Asymptotics of discrete Riesz $d$-polarization on subsets of $d$-dimensional manifolds

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Abstract

We prove a conjecture of T. Erdélyi and E.B. Saff, concerning the form of the dominant term (as $N \to \infty$) of the $N$-point Riesz $d$-polarization constant for an infinite compact subset $A$ of a $d$-dimensional $C^1$-manifold embedded in $\mathbb{R}^m$ ($d \leq m$). Moreover, if we assume further that the $d$-dimensional Hausdorff measure of $A$ is positive, we show that any asymptotically optimal sequence of $N$-point configurations for the $N$-point $d$-polarization problem on $A$ is asymptotically uniformly distributed with respect to $\mathcal{H}_d|_A$.

These results also hold for finite unions of such sets $A$ provided that their pairwise intersections have $\mathcal{H}_d$-measure zero.

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1 Introduction

Let \( \omega_N = \{x_1, \ldots, x_N\} \) denote a configuration of \( N \) (not necessarily distinct) points in the \( m \)-dimensional Euclidean space \( \mathbb{R}^m \) (such configurations are known as multisets, however, we will still use the word configurations). For an infinite compact set \( A \subset \mathbb{R}^m \) and \( s > 0 \), we define the following quantities:

\[
M^s_N(A) := \max_{\substack{\omega_N \subset A \\ \#\omega_N = N}} M^s(\omega_N; A),
\]

where \( \#\omega_N \) stands for the cardinality of the multiset \( \omega_N \). Following [5], we will call \( M^s_N(A) \) the \( N \)-point Riesz \( s \)-polarization constant of \( A \). The quantity \( M^s_N(A) \) is also known as the \( N \)th \( L_s \) Chebyshev constant of the set \( A \) (cf. e.g. [2]). We will call an \( N \)-point configuration \( \omega_N \subset A \) optimal for \( M^s_N(A) \) if it attains the maximum on the right-hand side of (1).

It is not difficult to verify that for a fixed vector \( x_N := (x_1, \ldots, x_N) \) in \( A^N \) (the \( N \)-th Cartesian power of \( A \)), the potential function \( f(y) := \sum_{i=1}^N |y - x_i|^{-s}, s > 0, \) is lower semi-continuous in \( y \) on the set \( A \) and the function \( g(x_N) := M^s(x_N; A), s > 0, \) is upper semi-continuous in \( x_N \) on \( A^N \). So, the function \( f(y) \) attains its minimum on \( A \) and the function \( g(x_N) \) attains its maximum on \( A^N \); i.e. an optimal configuration in (1) exists when \( A \) is an infinite compact set.

The \( N \)-point Riesz \( s \)-polarization constant was earlier considered by M. Ohtsuka in [16]. In particular, he showed that for any infinite compact set \( A \subset \mathbb{R}^m \), the following limit, called the Chebyshev constant of \( A \), exists as an extended real number:

\[
\mathcal{M}^s(A) := \lim_{N \to \infty} \frac{M^s_N(A)}{N}.
\]

Moreover, he showed that \( \mathcal{M}^s(A) \geq W^s(A) \), where \( W^s(A) \) is the Wiener constant of \( A \) corresponding to the same value of \( s \). Later, Chebyshev constants were studied in [6] and [8] and used to study the so-called rendezvous or average numbers in [7] and [8]. In particular, it follows from [6] Theorem 11 that \( \mathcal{M}^s(A) = W^s(A) \) whenever the maximum principle is satisfied on \( A \) for the Riesz \( s \)-potential. More information on the properties of the Wiener constant can be found, for example, in the book [12].
The optimality of \( N \) distinct equally spaced points on the circle for the Riesz \( s \)-polarization problem was proved by G. Ambrus in [1] and by G. Ambrus, K. Ball, and T. Erdélyi in [2] for \( s = 2 \). T. Erdélyi and E.B. Saff [3] established this for \( s = 4 \). For arbitrary \( s > 0 \), this result was proved by D.P. Hardin, A.P. Kendall, and E.B. Saff [9] (paper [14] earlier established this result for \( N = 3 \)). Some problems closely related to polarization were considered in [15].

Let \( H_d \) be the \( d \)-dimensional Hausdorff measure in \( \mathbb{R}^m \) normalized so that the copy of the \( d \)-dimensional unit cube embedded in \( \mathbb{R}^m \) has measure 1. The inequality \( M^s(A) \geq W^s(A) \) implies that on any infinite compact set \( A \) of zero \( s \)-capacity (i.e., when \( W^s(A) = \infty \)), the limit \( M^s(A) \) is infinite. This means that the \( N \)-point Riesz \( s \)-polarization constant \( M^s_N(A) \) grows at a rate faster than \( N \). In particular, it was proved by T. Erdélyi and E.B. Saff [5] Theorem 2.4] that for a compact set \( A \) in \( \mathbb{R}^m \) of positive \( d \)-dimensional Hausdorff measure, one has \( M^d_N(A) = O(N \ln N) \), \( N \to \infty \), and \( M^d_N(A) = O(N^{s/d}) \), \( N \to \infty \), for every \( s > d \). The order estimate for \( s = d \) is sharp when \( A \) is contained in a \( d \)-dimensional \( C^1 \)-manifold and the order estimate for \( s > d \) is sharp when \( A \) is \( d \)-rectifiable (see [5] Theorem 2.3). We remark that the case \( d = 1 \) of these order estimates when \( A \) is a circle was obtained in [2].

Furthermore, when \( A \) is the unit ball \( B^d \) in \( \mathbb{R}^d \) or the unit sphere \( S^d \) in \( \mathbb{R}^{d+1} \), paper [5] proves that

\[
\lim_{N \to \infty} \frac{M^d_N(B^d)}{N \ln N} = 1, \quad d \geq 1,
\]

and

\[
\lim_{N \to \infty} \frac{M^d_N(S^d)}{N \ln N} = \frac{\beta_d}{\mathcal{H}_d(S^d)}, \quad d \geq 2,
\]

where \( \beta_d \) denotes the volume of the \( d \)-dimensional unit ball \( B^d \).

When \( A \) is an infinite compact subset of a \( d \)-dimensional \( C^1 \)-manifold, T. Erdélyi and E.B. Saff [5] also show that

\[
\liminf_{N \to \infty} \frac{M^d_N(A)}{N \ln N} \geq \frac{\beta_d}{\mathcal{H}_d(A)}
\]

and conjecture that the limit of the sequence on the left-hand side of (5) exists and equals the right-hand side.

Another interesting fact established in [5] is that \( M^s_N(B^d) = N \) for every \( N \geq 1 \) and \( 0 < s \leq d - 2 \) (the maximum principle does not hold for the Riesz \( s \)-potential in the case \( 0 < s < d - 2 \)).
A more detailed review of results on polarization can be found, for example, in the papers [2], [5], [6], and [8].

The polarization problem is related to the discrete minimal Riesz energy problem described below. For a set $X_N = \{x_1, \ldots, x_N\}$ of $N \geq 2$ pairwise distinct points in $\mathbb{R}^m$, we define its Riesz $s$-energy by

$$E_s(X_N) := \sum_{1 \leq j \neq k \leq N} \frac{1}{|x_j - x_k|^s},$$

and the minimum $N$-point Riesz $s$-energy of a compact set $A \subset \mathbb{R}^m$ is defined as

$$E_s(A, N) := \min_{X_N \subset A, \#X_N = N} E_s(X_N).$$

D.P. Hardin and E.B. Saff proved in [11] (see also [10]) that if $A$ is an infinite compact subset of a $d$-dimensional $C^1$-manifold embedded in $\mathbb{R}^m$ (see Definition 2.1), then

$$\lim_{N \to \infty} \frac{E_d(A, N)}{N^2 \ln N} = \frac{\beta_d}{\mathcal{H}_d(A)}. \quad (6)$$

Furthermore, if $A$ is as in above condition and $\mathcal{H}_d(A) > 0$, then for any sequence $X_N = \{x_{k,N}\}_{k=1}^N$, $N \in \mathbb{N}$, of asymptotically $d$-energy minimizing $N$-point configurations in $A$ in the sense that

$$\lim_{N \to \infty} \frac{E_d(X_N)}{E_d(A, N)} = 1,$$

we have

$$\frac{1}{N} \sum_{i=1}^N \delta_{x_{i,N}} \rightharpoonup \frac{\mathcal{H}_d(\cdot)|_A}{\mathcal{H}_d(A)} \quad N \to \infty, \quad (7)$$

in the weak* topology of measures (see Section 2 for the definition). Here $\delta_x$ denotes the unit point mass at the point $x$.

The dominant term of the minimum $s$-energy on $d$-rectifiable closed sets in $\mathbb{R}^m$ ($s > d$) as well as relation (7) for asymptotically optimal sequences of $N$-point configurations were obtained in [11] and [3]. (In the case $d = 1$ these results were earlier established for curves in [13]).

\footnote{The results and techniques of [11] in fact, yield relations (6) and (7) under a more general assumption that $A$ is a compact set in $\mathbb{R}^m$ which for every $\epsilon > 0$ can be partitioned into finitely many subsets bi-Lipschitz homeomorphic to some sets from $\mathbb{R}^d$ with constant $1 + \epsilon$ and having boundaries relative to $A$ of $\mathcal{H}_d$-measure zero (see [4]).}
Relations (6) and (7) have recently been extended by D.P. Hardin, E.B. Saff, and J.T. Whitehouse to the case of \( A \) being a finite union of compact subsets of \( \mathbb{R}^m \) where each compact set is contained in some \( d \)-dimensional \( C^1 \)-manifold in \( \mathbb{R}^m (d \leq m) \) and the pairwise intersections of such compact sets have \( H_d \)-measure zero. These authors observed that the methods of [13] could be applied (see [4]).

A detailed review of known results on discrete minimum energy problems can be found, for example, in the book [4].

2 Notation and definitions

In this section we will mention the main definitions used in the paper. For a subset \( K \subset A \), we will denote by \( \partial_A K \) the boundary of \( K \) relative to \( A \).

We say that a sequence \( \{\mu_n\}_{n=1}^\infty \) of Borel probability measures in \( \mathbb{R}^m \) converges to a Borel probability measure \( \mu \) in the weak* topology of measures (and write \( \mu_n \rightharpoonup \mu, \ n \to \infty \)) if for every continuous function \( f : \mathbb{R}^m \to \mathbb{R} \),

\[
\int f \, d\mu_n \to \int f \, d\mu, \quad n \to \infty. \tag{8}
\]

**Remark 2.1.** It is well known that to prove (8) when \( \mu \) and all the measures \( \mu_n \) are supported on a compact set \( A \subset \mathbb{R}^m \), it is sufficient to show that

\[
\mu_n(K) \to \mu(K), \quad n \to \infty,
\]

for every closed subset \( K \) of \( A \) with \( \mu(\partial_A K) = 0 \).

It will be convenient to use throughout this paper the following definition of a \( d \)-dimensional \( C^1 \)-manifold in \( \mathbb{R}^m \) (see, for example, [17], Chapter 5).

**Definition 2.1.** A set \( W \subset \mathbb{R}^m \) is called a \( d \)-dimensional \( C^1 \)-manifold **embedded in** \( \mathbb{R}^m \), \( d \leq m \), if every point \( y \in W \) has an open neighborhood \( V \) relative to \( W \) such that \( V \) is homeomorphic to an open set \( U \subset \mathbb{R}^d \) with the homeomorphism \( f : U \to V \) being a \( C^1 \)-continuous mapping and the Jacobian matrix

\[
J^f_x := \begin{bmatrix}
\nabla f_1(x) \\
\vdots \\
\nabla f_m(x)
\end{bmatrix}
\]

of the function \( f \) having rank \( d \) at any point \( x \in U \) (here \( f_1, \ldots, f_m \) denote the coordinate mappings of \( f \)).
Finally, we call a sequence \( \{ \omega_N \}_{N=1}^{\infty} \) of \( N \)-point configurations on \( A \) asymptotically optimal for the \( N \)-point \( d \)-polarization problem on \( A \) if
\[
\lim_{N \to \infty} \frac{M^d(\omega_N; A)}{M^d_N(A)} = 1.
\]

### 3 Main results

In this paper we extend relation (3) to the case of an arbitrary infinite compact set in \( \mathbb{R}^d \) and relation (4) to the case when \( A \) is an infinite compact subset of a \( d \)-dimensional \( C^1 \)-manifold embedded in \( \mathbb{R}^m \) where \( m > d \) or a finite union of such sets provided that their pairwise intersections have \( \mathcal{H}_d \)-measure zero. Under additional assumption that \( \mathcal{H}_d(A) > 0 \), we also determine the weak* limiting distribution of asymptotically optimal \( N \)-point configurations for the \( N \)-point \( d \)-polarization problem on these classes of sets. Relation (9) below proves the conjecture made by T. Erdélyi and E.B. Saff in [5, Conjecture 2].

**Theorem 3.1.** Let \( A = \bigcup_{i=1}^{l} A_i \) be an infinite subset of \( \mathbb{R}^m \), where each set \( A_i \) is a compact subset contained in some \( d \)-dimensional \( C^1 \)-manifold in \( \mathbb{R}^m \), \( d \leq m \), and \( \mathcal{H}_d(A_i \cap A_j) = 0, 1 \leq i < j \leq l \). Then
\[
\lim_{N \to \infty} \frac{M^d_N(A)}{N \ln N} = \frac{\beta_d}{\mathcal{H}_d(A)}.
\]

Furthermore, under an additional assumption that \( \mathcal{H}_d(A) > 0 \), if \( \omega_N = \{ x_{i,N} \}_{i=1}^{N}, N \in \mathbb{N} \), is a sequence of asymptotically optimal configurations for the \( N \)-point \( d \)-polarization problem on \( A \), then in the weak* topology of measures we have
\[
\frac{1}{N} \sum_{i=1}^{N} \delta_{x_{i,N}} \overset{*}{\rightarrow} \frac{\mathcal{H}_d(\cdot|A)}{\mathcal{H}_d(A)}, \quad N \to \infty.
\]

**Remark 3.1.** Note that the conditions imposed on the set \( A \) imply \( \mathcal{H}_d(A) < \infty \). Moreover, if \( \mathcal{H}_d(A) = 0 \), then the limit in (9) is understood to be \( \infty \).

To establish Theorem 3.1 we will use the result proved in Section 4, Lemma 5.1 and Proposition 5.2.
4 Upper estimate

For a compact set \( A \subset \mathbb{R}^m \), define the quantity

\[
\overline{\alpha}_d(A; \varepsilon) := \sup_{0 < r \leq \varepsilon} \sup_{x \in A} \frac{H_d(B(x, r) \cap A)}{\beta_d r^d}.
\]  \( (11) \)

Let also

\[
\underline{h}_d(A) := \liminf_{N \to \infty} \frac{M^d_N(A)}{N \ln N} \quad \text{and} \quad \overline{h}_d(A) := \limsup_{N \to \infty} \frac{M^d_N(A)}{N \ln N}.
\]

The main result of this section is given below.

**Theorem 4.1.** Let \( d, m \in \mathbb{N}, d \leq m, \) and \( A \subset \mathbb{R}^m \) be a compact set with \( 0 < \mathcal{H}_d(A) < \infty \), containing a closed subset \( B \) of zero \( \mathcal{H}_d \)-measure such that every compact subset \( K \subset A \setminus B \) satisfies

\[
\lim_{\varepsilon \to 0^+} \overline{\alpha}_d(K; \varepsilon) \leq 1.
\]  \( (12) \)

Then

\[
\overline{h}_d(A) \leq \frac{\beta_d}{\mathcal{H}_d(A)}.
\]  \( (13) \)

If an equality holds in (13), then any infinite sequence \( \omega_N = \{x_{k,N}\}_{k=1}^N, N \in \mathcal{N} \subset \mathbb{N}, \) of configurations on \( A \) such that

\[
\lim_{N \to \infty} \frac{M^d(\omega_N; A)}{N \ln N} = \frac{\beta_d}{\mathcal{H}_d(A)}
\]  \( (14) \)

satisfies

\[
\frac{1}{N} \sum_{i=1}^N \delta_{x_{i,N}} \Rightarrow \frac{d \mathcal{H}_d(x)}{\mathcal{H}_d(A)}, \quad N \ni N \to \infty.
\]  \( (15) \)

We precede the proof of Theorem 4.1 with the following auxiliary statements.

**Lemma 4.1.** Let \( 0 < R \leq r, D \subset \mathbb{R}^m \) be a compact set with \( \mathcal{H}_d(D) < \infty, \) \( d \in \mathbb{N}, d \leq m, \) and \( y \in D. \) Then

\[
\int_{D \setminus B(y, R)} \frac{d \mathcal{H}_d(x)}{|x - y|^d} \leq r^{-d} \mathcal{H}_d(D) + \beta_d \overline{\alpha}_d(D; r) \ln \left( \frac{r}{R} \right)^d.
\]
Proof. We have

\[
\int_{D \setminus B(y,R)} \frac{d\mathcal{H}_d(x)}{|x-y|^d} = \int_0^\infty \mathcal{H}_d\{x \in D \setminus B(y,R) : |x-y|^{-d} > t\} dt \\
= \int_0^\infty \mathcal{H}_d\{x \in D \setminus B(y,R) : t^{-1/d} > |x-y|\} dt \\
\leq \int_0^{R^{-d}} \mathcal{H}_d(B(y,t^{-1/d}) \cap D) dt \\
\leq r^{-d}\mathcal{H}_d(D) + \int_{R^{-d}}^{r^{-d}} \mathcal{H}_d(B(y,t^{-1/d}) \cap D) dt \\
\leq r^{-d}\mathcal{H}_d(D) + \beta_d \int_{R^{-d}}^{r^{-d}} \alpha_d(D;r)t^{-1} dt \\
= r^{-d}\mathcal{H}_d(D) + \beta_d\alpha_d(D;r) \ln \left( \frac{r}{R} \right),
\]

which completes the proof. \(\square\)

Lemma 4.2. Let \(d, m \in \mathbb{N}\), \(d \leq m\), and \(A \subset \mathbb{R}^m\) be a compact set with \(0 < \mathcal{H}_d(A) < \infty\), containing a closed subset \(B\) of zero \(\mathcal{H}_d\)-measure such that every compact subset of the set \(A \setminus B\) satisfies (12). Then for any infinite sequence \(\{\omega_N\}_{N \in \mathbb{N}}\), \(N \subset \mathbb{N}\), of \(N\)-point configurations on the set \(A\), the inequality

\[
\frac{\mathcal{H}_d(K)}{\beta_d} \cdot \liminf_{N \to \infty} \frac{M^d(\omega_N; A)}{N \ln N} \leq \liminf_{N \to \infty} \frac{\#(\omega_N \cap K)}{N} \tag{16}
\]

holds for any compact subset \(K \subset A\) with \(\mathcal{H}_d(K) > 0\) and \(\mathcal{H}_d(\partial A K) = 0\).

Proof. Without loss of generality, we can assume that \(B \neq \emptyset\) since in the case \(B = \emptyset\) we can also use as \(B\) any non-empty compact subset of \(A\) with \(\mathcal{H}_d(B) = 0\).

Let \(x_{1,N}, \ldots, x_{N,N}\) be the points in the configuration \(\omega_N\), \(N \in \mathbb{N}\), and let \(K \subset A\) be any compact subset of positive \(\mathcal{H}_d\)-measure such that \(\mathcal{H}_d(\partial A K) = 0\). Denote

\[
K_{\rho} := \{x \in K : \text{dist}(x, B \cup \partial A K) \geq \rho\}, \quad \rho > 0.
\]

Choose an arbitrary number \(\rho > 0\) such that \(\mathcal{H}_d(K_{2\rho}) > 0\). Let \(r > 0\) be any number such that \(2\beta_dr^d < \mathcal{H}_d(K_{2\rho})\). For each \(j = 1, \ldots, N\), define the set

\[
D_{j,N} := K_{2\rho} \setminus B(x_{j,N}, rN^{-1/d}) \quad \text{and let} \quad D_N := \bigcap_{j=1}^N D_{j,N}.
\]
Notice that \( \text{dist}(K_{2\rho}, K \setminus K_{\rho}) \geq \rho > 0 \). Furthermore, \( \text{dist}(K_{2\rho}, A \setminus K) > 0 \).

Indeed, if there were sequences \( \{x_n\} \) in \( K_{2\rho} \) and \( \{y_n\} \) in \( A \setminus K \) such that \(|x_n - y_n| \to 0, \ n \to \infty\), then by compactness of \( K_{2\rho} \) and \( A \) there would exist subsequences \( \{x_{n_k}\} \) and \( \{y_{n_k}\} \) having the same limit \( z \in K_{2\rho} \). Since \( \{y_{n_k}\} \subset A \setminus K \) the point \( z \) must belong to \( \partial A \), which contradicts to the definition of the set \( K_{2\rho} \). Thus, we have

\[
 h := \text{dist}(K_{2\rho}, A \setminus K_{\rho}) = \min \{\text{dist}(K_{2\rho}, K \setminus K_{\rho}), \text{dist}(K_{2\rho}, A \setminus K)\} > 0. 
\]

Choose \( N \in \mathcal{N} \) to be such that \( rN^{-1/d} < h \) and \( \overline{\pi}_d(K_{\rho}; rN^{-1/d}) \leq 2 \) (such \( N \) exists since \( K_{\rho} \) is a compact subset of \( A \setminus B \), and by assumption, satisfies \( \lim_{N \to \infty} \overline{\pi}_d(K_{\rho}; rN^{-1/d}) \leq 1 \)). Then

\[
 \mathcal{H}_d(D_N) = \mathcal{H}_d \left( K_{2\rho} \setminus \bigcup_{j=1}^{N} B(x_{j,N}, rN^{-1/d}) \right) 
\]

\[
 = \mathcal{H}_d \left( K_{2\rho} \setminus \bigcup_{x_{j,N} \in K_{\rho}} B(x_{j,N}, rN^{-1/d}) \right) 
\]

\[
 \geq \mathcal{H}_d(K_{2\rho}) - \sum_{x_{j,N} \in K_{\rho}} \mathcal{H}_d \left( K_{\rho} \cap B(x_{j,N}, rN^{-1/d}) \right) 
\]

\[
 \geq \mathcal{H}_d(K_{2\rho}) - \beta_d r^d \frac{\#(\omega_N \cap K_{\rho}) \cdot \overline{\pi}_d(K_{\rho}; rN^{-1/d})}{N} 
\]

\[
 \geq \mathcal{H}_d(K_{2\rho}) - \beta_d r^d \overline{\pi}_d(K_{\rho}; rN^{-1/d}) \geq \mathcal{H}_d(K_{2\rho}) - 2 \beta_d r^d =: \gamma_{r, \rho} > 0. 
\]

Let \( \tilde{D}_{j,N} := K_{\rho} \setminus B(x_{j,N}, rN^{-1/d}) \). Then

\[
 M^d(\omega_N; A) = \min_{x \in A} \sum_{j=1}^{N} \frac{1}{|x - x_{j,N}|^d} 
\]

\[
 \leq \frac{1}{\mathcal{H}_d(D_N)} \sum_{j=1}^{N} \int_{D_N} \frac{d\mathcal{H}_d(x)}{|x - x_{j,N}|^d} \leq \frac{1}{\gamma_{r, \rho}} \sum_{j=1}^{N} \int_{\tilde{D}_{j,N}} \frac{d\mathcal{H}_d(x)}{|x - x_{j,N}|^d} 
\]

\[
 \leq \frac{1}{\gamma_{r, \rho}} \left( \sum_{x_{j,N} \in K_{\rho}} \int_{\tilde{D}_{j,N}} \frac{d\mathcal{H}_d(x)}{|x - x_{j,N}|^d} + \sum_{x_{j,N} \in A \setminus K_{\rho}} \int_{\tilde{D}_{j,N}} \frac{d\mathcal{H}_d(x)}{|x - x_{j,N}|^d} \right). 
\]
Taking into account Lemma 4.1 with $R = rN^{-1/d}$ and $D = K_\rho$ and the fact that $\text{dist}(D_{j,N}, A \setminus K_\rho) \geq \text{dist}(K_{2\rho}, A \setminus K_\rho) = h > 0$, we will have

$$M^d(\omega_N; A) \leq \frac{1}{\gamma_{r,\rho}} \left( \#(\omega_N \cap K_\rho) \left( \frac{\mathcal{H}_d(K_\rho)}{r^d} + \beta_d \mathcal{P}_d(K_\rho; r) \ln N \right) + \sum_{x_i \in A \setminus K_\rho} \frac{\mathcal{H}_d(D_{i,N})}{h^d} \right).$$

Consequently,

$$\frac{M^d(\omega_N; A)}{N \ln N} \leq \frac{1}{\gamma_{r,\rho}} \left( \#(\omega_N \cap K_\rho) \left( \frac{\mathcal{H}_d(K_\rho)}{r^d \ln N} + \beta_d \mathcal{P}_d(K_\rho; r) \right) + \frac{\mathcal{H}_d(A)}{h^d \ln N} \right).$$

Passing to the lower limit in (17) we will have

$$\tau := \liminf_{N \to \infty} \frac{M^d(\omega_N; A)}{N \ln N} \leq \frac{\beta_d \mathcal{P}_d(K_\rho; r)}{\mathcal{H}_d(K_{2\rho}) - 2\beta_d r^d} \liminf_{N \to \infty} \frac{\#(\omega_N \cap K_\rho)}{N}.$$
and inequality (13) follows.

Assume now that \( h_d(A) = \beta_d \mathcal{H}_d(A)^{-1} \) and let \( \{\omega_N\}_{N \in \mathbb{N}}, \mathcal{N} \subset \mathbb{N} \), be any infinite sequence of \( N \)-point configurations on \( A \) satisfying (14). For any closed subset \( D \subset A \) with \( \mathcal{H}_d(D) > 0 \) and \( \mathcal{H}_d(\partial_A D) = 0 \), by Lemma 4.2 we have

\[
\liminf_{N \to \infty} \frac{\#(\omega_N \cap D)}{\mathcal{N}} \geq \frac{\mathcal{H}_d(D)}{\beta_d} \limsup_{N \to \infty} \frac{M_d(\omega_N; A)}{N \ln N} = \frac{\mathcal{H}_d(D)}{\mathcal{H}_d(A)}.
\]

(18)

Let now \( P \subset A \) be any closed subset of zero \( \mathcal{H}_d \)-measure. Show that

\[
\lim_{N \to \infty} \frac{\#(\omega_N \cap P)}{\mathcal{N}} = 0.
\]

(19)

If \( P = \emptyset \), then (19) holds trivially. Let \( P \neq \emptyset \). Since \( \mathcal{H}_d(A) < \infty \), for every \( \epsilon > 0 \), there are at most finitely many numbers \( \delta > 0 \) such that the set \( P[\delta] := \{x \in A : \text{dist}(x, P) = \delta\} \) has \( \mathcal{H}_d \)-measure at least \( \epsilon \). This implies that there are at most countably many numbers \( \delta > 0 \) such that \( \mathcal{H}_d(P[\delta]) > 0 \). Denote also \( P_{\delta} = \{x \in A : \text{dist}(x, P) \geq \delta\}, \delta > 0 \). Then there exists a positive sequence \( \{\delta_n\} \) monotonically decreasing to 0 such that every set \( \partial_A P_{\delta_n} \subset P[\delta_n] \) has \( \mathcal{H}_d \)-measure zero. Since \( P_{\delta_n} \) is closed and \( \mathcal{H}_d(P_{\delta_n}) > 0 \) for every \( n \) greater than some \( n_1 \), in view of (18), we have

\[
\liminf_{N \to \infty} \frac{\#(\omega_N \cap (A \setminus P))}{\mathcal{N}} \geq \liminf_{N \to \infty} \frac{\#(\omega_N \cap P_{\delta_n})}{\mathcal{N}} \geq \frac{\mathcal{H}_d(P_{\delta_n})}{\mathcal{H}_d(A)}, \quad n > n_1.
\]

Since \( \mathcal{H}_d(P_{\delta_n}) \to \mathcal{H}_d(A \setminus P) = \mathcal{H}_d(A), \ n \to \infty \), we have

\[
\lim_{N \to \infty} \frac{\#(\omega_N \cap (A \setminus P))}{\mathcal{N}} = 1,
\]

which implies (19).

Since the set \( A \setminus D \) is also a closed subset of \( A \) and \( \mathcal{H}_d(\partial_A (A \setminus D)) = \mathcal{H}_d(\partial_A D) = 0 \), by (18) and (19) (with \( P = \partial_A D \)) we have

\[
\limsup_{N \to \infty} \frac{\#(\omega_N \cap D)}{\mathcal{N}} = 1 - \liminf_{N \to \infty} \frac{\#(\omega_N \cap (A \setminus D))}{\mathcal{N}}
\]

\[
= 1 - \liminf_{N \to \infty} \frac{\#(\omega_N \cap A \setminus D)}{\mathcal{N}} \leq 1 - \frac{\mathcal{H}_d(A \setminus D)}{\mathcal{H}_d(A)} = \frac{\mathcal{H}_d(D)}{\mathcal{H}_d(A)}.
\]
Thus,

$$\lim_{N \to \infty} \frac{\#(\omega_N \cap D)}{N} = \frac{\mathcal{H}_d(D)}{\mathcal{H}_d(A)}$$  \hspace{1cm} (20)

for any closed subset $D \subset A$ with $\mathcal{H}_d(D) > 0$ and $\mathcal{H}_d(\partial A) = 0$. In view of (19) relation (20) also holds when $D \subset A$ is closed and $\mathcal{H}_d(D) = 0$. Then in view of Remark 2.1 we have (15). \hfill \square

5 Auxiliary statements

We will show in this section that for every set $A$ satisfying the assumptions of Theorem 3.1, the assumptions of Theorem 4.1 necessarily hold.

Proposition 5.1. Let $A$ be a compact subset of a $d$-dimensional $C^1$-manifold embedded in $\mathbb{R}^m$, $d \leq m$. Then for such a set $A$,

$$\lim_{\varepsilon \to 0^+} \alpha_d(A; \varepsilon) \leq 1.$$ \hspace{1cm} (21)

The proof of this statement is given in the Appendix.

Lemma 5.1. Let $A = \cup_{i=1}^l A_i$, where each set $A_i$ is a compact set contained in some $d$-dimensional $C^1$-manifold in $\mathbb{R}^m$, $d \leq m$, and $\mathcal{H}_d(A_i \cap A_j) = 0$, $1 \leq i < j \leq l$. Then there is a compact subset $B \subset A$ such that every compact subset $K \subset A \setminus B$ satisfies

$$\lim_{\varepsilon \to 0^+} \alpha_d(K; \varepsilon) \leq 1.$$  \hspace{1cm} (22)

Proof. Denote $B := \bigcup_{1 \leq i < j \leq l} A_i \cap A_j$. Let $K \subset A \setminus B$ be a compact subset. Then

$$\delta_0 := \min_{1 \leq i < j \leq l} \text{dist}(A_i \cap K, A_j \cap K) > 0.$$  

Choose any $\varepsilon \in (0, \delta_0)$. Choose also arbitrary $r \in (0, \varepsilon)$ and $x \in K$. We have $x \in A_i$ for some $1 \leq i \leq l$ and $x \notin A_j$ for every $j \neq i$. Since $r < \delta_0$, we have $B(x, r) \cap K \subset B(x, r) \cap A_i$ and consequently,

$$\frac{\mathcal{H}_d(B(x, r) \cap K)}{\beta_d r^d} \leq \frac{\mathcal{H}_d(B(x, r) \cap A_i)}{\beta_d r^d} \leq \max_{1 \leq j \leq l} \alpha_d(A_j; \varepsilon).$$

Consequently,

$$\alpha_d(K; \varepsilon) = \sup_{r \in (0, \varepsilon)} \sup_{x \in K} \frac{\mathcal{H}_d(B(x, r) \cap K)}{\beta_d r^d} \leq \max_{1 \leq j \leq l} \alpha_d(A_j; \varepsilon).$$
Since each $A_i$ is a compact subset of a $d$-dimensional $C^1$-manifold, by Proposition 5.1, we have $\lim_{\epsilon \to 0^+} \alpha^d(A_i; \epsilon) \leq 1$, $i = 1, \ldots, l$. Then in view of (22) we have $\lim_{\epsilon \to 0^+} \alpha^d(K; \epsilon) \leq 1$.

The following proposition is a part of the result by D.P. Hardin, E.B. Saff, and J.T. Whitehouse mentioned at the end of Section 1. For completeness, we will reproduce its proof.

**Proposition 5.2.** Let $A = \bigcup_{i=1}^{l} A_i$, where each $A_i$ is a compact set contained in some $d$-dimensional $C^1$-manifold in $\mathbb{R}^m$ and $\mathcal{H}^d(A_i \cap A_j) = 0$, $1 \leq i < j \leq l$. Then

$$g_d(A) := \liminf_{N \to \infty} \frac{E_d(A, N)}{N^2 \ln N} \geq \frac{\beta_d}{\mathcal{H}^d(A)}.$$ 

**Proof.** Since every set $A_i$ is a compact subset of a $d$-dimensional $C^1$-manifold, in view of Theorem 2.4 in [11], there holds $g_d(A_i) \geq \beta_d \mathcal{H}^d(A_i)^{-1}, i = 1, \ldots, l$. In view of inequality (34) from Lemma 3.2 in [11], we then have

$$g_d(A) = g_d\left(\bigcup_{i=1}^{l} A_i\right) \geq \left(\sum_{i=1}^{l} g_d(A_i)^{-1}\right)^{-1} \geq \left(\frac{1}{\beta_d} \sum_{i=1}^{l} \mathcal{H}^d(A_i)\right)^{-1} = \frac{\beta_d}{\mathcal{H}^d(A)},$$

which yields the desired inequality. \qed

**6 Proof of Theorem 3.1**

**Proof.** The proof of the lower estimate in (9) will repeat the proof of inequality (2.9) in [5]. It is known that (see [5], [6], or [8]) for any infinite compact set $A \subset \mathbb{R}^m$,

$$M^s_N(A) \geq \frac{1}{N - 1} E_s(A, N), \quad N \geq 2, \quad s > 0. \tag{23}$$

Then Proposition 5.2 and inequality (23) give the lower estimate for $M^d_N(A)$:

$$\liminf_{N \to \infty} \frac{M^d_N(A)}{N \ln N} \geq \liminf_{N \to \infty} \frac{E_d(A, N)}{(N - 1)N \ln N} \geq \frac{\beta_d}{\mathcal{H}^d(A)}.$$ 

Note that if $\mathcal{H}^d(A) = 0$, then $\lim_{N \to \infty} M^d_N(A) / (N \ln N) = \infty$. 13
Now, assume that $H_d(A) > 0$. In view of Lemma 5.1 and Remark 3.1, the set $A$ satisfies the assumptions of Theorem 4.1. Consequently
\[
\limsup_{N \to \infty} \frac{M^d_N(A)}{N \ln N} \leq \frac{\beta_d}{H_d(A)}.
\]
This implies (9).

Every sequence $\{\omega_N\}_{N=1}^\infty$ of $N$-point configurations, which is asymptotically optimal for the $N$-point $d$-polarization problem on $A$ must satisfy (14) with $N = N$. Since $H_d(A) = \beta_d H_d(A)^{-1}$, by Theorem 4.1 we obtain (10).

7 Appendix

In this part of the paper we prove Proposition 5.1.

We say that a set $B$ in $\mathbb{R}^m$ is bi-Lipschitz homeomorphic to a set $D \subset \mathbb{R}^n$ with a constant $M \geq 1$, if there is a mapping $\varphi : B \to D$ such that
\[
\frac{1}{M} |x - y| \leq |\varphi(x) - \varphi(y)| \leq M |x - y|, \quad x, y \in B.
\]

Lemma 7.1. Let $U \subset \mathbb{R}^d$ be a non-empty open set and $f : U \to \mathbb{R}^m$, $m \geq d$, be an injective $C^1$-continuous mapping such that its inverse $f^{-1} : f(U) \to U$ is continuous and the Jacobian matrix
\[
J^f_x := \begin{bmatrix}
\nabla f_1(x) \\
\vdots \\
\nabla f_m(x)
\end{bmatrix}
\]

of $f$ has rank $d$ at any point $x \in U$. Then for every $\epsilon > 0$ and every point $y_0 \in f(U)$, there is a closed ball $B$ centered at $y_0$ such that the set $B \cap f(U)$ is bi-Lipschitz homeomorphic to some compact set in $\mathbb{R}^d$ with a constant $1 + \epsilon$.

Proof of Lemma 7.1. Let $x_0 \in U$ be the point such that $f(x_0) = y_0$. Choose any $\epsilon > 0$ and let $\delta = \delta(x_0, \epsilon) > 0$ be such that $B[x_0, \delta] \subset U$ and
\[
|\nabla f_i(x) - \nabla f_i(x_0)| < \epsilon, \quad x \in B[x_0, \delta], \ i = 1, \ldots, m.
\]

Let $x, y \in B[x_0, \delta]$ be two arbitrary points. Define the function $g_i(t) := f_i(x + t(y - x))$, $t \in [0, 1]$. Then there exists $\xi_i \in (0, 1)$ such that
\[
f_i(y) - f_i(x) = g_i(1) - g_i(0) = g_i'(\xi_i) = \nabla f_i(z_i) \cdot (y - x)
\]
\[ \nabla f_i(x_0) \cdot (y - x) + (\nabla f_i(z_i) - \nabla f_i(x_0)) \cdot (y - x), \]

where \( z_i = x + \xi_i(y - x), \ i = 1, \ldots, m. \) Since \( z_i \in B[x_0, \delta], \) we have

\[
|f_i(y) - f_i(x) - \nabla f_i(x_0) \cdot (y - x)|
\]

\[
= |(\nabla f_i(z_i) - \nabla f_i(x_0)) \cdot (y - x)| \leq \epsilon |y - x|, \quad i = 1, \ldots, m,
\]

and hence (we treat \( x \) and \( y \) as vector-columns below),

\[
|f(y) - f(x) - J_{x_0}^f(y - x)| \leq \epsilon \sqrt{m} |y - x|, \quad x, y \in B[x_0, \delta]. \tag{25}
\]

Since the matrix \( J_{x_0}^f \) has rank \( d \), for every standard basis vector \( e_i \) from \( \mathbb{R}^d \), there is a vector \( v_i \in R^m \) such that \( (J_{x_0}^f)^T v_i = e_i, \ i = 1, \ldots, d, \) where \( (J_{x_0}^f)^T \) denotes the transpose of the matrix \( J_{x_0}^f \). Then the \( d \times m \) matrix \( Z := [v_1, \ldots, v_d]^T \) satisfies \( Z J_{x_0}^f = I_d \), where \( I_d \) is the \( d \times d \) identity matrix. Taking into account (25) we have

\[
|f(y) - f(x) - J_{x_0}^f(y - x)| \leq \epsilon \sqrt{m} \left| Z J_{x_0}^f (y - x) \right|
\]

\[
\leq \epsilon \sqrt{m} \|Z\| \left| J_{x_0}^f (y - x) \right|, \quad x, y \in B[x_0, \delta],
\]

where \( \|Z\| := \max \{|Z u| : u \in \mathbb{R}^m, \ |u| = 1\} \). Consequently,

\[
(1 - \epsilon \sqrt{m} \|Z\|) \left| J_{x_0}^f (y - x) \right| \leq |f(y) - f(x)|
\]

\[
\leq (1 + \epsilon \sqrt{m} \|Z\|) \left| J_{x_0}^f (y - x) \right|, \quad x, y \in B[x_0, \delta].
\]

Let \( u_1, \ldots, u_d \) be an orthonormal basis in the subspace \( H \) of \( \mathbb{R}^m \) spanned by the columns of the matrix \( J_{x_0}^f \) and let \( D := [u_1, \ldots, u_d] \) be the \( m \times d \) matrix with columns \( u_1, \ldots, u_d \). Since the columns of \( J_{x_0}^f \) also form a basis in \( H \), there exists an invertible \( d \times d \) matrix \( Q \) such that \( D = J_{x_0}^f Q \).

Let \( V \subset \mathbb{R}^d \) be the open set such that \( \Phi(V) = B(x_0, \delta) \), where \( \Phi : \mathbb{R}^d \to \mathbb{R}^d \) is the linear mapping given by \( \Phi(v) = Qv \). Since the columns of the matrix \( D \) are orthonormal, for every \( u, v \in V \), we will have

\[
|f \circ \Phi(u) - f \circ \Phi(v)| = |f(Qu) - f(Qv)|
\]

\[
\leq (1 + \epsilon \sqrt{m} \|Z\|) \left| J_{x_0}^f Q(u - v) \right| = (1 + \epsilon \sqrt{m} \|Z\|) \left| D(u - v) \right|
\]

\[
= (1 + \epsilon \sqrt{m} \|Z\|) \left| u - v \right|.
\]
Similarly,

\[ |f \circ \Phi(u) - f \circ \Phi(v)| \geq (1 - \epsilon \sqrt{m} \|Z\|) |u - v|, \quad u, v \in V, \]

which implies that for \(0 < \epsilon < (\sqrt{m} \|Z\|)^{-1}\), the restriction of the mapping \(\psi := f \circ \Phi\) to the set \(V\) is a bi-Lipschitz mapping onto the set \(f(\Phi(V)) = f(B[x_0, \delta])\) with constant \(M_\epsilon := \max\{1 + \epsilon \sqrt{m} \|Z\|, (1 - \epsilon \sqrt{m} \|Z\|)^{-1}\}\).

Since \(f\) is a homeomorphism of \(U\) onto \(f(U)\), the set \(f(B(x_0, \delta))\) is open relative to \(f(U)\). Then there is a closed ball \(B\) in \(\mathbb{R}^m\) centered at \(y_0 = f(x_0)\) such that \(B \cap f(U) \subset f(B(x_0, \delta))\). Then the set \(B \cap f(U) = B \cap f(B[x_0, \delta])\) is bi-Lipschitz homeomorphic (with constant \(M_\epsilon\)) to the set

\[ V_1 := \psi^{-1}(B \cap f(U)) = \psi^{-1}(B \cap f(B[x_0, \delta])), \]

which is compact in \(\mathbb{R}^d\). Since \(M_\epsilon \to 1\) as \(\epsilon \to 0^+\), the assertion of the lemma follows. \(\square\)

**Proof of Proposition 5.1.** Let \(W\) denote the \(d\)-dimensional \(C^1\)-manifold that contains \(A\) and let \(\epsilon > 0\) be arbitrary. In view of Definition 24 for every point \(x \in W\), there is an open neighborhood \(V_x\) of \(x\) relative to \(W\) which is homeomorphic to an open set \(U_x \subset \mathbb{R}^d\) such that the homeomorphism \(f : U_x \to V_x\) is a \(C^1\)-continuous mapping and the Jacobian matrix \(J^f_u\) (see the definition \(J^f_u\) in [24]) has rank \(d\) for every \(u \in U_x\). There is also a number \(\epsilon_x > 0\) such that \(B(x, \epsilon_x) \cap W \subset V_x\). By Lemma 7.1, there is a number \(0 < \delta(x) < \epsilon_x/2\) such that the set \(B[x, 2\delta(x)] \cap W = B[x, 2\delta(x)] \cap f(U_x)\) is bi-Lipschitz homeomorphic to a compact set \(D_x\) from \(\mathbb{R}^d\) with constant \(1 + \epsilon\).

Since \(A\) is compact, the open cover \(\{B(x, \delta(x))\}_{x \in A}\) has a finite subcover \(\{B(x_i, \delta(x_i))\}_{i=1}^p\).

Denote \(\delta_i := \min_{j=1, \ldots, p} \delta(x_j)\). Let \(x\) be any point in \(A\) and \(r \in (0, \delta_i]\). There is an index \(i\) such that \(x \in B(x_i, \delta(x_i))\). Since \(B(x, r) \cap A \subset B[x_i, 2\delta(x_i)] \cap W\), the set \(B(x, r) \cap A\) is bi-Lipschitz homeomorphic to a set \(D_i \subset D_{x_i}\) with constant \(1 + \epsilon\). If \(\varphi : B(x, r) \cap A \to D_i\) denotes the corresponding bi-Lipschitz mapping, we have \(D_i \subset B(\varphi(x), (1 + \epsilon)r)\). Then

\[ H_d(B(x, r) \cap A) \leq (1 + \epsilon)^d \mathcal{L}_d(D_i) \leq \beta_d r^d (1 + \epsilon)^{2d}. \]

Consequently,

\[ \overline{\sigma}_d(A; \delta_i) = \sup_{r \in (0, \delta_i]} \sup_{x \in A} \frac{H_d(B(x, r) \cap A)}{\beta_d r^d} \leq (1 + \epsilon)^{2d}, \]

which implies that \(\lim_{\delta \to 0^+} \overline{\sigma}_d(A; \delta) \leq 1\). \(\square\)
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