Background field formalism for chiral matter and gauge fields conformally coupled to supergravity

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Abstract
We expand the generic model involving chiral matter, super Yang-Mills gauge fields, and supergravity to second order in the gravity and gauge prepotentials in a manifestly covariant and conformal way. Such a class of models includes conventional chiral matter coupled to supergravity via a conformal compensator. This is a first step toward calculating one-loop effects in supergravity in a way that does not require a perturbative expansion in the inverse Planck scale or a recourse to component level calculations to handle the coupling of the Kähler potential to the gravity sector. We also consider a more restrictive model involving a linear superfield in the role of the conformal compensator and investigate the similarities it has to the dual chiral model.
1 Introduction

The background approach to quantization has a long pedigree in superspace approaches to supergravity. The important work of Grisaru and Siegel [1, 2] (extended later by Grisaru and Zanon [3, 4, 5] to include off-shell background fields) showed how to expand old minimal Poincaré supergravity in terms of fundamental quantum variations about a classical background, but they restricted their consideration to old minimal supergravity alone. This is difficult enough to do given the constrained supergeometry, and its quantization requires the introduction of not only Fadeev-Popov ghosts but also ghosts for ghosts, Nielsen-Kallosh ghosts [6], and “hidden” ghosts [7] which a casual application of the Fadeev-Popov procedure might miss. The on-shell one-loop gauge-fixed quantum Lagrangian was found which allows certain simple calculations as well as the construction of covariant Feynman rules to handle more general theories perturbatively. This story is by now textbook material [8].

However, the calculation of even one-loop effects involving not only supergravity but also chiral matter and gauge fields has to our knowledge never been comprehensively undertaken in superspace. Part of this is undoubtedly the difficulty in dealing with not only the constrained structure of supergravity in superspace but also the Brans-Dicke coupling of chiral matter to the superspace Einstein-Hilbert term. In a purely Poincaré approach, this last feature requires either a component space Weyl rescaling [9] or the introduction of $U(1)$ superspace and a superfield Weyl rescaling [10]. In this respect, it is almost more straightforward to work at the component level and then to extract superspace results from the component ones. A conformal approach at the superfield level seems a more feasible method, and that is the approach we take here.

We have begun a program to attempt the calculation of one-loop corrections to an arbitrary chiral model coupled to super Yang-Mills and supergravity within superspace, thus maintaining manifest supersymmetry at all stages. In order to deal ultimately with the conformal coupling of the canonical Kähler potential in the Einstein-Hilbert term, we have shown how, in a previous work, to extend the structure group of Poincaré superspace to include the superconformal group [11]. The new conformally covariant derivatives possess an algebra which is identical to that of gauge theories: their curvatures are expressed in terms of “gaugino” superfields $W_\alpha$ and $W_\dot{\alpha}$ valued in the superconformal group, which obey a generalized chirality condition (2.2) as well as a Bianchi identity (2.3). The selection of a number of curvature constraints eliminate most of these superfields, and the ones which remain may all be described by the single chiral superfield $W_{\alpha \dot{\beta} \gamma}$, the chiral spinor field strength of conformal supergravity. The conformally covariant derivatives and their curvatures all transform covariantly under the superconformal algebra, which simplifies the calculation of superscale transformations considerably.

Were it not for the constraints on the $W_\alpha$, the structure of the theory would be quite easy to solve. In analogy with Yang-Mills, one would expect unconstrained prepotentials $V^A$, one for each member of the superconformal algebra. The constraints on the curvatures clearly must eliminate most of these prepotentials since a large volume of literature (see for example the textbooks [8, 12] as well as the original work [13]) shows that the fundamental quanta of old minimal Poincaré supergravity are the superfields $H^M = (H^m, H^\mu, H_\mu)$ and a chiral compensator $\sigma$, with a gauge invariance allowing one to algebraically eliminate $H^\mu$ and $H_\mu$. We will not attempt to solve the constraints on the full prepotentials here. Rather, as our interest is in performing one-loop calculations in a classical background, we will focus on calculating the allowed deformations of the prepotentials which preserve the curvature constraints. The degrees of freedom must, of course, be the same in either approach.
This paper is composed of three sections. In the first, we establish that the theory, like Yang-Mills, is defined in terms of prepotentials. We study arbitrary first order deformations of the prepotentials and solve for the form that leave the constraints invariant to first order. In the second section, we consider two physical actions, one involving the arbitrary coupling of chiral superfields to supergravity and the other involving the minimal linear compensator model with a Kähler potential. We construct their first order variations in terms of their fundamental quanta about a classical background and demonstrate that they possess a common structure. In the third section, we proceed to second order and present the second order variation of the action for both models, which is sufficient (after gauge fixing) for one-loop computations.

2 Prepotential formulation of conformal superspace

The algebra of the conformally covariant derivatives are

\[ \{\nabla_\alpha, \nabla_\beta\} = 0, \quad \{\nabla_\dot{\alpha}, \nabla_\dot{\beta}\} = 0 \]

\[ \{\nabla_\alpha, \nabla_\dot{\alpha}\} = -2i\nabla_{\dot{\alpha}\dot{\alpha}} \]

\[ \{\nabla_\beta, \nabla_{\dot{\alpha}\dot{\alpha}}\} = -2i\epsilon_{\dot{\beta}\dot{\alpha}} W_\alpha, \quad \{\nabla_{\dot{\beta}}, \nabla_{\dot{\alpha}\dot{\alpha}}\} = -2i\epsilon_{\dot{\beta}\dot{\alpha}} W_\alpha \]  

(2.1)

where \( W_\alpha \) are the “gaugino superfields” for the superconformal group. These superfields are covariantly chiral in the sense that

\[ \{\nabla_\dot{\alpha}, W_\alpha\} = 0, \quad \{\nabla_\alpha, W_{\dot{\alpha}}\} = 0 \]  

(2.2)

and obey the Bianchi identity

\[ \{\nabla_\alpha, W_\alpha\} = \{\nabla_\dot{\alpha}, W^{\dot{\alpha}}\} \]  

(2.3)

The structure is clearly reminiscent of Yang-Mills, except for two differences: the gauge generators \( X_B \) do not commute with the covariant derivatives \( \{X_B, \nabla_A\} \neq 0 \), and most of the \( W_\alpha \) are constrained to vanish. The combination of the constraints and the Bianchi identities then allow one to solve for the non-vanishing \( W_\alpha \) all in terms of the single chiral superfield \( W_{\alpha\beta\gamma} \).

The structure of the covariant derivatives of conformal supergravity allows a solution in terms of prepotentials that is identical in its structure to that of gauge theories. For example, (2.1) implies the existence of a chiral (+) and an antichiral (-) gauge where

\[ \nabla^{\dot{\alpha}(+)} = \partial^{\dot{\alpha}} = T \nabla^{\dot{\alpha}} T^{-1}, \quad \nabla^{(-)} = \partial_\alpha = \bar{T} \nabla_\alpha \bar{T}^{-1} \]  

(2.4)

where \( T \) and \( \bar{T} \) represent the superconformal gauge transformations taking us from an arbitrary gauge to the two special ones. Inverting these formulae gives

\[ \nabla_\alpha = \bar{T}^{-1} \partial_\alpha \bar{T}, \quad \nabla_{\dot{\alpha}} = T^{-1} \partial_{\dot{\alpha}} T \]  

(2.5)

which serve to encode the details of the connections in an arbitrary gauge in terms of a complex gauge prepotential \( T \).

It is clear that the special gauges \( T \) and \( \bar{T} \) are ill-defined up to transformations of the form

\[ T \rightarrow CT, \quad \bar{T} \rightarrow \bar{C} \bar{T} \]  

(2.6)
where $C$ is chiral ($[\partial_\alpha, C] = 0$) and $\bar{C}$ is antichiral ($[\partial_\alpha, \bar{C}] = 0$). In addition, they transform under gauge transformations as

$$T \to TG^{-1}, \quad \bar{T} \to \bar{T}G^{-1} \quad (2.7)$$

Putting these two transformations together gives a combined gauge/chiral transformation of the form

$$T \to CTG^{-1}, \quad \bar{T} \to \bar{C}TG^{-1} \quad (2.8)$$

It is convenient to define the object $U \equiv \bar{T}T^{-1}$, which represents the gauge transformation from the chiral to the antichiral gauge. That is, $\nabla^{(-)}_A = U \nabla^{(+)}_A U^{-1}$. Applying this formula and its inverse in the cases where the covariant derivative is simple leads to

$$\nabla^{(-)}_\alpha = \partial_\alpha, \quad \nabla^{(-)}_{\bar{\alpha}} = U\partial_\alpha U^{-1}$$
$$\nabla^{(+)}_\alpha = U^{-1}\partial_\alpha U, \quad \nabla^{(+)}_{\bar{\alpha}} = \partial_{\bar{\alpha}} \quad (2.9)$$

$U$ is invariant under the full gauge transformations but transforms under chiral gauge transformations as

$$U \to \bar{C}UC^{-1}. \quad (2.10)$$

A (covariantly) chiral superfield $\Phi$ is a superfield constrained to obey $\nabla_\alpha \Phi = 0$. This is not in practice a difficult constraint to satisfy. In the chiral gauge, we define the conventionally chiral superfield $\phi$ by $\phi \equiv \Phi^{(+)}$. The chirality condition is then simply the analytic statement that $\phi = \phi(x, \theta)$ is independent of $\bar{\theta}$. In any other gauge, we have

$$\Phi = T^{-1}\Phi^{(+)} = T^{-1}\phi \quad (2.11)$$

While $\Phi$ transforms under a gauge transformation as $\Phi \to G\Phi$, the conventionally chiral $\phi$ transforms as $\phi \to C\phi$ where $C$ is the chiral gauge transformation parameter. One may make an analogous statement about antichiral superfields:

$$\Phi^\dagger = \bar{T}^{-1}\Phi^{(-)} = \bar{T}^{-1}\bar{\phi} \quad (2.12)$$

Under a gauge transformation, $\Phi$ and $\Phi^\dagger$ transform covariantly while $\phi$ and $\bar{\phi}$ transform as

$$\phi \to C\phi, \quad \bar{\phi} \to \bar{C}\bar{\phi} \quad (2.13)$$

The canonical kinetic action for $\Phi$ can be rewritten in terms of the conventionally chiral superfields

$$\int E \Phi^\dagger \Phi = \int E (\bar{T}^{-1}\bar{\phi})(T^{-1}\phi) \quad (2.14)$$

Since the action is gauge-invariant (provided $\Phi$ is of scaling dimension $\Delta = 1$), we may perform a gauge transformation with parameter $G = \bar{T}$; this gives

$$\int E \bar{\phi}(\bar{T}T^{-1}\phi) = \int E \bar{\phi}(U\phi) \quad (2.15)$$

The equality of the above two statements is formally equivalent to $\bar{T}T = \bar{T}^{-1}$ where transposition is understood as moving the gauge generator off one term and onto another. (An
integration by parts, of course, has the same property.) One may use this to adopt a notation where the kinetic term is written as

\[ \Phi^i \Phi = \bar{\phi} U \phi \]  \hspace{1cm} (2.16)

where \( U \) may be understood as acting either to the right (as \( U \)) or to the left (as \( U^{-1} \)).

It is often useful to work in a Hermitian gauge. We denote such a gauge by \((0)\); it is easily found by interpolating between the chiral and antichiral gauges:

\[ \nabla^{(0)} = U^{-1/2} \partial \alpha U^{1/2}, \quad \nabla^{(0)} = U^{1/2} \partial \dot{\alpha} U^{-1/2} \]  \hspace{1cm} (2.17)

We note that it is often useful to represent \( U \) in an exponential form. We choose to define the superfield \( V^A \) by

\[ U = \exp(-2iV^A X_A) \]  \hspace{1cm} (2.18)

Under this definition, \( V^A \) is Hermitian and represents the superconformal analogue of the gauge prepotential. If the constraints \( (2.1) \) were the sole constraints on the geometry, the prepotentials \( V^A \) would be unconstrained. However, certain of the gaugino superfields \( \mathcal{W}_\alpha \) are constrained to vanish, which serves to implicitly define some of the \( V^A \) in terms of the others. Experience in Poincaré geometry tells us that \( V^a \) is the unconstrained object out of which the others are defined.\(^1\) We will not be concerned, however, with presenting a full solution of the constraints. Rather, as we are more concerned with one loop calculations around a classical background, we will seek to construct the \( V^A \) associated with the quantum deformations themselves.

### 2.1 Quantum deformations of conformal geometry

The standard recipe for quantum calculations in supergravity involves splitting the geometry into a background geometry and quantum fluctuations about that background. Since the gauge connections are encoded in \( T \) and \( \bar{T} \) (and thereby in \( U \)), splitting the former into a background and quantum contribution is accomplished by doing the same with the latter. The method of splitting we will adopt is

\[ T \rightarrow TT_Q, \quad \bar{T} \rightarrow \bar{T}\bar{T}_Q \]  \hspace{1cm} (2.19)

which corresponds to

\[ \nabla_\alpha \rightarrow \bar{T}^{-1}_Q \nabla_\alpha \bar{T}_Q, \quad \nabla_{\dot{\alpha}} \rightarrow T_Q^{-1} \nabla_{\dot{\alpha}} T_Q. \]  \hspace{1cm} (2.20)

The new covariant derivatives can then be constructed perturbatively out of the old ones. Similarly, chiral superfields transform under these variations as

\[ \Phi \rightarrow T^{-1}_Q \Phi, \quad \bar{\Phi} \rightarrow \bar{T}^{-1}_Q \bar{\Phi} \]  \hspace{1cm} (2.21)

The prepotentials transform under the combined chiral and supergauge transformations as

\[ TT_Q \rightarrow CTT_Q G^{-1}, \quad \bar{T}\bar{T}_Q \rightarrow \bar{C}\bar{T}\bar{T}_Q G^{-1}. \]  \hspace{1cm} (2.22)

\(^1\) In the literature, \( V^a \) is usually replaced with \( H^m \) and would be defined from the above with the coordinate derivative \( \partial_M \) replacing the covariant \( \nabla_A \) in the set of generators.
Just as in the component case, the gauge transformation can be interpreted as either a background or a quantum transformation. As a background transformation, we take $T$ and $\bar{T}$ to transform as

$$ T \rightarrow CTG^{-1}, \quad \bar{T} \rightarrow \bar{C}\bar{T}G^{-1}. \quad (2.23) $$

and the quantum prepotentials to transform homogeneously

$$ T_Q \rightarrow GT_QG^{-1}, \quad \bar{T}_Q \rightarrow G\bar{T}_QG^{-1} \quad (2.24) $$

In practice, we will leave the background gauge unspecified; indeed, we will attempt to maintain background gauge covariance at all times.

As a quantum transformation, $T$ is invariant and $T_Q$ transforms as

$$ T_Q \rightarrow C_QT_QG^{-1}, \quad \bar{T}_Q \rightarrow \bar{C}_Q\bar{T}_QG^{-1} \quad (2.25) $$

where $C_Q \equiv T^{-1}CT$ and $\bar{C}_Q \equiv \bar{T}^{-1}\bar{C}\bar{T}$ are chiral and antichiral operators, obeying respectively

$$ 0 = [\nabla_{\dot{\alpha}}, C_Q] = [\nabla_{\dot{\alpha}}, C_Q] \quad (2.26) $$

Henceforth, we will be concerned only with quantum transformations. The supergauge freedom of $G_Q$ can be eliminated by choosing to work in quantum chiral, antichiral, or Hermitian gauge.

We prefer to work in a gauge which maintains manifest Hermiticity at all times, though it may occasionally be more cumbersome, so we choose the last of these gauges. To go to quantum Hermitian gauge, one takes $G_Q = \bar{T}_Q^{-1}U_Q^{1/2} = T_Q^{-1}U_Q^{-1/2}$ where $U_Q \equiv \bar{T}_QT_Q^{-1}$.

This yields $T_Q = U_Q^{-1/2}, \bar{T}_Q = U_Q^{1/2}$, giving

$$ \nabla_{\dot{\alpha}}' = U_Q^{-1/2}\nabla_{\dot{\alpha}}U_Q^{1/2}, \quad \nabla'_{\dot{\alpha}} = U_Q^{1/2}\nabla_{\dot{\alpha}}U_Q^{-1/2} \quad (2.27) $$

for the covariant derivatives and

$$ \Phi' = U_Q^{1/2}\Phi, \quad \bar{\Phi}' = U_Q^{-1/2}\bar{\Phi} \quad (2.28) $$

for the chiral and antichiral superfields. The residual gauge transformation acts on $U_Q$ as

$$ U_Q \rightarrow \bar{C}_QU_QC_Q^{-1} \quad (2.29) $$

Quantum chiral gauge consists of making the quantum gauge choice $T_Q = 1, \bar{T}_Q = U_Q$.

In this approach, $\nabla_{\dot{\alpha}}$ remains unchanged under quantum deformations of the geometry and so chiral superfields remain unchanged. Quantum antichiral gauge is analogously constructed.

It is worth noting the relation between $U_Q$ and $U'$ in background Hermitian gauge:

$$ U' = \bar{T}'T'^{-1} = \bar{T}\bar{T}_QT_Q^{-1}T^{-1} = \bar{T}U_QT^{-1} = U^{1/2}U_QU^{1/2} \quad (2.30) $$
2.2 Conformally covariant quantum prepotentials

The perturbative quantum prepotentials are the Hermitian superfields $V$ defined by

$$U_Q = \exp \left(-2iV^B \nabla_B - 2iV^b X_b\right) \quad (2.31)$$

To maintain general covariance, we have chosen to parametrize the quantum prepotentials in terms of the background covariant derivatives $\nabla_B$ rather than the coordinate derivatives. The factor of $-2$ is conventional and the $i$ is so that the superfields $V^B$ have the obvious Hermiticity conditions – for example,

$$(V^b)^\dagger = V^b, \quad (V^\alpha)^\dagger = V^\dot{\alpha} \quad (2.32)$$

These superfields are chosen to transform under the action of the group generators as

$$X_b V^A = -V^C f^A_{CB}$$

where $A$ and $C$ run over all indices and $f^A_{CB}$ are the structure constants as defined in [11]. We thus have a conformally covariant set of quantum prepotentials.

For the generators $D$ and $A$, the $V$'s transform contravariantly as their index indicates. Thus $V^a$ (like $e^m_a$) has scaling and $U(1)_R$ weights $(\Delta, w) = (-1, 0)$, $V^\alpha$ (like $\psi^m_\alpha$) has weights $(-1/2, +1)$, but $V(K)^\alpha$ has weights $(+1/2, -1)$. For the Lorentz generators, the $V$'s transform as their indices indicate. Only special conformal transformation properties are not obvious. Recall the action of $K$ on a group element $g = (\xi, \omega, \Lambda, w, \epsilon)$ is

$$K_B \xi^A = -\frac{1}{2} C^A_B \epsilon^C \xi^c, \quad \frac{1}{2} (K_B \omega^{de}) M_{cd} = -2 \xi^C M_{CB}$$

$$K_B \Lambda = -2(-)^B \xi_B, \quad K_B w = -3i \xi_B w(B)$$

$$K_B \epsilon^A = -\lambda(A) \Lambda \delta^{(A)}_B + iw(A) \omega_B^A + \omega^A_B + \epsilon^C C_{CB} A - \frac{1}{2} \xi^C C_{CB} A (-)^{BA} \quad (2.34)$$

where we have used the notation of [11]. Since the prepotentials are group elements, they must have these same transformation properties, and since the special conformal generator acts quite like an antiderivative, these formulae encapsulate a good deal of information. By inspection, one can easily see that only $V^a$ is conformally primary. This isn’t too great of a surprise, since the prepotential of conformal supergravity is a real superfield $H^m$, and $V^a$ is its obvious quantum variation. All other objects should in principle be given as derivatives of $V^a$ or otherwise be pure gauge artifacts. Using the special conformal transformation rules, it is possible to rewrite each of the prepotentials as derivatives of $V^a$ plus some remaining conformally primary object.

As an example, note that $V^\alpha$ obeys

$$S^\delta V^\alpha = -iV^\delta \alpha, \quad S_\beta V^\alpha = K_\beta V^\alpha = 0$$

This is easily solved by

$$V^\alpha = -\frac{i}{8} \nabla_\phi V^\delta \alpha + \tilde{V}^\alpha$$

\footnote{Notational consistency would demand that the $V$’s be subscripted with $Q$’s to denote that they are quantum prepotentials. Since we will never again mention the background prepotentials, it is easier to suppress the $Q$ for a less cluttered notation.}
where $\tilde{V}^\alpha$ is some conformally primary superfield. The other conditions are not all nearly so easy to solve, but the answer is straightforward to check. One finds
\begin{equation}
V^\alpha = -i 8 \nabla_\phi V^\alpha \tilde{\phi} + \tilde{V}^\alpha \tag{2.35}
\end{equation}
\begin{equation}
V_{\dot{\alpha}} = -i 8 \nabla_\phi V_{\dot{\phi} \dot{\alpha}} + \tilde{V}_{\dot{\alpha}} \tag{2.36}
\end{equation}
\begin{equation}
V(D) = \frac{1}{2} \nabla_c V^c + \frac{1}{2} \nabla^\alpha V_\alpha + \frac{1}{2} \nabla_\dot{\alpha} \tilde{V}^\dot{\alpha} + \tilde{V}(D) \tag{2.37}
\end{equation}
\begin{equation}
V(A) = -\frac{1}{4} \Delta_c V^c - \frac{3i}{4} (\nabla^\alpha V_\alpha - \nabla_\dot{\alpha} \tilde{V}^\dot{\alpha}) + \tilde{V}(A) \tag{2.38}
\end{equation}
\begin{equation}
V(M)_{\beta \alpha} = +\frac{1}{2} \nabla_{\{\beta \dot{V}_\alpha\} \tilde{\phi}} + \frac{i}{8} \nabla^\phi \nabla_{\{\beta \dot{V}_\alpha\} \tilde{\phi}} + \tilde{V}(M)_{\beta \alpha} \tag{2.39}
\end{equation}
\begin{equation}
V(M)_{\dot{\beta} \dot{\alpha}} = -\frac{1}{2} \nabla^\beta \tilde{V}_\dot{\alpha} + \frac{i}{8} \nabla^\phi \nabla_{\{\beta \dot{V}_\dot{\alpha}\} \tilde{\phi}} + \tilde{V}(M)_{\dot{\beta} \dot{\alpha}} \tag{2.40}
\end{equation}
where we have defined
\begin{equation}
[\nabla_\alpha, \nabla_\dot{\alpha}] = -2 \Delta_\alpha \dot{\alpha} \tag{2.41}
\end{equation}
These prepotential formulae will be the most useful to us. We have given them both in terms of the conformally non-primary $V^\alpha$ and the primary $\tilde{V}^\alpha$. The other tilded objects are similarly primary.

For completeness, we include also the special conformal prepotentials, which are a little messier and which we will not have a great deal of use for in what follows:
\begin{equation}
V(K)_{\alpha} = +\frac{1}{8} \nabla^2 V_\alpha - \frac{1}{4} \nabla^\phi \nabla_\phi V_\alpha + \frac{i}{96} \nabla^2 \nabla_\phi V_\alpha \tilde{\phi} + \frac{1}{24} \nabla^\alpha \nabla_{\beta \dot{\alpha}} \tilde{V}^\beta \tilde{\beta} + \tilde{V}(K)_{\alpha} \tag{2.42}
\end{equation}
\begin{equation}
V(K)_{\dot{\alpha}} = +\frac{1}{8} \nabla^2 \tilde{V}_{\dot{\alpha}} - \frac{1}{4} \nabla_\phi \nabla_\phi \tilde{V}_{\dot{\alpha}} + \frac{i}{96} \nabla^2 \nabla_\phi \tilde{V}_{\dot{\alpha} \tilde{\phi}} + \frac{1}{24} \nabla_\dot{\alpha} \nabla_{\beta \dot{\alpha}} \tilde{V}^\beta \tilde{\beta} + \tilde{V}(K)_{\dot{\alpha}} \tag{2.43}
\end{equation}
The objects $\tilde{V}(K)_{\alpha}$ are not themselves fully primary, but are related to $\tilde{V}(D)$, $\tilde{V}(A)$, and $\tilde{V}(M)_{\beta \alpha}$ by the action of $S_{\beta}$. When these latter objects vanish, $\tilde{V}(K)_{\alpha}$ is itself primary.

In addition, when we consider Yang-Mills theories, we will also need the prepotential $\Sigma'$, the Yang-Mills prepotential associated with the Yang-Mills generator $X_r$. It is naturally conformally primary.

We emphasize that the separation we have made above is entirely dictated by conformality concerns; the tilded objects we have introduced are defined by the above equations. We will very quickly find that they are constrained to be pure gauge artifacts. To demonstrate this, we require two new pieces of information: the form of the chiral gauge transformations and the first-order solution of the supergravity constraints.
2.3 Chiral gauge transformations

In choosing to work in quantum Hermitian gauge, we have exhausted the full supergroup gauge transformation, but the chiral transformations remain. Recall they are given by

\[ U_Q \rightarrow \tilde{C}_Q U_Q C_Q^{-1} \]

(2.44)

where \( C_Q \) obeys a chirality condition, \( [\nabla^\dagger, C_Q] = 0 \). If we define \( U_Q \equiv \exp(-2iV) \), \( C_Q^{-1} \equiv \exp(-2i\Lambda) \), and \( \tilde{C}_Q \equiv \exp(-2i\tilde{\Lambda}) \), then the above transformation rule is equivalent (for infinitesimal \( \Lambda \)) to

\[ \delta V = \Lambda + \tilde{\Lambda} - i[V, \Lambda - \tilde{\Lambda}] + O(V^2) \]

(2.45)

Writing \( \Lambda = \xi^A \nabla_A + \frac{1}{2} \omega^{ab} M_{ab} + \Lambda D + w A + \epsilon^B K_B \), we can solve for the conditions that these various parameters must obey:

\[ \xi_{a\dot{a}} = -\nabla_{\dot{a}} L_{a\alpha}, \quad \xi_{a} = \frac{i}{8} \bar{\nabla}^2 L_{a\alpha}, \quad \xi_{\dot{a}} = \text{arbitrary} \]

\[ \Lambda = -\frac{1}{2} \nabla^\dagger \xi_a + \phi(D), \quad w = -\frac{3i}{4} \nabla^\dagger \xi_a + \frac{i}{2} \phi(D) \]

\[ \omega_{a\dot{a}} = \frac{1}{2} \nabla_{\{a} \xi_{\dot{a}}\}, \quad \omega_{a\beta} = -2i L^\gamma W_{\gamma a\beta} + \phi(M)_{a\beta} \]

\[ \epsilon_{a} = \frac{1}{8} \bar{\nabla}^2 \xi_a, \quad \epsilon_{\dot{a}} = 2i L^\phi \nabla^\gamma W_{\gamma \phi a} + \psi(K)_{a}, \quad \epsilon_{(a\dot{a})} = \frac{1}{2} i L^\phi \nabla_{\dot{a}} \gamma W_{\gamma \phi a} + i \nabla_{\dot{a}} \psi(K)_{a} \]

(2.46)

In the above formulae \( \{a\dot{a}\} \) denotes the (unnormalized) symmetric sum \( a\dot{a} + \dot{a}a \). The superfields \( \phi(D) \) and \( \phi(M)_{a\beta} \) are chiral, \( \psi(K)_{a} \) is complex linear, \( \xi_{a\dot{a}} \) is arbitrary, but none of these four is primary. \( L_{a\alpha} \) is both primary and arbitrary. As with the prepotentials, we may rewrite the non-primary operators as derivatives of primary ones plus some new primary object. Doing so gives

\[ \xi_{a\dot{a}} = -\nabla_{\dot{a}} L_{a\alpha}, \quad \xi_{a} = \frac{i}{8} \bar{\nabla}^2 L_{a\alpha}, \quad \xi_{\dot{a}} = -\frac{i}{8} \nabla_{\beta} \nabla_{\dot{a}} L^\beta + \bar{\xi}_{\dot{a}} \]

\[ \Lambda = -\frac{1}{2} \nabla^\dagger \xi_a - \frac{i}{16} \bar{\nabla}^2 \nabla_{\beta} L^\beta + \bar{\phi}(D), \quad w = -\frac{3i}{4} \nabla^\dagger \xi_a + \frac{1}{32} \bar{\nabla}^2 \nabla_{\beta} L^\beta + \frac{i}{2} \phi(D) \]

\[ \omega_{a\dot{a}} = \frac{1}{2} \nabla_{\{a} \xi_{\dot{a}}\}, \quad \omega_{a\beta} = -2i L^\gamma W_{\gamma a\beta} - \frac{i}{16} \bar{\nabla}^2 \nabla_{\{a} L_{\beta\}} + \bar{\phi}(M)_{a\beta} \]

(2.47)

We have not included the terms corresponding to \( \epsilon(K) \) since they are fairly messy and we don’t actually have much use for these specific formulae in what follows.

The useful part of the above formulae is to note the correspondence between the tilded gauge objects and the tilded prepotentials. For example, if we could show that \( \tilde{V}(K)_{a} \) were constrained to be complex linear, then it is a pure gauge artifact, cancelling against \( \psi(K)_{a} \). Similarly, if we could show that \( \tilde{V}(M)_{a\beta} \) were chiral, we could cancel it against \( \dot{\phi}(M)_{a\beta} \). Clearly \( \tilde{V}_{a\dot{a}} \) already corresponds to \( \xi_{a\dot{a}} \). To eliminate \( \tilde{V}(D) \) and \( \tilde{V}(A) \), we would need to show that they can be related to the appropriate sum (or difference) of a chiral and an antichiral field – in this case, \( \dot{\phi}(D) \) and its conjugate. Provided these constraints can be enforced, the theory becomes one entirely of \( V^a \).

We should check that the number of degrees of freedom work out. \( V^a \) itself consists of 32 bosonic and 32 fermionic degrees of freedom. The gauge degree of freedom \( L_{a\alpha} \),
however, also seems to have $32 + 32$ components. The solution to this puzzle is that $L_\alpha$ has weight $(-3/2, -1)$ which has precisely the ratio necessary to accommodate a primary chiral superfield. We will find in physical models, in fact, that $L_\alpha$ itself possesses a gauge symmetry of $L_\alpha \rightarrow L_\alpha + \phi_\alpha$, where $\phi_\alpha$ has $8 + 8$ components. Since it is a second order gauge degree of freedom (ie. a gauge degree of freedom for a gauge degree of freedom), these components contribute positively to the counting. Put more simply,

$$32 + 32 - (32 + 32 - (8 + 8)) = 8 + 8$$

which is the right number for conformal supergravity. It is interesting that the physical degrees of freedom of conformal supergravity coincide with those of a chiral spinor.

For completeness, we also include the Yang-Mills variation:

$$\Lambda^r = i L^\beta W^r_\beta + \tilde{\Lambda}^r$$

(2.48)

where $\tilde{\Lambda}^r$ is chiral. Note that because we have included $\Sigma^r$ with the supergravity prepotentials, its chiral gauge variation includes a term coming from supergravity, in addition to the usual chiral superfield.

### 2.4 First-order constraint solution

We next turn to the task of solving the supergravity constraints to first order. Because conformal supergravity is characterized by conventional constraints as in super Yang-Mills, the curvatures are entirely described by “gaugino” superfields $W_\alpha$ which are given by the commutators

$$[\nabla_\alpha, \nabla_{\beta\dot{\beta}}] = -2i\epsilon_{\alpha\beta} W_{\beta}, \quad [\nabla_\dot{\alpha}, \nabla_{\dot{\beta}\beta}] = -2i\epsilon_{\dot{\alpha}\dot{\beta}} J W_{\beta}$$

(2.49)

These are superfields which obey a chirality condition, $\{\nabla_\dot{\alpha}, W_\beta\} = 0$. The constraints of conformal supergravity involve imposing $W_\alpha(P)^B = W_\alpha(D) = W_\alpha(A) = 0$. From these it follows that $W_\alpha(M)^{\dot{\beta}\dot{\gamma}} = 0$ and $W_\alpha(K)_{\dot{\alpha}} = 0$ and that all the remaining curvatures can be expressed in terms of the single chiral superfield $W_{\alpha\beta\gamma}$.

The chiral superfield $W_\alpha$ can be defined by

$$8W_\alpha = [\nabla_\dot{\alpha}, \{\nabla^{\dot{\alpha}}, \nabla_\alpha\}] = +2i[\nabla^{\dot{\alpha}}, \nabla_{\alpha\dot{\alpha}}]$$

(2.50)

Varying this object to first order involves varying each of the covariant derivatives on the right side. The easiest way to handle this is to adopt a chiral quantum gauge where we force all of the quantum variation onto $\nabla_\alpha$ and leave $\nabla_\dot{\alpha}$ unchanged. If the gaugino superfield vanishes in this gauge, it vanishes in any gauge, including quantum Hermitian gauge. (This is equivalent to doing the variation in Hermitian gauge and then performing a quantum prepotential-dependent gauge transformation.)

Thus,

$$\delta_c \nabla_\alpha = [2iV, \nabla_\alpha], \quad \delta_c \nabla_\dot{\alpha} = 0$$

(2.51)

where the subscript $c$ denotes that the quantum gauge is chiral.

Note first that the Hermitian quantum variation of $\nabla_\alpha$ is

$$\delta \nabla_\alpha = [iV, \nabla_\alpha] \equiv -H_\alpha^B X_B = -H_\alpha^B \nabla_B - \Omega_\alpha(M) - \Lambda_\alpha D - \omega_\alpha A - J_\alpha^B K_B$$

(2.52)
The variation of the bosonic derivative is rather easy to calculate in chiral gauge. One finds

$$H_{\alpha}^\beta = +i\nabla_\alpha V^\beta - iV(M)_{\alpha}^\beta - \frac{i}{2}V(D)\delta_\beta^\alpha - V(A)\delta_\beta^\alpha \quad (2.53)$$

$$H_{\alpha\beta} = +i\nabla_\alpha V_\beta \quad (2.54)$$

$$H_{(\alpha\beta)} = +i\nabla_\alpha V_{(\beta)} + 4\epsilon_{\alpha\beta} V_\beta \quad (2.55)$$

$$\Omega_\alpha(M) = +iV^\beta R_{\alpha\beta}(M) + i\nabla_\alpha V(M) + 2iV(K)^\beta M_{\beta\alpha} \quad (2.56)$$

$$\Lambda_\alpha = +i\nabla_\alpha V(D) + 2iV(K)_\alpha \quad (2.57)$$

$$\omega_\alpha = +i\nabla_\alpha V(A) + 3V(K)_\alpha \quad (2.58)$$

$$J_\alpha^\beta = +i\nabla_\alpha V(K)^\beta \quad (2.59)$$

$$J_{\alpha\beta} = +i\nabla_\alpha V(K)_{\beta} + iV^c R_{\alpha\beta}(K)_{\beta} + V(K)_{\alpha\beta} \quad (2.60)$$

$$J_\alpha^b = +i\nabla_\alpha V(K)^b + iV^c R_{\alpha\beta}(K)^b \quad (2.61)$$

In the chiral gauge we are using, the variation of $\nabla_\alpha$ is simply twice this:

$$\delta_\epsilon \nabla_\alpha = -2H_{\alpha}^B \nabla_B - 2H_{\alpha}^b X_\beta \quad (2.62)$$

The variation of the bosonic derivative is rather easy to calculate in chiral gauge. One finds

$$\delta_\epsilon \nabla_\alpha a\dot{\alpha} = -i\nabla_\alpha H_{\alpha}^B X_B - i\nabla_\dot{\alpha} H_{\dot{\alpha}}^b X_\beta - 2H_\alpha^\beta \nabla_\beta \dot{\alpha} + H_{(\alpha\dot{\alpha})} \omega_\dot{\alpha} + iH_{\alpha}^b f_{\beta\dot{\alpha}}^D X_D \quad (2.63)$$

$\delta W$ is then given by

$$4\delta W_\alpha = -\nabla^2 H_{\alpha}^B X_B + 4i\nabla_\alpha H_{\alpha}^\beta \nabla_\beta \dot{\alpha} + \left(2i\nabla^\beta H_{(\beta\dot{\alpha})} + 8H_{\alpha\beta}\right) \omega_\beta + \left(2\nabla_\alpha H_{\alpha}^b - H_{\alpha\beta} f_{\alpha\beta}^D \right) f_{\beta\dot{\alpha}}^D X_D \quad (2.64)$$

We begin the analysis by considering the constraints imposed on the prepotentials by $W_\alpha(P) = 0$. These amount to two conditions, which we write as

$$\nabla^2 H_{(\alpha\dot{\alpha})} = 8i\nabla_\dot{\alpha} H_{\alpha\beta} \quad (2.65)$$

$$8J_{\alpha\dot{\alpha}} = -\nabla^2 H_{\alpha\dot{\alpha}} - \nabla_\alpha \Lambda_\alpha - 2i\nabla_\dot{\alpha} \omega_\alpha + 2\nabla^\dot{\alpha} \omega_{\alpha\dot{\alpha}} \quad (2.66)$$

The second of these amounts to a definition of $V(K)_{a\dot{\alpha}}$, on which $J_{a\dot{\alpha}}$ linearly depends. (There is a third condition that we haven’t listed which is a trivial consequence of the first.)

Choosing $W_\alpha(D)$ and $W_\alpha(A)$ to vanish amount to the condition

$$\nabla^2 \Lambda_\alpha = -\frac{2i}{3} \nabla^2 \omega_\alpha \quad (2.67)$$

All other conditions on the $W_\alpha$’s follow from these three.

The third condition, (2.67), is the easiest to immediately evaluate. Using the above definitions for $\Lambda_\alpha$ and $\omega_\alpha$ leads to

$$0 = \nabla^2 \left(i\nabla_\alpha V(D) - \frac{2}{3} \nabla_\alpha V(A) + 4iV(K)_\alpha \right)$$

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Inserting the definitions of the $V$'s in terms of the $\tilde{V}'$s, we discover a nice surprise. The above condition reduces to

$$0 = \nabla^2 \left( i\nabla_\alpha \tilde{V}(D) - \frac{2}{3} \nabla_\alpha \tilde{V}(A) + 4i\tilde{V}(K)_a \right)$$  (2.68)

The first condition, (2.65), is the next easiest to check. Again using the $\tilde{V}$'s we can conclude that

$$0 = \nabla_\beta \tilde{V}(M)_{\beta\alpha}$$  (2.69)

$$0 = \nabla_\beta \left( \frac{i}{2} \tilde{V}(D) + \tilde{V}(A) \right)$$  (2.70)

The first of these implies that $\tilde{V}(M)_{\beta\alpha}$ is chiral and therefore pure gauge: it is in one-to-one correspondence with its chiral gauge parameter $\tilde{\phi}(M)_{\beta\alpha}$. We can therefore choose $\tilde{V}(M)$ to vanish. The second equation implies that

$$\tilde{V}(D) - 2i\tilde{V}(A) = 2\tilde{\phi}(D)$$

Together with its conjugate, this implies that $\tilde{V}(D)$ and $\tilde{V}(A)$ are the real and imaginary parts of a chiral superfield $\tilde{\phi}(D)$. Since this also precisely overlaps with their gauge degrees of freedom, we can similarly choose $\tilde{V}(D)$ and $\tilde{V}(A)$ to vanish.

This last point is an important one. In a theory with a conformal compensator $\Phi_0$ of unit scaling dimension and matter fields $\Phi^i$ of vanishing scaling dimension, the quanta of $\Phi_0$ are indistinguishable from the chiral degree of freedom $\tilde{\phi}(D)$. Both have an equally valid claim to be the chiral quanta which together with $V^a$ make up the quanta of Poincaré supergravity, while the other is the pure gauge degree of freedom. From our point of view, it is almost always more sensible to remove $\tilde{\phi}(D)$ immediately. If desired, it can be restored by undoing the chiral scale transformation.

Whether or not we choose to eliminate $\tilde{\phi}(D)$, the condition that $\tilde{V}(D)$ and $\tilde{V}(A)$ are made up of a sum and a difference of a chiral and an antichiral superfield together with (2.68) implies that

$$\nabla^2 \tilde{V}(K)_a = 0$$  (2.71)

This means that $\tilde{V}(K)_a$ is a complex linear superfield and so it too is in perfect correspondence with its gauge degree of freedom and so can be taken to vanish.

We return now to the second condition, (2.63). This boils down to

$$V(K)_{a\dot{a}} = -i\nabla_a V(K)_{\dot{a}} - i\nabla_{\dot{a}} V(K)_a + \frac{i}{8} \nabla_\alpha \nabla^2 V_{\dot{a}} + \frac{i}{8} \nabla_\alpha \nabla^2 V_a + \frac{1}{32} \hat{\Delta}_D V_{a\dot{a}}$$  (2.72)

where we have defined

$$\hat{\Delta}_D V_{a\dot{a}} = \nabla^2 V_{a\dot{a}} + 16\nabla_\gamma W_{\alpha\beta\gamma}^a V_{a\beta\dot{a}} + 16 W_{\alpha\beta\gamma}^a \nabla_\gamma V_{a\beta\dot{a}}$$  (2.73)

One can show that $\hat{\Delta}_D V_{a\dot{a}}$ is Hermitian.

Before moving on, we note here the chiral variation of the conformal supergravity field strength in the chiral gauge where $\tilde{V}(D)$, $\tilde{V}(M)$, and $\tilde{V}(A)$ vanish:

$$\delta_c W_{\alpha\beta\gamma} = \sum_{(a\beta\gamma)} \frac{i}{96} \nabla^2 \nabla\tilde{\phi}_a \nabla_\beta \nabla_\gamma \tilde{\phi}$$  (2.74)

We have discovered how to use the Yang-Mills-like features of the conformal supergravity algebra to extract the geometric quanta at first order. We turn next to some specific physical models.
3 Two physical models at first order

3.1 Linear compensator model

Although we will be most concerned with an arbitrary chiral model, we will first consider a simpler model. The minimally coupled linear compensator model with a Kähler potential consists of a D-term action of two terms

$$S = S_G + S_K. \quad (3.1)$$

The Einstein-Hilbert term is contained within the first term

$$S_G = \int ELV_R \equiv 3 \int EL \log(L/\Phi_0 \bar{\Phi}_0) \quad (3.2)$$

where $L$ is the linear compensator and $\Phi_0$ is a chiral superfield of scaling dimension 1, whose presence is almost solely to make the argument of the logarithm conformally invariant, as a redefinition

$$\Phi_0 \rightarrow e^\Lambda \Phi_0$$

for chiral $\Lambda$ leaves the action invariant due to the linearity condition of $L$. In the gauge where $L = 1$, this has the form of a Fayet-Iliopoulos for the supergravity $U(1)_R$.

The coupling of chiral matter to the theory is contained within the second term

$$S_K = \int ELK \quad (3.3)$$

where $K$ is the Kähler potential, a dimension zero Hermitian function of chiral and antichiral superfields which possesses a symmetry

$$K \rightarrow K + F + \bar{F}, \quad (3.4)$$

also a consequence of the linearity of $L$.

We could also include Fayet-Iliopoulos terms for Yang-Mills fields by introducing them as $\int EL \text{Tr} V$ where $V$ is the gauge prepotential. In fact, one can likewise view $S_K$ as essentially being the FI term for a $U(1)_K$ symmetry. One would then naturally combine all these to give the single term

$$-3 \int EL \log \left( \Phi_0 e^{-(K+V)/3} \bar{\Phi}_0 / L \right) \quad (3.5)$$

which can be understood as a sum of the FI terms for the Yang-Mills, Kähler, and $U(1)_R$ gauge sectors. We will exclude from our discussion Yang-Mills FI terms and treat the supergravity and Kähler sectors separately.

In order to proceed, we need to determine the transformation of the various quantities. We will work in the gauge where $\bar{V}(D) = \bar{V}(A) = \bar{V}(M) = \bar{V}(K) = 0$. The non-primary object $V^\alpha$ we will leave for the moment unfixed and specify a gauge for it later.

The first order variation of $E$ is

$$\delta E = H^\alpha_\alpha + H^\alpha \bar{\alpha} + H^a_a = -3i \bar{\nabla}^\alpha V_\alpha + 3i \bar{\nabla}_\bar{\alpha} V^\bar{\alpha} - \Delta_b V^b - 4V(A) = 0 \quad (3.6)$$
This is an initially surprising result, but it is owed to our working in a conformal theory. For example, in a component four dimensional theory, the first order variation of $\sqrt{g}$ is the trace of the graviton perturbation, which is the conformal mode of the graviton. We could set the scaling gauge in such a theory by forcing the conformal mode to vanish. This is something of a shell game, however, since the conformal mode of the graviton is essentially the same object as the conformal compensator in such a theory. In the current theory, the role of the “conformal mode” of the graviton will be taken up by the linear compensator (and later the chiral compensator) and so $\delta E = 0$ here.

The first order variation of a chiral superfield $\Phi$ of scaling dimension $\Delta$ and $U(1)_R$ weight $2\Delta/3$ is given in Hermitian gauge by

$$
\delta \Phi = -iV^B X_B \Phi + \delta_c \Phi
$$

where we define $\delta_c \Phi \equiv \eta$ as the variation in chiral gauge.

We next note that $L$ may be written

$$
L = \nabla^\alpha \Phi_\alpha + \nabla_\alpha \Phi^\dagger_\alpha
$$

in terms of chiral primary superfields $\Phi_\alpha$ of weight $(3/2,1)$. The variation of $\nabla^\alpha \Phi_\alpha$ is given by

$$
\delta(\nabla^\alpha \Phi_\alpha) = -i\nabla^\beta(V_\beta \nabla^\alpha \Phi_\alpha) + i\nabla^\beta(V^\beta \nabla^\alpha \Phi_\alpha) - \Delta_b(V^b \nabla^\alpha \Phi_\alpha) + 2V^\alpha \bar{\nabla}_\alpha \Phi_\alpha
$$

$$
+ \frac{1}{4} \nabla_\alpha \nabla^2(V^\alpha \Phi_\alpha) + \nabla^\alpha(\delta_c \Phi_\alpha)
$$

$$
- i\Sigma \nabla^\alpha \Phi_\alpha - 2i(\nabla^\alpha \Sigma^\nu)X_\nu \Phi_\alpha
$$

Assuming $\Phi_\alpha$ to be a gauge singlet, we can write the variation of $L$ as

$$
\delta L = \mathcal{L} - i\nabla^\beta(V_\beta L) + i\nabla^\beta(V^\beta L) - \Delta_b(V^b L)
$$

where

$$
\mathcal{L} \equiv \nabla^\alpha \left( \delta_c \Phi_\alpha - \frac{1}{4} \nabla^2(V^{\alpha \dagger} \Phi_\dagger_\alpha) \right) + \text{h.c.} \equiv \nabla^\alpha \eta_\alpha + \text{h.c.}
$$

$\eta_\alpha$ is a weight $(3/2,1)$ chiral primary superfield, which we have defined to depend on both $\Phi_\alpha$ and $\Phi^\dagger_\alpha$ so as to simplify the formula.

After several integrations by parts, one can show that

$$
\delta S_G = \int E \left( \mathcal{L} V_\alpha - 2V^b \Delta_b L + \frac{3}{2L} V^{\alpha \dagger} \nabla_\alpha L \nabla_\dagger \alpha L \right)
$$

We may define a new weight $(0,0)$ primary superfield $G_b$ by

$$
G_b \equiv \frac{1}{2} L^{-1} \Delta_b L - \frac{3}{8L^2} \nabla_\alpha L \nabla_\dagger \alpha L = -L^{1/2} \Delta_b L^{-1/2}
$$

So that

$$
\delta S_G = \int E \left( \mathcal{L} V_\alpha - 4V^b G_b \right)
$$
One can similarly work out the structure of $S_K$. Skipping details (the most difficult of which is an integration by parts) one finds

$$
\delta S_K = \int EL \left( K_i \eta^i + K_j \eta^j + V^b K_b + \Sigma^r K_r \right) + \int ELK
$$

(3.15)

where

$$
K_{\alpha \dot{\alpha}} \equiv K_{ij} \nabla_\alpha \Phi^i \nabla_{\dot{\alpha}} \Phi^j
$$

(3.16)

$$
K_r \equiv -i K_i X_r \Phi^i + i K_j X_r \Phi^j
$$

(3.17)

Both $K_\alpha$ and $K_r$ are conformally primary.

Combining these two variations gives

$$
\delta S = \int E \left[ LV^b (-4 G_b + K_b) + L \Sigma^r K_r + L K_i \eta^i + L K_j \eta^j + \mathcal{L}(V_R + K) \right]
$$

(3.18)

This is a surprisingly compact expression. When $L$ is gauged to 1, $G_b$ becomes the Poincaré superfield of the same name and represents the pure supergravity contribution to the energy-momentum tensor. $K_b$ represents the matter contribution to the energy-momentum tensor, and $K_r$ is the matter contribution to the gauge current.

3.1.1 Gauge invariance of the linear compensator model

The first feature we should observe about our linear compensator model is that at first order it is independent of $V^\alpha$ and $V_{\dot{\alpha}}$. This is certainly sensible since these are gauge degrees of freedom and should certainly not have any equations of motion associated with themselves.

The dynamical theory would seem to consist of $V^\alpha$ and $\Sigma^r$ — the Hermitian superfields associated with the graviton and gauge multiplets — as well as the matter superfield $\eta^i$ and $\bar{\eta}^\dot{j}$ and the linear compensator variation $\mathcal{L}$. We recall that $V^\alpha$ transforms under the quantum chiral gauge transformation as

$$
\delta V_{\alpha \dot{\alpha}} = \nabla_\alpha \bar{L}_{\dot{\alpha}} - \nabla_{\dot{\alpha}} L_\alpha
$$

(3.19)

Under the $L_\alpha$ transformation, a chiral superfield transforms as

$$
\Phi' = C \Phi
$$

(3.20)

Differentially, this reads

$$
\delta \eta = 2i \Lambda \Phi = 2i \xi^\alpha \nabla_\alpha \Phi + 2i \xi_{\dot{\alpha}} \nabla_{\dot{\alpha}} \Phi + 2i \Lambda \Delta \Phi - \frac{4}{3} \omega \Delta \Phi + 2i \Lambda^r X_r \Phi
$$

(3.21)

where $\Delta$ is the scaling dimension of $\Phi$. Plugging in the values for superfields, we find

$$
\delta \eta = -\frac{1}{4} \bar{\nabla}^2 (L^\alpha \nabla_\alpha \Phi) - \frac{\Delta}{12} (\bar{\nabla}^2 \nabla^\beta L_\beta) \Phi + 2i \bar{\Lambda}^r X_r \Phi
$$

(3.22)

The gauge superfield $\Sigma^r$ transforms as

$$
\delta \Sigma^r = \bar{\Lambda}^r + \bar{\Lambda}^\dot{r} + i L^\beta W^r_\beta + i L_\beta \bar{W}^\dot{r}
$$

(3.23)

The quantum linear compensator varies as

$$
\delta \mathcal{L} = \frac{1}{4} \nabla^\alpha \bar{\nabla}^2 (L_\alpha L) + h.c.
$$

(3.24)

Note that this last expression depends on $\Phi_\alpha$ only implicitly via $L$.

One can check that the first-order action is invariant under this first-order shift in the quantum superfields, as it must be by construction.
3.2 Arbitrary chiral model

The minimal linear compensator model is notable for the clean decoupling of the gravitational and matter terms of the action, which gives a corresponding decoupling of their contributions to the gravitational current. The arbitrary chiral model will not be so immediately simple to evaluate, but we will find its first order variation shares the same features.

The chiral model classically dual to the minimal linear compensator model with a Kähler potential $K$ is

$$S = -3 \int E \Phi \bar{\Phi} e^{-K/3}$$

(3.25)

This action encapsulates not only the pure gravity effects (denoted $S_G$ in the linear model) but also kinetic matter terms (denoted $S_K$). Here $\Phi_0$ is a weight $(1, 2/3)$ conformally primary chiral superfield and $K$ is as before a Hermitian function of weight $(0, 0)$ chiral and antichiral superfields. A canonically normalized Einstein-Hilbert term is found in the gauge $\Phi_0 \bar{\Phi}_0 = e^{K/3}$.

The above D-term is a special case of a more general theory involving an arbitrary set of chiral superfields of arbitrary weights, $\Phi_i^\alpha$ and their conjugates. The factor of $-3$ is necessary so that the gauge $Z = 1$ gives a canonical Einstein-Hilbert term. The proof of this is straightforward. Using the scaling and $U(1)_R$ weights of $Z$,

$$DZ = 2Z = Z_i \Delta_i \Phi^i + Z_j \bar{\Phi}^j$$

$$-\frac{3i}{2} AZ = 0 = Z_i \Delta_i \Phi^i - Z_j \bar{\Phi}^j$$

and that the Einstein-Hilbert term is contained within

$$-3[Z]_D = -3 \left[ Z_j \mathcal{P} \Phi^j + \ldots \right]_F = -3Z_j \bar{\mathcal{P}} \Phi^j + \ldots = -3Z_j \Box \Phi^j + \ldots$$

where $\mathcal{P} = -\nabla^2/4$, $\bar{\mathcal{P}} = -\nabla^2/4$ and $\Box$ are superconformal. That $\Box$ is superconformal means it contains $\mathcal{R}/6$ weighted by the scaling dimension of the field on which it acts, and so it is easy to see that the Einstein-Hilbert term is

$$-3[Z]_D \ni -\frac{1}{2} \mathcal{R} Z_j \Delta_j \Phi^j = -\frac{Z}{2} \mathcal{R}$$

The gauge $Z = 1$ then corresponds to a canonical Einstein-Hilbert term.

Since $\delta E = 0$, we concern ourselves only with the first order variation of $Z$:

$$\delta Z = Z_i (\eta^i - iV \Phi^i) + Z_j (\bar{\eta}^j + i\bar{V} \bar{\Phi}^j)$$

$$= Z_i \eta^i + Z_j \bar{\eta}^j - iZ_i \Sigma^r X_r \Phi^i + iZ_j \Sigma^r X_r \bar{\Phi}^j - iZ_i V^b \nabla_b \Phi^i + iZ_j V^b \nabla_b \bar{\Phi}^j$$

$$- iV^\alpha \nabla_\alpha Z + iV_\alpha \bar{\nabla}_\alpha Z + \frac{4}{3} V(A) Z$$

(3.27)
Plugging in the value of $V(A)$ gives

$$
\delta Z = Z_t \eta^i + Z_j \eta^j - iZ_t \Sigma^r X_r \Phi^i + iZ_j \Sigma^r X_r \tilde{\Phi}^j - iZ_t V^b \nabla_b \Phi^i + iZ_j V^b \nabla_b \tilde{\Phi}^j
\right. \\
\left. + i\nabla_\alpha (V^\alpha Z) - i\tilde{\nabla}^\alpha (V_\alpha Z) - \frac{1}{3} \Delta_b V^b Z
\right) \quad (3.28)
$$

The two terms in the last line which appear to vanish as total derivatives actually do not. To see why, note that the actual statement of a vanishing total derivative involves only the coordinate derivative:

$$
0 = \partial_M (E E_\alpha M V^\alpha Z) = \nabla_M (E E_\alpha M V^\alpha Z) + h_M X_b (E E_\alpha M V^\alpha Z)
$$

The term involving the connection usually vanishes by gauge invariance; however, in this case $V^\alpha$ is not conformally invariant (though the other terms in the parentheses are), and so the second term yields

$$
Ef_\alpha \delta_\alpha (V^\alpha Z) = E (-i f_\alpha \delta_\alpha V^\alpha Z)
$$

Evaluating the first term yields

$$
E \left( \nabla_\alpha (V^\alpha Z) + T_\alpha B V^\alpha Z \right)
$$

The trace of the torsion tensor vanishes, which leads to the identity

$$
i \nabla_\alpha (V^\alpha Z) = -f_\alpha \delta_\alpha V^\alpha Z + \text{t.d.}
$$

Integrating by parts on the $\Delta_b V^b$ term gives the same explicit connections but with the opposite sign, yielding

$$
\delta S = -3Z_t \eta^i - 3Z_j \eta^j + 3iZ_t \Sigma^r X_r \Phi^i - 3iZ_j \Sigma^r X_r \tilde{\Phi}^j + V^b \left( \Delta_b Z + 3iZ_t \nabla_b \Phi^i - 3iZ_j \nabla_b \tilde{\Phi}^j \right)
\right) \quad (3.29)
$$

There are several annoying features of this expression. One is that the terms involving $V^b$ are not individually conformally invariant. Another is that in the linear compensator model, we had a clear factor of $L$ out front of all the terms which we could gauge to one. Here we would like to gauge $Z = 1$ to arrive at the supergravity of Binetruy, Girardi, and Grimm [10], but none of the terms possess an explicit $Z$ out front. We can deal with both of these issues by the following field redefinition:

$$
K = -3 \log Z
$$

$K$ is a superfield which transforms non-linearly under a conformal transformation. If we choose $Z = \Phi_0 \Phi_0 e^{-K/3}$, we see that this $K$ is essentially the same object as the canonical Kähler potential:

$$
K = K - 3 \log (\Phi_0 \Phi_0)
$$

The advantage of this definition is that we may now rewrite $\delta S$ as

$$
\delta S = Z \left( K \eta^i + K_j \eta^j + \Sigma^r K_r + V^b \left( -4G_b + K_b \right) \right) \quad (3.31)
$$
where we have defined

\[ G_b \equiv -Z^{1/2} \Delta_b Z^{-1/2} \]  

(3.32)

\[ K_{\alpha \dot{\alpha}} \equiv K_{i j} \nabla_\alpha \Phi^i \nabla_{\dot{\alpha}} \Phi^j \]  

(3.33)

\[ K_r \equiv -i K_{i r} \Phi^i + i K_{\bar{j} r} \bar{\Phi}^\bar{j} \]  

(3.34)

If we choose \( Z = \Phi_0 \bar{\Phi}_0 e^{-K/3} \), then we find

\[ \delta S = Z \left( K_i \eta^i + K_{\bar{j}} \bar{\eta}^{\bar{j}} + \Sigma^r K_r + V^b (-4 G_b + K_b) - \frac{3 \eta_0}{\Phi_0} - \frac{3 \bar{\eta}_0}{\bar{\Phi}_0} \right) \]  

(3.35)

and the chiral first-order action is superficially the same as the linear one except for the exchange of the \( L \) sector for the \( \eta_0 \) sector and the exchange of the \( L \) compensator for \( Z \).

The importance of this observation is that it simplifies the task of finding the second-order action for both of these theories. Rather than treating each individually, we can focus on their common features and only worry about where they specifically differ.

Let us consider several other terms that we might like to include in both of these models.

### 3.3 Superpotential terms

A superpotential term is a chiral action \( S_P \) defined as

\[ S_P = \int \mathcal{E} P + \text{h.c.} \]  

(3.36)

where \( P \) is some chiral superfield of weight \((3, 2)\). For the simplest chiral compensator model, \( P = \Phi_0 W \) where \( W \) is the object one normally calls the superpotential. Because we’re interested in linear compensator models as well as the general chiral model, we will use the more generic name \( P \) to denote this F-term superfield Lagrangian.

Since the superpotential terms involve purely chiral and antichiral actions, we can use the quantum chiral and antichiral gauges to describe them. We note that

\[ \delta_c \mathcal{E} = H^a_{\alpha} + H^a_{\dot{\alpha}} = 0 \]  

(3.37)

in quantum chiral gauge, so only the chiral variation of the integrand remains. The variation of the superpotential term is then simply

\[ \delta_c S_P = \int \mathcal{E} P_i \eta^i + \text{h.c.} \]  

(3.38)

implying that the superpotential plays no role in the pure conformal supergravity equations of motion. (That it plays a role in Poincaré supergravity arises because of the presence of the chiral compensator.)

### 3.4 Yang-Mills terms

The Yang-Mills term we will consider is

\[ S_{YM} = \frac{1}{4} \int \mathcal{E} f_{rs} \omega^a \omega^s_a + \text{h.c.} \]  

(3.39)
where \( f_{rs} \) is a holomorphic covariant gauge coupling. In the simplest of cases, \( f_{rs} = \delta_{rs} \), but we will for the moment allow for a more generic holomorphic coupling.

As before, one finds quantum chiral gauge the simplest for the chiral action. Using

\[
\delta_c W_\alpha^r = -\frac{i}{4} \bar{\nabla}^2 \nabla_\alpha \Sigma^r - \frac{1}{4} \bar{\nabla}^2 \left( V_{\alpha \beta} \bar{W}^{\beta r} \right)
\]

(3.40)

as well as

\[
\delta_c f_{rs} = f_{rs,i} \eta^i
\]

one immediately finds

\[
\delta S_{YM} = \int \mathcal{E} \left( \frac{1}{4} f_{rs,i} \eta^i W^{ar} W_\alpha^s - \frac{i}{8} f_{rs} W^{ar} \bar{\nabla}^2 \nabla_\alpha \Sigma^s - \frac{1}{8} f_{rs} W^{ar} \bar{\nabla}^2 (V_{\alpha \beta} \bar{W}^{\beta s}) \right) + \text{h.c.}
\]

\[
= \int \mathcal{E} \left( \frac{1}{4} f_{rs,i} \eta^i W^{ar} W_\alpha^s \right) + \int E \left( \frac{i}{2} f_{rs} W^{ar} \nabla_\alpha \Sigma^s + \frac{1}{2} f_{rs} V_{\alpha \alpha} W^{ar} \bar{W}^{\alpha s} \right) + \text{h.c.}
\]

(3.42)

There is the possibility of introducing the Yang-Mills interactions by requiring the linear compensator \( L \) to obey the modified linearity conditions

\[
\bar{\nabla}^2 L = 2k \text{Tr} (W^a W_\alpha), \quad \nabla^2 L = 2k \text{Tr} (\bar{W}^a \bar{W}_\alpha)
\]

Then Yang-Mills interactions can be made part of the structure of superspace when the compensator is gauged to 1. This tends to introduce non-holomorphic gauge couplings. We will avoid this possibility for now and restrain ourselves to the normal holomorphic Yang-Mills terms.

### 3.5 Generic first-order structure

We summarize the generic structure that the arbitrary chiral model and the minimal linear compensator models possess. The common part of the first order action consists of a sum of four terms. They are:

\[
(\delta S)_G = \left[ -4X V^b G_b \right]_D
\]

\[
(\delta S)_K = \left[ X (V^b K_b + \Sigma^r K_r + \eta^i K_i + \bar{\eta}^j \bar{K}_j) \right]_D
\]

\[
\delta S_P = \left[ \eta^i P_i \right]_F + \text{h.c.}
\]

\[
\delta S_{YM} = [V^a Y_a + \Sigma^r Y_r]_D + \left[ \eta^i Y_i \right]_F + \left[ \bar{\eta}^j \bar{Y}_j \right]_F
\]

(3.43)\( \quad \) (3.44)\( \quad \) (3.45)\( \quad \) (3.46)

where \( X \) is the compensator \( (L \) or \( Z \) and

\[
G_b \equiv -X^{1/2} \Delta_b X^{-1/2}
\]

\[
K_{\alpha \dot{\alpha}} \equiv K_{ij} \nabla_\alpha \Phi^i \nabla_{\dot{\alpha}} \bar{\Phi}^j
\]

\[
\mathcal{K}_r \equiv -i K_i X_r \Phi^i + i K_{\dot{j}} X_r \bar{\Phi}^\dot{j}
\]

\[
\mathcal{Y}_i \equiv \frac{1}{4} f_{rs,i} W^{ar} W_\alpha^s
\]

\[
\mathcal{Y}_{\alpha \dot{\alpha}} \equiv -(f_{rs} + \tilde{f}_{rs}) W_\alpha^r \bar{W}_\alpha^s
\]

\[
\mathcal{Y}_r \equiv -\frac{i}{2} \bar{\nabla}^\alpha (f_{rs} W_\alpha^s) + \text{h.c.}
\]

(3.47)\( \quad \) (3.48)\( \quad \) (3.49)\( \quad \) (3.50)\( \quad \) (3.51)\( \quad \) (3.52)
We will find use to denote $G_{rs} \equiv f_{rs} + \bar{f}_{rs}$. Then the last two equations above may be written

$$
\mathcal{Y}_{r\dot{a}} \equiv -G_{rs}W^r_{a} \bar{W}_{s}^a \\
\mathcal{Y}_r \equiv -\frac{i}{2}(\nabla^a G_{rs})W^s_{a} + \frac{i}{2}(\nabla_{\dot{a}} G_{rs})\bar{W}^{\dot{a}}s - \frac{i}{2}G_{rs} \nabla^a W^s_{a}
$$

using $\nabla^a W^r_{a} = \nabla_{\dot{a}} \bar{W}^r_{\dot{a}}$.

The equations of motion amount to

$$
0 = -4XG_b + XK_b + \mathcal{Y}_b \\
0 = K_r + \mathcal{Y}_r \\
0 = -\frac{1}{4}\nabla^2(XK_i) + P_i + \mathcal{Y}_i
$$

(3.53)

(3.54)

(3.55)

For the linear compensator model, there is the additional term

$$
\delta S_L = [\mathcal{L}(V_R + K)]_D
$$

(3.56)

along with that model’s equation of motion

$$
0 = \nabla^2 \nabla_{a}(V_R + K) = \nabla^2 \nabla_{\dot{a}}(V_R + K)
$$

(3.57)

which implies that $V_R = -K$ up to the real part of a chiral superfield.

The structure we have identified here is actually more general than this treatment indicates. The same features persist in arbitrary models involving any number of linear and chiral superfields. A brief discussion of the first order variation of an arbitrarily coupled linear superfield is given in Appendix A.

4 Going to second order

In order to construct a one-loop effective action, we require the action to second order in the quantum deformations. The simplest way to do this is a sort of bootstrap: vary our first order expression again to first order.

However, doing so immediately tends to produce a nasty set of terms involving many derivatives of the compensator $X$ for the graviton’s action. The reason is easy to see: the action for the graviton is hidden within the action for the compensator. In addition to a term $XV^a \Box V_a$, there would be a host of terms involving derivatives of $X$ needed in order to make this expression invariant under special conformal transformations. One way to simplify this would be to eliminate many of these terms by choosing a gauge where $X$ is constant and then degauging to Poincaré derivatives. Unfortunately this sacrifices the conformal invariance of the classical action before quantization has even taken place. A better approach would be to introduce conformally invariant derivatives, with respect to which $X$ is covariantly constant. These would compactly encode the many terms involving derivatives of $X$ in conformally invariant combinations. It is to this construction that we now turn.
4.1 A brief interlude: conformally invariant (or compensated) derivatives

4.1.1 Definition

In the preceding discussion, we introduced the conformally primary superfield \( G_b \) which was defined in terms of the dimension 2 compensator \( X \). When \( X \) is gauged to unity and the conformally covariant derivatives are themselves “degauged”, the object \(-X^{1/2}\Delta_b X^{-1/2}\) reduces simply to the Poincaré superfield \( G_b \), but the existence of this conformally primary combination means we may identify the equivalent of \( G_b \) even in the conformal theory. We may similarly identify other Poincaré equivalents and thereby perform something very much like a degauging while still maintaining the underlying conformal invariance.

We begin with \( X \), a primary Hermitian superfield with \( \Delta = 2 \) and \( w = 0 \). Define \( U = \log X \) so that under scalings, \( U \) transforms nonlinearly into a constant, here \( DU = 2 \). Then we define the compensator-assisted derivatives as

\[
\mathcal{D}_\alpha \equiv \nabla_\alpha - \frac{1}{2} \nabla_\alpha U D - \frac{1}{2} \nabla^\beta U M_{\beta\alpha} + \frac{3i}{4} \nabla_\alpha U A \tag{4.1}
\]

\[
\mathcal{D}^{\dot{\alpha}} \equiv \nabla^{\dot{\alpha}} - \frac{1}{2} \nabla^{\dot{\alpha}} U D - \frac{1}{2} \nabla_\beta U M^{\beta\dot{\alpha}} - \frac{3i}{4} \nabla^{\dot{\alpha}} U A \tag{4.2}
\]

These new derivatives are constructed so that when they act on a conformally primary object, the result is conformally primary.

We are not the first to construct these objects. Kugo and Uehara, in their treatment of conformal supergravity [14], constructed these operators almost immediately out of the covariant derivatives, dubbing these the \( u \)-assisted derivatives, where \( u \) denoted the compensator being used. Their motivation seemed to be the desire for operators that would act on conformally primary superfields to generate more conformally primary superfields. In that sense, these new operators are special conformal invariant rather than covariant.

The purely undotted objects have a new algebra

\[
\{\mathcal{D}_\beta, \mathcal{D}_\alpha\} = \frac{1}{2} \left( \nabla^2 U + \nabla^\gamma U \nabla_\gamma U \right) M_{\beta\alpha} = \frac{1}{2} \frac{1}{X} \nabla^2 X M_{\beta\alpha} = -4\overline{R} M_{\beta\alpha} \tag{4.3}
\]

Similarly,

\[
\{\mathcal{D}^{\dot{\beta}}, \mathcal{D}^{\dot{\alpha}}\} = -4\overline{R} M^{\dot{\beta}\dot{\alpha}} \tag{4.4}
\]

where we have defined

\[
R \equiv -\frac{1}{8X} \nabla^2 X, \quad \overline{R} \equiv -\frac{1}{8X} \nabla^2 X \tag{4.5}
\]

From these definitions, \( R \) possesses scaling and \( U(1)_R \) weights \((\Delta, w) = (1, +2)\) and \( \overline{R} \) the weights \((1, -2)\). It is straightforward to show that in the limit where we gauge fix \( X \) to unity, these \( R \)’s become the \( R \)’s of Poincaré supergravity. However, these versions are more useful since they are also conformally invariant by nature of the fact that the new covariant derivatives are themselves conformally invariant. Furthermore, one may show that they are chiral with respect to the new derivatives:

\[
\mathcal{D}^{\dot{\alpha}} R = 0, \quad \mathcal{D}_\alpha \overline{R} = 0. \tag{4.6}
\]
It is straightforward to guess the form of the analogues of $G_c$ and $X_\alpha$. Demanding that the definition of $G_c$ match when $X$ is fixed to unity (and also be conformally invariant) gives

$$G_{\alpha\dot{\alpha}} = -\frac{1}{4} \{\nabla_\alpha, \nabla_\dot{\alpha}\} U + \frac{1}{4} \nabla_\alpha U \nabla_\dot{\alpha} U = \frac{1}{2} X^{1/2} \{\nabla_\alpha, \nabla_\dot{\alpha}\} X^{-1/2} \tag{4.7}$$

which is as we have defined it before. Defining $X_\alpha$ as $D_\alpha R - D_\dot{\alpha} G_{\alpha\dot{\alpha}}$ leads to

$$X_\alpha = \frac{3}{8} \tilde{\nabla}^2 \nabla_\alpha U, \quad X^{\dot{\alpha}} = \frac{3}{8} \tilde{\nabla}^2 \nabla^{\dot{\alpha}} U \tag{4.8}$$

which is conformally invariant automatically.

We briefly pause to note the following features. If $X = \Phi_0 = \Phi_0 e^{-K/3}$,

$$X_\alpha = -\frac{1}{8} \tilde{\nabla}^2 \nabla_\alpha K = -\frac{1}{8} (\tilde{D}^2 - 8 R) D_\alpha K$$

as in Kähler $U(1)$ supergravity. Similarly, if $X = L$, then $R = 0$ as in new minimal supergravity.

We next define the bosonic derivative $D_{\alpha\dot{\alpha}}$ by the anti-commutator

$$\{D_\alpha, D^{\dot{\alpha}}\} = -2 i D_\alpha^{\dot{\alpha}} - \lambda G^{\dot{\alpha} \dot{\beta}} M_{\beta \alpha} + \lambda G_{\alpha \dot{\beta}} M^{\dot{\beta} \dot{\alpha}} + 3 i \lambda G_{\alpha \dot{\alpha}} A \tag{4.9}$$

We have introduced into this definition a parameter $\lambda$ which parametrizes how much of the various bosonic connections of $D_\alpha$ is stored in the additional “curvatures” on the right hand side. $\lambda = 1$ corresponds to the standard $U(1)$ supergravity of Binétruy, Girardi, and Grimm [10] and what is achieved by straightforwardly degauging from conformal to Poincaré supergravity [11]. $\lambda = 0$ corresponds to a redefinition of that theory so that the $\alpha \dot{\alpha}$ curvatures are trivial. (This is the choice made in [8] and [12].) The latter has the simplest-looking curvatures overall, but it introduces a nonzero torsion $T_{\epsilon \delta \alpha}$ proportional to the dual of $G_\alpha$, which leads to a bosonic Riemann curvature tensor lacking the common symmetries and with an auxiliary superfield hiding within the spin connection. For this reason $\lambda = 0$ seems to be ill-suited for component calculations; however, for the pure superfield manipulations we perform here, it leads to a simpler algebra for the covariant derivatives. The two definitions are completely equivalent, of course, and differ only in the definition of the bosonic connections.

These definitions lead to

$$D_\alpha^{\dot{\alpha}} \equiv \nabla_\alpha^{\dot{\alpha}} - \frac{i}{2} \nabla_\alpha U D^{\dot{\alpha}} - \frac{i}{2} \nabla^{\dot{\alpha}} U D_\alpha - \frac{1}{2} \nabla_\alpha \nabla^{\dot{\alpha}} U + \left( \frac{3}{8} \nabla_\alpha, \nabla^{\dot{\alpha}} U + \frac{3\lambda}{2} G_\alpha^{\dot{\alpha}} \right) A$$

$$+ \left( -\frac{i}{4} \nabla_\alpha \nabla_\beta - \frac{i\lambda}{2} G_{\alpha \beta} \right) M^{\dot{\beta} \dot{\alpha}} + \left( -\frac{i}{4} \nabla^{\dot{\alpha}} \nabla^{\dot{\beta}} U + \frac{i\lambda}{2} G^{\dot{\beta} \dot{\alpha}} \right) M_{\beta \alpha} \tag{4.10}$$

The newly-defined curvatures are straightforward to work out. For the bosonic-fermionic
curvatures,

\[ T_{\gamma(\beta\delta)} = -2i\epsilon_{\gamma\beta\varepsilon_{\beta\delta}} R \]  

(4.11)

\[ T_{\gamma(\beta\delta)} = i\lambda G_{\gamma\beta} \epsilon_{\beta\alpha} - 2i(1 - \lambda) G_{\alpha\beta} \epsilon_{\gamma\beta} \]  

(4.12)

\[ F_{\beta(a\dot{a})} = -\frac{3\lambda}{2} D_{\beta} G_{a\dot{a}} - \epsilon_{\beta\alpha} X_{\dot{a}} \]  

(4.13)

\[ R_{\delta(\gamma)} = \sum_{\beta\alpha} \left[ i\epsilon_{\delta\gamma} D_{\beta} G_{\alpha\gamma} + \frac{i\lambda}{2} D_{\delta} G_{\beta\gamma} \epsilon_{\gamma\alpha} - i\epsilon_{\delta\beta} \epsilon_{\gamma\alpha} D_{\delta} \tilde{R} \right] \]  

(4.14)

\[ R_{\delta(\gamma)} = 4i\epsilon_{\delta\gamma} W_{\gamma\dot{a}} + \sum_{\beta\dot{a}} \epsilon_{\gamma\dot{a}} \left[ \frac{i}{3} \epsilon_{\delta\gamma} \bar{X}_{\dot{a}} + \frac{i\lambda}{2} D_{\delta} G_{\beta\dot{a}} \right] \]  

(4.15)

Note that these curvatures simplify a fair amount by choosing \( \lambda = 0 \).

The bosonic torsions are

\[ T_{(\beta\dot{\beta})(a\dot{a})} \gamma D_{\gamma} = -2\epsilon_{\beta\dot{a}} W_{\beta\alpha\gamma} D_{\gamma} - \frac{1}{2} \epsilon_{\beta\dot{a}} D_{(\beta} R_{\alpha)(\gamma} - \frac{1}{6} \epsilon_{\beta\dot{a}} X_{(\beta} D_{\alpha)} - \frac{1}{2} \epsilon_{\beta\dot{a}} D_{(\beta} G_{\alpha)\gamma} D_{\gamma} \]  

(4.16)

\[ T_{(\beta\dot{\beta})(a\dot{a})} \gamma D_{\gamma} = -2\epsilon_{\beta\dot{a}} W_{\beta\alpha\gamma} D_{\gamma} + \frac{1}{2} \epsilon_{\beta\dot{a}} D_{(\beta} \bar{R}_{\alpha)(\gamma} + \frac{1}{6} \epsilon_{\beta\dot{a}} \bar{X}_{(\beta} D_{\alpha)} + \frac{1}{2} \epsilon_{\beta\dot{a}} D_{(\beta} G_{\alpha)\gamma} D_{\gamma} \]  

(4.17)

\[ T_{(\beta\dot{\beta})(a\dot{a})} \gamma D_{\gamma} = -2i(1 - \lambda) G_{\beta\alpha} D_{\alpha\beta} + 2i(1 - \lambda) G_{a\beta} D_{\beta\dot{a}} \]  

(4.18)

Note the last torsion vanishes for \( \lambda = 1 \).

The part of the Riemann tensor acting on spinor indices is

\[ \frac{1}{2} R_{(\beta\dot{\beta})(a\dot{a})} M^{\phi\gamma} \equiv \epsilon_{\beta\dot{a}} \sum_{\beta\alpha} \left( \frac{1}{2} D_{\beta} W_{\alpha\phi\gamma} M^{\gamma\phi} + \frac{1}{12} D_{\beta} X^{\gamma} M_{\alpha\gamma} - \frac{1}{8} D^{2} R M_{\beta\alpha} + 2 R R M_{a\beta} \right) \]  

\[ - \frac{1}{4} \epsilon_{\beta\alpha} D_{(\beta} \gamma G_{\phi\alpha)} M^{\gamma\phi} - \frac{i\lambda}{2} D_{\beta} G_{\phi\alpha} M_{\beta\alpha} + \frac{i\lambda}{2} D_{\alpha\beta} G_{\phi\alpha} M_{\phi\beta} \]  

\[ - \frac{\lambda^2}{2} G_{\beta\alpha} G^{\beta\alpha} M_{\phi\alpha} + \frac{\lambda^2}{2} G_{a\beta} G^{a\beta} M_{\phi\beta} \]  

\[ + \frac{1}{2} (\lambda^2 - \lambda) \epsilon_{\beta\alpha} G_{\phi\beta} G^{\phi\beta} M_{\beta\alpha} \]  

(4.19)

The other half can be found by Hermitian conjugation.

The remaining \( U(1) \) curvature is

\[ F_{(\beta\dot{\beta})(a\dot{a})} = -\frac{3\lambda}{2} D_{(\beta\dot{\beta})} G_{(a\dot{a})} \]  

(4.20)

Again note the simplifications which occur for the choice \( \lambda = 0 \).

### 4.1.2 Deformation

The compensated derivatives (for \( \lambda = 0 \)) can be compactly written as

\[ D_{\alpha} \equiv \nabla_{\alpha} + \frac{1}{4} \left( \nabla_{\beta} U \right) \{ S_{\beta}, Q_{\alpha} \}, \quad D^{\dot{a}} \equiv \nabla^{\dot{a}} + \frac{1}{4} \left( \nabla_{\beta} U \right) \{ \bar{S}_{\beta}, \bar{Q}^{\dot{a}} \} \]  

\[ D_{a\dot{a}} \equiv \frac{i}{2} \{ D_{\alpha}, D_{\dot{a}} \} \]
provided we restrict them to only act on conformally primary objects. It is in this form that it is easiest to demonstrate that if $\Psi$ is primary, so is $D_\alpha \Psi$ where $\Psi$ possesses arbitrary weights and Lorentz indices.

We have previously argued that to first order the spinor derivatives vary (in Hermitian quantum gauge) as $\delta \nabla_\alpha = [iV, \nabla_\alpha]$ and $\delta \nabla_\dot{\alpha} = [-iV, \nabla_\dot{\alpha}]$, where we had expanded

$$V \equiv V^A \nabla_A + V^\dot{\alpha} \nabla_\dot{\alpha}$$

It follows then that the compensated spinor derivatives should vary as

$$\delta D_\alpha = [iV, D_\alpha] + \frac{1}{4} \nabla_\beta (-iV U + \delta U) \{S_\beta, Q_\alpha\}$$

where we have substituted $D$ for $\nabla$ in the commutator. Note that $(-iV U + \delta U)$ is conformally primary of dimension zero, and so we may replace the $\nabla_\beta$ acting on it with $D_\beta$.

Further simplifications arise if we choose to expand $V$ in terms of the compensated derivative rather than the covariant derivative:

$$V^a = V^A \nabla_A + V^\dot{\alpha} \nabla_\dot{\alpha} = V^A D_A + V^\dot{\alpha} \nabla_\dot{\alpha}$$

One may check that the $V^a$'s are now conformally primary objects. In particular, it is easy to show (by considering the variation of a chiral superfield of vanishing weight for example) that

$$V'^a = V^a, \quad V'^{\dot{\alpha}} = \tilde{V}^{\dot{\alpha}}$$

where $V'^a \equiv -\frac{i}{8} D_\phi V'^{\phi a} + \tilde{V}'^a$. Then provided we define a theory entirely in terms of $V^a$ and $\tilde{V}^{\dot{\alpha}}$, we can make use of these conformally invariant derivatives when we calculate deformations of the quantum theory.

Henceforth we suppress the primes and trade the conformally covariant prepotentials for the conformally invariant (or compensated) ones. One can show that

$$V(D) = \frac{1}{2} D_b V^b + \frac{1}{2} D^a V_a + \frac{1}{2} D_\dot{\alpha} V^{\dot{\alpha}} + \tilde{V}(D)$$

$$V(A) = -\frac{1}{4} \Delta_\dot{\alpha} V^{\dot{\alpha}} + V^b G_b - \frac{3i}{4} D^a V_a + \frac{3i}{4} D_\dot{\alpha} V^{\dot{\alpha}} + \tilde{V}(A)$$

$$V(M)_{\beta\alpha} = +\frac{1}{2} D_{(\beta} V_{\alpha)} + \frac{i}{8} D^\phi D_{(\beta} V_{\alpha)} + \frac{i}{2} V_{(\alpha\dot{\phi} G_{\beta})} \phi + \tilde{V}(M)_{\beta\alpha}$$

$$V(M)_{\dot{\beta}\dot{\alpha}} = +\frac{1}{2} D_{(\dot{\beta}} V_{\dot{\alpha})} - \frac{i}{8} D^\phi D_{(\dot{\beta}} V_{\dot{\alpha})} + \frac{i}{2} V_{(\dot{\alpha}\dot{\phi} G_{\dot{\beta})}} + \tilde{V}(M)_{\dot{\beta}\dot{\alpha}}$$

Note the forms are quite similar to what we had in (2.35), except for the appearance of the new superfield $G_b$. We have also introduced the conformally invariant operator $\Delta_\alpha = -\frac{1}{2} \{D_\alpha, \dot{D}_\dot{\alpha}\}$.

Since $U$ obeys $D_A U = 0$, it follows that

$$\delta D_\alpha = [iV, D_\alpha] + \frac{1}{4} \nabla_\beta (-2iV(D) + \delta U) \{S_\beta, Q_\alpha\}$$
from which we may derive the variations of each of the spinor connections. We find

\[ H_{\alpha \beta} = iD_\alpha V_\beta - iV^c T_{ac\beta} - iV(M)_{\alpha \beta} + \frac{i}{2} V(D) \epsilon_{\alpha \beta} + V(A) \epsilon_{\alpha \beta} \]

\[ H_{\dot{\alpha} \dot{\beta}} = iD_{\dot{\alpha}} V_{\dot{\beta}} - iV^c T_{\dot{a}c\dot{\beta}} \]

\[ H_{\alpha(\beta \dot{\beta})} = iD_\alpha V_{\beta \dot{\beta}} + 4V_\beta \epsilon_{\alpha \beta} \]

\[ \Lambda_\alpha = \frac{1}{2} D_\alpha (\delta U) \]

\[ \omega_\alpha = iD_\alpha V(A) - iV^b F_{ab} - \frac{3}{2} D_\alpha V(D) - \frac{3i}{4} D_\alpha \delta U \]

\[ \Omega_\alpha(M) = iD_{\alpha} V(M) + 4i \bar{R} V^c M_{\beta \alpha} - iV^b R_{ab}(M) - iD^\beta V(D) M_{\beta \alpha} + \frac{1}{2} D^c \delta U M_{\beta \alpha} \quad (4.28) \]

and for their conjugates

\[ H_{\dot{\alpha} \dot{\beta}} = -iD_{\dot{\alpha}} V_{\dot{\beta}} + iV^c T_{\dot{a}c\dot{\beta}} \]

\[ H_{\alpha(\beta \dot{\beta})} = -iD_\alpha V_{\beta \dot{\beta}} + 4V_{\beta} \epsilon_{\alpha \beta} \]

\[ \Lambda_{\dot{\alpha}} = \frac{1}{2} D_{\dot{\alpha}} (\delta U) \]

\[ \omega_{\dot{\alpha}} = -iD_{\dot{\alpha}} V(A) + iV^b F_{ab} - \frac{3}{2} D_{\dot{\alpha}} V(D) + \frac{3i}{4} D_{\dot{\alpha}} \delta U \]

\[ \Omega_{\dot{\alpha}}(M) = -iD_{\dot{\alpha}} V(M) - 4i \bar{R} V^c M_{\dot{\beta} \alpha} + iV^b R^{\dot{c} \dot{b}}(M) + iD_{\dot{\beta}} V(D) M_{\dot{\beta} \alpha} + \frac{1}{2} D_{\dot{c}} \delta U M_{\dot{\beta} \alpha} \quad (4.29) \]

The variation of the bosonic derivatives is straightforward to work out from the above results. Using these, one may for example work out the variations of the superfields \( G_{\alpha \dot{\alpha}} \) and \( R \) in the language of these compensated derivatives. For \( R \), it is actually easier to work in the original theory at first. Recall the chiral variation of an arbitrary superfield \( \Psi \) can be defined by

\[ \delta_c \Psi = \delta \Psi + iV \Psi \quad (4.30) \]

which generalizes the case where \( \Psi \) is itself chiral. Then the chiral variation of \( R \) is

\[ \delta_c R = -\frac{1}{8X} \nabla^2 (X \delta U) + \frac{1}{8X} \delta_c U \nabla^2 X = -\frac{1}{8} D^2 \delta_c U \quad (4.31) \]

Similarly, the chiral variation of \( X_\alpha \) is

\[ \delta_c X_\alpha = \frac{3}{8} \nabla^2 (\nabla_\alpha \delta U + 2iV \nabla_\alpha U - i \nabla_\alpha (VU)) \]

\[ = \frac{3}{8} (D^2 - 8R)(D_\alpha \delta U + 2iZ_\alpha - 2iD_\alpha V(D)) \quad (4.32) \]

where

\[ Z_\alpha \equiv V \nabla_\alpha U \]

\[ = -\frac{1}{2}(D^2 - 12\bar{R})V_\alpha + \frac{1}{2} D^\beta D_\alpha V_\beta \]

\[ - \frac{1}{6} D_\alpha D_\beta D_\gamma V^{\beta \gamma} + \frac{i}{3} D_\alpha (G_{\beta \dot{\gamma}} V^{\beta \dot{\gamma}}) + \frac{i}{24}(D^2 - 12\bar{R})D^\beta V_{\beta \alpha} \]

\[ + D^\beta (\bar{R} V_{\alpha \dot{\beta}}) + \frac{i}{3} X^\beta V_{\alpha \dot{\beta}} \quad (4.33) \]
Calculating $\delta G_{a\dot{a}}$ is a bit more difficult since its definition in terms of $X$ necessarily involves both dotted and undotted spinor derivatives in a symmetric fashion. The most straightforward way to proceed seems to be to work out its variation by calculating the variation of the torsion component $\delta T_{\gamma\beta\alpha}$. This gives the following rather complicated expression:

$$\delta G_{a\dot{a}} = -\frac{1}{4} [D_a, D_{\dot{a}}] \delta U - H_{a\dot{a}} b G_b - i V^{\beta} D_{\beta} G_{a\dot{a}} - i V^{\dot{\beta}} D_{\dot{\beta}} G_{a\dot{a}}$$

$$- \frac{1}{2} \Delta_{a\dot{a}} \Delta_b V^b - \frac{1}{2} D_{a\dot{a}} D_b V^b - \frac{1}{32} (D^2 - 8R)D + \text{h.c.}) V_{a\dot{a}}$$

$$+ \frac{1}{2} (D^\gamma V_{a\dot{a}}) W_{\gamma\dot{a}a} + \frac{1}{2} (D_{\dot{\gamma}} V_{a\dot{a}}) W_{\dot{\gamma}a\dot{a}} - \frac{1}{2} \Delta_{a\dot{a}} (V^b G_b)$$

$$+ \frac{1}{8} D^\beta V_{\beta\dot{a}} D_a R + \frac{1}{8} D_a V_{\beta\dot{a}} D^\beta R + \frac{1}{6} D_{(\beta} V_{\alpha)\dot{a}} X^\beta$$

$$- \frac{1}{8} D^\dot{\beta} V_{\beta\dot{a}} D_a R - \frac{1}{8} D_{\dot{a}} V_{\beta\dot{a}} D^\dot{\beta} R - \frac{1}{6} D_{(\dot{\beta} \dot{a})\alpha} X^\beta$$

$$- \bar{R} R V_{a\dot{a}} - \frac{1}{4} V_{a\dot{a}} D^\beta X^\beta$$

where we have defined

$$\tilde{\delta} U \equiv \delta U + i D^\beta V_{\beta} - i D_{\dot{\beta}} V_{\dot{\beta}} + \Delta_b V^b.$$ (4.34)

For the linear compensator model, $\tilde{\delta} U = L^{-1} L$, but for the generic chiral model

$$\tilde{\delta} U = -\frac{1}{3} \left( K_i \bar{\eta}^i + K_{j\dot{a}} \bar{\eta}^j - 2 \Delta_b V^b - 4 V^b G_b + V^b K_b \right)$$ (4.35)

The expression for $\delta G_{a\dot{a}}$ involves a combination of the supergravity potentials that has been succinctly combined into $H_{a\dot{a}} b$, which is the deformation of the bosonic vierbein. It can be calculated from

$$\delta D_a = -H_{a\dot{a}} b D_{\dot{a}} - H_a b X_b,$$

the left hand side of which can itself be calculated easily from $\delta D_a$ and $\delta D_{\dot{a}}$. The reason for collecting these terms in this way is that we will eventually find they cancel out.

Rearranging a number of terms leads to

$$\delta G_{a\dot{a}} = \frac{1}{2} \Delta_{a\dot{a}} \tilde{\delta} U - H_{a\dot{a}} b G_b - i V^{\beta} D_{\beta} G_{a\dot{a}} - i V^{\dot{\beta}} D_{\dot{\beta}} G_{a\dot{a}}$$

$$- \frac{1}{2} \Delta_{a\dot{a}} \Delta_b V^b - \frac{1}{2} D_{a\dot{a}} D_b V^b - \frac{1}{32} (D^2 - 8R)D + \text{h.c.}) V_{a\dot{a}}$$

$$+ \frac{1}{2} (R D^2 + \bar{R} D^2) V_{a\dot{a}} + (D^\gamma V_{a\dot{a}}) W_{\gamma\dot{a}a} + (D_{\dot{\gamma}} V_{a\dot{a}}) W_{\dot{\gamma}a\dot{a}}$$

$$- G^b \Delta_b V_{a\dot{a}} - (\Delta_{a\dot{a}} V^b) G_b - \Delta_b (V^b G_{a\dot{a}}) + (\Delta_b V^b) G_{a\dot{a}} + \frac{1}{2} V^b \Delta_b G_{(a\dot{a})}]$$

$$+ \frac{1}{2} D^\beta V_{\beta\dot{a}} (D_{\dot{a}} R - \frac{1}{3} X_{\dot{a}}) + \frac{1}{12} D^\beta V_{a\dot{a}} X_{\dot{a}}$$

$$- \frac{1}{2} D^\beta V_{\beta\dot{a}} (D_{\dot{a}} R - \frac{1}{3} X_a) + \frac{1}{12} D_{\beta} V_{a\dot{a}} X^\beta$$

$$- \bar{R} R V_{a\dot{a}} - \frac{1}{8} V_{a\dot{a}} (D^\beta X_{\beta} + \text{h.c.}) + \frac{i}{4} V_{a\dot{a}} D^\beta G_{a\beta} - \frac{i}{4} V_{a\beta} D^{\dot{\beta}} G_{\beta a}$$ (4.36)
4.2 Proceeding to second order

We would like to proceed to second order so that we can perform one-loop calculations. The immediate difficulty we face is that we solved our constraints only to first order. For example, $\tilde{V}(A)$ might also involve some second order object of the form $V^a\mathcal{O}_a^b V_b$ where $\mathcal{O}_a^b$ is some conformally invariant operator. Then in analyzing the variations of the $W$’s, we should have worked to second order in $V^a$ to find out if any such object exists.

There are two approaches one could take at this point. One would be to return to the original analysis and redo it to second order and determine what modifications are necessary. The second approach is to use our ability to take first order variations and to vary to first order the first order action that we already have – thereby bootstrapping to second order. This is possible since our first order solution was not dependent on any specific origin point on the constraint surface of conformal supergravity; it merely required that we remain somewhere on that surface.

This latter approach is the one we will take. The main difficulty is figuring out how to vary the quantum superfields $V^a$ and $\Sigma^r$. On the one hand, varying these only shifts the action by a term proportional to the equations of motion, so it’s not an immediate issue if we choose to work on shell. On the other, if there is some sort of natural variation of these objects, then we can possibly simplify the second-order action without the need to apply the equations of motion.

We begin by considering a primary chiral superfield of vanishing weights. In this way its variation can be defined solely in terms of $V^a$ and $\tilde{V}^\alpha$. Then varying $\Phi$ in the most natural way amounts to

$$
\Phi' = e^{-iV}(\Phi + \eta) = \Phi - iV\Phi + \eta - \frac{1}{2}V^2\Phi - iV\eta + \mathcal{O}(V^3)
$$

(4.37)

where we have stopped the expansion at second order. Demanding that the second order terms agree with the first order variation of the first order terms gives

$$
\delta\eta = -iV\eta, \quad \delta(V\Phi) = -iV^2\Phi
$$

(4.38)

(In the calculation one must include an additional factor of 2 since the second variation is generated from half of the first order variation squared.) The first is a perfectly sensible definition (it amounts to $\delta_c\eta = 0$) and the second implies for the variations of $V^a$ and $\Sigma^r$

$$
\delta V_{a\dot{a}} = -8V_a V_{\dot{a}} + iV^{\beta}D_{\beta}V_{a\dot{a}} - iV_{\beta}D^{\beta}V_{a\dot{a}} + V^b H_{b(a\dot{a})}
$$

$$
\delta \Sigma = iV^{a}D_{a}\Sigma - iV_{a}D^{a}\Sigma + 2iV^{\alpha}V^{b}F_{b\alpha} + 2iV^a V^b F_{b\dot{a}} - V^a \Delta_a \Sigma
$$

$$
+ V^{\dot{a}\alpha} \left(-\frac{1}{2}D_{\alpha}V^{b}\delta_{b\dot{a}} + \frac{1}{2}D_{\dot{a}}V^{b}\delta_{b\alpha} - \frac{1}{4}V^{b}D_{\alpha}F_{b\dot{a}} + \frac{1}{4}V^{\dot{b}}D_{\dot{a}}F_{b\alpha} \right)
$$

(4.39)

In the last equation we have suppressed the $r$ index to simplify notation.

Note that $\delta V_a \ni V^b H_b^a$ and $\delta G_a = -H_b^a G_b$, and so there will be no $H_b^a$ in terms like $\delta(V^a G_a)$. We will similarly identify the combination $H_b^a$ in the variation of $K_a$ and $Y_a$ so that this cancellation occurs for these terms as well.

4.2.1 Variation of the $\eta$ term

Beginning with

$$
\delta_\eta S = \int \mathcal{E} \eta^i \left( X^{\mathcal{P}} K_i + P_i + Y_i \right) + \text{h.c.}
$$

(4.40)
we consider the effect of a second variation. Given the presence of \( \eta^i \), it is most sensible to work in quantum chiral gauge where \( \eta^i \) has no further variation.

Taking the superpotential term, one finds simply

\[
\delta \delta \eta S \ni \int \mathcal{E} \eta^i \eta^j P_{ij}
\]  
(4.41)

The gauge field term is a bit more complicated:

\[
\delta \delta \eta S \ni \int \mathcal{E} \eta^i \left( \frac{1}{4} \eta^i f_{rs,ij} W^{\alpha r} W^s_\alpha + \frac{1}{2} f_{rs,ij} W^{\alpha r} \delta_c W^s_\alpha \right)
\]  
(4.42)

Plugging in \( \delta_c W^s_\alpha \) gives

\[
\delta \delta \eta S \ni \frac{1}{4} \int \mathcal{E} \eta^i \eta^j f_{rs,ij} W^{\alpha r} W^s_\alpha + \frac{1}{2} \int \mathcal{E} \eta^i f_{rs,ij} W^{\alpha r} (i \nabla_\alpha \Delta^s + V_{a0} \bar{W}^a_{i\alpha})
\]  
(4.43)

The term involving \( X \) and \( K_i \) is the most difficult to deal with. We rewrite it as a full superspace integral and then take the chiral quantum variation.

\[
\delta \delta \eta S \ni \delta_c X \eta^i K_i + X \eta^j K_{ij} \left( \bar{\eta}^j + 2iV B \bar{\Phi}^j \right) + X \eta^i K_{ij} \eta^j
\]  
(4.44)

The last term we will consider in tandem with \( P_{ij} \). The second term can be simplified by noting that when \( X_B = D \) or \( A \), the result simplifies. First note

\[
DK_i = -\Delta K_i = +K_{ij} \Delta_j \Phi^i + K_{ij} \bar{\Phi}^j
\]  
(4.45)

\[
\frac{3}{2} AK_i = \Delta K_i = -K_{ij} \Delta_j \phi^i + K_{ij} \bar{\Phi}^j
\]  
(4.46)

which together imply

\[
0 = K_{ij} \Delta_j \Phi^i.
\]  
(4.47)

This gives

\[
K_{ij} \eta^i \bar{\eta}^j + 2i \eta^j K_{ij} \left( V^b D_b + \Sigma^r X_r \right) \bar{\Phi}^j + 2i V_{\dot{a}} D^\dot{a} (\eta^i K_i)
\]  
(4.48)

Next we observe that \( \delta_c X \) is equivalent to

\[
\delta_c X = X \delta U + iV X = X \delta U - iX \mathcal{D}^\dot{a} V_{\dot{a}} + iX \mathcal{D}^\dot{a} V_{\dot{a}} - X \Delta_b V^b + 2iV(D) X
\]  
(4.49)

where we have used \([4.33]\) again. Plugging this in and using several integrations by parts, we can show that the total variation of this term is

\[
\delta \delta \eta S \ni X \left( i \mathcal{D}_b V^b K_i \eta^i - \Delta_b V^b K_i \eta^i + 2i \eta^i K_{ij} (V + \Sigma) \bar{\Phi}^j + \delta U K_i \eta^i + K_{ij} \eta^i \eta^j + K_{ij} \eta^i \bar{\eta}^j \right)
\]  
(4.50)

The combination \( V + \Sigma \) is shorthand for \( (V^b D_b + \Sigma^r X_r) \). Note that the terms involving \( V^a \) and \( V_{\dot{a}} \) have dropped out. We can simplify this expression by combining the first two terms and then integrating by parts. The result is

\[
\delta \delta \eta S \ni X \left( -\frac{1}{2} V^a D_a (K_{ij} \eta^i) \bar{D}_\alpha \bar{\Phi}^j + 2i \eta^i K_{ij} (V + \Sigma) \bar{\Phi}^j + \delta U K_i \eta^i + K_{ij} \eta^i \eta^j + K_{ij} \eta^i \bar{\eta}^j \right)
\]  
(4.51)

\[\text{We have written this and many subsequent D-terms without an overall } \int E \text{ or with the brackets } [ ]_D \text{ to keep the formulae from growing cluttered.}\]
Combining this with everything else yields
\[
\delta \delta \eta S = \left[ n^i (\mathcal{P} X K_{ij} + P_{ij} + \mathcal{Y}_{ij}) \eta^j \right]_F + \text{h.c.}
\]
\[
+ \left[ X \eta^j K_{ij} \bar{\eta}^\beta + X \delta U \eta^j K_i + \eta^j (X K_{i,r} + \mathcal{Y}_{i,r}) \Sigma^r + XV^a K_a,\eta^i + V^\alpha \mathcal{Y}_{a,\eta^i} \right] D + \text{h.c.}
\]
\[(4.52)\]
where we have defined
\[
\mathcal{K}_{i,j} \equiv 2i K_{ij} X \Phi^j
\]
\[(4.53)\]
\[
\mathcal{Y}_{i,r} \equiv \frac{i}{2} f_{r,s} W^{a,s} \mathcal{D}_a = \frac{i}{2} G_{r,s} W^{a,s} \mathcal{D}_a
\]
\[(4.54)\]
\[
\mathcal{K}_{a \dot{a}, i} \eta^i \equiv -\mathcal{D}_a \bar{\Phi}^j \mathcal{D}_a (K_{ij} \eta^j)
\]
\[(4.55)\]
\[
\mathcal{Y}_{a \dot{a}, i} \equiv -f_{r,s} W^a W^s_{\dot{a}} = -G_{r,s} W^a W^s_{\dot{a}}
\]
\[(4.56)\]

4.2.2 Variation of the \( \Sigma \) term

The \( \Sigma \) term is
\[
\int E \Sigma'(\mathcal{Y}_r + X K_r)
\]
where we recall
\[
\mathcal{Y}_r \equiv -\frac{i}{2} \nabla^\alpha (f_{r,s} W^a s) + \text{h.c.}
\]
\[
K_r \equiv -i K_i X \Phi^i + i K_j X \Phi^j
\]

The variation of the first term is given by using the formula
\[
\delta (\nabla^\alpha \Phi_a) = -i \nabla^\beta (V^b \nabla^\alpha \Phi_a) + i \nabla^\beta (V^\alpha \nabla^\alpha \Phi_a) - \Delta_b (V^b \nabla^\alpha \Phi_a) + 2V^\alpha \nabla^\alpha \Phi_a
\]
\[
- \nabla^\alpha \nabla^\alpha \Phi_a - 2i (\nabla^\alpha \Sigma^r) X_r \Phi_a + \frac{1}{4} \nabla_{\dot{a}} \nabla^2 (V^\alpha \Phi_a) + \nabla^\alpha (\delta c \Phi_a)
\]
\[(4.57)\]
where \( \Phi_a \) is an arbitrary chiral spinor superfield. This is written in terms of the old \( V_\beta \) and \( V^\beta \). Exchanging for the new conformally invariant ones gives
\[
\delta (\nabla^\alpha \Phi_a) = -i D^\beta (V^\alpha \nabla^\beta \Phi_a) + i D^\beta (V^\alpha \nabla^\alpha \Phi_a) - \Delta_b (V^b \nabla^\alpha \Phi_a) + 2V^\alpha \nabla^\alpha \Phi_a
\]
\[
- \nabla^\alpha \nabla^\alpha \Phi_a - 2i (D^\alpha \Sigma^r) X_r \Phi_a + \frac{1}{4} \nabla_{\dot{a}} \nabla^2 (V^\alpha \Phi_a) + D^\alpha (\delta c \Phi_a)
\]
\[(4.58)\]
In this formula, we have mixed conventions with \( \nabla \)'s and \( D \)'s appearing in the same expression. Every isolated \( \nabla \alpha \) (or \( \nabla \dot{a} \)) here is equivalent to \( D_\alpha \) (or \( D_{\dot{a}} \)), while \( \nabla^2 \) is equivalent to \( D^2 - 8R \). \( \Delta_b \) is in terms of \( D \) and this will remain the case for the rest of this work.

Applying this formula to \( \mathcal{Y}_r \) gives
\[
\delta \mathcal{Y}_r = -i D^\beta (V_\beta \mathcal{Y}_r) + i D^\beta (V^\beta \mathcal{Y}_r) - \Delta_b (V^b \mathcal{Y}_r)
\]
\[
+ i V^{\alpha a} W^a s f_{s r} t f_{t u} + i V^{\alpha a} W^{a,s} \hat{W}^{a u} f_{s r} t f_{t u}
\]
\[
+ \frac{i}{8} \nabla^\alpha \nabla^2 (V_{a s} G_{r s} \hat{W}^{a r}) - \frac{i}{8} \nabla_{\dot{a}} \nabla^2 (V^{\alpha a} G_{r s} W^a s)
\]
\[
+ i \Sigma^a s f_{s r} t \hat{\mathcal{Y}} + D^a \Sigma^a s f_{s r} t W^a u f_{t u} - D_{\dot{a}} \Sigma^a s f_{s r} t \hat{W}^{a u} f_{t u}
\]
\[
- \frac{1}{8} \nabla^\alpha (f_{s r} \nabla^2 \nabla \Sigma^a s) - \frac{1}{8} \nabla_{\dot{a}} (f_{s r} \nabla^2 \nabla^{a} \Sigma^s)
\]
\[
- \frac{i}{2} D^a (\eta^j f_{r s} W^a s) - \frac{i}{2} D_{\dot{a}} (\bar{\eta}^j f_{r s} \hat{W}^{a s})
\]
\[(4.59)\]
Including the variation of $\Sigma$ and integrating by parts gives

\[
\delta(\Sigma^r \mathcal{Y}_r) = \left( 2i V^a D_a \Sigma^r - 2i V_a D^a \Sigma^r + 2i V^a V^b F_{ba} + 2i V^a V^b F_{ba} \right) \mathcal{Y}_r \\
+ V^\alpha \left( -\frac{1}{2} D_a V^b F_{ba} + \frac{1}{2} D_a V^b F_{ba} - \frac{1}{4} V^b D_a F_{ba} + \frac{1}{4} V^b D_a F_{ba} \right) \mathcal{Y}_r \\
- 2V^a (\Delta_a \Sigma^r) \mathcal{Y}_r \\
+ \Sigma^r V^a \mathcal{Y}_{a,r} + \frac{i}{8} \Sigma^r \nabla^a \nabla^2 (V_{a\alpha} G_{rs} W_{\alpha r}) - \frac{i}{8} \Sigma^r \nabla_\alpha \nabla^2 (V^\alpha G_{rs} W^s_{\alpha r}) \\
+ \Sigma^r D^a \Sigma^s f_{sr} W^a \mathcal{Y}_{a,tu} - \Sigma^r D_\alpha \Sigma^s f_{sr} W^\alpha \mathcal{Y}_{a,tu} \\
- \frac{1}{8} \Sigma^r \nabla^a \left( f_{rs} \nabla^2 \nabla^a \Sigma^s \right) \\
+ \eta^j \mathcal{Y}_{ir} \Sigma^r + \bar{\eta}^j \mathcal{Y}_{jr} \Sigma^r 
\] (4.60)

where we have defined

\[
\mathcal{Y}_{a\alpha,r} \equiv -2i \left( \bar{W}^a \bar{W}^a f_{sr} \mathcal{Y}_{a,tu} + W^a \bar{W}^a f_{sr} \mathcal{Y}_{a,tu} \right) 
\] (4.61)

Varying $K_r$ gives

\[
\delta K_r = -i V^\beta D_\beta K_r + i V_\beta D^\beta K_r + \eta^j K_{ir} + \bar{\eta}^j K_{jr} \\
+ 2K_{ij} (V + \Sigma) \Phi^i X_r \bar{\Phi}^j + 2K_{ij} X_r \Phi^i (V + \Sigma) \bar{\Phi}^j 
\] (4.62)

where again

\[
V + \Sigma \equiv V^b D_b + \Sigma^r X_r 
\]

Including the variation of $X$ and $\Sigma^r$ gives

\[
X^{-1} \delta(\Sigma^r X K_r) = \left( 2i V^a D_a \Sigma^r - 2i V_a D^a \Sigma^r + 2i V^a V^b F_{ba} + 2i V^a V^b F_{ba} \right) K_r \\
+ V^\alpha \left( -\frac{1}{2} D_a V^b F_{ba} + \frac{1}{2} D_a V^b F_{ba} - \frac{1}{4} V^b D_a F_{ba} + \frac{1}{4} V^b D_a F_{ba} \right) K_r \\
- V^a (\Delta_a \Sigma^r) K_r - (\Delta_b V^b) \Sigma^r K_r + \delta U \Sigma^r K_r \\
+ 4K_{ij} \Sigma \Phi^i \Sigma \bar{\Phi}^j + 2K_{ij} V^i \Sigma \bar{\Phi}^j + 2K_{ij} \Sigma \Phi^i V \bar{\Phi}^j \\
+ \Sigma^r K_{i,r} \eta^i + \Sigma^r K_{j,r} \bar{\eta}^j 
\] (4.63)

4.2.3 Variation of the $V^a$ term

The $V^a$ term is

\[
\left[ V^b (-4X G_b + X K_b + \mathcal{Y}_b) \right]_D 
\] (4.64)

We require the variations of $G_{a\alpha}$, $K_{a\alpha}$, and $\mathcal{Y}_{a\alpha}$ in order to continue.

The variation of $G_{a\alpha}$ contains the graviton kinetic term. We have already worked this
out in (4.36), but we rewrite it here in the compact and useful form

\[ X^{-1} \delta(XG_{\alpha\dot{\alpha}}) = \delta U G_{\alpha\dot{\alpha}} + \frac{1}{2} \Delta_{\alpha\dot{\alpha}} \delta U - H_{\alpha \beta} G_{\dot{\beta}} - iD^\beta (V_{\dot{\beta}} G_{\alpha\dot{\alpha}}) + iD_\beta (V^\dot{\beta} G_{\alpha\dot{\alpha}}) \]

\[- \frac{1}{2} \Delta_{\alpha\dot{\alpha}} \Delta_b V^b - \frac{1}{2} D_{\alpha\dot{\alpha}} D_b V^b - \frac{1}{32} (D^2, D^2) V_{\alpha\dot{\alpha}} + \frac{1}{2} \Box V_{\alpha\dot{\alpha}} \]

\[- G_{\alpha\dot{\alpha}} \Delta_b V^b - \Delta_{\alpha\dot{\alpha}} (V^b G_b) + \frac{1}{2} D^\beta (R \dot{D}_\beta V_{\alpha\dot{\alpha}}) + \frac{1}{2} D_\beta (\dot{R} \dot{D}^\beta V_{\alpha\dot{\alpha}}) \]

\[- \frac{1}{2} D_a V_{\alpha} \delta X_\beta + \frac{1}{2} D_\alpha V_{\dot{\beta}} X_\beta - \frac{1}{2} V_b D_c G_a c \delta \sigma^a_{\alpha\dot{\alpha}} \]

\[- \frac{1}{8} V^{\dot{\alpha\alpha}} (D^2 R + D^2 \bar{R}) - R \dot{R} V_{\alpha\dot{\alpha}} + \frac{1}{2} V^{\dot{\alpha\alpha}} \Delta_b G_b + \frac{1}{2} V^b \Delta_{(\alpha\dot{\alpha})} G^b \]

(4.65)

where we have defined

\[ \Box V_{\alpha\dot{\alpha}} \equiv \Box V_{\alpha\dot{\alpha}} - \frac{1}{2} D^\beta (G_{\beta\dot{\beta}} D^\dot{\beta} V_{\alpha\dot{\alpha}}) + \frac{1}{2} D^\gamma V^\dot{\beta} \gamma_{(\beta\dot{\beta})(\gamma)} + \frac{1}{2} D^\gamma V^\dot{\beta} \gamma_{(\beta\dot{\beta})(\alpha\dot{\alpha})} \]

(4.66)

\[ W_{\gamma\beta} \] and its conjugate are defined by

\[ R_{\delta(\gamma\beta)} = 2i \epsilon_{\delta\gamma\beta} W_{\gamma\beta} \]

\[ R_{\delta(\gamma\beta)} = 2i \epsilon_{\delta\gamma\beta} W_{\gamma\beta} \]

(4.67)

The variation we need is

\[ X^{-1} \delta(-4X^n V^n G_a) = -8i V^\beta D_\beta V^a G_a + 8i V^a D^\alpha G_a - 16 V^n V^a G_{\alpha\dot{\alpha}} \]

\[- 4 \delta U (V^a G_a) - 2 \Delta_a V^a \delta U \]

\[ + 2 (\Delta_b V^b)^2 - 2 (D_b V^b)^2 - \frac{1}{8} D^2 V^{\alpha\dot{\alpha}} D^2 V_{\alpha\dot{\alpha}} + V^{\dot{\alpha}\alpha} \Box V_{\alpha\dot{\alpha}} \]

\[ + 8 V^n G_a \Delta_b V^b \]

\[ + V^{\dot{\alpha}\alpha} D_{\alpha} V_{\beta} \dot{D}_\beta V_{\alpha\dot{\alpha}} + V^{\dot{\alpha}\alpha} D_{\alpha} V_{\dot{\beta}} \dot{X}_\beta \]

\[ - V^{\dot{\alpha}\alpha} D_{\alpha} V_{\beta} \dot{X}_\beta \]

\[ - \frac{1}{4} V^{\dot{\alpha}\alpha} V_{\alpha\dot{\alpha}} (D^2 R + D^2 \bar{R}) - 2 R \dot{R} V^{\dot{\alpha}\alpha} V_{\alpha\dot{\alpha}} + V^{\dot{\alpha}\alpha} V_{\alpha\dot{\alpha}} \Delta_b G^b \]

(4.68)

Note that the combination \( H_{\alpha \beta} \) cancels out of the expression.

Turning to the variation of the matter term, we begin by noting that \( K_{\alpha\dot{\alpha}} \) may be written a number of equivalent ways

\[ K_{\alpha\dot{\alpha}} = K_{\alpha\beta} \nabla_{\alpha} \Phi^i \nabla_{\dot{\alpha}} \Phi^{\dot{i}} = \nabla_{\alpha} \Phi^i \nabla_{\dot{\alpha}} K_i = \nabla_{\dot{\alpha}} K_{\dot{\beta}} \nabla_{\alpha} \Phi^{\dot{\beta}} \]

\[ = K_{i\beta} D_{\alpha} \Phi^i D_{\dot{\alpha}} \Phi^{\dot{\beta}} = D_{\alpha} \Phi^i D_{\dot{\alpha}} K_i = D_{\dot{\alpha}} K_{\dot{\beta}} D_{\alpha} \Phi^{\dot{\beta}} \]

(4.69)
where again we have collected a number of terms into the combination \( H_a^b \). The object \( X^{(K)}_\beta \) is defined as

\[
X^{(K)}_\beta \equiv -\frac{1}{8} \tilde{\nabla}^2 \nabla_\beta \mathcal{K}
\]  

In the chiral model, this can further be identified as the \( U(1) \) spinor field strength \( X_\beta \).

Including the variation of the compensator and \( V^a \) gives

\[
X^{-1}\delta(X V^a \mathcal{K}_a) = 2i V^\beta D_\beta V^a \mathcal{K}_a - 2i V^\beta D^\delta V^a \mathcal{K}_a + 4V^a V^\dot{a} \mathcal{K}_{a\dot{a}} \\
+ \bar{\delta} U V^a \mathcal{K}_a - 2V^a \mathcal{K}_a \Delta_b V^b \\
+ 2K_{ij}(V + \Sigma) \Phi^i \bar{\Phi}^j + 2K_{ij}(V + \Sigma) \bar{\Phi}^j \Phi^i \\
- \Delta_b V^b (\Sigma' \mathcal{K}_r) + \nu^a (\Delta_a \Sigma') \mathcal{K}_r \\
- V^\dot{a}a D_\dot{a}V^\beta X^{(K)}_\beta - V^{\dot{a}a} D_\dot{a}V^\beta X^{(K)}_{\dot{\beta}} \\
+ \frac{1}{4} V^{\dot{a}a} D_\dot{a} \Phi^i D_i (\mathcal{K}_{ij} \bar{\eta}^j) + \frac{1}{2} V^{\dot{a}a} D_\dot{a} \Phi^j D_a (\mathcal{K}_{ij} \eta^j)
\]

The term arising from varying the Yang-Mills piece is fairly complicated. One finds

\[
\delta \mathcal{Y}_{a\dot{a}} = -H_{a\dot{a}}^b \mathcal{Y}_b - (\Delta_{a\dot{a}} V^b) \mathcal{Y}_b - iD^\beta (V_\beta \mathcal{Y}_{a\dot{a}}) + iD_\beta (V^\beta \mathcal{Y}_{a\dot{a}}) \\
+ \frac{1}{4} D^\dot{a} D_{\dot{a}}(V_\beta) \mathcal{Y}_\beta - \frac{1}{4} D^\beta D_{\beta}(V_\dot{a}) \mathcal{Y}_{\dot{a}} \\
- V^\beta \beta G_{a\beta} \mathcal{Y}_{\dot{a}} - V^\dot{a} \beta G_{a\dot{a}} \mathcal{Y}_{\beta} \\
+ \frac{1}{4} G_{rs} \bar{\nabla}^2 (V_{a\beta} W^{\beta r}) W_a^{\dot{s}} - \frac{1}{4} G_{rs} \nabla^2 (V_{a\beta} W^{\beta r}) W_a^{\dot{s}} \\
i V^\beta D_b (f_{rs} W_{a\dot{r}}) W_a^{\dot{s}} + iV^\beta D_b W_{a\dot{r}} W_a^{\dot{s}} f_{rs} - iV^\beta W_{a\dot{r}} f_{rs} D_b W_a^{\dot{s}} - iV^\beta W_{a\dot{r}} D_b (f_{rs} W_a^{\dot{s}}) \\
- (G_{rs, \dot{r}} \eta^j + G_{rs, \dot{r}} W_a^{\dot{s}}) W_{a\dot{a}}^{\dot{s}} \\
- 2i f_{rs} W_{a\dot{r}} \Sigma W_a^{\dot{s}} + 2i f_{rs} (\Sigma W_a^{\dot{s}}) W_a^{\dot{s}} + \frac{i}{4} G_{rs} \bar{\nabla}^2 \nabla_\alpha \Sigma^r W_a^{\dot{s}} - \frac{i}{4} G_{rs} \nabla^2 \nabla_\dot{\alpha} \Sigma^r W_a^{\dot{s}}
\]
A number of somewhat complicated looking terms have been introduced in the first few lines, partly because the $H_a^b$ term is not generated here as readily as in $\delta G_a$ and $\delta K_a$. A more convenient arrangement of the above terms is given by

\[
\delta \mathcal{Y}_{\alpha \alpha} = - H_{\alpha \alpha}^b \mathcal{Y}_b - i D^b (V_b \mathcal{Y}_{\alpha \alpha}) + i D_{\beta} (V^\beta \mathcal{Y}_{\alpha \alpha}) \\
- 4(D_b V_c) G_{\gamma(\alpha \alpha) \beta \delta} + D^\beta (G_{\alpha \alpha}^\gamma D^\delta V_{\alpha \alpha}) - D^\delta (G_{\alpha \alpha}^\gamma D^\beta V_{\alpha \alpha}) \\
+ 2 D^\beta (R^\gamma D^\delta V_{\alpha \alpha}) + 2 D_{\beta} (\bar{R}^\gamma D^\beta V_{\alpha \alpha}) - D^\gamma V^\beta \mathcal{Y}^\gamma_{\gamma(\beta \beta)(\alpha \alpha)} - D^\delta V^\beta \mathcal{Y}^\delta_{\gamma(\beta \beta)(\alpha \alpha)} \\
+ 2 \left( - \frac{1}{2} D_a V^b F_{b \alpha} + \frac{1}{2} D_a V^b F_{b \alpha} - \frac{1}{4} V^b D_a F_{b \alpha} + \frac{1}{4} V^b D_a F_{b \alpha} \right) \mathcal{Y}_r \\
- 2 V^\beta \mathcal{Y}^\beta_{\gamma(\alpha \alpha)(\beta \beta)} - (G_{rs,i}^j + G_{rs,j}^i) W^r W_{\alpha}^s \\
- 2 i f_{rs} W^s_{\alpha} \bar{\Sigma} W_{\alpha}^s + 2 i f_{rs} (\Sigma W^r_{\alpha}) W_{\alpha}^s + \frac{i}{4} G_{rs} \bar{\nabla}^2 \nabla_{\alpha} \Sigma_{\alpha} \bar{\nabla} W_{\alpha}^s - \frac{i}{4} G_{rs} \bar{\nabla}^2 \nabla_{\alpha} \Sigma W_{\alpha}^s
\]

where we have made a number of definitions. In particular,

\[
R^\gamma_{\gamma(\alpha \alpha)(\beta \beta)} = \frac{1}{16} G_{rs} W_{\phi}^r W_{\phi}^s \\nR^\gamma_{\gamma(\alpha \alpha)(\beta \beta)} = \frac{1}{16} G_{rs} \bar{\Sigma} W_{\alpha}^s \\
G_{\alpha \alpha}^\gamma = \frac{1}{4} G_{rs} W_{\alpha}^r W_{\alpha}^s
\]

These definitions should not be taken more seriously than just serving as convenient names. $R^\gamma_{\gamma(\alpha \alpha)(\beta \beta)}$, for example, is not chiral unless the gauge couplings are trivial. We have simply identified these combinations since they seem like they shall combine nicely with actual objects of those names in the graviton propagator. In addition, we have written “curvature” terms which will also combine with the similar term in $\square_V$:

\[
W^\gamma_{\gamma(\alpha \alpha)(\beta \beta)} = \frac{1}{4} \epsilon_{\alpha \beta} \sum_{\gamma \delta} \left( \epsilon_{\alpha \gamma} W_{\beta}^r D_{\phi}^\gamma (G_{rs} W_{\phi}^s) + \epsilon_{\alpha \gamma} W_{\beta}^r D_{\phi}^\gamma (G_{rs} W_{\phi}^s) \\
- \epsilon_{\alpha \gamma} G_{rs} W_{\beta}^r (DW)^s + G_{rs} W_{\alpha}^r D_{\gamma} W_{\beta}^s \right)
\]

\[
W^\gamma_{\gamma(\alpha \alpha)(\beta \beta)} = - \frac{1}{4} \epsilon_{\alpha \beta} \sum_{\gamma \delta} \left( \epsilon_{\alpha \gamma} W_{\beta}^r D_{\phi}^\gamma (G_{rs} W_{\phi}^s) + \epsilon_{\alpha \gamma} W_{\beta}^r D_{\phi}^\gamma (G_{rs} W_{\phi}^s) \\
- \epsilon_{\alpha \gamma} G_{rs} W_{\beta}^r (DW)^s - G_{rs} W_{\alpha}^r D_{\gamma} W_{\beta}^s \right)
\]

\[\text{It is plausible, although we haven’t explored this possibility deeply yet, that if the linear compensator is coupled to the Chern-Simons term for the gauge sector, then the superfields $R$ and $G$ defined in terms of $L$ will pick up contributions of the above form for the case $G_{rs} \propto \delta_{rs}$.}\]
as well as the “potential” term

\[ V^{\beta\gamma} y_{(\alpha\delta)(\beta\gamma)} \equiv - V^b G_b y_{\alpha\delta} - G_{\alpha\delta} V^b y_b + V_{\alpha\delta} G_y y_b \]

\[ + \frac{1}{4} V^{\beta\gamma} (D_\beta W_{\alpha r} D_\gamma W^r_{\alpha} + D_\beta W^r_{\alpha} D_\gamma W_{\alpha r}) - \frac{1}{2} V^b \Delta_b y_{\alpha\delta} \]

\[ - \frac{1}{8} V_{\alpha\beta}(D^2 - 8\bar{R})W^{\beta\gamma} G_{rs} W_{\alpha s} + \frac{1}{8} V_{\alpha\beta}(D^2 - 8\bar{R})W^{\beta\gamma} G_{rs} W_{\alpha s} \]

\[ + \frac{1}{8} V_{\alpha\beta} D_\alpha W^{\beta\gamma} \gamma (f_{rs} W_{\gamma s}) + \frac{1}{8} V_{\alpha\beta} D_\alpha W^{\beta\gamma} \gamma (f_{rs} W_{\gamma s}) \]

\[ - \frac{1}{8} V_{\alpha\beta} D_\alpha \bar{W}^{\beta\gamma} \gamma (f_{rs} W_{\gamma s}) - \frac{1}{8} V_{\alpha\beta} D_\alpha \bar{W}^{\beta\gamma} \gamma (f_{rs} W_{\gamma s}) \quad (4.78) \]

These look like they could be defined in terms of the new \( R^\gamma \) and \( G^\gamma \) objects we have mentioned before, but we will avoid doing so explicitly.

The combination we need is

\[ \delta(V^\gamma y_a) = 2i V^\beta D_\beta V^a y_a - 2i V^\beta D_\beta V^a y_a + 4V^a V^\gamma y_{a\delta} \]

\[ - 4V^a (D_\gamma V_c) G^\gamma_{\delta} \epsilon_{abcd} \]

\[ - \frac{1}{2} V^{\alpha\alpha} D^\beta (G^\gamma \beta \gamma D^\delta y_{a\delta}) + \frac{1}{2} V^{\alpha\alpha} D^\beta (G^\gamma \beta \gamma D^\delta y_{a\delta}) \]

\[ - V^{\alpha\alpha} D^\beta (R^\gamma D_\beta y_{a\delta}) - V^{\alpha\alpha} D^\beta (R^\gamma D_\beta y_{a\delta}) \]

\[ + \frac{1}{2} V^{\alpha\alpha} D^\delta V^{\beta\gamma} W^\gamma_{\gamma(\beta\delta)(\alpha\delta)} + \frac{1}{2} V^{\alpha\alpha} D^\delta V^{\beta\gamma} W^\gamma_{\gamma(\beta\delta)(\alpha\delta)} \]

\[ - V^{\alpha\alpha} \left( \frac{1}{2} D_\alpha V^b F_{b\alpha} + \frac{1}{2} D_\alpha V^b f_{b\alpha} - \frac{1}{4} V^b D_\alpha f_{b\alpha} + \frac{1}{4} V^b D_\alpha f_{b\alpha} \right) \gamma \]

\[ + V^{\alpha\alpha} V^{\beta\gamma} y_{(a\delta)(\beta\gamma)} + \frac{1}{2} V^{\alpha\alpha} (G_{rs,ij} \eta^i + G_{rs,j} \eta^j) \gamma W^r_{\alpha s} \]

\[ + iV^{\alpha\alpha} f_{rs} W^r_{\alpha s} + iV^{\alpha\alpha} f_{rs} (\Sigma W^r_{\alpha s}) \bar{W}^s_{\alpha} - \frac{i}{8} V^{\alpha\alpha} G_{rs} \nabla^2 \nabla^s \Sigma^r \bar{W}^s_{\alpha} + \frac{i}{8} V^{\alpha\alpha} G_{rs} \nabla^2 \nabla^s \Sigma^r \bar{W}^s_{\alpha} \quad (4.79) \]

### 4.2.4 Variation of the \( \mathcal{L} \) term

In the simple linear compensator model, there is one additional term – that involving \( \mathcal{L} \).

Beginning with

\[ S_L = [\mathcal{L}(V_R + K)]_D \quad (4.80) \]

one varies it to find

\[ \delta S_L = \mathcal{L} \left( 3\frac{\mathcal{L}}{L} - 2\Delta_b V^b + V^b (K_b - 4G_b) + K_i \eta^i + K_j \eta_j^j + \Sigma^r K_r \right) \quad (4.81) \]

### 4.3 Summary

We will break down our results into various sectors.

The terms involving just the chiral (and antichiral) quanta are

\[ S_{\eta i} = \left[ \eta^i X K_{ij} \bar{\eta}^j \right]_D + \left[ \eta^i (\mathcal{P}(X K_{ij}) + P_{ij} + \mathcal{V}_{ij}) \eta^j \right] F + \text{h.c.} \]

33
The terms involving chiral and gauge fields are

\[ S_{\eta \Sigma} = 4i\eta^j K_{ij} \Phi^j \Sigma^r + i\eta^i f_{rs,i} W^{\alpha s} \nabla_\alpha \Sigma^r + \text{h.c.} \]

\[ = 2\eta^i (X\mathcal{K}_{ir} + \mathcal{Y}_{ir}) \Sigma^r + \text{h.c.} \]

The terms involving chiral and gravity fields are

\[ S_{\eta V} = +V^{\dot{\alpha} \dot{r}} D_{\dot{a}} \Phi^j \Sigma^r (X\mathcal{K}_{ij} \eta^j) + V^{\dot{\alpha} \alpha} W^s_{\dot{a}} \bar{W}^s_{\dot{a}} f_{rs, i} \eta^i + \text{h.c.} \]

\[ = 2V^\alpha \left( X\mathcal{K}_{a,i} \eta^j + \mathcal{Y}_{a,i} \right) \eta^i + \text{h.c.} \]

The terms involving gravity and gauge fields are

\[ S_{\Sigma V} = \left( 2iV^\alpha D_{\dot{a}} \Sigma^r - 2iV_{\dot{a}} D^\alpha \Sigma^r \right) (X\mathcal{K}_r + \mathcal{Y}_r) \]

\[ - 2V^\alpha (\Delta \Sigma^r) \mathcal{K}_r - 2X (\Delta V^b) \Sigma^r \]

\[ + \frac{i}{4} V_{\alpha a} G_{rs} W^{\dot{a} s} \nabla^2 \nabla^a \Sigma^r - \frac{i}{4} V^{\dot{\alpha} \dot{a}} G_{rs} W^a \nabla^2 \nabla_a \Sigma^r \]

\[ + 4X\mathcal{K}_{ij} \Phi^j \Sigma^r \eta^i + 4X\mathcal{K}_{ij} \Sigma^i \Phi^j \]

\[ + 2iV^{\dot{\alpha} \dot{a}} f_{rs} W^a \bar{W}^a - 2iV^{\dot{\alpha} \dot{a}} \bar{f}_{rs} (\Sigma^r W_{\dot{a}}) W_{\dot{a}} \]

(4.82)

In the last two lines, we use a single \( \Sigma \) to denote \( \Sigma^r X_r \) acting to the right. It seems reasonable to rearrange the second line of \( S_{\Sigma V} \) so that it is proportional to the equation of motion.

\[ S_{\Sigma V} = \left( 2iV^\alpha D_{\dot{a}} \Sigma^r - 2iV_{\dot{a}} D^\alpha \Sigma^r \right) (X\mathcal{K}_r + \mathcal{Y}_r) \]

\[ - 2(\Delta V^b) \Sigma^r (X\mathcal{K}_r + \mathcal{Y}_r) \]

\[ + \frac{i}{4} V_{\alpha a} G_{rs} W^{\dot{a} s} \nabla^2 \nabla^a \Sigma^r - \frac{i}{4} V^{\dot{\alpha} \dot{a}} G_{rs} W^a \nabla^2 \nabla_a \Sigma^r \]

\[ - \Sigma^r D^\alpha V_{\alpha a} \mathcal{K}_r + \Sigma^r D^\dot{a} V_{\dot{a} a} \mathcal{Y}_r \]

\[ - 2\Sigma^r V^a \Delta a \mathcal{Y}_r \]

\[ + 4X\mathcal{K}_{ij} \Phi^j \Sigma^r \eta^i + 4X\mathcal{K}_{ij} \Sigma^i \Phi^j \]

\[ + 2iV^{\dot{\alpha} \dot{a}} f_{rs} W^a \bar{W}^a - 2iV^{\dot{\alpha} \dot{a}} \bar{f}_{rs} (\Sigma^r W_{\dot{a}}) W_{\dot{a}} \]

(4.83)

The term with three spinor derivatives can be rearranged so that it is proportional to \( D^\alpha V_{\alpha \dot{a}} (D^2 - 8R) \Sigma^r G_{rs} \bar{W}^\alpha s \), which can be cancelled if we introduce a Gaussian smearing with the gauge fixing functions \( D^\alpha V_{\alpha \dot{a}} \) for the gravity sector and \( (D^2 - 8R) \Sigma^r \) for the gauge sector, which is the standard approach. [8]

Next we turn to the pure gauge sector. We find

\[ S_{\Sigma \Sigma} = 4X\mathcal{K}_{ij} \Sigma^i \Sigma^j \Phi^j + \Sigma^r D^\alpha \Sigma^s f_{sr} f_{tu} W^a t W^a t - \Sigma^r D^\dot{a} \Sigma^s f_{sr} \bar{W}^\dot{a} t \bar{f}_{tu} \]

\[ - \frac{1}{8} \Sigma^r \nabla^a (f_{rs} \nabla^2 \nabla_a \Sigma^s) - \frac{1}{8} \Sigma^r \nabla_\dot{a} (\bar{f}_{rs} \nabla^2 \nabla_\dot{a} \Sigma^s) \]

(4.84)

It is conspicuous that for arbitrary holomorphic \( f_{rs} \), the last term yields the three spinor derivative term \( \Sigma^r (\nabla^\alpha f_{rs}) \nabla^2 \nabla_\alpha \Sigma^s \) which it does not seem possible to remove by a smeared gauge. It is not strictly speaking problematic to have a third order spinor derivative term (as it is still less divergent than the pure kinetic term and so can in principle be treated at least perturbatively), but it will lead to a more complicated one-loop analysis.
In any case, it is useful to rearrange the kinetic term into a form involving chiral projections of $\Sigma$. We use the identity
\[
\frac{1}{8} \Sigma \nabla^\alpha (f \nabla^2 \nabla_\alpha \Sigma) + \text{h.c.} = (\mathcal{D}_\alpha \Sigma) G(\mathcal{D}_\alpha \Sigma) + \frac{1}{8} (\mathcal{D}^2 - 8R) \Sigma G(\mathcal{D}^2 - 8\bar{R}) \Sigma
\]
\[
+ \left( \frac{1}{8} \mathcal{D}_\alpha \Sigma \mathcal{D}^\alpha f(\mathcal{D}^2 - 8\bar{R}) \Sigma + \text{h.c.} \right)
- 8RR \Sigma G \Sigma + \frac{1}{2} \Sigma \mathcal{D}^\alpha f \Sigma \mathcal{D}_\alpha R + \frac{1}{2} \Sigma \mathcal{D}_\alpha f \Sigma \mathcal{D}^\alpha \bar{R} + \frac{1}{2} \Sigma G \Sigma (\mathcal{D}^2 R + \mathcal{D}^2 \bar{R})
- \mathcal{D}_\alpha \Sigma G^{\alpha \alpha} G \mathcal{D}_\alpha \Sigma + \frac{i}{4} \mathcal{D}^\alpha \Sigma \mathcal{D}^\alpha (\mathcal{D}_{\alpha \dot{a}} f - \mathcal{D}_{\alpha \dot{a}} \bar{f})
+ \mathcal{D}^\alpha \Sigma \bar{f}(W_{\alpha} \Sigma) - \mathcal{D}_{\dot{a}} \Sigma f(W_{\alpha} \Sigma)
+ \frac{i}{4} \mathcal{D}^\dot{a} \mathcal{D}^\alpha f \mathcal{D}_{\alpha \dot{a}} \Sigma + \frac{i}{4} \mathcal{D}_{\alpha} \Sigma \mathcal{D}_{\alpha} \bar{f} \mathcal{D}^\alpha \Sigma
\]
(4.85)

In the above, we have suppressed all gauge indices for the sake of a less cluttered notation. They should be contracted in the obvious way, taking care to note that $(W_{\alpha} \Sigma) = -W_{\alpha}^{\Sigma} t^s f_{t^s}$. We have also chosen to integrate certain terms by parts so that the result is manifestly symmetric.

It is useful to define a generalized d’Alembertian for $\Sigma$ based on the above formula. We choose
\[
\Box_V \Sigma^s \equiv \mathcal{D}^\alpha (G_{rs} \mathcal{D}_\alpha \Sigma^s) - \frac{1}{2} \mathcal{D}^{[\alpha} (G_{\alpha \beta} G_{rs} \mathcal{D}^{\dot{\beta}]} \Sigma^s) + \mathcal{D}^\alpha \Sigma^s G_{su} W_{\alpha} f_{rt} u - \mathcal{D}_{\dot{a}} \Sigma^s G_{su} \bar{W}^{\dot{a}t} f_{rt} u
\]
(4.86)

so that in compacted notation
\[
\Sigma \Box_V \Sigma = \mathcal{D}^\alpha (G \mathcal{D}_\alpha \Sigma) - \frac{1}{2} \mathcal{D}^{[\alpha} (G_{\alpha \beta} G \mathcal{D}^{\dot{\beta}]} \Sigma) - \mathcal{D}^\alpha \Sigma G W_{\alpha} \Sigma + \mathcal{D}_{\dot{a}} \Sigma G \bar{W}^{\dot{a} \dot{b}} \Sigma
\]
(4.87)

This is a generalization of the scalar d’Alembertian $\Box_V$ discussed in [12], generalized to a superfield $\Sigma$ with a nontrivial gauge sector with corresponding gaugino superfield $W_{\alpha}$. The form of this operator also inspired the definition of $\Box_V V_{\alpha \dot{a}}$ for the gravity sector.

We may then write
\[
S_{\Sigma \Sigma} = \Sigma \Box_V \Sigma - \frac{1}{8} (\mathcal{D}^2 - 8R) \Sigma G(\mathcal{D}^2 - 8\bar{R}) \Sigma
\]
\[
- \frac{1}{2} \Sigma G \Sigma (\mathcal{D}^2 - 8\bar{R}) R - \frac{1}{2} \Sigma G \Sigma (\mathcal{D}^2 - 8R) \bar{R} + 4X K_{\dot{i}j} \Sigma \Phi \dot{i} \Sigma \Phi \dot{j}
\]
\[
- \left( \frac{1}{8} \mathcal{D}_\alpha \Sigma \mathcal{D}^\alpha f(\mathcal{D}^2 - 8\bar{R}) \Sigma + \text{h.c.} \right)
- \frac{1}{2} \Sigma \mathcal{D}^\alpha f \Sigma \mathcal{D}_\alpha R - \frac{1}{2} \Sigma \mathcal{D}_\alpha f \Sigma \mathcal{D}^\alpha \bar{R} - \frac{i}{4} \mathcal{D}^\alpha \Sigma \mathcal{D}^\alpha (\mathcal{D}_{\alpha \dot{a}} f - \mathcal{D}_{\alpha \dot{a}} \bar{f})
- \frac{i}{4} \mathcal{D}^\dot{a} \mathcal{D}^\alpha f \mathcal{D}_{\alpha \dot{a}} \Sigma - \frac{i}{4} \mathcal{D}_{\alpha} \Sigma \mathcal{D}_{\alpha} \bar{f} \mathcal{D}^\alpha \Sigma
+ \Sigma^s \mathcal{D}^\alpha \mathcal{D}^s W_{\alpha} u(X_r f_{su}) - \Sigma^s \mathcal{D}_{\dot{a}} \Sigma^s \bar{W}^{\dot{a} \dot{b}} u(X_r \bar{f}_{su})
\]

Note the last line involves the gauge generator acting on the holomorphic gauge couplings. If these are taken to be proportional to the identity, then the last line will vanish.
We turn finally to the pure gravity sector. The terms are quite numerous:

\[ S_{VV} = \left( 2iV^a V^b F_{ba} + 2iV^a V^b F_{ab} \right) (XK_r + Y_r) \]

\[ + 2iV^\beta D^\alpha V^a - 2iV^\beta D^{\hat{\alpha}} V^a + 4V^a V^{\hat{\alpha}} \sigma^{\alpha \hat{\alpha}} \left( -4XG_a + XK_a + Y_a \right) \]

\[ + 2X(\Delta b V^b)^2 - 2X(D_b V^b)^2 - \frac{X}{8} D^2 V^{\hat{\alpha} \alpha} D^2 V_{a \hat{\alpha}} + XV^{\hat{\alpha} \alpha} \Box V_{a \hat{\alpha}} \]

\[ - 2\Delta_b V^b \left( XV^a K_a - 4XV^a G_a \right) \]

\[ + XV^{\hat{\alpha} \alpha} D^b (RD^\beta V_{a \hat{\alpha}}) + XV^{\hat{\alpha} \alpha} D^b (\tilde{R} \tilde{D}^{\hat{\beta}} V_{a \hat{\alpha}}) \]

\[ - V^{\hat{\alpha} \alpha} D^b (R^Y D^\beta V_{a \hat{\alpha}}) - V^{\hat{\alpha} \alpha} D^b (\tilde{R} \tilde{D}^{\hat{\beta}} V_{a \hat{\alpha}}) \]

\[ - \frac{1}{2} V^{\hat{\alpha} \alpha} D^b (G^Y_{\beta \gamma} D^{\hat{\beta}} V_{a \hat{\alpha}}) + \frac{1}{2} V^{\hat{\alpha} \alpha} D^b (G^Y_{\alpha \hat{\alpha}} D^{\beta} V_{a \hat{\alpha}}) \]

\[ - 4V^a (D_b V_c) G^Y_{\alpha \beta \gamma \delta} \]

\[ + XV^{\hat{\alpha} \alpha} D_{(a} V_{\beta} \hat{\beta}_{b)} - XV^{\hat{\alpha} \alpha} D_{(a} V_{\beta} \hat{\gamma}_{b)} \]

\[ + \frac{1}{2} V^{\hat{\alpha} \alpha} D^b V^{\beta \beta} W^Y_{\gamma (\beta \beta)(a \hat{\alpha})} + \frac{1}{2} V^{\hat{\alpha} \alpha} D^b V^{\beta \beta} W^Y_{\gamma (\beta \beta)(a \hat{\alpha})} \]

\[ - 2X \tilde{R} \tilde{V} \tilde{V}^{\hat{\alpha} \alpha} V_{a \hat{\alpha}} + 4XK_{ij} V \tilde{\Phi}^i V \tilde{\Phi}^j + V^{\hat{\alpha} \alpha} V^{\beta \beta} Y_{(a \hat{\alpha})(\beta \hat{\beta})} \] (4.88)

We have defined

\[ \hat{\mathcal{K}}_a = \begin{cases} 
-\frac{1}{8} \nabla^2 \nabla_a K - X_a & \text{for the simple linear compensator model} \\
0 & \text{for the arbitrary chiral model} 
\end{cases} \] (4.89)

We have until now left the gauge for \( V^\alpha \) unspecified. Inspection of its appearance in all the terms shows that it is always proportional to the equations of motion, so if we work with the background on-shell then the gauge of \( V^\alpha \) (at least to one-loop order) is physically irrelevant. We will still choose the particular gauge \( V^\alpha = 0 \) for definiteness.

The above represent the common features of the linear and chiral models. They also each have a term involving \( \tilde{\delta}U \):

\[ S_{SU,=} = \tilde{\delta}U \left( XK_i \eta^i + XK_j \tilde{\eta}^j + XK_r \Sigma^r - 4XV^b G_b + XV^b K_b - 2X \Delta_b V^b \right) \]

Depending on the model, the variation of the compensator may be quite different. The simple linear compensator model has

\[ \tilde{\delta}U = L^{-1} \mathcal{L} \]

while the arbitrary chiral model possesses

\[ \tilde{\delta}U = -\frac{1}{3} \left( K_i \eta^i + K_j \tilde{\eta}^j + K_r \Sigma^r - 4V^b G_b + V^b K_b - 2\Delta_b V^b \right) \]

In addition, for the linear compensator model there are the terms arising from varying (4.81):

\[ S_{L,=} = \mathcal{L} \left( 3\frac{\mathcal{L}}{L} - 2\Delta_b V^b + V^b (K_b - 4G_b) + K_i \eta^i + K_j \tilde{\eta}^j + \Sigma^r K_r \right) \] (4.90)
Combining these two effects gives the second order action for the linear compensator model

\[ S^{(2)}_{\text{linear}} = S_{VV} + S_{\Sigma V} + S_{\Sigma \Sigma} + S_{\eta V} + S_{\eta \Sigma} + S_{\eta \eta} \]

\[ + 3 \frac{L^2}{L} + 2 \mathcal{L} \left( K_i \eta^i + K_j \bar{\eta}^j + \Sigma^r K_r + V^b (K_b - 4G_b) - 2 \Delta_b V^b \right) \]

(4.91)

For the chiral model, we find

\[ S^{(2)}_{\text{chiral}} = S_{VV} + S_{\Sigma V} + S_{\Sigma \Sigma} + S_{\eta V} + S_{\eta \Sigma} + S_{\eta \eta} \]

\[ - \frac{X}{3} \left( K_i \eta^i + K_j \bar{\eta}^j + \Sigma^r K_r + V^b (K_b - 4G_b) - 2 \Delta_b V^b \right)^2 \]

(4.92)

For reference, we include here their first order variations as well:

\[ S^{(1)}_{\text{chiral}} = \left[ V^a (XK_a - XG_a + \mathcal{Y}_a) + \Sigma^r (XK_r + \mathcal{Y}_r) \right]_D \]

\[ + \left[ \bar{\eta}^i (\mathcal{P}K_i + P_i + \mathcal{Y}_i) \right]_F + \left[ \bar{\eta}^j (\bar{\mathcal{P}}K_j + \bar{P}_j + \mathcal{Y}_j) \right]_F \]

\[ S^{(1)}_{\text{linear}} = \left[ V^a (XK_a - XG_a + \mathcal{Y}_a) + \Sigma^r (XK_r + \mathcal{Y}_r) \right]_D \]

\[ + \left[ \eta^i (\mathcal{P}K_i + P_i + \mathcal{Y}_i) \right]_F + \left[ \bar{\eta}^j (\bar{\mathcal{P}}K_j + \bar{P}_j + \mathcal{Y}_j) \right]_F \]

\[ + \left[ L(V_R + K) \right]_D \]

(4.93)

(4.94)

Their respective actions to second order in the quantum fields are then given by

\[ S^{(0)}_{\text{chiral}} + S^{(1)}_{\text{chiral}} + \frac{1}{2} S^{(2)}_{\text{chiral}} \quad (4.95) \]

\[ S^{(0)}_{\text{linear}} + S^{(1)}_{\text{linear}} + \frac{1}{2} S^{(2)}_{\text{linear}} \quad (4.96) \]

When we consider that the linear compensator model is classically dual to a special case of the arbitrary chiral model, it becomes perhaps unsurprising that their quantum actions should have so many features in common. This commonality is enough for us to ask whether the two theories might actually be equivalent at the one-loop level, at least on-shell. One can in fact make a rather straightforward argument, based on the existing proofs of equivalence for chiral spinors and chiral scalars [12, 15] that the two effective actions should be equivalent on-shell at one-loop. There is a subtlety, however, due to the inability to nicely define the path integration for a generic chiral superfield. We will return to this issue in a subsequent paper.

5 Conclusion

The formulae listed above constitute the end of the algebraic manipulations necessary to produce a suitable action quadratic in the quantum superfields of supergravity, super Yang-Mills, and chiral matter. Further steps are necessary to produce one-loop results.

The first step is obviously to perform a gauge-fixing of the gravity and gauge sectors. Part of the procedure here will involve deciding just how to do it. Even if we choose a smeared gauge and aim for only \(1/p^2\) propagators (as was the guiding principle in [4]), we have the option of removing certain terms in \(S_{\Sigma V}\) or \(S_{VV}\) involving operators of dimension less than two. Any choice must, of course, be physically equivalent to any other, but certain
calculational simplifications may occur only one way. It is possible that the dual formulation with the linear compensator can play a role in helping us find the simplest gauge choice due to the way in which it decouples the matter and gravitational sectors, but we have not finished exploring other options yet.

The second is to actually perform the resulting path integrals. For background field calculations, one generally prefers a method which is non-perturbative, such as the Schwinger proper time method or the derivative expansion. Such a procedure here is a bit more difficult since while the gauge and gravity sectors involve generalized Laplacians, the chiral sector involves Dirac-like operators. If the couplings between these sectors do not vanish, some amount of perturbation seems necessary, since the determinant of an operator with a diagonal consisting of Laplace and Dirac operators is difficult to deal with without separating out the two sectors. We hope to explore these two steps soon.
A Arbitrary linear and chiral superfield models at first order

We have expanded the actions for arbitrary chiral models to second order in the quantum fields to enable quantization. The structure they possess is fairly interesting and is reflected in the minimal model of a linear compensator coupled to supergravity and a Kähler potential. We will briefly consider the generalization to an arbitrary coupling of a linear superfield $L$ to chiral multiplets $\Phi^i$ in the context of conformal supergravity. Although we will assume only a single linear superfield $L$, the generalization to several is straightforward.

The interesting part will be contained in the $D$-term action

$$S_D = -3 \int E \mathcal{F}(L, \Phi^i, \bar{\Phi}^j)$$

The $-3$ is chosen so that if $\mathcal{F}$ is independent of $L$, a canonical Einstein-Hilbert term is reproduced for the choice $\mathcal{F} = 1$. Observing that

$$D\mathcal{F} = 2\mathcal{F} = \mathcal{F}_i \Delta_i \Phi^i + \mathcal{F}_j \Delta_j \bar{\Phi}^j + 2\mathcal{F}_L L$$

and that the Einstein-Hilbert term is contained within

$$S_D \equiv -\frac{3}{2} \mathcal{F}_i \Box \Phi^i - \frac{3}{2} \mathcal{F}_j \Box \bar{\Phi}^j$$

where $\Box$ are superconformal (and thus contain $\mathcal{R}/6$ weighted by the scaling dimension of the field on which $\Box$ acts), it is easy to see that the normalization of the Einstein-Hilbert term is

$$X = \frac{1}{2} \mathcal{F}_i \Delta_i \Phi^i + \frac{1}{2} \mathcal{F}_j \Delta_j \bar{\Phi}^j = \mathcal{F} - L \mathcal{F}_L$$

It is clear that the field multiplying the Einstein-Hilbert term is the proper conformal compensator to use for our theory, so we have chosen to label the above combination as $X$.

Expanding $S_D$ to first order in quantum fields using the tools we have developed is straightforward. One finds

$$\delta S_D = 3i \nabla^a (V_a \mathcal{F}) - 3i \nabla_a (V^a \mathcal{F}) + 3 \Delta_b (V^b \mathcal{F}) \mathcal{F}_L + (\Delta_b V^b)(\mathcal{F} - L \mathcal{F}_L)$$

$$+ 3i V^b \left( \mathcal{F}_i \nabla_b \Phi^i - \mathcal{F}_j \nabla_b \bar{\Phi}^j \right) + 3i \Sigma' \left( \mathcal{F}_i X_r \Phi^i - \mathcal{F}_j X_r \bar{\Phi}^j \right) - 3 \mathcal{F}_L \mathcal{L}$$

where $\Delta_b$ is conformally covariant, as are all the other derivatives. Integrating by parts (and taking care that the special conformal connections vanish) gives

$$\delta S_D = + 3 V^b \Delta_b \mathcal{F}_L + V^b \Delta_b (\mathcal{F} - L \mathcal{F}_L) + 3i V^b \left( \mathcal{F}_i \nabla_b \Phi^i - \mathcal{F}_j \nabla_b \bar{\Phi}^j \right)$$

$$+ 3i \Sigma' \left( \mathcal{F}_i X_r \Phi^i - \mathcal{F}_j X_r \bar{\Phi}^j \right) - 3 \mathcal{F}_L \mathcal{L}$$

Using

$$V^b \Delta_b \mathcal{F} = V^b \mathcal{F}_L \Delta_b L - i V^b \mathcal{F}_i \nabla_b \Phi^i + i V^b \mathcal{F}_j \nabla_b \bar{\Phi}^j$$

$$+ \frac{1}{2} \mathcal{F}_{ij} \nabla_{\alpha} \Psi^i \nabla_{\alpha} \Psi^j$$

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where $\Psi'^{i}$ denotes the set $(\Phi^{i}, L)$ and $\Psi'^{J}$ the set $(\bar{\Phi}^{j}, L)$, we can write the variation as

\[
\delta S_D = -2 V^b \Delta_b (\mathcal{F} - L \mathcal{F}_L) + \frac{3}{2} V^{\dot{\alpha} \alpha} \mathcal{F}_{i j} \nabla_{\alpha} \Phi^{i} \nabla_{\dot{\alpha}} \bar{\Phi}^{j} - \frac{3}{2} V^{\dot{\alpha} \alpha} \mathcal{F}_{L L} \nabla_{\alpha} L \nabla_{\dot{\alpha}} L
\]
\[
+ 3i \Sigma' \left( \mathcal{F}_{i} X_{\Phi}^{i} - \mathcal{F}_{j} X_{\Phi}^{j} \right) - 3 \mathcal{F}_{L L} \mathcal{L}
\]

This form is immediately reminiscent of that we have discussed before. Since $X \equiv \mathcal{F} - L \mathcal{F}_L$ is to be identified as the compensator, we define $G_{a} \equiv -X^{1/2} \Delta_{a} X^{-1/2}$ as before. This immediately yields

\[
\delta S_D = -4 X V^{b} G_b + \frac{3}{2} V^{\dot{\alpha} \alpha} \mathcal{F}_{i j} \nabla_{\alpha} \Phi^{i} \nabla_{\dot{\alpha}} \bar{\Phi}^{j} - \frac{3}{2} V^{\dot{\alpha} \alpha} \mathcal{F}_{L L} \nabla_{\alpha} L \nabla_{\dot{\alpha}} L - \frac{3}{2} V^{\dot{\alpha} \alpha} X^{-1} \nabla_{\alpha} X \nabla_{\dot{\alpha}} X
\]
\[
+ 3i \Sigma' \left( \mathcal{F}_{i} X_{\Phi}^{i} - \mathcal{F}_{j} X_{\Phi}^{j} \right) - 3 \mathcal{F}_{L L} \mathcal{L}
\]

To maintain the analogy, we should make the identifications

\[
K_{a \dot{a}} \equiv -3 X^{-1} \mathcal{F}_{i j} \nabla_{\alpha} \Phi^{i} \nabla_{\dot{\alpha}} \bar{\Phi}^{j} + 3 X^{-1} \mathcal{F}_{L L} \nabla_{\alpha} L \nabla_{\dot{\alpha}} L + 3 X^{-2} \nabla_{\alpha} X \nabla_{\dot{\alpha}} X
\]
\[
K_{r} \equiv +3i X^{-1} \left( \mathcal{F}_{i} X_{\Phi}^{i} - \mathcal{F}_{j} X_{\Phi}^{j} \right)
\]

which would give

\[
\delta S_D = -4 X V^{b} G_b + X V^{b} K_{b} + X \Sigma' K_{r} - 3 \mathcal{F}_{L L} \mathcal{L}
\]

We would like to think of terms involving $V^{a}$ to consist of a “supergravity term” $G_{b}$ and the “matter term” $K_{b}$, so it is sensible to expand $K_{b}$ out entirely in terms of the fields. We find

\[
K_{a \dot{a}} = -3 \nabla_{\alpha} \Phi^{i} \nabla_{\dot{\alpha}} \bar{\Phi}^{j} \left( X^{-1} \mathcal{F}_{i j} - X^{-2} X_{i} X_{j} \right)
\]
\[
+ 3 \nabla_{\alpha} \Phi^{i} \nabla_{\dot{\alpha}} L \left( X^{-2} X_{i} X_{L} \right) + 3 \nabla_{\alpha} L \nabla_{\dot{\alpha}} \Phi^{j} \left( X^{-2} X_{L} X_{j} \right)
\]
\[
+ 3 \nabla_{\alpha} L \nabla_{\dot{\alpha}} L \left( X^{-1} \mathcal{F}_{L L} + X^{-2} X_{L} X_{L} \right)
\]

where $X_{i} = \mathcal{F}_{i} - L \mathcal{F}_{L i}$, $X_{j} = \mathcal{F}_{j} - L \mathcal{F}_{L j}$ and $X_{L} = -L \mathcal{F}_{L L}$.

Before moving on, we should make one more generalization. Up until now we have assumed $L$ to be a normal linear multiplet. However, we may instead choose for $L$ to obey the modified linearity condition

\[
\mathcal{P} L = -\frac{1}{4} \nabla^{2} L = -\frac{1}{2} k \text{Tr}(W^{a} W_{a}) \quad (A.1)
\]

This amounts to choosing $L = L_{0} + k \Omega$, where $L_{0}$ is a normal linear superfield and $\Omega$ is the Chern-Simons superfield $[10]$. $L$ is chosen to be gauge invariant, so the gauge transformation of $\Omega$, which is itself a linear superfield, must be cancelled by the transformation of $L_{0}$.

The Yang-Mills term then receives contributions from the D-term of $\mathcal{F}$:

\[
-3 \int E \mathcal{F} = \frac{3k}{4} \int \mathcal{E} \left( \mathcal{F}_{L} \text{Tr}(W^{a} W_{a}) + \ldots \right) + \text{h.c.} \quad (A.2)
\]

This contributes to $f_{r s}$ (effectively) a non-holomorphic factor of $3 k \delta_{r s} \mathcal{F}_{L}$ and thus to $G_{r s}$ a factor of $6 k \delta_{r s} \mathcal{F}_{L}$. 

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The quanta of $L$ which we previously denoted $\mathcal{L}$ should now be understood as

$$\mathcal{L} = \mathcal{L}_0 - ik\nabla^\alpha(W_\alpha \Sigma) - ik\nabla_\dot{\alpha}(\dot{W}^\alpha \Sigma) + i\kappa(\nabla^\alpha \dot{W}_\alpha)\Sigma - k\nabla^{\dot{\alpha}}\dot{W}_\dot{\alpha}$$

where $\mathcal{L}_0$ is linear. This formula is determined by requiring the chiral quantum variation of both sides of (A.1) to coincide.

One can easily check that

$$-3F_L\mathcal{F}_L = -3F_L\mathcal{F}_L \mathcal{L}_0 + 3ikF_L\nabla^\alpha(W_\alpha \Sigma) + 3ikF_L\nabla_\dot{\alpha}(\dot{W}^\alpha \Sigma) - 3ikF_L(\nabla^\alpha \dot{W}_\alpha)\Sigma + 3kF_LV^{\dot{\alpha}}\dot{W}_\dot{\alpha}$$

where

$$\mathcal{F}_L = -3ik(\nabla^\alpha F_L)W_{ar} - 3ik(\nabla_\dot{\alpha} F_L)\dot{W}_\dot{ar} - 3ikF_L(\nabla^\alpha \dot{W}_\alpha)_r$$

This agrees with the previous definition for these objects provided we rewrite them solely in terms of $G_{rs} = f_{rs} + \tilde{f}_{rs}$. Then taking into account the contribution from the linear multiplet gives $G_{rs} = f_{rs} + \tilde{f}_{rs} + 6F_L\kappa\delta_{rs}$.

The first order structure can then be written

$$\delta S_D = V^b(-4XG_b + X K_b + \mathcal{Y}_b) + \Sigma^r(X K_r + \mathcal{Y}_r) - 3\mathcal{F}_L\mathcal{L} - 3\mathcal{F}_i\eta^i - 3\mathcal{F}_{j\bar{j}}\bar{\eta} \bar{\eta} \quad (A.3)$$

where we have included also the chiral superfield variations.

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