Binary Codes with Locality for Four Erasures

S. B. Balaji, K. P. Prasanth and P. Vijay Kumar, Fellow, IEEE
Department of Electrical Communication Engineering, Indian Institute of Science, Bangalore.
Email: balaji.profess@gmail.com, prasanthkp231@gmail.com, pvk1729@gmail.com

Abstract

In this paper, binary codes with locality for four erasures are considered. An upper bound on the rate of this class of codes is derived. An optimal construction for codes meeting the bound is also provided. The construction is based on regular bipartite graphs of girth 6 and employs the sequential approach of locally recovering from multiple erasures. An extension of this construction that generates codes which can sequentially recover from five erasures is also presented.

Index Terms

Distributed storage, codes with locality, sequential repair, multiple erasures.

I. INTRODUCTION

An \([n, k]\) code is said to have have locality \(r\) if each of the \(n\) code symbols of \(C\) can be recovered by accessing at most \(r\) other code symbols. Equivalently, there exist \(n\) codewords \(h_1 \cdots h_n\) in the dual code \(C^\perp\) such that \(c_i \in \text{supp}(h_i)\) and \(|\text{supp}(h_i)| \leq r + 1\) for \(1 \leq i \leq n\) where \(c_i\) denote the \(i\)th code symbol of \(C\) and \(\text{supp}(h_i)\) denote the support of the codeword \(h_i\).

a) Codes with Sequential Recovery: An \([n, k]\) code is defined as a code with sequential recovery \([1]\) from \(t\) erasures having locality \(r\) if for any set of \(s \leq t\) erased symbols, \(\{x_1, \cdots, x_s\}\), there is an arrangement of these \(s\) symbols (say) \(\{x_i, \cdots, x_{j}\}\) such that there are \(s\) codewords \(\{h_1, \cdots, h_s\}\) in the dual of the code, each of Hamming weight \(\leq r + 1\), with \(i_j \in \text{supp}(h_j)\) and \(\text{supp}(h_j) \cap \{i_{j+1}, \cdots, i_s\} = \emptyset\), \(\forall 1 \leq j \leq s\). The parameter \(r\) is the locality parameter and we will formally refer to this class of codes as \((n, k, r, t)_{\text{seq}}\) codes. When the parameters \((n, k, r, t)\) are clear from the context, we will simply refer to a code in this class as a code with sequential recovery.

A. Background

In \([2]\) P. Gopalan et al. introduced the concept of codes with locality (see also \([3], [4]\)), where an erased code symbol is recovered by accessing a small subset of other code symbols. The size of this subset denoted by \(r\) is typically much smaller than the dimension of the code, making the repair process more efficient compared to MDS codes. The authors of \([2]\) considered codes that can locally recover from single erasures (see also \([2], [5], [6], [7]\)).

The sequential approach introduced by Prakash et al. \([8]\) is one of the many approaches to locally recover from multiple erasures. Codes employing this approach have been shown to be better in terms of rate and minimum distance (see \([8], [9], [10], [11]\)). The authors of \([8]\) considered codes that can sequentially recover from two erasures (see also \([11]\)). Codes with sequential recovery from three erasures can be found discussed in \([11], [11]\).

Alternate approaches for local recovery from multiple erasures can be found in \([6], [12], [13], [14], [15], [16], [17], [18], [17], [19], [20], [21]\).

B. Our Contributions

In this paper, binary codes with locality for four erasures are considered. An upper bound on the rate of this class of codes is derived. A construction for codes achieving this bound with equality using the sequential approach is also provided. The construction uses regular bipartite graphs of girth 6. Finally, we show that this construction can be easily modified to generate codes with sequential recovery for five erasures.

P. Vijay Kumar is also an Adjunct Research Professor at the University of Southern California. This research is supported in part by the National Science Foundation under Grant 1421848 and in part by an India-Israel UGC-ISF joint research program grant.

S. B. Balaji would like to acknowledge the support of TCS research scholarship program.
II. Upper Bound on Rate

The following theorem gives an upper bound on the rate of binary $(n, k, r, 4)_{\text{seq}}$ codes.

**Theorem 1.** The rate of a binary $(n, k, r, 4)_{\text{seq}}$ code satisfies the following upper bound:

$$\frac{k}{n} \leq \frac{r^2}{r^2 + 2r + 2}.$$ 

**Proof:** Let $C$ be a binary $(n, k, r, 4)_{\text{seq}}$ code. Let,

$$B = \text{span} \left( c \in C^\perp, |\text{supp}(c)| \leq r + 1 \right).$$

i.e., $B_0$ is the span of all local parity-checks (those parity-checks having Hamming weight $\leq r + 1$). Let $m$ denote the dimension of the subcode $B$. We have

$$n - k \geq m.$$  \hspace{1cm} (2)

Choose $m$ linearly independent vectors from the set $\{c \in B, |\text{supp}(c)| \leq r + 1\}$, and form an $(m \times n)$ matrix $H'$ with these $m$ vectors in its rows. Let $s_1$ denote the number of columns of $H$ having a Hamming weight of $1$ and $s_2$ denote the number of columns having a Hamming weight of $2$. Permute the rows and columns of the binary matrix $H'$ to get a matrix $H$ as shown in (3). Note that this permutation does not affect the rank of the matrix or the Hamming weights of its rows/columns.

$$H = \begin{bmatrix} D_{s_1} & A & 0 \\ 0 & B & 0 \\ C & \end{bmatrix}$$ \hspace{1cm} (3)

The sub-matrix $[D_{s_1}]$ is an $(m \times s_1)$ matrix comprising the $s_1$ columns of $H$ having weight $1$. Note that out of the $s_1$ columns having weight $1$, no two columns can have a '1' in the same row, as the those columns would form a set of two linearly dependent columns preventing local recovery from $2$ erasures. Therefore, WLOG we can assume that $D_{s_1}$ is an $(s_1 \times s_1)$ matrix with $s_1$ ones on its diagonal and zeros elsewhere.

The sub matrix $[\frac{A}{B}]$ comprises those columns of $H$ having weight $2$ with one non zero element in the first $s_1$ rows and the second non zero element in the next $(m - s_1)$ rows. i.e., $A$ and $B$ are matrices having columns of Hamming weight one each. Note that among the columns having weight $2$, no column can have both its non zero entries in the first $s_1$ rows, as that column along with a set of $2$ columns of $D_{s_1}$ will form a set of three linearly dependent vectors, preventing local recovery from $3$ erasures. Hence, the remaining columns of $H$ which have a Hamming weight of $2$ form the sub matrix $[\frac{0}{C}]$. Each column of $C$ has Hamming weight $2$. The sub-matrix $D$ contains all columns of $H$ having Hamming weight three or more.

Let $s_{21}$ denote the number of columns of $[\frac{A}{B}]$ and $s_{22}$ denote that of $[\frac{0}{C}]$. Therefore, $s_{21} + s_{22} = s_2$.

Consider the matrix $B$. Assume that there is a row in $B$ with more than one non zero element. Consider the two columns where this row has its two non-zero elements. It is straightforward to see that these two columns, along with a set of at most two columns from $D_{s_1}$ forms a non zero set of at most $4$ linearly dependent columns, making local recovery from $4$ erasures impossible. Hence each row in $B$ can have at most one non zero element. It follows that

$$s_{21} \leq m - s_1.$$ 

Assume that $$s_{21} = m - s_1 - p,$$ \hspace{1cm} where $0 \leq p \leq m - s_1$.

Each of the $(m - s_1 - p)$ columns of $A$ has only one non-zero element. Each of the $s_1$ rows of $A$ can have at most $r$ non zero elements.

$$\therefore s_1 \geq \frac{m - s_1 - p}{r}$$ 

$$s_1 \geq \frac{m - p}{r + 1}$$ \hspace{1cm} (4)

The number of non zero elements in the sub-matrix $[B|C]$ is upper bounded by $(m - s_1)(r + 1)$. The sub-matrix $B$ consists of $m - s_1 - p$ non-zero elements (one in each column). Therefore the number of non zero elements in $C$ is upper bounded by $(m - s_1)(r + 1) - (m - s_1 - p)$. Each column of $C$ has $2$ non zero elements. Thus we have

$$s_{22} \leq \frac{(m - s_1)(r + 1) - (m - s_1 - p)}{2} = \frac{(m - s_1)r + p}{2}$$

$$\therefore s_2 = s_{21} + s_{22} \leq (m - s_1 - p) + \frac{(m - s_1)r + p}{2}$$ \hspace{1cm} (5)

Consider the matrix $H$. Each of the $m$ rows of $H$ can have at most $(r + 1)$ non-zero values. Thus we have:

$$s_1 + 2s_2 + 3(n - s_1 - s_2) \leq m(r + 1)$$ 

$$3n - 2s_1 - s_2 \leq m(r + 1)$$
Substituting the upper bound on \( s_2 \) in (5), in the above inequality gives:

\[
3n + s_1 \left( \frac{r}{2} - 1 \right) + \frac{p}{2} \leq m \left( \frac{3r}{2} + 2 \right).
\]

Substituting the lower bound on \( s_1 \) in (4), in the above inequality gives:

\[
3n + \frac{m - p}{r + 1} \left( \frac{r}{2} - 1 \right) + \frac{p}{2} \leq m \left( \frac{3r}{2} + 2 \right),
\]

\[
3n \leq m \left( \frac{3r}{2} + 2 - \frac{r - 2}{2(r + 1)} \right) + p \left( \frac{r - 2}{2(r + 1)} - \frac{1}{2} \right).
\]

The coefficient of \( p \) in the above inequality is negative for \( r \geq 1 \).

\[
\therefore 3n \leq m \left( \frac{3r}{2} + 2 - \frac{r - 2}{2(r + 1)} \right)
\]

From (2) and (6), we have:

\[
\frac{k}{n} \leq 1 - \frac{m}{n} \leq \frac{r^2}{r^2 + 2r + 2}.
\]

**III. Optimal Construction**

In this section we provide a construction that generates \((n, k, r, 4)_{\text{seq}}\) codes that achieve the upper bound on rate given in Theorem 1 with equality.

**Construction 1.** Consider an \( r \)-regular bipartite graph on \( 2L \) nodes, having girth at least 6. The edges of this bipartite graph represent information symbols of the code. A node in the graph represents a code symbol which is the parity of the \( r \) information symbols corresponding to the \( r \) edges connected to it. Consider \( r \) copies of this graph. Let \( n^{(i)}_l \) represent the \( l \)-th node of the \( i \)-th bipartite graph, \( 1 \leq l \leq 2L \), \( 1 \leq i \leq r \). The nodes in one copy of the bipartite graph are labeled as follows. Nodes labeled \( n^{(i)}_1 \ldots n^{(i)}_{L} \) are the nodes appearing on one side, say the left side, of the \( i \)-th bipartite graph and nodes \( n^{(i)}_{L+1} \ldots n^{(i)}_{2L} \) appear on the other side, (i.e., on the right side). Consider the set \( \{n^{(1)}_1, n^{(2)}_1, \ldots n^{(r)}_1\} \), the set of \( r \) nodes appearing in the first position in the \( r \) bipartite graphs. Nodes in this set are connected to a new node \( N_1 \) added to the graph. These connections are represented by using dotted lines as the corresponding edges do not represent code symbols. Node \( N_1 \) represents a new code symbol that holds the parity of the \( r \) parity symbols represented by the \( r \) nodes connected to \( N_1 \).

A total of \((2L)\) new nodes \( \{N_1 \ldots N_{2L}\} \) are added to the graph in a similar manner. Node \( N_j \) represents the code symbol that stores the parity of the \( r \) symbols represented by the nodes \( \{n^{(i)}_j, n^{(i)}_{j+1}, \ldots n^{(i)}_{2L}\}, 1 \leq j \leq 2L \), which are connected to \( N_j \).

Altogether the graph has \((Lr^2)\) solid edges and \((2Lr + 2L)\) nodes. Thus the code has \((Lr^2)\) information symbols and \((2Lr + 2L)\) parity symbols. Hence the code has a rate of \( \frac{r^2}{r^2 + 2r + 2} \), which matches the upper bound in Theorem 1.

An example code for the case \( r = 3, L = 7 \) is provided in Fig. 1.

![Fig. 1: An example code obtained from Construction 1 for \( r = 3, L = 7 \). Solid edges represent information symbols. Nodes represent parity symbols.](image-url)
Claim 1. A code generated by Construction 2 is an \((n, k, r, 4)_{\text{seq}}\) code.

Proof: Assume that \(x\) number of information symbols are erased and \(4-x\) parity symbols are erased. We consider the cases \(x=0, x=1, x=2, x=3\) separately and prove that sequential recovery is possible in each case. Throughout the proof we use the terms 'nodes' (or 'edges') and 'code symbols' interchangeably. Hence, whenever it is mentioned that a node \(n_{l}\) is erased, it means that the corresponding parity symbol is erased. Similarly, an edge is erased means that the corresponding information symbol is erased.

(I) \(x=0\): Each node \(n_{l}^{(i)}, 1 \leq l \leq 2L, 1 \leq i \leq r\) in the bipartite graph can be recovered locally from the edges (note that none of the edges are erased in this case). Once the erasures among nodes \(n_{l}^{(i)}, 1 \leq l \leq 2L, 1 \leq i \leq r\) are recovered, the erasures among the nodes \(N_1 \cdot \cdot \cdot N_{2L}\) can be recovered locally using nodes \(n_{l}^{(i)}, 1 \leq l \leq 2L, 1 \leq i \leq r\). Hence, sequential recovery is possible in this case.

(II) \(x=1\): Let \(n_{l}^{(i)}\) and \(n_{l'}^{(i)}\) be the two nodes appearing on the two ends of the one edge that is erased. Then, the erased edge can be recovered using two parity checks \(P_l\) and \(P_{l'}\) where, \(P_l\) is the parity-check involving node \(n_{l}^{(i)}\) and the edges connected to \(n_{l}^{(i)}\), and \(P_{l'}\) is the parity-check involving node \(n_{l'}^{(i)}\) and the edges connected to \(n_{l'}^{(i)}\). No more information symbol (i.e., edge) erasures are possible in this case. Hence the erased information symbol can be recovered locally unless both \(n_{l}^{(i)}\) and \(n_{l'}^{(i)}\) are erased.

Consider the case when both \(n_{l}^{(i)}\) and \(n_{l'}^{(i)}\) are erased, so that the erased information symbol cannot be recovered using \(P_l\) and \(P_{l'}\). Node \(n_{l}^{(i)}\) can be recovered using the parity-check involving node \(N_l\) and the \(r\) nodes connected to it. Similarly, \(n_{l'}^{(i)}\) can be recovered using the parity-check involving node \(N_{l'}\) and the \(r\) nodes connected to it. Also, note that these two parity-checks have disjoint support sets. Hence, irrespective of the fourth erasure, either \(n_{l}^{(i)}\) or \(n_{l'}^{(i)}\) can be recovered. Subsequently, the remaining erasures can be repaired sequentially.

(III) \(x=2\): We consider two different cases.

(a) Assume that both erased edges belong to the same bipartite graph, say the \(i^{th}\) bipartite graph. Note that any pair of edges in a bipartite graph can have at most one node in common. Hence, out of the \(2L\) nodes \(n_{l}^{(i)}, 1 \leq l \leq 2L\), there are at least two nodes that are connected to only one of the two erased edges. Let those nodes be \(n_{l}^{(i)}\) and \(n_{l'}^{(i)}\), \(l \neq l'\). Using similar arguments as in case (II), it can be proved that sequential recovery is possible in this case.

(b) Assume that the two erased edges belong to two different bipartite graphs. The edges are connected to \(4\) distinct nodes. No more edges can be erased in this case. Hence at least one of the erased edges can be recovered unless the \(4\) nodes are erased (which is not possible as it would make the total number of erasures 6). Once an erased edge is recovered, the remaining erasures can be repaired sequentially.

(IV) \(x=3\): The bipartite graph considered here has a girth of at least 6. Hence the code can recover from upto five (i.e., \((girth - 1)\), see [9]) information symbol erasures sequentially, if none of the nodes \(n_{l}^{(i)}, 1 \leq l \leq 2L, 1 \leq i \leq r\) are erased. Therefore, assume that a node \(n_{l}^{(i)}\) is erased. But this node can be recovered using the parity check involving \(N_l\) and the \(r\) nodes connected to it. Subsequently, the \(3\) erased information symbols can be recovered.

(V) \(x=4\): The bipartite graph considered here has a girth of at least 6. Hence, even when \(4\) edges are erased, there is at least one node that is connected to exactly one of the erased edges. Hence sequential recovery is possible.

IV. Construction for Five Erasures

The construction presented in this section generates an \((n, k, r, 5)_{\text{seq}}\) code by adding a few parity symbols to the \((n, k, r, 4)_{\text{seq}}\) code obtained from Construction II.

Construction 2. Consider a code obtained from Construction \(\text{II}\) Let

\[
A_i = \{n_{1}^{(i)} \cdot \cdot \cdot n_{L}^{(i)}\},
\]

\[
B_i = \{n_{L+1}^{(i)} \cdot \cdot \cdot n_{2L}^{(i)}\}.
\]

Consider the \(2L\) nodes \(N_1 \cdot \cdot \cdot N_{2L}\). Each of these \(2L\) nodes store the parity of \(r\) nodes in a fixed position in the \(r\) copies of the bipartite graph. Out of these \(2L\) nodes, let \(N_1 \cdot \cdot \cdot N_L\) represent the nodes connected to the nodes on one side of the bipartite graphs (see Fig. 2), i.e., nodes in \(\bigcup_{i=1}^{L} A_i\). The set \(\{N_1 \cdot \cdot \cdot N_L\}\) is partitioned into \(\left\lceil \frac{L}{r} \right\rceil\) sets of size \(r\) and one set of size \(L \mod r\). The nodes in each partition are connected to one of the \(\left\lceil \frac{L}{r} \right\rceil\) new nodes added to the graph. Each of the \(\left\lceil \frac{L}{r} \right\rceil\) new nodes represent the parity of the nodes connected to it. All five erasure patterns that the \((n, k, r, 4)_{\text{seq}}\) code cannot correct are of the form shown in Fig. 2. The new parity nodes added to the code can help in correcting each of this erasure patterns. Hence we obtain an \((n, k, r, 5)_{\text{seq}}\) code having a rate of \(\frac{Lr^2}{Lr^2 + 2Lr + 2L + \left\lceil \frac{L}{r} \right\rceil}\).
Fig. 2: An example code obtained from Construction 2 for $r = 3$, $L = 7$. 'X' indicates a typical five erasure pattern that the $(n, k, r, 4)$ seq code in Construction 1 cannot correct.

REFERENCES

[1] S. B. Balaji, K. P. Prasanth, and P. V. Kumar, “Binary codes with locality for multiple erasures having short block length,” CoRR, 2016. [Online]. Available: http://arxiv.org/abs/1601.07122

[2] P. Gopalan, C. Huang, H. Simidi, and S. Yekhanin, “On the Locality of Codeword Symbols,” IEEE Trans. Inf. Theory, vol. 58, no. 11, pp. 6925–6934, Nov. 2012.

[3] D. Papailiopoulos and A. Dimakis, “Locally repairable codes,” in Information Theory Proceedings (ISIT), 2012 IEEE International Symposium on, July 2012, pp. 2771–2775.

[4] F. Oggier and A. Datta, “Self-repairing homomorphic codes for distributed storage systems,” in INFOCOM, 2011 Proceedings IEEE, April 2011, pp. 1215–1223.

[5] C. Huang, M. Chen, and J. Li, “Pyramid codes: Flexible schemes to trade space for access efficiency in reliable data storage systems,” in Network Computing and Applications, 2007. NCA 2007. Sixth IEEE International Symposium on, July 2007, pp. 79–86.

[6] G. Kanath, N. Prakash, V. Lalitha, and P. Kumar, “Codes with local regeneration,” in Information Theory and Applications Workshop (ITA), 2013, pp. 1–5.

[7] I. Tamo and A. Barg, “A family of optimal locally recoverable codes,” IEEE Trans. Inf. Theory, vol. 60, no. 8, pp. 4661–4676, 2014.

[8] N. Prakash, V. Lalitha, and P. V. Kumar, “Codes with locality for two erasures,” CoRR, vol. abs/1401.2422, 2014. [Online]. Available: http://arxiv.org/abs/1401.2422

[9] A. Rawat, A. Mazumdar, and S. Vishwanath, “On cooperative local repair in distributed storage,” in Information Sciences and Systems (CISS), 2014 48th Annual Conference on, 2014, pp. 1–5.

[10] W. Song and C. Yuen, “Binary locally repairable codes - sequential repair for multiple erasures,” CoRR, vol. abs/1511.06034, 2015. [Online]. Available: http://arxiv.org/abs/1511.06034

[11] ——, “Locally repairable codes with functional repair and multiple erasure tolerance,” CoRR, vol. abs/1507.02796, 2015. [Online]. Available: http://arxiv.org/abs/1507.02796

[12] W. Song, S. H. Dau, C. Yuen, and T. Li, “Optimal locally repairable linear codes,” Selected Areas in Communications, IEEE Journal on, vol. 32, no. 5, pp. 1019–1036, May 2014.

[13] I. Zhang, X. Wang, and G. Ge, “Some improvements on locally repairable codes,” CoRR, vol. abs/1506.04822, 2015. [Online]. Available: http://arxiv.org/abs/1506.04822

[14] P. Huang, E. Yaakobi, H. Uchikawa, and P. H. Siegel, “Binary linear locally repairable codes,” CoRR, vol. abs/1511.06960, 2015. [Online]. Available: http://arxiv.org/abs/1511.06960

[15] I. Tamo, A. Barg, and A. Frolov, “Bounds on the parameters of locally recoverable codes,” CoRR, vol. abs/1506.07196, 2015. [Online]. Available: http://arxiv.org/abs/1506.07196

[16] J.-H. Kim, M.-Y. Nam, and H.-Y. Song, “Binary locally repairable codes from complete multipartite graphs,” in Information and Communication Technology Convergence (ICTC), 2015 International Conference on, Oct 2015, pp. 1093–1095.

[17] L. Shen, F. Fu, and X. Guang, “On the locality and availability of linear codes based on finite geometry,” IEICE Transactions, vol. 98-A, no. 11, pp. 2354–2355, 2015.

[18] A. Wang and Z. Zhang, “Repair locality from a combinatorial perspective,” in Proc. IEEE Int. Symp. Inf. Theory (ISIT), 2014, pp. 1972–1976.

[19] L. Panni-Juarez, H. Hollmann, and F. Oggier, “Locally repairable codes with multiple repair alternatives,” in Information Theory Proceedings (ISIT), 2013 IEEE International Symposium on, July 2013, pp. 892–896.

[20] A. S. Rawat, D. S. Papailiopoulos, A. G. Dimakis, and S. Vishwanath, “Locality and availability in distributed storage,” CoRR, vol. abs/1402.2011, 2014. [Online]. Available: http://arxiv.org/abs/1402.2011

[21] A. Wang and Z. Zhang, “Repair locality with multiple erasure tolerance,” IEEE Trans. Inf. Theory, vol. 60, no. 11, pp. 6979–6987, 2014.