Quantum resource theory is a cutting-edge tool used to study practical implementations of quantum mechanical principles under realistic operational constraints. It does this by modelling quantum systems as restricted classes of possible or permissible experimental operations. Modal logic provides a formal tool for studying possibility and impossibility in a completely general logical setting. Here, I show that quantum resource theories may be functorially translated into models of variable-domain S4 modal logic in a way that provides a new class of formal techniques for exploring quantum resource-theoretic problems. I then extend this functorial relationship to an injective one by adding structure to these logical models to reflect the convertibility preorder of resources in the underlying resource theory. I conclude by discussing how this viewpoint may be deployed concretely.

The operationalization of quantum theory to a theory centered around agent-based informational measures and operational constraints has proven to be one of the most significant conceptual advancements in the practical deployment of the theory for practical applications. Indeed, operational quantum information theory provides the backbone for most applications in quantum cryptography, computing, communication, and many other diverse settings Nielsen and Chuang [2010], Wilde [2013], Bruss and Leuchs [2019]. Moreover, this operationalization resembles the historical operationalization of thermodynamics, wherein thermodynamic phenomena were first understood in terms of human intervention on particular physical systems. In this way, there is a natural relation between quantum information and thermodynamics, which have merged to form modern quantum thermodynamics (see, for instance Duffner and Campbell [2019]), which is itself a rapidly developing field in close contact with quantum resource theory Ng and Woods [2018].

Quantum resource theory essentially takes this lesson – of formulating quantum theory in terms of constrained classes of physical operations – and generalizes it to a unified formalism wherein this pragmatically-oriented view of quantum theory may be realized. This formalism is very powerful and has been successfully applied to in many diverse settings to study entanglement Vedral et al. [1997], non-Gaussianity Genoni and Paris [2010] coherence Baumgratz et al. [2014], computation Veitch et al. [2014], and contextuality Amaral [2019], among other things.

This reduction of quantum theory to constrained classes of interventions is essentially a selection of possible operations which are allowed to be carried out. Such a stipulation carries with it a dual notion of disallowed or impossible operations. These notions of possibility and impossibility – modality in general – are essentially the important features of the theory.

In a completely separate domain of research (mathematical and philosophical logic) questions about the general formalization of possibility, necessity, and modality, have been studied in great detail using modal logic. On the philosophical side, modal logic has been used to provide formal insights into many philosophically interesting questions Hintikka [1962], Lewis [1973], Kripke [2012], Sider [2010], Burgess [2012]. In mathematics literature, modal logic has been realized as being in close connection with intuitionistic logic Gödel [1986] and has been explored using topos theory Awodey et al. [2014], Goldblatt [2006] topological semantics Kremer and Mints [2005], Awodey and Kishida [2008], homotopy type theory Univalent Foundations Program [2013], Corfield [2020] and many other such
contexts Goldblatt [2003] proving it to be a rich source of mathematical structure.

Recently, modal logic has been successfully deployed in the study of a variety of foundational aspects of quantum theory Nurgalieva and del Rio [2019], Boge [2019]. I here seek to make a further contribution to this growing view that modal logic may exploited to make amenable particular issues in quantum information and quantum foundations. Specifically, I here show that the modality present in quantum resource theories (namely, in the scope of possible interventions which characterize a quantum resource theory) is in fact enough to recover the majority of their mathematical structure. That is, I show that there is a manner of functorially interpreting quantum resource theories as models of variable-domain modal logic. By then considering such variable-domain models with an added preorder structure on their global domains (induced by the convertibility preorder on quantum states from quantum resource theory), I show that the given functor may be extended to be injective. I then provide some preliminary results indicating just how this correspondence may be used to pose resource-theoretic questions in the language of modal logic.

Given the intellectual distance between quantum resource theory and modal logic, I begin with a short introduction to both.

1 Quantum Resource Theories

Quantum resource theory provides a formalism for studying quantum mechanical protocols under different kinds of operational constraints. Briefly, a Quantum Resource Theory (QRT) is a collection of permitted operations on specified quantum systems; states which may be generated using only those permitted operations are called free states, and those which may not are called resources.

The usual example of a QRT is where two observers are situated in separate laboratories and they are only able to communicate via classical channels. Then no matter what local quantum mechanical operations they perform, they can never create a quantum state which jointly entangles their separate laboratories. Thus, entanglement is a resource. However, if they share such an entangled state to begin with, then still using their local and classical operations, they can carry out teleportation protocols which would otherwise be impossible, hence why such a resource is resourceful.

In this basic setting, there is a relevant collection of Hilbert spaces as \( H = \{ \mathbb{C}, H^A, H^B, H^A \otimes H^B \} \) where \( H^A \) and \( H^B \) are the lab Hilbert spaces of the two observers, respectively, and a class of permissible operations (i.e. quantum channels) \( \mathcal{O} \) which consists of local operations and classical communication (LOCC). Essentially, any channel may be permitted on \( H^A \) or \( H^B \), but only classical communication operations are permitted between the two; thus any operation on \( H^A \otimes H^B \) must be separable, among other things Chitambar et al. [2014].

This example provides the basic formula for describing a quantum resource theory; first, one specifies the Hilbert spaces of the systems under consideration (this step is usually left implicit), and then they describe the class of permitted operations on these systems in terms of quantum channels. Once this has been done, there is a class of free states which may be generated using only those permitted operations. Then all other states on the relevant systems are deemed resources (in the above example, for instance, entangled states are resources).

QRTs operationalize quantum theory by posing questions about quantum systems purely in the pragmatic language of interventions which may be carried out in the laboratory. Indeed, while the above heuristic referred to quantum states, the Choi-Jamiołkowski isomorphism ensures the existence of a full channel-state duality Jamiołkowski [1972], Choi [1975], Jiang et al. [2013]. Thus, it is enough to speak entirely in the language of channels (channels which ‘prepare’ certain states are given as channels from \( \mathbb{C} \) to the relevant Hilbert space).

The usual Hilbert space formalism of quantum resource theories Chitambar and Gour [2019] characterizes particular QRTs in terms of (i) the Hilbert spaces on which the relevant quantum systems are defined, (ii) the available channels by which hypothetical agents are taken to be allowed to intervene on those systems, and (iii) the usual collection of states which may be freely prepared and manipulated via those channels. I proceed with a few definitions.

**Definition 1.0.1.** Let \( H \) be a Hilbert space. A
state $\rho \in B(H)$ is any positive semi-definite, self-adjoint operator with $Tr\rho = 1$.

The collection of all states on $H$ is denoted $S(H)$.

**Definition 1.0.2.** A quantum channel between systems $A$ and $B$ defined on Hilbert spaces $H^A$ and $H^B$ respectively is a completely positive trace preserving (CPTP) map $\Phi : B(H^A) \rightarrow B(H^B)$.

A map $\Phi$ is positive on $B(H)$ if it takes positive operators to positive operators. It is completely positive if for every $k \geq 1$, the induced map $\Phi_k : M_{k \times k}(B(H^A)) \rightarrow M_{k \times k}(B(H^B))$ which takes $T_{ij} \mapsto \Phi(T_{ij})$ is positive.

**Definition 1.0.3.** Given a collection $\mathcal{O}$ of quantum channels over a particular class of Hilbert spaces $H$, the induced free states $F$ are the states $\rho \in S(H)$ for any $H \in H$ such that there exists some $\Phi \in \mathcal{O}$ with $\Phi : C \rightarrow B(H)$ and $\rho \in Im(\Phi)$.

The shorthand notation $\mathcal{O}(A \rightarrow B)$ shall be adopted to denote channels between of the form $\Phi : B(H^A) \rightarrow B(H^B)$. Likewise, $\mathcal{O}(A)$ shall denote $\mathcal{O}(A \rightarrow A)$ and $1_A$ shall denote the identity channel in $\mathcal{O}(A)$.

When QRTs are defined, $H$ is not usually considered an explicit feature of the theory. Indeed, $H$ is often taken to be quite large, and perhaps closed under Hilbert space tensor products, and so on. However, for the discussion to follow, we shall see that this implicit feature must be made explicit.

**Definition 1.0.4.** A quantum resource theory is a triple $\langle H, O, F \rangle$ of quantum channels and free states over a collection of specified Hilbert spaces $H = \{H^\alpha\}$ such that, for every Hilbert space $H^\alpha \in H$, $1_{\alpha} \in O(\alpha)$ and if $\Phi \in O(A \rightarrow B)$ and $\Psi \in O(B \rightarrow C)$, then $\Psi \circ \Phi \in O(A \rightarrow C)$.

These conditions on $O$ ensure that doing nothing is a permitted operation (given by the identity channel), and any two operations may be composed with one another. In a sense, we see already that QRTs must be reflexive (you can always ‘transform’ a state to itself) and transitive (under channel composition). This will serve to be useful in the discussion of modal logic to come.

The resource states of a QRT are given by the states in $R = \bigcup_\alpha S(H^\alpha) - F$. Quantum resource theories under this construal may be isomorphic to one another in the following sense:

**Definition 1.0.5.** Two quantum resource theories $\langle H, O, F \rangle$ and $\langle H', O', F' \rangle$ are isomorphic if there exists a set isomorphism $\varphi_H : H \rightarrow H'$ where $\varphi_H(H^\alpha) \equiv H$ (with $\varphi_A \rightarrow_A : H^A \rightarrow \varphi_H(H^A)$ denoting the induced Hilbert space isomorphism) such that $\Phi \in O(A \rightarrow B)$ if and only if $\varphi_B \rightarrow B' \circ \Phi \circ \varphi_A \rightarrow A \in O'$.

This essentially means that two QRTs $X$ and $Y$ are isomorphic when their collections of Hilbert spaces may be paired up with each other in such a way that every Hilbert space in $X$ is paired with an isomorphic one in $Y$, and the quantum channels in $X$ can be pushed forward to the channels in $Y$. This is a fairly string notion of isomorphism. For a pair of such isomorphic QRTs, $O(A' \rightarrow B')$ shall denote $O(\varphi_H(H^A) \rightarrow \varphi_H(H^B))$ where $\varphi_H$ is the set bijection between their respective classes of Hilbert spaces.

Given QRTs $A = (H, O, F)$ and $B = (H', O', F')$, we say that $B$ is a sub-QRT of $A$ and write $B \subseteq A$ when $H' \subseteq H$ and $O' = O \cap \mathcal{H}'$ (and hence $F' \subseteq F \cap (\bigcup_{H^\alpha \in H} S(H^\alpha))$, as determined by the restricted class of channels). If $B \subseteq A$, then a projection $\pi : A \rightarrow B$ is a map which identically restricts $H$ to $H'$ and preserves the surviving channel structures.

If one were to view a QRT as a graph whose vertices are the elements of $H$ and whose edges are the channels in $O$, a projection then looks like a restriction to a subgraph. With this heuristic in mind, given a QRT, a projection determines a unique sub-QRT, but there may be multiple distinct projections onto sub-QRTs which are isomorphic to one another.

Likewise, one may define an inclusion $i : B \rightarrow A$ which identically maps the QRT $B$ (understood as a separate QRT) into itself (as a sub-QRT of $A$). Once again, there may be multiple distinct but isomorphic inclusions of a particular QRT into another.

With these considerations in mind, we may define a category whose objects are isomorphism classes of QRTs, and whose arrows are inclusions into larger QRTs and projection down onto sub-QRTs, which I shall denote $\textbf{QRT}$.

In addition to these basic definitions, a popular direction which has been pursued in the abstract study of quantum resource theories is the study of

\footnote{Note that there is a natural inverse of this isomorphism given by $\varphi_A \rightarrow A$.}
of resource monotones and resource convertibility. Often, one is not so interested in the particular sequence of operations necessary to manipulate resource in a particular way. Rather, they are concerned with understanding which resources they may create provided they already have some other resource. Ordering resources by their convertibility provides insight into how operationally ‘valuable’ a particular resource is in a given context.

There is a natural preorder Chitambar and Gour [2019] on resource states which reflects exactly this fact; given a QRT \( \langle H, O, F \rangle \) with a states \( S \), for any \( \rho, \sigma \in S \), we write \( \rho \overset{\Phi}{\rightarrow} \sigma \) if there is some \( \Phi \in O \) with \( \sigma = \Phi(\rho) \). Then \( \overset{\Phi}{\rightarrow} \) is a preorder on \( S \) which is closed on the class of resource states \( R \) in the sense that if \( \sigma \) is a resource state, so too is \( \rho \). Given this preorder, one may then study the monotones – order respecting real functions over the class of resource states – of a given resource theory which indicate other qualitative features of the theory.

Coeke et al. have constructed a full-blown categorical articulation of a general theory of resource convertibility using commutative pre-ordered monoids Coeke et al. [2016]. They claim that any manner of measuring resources is essentially the same as describing features of this preorder structure. We shall see in Theorem 3.1 that this preorder structure, if added to the modal logic framework described below, is sufficient to completely recover the full structure of any QRT up to isomorphism without reference to the Hilbert space formalism.

I now introduce the basic features of variable-domain modal logic.

2 Modal Logic

Modal logic provides a formal setting wherein philosophers and logicians alike are able to speak formally about possibility and necessity. Essentially, modal logic proceeds by carrying the usual syntactic constructions for classical logic with an added possible-worlds structure and the introduction of additional ‘modal’ operators. These possible worlds represent copies of the underlying classical logic which may be semantically distinct from each other. The way in which these possible worlds are connected to each other then provides a natural interpretation of modal terms using the modal operators. I should note that, while the term ‘possible worlds’ may seem very mystical, it merely refers to a particular sort of formal semantics.

Noting that any classical logical connective may be expressed in terms of any other with suitable use of brackets and negation \( \neg \), I here suppose for simplicity that the only connective symbol in the logical language to be considered is the conditional \( \rightarrow \). Likewise, I suppose the usual underlying classical propositional logic axioms, and take modus ponens to be the only classical rule of inference (there will be an added modal axiom and rule of inference as well).

Possible worlds are constructed in the following manner: one creates a collection of ‘worlds’ \( \mathcal{W} \) and a binary ‘accessibility’ relation \( \mathcal{R} \) on \( \mathcal{W} \), and adds two new symbols \( \Box \) (the ‘necessity’ operator) and \( \Diamond \) := \( \neg \Box \neg \) (the ‘possibility’ operator) to the language. These new operators are interpreted such that \( \Box \phi \) means ‘necessarily \( \phi \)’ and \( \Diamond \phi \) means ‘possibly \( \phi \).’ These operators are connected to the possible worlds as follows: \( \Diamond \phi \) is true at a world \( w \in \mathcal{W} \) (that is, \( \phi \) is possible in \( w \)) if there is a world \( u \) which is accessible to \( w \) wherein \( \phi \) is true. Necessity of \( \phi \) (the formula \( \Box \phi \)) is interpreted in a similar manner, but instead requiring that \( \phi \) is true in all worlds which are accessible to \( w \).

Truth valuation then occurs at each world separately (though in the modal logic setting considered here, atomic symbols will have a global truth value\(^5\)). In this article, I am concerned with Variable-Domain Modal Logic (VDML), which may readily be extended to fully quantified predicate modal logic [Sider, 2010, pp. 308 – 314]. Loosely following Sider [2010], a model of this system is defined as follows:

**Definition 2.0.1.** A VDML-model is a 5-tuple \( \mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{D}, \mathcal{D}, \Phi \rangle \) where:

- \( \mathcal{W} \) is a non-empty set (called possible worlds).
- \( \mathcal{R} \) is a binary relation on \( \mathcal{W} \) (an inter-word accessibility relation).
- \( \mathcal{D} \) is a non-empty set (a global domain of atomic propositional symbols).

\(^5\)There are many different kinds of modal logic, some of which have world-dependent truth interpretations for atomic symbols, but we are concerned here only with variable-domain modal semantics which uses rigid designators.
• \( \mathcal{D} : \mathcal{W} \rightarrow \mathcal{P}(\mathcal{D}) \) is a function that assigns a sub-domain \( \mathcal{D}_w \) to every world \( w \in \mathcal{W} \).

• \( \mathcal{I} : \mathcal{D} \rightarrow \{0, 1\} \) is a truth interpretation function on atomic symbols.

The domain \( \mathcal{D}_w \) of a world \( w \) is essentially the restriction of the language of logical discourse available at that world. For formulas which do not contain modal operators, the syntax of this system is given by the usual one for classical propositional logic. When modal operators are present, there is one additional axiom schema for deductions, called the \( K \) axiom:

\[
\vdash \Box(\phi \rightarrow \psi) \rightarrow (\Box \phi \rightarrow \Box \psi)
\]

for all formulas \( \phi \) and \( \psi \). Likewise, there is one additional rule of inference, called the necessitation rule:

\[
\phi \vdash \Box \phi
\]

Validity in this system, which is defined at a particular world, is then given by the following valuation function.

**Definition 2.0.2.** In a model \( \mathcal{M} \), for any atomic symbol \( \alpha \in \mathcal{D} \) and formulas \( \phi \) and \( \psi \) at any world \( w \in \mathcal{W} \), the valuation function \( V_M \) is given inductively by

- \( V_M(\alpha, w) = \mathcal{I}(\alpha) \).
- \( V_M(\neg \phi, w) = 1 \) iff \( V_M(\phi, w) = 0 \).
- \( V_M(\phi \rightarrow \psi, w) = 1 \) iff either \( V_M(\phi, w) = 0 \) or \( V_M(\psi, w) = 1 \).
- \( V_M(\Box \phi, w) = 1 \) iff for ever \( v \in \mathcal{W} \), if \( (w, v) \in \mathcal{R} \), then \( V_M(\phi, v) = 1 \).
- \( V_M(\Diamond \phi, w) = 1 \) iff there exists some \( v \in \mathcal{W} \) with \( (w, v) \in \mathcal{R} \) and \( V_M(\phi, v) = 1 \).

If some formula \( \phi \) is valid in a modal \( \mathcal{M} \) at all worlds of some model \( \mathcal{M} \), we write \( \models_{\mathcal{M}} \phi \). If \( \phi \) is valid in all worlds of all models (for instance, tautologies of classical propositional logic), then we simply write \( \models \phi \). If \( \phi \) is valid in a model \( \mathcal{M} \) whenever \( \psi \) is valid, we may write \( \psi \models_{\mathcal{M}} \phi \).

It should be noticed that these models do not include quantification or predication. However, such features may readily be added at a layer of the semantics and syntax which is detached from the underlying structure which is needed for comparison with quantum resource theory. I thus do not define all of this here for the sake of brevity, but note that it is generically possible. VDML-models may be isomorphic in the following way:

**Definition 2.0.3.** Two VDML-models \( \mathcal{M} = (\mathcal{W}, \mathcal{R}, \mathcal{D}, \mathcal{I}) \) and \( \mathcal{M}' = (\mathcal{W}', \mathcal{R}', \mathcal{D}', \mathcal{I}') \) are isomorphic \( (\mathcal{M} \cong \mathcal{M}') \) if there exists a pair of bijections \( \varphi_W : \mathcal{W} \rightarrow \mathcal{W}' \) and \( \varphi_D : \mathcal{D} \rightarrow \mathcal{D}' \) with

- \( (w, u) \in \mathcal{R} \) iff \( (\varphi_W(w), \varphi_W(u)) \in \mathcal{R}' \).
- \( \varphi_D(\mathcal{D}_w) = \mathcal{D}'_{\varphi_W(w)} \)
- \( \mathcal{I}(\alpha) = \mathcal{I}'(\varphi_D(\alpha)) \) for all \( \alpha \in \mathcal{D} \)

This final condition ensures that the truth valuation is equivalent in isomorphic models. In the comparison with quantum resource theories, this will turn out to be necessary in encoding the free states of a QRT in a logical model.

Given a pair of VDML-models \( \mathcal{M} \) and \( \mathcal{M}' \), we say that \( \mathcal{M}' \) is a sub-model of \( \mathcal{M} \) and write \( \mathcal{M}' \subseteq \mathcal{M} \) when \( \mathcal{W}' \subseteq \mathcal{W} \), \( \mathcal{R}' = \mathcal{R} \setminus (\mathcal{W}' \times \mathcal{W}') \), \( \mathcal{D}' = \mathcal{D}_w \) for all \( w \in \mathcal{W}' \), and \( \mathcal{I}'(\alpha) = 1 \) for \( \alpha \in \mathcal{D}' \) only if \( \mathcal{I}(\alpha) = 1 \).

A VDML-model is called a \( \text{VDS}_4 \)-model if the accessibility relation \( \mathcal{R} \) is both reflexive and transitive. The collection of all isomorphism classes of \( \text{VDS}_4 \)-models form a category whose arrows are inclusions into larger models and projections onto sub-models, which I shall denote \( \text{VDS}_4 \).

### 3 Functoriality

Here, I show that the class of all quantum resource theories is related to the class of all \( \text{VDS}_4 \)-
models by a functor $F : \text{QRT} \to \text{VDS4}$, determine how close it is to being injective and surjective. I then show that the so-called convertibility preorder of states in a quantum resource theory provides enough additional structure to a VDS4-model to single it out uniquely (up to isomorphism). Thus, I construct an injective functor into the category of VDS4-models with additional preorder structure added. I use this to show that the several intuitions about quantum resource theory in general translate cleanly into the modal logic framework.

Before doing this, I pause to make clear just why this procedure is natural. Essentially, the Hilbert spaces of a QRT correspond to particular quantum systems which agents may intervene upon. Thus, there is a sense in which they may be treated as individual, physically distinct ‘worlds’ whose underlying logical variables are the states on those worlds. However, the channels which these quantum agents have access to may allow them to communicate between different systems. In this way, there is a certain notion of inter-world accessibility. Thus, the possible-worlds semantics is a natural tool with which to model this phenomenon. In general, totally distinct systems have logically distinct states, and so the ‘logical’ language of studying each ‘world’ ought to be distinct. However, given that some systems may be viewed as subsystems, it is natural to suppose that these ‘logical variables’ are not completely separate, but may overlap. It is for this respect that a variable domain approach is natural. I now make this correspondence precise with a functor I shall denote by $F$.

Let $F$ be defined as follows. Given a QRT $(H, O, F)$, let $\mathcal{H} = H$ and take $\mathcal{D}_H = S(H)$. Then take $\mathcal{O} = \{\langle \mathcal{H}^A, \mathcal{H}^B \rangle | (\exists \Phi \in O(A \to B))\}$. Take the truth valuation function to then be $\mathcal{I}_F(\rho) = 1$ iff $\rho \in F$. Take $\mathcal{D}_H = \bigcup_{H \in H} \mathcal{D}_H$ and thus $\mathcal{D}_H(H) = \mathcal{D}_H$. Then define $F$ by

$$F((H, O, F)) := (\mathcal{H}, \mathcal{O}, \mathcal{D}_H, \mathcal{D}_H, \mathcal{I}_F).$$

It is easy to check that the image of $F$ is a VDML-model.

**Theorem 3.1.** The map $F : \text{QRT} \to \text{VDS4}$ is a functor.

**Proof.** Since $1_\alpha \in O$ for all $\mathcal{H}^\alpha \in H$, and since $O$ is transitive, we see that the constructed accessibility relation $\mathcal{R}$ is reflexive and transitive, whence $F((H, O, F))$ is a VDS4-model (and not just a VDML-model). Thus, functoriality of $F$ amounts to showing (i) that the image of $F$ on two isomorphic QRTs are isomorphic as VDS4-models (since objects are defined as isomorphism classes), and (ii) that for any arrow $f \in \text{QRT}$, $F$ induces an arrow $F(f) \in \text{VDS4}$.

For (i), let $X$ and $Y$ be QRTs and suppose $X = (H, O, F) \cong (H', O', F') = Y$. If we denote

$$F(X) = (\mathcal{H}, \mathcal{O}, \mathcal{D}_H, \mathcal{D}_H, \mathcal{I}_F)$$

$$F(Y) = (\mathcal{H}', \mathcal{O'}, \mathcal{D}_H', \mathcal{D}_H', \mathcal{I}_F')$$

we see that QRT isomorphism of $X$ and $Y$ implies the existence of some bijection $\varphi_H$ which defines an isomorphism between $\mathcal{H}$ and $\mathcal{H}'$ with $\varphi_H(\mathcal{H}) \cong \mathcal{H}'$ such that $\Phi \in O(A \to B)$ if and only if $\varphi_{B \to B'} \circ \Phi \circ \varphi_{A' \to A} \in O'(A' \to B')$. Immediately, we note that $\varphi_H$ defines an isomorphism between the collections of worlds (Hilbert spaces) of the respective models related by $F$. Moreover, the Hilbert space isomorphisms $\varphi_{A \to A'}$ induced by $\varphi_H$ lift to isomorphisms between $B(\mathcal{H}^A)$ and $B(\varphi_H(\mathcal{H}^A))$, and hence between their sets of states. Then $\varphi_{A \to A'}$ is an isomorphism between domains; we have for every $\mathcal{H}^A \in H$,

$$\varphi_{A \to A'}(\mathcal{D}_{\mathcal{H}^A}) = \varphi_{A \to A'}(S(\mathcal{H}^A))$$

$$= S(\varphi_H(\mathcal{H}^A))$$

$$= \mathcal{D}_{\varphi_H(\mathcal{H}^A)}$$

so the respective sub-domains (and hence the respective super-domains) are equivalent in the appropriate sense.

Finally, if $(\mathcal{H}^A, \mathcal{H}^B) \in \mathcal{O}$, then there is some $\Phi \in O(A \to B)$ between $B(\mathcal{H}^A) \to B(\mathcal{H}^B)$. But then $\varphi_{B \to B'} \circ \Phi \circ \varphi_{A' \to A} \in O'(A' \to B')$, whence $\langle \varphi_H(\mathcal{H}^A), \varphi_H(\mathcal{H}^B) \rangle \in \mathcal{R}'$. But then all three conditions are satisfied ensuring that the resulting VDS4-models are isomorphic. Whence, $F(A) \cong F(B)$, so $F$ takes isomorphism classes of QRTs to isomorphism classes of VDS4-models.

I now show (ii). Clearly, for any QRT $X$, $F(1_X) = 1_{F(X)}$ since $F(1_X(X)) = F(X) = 1_{F(X)}F(X)$. All other arrows are either projections or inclusions. I start with projections. Suppose that $B \leq A$ and thus there is some projection arrow $\pi_1 : A \to B$. Then we have that $\pi_1(A) = B$ and so $F(\pi_1(A)) = F(B)$, whence the following diagram commutes:
where $\pi_2 = F(\pi_1)$. It remains to be shown that $\pi_2$ is an arrow in $\text{VDS}4$. But since $\mathcal{H}' \subseteq \mathcal{H}$ and $\mathcal{O}' = \mathcal{O} \upharpoonright \mathcal{H}'$, the induced worlds have $\mathcal{W}' = \mathcal{H}' \subseteq \mathcal{H} = \mathcal{W}$ and $\mathcal{R}' = \mathcal{R} \upharpoonright (\mathcal{W}' \times \mathcal{W}')$. Likewise, $\mathcal{D}_H = \mathcal{H} = \mathcal{D}_H'$ for any $H \in \mathcal{H}$ so the domain condition for $\text{VDS}4$ sub-models is satisfied. Moreover, since $\mathcal{O}' \subseteq \mathcal{O}$, we see that the only change to $\mathcal{F}$ in the projection to $\mathcal{F}'$ is that some channels get ‘turned off’, whence $\mathcal{F}' \subseteq \mathcal{F}$ and so $\mathcal{F}(\alpha) = 1$ only if $\mathcal{F}(\alpha) = 1$ for any $\alpha \in \mathcal{D}_H'$. Thus, $F(A) \leq F(B)$ and so $\pi_2$ is a projection map in $\text{VDS}4$.

Now suppose $\iota_1 : B \rightarrow A$ is an inclusion map. Then $\iota_1(B) = A$, and so the following diagram commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{F} & F(A) \\
\downarrow{\pi_1} & & \downarrow{\pi_2} \\
B & \xrightarrow{F} & F(B)
\end{array}
$$

where $\iota_2 = F(\iota_1)$. Identical reasoning as above shows that $\iota_2$ is an injection arrow in $\text{VDS}4$. Thus, $F$ takes arrows to arrows. Therefore, $F$ is a functor.

I now determine certain features of this functor. I first determine how close this functor is to being injective, and then how close it is to being surjective.

Theorem 3.2. Let $X = \langle \mathcal{H}, \mathcal{O}, \mathcal{F} \rangle$ and $Y = \langle \mathcal{H}', \mathcal{O}', \mathcal{F}' \rangle$ be two QRTs. Then $F(X) \cong F(Y)$ if and only if (i) $\mathcal{H} \cong \mathcal{H}'$, (ii) for each $\mathcal{H}'' \in \mathcal{H}$, $\mathcal{F}' = \varphi_{\alpha''} \circ (\mathcal{F})$, and (iii) for every $\Phi \in \mathcal{O}(A \rightarrow B)$, there exists a collection $\{\Psi_{\alpha} \subseteq \mathcal{O}'(A' \rightarrow B')_{\alpha} \}$ such that $\text{Im}(\Phi) = \bigcup_{\alpha} \text{Im}(\varphi_{B' \rightarrow B} \circ \Psi_{\alpha} \circ \varphi_{A \rightarrow A'})$.

This shows that $F$ is almost injective, but the last condition says that some of the information about the QRT is lost under $F$, whence injectivity fails. I will show later that this information is precisely the state convertibility preorder.

Proof. To begin, if $\mathcal{H} \not\cong \mathcal{H}'$, then the induced classes of worlds will have $\mathcal{W}_H \not\cong \mathcal{W}'_H$, whence $F(X) \not\cong F(Y)$. Likewise, if $\mathcal{F}' \not\cong \varphi_{\alpha''} \circ (\mathcal{F})$, then the induced truth valuations $\mathcal{J}_\mathcal{F}$ and $\mathcal{J}_\mathcal{F}'$ will disagree, whence $F(X) \not\cong F(Y)$. Finally, if condition (iii) fails, there will be some channel $\Phi \in \mathcal{O}(A \rightarrow B)$ for which $\Psi = \varphi_{B \rightarrow B'} \circ \Phi \circ \varphi_{A \rightarrow A'} \not\in \mathcal{O}'(A' \rightarrow B')$, whence $F(X) \not\cong F(Y)$. Therefore conditions (i) – (iii) are necessary.

Conversely, if conditions (i)–(iii) are met, then we see that condition (i) ensures that $\mathcal{W}_H \cong \mathcal{W}'_H$, and likewise, that the respective variable domains are also isomorphic, whence the VDML isomorphism conditions on $\mathcal{W}$, $\mathcal{D}$, and $\mathcal{Q}$ are satisfied. Condition (ii) then ensures that $\mathcal{J}_\mathcal{F} = \mathcal{J}_{\mathcal{F}'} \circ \varphi_{\alpha''} \circ (\mathcal{F})$ and so the isomorphism condition on $\mathcal{J}$ will be satisfied. Thus, all that needs to be shown is that $(\mathcal{H}_A, \mathcal{H}_B) \in \mathcal{R}_\mathcal{O}$ if and only if $(\varphi_{\mathcal{H}}(\mathcal{H}_A), \varphi_{\mathcal{H}}(\mathcal{H}_B)) \in \mathcal{R}_\mathcal{O}$.

However, two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ are related by $\mathcal{R}_\mathcal{O}$ if and only if there is some channel $\Phi \in \mathcal{O}(A \rightarrow B)$. But condition (iii) ensures that, whenever there is such a channel, there is at least one channel (possibly many in the set $\{\Psi_{\alpha} \}$) in $\mathcal{O}(A' \rightarrow B')$. Thus the requisite condition on the relation $\mathcal{R}$ are satisfied as well, whence $F(X) \cong F(Y)$. Therefore conditions (i) – (iii) are sufficient.

To see the kind of pathology which obstructs injectivity for $F$, consider the following form of counter-example. Let $\mathcal{H} = \{C, \mathcal{H}_A, \mathcal{H}_B\}$. Then there is one QRT on this collection of Hilbert spaces, $X$, for which $\mathcal{O}(A \rightarrow B)$ only includes one channel $\Phi$ (suppose that there are other channels out of $C$ to ensure that $F$ is non-empty). There may, however, be another QRT $Y$ which is identical to $X$ in every way (including the channels from $C$ which see the free states) except with $\Phi$ replaced with $\xi \circ \Phi$ where $\xi$ is some automorphism on $B(\mathcal{H}_A)$ such that $F$ is invariant under $\xi$. Then one can readily check $F(X) \cong F(Y)$, even though $A \not\cong B$ (because $\varphi_{B \rightarrow B'} \circ \Phi \circ \varphi_{A \rightarrow A'} \not\in \mathcal{O}(A' \rightarrow B')$), whence injectivity of $F$ is violated. However, $X$ and $Y$ still satisfy property (iii) from the above theorem. These free-state preserving automorphisms seem generically to be the only kind of behaviour which prevents injectivity.

I now determine the obstructions to surjectivity of $F$. In the $\text{VDS}4$ setting, for a given world $w$, let $T(w) := \{p \in \mathcal{D}_a | \mathcal{J}(p) = 1\}$ (i.e. the collection of atomic symbols which are true under the interpretation $\mathcal{J}$ in the domain of $w$). Then we have the following.

Theorem 3.3. Let $\mathcal{M} \in \text{VDS}4$. Then there exists some $X \in \text{QRT}$ such that $\mathcal{M} = F(X)$ only if for all $w, u \in \mathcal{W}$ with $\langle w, u \rangle \in \mathcal{R}$, either
if a VDS4-model $C \emptyset$ it, viewed as a world in the VDS4-model $D$ for $w \neq c$ such that $\langle c, w \rangle \in \mathcal{W}$ for all $w$ with $T(w) \neq \emptyset$.

It is here that we see how quantum resource-theoretic considerations limit modality.

Proof. Suppose first that $T(u) = \emptyset$ and that $T(w) \neq \emptyset$. Then for any QRT $X = \langle H, \mathcal{O}, \mathcal{F} \rangle$, $\mathcal{O}$ and $\mathcal{F}$ are fixed such that $\Phi(\mathcal{F} \cap S(H)) \cap \mathcal{R} = \emptyset$ for all $\Phi \in \mathcal{O}$ and $H \in H$. We see that $T(w) = \emptyset$ implies $\mathcal{F} \cap S(H_u) = \emptyset$ and so $S(H_u) \subseteq \mathcal{R}$ while $T(w) \neq \emptyset$ implies $\mathcal{F} \cap S(H_w) \neq \emptyset$.

But under the assumption that $\langle w, w \rangle \in \mathcal{R}$, there must be some channel $\Phi \in \mathcal{O}(w \rightarrow u)$. However, this channel necessarily is such that $\Phi(\mathcal{F} \cap S(H_u)) \cap \mathcal{R} \neq \emptyset$. If $\mathcal{F} \cap S(H_w) \neq \emptyset$, then there is some free state in $S(H_w) \neq \emptyset$, but all states in the image of $\Phi$ are resource states. Thus it is possible to transform a free state into a resource state, a contradiction. Therefore, any such $\Phi$ violates the necessary compatibility between $\mathcal{O}$ and $\mathcal{F}$. Thus, there can be no such QRT whose image under $F = M$.

I now look at the condition on the world $c$. Suppose $X = \langle H, \mathcal{O}, \mathcal{F} \rangle$ is some QRT. Then if $X$ has any free states, there must be a copy of $\mathcal{C} \in H$ such that, for any free state $\rho \in \mathcal{F}$ on any Hilbert space $H \in H$, there is a channel $\Phi \in \mathcal{O}(\mathcal{C} \rightarrow H)$ with $\rho \in \text{Im} \Phi$. Viewed as a world in $F(X)$, this distinct Hilbert space $\mathcal{C}$ has $|\mathcal{C}| = |\mathcal{S}(\mathcal{C})| = 1$ because there is only a single state on $\mathcal{C}$ (the identity operator). Additionally, taken in this way to be a separate world, we see that $\mathcal{D}_c \cap \mathcal{D}_H = S(\mathcal{C}) \cap S(H) = \emptyset$ for all other $H \in H$.

Then any such Hilbert space with free states on it, viewed as a world in the VDS4-model $F(X)$, must be accessible to $\mathcal{C}$ under $\mathcal{R}_\mathcal{O}$. But a Hilbert space $H \in H$ has free states if and only if $T(H) \neq \emptyset$ in the VDS4-model $F(X)$. Thus, viewed as a world, $\mathcal{C}$ is such a world $c$ in the resulting model $F(X)$ for any QRT $X$ with free states. Whence, if a VDS4-model $M = F(X)$ for some QRT $X$, it must have such a world $c$.

I now provide some preliminary heuristics for how this sort of modal language may be deployed for studying QRTs, after which I shall elaborate the formalism further to ensure injectivity using the convertibility preorder.

Theorem 3.4. For any QRT $X$ with VDS4-model $F(X)$, if $\rho \in S(\mathcal{H}^A)$ and $\Phi \in \mathcal{O}(A \rightarrow B)$ for some $\mathcal{H}^B$, then $\vdash_{F(X)} \rho \rightarrow \Diamond \Phi(\rho)$.

Proof. Whenever $\rho \in \mathcal{F}$, there is some $\Psi \in \mathcal{O}(\mathcal{C} \rightarrow A)$ with $\rho \in \text{Im}(\Psi)$. But then if $\Phi : \mathcal{O}(A \rightarrow B)$, $\Phi(\rho) \in \mathcal{F}$ as well by transitivity of $\mathcal{O}$. Hence, either $\rho \notin \mathcal{F}$, or both $\rho, \Phi(\rho) \in \mathcal{F}$. The existence of $\Phi$ as a member of $\mathcal{O}$ implies that $\langle \mathcal{H}^A, \mathcal{H}^B \rangle \in \mathcal{R}_\mathcal{O}$. From these facts, in the VDS4 model $F(X)$, $V(\rho, \mathcal{H}^A) = 1$ only if $V(\Phi(\rho), \mathcal{H}^B) = 1$. Then using accessibility, $V(\rho, \mathcal{H}^A) = 1$ only if $V(\Diamond \Phi(\rho), \mathcal{H}^A) = 1$. Thus $V(\rho \rightarrow \Diamond \Phi(\rho), \mathcal{H}^A) = 1$, and so $\vdash_{F(X)} \rho \rightarrow \Diamond \Phi(\rho)$ for any $\rho$. □

This reflects the modality inherent in the notion that free states are those which are possible to freely generate. By the same reasoning, we have the following:

Theorem 3.5. For any resource states $\rho$ and $\sigma$ is a QRT $X$, $\vdash_{F(X)} \Box \sigma \rightarrow \rho$.

It is worth noting, here, that necessity in the variable domain setting is understood to mean that some formula is true in all accessible worlds whose domains include all atomic symbols needed to define that formula.

Proof. If a state $\rho$ is a resource state in $X$, then $\mathcal{I}(\rho) = 0$ in the VDS4-model $F(X)$. Thus, $V_r(\neg \rho, \mathcal{H}^A) = 1$ in the world $\mathcal{H}^A$ that $\rho$ lives on. Then if $\sigma$ is also a resource state, we also have that $\mathcal{I}(\sigma) = 0$ and so $V_r(\neg \sigma, \mathcal{H}^B) = 1$ for any world $\mathcal{H}^B$ whose domain contains $\sigma$, and thus for any world. Then for any such world, if there is a channel $\Phi \in \mathcal{O}(A \rightarrow B)$, we have $\langle \mathcal{H}^A, \mathcal{H}^B \rangle \in \mathcal{R}_\mathcal{O}$ and so $V_r(\neg \rho \rightarrow \Diamond \neg \sigma) = 1$. Whence we have that $\vdash_{F(X)} \neg \rho \rightarrow \Diamond \neg \sigma$. Using contraposition, together with the identity $\neg \neg \neg = \Box$, this is equivalent to $\vdash_{F(X)} \Box \sigma \rightarrow \rho$. □

While the functor $F$ clearly allows for a new perspective on quantum resource theory in terms of modal logic, its non-injectivity and non-surjectivity limit its value for clarifying important issues. However, if QRT and VDS4 are appropriately extended with minimal additional structure, $F$ may then be extended to be injective too. Specifically, while the convertibility preorder of a QRT does not appear in its VDS4-model under $F$, if it is specified, it makes the functorial
correspondence between QRTs and VDS4-models injective. I demonstrate this now.

Suppose a QRT $X = \langle H, O, F \rangle$ is given and that the preorder $\leq_X$ is specified (in principle it may be read off of $X$ directly from $\mathcal{O}$, however, this information is ignored by $F$, so I here mean that we suppose this data is ‘stored’ elsewhere). Then $\leq_X$ is a preorder on $\bigcup_a \mathcal{S}(\mathcal{H}^a) = \mathcal{D}_H$ (for $H = \{\mathcal{H}^a\}$), the global domain of $F(X)$ as a VDS4-model.

We may define a pair of new categories, $\text{QTR}^*$ and $\text{VDS4}^*$ whose objects are isomorphism classes of QRTs with their convertibility preorders on the states of each Hilbert space in $H$, and VDS4-models together with a preorder on the global domain $\mathcal{D}$, respectively. That is, objects of $\text{QRT}^*$ are isomorphism classes of pairs $\langle X, \leq_X \rangle$ for QRTs $X$ and objects of $\text{VDS4}^*$ are isomorphism classes of pairs $\langle M, \leq \rangle$ for VDS4-models $M$ with $\leq$ a preorder on the global domain of $M$.

Isomorphism in these categories are just isomorphisms on the unstarred categories with the additional requirements that the preorders are order-isomorphic under the given isomorphism. Projections and inclusions may be defined in the usual manner by restricting the preorder to an appropriate subset or embedding it in a larger preorder. If we extend $F$ to a new functor $F^* : \text{QTR}^* \to \text{VDS4}^*$ which respects this preorder in the right way, then in fact $F^*$ ends up being properly injective.

Given a QRT $X$ with a preorder $\leq_X$, we may define the image of $F^*$ on $\langle X, \leq_X \rangle$ by $F^*(\langle X, \leq_X \rangle) = \langle F(X), \preceq \rangle$ where $\preceq$ is defined as $\rho \preceq \sigma$ (where $\rho$ and $\sigma$ are viewed as symbols in the domain $\mathcal{D}_H$ of $F(X)$) if and only if $\rho \leq_X \sigma$, viewed as states in the QRT $X$. The functoriality of $F$ from Theorem 3.1 is enough to show that this is a functor as well.

We have the following theorem.

**Theorem 3.6.** $F^*$ is an injective functor.

**Proof.** Let $X = \langle H, O, F \rangle$ and $Y = \langle H, O', F' \rangle$. Suppose $\langle X, \leq_X \rangle \not\sim \langle Y, \leq_Y \rangle$, it suffices to show that $F^*(\langle X, \leq_X \rangle) \not\sim F^*(\langle Y, \leq_Y \rangle)$. If $X \simeq Y$, then $\leq_X$ and $\leq_Y$ are clearly order-isomorphism, whence the images under $F^*$ are isomorphic. If $X \not\simeq Y$, then it has previously been established in Theorem 3.2 that either $F(X) \not\simeq F(Y)$, or else there is some $\Phi \in \mathcal{O}(A \to B)$ for which there does not a collection $\{\Psi_\alpha\} \subseteq \mathcal{O}(A' \to B')$ satisfying $\text{Im}(\Phi) = \bigcup_\alpha \text{Im}(\varphi_{B'\to B} \circ \Psi_\alpha \circ \varphi_{A\to A'})$.

If $F(X) \not\simeq F(Y)$, then $F^*(\langle X, \leq_X \rangle) \not\simeq F^*(\langle Y, \leq_Y \rangle)$, and we are done. Thus, suppose $F(X) \simeq F(Y)$. The nonexistence of channels just described then impacts the preorders in the following way. Given the channel $\Phi \in \mathcal{O}$, there is some $\sigma = \Phi(\rho)$ for some $\rho$ such that $\sigma \not\in \text{Im}(\varphi_{B'\to B} \circ \Psi \circ \varphi_{A\to A'})$ for any $\Psi \in \mathcal{O}(A' \to B')$. Thus, $\rho \leq_X \sigma$, but $\varphi_{A\to A'}(\rho) \not\sim \varphi_{B'\to B'}(\sigma)$, whence $\leq_X$ and $\leq_Y$ are not order-isomorphism. Thus, the preorders on VDS4-model domains induced under $F^*$ will fail to be order-isomorphic. Whence, $F^*(\langle X, \leq_X \rangle) \not\simeq F^*(\langle Y, \leq_Y \rangle)$ as needed. Therefore, $F^*$ is injective.

One final note which is worth making explicit is that, while it is common to consider only finite-dimensional QRTs, there were no dimensionality assumptions made here; indeed, there wasn’t even a cardinality assumption made with respect to the Hilbert space dimension, whence separability is not assumed. This is important because the channel-state duality exploited for the definition of QRTs is generalizable to infinite dimensional settings (see, for instance, Holevo [2011]), and there have been approaches to quantum mechanics in more model-theoretically exotic settings (e.g. Benci et al. [2019]) so one should strive to provide a description of QRTs which is capable of capturing these generalizations. QRTs over a finite collection of finite-dimensional systems have corresponding VDS4 models with finite global domains (the $\mathcal{D}_H$’s). If $H = \{\mathcal{H}^a\}$ consists of countably many separable Hilbert spaces, then the global domain will obey $|\mathcal{D}| \leq 2^{N_0}$ with equality when there is at least one Hilbert space of dimension greater than one (whence the collection of states on that space becomes uncountable as it is closed under convex combinations of its pure states). Higher cardinality domains may be needed in the non-separable cases.

**4 Discussion**

The construction provided here connects quantum resource theories to variable-domain modal logic in such a way that quantum states are understood as atomic symbols, truth values are
understood to refer to the experimental ability to produce those quantum states, and quantum channels are interpreted as modes of accessibility between different quantum systems. I now illustrate why this modal logical representation is valuable for gaining a practical understanding of quantum resource theories.

One immediate justification for the naturalness of this presentation of QRTs is that, provided the VDS4-models carry a pre-ordering on their global domains (i.e. they live in \textbf{VDS}4*), injectivity of \(F^*\) ensures that all of the QRT structure is preserved. Thus, this representation is, in a sense, lossless with respect to its encoding of the qualitative features of the QRTs in question. In this way, there isn’t a downside to using the modal logic representation provided here. However, there are several obvious benefits. First, the modal logic framework doesn’t rely on the full Hilbert space and operator-theoretic structure of the usual QRT formalism. Thus, it is mathematically must simpler. One could view the image of \(F^*\) in the category \textbf{VDS}4* as a mathematical reduction of quantum resource theory, carrying forward only the necessary features of the theory.

Another valuable feature of this framework is that the language of modal logic provides a new way to state features of QRTs. As an example of this, let us consider so-called resource destroying QRTs. These are QRTs for which the collection of channels \(\mathcal{C}\) may include a channel which takes resource states to free states (the converse, of course, is prohibited by the definition of a resource). In the context of resource preserving QRTs, for any such theory \(X\) it is the case that if \(\rho \not\in \mathcal{F}\), then \(\Phi(\rho) \not\in \mathcal{F}\) too. Thus, in the modal logic setting, it is true that \(\models_{F(X)} \neg \Phi(\rho) \rightarrow \neg \rho \rightarrow \neg \Phi(\rho)\). Applying contraposition, one obtains \(\models_{F(X)} \Phi(\rho) \rightarrow \rho\). This is the converse of theorem 3.4. Therefore, theorem 3.4 may be extended to a bi-conditional exactly if \(X\) is resource-preserving. Resource destroying QRTs therefore are those theories for which theorem 3.4 cannot be inverted. Thus, the class of resource-destroying QRTs corresponds exactly to those VDS4-models in the image of \(F\) which fail to be models of the extended logical theory which takes the bi-conditional form of 3.4 as an axiom.

There is another useful property which is possessed by the variable domain semantics provided: it be readily extended to one with full quantification and predication. Indeed, variable domains were constructed specifically to allow modal propositional logic to be extended to a full first-order theory Garson [2001]. Essentially, in order to extend VDM model to one which is capable of handling quantification and predicates, one needs to add semantical technology for substituting elements of the domain at a world for bound and free variables in quantified expressions. Once this has been done, one may begin carrying out deductions and proving validity of predicated and quantified formulas. I do not expand the details of this procedure here. However, I shall now discuss how it may be exploited to provide a model theoretic interpretation of classes of QRTs.

Let us consider convex QRTs. These QRTs satisfy the condition that, given any two free states \(\rho\) and \(\sigma\) on some shared Hilbert space, the convex sum \(p\rho + (1 - p)\sigma\) is also a free state for all \(0 \leq p \leq 1\). However, convexity may be cast in terms of (quantified) predicate formulas in the VDS4 setting. Let \(\{C_p\}\) be a class of two-place predicates indexed by \(p \in [0, 1]\) such that \(C_p[\rho_1, \rho_2]\) is true exactly when either (i) \(\rho_1, \rho_2 \in \mathcal{F}\) and \(pp_1 + (1 - p)\rho_2 \in \mathcal{F}\) or (ii) one of \(\rho_1\) or \(\rho_2\) is not in \(\mathcal{F}\). Then to show that a particular theory \(X\) is convex is equivalent in the modal logic setting to proving \(\models_{F(X)} (\forall \rho_1)(\forall \rho_2)(C_p[\rho_1, \rho_2])\) for each value of \(p\). Whence, this may be used in proofs of other QRT results in the modal logical setting.

Another fact to take not of is that the class of all convex QRTs is a sub-collection of the class of VDS4-models which satisfy the extended logical theory which takes \((\forall \rho_1)(\forall \rho_2)(C_p[\rho_1, \rho_2])\) as an axiom schema (for all \(p \in [0, 1]\)). Namely, it is the sub-collection which satisfy the conditions of Theorem 3.3. Then by studying the axiomatic extension of VDS4 by these axioms, one may demonstrate features of all such VDS4 models and thus, a fortiori all convex QRTs. The same sort of analysis may be applied to affine QRTs as well.

In short, the specific details of a particular QRT, where much of the richness of the theory lies, may be understood in terms of the validity of certain quantified predicate formulas and formulas involving modal operators in the modal logic setting. Then the properties of these resource theories may be viewed in a model-theoretic way, with no reference to the underlying Hilbert space
structure. QRTs which deal with complex aspects of quantum thermodynamics or quantum communication protocols, for instance, may be regarded as specific models in VDS4 which have a particular collection of predicated or modal formulas as tautologies. In this way, the functorial relation between quantum resource theories and variable-domain modal logic is such that it allows for a new class of tools for the explorations of operational theory.

5 Conclusions

Here, I presented a basic introduction to both quantum resource theory as an abstract theory of quantum channels, and variable-domain modal logic as a logic of possibility established over a so-called ‘possible worlds’ semantics. Using these constructions, I defined the categories, $\text{QRT}$ and $\text{VDS4}$, of isomorphism classes of quantum resource theories and models of variable-domain S4 modal logic, respectively, where the arrows in both cases were inclusions and projections onto other quantum resource theories and logical models, respectively. With this machinery in place, a functor $F : \text{QRT} \to \text{VDS4}$ was explicitly constructed and characterized, indicated exactly how close it is to being injective and surjective, and diagnosing precisely which features of quantum resource theory it erases, thereby establishing a strong relation between the formalisms of these otherwise disparate subjects. By then including state convertibility data in the form of a preorder to construct the categories $\text{QRT}^\ast$ and $\text{VDS4}^\ast$, it was shown that $F$ could be extended to an injective functor $F^\ast$. Finally, it was shown that the modal framework allows certain common intuitions about the general structure of quantum resource theories to be expressed and proven in the modal logic setting. The possibility of applying this functorial relation to the further elaboration of particular quantum resource theories as models of certain axiomatically extended logical theories was discussed.

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