GAUGING A NON-SEMI-SIMPLE WZW MODEL

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ABSTRACT

We consider gauged WZW models based on a four dimensional non-semi-simple group. We obtain conformal $\sigma$-models in $D = 3$ spacetime dimensions (with exact central charge $c = 3$) by axially and vectorially gauging a one-dimensional subgroup. The model obtained in the axial gauging is related to the $3D$ black string after a correlated limit is taken in the latter model. By identifying the CFT corresponding to these $\sigma$-models we compute the exact expressions for the metric and dilaton fields. All of our models can be mapped to flat spacetimes with zero antisymmetric tensor and dilaton fields via duality transformations.

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1. Introduction

A large class of Conformal Field Theories (CFT’s) can be constructed by using current algebras \[1\][2]. Models where the full symmetry of the action is realized in terms of current algebras are the WZW models \[3\] based on a group \(G\). By gauging an anomaly free subgroup \(H\) of \(G\) we obtain new conformal theories or coset models \(G/H\) \[2\][4][5]. Both WZW and gauged WZW models \[3\] are very important in understanding String theory since they provide exact solutions to it and the current algebra description makes the theories solvable. Compact groups were used in the construction of such models in String compactifications as internal theories where the space-time was taken to be Minkowskian. Later non-compact groups were used for non-compact current algebras and coset models that provided exact String models in \textit{curved space-time} \[7\][8]. Explicit constructions of the geometries corresponding to gauged WZW models was done in the recent years (see for example \[9\][10][11][12]) including all order corrections to the semiclassical (leading order) results (see for example \[13\][14]).

So far the attention has been concentrated to the cases where simple or semi-simple groups were used for reasons that will become obvious. In a recent paper Nappi and Witten \[15\] showed how to write a WZW action for a non-semi-simple group. They considered the algebra \(E_2^c\), i.e. the 2D Euclidean algebra with a central extension operator \(T\). The algebra for the generators \(T_A = \{P_1, P_2, J, T\}\) is\[1\]

\[
\begin{align*}
[J, P_i] &= \epsilon_{ij} P_j , \\
[P_i, P_j] &= \epsilon_{ij} T , \\
[T, J] &= [T, P_i] = 0 .
\end{align*}
\] (1.1)

Unlike the case of semi-simple algebras here the quadratic form \(\Omega_{AB} = f_{AC}^D f_{BD}^C\) is degenerate, i.e. its determinate is zero. This makes the straightforward writing of the corresponding WZW action problematic. However one can still resolve this problem by considering another quadratic form which satisfies the properties a) \(\Omega_{AB} = \Omega_{BA}\), b)\[1\]

\(^1\) This algebra appeared before in the context of contraction of Lie groups \[16\] and in studies of \((1 + 1)\)-dimensional gravity \[17\].
\( f^D_{AB} \Omega_{CD} + f^D_{AC} \Omega_{BD} = 0 \) and c) is non-degenerate, i.e. the inverse matrix \( \Omega^{AB} \) obeying \( \Omega^{AB} \Omega_{BC} = \delta^A_C \) exists. The first and the second properties ensure the existence of the quadratic and the Wess-Zumino term in the WZW action and the third one gives a way to properly lower and raise indices. The most general such quadratic form is \( \Omega^{AB} \)

\[
\Omega_{AB} = k \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & b & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}.
\]

Then parametrizing the group element as (summation over repeated indices is implied)

\[
g = e^{a_i P_i} e^{u^J + v^T},
\]

the corresponding WZW action takes the form \( S(u, v, a_i) = \frac{k}{2\pi} \int d^2 z \left( \partial a_i \bar{\partial} a_i + \partial u \bar{\partial} v + \partial v \bar{\partial} u + b \partial u \bar{\partial} u + \epsilon_{ij} \partial a_i a_j \bar{\partial} u \right) \).

From this action one can easily read off the corresponding metric and antisymmetric fields which represent a monochromatic plane wave. The corresponding exact CFT was identified as a solution of the Master equation of the generalized virasoro construction of \( SU(2) \otimes \mathbb{R} \) with central charge \( c = 4 \). The 1-loop solution is in fact exact since there are not any higher loop diagrams that can even be drawn \( \left[ 15 \right] \). In fact there is an alternative way to understand the absence of higher loop corrections. If we change variables as \( a_1 = x_1 + x_2 \cos u, \quad a_2 = x_2 \cos u, \quad v \rightarrow v + \frac{1}{2} x_1 x_2 \sin u, \)

then the action \( S(u, v, a_i) \) reads

\[
S = \frac{k}{2\pi} \int d^2 z \left( \partial x_1 \bar{\partial} x_1 + \partial x_2 \bar{\partial} x_2 + 2 \cos u \partial x_1 \bar{\partial} x_2 + b \partial b \bar{\partial} u + 2 \partial u \bar{\partial} v \right).
\]

In this form it can easily be shown that it is equivalent to a correlated limit of the WZW action for \( SU(2) \otimes \mathbb{R} \). For the latter model, if we parametrize the \( SU(2) \) group element as

\[
g = e^{a_1^L \theta_L} e^{a_2 \phi} e^{a_2^R \theta_R}
\]
and the translational factor $\mathbb{R}$ in terms of the time-like coordinate $y$, the action is

$$S = \frac{k'}{4\pi} \int d^2z \left( \partial\theta_L \bar{\partial}\theta_L + \partial\theta_R \bar{\partial}\theta_R + \partial\phi \bar{\partial}\phi + 2 \cos \phi \partial\theta_L \bar{\partial}\theta_R - \partial y \bar{\partial} y \right). \quad (1.7)$$

If we define $k' = 2k/\epsilon$, $\theta_L = \sqrt{\epsilon} x_1$, $\theta_R = \sqrt{\epsilon} x_2$, $\phi = \epsilon v + u$, $y = (1 - \epsilon b/2) u$ and take the limit $\epsilon \to 0$ the action (1.7) becomes identical to (1.6). The absence of higher loop corrections in (1.6) is then attributed to the fact that such corrections are also absent in the WZW action for $SU(2) \otimes \mathbb{R}$ (except for a trivial overall shift in the value of $k'$). Notice also that after the rescaling the periodic variables $\theta_L$ and $\theta_R$ take values, as $x_1$ and $x_2$, in the whole real line.

As in the case of WZW models based on simple or semi-simple groups one can obtain new conformal models by gauging anomaly free subgroups of (1.4). In this paper we consider the gauging of the WZW model of (1.3) with respect to the generator $J$ of the Cartan subalgebra. In section 2 we consider the axial, vector and chiral gauging cases. It will be shown that our $\sigma$-models can also be obtained if a specific limit in the 3D charged and neutral black string models based on $SL(2, \mathbb{R}) \otimes \mathbb{R}/\mathbb{R}$ are taken. In particular the limit is such that it ‘explores’ the region around the curvature singularities present in the latter models. By performing a duality transformation we show that all of our solutions can be mapped to flat spacetime solutions with zero antisymmetric tensor and constant dilaton showing that these singularities are harmless from the point of view of String theory. In section 3 we are identifying the exact CFT corresponding to the $\sigma$-model solution of the previous section as a particular case of the models of [19 20] with $c = 3$. Using the Hamiltonian for this CFT we compute the exact expressions for the metric and dilaton fields. We end this paper with some concluding remarks in section 4.

2. Axial, vector and chiral gauging

In this section we consider different gaugings of the WZW model for $E_2^c$ with respect to the $U(1)$ subgroup generated by $J$, i.e. $E_2^c/U(1)$. A more general gauging of the linear combination $J + \lambda T$ turns out to give an identical to the $\lambda = 0$ case results, up to a shift in the value of the constant $b$ in the expressions below (this has its origin in the fact that the algebra (1.4) is invariant under the redefinition $J \to J + \lambda T$).
2.1. Axial gauging

Consider first the case of the axial gauging (it turns to be the most interesting one) which is not anomalous since the subgroup generated by $J$ is abelian. In this case the gauged WZW action is (see for instance \cite{9,21})

$$S_{\text{axial}} = k \left[ I(hg \bar{h}) - I(h^{-1} \bar{h}) \right]$$

$$h = e^{-J\phi}, \quad \bar{h} = e^{-J\bar{\phi}}.$$  \hfill (2.1)

The action $S_{\text{axial}}$ is invariant under the following gauge transformations

$$\delta a_i = -\epsilon_{ij}a_j \epsilon, \quad \delta u = 2 \epsilon, \quad \delta v = 0, \quad \delta \phi = \delta \bar{\phi} = \epsilon, \quad \epsilon = \epsilon(z, \bar{z}).$$ \hfill (2.2)

The easiest way to realize that is to use the commutation relations (1.1) in order to rewrite the above action (2.1) in the form

$$S_{\text{axial}} = k \left[ I(e^{a_i' P_i} e^{u'J + v'T}) - I(e^{u''J}) \right],$$ \hfill (2.3)

where we have defined

$$a_i' = (\cos \phi \delta_{ij} + \sin \phi \epsilon_{ij})a_j, \quad u' = u - \phi - \bar{\phi}, \quad v' = v, \quad u'' = \phi - \bar{\phi}.$$ \hfill (2.4)

Gauge invariance of (2.3) under (2.2) is manifest upon realizing that $\delta a_i' = 0$. Using the explicit from of the action for the WZW model (1.4), and the formulae

$$\partial a_i' \partial a_i' = \partial a_i \partial a_i + a_i a_i \partial \phi \partial \bar{\phi} + \epsilon_{ij}(\partial \phi \partial a_i a_j + \partial a_i a_j \partial \phi)$$

$$\epsilon_{ij} \partial a_i' a_j' = \epsilon_{ij} \partial a_i a_j + a_i a_i \partial \phi,$$ \hfill (2.5)

one can cast $S_{\text{axial}}$ in (2.1), in the usual form of a gauged WZW model

$$S_{\text{axial}} = \frac{k}{2\pi} \int d^2z \left( \partial a_i \partial a_i + \partial u \partial v + \partial v \partial u + b \partial u \partial \bar{u} + \epsilon_{ij} \partial a_i a_j \partial \bar{u} ight.$$

$$\left. + A \left( \epsilon_{ij} \partial a_i a_j - 2\partial v + (a_i a_i - 2b)\partial u \right) - (\epsilon_{ij} \partial a_i a_j + 2\partial v + 2b\partial u) \bar{A} + (4b - a_i a_i) A \bar{A} \right),$$ \hfill (2.6)
where we have defined the gauged fields as $A = \partial \phi$ and $\bar{A} = \bar{\partial} \bar{\phi}$. To obtain the $\sigma$-model we have to fix the gauge and integrate over the gauge fields. A convenient gauge choice is $a_1 = 0$. The resulting $\sigma$-model action is (we will denote $\rho = a_2$)

$$S = \frac{k}{2\pi} \int d^2z \left( \partial \rho \bar{\partial} \rho + \frac{4}{4b - \rho^2} (-\partial v \bar{\partial} v + \frac{b}{4} \rho^2 \partial u \bar{\partial} u) + \frac{4b}{4b - \rho^2} (\partial v \bar{\partial} u - \partial u \bar{\partial} v) \right) \quad (2.7)$$

Let us assume that $b > 0$ and define the rescaled variables $r = \frac{\rho}{2\sqrt{b}}$, $x = v/\sqrt{b}$, $y = \sqrt{b} u$. Then the metric, the antisymmetric tensor one reads off from the action (2.7) and the dilaton induced from integrating out the gauge fields are given by

$$ds^2 = 4b \, dr^2 + \frac{1}{r^2 - 1} \, dx^2 - \frac{r^2}{r^2 - 1} \, dy^2$$

$$B_{xy} = -\frac{1}{r^2 - 1}, \quad B_{xr} = B_{yr} = 0$$

$$\Phi = \ln(r^2 - 1) + \text{const.}$$

Let us verify that the above fields do in fact solve the equations for conformal invariance in the 1-loop approximation. In $D = 3$ the antisymmetric field strength $H_{\mu \nu \rho} = 3 \partial_\rho B_{\mu \nu}$ has only one component which can be parametrized in terms of a scalar $H$ as $H_{\mu \nu \rho} = \epsilon_{\mu \nu \rho} H$.

In our case

$$H = \frac{1}{\sqrt{G}} \partial_r B_{xy} = -\frac{i}{\sqrt{b}} \frac{1}{r^2 - 1}.$$

Then in $D = 3$ the 1-loop equations for conformal invariance (see for instance [24]) can be written as [25]

$$\bar{\beta}^G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} H^2 G_{mn} - D_\mu D_\nu \Phi = 0$$

$$\bar{\beta}^B_{\nu \lambda} \epsilon^{\nu \lambda}_\mu = -e^{-\Phi} \partial_\mu (e^{\Phi} H) = 0$$

$$\bar{\beta}^\Phi = \frac{1}{6} (3 - c) + \frac{\alpha'}{4} [D^2 \Phi + (\partial_\mu \Phi)^2 - H^2] = 0.$$  

---

2 This gauge fixing introduces a Faddeev-Popov factor $(FP) \sim a_2$ in the path integral measure. This factor should combine with the original measure in the path integral for the gauged WZW model (in our case is the flat one) to give $e^{\Phi} \sqrt{G}$ [22] [23]. The fact that this is indeed the case can be verified using the appropriate expressions below.
We can explicitly compute the relevant tensors

\[
R_{rr} = -2 \frac{r^2 + 2}{(r^2 - 1)^2}, \quad R_{xx} = -\frac{1}{2b} \frac{r^2 + 1}{(r^2 - 1)^3}, \quad R_{yy} = \frac{1}{b} \frac{r^2}{(r^2 - 1)^3},
\]

\[
D_r D_r \Phi = -2 \frac{r^2 + 1}{(r^2 - 1)^2}, \quad D_x D_x \Phi = -\frac{1}{2b} \frac{r^2}{(r^2 - 1)^3}, \quad D_y D_y \Phi = \frac{1}{2b} \frac{r^2}{(r^2 - 1)^3},
\]

with the off-diagonal elements being zero and the scalars

\[
R = -\frac{1}{2b} \frac{2r^2 + 5}{(r^2 - 1)^2}, \quad D^2 \Phi = -\frac{1}{b} \frac{r^2 + 1}{(r^2 - 1)^2}.
\]

One then can easily verify that the equations (2.9) are indeed satisfied with central charge \(c = 3\). The signature of the spacetime in (2.8) is \((+, +, -)\). If \(b < 0\) one has to analytically continue \((r, x, y) \to (ir, ix, iy)\). Then the spacetime in (2.8) has signature \((+, +, +)\) and no singularity at all. The background defined in (2.8) is related to the corresponding one of the \(D = 3\) charged black string based on the coset model \(SL(2, \mathbb{R}) \otimes \mathbb{R}/\mathbb{R}\) through a limiting procedure. The expressions for the latter are \[10\]

\[
ds^2 = \frac{1}{\alpha'} (\frac{z - q_0 - 1}{z} \ dt^2 + \frac{z - q_0}{z} \ dx^2 + \frac{dz^2}{4(z - q_0 - 1)(z - q_0)}),
\]

\[
B_{xt} = \frac{1}{\alpha'} \sqrt{\frac{q_0(q_0 + 1)}{z}}, \quad \Phi = \ln z + \text{const.}
\]

If we take the following correlated limits \(z \to \epsilon (r^2 - 1), \ q_0 \to -1 - \epsilon, \ x \to \sqrt{\epsilon} \alpha' \ x, \ t \to -\sqrt{\alpha'} y\) with \(\alpha' \equiv \epsilon / 4b\) and \(\epsilon \to 0\) then (2.12) becomes (2.8). One might wonder what the physical meaning of such a correlated limit is. It is known \[10\] that the black string geometry (2.12) has a curvature singularity at \(z = 0\). Our limit corresponds to ‘magnifying’ the region around \(z = 0\). The outer and inner horizons for \(z = q_0 + 1\) and

\[3\] It is interesting to note that there is another limiting procedure one can follow in (2.12) (or by analytically continue (2.8)) leading to a solution with a different physical interpretation. Namely if \(z \to \epsilon (t^2 + 1), \ q_0 \to \epsilon, \ x \to \sqrt{\alpha'} x, \ t \to \sqrt{\alpha' \epsilon} y\) with \(\alpha' = \epsilon / b\) and \(\epsilon \to 0\) then (2.12) becomes \(ds^2 = -bdt^2 + (t^2 + 1)^{-1}(dy^2 + t^2 dx^2), \ B_{xy} = 1/(t^2 + 1)\) and \(\Phi = \ln(t^2 + 1)\). The above metric has the cosmological interpretation of an expanding and recollapsing Universe. String backgrounds obtained by considering various limiting procedures on already existing solutions can also be found in \[26\].
\( z = q_0 \) in the metric (2.12) dissapear in the above limit exactly because we ‘look’ at the region close to \( z = 0 \) (already inside the inner horizon). Let us note that if we had taken the (straightforward) limit of zero axionic charge \( q_0 \to -1 \) (essentially zero embedding of \( H = \mathbb{R} \) into the \( \mathbb{R} \) factor in \( G \) – up to an analytic continuation) we would have obtained the \( \sigma \)-model for the \( SL(2, \mathbb{R})/U(1) \otimes \mathbb{R} \) (the Euclidean black hole times a translation). The metric defined in (2.8) has a time-like curvature singularity at \( r^2 = 1 \) (corresponding to the black string singularity at \( z = 0 \)). However, from the point of view of string theory this is not very harmful because next we will show that the fields in (2.8) are related to the \( D = 3 \) Minkowski space-time (with constant dilaton and zero antisymmetric tensor) by means of a duality transformation. In contrast the background (2.12) is dual to the neutral black string (with \( q_0 = 0 \)) which is the 2D black hole times a \( U(1) \). The duality transformations for the case of one isometry read (the coordinate system is \((x^0, x^a)\)) \[27\]

\[
\begin{align*}
\tilde{G}_{00} &= \frac{1}{G_{00}}, & \tilde{g}_{0a} &= \frac{B_{0a}}{G_{00}}, & \tilde{G}_{ab} &= G_{ab} - \frac{G_{0a}G_{0b} - B_{0a}B_{0b}}{G_{00}} \\
\tilde{B}_{0a} &= \frac{G_{0a}}{G_{00}}, & \tilde{B}_{ab} &= B_{ab} - \frac{G_{0a}B_{0b} - G_{0b}B_{0a}}{G_{00}} \\
\tilde{\Phi} &= \Phi + \ln G_{00}.
\end{align*}
\]

Applying this transformation to the background in (2.8) for the isometry in the \( x \)-direction we obtain

\[
d\tilde{s}^2 = 4b \, dr^2 + (r^2 - 1) \, dx^2 - dy^2 - 2 \, dxdy
\]

\[\tilde{B}_{\mu\nu} = 0, \quad \tilde{\Phi} = \text{const.} .\]  

(2.14)

After we make the shift of the coordinate \( y \to y - x \) we see that the metric becomes the Minkowski metric in three dimensions, i.e. \( d\tilde{s}^2 = 4b \, dr^2 + r^2 dx^2 - dy^2 \). If we apply (2.13) for the isometry along the \( y \)-direction we get (we also have to shift \( x \to x + y \))

\[
d\tilde{s}^2 = 4b \, dr^2 + dy^2 - \frac{1}{r^2} \, dx^2
\]

\[\tilde{B}_{\mu\nu} = 0, \quad \tilde{\Phi} = \ln r^2 + \text{const.} .\]  

(2.15)

Obviously this solution can also be mapped into the background corresponding to the flat Minkowski space-time with zero antisymmetric tensor and constant dilaton by performing a duality transformation in (2.15) along the \( x \)-direction.
2.2. Vector and chiral gaugings

Let us now consider the case of the vector gauging. In this case the action is

\[ S_{\text{vector}} = k \left[ I(h^{-1}g\bar{h}) - I(h^{-1}\bar{h}) \right] \]

\[ h = e^{-J\phi}, \quad \bar{h} = e^{-J\bar{\phi}}. \]  

(2.16)

As in the case of the axial gauging it is convenient to cast (2.16) in the form

\[ S_{\text{vector}} = k \left[ I(e^{a'_i P_i}e^{u'J+\nu'T}) - I(e^{u''J}) \right], \]

(2.17)

but now with definitions

\[ a'_i = (\cos \phi \delta_{ij} - \sin \phi \epsilon_{ij})a_j, \quad u' = u + \phi - \bar{\phi}, \quad v' = v, \quad u'' = \phi - \bar{\phi}. \]

(2.18)

Then it is easy to check that the action (2.16) (or equivalently (2.17)) is invariant under the vector-like gauge transformations

\[ \delta a_i = \epsilon_{ij}a_j \epsilon, \quad \delta u = \delta v = 0, \quad \delta \phi = \delta \bar{\phi} = \epsilon, \quad \epsilon = \epsilon(z, \bar{z}). \]

(2.19)

Then

\[ S_{\text{vector}} = \frac{k}{2\pi} \int d^2z \left( \partial a_i \partial a_i + \partial u \partial \bar{v} + \partial v \partial \bar{u} + b \partial u \partial \bar{u} + \epsilon_{ij} \partial a_i \partial a_j \partial u \right. \]

\[ \left. + A (2\partial \bar{v} + (2b - a_i a_i) \partial u - \epsilon_{ij} \partial a_i a_j) - (2\partial v + 2b \partial u + \epsilon_{ij} \partial a_i a_j) \bar{A} + a_i a_i A \bar{A} \right). \]

(2.20)

The result of the integration over the gauge fields is (As before we fix the gauge by choosing \( a_1 = 0 \) and we denote \( \rho = a_2 \). We also perform the shifting \( v \to v - bu \))

\[ ds^2 = d\rho^2 - b \, du^2 + \frac{4}{\rho^2} \, dv^2 \]

\[ B_{\mu\nu} = 0, \quad \Phi = \ln \rho^2 + \text{const}. \]

(2.21)

Perhaps this simple result should have been expected since not only \( v \) but also \( u \) is inert under the vector transformations (2.19). It represents the dual space-time to the Euclidean space-time in 2D in polar coordinates [28] plus a time-like coordinate. Thus, the coset \( E_2^c/U(1) \) provides also a exact CFT description for the space (2.21) as well. In view of the
fact that the background in (2.8) can be obtained from that for the charged black string (2.12) by taking a correlated limit, one might expect that (2.21) may be derived from the dual of the charged black string [29], namely $SL(2, \mathbb{R})/\mathbb{R} \otimes \mathbb{R}$, via a similar limiting procedure. One can easily verify that (again the limit corresponds to a ‘magnification’ of the region around the 2D black hole singularity at $uv = 1$). Notice also that (2.21) is dual to an analytically continued version of (2.13).

Let us consider briefly the case of chiral gauging which, generically, gives rise to $\sigma$-models that are inequivalent with the corresponding models one obtains in the usual cases of axial and vector gaugings. Nevertheless, they will also be conformally invariant because the action one starts with, before integrating out the gauge fields, can be written as the sum of three independent WZW actions, as it was discussed in [30]. Namely, in the chiral gauging case the action has the form [31]

$$S_{\text{chiral}} = k \left[ I(hg\bar{h}) - I(h) - I(\bar{h}) \right]$$

$$= \frac{k}{2\pi} \int d^2 z \left( \partial a_i \bar{\partial} a_i + \partial u \bar{\partial} v + \partial v \bar{\partial} u + \epsilon_{ij} \partial a_i a_j \bar{\partial} u 
+ A \left( \epsilon_{ij} \bar{\partial} a_i a_j - 2 \bar{\partial} v + (a_i a_i - 2b) \bar{\partial} u \right) - (\epsilon_{ij} \partial a_i a_j + 2 \partial v + 2b \bar{\partial} u) \bar{A} + (2b - a_i a_i) A \bar{A} \right).$$ (2.22)

The above action is invariant under the chiral gauge transformations

$$\delta a_i = -\epsilon_{ij} a_j \epsilon, \quad \delta u = \epsilon + \bar{\epsilon}, \quad \delta v = 0, \quad \delta \phi = \epsilon, \quad \delta \bar{\phi} = \bar{\epsilon}, \quad \epsilon = \epsilon(z), \quad \bar{\epsilon} = \bar{\epsilon}(\bar{z}).$$

(2.23)

After integrating over $A, \bar{A}$ and make the shift $v \rightarrow v - \frac{b}{2} u$ we obtain

$$ds^2 = \frac{1}{a^2 - 2b} \left( (a_i a_i)^2 - 2b a_i a_i - 2b \epsilon_{ij} a_i a_j du + 4dv^2 + b(b - a^2) d\tilde{u}^2 \right)$$

$$B_{uv} = \frac{a^2}{a^2 - 2b}, \quad B_{iv} = \frac{2}{a^2 - 2b} \epsilon_{ij} a_j$$

$$\Phi = \ln(a^2 - 2b) + \text{const.},$$

(2.24)

where $a^2 \equiv a_i a_i$. Because there is no true gauge invariance in (2.22) (the parameters of the transformation in (2.23) are holomorphic or antiholomorphic) this is a $D = 4 \sigma$-model.
However by changing variables as: \( a_1 = r \cos \theta, \ a_2 = r \sin \theta \) and after a few rescalings of the variables and the shifting \( \theta \rightarrow \theta + u/2 \) one discovers that (2.24) reduces to the \( \sigma \)-model obtained in the axial gauging plus the action for a free boson corresponding to a time-like (space-like) coordinate if \( b > 0 \) (\( b < 0 \)). The spacetime has signature \((+,+,−\text{sign}(b),−\text{sign}(b))\). This is similar to a relationship between axially gauged and chiral gauged WZW models found in [30] for the case of simple groups and if the subgroup is an abelian one.

3. Operator method

In general there are \( \alpha' \sim 1/k \) corrections to the semiclassical expressions for the \( \sigma \)-model fields one obtains by integrating out the gauge fields, as we have done so far, in the gauged WZW action.\( ^4 \) One can find the exact expressions for the metric and dilaton fields we have obtained in (2.8), (2.21) by identifying the underlying exact CFT. As in was noted in [15] using the OPE expansions

\[
P_i P_j \sim \frac{\epsilon_{ij} T}{z - w} + \frac{\delta_{ij}}{(z - w)^2}, \quad JP_i \sim \frac{\epsilon_{ij} P_j}{z - w},
\]

\[
JJ \sim \frac{b}{(z - w)^2}, \quad JT \sim \frac{1}{(z - w)^2}, \quad TT \sim 0,
\]

one can show that the stress energy tensor defined as

\[
T_G = \frac{1}{2} : (P_i P_i + JT + TJ + (1 - b)T^2) :
\]

satisfies the Virasoro algebra with central charge \( c = 4 \). In fact this corresponds to a solution of the Master equation of [19][20]. It also obvious that \( T_H = \frac{1}{2\kappa} : J^2 : \) satisfies the

\( ^4 \) There exist regularization schemes in conformal perturbation theory in which the semiclassical results for \( SL(2, \mathbb{R})/\mathbb{R} \) and \( SL(2, \mathbb{R}) \otimes \mathbb{R}/\mathbb{R} \) solve the \( \beta \)-function equations to two loop order in perturbation theory [32][25] (for a different argument that such a scheme should exist see [33]). However, this is far from being a general conclusion for all gauged WZW models. Nevertheless, the exact expressions, as ones obtains them by making contact with the exact coset CFT, are needed in order to correctly describe the Klein-Gordon-type of equations for the string modes [32][23].
Virasoro algebra with central charge $c = 1$. The difference $T_{G/H} = T_{G} - T_{H}$ satisfies the same algebra with $c = 3$. What is also true is that $T_{G/H} J \sim 0$. The regularity of the last OPE makes it possible to gauge the corresponding symmetry.

It is convenient to express the zero modes of the holomorphic currents in (3.1) (and of the corresponding anti-holomorphic ones) as first order differential operators. We compute the ‘left’ and ‘right’ Cartan forms

$$g^{-1} dg = (\cos u \, da_j - \sin u \, \epsilon_{ij} da_i) \, P_j + du \, J + (dv + \frac{1}{2} \epsilon_{ij} da_i a_j) \, T \quad (3.3)$$

$$dgg^{-1} = (da_j - \epsilon_{ij} a_i \, du) \, P_j + du \, J + (dv - \frac{1}{2} \epsilon_{ij} da_i a_j - \frac{1}{2} a_i a_i \, du) \, T .$$

From the above we compute the following matrices defined as

$$L_{M}^{A} = \begin{pmatrix} \cos u & -\sin u & 0 & \frac{a_2}{2} \\ \sin u & \cos u & 0 & -\frac{a_1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad R_{M}^{A} = \begin{pmatrix} -1 & 0 & 0 & \frac{a_2}{2} \\ 0 & -1 & 0 & -\frac{a_1}{2} \\ -a_2 & a_1 & -1 & \frac{1}{2}(a_1^2 + a_2^2) \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (3.4)$$

and their inverses

$$L_{A}^{M} = \begin{pmatrix} \cos u & \sin u & 0 & -\frac{1}{2}(a_2 \cos u - a_1 \sin u) \\ -\sin u & \cos u & 0 & \frac{1}{2}(a_1 \cos u + a_2 \sin u) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad R_{A}^{M} = \begin{pmatrix} -1 & 0 & 0 & -\frac{a_2}{2} \\ 0 & -1 & 0 & \frac{a_1}{2} \\ a_2 & -a_1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (3.5)$$

The first order differential operators defined as $J_{A} = L_{A}^{M} \partial_{M}$, $\bar{J}_{A} = R_{A}^{M} \partial_{M}$ satisfy the commutation relations (1.1) and commute with each other. Explicitly they are given by

$$J_{P} = (\cos u \, \delta_{ij} + \sin u \, \epsilon_{ij}) \partial_{a_j} + \frac{1}{2} (\sin u \, \delta_{ij} - \cos u \, \epsilon_{ij}) a_j \partial_{v} , \quad J_{J} = \partial_{u} \, , \quad J_{T} = \partial_{v}$$

$$\bar{J}_{P} = -\partial_{a_i} - \frac{1}{2} \epsilon_{ij} a_j \partial_{v} \, , \quad \bar{J}_{J} = -\epsilon_{ij} a_i \partial_{a_j} - \partial_{u} \, , \quad \bar{J}_{T} = -\partial_{v} \, .$$

As usual the metric and the dilaton in any WZW model (gauged or not) can be deduced by comparing $[13][14]$ $HT = (L_0 + \bar{L}_0)T$ with

$$HT = -\frac{1}{\sqrt{G}} e^{\Phi} \partial_M \sqrt{G} \, e^{\Phi} \, G^{MN} \partial_N T \, ,$$

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where $H$ is the Hamiltonian of the corresponding CFT, $L_0$ and $\bar{L}_0$ are the zero modes of the holomorphic and antiholomorphic stress energy tensors and $T$ denotes tachyonic states of the theory annihilated by the positive modes of the holomorphic and antiholomorphic currents. For the case of the $D = 4$ WZW model the tachyon depends on all of the four group parameters, i.e. $T = T(a_1, a_2, u, v)$. In that case the forementioned comparison gives for the inverse metric

$$G^{MN} = \begin{pmatrix}
1 & 0 & 0 & -\frac{a_2}{2} \\
0 & 1 & 0 & \frac{a_1}{2} \\
0 & 0 & 0 & \frac{a_1}{4} + \frac{a_2}{4} + 1 - b \\
-\frac{a_2}{2} & \frac{a_1}{2} & 1 & \frac{a_1}{4} + \frac{a_2}{4} + 1 - b
\end{pmatrix} \tag{3.7}$$

which upon inverting it gives the metric corresponding to the $\sigma$-model (1.4), but with $b \to b - 1$. One might think of the shifting as a quantum correction to the semiclassical result in (1.4). However, this it does not really matter since one can absorb $b$ into a redefinition of $v$ (however it will be important for the gauged models that we will shortly consider). The dilaton turns out to be constant in this case as expected. For the $D = 3$ model case one should choose gauge invariant tachyonic states $T$ because of the gauge symmetry [14]. For the case of the axial gauging we have the constraint

$$(J_J - \bar{J}_J)T = 0 \quad \Rightarrow \quad T = T(\rho^2 = a_i a_i , v , x = a_1 \cos \frac{u}{2} + a_2 \sin \frac{u}{2}) \tag{3.8}.$$ 

Then the result for the inverse metric is (with $X^\mu = \{v, x, \rho\}$)

$$G^{\mu\nu} = \begin{pmatrix}
\frac{x^2}{4} - b + 1 & 0 & 0 \\
0 & 1 + \frac{1}{4b}(x^2 - \rho^2) & \frac{x}{\rho} \\
0 & \frac{x}{\rho} & \frac{1}{4b}
\end{pmatrix} \tag{3.9}.$$ 

After we invert it and change variables as $x = \rho \cos \frac{u}{2}$ we obtain the metric and dilaton

$$ds^2 = d\rho^2 + \frac{4}{\rho^2 - 4b + 4} \frac{du^2}{\rho^2 - 4b} \frac{b \rho^2}{\rho^2 - 4b} du^2 \tag{3.10}.$$ 

$$\Phi = \frac{1}{2} \ln\left((\rho^2 - 4b)(\rho^2 - 4b + 4)\right) + \text{const.} .$$ 

The above expressions become equivalent to the ones in (2.8) for large $b$. To obtain the result of the vector gauging one has to impose the constraint

$$(J_J + \bar{J}_J)T = 0 \quad \Rightarrow \quad T = T(\rho^2 = a_i a_i , u , v) \tag{3.11}.$$
In this case one obtains

\[ ds^2 = d\rho^2 + \frac{4dv^2}{\rho^2 + 4} - bdu^2, \quad \Phi = \frac{1}{2} \ln(\rho^2(\rho^2 + 4)) + \text{const.} \quad (3.12) \]

These results become equivalent to the ones in (2.21) after we rescale \( \rho \to b\rho \) and take the large \( b \) limit. For all cases, namely the \( D = 4 \) WZW and the \( D = 3 \) gauged WZW models one can verify that indeed the physical condition for closed strings \( (L_0 - \bar{L}_0)T = 0 \) is obeyed. It can be shown that (3.10) and (3.12) are related to the exact expressions for the metric and dilaton of the 3D charged black string [34] and the 2D black hole (times a free boson) [13] through a limiting procedure similar to the one we described in the previous section. Therefore their conformal invariance has already been checked against perturbation theory in [25] [35] [36].

4. Concluding remarks and discussion

We considered various gaugings of a one dimensional subgroup of a WZW model based on a four dimensional non-semi-simple group. We explicitly demonstrated that the three dimensional \( \sigma \)-models we have obtained can be derived by taking a correlated limit of models of gauged WZW based on semi-simple groups. In particular the limit we took corresponds to ‘magnifying’ or blowing up the region around the curvature singularities in the latter models. We show that our backgrounds, although they still have curvature singularities, can be mapped to flat spacetimes via duality transformations which renders these singularities harmless. In contrast the singularities of the original models which were based on semi-simple groups were more severe (although still not peculiar from the CFT point of view [1] [2]) in the sense that duality transformations cannot remove them, i.e. both the charged and the neutral black strings have curvature singularities.

\footnote{To compute the exact antisymmetric tensor one needs to use an effective action approach. Assuming that in our case it can also be obtained as a limit of the corresponding expression in the charged black string case one can show that both prescriptions of [25] give for it the semiclassical expression of (2.8).}
It will be interesting to construct other WZW and gauged WZW models based on non-
semi-simple groups. If the conclusions we drew by considering the particular examples of
this paper are generalizable certain gauged WZW models based on non-semi-simple groups
describe the geometry of gauged WZW models based on simple or semi-simple groups close
to the curvature singularities the latter models have. Then by duality transformations
we might be able to map these space (which still have curvature singularities) to non-
singular ones. This in turn is very important in order to understand better gravitational
singularities in the context of String theory.

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Noted added

1). While finishing the typing of the paper we received ref.[37], which contains some
overlapping materials with the present work. In particular gauging of the same WZW
model was considered, but with a subgroup generated by $P_1$ instead of $J$. The resulting
$\sigma$-model is different than ours but of course it also has $c = 3$. We were also informed about
some relevant work in [38].

2). After this paper was submitted for publication the paper [39] appeared where a large
class of WZW models based on non-semi-simple groups was constructed as a particular
contraction of the WZW model for $G \otimes H$. The model of [15] corresponds to $G = SO(3)$
and $H = SO(2)$. That explains why the action (1.6) can be obtained from (1.7) through a
limiting procedure. The gauging of the generator $J$ we have been considering corresponds
to a gauging of the total subgroup current in $G \otimes H$ in the models of [39]. That again gives
an explanation of the relation between, for instance, (2.8) and (2.12). The generalization
of the present work to cover gauged WZW models based on the non-semi-simple models
of [39] is currently under investigation. Also the paper [40] appeared where an explicit
expression for the WZW action based on the non-semi-simple group $E_{d}^c$, i.e. a central
extension of the Euclidean group in $d$-dimensions, was given.
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