Non-Equilibrium Duality

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The dual ‘worldline’, or ‘string’, description of adiabatic Ginzburg-Landau field theory near a phase transition in terms of quasi-Brownian strings and loops is well understood. In reality, the implementation of a transition is intrinsically non-equilibrium. We sketch how time-dependent Ginzburg-Landau theory leads to a modified dual string picture in which a causal bound on the growth of unstable string prevents the uncontrolled proliferation of string (the Shockley-Hagedorn transition) suggested by the adiabatic approximation.

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From the early days of lattice models \cite{1} it has been appreciated that time-independent Ginzburg-Landau (TIGL) theory has a dual ‘worldline’ representation in terms of near-Brownian strings and loops. More details are given in Refs. \cite{2,3}.

The main ingredient in all these calculations is that they are performed for systems in equilibrium, in which the temperature \(T\) is fixed at values close to the critical temperature \(T_c\). For the case that interests us, that of continuous transitions, the instabilities that characterise the transitions are, in this adiabatic dual picture, a consequence of the uncontrolled proliferation of string.

In practice, transitions occur in a finite, often short, time in which the temperature \(T\) crosses \(T_c\) from its initial value. The adiabatic approximation is only valid away from the transition, when the field can keep in step with the changing environment. Close to the transition the adiabatic regime is replaced by an impulse regime, in which the field falls out of step with what its equilibrium behaviour would be, and freezes in \cite{2,3}. Just as the correlation length, in reality, is unable to diverge because there is not enough time, so we would expect a realistic dual worldline description in terms of strings to show a limit to their proliferation as we pass through a transition.

The question that we shall begin to address here is how the intrinsically non-adiabatic transition has a dual representation in terms of strings and loops.

However, we begin with a brief recapitulation of equilibrium dual theory. The simplest condensed matter TIGL theory is that of a single complex field \(\phi\), with Ginzburg-Landau free energy

\[
F(T) = \int d^3x \left( \frac{\hbar^2}{2m} |\nabla \phi|^2 + \alpha(T)|\phi|^2 + \beta|\phi|^4 \right)
\]

in which the chemical potential \(\alpha(T) = \alpha_0 \epsilon(T)\), where \(\epsilon(T) = (T/T_c - 1)\), vanishes at \(T_c\). Such an energy provides a reasonable description of superfluid \(^4\text{He}\), a simplified model for \(^3\text{He}\) and a good description of the scalar sector of low-\(T_c\) superconductors.

It is convenient to work in spatial units of \(\xi_0 = \sqrt{\hbar^2/2m\alpha_0}\) when, on rescaling the field, self-coupling and temperature,

\[
\tilde{F}(T) = \int d^3x \left( \frac{1}{2} |\nabla \phi|^2 + \epsilon(T)|\phi|^2 + \tilde{\beta}|\phi|^4 \right).
\]

At temperature \(T_0 > T_c\) the free-field correlation function that follows from

\[
\tilde{F}_0(T_0) = \int d^3x \left( |\nabla \phi|^2 + \epsilon(T_0)|\phi|^2 \right)
\]

is

\[
\langle \phi(x)\phi^*(0) \rangle = G_0(r) = \int d^3k \, e^{i\textbf{k}\cdot\textbf{x}} P(k),
\]

\((r = |\textbf{x}|)\) in which the power spectrum

\[
P(k) = \frac{1}{k^2 + \epsilon(T_0)} = \int_0^\infty d\tau \, e^{-r^2 \tau} e^{-\tau\epsilon(T_0)}
\]

has the usual representation in terms of the Schwinger proper-time \(\tau\). In turn, this gives

\[
G_0(r) = \int_0^\infty d\tau \left( \frac{1}{4\pi\tau} \right)^{3/2} e^{-r^2/4\tau} e^{-\tau\epsilon(T_0)}. \tag{1}
\]

\(G_0(0)\), necessary for loops, diverges from the UV singularities at \(\tau = 0\), but these can be regulated with a fixed cutoff. We choose a cutoff of order unity in units of \(\xi_0\), which will be implicit throughout.

The dual picture is obtained by observing that \((1/4\pi\tau)^{3/2} e^{-r^2/4\tau}\) is the probability distribution for a Brownian ‘worldline’ or, more usefully, a ‘polymer’, path \(\textbf{x}(\tau)\), parametrised by \(\tau\), having its endpoints separated by a distance \(r\). Specifically, it is expressible as the sum over such paths

\[
\int_{\textbf{x}(0)=0}^{\textbf{x}(\tau)=\textbf{x}} \mathcal{D}\textbf{x} \exp \left[-\int_0^\tau d\tau' \frac{1}{4\tau'} \frac{d^2\textbf{x}}{d\tau'^2} \right].
\]

If we think of the path as having step length unity (in units of \(\xi_0\)) then \(\tau\) is proportional to the length of the
path \[3\] and we shall use \(\tau\) and path length synonymously.

Then \(G_0(r)\) is the sum over string paths of all lengths,

\[
G_0(r) = \int_0^\infty d\tau \int_{x(0)=0}^{\infty} Dx \, e^{-S_{eq}[x;\tau,\epsilon(T)]},
\]

where \(S_{eq}[x;\tau,\epsilon(T)]\) is the equilibrium Euclidean action

\[
S_{eq}[x;\tau,\epsilon(T)] = \int_0^\tau d\tau' \left( \frac{1}{4} \left( \frac{dx}{d\tau'} \right)^2 + \epsilon(T) \right).
\]

The free-field partition function

\[
Z = \int D\phi D\phi^* e^{-F_0} = \exp[-\tau r \ln(-\nabla^2 + \epsilon(T))]
\]

is just that of a gas of free orientable polymer or string loops, whose lengths are parametrised by \(\tau\), with one-loop partition function

\[
\ln Z = \int_0^\infty d\tau \frac{\tau^{3/2}}{(4\pi\tau)^{3/2}} e^{-\tau \epsilon(T_0)}.
\]

\(\epsilon(T_0)\) is understood as the energy/unit length or tension of the string. The factor \(\propto \tau^{-3/2}\) in \([2]\) is the length distribution for Brownian strings of length \(\tau\). As we drive \(T_0 \to T_c\), it is easier to create string. If, formally, we take \(T_0 \to 0\) below \(T_c\), then the onset of 'negative' tension makes long loops overwhelmingly favoured and the condensation of loops that follows is understood by condensed matter physicists as a Shockley-Feynman transition, and by quantum field theorists as a Hagedorn transition.

The average loop length is

\[
\langle r \rangle = -\frac{d \ln Z}{d \epsilon(T_0)} = \int_0^\infty d\tau \frac{\tau^{3/2}}{(4\pi\tau)^{3/2}} e^{-\tau \epsilon(T_0)}.
\]

Yet again, forcing \(T\) below \(T_c\) makes \(\langle r \rangle\) diverge from the IR divergence of the integrals at large \(\tau\). A formal IR cutoff at \(\tau = \tau_{max}\) leads to \(\langle r \rangle = O(\tau_{max})\).

The incorporation of the \(\beta|\phi|^4\) field self-interaction into equilibrium dual string theory is implemented by the introduction of a repulsive steric string interaction at the points where paths or strings cross \([3]\). Although this reduces the weight of string configurations, qualitatively the transition still occurs in this adiabatic picture because of the proliferation of strings.

In practice, a change of phase is enforced by changing the sign of \(\epsilon(t)\), either by reducing \(T\), or by varying the critical temperature \(T_c\). Experiments with superconductors and (the neutron bombardment of) \(^3\)He do the former, while pressure quenches of \(^4\)He do the latter. If we write

\[
\epsilon(t) = \epsilon(T(t)) = \frac{T(t)}{T_c(t)} - 1,
\]

the adiabatic approximation for the 'free' correlation function \(G_0(r,t)\) at time \(t\) is

\[
G_0^{ad}(r,t) = \int_0^\infty d\tau \int_{x(0)=0}^{\infty} Dx \, e^{-S_{eq}[x;\tau,\epsilon(t)]},
\]

in which we just make a straightforward substitution of the equilibrium \(\epsilon(T_0)\) with \(\epsilon(t)\) in \(S_{eq}[x;\tau,\epsilon(T_0)]\). This is treating the equilibrium pictures as a series of snapshots that can be run together as a continuous film. In this film string lengths increase uncontrollably as we cross the transition. In particular, \(G_0^{ad}(r,t)\) of \([4]\) is simply calculable as the Yukawa correlator

\[
G_0^{ad}(r) = \frac{1}{4\pi r} e^{-r/\xi(t)},
\]

where, on rescaling, \(\xi(t) = \xi_0/\sqrt{\epsilon(t)}\) diverges as \(T(t) \to T_c\). This cannot be the case, since causality alone prevents the correlation length diverging in a finite time. In terms of the dual picture this implies that the production of an infinity of string in a finite time is equally prohibited. The question is, how?

To answer this, we adopt the time-dependent Landau-Ginzburg (TDLG) equation for \(\bar{F}\),

\[
\frac{1}{\Gamma} \frac{\partial \phi}{\partial t} = -\frac{\delta \bar{F}}{\delta \phi} + \eta,
\]

where \(\eta\) is Gaussian thermal noise, satisfying

\[
\langle \eta(x, t)\eta^*(y, t') \rangle = 2T(t)\Gamma \delta(x - y)\delta(t - t').
\]

Let us continue to consider the free-field case, \(\bar{F} = F_0\). In \([3]\) the natural unit of time is \(\tau_0 = 1/\alpha_0 \Gamma\) and, in units of \(\tau_0\) and \(\xi_0\), Eq. \([4]\) becomes

\[
\phi(x, t) = -[\nabla^2 + \epsilon(t)]\phi(x, t) + \tilde{\eta}(x, t).
\]

where \(\tilde{\eta}\) is the renormalised noise. It is straightforward to show that the equal-time correlation function is now

\[
\langle \phi(x, t)\phi^*(y, t) \rangle = G_0(r, t) = \int \delta^3 k \, e^{ik \cdot x} P(k, t).
\]

in which the power spectrum \(P(k, t)\) has a representation in terms of the Schwinger proper-time \(\tau\) as

\[
P(k, t) = \int_0^\infty d\tau \bar{T}(t - \tau/2) e^{-\tau k^2} e^{-\int_0^\tau d\tau' \epsilon(t - \tau'/2)},
\]

where \(\bar{T}\) is the renormalised temperature. In turn, this gives

\[
G_0(r, t) = \int_0^\infty d\tau \bar{T}(t - \tau/2) \left( \frac{1}{4\pi\tau} \right)^{3/2} e^{-\tau r^2/4\tau} e^{-\int_0^\tau d\tau' \epsilon(t - \tau'/2)}.
\]
This differs significantly from \([3]\), which would replace \(\epsilon(t - \tau' / 2)\) in \([3]\) with \(\epsilon(t)\). Further, because of \(\bar{\epsilon}(t - \tau' / 2)\), the paths are no longer Brownian in their length distribution.

However, \(G_0(r, t)\) can still be expressed in terms of paths, as

\[
G_0(r, t) = \int_0^{\infty} d\tau \int_\mathbb{R}^d dx e^{-S_{\{x, \tau, \epsilon(t)\}}}.
\]

\(S[x, \tau, t]\) is not the equilibrium action \(S_{eq}[x, \tau, \epsilon(t)]\), but

\[
S[x, \tau, t] = \int_0^\infty d\tau' \left( \frac{1}{4 \pi \tau'} \right)^{3/2} e^{-\tau'/4} e^{-\tau(t - \tau')/2}.
\]

Unlike the case for the adiabatic approximation \([4]\), the local tension \(\epsilon(t - \tau'/2)\) is dependent on both the time \(t\) and on the position \(\tau\) along them. As a result the integrated tension \(\int_0^T d\tau' \epsilon(t - \tau'/2)\) does not vanish at the transition time. Since it is the uniform vanishing of tension which triggers the avalanche of string production we see already that it will not happen in this case.

As a simple, if unrealistic, demonstration that transitions implemented in a finite time do not impose singular dual behaviour we consider the case of an instantaneous quench at time \(t = 0\) from a temperature \(T_0\) above the critical temperature \(T_c\) to absolute zero. That is, \(\epsilon(t) = \epsilon(T_0)\theta(-t)\).

Simple calculation shows that \(G_0(r, t)\) takes the form of \([3]\) for \(t < 0\), whereas for \(t > 0\)

\[
G_0(r, t) = e^{2\epsilon(T_0)/T_0} \int_0^\infty d\tau \int_\mathbb{R}^d dx e^{-S_{eq}[x, \tau, \epsilon(T_0)]} \tag{9}
\]

\[
= e^{2\epsilon(T_0)/T_0} \int_0^\infty d\tau \left( \frac{1}{4 \pi \tau} \right)^{3/2} e^{-\tau^2/4\tau} e^{-\tau(T_0)}.
\]

After the transition we have a representation in terms of positive tension paths at the initial temperature \(T_0\). The unstable (negative tension) paths, for which \(\tau < 2t\), are totally excluded. Instead, the instabilities are encoded in the non-singular exponential prefactor.

This lack of IR singular behaviour is made even more explicit in a saddle-point approximation for \(G_0(r, t)\). For \(\tau^2 > 4\epsilon(T_0)t\) we find

\[
G_0(r, t) \sim e^{2\epsilon(T_0)} \frac{1}{4\pi t} e^{-\tau^2/\xi},
\]

where \(\xi = \xi(T_0)\) remains frozen in at its initial equilibrium value. Unlike \(\xi_{ad}(t)\) there is no divergence of \(\xi\) as we cross \(T = T_c\) in this abrupt way.

For the equilibrium theory, \(\langle \tau \rangle\) of \([3]\) can also be expressed as

\[
\langle \tau \rangle = \frac{G_0(0)}{2G_0''(0)},
\]

where the prime denotes differentiation with respect to \(r\). If we adopt the same definition out of equilibrium the benign effect of the quench to \(T = 0\) is again apparent in that \(\langle \tau \rangle\) agrees with \([3]\) for \(t < 0\), but is

\[
\langle \tau \rangle_t = \frac{\int_0^\infty d\tau \left( \frac{1}{4 \pi \tau} \right)^{3/2} e^{-\tau(T_0)}}{\int_0^\infty d\tau \left( \frac{1}{4 \pi \tau} \right)^{3/2} e^{-\tau(T_0)}} \tag{10}
\]

for \(t > 0\). The exponential prefactors have cancelled to reproduce the equilibrium result \([3]\), again at the initial temperature \(T = T_0\), but for the absence of unstable loops with length less than \(2t\), to which there is now no reference. Because of the exponential damping of long strings the dominant string length is \(\tau = 2t\).

There is no question of an IR divergence of loop length as naively suggested by the equilibrium theory. We understand the stability of loops with length \(\tau > 2t\) in \([10]\) as a causal bound. In our units, negative tension can only propagate at speed \(c = 1\), which happens to be the cold speed of sound in the \(\phi\) field.

This is reinforced by an extension to non-zero final temperature \(T_f\)

\[
\epsilon(t) = \epsilon(T_0) > 0 \quad \text{if} \quad t < 0,
\]

\[
= \epsilon(T_f) < 0 \quad \text{if} \quad t > 0.
\]

For \(t > 0\) we now find

\[
G_0(r, t) = e^{2\epsilon(T_0 - T_f)/T_f} \int_0^\infty d\tau \int_\mathbb{R}^d dx e^{-S_{eq}[x, \tau, \epsilon(T_0)]} \tag{11}
\]

\[
+ \frac{T_f}{T_0} \int_0^{2t} d\tau \int_\mathbb{R}^d dx e^{-S_{eq}[x, \tau, \epsilon(T_f)]}.
\]

We see explicitly how the limited length \(\tau < 2t\) of string with negative tension \(\epsilon(T_f) < 0\) prevents such string from giving a divergent contribution. Small unstable loops are no longer precluded, but their contribution is finite. The dominant length remains \(\tau = 2t\).

Although an instantaneous quench is impossible, more general quenches show similar qualitative behaviour. Specifically, suppose that \(\epsilon(t)\) decreases monotonically, with a single zero \(\epsilon(0) = 0\) at time \(t = 0\). As before, only strings of limited length \(\tau < 2t\), for which \(\epsilon(t - \tau') < 0\), have negative tension. However, for \(\tau > 2t\), for which \(\epsilon(t - \tau') > 0\), strings have segments with negative and positive tension. There is a causal bound \(c = 1\) on the speed at which instability can propagate along a string. The string with negative tension, with a contribution that is independent of \(\tau\), gives a prefactor growing at least exponentially. The effect is to leave only stable string of length \(\tau > 2t\) in the \(\tau\) integral for \(\tau > 2t\). This boundedness on unstable string prevents the unlimited production of string suggested by the adiabatic approximation.
The exponential growth of $G_0(r, t)$ with time in (10), and more generally, can only be accommodated for a very short period, since $\langle |\phi|^2 \rangle = G_0(0, t)$ must be constrained by the value of the order parameter $\langle |\phi| \rangle$ after the transition, equal to $\sqrt{\beta^{-1}/2}$ in the absence of corrections. A rough guide to the maximum time for which the free-field approximation is valid is that $G_0(0, t) = \beta^{-1}/2$. As $t$ approaches $\beta$, and thereafter, the reaction of the field with itself will cut off the exponential growth.

It is difficult to see how the dual string picture will survive at later times without further approximation. One indication is through a mean-field (or large-N) approach [9]. In this approach (7) is replaced by

$$\langle |\phi|^2 \rangle = G(t) + \bar{G}(0, t),$$

where $G(0, t)$ is determined self-consistently from (11). The coefficient $p$ depends on whether we adopt a mean-field (Hartree) approximation or a large-N limit for $N = 2$.

The assumed single zero of $\epsilon(t)$ at $t = 0$ will lead to a zero of $\epsilon_{eff}(t)$ at $t \approx 0$, and the previous analysis applies; with only segments no longer than $2t$ with negative tension, there is no singular behaviour as we cross the transition.

To have a quantitative estimate of the effect of backreaction we make a Gaussian approximation for $G(0, t)$ by expanding about the zero of $\epsilon_{eff}(t)$. Assuming that the quench is not too rapid, we find that, for $t \leq \beta$, the average loop length is

$$\langle \tau \rangle_t \approx \int_0^\infty d\tau \left( T(t-\tau/2 \langle 4\pi \rangle)^2 \right) e^{-(\tau-2t)^2/\langle |\phi|^2 \rangle}/4 \int_0^\infty d\tau \left( T(t-\tau/2 \langle 4\pi \rangle)^2 \right) e^{-(\tau-2t)^2/\langle |\phi|^2 \rangle}/4 \int_0^\infty d\tau \left( T(t-\tau/2 \langle 4\pi \rangle)^2 \right) e^{-(\tau-2t)^2/\langle |\phi|^2 \rangle}/4.$$

As before, the prefactors encoding the unstable strings cancel approximately, and we need no information about the self-consistent mass, but for the fact that $\epsilon_{eff}(t) \rightarrow 0$ at large times to stop $G(0, t)$ from growing. We see the peak at $\tau = 2t$ in the length distribution moving clear of the UV endpoint behaviour. This continues for later times and $\langle \tau \rangle_t = O(t)$ once the temperature has become low enough that $T(\tau)$ has suppressed the UV singular behaviour. Details are given in our earlier work [8].

We stress that this discussion has been restricted to the 'first quantised' dual representation of the Ginzburg-Landau theory. However, it is a familiar result from a different viewpoint. For a linear system like (11) it can be shown [10] that the mean loop length $\langle \tau \rangle_t$ satisfies

$$\langle \tau \rangle_t = \frac{1}{4\pi n(t)},$$

where $n(t)$ is the density of line zeroes of the complex field $\phi$. The relevance of this is that, at later times, the global vortices of this $U(1)$ scalar theory can be identified by the line zeroes of their cores [12]. The linear growth of dual loop lengths with time corresponds to a $t^{1/2}$ behaviour for $\phi$-field line-zero separation and, when vortices are well-defined, vortex separation [12]. This scaling behaviour can be justified [13] for the decay of vortices of $^4He$, and is used to determine initial vortex densities at the $^4He$ transition [4].

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