Noether Symmetries of Some Homogeneous Universe Models in Curvature Corrected Scalar-Tensor Gravity

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Abstract

We explore Noether gauge symmetries of FRW and Bianchi I universe models for perfect fluid in scalar-tensor gravity with extra term $R^{-1}$ as curvature correction. Noether symmetry approach can be used to fix the form of coupling function $\omega(\phi)$ and the field potential $V(\phi)$. It is shown that for both models, the Noether symmetries, the gauge function as well as the conserved quantity, i.e., the integral of motion exist for the respective point like Lagrangians. We determine the form of coupling function as well as the field potential in each case. Finally, we investigate solutions through scaling or dilatational symmetries for Bianchi I universe model without curvature correction and discuss its cosmological implications.

Keywords: Homogeneous universe; Noether symmetry; Scalar-tensor gravity.

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1 Introduction

The existence of dark energy (DE) and its role on the expansion history of the universe has become a center of interest for the researchers. It is a mysterious type of energy having negative pressure that is believed to be a basic reason for the rapid expanding behavior of the universe [1, 2]. For the description of its cryptical nature, there are mainly two kinds of efforts: modified matter approach like quintessence, Chaplygin gas, phantom, quintom, tachyon etc. [3] and the modified gravity (due to some extra degrees of freedom) including $f(R)$ gravity, scalar-tensor theory, $f(T)$ gravity etc. [4]. Although the modified matter approach has many novel features but this is not fully free from ambiguities. The modified gravity approach is considered to be more appropriate in this respect.

The dominant presence of DE in the universe leads to numerous theoretical problems like cosmic coincidence and fine-tuning problems [5]. Scalar-tensor theories are proved to be important efforts in the investigation of DE problem as well as various cosmic issues like the early and late time behavior of the universe and inflation [6]. The phenomenon of cosmic acceleration can be better described by introducing some sub-dominant terms of geometric origin like inverse of the Ricci scalar in the Einstein-Hilbert action. The simplest action with such modification is defined as [7]

$$S = \frac{1}{8\pi G} \int \sqrt{-g} \left( R - \frac{\mu_0^4}{R} \right) d^4 x,$$

where $R$ is the Ricci scalar, $G$ is the gravitational constant and $\mu_0$ is an arbitrary non-zero constant. In order to be consistent with observations and physical constraints, the action of scalar-tensor theories, in particular, Brans-Dicke (BD) theory can be modified in the following form [8]

$$S = \int \left[ \phi (R - \frac{\mu_0^4}{R}) + \frac{\omega(\phi)}{\phi} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) + L_m \right] \sqrt{-g} d^4 x,$$

where $\phi$ is the scalar field, $\omega(\phi)$ is the BD coupling function, $V(\phi)$ is the field potential, $L_m$ is the matter part of the Lagrangian and $\nabla_\mu$ indicates the covariant derivative.

In the cosmological contexts, there are two types of Noether symmetry techniques available in literature [9]. Symmetries which are obtained by setting the Lie derivative of the Lagrangian to zero are called Noether symmetries. The second technique is related with the more general symmetries
known as Noether gauge symmetries (containing the Noether symmetries as a subcase) which involve non-zero gauge function. Noether symmetries have many significant applications in cosmology and theoretical physics. In particular, the existence of Noether symmetries leads to a specific form of coupling function and the field potential in scalar-tensor theories.

Physically, symmetries lead to the existence of conserved quantities while on mathematical grounds, these reduce dynamics of the system due to the presence of cyclic variables [10]. Using Noether symmetry technique, the homogeneous universe models like FRW and Bianchi models have been discussed in $f(R)$ and scalar-tensor gravity [11]. Motavali and Golshani [12] explored the form of coupling function and the field potential for FRW universe model using Noether symmetries. Camci and Kucukakca [13] evaluated Noether symmetries for Bianchi I, III and Kantowski-Sachs spacetimes and discussed some field potentials. In recent papers [14], we have explored approximate Lie and Noether symmetries of some black holes and colliding plane waves in the framework of GR.

Kucukakca and Camci [15] have obtained the function $f(R)$ and the scale factor using Noether symmetry approach in Palatini $f(R)$ theory. Capozziello et al. [16] have discussed non-static spherically symmetric solutions in $f(R)$ gravity via Noether symmetry analysis. Shamir et al. [17] have investigated Noether symmetries and the respective conserved quantities for FRW and general static spherically symmetric spacetimes in $f(R)$ gravity. Jamil et al. [18] have discussed Noether gauge symmetries and the respective conserved quantities with different forms of potential for Bianchi I (BI) universe model in generalized Saez-Ballester scalar-tensor gravity. Kucukakca et al. [19] have explored BI universe model through Noether symmetry analysis with degeneracy condition of the Lagrangian and concluded that their results are consistent with the observations. Motavali et al. [8] calculated the Noether symmetries of the Lagrangian with an extra curvature term for FRW universe model.

Consider the point transformations (invertible transformations of “generalized positions”) that depend only upon one infinitesimal parameter $\sigma$, i.e., $Q^i = Q^i(q^j, \sigma)$ which can generate one-parameter Lie group [11, 16, 18]. The vector field with unknowns $\alpha^i$ defined by

$$X = \alpha^i(q^j) \frac{\partial}{\partial q^i} + \left[ \frac{d}{d\lambda}(\alpha^i(q^j)) \right] \frac{\partial}{\partial q^i}$$

is said to be a Noether symmetry for the dynamics derived by the Lagrangian
if it leaves the Lagrangian invariant, that is, $L_X L = 0$. In this case, the Euler-Lagrange equations and the constant of motion can be written as

$$\frac{d}{d\lambda}(\frac{\partial L}{\partial \dot{q}^i}) - \frac{\partial L}{\partial q^i} = 0, \quad \vartheta = \alpha^i \frac{\partial L}{\partial q^i}. \quad (3)$$

Noether gauge symmetries are the generalization of Noether symmetries (as it is expected that they contain some extra symmetries). Consider a vector field $X$ as

$$X = \tau(t, q^i) \frac{\partial}{\partial t} + \eta^j(t, q^i) \frac{\partial}{\partial q^j}$$

and its first-order prolongation is defined as

$$X^{[1]} = X + (\eta^j_t + \eta^j_s \dot{q}^i - \tau^j_s \dot{q}^i - \tau^j \dot{q}^i \dot{q}^j) \frac{\partial}{\partial \dot{q}^j}.$$  

Here $\tau$ and $\eta^j$ are the unknown functions to be determined and $t$ is the affine parameter. The vector field $X$ is said to be Noether gauge point symmetry of the Lagrangian $L(t, q^i, \dot{q}^i)$, if there exists a function (known as gauge term) $G(t, q^i)$ such that the following condition is satisfied

$$X^{[1]} L + (D_t \tau)L = D_t G; \quad D_t = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i}. \quad (4)$$

Here $D_t$ is the total derivative operator.

There are two physical frames available in literature: Einstein and Jordan frames which are related with each other by a conformal transformation ($\tilde{g} = e^{2\Omega} g$). It is argued that both these frames are equivalent on mathematical as well as physical grounds in the classical gravity regime where the conformal mapping is well defined. The compatibility of the Noether symmetries and the conformal transformations have been discussed in literature [20]. It is proved that the Noether point symmetry if exists, it remains preserved under the conformal transformations.

In the present paper, we evaluate Noether gauge symmetries of the non-vacuum point like Lagrangian for FRW universe model and then extend to locally rotationally symmetric (LRS) BI universe model, the simplest generalization of FRW universe. The paper is designed in the following manner. In section 2, we evaluate Noether gauge symmetries for FRW universe model with correction term. Section 3 provides Noether as well as Noether gauge symmetries for BI universe model with correction term. In section 4, we discuss BI solutions using scaling or dilatation symmetries without correction term. Finally, we present an outlook in the last section.
2 Noether Gauge Symmetries for FRW Universe Model

For the sake of simplicity, we take $\phi = \varphi^2$ and $\mu_0^4 = -\mu$. Thus the action for scalar-tensor gravity with extra curvature term [2] can be written as

$$S = \int \left[ \varphi^2 (R + \frac{\mu}{R}) + 4\omega(\varphi) g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi - V(\varphi) + L_m \right] \sqrt{-g} d^4x. \quad (5)$$

The homogeneous, non-flat FRW universe model is given by

$$ds^2 = dt^2 - a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right], \quad (6)$$

where $a(t)$ is the scale factor and $k(= 0, \pm 1)$ is the curvature index. The matter part of the Lagrangian is described by the perfect fluid

$$T_{\mu\nu} = (\rho + P) u_\mu u_\nu - Pg_{\mu\nu}, \quad (7)$$

where $\rho$, $P$ and $u_\mu$ denote the energy density, pressure and four velocity, respectively. The equation of state (EoS) for perfect fluid is $P = \epsilon \rho$, where $\epsilon$ is the EoS parameter. The energy conservation leads to $\rho = \rho_0 a^{-3(1+\epsilon)}$ and hence the pressure becomes $P = \epsilon \rho_0 a^{-3(1+\epsilon)}$.

We are interested in the Noether gauge symmetries (non-zero gauge function) of FRW model with perfect fluid matter contents. For this purpose, the point like Lagrangian constructed by the partial integration of the action [5] is [8]

$$\mathcal{L} = 2a^3 \varphi^2 \mu q + 6(\mu q^2 - 1)(2a^2 \varphi \dot{a} \dot{\varphi} + \varphi^2 a \dot{a}^2) + 12\mu \varphi^2 a^2 q \dot{a} \dot{q} - 6ka\varphi^2(\mu q^2 - 1) + a^3(4\omega(\varphi) \dot{\varphi}^2 - V(\varphi)) + \rho_0 \epsilon a^{-3\epsilon}. \quad (8)$$

In this case, the configuration space is given by $(t, a, \varphi, q)$, consequently the Lagrangian is defined as $\mathcal{L} : TQ \to \mathbb{R}$, where $TQ = (t, a, \varphi, q, \dot{a}, \dot{\varphi}, \dot{q})$ is the respective tangent space and $\mathbb{R}$ is the set of real numbers. The first-order prolongation of the symmetry generator is given by

$$X^{[1]} = \tau(t, a, \varphi, q) \frac{\partial}{\partial t} + \alpha(t, a, \varphi, q) \frac{\partial}{\partial a} + \beta(t, a, \varphi, q) \frac{\partial}{\partial \varphi} + \gamma(t, a, \varphi, q) \frac{\partial}{\partial q} + \alpha_t(t, a, \varphi, q) \frac{\partial}{\partial \dot{a}} + \beta_t(t, a, \varphi, q) \frac{\partial}{\partial \dot{\varphi}} + \gamma_t(t, a, \varphi, q) \frac{\partial}{\partial \dot{q}},$$
where $\alpha$, $\beta$ and $\gamma$ are unknown functions to be determined. Moreover,

$$
\alpha_t = D_t\alpha - \dot{a}D_t\tau, \quad \beta_t = D_t\beta - \dot{q}D_t\tau, \quad \gamma_t = D_t\gamma - \ddot{q}D_t\tau,
$$

where

$$
D_t = \frac{\partial}{\partial t} + \dot{a} \frac{\partial}{\partial a} + \dot{\varphi} \frac{\partial}{\partial \varphi} + \ddot{q} \frac{\partial}{\partial q}.
$$

Substituting these values with Eq. (8) in Eq. (4), we get the following system of determining equations

$$
\begin{align*}
\tau_q &= 0, \quad \tau_a = 0, \quad \tau_\varphi = 0, \\
12(\mu a^2 - 1)\alpha^2 \varphi \alpha_t + 8a^3 \omega(\varphi)(t)\beta_t + \tau_\varphi(2a^3 \varphi^2 \mu q - a^3 V(\varphi)) \\
&- 6k a \varphi^2 (\mu q^2 - 1) = G_\varphi, \\
12(\mu q^2 - 1)\alpha^2 \varphi a t + 12(\mu q^2 - 1)\alpha^2 \varphi \beta t + 12 \mu q a^2 \varphi^2 \gamma t + (2a^3 \varphi^2 \mu q \\
&- a^3 V(\varphi) - 6k a \varphi^2 (\mu q^2 - 1))\tau_a = G_a, \\
12(\mu q^2 - 1)a^2 \varphi^2 \alpha t + (2a^3 \varphi^2 \mu q - a^3 V(\varphi) - 6k a \varphi^2 (\mu q^2 - 1))\tau_t = G_q,
\end{align*}
$$

$$
\begin{align*}
6a^2 \varphi^2 \mu q \alpha - 6k \varphi^2 (\mu q^2 - 1)\alpha - 3a^2 \alpha V(\varphi) + 4a^3 \mu q \beta \alpha - 12k a \beta \varphi (\mu q^2 \\
- 1) - a^3 \beta V'(\varphi) + 2a^3 \mu \varphi^2 \gamma - 12k a \mu \varphi^2 q a \gamma + (2a^3 \mu q \varphi^2 - 6k a \varphi^2 (\mu q^2 \\
- 1) - a^3 V(\varphi))\tau_t - 3\rho_0 \epsilon^2 a^{- (1 + 3\epsilon)} \alpha = G_t, \\
24a \alpha \varphi (\mu a q^2 - 1) + 12a \beta \varphi (\mu q^2 - 1) + 24a \varphi \mu q \gamma + 12(\mu q^2 - 1) a^2 \varphi a \alpha \\
+ 12(\mu q^2 - 1) a^2 \varphi a \varphi + 12a \varphi (\mu q^2 - 1) \beta \varphi + 12 \mu q a^2 \varphi^2 \gamma a + 8a^3 \omega(\varphi) \beta a \\
- 12(\mu q^2 - 1) a^2 \varphi \tau_t = 0,
\end{align*}
$$

$$
\begin{align*}
6(\mu q^2 - 1)\alpha \varphi^2 + 12a \beta \varphi (\mu q^2 - 1) + 12a \varphi \mu q \varphi^2 \gamma + 12(\mu q^2 - 1) a^2 \varphi^2 \alpha a \\
+ 12(\mu q^2 - 1) a^2 \varphi \beta a + 12 \mu q a^2 \varphi^2 \gamma a - 6a \varphi^2 (\mu q^2 - 1) \tau_t = 0, \\
24 \mu q a \varphi^2 \alpha + 24 \mu q a^2 \varphi^2 \beta + 12a^2 \varphi^2 \gamma + 12 \mu q a^2 \varphi^2 \alpha a + 12a^2 \varphi (\mu q^2 - 1) \alpha q \\
+ 12a^2 \varphi (\mu q^2 - 1) \beta q + 12 \mu q a^2 \varphi^2 \gamma q - 12 \mu q a^2 \varphi^2 \tau_t = 0, \\
12a^2 \omega(\varphi) \alpha + 4a^3 \beta \omega'(\varphi) + 12a^2 \varphi (\mu q^2 - 1) \alpha \varphi + 8a^3 \omega(\varphi) \beta \varphi \\
- 4a^3 \omega(\varphi) \tau_t = 0,
\end{align*}
$$

where $G = G(t, a, \varphi, q)$. 

6
This is a system of 11 partial differential equations (PDEs) which we solve simultaneously for the unknown functions \((\tau, \alpha, \beta, \gamma, G)\). The coupling function and the field potential both are also unknown and we specify their forms by the existence of Noether symmetries. Integration of Eq. (19) implies \(\alpha = \alpha_1(t, a, \varphi)\). Since the above system of PDEs is difficult to solve, therefore we take the ansatze for the functions \(\alpha_1\) and \(\beta\) as

\[
\alpha_1 = \alpha_0 t^{n_1} a^n \varphi^m, \quad \beta = \beta_0(q) t^{l_1} a^l \varphi^s,
\]

where \(\beta_0\) is an arbitrary function and \(\alpha_0, n, m, n_1, l, l_1, s\) are the parameters to be determined. Substituting these values in Eq. (18), it follows

\[
\beta_0(q) = -\frac{3}{4} \mu \left( \frac{m \alpha_0}{\omega_0} \right) q^2 + c_1, \quad \omega(\varphi) = \omega_0 \varphi^{m-s+1}, \quad n = l + 1, \quad n_1 = l_1,
\]

where \(c_1\) and \(\omega_0\) are constants. Equation (17) leads to

\[
\tau = c_2, \quad s = m + 1, \quad \omega_0 = 1 + \frac{4c_1}{3a_0}, \quad m = 1.
\]

Consequently, Eq. (20) takes the form

\[
\alpha_1 = \alpha_0 a^n \varphi t^{n_1}, \quad \beta = (c_1 - \frac{3\alpha_0 \mu}{4\omega_0} q^2) \varphi^2 a^{n_1} t^{n_1}.
\]

Equation (16) implies that

\[
\gamma(t, a, \varphi, q) = f(q) a^{n_1} t^{n_1} \varphi + \frac{g_1(t, a, \varphi)}{q},
\]

where \(f(q) = \frac{3\alpha_0}{2\omega_0} \left( \frac{\omega_0}{4} - \frac{\mu}{2} \right) - q \alpha_0 + \frac{3\mu \alpha_0 q^2}{8\omega_0} - q c_1 - \frac{\alpha_0 q^2}{2} + g_1\) and \(g_1\) is an integration function. Inserting these values in Eqs. (14) and (15), it follows that

\[
\gamma = f(q) a^{n_1} t^{n_1} \varphi + \frac{g_3(t)}{aq^2},
\]

where \(g_3\) is an integration function.

Moreover, the following constraints should be satisfied

\[
n f(q) = \frac{1}{2} - \mu q^2 \left[ \alpha_0 (1 + 2n) + 2n (c_1 - \frac{3\alpha_0 \mu}{4\omega_0} q^2) \right],
\]

\[
(\mu q^2 - 1)(3 + n) \alpha_0 + 3(c_1 - \frac{3\alpha_0 \mu}{4\omega_0} q^2) + \frac{2\omega_0}{3} (n - 1)(c_1 - \frac{3\alpha_0 \mu}{4\omega_0} q^2) + 3\mu q f = 0.
\]
Integration of Eq. (12) yields
\[ G(t, a, \varphi, q) = 12n_1\alpha_0\varphi^3t^{n_1-1}a^{n+2}(\frac{\mu q^3}{3} - q) + h_1(t, a, \varphi), \]
where \( h_1 \) is an integration function. Further, Eqs. (10) and (11) lead to
\[ G = 12n_1\alpha_0 q(\frac{\mu q^2}{3} - 1)a^{n+2}\varphi^3t^{n_1-1} + 6a^2\mu g_{3,t} + h_3(t) \]
with the constraints
\[ 12(\mu q^2 - 1)\alpha_0 n_1 + 8\omega_0(c_1 - \frac{3\alpha_0 \mu}{4\omega_0} q^2)n_1 - 36n_1\alpha_0 q(\frac{\mu q^2}{3} - 1) = 0, \tag{24} \]
\[ 12(\mu q^2 - 1)\alpha_0 n_1 + 12(\mu q^2 - 1)\alpha_0 n_1(c_1 - \frac{3\alpha_0 \mu}{4\omega_0} q^2) + 12\mu q f n_1 \]
\[ = 12n_1(n + 2)\alpha_0 q(\frac{\mu q^2}{3} - 1). \tag{25} \]
Finally, Eq. (13) yields
\[ 6\mu \alpha_0 a^{n+2}\varphi^3t^{n_1} - 6k(\mu q^2 - 1)\alpha_0 a^{n+2}\varphi^3t^{n_1} - 3V(\varphi)\alpha_0 a^{n+2}\varphi t^{n_1} + 4\mu q(c_1 - \frac{3\alpha_0 \mu}{4\omega_0} q^2)a^{n+2}\varphi^3t^{n_1} - a^{n+2}\varphi^2t^{n_1}(c_1 - \frac{3\alpha_0 \mu}{4\omega_0} q^2)dV - 2\varphi^3 a^{n+2}f \mu t^{n_1} + \frac{2a^2 \mu g_3(t)}{q} - 12kq \mu a^{n+1}t^{n_1} f \varphi^3 \]
\[ -12k \mu g_3(t) - 3\rho_0 \epsilon^2 \alpha_0 \varphi t^{n_1} a^{-(1+3\epsilon)+n} = 12n_1(n_1 - 1)\alpha_0 q(\frac{\mu q^2}{3} - 1) \]
a^{n+2}\varphi^3t^{n_1-2} + 6a^2\mu g_{3,tt} + h_{3,t}. \]
This equation will be satisfied if \( n_1 = 1 \) with the following constraints
\[ -12k \mu g_3(t) + 2a^2 \mu g_3(t) = 6a^2 \mu g_{3,tt} + h_3,t, \tag{26} \]
\[ -\frac{3\alpha_0 V(\varphi)}{\varphi^2} - \frac{dV}{\varphi} d\varphi = 0, \tag{27} \]
\[ 6\alpha_0 \mu - 6k(\mu q^2 - 1)\alpha_0 a^{-2} + 4\rho_3 \mu - 12k(\mu q^2 - 1)a^{-2} \]
\[ -12k q f a^{-2} \mu - \frac{3\rho_0 \epsilon^2 a^{-3(1+\epsilon)+\alpha_0}}{\varphi^2} = 0. \tag{28} \]

8
Integration of Eqs. (26) and (27) yields

\[
g_3(t) = c_3 \exp\left(\sqrt{\frac{\mu}{3}} t\right) + c_4 \exp\left(-\sqrt{\frac{\mu}{3}} t\right),
\]

\[
h_3(t) = -12k\mu \sqrt{\frac{\mu}{3}} \left[ c_3 \exp\left(\sqrt{\frac{\mu}{3}} t\right) + c_4 \exp\left(-\sqrt{\frac{\mu}{3}} t\right) \right] + c_5,
\]

\[
V(\varphi) = c_6 \varphi^{3\alpha_0},
\]

where \( c_3, c_4, c_5 \) and \( c_6 \) are constants of integration.

Now we can discuss Eq. (28) for the following two cases, i.e., when \( \epsilon = 0 \) or \( \alpha_0 = 0 \). If \( \epsilon = 0 \), then pressure becomes zero and matter distribution will be the dust dominated fluid. Moreover, Eq. (28) leads to the following constraints

\[
6q\alpha_0\mu + 4q\beta_0\mu + 2f\mu = 0,
\]

\[
-6k(\mu q^2 - 1)\alpha_0 + 12k(\mu q^2 - 1) - 12kfq\mu = 0. \tag{29}
\]

In this case, the solution turns out to be

\[
\alpha = \alpha_1 = \alpha_0 a^n \varphi t, \quad \beta = (c_1 - \frac{3\alpha_0 \mu q^2}{4\omega_0}) \varphi^2 a^{n-1} t, \quad \tau = c_2, \quad V = c_6 \varphi^{3\alpha_0},
\]

\[
\gamma = \frac{3\alpha_0}{2\omega_0} \left( \frac{\mu q^3}{4} - \frac{q}{2} \right) - q\alpha_0 + \frac{3\mu\alpha_0 q^3}{8\omega_0} - qc_1 - \frac{\alpha_0 n q}{2} a^{n-1} t \varphi
\]

\[
+ \frac{c_3 \exp\left(\sqrt{\frac{\mu}{3}} t\right) + c_4 \exp\left(-\sqrt{\frac{\mu}{3}} t\right)}{aq\varphi^2},
\]

\[
G = 12q\alpha_0 \left( \frac{\mu q^2}{3} - 1 \right) a^{n+2} \varphi^3 + \frac{6a^2\mu^{3/2}}{\sqrt{3}} \left[ c_3 \exp\left(\sqrt{\frac{\mu}{3}} t\right) - c_4 \exp\left(-\sqrt{\frac{\mu}{3}} t\right) \right]
\]

\[
-12k\mu \sqrt{\frac{\mu}{3}} \left[ c_3 \exp\left(\sqrt{\frac{\mu}{3}} t\right) + c_4 \exp\left(-\sqrt{\frac{\mu}{3}} t\right) \right] + c_5.
\]

Consequently, the symmetry generator is

\[
X = c_2 \frac{\partial}{\partial t} + \alpha_0 a^n \varphi t \frac{\partial}{\partial a} + (c_1 - \frac{3\alpha_0 \mu q^2}{4\omega_0} q^2) \varphi^2 a^{n-1} t \frac{\partial}{\partial \varphi} + (f(q)a^{n-1} t \varphi
\]

\[
+ \frac{c_3 \exp\left(\sqrt{\frac{\mu}{3}} t\right) + c_4 \exp\left(-\sqrt{\frac{\mu}{3}} t\right)}{aq\varphi^2} \frac{\partial}{\partial q}.
\]
The corresponding conserved quantity becomes

\[ I = c_2(2a^3\varphi^2\mu q + 6(\mu q^2 - 1)(2a^2\varphi\dot{a}\varphi + \varphi^2\dot{a}\dot{a}) + 12\mu\varphi^2a^2q\dot{a}\dot{q} - 6ka\varphi^2(\mu q^2 - 1) + a^3(4(1 + \frac{4c_1}{3\alpha_0})\varphi^2 - c_6\varphi^{3\alpha_0}) + (\alpha_0 a^n\varphi t - c_2\dot{a})(6(\mu q^2 - 1)(2a^2\varphi\dot{\phi} + 2a\varphi^2\dot{a}^2) + 12\mu qa^2\varphi^2\dot{q}) + ((c_1 - \frac{3\alpha_0\mu}{4\omega_0} q^2)\varphi^2 a^{n-1} - c_2\dot{a})(12(\mu q^2 - 1)a^2
\varphi\dot{a} + 8a^3\varphi\omega_0) + (f(q)a^{n-1}t\varphi + \frac{1}{aq\varphi^2}(c_3 \exp(\sqrt{\frac{\mu}{3}}t) + c_4 \exp(-\sqrt{\frac{\mu}{3}}t)) - c_2\dot{q})(12\mu\varphi^2a^2q\dot{a}) - 12qa_0(\frac{\mu q^2}{3} - 1)a^{n+2}\varphi^3 - \frac{6a^2\mu^{3/2}}{3}[c_3 \exp(\sqrt{\frac{\mu}{3}}t) - c_4 \exp(-\sqrt{\frac{\mu}{3}}t)] - c_5.\]

For the flat universe, the constraint (23) restricts \( q \) to be constant say \( q_0 \) as follows

\[ q^2 = q_0^2 = \frac{(3 + n)\alpha_0 + c_1 - 2/3(n - 1)\omega_0c_1}{(1 - 3n)\mu}. \]

Equations (22), (24), (25) and (29) are four constraints that can be used to restrict the parameters \( \mu, \alpha_0, c_1 \) and \( n \). Notice that \( q = 1/R \), where \( R \) is the Ricci scalar, which turns out to be constant, i.e., \( R = 1/q_0 \). This is in agreement with Noether theorem according to which, when a cyclic variable is identified, Noether symmetry appears and, in the present case, the combination \( R = 1/q_0 \) is constant which corresponds to constant scalar curvature solution. Its physical meaning is that the Noether symmetry generator exists for the solutions with constant curvature like de Sitter solutions.

It is found that Noether symmetry generator exists for \( \epsilon = 0 \) and the respective gauge function turns out to be a dynamical quantity. Moreover, the potential is a dynamical quantity given by a power law form while the BD coupling is a constant quantity. The behavior of the field potential depends upon the constant \( \alpha_0 \) (for \( \alpha_0 > 0 \), the field potential behaves as a positive power law while \( \alpha_0 < 0 \) leads to inverse power law potential). Such field potentials have been used to discuss many cosmological issues in literature [22]. The existence of Noether gauge symmetries yields the conserved quantity, i.e., Noether charge exists which can be used to reduce the complexity of the Euler-Lagrange equations.
For the second case ($\alpha_0 = 0$), the field potential turns out to be constant, i.e., $V_0 = c_6$ and $\alpha = 0$, also, $\omega$ is diverging, i.e., $\omega \rightarrow \infty$, hence we neglect this choice.

3 Noether Gauge Symmetries for LRS Bianchi I Universe Model

Here, we calculate the Noether and Noether gauge symmetries of the LRS BI spacetime. The LRS BI universe with scale factors $A$ and $B$ is defined by the line element [21]

$$ds^2 = dt^2 - A^2(t)dx^2 - B^2(t)(dy^2 + dz^2).$$

The dynamical constraint evaluated in terms of the Ricci scalar follows

$$R - 2[\frac{\ddot{A}}{A} + 2\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + 2\frac{\dot{A}\dot{B}}{AB}] = 0.$$

Using the Lagrange multiplier approach, the action can be written as

$$S = \int \left[ \varphi^2(R + \frac{\mu}{R}) + 4\dot{\varphi}^2\omega(\varphi) - V(\varphi) + \chi(R - 2\frac{\dot{A}}{A} - 4\frac{\dot{B}}{B} - 2\frac{\dot{B}^2}{B^2}) - 4\frac{\dot{A}\dot{B}}{AB} \right] + L_m] (AB^2) d^4x.$$

Here $\chi$ is the Lagrange multiplier parameter. Varying this action with respect to the Ricci scalar, the parameter $\chi$ turns out to be

$$\chi = \varphi^2(\mu R^{-2} - 1).$$

The matter part of the Lagrangian is described by the perfect fluid (as defined in the previous section). We consider the matter dominated universe for which the matter part of the Lagrangian is given by $\mathcal{L}_m = \rho_0(AB^2)^{-1}$. By substituting the respective values in the action (31), it follows

$$S = \int \left[ 2\mu AB^2\varphi^2 + 4AB^2\omega\dot{\varphi}^2 - AB^2V - \varphi^2(\mu q^2 - 1)(2\dot{A}B^2 + 4B\dot{A}\dot{B}) \right] - \varphi^2(\mu q^2 - 1)(2\dot{A}B^2 + 4AB\dot{B}) + \rho_0 dt.$$
The energy function related with the Lagrangian of the Lagrangian as

\[ \mathcal{L} = 2\mu qA B^2 \varphi^2 + 4A B^2 \omega(\varphi) \varphi^2 - A B^2 V + 2 \varphi^2 (\mu q^2 - 1) A \dot{B}^2 + 4B^2 \varphi(\mu q^2 - 1) \dot{\varphi} \dot{B} + 8A B \varphi(\mu q^2 - 1) \dot{\varphi} \dot{B} + 8 A B \mu \varphi^2 \dot{q} \dot{q} + \rho_0. \] (32)

The Euler-Lagrange equations [3] for this Lagrangian become

\[ 8(\mu q^2 - 1) B \varphi \dot{B} \dot{\varphi} + 4(\mu q^2 - 1) B^2 \dot{\varphi}^2 + 8 \mu q B^2 \dot{\varphi} \dot{q} + 4(\mu q^2 - 1) \varphi B^2 \dot{\varphi} + 8 \mu q B^2 \varphi \dot{q} + 4 \mu q B^2 \dot{q}^2 + 4 \mu q \varphi B^2 \dot{q} + 2(\mu q^2 - 1) \varphi^2 \dot{B}^2 + 4(\mu q^2 - 1) B \varphi^2 \dot{B} - 2 \mu q B^2 \varphi^2 - 4B^2 \omega(\varphi) \varphi^2 + B^2 V(\varphi) = 0, \] (33)

\[ 4(\mu q^2 - 1) \varphi A \dot{B} + 4(\mu q^2 - 1) \varphi^2 B \dot{A} + 8(\mu q^2 - 1) \varphi A B \dot{\varphi} + 8 \mu A B \varphi^2 q \dot{q} + 8 \mu A q \varphi^2 \dot{q} B + 32 \mu q A B \dot{q} \dot{\varphi} + [8(\mu q^2 - 1) A B - 8A B \omega(\varphi)] \dot{\varphi}^2 - 4 q \mu A B \varphi^2 + 2 A B V(\varphi) = 0, \] (34)

\[ 4A B^2 \varphi^2 \frac{d\omega}{d\varphi} + 4A \varphi(\mu q^2 - 1) \dot{B}^2 + A B^2 \frac{dV}{d\varphi} + 8A B \varphi(\mu q^2 - 1) \dot{B} + 4B^2 \varphi \times(\mu q^2 - 1) \dot{A} + 8A B \omega(\varphi) \dot{\varphi} + 8B \varphi(\mu q^2 - 1) \dot{\varphi} B + 8B^2 \dot{\varphi} \omega(\varphi) + 16A B \dot{\varphi} \dot{B} \omega(\varphi) - 4q \mu A B^2 \varphi = 0. \] (35)

These equations exhibit dynamics of the spatial components of the Einstein field equations as well as scalar wave equation for BI universe model. Another constraint on the variables can be determined from the Ricci scalar given by

\[ \frac{1}{q} = 2 \left[ \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{B}^2}{B^2} + 2 \frac{A \ddot{B}}{A B} \right]. \]

The energy function related with the Lagrangian \( \mathcal{L} \) is defined as [8]

\[ E_{\mathcal{L}} = \dot{A} \frac{\partial \mathcal{L}}{\partial A} + \dot{B} \frac{\partial \mathcal{L}}{\partial B} + \dot{q} \frac{\partial \mathcal{L}}{\partial \dot{q}} + \dot{\varphi} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} - \mathcal{L} \]

Inserting the respective values in this energy function and after simplification, it can be written as

\[ \frac{(\mu q^2 - 1) \varphi^2}{6} \frac{\dot{B}^2}{B} + \omega(\varphi) \varphi^2 + \frac{(\mu q^2 - 1) \varphi \dot{\varphi}}{3} \frac{\dot{A}}{A} + \frac{2(\mu q^2 - 1) \varphi \dot{\varphi}}{3} \frac{\dot{B}}{B} + \frac{(\mu q^2 - 1) \varphi^2 \dot{q} \dot{q}}{3} \frac{A \ddot{B}}{A B} + 2 \mu \varphi^2 q \dot{q} \dot{B} + \frac{2 \mu \varphi^2 q \dot{q} \dot{B}}{3} \frac{A \ddot{B}}{A B} - \frac{q \varphi^2 \mu}{6} + \frac{V(\varphi)}{12} - \frac{\rho_0}{12} = 0. \] (36)
In this configuration, the total derivative operator $D_{\tau, \alpha, \beta, \gamma}$ defined by

$$X = \tau \frac{\partial}{\partial t} + \alpha \frac{\partial}{\partial A} + \beta \frac{\partial}{\partial B} + \gamma \frac{\partial}{\partial \varphi} + \delta \frac{\partial}{\partial q} + \alpha \frac{\partial}{\partial A} + \beta \frac{\partial}{\partial B} + \gamma \frac{\partial}{\partial \varphi} + \delta \frac{\partial}{\partial q},$$

where $\tau, \alpha, \beta, \gamma$ and $\delta$ are unknown functions to be determined. Moreover,

$$\alpha_t = D_t \alpha - \dot{A} D_t \tau, \quad \beta_t = D_t \beta - \dot{B} D_t \tau, \quad \gamma_t = D_t \gamma - \dot{\varphi} D_t \tau, \quad \delta_t = D_t \delta - \dot{q} D_t \tau.$$ 

In this configuration, the total derivative operator $D_t$ is

$$D_t = \frac{\partial}{\partial t} + \dot{A} \frac{\partial}{\partial A} + \dot{B} \frac{\partial}{\partial B} + \varphi \frac{\partial}{\partial \varphi} + q \frac{\partial}{\partial q}.$$ 

Using all these values in Eq. (41), the system of determining equations will become

$$\tau_q = 0, \quad \tau_\varphi = 0, \quad \tau_A = 0, \quad \tau_B = 0, \quad (37)$$

$$B^2\alpha(2\mu\varphi^2 - V(\varphi)) + AB\beta(4\mu\varphi^2 - 2V(\varphi)) + AB^2\gamma(4\mu\varphi - V'(\varphi))$$

$$+ 2AB^2\varphi^2\delta - \epsilon^2 \rho_0 (AB^2)^{(1+t)}(B^2\alpha + 2AB\beta) = G_t, \quad (38)$$

$$4B^2 q\varphi^2 \alpha + 8ABq\mu\varphi^2 \beta = G_q, \quad (39)$$

$$4\varphi^2(\mu q^2 - 1)B\alpha + 4(\mu q^2 - 1)\varphi^2\beta + 8AB\varphi(\mu q^2 - 1)\gamma$$

$$+ 8AB\varphi^2 \mu q \delta = G_B, \quad (40)$$

$$4\varphi^2(\mu q^2 - 1)B\beta + 4\mu B^2\varphi^2 q \delta + 4(\mu q^2 - 1)\varphi B^2\gamma = G_A, \quad (41)$$

$$8AB\varphi(\mu q^2 - 1) + 8AB\varphi(\varphi)\gamma = 4(\mu q^2 - 1)\varphi B^2\alpha = G_\varphi, \quad (42)$$

$$4\varphi^2(\mu q^2 - 1)B\beta + 4\mu B^2\varphi^2 q \delta + 4(\mu q^2 - 1)\varphi B^2\gamma = 0, \quad (43)$$

$$4B^2\varphi^2 \alpha q + 8AB\varphi^2 \mu q \beta = 0, \quad (44)$$

$$4B^2\alpha \omega(\varphi) + 8AB\omega(\varphi)\beta + 4AB^2\omega'(\varphi)\gamma + 4(\mu q^2 - 1)\varphi B^2\alpha \varphi$$

$$+ 8AB(\mu q^2 - 1)\varphi \beta + 8AB^2\omega(\varphi)\gamma - 8AB^2\omega(\varphi)\gamma = 0, \quad (45)$$

$$2\alpha\varphi^2(\mu q^2 - 1) + 4A\varphi(\mu q^2 - 1) + 4q\mu\varphi^2 A \delta + 4\varphi^2(\mu q^2 - 1)B \alpha_B$$
\begin{equation}
+4\varphi^2(\mu q^2 - 1)A\beta_B - 4\varphi^2(\mu q^2 - 1)A\tau_t + 8ABq\mu\varphi^2\delta_B
+8AB(\mu q^2 - 1)\varphi\gamma_B = 0,
\end{equation}
\begin{equation}
4\varphi^2(\mu q^2 - 1)\beta + 8\varphi(\mu q^2 - 1)B\gamma + 8\mu q B\varphi^2\delta + 4\varphi^2(\mu q^2 - 1)B\alpha_A
-4\varphi^2(\mu q^2 - 1)B\tau_t + 4\varphi^2(\mu q^2 - 1)A\beta_A + 4\varphi^2(\mu q^2 - 1)B\beta_B - 4\varphi^2
\times(\mu q^2 - 1)B\tau_t + 8AB\varphi^2\mu\delta_A + 4B^2q\mu\varphi^2\delta_B + 8AB(\mu q^2 - 1)\varphi\gamma_A
+4(\mu q^2 - 1)B^2\varphi\gamma_B = 0,
\end{equation}
\begin{equation}
8B\mu\varphi^2\omega(\varphi)\gamma_A + 4B^2(\mu q^2 - 1)\varphi\gamma_A = 0,
\end{equation}
\begin{equation}
8B\gamma_B = 0 \text{ and can be found by the system of determining equations (37)-(52) with } G = 0 \text{ and } \tau = 0. \text{ In this case, all the functions } \alpha, \beta, \gamma \text{ and } \delta \text{ are independent of time. Integration of Eq.}(37) \text{ yields } \tau = \tau(t). \text{ For the sake of simplicity, we take the ansatz for unknowns } \alpha, \beta \text{ and } \gamma \text{ as}
\end{equation}
\begin{equation}
\alpha = A^a B^b \varphi^c q_0(q), \quad \beta = A^f B^g \varphi^h q_1(q), \quad \gamma = A^m B^n \varphi^p q_2(q).
\end{equation}

Here \(a, b, c, f, g, h, m, n\) and \(p\) are parameters to be determined, while \(q_0, q_1\) and \(q_2\) are unknown functions of variable \(q\). We would like to find the functions \(\delta, V\) and \(\omega\) by requiring the existence of Noether symmetries. From Eq.(44), it follows that
\begin{equation}
a = f + 1, \quad g = b + 1, \quad c = h, \quad q_0 = -2q_1,
\end{equation}
and hence

\[ \alpha = -2A^{f+1}B^{-1}\varphi^h q_1(q), \quad \beta = A^f B^g \varphi^h q_1(q), \quad \gamma = A^m B^n \varphi^p q_2(q). \quad (54) \]

Equations (45) and (52) imply that \( \omega(\varphi) = \frac{c_2}{\varphi^p} \) and \( q_2 = c_1 \), respectively, where \( c_1 \) and \( c_2 \) are integration constants. Equation (43) yields

\[ \delta = \frac{(1 - \mu q^2)}{\mu q} [A^f B^g \varphi^h q_1 + c_1 A^m B^n \varphi^{p-1}] + h_1(B, q, \varphi), \]

where \( h_1 \) is an integration function. Further, Eq. (46) leads to \( h_1 = \frac{h_2(q, \varphi)}{\sqrt{B}} \) and \( g = 2/3 \), hence

\[ \alpha = -2q_1 A^{f+1} B^{-1/3} \varphi^h, \quad \beta = q_1 A^f B^{2/3} \varphi^h, \quad \gamma = A^m B^n \varphi^p c_1, \]

\[ \delta = \frac{1 - \mu q^2}{\mu q} [A^f B^{-1/3} \varphi^h q_1 + c_1 A^m B^n \varphi^{p-1}] + \frac{h_2(q, \varphi)}{\sqrt{B}}. \]

Equation (47) implies that \( h_2 = 0 \) and \( f = -2/3 \). Moreover, Eq. (49) leads to \( m = -2/3, \ n = -1/3, \ h = -(1 + p) \) and \( q_1 = \frac{2c_1 c_2}{3(1 - \mu q^2)} \); thus

\[ \alpha = -2(\frac{2c_1 c_2}{3(1 - \mu q^2)}) A^{1/3} B^{-1/3} \varphi^{-(1+p)}, \quad \beta = \frac{2c_1 c_2}{3(1 - \mu q^2)} A^{-2/3} B^{2/3} \varphi^{-(1+p)}, \]

\[ \gamma = A^{-2/3} B^{-1/3} c_1 \varphi^p, \quad \delta = \frac{(1 - \mu q^2)}{\mu q} [A^{-2/3} B^{-1/3} \varphi^{-(1+p)} - \frac{2c_1 c_2}{3(1 - \mu q^2)} + c_1 A^{-2/3} B^{-1/3} \varphi^{p-1}]. \quad (55) \]

Inserting these values in Eq. (48), we obtain either \( c_1 c_2 = 0 \) or \( \mu = 0 \). Since \( \mu \neq 0 \), so \( c_1 c_2 = 0 \). When we take \( c_1 \neq 0, \ c_2 = 0 \), it follows that

\[ \alpha = 0, \quad \beta = 0, \quad \omega = 0, \quad \gamma = c_1 \varphi^p A^{-2/3} B^{-1/3}, \]

\[ \delta = \frac{1 - \mu q^2}{\mu q} c_1 A^{-2/3} B^{-1/3} \varphi^{p-1}. \quad (56) \]

Equation (48) leads to \( V(\varphi) = \frac{c_1 \varphi^2}{2} + c_4 \) and \( q = q_0 = \frac{c_3 \pm \sqrt{c_3^2 - 8\mu}}{4\mu} \), where \( c_3 \) and \( c_4 \) are integration constants. In this case, the symmetry generator follows

\[ X_1 = \varphi^p A^{-2/3} B^{-1/3} \frac{\partial}{\partial \varphi} + \frac{1 - \mu q^2}{\mu q} A^{-2/3} B^{-1/3} \varphi^{p-1} \frac{\partial}{\partial q}. \quad (57) \]
which yields only one symmetry and the respective constant of motion is zero, i.e., $I_1 = 0$. This shows that for Noether symmetries of the point like Lagrangian exists but there is no non-trivial conserved quantity.

For Noether gauge symmetries, we consider the full symmetry generator $(\tau \neq 0, \ G \neq 0)$ in which the unknown functions are dependent on time. Proceeding in the similar way, Eqs.(37) and (43)-(52) lead to

$$\tau = c_3, \ \alpha = 0, \ \beta = 0, \ \gamma = c_1 t^l \varphi^p, \ \delta = \frac{1 - \mu q^2}{\mu q} c_1 t^l \varphi^{p-1}, \ \omega = \frac{c_4 \varphi^{2p}}{\varphi_{2p}}.$$  (58)

Equations (38)-(42) yield $l = 0$ and $q = q_0$ with

$$V(\varphi) = \frac{1 + 3q_0^2 \mu}{2q_0} \varphi^2, \quad G = c_5.$$  

Thus there exist two symmetry generators given by

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \varphi^p \frac{\partial}{\partial \varphi} + \left(\frac{1 - \mu q^2}{\mu q}\right) \varphi^{p-1} \frac{\partial}{\partial q}.$$  

The constant of motion, i.e., the integral of motion can be written as

$$I = \tau \mathcal{L} + (\alpha - \dot{A} \tau) \frac{\partial \mathcal{L}}{\partial A} + (\beta - \dot{B} \tau) \frac{\partial \mathcal{L}}{\partial B} + (\gamma - \dot{\varphi} \tau) \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} + (\delta - \dot{q} \tau) \frac{\partial \mathcal{L}}{\partial \dot{q}} - G.$$  

In this case, these are given by

$$I_1 = 8AB^2c_4\dot{\varphi}\varphi^{-p} - \frac{c_5}{2},$$  

$$I_2 = 2q_0 \mu AB^2 \varphi^2 - 4AB^2 \dot{\varphi}^2 c_4 \varphi^{-2p} - AB^2 \left(\frac{1 + 3q_0^2 \mu}{2q_0}\right) \varphi^2 - 2(\mu q_0^2 - 1) \varphi^2 \dot{A} \dot{B}^2 - 4(\mu q_0^2 - 1) (B^2 \dot{\varphi} \dot{A} \dot{\varphi} + B \varphi^2 \dot{A} \dot{B} + 2AB \varphi \dot{B} \varphi) + \rho_0 - \frac{c_5}{2}.$$  

It can be concluded that for the point-like Lagrangian of BI universe model, there is only one Noether symmetry generator while two Noether gauge symmetry generators exist. The existence of Noether symmetries allows zero BD coupling function and the quadratic field potential. Such field potentials have widely been used in literature [22] to discuss many cosmological problems in the context of scalar tensor gravity. Since, $\omega = 0$, therefore these symmetries may correspond to the symmetries of pointlike Lagrangian of BI universe in Palatini $f(R)$ gravity [24].
It is found that the Noether charge is zero in the case of Noether symmetries. However, the existence of Noether gauge symmetries leads to dynamical BD coupling parameter (in the form of inverse power law) with quadratic potential. It is observed that the behavior of BD coupling function depends upon the parameter $p$, for $p > 0$, the BD coupling becomes divergent at $\varphi = 0$ while for $p < 0$, it turns out to be zero there. Moreover, in this case, the gauge function turns out to be constant and the Noether charge, i.e., the conserved quantities exist. The EoS parameter for this configuration is given by

$$
\omega_\varphi = \frac{\omega_{x\varphi} + 2\omega_{y\varphi}}{3} = \frac{1}{3}[3\mu q_0 \varphi^2 - 3/2V(\varphi) + 6\omega(\varphi)\varphi^2 - 6\varphi^2(\mu q_0^2 - 1)
- 6\varphi\varphi(\mu q_0^2 - 1) - 4(\mu q_0^2 - 1)\varphi(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B})][\mu q_0^2 - \frac{V(\varphi)}{2} - 6\omega(\varphi)\varphi^2
- 2(\mu q_0^2 - 1)\varphi(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B})]^{-1},
$$

where $V$ and $\omega$ are given in previous cases (found by the Noether symmetry analysis). Following [18], we have tried to plot this expression using Maple software but due to highly non-linear terms present in the field equations with $A$, $B$ and $\varphi$ as unknowns, it is not possible to have the plot of this expression (basically, Maple could not convert the expressions into explicit first-order system of DEs).

## 4 Bianchi I Solutions Using Scaling Symmetries

In this section, we discuss BI solutions by taking $\mu = 0$ in the Lagrangian with constant BD parameter. In canonical form, the action can be written as

$$
S = \int \sqrt{-g}[\frac{1}{8\omega} \varphi^2 R - \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + V_0 \varphi^2 + \mathcal{L}_m] d^4 x. \quad (59)
$$

Here $\omega$ is a constant BD parameter and the field potential is taken to be $V = V_0 \varphi^2$. Moreover, the matter distribution is taken as the perfect fluid.
The corresponding field equations are

\[
\frac{\varphi^2}{4\omega} \left(2\frac{\dot{A}}{AB} + \frac{\dot{B}}{B}\right) + \frac{\dot{\varphi}^2}{2} + V_0\varphi^2 + \frac{1}{2\omega} \left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right)\varphi \dot{\varphi} = \rho, \tag{60}
\]

\[
-\frac{\varphi^2}{4\omega} \left(2\frac{\dot{B}}{B} + \frac{\dot{B}^2}{AB}\right) - \frac{1}{\omega} \frac{\dot{B}}{B} \varphi - \frac{1}{2\omega} \ddot{\varphi} \varphi - V_0\varphi^2 + (1/2 - 1/2\omega)\dot{\varphi}^2 = P, \tag{61}
\]

\[
-\frac{\dot{\varphi}^2}{4\omega} \left(\frac{\dot{B}}{B} + \frac{\dot{A}}{A} + \frac{\dot{A} \dot{B}}{AB}\right) - \frac{1}{2\omega} \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)\dot{\varphi} - \frac{1}{2\omega} \ddot{\varphi} - V_0\varphi^2 + \frac{1}{2 - 2\omega} \dot{\varphi}^2 = P, \tag{62}
\]

\[
\ddot{\varphi} + \left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right) \dot{\varphi} + \left[\frac{\dot{V}_0}{2\omega} \left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B} + \frac{\dot{B}^2}{B^2} + 2\frac{\dot{A} \dot{B}}{AB}\right)\right] \varphi = 0. \tag{63}
\]

The equation of continuity is

\[
\dot{\rho} + \left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right)(\rho + P) = 0. \tag{64}
\]

Due to complexity of the system, we use the physical relationship between the scale factors, i.e., \(A = B^m; \ m \neq 1\) \[21\]. This condition is originated by the assumption that the ratio of shear scalar to expansion scalar is constant. Consider the power law form for the density \(\rho = \rho_0(AB^2)^\varepsilon/3\), and hence pressure \(P = -\frac{\varepsilon+3}{3}\rho\). Here \(\varepsilon\) is any parameter that acts as an equation of state parameter and its different values can classify different phases of the universe, e.g., \(\varepsilon = -3\) implies matter dominated era and \(\varepsilon = 0\) yields dark energy dominated universe. We take dependent variables \(H_2 = H_2(B)\) and \(F = F(B_2)\) with \(B\) as independent variable and introduce the following notations in the system of equations (60)-(63)

\[
F = \frac{\dot{\varphi}}{\varphi}, \quad H_2 = \frac{\dot{B}}{B}, \quad H_1 = m\frac{\dot{B}}{B}, \quad \dot{H}_2 = BH_2H_2', \quad \dot{H}_1 = mBH_2H_2'.
\]

\[
\frac{\dot{B}}{B} = BH_2H_2' + H_2'^2, \quad \ddot{\varphi} = BH_2F' + F^2,
\]

where prime indicates derivative with respect to \(B\). This leads to

\[
H_2^2 + (F^2 + 2V_0) \frac{2\omega}{1 + 2m} + \frac{2(m + 2)}{2m + 1} H_2 F = \frac{4\omega \rho_0 B^{\varepsilon(m+2)/3}}{(1 + 2m)\varphi^2}, \tag{65}
\]
\[
\frac{2BH_2}{3}(F' + H_2') + H_2^2 + \frac{4H_2F}{3} + \frac{2(2 - \omega)F^2}{3} + \frac{4V_0\omega}{3} = -\frac{4\omega P}{3}\varphi^2 \quad (66)
\]

\[
H_2^2 + \frac{1}{(m^2 + m + 1)}[BH_2((m + 1)H_2' + 2F') + 2(1 + m)H_2F + 4V_0\omega + 2(2 - \omega)F^2] = -\frac{4P\omega}{(m^2 + m + 1)\varphi^2}, \tag{67}
\]

\[
BH_2\left[\frac{\omega}{m + 2}F' + \frac{H_2'}{2}\right] + \frac{H_2^2}{2} \left(\frac{m^2 + 2m + 3}{m + 2}\right) + \omega H_2 F + \frac{\omega}{m + 2}(F^2 + 2V_0) = 0. \tag{68}
\]

Also, the evolution of energy density is given by

\[
\dot{\rho} = \varepsilon \rho_0 \frac{(m + 2)H_2\rho}{3}.
\]

In the above system of five field equations, only three equations are independent. We shall use Eqs. (64), (65) and (68) as independent equations. By taking the time derivative of Eq. (65), and after some manipulation, it becomes

\[
[H_2^2 + \frac{m + 2}{1 + 2m}FH_2]H_2' + \frac{F'}{1 + 2m}[2\omega FH_2 + (m + 2)H_2^2] = \frac{\varepsilon(m + 2)H_2^3}{6B} + \frac{F^2\omega}{B^2(1 + 2m)}(\varepsilon\omega - 6) \frac{H_2^2(m + 2)}{3B(1 + 2m)} + \frac{2V_0\omega}{B(1 + 2m)}\left[\varepsilon(m + 2)H_2 - \frac{2F}{3}\right]. \tag{69}
\]

We can write equations for the unknowns \(H_2'\) and \(F'\) after some manipulation from Eqs. (65) and (69) as follows

\[
BH_2 \frac{dH_2}{dB}\left[\frac{2(\omega - 1) + m(4\omega - 1)}{2(m + 2)(1 + 2m)}\right] = H_2^2\left[\frac{\varepsilon\omega}{6} + \frac{m^2 + 2m + 3}{2(1 + 2m)}\right] + F^2\omega \times \left[\frac{(\varepsilon + 6)\omega - 3}{3(1 + 2m)}\right] + \omega FH_2\left[\frac{\varepsilon(m + 2)^2 + 3(m - 1)}{3(1 + 2m)}\right] + \frac{2V_0\omega}{3(1 + 2m)}(\varepsilon\omega + 3), \tag{70}
\]

\[
BH_2\left[\frac{m + 2}{2(1 + 2m)} + \frac{\omega}{m + 2}\right] \frac{dF}{dB} = H_2^2\left[\frac{\varepsilon(m + 2)^2 + 6(m^2 + 2m + 3)}{12(m + 2)}\right]
\]

19
These form a closed system of equations (two equations involving two unknowns). We adopt the analysis of classical Lie groups \[25\] to find solution of Eqs.\((70)\) and \((71)\). Since these equations are quite similar to the equations exhibiting scaling or dilatational symmetries, therefore we assume a vector field

\[
X = B\alpha \frac{\partial}{\partial B} + \beta F \frac{\partial}{\partial F} + \gamma H_2 \frac{\partial}{\partial H_2}
\]

generated by a scaling group of mappings given by

\[
\tilde{B} = \lambda^\alpha B, \quad \tilde{A} = \lambda^{m\alpha} A, \quad \tilde{F} = \lambda^\beta F, \quad \tilde{H}_2 = \lambda^\gamma H_2, \quad \tilde{H}_1 = m\lambda^\gamma H_2.
\]

The invariance of Eqs.\((70)\) and \((71)\) under the above transformation provides two different cases:

- The scalar field is massive, i.e., \(V_0 \neq 0\) which implies that \(\beta = \gamma = 0\) and \(\alpha = 1\). This allows the form of generator given by \(X_1 = B\frac{\partial}{\partial B}\).

- The scalar field is massless, i.e., \(V_0 = 0\) which yields two choices of the parameters and consequently two symmetry generators exist. (i) \(\beta = \gamma = 0\) with \(\alpha = 1\) implies \(X_1 = B\frac{\partial}{\partial B}\) (ii) \(\beta = \gamma = 1\) and \(\alpha = 0\), leading to \(X_2 = H_2 \frac{\partial}{\partial H_2} + F\frac{\partial}{\partial F}\).

In the first case, there is only one symmetry generator with basis of invariants \(\{H_2, F\}\). This symmetry ensures that the solution is in the form of invariants given by \(F = F(H_2)\). In order to find the solution, the number of known symmetries should be equal to the order of DE. Since Eqs.\((70)\) and \((71)\) lead to \(\frac{dF}{dH_2} = K(H_2, F)\), i.e., first-order differential with no more known symmetries, so the integration in quadratures will not be possible. We construct a solution by imposing the invariance of \(F\) and \(H_2\), i.e., \(\frac{dH_2}{dB} = 0\) and \(\frac{dF}{dB} = 0\). Equations \((70)\) and \((71)\) become quadratic equations for \(H_2\) and \(F\). The roots of these equations are quite lengthy. To get insights, we choose \(m = 2\), \(U_0 = 2\) and \(\varepsilon = -3\) (matter dominated phase) which yields
four roots as

\[
H_2 = \pm [\omega (11264 - 15936\omega + 28912\omega^2 - 4940\omega^3 - 9600\omega^4 - 1600\omega^5) \\
\pm 20\omega^2 (3748096 - 11585024\omega + 16537312\omega^2 - 11402336\omega^3 + 3126193 \\
\times \omega^4 - 557336\omega^5 + 197920\omega^6 + 92800\omega^7 + 6400\omega^8)^{1/2} [2000\omega^5 \\
+ 1400\omega^4 - 83035\omega^3 + 111598\omega^2 - 6512\omega - 15488]^{-1}].
\]

Likewise, there exist four roots for \(F\) given by

\[
F = \pm [\omega (25696 - 53008\omega + 29310\omega^2 - 25600\omega^3 + 4000\omega^4) \pm 10\omega (3748096 \\
- 24873728\omega + 64116448\omega^2 - 8571569\omega^3 + 65210905\omega^4 - 2517980\omega^5 \\
+ 8538400\omega^5 - 1904000\omega^6 + 160000\omega^7)^{1/2} [2000\omega^5 + 1400\omega^4 - 83035 \\
\times \omega^3 + 111598\omega^2 - 6512\omega - 15488]^{-1}]^{-1/2}.
\]

Consequently, the scale factor \((H_2 = \text{constant})\) and scalar field \((F = \text{constant})\) turn out to be \(B = B_0 \exp(H_2 t)\) and \(\varphi = \varphi_0 \exp(F_i t)\), respectively. Here \(H_2\) and \(F_i\) are the roots given above, while \(B_0\) and \(\varphi_0\) are present values of the scale factor and the scalar field, respectively. The present values can be restricted by using Eq.(65) as follows

\[
\varphi_0 = \sqrt{\frac{4\omega \rho_0 B_0^{-4}}{(5H_2^2 + 2\omega F_i^2 + 8\omega + 8H_2 F_i)}}.
\]

The plots for the scale factor and density function \((\rho(t) = \rho_0 B_0^4 e^{4\omega H_2 t}/3)\) are shown in Figure 1. Since the scalar field is massive, therefore \(\omega\) can take any value satisfying \(\omega > -3/2\) to avoid ghost instabilities [24]. Figure 1(a) indicates that the scale factor increases for \((-,-)\) and \((+,+)\) roots, while decreases to zero for \((-,+)\) and \((+,-)\) roots. Figure 1(b) shows that the behavior of density is exactly opposite to that of the scale factor. This means that only for \((-,-)\) and \((+,+)\) roots, the constructed model shows expanding behavior with decreasing energy density which is consistent with the recent observations. The scalar field exhibits a similar behavior as shown in Figure 2(a).

For \((-,-)\) and \((+,+)\) roots, the scalar field is expanding, while for \((+,+)\) and \((-,-)\), the scalar field is contracting with the passage of time. This indicates that the scalar field plays a dominant role in the later phase of cosmic expansion. The volume of the universe is given by \(V(t) = B^4 = \)
Figure 1: Plots (a) and (b) show the scale factor and energy density versus time $t$, respectively. Here red and green correspond to $(+, +)$ and $(-, -)$ roots with $\omega = 1.5$, respectively, while yellow and purple lines indicate $(+, -)$ and $(-, +)$ roots with $\omega = 1.2$, respectively.

Figure 2: Plots (a) and (b) represent the volume of the universe and scalar field versus time $t$, respectively. Here red and green correspond to $(+, +)$ and $(-, -)$ roots with $\omega = 1.5$, respectively, while yellow and purple lines indicate $(+, +)$ and $(-, -)$ roots with $\omega = 1.2$, respectively.

Figure 3: Plots (a) and (b) show the EoS parameters for scalar field $\omega_\phi$, versus time. Here red and green lines correspond to $(-, +)$ and $(+, -)$ roots with $\omega = 200$, respectively.
$B_0^4 \exp(4H_2t)$ which shows that the universe expands exponentially for different $H_{2i}$ roots as shown in Figure 2(b) (indicating infinite volume in future). Moreover, the average scalar factor, $a = B_0^{4/3} \exp(H_{2i}/3)$, leads to negative value of the deceleration parameter, which shows the rapid expansion in the universe consistent with the observations. Figure 3 represents the EoS parameter for scalar field $\omega_\phi = p_\phi/\rho_\phi$ versus time. These indicate that the universe model lies in the phantom phase for all values of time which is in agreement with the recent rapid expanding behavior of the universe.

In the second case, we take $V_0 = 0$ (massless scalar field). Since there exist two commuting symmetries $X_1$ and $X_2$, so we have first-order DE $rac{dE}{dH_2} = K(F, H_2)$ with one known symmetry. Consequently, the integration in quadrature will be possible and then Eqs.(70) and (71) yield

$$-\frac{dH_2}{H_2} = [\frac{\varepsilon \omega}{6} + \frac{m^2 + 2m + 3}{2(1 + 2m)} + C^2 \omega (\frac{\varepsilon + 6)\omega - 3}{3(1 + 2m)} + \omega G (\frac{\varepsilon + 6}{3(1 + 2m)} - 1 + 2\omega + \frac{m^2 + 2m + 3}{1 + 2m})]^{-1} dG,$$

where $G = \frac{F}{H_2}$. The solution of this equation yields directional Hubble parameter in terms of new parameter $G$. Since the scalar field is massless, so BD coupling parameter should satisfy the range $\omega \geq 40,000$ as suggested by the solar system experiments [24]. For the present era, its solution can be written as

$$H_2 = H_{2,0}[(G - 0.004916)^{0.000024}(G + 19512.79)^{0.99995}(G - 0.005085)^{0.0000256}],$$

where we have taken $m = 0.5$, $w = 40,000$ and $H_{2,0}$ indicates present value
of the parameter $H_2$. From Eq.(70), the scale factor becomes

$$
- \frac{dB_2}{B_2} = \frac{2(\omega - 1) + m(4\omega - 1)}{2(1 + 2m)(m + 2)} \{ \frac{2(\omega - 1) + m(4\omega - 1)}{(m + 2)^2 - 2\omega(1 + 2m)} \} \{ \frac{\varepsilon(m + 2)^2}{12(m + 2)} 
+ \frac{6(m^2 + 2m + 3)}{12(m + 2)} \} + G^2 \left( \frac{(m + 2)(\varepsilon \omega - 6)}{6(1 + 2m)} + \frac{\omega}{m + 2} + \frac{\omega(m + 2)}{1 + 2m} \right) 
+ G \{ \frac{2(\omega - 1) + m(4\omega - 1)}{(m + 2)^2 - 2\omega(1 + 2m)} \} \{ \frac{\varepsilon(m + 2)^2}{3(1 + 2m)} - 1 + 2\omega + \frac{m^2 + 2m + 3}{1 + 2m} \} 
- \left( \frac{\varepsilon \omega}{6} + \frac{m^2 + 2m + 3}{2(1 + 2m)} \right) - G^2 \omega \left( \frac{\varepsilon + 6}{3(1 + 2m)} \right) - \omega G \frac{\varepsilon(m + 2)^2}{3(1 + 2m)} 
+ \frac{3(m - 1)}{3(1 + 2m)} \}^{-1}dG.
$$

In the present era, for the choice $m = 0.5$ and $\omega = 40,000$, the solution can be written as

$$
B = B_0[(G - 0.004916)^{0.000024}(G + 19512.79)^{0.99995}(G - 0.005085)^{0.0000256} - 15999.8].
$$

Further, the scalar field can be determined by the relationship $\frac{d\varphi}{\varphi} = G \frac{da}{a}$, which follows from $F = \dot{\varphi} = GH$. The corresponding scalar field takes the form

$$
\varphi = \varphi_0 \exp(-15999.8G)[(0.00508 - G)^{1.3008 \times 10^{-7}}(G + 0.004916)^{-1.2008 \times 10^{-7}} 
\times (G + 19512.8)^{-1.9511.8} - 15999.8].
$$

The energy density is given by

$$
\rho = \rho_0 B_0^{-2.5}[(G - 0.004916)^{0.000024}(G + 19512.79)^{0.99995}(G 
- 0.005085)^{0.0000256}]^{2.5 \times 15999.8}.
$$

The symbols with 0 subscript indicate the present values. The time related with the solution can be calculated by the expression $t - t_0 = \int \frac{da}{aH}$. For particular choice of parameters, it becomes

$$
t - t_0 = \int [0.800207(G + 5.62517 \times 10^{-6})(G + 0.975609)](\ln[(G + 0.0035)^{0.000012}(G + 26655.8)^{0.99997}(G - 0.0036)^{0.000012}])(G 
- 0.00508)(G + 0.004916)(G + 19512.8)^{-1}]dG.
$$
By inverting this expression, the parameter $G$ can be determined in terms of time and hence the scale factor and the scalar field. The above solutions are parametric solutions in new variable $G$, i.e., $a = a(G)$ and $\varphi = \varphi(G)$.

The above expression can be evaluated numerically and then by the obtained set of data points, we can interpolate the function $G(t)$. By adopting this procedure, we interpolate the function $G(t)$ using polynomial interpolation and is given by

$$G(t) = 63578.70834t^4 - 63719.54167t^3 + 22535.2479t^2 - 859.1271t + 0.0059,$$

where we have used the initial condition $G(0) = 0.0059$. Using this value of $G$, all the expressions like energy density, scale factors, scalar field and Hubble parameter can be discussed versus time. It can be observed that the obtained model is not free from singularities as the scalar field and scale factor become singular for some particular values of $G$. Moreover, as $G \to 0$, all these quantities remain finite while as $G \to \infty$, only the scalar field and the scale factor turn out to be zero. In this case, the deceleration parameter is given by

$$q = -1 - \frac{3}{m + 2} \left( \frac{\dot{H}_2}{H_2^2} \right),$$

where $m = 0.5$, while $H_2$ and $G$ are given by Eqs. (72) and (73), respectively. Figure 4 shows that the deceleration parameter remains negative for all values of time which yields the accelerated expansion of the universe model and is well-consistent with the recent observations.
5 Summary

The modified theories of gravity with action involving positive or negative powers of curvature as an extra term can lead to a better description of phenomenon of initial cosmic inflation and the late-time cosmic acceleration. The main objective of this paper is to evaluate the Noether and Noether gauge symmetries for some homogeneous universe models in the framework of non-vacuum scalar-tensor gravity with inverse curvature correction term, i.e., $R^{-1}$. For this purpose, we have applied the Noether gauge symmetry analysis to non-flat FRW universe model. Furthermore, we have discussed the Noether and Noether gauge symmetries for BI universe model. In both cases, the matter part of Lagrangian has been taken as perfect fluid. We have constructed the field potential and the coupling function by requiring the existence of Noether symmetries.

In the case of FRW universe model, we have a system of 11 PDEs for Noether gauge symmetries of the constructed point like Lagrangian. In literature [8], the Noether symmetries of the same Lagrangian in vacuum has been discussed, where the model with dust matter seem to be more physical, but the Noether symmetries cannot always exist. We have extended this work by exploring more general symmetries, i.e., Noether gauge symmetries by introducing perfect fluid matter part in the Lagrangian. We have found that the Noether symmetry generator exists with non-zero gauge function in matter dominated phase. It is seen that the gauge term turns out to be a dynamical quantity and the integral of motion associated with the dynamics of Lagrangian exist. We have also specified the form of BD coupling function and the field potential. In this case, the BD coupling function turns out to be a constant quantity, while the field potential is given by the power law form.

Next, it is shown that the Noether as well as Noether gauge symmetries exist for the flat BI universe model with perfect fluid using point like Lagrangian with curvature corrected term. The existence of Noether symmetry generator allows zero coupling function and quadratic potential with zero integral of motion. The Noether gauge symmetry generators yield the constant gauge function with quadratic potential and variable BD parameter, $\omega = c_4/\varphi^{2p}$. The behavior of BD coupling parameter is dependent on the parameter $p$. We have also determined the respective conserved quantities in this case.

Finally, we have evaluated the BI solutions using scaling or dilatational
symmetries. Since it is difficult to find the BI cosmological solutions in the curvature corrected configuration, so we take $\mu = 0$ in the action, i.e., zero curvature correction term and constant BD coupling parameter. For this purpose, two cases have been taken into account. In the first case, it is seen that both the scale factor and the scalar field evolve exponentially yielding deceleration parameter $q = -1$ which is compatible with the observations and inflationary scenario. Furthermore, the EoS parameter for scalar field turns out to be negative $\omega_{\phi} < -1$ for $(-, +)$ and $(+, -)$ roots only that confirms the accelerating phase of the universe model. The graphs of scale factor and energy density have also been given. In the second case, there exist two symmetry generators and consequently, the integration in quadrature is possible. By introducing a new parameter $G = F/H^2$, the forms of scale factor, scalar field, directional Hubble parameters and energy density have been calculated. It is observed that the obtained solution is parametric in the variable $G$. The relation of new variable $G$ in terms of time has also been given. By solving the function $G$ numerically using polynomial interpolation, all the cosmological parameters can be discussed versus time. In this respect, the plot of the deceleration parameter versus time has been given which shows that the parameter takes negative values for all values of time. This is well-consistent with the current rapid expanding behavior of the universe. It would be interesting to discuss the cylindrically or plane symmetric models using Noether symmetry analysis in the framework of scalar-tensor gravity with curvature correction.

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