Breached pairing in trapped three-color atomic Fermi gases

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We introduce an exactly solvable model for trapped three-color atom gases. Applications to a cigar-shaped trapped cold fermions reveals a complex structure of breached pairing phases. We find two competing superfluid phases at weak and intermediate couplings, each one with two color pair condensates, that can be distinguished from density profile measurements.

PACS numbers: 05.30.Fk, 02.30.Ik, 03.75.Mn, 03.75.Ss

Macroscopic coherent phenomena in matter, such as superconductivity and superfluidity, are deep manifestations of wave mechanics with consequences not only in technology but also for the fundamental understanding of vacuum condensates in the standard model. Color superconductivity is predicted to occur in quark matter at sufficiently high density and low temperatures \cite{1}. Quarks, having three different colors (red, green, blue) and a strong attractive interaction, allow for more diverse pairing patterns compared to the SU(2) Cooper pairing in classic metallic superconductors. Such diversity, likewise, makes it hard to establish the particular pairing symmetry favored by nature. With the advent of ultracold trapped Fermi gases a window of opportunities has opened to address some of these fundamental questions, at least, in a qualitative fashion (see for example \cite{2}). One can certainly manipulate different atomic species and hyperfine states to effectively generate multicolor Fermi gases with attractive interactions.

It is the goal of this paper to investigate the superfluid behavior of an \textit{imbalanced} three-color Fermi gas by means of an exactly-solvable pairing model of the Richardson-Gaudin (RG) type, derived from the quadratic invariants of the SO(6) RG model \cite{3, 4}. Previous studies using standard mean-field \cite{5}, density matrix renormalization group \cite{6}, or Bethe ansatz \cite{7} techniques concentrated on the competition between a trionic or barionic phase and a color superfluid phase. A main result of our work, from the standpoint of collective behavior, is the competition between \textit{breached-pair} (BP) and \textit{unbreached-pair} (UP) superfluid phases in a polarized multicolor Fermi gas. As in the two-color case, where density profiles have been recently investigated experimentally \cite{8} and theoretically \cite{9}, an analogue of the BP or Sarma phase \cite{10, 11} appear in multicolor polarized Fermi gases. In this case we find a complex structure of breached pairing as well as the coexistence of two pair condensates. While the possibility of coexistence of several superfluid phases has been suggested in \cite{12} using an LDA theory, we predict the existence two distinct \textit{color fermionic condensates}. Within our model this is a genuine effect, although care must be exercised when contrasted to experiments since interactions not included in our model could make this phase unstable against the formation of a fraction of bound trimers in the strong coupling limit. However, population imbalance as well as the experimental realization of a stable three-color atomic gas with different atomic masses and/or different Feshbach resonances, as recently reported in experiments with balanced mixtures of three components \cite{13} \textsuperscript{6}Li atoms \cite{14}, could stabilize it.

Consider the SU(3) color-symmetric Hamiltonian

\begin{equation}
H = \sum_{i}^{L} \varepsilon_{i} N_{i} - g \sum_{i}^{L} \sum_{\alpha} A_{i\alpha}^{\dagger} A_{i\alpha}^{} \quad (1)
\end{equation}

for \(L\) levels of energy \(\varepsilon_{i}\), where \(\alpha=(\text{red}(R), \text{green}(G), \text{blue}(B))\) is the color index, \(N_{i} = \sum_{\alpha} N_{i\alpha}\) is the number operator of the orbit \(i\), \(A_{i\alpha}^{\dagger} = \sum_{\beta, \gamma} \varepsilon_{\alpha \beta \gamma} a_{i\beta}^{\dagger} a_{i\gamma}^{}\), \(A_{i\alpha} = (A_{i\alpha}^{\dagger})^{\dagger}\), are the pair creators, and \(g > 0\) the pairing strength. Here \(a_{i\beta}^{\dagger}\) creates a (canonical) fermionic atom in level \(i\) with color \(\beta\), and \(\varepsilon_{\alpha \beta \gamma}\) is the completely antisymmetric tensor in color space. The SO(6) algebra (15 generators) is completed by the 9 particle-hole operators \(C_{i,\alpha \beta} \equiv a_{i\alpha}^{\dagger} a_{i\beta}\). These 9 operators, which include the number operators for the three different colors in the level \(i\), \(N_{i\alpha} = C_{i,\alpha \alpha}\), close an U(3) subalgebra of SO(6).

The Hamiltonian \cite{11} has equal pairing strengths \(g\) (or scattering lengths), and equal single particle energies (masses) for the three colors. It describes a three-color Fermi gas with attractive contact interactions in the low-density limit. Moreover, the SU(3) symmetry is preserved and, thus, the eigenstates are organized in degenerate SU(3) multiplets. The SU(3) symmetry, however, may be broken by choosing a different combination of integrals of motion and, for example, one can generate an integrable model of atoms with unequal masses \cite{15}.

The exact solution of the SO(6) RG model, being an algebra of rank three, depends on three sets of spectral parameters. The first set includes the usual pair energies \(e_{\alpha}\) of the SO(6) algebra, while the other two sets, composed of the spectral parameters \(\omega_{\alpha}\) and \(\gamma_{\alpha}\), are associated with the SU(3) subalgebra of SO(6). The complete set of spectral parameters satisfy the generalized...
Richardson equations

\[
\sum_{\beta(\neq \alpha)}^{M} \frac{2}{\epsilon_{\beta} - \epsilon_{\alpha}} - \sum_{\beta}^{N_{B}} \frac{1}{\omega_{\beta} - \epsilon_{\alpha}} + \sum_{i}^{L} \frac{\nu_{i} - 1}{2\varepsilon_{i} - \epsilon_{\alpha}} = -\frac{1}{4g}
\]

\[
\sum_{\beta(\neq \alpha)}^{N_{B}} \frac{2}{\epsilon_{\beta} - \omega_{\alpha}} - \sum_{\beta}^{M} \frac{1}{\omega_{\beta} - \epsilon_{\alpha}} - \sum_{i}^{Q} \frac{1}{\gamma_{\beta} - \omega_{\alpha}} = 0 \quad (2)
\]

\[
\sum_{\beta}^{Q} \frac{2}{\gamma_{\beta} - \gamma_{\alpha}} - \sum_{\beta(\neq \alpha)}^{N_{B}} \frac{1}{\omega_{\beta} - \gamma_{\alpha}} - \sum_{i}^{L} \frac{\nu_{i}}{2\varepsilon_{i} - \gamma_{\alpha}} = 0.
\]

The number of spectral parameters in each set is determined by the number of particles of each color $N_{\alpha}$, and by the total seniority quantum number $\nu$: $M = (N - \nu)/2$ and $Q = (N_{B} + N_{R} - N_{G} + \nu)/2$, where we have assumed, without loss of generality, $N_{G} \geq N_{R} \geq N_{B}$. The seniority of level $i$, $\nu_{i}$, counts the number of unpaired fermions, and is defined from $A_{\alpha}|\nu_{i}\rangle = 0$, $N_{i}|\nu_{i}\rangle = \nu_{i}|\nu_{i}\rangle$, $\nu_{i} = 0, 1$. The total seniority is $\nu = \sum_{i} \nu_{i}$. Hamiltonian (1) preserves the seniority since it can create or destroy pairs of particles conserving the number parity of the level, i.e., for a given configuration each level has an even (odd) number of particles where $\nu_{i} = 0$ (1).

The eigenvalues of Hamiltonian (1) are

\[
E = \sum_{\alpha}^{M} e_{\alpha} + \sum_{i=1}^{L} \varepsilon_{i} \nu_{i}, \quad (3)
\]

and only depend on the parameters $e_{\alpha}$. The corresponding eigenfunctions, though, are determined by the three sets of parameters. The solutions of the Richardson equations defines a basis which spans the complete many-body Hilbert space of the system.

Assume a polarized gas composed of $N = N_{G} + N_{R} + N_{B}$ fermionic atoms. In the weak coupling limit the energy levels are filled up to the Fermi energy for each color $\varepsilon_{N_{\alpha}}$, a situation depicted in Fig. 1 A for $N_{G} = 80$, $N_{R} = 50$ and $N_{B} = 20$. In this case the seniorities $\nu_{i}$ are equal to 1 for $i \leq 20$ and 50 $< i \leq 80$, and they are 0 for $20 < i \leq 50$ and $i > 80$, defining a clear separation of the Hilbert space into regions of odd particle states ($\nu_{i} = 1$) and regions of even particle states ($\nu_{i} = 0$). When the pairing interaction $g$ is switched on, $R-B$ pairs from the first region ($i \leq 20$) cannot scatter to the second region ($20 < i \leq 50$) due to Pauli blocking, and they have to jump this forbidden region to scatter into the third region $50 < i \leq 80$. Analogously, $G-R$ pairs of the second region have to jump the third forbidden region to scatter into the fourth region ($i > 80$). This configuration, that we call BP state, turns out to be the ground state (GS) at weak coupling. For larger values of $g$ other configurations compete with the BP state. Those configurations, at the cost of increasing their kinetic energy, reduce the effect of Pauli blocking, therefore, facilitating the pair scattering into interior level regions. In particular, the UP state depicted in panel B has no blocked interior region, and will be the GS of the system at strong coupling.

![FIG. 1: Occupation number for the BP state (A) and the UP state (B).](image)

![FIG. 2: Phase diagram of a 1D partially polarized trapped Fermi gas with $N = 150$, $N_{R} = 50$, $P = (N_{G} - N_{B})/(N_{G} + N_{B})$, and $L = 500$.](image)

Although Hamiltonian (1) is exactly solvable in any dimension, for simplicity we will consider a system of $N = 150$ ($N_{R} = 50$) fermionic atoms trapped by a 1D harmonic potential (of frequency $\omega$) with an energy cutoff at $E_{\text{cut}} = 500h\omega$, implying $L = 500$ threefold degenerate single particle levels. In the weak coupling limit, the Richardson equations (2) decouple into independent sets of equations, each one related to the single particle level, partially or fully occupied, as discussed above. These equations can be solved analytically producing three sets of spectral parameters. These parameters are used as an initial seed in an iterative procedure in which the coupling constant $g$ is systematically increased by using the
solution of the previous step as the new initial guess. In this way the initial solution at weak coupling is evolved up to the desire value of $g$. We performed extensive calculations to determine the quantum phase diagram of this system as a function of the pairing strength $g$ and the polarization $P = (N_G - N_B)/(N_G + N_B)$.

Two color superfluid phases emerge as a function of color asymmetry and pairing strength (see fig. 2). A first order quantum phase transition, due to level crossing, separates the BP and UP superfluid phases, which are labeled by different sets of seniority quantum numbers. On the other hand, there is a smooth crossover between the two superfluid phases and a normal, fluctuation-dominated state depicted by a thick grey line [16]. The normal Fermi-liquid-like state is dominated by pairing fluctuations which are fully taken into account by the exact solution. We adopted the criterium that the normalized state depicted by a thick grey line [16].

FIG. 3: Spectral parameters of the BP state with $P = 0.8$ and the UP state with $P = 0.4$, for $g = 0.16$.

The BP state has an inner condensate of $10\%$ of the pair energies $e_\alpha$ describes $G-R$ Cooper pairs, while the lower arc corresponds to $R-B$ Cooper pairs. An analysis in terms of the eigenvalues of the two-body density matrix would lead to a macroscopic eigenvalue of the $G-R$ and the $R-B$ pair density matrices and no macroscopic eigenvalue in the $G-B$ pair density matrix. The appearance of these two condensates will be reflected in the occupation probabilities that can be calculated using the Hellmann-Feynman theorem on the integrals of motion as will be explained in a forthcoming paper [18].

The occupation probabilities, $(N_i)$, for both states are depicted in fig. 4. In the BP state panel we see how the $G-R$ pairs avoid the region $50 < i \leq 90$, Pauli blocked by the $G$ atoms, to scatter off into the region $i > 90$. Analogously, though less evident due to the smaller number of pairs, the $10$ $R-B$ pairs avoid the region $10 < i \leq 50$, blocked by the $R$ atoms, to scatter off into the region $i > 50$. We may realize that in the latter case the blocking is not perfect, unlike in the two-component gases, because of the depletion of the $R$ atoms in the $10 < i \leq 50$ region due to $G-R$ pairing. The $G-B$ pairing is prevented from being realized due to two consecutive blocked regions. On the contrary, in the UP state pairs do not have to avoid blocked regions. The $G-R$ pairs in the $50 < i \leq 70$ region scatter off into the regions $i > 70$, and $R-B$ pairs in the region $i \leq 30$ scatter off into the region $30 < i \leq 50$ as well as into the depleted region $i > 50$. These two different physical scenarios manifest in the arc geometry of the spectral parameters in fig. 3.

While the BP state has an inner condensate of $10$ $R-B$. FIG. 4: (Color online) Occupation numbers of the BP state with $P = 0.8$ and the UP state with $P = 0.4$, for $g = 0.16$. In the inset we display the different color contributions to $(N_i)$. 

In both cases the pair energies form two separate arcs in the complex plane, indicating the existence of two color fermionic pair condensates. In the strong coupling limit the two arcs coalesce into a single arc corresponding to a single BEC condensate with negative real part of their pair energies. The lower arc is overlapping with arcs of the two other spectral parameters $\omega_\alpha$ and $\gamma_\alpha$ which account for the couplings in the SU(3) color subspace. The interpretation is that the upper arc of isolated pair energies $e_\alpha$ would lead to a macroscopic eigenvalue of the $G-R$ and the $R-B$ pair density matrices and no macroscopic eigenvalue in the $G-B$ pair density matrix. The appearance of these two condensates will be reflected in the occupation probabilities that can be calculated using the Hellmann-Feynman theorem on the integrals of motion as will be explained in a forthcoming paper [18].
and an outer condensate of 40 $G$-$R$ pairs, the UP state has 30 condensed $R$-$B$ and 20 condensed $G$-$R$ pairs.

An experimental way to uncover the nature of the color superfluid correlations consists in measuring the density clouds of the trapped Fermi gas. Figure 5 shows the radial density profiles for these two states normalized to the clouds of the trapped Fermi gas. Figure 5 shows the radial density profiles for these two states normalized to the clouds of the trapped Fermi gas. Figure 5 shows the radial density profiles for these two states normalized to the clouds of the trapped Fermi gas. Figure 5 shows the radial density profiles for these two states normalized to the clouds of the trapped Fermi gas. Figure 5 shows the radial density profiles for these two states normalized to the clouds of the trapped Fermi gas.

FIG. 5: (Color online) Radial density profiles for the BP and UP states for $g = 0.16$. The dash curve corresponds to the Thomas-Fermi approximation for the $R$ atoms. The insets show the density differences between the $R$ and $B$ (dash dark gray), and $G$ and $R$ (solid light gray) species.

One can think about measuring the fraction of each of the two color pair condensates by exploiting the differences in the color-pair dependent Feshbach resonances. It would then be possible to use the ramp technique as described in [14]. sweeping the magnetic field such that only one class of pairs are transformed into bound molecules, allowing for the determination of the corresponding fraction of the condensate. A bigger ramp would then transform all pairs into molecules, therefore, allowing measurement of the complete fraction of the condensate.

We acknowledge fruitful discussions with G. G. Dussel. This work was supported in part by the grant FIS2006-12783-C03-01 of the Spanish DGI. B.E. was supported by the Spanish CE-CAM.

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