No-Go Theorems for Quantum Resource Distillation

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Distillation, which generally means converting noisy states into better approximations of some valuable pure states by low-cost operations, is a fundamental type of resource trading protocol that plays major roles in quantum computation and information processing. In this work, we establish a series of fundamental limitations to the efficiency of distillation with finite resources which apply to any reasonable resource theory, using an information-theoretic tool called the quantum hypothesis testing relative entropy. We show that, for highly generic noisy inputs and any target pure resource state, there is an insuperable limitation to the accuracy (lower bound on the error) of any distillation protocol, even probabilistic ones. More explicitly, we prove an "uncertainty relation" between the accuracy and success probability, which indicates a forbidden regime that no protocol with the corresponding parameters can exist. We use these no-go theorems to derive explicit bounds which imply that the total overhead i.e. the number of noisy inputs required to distill a pure resource state up to small error $\epsilon$ scale at least as $\Omega(\log(1/\epsilon))$. In particular, we establish specific lower bounds on the cost of magic state distillation, a task crucial to fault-tolerant quantum computation (which address an open problem raised by Hastings and Haah [PRL 120, 050504 (2018)]). Finally, we showcase a basic no-go result for simulating unitary resource channels to demonstrate that the analogous understandings may extend to channel resource theories.

I. INTRODUCTION

The field of quantum information takes a pragmatic approach to the mysteries of quantum mechanics, seeking to obtain a better understanding of which information processing tasks can or cannot be accomplished in this new arena. A landmark result is the no-cloning theorem [1, 2], which states that there is no universal quantum operation duplicating an unknown quantum state. The discovery of this no-go result represents a key advance in our understanding of quantum mechanics, and has profound impact on the development of quantum information and computation. On the one hand, the result prevents several important extensions of classical results such as classical error correction, which for some time was thought to be a fatal limitation for practical quantum computing. But on the other hand, it also stimulates the invention of quantum error correcting codes [3, 4], and lays the foundation for plenty of other major quantum applications, such as the BB84 protocol in quantum cryptography [5].

This work focuses on another fundamental task in quantum information processing that plays particularly important roles in practical scenarios, and reveals its limitations through a host of no-go theorems which prohibit the existence of any allowed operation achieving certain goals, in analogy to the no-cloning theorem. This task is often called resource distillation, which aims at "purifying" the resource encoded in noisy quantum systems. In practice, quantum states prepared from experiments or stored in memories are unavoidably contaminated by various kinds of noises, making them unreliable for direct usage. Therefore, distillation comes into play as a standard procedure which helps us extract high-quality resources better suited for application. Most notably, the distillation of entanglement [6–8], coherence [9–11] and magic states [12] have been extensively studied as key subroutines in quantum communication and computation. Understanding the limitations and costs of distillation tasks, especially in the finite-copy (one-shot) setting, are thus of great theoretical and practical importance.

To address this problem in a simple and general manner, we shall use the language of resource theory (see [13] for an introduction of this framework): A state resource theory is defined by free states (in contrast to resource states) and free operations, with a golden rule that any free operation can only map a free state to another free state. This simple rule selects the largest possible set of free operations allowed in resource manipulation such as distillation, since any other operation can by definition create resource and thus trivializing the theory. Whether some resource state can be transformed into another via certain free operations is a fundamental type of problem in quantum information theory. Practical distillation tasks mostly concern the following case: Is it possible to convert a finite amount of mixed (noisy) states to pure resource states to arbitrary accuracy by free operations?

We first give a negative answer to the above question in general, manifesting that the production of any pure resource state with arbitrarily small error from a full-rank noisy state, however weak this target state is, is inherently prohibited by the golden rule embedded in any resource theory. More formally, we establish quantitative bounds which limit the achievable accuracy for free operations that work with a certain probability. It turns out that there is a non-trivial trade-off between the accuracy and success probability, akin to the uncertainty relations. The proofs follow from analyzing the peculiar properties of the hypothesis testing relative entropy monotone, a quantity known to characterize the efficiency of one-shot distillation in many cases [11, 14–20] but not studied in great depth. Using the above results, we find lower bounds on the
overhead of distillation, namely, the number of copies of a certain primitive state needed. As a particularly important application, we provide specific lower bounds for the cost of magic state distillation [12], which is estimated to be a dominant source of resource cost in current fault-tolerant quantum computation schemes. The consequent limitations to the common distillation protocols based on quantum error correction are discussed in relation to key progresses in the search for better codes [21–24]. Lastly, we prove a no-go theorem for the simulation of unitary resource channels, which is analogous to the distillation of pure states, in accordance with the recent interest in extending conventional resource theory approaches for quantum states to quantum channels (see e.g. [25–40]).

II. NO-GO THEOREMS FOR DISTILLATION

In this section, we establish fundamental limits to the achievable accuracy of any such distillation procedure in terms of certain properties of the input and output states. Our arguments make use of the following information-theoretic quantities. For any two quantum state \(\rho, \sigma\), their quantum hypothesis testing relative entropy is defined as

\[
D^r_H(\rho||\sigma) = -\log \beta_r(\rho||\sigma),
\]

where

\[
\beta_r(\rho||\sigma) = \min \left\{ \frac{1}{2} \| \text{Tr} M \sigma - \text{Tr} \rho M \|_1 : 0 \leq M \leq 1 \right\}.
\]

The fidelity between two quantum states \(\rho\) and \(\sigma\) is defined as

\[
F(\rho, \sigma) = \| \sqrt{\rho} \sqrt{\sigma} \|_1^2.
\]

Denote the maximum overlap between \(\rho\) and the set of free states \(\mathcal{F}\) as

\[
f_0 = \max_{\omega \in \mathcal{F}} F(\rho, \omega).
\]

Let us first consider the deterministic distillation scenario, where the transformation process is required to succeed with certainty and thus represented by a free CPTP map.

**Theorem 1 (No-go, deterministic)** Given any full rank primitive state \(\rho \notin \mathcal{F}\) and any pure target resource state \(\psi \notin \mathcal{F}\), there is no free operation transforming \(\rho\) to \(\psi\) with fidelity \(1 - \epsilon\) within \(\epsilon(\rho, \psi)\) error, where \(\epsilon(\rho, \psi) = \lambda_{\text{min}}(\rho) (1 - f_0) > 0\) and \(\lambda_{\text{min}}(\rho)\) is the smallest non-zero eigenvalue of \(\rho\).

The proof of this theorem will use the following continuity bound of the hypothesis testing relative entropy. We delegate its detailed proof to the Supplemental Material.

**Lemma 2** For any full rank state \(\rho\) and any quantum state \(\sigma\), their quantum hypothesis testing relative entropy is continuous around \(\epsilon = 0\). That is, for any \(0 \leq \epsilon < \lambda_{\text{min}}(\rho)\) where \(\lambda_{\text{min}}(\rho)\) is the smallest eigenvalue of \(\rho\), it holds that

\[
0 \leq D^r_H(\rho||\sigma) \leq \log \frac{\lambda_{\text{min}}(\rho)}{\lambda_{\text{min}}(\rho) - \epsilon}.
\]

**Proof of Theorem 1:** The proof is given by contradiction. Suppose there is a free operation \(\mathcal{E}\) such that \(\mathcal{E}(\rho) = \sigma\) and \(\text{Tr} \varphi \sigma \geq 1 - \epsilon\) with \(\epsilon < \epsilon(\rho, \psi)\). Consider the quantum test \((\psi, I - \psi)\), we have \(D^r_H(\sigma||\omega) \geq -\log \text{Tr} \varphi \omega\) for all \(\omega \in \mathcal{F}\).

Then the following chain of inequalities holds

\[
-\log f_\psi \leq \min_{\omega \in \mathcal{F}} D^r_H(\sigma||\omega) = \min_{\omega \in \mathcal{F}} D^r_H(\mathcal{E}(\rho)||\omega) \leq \min_{\omega \in \mathcal{F}} D^r_H(\mathcal{E}(\rho)||\mathcal{E}(\omega)) = \min_{\omega \in \mathcal{F}} D^r_H(\rho||\omega) \leq \log (\lambda_{\text{min}}(\rho)/(\lambda_{\text{min}}(\rho) - \epsilon))
\]

where the second line follows from the assumption that \(\mathcal{E}(\rho) = \sigma\), the third line follows since it is minimized over a smaller set due to \(\mathcal{E}(\omega) \in \mathcal{F}\), \(\forall \omega \in \mathcal{F}\), the fourth line follows from the data-processing inequality of \(D^r_H\) [41], the last line follows from the continuity bound in Lemma 2 (applicability guaranteed by the assumption \(\epsilon < \epsilon(\rho, \psi)\)). A simple calculation gives us \(\epsilon \geq \lambda_{\text{min}}(\rho) (1 - f_0)\), which forms a contradiction with our assumption.

**Theorem 3 (No-go, probabilistic)** Given any full rank primitive state \(\rho \notin \mathcal{F}\) and any pure target resource state \(\psi \notin \mathcal{F}\), there is a nontrivial upper bound on the accuracy of distillation that no free probabilistic protocol can exceed, which decreases as the success probability grows. More explicitly, the following trade-off relation between the transformation fidelity \(1 - \epsilon\) and the success probability \(p\) must hold:

\[
\epsilon \geq \frac{\epsilon(\rho, \psi)}{1 + R(\rho)} = \frac{\lambda_{\text{min}}(\rho)(1 - f_0)}{1 + R(\rho)}.
\]

The proof of this theorem will use another two Lemmas (proofs in the Supplemental Material).

**Lemma 4** For any two flagged quantum states \(\rho_i = p_i |0\rangle\langle 0| \otimes \sigma_i + (1 - p_i) |1\rangle\langle 1| \otimes \omega_i\) with \(i \in \{1, 2\}\) and \(p_i \in [0, 1]\), it holds that \(\beta_r(\rho_1||\rho_2) \leq 2^{1 - R(\rho)} \beta_r(\sigma_1||\sigma_2) + (1 - 2^{R(\rho)}) \beta_r(\omega_1||\omega_2)\).

**Lemma 5** For any linear suboperation \(\mathcal{L}\), there exists a free state \(\omega \in \mathcal{F}\) such that \(\text{Tr} \mathcal{L}(\omega) \geq (1 + R(\rho))^{-1} \text{Tr} \mathcal{L}(\rho)\), where \(R(\rho) \equiv \min \{s : \exists \sigma, s \geq 0, s(\rho + s\sigma)/(1 + s) \in \mathcal{F}\}\) is the (generalized) robustness of state \(\rho\).
Proof of Theorem 3: Suppose there is a probabilistic protocol \( E_{A\rightarrow F}(\rho_A) = |0\rangle\langle 0|_F \otimes \mathcal{L}_{A\rightarrow B}(\rho_A) + |1\rangle\langle 1|_F \otimes \mathcal{G}_{A\rightarrow B}(\rho_A) \) such that \( \mathcal{L}_{A\rightarrow B}(\rho_A) = p_\sigma B \) and \( \text{Tr} \rho \psi \geq 1-\varepsilon \) with \( \varepsilon < p(1 + R(p))^{-1}e(\rho, \psi) \). Based on Lemma 5, let us consider a free state \( \omega_1 \) such that \( \text{Tr} \mathcal{L}(\omega_1) \geq (1 + R(p))^{-1} \text{Tr} \mathcal{L}(\rho) \) and take \( \omega_2 = \mathcal{L}(\omega_1)/\text{Tr} \mathcal{L}(\omega_1) \). Then we have

\[
    f_\psi = \max_{\omega \in \mathcal{F}} \text{Tr} \omega \psi \geq \text{Tr} \omega_2 \psi \geq \beta_\varepsilon(\sigma \| \omega_2),
\]

where the last inequality follows by considering the quantum test \( |\psi, 1 - \psi \rangle \). On the other hand, we have

\[
    \beta_\varepsilon(\sigma \| \omega_2) = \beta_\varepsilon \left( \frac{\mathcal{L}(\rho)}{\text{Tr} \mathcal{L}(\rho)} \left| \mathcal{L}(\omega_1) \right| \text{Tr} \mathcal{L}(\omega_1) \right)
\]

\[
    \geq \frac{\beta_\varepsilon(\epsilon(\rho) \| \epsilon(\omega_1)) - (1 - \text{Tr} \mathcal{L}(\omega_1))}{\text{Tr} \mathcal{L}(\omega_1)}
\]

\[
    \geq \frac{\beta_\varepsilon(\epsilon(\rho)) - (1 - \text{Tr} \mathcal{L}(\omega_1))}{\text{Tr} \mathcal{L}(\omega_1)}
\]

\[
    \geq \frac{(\lambda_{\min}(\rho) - \epsilon)/\lambda_{\max}(\rho) - (1 - \text{Tr} \mathcal{L}(\omega_1))}{\text{Tr} \mathcal{L}(\omega_1)},
\]

where the second line follows from Lemma 4, the third line follows from the data-processing inequality, and the last line follows from the continuity bound in Lemma 2 (applicability guaranteed by the assumption \( \epsilon < p(1 + R(p))^{-1}e(\rho, \psi) \)) and the assumption that \( p \) is full rank. Combining (11) with (7), we have \( \epsilon \geq \text{Tr} \mathcal{L}(\omega_1) e(\rho, \psi) \). Recall that \( \omega_1 \) is defined as a free state such that \( \text{Tr} \mathcal{L}(\omega_1) \geq (1 + R(p))^{-1} \text{Tr} \mathcal{L}(\rho) = (1 + R(p))^{-1}p \), we obtain \( \epsilon \geq p(1 + R(p))^{-1}e(\rho, \psi) \), which forms a contradiction with our assumption.

Note that a slightly weaker version of Theorem 1 for deterministic distillation is recovered by letting \( p = 1 \). It is also possible to get rid of the \( (1 + R(p))^{-1} \) factor and obtain a stronger bound that covers Theorem 1 under certain restrictions. For example, suppose the theory admits a resource destroying channel \[42]\Lambda, and the allowed free suboperations are those commuting with \( \Lambda \) (such as dephasing–covariant incoherent suboperations for coherence theory [10]). Then for any free suboperation \( \mathcal{L}, \) it holds that \( \text{Tr} \mathcal{L}(\rho) = \text{Tr} \Lambda \circ \mathcal{L}(\rho) = \text{Tr} \mathcal{L} \circ \Lambda(\rho) \), which indicates that there always exists a free state \( \omega = \Lambda(\rho) \) such that \( \text{Tr} \mathcal{L}(\omega) = p \), and therefore the bound reduces to \( \epsilon/p \geq \epsilon(\rho, \psi) \).

In particular, by letting \( \epsilon = 0 \) we directly obtain the following no-go result for ideal distillation:

**Corollary 6** It is impossible to exactly distill a pure target resource state from a full rank primitive state, even probabilistically.

**Remarkably**, the noisy primitive state could “look” much more resourceful (or more precisely, takes much greater value in terms of other resource monotones) than the target state, yet the possibility of an arbitrarily accurate probabilistic transformation is ruled out. An illustrative toy example in terms of qubit coherence theory is given in Fig. 2, where \( \rho \) is a slightly noisy version of the maximally coherent state \( |+\rangle \) and \( \psi \) is a pure target state very close to basis state \( |1\rangle \). It is clear that commonly used coherence measures such as the minimum relative entropy or trace distance to incoherent states assign much greater value to \( \rho \) than to \( \psi \), which may lead to the false intuition that \( \rho \) can be used to trade for \( \psi \). This highlights the special role of the hypothesis testing relative entropy monotone among all resource measures, especially in characterizing distillation tasks.

**III. RESOURCE OVERHEAD**

The setting of distillation tasks in practice is usually the following: One has access to a certain primitive resource state; The goal is to distill certain “standard” pure resource states to
some desired accuracy while consuming as few copies of the primitive states as possible. Therefore, the amount of primitive states needed to accomplish the desired distillation, namely the \textit{resource cost or overhead}, is a key figure of merit for distillation protocols. To state the most general result, we consider error on the entire output state (which could be a collection of unit states) for now. As we now show, our no-go theorems induce fundamental lower bounds on the total overhead of distillation, when the target error is small.

\textbf{Theorem 7 (Total overhead)} Consider the task of distilling some pure target resource state \( \psi \) with error at most \( \epsilon \), from \( n \) copies of primitive state \( \hat{\rho} \). For sufficiently small \( \epsilon \) and any full rank \( \hat{\rho} \), there does not exist any free operation that accomplishes the task if the following is not satisfied:

\[
n \geq \log \left( \frac{1 - f_{\hat{\rho}}}{\lambda_{\min}(\hat{\rho})} \right).
\]

Furthermore, for sufficiently small \( \epsilon \) and any full rank state \( \hat{\rho}' \), there does not exist any probabilistic protocol with success probability \( p \) that accomplishes the task if the following is not satisfied:

\[
n \geq \log \left( \frac{(1 - f_{\hat{\rho}})p}{\lambda_{\min}(\hat{\rho})} \right). \tag{13}
\]

\textit{Proof of Theorem 7}: Follows by transforming the no-go theorems with input state \( \hat{\rho}^\otimes n \). Notice that \( \lambda_{\min}(\hat{\rho}^\otimes n) = \lambda_{\min}(\hat{\rho})^n \).

For the deterministic case, Theorem 1 implies that for any full rank \( \hat{\rho} \) and \( \epsilon < \epsilon(\hat{\rho}^\otimes n, \psi) \), we have

\[
\epsilon \geq \lambda_{\min}(\hat{\rho}^\otimes n)(1 - f_{\hat{\rho}}) = \lambda_{\min}(\hat{\rho})^n(1 - f_{\hat{\rho}}). \tag{14}
\]

This directly translates to Eq. (12).

To obtain the claimed bound for the probabilistic case, notice the following: By the definition of \( R(\hat{\rho}) \), there exists some state \( \tau \) such that \( \hat{\rho} + R(\hat{\rho})\tau = (1 + R(\hat{\rho}))\omega \) where \( \omega \in \mathcal{F} \). By expanding this equation, we obtain

\[
\omega^\otimes n = \frac{1}{1 + R(\hat{\rho})} \hat{\rho}^\otimes n + \frac{(1 + R(\hat{\rho}))^n - 1}{(1 + R(\hat{\rho}))^n} \tau',
\]

where \( \omega^\otimes n \in \mathcal{F} \) axiomatically [43], and \( \tau' \) is a density operator. Therefore, \( 1 + R(\hat{\rho})^\otimes n \leq (1 + R(\hat{\rho}))^n \). Now, by Theorem 3, for any full rank state \( \hat{\rho}' \) and \( \epsilon < p(1 + R(\hat{\rho})^\otimes n)^{-1} \epsilon(\hat{\rho}^\otimes n, \psi) \), we have

\[
\epsilon/p \geq \frac{\lambda_{\min}(\hat{\rho}^\otimes n)(1 - f_{\hat{\rho}})}{1 + R(\hat{\rho}^\otimes n)} \geq \lambda_{\min}(\hat{\rho}')^n(1 - f_{\hat{\rho}'})/(1 + R(\hat{\rho}'))^n. \tag{16}
\]

This directly translates to Eq. (13). \( \square \)

The above theorem indicates that for distillation protocols which succeed with at least a constant probability (that does not vanish when reducing the target \( \epsilon \)), the total overhead must scale at least as \( \Omega(\log(1/\epsilon)) \) as \( \epsilon \to 0 \).

\textbf{IV. MAGIC STATE DISTILLATION}

Optimizing the resource costs for implementing a universal set of logical gates fault-tolerantly is a crucial problem for realizing scalable quantum computation [44–46]. A leading scheme for fault-tolerance is based on the so-called \textit{magic states} [12]. Here the so-called Clifford operations are considered “cheap” since they admit fault-tolerant implementations thanks to stabilizer codes [44, 47–49], but their computational power is very limited—due to the celebrated Gottesman-Knill theorem, they can even be efficiently simulated by classical computers [45, 50, 51]. To achieve universal quantum computation, one needs non-Clifford gates such as \( T = \text{diag}(1, e^{i\pi/4}) \).

A standard approach is to distill high-quality pure magic state \( |T\rangle = ((0) + e^{i\pi/4}|1\rangle)/\sqrt{2} \) from sufficiently many dirty magic states, which can be directly used to emulate a logical \( T \)-gate via state injection [52]. Unfortunately, the production of any magic state requires non-Clifford operations, and to achieve sufficiently low error the amount of these resources needed is substantial, constituting a major source of the cost of such fault-tolerance schemes. Therefore, understanding the ultimate limitations to the efficiency of this distillation component is crucial.

We now address this problem by tailoring our general results to the practical magic state distillation settings and establish explicit lower bounds on the overhead. (Note that the resource theory ideas have advanced our understanding of magic states and quantum computation in various other ways [18, 28, 29, 53–55].) Known protocols for magic state distillation are commonly based on concatenating error correction subroutines using stabilizer codes to (probabilistically) produce an output with sufficiently high quality upon passing the syndrome measurements. Note that such procedures alone cannot create magic states, following the golden rule. Depending on the codes being used, the output could take the form of a large state with each marginal sufficiently close to a unit target state, in which case we are also interested in the average overhead, i.e., the total overhead divided by the number of marginals. Here we only showcase the \( T \)-state result, but the bounds for other useful magic states (see e.g. [24]) can be similarly obtained by plugging in corresponding parameters.

\textbf{Theorem 8} Consider the following general form of magic state distillation task: given \( n \) copies of full rank primitive magic states \( \hat{\rho} \), output an \( m \)-qubit state \( \tau \) such that \( \text{Tr} \tau |T\rangle = \langle T|\tau = 1 - \epsilon, \forall i = 1, \ldots, m \) where \( \tau_i = T_\tau \tau \) is the \( i \)-th qubit. Then for sufficiently small \( \epsilon \), the average overhead of any probabilistic protocol that succeeds with probability \( p \) must obey

\[
C := n/m \geq \frac{\log \left( \frac{1 + R(\hat{\rho})}{\lambda_{\min}(\hat{\rho})} \right)}{m} \frac{(4 - 2\sqrt{2})m - 1}{(4 - 2\sqrt{2})m}p. \tag{17}
\]

\textit{Proof of Theorem 8}: By applying the union bound, we have \( \langle T^\otimes m |\tau| T^\otimes m \rangle \geq 1 - m\epsilon \). Also notice that \( f_T^\otimes m = (4 - 2\sqrt{2})^{-m} \) [19, 56–58]. By plugging everything into Eq. (13) we obtain the claimed bound. \( \square \)

In the analyses of magic state distillation protocols, one is particularly interested in the exponent \( y \) in the asymptotic average overhead \( \Omega(\log^y(1/\epsilon)) \). A subtlety of our lower bound is that the output size \( m \) could depend on the target \( \epsilon \) for specific protocols. Thus, to understand the scaling, one needs
to take into account the behavior of \( m \) as well. There are two key implications of our bound to code-based distillation protocols. Assuming non-vanishing success probability (the passing probability of deeper rounds of concatenation converges sufficiently fast to one), then: (i) It is impossible to construct a protocol with sublogarithmic average overhead (\( \gamma < 1 \)) with any \([n, k, d]\) code such that \( k \leq d \). This can be seen by plugging \( m = \kappa^2 \) and \( \log(1/\epsilon) \sim d^2 \) into Eq. (17). This in particular implies a \( \gamma \geq 1 \) bound for \( k = 1 \) codes, in response to a problem raised in [22]. Note that the best known such codes allow \( \gamma \rightarrow 2 \) [23, 24], so there is still a gap. (ii) Any \( \gamma < 1 \) protocol must have a scale (size of input and output) that diverges under concatenation. It was actually conjectured that no codes allowing \( \gamma < 1 \) exist [21], but such codes were recently found [22] (see also [59]). There, indeed, the codes employed have \( k > d \). So although the average overhead of such a protocol is a constant, its output size must grow rapidly as we reduce \( \epsilon \), which could blow up the overall cost. Our results indicate that this is inevitable.

\[ \]  

V. CHANNEL THEORY

As a basic result for the channel resource theory setting [31, 39, 40], we now show that there does not exist any free superchannel that perfectly transforms a noisy channel into a unitary resource channel, which is analogous to pure resource states. Here, again, the only constraint on the allowed superchannels is the golden rule of no resource generation.

**Theorem 9 (Channel no-go)** Given any primitive channel \( N_{A \rightarrow B} \notin \mathcal{O}(A \rightarrow B) \) with free component (i.e. can be written as \( N = pE + (1 - p)M \), \( p \in (0, 1) \) where \( E \in \mathcal{O}(A \rightarrow B) \) and \( M \) is any channel), and any target unitary resource channel \( \mathcal{U}_{C \rightarrow D} \notin \mathcal{O}(C \rightarrow D) \), there is no free superchannel \( \Pi \) transforming \( N_{A \rightarrow B} \) to \( \mathcal{U}_{C \rightarrow D} \).

Proof can be found in the Supplemental Material. A straightforward implication of this result is that the zero-error quantum capacity of a generic quantum channel, such as the quantum depolarizing channel, is zero. More comprehensive studies of the channel setting will be left for future work.

VI. CONCLUDING REMARKS

This work reveals universal limitations to the accuracy and efficiency of quantum resource distillation tasks through investigating the properties of the hypothesis testing relative entropy. Our results do not rely on any particular structure of the resource theory, and only require very mild assumptions on the primitive state that hold in most practically interesting cases. All bounds depend only on very few parameters that concisely encode relevant properties of the noise, the target state, and the resource theory, and are thus easy to analyze. Just as how the no-cloning theorem essentially stem from linearity, a fundamental feature of quantum mechanics, our results are succinct consequences of the golden rule of quantum resource theory at bottom. An important future work is to investigate to what extent the bounds on the distillation fidelity and asymptotic overhead can be approached. It would also be interesting to further understand the approximate and probabilistic regimes of unitary channel simulation, due to its connections to the fields of quantum Shannon theory, gate and circuit synthesis etc.

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Supplemental Material: No-Go Theorems for Quantum Resource Distillation

This supplemental material provides a more detailed analysis and proofs of some results omitted in the main text.

1. TECHNICAL LEMMAS

Lemma 2 For any full rank states $\rho$ and any quantum state $\sigma$, their quantum hypothesis testing relative entropy is continuous around $\epsilon = 0$. That is, for any $0 \leq \epsilon < \lambda_{\min}(\rho)$ where $\lambda_{\min}(\rho)$ is the smallest eigenvalue of $\rho$, it holds that

$$0 \leq D^r_H(\rho || \sigma) \leq \log \frac{\lambda_{\min}(\rho)}{\lambda_{\min}(\rho) - \epsilon}.$$ 

Proof. Suppose $M$ is an optimal measurement operator that attains $D^r_H(\rho || \sigma)$. Then we have $0 \leq M \leq 1$, $\text{Tr} \rho M \geq 1 - \epsilon$ and $D^r_H(\rho || \sigma) = - \log \text{Tr} \rho M \sigma$. Denote the non-zero eigenvalues of $\rho$ and $M$ as $\{\lambda_i\}_{i=1}^d$ and $\{m_j\}_{j=1}^k$, which are both sorted in a non-increasing order. Let $\lambda_{\min} = \min_i \lambda_i$ and $m_{\min} = \min_j m_j$. Since $\rho$ is full rank, we have $k \leq d$. We first argue that $k = 1$, i.e., $M$ is full rank. Suppose $k < d$, then we have

$$1 - \epsilon \leq \text{Tr} \rho M \leq \sum_{i=1}^k \lambda_i m_i \leq \sum_{i=1}^k \lambda_i \leq 1 - \lambda_{\min},$$

where the second inequality follows from the von Neumann’s trace theorem [60, Theorem 7.4.1.1], and the third inequality follows since $m_i \leq 1$. Therefore, Eq. (S1) contradicts to the assumption that $\epsilon < \lambda_{\min}$ and thus $k = d$. Similar to Eq. (S1), we have

$$1 - \epsilon \leq \text{Tr} \rho M \leq \sum_{i=1}^d \lambda_i m_i \leq \lambda_{\min} m_{\min},$$

where the last inequality follows from the rearrangement inequality and the fact that $m_i \leq 1$. This implies $m_{\min} \geq 1 - \epsilon / \lambda_{\min}$. Then we have $M \geq m_{\min} \mathbb{1} \geq (1 - \epsilon / \lambda_{\min}) \mathbb{1}$, and thus

$$D^r_H(\rho || \sigma) = - \log \text{Tr} M \sigma \leq - \log \text{Tr}(1 - \epsilon / \lambda_{\min}) \sigma = \log \frac{\lambda_{\min}(\rho)}{\lambda_{\min}(\rho) - \epsilon}.$$ (S3)

This completes the proof.

Lemma 4 For any two flagged quantum states $\rho_i = p_i |0\rangle \langle 0 | \otimes \sigma_i + (1 - p_i) |1\rangle \langle 1 | \otimes \omega_i$ with $i \in \{1, 2\}$ and $p_i \in [0, 1]$, it holds that $\beta_{\epsilon}(\rho_1 || \rho_2) \leq p_2 \beta_{\epsilon}(\sigma_1 || \sigma_2) + (1 - p_2) \beta_{\epsilon}(\omega_1 || \omega_2)$.

Proof. Suppose $\beta_{\epsilon}(\sigma_1 || \sigma_2)$ and $\beta_{\epsilon}(\omega_1 || \omega_2)$ are achieved by optimal measurement operators $M$ and $N$ respectively. Then we can take $Q = |0\rangle \langle 0 | \otimes M |1\rangle \langle 1 | \otimes N$. It is clear that $0 \leq Q \leq 1$ and $\text{Tr} Q \rho_1 = p_1 \text{Tr} M \rho + (1 - p_1) \text{Tr} N \omega \geq p_1 (1 - \epsilon) + (1 - p_1) (1 - \epsilon) = 1 - \epsilon$. So $Q$ is a feasible measurement operator for $\beta_{\epsilon}(\rho_1 || \rho_2)$. Thus we have

$$\beta_{\epsilon}(\rho_1 || \rho_2) \leq \text{Tr} Q \rho_2 = p_2 \text{Tr} M \sigma_2 + (1 - p_2) \text{Tr} N \omega_2 = p_2 \beta_{\epsilon}(\sigma_1 || \sigma_2) + (1 - p_2) \beta_{\epsilon}(\omega_1 || \omega_2),$$

which completes the proof.

Lemma 5 For any linear suboperation $L$, there exists a free state $\omega \in \mathcal{F}$ such that $\text{Tr} L(\omega) \geq (1 + R(\rho))^{-1} \text{Tr} L(\rho)$, where $R(\rho) \equiv \min \{s | \exists \sigma, s \geq 0, \text{ s.t. } (\rho + s \sigma)/(1 + s) \in \mathcal{F} \}$ is the (generalized) robustness of state $\rho$.

Proof. By definition of $R(\rho)$, there exists $\omega \in \mathcal{F}$ such that $\omega = \frac{1}{1 + R(\rho)} \rho + \frac{R(\rho)}{1 + R(\rho)} \sigma$. By linearity of $L$,

$$\text{Tr} L(\omega) = \frac{1}{1 + R(\rho)} \text{Tr} L(\rho) + \frac{R(\rho)}{1 + R(\rho)} \text{Tr} L(\sigma),$$

and thus the desired bound directly follows.
II. RESOURCE THEORY OF QUANTUM CHANNELS

Analogous to the resource theories of quantum states, a resource theory of quantum channels can also be built upon two basic ingredients: the set of free channels \( \mathcal{O} \) and the set of free superchannels (maps from channels to channels) \( \Theta \), with the golden rule \( \Theta(\mathcal{O}) \subseteq \mathcal{O} \) which selects all superchannels that can be possibly allowed (including those admitting implementations by free combs considered in [40]). Now the general question becomes whether there exists a free superchannel \( \Pi \in \Theta \) that maps one quantum channel \( \mathcal{N} \) to another quantum channel \( \mathcal{M} \), i.e., \( \Pi(\mathcal{N}_{A\rightarrow B}) = \mathcal{M}_{C\rightarrow D} \). If so, we say that \( \mathcal{M} \) can be simulated by \( \mathcal{N} \).

In analogy to the state distillation tasks where one aims to turn a noisy state into a pure one, here we want to turn a noisy channel into a unitary one, which preserves information. Below we show an elementary channel version of the no-go results, which says that perfect simulation of unitary channels are generically impossible. Similarly, we say a quantum channel \( \mathcal{N} \) contains a free component if there exists free channel \( \mathcal{E} \in \mathcal{O} \) and another quantum channel \( \mathcal{M} \) such that \( \mathcal{N} = p\mathcal{E} + (1-p)\mathcal{M} \) with \( p > 0 \).

**Theorem 9 (Channel no-go)** Given any primitive channel \( \mathcal{N}_{A\rightarrow B} \notin \mathcal{O}(A \rightarrow B) \) with free component (i.e. can be written as \( \mathcal{N} = p\mathcal{E} + (1-p)\mathcal{M}, p \in (0,1) \) where \( \mathcal{E} \in \mathcal{O}(A \rightarrow B) \) and \( \mathcal{M} \) is any channel) and any target unitary resource channel \( \mathcal{U}_{C\rightarrow D} \notin \mathcal{O}(C \rightarrow D) \), there is no free superchannel \( \Pi \) transforming \( \mathcal{N}_{A\rightarrow B} \) to \( \mathcal{U}_{C\rightarrow D} \).

If our target \( \mathcal{U} \) is a noiseless quantum channel, the channel transformation task will correspond to the well-studied channel capacity problem (see e.g. [32, 61]). A straightforward implication of the above result shows that the zero-error quantum capacity of a generic quantum channel, such as the quantum depolarizing channel, is zero.

The proof of this no-go result requires the definition of the channel’s min-relative entropy and its monotonicity under superchannels. The channel’s min-relative entropy is defined as

\[
D_{\text{min}}(\mathcal{N} || \mathcal{M}) \equiv \sup_{\rho_A} D_{\text{min}}(\mathcal{N}_{A\rightarrow B}(\rho_{AA'}) || \mathcal{M}_{A\rightarrow B}(\rho_{AA'})) = \sup_{\rho_A} D_{\text{min}}(\mathcal{N}_{A\rightarrow B}(\phi_{AA'}) || \mathcal{M}_{A\rightarrow B}(\phi_{AA'}))
\]

where \( \phi_{AA'} \) is a purification of \( \rho_{AA'} \). The second equality follows from the joint-convexity of \( D_{\text{min}} \). The monotonicity of the channel’s min-relative entropy holds as follows:

**Lemma S1** For any superchannel \( \Pi \) and quantum channels \( \mathcal{N}_{A\rightarrow B} \) and \( \mathcal{M}_{A\rightarrow B} \), it holds that \( D_{\text{min}}(\Pi(\mathcal{N}) || \Pi(\mathcal{M})) \leq D_{\text{min}}(\mathcal{N} || \mathcal{M}) \).

**Proof.** Note that any superchannel \( \Pi \) can be implemented by pre- and post- quantum processings as \( \Pi(\mathcal{N}_{A\rightarrow B}) = \mathcal{D}_{BR\rightarrow D} \circ (\mathcal{N}_{A\rightarrow B} \otimes \mathbb{I}_R \otimes \mathbb{1}_R) \circ \mathcal{E}_{C\rightarrow AR} \), where \( \mathcal{E}, \mathcal{D} \) are channels [62]. Suppose \( \Pi \) is realized by \( \{\mathcal{E}, \mathcal{D}\} \) and the optimal solution of \( D_{\text{min}}(\Pi(\mathcal{N}) || \Pi(\mathcal{M})) \) is achieved by state \( \rho_{CE} \). Denote \( \sigma_{ARE} = \mathcal{E}_{C\rightarrow AR}(\rho_{CE}) \). Then we have the following chain of inequalities,

\[
D_{\text{min}}(\Pi(\mathcal{N}) || \Pi(\mathcal{M})) = D_{\text{min}}(\mathcal{D}_{BR\rightarrow D} \circ \mathcal{N}_{A\rightarrow B}(\sigma_{ARE}) || \mathcal{D}_{BR\rightarrow D} \circ \mathcal{M}_{A\rightarrow B}(\sigma_{ARE})) \leq D_{\text{min}}(\mathcal{N}_{A\rightarrow B}(\sigma_{ARE}) || \mathcal{M}_{A\rightarrow B}(\sigma_{ARE})) \leq D_{\text{min}}(\mathcal{N} || \mathcal{M}),
\]

where the first inequality follows from the data-processing inequality of \( D_{\text{min}} \), and the second inequality follows by definition. ■

**Proof of Theorem 9:** We prove this by contradiction. Suppose there is a free superchannel \( \Pi \) such that \( \Pi(\mathcal{N}) = \mathcal{U} \). Denote \( J_{\mathcal{U}} \) as the corresponding Choi state of \( \mathcal{U} \). Since \( \mathcal{U} \) is a unitary channel, we know that \( J_{\mathcal{U}} \) is a pure state. By the assumption of \( \mathcal{U} \notin \mathcal{O} \), we have for any \( \mathcal{E} \in \mathcal{O} \) it holds \( D_{\text{min}}(\mathcal{U} || \mathcal{E}) \geq D_{\text{min}}(J_{\mathcal{U}} || J_{\mathcal{E}}) = -\log \text{Tr} J_{\mathcal{U}} J_{\mathcal{E}} > 0 \). On the other hand, suppose \( \mathcal{N} = p\mathcal{E} + (1-p)\mathcal{M} \) with \( \mathcal{E} \in \mathcal{O} \), we have \( D_{\text{min}}(\mathcal{N} || \mathcal{E}) = D_{\text{min}}(p\mathcal{E} + (1-p)\mathcal{M} || \mathcal{E}) = 0 \). Then we have

\[
0 < \min_{\mathcal{E} \in \mathcal{O}} D_{\text{min}}(\mathcal{U} || \mathcal{E}) \leq \min_{\mathcal{E} \in \mathcal{O}} D_{\text{min}}(\mathcal{U} || \Pi(\mathcal{E})) = \min_{\mathcal{E} \in \mathcal{O}} D_{\text{min}}(\Pi(\mathcal{N}) || \Pi(\mathcal{E})) \leq \min_{\mathcal{E} \in \mathcal{O}} D_{\text{min}}(\mathcal{N} || \mathcal{E}) = 0,
\]

where the second inequality follows since the minimization is restrict to \( \mathcal{E} \in \Pi(\mathcal{O}) \) on the r.h.s., and the third inequality follows from the monotonicity in Lemma S1. This forms a contradiction. □

The approximate and probabilistic versions are left for future work.