Infinity, self-similarity, and continued fractions in physics: applications to resistor network puzzles

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Abstract

Puzzles involving infinite networks of resistors are an engaging way for students to explore the idea of infinity and self-similarity in physics. Recently K Atkin has described one such puzzle, alongside a solution based on an equivalent finite network (2022 Phys. Educ. 57 025015). Here we present a generalisation of this problem which showcases an important perspective on infinity as the limit of a process, and an alternative method of solution using continued fractions. We illustrate how our method can be used to devise new network puzzles suitable for class challenge problems.

Keywords: resistor networks, physics puzzles, infinity, self-similarity, continued fractions

In a recent article [1], K. Atkin describes a physics puzzle involving an infinite network of resistors $R$ configured between two terminals A and B in a repeating 'series-parallel' pattern (see figure 1(a)). The challenge of this puzzle is to determine the effective resistance $R_*$ of the network between the two terminals [1, 2].

As with conventional approaches to this problem [2], Atkin’s solution exploits the self-similarity of the network to construct an equivalent finite network with total resistance $R_*$ (figure 1(b)). It then follows from this finite network that $R_*$ must satisfy:

$$R_* = R + \left( \frac{1}{R} + \frac{1}{R_*} \right)^{-1}, \quad (1)$$

that is,

$$R_*^2 - RR_* - R^2 = 0. \quad (2)$$

Hence, by choosing the positive root of this quadratic, one obtains the solution:

$$R_* = \varphi R, \quad \text{where} \quad \varphi = \left( \frac{1 + \sqrt{5}}{2} \right) \quad (3)$$

is the golden ratio.
Two perspectives on the puzzle: (a) an infinite network of identical resistors, with total resistance $R^*$, and (b) a finite network with equivalent total resistance $R^*$. Generalised infinite network of resistors, with resistances expressed relative to $R$. The infinite network of figure 1 corresponds to setting $a_1 = a_2 = a_3 = \cdots = 1$. Sequential assembly of the infinite resistor network of figure 2 using: (a) $n = 1$ resistor; (b) $n = 2$ resistors; (c) $n = 3$ resistors; and (d) $n = 4$ resistors. Thus, the network with $n + 1$ resistors is obtained by adding a corresponding resistor to the network with $n$ resistors. Figure 2 represents the limit of this process as $n \rightarrow \infty$. The puzzle described by Atkin is an engaging context for students to explore ideas about infinity and self-similarity in physics [1, 3], and in our experience invariably ignites lively class debate [4, 5]. As with all good puzzles, however, proper understanding of the method of solution typically induces students to ask about related systems. In particular, the premise of using identical resistors—on which the argument by self-similarly is based—usually prompts the question: *how does one solve a generalised formulation of the puzzle in which the resistors are assumed to be different?* To answer this question, let us consider the infinite network of resistors depicted in figure 2. In this network, the resistors along the top rail have resistances $a_1 R, a_3 R, a_5 R$, etc while the resistors joining the two rails have resistances $R/a_2, R/a_4, R/a_6$, etc where the coefficients $a_i$ (with $i = 1, 2, \ldots$) are numbers without units. Each resistor is thus given an arbitrary resistance relative to $R$ by selecting different values for the $a_i$. The effective resistance $R^*$ between the terminals A and B in this generalised problem may then be determined as follows. We suppose that the network is assembled by adding one resistor at a time, i.e. as the limit of a process involving a sequence of finite networks (see figure 3). In this way, if we denote $R_n$ as the resistance between the terminals when the network comprises $n$ resistors, then $R^*$ is given by:

$$R^* = \lim_{n \rightarrow \infty} R_n.$$  

(4)

According to figure 3(a), when the network contains one resistor only ($n = 1$), the total effective resistance $R_1$ satisfies:

$$\frac{R_1}{R} = a_1.$$  

(5)

Thence, adding a second resistor ($n = 2$), we find by figure 3(b) that the total effective resistance $R_2$ satisfies:

$$\frac{R_2}{R} = a_1 + \frac{1}{a_2}.$$  

(6)
Similarly, with a third resistor \((n = 3)\), we have from figure 3(c) that:

\[
\frac{R_3}{R} = a_1 + \frac{1}{a_2 + \frac{1}{a_3}}. \tag{7}
\]

Each of the expressions in equations (5)–(7) is known as a continued fraction. Notice in general that the expression for \(R_{n+1}\) is obtained from the expression for \(R_n\) by making the transformation:

\[
a_n \rightarrow a_n + \frac{1}{a_{n+1}}. \tag{8}
\]

Thus, in the case of the infinite network of resistors \((n \rightarrow \infty)\), it follows that if the limiting process converges, then the total effective resistance \(R_\ast\) may be expressed as:

\[R_\ast = r_\ast R,\]

where

\[r_\ast = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \ddots}}} \tag{9}\]

is an infinite continued fraction. As we explore in the examples below, this solution for the generalised network (figure 2) provides an alternative solution to the original puzzle, and can be used to devise related networks suitable for class challenge problems.

**Example 1: an infinite network yielding the golden ratio \(\varphi\)**

To demonstrate how equation (9) can be used to solve the original puzzle described by Atkin [1], observe that the network in figure 1 is a special case of the general infinite network with \(a_1 = a_2 = \cdots = 1\) (see figure 2). Thus, by equation (9), the effective resistance of the original infinite network is:

\[R_\ast = r_\ast R, \text{ where } r_\ast = 1 + \frac{1}{1 + \frac{1}{1 + \ddots}}. \tag{10}\]

Here we can determine the value of the continued fraction informally by noting:

\[r_\ast = 1 + \frac{1}{r_\ast}, \text{ that is } r_\ast^2 - r_\ast - 1 = 0. \tag{11}\]

Indeed, solving this quadratic, and retaining the positive root only, we recover our original solution from equation (3), i.e.

\[R_\ast = r_\ast R, \text{ where } r_\ast = \varphi = \left(\frac{1 + \sqrt{5}}{2}\right). \tag{12}\]

is the golden ratio. Equation (10) is thus the continued fraction representation for \(\varphi\) [6].

**Example 2: an infinite network yielding \(\sqrt{2}\)**

In our experience it is helpful for students to consolidate their understanding of the original puzzle by adapting the methods of solution described above to related networks [4, 7]. To this end, we have found that by beginning with a known continued fraction, equation (9) can be used ‘in reverse’ to generate new puzzles suitable for class problem solving sessions [8].

Figure 4 depicts a network puzzle devised in this way to yield effective resistance \(R_\ast = \sqrt{2} R\). To see this, observe that the network is obtained from figure 2 by selecting:

\[a_1 = 1 \quad \text{and} \quad a_2 = a_3 = a_4 = \cdots = 2, \tag{13}\]

such that its effective resistance \(R_\ast\) is (see equation (9)):

\[R_\ast = r_\ast R, \text{ where } r_\ast = 1 + \frac{1}{2 + \frac{1}{2 + \ddots}}. \tag{14}\]

Thus, because the continued fraction satisfies:

\[r_\ast = 1 + \frac{1}{(1 + r_\ast)}, \text{ i.e., } r_\ast^2 = 2, \tag{15}\]

we have,

\[R_\ast = r_\ast R, \quad \text{with } r_\ast = \sqrt{2}. \tag{16}\]
Figure 4. Infinite network of resistors yielding total effective resistance \(R^* = \sqrt{2}R\).

Figure 5. Two resistor networks yielding \(R^* = \sqrt{2}R\): (a) the infinite network of figure 4 constructed using identical resistors \(R\) only; and (b) an equivalent finite network.

In our experience running sessions with foundation year (high-school level) and first-year undergraduate students, this network for \(\sqrt{2}\) is particularly suitable for two reasons [4]. First, the value of the continued fraction can be determined using elementary algebra. Second, by exploiting the rules for combining two resistors \(R\) in series and parallel, students can be tasked with the extra challenge of designing an alternative version of the network using identical resistors \(R\) only (see figure 5(a)). Indeed, this alternative form for the network presents a neat modification of the problem, and can be used as a way of encouraging students to think about strategies for adapting arguments from self-similarity. For example, the network can be represented in an equivalent finite form as in figure 5(b), in which case one finds:

\[
R^* = R + \left( \frac{1}{R} + \frac{1}{R + R^*} \right)^{-1}. \tag{17}
\]

Solving the quadratic obtained from this equation yields \(R^* = \sqrt{2}R\) as above.

Example 3: an infinite network to yield \(\pi\)

An alternative way for students to explore equation (9) is to be tasked with a ‘physics jeopardy’ type problem in which they are given the solution for the resistance of an infinite network (in the form of a continued fraction), and challenged to deduce the corresponding configuration of resistors [4, 9].

Challenge problem: an infinite resistor network for \(\pi\)

An infinite network of identical resistors \(R\) has effective total resistance \(R^* = \pi R\), where \(r_1 = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ddots}}}}\).

By exploiting the rules for combining resistors in series and parallel, adopt the continued fraction method to devise and sketch a corresponding infinite network of identical resistors with effective total resistance \(R^* = \pi R\).

Figure 6 presents a challenge problem suitable for this purpose which we have devised from the continued fraction for \(\pi\) given by [10]:

\[
\frac{\pi}{2} = 1 + \frac{1}{1 + \frac{1}{\frac{1}{2} + \frac{1}{\frac{1}{3} + \frac{1}{\frac{1}{4} + \frac{1}{\frac{1}{5} + \ddots}}}}}. \tag{18}
\]

To solve this problem, we begin by forming a network with resistance \(R^* = (\pi/2)R\); comparing equations (9) and (18), this may be done as in figure 2 by selecting:

\[
[a_1, a_2, a_3, a_4, a_5, a_6, a_7, \ldots] = \left[1, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots \right]. \tag{19}
\]
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Figure 7. Infinite network with total effective resistance $R_\ast = (\pi/2)R$.

Figure 8. Infinite network of identical resistors $R$. The effective total resistance between the terminals A and B is $R_\ast = (\pi/2)R$; it therefore follows that the effective total resistance $R_\pi$ between the terminals A and C is $R_\pi = 2R_\ast = \pi R$.

Once this network has been constructed (see figure 7), it may then be rebuilt using identical resistors only: one simply replaces those resistors that are integer multiples of $R$ with resistors in series, and those resistors that are integer divisors of $R$ with resistors in parallel. In this way one obtains an infinite network of identical resistors $R$ with effective total resistance $R_\pi = (\pi/2)R$, as shown in figure 8. Duplicating the network then yields an infinite network with total resistance $R_\pi = 2R_\pi = \pi R$ as required.

Summary

Recently Atkin has described a physics puzzle involving an infinite network of identical resistors as an engaging way of illustrating arguments by self-similarity in physics [1]. Here we have presented a generalisation of this puzzle to account for the possibility of non-identical resistors, and an alternative method of solution based on continued fractions. Our method offers a different perspective on ‘infinity in physics’ as the limit of process [3], whilst simultaneously introducing students to the fascinating subject of continued fractions. In our experience with foundation year (high-school level) students and first-year undergraduates [4], this approach provides a convenient framework for generating new puzzles suitable for class exercises, including ‘physics jeopardy’ type problems [9]. We look forward to developing physical demonstrations of these networks as student projects.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

References

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[4] Problem solving sessions on infinite network puzzles run for foundation year (high-school level) students, and first-year undergraduates at the University of York
[5] Students are not usually able to ‘crack’ the puzzle independently; however, many reason correctly that $R_\pi$ must lie somewhere between $R$ and $2R$, and can be guided towards a solution upon realising that part of the answer involves exploiting the network’s self-similarity
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Continued fractions for square roots are good candidates for such problems. For example, the square root \( r = \sqrt{N} \) of a natural number \( N = 2, 3, 4, \ldots \) may be expressed in the form
\[
r = 1 + \frac{(N-1)}{2 + \frac{(N-1)}{2 + \frac{(N-1)}{2 + \cdots}}},
\]
because the continued fraction satisfies:
\[
r = 1 + \frac{(N-1)}{1 + r}, \quad \text{i.e.,} \quad r^2 = N.
\]
Continued fractions obtained in this way can be written in the form of equation (9) after cancelling appropriate factors of \((N-1)\) within the fraction; the resulting form for the continued fraction may then be used to generate new puzzles similar to that of example 2.

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