Self-preservation of large-scale structures in Burgers’ turbulence.

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We investigate the stability of large-scale structures in Burgers’ equation under the perturbation of high wave-number noise in the initial conditions. Analytical estimates are obtained for random initial data with spatial spectral density $k^n, n < 1$. Numerical investigations are performed for the case $n = 0$, using a parallel implementation of the Fast Legendre Transform.

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The appearance of ordered structures is possible for nonlinear media with dissipation\[3\]. These structures often form a set of cells with regular behaviour, alternating with randomly localised zones of dissipation. One nonlinear dissipative system with such behaviour is the well-known Burgers’ equation

\[
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = \mu \frac{\partial^2 v}{\partial x^2}; \quad v(x, t = 0) = v_0(x)
\]

where \(\mu\) is the viscosity coefficient. The solutions of Burgers’ equation with random initial conditions display two mechanisms, which are also inherent to real turbulence: nonlinear transfer of energy through the spectrum, and viscous damping in the small scale region. Equation (1) was proposed by Burgers\[2\] as one-dimensional model of fluid turbulence. It has since been shown to arise in a large variety of non-equilibrium phenomena, when parity invariance holds\[10\]. Burgers’ equation have applications to nonlinear acoustics, nonlinear waves in thermoelastic media, to modelling of the formation of large-scale structures in the Universe, and to many other systems where dispersion is negligibly small compared with nonlinearity (see \[8\] and references therein).

Burgers’ turbulence is an example of strong turbulence, the properties of which are determined by the strong interaction of large number of harmonic waves. In Burgers’ turbulence, due to the coherent interaction of harmonics, saw-tooth shock waves are formed, which may be treated as a gas of strong local interaction of particles\[13, 8\]. The collisions of shock fronts leads to their merging, which is analogous to inelastic collisions of particles, and to an increase of the integral scale of turbulence.

In the limit of zero viscosity the solution of (1) for the velocity field is given by

\[
v(x, t) = \frac{x - y(x, t)}{t}; \quad v(x, t) = -\frac{\partial S(x, t)}{\partial x}
\]

where \(y(x, t)\) is where is realised the absolute maximum of the function

\[
G(x, y, t) = S_0(y) - \frac{(x - y)^2}{2t}; \quad S_0(x) = \int^x v_0(y)dy
\]

By analogy between the solutions of Burgers’ equation and the flow of particles\[8\], we shall call \(S_0(y)\) the initial action, and \(S(x, t) = G(x, y(x, t), t)\) the action at time \(t\). This solution can be computed by a Legendre transform of initial data\[12\]. We shall assume that the spectrum of random initial velocity field has a following form:

\[
g_0(k) = \alpha_n^2 k^n b_0(k) \quad b_0(k < k_0) = 1, \quad b_0(k > k_0) = 0
\]

The curvature \(K_s\) of the initial action in (1) may then be estimated as

\[
K^2 = <(S''_0(y))^2> = \alpha_n^2 k_0^{n+3}/(n + 3) = \sigma_0^2/l_{\min}^2, \sigma_0^2 = <v^2> = \alpha_n^2 k_0^{n+1}/(n + 1)
\]

Taking into account that the curvature of parabola in (3) is 1/t, we get that when \(t \gg t_n = l_{\min}/\sigma_0 = K^{-1}_s\) the curvature of the parabola is much smaller than the curvature of the initial action. This implies that the global maximum of \(G(x, y, t)\) is in the neighbourhood of the local maximum of \(S_0(y)\), and that \(y(x, t)\) is a stepwise non-decreasing function of \(x\). Thus the velocity field has a universal behaviour

\[
v(x, t) = \frac{x - y_k}{t}; \quad x_{k-1} < x < x_k
\]
in each cell between shock positions $x_k$. Each cell is characterized by two numbers: the inverse Lagrangian coordinate $y_k$, and the value of initial action in the cell $S_k = S_0(y_k)$. The dissipation zone is in this case at the shock front, and hence have a zero width. The positions of the shocks are determined by the equality at the point $x_k$ of the values of two absolute maxima $G(x_k, y_{k-1}, t) = G(x_k, y_k, t)$:

$$x_k = \frac{y_k + y_{k-1}}{2} + V_k t; \quad V_k = \frac{S_0(y_{k-1}) - S_0(y_k)}{y_k - y_{k-1}}$$

(7)

It is easy to see that the rate of collisions of shocks depends on the asymptotic behaviour of the structure function of initial action

$$d_s(\rho) = <[S_0(x + \rho) - (S_0(x)]^2 > \approx \begin{cases} \alpha_n^2 \rho^{1-n}, & \text{if } n < 1 \\ 2\sigma_s^2, & \text{if } n > 1 \end{cases}$$

(8)

Here $\sigma_s^2 = < S_0^2 >$. We can estimate the integral scale of turbulence $l(t) = |x - y|$ from the condition that the parabola and the initial action are of the same order:

$$d_s[l(t)]^{1/2} \approx l^2/t$$

(9)

From (8,9) it therefore follows that we have two different types of growth for the scale $l(t)$:

$$l(t) \approx \begin{cases} (\alpha t)^{2/(n+3)}, & \text{if } n < 1 \\ (\sigma_s t)^{1/2}, & \text{if } n > 1 \end{cases}$$

(10)

It was shown [8] (see also [12]) that the common feature for both types of initial spectra, is the existence of self-similarity of statistical properties of solutions, which are determined by only one scale $l(t)$. From (10) one can see that in the case $n < 1$, the behaviour of Burgers’ turbulence at time $t$ is determined by the local large-scale behaviour of the initial spectrum. This was the motivation for a simple qualitative model of Burgers’ turbulence (at $n < 1$) as a discrete infinite set of modes - the spatial harmonics $k_m = k_0 \gamma^{-m}(\gamma \gg 1)$, sufficiently spaced in the spatial spectrum [6, 5]. The amplitudes $A_m^2$ of harmonics were chosen from the condition that the mean spectral density of harmonics in the interval $\Delta_m = k_{m+1} - k_m$ was identical to the random noise spectral density: $A_m^2 = g_0(k_m) \Delta_m$. Assuming that the influence of large-wavenumber components on large-scale one is small, it was obtained that the growth of average scale for this discrete model is the same as in the case of continuous spectrum given by (10). The idea of independent evolution of large-scale structures has also been used to obtain long time asymptotics of the cylindrical Burgers’ equation [4].

It is possible to show that the assumption of relatively independent evolution of the large-scale structures is valid for the continuous model of Burgers’ turbulence as well. In [7] it was obtained analytically that the interaction of a regular positive pulse with noise in Burgers’ equation does not change the evolution of the large-scale structure of the pulse, when the value of initial action of the pulse is greater than the dispersion of the noise action. This result was confirmed by numerical simulations based on the asymptotic solution (2,3) of Burgers’ equation. Here we shall give some simple estimations of this effect for the random perturbations with initial spectrum (4) and illustrate it by results of numerical experiment for the case of truncated white noise ($n=0$ in (10)).

Numerical experiments were performed using a parallel version of Fast Legendre Transform algorithm implemented on a Connection Machine CM-200 [12, 4]. The algorithm
uses the property that the inverse Lagrangian function \( y(x, t) \) is a non-decreasing function of \( x \), and specific low-level instructions on the Connection Machine, that allows one to perform simultaneously partial maximization operations over divisions of the range, that are only known at run-time. The most primitive computation of the maximum of \( f \) takes for \( N \) points \( O(N^2) \) operations on a sequential computer. The serial Fast Legendre Transform takes \( O(N \log N) \) operations in 1D (and \( O(N^2 \log^2 N) \) operations for data on a \( N \times N \) grid in 2D [11]). Our Parallel Fast Legendre Transform takes \( O(\log^2 N) \) operations on an ideal parallel computer with unlimited number of physical processors, connected in hypercube network (this is the type of connection used on the CM-2 and CM-200 models). The Connection Machine simulates an arbitrary number of processors with a finite number (on our machine 8192). Our algorithm will therefore eventually go linearly with the ratio of initialized gridpoints \( (N) \), to the actual number of physical processors.

For the class of one-dimensional problems we have considered here, the computational time including input, output and Connection Machine initialization is much less then one minute for total number of points up to \( N = 2^{20} \).

We shall consider the evolution of two initial random perturbations \( v_0(x) \) and \( \tilde{v}_0(x) \):

\[
\tilde{v}_0(x) = v_0(x) + v_h(x)
\]  

(11)

The power spectra of both processes are described by [4] with \( n < 1 \), but the process \( v_0(x) \) has spectral components in the range \([0, k_s]\), while the process \( \tilde{v}_0(x) \) in the range \([0, k_0]\) with \( k_0 \gg k_s \). Restricted to the range \([0, k_s]\) the two processes are identical, \( v_h(x) \) being their difference in the large wave-number range \([k_s, k_0]\). The correlation coefficient between \( v_0 \) and \( \tilde{v}_0 \) is \( r_0 = (k_s/k_0)^{(n+1)/2} \ll 1 \). In particular, we made calculations for \( k_0^{(1)}/k_s = 2^2 \) and \( k_0^{(2)}/k_s = 2^6 \).

Three initial realisations, as described above, are shown on Fig.1. At \( t \gg t_n \approx 1/(ak_s^{(n+3)/2}) \) all three processes are transformed to a sequence of triangular pulses with the universal behaviour inside the cells according to [4]. The solution \( v(x, t) \) will be stable relative to high wave-number perturbation \( v_h(x) \), if the fluctuations of both the inverse Lagrangian coordinates \( \Delta y_k = \tilde{y}_k - y_k \) and the shock positions \( \Delta x_k = \tilde{x}_k - x_k \), are small respectively to integral scale of turbulence \( l(t) \). While the asymptotic properties of Burgers’ turbulence are determined by the behaviour of initial action, the values of disturbances \( \Delta x_k \) and \( \Delta y_k \) will be determined by the value of dispersion \( \sigma_{hS}^2 \) of perturbation action \( S_h(x) = \int x v_h(y)dy \):

\[
\sigma_{hS}^2 = \langle S_h^2 \rangle = \alpha_n \frac{\alpha_n^2}{(1-n)k_s^{1-n}} \left( 1 - \left( \frac{k_s}{k_0} \right)^{1-n} \right)
\]  

(12)

and its correlation scale \( l_s \approx (k_s k_0)^{-1/2} \). The main divergence between \( \tilde{v} \) and \( v \) is due to the different velocities of shocks \( V_k \) and \( \tilde{V}_k \), and these errors increase in time. From [4] one sees that the fluctuations of velocity are determined by the fluctuations of the action \( S_h(y) \) in the neighbourhood of local maximum of \( S_0(y) \). This problem is similar to the analysis of statistical properties of Burgers’ turbulence with stationary initial action [4]. Thus we get that the relative deviation of shock positions is equal to

\[
\epsilon(t) = \frac{\langle (\Delta X_k^2)^{1/2} \rangle}{l(t)} \approx \frac{\sigma_{hS} t}{l^2(t)}
\]  

(13)
which means that the behaviour of $\epsilon(t)$ depends on the growth law of the integral scale $l(t)$. From\(10\),\(12\),\(13\) we get that in the case $n < 1$ and $k_0 \gg k_*$:

$$\epsilon(t) \sim \frac{1}{(k_* l(t))^{1-\frac{n}{2}}} \ln \left( \frac{k_0}{k_*} \right) \sim \frac{1}{t^{\frac{n}{2(n+3)}}}$$  \hspace{1cm} (14)

Here we have taken into account that for $k_0 \gg k_*$, the value of absolute maximum of the random process $S_h(y)$ in the vicinity of local maximum of $S_0(y)$, has a double logarithmic distribution with average value $\sim \sigma_h S (\ln (k_0/k_*))^{1/2}$, and dispersion decreasing with correlation scale $l_s$, proportional to $\ln (k_0/k_*)$\(5\),\(14\). It may be that one shock in $v$ corresponds to a cluster of shocks in $\tilde{v}$, but then they are spread out over a distance no more than $\epsilon(14)$. Therefore one can say that the large scale structures of Burgers’ turbulence are self-preserving due to multiple merging of shocks. The stability of large-scale structures is illustrated by results of numerical calculations, presented on Fig. 2.

Three realisations of velocity field are shown, corresponding to initial conditions on Fig.1 with cutoff wavenumbers equal respectively $k_*$, $k_0^{(1)}$ and $k_0^{(2)}$. One can see that the large-scale behaviour of all those realizations is similar, and only fine structure weekly depends on cutoff wavenumber. The correlation coefficient between the velocity fields with cutoff wavenumbers $k_*$ and $k_0^{(2)}$ increase from the value 0.035 for initial perturbation (Fig. 1) up to the value 0.92 on the stage of developed shocks (Fig. 2). This become even more clear when comparing the corresponding initial and final actions, presented on Fig. 3 and Fig.4. There one sees that the action for higher cutoff values can be obtained from the smaller one approximately by a shift along the vertical axis. This is in accordance with estimations following from double logarithmic distribution of $S_h(y)$.

In conclusion, we would like to stress that the stability of large-scale structures relative to large wave-number perturbations, is similar to the independence of asymptotic evolution of Burgers’ turbulence (with $n < 1$) of the value of viscosity coefficient $\mu$\(3\). We may thus interprete the action of small-scale perturbations on solution of Burgers’ equation as a turbulent viscosity, effectively changing $\mu$, but leaving the initial conditions invariant\(4\),\(14\).

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Fig. 3. Realizations of the action fields with cutoff wave-numbers $k_\ast(1)$, $k_0^{(1)}(2)$ and $k_0^{(2)}(3)$ at $t \gg t_n$ and initial action for wave-number $k = k_\ast(4)$.
Fig. 4. Realizations of the action fields with cutoff wave-numbers $k_*(1)$ and $k_0(2)$ at $t \gg t_n$ and initial actions for the same wave-numbers (3,4).