Quantum Permanents and Quantum Hafnians
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Abstract. Analogous to the quantum general linear group, a quantum group is investigated on which the quantum determinant is shown to be equal to the quantum permanent. The quantum Hafnian is then computed by a closely related quantum permanent. Similarly the quantum Pfaffian is proved to be identical to the quantum Hafnian on the quantum algebra.

1. Introduction

In revealing the correspondence between the bosonic and fermionic versions of the Wick formula, Caianiello \[C\] introduced the Hafnian of a symmetric matrix \( C \) of even dimension as

\[
Hf(C) = \frac{1}{n!} \sum_{\sigma \in \Pi'} c_{\sigma(1)} c_{\sigma(2)} c_{\sigma(3)} c_{\sigma(4)} \cdots c_{\sigma(2n-1)} c_{\sigma(2n)},
\]

where \( \Pi' \) consists of all permutations \( \sigma \in S_{2n} \) such that \( \sigma(2i-1) < \sigma(2i), i = 1, \ldots, n \). It is known that the Hafnian has several properties similar to the Pfaffian \[LT\]. We will add one more identity \( Hf(C) = \mathrm{per}(A) \) for a closely related matrix \( A \) (cf. Theorem 3.3), which appears to be fundamental as one notes that \( Hf(C) \neq \sqrt{\mathrm{per}(C)} \) in general. Here \( \mathrm{per}(C) \) is the permanent of \( C \) \[L\] and sometimes is referred as the positive determinant \[M\] and defined by changing all the signs to +1 in the definition of \( \det(C) \).

In \[JZ\] we define the quantum Pfaffian of an even dimensional matrix \( B = (b_{ij}) \) of noncommutative entries under some quadratic relations (called \( q \)-Maya relations) as

\[
Pf_{q}(B) = \frac{1}{[n]_{q^{4}}} \sum_{\sigma \in \Pi'} (-q)^{l(\sigma)} b_{\sigma(1)} b_{\sigma(2)} b_{\sigma(3)} b_{\sigma(4)} \cdots b_{\sigma(2n-1)} b_{\sigma(2n)}.
\]

One can also formally define the \( q \)-Hafnian of a quantum symmetric matrix by the analogous formula to \[L.2\].

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Classically the relationship between Pfaffians and Hafnians goes much deeper than formal similarity. On the enveloping algebras of the orthogonal and symplectic Lie algebras, Molev and Nazarov [MN] found the interesting correspondence between Pfaffians and Hafnians in the study of Capelli identities. The goal of this note is to show that in the coordinate ring of certain generalized general linear quantum group, both the quantum Pfaffian and the quantum Hafnian can be defined. Moreover, we show that they are actually equal to each other on the special quantum group. This result is a consequence of the identity

\[(1.3) \quad \det_q(A) = \per_q(A)\]

on this quantum group (Theorem 2.1). Here \(\per_q\) is formally defined as \(\det_q\) by changing all \(-1\) to \(1\) (cf. [V]).

The new quantum group is a special root-of-unity case of the two-parameter quantum general linear group considered in [JL]. On this quantum group we will introduce a similar concept of the quantum Pfaffian and quantum Hafnian. In particular, we will show that the quantum Hafnian \(Hf_q\) also satisfies the fundamental identity

\[(1.4) \quad Hf_q(B') = \per_q(A)\]

for a closely related quantum symmetric matrix \(B'\) in terms of \(A\) (see Theorem 3.4). This identity would be a bosonic version of the identity between the quantum Pfaffian and quantum determinant proved in [JZ] if the latter is taken as the fermionic case. We expect that the new quantum group can provide a context where one can see more bosonic and fermionic identities in the quantum multilinear algebra.

2. \(q\)-Determinants and \(q\)-Permanents

2.1. Quantum determinants and permanents. Let \(q\) be a fixed generic number in the complex field \(\mathbb{C}\). The unital algebra \(A_q\) is an associative complex algebra generated by \(a_{ij}\), \(1 \leq i, j \leq n\) subject to the relations:

\[(2.1) \quad a_{ik}a_{il} = qa_{il}a_{ik},\]

\[(2.2) \quad a_{ik}a_{jk} = -qa_{jk}a_{ik},\]

\[(2.3) \quad a_{jk}a_{il} = -a_{il}a_{jk},\]

\[(2.4) \quad a_{ik}a_{jl} - a_{jl}a_{ik} = (q + q^{-1})a_{il}a_{jk},\]

where \(i < j\) and \(k < l\). The algebra \(A_q\) is a bialgebra under the comultiplication \(\Delta: A_q \rightarrow A_q \otimes A_q\) given by

\[(2.5) \quad \Delta(a_{ij}) = \sum_k a_{ik} \otimes a_{kj},\]

and the counit given by \(\eta(a_{ij}) = \delta_{ij}\). This bialgebra is formally both similar and different from the quantum coordinate ring of the general linear group (cf. [FRT], [LS]).
The quantum (row)-determinant and quantum (column)-permanent of $A$ are defined as follows.

\begin{align}
\det_q(A) &= \sum_{\sigma \in S_n} (-q)^{l(\sigma)} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}, \\
\per_q(A) &= \sum_{\sigma \in S_n} q^{l(\sigma)} a_{\sigma(1),1} \cdots a_{\sigma(n),n}.
\end{align}

It is easy to see that both are group-like elements:

\begin{align*}
\Delta(\det_q(A)) &= \det_q(A) \otimes \det_q(A), \\
\Delta(\per_q(A)) &= \per_q(A) \otimes \per_q(A).
\end{align*}

In fact, we have the following result.

**Theorem 2.1.** In the bialgebra $A_q$, one has that

\[ \det_q(A) = \per_q(A), \]

and $\det_q(A)a_{ij} = (-1)^{i+j}a_{ij} \det_q(A)$.

To show this result, we use the technique of $q$-forms [Ma, JZ] and introduce two copies of commuting quantum exterior algebras. The first one is $\Lambda = \mathbb{C}[x_1, \ldots, x_n]/I$, where $I$ is the ideal $< x_i^2, qx_i, x_j - x_jx_i | 1 \leq i < j \leq n >$ and write $x \wedge y = x \otimes y \mod I$. Then we have that

\begin{align}
\delta &= x_j \wedge x_i = qx_i \wedge x_j, \\
\epsilon &= x_i \wedge x_i = 0,
\end{align}

where $i < j$. The algebra $\Lambda$ is naturally $\mathbb{Z}_{n+1}$-graded by $\deg(x_i) = 1$ and the top degree subspace is one-dimensional and spanned by $x_1 \wedge \cdots \wedge x_n$. Assume that $a_{ij}$'s commute with $x_i$'s, and let $\delta_i = \sum_{j=1}^{n} a_{ij} x_j$, then $\delta_i$ also satisfy (2.8-2.9). Therefore

\[ \delta_1 \wedge \cdots \wedge \delta_n = \per_q(A)x_1 \wedge \cdots \wedge x_n. \]

The second quantum exterior algebra $\Lambda'$ is the unital associative algebra $\mathbb{C}[y_1, \ldots, y_n]/J$, where $J$ is the ideal $< y_i^2, qy_iy_j + y_jy_i | 1 \leq i < j \leq n >$. Using similar convention as $x_i$'s, the relations are

\begin{align}
\epsilon &= y_j \wedge y_i = -qy_i \wedge y_j, \\
\epsilon &= y_i \wedge y_i = 0,
\end{align}

where $1 \leq i < j \leq n$. The algebra $\Lambda'$ is also $\mathbb{Z}_{n+1}$-graded by $\deg(x_i) = 1$ with the top degree 1-dimensional subspace spanned by the single vector $y_1 \wedge \cdots \wedge y_n$. Assume that $y_i$'s commute with $a_{ij}$ and set $\partial_i = \sum_{j=1}^{n} a_{ij} y_j$, then $\partial_i$ satisfy the quantum exterior relations (2.11-2.12). Subsequently

\[ \partial_1 \wedge \cdots \wedge \partial_n = \det_q(A)y_1 \wedge \cdots \wedge y_n. \]

These properties can be pursued as in the usual quantum coordinate ring. In fact, the following result holds.
Proposition 2.2. The entries of the matrix $A$ satisfy the relations (2.14) of the quantum algebra $A_q$ if and only if $\delta_i = \sum_j a_{ij}x_j$ and $\partial_i = \sum_j a_{ij}y_j$ satisfy the relations (2.8-2.9) and (2.11-2.12) respectively.

We can now prove Theorem 2.1. Suppose that $\Lambda$, $\Lambda'$ and $A$ commute with each other and generate a $\mathbb{Z}_{n+1} \times \mathbb{Z}_{n+1}$-bigraded algebra $A = A_q \otimes \Lambda \otimes \Lambda'$ with $A_{(i,j)} = A_q \otimes \Lambda_i \otimes \Lambda'_j$, where $\Lambda_i$ (resp. $\Lambda'_j$) is the degree $i$ subspace of $\Lambda$ (resp. $\Lambda'$). We will simply write the general monomial element $a \otimes x \otimes y$ as $axy$, where $a \in A_q$, $x \in \Lambda$, $y \in \Lambda'$. Consider the following special linear element $\Omega$ in $A_q \otimes \Lambda \otimes \Lambda'$:

$$\Omega = \sum_{i,j=1}^n a_{ij}x_i y_j \in A_{(1,1)}.$$

Let $\omega_i = x_i \partial_i = \sum_{j=1}^n a_{ij}x_i y_j \in A_{(1,1)}$. If follows from (2.8-2.9) and the commutation relations of $\partial_i$ that

\begin{align*}
(2.14) & \quad \omega_i \wedge \omega_i = 0, \quad i = 1, \ldots, n, \\
(2.15) & \quad \omega_j \wedge \omega_i = -q^2 \omega_i \wedge \omega_j, \quad 1 \leq i < j \leq n.
\end{align*}

Note that $\Omega = \sum_{i=1}^n \omega_i$. Using (2.14, 2.15), we have that

$$\wedge^n \Omega = \left( \sum_{\sigma \in S_n} (-q^2)^{l(\sigma)} \omega_1 \wedge \cdots \wedge \omega_n \right)$$

$$= \left[ [n] - q^2 \right] \left( \frac{1}{x_1 \cdots x_n} \right) \left( \partial_1 \wedge \cdots \wedge \partial_n \right)$$

$$= \left[ [n] - q^2 \right] \text{det}_q(A) x_1 \wedge \cdots \wedge x_n y_1 \wedge \cdots \wedge y_n,$$

where $[n]_v = [n]_v \cdots [1]_v$, $[n]_v = 1 + v + \cdots + v^{n-1}$ for any variable $v$, and $l(\sigma)$ is the number of inversions of the permutation $\sigma$. Similarly we can write $\Omega = \sum_{i=1}^n \omega'_i$ with $\omega'_i = \delta_i y_i = \sum_{j=1}^n a_{ji}x_j y_i$. It is easy to see that the elements $\omega'_i$ satisfy the same quantum exterior algebra (2.14, 2.15). Thus

$$\wedge^n \Omega = \left[ [n] - q^2 \right] \omega'_1 \wedge \cdots \wedge \omega'_n$$

$$= \left[ [n] - q^2 \right] \left( \delta_1 \wedge \cdots \wedge \delta_n \right) \frac{1}{y_1 \cdots y_n}$$

$$= \left[ [n] - q^2 \right] \text{per}_q(A) x_1 \wedge \cdots \wedge x_n y_1 \wedge \cdots \wedge y_n.$$

Subsequently we get the first identity of Theorem 2.1

$$\text{det}_q(A) = \text{per}_q(A).$$

For a pair of $r$ indices $i_1, \ldots, i_r$ and $j_1, \ldots, j_r$, we define the quantum (row)-minor $A^{i_1 \cdots i_r}_{j_1 \cdots j_r}$ as in (2.6). Then for a fixed permutation $i_1 < \cdots < i_r, i_{r+1} < \cdots < i_n$ of $n$ of $n$ and $1 \leq j_1, \ldots, j_r \leq n$, the following Laplace expansion can be proved similarly as in $GL_q(n)$:

$$\text{det}_q(A) = \sum_{j_1 < \cdots < j_r, j_{r+1} < \cdots < j_n} (-q)^{j_1 + \cdots + j_r - (i_1 + \cdots + i_r)} A^{i_1 \cdots i_r}_{j_1 \cdots j_r} A^{i_{r+1} \cdots i_n}_{j_{r+1} \cdots j_n}.$$
In particular we have that for a fixed $i$

$$\det_q(A) = \sum_{j=1}^{n} (-q)^{j-i} a_{ij} A_{j}^{\hat{i}}, \quad (2.16)$$

where $\hat{i}$ means the indices $1, \cdots, i-1, i+1, \cdots, n$. Then the quasi-commutativity of $\det_q$ can be proved inductively by using the Laplace expansion (2.16). Thus Theorem 2.1 is proved.

2.2. Quantum group $\mathcal{A}_q$. By defining the antipode

$$S(a_{ij}) = (-q)^{i-j} A_{i}^{\hat{j}} \det_q^{-1},$$

one can check that the bialgebra $\mathcal{A}_q[\det_q^{-1}]$ is a Hopf algebra and defines a new quantum group in the sense of Drinfeld. In fact, $AS(A) = S(A)A = I$ follows from the Laplace expansion for the row determinant and column permanent respectively as well as the anti-commutation relations (2.3).

As for the quantum permanent, we also have the Laplace expansion for a fixed permutation $i_1 < \cdots < i_r, i_{r+1} < \cdots < i_n \in S_n$

$$\text{per}_q(A) = \sum_{j_1 < \cdots < j_r, j_{r+1} < \cdots < j_n} q^{(j_1 + \cdots + j_r) - (i_1 + \cdots + i_r)} pA_{j_1 \cdots j_r} A_{i_1 \cdots i_r},$$

where $pA$ refers to the $q$-minor permanent of $A$ in the concerned rows and columns. In particular, we also have that for a fixed $i$

$$\text{per}_q(A) = \sum_{j=1}^{n} q^{j-i} a_{ji} pA_{i}^{\hat{j}}. \quad (2.17)$$

In [JL] the generic two-parameter quantum linear group was studied. Our new quantum group appears to be a root of unity case there, however one needs to be careful as some of the arguments are not applicable in this degenerate case.

We remark that our current example offers new identities to the quantum multilinear algebra studied in [JZ] (see [LT] for several interesting identities in the classical situation; also see [TT, Z, HH] on usual quantum groups). We also remark that if $q$ is an indeterminate and $F$ is a field of $\text{char} = 0$, then the quantum group $\mathcal{A}_q$ can be defined over the rational field $F(q)$, and all results in the paper hold in that situation.

3. $q$-Pfaffians and $q$-Hafnians

3.1. $q$-Pfaffians. We recall the notion of the quantum Pfaffian $\text{Pf}_q$ following [JZ]. Let $\mathcal{B}_q$ be the unital associative algebra generated by $b_{ij}$, $1 \leq i < j \leq 2n$ subject to the quadratic relations

$$b_{ij}b_{kl} - qb_{ik}b_{jl} + q^{2}b_{il}b_{jk} = b_{kl}b_{ij} - q^{-1}b_{jl}b_{ik} + q^{-2}b_{jk}b_{il}, \quad (3.1)$$

for $1 \leq i < j < k < l \leq 2n$. We refer them as the $q$-Maya relations or $q$-Plücker relations. Assume that $b_{ij}$ commute with the algebra $\Lambda'$ generated by $y_1, \cdots, y_{2n}$, and consider the algebra $\mathcal{B}_q \otimes \Lambda'$.
Let $\Omega = \sum_{1 \leq i < j \leq 2n} b_{ij}(y_i \wedge y_j) \in B_q \otimes \Lambda'$. The quantum Pfaffian is defined by
\begin{equation}
\wedge^n \Omega = [n]_q! \text{Pf}_q(B) y_1 \wedge y_2 \wedge \cdots \wedge y_{2n}.
\end{equation}
Explicitly the quantum Pfaffian is given by [JZ]:
\begin{equation}
\text{Pf}_q(B) = \frac{1}{[n]_q!} \sum_{\sigma \in \Pi} (-q)^{l(\sigma)} b_{\sigma(1)\sigma(2)} b_{\sigma(3)\sigma(4)} \cdots b_{\sigma(2n-1)\sigma(2n)},
\end{equation}
where $\Pi$ is the set of permutations $\sigma$ of $2n$ such that $\sigma(1) < \sigma(2), \ldots, \sigma(2n-1) < \sigma(2n)$. The subset $\Pi$ restricts further that $\sigma(1) < \sigma(3) < \cdots < \sigma(2n-1)$.

In the quantum coordinate ring $A_q(2n)$, we let for $1 \leq i < j \leq 2n$
\begin{equation}
b_{ij} = \sum_{m=1}^{n} A_{i,j}^{2m-1,2m} = \sum_{m=1}^{n} (a_{2m-1,i} a_{2m,j} - q a_{2m-1,j} a_{2m,i}),
\end{equation}
then the elements $b_{ij}$ satisfy the $q$-Maya relations (3.1). Furthermore, additional quantum anti-symmetric relations are also held by $b_{ij}$. It was shown [JR] that $\text{Pf}_q(b_{ij}) = \text{det}_q(a_{ij})$ by using representation theory of the quantum enveloping algebra. Here we give an independent proof on the new quantum group.

**Theorem 3.1.** On the quantum coordinate ring $A_q(2n) = \langle a_{ij} | 1 \leq i, j \leq 2n \rangle/ \sim$ one has that
\[ \text{Pf}_q(B) = \text{det}_q(A), \]
where $b_{ij}$ are defined by (3.3).

**Proof.** Consider the 2-form
\begin{equation}
\Omega = \partial_1 \wedge \partial_2 + \partial_3 \wedge \partial_4 + \cdots + \partial_{2n-1} \wedge \partial_{2n},
\end{equation}
where $\partial_i = \sum_{j=1}^{2n} a_{ij} y_j$, $(i = 1, 2, \ldots, 2n)$ obey (2.11 2.12). Note that $\partial_{2i-1} \wedge \partial_{2i}$ $(1 \leq i \leq n)$ form the $q$-exterior algebra (2.8 2.9) with $q$ replaced by $q^2$. Therefore
\[ \wedge^n \Omega = [n]_q! \partial_1 \wedge \cdots \wedge \partial_{2n} = [n]_q! \text{det}_q(A) y_1 \wedge \cdots \wedge y_{2n}, \]
where the last identity uses the wedge formulation of $\text{det}_q$. On the other hand, it follows from the quantum exterior relations of $y_i's$ that
\[ \Omega = \sum_{1 \leq i < j \leq 2n} b_{ij} y_i \wedge y_j. \]
By our wedge definition of $\text{Pf}_q$, we immediately obtain that
\[ \wedge^n \Omega = [n]_q! \text{Pf}_q(B) y_1 \wedge \cdots \wedge y_{2n}. \]
Therefore, $\text{det}_q(A) = \text{Pf}_q(B)$. \qed
### 3.2. $q$-Hafnians

Let $B'_q$ be the algebra generated by $b'_{ij}$ for $1 \leq i < j \leq 2n$ modulo the ideal generated by the $q$-Maya relations

\begin{equation}
(3.7) \quad b'_{ij}b'_{kl} + q b'_{ik}b'_{jl} + q^2 b'_{il}b'_{jk} = b'_{ki}b'_{lj} + q^{-1} b'_{kj}b'_{il} + q^{-2} b'_{jk}b'_{il},
\end{equation}

where $1 \leq i < j < k < l \leq 2n$.

We define the $q$-Hafnian by

\begin{equation}
(3.8) \quad Hf_q(B') = \sum_{\sigma \in \Pi} q^{l(\sigma)} b'_{\sigma(1), \sigma(2)} \cdots b'_{\sigma(2n-1), \sigma(2n)}.
\end{equation}

We can also evaluate $Hf_q$ inductively as follows. For $n = 2$, $Hf_q(B'[1, 2]) = b'_{12}$, then

\begin{equation}
(3.9) \quad Hf_q(B') = \sum_{j=2}^{2n} q^{j-2} Hf_q(B'[1, j]) Hf_q(B'[2, 3, \ldots, \hat{j}, \ldots, 2n]).
\end{equation}

Similarly to the quantum Pfaffian, we have that

\[ \sum_{\sigma \in \Pi'} q^{l(\sigma)} b'_{\sigma(1)\sigma(2)} \cdots b'_{\sigma(2n-1)\sigma(2n)} = [n]_q! Hf_q(B'), \]

which is an easy consequence of the following lemma (proved by the method of [JZ]).

**Lemma 3.2.** One has that

\[ Hf_q B'[1, \ldots, 2n] = [n]^{-1}_q \sum_{i<j} q^{i+j-3} Hf_q(B'[i,j]) Hf_q B'([1, \ldots, \hat{i}, \ldots, \hat{j}, \ldots, 2n]). \]

Similar to the $q$-Pfaffian, we can also use a volume form to evaluate the $q$-Hafnian. The following result is immediate from our discussion (cf. [JZ]).

**Theorem 3.3.** Let $\Omega = \sum_{1 \leq i < j \leq 2n} b'_{ij}(x_i \wedge x_j)$ and $b'_{ij}$ satisfy the $q$-Maya relations (3.7), then one has

\[ \wedge^n \Omega = [n]_q! Hf_q(B') x_1 \wedge \cdots \wedge x_{2n}. \]

We now come to the fundamental identity between the quantum Hafnian and quantum permanent.

**Theorem 3.4.** In the quantum coordinate ring $A_q(2n)$ we have

\[ Hf_q(B') = \text{per}_q(A). \]

where $b'_{ij} = \sum_{m=1}^{n} p A_{2m-1, 2m} = \sum_{m=1}^{n} (a_{i, 2m-1} a_{j, 2m} + qa_{j, 2m-1} a_{i, 2m}), i < j$.

**Proof.** This is proved by a similar method. Consider the two-form

\begin{equation}
(3.10) \quad \Omega = \delta_1 \wedge \delta_2 + \delta_2 \wedge \delta_3 + \cdots + \delta_{2n-1} \wedge \delta_{2n},
\end{equation}

where $\delta_i = \sum_{j=1}^{2n} a_{ji} x_j, (i = 1, 2, \ldots, 2n)$. On the one hand, one just verifies that

\[ \wedge^n \Omega = [n]_q! \text{per}_q(a_{ij}) x_1 \wedge x_2 \wedge \cdots \wedge x_{2n}. \]
On the other hand, the 2-form $\Omega$ is also expressed as

$$\Omega = \sum_{1 \leq i < j \leq 2n} b'_{i,j} x_i \wedge x_j,$$

so we also have

$$\wedge^n \Omega = [n]_q^n !Hf(B') x_1 \wedge x_2 \wedge \cdots \wedge x_{2n},$$

Therefore $\text{per}_q(A) = Hf_q(B')$. □

Since $\det_q(A) = \text{per}_q(A)$ on $A_q$, we have the following result.

**Theorem 3.5.** Let $b_{ij} = \sum_{m=1}^n (a_{2m-1,i}a_{2m,j} - qa_{2m-1,j}a_{2m,i})$ and $b'_{ij} = \sum_{m=1}^n (a_{i,2m-1}a_{j,2m} + qa_{j,2m-1}a_{i,2m})$, then one has that

$$\text{Pf}_q(B) = Hf_q(B')$$

on the quantum group $A_q$.

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**References**

[C] E. R. Caianiello, *Combinatorics and renormalization in quantum field theory*, Frontiers in Physics, W. A. Benjamin, Inc., Reading-London-Amsterdam, 1973.

[FRT] L. Faddeev, Yu. Reshetikhin, Tacktajan, *Quantization of Lie groups and Lie algebras*, Algebraic analysis, Vol. I, pp. 129–139, Academic Press, Boston, MA, 1988.

[HH] M. Hashimoto, T. Hayashi, *Quantum multilinear algebra*. Tohoku Math. J. (2) 44 (1992), 471–521.

[JR] N. Jing, R. Ray, *Zonal polynomials and quantum antisymmetric matrices*. Bull. Inst. Math. Acad. Sinica (N.S.) 7 (2012), 1–31.

[LL] N. Jing, M. Liu, *R-matrix realization of two-parameter quantum group $U_{r,s}(gl_n)$*. Commun. Math. Stat. 2 (2014), no. 3-4, 211–230.

[JZ] N. Jing, J. Zhang, *Quantum Pfaffians and hyper-Pfaffians*, Adv. in Math. 265 (2014), 336–361.

[LS] T. Levasseur and J. T. Stafford, *The quantum coordinate ring of the special linear group*, J. Pure Appl. Algebra 86 (1993), 181–186.

[L] D. E. Littlewood, *The theory of group characters and matrix representations of groups*. Oxford Univ. Press, New York, 1940.

[LT] J.-G. Luque and J. Y. Thibon, *Pfaffian and Hafnian identities in shuffle algebras*, Adv. in Appl. Math. 29 (2002), 620–646.

[Ma] Yu. I. Manin, *Notes on quantum groups and quantum de Rham complexes*. Teoret. Mat. Fiz. 92 (1992), 425–450; translation in Theoret. and Math. Phys. 92 (1992), 997–1023.

[M] H. Minc, *Permanents*, Encyclopedia of Mathematics and its Applications, Vol. 6, Addison-Wesley Publishing Co., Reading, Mass., 1978.

[MN] A. Molev and M. Nazarov, *Capelli identities for classical Lie algebras*, Math. Ann. 313 (1999), 315–357.

[TT] E. Taft, J. Towber, *Quantum deformation of flag schemes and Grassmann schemes. I. A $q$-deformation of the shape-algebra for $GL(n)$*, J. Algebra 142 (1991), 1–36.

[V] D. Vere-Jones, *A generalization of permanents and determinants*, Lin. Alg. Appl. 111 (1988), 119–124.
J. J. Zhang, *The quantum Cayley-Hamilton theorem*, J. Pure Appl. Algebra 129 (1998), 101–109.

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