Chirality and parity in a first-quantized representation

C. D. Fosco *

Centro Atómico Bariloche - Instituto Balseiro, 
Comisión Nacional de Energía Atómica 
8400 Bariloche, Argentina.

Abstract

We study the realization of chiral and parity transformations within a particle-like path-integral representation for Dirac fields, showing how those transformations can be implemented in a natural way within the formalism. We then obtain a representation for the chiral fermion propagator and determinant within this framework, and also formulate a way to define the average of the propagator over a random mass in $d = 2n$ dimensions.

1 Introduction

Some important Quantum Field Theory objects can be conveniently represented in terms of particle-like path-integrals, something which has been known since a long time ago [1, 2]. This may be more appealing, from the physical point of view, than the standard ‘second quantized’ picture; besides, some approximation schemes suggest themselves in a clearer fashion. Some successful applications of these representations appeared, for example, within the context of the infrared approximation to the evaluation of the full Dirac propagator in $QED$ [3, 4, 5]. They have also been applied to many other interesting problems, since its use provides a framework which often becomes very convenient for the introduction of non-standard calculation techniques [6, 7, 8, 9].

*Electronic address: fosco@cab.cnea.gov.ar
For the case of non-zero spin fields, different proposals for the construction of an integral over first-quantized trajectories have been advanced. Since they usually involve different sets of variables, the task of relating them is far from trivial, unless it is undertaken at a purely formal level.

In this article, we build on the results presented in [10], which dealt with the particular case of a path-integral representation for Dirac fields, first introduced by Migdal in [11]. We first consider the realization of symmetries within this framework: in particular, we show how chiral transformations can be introduced at the level of the ‘particle’ trajectories that have to be integrated out in order to derive the propagator. We then show how chiral fermion fields can be described within this formulation, and how this representation may be used to define chiral fermion propagators and determinants.

We conclude with the consideration of a random mass, which is a pathological situation if considered under the light of a representation based on [11]. It is, however, an interesting situation, and we show how the problems can be dealt with and a representation valid for this case be derived.

This article is organized as follows: in section 2 we discuss chiral and parity transformations within the context of Migdal’s representation. In section 3 we present a derivation of the expressions for the propagator and effective action of a chiral fermion in an even-dimensional spacetime.

Section 4 deals with the definition of the path integral for the case of a random mass field, and in section 5 we present our conclusions.

## 2 Chiral and parity transformations

Let us briefly review the main properties of the particle path-integral representation, as presented in [10], for the case of a massive Dirac fermion in $d$ Euclidean dimensions. To be specific, we consider a theory where the fermionic action, $S_f$, is defined by:

$$ S_f(\bar{\psi}, \psi; A) = \int d^d x \bar{\psi} D \psi $$  \hspace{1cm} (1)

where the Dirac operator $D$, (defined including the mass term) is given by

$$ D = \not{D} + m, \text{ with } \not{D} = \gamma_\mu D_\mu \text{ and } D_\mu = \partial_\mu + ie A_\mu. $n A_\mu \text{ is an Abelian gauge field. } $n For the case of a non-Abelian group $G$, we replace $ie A_\mu \to gA_\mu$ in the covariant derivative. Here $g$ is the coupling constant for the non-Abelian gauge theory, and $A_\mu$ belongs to the Lie algebra of $G$ ($A_\mu$ is anti-Hermitian).

The $\gamma$-matrices are chosen to be Hermitian, and to verify the anticommutation relations $\{ \gamma_\mu, \gamma_\nu \} = 2 \delta_{\mu\nu}$. 

2
When the number $d$ of spacetime dimensions is even ($d = 2n$), it is convenient to introduce a $\gamma_s$ matrix, the generalization of $\gamma_5$ to $n \neq 2$, such that $\{\gamma_a, \gamma_\mu\} = 0$, $\forall \mu$. Following the conventions of [12], we assume that $\gamma_s$ is Hermitian, and that $\gamma_s^2 = 1$.

The Dirac propagator $G(x, y; m)$ is the kernel of the inverse of the operator defining the quadratic form in $S_f$, equation (1), namely:

$$G(x, y; m) = \langle \psi(x) \bar{\psi}(y) \rangle = \langle x | D^{-1} | y \rangle, \quad (2)$$

where we have adopted Schwinger’s convention: $\langle x | T | y \rangle$ for $T(x, y)$, the kernel of any operator $T$ in coordinate space. The dependence of $G$ on the mass has been explicitly written, and we have omitted the spinorial indices, although it should be evident from the context that $G$ is a matrix ($2 \times 2$ for $d = 2$, $3$ and $4 \times 4$ when $d = 4$).

When the mass $m$ is constant and positive, a path-integral representation associated to $G(x, y; m)$, can be introduced by setting:

$$G(x, y; m) = \int_0^\infty dT K(x, y; m, T) \quad (3)$$

where

$$K(x, y; m, T) = \langle x | \exp[-T(\not{D} + m)] | y \rangle \quad (4)$$

has the path-integral representation:

$$K(x, y; m, T) = \int_{x(0) = y}^{x(T) = x} Dp Dx e^{\int d\tau (ip \cdot \dot{x} - m)} \Phi(T) e^{-ie \int_0^T d\tau A}, \quad (5)$$

with

$$\Phi(T) = \mathcal{P}[e^{-ie \int_0^T d\tau p(\tau)}]. \quad (6)$$

There is an analogous path-integral like expression for $\Gamma(A, m)$, the 1-loop contribution to the effective action due to the fermions:

$$\Gamma(A, m) = -\ln \det D, \quad (7)$$

where we have omitted an infinite additive constant (that corresponds to the determinant of the free Dirac operator). The path integral enters into the game through the representation:

$$\Gamma(A, m) = \int_0^\infty \frac{dT}{T} \int d^dx \text{tr} K(x, x; m, T), \quad (8)$$

with $K$ defined as in (4).
On the other hand, if the mass \( m \) is negative, it is still possible to represent \( G(x, y; m) \) in a similar way:

\[
G(x, y; m) = - \int_{-\infty}^{0} dT \, K(x, y; m, T),
\]

while for \( \Gamma \) the corresponding formula reads:

\[
\Gamma(A, m) = \int_{0}^{T} dT \, \int d^d x \, \text{tr} \, K(x, x; m, T).
\]

The effect of global chiral transformations on Dirac operators is of course well known, and can be obtained from the fermionic action (1), by performing the corresponding transformations on the fermionic fields. The chiral properties of \( D \) are reflected in the effect of a \( \gamma_s \) transformation, for example, by considering:

\[
D' = \gamma_s D \gamma_s,
\]

which corresponds to the discrete transformation of the fermionic fields:

\[
\psi'(x) = \gamma_s \psi(x), \quad \bar{\psi}'(x) = \bar{\psi}(x) \gamma_s.
\]

For the propagator, we have

\[
\gamma_s [G(x, y; m)]^{-1} \gamma_s = -[G(x, y; -m)]^{-1}.
\]

Let us see how these transformations are realized at the level of the path integral representation for \( G \), assuming \( m > 0 \). Clearly, the only effect of the transformations (12) is to change the sign of the exponent of the path-ordered exponential:

\[
\gamma_s G(x, y; m) \gamma_s = \int_{0}^{\infty} dT \int_{x(0)=y}^{x(T)=x} DpDx \, e^{\int_{0}^{T} d\tau [ip \cdot \dot{x} - m]}
\]

\[
\times \mathcal{P} \left[ e^{+i \int_{0}^{T} d\tau p} e^{-ie \int_{0}^{T} d\tau \dot{x} \cdot A} \right].
\]

To relate the previous expression to the path integral representation for \( G \), we perform a reparametrization of the variables; \( \tau \to -\tau \), under which

\[
\bar{p}(\tau) = p(-\tau), \quad \bar{x}(\tau) = x(-\tau),
\]

and

\[
\gamma_s G(x, y; m) \gamma_s = \int_{0}^{\infty} dT \int_{\bar{x}(0)=y}^{\bar{x}(-T)=x} D\bar{p}D\bar{x} \, e^{\int_{0}^{-T} d\tau [i\bar{p} \cdot \dot{\bar{x}} + m]}
\]
\[ \mathcal{P}[e^{-i \int_{-T}^{T} d\tau \tilde{p}}] e^{-ie \int_{-T}^{T} d\tau \dot{\tilde{x}}(\tau) \cdot A[\tilde{x}]} . \quad (16) \]

Then we change variables: \( T \to -T \) in the integration over \( T \), to obtain:

\[ \gamma_s G(x, y; m) \gamma_s = -\int_{-\infty}^{\infty} dT \int_{\tilde{x}(0)=y}^{\tilde{x}(T)=x} \mathcal{D}\tilde{p}\mathcal{D}\tilde{x} e^{i \int_{-T}^{T} d\tau [i\tilde{p} \cdot \dot{\tilde{x}} + m]} \]

or:

\[ \mathcal{P}[e^{-i \int_{0}^{T} d\tau \tilde{p}}] e^{-ie \int_{0}^{T} d\tau \dot{\tilde{x}}(\tau) \cdot A[\tilde{x}]} , \quad (17) \]

Recalling (9), and that we had assumed that \( m > 0 \), we conclude that (13) holds true.

On the other hand, parity transformations in an odd-dimensional Euclidean spacetime can be represented by:

\[ \psi(x) \to i\psi(-x) \quad \bar{\psi} \to i\bar{\psi}(-x) , \quad (19) \]

so that the parity transformed of \( \mathcal{D} \) differs only in the sign of its mass with \( \mathcal{D} \). For the propagator, a similar calculation to the one for the \( \gamma_s \) case leads to a relation between \( G \) and its parity transformed \( G' \):

\[ [G(x, y; m)]' = -\int_{-\infty}^{0} dT \mathcal{K}(x, y; -m, T) = G(x, y; -m) , \quad (20) \]

while for \( \Gamma \) we have

\[ [\Gamma(A, m)]' = \int_{-\infty}^{0} \frac{dT}{T} \int d^4x \text{tr} \mathcal{K}(x, x; -m, T) . \quad (21) \]

We can derive similar relations involving \( \gamma_s \) transformations in even numbers of dimensions. For example, due to the cyclic property of the trace, in \( d = 2n \) we have

\[ \Gamma_{d=2n}(A, m) = \int_{0}^{\infty} \frac{dT}{T} \int d^4x \text{tr} [\gamma_s \mathcal{K}(x, x; m, T) \gamma_s] , \quad (22) \]

and by a similar argument to the one used in (17), we see that the only change appears in the sign of \( p \) in the path-ordered factor. Thus we may write an equivalent but more symmetric expression

\[ \Gamma_{d=2n}(A, m) = \int_{0}^{\infty} \frac{dT}{T} \int d^4x \text{tr} [\mathcal{K}_{sym}(x, x; m, T)] , \quad (23) \]
with:

\[ K_{\text{sym}}(x, y; m, T) = \int_{x(0)=y}^{x(T)=x} DpDx \ e^{i \int d\tau (ip\cdot \dot{x} - m)} \Re[\Phi(T)] e^{-ie \int_0^T d\tau \dot{x} \cdot A}, \]  

(24)

where the real part of \( \Phi(T) \) may of course be also represented as:

\[ \Re[\Phi(T)] = \mathcal{P} \cos \left[ \int_0^T d\tau \dot{\rho}(\tau) \right]. \]  

(25)

On the other hand, we can also obtain a representation for the parity-odd part of the effective action (in an odd number of spacetime dimensions). This object, \( \Gamma_{\text{odd}} \), is given explicitly by:

\[ \Gamma_{\text{odd}}(A, m) = \int_0^\infty dT \int d^dx \text{tr} \left[ K_{\text{odd}}(x, x; m, T) \right], \]  

(26)

with

\[ K_{\text{odd}}(x, y; m, T) = i \int_{x(0)=y}^{x(T)=x} DpDx \ e^{i \int d\tau (ip\cdot \dot{x} - m)} \Im[\Phi(T)] e^{-ie \int_0^T d\tau \dot{x} \cdot A}. \]  

(27)

\( \Gamma_{\text{odd}} \) is proportional to the imaginary part of the effective action, since \( \Gamma_{\text{odd}} = i \Im(\Gamma) \).

Taking advantage of (21), for any value of the sign of \( m \), we have:

\[ \Gamma_{\text{odd}} = \frac{1}{2} \text{sign}(m) \left[ \int_0^\infty \frac{dT}{T} \int d^dx \text{tr} K(x, x; m, T) - \int_{-\infty}^0 \frac{dT}{T} \int d^dx \text{tr} K(x, x; -m, T) \right], \]  

(28)

or:

\[ \Gamma_{\text{odd}} = \text{sign}(m) \int_{-\infty}^\infty \frac{dT}{2|T|} \int d^dx \text{tr} K(x, x; 0, T). \]  

(29)

Although this is in principle valid for any constant value of the mass, the difficulties to generalize the result to dynamical or random masses should be evident. This problem is discussed in section 4.

The generalization of the previous expressions to the non-Abelian case amounts to replacing everywhere the Wilson line factor for the corresponding non-Abelian object. Namely,

\[ \exp[-ie \int_0^T d\tau \dot{x}_\mu(\tau) A_\mu(x(\tau))] \to \mathcal{P} \exp[-g \int_0^T d\tau \dot{x}_\mu(\tau) A_\mu(x(\tau))]. \]  

(30)
3 Chiral fermions

The matrix Φ(T), defined in (30), verifies the ‘evolution equation’:

\[ i\partial_T \Phi(T) = \not p(T)\Phi(T) \quad T \in [0, \infty) \, . \] (31)

To solve it for Φ(T), we introduce its four ‘chiral’ components, in the following way

\[ \Phi_{LL}(T) = P_L \Phi(T) P_L, \quad \Phi_{LR}(T) = P_L \Phi(T) P_R \]
\[ \Phi_{RL}(T) = P_R \Phi(T) P_L, \quad \Phi_{RR}(T) = P_R \Phi(T) P_R \, , \] (32)

where \( P_L \equiv \frac{1 + \gamma_5}{2} \).

Of course, we have:

\[ \Phi(T) = \Phi_{LL}(T) + \Phi_{LR}(T) + \Phi_{RL}(T) + \Phi_{RR}(T) \, , \] (33)

and this kind of decomposition holds also true for the propagator and effective action, since both depend linearly on \( \Phi(T) \).

The equation of motion (31) is then equivalent to a system of four equations:

\[ i\partial_T \Phi_{LL}(T) = P_L \not p(T)\Phi_{RL}(T), \quad i\partial_T \Phi_{LR}(T) = P_L \not p(T)\Phi_{RR}(T) \]
\[ i\partial_T \Phi_{RL}(T) = P_R \not p(T)\Phi_{LL}(T), \quad i\partial_T \Phi_{RR}(T) = P_R \not p(T)\Phi_{LR}(T) \, , \] (34)

which form two decoupled pairs; namely \( \Phi_{LL} \) only mixes with \( \Phi_{RL} \), and \( \Phi_{RR} \) with \( \Phi_{LR} \). For a Dirac field, the initial condition is \( \Phi(0) = 1 \); it implies \( \Phi_{LL}(0) = P_L, \Phi_{RR}(0) = P_R, \) and \( \Phi_{LR}(0) = \Phi_{RL}(0) = 0 \).

The solution to these equations may be found by transforming each pair of coupled equations into an equivalent system of (coupled) integral equations. The fact that there is no coupling between pairs, allows for the introduction of chiral fermions, when the proper initial conditions for the \( \Phi \)'s are introduced.

For example, from the first and third differential equations, plus the initial conditions \( \Phi_{LL}(0) = 1 \) and \( \Phi_{RL} = 0 \), we obtain:

\[ \Phi_{LL}(T) = P_L - i \int_0^T d\tau P_L \not p(\tau)\Phi_{RL}(\tau) \]
\[ \Phi_{RL}(T) = - i \int_0^T d\tau P_R \not p(\tau)\Phi_{LL}(\tau) \, . \] (35)

This allows us to solve for \( \Phi_{LL}(T) \) and \( \Phi_{RL}(T) \), while the other two components, \( \Phi_{RR}(T) \) and \( \Phi_{LR}(T) \), vanish identically for all \( T \) when the initial
conditions $\Phi_{RR}(0) = 0$ and $\Phi_{LR} = 0$ are imposed. Note that these two conditions could be inconsistent on a Dirac fermion. Besides, note that we use the term ‘initial’ here for the $T$ evolution, which is completely different to the time ($x_0$) evolution.

From (35), we then derive an equation involving only $\Phi_{LL}$

$$\Phi_{LL}(T) = P_L - \int_0^T d\tau \int_0^\tau d\tau' P_L \not{\!p}(\tau)P_R \not{\!p}(\tau') \Phi_{LL}(\tau') ,$$

which may be formally solved in terms of a series:

$$\Phi_{LL}(T) = P_L - \int_0^T d\tau \int_0^\tau d\tau' \Lambda_{\mu\nu}^{(+)} p^\mu(\tau) p^\nu(\tau') P_L \ldots$$

$$+ (-1)^n \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{2n-2}} d\tau_{2n-1} \int_0^{\tau_{2n-1}} d\tau_{2n} \Lambda_{\mu_1\mu_2}^{(+)} p^\mu_1(\tau_1) p^\mu_2(\tau_2) \ldots \Lambda_{\mu_{2n-1}\mu_{2n}}^{(+)} p^\mu_{2n-1}(\tau_{2n-1}) p^\mu_{2n}(\tau_{2n}) P_L + \ldots$$

where $\Lambda_{\mu\nu}^{(+)} = P_L \gamma_\mu \gamma_\nu$. In $1+1$ dimensions, $\Lambda_{\mu\nu}^{(+)} \equiv \delta_{\mu\nu} + i\epsilon_{\mu\nu}$, so that $\Phi_{LL}(T)$ may be found, in that particular case, by solving equations that involve only scalar and pseudoscalar functions.

Let us examine the particular case of a free ($e = 0$) propagator. The functional integral over $x_\mu$ in $K$ yields a $\delta$ function of $\not{\!p}_\mu$. In this situation, $\Phi_{LL}(T)$ may be explicitly evaluated:

$$\Phi_{LL}(T) = \cos(T \not{\!p}) P_L ,$$

where $p = \sqrt{\not{\!p}_\mu \not{\!p}_\mu}$. For $\tilde{G}_{LL}(p)$, the corresponding component of the propagator, an application of (3), (4) and (5) (in Fourier space) yields:

$$\tilde{G}_{LL}(p) = \int_0^\infty dT e^{-mT} \cos(pT) P_L = \frac{m}{p^2 + m^2} P_L ,$$

as it should be.

For $\Phi_{RL}(T)$, on the other hand, we obtain:

$$\Phi_{RL}(T) = -i \frac{\not{\!p}}{\not{\!p}} \sin(pT) P_L$$

which yields:

$$\tilde{G}_{RL}(p) = -i \frac{\not{\!p}}{\not{\!p}} \int_0^\infty dT e^{-mT} \sin(pT) = \frac{-i \not{\!p}}{p^2 + m^2} P_L ,$$
which becomes the chiral fermion propagator in the \( m \to 0 \) limit.

The chiral fermion propagator, \( G_{RL} \), is then obtained in the general \( (e \neq 0) \) case, by considering only the \( \Phi_{RL} \) component in the path-integral for \( \mathcal{K} \) and taking the \( m \to 0 \) limit; namely:

\[
G_{RL}(x, y) = \lim_{m \to 0} \int_0^\infty dT \mathcal{K}_{RL}(x, y; m, T) \quad (42)
\]

where

\[
\mathcal{K}_{RL}(x, y; m, T) = \int_{x(0) = y}^{x(T) = x} DpDx \ e^{i \int_0^T d\tau (ip \dot{x} - m)} \Phi_{RL}(\tau) e^{-ie \int_0^T d\tau \dot{x} \cdot A}. \quad (43)
\]

It is interesting to check that, when \( m \to 0 \), \( G_{LL} \to 0 \), while \( G_{RL} \) tends to the free chiral propagator. Regardless of the fact that \( G_{LL} \to 0 \), \( \Phi_{LL} \) has a non-zero limit for \( m \to 0 \). This is related, in our context, to the fact that in order to regulate a chiral fermion one has to introduce both chiralities. Indeed, if the UV divergences that arise for small values of \( T \) are regulated by distorting the integral over \( T \), this will generally introduce a non-vanishing \( G_{LL} \).

An entirely analogous derivation allows one to define a propagator for the opposite chirality. Indeed, one imposes now the conditions \( \Phi_{LL}(0) = \Phi_{RL}(0) = 0 \). Then:

\[
\Phi_{RR}(T) = P_R - \int_0^T d\tau \int_0^\tau d\tau' P_R \hat{\Phi}(\tau) P_L \hat{\Phi}(\tau') \quad (44)
\]

and

\[
\Phi_{LR}(T) = -i \int_0^T d\tau P_L \hat{\Phi}(\tau). \quad (45)
\]

Regarding \( \Phi_{RR}(T) \), we have the integral equation:

\[
\Phi_{RR}(T) = 1 - \int_0^T d\tau \int_0^\tau d\tau' \lambda_{\mu}^{-i} p_{\mu}(\tau) \lambda_{\nu}^{-i} p_{\nu}(\tau') \Phi_{RR}(\tau') \quad (46)
\]

with \( \lambda_{\mu}^{-i} \equiv P_R \gamma_\mu \gamma_\nu \) (in 1 + 1 dimensions, \( \lambda_{\mu}^{-i} \equiv \delta_{\mu\nu} - i \epsilon_{\mu\nu} \)). It of course also admits a formal series solution like the one for \( \Phi_{LL} \), although with the substitution: \( \Lambda^{(+)} \to \Lambda^{(-)} \).

The formal series that solve the equations for \( \Phi_{LL} \) and \( \Phi_{RR} \), relevant to the construction of the two chiral propagators, can be written in terms of path-ordering operators; indeed:

\[
\Phi_{LL}(T) = \mathcal{P} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left[ \int_0^T d\tau \int_0^\tau d\tau' p_{\mu}(\tau) \lambda_{\mu}^{(+)} p_{\nu}(\tau') \right]^n P_L
\]

\[
\Phi_{RR}(T) = \mathcal{P} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left[ \int_0^T d\tau \int_0^\tau d\tau' p_{\mu}(\tau) \lambda_{\mu}^{(-)} p_{\nu}(\tau') \right]^n P_R. \quad (47)
\]
A square root can be introduced, just in order to have more compact expressions:

\[ \Phi_{LL}(T) = \mathcal{P} \cos \sqrt{\int_{0}^{T} d\tau \int_{0}^{T} d\tau' p_\mu(\tau) \Lambda_{\mu\nu}^{(\pm)} p_\nu(\tau')} \ P_L. \]  

(48)

It should be noted that, in the previous expressions, the \( \mathcal{P} \) operator acts also on the functions that are contracted with a matrix \( \Lambda_{\mu\nu}^{(\pm)} \); for example:

\[ \mathcal{P} \left[ p_\mu(\tau) \Lambda_{\mu\nu}^{(\pm)} p_\nu(\tau') \right] = \theta(\tau - \tau') p_\mu(\tau) \Lambda_{\mu\nu}^{(\pm)} p_\nu(\tau') \]

\[ + \theta(\tau' - \tau) p_\mu(\tau') \Lambda_{\mu\nu}^{(\pm)} p_\nu(\tau) . \]  

(49)

For a general product, the \( \mathcal{P} \) operator acts by ordering the \( p' \)'s according to their arguments, and connecting them pairwise with the \( \Lambda \) matrix (i.e., connecting each consecutive pair of \( p' \)'s).

In 1 + 1 dimensions, we can make use of the identities:

\[ p_\mu(\tau) \Lambda_{\mu\nu}^{(+)} p_\nu(\tau') = \bar{p}(\tau) p(\tau') , \quad p_\mu(\tau) \Lambda_{\mu\nu}^{(-)} p_\nu(\tau') = p(\tau) \bar{p}(\tau') \]  

(50)

where \( p := p_0 + i p_1 \) and \( \bar{p} := p_0 - i p_1 \), to write the sum of the series in the following way:

\[ \Phi_{LL}(T) = \mathcal{P} \cos \sqrt{\int_{0}^{T} d\tau \int_{0}^{T} d\tau' \bar{p}(\tau) p(\tau')} P_L \]

\[ \Phi_{RR}(T) = \mathcal{P} \cos \sqrt{\int_{0}^{T} d\tau \int_{0}^{T} d\tau' p(\tau) \bar{p}(\tau')} P_R . \]  

(51)

We conclude this section by presenting the expression for effective actions \( \Gamma_{L,R} \) corresponding to the chiral fermion determinant of the given chirality:

\[ \Gamma_{L,R}(A) = \lim_{m \to 0} \int_{0}^{\infty} \frac{dT}{T} \int d^d x \ \text{tr} \mathcal{K}_{LL,RR}(x, x; m, T) , \]  

(52)

with

\[ \mathcal{K}_{LL,R}(x, y; m, T) = \int_{x(0)=y}^{x(T)=x} DpD\bar{p} e^{\int d\tau (ip\dot{x} - m)} \Phi_{LL,R}^{(\pm)}(T) e^{-ie \int_{0}^{T} d\tau \dot{x} \cdot A} . \]  

(53)

The decomposition of \( \Phi(T) \) into its components can also be applied to the derivation of the effect of continuous chiral transformations in this representation. Indeed, the transformations:

\[ \psi(x) \to e^{i\alpha(x)\gamma_s} \psi(x) , \quad \bar{\psi}(x) \to \bar{\psi}(x) e^{i\alpha(x)\gamma_s} \]  

(54)
lead to the following transformations at the level of $\Phi(T)$:

\[
\begin{align*}
\Phi_{LL}(T) &\to e^{i\alpha(x)}\Phi_{LL}(T)e^{i\alpha(y)} & \Phi_{RL}(T) &\to e^{-i\alpha(x)}\Phi_{RL}(T)e^{i\alpha(y)} \\
\Phi_{RR}(T) &\to e^{-i\alpha(x)}\Phi_{RR}(T)e^{-i\alpha(y)} & \Phi_{LR}(T) &\to e^{i\alpha(x)}\Phi_{LR}(T)e^{-i\alpha(y)}
\end{align*}
\]

(55)

where $x$ and $y$ denote the arguments of $K(x, y; m, T)$.

4 Random mass

It is interesting to see how a random mass in $d$ dimensions may also be described within this formalism. By ‘random mass’ we mean that the mass $m$ in the Dirac operator is now a function $M(x)$, and that one wants to functionally average over the configurations $M(x)$, with a weight function. To be concrete, we will consider Gaussian averages $\langle \cdots \rangle_M$ defined by:

\[
\langle \cdots \rangle_M = \int D M \cdots e^{-\frac{1}{2g}\int dx |M(x)|^2}
\]

(56)

where $g$ is a positive constant. The main difficult to confront from the point of view of the path-integral representation is that a varying mass will take both positive and negative values, and the representation we are using depends explicitly on the sign of the mass. A naïve application of the previous average formula to the representation for a positive mass, say, yields a contribution that blows up when integrating over $T$.

One way to avoid this problem, is to consider instead the inverse of $H_D \equiv \gamma_s D$. The inverse of $H_D$ is obviously related to the one of $D$, since $D^{-1} = H_D^{-1} \gamma_s$. However, it is much easier to average over the inverse of $H_D$. To see this, we consider the representation:

\[
\langle x| H_D^{-1}|y \rangle = i \int_0^\infty dT \langle x| U(T)|y \rangle
\]

(57)

where $U(T)$ is the evolution operator in real time for $H_D$, which is a Dirac Hamiltonian in $d$ spatial dimensions. More explicitly

\[
H_D = \gamma_s (\slash{D} + M(x))
\]

(58)

and we assume a $-i\varepsilon$ is added to $H_D$ to pick up the proper contribution in the $T$ integral.

Note that $\gamma_s$ plays the role of Dirac’s ‘$\beta$’ matrix, while the ‘$\alpha$’ matrices are given by: $\gamma_s \gamma_\mu = i\alpha_\mu$. The kernel $K$ for $U(T)$ can then be represented by a path integral:

\[
K(x, y; M, T) = \langle x| U(T)|y \rangle
\]
\[ = \int_{x(0)=y}^{x(T)=x} DpDx \ e^{i \int d\tau p \cdot \dot{x} \Phi(T) \ e^{-ie \int_0^T d\tau \dot{x}_\mu A_\mu}}, \] (59)

where

\[ \Phi(T) = \mathcal{P} e^{-i \int_0^T d\tau [\alpha_\mu p_\mu(\tau) + \gamma_\mu M(x(\tau))]}, \] (60)

and the \( \mu \) index runs from 0 to \( d-1 \). The \( \Phi \) factor contains the mass multiplied by a Hermitian matrix, and is so in a position analogous to the momentum.

The average over the mass can now be performed by introducing a local representation for the factor \( \Phi(T) \), like the one considered in \[10\], obtained by adding Grassmann variables. For the interesting case of 2 dimensions, the expression is simple, since

\[ G(x, y; M) = i \gamma_s \int_0^\infty dT \int_{x(0)=y, \xi(0)=\xi}^{x(T)=x, \xi(T)=\xi} DpDx D\xi D\bar{\xi} \ \text{exp}\[-S(p, x, \xi, \bar{\xi}; T)\] \] (61)

where

\[ S(p, x, \xi, \bar{\xi}; T) = \bar{\xi}(T)\xi(T) + \int_0^T d\tau \ [-ip \cdot \dot{x} + \bar{\xi}\dot{\xi} - i(p_0 + ip_1)\xi - i(p_0 - ip_1)\bar{\xi} + ie\dot{x} \cdot A]. \] (62)

The presence of an imaginary exponent allows us to average over \( M \), obtaining:

\[ \langle G(x, y; M) \rangle_M = i \int_0^\infty dT \int_{x(0)=y, \xi(0)=\xi}^{x(T)=x, \xi(T)=\xi} DpDx D\xi D\bar{\xi} \ \text{exp}\[-S_{eff}(p, x, \xi, \bar{\xi}; T)\] \] (63)

where

\[ S_{av}(p, x, \xi, \bar{\xi}; T) = \bar{\xi}(T)\xi(T) + \int_0^T d\tau \ [-ip \cdot \dot{x} + \bar{\xi}\dot{\xi} - i(p_0 + ip_1)\xi - i(p_0 - ip_1)\bar{\xi} + ie\dot{x} \cdot A] - 2g \int_0^T \int_0^T d\tau d\tau' (\bar{\xi}\xi)(\tau) \delta(x(\tau) - x(\tau'))(\bar{\xi}\xi)(\tau'). \] (64)
5 Conclusions

We have derived expressions for the realization of parity and chiral transformations within a particle-like representation, and for the propagator and determinant of a chiral fermion field. Although we presented the results for the case of an Abelian gauge field, their non-Abelian generalizations proceed simply by making the substitution of equation (30).

We wish to point out that, when one considers a propagator in an external (non-dynamical) field, a particle-like representation is more natural than a second quantized functional integral. The reason is that, to represent that kind of propagator, the path integral cannot contain physical field degrees of freedom. It is rather an integral over particle-like trajectories. Indeed, $G(x, y; m)$ can be obtained from a path integral over Grassmann fields $\psi, \bar{\psi}$ as follows:

$$G(x, y; m) = [\det \mathcal{D}]^{-1} \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \psi(x) \bar{\psi}(y) e^{-\int d^4x \bar{\psi} \mathcal{D} \psi}, \quad (65)$$

where the $\det^{-1}$ factors must be present in order to cancel the det which comes from the Grassmann integral. Of course, this is not a local representation, precisely because of the determinant. In order to localize it, one is forced to introduce bosonic spinorial fields $\varphi, \bar{\varphi}$, so that:

$$G(x, y; m) = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}\varphi \mathcal{D}\bar{\varphi} \psi(x) \bar{\psi}(y) e^{-\int d^4x (\bar{\psi} \mathcal{D} \psi + \bar{\varphi} \mathcal{D} \varphi)} . \quad (66)$$

And finally, one can write the even more symmetric expression:

$$G(x, y; m) = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}\varphi \mathcal{D}\bar{\varphi} \frac{1}{2} [\psi(x) \bar{\psi}(y) + \varphi(x) \bar{\varphi}(y)] e^{-\int d^4x (\bar{\psi} \mathcal{D} \psi + \bar{\varphi} \mathcal{D} \varphi)} . \quad (67)$$

This expression is invariant under two supersymmetry transformations $s, \bar{s}$ that act on the fields as follows:

$$s\psi(x) = \varphi(x) \quad s\bar{\varphi}(x) = \bar{\psi}(x) \quad (68)$$

$$\bar{s}\bar{\psi}(x) = \bar{\varphi}(x) \quad \bar{s}\varphi(x) = -\psi(x) \quad (69)$$

and all the other transformations equal zero. It is evident that both $s$ and $\bar{s}$ are nilpotent ($s$ and $\bar{s}$ are assumed to anticommute with $\psi, \bar{\psi}$). The existence of these $BRST$-like symmetries implies, by the Parisi-Sourlas mechanism [13], that there are no physical field degrees of freedom for the fields.
Namely, the only remaining physical degrees of freedom can only be zero-dimensional; i.e., particle-like.

Regarding the expression for the average over the random mass, we point out to the fact that it is possible, in principle, to generalize the procedure to any even number of dimensions, since the Dirac algebra may be also dealt with by the introduction of the proper raising and lowering operators.

Acknowledgments

The author thanks Prof. J. Sánchez-Guillén for carefully reading the manuscript, and making useful comments. He also acknowledges the financial support of CONICET (Argentina) and Fundación Antorchas.

References

[1] J. Schwinger, Phys. Rev. 82, 664 (1951).
[2] R. P. Feynman, Phys. Rev. 76, 749 (1949).
[3] A. I. Karanikas and C. N. Ktorides, Phys. Rev. D 52, 5883 (1995).
[4] A. I. Karanikas, C. N. Ktorides and N. G. Stefanis, Phys. Rev. D 52, 5898 (1995).
[5] J. Sanchez-Guillen and R. A. Vazquez, Phys. Rev. D 65, 105001 (2002).
[6] D. Kosower, B. H. Lee and V. P. Nair, Phys. Lett. B 201, 85 (1988).
[7] Z. Bern and D. A. Kosower, Phys. Rev. D 38, 1888 (1988).
[8] M. J. Strassler, Nucl. Phys. B 385, 145 (1992).
[9] C. Schubert, Phys. Rept. 355 (2001) 73.
[10] C. D. Fosco, J. Sanchez-Guillen and R. A. Vazquez, arXiv:hep-th/0310191.
[11] A. A. Migdal, Nucl. Phys. B265 [FSI5], 594 (1986). See also, M.B. Halpern, A. Jevicki, and P. Senjanovic, Phys. Rev. D16 (1977) 2476.
[12] J. Zinn-Justin, Quantum Field Theory and Critical Phenomena, Oxford Science Publications, 4th. Ed., (2002).
[13] G. Parisi and N. Sourlas, Phys. Rev. Lett. 43, 744 (1979).