{equation}

An important tool for our developments is given by the next proposition.

**Proposition 3.1.** Fix $R > 0$. Let $X$ be a $\mathbb{C}^2$-valued holomorphic function in $\mathbb{D}_R$ solution of (3.7) in $\mathbb{D}^*_R$ and vanishing at $z = 0$. Then $X = 0$ in $\mathbb{D}_R$.

**Proof.** Let $R > 0$ and $X$ be as in the Proposition 3.1. We have then
\begin{equation}
z X'(z) = -2\gamma J \times X(z) + z \Lambda(z) \times X(z),
\end{equation}
for all $z \in \mathbb{D}^*_R$. Here $\Lambda$ is the function defined by (3.4) and $J := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Since $JX(0) = 0$ we see that (3.8) holds for $z = 0$ as well.

In other hand, by analyticity argument, there exists a sequence $A_p := A_p(x, y) \in \mathbb{C}^2$, $p \in \mathbb{Z}^+$ satisfying $X(z) = \sum_{p=0}^{+\infty} A_p z^p$ for all $z \in \mathbb{D}_R$ and an easy calculation shows that (3.8) equivalents to
\begin{equation}
\sum_{p=1}^{+\infty} p A_p z^p = 2\gamma J \sum_{p=0}^{+\infty} A_p z^p + \sum_{p=0}^{+\infty} C_p z^{p+1},
\end{equation}
where $C_p = \sum_{k=0}^{p} B_{k} A_{p-k}$ and the matrices $B_p$ are given by (3.5).
View of (2.2) we see that $B_0 = 0$ and then (3.9) rewrites
\[
\sum_{p=1}^{+\infty} pA_p z^p = 2\gamma J \sum_{p=0}^{+\infty} A_p z^p + \sum_{p=1}^{+\infty} C_p z^p,
\]
for all $z \in \mathbb{D}_R$, which in turn, since $JA_0 = 0$, equivalents to
\[
A_p = (p + 2\gamma J)^{-1} \sum_{k=0}^{p-1} B_{p-k-1} A_k, \quad (p \geq 1).
\]
Note that by our assumption on the parameter function $k$, the matrix $(p + 2\gamma J)^{-1}$ is well defined and an induction yields $A_p = 0$ for all $p \in \mathbb{Z}^+$.

The next theorem is the last rock in our developments and is the fundamental one.

**Theorem 3.1.** There exists a constant $\delta \geq 1$ not depending on $(x, y)$ and a $C^2$-valued holomorphic function $Q$ in $\mathbb{D}^\ast \frac{1}{\gamma (x, y)}$ vanishing at $z = 0$ and solution of (3.6) in $\mathbb{D}_\rho^\ast \frac{1}{\gamma (x, y)}$.

$Q$ is defined by $Q(z) = \sum_{p=1}^{+\infty} A_p z^p$ where $A_p := A_p(x, y) \in \mathbb{C}^2$ given by:
\[
A_0 := \left( \frac{2n}{\gamma}, 0 \right), \quad A_p := \left[ \frac{1}{p} \begin{array}{c} 0 \\ 1 \end{array} \right] \sum_{i=0}^{p-1} B_{p-i-1} A_i, \quad (p \geq 1).
\]
Here the matrices $B_p$ are given by (3.5).

The coefficient $A_p$ satisfies
\[
\|A_p\| \leq \frac{2n}{|\gamma|} (\delta a(x, y))^p.
\]
for all $p \in \mathbb{Z}^+$. Moreover, each $C^2$-valued holomorphic function in $\mathbb{D}_\rho$, $\rho > 0$, solution of (3.6) in $\mathbb{D}_\rho^\ast$, vanishing at $z = 0$ coincides with $Q$ in the disk $\mathbb{D}_\rho \cap \mathbb{D}^\ast \frac{1}{\gamma (x, y)}$.

**Proof.** Assume for now that $Q$ exists and is as in Theorem 3.1. Then since $Q(0) = 0$, the last part of Theorem 3.1 follows immediately from Proposition 3.1.

Now we will proceed to construct the function $Q$. First of all note that the relationship (3.11) define a unique sequence $A_p \in \mathbb{C}^2$ depending only in $(x, y)$ and, view of (3.5), an easy induction shows that the components of $A_p(x, y)$, $p \in \mathbb{Z}^+$ are in fact homogeneous polynomials of degree $p$ with respect to $x$ and $y$.

Further, it is easy to see from (3.5) that
\[
\|B_p\| \leq |\gamma| (a(x, y))^{p+1},
\]
for all $p \in \mathbb{Z}^+$ and then
\[
\|A_p\| \leq \frac{\delta}{p} \sum_{i=0}^{p-1} (a(x, y))^{p-i} \|A_i\|,
\]
where we have set
\[
\delta := 2 |\gamma| \sup_{p \in \mathbb{Z}^+} \left\{ 1, \frac{p}{|p + 2\gamma|} \right\}.
\]
Note that up to consider $\max(1, \delta)$, we may and will suppose that $\delta \geq 1$.

Suppose for a while that $a(x, y) > 0$. Then since $\delta \geq 1$, the inequality (3.14) yields
\[
\frac{\|A_p\|}{(\delta a(x, y))^p} \leq \frac{1}{p} \sum_{i=0}^{p-1} (\delta a(x, y))^{i}\|A_i\|^{1/i},
\]
An Integral Expression for the Dunkl Kernel in the Dihedral Setting

and an immediate induction gives

\[ \|A_p\| \leq (\delta a(x, y))^p \|A_0\|, \]

for all \( p \in \mathbb{Z}^+ \). View of (3.14), we see that (3.15) holds also when \( a(x, y) = 0 \).

Further, the function \( Q \) defined by \( Q(z) = \sum_{p=1}^{\infty} A_p z^p \) is holomorphic in \( D_{\frac{1}{\delta a(x, y)}} \), vanishes at \( z = 0 \) and the same arguments used in Proposition 3.1 show that \( Q \) is solution of (3.6) in \( D^{*}_{\frac{1}{\delta a(x, y)}} \), which completes the proof.

The next result solves the group differential system in the case where \( x \) or \( y \) is \( \sigma \)-invariant which allows us, in section 4, to obtain the explicit expression of the Dunkl kernel in this context.

**Corollary 3.2.** Assume that \( x \) or \( y \) is \( \sigma \)-invariant. Then the function \( Q \) defined in Theorem 3.1 is given by

\[ Q(z) = -\frac{2}{k} \left( 1 - \frac{1}{\prod_{i=0}^{n-1} (1 - z \langle r^i x, y \rangle)^{-k}, 0} \right), \quad z \in D_{\frac{1}{\delta a(x, y)}} \]

for some \( \delta \geq 1 \), where we make use of the principal determination of the \( z \mapsto (1 - z)^{-k} \) in the unit disk \( \mathbb{D} \).

**Proof.** Suppose that \( x \) is \( \sigma \)-invariant, then we get \( g_s(z) = 0 \) and

\[ g(z) = 2 \sum_{i=0}^{n-1} \frac{\langle r^i x, y \rangle}{1 - z \langle r^i x, y \rangle}, \]

where \( g \) and \( g_s \) are given by (3.2) and (3.3). In this case, it is easy to verify that the \( \mathbb{C}^2 \)-holomorphic function

\[ X(z) := -\frac{2}{k} \left( 1 - \frac{1}{\prod_{i=0}^{n-1} (1 - z \langle r^i x, y \rangle)^{-k}, 0} \right), \]

is solution of (3.6) in \( D_{\frac{1}{\delta a(x, y)}} \) with \( X(0) = 0 \).

Since \( g_s(z, x, y) = g_s(z, y, x) \) and \( g(z, x, y) = g(z, y, x) \) then the same result holds when \( \sigma y = y \). Now Theorem 3.1 ends the proof.

### 4. An Integral Expression for the Dunkl Kernel

This section is the last step towards the main result of this paper. We will gather here all the work done in the previous sections to derive out an integral expression for the polynomials \( E_m, m \in \mathbb{Z}^+ \), which, as a consequence, provides an integral expression for the Dunkl kernel. We mention here that in [2], an integral representation of \( E_k \) was obtained for the rank-two root system \( A_2 \). Also, based on the results of [12] and [11, Chapter 3], the authors in [10] present another version of [2].

Let \( Q(z, x, y) := (Q_1(z, x, y), Q_2(z, x, y)) \in \mathbb{C}^2 \) be as in Theorem 3.1 and set

\[ \Phi(z, x, y) := \frac{2n}{\gamma} + Q_1(z, x, y) - Q_2(z, x, y). \]

Our first main result is the generating series of the polynomials \( E_m \) given below.
Theorem 4.1. There exists a constant $\delta \geq 1$ such that for all $x, y \in \mathbb{R}^2$ and $0 < \rho < \frac{1}{\delta a(x, y)}$ we have
\[
E_m(x, y) = \frac{\gamma/2n}{(1 + \gamma)_m} \frac{1}{2\pi i} \int_{|z| = \rho} \frac{\Phi(z, x, y)}{1 - z \langle x, y \rangle} z^{m+1}. \tag{4.2}
\]
Here $\Phi$ is defined by (4.1). By consequence we get
\[
\frac{\Phi(z, x, y)}{1 - z \langle x, y \rangle} = \frac{2n}{\gamma} \sum_{m=0}^{+\infty} (1 + \gamma)_m E_m(x, y). \tag{4.3}
\]

Proof. We use notations (2.6) and (2.7) where, once again, we omit the dependance on $(x, y)$. Consider the functions $f$, $f_s$, $F$ and $F_s$ defined by their respective power series where as follows:
\[
f(z) = \sum_{j=1}^{\infty} \langle Y_j, DW \rangle z^j, \quad f_s(z) = \sum_{j=1}^{\infty} \langle Y_j, DW_s \rangle z^j.
\]
\[
F(z) = \sum_{j=1}^{\infty} \frac{1}{j} \langle Y_{j-1}, DW \rangle z^j, \quad F_s(z) = \sum_{j=1}^{\infty} \frac{1}{j + 2\gamma} \langle Y_{j-1}, DW_s \rangle z^j.
\]

If we let $\delta = \delta(n, k) := \max_{m \in \mathbb{Z}_+} (\|A_m\| + \|B_m\|)$, where the matrices $A_m$ and $B_m$ are given by (2.9) and (2.10), then we see from (2.11) and (2.8) that
\[
\|Y_{m+1}\| \leq \delta a(x, y) \|Y_m\|,
\]
for all $m \in \mathbb{Z}^+$. We deduce that
\[
\|Y_m\| \leq (\delta a(x, y))^m \|Y_0\| = (\delta a(x, y))^m,
\]
which leads to
\[
|\langle Y_{j-1}, DW \rangle| \leq (\delta a(x, y))^j \quad \text{and} \quad |\langle Y_{j-1}, DW_s \rangle| \leq (\delta a(x, y))^j,
\]
for all $j \geq 1$. This shows in particular that $F$ and $F_s$ are holomorphic functions in $\mathbb{D}^{\frac{1}{\delta a(x, y)}}$ satisfying $F(0) = F_s(0) = 0$. Further, a straightforward calculations yields
\[
F'(z) = f(z), \quad F'_s(z) = f_s(z) - \frac{2\gamma}{z} F_s(z)
\]
for all $z \in \mathbb{D}^{\frac{1}{\delta a(x, y)}}$. From the equations (2.15) and (2.16) we see that $(F, F_s)$ is solution of the differential system (3.6) in $\mathbb{D}^{\frac{1}{\delta a(x, y)}}$.

Note that, up to increase $\delta$, one gets by use of Theorem 3.1 that $(F(z), F_s(z)) = Q(z)$ for all $z \in \mathbb{D}^{\frac{1}{\delta a(x, y)}}$.

By their definitions, we have for all $j \geq 1$ and for all $0 < \rho < \frac{1}{\delta a(x, y)}$ that
\[
\frac{1}{j} \langle Y_{j-1}, DW \rangle = \frac{1}{2\pi i} \int_{|z| = \rho} \frac{F(z)}{z^{j+1}} dz = \frac{1}{2\pi i} \int_{|z| = \rho} \frac{G_1(z)}{z^{j+1}} dz,
\]
and
\[
\frac{1}{j + 2\gamma} \langle Y_{j-1}, DW_s \rangle = \frac{1}{2\pi i} \int_{|z| = \rho} \frac{F_s(z)}{z^{j+1}} dz = \frac{1}{2\pi i} \int_{|z| = \rho} \frac{G_2(z)}{z^{j+1}} dz.
\]
All of this yields to
\[
Y_m = D^m W + \frac{\gamma}{2n} \sum_{j=1}^{m} \left( \frac{1}{2i\pi} \int_{|z|=\rho} \frac{G_1(z)}{z^{j+1}} dz \right) D^{m-j} W
- \frac{\gamma}{2n} \sum_{j=1}^{m} \left( \frac{1}{2i\pi} \int_{|z|=\rho} \frac{G_2(z)}{z^{j+1}} dz \right) D^{m-j} W.
\]

Finally, taking the scalar product with \((1, 0, \ldots, 0) \in \mathbb{R}^{2n}\) for both members above, we get
\[
(1 + \gamma) m E_m(x, y) = \langle x, y \rangle^m + \frac{\gamma}{2n} \sum_{j=1}^{m} \left( \frac{1}{2i\pi} \int_{|z|=\rho} \frac{G_1(z) - G_2(z)}{z} dz \right) \langle x, y \rangle^{m-j}
= \langle x, y \rangle^m + \frac{\gamma}{2n} \sum_{j=1}^{m} \left( \frac{1}{2i\pi} \int_{|z|=\rho} \frac{\Phi(z)}{z^{j+1}} dz \right) \langle x, y \rangle^{m-j}.
\]

Whence
\[
(1 + \gamma) m E_m(x, y) = \frac{\gamma}{2n} \sum_{j=0}^{m} \left( \frac{1}{2i\pi} \int_{|z|=\rho} \frac{\Phi(z)}{z^{j+1}} dz \right) \langle x, y \rangle^{m-j}.
\]

The last equation is obtained noting that \(\frac{1}{2i\pi} \int_{|z|=\rho} \frac{\Phi(z)}{z} dz = \Phi(0) = \frac{2\pi}{\gamma}\). The result follows noting that the right member in (4.4) is nothing but the coefficient of order \(m\) in the Cauchy product of \(\frac{\gamma}{2n}\Phi(z)\) with the power series \(\frac{1}{1-\gamma z}\),
\[
\sum_{j=0}^{\infty} \langle x, y \rangle^j z^j.
\]

In the sequel we will define \(\mathcal{J} := \mathcal{J}(k, n)\) as the set of all real numbers \(\delta \geq 1\) satisfying Theorem 4.1. View of Theorem 4.1 and the analyticity of the function \(z \mapsto \Phi(z)\), it is not hard to see that actually \(\mathcal{J}\) is an interval of the form \((\inf \mathcal{J}, +\infty]\).

For \(\delta \in \mathcal{J}\), \(x, y \in \mathbb{R}^2\) and \(0 < \rho < \frac{1}{3\delta(x, y)}\) define the kernel
\[
K(t, \delta, \rho, x, y) := \frac{\gamma^2}{2n} \frac{1}{2i\pi} \int_{|z|=\rho} \frac{\Phi(z, x, y)}{z(1-z \langle x, y \rangle)} e^{t/z} dz,
\]
where \(\Phi\) is defined by (4.1). Our integral representation for the Dunkl kernel now reads:

**Theorem 4.2.** Assume that \(\Re(\gamma) > 0\). Then for all \(\delta \in \mathcal{J}\), \(x, y \in \mathbb{R}^2\) and \(0 < \rho < \frac{1}{3\delta(x, y)}\) we have
\[
E_k(x, y) = \int_0^1 (1 - t)^{\gamma - 1} K(t, \delta, \rho, x, y) dt,
\]
where the kernel \(K\) is given by (4.5).

**Proof.** Assume that \(k \in M^*\) and fix \(\delta \in \mathcal{J}\), \(x, y \in \mathbb{R}^2\). Let \(\rho\) be any positive real number such that \(\rho < \frac{1}{3\delta(x, y)}\). Appealing to (12) and Theorem 4.1 we have
\[
E_k(x, y) = \sum_{m=0}^{\infty} E_m(x, y)
= \frac{\gamma}{2n} \frac{1}{2i\pi} \int_{|z|=\rho} \frac{\Phi(z, x, y)}{z(1-z \langle x, y \rangle)} \left( \sum_{m=0}^{\infty} \frac{1}{(1+\gamma) m z^m} \right) dz.
\]
Now if we assume further that $\text{Re}(\gamma) > 0$, then from [1 p. 505], we get
\[
E_k(x, y) = \frac{\gamma^2}{2n} \frac{1}{2\pi i} \int_{|z| = \rho} \frac{\Phi(z, x, y)}{z(1 - z(x, y))} \left(\int_0^1 (1 - t)^{-1} e^{\gamma t} dt\right) dz.
\]
which completes the proof. \(\square\)

5. Applications

This section gives two important applications of the integral representation established in the previous section for the polynomials $E_m$, $m \in \mathbb{Z}^+$. Namely, we will derive the explicit expression of $E_m(x, y)$ when one of its arguments $x$ or $y$ is $\sigma$-invariant and we will also give sharp estimates for the Dunkl kernel when $\text{Re}(\gamma) > -\nu$ and $\nu \in \mathbb{Z}_p$.

We begin by the explicit expression, and for simplicity, we write here $\gamma/n = k$ when it is needed.

**Theorem 5.1.** Let $x, y \in \mathbb{R}^2$. Assume that $x$ or $y$ is $\sigma$-invariant then for all nonnegative integer $m$ we have
\[
E_m(x, y) = \frac{1}{(1 + \gamma)^m} \sum_{j=0}^m \left( \sum_{\nu_0 + \cdots + \nu_{n-1} = j} \frac{n!}{\nu_1!} \Phi(r_i x, y) \right) x^{m-j}.
\]

**Proof.** Assume that $x$ is $\sigma$-invariant. From Corollary 3.2 and (4.1), we see that
\[
\Phi(x, y) = \frac{2}{k} \prod_{i=0}^{n-1} (1 - z(r_i x, y))^{-k} = \frac{2}{k} \prod_{i=0}^{n-1} \sum_{\nu_i = 0}^\infty \frac{(k)_{\nu_i}}{\nu_i!} (r_i x, y)^{\nu_i} z^{\nu_i}.
\]
Whence
\[
\Phi(x, y) = \frac{2}{k} \sum_{m=0}^{\infty} \left( \sum_{\nu_0 + \cdots + \nu_{n-1} = m} \frac{n!}{\nu_1!} (r_i x, y)^{\nu_i} \right) z^m
\]
and Theorem 4.1 ends the proof. \(\square\)

The next theorem gives new sharp estimates for the homogeneous components of the Dunkl kernel $E_k$ when $\text{Re}(\gamma) > -\nu$, $\nu \in \mathbb{Z}_p$, which are used next to derive appropriate ones for $E_k$. We point out here that in [13] and [14] other concise estimates for $E_k$ are given when $k$ is nonnegative or with nonnegative real part respectively.

**Theorem 5.2.** Assume that $\text{Re}(\gamma) > -\nu$ where $\nu$ is a nonnegative integer. Then there exists a constant $\delta \geq 1$ and a positive constant $C := C(n, k, \nu)$ such that
\[
|E_m(x, y)| \leq C(n, k, \nu) \left(\frac{\delta a(x, y)}{m!}\right)^m, \quad (m \geq 1),
\]
for all $x \in \mathbb{R}^2$, $y \in \mathbb{C}^2$.

**Proof.** First of all by analyticity argument, Theorem 4.1 holds for $x \in \mathbb{R}^2$ and $y \in \mathbb{C}^2$ as well. In other hand, recalling the definition of the function $Q$ given by Theorem 3.1 we know that there exists a constant $\delta \geq 1$ such that
\[
\|Q(z)\| \leq \sum_{p=1}^{\infty} \|A_p\| |z|^p \leq \sum_{p=1}^{\infty} \left( \frac{\delta a(x, y)}{m!} \right)^p |z|^p \|A_0\| = \frac{\delta a(x, y) |z|}{1 - \delta a(x, y) |z|} \|A_0\|,
\]
whenever \( \delta a(x, y) \mid z \mid < 1 \). Fix this constant \( \delta \). Since \( A_0 = \left( \frac{2n}{\gamma}, 0 \right) \), this entails
\[
|\Phi(z)| \leq \left| \frac{2n}{\gamma} + Q_1(z) \right| + |Q_2(z)| \leq \frac{1}{1 - \delta a(x, y) \mid z \mid} \| A_0 \| + \frac{\delta a(x, y) \mid z \mid}{1 - \delta a(x, y) \mid z \mid} \| A_0 \|.
\]

Whence
\[
|\Phi(z)| \leq \frac{4n/|\gamma|}{1 - \delta a(x, y) \mid z \mid},
\]
whenever \( \delta a(x, y) \mid z \mid < 1 \). From Theorem 4.1 we infer that
\[
|E_m(x, y)| \leq \frac{2}{|(1 + \gamma)_m|} \frac{1}{(1 - \rho \delta a(x, y))^2} \rho^m,
\]
for all \( \rho > 0 \) such that \( \rho \delta a(x, y) < 1 \). Put \( \rho = \frac{1}{\delta a(x, y)} \), to get that
\[
|E_m(x, y)| \leq \frac{2}{|(1 + \gamma)_m|} \frac{(\delta a(x, y))^m}{t_m(1 - t)^2},
\]
for all \( t \in [0, 1] \). Taking the infimum over all \( t \in [0, 1] \), we obtain
\[
|E_m(x, y)| \leq \frac{e^2 \delta a(x, y)}{m!} \frac{(m + 2)^2}{(1 + \gamma)_m},
\]
for all \( m \in \mathbb{Z}^+ \). Now, assume that \( \text{Re}(\gamma) > -\nu \) for some \( \nu \in \mathbb{Z}^+ \) fix a positive integer \( m \). If \( \nu = 0 \), then it is clear that \( |(1 + \gamma)_m| \geq m! \). Suppose that \( \nu \geq 1 \), then we get
\[
|(1 + \gamma)_m| \geq (m - \nu)! \prod_{j=1}^{\nu} |\gamma + j|,
\]
for all \( m \geq \nu + 1 \), and then for both cases, from (5.2) we may write
\[
|E_m(x, y)| \leq C_1 \frac{(m + 2)^{2+\nu}}{m!} (\delta a(x, y))^m \leq C_2 \frac{m^{2+\nu}}{m!} (\delta a(x, y))^m
\]
for all \( m \geq \nu + 1 \), for some positive constants \( C_1, C_2 \) depending only on \( n, k \) and \( \nu \). To complete the proof it suffices to choose a positive constants \( C_3 \) such that
\[
\frac{e^2 \delta a(x, y)}{2} \frac{(m + 2)^2}{|(1 + \gamma)_m|} \leq C_3 \frac{m^{2+\nu}}{m!}
\]
for all \( m = 1, \ldots, \nu + 1 \).

\[\square\]

**Corollary 5.1.** Assume that \( \text{Re}(\gamma) > -\nu \) where \( \nu \) is a positive integer. Then there exists a constant \( \delta \geq 1 \) and a positive constant \( C := C(n, k, \nu) \) such that
\[
|E_k(x, y)| \leq C(|\delta a(x, y) + 1|)^{\nu+2} e^{\delta a(x, y)},
\]
for all \( x \in \mathbb{R}^2 \) and \( y \in \mathbb{C}^2 \).

**Proof.** Using the fact that \( E_k(x, y) = \sum_{m=0}^{\infty} E_m(x, y) \) the result of Theorem 5.2, we see that there exists \( \delta \geq 1 \) and a positive constant \( C := C(n, k, \nu) \) such that
\[
|E_k(x, y)| \leq C \sum_{m=0}^{\infty} \frac{(m + 1)^{\nu+2}}{m!} (\delta a(x, y))^m,
\]
for all \( x \in \mathbb{R}^2, y \in \mathbb{C}^2 \). Noting that
\[
(m + 1)^{\nu+2} \leq (m + \nu + 2)(m + \nu + 1) \times \cdots \times (m + 1),
\]
for all $m \in \mathbb{Z}^+$, one gets for some positive constant $C$,

$$|E_k(x, y)| \leq C \left| \frac{\partial^{\nu+2}}{\partial z^{\nu+2}} (z^{\nu+2} e^{z}) \right|_{z=\delta a(x, y)},$$

hence for some constant $C$, that changes from line to line, we have

$$|E_k(x, y)| \leq C ((\delta a(x, y))^{\nu+2} + \cdots + \delta a(x, y) + 1) e^{\delta a(x, y)},$$

for all $x \in \mathbb{R}^2$, $y \in \mathbb{C}^2$, from which the proof follows. 

\[\square\]

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