Corrigendum to “Knot Floer homology detects fibred knots”

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Abstract

We correct a mistake on the citation of JSJ theory in [4]. Some arguments in [4] are also slightly modified accordingly.

An important step in [4] uses JSJ theory [2, 3] to deduce some topological information about the knot complement when the knot Floer homology is monic, see [4, Section 6]. The version of JSJ theory cited there is from [1]. However, as pointed out by Kronheimer, the definition of “product regions” in [1] is not the one we want. In this note, we will provide the necessary background on JSJ theory following [2]. Some arguments in [4] will then be modified.

Definition 1. An \(n\)-manifold pair is a pair \((M, T)\) where \(M\) is an \(n\)-manifold and \(T\) is an \((n-1)\)-manifold contained in \(\partial M\). A 3-manifold pair \((M, T)\) is irreducible if \(M\) is irreducible and \(T\) is incompressible. An irreducible 3-manifold pair \((M, T)\) is Haken if each component of \(M\) contains an incompressible surface.

Definition 2. \([2]\) Page 10\) A compact 3-manifold pair \((S, T)\) is called an I-pair if \(S\) is an I-bundle over a compact surface, and \(T\) is the corresponding \(\partial I\)-bundle. A compact 3-manifold pair \((S, T)\) is called an \(S^1\)-pair if \(S\) is a Seifert fibred manifold and \(T\) is a union of Seifert fibres in some Seifert fibration. A Seifert pair is a compact 3-manifold pair \((S, T)\), each component of which is an I-pair or an \(S^1\)-pair.

Definition 3. \([2]\) Page 138\) A characteristic pair for a compact, irreducible 3-manifold pair \((M, T)\) is a perfectly-embedded Seifert pair \((\Sigma, \Phi) \subset (M, \text{int}(T))\) such that if \(f\) is any essential, nondegenerate map of an arbitrary Seifert pair \((S, T)\) into \((M, T)\), \(f\) is homotopic, as a map of pairs, to a map \(f'\) such that \(f'(S) \subset \Sigma\) and \(f'(T) \subset \Phi\).

The definition of a perfectly-embedded pair can be found in [2, Page 4]. We note that the definition requires that \(\Sigma \cap \partial M = \Phi\), so \(\Sigma\) is disjoint from \(\partial M - T\).

The main result in JSJ theory is the following theorem.
Theorem 4 (Jaco–Shalen [2], Johannson [3]). Every Haken 3–manifold pair \((M, T)\) has a characteristic pair. This characteristic pair is unique up to ambient isotopy relative to \((\partial M – \text{int}(T))\).

Definition 5. Let \((M, \gamma)\) be a sutured manifold. A 3–manifold pair \((P, Q) \subset (M, R(\gamma))\) is a product pair if \(P = F \times [0,1], Q = F \times \{0,1\}\) for some compact surface \(F\), and \(F \times 0 \subset R_-(\gamma), F \times 1 \subset R_+(\gamma)\). We also require that \(P \cap A = \emptyset\) or \(A\) for any annular component \(A\) of \(\gamma\). A product pair is gapless if no component of its exterior is a product pair.

Definition 6. Suppose \((M, \gamma)\) is a taut sutured manifold, \((\Sigma, \Phi)\) is the characteristic pair for \((M, R(\gamma))\). The characteristic product pair for \(M\) is the union of all components of \((\Sigma, \Phi)\) which are product pairs. A maximal product pair for \(M\) is a gapless product pair \((P, Q)\) such that it contains the characteristic product pair, and if \((P', Q') \supset (P, Q)\) is another gapless product pair, then there is an ambient isotopy relative to \(\gamma\) that takes \((P', Q')\) to \((P, Q)\).

The existence of maximal product pairs follows from the definition, although the uniqueness is not guaranteed. The exterior of a maximal product pair is also a sutured manifold. By definition the exterior does not contain essential product annuli or essential product disks.

Now we are ready to modify the arguments in [4]. The next theorem is a reformulation of [4, Theorem 6.2]. The proof is not changed though.

Theorem 6.2' Suppose \((M, \gamma)\) is an irreducible balanced sutured manifold, \(\gamma\) has only one component, and \((M, \gamma)\) is vertically prime. Let \(E\) be the subgroup of \(H_1(M)\) spanned by the first homologies of product annuli in \(M\). If \(\overline{HFS}(M, \gamma) \cong \mathbb{Z}\), then \(E = H_1(M)\).

Corollary 7. In the last theorem, suppose \((\Pi, \Psi)\) is the characteristic product pair for \(M\), then the map

\[ i_* : H_1(\Pi) \to H_1(M) \]

is surjective.

Proof. We recall that such an \(M\) is a homology product [4, Proposition 3.1].

Suppose \((\Sigma, \Phi)\) is the characteristic pair for \((M, R(\gamma))\), then any product annulus can be homotoped into \((\Sigma, \Phi)\) without crossing \(\gamma\). Let \(\Phi_+ = (\Phi \cap R_+(\gamma)) \subset \text{int}(R_+(\gamma))\). Theorem 6.2' implies that the map \(H_1(\Phi_+) \to H_1(R_+(\gamma))\) is surjective, so \(\partial \Phi_+\) consists of separating circles in \(R_+(\gamma)\). If a component \((S, T)\) of \((\Sigma, \Phi)\) is an \(S^1\)–pair, then \(T \cap R_+(\gamma)\) consists of annuli by definition. We conclude that each annulus is null-homologous in \(H_1(R_+(\gamma))\).

Suppose a product annulus \(A\) contributes to \(H_1(M)\) nontrivially, and it can be homotoped into a component \((\sigma, \varphi)\) of \((\Sigma, \Phi)\). Given the result from the last paragraph, this \((\sigma, \varphi)\) cannot be an \(S^1\)–pair. It is neither a twisted \(I\)–bundle since the two components of \(\partial A\) are contained in different components of \(R(\gamma)\). So \((\sigma, \varphi)\) must be a trivial \(I\)–bundle, and the two components of \(\varphi\) lie in different components of \(R(\gamma)\). In other words, \((\sigma, \varphi)\) is a product pair. Now our desired result follows from Theorem 6.2'.
The following proof of the main theorem in [4] is only slightly changed. Basically we use “maximal product pair” here instead of the wrong notion “characteristic product region” in [4].

Proof of [4, Theorem 1.1]. Suppose \((M, \gamma)\) is the sutured manifold obtained by cutting open \(Y \setminus \text{int}(\text{Nd}(K))\) along \(F\), \((P, Q)\) is a maximal product pair for \(M\).

We need to show that \(M\) is a product. By [4, Proposition 3.1], \(M\) is a homology product. Moreover, by [4, Theorem 4.1], we can assume \(M\) is vertically prime.

If \(M\) is not a product, then \(M \setminus P\) is nonempty. Thus there exist some product annuli in \((M, \gamma)\), which split off \(P\) from \(M\). Let \((M', \gamma')\) be the remaining sutured manifold. By definition \((P, Q)\) contains the characteristic product pair for \(M\). Corollary 7 then implies that the map \(H_1(P) \to H_1(M)\) is surjective. So \(R_\pm(\gamma')\) are planar surfaces, and \(M' \cap P\) consists of separating product annuli in \(M\). Since we assume that \(M\) is vertically prime, \(M'\) must be connected. (See the first paragraph in the proof of [4, Theorem 5.1] for this.) Moreover, \(M'\) is also vertically prime. By [4, Theorem 5.1], \(HFS(M', \gamma') \cong \mathbb{Z}\).

We add some product 1–handles to \(M'\) to get a new sutured manifold \((M'', \gamma'')\) with \(\gamma''\) connected. By [4, Proposition 2.9], \(HFS(M'', \gamma'') \cong \mathbb{Z}\). It is easy to see that \(M''\) is also vertically prime. [4, Proposition 3.1] shows that \(M''\) is a homology product.

Let \(H\) be one of the product 1–handles added to \(M'\) such that \(H\) connects two different components of \(\gamma'\). By Theorem 6.2', there is at least one product annulus \(A\) in \(M''\), such that \(A\) cannot be homotoped to be disjoint from the cocore of \(H\). Isotope \(A\) if necessary, we find that at least one component of \(A \cap M'\) is an essential product disk in \(M'\), a contradiction to the assumption that \((P, Q)\) is a maximal product pair.

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