Limiting behavior of the Jeffreys
Power-Expected-Posterior Bayes Factor in
Gaussian Linear Models

D. Fouskakis∗ and I. Ntzoufras†

Summary: Expected-posterior priors (EPP) have been proved to be extremely useful for testing
hypothesis on the regression coefficients of normal linear models. One of the advantages of using
EPPs is that impropriety of baseline priors causes no indeterminacy. However, in regression
problems, they based on one or more training samples, that could influence the resulting posterior
distribution. The power-expected-posterior priors are minimally-informative priors that diminishing
the effect of training samples on the EPP approach, by combining ideas from the power-prior
and unit-information-prior methodologies. In this paper we show the consistency of the Bayes fac-
tors when using the power-expected-posterior priors, with the independence Jeffreys (or reference)
prior as a baseline, for normal linear models under very mild conditions on the design matrix.

Keywords: Bayesian variable selection; Bayes factors; Consistency; Expected-posterior priors;
Gaussian linear models; Objective model selection methods; Power-expected-posterior priors; Power
prior; Training sample; Unit-information prior.

1 Introduction

Pérez and Berger (2002) developed priors for use in model comparison, through utilization of the
device of “imaginary training samples” (Good, 2004, Spiegelhalter and Smith, 1988, Iwaki, 1997).
They defined the expected-posterior prior (EPP) as the posterior distribution of a parameter vector
for the model under consideration, averaged over all possible imaginary samples \(y^∗\) coming from
a “suitable” predictive distribution \(m^∗(y^∗)\). Hence the EPP for the parameters of any model
\(M_ℓ \in \mathcal{M}\), with \(\mathcal{M}\) denoting the model space, is

\[
π^E_ℓ(θ_ℓ) = \int π^N_ℓ(θ_ℓ|y^∗) m^∗(y^∗) \, dy^∗,
\]

(1)

where \(π^N_ℓ(θ_ℓ|y^∗)\) is the posterior of \(θ_ℓ\) for model \(M_ℓ\) using a baseline prior \(π^N_ℓ(θ_ℓ)\) and data \(y^∗\).

An attractive option for \(m^∗\) arises from selecting a “reference” or “base” model \(M_0\) for the
training sample and defining \(m^∗(y^∗) = m^N_0(y^∗) \equiv f(y^∗|M_0)\) to be the prior predictive distribution,
evaluated at \(y^∗\), for the reference model \(M_0\) under the baseline prior \(π^N_0(θ_0)\). In the

∗D. Fouskakis is with the Department of Mathematics, National Technical University of Athens, Zografou Cam-
pus, Athens 15780 Greece; email fouskakis@math.ntua.gr
†I. Ntzoufras is with the Department of Statistics, Athens University of Economics and Business, 76 Patision
Street, Athens 10434 Greece; email ntzoufras@aeub.gr
variable-selection problem that we consider in this paper, following the skeptical-prior approach described by Spiegelhalter, Abrams and Myles (2004, Section 5.5.2), the constant model (with no predictors) is a good reference model. This selection makes calculations simpler, and additionally makes the EPP approach essentially equivalent to the arithmetic intrinsic Bayes factor approach of Berger and Pericchi (1996).

One of the advantages of using EPPs is that impropriety of baseline priors causes no indeterminacy. There is no problem with the use of an improper baseline prior \( \pi^{\ell}_Y(\theta_\ell) \) in \(m\); the arbitrary constants cancel out in the calculation of any Bayes factor. However, in regression problems, EPPs are based on one or more training samples, that could influence the resulting posterior distribution.

To diminish the effect of training samples on the EPP approach and simultaneously to produce a minimally-informative prior, Fouskas, Ntzoufras and Draper (2013) introduce the power-expected-posterior priors (PEP), by combining ideas from the power-prior approach of Ibrahim and Chen (2000) and the unit-information-prior approach of Kass and Wasserman (1995). As a first step, the likelihoods involved in the EPP distribution are raised to the power \(1/\delta\) and density-normalized. This power parameter \(\delta\) is set equal to the size of the training sample \(n^*\), to represent information equal to one data point. Regarding the size of the training sample, \(n^*\), this is set equal to the sample size \(n\); in this way the selection of a training sample and its effects on the posterior model comparison is completely avoided.

In what follows, we examine variable-selection problems in Gaussian regression models. Thus, for any model \(M_\ell\), with parameters \(\theta_\ell = (\beta_\ell, \sigma_\ell^2)\), the likelihood is specified by

\[
(Y|X_\ell, \beta_\ell, \sigma_\ell^2, M_\ell) \sim N_n(\mu_\ell, \sigma_\ell^2 I_n),
\]

where \(Y = (Y_1, \ldots, Y_n)\) is a vector containing the (real-valued) responses for all subjects, \(X_\ell\) is an \(n \times d_\ell\) design matrix containing the values of the explanatory variables in its columns, \(I_n\) is the \(n \times n\) identity matrix, \(\beta_\ell\) is a vector of length \(d_\ell\) summarizing the effects of the covariates in model \(M_\ell\) on the response \(Y\) and \(\sigma_\ell^2\) is the error variance. Furthermore we denote the imaginary/training data set \(y^*\), their size by \(n^*\), and the corresponding imaginary design matrix by \(X^*\) of size \(n^* \times (p + 1)\), where \(p\) denotes the total number of available covariates. Following the PEP methodology we set \(n^* = n\) and \(X^* = X\).

For any \(M_\ell \in M\), we denote by \(\pi^N_\ell(\beta_\ell, \sigma_\ell^2|X^*_\ell)\) the baseline prior for model parameters \(\beta_\ell\) and \(\sigma_\ell^2\). Then the power-expected-posterior (PEP) prior \(\pi^{\ell}_{PEP}(\beta_\ell, \sigma_\ell^2|X^*_\ell, \delta)\) takes the following form:

\[
\pi^{\ell}_{PEP}(\beta_\ell, \sigma_\ell^2|X^*_\ell, \delta) = \pi^N_\ell(\beta_\ell, \sigma_\ell^2|X^*_\ell) \int m^N_0(y^*|X^*_\ell, \delta) f(y^*|\beta_\ell, \sigma_\ell^2, M_\ell; X^*_\ell, \delta) dy^*,
\]

where \(f(y^*|\beta_\ell, \sigma_\ell^2, M_\ell; X^*_\ell, \delta)\) is the EPP likelihood raised to the power of \(1/\delta\) and density-normalized, i.e.,

\[
f(y^*|\beta_\ell, \sigma_\ell^2, M_\ell; X^*_\ell, \delta) = \frac{f(y^*|\beta_\ell, \sigma_\ell^2, M_\ell; X^*_\ell, \delta) \pi^{\ell}_{PEP}(\beta_\ell, \sigma_\ell^2|X^*_\ell, \delta)}{\int f(y^*|\beta_\ell, \sigma_\ell^2, M_\ell; X^*_\ell, \delta) dy^*} = \frac{f_{N_n}(y^*; X^*_\ell\beta_\ell, \sigma_\ell^2 I_{n^*})}{\int f_{N_n}(y^*; X^*_\ell\beta_\ell, \sigma_\ell^2 I_{n^*}) dy^*};
\]

here \(f_{N_n}(y; \mu, \Sigma)\) is the density of the \(d\)-dimensional Normal distribution with mean \(\mu\) and covariance matrix \(\Sigma\), evaluated at \(y\).

The distribution \(m^N_0(y^*|X^*_\ell, \delta)\) appearing in (3) is the prior predictive distribution (or the marginal likelihood), evaluated at \(y^*\), of model \(M_\ell\) with the power likelihood defined in (4) under
the baseline prior \( \pi^N_\ell(\beta_\ell, \sigma^2_\ell | X^*_\ell) \), i.e.,

\[
m_{\ell}(y^* | X^*_\ell, \delta) = \int \int f_{N_{\ell}^*}(y^* ; X^*_\ell \beta_\ell, \delta \sigma^2_{\ell} I_{\ell}^*) \pi^N_\ell(\beta_\ell, \sigma^2_\ell | X^*_\ell) \, d\beta_\ell \, d\sigma^2_\ell.
\]

(5)

Here we use the independence Jeffreys prior (or reference prior) as the baseline prior distribution. Hence for any \( M_\ell \in \mathcal{M} \) we have

\[
\pi^N_\ell(\beta_\ell, \sigma^2_\ell | X^*_\ell) = \frac{c_\ell}{\sigma^2_\ell},
\]

(6)

where \( c_\ell \) is an unknown normalizing constant; we refer to the resulting PEP prior as \( J\text{-PEP} \).

2 The conditional J-PEP prior distribution

In the following, we denote by

\[
H_\ell = X_\ell (X_\ell^T X_\ell)^{-1} X_\ell^T \quad \text{and} \quad P_\ell = I_n - H_\ell
\]

and the corresponding measures based on \( X^*_\ell \) by \( H^*_\ell \) and \( P^*_\ell \), respectively.

Under (6), the corresponding marginal likelihood with response data \( y^* \) explanatory variables \( X^*_\ell \) and likelihood function raised to the power of \( 1/\delta \) is given by

\[
m_{\ell}^N(y^* | X^*_\ell, \delta) = c_\ell \pi^{\frac{1}{2}(d_l - n^*)} |X^*_\ell^T X^*_\ell|^{-1/2} \Gamma \left( \frac{n^* - d_l}{2} \right) RSS^*_{\ell}^{-\frac{n^* - d_l}{2}},
\]

where \( RSS^*_\ell \) is the residual sum of squares given by \( RSS^*_\ell = y^{*T} P^*_\ell y^* \).

The J-PEP prior for the parameters of model \( M_\ell \) is given by

\[
\pi^{J\text{-PEP}}_\ell(\beta_\ell, \sigma^2_\ell | X^*_\ell, \delta) = \int \pi^{N}_\ell(\beta_\ell, \sigma^2_\ell | y^* ; X^*_\ell, \delta) m_{\ell}^N(y^* | X^*_\ell, \delta) \, dy^*
\]

\[
= \int \int f(y^* | \beta_\ell, \sigma^2_\ell, M_\ell ; X^*_\ell) \pi^{N}_\ell(\beta_\ell, \sigma^2_\ell | X^*_\ell) m_{\ell}^N(y^* | X^*_\ell, \delta) \, dy^*
\]

\[
= \int \int \left[ \int \left[ f(y^* | \beta_\ell, \sigma^2_\ell, M_\ell ; X^*_\ell, \delta) \pi^{N}(\beta_\ell, \sigma^2_\ell | X^*_\ell) \, dy^* \right] \right.
\]

\[
\times \pi^{N}_\ell(\beta_\ell, \sigma^2_\ell | X^*_\ell) \, d\beta_\ell \, d\sigma^2_\ell
\]

\[
= \int \int \pi^{J\text{-PEP}}_\ell(\beta_\ell, \sigma^2_\ell | \beta_0, \sigma^2_0 ; X^*_\ell, \delta) \pi^{N}_\ell(\beta_\ell, \sigma^2_\ell | X^*_\ell) \, d\beta_0 \, d\sigma^2_0
\]

with the conditional J-PEP prior given by

\[
\pi^{J\text{-PEP}}_\ell(\beta_\ell, \sigma^2_\ell | \beta_0, \sigma^2_0 ; X^*_\ell, \delta) = \int \int \frac{f_{N_{\ell}^*}(y^* ; X_\ell \beta_\ell, \delta \sigma^2_{\ell} I_{\ell}^*) f_{N_{\ell}^*}(y^* ; X_0 \beta_0, \delta \sigma^2_{\ell} I_{\ell}^*)}{c_\ell \pi^{\frac{1}{2}(d_l - n^*)} |X^*_\ell^T X^*_\ell|^{-1/2} \Gamma \left( \frac{n^* - d_l}{2} \right) RSS^*_{\ell}^{-\frac{n^* - d_l}{2}}} \, dy^*
\]

\[
= \frac{\pi^{-\frac{1}{2}(d_l - n^*)}}{\sigma^2_\ell \Gamma \left( \frac{n^* - d_l}{2} \right)} |X^*_\ell^T X^*_\ell|^{1/2} \int \int RSS^*_{\ell}^{-\frac{n^* - d_l}{2}} f_{N_{\ell}^*}(y^* ; X_\ell \beta_\ell, \delta \sigma^2_{\ell} I_{\ell}^*) f_{N_{\ell}^*}(y^* ; X_0 \beta_0, \delta \sigma^2_{\ell} I_{\ell}^*) \, dy^*
\]

(7)
where $\bar{\beta}_0 = (\beta_0^T, {0}_{d_\ell-d_0})^T$ and $0_k$ being a vector of zeros of length $k$. The product of the two normal densities involved in the integrand is given by

$$f_{N_{n^*}}(y^*; X_\ell \beta_\ell, \delta \sigma_\ell^2 I_{n^*}) = f_{N_{n^*}}(y^*; X_\ell \bar{\beta}_0, \delta \sigma_0^2 I_{n^*}) =$$

$$(2\pi)^{-\frac{n^*+d_\ell}{2}} [\delta(\sigma_0^2 + \sigma_\ell^2)]^{-\frac{n^*+d_\ell}{2}} |X_\ell^* X_\ell^*|^{-1/2} f_{N_{n^*}}(y^*; E^{-1}D, E^{-1}) \times f_{N_{d_\ell}}(\beta_\ell; \bar{\beta}_0, \delta(\sigma_0^2 + \sigma_\ell^2)(X_\ell^* X_\ell^*)^{-1})$$

(8)

with

$$E = \left( \frac{\sigma_\ell^2 + \sigma_0^2}{\delta \sigma_0 \sigma_\ell} \right) I_{n^*} \text{ and } D = \frac{1}{\delta \sigma_0^2} X_\ell^T \bar{\beta}_0 + \frac{1}{\delta \sigma_\ell^2} X_\ell^T \beta_\ell = \frac{1}{\delta} X_\ell^* \left( \frac{\sigma_\ell^2 + \sigma_0^2}{\delta \sigma_\ell^2} \bar{\beta}_0 + \frac{\sigma_0^2}{\delta \sigma_0^2} \beta_\ell \right).$$

Note that (8) was obtained from the property

$$f_{N_n}(y; X\Theta_1, A_1) f_{N_n}(y; X\Theta_2, A_2) = (2\pi)^{-\frac{n}{2}} |A_1 + A_2|^{-1/2} |X_\ell^T (A_1 + A_2)^{-1} X_\ell|^{-1/2} \times f_{N_n}(y; E^{-1}D, E^{-1}) f_{N_n}(\Theta_1; \Theta_2, A_1 + A_2)$$

(9)

with

$$E = A_1^{-1} + A_2^{-1} \text{ and } D = A_1^{-1} X\Theta_1 + A_2^{-1} X\Theta_2.$$  

Expression (9) can be easily obtained using the identity:

$$(y - X\Theta_1)^T A_1^{-1} (y - X\Theta_1) + (y - X\Theta_2)^T A_2^{-1} (y - X\Theta_2) =$$

$$= y^T E y - 2y^T (A_1^{-1} X\Theta_1 + A_2^{-1} X\Theta_1) + \Theta_1^T + X_\ell^T A_1^{-1} X\Theta_1 + \Theta_2^T + X_\ell^T A_2^{-1} X\Theta_2$$

$$= [C^T y - C^{-1} D]^T [C^T y - C^{-1} D] + (\Theta_2 - \Theta_1)^T X_\ell (A_1 + A_2)^{-1} X(\Theta_2 - \Theta_1),$$

with $C$ a $n \times n$ lower triangular matrix (the Cholesky decomposition) with non zero elements in the diagonal such that $E = CC^T$.

Replacing (8) in (7), we obtain

$$\rho^T_{\ell - PEP}(\beta_\ell, \sigma_\ell^2 | \beta_0, \sigma_0^2; X_\ell, \delta) = \frac{(d_\ell-n^*) - \frac{1}{2}(d_\ell-n^*)}{2} |X_\ell^T X_\ell^*|^{1/2} (2\pi)^{-\frac{n^*+d_\ell}{2}} |X_\ell^* X_\ell^*|^{-1/2} \times f_{N_{n^*}}(\beta_\ell; \bar{\beta}_0, \delta(\sigma_0^2 + \sigma_\ell^2)(X_\ell^* X_\ell^*)^{-1})$$

$$\times \int (y^* T P_\ell y^*)^{\frac{n^*+d_\ell}{2}} f_{N_{n^*}}(y^*; E^{-1}D, E^{-1}) dy^*. \quad (10)$$

We set

$$z = E^{1/2} (y^* - E^{-1}D)^\ell = \zeta^{1/2}(y^* - X_\ell \Gamma)$$

where $\zeta = \left( \frac{\sigma_\ell^2 + \sigma_0^2}{\delta \sigma_0 \sigma_\ell} \right)$ and $\Gamma = (\zeta \delta)^{-1} \left( \frac{\sigma_\ell^2 + \sigma_0^2}{\delta \sigma_\ell^2} \bar{\beta}_0 + \frac{\sigma_0^2}{\delta \sigma_0^2} \beta_\ell \right)$. Therefore we have $y^* = \zeta^{-1/2} z + X_\ell \Gamma$, $dy^* = \zeta^{-n^*/2}dz$ and

$$f_{N_{n^*}}(y^*; E^{-1}D, E^{-1}) dy^* = f_{N_{n^*}}(z; 0, I_{n^*}) dz$$
since the term $\zeta^{-n^*/2}$, coming from the Jacobian of the transformation, cancels out with the determinant of the variance, that is $|E|^{1/2} = \zeta^{n^*/2}$. Moreover,

$$y^* P_\ell^* y^* = (\zeta^{-1/2} z + X_\ell^* \Gamma)^T P_\ell^* (\zeta^{-1/2} z + X_\ell^* \Gamma)$$
$$= \zeta^{-1} z^T P_\ell^* z + \zeta^{-1/2} z^T P_\ell^* X_\ell^* \Gamma + \Gamma^T X_\ell^* P_\ell^* \zeta^{-1/2} z + \Gamma^T X_\ell^* P_\ell^* X_\ell^* \Gamma$$
$$= \zeta^{-1} z^T P_\ell^* z$$

(11)

since $X_\ell^* P_\ell^* = P_\ell^* X_\ell^* = 0$.

Returning back to (10) we obtain

$$\pi_{\ell}^{\text{J-PEP}}(\beta_\ell, \sigma_\ell^2 | \beta_0, \sigma_0^2; X_\ell^*, \delta) =$$

$$= 2^{-n^* - d_\ell} \left[ \sigma_\ell^2 \Gamma \left( \frac{n^* - d_\ell}{2} \right) \right]^{-1} \left[ \delta(\sigma_\ell^2 + \sigma_0^2) \right]^{-\frac{n^* - d_\ell}{2}} f_{N_{n^*}} \left( \beta_\ell; \overline{\beta}_0, \delta(\sigma_\ell^2 + \sigma_0^2)(X_\ell^* T X_\ell^*)^{-1} \right)$$

$$\times \zeta^{-\frac{n^* - d_\ell}{2}} \int (z^T P_\ell^* z)^{\frac{n^* - d_\ell}{2}} f_{N_{n^*}} (z; 0, I_{n^*}) dz$$

$$= 2^{-n^* - d_\ell} \left[ \Gamma \left( \frac{n^* - d_\ell}{2} \right) \right]^{-1} \left[ \delta(\sigma_\ell^2 + \sigma_0^2) \right]^{-\frac{n^* - d_\ell}{2}} \left( \sigma_\ell^2 + \sigma_0^2 \right)^{-\frac{n^* - d_\ell}{2}}$$

$$\times f_{N_{n^*}} \left( \beta_\ell; \overline{\beta}_0, \delta(\sigma_\ell^2 + \sigma_0^2)(X_\ell^* T X_\ell^*)^{-1} \right) 2^{-\frac{n^* - d_\ell}{2}} \frac{\Gamma(n^* - d_\ell)}{\Gamma(e^* - d_\ell)}$$

since

$$E \left[ (z K z)^r \right] = 2^r \frac{\Gamma(h + r/2)}{r/2}$$

where $K$ is idempotent and $z \sim N(0, I_n)$ and therefore $z K z \sim \chi^2_r$; where $r$ is the rank of $K$.

Therefore (12) becomes

$$\pi_{\ell}^{\text{J-PEP}}(\beta_\ell, \sigma_\ell^2 | \beta_0, \sigma_0^2; X_\ell^*, \delta) = \frac{\Gamma(n^* - d_\ell)}{\Gamma(e^* - d_\ell)} \left( \sigma_\ell^2 + \sigma_0^2 \right)^{-\frac{n^* - d_\ell}{2}}$$

$$\times f_{N_{n^*}} \left( \beta_\ell; \overline{\beta}_0, \delta(\sigma_\ell^2 + \sigma_0^2)(X_\ell^* T X_\ell^*)^{-1} \right)$$

(12)

3 The J-PEP Bayes factor

The Bayes factor of any model $M_\ell$ versus the reference model $M_0$, under the J-PEP-prior approach, is given by

$$BF_{\ell M_0}^{\text{J-PEP}} = \frac{\int f_{N_{n^*}}(y; X_\ell \beta_\ell, \sigma_\ell^2 I_n) \pi_{\ell}^{\text{J-PEP}}(\beta_\ell, \sigma_\ell^2 | X_\ell^*, \delta) d\beta_\ell d\sigma_\ell^2}{\int f_{N_{n^*}}(y; X_0 \beta_0, \sigma_0^2 I_n) \pi_{0}^{\text{J-PEP}}(\beta_0, \sigma_0^2 | X_0^*) d\beta_0 d\sigma_0^2}$$

with the denominator given by

$$m_0^N(y | X_0) = c_0 \frac{1}{2^{(d_0 - n)}} |X_0^T X_0|^{-1/2} \Gamma \left( \frac{n - d_0}{2} \right) RSS_0^{\frac{n - d_0}{2}}.$$
Using (12), the nominator is given by

\[ m_{\ell}^{J-PEP}(y|X_\ell, X_\ell^*, \delta) = \int \int \int \int f_{N_\ell}(y; X_\ell \beta_\ell, \sigma^2_\ell I_n) \pi^{PE}_\ell (\beta_\ell, \sigma^2_\ell | \beta_0, \sigma^2_0; X_\ell^*, \delta) \pi^N_0 (\beta_0, \sigma^2_0 | X_0^*) d\beta_\ell d\sigma^2_\ell d\beta_0 d\sigma^2_0 \]

= \int \int \int \int \frac{c_0}{\sigma^2_0} C_\ell f_{N_\ell}(y; X_\ell \beta_\ell, \sigma^2_\ell I_n) f_{N_\ell} (\beta_\ell; \overline{\beta}_0, \delta (\sigma^2_0 + \sigma^2_\ell) (X_\ell^T X_\ell)^{-1}) d\beta_\ell d\sigma^2_\ell d\beta_0 d\sigma^2_0,

with

\[ C_\ell = (\sigma^2_0)^{-\frac{n^*-d_\ell}{2}} (\sigma^2_\ell)^{-\frac{n^*-d_\ell}{2} - 1} \left( 1 + \frac{\sigma^2_\ell}{\sigma^2_0} \right)^{-\frac{(n^*-d_\ell)}{2}}. \]

Integrating out \( \beta_\ell \), we obtain

\[ m_{\ell}^{J-PEP}(y|X_\ell, X_\ell^*, \delta) = \int \int \int \frac{c_0}{\sigma^2_0} C_\ell \left[ f_{N_\ell}(y; X_\ell \overline{\beta}_0, \Sigma') \right] d\beta_0 d\sigma^2_\ell d\sigma^2_0, \]

with

\[ \Sigma' = \sigma^2_\ell I_n + \delta (\sigma^2_0 + \sigma^2_\ell) X_\ell (X_\ell^T X_\ell)^{-1} X_\ell^T. \]

The above expression was obtained using the following formula:

\[ \int f_{N_\ell}(y; X \Theta_1, A_1) f_{N_\ell}(X \Theta_2, A_2) d\Theta_1 = f_{N_\ell}(y; X \Theta_2, A_1 + X A_2 X^T). \]

Moreover,

\[ m_{\ell}^{J-PEP}(y|X_\ell, X_\ell^*, \delta) = \int \int \int \frac{c_0}{\sigma^2_0} C_\ell \left[ f_{N_\ell}(y; X_\ell \overline{\beta}_0, \Sigma') \right] d\beta_0 d\sigma^2_\ell d\sigma^2_0 \]

= \int \int \int \frac{c_0}{\sigma^2_0} C_\ell \left[ f_{N_\ell}(y; X_0 \beta_0, \Sigma') \right] d\beta_0 d\sigma^2_\ell d\sigma^2_0

= \int \int \int \frac{c_0}{\sigma^2_0} C_\ell \left[ (2\pi)^{-\frac{n^*-d_\ell}{2}} |\Sigma'|^{-1/2} |X_0^T \Sigma'^{-1} X_0|^{-1/2} \exp \left\{ -\frac{1}{2} y^T A \Sigma y \right\} \right] d\sigma^2_\ell d\sigma^2_0,

where

\[ A \Sigma = \Sigma'^{-1} - \Sigma'^{-1} X_0 \left[ X_0^T \Sigma'^{-1} X_0 \right]^{-1} X_0^T \Sigma'^{-1} \]

since

\[ \int f_{N_\ell}(y; X \Theta, A) d\Theta = (2\pi)^{-\frac{n^*}{2}} |A|^{-1/2} |X^T A^{-1} X|^{-1/2} \]

\[ \times \exp \left\{ -\frac{1}{2} y^T \left[ A^{-1} - A^{-1} X (X^T A^{-1} X)^{-1} X^T A^{-1} \right] y \right\} \]

with \( X \) being a \( n \times p \) matrix.
Substituting expression (13), we obtain

\[
m^{J-PEP}_\ell(y|X_\ell, X_\ell^*, \delta) = \int \int \frac{c_0}{\sigma_0^2} \left( \frac{\sigma^2}{\sigma_0^2} \right)^{n^* - d_\ell} \left( \frac{\sigma^2}{\sigma_0^2} \right)^{- \frac{1}{2}} \left( 1 + \frac{\sigma^2}{\sigma_0^2} \right)^{- (n^* - d_\ell)} \frac{\Gamma(n^* - d_\ell)}{\Gamma\left( \frac{n^* - d_\ell}{2} \right)} \times \left( 2\pi \right)^{- \frac{n^* - d_\ell}{2}} |\Sigma'_\ell|^{-1/2} |X_0^T \Sigma'^{-1} X_0|^{-1/2} \exp \left\{ \frac{1}{2} y^T A_\Sigma y \right\} \right] \, \sigma^2 \, d\sigma^2,
\]

\[
= c_0 (2\pi)^{- \frac{n^* - d_\ell}{2}} \frac{\Gamma(n^* - d_\ell)}{\Gamma\left( \frac{n^* - d_\ell}{2} \right)} \int \int \left( \sigma^2 \right)^{- \frac{1}{2}} \left( \frac{\sigma^2}{\sigma_0^2} \right)^{- \frac{1}{2}} \left( 1 + \frac{\sigma^2}{\sigma_0^2} \right)^{- (n^* - d_\ell)} \times |\Sigma'_\ell|^{-1/2} |X_0^T \Sigma'^{-1} X_0|^{-1/2} \exp \left\{ \frac{1}{2} y^T A_\Sigma y \right\} \right] \, \sigma^2 \, d\sigma^2.
\]

(14)

We now set

\[
r = \sqrt{\sigma^2_0 + \sigma^2_\ell} \quad \text{and} \quad \phi = \arctan \left( \frac{\sigma^2_\ell}{\sigma^2_0} \right)
\]

for \( r \in [0, +\infty) \) and \( \phi \in [0, \pi/2] \). The inverse transformations are given by

\[
\sigma^2_0 = r^2 \cos^2 \phi \quad \text{and} \quad \sigma^2_\ell = r^2 \sin^2 \phi
\]

(15)

while the Jacobian is

\[
J(r, \phi) = \begin{vmatrix}
\frac{\partial \sigma^2_0}{\partial r} & \frac{\partial \sigma^2_\ell}{\partial r} \\
\frac{\partial \sigma^2_0}{\partial \phi} & \frac{\partial \sigma^2_\ell}{\partial \phi}
\end{vmatrix} = \begin{vmatrix}
\frac{\partial (r^2 \cos^2 \phi)}{\partial r} & \frac{\partial (r^2 \cos^2 \phi)}{\partial \phi} \\
\frac{\partial (r^2 \sin^2 \phi)}{\partial r} & \frac{\partial (r^2 \sin^2 \phi)}{\partial \phi}
\end{vmatrix} = \begin{vmatrix}
2r \cos^2 \phi & -2r^2 \cos \phi \sin \phi \\
2r \sin^2 \phi & 2r^2 \sin \phi \cos \phi
\end{vmatrix}
= 4r^3 \sin \phi \cos \phi \left( \cos^2 \phi + \sin^2 \phi \right) = 4r^3 \sin \phi \cos \phi.
\]

(16)

Then, the matrix \( \Sigma'_\ell \) becomes equal to

\[
\Sigma'_\ell = \sigma^2_\ell I_n + \delta (\sigma^2_\ell + \sigma^2_0) X_\ell (X^*_\ell X^T_\ell)^{-1} X^T_\ell = r^2 \sin^2 \phi \, I_n + r^2 \delta X_\ell (X^*_\ell X^T_\ell)^{-1} X^T_\ell = r^2 B(\phi)
\]

(17)

with \( B(\phi) \) being a \( n \times n \) matrix given by

\[
B(\phi) = \sin^2 \phi \, I_n + \delta X_\ell (X^*_\ell X^T_\ell)^{-1} X^T_\ell
\]

(18)

while \( A_\Sigma \) can be rewritten as

\[
A_\Sigma = \Sigma'^{-1}_\ell - \Sigma^{-1}_\ell X_0 \left[ X^T_0 \Sigma'^{-1}_\ell X_0 \right]^{-1} X^T_0 \Sigma^{-1}_\ell
\]

\[
= r^{-2} B^{-1}(\phi) - r^{-2} B^{-1}(\phi) X_0 \left[ X^T_0 r^{-2} B^{-1}(\phi) X_0 \right]^{-1} X^T_0 r^{-2} B^{-1}(\phi)
\]

\[
= r^{-2} \left[ B^{-1}(\phi) - B^{-1}(\phi) X_0 A^{-1}(\phi) X^T_0 B^{-1}(\phi) \right]
\]

with

\[
A(\phi) = X^T_0 B^{-1}(\phi) X_0
\]

(19)

being a \( d_0 \times d_0 \) matrix. Moreover, we have that

\[
y^T A_\Sigma y = r^{-2} D(\phi)
\]

(20)
with
\[ D(\phi) = y^T [B^{-1}(\phi) - B^{-1}(\phi)X_0 A^{-1}(\phi)X_0^T B^{-1}(\phi)] y \]  
(21)

being a scalar. Finally, the first three terms in the integrand of (14) can be written as
\[
\left( \frac{\sigma_0^2}{\sigma^2} \right)^{(n^*-d_\ell)} \frac{1}{2} \left( 1 + \frac{\sigma_0^2}{\sigma^2} \right)^{-(n^*-d_\ell)}
\]

being a scalar. Finally, the first three terms in the integrand of (14) can be written as
\[
= (r^2 \cos^2 \phi)^{-n^-d_\ell} \left( \frac{\sin^2 \phi}{\cos^2 \phi} \right)^{-n^-d_\ell} \left( \frac{r^2 \cos^2 \phi + r^2 \sin^2 \phi}{r^2 \cos^2 \phi} \right)^{(n^*-d_\ell)}
\]

Using the transformation (15) and the corresponding Jacobian given by (16), as well as expressions (17), (20) and (22), the marginal likelihood (14) now becomes
\[
m_{\ell}^{J-PEP}(y|X_\ell, X_\ell^*, \delta) = \]
\[
= c_0(2\pi)^{-\frac{n-d_0}{2}} \int_0^{\pi/2} \int_0^\infty \frac{r^{-4}(\sin \phi \cos \phi)^{n^*-d_\ell-2}}{|r^2B(\phi)|^{1/2}|r^{-2}X_0^T B^{-1}(\phi)X_0|^{1/2}}
\]
\[
\times \exp \left\{ -\frac{1}{2} r^{-2}D(\phi) \right\} 4r^3 \sin \phi \cos \phi \ dr d\phi
\]
\[
= 4c_0(2\pi)^{-\frac{n-d_0}{2}} \int_0^{\pi/2} \frac{(\sin \phi \cos \phi)^{n^*-d_\ell-1}}{|B(\phi)|^{1/2}|A(\phi)|^{1/2}} \int_0^\infty \frac{r^{-n+d_0-1}}{r^2} \exp \left\{ -\frac{1}{2} r^{-2}D(\phi) \right\} \ dr d\phi.
\]
(23)

We now set \( w = 1/r \) (\( \leftrightarrow r = w^{-1} \) and \( dr = (-1)w^{-2}dw \), resulting in
\[
m_{\ell}^{J-PEP}(y|X_\ell, X_\ell^*, \delta) = 4c_0(2\pi)^{-\frac{n-d_0}{2}} \int_0^{\pi/2} \frac{(\sin \phi \cos \phi)^{n^*-d_\ell-1}}{|B(\phi)|^{1/2}|A(\phi)|^{1/2}} \int_0^\infty \frac{w^{n-d_0+1}}{w^{-2}} \exp \left\{ -\frac{1}{2} w^2D(\phi) \right\} \ w^{-2}dw d\phi.
\]
\[
= 4c_0(2\pi)^{-\frac{n-d_0}{2}} \int_0^{\pi/2} \frac{(\sin \phi \cos \phi)^{n^*-d_\ell-1}}{|B(\phi)|^{1/2}|A(\phi)|^{1/2}} \int_0^\infty \frac{w^{n-d_0-2} w}{D(\phi)^{-1}} \exp \left\{ -\frac{w^2}{2D(\phi)^{-1}} \right\} dw d\phi.
\]
\[m_{\ell}^{J-PEP}(y | X_\ell, X^*_\ell, \delta) = \]

\[= 4c_0(2\pi)^{-\frac{n-k}{2}} \frac{\Gamma \left( \frac{n^* - d_0}{2} \right)}{\Gamma \left( \frac{n^* - d_0}{2} \right)^{2}} \frac{\Gamma \left( \frac{n^* - d_\ell}{2} \right)}{\Gamma \left( \frac{n^* - d_\ell}{2} \right)^{2}} \frac{\pi/2}{\int_0^{\pi/2} (\sin \phi \cos \phi)^{n^* - d_\ell} \frac{\Gamma \left( \frac{n^* - d_0}{2} \right)}{\Gamma \left( \frac{n^* - d_0}{2} \right)^{2}} f_R(w; D(\phi)^{-1}) \, dw \, d\phi, \]

where \( f_R(w; s^2) \) is the density function of the Rayleigh distribution with scale parameter \( s^2 \) (which here is equal to \( D(\phi)^{-1} \)) and variance \( s^2(4 - \pi)/2 \). Moreover, by \( E_R(w^k; s^2) \) we denote the corresponding \( k^{th} \) moment about zero which is given by \( s^2 k^2/2 \Gamma(1 + k/2) \). Therefore we have:

\[B F_{\ell 0}^{J-PEP} = \]

\[= \frac{2c_0(2\pi)^{-\frac{n-k}{2}} \Gamma \left( \frac{n^* - d_\ell}{2} \right) \Gamma \left( \frac{n^* - d_0}{2} \right) \pi/2}{\int_0^{\pi/2} (\sin \phi \cos \phi)^{n^* - d_\ell} \frac{\Gamma \left( \frac{n^* - d_0}{2} \right)}{\Gamma \left( \frac{n^* - d_0}{2} \right)^{2}} f_R \left( w; D(\phi)^{-1} \right) \, dw \, d\phi. \]

Under the J-PEP approach we set \( (X^*_\ell X_\ell)^{-1} = (X_\ell^T X_\ell)^{-1} \), \( n^* = n \) and \( \delta = n \) and thus

\[B(\phi) = \sin^2 \phi I_n + \delta X_\ell (X_\ell^T X_\ell)^{-1} X_\ell^T = \sin^2 \phi I_n + \delta H_\ell. \]
Moreover,

\[
B^{-1}(\phi) = \left[ \sin^2 \phi I_n + \delta H_\ell \right]^{-1} = \frac{1}{\sin^2 \phi} \left[ I_n + \frac{\delta}{\sin^2 \phi} X_\ell (X_\ell^T X_\ell)^{-1} X_\ell^T \right]^{-1} = \frac{1}{\sin^2 \phi} \left[ I_n - \frac{\delta}{\sin^2 \phi} \frac{\sin^2 \phi}{\delta + \sin^2 \phi} H_\ell \right] = \frac{1}{\sin^2 \phi} \left[ I_n - \frac{\delta}{\delta + \sin^2 \phi} H_\ell \right] = \frac{\delta}{\sin^2 \phi (\delta + \sin^2 \phi)} P_\ell + \frac{1}{\delta + \sin^2 \phi} I_n \right]^{-1} \]  

(25)

resulting in

\[
|B(\phi)| = |\sin^2 \phi I_n + \delta H_\ell| = (\sin^2 \phi)^n \left| I_n + \frac{\delta}{\sin^2 \phi} H_\ell \right| = (\sin^2 \phi)^n \left| I_{d_\ell} + \frac{\delta}{\sin^2 \phi} (X_\ell^T X_\ell) (X_\ell^T X_\ell)^{-1} \right| = (\sin^2 \phi)^{n-d_\ell} (\delta + \sin^2 \phi)^{d_\ell}.
\]

Also \( y^T B^{-1}(\phi) y = \frac{\delta}{\sin^2 \phi (\delta + \sin^2 \phi)} y^T [I_n - H_\ell] y + \frac{1}{\delta + \sin^2 \phi} y^T y = \frac{1}{\sin^2 \phi} \left( \frac{\delta}{\sin^2 \phi} RSS_\ell + y^T y \right). \) From \((19), A(\phi)\) is now given by

\[
A(\phi) = X_0^T B^{-1}(\phi) X_0 = \frac{1}{\sin^2 \phi} X_0^T \left[ I_n - \frac{\delta}{\delta + \sin^2 \phi} H_\ell \right] X_0 = \frac{1}{\sin^2 \phi} \left[ X_0^T X_0 - \frac{\delta}{\delta + \sin^2 \phi} X_0^T H_\ell X_0 \right] = \frac{1}{\delta + \sin^2 \phi} X_0^T X_0 \]

since \( H_\ell \) is idempotent and \( X_0^T H_\ell = X_0 \) for any model \( M_0 \) nested in \( M_\ell \). This comes from the blockwise formula where for any \( X_\ell = [X_0, X_\ell^0] \) we have

\[
H_\ell = H_0 + H_{(I_n - H_0) X_\ell^0} \quad \Leftrightarrow \quad X_0^T H_\ell = X_0^T H_0 + X_0^T H_{P_0 X_\ell^0} = X_0^T + X_0^T P_0 X_\ell^0 \left[ (P_0 X_\ell^0)^T P_0 X_\ell^0 \right]^{-1} (P_0 X_\ell^0)^T \]

\[
= X_0^T + (X_0^T - X_0^T H_0) X_\ell^0 \left[ (P_0 X_\ell^0)^T (P_0 X_\ell^0) \right]^{-1} (P_0 X_\ell^0)^T = X_0^T \]

Therefore \( |A(\phi)| = (\delta + \sin^2 \phi)^{-d_0} |X_0^T X_0| \) and \( X_0 A^{-1}(\phi) X_0 = (\delta + \sin^2 \phi) H_0 \). From \((21), we obtain
that

\[
D(\phi) = y^T B^{-1}(\phi) y - y^T B^{-1}(\phi) X_0 A^{-1}(\phi) X_0^T B^{-1}(\phi) y
\]

= \frac{1}{\delta + \sin^2 \phi} \left( \frac{\delta}{\sin^2 \phi} RSS_\ell + y^T y \right) - y^T B^{-1}(\phi) [(\delta + \sin^2 \phi) H_0] B^{-1}(\phi) y

= \frac{1}{\delta + \sin^2 \phi} \left( \frac{\delta}{\sin^2 \phi} RSS_\ell + y^T y \right) - (\delta + \sin^2 \phi) y^T

\frac{1}{\sin^2 \phi} \left( I_n - \frac{\delta}{\delta + \sin^2 \phi} H_\ell \right) H_0 \left( I_n - \frac{\delta}{\delta + \sin^2 \phi} H_\ell \right) y

= \frac{1}{\delta + \sin^2 \phi} \left( \frac{\delta}{\sin^2 \phi} RSS_\ell + y^T y \right)

- \frac{\delta + \sin^2 \phi}{\sin^4 \phi} y^T \left( I_n - \frac{\delta}{\delta + \sin^2 \phi} H_\ell \right) H_0 \left( I_n - \frac{\delta}{\delta + \sin^2 \phi} H_\ell \right) y

= \frac{1}{\delta + \sin^2 \phi} \left( \frac{\delta}{\sin^2 \phi} RSS_\ell + y^T y \right)

- \frac{\delta + \sin^2 \phi}{\sin^4 \phi} y^T \left( H_0 - \frac{\delta}{\delta + \sin^2 \phi} H_\ell H_0 - \frac{\delta}{\delta + \sin^2 \phi} H_\ell H_0 H_\ell + \left[ \frac{\delta}{\delta + \sin^2 \phi} \right]^2 H_\ell H_\ell H_\ell \right) y

\begin{array}{c}
\text{(H}_0H_\ell=H_0) \\
\end{array}

= \frac{1}{\delta + \sin^2 \phi} \left( \frac{\delta}{\sin^2 \phi} RSS_\ell + y^T y \right) - \frac{\delta + \sin^2 \phi}{\sin^4 \phi} \left[ \frac{\sin^2 \phi}{\delta + \sin^2 \phi} \right]^2 y^T H_0 y

= \frac{1}{\delta + \sin^2 \phi} \left( \frac{\delta}{\sin^2 \phi} RSS_\ell + y^T y - y^T H_0 y \right)

= \frac{1}{\delta + \sin^2 \phi} \left( \frac{\delta}{\sin^2 \phi} RSS_\ell + RSS_0 \right).

By substituting the above equations in (24), we obtain

\[
B_{\ell_0}^{J-P\text{EP}} = 2 \frac{\Gamma (n-d_\ell)}{\Gamma (\frac{n-d_\ell}{2})^2 |X_0^T X_0|^{\frac{1}{2}} RSS_0^{\frac{n-d_\ell}{2}} \int_0^{\pi/2} (\sin \phi \cos \phi)^{n-d_\ell-1} (\sin \phi)^{d_\ell} |B(\phi)| |A(\phi)|^{1/2} |D(\phi)|^{(n-d_\ell)/2} d\phi
\]

= 2 \frac{\Gamma (n-d_\ell)}{\Gamma (\frac{n-d_\ell}{2})^2 |X_0^T X_0|^{\frac{1}{2}} RSS_0^{\frac{n-d_\ell}{2}} \int_0^{\pi/2} (\sin \phi \cos \phi)^{n-d_\ell-1} (\sin \phi)^{d_\ell} (\frac{n}{\sin^2 \phi} RSS_\ell + RSS_0)^{\frac{n-d_\ell}{2}} d\phi

= 2 \frac{\Gamma (n-d_\ell)}{\Gamma (\frac{n-d_\ell}{2})^2 \int_0^{\pi/2} (\sin \phi \cos \phi)^{n-d_\ell-1} (\sin \phi)^{d_\ell} (\frac{n}{\sin^2 \phi} RSS_\ell + sin^2 \phi)^{\frac{n-d_\ell}{2}} d\phi

= 2 \frac{\Gamma (n-d_\ell)}{\Gamma (\frac{n-d_\ell}{2})^2 \int_0^{\pi/2} (\sin \phi \cos \phi)^{n-d_\ell-1} (\sin \phi)^{d_\ell} (\frac{n}{\sin^2 \phi} RSS_\ell + sin^2 \phi)^{\frac{n-d_\ell}{2}} d\phi

= 2 \frac{\Gamma (n-d_\ell)}{\Gamma (\frac{n-d_\ell}{2})^2 \int_0^{\pi/2} (\sin \phi \cos \phi)^{n-d_\ell-1} (\sin \phi)^{d_\ell} (n RSS_\ell + sin^2 \phi)^{\frac{n-d_\ell}{2}} d\phi

(26)
For large $n$, we can write

\[(n + \sin^2 \phi)^{\frac{n-d_\ell}{2}} = (n + \sin^2 \phi)^\frac{n}{2} (n + \sin^2 \phi)^{-\frac{d_\ell}{2}} = n^{\frac{n}{2}} \left(1 + \frac{\sin^2 \phi/2}{n/2}\right)^{\frac{n}{2}} (n + \sin^2 \phi)^{-\frac{d_\ell}{2}}\]

\[\approx n^{\frac{n}{2}} (n + \sin^2 \phi)^{-\frac{d_\ell}{2}} \exp \left(\frac{\sin^2 \phi}{2}\right)\]

\[\approx n^{\frac{n-d_\ell}{2}} \exp \left(\frac{\sin^2 \phi}{2}\right).\]

Similarly,

\[\left(\frac{n \text{RSS}_\ell}{\text{RSS}_0} + \sin^2 \phi\right)^{\frac{n-d_0}{2}} = \left[\frac{n \text{RSS}_\ell}{\text{RSS}_0}\right]^{\frac{n-d_0}{2}} \left(1 + \frac{\frac{1}{2} \sin^2 \phi \text{RSS}_0}{\text{RSS}_\ell}ight)^{\frac{n}{2}} \left(1 + \frac{\sin^2 \phi \text{RSS}_0}{n}\right)^{-\frac{d_0}{2}}\]

\[\approx \left[\frac{n \text{RSS}_\ell}{\text{RSS}_0}\right]^{\frac{n-d_0}{2}} \exp \left(\frac{\frac{1}{2} \sin^2 \phi \text{RSS}_0}{\text{RSS}_\ell}\right).\]

Moreover, for large $z$ we have

\[\log \Gamma(z) \approx \left(z - \frac{1}{2}\right) \log z - z + \frac{1}{2} \log(2\pi).\]

Hence

\[\log \Gamma(n - d_\ell) \approx \left(n - d_\ell - \frac{1}{2}\right) \log(n - d_\ell) - (n - d_\ell) + \frac{1}{2} \log(2\pi)\]

\[\log \Gamma\left(\frac{n - d_\ell}{2}\right) \approx \left(n - d_\ell - \frac{1}{2}\right) \log\left(\frac{n - d_\ell}{2}\right) - \left(n - d_\ell - \frac{1}{2}\right) + \frac{1}{2} \log(2\pi)\]

\[\log \Gamma(n - d_\ell) - 2 \log \Gamma\left(\frac{n - d_\ell}{2}\right) \approx \left(n - d_\ell - \frac{1}{2}\right) \log(n - d_\ell) - (n - d_\ell) + \frac{1}{2} \log(2\pi) - 2\left(n - d_\ell - \frac{1}{2}\right) \log\left(\frac{n - d_\ell}{2}\right) + 2 \left(n - d_\ell - \frac{1}{2}\right) - \frac{1}{2} \log(2\pi)\]

\[\approx \frac{1}{2} \log(n - d_\ell) - \frac{1}{2} \log(2\pi) + (n - d_\ell - 1) \log 2\]

\[\approx \frac{1}{2} \log(n) + n \log 2.\]
From the above we obtain that

\[
\log B_{\ell_0}^{J-PEP} \approx \frac{1}{2} \log(n - d_\ell) - \frac{1}{2} \log(2\pi) + (n - d_\ell) \log 2 \\
+ \log \int_0^{\pi/2} (\sin \phi)^{n-d_\ell-1}(\cos \phi)^{n-d_\ell-1} n^{\frac{n-d_\ell}{2}} \exp \left(\frac{\sin^2 \phi}{2}\right) d\phi \\
\approx \frac{1}{2} \log(n - d_\ell) - \frac{1}{2} \log(2\pi) + (n - d_\ell) \log 2 \\
- \frac{n - d_0}{2} \log \frac{RSS_\ell}{RSS_0} + \log \int_0^{\pi/2} (\sin \phi)^{n-d_\ell-1}(\cos \phi)^{n-d_\ell-1} \exp \left(\frac{\sin^2 \phi}{2}\right) d\phi \\
\approx \frac{1}{2} \log(n - d_\ell) - \frac{1}{2} \log(2\pi) + (n - d_\ell) \log 2 - \frac{d_\ell - d_0}{2} \log n \\
- \frac{n - d_0}{2} \log \frac{RSS_\ell}{RSS_0} + \log \int_0^{\pi/2} (\sin \phi)^{n-d_\ell-1}(\cos \phi)^{n-d_\ell-1} \exp \left(\frac{\sin^2 \phi}{2}\right) d\phi \\
\approx \frac{1}{2} \log n + n \log 2 - \frac{d_\ell - d_0}{2} \log n - \frac{n - d_0}{2} \log \frac{RSS_\ell}{RSS_0} \\
\]  

(27)
since the integral

\[
\int_0^{\pi/2} \frac{(\sin \phi)^{n-d_\ell-1}(\cos \phi)^{n-d_\ell-1} \exp \left(\frac{\sin^2 \phi}{2}\right)}{\exp \left(\frac{\sin^2 \phi \cdot RSS_0}{RSS_\ell}\right)} d\phi \leq \int_0^{\pi/2} \exp \left(\frac{\sin^2 \phi}{2}\right) \left[1 - \frac{RSS_0}{RSS_\ell}\right] d\phi
\]

when \(n > d_0 + 1\) and \(n > d_\ell + 1\). The latter integral has a finite value for all \(n\) according to Casella, Girón, Martínez and Moreno (2003, p.1216) Hence the interval involved in the \(B_{\ell_0}^{PEP}\) has also a finite value for all \(n\).

If we compare any two models \(M_\ell\) versus \(M_k\) (both of them different than the reference model) we have that

\[
-2 \log B_{\ell k}^{PEP} \approx n \log \frac{RSS_\ell}{RSS_k} + (d_\ell - d_k) \log n = BIC_\ell - BIC_k .
\]

(28)

Therefore the J-PEP approach has the same asymptotic behavior as the BIC-based variable-selection procedure. The following Lemma is a direct result of (28) and of Theorem 4 of Casella et al. (2003).

**Lemma 1:** Let \(M_\ell \in \mathcal{M}\) be a normal regression model of type (2) such that

\[
\lim_{n \to \infty} \frac{X_T(I_{n} - X_\ell(X_\ell^T X_\ell)^{-1} X_\ell^T)X_T}{n} \text{ is a positive semidefinite matrix,}
\]

with \(X_T\) being the design matrix of the true data generating regression model \(M_T \neq M_\ell\). Then, the variable selection procedure based on J-PEP Bayes factor is consistent since \(BF_{J-PEP}^\ell \to 0\) as \(n \to \infty\).
4 Discussion

Under the *power-expected-posterior priors* (PEP) approach, ideas from the power-prior and unit-information-prior methodologies are combined. As a result the resulting priors are minimally-informative and additionally the effect of training samples that is a big issue on the expected-posterior prior approach is diminishing. When using the independence Jeffreys (or reference) prior as a baseline prior for normal linear models we prove that PEP approach has the same asymptotic behavior as the BIC-based variable-selection procedure. Therefore under very mild conditions on the design matrix is a consistent variable selection technique.

References

Berger, J. and Pericchi, L. (1996), The intrinsic Bayes factor for linear models, in J. Bernardo, J. Berger, A. Dawid, and A. Smith, eds., *Bayesian Statistics*, Vol. 5, Oxford University Press, pp. 25–44.

Casella, G., Girón, F., Martínez, M. and Moreno, E. (2009), ‘Consistency of Bayesian procedures for variable selection’, *Annals of Statistics*, 37, 1207–1228.

Fouskakis, D., Ntzoufras, I. and Draper, D. (2013), ‘Power-expected-posterior priors for variable selection in Gaussian linear models, submitted’.

Good, I. (2004), *Probability and the Weighting of Evidence*, Haffner, New York, USA.

Ibrahim, J. and Chen, M. (2000), ‘Power prior distributions for regression models’, *Statistical Science*, 15, 46–60.

Iwaki, K. (1997), ‘Posterior expected marginal likelihood for testing hypotheses’, *Journal of Economics, Asia University*, 21, 105–134.

Kass, R. and Wasserman, L. (1995), ‘A reference Bayesian test for nested hypotheses and its relationship to the Schwarz criterion’, *Journal of the American Statistical Association*, 90, 928–934.

Pérez, J. and Berger, J. (2002), ‘Expected-posterior prior distributions for model selection’, *Biometrika*, 89, 491–511.

Spiegelhalter, D., Abrams, K. and Myles, J. (2004), *Bayesian Approaches to Clinical Trials and Health-Care Evaluation*, Statistics in Practice, Wiley, Chichester, UK.

Spiegelhalter, D. and Smith, A. (1988), ‘Bayes factors for linear and log-linear models with vague prior information’, *Journal of the Royal Statistical Society B*, 44, 377–387.