ON METRIC CONNECTIONS WITH TORSION ON THE COTANGENT BUNDLE WITH MODIFIED RIEMANNIAN EXTENSION

LOKMAN BILEN AND AYDIN GEZER

Abstract. Let \( M \) be an \( n \)-dimensional differentiable manifold equipped with a torsion-free linear connection \( \nabla \) and \( T^* M \) its cotangent bundle. The present paper aims to study a metric connection \( \overset{\sim}{\nabla} \) with nonvanishing torsion on \( T^* M \) with modified Riemannian extension \( g_{\nabla,c} \). First, we give a characterization of fibre-preserving projective vector fields on \( (T^* M, g_{\nabla,c}) \) with respect to the metric connection \( \overset{\sim}{\nabla} \). Secondly, we study conditions for \( (T^* M, g_{\nabla,c}) \) to be semi-symmetric, Ricci semi-symmetric, \( \overset{\sim}{Z} \) semi-symmetric or locally conharmonically flat with respect to the metric connection \( \overset{\sim}{\nabla} \). Finally, we present some results concerning the Schouten-Van Kampen connection associated to the Levi-Civita connection \( \nabla \) of the modified Riemannian extension \( g_{\nabla,c} \).

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1. Introduction

Let \( (M, \nabla) \) be an \( n \)-dimensional differentiable manifold equipped with a torsion-free linear connection \( \nabla \). Denote by \( T^* M \) the cotangent bundle of \( M \) and let \( \pi \) be the natural projection \( T^* M \to M \). The vertical distribution \( V \) on \( T^* M \) (\( V \) is the kernel of the submersion \( T^* M \to M \)), which is the integrable distribution. If \( M \) is a paracompact manifold there exists a \( \mathcal{C}^\infty \)-distribution \( H \) on \( T^* M \) which is supplementary to the vertical distribution \( V \), such as the Whitney sum \( TT^* M = HT^* M \oplus VT^* M \) holds.

For the torsion-free linear connection \( \nabla \) on \( M \), the cotangent bundle of \( M \), \( T^* M \), can be endowed with a pseudo-Riemannian metric \( g_{\nabla} \) of neutral signature, called the Riemannian extension of \( \nabla \), given by

\[
\begin{align*}
\mathcal{G}_{\nabla}(H X, H Y) &= 0 \\
\mathcal{G}_{\nabla}(H X, V \omega) &= \mathcal{G}_{\nabla}(V \omega, H X) = \omega(X) \\
\mathcal{G}_{\nabla}(V \omega, V \theta) &= 0
\end{align*}
\]

where \( H X \) and \( H Y \) denote the horizontal lifts of the vector fields \( X \) and \( Y \), and \( V \omega \) and \( V \theta \) denote the vertical lifts of the covectors (1-forms) \( \omega \) and \( \theta \). Thus, the Riemannian extension of \( (M, \nabla) \) is a pseudo-Riemannian manifold \( (T^* M, g_{\nabla}) \). Riemannian extensions were first defined and studied by Patterson and Walker \[16\] and then investigated in Afifi \[2\]. Moreover, Riemannian extensions were also considered in Garcia-Rio et al. \[7\] in relation to Osserman manifolds (see also Derdzinski \[3\]). For further references relation to Riemannian extensions, see \[1\] \[6\] \[10\] \[15\] \[21\] \[22\] \[23\]. Classical Riemannian extensions have been generalized in...
several ways, see, as an example [13]. In [3, 4], the authors introduced another generalization which is called modified Riemannian extension. For a symmetric (0, 2)-tensor field $c$ on $(M, \nabla)$, this metric is given by $\mathcal{g}_{\nabla, c} = \mathcal{g}_{\nabla} + \pi^* c$, that is,

\[
\begin{align*}
\mathcal{g}_{\nabla, c}(H^X, H^Y) &= c(X, Y) \\
\mathcal{g}_{\nabla, c}(H^X, V^\omega) &= \mathcal{g}_{\nabla, c}(V^\omega, H^X) = \omega(X) \\
\mathcal{g}_{\nabla, c}(V^\omega, V^\theta) &= 0.
\end{align*}
\]

In this paper, we consider a metric connection $\bar{\nabla}$ with nonvanishing torsion on the cotangent bundle $T^*M$ with modified Riemannian extension $\mathcal{g}_{\nabla, c}$. First, we give a necessary and sufficient condition for a vector field on $(T^*M, \mathcal{g}_{\nabla, c})$ to be fibre-preserving projective vector field on $T^*M$ with respect to the metric connection $\bar{\nabla}$. This condition is represented by a set of relations involving certain tensor fields on $M$. Secondly, we investigate the conditions for the cotangent bundle $(T^*M, \mathcal{g}_{\nabla, c})$ to be semi-symmetric, Ricci semi-symmetric, $\tilde{Z}$ semi-symmetric and locally conharmonically flat with respect to the metric connection $\bar{\nabla}$. Finally, we show that the Schouten-Van Kampen connection associated to the Levi-Civita connection $\nabla$ of the modified Riemannian extension $\mathcal{g}_{\nabla, c}$ is equal to the horizontal lift $H^X$ of the torsion-free linear connection $\nabla$ to $T^*M$ and present a result concerning the curvature tensor of the Schouten-Van Kampen connection.

The manifolds, tensor fields and geometric objects we consider in this paper are assumed to be differentiable of class $C^\infty$. Einstein’s summation convention is used, the range of the indices $h, i, j, k, l, m, r, \ldots, n$, being always $\{1, \ldots, n\}$.

2. Preliminaries

We refer to [24] for further details concerning the material of this section. Let $M$ be an $n$-dimensional differentiable manifold with a torsion-free linear connection $\nabla$ and denote by $\pi : T^*M \to M$ its cotangent bundle with fibres the cotangent spaces to $M$. Then $T^*M$ is a $2n$-dimensional smooth manifold and some local charts induced naturally from local charts on $M$, may be used. Namely, a system of local coordinates $\left( U, x^i \right)$, $i = 1, \ldots, n$ on $M$ induces on $T^*M$ a system of local coordinates $\left( \pi^{-1}(U), x^i, x^\tau = p_i \right)$, $\bar{i} = n + i = n + 1, \ldots, 2n$, where $x^\tau = p_i$ is the components of covectors $p$ in each cotangent space $T^*_xM$, $x \in U$ with respect to the natural coframe $\{dx^i\}$.

Let $X = X^i \frac{\partial}{\partial x^i}$ and $\omega = \omega_i dx^i$ be the local expressions in $U$ of a vector field $X$ and a covector field $\omega$ on $M$, respectively. Then the vertical lift $V^\omega$ of $\omega$, the horizontal lift $H^X$ of $X$ are given, with respect to the induced coordinates, by

\[ V^\omega = \omega_i \partial_\tau, \]

and

\[ H^X = X^i \partial_i + p_h \Gamma^h_{ij} X^j \partial_\tau \]

where $\partial_i = \frac{\partial}{\partial x^i}$, $\partial_\tau = \frac{\partial}{\partial \tau}$ and $\Gamma^h_{ij}$ are the coefficients of $\nabla$ on $M$.

Next, we can introduce a frame field on each induced coordinate neighborhood $\pi^{-1}(U)$ of $T^*M$. It is called the adapted frame and consists of the following $2n$
linearly independent vector fields $\{E_\beta\} = \{E_j, E_\beta\}$:

$$
\begin{align*}
E_j &= \partial_j + p_a \Gamma^a_{hj} \partial_h \\
E_\beta &= \partial_\beta.
\end{align*}
$$

The indices $\alpha, \beta, \gamma, ... = 1, ..., 2n$ indicate the indices with respect to the adapted frame. The Lie brackets of the adapted frame of $T^*M$ satisfy the following identities:

$$
\begin{align*}
[E_i, E_j] &= p_s R_{ijl}^s E_l, \\
[E_i, E_\beta] &= -\Gamma_{il}^j E_l, \\
[E_\beta, E_\gamma] &= 0,
\end{align*}
$$

where $R_{ijl}^s$ denote the coefficients of the curvature tensor $R$ of $\nabla$ on $M$.

With respect to the adapted frame $\{E_\beta\}$, the vector fields $V\omega$ and $^HX$ on $T^*M$ has the components

$$
V\omega = \begin{pmatrix} 0 \\ \omega_j \end{pmatrix} \quad \text{and} \quad ^HX = \begin{pmatrix} X^j \\ 0 \end{pmatrix}.
$$

3. The metric connection with nonvanishing torsion on the cotangent bundle with respect to modified Riemannian extension

Let us consider $T^*M$ equipped with the modified Riemannian extension $\bar{g}_\nabla, c$ for a given torsion-free connection $\nabla$ on $M$. In adapted frame $\{E_\beta\}$, the modified Riemannian extension $(\bar{g}_\nabla, c)_{\beta\gamma}$ and its inverse $(\bar{g}_\nabla, c)_{\beta^\gamma}$ have in the following forms:

$$
(\bar{g}_\nabla, c)_{\beta\gamma} = \begin{pmatrix} c_{ij} & \delta_i^j \\ \delta_j^i & 0 \end{pmatrix},
$$

$$
(\bar{g}_\nabla, c)^{\beta\gamma} = \begin{pmatrix} 0 & \delta_j^i \\ \delta_i^j & -c_{ij} \end{pmatrix}
$$

where $c_{ij}$ are the components of the symmetric $(0, 2)$--tensor field $c$ on $(M, \nabla)$.

For the Levi-Civita connection $\nabla$ of the modified Riemannian extension $\bar{g}_\nabla, c$, we get:

**Proposition 1.** Let $\nabla$ be a torsion-free linear connection on $M$ and $T^*M$ be the cotangent bundle with the modified Riemannian extension $\bar{g}_\nabla, c$ over $(M, \nabla)$. The Levi-Civita connection $\nabla$ of $(T^*M, \bar{g}_\nabla, c)$ is given by

$$
\begin{align*}
\nabla_{E_j} E_{\beta} &= 0, \\
\nabla_{E_\beta} E_j &= 0, \\
\nabla_{E_j} E_{\beta} &= -\Gamma_{h}^{ij} E_h, \\
\nabla_{E_\beta} E_\gamma &= \Gamma_{ij}^{\beta \gamma} E_i + \{p_s R_{hji}^s + \frac{1}{2}(\nabla_i c_{jh} + \nabla_j c_{ih} - \nabla_h c_{ij})\} E_h,
\end{align*}
$$

with respect to the adapted frame $\{E_\beta\}$, where $\Gamma_{ij}^{\beta \gamma}$ and $R_{hji}^s$ respectively denote components of $\nabla$ and its curvature tensor field $R$ on $M$ (see, [8]).
If there is a Riemannian metric $g$ on $M$ such that $\nabla g = 0$, then the connection $\nabla$ is a metric connection, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection. The Levi-Civita connection $\nabla$ of the modified Riemannian extension $\mathcal{F}_{\nabla,c}$ on $T^*M$ is the unique connection which satisfies $\nabla_{\alpha}(\mathcal{F}_{\nabla,c})_{\beta\gamma} = 0$ and has a zero torsion. Now we are interested in a metric connection $\tilde{\nabla}$ of the modified Riemannian extension $\mathcal{F}_{\nabla,c}$ whose torsion tensor $\tilde{T}_{\alpha\beta}$ is skew-symmetric in the indices $\gamma$ and $\beta$. Metric connection with nonvanishing torsion on Riemannian manifolds were introduced by Hayden [9]. We denote components of the metric connection $\tilde{\nabla}$ by $\tilde{\Gamma}^\gamma_{\alpha\beta}$. The metric connection $\tilde{\nabla}$ satisfies

$$\tilde{\nabla}_{\alpha}(\mathcal{F}_{\nabla,c})_{\beta\gamma} = 0$$

and

$$\tilde{\Gamma}^\gamma_{\alpha\beta} - \tilde{\Gamma}^\gamma_{\beta\alpha} = \tilde{T}^\gamma_{\alpha\beta}.$$

When the above equation is solved with respect to $\tilde{\Gamma}^\gamma_{\alpha\beta}$, one finds the following solution [9]

$$\tilde{\Gamma}^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta} + \tilde{U}^\gamma_{\alpha\beta},$$

where $\Gamma^\gamma_{\alpha\beta}$ is the components of the Levi-Civita connection $\nabla$ of the modified Riemannian extension $\mathcal{F}_{\nabla,c}$,

$$\tilde{U}^\gamma_{\alpha\beta} = \frac{1}{2}(\tilde{T}^\gamma_{\alpha\beta} + \tilde{T}^\gamma_{\beta\alpha} + \tilde{T}^\gamma_{\gamma\beta\alpha}).$$

If we choose the torsion tensor $\tilde{T}$ as

$$\begin{cases} \tilde{T}^\gamma_{ij} = -p_s R^s_{ijr}, \\ \text{otherwise} = 0, \end{cases}$$

with the help of (3.6), from (3.5), we find non-zero component of $\tilde{U}^\gamma_{\alpha\beta}$ as follows:

$$\tilde{U}^\gamma_{ij} = p_s R^s_{jhi}.$$

with respect to the adapted frame. In view of (3.6) and (3.4), we have the following proposition.

**Proposition 2.** Let $\nabla$ be a torsion-free linear connection on $M$ and $T^*M$ be the cotangent bundle with the modified Riemann extension $\mathcal{F}_{\nabla,c}$ over $(M, \nabla)$. The metric connection $\tilde{\nabla}$ on $T^*M$ with respect to the modified Riemannian extension $\mathcal{F}_{\nabla,c}$ satisfies

$$\begin{cases} \tilde{\nabla}_{E_x} E_x = 0, & \tilde{\nabla}_{E_x} E_j = 0, \\ \tilde{\nabla}_{E_y} E_x = -\Gamma^h_{ij} E^s, \\ \tilde{\nabla}_{E_x} E_j = \Gamma^h_{ij} E_h + \frac{1}{2}(\nabla_{E_{jhi}} + \nabla_{E_{jhi}} - \nabla_{E_{ijh}}) E^s, \end{cases}$$

with respect to the adapted frame $\{E_\beta\}$.

The horizontal lift $H\nabla$ of the torsion-free linear connection $\nabla$ on $M$ to $T^*M$ is characterized the following conditions:

$$\begin{cases} H\nabla_{E_{\omega}} V \theta = 0, & H\nabla_{E_{\omega}} H Y = 0 \\ H\nabla_{E_{X}} V \theta = V(\nabla_{E_{X}} \theta), & H\nabla_{E_{X}} H Y = H(\nabla_{E_{X}} Y) \end{cases}$$
for all vector fields $X, Y$ and covector fields $\omega, \theta$ on $M$ ([24], p. 287). In the adapted frame, the followings satisfy (see, also [1])

$$
\begin{align*}
H^i \nabla_E T^i_j &= 0, \quad H^i \nabla_{E_i} E_j = 0, \\
H^i \nabla E_i, E_j = -\Gamma^i_{h j} E_h, \quad H^i \nabla E_i, E_j = \Gamma^h_{i j} E_h.
\end{align*}
$$

From these formulas, we can readily deduce:

**Proposition 3.** Let $\nabla$ be a torsion-free linear connection on $M$ and $T^*M$ be the cotangent bundle with the modified Riemann extension $\mathfrak{g}_{\nabla, c}$ over $(M, \nabla)$. The metric connection $\tilde{\nabla}$ on $T^*M$ of the modified Riemannian extension $\tilde{\mathfrak{g}}_{\nabla, c}$ coincides with the horizontal lift $H^i \nabla$ of the torsion-free linear connection $\nabla$ on $M$ if and only if the components $c_{ij}$ of $c$ satisfy the condition

$$
\nabla_i c_{jh} + \nabla_j c_{ih} - \nabla_h c_{ij} = 0.
$$

3.1. **Projective vector fields on the cotangent bundle with respect to the metric connection $\tilde{\nabla}$.** Given a linear connection $\nabla$ on a manifold $M$, a vector field $V$ is said to be a projective vector field if there exists a 1-form $\theta$ such that

$$(L_V \nabla)(X, Y) = \theta(X)Y + \theta(Y)X$$

for any pair of vector fields $X$ and $Y$ on $M$. In particular, if $\theta = 0$, $V$ is an affine Killing vector field.

Let $\tilde{V}$ be a vector field on $T^*M$ and $(v^h, v^\ell)$ its the components with respect to the adapted frame $\{E_\beta\}$. The components $v^h$ and $v^\ell$ are said to be the horizontal components and vertical components of $\tilde{V}$, respectively. As is known, a vector field is called a fibre-preserving vector field if and only if its horizontal components depend only on the variables $(x^h)$. Hence, one can easily say that every fibre-preserving vector field $\tilde{V}$ on $T^*M$ induces a vector field $V$ with components $(v^h)$ on the base manifold $M$.

By straightforward calculations, we have the following.

**Lemma 1.** Let $\tilde{V}$ be a fibre-preserving vector field on $T^*M$ with components $(v^h, v^\ell)$. The Lie derivatives of the adapted frame satisfy

$$
\begin{align*}
i) L_{\tilde{V}} E_i &= -(E_i v^k) E_k - \left( v^\alpha p_\alpha R_{\tau ak} \right) E_k, \\
ii) L_{\tilde{V}} E^*_i &= -(v^\alpha \Gamma^i_{ak} + E_k v^\ell) E^*_i,
\end{align*}
$$

where $L_{\tilde{V}}$ denotes the Lie derivation with respect to $\tilde{V}$.

The general forms of fibre-preserving projective vector fields on $T^*M$ with respect to the metric connection $\tilde{\nabla}$ are given by:

**Theorem 1.** Let $\nabla$ be a torsion-free linear connection on $M$ and $T^*M$ be the cotangent bundle with the modified Riemannian extension $\mathfrak{g}_{\nabla, c}$ over $(M, \nabla)$. Then a vector field $\tilde{V}$ is a fibre-preserving projective vector field with associated 1-form $\tilde{\theta}$ on $T^*M$ with respect to the metric connection $\tilde{\nabla}$ if and only if the vector field $\tilde{V}$ is defined by

$$(3.8) \quad \tilde{X} = H V + V B + \gamma A,$$

where the vector field $V = (v^h)$, the covector field $B = (B_h)$, the $(1, 1)$-tensor field $A = (A^h_i)$ and the associated 1-form $\tilde{\theta}$ satisfy
\( (i) \quad \bar{\theta} = \theta_i dx^i, \)

\( (ii) \quad \nabla_j A_k^i = \theta_j \delta_k^i - v^a R_{jak}^i, \)

\( (iii) \quad L_V \Gamma^h_{ij} = \theta_i \delta^h_j + \theta_j \delta^h_i, \)

\( (iv) \quad \nabla_j \nabla_j B_i + R_{ij}^a B_a + \frac{1}{2} v^a \nabla_a M_{ij} + \frac{1}{2} (\nabla_j v^a) M_{ial} + \frac{1}{2} (\nabla_j v^a) M_{ial} \)

\[ + \frac{1}{2} (\nabla_i v^a) M_{ajl} - A_j^a M_{ija} = 0 \] (\( M_{ija} := \nabla_i c_{jl} + \nabla_j c_{il} - \nabla_{ijl} \))

\( (v) \quad \nabla_i \nabla_j A_k^a + R_{ij}^a A_a^k - R_{ajl}^s A_l^a + v^a \nabla_i R_{jal}^s + (\nabla_i v^a) R_{jal}^s = 0. \)

**Proof.** A fibre-preserving vector field \( \bar{V} = v^h E_k + v^a E^i_a \) on \( T^* M \) is a fibre-preserving projective vector field if and only if there exist a 1-form \( \theta \) with components \((\bar{\theta}_i, \bar{\theta}_j)\) on \( T^* M \) such that

\[
(L_{\bar{V}} \nabla)(\tilde{Y}, \bar{Z}) = L_{\bar{V}}(\nabla_\tilde{Y} \bar{Z}) - \nabla_{(L_{\bar{V}} \bar{Y})} \bar{Z} = \tilde{\theta}(\bar{Y}) \bar{Z} + \bar{\theta}(\bar{Z}) \tilde{Y}
\]

for any vector fields \( \tilde{Y} \) and \( \bar{Z} \) on \( T^* M \).

Putting \( \tilde{Y} = E_{i}, \bar{Z} = E_{j} \) in (3.9), we get

\[
E_{i} \left( E_{j} v^k \right) E_{k} = \theta_{i} E_{j} + \theta_{j} E_{i}.
\]

Putting \( \tilde{Y} = E_{i}, \bar{Z} = E_{j} \) in (3.9), we find

\[
\theta_{i} = 0
\]

and

\[
v^a R_{jai} + E_{i} \left( E_{j} v^k \right) - (E_{j} v^a) \Gamma_{jai}^a = \theta_{j} \delta_{i}^k.
\]

In view of (3.11), (3.10) reduces to

\[
E_{i} \left( E_{j} v^k \right) E_{k} = 0
\]

from which it follows that

\[
v^k = p_{a} A_{k}^a + B_{k}
\]

where \( A_{k}^a \) and \( B_{k} \) are certain functions which depend only on the variables \((x^h)\).

The coordinate transformation rule implies that \( A \) is a \((1,1)\)–tensor field with components \((A_{j}^a)\) and \( B \) is a covector field with components \((B_{k})\). Hence, the fibre-preserving projective vector field \( \bar{V} \) on \( T^* M \) can be written in the form:

\[
\bar{V} = v^k E_k + v^a E^i_a = v^k E_k + \{p_{a} A_{k}^a + B_{k}\} E^a_k
\]

where \( \gamma \) is an operator applied to the \((1,1)\)–tensor field \( A \) and expressed locally \( \gamma A = (p_{a} A_{k}^a) E^a_k \) (for details related to the operator \( \gamma \), see [24], p.12 – 13).

Substitution (3.13) into (3.12) gives

\[
v^a R_{jai} + \nabla_j A_{k}^i = \theta_{j} \delta_{k}^i.
\]

Contracting \( i \) and \( k \) in (3.14), we have

\[
\theta_{j} = \frac{1}{n} \nabla_j A_{k}^k.
\]

Finally, putting \( Y = E_{i}, Z = E_{j} \) in (3.9), we obtain

\[
L_V \Gamma_{ij}^h = \theta_{i} \delta_{j}^h + \theta_{j} \delta_{i}^h.
\]
The curvature tensor $\tilde{R}$ of the metric connection $\tilde{\nabla}$ on $T^*M$ is obtained from the formula

$$\tilde{R}(E_\alpha, E_\beta)E_\gamma = \tilde{\nabla}_{E_\alpha} \tilde{\nabla}_{E_\beta} E_\gamma - \tilde{\nabla}_{E_\beta} \tilde{\nabla}_{E_\alpha} E_\gamma - \tilde{\nabla}_{[E_\alpha, E_\beta]} E_\gamma$$

with respect to the adapted frame. For the curvature tensor $\tilde{R}$ of the metric connection $\tilde{\nabla}$, with the help of (2.1) and (3.7), we have:
Proposition 4. Let $\nabla$ be a torsion-free linear connection on $M$ and $T^*M$ be the cotangent bundle with the modified Riemann extension $\mathcal{F}_{\nabla,c}$ over $(M, \nabla)$. The curvature tensor $\tilde{R}$ of the metric connection $\tilde{\nabla}$ on $T^*M$ satisfies the following conditions:

$$
\tilde{R}(E_i, E_j)E_k = R_{ijk}^h E_h \\
+ \frac{1}{2} \{ \nabla_i (\nabla_k c_{jh} - \nabla_h c_{jk}) - \nabla_j (\nabla_k c_{ih} - \nabla_h c_{ik}) \\
- R_{ijk}^m c_{mnh} - R_{ijh}^m c_{km} \} E_{\pi} \\
\tilde{R}(E_i, E_j)E_{\pi} = R_{jih}^k E_k, \\
\tilde{R}(E_i, E_j)E_k = 0, \tilde{R}(E_i, E_j)E_{\pi} = 0, \tilde{R}(E_i, E_j)E_{\pi} = 0
$$

with respect to the adapted frame $\{E_{\beta}\}$.

Let $\tilde{X}$ and $\tilde{Y}$ be vector fields of $T^*M$. The curvature operator $\tilde{R}(\tilde{X}, \tilde{Y})$ is a differential operator on $T^*M$. Similarly, for vector fields $X$ and $Y$ of $M$, $R(X, Y)$ is a differential operator on $M$. Now, we operate the curvature operator $\tilde{R}(\tilde{X}, \tilde{Y})$ to the curvature tensor $\tilde{R}$. That is, for all $\tilde{Z}, \tilde{W}$ and $\tilde{U}$, we consider the condition $(\tilde{R}(\tilde{X}, \tilde{Y})\tilde{R})(\tilde{Z}, \tilde{W})\tilde{U} = 0$. In the case, we shall call the cotangent bundle $T^*M$ as semi-symmetric with respect to the metric connection $\tilde{\nabla}$.

In the adapted frame $\{E_{\beta}\}$, the tensor $(\tilde{R}(\tilde{X}, \tilde{Y})\tilde{R})(\tilde{Z}, \tilde{W})\tilde{U}$ is locally expressed as follows:

$$
((\tilde{R}(\tilde{X}, \tilde{Y})\tilde{R})(\tilde{Z}, \tilde{W})\tilde{U})_{\alpha\beta\gamma\theta\sigma}^\varepsilon \\
= \tilde{R}_{\alpha\beta\tau} \tilde{R}_{\gamma\theta\varepsilon} - \tilde{R}_{\alpha\beta\varepsilon} \tilde{R}_{\gamma\theta\tau} - \tilde{R}_{\alpha\beta\theta} \tilde{R}_{\gamma\tau\varepsilon} - \tilde{R}_{\alpha\beta\varepsilon} \tilde{R}_{\gamma\theta\tau}.
$$

Similarly, in local coordinates,

$$
((R(X, Y)R)(Z, W)U)_{ijklm}^n = R_{ijp}^n R_{klm}^p - R_{ijp}^n R_{klm}^p - R_{ijp}^n R_{klm}^p - R_{ijm}^p R_{klp}^n.
$$

Theorem 2. Let $\nabla$ be a torsion-free linear connection on $M$ and $T^*M$ be the cotangent bundle with the modified Riemann extension $\mathcal{F}_{\nabla,c}$ over $(M, \nabla)$. Under the assumption that $\nabla_i (\nabla_k c_{jh} - \nabla_h c_{jk}) - \nabla_j (\nabla_k c_{ih} - \nabla_h c_{ik}) - R_{ijk}^m c_{mnh} - R_{ijh}^m c_{km} = 0$, where $R$ is the curvature tensor of $\nabla$, the cotangent bundle $T^*M$ is semi-symmetric with respect to the metric connection $\tilde{\nabla}$ if and only if the base manifold $M$ is semi-symmetric with respect to $\nabla$.

Proof. We consider the conditions $(\tilde{R}(\tilde{X}, \tilde{Y})\tilde{R})(\tilde{Z}, \tilde{W})\tilde{U} = 0$ for all vector fields $\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}$ and $\tilde{U}$ on $T^*M$.

For all cases $\alpha = (i, l), \beta = (j, m), \gamma = (k, \pi), \theta = (l, \pi), \sigma = (m, \pi)$ and $\varepsilon = (h, \pi)$ in (3.10), the non-zero components of the tensor $((\tilde{R}(\tilde{X}, \tilde{Y})\tilde{R})(\tilde{Z}, \tilde{W})\tilde{U})_{\alpha\beta\gamma\theta\sigma}^\varepsilon$ are
as follows:

\begin{equation}
(3.16) \quad \left((\tilde{R}(\tilde{X}, \tilde{Y})\tilde{R})(\tilde{Z}, \tilde{W})\tilde{U}\right)_{ijklm}^h
= \tilde{R}_{ijp}^p \tilde{R}_{klm}^p + \tilde{R}_{ijp}^p \tilde{R}_{klm}^h - \tilde{R}_{ijk}^p \tilde{R}_{pklm}^h - \tilde{R}_{ijkl}^p \tilde{R}_{plm}^h - \tilde{R}_{ijkl}^p \tilde{R}_{pklm}^h
- \tilde{R}_{ijkl}^p \tilde{R}_{klm}^p - \tilde{R}_{ijkm}^p \tilde{R}_{klp}^h - \tilde{R}_{ijkl}^p \tilde{R}_{klp}^h
= ((R(X, Y)R)(Z, W)U)_{ijklm}^h.
\end{equation}

ii) \quad \left((\tilde{R}(\tilde{X}, \tilde{Y})\tilde{R})(\tilde{Z}, \tilde{W})\tilde{U}\right)_{ijklm}^h

iii) \quad ((\tilde{R}(\tilde{X}, \tilde{Y})\tilde{R})(\tilde{Z}, \tilde{W})\tilde{U})_{ijklm}^h

If we assume that

\[
\tilde{R}_{ijk}^h = \nabla_i(\nabla_k c_{jh} - \nabla_h c_{jk}) - \nabla_j(\nabla_k c_{ih} - \nabla_h c_{ik}) - R_{ijk}^m c_{mh} - R_{ijh}^m c_{km} = 0,
\]

then it follows from (3.16) that \((\tilde{R}(\tilde{X}, \tilde{Y})\tilde{R})(\tilde{Z}, \tilde{W})\tilde{U} = 0\) if and only if \((R(X, Y)R)(Z, W)U = 0\). This completes the proof. \(\square\)

Denote by \(\tilde{R}_{\alpha\beta} = \tilde{R}_{\alpha\beta}^\sigma\) the contracted curvature tensor (Ricci tensor) of the metric connection \(\tilde{\nabla}\). The only non-zero component of \(\tilde{R}_{\alpha\beta}\) is as follows: \(\tilde{R}_{ij} = \tilde{R}_{ij}\), where \(\tilde{R}_{ij}\) denote the components of the Ricci tensor of \(\nabla\) on \(M\). Now we prove the following theorem.

**Theorem 3.** Let \(\nabla\) be a torsion-free linear connection on \(M\) and \(T^*M\) be the cotangent bundle with the modified Riemannian extension \(\nabla_{\gamma,c}\) over \((M, \nabla)\). The cotangent bundle \(T^*M\) is Ricci semi-symmetric with respect to the metric connection \(\nabla\) if and only if the base manifold \(M\) is Ricci semi-symmetric with respect to \(\nabla\).

**Proof.** The tensor \((\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Ric})(\tilde{Z}, \tilde{W})\) has the components

\begin{equation}
((\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Ric})(\tilde{Z}, \tilde{W}))_{\alpha\beta\gamma\theta} = \tilde{R}_{\alpha\beta\gamma}^\varepsilon \tilde{R}_{\varepsilon\theta} + \tilde{R}_{\alpha\beta\theta}^\varepsilon \tilde{R}_{\varepsilon\gamma}
\end{equation}

with respect to the adapted frame \(\{E_j\}\).

Choosing \(\alpha = i, \beta = j, \gamma = k, \theta = l\) in (3.10), we find

\[
((\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Ric})(\tilde{Z}, \tilde{W}))_{ijkl} = \tilde{R}_{ijk}^p \tilde{R}_{pl} + \tilde{R}_{ijl}^p \tilde{R}_{kp}
= \tilde{R}_{ijp}^k \tilde{R}_{pl} + \tilde{R}_{ijp}^k \tilde{R}_{klp}
= ((R(X, Y)Ric)(Z, W))_{ijkl},
\]

all the others being zero. This finishes the proof. \(\square\)
For the scalar curvature $\bar{r}$ of the metric connection $\nabla$ with respect to $\bar{g}$, we find

$$\bar{r} = \bar{R}_{\alpha\beta}(\bar{g})_{\alpha\beta} = 0.$$ 

Thus we have the following theorem.

**Theorem 4.** Let $\nabla$ be a torsion-free linear connection on $M$ and $T^* M$ be the cotangent bundle with the modified Riemannian extension $\bar{g}_{\nabla, c}$ over $(M, \nabla)$. The scalar curvature of the cotangent bundle $T^* M$ with the metric connection $\nabla$ with respect to $\bar{g}_{\nabla, c}$ vanishes.

Next, we shall apply the differential operator $R(\vec{X}, \vec{Y})$ to the torsion tensor $\bar{T}$ of the metric connection $\nabla$.

**Theorem 5.** Let $\nabla$ be a torsion-free linear connection on $M$ and $T^* M$ be the cotangent bundle with the modified Riemannian extension $\bar{g}_{\nabla, c}$ over $(M, \nabla)$. Then $R(\vec{X}, \vec{Y}) \bar{T} = 0$ for all vector fields $\vec{X}$ and $\vec{Y}$ on $T^* M$, where $\bar{T}$ is the torsion tensor of the metric connection $\nabla$ if and only if the base manifold $M$ is semi-symmetric with respect to $\nabla$.

**Proof.** The differential operator $R(\vec{X}, \vec{Y})$ applied the torsion tensor $\bar{T}$ of the metric connection $\nabla$ is in the form:

$$((R(\vec{X}, \vec{Y}) \bar{T})(\vec{Z}, \vec{W}))_{\alpha\beta\gamma\theta} = R_{\alpha\beta\tau}^{\varepsilon} \bar{T}_{\gamma\theta}^{\tau} - R_{\alpha\beta\gamma}^{\varepsilon} \bar{T}_{\tau\theta}^{\varepsilon} - R_{\alpha\beta\gamma}^{\varepsilon} \bar{T}_{\tau\gamma}^{\varepsilon}$$

with respect to the adapted frame $\{E_{\alpha}\}$. It follows immediately that

$$
\begin{cases}
((R(\vec{X}, \vec{Y}) \bar{T})(\vec{Z}, \vec{W}))_{ijkl} = \bar{R}_{ijkl}^{m} \bar{T}_{mkl} + \bar{R}_{ijkl}^{n} \bar{T}_{mnl}
- \bar{R}_{ijkm}^{m} \bar{T}_{mnl} - \bar{R}_{ijkl}^{m} \bar{T}_{km} - \bar{R}_{ijkl}^{n} \bar{T}_{klm}
= p_s(R_{ijkm}^{n} \bar{T}_{klm} + R_{ijkl}^{m} \bar{T}_{km} + R_{ijkm}^{n} \bar{T}_{klm})
= -p_s((R(X,Y)R)(Z,W)U)_{ijkl}
\end{cases}
$$

which finishes the proof. \qed

On an $n$-dimensional Riemannian manifold $(M, g)$, it was defined a generalized $(0, 2)$-symmetric $Z$ tensor given by $\bar{Z}$

$$Z(X,Y) = \text{Ric}(X,Y) + \phi g(X,Y)$$

for all vector fields $X$ and $Y$ on $M$, where $\phi$ is an arbitrary scalar function. Analogous to this definition, it may be locally define generalized $Z$ tensor on $(T^* M, \bar{g}_{\nabla, c})$ with respect to the metric connection $\nabla$ as follows:

$$\bar{Z}_{\alpha\beta} = R_{\alpha\beta} + \phi(\bar{g})_{\alpha\beta}. $$

Putting the values of $\bar{R}_{\alpha\beta}$ and $\bar{g}_{\nabla, c}$ in the above equation, we have the non-zero components

$$(3.18)\ 
\begin{align*}
\bar{Z}_{ij} &= R_{ij} + \phi c_{ij}, \\
\bar{Z}_{\tau j} &= \phi \delta_{\tau j}^i, \\
\bar{Z}_{\tau} &= \phi \delta_{\tau j}^j.
\end{align*}$$

We can state the following theorem.
Theorem 6. Let $\nabla$ be a torsion-free linear connection on $M$ and $T^*M$ be the cotangent bundle with the modified Riemannian extension $\tilde{\nabla}_{\nabla,c}$ over $(M, \nabla)$. The cotangent bundle $T^*M$ is $\tilde{Z}$ semi-symmetric with respect to the metric connection $\tilde{\nabla}$ if and only if the base manifold $M$ is Ricci semi-symmetric with respect to the metric $\nabla$.

Proof. The tensor $(\tilde{R}(\tilde{X}, \tilde{Y}), \tilde{Z})(\tilde{Z}, \tilde{W})$ has the components

$$((\tilde{R}(\tilde{X}, \tilde{Y}), \tilde{Z})(\tilde{Z}, \tilde{W}))_{\alpha\beta\gamma\theta} = \tilde{R}_{\alpha\beta\gamma}Z_{\varepsilon\theta} + \tilde{R}_{\alpha\beta\delta}Z_{\gamma\varepsilon}$$

with respect to the adapted frame $\{E_\beta\}$.

By choosing $\alpha = (i, \tilde{i})$, $\beta = (j, \tilde{j})$, $\gamma = (k, \tilde{k})$ and $\theta = (l, \tilde{l})$ in (3.19), in view of (3.18) we find the only non-zero component

$$((\tilde{R}(\tilde{X}, \tilde{Y}), \tilde{Z})(\tilde{Z}, \tilde{W}))_{ijkl} = \tilde{R}_{ijl}hZ_{hl} + \tilde{R}_{ijl}h\pi_{\tilde{k}l} + \tilde{R}_{ijl}h\tilde{Z}_{\tilde{k}l}$$

$$= R_{ijl}h(R_{hl} + \phi_{hl}) + \frac{1}{2}\{\nabla_i(\nabla_kc_{jh} - \nabla_hc_{jk})$$

$$- \nabla_j(\nabla_kc_{ih} - \nabla_hc_{ik}) - R_{ijl}^{m}c_{mh} - R_{ijh}^{m}c_{km}\}\pi_{\tilde{k}l}$$

$$+ R_{ijl}h(R_{kh} + \phi_{kh}) + \frac{1}{2}\{\nabla_i(\nabla_c_{jh} - \nabla_hc_{ji})$$

$$- \nabla_j(\nabla_i_{ch} - \nabla_hc_{il}) - R_{ijh}^{m}c_{m_{kh}} - R_{ijh}^{m}c_{m_{kh}}\}\pi_{\tilde{k}l}$$

$$= R_{ijl}hR_{hl} + R_{ijl}hR_{kh}$$

$$= (R(X, Y)R_{ic})_{ijkl},$$

from which the proof follows. \hfill \Box

3.3. Conharmonic Curvature tensor on the cotangent bundle with respect to the metric connection $\nabla$. We recall that the conharmonic curvature tensor $V$ on an $n$–dimensional Riemannian manifold $(M, g)$ is defined as a $(4, 0)$–tensor by the formula

$$V_{ijkl} = R_{ijkl} - \frac{1}{n - 2}[R_{jikl} - R_{iklj} - R_{ijlk} + R_{iljk}],$$

where $R_{ijkl}$ and $R_{ij}$ are respectively the components of the Riemannian curvature tensor and the Ricci tensor. The conharmonic curvature tensor was first introduced by Ishii (see, [12]). A Riemannian manifold whose conharmonic curvature tensor vanishes is called conharmonically flat.

Analogous to the conharmonic curvature tensor with respect to a Levi–Civita connection $\nabla$, it may be given the conharmonic curvature tensor $\tilde{V}$ on $T^*M$ with respect to the metric connection $\tilde{\nabla}$ as follows:

$$\tilde{V}_{\alpha\beta\gamma\varepsilon} = \tilde{R}_{\alpha\beta\gamma}Z_{\varepsilon\gamma} - \frac{1}{2(n - 1)} \left[R_{\beta\gamma}(\tilde{g}_{\nabla,c})_{\alpha\varepsilon} - \tilde{R}_{\alpha\gamma}(\tilde{g}_{\nabla,c})_{\beta\varepsilon} - \tilde{R}_{\beta\varepsilon}(\tilde{g}_{\nabla,c})_{\alpha\gamma} + \tilde{R}_{\alpha\varepsilon}(\tilde{g}_{\nabla,c})_{\beta\gamma}\right].$$

Next we prove the following theorem:

Theorem 7. Let $\nabla$ be a torsion-free linear connection on $M$ and $T^*M$ be the cotangent bundle with the modified Riemannian extension $\tilde{g}_{\nabla,c}$ over $(M, \nabla)$. The cotangent bundle $T^*M$ is locally conharmonically flat with respect to the metric
connection \( \tilde{\nabla} \) if and only if the base manifold \( M \) is Ricci flat with respect to \( \nabla \) and the components \( c_{ij} \) of \( c \) satisfy the condition

\[
\nabla_i(\nabla_k c_{jh} - \nabla_h c_{jk}) - \nabla_j(\nabla_k c_{ih} - \nabla_h c_{ik}) + R_{ijk}^m c_{mh} - R_{ijh}^m c_{km} = 0,
\]

where \( R_{ijk}^m \) denote the components of the curvature tensor \( R \) of \( \nabla \).

**Proof.** If the components of the curvature tensor \( \tilde{R} \) of the metric connection \( \tilde{\nabla} \) on \( T^*M \) satisfy the following equations:

\[
(3.20a) \quad \tilde{R}_{\alpha\beta\gamma\varepsilon} = \frac{1}{2(n-1)} \left[ \tilde{R}_{\beta\gamma}(\nabla_{\varepsilon},c)_{\alpha\varepsilon} - \tilde{R}_{\alpha\gamma}(\nabla_{\varepsilon},c)_{\beta\varepsilon} - \tilde{R}_{\beta\varepsilon}(\nabla_{\alpha},c)_{\beta\gamma} + \tilde{R}_{\alpha\varepsilon}(\nabla_{\gamma},c)_{\beta\gamma} \right],
\]

then \( T^*M \) is said to be locally conharmonically flat with respect to the metric connection \( \tilde{\nabla} \).

On lowering the upper index in the proposition 4, we obtain the components of the \((0,4)\)-curvature tensor of the metric connection \( \tilde{\nabla} \) as follows:

\[
\left\{ \begin{array}{c}
\tilde{R}_{ijkl} = +\frac{1}{2} \left( \nabla_i(\nabla_k c_{jl} - \nabla_l c_{jk}) - \nabla_j(\nabla_k c_{il} - \nabla_l c_{ik}) \right. \\
R_{ijkl}^m c_{ml} - R_{ijl}^m c_{km}
\end{array} \right.
\]

Putting the values of \( \tilde{R}_{\alpha\beta\gamma\varepsilon}, \tilde{R}_{\alpha\beta} \) and \( (\nabla_{\varepsilon},c)_{\beta\varepsilon} \) respectively in (3.20a), we have

\[
(3.21) \quad \nabla_i(\nabla_k c_{jl} - \nabla_l c_{jk}) - \nabla_j(\nabla_k c_{il} - \nabla_l c_{ik}) + R_{ijk}^m c_{ml} - R_{ijl}^m c_{km}
= \frac{1}{2(n-1)}(R_{jk} c_{il} - R_{ik} c_{jl} - R_{jl} c_{ik} + R_{il} c_{jk})
\]

\[
(3.22) \quad R_{ijk}^l = \frac{1}{2(n-1)}(R_{jk}^l d_{ij}^l - R_{ik}^l d_{ij}^l) \\
-R_{ijl}^k = \frac{1}{2(n-1)}(R_{il}^k d_{jk}^k - R_{jl}^k d_{jk}^k)
\]

Contraction \( i \) and \( l \) in (3.22) gives

\[
R_{ijk}^l = \frac{1}{2(n-1)}(R_{jk}^l d_{ij}^l - R_{ik}^l d_{ij}^l) \\
R_{jk} = \frac{1}{2(n-1)}(nR_{jk} - R_{jk}) \\
R_{jk} = \frac{1}{2(n-1)}R_{jk}(n-1) \\
R_{jk} = 0,
\]

that is, the torsion-free linear connection \( \nabla \) is Ricci flat. In the case, from (3.21), it follows that

\[
\nabla_i(\nabla_k c_{jl} - \nabla_l c_{jk}) - \nabla_j(\nabla_k c_{il} - \nabla_l c_{ik}) + R_{ijk}^m c_{ml} - R_{ijl}^m c_{km} = 0.
\]

\( \square \)
4. The Schouten-van Kampen connection associated to the Levi-Civita connection of the modified Riemannian extension

The Schouten-Van Kampen connection has been introduced in [17] for a study of non-holonomic manifolds. The Schouten-Van Kampen connection associated to the Levi-Civita connection of the modified Riemannian extension $\nabla$ and adapted to the pair of distributions $(H, V)$ are defined by

\[ \nabla_X Y = H(\nabla_X H Y) + V(\nabla_X V Y) \]

for all vector fields $X$ and $Y$, where $V$ and $H$ are the projection morphism of $TT^*M$ on $VT^*M$ and $HT^*M$ respectively. The formula (4.1) for $\nabla^*$ has been first given by Ianus (see, [11]). By using (4.1) and (3.3), the Schouten-Van Kampen connection associated to the Levi-Civita connection $\nabla$ of the modified Riemannian extension $\nabla_{\mathcal{V},c}$ are locally given by the following formulas:

\[
\begin{align*}
\nabla_{E_i} E_j &= 0, \\
\nabla_{E_i} E_j &= -\Gamma^h_{ij} E_h, \\
\nabla_{E_i} E_j &= \Gamma^h_{ij} E_h,
\end{align*}
\]

which are the components of the horizontal lift $^H \nabla$ of the torsion-free linear connection $\nabla$. Hence we get:

**Proposition 5.** Let $\nabla$ be a torsion-free linear connection on $M$ and $T^*M$ be the cotangent bundle with the modified Riemannian extension $\nabla_{\mathcal{V},c}$ over $(M, \nabla)$. The Schouten-Van Kampen connection $\nabla^*$ associated to the Levi-Civita connection $\nabla$ of the modified Riemannian extension $\nabla_{\mathcal{V},c}$ and the horizontal lift $^H \nabla$ of the torsion-free linear connection $\nabla$ to $T^*M$ coincide to each other.

In view of Proposition 3, Proposition 5, Theorem 2 and its proof, it immediately follows the final result.

**Theorem 8.** Let $\nabla$ be a torsion-free linear connection on $M$ and $T^*M$ be the cotangent bundle with the modified Riemannian extension $\nabla_{\mathcal{V},c}$ over $(M, \nabla)$. The cotangent bundle $T^*M$ is semi-symmetric with respect to the Schouten-Van Kampen connection $\nabla^*$ associated to the Levi-Civita connection $\nabla$ of the modified Riemannian extension $\nabla_{\mathcal{V},c}$ if and only if the base manifold $M$ is semi-symmetric with respect to $\nabla$.

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