Black Hole Thermodynamics and the Factor of 2 Problem

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Abstract

We show that the recent tunneling formulas for black hole radiation in static, spherically symmetric spacetimes follow as a consequence of the first law of black hole thermodynamics and the area-entropy relation based on the radiation temperature. A tunneling formula results even if the radiation temperature is different from the one originally derived by Hawking and this is discussed in the context of the recent factor of 2 problem. In particular, it is shown that if the radiation temperature is higher than the Hawking temperature by a factor of two, thermodynamics then leads to a tunneling formula which is exactly the one recently found to be canonically invariant.

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INTRODUCTION

In 1975, Hawking discovered [1] the remarkable fact that black holes radiate a thermal spectrum of particles and that the temperature of this radiation depends on the surface gravity $\kappa$ of the black hole. This discovery quantified the connections that had been found [2] between black holes and thermodynamics which became the laws of black hole thermodynamics [19], summarized as follows:

(0) The temperature of the black hole will be in equilibrium with the outside temperature. Since the first law of thermodynamics stated below gives $T^{-1} = \partial S/\partial E$ as the black hole temperature, this must be also equal to the temperature of the radiation.

(1) For a static black hole [20] of mass $M$, $dM = \kappa 8\pi dA$, which is analogous to the usual first law of thermodynamics $dE = T dS$ if we identify $\kappa 8\pi dA = T dS$.

(2) The entropy of a black hole is proportional to its horizon area and the sum of the entropy of the black hole and the ordinary entropy of the matter outside the black hole will never decrease.

(3) The temperature of the black hole is always greater than or equal to zero.

The rate of particle emission from the horizon is then proportional to the change in the black hole entropy $\Gamma \sim e^{\beta \Delta E} = e^{\Delta S}$ where $\beta = T^{-1}$.

Since these discoveries it remained somewhat mysterious where the particles constituting the radiation come from. A physical picture suggested by Hartle and Hawking [3] is that it comes from vacuum fluctuations tunneling through the horizon of the black hole and this viewpoint has been adopted by many authors. However, the original derivation was not directly connected with this viewpoint. Recently Parikh and Wilczek [4] and Volovik [5] have made the connection by calculating the the particle flux in a tunneling picture and showing that the temperatures agree with Hawking’s original results. Since then many calculations [6] have been performed using this method verifying that gives the correct temperatures in many different backgrounds.

However, Hawking’s result that radiation must be emitted by a black hole implies a connection with thermodynamics regardless of the actual value of the radiation temperature. So in what follows we will assume the validity of black hole thermodynamics and show that it
leads to a tunneling formula for the radiation but that the specific tunneling formula arrived at depends on the definition of the entropy of a black hole and so depends on exactly how the temperature is related to the surface gravity at the horizon. In particular, if the radiation temperature were different from that found by Hawking, the entropy-area law would change accordingly and this would change the resulting tunneling formula.

TUNNELING METHODS

There has recently appeared two different tunneling methods to calculate the Hawking temperature. Both formulas come from a semi-classical approximation with a scalar field on a curved background to calculate the tunneling amplitude but they differ by a factor of 2 in the resulting temperature. In the first method, called the null geodesic method, one uses Hamilton’s equation on null geodesics [4] to calculate the imaginary part of the action for particles tunneling across the horizon. This method gives the same temperatures as Hawking’s original calculation. The second method which we will call canonically invariant tunneling, one uses a particular anzatz for the action and then solves the Hamilton-Jacobi equations to find the imaginary part [7, 8]. This method leads to a formula that is slightly different from the null geodesic one in that it is canonically invariant and that it gives a temperature which is higher than the Hawking temperature by a factor of 2 [21].

The idea behind the tunneling methods can be seen by considering the space inside, but nearby, the event horizon of the black hole. Vacuum fluctuations can then occur and the quantity \( \xi \cdot p + \xi \cdot \overline{p} = 0 \) for particle four-momenta \( p \) and \( \overline{p} \) and Killing vector \( \xi \) must be conserved. Since \( \xi \) is time-like outside the horizon and space-like inside, the energy of a particle inside the horizon can be negative [9]. The horizon can then move inward with the initial and final horizon positions constituting the tunneling barrier.

The tunneling rate for particles through the event horizon using the null geodesic method is

\[
\Gamma = e^{-2\text{Im} \int_{r_{in}}^{r_{out}} p_r dr},
\]

(1)

where \( p_r \) is the momentum conjugate to \( r \) and \( r_{in} > r_{out} \) are the initial and final event horizon radii respectively. The canonically invariant tunneling method give a rate of

\[
\Gamma = e^{-\text{Im} \int p_r dr}.
\]

(2)
Given a metric with horizon and using Hamilton’s equation of motion, \( \dot{r} = \frac{dE}{dP} \bigg|_r \), one can calculate \( \Gamma \) and compare with \( \Gamma = e^{-\beta E} \) to extract the radiation temperature \( T = \beta^{-1} \).

In the next section we will show that the first law of black hole thermodynamics, along with the usual definitions of the entropy and surface gravity, leads to a tunneling formula to leading order in energy.

**THERMODYNAMICS AND TUNNELING**

In order to allow for an arbitrary temperature we will assume that \( T = \eta T_H \) where \( T_H \) is the original Hawking temperature and thus \( \eta \) is an arbitrary positive factor. The restriction to \( \eta = 1 \) gives the original Hawking temperature, \( \eta = 2 \) gives a temperature twice as high as found in the canonically invariant technique, and \( \eta = \frac{1}{2} \) would be the choice for the fluid analogues with co-tunneling [10].

We consider a general class of static, spherically symmetric spacetimes of the form

\[
ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\Omega^2,
\]

where the horizon \( r = r_H \) is given by \( f(r_H) = g(r_H) = 0 \).

The metric has a coordinate singularity at the horizon which will remove by transforming to Painlevé coordinates. Let \( dt \to dt - \Delta(r)dr \) and fix \( \Delta(r) \) so as to eliminate the singularity at \( r = r_H \) while requiring the spatial sections at any particular time to look like those of Minkowski space (so that the coefficient to \( dr^2 \) is 1), the result is

\[
ds^2 = -f(r)dt^2 + 2f(r) \sqrt{\frac{1 - g(r)}{f(r)g(r)}} dt dr + dr^2 + r^2 d\Omega^2.
\]

Test particles moving radially in this background will travel along null geodesics given by

\[
\frac{dr}{dt} \equiv \dot{r} = \sqrt{\frac{f(r)}{g(r)}} \left( \pm 1 - \sqrt{1 - g(r)} \right)
\]

where the positive (negative) sign gives outgoing (incoming) radial geodesics.

The surface gravity of the black hole is defined in terms of a time-like Killing vector \( \xi \) as \( \nabla_\xi \xi = \kappa \xi \), which reduces to a Christoffel component for our choice of metric

\[
\kappa = \Gamma^0_{00} = \frac{1}{2} \sqrt{\frac{1 - g(r)}{f(r)g(r)}} \frac{df(r)}{dr} \bigg|_{r = r_H}.
\]
The temperature of a black hole is related to the surface gravity $\kappa$ according to $T = \frac{\eta \kappa}{2\pi}$ where we have included our factor of $\eta$ described above. The entropy [11, 12] is then proportional to the area, $A$, of the event horizon via the first law of black hole thermodynamics as $S = \frac{A}{4\eta} = \frac{\pi r_H^2}{\eta}$. The factor of $\eta$ cancels out in the combination $TdS$ to reduce to the correct form of the first law of thermodynamics in terms of surface gravity. The temperature is then related to the entropy via $\frac{\partial S}{\partial E} = \frac{1}{T}$, where $E$ is the total energy.

We now take the thermodynamic quantities given above and apply them to the region near the horizon. The derivative of the entropy with respect to energy is

$$\frac{dS}{dE} = 2\pi r_H \frac{dr_H}{dE}$$

(7)

and if the energy of the black hole changes from $E_i$ to $E_f$, the corresponding change in the entropy is

$$\Delta S = \int_{E_i}^{E_f} 2\pi r_H \frac{dr_H}{dE} dE.$$  

(8)

The quantities $f(r)$ and $g(r)$ of our metric are zero at the horizon and so their expansion in powers of $r - r_H$ are

$$f(r) = f'(r_H)(r - r_H) + \cdots,$$

$$g(r) = g'(r_H)(r - r_H) + \cdots.$$  

(9)

We will only consider metrics where $f'(r_H)$ and $g'(r_H)$ are non-zero. This encompasses most of the important black hole metrics. In the case of metrics for which one of $f'(r_H)$ or $g'(r_H)$ is still zero, for example extremal black holes [13], the following analysis would need to be slightly modified. By inserting these expressions into (5) and choosing the positive sign since we are looking at the ‘outward’ geodesics we can write the near horizon radial geodesic equation as

$$\dot{r} = \frac{1}{2} \sqrt{f'(r_H)g'(r_H)(r - r_H)}.$$  

(10)

Since $f'(r_H)$ and $g'(r_H)$ are non-zero we can invert this equation to give

$$r - r_H = \frac{2\dot{r}}{\sqrt{f'(r_H)g'(r_H)}}.$$  

(11)

The surface gravity is found to be $\kappa = \frac{1}{2} \sqrt{f'(r_H)g'(r_H)}$, which gives the temperature as

$$T = \frac{\eta \sqrt{f'(r_H)g'(r_H)}}{4\pi}.$$  

(12)
For a small path from \( r_i \) to \( r_f \) containing \( r_H \) we make the connection to tunneling with the identity

\[
\text{Im} \int_{r_i}^{r_f} \frac{1}{r - r_H} dr = -\pi
\]

so that the horizon radius can be written as

\[
r_H = -\text{Im} \frac{1}{\pi} \int_{r_i}^{r_f} \frac{r_H}{r - r_H} dr
\]

and thus we can write our entropy change in the mathematically equivalent form

\[
\Delta S = -\text{Im} \int_{E_i}^{E_f} \int_{r_i}^{r_f} \frac{2r_H}{r - r_H} \frac{dr_H}{dE} dr dE.
\]

Now using (11) this becomes

\[
\Delta S = -\text{Im} \int_{E_i}^{E_f} \int_{r_i}^{r_f} \sqrt{f'(r_H)g'(r_H)} \frac{r_H}{r} \frac{dr_H}{dE} dr dE.
\]

Equations (7), and (12) along with the first law give

\[
r_H \frac{dr_H}{dE} = \frac{2}{\eta \sqrt{f'(r_H)g'(r_H)}}
\]

and our change in entropy (16) becomes

\[
\Delta S = -\frac{2}{\eta} \text{Im} \int_{m-\omega}^{m} \int_{r_i}^{r_f} \frac{dr}{\hat{r}} dE.
\]

where we have written the initial energy as the mass \( m \) and the final energy as \( m - \omega \) where \( \omega \) is interpreted as the energy radiated. To first order in \( \omega \) the right hand side is identical, aside from our factor of \( \eta \), to the expression for a particle tunneling through the event horizon on a null geodesic in the \( s \)-wave WKB approximation given in (11) above. Therefore we can interpret \( \omega \) as the energy of a tunneling particle. On the other hand, from the tunneling point of view

\[
\dot{r} = \dot{r}(r, m - \omega) = \dot{r}(r, m) + \mathcal{O}(\omega)
\]

where the higher order corrections lead to non-thermal corrections to the black body Hawking spectrum. Thus we have seen that, to first order in the energy of the tunneling particle, the laws of black hole thermodynamics lead directly to the following relation

\[
\Gamma \sim e^{\Delta S} = e^{-\frac{2}{\eta} \text{Im} \int_{r_{in}}^{r_{out}} p \, dr},
\]

in which \( \eta = 1 \) gives the null geodesic tunneling formula [4, 5] and \( \eta = 2 \) is equivalent to the canonically invariant tunneling formula \( \Gamma \sim e^{-\text{Im} \int p \, dr} \) [7, 14]. It is important to notice that
the formula follows mathematically from the entropy-area relation without using quantum field theory. The semi-classical field theory derivations from tunneling give similar formulas which differ in the choice of $\eta$. Once a tunneling formula has been derived, the temperature can be read off of it through $\eta$ and so the tunneling formulas amount to predictions for the factor of $\eta$.

As it stands we have two different tunneling formulas implying two different temperatures. In order to see which formula is correct, one needs to calculate the entropy of a black hole in an independent way and then compare with $S = \frac{A}{4\eta}$ to extract $\eta$. The canonically invariant tunneling prediction is that $\eta = 2$ whereas the null geodesic method predicts $\eta = 1$.

**SOME CONSEQUENCES OF THE TUNNELING FORMALISM**

We have found that the integrand found in the tunneling method is always related to the entropy as $\int \frac{\partial S}{\partial m'} dm'$ [15]. This relation is not obvious from the tunneling picture alone, but with the connection to thermodynamics established the reason is just the one given by Hawking when he wrote down the same formula in 1976 [11].

Given our present demonstration one can now calculate the self-gravitation corrections to the Hawking temperature in an easy way, directly from the expression for the entropy. For example, in the Schwarzschild case the entropy is $S = \frac{\pi r_H^2}{\eta}$, where $r_H = 2m$. Thus, writing $m' = m - \omega$ we have

$$\Delta S = \int \frac{\partial S}{\partial m'} (-d\omega) = -\frac{8\pi \omega}{\eta} \left( m - \frac{\omega}{2} \right)$$

which, for $\eta = 1$, is the expression derived in [4]. As a less trivial example the Reissner-Nordström black hole with line element

$$ds^2 = -f(r) \, dt^2 + f(r)^{-1} \, dr^2 + r^2 d\Omega^2$$

where $f(r) = \left( 1 - \frac{2m}{r} + \frac{q^2}{r^2} \right)$ and the radii of the inner and outer horizons are given by $r_{\pm} = m \pm \sqrt{m^2 - q^2}$. The entropy is $S = \frac{\pi r_{\pm}^2}{\eta}$ and so with $m' = m - \omega$ we find a formula for the change in entropy including self-gravitation corrections

$$\Delta S = -\frac{2\pi}{\eta} \left\{ m \sqrt{m^2 - q^2} + 2\omega(m - \omega/2) \\ - (m - \omega) \sqrt{(m - \omega)^2 - q^2} \right\}.$$
From which the rate $\Gamma \sim e^{\Delta S}$ follows giving the radiation temperature for $\omega = 0$ as

$$T = \frac{\eta}{2\pi} \frac{\sqrt{m^2 - q^2}}{m + \sqrt{m^2 - q^2}}$$

(24)

Notice that in the extreme case $q^2 = m^2$ the Hawking temperature vanishes, which implies that the tunneling amplitude also vanishes. This is consistent with cosmic censorship. However we should check that this is really true, since our derivation is not valid in the extremal case due to the fact that the integrand is no longer just a simple pole. Another reason to check the derivation for the extremal case is that, when self gravity corrections are included in the extremal limit our naive formula for the amplitude no longer seems to vanish. This may incline one to believe that cosmic censorship may be violated by higher order quantum corrections. In fact one can easily see that it does still vanish. In the near extremal case, with outer horizon at $r = r_H$ and inner horizon at $r = r_H - \epsilon$ the tunneling amplitude is $\Gamma \sim e^{-\alpha/\epsilon}$ for constant $\alpha > 0$ and thus vanishes in the extremal limit. In the exact extremal case one can use the fact that

$$\frac{df(z_0)}{dz_0} = \oint_C \frac{f(z)}{(z - z_0)^2}dz$$

(25)

to see that the integral vanishes exactly. This occurs before we even complete the $dE$ integral and so quantum corrections should not alter the result and we find that cosmic censorship is not violated by tunneling.

THE FACTOR OF 2 PROBLEM

We have shown that to first order in the energy of the tunneling particle the tunneling picture follows from the first law of black hole thermodynamics and the entropy-area relation. The specific form of the tunneling formula then follows from the specific value of the radiation temperature. The choice $\eta = 1$ gives the null geodesic tunneling formula, used by many authors, corresponding to the original Hawking temperature.

However, Chowdhury [14] in a recent analysis of the null geodesic tunneling formula has shown that it is not invariant under canonical transformations, but that the same formula with a factor of $1/2$ in the exponent is canonically invariant. It seems clear physically, as Chowdhury argues, that one should use the canonically invariant formula $\Gamma \sim e^{-\text{Im} \int pdr}$. This formula reduces to our $\eta = 2$ formula for black hole horizons because a horizon is a
one-way tunneling barrier and so the transmission coefficient for the half of the path which
is directed inward is equal to $1 \ [22]$. This is interesting in light of the recent results of
Akhmedov et.al. who showed that the temperatures found using the canonically invariant
tunneling formula differ by a factor of 2 from the Hawking results in several backgrounds \[7\].
Nakamura \[16\] has recently offered an explanation of this difference in terms of the alternate
vacuum used by the tunneling methods and this may solve the factor of 2 problem, but
if so it it raises a further question. How is it that, when the null geodesic method (being
a tunneling method) uses the alternate vacuum, it still reproduces the original Hawking
temperatures without the factor of 2?

Our present demonstration shows that the paradox can be resolved if the black hole
temperature really is a factor of two higher than that originally given by Hawking. This
would mean $\eta = 2$ and the black hole entropy would be $S = A/8$ instead of the usual
relation, leading to a different formula for the tunneling. In fact, the formula that results
is exactly the canonically invariant formula, $\Gamma \sim e^{-\text{Im} \int p dr}$, given by Akhmedov et.al and
Chowdhury.

We conclude that setting $\eta = 2$ makes the tunneling formula consistent with Nakamura’s
result for tunneling vacua, consistent with Chowdhury’s result on the canonical invariance
and consistent with the tunneling derivation of Akhmedov et.al. However, the resulting
formula contradicts Hawking’s original value of the black hole radiation temperature. The
tunneling methods each give a different prediction for the factor of $\eta$ in the entropy-area
relation and therefore it is necessary to calculate the black hole entropy in a completely
independent way to find out which is correct. There are different groups working on calculat-
ing the entropy using stringy or loopy microstates \[17, 18\] and it is hoped that they will
independently fix $\eta$.

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[19] See section V of Wald (1979) [2].

[20] This formula has the usual generalization for the cases of charged and spinning black holes [11].

[21] Temperatures which differ from the Hawking temperature have been seen to occur also in fluid analogues to black holes sometimes called sonic black holes or dumb holes where the cotunneling of quasiparticles leads to temperatures which differ from the Hawking temperature by a factor of $\frac{1}{2}$ [10].

[22] See equation (23) in [14].