Abstract. We prove the LeBrun-Salamon Conjecture in low dimensions. More precisely, we show that a contact Fano manifold $X$ of dimension $2n + 1$ that has reductive automorphism group of rank at least $n - 2$ is necessarily homogeneous. This implies that any positive quaternion-Kahler manifold of real dimension at most 16 is necessarily a symmetric space, one of the Wolf spaces. A similar result about contact Fano manifolds of dimension at most 9 with reductive automorphism group also holds.

The main difficulty in approaching the conjecture is how to recognize a homogeneous space in an abstract variety. We contribute to such problem in general, by studying the action of algebraic torus on varieties and exploiting Białynicki-Birula decomposition and equivariant Riemann-Roch theorems. From the point of view of $T$-varieties (that is, varieties with a torus action), our result is about high complexity $T$-manifolds. The complexity here is at most $\frac{1}{2}(\dim X + 5)$ with $\dim X$ arbitrarily high, but we require this special (contact) structure of $X$. Previous methods for studying $T$-varieties in general usually only apply for complexity at most 2 or 3.

Dedicated to Andrzej Szczepan Białynicki-Birula.

2010 Mathematics Subject Classification. Primary: 14L30; Secondary: 53C26, 53D10, 14J45, 14M17, 22E46.

Key words and phrases. Fano manifolds, quaternion-Kahler manifolds, complex contact manifolds, algebraic torus action, homogeneous spaces, adjoint action, localization in K-theory.

The authors are supported by the Polish National Science Center (NCN) project Algebraic Geometry: Varieties and Structures, 2013/08/A/ST1/00804. Buczyński is also supported by a scholarship of Polish Ministry of Science and by the NCN project Complex contact manifolds and geometry of secants, 2017/26/E/ST1/00231. Wiśniewski and Weber are supported by NCN project Algebraic Torus Action: Geometry and Combinatorics 2016/23/G/ST1/04828.
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1. Introduction

A complex manifold $X$ of dimension $2n + 1$ (with $n \geq 1$) is called a contact manifold if there exists a rank $2n$ vector subbundle $F \subset TX$ of the tangent bundle with a short exact sequence:

$$0 \to F \to TX \xrightarrow{\theta} L \to 0,$$

such that the derivative $d\theta|_F: \bigwedge^2 F \to L$ of the twisted form $\theta \in H^0(\Omega^1 X \otimes L)$ is nowhere degenerate. Moreover, we say $X$ is a contact Fano manifold, if in addition it is projective and $\bigwedge^{2n+1} TX \simeq L^\otimes(n+1)$ is an ample line bundle.

The geometry of complex contact manifolds attracts a lot of attention for a few notable reasons. It naturally generalizes the real case that appears in classical mechanics. It is motivated by a problem from Riemannian geometry, namely the Berger classification of all manifolds by their holonomy group \[7\]. Classification of nonsymmetric positive quaternion-Kähler manifolds (so one of the building blocks in the Berger list; in fact, this is the only building block, for which there is no compact example known) is equivalent to the classification of contact Fano manifolds admitting a Kähler-Einstein metric \[55\]. Further, contact manifolds connect geometry and representation theory via a version of a moment map \[6\], \[45\]. Finally, such varieties produce a fertile test ground for tools of higher dimensional algebraic geometry, such as Minimal Model Program, minimal rational curves, vector bundles. It is also strictly related to problems in Riemannian geometry, Kähler-Einstein metrics, non-compact hyperkähler manifolds, algebraic group actions and homogeneous spaces, dual varieties, and Legendrian varieties.

1.1. The LeBrun-Salamon conjecture in dimensions 12 and 16. A major open question in this area is the classification of projective contact manifolds. It is known that they all fit in one of three cases \[40\], \[25\], \[60\], \[44\]: if $X$ is a projective contact manifolds of dimension $2n + 1$ with the line bundle $L$ as above, then either

(i) $X = \mathbb{P}(T^*Y)$ for a projective manifold $Y$ of dimension $n + 1$ with $L \simeq \mathcal{O}_{\mathbb{P}(T^*Y)}(1)$, or
(ii) $X = \mathbb{P}^{2n+1}$ with $L = \mathcal{O}_{\mathbb{P}^{2n+1}}(2)$, or
(iii) $X$ is a contact Fano variety with $\text{Pic} \, X = \mathbb{Z}L$.

Therefore it remains to classify the last case, and the LeBrun-Salamon conjecture \[48\] claims that they are necessarily rational homogeneous spaces, more specifically, the adjoint varieties (see for instance \[17\].
Table 1 on p. 9 for more details). This conjecture has a reinterpretation in terms of twistor spaces of quaternion-Kähler manifolds: if $\mathcal{M}$ is a compact simply connected quaternion-Kähler manifold of real dimension $4n$ with positive scalar curvature, then $\mathcal{M}$ is (conjecturally) a symmetric space, and more specifically one of the Wolf spaces. Both versions of the conjecture are proven for $n = 1$ [36, 61], and $n = 2$ [53, 26]. It was also claimed for $n = 3$ by [34], however, see [35].

One consequence of the results of this article is the proof of the LeBrun-Salamon conjecture for $n = 3$ and $n = 4$.

**Theorem 1.1.** Suppose $\mathcal{M}$ is a compact simply connected quaternion-Kähler manifold of real dimension $4n$ with positive scalar curvature. If $n = 3$ or $n = 4$, then $\mathcal{M}$ is isometric (up to rescaling) to a symmetric space, one of the Wolf spaces: either the quaternion projective space $\mathbb{HP}^3$ or $\mathbb{HP}^4$, or the complex Grassmannian $Gr(\mathbb{C}^2, \mathbb{C}^5)$ or $Gr(\mathbb{C}^2, \mathbb{C}^6)$, or the real Grassmannian of oriented subspaces $\widetilde{Gr}(\mathbb{R}^4, \mathbb{R}^7)$ or $\widetilde{Gr}(\mathbb{R}^4, \mathbb{R}^8)$.

Also a slightly weaker version of the conjecture for complex contact Fano manifolds holds in (complex) dimensions 7 and 9.

**Theorem 1.2.** Suppose $X$ is a contact Fano manifold of dimension $d = 2n + 1$ with $3 \leq d \leq 9$ (equivalently, $1 \leq n \leq 4$), whose automorphism group is reductive. Then $X$ is a homogeneous space, explicitly, one of the following manifolds:

- a (complex) projective space $\mathbb{P}^d$ for $d = 3, 5, 7, 9$, or
- a projectivization of a cotangent bundle $\mathbb{P}(T^*\mathbb{P}^{n+1})$ for $n = 2, 3, 4$, or
- the 5-dimensional adjoint variety of $G_2$, or
- the Grassmannian of projective lines on a smooth quadric hypersurface $Gr(\mathbb{P}^1, \mathbb{Q}^{n+2})$ for $n = 3, 4$.

1.2. Contact Fano manifolds with reductive group of automorphisms of high rank. Let $X$ be a complex projective manifold of dimension $d$ with an ample line bundle $L$. We assume that $X$ admits an action of an algebraic torus $H$ of rank $r$ and the map $H \to Aut(X)$ has at most finite kernel. Our main interest is when $d = 2n + 1$ and $X$ is a contact Fano manifold as briefly defined above.

The main result of the present paper is the following:

**Theorem 1.3.** Let $X$ be a contact Fano manifold of dimension $2n + 1$, whose group of automorphisms $G$ is reductive and contains an algebraic torus $H$ of rank $n - 2$. Then $X$ is a homogeneous space. The complete list of all such manifolds is given in Table 1.
Because of the twistor construction the above theorem is related to results for quaternion-Kähler manifolds, see the surveys [54], [1], or [19], as well as [6], [10], [28], [29], [41] and references therein. A consequence of Theorem 1.3 is an analogous statement on the isometries of a quaternion-Kähler manifold.

**Theorem 1.4.** Let $\mathcal{M}$ be a positive quaternion-Kähler manifold of dimension $4n$. If the isometry group $\text{Isom}(\mathcal{M})$ has rank at least $n-2$, then $\mathcal{M}$ is isometric to one of the Wolf spaces.

See Table 2 for a comparison of Theorem 1.4 with earlier results in this direction. In particular, our result is the strongest known for $\dim \mathcal{M} \leq 36$, but weaker than [29] for dimension at least 48.

### 1.3. Projective manifolds with torus action.

In order to prove Theorem 1.3 we propose a new approach to algebraic manifolds with...
an action of a complex torus $(\mathbb{C}^\ast)^r$. This treatment might be of independent interest and it is described in details in Sections 2 and 3, which never mention “contact manifolds” explicitly. Our focus is on the components of fixed-point sets, some related polytopes, and a notion of compass, which locally describes the action near a fixed point. The narration is built in a way that imitates the classical correspondence between geometry of toric varieties and combinatorics of convex bodies, see for instance [24]. We briefly review this correspondence in the following paragraph.

Let $X$ be a smooth toric projective variety and $L$ an ample line bundle on $X$. Denote by $H = (\mathbb{C}^\ast)^{\dim X}$ the algebraic torus whose action on $X$ has an open orbit. The space sections of $L$ decomposes into eigenspaces of the action of $H$:

$$H^0(X, L) = \bigoplus_{u \in M \cap \Delta} \mathbb{C}_u,$$

where $M$ is the lattice of characters of $H$ and $\Delta = \Delta(X, L)$ is a lattice polytope in $M_{\mathbb{R}}$. Vertices of $\Delta$ can be identified with fixed points in $X^H$ and for any vertex $v \in \Delta$ the variety $\text{Spec}(\mathbb{C}[\mathbb{R}_{\geq 0}(\Delta - v) \cap M])$ is an affine neighborhood of the respective fixed point. The polytope $\Delta$ can be also obtained via the moment map.

In a more general situation, suppose $X$ is a projective manifold $X$ and again $L$ is an ample line bundle on $X$. Also assume a torus $H \simeq (\mathbb{C}^\ast)^r$ acts on $X$, and $\mu$ is a linearization of the action on $L$. From this data one can construct two polytopes: $\Gamma$, which comes from weights of global sections of $L$ (unlike in the toric case, the multiplicities do not need to be equal to 1), and $\Delta$, which comes from the linearization of the action on $L$ at fixed points. In nice situations the two polytopes coincide, see for instance Lemmas 2.4 and 4.7 and Proposition 3.9.
Another important ingredient is the compass which is analogous to the affine open neighborhood interpretation in the case of a toric variety. It encodes the characters of the action of $H$ on the (co)tangent space at a fixed point. There is a strict relation between the compass at a fixed point corresponding to $v \in \Delta$, and the polytopes $\Delta$ and $\Gamma$. Namely, the elements of the compass must be contained in the cone generated by $\Delta - v$, and (again, in a sufficiently nice situation) the cone and a related semigroup of lattice points associated to sections of multiples of $L$ must be generated by the elements of the compass.

The tool box we use includes Białynicki-Birula decomposition, Theorem 3.1, and the localization for torus action, Theorem A.1. As a result we develop new criteria to recognize a homogeneous space in some relatively abstract variety, by analyzing the fixed locus $X^H$ and the combinatorics of $\Delta$, $\Gamma$ and the compass — see for instance Propositions 2.24 and 3.12, Lemmas 2.22 and 3.15, and Corollary 3.14. The statements in this article are tailor made to the applications for contact Fano manifolds, but the potential of the methods is more general and yet to be discovered.

1.4. Content of the paper and an outline of the proof. In Section 2 of the article we consider a general situation when a torus $H$ acts on a projective manifold $X$, as described in Subsection 1.3. We introduce combinatorial objects that encode a lot of information about the action, and explain how these objects change, when we restrict the action to a smaller subtorus, or to an invariant submanifold, or if we replace the polarization by its tensor power, etc.

In Section 3 we describe the BB-decomposition (Theorem 3.1) and derive its applications relevant to the content of this article. The case of primary interest is when Pic $X = \mathbb{Z}$ and rank of $H$ is not too small so that we can use action of its subtori (downgrading) and the action of quotient tori on the fixed-point sets of these smaller tori (restriction). We relate the properties of fixed-point components to those of $X$, see Lemmas 3.4 and 3.6. The most accessible components of $X^H$ are the extremal components, which are associated to vertices of a polytope $\Delta$.

In Section 4 we use these tools to deal with contact manifolds. In the case considered in Theorem 1.3, when the rank of the torus is large with respect to the dimension of the contact manifold, two convex polytopes $\Gamma$ and $\Delta$ associated to the torus action coincide by Lemma 4.7. Here $\Gamma$ is a convex hull of weights of sections of the line bundle $L$. Results of [6] imply that $\Gamma$ is the convex hull of the roots of the group $G$ of contactomorphism of $X$. By contrast, the other polytope $\Delta$ is full dimensional in the character lattice of $H$. Therefore,
$G$ must be semi-simple, as we show in Lemma 4.6 and eventually simple by Proposition 4.8. Thus $\Delta(X, L, H)$ is the convex hull of roots in the space of weights of a simple group $G$, known as the root polytope of the group $G$.

In Section 5, we list a bunch of assumptions that are implied by the hypotheses of Theorem 1.3 and the results of earlier sections. Then we show Theorem 5.3 that under the above assumptions, there is only a short list of possible groups of automorphisms, or the dimension is too large to remain consistent with Theorem 1.3. In the proof we run a case-by-case analysis of root polytopes of simple groups, exploiting heavily the theory introduced in Sections 2 and 3 in very explicit situations. An important criterion is Proposition 2.24, which allows to recognize a homogeneous manifold from the properties of an action of the maximal torus.

In Section 6 we conclude the proofs of all theorems mentioned in the introduction and we review the relevant literature. In particular, we slightly strengthen a theorem of Salamon to show that the dimension of the automorphisms group of a contact Fano manifold in dimension 7 or 9 is bounded from below by 5 or 8, respectively (Theorem 6.1). We use this to eliminate the last remaining case left in the proof of Theorem 1.3. We also explain how the classification of contact Fano manifolds in dimension 7 and 9 follows (Theorem 1.2), and how to derive from the literature the appropriate results about quaternion-Kähler manifolds (Theorems 1.1 and 1.4).

In Appendix A we give an exposition of the Localization Theorem for torus action (Theorem A.1). It is derived from classical works by Atiyah–Bott, Grothendieck, Atiyah–Singer, and Berline–Vergne on localization in cohomology. In addition, we apply it to prove Corollary A.3, a special case, when the fixed-point locus consist of isolated points and curves only. This set-up provides a clear and manageable tool for calculating characters of torus representation on the space of sections of equivariant ample line bundles. Then the combinatorial data discussed in Section 2 are enough to determine the space of sections of a line bundle as an $H$-representation. The corollary is applied in the proofs in Subsection 2.6.

1.5. Notation. The following notation is used throughout the article.

- $X$ is a connected projective manifold over complex numbers of dimension $d$. In Sections 4 and 5 we will additionally assume that $X$ admits contact structure and $d = 2n + 1$. 
L is an ample line bundle over $X$. (We frequently assume that $\text{Pic } X = \mathbb{Z} \cdot L$.)

- If $X$ admits a contact structure (in Sections 4 and 5), then $\theta \in H^0(X, \Omega_X \otimes L)$ is a contact form on $X$ (see Section 4 for more details).

- $H$ denotes an algebraic torus acting on $X$, and $M$ is the lattice of characters (or weights) of $H$. We assume both $H$ and $M$ are of rank $r$. (We frequently suppose the action of $H$ is almost faithful, that is $H \rightarrow \text{Aut}(X)$ has finite kernel, or equivalently, there are only finitely many group elements that act trivially on $X$.) By a minor abuse we will shamelessly identify weights and characters throughout Sections 2–6. In Appendix A we will also use characters of representations of $H$ and we are careful to distinguish between weights and characters.

- $G$ is a connected reductive group with a maximal torus $H$, and $G$ acts almost faithfully on $X$.

- By $\mu$ or $\mu_L$ we denote a linearization of the action of $G$ or $H$ on the line bundle $L$. $\Gamma(L) = \Gamma(X, L, H, \mu)$ is the polytope of sections in $M_{\mathbb{R}}$, that is, the convex hull of the set of weights (eigenvalues) of the action of $H$ on $H^0(X, L)$, see Subsection 2.1 for details.

- $\mathcal{R}(X, L)$ is the ring of sections of $L^{\otimes m}, m \geq 0$, graded by $M \times \mathbb{Z}_{\geq 0}$.

- $X^H$ we denote the set of fixed points of the action of $H$.

- $\Delta(L) = \Delta(X, L, H, \mu)$ is the polytope of fixed points in $M_{\mathbb{R}}$, which is the convex hull of the characters $\mu(Y)$ with which $H$ acts on $L$ over a fixed-point component $Y \subset X^H$, see Subsection 2.1 for details. A component of $X^H$ associated to a vertex of $\Delta(L)$ is called extremal.

- The compass $C(Y, X, H)$ of the action of $H$ on a component $Y \subset X^H$ is the set of (nontrivial) characters $\nu_1, \ldots, \nu_{\dim X - \dim Y}$ (possibly, with repetitions) associated to the linearization of the action of $H$ on the conormal bundle of $Y$ in $X$, see Subsection 2.3 for details.

- For a semi-simple group $G$ by $\Delta(G)$ we denote the root polytope of $G$. That is, $\Delta(G)$ is the polytope in the lattice of characters of the maximal torus $H$, which is the convex hull of roots of $G$. If $G$ is simple and simply connected of type $A_r, B_r, \ldots, G_2$ then we also write $\Delta(A_r)$, etc.

Acknowledgments. We would like to thank Klaus Altmann, Roger Bielawski, and Aleksandra Borówka for many interesting conversations. We are also grateful to the participants of workshops in Oberwolfach (November 3–9, 2013) and in Levico Terme (November 2–6, 2015) for inspiring talks and discussions. We thank Michel Brion, Joseph Landsberg, Eleonora Romano, Uwe Semmelmann, and anonymous referees for their comments on the initial versions of the article.
2. Torus action and combinatorics

2.1. Polytopes of sections and of fixed points. We consider an ample line bundle \( L \) over \( X \) with an action of an algebraic torus \( H \). We will assume that \( H \to \text{Aut}(X) \) has at most finite kernel.

There exists a linearization \( \mu = \mu_L \) of the action of \( H \) on \( L \), [43, Prop. 2.4, and the following Remark]. Any two linearizations differ by a character in \( M \). For more details see [15]. To each \( p \in X^H \) we associate \( \mu(p) \in M \), the weight of the action of \( H \) on the fiber \( L_p \).

Let \( X^H = Y_1 \sqcup \cdots \sqcup Y_s \) be a decomposition of the fixed-point locus into connected components. Then each \( Y_i \) is smooth [49] and if \( y_1 \) and \( y_2 \) belong to the same connected component \( Y \) then \( \mu(y_1) = \mu(y_2) \) and we will denote this by \( \mu(Y) \). This determines lattice points associated to characters \( \mu(Y_1), \ldots, \mu(Y_s) \in M \). By \( \tilde{\Delta}(X, L, H, \mu) \) we denote the set of these characters and we define a polytope of fixed points \( \Delta(X, L, H, \mu) \) in \( M_\mathbb{R} = M \otimes \mathbb{R} \) as the convex hull of these points. We will frequently simplify the notation and when referring to \( \tilde{\Delta} \) or \( \Delta \) we will skip the elements of the quadruple \( (X, L, H, \mu) \) which are obvious from the context.

In the literature (for instance [14], [21]) polytope \( \Delta \) is also called the moment polytope and arises via the moment map. However, in the setting of contact manifolds, the meaning of moment map is ambiguous, as a different map is also called the moment map, see [6]. Thus to avoid confusion we refrain from using the word moment in either context. We also want to stress the difference between the two polytopes \( \Delta \) and \( \Gamma \) which are built on the fixed points and on the sections of possibly not spanned line bundle.

A component \( Y \subset X^H \) is called extremal if \( \mu(Y) \) is a vertex of \( \Delta(L) \).

A linearization \( \mu \) determines a decomposition into eigenspaces of the \( H \) action:

\[
\mathbb{H}^0(X, L) = \bigoplus_{u \in \tilde{\Gamma}(X, L, H, \mu)} \mathbb{H}^0(X, L)_u,
\]

where \( \mathbb{H}^0(X, L)_u \) denotes the eigenspace on which \( H \) acts with the weight \( u \in M \) and \( \tilde{\Gamma}(X, L, H, \mu_L) \) is the set of the weights (eigenvalues) of the action of \( H \) on \( \mathbb{H}^0(X, L) \). We define a lattice polytope of sections \( \Gamma(X, L, H, \mu) \) in \( M_\mathbb{R} \) as the convex hull of \( \tilde{\Gamma}(X, L, H, \mu) \). Again, we will frequently simplify the notation and skip the elements of the quadruple \( (X, L, H, \mu) \) which are obvious from the context.

As a digression, we remark that if \( L \) is very ample then \( \Gamma(L) \) is the image of the moment map, see [42, Ch. 2].
Observe that a linearization of $L$ determines a linearization of $L^\otimes m$ for $m \in \mathbb{Z}$. Therefore the action of $H$ determines the grading of the ring of sections

$$R(X, L) = \bigoplus_{m \geq 0} H^0(X, L^\otimes m) = \bigoplus_{m \geq 0} \bigoplus_{u \in M} H^0(X, L^\otimes m)_u. \quad (2.1)$$

We introduce the weight cone of $R(X, L)$, denoted by $\hat{\Gamma}(L) \subset M_\mathbb{R} \times \mathbb{R}$ and defined as

$$\hat{\Gamma}(L) = \mathbb{R}_{\geq 0} \cdot \left( \frac{\Gamma(L^\otimes m)}{m} \times \{1\} \right) = \mathbb{R}_{\geq 0} \cdot (\Gamma(L^\otimes m) \times \{m\}) \quad (2.2)$$

where $m$ is sufficiently large. This is a polyhedral cone, since $R(X, L)$ is finitely generated. If $\gamma \subset \hat{\Gamma}(L)$ is a ray (that is, a one-dimensional face), then we can define a homogeneous ideal in $R(X, L)$:

$$J_\gamma = \bigoplus_{m \geq 0} \bigoplus_{u \notin \gamma} H^0(X, L^\otimes m)_u. \quad (2.3)$$

The ideal $J_\gamma$ defines a closed $H$-invariant subset of $X$. Note that this subset is non-empty, since the Hilbert polynomial of $R(X, L)/J_\gamma$ is equal to $\dim H^0(X, L^\otimes m)_u$ for large values of $m$, which is non-zero. We will briefly use this set in the proof of Lemma 2.4, and then discuss it in more details in Corollary 2.8.

**Lemma 2.4.** In the situation above the following holds:

1. $m\Delta(L) = \Delta(mL)$ and $m\Gamma(L) \subseteq \Gamma(mL)$ for any integer $m > 0$,
2. $\hat{\Gamma}(L) \subset \Delta(L)$,
3. if $L$ is base point free then $\Delta(L) = \Gamma(L)$.

**Proof.** The first part follows directly from the definitions.

In the proof we will consider the following situation. Fix any integer $m > 0$, and suppose $y \in X^H$ is a fixed point that is not contained in the base point locus of $L^\otimes m$. Then we have the short exact sequence of $H$-modules:

$$0 \longrightarrow H^0(X, L^\otimes m \otimes m_y) \longrightarrow H^0(X, L^\otimes m) \longrightarrow L^\otimes m_y \longrightarrow 0, \quad (2.5)$$

where $m_y$ is the maximal ideal of $y$. Note that the sequence admits an $H$-equivariant splitting.

Now we argue for the second item. Assume by contradiction that $\hat{\Gamma}(L)$ is not contained in $\Delta(L)$. Then there exists a 1-dimensional face $\gamma$ of the weight cone $\hat{\Gamma}(L)$ such that the intersection $\gamma \cap M_\mathbb{R} \times \{1\}$ is not contained in $\Delta(L) \times \{1\}$. Consider the closed non-empty $H$-invariant subset of $X$ defined by the ideal $J_\gamma$ as in Equation (2.3). We claim, that the subset does not contain any $H$-fixed point. Indeed,
suppose $y$ is such a fixed point. By the equivariant splitting in the exact sequence (2.5) there exists a homogeneous section of $L^\otimes m$ (for $m$ sufficiently large) that does not vanish at $y$. Its degree determines a character in $\Delta(L^\otimes m) = m\Delta(L)$. Since $\gamma \cap \Delta(L) = \emptyset$, the section must be in $J_{\gamma}$. In other words, the section vanishes at $y$, a contradiction, which proves the claim. This in turn concludes the proof of (2), since $X$ is projective and there is no closed $H$-invariant subset of $X$ with no fixed points.

For the third part, by (2) we only have to prove $\Delta(L) \subset \Gamma(L)$. Using the equivariant splitting in the exact sequence (2.5) for $m = 1$ we find a homogeneous section that has the degree equal to $\mu(y)$, proving the inclusion.

\textbf{Corollary 2.6.} In the notation as above, suppose $H$ acts almost faithfully on $X$. Then $\Delta(L) \subset M_R$ is a lattice polytope of full dimension (that is, $\dim \Delta(L) = \dim M_R = \dim H$).

\textbf{Proof.} If $mL$ is very ample and defines an embedding of $X$ into a projective space $\mathbb{P}^N$, then $\Gamma(X, mL) = \Gamma(\mathbb{P}^N, O(1))$ is of full dimension since the action on $X$ and also on $\mathbb{P}^N$ is almost faithful. Thus $\Delta(mL)$ is of full dimension by Lemma 2.4(3) and also $\Delta(L)$ is of full dimension by Lemma 2.4(1). □

\textbf{Lemma 2.7.} Suppose $X'$ is a projective manifold, $L'$ is a line bundle, a torus $H$ acts on $X'$ with a linearization $\mu_{L'}$ of the action. Further assume $X \subset X'$ is a closed $H$-invariant submanifold, and $L = L'|_X$, and the linearization $\mu_L$ is the restriction of $\mu_{L'}$ to $X$. Then:

1. For all $y \in X^H$ we have $\mu_L(y) = \mu_{L'}(y)$.
2. $\Delta(X, L, H, \mu_L) \subset \Delta(X', L', H, \mu_{L'})$.
3. In the special case, where $X' = \mathbb{P}(H^0(X, L)^*)$, $L' = O(1)$, and the embedding $X \subset X'$ is given by the linear system, $\Delta(X', L', H, \mu_{L'}) = \Delta(X, L, H, \mu_L)$.

The proof is straightforward from definitions and, for Item 3, also using Lemma 2.4(3). We ommit the details.

As shown in Lemma 2.4 for sufficiently large $m$ the polytopes satisfy $m\Delta(L) = \Delta(L^\otimes m) = \Gamma(L^\otimes m)$. Thus the vertices of $\Delta(L)$ are equal to the generators of rays of the weight cone $\hat{\Gamma}(L)$ defined in Equation (2.2). Let $v \in \Delta(L)$ be a vertex and let $\gamma_v$ be the corresponding ray of $\Gamma(L)$. Consider the homogeneous ideal $J_{\gamma_v} \subset \mathcal{R}(X, L)$ as in Equation (2.3).

\textbf{Corollary 2.8.} The ideal $J_{\gamma_v}$ is radical and defines a closed subset of $X$ consisting of all those fixed points $y \in X^H$ that $\mu(y) = v$. 
Proof. Suppose $s \in \mathcal{R}(X, L)_v$ is a homogeneous element of the section ring such that $s^m \in \mathcal{J}_v$ for some integer $m > 0$. Then $mu \not\in \gamma_v$, hence $u \not\in \gamma_v$ and $s \in \mathcal{J}_v$. That is, $\mathcal{J}_v$ is radical as claimed.

We consider $m \gg 0$ such that $L^{\otimes m}$ is very ample and it equivariantly embeds $X$ into a projective space $\mathbb{P}^N = \mathbb{P}^0(X, L^{\otimes m})^*$ with the action of $H$ determined by the decomposition

$$H^0(X, L^{\otimes m}) = \bigoplus_{u \in \Gamma(L^{\otimes m})} H^0(X, L^{\otimes m})_u.$$

Suppose $y \in X \subset \mathbb{P}^N$ is such that all sections from $\mathcal{J}_v$ vanish on $y$. In particular, all sections from $\mathcal{J}_v \cap H^0(X, L^{\otimes m})$ vanish on $y$, thus $y \in \mathbb{P}((H^0(X, L^{\otimes m})^*)_{mv})$. All points in this subspace are fixed by the torus action, and by Lemma 2.7 the corresponding character is

$$mv = \mu_{\mathbb{P}^N(1)}(y) = \mu_{L^{\otimes m}}(y) = m\mu_L(y).$$

Conversely, suppose $y \in X$ is a fixed point of the torus action. Then $y \in \mathbb{P}^N$ is in a fixed-point locus, and the character $\mu_L(y)$ forces it to be in the linear space $\mathbb{P}((H^0(X, L^{\otimes m})^*)_{mv})$. Thus $\mathcal{J}_v_{mv}$ vanishes on $y$. The same happens for all sufficiently large $m$, and since the ideal is radical, whole $\mathcal{J}_v$ vanishes on $y$ as claimed. □

2.2. Reduction of torus action: downgrading and restriction. Let $H_1 \subset H$ be a subtorus with the quotient torus $H_2 = H/H_1$. Clearly the linearization $\mu$ of the action of $H$ on $L$ defines the linearization $\mu_1$ of the action of $H_1$. We have an exact sequence of the respective lattices of characters

$$0 \rightarrow M_2 \xrightarrow{\iota} M \xrightarrow{\pi} M_1 \rightarrow 0 \quad (2.9)$$

The following lemma describes the situation of downgrading of the action of $H$ to $H_1$ and the properties of the restriction of the action of $H$ to components of $X^{H_1}$ which yields the action of the quotient torus $H_2$.

Lemma 2.10. Let $H$’s, $M$’s and $\mu$’s be as above.

1. The projection $\pi: M \rightarrow M_1$ restricts to surjective maps of polytopes

$$\Gamma(X, L, H, \mu) \rightarrow \Gamma(X, L, H_1, \mu_1)$$

and

$$\Delta(X, L, H, \mu) \rightarrow \Delta(X, L, H_1, \mu_1).$$

2. Let $Y_1 \subset X^{H_1}$ be a connected component of the fixed-point locus of $H_1$. Then $Y_1$ is $H$ invariant and the action of $H$ on $Y_1$ descends to the action of the quotient $H_2$. The linearization $\mu$ of $L|_{Y_1}$ can be twisted by $\mu_1(Y_1) \in M$ which is any preimage of $\mu_1(Y_1) \in M_1$, and
this twisted linearisation $\mu - \mu_1(Y_1)$ induces a linearization $\mu_2$ of the action of $H_2$ on $L|_{Y_1}$. (We stress that $\mu_2$ depends on the choice of the preimage $\mu_1(Y_1) \in M$).

(3) Moreover, $Y_1^{H_2} = X^H \cap Y_1$ and

$$\tilde{\Delta}(Y_1, L|_{Y_1}, H_2, \mu_2) \subseteq \pi^{-1}(\mu_1(Y_1)) \cap \tilde{\Delta}(X, L, H, \mu) - \mu_1(Y_1).$$

Proof. The claims follow directly from definitions. Perhaps the only not completely trivial check is that $Y_1$ is $H$-invariant, which follows from commutativity and connectedness of $H$. Indeed, if $h \in H$ and $y_1 \in Y_1$, then for all $h_1 \in H_1$ we have $h_1 \cdot (h \cdot y_1) = h \cdot (h_1 \cdot y_1) = h \cdot y_1$. That is $h \cdot y_1 \in X^{H_1}$, and by connectedness of $H$, it must belong to the same connected component as $y_1$, which is $Y_1$.

The choice of $\mu_1(Y_1)$ ensures that the linearization $\mu - \mu_1(Y_1)$ of the action of $H$ is trivial after the restriction to $H_1$. $\square$

2.3. The compass. For every $y \in X^H$ we have a natural linearization of the action of $H$ on the cotangent space $T_yX^* = m_y/m_2^2$. This determines the set of weights of this action $\nu_1(y), \ldots, \nu_d(y) \in M$ (possibly with repetitions) which is the same for every $y$ in a connected component $Y \subset X^H$, hence denoted by $\nu_1(Y), \ldots, \nu_d(Y)$. The number of zero weights among $\nu_i(Y)$ is equal to $\dim Y$. The set of non-zero $\nu_i(Y)$’s, with possibly multiple entries, will be called the compass of the component $Y$ in $X$ with respect to the action of $H$ and denoted $C(Y, X, H)$. We will usually write $C(Y, X, H) = (\nu_1^{a_1}, \ldots, \nu_k^{a_k})$ where $\nu_i$’s are pairwise different elements of $M$ and the positive integers $a_i$ denote the number of occurrences of $\nu_i$ in the compass.

In the literature a similar notion of moment graph is discussed [13, 30]. In some references it is also called the GKM-graph [33]. Under the assumption that there are only isolated fixed points and finitely many one dimensional orbits, the behaviour of this graph encodes in particular the mutual relations between elements of compasses of various fixed points.

We first discuss a special case when the torus $H$ has dimension $1$.

Example 2.11. Let $H = \mathbb{C}^*$ be a torus acting on a vector space $V$ with weights $a_1, \ldots, a_s$, where $a_1 < \cdots < a_s$, and the associated eigenspaces $V_i$, $\dim V_i = d_i$. The action of $H$ on $X = \mathbb{P}(V)$ has $s$ fixed-point components $Y_i = \mathbb{P}(V_i)$ of dimension $\dim Y_i = d_i - 1$. There is a natural linearization $\mu$ of the action of $H$ on $L = \mathcal{O}(1)$ (so that $H^0(L) = V^*$) such that $\mu(\mathbb{P}(V_i)) = -a_i$. Then $C(Y_i, X, H) = ((a_i - a_1)^{d_1}, \ldots, (a_i - a_s)^{d_s})$ where the sequence omits the zero terms $(a_i - a_i)^{d_i}$. The extremal fixed-point components of the action of $H$ are $Y_1$.
and \( Y_s \). They coincide with those for which the sign of all the elements of the compass is the same. They are called the source and the sink of this action, as a general orbit “flows” from the highest weight \( \mu(Y_1) \) to the lowest weight \( \mu(Y_s) \).

**Lemma 2.12.** Let \( H = \mathbb{C}^* \) be the torus acting almost faithfully on a projective manifold \( X \) with an ample line bundle \( L \). Suppose \( v \) is a vertex of \( \Delta \) and \( Y \subset X^{\mathbb{C}^*} \) is a corresponding extremal component. Then:

1. if \( v \) is such that \( \Delta \subset v + \mathbb{R}_{\geq 0} \) then \( C(Y,X,H) \supset \mathbb{Z}_{>0} \) (in this situation, we say the extremal component \( Y \) is a sink),
2. if \( v \) is such that \( \Delta \subset v + \mathbb{R}_{\leq 0} \) then \( C(Y,X,H) \subset \mathbb{Z}_{<0} \) (in this situation, we say the extremal component \( Y \) is a source).

**Proof.** By passing to a multiple of \( L \) we only rescale \( \Delta \) (see Lemma 2.4(1)), thus we may assume that \( L \) is very ample. Then \( X \) is embedded equivariantly in the projective space on which the torus \( H \) acts as in Example 2.11 with \( V = H^0(X,L)^* \). By Lemma 2.7(3) we have the equality \( \Delta(\mathbb{P}(V),\mathcal{O}(1),\mathbb{C}^*) = \Delta(X,L,\mathbb{C}^*) \). Moreover, we have \( C(Y,X,H) \subset C(Y',\mathbb{P}(V),H) \) by definition of the compass (where \( Y' \) is the component of \( \mathbb{P}(V)^H \) containing \( Y \)). Thus the claim follows from Example 2.11. \( \square \)

Lemma 2.12 also has a partial converse. Namely, if \( Y \subset X^{\mathbb{C}^*} \) is a component of the fixed-point locus and the compass consists of only positive or only negative elements, then \( Y \) is an extremal component. To show this one uses Bialynicki-Birula decomposition (Theorem 3.1), but we will not use this statement here, so we skip the proof.

We have the following extension of Lemma 2.10.

**Lemma 2.13.** In the situation of Lemma 2.10 take connected components \( Y_1 \subset X^{H_1} \) and \( Y \subset Y_1 \cap X^H \). Then

\[
\mathcal{C}(Y_1,X,H_1) = \pi(\mathcal{C}(Y,X,H)) \quad \text{and} \quad \mathcal{C}(Y,Y_1,H_2) = \mathcal{C}(Y,X,H) \cap \ker \pi.
\]

**Proof.** Again, the proof is a direct consequence of the definitions. \( \square \)

**Corollary 2.14.** Let \( Y \subset X^H \) be a fixed-point component of the action of an algebraic torus \( H \) and \( \nu \in \mathcal{C}(Y,X,H) \) an element in the compass of \( H \) at \( Y \). Then there exists a component \( Y'' \subset X^H \) and \( \lambda \in \mathbb{Q}_{>0} \) such that \( \mu(Y'') = \mu(Y) + \lambda \cdot \nu \). In particular

\[
(\mu(Y) + \mathbb{Q}_{>0} \cdot \nu) \cap \Delta(X,L,H,\mu) \cap M \neq \emptyset.
\]
Proof. We restrict the action of \( H \) to the action of a torus \( H_1 \) obtained by projection of \( M \) along \( \nu \). Then there exists a component \( Y_1 \subset X^{H_1} \) which contains \( Y \). The action of the 1-dimensional torus associated to \( \nu \) on \( Y_1 \) has \( Y \) as a fixed-point component by Lemma 2.10(2) and \( \nu \) is in its compass by Lemma 2.13. Thus the claim of the corollary follows from the rank one case discussed in Lemma 2.12. \( \square \)

The usefulness of the notion of compass (particularly for the case of rank of \( H \) at least 2) lies in the ease of controlling the dimensions and fixed points. We illustrate it in details on specific applications in Section 5. Here we mention a general observation which allows us to even better control the directions and multiplicities of the compass.

Lemma 2.15. Suppose \( Y \subset X^H \) is a fixed-point component of dimension \( a \), and that the compass of \( Y \) in \( X \) contains \( b > 0 \) elements (counting with multiplicities) on some linear subspace \( W \subset M_\mathbb{R} \). Then for every \( \nu \in \mathcal{C}(Y,Y,Y) \cap W \) there exists a fixed-point component \( Y' \subset X^H \) such that:

- \( \mu(Y') - \mu(Y) \in \nu \cdot \mathbb{Q}_{>0} \),
- \( a' + b' = a + b \), where \( a' = \dim Y' \) and \( b' \) is the number of elements of \( \mathcal{C}(Y',X,H) \cap W \) (with multiplicities).

In particular, \( \dim Y < a' + b' \), where we can take \( a' \) and \( b' \) as in the second item, maximizing over all suitable \( Y' \).

Proof. Consider the subtorus \( H_1 \subset H \) which corresponds to the projection \( M \to M/(W \cap M) \). Let \( Y_1 \subset X^{H_1} \) be the fixed-point component containing \( Y \) and let \( H_2 = H/H_1 \). By Lemma 2.13 the compass \( \mathcal{C}(Y,Y_1,H_2) \) consists of \( b \) elements, in particular, \( \dim Y_1 = a + b > \dim Y \). By Corollary 2.14 there is a component \( Y' \neq Y \) of \( Y_1^{H_2} \) with \( \mu_2(Y') - \mu_2(Y) \in \nu \cdot \mathbb{Q}_{>0} \). We have \( \dim Y' + \#\mathcal{C}(Y',Y_1,H_2) = \dim Y_1 \) and \( Y' \) is a component of \( X^H \). By Lemma 2.10 we must have \( \mu_2(Y) - \mu_2(Y') = \mu(Y) - \mu(Y') \) so the first item is satisfied. Therefore, by Lemma 2.13 we have \( \mathcal{C}(Y',Y_1,H_2) = \mathcal{C}(Y',X,H) \cap W \) and the itemized properties hold. \( \square \)

Let \( y \in Y \subset X^H \) be a fixed point in a connected component \( Y \) with \( \dim Y = d_Y \). Then, up to an étale cover, the action of \( H \) can be diagonalized with weights associated to the compass of \( H \) at \( Y \), see [49, Lemma on p. 96]. That is, there exists a coordinate system \( x_1, \ldots, x_{d_Y} \) around \( y = \{ x_i = 0, i = 1, \ldots, d_Y \} \) such that for \( t \in H \) it holds

\[
(2.16) \quad t \cdot (x_1, \ldots, x_{d_Y}) = (t^{\mu_1} x_1, \ldots, t^{\mu_{d_Y}} x_{d-Y}, x_{d-Y+1}, \ldots, x_{d_Y}),
\]
where $C(Y, X, H) = (\nu_1, \ldots, \nu_{d-d_Y})$ and $t^\nu$ denotes the application of the character $\nu: M \to \mathbb{C}^*$ to $t \in H$. (In coordinates, if $t = (t_1, \ldots, t_r) \in H \simeq (\mathbb{C}^*)^r$, and $\nu = (\nu^1, \ldots, \nu^r) \in M \simeq \mathbb{Z}^r$, then $t^\nu = t_1^{\nu^1} \cdots t_r^{\nu^r}$.) In other words, the functions $x_i$ are graded by elements of $M$, with $\deg x_i = \nu_i$, for $i \leq d - d_Y$ and $\deg x_i = 0$ for $i > d - d_Y$.

**Lemma 2.17.** Let $D_\sigma \subset X$ be an $H$-invariant divisor associated to an $H$-equivariant section $\sigma \in H^0(X, L)_u$, where $u \in M$. Suppose that $y$ and $Y$ are as above. Then in the above local coordinates near $y$ the divisor $D_\sigma$ is described as the zero set of a homogeneous power series $f \in \mathbb{C}[x_1, \ldots, x_d]$ of degree $u - \mu(Y)$ with respect to the grading introduced above.

**Proof.** Let $z$ be a local coordinate in the fiber of $L$ around $y$. Then, locally around $y$ the bundle can be trivialized and the section $\sigma$ can be presented as $\sigma(x) = f(x) \cdot z$ where $f$ is a power series in $H$-homogeneous coordinates introduced above. By our assumption $\sigma$ is an eigenvector of the action of $H$ associated to the weight $u$ while $H$ acts on $z$ with the weight $\mu(Y)$. Thus $H$ acts on $f$ with weight $u - \mu(Y)$ and the claim follows. \qed

We remark that the power series $f$ appearing in the lemma can be chosen to be convergent in some neighbourhood of 0. We will not use this stronger statement.

**Corollary 2.18.** For every $u \in \tilde{\Gamma}(X, H, L, \mu)$ and a component $Y \subset X^H$ there exist non-negative integers $a_i$ such that

$$u - \mu(Y) = \sum a_i \nu_i$$

where $\nu_i$ are in the compass $C(Y, X, H)$.

**Proof.** The equality follows from calculating the degree of $f$ from the previous lemma, where $0 \neq \sigma \in H^0(X, L)_u$, and $f$ is the local equation of the zero set of $\sigma$. \qed

**Corollary 2.19.** Let $L$ be an ample line bundle on a manifold $X$ with an action of a torus $H$. Then the dimension of $X$ is bigger or equal to the maximal number of edges from any vertex of $\Delta(X, H, L)$.

**2.4. Examples.** Projective toric varieties with the action of the "big" torus or its downgrading to a smaller subtorus are the most natural examples to illustrate the objects that we have discussed so far. Then the relations between geometry of torus actions, including fixed points and orbits, and the properties of polytopes of sections, are described in a lot of details in classical works and modern textbooks, such as [24].
Quadrics are the next natural class of examples.

**Example 2.20.** The torus $H = (\mathbb{C}^*)^r$ with coordinates $(t_1, \ldots, t_r)$ acts on $\mathbb{C}^{2r+1}$:
\[(t_1, \ldots, t_r) \cdot (z_0, z_1, z_2, \ldots, z_{2r-1}, z_{2r}) = (z_0, t_1z_1, t_1^{-1}z_2, \ldots, t_rz_{2r-1}, t_r^{-1}z_{2r}) \].

The action of $H$ descends to an almost faithful action on the quadric $Q^{2r-1} \subset \mathbb{P}^{2r}$ given by the equation
\[z_0^2 + z_1z_2 + \cdots + z_{2r-1}z_{2r} = 0.\]

The action has $2r$ isolated fixed points. If $M$ is the lattice of characters of $H$ with the basis $e_1, \ldots, e_r$, then
\[\Delta(Q^{2r-1}, O(1), H) = \text{conv}(\pm e_1, \ldots, \pm e_r)\]
and we see that all fixed points are extremal. We will see in Corollary 3.14 that this shape of $\Delta$ with some additional properties essentially identifies a quadric hypersurface. The compass of $H$ at the fixed point associated to the character $e_i$ consists of $-e_i$ and $\pm e_j - e_i$, for $j \neq i$. Note that the compass generates the semigroup $\mathbb{R}_{\geq 0}(\Delta - e_i) \cap M$.

The case of even dimensional quadric is similar. The torus $H$ acts on $\mathbb{C}^{2r}$
\[(t_1, \ldots, t_r) \cdot (z_1, z_2, \ldots, z_{2r-1}, z_{2r}) = (t_1z_1, t_1^{-1}z_2, \ldots, t_rz_{2r-1}, t_r^{-1}z_{2r}) \].

The action of $H$ descends to an action of the quotient torus $H' = H/\langle(-1, \ldots, -1)\rangle$ on the quadric $Q^{2r-2} \subset \mathbb{P}^{2r-1}$ given by equation
\[z_1z_2 + \cdots + z_{2r-1}z_{2r} = 0\]

The action has $2r$ isolated fixed points. As before, $M = \mathbb{Z}^r$ generated by $e_i$’s and $M' \subset M$ is an index 2 sublattice of vectors $\sum_i a_i e_i$ such that $\sum_i a_i$ is even. Now
\[\Delta(Q^{2r-2}, O(1), H) = \text{conv}(\pm e_1, \ldots, \pm e_r)\]
and the compass of $H$ at the fixed point associated to the character $e_i$ consists of $\pm e_j - e_i$, for $j \neq i$. Note that the compass generates $\mathbb{R}_{\geq 0}(\Delta - e_i) \cap M'$.

In the present paper we will attempt to recognize homogeneous spaces $X$ with respect to some simple group $G$ only looking at the properties of the action of the maximal torus $H \subset G$ (or even a smaller torus) on $X$. For the rest of this subsection we assume (as an example) that $X$ is homogeneous with respect to $G$ in order to illustrate some characteristic properties of this situation. Then the action of $G$ (perhaps after taking a finite cover) admits a unique linearization...
and $H^0(X, L)$ is a $G$-module (representation) with associated set of $H$-weights $\widetilde{\Gamma}(X, H, L)$. In fact, the case of quadrics discussed above arises from the standard representations of $SO(2r)$ and $SO(2r+1)$.

The case of main interest for us is when $X$ is the closed orbit of $G$ in $\mathbb{P}(g)$, $g$ denoting the Lie algebra of $G$, and the group acts via the adjoint action. Then we have the root decomposition $g = h \oplus \bigoplus_{u \in \mathcal{R}} C_u$, where $h$ denotes the tangent space of $H$ on which the adjoint action is trivial and $\mathcal{R}$ is the set of roots. If $L$ denotes the restriction of $\mathcal{O}(1)$ from $\mathbb{P}(g)$ to $X$ then $\Gamma(X, L, H) = \Delta(X, L, H)$ is the root polytope $\Delta(G)$ which, by definition, is the convex hull of roots in the $M_\mathbb{R}$, with $M$ denoting the space of weights of $G$.

The following example illustrates the concept of downgrading and restricting the action of the maximal torus in the group $SO(7)$ (or more precisely, $Spin(7)$, the double cover of $SO(7)$).

**Example 2.21.** The figure on the left hand side presents the root system $B_3$ inscribed in the unit cube. Long roots are denoted by $\bullet$ and short roots by $\circ$. The figure on the right presents the polytope $\Delta(B_3)$. The only fixed points of the action of the 3-dimensional maximal torus $H \subset Spin(7)$ on $X = Gr(\mathbb{P}^1, Q^5)$ are extremal. There are 12 fixed points associated to long roots which are vertices of $\Delta$.

The next figure shows a downgrading of the action to a 2-dimensional torus: we take the planar projection along the diagonal in the cube. The number of the fixed points does not change but now only six of them are extremal.
If we further downgrade to 1-dimensional torus action then we get fixed-point set consisting of 5 components: 2 isolated points, \( \mathbb{P}^1 \) and two copies of \( \mathbb{P}^1 \times \mathbb{P}^1 \). The restriction of \( L \) to \( \mathbb{P}^1 \) is \( \mathcal{O}(2) \) and to each of \( \mathbb{P}^1 \times \mathbb{P}^1 \) is \( \mathcal{O}(2,1) \).

2.5. Root polytopes of exceptional Lie groups of types \( E \) and \( F \). Consider the root systems of type \( E_6, E_7, E_8 \) or \( F_4 \). Let \( M \) be the weight lattice of the simple Lie group/algebra of such type and \( \Delta(\cdot) \) be the convex hull of the roots from the system. We have the following restrictions on the dimensions of projective manifolds, which admit an action of a torus \( H \) corresponding to \( M \), such that the action resembles the adjoint action of \( H \) on the Lie algebra.

**Lemma 2.22.** Let \( X \) be a manifold with an ample line bundle \( L \). Suppose that \( X \) admits an almost faithful action of a torus \( H \) such that all extremal fixed-point components are isolated points.

1. if \( \Delta(X, L, H) = \Delta(E_6) \) then \( \dim X \geq 20 \),
2. if \( \Delta(X, L, H) = \Delta(E_7) \) then \( \dim X \geq 32 \),
3. if \( \Delta(X, L, H) = \Delta(E_8) \) then \( \dim X \geq 56 \),
4. if \( \Delta(X, L, H) = \Delta(F_4) \) and the set \( \tilde{\Gamma}(X, L, H) \) of weights of the action of \( H \) on \( H^0(X, L) \) contains all roots of \( F_4 \) then \( \dim X \geq 14 \).

**Proof.** We use the information from [12] Tables 5–8. By Corollary 2.19 the bound on the dimension of \( X \) for the cases \( E_i \) comes from the number of edges from a vertex of the respective polytope \( \Delta(E_i) \) which was calculated by *magma* [11].

In the case of \( F_4 \) the number of edges at a vertex is only 8, so it is not sufficient to prove the claim as stated. However, by Corollary 2.18 we need more elements in the compass. We use notation from [12] Table 8 and consider a long root \( v = e_1 + e_2 \) which is a vertex of \( \Delta(F_4) \). The edges of the cone \( \mathbb{R}_{\geq 0} \cdot (\Delta(F_4) - v) \) are spanned by \( \pm e_j - e_i \), with \( i = 1, 2 \) and \( j = 3, 4 \). Additionally, taking the short roots and subtracting \( v \) we get \( -e_1, -e_2 \) and \( (\pm e_3 \pm e_4 - e_1 - e_2)/2 \) which makes the total of 14.
note that all these points lie on a hyperplane \( \{ u : (e_1^* + e_2^*)(u) = -1 \} \) hence they are the minimal set which satisfies the condition for the compass from Corollary 2.18.

2.6. Comparing \( G \)-varieties. One of the difficulties in proving the cases of the LeBrun-Salamon Conjecture (see Subsection 1.1) is how to recognize a relatively abstract variety as a homogeneous space. The combination of torus action tools we reviewed so far and the localization for equivariant K-theory discussed in Appendix A result in the following relatively easy to test criterion. This arises as an application of an easy case of Corollary A.3 phrased explicitly as Proposition 2.23. A more refined use of this statement might lead to more powerful criteria, and should be a topic of further investigation. For our applications in Section 5 the following propositions are sufficient. Roughly, for a simple Lie group action on a projective manifold we look at the action of the maximal torus. If it has only finitely many fixed points, and the compasses can be compared to compasses of a similar action on a homogeneous space, then the manifold is isomorphic to the homogeneous space.

The proposition below explains, that if the action of \( H \) on \( X \) and \( X' \) and line bundles \( L \) and \( L' \) has the same combinatorial data, then the spaces of sections are isomorphic as \( H \)-representations.

**Proposition 2.23.** Suppose \( X \) and \( X' \) are two projective manifolds with an action of a torus \( H \) on both of them, such that \( X^H = \{ y_1, \ldots, y_k \} \) and \( (X')^H = \{ y'_1, \ldots, y'_k \} \) consist of isolated points, and are of the same cardinality. Assume \( L \) and \( L' \) are two ample line bundles on \( X \) and \( X' \), respectively, both with linearizations \( \mu \) and \( \mu' \), and both with vanishing higher cohomologies (for instance, if \( X \) and \( X' \) are Fano). If for all \( i \in \{1, \ldots, k\} \) both conditions hold:

- the compasses agree \( C(y_i, X, H) = C(y'_i, X', H) \), and
- the characters agree \( \mu(y_i) = \mu'(y'_i) \),

then \( H^0(X, L) \simeq H^0(X', L') \) as representations of \( H \).

**Proof.** The fixed points and related combinatorial data agree, so Corollary A.3 in Appendix A implies that also the equivariant Euler characteristic must agree \( \chi^H(X, L) = \chi^H(X', L') \). Since higher cohomologies of both \( L \) and \( L' \) vanish, the isomorphism classes of the representations \( H^0(X, L) \) and \( H^0(X', L') \) are uniquely determined by the equivariant Euler characteristic.

In the next proposition under much stronger assumptions we show that not only spaces of sections are isomorphic but actually also \( X \) is isomorphic to \( X' \).
Proposition 2.24. Suppose that a semisimple group $G$ with a maximal torus $H$ acts on a projective manifold $X$. Assume in addition that the restricted action of $H$ has only finitely many fixed points, $X^H = \{y_1, \ldots, y_k\}$. Denote by $C_i = \{\nu_1(y_i), \ldots, \nu_d(y_i)\}$ the compass of the action of $H$ at $y_i$. Let $L$ be an ample line bundle on $X$ and $\mu$ a linearization of the action of $G$ on $L$. If there is a $G$-homogeneous manifold $(X', L')$ with $(X')^H = \{y'_1, \ldots, y'_k\}$ and linearization $\mu'$ on $L'$ such that $C(y_i, X', H') = C_i$ and $\mu'(y'_i) = \mu(y_i)$, then there exists an isomorphism $(X', L') \cong (X, L)$.

Proof. By passing to a multiple of $L$ (and taking the same multiple of $L'$) we may assume that $L$ is very ample with no higher cohomology. By Proposition 2.23, $H^0(X, L)$ and $H^0(X', L')$ are isomorphic as $H$-modules. Thus, by the representation theory, they are isomorphic as $G$-modules \cite[Thm 23.24(b)]{31}. Being homogeneous, $X' \subset \mathbb{P}(H^0(X', L')^*)$ is the unique closed orbit of the action of the semisimple group $G$ on $\mathbb{P}(H^0(X', L')^*)$. Therefore, via the isomorphism $H^0(X', L') \cong H^0(X, L)$, the homogeneous space $X'$ is contained in the $G$-invariant $X \subset \mathbb{P}(H^0(X, L)^*)$. Finally, since the dimensions are encoded in the number of elements of any compass, we must have $\dim X = \dim X' = d$ and thus $X = X'$.

Proposition 2.25. Suppose a torus $H$ of rank $r$ acts on $(X, L)$ in such a way that $X^H = \{y_1, \ldots, y_k\}$ is a finite set. Then the rational function

$$F = \sum_{i=1}^{k} t^{\mu(y_i)} \prod_{\nu \in C(y_i, X, H)} (1 - t^\nu)$$

in variables $t_1, \ldots, t_r$ is a Laurent polynomial in these variables.

Proof. Since $X$ is projective Corollary A.3 of Appendix A applies. The character of the finite dimensional $H$ representations $H^\alpha(L)$ is necessarily a Laurent polynomial. The function $F$ is equal to the equivariant Euler characteristic of $L$, that is, a sum with signs of the characters above, hence it also is a Laurent polynomial.

3. Applications of Białynicki-Birula decomposition

In what follows we will reduce the action of a higher dimensional torus $H$ to a suitably chosen 1-parameter subgroup of $H$. We note that by \cite[Lemma 2.3]{9}, see also Lemma 2.13, a sufficiently general choice of the 1-parameter subgroup does not change the set of fixed points. However, we will frequently be interested in choosing a special
1-parameter subgroup whose fixed-point locus is larger than that of $H$, see also Lemma 2.10.

We use BB-decomposition as discussed in [22], for the original exposition see [9].

**Theorem 3.1.** Suppose $\Lambda = \mathbb{C}^*$ with a coordinate $t$ acts almost faithfully on $X$, and $L$ is an ample line bundle with a linearization $\mu$ of the action of $\Lambda$. Take the decomposition of the fixed-point set into the connected components $X^\Lambda = Y_1 \sqcup \cdots \sqcup Y_s$. For every $Y_i$ by $\nu^\pm (Y_i)$ denote the number of positive and negative characters in the compass $C(Y_i, X, \Lambda)$. Let us define

\[ X_i^+ = \{ x \in X : \lim_{t \to 0} t \cdot x \in Y_i \} \quad \text{and} \quad X_i^- = \{ x \in X : \lim_{t \to \infty} t \cdot x \in Y_i \}. \]

Then the following holds:

- $X_i^\pm$ are locally closed subsets and $X = X_1^+ \sqcup \cdots \sqcup X_s^+ = X_1^- \sqcup \cdots \sqcup X_s^-$,
- There are unique $X_i^\pm$-cells associated to the largest/smallest value of $\mu(Y_i)$. Such cell is necessarily dense and the corresponding $Y_i$ is called the source or sink, respectively.
- The natural map $X_i^\pm \to Y_i$ is algebraic and is a $\mathbb{C}^{\nu^\pm (Y_i)}$ fibration, which implies the decomposition in homology

\[
H_m(X, \mathbb{Z}) = \bigoplus_i H_{m-2\nu^+ (Y_i)}(Y_i, \mathbb{Z}) = \bigoplus_i H_{m-2\nu^- (Y_i)}(Y_i, \mathbb{Z}).
\]

We note that the source and sink are extremal (in the sense of Subsection 2.1) fixed-point components of the action of $\Lambda$. Conversely, given an action of a higher dimensional torus $H$ on $X$ and a vertex $v$ of $\Delta(X, L, H)$ we can choose $\lambda \in \text{Hom}(M, \mathbb{Z})$ such that the affine hyperplane $\lambda^+ + v$ in $M_{\mathbb{R}}$ meets $\Delta(X, L, H)$ at a vertex $v$. Then the extremal fixed-point component of the action of $H$ associated to $v$ becomes a source or sink of the respective 1-dimensional subtorus $\Lambda \hookrightarrow H$. Thus we have the following observation.

**Lemma 3.2.** In the situation above the following holds:

1. $Y_i \subset X^H$ is extremal if and only if there exists a 1-parameter subgroup $\Lambda$ such that $Y_i$ is a source (or sink) of $\Lambda$.

2. For every vertex $v$ of $\Delta(L)$ there is a unique component $Y$ of $X^H$ with $\mu(Y, L, H) = v$. In particular, there is a bijection between extremal fixed-point components and the vertices of $\Delta(L)$.

The uniqueness of the extremal components implies a strengthening of the inclusion in Lemma 2.10(3). To explain this, for any face $\delta \subset \Delta(L)$ let $H_1 \subset H$ be the subtorus corresponding to a projection
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\[ \pi: M \to M_1 \] which contracts \( \delta \) to a point \( v \in M_1 \) and \( v \notin \pi(\Delta(L) \setminus \delta) \). We define \( Y_\delta \) to be the (unique!) component of \( X^{H_1} \) corresponding to \( v \). The downgraded torus \( H_2 = H/H_1 \) acts on \( Y_\delta \) as in Subsection 2.2.

**Lemma 3.3.** For any face \( \delta \subset \Delta(X, L, H) \) and with the notation as above we have

\[ \Delta(Y_\delta, L|_{Y_\delta}, H_2) = \Delta(X, L, H) \cap \delta - w, \]

where \( w = \mu_1(Y_\delta) \in M \) is any lattice point shifting \( \delta \) into \( M_2 \otimes \mathbb{R} \), as in Lemma 2.10. In particular,

\[ \Delta(Y_\delta, L|_{Y_\delta}, H_2) = \Delta(X, L, H) \cap \delta - w. \]

**Proof.** The inclusion “\( \subset \)” is shown in Lemma 2.10(3). So pick \( y \in X^{H_1} \) such that \( \mu(y) \in \delta \). We have to show that \( y \in Y_\delta \). Indeed, clearly, \( y \in X^{H_1} \) and \( \mu_1(y) = v \). Therefore by the uniqueness in Lemma 3.2 we must have \( y \in Y_\delta \). \( \square \)

### 3.1. BB-decomposition for \( \text{Pic} X = \mathbb{Z} \)

In the case \( \text{Pic} X = \mathbb{Z} \) we have the following.

**Lemma 3.4.** Let a 1-parameter group \( \Lambda \) act almost faithfully on \( X \), as above. Assume in addition that \( \text{Pic} X = \mathbb{Z} \cdot L \) and \( Y_0 \subset X^{\Lambda} \) is the source of the action of \( \Lambda \). Then \( X \) is Fano and

(1) either \( \dim Y_0 > 0 \) and

\begin{itemize}
  \item \( Y_0 \) is Fano with \( \text{Pic} Y_0 = \mathbb{Z} \cdot L \), and
  \item the complement of the BB-cell \( X_0^+ \) is of codimension at least 2 in \( X \),
\end{itemize}

(2) or \( Y_0 \) is a point and

\begin{itemize}
  \item \( X_0^+ \) is an affine space with the linear action of \( \Lambda \) associated to weights in \( C(Y_0, X, \Lambda) = (\nu_1, \ldots, \nu_d) \) with all \( \nu_i \) negative.
  \item \( D = X \setminus X_0^+ \) is an irreducible divisor which is in the linear system \( |L| \),
  \item there exists a unique fixed-point component \( Y_1 \subset X \) such that \( \mu(Y_1) \) is minimal in \( \Delta(X, L, \Lambda) \setminus \mu(Y_0) \),
  \item the respective BB-cell \( X_1^+ \) associated to \( Y_1 \) is dense in \( D \).
\end{itemize}

**Proof.** The manifold \( X \) is generically covered by non-trivial orbits of the action of \( \Lambda \) whose closures are rational curves, thus it is uniruled. Since \( \text{Pic} X = \mathbb{Z} \) this implies \( X \) is Fano. If \( \dim Y_0 > 0 \), then \( X \setminus X_0^+ \) is of codimension \( \geq 2 \) in \( X \): Indeed, its intersection with \( Y_0 \) is zero, and every effective divisor is ample, thus \( X \setminus X_0^+ \) cannot contain any divisor of \( X \).

The rest of the lemma follows by BB-decomposition, see Theorem 3.1 or 22 Thm 4.2 and Thm 4.4]. To prove that \( Y_0 \) is Fano
If its dimension is $\geq 1$ we use rational curves again. Namely, we can choose a rational curve in $X$ which does not meet $X \setminus X_0^+$. Hence by the map $X_0^+ \to Y_0$ we have a rational curve in $Y_0$. □

If $Y \subset X^H$ is an extremal component of the fixed-point locus then by Corollary 2.8 we have

\begin{equation}
\bigoplus_{u \neq \mu(Y)} \mathbb{H}^0(X, L_u) = \ker \left( \mathbb{H}^0(X, L) \longrightarrow \mathbb{H}^0(Y, L|_Y) \right).
\end{equation}

**Lemma 3.6.** Let $X$ be a projective manifold with an ample line bundle $L$ and an almost faithful action of an algebraic torus $H$. Moreover assume $\text{Pic} \ X = \mathbb{Z}$. If $Y \subset X^H$ is the extremal fixed-point component associated to a vertex $\mu(Y) \in \Delta(L)$ then the restriction $\mathbb{H}^0(X, L) \longrightarrow \mathbb{H}^0(Y, L|_Y)$ is surjective and $\mathbb{H}^0(X, L)_{\mu(Y)} = \mathbb{H}^0(Y, L|_Y)$. Therefore

$\mathcal{R}(Y, L|_Y) \cong \mathcal{R}(X, L)/\mathcal{J}_{\gamma_{\mu(Y)}}$,

where $\mathcal{J}_{\gamma_{\mu(Y)}}$ is the ideal defined in (2.3), and $\gamma_{\mu(Y)}$ is the ray of the weight cone $\tilde{\Gamma}(L)$ corresponding to the vertex of $\mu(Y)$ as in Subsection 2.7.

**Proof.** We choose a suitable 1-parameter subgroup of $H$ which does not change the extremal component $Y$. Up to the inverse of the action, by Theorem 3.1, $Y$ is associated to the maximal cell $X_0^+$ which is dense in $X$ and admits a fibration (retract) $p_0 : X_0^+ \to Y \hookrightarrow X_0^+$.

If $Y$ is a point then our statement follows by Lemma 3.4(2). Indeed, there is a divisor in the linear system of $L$, which does not contain $Y$. Thus the restriction map in (3.5) is surjective, and all the other claims follow.

Also, if $\dim Y > 0$ then any section in $\mathbb{H}^0(Y, L|_Y)$ lifts via $p_0 : X_0^+ \to Y$ to a section $\mathbb{H}^0(X_0^+, L)$, and the complement of $X_0^+$ is of codimension $\geq 2$ in $X$ by Lemma 3.4(1). Therefore the section extends uniquely to $X$ and the map is $\mathbb{H}^0(X, L) \longrightarrow \mathbb{H}^0(Y, L|_Y)$ is surjective. Equation (3.5) shows that $\mathcal{R}(Y, L|_Y) \cong \mathcal{R}(X, L)/\mathcal{J}_{\gamma_{\mu(Y)}}$. □

### 3.2. Extending divisors and equality of polytopes.

**Lemma 3.7.** Suppose $Y$ is a Fano manifold of dimension at most 3 and $L_Y$ is an ample line bundle on $Y$ such that $\text{Pic} \ Y = \mathbb{Z} \cdot L$. Then $h^0(L_Y) \geq 2$ or $Y$ is a point.

**Proof.** If $\dim Y \leq 2$, then the statement is clear. If $\dim Y = 3$, then [37 Prop. (1.3)(ii)] gives a formula for the Hilbert polynomial of
\[-K_Y: \]
\[h^0(\mathcal{O}_Y(-mK_Y)) = \frac{m(m+1)(2m+1)}{12}(-K_Y)^3 + 2m + 1.\]

Since \(L = -\frac{1}{i}K_Y\) for a positive integer \(i\), and by Serre vanishing the formula above also works for fractional \(m\), we must have \(h^0(L) > 1\). \(\square\)

We have the following immediate corollaries.

**Corollary 3.8.** If \(\text{Pic } X = \mathbb{Z} \cdot L\) and every extremal component of \(X^H\) is of dimension \(\leq 3\) then \(\Gamma(L) = \Delta(L)\).

**Proof.** If \(\dim Y \leq 3\), then \(H^0(Y, L) \neq 0\) by Lemma 3.7. Therefore, if \(Y \subset X^H\) is an extremal fixed-point component then by Lemma 3.6 there exists a section in \(H^0(X, L)\) which is an eigenvalue of the action of \(H\). Moreover, it does not vanish identically on \(Y\), and its weight is \(\mu(Y)\). Thus \(\mu(Y) \in \Gamma(L)\). This shows \(\Delta(L) \subset \Gamma(L)\), while the opposite inclusion is shown in general in Lemma 2.4(2). \(\square\)

**Proposition 3.9.** Suppose that a torus \(H\) of rank \(r\) acts almost faithfully on the projective manifold \(X\) of dimension \(d\) with \(\text{Pic } X = \mathbb{Z} \cdot L\). If \(d \leq r + 4\) then \(\Gamma(L) = \Delta(L)\).

**Proof.** By Corollary 3.8 we will be done if we prove that every extremal component of \(X^H\) is of dimension \(\leq 3\).

For a vertex \(v \in \Delta(L)\) and we choose a flag of linear spaces \(V_0 = \{0\} \supseteq \cdots \supseteq V_{r-1} \subset M_{\mathbb{R}}\) such that \((v + V_i) \cap \Delta(L)\) is an \(i\)-dimensional face of \(\Delta(L)\). The flag determines a sequence of subtori

\[H = H^0 \supseteq H^1 \supseteq \cdots \supseteq H^{r-1}\]

where \(H^i\) is of dimension \(r - i\) associated to the quotient \(M/(M \cap V_i)\). For each such subtorus \(H_i\) one has an irreducible variety \(Y_i\), where

\[Y = Y^0 \subsetneq Y^1 \subsetneq \cdots \subsetneq Y^{r-1},\]

and \(Y^i\) is an extremal component of \(X^{H^i}\). Note that \(Y^i \neq Y^{i+1}\) because \(Y^{i+1}\) contains not only \(Y^i\) but also some other extremal components of \(X^H\) associated to vertices of \(\Delta(L)\) which are in the face \(V^{i+1} \cap \Delta(L)\) but are not in the face \(V^i \cap \Delta(L)\). Therefore \(\dim Y^{r-1} \geq \dim Y^0 + r - 1\).

By our assumptions \(H^{r-1}\) acts almost faithfully on \(X\) and if \(Y^{r-1} \subset X^{H^{r-1}}\) is a divisor then it is in \(|L|\) and the action of \(H^{r-1}\) determines a non-trivial section of \(TX \otimes L^{-1}\) and thus \(X \cong \mathbb{P}^d\), by a result of Wahl, 59.

Since the result is true for \(X = \mathbb{P}^d\) by Lemma 2.4(3) we may assume \(\dim Y^{r-1} \leq d - 2\) and therefore

\[\dim Y^0 \leq \dim Y^{r-1} - r + 1 \leq d - r - 1\]
and we are in situation of Proposition 3.9.

We also note that the property \( \Gamma = \Delta \) propagates to extremal components corresponding to subfaces.

**Lemma 3.10.** Suppose \( \text{Pic} \, X = \mathbb{Z} \), \( L \) is an ample line bundle on \( X \), and a torus \( H \) acts on \( (X, L) \) in such a way that \( \Gamma(X, H, L) = \Delta(X, H, L) \). Let \( \delta \subset \Delta(X, H, L) \) be a proper face, \( H' \subset H \) a subtorus corresponding to a projection \( M \to M' \) that contracts \( \delta \) to a point \( v \in M' \) and no other face is contracted to \( v \). Let \( Y_\delta \) be the extremal component of \( X^{H'} \) corresponding to \( v \). Then also for the action of \( H/H' \) on \( Y_\delta \) the section and fixed-point polytopes are equal:

\[
\Delta(Y_\delta, H/H', L|_{Y_\delta}) = \Gamma(Y_\delta, H/H', L|_{Y_\delta}) = \delta - w,
\]

where \( w \) is any lattice point in the affine span of \( \delta \) (the choice of the shift by \( w \) corresponds to the choice of the linearization as in Lemma 2.10).

**Proof.** Suppose \( W \subset M \otimes \mathbb{R} \) denotes the affine span of \( \delta \) and set \( Y := Y_\delta \) for brevity. We have the following inclusions:

\[
\begin{align*}
\Delta(Y, L|_Y, H/H') & \overset{\text{Lem. 2.10(3)}}{=} \delta - w \\
& = \Gamma(X, L, H) \cap W - w \\
& = \text{conv}(\tilde{\Gamma}(X, L, H) \cap W - w) \\
& \overset{\text{Lem. 3.6}}{=} \text{conv}(\tilde{\Gamma}(Y, L|_Y, H/H')) \\
& = \Gamma(Y, L|_Y, H/H') \\
& \overset{\text{Lem. 2.4(2)}}{=} \Delta(Y, L|_Y, H/H').
\end{align*}
\]

Therefore all the inclusions are equalities, as claimed in the lemma. \( \square \)

**3.3. Fano manifolds, projective spaces and quadrics.** Suppose \( X \) is a projective manifold with an almost faithful action of a nontrivial torus \( H \). Then the line bundle \( \det TX \) has a natural linearization coming from that of \( TX \). Recall, that the **index** of a Fano manifold is the maximal positive integer, such that \( \det TX \cong L^{\otimes t} \) for an ample line bundle \( L \).

**Lemma 3.11.** Let \( \mu \) be the natural linearization of \( L^{\otimes t} = \det TX \). Then for every fixed-point component \( Y_i \subset X^H \) we have

\[
\mu_{L^{\otimes t}}(Y_i) = - \sum_{\nu_j \in \mathcal{C}(Y_i)} \nu_j
\]
PROOF. For every $y \in Y$, the character of the action of $H$ on $\det T_yX$ is the sum of weights of the action on $T_yX$. Hence the claim. □

PROPOSITION 3.12. Suppose that a projective manifold $X$ of dimension $d$ with $\text{Pic} X \cong \mathbb{Z}$ admits a nontrivial action of a 1-dimensional torus $H$. Assume that $\Delta(L) = [0, 2] \subset M = \mathbb{R}$ and the extremal fixed-point components are of dimension zero. Then one of the following holds:

(1) $d = 1$ and $(X, L)$ is either $(\mathbb{P}^1, O(1))$ or $(\mathbb{P}^1, O(2))$,

(2) $d \geq 2$ and $(X, L) = (\mathbb{P}^d, O(1))$ and in some coordinates $[z_0, \ldots, z_d]$ the action of $H$ has weights $(0, 1, \ldots, 1, 2)$,

(3) $d \geq 3$ and $(X, L) = (\mathbb{Q}^d, O(1))$ and for some equivariant embedding $\mathbb{Q}^d \hookrightarrow \mathbb{P}^{d+1}$ in which $\mathbb{Q}^d$ has equation $z_0z_{d+1} + z_1z_d + \cdots = 0$ the action of $H$ on $\mathbb{P}^{d+1}$ has weights $(0, 1, \ldots, 1, 2)$.

PROOF. $X$ is Fano by Lemma 3.4. Let $\mu$ denote the linearization of $L$ such that $\Delta(L, \mu) = [0, 2]$ and by $y_0, y_2 \in X^H$ denote the extremal fixed points associated to, respective endpoints of $\Delta(L, \mu)$. If $\mu'$ is the natural linearization of $\det TX = L^{\otimes \iota}$ then

$$\mu'(y_0) = -\sum_{\nu_j \in C(y_0)} \nu_j \leq -d \quad \mu'(y_2) = -\sum_{\nu_j \in C(y_2)} \nu_j \geq d$$

by Lemma 3.11. In particular, $\mu'(y_2) - \mu'(y_0) \geq 2d$. On the other hand $\Delta(L^{\otimes \iota}, \mu_{L^{\otimes \iota}}) = [0, 2\iota]$. Since these two linearizations differ by a constant in $M$, it follows that $\iota \geq d$. By the Kobayashi and Ochiai characterization [44] the pair $(X, L)$ is either $(\mathbb{P}^d, O(1))$ or $(\mathbb{Q}^d, O(1))$. The rest of the claim follows by a straightforward verification. □

COROLLARY 3.13. In the situation of Proposition 3.12 the compass at the extremal fixed point associated to $0$ is $(1^{d-1}, 2)$ in the case (2) and $(1^d)$ in the case (3).

The following special case will be relevant to our investigations in Section 5.

COROLLARY 3.14. Let $H$ be a torus of rank $r$ with a basis $x_1, \ldots, x_r$ of the lattice of characters $M \cong \mathbb{Z}^r$. Suppose that $H$ acts almost faithfully on a projective manifold $X$ of dimension at least 2, such that $\text{Pic} X = \mathbb{Z} \cdot L$, and all extremal fixed-point components of $X^H$ are isolated points. If $\Delta(X, H, L) = \text{conv}(x_1, -x_1, \ldots, x_r, -x_r)$ and the compass of the action of $H$ at any extremal fixed point corresponding to the vertex $\pm x_i$ of $\Delta(X, H, L)$ does not contain $\mp 2x_i$, then $X \cong \mathbb{Q}^d$ where $d = 2r + \dim(H^0(X, L)_0) - 2$. 
Proof. We note that for every $i$ the projection $M \to \mathbb{Z} \cdot x_i$ yields the situation as in Proposition 3.12 hence the claim follows from Corollary 3.13. \hfill \QED

The last result in this sub-section seems to be known to experts but we were not able to find a proper reference for it.

**Lemma 3.15.** Let $X$ be a projective manifold with an action of 1-dimensional torus $H$. If the fixed-point set has two components $X^H = Y_0 \sqcup Y_1$ and $Y_0$ is a point then $X \cong \mathbb{P}^d$ and $Y_1 \cong \mathbb{P}^{d-1}$.

**Proof.** By the Bialynicki-Birula decomposition of cohomology in Theorem 3.1, we see that $\dim Y_1 = \dim X - 1$ and the second Betti number of $X$ is 1. Hence $Y_1$ is an ample divisor and the vector field tangent to the action of $H$ vanishes along it. Thus the result follows from [59]. \hfill \QED

**Example 3.16.** Assume $X$ is a smooth connected projective surface with an action of 1-dimensional torus $\Lambda$ and an ample linearized line bundle $L$. Suppose that $\Delta(X, L, \Lambda)$ is an interval $[0, 3] \subset M_\mathbb{R}$ and $X^\Lambda$ consists of isolated fixed points only. Moreover, suppose $\Gamma(X, L, \Lambda) = \Delta(X, L, \Lambda)$ and for all $y \in X^\Lambda$

$$C(y, X, \Lambda) = \begin{cases} (1^2) & \text{if } \mu(y) = 0, \\ (-1, 1) & \text{if } \mu(y) \in (1, 2), \text{ and} \\ ((-1)^2) & \text{if } \mu(y) = 3, \end{cases}$$

Then $X^\Lambda$ consist of 4 points, one for each integral point of $\Delta(X, L, \Lambda)$.

We provide two proofs of the example, illustrating the strength of various techniques presented in Sections 2, 3 and Appendix A. A reader who is not fond of the geometric analysis presented in the first proof will certainly appreciate a short argument based on localization theorem which however does not explain the geometry as thoroughly as the first argument does.

**Proof using BB-decomposition.** By Lemma 3.2 there is a unique extremal point $y_i$ for each vertex $i \in \{0, 3\}$. Note that these are not the only fixed points by Theorem 3.1 as then $X$ would be a one point compactification of $X_0^+ \simeq \mathbb{C}^2$, which is impossible. Without loss of generality, suppose there is $y_2 \in X^\Lambda$ with $\mu(y_2) = 2$.

Since $\Gamma(X, L, \Lambda) = \Delta(X, L, \Lambda)$ there are sections $\sigma_0$ and $\sigma_3$ of $L$ which have weights 0 and 3 respectively. Let $D_0$ and $D_3$ be the corresponding divisors. Consider the local defining equations $f_{0,y}$ and $f_{3,y}$ of $D_0$ and $D_3$ at a fixed point $y$, as in Lemma 2.17. The weights of $f_{0,y_0}$ and $f_{3,y_3}$ are 0, while all the weights of the remaining $f_{i,y}$ are nonzero.
and equal to $i - \mu(y)$ (for $i = 0$ or $i = 3$). In particular, the local expression of $\sigma_0$ near a fixed point $y \neq y_0$ is equal to $f_{0,y}$, which is homogeneous and nonconstant, thus vanishing at $y$. Therefore $D_0$ contains all fixed points except $y_0$, and analogously $D_3$ contains all fixed points except $y_3$. Moreover, $D_0$ near $y_3$ is defined by $f_{0,y_3}$, which is a homogeneous polynomial of degree $-3$ in two variables both of degree $-1$. Thus $D_0$ is a union of 3 orbits of $\Lambda$ (perhaps with multiplicities).

Now consider any point $y'_2 \in X^\Lambda$ such that $\mu(y'_2) = 2$ (by our choice, there is at least one such point). By the local coordinates as in Lemma 2.17 the neighbourhood of $y'_2$ is isomorphic to (up to an étale cover) $\text{Spec} \mathbb{C}[x_1, x_2]$ where the action of $\Lambda$ has weights 1 and $-1$. In particular, there are only two one dimensional orbits in $X$, whose closures $C_{y'_2,-1}$ and $C_{y'_2,1}$ contain $y'_2$, and one of them, say $C_{y'_2,1}$, contains also $y_3$ in its closure. The divisor $D_0$ near $y'_2$ is defined by $f_{0,y'_2} \in \mathbb{C}[x_1, x_2]$ of degree $-2$, thus $D_0$ is equal near $y'_2$ to $aC_{y'_2,-1} + (a + 2)C_{y'_2,1}$ for some nonnegative integer $a$. In particular, looking again near $y_3$ (where the multiplicity is 3 as shown above) we see that $a \leq 1$, and that $D_0$ near $y_3$ has a component of multiplicity at least 2. Therefore, there can be at most one point $y'_2$ with $\mu(y'_2) = 2$, namely $y_2$, as any other such fixed point would lead to another component of multiplicity 2 near $y_3$, contradicting its multiplicity 3 at $y_3$.

If $a = 0$, then $D_0$ has another component (closure of an orbit) with multiplicity 1 near $y_3$. The other point in the closure of the orbit is a fixed point, which is not equal to $y_0$ (because $y_0 \notin D_0$), nor $y_2$ (because, $C_{y_2,1}$ is the only orbit, whose closure contains both $y_2$ and $y_3$), nor $y_3$ (by the local description of the action as in Equation (2.16)). Thus there must exist a point $y_1 \in X^\Lambda$ with $\mu(y_1) = 1$. Similarly, if $a = 1$, then $C_{y_2,-1}$ is contained in $D$ with multiplicity 1 and the other end of $C_{y_2,-1}$ must be a point $y_1$ as above. In both cases, we can swap the roles of 0 and 3, and argue in the same way, to show that $y_1$ is a unique point with $\mu(y_1) = 1$, which concludes the proof. 

PROOF USING LOCALISATION THEOREM. There are unique fixed points corresponding to 0 and 3 by Lemma 3.2. Suppose that there are $a$ fixed points corresponding to 1 and $b$ fixed points corresponding to 2. Then the rational function from Proposition 2.25 gives:

$$F = \frac{1}{(1-t)^2} + \frac{at}{(1-t^{-1})(1-t)} + \frac{bt^2}{(1-t^{-1})(1-t)} + \frac{t^3}{(1-t^{-1})^2}$$

$$= \frac{1 - at^2 - bt^3 + t^5}{(t-1)^2}$$
Thus by the same proposition, $F$ must be a Laurent polynomial, in particular, the numerator must have a double root at 1. It is straightforward to check that this happens if and only if $a = b = 1$. □

We remark that in the situation of Example 3.16 the surface $X$ must be isomorphic either to $\mathbb{P}^1 \times \mathbb{P}^1$ or the Hirzebruch surface $\mathbb{F}_2$. Indeed, by the cohomology description in Theorem 3.1 the variety $X$ is a rational surface with $\text{Pic} X \simeq \mathbb{Z}^2$, thus it is a Hirzebruch surface $\mathbb{F}_a$ for some $a$. Using the toric geometry one shows that $a = 0$ or $a = 2$. We leave out the details as we are not going to use this observation.

4. Torus actions on contact manifolds

Throughout this section we assume that $X$ is a contact Fano manifold of dimension $d = 2n + 1$, with a contact form $\theta \in H^0(X, \Omega_X(L))$ which determines the exact sequence of vector bundles on $X$:

$$0 \longrightarrow F \longrightarrow TX \longrightarrow L \longrightarrow 0.$$ 

The form $d\theta|_F \in H^0(X, \bigwedge^2 F \otimes L)$ determines a nondegenerate bilinear skew symmetric pairing $F \times F \longrightarrow L$. See [17] Sect. E.3 and Chapt. C] and references therein for introduction and more details about contact manifolds, the contact distribution $F$, the contact form $\theta$, the contact line bundle, and the skew-pairing $d\theta|_F$.

By a slight abuse of standard notation we will assume that any smooth projective curve is a contact manifold as well. In this case $n = 1$, $F = 0 \subset TX = L$, and $X$ is Fano if and only if $X \simeq \mathbb{P}^1$. This is need for a uniform statement of Corollary 4.4 and consequently also in multiple proofs in Section 5, where the case of $n = 0$ is critical.

We will use the set up introduced in [6] or [17] Sect. E.3.2. We consider a connected nontrivial reductive group $G$ of automorphims of $X$. In particular, we will consider a maximal torus $H \subset G$. The tangent action of $G$ and of $H$ on $TX$ induces a canonical linearization $\mu_0$ of $L$.

We remark that unless $X \simeq \mathbb{P}^{2n+1}$, the action of $G$ always preserves the contact distribution $F$, see [38] Cor. 4.5, since the contact structure on $X$ is unique. Our main interest is in the case $\text{Pic} X = \mathbb{Z} \cdot L$, which in particular excludes the case of $\mathbb{P}^{2n+1}$. In fact, from Subsection 4.2 onwards we will suppose that $\text{Pic} X = \mathbb{Z} \cdot L$, but in [4.1 we need this more general setup for applications in Section 5. There we will also consider contact submanifolds of $X$, which arise as fixed points of a torus action (see Corollary 4.4) and they do not necessarily satisfy $\text{Pic} X = \mathbb{Z} \cdot L$. 
4.1. Fixed points components of contact automorphisms.

**Lemma 4.1.** Assume that $H$ is a torus acting on a contact manifold $X$ and preserving the contact structure. For a fixed point $y \in X^H$ we consider $\nu_i(y) \in M$, the weights of the action of $H$ on $T^*_yX$. Then after a renumbering, for $i = 1, \ldots, n$, the weights satisfy the following equalities in $M$:

$$\nu_i + \nu_{i+n} = \nu_0$$

and $\mu_0(y) = -\nu_0$ is the weight of the action of $H$ on $L_y$.

**Proof.** The claim follows from the duality $F \simeq F^* \otimes L$ defined by the nondegenerate form $d\theta|_F$. \qed

**Corollary 4.2.** In the situation of the previous lemma suppose in addition that $Y \subset X^H$ is an extremal fixed-point component. If the action of $H$ on $X$ is nontrivial, then $\mu_0(Y) \neq 0$.

**Proof.** We can reduce the action to a 1-dimensional torus for which $Y$ is the source (or sink) and thus all elements in the compass of $Y$ are negative (or positive). Thus $\mu_0(Y) = 0$ contradicts the previous lemma. \qed

Now, in view of Lemma 4.1, as immediate consequences we get the following description of fixed-point components $Y \subset X^H$ depending on whether $\mu_0(Y) = 0$ or not.

**Corollary 4.3.** In the situation of Lemma 4.1 assume that $Y \subset X^H$ is a component such that $\mu_0(Y) \neq 0$. Then $Y$ is isotropic with respect to $\theta$, that is

$$TY \hookrightarrow F|_Y \hookrightarrow TX|_Y$$

and the form $d\theta$ is zero on $TY$. In particular, $\dim Y \leq n$ and, in fact, $\dim Y + 1$ is equal to the multiplicity of the weight $-\mu_0(Y)$ in the compass $C(Y,X,H)$.

**Corollary 4.4.** In the situation of Lemma 4.1 assume that $Y \subset X^H$ is a component such that $\mu_0(Y) = 0$. Then $Y$ is a contact manifold, with a contact form defined by the restriction of $\theta$ (in particular, $L|_Y$ is the contact line bundle on $Y$, that is the quotient of $TY$ by the contact distribution of $Y$) and $d\theta|_F$ induces bilinear pairing on the normal bundle $N_{Y/X} \times N_{Y/X} \rightarrow L|_Y$.

4.2. Adjoint representation. In addition to the assumptions listed at the beginning of this section, from now on we suppose $\text{Pic}X = \mathbb{Z} \cdot L$. Let $g \subset \mathfrak{h}(X, TX)$ denote the Lie algebra of vector fields tangent to the action of $G$. 
Lemma 4.5. The canonical $G$-linearization $\mu_0$ of $L$ makes $H^0(X, L)$ into a representation of $G$ isomorphic to $H^0(X, TX)$. If in addition $G = \text{Aut}(X)$, then the set and multiplicities of weights in $\Gamma(X, L, H, \mu_0)$ coincide with the set and multiplicities of weights of the adjoint representation of $G$.

Proof. The first statement is discussed in [6] Sect. 1.2 or in [17], Thm E.13, Cor. E.14. The second claim follows, as then $H^0(X, TX) = \mathfrak{g}$. □

Lemma 4.6. Let $X$ be a projective contact manifold with $\text{Pic} X = \mathbb{Z} \cdot L$ as above. Suppose that $G = \text{Aut}(X)$ is a reductive group with a maximal torus $H$. If $\Gamma(X, L, H, \mu_0) = \Delta(X, L, H, \mu_0)$ then $G$ is semisimple.

Proof. By Lemma 4.5 the polytope $\Gamma(L)$ is the convex hull of the weights of the adjoint action of $H$ on $\mathfrak{g}$. Hence $\Gamma(L)$ is of maximal dimension if and only if $G$ is semisimple. On the other hand, since the action of $H$ is almost faithful, the polytope $\Delta(L)$ is of maximal dimension by Corollary 2.6. Therefore $G$ is semisimple. □

4.3. Semisimple groups of automorphisms of high rank. As before $X$ is a contact Fano manifold of dimension $2n + 1$ with $\text{Pic} X = \mathbb{Z} \cdot L$ and $-K_X = (n + 1)L$. By $G$ we denote the automorphism group of $X$ and we assume it is reductive of rank $r$.

Lemma 4.7. Let $X$ be a contact Fano manifold of dimension $2n + 1$ and $\text{Pic} X = \mathbb{Z} \cdot L$. Suppose that the group $G$ of automorphisms of $X$ is reductive with a maximal torus $H$ of rank $r$. If $r + 2 \geq n$ then $\Delta(X, L, H, \mu_0) = \Gamma(X, L, H, \mu_0)$ and all extremal fixed-point components of $X^H$ are isolated points.

Proof. The arguments are similar to the proof of Proposition 3.9. By Corollary 3.8 the first statement will follow if we prove that every extremal component $Y^0$ of $X^H$ is of dimension $\leq 3$. Indeed, as in the proof of Proposition 3.9 we construct a flag of submanifolds $Y^0 \subsetneq Y^1 \subsetneq \cdots \subsetneq Y^{r-1}$ and by Corollary 4.3 we have $\dim Y^{r-1} \leq n$ hence $\dim Y^0 \leq n - (r - 1) \leq 3$.

Now we know that $\dim Y^0 \leq 3$ and $Y^0$ is Fano with $\text{Pic} Y^0 = \mathbb{Z} \cdot L_{|Y^0}$ by Lemma 3.4. Moreover, by Lemma 3.6 we know that $\dim H^0(Y^0, L_{|Y^0})$ is the same as the multiplicity of the respective root in the weights of the adjoint representation of $G$, hence it is one. Finally, $\dim H^0(Y^0, L_{|Y^0}) = 1$ implies that $Y^0$ is a point by Lemma 3.7. □
Proposition 4.8. Let $X$ be a contact Fano manifold of dimension $2n + 1$ and $\text{Pic} X = \mathbb{Z} \cdot L$. Suppose that the group $G$ of automorphisms of $X$ is reductive of rank $r \geq n - 2$. Then $G$ is simple.

Proof. Let $H$ be the maximal torus of $G$. By Lemma 4.7 the polytopes $\Gamma(X, L, H, \mu_0)$ and $\Delta(X, L, H, \mu_0)$ are equal. Lemma 4.6 thus implies that $G$ is semisimple. Up to a finite cover we can write a decomposition of $G$ into simple factors $G = G_1 \times \cdots \times G_s$. Correspondingly, the maximal torus and character lattice decompose as $H = H_1 \times \cdots \times H_s$ and $M = M_1 \oplus \cdots \oplus M_s$. Here $H_i$ is a maximal torus in $G_i$ and $M_i$ is the lattice of weights of $G_i$, hence also the lattice of characters of $H_i$. If $R_i$ is a root system of $G_i$ in $M_i \hookrightarrow M$ then $\Delta(L)$ is the convex hull of $\bigcup R_i \subset M$.

Suppose that $s > 1$. For $i = 1, 2$ take $u_i \in R_i \subset M_i \hookrightarrow M$ which is the vertex of the root polytope $\Delta(G_i) \subset M_i$. Then $\mathbb{R}_{\geq 0}(u_2 - u_1)$ is a ray of the cone $\mathbb{R}_{\geq 0}(\Delta(X, L) - u_1)$. The element $(u_2 - u_1)$ is primitive in the semigroup $\mathbb{R}_{\geq 0}(u_2 - u_1) \cap M$ unless both $G_1$ and $G_2$ are groups of type $A_1$ or $C$ and then $\mathbb{R}_{\geq 0}(u_2 - u_1) \cap M$ is generated by $(u_2 - u_1)/2$.

In the former case, by Corollary 2.18 the element $(u_2 - u_1)$ is in the compass of the extremal fixed-point component $Y_1$ associated to $u_1$. Then, however by Lemma 4.1 $-u_2 = -u_1 - (u_2 - u_1)$ should be contained in $C(Y_1, X, H)$ as well, hence by Corollary 2.14 we get $u_1 - \lambda u_2 \in \Delta(X, L)$ for a positive integer $\lambda$, a contradiction.

If both $G_1$ and $G_2$ are of type $A_1$ or $C$ then we consider the projection of $M_R$ along the direction of the edge spanned by $u_1$ and $u_2$. That is we consider the reduction of the action of the torus $H$ to $H'$ associated to the $\mathbb{Z} \cdot (u_2 - u_1)/2 \hookrightarrow M \to M'$. Let $Y'$ be the extremal component of $X^{H'}$ corresponding to the vertex, which is the image of the edge spanned by $u_1$ and $u_2$. The manifold $Y'$ admits an almost faithful $\mathbb{C}^* = H/H'$ action with $\Delta(Y', L|_{Y'}, \mathbb{C}^*)$ equal to an interval of length 2 by Lemma 2.1(13). Since $\text{dim} Y' > 0$, by Lemma 3.4(1) we also have $\text{Pic} Y' = \mathbb{Z} \cdot L|_{Y'}$. Moreover, the extremal components of $(Y')^{\mathbb{C}^*}$ are the extremal components of $X^H$ corresponding to vertices $u_1$ and $u_2$, thus they are points. Thus $Y'$ with the action of $\mathbb{C}^*$ satisfies the assumptions of Proposition 3.12 and the pair $(Y', L|_{Y'})$ is isomorphic either to $(\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}(1))$ for $d \geq 1$ or to $(\mathbb{Q}^d, \mathcal{O}_{\mathbb{Q}^d}(1))$ for $d \geq 3$. By Lemma 3.6 we get $\text{dim} H^0(Y', L|_{Y'}) = 2$, hence $(Y', L|_{Y'}) \simeq (\mathbb{P}^1, \mathcal{O}(1))$. But then $(u_2 - u_1)/2$ is not in the compass of the action of $\mathbb{C}^*$ on $Y'$ at the fixed point associated to $u_1$. By Lemma 2.13 the vector $(u_2 - u_1)/2$ is not in the compass of the action of $H$ on $X$ at the same point either. Therefore, we get a contradiction as in the previous case. \qed
5. Contact manifolds with a simple automorphisms group

In this section, we always suppose the following assumptions hold.

**Assumptions 5.1** (contact manifolds with an action of torus).

1. $X$ is a contact Fano manifold of dimension $2n + 1$, with a contact form $\theta \in H^0(X, \Omega_X \otimes L)$. In particular, $L$ is an ample line bundle on $X$.
2. $\text{Pic} \, X = \mathbb{Z} \cdot L$.
3. There exists a simply-connected simple group $G$ of rank $r$ with a maximal torus $H$, such that $H$ acts on $X$. $M$ is the lattice of weights of $G$, thus also the lattice of characters of $H$.
4. $\Delta(X, L, H) = \Delta(G)$, where $\Delta(G)$ is the convex hull of roots.
5. all extremal fixed-point components in $X^H$ are isolated points thus, in particular, $\Delta(X, L, H) = \Gamma(X, L, H)$ by Corollary 3.8 and for every vertex $v$ of $\Delta(X, L, H)$ the dimension of $\Gamma(X, L, H)_v$ is 1 by Lemma 3.6.

Moreover, our main interest is in the stronger version of these assumptions.

**Assumptions 5.2** (contact manifolds with simple automorphism group). In addition to Assumptions 5.1 we suppose:

1. The identity component of the group of automorphisms of $X$ is a simple group. The group $G$ from (3) is the universal cover of that simple group.
2. $H^0(X, L)$, as a representation of $G$, can be identified with $\mathfrak{g}$, the Lie algebra of $G$, with the adjoint action of $G$. Hence $\widetilde{\Gamma}(X, L, H)$ contains roots of $G$, each with multiplicity 1 and $0 \in \widetilde{\Gamma}(X, L, H)$ with multiplicity $r$.

These assumptions are motivated by the hypothesis of Theorem 1.3 together with results of Section 4. The details are discussed in Subsection 6.2. As we will see, the role of $G$ is rather decorative, but, instead, the action of $H$ rules the roost. The importance of $G$ is mainly in Item (7) which forces the weights of $H$ to be prescribed by the simple root system. We use the weaker set of assumptions to be able to say something explicit about some of the existing homogeneous cases, but with an action of a subtorus of the maximal torus, which highly resembles a general situation for a smaller group (see Subsection 5.5).

**Theorem 5.3.** Suppose Assumptions 5.2 are satisfied. Then one of the following holds:

- $X \cong Gr(\mathbb{P}^1, \mathbb{Q}^{n+2})$ and $G$ is of type $B_{\frac{n+3}{2}}$ or $D_{\frac{n+4}{2}}$ (depending on the parity of $n$), or
• $G$ is of type $G_2$ and either $X$ is the (homogeneous) 5-dimensional adjoint variety of $G_2$ or $\dim X \geq 11$, or
• $\dim X \geq 11$ and $G$ is of type $A_2$, or
• $\dim X \geq 21$ and $G$ is of type $E_6$, or
• $\dim X \geq 33$ and $G$ is of type $E_7$, or
• $\dim X \geq 57$ and $G$ is of type $E_8$, or
• $\dim X \geq 15$ and $G$ is of type $F_4$, or
• $G$ is of type $A_1$.

A simple Lie group is of one of the types $A_r$ (for $r \geq 1$), $C_r$ (for $r \geq 2$), $B_r$ (for $r \geq 3$), $D_r$ (for $r \geq 4$), $E_6$, $E_7$, $E_8$, $F_4$, or $G_2$. The goal of this section is to analyze case-by-case all of them and figure out which of them can appear under the assumptions above. The cases $A_r$ are excluded in Subsection 5.2, the cases $C_r$ in Subsection 5.3, while the cases $B_r$ and $D_r$ are analyzed in Subsection 5.4. The exceptional cases $E_r$ and $F_4$ follow from Lemma 2.22 (and the fact, that for a contact manifold $X$ its dimension is odd). Finally, the types $A_2$ and $G_2$, which require much more attention than the other ones, are treated in Subsection 5.5. Moreover, in Corollary 6.3 using a different method we strengthen the last case of type $A_1$, showing that then $\dim X \geq 11$, analogously to the $A_2$ and $G_2$ cases.

5.1. General techniques. The highlights of the general strategy for the cases are as follows. Firstly, we consider subtori of $H$ associated to hyperplanes supporting faces and use Lemma 3.6 to understand the extremal fixed-point components of these actions. In particular, we use Proposition 3.12 and its corollaries. Secondly, we study the compass of $H$ at any extremal fixed point of $X_H$ and apply Lemma 4.1 and Corollaries 2.14 and 2.18.

We gather a few observations that we exploit in all cases. For every face $\delta$ of $\Delta(G)$ we denote by $Y_\delta$ the corresponding extremal component of $X_H'$, where $H' \subset H$ is the subtorus corresponding to the quotient of $M$ by the linear space parallel to $\delta$.

Lemma 5.4. Suppose Assumptions hold. For every proper face $\delta$ of $\Delta(G)$, that is not a vertex, we have $\text{Pic} Y_\delta = \mathbb{Z} \cdot L|_{Y_\delta}$. For the action of the quotient torus $H/H'$ on $Y_\delta$ the polytopes of sections and fixed points, $\Gamma(Y_\delta, H/H', L|_{Y_\delta})$ and $\Delta(Y_\delta, H/H', L|_{Y_\delta})$, coincide and both are equal to $\delta$ up to a shift by a lattice vector in the affine span of $\delta$. In particular, $h^0(L|_{Y_\delta})$ is at least the number of vertices of $\delta$. If $\delta$ has no lattice points other than the vertices, then equality holds. If Assumptions hold, then $h^0(L|_{Y_\delta})$ is equal to the number of roots of $G$ contained in $\delta$. 
Proof. The claim on $\text{Pic} Y_\delta = \mathbb{Z} \cdot L|_{Y_\delta}$ follows from Assumption 5.1(2) and Lemma 3.4(1). The equality of polytopes follows from Lemma 3.10. Finally, the statements on $h^0(L|_{Y_\delta})$ are implied by Lemma 3.6 and Assumption 5.2(7). □

We note that for most simple groups we have one of the following properties of an edge $\delta$ of $\Delta(G)$.

\begin{itemize}
    \item[$\star_0$] There is no lattice point of $\delta$ other than the end points.
    \item[$\star_1$] There is exactly one lattice point of $\delta$ other than the end points and this middle lattice point is a root of $G$.
\end{itemize}

It can be verified that (except in the case of type $A_1$) either ($\star_0$) or ($\star_1$) holds for all edges of the root polytope. We check this for the relevant cases in the subsequent subsections. Now we list some consequences.

Lemma 5.5. Under Assumptions 5.1, consider an edge $\delta$ of the polytope $\Delta$ and let $Y_\delta$ be the corresponding extremal component as above.

\begin{enumerate}
    \item If $\delta$ satisfies ($\star_0$), then $(Y_\delta, L|_{Y_\delta}) \simeq (\mathbb{P}^1, \mathcal{O}(1))$.
    \item If Assumptions 5.2 hold and $\delta$ satisfies ($\star_1$), then $(Y_\delta, L|_{Y_\delta}) \simeq (\mathbb{P}^2, \mathcal{O}(1))$.
\end{enumerate}

Proof. The 1-dimensional quotient torus $\Lambda$ corresponding to the line in $M_\mathbb{R}$ parallel to $\delta$ acts on $Y_\delta$ and the polytope $\Delta(Y_\delta, L|_{Y_\delta}, \Lambda)$ is equal to $\delta$ by Lemma 2.10(3) and Lemma 5.4. Also by Lemma 5.4 we must have $\text{Pic} Y_\delta = \mathbb{Z} \cdot L$ and $h^0(L|_{Y_\delta}) = 2$ or 3. Finally, the claim follows from Proposition 3.12. □

One of the ideas to exclude many of the cases is to study the compass of an extremal fixed-point component in $X$. This way we can restrict the dimension of $X$ and many other of its properties.

Corollary 5.6. Under Assumptions 5.1, pick a vertex $v \in \Delta$ and suppose $y \in X^H$ is the corresponding fixed point. Let $\mathcal{C} = \mathcal{C}(y, X, H)$ be the compass of $y$ in $X$. Then:

\begin{enumerate}
    \item If $\delta \subset \Delta$ is an edge containing $v$ and satisfying ($\star_0$) (or ($\star_1$) if in addition Assumptions 5.2 are satisfied), then for every lattice point $m \in \delta \setminus \{v\}$ we have $m - v \in \mathcal{C}$ with multiplicity exactly 1. Moreover, for any lattice point $m'$ in the line containing $\delta$, the vector $m' - v$ is in the compass if and only if $m' \in \delta \setminus \{v\}$.
    \item The compass $\mathcal{C}$ contains $-v$ with multiplicity exactly 1.
    \item Consider a convex cone $\sigma \subset M_\mathbb{R}$ generated by the shift $\Delta - v$. Then the elements of the compass $\mathcal{C}$ are contained in the set of lattice points of the intersection $\sigma \cap (-v - \sigma)$.
\end{enumerate}
Proof. To see (1) consider \( Y_\delta \subset X \). By Lemma 2.13 it is enough to prove \( m - v \in C(y, Y_\delta) \) with multiplicity 1. If \( \delta \) satisfies \((\star_0)\) then the only possible \( m \) is the end point of \( \delta \), and \( Y_\delta \simeq \mathbb{P}^1 \) by Lemma 5.5. In particular, there is only one element of the compass and it must be equal to \( v' - v \), where \( v' \in \delta \) is the other vertex. If \( \delta \) satisfies \((\star_1)\), then there are two possible \( m \), the end point of \( \delta \) and the middle point, and both are roots of \( G \). By Lemma 5.5 we have \( Y_\delta \simeq \mathbb{P}^2 \) with \( L|_{Y_\delta} \simeq \mathcal{O}(1) \) and by Corollary 3.13 there are exactly two elements of the compass equal to the two possible values of \( m - v \).

Item (2) follows from the short exact sequence of vector spaces

\[ 0 \to L_y^* \to T^*_y X \to F_y^* \to 0, \]

because \(-v\) is the weight of the action on \( L_y^* \), and from the duality in Lemma 4.1.

To prove Item (3), we note that the compass is contained in \( \sigma \) by Corollary 2.14. It is also contained in \(-v - \sigma\) by Lemma 4.1. □

Lemma 5.7. Under Assumptions 5.1 suppose in addition that \( 0 \in \tilde{\Delta} \), that is there exists a fixed-point component \( Y \subset X^H \) with \( \mu_0(Y) = 0 \). Then \( \tilde{\Delta} \) also contains another point, which is nonzero, and not a vertex of \( \Delta \).

Proof. By Corollary 4.4 the manifold \( Y \) is contact, in particular, \( \dim Y \geq 1 \). Let \( \nu \in C(Y, X, H) \) be any element and define \( W \) to be the linear span of \( \nu \). By Lemma 2.15, there is another component \( Y' \subset X^H \), such that \( \mu(Y') \in W \) and \( \dim Y' + b' \geq 2 \), where \( b' \) is the number of elements in \( C(Y', X, H) \cap W \). But if \( Y' \) was an extremal component, then \( \dim Y' = 0 \) by Assumption 5.1 and \( b' = 1 \) by Corollary 5.4, a contradiction. Thus \( Y' \) is not extremal, showing the claim of the lemma. □

5.2. Case of \( A_r \) with \( r \geq 3 \). We use the notation consistent with Table 1. That is, the roots of \( A_r \) are located on a hyperplane \( \sum_{i=0}^r e_i^* = 0 \) in a Euclidean space \( \mathbb{R}^{r+1} \) with an orthonormal basis \( e_0, \ldots, e_r \), where by \( e_i^* \) we denote the dual functionals. The roots are \( e_i - e_j, i \neq j \), and they are the vertices of \( \Delta(A_r) \). The lattice of weights is generated by the roots and \( e_0 - (e_0 + \cdots + e_r)/(r + 1) \). All facets of \( \Delta(A_r) \) can be described as follows: take a proper nonempty subset of indices \( J \subset \{0, \ldots, r\} \) and define the facet \( \delta(J) \) as

\[ \delta(J) = \Delta(A_r) \cap \left\{ u \in \mathbb{M}_R : \left( \sum_{i \in J} e_i^* \right)(u) = 1 \right\}. \]

Note that \( e_i - e_j \in \delta(J) \) if and only if \( i \in J \) and \( j \notin J \). Our main interest is in the face \( \delta(\{0,1\}) \) and the hyperplane \( \mathbb{R}^{r-1} \) containing this
Therefore each face, given by the equation \( e_0^* + e_1^* = 1 \) (together with the omnipresent equation \( \sum_{i=0}^r e_i^* = 0 \)).

**Lemma 5.8.** Suppose \( r \geq 2 \). Then all weight lattice points in \( \delta(\{0,1\}) \) are the vertices, except in the case \( r = 3 \), when there is in addition one interior lattice point \( \frac{1}{2}(e_0 + e_1 - e_2 - e_3) \). In particular, all edges of \( \Delta \) satisfy \( [\star_0] \).

**Proof.** Any point in the weight lattice is of the form \( v = \sum a_i e_i + b(e_0 - (e_0 + \cdots + e_r)/(r + 1)) \) for some integers \( a_i \) satisfying \( \sum a_i = 0 \), and \( b \in \{0, \ldots, r - 1\} \). If \( b = 0 \) and \( v \) is in \( \delta(\{0,1\}) \), then it is straightforward to verify that \( v \) is one of the vertices \( e_0 - e_j \) or \( e_1 - e_j \) (\( j \notin \{0,1\} \)). Thus assume \( b \in \{1, \ldots, r - 1\} \). The value \((e_0^* + e_1^*)(v)\) is a sum of an integer and \( \frac{b(r-1)}{r+1} \). Thus if this value is an integer, then \( 2b \equiv 0 \mod (r + 1) \), in particular, \( r \) must be odd, say equal to \( 2k + 1 \). Then by our assumptions \( b = k + 1 \) and \( k \geq 1 \).

So assume \( v \in \delta(\{0,1\}) \) so that \( 1 = a_0 + a_1 + \frac{2k(k+1)}{2k+2} = a_0 + a_1 + k \). Moreover, \( v \) satisfies the inequalities of \( \Delta(A_r) \), that is for all proper subsets \( J \subset \{0, \ldots, r\} \) we have \( \sum_{i \in J} e_i^*(v) \leq 1 \). The relevant inequalities are those for \( J \) equal to one of the following: \( \{0\} \), \( \{1\} \), \( \{0,1,i\} \), or \( \{0, \ldots, r\} \setminus \{i\} \). The first two inequalities give:

\[
a_0 + k + \frac{1}{2} \leq 1 \quad \text{and} \quad a_1 - \frac{1}{2} \leq 1.
\]

Combining them with the equation \( a_0 + a_1 = 1 - k \) and the assumption that \( a_i \) are integers we get \( a_0 = -k \) and \( a_1 = 1 \). Then

\[
v = \frac{1}{2}e_0 + \frac{1}{2}e_1 + \sum_{i=2}^{2k+1} (a_i - \frac{1}{2})e_i.
\]

Applying the remaining two inequalities we get

\[
1 + a_i - \frac{1}{2} \leq 1 \quad (\Rightarrow a_i \leq 0)
\]

\[
1 + \sum_{j=2}^{2k+1} (a_j - \frac{1}{2}) - (a_i - \frac{1}{2}) \leq 1 \quad (\Rightarrow a_i \geq 0).
\]

\( \quad \text{= 0 since } \sum_{i=0}^{2k+1} a_i = 0 \)

Therefore each \( a_i \) for \( i \geq 2 \) is zero, and

\[
v = \frac{1}{2}(e_0 + e_1 - \sum_{i=2}^{2k+1} e_i),
\]

which satisfies the omnipresent equation \( \sum_{i=0}^{2k+1} e_i^* = 0 \) if and only if \( k = 1 \), that is \( r = 3 \), and the proof is concluded.
PROPOSITION 5.9. Suppose that \((X, L)\) and \((G, H)\) satisfy Assumptions 5.3 with \(G\) of type \(A_r\). Then \(r \leq 3\). If, in addition, they satisfy Assumptions 5.2, then \(r \leq 2\).

PROOF. Suppose by contradiction that \(r \geq 3\). We take a facet \(\delta(\{0, 1\})\) associated to \(J\) consisting of two indices, 0 and 1. The vertices of this facet are \(e_0 - e_i\) and \(e_1 - e_i\), with \(i = 2, \ldots, r\). It follows that \(\delta(\{0, 1\})\) (without taking the lattice structure into account) has the same combinatorial structure as a product of an interval and a simplex of dimension \(r - 2\). In particular, \(\delta\) has \(2(r-1)\) vertices. Fix a vertex of \(\delta\), say \(e_0 - e_2\). The edges of \(\delta\) that contain this vertex are those connecting it to vertices \(e_1 - e_2\) and \(e_0 - e_i\) (for \(i \in \{3, \ldots, r\}\)). In particular, there are \(r - 1\) of them.

We will run the torus action program on the manifold
\[Y = Y_{\delta(\{0,1\})} \subset X\]
associated to the facet \(\delta(\{0,1\})\). It admits an almost faithful action of a torus of rank \(r - 1\) and the compass of a fixed point \(y\) associated to a vertex \(e_0 - e_2\) is the intersection of the compass of \(X\) at \(y\) and the hyperplane parallel to the facet. In particular, by Corollary 5.6(1) there are \(r - 1\) elements of the compass coming from the edges, all of them of multiplicity 1.

We claim that the compass of \(y\) in \(X\) or in \(Y\) does not contain the vectors of the form \(a((e_1 - e_i) - (e_0 - e_2))\) for any \(i \in \{3, \ldots, r\}\) and \(a \geq 1\). Indeed, the symmetric counterpart (see Corollary 5.6(3) or Lemma 4.1) of this vector is equal to
\[-(e_0 - e_2) - a((e_1 - e_i) - (e_0 - e_2)) = a(e_i - e_1) + (a - 1)(e_0 - e_2),\]
which does not belong to the cone \(\sigma = \mathbb{R}_{\geq 0}(\Delta(A_r)-(e_0-e_2))\). This last claim can be seen using the inequalities of \(\sigma\), one of them is \(e_0^* + e_i^* \leq 0\).

Suppose \(r \geq 4\). By Lemma 5.8, the only lattice points in \(\delta\) are the vertices. Thus by Corollary 2.14, any element of the compass must be of the form \(a(v - (e_0 - e_2))\) for some vertex \(v\) and \(a \geq 1\). Therefore there are only \(r - 1\) elements of the compass, \(\dim Y = r - 1\), and thus \(Y\) is a (smooth, projective) toric variety with \(\text{Pic} Y = \mathbb{Z} \cdot L|_Y\) and \(h^0(L|_Y) = 2(r - 1)\) by Lemma 5.4. But the only smooth complete toric variety of dimension \(r - 1\) with \(\text{Pic} Y \cong \mathbb{Z}\) is \(\mathbb{P}^{r-1}\), and the ample generator of the Picard group has an \(r\)-dimensional space of sections. Therefore, \(2(r-1) = r\), a contradiction with our assumption that \(r \geq 4\).

It remains to consider the case of \(r = 3\), which has an additional lattice point in the relative interior of \(\delta\), as shown in Lemma 5.8 and illustrated on the top left in Figure 1. By Corollary 2.14 the compass of \(y\) in \(Y\) must be contained in the union of three half lines starting at
the vertex $e_0 - e_2$ shifted to 0 and passing through the lattice points of $\delta$. By the above considerations, the only possibilities are the three vectors indicated on the top right in Figure 1.

Downgrading the action of the two dimensional torus by a vertical squeeze of the lattice (bottom of Figure 1), we are in the situation of Proposition 3.12. Thus the manifold $Y$ is either a projective space or a quadric of dimension at least 3. By Lemma 5.4 the space of sections $H^0(Y, L|_Y)$ is 4-dimensional, thus $Y \simeq \mathbb{P}^3$. (Note that here we exploit Assumption 5.2(7)). The compass of $y$ in $Y$ after the downgrading must be just the vector of length 1 with some multiplicity by Lemma 2.13. This is in contradiction with Corollary 3.13 for $\mathbb{P}^3$ and concludes the proof. \qed

5.3. Case of $C_r$ with $r \geq 2$.

Lemma 5.10. Assume $X$, $L$, $G$, and $H$ satisfy Assumptions 5.2. Then $G$ is not of type $C_r$ (for any $r \geq 2$).

Proof. We use information from [12 Table 3]. The polytope $\Delta(C_r)$ is in a Euclidean space $\mathbb{R}^r$ with a basis $e_i, i = 1, \ldots, r$. The vertices of $\Delta(C_r)$ are long roots $\pm 2e_i$, each edge of $\Delta(C_r)$ contains a short root $\pm e_i \pm e_j, i \neq j$ and the roots are the only weights contained in the edges. In other words, the condition $(*)$ is satisfied. By Lemma 5.5 the manifolds corresponding to edges are isomorphic to $\mathbb{P}^2$, ...
and by Corollary 5.6(1) the difference $2(e_2 - e_1)$ is in the compass of the fixed point corresponding to $2e_1$. This is in a contradiction with Corollary 5.6(3), thus $G$ cannot be of type $C_r$. \hfill $\square$

5.4. Cases of $B_r$ and $D_r$. To avoid an overlap with the previous cases we assume $r \geq 3$ for the case $B_r$ and $r \geq 4$ for the case $D_r$. We use Table 2 and 4. We take the lattice $\mathbb{Z}^r$ generated by $e_1, \ldots, e_r$ and define $M = \sum_i \mathbb{Z} e_i + \mathbb{Z}(\sum_i e_i)/2$. Then $\Delta = \Delta(B_r) = \Delta(D_r)$ has vertices $\pm e_i \pm e_j$ for $i \neq j$. The vertex $\pm e_i \pm e_j$ is connected by an edge to $\pm e_r \pm e_s$ if they have one common index in pairs $(i,j)$ and $(r,s)$ and the same sign for this index. There are distinguished facets of $\Delta$ defined as $\delta_i^\pm := \{u \in M_R : e_i^*(u) = \pm 1\}$ with vertices $e_i \pm e_j$, $j \neq i$ and $-e_i \pm e_j$, $j \neq i$, respectively. More generally, the inequalities defining $\Delta$ include:

$$\pm e_i^* \leq 1 \text{ and } \sum_i (\pm e_i^*) \leq 2$$

These give $2r + 2^r$ linear inequalities which can be briefly described as $|e_i^*| \leq 1$ and $\sum_i |e_i^*| \leq 2$. It is straightforward to see that each inequality is the supporting inequality of a facet, that is these are minimal inequalities. It can also be verified that these are all inequalities of $\Delta$, but we are not going to use this last statement.

All the non-zero weights contained in $\Delta$ are the following:

- the vertices of $\Delta$, which are all roots in the case $D_r$, or long roots in the case $B_r$,
- points $\pm e_i$ which lie on the facets $\{u \in M : e_i^*(u) = \pm 1\}$ (and which in the case $B_r$ are the short roots), and
- for $r \leq 4$ points $(\sum_{i=1}^r \pm e_i)/2$ which for $r = 3$ lie in the interior of $\Delta$ and for $r = 4$ they lie on the facets of type $\{u \in M_R : \sum_i \pm e_i^*(u) = 2\}$.

**Lemma 5.11.** Suppose $r \geq 3$ and fix a vertex of $\Delta$, say $e_1 + e_2$. Let $\sigma = \mathbb{R}_{\geq 0} \cdot (\Delta - (e_1 + e_2))$, as in Corollary 5.6(3). Then every lattice point $v$ in $\sigma \cap (-e_1 - e_2 - \sigma)$ satisfies at least one of the following:

- $v$ is contained in at least one of the hyperplanes $e_i^* = 0$ or $e_2^* = 0$,
- $v = -e_1 - e_2$,
- $r \leq 4$ and $v = -\frac{1}{2}(e_1 + e_2 \pm e_3 \pm e_4)$ (if $r = 4$) or $v = -\frac{1}{2}(e_1 + e_2 \pm e_3)$ (if $r = 3$).

**Proof.** Some inequalities of $\sigma$ are easily obtained from the inequalities of $\Delta - (e_1 + e_2)$. In particular, for all $v \in \sigma$ we have

(i) $e_i^*(v) \leq 0$,
(ii) $e_2^*(v) \leq 0$. 


(iii) \((e_1^* + e_2^* + \sum_{i=3}^r \pm e_i^*)(v) \leq 0\) for any choices of signs \(\pm\).

Similarly, the inequalities of \(-e_1 - e_2 - \sigma\) include:

(iv) \(e_1^*(v) \geq -1\),

(v) \(e_2^*(v) \geq -1\),

(vi) \((e_1^* + e_2^* + \sum_{i=3}^r \pm e_i^*)(v) \geq -2\) for any choices of signs \(\pm\).

In particular, the only possible values of \(e_1^*(v)\) for a lattice point \(v \in M\) in the intersection of the two cones are \(0, -\frac{1}{2}, \) or \(-1\) (and similarly for \(e_2^*(v)\)). The case of one of them equal to \(0\) is the first item in the lemma. Thus assume otherwise, that both \(e_1^*(v)\) and \(e_2^*(v)\) are non-zero. By the construction of \(M\), either all coordinates of \(v\) are integral, or all are congruent to \(\frac{1}{2}\) modulo \(1\). In particular, \(e_1^*(v) = e_2^*(v) \in \{ -\frac{1}{2}, -1 \}\).

If \(e_1^*(v) = e_2^*(v) = -1\), then by (vi) we must have \(\sum_{i=3}^r \pm e_i^*(v) \geq 0\) for any choices of signs. Thus \(e_i^*(v) = 0\) for all \(i \in \{3, 4, \ldots, r\}\), that is \(v = -e_1 - e_2\) as in the second item of the lemma.

Finally, consider the case \(e_1^*(v) = e_2^*(v) = -\frac{1}{2}\), and all the other coordinates are also non-integral. We must have \(-1 \leq \sum_{i=3}^r \pm e_i^*(v) \leq 1\), thus \(r \leq 4\) and \(e_i^*(v) = \pm \frac{1}{2}\) for \(i = 3\) and \(i = 4\) (when \(r = 4\)). \(\square\)

**Lemma 5.12.** Suppose \((X, L)\) and \((G, H)\) satisfy Assumptions [5.2] with \(G\) of type \(B_r\), \(r \geq 3\) or \(D_r\), \(r \geq 4\). Then the following conditions hold:

1. the source/sink of the action on \(X\) of the 1-dimensional subtorus associated to the projection \(e_i^*: M \to \mathbb{Z} \cdot \frac{1}{2}\) is the quadric \(Q^{2r-3}\) in the case \(B_r\) and quadric \(Q^{2r-4}\) in the case \(D_r\),
2. the middle point \(\pm e_i\) of the facet \(e_i^* = \pm 1\) does not correspond to any fixed-point component of \(X^H\), that is \(\pm e_i \notin \Delta(X, L, H)\),
3. for \(r = 4\) the source/sink of the action on \(X\) of the 1-dimensional subtorus associated to the projection \(\sum_{i=1}^4 e_i^*: M \to \mathbb{Z}\) is the quadric \(Q^4\).

**Proof.** In (1) fix \(i = 1\) and consider the source, that is the fixed-point component corresponding to \(1 \in \mathbb{Z} \cdot \frac{1}{2}\). The projection \(e_1^*\) maps the facet \(\delta_1^+ = \text{conv}(e_1, e_2, j \neq 1)\) to \(\{1\}\). Thus the source is just \(Y := Y_{\delta_1^+}\). After shifting \(e_1\) to \(0\), the polytope \(\delta_1^+\) becomes the convex hull of \(\{\pm e_2, j \neq 1\}\), and the lattice \(M\) intersected with the linear span of \(e_2\) (for \(j \neq 1\)) is equal to the group generated by the \(e_j\)’s.

Therefore \(\Delta(Y, L, (\mathbb{C}^*)^{r-1}) = \text{conv}(\pm e_2, j \neq 1)\) by Lemma [5.4] and by Corollary [3.14] the manifold \(Y\) is a quadric hypersurface. The dimension of the quadric follows from Lemma [5.4].

To see (2) note that by Lemma [3.3] the middle point corresponds to a fixed point in \(X^H\) (that is, the middle point is in \(\Delta(X, L, H)\))...
if and only if the point is in \( \tilde{\Delta}(Y_{δ^+}, L|Y_{δ^+}^*, (\mathbb{C}^*)^{r-1}) \). But by the above arguments, \( Y_{δ^±} \) is a quadric with the automorphisms group of rank \( r-1 \), thus the action of \((\mathbb{C}^*)^{r-1}\) must be (up to a finite cover) the standard action of the maximal torus on a quadric, which has no non-extremal fixed points, see Subsection 2.4. Thus \( \pm e_i \notin \tilde{\Delta}(X, L, H) \).

Now we restrict to the case \( r = 4 \) in order to show (3). Say, consider only the source \( Y_δ \) for the facet \( δ := \Delta \cap \{ u \in M_\mathbb{R} : (e_1^* + e_2^* + e_3^* + e_4^*)(u) = 2 \} \). The lattice points on \( δ \) are the 6 vertices \( e_i + e_j \) and the interior point \( \frac{1}{2}(e_1 + e_2 + e_3 + e_4) \). Shifting \( \frac{1}{2}(e_1 + e_2 + e_3 + e_4) \) to the origin, and choosing \( \frac{1}{2}(e_1 + e_2 - e_3 - e_4), \frac{1}{2}(e_1 - e_2 + e_3 - e_4), \frac{1}{2}(e_1 - e_2 - e_3 + e_4) \) as the basis of the 3-dimensional sublattice containing the lattice span of \( δ \), this facet becomes a 3-dimensional polytope such as in Corollary 3.14. Thus \( Y_δ \) is a quadric and by counting the number of sections in Lemma 5.4 we get \( \dim Y_δ = 4 \), proving the claim. \( \square \)

**Lemma 5.13.** If \((X, L)\) and \((G, H)\) satisfy Assumption 5.2 with \( G \) of type \( B_r \) and \( r \geq 3 \) or \( D_r \) and \( r \geq 4 \). Then:

\[
\dim X = \begin{cases} 
\dim Gr(\mathbb{P}^1, Q^{2r-3}) = 4r - 5 & \text{in the case } B_r \\
\dim Gr(\mathbb{P}^1, Q^{2r-4}) = 4r - 7 & \text{in the case } D_r,
\end{cases}
\]

- the only points in \( \tilde{\Delta} \) are the vertices of \( \Delta \),
- for a fixed type of the group, the compass of any extremal component \( y \in X \) corresponding to a fixed vertex of \( \Delta \) is independent of \( X \).

**Proof.** We count the elements of the compass at the fixed point \( y \in X^H \) associated to the vertex \( e_1 + e_2 \). By Corollary 5.6(3) and Lemma 5.11 any element \( ν \) of the compass must be of one of the following types:

- \( ν \) is contained in at least one of the hyperplanes \( e_1^* = 0 \) or \( e_2^* = 0 \), or
- \( ν = -e_1 - e_2 \), or
- \( r = 4 \) and \( ν = -\frac{1}{2}(e_1 + e_2 \pm e_3 \pm e_4) \).
- \( r = 3 \) and \( ν = -\frac{1}{2}(e_1 + e_2 \pm e_3) \).

If \( ν \) is of the first type, from Lemmas 2.13 and 5.12(1) we conclude that \( ν \) has multiplicity 1 in the compass and is one of the \( \pm e_i - e_2 \) and \( \pm e_1 - e_1 \), for \( i \geq 3 \) and in the case \( B_r \) also \( -e_2 \) and \( -e_1 \). This gives a total of \( 4(r-2) \) elements in the case \( D_r \) and 2 more elements in the case \( B_r \).

If \( ν = -(e_1 + e_2) \), then it is of the second type, and thus \( ν \) also has multiplicity 1 in the compass by Corollary 5.6(2). Thus to show
the dimension part of the lemma, we have to prove, that the third or fourth type of points does not appear in the compass.

Note that, the remaining claims about \( \tilde{\Delta} \) and the compass at \( e_1 + e_2 \) also follow, once we show that the third and fourth types do not appear. Indeed, by Corollary 2.14, the only other candidates for points in \( \tilde{\Delta} \) are the midpoints of the faces \( \pm e_i \), which are excluded by Lemma 5.12(2), or \( 0 \in \Delta \), which is excluded by Lemma 5.7. The statement about compass is straightforward from the above calculations.

Thus there is nothing left to prove for \( r > 4 \). First assume \( r = 4 \). Say we want to show that \( \nu = -\frac{1}{2}(e_1 + e_2 + e_3 + e_4) \) is not in the compass (the proof for the other cases is analogous). This point is the middle point of the facet \( \delta \) with the supporting hyperplane \( e_1^* + e_2^* + e_3^* + e_4^* = 2 \) as in Lemma 5.12(3). In particular, \( \dim Y_\delta = 4 \), and there are already 4 distinct elements of the compass of the form \( e_3 - e_1, e_3 - e_2, e_4 - e_1, \) and \( e_4 - e_2 \) that are parallel to this face. Thus by Lemma 2.13, they are all in the compass of \( y \) in \( Y_\delta \), hence they are the only elements of this compass, and \( \nu \) is not in either of the compasses \( C(y, Y_\delta, (\mathbb{C}^*)^3) \) or \( C(y, X, H) \).

Finally, we treat the most delicate case, \( r = 3 \) and \( G \) of type \( B_3 \). The following diagram presents the root polytope \( \Delta = \Delta(B_3) \) with \( • \) denoting the long roots (vertices) and \( ◦ \) denoting the short roots of \( B_3 \). The doubled line segments indicate the six elements of the compass at one of the extremal fixed points, we ignored the element pointing towards the center of the polytope.

Let \( \nu = -\frac{1}{2}(e_1 + e_2 + e_3) \) and denote by \( m \) the multiplicity of \( \nu \) in the compass \( C(y, X, H) \). By the duality Lemma 4.1 the multiplicity of \( \nu' := -\frac{1}{2}(e_1 + e_2 - e_3) \) is also equal to \( m \) and we aim to show \( m = 0 \). We assume \( m > 0 \) and argue to get a contradiction.

The next diagram is a cross section of the root polytope \( \Delta \) (dotted line segments) by the plane \( e_1^* - e_2^* = 0 \). Now \( \otimes \) denotes the zero weight.
Consider the 1-dimensional subtorus $H' \subset H$ associated to the projection $e_1^* - e_2^* : M \to \mathbb{Z} \cdot \frac{1}{2}$. Note that $0 \in \tilde{\Delta}(X, L, H')$, as the fixed-point set $X^{H'}$ contains in particular $y$. Let $Y_0 \subset X^{H'}$ be the component that contains $y$. By Corollary 4.4 the variety $Y_0$ is a contact manifold with an action of the 2-dimensional torus $H/H'$. Moreover, by Lemma 2.13 the compass $\mathcal{C}(y, Y_0, H/H')$ consists of $\nu$ and $\nu'$ both with multiplicity $m$, and $-e_1 - e_2$ with multiplicity 1. In particular, dim $Y_0 = 2m + 1$.

We claim that the solid line segments on the above figure are the boundary of the fixed-point polytope $\Delta(Y_0, L|Y_0, H/H')$. Indeed, $\Delta(Y_0, L|Y_0, H/H')$ is a convex hull of some of the lattice points in $\tilde{\Delta}(X, L, H) \cap \{e_1^* - e_2^* = 0\}$ by Lemma 2.10(3). Note however, that $\pm e_3$ does not belong to $\tilde{\Delta}(X, L, H)$ by Lemma 5.12(2). Hence the $\Delta(Y_0, L|Y_0, H/H')$ is a subset of the region bounded by the solid lines. Moreover, the compass at $y$ together with Corollary 2.14 indicates that there must be fixed-point components $Z$ and $Z'$ corresponding to $\nu$ and $\nu'$, respectively. Now move to the compass at $Z$ (or $Z'$). By the duality in Lemma 4.1 the next edge must also be included in $\Delta(Y_0, L|Y_0, H/H')$, which shows the claim.

From Lemma 3.15 applied to the action of 1-dimensional torus associated to the projection along the respective edge we conclude that the extremal fixed-point components in $Y_0$ associated to $^*$’s are of dimension $m - 1$ (in fact they are $\mathbb{P}^{m-1}$’s). By Lemma 4.1 applied to $Y_0$ the compass $\mathcal{C}(Y_1, Y_0, H')$ of $H'$ on the extremal fixed-point component $Y_1$ associated to, say $(e_1 + e_2 + e_3)/2$ contains $-(e_1 + e_2 + e_3)/2$ with multiplicity $m$ and $(e_1 + e_2 - e_3)/2, -e_1 - e_2$, both with multiplicity 1, coming from the decomposition

$$-\frac{1}{2}(e_1 + e_2 + e_3) = (-e_1 - e_2) + \frac{1}{2}(e_1 + e_2 - e_3).$$

Hence by Lemma 2.13 the compass $\mathcal{C}(Y_1, X, H)$ contains $m + 2$ elements listed above and, possibly, a few other elements from outside.
the plane $e_1^* - e_2^* = 0$. Such an element $w$ must satisfy the following conditions:

- the halfline $(e_1 + e_2 + e_3)/2 + w \cdot \mathbb{R}_{>0}$ must intersect the set $\bar{\Delta}$ (Corollary 2.14),
- the dual element $-(e_1 + e_2 + e_3)/2 - w$ must also be in the compass (Lemma 4.1), hence satisfies the above.

There are 27 lattice points in $\Delta$, out of which at most 21 are in $\bar{\Delta}$ (Lemma 5.12(2)). Further removing the points on the plane $e_1^* - e_2^* = 0$ we are left with 14 points. Listing them and explicitly checking if they satisfy the second condition above, we are left only with 4 candidates, coming in two pairs:

\[
\frac{1}{2}(e_1 - e_2 + e_3) \text{ and } (-e_1 - e_3); \quad \frac{1}{2}(-e_1 + e_2 + e_3) \text{ and } (-e_2 - e_3).
\]

The first pair is contained in the plane $e_1^* - e_3^* = 0$, and the second is contained in the plane $e_2^* - e_3^* = 0$, and they are both analogous to the pair of $\frac{1}{2}(e_1 + e_2 - e_3)$ and $(-e_1 - e_2)$ in the plane $e_1^* - e_2^* = 0$. In particular, the same proof as above shows that the multiplicity of these four vectors in the compass is equal to 1. Therefore $\dim X \leq \dim Y_1 + m + 6 = 2m + 5$, a contradiction. □

**Corollary 5.14.** Let $X$ be a contact Fano manifold which satisfies Assumptions [5.2] with $G$ of type $B_r$ or $D_r$. Then $X$ is isomorphic to the Grassmanian of lines on the quadric $Q^{2r-1}$ or $Q^{2r-2}$, respectively.

**Proof.** In Lemma 5.13 we showed that the fixed-point locus $X^H$ consists of extremal fixed points only and the compass at each of these points is determined uniquely. The respective Grassmannian of lines on the quadric of matching dimension satisfies Assumptions [5.2]. Thus the result follows by Proposition 2.24. □

Note that in the above corollary for the first time in Section 5 we used the whole action of $G$, not only the action of $H$.

5.5. **Case of $A_2$ and $G_2$**. In this subsection we deal with the action of groups of types $A_2$ (that is $SL(3)$) or $G_2$ on a contact manifold $X$ of dimension 7 or 9. Below we illustrate the weights of $G = SL(3)$ in the root polytope $\Delta(A_2)$. The same polytope and weight lattice is obtained for the group $G$ of type $G_2$. The vertices of $\Delta(A_2)$ are denoted by $\bullet$ and labeled clockwise by $\alpha_i$, the center $\otimes$ is the zero weight and the other weights inside $\Delta(A_2)$ are denoted by $\circ$ and labeled by $\beta_i$ also clockwise. The indices of $\alpha$’s and $\beta$’s are considered modulo 6. The fixed-point components associated to $\alpha$’s are extremal. The fixed-point components associated to $\beta$’s will be called inner and the ones...
associated to the zero weight will be called central. The dotted line segment illustrates the fixed-point locus of some 1-parameter subgroup of $H \subset G$ associated to the respective projection $M \to \mathbb{Z}$ on which we have the action of the quotient $\mathbb{C}^*$.

Throughout we work with Assumptions 5.1, with $G$ of type $A_2$ or $G_2$. Denote by $y_{\alpha_i} \in X^H$ the unique extremal point with $\mu(y_{\alpha_i}) = \alpha_i$.

In the proofs below we will exploit the downgrading mentioned above and its consequences. Thus fix $i \in \{0, \ldots, 5\}$ and let $H' = H'_i$ be the subtorus corresponding to the projection $\pi = \pi_i: M \to \mathbb{Z}$ which maps all of $\alpha_{i-1}, \beta_i, \beta_{i+1}, \alpha_{i+1}$ to $1 \in \mathbb{Z}$. For $i = 0$, the dotted line in the figure passes through all these points. By the last item of Lemma 3.4(2) there is a unique connected component $Y = Y_i \subset X^{H'}$ which contains the extremal fixed points $y_{\alpha_i \pm 1}$ and all inner components of $X^H$ associated to $\beta_i$ and $\beta_{i+1}$. This manifold $Y$ admits the restricted action of a 1-dimensional torus $H/H'$ with

$$\Delta(Y, L|_Y, H/H') = \Gamma(Y, L|_Y, H/H') = [0, 3]$$

where 0 represents $\alpha_{i-1}$ and 3 represents $\alpha_{i+1}$. Using Lemma 2.13 and the compass calculation at $y_{\alpha_{i-1}}$ in Lemma 5.15 below we see that $\dim Y = 2$ if $\dim X = 7$ and $\dim Y = 3$ if $\dim X = 9$.

**Lemma 5.15.** Under Assumptions 5.1 with the group $G$ of type $A_2$ or $G_2$, suppose in addition that dimension of $X$ is either 7 or 9. Then the following holds:

1. the extremal components of $X^H$ are points and the compass of the action of $H$ at a point associated to character $\alpha_i$ consists of the following characters:
   - $\alpha_i - \alpha_i, \alpha_{i+1} - \alpha_i, -\alpha_i$, all three of multiplicity 1 and
   - $\beta_i - \alpha_i, \beta_{i+1} - \alpha_i$, both with multiplicity 2 if $\dim X = 7$ or multiplicity 3 if $\dim X = 9$,

2. inner components of $X^H$ are points if $\dim X = 7$ or points or curves if $\dim X = 9$; the compass of the action of $H$ at a component associated to the weight $\beta_i$ consists of the following characters:
\[ \alpha_i - \beta_i, \alpha_{i-1} - \beta_i, \text{ both with multiplicity 1}, \]
\[ \beta_j - \beta_i, \text{ where } j \neq i, i+3, \text{ all with multiplicity 1, except if } \dim X = 9 \text{ and the fixed-point component is a point in which case } \beta_{i \pm 1} - \beta_i \text{ are of multiplicity 2}, \]
\[ \beta_i \text{ with multiplicity 1 if } \dim X = 7 \text{ or multiplicity 1 or 2 if } \dim X = 9, \text{ and dimension of the component is 0 or 1, respectively,} \]

(3) There is no central component of \( X^H \).

**Proof.** Part (1) is proven with the usual methods, as in previous subsections. Specifically, the multiplicities of \( \alpha_{i-1} - \alpha_i, \alpha_{i+1} - \alpha_i, -\alpha_i \) follow from Corollary 5.6. The same corollary, part (3) show that \( \beta_i - \alpha_i \) and \( \beta_{i+1} - \alpha_i \) are the only other candidates for elements of the compass. Their multiplicities must be equal by the duality of Lemma 4.1. Thus the statement follows from the dimension calculation.

For Part (2) we note that using the duality (Lemma 4.1) and Corollary 2.14, for every \( \nu \) in the compass of any component \( Z \) corresponding to \( \beta_i \), we must have both halflines \( \beta_i + \mathbb{Q}_{>0} \cdot \nu \) and \( \beta_i + \mathbb{Q}_{>0} \cdot (-\nu - \beta_i) \) contain a lattice point of \( \Delta \). This condition shows, that no other vector than those 7 vectors listed in the statement can be in the compass. It remains to determine the multiplicities. For

\[ \nu \in \{ \alpha_i - \beta_i, \alpha_{i-1} - \beta_i, \beta_{i+1} - \beta_i, \beta_{i-1} - \beta_i, \beta_{i+2} - \beta_i, \beta_{i-2} - \beta_i, -\beta_i \} \]

denote by \( b_{Z,\nu} \) the multiplicity of \( \nu \) in the compass. We always have \( b_{Z,-\beta_i} = \dim Z + 1 \).

Consider the component \( Y = Y_i \), the extremal fixed points \( y_{\alpha_i} \), the projection \( \pi = \pi_i \), and the corresponding subtorus \( H' = H'_i \subset H \) as introduced above. By Lemma 2.13 applied twice we have

\[ \{-2, -1^n, 1\} = \pi(C(y_i, X, H)) = C(Y, X, H') = \pi(C(Z, X, H)), \]

where \( n = 3 \) for \( \dim X = 7 \) and \( n = 4 \) for \( \dim X = 9 \). The only candidate for a member in the compass \( C(Z, X, H) \), that is mapped to 1 via \( \pi \) is \( \alpha_i - \beta_i \). Therefore, \( b_{Z,\alpha_i - \beta_i} = 1 \), and the same proof with a different choice of \( \pi = \pi_{i-1} \) shows that \( b_{Z,\alpha_{i-1} - \beta_i} = 1 \). By the duality (Lemma 4.1), also \( \beta_{i+2} - \beta_i \) and \( \beta_{i-2} - \beta_i \) have multiplicity 1, while \( \beta_{i+1} - \beta_i \) and \( \beta_{i-1} - \beta_i \) have equal multiplicity. The only claim in (2) left to prove is that \( b_{Z,\beta_{i+1} - \beta_i} \geq 1 \), which follows from Białynicki-Birula Decomposition (Theorem 3.1) applied to the action of \( H/\mathbb{C}^* \) on \( Y \). Indeed, if \( b_{Z,\beta_{i+1} - \beta_i} = 0 \), then the respective BB-cells would be of dimension equal to \( \dim Y \), which is impossible, as \( \pi(\beta_i) = 1 \) is not the smallest nor the largest in the \( \Delta(X, L, H'_i) = [-2, 2] \).

To prove (3) we argue by contradiction and assume that we have a central component \( Z \). The compass of \( Z \) in \( X \) with respect to \( H \) is
symmetric: if it contains a vector $\nu$, then it also contains $-\nu$ with the same multiplicity (Lemma 4.1). Consider the projection of $M$ along $\nu \in C(Z, X, H)$ and the corresponding subtorus $H' \subset H$. Pick the component $Y \subset X^{H'}$ which contains $Z$. An extremal component $Y' \subset Y^{H'/H'}$ of the fixed-point set of the quotient torus $H/H'$ action on $Y$ is a component of $X^H$ by Lemma 2.10(3) and it is not a central component. Hence by Parts (1) and (2) we know that $\dim Y' \leq 1$ and the compass $C(Y', Y, H/H')$ has at most 2 elements (and 2 elements are possible only if $\dim X = 9$). We also know that $\dim Z$ and $\dim Y$ are odd by Corollary 4.4.

We conclude that the only possibility is $\dim X = 9$, $\dim Y = 3$ and $\dim Z = 1$ and $\nu = \beta_i$ for some $i$, with multiplicity 1. Thus the compass $C(Z, X, H)$ consists of the $\beta_i$’s, each with multiplicity at most one which is impossible because it should consist of $8 = \dim X - \dim Z$ vectors, while there are only 6 of the $\beta_i$’s.

For a while now we restrict to the case $\dim X = 7$.

**Lemma 5.16.** Under Assumptions 5.1, suppose in addition $G$ is of type $A_2$ or $G_2$ and $\dim X = 7$. Then there exists a unique inner fixed point $y_{\beta_i}$ for every $i \in \{0, \ldots, 5\}$.

**Proof.** We consider $Y = Y_i$ with the action of $H/H' = H/H'_i \simeq \mathbb{C}^*$ as introduced above. Lemma 5.15 together with Lemma 2.10 show that $Y$ is a surface and its polytope of fixed points $\Delta(Y, L|_Y, \mathbb{C}^*)$, polytope of sections $\Gamma(Y, L|_Y, \mathbb{C}^*)$ and the compasses $C(y, Y, \mathbb{C}^*)$ are exactly as in Example 3.16. Thus there is exactly 1 fixed point in $Y^{H'/H'}$ corresponding to $\beta_i$. But all fixed points corresponding to $\beta_i$ are contained in $Y$, so there is exactly 1 fixed point in $X^H$ corresponding to $\beta_i$, as claimed.

The following lemma is a straightforward explicit verification.

**Lemma 5.17.** Consider the 7-dimensional contact manifold $X = Gr(\mathbb{P}^1, Q^5)$. A three-dimensional torus acts on $X$, and the lattice and polytopes corresponding to type $B_3$ are described in Subsection 3.4. Using the basis notation as in that subsection, consider a downgrading to a two dimensional torus $H$, corresponding to a projection $\mathbb{Z}^3 + (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})\mathbb{Z} \to M$, with the kernel generated by $(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$. Then the pair $(X, H)$ satisfies Assumptions 5.1 with $G$ of type $G_2$. In particular, its fixed points $X^H$ and compasses are described in Lemmas 5.15 and 5.16.

We remark that the torus downgrading comes from the usual embedding of the $G_2$-group in $SO(7)$.
Proposition 5.18. Under Assumptions 5.1 with \( \dim X = 7 \) and \( G \) of type \( A_2 \) or \( G_2 \), we have \( h^0(L) = \dim SO(7) = 21 \). In particular, \( X \) of dimension 7 and \( G \) of type \( A_2 \) or \( G_2 \) cannot satisfy Assumptions 5.2.

Proof. Let \( X' = \text{Gr}(\mathbb{P}^1, \mathbb{Q}^5) \). By Lemma 5.17 both Lemmas 5.15 and 5.16 apply to \( X' \) equally well as to \( X \). There are only finitely many fixed points \( X^H \) and \( (X')^H \) by Assumption 5.1(5) and Lemma 5.15. For each \( X \) and \( X' \) there are unique fixed points with \( \mu(y) = \alpha_i \) (Lemma 3.2), or \( \beta_i \) (Lemma 5.16), and there is no fixed point corresponding to \( 0 \in M \). Moreover, the compasses at these fixed points are uniquely determined by Lemma 5.15. Therefore by Proposition 2.23 we must have \( h^0(X, L) \simeq h^0(X', L') = 21 \) as claimed.

From now on we treat the case \( \dim X = 9 \).

Lemma 5.19. Under Assumptions 5.1 with \( \dim X = 9 \) and \( G \) of type \( A_2 \) or \( G_2 \), the fixed-point locus \( X^H \) has no component of dimension 1.

Proof. Suppose by contradiction that \( Z \subset X^H \) is a fixed curve. By Lemma 5.15 we must have \( \mu(Z) = \beta_i \) for some \( i \). We argue in several steps, using restriction and downgrading the action with respect to two different subtori \( \mathbb{C}^* \subset H \).

Step 1, in which we show \( Z \simeq \mathbb{P}^1 \) and \( L|_Z \simeq \mathcal{O}_{\mathbb{P}^1}(2) \). Here we consider the one parameter subgroup \( H' \subset H \) which corresponds to the projection \( M \to \mathbb{Z} \) with kernel generated by \( \beta_i \). Let \( Y' \subset X^{H'} \) be the component containing \( Z \). In particular, \( Y' \) is a contact Fano manifold by Corollary 4.4 with the contact line bundle \( L_{|Y'} \). The fixed points and compass calculations in Lemma 5.15 (and using Lemma 2.13) show \( \dim Y' = 3 \) and there is no component of \( (Y')^{H'/H'} \) corresponding to \( 0 \in \mathbb{Z} \cdot \beta_i \). Moreover, there are exactly two components of \( (Y')^{H'/H'} \), corresponding to the extremal values \( \beta_i \) and \( -\beta_i \), respectively (Lemma 3.2). The component corresponding to \( \beta_i \) is \( Z \), and the other component is either a point or a curve (Lemma 5.15(2)). Thus by the homology description in Theorem 3.1 the manifold \( Y \) has topological Euler characteristic at most 4. Any contact Fano 3-fold must be isomorphic to a projective space \( \mathbb{P}^3 \) or to the projectivisation of a cotangent bundle \( \mathbb{P}(T^*\mathbb{P}^2) \) (this result was originally claimed by [61], see [13], §1.3 for a detailed historical discussion). Given the restriction on the Euler characteristic, \( Y' \simeq \mathbb{P}^3 \) and therefore \( L|_{Y'} \simeq \mathcal{O}_{\mathbb{P}^3}(2) \). Since \( Z \) is an eigenspace of a torus action, \( Z \) is a line on \( \mathbb{P}^3 \), in particular, \( Z \simeq \mathbb{P}^1 \) and also \( L|_Z \simeq \mathcal{O}_{\mathbb{P}^1}(2) \).
Step 2, in which we review our notation for another $\mathbb{C}^*$-action on $X$. We switch our attention to $Y = Y_{i-1}$, the submanifold of dimension 3, which arises from the downgrading of the action as described in the introductory paragraphs of this subsection. The quotient $\mathbb{C}^*$ acts on $Y$ and $\Delta(Y, L|_Y, \mathbb{C}^*) = \Gamma(Y, L|_Y, \mathbb{C}^*) = [0, 3]$ with $\beta_i$ corresponding to $2 \in [0, 3]$. The setting is similar to the proof of Example 3.16 since $\Gamma(Y, L|_Y, \mathbb{C}^*) = \Delta(Y, L|_Y, \mathbb{C}^*)$ there are sections $\sigma_0$ and $\sigma_3$ of $L|_Y$ which have weights 0 and 3 respectively. Let $D_0$ and $D_3$ be the corresponding divisors. By Lemma 2.17, the divisor $D_0$ contains all fixed points of $Y$ except $y_{\alpha_{i-1}}$ (the fixed point corresponding to 0), and analogously $D_3$ contains all of them except $y_{\alpha_{i+1}}$ corresponding to 3. Consider the local defining equations $f_{0,z}$ and $f_{3,z}$ of $D_0$ and $D_3$ at a general point $z \in Z$, as in Lemma 2.17. Denote the local coordinate ring by $\mathbb{C}[s, t, u]$ with weights of $s, t, \text{and } u$ equal to $-1, 0, \text{and } 1$, respectively.

Step 3, in which we analyze the divisor $D_0$. The weight of $f_{0,z}$ is 2, thus $f_{0,z} = g(t, su) \cdot u^2$ for some power series $g$ in two variables. Thus $D_0$ has a component $U$ with multiplicity at least 2 corresponding to $u = 0$ near $z$. By Theorem 3.1, $U$ must coincide with the closure of the cell corresponding to the fixed-point component $Z$, and its closure contains only one extra point, the extremal point $y_{\alpha_{i+1}}$. In particular, $U$ is smooth except perhaps at $y_{\alpha_{i+1}}$. Looking at the local equation of $D_0$ near $y_{\alpha_{i+1}}$, we see that $U$ is given by a weight $-1$ equation in a coordinate ring with all negative weights (in fact all the weights are equal to $-1$ by the usual compass calculation, but we will not use it). Therefore $U$ is also smooth at $y_{\alpha_{i+1}}$, and by the previous arguments $U$ is smooth everywhere. Moreover $y_{\alpha_{i+1}}$ has a neighbourhood in $U$ isomorphic with $\mathbb{A}^2$, and $U \setminus \mathbb{A}^2 = Z \simeq \mathbb{P}^1$. Therefore $U \simeq \mathbb{P}^2$ and $Z$ is a line on this $\mathbb{P}^2$ and by Step 1 the line bundle $L|_U \simeq \mathcal{O}_{\mathbb{P}^2}(2)$. Denote by $K \subset U$ the line which is the closure of the orbit of $\mathbb{C}^*$ containing $z$ and $y_{\alpha_{i+1}}$.

Step 4, in which we analyze the divisor $D_3$ and conclude the proof. We aim to calculate the intersection number $D_3.K$ (which should be equal to 2) by Step 3, since $D_3$ is in the linear system of $L$, and $L|_K \simeq \mathcal{O}_{\mathbb{P}^2}(2)$ and arrive at a contradiction. The local equation $f_{3,z}$ is not divisible by $u$ because $D_3$ does not contain $\alpha_{i+1}$. The weight of $f_{3,z}$ is $-1$ by Lemma 2.17, hence $f_{3,z} = s \cdot h(t, su)$ for some power series $h$ in two variables, which is not divisible by the second variable. In particular, $D_3$ near $z$ is equal to the sum of $(s = 0)$ and $(h(t, su) = 0)$, where the second divisor does not vanish on all of $Z$. Since $z$ was chosen as a general point of $Z$, $h$ is invertible near 0, and $D_3$ is equal to $(s = 0)$ near $z$. In particular, $D_3.K = 1$, a contradiction. \[ \Box \]
Lemma 5.20. Under Assumptions \[5.1\] with \( \dim X = 9 \) and \( G \) of type \( A_2 \) or \( G_2 \), for every \( i \) there are exactly 3 fixed points of \( X^H \) corresponding to \( \beta_i \).

Proof. Fix \( i \in \{0, \ldots, 5\} \), and as before consider the smooth 3-fold \( Y = Y_i \) with a \( \mathbb{C}^* \) action and containing all the fixed points associated to \( \alpha_{i-1}, \beta_i \) and \( \beta_{i+1}, \alpha_{i-1} \). By Lemma 5.19 there are only \( a \) fixed points associated to \( \beta_i \) and \( b \) fixed points associated to \( \beta_{i+1} \) for some (finite) integers \( a, b \geq 0 \). We consider the rational function in one variable \( t \):

\[
F(t) = \sum_{y \in Y^H} \frac{t^{\mu(y)}}{\prod_{\nu \in \mathcal{C}(y,X,H)} (1 - t^\nu)}
= \frac{1}{(1-t)^3} + a \cdot \frac{t}{(1-t)^2(1-t^{-1})} + b \cdot \frac{t^2}{(1-t)(1-t^{-1})^2} + \frac{t^3}{(1-t^{-1})^3}
= \frac{t^6 - bt^4 + at^2 - 1}{(t-1)^3}.
\]

The second equality follows from Lemmas \[5.15\] and \[2.13\]. By Proposition \[2.25\] the rational function \( F \) must be a Laurent polynomial in \( t \). In particular, \( t = 1 \) must be a root of the numerator, thus \( a = b \). With \( a = b \), dividing out both numerator and denominator by \( (t-1) \) we get:

\[
F(t) = \frac{(t+1)(t^4 + (1-a)t^2 + 1)}{(t-1)^2}
\]

Again, \( t = 1 \) must be a root of the numerator, thus \( a = b = 3 \), and \( F = (1 + t)^3 \). So there are exactly 3 fixed points for each \( \beta_i \). \( \square \)

Now we check the model case also satisfies Assumptions \[5.1\] analogously to Lemma \[5.17\].

Lemma 5.21. Consider the 9-dimensional contact manifold \( X = Gr(P^1, Q^6) \). A four-dimensional torus acts on \( X \), and the lattice and polytopes corresponding to type \( D_4 \) are described in Subsection \[3.4\]. Using the basis notation as in that subsection, consider a downgrading to a two dimensional torus \( H \), corresponding to a projection

\[
\mathbb{Z}^4 + \left\langle \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\rangle \mathbb{Z} \rightarrow M,
\]

with the kernel generated by \( \left\langle \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right\rangle \) and \( (0,0,0,1) \). Then the pair \( (X,H) \) satisfies Assumptions \[5.1\] with \( G \) of type \( G_2 \).
The proof is a straightforward and explicit calculation. The torus embedding is the restriction of standard embedding: \( G_2 \hookrightarrow SO(7) \hookrightarrow SO(8) \). As a conclusion, we show the analogue of Proposition 5.18 in dimension 9.

**Proposition 5.22.** Under Assumptions 5.1 with \( \dim X = 9 \) and \( G \) of type \( A_2 \) or \( G_2 \), we have \( \dim \text{Aut}(X) = h^0(L) = \dim SO(8) = 28 \). In particular, \( X \) of dimension 9 and \( G \) of type \( A_2 \) or \( G_2 \) cannot satisfy Assumptions 5.2.

**Proof.** Let \( X' = Gr(\mathbb{P}^1, Q^6) \). By Lemma 5.21 all Lemmas 5.15, 5.19 and 5.20 apply both to \( X \) and \( X' \). There are only finitely many fixed points \( X^H \) and \( (X')^H \) by Assumption 5.1(5) and Lemmas 5.15, 5.19. For each \( X \) and \( X' \) there are unique fixed points with \( \mu(y) \) equal to \( \alpha_i \) (Lemma 3.2), and three fixed points with \( \mu(y) \) equal to \( \beta_i \) (Lemma 5.20), and there is no fixed point corresponding to \( 0 \in M \). Moreover, the compasses at these fixed points are uniquely determined by Lemma 5.15. Therefore by Proposition 2.23 we must have \( h^0(X, L) = h^0(X', L') = 28 \) as claimed. This is in contradiction with Assumption 5.2(7). □

This concludes the proof of Theorem 5.3 as we have already analyzed all the possible cases.

### 6. Proofs and concluding remarks

Having all the technical statements done, we conclude the article with gathering known results from the literature and applying our lemmas to show main results of the article. We provide a lower bound on the dimension of automorphism group of a low dimensional contact Fano manifold, following [55]. We further discuss the main theorems listed in the introduction, providing the necessary cross-references and citations. We also provide a brief overview of the consequences of twistor construction for positive quaternion-Kähler manifolds.

#### 6.1. Dimension of the automorphism group of a contact Fano manifold.

In this subsection we provide a lower bound on the dimension of the group of automorphism of a contact Fano manifold of dimension 7 or 9. The argument is analogous to [55] Thm. 7.5], whose statement is slightly weaker, as it only concerns quaternion-Kähler manifolds, and these correspond to contact Fano manifolds with Kähler-Einstein metric. In addition we rely on [39], Cor. 1.2] and Bogomolov–Gieseker inequality. Although it is possible to perform the calculations by hand (as shown in [55]), for brevity we refer to computer calculations carried out in magma [11].
Theorem 6.1. Suppose $X$ is a contact Fano manifold.

- If $\dim X = 7$, then $\dim(\text{Aut } X) \geq 5$.
- If $\dim X = 9$, then $\dim(\text{Aut } X) \geq 8$.

Note that (unlike in Theorem 1.3) we do not need to assume that $\text{Aut}(X)$ is reductive.

Proof. The statement holds true if $X$ is one of $\mathbb{P}^7$, $\mathbb{P}(T^*\mathbb{P}^4)$, $\mathbb{P}^9$, or $\mathbb{P}(T^*\mathbb{P}^5)$, so without loss of generality we may assume $\text{Pic } X \simeq \mathbb{Z} \cdot L$.

Therefore, $\dim(\text{Aut } X) = h^0(X, L)$ by Lemma 4.5 and the latter is equal to $\chi(X, L)$ by Kodaira vanishing since $L$ is ample and $X$ is Fano. Thus we have to show that $\chi(X, L)$ is at least 5 or 8 respectively.

We calculate the Hilbert polynomial $p(m) = \chi(X, L^m)$ using Hirzebruch-Riemann-Roch Theorem:

$$p(m) = \int_X td(TX)ch(L^m).$$

The Todd class is multiplicative $td(TX) = td(F)td(L)$ (by the short exact sequence $0 \to F \to TX \to L \to 0$), and the Chern classes of $F$ satisfy the symmetry property arising from the isomorphism $F \simeq F^* \otimes L$. This determines the odd Chern classes of $F$ in terms of the even Chern classes and $c_1(L)$. Explicitly, if $\dim X = 7$, then

$$c_1(F) = 3c_1(L),$$
$$c_3(F) = 2c_2(F)c_1(L) - 5c_1(L)^3,$$
$$c_5(F) = c_4(F)c_1(L) - c_2(F)c_1(L)^3 + 3c_1(L)^5,$$

and if $\dim X = 9$, then

$$c_1(F) = 4c_1(L),$$
$$c_3(F) = 3c_2(F)c_1(L) - 14c_1(L)^3,$$
$$c_5(F) = 2c_4(F)c_1(L) - 5c_2(F)c_1(L)^3 + 28c_1(L)^5,$$
$$c_7(F) = c_6(F)c_1(L) - c_4(F)c_1(L)^3 + 3c_2(F)c_1(L)^5 - 17c_1(L)^7.$$

Thus, the formula for the polynomial $p(m)$ is an explicit polynomial in $m$ and the Chern classes $c_1(L), c_2(F), c_4(F), c_6(F)$ and (in the case $\dim X = 9$) $c_8(F)$. There are extra conditions to impose on this polynomial, implied by Kodaira vanishing: $p(-2) = p(-1) = 0$ and $p(0) = 1$ (in the case $\dim X = 7$, the condition $p(-2) = 0$ is vacuous, as it follows from the symmetry, or Serre duality). Combining all of those identities and denoting $d = \deg(X, L) = c_1(L)^{\dim X}$ (the self
intersection of $L$) we get that:

if $\dim X = 7$, then

$$c_1(TX)^2 c_1(L)^5 = 16 \delta, \quad c_2(TX) c_1(L)^5 = 4 \delta + 12 p(1) - 48,$$

if $\dim X = 9$, then

$$c_1(TX)^2 c_1(L)^7 = 25 \delta, \quad c_2(TX) c_1(L)^7 = 9 \delta + 24 p(1) - 168.$$

The vector bundle $TX$ is stable by [39, Cor. 1.2], hence by Bogomolov–Gieseker inequality [46, Thm 0.1] we have:

$$(2 \dim X \cdot c_2(TX) - (\dim X - 1) \cdot c_1(TX)^2) \cdot c_1(L)^{\dim X - 2} \geq 0.$$ 

Substituting the explicit expressions and rearranging terms we get

$$p(1) \geq 4 + \frac{5}{21} \delta \quad \text{or} \quad p(1) \geq 7 + \frac{19}{216} \delta,$$

respectively, which proves the claim of the theorem.

The calculations were performed using the Chern classes package for magma available at [16].

Remark 6.2. The Hilbert polynomials of $p(m)$ from the proof of Theorem 6.1 are equal to:

$$p(m) = \delta \binom{m+7}{7} - 2 \delta \binom{m+6}{6} + (\delta + p(1) - 4) \binom{m+5}{5}$$

$$- (p(1) - 4) \binom{m+4}{4} + \binom{m+3}{3}, \text{ or}$$

$$p(m) = \delta \binom{m+9}{9} - \frac{5}{2} \delta \binom{m+8}{8} + (2 \delta + 2 p(1) - 14) \binom{m+7}{7}$$

$$- \left(\frac{1}{2} \delta + 3 p(1) - 21\right) \binom{m+6}{6} + (p(1) - 5) \binom{m+5}{5} - \binom{m+4}{4},$$

for $\dim X = 7$ or 9, respectively. In particular, if $\dim X = 9$, then the degree $\delta$ is even. For $\dim X \geq 11$ the information from Kodaira vanishing and Bogomolov–Gieseker inequality is not enough to determine the bounds on $h^0(L)$ in terms of the degree of $(X, L)$. If $\dim X = 11$, then we have:

$$p(m) = \delta \binom{m+11}{11} - 3 \delta \binom{m+10}{10} + (3 \delta - 8 p(1) + p(2) + 27) \binom{m+9}{9}$$

$$- (\delta - 16 p(1) + 2 p(2) + 54) \binom{m+8}{8} - (7 p(1) - p(2) - 21) \binom{m+7}{7}$$

$$- (p(1) - 6) \binom{m+6}{6} + \binom{m+5}{5},$$

and

$$11 p(2) + 297 \geq 88 p(1) + 4d.$$

We may strengthen the statement of the final item of Theorem 5.3.

Corollary 6.3. Suppose Assumptions 5.2 are satisfied with $G$ of type $A_1$. Then $\dim X \geq 11$. 

PROOF. If \( \dim X = 5 \) or \( \dim X = 3 \) by theorems of [26] and [61, 20], either Assumptions 5.1(1) and (2) fail to simultaneously hold, or \( G \) is of type \( G_2 \), a contradiction. Since \( \dim G = 3 \), if \( \dim X = 7 \) or \( \dim X = 9 \), then we get a contradiction from Theorem 6.1.

6.2. Classification results for contact manifolds. We gather the results from earlier sections and the literature in order to prove Theorems 1.2 and 1.3 concerning the classification of contact Fano manifolds with reductive automorphism group. In addition, we assume that either the dimension of the manifold is small, or rank of the automorphism group is sufficiently large.

PROOF OF THEOREM 1.3. Here we assume that \( X \) is a contact Fano manifold of dimension \( 2n + 1 \) and \( G = \text{Aut}(X) \) is reductive of the rank at least \( n - 2 \). We claim that \( X \) is a homogeneous space. If \( \dim X \leq 5 \) then the theorems of [26] and [61, 20] imply the statement. If \( \text{Pic} X \not\cong \mathbb{Z} \), then the theorems of [40, 25] imply that \( X \cong \mathbb{P}(T^*M) \) for some projective manifold \( M \). Further, since \( X \) is Fano, \( TM \) is an ample vector bundle and \( M \cong \mathbb{P}^{n+1} \) by [51], that is \( X \cong \mathbb{P}(T^*\mathbb{P}^{n+1}) \), which is homogeneous, as claimed. Now suppose \( \text{Pic} X = \mathbb{Z} \), but \( L \) does not generate \( \text{Pic}(X) \), say \( L \cong (L')^a \) for a line bundle \( L' \) and an integer \( a > 1 \). Then \( -K_X \cong L^{n+1} \cong (L')^{an+a} \), thus \( X \) is Fano of index at least \( 2n + 2 \). By [44], \( X \cong \mathbb{P}^{2n+1} \), in particular \( X \) is homogeneous.

Therefore, it remains to treat the case when \( \text{Pic} X = \mathbb{Z} \cdot L \). By Proposition 4.8 the group \( G \) is simple and we claim that Assumptions 5.1 and 5.2 are satisfied. Indeed, (1), (2), (3), (6) are immediate. Items (5) and (7) hold by Lemma 4.7 while (4) follows from Lemma 4.5 and \( \Delta(X, L, H) = \Gamma(X, L, H) \) from (5).

Thus we can apply Theorem 5.3. Most of the cases of that theorem have too large dimension compared to the rank of \( G \). The only case left which is not a homogeneous space is \( G \) of type \( A_1 \), which is treated in Corollary 6.3. This shows that \( X \) must be a homogeneous space, and the explicit list in Table 1 arises from comparing the rank of a simple group and the dimension of the corresponding adjoint variety.

PROOF OF THEOREM 1.2. Let \( X \) be a contact Fano manifold of dimension at most 9. If \( \dim X = 3 \) or \( \dim X = 5 \) then [61, 20] and [26] solved the problem. If \( \dim X = 7 \) or 9, then \( \dim \text{Aut}(X) \geq 5 \) by Theorem 6.1. Therefore, the rank of \( \text{Aut}(X) \) must be at least 2, and we are in a position to apply Theorem 1.3 which shows the claim.

6.3. Classification results for quaternion-Kähler manifolds. In this section we provide references that connect our results about contact Fano manifolds to the claims about quaternion-Kähler manifolds.
For the details of the construction of the twistor space we follow [55]. See also the more precise references in the proofs below.

**Theorem 6.4.** The twistor construction determines a bijection between the following sets:

- The set of contact Fano manifolds admitting a Kähler-Einstein metric (up to an algebraic isomorphism),
- The set of positive quaternion-Kähler manifolds (up to a conformal diffeomorphism).

**Proof.** Let $X$ be the twistor space [55, §2] of a quaternion-Kähler manifold $M$. Then $X$ is a complex, contact manifold [55, Thms 4.1, 4.3]. If $M$ is in addition a positive quaternion-Kähler manifold, then the twistor space is Fano, in particular, projective [55, Cor. 6.2], and admits a Kähler-Einstein metric [55, Thm 6.1]. By [48, Thm 3.2] the map from the set of positive quaternion-Kähler manifolds up to a conformal diffeomorphisms into the set of contact Fano manifolds admitting a Kähler-Einstein metric up to an algebraic isomorphism is well defined and injective. It is also surjective by [47, Thm A], completing the proof. □

**Theorem 6.5.** Except in the case $M \simeq \mathbb{H}P^n$ and $X \simeq \mathbb{C}P^{2n+1}$, the real dimension of the isometry group of a positive quaternion Kähler manifold $M$ is equal to complex dimension of the group of automorphism of its twistor space $X$. In fact, the connected component of the latter group is the complexification of the connected component of the first one.

**Proof.** By [55] Lem. 6.4, 6.5] the dimension of the isometry group is equal to $h^0(X, L)$, where $L$ is the contact line bundle as in Sections § and § (Note the traditional discrepancy between the notation in algebraic geometry papers such as this one, and quaternion-Kähler papers such as [55], where $L$ usually denotes the “half” of the contact line bundle, which is defined only locally, but whose square is equal to the contact line bundle.) Further, $h^0(X, L)$ is equal to the dimension of the group of contact automorphisms of $X$ (see [6 Prop. 1.1] or [17, Cor.E.14(i)]). Finally, the claim in most of the cases follows from the uniqueness of the contact structure [38, Cor. 4.5] or [17, Cor.E.14(ii)]]. It remains to verify the claim explicitly in the only contact Fano case with Pic $X \neq \mathbb{Z}$, i.e. $\mathbb{P}(T^*\mathbb{P}^{n+1})$, which is straightforward.

To see that Isom($M$) complexifies to Aut($X$), note that the naturality of the twistor construction gives an embedding Isom($M$) $\hookrightarrow$
Theorem 6.6. Let $X$ be the twistor space of a positive quaternion-Kähler manifold $\mathcal{M}$. Then $\text{Aut } X$ is reductive.

Proof. By [55, Thm 6.1] the manifold $X$ is Fano and admits a Kähler-Einstein metric. Thus by [50, Thm. 1] the automorphism group is reductive.

We are now ready to put together the building blocks and prove Theorems 1.1 and 1.4.

Proof of Theorem 1.4. Let $\mathcal{M}$ be a positive quaternion-Kähler manifold of dimension $4n$, and suppose the isometry group $\text{Isom}(\mathcal{M})$ has rank at least $n - 2$. Let $X$ be the twistor space of $\mathcal{M}$. The manifold $X$ is contact Fano of dimension $2n + 1$ and it admits a Kähler-Einstein metric by Theorem 6.4. By Theorem 6.6 the group of automorphisms of $X$ is reductive. By Theorem 6.5 the rank of $\text{Aut } X$ is at least $n - 2$, too. Therefore $X$ is an adjoint variety by Theorem 1.3. Since the adjoint varieties are exactly the twistor spaces of Wolf spaces, by Theorem 6.4 the manifold $\mathcal{M}$ must be one of the Wolf spaces, as claimed.

Proof of Theorem 1.1. Let $\mathcal{M}$ be a positive quaternion-Kähler manifold of dimension 12 or 16 and let $X$ be the twistor space of $\mathcal{M}$. Since $X$ is a contact Fano manifold of dimension 7 or 9 (respectively) admitting a Kähler-Einstein metric by Theorem 6.4 and by Theorem 6.6 the group of automorphisms of $X$ is reductive. Therefore $X$ is an adjoint variety by Theorem 1.2. Thus $\mathcal{M}$ must be one of the Wolf spaces by Theorem 6.4.

Appendix A. Riemann-Roch and localization in $K$-theory
(by Andrzej Weber)

Suppose an algebraic torus $H \simeq (\mathbb{C}^*)^r$ acts on an algebraic variety $X$. An algebraic vector $H$-bundle over $X$ defines an element in topological equivariant $K$-theory $K^H(X)$ defined by Segal [56], as well as in the algebraic equivariant K-theory, see [52], [57], [27]. Which theory we use is irrelevant for us. For an accessible overview of algebraic equivariant $K$-theory we refer the reader to [23], §5. The Localization Theorem provides a formula for the equivariant Euler characteristic $\chi^H$, which is an element of the equivariant $K$-theory of a point $K^H(pt)$. This ring is just the representation ring

$$R(H) \simeq \mathbb{Z}[t_1, t_1^{-1}, t_2, t_2^{-1}, \ldots, t_r, t_r^{-1}].$$
In the appendix we use the following notation: fix an integral basis
\[ x_1, x_2, \ldots, x_r \in M \subset \mathfrak h^* \]
of the dual of Lie algebra of \( H \). Elements of \( M \) are called weights. Denote by \( t_1, t_2, \ldots, t_r \) the corresponding characters. That is, the image of \( x_i \) under the identification of \( M \) (with additive notation) with \( \text{Hom}(H, \mathbb C^*) \) (with multiplicative notation) is denoted by \( t_i \). For a weight \( \nu = \sum_{i=1}^r a_i x_i \) the corresponding character is denoted by \( t^\nu = \prod_{i=1}^r t_i^{a_i} \).

**Theorem A.1.** Assume that \( X \) is a smooth compact complex \( H \)-manifold and \( E \) is an \( H \)-vector bundle. Then the equivariant Euler characteristic is equal to
\[
\chi^H(X; E) = \sum_{y_i} p_! \left( \frac{E|_{Y_i}}{\lambda_{-1}(N^*_{Y_i})} \right) \in R(H).
\]
Here \( X^H = \bigcup_{i \in I} Y_i \) is the decomposition into connected components, \( \lambda_{-1}(N^*_{Y_i}) \) is the equivariant Euler class of the normal bundle to \( Y_i \). The map \( p_i : Y_i \to \text{pt} \) is the constant map and \( p_! \) is the push forward in the equivariant \( K \)-theory.

We will explain what are the objects appearing in the theorem below, but first let us derive a corollary:

**Corollary A.3.** Suppose \( E = L \) is a line bundle and \( \mu \) is a linearization of the action of \( H \) on \( L \). Assume the fixed points \( X^H \) consist of isolated points \( y_1, y_2, \ldots, y_k \) and curves \( C_{k+1}, C_{k+2}, \ldots, C_{\ell} \). The genus of \( C_i \) is denoted by \( g_i \). Suppose
\begin{enumerate}
\item for \( i = 1, 2, \ldots, k \): \( \mu_i = \mu(y_i) \) is the weight of the action of \( H \) on \( L|_{y_i} \) and \( \nu_{i,j} \) are the weights of \( H \) on \( T^*_{y_i} X \);
\item for \( i = k + 1, k + 2, \ldots, \ell \): \( \mu_i = \mu(C_i) \) is the weight of the action of \( H \) on \( L|_{C_i} \), \( d_i \) — the degree of \( L|_{C_i} \); the conormal bundle to \( C_i \) decomposes into a direct sum \( N^*(C_i) = \bigoplus \{N^*(i, j)\} \), and suppose that the action of \( H \) on the summand \( N^*(i, j) \) is of the weight \( \nu_{i,j} \); let \( rk(N^*(i, j)) = r_{i,j} \) and let \( \int_{C_i} c_1(N^*(i, j)) = n_{i,j} \).
\end{enumerate}
Then the equivariant Euler characteristic \( \chi^H(X, L) \) is equal to
\[
\sum_{i=1}^k \frac{t^{\mu_i}}{1 - t^{\nu_{i,j}}} + \sum_{i=k+1}^\ell \frac{t^{\mu_i}}{1 - t^{\nu_{i,j}}} \left( 1 - g_i + d_i + \sum_{j} \frac{n_{i,j}}{1 - t^{\nu_{i,j}} - 1} \right).
\]
The notation \( \mu(y_i), \mu(C_i) \) above is consistent with Sections 2-5. The characters \( \nu_{i,j} \) form the compass of \( y_i \) or \( C_i \) in \( X \) with respect to the action of \( H \).
Corollary A.3 is applied in Propositions 2.23, 2.25 and Example 3.16 in the situation when the fixed-point set is finite. We consider here the case when $X^H$ is of dimension $\leq 1$ having in mind further applications, and also to point out which invariants of the fixed-point components are relevant to compute the space of global sections.

It is hard to trace the first appearance of Theorem A.1. Let us review what is present in literature. In a paper of Atiyah-Bott from the Wood-Hole conference [3], the result in the case of isolated fixed points is given. The formula was then repeated by Grothendieck [32, Cor. 6.12] and Nielsen [52, §4.7]. For the case of nonisolated fixed points we quote Atiyah-Singer paper on equivariant Index Theorem. The relevant theorem is called there “Holomorphic Lefschetz Theorem” [21, (4.6)]. In [5], where finite group actions are studied, such a kind of formula is called Lefschetz-Riemann-Roch. One can also apply widely known Atiyah-Bott-Berline-Vergne localization in equivariant cohomology [4, 8]. Equivariant Riemann-Roch theorem allows to deduce localization in equivariant K-theory from localization in equivariant cohomology. For localization theorems in algebraic equivariant K-theory see [57, Theorem 2.1], [23, Theorem 5.11.7].

When we identify the direct image of sheaves in homological algebra with the push-forward in homotopy theory then the statement of Theorem A.1 follows from [58], which is valid for any complex-oriented generalized cohomology theory. Equally well we can deduce Theorem A.1 from [27, Theorem 4.3(b)].

To show Theorem A.1 in the form presented here, consider the equivariant Chern character $\text{ch}^H$. It maps equivariant $K$-theory to equivariant cohomology, in particular

$$\text{(A.4)} \quad \text{ch}^H : K^H(pt) \longrightarrow \hat{H}^*_H(pt; \mathbb{Q}) = \prod_{i \geq 0} H^i_H(pt; \mathbb{Q}).$$

For a weight $\nu = \sum_{i=1}^n \nu_i x_i$ the image $\text{ch}^H(t^\nu) = \exp(\nu) \in \hat{H}^*_H(pt; \mathbb{Q})$ will be denoted by $t^\nu$ again, in order to maintain a brevity of formulas. We use [52, §4.10], which expresses $\chi^H(X,E)$ as:

$$\text{(A.5)} \quad \chi^H(X,E) = \sum_{i \in T} \int_{Y_i} \frac{td(Y_i) \cdot \text{ch}^H(E|_{Y_i})}{\text{ch}^H \lambda_{-1}(N_{Y_i}^*)},$$

where $E \mapsto \lambda_{-1}(E)$ is the multiplicative transformation of $K$-theory such that for a line bundle $L$

$$\lambda_{-1}(L) = 1 - L.$$

A part of the proof of localization theorem is to show that $\lambda_{-1}(N_{Y_i}^*)$ is invertible after a suitable localization. Note that the action of the
torus $H$ on the base of the bundle $N_{Y_i}^*$ is trivial. If $L$ is an invariant line subbundle of $N_{Y_i}^*$ such that $H$ acts with weight $\nu$ on the fibers of $L$ (the weight is necessarily nonzero) then

$$
(A.6) \quad ch^H \lambda_{-1}(L)^{-1} = \frac{1}{1 - ch^H(L)} = \frac{1}{1 - t^\nu (1 + c_1(L) + \ldots)} = \frac{1}{1 - t^\nu} \cdot \frac{1}{1 - \frac{t^\nu}{1 - t^\nu} c_1(L) + \ldots} = \frac{1}{1 - t^\nu} \left(1 + \frac{c_1(L)}{t^\nu - 1} + \ldots\right).
$$

The integrals in (A.5) can be rewritten as

$$
(A.7) \quad \int_{Y_i} td(Y_i) \cdot ch^H \left( \frac{E_{Y_i}}{\lambda_{-1}(N_{Y_i}^*)} \right).
$$

The expression $\frac{E_{Y_i}}{\lambda_{-1}(N_{Y_i}^*)}$ belongs to $S^{-1}K^H(Y_i) \simeq S^{-1}R(H) \otimes K(Y_i)$, where $S$ is the multiplicative system generated by $1 - t^\nu$, for all $\nu$ which are the weights of $H$-action on $N_{Y_i}^*$. Formally the nonequivariant Chern character is a map

$$
ch: K(Y_i) \to H^*(Y_i).
$$

The Chern character defined for the representation ring

$$
ch^H: R[H] \to \mathbb{Q}[x_1, x_2, \ldots x_r]
$$

$$
t_i \mapsto exp(x_i),
$$

coincides with (A.4). We obtain an extension

$$
ch^H: S^{-1}R(H) \otimes K(Y_i) \to (ch^H(S))^{-1}\mathbb{Q}[x_1, x_2, \ldots x_r] \otimes H^*(Y_i).
$$

Now we apply the Hirzebruch-Riemann-Roch theorem for the extended Chern character appearing in (A.7) and conclude that

$$
(A.8) \quad \chi^H(X; E) = \sum_{i \in I} p_i! \left( \frac{E_{Y_i}}{\lambda_{-1}(N_{Y_i}^*)} \right).
$$

This way we have identified the Nielsen formula (A.5) with the statement of Theorem A.1.

Corollary A.3 follows directly from (A.5).

**Proof of Corollary A.3** Assume that each fixed-point component of $X^H$ is either a point or a curve. The contributions to $\chi^H(X; L)$ in (A.5) coming from the components $Y_i$ are the following:

If $Y_i = \{y_i\}$ is a point and $\nu_{i,j}$ are the cotangent weights (compass) then

$$
\lambda_{-1}(N_{y_i}^*)^{-1} = \prod_{j=1}^{\dim X} \frac{1}{1 - t^{\nu_{i,j}}}
$$
because $N^*_{y_i} = T^*_{y_i}X$, which is an equivariant bundle over a point. Then
\[
\int_{Y_i} \frac{td(Y_i) \cdot ch^H(L|_{Y_i})}{ch^H \lambda_{-1}(N^*_{Y_i})} = \frac{t^{\mu_i}}{\prod_{j=1}^{\dim X}(1 - t^{\nu_{i,j}})}.
\]

Suppose $Y_i = C_i$ is a curve. Let $[p]$ be the generator of $H^2(C_i)$. Let us decompose the conormal bundle $N^*(C_i) = \bigoplus N^*(i, j)$, and suppose that the action of $H$ on the summand $N^*(i, j)$ is of the weight $\nu_{i,j}$ and $rk(N^*(i, j)) = r_{i,j}$. Then by (A.6)
\[
\lambda_{-1}(N^*(Y_i))^{-1} = \prod_j \frac{1 + \frac{n_{i,j}}{t^{\nu_{i,j}-1}}[p]}{(1 - t^{\nu_{i,j}})^{r_{i,j}}},
\]
where $c_1(N^*(i, j)) = n_{i,j}[p]$ is the first Chern class. The Todd class of the curve $C_i$ is equal to $td(Y_i) = 1 + (1 - g_i)[p]$. Assume that the action of $H$ on $L|_{C_i}$ is of weight $\mu_i$. Let $d_i$ be the degree of $L$ on $C_i$. The integral over $Y_i$ is equal to the homogenous component of degree one (with respect to $[p]$) of the product
\[
(1 + (1 - g_i)[p]) \cdot t^{\mu_i}(1 + d_i[p]) \cdot \prod_j \frac{1 + \frac{n_{i,j}}{t^{\nu_{i,j}-1}}[p]}{(1 - t^{\nu_{i,j}})^{r_{i,j}}}.
\]
We obtain the contribution
\[
\frac{t^{\mu_i}}{\prod_j (1 - t^{\nu_{i,j}})^{r_{i,j}}} \left(1 - g_i + d_i + \sum_j \frac{n_{i,j}}{t^{\nu_{i,j}-1}}\right).
\]

□

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