Modern light on ancient feud: Robert Hooke and Newton’s graphical method

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Abstract

The feud between Robert Hooke and Isaac Newton has remained ongoing even after 300 years, over whether Newton should have acknowledged Hooke’s influence on his graphical method of constructing planet orbits, the celebrated Proposition 1, Theorem 1 of the Principia. The drama has escalated in recent decades, with a claim that Hooke may have used the same method and obtained an elliptical orbit for a linear force, a feat that some considered Newton never did for the inverse-square force. Modern understanding of Newton’s graphical method as a symplectic integrator can now shed light on whether this claim is creditable. This work, based on knowing the Hamiltonian of the symplectic integrator, deduced the analytical orbit corresponding to Newton’s graphical construction. A detailed comparison between this analytical orbit and Hooke’s drawing shows that it is unlikely that Hooke had used Newton’s graphical method and obtained the correct orbit.
I. INTRODUCTION

Historians of science have continued to debate Robert Hooke’s influence on Newton’s formulation of orbital dynamics\(^1\text{–}9\). According to a series of well-studied correspondences\(^10\), Hooke wrote a letter to Newton on November 24, 1679, asking for his thought on Hooke’s hypothesis of “compounding” the “direct motion by the tangent” of the planet and the “attractive motion” toward the central body. Newton replied that he did not hear about Hooke’s hypothesis prior to this letter, a claim that some have viewed as “evasive”\(^11\), “disingenuous”\(^12\), or even possibly a “bare-face lie”\(^13\). In 1684 Newton deposited the *De Motu*, an initial draft of the *Principia*, with the Royal Society, which Hooke has access. In the *De Motu*, Newton showed a graphical construction of a central force orbit incorporating Kepler’s equal area law, which later became Proposition 1, Theorem 1 of the *Principia*. (See Fig.1) To Hooke, and to many of his later supporters\(^3\text{–}9,14\), this graphical construction seemed the exact embodiment of Hooke’s idea of compounding tangential inertial motion with a central attractive force, yet Newton gave Hooke no credit whatsoever.

Newton’s action was probably due to his strong and negative reaction to Hooke’s interim accusation of him plagiarizing the inverse-square law. A year before the publication of the *Principia*, Edmond Halley wrote to Newton on May 22, 1686, relating that “... Mr Hooke has some pretension upon the invention of the rule of decrease of gravity being reciprocally as the squares of the distances from the center. He says you had the notion from him, ... Mr Hook seems to expect you should make some mention of him, in the preface [of the *Principia*]...”\(^10\) Newton was furious and wrote back on June 20, 1686 that “he [Hooke] has done nothing and yet written in such a way as if he knew and had sufficiently hinted all but what remained to be determined by the drudgery of calculations and observations, excusing himself from that labour by reason of his other business: whereas he should be rather have excused himself by reason of his inability.”\(^10\)

This acrimony not only persisted in Hooke’s and Newton’s life time, but flared up dramatically in 1989 when Pugliese\(^5\) found, among Hooke’s unpublished manuscripts, a drawing, Fig.2, dated September 1, 1685, 10 months after the *De Motu*, but nearly two years before the publication of the *Principia*, seemingly showing an elliptical orbit due to a central linear (“Hooke’s law”) force! Nauenberg\(^14,15\) has seized upon this diagram as the basis for re-assessing Hooke’s mathematical “ability”, arguing that Hooke has achieved something even
Newton did not, that of using a graphical method to solve an inverse problem of determining the orbit from a given force. A claim that has since repeated by Cooper\textsuperscript{16}.

However, Nauenberg’s conclusion was disputed by Erlichson\textsuperscript{17}, who argued that Hooke had simply constructed an ellipse geometrically by uniformly squeezing a circle and found that the resulting vertices can be fitted according to Newton’s graphical method for a linear force. Therefore, at most, what Hooke has solved is the direct problem, that given a centered elliptical orbit, the force can be determined to be linear.

It is difficult even for experts to evaluate each side of the argument. Clearly, Hooke has drawn a circumscribing circle in Fig.\textsuperscript{2}. His orbital vertices $\alpha$, $\beta$, $\kappa$, etc., were marked precisely at midpoints\textsuperscript{18} of chord lines drawn between the central vertical to points on the circle’s circumference, a known method (an affine transformation) of constructing an ellipse. This fact was admitted as possible by Nauenberg in his original article\textsuperscript{14} and in his
FIG. 2. Hooke's unpublished manuscript, taken from Ref. 5, showing his construction of an ellipse whose vertices $\alpha$, $\beta$, and $\kappa$, etc., were marked at half the distances between the central vertical and the circumscribing circle. For a more detailed description, see Nauenberg$^{14}$ and Erlichson$^{17}$. 
rebuttal\textsuperscript{22} to Erlichson. However, Hooke’s text accompanying the diagram did mention that each impulse produced by the attractive body is proportional to the radial distance.

A related puzzle is that if Hooke had been successful in graphing the orbit of a linear force, why didn’t he apply it to the all-important inverse-square force? Many of Hooke’s supporters\textsuperscript{7,9,14} believed that he did, but those drawings were simply lost. When referring to Fig.2 Purrington\textsuperscript{21} ventured that “It is limited to the case of a linear force, and while no similar calculation is known for the crucial case of the inverse-square force, such a proof would be not terribly more difficult, and it does not require a great leap of faith to suppose that such a proof by Hooke once existed.”

Fortunately, modern understanding of Newton’s graphical construction as a symplectic integrator can now shed light on these issues. A symplectic integrator is a canonical transformation which seeks to integrate Hamilton’s equation to obtain the system’s coordinate and momentum as a function of time. The crucial point about a symplectic integrator is that, the trajectory it produces, while only approximate for the original Hamiltonian it intended to solve, is exact for a Hamiltonian which is the original Hamiltonian plus some error terms. The key contribution of this work is to show that, when Newton’s graphical method is applied to a linear force, its corresponding Hamiltonian with error terms remained harmonic and exactly solvable. This means that the trajectory produced by Newton’s graphical method can be determined analytically. By comparing this analytical solution to Hooke’s construction, one can decide whether Hooke has applied Newton’s method and obtained the correct elliptical orbit. Similarly, one can repeat the analysis for the inverse-square force, and ascertain whether Newton’s graphical method can produce an ellipse with the force at one of its focus. In short, this work substantiated many of Erlichson’s\textsuperscript{17} original criticisms with precise analytical findings made possible by the modern development of symplectic integrators.

In the following, Sect.\textsuperscript{II} reviews Newton’s graphical construction, explain how its geometric formulation is translated into algebraic expressions, and why it is a symplectic integrator. Sect.\textsuperscript{III} gives the analytical form of the elliptical orbit produced by Newton’s graphical method for a linear force. Sect.\textsuperscript{IV} compares Newton’s graphical construction of an upright symmetric orbit, similar to the one found in Hooke’s manuscript. Sect.\textsuperscript{V} repeats the orbit construction for the inverse square force. Sect.\textsuperscript{VI} summaries key conclusions of this work.
II. NEWTON’S GRAPHICAL CONSTRUCTION AS A SYMPLECTIC INTEGRATOR

Newton’s graphical construction, the celebrated Proposition 1, Theorem 1 of the *Principia* is shown in Fig.1. In order to fully appreciate its relationship to symplectic integrators, it is necessary to review details of its construction. Newton stated that when a planet is initially moving at a given velocity, it will move from A to B in a given time interval. If there were no force acting on it, then by Newton’s first law, it will continue to move in the same direction in the next time interval from B to c, with distance Bc equal to AB. Triangles SAB and SBc (S=Sun) then have equal area; since their bases AB and Bc are of equal length, and both have the same apex S. However, if at B, the planet is acted on by a force impulse directed toward S along BS, instantaneously changing the planet’s velocity so that it is moving along BC instead, then it will arrive at C. The position C (this is the key point), is along the new direction BC such that cC is parallel to BS. Since cC is parallel to BS, triangles SBc and SBC have the same perpendicular height to their common base SB. Therefore, their areas are equal. It then follows that the area SBC equals area SAB, since both are equal to area SBc. Repeating this process for D, E, etc., shows that for any central force, the discrete trajectory ABCDE will sweep out equal areas SAB, SBC, SCD, SDE, etc., at equal time intervals, thus proving Kepler’s second law for any time interval.

One can illustrate this construction concretely in Fig.3 for a central linear force with acceleration exactly equal to the radial distance. One can take the initial position as \( \mathbf{r}_0 = (0, 2) \), initial velocity \( \mathbf{v}_0 = (1, 0) \) and time step \( \Delta t = 1 \). Red dots A, B, C, D, E, and F are discrete positions generated by the the analytical forms (1) and (2) given below, which anyone can plainly see as precisely matching Newton’s graphical construction as follow:

Starting at position A, the planet moves at constant velocity \( v_{x_0} = 1 \) for time \( \Delta t = 1 \) to position B. If there were no force, the planet would have drifted at the same velocity to c in another \( \Delta t \) time interval. However, at B, the central force exerted an impulse exactly equal to the distance BO, thereby changing velocity direction from AB to BC instantaneously. Because of this new velocity, the planet actually moved from B to C in time interval \( \Delta t \). The position C is precisely along the new trajectory Bd such that cC is parallel to BO. With the new velocity BC, if there were no force, the planet would again drift on from C to d. However, the impulse CO at C, changes the velocity BC to CD, and the planet actually
FIG. 3. Red dots are orbital positions produced by Newton’s graphical method for a linear central force at $\Delta t = 1$. The green ellipse is its analytical description. The black ellipse is the correct orbit. See text for details.

arrived at D, and so on. The unit triangular areas OAB, OBC, OCD, ODE, OEF, OFA, sweep out by the polygonal trajectory ABCDEF, are obviously equal. The black and green ellipses will be discuss below.

Newton’s graphical construction consisted of iterating two steps: 1) Drifting at a constant velocity for time $\Delta t$ and 2) Changing the velocity according to impulse $a\Delta t$ instantaneously. These are just Newton’s first and second laws respectively. As first noted by Nauenberg\textsuperscript{14}, these two steps, occurring at one time interval $\Delta t$, can be identified analytically as

\begin{align*}
    r' &= r + v\Delta t \\
    v' &= v + a(r')\Delta t.
\end{align*}
For a central linear or inverse square force, \( a(r) = f(r)r \), with \( f(r) = -\omega^2 r \) or \( f(r) = -k/r^3 \) respectively. Note that the updated position \( r' \) is used immediately in \( a(r') \) to update the velocity. This *sequential* updating is at the heart of Newton’s graphical method, clearly shown in Fig.[1] but easily overlook. The force impulse is evaluated at B, *after* drifting from A, and evaluated at C, *after* drifting from B, and so on. Newton singles out the drift step (1) first, then follow by the change in velocity (2) *together* with the next iteration of (1). That is, Newton iterates his algorithm at successive \( \Delta t \) to the rhythm of (1), (2)(1), (2)(1), etc., always ending with (1) drifting into the final position with the updated velocity. For example, if (2) were followed by by (1) again, one would get

\[
    r'' = r' + v'\Delta t = r + v\Delta t + v\Delta t + (a(r')\Delta t)\Delta t, 
\]

(3)
corresponding to the initial position \( r \) drifted \( 2\Delta t \) at the original velocity \( v \), then the impulse (the change of velocity) \( a(r')\Delta t \) is evaluated at the previous position \( r' \), and drift for a time \( \Delta t \) to the final position. This *precisely* corresponds to the initial drift from A to c in Fig.[1] then correcting the trajectory from c to C. Note that the trajectory correction (or deviation) \( cC = (a(r')\Delta t)\Delta t \), is parallel to the impulse \( a(r')\Delta t \) evaluated at the *prior* position \( r' \).

Equations (1) and (2) also gives an analytical proof of Kepler’s equal area law. Referring to Fig.[1] let \( SA = r \) and \( AB = v\Delta t \), then area \( SAB = |r \times v\Delta t|/2 \). After the updating according to (1) and (2), the area \( SBC = |r' \times v'\Delta t|/2 \) will be equal to that of \( SAB \), since

\[
    r' \times v' = r' \times (v + f(r')r'\Delta t) = (r + v\Delta t) \times v = r \times v, 
\]

(4)
independent of the actual form of \( f(r) \). Thus, as it is well known, Kepler’s equal area law is just angular momentum conservation.

This equal area law is is not evident in Hooke’s notion of “compounding” inertial and attractive motions. For example, if one were to replace (2) by \( v' = v + a(r)\Delta t \), which evaluates the force at the old position \( r \), and *not* at the updated position \( r' \), then the result would be the *disastrous* Euler algorithm\(^{22}\) with no conservation of angular momentum, and for a non-constant force, unstable at any \( \Delta t \), no matter how small. Yet, such a mathematical algorithm would still be consistent with Hooke’s notion of “compounding” inertial motion with an attractive force. Therefore, *for orbital motion around a central force*, what is needed, but *missing* from Hooke’s hypothesis, is a special way of “compounding” that would conserve angular momentum, *i.e.*, respecting Kepler’s area law. Even *if* he had been influenced by
Hooke’s idea of “compounding”, Newton’s way of “compounding”, of incorporating Kepler’s second law, is entirely due to his own genius.

Newton’s graphical construction, as described earlier, and obvious from its analytical form (1)-(2), is that it is sequential. As emphasized in Ref.22, this is a defining hallmark of being a symplectic integrator. As a matter of fact, it is one of two fundamental, long known first order symplectic integrator.22 (The other one, Cromer’s algorithm,23 is the sequential updating of (2) followed by (1).) When Nauenberg14 first wrote down (1)-(2), he did not know that they corresponded to a symplectic integrator. Those who knew (1)-(2) as a symplectic integrator were unaware of its connection to Newton’s graphical construction. This work, aware of this connection, also knows the Hamiltonian which govern the trajectories produced by (1)-(2).

If the original motion is governed by the Hamiltonian $H = p^2/2m + V(r)$ then the trajectory generated by Newton’s algorithm (1)-(2) is governed by the approximate Hamiltonian,24

$$H_A = H - \frac{1}{2} \mathbf{p} \cdot \mathbf{a}(r) \Delta t + O(\Delta t^2),$$

(5)

where $\mathbf{v} = \mathbf{p}/m$ and $\mathbf{a}(r) = \mathbf{F}(r)/m$. For the harmonic potential $V(r) = m\omega^2r^2/2$, the above Hamiltonian with only the first order $\Delta t$ term, is exact for Newton’s algorithm. (All higher order $\Delta t$ terms only sum to an overall multiplicative constant25,26.) Thus, as will be shown in the next Section, for a linear central force, the trajectory produced by Newton’s graphical method is known analytically.

III. ELLIPTICAL ORBITS OF A LINEAR FORCE

Consider a central linear force in two dimension, the Hamiltonian with unit mass is

$$H = \frac{1}{2} \left( \mathbf{v}^2 + \omega^2 \mathbf{r}^2 \right),$$

(6)

with trajectory given by

$$\mathbf{r}(t) = \mathbf{r}_0 \cos(\omega t) + \frac{\mathbf{v}_0}{\omega} \sin(\omega t).$$

(7)

For $\mathbf{r}_0 = (0, y_0)$, $\mathbf{v}_0 = (v_{x0}, 0)$ one has

$$x(t) = \frac{v_{x0}}{\omega} \sin(\omega t), \quad y(t) = y_0 \cos(\omega t),$$

(8)

which is the elliptical orbit

$$\left( \frac{x}{v_{x0}/\omega} \right)^2 + \left( \frac{y}{y_0} \right)^2 = 1.$$  

(9)
For \( \omega = 1 \), \( v_{x0} = 1 \) and \( y_0 = 2 \), this is the exact orbit \( x^2 + (y/2)^2 = 1 \), the black ellipse, shown in Fig.3.

The Hamiltonian (5) corresponding to Newton’s algorithm is

\[
H_A = \frac{1}{2}(\mathbf{v}^2 + \omega^2 \mathbf{r}^2 + \Delta t \omega^2 \mathbf{v} \cdot \mathbf{r})
\]

\[
= \frac{1}{2}\left((\mathbf{v} + \frac{\omega^2 \Delta t}{2} \mathbf{r})^2 + \omega^2(1 - \frac{\omega^2 \Delta t^2}{4})\mathbf{r}^2\right),
\]

which can be rewritten as

\[
H_A = \frac{1}{2}\left(\tilde{\mathbf{v}}^2 + \Omega^2 \mathbf{r}^2\right),
\]

with \( \tilde{\mathbf{v}} = \mathbf{v} + \omega^2 \Delta t \mathbf{r}/2 \) and \( \Omega = \sqrt{1 - \omega^2 \Delta t^2/4} \). Therefore as long \( |\Delta t| < 2/\omega \), this remains a harmonic oscillator with angular frequency \( \Omega \) and trajectory

\[
\mathbf{r}(t) = \mathbf{r}_0 \cos(\Omega t) + \frac{\mathbf{v}_0}{\Omega} \sin(\Omega t),
\]

where \( \mathbf{v}_0 = \mathbf{v}_0 + \omega^2 \Delta t \mathbf{r}_0/2 \). Here, \( H_A \) is the Hamiltonian exactly conserved by Newton’s algorithm (1)(2) in that \( H_A(\mathbf{r}', \mathbf{v}') = H_A(\mathbf{r}, \mathbf{v}) \).

For a given \( \Delta t \), and with the same initial conditions as above, the trajectory generated by Newton’s algorithm (12) is

\[
\begin{align*}
x(t) &= \frac{v_{x0}}{\Omega} \sin(\Omega t), \\
y(t) &= y_0 \cos(\Omega t) + \frac{\omega^2 \Delta t y_0}{2\Omega} \sin(\Omega t),
\end{align*}
\]

which is a closed elliptical orbit

\[
\left(\frac{x}{v_{x0}/\Omega}\right)^2 + \left(\frac{y - (\omega^2 \Delta t y_0/2\Omega)}{y_0}\right)^2 = 1.
\]

Thus Newton’s graphical construction, when applied to a linear central force, will always yield orbital positions on a closed ellipse, as long as \( |\Delta t| < 2/\omega \).

This surprisingly result is the key contribution this work. Normally, when one applies a numerical method, one has no expectation that it will yield anything resembling the exact result at a large value of \( \Delta t \). To illustrate this we compare Newton’s graphically constructed orbit in Fig.4 with trajectories produce by other well-known algorithms.

That’s why Erlichson remarked that, had Hooke used Newton’s method to solve the inverse problem, “then there would be no requirement that the vertex points lie on-orbit [on
FIG. 4. Comparing Newton’s graphical method (red) with Euler’s algorithm (blue), second-order Runge-Kutta algorithm RK2 (orange) and fourth-order Runge-Kutta algorithm RK4 (light blue) at $\Delta t = 1$ for 18 iterations. Only Newton’s graphical construction remains a closed orbit.

...the elliptical orbit]”. He therefore thought that Hooke’s results were too good to be true, and can’t possibly from using Newton’s method. However, Nauenberg, also observed that “In general, a graphical construction can give only an approximation to such a curve, but in this case, the resulting vertex points of the polygon lie on an ellipse.” Since Hooke’s construction is consistent with using a linear force, Nauenberg drawn the opposite conclusion that Hooke had produced an elliptical orbit using Newton’s graphical method. Both conclusions were wrong because both tacitly assumed that the ellipse drawn by Hooke is the correct orbit for the linear force. They did not anticipate the result of this work, that Newton’s graphical method, when applied to a linear force, will always yield positions on a closed ellipse, but that ellipse is not the correct elliptical orbit!
FIG. 5. linear force orbit (red dots) generated by Newton’s graphical method at $\Delta t = 1$ with a pair of symmetric initial positions A and B.

That ellipse, (14), is shown in Fig 8 as the green ellipse encompassing the polygonal orbit ABCDEF. This green ellipse approaches the correct black ellipse only in the limit of $\Delta t \to 0$. At any finite $\Delta t$, this ellipse has a characteristic “tilt” from the vertical because its first step $r_1 = (v_x \Delta t, y_0)$ is asymmetric with respect to the $y$-axis. Therefore, as originally noted by Erlichson, had Hooke just applied Newton’s graphical method, using similar initial conditions, he would not have gotten an upright, left-right symmetric ellipse (to be discussed in the next Section). The reason why Hooke did not choose such a natural starting point is clear; he did not apply Newton’s graphical method.
IV. THE UPRIGHT SYMMETRIC ELLIPSE

Instead, Hooke chooses the initial position to be \( \mathbf{r}_0 = (-v_x \Delta t/2, y_0) \), so that after step (1) of Newton’s algorithm, \( \mathbf{r}_1 = (v_x \Delta t/2, y_0) \). This then guarantees that the first two vertices of the polygonal orbit will be symmetric with respect to the \( y \)-axis. The resulting orbit from (12) is then given by

\[
\begin{align*}
\omega^2 x(t) \frac{v_x}{v_{x0}} &= -\frac{\omega^2 \Delta t}{2} \cos(\Omega t) + \Omega \sin(\Omega t), \\
\Omega \frac{y(t)}{y_0} &= \Omega \cos(\Omega t) + \frac{\omega^2 \Delta t}{2} \sin(\Omega t).
\end{align*}
\]

(15)

The cross terms cancel when both sides are squared and added. Since \( \Omega^2 + \omega^4 \Delta t^2/4 = \omega^2 \), this gives remarkably,

\[
\left( \frac{x}{v_{x0}/\omega} \right)^2 + \left( \frac{y}{y_0/\sqrt{1 - \omega^2 \Delta t^2/4}} \right)^2 = 1,
\]

(16)

which differs from the exact orbit of (9) only by the factor \((1 - \omega^2 \Delta t^2/4)^{1/2}\)! Such a starting condition not only produces an upright symmetric ellipse, but also dramatically improved the algorithm from first to second order in \( \Delta t \)! There is now no “tilt” proportional to \( \Delta t \). Using the same parameter values as in Fig.3, the algorithm’s orbit (16) is shown as the green ellipse ABCDEF in Fig.5. This green ellipse is now much closer to the black ellipse, but it is still not the correct orbit! The discrete graphical constructed points are again A, B, C, D, E and F. It is now even more obvious that Kepler’s equal area law is obeyed: the areas swept out are equal area segments of the hexagon ABCDEF.

However, the choice of \( \Delta t = 1 \), which divided the algorithm’s orbit exactly into six segments, is rather special. Although the orbit of the algorithm is determined by \( \Omega \), the actual angular velocity of the algorithm is given by \( \Omega_A \), defined by

\[
\cos(\Omega_A \Delta t) = 1 - \omega^2 \Delta t^2/2.
\]

(17)

Therefore, if one were to choose \( \Delta t \) such that \( T/\Delta t = n \), where the period \( T = 2\pi/\Omega_A \), then one must have \( \Omega_A \Delta t = 2\pi/n \) or

\[
\Delta t = \frac{1}{\omega} \sqrt{2 \left(1 - \cos \left(\frac{2\pi}{n}\right)\right)},
\]

(18)

which gives \( \Delta t = 1 \) for \( \omega = 1 \) and \( n = 6 \). For any other arbitrary value of \( \Delta t \), say \( \Delta t = 0.8 \), \( \Delta t \) would not divide the period evenly. This is shown in Fig.6. The positions
produced by the graphical method still lie on the predicted green ellipse (16), but are no longer left-right symmetric. This is another of Erlichson’s criticism. Therefore, if Hooke were simply applying Newton’s graphical method, he would have gotten, either the tilted polygonal orbit of Fig. 3, or an upright but left-right asymmetric polygonal orbit of Fig. 6. The fact that Hooke is marking off parallel chord lines in Fig. 2 shows that he is using the circumscribing circle to construct left-right symmetric elliptical vertices. But such vertices are highly unlikely, because unless $\Delta t$ is specifically chosen according to (18), the resulting orbital vertices cannot be left-right symmetric.

Nauenberg also claimed that Hooke’s diagram (Fig. 2) proved the area law differently from Newton. Newton’s graphical proof of Kepler’s area law is general, true for any central force. Hooke’s demonstration of the area law is specific to his ellipse construction. This is because Hooke is uniformly squeezing the circle; areas of pie-like slices on the circumscribing

FIG. 6. Same as Fig. 5 but with $\Delta t = 0.8$. 
circle are all squeezed by the same factor into areas of pie-like slices on the ellipse. Therefore, if the original slices on the circle are of equal area, then the squeezed slices on the ellipse are also of equal area. This is just a geometric consequence of Hooke’s construction, not a general embodiment of Kepler’s equal area law.

V. THE INVERSE-SQUARE FORCE

What happens when Newton’s graphical method is applied to the gravitational force? Why didn’t Newton use it to compute Kepler’s orbit? Since Newton has claimed in Corollary 1 of Book 1’s Proposition 13, that an inverse-square force would yield an elliptical orbit, he would not have felt the need to use his graphical method to prove anything else except Kepler’s second law. However, others, such as Weinstock, have argued that Newton’s Corollary 1 is not a proof, and that Newton never solved the inverse problem for the inverse square force in the *Principia*. But is it possible that Hooke could have used the same graphical method to produce an orbit for the inverse-square force, as suggested by Purrington?

Nauenberg has shown that if $\Delta t$ is too large, the orbit will not be elliptical near the force center. Here, we can again study the trajectory produced Newton’s graphical method in greater details by analyzing the approximate Hamiltonian. For the inverse-square force, the approximate Hamiltonian is

$$H_A = \frac{1}{2} \mathbf{v}^2 - \frac{k}{r} + \frac{\Delta t}{2} \frac{k \mathbf{r} \cdot \mathbf{v}}{r^3} + O(\Delta t^2).$$

In the linear force case, the first-order error term ($\propto \mathbf{v} \cdot \mathbf{r}$) is less singular (as $r \to \infty$) than the original potential ($\propto r^2$), and therefore the approximate Hamiltonian can still be redefined as harmonic with a potential $\propto r^2$. This then produces at most, a tilted, but still closed elliptical orbit. By contrast, for the gravitational force, the first-order $\Delta t$ term above ($\propto \mathbf{v} \cdot \mathbf{r}/r^3$) is more singular (as $r \to 0$) than the original potential ($\propto 1/r$) and can never be redefined so that the approximate Hamiltonian is having a potential $\propto 1/r$. By Bertrand’s theorem, if the potential is neither $r^2$ nor $1/r$ exactly, then the orbit cannot be closed. Thus when applied to the gravitational force, Newton’s graphical method, even if $\Delta t$ is sufficiently small so that the trajectory can go around the force center, will not yield a closed orbit. The orbit will remain open and just precess. For the kind of geometrical
FIG. 7. Inverse-square force orbits generated by Newton’s graphical method at $\Delta t = 0.5$ (red) $\Delta t = 0.2$ (blue-green) and $\Delta t = 0.01$ (blue). See text for details.

construction that can be done by hand, one would not get a recognizable ellipse. This is illustrated in Fig. 7 for $\Delta t = 0.5$, $k = 1$ and a symmetrical start with initial velocity reduced to $v_{x0} = 0.5$. The exact orbit with eccentricity $1/2$ given by

$$
\left(\frac{x}{2/\sqrt{3}}\right)^2 + \left(\frac{y - 2/3}{4/3}\right)^2 = 1
$$

(20) can be produced by Newton’s algorithm (1)-(2) with $\Delta t = 0.01$. The red dot trajectory generated by $\Delta t = 0.5$ also coincide with the correct orbit as it proceeds down the right side of the ellipse. However, as noted by Nauenberg, this is no longer the case after the trajectory passes below the force center at the origin. Further iterations show that the orbit precesses counter-clockwise. After $\approx 5$ periods, the semi-major axis has rotated $90^\circ$. (The precession shown is actually due to the $\Delta t^2$ terms (not shown) in (19) because the $\Delta t$ term
contribution vanishes after each period. To verify this $\Delta t^2$-dependence, Fig.7 also shows that the orbital precession for $\Delta t = 0.2$ is roughly $(0.5/0.2)^2 \approx 6$ times smaller than that of $\Delta t = 0.5$.)

Thus when Newton’s graphical method is applied, the resulting trajectory differs greatly depending on whether the central force is linear or inverse-square. For the inverse square force a closed orbit is possible only for very small time steps, such as $\Delta t = 0.01$, as shown in Fig.7. However, such a time step is probably 50 times too small to be drawn by hand. Most likely, a hand construction would resemble the red dot orbit in Fig.7 corresponding to $\Delta t = 0.5$. In this case, the orbit will not close. Therefore, contrary to Purrington’s musing, it is also unlikely that Hooke could have constructed by hand, a closed elliptical orbit for the inverse square force.

VI. CONCLUSIONS

This work, by knowing the Hamiltonian which governs the trajectory of a symplectic integrator, has shown that when Newton’s graphical method is applied to a central linear force, the resulting trajectory is always an centered ellipse, just not the correct elliptical orbit. By carefully comparing this analytically ellipse to Hooke’s drawing, this work substantiated Erlichson’s conclusion that Hooke was trying to repeat what he had done before with a circular orbit, that he was trying to follow an elliptical orbit in order to determine the required force. He constructed an elliptical orbit, then found that, consistent with Newton’s graphical construction, the required force is linear. This solves the direct, not the inverse problem. Moreover, even this success is fortuitous; because by the same result above, it is a peculiar property of Newton’s graphical method that any centered ellipse can be fitted to a linear force.

When the same analysis is applied to the inverse square force, the result is drastically different. When Newton’s graphical method is applied by hand to the inverse square force, it is unlikely to yield a closed, elliptical orbit. This is the more likely reason why no similar drawing for the inverse square force can be found in Hooke’s possession.
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