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Small ball probabilities for certain gaussian fields.

1. Introduction and results.

We study the behavior of the tail probabilities \( P(V^2 < r) \) as \( r \to 0 \), where \( V^2 \) is defined by the following double sum

\[
V^2 = \pi^{-4} \sum_{i,j \geq 1} \left( (i + b) (j + \delta) \right)^{-2} \xi_{ij}^2,
\]

(1.1)

where \( \{\xi_{ij}\} \) are independent standard normal random variables, and \( b \) and \( \delta \) are constants: \( b > -1, \delta > -1 \).

According to the well-known Karhunen — Loève expansion, the sum in (1.1) can be considered as the squared \( L_2 \)-norm of a 2-parameter Gaussian random field, which has the covariance of a "tensor product" type (see, for instance, [6] and [7]). The classical examples of such fields are the standard Brownian sheet \( W(t,s) \), i.e. \( V^2 = \int_0^1 \int_0^1 W^2(t,s) \, dt \, ds \), the Brownian pillow and the pillow-sleep, or the Kiefer field \( (b = \delta = -1/2, b = \delta = 0 \) and \( b = 0, \delta = -1/2 \), respectively).

We remark that small deviations for Gaussian random fields in the Hilbert norm have not been studied as extensively as their one-parameter counterparts. In [4] — [9], the logarithmic \( L_2 \) small ball asymptotics for some two-parameter and multi-parameter Gaussian random fields were obtained. Concerning sharper estimates, the only work we know is [5], where small deviations in \( L_2 \)-norm for the Brownian sheet and its integrated counterparts were thoroughly examined. Therefore, a more detailed study of the tail probabilities \( P(V^2 < r) \) is of interest, even though the sum (1.1) has a special form.

In order to formulate our results, we need to introduce some notation.

Let us introduce functions \( B_i = B_i(b, \delta), b > -1, \delta > -1, (1 \leq i \leq 3) \) as follows:

\[
B_1 = 1 + \Psi(1 + b) + \Psi(1 + \delta), \quad B_2 = (1 + 2b) (1 + 2 \delta)/2, \\
B_3 = (1 + \delta + b) \log(2\pi) - (1 + 2 \delta) \ln \Gamma(1 + b) - (1 + 2b) \ln \Gamma(1 + \delta).
\]

(1.2)

Here \( \Psi(x) \) is the logarithmic derivative of the gamma — function \( \Gamma(x) \). Let us set

\[
B = \pi^{-1} (B_1 + 2 \ln \pi), \quad C = B_2, \quad D = B_3 - 2B_2 \ln \pi.
\]

(1.3)
Next, we introduce polynomials $\pi_k = \pi_k(s), k \geq 0$, such that $\pi_0 = 1$, $\pi_1 = s$, and for each $k \geq 2$ the function $\pi_{k+1}$ is given by the following recursive relation

$$\pi_{k+1} = \pi_k - \frac{k-1}{2k} \sum_{l=1}^{k} \pi_l \pi_{k+1-l}, \ k \geq 1. \quad (1.4)$$

So, $\pi_2 = s$, $\pi_3 = s - s^2/2$, $\pi_4 = s - 3s^2/2 + s^3/3$.

Remark 1. The polynomials $\pi_m(s)$ can be determined explicitly:

$$\pi_m(s) = \sum_{k=1}^{m-1} s^k a_{k-1,m-k}, \ m \geq 2. \quad (1.5)$$

Here the coefficients $a_{jl} = s(j + l, l)/(j + 1)!$, and the values $s(n, k)$ are the Stirling numbers of the first kind, i.e. (see [2, V]):

$$\frac{1}{k!} \ln^k(1 + t) = \sum_{n \geq k} s(n, k) \frac{t^n}{n!}.$$

In particular, the coefficients in (1.5) at $s$ and $s^{m-1}$ are 1 and $(-1)^m/(m - 1)$, respectively.

Moreover, the coefficients $a_{jl}$ may be calculated using the following formulas:

$$a_{0l} = 1, \ a_{j0} \big|_{j \geq 1} = 0 \quad \text{and} \quad a_{jl} = \frac{(j + l)!}{(j + 1)!} \left( -1 \right)^j \sum_{k=1}^{\min(j, l)} \left( \sum_{l=k}^{\lfloor \frac{j}{l} \rfloor} \frac{1}{(l-k)!} \prod_{m=1}^{j} \frac{(m+1)^{-k_m}}{k_m!} \right), \ j, l \geq 1, \quad (1.6)$$

where the summation is taken over all non-negative integer solutions $(k_1, \ldots, k_j)$ of the equations $1 \cdot k_1 + \cdots + j \cdot k_j = j$ and $k_1 + \cdots + k_j = k$, and as a result,

$$|a_{jl}| \leq \frac{1}{j + 1} 4^{j+l-1}. \quad (1.7)$$

We check Remark 1 in Section 2 (see Lemma 2 and below).

For each $r \in (0, 1)$, we introduce the following notation:

$$s = \ln |\ln r| - \Psi(1 + b) - \Psi(1 + \delta) - \ln (2\pi^3), \ \bar{\pi}_m = \pi_m(s), \ m \geq 0. \quad (1.8)$$
Here $\Psi$ and $\pi_m$ occur respectively in (1.2) and (1.4). Further, let us define the function $\varepsilon(r)$ as follows:

$$
\varepsilon(r) = \sum_{m \geq 1} p_m(\bar{s}) |\ln r|^{-m},
$$

(1.9)

where

$$
p_1(\bar{s}) = 2\bar{s} - 2, \quad p_2(\bar{s}) = \bar{s}^2, \quad p_m(\bar{s}) = -\frac{2}{m-2} (\bar{\pi}_m - \bar{\pi}_{m-1}), \quad m \geq 3.
$$

Now, using the notation in (1.2), (1.3), (1.8) and (1.9), we are able to formulate the main result.

**Theorem 1.** Let $\varkappa(r) = (8\pi^2 r)^{-1} \ln^2 r$, $c_0 = \left(\pi 2^{C/2} e^D\right)^{-1/2}$, $c_1 = C/4$, $c_2 = -(C + 2)/4$, where $C$ and $D$ are introduced in (1.3). Then as $r \to 0$

$$
P(V^2 < r) = c_0 r^{c_1} \varkappa^2(r) e^{-\varkappa(r) (1 + \varepsilon(r))} \left(1 + O\left(\frac{\ln |\ln r|}{|\ln r|}\right)\right),
$$

(1.10)

and, in addition,

$$
\frac{d}{dr} P(V^2 < r) = \frac{\varkappa(r)}{r} P(V^2 < r) \left(1 + O\left(\frac{\ln |\ln r|}{|\ln r|}\right)\right).
$$

(1.11)

Moreover, for any $k \geq 2$

$$
\varepsilon(r) = \sum_{m=1}^{k} p_m(\bar{s}) |\ln r|^{-m} + O\left((\ln |\ln r|)^k/|\ln r|^{k+1}\right).
$$

(1.12)

Note that the remainders in (1.10) – (1.12) are optimal.

**Corollary 1.** For any $k \geq 2$ as $r \to 0$

$$
\frac{8\pi^2 r}{\ln^2 r} |\log P(V^2 < r)| = 1 + \sum_{m=1}^{k} p_m(\bar{s}) |\ln r|^{-m} + O\left((\ln |\ln r|)^k/|\ln r|^{k+1}\right).
$$

In particular (see (1.8)),

$$
\frac{8\pi^2 r}{\ln^2 r} |\log P(V^2 < r)| = 1 + 2 (\bar{s} - 1) |\ln r|^{-1} + \bar{s}^2 |\ln r|^{-2} + O\left((\ln |\ln r|)^2/|\ln r|^3\right).
$$

(1.13)
Note that (1.13) for $b = \delta = -1/2$ refines [5, Example 5.4 (d=2)] and [14, Example 2] (simultaneously correcting the constant in [14, (1.20)]).

**Corollary 2.** Let (see (1.1))

$$
\tilde{V}^2 = \pi^{-4} \sum_{i,j \geq 1} \left( ((i + b)^2 - q) ((j + \delta)^2 - \tau) \right)^{-1} \xi_{ij}^2,
$$

where $q < 1 + b$, $\tau < 1 + \delta$.

Then

$$
P(\tilde{V}^2 < r) \sim C_{\text{dist.}} P(V^2 < r), \ r \to 0.
$$

Here

$$
C_{\text{dist.}} = \left( \frac{\Gamma(1 + b + \sqrt{q}) \Gamma(1 + b - \sqrt{q}) \Gamma(1 + \delta + \sqrt{\tau}) \Gamma(1 + \delta - \sqrt{\tau})}{\Gamma^2(1 + b) \Gamma^2(1 + \delta)} \right)^{1/2}.
$$

In the conclusion of this section we present the results of application of Theorem 1 to the classical cases mentioned earlier. Here, the function $\varepsilon(r) = \varepsilon(r, \tilde{s})$ is defined in (1.9), $C_e$ is the Euler constant, and $\kappa(r) = (8\pi^2 r)^{-1} \ln^2 r$.

**Theorem 2.** Set $\tilde{s} = \ln |\ln r| + 2 C_e - 3 \ln \pi$. Then as $r \to 0$

1) The Brownian sheet ($\tilde{s} = \tilde{s} + 3 \ln 2$)

$$
P(V^2|_{b=\delta=-1/2} < r) = \frac{e^{-\kappa(r)(1+\varepsilon(r))}}{\sqrt{\pi \kappa(r)}} \left( 1 + O \left( \frac{\ln |\ln r|}{|\ln r|} \right) \right).
$$

2) The Brownian pillow ($\tilde{s} = \tilde{s} - \ln 2$)

$$
P(V^2|_{b=\delta=0} < r) = r^{1/8} \left( 2 \kappa(r)^{-5/8} e^{-\kappa(r)(1+\varepsilon(r))} \right) \frac{1}{\sqrt{\pi}} \left( 1 + O \left( \frac{\ln |\ln r|}{|\ln r|} \right) \right).
$$

3) The Kiefer field ($\tilde{s} = \tilde{s} + \ln 2$)

$$
P(V^2|_{b=-1/2, \delta=0} < r) = 2^{-1/4} \frac{e^{-\kappa(r)(1+\varepsilon(r))}}{\sqrt{\pi \kappa(r)}} \left( 1 + O \left( \frac{\ln |\ln r|}{|\ln r|} \right) \right).
$$

2. Lemmas.
For \( b > -1 \) and \( \delta > -1 \) denote
\[
\lambda(t) = (t + b)^{-1}, \quad t \geq 1; \quad \lambda_n = \lambda(n),
\]
\[
u(x) = \ln \left( |\Gamma(1 + \delta + ix)|^2 / \Gamma^2(1 + \delta) \right),
\]
\[
J(\gamma) = J(\gamma; b, \delta) = \sum_{i \geq 1} u(\lambda_i \gamma).
\] (2.1)

**Lemma 1.** If \( \gamma \to \infty \) then
\[
J(\gamma) = -\pi \gamma \ln \gamma + B_1 \pi \gamma - B_2 \ln \gamma - B_3 + O(\gamma^{-1}),
\]
\[
\gamma J'(\gamma) = -\pi \gamma \ln \gamma + (B_1 - 1) \pi \gamma - B_2 + O(\gamma^{-1}),
\]
\[
\gamma^2 J''(\gamma) = -\pi \gamma + B_2 + O(\gamma^{-1}),
\] (2.2)

where \( B_i \) are defined in (1.2).

**The proof of Lemma 1.**
From [1, (8.341.1)] provided \( z = 1 + \delta + ix \) it follows that
\[
u(x) = v_\delta + \left( 1/2 + \delta \right) \log (x^2 + (1 + \delta)^2) - 2x \arctan x_\delta + 1/x_\delta + 2 R(x),
\] (2.3)

with
\[
v_\delta = \log \frac{2 \pi}{\Gamma^2(1 + \delta)}, \quad x_\delta = x/(1 + \delta),
\]
\[
R(x) = \int_0^\infty \left( \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right) e^{-t(1+\delta)} \cos tx \, \frac{dt}{t}.
\] (2.4)

Hence, \( \nu(x) = v(x) + g(x) \) and
\[
v(x) = v_\delta + (1 + 2\delta) \log x - \pi x,
\]
\[
g(x) = \left( 1/2 + \delta \right) \log (1 + x_\delta^{-2}) - 2x \arctan x_\delta - \pi/2 + 1/x_\delta + 2 R(x).
\] (2.5)

Note (see [1, 8.343.1]) that
\[
R^{(k)}(x) = O(x^{-k-2}), \quad g^{(k)}(x) = O(x^{-k-2}) \quad (k \geq 0), \quad x \to \infty.
\] (2.6)

Using the Euler – Maclaurin formula of the second order, we find
\[
J(\gamma) = \int_1^\infty u(\gamma \lambda(t)) \, dt + \frac{1}{2} u(\gamma \lambda_1) + \Delta(\gamma) + \bar{\Delta}(\gamma),
\] (2.7)
where

$$\Delta(\gamma) = \sum_{j \geq 1} \int_0^1 \frac{t - t^2}{2} (v(\gamma \lambda(t+j)))''_{tt} dt, \quad \bar{\Delta}(\gamma) = \sum_{j \geq 1} \int_0^1 \frac{t - t^2}{2} (g(\gamma \lambda(t+j)))''_{tt} dt.$$ 

In this case by (2.1), (2.5) и (2.6)

$$|\bar{\Delta}(\gamma)| \leq \frac{1}{8} \int_1^\infty |(g(\gamma \lambda(t)))''_{tt}| dt \leq A_1 \gamma^{-1} \quad (2.8)$$

with the constant

$$A_1 = \frac{1}{8} \int_0^\infty (2y |g'(y)| + y^2 |g''(y)|) dy < \infty.$$

Put

$$s(t) = -(1 + 2 \delta) \log t - \gamma \pi t^{-1},$$

$$U_n = -\frac{1}{2} s(n+b) + \sum_{j=1}^n s(j+b) - \int_1^n s(t+b) dt. \quad (2.9)$$

According to (2.5), $v(\gamma \lambda(t))'_{tt} = s''(t+b)$, and therefore, by the Euler – Maclaurin formula again,

$$\Delta(\gamma) = \sum_{j \geq 1} \int_0^1 \frac{t - t^2}{2} s''(t+j) dt = -\frac{1}{2} s(1+b) + \lim U_n. \quad (2.10)$$

Straightforward calculations show that

$$\Delta(\gamma) = \gamma \pi (A_2(b) - A_3(b)) + (1 + 2 \delta) (A_4(b) - A_5(b)), \quad (2.11)$$

where

$$A_2(b) = (2(1+b))^{-1} - \ln (1+b),$$

$$A_4(b) = 1 + b - (1/2 + b) \ln (1+b),$$

$$A_3(b) = \lim \left( \sum_{j=1}^n \frac{1}{j+b} - \ln n \right), \quad (2.12)$$

$$A_5(b) = \lim \left( \sum_{j=1}^n \ln (j+b) + n - (n + 1/2 + b) \ln n \right).$$
The limits $A_3(b)$ and $A_5(b)$ may be calculated. So,

$$A_3(b) = A_3(0) - b \sum_{j \geq 1} 1/(j + b) = -\Psi(1 + b) \quad (2.13)$$

(see [1, 8.362.1, 8.366.1, 8.366.2, 8.367.1, 8.367.2]).

Similarly, taking into account [1, 8.322] (and the notation (2.3)), we can show that

$$A_5(b) = A_5(0) + b C_{\varepsilon} + \sum_{j \geq 1} \left( \ln (1 + b/j) - b/j \right) = \frac{1}{2} \log \frac{2\pi}{\Gamma^2(1 + b)} = \frac{1}{2} v_b. \quad (2.14)$$

Continue to analyze the equality (2.7). From (2.1) and (2.3) – (2.6) it follows that

$$u(\gamma \lambda_1) = v_\delta + \frac{(1 + 2\delta) \log (\gamma/(1 + b)) \pi \gamma/(1 + b) + O(\gamma^{-2})}{\gamma \to \infty}. \quad (2.15)$$

Further,

$$\int_1^{\infty} u(\gamma \lambda(t)) dt = \gamma \int_0^{\gamma \lambda_1} u(x) \frac{dx}{x^2} = -\frac{u(\gamma \lambda_1)}{\lambda_1} + \gamma \int_0^{\gamma \lambda_1} u'(x) \frac{dx}{x}. \quad (2.16)$$

We have (see (2.3)),

$$u'(x) = -2 \arctan x_\delta - \frac{1}{1 + \delta} \frac{x_\delta}{1 + x^2_\delta} + 2 R'(x),$$

and hence

$$\int_0^{\gamma \lambda_1} u'(x) \frac{dx}{x} = -2 \int_0^{\gamma \lambda_1} \arctan x_\delta \frac{dx_\delta}{x_\delta} - \frac{1}{1 + \delta} \arctan (\gamma \lambda_1/(1 + \delta)) -$$

$$2 \int_{\gamma \lambda_1}^{\infty} R'(x) \frac{dx}{x} + 2 \int_0^{\infty} R'(x) \frac{dx}{x}. \quad (2.17)$$
In this case (see (2.4) [1, 8.341.1] and (2.6)),

\[
\int_0^\infty \frac{R'(x)}{x} \, dx = -\int_0^\infty \left( \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right) e^{-t(1+\delta)} \int_0^\infty \sin tx \, dx \, dt = \left( \Psi(1 + \delta) - \log (1 + \delta) + \frac{1}{2(1 + \delta)} \right) \frac{\pi}{2}, \tag{2.18}
\]

\[
\int_0^{\gamma_{\lambda_1}} \arctan x_{\delta} \frac{dx_{\delta}}{x_{\delta}} = \frac{\pi}{2} \log \bar{\gamma} + \frac{1}{\bar{\gamma}} + O(\gamma^{-3}),
\]

\[
\arctan (\gamma_{\lambda_1}/(1 + \delta)) = \frac{\pi}{2} - \frac{1}{\bar{\gamma}} + O(\gamma^{-3}),
\]

with \( \bar{\gamma} = \gamma_{\lambda_1}/(1 + \delta) = \gamma/((1 + \delta)(1 + b)) \).

Combining (2.7) (2.8) (2.11) – (2.18), we get the first relation in (2.2).

To obtain the other assertions in (2.2) one can differentiate the right-hand side of (2.7) in \( \gamma \) and use (2.6), (2.11) (2.16).

Lemma 1 is proved.

Further, let us give one more result, which will used in what follows. Let \( d > 0, \ 0 < \varepsilon < d/e \). Consider the equation

\[
\ln (d y)/y = \varepsilon, \ y > e/d, \tag{2.19}
\]

and examine its solution \( y = y(\varepsilon) \) for small positive \( \varepsilon \).

**Lemma 2.** The solution of the equation (2.19) for all \( \varepsilon \) small enough can be written as absolutely convergent series

\[
y = \frac{\xi}{\varepsilon} \left( 1 + \sum_{j,l \geq 0} a_{jl} s_d^{j+1} \xi^{-l-j-1} \right), \ \xi = -\ln \varepsilon, \ s_d = \ln d \xi, \tag{2.20}
\]

where the coefficients \( a_{jl} \) are defined in (1.6).

Presumably, the representation (2.20) may be deduced from Theorem 2 in [3]. We provide a shorter proof below.

**Proof of Lemma 2.**
Put (see (2.20))
\[ y = \frac{\xi}{\varepsilon} (1 + v), \quad g(v) = \ln (1 + v)/v, \quad \sigma = 1/\xi, \quad \tau = s_d/\xi. \tag{2.21} \]

Then (2.20) is equivalent to the equation
\[ v - \sigma \ln (1 + v) = v(1 - \sigma g(v)) = \tau. \]

From this and from Lagrange’s formula it follows that
\[ v = \sum_{j \geq 1} d_j \tau^j, \tag{2.22} \]
for all \( \varepsilon \) small enough with
\[ d_j = \frac{1}{j!} \lim_{t \to 0} \frac{d^{j-1}}{dt^{j-1}} (1 - \sigma g(t))^{-j}. \]

We have,
\[ (1 - \sigma g(t))^{-j} = 1 + \sum_{l \geq 1} C_{j+l}^l \sigma^l g^l(t), \]
and hence
\[ d_{j+1} = \frac{1}{(j + 1)!} \sum_{l \geq 0} \sigma^l C_{j+l}^l \lim_{t \to 0} (g^l(t))^{(j)}, \quad j \geq 0. \tag{2.23} \]

Taking into account that according to the definition of the Stirling numbers of the first kind,
\[ \lim_{t \to 0} (g^l(t))^{(j)} = s(j + l, l)/C_{j+1}^l, \]
we obtain the first assertion of Remark 1. The second statement, i.e. the equality (1.6), follows from (2.21) – (2.23) and [12, Ch. VI, Lemma 3]. This also implies
\[ |a_{jl}| \leq \frac{1}{j + 1} C_{j+l}^l \frac{1}{2} C_{j+l-1}^j, \]
and thus the estimate (1.9) holds.

Lemma 2 and, simultaneously, remark 1 are completely proved.

Equation (2.20), obviously, can be rewritten in the form
\[ y = \frac{\xi}{\varepsilon} \left( 1 + \sum_{m \geq 1} \pi_m(s_d) \xi^{-m} \right), \tag{2.24} \]
where the polynomials $\pi_m(s)$ are defined in (1.5), and also can be calculated using the recursive relation (1.4).

Check out the latter fact. Set $f = f(z) = \sum_{m \geq 1} \pi_{m+1}(s) z^m$.

From (2.19) and (2.24) the formal equality $f = \ln (1 + s z + z f)$ follows. Differentiating it in $z$, we find $(1 + s z + z f) f' = s + (z f)'$. Equating the coefficients of the identical powers at $z$ from the lefthand side and from the righthand side, we get

$$\pi_{k+1} = \pi_k - \frac{1}{k} \sum_{i=1}^{k-1} i \pi_{k-i} \pi_{i+1}, \quad k \geq 1.$$  

Taking into account the fact that the sum is equal to $\sum_{i=0}^{k-2} (k-i-1) \pi_{k-i} \pi_{i+1}$, we obtain (1.4).

**Lemma 3.** Let $\xi_j$ and $\lambda_j$, $j \geq 1$ denote independent standard Gaussian random variables and positive numbers, respectively.

Assuming that $\sum_{j \geq 1} \lambda_j < \infty$, put $S = \sum_{j \geq 1} \lambda_j \xi_j^2$ and for $h > 0$ denote

$$L(h) = -\frac{1}{2} \sum_{j \geq 1} \ln (1 + 2h\lambda_j), \quad \tau^2(h) = h^2 L''(h).$$

Then for any $0 < r < E_S$

$$P(S < r) = \frac{1}{\tau(h) \sqrt{2\pi}} e^{L(h)+hr} (1 + \theta_1 \tau^{-2}(h)), \quad \frac{d}{dr} P(S < r) = h P(S < r) (1 + \theta_2 \tau^{-2}(h)),$$

where $h$ is the unique solution of the equation

$$L'(h) + r = 0, \quad (2.25)$$

and $|\theta_i|$ is bounded by a constant.

Recall that

$$L(h) + hr = \inf_{u \geq 0} (L(u) + ur). \quad (2.26)$$

Lemma 3 follows from [13, Corollary 1].

3. Proofs.
It is known (see [1, 8.326.1]) that
\[
\frac{\Gamma^2(1 + \delta)}{\left| \Gamma(1 + \delta + i x) \right|^2} = \prod_{j \geq 1} \left( 1 + \frac{x^2}{(j + \delta)^2} \right).
\]

Denoting for convenience \( h \) by \( \gamma^2 / 2 \), we get from here (see the notation in (1.1) and (2.1))
\[
L(h) = \log \mathbb{E} e^{-hV^2} = \frac{1}{2} J(\pi^{-2}\gamma; b, \delta) = \frac{1}{2} I(\gamma). \tag{3.1}
\]

We set (see (1.3))
\[
\bar{I}(\gamma) = -\pi^{-1} \gamma \ln \gamma + B \gamma - C \ln \gamma - D. \tag{3.2}
\]

By Lemma 1, as \( \gamma \to \infty \),
\[
I(\gamma) = \bar{I}(\gamma) + O(\gamma^{-1}),
\]
\[
\gamma I'(\gamma) = \gamma \bar{I}'(\gamma) + O(\gamma^{-1}) = -\pi^{-1} \gamma \ln \gamma + (B - \pi^{-1}) \gamma - C + O(\gamma^{-1}),
\]
\[
\gamma^2 I''(\gamma) = \gamma^2 \bar{I}''(\gamma) + O(\gamma^{-1}) = -\pi^{-1} \gamma + C + O(\gamma^{-1}). \tag{3.3}
\]

Now consider the equation (2.25) or (see (3.1))
\[
\frac{1}{2} I'(\gamma) + \gamma r = 0. \tag{3.4}
\]

From (3.3) it follows, in particular, that the solution of the equation (3.4) is such that
\[
\gamma \sim \gamma_0 = \gamma_0(r) = \frac{\left| \ln r \right|}{2\pi r}, \quad r \to 0. \tag{3.5}
\]

We have (see (3.1), (3.2) and (2.26)) for positive \( \varepsilon \) small enough
\[
2 \inf_{u > 0} (L(u) + ur) = \min_{1-\varepsilon < s/\gamma_0 < 1+\varepsilon} (I(s) + s^2r)
\]
\[
= \min_{1-\varepsilon < s/\gamma_0 < 1+\varepsilon} (\bar{I}(s) + s^2r) + O(\gamma_0^{-1})
\]
\[
= \bar{I}(\bar{\gamma}) + \bar{\gamma}^2r + O(\gamma_0^{-1}), \quad r \to 0,
\]
where \( \bar{\gamma} = \bar{\gamma}(r) \to \infty \) as \( r \to 0 \) is the solution of the equation
\[
\frac{1}{2} \bar{I}'(\bar{\gamma}) + \bar{\gamma} r = 0. \tag{3.7}
\]
Put (see (1.3) and (3.2))
\[ \tilde{I}_0(\gamma) = \bar{I}(\gamma) + C \ln \gamma, \]  
and let \( \tilde{\gamma} (\tilde{\gamma} \to \infty, \ r \to 0) \) satisfy the condition
\[ \frac{1}{2} \tilde{I}_0'(\tilde{\gamma}) + \tilde{\gamma} r = 0, \quad \text{or} \quad \frac{\log \tilde{\gamma} + 1 - \pi B}{2 \pi \tilde{\gamma}} = r. \]

Now we compare the solutions of equations (3.7) and (3.9). Note preliminary (see (3.5)) that \( \tilde{\gamma} \sim \tilde{\gamma} \sim \gamma_0 \) as \( r \to 0 \).

We have by (3.2) and (3.3)),
\[ 0 = \tilde{\gamma} \tilde{I}'(\tilde{\gamma}) - \tilde{\gamma} \tilde{I}_0'(\tilde{\gamma}) = \tilde{\gamma} (\tilde{I}'(\tilde{\gamma}) - \tilde{I}_0'(\tilde{\gamma})) + \tilde{\gamma} (\tilde{I}_0'(\tilde{\gamma}) - \tilde{I}_0'(\tilde{\gamma})) - (\tilde{\gamma} - \tilde{\gamma}) \tilde{I}_0'(\tilde{\gamma}) \]
\[ = -\tilde{\gamma} \frac{C}{\tilde{\gamma}} + \frac{\tilde{\gamma} - \tilde{\gamma}}{\tilde{\gamma}} \left( \tilde{\gamma}^2 \tilde{I}_0''(\tilde{\gamma}) \right) \]
\[ = (-1 + o(1)) C + (\pi - 1 + o(1)) \frac{\tilde{\gamma} - \tilde{\gamma}}{\tilde{\gamma}} \chi(r), \]
where \( \chi(r) = \gamma_0 \ln \gamma_0. \)

This implies
\[ \frac{\tilde{\gamma} - \tilde{\gamma}}{\tilde{\gamma}} = O \left( |C|/\chi(r) \right), \ r \to 0. \]

Next, by (3.9)
\[ \tilde{I}(\tilde{\gamma}) + \tilde{\gamma}^2 r = -C \ln \tilde{\gamma} + (\tilde{I}_0(\tilde{\gamma}) + \tilde{\gamma}^2 r) \]
\[ = -C \ln \gamma + (\tilde{I}_0(\tilde{\gamma}) + \tilde{\gamma}^2 r) + \frac{1}{2} (\tilde{\gamma} - \tilde{\gamma})^2 \left( 2 r + \tilde{I}_0''(\gamma) \right) \]
Therefore, taking into account (3.9) and (3.10), we obtain for \( r \to 0 \)
\[ \tilde{I}(\gamma) + \tilde{\gamma}^2 r = -C \ln \gamma + (\tilde{I}_0(\gamma) + \tilde{\gamma}^2 r) + O \left( 1/\chi(r) \right), \]
and similarly
\[ \tilde{\gamma}^2 \tilde{I}_0''(\tilde{\gamma}) - \tilde{\gamma} \tilde{I}_0'(\tilde{\gamma}) = 2C + (\tilde{\gamma}^2 \tilde{I}_0''(\tilde{\gamma}) - \tilde{\gamma} \tilde{I}_0'(\tilde{\gamma})) = \tilde{\gamma}^2 \tilde{I}_0''(\tilde{\gamma}) - \tilde{\gamma} \tilde{I}_0'(\tilde{\gamma}) + O \left( 1 \right). \]

Now let us examine the behavior of \( \tau^2(h) = h^2 L''(h) \) as \( h \to \infty \).
It is obvious (see (3.1) and (3.3)) that
\[ \tau^2(h) \bigg|_{h=\gamma^2/2} = \frac{1}{8} (\gamma^2 \tilde{I}''(\gamma) - \gamma \tilde{I}'(\gamma)) = \frac{1}{8} (\gamma^2 \tilde{I}''(\gamma) - \gamma \tilde{I}'(\gamma)) + O \left( \gamma^{-1} \right), \ \gamma \to \infty. \]
Therefore, if $\gamma$ and $\tilde{\gamma}$ satisfy the equations (3.7) and (3.9), respectively (and $h = \gamma^2/2$), then

$$\tau^2(h) \sim \chi(r), \ r \to 0,$$

(3.13)

and, taking into account (3.10), (3.12),

$$\tau^2(h) = \frac{1}{8} (\tilde{\gamma}^2 \tilde{I}_0''(\tilde{\gamma}) - \tilde{\gamma} \tilde{I}_0'((\tilde{\gamma})) + O(1), \ r \to 0. \quad (3.14)$$

Denote

$$Q(r) = -\frac{1}{2} (\tilde{I}_0((\tilde{\gamma}) + \tilde{\gamma}^2 r), \quad q(r) = \frac{1}{8} (\tilde{\gamma}^2 \tilde{I}_0''(\tilde{\gamma}) - \tilde{\gamma} \tilde{I}_0'((\tilde{\gamma})).$$

By (3.2), (3.8) и (3.9)

$$Q(r) = \frac{1}{2} (\tilde{\gamma}^2 r - \pi^{-1} \tilde{\gamma} + D), \quad q(r) = \frac{1}{8} (2 \tilde{\gamma}^2 r - \pi^{-1} \tilde{\gamma}), \quad (3.15)$$

and, hence (see (3.6) and (3.11))

$$\inf_{u>0} (L(u) + ur) = -Q(r) - C_2 \ln \tilde{\gamma} + O(1/\gamma_0), \ r \to 0.$$

This relation and Lemma 3 with $S = V^2$, as well as relations (2.26), (3.13) - (3.15), imply

$$P(V^2 < r) = (2\pi \tilde{\gamma} C q(r))^{-1/2} e^{-Q(r)} \left(1 + O(\gamma_0^{-1})\right),$$

$$\frac{d}{dr} P(V^2 < r) = \frac{\tilde{\gamma}^2}{2} P(V^2 < r) \left(1 + O(\gamma_0^{-1})\right), \ r \to 0, \quad (3.16)$$

provided $\tilde{\gamma}$ satisfies the equation (3.9) and $\gamma_0$ is defined in (3.5).

Continue our reasoning.

Let $y = 2 \pi \tilde{\gamma}$, $d = \pi^{-1} e^{1 - \pi B} / 2$, $r = \varepsilon$. Then (see (3.5) and (3.8)), the equation (3.9) has the form (2.19).

Hence, using Lemma 2 ((2.24)), we obtain

$$2\tilde{\gamma} r = \pi^{-1} \xi \left(1 + \sum_{m \geq 1} \bar{\pi}_m \xi^{-m}\right), \quad \xi = |\ln r|,$$

$$\bar{\pi}_m = \pi_m(\bar{s}), \quad \bar{s} = \ln \xi + 1 - \pi B - \ln (2\pi). \quad (3.17)$$
Put $\tilde{\pi}_m = \sum_{l=0}^m \tilde{\pi}_l \tilde{\pi}_{m-l}$ ($\tilde{\pi}_0 = 1$). We have,

$$\tilde{\pi}_1 = 2 \bar{s}, \quad \tilde{\pi}_2 = \tilde{\pi}_3 = 2 \bar{s} + \bar{s}^2,$$

and (see (1.4))

$$\tilde{\pi}_m = -\frac{2}{m-2} (\tilde{\pi}_m - (m-1) \tilde{\pi}_{m-1}), \quad m \geq 3. \quad (3.19)$$

By (3.17)

$$(2 \tilde{\gamma} r)^2 = \pi^{-2} \xi^2 \left(1 + \tilde{\epsilon}(r)\right), \quad \tilde{\epsilon}(r) = \sum_{m \geq 1} \tilde{\pi}_m \xi^{-m}. \quad (3.20)$$

Using (3.15), (3.17) and (3.20), we find

$$Q(r) = \frac{1}{2} D + \frac{\pi^{-2} \xi^2}{8r} \left(1 + \varepsilon(r)\right), \quad \varepsilon(r) = \sum_{m \geq 1} (\tilde{\pi}_m - 2 \tilde{\pi}_{m-1}) \xi^{-m},$$

$$q(r) = \frac{\xi^2}{16 \pi^2 r} \left(1 + \varepsilon_q(r)\right), \quad \varepsilon_q(r) = \sum_{m \geq 1} (\tilde{\pi}_m - \tilde{\pi}_{m-1}) \xi^{-m}. \quad (3.21)$$

The relations (3.16), (3.20) and (3.21) lead to the following result.

**Proposition 1.** Let

$$\kappa(r) = \frac{\ln^2 r}{8 \pi^2 r}, \quad K = \left(\pi \frac{2^{C/2} e^D}{C/2}\right)^{-1/2}.$$

Then as $r \to 0$

$$P(V^2 < r) = K r^{C/4} \kappa^{-(C+2)/4}(r) e^{-\kappa(r) (1+\varepsilon(r))} \left(1 + \delta(r) + O(r/|\ln r|)\right),$$

$$\frac{d}{dr} P(V^2 < r) = K r^{C/4-1} \kappa^{(2-C)/4}(r) e^{-\kappa(r) (1+\varepsilon(r))} \left(1 + \delta(r) + \tilde{\varepsilon}(r) + O(r/|\ln r|)\right),$$

where

$$\delta(r) = \left((1 + \varepsilon_q(r)) (1 + \tilde{\varepsilon}(r))^{C/2}\right)^{-1/2} - 1.$$

The function $\delta(r)$ can be written as an absolutely convergent series with the structure which is similar to the representation of $\tilde{\varepsilon}(r)$ (see (3.20)). In particular,

$$\delta(r) = \frac{1}{2} \left(1 - (C+2) \bar{s}\right)/|\ln r| + O\left(\frac{|\ln |\ln r|}{|\ln r|}\right)^2. \quad (3.22)$$
The relations (1.10) and (1.11) of Theorem 1 follow from (3.22) and Proposition 1 (recall that $\Psi(1) = -C_e$, $\Psi(1/2) = -C_e - 2 \ln 2$). The relation (1.12) is easily verified with the help of (1.5)–(1.7).

Corollary 2 follows from the comparison theorem of [8] and the well-known representation of gamma-function

$$\frac{1}{\Gamma(1 + z)} = e^{C_e z} \prod_{n \geq 1} (1 + z/n) e^{-z/n}.$$ 

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REFERENCES

1. I.S. Gradshteyn, I.M. Ryzhik, Tables of integrals, sums, series and products, 5th ed. Moscow, Nauka, 1971 (in Russian). English transl.: Table of integrals, series, and products. Corr. and enl. ed. by Alan Jeffrey. New York London Toronto: Academic Press (Harcourt Brace Jovanovich, Publishers), 1980.

2. L. Comtet, Advanced Combinatorics (Reidel, 1974)).

3. L. Comtet, Inversion de $y^\alpha e^y$ et $y \log^\alpha y$ au moyen des nombres de Stirling, Comptes Rendus de l’Academie Scientifique, Ser. Mathematiques, 270(1970), 1085–8.

4. E. Csáki, On small values of the square integral of a multiparameter Wiener process., In: Statistics and Probability, Proc. of the 3rd Pannonian Symp. on Math. Stat. D.Reidel, Boston, 1982, 19–26.

5. J.A. Fill, F. Torcaso, Asymptotic analysis via Mellin transforms for small deviations in L2-norm of integrated Brownian sheets, Probab. Theory Relat. Fields 130 (2003), 259–288.

6. A. Karol’, A. Nazarov, Ya. Nikitin, Small ball probabilities for Gaussian random fields and tensor products of compact operators, Trans. AMS, 360 (2008), N3, 1443–147.

7. A. Karol’, A. Nazarov, Small ball probabilities for smooth Gaussian fields and tensor products of compact operators, Mathematische Nachrichten 287 (2014), 595–609,
8. W.V. Li, Comparison results for the lower tail of Gaussian seminorms, J. Theor. Probab. 5(1) (1992), 1–31.

9. A.I. Nazarov and Ya.Yu. Nikitin, Logarithmic L2-small ball asymptotics for some fractional Gaussian processes, Theor. Probab. and Appl., 49 (2004), 695–711.

12. V.V. Petrov, Sums of independent random variables, Nauka, Moscow 1972 (in Russian).

13. L. V. Rozovsky, On gaussian measure of balls in a hilbert space, Theory Probab. Appl., 53(2) (2009), 357-364.

14. L. V. Rozovsky, Small deviation probabilities for weighted sums of independent random variables with a common distribution that can decrease at zero fast enough, Statistics and Probability Letters, 117 (2016), pp. 196–200.