Exact analysis of the spectral properties of the anisotropic two-bosons Rabi model

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Abstract

We introduce the anisotropic two-photon Rabi model in which the rotating and counter rotating terms enters the Hamiltonian with two different coupling constants. Eigenvalues and eigenvectors are studied with exact means. We employ a variation of the Braak method based on Bogolubov rotation of the underlying $su(1, 1)$ Lie algebra. Accordingly, the spectrum is provided by the analytical properties of a suitable meromorphic function. Our formalism applies to the two-modes Rabi model as well, sharing the same algebraic structure of the two-photon model. Through the analysis of the spectrum, we discover that the model displays close analogies to many-body systems undergoing quantum phase transitions.

Keywords: quantum Rabi models, exact solutions, quantum technologies

(Some figures may appear in colour only in the online journal)

1. Introduction

Two-photon Rabi type models serve the study of various quantum effects in systems of bosonic fields coupled to a set of localized levels. Its simplest instance traces back to the theory of micromaser \cite{1, 2}. In such a quantum optics set up, three atomic levels are coupled each other through a two-boson field in a cascade transition $|g\rangle \rightarrow |i\rangle \rightarrow |e\rangle$, with
the energy difference $\omega_{eg}$ set to twice that of boson field frequency $\omega$, and the intermediate state being strongly detuned from $\omega_{ei}$ and $\omega_{ig}$. After adiabatically eliminating the intermediate state, and neglecting the Stark shift, one arrives at the two-photon Rabi model [3]. Phenomenologically, such models can describe two-level atoms interacting with squeezed light [4]. More recently, models of these type emerged in different contexts of quantum technology. Indeed, the results show them to be effective models for the description of quantum dots inserted in QED micro cavities [5]. For such systems, the localized levels are provided by the excitonic levels of the quantum dot; remarkably, two photon process in quantum dots were observed through photoluminiscence experiments [6]. The two-mode Rabi models also emerge in circuit QED involving superconducting qubits in the ultra strong regime in which non-linear couplings become realistic [7–9] and in ion traps [10, 11]. Finally, a two-mode boson field can be implied in the dynamics of a charged particle in a magnetic field [26].

In the weak coupling regime between the bosonic degree of freedom and the localized levels (that can be modelled by a spin degree of freedom), the physics of the system can be described by employing the so-called rotating-wave approximation (RWA) [12] that neglects the so-called counter-rotating terms in the Hamiltonian. In such a regime, eigen-energies and eigenstates of the two-boson Rabi model can be obtained by resorting to a similar logic working for the Jaynes–Cummings model. For the application of the algebraic Bethe ansatz to a slightly modified model (but with no counter-rotating terms) see [14].

In most, if not all, the scenarios depicted above the ‘physical working point’ is within the strong coupling regime, in which the RWA is a poor approximation. The limits of the RWA for the two-boson Rabi models were investigated numerically in [13].

Here, we deal with the exact energy spectrum and eigenstates of the two-photon Rabi model beyond the RWA. Recently, the exact eigenenergies and eigenstates of the system were obtained in the ‘isotropic’ case in which the rotating terms and counter-rotating terms enter the Hamiltonian with the same parameter [15–17]. The solution is obtained applying a variation of the procedure that Braak recently devised for the single photon Rabi model [18] (see also [19, 20]).

In this paper, we introduce a class of anisotropic version of the two-mode and two-photon Rabi model, where the rotating and counter rotating terms enter the Hamiltonian with two different parameters. As in the single-mode case, our model enjoys a $Z_2$ parity symmetry. Relying on such symmetry, we work out the exact solution of both the anisotropic two-photon and the two-mode Rabi model by applying the Bogolubov operator method [20, 21]. We will demonstrate that the energy levels of such models change as function of a suitable control parameter with a similar effects displayed in quantum phase transitions of many-body systems. A discontinuous transition is controlled by the anisotropic parameter. For higher values of the spin-boson strength, we find a condensation of levels breaking implying a super-radiance phenomenon.

The paper is organized as follows. In section 2, the mathematical implications arising from the relation between the $su(1, 1)$ Lie algebra and the two-photon Rabi model is discussed. In section 3, we detail our approach for the anisotropic two-photon Rabi model. In section 4, we discuss the kind of ‘critical behavior’ that we found in the energy spectrum of the model. In section 5, we draw our conclusions and future directions. In appendix A, we discuss a specific circuit QED realization of the model Hamiltonian. In appendix B, we sketch the application of our approach to the two-mode Rabi model.
2. The two-photon Rabi model: $su(1, 1)$ and $Z_2$ symmetry

The Hamiltonian reads
\[ H_{2\text{-ph}} = \omega a^\dagger a + \Delta \sigma_z + g(\sigma^+ a^2 + \sigma^- a^\dagger^2) + \lambda g(\sigma^+ a^2 + \sigma^- a^\dagger^2), \] (1)
where $\omega$ fixes the energy of the bosonic field and $g$ is the coupling constant, $\lambda$ being the anisotropic parameter characterizing the different roles rotating and counter-rotating spin-boson interactions play in the dynamics.

The features of the Hamiltonian above are intimately connected with $su(1, 1)$ Lie algebra realized by
\[ K_0 = \frac{1}{2}(a^\dagger a + \frac{1}{2}), \quad K_+ = \frac{1}{2}a^2, \quad K_- = \frac{1}{2}a^\dagger, \] (2)
and obeying to
\[ [K_0, K_\pm] = \pm K_\pm, \quad [K_+, K_-] = -2K_0, \] (3)
with invariant Casimir operator
\[ C = K_+ K_- + K_0 (1 - K_0) = \kappa (1 - \kappa), \] (4)
in which $\kappa$ labels the group representations. $K_\pm$ and $K_0$ are off-diagonal and diagonal operators in the Cartan basis, respectively. Relations (3) can be realized through first order differential operators acting on a suitable functional space:
\[ K_0 \leftrightarrow z \frac{d}{dz} + \kappa, \quad K_+ \leftrightarrow z^2 \frac{d}{dz} + 2\kappa z, \quad K_- \leftrightarrow \frac{d}{dz}. \] (5)

The isomorphism above can be seen as a ‘Bargman realization’ for $su(1, 1)$ (see for example [22]). We remark that $su(1, 1)$ is a non-compact algebra and therefore the representation space is a non trivial manifold. In the $su(1, 1)$ specific case, the representation space is isomorphic to an hyperboloid. Different representations correspond to a discrete series in the upper or lower branch of the hyperboloid, often denoted as $D^{\pm}_k$. In the case of the two-boson case considered here, $\kappa = \frac{1}{4}, \frac{3}{4}$ for even or odd eigenvalues of the Fock number operators, respectively (see the appendix). Another possible, but inequivalent, representation is the continuous one, labeled by the eigenvalues of $K_\pm$ [23]). As we shall see in the next sections, the computation of the exact solution of the model (B.1) involves the diagonalization (inner automorphisms) in $su(1, 1)$. We will choose a representation in which $K_0$ is diagonal. Correspondingly, certain constraints will emerge in the solution (see (15)). In appendix, we discuss the spectrum beyond such constraints.

The Hamiltonian (1) enjoys a $Z_2$ symmetry:
\[ \Pi_{2\text{-ph}} = e^{i \frac{\pi}{2}(\sigma^+ a^\dagger a + \sigma^- a^\dagger^2)} = -\sigma_z \left[ \cos \left( \frac{\pi}{2} a^\dagger a \right) + i \sin \left( \frac{\pi}{2} a^\dagger a \right) \right], \] (6)
whose eigenvalues belong to a four-dimensional manifold spanned by $p = \pm 1$ and $\pm i$. Such a manifold is defined by the four irreducible subspaces of the $su(1, 1)$ for two photons. Incidentally, we note that the model with $Z_2$ symmetry breaking, i.e. with $\sigma_z$ term added to the Hamiltonian, can also be solved by our method [24].

The following observation is important for the procedure to analyze the eigen-system of (1). Despite, the (unitary) transformation
\[ a^\dagger \rightarrow ia^\dagger, \quad a \rightarrow -ia \]  
(7)

is not a symmetry of the Hamiltonian, it defines an automorphism of su(1, 1). In this way, (7) induces a specific redundancy of the eigenvectors of (1), that we will exploit in our approach. Such issue is more evident by spelling out the spin basis in (1)

\[
(H) = \begin{pmatrix}
\omega a^\dagger a + \Delta & g(a^2 + \lambda a^2)\\
g(a^2 + \lambda a^2) & \omega a^\dagger a - \Delta
\end{pmatrix}
\]  
(8)

and applying a rotation of the basis: \[ W = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \]

\[ W^\dagger (H) W = \frac{1}{2} \begin{pmatrix} 2\omega a^\dagger a + g(1 + \lambda)(a^2 + a^2) & -2\Delta - g(\lambda - 1)(a^2 - a^2) \\ -2\Delta + g(\lambda - 1)(a^2 - a^2) & 2\omega a^\dagger a - g(1 + \lambda)(a^2 + a^2) \end{pmatrix}. \]

(9)

By inspection of the equation above, we see that its eigen-system

\[ W^\dagger (H) W \begin{pmatrix} v_1^\prime \\ v_2^\prime \end{pmatrix} = E \begin{pmatrix} v_1^\prime \\ v_2^\prime \end{pmatrix} \]  
(10)

is left invariant by applying the transformation (7) and swapping \( v_1 \) and \( v_2 \).

Summarizing: the \( su(1, 1) \) automorphism (7) implies the existence of two sets of eigenstates of (1): the original ones \( (v_1, v_2)^T \) and \( (v_2^\prime, v_1^\prime)^T \) where \( (v_1^\prime, v_2^\prime)^T = W^\dagger (v_1, v_2)^T \).

The role of (7) in the Bogolubov scheme that we will adopt below is played by the transformation \( z \rightarrow -z \) in the \( su(1, 1) \)-Bargman realization (5) of the spectral problem for (8). In this way, the redundancy property of the eigenvectors involves \( (v_1(z), v_2(z))^T \) and \( (v_2^\prime(-z), v_1^\prime(-z))^T \). Indeed, the resulting structure of the Hilbert space implies that the domain of analyticity of the eigenvectors \( (v_1(z), v_2(z))^T \) can be extended to the whole complex plane.

In the next section, we shall see how such a property will be exploited to construct a meromorphic function \(-\mathcal{G}(z)\)- whose analytical structure provides the spectrum of the Hamiltonian (1).

3. Exact analysis of the eigensystem

In this section, we will provide the exact analysis of spectrum and eigenstates of the anisotropic two-photon Rabi model equation (1). We will be formulating an ansatz for the eigenvectors expressed as a series expansion that we eventually determine through recurrence relations. It turns out that such recurrence relations are easier to solve if we preventively rotate the spin axes suitably: \( (H)^\prime = U^\dagger (H) U \), with

\[
U = \begin{pmatrix}
\cos \beta & -\sin \beta \\
\sin \beta & \cos \beta
\end{pmatrix}
\]

\[ (H)^\prime = \begin{pmatrix}
\omega a^\dagger a + p + r(a^2 + a^2) & -q + sa^2 + ta^2 \\
-q + sa^2 + ta^2 & \omega a^\dagger a - p - r(a^2 + a^2)
\end{pmatrix}
\]  
(11)

where \( p = \Delta \cos 2\beta \), \( q = \Delta \sin 2\beta \), \( r = \frac{\sin 2\beta}{2}(1 + \lambda)g \), \( s = (\cos^2 \beta - \lambda \sin^2 \beta)g \), and \( t = (\lambda \cos^2 \beta - \sin^2 \beta)g \).

Next, we perform a rotation in the \( su(1, 1) \) algebra by using a Bogolubov transformation [24],

\[
\alpha = ua + va^\dagger, \quad \alpha^\dagger = ua^\dagger + va,
\]  
(12)

where \( [\alpha, \alpha^\dagger] = 1 \) if \( u^2 - v^2 = 1 \). Because of the anisotropy, we notice that we cannot get rid of the non-Cartan generators of \( su(1, 1) \) in all the matrix elements. Nevertheless, we can fix
the parameters \(u, v\), and \(\beta\) in such a way that the upper (lower) off-diagonal element of \((H)\)' is a generalized lowering (raising) operator in \(su(1, 1)\). Such a choice will simplify the solution of the recurrence relations below (see (19), (20) and (22)). The Hamiltonian, then, reads

\[
(H)' = \begin{pmatrix}
\omega \eta (\alpha^1 + \frac{1}{2}) - \frac{\eta}{\sqrt{2}} + \Delta \cos 2\beta & -\Delta \sin 2\beta - r(1 - \lambda) \xi (2\alpha^1 + 1) + (1 - \lambda)\eta \beta^2 \\
-\Delta \sin 2\beta - r(1 - \lambda) \xi (2\alpha^1 + 1) + (1 - \lambda)\eta \beta^2 & \omega (\frac{\eta}{\sqrt{2}} - \eta (\alpha^1 + \frac{1}{2}) - \frac{\eta}{\sqrt{2}} - \Delta \cos 2\beta - \frac{\eta}{\sqrt{2}} (\alpha^2 + \alpha^1^2))
\end{pmatrix}
\]

with

\[
u = \frac{1 + \eta}{2\eta}, \quad \cos 2\beta = \frac{1 - \lambda}{1 + \lambda}, \quad (13)
\]

where

\[
\eta = \sqrt{\frac{1 - (1 + \lambda)^2 g^2 / \omega^2}{1 - (1 - \lambda)^2 g^2 / \omega^2}}, \quad (14)
\]

with the condition

\[
|g| < \frac{\omega}{1 + \lambda}. \quad (15)
\]

As announced in section 2, the constraint (15) arises because of the non trivial topology of the \(su(1, 1)\) representation space.

The Fock basis for the new bosonic operators \(\alpha\) is spanned by \(|m\rangle_{\alpha} = (\alpha^1)^m|0\rangle_{\alpha}, m = 0, 1, 2, ...,\) where the vacuum \(\alpha|0\rangle_{\alpha} = 0\) is

\[
|0\rangle_{\alpha} = \frac{1}{\sqrt{u}} \sum_{n=0}^{\infty} (-v/u)^n \sqrt{(2n)!} |2n\rangle. \quad (16)
\]

We observe that, although the normalization coefficient can be fixed by the series expansion of \(1/\sqrt{u}\), we work with non-normalized Fock basis (the normalization constant does not change the results).

We are now ready to formulate the ansatz for the eigenvectors of \((H)\)'

\[
(H)' \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = E \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad (17)
\]

\[
\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \sum_{m=0}^{\infty} L_m |m\rangle_{\alpha} \\ \sum_{m=0}^{\infty} K_m |m\rangle_{\alpha} \end{pmatrix}, \quad (18)
\]

where \(K_m\) and \(L_m\) are coefficients to be determined.

Plugging (18) into (11), we obtain,

\[
\sum_{m=0}^{\infty} f_m L_m |m\rangle_{\alpha} - \sum_{m=0}^{\infty} d_m K_m |m\rangle_{\alpha} + (1 - \lambda)g \sum_{m=0}^{\infty} m(m - 1)K_m |m - 2\rangle_{\alpha} = 0, \quad (19)
\]

\[- \sum_{m=0}^{\infty} d_m L_m |m\rangle_{\alpha} + (1 - \lambda)g \sum_{m=0}^{\infty} L_m |m + 2\rangle_{\alpha} + \sum_{m=0}^{\infty} \left[ \omega \left( \frac{\eta}{\sqrt{2}} - \eta \right) (m + \frac{1}{2}) - \omega \left( \frac{\eta}{\sqrt{2}} - p - E \right) K_m |m\rangle_{\alpha} - \frac{2\eta}{\sqrt{2}} \sum_{m=0}^{\infty} K_m |m + 2\rangle_{\alpha} + m(m - 1)(m - 2) \right] = 0, \quad (20)
\]
where $d_m = q + r(1 - \lambda)(2m + 1)\frac{\lambda}{2}$.

By inspection of equation (19), we obtain

$$L_m = \frac{d_m K_m - (1 - \lambda)(m + 2)(m + 1)gK_{m+2}}{f_m},$$

leading to a closed equation (20) for $K_m$:

$$a_m K_{m+2} = b_m K_m + c_m K_{m-2},$$

$$a_m = \left[ -\frac{d_m(1 - \lambda)}{f_m} + \frac{2r}{\eta} \right] (m + 2)(m + 1),$$

$$b_m = -\frac{d_m^2}{f_m} \left( \frac{(1 - \lambda)^2 m(m - 1)g^2}{f_{m-2}} + \omega \left( \frac{2}{\eta} - \eta \right) \left( m + \frac{1}{2} \right) - \omega' - p - E, \right.$$\n
$$c_m = \frac{(1 - \lambda)gd_{m-2}}{f_{m-2}} - \frac{2r}{\eta}. \quad (22)$$

With a similar logic employed in [18, 20, 24], we want to extract the eigenvalues of $H$ by looking at the analytical structure of a meromorphic function $G(z)$; such function is constructed by imposing that the eigenvectors of the Hamiltonian are indeed analytic in whole complex plane.

In our scheme, $G(z)$ is constructed resorting the property of the eigen-system under the application of the $su(1, 1)$ automorphism (7). Namely, the transformation (7) on the new bosonic operators $\alpha, \alpha^\dagger$ leaves the Hamiltonian’s spectrum invariant, with eigenvectors changing as

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = C \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = U^\dagger \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \quad (23)$$

Resorting to the parity symmetry operator $\Pi_{2-\text{ph}}$, in the even Fock space of $\alpha$, the constant $C = \pm 1$, while it is $C = \pm i$, for Fock states with odd parity. The transcendental functions in even and odd parity sectors are

$$G_{\alpha, \pm}^e = \langle 0 | [\varphi_2 \mp \varphi_1], \quad (24)$$

$$G_{\alpha, \pm}^o = \langle 1 | [-i\varphi_2 \mp \varphi_1], \quad (25)$$

where $G_{\alpha, \pm}^e$ is defined real by multiplying by the factor $-i$. Indeed, even and odd functions can be considered on equal footing multiplying by the overall factor $D_m, G_{\alpha, \pm}^e \sim G_{\alpha, \pm}^o$. Dropping the even and odd superscripts ‘o’ and ‘e’, we have,

$$G_{\alpha, +} = \sum_{m=0}^{\infty} (\cos \beta L_m + \sin \beta K_m)D_m,$$

$$G_{\alpha, -} = \sum_{m=0}^{\infty} (\sin \beta L_m + \cos \beta K_m)D_m. \quad (26)$$

Here

$$D_{2k} = \langle 0 | \alpha^{2k} | 0 \rangle_\alpha = \langle 0 | \varphi_2^{12k} | 0 \rangle_\varphi = \frac{1}{\sqrt{\pi}} \frac{(2k)!}{2^k k!} \frac{\nu^k}{\bar{\nu}^k},$$

$$D_{2k+1} = \langle 1 | \alpha^{2k+1} | 1 \rangle_\alpha = -i \langle 1 | \varphi_2^{12k} | 1 \rangle_\varphi = \frac{1}{\bar{\nu}^{k+2}} \frac{(2k + 1)!}{2^k k!} \frac{\nu^k}{\bar{\nu}^k},$$

$$\begin{align*}
D_{2k} &= \langle 0 | \alpha^{2k} | 0 \rangle_\alpha = \langle 0 | \varphi_2^{12k} | 0 \rangle_\varphi = \frac{1}{\sqrt{\pi}} \frac{(2k)!}{2^k k!} \frac{\nu^k}{\bar{\nu}^k},
D_{2k+1} &= \langle 1 | \alpha^{2k+1} | 1 \rangle_\alpha = -i \langle 1 | \varphi_2^{12k} | 1 \rangle_\varphi = \frac{1}{\bar{\nu}^{k+2}} \frac{(2k + 1)!}{2^k k!} \frac{\nu^k}{\bar{\nu}^k},
\end{align*}$$

where $d_m = q + r(1 - \lambda)(2m + 1)\frac{\lambda}{2}$.
where \( k = 0, 1, 2, \ldots \). The results are consistent with the isotropic two-photon Rabi model ones, \( \lambda = 1 \) [20].

We comment that the construction of \( G \)-functions as in (24) lies ultimately on the fact that the spectral problem for the model (8) can be recast (through a differential realization of \( su(1, 1) \)) to a differential equation of the Heun type. The analytical properties of the \( G \)-functions, giving in turn the eigenvalues of the model, correspond to specific conditions of analyticity that Heun functions must fulfill to be indeed well behaved solution of the spectral differential equation [18]. See [25] for the derivation of the \( G \)-functions as properties of the Heun functions.

As for the single-photon Rabi model [18], the poles of \( G(z) \) provide the eigenvalues of the uncoupled bosonic mode, \( \Delta = 0 \). This is obtained by putting \( a_m = 0 \) in equation (22):

\[
    E_{\lambda, m}^{\text{pole}} = \omega \eta' \left( m + \frac{1}{2} \right) - \frac{\omega}{2}
\]

where \( \eta' = \eta \left[ 1 - (1 - \lambda^2) \frac{g^2}{\omega^2} \right] \), and \( m = 0, 2, 4, \ldots \) for \( G_{\lambda}^+ \), \( m = 1, 3, 5, \ldots \) for \( G_{\lambda}^- \), respectively, as shown in figure 1. In the full fledged interacting case, the ‘regular spectrum’ are given by the zeros of \( G_{\lambda, \pm}^{\text{pole}} \) as \( E_{\lambda, \pm}^{\text{pole}} \). The ‘irregular spectrum’ providing the well known isolated integrability conditions in the parameter space \( \Delta, g \), known as Juddian points, entails \( K_{m+1}(E_{\lambda, m}^{\text{pole}}) = 0 \), for \( m = 0, 1, 2, \ldots \). We comment that the Juddian points can be equivalently obtained from equation (22) for \( a_m = 0 \): \( b_m K_m(E_{\lambda, m}^{\text{pole}}) + c_m K_{m-2}(E_{\lambda, m}^{\text{pole}}) = 0 \), holding for both odd and even \( m \) (implying, then, two different sets of coefficients). The Juddian isolated integrability points emerge as a degeneracy point between odd and even sectors.

The first Juddian solution can be obtained from \( b_0 = 0 \) and \( b_1 = 0 \),

\[
    \Delta = (1 - \lambda^2) \frac{g^2}{2\omega^2}, \quad (1 - \lambda^2) \frac{3g^2}{2\omega^2}.
\]

![Figure 1. Function \( G_{\lambda, \pm}^{\text{pole}} \) of two-photon anisotropic Rabi model for \( \omega = 1, \Delta = 0.2, g = 0.3, \lambda = 0.25 \) for even (a) odd (b) Fock subspaces, respectively. The blue (green) lines are \( G_{\lambda}^+ \) (\( G_{\lambda}^- \)), and red (purple) dashed lines are \( G_{\lambda}^- \) (\( G_{\lambda}^+ \)), the poles of \( G_{\lambda}^{\text{pole}} \) have labeled by dotted lines.](image-url)
For $m = 0, 1$, respectively. For the parameter in figure 2, $\Delta = 0.2$, $\lambda = 0.25$, we get $g_0 = 4\sqrt{2}/5 \approx 0.6532$, $E_{\text{pole}}^0 = 1/2 \sqrt{19/75} - 1/2 \approx -0.2483$, and $g_1 = 4\sqrt{2}/15 \approx 0.3771$, $E_{\text{pole}}^1 = 3/2 \sqrt{161/225} - 1/2 \approx 0.7689$.

4. Criticality in the energy levels

Despite our model describes a coupling between a bosonic field and a two levels system in ‘zero dimension’, the energy levels of the system display a ‘critical’ behavior resembling very much that one occurring in quantum phase transitions of many-body systems.

4.1. Discontinuity of entanglement entropy

Let us consider the wave function

$$\Psi_C = \begin{pmatrix} C \varphi_1 \\ \varphi_1 \end{pmatrix}, \text{ or } \Psi_C = \begin{pmatrix} \varphi_2 \\ C \varphi_2 \end{pmatrix},$$

(29)

where the constant $C = \pm 1$ or $C = \pm i$. In each parity sector, $\Psi_C$ reads as in (B.19). At parity degenerate point, the wave function with fixed parity $C$ can be determined as follows

$$\Psi_C = \frac{\sum_{m=0}^M CL_m |m\rangle_\alpha + K_m |m\rangle_\alpha}{\sum_{m=0}^M CK_m |m\rangle_\alpha + L_m |m\rangle_\alpha},$$

(30)

where $M$ is the truncated number, and $|m\rangle_\alpha$ is the Fock basis once the transformation $a \to ia$ is performed. For our anisotropic model, the first energy crossing point corresponds to $K_{m+1}(E_{\text{pole}}^0) = 0$, $m = 0, 1$:

$$K_0 = 1, \quad L_0 = \tan 2\beta,$$

(31)

and the corresponding wave function is
\[ \Phi = \left( \tan 2\beta |0\rangle_\alpha \right). \] (32)

Therefore, the wave functions of the \( Z_2 \) Hamiltonian are
\[ \Psi = W^\dagger U \Phi \propto \begin{pmatrix} (\cos \beta + \sin \beta) |0\rangle_\alpha \\ (\cos \beta - \sin \beta) |0\rangle_\alpha \end{pmatrix}, \] (33)

and
\[ \overline{\Psi} \propto \begin{pmatrix} (\cos \beta - \sin \beta) |0\rangle_\alpha \\ (\cos \beta + \sin \beta) |0\rangle_\alpha \end{pmatrix}. \] (34)

The eigenfunction labeled by parity \( C \) can be written as,
\[ \Psi_C = C\Psi + \overline{\Psi}. \] (35)

The entanglement entropy is
\[ S = -\text{tr} \rho \log \rho, \] (36)
where \( \rho \) is the reduced density matrix obtained by tracing out the bosonic degrees of freedom \( \rho = \text{tr}_{\text{bosonic}} (|\Psi\rangle \langle \Psi|) \) [27]. For the first two levels: \( S = -\sum_{i=1}^2 \lambda_i \log_2 (\lambda_i) \), where \( \lambda_i \) is the eigenvalue of \( \rho \). From figure 3, we can see that the entanglement entropy of the ground state jumps at a critical point. The size of the jump depends on anisotropic parameter \( \lambda \).

We comment that the discontinuity displayed by the entanglement occurs with the same mechanism (level crossing) in which first order phase transitions occur in many body systems [27–30]. We also comment that the parity of the ground state changes at this critical point.

4.2. Energy levels condensation

With a logic that is similar to the one applied to second order quantum phase transitions, we analyze the behavior of the energy spectrum of \( H \) for spin-boson coupling in the neighborhood of \( |g|/\omega = g_c = \frac{1}{1+\lambda} \) (with no lack of generality, we set \( g > 0, \lambda > 0, \) and \( \omega = 1 \)). Indeed, by
inspection of equation (14), we notice that $\eta = 0$ at $g_c$. This implies that all the energy levels 'condense' at $E_0 = -\frac{1}{2}$, un-respective of $m$, therefore with a clear parity symmetry breaking.

The transcendental function $G_{\lambda,0}^{n,0}$, in turn, displays a pole of higher order. This scenario indicates that $\langle a \rangle$ can be non-vanishing at $g = g_c$ meaning that the bosonic mode is macroscopically occupied. Therefore, such energy condensation implies super-radiance.

Although we cannot analyze the energy for $g > g_c$ with our exact solution (because of the topological constraint coming from the non-compactness of $su(1,1)$ algebra discussed in section 2), the numerical data shown in figure 4, indicate that the low energy levels decrease linearly, with a slope depending on the total bosonic number as $-\frac{N^x}{2}$ with $x \sim 1.1$. If $N \to \infty$, the curve will be a vertical line. Some insight in the limit of large $n$ can be acquired by resorting to the RWA. Accordingly, the lowest energy is $E_0 = (n + 1)\omega/2 - \sqrt{(\omega - \Delta)^2 + g^2(n + 1)(n + 2)} \approx (n + 1)\omega/2 - g(n + \frac{1}{2}) \approx n(g_c - g)$, in the limit of $n \gg 1$ (the term of $\omega - \Delta$ can be neglected in the resonant case). This indicates that for $g > g_c$, the ground state energy decreases as $E_0 \approx -n\omega$ for large $n$ (figure 4(a)).

In figure 4(b), where $g = 0.85$ which is larger than $g_c$, the photon number probability of ground state appears to be equally distributed among a large number of modes $n$. Accordingly, the entanglement entropy saturates the bound 1, as shown in figure 3.

We remark that such a parity change occurs also in the anisotropic Rabi model proposed in [24].

Similar findings were recently reported for the isotropic Rabi model by Plenio and coworkers [33].

5. Discussions and summary

In this paper we studied a two-boson spin-boson model of the Rabi type in which the rotating and counter-rotating spin-boson couplings enter the Hamiltonian with two different parameters. The exact solution of the model is provided through a variation of the Braak method.
We remark that our formalism can be straightforwardly applied to two-modes Rabi models enjoying the same algebraic structure $su(1, 1) \otimes Z_2$ of the two-boson case (see the appendix). The case with explicit parity symmetry breaking (inclusion of a term proportional to $\sigma_x$) could be approached within our scheme with minor changes [24].

We comment how the spectrum of the model is found modulo certain restriction in the system parameters (equation (15)). Ultimately, such a constraint arises because the procedure leading to the exact solution involves a diagonalization in $su(1, 1)$ which is a non-compact Lie algebra. In order to go beyond such restriction, one should use the continuous representation of Limblad-Nagel instead of the discrete one [23]. To this route, a separate study will be dedicated.

Remarkably, despite how the Hamiltonian describes just two interacting degrees of freedom, the energy spectrum of the system displays characteristic quantum critical features. We believe that this is possible because of the specific $Z_2$ parity symmetry of the system. Specifically, a ‘first order quantum phase transition’ occurs because two states belonging to different parity sectors with the lowest energies cross each other. Such a level crossing is visible as a discontinuity of the entanglement entropy (obtained tracing out the bosonic degree of freedom). The other transition occurs at higher values of the spin-boson coupling strength. It is displayed as a parity symmetry breaking transition allowing a macroscopic occupation of the bosonic mode with a ‘off diagonal order’ $\langle a \rangle \neq 0$. By exact analysis of the entanglement entropy we established that the energy condensation occurs continuously, sharing similarities with the super-radiance phenomenon. Finding the critical properties of the system within the exact formalism defines a major direction for future investigation.

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Appendix A. A realization of the anisotropic two-photon Rabi model in circuit QED

In this section, we provide details on how a two-photon Hamiltonian can be realized through circuit QED.

We refer to the phase qubit, interacting with a microwave field [31, 32]. For such a system, three or more levels participate to the quantum dynamics. Phase qubit can be realized with a Josephson junction in a high-inductance superconducting loop biased with a flux sufficiently large that the phase across the junction sees a potential analogous to that found for the current-biased junction. The energy differences between the ground and first excited states $\omega_{01}$, and the first and the second excitation $\omega_{12}$ are different each other. By tuning the boson field frequency to half of the difference between the ground and second excitation energy, the first excitation is excluded from the dynamics. The transition between the ground and the second excitations can occur by two photons. In this regime, the Hamiltonian of the magnetic flux-biased phase qubit can be written as

$$H_{ph} = \omega_m a^\dagger a + \omega_q \sigma^+ \sigma^- + g (\cos \theta \sigma_z - \sin \theta \sigma_x) (a^2 + a^\dagger 2).$$
Experiments nowadays can access the ultra-strong coupling regime. Indeed, typical numbers involved in the experiments are $\omega_{01}/2\pi = 5.5$ GHz, $\omega_{12} = 4.5$ GHz, and coupling constant $g = 0.1$ GHz. These parameters imply a detectable Bloch–Siegert shift $\sim g^2/\Delta = 1$ MHz.

Anisotropies in the spin-boson coupling can arise from mutual inductance between the phase qubit and the controlling SQUID. In this case the Hamiltonian of the circuit is $H_{ph} + M(a^2 - a^\dagger)^2$, which, apart from the $g \cos \theta \sigma_z (a^2 + a^\dagger)^2$, term can be recast to the anisotropic two-boson Rabi model equation (1).

Appendix B. The anisotropic 2-mode Rabi model

The Hamiltonian for the two-mode Rabi model reads

$$H_{2-m} = a_1^\dagger a_1 + a_2^\dagger a_2 + \Delta \sigma_z + g(\sigma^+ a_1 a_2 + \sigma^- a_1^\dagger a_2^\dagger) + g\lambda(\sigma^+ a_1 a_2^\dagger + \sigma^- a_1^\dagger a_2).$$

With two modes in this Hamiltonian, the $su(2, 1)$ Lie algebra is spanned by

$$K_0 = \frac{1}{2}(a_1^\dagger a_1 + a_2^\dagger a_2 + 1), \quad K_+ = a_1^\dagger a_2, \quad K_- = a_1 a_2.$$

Here, $\kappa = \frac{1}{2}, 1, \frac{3}{2}, \ldots, \kappa > 0$ for two-mode case, every $\kappa$ corresponds an irreducible subspace. These subspace can also be written as $\{|n_0 + n, n\}$, $n_0, n = 0, 1, 2, \ldots, n_0 \geq n_2$ is the Fock state of the boson operator $a_1, a_2$. We note that $n_1 \leq n_2$ give a set of equivalent subspace, $\{|n, n_0 + n\}$, giving degenerate eigenvalues and eigenfunctions; in the following we will work with the first choice. Thus we can also recognize $n_0$ as the subspace index, the relation of $n_0$ and $\kappa$ is $n_0 = 2\kappa - 1$.

The Hamiltonian enjoys the $Z_2$ symmetry generated by

$$\Pi_{2-m} = e^{i\pi[a_1^\dagger a_1 + \frac{1}{2}(\sigma_z + 1)]} = -\sigma_z \cos(\pi a_1^\dagger a_2^\dagger)$$

which have value $p = \pm 1$ in the corresponding subspaces of $\kappa$.

B.1. Exact solution of the anisotropic 2-mode Rabi model

The exact analysis of the two mode Rabi model (B.1) proceeds along very similar lines to those followed for the two-boson case. We apply $U$ and the Bogolubov transformation:

$$U^\dagger (H) U = \begin{pmatrix} \omega_1 a_1 + \omega a_2^\dagger a_2 + p + r(a_1 a_2 + a_1^\dagger a_2^\dagger) & -q + sa_1 a_2 + ta_1^\dagger a_2^\dagger \\ -q + sa_1^\dagger a_2^\dagger + ta_1 a_2 & \omega_1 a_1 + \omega a_2^\dagger a_2 - p - r(a_1 a_2 + a_1^\dagger a_2^\dagger) \end{pmatrix}$$

$$b_1 = ua_1 + va_2^\dagger, \quad b_2 = ua_2 + va_1^\dagger,$$

with

$$|g| < \frac{2\omega}{1 + \lambda},$$

we can get

$$u = \sqrt{\frac{1 + \zeta}{2\zeta}}, \quad v = \sqrt{\frac{1 - \zeta}{2\zeta}}, \quad \cos 2\beta = \frac{1 - \lambda}{1 + \lambda},$$

(B.6)
where
\[
\zeta = \sqrt{\frac{1 - (1 + \lambda)^2 g^2 / 4\omega^2}{1 - (1 + \lambda)^2 g^2 / 4\omega^2}}.
\]  

(B.7)

The vacuum state can be found in \( b_1 b_2 |n_0, 0\rangle_b = 0 \), as
\[
|n_0, 0\rangle_b = \frac{1}{u^{n_0+1}} \sum_{n=0}^{\infty} \left( -\frac{v}{u} \right)^n \sqrt{\frac{(n_0 + n)!}{n_0!}} |n_0 + n, n\rangle,
\]

(B.8)

which also have the property \( b_1 |n_0, 0\rangle_b = \sqrt{n_0} |n_0 - 1, 0\rangle_b \). Here, \( 1/u^{n_0+1} \) is the normalized coefficient, which can be found with the Taylor expansion,
\[
\frac{1}{(1 - x)^{n_0+1}} = 1 + (n_0 + 1)x + \frac{(n_0 + 2)(n_0 + 1)}{2} x + \cdots,
\]

where \( x = v^2 / u^2 \). Actually, \( |n_0, 0\rangle_b \) is a two-mode squeezed vacuum state, which can be generated by the 2-mode squeeze operator \( S_2(\xi) \) on the vacuum
\[
|n_0, 0\rangle_b = S_2(\xi) |n_0, 0\rangle = \exp(\xi a_1^\dagger a_2^\dagger - \xi^* a_1 a_2) |n_0, 0\rangle,
\]

where \( \eta = e^{i\xi/2} \) and \( v = -\xi e^{-i\xi/2} \). In quantum optics \( S_2(\xi) \) is associated with degenerate parametric amplification [1].

The Fock states can be constructed as
\[
|n_0 + m, m\rangle_b = (h_1 b_2^\dagger)^m |n_0, 0\rangle_b.
\]

(B.9)

Note that, the normalized coefficient \( \sqrt{\frac{m!}{(n_0+m)!}} \) is eliminated for the simplicity.

The ansatz for the eigenstates of \( (H) \) is
\[
\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \frac{1}{\sqrt{\sum_{m=0}^{\infty} L_m |n_0 + m, m\rangle_b}} \sum_{m=0}^{\infty} \sum_{m=0}^{\infty} K_m |n_0 + m, m\rangle_b.
\]

(B.10)

Because of the \( Z_2 \) symmetry, the constant \( C \) has only two values ±1. So we can construct the transcendental function as:
\[
G_{\lambda, \pm}^{n_0} = \langle n_0, 0 | [\phi_2 \mp \phi_1] \rangle,
\]

(B.11)

using the coefficients above we can obtain G-function as:
\[
G_{\lambda, +}^{n_0} = \sum_{m=0}^{\infty} (\cos \beta L_m + \sin \beta K_m) D_m,
\]
\[
G_{\lambda, -}^{n_0} = \sum_{m=0}^{\infty} (\sin \beta L_m + \cos \beta K_m) D_m.
\]

(B.12)

Here the coefficients \( D_m \) is as follows
\[
D_m = \langle n_0, 0 \mid (b_1 b_2^\dagger)^m \mid n_0, 0 \rangle_b = \langle n_0, 0 \mid (b_1^\dagger b_2^\dagger)^m \mid n_0, 0 \rangle_b
\]
\[
= \frac{1}{u^{n_0+1}} \left( \frac{v}{u} \right)^m \frac{(n_0 + m)!}{n_0!}.
\]

(B.13)

The analytical property of \( G_{\lambda, \pm}^{n_0} \) is similar to it in the isotropic case, as shown in figure B1. The poles of \( G_{\lambda, \pm}^{n_0} \) can be found to be similar to that for two-photon case.
where $\zeta' = \zeta \left[1 - (1 - \lambda)^2 \frac{g}{4\omega}\right]$. The energy spectra has divided into two parts, one is the regular case, which corresponds to the zeros of $G_{\lambda,m}^0$, and red dashed lines are $G_{\lambda,m}^0$, the poles of $G_{\lambda,m}^0$ have labeled by dotted lines.

$$E_{\lambda,m}^{\text{pole}} = \omega \zeta' (n_0 + 2m + 1) - \omega$$

(B.14)

As shown in figure B2, we can exactly determine the first level-crossing points at subspace $n_0 = 0, 1$ for parameter $\Delta = 0.2, \lambda = 0.5$. In $n_0 = 0$ subspace, it is $g = \frac{4\Delta\omega}{n_0 + 1}(1 - \lambda^2)$.

(B.16)

As shown in figure B2, we can exactly determine the first level-crossing points at subspace $n_0 = 0, 1$ for parameter $\Delta = 0.2, \lambda = 0.5$. In $n_0 = 0$ subspace, it is $g = \frac{4\Delta\omega}{n_0 + 1}(1 - \lambda^2)$.

As shown in figure B2, we can exactly determine the first level-crossing points at subspace $n_0 = 0, 1$ for parameter $\Delta = 0.2, \lambda = 0.5$. In $n_0 = 0$ subspace, it is $g = \frac{4\Delta\omega}{n_0 + 1}(1 - \lambda^2)$. And $g = \frac{2n_0^2}{\sqrt{15}} \approx 0.7303, E = \sqrt{\frac{2n_0^2}{15}} - 1 \approx 0.6452$ for $n_0 = 1$ case.

The wavefunctions at these degenerate points will be discussed in detail later.

B.2. Entanglement entropy

For example, in isotropic 2-mode and 2-photon case, the first level crossing occurs in $m = 1$,

$$K_0 = 1, \quad L_0 = -\frac{\Delta}{2\omega\eta}$$

(B.17)

So the eigenfunction can be written as
\[ \Psi \equiv (L_0 |0\rangle_b, \quad \text{or} \quad (K_0 |\bar{0}\rangle_b) , \]  
where \(|\bar{0}\rangle_b\) is the bosonic vacuum state after the transformation \(b^\dagger \rightarrow ib^\dagger\). Using the parity symmetry, they can be written as

\[ \Psi_C = \begin{pmatrix} C L_0 |0\rangle_b + K_0 |\bar{0}\rangle_b \\ CK_0 |0\rangle_b + L_0 |\bar{0}\rangle_b \end{pmatrix}, \]  
where \(C = \pm 1, \pm i\), the two eigenfunctions are orthogonal to each other.

Similarly as in two-photon case, here we can also find that the entanglement entropy of the ground state experiences a discontinuity at a critical point, see figure B3.
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