The Stretch Factor of Hexagon-Delaunay Triangulations

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Abstract

The problem of computing the exact stretch factor (i.e., the tight bound on the worst case stretch factor) of a Delaunay triangulation has been open for more than three decades. Over the years, a series of upper and lower bounds on the exact stretch factor have been obtained but the gap between them is still large. An alternative approach to solving the problem is to develop techniques for computing the exact stretch factor of “easier” types of Delaunay triangulations, in particular those defined using regular-polygons instead of a circle. Tight bounds exist for Delaunay triangulations defined using an equilateral triangle [6] and a square [2]. In this paper, we determine the exact stretch factor of Delaunay triangulations defined using a hexagon instead of a circle: It is 2. We think that the techniques we have developed may prove useful in future work on computing the exact stretch factor of classical Delaunay triangulations.

1 Introduction

In this paper we consider the problem of computing a tight bound on the worst case stretch factor of a Delaunay triangulation.

Given a set \( P \) of points in the plane, a triangulation \( T \) on \( P \) is a Delaunay triangulation if no point of \( P \) lies inside any circle circumscribing a triangle of \( T \). In this paper, we refer to Delaunay triangulations defined using the circle as Circle-Delaunay triangulations. The Circle-Delaunay triangulation \( T \) of \( P \) is a plane subgraph of the complete, weighted Euclidean graph \( E_P \) on \( P \) in which the weight of an edge is the Euclidean distance between its endpoints. Graph \( T \) is also a spanner, defined as a subgraph of \( E_P \) with the property that the distance in the subgraph between any pair of points is no more than a constant multiplicative ratio of the distance in \( E_P \) between the points. The constant ratio is referred to as the stretch factor (or spanning ratio) of the spanner.

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1Assumes that the points are in general position, which is discussed in Section 2
The problem of computing a tight bound on the worst case stretch factor of the $\odot$-Delaunay triangulation has been open for more than three decades. In the 1980s, when $\odot$-Delaunay triangulations were not known to be spanners, Chew considered related, “easier” structures. In 1986 [5], Chew proved that a $\Box$-Delaunay triangulation—defined using a fixed-orientation square instead of a circle—is a spanner with stretch factor at most $\sqrt{10}$. Following this, Chew proved that the $\triangle$-Delaunay triangulation—defined using a fixed-orientation equilateral triangle—has a stretch factor of 2 [6]. Significantly, this bound is tight: one can construct $\triangle$-Delaunay triangulations with stretch factor arbitrarily close to 2. Finally, Dobkin et al. [7] showed that the $\odot$-Delaunay triangulation is a spanner as well. The bound on the stretch factor they obtained was subsequently improved by Keil and Gutwin [8] as shown in Table 1. The bound by Keil and Gutwin stood unchallenged for many years until Xia recently improved the bound to below 2 [9]. On the lower bound side, some progress has been made on bounding the worst case stretch factor of a $\odot$-Delaunay triangulation. The trivial lower bound of $\pi/2 \approx 1.5707$ has been improved to 1.5846 [4] and then to 1.5932 [10].

After three decades of research, we know that the worst case stretch factor of $\odot$-Delaunay triangulations is somewhere between 1.5932 and 1.998. Unfortunately, the techniques that have been developed so far seem inadequate for proving a tight stretch factor bound. Rather than attempting to improve further the bounds on the stretch factor of $\odot$-Delaunay triangulations, we follow an alternative approach. Just like Chew turned to $\triangle$- and $\Box$-Delaunay triangulations to develop insights useful for showing that $\odot$-Delaunay triangulations are spanners, we make use of Delaunay triangulations defined using regular polygons to develop techniques for computing tight stretch factor bounds. Delaunay triangulations based on regular polygons are known to be spanners (Bose et al. [3]). Tight bounds are known for $\triangle$-Delaunay triangulations [6] and also for $\Box$-Delaunay triangulations (Bonifaci et al. [2]).

Table 1: Key stretch factor upper bounds (tight bounds are bold).

| Paper     | Graph            | Upper Bound            |
|-----------|------------------|------------------------|
| [7]       | $\odot$-Delaunay | $\pi(1 + \sqrt{5})/2 \approx 5.08$ |
| [8]       | $\odot$-Delaunay | $4\pi/(3\sqrt{3}) \approx 2.41$ |
| [9]       | $\odot$-Delaunay | 1.998                  |
| [1]       | $\triangle$-Delaunay | 2                     |
| [5]       | $\Box$-Delaunay   | $\sqrt{10} \approx 3.16$ |
| [2]       | $\Box$-Delaunay   | $\sqrt{4 + 2\sqrt{2}} \approx 2.61$ |
| [This paper] | $\odot$-Delaunay | 2                     |

2 Defined more precisely in Section 2.
This paper is part of a project that builds on these results to extend them to Delaunay triangulations defined using other regular polygons. The motivation is that the work results in the development of techniques that may be useful for proving a tight bound on the stretch factor of \( \odot \)-Delaunay triangulations.

In this paper, we show that the worst case stretch factor of \( \odot \)-Delaunay triangulations is two. We think that the techniques we have developed to obtain this result are at least as important as the result.

2 Preliminaries

We consider a finite set \( P \) of points in the two-dimensional plane with an orthogonal coordinate system. The \( x \)- and \( y \)-coordinates of a point \( u \) will be denoted by \( x(u) \) and \( y(u) \), respectively. The Euclidean graph \( \mathcal{E}_P \) of \( P \) is the complete weighted graph embedded in the plane whose nodes are identified with the points of \( P \). For every pair of nodes \( u \) and \( w \), the edge \((u, w)\) represents the segment \([u w]\) and the weight of \((u, w)\) is the Euclidean distance between \( u \) and \( w \) which is 
\[
d_2(u, w) = \sqrt{d_x(u, w)^2 + d_y(u, w)^2}
\]
where \( d_x(u, w) \) (resp. \( d_y(u, w) \)) is the absolute value of the difference between the \( x \) (resp. \( y \)) coordinates of \( u \) and \( w \).

Let \( T \) be a subgraph of \( \mathcal{E}_P \). The length of a path in \( T \) is the sum of the weights of the edges of the path and the distance \( d_T(p, q) \) in \( T \) between two points \( p \) and \( q \) is the length of the shortest path in \( T \) between them. \( T \) is a \( t \)-spanner for some constant \( t > 0 \) if for every pair of points \( p, q \) of \( P \), 
\[
d_T(p, q) \leq t \cdot d_2(p, q).
\]
The constant \( t \) is referred to as the stretch factor of \( T \).

We define a family of spanners to be a set of graphs \( T^P \), one for every finite set \( P \) of points in the plane, such that for some constant \( t > 0 \), every \( T^P \) is a \( t \)-spanner of \( \mathcal{E}_P \). We say that the stretch factor \( t \) is exact or tight for the family (or that the worst case stretch factor is \( t \)) if for every \( \epsilon > 0 \) there exists a set of points \( P \) such that \( T^P \) is not a \((t - \epsilon)\)-spanner of \( \mathcal{E}_P \).

The families of spanners we consider are a type of Delaunay triangulation on a set \( P \) of points in the plane. A triangulation \( T \) on \( P \) is a \( \odot \)-Delaunay triangulation if no point of \( P \) lies inside any circle circumscribing a triangle of \( T \). (This definition assumes that points are in general position which in the case of \( \odot \)-Delaunay triangulations means that no four points of \( P \) are co-circular.) If, in the definition, circle is replaced by fixed-orientation square (e.g., a square whose sides are axis-parallel) or by fixed-orientation equilateral triangle then different triangulations are obtained: the \( \Box \)- and the \( \triangle \)-Delaunay triangulations.

If, in the definition of \( \odot \)-Delaunay triangulation, we change circle to fixed-orientation regular hexagon, then a \( \odot \)-Delaunay triangulation is obtained. In this paper we focus on such triangulations. While any fixed orientation of the hexagon is possible, we choose w.l.o.g. the orientation that has two sides of the hexagon parallel to the \( y \)-axis as shown in Fig. 1(a). In the remainder of the paper, hexagon will always refer to a regular hexagon with such an orientation.
We find it useful to label the vertices of the hexagon N, NE, SE, S, SW, and NW, in clockwise order and starting with the top one. We also label the sides NE, E, SE, SW, W, and NW as shown in Fig. 1-(a).

The definition O-Delaunay triangulation assumes that no four points lie on the boundary of an empty hexagon (a hexagon whose interior contains no point of \( P \)). Our arguments also assume that no two points lie on a side of an empty hexagon. The general position assumption we therefore make in this paper consists of the above two restrictions. That said, we note that because the plane can be rotated to ensure a given set of points is in general position, our result apply to all sets of points.

We end this section with a lower bound, by Bonichon [1], on the worst case stretch factor of O-Delaunay triangulations.

**Lemma 2.1.** For every \( \varepsilon > 0 \), there exists a set \( P \) of points in the plane such that the O-Delaunay triangulation on \( P \) has stretch factor at least \( 2 - \varepsilon \).

**Proof.** Let \( k \) be some positive integer and let points \( p = p_0, q = q_k, p_k, \) and \( q_0 \) have coordinates \((0,0)\), \((1,\frac{1}{\sqrt{3}})\), \((\delta,\frac{2}{\sqrt{3}} - \sqrt{3}\delta)\), and \((1-\delta, -\frac{1}{\sqrt{3}} + \sqrt{3}\delta)\), respectively, where \( \delta = \frac{1}{k+2} \) (see Fig. 1-(b)). Additional \( k-1 \) points \( p_1, \ldots, p_{k-1} \) are placed on line segment \([p_0p_k]\) and another \( k-1 \) points \( q_1, \ldots, q_{k-1} \) on line segment \([q_0q_k]\) so that all segments \([p_{i-1}p_i]\) and \([q_{i-1}q_i]\), for \( i = 1, \ldots, k \), have equal length.

For every \( i = 1, 2, \ldots, k \), if \( H_i \) is the hexagon of minimum width \( 1 - \delta \) with \( p_{i-1} \) and \( p_i \) on its W and NW sides, then \( q_{i-1} \) is exactly the SE vertex of \( H_i \) (e.g., refer to \( p_0, p_1, q_0, \) and \( H_1 \) in Fig. 1-(c)). This means that all points \( q_j \) with \( j \neq i - 1 \) as well as all points \( p_j \) with \( j \neq i - 1 \) lie outside of \( H_i \). Therefore, for every \( i = 1, 2, \ldots, k \), points \( p_{i-1}, p_i, \) and \( q_{i-1} \) define a triangle in the O-Delaunay triangulation \( T \) on \( P \). A similar argument shows that for every \( i = 1, 2, \ldots, k \), points \( q_{i-1}, q_i, \) and \( p_i \) define a triangle in \( T \) and so the triangulation is as shown in Fig. 1-(b).

![Figure 1: (a) The hexagon orientation and the side and vertex labels that we use (b) A O-Delaunay triangulation with points \( p, q, p_k, \) and \( q_0 \) having coordinates \((0,0)\), \((1,\frac{1}{\sqrt{3}})\), \((\delta,\frac{2}{\sqrt{3}} - \sqrt{3}\delta)\), and \((1-\delta, -\frac{1}{\sqrt{3}} + \sqrt{3}\delta)\), respectively. (c) A closer look at the bottom faces of this triangulation.](image)
A shortest path from \( p \) to \( q \) in \( T \) is, for example, \( p = p_0, p_1, \ldots, p_k, q \) and so \( d_T(p, q) = d_2(p_0, p_k) + d_2(p_k, q_k) \) which tends to \( \frac{4}{\sqrt{3}} \) from below as \( \delta \to 0 \) or \( k \to \infty \). The distance between \( p \) and \( q \), on the other hand is \( 2\sqrt{3} \). Therefore, for any \( \varepsilon > 0 \), it is possible to choose \( k, \delta \), and a set \( P \) of points such that the \( \Sigma \)-Delaunay triangulation on \( P \) has stretch factor at least \( 2 - \varepsilon \). \( \square \)

3 Main result

The following is the main result of this paper:

**Theorem 3.1.** The stretch factor of a \( \Sigma \)-Delaunay triangulation is at most \( 2 \).

We prove Theorem 3.1 by showing that between any two points \( s \) and \( t \) of a set of points \( P \) there is, in the \( \Sigma \)-Delaunay triangulation \( T \) on \( P \), a path from \( s \) to \( t \) of length at most \( 2d_2(s, t) \). Let \( m \) be the slope of the line passing through \( s \) and \( t \). W.l.o.g., we assume that \( s \) has coordinates \((0, 0)\). Thanks to the hexagon’s rotational and reflective symmetries, we can rotate the plane around \( s \) and possibly reflect the plane with respect to the \( x \)-axis to ensure that \( t \) is in the positive quadrant of the coordinate system and \( 0 \leq m \leq \frac{1}{\sqrt{3}} \). Given this assumption, our main theorem will follow from:

**Lemma 3.2.** For every pair of points \( s, t \in P \) with \( 0 \leq m \leq \frac{1}{\sqrt{3}} \):

\[
d_T(s, t) \leq \max\{\frac{5}{\sqrt{3}} - 1, \sqrt{3} + m\}d_x(s, t).
\]  

(1)

Before we prove this lemma, we show that it implies the main theorem.

*Proof of Theorem 3.1.* W.l.o.g., we assume that \( s \) has coordinates \((0, 0)\), that \( t \) lies in the positive quadrant, and that \( d_2(s, t) = 1 \). With this assumption and because \( m \leq \frac{1}{\sqrt{3}} \), we only need to show that \( d_T(s, t) \leq 2 \) when \( \frac{\sqrt{3}}{2} \leq x(t) = d_x(s, t) \leq 1 \). By Lemma 3.2 either \( d_T(s, t) \leq (\frac{5}{\sqrt{3}} - 1)d_x(s, t) \leq (\frac{5}{\sqrt{3}} - 1) \leq 2 \) or

\[
d_T(s, t) \leq (\sqrt{3} + m)d_x(s, t) = \sqrt{3}d_x(s, t) + d_y(s, t) = \sqrt{3}d_x(s, t) + \sqrt{1 - d_x(s, t)^2}
\]

which is maximized, over the interval \([\frac{\sqrt{3}}{2}, 1]\), at \( d_x(s, t) = \frac{\sqrt{3}}{2} \) giving \( d_T(s, t) \leq 2 \). \( \square \)

We now turn to the proof of Lemma 3.2. We assume that \( s \) has coordinates \((0, 0)\) and that \( t \) lies in the positive quadrant. While most of our arguments in this section assume that the slope \( m \) of the line through \( s \) and \( t \) satisfies \( 0 \leq m \leq \frac{1}{\sqrt{3}} \), not all do so we do not make this assumption here. We start by noting that if there is a point \( p \) of \( P \) on the segment \([st]\) then (1) would follow if Lemma 3.2 holds for the pairs of points \( s, p \) and \( p, t \); it is therefore sufficient to prove Lemma 3.2 for the case when no point of \( P \) other than \( s \) and
We also set \( T_1, T_2, T_3, \ldots, T_n \) be the sequence of triangles of triangulation \( T \) that line segment \([st]\) intersects when moving from \( s \) to \( t \) (refer to Fig. 2). We abuse our notation slightly and from now on use \( T \) to refer to the union of all triangles \( T_i \); our arguments make use of no points or edges other than the ones in this union. Let \( u_1 \) and \( l_1 \) be the vertices of \( T_1 \) other than \( s \), with \( u_1 \) lying above segment \([st]\) and \( l_1 \) and lying below. Every triangle \( T_i \), for \( 1 < i < n \), intersects line segment \([st]\) twice; let \( u_i \) and \( l_i \) be the endpoints of the edge of \( T_i \) that intersects segment \([st]\) last, when moving on segment \([st]\) from \( s \) to \( t \), with \( u_i \) being above and \( l_i \) being below segment \([st]\). Note that for \( 1 < i < n \):

- either \( u_i = u_{i-1} \) and \( T_i = \triangle(u_i, l_i, l_{i-1}) \), in which case we call \( u_i = u_{i-1} \) the base vertex of \( T_i \) and \( l_{i-1} \) and \( l_i \) the left and right vertices of \( T_i \) (e.g., in Fig. 2) \( u_6 \) is the base and \( l_5 \) and \( l_6 \) are the left and right vertices of \( T_6 \),
- or \( l_i = l_{i-1} \) and \( T_i = \triangle(u_{i-1}, u_i, l_i) \), in which case we call \( l_i = l_{i-1} \) the base vertex of \( T_i \) and \( u_{i-1} \) and \( u_i \) the left and right vertices of \( T_i \).

We also set \( u_0 = l_0 = s \), and call \( s \) the left vertex and \( l_1 \) and \( u_1 \) the right vertices of \( T_1 \), and we set \( u_n = l_n = t \), and call \( t \) the right vertex and \( l_{n-1} \) and \( u_{n-1} \)

\[
\begin{align*}
\text{Figure 2: Triangles } T_1, T_2, \ldots, T_{12} \text{ are visited in that order when moving from } s \text{ to } t \text{ along the dotted segment } [st]. \text{ The vertices of each triangle } T_i \ (u_{i-1}, u_i, l_{i-1}, l_i, \text{ two of which are equal}) \text{ lie on the boundary of the hexagon } H_i. \text{ The length of the dashed piecewise-linear curve from } u_9 \text{ to } l_{11} \text{ (consisting of two vertical segments and a third with slope } -\frac{1}{\sqrt{3}}) \text{ is } \sqrt{3}d_{xy}(u_2, l_{11}) -(y(u_2) - y(l_{11})); \text{ the path } u_2 = u_3, u_4, u_5 = u_6, u_7 = u_8, l_8 = l_9 = l_{10}, l_{11} \text{ is gentle because its length is less than that.}
\end{align*}
\]
the left vertices of $T_n$. Let $U$ and $L$ be the sets of all point labels $u_i$ and $l_i$, respectively. We can assume a partial ordering of the point labels in $U \cup L$ that is the transitive closure of the relations 1) $u_i \leq u_j$ if $i \leq j$, 2) $l_i \leq l_j$ if $i \leq j$, 3) $u_i < l_i$ if $x(u_i) < x(l_i)$, and 4) $l_i < u_i$ if $x(l_i) < x(u_i)$. Finally, for $1 \leq i \leq n$, we define $H_i$ to be the empty hexagon having the vertices of $T_i$ on its boundary.

We now introduce some key definitions. A path between points $l_i \in L$ and $u_j \in U$ is gentle if the length of the path is not greater than $\sqrt{3}d_x(u_j, l_i) - (y(u_j) - y(l_i))$ (see Fig. 2.) A gentle path is a low cost path that, under the right conditions, will be used as a subpath of a path from $s$ to $t$ that meets the bound $1$. If a gentle path consists of a single edge (e.g., $(u_0, l_1)$ and $(u_8, l_9)$ in Fig. 2) we refer to the edge as a gentle edge. Note that an edge is gentle if and only if the slope of the line going through its endpoints is between $-\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$.

**Lemma 3.3.** If $0 \leq m \leq \frac{1}{\sqrt{3}}$ then:

- $u_i$ lies on the SW side of hexagon $H_{i+1}$ then $(u_i, l_{i+1})$ is gentle
- $l_i$ lies on the NW side of $H_{i+1}$ then $(l_i, u_{i+1})$ is gentle
- $u_i$ lies on the SE side of $H_i$ or if $l_i$ lies on the NE side of $H_i$ then $(u_i, l_i)$ is gentle

Note, for example, that point $u_8$ lies on the SW side of hexagon $H_9$ in Fig. 2 and that edge $(u_8, l_9)$ is gentle. Gentle edges that satisfy one of the conditions of Lemma 3.3 will be called short gentle edges.

**Proof.** If $u_i$ lies on the SW side of some hexagon $H_{i+1}$ then, since $0 \leq m \leq \frac{1}{\sqrt{3}}$, either $u_i = u_{i+1}$ and $l_i$ and $l_{i+1}$ must lie on the SE and E side of $H_{i+1}$, respectively, or $l_i = l_{i+1}$ must lie on the SE or E side of $H_{i+1}$. Either way, the slope of the line going through $u_i$ and $l_{i+1}$ must be between $-\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$. The remaining three cases are handled similarly. \qed

A left (resp., right) vertex of $T_i$ lying on the W (resp., E) side of $H_i$ will be referred to as the left (resp., right) induction point of $T_i$ or $H_i$. In Fig. 2 for example, $u_0 = l_0$, $l_2$, and $u_9$ are left induction points of $T_1$, $T_3$, and $T_{10}$, respectively. We will also say that $s$ is a left induction point of $T_1$ and $t$ is a right induction point of $T_n$ regardless of the side of $H_1$ and $H_n$ they lie on. Note that, by Lemma 3.3, if $s$ lies on the NW or SW side of $H_1$ then $(s = l_0, u_1)$ or $(s = u_0, l_1)$, respectively, is a short gentle edge; similarly, if $t$ lies on the NE or SE side of $H_n$ then $(t = l_n, u_{n-1})$ or $(t = u_n, l_{n-1})$, respectively, is a short gentle edge.

Let $T_{ij}$, where $i \leq j$, be the union of $T_i, T_{i+1}, \ldots, T_j$. We say that $T_{ij}$ is standard if the left vertex of $T_i$ is a left induction point of $T_i$, the right vertex of $T_j$ is a right induction point of $T_j$, and neither the base vertex of $T_i$ nor the base vertex of $T_j$ is the endpoint of a gentle path in $T_{ij}$. Note that $T_{1n}$ is trivially standard because $s$ is a left induction point of $T_1$, $t$ is a right induction point of $T_n$, and $T_1$ and $T_n$ have no base vertices.
Let $T_{ij}$ be standard for some $i, j$ such that $1 \leq i \leq j \leq n$. A gentle path in $T_{ij}$ from $p$ to $q$, where $p$ occurs before $q$, is canonical in $T_{ij}$ (or simply canonical if $T_{ij}$ is clear from the context) if $p$ is a right induction point of $T_{ij}$ for some $i' \geq i$ or $p$ is the left induction point of $T_j$, and if $q$ is a left induction point of $T_j'$ for some $j' \leq j$ or $q$ is the right induction point of $T_j$. For example, the gentle path $u_2 = u_3, u_4, u_5 = u_6, u_7 = u_8, l_2 = l_9 = l_{10}, l_{11}, l_{12}$ in Fig. 2 is canonical.

The key technical ingredient for the proof of Lemma 3.2 is stated next and proven in the next section. Note that this lemma does not have a restriction on the slope of the line passing through $s$ and $t$ other than being non-negative.

**Lemma 3.4** (The Key Lemma). If $T_{ij}$ contains no gentle edge, $T_i$ has a vertex $p$ on the $W$ side of $H_i$, and $T_j$ has vertex $q$ on the $E$ side of $H_j$ then there exists a path in $T_{ij}$ from $p$ to $q$ of length at most $\frac{1}{\sqrt{3}}d_x(p, q)$. If, in addition, $T_{ij}$ contains no gentle path then there exists a path in $T_{ij}$ from $p$ to $q$ of length at most $(\frac{5}{\sqrt{3}} - 1)d_x(p, q)$.

The remainder of this section is devoted to using this lemma to prove Lemma 3.2. We first make use of the first part of Lemma 3.4 to gain the following insight:

**Lemma 3.5.** Let $0 \leq m \leq \frac{1}{\sqrt{3}}$. If $T_{ij}$ is standard and contains a gentle path then the path can be extended to a canonical gentle path in $T_{ij}$.

Before we prove this lemma, we provide a high level overview of the approach we take to prove Lemma 3.2 If $T = T_{1n}$ does not contain a gentle path then the proof follows from the second part of Lemma 3.4. If $T = T_{1n}$ contains a gentle path then by Lemma 3.5 it must contain a canonical gentle path $P$, say, for example, from $u_i$ to $l_{j-1}$ where $0 < i < j < n$. We assume that $P$ is maximal in the sense that it is not a subpath of any other gentle path in $T_{1n}$. Since $P$ is canonical, $u_i$ is a right induction point of $T_i$ and $l_{j-1}$ is a left induction point of $T_j$. The maximality of $P$ will guarantee that neither $T_{1i}$ nor $T_{jn}$ contains a gentle path whose endpoint is the base vertex of $T_i$ or the base vertex of $T_j$, respectively. Therefore $T_{1i}$ and $T_{jn}$ are standard and we then proceed by induction to prove a slightly more general version of Lemma 3.2 for $T_{1i}$ and $T_{jn}$. The obtained bounds are combined with the bound on the length of $P$ to complete the proof of Lemma 3.2.

**Proof of Lemma 3.5.** Case A. We first prove the claim in the case when $T_{ij}$ contains no short gentle edge. Because $T_{ij}$ is standard and by Lemma 3.3 the left vertex of $T_i$, whether it is $l_{i-1}$ or $u_{i-1}$ (or $s = l_0 = u_0$ if $i = 1$), lies on the $W$ side of $H_i$ and the right vertex of $T_j$, whether it is $l_j$ or $u_j$ (or $t = u_j = l_j$ if $j = n$), lies on the $E$ side of $H_j$.

Suppose that $T_{ij}$ contains a gentle path between $u_r \in U$ and $l_s \in L$. W.l.o.g., we assume that $u_r$ is before $l_s$ (i.e., $s > r$ or $(s = r$ and $x(u_r) < x(l_s)$)), as the case when $l_s$ is before $u_r$, can be argued using a symmetric argument. Consider the sequence of points $u_{i-1}, u_i, u_{i+1}, \ldots, u_r$ and let $p = u_{r'}$ be the last point that is a right induction point (of $H_{i'}$) or $p = u_{r'} = u_{i-1}$ if there are no right
induction points in the sequence. Note that in the second case, \( p \) cannot be a base vertex of \( T_i \) (since \( T_{ij} \) is standard) and so \( p \) must be the left induction point of \( T_i \). The path \( u_{r'}, \ldots, u_r \) consists of edges \((u_{x-1}, u_x)\) for every \( x \) such that \( r' < x \leq r \) and \( u_x \) is a right vertex of \( H_x \). Because \( T_{ij} \) contains no short edges and the fact that \( u_x \) cannot lie on the E or SE side of \( H_x \), the endpoints \( u_{x-1} \) and \( u_x \) lie on the W or NW and NW or NE sides, respectively, of \( H_x \). This implies that \( x(u_{x-1}) < x(u_x) \) and that the length of \((u_{x-1}, u_x)\) is bounded by \( \sqrt{3d_x(u_{x-1}, u_x) - (y(u_{x-1}) - y(u_x))} \), as illustrated in Fig. 3. Therefore, the length of the path \( p = u_{r'}, \ldots, u_r \) (path \( P \)) in \( T_{ij} \) is at most \( \sqrt{3d_x(p, u_r) - (y(p) - y(u_r))} \).

![Figure 3](image.png)

Figure 3: Illustration of case A in the proof of Lemma 3.5. The case \( r = s \) is shown. For every \( x \) such that \( r' < x \leq r \) and \( u_x \) is the right vertex of \( H_x \), hexagon \( H_x \) and the edge \((u_{x-1}, u_x)\) are shown in red; the length of \((u_{x-1}, u_x)\) is bounded by \( \sqrt{3d_x(u_{x-1}, u_x) - (y(u_{x-1}) - y(u_x))} \), a value equal to the total length of the two intersecting, red, dashed segments out of \( u_{x-1} \) and \( u_x \). Dashed segments are also used to represent the upper bounds on the lengths of the path \( p = u_{r'}, \ldots, u_r \) (in red), the edge \((u_r, u_s)\) (in blue), and the path \( l_s, l_{s+1}, \ldots, l_{s'} = q \) (in red).

Similarly, we consider the sequence of points \( l_s, l_{s+1}, \ldots, l_j \) and set \( q = l_{s'} \) to be the first that is a left induction point (of \( H_{x+1} \)) or \( q = l_{s'} = l_j \) if there are no left induction points in the sequence. For every \( x \) such that \( s \leq x < s' \), if \( l_x \) is a left vertex of \( H_{x+1} \) then edge \((l_x, l_{x+1})\) has endpoints \( l_x \) and \( l_{x+1} \) lying on the SW or SE and SE or E sides, respectively, of \( H_{x+1} \). This implies that \( x(l_x) < x(l_{x+1}) \) and that the length of \((l_x, l_{x+1})\) is bounded by \( \sqrt{3d_x(l_x, l_{x+1}) - (y(l_x) - y(l_{x+1}))} \). Therefore, the length of the path \( l_s, l_{s+1}, \ldots, l_{s'} = q \) (path \( Q \)) is at most \( \sqrt{3d_x(l_s, q) - (y(q) - y(l_s))} \), as shown in Fig. 3. By combining the gentle path from \( u_r \) to \( l_s \) with \( P \) and \( Q \), we obtain a canonical gentle path from \( p \) to \( q \).

**Case B.** We now consider the case when \( T_{ij} \) contains a short gentle edge. We assume w.l.o.g. that the gentle edge has an endpoint \( u_r \) that lies on the SW side of \( H_{r+1} \) and also that the gentle edge is the first such edge (in the sense that for all \( x \) such that \( i - 1 \leq x < r \), \( u_x \) does not lie on the SW side of \( H_{x+1} \)). (For the case when \( u_r \) lies on the SE side of \( H_r \) we would consider the last such
edge and use an argument that is symmetric to the one we make below; the cases when \( l_r \) is on the NW side of \( H_{r+1} \) or on the NE side of \( H_r \) are symmetric to the cases when \( u_r \) is on the SW side of \( H_{r+1} \) and the SE side of \( H_r \).

Let the gentle edge under consideration be \((u_r, l_{r+1})\) and let \( p \) and \( q \) be as defined in Case A. Because of our assumption that \((u_r, l_{r+1})\) is first, the case A arguments can be applied to bound the distance from \( p \) as defined in Case A. Because of our assumption that \((u_r, l_{r+1})\) is first, the case A arguments can be applied to bound the distance from \( p \) to \( u_r \) with \( \sqrt{3}d_x(p, u_r) - (y(p) - y(u_r)) \). We cannot do the same to bound the distance from \( u_r \) to \( q = \sigma u_r \) because it is possible that for some \( x \) such that \( r + 1 < x < s' \), \( l_x \) lies on the NE side of \( H_x \) and \( x(l_{x-1}) > x(l_x) \). So we proceed instead with induction and prove that

\[
d_T(u_r, l_x) \leq \sqrt{3}d_x(u_r, l_x) - (y(u_r) - y(l_x)) \text{ for every } x \text{ such that } r < x < s' \tag{2}
\]

which will complete the proof.

The base case \( x = r + 1 \) holds because \((u_r, l_{r+1})\) is gentle. For the induction step, we assume that \( d_T(u_r, l_x) \leq \sqrt{3}d_x(u_r, l_x) - (y(u_r) - y(l_x)) \) for all \( x \) such that \( r < x < s \). If \( l_x = l_{x-1} \), the claim holds. Otherwise, \( l_x \) is a right vertex of \( H_x \). If \( l_x \) lies on the E or SE side of \( H_x \) then \( x(l_{x-1}) < x(l_x) \) and the length of \((l_{x-1}, l_x)\) is less than \( \sqrt{3}d_x(l_{x-1}, l_x) - (y(u_{x-1}) - y(u_x)) \) and the inductive claim again easily holds.

If, however, \( l_s \) lies on the NE side of \( H_s \) then \((u_s, l_s)\) must be a (short) gentle edge by Lemma 3.3 (as illustrated in Fig. 4(a)). We consider indexes \( r + 1, \ldots, s - 1 \) and set \( t \) to be the last one such that edge \((l_t, u_t)\) has positive slope, if one exists. We consider two subcases:

**Case B.1.** If no \((l_t, u_t)\), for \( r < t < s \) has positive slope (as is the case in Fig. 4(a)) we consider the transformation \( T'_{(r+1)s} \) of \( T_{(r+1)s} \) obtained by rotating the plane clockwise by an angle of \( \pi/3 \) and then reflecting the plane with respect to the \( x \)-axis, as illustrated in Fig. 4(b). Let \( T'_t, T'_t, \ldots, T'_s \) be the corresponding transformations of triangles \( T_{t+1}, \ldots, T_s \) and let \( d'_x(u_r, l_t) \) be the difference between the abscissas of \( l_t \) and \( u_r \) in the transformed plane. We note that \( u_r \) is on the W side of \( T'_{t+1} \) and \( l_t \) is on the E side of \( T'_t \). Because the slope of the line through \( u_r \) and \( l_t \) is at most \( 1/3 \) in \( T_{(r+1)s} \), it follows that the slope of the line through \( u_r \) and \( l_t \) in \( T'_{(r+1)s} \) is at least \( 1/\sqrt{3} \). Finally, we note that if there is a gentle edge in \( T'_{(r+1)s} \), it would correspond to an edge with positive slope in \( T_{(r+1)s} \). So Lemma 3.3 can be applied to \( T'_{(r+1)s} \) which we use to bound the distance from \( u_r \) to \( l_t \) in \( T'_{(r+1)s} \) (and thus in \( T_{(r+1)s} \) as well) by \( \sqrt{3}d'_x(u_r, l_t) \). Because the slope of the line through \( u_r \) and \( l_t \) is at least \( 1/\sqrt{3} \), this last quantity is less than the length of the dashed piecewise-linear curve in Fig. 4(b) consisting of a vertical segment out of \( u_r \) followed by a segment with slope \(-1/\sqrt{3}\) to \( l_t \). The length of the dashed curve is exactly \( \sqrt{3}d_x(u_r, l_t) - (y(u_r) - y(l_t)) \) as illustrated in Fig. 4(a).

**Case B.2.** If \((l_t, u_t)\) exists, then, because \( t \) is last, for every \( x \) such that \( t < x < s \), \( u_x \) cannot lie on the E or SE side of \( H_x \) (because otherwise \((l_x, u_x)\) would have to have positive slope). Also, by induction, the length of
on the SW side of $H$ also in (a)) is an upper bound on the length of a path in $T$ from case A we obtain

We use induction to prove a more general statement: if

Proof of Lemma 3.2. Together with the inductive hypothesis bound on $T_{(r+1)s}$, the length of the red, dashed segments (shown also in (a)) is an upper bound on the length of a path in $T_{(r+1)s}$ from $u_r$ to $l_s$.

the path $u_r, l_r, l_{r+1}, \ldots, l_t$ is at most $d_T(u_r, l_t) \leq \sqrt{3}d_x(u_r, l_t) - (y(u_r) - y(l_t))$.

We now have two cases to consider.

Case B.2.i. If we also have that no $u_x$ lies on the SW side of $H_x$, where $t < x < s$ then, using the arguments from case A, we can bound the length of the path $u_t, u_{t+1}, \ldots, u_s, l_s$ with $\sqrt{3}d_x(u_t, l_s) - (y(u_t) - y(u_s))$. Putting everything together, we get:

$$d_T(u_r, l_s) \leq d_T(u_r, l_t) + d_T(l_t, u_t) + d_T(u_t, l_s)$$
$$\leq \sqrt{3}d_x(u_r, l_t) - (y(u_r) - y(l_t)) + \sqrt{3}d_x(l_t, u_t) - (y(l_t) - y(u_t))$$
$$+ \sqrt{3}d_x(u_t, l_s) - (y(u_t) - y(l_s))$$
$$\leq \sqrt{3}d_x(u_r, l_s) - (y(u_r) - y(l_s)).$$

Case B.2.ii. Finally, suppose that $u_{t'}$, for some $t'$ such that $t < t' < s$, lies on the SW side of $H_{t'+1}$ and let’s assume that $t'$ is leftmost (in the sense that no $u_x$ lies on the SW side of $H_{x+1}$ for $x$ such that $t < x < t'$). Using arguments from case A we obtain $d_T(u_t, u_{t'}) \leq \sqrt{3}d_x(u_t, u_{t'}) - (y(u_t) - y(u_{t'}))$. Using the approach from case B.1 we get $d_T(u_{t'}, l_s) \leq \sqrt{3}d_x(u_{t'}, l_s) - (y(u_{t'}) - y(l_s))$. Together with the inductive hypothesis bound on $d_T(u_t, l_t)$ and $d_T(l_t, u_t) \leq \sqrt{3}d_x(l_t, u_t) - (y(l_t) - y(u_t))$, we complete the proof of Lemma 3.2.

We now use this last lemma and Lemma 3.4 to prove Lemma 3.2.

Proof of Lemma 3.4. We use induction to prove a more general statement: if
Thus:

\[ d_{T_{ij}}(p, q) \leq \max\{ \frac{5}{\sqrt{3}} - 1, \sqrt{3} + m \} d_x(p, q) \]

where \( m = \text{slope}(s, t) \). Note that the claim in Lemma 3.2 is a special case of this statement when \( i = 1 \) and \( j = n \).

We proceed by induction on \( j-i \). If \( T_{ij} \) is standard and there is no gentle path in \( T_{ij} \) (the base case) then, by Lemma 3.4, we have \( d_T(p, q) \leq (\frac{5}{\sqrt{3}} - 1)d_x(p, q) \).

If \( T_{ij} \) is standard and there is a gentle path in \( T_{ij} \), then, by Lemma 3.5 there exist points \( u_{i'} \) and \( l_{j'} \) in \( T_{ij} \) such that there is a canonical gentle path between \( u_{i'} \) and \( l_{j'} \) in \( T_{ij} \); w.l.o.g. we assume that \( u_{i'} \) is before \( l_{j'} \), i.e. \( i - 1 \leq i' \leq j' \leq j \). We also assume that the canonical path from \( u_{i'} \) to \( l_{j'} \) is maximal in the sense that it is not a proper subpath of a gentle path. So, \( d_T(u_{i'}, l_{j'}) \leq \sqrt{3}d_x(u_{i'}, l_{j'}) - (y(u_{i'}) - y(l_{j'})) \). Since \( u_{i'} \) is either \( s \) or above \( st \) and \( l_{j'} \) is either \( t \) or below \( st \), it follows that \( -(y(u_{i'}) - y(l_{j'})) \leq md_x(u_{i'}, l_{j'}) \).

Therefore, \( d_T(u_{i'}, l_{j'}) \leq (\sqrt{3} + m)d_x(u_{i'}, l_{j'}) \).

Since the gentle path from \( u_{i'} \) to \( l_{j'} \) is canonical, either \( u_{i'} \) is a right induction point of \( H_{i'} \) and \( i' \geq i \) or \( u_{i'} = u_{i-1} \). In the first case, because \( u_{i'} \) is on the E side of \( H_{i'} \) the base vertex \( l_{i'-1} = l_{i'} \) of \( T_{i'} \) must satisfy \( x(l_{i'}) < x(u_{i'}) \). If \( l_{i'} \) is the endpoint of a gentle path in \( T_{i'} \) from, say, point \( u_{i'} \) then we would have

\[
d_{T_{ij}}(u_{i'}, l_{j'}) \leq d_{T_{ij}}(u_{i'}, l_{i'}) + d_x(l_{i'}, u_{i'}) + d_{T_{ij}}(u_{i'}, l_{j'}) \\
\leq \sqrt{3}d_x(u_{i'}, l_{i'}) - (y(u_{i'}) - y(l_{i'})) + \sqrt{3}d_x(l_{i'}, u_{i'}) - (y(l_{i'}) - y(u_{i'})) \\
+ \sqrt{3}d_x(u_{i'}, l_{j'}) - (y(u_{i'}) - y(l_{j'})) \\
\leq \sqrt{3}d_x(u_{i'}, l_{j'}) - (y(u_{i'}) - y(l_{j'})).
\]

This contradicts the maximality of the canonical gentle path from \( u_{i'} \) to \( l_{j'} \). This means that \( l_{i'} \) is not the endpoint of a gentle path in \( T_{i'} \). Therefore \( T_{i'} \) is standard, the inductive hypothesis applies, and \( d_T(p, u_{i'}) \leq \max\{ \frac{5}{\sqrt{3}} - 1, \sqrt{3} + m \} d_x(p, u_{i'}) \). In the second case, because \( T_{ij} \) is standard, \( u_{i'} \) cannot be the base vertex of \( T_{i} \) and so \( u_{i'} = p \). The inequality \( d_T(p, u_{i'}) \leq \max\{ \frac{5}{\sqrt{3}} - 1, \sqrt{3} + m \} d_x(p, u_{i'}) \) holds trivially. Similarly, \( d_T(l_{j'}, q) \leq \max\{ \frac{5}{\sqrt{3}} - 1, \sqrt{3} + m \} d_x(l_{j'}, q) \).

Thus:

\[
d_T(p, q) \leq d_T(p, u_{i'}) + d_T(u_{i'}, l_{j'}) + d_T(l_{j'}, q) \\
\leq \max\{ \frac{5}{\sqrt{3}} - 1, \sqrt{3} + m \} (d_x(p, u_{i'}) + d_x(l_{j'}, q)) + (\sqrt{3} + m)d_x(u_{i'}, l_{j'}) \\
\leq \max\{ \frac{5}{\sqrt{3}} - 1, \sqrt{3} + m \} d_x(p, q)
\]
4 Proof of the key lemma

This section is devoted to the proof of Lemma 3.4. We prove it in its full generality but, in order to simplify the notation, we set $T_{ij} = T_{1n}$ and assume that $p$ (not necessarily $s$) lies on the W side of hexagon $H_1$ and $q$ (not necessarily $t$) lies on the E side of hexagon $H_n$.

We start by defining functions $H(x)$, $u(x)$, and $\ell(x)$ for $x(p) \leq x \leq x(q)$. If $c_i$ is the center of hexagon $H_i$, for $i = 1, \ldots, n$, we define $H(x(c_i)) = H_i$. We also define $u(x(c_i))$ and $\ell(x(c_i))$ to be $u_i$ and $l_i$, respectively. Then, for every $i = 1, \ldots, n - 1$ and $x$ such that $x(c_i) < x < x(c_{i+1})$, we define $H(x)$ to be the hexagon whose center has abscissa $x$ and that has points $u_i$ and $l_i$ on its boundary; we also define $u(x)$ to be $u_i$ and $\ell(x)$ to be $l_i$ (see Fig. 5). Intuitively, $H(x)$ for $x(c_i) < x < x(c_{i+1})$ models the hexagon $H_i$ as it is “pushed” through $u_i$ and $l_i$ up until it becomes $H_{i+1}$. We note that $H(x)$ is uniquely defined because $T_{1n}$ contains no gentle edges and so $u(x)$ and $\ell(x)$ cannot lie on the E and W sides of $H(x)$. As we will soon see, function $H(x)$ has a specific growth pattern that depends on what sides of $H(x)$ points $u(x)$ and $\ell(x)$ lie on. In order to simplify our presentation, we define $H(x) = H(x(c_1))$ when $x(p) \leq x < x(c_1)$ and $x(c_n) < x \leq x(q)$ in a way that fits that pattern. Let $w$ be the SW vertex of $H_1 = H(x(c_1))$ and let $H^*$ be the hexagon with $p$ and $w$ as its NW and SW vertices, respectively. Let $c^*$ be the center of $H^*$. We define $H(x)$ to be the hexagon whose center is collinear with $p$ and $c^*$ and asumme $u(x)$ to be the hexagon whose center is collinear with $p$ and $c_1$ and has abscissa $x$ and that has, when $x(p) \leq x < x(c^*)$, point $p$ as its NW vertex; we also define $u(x) = \ell(x) = p$. When $x(c^*) \leq x < x(c_1)$, we define $H(x)$ to be the hexagon whose center is collinear with $c^*$ and $c_1$ and has abscissa $x$ and that has point $w$ as its SW vertex; we also define $u(x) = p$ and $\ell(x) = w$. We define $H(x)$ when $x(c_n) < x \leq x(q)$ in a symmetric fashion. Next, we define $r(x)$ to be the minimum radius of $H(x)$. Note that hexagons $H(x(p))$ and $H(x(q))$ both have

![Figure 5](image-url)

Figure 5: For $x$ such that $x(c_i) < x < x(c_{i+1})$, $H(x)$ is the hexagon whose center has abscissa $x$ and that has points $u_i = u(x)$ and $l_i = \ell(x)$ on its boundary; $r(x)$ is the minimum radius of $H(x)$ and $w(x) = x - r(x)$ and $e(x) = x + r(x)$. $N(x)$ and $S(x)$ are the N and S vertices of $H(x)$.
Finally, we define the potential function $x$ and for $P(x)$ growth rate. $u$ by the distance from $d$ otherwise (see Fig. 6). We use the shorthand notation $d$ this distance is exactly $d$ upper bound for function $P(x)$ such, and a similar observation about each edge $(u_{i-1}, u_i)$ lying on the boundary of $H(x) = H_i$, is bounded by $d_N(u_{i-1}, x) - d_N(u_i, x)$.

radius 0 and define their centers to be $c_0 = p$ and $c_{n+1} = q$. We also extend the notation $T_{ij}$ to include $T_{10} = c_0$. We define $N(x)$ and $S(x)$ to be the N and S vertex, respectively, of $H(x)$. Finally, we define functions $w(x) = x - r(x)$ and $e(x) = x + r(x)$ that keep track of the abcissa of the W and E sides, respectively, of $H(x)$ (refer to Fig. 5).

We define next functions that we will use to bound the length of the shortest path from $p$ to $q$. We define function $d_N(u, x)$ (resp., $d_S(l, x)$) to be the signed shortest distance from a point $u \in U$ (resp., $l \in L$) lying on a side of $H(x)$ to the N (resp., S) vertex of $H(x)$ when traveling along the perimeter of $H(x)$; the sign is positive if $u$ (resp., $l$) lies on one of the W sides of $H(x)$ and negative otherwise (see Fig. 6). We use the shorthand notation $d_N(x)$ (resp., $d_S(X)$) if $x(c_i) \leq x < x(c_{i+1})$ and $u = u_i$ (resp., $l = l_i$).

The key functions $U(x)$ and $L(x)$ are defined as follows for $x(p) = x(c_0) \leq x \leq x(c_{n+1})$:

\[
\begin{align*}
U(x) &= d_{T_{i1}}(p, u_i) + d_N(x) \quad \text{when } x(c_i) \leq x < x(c_{i+1}) \\
L(x) &= d_{T_{1i}}(p, l_i) + d_S(x) \quad \text{when } x(c_i) \leq x < x(c_{i+1})
\end{align*}
\]

and for $x = x(c_{n+1}) = x(q)$

\[
\begin{align*}
U(x(q)) &= d_{T_{1n}}(p, q) \\
L(x(q)) &= d_{T_{1n}}(p, q)
\end{align*}
\]

Finally, we define the potential function $P(x)$ to be $U(x) + L(x)$. We note that $P(x(q))$ is exactly twice the distance in $T_{1n}$ from $p$ to $q$. Our goal is to compute an upper bound for function $P(x)$. We will do this by bounding its (average) growth rate.

The length of each edge $(u_{i-1}, u_i)$ (assuming $u_{i-1} \neq u_i$) can be bounded by the distance from $u_{i-1}$ to $u_i$ when traveling clockwise along the sides of $H_i$. This distance is exactly $d_N(u_{i-1}, x(c_i)) - d_N(u_i, x(c_i))$ as illustrated in Fig. 6 (c). This, and a similar observation about each edge $(l_{i-1}, l_i)$, motivates the
following definition of functions $\bar{U}(x)$ and $\bar{L}(x)$ that we will use to bound the length of paths connecting adjacent points in $U$ or adjacent points in $L$. When $x(c_i) \leq x \leq x(c_{i+1})$, we define

$$\bar{U}(x) = \begin{cases} d_N(x) & \text{if } i = 0 \\ \sum_{j=0}^{i-1} (d_N(u_j, x(c_{j+1})) - d_N(u_{j+1}, x(c_{j+1}))) + d_N(x) & \text{if } i > 0 \end{cases}$$

$$\bar{L}(x) = \begin{cases} d_S(x) & \text{if } i = 0 \\ \sum_{j=0}^{i-1} (d_S(l_j, x(c_{j+1})) - d_S(l_{j+1}, x(c_{j+1}))) + d_S(x) & \text{if } i > 0 \end{cases}$$

Note that the functions $\bar{U}(x)$ and $\bar{L}(x)$ are continuous from $x(c_0)$ to $x(c_{n+1})$. On the other hand, $U(x)$ is discontinuous at $x = x(c_i)$ if $u(x(c_i)) = u_i \neq u_{i-1}$; similarly, $L(x)$ is discontinuous at $x = x(c_i)$ if $\ell(x(c_i)) = l_i \neq l_{i-1}$. We will show that outside of those points of discontinuity, $U(x)$ and $L(x)$, and therefore $P(x)$ as well, are monotone increasing, piecewise-linear functions. Before doing so, we prove:

**Lemma 4.1.** Functions $U(x)$, $L(x)$, and $P(x)$ do not increase at points of discontinuity.

**Proof.** Let $x^* = x(c_i)$ be a point of discontinuity for function $U(x)$. Then

$$U(x) = d_{T_i}(p, u_i) + d_N(u_i, x^*)$$

$$\leq d_{T_{i-1}}(p, u_i) + d_N(u_{i-1}, u_i) + d_N(u_i, x^*)$$

$$\leq d_{T_{i-1}}(p, u_i) + d_N(u_{i-1}, x^*) - d_N(u_i, x^*) + d_N(u_i, x^*)$$

$$= d_{T_{i-1}}(p, u_i) + d_N(u_{i-1}, x^*)$$

The last term is the limit for $U(x)$ when $x \to x^*$ from the left and so the claim holds for $U(x)$. The claim for $L(x)$ holds using an equivalent argument, and the claim for $P(x)$ follows from $P(x) = L(x) + U(x)$. \qed

Between points of discontinuity, the terms $d_{T_i}(p, u_i)$ and $d_{T_i}(p, l_i)$ in the definitions of $U(x)$ and $L(x)$, respectively, are constant. This means that the rate of growth of functions $U(x)$ and $L(x)$ between points of discontinuity is determined solely by the terms $d_N(x)$ and $d_S(x)$, respectively. The same also holds for functions $\bar{U}(x)$ and $\bar{L}(x)$. Functions $d_N(x)$ and $d_S(x)$ are monotone increasing piecewise-linear functions whose rate of increase depends on the placement of $u(x)$ and $\ell(x)$ on the sides of $H(x)$. In order to capture precisely this rate of growth, we label the sides of hexagon $H(x)$, in counterclockwise order starting from the NW side, with integer labels 0 through 5. We define the transition function $t(x)$ to be transition $t_{ij}$ if $\ell(x)$ lies in the interior of side $i$ and $h(x)$ lies in the interior of side $j$. We use the wildcard notations $t_{*j}$ (resp., $t_{i*}$) to refer to any transition with $u(x)$ on side $j$ (resp., with $\ell(x)$ on side $i$) of $H(x)$.

**Lemma 4.2.** If $T_{ij}$ contains no gentle edge then $t(x)$, when defined, is one of $t_{10}, t_{15}, t_{21}, t_{20}, t_{25}, t_{24}, t_{31}, t_{30}, t_{35}, t_{34}, t_{40}, t_{45}$. Furthermore, functions $d_N(x)$,
\(y(N(x)), d_S(X), y(S(x)), x(x), w(x), \text{ and } e(x)\) are, between points of discontinuity, monotone increasing piecewise-linear functions with the following growth rates where defined:

| \(t(x)\) | \(t_{10}\) | \(t_{15}\) | \(t_{21}\) | \(t_{25}\) | \(t_{24}\) | \(t_{31}\) | \(t_{30}\) | \(t_{35}\) | \(t_{34}\) | \(t_{40}\) | \(t_{45}\) |
|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| \(\Delta P(x) / \Delta x\) | \(\frac{6}{\sqrt{3}}\) | \(\frac{6}{\sqrt{3}}\) | \(\frac{6}{\sqrt{3}}\) | \(\frac{6}{\sqrt{3}}\) | \(\frac{6}{\sqrt{3}}\) | \(\frac{6}{\sqrt{3}}\) | \(\frac{6}{\sqrt{3}}\) | \(\frac{6}{\sqrt{3}}\) | \(\frac{6}{\sqrt{3}}\) | \(\frac{6}{\sqrt{3}}\) | \(\frac{6}{\sqrt{3}}\) |
| \(\Delta d_N(x) / \Delta x\) | \(\frac{2}{\sqrt{3}}\) | \(\frac{2}{\sqrt{3}}\) | \(\frac{2}{\sqrt{3}}\) | \(\frac{2}{\sqrt{3}}\) | \(\frac{2}{\sqrt{3}}\) | \(\frac{2}{\sqrt{3}}\) | \(\frac{2}{\sqrt{3}}\) | \(\frac{2}{\sqrt{3}}\) | \(\frac{2}{\sqrt{3}}\) | \(\frac{2}{\sqrt{3}}\) | \(\frac{2}{\sqrt{3}}\) |
| \(\Delta y(N(x)) / \Delta x\) | \(\frac{1}{\sqrt{3}}\) | \(-\frac{1}{\sqrt{3}}\) | \(\frac{3}{\sqrt{3}}\) | \(-\frac{5}{\sqrt{3}}\) | \(-\frac{5}{\sqrt{3}}\) | \(-\frac{5}{\sqrt{3}}\) | \(-\frac{5}{\sqrt{3}}\) | \(-\frac{5}{\sqrt{3}}\) | \(-\frac{5}{\sqrt{3}}\) | \(-\frac{5}{\sqrt{3}}\) | \(-\frac{5}{\sqrt{3}}\) |
| \(\Delta d_S(x) / \Delta x\) | \(\frac{4}{\sqrt{3}}\) | \(\frac{6}{\sqrt{3}}\) | \(\frac{6}{\sqrt{3}}\) | \(\frac{6}{\sqrt{3}}\) | \(\frac{6}{\sqrt{3}}\) | \(\frac{6}{\sqrt{3}}\) | \(\frac{6}{\sqrt{3}}\) | \(\frac{6}{\sqrt{3}}\) | \(\frac{6}{\sqrt{3}}\) | \(\frac{6}{\sqrt{3}}\) | \(\frac{6}{\sqrt{3}}\) |
| \(\Delta y(S(x)) / \Delta x\) | \(\frac{3}{\sqrt{3}}\) | \(-\frac{5}{\sqrt{3}}\) | \(-\frac{1}{\sqrt{3}}\) | \(-\frac{1}{\sqrt{3}}\) | \(-\frac{1}{\sqrt{3}}\) | \(-\frac{1}{\sqrt{3}}\) | \(-\frac{1}{\sqrt{3}}\) | \(-\frac{1}{\sqrt{3}}\) | \(-\frac{1}{\sqrt{3}}\) | \(-\frac{1}{\sqrt{3}}\) | \(-\frac{1}{\sqrt{3}}\) |
| \(\Delta d(x) / \Delta x\) | \(1\) | \(1\) | \(\frac{3}{2}\) | \(1\) | \(1\) | \(\frac{3}{2}\) | \(1\) | \(1\) | \(\frac{3}{2}\) | \(1\) | \(1\) |
| \(\Delta w(x) / \Delta x\) | \(0\) | \(0\) | \(\frac{1}{2}\) | \(1\) | \(2\) | \(0\) | \(1\) | \(\frac{3}{2}\) | \(2\) | \(2\) | \(2\) |
| \(\Delta e(x) / \Delta x\) | \(2\) | \(2\) | \(2\) | \(0\) | \(2\) | \(0\) | \(2\) | \(\frac{3}{2}\) | \(0\) | \(0\) | \(0\) |

We note that the first part of Lemma 3.4 follows from this lemma.

Proof. The first part follows from the fact that gentle edges have slope between \(-\frac{1}{\sqrt{3}}\) and \(\frac{3}{\sqrt{3}}\). The growth rates for transitions \(t_{25}, t_{10}, \text{ and } t_{15}\) follow from elementary geometric constructions in Fig. 7. The geometric constructions for the remaining transitions are similar. \qed

Note that we have not defined \(t(x)\) at values of \(x\) when \(l(x)\) or \(u(x)\) is a vertex of \(H(x)\), which is when \(\bar{U}(x)\) or \(\bar{L}(x)\) is not smooth and differentiable.

Figure 7: Constructions demonstrating Lemma 4.2 for transitions (a) \(t_{25}\), (b) \(t_{10}\), and (c) \(t_{15}\). In all three cases the growths shown are with respect to \(\Delta(x) = 1\). \(\Delta r(x)\) can be obtained from \(\frac{1}{2}(\Delta e(x) - \Delta w(x))\) and \(\Delta P(x)\) from \(\Delta d_N(x) + \Delta d_S(x)\).
what follows, for clarity of presentation we will sometimes abuse our definition of \( t(x) \) to include such points.

We will refer to transitions \( t_{15}, t_{24}, t_{31}, t_{40} \) as bad. Note that when \( t(x) \) is not bad, the growth rate of \( P(x) \) is at most \( \frac{6}{\sqrt{3}} \), well under the desired growth rate of \( 2 \left( \frac{5}{\sqrt{3}} \right) - 1 \). Our goal is to amortize the “extra” growth rate of \( \frac{2}{\sqrt{3}} \) of \( P(x) \) when \( t(x) \) is bad so that the average growth rate of \( P(x) \) is at most \( 2 \left( \frac{5}{\sqrt{3}} \right) - 1 \).

We first define intervals that contain only one type of bad transition and over which some of the extra cost of that bad transition will be amortized.

**Definition 4.1.** Given \( x_l, x_r \) such that \( x_0 < x_l < x_r \leq x_{n+1} \), interval \([x_l, x_r]\) is said to be a

- \( t_{15} \)-interval if \( t(x_l) = t_{15} \) and \( t(x) \neq t_{s_1} \) for all \( x_l < x < x_r \) and a strict \( t_{15} \)-interval if, in addition, \( t(x) \neq t_{40}, t_{s_4} \) when \( x_l < x < x_r \).
- \( t_{31} \)-interval if \( t(x_l) = t_{31} \) and \( t(x) \neq t_{1_4} \) for all \( x_l < x < x_r \) and a strict \( t_{31} \)-interval if, in addition, \( t(x) \neq t_{24}, t_{4_4} \) when \( x_l < x < x_r \).
- \( t_{24} \)-interval if \( t(x_r) = t_{24} \) and \( t(x) \neq t_{4_4} \) for all \( x_l < x < x_r \) and a strict \( t_{24} \)-interval if, in addition, \( t(x) \neq t_{31}, t_{1_4} \) when \( x_l < x < x_r \).
- \( t_{40} \)-interval if \( t(x_r) = t_{40} \) and \( t(x) \neq t_{s_4} \) for all \( x_l < x < x_r \) and a strict \( t_{40} \)-interval if, in addition, \( t(x) \neq t_{15}, t_{s_1} \) when \( x_l < x < x_r \).

We introduce some final bit of notation. Given \( u(x) = u_i \) and \( \ell(x) = l_i \) for some \( x \), we define \( f(x) \) and \( b(x) \) to be \( x(u_i) \) and \( x(l_i) \), respectively, when \( x(u_i) > x(l_i) \), and \( x(l_i) \) and \( x(u_i) \), respectively, when \( x(u_i) \leq x(l_i) \). Finally, we define \( \theta_f(x) = f(x) - x \) and \( \theta_b(x) = x - b(x) \). The following lemma bounds the amount of time certain transitions, including the bad transition, can occur within a bad transition interval (see Fig. 5).

**Lemma 4.3.** Let \( T_{1n} \) have no gentle paths, let \( x_0 < x_l < x_r \leq x_{n+1} \), and let \( z_{ij} \) be the amount of time within interval \([x_l, x_r]\) spent in transition \( t_{ij} \). If \([x_l, x_r]\) is

- a \( t_{15} \)-interval then \( z_{s_5} + z_{s_4} + (\sqrt{3} - 1)\theta_f(x_l) \leq \sqrt{3}r(x_l) \)
- a \( t_{31} \)-interval then \( z_{3_5} + z_{4_4} + (\sqrt{3} - 1)\theta_f(x_l) \leq \sqrt{3}r(x_l) \)
- a \( t_{24} \)-interval then \( z_{1_4} + z_{2_4} + (\sqrt{3} - 1)\theta_b(x_r) \leq \sqrt{3}r(x_r) \)
- a \( t_{40} \)-interval then \( z_{s_0} + z_{s_4} + (\sqrt{3} - 1)\theta_b(x_r) \leq \sqrt{3}r(x_r) \)

**Proof.** We assume that \([x_l, x_r]\) is a \( t_{15} \)-interval (the other cases follow by symmetry). We also assume that \( t(x_r) \neq t_{s_0} \) implying that \( u(x_r) \) lies either on the N vertex, side 5, the NE vertex, or side 4 of \( H(x_r) \) (or, with some abuse of the definition of \( t(x) \), \( t(x_r) = t_{s_5}, t_{s_4} \)). (If that is not the case, we simply consider the shorter interval \([x_l, x_q]\) where \( x_q \) is such that \( u(x_q) \) is the N vertex of \( H(x_q) \) and \( t(x) = t_{s_0} \) for \( x_q < x < x_r \).)
Figure 8: Illustration of Lemma 4.3 for the case of a $t_{15}$-interval. The lemma gives a bound, assuming that $T_n$ contains no gentle path, on the amount of time within the interval $[x_l, x_r]$ spent in transitions $t_5$ (which includes bad transition $t_{15}$) and $t_{14}$. These are the transitions that have $u(x)$ on the NE or E side of $H(x)$.

Consider the path in $T_n$ from $\ell(x_l)$ to $u(x_r)$ that starts with the edge $(\ell(x_l), u(x_l))$ and then visits the vertices $u(x_l) = u_t, u_{t+1}, \ldots, u_{t+s} = u(x_r)$ in order (refer to Fig. 5). Since $T_n$ contains no gentle paths, the following holds:

$$d_2(\ell(x_l), u(x_l)) + \sum_{i=1}^{i+s-1} d_2(u_t, u_{t+1}) + (y(u(x_r)) - y(\ell(x_l))) \geq \sqrt{3}d_x(\ell(x_l), u(x_r))$$

(3)

The length of edge $(\ell(x_l), u(x_l))$ is at most the length of line segment $[wu(x_l)]$ where $w$ is the SW vertex of $H(x_l)$, and the length of $[wu(x_l)]$ is at most $\frac{2r(x_l)}{\cos(\Theta)}$, where $\Theta$ is the angle $\angle N(x_l)wu(x_l)$. Note that $0 \leq \Theta \leq \frac{\pi}{3}$. So we have:

$$d_2(\ell(x_l), u(x_l)) \leq \frac{2r(x_l)}{\cos(\Theta)}$$

(4)

Note also that $d_y(\ell(x_l), N(x_l)) \leq \sqrt{3}r(x_l)$ and so, by Lemma 4.2

$$y(u(x_r)) - y(\ell(x_l)) = (y(N(x_l)) - y(x_l)) + (y(N(x_r)) - y(N(x_l)))$$

$$- (y(N(x_r) - y(u(x_r))))$$

$$\leq \sqrt{3}r(x_l) + \frac{1}{\sqrt{3}}z_{s0} - \frac{1}{\sqrt{3}}z_{s5} - \frac{3}{\sqrt{3}}z_{s34} - \frac{5}{\sqrt{3}}z_{24}$$

$$- (\frac{1}{\sqrt{3}}(x(u(x_r)) - x_r) + \max\{0, y(v) - y(u(x_r))\})$$

(5)

where $v$ is the NE vertex of $H(x_r)$. The max term in (5) is 0 or positive depending on whether $u(x_r)$ is on side 5 or 4, respectively, of $H(x_r)$. Each edge $(u_t, u_{t+1})$ on the path $u(x_l) = u_t, u_{t+1}, \ldots, u_{t+s} = u(x_r)$ can be bounded by $d_N(u_t, x(c_{t+1})) - d_N(u_{t+1}, x(c_{t+1}))$ and so:

$$\sum_{i=1}^{i+s-1} d_2(u_t, u_{t+1}) \leq \sum_{i=1}^{i+s-1} (d_N(u_t, x(c_{t+1})) - d_N(u_{t+1}, x(c_{t+1})))$$
\[ = (\bar{U}(x_r) - d_N(x_r)) - (\bar{U}(x_l) - d_N(x_l)) \]
\[ = d_N(x_l) + (\bar{U}(x_r) - \bar{U}(x_l)) - d_N(x_r) \]
\[ = -\frac{2}{\sqrt{3}} \theta_f(x_l) + \left( \frac{2}{\sqrt{3}} (z_{w0} + z_{w5}) + \frac{4}{\sqrt{3}} z_{34} + \frac{6}{\sqrt{3}} z_{24} \right) \]
\[ + \left( \frac{2}{\sqrt{3}} (x(u(x_r)) - x_r) + \max\{0, y(x) - y(u(x_r))\} \right) \]

Substituting the left-hand side of (3) with (4), (5), and (6) gives

\[ u(s) = \left( \frac{2}{\sqrt{3}} + \sqrt{3} \right) x_l - \frac{2}{\sqrt{3}} \theta_f(x_l) + \frac{3}{\sqrt{3}} z_{w0} + \frac{1}{\sqrt{3}} (z_{w5} + z_{w4} + x(u(x_r)) - x_r) \]
\[ \geq \sqrt{3}(x_l + x(u(x_r)) - x_l) \]

where we use the fact that \( x(l(x_l)) = x_l - r(x_l) \). This then implies

\[ \left( \frac{2}{\cos(\Theta)} + \sqrt{3} \right) x_l - \frac{2}{\sqrt{3}} \theta_f(x_l) + \frac{3}{\sqrt{3}} z_{w0} + \frac{1}{\sqrt{3}} (z_{w5} + z_{w4} + \theta_f(x_l)) \geq \sqrt{3}(x_l + x_r - x_l) \]

which can in turn be simplified to give

\[ \sqrt{3} x_l \geq \cos(\Theta)(z_{w4} + z_{w5}) + \cos(\Theta) \theta_f(x_l) \]
\[ = z_{w4} + z_{w5} + (\cos(\Theta) - 1)(z_{w4} + z_{w5}) + \cos(\Theta) \theta_f(x_l) \]
\[ \geq z_{w4} + z_{w5} + (2 \cos(\Theta) - 1) \theta_f(x_l) \]
\[ \geq z_{w4} + z_{w5} + (\sqrt{3} - 1) \theta_f(x_l) \]

\[ \square \]

The following two lemmas amortize the “extra” growth (i.e., above \( \frac{\theta}{\sqrt{3}} \)) of bad transitions over two types of bad intervals (see Fig. 9).

**Lemma 4.4.** Given the assumptions of Lemma 4.3, if \([x_l, x_r]\) is a \( t_{15} \)- (resp., \( t_{31} \), \( t_{40} \), \( t_{24} \)) interval, if \( t(x) \neq t_{40} \) (resp., \( t_{24} \), \( t_{15} \), \( t_{31} \)) when \( x_l \leq x \leq x_r \), and if \( t(x_r) = t_{40} \) (resp., \( t_{24} \), \( t_{15} \), \( t_{31} \)) then

\[ \frac{2}{\sqrt{3}} z_{15}, \frac{2}{\sqrt{3}} z_{31} \leq \left( \frac{2}{\sqrt{3}} - 1 \right) (e(x_r) - w(x_l) - 2 \theta_f(x_l)) \]
\[ \leq \left( \frac{4}{\sqrt{3}} - 2 \right) (x_r - w(x_l) - \theta_f(x_l)) \]

\[ \frac{2}{\sqrt{3}} z_{24}, \frac{2}{\sqrt{3}} z_{40} \leq \left( \frac{2}{\sqrt{3}} - 1 \right) (e(x_r) - w(x_l) - 2 \theta_b(x_l)) \]
\[ \leq \left( \frac{4}{\sqrt{3}} - 2 \right) (e(x_r) - x_l - \theta_b(x_r)) \]
The fact that Lemma 4.4: the satisfying the conditions of the lemma then

Proof. We prove the lemma for the case of a $t_{15}$-interval; the other 3 cases can be seen to follow by symmetry. By Lemma 4.2, if $[x_r]$ is a $t_{15}$-interval then

$$r(x_l) + z_{15} + z_{10} + \frac{1}{2}z_{20} = r(x_r) + \frac{1}{2}z_{35} + z_{45} + z_{24} + z_{34}$$  \hspace{1cm} (9)

Since $t(x) \neq t_{s1}$ for $x_r \leq x \leq e(x_r)$, the interval $[x_l, e(x_r)]$ is a $t_{15}$-interval such that $t(x) = t_{s5}$ or $t(x) = t_{s4}$ when $x_r \leq x \leq e(x_r)$ and therefore, by Lemma 4.3,

$$r(x_r) + z_{s5} + z_{s4} + (\sqrt{3} - 1)\theta_f(x_l) \leq \sqrt{3}r(x_l)$$ which, together with (9), gives

$$r(x_r) + z_{s5} + z_{s4} + (\sqrt{3} - 1)\theta_f(x_l) \leq \sqrt{3}(r(x_r) - z_{15} + \frac{1}{2}z_{35} + z_{45} + z_{34} + z_{24})$$

which implies

$$(\sqrt{3} + 1)z_{15} \leq (\sqrt{3} - 1)(r(x_r) + z_{45} + z_{24} + z_{34} - \theta_f(x_l))$$

and also

$$\frac{2}{\sqrt{3}}z_{15} \leq (\frac{4}{\sqrt{3}} - 2)(r(x_r) + z_{45} + z_{24} + z_{34} - \theta_f(x_l)).$$  \hspace{1cm} (10)

The fact that $e(x_r) - w(x_l) = r(x_l) + z_{s0} + z_{s4} + z_{s5} + r(x_s)$ together with (9) gives us $r(x_r) + z_{45} + z_{24} + z_{34} \leq \frac{1}{2}(e(x_r) - w(x_l))$ and so (10) implies inequality (7). Because $e(x_r) - w(x_l) \leq 2(x_r - w(x_l))$, inequality (8) follows.

Figure 9: Illustration of Lemmas 4.4 and 4.5 for the case of a $t_{15}$-interval. (a) Lemma 4.4: the $\frac{2}{\sqrt{3}}$ extra cost of the bad transition $t_{15}$ within the interval $[x_l, x_r]$ is amortized over the interval $[w(x_l) + \theta_f(x_l), x_r]$ (shown in bold) as cost $(\frac{4}{\sqrt{3}} - 2)$. (b) Lemma 4.5: the $\frac{2}{\sqrt{3}}$ extra cost of the bad transition $t_{15}$ within the interval $[x_l, x_r]$ is amortized over the interval $[w(x_l) + \theta_f(x_l), w(x_r)]$ as cost $(\frac{4}{\sqrt{3}} - 2)$.

**Lemma 4.5.** Given the assumptions of Lemma 4.3, if $[x_l, x_r]$ is a $t_{15}$- (resp., $t_{31}$-, $t_{40}$-, $t_{24}$-) interval, if $t(x) \neq t_{40}$ (resp., $t_{24}$, $t_{15}$, $t_{31}$) when $x_l \leq x \leq x_r$, if $t(x_r) = t_{s1}$ (resp., $t_{s1}$, $t_{s4}$, $t_{44}$) then

$$\frac{2}{\sqrt{3}}z_{15} - \frac{1}{\sqrt{3}}(z_{20} + z_{30}),$$  \hspace{1cm} (11)
\[
\frac{2}{\sqrt{3}} z_{21} - \frac{1}{\sqrt{3}} (z_{20} + z_{23}) \leq \left( \frac{2}{\sqrt{3}} - 1 \right) (w(x_r) - w(x_l) - 2\theta_f(x_l)) \tag{12}
\]
\[
\frac{2}{\sqrt{3}} z_{40} - \frac{1}{\sqrt{3}} (z_{25} + z_{35}), \tag{13}
\]
\[
\frac{2}{\sqrt{3}} z_{24} - \frac{1}{\sqrt{3}} (z_{30} + z_{35}) \leq \left( \frac{2}{\sqrt{3}} - 1 \right) (e(x_r) - e(x_l) - 2\theta_b(x_r)) \tag{14}
\]

**Proof.** We prove the lemma for the case of a \(t_{15}\)-interval only; the other 3 cases can be seen to follow by symmetry. Then, since \([x_l, x_r]\) is a \(t_{15}\)-interval and \(t(x_r) = t_{41}\), point \(u(x_r)\) must be the NW point of \(H(x_r)\). The point \(u(x_r)\) must lie on side 0 of \(H(x)\) for \(w(x_r) \leq x \leq x_r\) and so \(t(x) = t_{40}\) when \(w(x_r) \leq x \leq x_r\). This, together with Lemma 4.2 and the fact that \(t(x) \neq t_{40}\) when \(w(x_r) \leq x \leq x_r\), implies that \(r(w(x_r)) \leq \frac{1}{2}(z_{20} + z_{30}) \leq z_{20} + z_{30}\). It also implies that all \(t_{45}\) and \(t_{48}\) transitions in interval \([x_l, x_r]\) occur within interval \([t_{15}, w(x_r)]\) and so \(w(x_r) - w(x_l) \geq r(x_l) + z_{45} + z_{44}\). Since, by Lemma 4.2 \(r(x_l) + z_{15} \leq r(w(x_r)) + z_{24} + \frac{1}{2}z_{35} + z_{34} + z_{45}\), it follows that \(z_{24} + z_{34} + z_{25} + z_{35} + z_{45} \geq r(x_l) + z_{15} - r(w(x_r)) \geq r(x_l) + z_{15} - (z_{20} + z_{30})\). Therefore, \(w(x_r) - w(x_l) \geq 2r(x_l) + 2z_{15} - (z_{20} + z_{30})\) and:
\[
\left( \frac{4}{\sqrt{3}} - 2 \right) (r(x_l) + z_{15} - \theta_f(x_l)) - \left( \frac{2}{\sqrt{3}} - 1 \right) (z_{20} + z_{30})
\]
\[
\leq \left( \frac{2}{\sqrt{3}} - 1 \right) (w(x_r) - w(x_l) - 2\theta_f(x_l))
\]

By Lemma 4.3 we have \(z_{45} + z_{44} + (\sqrt{3} - 1)\theta_f(x_l) \leq \sqrt{3} r(x_l)\) and by Lemma 4.2 we have \(r(x_l) + z_{15} - (z_{20} + z_{30} + z_{45} + z_{24} + z_{34}) \leq r(w(x_r)) \leq z_{20} + z_{30}\). Combining these gives
\[
2z_{15} + (\sqrt{3} - 1)\theta_f(x_l) \leq (\sqrt{3} - 1) r(x_l) + (z_{20} + z_{30})
\]
or
\[
\left( \frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}} \right) z_{15} - (1 - \frac{1}{\sqrt{3}}) (z_{20} + z_{30}) \leq \left( \frac{4}{\sqrt{3}} - 2 \right) (r(x_l) - \theta_f(x_l))
\]

By plugging this inequality into (11), we obtain (12).

We define a time (coordinate) \(x\) in the interval \([x(c_0), x(c_{n+1})]\) to be left critical or right critical if \(x\) is the left boundary of a maximal strict \(t_{15}\)- or \(t_{31}\)-interval or the right boundary of a maximal strict \(t_{24}\)- or \(t_{40}\)-interval, respectively. We also define the start and end coordinates \(x(c_0)\) and \(x(c_{n+1})\) to be both left and right critical. A time \(x\) is said to be critical if it is left or right critical. The following lemma makes use of the bad transition growth amortization Lemmas 4.4 and 4.5 to inductively prove that the growth rate of \(P(x)\) is at most \(\frac{10}{\sqrt{3}} - 2\) on average:

**Lemma 4.6.** Let \(T_{1n}\) have no gentle path and let \(x\) be a critical point. Then:

- if \(x\) is a left critical point then \(P(x) \leq \left( \frac{10}{\sqrt{3}} - 2 \right) (w(x) + \theta_f(x)) + \frac{6}{\sqrt{3}} (r(x) - \theta_f(x))\)

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implies that transitions spent in transition \( x \) the right boundary of this interval then we assume that \( x \).

We proceed by induction, using the left to right ordering of critical points. The first critical point (i.e., the base case) is \( x(q_0) \). Since \( w(x(q_0)) = \theta_f(x(q_0)) = e(x(q_0)) = \theta_b(x(q_0)) = r(x(q_0)) = P(x(q_0)) = 0 \), the claim holds. We assume now that the claim holds for critical point \( x_1 \) and prove that it holds for the next critical point \( x_r \).

We first consider the case when \( x_l \) and \( x_r \) are both left critical points. This implies that transitions \( t_{34} \) and \( t_{40} \) do not occur within interval \([x_l, x_r]\). W.l.o.g. we assume that \( x_l \) is the left boundary of a maximal strict \( t_{15} \)-interval; if \( x_q \) is the right boundary of this interval then \( x_q \leq x_r \), \( t(x_q) = t_{s1} \) or \( t(x_q) = t_{34} \), and \( t(x) \) is not bad when \( x_q < x < x_r \). Let \( z_{ij} \) be the time within interval \([x_l, x_q]\) spent in transition \( t_{ij} \), for every transition \( t_{ij} \), and let \( z = \sum z_{ij} = x_q - x_l \). If \( t(x_q) = t_{s1} \), by Lemma 4.2, we have:

\[
\Delta P(x_l, x_r) \leq \frac{8}{\sqrt{3}}(z_{15}) + \frac{4}{\sqrt{3}}(z_{20} + z_{30}) + \frac{6}{\sqrt{3}}(z - z_{15} - z_{20} - z_{30} + x_r - x_q)
\]

The last inequality follows from Lemma 4.5. Using \( w(x_q) \leq w(x_r) \), we now apply the inductive hypothesis:

\[
P(x_r) = P(x_l) + \Delta P(x_l, x_r)
\]

which implies the claim. If \( t(x_q) = t_{34} \) then, by Lemma 4.2, we have:

\[
\Delta P(x_l, x_r) \leq \frac{8}{\sqrt{3}}(z_{15}) + \frac{6}{\sqrt{3}}(z - z_{15} + x_r - x_q)
\]

The last inequality follows from Lemma 4.4. Now, since \( t(x_q) = t_{34} \), then
\[ x_q \leq b(x_q) \leq b(x_r) = w(x_r). \] We now apply the inductive hypothesis:

\[
P(x_r) = P(x_l) + \Delta P(x_l, x_r)
\]

\[
\leq \left( \frac{10}{\sqrt{3}} - 2 \right)(w(x_l) + \theta_f(x_l)) + \frac{6}{\sqrt{3}}(r(x_l) - \theta_f(x_l))
\]

\[
+ \frac{6}{\sqrt{3}}(x_r - x_l) + \left( \frac{4}{\sqrt{3}} - 2 \right)(x_q - w(x_l) - \theta_f(x_l))
\]

\[
\leq \left( \frac{10}{\sqrt{3}} - 2 \right)w(x_r) + \frac{6}{\sqrt{3}}r(x_r)
\]

where the last inequality implies the claim. The case when \( x_l \) and \( x_r \) are both right critical points can be handled using a symmetric argument. Since \( x(c_{n+1}) \) is both left and right critical, we can handle it with one of these two arguments.

If \( x_l \) is right critical and \( x_r \) is left critical then no bad transitions occur within the interval \([x_l, x_r]\) and \( \Delta P(x_l, x_r) \leq \frac{6}{\sqrt{3}}(x_r - x_l) \). If \( e(x_l) - \theta_b(x_l) \leq w(x_r) + \theta_f(x_r) \):

\[
P(x_r) = P(x_l) + \Delta P(x_l, x_r)
\]

\[
\leq \left( \frac{10}{\sqrt{3}} - 2 \right)(e(x_l) - \theta_b(x_l)) - \frac{6}{\sqrt{3}}(r(x_l) - \theta_b(x_l)) + \frac{6}{\sqrt{3}}(x_r - x_l)
\]

\[
\leq \left( \frac{10}{\sqrt{3}} - 2 \right)(w(x_r) + \theta_f(x_r)) + \frac{6}{\sqrt{3}}(r(x_r) - \theta_f(x_r))
\]

If \( e(x_l) - \theta_b(x_l) > w(x_r) + \theta_f(x_r) \), consider the interval \( I = [w(x_r) + \theta_f(x_r), e(x_l) - \theta_b(x_l)] \). We argue that this interval is contained within interval \([x_l, x_r]\) and that \( \Delta P \) has a growth rate of just \( \frac{1}{\sqrt{3}} \) in that interval. To do this, we assume w.l.o.g. that \( t(x_l) = t_{40} \). Suppose that \( u(x_l) \) is not the point on the W side of \( H(x_r) \) (i.e., \( b(x_l) \neq b(x_r) = w(x_r) \)). Then either the point on the W side of \( H(x_r) \) is a point in \( L \) and \( x_r > b(x_r) \geq f(x_l) = e(x_l) \) or the point on the W side of \( H(x_r) \) is a point in \( U \) and \( x_r = b(x_r) + r(x_r) \geq x_l + r(x_l) = e(x_l) \). In either case \( x_r \geq e(x_l) > e(x_l) - \theta_b(x_l) \). If \( b(x_l) = b(x_r) \) then \( t(x_l) = t_{40} \) for all \( x \in [x_l, x_r] \) which, using Lemma \ref{Lemma:CriticalPoints} implies that \( x_r - x_l > r(x_l) - \theta_b(x_l) \), or \( x_r > e(x_l) - \theta_b(x_l) \). A symmetric argument can be used to show that \( x_l < w(x_r) + \theta_f(x_r) \) and therefore interval \( I \) is contained within \([x_l, x_r]\).

For every \( x \) in interval \( I, x \geq w(x_r) = b(x_r) \geq b(x) \) (using the property: if \( x \leq x' \) then \( b(x) \leq b(x') \)) and \( x \leq e(x_l) \leq f(x_l) \leq f(x) \). Given that no bad transitions occur withing \([x_l, x_r]\), \( t(x) = t_{30} \) for \( x \in I \) and therefore \( \Delta P \) has a growth rate bounded by \( \frac{1}{\sqrt{3}} \) in \( I \).

\[
P(x_r) = P(x_l) + \Delta P(x_l, x_r)
\]

\[
\leq \left( \frac{10}{\sqrt{3}} - 2 \right)(e(x_l) - \theta_b(x_l)) - \frac{6}{\sqrt{3}}(r(x_l) - \theta_b(x_l))
\]

\[
+ \frac{4}{\sqrt{3}}|I| + \frac{6}{\sqrt{3}}(x_r - x_l - |I|)
\]

\[
\leq \left( \frac{10}{\sqrt{3}} - 2 \right)w(x_r) + \frac{6}{\sqrt{3}}r(x_r)
\]

\[\text{23}\]
We will show that in this case:

\[ \text{We consider the case when } t \text{ holds and it, together with inequality (16), gives:} \]

\[ \Delta P(x_l, x_r) \leq \left( \frac{10}{\sqrt{3}} - 2 \right)(w(x_r) + \theta_f(x_r)) + \left( \frac{8}{\sqrt{3}} - 2 \right)|I| + \frac{6}{\sqrt{3}}(x - e(x_l) + \theta_b(x_l) - I) \]

\[ \leq \left( \frac{10}{\sqrt{3}} - 2 \right)(w(x_r) + \theta_f(x_r)) + \frac{6}{\sqrt{3}}(x - w(x_r) - \theta_f(x_r) - I) \]

\[ \leq \left( \frac{10}{\sqrt{3}} - 2 \right)(w(x_r) + \theta_f(x_r)) + \frac{6}{\sqrt{3}}(x - \theta_f(x_r)) \]

Finally, we consider the case when \( x_l \) is left critical and \( x_r \) is right critical. We will show that in this case:

\[ \Delta P(x_l, x_r) \leq \left( \frac{4}{\sqrt{3}} - 2 \right)(r(x_l) - \theta_f(x_l) + r(x_r) - \theta_b(x_r)) + \left( \frac{10}{\sqrt{3}} - 2 \right)(x_r - x_l) \] (15)

Note that if this inequality holds then:

\[ P(x_r) = P(x_l) + \Delta P(x_l, x_r) \]

\[ \leq \left( \frac{10}{\sqrt{3}} - 2 \right)(w(x_l) + \theta_f(x_l)) + \frac{6}{\sqrt{3}}(x - \theta_f(x_l)) \]

\[ + \left( \frac{4}{\sqrt{3}} - 2 \right)(r(x_l) - \theta_f(x_l) + r(x_r) - \theta_b(x_r)) + \left( \frac{10}{\sqrt{3}} - 2 \right)(x_r - x_l) \]

\[ \leq \left( \frac{10}{\sqrt{3}} - 2 \right)(e(x_r) - \theta_b(x_r)) + \frac{6}{\sqrt{3}}(x - \theta_f(x_l)) \]

Let \( z_{ij} \) be the time within interval \([x_l, x_r]\) spent in transition \( t_{ij} \), for every transition \( t_{ij} \), and let \( z = \sum z_{ij} = x_r - x_l \). We assume w.l.o.g. that \( t(x_l) = t_{15} \). We consider the case when \( t(x_r) = t_{24} \) first; this means that \( t(x) \neq t_{31}, t_{40} \) when \( x_l \leq x \leq x_r \). Either \([x_l, x_r]\) is a \( t_{15}\)-interval with \( t(x_r) = t_{24} \), and so Lemma 4.4 applies to interval \([x_l, x_r]\), or \([x_l, x_q]\) is a \( t_{15}\)-interval for some \( x_q < x_r \) with \( t(x_q) = t_{21} \), and thus Lemma 4.4 applies to interval \([x_l, x_q]\). In the second case, since \( t(x) \neq t_{15} \) when \( x_q \leq x \leq x_r \) and \( w(x_q) < w(x_r) < e(x_r) \), it follows that

\[ \frac{2}{\sqrt{3}}z_{15} - \frac{2}{\sqrt{3}}(z_{20} + z_{30}) \leq \left( \frac{4}{\sqrt{3}} - 2 \right)((e(x_r) - w(x_l)) - \theta_f(x_l)) \] (16)

Note that this inequality holds for the first case as well. Similarly, either \([x_l, x_r]\) is a \( t_{24}\)-interval with \( t(x_l) = t_{15} \), and so Lemma 4.4 applies to interval \([x_l, x_r]\), or \([x_q', x_r]\) is a \( t_{24}\)-interval for some \( x_q' > x_l \) with \( t(x_{q'}) = t_{45} \), and thus Lemma 4.4 applies to interval \([x_{q'}, x_r]\). Either way, the inequality

\[ \frac{2}{\sqrt{3}}(z_{24} - z_{35}) \leq \left( \frac{4}{\sqrt{3}} - 2 \right)(e(x_r) - w(x_l) - \theta_b(x_r)) \]

holds and it, together with inequality (16), gives:

\[ \frac{2}{\sqrt{3}}(z_{15} + z_{24}) - \frac{2}{\sqrt{3}}(z_{20} + z_{30} + z_{25} + z_{35}) \leq \left( \frac{4}{\sqrt{3}} - 2 \right)(e(x_r) - w(x_l) - \theta_b(x_r) - \theta_f(x_l)). \]

Thus we have:

\[ \Delta P(x_l, x_r) \leq \frac{6}{\sqrt{3}}z + \frac{2}{\sqrt{3}}(z_{15} + z_{24}) - \frac{2}{\sqrt{3}}(z_{20} + z_{30} + z_{25} + z_{35}) \]
\[
\begin{align*}
\leq & \frac{6}{\sqrt{3}} z + \left( \frac{4}{\sqrt{3}} - 2\right)(e(x_r) - w(x_l) - \theta_b(x_r) - \theta_f(x_l)) \\
= & \left( \frac{4}{\sqrt{3}} - 2\right)(r(x_l) - \theta_f(x_l) + r(x_r) - \theta_b(x_r)) + \left( \frac{10}{\sqrt{3}} - 2\right)(x_r - x_l)
\end{align*}
\]

We now consider the case when \( t(x_l) = t_{15} \) and \( t(x_r) = t_{40} \). Note that this means that \( t(x) \neq t_{31}, t_{24} \) when \( x_l \leq x \leq x_r \). If
\[
\frac{2}{\sqrt{3}} (z_{15} + z_{40}) - \frac{2}{\sqrt{3}} (z_{20} + z_{30} + z_{25} + z_{35}) \leq \left( \frac{4}{\sqrt{3}} - 2\right)(e(x_r) - w(x_l) - \theta_b(x_r) - \theta_f(x_l))
\]

then the above argument can be applied to obtain (15). We will show next that if \( t(x) = t_{*1} \) or \( t(x) = t_{*4} \) for some \( x \) such that \( x_l < x < x_r \) then inequality (17) holds.

Consider the maximal \( t_{15} \)-interval \([x_l, x_q]\) and note that \( x_q \leq x_r \). If \( t(x_q) = t_{21} \) then Lemma 4.3 applies to interval \([x_l, x_q]\). If \( t(x_q) = t_{34} \) then Lemma 4.3 applies to interval \([x_l, x_q]\). Since \( w(x_q) \leq w(x_r) \leq e(x_r) \) it follows in both cases that
\[
\frac{2}{\sqrt{3}} z_{15} - \frac{1}{\sqrt{3}} (z_{20} + z_{30}) \leq \left( \frac{2}{\sqrt{3}} - 1\right)(e(x_r) - w(x_l) - 2\theta_f(x_l))
\]

Similarly, consider the maximal \( t_{40} \)-interval \([x_{q'}, x_r]\) and note that \( x_l \leq x_{q'} \). If \( t(x_{q'}) = t_{34} \) then Lemma 4.3 applies to interval \([x_{q'}, x_r]\). If \( t(x_{q'}) = t_{21} \) then Lemma 4.3 applies to interval \([x_{q'}, x_r]\). Since \( w(x_l) \leq w(x_{q'}) \leq e(x_{q'}) \) it follows in both cases that
\[
\frac{2}{\sqrt{3}} z_{40} - \frac{1}{\sqrt{3}} (z_{25} + z_{35}) \leq \left( \frac{2}{\sqrt{3}} - 1\right)(e(x_r) - w(x_l) - 2\theta_b(x_r))
\]

Finally, if \( t(x) = t_{*1} \) or \( t(x) = t_{*4} \) for some \( x \) in \([x_l, x_r]\), then \( t(x_q) \) is one of \( t_{21}, t_{34} \) and \( t(x_{q'}) \) is one of \( t_{21}, t_{34} \). Thus, we can combine (18) and (19) to obtain inequality (17).

We assume now that \( t(x) \neq t_{*1}, t_{*4} \). We can also assume that inequality (17) does not hold which implies
\[
\frac{2}{\sqrt{3}} (z_{15} + z_{40}) > \left( \frac{4}{\sqrt{3}} - 2\right)(z + r(x_r) + r(x_l) - \theta_b(x_r) - \theta_f(x_l))
\]

which, in turn implies:
\[
(2 - \frac{2}{\sqrt{3}})(z_{15} + z_{40}) > \left( \frac{4}{\sqrt{3}} - 2\right)(r(x_r) + r(x_l) - \theta_b(x_r) - \theta_f(x_l))
\]

Now, \( \Delta P(x_l, x_r) = U(x_r) - U(x_l) + L(x_r) - L(x_l) \). Because \( t(x) = t_{40}, t_{45} \) for all \( x \in [x_l, x_r] \), \( U(x_r) - U(x_l) \leq \frac{2}{\sqrt{3}}(x_r - x_l) \). In order to bound \( L(x_r) - L(x_l) \), we consider the path \( P \) from point \( l(x_l) \) that lies on side 1 of \( H(x_l) \) to point \( l(x_r) \) that lies on side 4 of \( H(x_r) \): \( l(x_l), u(x_l) = u_l, u_{l+1}, \ldots, u_{l+j} = u(x_r), l(x_r) \). Then, if \( |P| \) is the length of path \( P \):
\[
L(x_r) - L(x_l) \leq d_S(l(x_l), x_l) + |P| + d_S(l(x_r), x_r)
\]
\[
\leq d_S(\ell(x_1), x_1) + |(\ell(x_1), u(x_1))| - d_N(u(x_1), x_1) + \frac{2}{\sqrt{3}}(x_r - x_l) \\
- d_N(u(x_r), x_r) + |(\ell(x_1), u(x_1))| + d_S(\ell(x_r), x_r)
\]

Now, \(d_S(\ell(x_1), x_1)+|\ell(x_1), u(x_1)|-d_N(u(x_1), x_1)\) is bounded by \((2-\frac{2}{\sqrt{3}})(r(x_1)-\theta_f(x_1))\) and \(-d_N(u(x_r), x_r) + |\ell(x_1), u(x_1)| + d_S(\ell(x_r), x_r)\) is bounded by \((2-\frac{4}{\sqrt{3}})(r(x_r)-\theta_b(x_r))\). Combining the bounds on \(U(x_r)-U(x_1)\) and \(L(x_r)-L(x_1)\), we get

\[
\Delta P(x_1, x_r) \leq (2 - \frac{2}{\sqrt{3}})(r(x_1) + r(x_r) - \theta_f(x_1) - \theta_b(x_r)) + \frac{4}{\sqrt{3}}(x_r - x_l) \\
\leq (\frac{4}{\sqrt{3}} - 2)(r(x_1) - \theta_f(x_1) + r(x_r) - \theta_b(x_r)) + (\frac{10}{\sqrt{3}} - 2)(x_r - x_l)
\]

where the last inequality holds if

\[
(\frac{6}{\sqrt{3}} - 2)(z_{15} + z_{40}) > (4 - \frac{6}{\sqrt{3}})(r(x_r) + r(x_1) - \theta_b(x_r) - \theta_f(x_1)).
\]

This inequality is equivalent to \(20\). \(\square\)

We can now provide a proof of the Main Lemma. (Recall that at the beginning of this section we simplified the notation and set \(T_{1n}\) to refer to \(T_{ij}\)).

**Proof of Lemma \([4.4]\).** The first part follows from Lemma \([4.2]\). For the second part, note that \(x(q)\) is a critical point and that \(r(x(q)) = \theta_f(x(q)) = 0\) and \(w(x) = x(q)\). Therefore, by Lemma \([4.6]\) \(P(x(q)) \leq (\frac{10}{\sqrt{3}} - 2)x(q)\). Since \(2d_{T_{ij}}(p, q) = P(x(q))\), the lemma follows. \(\square\)

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