Quantum black-hole kinks

Pedro F. González-Díaz

Centro de Física "Miguel Catalán", Instituto de Matemáticas y Física Fundamental,
Consejo Superior de Investigaciones Científicas, Serrano 121, 28006 Madrid (SPAIN)
(March 10, 1996)

By allowing the light cones to tip over on hypersurfaces according to the conservation laws of an one-kink in static, Schwarzschild and five-dimensional black hole metrics, we show that in the quantum regime there also exist instantons whose finite imaginary action gives the probability of occurrence of the kink metric corresponding to single chargeless, nonrotating black holes taking place in pairs, joined on an interior surface, beyond the horizon, with each hole residing in a different universe. Evaluation of the thermal properties of each of the black holes in a neutral pair leads one to check that, to an asymptotic observer in either universe, each black hole is exactly the quantum-mechanically defined anti-black hole to the other hole in the pair. The independent quantum states of black holes in neutral pairs have been formulated by using the path integral method, and shown to be that of a harmonic oscillator. Our results suggest that the boundary condition of a single universe in the metauniverse is that this universe can never be self-contained and must always have at least one boundary which connects it to the rest of a self-contained metauniverse.

PACS number(s): 04.70.Bw, 04.70.Dy, 04.60.Gw

I. INTRODUCTION

The success of the thermodynamical analogy in black hole physics allows us to hope that this analogy may be even deeper, and that it is possible to develop a statistical-mechanical foundation of black hole thermodynamics [1]. However, it is not quite clear that black hole entropy may count the number of internal degrees of freedom. These would describe different internal states which may exist for the same values of the black hole external parameters. It might be [2] that in the state of thermal equilibrium the parameters for the internal degrees of freedom will depend on the temperature of the system in the universal way, thus cancelling all contributions which depend on the particular properties and number of internal fields.

Clearly, the most compelling argument in favour of the idea that black hole entropy counts the number of internal states has recently come from proposals of black hole pair creation [3–7]. By computing the exact action of the black hole pair instanton it has been shown [4] that the action for the case of nonextreme Reissner-Nordstrom black holes is smaller than that of the corresponding instanton action for the case of nonextreme Reissner-Nordstrom black hole pair instanton it has been shown [4] that the action for the case of nonextreme Reissner-Nordstrom black holes is smaller than that of the corresponding pair creation rate by exactly a factor of the black hole ent-

[7] to contribute importantly as well, the above analyses show two limitations. First of all, they are restricted to deal with Reissner-Nordstrom black holes. It would be of most interest if we could make a similar analysis in Schwarzschild black holes, where there is just an event horizon with unambiguous classical localization and one external parameter, i.e. the mass $M$. On the other hand, there seems to be no clear idea about where exactly the internal states may reside. They might be either inside the black hole or on the horizon and, in the event the first possibility applies, one may still wonder which region of the black hole interior does form the geometrical domain for states, so as exactly on what internal surface are identified the two black holes in a pair while preserving the appropriate contribution of the spacetime to the quantum construct.

The formation of neutral black-hole pairs with the two holes residing in the same universe is believed to be a highly suppressed process [8]. However, it has been suggested [9] that single black holes could evaporate completely, taking with them all the particles that fell in to form the black hole and the antiparticles to emit radiation, by going off through a wormhole whose other end opens up in other universe, forming another evaporating black hole which can be regarded to have been formed from the collapse of exactly the massive antifermions to the fermions that went in to form the first hole. One could thus regard each single Schwarzschild black hole in our universe as just a member of a neutral pair, with the other hole of the pair residing in other universe. In order to investigate the nature of such interniverse pairs, it appears important to know what is the quantum state of the wormhole that connects the two holes. If the wormhole is in a pure quantum state [10], the universe should
be self-contained with no real boundary through which information may escape from it to any other possible universes. If, on the contrary, wormholes are in mixed quantum states [11], then one would not expect the universe to be self-contained and its quantum state could not be factored out from the rest of the metauniverse.

This paper deals with these issues by considering the geometrical situation that results from regarding neutral black holes as gravitational topological defects which can move in spacetime but cannot be removed without cutting [12], when such defects are viewed as quantum geometrical constructs. As a consequence from imposing the kink metric in standard form to hold along the entire radial-coordinate interval from \( \infty \) to some interior nonzero surface, and invariance of the number of the resulting D-dimensional black hole kinks, we can obtain maximally-extended black hole kink metrics which can only be described by means of two coordinate patches for each one-kink. This extended kink metric can only be consistently defined in some quantum realm and represents two black holes, rather than one, with each hole being described in just one patch and the two patches identified just on the minimal single interior surface [13]. If we interpret each coordinate patch as the set of coordinates with which one just describes a single, distinct universe, to an observer in one of the asymptotically flat regions, the essential quantum nature of the black-hole kink interior would manifest in such a way that the observer can regard the whole of the kink geometry like though if it were a neutral black-hole pair, with each hole in the pair residing in a distinct universe. The two black holes of this pair are joined to each other at a given interior surface through a wormhole thinner than the corresponding Einstein-Rosen bridge [14].

We review D-dimensional black hole kinks and their connection to (D-1)-wormholes in Sec. II, and discuss the black-hole kink instantons that describe creation and annihilation of black-hole kinks in vacuum in Sec. III. Sec.IV deals with the quantum theory of black-hole kinks by using the Euclidean path integral approach. The probability that neutral black-hole pairs can occur as a consequence from the conservation of the topological charge of the kink is discussed in Sec. V, where the thermal emission of such pairs is also discussed, together with their quantization using both the Euclidean path integral approach and a generalized quantum theory in terms of a decoherence functional. It is obtained that single black holes behave like quantum harmonic oscillators whose frequency corresponds to the minimum possible internal energy compatible with the quantum De Broglie relation between energy and wavelength. A brief discussion on the possible cosmological implications of our results is included in Sec. VI, where a generalized boundary condition for a universe which is not self-contained is proposed. We summarize the main results in Sec. VII.

Finally, an Appendix is added which contains a calculation of the action for the Schwarzschild pairs. This action turns out to be smaller than that of the corresponding rate of pair creation by the Bekenstein-Hawking entropy factor. Throughout the paper natural units so that \( \hbar = c = G = k_B = \hbar \) are used.

II. THE BLACK-HOLE KINKS

The idea that we shall explore in this section is based on looking at the spacetime metric of a (D-1)-wormhole as the metric that results on the constant-time hypersurfaces corresponding to purely future directed or purely past directed light-cone orientations of a D-dimensional black-hole spacetime where we allow all possible light cone orientations compatible with the existence of a gravitational kink [15]. We shall restrict to the physically most interesting examples with D=4, which associated a Schwarzschild black hole to a three-dimensional wormhole, and with D=5, which corresponded to a five-dimensional Tangherlini black hole associated with a four-dimensional Tolman-Hawking wormhole [15].

We first briefly review the general topological concept of a kink and its associated topological charge. Let \( (\mathcal{M}, g_{ab}) \) be a given D-dimensional spacetime, with \( g_{ab} \) a Lorentz metric on it. One can always regard \( g_{ab} \) as a map from any connected D-1 submanifold \( \Sigma \subset \mathcal{M} \) into a set of timelike directions in \( \mathcal{M} \) [16]. Metric homotopy can then be classified by the degree of this map. This is seen by introducing a unit line field \( \{ u, -u \} \), normal to \( \Sigma \), and a global framing \( u_i : i=1,2,...,D-1 \), of \( \Sigma \). A timelike vector \( \mathbf{v} \) can then be written in terms of the resulting tetrad framing \( \{ n, u_i \} \) as \( v^a = n^a u_i + v^i u_i \), such that \( \sum_i (v^i)^2 = 1 \). Restricting to time orientable manifolds \( \mathcal{M}, \mathbf{v} \) then determines a map

\[
K : \Sigma \to S^{D-1}
\]

by assigning to each point of \( \Sigma \) the direction that \( \mathbf{v} \) points to at that point. This mapping allows a general definition of kink and kink number. Respect to hypersurface \( \Sigma \), the kink number (or topological charge) of the Lorentz metric \( g_{ab} \) is defined by [16]

\[
kink(\Sigma; g_{ab}) = \deg(K),
\]

so this topological charge measures how many times the light cones rotate all the way around as one moves along \( \Sigma \) [17].

In the case of an asymptotically flat spacetime the pair \( (\Sigma, g) \) will describe an asymptotically flat kink if \( \kink(\Sigma; g) \neq 0 \). All of the topological charge of the kink in the metric \( g \) is in this case confined to some finite compact region [17]. Outside that region all hypersurfaces \( \Sigma \) are everywhere spacelike. For the case of a spherically symmetric kink, to asymptotic observers, the compact region containing all of the topological charge coincides with the interior of either a black hole when the light cones rotate away from the observers (positive topological charge), or a white hole when the asymptotic observers "see" light cones rotating in the opposite direction, toward them (negative topological charge).
Topology changes, such as handles or wormholes, can occur in the compact region supporting the kink, but not outside it. All topology changes are actually allowed to happen in such a region. Therefore, in the case of spherically symmetric kinks, the supporting region should be viewed as an essentially quantum-spacetime construct. This is the view we shall assume throughout this paper.

A. The Schwarzschild Kink

We can take for the static, spherically symmetric metric of a three-dimensional wormhole

\[ ds^2 = (1 - \frac{2M}{r})^{-1}dr^2 + r^2d\Omega^2, \tag{2.1} \]

where \( d\Omega^2 \) is the metric on the unit two-sphere. Metric (2.1) describes a spacetime which (i) is free from any curvature singularity at \( r = 0 \), and (ii) possesses an apparent (horizon) singularity at \( r = 2M \) that is removable by a suitable coordinate transformation. The re-definition

\[ r = \frac{M}{2} \left( \frac{u + \mu}{u} \right)^2, \tag{2.2} \]

where \( \mu \) is an arbitrary scale, transforms metric (2.1) into

\[ ds^2 = \frac{M^2}{4} \left( \frac{u + \mu}{u} \right)^4 \left( \frac{4du^2}{u^2} + d\Omega^2 \right). \tag{2.3} \]

Along the complete \( u \)-interval, \((\infty, \mu)\), metric (2.3) varies from an asymptotic region at \( u = \infty \) to a minimum throat at \( u = \mu \) (i.e. at \( r = 2M \)).

On the other hand, metric (2.1) is in fact a constant time section, \( T = t_0 \), of a Schwarzschild black hole,

\[ ds^2 = -(1 - \frac{2M}{r})dT^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2d\Omega^2. \tag{2.4} \]

Metric (2.1) can likewise be regarded as being described by constant Euclidean time \( \tau = -iT \) sections of the Gibbons-Hawking instanton [18] associated to (2.4). Each of such three-wormhole would correspond to a given Einstein-Rosen bridge on this instaton, so that one of the two halves of the wormhole should then be described in the unphysical [19] exterior region created in the Kruskal extension of metric (2.4). In order to avoid the need of using such an unphysical region to describe a complete wormhole, we shall consider that metric (2.1) corresponds to a given fixed value of time \( T \) in the kink extension of (2.4).

We take for the metric that describes a spherically symmetric one-kink in four dimensions [12],

\[ ds^2 = -\cos 2\alpha (dt^2 - dr^2) \pm 2 \sin 2\alpha dtdr + r^2d\Omega^2, \tag{2.5} \]

where \( \alpha \) is the angle of tilt of the light cones, and the choice of sign in the second term depends on whether a positive (upper sign) or negative (lower sign) topological charge is being considered. An one-kink is ensured to exist if \( \alpha \) is allowed to monotonously increase from 0 to \( \pi \), starting with \( \alpha(0) = 0 \). Then metric (2.5) converts into (2.4) if we use the substitution

\[ \sin \alpha = \sqrt{\frac{M}{r}}, \tag{2.6} \]

and introduce a change of time variable \( t + g(r) = T \), with

\[ \frac{dg(r)}{dr} = \tan 2\alpha. \tag{2.7} \]

Now, since \( \sin \alpha \) cannot exceed unity, it follows that \( \infty \geq r \geq M \), so that \( \alpha \) varies only from 0 to \( \frac{\pi}{2} \). In order to have a complete one-kink gravitational defect, we need therefore a second coordinate patch to describe the other half of the \( \alpha \) interval, \( \frac{\pi}{2} \leq \alpha \leq \pi \).

The kink metric (2.5), which is defined by coordinates \( t, r, \theta, \phi \) and satisfy (2.6) and (2.7), restricts the Schwarzschild solution to cover only the region \( \infty \geq r \geq M \). If one wants to extend such a metric to describe the region beyond \( r = M \) as well, two procedures can in principle be followed: (i) if the compact support of the kink is assumed to be classical, then one lets \( \alpha \) continue to increase as \( r \) decreases from \( r = M \) until \( \alpha = \pi \) at \( r = 0 \) to produce a manifold which has a homotopically nontrivial light cone field and one kink. This procedure makes metric (2.5) and definitions (2.6) and (2.7) to hold asymptotically only, and since, classically, one should assume a continuous distribution of matter in the kink support, the momentum-energy tensor can be chosen to satisfy reasonable physical conditions such as the weak energy condition [12]. (ii) The second procedure can apply when one assumes the black hole interior (i.e. the supporting compact region of the kink) to be governed by quantum mechanics. The simplest quantum condition to be satisfied by the interior region supporting a black- or white-kink arises from imposing metric (2.5) to hold along the radial coordinate interval of the kink, i.e.: \( \infty \geq r \geq M \), rather than asymptotically only. Actually, in this case, the kink geometry should hold in the two coordinate patches which we need to describe the complete one-kink gravitational defect. The need for a second coordinate patch can most clearly be seen by introducing the new time coordinate

\[ \tilde{t} = t + h(r), \tag{2.8} \]

which transforms metric (2.5) into the standard metric [12]

\[ ds^2 = -\cos 2\alpha d\tilde{t}^2 \pm 2kd\tilde{t}dr + r^2d\Omega^2, \tag{2.9} \]

provided

\[ \frac{dh(r)}{dr} = \frac{dg(r)}{dr} - \frac{k}{\cos 2\alpha}. \tag{2.10} \]
with $k = \pm 1$ and the choice of sign in the second term of (2.9) again depending on whether a positive (upper sign) or negative (lower sign) topological charge is considered. The choice of sign in (2.10) is adopted for the following reason. The zeros of the denominator of $dh/dr = (\sin 2\alpha \mp 1)/\cos 2\alpha$ correspond to the two horizons where $r = 2M$, one per patch. For the first patch, the horizon occurs at $\alpha = \frac{\pi}{4}$ and therefore the upper sign is selected so that both $dh/dr$ and $h$ remain well defined and hence the kink is not lost in the transformation from (2.5) to (2.9). For the second patch the horizon occurs at $\alpha = \frac{3\pi}{4}$ and therefore the lower sign in (2.10) is selected. $k = +1$ will then correspond to the first coordinate patch and $k = -1$ to the second one.

Metric (2.9) can be transformed directly into the Schwarzschild metric (2.4) if we use (2.6) and the new coordinate transformation

$$\bar{t} = T - f(r),$$

(2.11)

where

$$\frac{df(r)}{dr} = \frac{k}{\cos 2\alpha}.$$  

(2.12)

We impose then the standard kink metric (2.9) to hold along $\infty \geq r \geq M$ on the two patches $k = \pm 1$. It will be seen in Sec. VB that the holding of this condition ultimately amounts to a black-hole quantum interior whose energy spectrum is given by $\frac{1}{2}(n + \frac{1}{2})$, $n = 0, 1, 2, \ldots$. The imposing of a kinky metric like (2.9) also in the interior of the hole (supporting region of the kink [17]), together with the resulting harmonic-oscillator spectrum, becomes a suitable quantization condition of such a region for the following reasons. For the first patch, the energy concentrated at $n = 0$ on the event horizon at $r = 2M$ will correspond to a pure vacuum energy, interpretable as a cosmological constant which is, in any case, compatible with the holding at $r = 2M$ of the kink metric (2.9), as such it happens with the similar kinky metric at the cosmological horizon in de Sitter space [20]. Subtracting then the zero-point energy corresponding to $n = 0$, along the radial coordinate interval $2M > r > M$ there is no spherical surface with nonzero energy and therefore this interval does not contribute the stress tensor $T_{\mu\nu}$. As one gets at $r = M$ on the first patch it would appear a delta-function-like concentration of positive energy on that surface corresponding to the quantum level $n = 1$. This would at first glance blatantly violate energy-momentum conservation. However, the continuity of the angle of tilt $\alpha$ at $\frac{\pi}{4}$ implies that the two coordinate patches are identified at exactly the surfaces $r = M$. Thus, since there would be an identical delta-function-like concentration of negative energy-momentum at $n = 1$ on $r = M$ in the second patch, the total stress tensor $T_{\mu\nu}$ will also be zero at the minimal surface $r = M$. Thus, although the Birkhoff’s theorem ensures [21] the usual Schwarzschild metric as the unique spherically symmetric solution to the four-dimensional vacuum Einstein equation, the violation of this classical result the way we have shown above implies an allowed quantized extension from it because this extension entails no violation of energy-momentum conservation at any interior spacelike hypersurface.

This result should be interpreted as follows. All what is left at length scales equal or smaller than the minimum size of the bridge (i.e. for $n \geq 1$, $r < M$) is some sort of quantized “closed” baby universe with maximum size $M$, whose zero total energy may be regarded as the sum of the opposite-sign eigenergies of two otherwise identical harmonic oscillators with the zero-point energy subtracted. The positive energy oscillator would play the role of the matter field part of a constrained Hamiltonian, $H = 0$, and the negative energy oscillator would behave like though it were the gravitational part of this Hamiltonian constraint. On the other hand, to an asymptotic observer in either patch, the above quantized kink geometry would look like that of a black hole if the topological charge is positive, and like that of a white hole if the topological charge is negative. In the latter case, to the asymptotic observer there would actually be a topological change by which an asymptotically flat space converts into asymptotically flat space plus a baby universe being branched off from it. Now, since from a quantum-mechanical standpoint white and black holes with the same mass are physically indistinguishable [22], it follows that to asymptotic observers the asymptotically flat space of black holes is physically indistinguishable from asymptotically flat space plus a baby universe, with such a baby universe living outside the realm of the two coordinate patches where the kink is defined, in the inaccessible region between $r = M$ and $r = 0$.

Metric (2.9) contains still the geodesic incompleteness at $r = 2M$ of metric (2.4). This incompleteness can be removed by the use of Kruskal technique. Thus, introducing the metric

$$ds^2 = -F(U,V)dUdV + r^2d\Omega_2^2,$$

(2.13)

in which

$$F = \frac{4M\cos 2\alpha}{\beta} \exp \left(-2\beta k \int_{\infty/M}^{r} \frac{dr}{\cos 2\alpha}\right),$$

(2.14)

$$U = \mp e^{\beta \bar{t}} \exp \left(2\beta k \int_{\infty/M}^{r} \frac{dr}{\cos 2\alpha}\right),$$

(2.15)

$$V = \mp \frac{1}{2\beta M} e^{-\beta \bar{t}},$$

(2.16)

where $\beta$ is an adjustable parameter which will be chosen so that the unphysical singularity at $r = 2M$ is removed, and the lower integration limit $\infty/M$ refers to the choices $r = \infty$ and $r = M$, depending on whether the first or second patch is being considered. Using (2.6) we obtain from (2.14)
\[ F = 4M \left( 1 - \frac{2M_M}{\beta} \right) \left( \frac{r}{M} \left( \frac{2M}{r} - 1 \right) \right)^{-4\beta kr}. \]

This expression would actually have some constant term coming from the lower integration limits \( \infty/M \). We have omitted at the moment such a term because it is canceled by the similar constant term which appears in the Kruskal coordinate \( U \) when forming the Kruskal metric from (2.13)-(2.16).

Unphysical singularities are then avoided if we choose
\[ \beta = \frac{1}{4kM}. \quad (2.17) \]
Whence
\[ F = \frac{16kM^3}{r} e^{-\pi \tau}, \quad (2.18) \]
\[ U = \mp \frac{2M-r}{M} e^{\mp \pi \tau}, \quad V = \mp \frac{k}{2} e^{-\mp \pi \tau}, \quad (2.19) \]
where \[ \bar{t} = t_0 - k \int_{\infty/M}^{\cos 2\alpha} \frac{dr}{\cos 2\alpha} \]
\[ = \bar{t}_0 - k \left( r - 2M \ln \left( \frac{M}{2M-r} \right) \right), \quad (2.20) \]
with the constant \( \bar{t}_0 \) being obtained from \( t_0 \) after absorbing the term arising from the lower integration limit \( \infty \) or \( M \), depending on whether the first or second patch is being considered. We finally obtain for the Kruskal metric of the Schwarzschild kink
\[ ds^2 = -\frac{32kM^3}{r} e^{-\pi \tau} dU dV + r^2 d\Omega_2^2. \quad (2.21) \]
Except for the sign parameter \( k \), this metric is the same as the Schwarzschild-Kruskal metric.

Because of continuity of the angle of tilt \( \alpha \) at \( \bar{r} \), the two coordinate patches can be identified to each other only on the surfaces at \( r = M \) \[ [13,15] \]. Such an identification should occur both on the original and the new regions created by the Kruskal extension, and represents a bridge that connects asymptotically flat regions of the two coordinate patches. Any \( T = \text{const.} \) section of this spacetime construct will then describe halves of a three-dimensional wormhole whose neck is now at \( r = M \), rather than \( r = 2M \). One can then describe the two halves of a complete wormhole just in the physical original regions of either patch \( k = +1 \) or patch \( k = -1 \).

The causal structure of the considered geometry could at first glance be thought of as being unstable due to mass-inflation caused by the unavoidable presence of a Cauchy horizon [23]: because quanta that enter the future event horizon at arbitrary late time suffers an arbitrarily large blue shift while propagating parallel to the Cauchy horizon, there will be in general a mass-inflation singularity along a part of the horizon in one patch caused by small fluctuations in the other. However, using the spherical shell approach in the lightlike limit [24] where a mass shell is allowed to move toward \( r = 0 \) in the field of an interior mass distribution, it can be shown that our kink model with quantized support prevents the occurrence of any mass-inflation singularity. In fact, any interior energy fluctuation in one patch is necessarily sign-reversed to the energy of the imploding shell in the other. Therefore, a mass increase must now occur in the expanding fluctuation shell, rather than in the imploding shell, and the mass variation of these two shells is nonsingular everywhere for \( r \geq M \), even at the collision radius where one would expect the mass singularity to occur. We actually expect that, at that radius, imploding and expanding gravitational masses are both finite with half and twice their respective asymptotic values.

**B. The D=5 Tangherlini Kink**

The static, spherically symmetric metric of a four-dimensional wormhole can be written as [25]
\[ ds^2 = (1 - \frac{M^2}{r^2})^{-1} dr^2 + r^2 d\Omega_2^2, \quad (2.22) \]
where \( d\Omega_2^2 \) is the metric on the unit three-sphere. As for the three-dimensional wormhole (2.1), we can regard (2.22) as a singularity-free spacetime with removable apparent singularity at \( r = M \), describing a four-dimensional wormhole. This is better seen by using the coordinate re-definition
\[ r = \frac{M}{2} \left( \frac{\nu}{\nu} + \frac{\nu}{\bar{v}} \right), \quad (2.23) \]
in which again \( \nu \) is an arbitrary scale measuring the size of the wormhole throat. Using (2.23), (2.22) becomes
\[ ds^2 = \frac{M^2}{4} \left( \frac{\bar{v}}{\nu} + \frac{\nu}{\bar{v}} \right)^2 \left( \frac{d\nu^2}{v^2} + d\Omega_3^2 \right). \quad (2.24) \]
Metric (2.24) describes the connection between two asymptotically flat regions through a four-dimensional wormhole with minimum throat at \( \nu \); i.e. at \( r = M \). Now, we regard metric (2.22) as a constant time section of the kink extension of the five-dimensional Tangherlini black-hole metric [26]
\[ ds^2 = -\left( 1 - \frac{M^2}{r^2} \right) dT^2 + \left( 1 - \frac{M^2}{r^2} \right)^{-1} dr^2 + r^2 d\Omega_3^2, \quad (2.25) \]
Also in this case metric (2.22) can be regarded as well as being described by constant Euclidean time \( \tau = -iT \).
sections of the five-dimensional black hole instanton, so that each of such wormholes would correspond to a given four-dimensional Einstein-Rosen bridge on this instanton, with one of the two wormhole halves necessarily being in the unphysical exterior region created in the Kruskal extension of (2.25). Also in this case, we shall avoid the need of using this unphysical region for the description of a complete wormhole by using the kink version of (2.25). The kink extension in this case has been considered in detail elsewhere [15]. Let us summarize the results in what follows. By using arguments similar to those leading to metric (2.9), we can write the kink extension of (2.25) as [13,15]

\[ ds^2 = -\cos 2\alpha dr^2 \mp 2kd\rho dr + r^2d\Omega_2^2, \]

(2.26)

with the choice of sign in the second term again referring to whether a positive (upper sign) or negative (lower sign) topological charge is being considered.

Metric (2.26) is nonstatic, requires two coordinate patches, \( k = \pm 1 \), and can be obtained from (2.25) by using the functional relation

\[ \sin \alpha = \frac{M}{\sqrt{2}r} \]

and the same time transformation as in (2.6) and (2.7). It also contains a geodesic incompleteness at the horizon \( r = M \) which again is removable by the Kruskal technique. Imposing metric (2.26) to hold along the interval \( \infty \geq r \geq \frac{M}{\sqrt{2}} \) in the two patches and using arguments similar to those in Sec.VB for a convenient definition of the propagator through the corresponding complete wormhole, leads again to a black hole spectrum that allows a consistent quantum interpretation of the interior compact region supporting the kink which is analogous to that for the four-dimensional example considered in the precedent subsection. In this case, the Kruskal coordinate becomes finally [15]

\[ U = \mp e^{\frac{k}{2}r} e^{\frac{r}{\sqrt{2}M}} \left( \frac{r - M}{r + M} \right), \quad V = \mp \frac{k}{2} e^{-\frac{k}{\sqrt{2}M}r}, \]

(2.27)

with [15]

\[ \bar{t} = \bar{t}_0 - k \left( r + \frac{M}{2} \ln \left( \frac{r - M}{r + M} \right) \right), \]

(2.28)

where the constant \( \bar{t}_0 \) is obtained from \( \bar{t}_0 = T = \text{Const.} \), following the same procedure as in (2.20); i.e.: now by absorbing into \( \bar{t}_0 \) the respective constant term coming from the two distinct integration lower limits \( \infty \) and \( \frac{M}{\sqrt{2}} \) corresponding to the two coordinate patches. The maximally-extended metric for a Tangherlini five-dimensional kink becomes then

\[ ds^2 = -\frac{kM^2(r + M)^2}{r^2} e^{-\frac{r}{\sqrt{2}M}} dUdV + r^2d\Omega_2^2. \]

(2.29)

Again, this metric describes a spacetime which is nonsingular. Continuity of the angle of tilt at \( \alpha = \frac{\pi}{2} \) implies the nonexistence of any surface with \( r < \frac{M}{\sqrt{2}} \), and the identification of the surfaces at \( r = \frac{M}{\sqrt{2}} \) of the two coordinate patches, both on the original regions described by metrics (2.26) and (2.29), and on the new regions created by the the Kruskal extension. Thus, one can continue the asymptotically flat spacetime described in patch \( k = +1 \) into another asymptotically flat spacetime described in patch \( k = -1 \) by means of a bridge at \( r = \frac{M}{\sqrt{2}} \). Every \( T = \bar{t}_0 \) constant section of the resulting spacetime will thus represent halves of a Tolman-Hawking wormhole in four dimensions whose neck is now at \( r = \frac{M}{\sqrt{2}} \), rather than \( r = M \). Thus, also in this case, all wormhole halves can be described in the physical regions of either patch \( k = +1 \) or patch \( k = -1 \).

**C. Travelling Through Wormholes**

If we replace the expression of \( \bar{t} \) given by (2.20) and (2.28) in the correponding Kruskal coordinates \( U,V \), we recover the usual expressions for these coordinates outside the horizon [27] for both coordinate patches in terms of \( r \) and \( \bar{t}_0 \) (or its nonconstant version \( T \)). In principle, one could therefore simply continue in terms of these \( U,V \) themselves inside the horizon to obtain Kruskal coordinates which are real everywhere. However, the need for two coordinate patches (which continuously follow one another as the tilt angle is monotonously varies from 0 to \( \pi \)) to describe an one-kink makes it impossible to have coordinates \( U,V \) which are real everywhere in the two patches while keeping the original metric (2.4), or (2.25), Lorentzian on the two patches. In fact, it is easy to check that the lower integration limit one should use to get time \( \bar{t} \) in one patch is necessarily different from the limit that must be used at the same side of the horizon in the other patch. Since all that is physically required is having a real nonsingular metric in the range \( \infty \geq r > 0 \), and the times \( \bar{t}_0 \) and \( T \) in (2.20) and (2.28) can by no means be kept real in the same \( r \)-regions of the two patches simultaneously, instead of continuing \( U,V \), we shall continue time \( \bar{t} \) itself while keeping the Kruskal metric real and expressions (2.19) and (2.27) for \( U,V \) in terms of \( \bar{t}_0 \) or \( T \) valid everywhere. Different continuations of time \( \bar{t} \) that lead either to some imaginary sectors for coordinates \( U,V \), or to instantonic transitions with \( U,V \) real everywhere, will respectively be discussed in what follows and in Sec. III.

As it was pointed out before, the constant \( \bar{t}_0 \) is obtained by adding to \( \bar{t}_0 \) the distinct constants coming from the lower integration limits in (2.20) and (2.28), and is the same in the two coordinate patches [15]. If \( \bar{t}_0 \) would take on real values in (2.20), i.e. if one considers metric (2.4) to be complex on patch \( k = +1 \) and Lorentzian in patch \( k = -1 \), then though the Kruskal metric (2.21) remains real and unchanged, coordinates \( U \), \( V \) become both imaginary outside the black hole and both real inside the black hole, in the two coordinate patches. By
taking into account that the lower integration limit in (2.20) gives rise for \( k = +1 \) to a complex constant whose imaginary part is \( 2\pi i M \), if we continue the chosen real value of \( \tilde{t}_0 \) so that \( \tilde{t}_0 \rightarrow \tilde{t}_0 - 2\pi i M \), metric would be Lorentzian on the first patch and complex in the second one. On the two patches then coordinates \( U, V \) would be both real outside the horizon and both imaginary inside the horizon, while the Kruskal metric (2.21) remained real and unchanged. It follows that the spacetime description of three-dimensional wormholes, each with neck at \( r = M \) and two asymptotic physical regions, requires two different four-dimensional black-hole spacetimes, one in each patch, and therefore every complete three-dimensional wormhole with neck at \( r = M \), rather than \( 2M \), cannot correspond to a single \( T = \text{const.} \) section of the same Schwarzschild metric in the two patches.

As for the five-dimensional black hole, we obtain that real values of \( \tilde{t}_0 \) correspond to a black hole metric (2.25) which is Lorentzian on patch \( k = +1 \) and complex on patch \( k = -1 \). In this case, although metric (2.29) remains real and unchanged, coordinates \( U, V \) are both real for \( r > M \) and imaginary beyond the horizon. If now \( \tilde{t}_0 \) is continued so that \( \tilde{t}_0 \rightarrow \tilde{t}_0 - \frac{i}{2} \pi M \), then metric (2.25) would be complex for patch \( k = +1 \) and Lorentzian for patch \( k = -1 \), with the Kruskal coordinates \( U, V \) being real only for \( r \leq M \), and imaginary otherwise. Thus, the spacetime description of a four-dimensional Tolman-Hawking wormhole with neck at \( r = \frac{M}{2} \) and two asymptotic physical regions also requires two different five-dimensional black-hole spacetimes, one per patch, and it is not possible to describe four-dimensional wormholes with neck at \( r = \frac{M}{2} \), rather than \( M \), as single \( T = \text{const.} \) sections of the same five-dimensional Tangherlini black-hole metric in the two patches.

It may be instructive to look at null geodesics traveling through the spacetime of these spacetimes from one asymptotic region to the other (Fig. 1). Let us consider, for example, the null geodesic which starts at spatial infinity on the original region described by metric (2.4) on the coordinate patch \( k = +1 \), crossing the horizon at \( r = 2M \) on the \( V \) axis to get in the original interior region. This geodesic will then cross the identified surface at \( r = M \) over into the original interior region of the second patch, sharing on such a surface a common value of time \( \tilde{t} \) for the two patches. Further propagation of the same geodesic first crossing the \( U \) axis of patch \( k = -1 \) to get in a new (unphysical) exterior region to reach spatial infinity would at first sight seem to be impossible as the crossing of the \( U \) axis would take an infinite time. Thus, the geodesic would seem to be trapped in the interior of the black hole of patch \( k = -1 \). However, since the local geometry is that of a Schwarzschild kink with standard metric (2.9), or (2.26), everywhere even at \( r = A \), it would actually take a generally finite amount of proper time with respect to such a metric for null geodesics to reach the past event horizon \( (\alpha = \frac{3\pi}{4}) \) of the second patch, as also it does entering the future event horizon \( (\alpha = \frac{\pi}{4}) \) of the first patch. The same effect will also be obtained for null geodesics in the five-dimensional spacetime. It will be shown in Sec. VA that the black hole in patch \( k = -1 \) must necessarily have been formed from exactly the antiparticles to the particles that went in to form the black hole in patch \( k = +1 \). Thus, on the identified surface \( r = A \) (with \( A = M \) for \( D=4 \) and \( A = \frac{M}{\sqrt{2}} \) for \( D=5 \), the positive energy of the null ray would be instantly annihilated, leading to a decrease of the absolute value of the black hole mass. Balance between the two black hole masses will be restored if, upon the arrival of the first ray at the black hole in the second patch, this would emit another ray with the same positive energy along an escaping path parallel to the \( U \) axis, into the original exterior region of the black hole in the second patch. The whole process will be equivalent to the simultaneous (in time \( \tilde{t} \) ) travel of a null ray with positive energy from the original spatial infinity in patch \( k = +1 \) and a null ray with the same but negative energy from the original spatial infinity in patch \( k = -1 \), both parallel to the \( U \) axes, which encounter and annihilate each other at the surface \( r = A \).

Inspection of Fig. 1 might suggest that null geodesics could be closed timelike curves that somehow loop back through the new regions created by the Kruskal extension. It is not difficult to see however that this cannot be the case since, though surfaces \( r = A \) in the two patches are identified, the identification of any surfaces at \( r = \infty \) of the two patches is disallowed, so null geodesics can never complete a closed itinerary.

### III. THE BLACK-HOLE KINK INSTANTONS

The Euclidean section of the Schwarzschild solution is asymptotically flat and nonsingular because it does not contain any points with \( r < 2M \). Thus, the curvature singularity does not lie on the Euclidean section. Here I shall consider the instantons that can be associated with the black hole kinks, and show that their Euclidean sections can be extended beyond the horizon down to the surface \( r = A \).

The Euclidean continuation of the metrics which contain one kink should be obtained by putting

\[
\tilde{t} = i\tilde{\tau}.
\]

Using (2.6) and (2.7) we then have

\[
d\tilde{\tau} = -i dT + \frac{ik}{\cos 2\alpha} dr,
\]

which is valid both for \( D=4 \) and \( D=5 \) black holes. This Euclidean continuation would give rise to metrics which are positive definite if we choose either the usual continuation \( T = i\tau \), for \( r \geq 2M \), or the new Euclidean continuation \( r = -ip, M = -ip \), for \( r < 2M \), where \( r \) becomes timelike, and we transform a space coordinate.
into a time coordinate. In the first case, restricting to
D=4, metric (2.9) becomes
\[ ds^2 = \cos 2\alpha d\bar{r}^2 + 2ikd\bar{r}d\rho + r^2d\Omega_2^2. \]  
(3.3)
This corresponds to the usual Euclidean subsection \( r \geq M \),
\[ ds^2 = \left(1 - \frac{2M}{r}\right)d\tau^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2d\Omega_2^2, \]  
(3.4)
and can be maximally-extended to the Kruskal metric\[ ds^2 = -\frac{32M^3k}{r}e^{-\frac{2\mu}{r}}d\bar{U}d\bar{V} + r^2d\Omega_2^2, \]  
(3.5)
where
\[ \bar{U} = \mp e^{\frac{2\mu}{r}}e^{\frac{2\mu}{r}} \left(\frac{2M - r}{M}\right), \quad \bar{V} = \mp \frac{k}{2}e^{-\frac{2\mu}{r}}. \]  
(3.6)
In order for the new continuation \( r = -i\rho, M = -i\mu \) to
give rise to a metric which is positive definite for \( r < 2M \),
we would also continue the angular polar coordinates
\[ x = -X = -\rho \sinh \Theta \cos \phi, \quad y = -Y = -\rho \sin \Theta \sin \phi, \]
so that \( r = \sqrt{(x^2 + y^2 + z^2)} = \pm i\rho \) and \( |Z| \geq \rho \). Therefore,
we in fact have \( \phi = \text{arccosh} \frac{x}{r} = \text{arctan} \frac{y}{\rho} \), and
\[ \theta = \text{arccos} \frac{r}{\rho} = \pm i\Theta, \]  
with \( \Theta = \cos^{-1} \frac{x}{\rho} \). Hence, the metric
on the unit two-sphere \( d\Omega_2^2 \) should transform as
\[ d\Omega_2 = id\Omega_2^2 = i(d\Theta^2 + \sinh^2 \Theta d\phi^2)^{\frac{1}{2}}. \]
The choice of the minus sign for the Euclidean continuation
of both the radial coordinate \( r \) and the polar angle \( \theta \)
would allow us to have the same action continuation
as that corresponding to the continuation \( T = i\tau \); i.e.
\( S = iI \), where \( S \) and \( I \) are the Lorentzian and Euclidean action,
respectively, since the scalar curvature transforms as
\( R(r) = -R(\rho) \) under continuation \( r = -i\rho \).
Thus, for the continuation \( r = -i\rho, M = -i\mu, d\Omega_2 = id\Omega_2^2 \) for \( r < 2M \), metric (2.9) becomes
\[ ds^2 = \cos 2\alpha d\bar{r}^2 + 2kd\bar{r}d\rho + r^2d\Omega_2^2, \]  
(3.7)
which corresponds to the new Euclidean subsection \( 2M > r > M \), with positive definite metric
\[ ds^2 = (\frac{2\mu}{\rho} - 1)dT^2 + (\frac{2\mu}{\rho} - 1)^{-1}d\rho^2 + r^2d\omega_2^2, \]  
(3.8)
and can be maximally-extended to the Kruskal metric
\[ ds^2 = \frac{32\mu^3k}{\rho}e^{-\frac{2\mu}{r}}d\bar{U}d\bar{V} + r^2d\Omega_2^2, \]  
(3.9)
where in this case
\[ \bar{U} = \mp e^{\frac{2\mu}{r}}e^{\frac{2\mu}{r}} \left(\frac{2\mu - \rho}{\mu}\right), \quad \bar{V} = \mp \frac{k}{2}e^{-\frac{2\mu}{r}}. \]  
(3.10)
Using a positive definite metric such as (3.8) leads however
to the problem that the azimuthal angle \( \theta \) is a periodic variable only outside the horizon. In this case the transverse two-manifold, which is a two-sphere of positive scalar curvature outside the Euclidean horizon, becomes a hyperbolic plane of negative scalar curvature inside the horizon and any boundary at finite geodesic distance inside the horizon is no longer compact. In particular, the boundary at \( r = M \) would then have the noncompact topology \( S^1 \times R^2 \), with \( S^1 \) corresponding to time \( T \). One cannot identify this geometry at \( \rho = 2\mu \) with the geometry at \( r = 2M \) corresponding to the compact topology \( S^1 \times S^2 \) of the Gibbons-Hawking instanton [18]. Nevertheless, avoidance of the spacetime singularities in the calculation of the black hole action does not actually require having a positive definite metric in our spacetime kinks. Indeed, the "tachyonic continuation" of the signature \( ++- \) (which is --- ++ ) that corresponds to a real azimuthal periodic variable \( \theta \) also inside the horizon and includes the same metrics as (3.7), (3.8) and (3.9) but with the sign for the polar coordinate terms reversed, can not only avoid singularities but erase them even at \( r = 0 \). In order to see this, let us consider the new variables \( y + z = U \) and \( y - z = V \) in the Kruskal metric (2.21) which then becomes
\[ ds^2 = -\frac{32kM^3}{r}e^{-\frac{2\mu}{r}}(dy^2 - dz^2) + r^2d\Omega_2^2, \]  
(3.11)
with
\[ y^2 - z^2 = k e^{\frac{2\mu}{r}} \left(1 - \frac{r}{2M}\right) \]  
(3.12)
\[ y + z = k e^{\frac{2\mu}{r}} e^{\frac{2\mu}{r}} \left(\frac{r}{2M} - 1\right). \]  
(3.13)
The singularity at \( r = 0 \) lies on the surfaces \( y^2 - z^2 = k \). Although this singularity cannot be removed by any coordinate changes, it can be avoided by defining either a new coordinate \( \zeta = iy \) or a new coordinate \( \xi = iz \). For the first choice the metric takes the Euclidean form
\[ ds^2 = \frac{32kM^3}{r}e^{-\frac{2\mu}{r}}(dz^2 + d\zeta^2) + r^2d\Omega_2^2, \]  
(3.14)
which is positive definite in the patch \( k = +1 \) and has in fact signature ---- + in the patch \( k = -1 \). The radial coordinate is then defined by
\[ z^2 + \zeta^2 = k e^{\frac{2\mu}{r}} \left(\frac{r}{2M} - 1\right). \]  
(3.15)
On the case of the positive definite metric which corresponds to the trace of the second fundamental form of the boundary surface integrals corresponding to the fixed boundaries, vanishes, the action can be written only in terms of the Riemannian metric. Then, from (2.20) and (3.13) we obtain

\[ z - i\zeta = \pm (z^2 + \zeta^2)^{\frac{1}{2}} e^{\frac{i\pi}{4}}. \] (3.16)

It follows that for this time continuation \( \tau \) is periodic with period \( 8\pi kM \). On this nonsingular Euclidean section (see Fig. 2a), \( \tau \) has then the character of an angular coordinate which rotates about the "axis" \( r = 2M \) clockwise in patch \( k = +1 \), and anticlockwise about the "axis" \( r = 0 \) in patch \( k = -1 \). Any boundary \( \partial M_k \) in this Euclidean section has topology \( S^1 \times S^2 \) and so is compact in both coordinate patches. Since the scalar curvature \( R \) vanishes, the action can be written only in terms of the surface integrals corresponding to the fixed boundaries. This action can be written

\[ I_k = \frac{1}{8\pi} \int_{\partial M_k} d^3x K_k, \] (3.17)

where \( K_k = K - \frac{1}{2}(1 + k)K^0 \), \( K \) being the trace of the second fundamental form of the boundary, and \( K^0 \) the trace of the second fundamental form of the boundary imbedded in flat space. This action was evaluated \([16]\) in the case of the positive definite metric which corresponds to \( k = +1 \). It is \( I_{+1}(M) = 4\pi iM^2 \). In the case \( k = -1 \), fixing the boundary at the surface \( r = A = M \), we also have

\[ I_{-1}(M) = \frac{1}{8\pi} \int_{\partial M_{-1}} K d\Sigma \]

\[ = -4\pi i(2r - 3M) \mid_{r=M} = 4\pi iM^2. \]

For the second choice of coordinates, \( \xi = iz \), metric (3.11) takes the form

\[ ds^2 = -\frac{32kM^3}{r} e^{-\frac{r}{2M}} (dy^2 + d\xi^2) + r^2 d\Omega_2^2, \] (3.18)

which is positive definite in patch \( k = -1 \) and has again signature \(-++\) in patch \( k = +1 \). The radial coordinate is now defined by

\[ y^2 + \xi^2 = k e^{-\frac{r}{2M}} \left( 1 - \frac{r}{2M} \right), \] (3.19)

so that on the section on which \( y \) and \( \xi \) are both real (the usual Euclidean section for patch \( k = +1 \)) \( r \) will be in the interval \( \frac{1}{2} \leq r \leq 1 \) on patch \( k = +1 \), and greater or equal to 1 on patch \( k = -1 \). We define now the imaginary \( r \) and \( M \) by \( r = -i\rho \) and \( M = -i\mu \), keeping \( T \) and the azimuthal coordinate \( \theta \) real. In order for this definition to be compatible with the coordinate transformation \( \xi = iz \), it should leave metric (3.18) formally unchanged. For this to be accomplished one must also continue the line element \( ds \) itself, namely \( ds = -i\sigma \) instead of the azimuthal angle \( \theta \). This requirement becomes most natural if we recall that the interval \( ds \) has the same physical dimension as that of \( r \) and \( M \), and that the "tachyonic" mass \( \mu \) should be associated with an imaginary relativistic interval. Then, from (2.20) and (3.13) we obtain

\[ y - i\xi = \pm(y^2 + \xi^2)^{\frac{1}{2}} e^{\frac{i\pi}{4}}. \] (3.20)

It is now the Lorentzian time \( T \) which becomes periodic with period \( 8\pi k\mu \). On this new nonsingular Euclidean section (see Fig. 2b), \( T \) would have the character of an angular coordinate which rotates about the "axis" \( \rho = 0 \) clockwise in the patch \( k = +1 \), and anticlockwise about the "axis" \( \rho = 2\mu \) in the patch \( k = -1 \). In such a new section, the action is given by (3.17), where now \( K_k = K - \frac{1}{2}(1 - k)K^0 \). On the patch \( k = +1 \), we have \([18]\)

\[ I_{+1}(\mu) = \frac{1}{8\pi} \int_{\partial M_{+1}} K d\Sigma \]

\[ = 4\pi i(M(2r - 3M) \mid_{r=M} = -4\pi iM^2 = 4\pi i\mu^2. \]

In the patch \( k = -1 \), taking \( K^0 = \frac{1}{2} \) and following Gibbons and Hawking \([18]\), we obtain the action \( I_{-1}(\mu) \) which turns out to be the same as \( I_{+1}(\mu) \).

Thus, on the coordinate patch \( k = +1 \), the Euclidean continuation (3.1) of the time coordinate \( \tilde{t} \) of the kink metric contains both the continuation for time \( T \), \( T = i\tilde{t} \), where the apparent singularity at \( r = 2M \) is like the irrelevant singularity at the origin of the polar coordinates provided that \( \frac{r}{2M} \) is regarded as an angular variable and is identified with period \( 2\pi \) \([18]\), and a new continuation \( r = -i\rho \), which also implies "tachyonic" continuations \( M = -i\mu \) and \( ds = -i\sigma \), where the curvature singularity at \( \rho = 0 \) becomes again like a harmless polar-coordinate singularity provided that \( \frac{T}{4\mu} \) is regarded as an angular variable and is identified with period \( 2\pi \). The transverse two-manifold is now a compact two-sphere both outside and inside the Euclidean horizon and any boundaries have compact topology \( S^1 \times S^2 \), with \( S^1 \) corresponding to \( \tau \) outside the horizon and to \( T \) inside the horizon. Since these topological products are compact, have the same Euler characteristic and are both orientable, they are homeomorphic to each other with a continuous mapping between them. Therefore, one can identify the two corresponding geometries at the Euclidean horizons \( r = 2M \) and \( r = 2\mu \) which, respectively, \( \tau \) rotates about at zero geodesic distance and is the geodesic distance at which \( T \) rotates about \( \rho = 0 \). The Gibbons-Hawking instanton can then be extended beyond the Euclidean horizon down to just the boundary surface at \( r = M \) \( (\rho = \mu) \) where the first and second
patches must be somehow joined onto each other (Fig. 2c). The resulting Euclidean section does not contain any points with \( r < M \) and therefore the curvature singularity is still avoided, as it also is in the baby universe sector \( (\rho < \mu) \) due to the periodic nature of the instantonic time \( T \). The spacetime of the extended instanton covers the entire domain of the coordinate patch \( k = +1 \) and that of the baby universe is outside the two coordinate patches.

Euclidean continuation (3.1) on coordinate patch \( k = -1 \) leads to the same instantonic sections as for patch \( k = +1 \), but now \( T = iT \) corresponds to the section inside the horizon \( r = 2M \) up to \( r = M \), and \( r = -i\rho \), \( M = -i\mu \), \( ds = -id\sigma \) define the section outside the horizon \( \rho = 2\mu \), with \( \tau \) and \( T \) respectively rotating about \( r = 0 \) and \( 2\mu \), anticlockwise in both cases. Since the boundaries at constant radial coordinates on both sides of the Euclidean horizon have compact topology \( S^3 \times S^2 \), the geometries at the Euclidean horizon \( (r = 2M \) and \( \rho = 2\mu) \) can also be identified, leading to an instanton which covers the entire coordinate patch \( k = -1 \).

On any \( \tau - r \) plane in the coordinate patch \( k = +1 \) we can define the amplitude \( \langle \tau_2 | \tau_1 \rangle \) to go from the surface \( \tau_1 \) to the surface \( \tau_2 \) which is dominated by the action \( I_1(M) = 4\pi M^2 \), corresponding to the circular sector limited by the times \( \tau_1 \) and \( \tau_2 \) on a circle centered at \( r = 2M \) with large radius \( r_0 \gg 2M \). Similarly, on the \( t - \rho \) plane in the patch \( k = +1 \) the amplitude \( \langle t_2 | t_1 \rangle \) to go from the surface \( t_1 \) to the surface \( t_2 \) is dominated by the action \( I_2(\mu) = 4\pi \mu^2 \) that corresponds to the sector limited by times \( t_1 \) and \( t_2 \) from \( \rho = \mu \) to \( \rho = 2\mu \) on a circle centered at \( \rho = 0 \). An asymptotic observer in patch \( k = +1 \) would interpret these results as providing the probability of the occurrence in the vacuum state of, respectively, a black hole with mass \( M \) or a white hole with mass \( \mu \). In the coordinate patch \( k = -1 \), the same observer would reach the same interpretation but for a black hole with mass \( \mu \) or a white hole with mass \( M \).

All of the above discussion can be readily extended to the case of the hole kinks in five dimensions. We then have similar instantons which contain both all points with \( r \geq M \) for \( T = iT \) and all points with \( \rho \geq \frac{M}{\sqrt{2}} \) for \( r = -i\rho \), \( M = -i\mu \), and \( ds = -id\sigma \) on the coordinate patch \( k = +1 \), and all points with \( \rho \geq \mu \) for \( r = -i\rho \), \( M = -i\mu \), and \( ds = -id\sigma \) and all points with \( M > r \geq \frac{M}{\sqrt{2}} \) for \( T = iT \) on the coordinate patch \( k = -1 \).

Any constant time section of these instantons (Fig. 2) would represent the half of respectively a three- and a four-dimensional wormhole either in the first or the second coordinate patch. The connection of one such wormhole halves in the first patch to other wormhole half in the second patch would take place on an equatorial surface \( r = A \) and produce a complete wormhole with two original asymptotic regions, one in patch \( k = +1 \) and the other in patch \( k = -1 \). This connection will be considered in more detail in Sec. V.

### IV. QUANTUM THEORY OF BLACK-HOLE KINKS

#### A. Conformal Structures and Foliations

In Fig. 3 we show the Penrose diagram for the four-dimensional black-hole kink, resulting from glueing at the identified surfaces on \( r = M \) the two coordinate patches, both on the original and new, unphysical regions. The spacelike Cauchy surfaces at constant time \( T = t_0 \) are labeled \( \Sigma^\pm_{p/v} \), depending on whether the original (physical, \( p \)) or new (unphysical, \( u \)) region of the first (+) or second (-) patch is being considered. These surfaces would correspond to Einstein-Rosen bridges [14] with topology \( R \times S^2 \) in each coordinate patch. A similar Penrose diagram can also be constructed for the case D=5.

For Lorentzian four-dimensional black-hole metric in the first coordinate patch \( (t_0 \) real), the Kruskal coordinates \( U, V \) are both pure imaginary for \( r < 2M \), and both real otherwise. Thus, starting at spatial infinity of the physical region, the \( t_0 \)-surfaces follow their way first as \( \Sigma^+_p \) or \( \Sigma^-_p \), approaching the bifurcation point \( U = V = 0 \). At that point, before continuing through the unphysical region, the surfaces can be analytically continued into the upper \( (k = +1) \) or lower \( (k = -1) \) imaginary plane \( U^*, V^* \), where \( U \rightarrow iU^*, V \rightarrow iV^*, 0 \leq U^*, V^* \leq 1 \), as the surface sector \( \Sigma^+_p \) or \( \Sigma^-_p \) to finally reach the upper \( (k = +1) \) or lower \( (k = -1) \) surface \( r \rightarrow A \); the surfaces then return to the bifurcation point through the lower \( (k = +1) \) or upper \( (k = -1) \) imaginary \( U^*, V^* \)-plane as the sector \( \Sigma^+_u \) or \( \Sigma^-_u \), and complete finally their itinerary along the real unphysical region as \( \Sigma^+_u \) or \( \Sigma^-_u \) (Fig. 3). This itinerary would require the identification of points on the upper surface at \( r = A \) with the corresponding points (obtained by reflecting in the bifurcation point \( U = V = 0 \)) on the lower surface also at \( r = A \), in each coordinate patch. It is worth noting that these identifications give rise to the kind of periodicity that would imply Hawking thermal effect [28] in each patch, which becomes now a mathematical requirement of the present model. Thus, since time \( \tilde{t} \) enters the Kruskal coordinates \( U, V \) in the mathematical form of the dimensionless exponent \( \frac{\hat{t}}{4\kappa M} \) for \( D=4 \), one should generalize expression (2.20) to read

\[
\tilde{t}_g = \tilde{t} + 2\pi \kappa ki M (1 - \kappa), \tag{4.1}
\]

where \( \tilde{t} \) is given by (2.20) and \( \kappa = \pm 1 \). For \( \kappa = -1 \), the points \( (\tilde{t} - 4\pi \kappa i M, \theta, \phi) \) on each patch are in fact the points on the same patch, obtained by reflecting in the bifurcation point, while keeping the Kruskal metric real and unchanged.

On the extended instanton studied in the precedent section, each surface

\[
\Sigma^\pm = \Sigma^\pm_p \cup \Sigma^\pm_u \tag{4.2}
\]

is spacelike everywhere in each coordinate patch, as their slope does not change sign along the paths. Surfaces
\( \Sigma^\pm \) allow therefore a complete foliation in each patch separately. Each of the \( \Sigma^+ \) represents a wormhole half (an Einstein-Rosen bridge plus half an internal throat) in the patch \( k = +1 \), and each of the \( \Sigma^- \) represents a similar wormhole half in the patch \( k = -1 \). The discrete isometry \( U \rightarrow -U, V \rightarrow -V \) transforms \( \Sigma^+ \) into itself, so that asymptotically flat physical regions are mapped onto asymptotically flat unphysical regions, and physical wormhole necks are mapped onto unphysical wormhole necks.

The points at given constant time \( T \) on the minimal surfaces at \( r = M \) of these wormhole halves in either patch cannot be joined to the corresponding minimal surfaces at the same constant time \( t_0 \) in the other patch. Therefore, although each coordinate patch admits a complete foliation by spacelike surfaces of constant \( T = t_0 \) on the instanton, it is not possible to foliate the entire spacetime corresponding to the complete Penrose diagram formed by joining the diagrams for the two patches with spacelike curves of constant \( T \).

One still might try a foliation in terms of surfaces of constant time \( t \) which is defined so that

\[
dT = dt + \tan 2\alpha dr.
\]

Then, by integrating (4.3) using the change of variable \( p = k \cot \alpha \) and \( \sin \alpha = \sqrt{\frac{M}{r}} \), we obtain

\[
\bar{t} = t + 4M \cot \alpha - Mk \csc^2 \alpha + 4Mk \ln \left( \left( \frac{\sin \alpha - k \cos \alpha}{\sin \alpha + k \cos \alpha} \right)^2 \right) + 2Mk\kappa (1 - \kappa) \pi,
\]

with \( \kappa = \pm 1 \) as in (4.1).

In terms of time \( t \), metric (2.4) is transformed into metric (2.5) which is not static and does not explicitly distinguish the two coordinate patches.

There are Cauchy surfaces of constant \( t \), both on the physical and unphysical regions. These surfaces go from spatial infinity to the surface \( r = M \), after crossing the horizon at \( r = 2M \) (Fig. 3), but are not spacelike everywhere as their slopes change sign at the event horizon. However, once the surfaces have reached the horizon from the physical region, they can be continued first along such a horizon, passing over the bifurcation point \( U = V = 0 \), until they intersect the crossings of the corresponding surfaces in the unphysical region with the horizon, to go then along the latter surfaces into unphysical spatial infinity. The resulting spacelike surfaces, \( \Sigma^{\pm}_t \), can therefore foliate the entire exterior regions of both coordinate patches, but not their interior regions.

A completely parallel treatment can be made starting with the Lorentzian five-dimensional black hole, taking now \( t_0 \) as a real quantity for \( k = +1 \). Also in this case the spacetime can be foliated on the instanton in each coordinate patch separately by surfaces of \( T = t_0 \) constant. Similarly, foliation by surfaces of constant \( t \), which enters a metric like (2.5) for a three-sphere, instead of a two-sphere, is only possible in the exterior regions of each coordinate patch, but not in their interiors.

### B. Pure Quantum States

Quantum states can be introduced only on those spacetime regions which are foliable by a family of spacelike surfaces [29]. However, neither the Cauchy surfaces \( \Sigma^\pm \) nor the Cauchy surfaces \( \Sigma^\pm_u \) can foliate the full spacetime which is covered by the coordinate patches when joined on the surface at \( r = A \), with \( A = M \) for \( D=4 \) or \( A = \frac{M}{\sqrt{2}} \) for \( D=5 \). Then, there could not exist well-defined quantum states for the whole of this spacetime construct, but only for each of the two spacetime patches separately.

Let us consider those situations for which the Kruskal coordinates \( U, V \) are both imaginary for \( A \leq r \leq r_h \), where either \( A = M \), \( r_h = 2M \), or \( A = \frac{M}{\sqrt{2}} \), \( r_h = M \), depending (on whether the \( D=4 \) case or the \( D=5 \) case is being considered. These situations would correspond to \( t_0 \) being complex with imaginary part \( 2\pi M \) in the \( D=4 \) case, and to \( t_0 \) being real in the case \( D=5 \). From the perspective of an observer in each of the asymptotic regions, one should divide the full spacetime at \( r = A \) in two disconnected patches. Each of the two resulting black holes, long after it is formed by an independent collapse will possess states which are describable by a static geometry and small perturbations. In addition to one of such physical black holes, there will also be at late times an unphysical black hole obtained by maximally extending the first black hole spacetime. The maximally-extended solution for a static black hole is what is known as an eternal black hole [30]. In this case, the data on the Cauchy surfaces \( \Sigma^\pm_u \) cannot influence the black hole exterior and therefore any perturbations would propagate to the future entirely inside the horizon. Thus, the \( \Sigma^\pm_u \) will lie in the Lorentzian interior of the black hole. The boundaries \( \Sigma^\pm_u = \Sigma^\pm_u \cup \Sigma^\pm_p \) are Einstein-Rosen bridges.

If we considered Cauchy surfaces \( \Sigma^\pm \) with the topology of just an Einstein-Rosen bridge, without any continuation into the physical black-hole interior, then one could define quantum states on surfaces \( \Sigma^\pm \) which foliate just the regions with \( r \geq r_h \). Such states may be given as a pure-state wave functional satisfying a no-boundary condition on the event horizon, where we in fact have no singularity or edge. This wave function would represent some ground state for the black hole and be given in terms of the (D-1)-geometry and regular matter fields on surfaces \( \Sigma^\pm \) with topology \( R \times S^{D-2} \). It therefore could be expressed as an Euclidean path integral of Hartle and Hawking over D-geometries and space-time matter fields bounded by \( \Sigma^\pm \) and D-dimensional asymptotically flat

...
and empty infinity. Such a proposal has in fact been already advanced by Barvinsky, Frolov and Zelnikov [31].

However, the analytic continuation of the kink time $\bar{t}$ allows one to extend the Euclidean section beyond $r = r_h$ up to $r = A$, so that the Gibbons-Hawking instanton would correspondingly be extended to embrace all points with $r \geq A$. The half of this extended instanton is depicted in Fig. 2c. Now the surface $\Sigma_{1}^{\pm}$ becomes one boundary of the Euclidean manifold, and the boundary at $r = A$ is the asymptotically flat and empty infinity. Such a proposal has in fact been advanced by Barvinsky, Frolov and Zelnikov [31].

$r = A$ is an asymptotic (D-1)-surface embedded in the asymptotically flat spacetime [32]. Therefore, we must supplement it with an additional boundary term for $K_{3}^{\pm}$ defined on an asymptotic (D-1)-surface embedded in the asymptotically flat spacetime. This term transforms like the surface term in (4.6) for the two Euclidean subsections and therefore renders the full action finite [32].

Explicit computation of the path integral (4.5) would require specifying the integration measure in it. This can be generally achieved by computing the path integral as a sum of all contributions coming from all asymptotically flat (D-1)-metrics which satisfy the Fadeev-Popov gauge-fixing conditions which remove the gauge freedom associated with invariance of the full action under arbitrary change of spacetime coordinates [33]. We do not try this calculation here. The wave functions $\Psi_{1}^{\pm}$ will be obtained by a different procedure in the next subsection.

One can also introduce density matrices $\rho_{p/u}^{\pm}$ for the quantum state of a single black hole or wormhole half on each patch as a path integral analogous to (4.5) on the whole instanton, rather than half the instanton, by tracing over the values of the matter field on either the entire surfaces $\Sigma_{u}^{\pm}$ or $\Sigma_{p}^{\pm}$. We have

$$\rho_{p/u}^{\pm} = Tr_{p/u} |\Psi_{1}^{\pm}\rangle \langle \Psi_{1}^{\pm}|$$

Like for the wave functional (4.5), the density matrices (4.7) would represent a pure state on each coordinate patch.

V. NEUTRAL BLACK-HOLE PAIRS

Let us analyse the possibility for the formation of neutral black-hole pairs which is suggested by the continuity of the angle of tilt of the light cones at $\alpha = \frac{\pi}{2}$ on the surfaces $r = A$, connecting a coordinate patch to another. Such surfaces would thus be identified on the Kruskal diagrams of Fig. 1, both on the original regions and the new, unphysical regions created by Kruskal extension. Each point on $r = A$ simultaneously belonging to both coordinate patches represents a (D-2)-surface defined at the same value of time $\bar{t}$ and different values of times $T$ and $\bar{t}$ in the two patches. These identifications amount to the existence of bridges or throats with size $A$, connecting the physical (unphysical) interior of a black hole in patch
\( k = +1 \) to the physical (unphysical) interior of another black hole in patch \( k = -1 \). It will be argued later on in this section that, from the perspective of an observer in either of the asymptotic regions, \( \partial M^2_\infty \) or \( \partial M^2_\infty \), if the topological charge of the kink is positive, the whole spacetime of the above geometrical construct can be regarded as a black-hole pair formed up by a black hole with positive mass \( M \) in the coordinate patch of the observer, and an anti-black hole with negative mass \(-M\) in the patch where the observer is not. If we look at the coordinates of each patch as describing a single universe that contains at least one black hole, then for asymptotic observers in either patch it would follow that a neutral black-hole pair could exist provided each black hole in the pair is in a different universe, and that the Newtonian interaction between the two holes of a four-dimensional pair would be accelerating these holes away from each other with a maximum force

\[
F = \frac{M^2}{[(\delta r)_{k=+1 \rightarrow k=-1}^2]^{1/2}} = \frac{1}{4},
\]

where Eqn. (A.10) of the Appendix has been used.

To an observer in an asymptotic universe, any knowledge about the existence and properties of a black hole in another universe, or about the meaning of the relative motion associated with force \( F \) is classically forbidden, since classically the two black holes cannot be brought into the asymptotic universe of the observer. Such a knowledge may still be allowed however if, as it is actually the case, the compact region beyond the horizon for the asymptotic observer, running classically from \( r = 2M \) to the curvature singularity, becomes the support of an one-kink because such a region, which is here considered to be essentially quantum-mechanical, actually contains all of the information which is classically forbidden, including the black hole with negative mass. This hole can be thus quantum-mechanically brought into the asymptotic universe of the observer.

Two neutral black holes are expected not to be pair created if they are both in the same universe. Even spontaneous creation of neutral black-hole pairs driven by the effective cosmological constant in an inflationary universe should be expected to be highly suppressed [8].

From the discussion in Sec. II it follows that there could be no family of spacelike surfaces of constant time \( T \) able to foliate the whole of the spacetime of one such black-hole pairs. Since points on surfaces \( r = A \) simultaneously belonging to the two coordinate patches are labeled by equal values of time \( t \), but not \( T \) or \( t \), (D-1)-wormholes defined on \( T = \text{const.} \) surfaces \( \Sigma^\pm \) cannot be continued from one coordinate patch to another at any surface, unless topology change transitions, including non-simply connected surfaces that extend down to \( r = A \) with topology more complicated than that of surfaces \( \Sigma^\pm \), sharing in common only the asymptotically flat behaviour at infinities, also contribute the path integral. Consequently, cutting at \( r = A \) on any surface \( \tilde{\Sigma} = \tilde{\Sigma}^+ \cup \tilde{\Sigma}^- \) would not actually divide the corresponding wormhole manifold in two disconnected parts. Therefore, the instanton associated to the creation or annihilation of neutral black-hole pairs should necessarily contain contributions from all those topologies, other than that of \( \tilde{\Sigma} \), allowing for smooth transitions from one black hole in the pair to another; i.e.: such an instanton would correspond to the topology of \( \Sigma = \partial M^2_\infty \cup \Sigma_{\text{pair}} \cup \partial M^2_\infty \), with

\[
\Sigma_{\text{pair}} = \Sigma^+ \cup \left( \sum \sigma_i \right) \cup \Sigma^-, \quad (5.1)
\]

where the \( \sigma_i \)'s represent inner surfaces whose topology, labeled \( i \), is different from that for a Einstein-Rosen bridge. Then, one could not factorize the full quantum state of black-hole pairs or complete wormholes in a product of pure-state wave functions [34]. In particular, for a black-hole pair state \( P \), we would generally have \( P \neq \Psi^+ \Psi^- \).

### A. Thermal radiation

In order to investigate further on the nature of these black-hole pairs, let us at this point consider the process of thermal emission of one such pairs by first rewriting the analytical expression for \( t \) given by (4.4) for \( D=4 \) in a more convenient form. We note that one would still recover (2.9) from the general Kruskal metric [15]

\[
ds^2 = -2F(U,V)dUdV + r^2d\Omega_2^2 \quad (5.2)
\]

if we re-define the Kruskal coordinates as follows:

\[
U = \bar{U} \pm \kappa e^{i\pi r} \kappa e^{i\pi r} \left( \frac{2M - r}{M} \right) \quad (5.3)
\]

\[
V = \bar{V} \pm \kappa e^{-i\pi r} \kappa e^{-i\pi r} \quad (5.4)
\]

where

\[
\bar{t}_c = \bar{t} + 4\pi iMk\kappa, \quad (5.5)
\]

with \( \kappa \) as defined in (4.1) and (4.4), and \( \bar{t} \) taken to be the real part of (4.4). This choice leaves expressions for \( UV = \bar{U}\bar{V}, F, r \) and the Kruskal metrics (2.21) real and unchanged. For \( \kappa = -1 \) Eqns. (2.19) become, respectively, the sign-reversed to (5.3) and (5.4); i.e the points \( (\bar{t} - 4\pi iMk_1, r, \theta, \phi) \) on the coordinate patches of Fig. 1 are the points on the new regions \( III_{k_1} \) or \( IV_{k_1} \), on the same figure, obtained by reflecting in the origins of the respective \( U, V \) planes, while keeping metric (2.21) and the time \( t \) real and unchanged. This leads to identifications of hyperbolae in the new, unphysical regions \( III_{k_1} \) and \( IV_{k_1} \) with hyperbolae in, respectively, original regions \( II_{k_1} \) and \( I_{k_1} \).
The evolution of a field along null geodesics as those in Fig. 1 can be described using a quantum propagator. If the field is scalar with mass \( m \), such a propagator will be the one used by Hartle and Hawking [35] which satisfies the Klein-Gordon equation
\[
(\Box_g^2 - m^2)G(x', x) = -\delta(x, x').
\] (5.6)

We note [35] that for metric (2.21) the propagator \( G(x', x) \) will be analytic on a strip of width \( 4\pi M \) precisely that is predicted by the imaginary constant component of (5.5), thus without any need of extending time \( t \) into the Euclidean region. Then, following Hartle and Hawking [35], the amplitude for detection of a detector sensitive to particles of a given energy \( E \), in regions \( I_+ \) and \( II_- \), would be proportional to
\[
\Pi_E = \int_{-\infty}^{\infty} dt_\epsilon e^{-\epsilon i \Pi_{E+} G(0, \vec{R}; t_\epsilon, \vec{R})},
\] (5.7)
where \( \vec{R} \) and \( \vec{\bar{R}} \) denote, respectively, \((r', \theta', \phi')\) and \((r, \theta, \phi)\). Since time \( t_\epsilon \) (but not \( t \)) already contains the imaginary constant term which is exactly required for the thermal effects to appear, we need now not make the time \( t \) complex. From (5.5) and (5.7) one can then write
\[
\Pi_E = e^{i\pi Mk_1 k_2 E} \int_{-\infty}^{\infty} dt_\epsilon e^{-\epsilon i \Pi_{E+} G(0, \vec{R}; t_\epsilon + 4\pi i Mk_1 k_2, \vec{R})}.
\] (5.8)

Let us next consider a point \( x' \) on the hyperbola \( r = M \) of region \( II_+ \), corresponding to patch \( k = +1 \). Since such a hyperbola should be identified with the hyperbola at \( r = M \) of region \( I_- \) of patch \( k = -1 \), the point \( x' \) can be taken to simultaneously belong to the two patches at the same value of time \( t_\epsilon \). Then, one can draw null geodesics starting at \( x' \) which connect such a point with different points \( x \) on either the original region \( I_+ \) of patch \( k = +1 \) or the original region \( II_- \) of patch \( k = -1 \). In the first case, we obtain from (5.8)
\[
P_{a+}^{I+}(E) = e^{-8\pi ME} P_{a+}^{I+}(E),
\] (5.9)
where \( P_{a+}^{I+}(E) \) denotes the probability for detector to absorb a particle with positive energy \( E \) from region \( I_+ \), and \( P_{a+}^{I+}(E) \) accounts for the similar probability for detector to emit the same energy also to region \( I_+ \), in the coordinate patch \( k = +1 \) corresponding to the first universe. An observer in the exterior original region of patch \( k = +1 \) will then measure an isotropic background of thermal radiation with positive energy, at the Hawking temperature \( T_{BH} = (8\pi M)^{-1} \).

For the path connecting \( x' \) with a point \( x \) on the original exterior region \( II_- \) of patch \( k = -1 \) in the other universe, we obtain for an observer in this region,
\[
P_a^{II-}(-E) = e^{+8\pi ME} P_{a-}^{II-}(-E).
\] (5.10)

According to (5.10), in the exterior region \( II_- \) of the second universe there will appear as well an isotropic background of thermal radiation, also at the Hawking temperature \( T_{BH} \), which is formed by exactly the antiparticles to the particles of the thermal bath detected in region \( I_+ \). For \( k = +1 \) we obtain similar hypersurface identifications as for \( k = -1 \). In this case, the identifications come about in the situation resulting from simply exchanging the mutual positions of the original regions \( I_k \) and \( II_k \), for, respectively, the new regions \( III_k \) and \( IV_k \), on the coordinate patches of Fig. 1, while keeping the sign of coordinates \( U, V \) unchanged with respect to those in (2.19); i.e. the points \((t + 4\pi Mk_1, r, \theta, \phi)\) on the so-modified regions are the points on the original regions \( I_k \) or \( II_k \), on the same patches, again obtained by reflecting in the origins of the respective \( U, V \) planes, while keeping metric (2.21) and the time \( t \) real and unchanged. Thus, expressions for the relations between probabilities of absorption and emission for \( k = +1 \) are obtained by simply replacing region \( I_+ \) for \( IV_+ \), region \( II_- \) for \( III_- \), and energy \( E \) for \(-E\) in (5.9) and (5.10), so that the same Hawking temperature \( T_{BH} \) is obtained in all the cases. Hence, “observers” will detect thermal radiation with energy \( E < 0 \) on region \( IV_+ \), and with energy \( E > 0 \) on region \( III_- \), in both cases at the Hawking temperature.

An evaporation process in patch \( k = +1 \) would then follow according to which a black hole in a universe will disappear completely, taking with it all the particles that fell in to form the black hole and the antiparticles to emit radiation, by going off through the wormhole of throat radius \( M \) whose other end, which opens up in other universe described in the patch \( k = -1 \), is another black hole which can be regarded to have been formed in patch \( k = -1 \) from the collapse of massive antifermions, and also evaporated, giving off exactly the antiparticles to the particles of the thermal radiation emitted by the first black hole in the first universe [9]. One can then represent neutral black-hole pairs as instantons with zero total energy. An asymptotic observer in one universe would interpret each such pairs as being formed up by one black hole with positive mass \( M \) in the same universe, and one anti-black hole with the corresponding negative mass - \( M \) in a different universe, both holes being connected to each other by a wormhole with size \( M \), and accelerated away from each other with maximum force \( F = 1/4 \).

What such an observer would actually observe out from the pair is only either a black hole with positive energy or a wormhole mouth opening to the observer’s universe.

Interpreting that the black hole in the “other universe” has negative mass without violating the well-established positive-energy theorems [36] can still be consistently accomplished as follows. Let \((M, g_{ab})\) be a D-dimensional asymptotically flat spacetime and let \( \Sigma \subset M \) be a spacelike hypersurface with the pair \((\Sigma, g)\) describing a kink with topological charge \( kink(\Sigma, g) = \pm 1 \), such that \((\Sigma, h_{ab}, K_{ab})\) is an asymptotically flat initial data set for the first coordinate patch only, where \( h_{ab} \) and \( K_{ab} \) are the D-1 metric and the second fundamental form on the
hypersurface Σ. Then, one can generally define the classical ADM mass (total mass) by [37]

\[ E = \frac{1}{16\pi} \lim_{r \to \infty} \sum_{\nu=1}^{D-1} \int \left( \frac{\partial h_{\mu\nu}}{\partial x^\mu} - \frac{\partial h_{\mu\nu}}{\partial x^\nu} \right) N^\nu dA, \quad (5.11) \]

in which \( r \) is the radial coordinate, the integral is taken over a sphere of constant radius \( r \), and \( N^\nu \) is the unit outward normal to this sphere. Choosing Σ to be asymptotically orthogonal to the timelike Killing field makes (5.11) agree with the Komar definition of mass [38] as the total outward force that must be exerted by the asymptotic observer to keep in place a mass density distributed on the given spherical surface.

To the asymptotic observer in one coordinate patch it is then clear that any mass distribution on surfaces everywhere in the same patch should be defined by (5.11) because, respect to that observer, time must flow toward the future for \( \alpha \leq \frac{\pi}{2} \), so any mass distribution satisfies the positive-mass theorems. However, for spherical distributions of mass in the other patch, the same observer would interpret ADM mass to be the sign-reversed to the one given by (5.11): since for such an observer time at any surface in the other patch of the kink flows toward the past, he would interpret any ADM mass in the other patch of the kink flows to the future for \( \alpha \leq \frac{\pi}{2} \), and vice versa. Any asymptotic observer in either patch would therefore attribute a conserved vanishing mass to either a neutral black- or white-hole pair, or a complete wormhole or a baby universe living outside the spacetime of the kink.

B. Mixed quantum states

Let us return to the definition of the black-hole pair quantum state. Since cutting at \( r = A \) on any surface \( \Sigma = \Sigma^+ \cup \Sigma^- \) would not divide the corresponding (D-1)-wormhole manifold into two parts, for the state \( P \) of a black-hole pair one should take a nonfactorizable density matrix that describes a statistical mixed state. This can be given by an Euclidean path integral [34]

\[ P = \int_C Dg D\phi e^{-\frac{1}{\hbar}S}, \quad (5.12) \]

over the class \( C \) of D-geometries and matter field configurations \( \phi \) which are bounded by the prescribed (D-1)-data on \( \Sigma^+ \cup \partial M^+_A \) and the D-dimensional asymptotically flat and empty infinity \( \partial M^+_\infty \), and the orientation reverse of the (D-1)-data on \( \Sigma^- \cup \partial M^-_A \), and the D-dimensional infinity \( \partial M^-_\infty \), \( \Sigma^+ \cup \partial M^+_A \) and \( \Sigma^- \cup \partial M^-_A \) together forming the inner boundary \( \Sigma \cup \partial M_A \), and \( \partial M^+_\infty \) and \( \partial M^-_\infty \) together forming the asymptotic boundary \( \partial M_\infty \); i.e.: we choose as total boundary of (5.12) \( \partial M = \bar{\Sigma} \cup \partial M_\infty \cup \partial M_A = \partial M^+ \cup \partial M^- \). This nonfactorizable, mixed density matrix can be obtained as the real part of the propagator, \( K(+, -) \), from the wave function \( \Psi^+ \) on \( \partial M^+ \) at Euclidean time \( \tau_{k=+1} \) to the wave function \( \Psi^- \) on \( \partial M^- \) at Euclidean time \( \tau_{k=-1} \). Assuming that a black hole possesses a discrete energy spectrum labeled by the index \( n \), this propagator can be written in the form

\[ K(+, -) = \sum_n \Psi^+_n \Psi^-_n e^{-E_n \Delta \tau}, \quad (5.13) \]

where \( \Delta \tau \) is the time separation between the two boundaries \( \partial M^+ \) and \( \partial M^- \), and \( E_n \) denotes eigenenergy of the assumed discrete spectrum of the black hole. Now, since the two boundaries may have any time separation \( \Delta \tau \), in order to obtain the density matrix \( P \) one has to integrate propagator (5.13) over all possible values of \( \Delta \tau \). We restrict in what follows to the four-dimensional case. Then, according to our discussion in Secs. II and III, and corresponding to the Euclidean continuation of time \( \tilde{t} \) in (2.20) for the extended instanton of Fig. 2c, the time separation \( \Delta \tau \) turns out to be always complex and given by

\[ \Delta \tau = \tau_1 - iT_1 \quad (5.14) \]

when \( T_1 \) is expressed in terms of \( M \) rather than \( \mu \). Since the exponent in (5.13) is then real in \( \tau_1 \) and imaginary in \( T_1 \), one should allow \( \tau_1 \) to vary from 0 to \( \infty \) and \( T_1 \) to vary along an one-fourth of its period; i.e.: from 0 to \( 2\pi M \). Inserting (5.14) into (5.13) and integrating over \( \Delta \tau \) with these integration limits, we obtain for the real part of the resulting expression,

\[ P \propto \sum_n \frac{\Psi^+_n \Psi^-_n}{E_n}. \quad (5.15) \]

Eqn. (5.15) can be consistently interpreted by considering that the black holes in the pair are in the quantum state specified by the wave function \( \Psi_n \) with a relative probability

\[ P_n \propto \frac{1}{E_n}. \quad (5.16) \]

This relative probability is or is not positive definite and convergent depending on the values of the eigenergies \( E_n \) allowed by the black-hole spectrum. Since time \( \tau \) is periodic with period \( 8\pi M \), choosing any arbitrary origin for time \( \tau \), common in the two patches, one can always write \( \tau_1 = 8\pi M N + \delta \tau \), where \( N \) is any positive integer and \( \delta \tau \) any time difference between 0 and \( 8\pi M \) which, in turn, should induce integration of (5.13) between 0 and \( 8\pi M \), i.e. along one new complete time period. \( \tau_1 = \infty \) would therefore correspond to repeating an infinite number of times the full period \( 8\pi M \). Then, periodicity of \( \tau \) should imply that the final physical situations one would achieve either by going along all the equivalent complete
complex integration paths \( \Delta \tau \), or after completing the integration path just along the imaginary part, \( T_1 \), of such paths, must be identified. Integration of (5.13) over \( T_1 \) from 0 to \( 2\pi M \) for \( \tau_1 = 0 \) yields a density matrix

\[
P_R = \sum_n \frac{\Psi^+_n \Psi^-_n \sin^2 (\pi M E_n)}{E_n}.
\]  

(5.17)

Equalizing then (5.15) and (5.17) we obtain for the black-hole spectrum

\[
E_n^\pm = \pm \frac{1}{M} \left( n + \frac{1}{2} \right),
\]  

(5.18)

i.e. the black holes behave like though they were harmonic oscillators with positive or negative frequency \( \omega^\pm = \pm \frac{1}{M} \) where, according to the discussion on black-hole thermal emission above, one must choose the sign depending on whether the first (1) or second (-) patch is being considered, i.e.

\[
\omega^k = \frac{k}{M}.
\]  

(5.19)

The energy associated to these frequencies can be taken as an estimate of the minimum possible energy inside the black hole which is compatible with the quantum relation between energy and wavelength [39]. It is worth noticing that if one defines the angular frequency \( \Omega^k = \omega^k \) and disregards zero-point energy, the number of quantum states in a black-hole pair,

\[
N = \frac{M}{\Omega^+} - \frac{M}{\Omega^-} = 4\pi M^2,
\]

becomes the Bekenstein-Hawking entropy formula. A procedure to compute the Euclidean action for a neutral black-hole pair is given in the Appendix. Following this procedure we obtain the Bekenstein-Hawking entropy in a more rigurous fashion, and can interpret it as a count procedure we obtain the Bekenstein-Hawking entropy formula. A black hole which is compatible with the quantum relation between energy and wavelength [39], such a conventional quantum theory is still unable to describe overall evolution in neutral black hole pairs. Then, one would either recourse to the nonunitary Euclidean density-matrix formalism of Sec. VB, or use a generalized approach to quantum theory [40]. We shall consider here the latter approach.

The generalized quantum theory we shall use consists of: (1) a set of fine-grained histories which allow for the most refined possible description of the system. For this set we take sequences of sets of one-dimensional projections \( Q \) on every member of the foliating families of spacelike surfaces in the exterior regions of patches \( k = \pm 1 \) and \( k = -1 \). (2) Allowed sets of coarse-grained alternative histories which generally are partitions of the fine-grained sets into mutually exclusive classes \( \{ C_\alpha \} \). Here, simple examples [40] of such coarse-grained sets in a Heisenberg-like picture can be chains of projections \( Q \) in both patches for spacelike surfaces \( \sigma^k \)

\[
C^{k'}_\alpha = Q^{s_\alpha}_\alpha (s^{k'}_\alpha) \ldots Q^{1}_{\alpha_1} (s^{k'}_{\alpha_1}), \quad \sigma^{\pm/-} > / < \sigma_0^{\pm/-},
\]  

(5.22)

where the sign or relation in \( ./ \) depends on whether the first or second patch is being considered, the \( \sigma_0 \) are spacelike surfaces just before the interior regions, the \( \alpha \)'s denote [40] specifications of the particular alternative in the set ("yes" or "no"), and \( \iota \) expresses that there may be different sets at the different times \( T \) or \( t \) labeling the spacelike surfaces \( \sigma^k \). The projections satisfy [40]

\[
Q^\alpha_{\alpha', \beta} = \delta_{\alpha, \beta}, \quad \sum_{\alpha_\alpha} Q^\alpha_{\alpha, \iota} = I.
\]  

(5.23)

The set of all possible sequences gives a set of alternative histories \( \{ C^{k'}_\alpha \} \) which satisfy

\[
\sum_{\alpha} C^{k'}_\alpha = I.
\]  

(5.24)
The projections \( Q^\alpha_{\alpha} \) should evolve unitarily according to conventional quantum theory in the exterior regions, but do not in the interior regions. (3) Since we want to preserve causality, one would allow for a re-definition of the final Hilbert space for the whole system compatible with unitarity. The evolution through the interior regions should then be given by a nonunitary operator \( X \) which (i) is derived from a transition matrix given by a sum-over-histories \([40]\)

\[
\langle \phi^- (x), \sigma^- | \phi^+ (x), \sigma^+ \rangle = \int_{[\phi^+, \phi^-]} d\phi \exp (iS[\phi(x)]),
\]

(5.25)

where \( \phi^\pm \equiv \phi^k \) denotes a definite spatial field configuration on the spacelike surface \( \sigma^\pm \equiv \sigma^k \), and \( S \) is the Lorentzian action functional for the given matter field, (ii) connects alternatives in the exterior regions of patch \( k = +1 \) to those in the exterior regions of patch \( k = -1 \) in such a way that a whole history is \( k \)-sum-over-histories \([40]\) and matter fields bounded by \( \partial M \) and the four-dimensional asymptotically flat and empty infinity. Thus, a density matrix which incorporates a no boundary condition on spacelike surfaces in only the universe \( k = +1 \) can be constructed from \( \Phi_0^+ \) by tracing over the values of the fields on the unphysical regions, \( \phi^\alpha_{\infty} \), relative to the single universe

\[
\rho = Tr_{\nu} | \Phi_0^+ \rangle \langle \Phi_0^+ |
\]

(5.28)

which gives the kernel of the propagation amplitude from the initial state \( \phi_0 \) on the right asymptotic infinity to \( \phi_u \) on the left asymptotic infinity, in Lorentzian space.

VI. COSMOLOGICAL IMPLICATIONS

The most popular boundary condition for the universe was suggested by Hawking in 1981 \([42]\). It can be stated by saying that the boundary condition of the universe is that it has no boundary, and so expressed the view that the single universe should be self-contained. One can regard the no boundary proposal as the cosmological version of the feature that the Euclidean section of the Schwarzschild solution is nonsingular because it does not contain any points with \( r < 2M \) and shows no boundary at the event horizon. Beyond the horizon, black holes are thought to be connected through wormholes to other universes. The cosmological version of the Gibbons-Hawking instanton ignores any aspects altogether of the existence of such a connection, and therefore it entails the self-containment of the universe.

There exists a more direct analogy between the cosmological no boundary proposal and the complete Euclidean section of the black hole interior discussed in Sec. III. Restricting to the kinkless case, where we do not shop off inside the horizon along the spacelike hypersurface of constant radius \( r = A \), but allow completion of the Euclidean section up to \( r = 0 \) just for the coordinate patch \( k = +1 \) (see Fig. 2b), and continuing in \( r, M \) and \( ds^2 \) with the coordinate choice \( \xi = iz \) as in Sec. III, we in fact obtain a black hole interior without any boundary or singularity, even in the Lorentzian sector. The cosmological version of this kinkless black-hole interior would then correspond to a self-contained universe as far as only one coordinate patch is required to describe it.

However, restricting to the kinkless black-hole interior analogy, and hence to cosmological scenarios which are self-contained, implies the lossing of at least some of the quantum richness arising in the kink framework. We will therefore consider the more general boundary conditions which may be suggested from the analogy with the complete black-hole one-kink. The Euclidean continuation of the \( D=4 \) black-hole kink metric leads to an instanton which contains no points with \( r < M \), rather
than $r < 2M$. At $r = M$ there exists no singularity but there is a boundary connecting the universe where the hole has emerged to other universes. Since this boundary does not divide the whole manifold in that of a single universe and the rest manifold, the universe can therefore no longer be self-contained. This suggests a different alternate outcome to the problem of the boundary conditions of the single universe. It is that the boundary condition of the universe is that there is no boundary other than that for the throat of a wormhole, i.e. the boundary $\partial M^\pm$. In this approach the notion that there is just one single universe should be replaced by a new notion which could be regarded as a necessary previous step in the way to the modern quantum cosmological view that there exists one single metauniverse [43], that is the set of all possible incipient, inflating and large universes allowed by quantum cosmology. The proposed boundary condition would simply express the feature that although the whole of the metauniverse may well be self-contained, any single universe in it should not.

The no boundary wave function of Barvinsky, Frolov and Zelnikov [31] for the ground state of black holes corresponds to a modified version of the no boundary ground-state wave functional for asymptotically flat three-geometries proposed by Hartle [44]. Similarly, the state (3.6) would correspond to a modified version of the wave function for Euclidean quantum wormholes. Thus, parallelly to as the cosmological version of the Barvinsky-Frolov-Zelnikov or Hartle asymptotically flat wave functions are not but the familiar Hartle-Hawking prescription [45] for the quantum state of a compact universe, the cosmological version of (3.6) would correspond to a picture in which the path integral is over Euclidean four-geometries and matter field configurations on spacetimes which are no longer compact, but possess a boundary $\partial M^\pm$ that connects the single universe to the other objects that may exist in a unique, self-contained metauniverse. Once again, one would emphasize that this is simply a proposal which cannot be deduced from some other principle, but just suggested by the physics of black hole kinks.

**VII. CONCLUSIONS**

As a previous step to the quantization of black holes, this paper examines the spacetime geometry and associated causal structure of black-hole kinks in four and five dimensions by enforcing the complete domain of such spacetimes to be described in terms of Finkelstein-McCollum standard kink metrics. Two coordinate patches are then required to represent a complete one-kink in the studied cases. These patches are joined at a spacelike hypersurface which is an interior common surface of two black holes, each of which lies on a different coordinate patch, without violating energy-momentum conservation or inducing mass-inflation processes that may destroy the causal structure associated with the considered geometry. We call the resulting construct a neutral black-hole pair.

Nonsingular instantons for each of the black holes in a kinky pair have been constructed both outside and inside the event horizon, and purely imaginary finite actions have been evaluated on the respective complexified sections. We interpret these actions as a measure of the probability of finding the corresponding kink metrics in the vacuum state.

The quantum state of the black holes in one such pairs has been formulated by using the Euclidean path integral approach, with a multiply connected boundary at the interior spacelike surface where the two coordinate patches are identified. By evaluating the independent thermal-radiation properties of the two black holes in the pair with respect to an asymptotic observer in one patch, it is shown that the black hole in the other patch behaves like though if it were formed only by negative energy in an amount whose absolute value equals that of the positive-energy black hole which always is in the same patch as the observer. This is regarded to be a consequence from the essential quantum nature of the pair in relation with the rotation of the light cones in the one-kink, and circumvents therefore any violation of the classical positive-energy theorems.

The quantum state that corresponds to either a black-hole pair or, equivalently, a complete wormhole has been also formulated using the Euclidean approach. This state turns out to be a nonfactorizable, mixed density matrix, rather than a pure-state wave functional, expressible as a propagator for the single black-hole wave function from some hypersurface in one patch to another hypersurface in the other patch. The use of the periodic properties of Euclidean time enabled us to show that the quantum state of each black hole in the pair must be given in terms of harmonic-oscillator wave functions for a fundamental frequency which is positive if the black hole is in the same coordinate patch as the asymptotic observer, and negative otherwise. Since the spacetime of a black-hole pair admits no complete foliation, in formulating the quantum theory we have also employed a generalized formalism in terms of a decoherence functional.

Finally, we have discussed some possible cosmological implications of our black-hole kink model, suggesting that the boundary condition for a single universe which is not self-contained is that there is no boundary other than those corresponding to the cross sections of wormholes.

**ACKNOWLEDGMENTS**

This work was supported by DGICYT under Research Project No PB94-0107-A.
APPENDIX A: EUCLIDEAN ACTION OF THE SCHWARZSCHILD PAIR

It appears interesting to calculate an exact expression for the Euclidean action of a Schwarzschild pair. The issue would relate with the problem of the origin of the black hole entropy. Pair production appears to be independent of Planck scale physics and therefore it should be an unambiguous consequence from quantum gravity \[7\].

The amplitude for production of nonrotating, chargeless black hole pairs can be calculated in the semiclassical approximation from the corresponding instanton action which is

\[
S = -\frac{1}{16\pi} \int d^4x \sqrt{\eta} R - \frac{1}{8\pi} \int d^3x \sqrt{\eta} K, \quad (A.1)
\]

where \(K\) denotes the trace of the second fundamental form, and \(h\) is the metric on the boundary. We wish to find the contribution of the two black holes to action (A.1). This contribution will correspond to subtracting from (A.1) the flat metrics that the instanton asymptotically approaches.

Since variation of the tilt angle \(\alpha\) must be continuous at \(r = M\), the above contribution can be evaluated from the corresponding enforced variation in the time parameter \(r = M\) that leads to a net variation of radial coordinate \((\delta r)\) also on \(r = M\). This variation would be computed with respect to a given coordinate patch, as one passes from that patch to the other, and leaves unchanged the flat metrics. It would correspond to a variation of action (A.1), \(\delta S\), which should be finite when evaluated at \(r = M\). In what follows we shall interpret \(\delta S|_{r=M}\) as the contribution of the two black holes to the instanton action (A.1).

Let us then evaluate \(\delta S\). Using Einstein equations we find

\[
\delta S = -\frac{1}{8\pi} \int d^3x \delta(\sqrt{\eta} h^{ij} K_{ij})
\]

\[
= -\frac{1}{8\pi} \int d^3x \left( \sqrt{\eta} (\delta h)_{ij}(K_{ij} - \frac{1}{2} h_{ij} K) + \sqrt{\eta} h^{ij} \delta K_{ij} \right). \quad (A.2)
\]

The last term in the rhs of (A.2) can be written

\[h^{ij} \delta K_{ij} = \frac{1}{2} h^{ij} \frac{\partial}{\partial t} h_{ij} = \frac{1}{2} \frac{\partial}{\partial x^l} h^{ij} \delta h_{ij} = \frac{1}{2} \frac{\partial A'}{\partial x^l},\]

with \(A' = h^{ij} \delta h_{ij}\) a contravariant vector, and the overhead dot meaning time derivative. Hence

\[h^{ij} \delta K_{ij} = \frac{1}{2} \frac{1}{\sqrt{\eta}} \frac{\partial}{\partial x^l}(\sqrt{\eta} A'),\]

so that

\[
\int d^3x \sqrt{\eta} h^{ij} \delta K_{ij} = \frac{1}{2} \int d^3x \frac{\partial}{\partial x^l}(\sqrt{\eta} A').
\]

This can be now converted into an integral over \(A'\) extended to the 2-surface surrounding the boundary. Since variations of the field are all zero on the boundary, this term must vanish and we have finally

\[
\delta S = -\frac{1}{16\pi} \int d^3x \sqrt{\eta} (\delta h^{ij})(2K_{ij} - h_{ik} K). \quad (A.3)
\]

For the Schwarzschild metric, we then have

\[
\delta S_1 = \frac{M}{\pi} \left[ \frac{1}{24} \rho^{-\frac{1}{2}} + \rho^{-\frac{1}{2}} + \arctan(\rho^{\frac{1}{2}}) \right] \times
\]

\[
\int_0^\pi d\theta \int_0^{2\pi} d\phi(\delta r), \quad (A.4)
\]

where \(\rho = \frac{2M}{3} - 1\) and \(\eta = 1\) for \(r < 2M\) and \(\eta = i\) for \(r > 2M\). However, this variation of the action is still divergent on the asymptotic boundary, and one should supplement it with an additional variation, \(\delta S_0\), which corresponds to those terms in (A.4) that diverge at infinity for finite \(\delta r\). It can be seen that \(\delta S_0 = \frac{2M}{\pi} \arctan(\rho^{\frac{1}{2}})\), so that

\[
\delta S = \delta S_1 - \delta S_0 = \frac{M}{\pi} \left[ \frac{1}{24} \rho^{-\frac{1}{2}} + \rho^{\frac{1}{2}} \right] \times
\]

\[
\int_0^\pi d\theta \int_0^{2\pi} d\phi(\delta r). \quad (A.5)
\]

Then,

\[
\delta S|_{S^2} = \left(-\frac{M}{24} + M\right) \int_0^{2\pi} d\phi(\delta r), \quad (A.6)
\]

where \(S^2\) denotes the two-sphere on \(r = M\).

We evaluate variation \((\delta r)\) at \(r = M\) by considering null geodesics that cross each other at exactly the surface \(r = M\), going always through original regions on the Kruskal diagrams for the two patches. For such geodesics we have \(t_{k=+1} = \bar{t}_{k=-1}\). Thus, if in passing from the original regions of patch \(k = +1\) to the original regions of patch \(k = -1\) time \(t\) remains constant, time \(t\) must change on \(r = M\) according to (Ref. Eqn. (4.4))

\[
\delta t|_{r=M} \equiv \bar{t}_{k=+1} - \bar{t}_{k=-1}\big|_{r=M} = 2M. \quad (A.7)
\]

In order to calculate the corresponding rate \(\frac{dt}{dr}\) from patch \(k = +1\), we note that

\[
\frac{dt}{dr}|_{r=M} = \left(\frac{dt}{dr} + \tan 2\alpha - \frac{1}{\cos 2\alpha}\right)|_{r=M} = 0. \quad (A.8)
\]

Hence,
\[
\frac{dt}{dr}\bigg|_{r=M} = -1. \tag{A.9}
\]

We have then
\[
(\delta r)_{k=+1 \to k=-1} = \delta t\bigg|_{r=M} \frac{dr}{dt}\bigg|_{r=M} = -2M. \tag{A.10}
\]

Hence, from (A.6) we finally obtain as the exact expression for the Euclidean action of a black-hole pair for an observer in the asymptotic region of patch \( k = +1 \)
\[
I = \delta S|_{r=M} = \frac{\pi}{6} M^2 - 4\pi M^2. \tag{A.11}
\]

The semiclassical production rate is then
\[
e^{-1} \sim \exp(-\frac{\pi}{6} M^2 + 4\pi M^2). \tag{A.12}
\]

We interpret now the two factors in (A.12). For an asymptotic observer in patch \( k = +1 \), the factor \( e^{-\frac{\pi}{6} M^2} \) should give the full rate of Schwarzschild black hole pair production in the gravitational field created by a body with mass \( M \) made of fermions in a universe, and another body with the same mass, but having exactly the antifermions to the fermions of the first body, in the other universe. This factor should arise from the nonfactorizability of the quantum state of a pair. It would also give the probability for such a state to be factorizable, or in other words, the probability that after cutting the full manifold at \( r = A \) on any surface \( \Sigma = \Sigma^+ \cup \Sigma^- \), it is divided into two topologically disconnected submanifolds, each for a single black hole in one coordinate patch.

The second factor in (A.12) gives the entropy of the black hole, \( S_{BH} = 4\pi M^2 = \frac{1}{4} A_{BH} \), where \( A_{BH} \) is the surface area of a single black hole. This entropy can also be obtained from the black hole temperature \( T_{BH} \) by insertion into the thermodynamic formula \( T^{-1}_{BH} = \frac{\delta S_{BH}}{\delta M} \), or by using the instantonic procedure for single black holes described in Sec. III.

Consistency of the above interpretation can only be achieved if we look at the factor \( e^{S_{BH}} \) as a count of the number of physically relevant black hole internal states residing in between the event horizon and the interior surface at \( r = M \). Positiveness of the full exponent in (A.12) leads, on the other hand, to the remarkable feature that although the rate of pair production is maximum for Planck-sized black holes, once one of such pairs is formed, the semiclassical probability (A.12) will tend to favour processes in which the mass of the black holes increases endlessly.

---

[1] G ’t Hooft, Nucl. Phys. B256 727 (1985); V. Frolov and I. Novikov, Phys. Rev. D48 4545 (1993); S. Carlip and J. Uglum, Phys. Rev. D50 2700 (1994).
[2] V. Frolov, Black Hole Entropy, hep-th/9412211 (1994).
[3] D. Garfinkle and A. Strominger, Phys. Lett. 256B 146 (1991).
[4] D. Garfinkle, S. Giddings and A. Strominger, Phys. Rev. D49 958 (1994).
[5] H.F. Dowker, J. Gauntlett, S. Giddings and G.T. Horowitz, Phys. Rev. D50 2662 (1994).
[6] S.W. Hawking, G.T. Horowitz and S.F. Ross, Phys. Rev. D51, 4302 (1995).
[7] G.T. Horowitz, in: Matter of Gravity, No. 5 (1995) 10.
[8] S.W. Hawking and S.F. Ross, Duality Between Electric and Magnetic Black Holes, hep-th/9504014.
[9] S.W. Hawking, Mod. Phys. Lett. A5, 145; 453 (1990).
[10] S.W. Hawking, Phys. Rev. D37, 904 (1988).
[11] P.F. González-Díaz, Nucl. Phys. B351, 767 (1991); Phys. Rev. D45, 499 (1992).
[12] D. Finkelstein and C.W. Misner, Ann. Phys. (N.Y.) 6 230 (1959); D. Finkelstein, in Directions in General Relativity I, eds. B.L. Hu, M.P. Ryan Jr. and C.V. Vishveshwara (Cambridge Univ. Press, Cambridge, 1993).
[13] D. Finkelstein and G. McCollum, J. Math. Phys. 16 2250 (1975).
[14] A. Einstein and N. Rosen, Phys. Rev. 48, 73 (1935).
[15] P.F. González-Díaz, Phys. Rev. D51, 7144 (1995).
[16] G.W. Gibbons and S.W. Hawking, Phys. Rev. Lett. 69, 1719 (1992).
[17] A. Chamblin, Kinks and singularities, DAMTP preprint R95/44.
[18] S.W. Hawking, in General Relativity: An Einstein Centenary Survey, eds. S.W. Hawking and W. Israel (Cambridge Univ. Press, Cambridge, UK, 1979); G.W. Gibbons and S.W. Hawking, Phys. Rev. D15, 2752 (1977).
[19] R.M. Wald, General relativity (The University Chicago Press, Chicago, USA, 1984).
[20] K.A. Dunn, T.A. Harriot and J. G. Williams, J. Math. Phys. 35, 4145 (1994).
[21] S.W. Hawking and G.F.R. Ellis, The Large Scale Structure of Space-Time (Cambridge University Press, Cambridge, UK, 1973).
[22] S.W. Hawking, Phys. Rev. D13, 191 (1976); D14, 2460 (1976).
[23] E. Poisson and W. Israel, Phys. Rev. D41, 1796 (1990); A. Ori, Phys. Rev. Lett. 67, 789 (1991); 68, 2117 (1992).
[24] C. Barrabès and W. Israel, Phys. Rev. D43, 1129 (1991).
[25] J.J. Halliwell and R. Laflamme, Class. Quant. Grav. 6, 1839 (1989).
[26] F.R. Tangherlini, Nuovo Cim. 27, 636 (1963).
[27] F. De Felice and C.J.S. Clarke, Relativity on Curved Manifolds (Cambridge University Press, Cambridge, UK, 1990).
[28] N. Pauchapakesan, in: Highlights in Gravitation and Cosmology, eds. B.R. Iyer, A. Kembhavi, J.V. Narlikar and C.V. Vishveshwara (Cambridge University Press, Cambridge, 1988).
[29] J.B. Hartle, Spacetime Quantum Mechanics and the Quantum Mechanics of Spacetime, USSBTH 92-21. Lectures given at the 1992 Les Houches Ecole d’été, Gravitation et Quantifications, July 1992.
[30] N.D. Birrell and P.C.W. Davies, Quantum Fields in Curved Space, (Cambridge Univ. Press, Cambridge, UK, 1982).
[31] A.O. Barvinsky, V.P. Frolov and A.I. Zelnikov, Phys. Rev. D51, 1741 (1995).
[32] L.J. Garay, Phys. Rev. D44, 1059 (1991).
[33] C. Teitelboim, Phys. Rev. D28, 297 (1983).
[34] S.W. Hawking, in: 300 Years of Gravitation, eds. S.W. Hawking and W. Israel (Cambridge Univ. Press, Cambridge, UK, 1987).
[35] J.B. Hartle and S.W. Hawking, Phys. Rev. D31 2188 (1976).
[36] R. Schoen and S.-T. Yau, Commun. Math. Phys. 79, 231 (1981); E. Witten, Commun. Math. Phys. 80, 381 (1981).
[37] R. Arnowitt, S. Deser and C.W. Misner, in: Gravitation: An Introduction to Current Research, ed. L. Witten (Wiley, New York, USA, 1962).
[38] A. Komar, Phys. Rev. 113, 934 (1959).
[39] P.C.W. Davies, Rep. Prog. Phys. 41, 1313 (1978).
[40] J.B. Hartle, Phys. Rev. D49, 6543 (1994).
[41] A. Anderson, Phys. Rev. D51, 5707 (1995).
[42] S.W. Hawking, in: Astrophysical Cosmology: Proceedings of the Study Week on Cosmology and Fundamental Physics; eds. H.A. Brück, G.V. Coyne and M.S. Longeir (Pontificiae Academiae Scintiorum Scripta Varia, Vatican City, 1982), p. 563.
[43] A. Vilenkin, Phys. Rev. Lett. 74, 846 (1995).
[44] J.B. Hartle, Phys. Rev. D29, 2730 (1984).
[45] J.B. Hartle and S.W. Hawking, Phys. Rev. D28, 2960 (1983).
Legends for figures

• Fig. 1: Kruskal diagrams for the two coordinate patches \((k = \pm 1)\) of the one-kink extended Schwarzschild metric. Each of these patches is regarded as providing the coordinates which describe a different universe. Points on the diagrams represent 2-spheres. The null geodesic discussed in the text is the straight line labelled \(a_1a_2a_3\) on the diagrams. The hyperbolae at \(r = M\) are identified on, respectively, the original regions \((II_+ \text{ and } I_-)\) and the new regions \((III_+ \text{ and } IV_-)\) created by the Kruskal extension.

• Fig. 2: The half of the black-hole instantons described in the text as depicted in the coordinate patch \(k = +1\). (a) Half of the Gibbons-Hawking instanton. The Cauchy surface \(\Sigma^+ = \Sigma^+_p \cup \Sigma^+_u\) is one boundary and spatial infinity \(\partial M^+_\infty\) is another. \(\Sigma^+\) represents an Einstein-Rosen bridge. The amplitude \(\langle \tau_1 | \tau_2 \rangle\) to go from surface \(\tau_1\) to the surface \(\tau_2\) is given by the action of the shaded sector. The Barvinsky-Frolov-Zelnikov no-boundary wave function for the black-hole quantum state discussed in Sec. IVB is defined on this half instanton. (b) Half of the interior instanton, \(0 \leq r \leq 2M\), discussed in Sec. III. The Cauchy surface \(\tilde{\Sigma}^+ = \tilde{\Sigma}^+_p \cup \tilde{\Sigma}^+_u\) is one boundary of this Euclidean manifold and the surface at the horizon \(\partial M^+_h\) is another. The amplitude \(\langle T_2 | T_1 \rangle\) to go from the surface \(T_1\) to the surface \(T_2\) is given by the action of the corresponding shaded sector. If we were living in the interior of a black hole with the mass of the universe, the Hartle-Hawking no boundary wave function of the universe could be defined on this half instanton. (c) Half of the black-hole kink instanton discussed in Secs. III, IV and V, covering both the exterior and the compact interior region supporting the kink which is bounded at \(r = A\) from below. The disjoint union of the Cauchy surface \(\tilde{\Sigma}^+ = \Sigma^+ \cup \Sigma^+\) with the inner surface \(\partial M^+_A\), which is the maximal analytical extension of the Einstein-Rosen bridge, is one boundary of this kinky Euclidean manifold, and spatial infinity \(\partial M^+_\infty\) is another. The arguments for the wave function \(\Psi^+\) are the boundary values of the quantum fields on the asymptotically flat part and on the \(r = A\) part of the analytically extended Einstein-Rosen bridge \(\tilde{\Sigma}^+\). A similar half-instanton can be constructed in patch \(k = -1\). Identification of the surfaces \(\partial M^+_A\) of these two half-instantons can only be made if a topology change is allowed to occur by virtue of which a baby universe is branched off from the kinky spacetime in such a way that the submanifolds of the two patches are no longer topologically disconnected to each other (see Sec. VB). In the figure, \(\xi = 2\), \(r_h = 2M\) and \(A = M\) for \(D=4\), and \(\xi = \frac{1}{2}\), \(r_h = M\) and \(A = \frac{M}{\sqrt{2}}\) for \(D=5\). The subscripts \(p\) and \(u\) respectively denote physical and unphysical regions.

• Fig. 3: Penrose diagram of the Schwarzschild black-hole kink. On the figure, surfaces at \(r = M\) with ends at \(\iota^+\) in the two coordinate patches are identified. Surfaces at \(r = M\) with ends at \(\iota^-\) should also be identified on the two patches, so that we have a unique and periodic conformal diagram. The shaded interior regions correspond to imaginary values of the Kruskal coordinates only for the tilded surfaces \(\tilde{\Sigma}\). A similar Penrose diagram can also be constructed for the \(D=5\) Tangherlini black-hole kink. The meaning of the different symbols on the figure is given in Secs. IV and V.