Asymptotic behavior of positive solutions to a degenerate elliptic equation in the upper half space with a nonlinear boundary condition

Zhuoran Du

Abstract

We consider positive solutions of the problem

\[
\begin{aligned}
-\text{div}(x_n^a \nabla u) &= 0 \quad \text{in} \ \mathbb{R}^n_+,
\frac{\partial u}{\partial \nu} &= u^q \quad \text{on} \ \partial \mathbb{R}^n_+,
\end{aligned}
\]

where \( a \in (-1,0) \cup (0,1) \), \( q > 1 \) and \( \frac{\partial u}{\partial \nu} := -\lim_{x_n \to 0^+} x_n^a \frac{\partial u}{\partial x_n} \). We obtain some qualitative properties of positive axially symmetric solutions in \( n \geq 3 \) for the case \( a \in (-1,0) \) under the condition \( q \geq \frac{n-a}{n+a-2} \). In particular, we establish the asymptotic expansion of positive axially symmetric solutions.

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Key words Asymptotic behavior, positive solutions, degenerate elliptic equation, half space.

1 Introduction

In this paper, we consider qualitative properties of positive solutions of the problem

\[
\begin{aligned}
-\text{div}(x_n^a \nabla u) &= 0 \quad \text{in} \ \mathbb{R}^n_+ = \{ x \in \mathbb{R}^n : x_n > 0 \},
\frac{\partial u}{\partial \nu} &= u^q \quad \text{on} \ \partial \mathbb{R}^n_+ = \{ x \in \mathbb{R}^n : x_n = 0 \},
\end{aligned}
\]

where \( a \in (-1,0) \cup (0,1) \), \( q > 1 \), \( x = (x', x_n) \in \mathbb{R}^{n-1} \times [0, +\infty) \), and \( \frac{\partial u}{\partial \nu} := -\lim_{x_n \to 0^+} x_n^a \frac{\partial u}{\partial x_n} \).

When the parameter \( a = 0 \), there had been many results about existence and qualitative properties of positive solutions to the problem \((2)\). For critical case \( q = \frac{n}{n-2} \), positive

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0College of Mathematics and Econometrics, Hunan University, Changsha 410082, PRC. E-mail: duzr@hnu.edu.cn
solutions are known to exist in dimensions \( n > 2 \). Moreover, any positive solution is of the form (see \([2]\) and \([14]\))

\[
   u(x) = \alpha |x - x^0|^{-(n-2)}, \quad \alpha > 0, \quad x^0 = (x^0_1, \ldots, x^0_n) \in \mathbb{R}^n, \quad x^0_n = -\frac{1}{n-2} \alpha \frac{x^0_1}{n-2}.
\]

Positive solutions do not exist for subcritical case \( q < \frac{n}{n-2} \) (see \([12]\)). Subsequently, Chipot-Chlebik-Fila-Shafrir \([3]\) obtained the existence of positive solutions of \((2)\) for the supercritical case \( q \geq \frac{n}{n-2} \). Recently, under the conditions \( n > 2 \) and \( q \geq \frac{n-a}{n+a-2} \), Harada \([11]\) established the asymptotic expansion and the intersection property of positive solutions to the problem \((2)\). Qui-Reichel \([15]\) proved the existence of a unique singular positive axial symmetric solution for \( q \geq \frac{n-a}{n+a-2} \) for \( n \geq 3 \).

It is known that the problem \((2)\) with \( a = 0 \) is related to the square root of Laplacian equation

\[
   (-\Delta)^{\frac{1}{2}} u = u^q, \quad \text{in} \, \mathbb{R}^{n-1}.
\]

Indeed, from the well-known Caffarelli-Silvestre extension in \([1]\) we know that if \( u \) is a positive solution of \((2)\), then a positive constant multiple of \( u(x', 0) =: v(x') \) satisfies \((3)\). Therefore the asymptotic expansion results in \([11]\) generalize the asymptotic expansion results of the corresponding semilinear elliptic equation with standard Laplacian operator

\[
   -\Delta u = u^q, \quad \text{in} \, \mathbb{R}^{n-1}.
\]

in \([9], [13]\) to the square root of Laplacian operator case.

In the general case \( a \in (-1, 0) \), Fang, Gui and the author in \([5]\) obtain existence of positive axial symmetric solutions in space dimensions \( n \geq 3 \) for \( q \geq \frac{n-a}{n+a-2} \). Further the author establish a unique singular positive axial symmetric solution \( \psi_\infty(x) \), for \( q \geq \frac{n-1}{n+a-2} \), in \( n \geq 3 \) for all \( a \in (-1, 0) \cup (0, 1) \) (see \([6]\)).

Naturally we hope to establish similar qualitative properties of the positive solution of \((2)\) for the general case \( a \in (-1, 0) \) as that for the case \( a = 0 \) in \([11]\). Our main results in this paper are stated as follows.

**Theorem 1.1** Let \( q \geq \frac{n-a}{n+a-2} \) and denote \( m_q = \frac{1}{q+1} \). Then there exists a family of positive axial symmetric solutions \( u_\beta(x) \) \( (u_\beta(0) = \beta > 0) \) of \((2)\) satisfying the following properties

(i) \( u_\beta(x) = \beta u_1(\beta \frac{x}{\beta^{-1}}) \);
(ii) \( u_\beta(x) \leq C(1 + |x|)^{-m_q} \);
(iii) \( \lim_{\beta \to \infty} u_\beta(x) = \psi_\infty(x) \) for \( x \in \mathbb{R}^n_+ \{0\}, q \neq \frac{n-a}{n+a-2} \).

The following Theorem is the asymptotic expansion for the JL-supercritical, JL-critical and JL-subcritical case (see Definition \([3,1]\)).
Theorem 1.2  (I) Let \( q \) be JL-supercritical or JL-critical. Then there exists \( \varepsilon > 0 \) such that for any \( \beta > 0 \) there exist \( C_1(\beta) < 0, C_2(\beta) \in \mathbb{R} \), such that the solutions \( u_\beta \) of (2) hold the asymptotic expansion for large \( r > 0 \)

\[
u_\beta(x) = \psi_\infty(x) + [C_1 + O(r^{-\varepsilon})]e_1(\theta)r^{-(m_q + \rho_1)}, \quad \text{for JL-supercritical case,}
\]

\[
u_\beta(x) = \psi_\infty(x) + [C_1 \ln r + C_2 + O(r^{-\varepsilon})]e_1(\theta)r^{-\frac{n+a-2}{2}}, \quad \text{for JL-critical case,}
\]

where we have used the polar coordinate \( |x'| = r \sin \theta, \; x_n = r \cos \theta \), and

\[
\rho_1 = \frac{(n + a - 2 - 2m_q) - \sqrt{(n + a - 2 - 2m_q)^2 + 4[m_q(n + a - 2 - m_q) + \lambda_1]}}{2}.
\]

Moreover the asymptotic expansion holds uniformly for \( \theta \in (0, \frac{\pi}{2}) \).

(II) Let \( q \) be JL-subcritical. Then the solutions \( u_\beta \) of (2) satisfies one of the following two asymptotic expansions for large \( r > 0 \)

1) there exists \( (c_1, c_2) \neq 0 \) and \( \varepsilon > 0 \) such that

\[
u_\beta(x) = \psi_\infty(x) + [c_1 \sin(K \ln r) + c_2 \cos(K \ln r) + O(r^{-\varepsilon})]e_1(\theta)r^{-\frac{n+a-2}{2}};
\]

2) there exists \( c > 0 \) and \( \varepsilon > 0 \) such that

\[
u_\beta(x) = \psi_\infty(x) + [c + O(r^{-\varepsilon})]e_2(\theta)r^{-(m_q + \rho_2)},
\]

where \( K > 0, \; \rho_2 > 0 \) are given by

\[
K = \frac{\sqrt{-(n + a - 2 - 2m_q)^2 - 4[m_q(n + a - 2 - m_q) + \lambda_1]}}{2},
\]

\[
\rho_2 = \frac{(n + a - 2 - 2m_q) + \sqrt{(n + a - 2 - 2m_q)^2 + 4[m_q(n + a - 2 - m_q) + \lambda_2]}}{2}.
\]

Moreover the asymptotic expansion holds uniformly for \( \theta \in (0, \frac{\pi}{2}) \).

Therefore, by using the Caffarelli-Silvestre extension, we know that our results generalize the results from the square root Laplacian operator to general fractional Laplacian operator

\[
(-\Delta)^s v = v^q, \quad \text{in } \mathbb{R}^{n-1},
\]

where \( s = \frac{1-a}{2} \in \left(\frac{1}{2}, 1\right) \), since \( a \in (-1, 0) \).

The paper is organized as follows. Section 2 contains some notations and an estimate of the infimum \( C_a \) in generalized trace Hardy inequality, which will be used in the definition of JL-critical exponent in Section 3. In section 4 estimates of eigenvalues of an eigenvalue problem are made, which will be used in asymptotic expansions for positive axial symmetric solutions. In section 5 we derive decay estimates and limit behavior of them. Section 6 contains the proof of asymptotic expansions for them.
2 Preliminaries

We say that a function $u$ is axially symmetric with respect to the $x_n$-axis, if it can be expressed by $u(x) = \tilde{u}(|x'|, x_n)$ for some function $\tilde{u}$. For axially symmetric function, one need to introduce the polar coordinates $(r, \theta)$

$$|x'| = r \sin \theta, \quad x_n = r \cos \theta, \quad \theta \in [0, \frac{\pi}{2}].$$

We first recall the main results in [5] and [6].

**Proposition 2.1** ([5]) Let $q \geq \frac{n-a}{n+a-2}$ and $a \in (-1,0)$. The problem (2) admits a positive axially symmetric solution $u(x)$ satisfying $u_\theta > 0$ and $u(0) = 1$.

In [6], the author consider the existence of positive singular solutions of the form

$$\psi_\infty(x) = V(\theta) r^{\frac{q-1}{p-1}}.$$

To obtain $V(\theta)$, one need to solve

$$\begin{cases}
    (\sin^{n-2} \theta \cos^a \theta V_\theta)_\theta = \gamma V \sin^{n-2} \theta \cos^a \theta & \theta \in (0, \frac{\pi}{2}), \\
    \lim_{\theta \to \frac{\pi}{2}} V_\theta(\theta) \cos^a \theta = V^q(\frac{\pi}{2}),
\end{cases}$$

where

$$\gamma := m_q(n + a - 2 - m_q) > 0.$$

**Proposition 2.2** ([6]) Let $q \geq \frac{n-1}{n+a-2}$ and $a \in (-1,0) \cup (0,1)$. The problem (5) admits a unique solution $V(\theta)$ satisfying $V_\theta(\theta) > 0$.

Next we introduce some notations that will be used. Let $S_{n+1} = \{ \omega = (\omega', \cos \theta) : \omega' \in \mathbb{R}^{n-1}, \theta \in [0, \frac{\pi}{2}), |\omega'|^2 + \cos^2 \theta = 1 \}$ be a half unit sphere. For $p > 1$, we define the space

$$L_{\text{sym}}^p(S_{n+1}) = \left\{ e : e(\omega) \text{ depends only on } \theta \text{ and } \int_0^{\frac{\pi}{2}} |e(\theta)|^p \sin^{n-2} \theta \cos^a \theta d\theta < +\infty \right\}.$$

We denote

$$||e(\theta)||_{p,a} := \left( \int_0^{\frac{\pi}{2}} |e(\theta)|^p \sin^{n-2} \theta \cos^a \theta d\theta \right)^{\frac{1}{p}}.$$

Further we define the space

$$\mathcal{H}_a^k(S_{n+1}) = \left\{ e \in L_{\text{sym}}^p(S_{n+1}) : ||e||_{\mathcal{H}_a^k(S_{n+1})} := \sum_{i=0}^k \|\partial_\theta^i e(\theta)\|_{p,a} < +\infty \right\}.$$
We also define the following operator on $S_{+}^{n-1}$ according to the norm $\| \cdot \|_{p,a}$

$$\Delta_{S,a} := \partial_{\theta \theta} + [(n-2) \cot \theta - a \tan \theta] \partial_{\theta}.$$ 

Note that

$$\sin^{-2} \theta \cos^a \theta \Delta_{S,a} e(\theta) = (\sin^{-2} \theta \cos^a \theta e_{\theta})_{\theta}.$$ 

We will establish the following elementary lemma.

**Lemma 2.1** For $a \in (-1, 0)$, we have

$$\int_{0}^{\frac{\pi}{2}} \sin^{n+a-2} \theta d\theta < \int_{0}^{\frac{\pi}{2}} \sin^{-2} \theta \cos^a \theta d\theta. \quad (6)$$

**Proof.** Since

$$\int_{0}^{\frac{\pi}{2}} \sin^{n+a-2} \theta d\theta = \int_{0}^{\frac{\pi}{4}} \sin^{n+a-2} \theta d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^{n+a-2} \theta d\theta$$

and

$$\int_{0}^{\frac{\pi}{2}} \sin^{-2} \theta \cos^a \theta d\theta = \int_{0}^{\frac{\pi}{2}} \sin^{-2} \theta \cos^a \theta d\theta + \int_{0}^{\frac{\pi}{2}} \cos^{-2} \theta \sin^a \theta d\theta,$$

we have

$$\int_{0}^{\frac{\pi}{2}} \sin^{n+a-2} \theta d\theta - \int_{0}^{\frac{\pi}{2}} \sin^{-2} \theta \cos^a \theta d\theta$$

$$= \int_{0}^{\frac{\pi}{2}} (\sin^{-2} \theta - \cos^{-2} \theta)(\sin^a \theta - \cos^a \theta) d\theta < 0.$$ 

The lemma is proved.

We introduce the space $H^1_a(\mathbb{R}^n_+):= \left\{ u : \int_{\mathbb{R}^n_+} (u^2 + |\nabla u|^2) x^n dx < +\infty \right\}$, and consider the following minimizing problem

$$C_a := \inf_{u \in H^1_a(\mathbb{R}^n_+)} \frac{\int_{\mathbb{R}^n_+} |\nabla u|^2 x^n dx}{\int_{\partial \mathbb{R}^n_+} |u|^2 |x'|^{a-1} d\sigma},$$

which plays an essential role to define a so-called JL-critical exponent in Section 3.

**Lemma 2.2** Let $a \in (-1, 0)$ and $n \geq 4$. We have

$$C_a > \frac{n + a - 3}{2}.$$
We use now the inequality
\[ \int_{\partial \mathbb{R}^n_+} |u|^2 |x'|^{a-1} dx' = (n-1) \omega_{n-1} \int_0^\infty u^2(\bar{r},0) \bar{r}^{n+a-3} d\bar{r}, \]
where \( \omega_{n-1} \) denotes the measure of the unit sphere in \( \mathbb{R}^{n-1} \). Since
\[ u^2(\bar{r},0) = -2 \int_0^\infty u(\bar{r},t) \partial_t u(\bar{r},t) dt, \]
Hence
\[ \int_{\partial \mathbb{R}^n_+} |u|^2 |x'|^{a-1} dx' = -2(n-1) \omega_{n-1} \int_0^\infty \int_0^\infty u(\bar{r},t) \partial_t u(\bar{r},t) \bar{r}^{n+a-3} d\bar{r} dt \]
\[ \leq 2(n-1) \omega_{n-1} \int_0^\infty \left( \int_0^\infty u^2(\bar{r},t) \bar{r}^{n+a-4} d\bar{r} \right)^{1/2} \left( \int_0^\infty (\partial_t u(\bar{r},t))^2 \bar{r}^{n+a-2} d\bar{r} \right)^{1/2} dt. \]
We use now the inequality
\[ \int_0^\infty u^2(\bar{r},t) \bar{r}^{n+a-4} d\bar{r} \leq \frac{4}{(n+a-3)^2} \int_0^\infty (\partial_r u(\bar{r},t))^2 \bar{r}^{n+a-2} d\bar{r}, \]
which follows from the generalized Hardy’s inequality
\[ \int_{\mathbb{R}^d} \frac{u^2}{|x|^{2(b+1)}} dx \leq \frac{4}{(d-2-2b)^2} \int_{\mathbb{R}^d} \frac{\nabla u|^2}{|x|^{2b}} dx. \]
We obtain
\[ \int_{\partial \mathbb{R}^n_+} |u|^2 |x'|^{a-1} dx' \]
\[ \leq \frac{4(n-1) \omega_{n-1}}{n+a-3} \int_0^\infty \left( \int_0^\infty (\partial_r u(\bar{r},t))^2 \bar{r}^{n+a-2} d\bar{r} \right)^{1/2} \left( \int_0^\infty (\partial_t u(\bar{r},t))^2 \bar{r}^{n+a-2} d\bar{r} \right)^{1/2} dt \]
\[ \leq \frac{2(n-1) \omega_{n-1}}{n+a-3} \int_0^\infty \int_0^\infty \left( (\partial_r u(\bar{r},t))^2 + (\partial_t u(\bar{r},t))^2 \right) \bar{r}^{n+a-2} d\bar{r} dt. \]  
(7)
We introduce the coordinates \((r, \theta)\)
\[ \bar{r} = r \sin \theta, \quad t = r \cos \theta. \]
Then one has
\[ \int_0^\infty \int_0^\infty \left( (\partial_r u(\bar{r},t))^2 + (\partial_t u(\bar{r},t))^2 \right) \bar{r}^{n+a-2} d\bar{r} dt \]
\[ = \int_0^\pi \left( \int_0^\infty r^{n+a-1} \left( \int_0^\pi |\nabla u(r \sin \theta, r \cos \theta)|^2 \sin^{n+a-2} d\theta \right) dr \right). \] 
(8)
We may further assume that $|\nabla u(r \sin \theta, r \cos \theta)|^2 = |\nabla u(r \cos \theta, r \sin \theta)|^2$. From this, using the similar argument of Lemma 2.1, we obtain
\[
\int_0^\infty r^{n+a-1} \left( \int_0^{\frac{\pi}{2}} |\nabla u(r \sin \theta, r \cos \theta)|^2 \sin^{n+a-2} \theta d\theta \right) dr < \int_0^\infty r^{n+a-1} \left( \int_0^{\frac{\pi}{2}} |\nabla u(r \sin \theta, r \cos \theta)|^2 \sin^{n-2} \theta d\theta \right) dr.
\]
(9)

Note that
\[
\int_0^\infty r^{n+a-1} \left( \int_0^{\frac{\pi}{2}} |\nabla u(r \sin \theta, r \cos \theta)|^2 \sin^{n-2} \cos^a \theta d\theta \right) dr = \frac{1}{(n-1)\omega_{n-1}} \int_{\mathbb{R}^n_+} |\nabla u|^2 x_n^a dx.
\]
(10)

Hence, from (7)-(10), we have
\[
\int_{\partial S^{n-1}_+} |u|^2 |x'|^{a-1} dx' \leq \frac{2}{n+a-3} \int_{\mathbb{R}^n_+} |\nabla u|^2 x_n^a dx.
\]

The proof of this lemma is complete.

**Remark 1** This lemma generalize the conclusion that $C_0 > \frac{n-3}{2}$ for the case $a = 0$, which is obtained by Dávila-Dupaigne-Montenegro in [4].

We define
\[
h_{n,a} := -\frac{(n+a-2)^2}{4}.
\]
Next we show that $h_{n,a}$ is the first eigenvalue of the following eigenvalue problem
\[
-\Delta_{S,a} e = \lambda e, \quad \theta \in (0, \frac{\pi}{2}), \quad \lim_{\theta \to \frac{\pi}{2}} e_\theta(\theta) \cos^a \theta = C_a e(\frac{\pi}{2}).
\]
(11)

To this end, we introduce the space
\[
H^1_a(S^{n-1}_+) := \left\{ e : \int_{S^{n-1}_+} (e^2 + |\nabla_S e|^2) d\omega = \int_{S^{n-1}_+} (e^2 + |\nabla_S e|^2) \sin^{n-2} \cos^a \theta d\theta d\omega' < +\infty \right\},
\]
where $\nabla_S e$ is a gradient of $e$ on $S^{n-1}_+$. We consider a minimizing problem
\[
C_S := \inf_{e \in H^1_a(S^{n-1}_+)} \frac{\|\nabla_S e\|_{L^2(S^{n-1}_+)}^2 - h_{n,a} \|e\|_{L^2(S^{n-1}_+)}^2}{\|e\|_{L^2(\partial S^{n-1}_+)}^2},
\]
(12)
where \( \|e\|_{L^2_a(S^{n-1}_+)}^2 := \int_{S^{n-1}_+} e^2(\theta) \sin^{n-2}\theta d\theta d\omega' \). For simplicity of notations, hereafter we set
\[
d\mu := \sin^{n-2}\theta d\theta, \quad e_B := e(\frac{\pi}{2}).
\]
By a trace inequality, it is verified that there exists a minimizer \( e \in H^1_a(S^{n-1}_+) \) of (12) satisfying \( e(\theta) > 0 \) and
\[
- \Delta e = h_{n,a} e, \quad \theta \in (0, \frac{\pi}{2}), \quad \lim_{\theta \to \frac{\pi}{2}} e(\theta) \cos\theta = C_S e_B. \tag{13}
\]
Since \( e(\theta) > 0 \), then \( h_{n,a} \) is the first eigenvalue of this problem. Therefore, for problem (11), to prove \( \lambda_1 = h_{n,a} \) we only need to show \( C_S = C_a \).

**Lemma 2.3** \( C_S = C_a \).

**Proof.** Let \( u \in C^1_c(\mathbb{R}^n_+) \). We have
\[
\int_{\partial \mathbb{R}^n_+} |u(x', 0)|^2|x'|^{n-1} dx' = \int_0^\infty \|u(r, \cdot)\|_{L^2(\partial S^{n-1}_+)}^2 r^{n+a-3} dr.
\]
By the definition of \( C_S \) and the generalized Hardy inequality, one has
\[
C_S \int_0^\infty \|u(r, \cdot)\|_{L^2(\partial S^{n-1}_+)}^2 r^{n+a-3} dr \leq \int_0^\infty \int_{S^{n-1}_+} (|\nabla u|^2 - h_{n,a} u^2) r^{n+a-3} d\omega dr \\
\leq \int_0^\infty \int_{S^{n-1}_+} (r^{-2} |\nabla u|^2 + u^2) r^{n+a-1} d\omega dr.
\]
Note that \( |\nabla u|^2 = u^2 + r^{-2} |\nabla S u|^2 \), we obtain
\[
C_S \int_0^\infty \|u(r, \cdot)\|_{L^2(\partial S^{n-1}_+)}^2 r^{n+a-3} dr \leq \int_{\mathbb{R}^n_+} |\nabla u|^2 x_n^a dx,
\]
which gives that \( C_a \geq C_S \).

Next we show \( C_a \leq C_S \). We need to construct a sequence \( \{u_i\}_{i \in \mathbb{N}} \subset H^1_a(\mathbb{R}^n_+) \) such that
\[
\lim_{i \to \infty} \frac{\int_{\mathbb{R}^n_+} |\nabla u_i|^2 x_n^a dx}{\int_{\partial \mathbb{R}^n_+} |u_i|^2 |x'|^{n-1} dx'} = C_S. \tag{14}
\]
To this end, we set
\[
u_i(x) = \begin{cases} 
  e(\theta) i^{(n+a-2)/2} & \text{if } r \in [0, 1/i), \\
  e(\theta) r^{(n+a-2)/2} & \text{if } r \in [1/i, 1), \\
  \chi(r) e(\theta) r^{(n+a-2)/2} & \text{if } r \in [1, \infty),
\end{cases} \tag{15}
\]
where $e(\theta)$ is a minimizer of (12) and $\chi(r)$ is a cut-off function such that $\chi(r) = 1$ for $r \in [0, 1]$ and $\chi(r) = 0$ for $r \in [2, \infty)$. Calculation shows that
\[
\int_{\mathbb{R}^n_+} |\nabla u|^2 x_n^a \, dx = C_S ||e||_{L^2(\partial S^a_+)}^2 (\ln i) + O(1)
\]
and
\[
\int_{\partial \mathbb{R}^n_+} |u(x', 0)|^2 |x'|^{a-1} \, dx' = \|e\|^2_{L^2(\partial S^a_+)} (\ln i) + O(1).
\]
Hence we obtain (14). We complete the proof of this lemma.

3 JL-critical exponent

We introduce
\[
J(q) := \inf_{u \in H^1_0(\mathbb{R}^n_+)} \frac{\int_{\mathbb{R}^n_+} |\nabla u|^2 x_n^a \, dx - q \int_{\partial \mathbb{R}^n_+} \psi_\infty^{-1} u^2 \, dx'}{\int_{\partial \mathbb{R}^n_+} |u|^2 |x'|^{a-1} \, dx'}.
\] (16)

Substituting the explicit expression of $\psi_\infty(x) = V(\theta) r \frac{\sin \theta}{\theta}$ into (16), we have
\[
J(q) = C_a - qV_B^q - 1.
\]

**Definition 3.1** We say that an exponent $q$ is JL-supercritical if $J(q) > 0$, JL-critical if $J(q) = 0$ and JL-subcritical if $J(q) < 0$.

**Remark 2** Hence, an exponent $q$ is JL-supercritical if $C_a > qV_B^q$, JL-critical if $C_a = qV_B^q$ and JL-subcritical if $C_a < qV_B^q$.

Next we will show that $q$ is JL-supercritical if $q$ and $n$ are large enough, and $q$ is JL-subcritical if $q$ is close to $\frac{n-a}{n+a-2}$.

**Lemma 3.1** There exists $n_0 \in N$ and $q_1 > \frac{n_0-a}{n_0+a-2}$ such that for $n \geq n_0$ and $q > q_1$ we have $C_a > qV_B^q$.

**Proof.** Integrating (5) over $(0, \frac{\pi}{2})$, we have
\[
V_B^q = \lim_{\theta \to \frac{\pi}{2}} V_\theta(\theta) \cos^a \theta = \gamma \int_0^{\frac{\pi}{2}} V(\theta) \, d\mu \leq \gamma \int_0^{\frac{\pi}{2}} V_B \, d\mu,
\] (17)
where in the last inequality we used the fact that $V_\theta > 0$. Hence we have
\[
qV_B^q - 1 \leq q\gamma \int_0^{\frac{\pi}{2}} \, d\mu = qm_q(n + a - 2 - m_q)I_{n,a},
\]
where $I_{n,a} := \int_0^{\frac{\pi}{2}} \, d\mu = \int_0^{\frac{\pi}{2}} \sin^{n-2} \cos^a \theta d\theta \to 0$ as $n \to +\infty$. Note that for $q > q_1$ we have $qm_q = q\frac{1-q}{2} \leq C$, where $C$ is a positive constant independence of $n$. On the other hand, by Lemma 2.2 one has $C_a > \frac{n+a-3}{2}$. So $qm_q(n + a - 2 - m_q)I_{n,a} < C_a$ for large $n$. Hence the result of this lemma is true.
Lemma 3.2 For \( n \geq 3 \), there exists \( q_0 \geq \frac{n-a}{n+a-2} \) such that \( C_a < qV_B^{q-1} \) for \( q \in \left[ \frac{n-a}{n+a-2}, q_0 \right) \). Furthermore, in these low dimensions \( n = 3, 4, 5, 6 \), we have \( C_a < qV_B^{q-1} \) for any \( q \geq \frac{n-a}{n+a-2} \).

Proof. By the explicit expression of \( \gamma \) and \( h_{n,a} \), it is easily seen that \( h_{n,a} \geq \gamma \) for \( q \in \left[ \frac{n-a}{n+a-2}, +\infty \right) \), moreover, the strict inequality holds unless \( q = \frac{n-a}{n+a-2} \). Let \( e(\theta) \) be a positive solution of (13) with \( e_B = V_B \). We set \( W(\theta) := e(\theta) - V(\theta) \), then we have

\[
\begin{cases}
W_{\theta\theta} + (n-2) \cot W_{\theta} - a \tan W_{\theta} \geq \gamma W & \text{in } (0, \frac{\pi}{2}), \\
W = 0 & \text{on } \{ \frac{\pi}{2} \}.
\end{cases}
\]  

We claim that

\[ W(\theta) \leq 0 \quad \text{in } (0, \frac{\pi}{2}). \]  

(19)

First we suppose that \( W(0) > 0 \). Note that \( \lim_{\theta \to 0} \frac{W_{\theta}(\theta)}{\tan \theta} = \lim_{\theta \to 0} \frac{W_{\theta}(\theta) - W(0)}{\theta} = W_{\theta\theta}(0) \). By this and (18), we have \( (n-1)W_{\theta}(0) \geq \gamma W(0) > 0 \). Since \( W(0) = 0 \), there exists \( \delta > 0 \) such that \( W_{\theta} > 0 \) in \( (0, \delta) \). Due to \( W(\frac{\pi}{2}) = 0 \), there exists \( \theta_0 \in (0, \frac{\pi}{2}) \) such that \( W(\theta_0) > 0 \), \( W_{\theta}(\theta_0) = 0 \) and \( W_{\theta\theta}(\theta_0) \leq 0 \). However, these contradict with (18). Hence we obtain \( W(0) \leq 0 \). Now we suppose (19) is false. Then there exists \( \theta_1 \in (0, \frac{\pi}{2}) \) such that \( W(\theta_1) > 0 \), \( W_{\theta}(\theta_1) = 0 \) and \( W_{\theta\theta}(\theta_1) \leq 0 \), which contradict with (18) again. Hence the claim (19) is proved.

Multiplying the equation in the following problem

\[-\Delta_{s,a} e = h_{n,a} e, \quad \theta \in (0, \frac{\pi}{2}), \quad \lim_{\theta \to \frac{\pi}{2}} e_{\theta}(\theta) \cos^a \theta = C_a e_B,\]

by \( \sin^{n-2} \cos^a \theta \) and integrating over \( (0, \frac{\pi}{2}) \), we obtain

\[ C_a e_B = h_{n,a} \int_0^{\frac{\pi}{2}} e(\theta) d\mu. \]  

(20)

From (17), (20) and the assumption \( e_B = V_B \), we have

\[ qV_B^{q-1} - C_a = V_B^{-1}(qV_B^q - C_a e_B) = V_B^{-1} \left( q\gamma \int_0^{\frac{\pi}{2}} V(\theta) d\mu - h_{n,a} \int_0^{\frac{\pi}{2}} e(\theta) d\mu \right). \]

From (19) we have

\[ qV_B^{q-1} - C_a \geq V_B^{-1}(q\gamma - h_{n,a}) \int_0^{\frac{\pi}{2}} e(\theta) d\mu. \]

Then to prove \( qV_B^{q-1} > C_a \), we need to show \( q\gamma > h_{n,a} \). From the explicit expression of \( \gamma \), we have

\[ q\gamma - h_{n,a} = \left( 1 - a + \frac{1 - a}{q - 1} \right) \left( (n + a - 2) - \frac{1 - a}{q - 1} \right) - \left( \frac{n + a - 2}{2} \right)^2. \]  

(21)
Now we choose \( q = \frac{n-a}{n+a-2} \) in (21), then we see that \( q \gamma - h_{n,a} > 0 \). Hence by a continuity principle, there exists \( q_0 > \frac{n-a}{n+a-2} \) such that \( q \gamma - h_{n,a} > 0 \) for \( q \in \left[ \frac{n-a}{n+a-2}, q_0 \right) \), which shows the first statement.

Next we prove the second statement. We regard the right-hand side of (21) as a function \( G \) of \( \frac{1-a}{q-1} \). Then \( q \geq \frac{n-a}{n+a-2} \) is equivalent to \( \frac{1-a}{q-1} \in (0, \frac{n+a-2}{2}] \). Hence

\[
\inf_{q \geq \frac{n-a}{n+a-2}} (q \gamma - h_{n,a}) = \inf_{\tau \in (0, \frac{n+a-2}{2}]} G(\tau),
\]

where \( G(\tau) := (1-a+\tau)((n+a-2) - \tau) - \frac{(n+a-2)^2}{4} \). Elementary computation shows that, for \( n = 3 \)

\[
\inf_{\tau \in (0, \frac{n+a-2}{2}]} G(\tau) = G(\frac{n+a-2}{2}) = \frac{1-a^2}{2} > 0,
\]
and for \( n = 4 \)

\[
\inf_{\tau \in (0, \frac{n+a-2}{2}]} G(\tau) = G(\frac{n+a-2}{2}) = \frac{(2+a)(1-a)}{2} > 0.
\]

For the case \( n = 5 \), if \( a \in (-1, -\frac{1}{3}] \), we have

\[
\inf_{\tau \in (0, \frac{n+a-2}{2}]} G(\tau) = G(\frac{n+a-2}{2}) = \frac{(3+a)(1-a)}{2} > 0,
\]
and if \( a \in (-\frac{1}{3}, 0) \), we have

\[
\inf_{\tau \in (0, \frac{n+a-2}{2}]} G(\tau) = G(0) = \frac{(3+a)(1-a)}{4} > 0.
\]

For the case \( n = 6 \), if \( a \in (-1, -\frac{2}{3}] \), we have

\[
\inf_{\tau \in (0, \frac{n+a-2}{2}]} G(\tau) = G(\frac{n+a-2}{2}) = \frac{(4+a)(1-a)}{2} > 0,
\]
and if \( a \in (-\frac{2}{3}, 0) \), we have

\[
\inf_{\tau \in (0, \frac{n+a-2}{2}]} G(\tau) = G(0) = \frac{(4+a)(-a)}{4} > 0.
\]

As for the case \( n \geq 7 \), we have \( \inf_{\tau \in (0, \frac{n+a-2}{2}]} G(\tau) = G(0) < 0 \).

The proof of this lemma is complete.
4 Estimate of eigenvalues

In this section we will make some estimates for the eigenvalues of the following problem.

\[- \Delta_{S,a}e = \lambda e, \quad \theta \in (0, \frac{\pi}{2}), \quad \lim_{\theta \to \frac{\pi}{2}} e_\theta(\theta) \cos^a \theta = qV_B^{q-1}e_B. \tag{23} \]

We denote by \( \lambda_i, e_i(\theta) \) the \( i \)-th normalized eigenfunction such that

\[ \int_0^{\frac{\pi}{2}} e_i(\theta)e_j(\theta)d\mu = \delta_{ij}, \quad e_iB > 0. \]

First we give estimates for the first eigenvalue for this problem.

**Lemma 4.1** \( \lambda_1 < -\gamma \) and

- \( \lambda_1 > -\frac{(n+a-2)^2}{4} \) if \( q \) is JL-supercritical;
- \( \lambda_1 = -\frac{(n+a-2)^2}{4} \) if \( q \) is JL-critical;
- \( \lambda_1 < -\frac{(n+a-2)^2}{4} \) if \( q \) is JL-subcritical.

**Proof.** For the first eigenvalue \( \lambda_1 \) of (23), it is characterized by

\[ \lambda_1 = \inf_{e \in \mathcal{H}_2(S^n_+)} \frac{\|\partial_\theta e\|^2_{2,a} - qV_B^{q-1}e_B^2}{\|e\|^2_{2,a}}. \tag{24} \]

Recall that \( V(\theta) \) is a positive solution of (5), then \( -\gamma \) is the first eigenvalue of the problem

\[- \Delta_{S,a}e = \lambda e, \quad \theta \in (0, \frac{\pi}{2}), \quad \lim_{\theta \to \frac{\pi}{2}} e_\theta(\theta) \cos^a \theta = V_B^{q-1}e_B. \]

Hence

\[ -\gamma = \inf_{e \in \mathcal{H}_2(S^n_+)} \frac{\|\partial_\theta e\|^2_{2,a} - V_B^{q-1}e_B^2}{\|e\|^2_{2,a}}. \tag{25} \]

By (24), (25) and the fact \( q > 1 \), we obtain that \( \lambda_1 < -\gamma \).

In Section 2 we have proved that \( h_{n,a} = -\frac{(n+a-2)^2}{4} \) is the first eigenvalue of (11), hence we have

\[ -\frac{(n+a-2)^2}{4} = \inf_{e \in \mathcal{H}_2(S^n_+)} \frac{\|\partial_\theta e\|^2_{2,a} - C_ae_B^2}{\|e\|^2_{2,a}}. \tag{26} \]

By Remark 2, (24) and (26), we obtain the rest results of this lemma.

We set

\[ \sigma := (n+a-2) - 2m_q. \]

Elementary computation shows that \( \sigma^2 + 4(\gamma + \lambda_1) = (n+a-2)^2 + 4\lambda_1 \). From this and Lemma 4.1, we obtain the following lemma.
Lemma 4.2 One has
\[\sigma^2 + 4(\gamma + \lambda_1) > 0 \quad \text{if } q \text{ is JL-supercritical;}\]
\[\sigma^2 + 4(\gamma + \lambda_1) = 0 \quad \text{if } q \text{ is JL-critical;}\]
\[\sigma^2 + 4(\gamma + \lambda_1) < 0 \quad \text{if } q \text{ is JL-subcritical.}\]

Next we show that the second eigenvalue of (23) is positive.

Lemma 4.3 It holds that \(\lambda_2 > 0\).

Proof. Let \(e_2(\theta)\) be a a corresponding eigenfunction with \(e_2(0) < 0\). By Strum’s comparison theorem, \(e_2(\theta)\) has just one zero in \((0, \frac{\pi}{2})\), which is denoted by \(\theta_0\). Then \(\partial_\theta e_2(\theta_0) \geq 0\). Clearly \(\partial_\theta e_2(\theta_0) \neq 0\). Hence \(\partial_\theta e_2(\theta_0) > 0\). Multiplying (23) by \(\sin^{n-2} \theta_0 \cos^a \theta_0\) and integrating over \((0, \theta_0)\), we obtain
\[\partial_\theta e_2(\theta_0) = -\frac{\lambda_2}{\sin^{n-2} \theta_0 \cos^a \theta_0} \int_0^{\theta_0} e_2(\theta) d\mu,\]
which gives that \(\lambda_2 > 0\).

5 Decay estimates and Limit behavior

In this section we will prove Theorem 1.1.

Recall that from Proposition 2.1, (2) admits a positive axial symmetric solution \(u(x)\) satisfying \(u(0) = 1\). We set \(u_1(x) := u(x)\) and
\[u_\beta(x) := \beta u_1(\beta^{\frac{q-1}{q-1}} x).\]
Then \(u_\beta(x)\) is a positive axial symmetric solution of (2) satisfying \(u_\beta(0) = \beta\). Hence Theorem 1.1 (i) is proved.

Now we will follow the idea in [11] to prove Theorem 1.1 (ii). We denote a positive axial symmetric solution of (2) as \(u(x) = u(r, \theta)\), which satisfies \(u_\theta > 0\). Then we need to prove the decay estimate
\[u(r, \theta) = u(x) \leq C(1 + |x|)^{-m_q}. \tag{27}\]

We define
\[v(t, \theta) := r^{\frac{q-a}{q-1}} u(r, \theta), \quad r := e^t.\]
Then \(v(t, \theta)\) satisfies
\[
\begin{aligned}
\left\{ 
\begin{array}{ll}
\nu_t + \nu v_t - \gamma v + \Delta_{S,a} v = 0 & \text{in } \mathbb{R} \times (0, \frac{\pi}{2}), \\
\lim_{\theta \to \frac{\pi}{2}} v \cos a \theta = v_B(t) & \text{on } \mathbb{R},
\end{array}
\right.
\end{aligned}
\tag{28}
\]
Lemma 5.1 Let \( v(t, \theta) \) be a positive axial symmetric solution of \( (28) \) satisfying \( v_\theta > 0 \). Then \( \tilde{v}(t) \) is bounded on \( \mathbb{R}_+ \).

In order to prove that \( v(t, \theta) \) is bounded on \( \mathbb{R}_+ \times (0, \frac{\pi}{2}) \), we need to establish an apriori estimate for the following problem

\[
- \text{div}(x^n a \nabla u) + c(x) x^n u = 0 \quad \text{in} \quad B^+_1,
\]

where \( B^+_1 := \{ x \in \mathbb{R}^n_+ : |x| < 1 \} \), \( D_1 := \{ x \in \partial \mathbb{R}^n_+ : |x| < 1 \} \). We will borrow the method used in Theorem 8.17 in [8] (see also in [11]) to prove the following apriori estimate.

Lemma 5.2 Let \( u(x) \in H^1_a(B^+_1) \) be a weak solution of \( (30) \) with \( F \in L^p(D_1) \) for some \( p > \frac{n-1}{1-a} \). Then there exists \( C > 0 \) depending on \( n, a, \| F \|_{L^p(D_1)} \) and \( \| c \|_{L^\infty(B^+_1)} \) such that

\[
\| u \|_{L^\infty(B^+/2)} \leq C \| u \|_{L^2(B^+_1)}.
\]

Proof. Let \( \chi(|x|) \) be a smooth cut-off function satisfying \( \chi(|x|) = 0 \) for \( |x| \geq 1 \) and set \( \tilde{c} = \| c \|_{L^\infty(B^+_1)} \). We denote \( u_+ = \max\{u, 0\}, \quad u_- = \max\{-u, 0\} \). Multiplying the equation in \( (30) \) by \( u_+^k \chi^2(k > 1) \) and integrating over \( B^+_1 \), we obtain

\[
\frac{4k}{(k+1)^2} \int_{B^+_1} |\nabla u_+^{(k+1)/2}|^2 \chi^2 x^n dx
\leq \int_{B^+_1} u_+^k |\nabla u_+||\nabla \chi^2 x^n dx + \int_{B^+_1} u_+^{k+1} \chi^2 x^n dx + \int_{D_1} F(x') u_+^{k+1} \chi^2 dx'.
\]

Let \( p' = \frac{p}{p-1} \) be the dual exponent of \( p \). Since \( p > \frac{n-1}{1-a} \), then \( 2 < 2p' < \frac{2(n-1)}{n+a-2} \). We denote \( 2_* := \frac{2(n-1)}{n+a-2} \). Then there exists a \( \zeta \in (0, 1) \) such that \( \frac{1}{2p'} = \frac{\zeta}{2} + \frac{1-\zeta}{2_*} \). By the Hölder inequality and an interpolation inequality, we have

\[
\int_{D_1} F(x') u_+^{k+1} \chi^2 dx' \leq \| F \|_{L^p(D_1)} \| u_+^{(k+1)/2} \chi \|_{L^{2p'}(D_1)}^2
\]

\[
\leq \| F \|_{L^p(D_1)} \left[ \varepsilon \| u_+^{(k+1)/2} \chi \|_{L^{2_*}(D_1)} + \varepsilon^{-\frac{1-\zeta}{2_*}} \| u_+^{(k+1)/2} \chi \|_{L^2(D_1)} \right]^2.
\]
that

\[ \int_{D_1} F(x') u_+^{k+1} \chi^2 dx' \leq \left\| F \right\|_{L^p(D_1)} \left[ 4\varepsilon^2 \left\| u_+^{(k+1)/2} \right\|_{H^2(B_1^+)} + C(\varepsilon) \left\| u_+^{(k+1)/2} \right\|_{L^2(B_1^+)} \right]. \]

From (31) and (32), choosing \( \varepsilon = \left( \left\| F \right\|_{L^p(D_1)} \right)^{-1} \frac{k}{2(k+1)^2} \), we obtain

\[ \int_{B_1^+} |\nabla (u_+^{(k+1)/2} \chi)|^2 a dx \leq C(n, a, \left\| F \right\|_{L^p(D_1)}, \bar{\varepsilon})(k+1)^2 \int_{B_1^+} u_+^{k+1} (\chi + |\nabla \chi|)^2 a dx. \]

From [10], we know that there exists some \( v > 2 \) such that for \( 2^* := \frac{2v}{v-2} \), one has

\[ \left\| u_+^{(k+1)/2} \chi \right\|_{L^v(B_1^+)} \leq C(n, a, \left\| F \right\|_{L^p(D_1)}, \bar{\varepsilon})(k+1) \left\| u_+^{(k+1)/2} (\chi + |\nabla \chi|) \right\|_{L^2(B_1^+)}. \]

It is now desirable to specify the cut-off function \( \chi \) more precisely. Let \( r_1, r_2 \) be such that \( \frac{1}{2} \leq r_1 < r_2 \leq \frac{3}{2} \) and set \( \chi \equiv 1 \) in \( B_{r_1}^+ \), \( \chi \equiv 0 \) in \( B_{r_2}^+ \setminus B_{r_1}^+ \) with \( \int_{B_{r_2}^+ \setminus B_{r_1}^+} |\nabla \chi| \leq \frac{2}{r_2-r_1} \). Writing \( \eta := \frac{v}{v-2} \), we then have from (33)

\[ \left\| u_+^{(k+1)/2} \right\|_{L^v(B_{r_1}^+)} \leq C(k+1) \left\| u_+^{(k+1)/2} \right\|_{L^2(B_{r_2}^+)}. \]

Let us introduce

\[ \Phi(\delta, r) := \left( \int_{B_{r_1}^+} |u_+|^\delta a dx \right)^{\frac{1}{\delta}}. \]

Due to the fact that \( \int_{B_{r_1}^+} a dx < \infty \), it is verified that

\[ \Phi(+\infty, r) = \lim_{\delta \to +\infty} \Phi(\delta, r) = \sup_{B_{r_1}^+} u_+, \quad \Phi(-\infty, r) = \lim_{\delta \to -\infty} \Phi(\delta, r) = \inf_{B_{r_1}^+} u_. \]

From (34) we have

\[ \Phi(\eta(k+1), r_1) \leq C(k+1) \frac{2^{\frac{k+1}{2}}}{(k+1)} \Phi((k+1), r_2). \]

This inequality can now be iterated to yield the desired estimates. Hence, taking \( \delta > 1 \), we set \( k+1 = \eta^i \delta \) and \( r_i = 2^{-i} + 2^{-i-1}, i = 0, 1, \ldots, \) so that, by (35) we have

\[ \Phi(\eta^i \delta, \frac{1}{2}) \leq (C\eta)^{2^i} \Phi(\delta, 1) = \tilde{C} \Phi(\delta, 1), \]

where \( \tilde{C} \) depending on \( \left\| F \right\|_{L^p(D_1)}, \bar{\varepsilon}, a, \eta, \delta \). Consequently, letting \( i \) tend to infinity, we have

\[ \left\| u_+ \right\|_{L^\infty(B_{r_1}^+)} \leq C \left\| u_+ \right\|_{L^\delta(B_{r_1}^+)}. \]
Let us now choose \( \delta = 2 \), then

\[
\|u_+\|_{L^\infty(B_{1/2}^+)} \leq C\|u_+\|_{L^2_2(B_{1/2}^+)}.
\]

By the same way, we can obtain the estimate for \( u_- \).

We complete the proof of this lemma.

**Remark 3** Similar estimate as that of Lemma 5.2 had been obtained by Fabes-Kenig-Serapioni [7] (Corollary 2.3.4).

**Lemma 5.3** Let \( v(t, \theta) \) be a positive axial symmetric solution of (28) satisfying \( v_\theta > 0 \). Then \( v(t, \theta) \) is bounded on \( \mathbb{R}_+ \times (0, \pi) \).

**Proof.** We apply a test function method to prove a boundedness of \( v(t, \theta) \). Let \( \chi(t) \) be a cut-off function with a compact support in \( \mathbb{R}_+ \). Multiplying (28) by \( v^{-\frac{1}{2}}(t, \theta)\chi^2 \sin^{-2} \theta \cos^a \theta \) and integrating on \( \mathbb{R}_+ \times (0, \pi) \), we obtain

\[
\int_\mathbb{R} \left( \int_0^{\pi} v^{-\frac{3}{2}}(v_t^2 + v_\theta^2) d\mu + v_B^{-1/2} \right) \chi^2 dt \leq C \int_\mathbb{R} \int_0^{\pi} v^{\frac{1}{2}}(\chi^2 + \chi_t^2) d\mu dt. \tag{36}
\]

We need to use the following trace inequality for a two dimensional domain \((a_1, a_2) \times (b_1, b_2)\)

\[
\int_{a_1}^{b_2} |\phi(x, b_2)|^\alpha dx \leq C_\alpha \left( \int_{a_1}^{b_2} \int_{b_1}^{b_2} [\phi^2 + \phi_{x_1}^2 + \phi_{x_2}^2] dx_1 dx_2 \right)^{\frac{\alpha}{2}}.
\]

where \( \alpha \geq 1 \) and \( C_\alpha > 0 \) is a constant depending on \( \alpha, a_2 - a_1 \) and \( b_2 - b_1 \). Applying this inequality with \( \phi(x_1, \xi_2) = v^{\frac{1}{2}}(\xi_1, \xi_2), a_1 = \tau - 1, a_2 = \tau + 1, b_1 = \frac{\pi}{4}, b_1 = \frac{\pi}{2}, \) then we obtain

\[
\int_{\tau - 1}^{\tau + 1} |v_B^{\frac{1}{2}}|^\alpha dt \leq C_\alpha \left( \int_{\tau - 1}^{\tau + 1} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} [v^{\frac{1}{2}} + v^{-\frac{3}{2}}(v_t^2 + v_\theta^2)] d\theta dt \right)^{\frac{\alpha}{2}}.
\]

We take a cut-off function \( \chi(t) \) such that \( \chi(t) = 1 \) if \( t \in [\tau - 1, \tau + 1] \) and \( \chi(t) = 0 \) if \( t \in \mathbb{R} \setminus (\tau - 2, \tau + 2) \). It is clear that \( \sin \theta \geq \frac{\sqrt{2}}{2} \) for \( \theta \in \left[ \frac{\pi}{4}, \frac{\pi}{2} \right] \), and \( \cos^a \theta > 1 \), since \( a \in (-1, 0) \).

Hence from (36), for \( \alpha \geq 1 \), we have

\[
\int_{\tau - 1}^{\tau + 1} |v_B^{\frac{1}{2}}|^\alpha dt \leq C_\alpha \left( \int_{\tau - 2}^{\tau + 2} \int_0^{\frac{\pi}{4}} v^{\frac{1}{2}} d\mu dt \right)^{\frac{\alpha}{2}} \leq C_\alpha \left( \int_{\tau - 2}^{\tau + 2} v dt \right)^{\frac{\alpha}{2}},
\]

where in the last inequality we have used Hölder inequality. Therefore, from Lemma 5.1, for \( \alpha \geq 1 \) there exists \( \tilde{C}_\alpha > 0 \) independent of \( \tau > 0 \) such that

\[
\int_{\tau - 1}^{\tau + 1} |v_B^{\frac{1}{2}}|^\alpha dt \leq \tilde{C}_\alpha.
\]
Then from Lemma 5.2, we deduce that \( v(t, \theta) \) is bounded on \( \mathbb{R}_+ \times (0, \frac{\pi}{2}) \).

The proof of this lemma is complete.

In the rest of this section, we will prove Theorem 1.1 (iii): the limit behavior of positive axial symmetric solutions.

Recall that \( v(t, \theta) \) satisfies (28). Note that \( \sigma, \gamma > 0 \) if \( q > \frac{n-a}{n+a-2} \). We define the following energy function \( E(t) \) associated with (28) by

\[
E(t) := \frac{1}{2} \| \partial_t v \|_{2,a}^2 - \frac{\gamma}{2} \| v \|_{2,a}^2 - \frac{1}{2} \| \partial_\theta v \|_{2,a}^2 + \frac{1}{q+1} (v_B(t))^{q+1}.
\]

Simple calculation shows that

\[
\partial_t E(t) = -\sigma \| \partial_v v \|_{2,a}^2.
\]

**Lemma 5.4** Let \( q > \frac{n-a}{n+a-2} \) and \( v(t, \theta) \) be a positive bounded solution of (28) with \( \lim_{t \to -\infty} E(t) = 0 \). Then one has

\[
\lim_{t \to +\infty} v(t, \theta) = V(\theta) \text{ in } C[0, \pi/2].
\]

*Proof.* From the boundary condition in (28), \( a \in (-1,0) \) and the result that \( v(t, \theta) \) is bounded on \( \mathbb{R} \times (0, \pi/2) \), we have \( v_0(t, \frac{\pi}{2}) = 0 \). The elliptic regularity theory assures a boundedness of \( \partial_t v(t, \theta) \) and \( \partial_\theta v(t, \theta) \). Hence we deduce that \( E(t) \) is bounded and

\[
0 < \sigma \int_0^\infty \int_0^{\pi/2} v_t^2 \,d\mu \,dt = \lim_{t \to -\infty} E(t) - \lim_{t \to \infty} E(t) = -\lim_{t \to \infty} E(t) < \infty. \tag{37}
\]

Let \( \{ t_i \}_{i \in \mathbb{N}} \) be any sequence satisfying \( \lim_{i \to \infty} t_i = +\infty \) and set \( v_i(t, \theta) = v(t + t_i, \theta) \). Then there exists a subsequence of \( \{ t_i \}_{i \in \mathbb{N}} \), which is still denoted by the same symbol such that \( v_i(t, \theta) \) converges to some function \( v_\infty(t, \theta) \) in \( C([-1, 1] \times [0, \pi/2]) \). By (37) we find that \( \partial_t v_\infty(t, \theta) \equiv 0 \). Then we denote \( v_\infty(t, \theta) =: v_\infty(\theta) \) and \( v_\infty(\theta) \) satisfies

\[
-\gamma v + \Delta S_{a,\sigma} v = 0 \quad \text{in } (0, \pi/2), \quad \lim_{\theta \to \pi/2} v_\theta \cos^a \theta = v_B^a.
\]

By \( \lim_{t \to -\infty} E(t) = 0 \) and (37), we have \( \lim_{t \to \infty} E(t) < 0 \), which yields that the limiting function \( v_\infty(\theta) \) is not a trivial one. Hence by Proposition 2.2, we deduce that \( v_\infty(\theta) \equiv V(\theta) \). Therefore we obtain the result that \( \lim_{t \to \infty} v(t_i, \theta) = V(\theta) \) in \( C[0, \pi/2] \) for any sequence \( \{ t_i \}_{i \in \mathbb{N}} \) converging to \( +\infty \). Hence

\[
\lim_{t \to \infty} v(t, \theta) = V(\theta) \quad \text{in } C[0, \pi/2].
\]

The lemma is proved.

*Proof of Theorem 1.1 (iii)* Recalling that \( \psi_\infty(x) = V(\theta) r^{\frac{a-1}{q-1}} \) and \( u_\beta(x) = \beta u_1(\beta^{\frac{a-1}{q-1}} x) \), we have

\[
|u_\beta(x) - \psi_\infty(x)| = |\beta u_1(\beta^{\frac{a-1}{q-1}} x) \rho_{\frac{a}{q-1}} - V(\theta) x^{\rho_{\frac{a}{q-1}}} - V(\theta) |x|^{\frac{a-1}{q-1}}.
\]
Note that \( \lim_{t \to -\infty} v(t, \theta) = 0 \), since \( u \) is bounded. Then the condition \( \lim_{t \to -\infty} E(t) = 0 \) in Lemma 5.4 is true. Hence by Lemma 5.4 we have
\[
\lim_{\beta \to +\infty} |u_\beta(x) - \psi_\infty(x)| = 0, \quad x \neq 0.
\]
We complete the proof of Theorem 1.1 (iii).

6 Asymptotic expansion

In this section, we will prove Theorem 1.2. We need to study the asymptotic behavior of
\[
w(t, \theta) = V(\theta) - v(t, \theta).
\]
Note that \( w(t, \theta) \) satisfies
\[
\begin{cases}
w_{tt} + \sigma w_t - \gamma w + \Delta_{S,a}w = 0 & \text{in } \mathbb{R} \times (0, \frac{\pi}{2}), \\
\lim_{\theta \to \frac{\pi}{2}} w_\theta \cos \theta = qv_B^{q-1}w_B + g(w_B) & \text{on } \mathbb{R},
\end{cases}
\]
where \( g(w) \) is given by
\[
g(w) = v_B^q - (v_B - w)^q - qv_B^{q-1}w.
\]
Since \( L^2_{\text{sym}}(S^{n-1}_+) \) is spanned by the eigenfunctions \( \{e_i(\theta)\}_{i \in \mathbb{N}} \) of (23), then \( w(t, \theta) \) can be expanded as
\[
w(t, \theta) = \sum_{i=1}^{\infty} z_i(t)e_i(\theta), \quad t \in \mathbb{R}.
\]
Multiplying the equation in (23) by \( e_i(\theta) \sin^{n-2} \theta \cos^a \theta \) and integrating with respect to \( \theta \) on \( (0, \frac{\pi}{2}) \), we obtain
\[
z''_i + \sigma z'_i - (\gamma + \lambda_i)z_i = f_i(t),
\]
where \( f_i(t) := -g(w_B(t))e_i B \).

We first consider the case \( i \geq 2 \). Recall that Lemma 4.3 shows that \( \lambda_i > 0 \) for \( i \geq 2 \). Hence the corresponding quadratic equation
\[
\rho^2 + \sigma \rho - (\gamma + \lambda_i) = 0
\]
to (39) admits two real roots
\[
\rho_i^\pm = -\sigma \pm \sqrt{\sigma^2 + 4(\gamma + \lambda_i)}. \quad (41)
\]
Note that \( \rho_i^- < 0 < \rho_i^+ \). From (11) we know that for \( i \geq 2 \)
\[
z_i(t) = z_i(0)e^{\rho_i^- t} - \frac{e^{\rho_i^- t}}{\sqrt{\sigma^2 + 4(\gamma + \lambda_i)}} \int_0^t (e^{-\rho_i^- s} - e^{-\rho_i^+ s})f_i(s)ds
\] 
\[
- \frac{e^{\rho_i^+ t} - e^{\rho_i^- t}}{\sqrt{\sigma^2 + 4(\gamma + \lambda_i)}} \int_t^\infty e^{-\rho_i^+ s}f_i(s)ds. \quad (42)
\]
For the case $i = 1$, by Lemma 4.2 (39) admits two real roots $\rho^-_1 < \rho^+_1 < 0$ if $q$ is JL-supercritical, just one root $\rho_1 = -\frac{\sigma}{2}$ if $q$ is JL-critical and admits no real roots if $q$ is JL-subcritical. Hence we obtain for the JL-supercritical case

$$z_1(t) = z_1(0)e^{\rho^+_1 t} - \frac{z'_1(0) - \rho^-_1 z_1(0)}{\sqrt{\sigma^2 + 4(\gamma + \lambda_1)}}(e^{\rho^+_1 t} - e^{\rho^-_1 t}) + \int_0^t \frac{1 - e^{(2\rho^-_1 + \sigma)(t-s)}}{\sqrt{\sigma^2 + 4(\gamma + \lambda_1)}}e^{\rho^-_1 (t-s)} f_1(s)ds,$$

for the JL-critical case

$$z_1(t) = z_1(0)e^{\rho_1 t} + (z'_1(0) - \rho_1 z_1(0))te^{\rho_1 t} + \int_0^t (t-s)e^{\rho_1 (t-s)} f_1(s)ds$$

and for the JL-subcritical case

$$z_1(t) = \frac{1}{K} \left( \frac{\sigma}{2} z_1(0) + z'_1(0) \right) (\sin Kt)e^{-\frac{\rho_1}{2} t} + z_1(0)(\cos Kt)e^{-\frac{\rho_1}{2} t} + \frac{1}{K} \int_0^t [(\sin Kt)(\cos Ks) - (\sin Ks)(\cos Kt)]e^{-\frac{\rho_1}{2} (s-t)} f_1(s)ds,$$

where $K$, given in Theorem 1.2, is the imaginary part of a root of $\rho^2 + \sigma \rho - (\gamma + \lambda_1) = 0$.

To prove Theorem 1.2, we need to establish the following proposition.

**Proposition 6.1** (i) If $q$ is JL-supercritical, then there exists $\xi_1 > 0$ and $\varepsilon$, $C > 0$ such that

$$\|w(t, \theta) - \xi_1 e^{\rho^+_1 t} e_1\|_{\infty} \leq Ce^{(\rho^+_1 - \varepsilon)t}, \quad t > 0;$$

(ii) If $q$ is JL-critical, then there exists $\xi_1 > 0$, $\xi_2 \in \mathbb{R}$ and $\varepsilon$, $C > 0$ such that

$$\|w(t, \theta) - (\xi_1 t + \xi_2)e^{-\sigma t/2} e_1\|_{\infty} \leq Ce^{(\sigma - \varepsilon)t/2}, \quad t > 0;$$

(iii) If $q$ is JL-subcritical, then one of the following two expansions holds.

(iii-1) there exist $(\xi_1, \xi_2) \neq 0$ and $\varepsilon$, $C > 0$ such that

$$\|w(t, \theta) - \xi_1(\sin Kt) + \xi_2(\cos Kt))e^{-\sigma t/2} e_1\|_{\infty} \leq Ce^{-(\sigma + \varepsilon)t/2}, \quad t > 0;$$

(iii-2) there exist $\xi \neq 0$ and $\varepsilon$, $C > 0$ such that

$$\|w(t, \theta) - \xi e^{\rho^-_1 t} e_2\|_{\infty} \leq Ce^{(\rho^-_1 - \varepsilon)t}, \quad t > 0.$$

As a consequence of Proposition 6.1, we immediately obtain Theorem 1.2.

**Proof of Theorem 1.2** We only show the proof of JL-supercritical case in Theorem 1.2, since the proof of the rest part is similar. Let $q$ be JL-supercritical and set $\rho_1 = |\rho^+_1|$. By Proposition 6.1 we obtain

$$\|v(t, \theta) - (V - \xi_1 e^{-\rho_1 t} e_1)\|_{\infty} \leq Ce^{-(\rho_1 + \varepsilon)t}, \quad t > 0.$$
Going back to the original variable, we obtain
\[ u(r, \theta) = V(\theta) r^{-m_\theta} - \xi_1 r^{-m_\theta - \rho_1} e_1(\theta) + O(r^{-m_\theta - \rho_1 - \epsilon}), \quad r \gg 1. \]

Plugging the explicit expression of \( \sigma \) and \( \gamma \) into (11), we obtain the explicit expression of \( \rho_1 \), which coincides with the expression in Theorem 1.2.

Now the remaining task for us is to prove Proposition 6.1.

First we show that \( w(t, \theta) \) decays exponentially as \( t \to +\infty \).

**Lemma 6.1** There exists \( \varepsilon, \quad C > 0 \) such that
\[
\| w(t, \cdot) \|_{2,a} \leq C e^{-\varepsilon t}, \quad t > 0.
\]

**Proof.** From (42), we obtain for \( i \geq 2 \)
\[
|z_i(t)| \leq |z_i(0)| e^{\rho_i^{-}t} + \frac{e^{\rho_i^{-} t}}{\lambda_i} \int_0^t e^{-\rho_i^{-} s} |f_i(s)| ds + \frac{e^{\rho_i^{+} t}}{\lambda_i} \int_t^\infty e^{-\rho_i^{+} s} |f_i(s)| ds
\]
\[
\leq |z_i(0)| e^{\rho_i^{+} t/2} + \frac{e^{\rho_i^{-} t/2}}{\sqrt{\lambda_i |\rho_i^{+}|}} \left( \int_0^t e^{-\rho_i^{-} s} |f_i(s)|^2 ds \right)^{1/2}
\]
\[
+ \frac{e^{\rho_i^{+} t/2}}{\sqrt{\lambda_i |\rho_i^{+}|}} \left( \int_t^\infty e^{-\rho_i^{+} s} |f_i(s)|^2 ds \right)^{1/2}.
\]

By a trace inequality
\[
|e_{iB}| \leq C \| e_i \|_{2,a}^{1/2} \| \partial_\theta e_i \|_{2,a}^{1/2},
\]
we have
\[
\| \partial_\theta e_i \|_{2,a}^2 = 2 \| \partial_\theta e_i \|_{2,a}^2 - \| \partial_\theta e_i \|_{2,a}^2 = 2(q V_B^{-1} e_i^2 + \lambda_i \| e_i \|_{2,a}^2) - \| \partial_\theta e_i \|_{2,a}^2
\]
\[
\leq 2 \lambda_i \| e_i \|_{2,a}^2 + C q V_B^{-1} \| e_i \|_{2,a} \| \partial_\theta e_i \|_{2,a} - \| \partial_\theta e_i \|_{2,a}^2
\]
\[
\leq 2 \lambda_i \| e_i \|_{2,a}^2 + C \| e_i \|_{2,a}^2 \leq C \lambda_i \| e_i \|_{2,a}^2.
\]
Therefore, since \( \| e_i \|_{2,a} = 1 \), we deduce that \( \| \partial_\theta e_i \|_{2,a}^2 \leq C \lambda_i \). Then we have
\[
|e_{iB}| \leq C \| e_i \|_{2,a}^{1/2} \| \partial_\theta e_i \|_{2,a}^{1/2} \leq C \lambda_i^{1/2}.
\]
Recalling that \( f_i(t) = -g(w_B(t)) e_{iB} \), we obtain
\[
|f_i(s)| \leq C \lambda_i^{1/2} w_B^2(s) \leq C \lambda_i^{1/2} \| w(s, \cdot) \|_{2,a} \| \partial_\theta w(s, \cdot) \|_{2,a}.
\]
Since \( |\rho_i^{-}|, \ |\rho_i^{+}| \geq C \sqrt{\lambda_i} \) for some \( C > 0 \), we have for \( i \geq 2 \)
\[
|z_i(t)|^2 \leq C |z_i(0)|^2 e^{2\rho_i^{+} t} + \frac{C}{\lambda_i} ( \int_0^t e^{\rho_i^{-} (t-s)} \varphi(s) \| w(s, \cdot) \|_{2,a}^2 ds
\]
\[
+ \int_t^\infty e^{-\rho_i^{+} (s-t)} \varphi(s) \| w(s, \cdot) \|_{2,a}^2 ds
\]
\[
\leq C |z_i(0)|^2 e^{2\rho_i^{+} t} + \frac{C}{\lambda_i} ( \int_0^t e^{\rho_i^{+} (t-s)} \varphi(s) \| w(s, \cdot) \|_{2,a}^2 ds
\]
\[
+ \int_t^\infty e^{-\rho_i^{+} (s-t)} \varphi(s) \| w(s, \cdot) \|_{2,a}^2 ds,
\]
where $\varphi(s) := \|\partial_\theta w(s, \cdot)\|_{2, a}^2$. Applying Strum’s comparison theorem, we can obtain the following estimate of the eigenvalues of problem [23]

$$C_1 \iota^2 \leq \lambda_i \leq C_2 \iota^2, \quad \forall \iota > 1,$$

where $C_1$, $C_2$ are positive constants independence of $i$. Hence we have

$$\sum_{i=2}^{\infty} |z_i(t)|^2 \leq C \|w(0, \cdot)\|_{2, a}^2 e^{2\rho_k^2 t} + C \left( \int_0^t e^{\rho_k^2(t-s)} \varphi(s) \|w(s, \cdot)\|_{2, a}^2 ds \right)$$

$$+ \int_t^\infty e^{-\rho_k^2(s-t)} \varphi(s) \|w(s, \cdot)\|_{2, a}^2 ds.$$ 

Similarly, from [43]-[45], we obtain

$$|z_1(t)|^2 \leq \begin{cases} C e^{2\rho_1 t} + C \int_0^t e^{\rho_1^2(t-s)} \varphi(s) \|w(s, \cdot)\|_{2, a}^2 ds & \text{if } q \text{ is } JL\text{-supercritical}, \\
C t^2 e^{2\rho_1 t} + C \int_0^t (t-s)^2 e^{\rho_1(t-s)} \varphi(s) \|w(s, \cdot)\|_{2, a}^2 ds & \text{if } q \text{ is } JL\text{-critical}, \\
C e^{-\sigma t} + C \int_0^t e^{-\sigma(t-s)} \varphi(s) \|w(s, \cdot)\|_{2, a}^2 ds & \text{if } q \text{ is } JL\text{-subcritical}.
\end{cases}$$

Hence there exists $\delta > 0$ such that

$$\|w(t, \cdot)\|_{2, a}^2 \leq C e^{-\delta t} + C \left( \int_0^t e^{-\delta(t-s)} \varphi(s) \|w(s, \cdot)\|_{2, a}^2 ds + \int_t^\infty e^{-\delta(s-t)} \varphi(s) \|w(s, \cdot)\|_{2, a}^2 ds \right)$$

By Lemma 6.2 stated below, we deduce the desired conclusion.

**Lemma 6.2** ([11]) Assume $\eta(t)$ and $\varphi(t)$ are positive continuous functions defined on $\mathbb{R}_+$ converging to zero as $t \to +\infty$. Moreover $\eta(t)$ satisfies

$$\eta(t) \leq C e^{-\delta t} + C \left( \int_0^t e^{-\delta(t-s)} \varphi(s) \eta(s) ds + \int_t^\infty e^{-\delta(s-t)} \varphi(s) \eta(s) ds \right), \quad t > 0$$

for some $\delta > 0$. Then there exists $\varepsilon > 0$ such that $\eta(t) \leq Ce^{-\varepsilon t}$.

Next we will show more precise decay rates of $\|w(t, \cdot)\|_{2, a}$.

**Lemma 6.3** There exists $C > 0$ such that for $t > 0$

$$\|w(t, \cdot)\|_{2, a}^2 \leq \begin{cases} C e^{\rho_1^2 t} & \text{if } q \text{ is } JL\text{-supercritical}, \\
C(1 + t)e^{\rho_1 t} & \text{if } q \text{ is } JL\text{-critical}, \\
C e^{-\sigma t/2} & \text{if } q \text{ is } JL\text{-subcritical}.
\end{cases}$$

**Proof.** Repeating the argument given in the proof of Lemma 6.1, we have

$$\sum_{i=k}^{\infty} |z_i(t)|^2 \leq C e^{2\rho_k^2 t} + C \left( \int_0^t e^{\rho_k^2(t-s)} \varphi(s) \|w(s, \cdot)\|_{2, a}^2 ds \right)$$

$$+ \int_t^\infty e^{-\rho_k^2(s-t)} \varphi(s) \|w(s, \cdot)\|_{2, a}^2 ds.$$
Since \( w(t, \cdot) \) is uniformly bounded in \( \mathcal{H}^2_0(S^{n-1}_t) \), by an interpolation inequality, it holds that
\[
\| \partial_\theta w(t, \cdot) \|_{2,a}^2 \leq C \| w(t, \cdot) \|_{2,a} \| w(t, \cdot) \|_{\mathcal{H}^2_0(S^{n-1}_t)} \leq C \| w(t, \cdot) \|_{2,a}.
\]
Recalling \( \varphi(s) = \| \partial_\theta w(s, \cdot) \|_{2,a}^2 \), by Lemma 6.1 we have for \( k \geq 2 \)
\[
\sum_{i=k}^{\infty} |z_i(t)|^2 \leq C[e^{2\rho_i t} + e^{\rho_i t} + e^{-3\varepsilon t}].
\]
From (42)-(45), we have that for \( 2 \leq i \leq k - 1 \)
\[
|z_i(t)| \leq C e^{\rho_i t} + C e^{\rho_i t} \int_0^t e^{-\rho_i s} e^{-3\varepsilon s/2} ds + C e^{\rho_i t} \int_t^\infty e^{-\rho_i s} e^{-3\varepsilon s/2} ds
\]
\[
\leq C e^{\rho_i t} + e^{-3\varepsilon t/2} \leq C e^{\rho_i t} + e^{-3\varepsilon t/2}
\]
and
\[
|z_1(t)| \leq \begin{cases} 
C e^{\rho_1 t} + e^{-3\varepsilon t/2} & \text{if } q \text{ is JL-supercritical,} \\
C[(1 + t) e^{\rho_1 t} + e^{-3\varepsilon t/2}] & \text{if } q \text{ is JL-critical,} \\
C e^{-\sigma t/2} + e^{-3\varepsilon t/2} & \text{if } q \text{ is JL-subcritical.}
\end{cases}
\] (46)
We choose \( k \geq 2 \) large enough such that \( |\rho_k| \geq 2|\rho_2| \), then we obtain
\[
\sum_{i=2}^{\infty} |z_i(t)|^2 \leq C[e^{2\rho_1 t} + e^{-3\varepsilon t}].
\] (47)

We first consider the JL-supercritical case. If \( \varepsilon \geq \frac{2|\rho_1|}{3} \), from (46)-(47) we know that this lemma is true. If \( \varepsilon < \frac{2|\rho_1|}{3} \), from (46)-(47) we know that
\[
\| w(t, \cdot) \|_{2,a} \leq C e^{-3\varepsilon t/2}.
\]
Repeating the above procedure, if \( \frac{2|\rho_1|}{3} > \varepsilon \geq \frac{4|\rho_1|}{9} \), from (46)-(47) (now the quantity \( \varepsilon \) in these inequalities should be changed into the quantity \( \frac{3\varepsilon}{2} \)), we know that this lemma is also true. If \( \varepsilon < \frac{4|\rho_1|}{9} \), we know that
\[
\| w(t, \cdot) \|_{2,a} \leq C e^{-9\varepsilon t/4}.
\]
After repeating such procedure finite times, we can obtain
\[
\| w(t, \cdot) \|_{2,a} \leq C e^{-\rho_i t}.
\]
Therefore we complete the proof of the JL-supercritical case. The JL-critical case and JL-subcritical case can be proved similarly.

Furthermore we see that the decay rate of \( \| w(t, \cdot) \|_{2,a} \) given in Lemma 6.3 is exact one.
Lemma 6.4 There exists $\xi_1, \xi_2 \in \mathbb{R}$ and $\varepsilon$, $C > 0$ such that for $t > 0$

\[
\|e^{-\rho_1^i t} w(t, \cdot) - \xi_1 e_1\|_{2,a} \leq C e^{-\varepsilon t} \quad \text{if } q \text{ is JL-supercritical,}
\]
\[
\|e^{\sigma t/2} w(t, \cdot) - (\xi_1 t + \xi_2) e_1\|_{2,a} \leq C e^{-\varepsilon t} \quad \text{if } q \text{ is JL-critical,}
\]
\[
\|e^{\sigma t/2} w(t, \cdot) - (\xi_1 \sin(K t) + \xi_2 \cos(K t)) e_1\|_{2,a} \leq C e^{-\varepsilon t} \quad \text{if } q \text{ is JL-subcritical.}
\]

Moreover the norm $\| \cdot \|_{2,a}$ can be replaced by the norm $\| \cdot \|_{\infty}$.

Proof. By Lemma 6.3 and (47), for the JL-supercritical case, we have

\[
\| \cdot \|_{a}
\]

Moreover the norm $\| \cdot \|_{2,a}$ can be replaced by the norm $\| \cdot \|_{\infty}$.

Hence it is sufficient to estimate $z_1(t)$. Similarly, we also only need to estimate $z_1(t)$ in the JL-critical and JL-subcritical case.

We first consider the JL-supercritical case. We set $z_1(t) = e^{\rho_1^i t} y_1(t)$. From (39), $y_1(t)$ solves

\[
y''_1 + (2\rho_1^+ + \sigma)y'_1 = e^{-\rho_1^i t} f_1.
\]

Then since $e^{(2\rho_1^+ + \sigma)t} [y''_1 + (2\rho_1^+ + \sigma)y'_1] = (e^{(2\rho_1^+ + \sigma)t} y'_1)' = e^{(\rho_1^+ + \sigma)t} f_1$, we have

\[
y'_1(t) = y'_1(0) e^{-(2\rho_1^+ + \sigma)t} + e^{-(2\rho_1^+ + \sigma)t} \int_0^t e^{(\rho_1^+ + \sigma)s} f_1(s) ds.
\]

Hence by Lemma 6.3 there exists $\varepsilon > 0$ such that $|y'_1(t)| \leq C e^{-\varepsilon t}$, which yields that $\lim_{t \to +\infty} y_1(t)$ exists. We denote $\xi_1 := \lim_{t \to +\infty} y_1(t)$. So we have

\[
|y_1(t) - \xi_1| \leq \int_t^\infty |y'_1(s)| ds \leq C e^{-\varepsilon t},
\]

which shows (48) for the JL-supercritical case.

For the JL-critical case, we set $z_1(t) = t e^{\rho_1^i t} y_1(t)$. From (39), $y_1(t)$ satisfies

\[
t y''_1 + 2 y'_1 = e^{-\rho_1^i t} f_1.
\]

This and the equation $t^2 y''_1 + 2t y'_1 = (t^2 y'_1)'$ give that

\[
y'_1(t) = \frac{1}{t^2} \int_0^t s e^{-\rho_1^i s} f_1(s) ds
\]

\[
= \frac{1}{t^2} \int_0^\infty s e^{-\rho_1^i s} f_1(s) ds - \frac{1}{t^2} \int_t^\infty s e^{-\rho_1^i s} f_1(s) ds,
\]

which gives that $y'_1 \in L^1([1, \infty))$. So $\lim_{t \to +\infty} y_1(t)$ exists and we denote $\xi_1 := \lim_{t \to +\infty} y_1(t)$. We set $\xi_2 := \int_0^\infty s e^{-\rho_1^i s} f_1(s) ds$. Then, from (51), we have

\[
|y_1(t) - \xi_1 + \frac{\xi_2}{t}| \leq C e^{-\varepsilon t}, \quad t > 0
\]

23
for some $\varepsilon > 0$, which shows (48) for the JL-critical case.

Finally we consider the JL-subcritical case. From (45), we have

$$ z_1(t) = \frac{1}{K} \left( \frac{\sigma}{2} z_1(0) + z_1'(0) \right) (\sin Kt)e^{-\frac{\sigma}{2}t} + z_1(0) (\cos Kt)e^{-\frac{\sigma}{2}t} $$

$$ + \frac{e^{-\sigma t/2}}{K} \left( (\sin Kt) \int_0^t (\cos Ks)e^{\sigma s/2} f_1(s) ds - (\cos Kt) \int_0^t (\sin Ks)e^{\sigma s/2} f_1(s) ds \right). $$

We set $\alpha_1 := \int_0^t (\cos Ks)e^{\sigma s/2} f_1(s) ds$ and $\alpha_2 := \int_0^t (\sin Ks)e^{\sigma s/2} f_1(s) ds$. From Lemma 6.3, we deduce that $|\alpha_1|, |\alpha_2| < +\infty$. Therefore we obtain

$$ \left| z_1(t) - \frac{1}{K} \left( \frac{\sigma}{2} z_1(0) + z_1'(0) + \alpha_1 \right) (\sin Kt)e^{-\frac{\sigma}{2}t} - \left( z_1(0) - \frac{\alpha_2}{K} \right) (\cos Kt)e^{-\frac{\sigma}{2}t} \right| $$

$$ \leq \frac{1}{K} e^{-\frac{\sigma}{2}t} \int_t^\infty e^{\frac{\sigma}{2}s} |f_1(s)| ds, \quad (52) $$

which shows (48) for the JL-subcritical case.

Now we consider the $L^\infty$-estimate. For the JL-supercritical case, we set $Y(t, \theta) := e^{-\rho_t^+ t} w(t, \theta) - \xi_1 e_1(\theta)$. Then we have

$$ \left\{ \begin{array}{l}
Y_t + (2\rho_1^+ + \sigma) Y_i + \lambda_1 Y + \Delta_{S_0} Y = 0 \quad \text{in} \quad \mathbb{R} \times (0, \frac{\pi}{2}), \\
\lim_{\theta \to +} Y_\theta \cos \theta = q e_{n-1}^- Y_B + e^{-\rho_t^+ t} g(w_B) \quad \text{on} \quad \mathbb{R}.
\end{array} \right. \quad (53) $$

By the similar argument as given in Lemma 3.2, we know that there exists $C > 0$ such that

$$ \sup_{t_1 \leq \tau \leq t+1} ||Y(\tau, \cdot)||_{L^\infty} \leq C \left( \int_{t_2}^{t_2+1} ||Y(\tau, \cdot)||_{L^\infty}^2 d\tau \right)^{\frac{1}{2}} + C \sup_{t_2 \leq \tau \leq t+2} e^{-\rho_t^+ \tau} ||g(w_B(\tau))||. $$

Since $|g(w_B(\tau))| \leq C w_B^2(\tau)$, by a trace inequality and an interpolation, we have

$$ |g(w_B(t))| \leq C w_B^2(t) \leq C ||\partial_t w(t, \cdot)||_{L^2} ||w(t, \cdot)||_{L^2} \leq C ||w(t, \cdot)||_{L^2}^\frac{1}{2} ||w(t, \cdot)||_{L^2}^\frac{3}{2}. $$

Therefore, since $||w(t, \cdot)||_{L^2(S_0^{n-1})}$ is bounded, we deduce that $|g(w_B(t))| \leq C e^{3\rho_t^+ t^2/2}$, which yields the $L^\infty$-estimate for the JL-supercritical case.

For the JL-critical and JL-subcritical case, the argument is similar, we omit them.

The proof of this lemma is complete.

**Lemma 6.5** Let $q$ be JL-supercritical, then $\xi_1 > 0$, where $\xi_1$ is the constant given in the previous lemma.

**Proof.** Recalling that $\lim_{t \to +\infty} y_1(t) = \xi_1$, we know that there exists a sequence $\{t_i\}_{i \in \mathbb{N}}$ such that

$$ \lim_{i \to +\infty} y_1(t_i) = \xi_1, \quad \lim_{i \to +\infty} y_1'(t_i) = 0. $$

24
By the convexity of the function $w^2$, we have $w^2 \geq 0$ if $w \neq 0$, which yields that $f_1(t) = -g(w_B(t))e_{1B} > 0$ if $w \neq 0$. This and the fact that $2\rho_1^+ + \sigma > 0$, we conclude that $\xi_1 > 0$.

We complete the proof of this lemma.

**Lemma 6.6** Let $q$ be JL-critical, then $\xi_1 > 0$.

**Proof.** Since (50) can be written as $(t^2y'_1)' = te^{-\rho_1 t}f_1$, integrating on $(1,t)$, we obtain

$$ty'_1(t) = t^{-1}y'_1(0) + t^{-1}\int_1^t e^{-\rho_1 s}f_1(s)ds, \quad t > 1,$$

which gives that $\lim_{t \to \infty} ty'_1(t) = 0$. As a consequence, since $\lim_{t \to \infty} y_1(t) = \xi_1$, integrating (50) on $(t,\infty)$, we obtain

$$-(ty'_1(t) + y_1(t)) + \xi_1 = \int_t^\infty e^{-\rho_1 s}f_1(s)ds.$$

Since $z_1(t) = te^{\rho_1 t}y_1(t)$, we have $ty'_1(t) + y_1(t) = (-\rho_1 z_1(t) + z'_1(t))e^{-\rho_1 t}$.

Taking $t \to 0$, we have

$$\xi_1 = (-\rho_1 z_1(0) + z'_1(0)) + \int_0^\infty e^{-\rho_1 s}f_1(s)ds. \tag{54}$$

Similarly, integrating (50) on $(-\infty,t)$, we obtain

$$ty'_1(t) + y_1(t) = \int_{-\infty}^t e^{-\rho_1 s}f_1(s)ds,$$

where we have used the facts that $\lim_{t \to -\infty} ty'_1(t) = 0$ and $\lim_{t \to -\infty} y_1(t) = 0$. Since $ty'_1(t) + y_1(t) = (-\rho_1 z_1(t) + z'_1(t))e^{-\rho_1 t}$, taking $t \to 0$, we have

$$(-\rho_1 z_1(0) + z'_1(0)) = \int_{-\infty}^0 e^{-\rho_1 s}f_1(s)ds. \tag{55}$$

Add (54) to (55), we conclude that

$$\xi_1 = \int_{-\infty}^\infty e^{-\rho_1 s}f_1(s)ds,$$

which yields $\xi_1 > 0$. The proof of this lemma is complete.

For the JL-subcritical case, if $(\xi_1, \xi_2) = 0$, where $(\xi_1, \xi_2)$ is given in (18), then we will establish the following expansion with a smaller error.
Lemma 6.7 Let $q$ be JL-subcritical and $(\xi_1, \xi_2) = 0$, then there exists $\xi < 0$ and $C, \varepsilon > 0$ such that
\[ \| e^{\rho_2 t} w(t, \theta) - \xi e_2 \|_{2,a} \leq Ce^{-\varepsilon t}. \]
Moreover the norm $\| \cdot \|_{2,a}$ can be replaced by the norm $\| \cdot \|_{\infty}$.

Proof. Since $(\xi_1, \xi_2) = 0$, from (57) and (52), we have $|z_1(t)| \leq Ce^{3\rho_2 t/2}$. Then, by the same argument as in the proof of Lemma 6.3, we obtain
\[ \sum_{i=3}^{\infty} |z_i(t)| \leq C[e^{\rho_2 t} + e^{3\rho_2 t/2}], \quad |z_2(t)| \leq C e^{\rho_2 t}. \tag{56} \]
Now we set $y_2(t) = e^{-\rho_2 t} z_2(t)$. Then $y_2(t)$ solves
\[ y'' + (2\rho_2 + \sigma)y' = e^{-\rho_2 t} f_2. \tag{57} \]
By (56) we know that $y_2(t)$ is bounded, so there exists a sequence $\{t_i\}_{i \in \mathbb{N}}$ such that $\lim_{i \to \infty} y_2(t_i) = 0$. Then we have
\[ -e^{(2\rho_2 + \sigma)t} y_2'(t) = \lim_{i \to \infty} e^{(2\rho_2 + \sigma)t} y_2'(t_i) - e^{(2\rho_2 + \sigma)t} y_2'(t) = \int_{t}^{\infty} e^{(\rho_2 + \sigma)s} f_2(s) ds. \]
Hence from (56), we deduce $|y_2'(t)| \leq Ce^{\rho_2 t/2}$. As a consequence, there exists $\xi \in \mathbb{R}$ such that $\lim_{t \to \infty} y_2(t_i) = \xi$ and
\[ \| e^{-\rho_2 t} w(t, \cdot) - \xi e_2 \|_{2,a} \leq \| e^{-\rho_2 t} w(t, \cdot) - e^{-\rho_2 t} z_2(t)e_2 \|_{2,a} + \| (e^{-\rho_2 t} z_2(t) - \xi)e_2 \|_{2,a} \leq \| e^{-\rho_2 t} [w(t, \cdot) - z_2(t)e_2] \|_{2,a} + \| (y_2(t) - \xi)e_2 \|_{2,a} \leq Ce^{-\varepsilon t}, \]
where in the last inequality we have used (56), the facts that $|z_1(t)| \leq Ce^{3\rho_2 t/2}$ and exponent decay of $y_2'(t)$. Integrating (57) on $(-\infty, \infty)$, we obtain
\[ (2\rho_2 + \sigma)\xi = \int_{-\infty}^{\infty} e^{-\rho_2 s} f_2(s) ds. \]
Since $2\rho_2 + \sigma < 0$ and $f_2(s) = -g(w_B(t))e_{2B} > 0$, we deduce that $\xi < 0$. Applying the same argument as given in Lemma 6.4, we obtain the $L^\infty$-estimate.

We complete the proof of this lemma.

Proof of Proposition 2.1 By Lemmas 6.4, 6.7, we immediately obtain Proposition 2.1.

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