NASH EQUILIBRIUM POINTS OF RECURSIVE NONZERO-SUM
STOCHASTIC DIFFERENTIAL GAMES WITH UNBOUNDED
COEFFICIENTS AND RELATED MULTIPLE
DIMENSIONAL BSDES

RUI MU
Center for Financial Engineering
Soochow University
Suzhou 215006, China

ZHEN WU*
School of Mathematics
Shandong University
Jinan 250100, China

(Communicated by Qi Lü)

Abstract. This paper is concerned with recursive nonzero-sum stochastic differential game problem in Markovian framework when the drift of the state process is no longer bounded but only satisfies the linear growth condition. The costs of players are given by the initial values of related backward stochastic differential equations which, in our case, are multidimensional with continuous coefficients, whose generators are of linear growth on the volatility processes and stochastic monotonic on the value processes. We finally show the well-posedness of the costs and the existence of a Nash equilibrium point for the game under the generalized Isaacs assumption.

1. Introduction. In this article, we discuss a recursive nonzero-sum stochastic differential game (NZSDG for short) under Markovian framework. Generally speaking, stochastic differential game theory deals with conflict or cooperate problems in a dynamic system which is influenced by multiple players. Let us introduce the setting of the problem briefly. Assume that we have a system which is described as follows:

\[ dx_t = \sigma(t,x_t)dB_t \quad \text{for} \quad t \leq T \quad \text{and} \quad x_0 = x, \quad (1) \]

where \( B \) is a Brownian motion. This system can also be controlled by two players which we represent by weak formulation of a stochastic differential equation (SDE for short):

\[ dx_t = f(t,x_t,u_t,v_t)dt + \sigma(t,x_t)dB_t^{u,v} \quad \text{for} \quad t \leq T \quad \text{and} \quad x_0 = x. \quad (2) \]
The process $B^{u,v}$ is a new Brownian motion generated from $B$ by applying Girsanov’s transformation. The precise analysis will be introduced in the following text. Processes $u = (u_t)_{t \leq T}$ and $v = (v_t)_{t \leq T}$ represent the control actions of the two players imposed on this system. Indeed, the controls are not free, which bring some costs for players. What we discussed is a recursive type of cost functional, which is defined by the initial value of the following backward stochastic differential equation (BSDE for short): for $i = 1, 2$,

$$y^{i,u,v}_t = g^i(x_T) + \int_t^T \left[z^{i,u,v}_s \sigma^{-1}(s, x_s) f(s, x_s, u_s, v_s) + h_i(s, x_s, y^{i,u,v}_s, u_s, v_s)\right] ds - \int_t^T z^{i,u,v}_s dB_s. \quad (3)$$

The costs are defined by $J^i(u, v) = y^{i,u,v}_0$ for players $i = 1, 2$, respectively. The objective of this game model is to find a Nash equilibrium point $(u^*, v^*)$ such that,

$$J^1(u^*, v^*) \leq J^1(u, v^*) \quad \text{and} \quad J^2(u^*, v^*) \leq J^2(u^*, v)$$

for any admissible control $(u, v)$. This is actually to say that both of the two players would like to minimize their costs and no one can cut more by unilaterally changing her own control.

In the following, let us discuss the main contribution of our work, as well as the main difference between recursive cost and the classical one. The concept of stochastic differential recursive utility has been considered by Duffie and Epstein in [3] which extends the classical utility. The recursive one involves instantaneous utility depending not only on instantaneous consumption rate but also on the future utility. The manner of using solutions of BSDEs to describe cost functionals of stochastic differential game is initially inspired by [5], where some formulations of recursive utilities and their properties are also discussed. In BSDE (3), If the functions $h_i$ are independent on parameters $y_i^t$, then by applying Girsanov’s transformation, the costs $J^i$ will be reduced to $E^{u,v} [g^i(x_T) + \int_0^T h_i(s, x_s, u_s, v_s)]$, which is the accumulation of the instantaneous cost $h_i$ and the terminal cost $g^i$. This is the classical structure of non-recursive cost functions as studied in [9], [11]. Some recursive optimal control problems are studied by [19]. There are also works study the zero-sum case of recursive game, such as [20]. Readers are referred to a series of works by Hamadène for research on classical NZSDGs without the recursive part, say [7, 8, 9] and the references therein. Our main contribution is that we study a nonzero-sum game with recursive cost which is defined by initial value of BSDE (3). Besides, assumptions on the coefficients are irregular. We finally show the existence of a Nash equilibrium point.

The method of BSDEs has been shown as an efficient tool to deal with the recursive nonzero-sum stochastic differential game, see the works by [13], [20] for example. A complete review on BSDEs theory as well as some applications are introduced in a survey paper by [18]. The connection of BSDE with NZSDG and some other popular methods to deal with game problem, such as partial differential equation, are presented in a celebrated survey paper by [2].

In the present paper, we study the recursive NZSDG through BSDE technique in the same line as [20]. However in [20], the drift function $f$ of the state process in (2) is bounded or almost equivalent to bounded one. This boundedness is important when we consider the related BSDEs since this guarantees the good Liptsitz property of the generator of the corresponding BSDE with respect to $z$ component. However, this restriction is too strict to some extent. Therefore, the motivation of our work
is to relax this limitation on $f$. To instead, we consider a drift $f$ which is of linear growth on the state process $x$. This has already been considered in some classical game problems without recursive part by [9] and [10]. To our knowledge, this general recursive case has not been studied in literatures. This is our main innovation. Besides, under appropriate assumptions on function $h$, we find that the generator of BSDE (3) is not regular, which is of stochastic linear growth on $z$ and stochastic monotonic on $y$. This BSDE is new and has not been studied before. We give the existence of solutions for BSDE (3) which provides the well-posedness of the cost function. This result is summarized as Theorem 3.1 which we mainly deal with in this article. Then, with the help of the generalized Isaccs condition and a kind of multiple dimensional BSDE (22) (whose existence of solutions has been show in [16]), we show the existence of NEP for this recursive NZSDG by applying comparison properties between BSDEs.

Finally, we point out that this work establishes a model involving only two players, however, it can be generalized to multiple players case following the same way without any difficulty.

The rest of this work is organized as follows:

In Section 2, we give the precise statement of the recursive game problem and some assumptions on coefficients. Section 3 is devoted to the well-posedness of cost functionals, i.e., the existence of solutions of BSDE (3). The idea is to take a partition of interval $[0, T]$ and firstly solve this BSDE in a small interval $[T - \delta, T]$, then, extend it backwardly to the whole interval. The existence of Nash equilibria is shown in Section 4. Finally, in Section 5, we give a simple one-dimensional example. We can clearly see that Nash equilibrium point exists for the recursive game under our assumptions.

2. Statement of the problem. In this section, we will give some basic notations, the preliminary assumptions throughout this paper, as well as the statement of the recursive nonzero-sum stochastic differential game. Let $T$ be fixed and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which we define a $d$-dimensional Brownian motion $B = (B_t)_{0 \leq t \leq T}$ with integer $d \geq 1$. Let us denote by $\mathcal{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$, the natural filtration generated by the process $B$ and augmented by $\mathcal{N}_\mathbb{P}$ the $\mathbb{P}$-null sets, i.e. $\mathcal{F}_t = \sigma\{B_s, s \leq t\} \vee \mathcal{N}_\mathbb{P}$.

Let $\mathcal{P}$ be the $\sigma$-algebra on $\mathcal{F}$-progressively measurable sets. Let $p \in [1, \infty)$ be a real constant and $t \in [0, T]$ be fixed. We then define the following spaces: $L^p = \{\xi: \mathcal{F}_t$-measurable and $\mathbb{R}^m$-valued random variable s.t. $E[|\xi|^p] < \infty\}$; $S_{p, T} = \{\varphi = (\varphi_s)_{s \leq T}: \mathcal{P}$-measurable and $\mathbb{R}^m$-valued s.t. $E[\sup_{s \in [t, T]} |\varphi_s|^p] < \infty\}$ and $\mathcal{H}_{p, T} = \{\varphi = (\varphi_s)_{s \leq T}: \mathcal{P}$-measurable and $\mathbb{R}^m$-valued s.t. $E[\int_t^T |\varphi_s|^2 ds] < \infty\}$. Hereafter, $S_{p, T}$ and $\mathcal{H}_{p, T}$ are simply denoted by $S_{p, T}$ and $\mathcal{H}_{p, T}$.

The following assumptions are in force throughout this paper. Let $\sigma$ be the function defined by: $\sigma: [0, T] \times \mathbb{R}^m \rightarrow \mathbb{R}^{m \times m}$ which satisfies the following assumption.

Assumption 1. (i): $\sigma$ is uniformly Lipschitz w.r.t $x$. i.e. there exists a constant $C_1$ such that, $\forall t \in [0, T]$, $\forall x, x' \in \mathbb{R}^m$, $|\sigma(t, x) - \sigma(t, x')| \leq C_1|x - x'|$.

(ii): $\sigma$ is invertible and bounded and its inverse is bounded, i.e., there exits a constant $C_\sigma$ such that $\forall (t, x) \in [0, T] \times \mathbb{R}^m$, $|\sigma(t, x)| + |\sigma^{-1}(t, x)| \leq C_\sigma$.

Remark 1 (Uniform elliptic condition). Under Assumption 1, we can verify that, there exists a real constant $\epsilon > 0$ such that for any $(t, x) \in [0, T] \times \mathbb{R}^m$, $\epsilon I \leq \sigma(t, x) \sigma^{-1}(t, x) \leq \epsilon^{-1} I$ where $I$ is the identity matrix of dimension $m$. 
Suppose that we have a system whose dynamic is described by a SDE as follows: for \((t,x) \in [0,T] \times \mathbb{R}^m\),
\[
\begin{align*}
X_{t,x}^s &= x + \int_t^s \sigma(u, X_{u,x}^x) dB_u, \; s \in [t,T]; \\
X_{t,x}^t &= x, \; s \in [0,t].
\end{align*}
\]  
(4)

The solution \(X = (X_{t,x}^x)_{s \leq T}\) exists and is unique under Assumption 1. (cf. [17], p.289). We recall here two well-known results associate to the integrability of the \(\sigma\)-function.

Moreover, for any \((t,x) \in [0,T] \times \mathbb{R}^m, \; p \geq 2\), it holds that, \(\mathbb{P}\)-a.s.
\[
\mathbb{E}\left[\sup_{0 \leq s \leq T} |X_{s,x}^t|^p\right] \leq C(1 + |x|^p),
\]  
where the constant \(C\) is only depend on the Lipschitz coefficient and the bound of \(\sigma\). In addition, for a constant \(\alpha \in (0,2)\), we also have, \(\mathbb{P}\)-a.s.
\[
\mathbb{E}\left[\exp\left(\sup_{0 \leq s \leq T} |X_{s,x}^t|^\alpha\right)\right] < \infty.
\]  
(6)

We consider a two players game model in this article for simplicity. The general multiple players case is a straightforward adaptation. Each of the two players imposes a control type strategy to this system. Let us now denote by \(U_1\) and \(U_2\) two compact metric spaces and let \(\mathcal{M}_1\) (resp. \(\mathcal{M}_2\)) be the set of \(\mathcal{P}\)-measurable processes \(u = (u_t)_{t \leq T}\) (resp. \(v = (v_t)_{t \leq T}\)) with values on \(U_1\) (resp. \(U_2\)). We denote by \(\mathcal{M}\) the set \(\mathcal{M}_1 \times \mathcal{M}_2\). Hereafter \(\mathcal{M}\) is called the set of admissible control.

We then introduce the following Borelian function \(f : [0, T] \times \mathbb{R}^m \times U_1 \times U_2 \rightarrow \mathbb{R}^m\) which satisfies:

**Assumption 2.** For any \((t,x) \in [0,T] \times \mathbb{R}^m\), \((u,v) \mapsto f(t, x, u, v)\) is continuous on \(U_1 \times U_2\). Moreover \(f\) is of linear growth w.r.t \(x\), i.e. there exists a constant \(C_f\) such that \(|f(t, x, u, v)| \leq C_f(1 + |x|)\), \(\forall (t, x, u, v) \in [0,T] \times \mathbb{R}^m \times U_1 \times U_2\).

For \((u,v) \in \mathcal{M}\), let \(\mathbb{P}_{t,x}^{u,v}\) be the measure on \((\Omega, \mathcal{F})\) defined as follows:
\[
d\mathbb{P}_{t,x}^{u,v} = \mathbb{E}_T\left[\exp\left(\int_0^T \frac{\sigma^{-1}(s, X_{s,x}^t) f(s, X_{s,x}^t, u_s, v_s) dB_s}{\mathbb{E}\left[\exp\left(\int_0^T \frac{\sigma^{-1}(s, X_{s,x}^t) f(s, X_{s,x}^t, u_s, v_s) dB_s}{\mu}\right)\right]_s}\right)ight] d\mathbb{P}
\]  
(7)

where for any \((\mathcal{F}_t, \mathbb{P})\)-continuous local martingale \(M = (M_t)_{t \leq T}\),
\[
\mathcal{E}(M) := (\exp\{M_t - (\langle M \rangle_t/2)\})_{t \leq T}.
\]  
(8)

The notation \(\langle \cdot \rangle\) denotes the quadratic variation process. By Assumptions 1 and 2, we know \(\mathbb{P}_{t,x}^{u,v}\) is a new probability on \((\Omega, \mathcal{F})\) (see Appendix A, [4] or [17], p.200). From Girsanov’s theorem ([6], pp.285-301), the process \(B_{t,x}^{u,v} := (B_s - \int_0^s \sigma^{-1}(r, X_{r,x}^t) f(r, X_{r,x}^t, u_r, v_r) dB_r)_{s \leq T}\) is a \((\mathcal{F}_s, \mathbb{P}_{t,x}^{u,v})\)-Brownian motion and the process \((X_{s,x}^t)_{s \leq T}\) satisfies the following SDE in weak formulation:
\[
\begin{cases}
\frac{dX_{s,x}^t}{ds} = f(s, X_{s,x}^t, u_s, v_s) ds + \sigma(s, X_{s,x}^t) dB_{s}^{u,v}, \; s \in [t,T]; \\
X_{t,x}^t = x, \; s \in [0,t].
\end{cases}
\]  
(9)

Actually, the process \((X_{s,x}^t)_{s \leq T}\) is not adapted with respect to the filtration generated by the Brownian motion \((B_{s}^{u,v})_{s \leq T}\). Therefore, it is known as the weak solution of (9). Besides, properties (5) and (6) hold true, as well, for the expectation under the probability \(\mathbb{P}_{t,x}^{u,v}\) (see [11], Lemma 3.3-(ii)).

Now, this system is controlled by two players through dynamic function \(f\). The control actions are not free, which bring the players corresponding costs, or payoffs.
in some circumstances. Before introducing the costs, we first present the following
Borelian functions $h_i$ (resp. $g^i$) : $[0, T] \times \mathbb{R}^m \times \mathbb{R} \times U_1 \times U_2$ (resp. $\mathbb{R}^m$) $\rightarrow \mathbb{R}$, $i = 1, 2$ which satisfy

**Assumption 3.** (i): For any $(t, x, y) \in [0, T] \times \mathbb{R}^m \times \mathbb{R}$, $(u, v) \mapsto h_i(t, x, y, u, v)$ is continuous on $U_1 \times U_2$, $i = 1, 2$. Moreover $h_i$ is of polynomial growth w.r.t. $x$ and of linear growth w.r.t. $y$ and satisfies the stochastic monotonic property on $y$, for $i = 1, 2$. i.e. there exist constants $C_h, \gamma \geq 0$ and $\alpha \in (0, 2)$ such that

$$
\forall (t, x, y, y_t, u, v) \in [0, T] \times \mathbb{R}^m \times \mathbb{R} \times U_1 \times U_2,$$

(a) $|h_i(t, x, y, u, v)| \leq C_h(1 + |x|^{\gamma} + |y|);$

(b) $(y - y_t)(h_i(t, x, y, u, v) - h_i(t, x, y_t, u, v)) \leq C_h(1 + |x|^{\gamma})|y - y_t|^2$;

(ii): The function $g^i, i = 1, 2$, is of polynomial growth with respect to $x$, i.e. there exist constants $C_g$ and $\gamma \geq 0$ such that $|g^i(x)| \leq C_g(1 + |x|^{\gamma}), \forall x \in \mathbb{R}^m,$ for $i = 1, 2$.

Now, let $x_0 \in \mathbb{R}^m$ be fixed. The costs (or payoffs) of the players for their controls $(u, v) \in \mathcal{M}$ are given by the initial values of related BSDEs. More precisely, we define:

$$J^i(u, v) = Y_0^{i, (u, v)}, \quad i = 1, 2 \quad (10)$$

and the process $(Y_t^{i, (u, v)})_{t \leq T}$ satisfies the following BSDE: for $i = 1, 2$,

$$Y_t^{i, (u, v)} = g^i(X_T^{0, x_0}) + \int_t^T \left[ Z_s^{i, (u, v)} \sigma^{-1}(s, X_s^{0, x_0}) f(s, X_s^{0, x_0}, u_s, v_s) \right.$$  

$$\left. + h_i(s, X_s^{0, x_0}, Y_s^{i, (u, v)}, u_s, v_s) \right] ds - \int_t^T Z_s^{i, (u, v)} dB_s. \quad (11)$$

Actually, BSDE (11) has solution in some appropriate space which will be shown in the next section (see Theorem 3.1). Therefore, the costs (10) are well-defined. However, for the integrity of the statement of the problem, we would like to go ahead and put the proof of the existence later. This manner of definition of the cost functional has already been considered in [20], [13]. Indeed, if $h_i$ is independent of $y$ component, which is the unrecursive case, we can express the value process of the BSDE (11) by $Y_t^{i, (u, v)} = \mathbb{E}_{0, x_0}^{u,v}[g^i(X_T^{0, x_0}) + \int_t^T h_i(s, X_s^{0, x_0}, u_s, v_s) ds | \mathcal{F}_t]$ where $\mathbb{E}_{0, x_0}$ is the expectation under the probability $P_{0, x_0}$. It is easy to check this conditional expectation is well-defined from the assumptions on $g^i$ and $h_i$. Apparently, in this case cost function is $J^i(u, v) = Y_0^{i, (u, v)} = \mathbb{E}_{0, x_0}^{u,v}[g^i(X_T^{0, x_0}) + \int_0^T h_i(s, X_s^{0, x_0}, u_s, v_s) ds]$. Since $\mathcal{F}_0$ is nothing but some null measure sets. Then functions $h_i$ and $g_i$ can be viewed as the instantaneous cost and the terminal cost respectively for player $i = 1, 2$. This situation is coincident with the classical nonzero-sum stochastic differential game model as in the work by Hamadène (see [8]).

Hereafter $\mathbb{E}_{0, x_0}^{u,v}$ (resp. $P_{0, x_0}^{u,v}$) will be simply denoted by $\mathbb{E}^{u,v}$ (resp. $P^{u,v}$).

What we concerned in this article is to find an admissible control $(u^*, v^*)$ such that

$$J^1(u^*, v^*) \leq J^1(u, v^*) \text{ and } J^2(u^*, v^*) \leq J^2(u^*, v), \forall (u, v) \in \mathcal{M}.$$

The control $(u^*, v^*)$ is called a Nash equilibrium point for the recursive NZSDG. It reads that each player chooses her best control, while, an equilibrium is a pair of controls, such that, when applied, no player will lower any cost by unilaterally changing her own control.
Now, we define the Hamiltonian functions $H_i, i = 1, 2$, of the game from $[0, T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^m \times U_1 \times U_2$ into $\mathbb{R}$ by: $H_i(t, x, y, p, u, v) = p \sigma^{-1}(t, x) f(t, x, u, v) + h_i(t, x, y, u, v)$.

Obviously, under Assumptions 2 and 3, $H_i$ satisfies the following hypothesis, for each $(t, x, y, z, u, v) \in \mathbb{R}$.

\[
\begin{cases}
|H_i(t, x, y, z, u, v)| \leq C_{\sigma} C_f(1 + |x|)|z| + C_h(1 + |x|^\gamma + |y|), \quad \gamma \geq 0; \\
(y - y')(H_i(t, x, y, z, u, v) - H_i(t, x, y', z, u, v)) \leq C_h(1 + |x|^\alpha)|y - y'|^2, \quad \alpha \in (0, 2); \\
|H_i(t, x, y, z, u, v) - H_i(t, x, y', u, v)| \leq C_{\sigma} C_f(1 + |x|)|z - z'|. 
\end{cases}
\]

(12)

For the existence of Nash equilibria, we also need the following assumption.

**Assumption 4 (Generalized Isaacs condition).** (i) There exist two Borelian applications $u^*, v^*$ defined on $[0, T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^m$, with values in $U_1$ and $U_2$, respectively, such that for any $(t, x, y, p, u, v) \in [0, T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^m \times U_1 \times U_2$, we have:

\[
H_1^*(t, x, y, p, q) = H_1(t, x, y, p, u^*(t, x, y, p, q), v^*(t, x, y, p, q)) \leq H_1(t, x, y, p, v^*(t, x, y, p, q))
\]

and

\[
H_2^*(t, x, y, p, q) = H_2(t, x, y, q, u^*(t, x, y, q, p, q), v^*(t, x, y, q, p, q)) \leq H_2(t, x, y, q, v^*(t, x, y, q, p, q)),
\]

(ii) the mapping $(y^1, y^2, p, q) \in \mathbb{R} \times \mathbb{R}^m \mapsto (H_1^{*}, H_2^{*})(t, x, y^1, y^2, p, q) \in \mathbb{R}$ is continuous for any fixed $(t, x) \in [0, T] \times \mathbb{R}^m$.

3. **Well-posedness of costs.** In this section, we focus on the well-posedness of costs $J^i(u, v)$ for admissible control $(u, v) \in \mathcal{M}$ and $i = 1, 2$. Precisely speaking, we need to show the existence of the solutions for BSDE (11) which we summarized as the following theorem.

**Theorem 3.1.** Under Assumptions 1, 2 and 3, BSDE (11) has solution $(Y^{i, (u, v)}, Z^{i, (u, v)})$ which belongs to $\mathcal{S}_T^2 \times \mathcal{H}_T^2$ for any $\bar{q} > 1$ and any admissible control $(u, v) \in \mathcal{M}, i = 1, 2$.

For simplicity, in this section, we omit the subscript $(u, v)$ and denote the pair $(Y^{i, (u, v)}, Z^{i, (u, v)})$ by $(Y^i, Z^i)$. We first provide an uniform priori estimate of the solution for BSDE (11). For this, we need the following result by [12], which related to the integrability of the Doléans-Dade exponential of $X^{i, x}$. Actually, we only need the following Lemma 3.2, Lemma 3.3 and Lemma 3.7, readers who are not interested in the technique proofs can skip the other lemmas and the proof process in the following subsection.

### 3.1. Integrability of Doléans-Dade exponential

**Lemma 3.2 (12).** Under Assumption 1, let $\varphi$ be a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^m)$-measurable application from $[0, T] \times \Omega \times \mathbb{R}^m$ to $\mathbb{R}^m$ which is uniformly of linear growth, that is, $\mathcal{P}$-a.s., $\forall (s, x) \in [0, T] \times \mathbb{R}^m, |\varphi(s, \omega, x)| \leq C_{\varphi}(1 + |x|)$. Then, there exists some $p_0 \in (1, 2)$ and a constant $C$, where $p_0$ depends only on $C_{\sigma}, C_{\varphi}, m$ while the constant $C$ depends only on $m$ and $p_0$, but not on $\varphi$, such that:

\[
E \left[ |E_T \left( \int_0^T \varphi(s, X^{i, x}_s) dB_s \right) |^{p_0} \right] \leq C,
\]

where the process $(E_t)_{t \leq T}$ is the density function defined in (8).
For the same function $\varphi$ in Lemma 3.2 and a fixed $t \in [0, T]$, let us now define a process $(\Gamma_{t,s})_{t \leq s \leq T}$ as follows:

$$d\Gamma_{t,s} = \Gamma_{t,s} \cdot \varphi(s, X_{s}^{t,x}) dB_{s}, \quad \forall s \in [t, T) \text{ and } \Gamma_{t,t} = 1, \ \Gamma_{t,T} = E_{T}/E_{t}. $$

Actually,

$$\Gamma_{t,s} := \Gamma_{t,s} \cdot \varphi(r, X_{r}^{t,x}) = e^{\int_{t}^{s} \varphi(r, X_{r}^{t,x}) dB_{r} - \frac{1}{2} \int_{t}^{s} \varphi^{2}(r, X_{r}^{t,x}) dr}, \quad \forall s \in [t, T]. \quad (13) $$

Then, the following lemma holds true by using the same mind as Lemma 3.2. We provide the proof here for readers' better understanding.

**Lemma 3.3.** Under Assumption 1, for the same $\varphi$ as in Lemma 3.2, there exists some $\delta \in (0, T)$ small enough, such that

$$E \left[ \sup_{T-\delta \leq t \leq T} |\Gamma_{T-\delta, t}|^{-p} \right] < \infty \text{ for any } p > 1. $$

To prove Lemma 3.3, we need the following lemmas.

**Lemma 3.4.** Under Assumption 1, let $M_{t} = \int_{0}^{t} \sigma(s, X_{s}^{t,x}) dB_{s}$ for each $t \leq T$, then for any $p > 1$, there exists a constant $C_{0}$ depending on $C_{\sigma}, T$ and $p$, such that,

$$|X_{s}^{t,x}|^{p} \leq C_{0} (1 + |x|^{p} + |M_{s}|^{p}) \text{ a.s.} $$

**Lemma 3.5.** If $B := (B_{t})_{t \leq T}$ is a $\mathbb{R}^{m}$-valued Brownian motion and $(\sigma_{t})_{t \leq T}$ is a $\mathbb{R}^{m}$-valued stochastic process such that $E \left[ \int_{0}^{T} |\sigma_{t}|^{2} dt \right] < \infty$, then $I(S(t))$ is a standard Brownian motion on $[0, R(T)]$ where $R(t) = \int_{0}^{t} |\sigma_{t}|^{2} dr < \infty$; $S(t) = \inf \{ s > 0, R(s) = t \}$ and $I(t) = \int_{0}^{t} \sigma_{s} dB_{s}$.

**Proof.** See [14] p.29.

**Lemma 3.6.** Let $B = (B_{t})_{t \geq 0}$ be a standard one dimensional Brownian motion. The law of $|B|$ has density $\frac{2}{\sqrt{2\pi t}} e^{-\frac{x^{2}}{2t}}$, $t \geq 0$. If $2\lambda t < 1$ for a constant $\lambda$, then, $E[|e^{\lambda \cdot |B_{t}|}] < \infty$.

We are now ready to provide the proof of Lemma 3.3.

**Proof of Lemma 3.3.** In this proof, the process $X_{s}^{t,x}$ is denoted simply by $X$. For a constant $\delta \in (0, T)$, let us define a stopping time

$$\tau_{N} := \inf \{ t \geq T - \delta, |\int_{T-\delta}^{t} \varphi(s, X_{s}) dB_{s}| \geq N \text{ or } \int_{T-\delta}^{t} |\varphi(s, X_{s})|^{2} ds \geq N \}. $$

For $p > 1$, since

$$E \left[ \int_{T-\delta}^{T} 1_{\tau_{N}}(t) |\Gamma_{T-\delta, t}(-p\varphi(s, X_{s}))|^{2} dt \right]$$

$$\leq E \left[ \int_{T-\delta}^{T+\tau_{N}} e^{2(\int_{T-\delta}^{s} -p\varphi(s, X_{s}) dB_{s} - \frac{1}{2} \int_{T-\delta}^{s} p^{2} |\varphi(s, X_{s})|^{2} ds)} \cdot |\varphi(t, X_{t})|^{2} dt \right]$$

$$\leq N e^{2pN},$$

then the process $(\int_{T-\delta}^{T+\tau_{N}} \Gamma_{T-\delta, s}(-p\varphi(s, X_{s})) dB_{s})_{T-\delta \leq t \leq T}$ is a $\mathcal{F}_{t}$-martingale.
Therefore, by Itô’s formula,
\[
E \left[ \Gamma_{T-\delta, T\wedge \tau_N} (-p \varphi(s, X_s)) \right] = 1 - E \left[ \int_{T-\delta}^{T\wedge \tau_N} \Gamma_{T-\delta, t\wedge \tau_N} (-p \varphi(s, X_s)) \cdot p \varphi(t, X_t) d\Gamma_t \right] = 1.
\]
We now define \( M_t := \int_{T-\delta}^{t\wedge \tau_N} \sigma(s, X_s) dB_s \) for each \( t \in [T-\delta, T] \). Then we obtain from the linear growth of \( \varphi \) and Lemma 3.4 that
\[
|\Gamma_{T-\delta, T\wedge \tau_N} (\varphi(s, X_s))|^{-p} = \Gamma_{T-\delta, T\wedge \tau_N} (-p \varphi(s, X_s)) \cdot e^{\frac{1}{2} (p^2 + p) \int_{T-\delta}^{t\wedge \tau_N} |\varphi(s, X_s)|^2 dt} \leq \Gamma_{T-\delta, T\wedge \tau_N} (-p \varphi(s, X_s)) \cdot e^{\frac{1}{2} (p^2 + p) \bar{C}(1 + |x|^2 + |M_{T\wedge \tau_N}|^2)}
\]
where the constant \( \bar{C} \) depends on \( T, C_0 \) and \( C_\varphi \).

Let the process \( B^N = (B_t^N)_{T-\delta \leq t \leq T} := (B_t - B_{T-\delta} - \int_{T-\delta}^{t\wedge \tau_N} -p \varphi(s, X_s) ds)_{t \leq T} \). Hence the process \( B^N \) is a Brownian motion under the probability \( P^N \) which satisfies that \( dB^N = \Gamma_{T-\delta, T\wedge \tau_N} (-p \varphi(s, X_s)) d\Gamma \). Let us denote \( M^N_t := \int_{T-\delta}^{t\wedge \tau_N} \sigma(s, X_s) dB^N_s \) for each \( t \in [T-\delta, T] \). Then
\[
M_t = M^N_t + \int_{T-\delta}^{t\wedge \tau_N} -p \sigma(s, X_s) \varphi(s, X_s) ds
\]
and from Assumption 1, the linear growth of \( \varphi \) and Lemma 3.4, we know,
\[
|M_t|^2 \leq 2|M^N_t|^2 + C^p \sigma^2 C_\varphi \int_{T-\delta}^{t\wedge \tau_N} (1 + |X_s|^2) ds \leq \bar{C} (|M^N_t|^2 + 1 + |x|^2 + \int_{T-\delta}^{t\wedge \tau_N} |M_s|^2 ds), \text{ for } t \in [T-\delta, T \wedge \tau_N],
\]
where the constant \( \bar{C} = 2 \vee (2p^2 C_\sigma^2 C_\varphi C_0 \delta) \). Thanks to Gronwall’s inequality, we have,
\[
|M_{T\wedge \tau_N}|^2 \leq \bar{C} \left( 1 + |x|^2 + |M^N_{T\wedge \tau_N}|^2 \right) e^{C \delta}.
\]
Back to (14) and take expectation on both sides, we obtain, there exists a constant which we still denoted by \( \bar{C} \) depending on \( C_0, C_\sigma, C_\varphi, p, m, T \), such that,
\[
E \left[ \Gamma_{T-\delta, T\wedge \tau_N} (\varphi(s, X_s)) \right]^{-p} \leq E^N \left[ e^{\frac{1}{2} (p^2 + p) \bar{C}(1 + |x|^2 + |M^N_{T\wedge \tau_N}|^2) e^{C \delta}} \right] \leq e^{\frac{1}{2} (p^2 + p) \bar{C} e^{C \delta}(1 + |x|^2)} E^N \left[ e^{\frac{1}{2} (p^2 + p) \bar{C} e^{C \delta} |M^N_{T\wedge \tau_N}|^2} \right]
\]
where \( E^N \) is the expectation under the probability \( P^N \). If \( \sigma_i(t) \) is the \( i \)-th row \((i = 1, 2, ..., m)\) of the matrix \( \sigma(t, X_t) \), then by a technique of splitting the stochastic integral into the integrals on random intervals, we get the following inequality,
\[
|M^N_T|^2 \leq \sum_{i=1}^{m} \left| \int_{T-\delta}^{T} \sigma_i(t) d\Gamma^N_t \right|^2 \leq \sum_{i=1}^{m} \left| \beta^N_i (R_i(T)) \right|^2,
\]
where \( \beta^N_i (t) = \int_{T-\delta}^{S_i(t)} \sigma_i(s) dB^N_s \) and \( S_i(t) = \inf \{ s \geq T-\delta : R_i(s) = t \} \) with \( R_i(s) = \int_{T-\delta}^{s} |\sigma_i(t)|^2 dt \).
It follows from Lemma 3.5 that $\beta_i^N$ is a Brownian motion on the random interval $[T - \delta, R_i(T)]$. Now Hölder’s inequality implies for a constant $\lambda$,
\[
E^N \left[ e^{\lambda |M^N_i|^2} \right] \leq E^N \left[ \prod_i e^{\lambda |\beta_i^N(R_i(T))|^2} \right] \leq \prod_i E^N \left[ \left( e^{m\lambda |\beta_i^N(R_i(T))|^2} \right)^{\frac{1}{p}} \right] = \prod_i E^N \left[ \left( e^{m\lambda |\beta_i^N(R_i(T))|^2} \right)^{\frac{1}{p}} \right]
\]
where $\beta$ is a scalar Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P}^N)$. Since $R_i(T) \leq \delta C_o^2$, then by Lemma 3.6, if $2\lambda m\delta(C_o)^2 < 1$, we have,
\[
E^N \left[ e^{\lambda |M^N_i|^2} \right] \leq E^N \left[ e^{m\lambda |\beta(\delta(C_o)^2)|^2} \right] \equiv e_0 < \infty.
\]
Now let $\lambda = \frac{1}{2}(p^2 + p)\bar{C}e\bar{C}^\delta$, the same $\bar{C}$ as in (16). Considering inequality (16) and the fact that $\delta < T$, we can conclude that if $(p^2 + p)\bar{C}e\bar{C}^\delta < \varepsilon = (m\delta C_o^2)^{-1}$ then
\[
E \left[ |\Gamma_{T-\delta, T \wedge \tau_N} (\phi(s, X_s))|^{-p} \right] \leq e^{\frac{1}{2}(1+|s|^2)}e_0^\frac{1}{4} = C.
\]
We can choose $\delta$ small enough, such that
\[
0 < \delta < \frac{1}{(p^2 + p)\bar{C}e\bar{C}^\delta m|C_o|^2} \quad \text{for any fixed } p > 1.
\]
Then Fatou’s lemma yields that there exists a $\delta \in (0, T)$ small enough such that $E \left[ |\Gamma_{T-\delta, T} (\phi(s, X_s))|^{-p} \right] < C$ holds true for any $p > 1$, with constant $C$ depending on $p, m$ but not on $\phi$.

Finally, BDG inequality yields that Lemma 3.3 is true. □

By exactly examining the proof of Lemma 3.3, we can also have the following result.

**Lemma 3.7.** Under Assumption 1, let $\phi$ satisfies the assumption in Lemma 3.2, then, there exists some $\delta \in (0, T)$ small enough, such that
\[
E \left[ \sup_{T - \delta \leq t \leq T} |\Gamma_{T-\delta, t}|^p \right] < \infty \quad \text{for any } p > 1.
\]

Now, we are ready to introduce the proof of Theorem 3.1.

3.2. **Proof of Theorem 3.1.** We begin with the integrability of $Y^i$, $i = 1, 2$ (the superscription $(u, v)$ is omitted for simplicity). Since the technique restriction as shown in Lemma 3.3, we will divide $[0, T]$ into small intervals, i.e., $[0, T] = [T - \delta, T] \cup [T - 2\delta, T - \delta] \cup \ldots \cup [0, T - (n - 1)\delta] = nT/\delta$ for small $\delta \in (0, T)$ and solve BSDE (11) in small time intervals backwardly. If the choice of $\delta$ is independent with the terminal value of the BSDE, then, with this time partition method, we can find the global solution on the whole time space. Next, we start with some $t \in [T - \delta, T]$. The following technique is inspired by [1]. We first take a linearization and take $i = 1$ for example. Let
\[
a_s = H_1(s, X^0_{s, x_0}, Y^1_{s}, Z^1_{s}, u_s, v_s) - H_1(s, X^0_{s, x_0}, 0, Z^1_{s}, u_s, v_s), \]
\[
b_s = H_1(s, X^0_{s, x_0}, 0, Z^1_{s}, u_s, v_s) - H_1(s, X^0_{s, x_0}, 0, 0, u_s, v_s) \cdot |Z^1_{s}|^2.
\]
Then, from BSDE (11), \((Y^1, Z^1)\) solves the following linear BSDE: \(\forall t \in [T - \delta, T],\)
\[
Y_t^1 = g^1(X_t^{0,x_0}) + \int_t^T [h_1(s, X_s^{0,x_0}, 0, u_s, v_s) + a_s \cdot Y_s^1 + b_s \cdot Z_s^1] ds - \int_t^T Z_s^1 dB_s.
\]
Let \(e_t = eB_t\), by Itô’s formula, we have, \(\forall t \in [T - \delta, T],\)
\[
e_t Y_t^1 = e_t g^1(X_T^{0,x_0}) + \int_t^T [e_s h_1(s, X_s^{0,x_0}, 0, u_s, v_s) + e_s b_s Z_s^1] ds - \int_t^T e_s Z_s^1 dB_s.
\]
It follows from Assumptions 1-(ii) and 2 that, \(|b_s| \leq C_\delta (1 + |X_s^{0,x_0}|)\). Once again, by Girsanov’s transformation, we have,
\[
e_t Y_t^1 = e_t g^1(X_T^{0,x_0}) + \int_t^T [e_s h_1(s, X_s^{0,x_0}, 0, u_s, v_s) ds - \int_t^T e_s Z_s^1 dB_s, \quad t \in [T - \delta, T],
\]
where \(B_t^* = B_t - B_T - \int_0^T b_s dr, s \in [T - \delta, T]\) is a Brownian motion under the probability \(P^*\) which is defined by \(dP^* = \Gamma_{T - \delta,t} (B_t) dB_t\), where \(\Gamma_{T - \delta,t}\) is defined in (13). Considering additionally that, \(a_s \leq C_\delta (1 + |X_s^{0,x_0}|)\) for \(\alpha \in (0, 2)\) which is obtained by Assumption 3-(i) on \(h_1\), therefore, we know \((e_t)_{t \leq T}\) has moments of any order by (6), the same with \(g^1\) and \(h_1\) following the estimate (5).

Besides, as we assume that BSDE (11) has solution and \(Z^1\) belongs to \(\mathcal{H}_{T - \delta, T}^p\) for some \(\delta \in (0, T)\) and any \(p > 1\), then for any \(q \in (1, p),\)
\[
E\left[\left(\int_{T - \delta}^T e_t^2 |Z_t^1|^2 ds \right)^{\frac{q}{2}}\right] \leq C \left(\sup_{T - \delta \leq s \leq T} e_t^q \left(\int_{T - \delta}^T |Z_s^1|^2 ds \right)^{\frac{q}{2}}\right) < \infty.
\]
This enables us to take the conditional expectation of (18) under the probability \(P^*,\) i.e., \(e_t Y_t^1 = E^*[e_t g^1(X_T^{0,x_0}) + \int_t^T [e_s h_1(s, X_s^{0,x_0}, 0, u_s, v_s) ds] \mid \mathcal{F}_t], t \in [T - \delta, T].\)
As we analyzed above, the conditional expectation is well-posed. More precisely,
\[
Y_t^1 = E^* \left[ e_t g^1(X_t^{0,x_0}) + \int_t^T e_s h_1(s, X_s^{0,x_0}, 0, u_s, v_s) ds \mid \mathcal{F}_t \right], \quad t \in [T - \delta, T].
\]
Let us now denote by \(\Gamma_{t,s}\) the process \((\Gamma_{t,s}(B_t))_{t \leq s \leq T}\) for fixed \(t \in [0, T]\) as in (13).
Equation (19) can be rewrite as: for \(t \in [T - \delta, T],\)
\[
Y_t^1 = \frac{1}{\Gamma_{T - \delta, T}} E \left[ \Gamma_{T - \delta, T} \left( e_t g^1(X_t^{0,x_0}) + \int_t^T e_s h_1(s, X_s^{0,x_0}, 0, u_s, v_s) ds \right) \mid \mathcal{F}_t \right].
\]
For any \(p > 1\), by applying Young’s inequality and conditional Jensen’s inequality, and considering the fact that functions \(g^1\) and \(h_1\) are of polynomial growth on \(x\) and Lemma 3.3, 3.7, we have
\[
E \left[ \sup_{T - \delta \leq t \leq T} |Y_t^1|^p \right] \leq C \left( E \left[ \sup_{T - \delta \leq t \leq T} \left| \Gamma_{T - \delta, t} \right|^{-q} \right] \right)^{\frac{p}{q}} \times \left\{ E \left[ \sup_{T - \delta \leq t \leq T} \Gamma_{T - \delta, T} \left( e_t g^1(X_t^{0,x_0}) + \int_t^T e_s h_1(s, X_s^{0,x_0}, 0, u_s, v_s) ds \right) \right]^p \right\}^{\frac{1}{p}} \leq C \{ E [\Gamma_{T - \delta, T}]^p \}^{\frac{p}{q}} \times \left\{ E \left[ \sup_{T - \delta \leq t \leq T} \Gamma_{T - \delta, T} \left( e_t g^1(X_t^{0,x_0}) + \int_t^T e_s h_1(s, X_s^{0,x_0}, 0, u_s, v_s) ds \right) \right]^p \right\}^{\frac{1}{p}}.
\]
with \( p < \hat{q} \) and \( \tilde{p} = \frac{\hat{p}}{q-p} < \tilde{p} \).

The uniform integrability of process \( Z^1 \) follows from Itô’s formula and the facts that \( Y^1 \in \mathcal{S}^p_{T-\delta,T} \) for any \( p > 1 \) from (20). Indeed, we have, for \( 1 < q < p, \)

\[
E\left[ \left( \int_{T-\delta}^T |Z^1_t|^2 \, dt \right)^{\frac{q}{2}} \right] \leq C \left( \sup_{T-\delta \leq t \leq T} (|Y^1_t|^q + |Y^0_t|^p) + \sup_{T-\delta \leq t \leq T} \left(1 + |X^0_t|^\frac{pq}{q-p} + |X^0_t|^q\right) \right)
\]

which is finite with the constant \( C \) depending only on \( C_h, C_\sigma, C_f, T, p \) and \( q \). The details are omitted here.

Let us summarize the estimates (20) and (21) as the following lemma.

**Lemma 3.8.** Under Assumptions 1 and 12, if for any admissible control \((u, v)\) \( \in \mathcal{M}, i = 1, 2, (Y^{i,(u,v)}, Z^{i,(u,v)}) \) are solutions to BSDE (11), such that, \( Y^{i,(u,v)} \in \mathcal{S}^p_{T-\delta,T} \) for some \( \delta \in (0, T) \) and any \( p > 1 \). Then, for any \( q \in (1, p), (Y^{i,(u,v)}, Z^{i,(u,v)}) \in \mathcal{S}^q_{T-\delta,T} \times \mathcal{H}^q_{T-\delta,T} \) and there exists \( \tilde{q} > q \) such that,

\[
E \left[ \left( \int_{T-\delta}^T |Z^i_{t-\delta} + \left( \int_{T-\delta}^T |Z^i_{t-\delta}|^2 \, dt \right)^{\frac{q}{2}} \right] \leq C \left( \left( E \left[ |g^1(X_{T-\delta}^0)|^q \right] \right)^{\frac{q}{q-p}} + \left( \left( E \left[ \left( \int_{T-\delta}^T h^i(s, X_s^0, 0, u_s, v_s) \, ds \right)^{\tilde{q}} \right] \right)^{\frac{q}{q-p}} \right) \right.
\]

which is obviously finite by considering \( g^i(x) \) and \( h^i(s,x,0,u,v) \) are of polynomial growth on \( x \) and the fact that \( X^0 \) has moments of any order. The constant \( C \) depends only on \( C_h, C_\sigma, C_f, T, q, \tilde{q} \).

We come back to

**Proof of Theorem 3.1.** For each \( n \geq 1 \), let us define the following stopping time:

\[
\tau_n = \inf \left\{ t \geq T-\delta, \int_{T-\delta}^t C^2_\sigma C^2_f (1+|X^0_s|^2) + C_h (1+|X^0_s|^\gamma) \, ds \geq n \right\} \wedge T, \delta \in (0, T).
\]

This stopping time is of stationary type and it will converge \( \mathbb{P} \)-a.s. to \( T \) as \( n \) tends to infinity. Let us set \( g^{1n}(x) = g^1(x)1_{g^1(x) \leq n} \). Then by the result of [15], we know that there exists a bounded process \( Y^{1n} \) and a process \( Z^{1n} \in \mathcal{H}^2_{T-\delta,T} \), which solve the following BSDE:\( \forall t \in [T-\delta, T], \)

\[
Y^{1n}_t = g^{1n}(X^0_{T-\delta}) + \int_t^T 1_{s \leq \tau_n} H_1(s, X^0_s, Y^{1n}_s, Z^{1n}_s, u_s, v_s) \, ds - \int_t^T Z^{1n}_s \, dB_s.
\]

Indeed, for any \((s, y, z, u, v) \in [T-\delta, T] \times \mathbb{R} \times \mathbb{R}^m \times U_1 \times U_2 \), we know,

\[
1_{s \leq \tau_n} H_1(s, X^0_s, y, z, u, v) \leq 1_{s \leq \tau_n} \left[ C_h (1+|X^0_s|^\gamma) + \frac{1}{2} C^2_\sigma C^2_f (1+|X^0_s|^2) \right] + \frac{1}{2} |z|^2
\]
and
\[ \int_{T - \delta}^{T} 1_{s \leq \tau_n} \left[ C_h (1 + |X_s^{0, x_0}|^\gamma) + \frac{1}{2} C_\sigma C_f^2 (1 + |X_s^{0, x_0}|^2) \right] ds \leq \frac{3n}{2}. \]

Then, Lemma 3.8 yields that, \((Y^{1n}, Z^{1n}) \in S_{T - \delta, T}^{q} \times \mathcal{H}_{T - \delta, T}^{q}\) uniformly with respect to \(n\) for any \(q > 1\).

Let us show that \((Y^{1n}, Z^{1n})_{n \geq 1}\) is a Cauchy sequence in \(S_{T - \delta, T}^{q} \times \mathcal{H}_{T - \delta, T}^{q}\) for all \(q > 1\). Let \(m, n\) be integers such that \(m > n > 1\) and let us set \(\delta Y = Y^{1m} - Y^{1n}\), \(\delta Z = Z^{1m} - Z^{1n}\). Then \((\delta Y, \delta Z)\) solves the BSDE: \(\forall t \in [T - \delta, T]\),

\[ \delta Y_t = g^{1m}(X_t^{0, x_0}) - g^{1n}(X_t^{0, x_0}) + \int_t^T F(s, X_s^{0, x_0}, \delta Y_s, \delta Z_s, u_s, v_s) ds - \int_t^T \delta Z_s dB_s, \]

where
\[
F(s, X_s^{0, x_0}, \delta y, \delta z, u, v) = \begin{cases} 
1_{s \leq \tau_n} [H_1(s, x, \delta y + Y_s^{1n}, \delta z + Z_s^{1n}, u, v) - H_1(s, x, Y_s^{1n}, Z_s^{1n}, u, v)] \\
+ 1_{\tau_n < s \leq \tau_m} H_1(s, x, Y_s^{1n}, Z_s^{1n}, u, v). \end{cases}
\]

The function \(F\) satisfies the hypothesis (12) and actually \(F(s, X_s^{0, x_0}, 0, 0, u, v) = -1_{\tau_n < s \leq \tau_m} H_1(s, X_s^{0, x_0}, Y_s^{1n}, Z_s^{1n}, u, v)\). Besides, for \(1 < q < \infty\), by Hölder inequality, we have,

\[
E \left[ \left( \int_{T - \delta}^{T} F(s, X_s^{0, x_0}, 0, 0, u_s, v_s) ds \right)^q \right] 
\leq C_q \left[ \left( \int_{\tau_n}^{T} |Y_s^{1n}|^2 ds \right)^{\frac{q}{2}} \left( \int_{\tau_n}^{T} C_\sigma C_f^2 (1 + |X_s^{0, x_0}|^2) ds \right)^{\frac{q}{2}} \right. 
+ \left. \left( \int_{\tau_n}^{T} C_h (1 + |X_s^{0, x_0}|^\gamma + |Y_s^{1n}|) ds \right)^{\frac{q}{2}} \right] 
\leq C_q \left[ \left( \int_{\tau_n}^{T} |Y_s^{1n}|^2 ds \right)^{\frac{q}{2}} \right] \left[ \left( \int_{\tau_n}^{T} C_\sigma C_f^2 (1 + |X_s^{0, x_0}|^2) ds \right)^{\frac{q\gamma}{2}} \right]^{\frac{2}{q\gamma}} 
+ E \left[ \left( \int_{\tau_n}^{T} C_h (1 + |X_s^{0, x_0}|^\gamma) ds \right)^{\frac{q}{2}} \right] + E \left[ \left( \int_{\tau_n}^{T} |Y_s^{1n}|^\gamma ds \right)^{\frac{q}{2}} \right]^{\frac{2}{q\gamma}} 
\]

which converges to 0 as \(n\) converges to infinity, considering \((Y^{1n}, Z^{1n}) \in S_{T - \delta, T}^{q} \times \mathcal{H}_{T - \delta, T}^{q}\) for \(q > 1\) and \(X_s^{0, x_0}\) has moments of any order, and \(\tau_n \to T\), \(P\)-a.s.

Since, \(g^{1m} - g^{1n} \to 0\) in \(\mathcal{L}^q\) for any \(q > 1\) as \(n, m \to \infty\), through Lemma 3.8, we obtain that, \((\delta Y, \delta Z) \to 0\) in \(S_{T - \delta, T}^{q} \times \mathcal{H}_{T - \delta, T}^{q}\) for \(q < \bar{q}\) as \(n, m \to \infty\). It is easy to check that the limit of this sequence is the solution to BSDE (11) for \(t \in [T - \delta, T]\).

Applying the same method, we will find the solution for BSDE (11) for \(t \in [T - 2\delta, T - \delta]\). Repeating the same method backwardly finitely many times, we finally find the solution for BSDE (11) on the global interval \([0, T]\).

4. Existence of Nash equilibria.

**Theorem 4.1.** Let us assume that, Assumptions 1, 2 and 3 are fulfilled, Then there exist two deterministic functions \(\xi^i(t, x)\), \(i = 1, 2\), with polynomial growth and two pairs of \(P\)-measurable processes \((Y^i, Z^i)\), \(i = 1, 2\), with values in \(\mathbb{R}^{1+m}\) such that:

For \(i = 1, 2\),

(a) \(P\)-a.s., \(\forall s \leq T\), \(Y^i_s = \xi^i(s, X_s^{0, x_0})\) and \(Z^i(t) := (Z^i_t(\omega))_{t \leq T}\) is \(dt\)-square integrable;

(b) For any \(s \leq T\),
\[
\begin{aligned}
\begin{cases}
-dY^i_t &= \mathcal{H}_t(s, X^0_{s^0}, Y^i_t, Z^i_t, (u^*, v^*) \mathbb{I}(s, X^0_{s^0}, Y^1_t, Y^2_t, Z^1_t, Z^2_t)) \, ds - Z^i_t \, dB_s, \\
Y^i_T &= g^i(X^0_{0^0}).
\end{cases}
\end{aligned}
\]

(22)

Besides, the control \((u^*, v^*)\mathbb{I}(s, X^0_{s^0}, Y^1_t, Y^2_t, Z^1_t, Z^2_t))_{s \leq T}\) is admissible and a Nash equilibrium point for the recursive NZSDG.

**Proof.** The existence of solutions for BSDE (22) has been shown in [16]. We focus on the second conclusion below.

For \(s \leq T\), let us set \(u^*_s = u^*(s, X^0_{s^0}, Y^1_t, Y^2_t, Z^1_t, Z^2_t)\) and \(v^*_s = v^*(s, X^0_{s^0}, Y^1_t, Y^2_t, Z^1_t, Z^2_t)\), then \((u^*, v^*) \in \mathcal{M}_1\). From the definition of costs (10), we obviously have, \(Y^1_0 = J^1(u^*, v^*)\).

Next let \(u\) be an arbitrary element of \(\mathcal{M}_1\) and let us show that \(Y^1 \leq Y^1(1, u^*)\), which yields \(Y^1_T = J^1(u^*, v^*) \leq Y^1_T = J^1(u^*, v^*)\).

The control \((u^*, v^*)\) is admissible and thanks to Theorem 3.1, there exists a pair of \(\mathcal{P}\)-measurable processes \(Y^1(1, u^*), Z^1(1, u^*) \in \mathcal{S}_{\mathcal{F}}^\#(d\mathbb{P}^{u^*, v^*}) \times \mathcal{H}_{\mathcal{T}}^\#(d\mathbb{P}^{u^*, v^*}),\) for any \(q > 1\), such that

\[
Y^1_T(1, u^*) = g^1(X^0_T) + \int_{t}^{T} H_t(s, X^0_{s^0}, Y^1_s(1, u^*), Z^1_s(1, u^*), u_s, v^*_s) \, ds
\]

\[
- \int_{t}^{T} Z^1_s(1, u^*) \, dB_s, \quad \forall t \leq T.
\]

(23)

Afterwards, we aim to compare \(Y^1\) and \(Y^1(1, u^*)\). So let us denote by

\[
\Delta Y = Y^1 - Y^1(1, u^*) \quad \text{and} \quad \Delta Z = Z^1 - Z^1(1, u^*).
\]

For \(k \geq 0\), we define the stopping time \(\tau_k\) as follows:

\[
\tau_k := \inf\{s \geq 0, |\Delta Y_s| + \int_{0}^{s} |\Delta Z_r|^2 \, dr \geq k\} \land T.
\]

The sequence of stopping times \((\tau_k)_{k \geq 0}\) is of stationary type and converges to \(T\). Next setting \(a_t = qC_h \int_{t}^{T} \mathbb{I}(X^0_{r}) \, dr, \alpha \in (0, 2)\) and applying Itô-Meyer formula to \(e^{a_t}(|\Delta Y_t|^q)\) \((1 < q < \bar{q})\) between \(t \land \tau_k\) and \(t \land \tau_k\), we obtain: \(\forall t \leq T\),

\[
e^{a_t}|\Delta Y_{t \land \tau_k}|^q + c(q) \int_{t \land \tau_k}^{T} e^{a_s}|(\Delta Y_s)|^q |\Delta Y_s|^2 \, ds
\]

\[
+ \int_{t \land \tau_k}^{T} qC_h (1 + |X^0_{s^0}|^\alpha) e^{a_s} |\Delta Y_s|^q \, ds
\]

\[
= e^{a_t}|\Delta Y_{t \land \tau_k}|^q + \int_{t \land \tau_k}^{T} e^{a_s} |(\Delta Y_s)|^q 1_{\Delta Y_s > 0} \times
\]

\[
\times [H^1_t(s, X^0_{s^0}, Y^1_s, Z^1_s, u_s, v^*_s) - H^1_t(s, X^0_{s^0}, Y^1_s, Z^1_s, u_s, v^*_s) - H^1_t(s, X^0_{s^0}, Y^1_s, Z^1_s, u_s, v^*_s)]
\]

\[
\leq 1_{\Delta Y_s > 0} |\Delta Z_s|^q 1_{\Delta Y_s > 0} \Delta Z_s dB_s,
\]

where \(c(q) = \frac{q(\bar{q} - 1)}{2}\). Since

\[
1_{\Delta Y_s > 0} [H^1_t(s, X^0_{s^0}, Y^1_s, Z^1_s, u_s, v^*_s) - H^1_t(s, X^0_{s^0}, Y^1_s, Z^1_s, u_s, v^*_s)]
\]

\[
= 1_{\Delta Y_s > 0} [H^1_t(s, X^0_{s^0}, Y^1_s, Z^1_s, u_s, v^*_s) - H^1_t(s, X^0_{s^0}, Y^1_s, Z^1_s, u_s, v^*_s)]
\]

\[
+ H^1_t(s, X^0_{s^0}, Y^1_s, Z^1_s, u_s, v^*_s) - H^1_t(s, X^0_{s^0}, Y^1_s, Z^1_s, u_s, v^*_s)]
\]

\[
\leq 1_{\Delta Y_s > 0} |\Delta Z_s|^q 1_{\Delta Y_s > 0} f(s, X^0_{s^0}, u_s, v^*_s) + C_h (1 + |X^0_{s^0}|^\alpha) |(\Delta Y_s)|^q
\]
which is obtained by the generalized Isaacs' assumption 4 and the monotonic assumption (3)-(i) on \( h_1 \). Then, we have that, for any \( t \leq T \),

\[
|e^{\alpha t + r_\tau} (\Delta Y_{t\wedge \tau_k})^+|^q \leq e^{\alpha_{\tau_k}} |(\Delta Y_{\tau_k})^+|^q - q \int_{t \wedge \tau_k}^{\tau_k} e^{\alpha_s} |(\Delta Y_s)^+|^q - 1_{\Delta Y_s > 0} \Delta Z_s dB^{u,v^*}_s.
\]

By definition of the stopping time \( \tau_k \) and the fact that \( e^{\alpha_s} \) has moment of any order since \( \alpha \in (0, 2) \), we have

\[
E^{u,v^*}[\int_{t \wedge \tau_k}^{\tau_k} |(\Delta Y_s)^+|^q - 1_{\Delta Y_s > 0} \Delta Z_s dB^{u,v^*}_s] = 0.
\]

Then taking expectation on both sides of (24) under the probability \( P^{u,v^*} \), we obtain,

\[
E^{u,v^*}[e^{\alpha_{\tau_k}} (\Delta Y_{t\wedge \tau_k})^+|^q] \leq E^{u,v^*}[e^{\alpha_{\tau_k}} (\Delta Y_{\tau_k})^+|^q]
\]

Next taking into account of \( Y^{1(u,v^*)} \in S^2_T(dP^{u,v^*}) \) and \( Y^1 \) has a representation through \( \xi^1 \) which is deterministic and of polynomial growth, we deduce that,

\[
E^{u,v^*}[\sup_{s \leq T} e^{\alpha_s} \left(|Y^{1(u,v^*)}_s| + |Y^{1}_s|^q\right)] 
\leq E^{u,v^*}[e^{q \sup_{s \leq T} a_s}] + E^{u,v^*}[\sup_{s \leq T} (|Y^{1(u,v^*)}_s| + |Y^{1}_s|^q)] < \infty.
\]

The sequence \( (e^{\alpha_{\tau_k}} (\Delta Y_{t\wedge \tau_k})^+|^q) \) converges to \( e^{\alpha_T} (\Delta Y_T)|^q = 0 \), \( P^{u,v^*}\)-a.s., as \( k \to \infty \), then, it also converges to 0 in \( L^1(dP^{u,v^*}) \) since it is uniformly integrable by the above inequality.

Therefore, by passing \( n \) to the limit on both sides of (25) and using Fatou's lemma, we can show that \( E^{u,v^*}[e^{\alpha t} (\Delta Y_t)|^q] = 0 \), \( \forall t \leq T \), which implies that \( Y^1_t \leq Y^{1(u,v^*)} \), \( P \)-a.s., since the probability \( P^{u,v^*} \) and \( P \) are equivalent. Thus, \( Y^1_0 = J^1(u^*, v^*) \leq Y^{1(u,v^*)}_0 = J^1(u, v^*) \). In the same way, we can show that for any arbitrary element \( v \in M_2 \), \( Y^2_0 = J^2(u^*, v^*) \leq Y^{2(u,v^*)}_0 = J^2(u^*, v) \), which tell us that the pair \( (u^*, v^*) \) is a Nash equilibrium point for this game problem.

5. An example. In this section, we will give a simple example which satisfies our assumptions. We can clearly see that the Nash equilibrium point exists for this recursive game.

Let us consider a one-dimensional game problem and we assume the admissible control \((u, v)\) takes value from \( U_1 \times U_2 = [-1, 1] \times [-1, 1] \). In addition, in BSDE (11), let us take \( \sigma(t, x) = 1, f(t, x, u, v) = 1, h_1(t, x, y, u, v) = yu + v^2, h_2(t, x, y, u, v) = yv + u^2 \) and \( g^i(x) = x, i = 1, 2 \). We can verify that Assumptions 1-3 are satisfied. Besides, the recursive type costs \( J^i(u, v) \) of the players are defined by the initial values \( Y^{i(u,v)}_0 \) of the following BSDEs: for \( i = 1, 2, \)

\[
\begin{align*}
Y^{1(u,v)}_t &= X_T^{0,x_0} + \int_t^T \left[Z^{1(u,v)}_s + Y^{1(u,v)}_s u_s + v^2_s\right] ds - \int_t^T Z^{1(u,v)}_s dB_s; \quad \\
Y^{2(u,v)}_t &= X_T^{0,x_0} + \int_t^T \left[Z^{2(u,v)}_s + Y^{2(u,v)}_s v_s + u^2_s\right] ds - \int_t^T Z^{2(u,v)}_s dB_s.
\end{align*}
\]

From Theorem 3.1, the cost is well-posed. We then obtain Hamiltonians as follows: \( H_1(t, x, y_1, p, u, v) = p + y_1 u + v^2 \) and \( H_2(t, x, y_2, q, u, v) = q + y_2 v + u^2 \).
Then, it is not difficult to deduce that

$$u^*(y_1) = (-1) \times 1_{\{y_1 \geq 0\}} + 1 \times 1_{\{y_1 < 0\}}$$

and

$$v^*(y_2) = (-1) \times 1_{\{y_2 \geq 0\}} + 1 \times 1_{\{y_2 < 0\}}$$

are admissible controls which satisfy the generalized Isaacs assumption 4-(i). As we can see, the control \((u^*, v^*)\) is discontinuous on \((y_1, y_2)\). However, considering

$$y_1 u^*(y_1) = -|y_1|, \quad y_2 v^*(y_2) = -|y_2|$$

and

$$|u^*(y_1)|^2 = |u^*(y_1)|^2 = 1$$

are continuous functions on \((y_1, y_2)\), therefore, the mapping \((y_1, y_2, p, q) \mapsto (H_1, H_2)(y_1, y_2, p, q)\) is continuous with

$$H_1(y_1, y_2, p, q) = p + y_1 u^*(y_1) + |u^*(y_2)|^2$$

and

$$H_2(y_1, y_2, p, q) = q + y_2 v^*(y_2) + |v^*(y_1)|^2,$$

which tell us Assumption 4-(ii) is satisfied.

Finally, from Theorem 4.1, we know \((u^*(Y^1), v^*(Y^2))\) is one pair of Nash equilibrium point of this game, where \((Y^1, Y^2)\) are solutions of the following BSDEs:

$$
\begin{aligned}
Y^1_t &= X^0_t, X^0_t + \int_t^T [Z^1_s + Y^1_s u^*(Y^1_s) + |u^*(Y^2_s)|^2] \, ds - \int_t^T Z^1_s dB_s; \\
Y^2_t &= X^0_t, X^0_t + \int_t^T [Z^2_s + Y^2_s v^*(Y^2_s) + |v^*(Y^1_s)|^2] \, ds - \int_t^T Z^2_s dB_s.
\end{aligned}
$$

Acknowledgments. We are grateful to the anonymous referees for their useful comments and suggestions.

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Received February 2016; revised January 2017.

*E-mail address*: rmu@suda.edu.cn

*E-mail address*: wuzhen@sdu.edu.cn