RING HOMEOMORPHISMS AND PRIME ENDS

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Abstract. We show that every homeomorphic $W^{1,1}_{\text{loc}}$ solution $f$ of a Beltrami equation $\overline{\partial} f = \mu \partial f$ in a domain $D \subseteq \mathbb{C}$ is the so-called ring $Q$–homeomorphism with $Q(z) = K_T^Q(z, z_0)$ where $K_T^Q(z, z_0)$ is the tangent (angular) dilatation quotient of the equation with respect to an arbitrary point $z_0 \in \overline{D}$. In this connection, we develop the theory of the boundary behavior of the ring $Q$–homeomorphisms with respect to prime ends. On this basis, we show that, for wide classes of degenerate Beltrami equations $\overline{\partial} f = \mu \partial f$, there exist regular solutions of the Dirichlet problem in arbitrary simply connected domains in $\mathbb{C}$ and pseudoregular and multivalent solutions in arbitrary finitely connected domains in $\mathbb{C}$ with boundary datum $\varphi$ that are continuous with respect to the topology of prime ends.

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1. Introduction

Let $D$ be a domain in the complex plane $\mathbb{C}$ and $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. (almost everywhere) in $D$. A Beltrami equation is an equation of the form

$$f_z = \mu(z) f_z$$

where $f = \frac{\partial f}{\partial z} = \frac{f_x + if_y}{2}$, $f = \frac{\partial f}{\partial \bar{z}} = \frac{f_x - if_y}{2}$, $z = x + iy$, and $f_x$ and $f_y$ are partial derivatives of $f$ in $x$ and $y$, correspondingly.

Boundary value problems for the Beltrami equations are due to the well-known Riemann dissertation in the case of $\mu(z) = 0$ and to the papers of Hilbert (1904, 1924) and Poincare (1910) for the corresponding Cauchy–Riemann system.

The classic Dirichlet problem for the Beltrami equation (1.1) in a domain $D \subset \mathbb{C}$ is the problem on the existence of a continuous function $f : D \to \mathbb{C}$ having partial derivatives of the first order a.e., satisfying (1.1) a.e. and such that

$$\lim_{z \to \zeta} \Re f(z) = \varphi(\zeta) \quad \forall \zeta \in \partial D$$

for a prescribed continuous function $\varphi : \partial D \to \mathbb{R}$, see, e.g., [1] and [49].

The function $\mu$ is called the complex coefficient and

$$K_{\mu}(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|}$$

the dilatation quotient of the equation (1.1). The Beltrami equation (1.1) is said to be degenerate if $\text{ess sup } K_{\mu}(z) = \infty$.

The existence of homeomorphic $W^{1,1}_{\text{loc}}$ solutions was recently established to many degenerate Beltrami equations, see, e.g., [10] and [24] and references therein. The theory of boundary behavior of homeomorphic solutions and of the Dirichlet problem for a wide circle of degenerate Beltrami equations in Jordan domains was developed in [16], [17], [18] and [34].

To study the similar problem in domains with more complicated boundaries we need to apply the theory of prime ends by Caratheodory, see, e.g., his paper [3] or Chapter 9 in monograph [4].

The main difference in the case is that $\varphi$ should be already a function of a boundary element (prime end $P$) but not of a boundary point. Moreover, (1.2) should be replaced by the condition

$$\lim_{n \to \infty} \Re f(z_n) = \varphi(P)$$

for all sequences of points $z_n \in D$ converging to prime ends $P$ of the domain $D$. Note that (1.4) is equivalent to the condition that

$$\lim_{z \to P} \Re f(z) = \varphi(P)$$

along any ways in $D$ going to the prime ends $P$ of the domain $D$.

Later on, $D_P$ denotes the completion of the domain $D$ by its prime ends and $E_D$ denotes the space of these prime ends, both with the topology of prime ends described in Section 9.5 of monograph [4]. In addition, continuity of mappings $f : \overline{D}_P \to \overline{D}_P$ and boundary functions $\varphi : E_D \to \mathbb{R}$ should mean with respect to the given topology of prime ends.
Remark 1.1. This topology can be described in terms of metrics. Namely, as known, every bounded finitely connected domain $D$ in $\mathbb{C}$ can be mapped by a conformal mapping $g_0$ onto the so-called circular domain $D_0$ whose boundary consists of a finite collection of mutually disjoint circles and isolated points, see, e.g., Theorem V.6.2 in [8]. Moreover, isolated singular points of bounded conformal mappings are removable by Theorem 1.2 in [3] due to Weierstrass. Hence isolated points of $\partial D$ correspond to isolated points of $\partial D_0$ and inversely.

Reducing this case to the Carathéodory theorem, see, e.g., Theorem 9.4 in [1] for simple connected domains, we have a natural one-to-one correspondence between points of $\partial D_0$ and prime ends of the domain $D$. Determine in $\overline{D}_P$ the metric $\rho_0(p_1, p_2) = |\tilde{g}_0(p_1) - \tilde{g}_0(p_2)|$ where $\tilde{g}_0$ is the extension of $g_0$ to $\overline{D}_P$ just mentioned.

If $g_*$ is another conformal mapping of the domain $D$ on a circular domain $D_*$, then the corresponding metric $\rho_*(p_1, p_2) = |\tilde{g}_*(p_1) - \tilde{g}_*(p_2)|$ generates the same convergence in $\overline{D}_P$ as the metric $\rho_0$ because $g_0 \circ g_*^{-1}$ is a conformal mapping between the domains $D_*$ and $D_0$ that is extended to a homeomorphism between $\overline{D}_*$ and $\overline{D}_0$, see, e.g., Theorem V.6.1′ in [8]. Consequently, the given metrics induce the same topology in the space $\overline{D}_P$.

This topology coincides with topology of prime ends described in inner terms of the domain $D$ in Section 9.5 of [4]. Later on, we prefer to apply the description of the topology of prime ends in terms of the given metrics because it is more clear, more convenient and it is important for us just metrizability of $\overline{D}_P$. Note also that the space $\overline{D}_P$ for every bounded finitely connected domain $D$ in $\mathbb{C}$ with the given topology is compact because the closure of the circular domain $D_0$ is a compact space and by the construction $\tilde{g}_0 : \overline{D}_P \to D_0$ is a homeomorphism.

Applying the description of the topology of prime ends in Section 9.5 of [4], we reduce the case of bounded finitely connected domains to Theorem 9.3 in [4] for simple connected domains and obtain the following useful fact.

Lemma 1.2. Each prime end $P$ of a bounded finitely connected domain $D$ in $\mathbb{C}$ contains a chain of cross–cuts $\sigma_m$ lying on circles $S(z_0, r_m)$ with $z_0 \in \partial D$ and $r_m \to 0$ as $m \to \infty$.

Given a point $z_0$ in $\mathbb{C}$, we apply here the more refined quantity than $K_\mu(z)$:

$$K^T_\mu(z, z_0) = \frac{1 - \frac{z-z_0}{\bar{z}-\bar{z}_0} \mu(z)^2}{1 - |\mu(z)|^2}$$

that is called the tangent dilatation quotient of the Beltrami equation (1.1) with respect to $z_0$, see, e.g., [38], cf. the corresponding terms and notations in [?, 9, ?] and [31]. The given term was first introduced in [98] and its geometric sense was described in [34], see also [24], Section 11.3. Note that

$$K_\mu^{-1}(z) \leq K^T_\mu(z, z_0) \leq K_\mu(z) \quad \forall \ z \in D \quad \forall \ z_0 \in \mathbb{C}$$

and the given estimates are precise. The quantity (1.6) takes into account not only the modulus of the complex coefficient $\mu$ but also its argument.

Throughout this paper, $B(z_0, r) = \{ z \in \mathbb{C} : |z - z_0| < r \}$, $D = B(0, 1)$, $S(z_0, r) = \{ z \in \mathbb{C} : |z - z_0| = r \}$, $R(z_0, r_1, r_2) = \{ z \in \mathbb{C} : r_1 < |z - z_0| < r_2 \}$. 


2. Regular domains

First of all, recall the following topological notion. A domain $D \subset \mathbb{C}$ is said to be locally connected at a point $z_0 \in \partial D$ if, for every neighborhood $U$ of the point $z_0$, there is a neighborhood $V \subseteq U$ of $z_0$ such that $V \cap D$ is connected. If this condition holds for all $z_0 \in \partial D$, then $D$ is said to be locally connected on $\partial D$. For a domain that is locally connected on $\partial D$, there is a natural one-to-one correspondence between prime ends of $D$ and points of $\partial D$ and the topology of prime ends coincides with the Euclidean topology. Note that every Jordan domain $D$ in $\mathbb{C}$ is locally connected on $\partial D$, see, e.g., [50], p. 66.

Now, recall that the (conformal) modulus of a family $\Gamma$ of paths $\gamma$ in $\mathbb{C}$ is the quantity

$$M(\Gamma) = \inf_{\varrho \in \text{adm} \Gamma} \int_{\mathbb{C}} \varrho^2(z) \, dm(z)$$

where a Borel function $\varrho : \mathbb{C} \to [0, \infty]$ is admissible for $\Gamma$, write $\varrho \in \text{adm} \Gamma$, if

$$\int_{\gamma} \varrho \, ds \geq 1 \quad \forall \, \gamma \in \Gamma.$$

Here $s$ is a natural parameter of the arc length on $\gamma$.

Later on, given sets $A$, $B$ and $C$ in $\mathbb{C}$, $\Delta(A, B; C)$ denotes a family of all paths $\gamma : [a, b] \to \mathbb{C}$ joining $A$ and $B$ in $C$, i.e. $\gamma(a) \in A$, $\gamma(b) \in B$ and $\gamma(t) \in C$ for all $t \in (a, b)$.

We say that $\partial D$ is weakly flat at a point $z_0 \in \partial D$ if, for every neighborhood $U$ of the point $z_0$ and every number $P > 0$, there is a neighborhood $V \subset U$ of $z_0$ such that

$$M(\Delta(E, F; D)) \geq P$$

for all continua $E$ and $F$ in $D$ intersecting $\partial U$ and $\partial V$. We say that $\partial D$ is weakly flat if it is weakly flat at each point $z_0 \in \partial D$.

We also say that a point $z_0 \in \partial D$ is strongly accessible if, for every neighborhood $U$ of the point $z_0$, there exist a compactum $E$ in $D$, a neighborhood $V \subset U$ of $z_0$ and a number $\delta > 0$ such that

$$M(\Delta(E, F; D)) \geq \delta$$

for all continua $F$ in $D$ intersecting $\partial U$ and $\partial V$. We say that $\partial D$ is strongly accessible if each point $z_0 \in \partial D$ is strongly accessible.

It is easy to see that if a domain $D$ in $\mathbb{C}$ is weakly flat at a point $z_0 \in \partial D$, then the point $z_0$ is strongly accessible from $D$. The following fact is fundamental, see, e.g., Lemma 5.1 in [20] or Lemma 3.15 in [24].

**Lemma 2.1.** If a domain $D$ in $\mathbb{C}$ is weakly flat at a point $z_0 \in \partial D$, then $D$ is locally connected at $z_0$.

The notions of strong accessibility and weak flatness at boundary points of a domain in $\mathbb{C}$ defined in [19], see also [20] and [35], are localizations and generalizations of the corresponding notions introduced in [22] and [23], cf. with the properties $P_1$ and $P_2$ by Väisälä in [48] and also with the quasiconformal accessibility and the quasiconformal flatness by Näkki in [27].
A domain $D \subset \mathbb{C}$ is called a quasiextremal distance domain, abbr. QED-domain, see [7], if
\begin{equation}
M(\Delta(E, F; \mathbb{C})) \leq K \cdot M(\Delta(E, F; D))
\end{equation}
for some $K \geq 1$ and all pairs of nonintersecting continua $E$ and $F$ in $D$.

It is well known, see, e.g., Theorem 10.12 in [48], that
\begin{equation}
M(\Delta(E, F; \mathbb{C})) \geq \frac{2}{\pi} \log \frac{R}{r}
\end{equation}
for any sets $E$ and $F$ in $\mathbb{C}$ intersecting all the circles $S(z_0, \rho)$, $\rho \in (r, R)$. Hence a QED-domain has a weakly flat boundary. One example in [24], Section 3.8, shows that the inverse conclusion is not true even in the case of simply connected domains in $\mathbb{C}$.

A domain $D \subset \mathbb{C}$ is called a uniform domain if each pair of points $z_1$ and $z_2 \in D$ can be joined with a rectifiable curve $\gamma$ in $D$ such that
\begin{equation}
s(\gamma) \leq a \cdot |z_1 - z_2|
\end{equation}
and
\begin{equation}
\min_{i=1, 2} s(\gamma(z_i, z)) \leq b \cdot \text{dist}(z, \partial D)
\end{equation}
for all $z \in \gamma$ where $\gamma(z_i, z)$ is the portion of $\gamma$ bounded by $z_i$ and $z$, see [26]. It is known that every uniform domain is a QED-domain but there exist QED-domains that are not uniform, see [7]. Bounded convex domains and bounded domains with smooth boundaries are simple examples of uniform domains and, consequently, QED-domains as well as domains with weakly flat boundaries.

It is also often met with the so-called Lipschitz domains in the mapping theory and in the theory of differential equations. Recall first that $\varphi : U \to \mathbb{C}$ is said to be a Lipschitz map provided $|\varphi(z_1) - \varphi(z_2)| \leq M \cdot |z_1 - z_2|$ for some $M < \infty$ and for all $z_1$ and $z_2 \in U$, and a bi-Lipschitz map if in addition $M^* |z_1 - z_2| \leq |\varphi(z_1) - \varphi(z_2)|$ for some $M^* > 0$ and for all $z_1$ and $z_2 \in U$. They say that $D$ in $\mathbb{C}$ is a Lipschitz domain if every point $z_0 \in \partial D$ has a neighborhood $U$ that can be mapped by a bi-Lipschitz homeomorphism $\varphi$ onto the unit disk $\mathbb{D}$ in $\mathbb{C}$ such that $\varphi(\partial D \cap U)$ is the intersection of $\mathbb{D}$ with the real axis. Note that a bi-Lipschitz homeomorphism is quasiconformal and, consequently, the modulus is quasiinvariant under such a mapping. Hence the Lipschitz domains have weakly flat boundaries.

3. BMO, VMO and FMO functions

Recall that a real-valued function $u$ in a domain $D$ in $\mathbb{C}$ is said to be of bounded mean oscillation in $D$, abbr. $u \in \text{BMO}(D)$, if $u \in L^1_{\text{loc}}(D)$ and
\begin{equation}
\|u\|_* := \sup_B \frac{1}{|B|} \int_B |u(z) - u_B| \, dm(z) < \infty,
\end{equation}
where the supremum is taken over all discs $B$ in $D$, $dm(z)$ corresponds to the Lebesgue measure in $\mathbb{C}$ and
\[ u_B = \frac{1}{|B|} \int_B u(z) \, dm(z). \]
We write \( u \in \text{BMO}_{\text{loc}}(D) \) if \( u \in \text{BMO}(U) \) for every relatively compact subdomain \( U \) of \( D \) (we also write \( \text{BMO} \) or \( \text{BMO}_{\text{loc}} \) if it is clear from the context what \( D \) is).

The class \( \text{BMO} \) was introduced by John and Nirenberg (1961) in the paper [15] and soon became an important concept in harmonic analysis, partial differential equations and related areas; see, e.g., [11] and [32].

A function \( \varphi \) in \( \text{BMO} \) is said to have \textit{vanishing mean oscillation}, abbr. \( \varphi \in \text{VMO} \), if the supremum in (3.1) taken over all balls \( B \) in \( D \) with \(|B| < \varepsilon\) converges to 0 as \( \varepsilon \to 0 \). \( \text{VMO} \) has been introduced by Sarason in [46]. There are a number of papers devoted to the study of partial differential equations with coefficients of the class \( \text{VMO} \), see, e.g., [5, 14, 25, 29] and [30].

Remark 3.1. Note that \( W^{1,2}(D) \subset \text{VMO}(D) \), see, e.g., [2].

Following [13], we say that a function \( \varphi : D \to \mathbb{R} \) has \textit{finite mean oscillation} at a point \( z_0 \in D \), abbr. \( \varphi \in \text{FMO}(z_0) \), if

\[
\lim_{\varepsilon \to 0} \frac{1}{B(z_0, \varepsilon)} \int_{B(z_0, \varepsilon)} |\varphi(z) - \bar{\varphi}_\varepsilon(z_0)| \, dm(z) < \infty,
\]

where

\[
\bar{\varphi}_\varepsilon(z_0) = \frac{1}{B(z_0, \varepsilon)} \int_{B(z_0, \varepsilon)} \varphi(z) \, dm(z)
\]

is the mean value of the function \( \varphi(z) \) over the disk \( B(z_0, \varepsilon) \). Note that the condition (3.2) includes the assumption that \( \varphi \) is integrable in some neighborhood of the point \( z_0 \). We say also that a function \( \varphi : D \to \mathbb{R} \) is of \textit{finite mean oscillation in} \( D \), abbr. \( \varphi \in \text{FMO}(D) \) or simply \( \varphi \in \text{FMO} \), if \( \varphi \in \text{FMO}(z_0) \) for all points \( z_0 \in D \). We write \( \varphi \in \text{FMO}(\mathcal{D}) \) if \( \varphi \) is given in a domain \( G \) in \( \mathbb{C} \) such that \( \mathcal{D} \subset G \) and \( \varphi \in \text{FMO}(G) \).

The following statement is obvious by the triangle inequality.

\textbf{Proposition 3.2.} If, for a collection of numbers \( \varphi_\varepsilon \in \mathbb{R} \), \( \varepsilon \in (0, \varepsilon_0] \),

\[
\lim_{\varepsilon \to 0} \frac{1}{B(z_0, \varepsilon)} \int_{B(z_0, \varepsilon)} |\varphi(z) - \varphi_\varepsilon| \, dm(z) < \infty,
\]

then \( \varphi \) is of finite mean oscillation at \( z_0 \).

In particular choosing here \( \varphi_\varepsilon \equiv 0 \), \( \varepsilon \in (0, \varepsilon_0] \), we obtain the following.

\textbf{Corollary 3.3.} If, for a point \( z_0 \in D \),

\[
\lim_{\varepsilon \to 0} \frac{1}{B(z_0, \varepsilon)} \int_{B(z_0, \varepsilon)} |\varphi(z)| \, dm(z) < \infty,
\]

then \( \varphi \) has finite mean oscillation at \( z_0 \).

Recall that a point \( z_0 \in D \) is called a \textbf{Lebesgue point} of a function \( \varphi : D \to \mathbb{R} \) if \( \varphi \) is integrable in a neighborhood of \( z_0 \) and

\[
\lim_{\varepsilon \to 0} \frac{1}{B(z_0, \varepsilon)} \int_{B(z_0, \varepsilon)} |\varphi(z) - \varphi(z_0)| \, dm(z) = 0.
\]

It is known that, almost every point in \( D \) is a Lebesgue point for every function \( \varphi \in L^1(D) \). Thus we have by Proposition 3.2 the following corollary.
Corollary 3.4. Every locally integrable function \( \varphi : D \to \mathbb{R} \) has a finite mean oscillation at almost every point in \( D \).

Remark 3.5. Note that the function \( \varphi(z) = \log(1/|z|) \) belongs to BMO in the unit disk \( \Delta \), see, e.g., [32], p. 5, and hence also to FMO. However, \( \varphi_\varepsilon(0) \to \infty \) as \( \varepsilon \to 0 \), showing that condition (3.5) is only sufficient but not necessary for a function \( \varphi \) to be of finite mean oscillation at \( z_0 \). Clearly, BMO\((D) \subset BMO_{\text{loc}}(D) \subset \text{FMO}(D) \) and as well-known \( \text{BMO}_{\text{loc}} \subset L^p_{\text{loc}} \) for all \( p \in [1, \infty) \), see, e.g., [15] or [32]. However, FMO is not a subclass of \( L^p_{\text{loc}} \) for any \( p > 1 \) but only of \( L^1_{\text{loc}} \). Thus, the class FMO is much more wide than BMO\(_{\text{loc}}\).

Versions of the next lemma has been first proved for BMO in [37]. For FMO, see the papers [13, 35, 39, 40] and the monographs [10] and [24].

Lemma 3.6. Let \( D \) be a domain in \( \mathbb{C} \) and let \( \varphi : D \to \mathbb{R} \) be a non-negative function of the class FMO\((z_0)\) for some \( z_0 \in D \). Then

\[
\int_{\varepsilon < |z - z_0| < \varepsilon_0} \frac{\varphi(z) \, dm(z)}{(|z - z_0| \log \frac{1}{|z - z_0|})^2} = O \left( \log \log \frac{1}{\varepsilon} \right) \quad \text{as} \quad \varepsilon \to 0
\]

for some \( \varepsilon_0 \in (0, \delta_0) \) where \( \delta_0 = \min(e^{-\varepsilon}, d_0) \), \( d_0 = \inf_{z \in \partial D} |z - z_0| \).

4. Beltrami equations and ring \( Q \)-homeomorphisms

The following notion was motivated by the ring definition of Gehring for quasi-conformal mappings, see, e.g., [6], and it is closely relevant with the Beltrami equations. Given a domain \( D \) in \( \mathbb{C} \) and a Lebesgue measurable function \( Q : \mathbb{C} \to (0, \infty) \), we say that a homeomorphism \( f : D \to \overline{\mathbb{C}} \) is a ring \( Q \)-homeomorphism at a point \( z_0 \in \partial D \) if

\[
M(\Delta(fC_1, fC_2; fD)) \leq \int_{A \cap D} Q(z) \cdot \eta^2(|z - z_0|) \, dm(z)
\]

for any ring \( A = A(z_0, r_1, r_2) \) and arbitrary continua \( C_1 \) and \( C_2 \) in \( D \) that belong to the different components of the complement of the ring \( A \) in \( \overline{\mathbb{C}} \) including \( z_0 \) and \( \infty \), correspondingly, and for any Lebesgue measurable function \( \eta : (r_1, r_2) \to [0, \infty] \) such that

\[
\int_{r_1}^{r_2} \eta(r) \, dr \geq 1.
\]

The notion was first introduced at inner points of a domain \( D \) in the work [38]. The ring \( Q \)-homeomorphisms at boundary points of a domain \( D \) have first been considered in the papers [41] and [42].

By Lemma 2.2 in [35] or Lemma 7.4 in [24], we obtain the following criterion for homeomorphisms in \( \mathbb{C} \) to be ring \( Q \)-homeomorphisms, see also Theorem A.7 in [24].

Lemma 4.1. Let \( D \) and \( D' \) be bounded domains in \( \mathbb{C} \) and \( Q : \mathbb{C} \to (0, \infty) \) be a measurable function. A homeomorphism \( f : D \to D' \) is a ring \( Q \)-homeomorphism
at $z_0 \in \overline{D}$ if and only if

$$M(\Delta (fS_1, fS_2; f)D) \leq \left( \int_{r_1}^{r_2} \frac{dr}{\|Q\|(z_0, r)} \right)^{-1}$$

$\forall r_1 \in (0, r_2), r_2 \in (0, d_0)$

where $S_i = S(z_0, r_i), i = 1, 2, d_0 = \sup_{z \in \partial D} |z - z_0|$ and $\|Q\|(z_0, r)$ is the $L_1$-norm of $Q$ over $D \cap S(z_0, r)$.

By Theorem 4.1 in [34] every homeomorphic $W^{1,1}_{\text{loc}}$ solution of the Beltrami equation (1.1) in a domain $D \subseteq \mathbb{C}$ is the so-called lower $Q$-homeomorphism at every point $z_0 \in \overline{D}$ with $Q(z) = K^T_{\mu}(z, z_0), z \in D$, and $Q(z) \equiv \varepsilon > 0$ in $\mathbb{C} \setminus D$. On the other hand, by Theorem 2 in [17] for a locally integrable $Q$, if $f : D \to D'$ is a lower $Q$-homeomorphism at a point $z_0 \in \overline{D}$, then $f$ is a ring $Q$-homeomorphism at the point $z_0$. Thus, we have the following conclusion.

**Theorem 4.2.** Let $f$ be a homeomorphic $W^{1,1}_{\text{loc}}$ solution of the Beltrami equation (1.1) in a domain $D \subseteq \mathbb{C}$ and $K_{\mu} \in L^1(D)$. Then $f$ is a ring $Q$-homeomorphism at every point $z_0 \in \overline{D}$ with $Q(z) = K^T_{\mu}(z, z_0), z \in D$.

In fact, it is sufficient to assume here instead of the condition $K_{\mu} \in L^1(D)$ that $K^T_{\mu}(z, z_0)$ is integrable along the circles $|z - z_0| = r$ for a.e. small enough $r$.

5. Continuous extension of ring $Q$-homeomorphisms

**Lemma 5.1.** Let $D$ and $D'$ be bounded finitely connected domains in $\mathbb{C}$ and let $f : D \to D'$ be a ring $Q_{z_0}$-homeomorphism at every point $z_0 \in \partial D$. Suppose that

$$\int_{D(z_0, \varepsilon, \delta_0)} Q_{z_0}(\cdot) \cdot \psi_{z_0, \varepsilon, \delta_0}^2(|\cdot - z_0|) \, dm(z) = o(I^2_{z_0, \varepsilon, \delta_0}(\varepsilon)) \quad \text{as } \varepsilon \to 0 \quad \forall z_0 \in \partial D$$

where $D(z_0, \varepsilon, \delta_0) = \{ z \in D : \varepsilon < |z - z_0| < \delta_0 \}$ for every small enough $0 < \varepsilon_0 < \delta_0$ is the set $z$ at which $\psi_{z_0, \varepsilon, \delta_0}(t) : (0, \infty) \to [0, \infty], \varepsilon \in (0, \varepsilon_0)$, is a family of (Lebesgue) measurable functions such that

$$0 < I_{z_0, \varepsilon, \delta_0}(\varepsilon) := \int_{\varepsilon}^{\delta_0} \psi_{z_0, \varepsilon, \delta_0}(t) \, dt < \infty \quad \forall \varepsilon \in (0, \varepsilon_0).$$

Then $f$ can be extended to a continuous mapping of $\overline{D}_P$ onto $\overline{D'}_P$.

**Proof.** By Remark 14 with no loss of generality we may assume that $D'$ is a circular domain and, thus, $\overline{D}_P = \overline{D}$. Let us first prove that the cluster set

$$L = C(P, f) := \left\{ \zeta \in \mathbb{C} : \zeta = \lim_{n \to \infty} f(z_n), z_n \to P, z_n \in D, n = 1, 2, \ldots \right\}$$

consists of a single point $\zeta_0 \in \partial D'$ for each prime end $P$ of the domain $D$.

Note that $L \neq \emptyset$ by compactness of the set $\overline{D}_P$ and $L$ is a subset of $\partial D'$, see, e.g., Proposition 2.5 in [33] or Proposition 13.5 in [24]. Let us assume that there is at least two points $\zeta_0$ and $\zeta_* \in L$. Set $U = B(\zeta_0, \rho_0) = \{ \zeta \in \mathbb{C} : |\zeta - \zeta_0| < \rho_0 \}$ where $0 < \rho_0 < |\zeta_* - \zeta_0|$. 
Let $\sigma_k$, $k = 1, 2, \ldots$, be a chain of cross-cuts of $D$ in the prime end $P$ lying on circles $S_k = S(z_0, r_k)$ from Lemma 1.2 where $z_0 \in \partial D$. Let $D_k$, $k = 1, 2, \ldots$ be the domains associated with $\sigma_k$. Then there exist points $\zeta_k$ and $\zeta_k^*$ in the domains $D'_k = f(D_k)$ such that $|\zeta_0 - \zeta_k| < \rho_0$ and $|\zeta_0 - \zeta_k^*| > \rho_0$ and, moreover, $\zeta_k \to \zeta_0$ and $\zeta_k^* \to \zeta_*$ as $k \to \infty$. Let $C_k$ be continuous curves joining $\zeta_k$ and $\zeta_k^*$ in $D'_k$. Note that by the construction $\partial U \cap C_k \neq \emptyset$, $k = 1, 2, \ldots$.

By the condition of strong accessibility of the point $\zeta_0$, there is a continuum $E \subset D'$ and a number $\delta > 0$ such that

$$M(\Delta(E, C_k; D')) \geq \delta$$

for all large enough $k$. Note that $C = f^{-1}(E)$ is a compact subset of $D$ and hence $d_0 = \text{dist}(z_0, C) > 0$. Let $\varepsilon_0 \in (0, d_0)$ be small enough from the hypotheses of the lemma. With no loss of generality, we may assume that $r_k < \varepsilon_0$ and $\delta$ holds for all $k = 1, 2, \ldots$.

Let $\Gamma_m$ be a family of all continuous curves in $D \setminus D_m$ joining the circle $S_0 = S(z_0, \varepsilon_0)$ and $\sigma_m$, $m = 1, 2, \ldots$. Note that by the construction $C_k \subset D'_k \subset D'_m$ for all $k \geq m$ and, thus, by the principle of minorization $M(f(\Gamma_m)) \geq \delta$ for all $m = 1, 2, \ldots$.

On the other hand, every function $\eta(t) = \eta_m(t) := \psi_{z_0, r_m, \varepsilon_0}(t)/I_{z_0, \varepsilon_0}(r_m)$, $m = 1, 2, \ldots$, satisfies the condition (1.2) and hence

$$M(f \Gamma_m) \leq \int_D Q(z) \cdot \eta_m^2(z) \, dm(z),$$

i.e., $M(f \Gamma_m) \to 0$ as $m \to \infty$ in view of (5.1).

The obtained contradiction disproves the assumption that the cluster set $C(P, f)$ consists of more than one point.

Thus, we have the extension $h$ of $f$ to $\overline{D}_P$ such that $C(E_D, f) \subset \partial D'$. In fact, $C(E_D, f) = \partial D'$. Indeed, if $\zeta_0 \in D'$, then there is a sequence $\zeta_n$ in $D'$ being convergent to $\zeta_0$. We may assume with no loss of generality that $f^{-1}(\zeta_n) \to P_0 \in E_D$ because $\overline{D}_P$ is compact, see Remark 1.1. Hence $\zeta_0 \in C(P_0, f)$.

Finally, let us show that the extended mapping $h : \overline{D}_P \to \overline{D}'$ is continuous. Indeed, let $P_n \to P_0$ in $\overline{D}_P$. If $P_0 \in D$, then the statement is obvious. If $P_0 \in E_D$, then by the last item we are able to choose $P_n^* \in D$ such that $\rho(P_n, P_n^*) < 1/n$ where $\rho$ is one of the metrics in Remark 1.1 and $|h(P_n) - h(P_n^*)| < 1/n$. Note that by the first part of the proof $h(P_n^*) \to h(P_0)$ because $P_n^* \to P_0$. Consequently, $h(P_n) \to h(P_0)$, too. □

**Theorem 5.2.** Let $D$ and $D'$ be bounded finitely connected domains in $\mathbb{C}$ and let $f : D \to D'$ be a ring $Q_{z_0}$-homeomorphism at every point $z_0 \in \partial D$. If

$$\int_0^{\varepsilon(z_0)} \frac{dr}{||Q_{z_0}||^2(r)} = \infty \quad \forall \ z_0 \in \partial D$$

where $0 < \varepsilon(z_0) < d(z_0) := \sup_{z \in D} |z - z_0|$ and

$$||Q_{z_0}||^2(r) := \int_{D \cap S(z_0, r)} Q_{z_0} \, ds,$$

then $f$ can be extended to a continuous mapping of $\overline{D}_P$ onto $\overline{D}'_P$. 


Proof. Indeed, condition (5.3) implies that

\[ \int_{0}^{\varepsilon_0} \frac{dr}{||Q_{z_0}(r)||} = \infty \quad \text{for all} \quad z_0 \in \partial D \quad \forall \quad \varepsilon_0 \in (0, \varepsilon(z_0)) \]

because the left hand side in (4.3) is not equal to zero, see Theorem 5.2 in [28], and hence by Lemma 4.1

\[ \int_{0}^{\varepsilon_0} \frac{dr}{||Q_{z_0}(r)||} < \infty . \]

On the other hand, for the functions

\[ \psi_{z_0, \varepsilon_0}(t) := \begin{cases} \frac{1}{||Q_{z_0}(t)||}, & t \in (0, \varepsilon_0), \\ 0, & t \in [\varepsilon_0, \infty), \end{cases} \]

we have by the Fubini theorem that

\[ \int_{D(z_0, \varepsilon_0)} Q_{z_0}(z) \cdot \psi_{z_0, \varepsilon_0}^2(|z - z_0|) \, dm(z) = \int_{\varepsilon_0}^{\infty} \frac{dr}{||Q_{z_0}(r)||} \]

and, consequently, condition (5.1) holds by (5.5) for all \( z_0 \in \partial D \) and \( \varepsilon_0 \in (0, \varepsilon(z_0)) \).

Here we have used the standard conventions in the integral theory that \( a/\infty = 0 \) for \( a \neq \infty \) and \( 0 \cdot \infty = 0 \), see, e.g., Section I.3 in [45].

Thus, Theorem 5.2 follows immediately from Lemma 5.1. \( \square \)

6. Extension of the inverse mappings to the boundary

The base for the proof on extending the inverse mappings of ring Q-homeomorphism by prime ends in the plane is the following fact on the cluster sets.

**Lemma 6.1.** Let \( D \) and \( D' \) be bounded finitely connected domains in \( \mathbb{C} \), \( P_0 \) and \( P_* \) be prime ends of \( D \), \( P_* \neq P_0 \). Denote by \( \sigma_m \), \( m = 1, 2, \ldots \), a chain of cross-cuts in \( P_0 \) from Lemma 1.1 lying on circles \( S(z_0, r_m) \), \( z_0 \in \partial D \), with associated domains \( d_m \). Suppose that \( Q \) is integrable over \( D \cap S(z_0, r) \) for a set \( E \) of numbers \( r \in (0, \delta) \) of a positive linear measure where \( \delta = r_{m_0} \) and \( m_0 \) is such that the domain \( d_{m_0} \) does not contain sequences of points converging to \( P_* \). If \( f : D \to D' \) is a ring Q-homeomorphism at the point \( z_0 \) and \( \partial D' \) is weakly flat, then

\[ C(P_0, f) \cap C(P_*, f) = \emptyset . \]

Note that in view of metrizability of the completion \( \overline{D}_F \) of the domain \( D \) with prime ends, see Remark 1.1, the number \( m_0 \) in Lemma 6.1 always exists.

**Proof.** Let us choose \( \varepsilon \in (0, \delta) \) such that \( E_0 := \{ r \in E : r \in (\varepsilon, \delta) \} \) has a positive linear measure. Such a choice is possible in view of subadditivity of the linear measure and the exhaustion \( E = \cup E_n \) where \( E_n = \{ r \in E : r \in (1/n, \delta) \} \), \( n = 1, 2, \ldots \). Note that by Lemma 1.1 for \( S_1 = S(z_0, \varepsilon) \) and \( S_2 = S(z_0, \delta) \),

\[ M(\Delta(fS_1, fS_2; fD)) < \infty . \]

Let us assume that \( C_0 \cap C_* \neq \emptyset \) where \( C_0 = C(P_0, f) \) and \( C_* = C(P_*, f) \). By the construction there is \( m_1 > m_0 \) such that \( \sigma_{m_1} \) lies on the circle \( S(z_0, r_{m_1}) \) with
The obtained contradiction disproves the assumption that
\[ \rho \leq M(f(\Gamma)) \]
for all continua \( E \) and \( F \) in \( D' \) intersecting the circles \( S(\zeta, \rho_0) \) and \( S(\zeta, \rho_*) \). However, these circles can be joined by continuous curves \( c_1 \) and \( c_2 \) in the domains \( f(d_0) \) and \( f(d_*) \), correspondingly, and, in particular, for these curves
\[ M_0 \leq M(\Delta(c_1, c_2; D')) \leq M(f(\Gamma)). \]
The obtained contradiction disproves the assumption that \( C_0 \cap C_* = \emptyset \). □

**Theorem 6.2.** Let \( D \) and \( D' \) be bounded finitely connected domains in \( \mathbb{C} \) and \( f : D \to D' \) be a \( Q_{\infty} \)-homeomorphism at every point \( z_0 \in \partial D \) with \( Q_{z_0} \in L^1(D \cap U_{z_0}) \) for a neighborhood \( U_{z_0} \) of \( z_0 \). Then \( f^{-1} \) can be extended to a continuous mapping of \( \overline{D'} \) onto \( \overline{D} \).

**Proof.** By Remark 1.2, we may assume with no loss of generality that \( D' \) is a circular domain, \( \overline{D'} = \overline{D} \); \( C(\zeta_0, f^{-1}) \neq \emptyset \) for every \( \zeta_0 \in \partial D' \) because \( \overline{D'} \) is metrizable and compact. Moreover, \( C(\zeta_0, f^{-1}) \cap \partial D = \emptyset \), see, e.g., Proposition 2.5 in [33] or Proposition 13.5 in [24].

Let us assume that there is at least two different prime ends \( P_1 \) and \( P_2 \) in \( C(\zeta_0, f^{-1}) \). Then \( \zeta_0 \in C(P_1, f) \cap C(P_2, f) \). Let \( z_1 \in \partial D \) be a point corresponding to \( P_1 \) from Lemma 1.2. Note that
\[ E = \{ r \in (0, \delta) : Q_{z_1}(D \cap S(z_1, r)) \in L^1(D \cap S(z_1, r)) \} \]
has a positive linear measure for every \( \delta > 0 \) by the Fubini theorem, see, e.g., [45], because \( Q_{z_1} \in L^1(D \cap U_{z_1}) \). The obtained contradiction with Lemma 0.1 shows that \( C(\zeta_0, f^{-1}) \) contains only one prime end of \( D \).

Thus, we have the extension \( g \) of \( f^{-1} \) to \( \overline{D} \) such that \( C(\partial D', f^{-1}) \subseteq \overline{D'} \) \( \setminus \) \( D \). Indeed, if \( P_0 \) is a prime end of \( D \), then there is a sequence \( z_n \) in \( D \) being convergent to \( P_0 \). We may assume without loss of generality that \( z_n \to z_0 \in \partial D \) and \( f(z_n) \to \zeta_0 \in \partial D' \) because \( \overline{D} \) and \( \overline{D'} \) are compact. Hence \( P_0 \in C(\zeta_0, f^{-1}) \).

Finally, let us show that the extended mapping \( g : \overline{D'} \to \overline{D} \) is continuous. Indeed, let \( \zeta_n \to \zeta_0 \) in \( \overline{D'} \). If \( \zeta_0 \notin D' \), then the statement is obvious. If \( \zeta_0 \in D' \), then take \( \zeta_n^* \in D' \) such that \( |\zeta_n - \zeta_n^*| < 1/n \) and \( \rho(g(\zeta_n), g(\zeta_n^*)) < 1/n \) where \( \rho \) is one of the metrics in Remark 1.2. Note that by the construction \( g(\zeta_n^*) \to g(\zeta_0) \) because \( \zeta_n^* \to \zeta_0 \). Consequently, \( g(\zeta_n) \to g(\zeta_0) \), too. □

**Theorem 6.3.** Let \( D \) and \( D' \) be bounded finitely connected domains in \( \mathbb{C} \) and \( f : D \to D' \) be a \( Q_{\infty} \)-homeomorphism at every point \( z_0 \in \partial D \) with the condition
\[ \int_0^{r(z_0)} \frac{dr}{||Q_{z_0}||(r)} = \infty \]
where $0 < \varepsilon(z_0) < d(z_0) = \sup_{z \in D} |z - z_0|$ and

\[
\|Q_{z_0}\|(r) = \int_{D \cap S(z_0, r)} Q_{z_0} \, ds.
\]

Then $f^{-1}$ can be extended to a continuous mapping of $D'$ onto $D$.

**Proof.** Indeed, condition (6.6) implies that

\[
\delta \int_0^\delta \frac{dr}{\|Q_{z_0}\|(r)} = \infty \quad \forall \ z_0 \in \partial D \quad \forall \ \delta \in (0, \varepsilon(z_0))
\]

because the left hand side in (4.3) is not equal to zero, see Theorem 5.2 in [28], and hence by Lemma 4.1

\[
\int_\delta^{\varepsilon(z_0)} \frac{dr}{\|Q_{z_0}\|(r)} < \infty.
\]

Thus, the set

\[
E = \{ r \in (0, \delta) : Q_{z_0}|_{D \cap S(z_0, r)} \in L^1(D \cap S(z_0, r)) \}
\]

has a positive linear measure for all $z_0 \in \partial D$ and all $\delta \in (0, \varepsilon(z_0))$. The rest of arguments is perfectly similar to one in the proof of Theorem 6.2. \[\square\]

### 7. Homeomorphic extension of ring $Q$–homeomorphisms

Combining Theorems 5.2 and 6.3 we obtain the following principal result.

**Theorem 7.1.** Let $D$ and $D'$ be bounded finitely connected domains in $\mathbb{C}$ and let $f : D \to D'$ be a ring $Q_{z_0}$-homeomorphism at every point $z_0 \in \partial D$. If

\[
\varepsilon(z_0) \int_0^{\varepsilon(z_0)} \frac{dr}{\|Q_{z_0}\|(r)} = \infty \quad \forall \ z_0 \in \partial D
\]

where $0 < \varepsilon(z_0) < d(z_0) := \sup_{z \in D} |z - z_0|$ and

\[
\|Q_{z_0}\|(r) := \int_{D \cap S(z_0, r)} Q_{z_0} \, ds,
\]

then $f$ can be extended to a homeomorphism of $D'$ onto $D$.

**Corollary 7.2.** In particular, the conclusion of Theorem 7.1 holds if

\[
q_{z_0}(r) = O \left( \log \frac{1}{r} \right) \quad \forall \ z_0 \in \partial D
\]

as $r \to 0$ where $q_{z_0}(r)$ is the average of $Q$ over the circle $|z - z_0| = r$.

Using Lemma 2.2 in [35], see also Lemma 7.4 in [24], by Theorem 7.1 we obtain the following general lemma that, in turn, makes possible to obtain new criteria in a great number.
Lemma 7.3. Let $D$ and $D'$ be bounded finitely connected domains in $\mathbb{C}$ and let $f : D \to D'$ be a ring $Q_{z_0}$-homeomorphism at every point $z_0 \in \partial D$ where $Q_{z_0}$ is integrable in a neighborhood of $z_0$. Suppose that

$$ \int_{D(z_0, \varepsilon, \varepsilon_0)} Q_{z_0}(z) \cdot \psi^2(|z - z_0|) \, dm(z) = o \left( I^2_{z_0, \varepsilon, \varepsilon_0}(\varepsilon) \right) \quad \text{as} \quad \varepsilon \to 0 \quad \forall \ z_0 \in \partial D $$

where $D(z_0, \varepsilon, \varepsilon_0) = \{ z \in D : \varepsilon < |z - z_0| < \varepsilon_0 \}$ for every small enough $0 < \varepsilon_0 < d(z_0) = \sup_{z \in D} |z - z_0|$ and where $\psi_{z_0, \varepsilon, \varepsilon_0}(t) : (0, \infty) \to [0, \infty]$, $\varepsilon \in (0, \varepsilon_0)$, is a family of (Lebesgue) measurable functions such that

$$ 0 < I_{z_0, \varepsilon, \varepsilon_0}(\varepsilon) := \int_{\varepsilon}^{\varepsilon_0} \psi_{z_0, \varepsilon, \varepsilon_0}(t) \, dt < \infty \quad \forall \ \varepsilon \in (0, \varepsilon_0). $$

Then $f$ can be extended to a homeomorphism of $\overline{D}_P$ onto $\overline{D'}_P$.

Remark 7.4. In fact, instead of integrability of $Q_{z_0}$ in a neighborhood of $z_0$, it is sufficient to request that $Q_{z_0}$ is integrable over $D \cap S(z_0, r)$ for a.e. $r \in (0, \varepsilon_0)$.

Note also that it is not only Lemma 7.3 follows from Theorem 7.1 under the given conditions on integrability of $Q_{z_0}$ but, inversely, Theorem 7.1 follows from Lemma 7.3 too, as it was shown under the proof of Theorem 5.2. Thus, Theorem 7.3 is equivalent under the given conditions to Lemma 7.3 but each of them is sometimes more convenient for applications than another one.

Finally, note that (7.4) holds, in particular, if

$$ \int_{D(z_0, \varepsilon_0)} Q_{z_0}(z) \cdot \psi^2(|z - z_0|) \, dm(z) < \infty \quad \forall \ z_0 \in \partial D $$

where $D(z_0, \varepsilon_0) = \{ z \in D : |z - z_0| < \varepsilon_0 \}$ and where $\psi(t) : (0, \infty) \to [0, \infty]$ is a locally integrable function such that $I_{z_0, \varepsilon, \varepsilon_0}(\varepsilon) \to \infty$ as $\varepsilon \to 0$. In other words, for the extendability of $f$ to a homeomorphism of $\overline{D}_P$ onto $\overline{D'}_P$, it suffices for the integrals in (7.6) to be convergent for some nonnegative function $\psi(t)$ that is locally integrable on $(0, \infty)$ but that has a non-integrable singularity at zero.

Choosing in Lemma 7.3 $\psi(t) := \frac{1}{t \log t}$ and applying Lemma 3.6, we obtain the next result.

Theorem 7.5. Let $D$ and $D'$ be bounded finitely connected domains in $\mathbb{C}$ and let $f : D \to D'$ be a ring $Q_{z_0}$-homeomorphism at every point $z_0 \in \partial D$ where $Q_{z_0}$ has finite mean oscillation at $z_0$. Then $f$ can be extended to a homeomorphism of $\overline{D}_P$ onto $\overline{D'}_P$.

Corollary 7.6. In particular, the conclusion of Theorem 7.3 holds if

$$ \lim_{\varepsilon \to 0} \int_{B(z_0, \varepsilon)} Q_{z_0}(z) \, dm(z) < \infty \quad \forall \ z_0 \in \partial D $$

Corollary 7.7. The conclusion of Theorem 7.5 holds if every point $z_0 \in \partial D$ is a Lebesgue point of the function $Q_{z_0}$.

The next statement also follows from Lemma 7.3 under the choice $\psi(t) = 1/t$. 
THEOREM 7.8. Let $D$ and $D'$ be bounded finitely connected domains in $\mathbb{C}$ and $f : D \to D'$ be a ring $Q_{z_0}$-homeomorphism at every point $z_0 \in \partial D$. If, for some $\varepsilon_0 = \varepsilon(z_0) > 0$,

\begin{equation}
\int_{\varepsilon < |z - z_0| < \varepsilon_0} Q_{z_0}(z) \frac{dm(z)}{|z - z_0|^2} = o \left( \left[ \log \frac{1}{\varepsilon} \right]^2 \right) \quad \text{as} \quad \varepsilon \to 0 \quad \forall z_0 \in \partial D
\end{equation}

then $f$ can be extended to a homeomorphism of $\overline{D}_P$ onto $\overline{D'}_P$.

REMARK 7.9. Choosing in Lemma 7.3 the function $\psi(t) = 1/(t \log 1/t)$ instead of $\psi(t) = 1/t$, (7.7) can be replaced by the more weak condition

\begin{equation}
\int_{\varepsilon < |z - z_0| < \varepsilon_0} Q_{z_0}(z) \frac{dm(z)}{|z - z_0| \log \frac{1}{|z - z_0|}} = o \left( \left[ \log \log \frac{1}{\varepsilon} \right]^2 \right)
\end{equation}

and (7.8) by the condition

\begin{equation}
q_{z_0}(r) = o \left( \frac{1}{r} \log \frac{1}{r} \right).
\end{equation}

Of course, we could give here the whole scale of the corresponding condition of the logarithmic type using suitable functions $\psi(t)$.

Theorem 7.8 has a magnitude of other fine consequences, for instance:

THEOREM 7.10. Let $D$ and $D'$ be bounded finitely connected domains in $\mathbb{C}$ and let $f : D \to D'$ be a ring $Q_{z_0}$-homeomorphism at every point $z_0 \in \partial D$ and

\begin{equation}
\int_{D \cap B(z_0, \varepsilon_0)} \Phi_{z_0} (Q_{z_0}(z)) \ dm(z) < \infty \quad \forall z_0 \in \partial D
\end{equation}

for $\varepsilon_0 = \varepsilon(z_0) > 0$ and a nondecreasing convex function $\Phi_{z_0} : [0, \infty) \to [0, \infty)$ with

\begin{equation}
\int_{\delta(z_0)}^{\infty} \frac{d\tau}{\tau \Phi_{z_0}^{-1}(\tau)} = \infty
\end{equation}

for $\delta(z_0) > \Phi_{z_0}(0)$. Then $f$ is extended to a homeomorphism of $\overline{D}_P$ onto $\overline{D'}_P$.

Indeed, by Theorem 3.1 and Corollary 3.2 in 43, (7.10) and (7.11) imply (7.1) and, thus, Theorem 7.10 is a direct consequence of Theorem 7.1.

COROLLARY 7.11. In particular, the conclusion of Theorem 7.10 holds if

\begin{equation}
\int_{D \cap B(z_0, \varepsilon_0)} e^{\alpha_0 Q_{z_0}(z)} \ dm(z) < \infty \quad \forall z_0 \in \partial D
\end{equation}

for some $\varepsilon_0 = \varepsilon(z_0) > 0$ and $\alpha_0 = \alpha(z_0) > 0$.

REMARK 7.12. By Theorem 2.1 in 43, see also Proposition 2.3 in 36, (7.11) is equivalent to every of the conditions from the following series:

\begin{equation}
\int_{\delta(z_0)}^{\infty} \frac{H_{z_0}'(t)}{t} \ dt = \infty, \quad \delta(z_0) > 0,
\end{equation}
(7.14) \[
\int_0^\infty \frac{dH_{z_0}(t)}{t} = \infty, \quad \delta(z_0) > 0,
\]
(7.15) \[
\int_0^\infty \frac{H_{z_0}(t)}{t^2} dt = \infty, \quad \delta(z_0) > 0,
\]
(7.16) \[
\int_0^\Delta(z_0) \frac{(1)}{t} dt = \infty, \quad \Delta(z_0) > 0,
\]
(7.17) \[
\int_{\delta_*(z_0)}^\infty \frac{dn}{H^{-1}_{z_0}(\eta)} = \infty, \quad \delta_*(z_0) > H_{z_0}(0),
\]
where
(7.18) \[
H_{z_0}(t) = \log \Phi_{z_0}(t).
\]

Here the integral in (7.13) is understood as the Lebesgue–Stieltjes integral and the integrals in (7.13) and (7.15)–(7.17) as the ordinary Lebesgue integrals.

It is necessary to give one more explanation. From the right hand sides in the conditions (7.13)–(7.17) we have in mind $+\infty$. If $\Phi_{z_0}(t) = 0$ for $t \in [0, t_*(z_0)]$, then $H_{z_0}(t) = -\infty$ for $t \in [0, t_*(z_0)]$ and we complete the definition $H'_{z_0}(t) = 0$ for $t \in [0, t_*(z_0)]$. Note, the conditions (7.12) and (7.15) exclude that $t_*(z_0)$ belongs to the interval of integrability because in the contrary case the left hand sides in (7.14) and (7.16) are either equal to $-\infty$ or indeterminate. Hence we may assume in (7.13)–(7.16) that $\delta(z_0) > t_0$, correspondingly, $\Delta(z_0) < 1/t(z_0)$ where $t(z_0) := \sup_{\Phi_{z_0}(t)=0} t$, set $t(z_0) = 0$ if $\Phi_{z_0}(0) > 0$.

The most interesting of the above conditions is (7.15) that can be rewritten in the form:
(7.19) \[
\int_{\delta(z_0)}^\infty \log \Phi_{z_0}(t) \frac{dt}{t^2} = \infty.
\]

Note also that, under every homeomorphism $f$ between domains $D$ and $D'$ in $\mathbb{C}$, there is a natural one-to-one correspondence between components of their boundaries $\partial D$ and $\partial D'$, see, e.g., Lemma 5.3 in [13] or Lemma 6.5 in [24]. Thus, if a bounded domain $D$ in $\mathbb{C}$ is finitely connected and $D'$ is bounded, then $D'$ is finitely connected, too.

Finally, note that if a domain $D$ in $\mathbb{C}$ is locally connected on its boundary, then there is a natural one-to-one correspondence between prime ends of $D$ and boundary points of $D$. Thus, if $D$ and $D'$ are in addition locally connected on their boundaries in theorems of Sections 7, then $f$ is extended to a homeomorphism of $\overline{D}$ onto $\overline{D'}$. We obtained before it similar results when $\partial D'$ was weakly flat which is a more strong condition than local connectivity of $D'$ on its boundary, see, e.g., [16] and [17].
As known, every Jordan domain $D$ in $\mathbb{C}$ is locally connected on its boundary, see, e.g., [50], p. 66. It is easy to see, the latter implies that every bounded finitely connected domain $D$ in $\mathbb{C}$ whose boundary consists of mutually disjoint Jordan curves and isolated points is also locally connected on its boundary.

Inversely, every bounded finitely connected domain $D$ in $\mathbb{C}$ which is locally connected on its boundary has a boundary consisting of mutually disjoint Jordan curves and isolated points. Indeed, every such a domain $D$ can be mapped by a conformal mapping $f$ onto the so-called circular domain $D^*$ bounded by a finite collection of mutually disjoint circles and isolated points, see, e.g., Theorem V.6.2 in [8], that is extended to a homeomorphism of $\overline{D}$ onto $\overline{D^*}$, see Remark 1.1.

8. Boundary behavior of homeomorphic solutions

On the basis of results in Sections 6 and 7, we obtain by Theorem 4.2 the corresponding results on the boundary behavior of solutions of the Beltrami equations.

**Theorem 8.1.** Let $D$ and $D'$ be bounded finitely connected domains in $\mathbb{C}$ and $f : D \to D'$ be a homeomorphic solution in the class $W^{1,1}_{\text{loc}}$ of (1.1) with $K^T_\mu(\cdot, z_0) \in L^1(D \cap B(z_0, \varepsilon_0))$ for every $z_0 \in \partial D$. Then $f^{-1}$ is extended to a continuous mapping of $\overline{D'}$ onto $\overline{D^*}$.

However, any degree of integrability of $K_\mu$ does not guarantee a continuous extendability of the direct mapping $f$ to the boundary, see, e.g., an example in the proof of Proposition 6.3 in [24]. Conditions for it have perfectly another nature. The principal relevant result is the following.

**Theorem 8.2.** Let $D$ and $D'$ be bounded finitely connected domains in $\mathbb{C}$ and $f : D \to D'$ be a homeomorphic solution in the class $W^{1,1}_{\text{loc}}$ of the Beltrami equation (1.1) with the condition

\[
\int_0^{\varepsilon_0} \frac{dr}{||K^T_\mu||(z_0, r)} = \infty \quad \forall \; z_0 \in \partial D
\]

where $0 < \varepsilon_0 = \varepsilon(z_0) < d(z_0) := \sup_{z \in D} |z - z_0|$ and

\[
||K^T_\mu||(z_0, r) = \int_{S(z_0, r)} K^T_\mu(z, z_0) \, ds .
\]

Then $f$ can be extended to a homeomorphism of $\overline{D'}$ onto $\overline{D^*}$.

Here and later on, we set that $K^T_\mu$ is equal to zero outside of the domain $D$.

**Corollary 8.3.** In particular, the conclusion of Theorem 8.2 holds if

\[
k^T_{z_0}(r) = O\left(\log \frac{1}{r}\right) \quad \forall \; z_0 \in \partial D
\]

as $r \to 0$ where $k^T_{z_0}(r)$ is the average of $K^T_\mu(z, z_0)$ over the circle $|z - z_0| = r$. 
LEMMA 8.4. Let $D$ and $D'$ be bounded finitely connected domains in $\mathbb{C}$ and $f : D \to D'$ be a homeomorphic solution in the class $W^{1,1}_{\text{loc}}$ of the Beltrami equation (1.1) with $K_\mu \in L^1(D)$ and

\begin{equation}
\int_{|z-z_0|<\varepsilon} K^T_\mu(z, z_0) \cdot |z-z_0|^2 dm(z) = o \left( I^2_{z_0}(\varepsilon) \right) \quad \forall \ z_0 \in \partial D \tag{8.4}
\end{equation}

as $\varepsilon \to 0$ where $0 < \varepsilon_0 < \sup_{z \in D} |z-z_0|$ and $\psi_{z_0, \varepsilon}(t) : (0, \infty) \to [0, \infty]$, $\varepsilon \in (0, \varepsilon_0)$, is a two-parametric family of measurable functions such that

\[ 0 < I_{z_0}(\varepsilon) := \int_{\varepsilon}^{\varepsilon_0} \psi_{z_0, \varepsilon}(t) \ dt < \infty \quad \forall \ \varepsilon \in (0, \varepsilon_0). \]

Then $f$ can be extended to a homeomorphism of $D_P$ onto $D'_P$.

**Theorem 8.5.** Let $D$ and $D'$ be bounded finitely connected domains in $\mathbb{C}$ and $f : D \to D'$ be a homeomorphic solution in the class $W^{1,1}_{\text{loc}}$ of the Beltrami equation (1.1) with $K^T_\mu(z, z_0)$ of finite mean oscillation at every point $z_0 \in \partial D$. Then $f$ can be extended to a homeomorphism of $D_P$ onto $D'_P$.

In fact, here it is sufficient for the function $K^T_\mu(z, z_0)$ to have a dominant of finite mean oscillation in a neighborhood of every point $z_0 \in \partial D$.

**Corollary 8.6.** In particular, the conclusion of Theorem 8.5 holds if

\begin{equation}
\lim_{\varepsilon \to 0} \int_{B(z_0, \varepsilon)} K^T_\mu(z, z_0) \ dm(z) < \infty \quad \forall \ z_0 \in \partial D . \tag{8.5}
\end{equation}

**Theorem 8.7.** Let $D$ and $D'$ be bounded finitely connected domains in $\mathbb{C}$ and $f : D \to D'$ be a homeomorphic solution in the class $W^{1,1}_{\text{loc}}$ of the Beltrami equation (1.1) with the condition

\begin{equation}
\int_{|z-z_0|<\varepsilon} K^T_\mu(z, z_0) \ dm(z) = o \left( \left( \log \frac{1}{\varepsilon} \right)^2 \right) \quad \forall \ z_0 \in \partial D . \tag{8.6}
\end{equation}

Then $f$ can be extended to a homeomorphism of $D_P$ onto $D'_P$.

**Remark 8.8.** Condition (8.6) can be replaced by the weaker condition

\begin{equation}
\int_{|z-z_0|<\varepsilon} K^T_\mu(z, z_0) \ dm(z) = o \left( \left( \log \log \frac{1}{\varepsilon} \right)^2 \right) \quad \forall \ z_0 \in \partial D . \tag{8.7}
\end{equation}

In general, here we are able to give a number of other conditions of logarithmic type. In particular, condition (8.3), thanking to Theorem 8.2, can be replaced by the weaker condition

\begin{equation}
k^T_{z_0}(r) = O \left( \log \frac{1}{r} \log \log \frac{1}{r} \right) . \tag{8.8}
\end{equation}

Finally, we complete the series of criteria with the following integral condition.
Theorem 8.9. Let $D$ and $D'$ be bounded finitely connected domains in $\mathbb{C}$ and $f : D \to D'$ be a homeomorphic solution in the class $W^{1,1}_{\text{loc}}$ of the Beltrami equation (1.1) with the condition

$$\int_{D \cap B(z_0, \varepsilon_0)} \Phi_{z_0}(K_T^\mu(z, z_0)) \, dm(z) < \infty \quad \forall \, z_0 \in \partial D$$

for $\varepsilon_0 = \varepsilon(z_0) > 0$ and a nondecreasing convex function $\Phi_{z_0} : [0, \infty) \to [0, \infty)$ with

$$\int_{\delta_0}^{\infty} \frac{d\tau}{\tau \Phi_{z_0}^{-1}(\tau)} = \infty$$

for $\delta_0 = \delta(z_0) > \Phi_{z_0}(0)$. Then $f$ is extended to a homeomorphism of $\overline{D}_P$ onto $\overline{D'}_P$.

Corollary 8.10. In particular, the conclusion of Theorem 8.9 holds if

$$\int_{D \cap B(z_0, \varepsilon_0)} e^{\alpha_0 K_T^\mu(z, z_0)} \, dm(z) < \infty \quad \forall \, z_0 \in \partial D$$

for some $\varepsilon_0 = \varepsilon(z_0) > 0$ and $\alpha_0 = \alpha(z_0) > 0$.

Remark 8.11. Note that condition (8.10) is not only sufficient but also necessary for a continuous extension to the boundary of all direct mappings $f$ with integral restrictions of type (8.9), see, e.g., Theorem 5.1 and Remark 5.1 in [21]. Recall also that condition (8.10) is equivalent to each of conditions (7.13)–(7.17).

9. Regular solutions for the Dirichlet problem

Recall that a mapping $f : D \to \mathbb{C}$ is called discrete if the pre-image $f^{-1}(y)$ of every point $y \in \mathbb{C}$ consists of isolated points and open if the image of every open set $U \subseteq D$ is open in $\mathbb{C}$.

For $\varphi(P) \not\equiv \text{const}$, $P \in E_D$, a regular solution of Dirichlet problem (1.4) for Beltrami equation (1.1) is a continuous discrete open mapping $f : D \to \mathbb{C}$ of the Sobolev class $W^{1,1}_{\text{loc}}$ with its Jacobian

$$J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2 \neq 0 \quad \text{a.e.}$$

satisfying (1.1) a.e. and condition (1.4) for all prime ends of the domain $D$. For $\varphi(P) \equiv c \in \mathbb{R}$, $P \in E_D$, a regular solution of the problem is any constant function $f(z) = c + ic'$, $c' \in \mathbb{R}$.

Theorem 9.1. Let $D$ be a bounded simply connected domain in $\mathbb{C}$ and let $\mu : D \to \mathbb{D}$ be a measurable function with $K_\mu \in L^1_{\text{loc}}$ and, moreover,

$$\int_0^{\delta(z_0)} \frac{dr}{||K_T^\mu||(z_0, r)} = \infty \quad \forall \, z_0 \in \overline{D}$$

for some $0 < \delta(z_0) < d(z_0) = \sup_{z \in \overline{D}} |z - z_0|$ and

$$||K_T^\mu||(z_0, r) := \int_{S(z_0, r)} K_T^\mu(z, z_0) \, ds.$$
Then the Beltrami equation \((1.1)\) has a regular solution \(f\) of the Dirichlet problem \((1.4)\) for every continuous function \(\varphi : E_D \to \mathbb{R}\).

Here and later on, we set that \(K^T_\mu\) is equal to zero outside of the domain \(D\).

**Corollary 9.2.** Let \(D\) be a bounded simply connected domain in \(\mathbb{C}\) and let \(\mu : D \to \mathbb{D}\) be a measurable function such that

\[
(9.3) \quad k^T_\mu(z) = O \left( \log \frac{1}{\varepsilon} \right) \quad \text{as} \quad \varepsilon \to 0 \quad \forall \ z_0 \in \overline{D}
\]

where \(k^T_\mu(z)\) is the average of the function \(K^T_\mu(z, z_0)\) over the circle \(S(z_0, \varepsilon)\).

Then the Beltrami equation \((1.1)\) has a regular solution \(f\) of the Dirichlet problem \((1.4)\) for every continuous function \(\varphi : E_D \to \mathbb{R}\).

**Remark 9.3.** In particular, the conclusion of Theorem 9.1 holds if

\[
(9.4) \quad K^T_\mu(z, z_0) = O \left( \log \frac{1}{|z - z_0|} \right) \quad \text{as} \quad z \to z_0 \quad \forall \ z_0 \in \overline{D}.
\]

**Proof of Theorem 9.1.** First of all note that \(E_D\) cannot consist of a single prime end. Indeed, all rays going from a point \(z_0 \in D\) to \(\infty\) intersect \(\partial D\) because the domain \(D\) is bounded, see, e.g., Proposition 2.3 in \([33]\) or Proposition 13.3 in \([24]\). Thus, \(\partial D\) contains more than one point and by the Riemann theorem, see, e.g., \(\Pi.2.1\) in \([8]\), \(D\) can be mapped onto the unit disk \(\mathbb{D}\) with a conformal mapping \(R\). However, then by the Carathéodory theorem there is one-to-one correspondence between elements of \(E_D\) and points of the unit circle \(\partial \mathbb{D}\), see, e.g., Theorem 9.6 in \([4]\).

Let \(F\) be a regular homeomorphic solution of equation \((1.1)\) in the class \(W^{1,1}_{1,1}\) which exists in view of condition \((9.3)\), see, e.g., Theorem 5.4 in paper \([38]\) or Theorem 11.10 in monograph \([24]\).

Note that the domain \(D^* = F(D)\) is simply connected in \(\overline{\mathbb{C}}\), see, e.g., Lemma 5.3 in \([13]\) or Lemma 6.5 in \([24]\). Let us assume that \(\partial D^*\) in \(\overline{\mathbb{C}}\) consists of the single point \(\{ \infty \}\). Then \(\overline{\mathbb{C}} \setminus D^*\) also consists of the single point \(\infty\), i.e., \(D^* = \mathbb{C}\), since if there is a point \(z_0 \in \mathbb{C}\) in \(\overline{\mathbb{C}} \setminus D^*\), then, joining it and any point \(\zeta\) in \(D^*\) with a segment of a straight line, we find one more point of \(\partial D^*\) in \(\mathbb{C}\), see, e.g., again Proposition 2.3 in \([33]\) or Proposition 13.3 in \([24]\). Now, let \(D^*\) denote the exterior of the unit disk \(\mathbb{D}\) in \(\mathbb{C}\) and let \(\kappa(\zeta) = 1/\zeta, \kappa(0) = \infty, \kappa(\infty) = 0\). Consider the mapping \(F_\kappa = \kappa \circ F : \overline{\mathbb{D}} \to \mathbb{D}_0\) where \(\mathbb{D} = F^{-1}(\mathbb{D}^*)\) and \(\mathbb{D}_0 = \mathbb{D} \setminus \{0\}\) is the punctured unit disk. It is clear that \(F_\kappa\) is also a regular homeomorphic solution of Beltrami equation \((1.1)\) in the class \(W^{1,1}_{1,1}\) in the bounded two-connected domain \(\overline{\mathbb{D}}\) because the mapping \(\kappa\) is conformal. By Theorem 8.2 there is a one-to-one correspondence between elements of \(E_D\) and \(0\). However, it was shown above that \(E_D\) cannot consists of a single prime end. This contradiction disproves the above assumption that \(\partial D^*\) consists of a single point in \(\overline{\mathbb{C}}\).

Thus, by the Riemann theorem \(D^*\) can be mapped onto the unit disk \(\mathbb{D}\) with a conformal mapping \(R_\kappa\). Note that the function \(g := R_\kappa \circ F\) is again a regular homeomorphic solution in the Sobolev class \(W^{1,1}_{1,1}\) of Beltrami equation \((1.1)\) which maps \(D\) to \(\mathbb{D}\). By Theorem 8.2 the mapping \(g\) admits an extension to a homeomorphism \(g_* : \overline{D}_\mu \to \overline{\mathbb{D}}\).
We find a regular solution of the initial Dirichlet problem (1.4) in the form 
\[ f = h \circ g \] where \( h \) is a holomorphic function in \( \mathbb{D} \) with the boundary condition 
\[ \lim_{z \to \zeta} \Re h(z) = \varphi(g_\nu^{-1}(\zeta)) \quad \forall \zeta \in \partial \mathbb{D}. \]
Note that we have from the right hand side a continuous function of the variable \( \zeta \).

As known, the analytic function \( h \) can be reconstructed in \( \mathbb{D} \) through its real part on the boundary up to a pure imaginary additive constant with the Schwartz formula, see, e.g., § 8, Chapter III, Part 3 in [12].

\[ h(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \varphi \circ g_\nu^{-1}(\zeta) \cdot \frac{\zeta + z}{\zeta - z} \cdot \frac{d\zeta}{\zeta}. \]

It is easy to see that the function \( f = h \circ g \) is a desired regular solution of the Dirichlet problem (1.4) for Beltrami equation (1.1). \( \square \)

Applying Lemma 2.2 in [35], see also Lemma 7.4 in [24], we obtain the following general lemma immediately from Theorem 9.1.

**Lemma 9.4.** Let \( D \) be a bounded simply connected domain in \( \mathbb{C} \) and \( \mu : D \to \mathbb{D} \) be a measurable function with \( K_\mu \in L^1(D) \). Suppose that, for every \( z_0 \in \mathbb{D} \) and every small enough \( \varepsilon_0 < d(z_0) := \sup_{z \in \mathbb{D}} |z - z_0| \), there is a family of measurable functions \( \psi_{z_0, \varepsilon, \varepsilon_0} : (0, \infty) \to [0, \infty] \), \( \varepsilon \in (0, \varepsilon_0) \) such that

\[ 0 < I_{z_0, \varepsilon_0}(\varepsilon) := \int_{\varepsilon_0}^{\varepsilon} \psi_{z_0, \varepsilon, \varepsilon_0}(t) \, dt < \infty \quad \forall \varepsilon \in (0, \varepsilon_0) \]

and

\[ \int_{D(z_0, \varepsilon, \varepsilon_0)} K_\mu^T(z, z_0) \cdot \psi_{z_0, \varepsilon, \varepsilon_0}^2(|z - z_0|) \, dm(z) = o(I_{z_0, \varepsilon_0}^2(\varepsilon)) \quad \text{as} \quad \varepsilon \to 0 \]

where \( D(z_0, \varepsilon, \varepsilon_0) = \{ z \in D : \varepsilon < |z - z_0| < \varepsilon_0 \} \). Then the Beltrami equation (1.7) has a regular solution \( f \) of the Dirichlet problem (1.4) for every continuous function \( \varphi : E_D \to \mathbb{R} \).

**Remark 9.5.** In fact, it is sufficient here to request instead of the condition \( K_\mu \in L^1(D) \) only a local integrability of \( K_\mu \) in the domain \( D \) and the condition \( \| K_\mu \|_{L^1(z_0, r)} \neq \infty \) for a.e. \( r \in (0, \varepsilon_0) \) at all \( z_0 \in \partial D \).

By Lemma 9.3 with the choice \( \psi_{z_0, \varepsilon}(t) \equiv 1/(t \log \frac{1}{t}) \) we obtain the following result, see also Lemma 8.6.

**Theorem 9.6.** Let \( D \) be a bounded simply connected domain in \( \mathbb{C} \) and let \( \mu : D \to \mathbb{D} \) be a measurable function with \( K_\mu \in L^1_{\text{loc}} \) and

\[ K_\mu^T(z, z_0) \leq Q_{z_0}(z) \in \text{FMO}(z_0) \quad \forall z_0 \in \overline{D}. \]

Then the Beltrami equation (1.7) has a regular solution \( f \) of the Dirichlet problem (1.4) for every continuous function \( \varphi : E_D \to \mathbb{R} \).

**Remark 9.7.** In particular, the hypotheses and the conclusion of Theorem 9.6 hold if either \( Q_{z_0} \in \text{BMO}_{\text{loc}} \) or \( Q_{z_0} \in W^{1,2}_{\text{loc}} \) because \( W^{1,2}_{\text{loc}} \subset \text{VMO}_{\text{loc}} \), see, e.g., [2].

By Corollary 8.3 we obtain from Theorem 9.6 the following statement.
COROLLARY 9.8. Let $D$ be a bounded simply connected domain in $\mathbb{C}$ and let $\mu : D \to \mathbb{D}$ be a measurable function with $K_\mu \in L^1_{\text{loc}}$ such that

\begin{equation}
\limsup_{\varepsilon \to 0} \int_{B(z_0, \varepsilon)} K_\mu^T(z, z_0) \, dm(z) < \infty \quad \forall \ z_0 \in \overline{D}.
\end{equation}

Then the Beltrami equation (1.7) has a regular solution $f$ of the Dirichlet problem (1.4) for every continuous function $\varphi : \partial D \to \mathbb{R}$.

REMARK 9.9. In particular, by (1.7) the conclusion of Theorem 9.6 holds if

\begin{equation}
K_\mu(z) \leq Q(z) \in \text{BMO}(\overline{D}).
\end{equation}

The next statement follows from Lemma 9.4 under the choice $\psi(t) = 1/t$, see also Remark 9.5.

THEOREM 9.10. Let $D$ be a bounded simply connected domain in $\mathbb{C}$ and let $\mu : D \to \mathbb{D}$ be a measurable function such that

\begin{equation}
\int_{\varepsilon < |z - z_0| < \varepsilon_0} K_\mu^T(z, z_0) \frac{dm(z)}{|z - z_0|^2} = o \left( \frac{1}{\varepsilon^2} \right) \quad \forall \ z_0 \in \overline{D}.
\end{equation}

Then the Beltrami equation (1.7) has a regular solution $f$ of the Dirichlet problem (1.4) for every continuous function $\varphi : \partial D \to \mathbb{R}$.

REMARK 9.11. Similarly, choosing in Lemma 9.4 $\psi(t) = 1/(t \log 1/t)$ instead of $\psi(t) = 1/t$, we obtain that condition (9.10) can be replaced by the condition

\begin{equation}
\int_{\varepsilon < |z - z_0| < \varepsilon_0} \frac{K_\mu^T(z, z_0) \, dm(z)}{\left( |z - z_0| \log \frac{1}{|z - z_0|} \right)^2} = o \left( \frac{1}{\varepsilon^2} \right) \quad \forall \ z_0 \in \overline{D}.
\end{equation}

Here we are able to give a number of other conditions of a logarithmic type. In particular, condition (9.3), thanking to Theorem 9.1, can be replaced by the weaker condition

\begin{equation}
k_{z_0}^T(\tau) = O \left( \frac{1}{r \log \log \frac{1}{r}} \right).
\end{equation}

Finally, by Theorem 9.1 applying also Theorem 3.1 in [43], we come to the following result.

THEOREM 9.12. Let $D$ be a bounded simply connected domain in $\mathbb{C}$ and let $\mu : D \to \mathbb{D}$ be a measurable function with $K_\mu \in L^1_{\text{loc}}$ and

\begin{equation}
\int_{D \setminus B(z_0, \varepsilon_0)} \Phi_{z_0}(K_\mu^T(z, z_0)) \, dm(z) < \infty \quad \forall \ z_0 \in \overline{D}
\end{equation}

for $\varepsilon_0 = \varepsilon(z_0) > 0$ and a nondecreasing convex function $\Phi_{z_0} : [0, \infty) \to [0, \infty)$ with

\begin{equation}
\int_{\delta_0}^{\delta} \frac{d\tau}{\tau \Phi_{z_0}(\tau)} = \infty
\end{equation}

for $\delta_0 = \delta(z_0) > \Phi_{z_0}(0)$. Then the Beltrami equation (1.7) has a regular solution $f$ of the Dirichlet problem (1.4) for every continuous function $\varphi : \partial D \to \mathbb{R}$. 
Remark 9.13. Recall that condition (9.14) is equivalent to each of conditions (7.13)–(7.17). Moreover, condition (9.14) is not only sufficient but also necessary to have a regular solution of the Dirichlet problem (1.4) for every Beltrami equation (1.1) with integral restriction (9.13) for every continuous function $\varphi : E_D \to \mathbb{R}$.

Indeed, by the Stoilow theorem on representation of discrete open mappings, see, e.g., [47], every regular solution $f$ of the Dirichlet problem (1.4) for Beltrami equation (1.1) with $K_\mu \in L^1_{\text{loc}}$ can be represented in the form of composition $f = h \circ F$ where $h$ is a holomorphic function and $F$ is a regular homeomorphic solution of (1.1) in the class $W^{1,1}_{\text{loc}}$. Thus, by Theorem 5.1 in [44] on the nonexistence of regular homeomorphic solutions of (1.1) in the class $W^{1,1}_{\text{loc}}$, if (9.14) fails, then there is a measurable function $\mu : D \to \mathbb{D}$ satisfying integral condition (9.13) for which Beltrami equation (1.1) has no regular solution of the Dirichlet problem (1.4) for any nonconstant continuous function $\varphi : E_D \to \mathbb{R}$.

Corollary 9.14. Let $D$ be a bounded simply connected domain in $\mathbb{C}$ and let $\mu : D \to \mathbb{D}$ be a measurable function with $K_\mu \in L^1_{\text{loc}}$ and

$$e^{\rho_0 K^T_\mu(z, z_0)} \, dm(z) < \infty \quad \forall \, z_0 \in \overline{D}$$

for some $\rho_0 = \varepsilon(z_0) > 0$ and $\rho_0 = \alpha(z_0) > 0$. Then the Beltrami equation (1.1) has a regular solution $f$ of the Dirichlet problem (1.4) for every continuous function $\varphi : E_D \to \mathbb{R}$.

10. Pseudoregular solutions in multiply connected domains

As it was first noted by Bogdan Bojarski, see, e.g., §6 of Chapter 4 in [49], the Dirichlet problem for the Beltrami equations, generally speaking, has no regular solution in the class of continuous (single–valued) in $\mathbb{C}$ functions with generalized derivatives in the case of multiply connected domains $D$. Hence the natural question arose: whether solutions exist in wider classes of functions for this case? It is turned out to be solutions for this problem can be found in the class of functions admitting a certain number (related with connectedness of $D$) of poles at prescribed points. Later on, this number will take into account the multiplicity of these poles from the Stoilow representation.

Namely, a pseudoregular solution of such a problem is a continuous (in $\overline{C}$) discrete open mapping $f : D \to \overline{\mathbb{C}}$ of the Sobolev class $W^{1,1}_{\text{loc}}$ (outside of poles) with its Jacobian $J_f(z) = |f_z|^2 - |f_\bar{z}|^2 \neq 0$ a.e. satisfying (1.1) a.e. and the boundary condition (1.4).

Arguing similarly to the case of simply connected domains and applying Theorem V.6.2 in [8] on conformal mappings of finitely connected domains onto circular domains and also Theorems 4.13 and 4.14 in [49], we obtain the following result.

Theorem 10.1. Let $D$ be a bounded $m$–connected domain in $\mathbb{C}$ with nondegenerate boundary components, $k \geq m - 1$ and $\mu : D \to \mathbb{D}$ be a measurable function with $K_\mu \in L^1_{\text{loc}}$ and

$$\delta(z_0) \int_0^\infty \frac{dr}{||K^T_\mu||(z_0, r)} = \infty \quad \forall \, z_0 \in \overline{D}$$
for some $0 < \delta(z_0) < d(z_0) = \sup_{z \in D} |z - z_0|$ and 
\[
||K^T_\mu||(z_0, r) := \int_{S(z_0, r)} K^T_\mu(z, z_0) \, ds.
\]

Then the Beltrami equation (1.1) has a pseudoregular solution $f$ of the Dirichlet problem (1.4) with $k$ poles at prescribed points in $D$ for every continuous function $\varphi : \overline{E_D} \to \mathbb{R}$.

Here, as before, we set $K^T_\mu$ to be extended by zero outside of the domain $D$.

**Corollary 10.2.** Let $D$ be a bounded $m$–connected domain in $\mathbb{C}$ with nondegenerate boundary components, $k \geq m - 1$ and $\mu : D \to \mathbb{D}$ be a measurable function with $K_\mu \in L^1_{\text{loc}}$ and
\[
(10.2) \quad k^T_{z_0}(\varepsilon) = O\left(\frac{1}{\log \frac{1}{\varepsilon}}\right) \quad \text{as} \quad \varepsilon \to 0 \quad \forall \ z_0 \in \overline{D}
\]

where $k^T_{z_0}(\varepsilon)$ is the average of the function $K^T_\mu(z, z_0)$ over the circle $S(z_0, \varepsilon)$.

Then the Beltrami equation (1.1) has a pseudoregular solution $f$ of the Dirichlet problem (1.4) with $k$ poles at prescribed points in $D$ for every continuous function $\varphi : \overline{E_D} \to \mathbb{R}$.

**Remark 10.3.** In particular, the conclusion of Theorem 10.1 holds if
\[
(10.3) \quad K^T_\mu(z, z_0) = O\left(\frac{1}{|z - z_0|}\right) \quad \text{as} \quad \varepsilon \to 0 \quad \forall \ z_0 \in \overline{D}.
\]

**Proof of Theorem 10.1.** Let $F$ be a regular solution of equation (1.1) in the class $W^{1,1}_{\text{loc}}$ that exists by condition (10.1), see, e.g., Theorem 5.4 in the paper \cite{38} or Theorem 11.10 in the monograph \cite{24}. Note that the domain $D^* = F(D)$ is $m$–connected in $\overline{C}$ and there is a natural one–to–one correspondence between components $\gamma_j$ of $\partial D$ and components $\Gamma_j$ of $\Gamma = \partial D^*$, $\Gamma_j = C(\gamma_j, F)$ and $\gamma_j = C(\Gamma_j, F^{-1})$, $j = 1, \ldots, m$, see, e.g., Lemma 5.3 in \cite{13} or Lemma 6.5 in \cite{24}. Moreover, by Remark 10.3, every subspace $E_j$ of $E_D$ associated with $\gamma_j$ consists of more than one prime end, even it is homeomorphic to the unit circle.

Next, no one of $\Gamma_j$, $j = 1, \ldots, m$, is degenerated to a single point. Indeed, let us assume that $\Gamma_{j_0} = \{\zeta_0\}$ first for some $\zeta_0 \in \mathbb{C}$. Let $r_0 \in (0, d_0)$ where 
\[
d_0 = \inf_{\zeta \in \Gamma \setminus \Gamma_{j_0}} \{\zeta - \zeta_0\}.
\]

Then the punctured disk $D_0 = \{\zeta \in \mathbb{C} : 0 < |\zeta - \zeta_0| < r_0\}$ is in the domain $D^*$ and its boundary does not intersect $\Gamma \setminus \Gamma_{j_0}$. Set $\tilde{D} = F^{-1}(D_0)$, then by the construction $\tilde{D} \subset D$ is a $2$–connected domain, $\tilde{D} \cap \gamma \setminus \gamma_{j_0} = \emptyset$, $C(\gamma_{j_0}, \tilde{F}) = \{\zeta_0\}$ and $C(\zeta_0, \tilde{F}^{-1}) = \gamma_{j_0}$, where $\tilde{F}$ is a restriction of the mapping $F$ to $D$. However, this contradicts Theorem 8.2 because, as it was noted above, $E_{j_0}$ contains more than one prime end.

Now, let assume that $\Gamma_{j_{\alpha}} = \{\infty\}$. Then the component of $\overline{C} \setminus D^*$ associated with $\Gamma_{j_{\alpha}}$, see Lemma 5.1 in \cite{13} or Lemma 6.3 in \cite{24}, is also consists of the single point $\infty$ because if the interior of this component is not empty, then choosing there an arbitrary point $\zeta_0$ and joining it with a point $\zeta_* \in D^*$ by a segment of a straight line we would find one more point in $\Gamma_{j_{\alpha}}$, see, e.g., Proposition 2.3 in \cite{33}, or Proposition 13.3 in \cite{24}.
Thus, applying if it is necessary an additional stretching (conformal mapping), with no loss of generality we may assume that \( D^* \) contains the exteriority \( \mathbb{D}_+ \) of the unit disk \( \mathbb{D} \) in \( \mathbb{C} \). Set \( \kappa(\zeta) = 1/\zeta \), \( \kappa(0) = \infty \), \( \kappa(\infty) = 0 \). Consider the mapping \( F_* = \kappa \circ F : \overline{\mathbb{D}} \to \mathbb{D}_0 \) where \( \overline{\mathbb{D}} = F^{-1}(\mathbb{D}_+) \) and \( \mathbb{D}_0 = \mathbb{D} \setminus \{0\} \) is the punctured unit disk. It is clear that \( F_* \) is also a homeomorphic solution of Beltrami equation \((1.1)\) of the class \( W_{1,1}^{\text{loc}} \) in 2–connected domain \( \overline{\mathbb{D}} \) because the mapping \( \kappa \) is conformal. Consequently, by Theorem 8.2 elements of \( E_{\gamma_0} \) should be in a one–to–one correspondence with 0. However, it was already noted, \( E_{\gamma_0} \) cannot consists of a single prime end. The obtained contradiction disproves the assumption that \( \Gamma_{\gamma_0} = \{\infty\} \).

Thus, by Theorem V.6.2 \([8]\), see also Remark 1.1 in \([7]\), \( D^* \) can be mapped with a conformal mapping \( R_* \) onto a bounded circular domain \( \mathbb{B}^* \) whose boundary consists of mutually disjoint circles. Note that the function \( g := R_* \circ F \) is again a regular homeomorphic solution in the Sobolev class \( W_{1,1}^{\text{loc}} \) for Beltrami equation \((1.1)\) that maps \( D \) onto \( \mathbb{B}^* \). By Theorem 8.2 the mapping \( g \) admits an extension to a homeomorphism \( g_* : \overline{\mathbb{B}} \to \overline{\mathbb{B}}^* \).

Let us find a solution of the initial Dirichlet problem \((1.4)\) in the form \( f = h \circ g \) where \( h \) is a meromorphic function in \( \mathbb{B}^* \) with the boundary condition
\[
\lim_{z \to \zeta} \Re h(z) = \varphi(g_*^{-1}(\zeta)) \quad \forall \zeta \in \partial \mathbb{B}^*
\]
and \( k \geq m - 1 \) poles corresponding under the mapping \( g \) to those at prescribed points in \( D \). Note that the function from the right hand side in \((10.4)\) is continuous in the variable \( \zeta \). Thus, such a function \( h \) exists by Theorems 4.13 and 4.14 in \([49]\). It is clear that the function \( f \) associated with \( h \) is by the construction a desired pseudoregular solution of the Dirichlet problem \((1.4)\) for Beltrami equation \((1.1)\).

Applying Lemma 2.2 in \([35]\), see also Lemma 7.4 in \([24]\), we obtain immediately from Theorem 10.1 the next lemma.

**Lemma 10.4.** Let \( D \) be a bounded \( m \)–connected domain in \( \mathbb{C} \) with nondegenerate boundary components, \( k \geq m - 1 \) and \( \mu : D \to \mathbb{D} \) be a measurable function with \( K_\mu \in L^1(D) \). Suppose that, for every \( z_0 \in \partial D \) and every small enough \( 0 < \varepsilon_0 < d(z_0) := \sup_{z \in \partial D} |z - z_0| \), there is a family of measurable functions \( \psi_{z_0, \varepsilon, \varepsilon_0} : (0, \infty) \to [0, \infty] \), \( \varepsilon \in (0, \varepsilon_0) \) such that
\[
(10.5) \quad 0 < I_{z_0, \varepsilon_0}(\varepsilon) := \int_\varepsilon^{\varepsilon_0} \psi_{z_0, \varepsilon, \varepsilon_0}(t) \, dt < \infty \quad \forall \varepsilon \in (0, \varepsilon_0)
\]
and
\[
(10.6) \quad \int_{\varepsilon < |z - z_0| < \varepsilon_0} K_\mu^T(z, z_0) \cdot \psi_{z_0, \varepsilon, \varepsilon_0}^2(\varepsilon) \, dm(z) = o(I_{z_0, \varepsilon_0}(\varepsilon)) \quad \text{as} \quad \varepsilon \to 0 .
\]

Then the Beltrami equation \((1.1)\) has a pseudoregular solution \( f \) of the Dirichlet problem \((1.4)\) with \( k \) poles at prescribed points in \( D \) for every continuous function \( \varphi : E \to \mathbb{R} \).
Remark 10.5. In fact, here it is sufficient to assume instead of the condition $K_\mu \in L^1(D)$ the local integrability of $K_\mu$ in the domain $D$ and the condition $\|K_\mu\|(z_0, r) \neq \infty$ for a.e. $r \in (0, \varepsilon_0)$ and all $z_0 \in \partial D$.

By Lemma 10.4 with the choice $\psi_{z_0}, \varepsilon(t) \equiv 1/t \log \frac{1}{t}$ we obtain the following result, see also Lemma 3.6.

Theorem 10.6. Let $D$ be a bounded $m$-connected domain in $\mathbb{C}$ with nondegenerate boundary components, $k \geq m - 1$ and $\mu : D \to \mathbb{D}$ be a measurable function with $K_\mu \in L^1(D)$ such that
\[
K_\mu^T(z, z_0) \leq Q_{z_0}(z) \in \text{FMO}(z_0) \quad \forall z_0 \in \partial D.
\]
Then the Beltrami equation (1.7) has a pseudoregular solution $f$ of the Dirichlet problem (1.4) with $k$ poles at prescribed points in $D$ for every continuous function $\varphi : E_D \to \mathbb{R}$.

Remark 10.7. In particular, the conclusion of Theorem 10.6 holds if either $Q_{z_0} \in \text{BMO}_{loc}$ or $Q_{z_0} \in \text{W}^{1,2}_{loc}$ because $\text{W}^{1,2}_{loc} \subset \text{VMO}_{loc}$, see, e.g., [2].

By Corollary 9.3 we have the next consequence of Theorem 10.6.

Corollary 10.8. Let $D$ be a bounded $m$-connected domain in $\mathbb{C}$ with nondegenerate boundary components, $k \geq m - 1$ and $\mu : D \to \mathbb{D}$ be a measurable function with $K_\mu \in L^1(D)$ such that
\[
\limsup_{\varepsilon \to 0} \frac{1}{\mu(z_0, \varepsilon)} \int_{B(z_0, \varepsilon)} K_\mu^T(z, z_0) \ dm(z) < \infty \quad \forall z_0 \in \partial D.
\]
Then the Beltrami equation (1.7) has a pseudoregular solution $f$ of the Dirichlet problem (1.4) with $k$ poles at prescribed points in $D$ for every continuous function $\varphi : E_D \to \mathbb{R}$.

Remark 10.9. In particular, by (1.7) the conclusion of Theorem 10.6 holds if
\[
K_\mu(z) \leq Q(z) \in \text{BMO}(\partial D)
\]

The following statement follows from Lemma 10.4 through the choice $\psi(t) = 1/t$, see also Remark 10.5.

Theorem 10.10. Let $D$ be a bounded $m$-connected domain in $\mathbb{C}$ with nondegenerate boundary components, $k \geq m - 1$ and $\mu : D \to \mathbb{D}$ be a measurable function such that
\[
\int_{\varepsilon < |z - z_0| < \varepsilon_0} K_\mu^T(z, z_0) \ dm(z) = o \left( \log 1 \right) \quad \forall z_0 \in \partial D.
\]
Then the Beltrami equation (1.7) has a pseudoregular solution $f$ of the Dirichlet problem (1.4) with $k$ poles at prescribed points in $D$ for every continuous function $\varphi : E_D \to \mathbb{R}$.

Remark 10.11. Similarly, choosing in Lemma 10.4 $\psi(t) = 1/(t \log 1/t)$ instead of $\psi(t) = 1/t$ we obtain that condition (10.10) can be replaced by the condition
\[
\int_{\varepsilon < |z - z_0| < \varepsilon_0} K_\mu^T(z, z_0) \ dm(z) = o \left( \log \log 1 \right) \quad \forall z_0 \in \partial D.
\]
Here we are able to give a number of other conditions of the logarithmic type. In particular, condition (10.22), thanking to Theorem 10.1, can be replaced by the weaker condition

(10.12)  
\[ k^T_{\varepsilon}(r) = O \left( \log \frac{1}{r} \log \log \frac{1}{r} \right). \]

Finally, by Theorem 10.1, applying also Theorem 3.1 in the paper [43], we come to the following result.

**Theorem 10.12.** Let \( D \) be a bounded \( m \)-connected domain in \( \mathbb{C} \) with nondegenerate boundary components, \( k \geq m - 1 \) and \( \mu : D \to \mathbb{D} \) be a measurable function with \( K_\mu \in L^1_{\text{loc}} \) such that

(10.13)  
\[ \int_{D \cap B(z_0, \varepsilon_0)} \Phi_{z_0}(K_\mu^{T}(z, z_0)) \, dm(z) < \infty \]

for \( \varepsilon_0 = \varepsilon(z_0) > 0 \) and a nondecreasing convex function \( \Phi_{z_0} : [0, \infty) \to [0, \infty) \) with

(10.14)  
\[ \int_{\delta_0}^{\infty} \frac{d\tau}{\tau \Phi_{z_0}^{-1}(\tau)} = \infty \]

for \( \delta_0 = \delta(z_0) > \Phi_{z_0}(0) \). Then the Beltrami equation \((1.1)\) has a pseudoregular solution \( f \) of the Dirichlet problem \((1.3)\) with \( k \) poles at prescribed points in \( D \) for every continuous function \( \varphi : E_D \to \mathbb{R} \).

Recall that condition (10.14) is equivalent to every of conditions \((7.13)\)–\((7.17)\).

**Corollary 10.13.** Let \( D \) be a bounded \( m \)-connected domain in \( \mathbb{C} \) with nondegenerate boundary components, \( k \geq m - 1 \) and \( \mu : D \to \mathbb{D} \) be a measurable function with \( K_\mu \in L^1_{\text{loc}} \) such that

(10.15)  
\[ \int_{D \cap B(z_0, \varepsilon_0)} e^{\alpha_0 K_\mu^{T}(z, z_0)} \, dm(z) < \infty \quad \forall \ z_0 \in \overline{D} \]

for some \( \varepsilon_0 = \varepsilon(z_0) > 0 \) and \( \alpha_0 = \alpha(z_0) > 0 \).

Then the Beltrami equation \((1.1)\) has a pseudoregular solution \( f \) of the Dirichlet problem \((1.3)\) with \( k \) poles at prescribed points in \( D \) for every continuous function \( \varphi : E_D \to \mathbb{R} \).

### 11. Multivalent solutions in finitely connected domains

In finitely connected domains \( D \) in \( \mathbb{C} \), in addition to pseudoregular solutions, the Dirichlet problem \((1.2)\) for the Beltrami equation \((1.1)\) admits multi-valued solutions in the spirit of the theory of multi-valued analytic functions. We say that a discrete open mapping \( f : B(z_0, \varepsilon_0) \to \mathbb{C} \), where \( B(z_0, \varepsilon_0) \subseteq D \), is a **local regular solution of the equation** \((1.1)\) if \( f \in W^{1,1}_{\text{loc}} \), \( J_f(z) \neq 0 \) and \( f \) satisfies \((1.1)\) a.e. in \( B(z_0, \varepsilon_0) \).

The local regular solutions \( f : B(z_0, \varepsilon_0) \to \mathbb{C} \) and \( f_* : B(z_*, \varepsilon_*) \to \mathbb{C} \) of the equation \((1.1)\) will be called extension of each to other if there is a finite chain of such solutions \( f_i : B(z_i, \varepsilon_i) \to \mathbb{C}, \ i = 1, \ldots, m, \) that \( f_1 = f_0, \ f_m = f_* \) and \( f_i(z) \equiv f_{i+1}(z) \) for \( z \in E_i := B(z_i, \varepsilon_i) \cap B(z_{i+1}, \varepsilon_{i+1}) \neq \emptyset, \ i = 1, \ldots, m - 1. \) A
collection of local regular solutions \( f_j : B(z_j, \varepsilon_j) \to \mathbb{C}, \ j \in J, \) will be called a **multivalent solution** of the equation (1.1) in \( D \) if the disks \( B(z_j, \varepsilon_j) \) cover the whole domain \( D \) and \( f_j \) are extensions of each to other through the collection and the collection is maximal by inclusion. A multi-valued solution of the equation (1.1) will be called a **multivalent solution of the Dirichlet problem** (1.2) if

\[
\lim_{z \to \zeta} u(z) = \varphi(\zeta) \quad \forall \, \zeta \to \partial D.
\]

As it was before, we assume later on that \( K_T^\mu(z_0, \cdot) \) is extended by zero outside of the domain \( D \).

The proof of the existence of multivalent solutions of Dirichlet problem (1.4) for Beltrami equation (1.1) in finitely connected domains is reduced to the Dirichlet problem for harmonic functions in circular domains, see, e.g., § 3 of Chapter VI in [8].

**Theorem 11.1.** Let \( D \) be a bounded finitely connected domain in \( \mathbb{C} \) with non-degenerate boundary components and \( \mu : D \to \mathbb{D} \) be a measurable function with \( K_\mu \in L^1_{loc} \) and

\[
\delta(z_0) \int_0^r \frac{dr}{||K_T^\mu||(z_0, r)} = \infty \quad \forall \, z_0 \in \partial D
\]

for some \( 0 < \delta(z_0) < d(z_0) = \sup_{z \in D} |z - z_0| \) and

\[
||K_T^\mu||(z_0, r) := \int_{S(z_0, r)} K_T^\mu(z, z_0) \, ds.
\]

Then Beltrami equation (1.1) has a multivalent solution of Dirichlet problem (1.4) for every continuous function \( \varphi : E_D \to \mathbb{R} \).

**Proof of Theorem 11.1.** Similarly to the first part of Theorem 11.1 it is proved that there is a regular homeomorphic solution \( g \) of Beltrami equation (1.1) mapping the domain \( D \) onto a circular domain \( D^* \) whose boundary consists of mutually disjoint circles. By Theorem 8.2 the mapping \( g \) admits an extension to a homeomorphism \( g^* : D \to D^* \).

As known, in the circular domain \( D^* \), there is a solution of the Dirichlet problem (11.2)

\[
\lim_{z \to \zeta} u(z) = \varphi(g^{-1}_*(\zeta)) \quad \forall \, \zeta \in \partial D^*
\]

for harmonic functions \( u \), see, e.g., § 3 of Chapter VI in [8]. Let \( B_0 = B(z_0, r_0) \) is a disk in the domain \( D \). Then \( B_0 = g(B_0) \) is a simply connected subdomain of the circular domain \( D^* \) where there is a conjugate function \( v \) determined up to an additive constant such that \( h = u + iv \) is a single-valued analytic function. The function \( h \) can be extended to, generally speaking multivalent, analytic function \( H \) along any path in \( D^* \) because \( u \) is given in the whole domain \( D^* \).

Thus, \( f = H \circ g \) is a desired multivalent solution of the Dirichlet problem (1.4) for Beltrami equation (1.1).

The hypotheses of the rest theorems and corollaries below imply the hypotheses of Theorem 11.1 as it was shown in the previous section.
Corollary 11.2. Let $D$ be a bounded finitely connected domain in $\mathbb{C}$ with nondegenerate boundary components and $\mu : D \to \mathbb{D}$ be a measurable function with $K_\mu \in L^1_{\text{loc}}$ and

$$k^T_{z_0}(\varepsilon) = O\left(\log \frac{1}{\varepsilon}\right) \quad \text{as} \quad \varepsilon \to 0 \quad \forall \ z_0 \in \overline{D}$$

(11.3)

where $k^T_{z_0}(\varepsilon)$ is the average of the function $K^T_\mu(z, z_0)$ over the circle $S(z_0, \varepsilon)$.

Then Beltrami equation (1.1) has a multivalent solution of Dirichlet problem (1.4) for every continuous function $\varphi : E_D \to \mathbb{R}$.

Remark 11.3. In particular, the conclusion of Theorem 11.1 holds if

$$K^T_\mu(z, z_0) = O\left(\log \frac{1}{|z - z_0|}\right) \quad \text{as} \quad z \to z_0 \quad \forall \ z_0 \in \overline{D}.$$ 

(11.4)

Applying Lemma 2.2 in [35], see also Lemma 7.4 in [24], we obtain immediately from Theorem 11.1 the next lemma.

Lemma 11.4. Let $D$ be a bounded finitely connected domain in $\mathbb{C}$ with nondegenerate boundary components and $\mu : D \to \mathbb{D}$ be a measurable function with $K_\mu \in L^1(D)$. Suppose that, for every $z_0 \in \overline{D}$ and every small enough $0 < \varepsilon_0 < d(z_0) := \sup_{z \in D} |z - z_0|$, there is a family of measurable functions $\psi_{z_0, \varepsilon, \varepsilon_0} : (0, \infty) \to [0, \infty]$, $\varepsilon \in (0, \varepsilon_0)$ such that

$$0 < I_{z_0, \varepsilon, \varepsilon_0}(\varepsilon) := \int_{\varepsilon}^{\varepsilon_0} \psi_{z_0, \varepsilon, \varepsilon_0}(t) \ dt < \infty \quad \forall \ \varepsilon \in (0, \varepsilon_0)$$

(11.5)

and

$$\int_{\varepsilon < |z - z_0| < \varepsilon_0} K^T_\mu(z, z_0) \cdot \psi_{z_0, \varepsilon, \varepsilon_0}(|z - z_0|) \ dm(z) = o(I^2_{z_0, \varepsilon, \varepsilon_0}(\varepsilon)) \quad \text{as} \quad \varepsilon \to 0.\quad \text{(11.6)}$$

Then Beltrami equation (1.1) has a multivalent solution of Dirichlet problem (1.4) for every continuous function $\varphi : E_D \to \mathbb{R}$.

Remark 11.5. In fact, here it is sufficient to assume instead of the condition $K_\mu \in L^1(D)$ the local integrability of $K_\mu$ in the domain $D$ and the condition $\|K_\mu\|_{(z_0, r)} \neq \infty$ for a.e. $r \in (0, \varepsilon_0)$ and all $z_0 \in \partial D$.

By Lemma 11.4 with the choice $\psi_{z_0, \varepsilon}(t) \equiv 1/t \log \frac{1}{t}$ we obtain the following result, see also Lemma 3.6.

Theorem 11.6. Let $D$ be a bounded finitely connected domain in $\mathbb{C}$ with nondegenerate boundary components and $\mu : D \to \mathbb{D}$ be a measurable function with $K_\mu \in L^1(D)$ such that

$$K^T_\mu(z, z_0) \leq Q_{z_0}(z) \in \text{FMO}(z_0) \quad \forall \ z_0 \in \overline{D}. \quad \text{(11.7)}$$

Then Beltrami equation (1.1) has a multivalent solution of Dirichlet problem (1.4) for every continuous function $\varphi : E_D \to \mathbb{R}$.

Remark 11.7. In particular, the conclusion of Theorem 11.6 holds if either $Q_{z_0} \in \text{BMO}_{\text{loc}}$ or $Q_{z_0} \in W^{1,2}_{\text{loc}}$ because $W^{1,2}_{\text{loc}} \subset \text{VMO}_{\text{loc}}$, see, e.g., [2].
By Corollary 11.3 we have the next consequence of Theorem 11.6:

**Corollary 11.8.** Let $D$ be a bounded finitely connected domain in $\mathbb{C}$ with nondegenerate boundary components and $\mu : D \to \mathbb{D}$ be a measurable function with $K_\mu \in L^1(D)$ such that

\[
\limsup_{\varepsilon \to 0} \int_{B(z_0, \varepsilon)} K^T_\mu(z, z_0) \, dm(z) < \infty \quad \forall \ z_0 \in \overline{D}.
\]

Then Beltrami equation (1.1) has a multivalent solution of Dirichlet problem (1.4) for every continuous function $\varphi : E_D \to \mathbb{R}$.

**Remark 11.9.** In particular, by (1.7) the conclusion of Theorem 11.6 holds if

\[
K_\mu(z) \leq Q(z) \in \text{BMO}(\overline{D})
\]

The following statement follows from Lemma 11.4 through the choice $\psi(t) = 1/t$, see also Remark 11.5.

**Theorem 11.10.** Let $D$ be a bounded finitely connected domain in $\mathbb{C}$ with nondegenerate boundary components and $\mu : D \to \mathbb{D}$ be a measurable function such that

\[
\int_{|z - z_0| < \varepsilon_0} K^T_\mu(z, z_0) \, dm(z) = o\left(\left[\log \frac{1}{\varepsilon_0}\right]^2\right) \quad \forall \ z_0 \in \overline{D}.
\]

Then Beltrami equation (1.1) has a multivalent solution of Dirichlet problem (1.4) for every continuous function $\varphi : E_D \to \mathbb{R}$.

**Remark 11.11.** Similarly, choosing in Lemma 11.4 $\psi(t) = 1/(t \log 1/t)$ instead of $\psi(t) = 1/t$ we obtain that condition (11.10) can be replaced by the condition

\[
\int_{|z - z_0| < \varepsilon_0} \frac{K^T_\mu(z, z_0) \, dm(z)}{|z - z_0|^2 \left|\log \frac{1}{|z - z_0|}\right|^2} = o\left(\left[\log \log \frac{1}{\varepsilon_0}\right]^2\right) \quad \forall \ z_0 \in \overline{D}.
\]

Here we are able to give a number of other conditions of the logarithmic type. In particular, condition (11.2), thanking to Theorem 11.1, can be replaced by the weaker condition

\[
k^T_{z_0}(r) = O\left(\log \frac{1}{r} \log \log \frac{1}{r}\right).
\]

Finally, by Theorem 11.12 applying also Theorem 3.1 in the paper [43], we come to the following result.

**Theorem 11.12.** Let $D$ be a bounded finitely connected domain in $\mathbb{C}$ with nondegenerate boundary components and $\mu : D \to \mathbb{D}$ be a measurable function with $K_\mu \in L^1_{\text{loc}}(D)$ such that

\[
\int_{D \cap B(z_0, \varepsilon_0)} \Phi_{z_0}(K^T_\mu(z, z_0)) \, dm(z) < \infty
\]
for \( \varepsilon_0 = \varepsilon(z_0) > 0 \) and a nondecreasing convex function \( \Phi_{z_0} : [0, \infty) \to [0, \infty) \) with

\[
(11.14) \quad \int_{\delta_0}^{\infty} \frac{d\tau}{\tau \Phi_{z_0}^{-1}(\tau)} = \infty
\]

for \( \delta_0 = \delta(z_0) > \Phi_{z_0}(0) \). Then Beltrami equation (1.1) has a multivalent solution of Dirichlet problem (1.4) for every continuous function \( \varphi : E_D \to \mathbb{R} \).

Recall that condition (11.14) is equivalent to every of conditions (7.13)–(7.17).

**Corollary 11.13.** Let \( D \) be a bounded finitely connected domain in \( \mathbb{C} \) with nondegenerate boundary components and \( \mu : D \to \mathbb{D} \) be a measurable function with \( K_\mu \in L_{1\text{loc}}^{1} \) such that

\[
(11.15) \quad \int_{D \cap B(z_0, \varepsilon_0)} e^{\alpha_0 K_\mu^+(z,z_0)} \ dm(z) < \infty \quad \forall \ z_0 \in \overline{D}
\]

for some \( \varepsilon_0 = \varepsilon(z_0) > 0 \) and \( \alpha_0 = \alpha(z_0) > 0 \).

Then Beltrami equation (1.1) has a multivalent solution of Dirichlet problem (1.4) for every continuous function \( \varphi : E_D \to \mathbb{R} \).

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