Nonequilibrium dynamics:  
a renormalized computation scheme  

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We present a regularized and renormalized version of the one-loop nonlinear relaxation equations that determine the non-equilibrium time evolution of a classical (constant) field coupled to its quantum fluctuations. We obtain a computational method in which the evaluation of divergent fluctuation integrals and the evaluation of the exact finite parts are cleanly separated so as to allow for a wide freedom in the choice of regularization and renormalization schemes. We use dimensional regularization here. Within the same formalism we analyze also the regularization and renormalization of the energy-momentum tensor. The energy density serves to monitor the reliability of our numerical computation. The method is applied to the simple case of a scalar \( \phi^4 \) theory; the results are similar to the ones found previously by other groups.

I. INTRODUCTION

Phase transitions in elementary particle physics have been studied over the last two decades in various contexts. The phase transitions of grand unified theories may lead to inflationary periods of the early universe \cite{1}; the electroweak phase transition can be based on a theory whose parameters - up to the Higgs sector - are meanwhile well known and studies both in perturbation theory \cite{2} and on the lattice \cite{3} show considerable progress in its understanding. The hadronic phase transition is being investigated both theoretically and experimentally \cite{4}.

For some applications it is sufficient to work in finite temperature quantum field theory, i.e. to consider only states of thermal equilibrium. For the inflationary phase transition, at least, a real time study involving nonequilibrium dynamics is necessary if one intends to describe the process of reheating. The simplest version of the inflationary scenario which consists in introducing a friction term \cite{5} has been shown recently \cite{6} to be on weak theoretical grounds. Linear and nonlinear relaxation have therefore been studied by various authors \cite{7,8}. The one-loop relaxation equations studied here do not seem to lead to a proper thermalization, presumably they have to be modified in such a way as to include the interactions among the fluctuating fields. However, they will certainly describe an initial stage of a more involved dynamical development. They present therefore a first step that has to be studied and well understood. Another subject to which nonequilibrium dynamics has been applied recently is the hadronic phase transition \cite{9} and the possible formation of disordered chiral condensates \cite{10,11}.

The computations of the nonlinear relaxation equations requires regularization and renormalization. Thus far \cite{12,13,14} noncovariant momentum cutoffs were used in such computations. While it would certainly not be an essential difficulty to replace those by a Pauli-Villars type regularization one may wish, in computations involving nonabelian gauge theories, to be able to use dimensional regularization as well. Furthermore it is more convenient to compute only convergent integrals instead of dealing with divergent integrals by varying their cutoff.

It is the aim of this work to present a computational method in which the process of regularization and renormalization is cleanly separated from the numerical computation of finite parts of the one-loop integrals. The method is similar to the one developed in \cite{15} which has been applied for the computation of one loop contributions to reaction rates associated with sphalerons \cite{16}, instantons \cite{17} and bounces \cite{18}. While in the cases mentioned the computations involved Euclidean Green functions which are behaved smoothly, here we will have to compute the trace of a Green function in Minkowski space, i.e. involving real time. The oscillating behaviour of such Green functions could present a new difficulty for the application of the computational scheme.

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In this first presentation of the method we have studied its practical numerical application only in the simplest possible model, a scalar field theory with a $\lambda \Phi^4$ self interaction. This theory has also been the subject of previous studies [9] and the results are similar. Especially it is found that the parametric resonance implied by an oscillating background field does not fully develop; back reaction damps the classical amplitude and the system relaxes to a new, essentially stationary oscillating phase.

The plan of this work is as follows: in section II we recall the basic definitions and relations; in section III we present the one-loop nonlinear relaxation equations; we prepare the regularization in section IV by expanding the fluctuation modes in orders of the vertex function governed by the classical field and by deriving the large momentum behaviour of the first terms; regularization is then straightforward, the renormalization requires some algebra, both are presented in section V; the formalism is discussed in section VI to the renormalization of the energy-momentum tensor; the numerical computation is discussed in section VII; we conclude in section VIII with a discussion of the numerical results and an outlook to more realistic and more general applications of the method.

II. BASIC RELATIONS

We restrict our study to self-interacting scalar $\phi^4$-theory without spontaneous symmetry breaking. The Lagrangean density is given by

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4.$$  (1)

We split the field $\Phi$ into its expectation value $\phi$ and the quantum fluctuations $\psi$:

$$\Phi(x, t) = \phi(t) + \psi(\vec{x}, t)$$  (2)

with

$$\phi(t) = \langle \Phi(\vec{x}, t) \rangle = \frac{\text{Tr} \rho(t)}{\text{Tr} \rho(t)}$$  (3)

where $\rho(t)$ is the density matrix of the system which satisfies the Liouville equation

$$i \frac{d \rho(t)}{dt} = [\mathcal{H}(t), \rho(t)].$$  (4)

The Lagrangean then takes the form

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$$  (5)

with

$$\mathcal{L}_0 = \frac{1}{2} \partial_\mu \psi \partial^\mu \psi - \frac{1}{2} m^2 \psi^2$$
$$+ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4,$$  (6)

$$\mathcal{L}_1 = \partial_\mu \psi \partial^\mu \phi - m^2 \psi \phi - \frac{\lambda}{4!} \phi^4 - \frac{\lambda}{6} \psi^3 \phi - \frac{\lambda}{4} \psi^2 \phi^2 - \frac{\lambda}{6} \psi \phi^3.$$  (7)

This decomposition will be used as the basis for a perturbative expansion. Though we will consider the so-called one-loop equations which are nonperturbative, we will need such a perturbative scheme in order to define the one-loop summation and its renormalization. Since we are dealing with a nonequilibrium system the appropriate expansion is the CTP formalism (see e.g. [19]). We will consider the zero temperature case only.

The Green function is defined by a $2 \times 2$ matrix

$$G(t, \vec{x}; t', \vec{x}') = \begin{cases} G^{++}(t, \vec{x}; t', \vec{x}') & G^{+-}(t, \vec{x}; t', \vec{x}') \\ G^{-+}(t, \vec{x}; t', \vec{x}') & G^{--}(t, \vec{x}; t', \vec{x}') \end{cases}$$  (8)

where
After a straightforward calculation one obtains

\[ -iG^{++}(t, \vec{x}; t', \vec{x}') = \langle T \Phi(t, \vec{x})\Phi(t, \vec{x}) \rangle \]  (9)
\[ -iG^{-+}(t, \vec{x}; t', \vec{x}') = \langle T \Phi(t, \vec{x})\Phi(t, \vec{x}) \rangle \]  (10)
\[ -iG^{+-}(t, \vec{x}; t', \vec{x}') = \langle T \Phi(t, \vec{x})\Phi(t, \vec{x}) \rangle \]  (11)
\[ -iG^{--}(t, \vec{x}; t', \vec{x}') = iG^{-+}(t, \vec{x}; t, \vec{x}) . \]  (12)

Here the brackets \( \langle \rangle \) refer to an expectation value with respect to the density matrix \( \rho \).

The Green functions can be decomposed as

\[ G^{++}(t, \vec{x}; t', \vec{x}') = G^>(t, \vec{x}; t', \vec{x}')\Theta(t - t') + G^<(t, \vec{x}; t', \vec{x}')\Theta(t' - t) \]  (13)
\[ G^{--}(t, \vec{x}; t', \vec{x}') = G^<(t, \vec{x}; t', \vec{x}')\Theta(t - t') + G^>(t, \vec{x}; t', \vec{x}')\Theta(t' - t) \]  (14)
\[ G^{+-}(t, \vec{x}; t', \vec{x}') = -G^<(t, \vec{x}; t', \vec{x}') \]  (15)
\[ G^{-+}(t, \vec{x}; t', \vec{x}') = -G^>(t, \vec{x}; t', \vec{x}') = -G^<(t', \vec{x}; t, \vec{x}) \]  (16)

with

\[ G^>(t, \vec{x}; t', \vec{x}') = i\langle \Phi(t, \vec{x})\Phi(t', \vec{x}') \rangle . \]  (17)

These equations apply to the exact Green functions. For the free Green functions \( G^>(t, \vec{x}; t', \vec{x}') \) is given explicitly by

\[ G^0(t, \vec{x}; t', \vec{x}') = \int \frac{d^3k}{(2\pi)^3} \frac{i}{2\omega_k} \exp(-i\omega_k(t - t') + i\vec{k} \cdot (\vec{x} - \vec{x}')) \]  (18)

with \( \omega_k = \sqrt{k^2 + m^2} \).

The vertices are obtained from \( \mathcal{L}_1 \) ordering the terms in powers of \( \psi \)

\[ \mathcal{L}_1 = (\partial_{\mu} \partial^\mu \phi - m^2 \phi - \frac{\lambda}{6} \phi^3)\psi - \frac{\lambda}{4} \psi^2 \phi^2 - \frac{\lambda}{6} \phi^3 \psi^2 - \frac{\lambda}{4!} \psi^4 . \]  (19)

Every term corresponds to a vertex. In the matrix formalism of the CTP formalism the vertex operators are given by

\[ i\Gamma_n(t, \vec{x}) = \left\{ -i\Gamma_n(t, \vec{x}), \begin{array}{c} 0 \\ 0 \end{array} \right\} . \]  (20)

The subscript \( n \) denotes the number of fluctuation fields \( \psi \) entering the vertex. We have

\[ i\Gamma_1 = i(\Box - m^2 - \frac{\lambda}{6} \phi^2(t))\phi(t) \]  (21)
\[ i\Gamma_2 = -\frac{\lambda}{2} \phi^2 \]  (22)
\[ i\Gamma_3 = -i\lambda \phi \]  (23)
\[ i\Gamma_4 = -i\lambda . \]  (24)

### III. EQUATIONS OF MOTION

The equation of motion for the field \( \phi(t) \) follows from the tadpole condition

\[ \langle \psi^+(x, t) \rangle = 0 . \]  (25)

After a straightforward calculation one obtains the equation

\[ \ddot{\psi}(t) + m^2 \phi(t) + \frac{\lambda}{6} \phi^3(t) + \frac{1}{12} \phi(t)G^{++}(0) = 0 \]  (26)

which is represented graphically in Fig. \[ 1 \]. Here \( G^{++} \) is the ++ matrix element of the exact Green function in the background field \( \phi(t) \). It can be expanded perturbatively as
\[
G = G_0 - G_0 \Gamma_2 G_0 + G_0 \Gamma_2 G_0 \Gamma_2 G_0 - ... = G_0 (1 + \Gamma_2 G_0)^{-1}.
\]

\(G^{++}(t, \vec{x})\) satisfies the differential equation:
\[
\left(\frac{\partial^2}{\partial t^2} - \Delta + m^2 + \frac{\lambda}{2} \phi^2(t)\right) G^{++}(x; x') = \delta^{(4)}(x - x').
\]

So we have effectively a time-dependent mass term
\[
m^2(t) = m^2 + \frac{\lambda}{2} \phi^2(t).
\]

Using translation invariance in \(\vec{x}\) we introduce the Fourier transform
\[
G^{++}_k(t, t') = \int d^3x \, G^{++}(t, \vec{x}; t', 0) \exp(-i\vec{k}\vec{x})
\]
and denote the associated energy as
\[
\omega_k(t) = \left[\vec{k}^2 + m^2 + \frac{\lambda}{2} \phi^2(t)\right]^\frac{1}{2}.
\]

We make the ansatz for \(G^{++}_k(t, t')\)
\[
G^{++}_k(t, t') = C^{-1} \left\{ U^+_k(t) U^-_k(t') \theta(t - t') + U^+_k(t') U^-_k(t) \theta(t' - t) \right\}
\]
where the \(U^\pm_k\) are solutions of the homogenous problem and \(C^{-1}\) is the Wronskian. The initial conditions are
\[
U^+_k(0) = 1 \quad \dot{U}^+_k(0) = -i\omega^0_k,
U^-_k(0) = 1 \quad \dot{U}^-_k(0) = i\omega^0_k
\]
with
\[
\omega^0_k = \left[\vec{k}^2 + m^2 + \frac{\lambda}{2} \phi^2(0)\right]^\frac{1}{2}.
\]

The complete Green function at equal times reads then
\[
G^{++}_k(t, t) = \frac{i}{2\omega^0_k} |U^+_k(t)|^2,
\]
where we have used \(U^-_k(t) = U^+_k(t)^*\).

The resulting equation of motion for the classical field \(\phi(t)\) and the mode functions \(U^+_k(t)\) are then
\[
\ddot{\phi}(t) + m^2 \phi(t) + \frac{\lambda}{6} \phi^3(t) + \frac{\lambda}{2} \phi(t) \int \frac{d^3k}{(2\pi)^3} \frac{|U^+_k(t)|^2}{2\omega^0_k} = 0
\]
\[
\left[\frac{d^2}{dt^2} + \vec{k}^2 + m^2 + \frac{\lambda}{2} \phi^2(t)\right] U^+_k(t) = 0
\]
\[
U^+_k(0) = 1 \quad \dot{U}^+_k(0) = -i\omega^0_k.
\]

We denote the fluctuation integral in (35) as
\[
\Delta M^2(t) = \frac{\lambda}{2} \int \frac{d^3k}{(2\pi)^3} \frac{|U^+_k(t)|^2}{2\omega^0_k}.
\]

It determines the back-reaction of the fluctuations onto the classical field \(\phi(t)\).

An important check for the consistency of our numerical analysis will be the conservation of energy. The energy density is given by
Calculating the trace over the Hamiltonian we obtain
\[
E = \frac{1}{2} \dot{\phi}^2(t) + V(\phi(t)) + \frac{\text{Tr} \mathcal{H} \rho(0)}{\text{Tr} \rho(0)}.
\]  

(39)

Using the equations of motion it is easy to see that the time derivative of the energy density vanishes. The equations obtained so far are yet formal since they contain divergent quantities. We will present their renormalized form in section IV.

IV. PERTURBATIVE EXPANSION

In order to prepare the renormalized version of the equations given in the previous section we introduce a suitable expansion of the mode functions \( U_k^\pm(t) \). Adding the term \( \frac{\lambda}{2} \phi^2(0) U_k^+(t) \) on both sides of the mode function equation it takes the form
\[
\left[ \frac{d^2}{dt^2} + (\omega_k^0)^2 \right] U_k^+(t) = -V(t) U_k^+(t)
\]

with
\[
V(t) := \frac{\lambda}{2} \left( \phi^2(t) - \phi^2(0) \right)
\]
\[
\omega_k^0 = \left[ k^2 + m^2 + \frac{\lambda}{2} \phi^2(0) \right]^{1/2}.
\]

(42)

Including the initial conditions (32) the mode functions satisfy the equivalent integral equation
\[
U_k^+(t) = e^{-i\omega_k^0 t} + \int_0^\infty dt' \Delta_{k,\text{ret}}(t-t') V(t') U_k^+(t')
\]

(43)

with
\[
\Delta_{k,\text{ret}}(t-t') = \frac{1}{\omega_k^0} \Theta(t-t') \sin(\omega_k^0 (t-t')).
\]

(44)

We separate \( U_k^+(t) \) into the trivial part corresponding to the case \( V(t) = 0 \) and a function \( f_k(t) \) which represents the reaction to the potential by making the ansatz
\[
U_k^+(t) = e^{-i\omega_k^0 t} (1 + f_k(t)).
\]

(45)

\( f_k(t) \) satisfies then the integral equation
\[
f_k(t) = \int_0^t dt' \Delta_{k,\text{ret}}(t-t') V(t') (1 + f_k(t')) e^{i\omega_k^0 (t-t')}
\]

(46)

and an equivalent differential equation
\[
\dot{f}_k(t) - 2i\omega_k^0 f_k(t) = -V(t) (1 + f_k(t))
\]

(47)

with the initial conditions \( f_k(0) = \dot{f}_k(0) = 0 \).

We expand now \( f_k(t) \) with respect to orders in \( V(t) \) by writing
\[ f_k(t) = f_k^{(1)}(t) + f_k^{(2)}(t) + f_k^{(3)}(t) + \ldots \]  
\[ = f_k^{(1)}(t) + f_k^{(2)}(t) \]  

where \( f_k^{(n)}(t) \) is of \( n \)'th order in \( V(t) \) and \( f_k^{(n)}(t) \) is the sum over all orders beginning with the \( n \)'th one. The \( f_k^{(n)} \) are obtained by iterating the integral equation (46) or the differential equation (47). The function \( f_k^{(1)}(t) \) is identical to the function \( f_k(t) \) itself which is obtained by solving (47), the function \( f_k^{(2)}(t) \) can be obtained as

\[ f_k^{(2)}(t) = \int_0^t dt' \Delta_{k,\text{ret}}(t-t')V(t')f_k^{(1)}(t')e^{i\omega_0^0(t-t')} \]  

or by solving the inhomogeneous differential equation

\[ f_k^{(2)}(t) - 2i\omega_0^0 f_k^{(2)}(t) = -V(t)f_k^{(1)}(t). \]  

Note that in this way one avoids the computation of \( f_k^{(2)}(t) \) via the small difference \( f_k(t) - f_k^{(1)}(t) \). This feature is especially important if deeper subtractions are required as in the case of fermion fields.

The order on the potential \( V(t) \) will determine the behaviour of the functions \( f_k^{(n)} \) at large momentum. We will give here the relevant leading terms for \( f_k^{(1)}(t) \) and \( f_k^{(2)}(t) \). We have

\[ f_k^{(1)}(t) = \frac{i}{2\omega_k^0} \int_0^t dt' (\exp(2i\omega_0^0(t-t')) - 1) V(t'). \]  

Integrating by parts we obtain

\[ f_k^{(1)}(t) = -\frac{i}{2\omega_k^0} \int_0^t dt' V(t') - \frac{1}{4(\omega_k^0)^2} V(t) + \frac{1}{4(\omega_k^0)^2} \int_0^t dt' \exp(2i\omega_0^0(t-t'))V(t') \]  

or, by another integration by parts

\[ f_k^{(1)}(t) = -\frac{i}{2\omega_k^0} \int_0^t dt' V(t') - \frac{1}{4(\omega_k^0)^2} V(t) + \frac{i}{8(\omega_k^0)^3} \dot{V}(t) \]  

\[ -\frac{i}{8(\omega_k^0)^3} \int_0^t dt' \exp(2i\omega_0^0(t-t'))\ddot{V}(t'). \]

Similarly we find for the leading behaviour of \( f_k^{(2)}(t) \)

\[ f_k^{(2)}(t) = -\frac{1}{4(\omega_k^0)^2} \int_0^t dt' \int_0^{t'} dt'' V(t')V(t'') + O((\omega_k^0)^{-3}). \]  

The leading terms of \( f_k^{(1)}(t) \) and \( f_k^{(2)}(t) \) in this expansion in powers of \( (\omega_k^0)^{-1} \) are the same as for \( f_k^{(1)}(t) \) and \( f_k^{(2)}(t) \) respectively.

Unlike a WKB ansatz for \( U_k^+(t) \) the expansion presented here can be easily extendend to coupled channel systems and higher orders in the expansion (if deeper subtractions are required).

V. RENORMALIZATION

The fluctuation term (38) occurring in the equation of motion (33) can then be written as
\[
\Delta M^2(t) = \frac{\lambda}{2} \int \frac{d^3k}{(2\pi)^3} \frac{|e^{-i\omega_k^0(t + f_k(t))}|^2}{2\omega_k^0}.
\]  

Inserting our expansion we obtain
\[
\Delta M^2(t) = \frac{\lambda}{2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k^0} \left\{ 1 + 2\text{Re}f_k^{(1)}(t) \right\} + \frac{\lambda}{2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k^0} \left\{ 2\text{Re}f_k^{(2)}(t) + |f_k^{(1)}(t)|^2 \right\}.
\]  

The second integral is convergent. Though \(\text{Re}f_k^{(2)}(t)\) and \(|f_k^{(1)}(t)|^2\) behave separately only as \((\omega_k^0)^{-2}\) one finds, using Eqs. (53) and (56), that this leading behaviour cancels among the two contributions. This integral can be computed numerically using the functions \(f_k^{(1)}\) and \(f_k^{(2)}\) obtained by solving (41) and (51). In the first integral we find a quadratic divergence which corresponds to the tadpole graph in \(\phi^4\)-theory and a logarithmic one associated with \(2\text{Re}f_k^{(1)}(t)\).

Indeed the real part of \(f_k^{(1)}(t)\) is obtained from (53) as
\[
2\text{Re}f_k^{(1)}(t) = -\frac{1}{2\omega_k^0} \left\{ V(t) - \int_0^t dt' \cos(2\omega_k^0(t - t'))V(t') \right\}.
\]  

The first term behaves as \((\omega_k^0)^{-2}\) and leads therefore to a logarithmic divergence in \(\Delta M^2\). The second one yields a finite contribution. It will be included into the finite part of \(\Delta M^2\).

So altogether we have
\[
\Delta M^2(t) = \frac{\lambda}{2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k^0} \left\{ 1 + 2\text{Re}f_k^{(1)}(t) \right\} + \Delta M_{\text{fin}}^2(t)
\]  

with
\[
\Delta M_{\text{fin}}^2(t) = \frac{\lambda}{2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k^0} \frac{1}{2(\omega_k^0)^2} \int_0^t dt' \cos(2\omega_k^0(t - t'))V(t')
\]
\[
+ \frac{\lambda}{2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k^0} \left\{ 2\text{Re}f_k^{(2)}(t) + |f_k^{(1)}(t)|^2 \right\}.
\]

The first term in (61) corresponds to the zeroth order diagram in Fig. 1. The second term in (60) and the first term in (61) together correspond to the first order diagram in Fig. 1. The second term in (61) sums up all the higher order diagrams. By this term-to-term equivalence of our expressions to diagrams of CTP perturbation theory we have achieved here a clean separation between finite quantities that can be computed numerically and the renormalization parts that are computed analytically up to Fourier transforms of the external sources. We consider this to be an important feature of our method, as the regularization may now be chosen freely, as required in order to maintain the symmetries of the theory.

The first divergent integral can be rewritten as
\[
\int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k^0} = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon},
\]  
an identity that holds in dimensional as well as in Pauli-Villars regularization. The integral is the one associated to the tadpole graph Fig. 2a and can be absorbed into the renormalization of the mass term. In the same way the second integral is equivalent to
\[
\int \frac{d^3k}{(2\pi)^3} \frac{1}{8(\omega_k^0)^3} = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \frac{-i}{(k^2 - m^2 + i\epsilon)^2}.
\]  

It is associated with the Feynman graph Fig. 2b which leads to a coupling constant renormalization. We will use dimensional regularization and renormalize the four point function at vanishing external momenta. We first rewrite the basic equation of motion, including appropriate counter terms, as
\[ \ddot{\phi}(t) + (m^2 + \delta m^2)\phi(t) + \frac{\lambda + \delta \lambda}{6} \mu^r \phi^3(t) + \Delta M^2(t)\phi(t) = 0. \] (64)

Next we separate from \( \Delta M^2(t)\phi(t) \) dimensionally regularized divergent terms

\[ \left\{ \frac{\lambda}{2} \phi(t) \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k^0} \right\}_{\text{reg}} = \mu^r \frac{\lambda}{2} \phi(t) \int \frac{d^4k}{(2\pi)^d} \frac{1}{2 \left[ k^2 + m^2 + \frac{\lambda^r}{2} \phi^2(0) \right]^{\frac{d}{2}}} \]

\[ = -\frac{\lambda m_0^2 \phi(t)}{32\pi^2} \left\{ \frac{2}{\epsilon} + \ln \frac{4\pi\mu^2}{m_0^2} - \gamma + 1 \right\} \] (65)

and

\[ \left\{ -\frac{\lambda}{2} \phi(t) V(t) \int \frac{d^3k}{(2\pi)^3} \frac{1}{4\omega_k^0} \right\}_{\text{reg}} = -\frac{\lambda^r}{8} \left( \frac{\mu^2}{\mu^2} \phi(t) \int \frac{d^4k}{(2\pi)^d} \left[ k^2 + m^2 + \frac{\lambda^r}{2} \phi^2(0) \right]^{\frac{d}{2}} \right) \]

\[ = -\frac{\lambda m_0^2 \phi(t)}{32\pi^2} \left\{ \frac{2}{\epsilon} + \ln \frac{4\pi\mu^2}{m_0^2} - \gamma \right\} \] (66)

where we have introduced

\[ m_0^2 := m^2 + \frac{\lambda}{2} \mu^r \phi^3(0). \] (67)

Recalling that \( V(t) = (\lambda/2)(\phi^2(t) - \phi^2(0)) \) these two expressions combine into

\[ -\delta m^2 \phi(t) - \frac{\delta \lambda \mu^r}{6} \phi^3(t) - \frac{\lambda}{32\pi^2} \ln \frac{m^2}{m_0^2} \left( m^2 \phi(t) + \frac{\lambda}{2} \phi^3(t) \right) - \frac{\lambda^2}{64\pi^2} \frac{\mu^r}{\mu^r} \phi^2(0) \phi(t) \] (68)

where we have defined the renormalization constants

\[ \delta m^2 = \frac{\lambda m^2}{32\pi^2} \left\{ \frac{2}{\epsilon} + \ln \frac{4\pi\mu^2}{m^2} - \gamma + 1 \right\} \] (69)

and

\[ \delta \lambda = \frac{3\lambda^2}{32\pi^2} \left\{ \frac{2}{\epsilon} + \ln \frac{4\pi\mu^2}{m^2} - \gamma \right\}. \] (70)

Note that these counterterms are independent of \( m_0 \) and therefore from the initial value of the field \( \phi(0) \), though the divergent integrals do depend on the initial condition.

The renormalized equation of motion now reads

\[ \ddot{\phi}(t) + (m^2 + \Delta m^2)\phi(t) + \frac{\lambda + \Delta \lambda}{6} \mu^r \phi^3(t) + \Delta M^2_{\text{fin}}(t)\phi(t) = 0 \] (71)

with the finite corrections

\[ \Delta m^2 = -\frac{\lambda m^2}{32\pi^2} \ln \frac{m^2}{m_0^2} - \frac{\lambda^2}{64\pi^2} \phi^2(0), \] (72)

\[ \Delta \lambda = -\frac{3\lambda^2}{32\pi^2} \ln \frac{m^2}{m_0^2}. \] (73)

This equation is in a form well suited for numerical computation.

**VI. THE ENERGY-MOMENTUM TENSOR**

The renormalization scheme applied so far concerned the Green functions of the theory. The energy density on the other hand is part of the energy-momentum tensor. It contains additional divergent terms which have to be defined by
a regularization and to be removed by new counter terms. These counter terms have been considered in the literature \[21\] \[24\]. If one considers a space with vanishing curvature one has only two possible counter terms besides the ones already introduced into the Lagrangean so that

\[
T_{\mu\nu}^{\text{ren}} = T_{\mu\nu}^{\text{reg}} + g_{\mu\nu} \left( \delta \Lambda + \frac{1}{2} \delta m^2 \phi^2(x) + \frac{\delta \lambda}{4!} \phi^4(x) \right) + A(g_{\mu\nu} \partial_{\sigma} \partial^\sigma - \partial_{\mu} \partial_{\nu}) \phi^2(x). \tag{74}
\]

In the case considered here the second term does not contribute to the energy density. It will be useful, however, to consider the entire energy-momentum tensor including the space components

\[
T_{ij} = -pg_{ij} \tag{75}
\]

where \( p \) is the pressure given formally (i. e. without regularization) by

\[
p = \dot{\phi}^2(t) + \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k^0} \left( |\vec{U}_k^+(t)|^2 + \frac{k^2}{3} |U_k^+|^2 \right) - \mathcal{E}. \tag{76}
\]

The constants \( A \) and \( \delta \lambda \) have been determined by various authors \[21\] \[24\] using different regularizations. Coleman and Jackiw use the covariant Pauli-Villars regularization and Christensen the point-splitting technique and heat kernel regularization. Cooper et al. find a way of dealing with the problems of a three-momentum cutoff. We will use the covariant dimensional regularization as we did above in regularizing the fluctuation integral.

In order to renormalize the energy we introduce the available counter terms into the unrenormalized expression \[40\] so that it reads now

\[
\mathcal{E} = \frac{1}{2} \dot{\phi}^2(t) + \frac{1}{2} (m^2 + \delta m^2) \phi^2(t) + \frac{\lambda + \delta \lambda}{4!} \mu^2 \phi^4(t) + \delta \Lambda \\
+ \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{\omega_k^0}{4} \left( 1 + 2 \Re f_k^{(1)}(t) + |f_k^{(1)}(t)|^2 \right) \right. \\
+ \frac{1}{4\omega_k^0} |\vec{f}_k^{(1)}(t)|^2 \\
- \frac{1}{2} \Re (\bar{f}_k^{(1)*}(t) + f_k^{(1)*}(t)) \right\}. \tag{77}
\]

While \( \delta \lambda \) and \( \delta m^2 \) have already been fixed above we have to determine now the ‘cosmological constant’ \( \delta \Lambda \); in order to do so we analyse the quartically divergent part of the fluctuation integral

\[
\mathcal{E}_{\text{quartic}} = \int \frac{d^3k}{(2\pi)^3} \frac{\omega_k^0}{2}. \tag{78}
\]

In dimensional regularization it becomes

\[
\mathcal{E}_{\text{quartic}} = -\frac{m_0^4}{64\pi^2} \left\{ \frac{2}{\epsilon} + \ln \frac{4\pi \mu^2}{m_0^2} + \frac{3}{2} - \gamma \right\}. \tag{79}
\]

This expression obviously depends on the initial condition via the ‘initial mass’ \( m_0 \). If we insert the expression \[67\] we find

\[
\mathcal{E}_{\text{quartic}} = - \frac{1}{64\pi^2} \left( m^4 + \lambda m^2 \phi^2(0) + \frac{\lambda^2}{4} \phi^4(0) \right) \left\{ \frac{2}{\epsilon} + \ln \frac{4\pi \mu^2}{m_0^2} + \frac{3}{2} - \gamma \right\}. \tag{80}
\]

The divergent parts proportional to \( \phi^2(0) \) and \( \phi^4(0) \) will be cancelled by divergent parts of the fluctuation integral (see below). We define here the ‘cosmological constant’ counter term by

\[
\delta \Lambda = \frac{m_0^4}{64\pi^2} \left( \frac{2}{\epsilon} + \ln \frac{4\pi \mu^2}{m_0^2} + \frac{3}{2} \right), \tag{81}
\]

again independent of the initial condition, and retain in the expression for the energy a finite term
Having considered the quartically divergent term in the fluctuation integral we turn now to the quadratic ones. One can show that the terms $\omega^2 \text{Re} f_k^{(1)}(t)/2$ and $-\text{Re}(if^{(1)*}(t))/2$ cancel. The term proportional to $V(t)^2/4\omega_k^2$ gives in dimensional regularization

$$\Delta \Lambda = -\frac{m^4}{64\pi^2} \ln \frac{m_0^3}{m^2}. \tag{82}$$

These terms have to be considered together with the logarithmically divergent ones which are proportional to $V(t)^2$. Their explicit form follows from an analysis of the leading behaviour. It combines to

$$\mathcal{E}_{\text{quad}} = \int \frac{d^3k}{(2\pi)^3} \frac{V(t)}{4\omega_k^2} = -\frac{\lambda m^2}{64\pi^2} \left( \phi(t)^2 - \phi(0)^2 \right) \left\{ \frac{2}{\epsilon} + \ln \frac{4\pi\mu^2}{m_0^2} - \gamma + 1 \right\}. \tag{83}$$

Instead of removing the complete terms $|\overline{f}_k^{(1)}(t)|^2/4\omega_k^0$ and $2V(t)\text{Re} \overline{f}_k^{(1)}(t)/4\omega_k^0$ from the fluctuation integral we just subtract the leading behaviour from the integrand, this is algebraically less cumbersome at the expense of inducing a small difference of large quantities (which was found to be tolerable). As before (see (68)) the quadratically and logarithmically divergent terms combine in such a way that the $m_0$ is replaced by $m$ in the divergent parts. The terms proportional to $\phi^2(t)$ and $\phi^4(t)$ are cancelled by the mass renormalization and coupling constant renormalization counter terms up to finite contributions. The terms proportional to $\phi^2(0)$ and $\phi^4(0)$ are cancelled by the corresponding terms occuring the quartically divergent part (79), again up to a finite remainder

$$\Delta \Lambda' = \frac{1}{128\pi^2} \left( \frac{\lambda^2}{4} \phi^4(0) - \lambda m^2 \phi^2(0) \right). \tag{85}$$

We finally obtain for the energy density the renormalized expression

$$\mathcal{E}_{\text{ren}} = \frac{1}{2} \dot{\phi}^2(t) + \frac{1}{2}(m^2 + \Delta m^2) \phi^2(t) + \frac{\lambda + \Delta \Lambda}{4!} \phi^4(t) + \Delta \Lambda + \Delta \Lambda'$$

$$+ \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{\omega_k^0}{2} (2\text{Re} f_k^{(1)}(t) + |\overline{f}_k^{(1)}(t)|^2) \right.$$  

$$+ \frac{1}{4\omega_k^0} |\overline{f}_k^{(1)}(t)|^2$$  

$$- \frac{1}{2} \text{Re}(f_k^{(1)*}(t) + i\overline{f}_k^{(1)}(t)f_k^{(1)*}(t))$$

$$+ \frac{V(t)}{4\omega_k^0} \left( 2\text{Re} f_k^{(1)}(t) + |\overline{f}_k^{(1)}(t)|^2 \right) + \frac{V^2(t)}{16\omega_k^3} \right\}. \tag{86}$$

In the expression (78) for the pressure the energy $\mathcal{E}$ has to be replaced by the renormalized one. This replacement absorbs all the counter terms proportional to $g_{\mu\nu}$ (see (73)). The remaining counter term has to be added and the constant $A$ has to be chosen so as to cancel the remaining divergencies. Replacing at the same time the mode functions $U_k^\pm$ by their expansion in terms of the $f_k^{(n)}$ we find

$$p_{\text{ren}} = -\mathcal{E}_{\text{ren}} + \phi^2(t) + A \frac{d^2}{dt^2} \phi(t) + p_{\text{fluct}} \tag{87}$$

with

$$p_{\text{fluct}} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \left\{ \left( \langle \omega_k^0 \rangle^2 + \langle \vec{k} \rangle^2/3 \right) \left( 1 + 2\text{Re} f_k^{(1)}(t) + |\overline{f}_k^{(1)}(t)|^2 \right) \right.$$  

$$+ |\overline{f}_k^{(1)}(t)|^2 - 2\omega_k^0 \text{Re} \left( 1 + f_k^{(1)*}(t) \right) |f_k^{(1)}(t)|^2 \right\}. \tag{88}$$

This expression looks at first sight hopelessly divergent and there is just one counter term to cancel these divergencies. Using dimensional regularization we find that the leading quartic divergence reduces to a finite term via
\[
\left\{ \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_k^0} (\omega_k^0)^2 + \frac{\vec{k}^2}{3} \right\}_{\text{reg}} = (\mu) \left\{ \int \frac{d^4 k}{(2\pi)^4} \frac{1}{2\omega_k^0} (\omega_k^0)^2 + \frac{\vec{k}^2}{3} \right\} = -\frac{m_0^4}{96\pi^2}.
\]

The terms leading to a quadratic divergence are
\[
\omega_k^0 \text{Re} f^{(1)}(t) - \text{Re}(i\dot{f}^{(1)*}(t)) + \frac{\vec{k}^2}{3\omega_k^0} \text{Re} f^{(1)}(t).
\]

Analyzing their leading behaviour one finds that the possible quadratic divergence reduces again to a finite result via
\[
\left\{ V(t) \int \frac{d^3 k}{(2\pi)^3} \frac{1}{6\omega_k^0} + m_0^2 V(t) \int \frac{d^3 k}{(2\pi)^3} \frac{1}{12(\omega_k^0)^3} \right\}_{\text{reg}} = -V(t) \frac{m_0^2}{48\pi^2} \left( \frac{2}{3} + \ln \frac{4\pi^2}{m_0^2} - \gamma + 1 \right) + V(t) \frac{m_0^2}{48\pi^2} \left( \frac{2}{3} + \ln \frac{4\pi^2}{m_0^2} - \gamma \right) = -V(t)m_0^2 \frac{3}{48\pi^2}.
\]

After evaluating these leading divergencies we find
\[
p_{\text{dust}} = -\frac{m_0^4}{96\pi^2} \frac{V(t)m_0^2}{48\pi^2} + \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_k^0} \left( (\omega_k^0)^2 + \frac{\vec{k}^2}{3} \right) \left[ \frac{1}{4(\omega_k^0)^3} \int_0^t dt \sin 2\omega_k^0(t-t') \dot{V}(t') + 2\text{Re} f_k^{(2)}(t) + |f_k^{(1)}(t)|^2 \right] + |f_k^{(1)}(t)|^2 \frac{1}{2\omega_k^0} \int_0^t dt \sin 2\omega_k^0(t-t') \dot{V}(t') + 2\text{Re}(-i\omega_k^0 f_k^{(1)}(t) - i\omega_k^0 f_k^{(1)}(t)f_k^{(1)*}(t)) \right\}.
\]

The analysis of the logarithmically divergent terms becomes now very cumbersome. After some algebra and integrations by parts one finds the logarithmically divergent term
\[
\dot{V}(t) \left\{ \int \frac{d^3 k}{(2\pi)^3} \frac{1}{8(\omega_k^0)^4} \left( (\omega_k^0)^2 + \frac{\vec{k}^2}{3} \right) - \frac{1}{4(\omega_k^0)^2} \right\}_{\text{reg}} = -\frac{\dot{V}(t)}{96\pi^2} \left\{ \mu^\prime \left( \frac{2}{3} + \ln \frac{4\pi^2}{m_0^2} - \gamma \right) + \frac{1}{3} \right\}.
\]

This determines the divergent part of the renormalization constant \(A\). The finite part can be chosen freely and constitutes a free parameter of the theory, used e.g. for the ‘improved’ energy-momentum tensor. We dispose of this freedom by choosing
\[
A = -\frac{\lambda}{192\pi^2} \left\{ \mu^\prime \left( \frac{2}{3} + \ln \frac{4\pi^2}{m_0^2} - \gamma \right) \right\}
\]

since then the expression in parentheses is just the standard equivalent of \(\ln(\Lambda^2/m^2)\) in Pauli-Villars regularization. The renormalized pressure then takes the final form
\[
p_{\text{ren}} = -\varepsilon_{\text{ren}} + \dot{\phi}^2(t) - \frac{m_0^4}{96\pi^2} - \frac{V(t)}{48\pi^2} m_0^2 - \frac{\dot{V}(t)}{288\pi^2} + \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_k^0} \left( (\omega_k^0)^2 + \frac{\vec{k}^2}{3} \right) (2\text{Re} f_k^{(2)}(t) + |f_k^{(1)}(t)|^2)
\]

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\[ + \left( \frac{1}{6(\omega_k^0)^2} - \frac{m_0^2}{24(\omega_k^0)^4} \right) \int_0^t dt \cos 2\omega_k^0(t - t') \bar{V}(t') \]
\[ + \left( \frac{1}{12(\omega_k^0)^2} + \frac{m_0^2}{24(\omega_k^0)^4} \right) \cos(2\omega_k^0 t) \bar{V}(0) \]
\[ + |f_k^{(1)}(t)|^2 - 2\omega_k^0 \Re(i f_k^{(2)*}(t) + i f_k^{(1)}(t) f_k^{(1)*}(t)) \right\} . \] (95)

VII. NUMERICAL ANALYSIS

As discussed in section 3 the computation of the fluctuation term \( \Delta M^2(t) \) which determines the back-reaction of the quantum fluctuations to the classical field \( \phi(t) \) can be reduced to the computation of the mode functions \( f_k^{(1)}(t) = f_k(t) \) and \( f_k^{(2)}(t) \) via (61). The differential equations satisfied by these functions, (61) and (62), have been solved using an improved Runge-Kutta-scheme with six steps [25]. We have also checked the accuracy of \( f_k^{(2)}(t) \) by computing it via the alternative expression (54).

The momentum integrals are finite; however, the convergence is not very good, as the integrand decreases only as an inverse power of \( \omega_k^0 \). Furthermore the integrand is oscillating rapidly. If we denote the upper limit of the momentum integration as \( K \) the integrals occurring in \( \Delta M^2 \), the energy \( E \) and the pressure \( p \) converge only as \( I(K) = I_\infty + C/(\omega_k^0)^2 \) to their value \( I_\infty \). We have determined the values \( I_\infty \) by fitting the integrals as a function of \( K \) to a function of this type. This way the numerical integration can be limited to moderate values of \( K \simeq 20 \div 40 \). A delicate part of the momentum integration is the region of small momenta. There the mode functions develop a parametric resonance before their back-reaction to the classical field \( \phi(t) \) suppresses the amplitude of the latter one and the system “shuts off” [10], i.e. reaches a stationary oscillating behaviour. We have chosen momentum intervals varying from \( \Delta k \simeq 10^{-4} \) at small momenta to \( \Delta k \simeq 1 \) for larger ones. The time steps were chosen as \( \Delta t \simeq 0.001 \).

The quality of the numerical procedure was monitored by the conservation of energy which was fulfilled with an accuracy of better than 3%. With the accuracy given above we had to compute the mode functions for 1000 values of momentum. The computations can be performed on a modern PC or small workstation, they take then several hours of CPU time.

VIII. RESULTS AND CONCLUSIONS

We have presented here a computational scheme for solving the one-loop nonlinear relaxation equations with a manifestly covariant regularization and renormalization.

The results are presented in Figs. 3 - 9 for two sets of parameters, similar to those of [10]: \( \lambda/8\pi^2 = 0.1 \), \( m = 1 \) (as the general unit) and \( \phi(0) = 5 \), \( \dot{\phi}(0) = 0 \) and \( \phi(0) = 1 \), \( \dot{\phi}(0) = 0 \), respectively. In Figs. 3 and 8 we display the classical amplitude \( \phi(t) \), in Figs. 4 and 9 we show the growth of the fluctuation integral \( \Delta M^2(t) \) (see (61)). Fig. 5 shows the absolute value of the integrand (to be integrated with the measure \( k^2 dk/\omega_k^0 \)) of the second fluctuation integral in (61) vs. \( \omega_k^0 \) in a double-logarithmic scale for the same set of parameters and for \( t = 37.5 \). The amplitude of the integrand is seen to fall off as \( (\omega_k^0)^{-4} \). In Fig. 6 we plot the classical and fluctuation energies as well as the total energy for the first parameter set. The pressure is displayed for the same parameter set in Fig. 7. Its average as \( t \to \infty \) is seen to be \( E/3 \) as found previously in [28]. These numerical results show the same qualitative and similar quantitative features as those obtained in [10]. An initial large amplitude oscillation leads after a short time interval - depending on the initial amplitude - to a strong excitation of the low momentum fluctuation modes as expected from a parametric resonance. Then after a considerable decrease of the classical amplitude an essentially stationary oscillating regime is reached. In contrast to a computation in Hartree-approximation in [10] we find that even for \( \phi(0) = 1 \) a considerable amount of energy is transferred to the fluctuation modes. The main difference between a small and large initial value of the classical field consists in the time that is needed for the fluctuations to build up.

The field theory we have considered here was a rather simple one; as discussed in [10] one would not expect the one-loop approximation to be a good here nor has the system been found previously to display thermalization. Here this simple model served as a toy example for demonstrating the method. More interesting systems are of course gauge theories with many light degrees of freedom. There the differential equations for the gauge field fluctuations form coupled systems. We should like to stress that the method as developed in sections [10] and [11] can be easily
generalized to deal with such coupled channel systems. This generalization is entirely analogous to the treatment of coupled channel one loop computations in [15].
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FIGURE CAPTIONS

FIG. 1. Diagrammatic representation of the one-loop equation [33].

FIG. 2. Renormalization parts: a) Tadpole diagram, Eq. (62), b) Fish diagram, Eq. (63).

FIG. 3. The classical amplitude $\phi(t)$ for $\lambda/8\pi^2 = 0.1$ and $\phi(0) = 5$.

FIG. 4. The fluctuation integral $\Delta M^2(t)$ for the same set of parameters.

FIG. 5. The fluctuation integrand as explained in the text. The absolute value of the integrand is plotted vs. $\omega^0_k$ on a double logarithmic scale. The straight line indicates a power behaviour as $(\omega^0_k)^{-4}$.

FIG. 6. The classical and fluctuation energies as a function of time for the same parameters: classical energy (short dashed line), the fluctuation energy (long dashed line) and total energy (solid line).

FIG. 7. The pressure as a function of time for the same parameter set. The horizontal line indicates the value $p = E/3$.

FIG. 8. The classical amplitude $\phi(t)$ for $\lambda/8\pi^2 = 0.1$ and $\phi(0) = 1$.

FIG. 9. The fluctuation integral $\Delta M^2(t)$ for the same set of parameters.
Fig. 1

Fig. 2
Fig. 6
Fig. 7
Fig. 8
Fig. 9