Quantum Metrology in Open Systems: A Dissipative Cramér-Rao Bound

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Estimation of parameters is a pivotal task throughout science and technology. Quantum Cramér-Rao bound provides a fundamental limit of precision allowed to achieve under quantum theory. For closed quantum systems, it has been shown how the precision estimation depends on the underlying dynamics. Here, we propose a general formulation for metrology scenarios in open quantum systems, aiming to relate the precision more directly to properties of the underlying dynamics. Specifically, we derive a Cramér-Rao bound for a fairly large class of open system dynamics, which is governed by a (time-dependent) dynamical semi-group map. We illustrate the utility of this scenario through three examples.

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Introduction.—Metrology and parameter estimation lie in the heart of science, and are prevalent in any aspect of technology. The basic task of identification or estimation of a set of unknown parameters essentially requires an inference from a pool of observed data about the parameters or the system to which they are attributed. As errors and imperfections are unavoidable in practice, increasing accuracy of the underlying tasks of data acquisition and inference—hence improving the quality of estimation—is an important goal of metrology [11]. Improving quality of measurement instruments and removing potential sources of systematic errors aside, statistics provides useful suggestions for enhancing metrology, such as increasing data size and repeated measurements on an ensemble of N ‘probe’ systems. Additionally (and more interestingly), the underlying physics of the system of interest may also dictate some restrictions or bounds on the ultimate achievable accuracy (usually described through a ‘Cramér-Rao inequality’ [2]), or even may offer new possibilities to exploit.

In quantum mechanics, measurements act differently than in classical systems. In addition, interactions with an environment or other systems as well as (quantum) correlations can each affect observed data [3], hence introduce new playing factors in estimation theory. For example, it has been shown that entanglement in a probe ensemble can be exploited to the advantage of a quantum metrology task [4], so that it enables the estimation error of $O(1/N)$ (the “Heisenberg limit”), in contrast to the classical statistical limit of $O(1/\sqrt{N})$ (the “shot-noise limit”). Alternatively, enabling $k$-body ($k \geq 2$) interactions among quantum probe systems has been shown to allow an error of $O(1/\sqrt{N^k})$ [5]; or, it has been argued that application of a suitable entangling operator may even offer an error as small as $O(2^{-N})$ [6] (beyond the Heisenberg limit). Moreover, nonclassicality has been examined as a potential resource for increasing the metrology resolution in quantum optics [7] (for a general framework of resource analysis, see, e.g., Ref. [8]). It thus seems natural to expect that some properties of quantum systems can be employed as a potential “resource” for quantum metrology. Numerous experiments have indeed demonstrated achievability of sub-shot-noise limit precision by using aspects of quantum mechanics; see, e.g., Refs. [9].

In open quantum systems, due to interaction with an environment, the underlying dynamics becomes ‘noisy.’ As a result, formulation and analysis of quantum estimation also becomes more involved [10, 11]. In general, dynamics of an open system can be described as $\varrho_S(t) = \text{Tr}_E[U_{SE}(t,t_0)\varrho_{SE}(t_0)U_{SE}^\dagger(t,t_0)]$, where $\varrho_{SE}$ is the state of the systems and environment (SE), and $U_{SE}(t,t_0)$ is the corresponding unitary evolution [12, 13]. Thereby one can argue that in general there may exist a flow of information between the system and the environment [14]. Under some conditions, this dynamics can feature quantum Markovian or non-Markovian properties [15, 16]. The former case typically appears when the environment has a small decoherence time during which correlations disappear, whereas in the latter correlations (both classical and/or quantum [17]) with the environment would form and persist. Such correlations are in practice inevitable, which necessitates investigation of noisy quantum metrology [10, 18–22], and may in turn offer new resources for enhancing estimation tasks. However, developing relatively general frameworks for open-system metrology is still needed and is of fundamental and practical importance.

Here, we first lay out a fairly general formalism for open quantum system metrology. This (re)formulation of the problem (e.g., cf. Ref. [10]) has this advantage that here precision of estimation is more directly related to the underlying dynamics; besides, it is in some sense analogous to the closed system formulation. In particular, we derive a quantum Cramér–Rao bound (QCRB) for open system dynamics generated through dynamical map with semigroup property. We next illustrate this setting through several examples. This first example shows that how induced correlations of probe quantum systems through a common environment may offer relatively higher precision for an estimation task in a sense akin to what many-body interactions enable. The other examples provide a comparison of our predicted precision with exact results in two estimation scenarios.

Open system dynamics.—Under some specific conditions, the dynamical equation describing the state of an open system $\varrho_S$ [defined on a Hilbert space $H_S$] reduces to $\partial_\tau \varrho_S(\tau) = L_\tau[\varrho_S(\tau)]$, or equivalently $\varrho_S(\tau) = \mathbf{T} e^{L_\tau[\varrho_S(\tau_0)]}$, in which $\mathbf{T}$ denotes time-ordering, and $L_\tau[\varrho] = -i[H_S(\tau), \rho] + \sum_k \eta_k(\tau)(A_k(\tau)\rho A_k^\dagger(\tau)-(1/2)(A_k^\dagger(\tau)A_k(\tau), \rho))$ (for some
set of operators \( \{ A_k(\tau) \} \) is the (Lindbladian) generator of the dynamical map, with \( H_S(\tau) \) being the system Hamiltonian up to a Lamb shift term (we omit subscript \( S \) henceforth). We have also assumed \( \hbar = 1 \). (In time-dependent) Markovian evolutions, we have \( \eta_k(\tau) \geq 0 \forall k, \tau \); while if some \( \eta_k \) becomes negative for some intervals, the associated dynamics would be non-Markovian \([12, 13, 15, 16]\). Let us assume that a set of unknown parameters \( x = (x_1, \ldots, x_f) \) are to be estimated in a quantum system subject to interaction with an environment. For simplicity of our analysis, here we consider the single-parameter case, while generalization of our framework to the multi-parameter case is also straightforward (see example II in the sequel). In the closed-system scenario, this parameter \( x \) is usually assumed to enter into the dynamics as a linear coupling in the Hamiltonian \( H(x) = \hbar x H \) acting on some known initial state. In the open-system scenario, similarly the devised dynamics would in general depend on \( x \) as \( \partial_x \phi(x, t) = \mathcal{L}_x(x) [\phi(x, \tau)] \). For our later use, we vectorize this equation, which yields \( \partial \rangle [\phi] = \mathcal{L}_x(x) [\phi] \), where \( \mathcal{L}_x \) is the matrix representation of \( \mathcal{L}_x \) \([23]\). Next we define the normalized pure state \( \tilde{\rho} = \frac{\langle \phi \rangle \langle \phi |}{\sqrt{\langle \phi | \phi \rangle}} \) in \( \mathcal{H}^{\otimes 2} \), and assume \( \mathcal{L}_x(x, \tau) = x(\tau) \mathcal{L} \), where \( \mathcal{L} \) does not depend on time; hence

\[
\tilde{\rho}(x, \tau) = e^{\int_0^\tau x(s)ds \mathcal{L}} \tilde{\rho}(0) e^{\int_0^\tau x(s)ds \mathcal{L}^\dagger} \frac{\langle \phi | \langle \phi |}{\sqrt{\langle \phi | \phi \rangle}}.
\]

The initial preparation \( \tilde{\rho}(0) \) may itself depend on \( x \), but here we do not assume such generality.

**QCRB for open system metrology.**—Given a data set \( D = \{ \gamma_i \} \) constituted from some measurement outcomes \( \gamma_i \) over \( N \) (identical) probe systems, an estimator \( x_{est}(D) \) is chosen for the true value \( x \). Repeating this scenario \( M \) times and averaging, the precision of the estimated \( x \), evaluated by \( \delta x = \sqrt{\text{var}(x)} \), is then fundamentally limited by the QCRB

\[
\delta x \geq 1/\sqrt{MF(Q)(x; N)}.
\]

Here, \( \text{var}(x) \) is the variance of any unbiased estimator \( x_{est}(D) \) (for which, by definition, \( \langle x_{est} \rangle = x \), with \( \langle \cdot \rangle \) denoting the average with respect to the underlying quantum probability distribution), and \( F(Q)(x; N) \) is the so-called “quantum Fisher information” \((QFI) [18, 24, 25]\). By assuming the state of each \( N \)-probe set to be \( \rho^{(N)}(x, \tau) \) (hereafter we omit superscript \( N \) for brevity) and assigning the corresponding symmetric logarithmic derivative \( L_{\tilde{\rho}} = \left( L_{\tilde{\rho}} \otimes 1 + 1 \otimes L_{\tilde{\rho}}^T \right) / 2 \), the QFI is defined as \( F(Q)(x, \tau; N) = \text{Tr}[\rho(x, \tau)L_{\tilde{\rho}}^{\otimes 2} \rho(x, \tau)] \).

We remind that in closed systems, with \( \rho(x, \tau) = e^{-i\tau H(x) / \hbar} \rho(0) e^{i\tau H(x) / \hbar} \), the spectral decomposition \( \rho = \sum r_i |r_i \rangle \langle r_i | \), and considering \( L_{\rho} = 2 \sum_{ij} r_i |r_i \rangle \langle r_i | r_j \rangle \langle r_j | r_i \rangle \), a direct relation between \( F(Q) \) and the interaction \( H \) is obtained. In particular, when \( H(x) = \hbar x H \) and \( \rho \) is pure, we have

\[
F(Q) = 4\tau^2 \text{Cov}_{\rho}(H, H).
\]

(with equality replaced with \( \leq \) for mixed \( \rho \)), where \( \text{Cov}_{\rho}(X, Y) \equiv \langle XY \rangle_{\rho} - \langle X \rangle_{\rho} \langle Y \rangle_{\rho} \) is the covariance function of a pair of observables \( X \) and \( Y \), which here is the very quantum standard deviation \( \Delta^2 H \) (with \( \langle \cdot \rangle_{\rho} \equiv \text{Tr} [\rho \cdot] \)). The resulting relation

\[
\delta x \geq 1/(2\tau \sqrt{M \text{Cov}_{\rho}(H, H)}) = 1/(2\tau \sqrt{M \Delta^2 H}),
\]

where \( \Delta H \equiv \sqrt{\Delta^2 H} \), is more in the spirit of an uncertainty-like relation \([24]\), and shows explicitly how the precision is dictated by the interaction. In open-system cases, however, deriving similar, direct relations is hardly possible since, e.g., calculating \( L_{\rho}(x, \tau) = 2 \int_0^\infty e^{-s\rho(x, \tau)} \partial_x \rho(x, \tau) e^{-s\rho(x, \tau)} ds \) is involved. Thus it is difficult to capture how interaction with an environment affects the QFI and the precision. To partially alleviate this issue, here we follow an alternative approach working with the vectorized state \( \tilde{\rho} \) instead, which enables a bound somewhat akin to Eq. (4)—with \( H \) replaced with \( L \). Although our method gives bounds on the QFI (not its exact value), we argue that this formalism still retains its utility in providing correct scaling for the estimation error, and show this explicitly in various examples.

Now from the symmetric logarithmic derivative \( L_{\tilde{\rho}} = 2\partial_x \tilde{\rho} \), one can define an associated QFI \( F(Q) \) by replacing \( \rho \) \([L_{\tilde{\rho}} \rho \) \( \rightarrow \langle \tilde{\rho} | L_{\tilde{\rho}} \tilde{\rho} \rangle \) in \( F(Q) \). After some straightforward algebra, using the dynamical equation Eq. (1), and assuming a linear \( x \)-dependence as \( \tilde{\rho}(x) = x \mathcal{L} \), it can be seen that

\[
\tilde{F}(Q) = \frac{4}{|\partial_x \ln \tau(x)|^2} \text{Cov}_{\tilde{\rho}}(\tilde{L}^1, \tilde{L}),
\]

where for the time-independent case we have \( 4\tau^2 \text{Cov}_{\tilde{\rho}}(\tilde{L}^1, \tilde{L}) \) instead. This relation is analogous to Eq. (3), where instead of the Hamiltonian we have the generator of the open dynamics.

The QFI \( \tilde{F}(Q) \) has a natural interpretation. Recall that \( F(Q) \) indeed emerges from the optimization of the Fisher information over all possible quantum measurements on the system \([23]\). Similarly then, \( \tilde{F}(Q) \) is obtained if any quantum measurement on the ‘system’ is allowed. Note, however, that a natural extension of the measurements in \( \mathcal{H} \) to \( \mathcal{H}^{\otimes 2} \) does not necessarily translate into most general measurements there. For example, a complete set of measurement \( \{ \Pi_i \} \) (with the properties \( \Pi_i \geq 0 \) and \( \sum_i \Pi_i = 1 \)), when extended simply as \( \tilde{\Pi}_i = |\tilde{\Pi}_i \rangle \langle \tilde{\Pi}_i | \), do not constitute a complete set in the sense that \( \sum_i \tilde{\Pi}_i \neq 1 \otimes 1 \) in general.

Let us see how \( \tilde{F}(Q) \) compares with \( F(Q) \). First we remark that, from vectorizing the very definition of the symmetric logarithmic derivative, we have \( L_{\tilde{\rho}} = L_{\rho} \otimes 1 + 1 \otimes L_{\rho}^T - \partial_x \ln \text{Tr}[\rho^2] \). This in turn yields the following expression:

\[
\tilde{F}(Q) = \frac{2}{\text{Tr}[\rho^2]} \left( \text{Tr}[\rho L_{\rho} \rho L_{\rho}] + \text{Tr}[\rho^2 L_{\rho}^2] - 2 \left( \frac{\text{Tr}[\rho L_{\rho}^2]}{\text{Tr}[\rho^2]} \right)^2 \right).
\]

This form is not directly related to \( F(Q) \). However, using \( \lambda_{\min}(X) \text{Tr}[Y] \leq \text{Tr}[XY] \leq \lambda_{\max}(X) \text{Tr}[Y] \) (valid for any
pair of positive matrices \( X \) and \( Y \) [here \( \lambda_{\min/\max}(X) \) denotes the minimum (maximum) eigenvalues of \( X \)], we obtain

\[
\frac{\text{Tr}[\rho^2]}{4\lambda_{\max}(\rho)} \bar{F}^{(Q)}(Q) \leq \frac{\text{Tr}[\rho^2]}{4\lambda_{\min}(\rho)} \bar{F}^{(Q)}(Q) + F(\rho),
\]

where \( F(\rho) \equiv (\text{Tr}[\rho^2 L])^2 / (\lambda_{\min}(\rho)\text{Tr}[\rho^2]) \). Note that the upper bound would be useless when \( \lambda_{\min}(\rho) = 0 \), and thus this case needs to be investigated separately. An example of such case is when the evolution is unitary with a pure initial state, that is, \( |\Psi(x, \tau)\rangle = U(x, \tau)|\Psi(0)\rangle \). Here, however, a significant simplification occurs due to \( \langle\Psi(x, \tau)|L_\rho|\Psi(x, \tau)\rangle = 0 \), whence Eq. (7) reduces to \( \bar{F}^{(Q)}(Q) = 2F(Q) \). Note that in this case the lower bound gives \( \bar{F}^{(Q)} \leq 4F(Q) \).

Equation (7) provides an upper and a lower bound on the exact QFI \( F^{(Q)}(Q) \). When finding the scaling of \( F^{(Q)}(Q) \) is our objective, it suffices to find the scaling of the lower bound of Eq. (7), since if this bound scales as \( O(N^p) \) (for some \( p \geq 0 \)), it is guaranteed that \( F^{(Q)}(Q) = O(N^q) \) with some \( q \geq p \). Putting everything together, in general we have

\[
1/F^{(Q)} \leq K/F^{(Q)},
\]

where

\[
K(\rho) = \begin{cases} 4\lambda_{\max}(\rho)/\text{Tr}[\rho^2]; & \rho \text{ mixed,} \\ 2; & \rho \text{ pure,} \end{cases}
\]

and for the latter case the inequality in Eq. (8) is replaced with equality.

A desirable property of \( F^{(Q)}(Q) \) is that for a separable estimation scenario with \( N \) product input states, we have \( F^{(Q)}(x, \tau; N) = N F^{(Q)}(x, \tau; 1) \). This property naturally carries over to \( \bar{F}^{(Q)}(x, \tau; N) \). Thus for this special case, at the left- and right-hand sides of Eq. (7), we must replace \( g[N\text{-probe state}] \) with \( g^{(1)} \) [single-probe state] and \( F^{(Q)}(x, \tau; N) \) with \( N \bar{F}^{(Q)}(x, \tau; 1) \), which exhibits the expected shot-noise scaling \( O(1/\sqrt{N}) \) for the estimation error.

Example I.—We assume \( N \) probe particles each of which only interacts with a common bath such that the interactions induce all possible \( k \)-body terms (Fig. 1) in the Lindbladian as follow:

\[
\mathcal{L}_\tau[\rho] = \sum_{i_1 \cdots i_k} \sigma_{i_1} \cdots \sigma_{i_k} \rho \sigma_{i_1} \cdots \sigma_{i_k} - C_{N,k} \rho,
\]

where \( \sigma_{i_j} \) are all the same Pauli matrix (e.g., \( \sigma^z \)), subscript \( i_j \) is the particle index, and the factor \( C_{N,k} = \binom{N}{k} \) counts the number of \( k \)-body operators. This is a natural generalization of the scenario considered in the closed-system context of Ref. [5].

We choose the initial state of the whole \( N \)-probe system to be the maximally entangled pure state \( \rho(0) = |\Psi\rangle \langle \Psi| \) (i.e., \( |\rho(0)\rangle = |\Psi\rangle |\Psi^*\rangle \)), where \( |\Psi\rangle = (|E_M\rangle \otimes |E_m\rangle) / \sqrt{2} \), and \( E_m \) is the smallest (largest) eigenvalue of \( \sigma \). For odd \( ks \), \( \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_k} \otimes \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_k} \langle |\Psi\rangle \otimes |\Psi^*\rangle \)

\[
\begin{align*}
\bar{F}^{(Q)}(x_1) & = \frac{N \tau^2 e^{-3\Gamma t/2}}{2 \cosh(\Gamma/2)}, \\
F^{(Q)}(x_1) & = \frac{N^2 \tau^2 e^{-3\Gamma t/2}}{2 \cosh(\Gamma/2)},
\end{align*}
\]

whereas the exact QFIs are argued to be [22, 29]

\[
\begin{align*}
F^{(Q)}(x_1) & = N \tau^2 e^{-2\Gamma t}, \\
F_e^{(Q)}(x_1) & = N^2 \tau^2 e^{-2\Gamma t}.
\end{align*}
\]

Here \( \Gamma(t) = \int_0^t x_2(s) \, ds \), \( ch = \cosh \), and \( sh = \sinh \). It is evident that the ratio of the bound [13] and the exact value

![Fig. 1. \( N \) probes, initially well isolated from each other, all interact with a common bath through two-body interactions \( H_{PB_1}, \) Here \( H_{PB_1} \) and \( H_B \) are the free Hamiltonians of probe 1 and the bath, respectively. These two-body interactions may induce a manybody quantum correlation among the probes [25, 27].](image-url)

\(|\Psi^+\rangle = |\Psi^\perp\rangle |\Psi^\perp\rangle \), where \( |\Psi^\perp\rangle = (|E_M\rangle \otimes |E_m\rangle) / \sqrt{2} \). It is straightforward to see that

\[
\bar{F}^{(Q)}(x_1) = \frac{e^{-2C_{N,k} \tau x} + 1}{4 \tau^2 C_{N,k} e^{-2C_{N,k} \tau x} + 1},
\]

An immediate implication of this relation is that for small values of the \( x \) parameter, noting \( C_{N,k} = O(N^k) \) for large \( N \), a polynomial precision in the estimation can be achieved.

Example II.—Consider a dephasing channel acting separately on an \( N \)-qubit system, described by \( \mathcal{L}_\tau[\rho] = ix_1 [H, \rho] + (1/2)x_2 (\sum_{m=1}^N \sigma_m \rho \sigma_m - N \rho) \), in which \( x_1 \) is the gap of the Hamiltonian \( H = \sum_{m=1}^N |1\rangle_m \langle 1| \), whose ground-state energy is zero [29]. We assume two different initial states; the product state \( |\Psi_p\rangle = [(|0\rangle + |1\rangle)] / \sqrt{2} \) with \( \sqrt{N} \rangle \) and the entangled “GHZ” state \( |\Psi_e\rangle = (|0\rangle \otimes |1\rangle) / \sqrt{2} \).

Estimation of \( x_1 \).—Using Eq. (8) and after some algebra, it is obtained that in the case of product (“p”) and entangled (“e”) states we have

\[
\begin{align*}
\bar{F}^{(Q)}(x_1) & = N \tau^2 e^{-\Gamma t}, \\
F^{(Q)}(x_1) & = N^2 \tau^2 e^{-2\Gamma t}.
\end{align*}
\]

Here \( \Gamma(t) = \int_0^t x_2(s) \, ds \), \( ch = \cosh \), and \( sh = \sinh \). It is evident that the ratio of the bound [13] and the exact value

![Fig. 1. N probes, initially well isolated from each other, all interact with a common bath through two-body interactions \( H_{PB_1}, \) Here \( H_{PB_1} \) and \( H_B \) are the free Hamiltonians of probe 1 and the bath, respectively. These two-body interactions may induce a manybody quantum correlation among the probes [25, 27].](image-url)
is always equal to \((1+e^{-\Gamma})^{-1}\); and for large \(N\), the ratio of the bound \((14)\) and the exact value \((16)\) goes to 1. Note that when \(x_2 = 0\), the ratios both are 1/2, which is consistent with what we expect in the unitary case \([\mathcal{F}^{(Q)} = 2\mathcal{F}^{(Q)}]\). Therefore, our framework correctly captures the scaling of the error in this example.

**Estimation of \(x_2\).**—Similar calculations yield

\[
\mathcal{F}_p^{(Q)}(x_2) = \frac{N e^{-\Gamma/2}}{\mathcal{K}} \frac{4[\partial_x \ln x_2]^2 \cosh(\Gamma/2)}{\tanh x_2^2} \tag{17}
\]

\[
\mathcal{F}_e^{(Q)}(x_2) = \frac{N^2 e^{-2\Gamma/2}}{\mathcal{K}} \frac{4[\partial_x \ln x_2]^2 \cosh(PT)\tanh(NT/2)}{\tanh x_2^2} \tag{18}
\]

On the other hand, here the exact QFIs are obtained as

\[
\mathcal{F}_p^{(Q)}(x_2) = \frac{1}{\mathcal{K}} \frac{N e^{-\Gamma}}{[\partial_x \ln x_2]^2 \tanh x_2^2} \tag{19}
\]

\[
\mathcal{F}_e^{(Q)}(x_2) = \frac{N^2 e^{-2\Gamma}}{\mathcal{K}} \frac{1}{[\partial_x \ln x_2]^2 \tanh(NT/2)} \tag{20}
\]

Again it is evident that the ratio of the bound \((17)\) and the exact value \((19)\) is always \((e^{2\Gamma} - e^{\Gamma})/(e^{2\Gamma} + 1)\); and the exact value \((20)\), for large \(N\), goes to 1, and the ratio of the bound. These results also exhibit correct scalings.

**Example III.**—Consider a lossy bosonic channel described by \(L_m[\rho] = x[xa^{\dagger}a - (\hat{n} + \rho\hat{n})]/2\), where \(a(a^{\dagger})\) is the bosonic annihilation (creation) operator, \(\hat{n} = a^{\dagger}a\), and \(x\) is the loss parameter. The Cramér-Rao bound for estimation of \(\varphi\), defined through \(\tan^2[\varphi(x, t)] = e^{\gamma t} - 1\), has been obtained as \(\delta \varphi \geq 1/(4\sqrt{x}\Gamma)\), and whereby \(\delta x \geq \sqrt{\delta \varphi} \Gamma / x(\Gamma)\), where \(\Gamma = \{\hat{n}\rho(0)\} [50]\). Particularly, it has been shown that Fock states are optimal for this estimation \([20]\). Here we revisit this example and demonstrate that the scaling of the error is captured correctly in our framework.

The evolution of this system, when the initial state is \(|\rho(0)\rangle = |N\rangle \langle N|\) (whence \(\pi = N\)), is given by

\[
|\varphi(x, t)\rangle = \sum_{m=0}^{N} \frac{s^{2mn} c^{2m-N}(N-m) C_{N,m}}{\sum_{m=0}^{N} s^{4m} c^{4m}(N-m) C_{N,m}^2} |N-m\rangle \langle N-m| \cdot \frac{1}{\mathcal{K}} \frac{N e^{-\Gamma}}{[\partial_x \ln x_2]^2 \tanh x_2^2} \tag{21}
\]

in which \(s = \sin \varphi\) and \(c = \cos \varphi\).

The analytic expression of \(\mathcal{K} / \mathcal{F}^{(Q)}\) can be found as

\[
\frac{\mathcal{K}}{\mathcal{F}^{(Q)}(\varphi)} = \left(1/4\right) \cot^2 \varphi \max_{0 \leq m \leq N} \left[C_{N,m} s^{2m} c^{2m-N} \right] \times \frac{1}{\sum_{m=0}^{N} s^{4m} c^{4m}(N-m) C_{N,m}^2 A_{N,m}^2} \left[\left(\sum_{m=0}^{N} s^{4m} c^{4m}(N-m) C_{N,m}^2 A_{N,m}\right)^2 - 1\right]^{-1}, \tag{22}
\]

where \(A_{N,m} = m(1 + \cot^2 \varphi) - N\). Using \(\mathcal{F}^{(Q)}(x) = (\partial_x \varphi)^2 \mathcal{F}^{(Q)}(\varphi)\), one can relate the upper bound for estimation of \(x\) to that of \(\varphi\). Figure 2 depicts \((\delta \varphi)_{\text{min}}\), which verifies that our bound gives the correct scaling of the error.

**Summary and outlook.**—Here we have outlined a fairly general formalism for open quantum system metrology. In this formulation, the precision of estimation is more directly related to the underlying dynamics, in some sense similar to the closed-system formulation. The core of this formalism is the derivation of a quantum Cramér-Rao bound for open system dynamics generated through dynamical map with the semigroup property. This setting was then illustrated through some examples. The first example has implied that it may be possible to exploit induced correlations of probe quantum systems through a common environment in order to achieve a relatively higher precision for an estimation task. Other two examples have illustrated that our bound could indeed give correct scaling of the estimation error.

Our formalism may introduce novel methods for utilizing some of the resources offered in open quantum dynamics, such as induced many-body correlations and memory, to hopefully enhance a quantum estimation task in the presence of noise. This in turn can open up possibilities for applications in, e.g., quantum sensing [13][31] and quantum control of optomechanical devices for advanced technologies [32].

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