Differential equations and duality
in massless integrable field theories at zero temperature

P. Fendley\textsuperscript{1} and H. Saleur\textsuperscript{2}

\textsuperscript{1}Department of Physics
University of Virginia
Charlottesville, VA 22901

\textsuperscript{2}Department of Physics
University of Southern California
Los Angeles, CA 90089-0484

Functional relations play a key role in the study of integrable models. We argue in this paper that for massless field theories at zero temperature, these relations can in fact be interpreted as monodromy relations. Combined with a recently discovered duality, this gives a way to bypass the Bethe ansatz, and compute directly physical quantities as solutions of a linear differential equation, or as integrals over a hyperelliptic curve. We illustrate these ideas in details in the case of the $c = 1$ theory, and the associated boundary sine-Gordon model.
Integrable models are one of the most widely studied areas of mathematical physics. The fact that there are an infinite number of conserved charges commuting with the Hamiltonian means that in principle physical quantities can be computed exactly. In practice, this task is usually quite difficult.

Indeed, there are surprisingly few techniques at our disposal. One of the most successful ones is actually a related set of techniques going by the name of the Bethe ansatz. In situations where the Bethe ansatz is applicable, one makes some technical assumptions about the wave function, and then can derive a set of equations for the zeroes of the wavefunction. The number of such zeroes goes to infinity in the continuum limit, and one can then use various density functions and the integral equations they satisfy. The equations obtained are rarely solvable in closed form but a great deal of quantitative information can be gained by studying them in various limits or by numerical solution.

Since the Bethe ansatz is usually technically formidable and often intractable, it is quite desirable to develop an alternative formulation. One method being explored in recent years is the use of functional relations or “fusion” relations for physical quantities. The existence of such relations is almost obvious, and based on very general properties of the Yang-Baxter equation and quantum groups. The assumptions made generally involve various analyticity properties of physical quantities, and can often be verified using the other approaches. Quite frequently, it can be shown that the Bethe ansatz equations follow from the fusion relations, but the latter seem more powerful, more convenient, or deeper.

Another method that appears promising is the use of duality. It has long been known that some quantities in certain integrable boundary field theories (for instance, the current in the boundary sine-Gordon model \([2]\), as well as the magnetizations in the Kondo model \([3]\)) do satisfy a weak to strong coupling duality. Moreover, inspired by work in supersymmetric gauge theories, it was shown in \([4,5,6]\) that these quantities can be expressed, at \(T = 0\), in terms of integrals on hyperelliptic curves. It would be quite useful to understand the precise mathematical framework underlying these properties. It seems possible that when combined with some sort of analyticity and maybe some other type of information, they can be used to bypass the Bethe ansatz, and lead directly to the determination of physical quantities in a way analogous to the supersymmetric gauge theories, as pioneered in \([7]\).

In this paper, we present some further progress along these lines, restricting to the simplest case of the boundary sine-Gordon model, and the underlying \(c = 1\) conformal
field theory. At zero temperature, we show that it is possible to interpret the fusion
relations as monodromy relations for the current (and magnetizations). This, combined
with duality and an assumption about the number of singularities in the plane of complex
couplings, leads to a direct determination of physical quantities that indeed bypasses the
Bethe ansatz.

A key ingredient in the following is the current. Although it was introduced on purely
physical grounds in [2], this quantity turns out indeed to also be central to our alternative,
quite mathematical, approach: in a nutshell, we will see that the current is the unique
solution of a differential equation that is regular at the origin, while for instance densities
of physical excitations can be expressed as monodromies around the other singularity (1 or
∞). The current is so special because it is exactly self-dual. Intuitively, the property arises
because the current is measuring charge transport, so only operators with charge affect its
computation (in contrast with quantities like the free energy or the magnetizations), and
they are strongly constrained by integrability [8].

For convenience, we summarize here the notations to be used in the following:

\[
z = -\frac{(t - 1)^{t-1}}{t^t} u^{2(t-1)}, \quad x \propto z^{-\frac{1}{2}}, \quad u = e^{A+\Delta-\theta}.
\]

(1)

with \( \Delta \) defined so that \( z(\theta = A) = -1 \).

1. Analytic properties of the current

Since in the case discussed in this paper, we do have the Bethe ansatz solution avail-
able, it is easiest and most concrete to use it to first illustrate the monodromy properties
and the result. In the next section, we will show how all of these results follow without
recourse to the Bethe ansatz.

1.1. The model and the current

The field theory discussed in this paper is usually known as the boundary sine-Gordon
model in its massless limit. It consists of a 1 + 1-dimensional boson \( \phi(x, \tau) \) on the half line
\( x \geq 0 \): there are no interactions in the bulk and the action is the usual \( (t/4\pi) \int dx d\tau (\partial \mu \phi)^2 \),
but on the boundary there is an interaction \( \lambda \cos[\phi(0)] \). In addition, there is a voltage
\( V \) coupled to the \( U(1) \) charge. The dimensionless couplings in this problem are \( u \propto V \lambda^{-t/(t-1)} \) (the coefficient of proportionality is non-universal and of little interest here)
and \( t \) (in the notation of \([2]\) \( u = e^{A + \Delta - \theta B} \) and \( t = 1/g \)). Although physical quantities are continuous in \( t \), many of the formal properties of the system are much easier to understand at rational or integer values of \( t \). This is familiar from the Bethe ansatz study of the XXZ spin chain, which, with appropriate boundary conditions, is a lattice discretization of this model.

The interaction violates charge conservation, so one can define the “current” describing this charge violation. This situation is discussed at length in \([2]\), where integral Bethe ansatz equations are derived for this current. Here we work with the normalized quantity \( \mathcal{I}(u) = t I/V \). At \( T = 0 \), the integral equations become linear, and the Wiener-Hopf technique allows one to find an integral expression for the current. This integral expression yields explicit weak-coupling and strong-coupling series expansions. In terms of the coupling the expansion of the current for \( u \) large, real and positive (the UV) is

\[
\mathcal{I}(u) = 1 - \sum_{n=1}^{\infty} a_n \left( \frac{1}{t} \right) u^{-2n(t-1)/t},
\]

and for \( u \) small (the IR) is

\[
\mathcal{I}(u) = \sum_{n=1}^{\infty} a_n(t) u^{2n(t-1)},
\]

where

\[
a_n(t) = \frac{(-1)^{n+1} \sqrt{\pi}}{n!} \frac{\Gamma(nt + 1)}{2 \Gamma \left[ \frac{3}{2} + n(t - 1) \right]}.
\]

As noted in \([2]\), a duality \( t \to 1/t \) is already apparent in the above expressions. It was shown in \([3]\) that these expressions for the current can be reformulated in terms of an integral on the hyperelliptic curve \( y^2 = x + x^t - u^2 \):

\[
\mathcal{I}(u) = 1 - \frac{i}{4u} \int_C \frac{dx}{y}
\]

where the contour \( C \) starts at the origin, loops around the branch point on the real axis (when \( u \) is real), and returns to the origin. This result is continuous in \( t \), but already we see a difference between \( t \) rational or irrational: the curve is of finite genus when \( t \) is rational and infinite genus when it is not. This expression means that the current obeys a differential (Picard-Fuchs) equation, whose order depends on the genus of the curve.
1.2. Generalized hypergeometric functions

To discuss the extension of this current to complex couplings, and the associated analytical properties, it is convenient to consider further the expansion (1.2). For simplicity, we restrict to the simplest case when \( t \) is an integer. We set

\[
z = -e^{-2\Delta(t-1)}u^{2(t-1)},
\]

where

\[
\Delta = \frac{-1}{2(t-1)}[t \ln t - (t-1) \ln(t-1)].
\]

The expansion (1.2) defines a unique analytic function for \(|z| < 1\). Introducing the generalized hypergeometric function

\[
_{p}F_{q}\left[a_1, \ldots, a_p; \rho_1, \ldots, \rho_q; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(\rho_1)_n \cdots (\rho_q)_n} \frac{z^n}{n!},
\]

and using the duplication formulas for \( \Gamma \) functions, one finds the simple expression

\[
\mathcal{I} = 1 - \frac{1}{2(t-1)} \left[ t^{-1} + \frac{t-2}{t-1} \ln \frac{1+z^{1/2}}{1-z^{1/2}} \right].
\]

For \( t = 2 \) for instance, we have

\[
\mathcal{I} = 1 - F \left( \frac{1}{2}, 1; \frac{3}{2}; z \right)
= 1 - \frac{1}{2z^{1/2}} \ln \frac{1+z^{1/2}}{1-z^{1/2}}.
\]

Like for usual hypergeometric functions, the analytic structure of the current in terms of the \( z \)-variable is quite simple. It has two singular points at \( z = 1 \) and \( z = \infty \), and there is a cut running from 1 to \( \infty \). The monodromies around 1 and infinity are identical, and the Riemann surface of the function has \( t \) sheets (except for \( t = 2 \) where there are logarithmic terms, as is obvious from (1.8)).

We will write the current for \(|z| > 1\) as

\[
\mathcal{I} = 1 - \sum_{n=1}^{\infty} a_n \left( \frac{1}{t} \right) u^{-2n(t-1)/t} = 1 - \sum_{n=1}^{\infty} a_n \left( \frac{1}{t} \right) \left( e^{-i\pi} e^{2(t-1)\Delta} z \right)^{-n/t},
\]
where \( 0 \leq \arg z < 2\pi \). This expansion, which we obtained originally from the Bethe ansatz and then from duality considerations, can also be deduced from standard references on the theory of generalized hypergeometric functions (Meijer’s G functions): see [9] for more details. As an example, let us recall the well-known result for \( t = 2 \):

\[
1 - F\left(\frac{1}{2}, 1; \frac{3}{2}; z\right) = 1 - \Gamma(3/2)\Gamma(1/2) \left( e^{-i\pi z} \right)^{-1/2} F\left(0, \frac{1}{2}; \frac{1}{2}; z^{-1}\right) - \frac{\Gamma(3/2)\Gamma(-1/2)}{\Gamma^2(1/2)} \left( e^{-i\pi z} \right)^{-1} F\left(1/2, 1; \frac{3}{2}; z^{-1}\right),
\]

where in fact \( F\left(0, \frac{1}{2}; \frac{1}{2}; z^{-1}\right) = 1 \). For \( t = 3 \), elementary calculations give:

\[
1 - 3F_2\left[\frac{1}{2}, \frac{3}{4}, 1; \frac{3}{5}, \frac{1}{4}, z\right] = 1 - c_1 \left( e^{-i\pi z} \right)^{1/3} F\left(\frac{7}{12}, \frac{1}{12}; \frac{2}{3}; z^{-1}\right) - c_2 \left( e^{-i\pi z} \right)^{2/3} F\left(\frac{1}{12}, \frac{5}{12}; \frac{4}{3}, z^{-1}\right) - c_3 \left( e^{-i\pi z} \right)^{-1} 3F_2\left[\frac{5}{3}, 1, \frac{1}{4}, \frac{4}{4}, \frac{1}{3}, \frac{3}{3}, z^{-1}\right],
\]

where the numerical constants are

\[
c_1 = \frac{\Gamma(1/3)\Gamma(3/4)\Gamma(5/4)}{\Gamma(5/12)\Gamma(11/12)},
\]

\[
c_2 = \frac{\Gamma(-1/3)\Gamma(3/4)\Gamma(5/4)}{\Gamma(1/12)\Gamma(7/12)},
\]

\[
c_3 = \frac{\Gamma(-1/3)\Gamma(-2/3)\Gamma(3/4)\Gamma(5/4)}{\Gamma(-1/4)\Gamma(1/4)\Gamma(1/3)\Gamma(2/3)}.
\]

### 1.3. Differential equations

Using standard results from the theory of generalized hypergeometric functions, it follows that for \( t \) integer, the current solves the following linear homogeneous differential equation of order \( t \)

\[
\left[ z \frac{d}{dz} \prod_{k=0}^{t-2} \left( z \frac{d}{dz} + \frac{3 + 2k}{2(t-1)} - 1 \right) - z \prod_{k=1}^{t} \left( z \frac{d}{dz} + \frac{k}{t} \right) \right] (1 - I) = 0.
\]

This can be verified using either (1.7) or (1.9). For example, for \( t = 3 \):

\[
\left[ -\frac{2}{9} + \left( \frac{15}{16} - \frac{38}{9} z \right) \frac{d}{dz} + (3z - 5z^2) \frac{d^2}{dz^2} + z^2(1 - z) \frac{d^3}{dz^3} \right] (1 - I) = 0.
\]
However, the expression (1.4) of $I$ in terms of a hyperelliptic curve actually requires that $I$ obey a differential equation of order $t - 1$ (the Picard-Fuchs equation). This is easy to see: taking $b$ derivatives with respect to $u$ gives integrals of the form

$$\int_C dx \frac{x^a}{y^{2b+1}}$$

where $a$ is some integer. If $a \geq t - 1$ or greater, this can be reduced to a sum of integrals with $a < t - 1$ by integrating by parts. Explicitly,

$$\int_C x^a \frac{dx}{y^{2b+1}} = \frac{1}{t} \int_C dx \frac{x^{a-t+1} t^{t-1} + 1}{y^{2b+1}} - \frac{1}{t} \int_C dx \frac{x^{a-t+1}}{y^{2b+1}}$$

The constant term arises because the contour is not closed: the integration by parts results in a non-vanishing surface term when $a = t - 1$. By using this relation, one can express the order $t - 1$ derivative of $I$ as a linear combination (with $u$-dependent coefficients) of the lower derivatives, so therefore $I$ satisfies a differential equation of order $t - 1$. Because of the constant term, this differential equation can have an inhomogeneous piece.

Therefore (1.13) must be a total derivative. This can easily be checked to be true directly: the reason can be traced back to the fact that the second product in (1.13) runs up to $k = t$. Let us illustrate this for $t = 3$: (1.14) reduces to

$$\left[ \frac{8z}{9} + \frac{1}{4} + 4z(2z - 1) \frac{dz}{dz} + 4z^2(z - 1) \frac{d^2}{dz^2} \right] (1 - I) = A_3. \tag{1.15}$$

The value of the constant $A_t$ is determined by the normalization of $I$ (here $A_3 = 1/4$). The situation is similar for other values of $t$: the current is a particular solution of an inhomogeneous differential equation of order $t - 1$ which reads

$$\left[ \prod_{k=0}^{t-2} \left( \delta + \frac{3 + 2k}{2(t - 1)} - 1 \right) - \sum_{k=1}^{t} b_k \frac{d^{k-1}}{dz^{k-1}} z^k \right] (1 - I) = A_t, \tag{1.16}$$

where

$$b_k = \frac{(-1)^k}{k!} \prod_{j=1}^{t} \left( \frac{j}{t} - k - 1 \right) - \sum_{j=1}^{k-1} \frac{(-1)^j}{j!} b_{k-j}, \quad k = 1, \ldots, t - 1, \quad b_t = 1$$

One obvious question is the significance of the other $t - 1$ solutions of the differential equation (1.13). These solutions are also necessarily solutions of (1.16) with $A_t = 0$. We
will see in the next section that, at \(t\) integer, they coincide with the densities of states of the various species of quasiparticles.

Mathematically, it is convenient to express these other \(t-1\) solutions in terms of the functions

\[
I_k(z) = \mathcal{I} \left[ e^{-i\pi(t-k)z} \right] - \mathcal{I} \left[ e^{i\pi(t-k)z} \right].
\]

(1.17)

defined for \(k = 1 \ldots t-1\) and \(|z| > 1\). In other words, we take the difference of the current on two successive sheets. The functions \(I_k\) are defined for \(|z| < 1\) by first using (1.17) for \(|z| > 1\), and then continuing around the singularity. The series expansions for \(z\) large are

\[
I_k = \frac{\sqrt{\pi}}{it} \sum_{n=1}^{\infty} \frac{(-1)^n}{\Gamma(n)} \frac{\Gamma \left( \frac{n}{2} \right)}{\Gamma \left[ \frac{3}{2} + n \left( \frac{1}{t} - 1 \right) \right]} \sin \left( n\pi \frac{t-k}{t} \right) \left( e^{-i\pi} e^{2(t-1)\Delta z} \right)^{-n/t}. \tag{1.18}
\]

Here it is assumed that \(0 \leq \arg z < 2\pi\). These coefficients differ from those in the expansion of the current only by the piece \(\sin(n\pi(t-k)/t)\). Since the solutions of the differential equation for \(|z|\) large are of the form

\[
z^{k/t} (a_k(1/t) + a_{k+t}(1/t)z + \ldots)
\]

it follows immediately that a basis of solutions of the differential equation consists of the \(t\) quantities \(\mathcal{I}(z), \mathcal{I}_{t-2p}(z)\) and \(\mathcal{I}_{t-2p-1} (e^{i\pi}z) \equiv \mathcal{I}_{t-2p-1}'(z)\).

For the monodromies to be discussed in the next section, we will also need the IR expansion valid at small \(z\). This does not follow instantly from (1.17), because the relation (1.17) does not hold at small \(z\). These expansions must be determined by continuing the expansions in (1.18) around the singularity at \(z = 1\) (this is easiest to do by writing a contour integral whose residues give the terms in (1.18); this form arises naturally in the Wiener-Hopf analysis). One finds

\[
\mathcal{I}_k = \frac{i\pi t}{2(t-1)} m_k \left( e^{-i\pi z} \right)^{-1/(2t-2)} + \frac{i\sqrt{\pi}}{t-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \cos \left( \pi \left( n + \frac{1}{2} \right) \frac{t-k-1}{t-1} \right) \left( e^{-i\pi} e^{2(t-1)\Delta z} \right)^{2n+1} \frac{\Gamma \left[ \frac{(2n+1)}{2} \right]}{\Gamma(n+1) \Gamma \left( \frac{(2n+1)t}{2(t-1)} \right)}, \quad k = 1, \ldots, t-2
\]

(1.19)

where \(m_k = 2 \sin(k\pi/(2t-2))\), \(G_+(0)\) and \(G_+(i)\) are some numerical constants (see [2] for their exact values). The function \(\mathcal{I}_{t-1}\) is given by (1.19) for \(k = t-1\) with an additional factor of \(1/2\) multiplying the right-hand-side. The radius of convergence of these expansions is the same as for the current: they converge for \(|z| > 1\) and \(|z| < 1\) respectively.
1.4. *t* rational

The situation is rather similar for rational *t*, and depends mostly upon the value of the numerator *Q* in *t* = *Q*/*P*. Here, we shall simply give the differential equation satisfied by the current:

\[
\left[ \prod_{k=0}^{P-1} (P\delta - k) \prod_{l=0}^{Q-P-1} \left( \delta + \frac{3 + 2k}{2(Q-P)} - 1 \right) - z \prod_{k=1}^{Q} \left( \delta + \frac{k}{Q} \right) \right] (1 - I) = 0, \tag{1.20}
\]

where now

\[
z = (-1)^P \frac{Q^Q}{(Q-P)^{Q-P}} u^{2(Q-P)}.	ag{1.21}
\]

This is a homogeneous differential equation of order *Q*. The fact that the current can be expressed in terms of the curve \(y^2 = x^P + x^{Q} - u^2\) means that this equation can be reduced to an inhomogeneous equation of order \(Q - 1\), just as in the case *t* integer.

1.5. Monodromies

The differential equation (1.13) is analytic in \(z\), so when one continues a solution around a singularity, one must obtain a solution upon return to the starting point. However, the continuation of a given solution is not necessarily the same solution: it only need be some linear combination of all the solutions. This linear combination is called a monodromy. Studying the monodromies is useful for understanding the solutions; in fact, if one knows all the monodromies, then one can reconstruct the differential equation automatically. In this section, we construct explicitly the monodromies of the solutions from the small- and large-*z* expansions. Later, we will reverse the order of the logic: we will derive the monodromies from general arguments and therefore infer the solutions.

The monodromies around infinity give the behavior of the solutions when we move \(z \to e^{2i\pi} z\) for fixed \(|z| > 1\). Since there are *t* solutions for the differential equation, the Riemann surface for the current is *t* sheeted, with a cut running from 1 to \(\infty\). The monodromies around infinity for the the whole set of solutions of the differential equation can be easily written by using the UV expansion. We have

\[
\begin{align*}
I \left( e^{2i\pi} z \right) &= I - I'_{t-1} \\
I'_{t-1} \left( e^{2i\pi} z \right) &= -I_{t-2} + I'_{t-3} \\
I_{t-2p} \left( e^{2i\pi} z \right) &= I'_{t-2p+1} - I_{t-2p} + I'_{t-2p+1} \\
I'_{t-2p-1} \left( e^{2i\pi} z \right) &= I'_{t-2p+1} - I_{t-2p} + I'_{t-2p-1} - I_{t-2p-2} + I'_{t-2p-3} \\
\cdots
\end{align*}
\tag{1.22}
\]
The last “boundary” terms depend on whether $t$ is odd or even. For instance, when $t = 2$

$$I(e^{2i\pi z}) = I(z) - I_1'(z)$$

$$I_1'(e^{2i\pi z}) = -I_1'(z)$$

(1.23)

and when $t = 3$,

$$I(e^{2i\pi z}) = I(z) - I_2'(z)$$

$$I_2'(e^{2i\pi z}) = -I_1(z)$$

$$I_1(e^{2i\pi z}) = I_2'(z) - I_1(z)$$

(1.24)

The monodromies around the origin are also quickly worked out from the IR expansions. The current in particular is regular at $z = 0$, and we have

$$I(e^{2i\pi z}) = I$$

$$I_{t-1}'(e^{2i\pi z}) = I_{t-1}' - I_{t-2} + I_{t-3}'$$

$$I_{t-2}'(e^{2i\pi z}) = 2I_{t-1}' + I_{t-3}' - I_{t-2}$$

$$I_{t-2p}'(e^{2i\pi z}) = I_{t-2p+1}' - I_{t-2p} + I_{t-2p-1}'$$

$$I_{t-2p-1}'(e^{2i\pi z}) = I_{t-2p+1}' - I_{t-2p} + I_{t-2p-1}' - I_{t-2p-2} + I_{t-2p-3}'$$

(1.25)

The last boundary terms depend on whether $t$ is odd or even. For instance, when $t = 2$

$$I(e^{2i\pi z}) = I$$

$$I_1'(e^{2i\pi z}) = -I_1'$$

(1.26)

and when $t = 3$,

$$I(e^{2i\pi z}) = I$$

$$I_1(e^{2i\pi z}) = 2I_2 - I_1$$

$$I_2'(e^{2i\pi z}) = I_2' - I_1$$

(1.27)

Finally, the monodromies around $z = 1$ follow from the other two, because going around a circle at large $z$ is equivalent to going around $z = 0$ and $z = 1$. Since the monodromies take a solution to a linear combination of the other solutions, we can represent them conveniently by matrices $M_\infty$, $M_0$ and $M_1$. The conventions for orienting the monodromies are usually taken so that one has $M_\infty = M_0M_1$. For instance, one has for $t = 3$,

$$M_1 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$
One checks on this example the general property that the monodromies of the current around \( z = 1 \) and \( z = \infty \) are identical, since the current has a trivial monodromy around the origin.

We observe that one has to be careful in performing the continuation of the \( I'_{t-2p-1} \) quantities for \(|z| < 1\), since the expressions \((1.18),(1.19)\) hold only for \(0 \leq \arg z < 2\pi\). But this continuation is easily worked out by using the previous relations.

A striking feature of the relations \((1.22),(1.25)\) is that the monodromies for the densities do not involve the current itself. This is because (or conversely, requires that) they satisfy a homogeneous linear differential equation \((1.16)\) of order \( t - 1 \), one order less than the hypergeometric equation \((1.13)\) we started with. This order \( t - 1 \) differential equation is the same one that the current satisfies, except that it does not have an inhomogeneous term.

### 2. A determination of the current from duality

In this section we will show that the monodromies derived above from the explicit Bethe ansatz solution can be obtained directly from general arguments, at least when \( t \) is rational. Therefore, the explicit solutions can be found without recourse to the Bethe ansatz.

As was discussed at length in [1,10], the partition functions \( Z_j \) of the spin \( j/2 \) Kondo model satisfy, as a consequence of the Yang Baxter equation, some general fusion properties. There are various ways to write those. A particularly convenient one here is based on the quantities \( Y_k \equiv Z_{k+1}Z_{k-1} \); one has then, in standard notations

\[
Y_k \left( q^{1/2}x \right) Y_k \left( q^{-1/2}x \right) = [1 + Y_{k+1}(x)] [1 + Y_{k-1}(x)],
\]

where \( q = e^{i\pi/t} \), and \( x \propto \lambda \propto z^{-\frac{1}{t}} \) (which is \( \propto e^{(1-g)\theta_B} \) in the notation of [2]). For \( t \) an integer, these relations hold only for \( k = 1, \ldots, t-3 \). Their closure is a slightly more delicate matter, that was discussed in detail in section 4 of [10]. It requires the introduction of a pair of objects, \( Y_{\pm} \), for which one has \( \frac{Y_+}{Y_-} = e^{\frac{\theta}{t}} \), and

\[
Y_{t-2} \left( q^{1/2}x \right) Y_{t-2} \left( q^{-1/2}x \right) = [1 + Y_{t-3}(x)] [1 + Y_+(x)] [1 + Y_-(x)],
\]

(2.2)

Together with

\[
Y_+ \left( q^{1/2}x \right) Y_+ \left( q^{-1/2}x \right) Y_- \left( q^{1/2}x \right) Y_- \left( q^{-1/2}x \right) = [1 + Y_{t-2}(x)]^2.
\]

(2.3)
Finally, we need the relation between the current and partition functions conjectured in [11]:

$$I\left(q^{1/2}x\right) - I\left(q^{-1/2}x\right) = -\frac{i\pi T}{t} x \partial_x \ln \frac{1 + Y_+}{1 + Y_-}. \quad (2.4)$$

This is true at any temperature $T$.

The claim we are making is that the relations (2.1),(2.2),(2.3) together with (2.4), and duality, are equivalent to the monodromy relations studied previously, and hence lead to a complete determination of the physical quantities in the problem of interest.

General considerations about perturbation near the UV and IR fixed point show that the quantities $Y$ should have a singularity at $x = 0$ and at $x = \infty$. In addition, there should be at least one other singularity. We have to make the hypothesis there is only one such singularity, which we set at $z = 1$ by convention. There are thus two cuts in the complex $z$ plane for our quantities, one from $-\infty$ to the origin, and one from 1 to $\infty$. The fusion relations are now simply interpreted as monodromy relations. To show this, we first observe that we can set $Y_k \equiv \exp \frac{\epsilon_k}{T} (\epsilon_k \text{ is necessarily positive to guarantee that the partition functions are not trivial, even in that limit})$. Writing the relations in terms of our usual variable $z$, and continuing them to the entire complex plane, leads to

$$\epsilon_k(e^{i\pi z}) + \epsilon_k(e^{-i\pi z}) = \epsilon_{k-1}(z) + \epsilon_{k+1}(z) \quad k = 1, \ldots, t - 3 \quad (2.5)$$

where we set $\epsilon_0 \equiv 0$. Thus if $I_k \propto z \frac{d\epsilon_k}{dz}$, these relations are equivalent to all but the first two monodromy relations in (1.22), (1.25).

Next, we set $Y_+ = e^{\epsilon_s/T}$ and $Y_- = e^{\epsilon_a/T}$. It is not clear now what the sign of these $\epsilon$’s is, since the $Y_\pm$ have no direct physical meaning as partition functions. To solve this difficulty, we need duality: the latter implies that the current should expand in powers of $z^n$ in the IR i.e. have no non-trivial monodromy around the origin. This implies that the right hand side of (2.4) vanishes, and thus, setting $Y_+ = e^{\epsilon_s/T}$ and $Y_- = e^{\epsilon_a/T}$, that both $\epsilon_s$ and $\epsilon_a$ have a positive real part and are equal for $|z| < 1$. As a result, the fusion relations for $|z| < 1$ close with

$$I_{t-2}(e^{i\pi z}) + I_{t-2}(e^{-i\pi z}) = I_{t-3} + 2I_{t-1}, \quad (2.6)$$

where

$$I_{t-1} \propto \frac{1}{2} \left( z \frac{d\epsilon_s}{dz} + z \frac{d\epsilon_a}{dz} \right), \quad \epsilon_s = \epsilon_a,$$

---

1 Recall that, as discussed in [3], this follows because the current is entirely determined by the leading irrelevant operator near the IR fixed point, unlike the magnetizations or $Y_k$’s.
and the coefficient of proportionality is the same as for $I_k$. No special relation is required for the current this time, since it has no monodromy around the origin,

$$I(e^{2i\pi z}) = I.$$  \hspace{1cm} (2.7)

Finally, the last relation in (2.3) gives

$$I_{t-1}(e^{i\pi z}) + I_{t-1}(e^{-i\pi z}) = I_{t-2}(z).$$  \hspace{1cm} (2.8)

In the UV on the other hand, the current does have a non-trivial monodromy, which implies that the right hand side of (2.4) does not vanish, and $\epsilon_s$ has a negative real part for $|z| > 1$. Therefore, for $|z| > 1$, the fusion relations close with

$$I_{t-2}(e^{i\pi z}) + I_{t-2}(e^{-i\pi z}) = I_{t-3} + I_{t-1},$$  \hspace{1cm} (2.9)

where now $I_{t-1} \propto z \frac{d\epsilon_s}{dz}$ (still with the same proportionality coefficient). In addition, (2.4) reads

$$I(e^{i\pi z}) - I(e^{-i\pi z}) = I_{t-1}.$$  \hspace{1cm} (2.10)

In both cases, it is easy to show that these relations are entirely equivalent to those of the previous section.

We thus see that all the monodromies are determined by the fusion relations, together with the relation between the current and the partition function (2.4). Knowing these relations is like knowing the monodromies, from which the differential equation follows. The current is then quickly identified as the solution that is regular at $z = 0$, as a result, again, of duality.

### 3. The other solutions

#### 3.1. The other solutions are quasiparticle densities

The physical significance of the other solutions to the differential equation is easy to establish when $t$ is integer. The $\epsilon_k$ arise in the Bethe ansatz as the energy it takes to create a single quasiparticle, and their derivatives correspond to the densities of these quasiparticles (or their holes).

These quasiparticles are familiar from the study of the sine-Gordon model, and consist of a soliton and antisoliton, along with $t - 2$ breathers, which are soliton-antisoliton bound
states. In the presence of the voltage, there are no breathers in the ground state at zero temperature, i.e. only “breather holes”. The energy $\epsilon_k$ required to create a $k^{th}$ breather of mass $m_k = 2 \sin\left(\frac{k\pi}{2(t-1)}\right)$ can be computed by the same technique as the one used to compute the current. The density of holes is simply related by $2\pi \rho_k^h(\theta) = \frac{de_k}{d\theta} \propto z\frac{d\epsilon_k}{dz}$, where, in the notations of [2], the rapidity $\theta$ of the particles is defined by the requirement that $\epsilon_k \propto m_ke^\theta$ at large $\theta$. It is related to $u$ via $u = e^{A+\Delta-\theta}$, $A$ is the Fermi rapidity; $z$ is still related with $u$ by (1.5), the UV region corresponds to $Re(\theta) > A$, the IR to $Re(\theta) < A$. The densities are precisely related to the $I_k$ we introduced earlier through the normalization

$$2\pi \rho_k^h = \frac{(t-1)V}{i\pi t} I_k \quad k = 1 \ldots t - 1.$$  

(3.1)

Here $V$, the physical voltage, is proportional to $e^A$ (see [2] for the exact value of this coefficient).

For $k = t - 1$, the meaning of formula (3.1) is that $2\pi \rho_{t-1}^h = 2\pi \rho_a^h$. In fact, for $\theta > A$, $\rho_s = 0$, but for $\theta < A$, $\rho_s = \rho_a^h$, so formula (3.1) gives this density too.

For $t = 2$, the particles do not interact, so there are no breathers. The density of states for free particles means that simply $2\pi \rho_a^h = e^\theta$, as is easy to establish for a free theory. In terms of the variable $z$, this goes as $\frac{1}{\sqrt{z}}$, which is precisely the second solution $z^{-1/2} F\left(0, \frac{1}{2}; \frac{1}{2}; z\right)$ of the hypergeometric equation corresponding to (3.8), as illustrated in (1.10). This agrees with the series expansions: from (3.8), we see that in the region $|z| > 1$,

$$I(e^{2i\pi z}) - I(z) = \sqrt{\pi} \sum_{n=1}^{\infty} (-1)^n i \sin\left(\frac{n\pi}{2}\right) \frac{\Gamma(1 + n/2)}{\Gamma(n + 1)\Gamma(3/2 - n/2)} \left(\frac{z}{4}\right)^{-n/2}$$

All terms here vanish except the term $n = 1$, giving $I(e^{2i\pi z}) - I(z) = -\frac{i\pi}{2z}$, and thus $I_1(z) = -\frac{\pi}{2z}$. Using that $e^A = \frac{V}{2}, e^\Delta = \frac{1}{4}$ for $t = 2$, this gives rise to $2\pi \rho_1^h = e^\theta$, as expected.

The fact that the $I_k$ are related to quasiparticle densities gives a physical significance to the radius of convergence for the expansions. In terms of $\theta$, the circle of convergence $|z| = 1$ intersects the real axis for $\theta = A$. This follows from the fact that $A$ is the Fermi rapidity, where some quantities are expected to be singular; for instance, the density for solitons vanishes for $\theta > A$ but is non zero for $\theta < A$.

Notice that, unlike the current, the other solutions have a singularity at the origin as well as at $z = 1$ and $z = \infty$. This has a simple physical interpretation in terms of the operators controlling the behavior of these quantities near the IR fixed point. As
mentioned in the introduction, the current is controlled only by the tunneling operator \( \cos t \Phi \) (\( \Phi \) the dual boson); the densities, like thermal properties, depend also on the local conserved quantities which are polynomials in the derivatives of \( \Phi \). [8]

For completeness, we finally would like to give the expansions for the quantities \( Z_{BSG} \) of [10]; one has

\[
T \frac{\partial}{\partial \theta} \ln Z_{BSG}(z, \pm iV/2T) = \pm V \frac{t - 1}{2t^2 \sqrt{\pi}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\Gamma(n)} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{3}{2} + n \left(\frac{1}{t} - 1\right)\right)} e^{\pm ni\pi/t} \frac{n^2}{t} \left( e^{-i\pi} e^{2(t-1)\Delta z} \right)^{-n/t}.
\] (3.2)

### 3.2. Representing the densities with the curve

The \( I_k \) and hence the densities can also be represented by integrals on the same hyperelliptic curve as the current in (1.4). This is very similar (and sometimes identical) to what happens for the higher-spin magnetizations in the Kondo problem, as discussed in [4,6]. For simplicity, we will discuss the case \( t \) odd only, for which it is convenient to make a change of variable \( x \to \frac{1}{x} \), so the current can be written

\[
1 - I = \frac{1}{4iu} \int_{C_0} \frac{x^{(t-3)/2} dx}{\sqrt{x^{t-1} + 1 - u^2 x^t}} \equiv \int_{C_0} \frac{dx}{y}.
\] (3.3)

Here the contour starts at infinity, loops around the branch point on the positive real axis, and goes back to infinity.

In general, the integrand has \( t \) branch points, which, for large \( u \) are approximately located at \( x_k = u^{-2/t} e^{2i\pi k/t} \), \( k = 0, \ldots, t - 1 \). One branch point is on the positive real axis, and at large \( u \) the others are symmetrically distributed in the complex plane. As a function of \( u \), the current has singularities when \( z = 1 \), an equation with \( 2(t-1) \) solutions. As \( u \) loops around one of the singularities, \( z \) loops around 1, and monodromies are picked up. Of course, as \( u \) goes around one of the singularities, the different branch points of the integrand are exchanged. As a result, or, more simply, by using formula (1.17), one can show the following. The dimensionless reduced densities for breathers with \( t - k \) even can be written simply as

\[
I_{t-2p}(z) = \int_{C_{p,t-p}} \frac{dx}{y},
\] (3.4)

where \( C_{p,t-p} \) loops around the singularities \( x_p \) and \( x_{t-p} \), i.e. the \((p+1)th\) cycle of the Riemann surface of the integrand, as shown on the figure (\( C_0 \equiv C_t \) can be considered the
Fig. 1: A choice of cuts and contours in the complex $x$ plane for $t = 5$. The integral around $C_0$ gives the current, while integrals around $C_{23}$ (resp. $C_{14}$) give $I_1$ (resp. $I_3$). For $g = \frac{1}{3}$ for instance, we thus see that the integral around one cycle is $I_1$, the density of the first breather, while the integral around the second cycle is the current.

It is also possible to express the dimensionless densities for breathers with $t - k$ odd by choosing different cycles. For instance, by writing

$$2i \sin \frac{n\pi}{t} = e^{-n\pi i/t} \left(e^{2n\pi i/t} - 1\right)$$

one can show that $I_{t-1} (e^{i\pi z})$ reads as (3.4) but with the integral taken along a contour $C_{10}$ that loops around $x_1$ and $x_0$. More generally, one has

$$I'_{t-2p-1} (z) = I_{t-2p-1} (e^{i\pi z}) = \int_{C_{p,p+1}} \frac{dx'}{y},$$

as illustrated in figure 2.

Similarly, $I_{t-2p-1} (e^{-i\pi z})$ reads as (3.5), but with a complex conjugate contour. As a result, fusion relations take a very simple form. For instance, one has $I_{t-3} (e^{i\pi z}) + I_{t-3} (e^{-i\pi z}) = I_{t-2} (z) + I_{t-4} (z)$, is illustrated on figure 3.

Also, the truncation of the fusion relations $I_{t-1} (e^{i\pi z}) + I_{t-1} (e^{-i\pi z}) = I_{t-2} (z)$ can be illustrated graphically in a similar fashion.

To conclude, we illustrate these considerations by the case $t = 2$. We now have only two branch points, and the integral of the differential form along the unique cycle is (going back to the original $x$ variable)

$$2 \frac{i}{4} \int_{x_0}^{x_1} \frac{dx}{u \sqrt{x + x^2 - u^2}} = -\frac{\pi}{2u},$$
Fig. 2: An other choice of cuts and contours in the complex $x$ plane for $t = 5$. The integral around $C_0$ gives the current, while integrals around $C_{12}$ (resp. $C_{34}$) give $I'_2$ (resp. $I'_2$).

Fig. 3: Graphical representation of the fusion relation $I_2 (e^{i\pi} z) + I_2 (e^{-i\pi} z) = I_3 (z) + I_1 (z)$ for $t = 5$. 
where \( x_0 \) and \( x_1 \) are the two branch points. On the other hand,

\[
\mathcal{I}_1(z) = \frac{2i\pi}{V} 2\pi \rho_\alpha = \frac{2i\pi}{V} e^\theta = \frac{i\pi}{2u},
\]

so \( \mathcal{I}_1(e^{i\pi} z) = -\pi/(2u) \) as required.

4. Conclusion and Questions

This paper we hope is a first step in finding new ways to solving integrable models. The case we have discussed can be thought of as the deformed \( SU(2)_1 \) case (when \( t = 1 \) the bulk conformal field theory is the \( SU(2)_1 \) WZW model). This has been well understood by many approaches, and so our results do shed light on the picture but don’t result in the computation of any new quantities. In the \( SU(2)_k \) case (the multi-channel quantum wire and multi-channel Kondo problem), the zero-temperature curves \( \mathbb{B} \) are known and the multi-channel Kondo problem fairly well understood (see \( \mathbb{[12]} \) and references within). The functional relations can easily be derived for most of the quantities, but the crucial one involving the current (the analog of \( (2.4) \)) is not yet known. In the \( SU(N)_k \) case, the problem is even murkier. The \( SU(N)_k \) Kondo problem is integrable \( \mathbb{[13]} \), but only in a special case has the zero-temperature magnetization been calculated. Moreover, it is not even known which if any \( SU(N)_k \) generalizations of the boundary sine-Gordon problem are integrable. We hope that our approach will be useful in understanding these problems.

More formally, observe that, although we were motivated by the study of the boundary sine-Gordon model, all the results obtained do have a well defined meaning within the \( c = 1 \) conformal field theory itself, and its integrable structure. It is tempting to speculate that monodromy and duality should play a crucial role in the study of this integrable structure, in particular that there should be hyperelliptic curves “naturally” associated with rational conformal field theories. More progress in understanding the general picture is clearly needed here.

Finally, we find it intriguing that the curves are so simple and continuous in \( t \), given that the differential equations change dramatically as \( t \) is varied. It is hard not to believe that there is a simple and deep reason underlying this, but we do not know what this is.

This research was supported by the NSF (DMR-9802813), the DOE (Outstanding Junior Investigator award) and by the Sloan Foundation (P.F.); and by the NYI program (NSF-PHY-9357207) and DOE grant DE-FG03-84ER40168 (H.S.).
References

[1] V. V. Bazhanov, S. Lukyanov, and A. B. Zamolodchikov, Comm. Math. Phys. 177 (1996) 381, hep-th/9412229; Comm. Math. Phys. 190 (1997) 247, hep-th/9604044; Comm. Math. Phys. 200 (1999) 297, hep-th/9805008; hep-th/9812094.

[2] P. Fendley, A.W.W. Ludwig and H. Saleur, Phys. Rev. B52 (1995) 8934, cond-mat/9503172

[3] V. Fateev and P.B. Wiegmann, Phys. Lett. 81A (1981) 179

[4] P. Fendley, hep-th/9804108, to appear in Advances in Theoretical and Mathematical Physics.

[5] P. Fendley, H. Saleur, Phys. Rev. Lett. 81 (1998) 2518, cond-mat/9804173

[6] P. Fendley and H. Saleur, “Hyperelliptic curves for multi-channel quantum wires and the multi-channel Kondo problem”, cond-mat/9809259

[7] N. Seiberg, E. Witten, Nucl. Phys. B426 (1994), 19; Nucl. Phys. B431 (1994) 484.

[8] F. Lesage and H. Saleur, “Perturbation of IR fixed points and duality in quantum impurity problems”, cond-mat/9812045.

[9] F. Smith, Bull. Am. Math. Soc. (1938) 429; Higher transcendental functions, Bateman Manuscript Project ed. by A. Erdelyi, (McGraw-Hill)

[10] P. Fendley, F. Lesage, H. Saleur, J. Stat. Phys. 85 (1996) 211, cond-mat/9510053.

[11] P. Fendley, F. Lesage, H. Saleur, J. Stat. Phys. 79 (1995) 799, hep-th/9409176.

[12] D. Cox and A. Zawadowski, Adv. Phys. 47 (1998) 599, cond-mat/9704103

[13] A.M. Tsvelick and P.B. Wiegmann, Adv. Phys. 32 (1983) 453; N. Andrei, K. Furuya and J. Lowenstein, Rev. Mod. Phys. 55 (1983) 331; N. Andrei and P. Zinn-Justin, Nucl. Phys. B528 (1998) 648, cond-mat/9801158.