ADMISSIBLE IDEALS FOR \( k \)-LINEAR CATEGORIES

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Abstract. We generalize the notion of an admissible ideal from path algebras to (small) \( k \)-linear categories that satisfy the Krull–Remak–Schmidt–Azumaya assumption. In our treatment we first prove some general results that are analogous to general results for path algebras and admissible ideals. We then cover generalizations of relations generated by paths of length two, which we call point relations, and more general length relations. We conclude the paper with several examples and an appendix containing further discussion on length relations.

1. Introduction

Quivers with relations play a fundamental role in the representation theory of finite dimensional algebras.

Every basic finite dimensional algebra \( \Lambda \) over an algebraically closed field \( k \) is isomorphic to a path algebra \( kQ/R \) of a finite quiver \( Q \) with admissible relations \( R \). Moreover, the modules of \( \Lambda \) are in bijection with the representations of \((Q, I)\) over \( k \). Without relations, we get a correspondence only to hereditary algebras \([G]\); see also \([ARS, Ch. III.1]\).

Non-hereditary algebras are central in most fields of modern representation theory of algebras. For one, higher homological algebra requires algebras of global dimension at least 2 \([J, Ky]\). There is a rich tradition of studying classes of non-hereditary algebras, such as gentle \([AH, AS]\), clannish \([C-B]\), Schur \([E]\), preprojective \([R]\), and self–injective \([SY]\) algebras.

Continuous quivers and their representations were first explicitly studied in \([IRT]\). They are a natural generalisation of quivers, replacing finite sets of vertices with uncountably infinite sets. In the process, one gains intuition about what characteristics of representation theory come from innate properties of algebraic structures, and what comes from the discrete examples that are usually studied.

One parameter persistence modules are often defined over the real line so that persistence modules coincide with pointwise finite-dimensional representations of a continuous quiver of type \( A \) (see, for example, \([CdSGO]\)). In \([BBOS]\) the authors consider \( m \times n \) rectangular grid quivers which have the commutativity relation on each square. The authors of \([BBH]\) study homological approximations in order to obtain new invariants of these representations (persistence modules).

Given the important role of quiver relations in the representation theory of finite-dimensional algebra, it is natural to ask if relations can be extended to the continuous setting. This has already been done in a restricted sense by the second author and Zhu in \([RZ]\). We give a more general definition that works with any underlying
quiver. To capture the full generality we actually go beyond quivers and consider categories instead.

In starting this work, we were motivated by two areas of study that we intend to lift to the continuous setting: gentle algebras and $d$-cluster-tilting subcategories. In gentle algebras, the relations appear in the definition, and are always generated by compositions of two arrows. This type of relations are generalized as point relations in Section 3.1. An important class of $d$-cluster-tilting subcategories appear in the module category of type A algebras, with relations consisting of all paths above a certain length $[V]$. This type of relations is generalized as length relations in Section 3.2.

1.1. Contributions. In Section 2, we give essential background, before stating our main definition.

**Definition 1.1** (Definition 2.12). Let $C$ be a category and $I$ an ideal in $C$. We say $I$ is admissible if the following are satisfied.

1. For each $f$ in $I$, there exists a finite collection of morphisms $g_1, \ldots, g_n$ not in $I$ such that $f = g_n \circ \cdots \circ g_1$.
2. For each nonzero endomorphism $f$, if $f$ is not an isomorphism then there exists $n \geq 2 \in \mathbb{N}$ such that $f^n \in I(X,X)$.

Section 2.4 gives some general results on the quotient category $C/I$, summarized here.

**Theorem 1.2** (Propositions 2.17 to 2.19). Let $C$ be a category. Let $I$ be an admissible ideal of morphisms.

1. If $C$ is connected, then $C/I$ is also connected.
2. If $C$ is Krull–Remak–Schmidt–Azumaya, then $C/I$ is also Krull–Remak–Schmidt–Azumaya.
3. If all endomorphism rings of $C$ are artinian, then
   $$\text{Rad}(C/I) = \text{Rad}(C)/I.$$ 

In Section 3 we give two important classes of relations that can generate admissible ideals. The first is point relations, which generalizes relations of length two. The idea is that certain paths through a vertex in the quiver are excluded, but others not. For an illustration see Figure 1.

\begin{figure}
\centering
\begin{minipage}{0.4	extwidth}
\begin{tikzpicture}
  \node (a) at (0,0) {$\circ$};
  \node (b) at (1,0) {$\bullet$};
  \node (c) at (2,0) {$\circ$};
  \draw (a) -- (b) -- (c);
\end{tikzpicture}
\caption{Discrete case}
\end{minipage} \hspace{1cm}
\begin{minipage}{0.4	extwidth}
\begin{tikzpicture}
  \node (a) at (0,0) {$P$};
  \node (b) at (2,0) {$P$};
  \draw (a) to [bend left = 60] (b);
\end{tikzpicture}
\caption{Continuous case}
\end{minipage}
\caption{An illustration of point relations in the discrete and continuous case. In both cases, the relations contain all paths passing through the point $P$. See Figure 2 on page 3 for an explanation of our drawing conventions.}
\end{figure}

**Theorem 1.3** (Theorem 3.5). Let $\{P_\alpha\}$ be an admissible collection of point relations in $C$, such that any cycles are either isomorphisms (and hence trivial) or contained in at least one $P_\alpha$. Then $I = \langle \bigcup_\alpha P_\alpha \rangle$ is an admissible ideal in $C$. 


The other class of relations we define are length relations. This is a generalisation of relations generated by paths containing at least $n$ arrows, where $n$ is a natural number.

**Theorem 1.4** (Theorem 3.18). A length relation generates an admissible ideal.

Section 4 contains multiple examples of how relations work, including a sketch of their Auslander–Reiten theory.

1.2. **Future Work.** The present paper is a precursor to future work on generalizations of non-hereditary structures. Of note, the authors will consider point relations, such as Example 3.7, that generalize gentle algebras. They will also study the modding out by length relations, such as Example 3.17(3), to generate higher cluster tilting subcategories.

1.3. **Acknowledgements.** The idea for this project was conceived at the Hausdorff Research Institute of Mathematics, KMJ visited JDR at Ghent University during this project, and JDR visited KMJ at Aarhus University during this project. The authors thank each of these institutions for their hospitality. KMJ is supported by the Norwegian Research Council via the project Higher Homological Algebra and Tilting Theory (301046). JDR is supported at Ghent University by BOF grant 01P12621. The authors would like to thank Jenny August, Raphael Bennett-Tennenhaus, Charles Paquette, Amit Shah, Emine Yıldırım, and Shijie Zhu for helpful discussions.

1.4. **Conventions.** We work over $\mathbb{k} = \mathbb{k}$ be a field of characteristic 0. By $\text{Vec}(\mathbb{k})$ and $\text{vec}(\mathbb{k})$ we denote the categories of $\mathbb{k}$-vector spaces and finite-dimensional $\mathbb{k}$-vector spaces, respectively. For a $\mathbb{k}$-algebra $\Lambda$, denote by $J(\Lambda)$ the Jacobson radical of $\Lambda$. We assume $\mathcal{C}$ is a $\mathbb{k}$-linear category.

Recall that a category $\mathcal{C}$ is called Krull–Remak–Schmidt–Azumaya if any object $X$ is isomorphic to a arbitrary sum $\bigoplus X_i$, where each $\text{End}_\mathcal{C} X_i$ is a local ring, which itself is unique up to isomorphism. If every object is instead isomorphic to a finite sum $\bigoplus_{i=1}^n X_i$ as above, we say $\mathcal{C}$ is Krull–Remak–Schmidt.

Finally, we consider (discrete) quivers, continuous generalizations of such quivers, and combinations of the two. When we draw an arrow, we use a thin line with an arrow head at the end to indicate the direction. When we draw a continuous line segment, we use a bold line with the arrow head in the middle to indicate the direction; see Figure 2.

\[ \bullet \longrightarrow \bullet \quad \bullet \longrightarrow \bullet \]

**Figure 2.** On the left, how we draw arrows. On the right, how we draw line segments.

2. **Definition and General Results**

2.1. **$\mathbb{k}$-linear categorization.**

**Definition 2.1.** Let $Q$ be a (finite) quiver and $\mathbb{k}Q$ its path algebra. Let $\mathcal{Q}$ be the category whose indecomposable objects are the vertices of $Q$ and morphisms between indecomposables $i$ and $j$ are given by

$$\text{Hom}_\mathcal{Q}(i, j) = e_j \mathbb{k}Q e_i.$$. 
The objects in $Q$ are finite direct sums of the indecomposables (and 0). The morphisms in $Q$ are given by extending bilinearly. We call $Q$ the $\mathbb{k}$-linear categorification of $Q$.

**Example 2.2.** Let $Q$ be the following quiver:

\[
\begin{array}{c}
1 \\
\downarrow^{\alpha_1} \quad \alpha_2 \\
2 \\
\downarrow^{\beta_1} \quad 3 \\
\downarrow^{\beta_2} \\
4
\end{array}
\]

Then the $\mathbb{k}$-linear categorification $Q$ is a category with indecomposable objects 1, 2, 3 and 4. The morphisms in $Q$ are given by paths in $Q$, so for example we have $\text{Hom}(1, 4) \cong \mathbb{k}^2$, while $\text{Hom}(4, 1) = 0$.

**Proposition 2.3.** There is a bijection between nonzero elements in $\mathbb{k}Q$ and nonzero morphisms in $Q$.

**Proof.** A non-zero element in $\mathbb{k}Q$ is a finite sum of paths in $Q$. We can therefore define a map $F$ from the elements in $\mathbb{k}Q$ to morphisms in $Q$ by specifying the action of $F$ on paths in $Q$. We let this mapping be determined by $\text{Hom}_Q(i, j) = e_j \mathbb{k}Q e_i$. This map is a bijection by bilinearity of $Q$.

**Lemma 2.4.** Let $\text{Mod}(Q)$ be the category of functors $Q \to \text{Vec}(\mathbb{k})$. Then there exists an isomorphism of categories $\Phi : \text{Mod}(Q) \to \text{Rep}(Q)$.

**Proof.** Let $F$ be a functor in $\text{Mod}(Q)$. We now define the corresponding representation $V = \Phi(F)$. Let $M$ be the representation of $Q$ over $\mathbb{k}$ where $V(i) = F(i)$ for each $i \in Q_0$. For a path $\rho$ in $Q$, let $V(\rho)$ be the $\mathbb{k}$-linear map $F(\rho)$.

Let $f : F \to G$ be a morphism in $\text{Mod}(Q)$. Then $\Phi(f) : \Phi(F) \to \Phi(G)$ is defined by the $f_i : F(i) \to G(i)$ for each $i \in Q_0$. Straightforward computations show that $\Phi$ respects composition and so it is a functor.

Define $\Phi^{-1} : \text{Rep}(Q) \to \text{Mod}(Q)$ in the following way. For a representation $V$ of $Q$, let $F = \Phi^{-1}(V)$ be determined by $F(i) = V(i)$, for each $i \in Q_0$, and $F(\rho) = V(\rho)$ for each path in $Q$. Morphisms are defined similarly. One may check $\Phi^{-1}\Phi$ and $\Phi\Phi^{-1}$ are the identity functors on $\text{Mod}(Q)$ and $\text{Rep}(Q)$, respectively. □

2.2. The Jacobson Radical.

**Definition 2.5.** Let $C$ be a category. The radical $\text{Rad}(C)$ of $C$ is the ideal consisting of

$$\text{Rad}_C(X, Y) := \{ f \in \text{Hom}_C(X, Y) \mid \forall g \in \text{Hom}_C(Y, X), f \circ g \in J(\text{End}_C(Y))\},$$

for each pair of objects $X, Y$ in $C$.

**Proposition 2.6** (Kr). Let $X$ and $Y$ be objects in $C$. Then $\text{Rad}_C(X, Y) = J(\text{Hom}_C(X, Y))$.

**Proposition 2.7.** Let $f : X \to Y$ be an morphism for indecomposable objects $X, Y$ in $C$. Then $f \in \text{Rad}(C)$ if and only if $f$ is not an isomorphism.

**Proof.** Suppose $f$ is not an isomorphism. If $C$ does not have cycles we are done. If $C$ has cycles, let $g : Y \to X$ be a nonzero morphism. Then $f \circ g \in J(\text{End}_C(Y))$ and so $f \in \text{Rad}_C(X, Y)$. Reversing the argument shows that if $f \in \text{Rad}_C(X, Y)$ then $f$ is not an isomorphism. □
Proposition 2.8. If $C$ is Krull–Remak–Schmidt, then $C/\text{Rad}(C)$ is semi-simple.

Proof. Let $C$ be a Krull–Remak–Schmidt category. Let $X$ and $Y$ be indecomposables in $C$ such that $X \not\cong Y$. Then $\text{Hom}_C(X,Y) = \text{Rad}_C(X,Y)$ and so $\text{Hom}_{C/\text{Rad}(C)}(X,Y) = 0$. Extending bilinearly we see $C/\text{Rad}(C)$ is semi-simple. \hfill $\square$

Remark 2.9. It follows immediately from Proposition 2.8 that if $Q$ is a finite acyclic quiver then $Q/\text{Rad}(Q)$ is semi-simple.

2.3. Admissible Ideals.

Definition 2.10 ([Kr]). Let $\{C_i\}_{i \in I}$ be a family of full additive subcategories of $C$. We have an orthogonal decomposition $\coprod_{i \in I} C_i$ of $C$ if every object $X$ in $C$ is isomorphic to a direct sum $\bigoplus_{i \in I} X_i$, where $X_i$ is an object of $C_i$, and for $X_i \in C_i$, $X_j \in C_j$ we have $\text{Hom}_C(X_i,X_j) = 0$ when $i \neq j$.

We say $C$ is connected if the only orthogonal decomposition of $C$ is the trivial one.

An ideal $\mathcal{I}$ of a category $C$ is a collection of morphisms in $C$ such that for any $f \in \mathcal{I}$ and for any $g$ and $h$ such that the composition $gfh$ is defined, the composition $gfh \in \mathcal{I}$. For an ideal $\mathcal{I}$ of $C$, we denote by $\mathcal{I}(X,Y)$ the morphisms in $\text{Hom}_C(X,Y) \cap \mathcal{I}$.

Remark 2.11. For an ideal $\mathcal{I}$ of $C$, the category $C/\mathcal{I}$ has the same objects as $C$. The morphisms of $C/\mathcal{I}$ are given by $\text{Hom}_C(X,Y)/\mathcal{I}(X,Y)$. A representation $V : C/\mathcal{I} \to \text{vec}$ is also a representation of $C$ by precomposition with the quotient functor $\pi$. Thus we obtain a representation $\bar{V} : C \xrightarrow{\pi} C/\mathcal{I} \xrightarrow{V} \text{vec}$. Hence we may consider the representations of $C/\mathcal{I}$ as a subcategory of the representations of $C$. In particular, representations of $C/\mathcal{I}$ are those representations $V$ of $C$ such that if $f \in \mathcal{I}$ then $V(f) = 0$.

Definition 2.12. Let $C$ be a $k$-linear, Krull–Remak–Schmidt–Azumaya category and $\mathcal{I}$ an ideal in $C$. We say $\mathcal{I}$ is admissible if the following are satisfied.

1. For each $f$ in $\mathcal{I}$, there exists a finite collection of morphisms $g_1, \ldots, g_n$ not in $\mathcal{I}$ such that $f = g_n \circ \cdots \circ g_1$.
2. For each indecomposable $X$ in $C$, the endomorphism ring $\text{End}_C(X)/\mathcal{I}(X,X)$ is finite-dimensional.

We remark that, in Definition 2.12(2), we do not want to require that there is some $n$ that works for all nonisomorphism endomorphisms $f$. See Example 4.6 for an explicit example why.

Lemma 2.13. Let $C$ be a Krull–Remak–Schmidt category and $\mathcal{I}$ an ideal in $C$. If $\mathcal{I}$ is admissible, then $\mathcal{I}$ is contained in the radical of $C$.

Proof. Let $f : X \to Y$ be a morphism in $\mathcal{I}$ between indecomposable objects. Then we know by Proposition 2.7 that if $f$ is not contained in the radical, it is an isomorphism. However, if $f$ is an isomorphism, we have $1_X \in \mathcal{I}$. Then every morphism to/from $X$ is in $\mathcal{I}$. Thus, if $1_x = g_n \circ \cdots \circ g_1$ for any composition, both $g_n, g_1 \in \mathcal{I}$, which contradicts condition Definition 2.12(1). Hence $\mathcal{I}(X,Y) \subseteq \text{Rad}(X,Y)$ for indecomposable $X,Y$. 

Now let \( X = \bigoplus_{i=0}^m X_i \) and \( Y = \bigoplus_{j=0}^n Y_j \), where each \( X_i \) and \( Y_j \) is indecomposable. Consider \( f \in \mathcal{I}(X,Y) \). We can rewrite \( f \) as \( f = (f_{ij}) \), where \( f_{ij} : X_i \rightarrow Y_j \). By composition with the canonical injections and projections, we see that \( f_{ij} \in \mathcal{I}(X_i,Y_j) \), so by the above, \( f_{ij} \in \operatorname{Rad}(X_i,Y_j) \). Then by linearity, \( f \in \operatorname{Rad}(X,Y) \). □

Let \( Q \) be a finite quiver and let \( \mathcal{I} \) be its \( \mathbb{k} \)-linear categorification. Suppose \( I \) is an ideal of the path algebra \( \mathbb{k}Q \). We show how to build an ideal \( \mathcal{I} \) in \( Q \) from \( I \).

From the definition of the \( \mathbb{k} \)-linear categorification, we know that each path in \( I \) corresponds to a non-zero morphism in \( Q \), see Proposition 2.3. We (naïvely) let \( \mathcal{I} \) be the set of morphisms obtained by mapping \( I \) to \( Q \). We now show that \( \mathcal{I} \) is an ideal of the category \( Q \).

By \( \mathbb{k} \)-linearity of \( Q \), it is enough to consider morphisms between indecomposable objects. Suppose \( f \in \mathcal{I}(i,j) \) for some \( i,j \in Q_0 = \text{Ind} \mathbb{Q} \), and let \( g : j \rightarrow k \) and \( h : l \rightarrow i \) be two nonzero morphisms in \( Q \). By Proposition 2.3 we know that \( f \) corresponds to an element \( \rho \) in \( e_j \mathbb{k}Q e_i \). Further, \( g \) corresponds to an element \( \psi \) in \( \mathbb{k}Q e_j \) and \( h \) corresponds to an element \( \phi \) in \( e_i \mathbb{k}Q \). Each of \( \rho, \phi, \psi \) are are sums of paths in \( Q \) from the respective source and to the respective target. Without loss of generality, due to \( \mathbb{k} \)-linearity, suppose each of \( \rho, \phi, \psi \) is a path in \( Q \). We see \( \psi \rho \phi \) is an element of \( I \) since \( I \) is a two-sided ideal containing \( \rho \). The image of the composition \( \psi \rho \phi \) is the composition \( g f h \), which must therefore be in \( \mathcal{I} \).

**Proposition 2.14.** Let \( Q \) be a finite quiver and \( I \) an ideal of \( \mathbb{k}Q \) as a path algebra. Let \( \mathcal{I} \) be the \( \mathbb{k} \)-linear category induced by \( I \) and let \( \mathcal{I} \) be the ideal induced by \( I \) in \( \mathcal{I} \). Then \( \mathcal{I} \) is an admissible ideal of \( Q \) as in Definition 2.12 if and only if \( I \) is an admissible ideal of \( \mathbb{k}Q \).

**Proof.** Let \( I \) be an admissible ideal of \( \mathbb{k}Q \) and \( \rho \in I \). We first prove \( \mathcal{I} \) satisfies property (1) of Definition 2.12. Without loss of generality, assume \( \rho \) is a path in \( Q \). Then \( \rho = \alpha_n \alpha_{n-1} \cdots \alpha_2 \alpha_1 \), where each \( \alpha_i \) is an arrow in \( Q \). Now, let \( f \) be the morphism in \( Q \) corresponding to \( \rho \) and \( g \) the morphism in \( Q \) corresponding to \( \alpha_i \), for each \( i \). Then we know each \( g_i \notin \mathcal{I} \) and have satisfied property (1) of Definition 2.12. Reversing the argument proves the converse.

If \( Q \) has no cycles then the proposition immediately holds for property (2) of Definition 2.12. So, suppose \( Q \) has at least one oriented cycle. Since \( I \subset \operatorname{Rad}^n(\mathbb{k}Q) \), for some \( n \geq 2 \), we see that \( \mathcal{I} \) must immediately satisfy property (2) of Definition 2.12.

Now suppose \( \mathcal{I} \) satisfies Definition 2.12(2). Since \( Q \) is finite, there are finitely many cycles. For each cycle \( \rho \) at each vertex \( i \), let
\[
m_{\rho} = \min\{m \mid \rho^m \in \mathcal{I}(i,i)\}.
\]
We know such an \( m_{\rho} \) exists since \( \operatorname{End}_Q(i) = \mathcal{I}(i,i) \) is finite-dimensional. Let \( n_{\rho} \) be \( m_{\rho} \) time the length of \( \rho \). Then let
\[
N = \max_{\rho} \{n_{\rho} \} \cup \{\text{length of longest path without cycles in } Q\}.
\]
Thus, \( \operatorname{Rad}^N(\mathbb{k}Q) \supset I \). This concludes the proof. □

**Remark 2.15.** The second half of the proof above can be extended to more general quivers. Suppose \( Q \) is the \( \mathbb{k} \)-linear categorification of a (not necessarily finite) quiver \( Q \) with finitely many cycles. Suppose that for each cycle \( \rho \) in the quiver with
corresponding morphism \( f_\rho : X \to X \), there is some \( n \geq 2 \) such that \( f_\rho \in \mathcal{I}(X,X) \).
Then \( \mathcal{I} \) satisfies criterion (2) in Definition 2.12.

For the majority of our examples, this will be the criterion we actually use.

**Example 2.16.** Consider the quiver from Example 2.2. Let \( I \) be the commutative ideal generated by \( \{ \alpha_2 \alpha_1 - \beta_2 \beta_1 \} \).

In the \( k \)-linear categorification, the relation \( \alpha_2 \alpha_1 - \beta_2 \beta_1 \) can be written as \( [ \alpha_2 \beta_2 ] [ - \alpha_1 \beta_1 ] \). The ideal generated by this morphism fulfills the criteria for being an admissible ideal.

### 2.4. General Results

**Proposition 2.17.** Let \( \mathcal{C} \) be connected, and let \( \mathcal{I} \) be an admissible ideal of morphisms. Then \( \mathcal{C}/\mathcal{I} \) is connected.

**Proof.** Assume towards a contradiction that \( \mathcal{C}/\mathcal{I} \) is not connected; then there exists a decomposition of \( \mathcal{C}/\mathcal{I} \) into mutually orthogonal subcategories \( \mathcal{C}_1', \ldots, \mathcal{C}_n' \). We can lift these subcategories to subcategories \( \mathcal{C}_1, \ldots, \mathcal{C}_n \) of \( \mathcal{C} \). As \( \mathcal{C} \) is connected, these subcategories cannot all be mutually orthogonal, so assume that there exists some morphism \( f : X_i \to X_j \), with \( X_i \in \mathcal{C}_1, X_j \in \mathcal{C}_j \) and \( i \neq j \). To preserve mutual orthogonality in \( \mathcal{C}/\mathcal{I} \), we must have \( f \in \mathcal{I} \). Then since \( I \) is an admissible ideal, we can write \( f = g_m \circ \cdots \circ g_1 \), where each of the \( g_1, \ldots, g_n \) are not in \( \mathcal{I} \).

Consider \( g_1 : X_i \to Y \) and denote its image in \( \mathcal{C}/\mathcal{I} \) by \( \overline{g_1} \). As \( \mathcal{C}/\mathcal{I} \) is not connected, we can write \( Y = \bigoplus_{i=1}^n Y_i \), with \( Y_i \in \mathcal{C}_i' \) and \( \overline{g_1} = (g_1^1, \ldots, g_1^n) \). Now, as \( \overline{g_1} \) is nonzero, \( g_1^k \) is nonzero for some \( k \). If \( k \neq i \), we have reached a contradiction. If \( k = i \), we can repeat the argument with \( \overline{g_2} \), eventually reaching a contradiction. \( \square \)

**Proposition 2.18.** Let \( \mathcal{C} \) be Krull–Remak–Schmidt–Azumaya and \( \mathcal{I} \) an admissible ideal. Then \( \mathcal{C}/\mathcal{I} \) is Krull–Remak–Schmidt–Azumaya.

**Proof.** Let \( X \) be an object in \( \mathcal{C}/\mathcal{I} \). Then \( X \) is an object in \( \mathcal{C} \), which we assume to be Krull–Remak–Schmidt–Azumaya. It follows that \( X = \bigoplus_\alpha X_\alpha \), where \( \text{End}_\mathcal{C}(X_\alpha) \) is local. If we can show that \( \text{End}_{\mathcal{C}/\mathcal{I}}(X_\alpha) \) is local, we are done.

We know that \( \text{End}_{\mathcal{C}/\mathcal{I}}(X_\alpha) = \text{End}_\mathcal{C}(X_\alpha)/\mathcal{I}(X_\alpha, X_\alpha) \). By property 1 of Definition 2.12, the identity on \( X_\alpha \) cannot be an element of \( \mathcal{I}(X_\alpha, X_\alpha) \), so \( \text{End}_{\mathcal{C}/\mathcal{I}}(X_\alpha) \) is a nonzero quotient ring of a local ring, which is local by the ideal correspondence theorem for quotient rings. \( \square \)

**Proposition 2.19.** Let \( \mathcal{C} \) be a category such that all endomorphism rings are artinian and \( \mathcal{I} \) an admissible ideal. Then

\[
\text{Rad}(\mathcal{C}/\mathcal{I}) = \text{Rad}(\mathcal{C})/\mathcal{I}.
\]

**Proof.** The equation \( \text{Rad}(\mathcal{C}/\mathcal{I}) = \text{Rad}(\mathcal{C})/\mathcal{I} \) holds if and only if \( \text{Rad}_\mathcal{C}(X,Y)/\mathcal{I} = \text{Rad}_\mathcal{C}(X,Y)/\mathcal{I} \) holds for all pairs of objects \( X,Y \in \mathcal{C} \).

First note that if \( \text{End}_\mathcal{C}(Y) \) is artinian, then

\[
\text{Rad}_\mathcal{C}(Y,Y)/\mathcal{I} = J(\text{End}_\mathcal{C}(Y))/\mathcal{I} = J(\text{End}_\mathcal{C}(Y) + \mathcal{I})/\mathcal{I} = J(\text{End}_{\mathcal{C}/\mathcal{I}}(Y)) = \text{Rad}_{\mathcal{C}/\mathcal{I}}(Y,Y).
\]
Now we can see that
\[ f \in \text{Rad}_C(X,Y)/I \iff f = f' + I(X,Y), \text{ with } f' \in \text{Rad}_C(X,Y) \]
\[ \iff f = f' + I(X,Y), \text{ s.t. } f' \circ g' \in J(\text{End}_C(Y)) \forall g' \in \text{Hom}_C(Y,X) \]
\[ \iff f \circ (g' + I(Y,X)) \in J(\text{End}_C(Y))/I \forall g' \in \text{Hom}_C(Y,X) \]
\[ \iff f \circ g \in J(\text{End}_{C/I}(Y)) \forall g \in \text{Hom}_{C/I}(Y,X) \]
\[ \iff f \in \text{Rad}_{C/I}(X,Y). \]

\[ \square \]

In particular, if \( C \) is \( k \)-linear and Krull–Remak–Schmidt, then all endomorphism rings are artinian and the above Proposition holds.

The following lemma is useful for our examples in Section 4.

**Lemma 2.20.** Let \( C \) be a small, \( k \)-linear, Krull–Remak–Schmidt–Azumaya category and \( I \) an admissible ideal in \( C \). Let \( \mathcal{J} \) be an admissible ideal in \( C/I \) and \( \tilde{\mathcal{J}} \) the set of morphisms \( f \) in \( C \) such that \( f + I \in \mathcal{J} \) in \( C/I \). Then the following hold:

1. \( \tilde{\mathcal{J}} \) contains \( I \).
2. \( \tilde{\mathcal{J}} \) is an ideal.
3. \( \tilde{\mathcal{J}} \) is admissible in \( C \).
4. \( (C/I)/\mathcal{J} \cong (C/I)/\tilde{\mathcal{J}} \).

**Proof.**

1. Let \( f \in I \). Then \( f \mapsto 0 \) in \( C/I \). Since all zero morphisms in \( C/I \) are in \( \mathcal{J} \), we see \( f \in \tilde{\mathcal{J}} \).

2. Let \( f : X \to Y \) be nonzero in \( \tilde{\mathcal{J}} \) and let \( g : Y \to Z \) be nonzero in \( C \). Then \( f + I \) and \( g + I \) are in \( \text{Mor}(C/I) \). So, \((g + I) \circ (f + I)\) is in \( \mathcal{J} \) and is equal to \( gf + I \). Then \( gf \in \tilde{\mathcal{J}} \).

3. Let \( f \in \tilde{\mathcal{J}} \). Then \( f + I \in \mathcal{J} \) and, by assumption, there exists \((g_1 + I), \ldots, (g_n + I)\) in \( \text{Mor}(C/I) \setminus \mathcal{J} \) such that
\[ f + I = (g_n + I) \circ (g_{n-1} + I) \circ \cdots \circ (g_2 + I) \circ (g_1 + I). \]

Then for each \( g_i \) there is \( h_i \in I \) such that \( g_i + h_i \mapsto g_i + I \) and
\[ f = (g_n + h_n) \circ (g_{n-1} + h_{n-1}) \circ \cdots \circ (g_2 + h_2) \circ (g_1 + h_1). \]

Since each \( g_i \notin \mathcal{J} \), we know each \( g_i + h_i \notin \tilde{\mathcal{J}} \) and so \( f \) is a finite composition of morphisms not in \( \tilde{\mathcal{J}} \). Additionally, for any nonzero, nonisomorphism endomorphism \( f \), we have \( f^n \in I \) for some \( n \in \mathbb{N} \). Then \( f^n \in \tilde{\mathcal{J}} \) by statement 1. Therefore, \( \tilde{\mathcal{J}} \) is admissible.

4. Recall \( \text{Ob}((C/I)/\mathcal{J}) = \text{Ob}(C/\tilde{\mathcal{J}}) \). We now produce a bijection between \( \text{Mor}((C/I)/\mathcal{J}) \) and \( \text{Mor}(C/\tilde{\mathcal{J}}) \) by producing bijections
\[ \phi_{X,Y} : \text{Hom}_{C/\tilde{\mathcal{J}}}(X,Y) \to \text{Hom}_{(C/I)/\mathcal{J}}(X,Y) \]
for each ordered pair \( X, Y \) of objects.

Let \( f + \tilde{\mathcal{J}} \in \text{Hom}_{C/\tilde{\mathcal{J}}}(X,Y) \). Then there exists \( g \in \tilde{\mathcal{J}} \subset \text{Mor}(C) \) such that
\[ f + g \mapsto f + \tilde{\mathcal{J}} \in \text{Mor}(C/\tilde{\mathcal{J}}). \]

If \( g \notin \mathcal{I} \) then \( f + g \mapsto f + \mathcal{I} \in \text{Mor}(C/I) \); otherwise \( f + g \mapsto f + g + \mathcal{I} \in \text{Mor}(C/I) \). In either case, \( f + g \mapsto f + \mathcal{J} \) in \( \text{Hom}_{(C/I)/\mathcal{J}}(X,Y) \).

We define \( \phi_{X,Y}(f + \tilde{\mathcal{J}}) := f + \mathcal{J} \).

It is immediate that \( \phi_{X,Y} \) is injective. Suppose \( f + \mathcal{J} \in \text{Hom}_{(C/I)/\mathcal{J}} \). Then there exists \( g + \mathcal{I} \) in \( \text{Hom}_{C/I}(X,Y) \) such that \( f + g + \mathcal{I} \mapsto f + \mathcal{J} \). Then there exists
so $h \in \text{Hom}_C(X,Y)$ such that $f + g + h \mapsto f + g + \mathcal{I}$. But this means $g + h \in \overline{\mathcal{J}}$ and so $f + (g + h) \mapsto f + \overline{\mathcal{J}}$ in $\text{Hom}_{C/\mathcal{J}}(X,Y)$. Thus, $\phi_{X,Y}$ is surjective. 

\section{Relations}

In this section we look at two types of admissible ideals: those generated by point relations (Section 3.1) and those generated by length relations (Section 3.2). These generalize a relation generated by a single path of length two and relations generated by all paths of a particular length, respectively.

\subsection{Point Relations}

Here we generalize relations generated by a single path of length 2 to a point relation (Definition 3.3) and prove this generates an admissible ideal (Theorem 3.5). We then give examples that point to a continuous version of a gentle algebra (Examples 3.6 and 3.7).

\begin{definition}
Let $f : X \to Y$ be a nonisomorphism between indecomposables in $\mathcal{C}$. A \textit{decomposition point} $Z$ in $\mathcal{C}$ is an indecomposable object such that there exists nonisomorphisms $g : X \to Z$ and $h : Z \to Y$ where $f = h \circ g$.
\end{definition}

\begin{definition}
Let $f : X \to Y$ be a nonisomorphism of indecomposables in $\mathcal{C}$, $Z$ a decomposition point of $f$, and $f = g \circ h$ such a decomposition. We call $f$ an \textit{acyclic morphism} if for all pairs $g' : X \to Z$ and $h' : Z \to Y$ such that $f = h' \circ g'$ then $h'$ and $g'$ are scalar multiples of $h$ and $g$, respectively.

Note that an acyclic morphism cannot be irreducible. (I.e., it must be a path of length at least 2 in a quiver.)
\end{definition}

\begin{definition}
Let $f$ be an acyclic morphism and $Z$ a decomposition point of $f$. Let $P$ be the set of all nonisomorphisms $g$ between indecomposables satisfying the following.
\begin{itemize}
  \item There exists $h_1$ and $h_2$ morphisms of indecomposables such that $f = h_2 \circ g \circ h_1$.
  \item We have $Z$ as a decomposition point of $g$.
\end{itemize}
Let $\mathcal{P}_{f,Z}$ be the ideal in $\mathcal{C}$ generated by $P$. We call $\mathcal{P}_{f,Z}$ the \textit{point relation through $Z$ by $f$}.
\end{definition}

\begin{definition}
Let $\{\mathcal{P}_\alpha\}$ be a collection of point relations in $\mathcal{C}$. We say $\{\mathcal{P}_\alpha\}$ is \textit{admissible} if each morphism of indecomposables appears in at most finitely-many $\mathcal{P}_\alpha$.
\end{definition}

\begin{theorem}
Let $\{\mathcal{P}_\alpha\}$ be an admissible collection of point relations in $\mathcal{C}$ and let $\mathcal{I} = \langle \bigcup_\alpha \mathcal{P}_\alpha \rangle$. Suppose also that for each indecomposable $X$ in $\mathcal{C}$, we have $\text{End}_{\mathcal{C}}(X)/\mathcal{I}(X,X)$ is finite dimensional. Then $\mathcal{I}$ is an admissible ideal.
\end{theorem}

\begin{proof}
We satisfy Definition 2.12(2) by assumption.

Now suppose $f \in \mathcal{I}$; we show that $f$ can be written as a finite composition of morphisms not in $\mathcal{I}$. Since $f$ is a finite sum of morphisms of indecomposables, we assume without loss of generality $f$ is a morphism of indecomposables. Then, by assumption there are at most finitely-many $\mathcal{P}_\alpha$ such that $f \in \mathcal{P}_\alpha$.

We proceed by induction beginning with $f$ is only in $\mathcal{P}_1$. Let $f_1$, $P_1$, and $Z_1$ be as in Definition 3.3. Then there exists morphisms $h_1$ and $h_2$ in $\text{Mor}(\mathcal{C})$ and $g \in P_1$ such that $f = h_2 \circ g \circ h_1$. Further, $g = h'_2 \circ h'_1$ where the target of $h'_1$ is $Z_\alpha$ and the source of $h'_2$ is $Z_\alpha$. So, let $g_1 = h'_1 \circ h_1$ and let $g_2 = h_2 \circ h'_2$. Note that neither $g_1$
Figure 3. The category considered in Example 3.7. The two copies of the real line have been drawn in different colours

nor $g_2$ is in $\mathcal{P}_1$. Further, neither $g_1$ nor $g_2$ is in $\mathcal{I}$ or else $f$ would be in another $\mathcal{P}_\alpha$ as well. Thus, we have our desired decomposition.

Now assume that if $f$ is in $n$ of the $\mathcal{P}_\alpha$, then $f$ is a finite composition of morphisms not in $\mathcal{I}$. Suppose $f$ is in $n+1$ of the $\mathcal{P}_\alpha$ and denote one of them by $\mathcal{P}_1$. Let $f_1$, $P_1$, and $Z_1$ be as before for $\mathcal{P}_1$. We find $g_1$ and $g_2$ as before, but they may be in $\mathcal{I}$. However, each $g_1$ and $g_2$ may only be in $n$ or fewer $\mathcal{P}_\alpha$ and so are a finite composition of morphisms not in $\mathcal{I}$. Therefore, $f$ is a finite composition of morphisms not in $\mathcal{I}$. □

Example 3.6 (Discrete quiver). Let $Q$ be a discrete quiver. Then any quadratic monomial relation in $Q$ corresponds to a point relation in the $k$-linear categorification of $Q$.

In particular, any gentle algebra can be obtained by considering a quiver with point relations.

Example 3.7 (Continuous “gentle”, crossing real lines). Consider two copies of the real line, labeled $\mathbb{R}$ and $\mathbb{R}'$. We label the numbers in $\mathbb{R}$ by $x$ and the numbers in $\mathbb{R}'$ by $x'$. Identify 0 and 0' and label the category of $k$-representations of the resulting partially ordered set by $\mathcal{C}$.

Let $\mathcal{P}$ be the point relation at 0 generated morphisms starting in $\mathbb{R}_{<0}$ and ending in $\mathbb{R}'_{>0}$. Dually, let $\mathcal{P}'$ be the point relation at 0 generated morphisms starting in $\mathbb{R}'_{<0}$ and ending in $\mathbb{R}_{>0}$. The collection $\{\mathcal{P}, \mathcal{P}'\}$ generates an admissible ideal. In later work, we will argue that this $\mathcal{C}$ with this ideal yields a continuous generalization of a gentle algebra.

Remark 3.8. If we do not assume that $\text{End}_{\mathcal{C}}(X)/\mathcal{I}(X,X)$ is finite dimensional in our hypothesis of Theorem 3.5, it is possible that we do not have an admissible ideal. See Example 4.7.

3.2. Length Relations. We now generalize relations generated by all paths of a certain length to length relations. To do this we define a way of measuring length in our category (Definitions 3.9 and 3.13) and provide examples (Examples 3.10 and 3.14). Then we define the length relations (Definition 3.15) and provide examples (Example 3.17) and prove that length relations generate admissible ideals (Theorem 3.18). In Appendix A we discuss the proof of Theorem 3.18 (Appendix A.1), why we require the specific setup that we have (Appendix A.2), and compare our notion of length to the notion of a metric on a category, introduced by Lawvere [L] (Appendix A.3).

Recall a commutative monoid $\Lambda$ is a set with an associative, commutative, binary operation $+: \Lambda \times \Lambda \to \Lambda$ and an identity $0$.

Definition 3.9. Let $\Lambda$ be a commutative monoid. We say $\Lambda$ is weakly Archimedean if it satisfies the following.

- There is a total order $\leq$ on $\Lambda$. 


• If \( \lambda \neq 0 \) then \( \lambda > 0 \).
• If \( \lambda_1 > \lambda_2 \) then, for any \( \lambda_3 \), we have \( \lambda_1 + \lambda_3 > \lambda_2 + \lambda_3 \) or \( \lambda_1 + \lambda_3 = \lambda_2 + \lambda_3 = \max \Lambda \).
• For all \( 0 < \lambda_1 < \lambda_2 \) in \( \Lambda \), there exists \( n \in \mathbb{N} \) such that
  \[
  n\lambda_1 := \lambda_1 +_\Lambda \lambda_1 +_\Lambda \ldots +_\Lambda \lambda_1 \geq \lambda_2.
  \]

**Example 3.10.** We give three examples, two of which the reader might expect.

1. The set \( \mathbb{N} \) with the usual total order and \( +_\mathbb{N} \) given in the usual way is weakly Archimedian.
2. The set \( \mathbb{R}_{\geq 0} \) with the usual total order and \( +_\mathbb{R} \) given in the usual way is weakly Archimedian.
3. Let \( \Lambda = \{0, 1, 2, \ldots, n-1, n, \infty\} \). Let \( +_\Lambda \) be given by
   \[
   \lambda_1 +_\Lambda \lambda_2 = \begin{cases} 
   \lambda_1 +_\mathbb{N} \lambda_2 & (\lambda_1 +_\mathbb{N} \lambda_2) \leq n \\
   \infty & \text{otherwise}.
   \end{cases}
   \]
   For the total order, we say \( 0 < 1 < 2 < \cdots < n-1 < n < \infty \). Then \( \Lambda \) is weakly Archimedian.

When the weakly Archimedean monoid structure is clear, we write + instead of \( +_\Lambda \).

**Definition 3.11.** Let \( \hat{\mathcal{C}} \) be a small category and ignore any additional structure. The *additive \( k \)-linearization* \( \mathcal{C} \) of \( \hat{\mathcal{C}} \) is the category such that

1. \( \text{Ob}(\mathcal{C}) \) contains finite direct sums of objects in \( \hat{\mathcal{C}} \) and
2. \( \text{Hom}_\mathcal{C}(X, Y) = \langle \text{Hom}_\hat{\mathcal{C}}(X, Y) \rangle_k \) for indecomposables \( X, Y \in \text{Ob}(\mathcal{C}) \).

The other \( \text{Hom} \) spaces and composition are given by extending bilinearly and using composition in \( \hat{\mathcal{C}} \).

Let \( \mathcal{C} \) be a small \( k \)-linear category. We say subcategory \( \hat{\mathcal{C}} \) of \( \mathcal{C} \) is a *stem category* of \( \mathcal{C} \) if \( \mathcal{C} \) is isomorphic to the additive \( k \)-linearization of \( \hat{\mathcal{C}} \).

This is similar to the use of “stem category” in [Bo]. We note that Bongartz considered stem categories of locally bounded categories. We say \( \mathcal{C} \) is *locally bounded* if, for each object \( X \in \text{Ob}(\mathcal{C}) \), we have

\[
\dim_k \left( \bigoplus_{Y \in \text{Ob}(\mathcal{C})} \text{Hom}_\mathcal{C}(X, Y) \oplus \text{Hom}_\mathcal{C}(Y, X) \right) < \infty.
\]

Immediately, we note that when we consider the \( k \)-linearization of \( \mathbb{R} \) as a category, this condition fails. In fact, many categories in the present paper are not locally bounded, especially those considered in Section 4.

**Remark 3.12.** If we consider a quiver \( Q \) as a category \( \hat{Q} \), then the additive \( k \)-linearization of \( \hat{Q} \) is precisely the \( k \)-linear categorification \( Q \) of \( Q \).

**Definition 3.13.** Let \( \mathcal{C} \) be a \( k \)-linear category, \( \hat{\mathcal{C}} \) be a stem category of \( \mathcal{C} \), and \( \Lambda \) a weakly archimedian monoid.

We say \( \mathcal{C} \) has *length in* \( \Lambda \) if there is a function \( \ell : \text{Mor}(\hat{\mathcal{C}}) \to \Lambda \) satisfying the following.

1. If \( f \in \text{Mor}(\hat{\mathcal{C}}) \) is an isomorphism then \( \ell(f) = 0 \).
Example 3.14. We give some existing examples of Definition 3.13.

(1) Let $Q$ be a quiver, $\hat{Q}$ the $k$-linear categorification of $Q$, and $\hat{\mathcal{C}}$ a category with length in $\Lambda$ with stem category $\hat{\mathcal{C}}$. Consider $\Lambda_1, \Lambda_2$ subsets of $\Lambda$ such that $\Lambda_1 \Pi \Lambda_2 = \Lambda$, $|\Lambda_1| \geq 2$, and for all $\lambda_1 \in \Lambda_1, \lambda_2 \in \Lambda_2$ we have $\lambda_1 < \lambda_2$. Then the set $\ell^{-1}(\Lambda_2)$ in $\hat{\mathcal{C}}$ generates an ideal $\mathcal{I}$ in $\mathcal{C}$. We call $\mathcal{I}$ a length relation.

Remark 3.16. It is possible that $\Lambda_1$ has no maximum element and $\Lambda_2$ has no minimum element. (Consider, for example, $\Lambda = \mathbb{Q}_{\geq 0}$.) Thus, we may not always be able to say that we are taking “paths longer than $\lambda$” for some $\lambda \in \Lambda$.

Example 3.17. We give three examples of length relations.

(1) Let $Q$ be a quiver and $\hat{Q}$ its $k$-linear categorification, which has length in $\mathbb{N}$ (Example 3.14(1)). Let $\hat{\mathcal{C}}$ be the stem category of $\hat{Q}$ seen, effectively, as $Q$ embedded in $\hat{Q}$. Let $\Lambda_1 = \{0, 1, 2\}$ and $\Lambda_2 = \{3, 4, 5, \cdots\}$. Then $\mathcal{I} = (\ell^{-1}(\Lambda_2))$ is the set of morphisms in $\hat{Q}$ generated by paths with length $\geq 3$ in $Q$.

(2) Any Nakayama algebra where the relations have constant length $l$ can be realized as the $k$-linear categorification of its underlying quiver with length relations of length $l$ in $\mathbb{N}$.

(3) Let $Q$ and $\hat{Q}$ be as in Example 3.14(2). Recall $Q$ has length in $\mathbb{R}_{\geq 0}$. Let $\Lambda_1 = [0, 4]$ and $\Lambda_2 = (4, +\infty)$. Then $(\ell^{-1}(\Lambda_2))$ is the set of morphisms in $Q$ of length strictly greater than 4.

Theorem 3.18. Let $\Lambda$ be a weakly Archimedean monoid, $\mathcal{C}$ a category with length in $\Lambda$ with stem category $\hat{\mathcal{C}}$, and $\mathcal{I}$ a length relation. If $\text{End}_{\hat{\mathcal{C}}}(X)$ is a finitely-generated monoid, for each $X \in \text{Ob}(\hat{\mathcal{C}})$, then $\mathcal{I}$ is an admissible ideal.

Proof. If $\mathcal{I} = \emptyset$ then condition (1) is vacuously satisfied. Assume $\mathcal{I} \neq \emptyset$, let $f \in \mathcal{I}$ such that $f \in \text{Mor}(\hat{\mathcal{C}})$, and let $\lambda \in \Lambda_1$ such that $\lambda > 0$. Then there is $n \in \mathbb{N}$ such that $n\lambda \geq \ell(f)$. Thus, there is some decomposition $f = g_n \circ \cdots \circ g_1$ where $\ell(g_i) \in \Lambda_1$ for each $g_i$. Thus, each $g_i$ is not in $\mathcal{I}$.

Since $\text{End}_{\hat{\mathcal{C}}}(X)$ is a finitely-generated monoid, let $m$ be the number of generators and let $\{f_i\}_{i=1}^m$ be the set of generators. Let

$$N = \max_i \{\min_n \{n\ell(f_i) \mid n\ell(f_i) \in \Lambda_2\}\}.$$
Then
\[ \dim_k(\operatorname{End}_C(X)/\mathcal{I}(X,X)) \leq m \cdot N + 1, \]
where we need the “+1” to account for the identity in \( \operatorname{End}_C(X) \). Therefore, \( \mathcal{I} \) is an admissible ideal. \( \square \)

4. Examples

Example 4.1. Consider the (discrete) quiver \( Q \) shown in Figure 4a. The relation \( \alpha_2 \alpha_1 - 2\beta_2 \beta_1 + 3\gamma_2 \gamma_1 \) generates an admissible ideal in the classical sense. In the \( k \)-linear categorification of \( Q \), we rewrite the relation as the composition of morphisms
\[
1 \begin{bmatrix} \alpha_1 \\ -2\beta_1 \\ 3\gamma_1 \end{bmatrix} \rightarrow 2 \oplus 3 \oplus 4 \begin{bmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{bmatrix} \rightarrow 5,
\]
which generates an admissible ideal.

Example 4.2. Consider the continuous analogue of Example 4.1, displayed in Figure 4b. We consider a similar relation \( \alpha - 2\beta + 3\gamma \). Let \( X, Y, \) and \( Z \) be points on the interior of the paths \( \alpha, \beta, \) and \( \gamma, \) respectively. Then \( \alpha = \alpha_2 \alpha_1 \) where \( \alpha_1 : 0 \rightarrow X \) and \( \alpha_2 : X \rightarrow 1 \). We similarly write \( \beta = \beta_2 \beta_1 \) and \( \gamma = \gamma_2 \gamma_1 \). Then the relation \( \alpha - 2\beta + 3\gamma \) can be written as the composition
\[
0 \begin{bmatrix} \alpha_1 \\ -2\beta_1 \\ 3\gamma_1 \end{bmatrix} \rightarrow X \oplus Y \oplus Z \begin{bmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{bmatrix} \rightarrow 1,
\]
and it generates an admissible ideal.

Example 4.3 (Real line with point relations on integers). Let \( Q \) be the additive \( k \)-linearization of \( \mathbb{R} \) as a category where paths move upwards. For a point \( c \in \mathbb{R} \), let the (unique) point relation at \( c \) be \( P_c \). The collection of point relations on the integers, \( \{ P_z \}_{z \in \mathbb{Z}} \), generates an admissible ideal by Theorem 3.5.

The “Auslander–Reiten space” of the representations of this quiver is shaped like a mountain range; it is a set of triangles joined at their bottom vertices, see Figure 5a.

Example 4.4 (Circle with length/Kupisch relations). Let \( Q \) be the additive \( k \)-linearization of a continuous quiver \( \hat{Q} \) of type \( \tilde{A} \) as in [HR]. Define \( \ell : \operatorname{Mor}(\hat{Q}) \rightarrow \mathbb{R}_{\geq 0} \) by \( \ell(f) = \phi - \theta + 2n\pi \) where \( f : e^{i\theta} \rightarrow e^{i\phi} \), and \( 0 \leq \phi - \theta < 2\pi \), and \( n \) is the number of full rotations around the circle at \( e^{i\phi} \) before moving to \( e^{i\phi} \). Then \( Q \) has

\[ \begin{array}{ccc}
1 & \xrightarrow{\alpha_1} & 2 \\
\downarrow \beta_1 & & \downarrow \beta_2 \\
3 & \xrightarrow{\gamma_1} & 4 \\
\end{array} \]

\( \text{(A)} \)

\[ \begin{array}{ccc}
0 & \xrightarrow{\alpha} & 1 \\
\downarrow \beta & & \downarrow \gamma \\
\end{array} \]

\( \text{(B)} \)

Figure 4. The quivers considered in Examples 4.1 and 4.2, respectively.
The "Auslander–Reiten space" of representations of the quiver from Example 4.3

The "Auslander–Reiten space" for representations of $\mathbb{R}$ modulo paths longer than $s$ and modulo paths that go through any $x \in r\mathbb{Z}$, where $r > s \in \mathbb{R}_{>0}$.

Figure 5. The "Auslander–Reiten spaces" of representations of the quivers in Examples 4.3 and 4.5

Figure 6. The circle with length relations as described in Example 4.4. In this figure, the relations have length $\frac{2\pi}{3}$.

length in $\mathbb{R}_{\geq 0}$. If $Q$ is acyclic, we may replace $\mathbb{R}_{\geq 0}$ with $\Lambda = [0, 2\pi) \cup \{\infty\}$ and define $+_\Lambda$ similarly to Example 3.10(3).

Now assume $\hat{Q}$ has cyclic orientation. Let $\kappa$ be a Kupisch function as in [RZ, Definition 3.9]. That is, $\kappa : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ is a function such that $\kappa(t) + t > t$ and $\kappa(t+1) = \kappa(t)$, for all $t \in \mathbb{R}$. This yields a map $S^1 \rightarrow \mathbb{R}_{\geq 0}$ where $S^1 = [0, 1]/\{0 \sim 1\}$. If $\kappa$ is constant with value $a$, then this yields a length relation where $\Lambda_1 = [0, a]$ and $\Lambda_2 = (a, +\infty)$. If $\kappa$ is not constant, then we do not have a length relation. However, if $\kappa$ does not have any separation points [RZ, Definition 4.2], then $\kappa$ still induces an admissible ideal.

Example 4.5 (Real line with length and point relations). Let $Q$ be the additive $k$-linearization of $\mathbb{R}$ as a category where paths move upwards. Let $r, s$ be positive real numbers and for each $x \in r\mathbb{Z} \subset \mathbb{R}$, let $P_x$ be the (unique) point relation in $Q$ through $x$ and $I$ the admissible ideal generated by $\bigcup_{x \in r\mathbb{Z}} P_x$. Let $J$ be the the length relation in $Q/I$ obtained by modding out by paths of length greater than $s$. 
By Theorems 3.5 and 3.18 with Lemma 2.20 we obtain an admissible ideal \( \tilde{J} \) given by the point relations at each \( x \in r\mathbb{Z} \) and paths of length greater than \( s \).

If \( r \leq s \) then \( C/I = C/\tilde{J} \) since we cannot have a morphism of length greater than \( r \) in \( C/I \) anyway. If \( r > s \) then we obtain paths of length less than or equal to \( s \) that do not pass through any \( x \in r\mathbb{Z} \). The “Auslander–Reiten space” for the case \( r > s \) is in Figure 5b; notice the similarity with Figure 5a.

**Example 4.6** (Complications with Cycles). For each \( n \in \mathbb{N} \), let \( C_n \) be circle whose radius is \( \frac{1}{2}e^{-n} \). Let \( C \) be the additive \( k \)-linearization of \( \mathbb{R} \leq 0 \wedge (\biguplus_{n \in \mathbb{N}} C_n) / \sim \), where \( \mathbb{R} \ni -n \sim 0 \in C_n \). See Figure 7 for a visual depiction.

We see that \( C \) has length in \( \mathbb{R} \geq 0 \). Let \( I \) be a length relation. Since our length is in \( \mathbb{R} \geq 0 \) we can say we are modding out by length \( > L \) or \( \geq L \) for some \( L > 0 \in \mathbb{R} \).

Notice that for each \( N \in \mathbb{N} \), there exists some \( C_n \) with radius \( r \) such that \( Nr < L \). Therefore, there is no natural number \( n \) such that for all nonisomorphism endomorphisms \( f \) we have \( f^n \in I \).

**Example 4.7** (Big wedge). Let \( \mathbb{C} \) be a cyclic continuous quiver of type \( \tilde{A} \) as in [HR]. Let \( \tilde{Q} = (\biguplus N \mathbb{C}) / \sim \) where we join all the copies of the \( \mathbb{C} \) together at one point. Denote the wedge point by \( X \). Let \( Q \) be the additive \( k \)-linearization of \( \tilde{Q} \). Let us discuss the construction of an admissible ideal out of point relations and a length relation.

Notice \( Q \) has length in \( \mathbb{R} \geq 0 \). However, since the endomorphism ring of \( X \) is an infinitely-generated monoid in \( \tilde{Q} \), we see that \( \text{End}_\mathbb{C}(X)/I(X,X) \) is not finite-dimensional. Instead, we must add a point relation on all but finitely-many different copies of \( \mathbb{C} \). If we do not have a point relation on all the cycles, we also need a length relation. Without such a combination, it is not possible to build an admissible ideal out of point relations and a length relation.

### 4.1. The Real Plane.

We now consider a continuous version of the grid quiver with commutativity relations as examined in [BBOS]. Let \( \mathbb{Q} \) be the category whose objects are points in \( \mathbb{R}^2 \).

Hom sets are given by considering paths made of up of horizontal and vertical line segments. For a pair \((x, y)\) and \((z, w)\), consider the set \( P_{x,y}^{z,w} \) of all finite sequences \( \{(x_i, y_i)\}_{i=1}^n \) such that

- \((x_1, y_1) = (x, y)\) and \((x_n, y_n) = (z, w)\),
- \(x_1 \leq x_2 \leq \cdots \leq x_n\) and \( y_1 \leq y_2 \leq \cdots \leq y_n\),
- \((x_i, y_i) \neq (x_{i+1}, y_{i+1})\) for all \( 1 \leq i < n \),
- \(x_1 = x_2\) or \(y_1 = y_2\),
• for all \(1 \leq i < n - 1\), if \(x_i = x_{i+1}\) then \(y_{i+1} = y_{i+2}\), and
• for all \(1 \leq i < n - 1\), if \(y_i = y_{i+1}\) then \(x_{i+1} = x_{i+2}\)

We define

\[
\text{Hom}_\hat{\mathcal{Q}}( (x, y), (z, w)) = P^z_w^{x,y}.
\]

Note that \(P^z_w^{x,y}\) may be empty. If either \(x > z\) or \(y > w\), then

\[
\text{Hom}_\hat{\mathcal{Q}}( (x, y), (z, w)) = \emptyset.
\]

If (i) \(x = z\) and \(y < w\) or (ii) \(x < z\) and \(y = w\), then

\[
\text{Hom}_\hat{\mathcal{Q}}( (x, y), (z, w)) = \{(x, y), (z, w)\}.
\]

If \((x, y) = (z, w)\) then

\[
\text{Hom}_\hat{\mathcal{Q}}( (x, y), (x, y)) = \{(x, y)\}.
\]

Composition is given by concatenating sequences and, if necessary, deleting a repeated term.

Let \(\mathcal{Q}\) be the additive \(k\)-linearization of \(\hat{\mathcal{Q}}\) and let \((x, y), (z, w) \in \mathbb{R}^2\) such that \(x < z\) and \(y < w\). Then, we define

\[
\mathcal{I}( (x, y), (z, w)) = \langle \{(x_i, y_i)\}_{i=1}^m - \{(x'_j, y'_j)\}_{j=1}^n | \{(x_i, y_i)\}_{i=1}^m \neq \{(x'_j, y'_j)\}_{j=1}^n \rangle.
\]

If \(x = z\) or \(y = w\), then \(\mathcal{I}( (x, y), (z, w)) = 0\).

Consider

\[
\{(x_i, y_i)\}_{i=1}^m - \{(x'_j, y'_j)\}_{j=1}^n \in \mathcal{I}( (x, y), (z, w))
\]

\[
\{(z_k, y_k)\}_{k=1}^p - \{(z'_i, z'_i)\}_{i=1}^q \in \mathcal{I}( (z, w), (u, v)).
\]

We show the composition is in \(\mathcal{I}( (x, y), (u, v))\).

\[
\{(z_k, y_k)\}_{k=1}^p - \{(z'_i, z'_i)\}_{i=1}^q \circ \{(x_i, y_i)\}_{i=1}^m - \{(x'_j, y'_j)\}_{j=1}^n
\]

\[
= \{(z_k, y_k)\} \circ \{(x_i, y_i)\} - \{(z_k, y_k)\} \circ \{(x'_j, y'_j)\}
\]

\[
- \{(z'_i, z'_i)\} \circ \{(x_i, y_i)\} + \{(z'_i, z'_i)\} \circ \{(x'_j, y'_j)\}.
\]

We know

\[
\{(z_k, y_k)\} \circ \{(x_i, y_i)\} \neq \{(z_k, y_k)\} \circ \{(x'_j, y'_j)\}
\]

and

\[
\{(z'_i, z'_i)\} \circ \{(x_i, y_i)\} \neq \{(z'_i, z'_i)\} \circ \{(x'_j, y'_j)\}.
\]

Thus, the composition is in \(\mathcal{I}( (x, y), (u, v))\).

Consider \(\{(x_i, y_i)\}_{i=1}^m - \{(x'_j, y'_j)\}_{j=1}^n \in \mathcal{I}( (x, y), (z, w))\). By assumption there is \(1 \leq k \leq m\) and \(1 \leq l \leq n\) such that \((x_k, y_k) \neq (x'_l, y'_l)\). Let

\[
f : (x, y) \to (x_k, y_k) \oplus (x'_l, y'_l) = \left[ \{(x_i, y_i)\}_{i=1}^k \right]
\]

\[
g : (x_k, y_k) \oplus (x'_l, y'_l) \to (z, w) = \left[ \{(x_i, y_i)\}_{i=k}^m \right] \left[ \{(x'_j, y'_j)\}_{j=1}^n \right] .
\]

Notice \(f \notin \mathcal{I}( (x, y), (x_k, y_k) \oplus (x'_l, y'_l))\), \(g \notin \mathcal{I}( (x_k, y_k) \oplus (x'_l, y'_l), (z, w))\), and

\[
\{(x_i, y_i)\}_{i=1}^m - \{(x'_j, y'_j)\}_{j=1}^n = g \circ f.
\]

Thus, every morphism in \(\mathcal{I}\) is given by a finite composition of morphisms not in \(\mathcal{I}\). Furthermore, there are no cycles in \(\mathcal{Q}\). Thus, \(\mathcal{I}\) is an admissible ideal.
The resulting $Q/\mathcal{I}$ is the category where the objects are finite direct sums of points in $\mathbb{R}^2$ and Hom spaces between points is given by

$$\text{Hom}_{Q/\mathcal{I}}((x,y),(z,w)) = \begin{cases} k & x \leq z \text{ and } y \leq w \\ 0 & \text{otherwise.} \end{cases}$$

This means that $Q/\mathcal{I}$ is the continuous generalization of (finite) discrete commutative grid quivers as examined in [BBOS].

**Appendix A. More on Length Relations**

In this appendix we first discuss why the proof of Theorem 3.18 fails if we do not require the weakly archimedian property or the finitely-generated monoid condition (Appendix A.1). Then we discuss why we require that every element $\lambda$ in a chosen $\Lambda$ must be the sum of finitely many elements $\lambda_1, \ldots, \lambda_n$ using an example (Appendix A.2). Finally, we compare our definition of a category $C$ having length in $\Lambda$ to the notion of a metric on the category $C$, originally introduced by Lawvere [L] (Appendix A.3).

**A.1. Discussion of Proof of Theorem 3.18.** We first discuss the weakly Archimedian property and then discussion the assumption on the monoid $\text{End}_{\hat{C}}(X)$.

Note the weakly Archimedian property is essential to the proof of Theorem 3.18. If $\Lambda$ were simply a commutative monoid that respected some total order, we would not be guaranteed either of the two properties of an admissible ideal. We present two examples that fail each of requirements (1) and (2) in Definition 2.12 without failing the other.

First we consider an example where a morphism $f \in \mathcal{I}$ may not be a finite composition of morphisms not in $\mathcal{I}$. Consider representations of $\mathbb{R} \cup \{+\infty\}$ with length of morphisms defined in the following way. Suppose $\Lambda = \mathbb{R}_{\geq 0} \cup \{\infty\}$ where $a +_R b$ is the standard addition when $a, b \in \mathbb{R}$ and $a +_R b = \infty$ if $a = \infty$ or $b = \infty$. The order on $\Lambda$ is the usual order on $\mathbb{R}$ and $x < _R \infty$ for all $x \in \mathbb{R}$. Suppose $\Lambda_2 = \{\infty\}$. Then each morphism $f : +_R \infty \to x$ has $\ell(f) = \infty$ but there is no finite composition $g_n \circ \cdots \circ g_1 = f$ where $\ell(g_i) < \infty$ for each $g_i$. Thus $\ell^{-1}(\Lambda_2)$ fails requirement (1) but satisfies requirement (2).

Now we consider an example where all powers of loops are not in $\mathcal{I}$. Let $Q$ be a quiver with loops and let $\Lambda = \mathbb{N} \cup \{\infty\}$. Let $Q$ be the categorification of $Q$ with length in $\Lambda$ by assigning $\ell(\alpha) = 1$ for each arrow in $Q$. Define $+$ to be the usual addition in $\mathbb{N}$ and $a + b = \infty$ if $a = \infty$ or $b = \infty$. Further, the total order is the usual order on $\mathbb{N}$ and $n < \infty$ for all $n \in \mathbb{N}$. Let $\Lambda_2 = \{\infty\}$ and note that, for each loop $\alpha$ and for every $n \geq 1 \in \mathbb{N}$, we have $\alpha^n \notin \ell^{-1}(\Lambda_2)$. Thus, $\ell^{-1}(\Lambda_2)$ fails requirement (2) but satisfies requirement (1).

In Example 4.7, we see that if we have an infinite wedge of circles, and our length relation is longer than the circumference of the circle, then the endomorphism ring of the wedge point in $Q/\mathcal{I}$ cannot be finite-dimensional. This is why we add the assumption that $\text{End}_C(X)$ is a finitely-generated monoid for all $X \in \text{Ob}(\hat{C})$.

**A.2. More on the Definition of Length Relation.** Here we explain why we need the definition that we have.

Let $Q$ be the semi-continuous quiver where we have the ascending real interval $[0,100]$ and an arrow $\alpha : -1 \to 0$ (see Figure 8a). If we want to put a length on
Q we run into problems. Notice the only possible length on a path \( x \to y \), for \( x, y \in [0, 100] \), is a variation on using \( y - x \).

We cannot give the discrete arrow \( \alpha \) some infinitesimal length. This would violate property (3) of Definition 3.9, since there is no finite sum of infinitesimal lengths that would yield a real length. We also cannot arbitrarily decide that two times the infinitesimal length (or any finite multiple) is equal to some real number, since our weakly archimedean monoid must be totally ordered and the addition must respect the order.

We cannot assign a real positive length to the arrow \( \alpha \), either. Suppose the length of the arrow is \( d \). Then, for any \( \lambda < d \in \mathbb{R}_{>0} \), we cannot satisfy (3) in Definition 3.13. If we ignored this problem, then we risk disconnecting our category if we mod out by a length less than \( d \).

However, there is hope. If we are careful, we can mod out by paths that feel too long. For example, one may check that the ideal generated by
\[
\{ x \to y | x, y \in [0, 100], y - x > 3 \} \cup \{ (x \to y) \circ \alpha | x = 0, y - x > 2 \}
\]
is an admissible ideal which makes the arrow feel as though it has length 1 when we want to mod out by paths whose lengths feel greater than 3.

Suppose we add an arrow \( \beta : 0 \to -1 \). Call this new semi-continuous quiver \( Q^+ \) (see Figure 8b). Then we may produce a new admissible ideal by adding to our previous generating set:
\[
\{ x \to y | x, y \in [0, 100], y - x > 3 \} \cup \{ (x \to y) \circ \alpha | x = 0, y - x > 2 \} \cup \{ (x \to y) \circ \alpha \beta | x = 0, y - x > 0, y \neq x \} \cup \{ \alpha \beta \alpha \beta, \beta \alpha \beta \alpha \}
\]
This again gives the feeling that the arrows have length 1 and feels like we’re modding out by paths of length greater than 3 without violating our definitions.

A.3. Metric versus Length. In [L], Lawvere introduced metrics on categories by formalising the relationship between the triangle inequality \( d(b, c) + d(a, b) \geq d(a, c) \), where \( d \) is some metric, and morphism composition \( \text{Hom}_C(B, C) \otimes \text{Hom}_C(A, B) \to \text{Hom}_C(A, C) \), where \( C \) is some category. Lawvere allows for some non-symmetric constructions. However, we note that this is not where our definition and Lawvere’s definition diverge.

We show that our example \( Q^+ \) from Appendix A.1 has a metric that does not have a compatible length function (Example A.1). Then we show there is a category \( \mathcal{X} \) with length \( \ell \) in \( Q \) that does not have a compatible metric (Example A.2).

Example A.1. Consider the example \( Q^+ \) in Appendix A.1. There is a canonical (symmetric) metric to put on \( Q^+ \) given by setting \( d(x, y) \) to \( |x - y| \) in \( \mathbb{R} \) and then
taking the different using the Euclidean metric of \( \mathbb{R}^n \) for direct sums of points. However, as we have noted, this is not a category with length in \( \mathbb{R} \).

**Example A.2.** Let \( \mathcal{X}' \) be the category obtained by considering disjoint closed intervals in \( \mathbb{Q} \) of the following infinite sequences of lengths:

\[
\{x_i\}_0^\infty = \{3, 3.1, 3.14, 3.141, 3.1415, 3.14159, \ldots\}
\]

\[
\{y_i\}_0^\infty = \{3, 2.9, 2.86, 2.859, 2.8585, 2.85841, \ldots\}
\]

Specifically, we take disjoint copies of the intervals \([0, x_i]\) and \([0, y_i]\) in \( \mathbb{Q} \), for each \( x_i \) and \( y_i \). Notice each \( y_i = 6 - x_i \). Furthermore, in \( \mathbb{R} \), the limit of \( \{x_i\} \) is \( \pi \) and the limit of \( \{y_i\} \) is \( 6 - \pi \). Thus, neither the limit of \( \{x_i\} \) nor the limit of \( \{y_i\} \) exists in \( \mathbb{Q} \).

Now, let \( \tilde{\mathcal{X}} \) be the result of

- identifying all \( 0 \in [0, x_i] \) and \( 0 \in [0, y_i] \) together and
- identifying all \( x_i \in [0, x_i] \) and all \( y_i \in [0, y_i] \) together.

By \( 0 \) we denote the equivalence class of all \( 0 \)'s and by \( X \) we denote the equivalence class of all \( x_i \)'s and \( y_i \)'s. Notice \( \text{Hom}_X(0, X) := \mathbb{N} \times \{0, 1\} \). For any other \( x, y \in \text{Ob}(\tilde{\mathcal{X}}) \), \( \text{Hom}_X(x, y) := \text{Hom}_X(x, y) \). Let \( \mathcal{X} \) be the additive \( \k \)-linearization of \( \tilde{\mathcal{X}} \).

Take \( \Lambda = \mathbb{Q} \). For any pair \( (x, y) \) in \( \mathcal{X} \) such that \( \text{Hom}_X(x, y) \neq 0 \) and \( (x, y) \neq (0, X) \), define \( \ell(f) = |x - y| \) for each \( f \in \text{Hom}_X(x, y) \). Let \( f \in \text{Hom}_X(0, X) \) be nonzero such that \( f \) is in copy \( (i, j) \) of \( \k \), where \( (i, j) \in \mathbb{N} \times 0, 1 \). Then set

\[
\ell(f) = \begin{cases} x_i & j = 0 \\ y_i & j = 1 \end{cases}
\]

We see this generates \( \ell \) such that \( \mathcal{X} \) has length in \( \mathbb{Q} \).

Let \( L \) be the set of lengths of paths from \( 0 \) to \( X \) in \( \mathcal{X} \). We see that, in \( \mathbb{Q} \), the set \( L \) has no infimum (or supremum) so we cannot use this value to define the distance between \( 0 \) and \( X \). If we define the length between \( 0 \) and \( X \) to be greater than \( 6 - \pi \), we see there are infinitely-many paths that cause the triangle inequality to fail. Anything less than \( 6 - \pi \) yields a distance incompatible with the length of the morphisms from \( 0 \) to \( X \). Thus we would need to use the infimum, \( 6 - \pi \), which does not exist in our \( \Lambda \). Thus, we cannot create a metric \( d \) on \( \mathcal{X} \) that is compatible with the length function \( \ell \).

It is important to note that our specific Example A.2 can be “fixed” by taking lengths in \( \mathbb{R} \) instead of \( \mathbb{Q} \). However, the point is to show that if we take an arbitrary \( \Lambda \) then there is no guarantee we can construct a compatible metric directly.

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