R-MATRIX REALIZATION OF TWO-PARAMETER QUANTUM GROUP \( U_{r,s}(\mathfrak{gl}_n) \)

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Abstract. We provide a Faddeev-Reshetikhin-Takhtajan’s RTT approach to the quantum group \( \text{Fun}(\text{GL}_{r,s}(n)) \) and the quantum enveloping algebra \( U_{r,s}(\mathfrak{gl}_n) \) corresponding to the two-parameter \( R \)-matrix. We prove that the quantum determinant \( \text{det}_{r,s}T \) is a quasi-central element in \( \text{Fun}(\text{GL}_{r,s}(n)) \) generalizing earlier results of Dipper-Donkin and Du-Parshall-Wang. The explicit formulation provides an interpretation of the deforming parameters, and the quantized algebra \( U_{r,s}(R) \) is identified to \( U_{r,s}(\mathfrak{gl}_n) \) as the dual algebra. We then construct \( n-1 \) quasi-central elements in \( U_{r,s}(R) \) which are analogues of higher Casimir elements in \( U_q(\mathfrak{gl}_n) \).

1. Introduction

Quantum groups were discovered as certain noncommutative and noncommutative Hopf algebras by Drinfeld [10] and Jimbo [15]. The standard definition of a quantum group \( U_q(\mathfrak{g}) \) is given as a \( q \)-deformation of universal enveloping algebra of a simple Lie algebra \( \mathfrak{g} \) generated by the Chevalley generators under the Serre relations based on the data coming from the corresponding Cartan matrix.

Faddeev, Reshetikhin and Takhtajan [12] gave another realization of the quantum groups using the solutions \( R \) of the Yang-Baxter equation:

\[
R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.
\]

They also studied the quantum function algebra \( \text{Fun}(\text{GL}_q(n)) \) for Lie group \( \text{GL}_n \) in [12]. In particular, the quantum determinant \( q\text{det}(T) \) for the quantum function algebra \( \text{Fun}(\text{GL}_q(n)) \) was introduced and proved to be a special central element. Furthermore, the authors studied the algebra \( U(R) \) as a dual algebra of the quantum function algebra and proved that \( U(R) \) is isomorphic to \( U_q(\mathfrak{g}) \). This approach can systematically provide a complete set of generators for the center of the quantum enveloping algebra \( U_q(\mathfrak{sl}_n) \):

\[
c_k = \sum_{\sigma,\sigma' \in S_n} (-q)^{l(\sigma)+l(\sigma')} l^+_{\sigma(1),\sigma'(1)} l^+_{\sigma(2),\sigma'(2)} \cdots l^+_{\sigma(k),\sigma'(k)} l^-_{\sigma(k+1),\sigma'(k+1)} \cdots l^-_{\sigma(n),\sigma'(n)},
\]

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where \( l_{ij}^{\pm} \) are the \( q \)-anlogs of the Weyl root vectors corresponding to the roots \( \pm(\epsilon_i - \epsilon_j) \).

Two-parameter general linear and special linear quantum groups were introduced by Takeuchi \[20\] in 1990. A special case of the quantum coordinate algebra was the subject of study in relationship with the Schur \( q \)-algebra \[8, 11\] where a quantum determinant was shown to be quasi-central. The two-parameter quantum enveloping algebras have later gained full attention after Benkart and Witherspoon’s works \[3, 4\] on the \((r, s)\)-deformed quantum algebras associated with \( \mathfrak{gl}_n \) and \( \mathfrak{sl}_n \), where the quantum R-matrix and the Drinfeld doubles were obtained (see also \[6\]). They further showed that the representation theory of the two-parameter quantum enveloping algebras can be similarly developed as the one-parameter case. In this context they established a two-parameter quantum Schur-Weyl duality (see also \[16\]). In \[17\] an interpretation of the two parameters was demonstrated through a combinatorial realization of the affine analog \[14\] of the quantum linear algebra.

The quantum inverse scattering method has been very successful in studying various quantum groups \[2\] such as the Yangian algebras \[19, 18\] and finite \( W \)-algebras \[1\]. For quantum matrix algebras the method has been particularly useful \[12, 13\]. In this paper we will generalize the Faddeev-Reshetikhin-Takahatajia (FRT) approach \[12\] to the two-parameter case by studying the two-parameter quantum linear groups and the quantum general/special linear algebras as dual Hopf algebras. The anti-symmetric tensors \[16\] associated to the R-matrix \[5\] help one to define 2-parameter quantum determinant \( det_{r,s} T \) and the quantum Casimir elements. If \( t_{ij} \) are the generators of the 2-parameter quantum monoid \( \text{Mat}_{r,s}(n) \), Using the FRT method we obtain that the row determinant and the column determinant are given respectively by

\[
det_{r,s} T = \sum_{\sigma \in S_n} (-s)^{l(\sigma)} t_{\sigma(1),1} \cdots t_{\sigma(n),n}
\]

\[
= \sum_{\sigma \in S_n} (-r)^{-l(\sigma)} t_{1,\sigma(1)} \cdots t_{n,\sigma(n)}.
\]

This means that the two parameters \( r, s \) exactly correspond to column permutations and row permutations respectively.

By a similar method to FRT approach, we are able to define naturally and systematically the Weyl root vectors or Gauss generators for \( U_{r,s}(\mathfrak{sl}_n) \) and their commutation relations can be compactly written as matrix equations in terms of the 2-parameter R-matrices. Moreover we can obtain the explicit formulae for the higher Casimir elements and show that they are quasi-central. In studying representation theory of 2-parameter quantum algebras, it seems to be more naturally to use the quasi-commutative Casimir elements instead of using the central elements given the similarity of the two representation theories.

The paper is organized as follows. In section 2 we study the two-parameter quantum function algebra \( Fun(GL_{r,s}(n)) \) and its two-parameter quantum determinant.
Different from the one-parameter case, we prove that the two-parameter quantum determinant \( \text{det}_{r,s}(T) \) is not a central element but a quasi-central element. In section 3, we study the algebra \( U(R) \) as the dual of \( \text{Fun}(GL_{r,s}(n)) \), and analogue to the one-parameter case, we give \( n - 1 \) quasi-central elements \( c_k \) in \( U(R) \) which generalize the classical higher Casimir elements in the center of the universal enveloping algebra. In section 4, we study the commutation relations between the Cartan-Weyl generators of \( U(R) \) by using the Gauss decomposition of the matrix \( L^\pm \) inside \( U(R) \). In section 5, we show that \( U_{r,s}(gl_n) \) in Drinfeld-Jimbo realization is isomorphic to \( U(R) \) in FRT realization.

2. Two-parameter quantum algebra \( \text{Fun}(GL_{r,s}(n)) \)

In this section we study the two-parameter quantum coordinate algebra \( \text{Fun}(GL_{r,s}(n)) \) associated to the general linear group \( GL_n \) using the FRT method.

Let \( V = C^n \), the \( n \)-dimensional complex space with the basis of column vectors \( e^i = (0, \cdots, 1, \cdots, 0)^T \). Let \( e_{ij} \) be the unit matrices acting on \( V \) so that \( e_{ij}e^k = \delta_{jk}e^i \).

The two-parameter quantum \( R \)-matrix \( \in \text{End}(V \otimes V) \) is given as

\[
R = \sum_{ijkl} R_{ij}^{kl} e_{ik} \otimes e_{jl} \\
= s \sum_{i=1}^n e_{ii} \otimes e_{ii} + rs \sum_{i>j} e_{ii} \otimes e_{jj} + \sum_{i<j} e_{ii} \otimes e_{jj} + (s-r) \sum_{i>j} e_{ij} \otimes e_{ji},
\]

which satisfies the well-known Yang-Baxter equation:

\[
R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},
\]

where \( R_{12} = R \otimes 1 \), and \( R_{ij} \) acts on the \( i \)th and \( j \)th copies of \( V \) inside \( V^\otimes 3 \).

Remark 2.1. When \( s = q = r^{-1} \), the R-matrix (2.1) is the one-parameter R-matrix considered in [12]. We remark that the R-matrix can not be re-scaled to a one-parameter R-matrix.

We set

\[
\hat{R} = PR = s \sum_{i=1}^n e_{ii} \otimes e_{ii} + rs \sum_{i<j} e_{ij} \otimes e_{ji} + \sum_{i>j} e_{ij} \otimes e_{ji} + (s-r) \sum_{i<j} e_{ii} \otimes e_{jj},
\]

where \( P \) is the permutation operator \( P(u \otimes v) = v \otimes u \). Then the Yang-Baxter equation (1.1) is equivalent to the braid relation

\[
\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}.
\]

Furthermore \( \hat{R} \) satisfies the following Hecke relation:

\[
(\hat{R} - s)(\hat{R} + r) = 0.
\]
For $r + s \neq 0$, $\hat{R}$ has the following spectral decomposition

$$\hat{R} = sP^+ - rP^-,$$

where $P^\pm$ are the idempotents such that $P^+P^- = P^-P^+ = 0$, $P^+ + P^- = 1$ and they are given by

$$P^+ = \frac{\hat{R} + r}{r + s},$$
$$P^- = \frac{-\hat{R} + s}{r + s}.$$

Following [12] we define the quantum algebra $A(R)$ of the matrix monoid for the two-parameter $R$-matrix (2.1).

**Definition 2.2.** The algebra $A(R)$ is an associative algebra generated by $t_{ij}$, $1 \leq i, j \leq n$ subject to the quadratic relations defined by

(2.4) \quad $RT_1T_2 = T_2T_1R$,

where $T_1 = \sum_{ij} t_{ij} \otimes e_{ij} \otimes 1$, $T_2 = \sum_{ij} t_{ij} \otimes 1 \otimes e_{ij} \in A(R) \otimes \text{End}(V^\otimes 2)$.

A representation of $A(R)$ is a linear map $T \longrightarrow A \in \text{End}(W)$ such that $RA_1A_2 = A_2A_1R$ on $\text{End}(W \otimes V^\otimes 2)$.

**Proposition 2.3.** [12] The algebra $A(R)$ is a bialgebra with comultiplication

$$\Delta(t_{ij}) = \sum_k t_{ik} \otimes t_{kj}, \quad i, j = 1, ..., n,$$

and co-unit $\epsilon$

$$\epsilon(t_{ij}) = \delta_{ij}, \quad i, j = 1, ..., n.$$

Suppose there are elements $t'_{ij}$, $1 \leq i, j \leq n$ such that

$$\sum_k t'_{ik}t_{kj} = \sum_k t_{ik}t'_{kj} = \delta_{ij} \cdot 1.$$

Then the algebra $\text{Fun}(GL_{r,s}(n))$ is then defined to be the associative algebra generated by $t_{ij}, t'_{ij}$ subject to aforementioned relations involving the generators. Clearly one has $T'T = TT' = 1$ in $\text{Fun}(GL_{r,s}(n))$. Moreover $\text{Fun}(GL_{r,s}(n))$ is a Hopf algebra with the antipode given by $S(t_{ij}) = t'_{ij}$. We remark that the elements $t'_{ij}$ will be shown to exist once we prove that the quantum determinant is a regular element in the ring $\text{Fun}(GL_{r,s}(n))$ as the elements $t'_{ij}$ can be solved by the quantum Cramer rule.
Example 2.4. The two-parameter quantum group $\text{Fun}(GL_{r,s}(2))$ is generated by $t_{ij}, \det_{r,s}^{\pm 1}$ subject to the relations

$$
t_{11}t_{12} = r^{-1}t_{12}t_{11}, \quad t_{11}t_{21} = st_{21}t_{11},
$$

$$
t_{21}t_{22} = r^{-1}t_{21}t_{22}, \quad t_{12}t_{22} = st_{22}t_{12},
$$

$$
t_{12}t_{21} = rst_{21}t_{12},
$$

$$
t_{11}t_{22} - t_{22}t_{11} = (s - r)t_{21}t_{12},
$$

$$
\det_{r,s} \det_{r,s}^{-1} = \det_{r,s}^{-1} \det_{r,s} = 1,
$$

$$
(\det_{r,s})t_{ij} = (rs)^{i-j}t_{ij}(\det_{r,s}),
$$

where

$$
\det_{r,s} = t_{11}t_{22} - st_{12}t_{21} = t_{22}t_{11} - rt_{12}t_{21}
$$

We now turn to the general quantum determinant of $\text{Fun}(GL_{r,s}(n))$ using the quantum inverse scattering method via anti-symmetric tensors (cf. [16]).

Let $V^*$ be the dual space of $V$ spanned by the row vectors $e_i = e_i^T$. The endomorphism space $\text{End}(V)$ is identified with $V \otimes V^*$, which has the natural basis elements $e_{ij} = e_i e_j := e_i \otimes e_j$.

Before introducing the quantum determinant, we introduce the $(r, s)$-deformed antisymmetric tensors $|\varepsilon\rangle \doteq |\varepsilon_n\rangle \in V^{\otimes n}, \langle \varepsilon | \doteq \langle \varepsilon_n | \in (V^*)^{\otimes n}$ as follows:

$$
\langle \varepsilon | \varepsilon \rangle = 1, \tag{2.5}
$$

$$
\langle \varepsilon | P_{k,k+1}^+ = 0, \quad k = 1, 2, \ldots, n - 1, \tag{2.6}
$$

$$
P_{k,k+1}^+ |\varepsilon\rangle = 0, \quad k = 1, 2, \ldots, n - 1. \tag{2.7}
$$

where $P_{k,k+1}^+ = 1 \otimes (k-1) \otimes P^+ \otimes 1 \otimes (n-k-1)$ with $P^+$ at the $(k, k+1)$-position. It is straightforward to prove that the above equations have a unique solution up to normalization, and the $(r, s)$-deformed antisymmetric tensors can be chosen as follows:

$$
\langle \varepsilon | = \frac{1}{[n]_{r,s}} \sum_{\sigma \in S_n} (-s)^{l(\sigma)} e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(n)}, \tag{2.8}
$$

$$
|\varepsilon\rangle = r^{\frac{n(n-1)}{2}} \frac{[s]}{[s-r]} \sum_{\sigma \in S_n} (-r)^{-l(\sigma)} e^{\sigma(1)} \otimes \cdots \otimes e^{\sigma(n)}, \tag{2.9}
$$

where, $[k]_{r,s} = \frac{s^k-r^k}{s-r}$, $[k]_{r,s}! = [1]_{r,s}[2]_{r,s} \cdots [k]_{r,s}$, and $e_i$ the i-th row vector, $e^j$ the j-th column vector.
In view of the RTT relations \([2.4]\) of \(A(R)\) and definition of \(P^+\), we see that 
\[
\langle \varepsilon | T_1 \cdots T_n \rangle = 0, \quad \varepsilon = 1, 2, \ldots, n-1,
\]

\[
P_{k,k+1}^+ T_1 \cdots T_n | \varepsilon \rangle = 0, \quad \varepsilon = 1, 2, \ldots, n-1.
\]

From the uniqueness of the solutions to Eq. \([2.6]–[2.7]\) it follows that
\[
\langle \varepsilon | T_1 \cdots T_n \rangle = c_T \langle \varepsilon | T_1 \cdots T_n \rangle,
\]
\[
T_1 \cdots T_n | \varepsilon \rangle = c^T | \varepsilon \rangle,
\]
where \(c_T, c^T \in \text{Fun}_{r,s}(GL_n)\). Subsequently it follows from \([2.5]\) that
\[
c_T = c^T = \langle \varepsilon | T_1 \cdots T_n | \varepsilon \rangle.
\]

**Definition 2.5.** The quantum determinant of \(\text{Fun}(GL_{r,s}(n))\) is defined to be the normalization factor
\[
\text{det}_{r,s} T = \langle \varepsilon | T_1 \cdots T_n | \varepsilon \rangle.
\]

To find an explicit formula of \(\text{det}_{r,s} T\) we consider the rank one tensor \(A_n = | \varepsilon \rangle \langle \varepsilon |\) acting on \((\mathbb{C}^n)^{\otimes n}\). It is easily seen that \(A_n^2 = A_n\). The following result gives some compact form of various commutation relations in \(\text{Fun}(GL_{r,s}(n))\).

**Proposition 2.6.** \(A_n T_1 T_2 \cdots T_n = T_1 T_2 \cdots T_n A_n = A_n T_1 T_2 \cdots T_n A_n = \text{det}_{r,s} T A_n\).

**Proof.** By definition \(T\) commutes with \(| \varepsilon \rangle\) or \(\langle \varepsilon |\). It follows from \(A_n^2 = A_n\) that
\[
A_n T_1 \cdots T_n = A_n T_1 \cdots T_n A_n
\]
\[
= | \varepsilon \rangle \langle \varepsilon | T_1 \cdots T_n | \varepsilon \rangle \langle \varepsilon |
\]
\[
= (\langle \varepsilon | T_1 \cdots T_n | \varepsilon \rangle | \varepsilon \rangle | \varepsilon |
\]
\[
= \text{det}_{r,s} T A_n.
\]

The other identities are proved similarly. \(\square\)

**Remark 2.7.** The quantum determinant is also expressed as a partial trace.
\[
\text{Tr}_{1,\ldots,n}(A_n T_1 \cdots T_n) = \text{Tr}_{1,\ldots,n}(| \varepsilon \rangle \langle \varepsilon | T_1 \cdots T_n)
\]
\[
= \text{Tr}_{1,\ldots,n}(\langle \varepsilon | T_1 \cdots T_n | \varepsilon \rangle)
\]
\[
= \text{det}_{r,s} T.
\]

In the following proposition, we give the explicit expression of the two-parameter quantum determinant \(\text{det}_{r,s} T\) in \(\text{Fun}(GL_{r,s}(n))\).

**Proposition 2.8.** For any fixed \(\eta \in S_n\) we have
\[
\text{det}_{r,s} T = (-s)^{-l(\eta)} \sum_{\sigma \in S_n} (-s)^{l(\sigma)} t_{\sigma(1),\eta(1)} \cdots t_{\sigma(n),\eta(n)}
\]
\[
= (-r)^{-l(\eta)} \sum_{\sigma \in S_n} (-r)^{l(\sigma)} t_{\eta(1),\sigma(1)} \cdots t_{\eta(n),\sigma(n)}.
\]
In particular,

\[
\det_{r,s} T = \sum_{\sigma \in S_n} (-s)^{l(\sigma)} t_{\sigma(1),1} \cdots t_{\sigma(n),n} \\
= \sum_{\sigma \in S_n} (-r)^{-l(\sigma)} t_{1,\sigma(1)} \cdots t_{n,\sigma(n)}.
\]

(2.14)

**Proof.** The two formulas of (2.13) are proved similarly, so we only consider the first one. Recall that Proposition 2.6 says that

\[
A_n T_1 \cdots T_n = \det_{r,s} T A_n.
\]

For any \( \eta \in S_n \) we apply the left-hand side to the column vector \( e^{\eta(1)} \otimes \cdots \otimes e^{\eta(n)} \):

\[
A_n T_1 \cdots T_n (e^{\eta(1)} \otimes \cdots \otimes e^{\eta(n)}) = A_n \sum_{i_1, \ldots, i_n} t_{i_1,\eta(1)} \cdots t_{i_n,\eta(n)} (e^{i_1} \otimes \cdots \otimes e^{i_n})
\]

(2.15)

\[
= \sum_{i_1, \ldots, i_n} t_{i_1,\eta(1)} \cdots t_{i_n,\eta(n)} |\varepsilon \rangle \langle \varepsilon| (e^{i_1} \otimes \cdots \otimes e^{i_n})
\]

\[
= \frac{1}{[n]_{r,s}!} \sum_{\sigma \in S_n} (-s)^{l(\sigma)} t_{\sigma(1),\eta(1)} \cdots t_{\sigma(n),\eta(n)} |\varepsilon \rangle \langle \varepsilon|.
\]

On the other hand we have

\[
\det_{r,s} T A_n (e^{\eta(1)} \otimes \cdots \otimes e^{\eta(n)}) = \frac{1}{[n]_{r,s}!} (-s)^{l(\eta)} \det_{r,s} T |\varepsilon \rangle.
\]

(2.16)

Paring with \( \langle \varepsilon \rangle \) to the right-hand sides of (2.15-2.16) we obtain that

\[
\det_{r,s} T = (-s)^{-l(\eta)} \sum_{\sigma \in S_n} (-s)^{l(\sigma)} t_{\sigma(1),\eta(1)} \cdots t_{\sigma(n),\eta(n)}.
\]

\[
\square
\]

**Remark 2.9.** One can also show the following

\[
\det_{r,s} T = \sum_{\sigma \in S_n} (-r)^{l(\sigma)} t_{\sigma(n),n} \cdots t_{\sigma(1),1}
\]

(2.17)

\[
= \sum_{\sigma \in S_n} (-s)^{-l(\sigma)} t_{n,\sigma(n)} \cdots t_{1,\sigma(1)}.
\]

When \( s = q = r^{-1} \), the two-parameter quantum determinant is reduced to the one-parameter quantum determinant

\[
\det_q T = \sum_{\sigma \in S_n} (-q)^{l(\sigma)} t_{1,\sigma(1)} \cdots t_{n,\sigma(n)}.
\]

Different from the one-parameter case, the two-parameter quantum determinant is not a central element, but a quasi-central element. It is still a regular element in the ring \( \text{Fun}(GL_{r,s}(n)) \), so the antipode is well-defined.
Theorem 2.10. The two-parameter quantum determinant $\det_{r,s} T$ is quasi-central in $\text{Fun}(GL_{r,s}(n))$. In fact,

\[(\det_{r,s} T)T = M^{-1} TM(\det_{r,s} T),\]

where

\[M = \begin{pmatrix}
(rs)^{n-1} & (rs)^{n-2} & \cdots & 1 \\
\end{pmatrix},
\]

which implies that $(\det_{r,s} T) t_{ij} = (rs)^{i-j} t_{ij} (\det_{r,s} T)$.

Before we prove the theorem, we need the following lemma.

Lemma 2.11. Let $M_{n+1} = 1^\otimes n \otimes M \in \text{End}(V)^{\otimes(n+1)}$, then $s M_{n+1} (A_n \otimes 1) = (A_n \otimes 1) R_{1,n+1} \cdots R_{n,n+1}$ on $V^{\otimes(n+1)}$.

Proof. We order the indices $(ij)$ lexicographically. As $R_{ij} = s - r$ when $i > j$, the matrix $R$ is an upper triangular block matrix with diagonal blocks given by

\[(2.19)\]

\[R^{ik}_{ij} = \begin{cases} 
s, & i = k; \\
rs, & i > k; \\
1, & i < k.
\end{cases}
\]

In the same manner $R_{1,n+1} R_{2,n+1} \cdots R_{n,n+1}$ is also an upper triangular block matrix with diagonal blocks given by

\[(2.20)\]

\[\sum_{i_1, \ldots, i_n, k} R^{i_1 k}_{i_1 i} \cdots R^{i_n k}_{i_n i} e_{i_1, i_1} \otimes \cdots \otimes e_{i_n, i_n} \otimes e_{k,k}.
\]

We then get that for $\langle \varepsilon | = | \varepsilon \rangle \otimes 1$

\[(2.21)\]

\[\langle \varepsilon | R_{1,n+1} \cdots R_{n,n+1} | \varepsilon \rangle = \sum_{k=1}^{n} \frac{r^{n(n-1)}}{[n]_{r,s}!} \sum_{\sigma \in S_n} (sr^{-1})^{l(\sigma)} R^{(1)k}_{\sigma(1)k} \cdots R^{(n)k}_{\sigma(n)k} (e_{k,k})_{n+1}.
\]

Moreover, by Eq. (2.19) we have $R^{(1)k}_{\sigma(1)k} \cdots R^{(n)k}_{\sigma(n)k} = s(rs)^{n-k}$ for any $\sigma \in S_n$. Therefore

\[(2.22)\]

\[\langle \varepsilon | R_{1,n+1} \cdots R_{n,n+1} | \varepsilon \rangle = \sum_{k=1}^{n} s(rs)^{n-k} (e_{k,k})_{n+1} = s M_{n+1}.
\]

Note that $A_n = | \varepsilon \rangle \langle \varepsilon |$, we then obtain that

\[(2.23)\]

\[(A_n \otimes 1) R_{1,n+1} \cdots R_{n,n+1} (A_n \otimes 1) = s M_{n+1} (A_n \otimes 1).
\]

On the other hand the map

\[T_i \mapsto R_{i,n+1}\]
defines a representation of \( Fun(\text{GL}_{r,s}(n)) \), so \( A_n T_1 \cdots T_n = T_1 \cdots T_n A_n \) implies that
\[
(A_n \otimes 1) R_{1,n+1} \cdots R_{n,n+1} (A_n \otimes 1) = (A_n \otimes 1)^2 R_{1,n+1} \cdots R_{n,n+1} \\
= (A_n \otimes 1) R_{1,n+1} \cdots R_{n,n+1} = sM_{n+1} (A_n \otimes 1).
\]

Now we prove Theorem 2.10.

**Proof.** For brevity we write \( A_n \) for \( (A_n \otimes 1) \) which is clear from the context. Using the RTT relations of \( Fun(\text{GL}_{r,s}(n)) \) to move \( T_{n+1} \), we have that
\[
A_n T_1 \cdots T_n T_{n+1} = A_n (R_{1,n+1} \cdots R_{n,n+1})^{-1} T_{n+1} T_1 \cdots T_n (R_{1,n+1} \cdots R_{n,n+1}).
\]

It follows from Lemma 2.11 that
\[
A_n T_1 \cdots T_n T_{n+1} = s^{-1} M_{n+1}^{-1} T_{n+1} A_n T_1 \cdots T_n (R_{1,n+1} \cdots R_{n,n+1}).
\]

Note that \( A_n T_1 \cdots T_n = A_n T_1 \cdots T_n A_n \) (Prop. 2.6). Applying Lemma 2.11 again we obtain that
\[
A_n T_1 \cdots T_n T_{n+1} = s^{-1} M_{n+1}^{-1} T_{n+1} A_n T_1 \cdots T_n A_n (R_{1,n+1} \cdots R_{n,n+1})
= M_{n+1}^{-1} T_{n+1} M_{n+1} A_n T_1 \cdots T_n.
\]

Taking partial trace \( Tr_{1,\ldots,n} \) (see Remark 2.7), we finally get that \( (det_{r,s}T) T = M^{-1} TM(det_{r,s}T) \). \( \square \)

### 3. FRT realization of two-parameter quantum groups

In this section we study the algebra \( U(R) \) as the dual Hopf algebra of \( Fun(\text{GL}_{r,s}(n)) \).

Consider the R-matrix \( R^{(+)} \in \text{End}(V \otimes V) \)
\[
R^{(+)} = PRP = R_{21} = s \sum_{i=1}^{n} e_{ii} \otimes e_{ii} + rs \sum_{i<j} e_{ii} \otimes e_{jj} + \sum_{i>j} e_{ii} \otimes e_{jj} + (s-r) \sum_{i<j} e_{ij} \otimes e_{ji},
\]

which is another solution of the Yang-Baxter equation (1.1).

**Definition 3.1.** \( U(R) \) is an associative algebra with generators \( l^{+}_{ij}, l^{-}_{ji}, 1 \leq i \leq j \leq n \) subject to the quadratic relations given by
\[
R^{(+)} L_{1}^{+} L_{2}^{+} = L_{2}^{+} L_{1}^{+} R^{(+)};
\]
\[
R^{(+)} L_{1}^{+} L_{2}^{-} = L_{2}^{-} L_{1}^{+} R^{(+)};
\]

where \( L_{1}^{\pm} = \sum_{i,j} l_{ij}^{\pm} e_{ij} \otimes 1 \), \( L_{2}^{\pm} = \sum_{i,j} l_{ij}^{\pm} 1 \otimes e_{ij} \) and \( L^{\pm} = (l_{ij}^{\pm}) \) \((1 \leq i, j \leq n)\) are invertible triangular matrices with \( l_{ij}^{+} = l_{ji}^{-} = 0 \) for \( 1 \leq j < i \leq n \).
Proposition 3.2. The algebra $U(R)$ is a Hopf algebra with comultiplication, antipode and counit given by

\begin{align}
\Delta(l^\pm_{ij}) &= \sum_k l^\pm_{ik} \otimes l^\pm_{kj}, \\
S(L^\pm) &= (L^\pm)^{-1}, \\
\epsilon(l^\pm_{ij}) &= \delta_{ij}.
\end{align}

As in the dual algebra case, let $\hat{R}^{(+)} = PR^{(+)} = \hat{R}_{21}$, then

$$\hat{R}^{(+)} = s \sum_{i=1}^n e_{ii} \otimes e_{ii} + rs \sum_{i>j} e_{ij} \otimes e_{ji} + \sum_{i<j} e_{ij} \otimes e_{ji} + (s-r) \sum_{i>j} e_{ii} \otimes e_{jj}.$$ 

Moreover it satisfies the Hecke relation

\begin{equation}
(\hat{R}^{(+)} - s)(\hat{R}^{(+)} + r) = 0.
\end{equation}

For this R-matrix $\hat{R}^{(+)}$, we introduce antisymmetric tensors $\langle \varepsilon^{(+)} |, | \varepsilon^{(+)} \rangle$ by the equations:

\begin{align}
\langle \varepsilon^{(+)} | \varepsilon^{(+)} \rangle &= 1, \\
\langle \varepsilon^{(+)} | (\hat{R}^{(+)}_{i,i+1} + r) | \varepsilon^{(+)} \rangle &= 0, \; i = 1, 2, \ldots, n-1, \\
(\hat{R}^{(+)}_{i,i+1} + r) | \varepsilon^{(+)} \rangle &= 0, \; i = 1, 2, \ldots, n-1.
\end{align}

These equations determine the antisymmetric tensors up to a constant. It is easy to see that $\langle \varepsilon^{(+)} |, | \varepsilon^{(+)} \rangle$ can be chosen as follows.

\begin{align}
\langle \varepsilon^{(+)} | &= \frac{1}{[n]_{r,s}!} \sum_{\sigma \in S_n} (-s)^{-l(\sigma)} e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(n)} \\
| \varepsilon^{(+)} \rangle &= s \frac{n(n-1)}{2} \sum_{\sigma \in S_n} (-r)^{l(\sigma)} e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(n)}.
\end{align}

Define the rank one matrix $A^{(+)}_n = | \varepsilon^{(+)} \rangle \langle \varepsilon^{(+)} |$. Explicitly we have that

\begin{equation}
A^{(+)}_n = s \frac{n(n-1)}{[n]_{r,s}!} \sum_{\sigma, \tau \in S_n} (-r)^{l(\tau)} (-s)^{-l(\sigma)} e_{\tau(1)} \otimes \cdots \otimes e_{\tau(n)} e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(n)}.
\end{equation}

The following result gives commutation relations among the Weyl generators of $U(R)$.

Proposition 3.3. In $U(R)$ the following identities are satisfied for $k = 1, 2, \ldots, n$:

$$A^{(+)}_n L_1^+ \cdots L_k^+ L_{k+1}^- \cdots L_n^- A^{(+)}_n = A^{(+)}_n L_1^+ \cdots L_k^+ L_{k+1}^- \cdots L_n^- = L_1^+ \cdots L_k^+ L_{k+1}^- \cdots L_n^- A^{(+)}_n.$$
Proof. Using the RTT defining relations (3.1-3.2), we have that
\[ \langle \varepsilon^{(+)} | L_1^+ \cdots L_k^+ L_{k+1}^- \cdots L_n^- (\hat{R}_{i,i+1}^{(+)}) + r \rangle = \langle \varepsilon^{(+)} | (\hat{R}_{i,i+1}^{(+)}) L_1^+ \cdots L_k^+ L_{k+1}^- \cdots L_n^-, \]
\[ (\hat{R}_{i,i+1}^{(+)}) L_1^+ \cdots L_k^+ L_{k+1}^- \cdots L_n^- | \varepsilon^{(+)} \rangle = L_1^+ \cdots L_k^+ L_{k+1}^- \cdots L_n^-(\hat{R}_{i,i+1}^{(+)}) | \varepsilon^{(+)} \rangle \]
for \( k = 1, 2, \ldots, n - 1 \).

By the uniqueness of the solution to (3.8-3.9) we can assume that
\[ \langle \varepsilon^{(+)} | L_1^+ \cdots L_k^+ L_{k+1}^- \cdots L_n^- = a_k \langle \varepsilon^{(+)} |, \]
\[ L_1^+ \cdots L_k^+ L_{k+1}^- \cdots L_n^- | \varepsilon^{(+)} \rangle = b_k | \varepsilon^{(+)} \rangle, \]
where \( a_k, b_k \in U(R) \). By the normalization (3.7) it follows that
\[ a_k = b_k = \langle \varepsilon^{(+)} | L_1^+ \cdots L_k^+ L_{k+1}^- \cdots L_n^- | \varepsilon^{(+)} \rangle. \]

Subsequently we get
\[ A_n^{(+)} L_1^+ \cdots L_k^+ L_{k+1}^- \cdots L_n^- = A_n^{(+)} = a_k A_n^{(+)}. \]

The following lemma will be needed to compute a partial trace in Theorem 3.5.

**Lemma 3.4.** In \( \text{End}(V)^{\otimes (n+1)} \) one has that
\[ (A_n^{(+)} \otimes 1) R_{1,n+1}^{(+)} \cdots R_{n,n+1}^{(+)} = s M_{n+1}'(A_n^{(+)} \otimes 1), \]
\[ R_{0,n}^{(+)} \cdots R_{0,1}^{(+)} (1 \otimes A_n^{(+)}) = s M_0 (1 \otimes A_n^{(+)}), \]
where the indices of \( V \) in \( V^{\otimes (n+1)} \) in the first (resp. second) identity are \( 1, \ldots, n+1 \) (resp. \( 0, 1, \ldots, n \)) and \( R_{ij} \) and \( M_{n+1} \) (resp. \( M_0 \)) are defined accordingly. Here
\[ M' = \begin{pmatrix} 1 & rs & & & rs^{n-1} \\ & \ddots & \ddots & \ddots \\ & & 1 & \ddots & rs^{n-2} \\ & & & \ddots & \ddots \\ & & & & 1 \end{pmatrix}, \]
and
\[ M = \begin{pmatrix} (rs)^{n-1} & \cdots & rs^{n-2} \\ & \ddots & \ddots \\ & & \ddots & \ddots \\ & & & \ddots & \ddots \\ & & & & 1 \end{pmatrix}. \]

Proof. These two identities are proved similarly as Lemma 2.11. Note that \( R^{(+)} \) is also a triangular block matrix with diagonal blocks given by
\[ (R^{(+)})_{ik}^{ik} = \begin{cases} rs, & i < k; \\ s, & i = k; \\ 1, & i > k. \end{cases} \]
Therefore for any $\sigma \in S_n$ we have $(R^+)_{i\sigma(1)}^{i\sigma(1)} \cdots (R^+)_{i\sigma(n)}^{i\sigma(n)} = s(rs)^{n-i}$. Subsequently for $\langle \varepsilon^{(+)} \rangle = 1 \otimes |\varepsilon^{(+)}\rangle \in (V^*) \otimes (n+1)$ and $|\varepsilon^{(+)}\rangle = 1 \otimes |\varepsilon^{(+)}\rangle \in V \otimes (n+1)$ one has as in (2.20)
\[
\langle \varepsilon^{(+)} | R^{(+)}_{0, n} \cdots R^{(+)}_{0, 1} | \varepsilon^{(+)} \rangle
\]
\[
= \sum_{i=1}^{n} \frac{s^{n(n-1)}}{[n]_{r,s}^i} \sum_{\sigma \in S_n} (s^{-1}r)^{l(\sigma)} (R^+)_{i\sigma(1)}^{i\sigma(1)} \cdots (R^+)_{i\sigma(n)}^{i\sigma(n)} (e_{ii})_0
\]
\[
= \sum_{i=1}^{n} s(rs)^{n-i} (e_{ii})_0 = sM_0.
\]
Applying $A^{(+)}_n = |\varepsilon^{(+)}\rangle \langle \varepsilon^{(+)}|$, we immediately get
\[
(1 \otimes A^{(+)}_n) R^{(+)}_{0, n} \cdots R^{(+)}_{0, 1} (1 \otimes A^{(+)}_n) = sM_0 (1 \otimes A^{(+)}_n).
\]
Now the map $U(R) \otimes (EndV)^{\otimes n} \to EndV \otimes (EndV)^{\otimes n}$ given by
\[
L^\pm_i \mapsto R^\pm_{0, i}
\]
is an algebra homomorphism, then the identity $A^{(+)}_n L^+_1 \cdots L^+_n = L^+_1 \cdots L^+_n A^{(+)}_n$ implies that
\[
(1 \otimes A^{(+)}_n) R^{(+)}_{0, n} \cdots R^{(+)}_{0, 1} (1 \otimes A^{(+)}_n) = (1 \otimes A^{(+)}_n) R^{(+)}_{0, n} \cdots R^{(+)}_{0, 1} = sM_0 (1 \otimes A^{(+)}_n).
\]

**Theorem 3.5.** The elements
\[
c_k = \sum_{\sigma, \sigma' \in S_n} (-s)^{l(\sigma)} (-r)^{-l(\sigma')} L^+_{\sigma(1), \sigma'(1)} \cdots L^+_{\sigma(k), \sigma'(k)} L^-_{\sigma(k+1), \sigma'(k+1)} \cdots L^-_{\sigma(n), \sigma'(n)}
\]
are quasi-central elements of $U(R)$. Explicitly, we have
\[
L^+ c_k = c_k M^{-1} L^+ M,
\]
and
\[
c_k L^- = M'^{-1} L^- M' c_k.
\]

**Proof.** The quantum Casimir elements can be expressed as traces. In fact
\[
c_k = \langle n | r_s | s^{n(1-n)} \varepsilon^{(+)} | L^+_{1} \cdots L^+_{k} L^-_{k+1} \cdots L^-_{n} \varepsilon^{(+)} \rangle
\]
\[
= \langle n | r_s | s^{n(1-n)} tr(A^{(+)}_n L^+_{1} \cdots L^+_{k} L^-_{k+1} \cdots L^-_{n})
\]
\[
= \langle n | r_s | s^{n(1-n)} tr(L^+_{1} \cdots L^+_{k} L^-_{k+1} \cdots L^-_{n} A^{(+)}_n)\rangle.
\]
We take an auxiliary copy $L^+$ in the zero position and consider
\[
L^+_0 L^+_1 \cdots L^+_k L^-_{k+1} \cdots L^-_{n} A^{(+)}_n.
\]
Moving $L^+_0$ to the extreme right by the RTT defining relations (3.1, 3.2), we have

$$L^+_0 L^+_1 \cdots L^+_k L^+_{k+1} \cdots L^+_n A^{(+)n}$$

$$(R_{0n}^{(+)} \cdots R_{01}^{(+)} - 1) L^+_1 \cdots L^+_k L^+_{k+1} \cdots L^+_n L^+_0 R_{0n}^{(+)} \cdots R_{01}^{(+)} A^{(+)n}$$

$$= s(R_{0n}^{(+)} \cdots R_{01}^{(+)} - 1) L^+_1 \cdots L^+_k L^+_{k+1} \cdots L^+_n A^{(+)n} L^+_0 M_0,$$

where the last identity uses Lemma 3.4.

Now we move $A^{(+)n}$ to the left by Proposition 3.3 and use Lemma 3.4 again:

$$s(R_{0n}^{(+)} \cdots R_{01}^{(+)} - 1) L^+_1 \cdots L^+_k L^+_{k+1} \cdots L^+_n A^{(+)n} L^+_0 M_0$$

$$= s(R_{0n}^{(+)} \cdots R_{01}^{(+)} - 1) A^{(+)n} L^+_1 \cdots L^+_k L^+_{k+1} \cdots L^+_n L^+_0 M_0$$

$$= A^{(+)n} M_0^{-1} L^+_1 \cdots L^+_k L^+_{k+1} \cdots L^+_n L^+_0 M_0$$

$$= A^{(+)n} L^+_1 \cdots L^+_k L^+_{k+1} \cdots L^+_n L^+_0 M_0 M_0^{-1} L^+_0 M_0.$$

Finally taking partial trace $Tr_{1 \cdots n}$ in $(\mathbb{C}^n)^{(n+1)}$ for the right-hand side, we obtain the result that $L^+_c k = c_k M^{-1} L^+_c M$. The identity (3.21) is proved similarly. \qed

Remark 3.6. When $rs = 1$, $c_k$ become the quantum Casimir elements in the center.

4. GAUSS DECOMPOSITION OF $L^\pm$

In this section we use the Gauss decomposition of $L^\pm$ to study the commuting relations among the quantum Cartan-Weyl generators. Gauss decomposition was used by Ding-Frenkel [7] to show that the RTT version of the quantum algebras $U_q(\mathfrak{gl}_n)$ is isomorphic to the Drinfeld-Jimbo realization.

**Proposition 4.1.** The matrices $L^\pm$ can be decomposed as follows.

$$(4.1) \quad L^+ = \begin{pmatrix} K^+_1 & K^+_2 & \cdots & K^+_n \\ & \ddots & \vdots & \vdots \\ & & K^+_n \end{pmatrix} \begin{pmatrix} 1 & E_{12} & \cdots & E_{1n} \\ & 1 & E_{23} & \vdots \\ & \vdots & \ddots & E_{n-1,n} \\ & & & 1 \end{pmatrix}$$

$$(4.2) \quad L^- = \begin{pmatrix} 1 & F_{21} & \cdots & \cdots \\ \cdots & \ddots & \vdots & \vdots \\ F_{n1} & \cdots & F_{n,n-1} & 1 \end{pmatrix} \begin{pmatrix} K^-_1 & K^-_2 & \cdots \\ & \ddots & \vdots & \vdots \\ & & K^-_n \end{pmatrix}$$

where the elements $K^\pm_i$ ($1 \leq i \leq n$), $E_{ij}, F_{ji}$ ($i < j$) are exclusively defined by $L^\pm$ through the equations.

The following relations are obtained by the RTT defining relations (3.1, 3.2) and the Gauss decomposition (4.1, 4.2).
Proposition 4.2. In $U(R)$ we have

\begin{align*}
K_i^\pm K_j^\pm &= K_j^\pm K_i^\pm, \\
K_i^\pm K_j^\mp &= K_j^\mp K_i^\pm.
\end{align*}

Next we compute the commutation relations between $K_i^\pm$ and $E_{j,j+1}$.

Proposition 4.3. In $U(R)$ we have that

\begin{align*}
K_i^+ E_{i,i+1} &= r E_{i,i+1} K_i^+, \\
K_i^+ E_{i-1,i} &= s E_{i-1,i} K_i^+, \\
K_i^+ E_{j,j+1} &= E_{j,j+1} K_i^+, \quad \text{for } i \neq j, j + 1; \\
K_i^- E_{i,i+1} &= s E_{i,i+1} K_i^-, \\
K_i^- E_{i-1,i} &= r E_{i-1,i} K_i^-, \\
K_i^- E_{j,j+1} &= E_{j,j+1} K_i^-, \quad \text{for } i \neq j, j + 1.
\end{align*}

Proof. Here we just prove Eqs. (4.5) and (4.7), as the other relations can be obtained similarly.

From the defining relation (3.1) of $U(R)$ it follows that

\[ l_{ii}^+ l_{i,i+1}^+ = r l_{i,i+1}^+ l_{ii}^+. \]

Plugging in the Gauss-decomposition (4.1), we get Eq. (4.5).

For $i \neq j, j + 1$ the defining relation (3.1) implies that

\[ l_{ii}^+ l_{j,j+1}^+ = l_{j,j+1}^+ l_{ii}^+, \]

which is equivalent to

\[ K_i^+ K_j^+ E_{j,j+1} = K_j^+ E_{j,j+1} K_i^+. \]

by using the Gauss decomposition of $L^+$. Finally the invertibility of $K_i^\pm$ gives Eq (4.7). \qed

Similarly we can obtain the commuting relations between $K_i^\pm$ and $F_{j+1,j}$.

Proposition 4.4. In $U(R)$ we have that

\begin{align*}
K_i^+ F_{i+1,i} &= r^{-1} F_{i+1,i} K_i^+, \\
K_i^+ F_{i,i-1} &= s^{-1} F_{i,i-1} K_i^+, \\
K_i^+ F_{j+1,j} &= E_{j+1,j} K_i^+, \quad \text{for } i < j; \\
K_i^- F_{i+1,i} &= s^{-1} F_{i+1,i} K_i^-, \\
K_i^- F_{i,i-1} &= r^{-1} F_{i,i-1} K_i^-, \\
K_i^- F_{j+1,j} &= E_{j+1,j} K_i^-, \quad \text{for } i < j.
\end{align*}

Now we compute the commuting relations between $E_{i,i+1}$ and $F_{j+1,j}$. 
Proposition 4.5. In $U(R)$ we have that
\begin{equation}
[E_{i,j+1}, F_{j+1,i}] = \delta_{ij}(r^{-1} - s^{-1})(K_{i+1}^{-}(K_{i+1}^{+})^{-1} - K_{i+1}^{+}(K_{i+1}^{-})^{-1}).
\end{equation}
Proof. It follows from (3.2) that
\begin{equation}
rs l_{i+1,j+1}^{\pm} l_{i,j}^{\mp} + (s - r)l_{i+1,j+1}^{\pm} l_{i,j}^{\mp} = l_{i+1,j+1}^{\pm} l_{i,j}^{\mp} + (s - r)l_{i+1,j+1}^{\mp} l_{i,j}^{\pm}.
\end{equation}
By Gauss decomposition of $L^\pm$ both sides of (4.18) can be written as
\[rs K_{i+1}^{+} E_{i,j+1} F_{j+1,i} K_{i}^{-} + (s - r) K_{i+1}^{+} K_{i}^{-} = F_{i+1,j} K_{i}^{-} K_{i}^{+} E_{i,j+1} + (s - r) K_{i+1}^{+} K_{i}^{-}.\]
Taking account of Eqs. (4.8) and (4.11), we see that the above is reduced to (4.17). \hfill \square

The analog of Serre relations is given below for $E_{ij}$.

Proposition 4.6. In $U(R)$ we have that
\begin{equation}
E_{i+1,j+1} E_{i,j+1} - (r + s) E_{i+1,j+1} E_{i,j+1} = 0,
\end{equation}
\begin{equation}
E_{i,j+1} E_{i+1,j+1} - (r + s) E_{i,j+1} E_{i+1,j+1} + rs E_{i+1,j+1} E_{i,j+1} = 0,
\end{equation}
\begin{equation}
E_{i,j+1} = E_{j+1,i}, \quad \text{if } |i - j| > 1.
\end{equation}
Proof. We first prove the following commutation relation.
\begin{equation}
E_{i+1,j+1} E_{i,j+1} = s^{-1} E_{i+1,j+1} E_{i,j+1} + (r^{-1} - s^{-1}) E_{i,j+1}.
\end{equation}
In fact, from the defining relation (3.1) it follows that
\[rs l_{i+1,j+1}^{\pm} l_{i,j+1}^{\mp} + (s - r) l_{i+1,j+1}^{\pm} l_{i,j+1}^{\mp} = rs l_{i+1,j+1}^{\mp} l_{i,j+1}^{\pm}.
\]
Plugging in the Gauss decomposition, this becomes
\begin{equation}
rs K_{i}^{+} E_{i,j+1} K_{i+1}^{+} E_{i+1,j} + (s - r) K_{i}^{+} K_{i+1}^{-} E_{i,j+1} + rs K_{i+1}^{+} K_{i}^{-} E_{i+1,j} = rs K_{i}^{+} E_{i+1,j} + (s - r) K_{i+1}^{+} K_{i}^{-} E_{i,j+1}.
\end{equation}
Then Eq. (4.22) is obtained by using Eqs. (4.6) and (4.7).

Multiplying $E_{i,j+1}$ from the left and the right of (4.22), we have that
\[E_{i+1,j+1}^{2} = s E_{i+1,j+1}^{2} - (r^{-1} - s^{-1}) E_{i+1,j+1} E_{i,j+1} - (r^{-1} s - 1) E_{i+1,j+1}^{2} = E_{i+1,j+1}^{2}.
\]
On the other hand, the defining relation (3.1) also gives that
\[E_{i+1,j+1}^{2} = E_{i+1,j+1}^{2}.
\]
Then (4.19) immediately follows. Eq. (4.20) is proved similarly. \hfill \square

In the same way we obtain the Serre relations for $F_{ij}$.

Proposition 4.7. In $U(R)$ we have
\begin{equation}
F_{i+1,j+1}^{2} = F_{i+1,j+1}^{2} - (r^{-1} + s^{-1}) F_{i+1,j+1} F_{i+1,j+1} F_{i,j+1} + r^{-1} s^{-1} F_{i+1,j+1}^{2} = 0,
\end{equation}
\begin{equation}
F_{i+1,j+1}^{2} = F_{i+1,j+1}^{2} - (r^{-1} + s^{-1}) F_{i+1,j+1} F_{i+1,j+1} + r^{-1} s^{-1} F_{i+1,j+1}^{2} = 0,
\end{equation}
\begin{equation}
F_{i+1,j+1} F_{i+1,j+1} = F_{i+1,j+1} F_{i+1,j+1}, \quad \text{if } |i - j| > 1.
\end{equation}
5. Isomorphism between the Quantum Group $U_{r,s}(\mathfrak{gl}_n)$ and $U(R)$

The current version of two-parameter quantum group was given by Benkart and Witherspoon [3] in terms of Chevalley generators and Serre relations in connection with the down-up algebras. In [4,5] they further developed the representation theory of the two-parameter quantum general and special linear algebras and constructed the corresponding $R$-matrix. We now identify their version with our FRT version given in earlier sections.

Let $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ be the orthonormal basis of the Euclidean space $\mathbb{C}^n$ with inner product $\langle \cdot, \cdot \rangle$. Let $\Pi = \{\alpha_j = \epsilon_j - \epsilon_{j+1} | j = 1, 2, \ldots, n - 1\}$ and $\Phi = \{\epsilon_i - \epsilon_j | 1 \leq i \neq j \leq n\}$. Then $\Phi$ realizes the root system of type $A_{n-1}$ with $\Pi$ a base of simple roots.

**Definition 5.1.** The algebra $U_{r,s}(\mathfrak{gl}_n)$ is a unital associated algebra over $\mathbb{C}$ generated by $e_i, f_j, (1 \leq j < n)$, and $a_i^{\pm 1}, b_i^{\pm 1} (1 \leq i \leq n)$ subject to the following relations.

- **R1:** $a_i^{\pm 1}, b_j^{\pm 1} (1 \leq i \leq n)$ commute with each other and $a_i a_i^{-1} = b_i b_i^{-1} = 1$;
- **R2:** $a_i e_j = r^{\langle \epsilon_i, \alpha_j \rangle} e_j a_i$, and $a_i f_j = r^{\langle \epsilon_i, \alpha_j \rangle} f_j a_i$;
- **R3:** $b_i e_j = s^{\langle \epsilon_i, \alpha_j \rangle} e_j b_i$, and $b_i f_j = s^{\langle \epsilon_i, \alpha_j \rangle} f_j b_i$;
- **R4:** $[e_i, f_j] = \frac{g_{i,j}}{r-s}(a_i b_{i+1} - a_{i+1} b_i)$;
- **R5:** $[e_i, e_j] = [f_i, f_j] = 0$ if $|i - j| > 1$;
- **R6:** $e_i^{e+1} e_j - (r + s)e_i e_{i+1} e_i + r s e_{i+1} e_i^2 = 0$;
- **R7:** $f_i^2 f_{i+1} - (r^{-1} + s^{-1}) f_i f_{i+1} f_i + r^{-1} s^{-1} f_{i+1} f_i^2 = 0$;

The algebra $U_{r,s}(\mathfrak{gl}_n)$ is a Hopf algebra such that $a_i^{\pm 1}, b_i^{\pm 1}$ are the group-like elements and the remaining Hopf algebra structure is given by

$$\delta(e_i) = e_i \otimes 1 + \omega_i \otimes e_i,$$

$$\delta(f_i) = f_i \otimes \omega_i + 1 \otimes f_i,$$

$$\varepsilon(e_i) = \varepsilon(f_i) = 0,$$

$$S(e_i) = -\omega_i^{-1} e_i,$$

$$S(f_i) = -f_i \omega_i^{-1}.$$

**Remark 5.2.** When $r = q = s^{-1}$, the Hopf algebra $U_{r,s}(\mathfrak{gl}_n)$ modulo the ideal generated by $b_i - a_i^{-1}, 1 \leq i \leq n$ is isomorphic to $U_q(\mathfrak{gl}_n)$.

Analogous with the one-parameter case, we have the following theorem.

**Theorem 5.3.** The mapping $\psi : U_{r,s}(\mathfrak{gl}_n) \rightarrow U(R)$ given by

$$e_i \mapsto \frac{r}{r-s} E_{i,i+1}, f_i \mapsto \frac{s}{s-r} F_{i+1,i}, a_1 \mapsto K_1^+, b_1 \mapsto K_1^-,$$
and

\[ a_i \mapsto K_i^+ \prod_{l=1}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l}, \]

\[ b_i \mapsto K_i^- \prod_{l=1}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l}, \]

is an isomorphism.

**Proof.** By Propositions 4.2, 4.6, and 4.7, relations R1, R5, R6, and R7 hold. We only need to check relations R2, R3, and R4.

First let consider R2. It follows from Proposition 4.3 that

\[ K_i^+ \prod_{l=1}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l} E_{i,i+1} = K_i^+ E_{i,i+1} \prod_{l=1}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l} \]

\[ = r E_{i,i+1} K_i^+ \prod_{l=1}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l}. \]

\[ K_i^+ \prod_{l=1}^{i-1} (K_{i-l}^-)^{(-1)^l} E_{i-1,i} \]

\[ = K_i^+(K_{i-1}^-)^{-1}(K_{i-1}^-)^{-1} E_{i-1,i} \prod_{l=2}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l} \]

\[ = r^{-1}s^{-1} K_i^+ E_{i-1,i} (K_{i-1}^-)^{-1}(K_{i-1}^-)^{-1} \prod_{l=2}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l} \]

\[ = r^{-1} E_{i-1,i} K_i^+ \prod_{l=1}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l}. \]

When \( j \neq i, i - 1 \), we have

\[ K_i^+ \prod_{l=1}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l} E_{j,j+1} = E_{j,j+1} K_i^+ \prod_{l=1}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l}. \]

Similarly we also have that

\[ K_i^+ \prod_{l=1}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l} F_{i+1,i} = r^{-1} F_{i+1,i} K_i^+ \prod_{l=1}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l}, \]

\[ K_i^+ \prod_{l=1}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l} F_{i,i-1} = r F_{i,i-1} K_i^+ \prod_{l=1}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l}, \]
and for $j \neq i, i - 1$

\[
K_i^+ \prod_{l=1}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l} F_{j+1,i} = F_{j+1,j} K_i^+ \prod_{l=1}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l}.
\]

Then relation R2 is satisfied. Relation R3 is proved similarly.

Next we consider relation R4. It from Proposition 4.5 that

\[
\frac{r}{r-s} \frac{s}{s-r} [E_{i,i+1}, F_{i+1,i}] = \frac{1}{r-s} (K_{i+1}^- (K_i^-)^{-1} - K_{i+1}^+ (K_i^+)^{-1}).
\]

It is easy to check that

\[
K_i^+ \prod_{l=1}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l} K_{i+1}^- \prod_{l=1}^{i} (K_{i+1-l}^+ K_{i+1-l}^-)^{(-1)^l} = K_{i+1}^- (K_i^-)^{-1},
\]

\[
K_i^+ \prod_{l=1}^{i} (K_{i+1-l}^+ K_{i+1-l}^-)^{(-1)^l} K_{i+1}^- \prod_{l=1}^{i-1} (K_{i-l}^+ K_{i-l}^-)^{(-1)^l} = K_{i+1}^+ (K_i^+)^{-1}.
\]

So R4 holds. Note that all the $E_{ij}, F_{ji}, i < j$, can be generated by $E_{i,i+1}, F_{i+1,i}$. Therefore $\psi$ is a surjective homomorphism.

The injectivity can be proved verbatim as in [7] for the one-parameter case. □

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References

[1] C. Briot and E. Ragoucy, RTT presentation of finite $W$-algebras, J. Phys. A 34 (2001), 7287-7310.
[2] V. Chari and A. Pressley, A guide to Quantum Groups, Cambridge Univ. Press, Cambridge, 1994.
[3] G. Benkart and S. Witherspoon, A Hopf structure for down-up algebras, Math. Z. 238 (2001), 523-553.
[4] G. Benkart and S. Witherspoon, Two-parameter quantum groups and Drinfeld doubles, Algebr. Represent. Theory 7 (2004), 261–286.
[5] G. Benkart and S. Witherspoon, Representations of two-parameter quantum groups and Schur-Weyl duality, Hopf algebras, Lecture Notes in pure and Appl. Math., 237 (2004), 65-92.
[6] N. Bergeron, Y. Gao and N. Hu, Drinfel’d doubles and Lusztig’s symmetries of two-parameter quantum groups, J. Algebra 301 (2006), 378–405.
[7] J. Ding and I. B. Frenkel, Isomorphism of two realizations of quantum affine algebra $U_q(\hat{\mathfrak{gl}}(n))$, Commun. Math. Phys. 156 (1993), 277–300.
[8] R. Dipper and S. Donkin, Quantum $GL_n$, Proc. London Math. Soc. (3) 63 (1991), no. 1, 165–211.
[9] V. Drinfeld, Hopf algebras and the quantum Yang-Baxter equation, Soviet Math. Dokl. 32 (1985), 254–258.
[10] V. Drinfeld, *Quantum Group*, Proc. ICM, Vol. 1, 2 (Berkeley, Calif., 1986), 798–820, Amer. Math. Soc., Providence, RI, 1987.
[11] J. Du, B. Parshall and J. Wang, *Two-parameter quantum linear groups and the hyperbolic invariance of q-Schur algebras*, J. London Math. Soc. (2) 44 (1991), 420–436.
[12] L. Faddeev, N. Reshetikhin and L. Takhtadzhyan, *Quantization of Lie groups and Lie algebras*, Leningrad Math. J. 1 (1990), 193–225.
[13] L. K. Hadjiivanov, A. P. Isaev, O. V. Ogievetsky, P. N. Pyatov and I. T. Todorov, *Hecke algebraic properties of dynamical R-matrices. Application to related quantum matrix algebras*, J. Math. Phys. 40 (1999), 427–448.
[14] N. Hu, M. Rosso and H. Zhang, *Two-parameter quantum affine algebra $U_{r,s}(\hat{sl}_n)$, Drinfeld realization and quantum affine Lyndon basis*, Commun. Math. Phys. 278 (2008), 453–486.
[15] M. Jimbo, *A q-difference analogue of $U(g)$ and the Yang-Baxter equation*, Lett. Math. Phys. 10 (1985), 63–69.
[16] N. Jing and M. Liu, *Fusion procedure for the two-parameter quantum algebra $U_{r,s}(\hat{sl}_n)$*, arXiv:1402.3665.
[17] N. Jing and H. Zhang, *Fermionic realization of two-parameter quantum affine algebra $U_{r,s}(\hat{sl}_n)$*, Lett. Math. Phys. 89 (2009), no. 2, 159–170.
[18] A. Molev, *Yangians and classical Lie algebras*, Math. Surv. and Monograph, 143. AMS, Providence, RI, 2007.
[19] A. Molev, M. Nazarov and G. Olshanskii, *Yangians and classical Lie algebras*, Russian Math. Surveys 51 (1996), 205–282.
[20] M. Takeuchi, *A two-parameter quantization of $GL(n)$*, Proc. Japan. Acad. 66 Ser. A (1990), 112–114.

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