Abstract

We review the concept of $\tau$-function for simple analytic curves. The $\tau$-function gives a formal solution to the 2D inverse potential problem and appears as the $\tau$-function of the integrable hierarchy which describes conformal maps of simply-connected domains bounded by analytic curves to the unit disk. The $\tau$-function also emerges in the context of topological gravity and enjoys an interpretation as a large $N$ limit of the normal matrix model.

1. Recently, it has been realized [1, 2] that conformal maps exhibit an integrable structure: conformal maps of compact simply connected domains bounded by analytic curves provide a solution to the dispersionless limit of the 2D Toda hierarchy. As is well known from the theory of solitons, solutions of an integrable hierarchy are represented by $\tau$-functions. The dispersionless limit of the $\tau$-function emerges as a natural object associated with the curves. In this paper we discuss the $\tau$-function for simple analytic curves and its connection to the inverse potential problem, area preserving diffeomorphisms, the Dirichlet boundary problem, and matrix models.

2. Inverse potential problem. Consider a closed analytic curve $\gamma$ in the complex plane and denote by $D_+$ and $D_-$ the interior and exterior domains with respect to the curve. The point $z = 0$ is assumed to be in $D_+$. Assume that the domain $D_+$ is filled homogeneously

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*Service de Physique Théorique, CEA-Saclay, 91191 Gif sur Yvette, France
†Department of Mathematics, Columbia University, New York, NY 10027, USA and Landau Institute for Theoretical Physics
‡Theoretical Division, MS-B213, LANL, Los-Alamos, NM 87545, USA
§James Franck Institute and Enrico Fermi Institute of the University of Chicago, 5640 S.Ellis Avenue, Chicago, IL 60637, USA and Landau Institute for Theoretical Physics
¶Joint Institute of Chemical Physics, Kosygina str. 4, 117334, Moscow, Russia and ITEP, 117259, Moscow, Russia

1A closed analytic curve is the curve which can be parametrized by a function $z \equiv x + iy = z(w)$, analytic in a domain which includes the unit circle $|w| = 1$
with electric charge, with a density which we set to be equal to 1. The potential $\Phi$ created by the charge obeys the equation

$$- \partial_z \partial_{\bar{z}} \Phi(z, \bar{z}) = \begin{cases} 1 & \text{if } z = x + iy \in D_+ \\ 0 & \text{if } z = x + iy \in D_- \end{cases}$$ (1)

The potential $\Phi$ can be written as an integral over the domain $D^+$:

$$\Phi(z, \bar{z}) = -\frac{2}{\pi} \int_{D^+} d^2 z' \log |z - z'|$$ (2)

In the exterior domain $D^-$, the potential is the harmonic function whose asymptotic expansion as $z \to \infty$ is given by

$$\Phi^-(z, \bar{z}) = -2t_0 \log |z| + 2\Re \sum_{k>0} \frac{v_k}{k} z^{-k},$$ (3)

where

$$v_k = \frac{1}{\pi} \int_{D^+} z^k d^2 z \quad (k > 0)$$ (4)

are the harmonic moments of the interior domain $D^+$ and

$$\pi t_0 = \int_{D^+} d^2 z$$ (5)

is its area. In the interior domain $D^+$, the potential (2) is equal to a function $\Phi^+$, which is harmonic up to the term $-|z|^2$. The expansion of this function around $z = 0$ is

$$\Phi^+(z, \bar{z}) = -|z|^2 - v_0 + 2\Re \sum_{k>0} t_k z^k$$ (6)

Here

$$t_k = -\frac{1}{\pi k} \int_{D^-} z^{-k} d^2 z \quad (k > 0)$$ (7)

are the harmonic moments of the exterior domain $D^-$ and

$$v_0 = \frac{2}{\pi} \int_{D^+} \log |z| d^2 z.$$ (8)

The two sets of moments (4) and (7) are related by the conditions that $\Phi^+ = \Phi^-$, $\partial_z \Phi^+ = \partial_z \Phi^-$ on the curve $\gamma$.

The inverse potential problem is to determine the form of the curve $\gamma$ given one of the functions $\Phi^+$ or $\Phi^-$, i.e. given one of the infinite sets of moments. We will choose as independent variables the area $\pi t_0$ and the moments of the exterior $t_k (k \geq 1)$. Under certain conditions, they completely determine the form of the curve as well as the moments $v_k (k \geq 0)$ (4). More precisely, $\{t_k\}_{k=0}^\infty$ is a good set of local coordinates in the space of analytic curves. For simplicity we assume in this paper that only a finite number of $t_k$ are non-zero. In this case the series (4) is a polynomial in $z$, $\bar{z}$ and, therefore, it gives the function $\Phi^+$ for $z \in D_+$. Note that $t_0$, $v_0$ are real quantities while all other moments are in general complex variables.
3. **Variational principle.** Consider the energy functional describing a charge with a density $\rho(z, \bar{z})$ in the background potential created by the homogeneously distributed charge with the density $+1$ inside the domain $D_+$ (I):

$$E\{\rho\} = -\frac{1}{\pi^2} \int \int d^2z d^2z' \rho(z, \bar{z}) \log |z - z'| \rho(z', \bar{z}') - \frac{1}{\pi} \int d^2z \rho(z, \bar{z}) \Phi(z, \bar{z}).$$  \hspace{1cm} (9)

The first term is the 2D “Coulomb” energy of the charge while the second one is the energy due to the background charge. Clearly, the distribution of the charge neutralizing the background charge gives the minimum to the functional: $\rho_0 = -1$ inside the domain and $\rho_0 = 0$ outside. At the minimum the functional is equal to minus electrostatic energy $E$ of the background charge:

$$E = \min_\rho E\{\rho\} = \frac{1}{\pi^2} \int_{D_+} d^2z \int_{D_+} d^2z' \log |z - z'| = -\frac{1}{2\pi} \int_{D_+} d^2z \Phi(z, \bar{z}).$$  \hspace{1cm} (10)

Varying over $\rho$ and then setting $\rho = -1$ inside the domain, we obtain eq.(6).

The first corollary of the variational principle is that the $E$ is a potential function for the moments. Eq. (6) suggests to treat $v_0$ and $t_k$ as independent variables, so moments of the interior, $v_k$, $k \geq 1$, and $t_0$ are functions of $v_0$ and $t_k$. Let us differentiate $E$ or $-E\{\rho\}$ at the extremum with respect to the parameters $v_0$, $t_k$. Since $\rho_0$ minimizes the functional, the derivative is equivalent to the partial derivative of $E$ at the fixed extremum $\rho$. This gives

$$\frac{\partial E}{\partial t_k} = v_k, \quad \frac{\partial E}{\partial t_k} = \bar{v}_k, \quad \frac{\partial E}{\partial v_0} = -t_0,$$  \hspace{1cm} (11)

where the partial derivative with respect to $t_k$ is taken at fixed $v_0$ and $t_j$ ($j \neq 0, k$). Therefore the differential $dE$ reads

$$dE = \sum_{k>0} (v_k dt_k + \bar{v}_k d\bar{t}_k) - t_0 dv_0.$$  \hspace{1cm} (12)

Let us note that the variational principle may be formulated in a number of different ways. One particular variational principle is suggested by the matrix model discussed in the Sec.9. In this case one consider a charged liquid in the potential

$$V(z, \bar{z}) = z\bar{z} + v_0 - \sum_{k>0} \left( t_k z^k + \bar{t}_k \bar{z}^k \right)$$  \hspace{1cm} (13)

defined everywhere on the plane and $v_0$ and $t_k$ are parameters. The energy of the charged liquid

$$E\{\rho, V\} = -\frac{1}{\pi^2} \int d^2z \int d^2z' \rho(z, \bar{z}) \rho(z', \bar{z}') \log |z - z'| + \frac{1}{\pi} \int d^2z \rho(z, \bar{z}) V(z, \bar{z}).$$  \hspace{1cm} (14)

reaches the minimum if the liquid forms a drop with the density $\rho_0 = -1$ bounded by the curve determined by parameters of the potential $v_0$ and $t_k$. For another version of the variational principle see [4].

4. **$\tau$-function.** It is more natural to treat the total charge $t_0$ rather than $v_0$ as an independent variable, i.e. to consider the variational principle at a fixed total charge.
\[ t_0 = \int \rho d^2 z. \] This is achieved via the Legendre transformation. Let us introduce the function \( F = E + t_0 v_0 \), whose differential is

\[ dF = \sum_{k>0} (v_k dt_k + \bar{v}_k d\bar{t}_k) + v_0 dt_0. \]  

(15)

We define the \( \tau \)-function as \( \tau = e^F \), so that

\[ \log \tau = \frac{1}{2\pi} \int_{D^+} d^2 z \Phi(z, \bar{z}) + t_0 v_0 = -\frac{1}{\pi} \int \int_{D^+} \log \left| \frac{1}{z} - \frac{1}{z'} \right| d^2 z d^2 z'. \]  

(16)

The \( \tau \)-function is a real function of the moments \( \{ t_0, t_1, t_2, \ldots \} \). Under the assumption that only a finite number of them are non-zero, we can substitute (6) into (16) and perform the term-wise integration. Taking into account that \( \frac{1}{4\pi} \int_{D^+} |z|^2 d^2 z = \frac{1}{2} t_0^2 + \sum_{k>0} k (t_k v_k + \bar{t}_k \bar{v}_k) \) (a simple consequence of the Stokes formula), we get the expression for the \( \tau \)-function in terms of \( t_k \) and \( v_k \):

\[ 2 \log \tau = -\frac{1}{2} t_0^2 + t_0 v_0 - \frac{1}{2} \sum_{k>0} (k-2)(t_k v_k + \bar{t}_k \bar{v}_k). \]  

(17)

Rephrasing (11) we get the main property of the \( \tau \)-function, which has been used as its definition in Ref. [2]

\[ \frac{\partial \log \tau}{\partial t_k} = v_k, \quad \frac{\partial \log \tau}{\partial \bar{t}_k} = \bar{v}_k, \quad \frac{\partial \log \tau}{\partial t_0} = v_0 \]  

(18)

where the derivative with respect to \( t_k \) is taken at fixed \( t_j (j \neq k) \).

Two immediate consequences of the very existence of the potential function are symmetry relations for the moments

\[ \frac{\partial v_k}{\partial t_n} = \frac{\partial v_n}{\partial t_k}, \quad \frac{\partial v_k}{\partial \bar{t}_n} = \frac{\partial \bar{v}_n}{\partial \bar{t}_k} \]  

(19)

and the quasi-homogeneity condition for the \( \tau \)-function:

\[ 4 \log \tau = -t_0^2 + 2t_0 \frac{\partial \log \tau}{\partial t_0} - \sum_{n>0} (n-2) \left( t_n \frac{\partial \log \tau}{\partial t_n} + \bar{t}_n \frac{\partial \log \tau}{\partial \bar{t}_n} \right). \]  

(20)

Apart from the term \( -t_0^2 \), this formula reflects the scaling of moments as \( z \to \lambda z \): \( t_k \to \lambda^{2-k} t_k \) \( (k \geq 0) \), \( v_k \to \lambda^{2+k} v_k \) \( (k \geq 1) \).

As an illustration we present the \( \tau \)-function of ellipse [2]. In this case only the first two moments \( t_1 \) and \( t_2 \) are nonzero:

\[ \log \tau = -\frac{3}{4} t_0^2 + \frac{1}{2} t_0^2 \log \left( \frac{t_0}{1 - 4|t_2|^2} \right) + \frac{t_0}{1 - 4|t_2|^2} \left( |t_1|^2 + t_1^2 \bar{t}_2 + \bar{t}_1^2 t_2 \right). \]

2The \( \tau \)-function for the ellipse (at \( t_1 = 0 \)) appeared in Ref. [2] as the limit of the Laughlin wave function or a planar limit of the free energy of normal matrix models, see Sec. 9.
5. **Schwarz function and generating function of the conformal map.** Consider a univalent conformal map of the exterior domain $D_-$ to the exterior of the unit disk and expand it in Laurent series:

$$w(z) = \frac{1}{r}z + \sum_{j=0}^{\infty} p_j z^{-j}, \quad (21)$$

where the coefficient $r$ is chosen to be real and positive. The series for the inverse map (from the exterior of the unit disk to $D_-$) has a similar form:

$$z(w) = rw + \sum_{j=0}^{\infty} u_j w^{-j}. \quad (22)$$

Chosen $w$ on the unit circle, eq. $(22)$ gives a parametrization of the curve. By the definition of an analytic curve, the map can be analytically continued to a strip-like neighborhood of the curve belonging to $D_+$. The continuation is given by the Riemann-Schwarz reflection principle (see e.g. [6]):

$$w = (\bar{w}(S(z)))^{-1}, \quad (23)$$

where $S(z)$ is the point reflected relative to the curve$^3$. Following [4], we call $S(z)$ the **Schwarz function** of the curve. Let us recall its construction. Write the equation for the curve $F(x, y) = 0$ in complex coordinates,

$$F(z + \bar{z}, z - \bar{z}) = 0,$$

and solve it with respect to $\bar{z}$. One gets the Schwarz function: $\bar{z} = S(z)$. The Schwarz function is analytic in a strip-like domain that includes the curve. On the curve the Schwarz function is equal to the complex conjugate argument. The main property of the Schwarz function is the obvious but important **unitarity condition**

$$\bar{S}(S(z)) = z \quad (24)$$

(the inverse function coincides with the complex conjugate function). In terms of a conformal map the Schwarz function is

$$S(z) = rw^{-1}(z) + \sum_{j=0}^{\infty} \bar{u}_j w^j(z). \quad (25)$$

Using the Schwarz function one can write the moments of the exterior and the interior domains (4,7) as contour integrals $^4$

$$t_n = \frac{1}{2\pi i} \oint_{\gamma} z^{-n} S(z) dz, \quad v_n = \frac{1}{2\pi i} \oint_{\gamma} z^n S(z) dz \quad (26)$$

Eq. (26) yields the Laurent expansion of the Schwarz function

$$S(z) = \sum_{k=1}^{\infty} k t_k z^{k-1} + \frac{t_0}{z} + \sum_{k=1}^{\infty} v_k z^{-k-1}. \quad (27)$$

Now let us define the **generating function** $\Omega(z)$, related to the Schwarz function by

$$S(z) = \partial_z \Omega(z). \quad (28)$$

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$^3$We use the notation: given an analytic function $f(z) = \sum_j f_j z^j$, we set $\bar{f}(z) = \sum_j \bar{f}_j z^j$.

$^4$This is due to a more general statement $\int_{D_\pm} f(z) d^2z = \pm \frac{1}{2\pi} \oint_{\gamma} f(z) S(z) dz$, where $f(z)$ is an analytic function in the domain $D_\pm$. 
The latter is given, according to (27), by the Laurent series

\[ \Omega(z) = \sum_{k=1}^{\infty} t_k z^k - \frac{1}{2} v_0 + t_0 \log z - \sum_{k=1}^{\infty} \frac{v_k}{k} z^{-k} \]  

(29)

It can be represented as \( \Omega(z) = \Omega^+(z) + \Omega^-(z) - \frac{1}{2} v_0 \), where \( \Omega^{\pm}(z) \) are analytic in \( D_\pm \) respectively:

\[ \Omega^+(z) = \frac{1}{\pi} \int_{D_-} \log(1 - \frac{z}{z'}) d^2 z' = \sum_{k=1}^{\infty} t_k z^k \]  

(30)

\[ \Omega^-(z) = \frac{1}{\pi} \int_{D_+} \log(z - z') d^2 z' = t_0 \log z - \sum_{k=1}^{\infty} \frac{v_k}{k} z^{-k} \]  

(31)

From (34) we see that \( \Phi^-(z, \bar{z}) = -2 \Re e \Omega^-(z) \) and \( \Phi^+(z, \bar{z}) = 2 \Re e \Omega^+(z) - v_0 - |z|^2 \). Contrary to the potentials \( \Phi^\pm \), the analytical functions \( \Omega^+ \) and \( -\Omega^- \) do not match each other on the curve. The discontinuity gives the value of the generating function restricted to the curve

\[ \Omega(z) = \frac{1}{2} |z|^2 + 2i A(z), \quad z \in \gamma \]  

(32)

where \( A(z) \) is the area of the interior domain bound by the ray \( \varphi = \arg z \) and the real axis. As a corollary, it is easy to show that variations of the \( \Omega(z) \) on the curve with respect to the real parameters \( t_0, \Re e t_k \) and \( \Im m t_k \) are purely imaginary. This allows one to apply the Riemann-Schwarz reflection principle to analytical continuation of

\[ H_k(z) = \partial_t_k \Omega(z), \quad \bar{H}_k(z) = -\partial_{\bar{t}_k} \Omega(z) \]  

(33)

and to prove the fundamental relations

\[ \partial_{t_0} \Omega(z) = \log w(z), \]  

(34)

\[ \partial_t_k \Omega(z) = \left( z^k(w) \right)_+ + \frac{1}{2} \left( z^k(w) \right)_0 \]  

(35)

\[ \partial_{\bar{t}_k} \Omega(z) = \left( S^k(z(w)) \right)_+ + \frac{1}{2} \left( S^k(z(w)) \right)_0 \]  

(36)

The symbols \( (f(w))_\pm \) mean a truncated Laurent series, where only terms with positive (negative) powers of \( w \) are kept, while \( (f(w))_0 \) is the constant term \( (w^0) \) of the series. Note that the derivatives in eqs. (34-36) are taken at fixed \( z \).

To prove (34), we first notice that

\[ \partial_{t_0} \Omega(z(w)) = \log z - \frac{\partial_{t_0} v_0}{2} + \text{negative powers in } z = \log wr - \frac{\partial_{t_0} v_0}{2} + \text{negative powers in } w. \]

Independently, one can show that \( \partial_{t_0} v_0 = 2 \log r \).

Then, using the Riemann-Schwarz reflection principle, we may write \( \partial_{t_0} \Omega(z(w)) \) also in the form \( \partial_{t_0} \bar{\Omega}(S(z(w))) \). Expanding the latter in \( S(z) \) and then, using expansion of (27) in \( w \), we have

\[ \partial_{t_0} \bar{\Omega}(S(z(w))) = \log S(z) - \frac{\partial_{t_0} v_0}{2} + \text{negative powers in } S(z) = \log w + \text{positive powers in } w. \]
Comparing both expansions, we conclude that \( \partial_{t_0} \Omega(z) = \log w(z) \). Similar arguments are used in the proof of (34) and (36).

6. *Dispersionless Hirota equation and the Dirichlet boundary problem.* Using the representation (18) of the moments \( v_k \) as derivatives of the \( \tau \)-function, one can express the conformal map \( w(z) \) (34) through the \( \tau \)-function:

\[
\log w = \log z - \partial_{t_0} \left( \frac{1}{2} \partial_{t_0} + \sum_{k \geq 1} z^{-k} \frac{1}{k} \partial_{t_k} \right) \log \tau. \tag{37}
\]

With the help of the \( \tau \)-function, eqs. (35, 36) can be similarly encoded as follows:

\[
\partial_z \partial_\zeta \log \left( \frac{w(z) - w(\zeta)}{z - \zeta} \right) = \frac{1}{(z - \zeta)^2} + \left( \sum_{k \geq 1} z^{-k-1} \partial_{t_k} \right) \left( \sum_{n \geq 1} \zeta^{-n-1} \partial_{t_n} \right) \log \tau \tag{38}
\]

\[
\partial_z \partial_\zeta \log \left( w(z) \bar{w}(\bar{\zeta}) - 1 \right) = \left( \sum_{k \geq 1} z^{-k-1} \partial_{t_k} \right) \left( \sum_{n \geq 1} \bar{\zeta}^{-n-1} \partial_{\bar{t}_n} \right) \log \tau. \tag{39}
\]

The derivation is similar to the one given in [8, 9] for the case of the KP hierarchy. Moreover, these equations in the integrated form are most conveniently written in terms of the differential operators

\[
D(z) = \sum_{k \geq 1} \frac{z^{-k}}{k} \partial_{t_k}, \quad \bar{D}(\bar{z}) = \sum_{k \geq 1} \frac{\bar{z}^{-k}}{k} \partial_{\bar{t}_k}. \tag{40}
\]

From (38, 39) one obtains:

\[
\log \frac{w(z) - w(\zeta)}{z - \zeta} = -\frac{1}{2} \partial_{t_0}^2 \log \tau + D(z)D(\zeta) \log \tau \tag{41}
\]

\[
-\log \left( 1 - \frac{1}{w(z)\bar{w}(\zeta)} \right) = D(z)\bar{D}(\bar{\zeta}) \log \tau \tag{42}
\]

Combining (37) and (41), one obtains the dispersionless Hirota equation (or the dispersionless Fay identity) for 2D Toda lattice hierarchy [4]:

\[
(z - \zeta) e^{D(z)D(\zeta) \log \tau} = ze^{-\partial_{t_0} D(z) \log \tau} - \zeta e^{-\partial_{\bar{t}_0} D(\zeta) \log \tau}. \tag{43}
\]

Eq. (43), after being expanded in powers of \( z \) and \( \zeta \), generates an infinite set of relations between the second derivatives \( \partial_n \partial_{\bar{t}_m} \log \tau \) of the \( \tau \)-function. Using [12] instead of (41), a similar equation for the mixed derivatives \( \partial_n \partial_{\bar{t}_m} \log \tau \) can be written:

\[
1 - e^{-D(z)\bar{D}(\bar{\zeta}) \log \tau} = \frac{1}{z\zeta} e^{\partial_{t_0}(\partial_{\bar{t}_0} + D(z) + \bar{D}(\bar{\zeta})) \log \tau} \tag{44}
\]

Let us conclude this section with two other forms of the dispersionless Hirota equation for the conformal map. They emphasize a relation between the Hirota equation and two fundamental objects of the classical analysis: the Green function of the Dirichlet problem\(^5\) and the Schwarz derivative.

\(^5\)This relation is pointed out to us by L. Takhtajan.
The Green function of the Dirichlet boundary problem for the Laplace operator in $D_-$ expressed through the conformal map $w(z)$ is:

$$G(z, \zeta) = \log \left| \frac{w(z) - w(\zeta)}{w(z)\bar{w}(\zeta) - 1} \right|$$  \hspace{1cm} (45)$$

Combining (41) and (42), and using the notation (40), we represent the Green function as follows:

$$2G(z, \zeta) = 2 \log |z - 1 - \zeta - 1| + \left( \partial_t + D(z) + \bar{D}(\zeta) \right) \left( \partial_t + D(\zeta) + \bar{D}(\zeta) \right) \log \tau$$  \hspace{1cm} (46)$$

This formula generalizes (37) since (46) becomes the real part of (37) as $\zeta \to \infty$. (As $\zeta \to \infty$, $G(z, \zeta) \to -\log |w(z)|$.) Note also that the real part of (37) can be written in the form

$$\Phi(z, \bar{z}) = -2t_0 \log |z| + \left( D(z) + \bar{D}(z) \right) \log \tau$$

where $\Phi$ is the potential (2) ($z \in D_-$).

The l.h.s. of eq.(41) generalizes the Schwarz derivative of the conformal map $T(z) \equiv w'''(z)w'(z) - \frac{3}{2} \left( \frac{w''(z)}{w'(z)} \right)^2 = 6 \lim_{z \to \zeta} \partial_z \partial_{\bar{z}} \log \frac{w(z) - w(\zeta)}{z - \zeta}$  \hspace{1cm} (47)$$

Taking the limit $\zeta \to z$ of both sides of (41), we get a relation between the Schwarz derivative and the $\tau$-function:

$$T(z) = 6z^{-2} \sum_{k,n \geq 1} z^{-k-n} \partial^2 \log \tau \frac{w(z) - w(\zeta)}{z - \zeta}$$  \hspace{1cm} (48)$$

The latter can be used as an alternative definition of the $\tau$-function.

7. Integrable structure of conformal maps. Eqs.(34-36) allow one to say that the differential

$$d\Omega = Sdz + \log w \, dt_0 + \sum_{k=1}^{\infty} (H_k dt_k - \bar{H}_k d\bar{t}_k)$$  \hspace{1cm} (49)$$

generates the set of Hamiltonian equations for deformations of the curve due to variation of $t_k$:

$$\partial_{t_k} S(z) = \partial_z H_k(z), \quad \partial_{\bar{t}_k} S(z) = -\partial_{\bar{z}} \bar{H}_k(z),$$  \hspace{1cm} (50)$$

where we set $H_0(z) = \log w(z)$. The equations are consistent due to commutativity of the flows:

$$\left( \partial_{t_j} H_k \right)_z = \left( \partial_{t_k} H_j \right)_z = \partial_j \partial_k \Omega(z)$$  \hspace{1cm} (51)$$

Equations (50) are more transparent being written in terms of canonical variables. The differential $d\Omega$ suggests that the pairs $\log w$, $t_0$ and $z(w)$, $S(z(w))$ are canonical and establishes the symplectic structure for conformal maps. Indeed, treating $w$ as an independent variable, one rewrites eq. (34) as

$$\{z(w), S(z(w))\} = 1$$  \hspace{1cm} (52)$$
where the Poisson bracket $\{,\}$ is with respect to $\log w$ and the area $t_0$ is defined as

$$\{f, g\} = w \frac{\partial f}{\partial w} \frac{\partial g}{\partial t_0} - w \frac{\partial g}{\partial w} \frac{\partial f}{\partial t_0}$$

(53)

where the derivatives with respect to $t_0$ are taken at fixed $t_k$ and $w$.

The other flows read

$$\frac{\partial z(w)}{\partial t_k} = \{H_k, z(w)\}$$

(54)

$$\frac{\partial S(z(w))}{\partial t_k} = \{H_k, S(z(w))\},$$

(55)

and similarly for the flows with respect to $\bar{t}_k$. Now the Hamiltonian functions $H_k$ and $\bar{H}_k$ are degree $k$ polynomials of $w$ and $w^{-1}$ respectively.

The consistency conditions (50) now take the form of the zero-curvature conditions:

$$\partial_{t_j} H_i - \partial_{t_i} H_j + \{H_i, H_j\} = 0,$$

(56)

$$\partial_{t_j} \bar{H}_i + \partial_{\bar{t}_i} H_j + \{\bar{H}_i, H_j\} = 0.$$

(57)

The infinite set of the Poisson-commutating flows form a Whitham integrable hierarchy [10]. Eqs. (54,55) are the Lax-Sato equations for the hierarchy. They generate an infinite set of differential equations for the coefficients (potentials) $u_j$ of the inverse conformal map (22). The first equation of the hierarchy is

$$\partial_{t_1}^2 \phi = \partial_{t_0} \exp(\partial_{t_0} \phi), \quad \partial_{t_0} \phi = \log r^2.$$  

(58)

The integrable hierarchy describing conformal maps is also known in the soliton literature as the dispersionless Toda lattice hierarchy, or SDiff(2) Toda hierarchy [11] (see the next section). The algebra SDiff(2) of area-preserving diffeomorphisms is the symmetry algebra of this hierarchy [11]. Eqs. (54-57) describe infinitesimal deformations of the curve such that the area $t_0$ is kept fixed.

The integrable hierarchy possesses many solutions. The particular solution relevant to conformal maps is selected by the subsidiary condition (52). This condition, known as dispersionless string equation, has already appeared in the study of the $c = 1$ topological gravity [11, 13, 14] and in the large $N$ limit of a model of normal random matrices [15]. The latter is discussed in Sec. 9.

8. Toda lattice hierarchy and its dispersionless limit. Below we review the two-dimensional Toda lattice hierarchy and show that its dispersionless limit gives the equations describing conformal maps (35,36,54,55).

The 2D Toda hierarchy is defined by two Lax operators

$$L = r(t_0) e^{\frac{h \phi}{2\alpha}} + \sum_{k=0}^{\infty} u_k(t_0) e^{-kh \frac{\phi}{2\alpha}}$$

(59)

Where the Poisson bracket of slit domains and special solutions to equations of hydrodynamic type (Benney equations) was first observed by Gibbons and Tsarev [12]
\[ L = e^{-\hbar \frac{\partial}{\partial t_0}} r(t_0) + \sum_{k=0}^{\infty} e^{k\hbar \frac{\partial}{\partial t_0}} \bar{u}_k(t_0) \]  

(60)

acting in the space of functions of \( t_0 \) where the coefficients \( u_j \) and \( \bar{u}_j \) are functions of \( t_0 \) and also of two independent sets of parameters ("times") \( t_k \) and \( \bar{t}_k \). Note that \( u_k \) and \( \bar{u}_k \) as well as \( t_k \) and \( \bar{t}_k \) in (58,60) are not necessarily complex conjugate to each other, although we choose them to be so.

The dependence of the coefficient \( u_k \) and \( \bar{u}_k \) on \( t_k \) and \( \bar{t}_k \) are given by the Lax-Sato equations:

\[ \hbar \frac{\partial L}{\partial t_k} = [H_k, L] \]  

(61)

\[ \hbar \frac{\partial L}{\partial \bar{t}_k} = [L, \bar{H}_k] \]  

(62)

and similar equations for \( \bar{L} \). The flows are generated by

\[ H_k = (L^k)_+ + \frac{1}{2}(L^k)_0 \]  

(63)

\[ \bar{H}_k = (\bar{L}^k)_- + \frac{1}{2}(\bar{L}^k)_0 \]  

(64)

where the symbol \((L^k)_\pm\) means positive (negative) parts of the series in the shift operator \( e^{\hbar \frac{\partial}{\partial t_0}} \). The first equation of the hierarchy is the Toda lattice equation

\[ \partial^2_{t_1 \bar{t}_1} \phi(t_0) = e^{\phi(t_0+\hbar)} - e^{\phi(t_0)} - e^{\phi(t_0-\hbar)} \]  

(65)

where \( r^2 = e^{\phi(t_0+\hbar)} - e^{\phi(t_0)} \).

The spectrum of the Lax operator is determined by the linear problem \( L \Psi = z \Psi \). The wave function \( \Psi \) is expressed through the \( \tau \)-function of the dispersionfull hierarchy (61,62) by the following formula:

\[ \Psi(z; t_0, t_1, t_2, \ldots) = \tau_{-1}(t_0, t_1, t_2, \ldots) z^{t_0/\hbar} e^{\frac{1}{\hbar} \sum_{k>0} t_k z^k} e^{\frac{1}{\hbar} \sum_{k>0} \frac{z^k}{k} \tau_k(t_0, t_1, t_2, \ldots)} \]  

(66)

Among many solutions of the hierarchy, one is of particular interest. It is selected by the string equation [16]

\[ [L, L] = \hbar \]  

(67)

This solution is known to describe the normal matrix model at finite size of matrices [15].

The dispersionless limit of the Toda hierarchy is a formal semi-classical limit \( \hbar \to 0 \). To proceed we notice that the shift operator \( W = e^{\hbar \frac{\partial}{\partial t_0}} \) obeys the commutation relation \([W, t_0] = \hbar W \). In the semiclassical limit it is supposed to be replaced by the canonical variable \( w \) with the Poisson bracket \( \{\log w, t_0\} = 1 \). The Lax operator then becomes a \( c \)-valued function which is identified with the inverse conformal map \( z(w) \) (22). Similarly, \( \bar{L} \) is identified with \( S(z(w)) \). In their turn, the Lax-Sato equations (61,62) are identified with eqs. (54,55) for the conformal map. In the same fashion the dispersionless limit of the string equation (67) is identified with eq.(52). The semiclassical limits of the wave function and the \( \tau \)-function give the generating function \( \Omega \) and the dispersionless \( \tau \)-function: \( \Psi \to e^{\Omega/\hbar}, \tau_\hbar \to e^{(\log \tau)/\hbar^2} \). Similarly, eq. (13) is a semiclassical limit of the Hirota equation for the \( \tau \)-function of the 2D Toda hierarchy.
9. The $\tau$-function of the conformal map as large $N$ matrix integral. The integrable structure of conformal maps is identical to the one observed in a class of random matrix models related to noncritical string theories. Moreover, there exists a random matrix model whose large $N$ limit reproduces exactly the $\tau$-function for analytic curves.

Consider the partition function of the ensemble of normal random $N \times N$ matrices $\mathbb{M}$, with the potential $\mathbb{V}$:

$$
\tau_h[t, \bar t] = \int dMdM^\dagger e^{-\frac{1}{\hbar}\text{Tr} V(M, M^\dagger)}
$$

(68)

A matrix is called normal if it commutes with its Hermitian conjugated $[M, M^\dagger] = 0$. Passing to the eigenvalues $\text{diag}(z_1, \ldots, z_N)$ of the matrix $M$, one obtains the measure of the integral in a factorized form $dM dM^\dagger \sim \prod_{i=1}^N dz_i d\bar z_i \prod_{k<j} (z_k - z_j)(\bar z_k - \bar z_j)$. Then the partition function is represents a two-dimensional Coulomb gas in the potential (13)

$$
\tau_h[t, \bar t] = \int \prod_{k=1}^N dz_k d\bar z_k \ e^{-\frac{1}{\hbar} V(z_k, \bar z_k)} \ \prod_{i<j} e^{2\log |z_i - z_j|}.
$$

(69)

To proceed to the large $N$ limit one introduces a parameter $t_0 = \hbar N$ and expresses the integrand in terms of density of eigenvalues as $e^{-\hbar^{-2} \mathcal{E}(\rho, V)}$, where $\mathcal{E}(\rho, V)$ is given by eq. (14). Then, the large $N$ ($\hbar \to 0$) limit yields to the variational principle of Sec.3. In the large $N$ limit the eigenvalues of the matrix homogeneously fill the domain $D_+$ bound by the curve, characterized by the harmonic moments $t_k$ and the area $t_0$ and leads to the $\tau$-function defined by eq. (13). Other objects introduced in Secs.3-7 can also be identified with expectation values of the matrix model. In particular the moments $v_k$ (eq.(3)) are

$$
v_k = \hbar \langle \text{Tr} M^k \rangle
$$

and $\Omega^- - \frac{1}{\hbar} v_0 = \hbar \langle \text{Tr} \log (z - M) \rangle$.

In order to identify the Lax operator, we follow [18, 13, 7]. Introduce the basis of orthogonal polynomials $P_n(z) = h_n z^n + \ldots (n \geq 0)$, by the orthonormality relations

$$
\langle m|n \rangle \equiv \int d^2z \ P_n(z) \ e^{-\frac{1}{\hbar} V(z, \bar z)} \ P_m(z) = \delta_{m,n}
$$

(70)

The polynomials are uniquely defined by the potential $V$ up to phase factors. It is easy to see that the $\tau$-function is given by the product of the coefficients $N!h_n h_{n-1} \ldots h_0$ of the highest powers of the polynomials $P_n(z) = h_n z^n + \ldots$. Then Lax operators $L$ and $\bar L$ appear as the operators $\langle m|z|n \rangle$ and $\langle m|\bar z|n \rangle$. Since $z P_n(z)$ can be expressed through polynomials of the degree not grater than $n$, one may represent $\langle m|z|n \rangle$ and $\langle m|\bar z|n \rangle$ in terms of shifts operators $W = e^{\hbar \frac{d}{d z}}$ in the form of (55, 60), where $r(t_0 = \hbar n) = h_n/h_{n+1}$.

Similar arguments allow one to identify the flows. Consider a variation of some operator $\langle m|O|n \rangle$ under a variation of $t_k$. We have $\hbar \partial_{t_k} \langle m|O|n \rangle = \langle m|[H_k, O]|n \rangle$, where $H_k = A_k - A_k^\dagger$ and $\langle m|A_k|n \rangle = \langle m|\partial_{t_k}|n \rangle$. Obviously $H_k = -L^k(W) +$ negative powers of $W$. Choosing $O$ to be $\bar L$ (see (14)) which consists on $W^{-1}$ and positive powers of $W$, one concludes that $H_k$ does not consists of negative powers of $W$. This brings us to eq.(13).

Earlier V. Kazakov pointed to us that the Lax equations (11, 62) are generated by the Hermitian 2-matrix model [17] with complex conjugated potentials. The latter and the normal matrix model have an identical $1/N$-expansion.
Finally, the operator \( D = \langle m|\hbar \partial_z |n \rangle \) is equal to

\[
D = \bar{L} - \sum_{k \geq 1} k t_k L^{k-1},
\]

The Heisenberg relation \([D, L] = \hbar\) prompts the string equation (71).

The matrix model also offers an effective method to derive eqs. (67) (see e.g. [17]).

10. \(\tau\)-function and spectral properties of the Dirichlet problem. This subject is under current study.

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