QUANTUM $N = 2$ SUPER $W_3^{(2)}$ ALGEBRA IN SUPERSPACE

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Abstract

We discuss the $N = 2$ extension of Polyakov-Bershadsky $W_3^{(2)}$ algebra with the generic central charge, $c$, at the quantum level in superspace. It contains, in addition to the spin 1 $N = 2$ stress tensor, the spins 1/2, 2 bosonic and spins 1/2, 2 fermionic supercurrents satisfying the first class nonlinear chiral constraints. In the $c \to \infty$ limit, the “classical” $N = 2 W_3^{(2)}$ algebra is recovered.

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1. Introduction

In recent years various (super)extensions of nonlinear $W$ algebras have been studied in two dimensional rational conformal field theory [1]. There exist very special algebras, quasi-superconformal algebras, which include the bosonic currents with noncanonical half-integer spins. One of the examples of such an algebra is the Polyakov-Bershadsky $W_{3}^{(2)}$ algebra [2, 3]. It is a bosonic analog of the linear $N = 2$ superconformal algebra (SCA) [4]: it contains two bosonic currents with noncanonical spins $3/2$, besides two bosonic currents with canonical spins 1 and 2. Quadratic nonlinearity in the operator product expansion (OPE) of the spin 3/2 currents in this algebra emerges from requiring the associativity.

In ref. [5], an $N = 2$ supersymmetric extension of this algebra has been constructed using the Polyakov’s soldering procedure at the classical level. This extension comprises four additional fermionic currents with non-canonical integer spins $(1, 1, 2, 2)$, besides $N = 2$ SCA and $W_{3}^{(2)}$ algebra as two different subalgebras. For the quantum case, it was further studied in [6]. A new feature of the quantum case is the necessity to include, in the r.h.s. of OPEs, extra composite currents which were not present in the classical case [5]. These are essential in order that OPEs between the twelve $N = 2$ $W_{3}^{(2)}$ currents form a closed set, in other words, satisfy the Jacobi identities. The currents of this $N = 2$ super $W_{3}^{(2)}$ algebra, both at the classical and quantum levels, cannot be arranged into $N = 2$ supermultiplets with respect to the original (manifest) $N = 2$ SCA because the numbers of currents with integer and half-integer spins do not match each other.

Recently it has been shown [7], at the classical level, that a $N = 2$ superfield formulation of this algebra can be achieved by exploring another, hidden $N = 2$ SCA. In [7], $N = 2$ $W_{3}^{(2)}$ algebra has been reformulated in terms of the spins $(1/2, 2)$ bosonic supercurrents and spins $(1/2, 2)$ fermionic ones satisfying the first class nonlinear chiral constraints, and a modified $N = 2$ spin 1 stress tensor $J(Z)$ (see the next section for notation). In [7] this $N = 2$ superfield formulation of classical $N = 2$ $W_{3}^{(2)}$ algebra has been reproduced in the framework of $N = 2$ superfield Hamiltonian reduction, starting from $N = 2$ super affine extension of the superalgebra $sl(3|2)$.

In the present article we show that the quantum $N = 2$ super $W_{3}^{(2)}$ algebra of ref. [6] also admits a similar superfield description with respect to the modified $N = 2$ SCA. We give the relevant SOPEs in the explicit form and compare them with the classical expressions of ref. [7].

2. $N = 2$ superfield structure of quantum $N = 2$ $W_{3}^{(2)}$ algebra

We start by recalling the basic points of ref. [7] that is closely related to our study.

$N = 2$ $W_{3}^{(2)}$ algebra in its original form [7] is generated by six bosonic currents $\{J_{w}, J_{s}, G^{+}, G^{-}, T_{w}, T_{s}\}$ and six fermionic ones $\{S_{1}, \bar{S}_{1}, S, \bar{S}, S_{2}, \bar{S}_{2}\}$ with the spins $\{1, 1, 3/2, 3/2, 2, 2\}$, respectively. A sum of $T_{w}, T_{s}$ and some appropriate composite currents is chosen as the Virasoro stress tensor.

In order to equalize the numbers of the half-integer and integer spins, the authors of [7] passed to another, twisted Virasoro stress tensor by making use of the fact that $\{G^{+}, G^{-}, S_{1}, \bar{S}_{1}, S, \bar{S}, S_{2}, \bar{S}_{2}\}$ have non-zero $u(1)$ charges associated with the spin 1 currents
$J_w$ and $J_s$. This properly twisted stress tensor is given by the expression:

$$T_s + T_w + \frac{4}{c} S_1 \bar{S}_1 - \frac{4}{c} J_s^2 + \frac{12}{c} J_w J_s - \frac{12}{c} J_w^2 - \partial J_s. \quad (1)$$

With respect to it the above eight currents possess, respectively, the following spins

$$(1/2, 5/2, 1/2, 3/2, 2, 1, 3/2, 5/2). \quad (2)$$

The spins of the remaining currents $\{J_s, J_w, T_s, T_w\}$ are the same as before, that is, $(1, 1, 2, 2)$. This way one succeeds in getting equal number of currents with integer and half-integer spins\(^{[7]}\).

Then, by adding two spins $(5/2, 3)$ composite bosonic currents and two fermionic ones of the same spins, the $N = 2 W^{(2)}_3$ algebra at the classical level can be rewritten in terms of the following five $N = 2$ supercurrents: a general spin 1 supercurrent $J(Z)$, spin 1/2 anti-chiral fermionic and bosonic supercurrents $G(Z)$ and $Q(Z)$, general spin 2 fermionic $F(Z)$ and bosonic $T(Z)$ ones\(^{[7]}\). To ensure the irreducible current content of the algebra, the latter two supercurrents should be subject to nonlinear chirality constraints\(^{[7]}\) the precise form of which for the quantum case will be given below. It can be found in\(^{[7]}\) how the $N = 2 W^{(2)}_3$ currents are spread over these five superfields. For what follows it will be important to remember that one independent combination of the original two $u(1)$ currents $J_w$ and $J_s$, $\bar{J}_s$, appears as the lowest component of the $N = 2$ stress tensor $J(Z)$, while another, $\bar{J}$, as the second, highest component in the anti-chiral superfield $G(Z)$.

From now on, we want to extend the consideration of ref.\(^{[7]}\) to the quantum case. The basic new feature is that now we should take care of Jacobi identities to all orders in contractions between the composite supercurrents.

As a starting point we take a natural assumption that the supercurrent $J(Z)$ generates the standard linear $N = 2$ SCA\(^{[7]}\)

$$J(Z_1)J(Z_2) = -\frac{1}{Z_{12}^2} 2c + \frac{\theta_{12} \bar{\theta}_{12}}{Z_{12}^2} J + \frac{\theta_{12}}{Z_{12}} \overline{D} J - \frac{\bar{\theta}_{12}}{Z_{12}} D J + \frac{\theta_{12} \bar{\theta}_{12}}{Z_{12}} \partial J, \quad (3)$$

where

$$\theta_{12} = \theta_1 - \theta_2, \quad \bar{\theta}_{12} = \bar{\theta}_1 - \bar{\theta}_2, \quad Z_{12} = z_1 - z_2 + \frac{1}{2} \left( \theta_1 \bar{\theta}_2 - \theta_2 \bar{\theta}_1 \right), \quad (4)$$

and $D, \overline{D}$ are the spinor covariant derivatives defined by

$$D = \frac{\partial}{\partial \theta} - \frac{1}{2} \theta \frac{\partial}{\partial z}, \quad \overline{D} = \frac{\partial}{\partial \bar{\theta}} - \frac{1}{2} \theta \frac{\partial}{\partial z}, \quad (5)$$

\(^{[1]}\)It is to the point here to remark that the common belief that the spin 1/2 currents can be factored out to yield a smaller nonlinear algebra\(^{[7]}\) is not true in the case under consideration\(^{[7]}\). The reason is that OPEs between spin 1/2 currents do not contain central terms the presence of which is crucial for these currents to be decoupled\(^{[7]}\). By the same reason one cannot decouple the spin 1/2 supercurrents in the superfield form of $N = 2$ super $W^{(2)}_3$.

\(^{2}\)By $Z$ we denote the coordinates of $1D N = 2$ superspace, $Z = (z, \theta, \bar{\theta})$.

\(^{3}\)Hereafter we do not write down the regular parts of SOPEs. All the supercurrents appearing in the right-hand sides of the SOPEs are evaluated at the point $Z_2$. Multiple composite currents are always regularized from the right to the left.
\{ \mathcal{D}, G \} = \frac{\partial}{\partial z}, \quad \{ \mathcal{D}, \mathcal{D} \} = \{ G, G \} = 0.

Let us also assume that all the SOPEs which in the classical case involve no nonlinearities retain their structure in the quantum case too (an analogous assumption in the component approach proved to be true [3]). Then the remaining four supercurrents have the above-mentioned spins with respect to this $N = 2$ SCA

\begin{align*}
J(Z_1)G(Z_2) &= -\frac{\theta_{12}}{Z_{12}^2} c + \frac{\theta_{12}}{Z_{12}^2} \frac{1}{2} G - \frac{1}{Z_{12}} G - \frac{\theta_{12}}{Z_{12}} \mathcal{D} G + \frac{\theta_{12}}{Z_{12}} \mathcal{D} G, \\
J(Z_1)Q(Z_2) &= \frac{\theta_{12}}{Z_{12}^2} \frac{1}{2} Q - \frac{1}{Z_{12}} Q - \frac{\theta_{12}}{Z_{12}} \mathcal{D} Q + \frac{\theta_{12}}{Z_{12}} \mathcal{D} Q, \\
J(Z_1)F(Z_2) &= \frac{\theta_{12}}{Z_{12}^2} \frac{1}{2} F + \frac{\theta_{12}}{Z_{12}} \mathcal{D} F - \frac{1}{Z_{12}} F + \frac{\theta_{12}}{Z_{12}} \mathcal{D} F, \\
J(Z_1)T(Z_2) &= \frac{\theta_{12}}{Z_{12}^2} \frac{1}{2} T + \frac{\theta_{12}}{Z_{12}} \mathcal{D} T - \frac{1}{Z_{12}} T + \frac{\theta_{12}}{Z_{12}} \mathcal{D} T. 
\end{align*}

Note that the $u(1)$ charges $-1$ of two anti-chiral supercurrents $G(Z)$ and $Q(Z)$ with respect to the $u(1)$ current $\bar{J}$ are strictly fixed by their chirality property

$$\mathcal{D} G(Z) = \mathcal{D} Q(Z) = 0.$$  \hfill (7)

While acting on both sides of the relevant SOPEs at the point $Z_2$, $\mathcal{D}$ should yield zero.

As a next step, we are led to consider the most general ansatz for the remaining SOPEs, such that it is consistent with symmetry under the interchange $Z_1 \leftrightarrow Z_2$, statistics, spins and the conservation of two $u(1)$ charges (the property that $Q(Z)$ and $F(Z)$ have nonzero $u(1)$ charges with respect to the $u(1)$ current $\bar{J}$ strictly fixes the SOPEs of these superfields with $\mathcal{D} G$ and hence with $G$ itself, see first two equations in [3]). After inclusion of all possible composite currents with undetermined structure constants we are left with more than 200 terms, in contrast to the classical consideration. In the component approach [3], we have experienced that the quantum Jacobi identities are not satisfied if one specializes only to those algebraic structures which are present at the classical level. Nonetheless, we will see that only three extra composite currents finally survive. Our approach to fixing the structure constants is extremely direct and straightforward: it goes by exploiting the Jacobi identities between the supercurrents.

As a final result we arrive at the following SOPEs which obey all the Jacobi identities except for $(T, T, T)$ \footnote{We have not been able to check the Jacobi identity of $(T, T, T)$ directly in package SOPEN2defs [10] using the desk-top. This check can be done in an indirect way, as discussed below.} for the generic value of the central charge

\begin{align*}
G(Z_1)Q(Z_2) &= -\frac{\theta_{12}}{Z_{12}^2} Q, \\
G(Z_1)F(Z_2) &= \frac{\theta_{12}}{Z_{12}^2} F, \\
G(Z_1)T(Z_2) &= -\frac{\theta_{12}}{Z_{12}^3} 2c - \frac{\theta_{12}}{Z_{12}^3} 2G - \frac{1}{Z_{12}^2} 2G + \frac{\theta_{12}}{Z_{12}^2} [J + 2\mathcal{D} G].
\end{align*}
\[
Q(Z_1)F(Z_2) = \frac{\theta_{12}}{Z_{12}^3} 2c + \frac{\theta_{12} \bar{\theta}_{12}}{Z_{12}^2} 2G + \frac{1}{Z_{12}^2} 2G - \frac{\theta_{12}}{Z_{12}^2} J + 2DG \]
\[
+ \frac{\theta_{12}}{Z_{12}^2} \frac{\bar{\theta}_{12}}{Z_{12}^2} \frac{1}{(-1 + 2c)} \left[ \frac{1}{2} (1 + 2c) \mathcal{D}J + 4GDG + 2JG - (-1 + 2c) \partial G \right] \]
\[
+ \frac{1}{Z_{12} (-1 + 2c)} \left[ (1 + 2c) \mathcal{D}J + 8GDG + 4JG + 2(-1 + 2c) \partial G \right],
\]

\[
Q(Z_1)T(Z_2) = -\frac{\theta_{12} \bar{\theta}_{12}}{Z_{12}^2} 2Q - \frac{1}{Z_{12}^2} 2Q + \frac{\theta_{12} \bar{\theta}_{12}}{Z_{12}^2} 2DQ \]
\[
+ \frac{1}{Z_{12}} \frac{\theta_{12}}{Z_{12}^2} \frac{1}{(-1 + 2c)} \left[ 4GDQ + 2JQ + 4DGQ + (3 - 2c) \partial Q \right] \]
\[
+ \frac{1}{Z_{12} (-1 + 2c)} \left[ 8GDQ + 4JQ + 8DGQ + 2(3 - 2c) \partial Q \right],
\]

\[
F(Z_1)T(Z_2) = \frac{1}{Z_{12}^2} 4F - \frac{\theta_{12}}{Z_{12}^2} \frac{1}{(1 + 2c)} (3 - 2c) \partial J - \frac{\theta_{12}}{Z_{12}^2} 2DF \]
\[
+ \frac{1}{Z_{12}^2} \frac{\bar{\theta}_{12}}{Z_{12}^2} \frac{1}{(-1 + 2c)} \left[ \frac{1}{16} GDF \right] \]
\[
+ 4JF + \frac{8(1 + 6c)}{(1 + 2c)} DGF + (-1 - 6c) \partial F \]
\[
+ \frac{1}{Z_{12}^2} \frac{\theta_{12}}{Z_{12}^2} \frac{1}{(-1 + 2c)} \left[ \frac{32}{(1 + 2c)} GDF \right] \]
\[
+ \frac{1}{Z_{12}^2} \frac{1}{(-1 + 2c)} \left[ 2(1 + 2c) \partial DF - 4JDF - 8DGDF + 4DF \right],
\]

\[
T(Z_1)T(Z_2) = \frac{1}{Z_{12}^2} 4T - \frac{\theta_{12}}{Z_{12}^2} \frac{1}{(1 + 2c)} (3 - 2c) \partial J - \frac{\theta_{12}}{Z_{12}^2} 2DT \]
\[
+ \frac{1}{Z_{12}^2} \frac{\bar{\theta}_{12}}{Z_{12}^2} \frac{1}{(-1 + 2c)} \left[ \frac{16}{(1 + 2c)} GDT + 4JT - \frac{16}{(1 + 2c)} QDF \right] \]
\[
+ \frac{8(1 + 6c)}{(1 + 2c)} DGT + \frac{8(1 + 6c)}{(1 + 2c)} DQF + (-1 - 6c) \partial T \]
\[
+ \frac{1}{Z_{12}^2} \frac{\theta_{12}}{Z_{12}^2} \frac{1}{(-1 + 2c)} \left[ \frac{32}{(1 + 2c)} GDT \right] \]
\[
+ \frac{1}{Z_{12}^2} \frac{1}{(-1 + 2c)} \left[ \frac{32}{(1 + 2c)} GDF \right] \]
\[
+ \frac{16}{(1 + 2c)} JQF - 4DJT - \frac{16}{(1 + 2c)} DGQF - 8 \partial GT - \frac{8(3 + 2c)}{(1 + 2c)} \partial QF \].
\[ + \frac{\theta_{12}}{Z_{12} (-1 + 2c)} \left[ 2(1 + 2c) \partial DT - 4JDT - 8DGDT + 4DJT + 8DQDF \right] \]
\[ + \frac{\theta_{12} \bar{\theta}_{12}}{Z_{12} (-1 + 2c)} \left[ 4J\partial T - 4\bar{\mathcal{D}}JDT + 16DG\partial T + \frac{16}{(1 + 2c)} \mathcal{D}JGT \right. \]
\[ + \frac{16}{(1 + 2c)} \mathcal{D}JQF + 16DQ\partial F + 16\partial DGT + 16\partial DQF - 2(1 + 2c)\partial^2 T \right] \]

Thus we have the complete structure of \( N = 2 \) quantum \( W_3^{(2)} \) algebra, \( (4) \), \( (5) \), \( (8) \), in \( N = 2 \) superspace.

Let us make several comments on this result.

The above SOPEs are consistent only on the shell of the constraints:

\[ A_1 \equiv \mathcal{D}F + \frac{4}{1 + 2c} (GF) = 0 \] \hspace{1cm} (9)

\[ A_2 \equiv \mathcal{D}T + \frac{4}{1 + 2c} (GT) + \frac{4}{1 + 2c} (QF) = 0 \] \hspace{1cm} (10)

which are the proper quantum version of the constraints of the classical case \([7, 8]\). We systematically used the above two constraints while fixing the structure constants in the SOPEs. We have checked that these constraints are first class as in the classical case, that is, the SOPEs between \( A_1 \) and \( A_2 \) (as well as their SOPEs with all supercurrents) vanish on the shell of constraints. The classical form of \( A_1, A_2 \) is recovered in the \( c \to \infty \) limit (properly implemented). Note that all the above SOPEs are by construction consistent with the linear anti-chirality conditions \((7)\).

The following correspondence between the component currents and superfields agrees with the previous results \([6]\),

\[ J_w = \frac{1}{6} (J + 4DG) \] \[ J_s = \frac{1}{2} (J + 2DG) \]

\[ G^+ = \frac{1}{\sqrt{2}} Q |, \quad G^- = -\sqrt{2}DF | \]

\[ T_w = \frac{1}{(2c + 3)} \left[ \frac{(1 - 2c)}{2} \mathcal{T} - \frac{(2c + 1)}{2} [\mathcal{D}, \mathcal{D}]J + \frac{2}{3} J + \frac{4}{3} JDG \right. \]
\[ \left. + \frac{8}{3} DGDG - 4DGJ + \partial J \right] | , \]

\[ T_s = \frac{1}{2} \left[ T + \partial J + 2\partial DG \right] | , \]

\[ S_1 = \frac{1}{\sqrt{2}} G |, \quad \bar{S}_1 = \frac{1}{\sqrt{2}} \mathcal{D}J | , \]

\[ S = -F |, \quad \bar{S} = -\mathcal{D}Q | \]

\[ S_2 = \frac{1}{\sqrt{2}} \left[ \frac{(2c + 1)}{2c - 1} \mathcal{D}J - \frac{(2c + 3)}{c(2c - 1)} GDG - \frac{(2c + 1)}{c(2c - 1)} JG + \partial G \right] | , \]

\[ \bar{S}_2 = \frac{1}{\sqrt{2}} \left[ \mathcal{D}T + \partial DJ - \frac{1}{c} J + \frac{3}{c} \mathcal{D}JDG \right] | , \] \hspace{1cm} (11)
where $|$ means the $\theta, \bar{\theta}$ independent part of corresponding composite operators. All four composite currents of spins $(5/2, 5/2, 3, 3)$ are given by

$$\mathcal{D}F \ |, \ \mathcal{D}T \ |, \ [\mathcal{D}, \mathcal{D}]T \ |, \ [\mathcal{D}, \mathcal{D}]F \ | \ .$$

By exploiting the constraint equations (10), these composite currents can be written through the basic twelve elementary currents defined in (11).

As was mentioned earlier, there is a trouble with checking Jacobi identity for $(T, T, T)$ directly in $N = 2$ superspace. This difficulty can be got round by the following argument. Using the correspondence (11) and resorting to the component results of [6] we are already guaranteed that the Jacobi identities

$$(T, T, T) \ |, \ (T, T, DT) \ |, \ (T, DT, DT) \ |, \ (DT, DT, DT) \ | \ (13)$$

are satisfied. Then other Jacobi identities containing $\mathcal{D}T \ |, [\mathcal{D}, \mathcal{D}]T \ |$ can be checked indirectly with making use of the above constraints, taking into account that the l.h.s. of these identities can be written as some composites involving $T \ |, DT \ |$. The latter property follows from the relations

$$[\mathcal{D}, \mathcal{D}]T \ | = -\frac{4}{(1 + 2c)} [GT + QF] \ |,$$

$$[\mathcal{D}, \mathcal{D}]T \ | = \left\{ \partial T + \frac{8}{(2c + 1)} [GDT - QDF - DGT - DQF] \right\} \ .$$

One observes that there are extra composite currents in the r.h.s. of (8) which do not appear at the classical level [7]:

$$F(Z_1)T(Z_2)_{q, corr.} = \frac{\theta_{12}}{Z_{12}^2} \frac{16}{(2c + 1)(2c - 1)} GDF,$$

$$T(Z_1)T(Z_2)_{q, corr.} = \frac{\theta_{12}}{Z_{12}^2} \frac{16}{(2c + 1)(2c - 1)} [GDT - QDF] \ .$$

Only these three currents survive among more than 200 ones we started with. These surviving terms vanish in the appropriate classical limit, $c \to \infty$, as it was discussed in [8]. Also it can be checked that after taking this limit all the structure constants present in the SOPEs (8) go into those appearing in the corresponding SOPEs of [7].

3. Conclusion

To summarize, as a generalization and development of previous findings [5 - 8] we have determined the full structure of quantum $N = 2$ super $W_3^{(2)}$ algebra in $N = 2$ superspace. It would be interesting to construct its free superfield realization which in components reduces to the "hybrid" field realization found in [8].

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References

[1] P. Bouwknegt, K. Schoutens, Phys. Reps. **223** (1993) 183

[2] A. Polyakov, Int. J. Mod. Phys. **A5** (1990) 833

[3] M. Bershadsky, Commun. Math. Phys. **139** (1991) 71

[4] M. Ademollo et. al., Phys. Lett. **B62** (1976) 105

[5] S. Krivonos and A. Sorin, “\(N = 2\) super-\(W_3^{(2)}\) algebra”, Preprint LNF-94/014 (P), [hep-th/9402064](http://arxiv.org/abs/hep-th/9402064), unpublished

[6] C. Ahn, S. Krivonos and A. Sorin, Mod. Phys. Lett. **A10** (1995) 1299, [hep-th/9501100](http://arxiv.org/abs/hep-th/9501100)

[7] E. Ivanov, S. Krivonos and A. Sorin, Mod. Phys. Lett. **A10** (1995) 2439, [hep-th/9505142](http://arxiv.org/abs/hep-th/9505142)

[8] C. Ahn, E. Ivanov and A. Sorin, “\(N = 2\) affine superalgebras and hamiltonian reduction in \(N = 2\) superspace”, Preprint JINR E2-95-343, [hep-th/9508003](http://arxiv.org/abs/hep-th/9508003)

[9] P. Goddard and A. Schwimmer, Phys. Lett. **B214** (1988) 209

[10] S. Krivonos and K. Thielemans, “A Mathematica package for computing \(N = 2\) super-field operator product expansions”, Preprint IMPERIAL-TP-95-96-13, [hep-th/9512029](http://arxiv.org/abs/hep-th/9512029)