Thermodynamics of Quasi Conformal Theories From Gauge/Gravity Duality

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We use gauge/gravity duality to study the thermodynamics of a generic almost conformal theory, specified by its beta function. Three different phases are identified, a high temperature phase of massless partons, an intermediate quasi-conformal phase and a low temperature confining phase. The limit of a theory with infrared fixed point, in which the coupling does not run to infinity, is also studied. The transitions between the phases are of first order or continuous, depending on the parameters of the beta function. The results presented follow from gauge/gravity duality; no specific boundary theory is assumed, only its beta function.

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1 Introduction

Gauge/gravity duality has been proposed to describe various aspects of SU($N_c$) gauge theories. The studies on thermodynamics \cite{1, 2, 3, 4} have mostly concentrated on QCD due to the existence of experimental data on ultrarelativistic heavy ion collisions and the existence of numerical lattice data \cite{5, 6, 7}. However, it is plausible that at larger energy scales other gauge theories may play a significant role. A prime example is Technicolor (TC) \cite{8} (for a review, see \cite{9}), which effectively replaces the fundamental scalar Higgs field by a $\bar{Q}Q$ composite.

The main purpose of this article is to lay a framework for the study of the thermodynamics of generic walking Technicolor-related theories within the framework of gauge/gravity duality. When the Standard Model (SM) degrees of freedom are included, this will, for example, be relevant for the expansion of the Universe through the electroweak phase transition. There is extensive literature on this in the framework of the Standard Model but only a limited amount within TC theories, on the thermal aspects \cite{10, 11, 12} and on relic dark matter \cite{13, 14}.

Of course, the details of the TC theory are unknown. A great advantage of the gauge/gravity duality approach is that this is not needed, duality is only used to compute expectation values in the boundary theory. This is also concretely manifest in applications of duality to condensed matter physics (see, for example, \cite{15}). The information on the boundary theory we will need is a knowledge of the beta function of its gauge coupling. This will contain a number of parameters which can be obtained if the dynamics of the underlying theory is known.

A general property of TC theories we shall assume is that they be of walking type. This assumes that there are two widely different energy scales, $\Lambda_T$ and $\Lambda_{ET} \sim 10^3 \Lambda_T$, between which the coupling constant of the theory evolves very slowly, the theory is almost conformal. Below $\Lambda_T$ and above $\Lambda_{ET}$ the coupling runs similarly to asymptotically free theories (like QCD). In (Extended) Technicolor theories $\Lambda_T = \Lambda_{TC} \approx 246$ GeV and $\Lambda_{ET} = \Lambda_{ETC}$. The need for walking behavior comes from the requirement that the successes of the Standard Model should not be spoiled: the contributions from new physics to the electroweak precision parameters should be small and the contributions to flavor changing neutral current interactions should be suppressed.

Concrete examples of walking TC theories with minimal matter content have been suggested in \cite{16, 17}. These have also been studied on the lattice \cite{18, 19, 20, 21, 22, 23, 24, 25, 28, 29}. Of course, it could be that the origin of flavor patterns and fermion masses is not Extended Technicolor (ETC) but some other extension of simple TC dynamics at higher energy scale \cite{30, 31, 32, 33}.

The beta-function ansatz we shall use is

$$\beta(\lambda) = -c\lambda^2 \frac{(1 - \lambda)^2 + e}{1 + a\lambda^3}, \quad \lambda = N_c g^2,$$

which is tuned to asymptotic freedom in the UV ($\lambda \to 0$) and to walking near $\lambda = 1$ if $e$ is small. A plot can be found in Fig. 8. The values of parameters $c, a, e$ will be described in detail later. The case $e = 0$ (for a plot, see Fig. 4) will play a special role: the theory then has an infrared fixed point (IRFP), already studied in this model in \cite{4}.
Using methods in [1, 2, 3, 4] we derive the intuitively viable and interesting finite temperature phase diagram shown later in Fig. 9. Namely, associated with the changes in the evolution of the coupling constant at $\Lambda_{ET}$ and $\Lambda_T$ we will find phase transitions with critical temperatures $T_{ET}$ and $T_T$, respectively. Above $T_{ET}$ we have deconfined partonic plasma consisting of (techni)quarks and (techni)gluons while below $T_T$ we have confined (techni)hadronic matter. Between these temperatures and stretching over a wide range, $T_T \ll T_{ET}$, we have a novel new phase of matter, quasi-conformal matter. This terminology is appropriate since over the extent of scales spanned by this phase we have $\beta(\lambda) \simeq 0$, and physics is almost conformal. For $e \neq 0$ the evolution never ends in the infrared fixed point and at least a small amount of conformality breaking will be always present.

Our approach here is 5 dimensional phenomenological bottom-up and the boundary theory is in a thermal state. Running couplings of the walking type at $T = 0$ have also been studied top-down starting from 10 dimensional supergravity solutions in [34, 35].

A beta function, of course, is scheme dependent. The consequences we derive, an equation of state and associated phase structure, are entirely physical. One can thus say that our model defines the regularisation scheme leading to the coupling constant and its beta function we start from.

The paper is organized so that in Sec. 2 we describe the gauge/gravity setup which we use. Our main analysis begins in Sec. 3 with a special case of the model beta function corresponding to a theory featuring a stable infrared fixed point, and in Sec. 4 we carry out the analysis for walking technicolor. In Sec. 5 we will discuss how our results can be interpreted within the phenomenological contexts of walking TC and ETC or unparticles, Sec. 6 contains conclusions.

2 The gravity dual

2.1 Equations

The gravity equations of the model are as follows [1, 2, 3, 4]. The model starts from a metric ansatz

$$ds^2 = b^2(z) \left[ -f(z)dt^2 + dx^2 + \frac{dz^2}{f(z)} \right]$$

plus a scalar field $\phi(z) = \log \lambda(z)$. The three functions $b(z), f(z)$ in the metric and the scalar field $\phi(z)$ in the gravity action (in standard notation)

$$S = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left[ R - \frac{4}{3} (\partial \mu \phi)^2 + V(\phi) \right]$$
are determined from the three equations (\(\dot{b} \equiv b'(z)\), etc.)

\[
6 \frac{\dot{b}^2}{b^2} + 3 \frac{\ddot{b}}{b} + 3 \frac{\dot{b} \dot{f}}{f} = \frac{b^2}{f} V(\phi),
\]

\[
6 \frac{\dot{b}^2}{b^2} - 3 \frac{\ddot{b}}{b} = \frac{4}{3} \phi^2,
\]

\[
\frac{\dot{f}}{f} + 3 \frac{\dot{b}}{b} = 0.
\]

Further, from the functions so evaluated, the beta function follows as

\[
\beta(\lambda) = b \frac{d\lambda}{db}, \quad \lambda(z) = e^{\phi(z)} \sim g^2 N_c.
\]

Thus \(\lambda(z)\) is the coupling and \(b(z)\) is its the energy scale. Note that the equation for the scalar field follows algebraically from (4)-(6).

We are interested in solutions which are asymptotically \((z \to 0)\) AdS_5, i.e., \(b(z \to 0) = L/z\) which have a zero at some \(z\), \(f(z_h) = 0\). These have an entropy and Hawking temperature \(4\pi T = -\dot{f}(z_h)\) and their field theory dual will be the thermal system we are searching for. When \(b = L/z\), (4) implies \(V(z = 0) = 12/L^2\), \(f(0) = 1\). Furthermore, we will always consider theories asymptotically free in the UV, i.e., \(\beta(\lambda \to 0) = -c\lambda^2\). Asymptotic freedom, asymptotic AdS_5 and (7) together imply \(dz/z = d\lambda/(-c\lambda^2)\), i.e.,

\[
\lambda(z \to 0) = \frac{1}{c \log(1/\Lambda z)},
\]

where \(\Lambda\) is a constant of integration of Eq.(7). One can, of course, add a 2-loop term in the beta function, but this will effectively just change \(\Lambda\) to a 2-loop \(\Lambda\). Note that (8) implies \(\phi(z \to 0) = -\log \log(1/z) \to -\infty\).

The constant \(\Lambda\) can also be related to the normalisation of \(b\) as follows. By integrating (7) one has

\[
\log \frac{b}{b_0} = \int_{\lambda_0}^{\lambda} \frac{d\lambda}{\beta(\lambda)} \to \log \frac{\mathcal{L}}{b_0 z} \approx \frac{1}{c \lambda}
\]

which together with (8) implies that

\[
\Lambda = \frac{b_0}{\mathcal{L}}.
\]

To solve Eqs. (4)-(6) one simplifies by introducing

\[
W = -\dot{b} / b^2.
\]

One ends up with the system of equations

\[
\dot{W} = 4bW^2 - \frac{1}{f} (W \dot{f} + \frac{1}{3} bV),
\]

\[
\dot{b} = -b^2 W,
\]

\[
\dot{\lambda} = \frac{3}{2} \lambda \sqrt{bW},
\]

\[
\ddot{f} = 3f \dot{b} W,
\]

(12)
which one can proceed to solve numerically when \( V \) is known. We do not know \( V \), we only
know the beta function. However, we can get a candidate \( V \) by first solving from (5) and (7)
\[
W(\lambda) = W(0) \exp \left(-\frac{4}{9} \int_0^\lambda d\lambda' \beta(\lambda) \right), \quad W(0) = \frac{1}{\mathcal{L}}.
\] (13)

Since the beta function is a property of the vacuum theory, \( f = 1 \), we can now find a candidate
potential simply by inserting (13) and (11) to (4) with \( f = 1 \):
\[
V(\lambda) = 12W^2(\lambda) \left[1 - \left(\frac{\beta}{3\lambda}\right)^2\right].
\] (14)

Apart from a slowly varying logarithmic term (see later Eq. 28), this is the potential we shall
use and see numerically that the solutions reproduce very accurately the beta function we
started from. For black hole solutions \( \beta(\lambda) \) as computed from (7) will show thermal effetcs
in the infrared, clearly, for example, \( \lambda < \lambda(z_h) \) (see later Fig. 8).

Expanding in the limit \( \lambda \to 0 \) with \( \beta(\lambda) = -c\lambda^2 \) one finds that \( V(\lambda = e^\phi) = 12/\mathcal{L}^2(1 + 
8c\lambda/9 + ...) \). This contrasts with many other scalar + gravity models having, at small \( z, V = 12/\mathcal{L}^2 - \frac{1}{2}m^2\phi^2 + ... \)

As an example, consider our model beta function (1). Integration of (13) gives
\[
\log[W(\lambda)/W(0)] = \frac{2c}{2\pi a} \left[2\sqrt{3}a^{1/3}(-2 + a^{1/3}(1 + e))\left(\arctan\left[\frac{-1 + 2a^{1/3}}{\sqrt{3}}\right] - \arctan\left[\frac{1}{\sqrt{3}}\right] \right) + 
\right. 
\left. a^{1/3}(2 + a^{1/3}(1 + e)) \log[(1 + a^{1/3}\lambda)^2/(1 - a^{1/3}\lambda + a^{2/3}\lambda^2)] + 2\log[1 + a\lambda^3]\right]
\equiv w(\lambda; c, a, e).
\] (15)

The limits of this rather untransparent expression near \( \lambda = 0,1,\infty \) are
\[
\log[W(\lambda)/W(0)] = \frac{4c}{9} \left[(1 + e)\lambda - \lambda^2 + \frac{1}{2}\lambda^3 + ...\right] \quad (16)
\]
\[
= \log w(1; c, a, e) + \frac{4ce}{9(1+a)} (\lambda - 1) + .. \quad (17)
\]
\[
= \frac{4c}{81a} \left[9\log(\lambda^{1/3}) + 2\pi\sqrt{3}(1 + e) \left(\frac{a^{1/3} - 2}{\lambda} + \frac{18}{\lambda} - \frac{9(1+e)}{2\lambda^2} + ...ight)\right]. (18)
\]

The general pattern is shown in Fig[1] linear increase at small \( \lambda \) and a transition to the
large-\( \lambda \) behavior \( W(\lambda) = \text{const} \cdot \lambda^{4c/(9a)} \), \( V(\lambda) = \text{const} \cdot \lambda^{8c/(9a)} \) around \( \lambda \sim 1 \).

2.2 Numerical integration

Numerical integration of (12) cannot start at \( z = 0 \) or at \( z_h, f(z_h) = 0 \). Instead, it is
convenient to start at some initial \( z = z_i = z_h - \varepsilon, \varepsilon = \text{some small number} \). Computing
analytically one finds that the initial values of various functions are
\[
\lambda_i = \lambda_h - \frac{3}{8} b_h^2 \frac{V'(\lambda_h)}{f_h} \varepsilon,
\]
\[
b_i = b_h + b_h^2 W_h \varepsilon,
\]
\[
W_i = W_h - \frac{1}{16 b_h^2} b_h^4 \lambda_h^2 (V'(\lambda_h))^2 \varepsilon,
\]
\[
f_i = f_h - f_h \varepsilon,
\]
\[
\dot{f}_i = \dot{f}_h - 3f_h b_h W_h \varepsilon,
\] (19)
Figure 1: The function $W(\lambda)$ for $c = 9, a = 6, e = 0.1, L = 1$. The dashed line is the large-$\lambda$ approximation (18).

where $\lambda_i \equiv \lambda(z_i), \lambda_h \equiv \lambda(z_h), \text{ similarly for the others. Among the five initial quantities at } z_h \text{ in (19) one firstly has } f_h = 0 \text{ by definition, the regularity of the } 1/f \text{ term in (12) requires}

$$W_h = \frac{b_h V(\lambda_h)}{3(-f_h)},$$

(20)

the value $b_h$ will be traded for the constant of integration $\Lambda$ in [8], the value of $\dot{f}_h$ will be traded for $T$ and, finally, $\lambda_h$ parametrises the different solutions.

Requiring that $-\dot{f}_h > 0 \text{ and } V'(\lambda_h) > 0$ one sees that $\lambda_i$ and $W_i$ are less than their values at horizon, the others are greater. This is as it should be since $\lambda(z), W(z)$ are monotonically increasing and $b(z), f(z)$ monotonically decreasing functions. Assume now we have a set of values of $\lambda_h, b_h, -\dot{f}_h$. For each of these a numerical integration of (12) produces a solution $\lambda(z), W(z), b(z), f(z)$ over some range $z_m < z < z_h, b(z_m)$ diverges. This solution is now processed as follows:

1. Scale $W(z_m)$ to the value one. Define a scaling factor $S_1 = W(z_m)$ and write $\lambda_1(z) = \lambda(z), W_1(z) = W(z)/S_1, b_1(z) = S_1 b(z), f_1(z) = S_1^2 f(z)$. This also scales $f_1(z_m) = 1$.

2. Shift $z_m$ to zero. Define $S_2 = z_m$ and write $\lambda_2(z) = \lambda_1(z + z_m), W_2(z) = W_1(z + z_m), b_2(z) = b_1(z + z_m), f_2(z) = f_1(z + z_m)$.

3. Scale $z$ so that for each solution, for any $\lambda_h$, Eq.(8) holds with some given $\Lambda$. The reason is that this is the only place where the solution is known analytically so that one use it to fix constants. We know that, for small $z$,

$$\lambda_2(z) = \frac{1}{-c \log(\Lambda_2 z)} = \frac{1}{-c \log(\Lambda(\Lambda_2 z/\Lambda))}. \tag{21}$$

Thus one sees that the proper scaling is $S_3 = \Lambda_2 / \Lambda$, leading to the new solution $\lambda_3(z) = \lambda_2(z/S_3), W_3(z) = W_2(z/S_3), b_3(z) = b_2(z/S_3)/S_3, f_3(z) = f_2(z/S_3)$. Here $\Lambda_2$ is the constant $\exp(-1/c \lambda_2(z))/z$. This set with index 3 is the final solution. Note that the value of $\lambda_h$ has remained unchanged and thus parametrises the solution.
Figure 2: Energy density and pressure/\(T^4\) for \(e = 0\) for a first order (left panel) and for a continuous transition (right panel). For a continuous transition also the interaction measure is given (multiplied by 2 for clarity); \(T_T\) then is defined as the location of its maximum. Parameter values are shown in the figure. The numbers needed to apply Eq. (26) to relate the \(T \to \infty\) and \(T \to 0\) limits are \(w(1; 11, 6, 0) = 3.998\), \(w(1; 9, 6, 0) = 3.108\).

Figure 3: Sound speed squared for \(e = 0\) for a first order (left panel) and for a continuous transition (right panel). For 1st order transition the dashed segments correspond to the supercooled and -heated branches in Fig. 2, the dotted segment to the unstable branch. Parameter values are shown in the figure. For a cross-over the dip in \(c_s^2\) and maximum of interaction measure need not coincide.

Thermodynamics can now be constructed as discussed in [2]. One chooses a numerical value for \(\Lambda\) (we used \(\Lambda = 1/200\)), a small UV value for \(\lambda(z)\) where [8] is valid (we used \(\lambda_{UV} = 0.02\)), a set of values for \(\lambda_h\) (we used \(0.021 < \lambda_h \lesssim 100\), for \(e = 0\) only values up to 1 are needed), integrates the equations as described above, calculates from the solution \(T = T(\lambda_h)\) and \(s = s(\lambda_h(T))\) and integrates \(p = p(T)\) from \(p'(T) = s(T)\). All the thermodynamics is then obtained from \(p(T)\) and its derivatives. Concrete expressions to be evaluated are in Eqs. (22)-(26) of [4].
Figure 4: The input beta function (parametrisation (1) with $e = 0$, continuous curve) and the output beta function, computed by inserting to (7) the numerical solution for $T = 0.55T_T$ or $T = 1.03T_T$ (black dashed curve) and for the smallest value of $T$ in the computation, $T \approx 0$ (dotdashed curve), for parameter values shown in the figure. The continuous and dotdashed curves coincide to the accuracy of the figure. The confinement line $- \frac{3}{2} \lambda$ is also shown.

3 Infrared fixed point

We begin with taking $e = 0$ in (1) and study the case where an infrared stable FP (IRFP) exists at $\lambda = 1$. The dilaton potential is given by (14), with $W$ as given by (15) with $e = 0$. Thermodynamics for this case has already been studied in [4] using a simpler beta function $\beta(\lambda) = -c\lambda^2(1 - \lambda)$. The approach towards $\lambda = 1$ is somewhat different, but the resulting thermodynamics is nevertheless qualitatively similar.

The coupling now runs from 0 in the UV to 1 in the IR. The theory is conformal in both ends and, accordingly, the transition is between black hole states in two asymptotically AdS$_5$ spaces with radii $L_{UV}$ and $L_{IR}$ defined by

$$\frac{1}{L_{UV}} = w(0; c, a, 0), \quad \frac{1}{L_{IR}} = w(1; c, a, 0),$$

where $w(\lambda; c, a, e)$ is given by (15). Due to the normalisation in (15),

$$\frac{L_{UV}}{L_{IR}} = w(1; c, a, 0) > 1.\quad (23)$$

The value of the parameter $a$ is inessential now that $\lambda$ is bounded, we take $a \approx 2c/3$ as later in the unbounded $e > 0$ case. The crucial parameter is $c$: the transition is of first order for large $c$ and weakens when $c$ is decreased. Below some threshold value the transition turns to a continuous one. The threshold value is approximately determined on whether the beta function crosses the confinement line $- \frac{3}{2} \lambda$; see Fig. 4.

Examples of the resulting equations of state and of the sound velocities squared $c_s^2 = dp/d\epsilon$ are shown in Figs. 2 and 3. For $c = 11$ one has a first order transition between a "gluonic" phase and between an "unparticle" phase, for $c = 9$ the transition is a cross-over, for $c \approx 9.95$
Figure 5: The temperature in units of $\Lambda$ computed for $c=3a/2=10$, $e=0.1$. The minima (dashed lines) are at $\lambda_h=0.248$ and $\lambda_h=44.9$. Crucial values are the two $\lambda_h$’s which have the same $T$ and pressure but are in different branches of decreasing $T$: these are $\lambda_h=0.173$ and $\lambda_h=0.617$. Finally, the quasiconformal $\rightarrow$ confining transition happens at $\lambda_h=13.81$: there $p$ goes to zero.

a 2nd order one. At $T \to \infty$ the model implies [4]

$$\frac{p}{T^4} \to \frac{L_{UV}^3 \pi^3}{4 G_5^4}$$

(24)

and the numerics is so normalised that

$$\frac{p}{T^4} \to \frac{\pi^2}{45}.$$  

(25)

The curves can simply be scaled to proper normalisation $p/T^4 = g_{\text{eff}} \pi^2/90$ if $g_{\text{eff}}$ is known. However, what is important is that (24) and (23) now imply that

$$\lim_{T \to 0} \frac{p}{T^4} = \frac{1}{w^3(1,c,a,0)} \lim_{T \to \infty} \frac{p}{T^4},$$

(26)

the effective number of degrees of freedom in the "unparticle" phase is determined by that in the "gluonic" phase and is less. The relevant numbers are $w(1;11,6,0)=3.998$, $w(1;9,6,0)=3.108$ and the explicit computations in Fig. 2 are seen to be in agreement with (26).

Since the potential $V(\lambda)$ was derived analytically from the $T=0$ field theory beta function [1], it is of interest to compare this input beta function with that one obtains by inserting numerically computed solutions for $\lambda(z)$, $b(z)$ to (7). These define a temperature dependent beta function. Fig. 4 shows the pattern: with decreasing $T$ the $T$-dependent beta function approximates the input beta function stepwise better and better until at $T \to 0$ the input beta function is obtained. The whole scheme is thus entirely consistent.
Figure 6: Energy density and $3p$ scaled by $T^4$ plotted vs $T/T_{ET}$ for the beta function (1) with $e = 0.1$, $c = 10$, $a = 2c/3$. The normalisation is such that at $T \gg T_{ET}$ both $\epsilon/T^4$ and $3p/T^4$ approach $\pi^2/15$. There is a first order transition at $T = T_{ET}$, metastable branches are dashed and the unstable branch is dotted. There is a second 1st order phase transition at $T = T_T \approx 0.0011T_{ET}$, the details are shown in the inset. In between, for $T_T < T < T_{ET}$ there is a quasiconformal phase with nearly constant $p/T^4$. The thick line below $T_T$ corresponds to the non-black hole low $T$ phase with $p = 0$.

4 Walking technicolor beta function

Consider now the beta function (1) for small but nonzero $e$. The potential $V(\lambda)$ is again evaluated from (14) with $W$ given by (15), but an additional modification is needed [1].

To fix the parameters we first note that, to model walking and quasiconformality, $e$ inherently must be some small number. We shall start with $e = 0.1$ and study the effects of varying $e$ at the end of this Section. Next, we shall choose $c/a = 3/2$. The motivation for this is that, as discussed in [36], a condition for confinement is that the equation

$$\beta(\lambda) + \frac{3}{2} \lambda = 0$$

have a solution. The existence of a $QQ$ condensate is a requisite for any TC model and at low $T$ confinement typically implies also the formation of a condensate. Thus we need confinement and (27) should have a solution at large $\lambda$. If $c/a = 3/2$, our beta function at large $\lambda$ is parallel to the line $\beta = -\frac{3}{2} \lambda$ and (27) does not have a solution at finite $\lambda$. However, as shown in [36], a solution is obtained if the potential (14) is modified by a logarithmic factor so that, at large $\lambda$, $V \sim \lambda^{4/3} (\log \lambda)^P$. At large $\lambda$ this effectively multiplies the beta function by a term $1 + 3P/(4 \log \lambda)$ and (27) has a solution. If $P = \frac{1}{2}$ one obtains a glueball spectrum with $M^2$ linear in a discrete index $n$. Though we now do not know what the analogue of the
glueball spectrum would be, we nevertheless stick to this property and take \( V \) to be
\[
V = 12W^2(\lambda) \left[ 1 - \beta^2/(9\lambda^2) \right] \left[ 1 + \frac{\alpha}{10} \sqrt{\log(1 + \lambda^4)} \right],
\]
(28)
with \( W \) computed from (13) and \( \beta \) in (1). The power \( \lambda^4 \) within the log is chosen so that it does not affect the leading small-\( \lambda \) behavior.

Thermodynamics computed from here for \( c = 3a/2 = 10, e = 0.1 \) is shown in Figs. 5-8. One again has a first order transition, and even two of them, for large \( c \). Crucial for this is that \( T(\lambda_h) \) have two minima, as shown explicitly in Fig. 5. We remind that \( \lambda_h \) parametrises numerical solutions of the gravity equations (4)-(6). Stable phases correspond to \( dT/d\lambda_h < 0 \) and with the structure in Fig. 5 one can have \( T \) equal in two different decreasing branches of \( T(\lambda_h) \), the precise location is determined by also pressure being the same in the two branches. Decreasing \( T \) from very large values one obtains a first order transition at \( T = T_{ET} \) with the 1st order structure shown in Fig. 6. Below \( T_{ET} \) there follows a long quasiconformal phase in which \( p/T^4 \) is almost constant. For the IRFP case in the previous Section this extended down to \( T = 0 \); now at about \( T \approx 0.01T_{ET} \) pressure starts decreasing and crosses \( p = 0 \) at some \( T = T_T = 0.001T_{ET} \). Below this the quasi conformal phase becomes metastable (see inset in Fig. 6) and the vacuum phase with \( p = 0 \) is the stable one. This is again a first order transition from a quasiconformal phase to a confining phase, the energy density \( \epsilon \) dropping suddenly to zero.

Fig. 7 shows the sound speed squared computed for the equation of state in Fig. 6. Finally, it is again of interest to compare the input beta function (1) with that what one obtains by inserting numerically computed solutions for \( \lambda(z), b(z) \) to (7). The outcome for this \( e \neq 0 \) case is shown in the left panel of Fig. 8. For finite \( T \) the computed beta functions terminate at some \( \lambda \), when \( T \) is decreased, the computed beta functions approximate the input one better and better. However, there is one small but very important difference: the input beta function (1) is parallel to the confinement line \( -3/2 \lambda \) at large \( \lambda \), but, due to the added logarithmic

![Figure 7: Sound speed squared for the equation of state in Fig. 6. The unstable region corresponds to \( c_s^2 < 0 \). \( T \) approaches the conformal limit 1/3 for \( T \gg T_{ET} \) and also near \( T_T \). Ultimately, as shown in the inset, \( c_s^2 \) drops there to zero.](image-url)
term in (28), the output beta function crosses the confinement line at the quasiconformal → confining transition. This explicitly shows the role played by this logarithmic term. The right panel of Fig. 8 shows the evolution of the coupling corresponding to the low $T$ output beta function of the left panel.

To outline the full phase diagram of this theory, we shall first choose fixed values in the first order transition region for the parameters $c$ and $a$. How the phase diagram then depends on the parameter $e$ which controls the departure from the limit of exactly infrared conformal theory is shown in Fig. 9. The phase structure along the $e = 0$ axis corresponds to the results of Sec. 3. At the origin, $T = e = 0$, the line of first order transitions, $T_T(e)$, terminates at a second order quantum transition, in which the vacuum theory becomes infrared conformal. At finite values of $e < \sim 0.5$ there are, as a function of temperature, three phases corresponding to confined, quasi-conformal and deconfined matter. When $e$ increases, the upper transition temperature $T_{ET}(e)$ decreases slowly, while the lower transition temperature $T_T(e)$ increases rapidly. At some critical $e_c$ these lines merge and the quasi-conformal phase disappears; for $e > e_c$ the phase structure will consist of a single transition line separating deconfined and confined phases. In the case depicted in the figure all transition lines correspond to a first order transition and hence the intersection is a triple point.

Full phase diagram would require adding more axes corresponding to $c$ and $a$. The inset in Fig. 9 shows for what values of $c$ the transition changes from a crossover to a 1st order one, for various $e$. 
Figure 9: The \((T, e)\) phase diagram for \(c = 3a/2 = 10\). The upper curve shows \(T_{ET}(e)\), the lower \(T_{T}(e)\), both normalized by \(T_{T}(0.1)\). The inset shows for what values of \(c\) for various \(e\) the upper ET transition is of first order, for \(c\) below the line the transition is a cross-over. The figure is qualitative in the sense that \(T_{ET}\) and \(T_{T}\) have been approximated by the values of the minima of \(T(\lambda_h)\). The line to the right of the triple point is the deconfinement transition line.

5 Discussion

Let us then discuss the implications of our results for lattice studies and cosmology. On the lattice one has only the generic TC gauge theory, in cosmology all the standard model degrees of freedom are involved.

In contrast to our approach here, where no microscopic dynamics is assumed but only the features present in the beta function of the theory affect the outcome, on the lattice one of course studies a specific SU\((N)\) gauge theory with some number of matter fields transforming under a given representation of the gauge group. In such theories, for given \(N\) and fermion representation, the departure from conformal behavior is controlled by the number of flavors. Generally then, the phase diagram in these cases is expected to be similar to the one in Fig. 9 via identification \(e \sim N_{f,\text{crit}} - N_f\), where \(N_{f,\text{crit}}\) denotes the critical number of flavors at which the SU\((N_c)\) gauge theory under consideration develops an infrared stable fixed point. As the number of flavors is decreased (\(e\) increases) the theory first becomes walking and then the conformal behavior vanishes. Lattice investigations of the \((T, N_f)\) phase diagram in any theory shown to possess an infrared stable fixed point at zero temperature for some \(N_{f,\text{crit}}\) would therefore provide more insight into the phases we have discussed in this paper.

Lattice studies of generic TC theories have so far only considered \(T = 0\). They involve all the complications of lattice fermions and are extremely demanding. Extending them to \(T > 0\) with asymmetric lattices will be all the more so.

To discuss consequences for cosmology we need to couple the strong dynamics with the SM fields. The possibilities are different for the cases of theories with walking coupling or an infrared stable fixed point, so we discuss them separately.
For the walking coupling the natural phenomenology framework is provided by TC. Then the scale \( \Lambda_T \) is identified with \( v_{\text{weak}} \approx 246 \) GeV and, if we assume ETC dynamics, the scale \( \Lambda_{ET} \) can be identified with \( \Lambda_{ETC} \). At high energies, above the scale \( \Lambda_{ETC} \), massless ETC gauge bosons mediate interactions between SM fermions and technifermions, while below \( \Lambda_{ETC} \) the ETC gauge bosons are massive due to dynamical symmetry breaking and Higgs mechanism associated with the ETC gauge symmetries. As a consequence, between the scales \( \Lambda_T \) and \( \Lambda_{ETC} \), the SM fields couple to the Technicolor theory via effective four-fermion operators

\[
\frac{g_{\text{ETC}}}{\Lambda_{ETC}^2} O_{\text{SM}} O_{\text{TC}}.
\]

Here the operators \( O_{\text{SM,TC}} \) represent bilinear fermion operators constructed from SM and TC fields.

At the scale \( \Lambda_T \) the TC dynamics result in the Higgs mechanism and the electroweak gauge bosons obtain their masses. In finite temperature this corresponds to the electroweak phase transition at \( T_T \). While the precise phenomenological details depend on the underlying gauge theory dynamics, the general features are as follows: The technifermions \( Q_L \) and \( Q_R \) are singlet under QCD but have usual electroweak quantum numbers. The electroweak symmetry is embedded into the global chiral symmetry of the technifermions so that the spontaneous breaking of this global symmetry results in correct breaking of \( SU_L(2) \times U_Y(1) \) into \( U_{em}(1) \). As the chiral symmetry of the techniquarks is spontaneously broken, TC is confined into TC singlet technihadrons. Note that this is an essential assumption in our approach: as we have described in detail, within the holographic framework we can identify the onset of confinement on the level of the beta function. Then, for the TC dynamics to operate as we imagine here, the underlying gauge dynamics have to be such that deconfinement and chiral symmetry restoration intertwine. Typically this is the case and the exception is the theory with adjoint fermions [37].

As a concrete model for TC dynamics one can therefore consider \( SU(N) \) with \( N_f \sim 4N \) as obtained from the ladder approximation and which is also compatible with recent lattice results [26]; however, see also [27]. With higher representations a possible model would be \( SU(3) \) with two flavors in the sextet representation. For \( SU(2) \) or \( SU(3) \) gauge theories with two adjoint flavors the dynamics is expected to be richer due to chiral symmetry restoration and deconfinement remaining as independent phase transitions with critical temperatures possibly widely separated.

The case of IRFP is related to very different phenomenology. Here we have only one scale, denote this by \( \Lambda_T \). Being exactly conformal in the infrared, the strongly coupled theory does not involve formation of a chiral condensate and hence cannot be used to break electroweak symmetry. Nevertheless, our results can in principle be applied to unparticle cosmology; see [39] for a brief review. Due to lack of knowledge on the form of the unparticle operators, precise value of the dynamical scale \( \Lambda_T \) and the nature of the Higgs sector we do not pursue the detail of this framework here.

Finally, we remark that the walking beta function [1] is a special case in a class of beta-functions parametrized as

\[
\beta(\lambda) = -\tilde{c}\lambda^2 + \frac{\lambda^2 + \tilde{f}\lambda + \tilde{e}}{1 + a_1\lambda + a_2\lambda^2 + a_3\lambda^3}
\]
In terms of this parametrization the ansatz in (1) corresponds to \( \tilde{c} = c, \tilde{f} = -2, \tilde{e} = 1 + e, a_3 = a \) and \( a_1 = a_2 = 0 \). Different parameter values lead to very similar results once the parameters are chosen so that the resulting beta-function has the three distinct regions corresponding to different evolutions of the coupling constant in regions separated by scales \( \Lambda_T \) and \( \Lambda_{ET} \). Therefore, for our investigation, the simple three parameter form (1) is convenient and sufficient. We have experimented with different functional forms which reproduce the essential features discussed in previous sections and lead to very similar results for finite temperature phase diagrams. We expect that the results we have obtained are generic for a walking-type beta-function.

6 Conclusions

We have, in this paper, studied the thermodynamics of a field theory with the beta function (1) containing a quasi conformal region in which the coupling varies very slowly, walks. As a limiting case, a theory with an infrared fixed point is also obtained. The basis of the computation is a bottom-up gravity dual with a metric ansatz and a dilaton. The results in this paper can be said to be a very concrete and productive application of gauge/gravity duality.

An essential property of the approach is that the details of the 4d boundary field theory need not be known, all of its properties are compressed in the beta function. This is reminiscent of the applications of gauge/gravity duality to condensed matter physics: there also the boundary theory is not written down, only expectation values are computed. The price one pays is that the beta function contains a number of unspecified parameters.

A central assumption in our analysis is that the knowledge of the functional form of \( \beta(\lambda) \) and the framework of [1] [2] [3] [4] provides a reasonable description of the thermodynamics of the underlying microscopic theory even though fermions in these theories are essential. The proper inclusion of flavor in various representations in the gauge/gravity correspondence setting, remains a challenge for theorists.

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References

[1] U. Gursoy, E. Kiritsis, L. Mazzanti and F. Nitti, “Holography and Thermodynamics of 5D Dilaton-gravity,” JHEP 0905, 033 (2009) [arXiv:0812.0792 [hep-th]].

[2] U. Gursoy, E. Kiritsis, L. Mazzanti and F. Nitti, “Improved Holographic Yang-Mills at Finite Temperature: Comparison with Data,” Nucl. Phys. B 820, 148 (2009) [arXiv:0903.2859 [hep-th]].
[3] J. Alanen, K. Kajantie and V. Suur-Uski, “A gauge/gravity duality model for gauge theory thermodynamics,” Phys. Rev. D 80, 126008 (2009) [arXiv:0911.2114 [hep-ph]].

[4] J. Alanen and K. Kajantie, “Thermodynamics of a field theory with infrared fixed point from gauge/gravity duality,” Phys. Rev. D, 81, 046003 (2010) [arXiv:0912.4128 [hep-ph]].

[5] G. Boyd, J. Engels, F. Karsch, E. Laermann, C. Legeland, M. Lutgemeier and B. Petersson, “Thermodynamics of SU(3) Lattice Gauge Theory,” Nucl. Phys. B 469, 419 (1996) [arXiv:hep-lat/9602007].

[6] B. Bringoltz and M. Teper, “The pressure of the SU(N) lattice gauge theory at large-N,” Phys. Lett. B 628 (2005) 113 [arXiv:hep-lat/0506034].

[7] M. Panero, “Thermodynamics of the QCD plasma and the large-N limit,” arXiv:0907.3719 [hep-lat].

[8] S. Weinberg, “Implications Of Dynamical Symmetry Breaking: An Addendum”, Phys. Rev. D 19, 1277 (1979); L. Susskind, “Dynamics Of Spontaneous Symmetry Breaking In The Weinberg-Salam Theory”, Phys. Rev. D 20, 2619 (1979).

[9] F. Sannino, “Conformal Dynamics for TeV Physics and Cosmology,” arXiv:0911.0931 [hep-ph].

[10] S. Nussinov, “Technocosmology: Could A Technibaryon Excess Provide A 'Natural' Missing Mass Candidate?,” Phys. Lett. B 165, 55 (1985).

[11] Y. Kikukawa, M. Kohda and J. Yasuda, “First-order restoration of SU(Nf) x SU(Nf) chiral symmetry with large Nf and Electroweak phase transition,” Phys. Rev. D 77, 015014 (2008) [arXiv:0709.2221 [hep-ph]].

[12] M. Jarvinen, T. A. Ryttov and F. Sannino, “The Electroweak Phase Transition in Ultra Minimal Technicolor” [arXiv:0903.3115 [hep-ph]]; J. M. Cline, M. Jarvinen and F. Sannino, “The Electroweak Phase Transition in Nearly Conformal Technicolor”, Phys. Rev. D 78, 075027 (2008) [arXiv:0808.1512 [hep-ph]].

[13] S. B. Gudnason, C. Kouvaris and F. Sannino, “Dark Matter from new Technicolor Theories”, Phys. Rev. D 74, 095008 (2006) [arXiv:hep-ph/0608055]; S. B. Gudnason, C. Kouvaris and F. Sannino, “Towards working technicolor: Effective theories and dark matter”, Phys. Rev. D 73, 115003 (2006) [arXiv:hep-ph/0603014].

[14] K. Kainulainen, K. Tuominen and J. Virkajarvi, “The WIMP of a minimal technicolor theory”, Phys. Rev. D 75, 085003 (2007) [arXiv:hep-ph/0612247]. K. Kainulainen, K. Tuominen and J. Virkajarvi, “Superweakly interacting dark matter from the Minimal Walking Technicolor,” JCAP 1002, 029 (2010) [arXiv:0912.2295 [astro-ph.CO]]. K. Kainulainen, K. Tuominen and J. Virkajarvi, “Naturality, unification and dark matter,” arXiv:1001.4936 [astro-ph.CO].

[15] M. Cubrovic, J. Zaanen and K. Schalm, Science 325, 439 (2009) [arXiv:0904.1993 [hep-th]].
[16] F. Sannino and K. Tuominen, “Techniorientifold,” Phys. Rev. D 71, 051901 (2005) [arXiv:hep-ph/0405209].

[17] D. D. Dietrich, F. Sannino and K. Tuominen, “Light composite Higgs from higher representations versus electroweak precision measurements: Predictions for LHC”, Phys. Rev. D 72, 055001 (2005) [arXiv:hep-ph/0505059]; D. D. Dietrich, F. Sannino and K. Tuominen, “Light composite Higgs and precision electroweak measurements on the Z resonance: An update”, Phys. Rev. D 73, 037701 (2006) [arXiv:hep-ph/0510217].

[18] S. Catterall and F. Sannino, “Minimal walking on the lattice”, Phys. Rev. D 76, 034504 (2007) [arXiv:0705.1664 [hep-lat]];

[19] L. Del Debbio, M. T. Frandsen, H. Panagopoulos and F. Sannino, “Higher representations on the lattice: perturbative studies”, JHEP 0806, 007 (2008) [arXiv:0802.0891 [hep-lat]].

[20] S. Catterall, J. Giedt, F. Sannino and J. Schneible, “Phase diagram of SU(2) with 2 flavors of dynamical adjoint quarks”, JHEP 0811, 009 (2008) [arXiv:0807.0792 [hep-lat]].

[21] A. J. Hietanen, J. Rantaharju, K. Rummukainen and K. Tuominen, “Spectrum of SU(2) lattice gauge theory with two adjoint Dirac flavours”, JHEP 0905, 025 (2009) [arXiv:0812.1467 [hep-lat]].

[22] A. J. Hietanen, K. Rummukainen and K. Tuominen, “Evolution of the coupling constant in SU(2) lattice gauge theory with two adjoint fermions”, Phys. Rev. D 80, 094504 (2009) [arXiv:0904.0864 [hep-lat]].

[23] Y. Shamir, B. Svetitsky and T. DeGrand, “Zero of the discrete beta function in SU(3) lattice gauge theory with color sextet fermions”, Phys. Rev. D 78, 031502 (2008) [arXiv:0803.1707 [hep-lat]]; T. DeGrand, Y. Shamir and B. Svetitsky, “Phase structure of SU(3) gauge theory with two flavors of symmetric-representation fermions”, Phys. Rev. D 79, 034501 (2009) [arXiv:0812.1427 [hep-lat]].

[24] L. Del Debbio, B. Lucini, A. Patella, C. Pica and A. Rago, “Conformal vs confining scenario in SU(2) with adjoint fermions”, [arXiv:0907.3896 [hep-lat]]; L. Del Debbio, A. Patella and C. Pica, “Higher representations on the lattice: numerical simulations. SU(2) with adjoint fermions”, [arXiv:0805.2058 [hep-lat]].

[25] Z. Fodor, K. Holland, J. Kuti, D. Nogradi and C. Schroeder, “Chiral properties of SU(3) sextet fermions”, [arXiv:0908.2466 [hep-lat]].

[26] T. Appelquist, G. T. Fleming and E. T. Neil, Phys. Rev. Lett. 100, 171607 (2008) [Erratum-ibid. 102, 149902 (2009)] [arXiv:0712.0609 [hep-ph]]; T. Appelquist, G. T. Fleming and E. T. Neil, Phys. Rev. D 79, 076010 (2009) [arXiv:0901.3766 [hep-ph]].

[27] Z. Fodor, K. Holland, J. Kuti, D. Nogradi and C. Schroeder, [arXiv:0911.2463 [hep-lat]].
[28] F. Bursa, L. Del Debbio, L. Keegan, C. Pica and T. Pickup, “Running of the coupling and quark mass in SU(2) with two adjoint fermions”, arXiv:0910.2562 [hep-ph].

[29] D. K. Sinclair and J. B. Kogut, “QCD thermodynamics with colour-sextet quarks,” arXiv:0909.2019 [hep-lat].

[30] E. H. Simmons, “Phenomenology of a Technicolor Model with Heavy Scalar Doublet”, Nucl. Phys. B 312, 253 (1989).

[31] M. Antola, M. Heikinheimo, F. Sannino and K. Tuominen, “Unnatural Origin of Fermion Masses for Technicolor,” arXiv:0910.3681 [hep-ph].

[32] M. Dine, A. Kagan and S. Samuel, “Naturalness in supersymmetry, or raising the supersymmetry breaking scale”, Phys. Lett. B 243, 250 (1990).

[33] M. Antola, S. Di Chiara, F. Sannino and K. Tuominen, “Minimal Super Technicolor,” arXiv:1001.2040 [hep-ph].

[34] C. Nunez, I. Papadimitriou and M. Piai, “Walking Dynamics from String Duals,” arXiv:0812.3655 [hep-th].

[35] D. Elander, C. Nunez and M. Piai, “A light scalar from walking solutions in gauge-string duality,” Phys. Lett. B 686, 64 (2010) [arXiv:0908.2808 [hep-th]].

[36] U. Gursoy, E. Kiritsis and F. Nitti, “Exploring improved holographic theories for QCD: Part II,” JHEP 0802, 019 (2008) [arXiv:0707.1349 [hep-th]].

[37] A. Mocsy, F. Sannino and K. Tuominen, “Confinement versus Chiral Symmetry,” Phys. Rev. Lett. 92, 182302 (2004) [arXiv:hep-ph/0308135].

[38] H. Georgi, “Unparticle Physics,” Phys. Rev. Lett. 98, 221601 (2007) arXiv:hep-ph/0703260.

[39] J. McDonald, “Unparticles: Interpretation and Cosmology,” arXiv:0805.1888 [hep-ph].