Hamiltonian formulation of Quantum Hall Skyrmions with Hopf term

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Abstract

We study the nonrelativistic nonlinear sigma model with Hopf term in this paper. This is an important issue because of its relation to the currently interesting studies in skyrmions in quantum Hall systems. We perform the Hamiltonian analysis of this system in $CP^1$ variables. When the coefficient of the Hopf term becomes zero we get the Landau-Lifshitz description of the ferromagnets. The addition of Hopf term dramatically alters the Hamiltonian analysis. The spin algebra is modified giving a new structure and interpretation to the system. We point out momentum and angular momentum generators and new features they bring in to the system.

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1. Introduction

It is well known that static and dynamical properties of ferromagnets are captured by the Landau-Lifshitz(LL) evolution equations\cite{1}. These equations can be obtained from LL Hamiltonian which is the continuum limit of the Heisenberg spin chain exploiting the modified Poisson brackets among the magnetisation fields. A lagrangian description is possible through the non relativistic non-linear sigma model(NLSM). This model has captured attention recently \cite{2} for the description of novel excitations known as skyrmions in a suitable quantum Hall regime. Conventional quantum Hall regime is one in which the magnetic field is strong enough that physical properties are robust against changes coming from mixing of Landau levels through interactions. Essentially one assumes that in this regime the Landau level separation $\hbar \omega_c$ is larger than any other energy scale in the system. For free electrons in magnetic field the Zeeman splitting ($g \mu_B B$) is of the same order as the Landau level separation\cite{3}. However electrons in conduction band have renormalised mass and $g$-factor. In the typical case of GaAs the the mass gets enhanced by a factor of 20 and the Zeeman splitting is reduced by a factor of 4\cite{4}. In such a case we have a Quantum Hall ferromagnet described by a nonrelativistic non-linear sigma model. It is well known that in such a system there are solitons arising from purely topological properties of the configuration space\cite{5}. It is an interesting question to ask whether the spin and statistics of these solitons can also obtained by topological considerations. Such a question has been studied extensively in relativistic NLSM and it is well known that the spin and statistics can be obtained through a well known Hopf term\cite{6}. Recently Chandar etal\cite{7} have studied the many electron system (2DEG) in a quantum Hall ferromagnetic regime and obtained a nonzero contribution to the Hopf term. This term turns out to be $\propto \nu$, the filling fraction. The role of this term has never been analysed in a Hamiltonian framework in a nonrelativistic context. On the other hand it is an important analysis to be performed since the symplectic structure of the system is far different from the relativistic context. The hamiltonian analysis has been studied earliar without the Hopf term\cite{8}.
In this paper, we study the implications of the addition of the Hopf term to the nonrelativistic NLSM. by carrying out the Hamiltonian analysis in a gauge independent manner. Recently this gauge independent scheme has been used in various models involving Chern-Simons(CS) term revealing the existence of fractional spin. This has certain advantages over gauge fixed approach. For example, the relevant symmetry is manifested right at the level of transformation properties of the basic fields, even if they are gauge variant. On the other hand the transformation property of these fields gets affected by the gauge fixing condition used. The transformation property of the gauge invariant objects however remains unaffected by gauge fixing. The underlying symmetry group is therefore uncovered only at the level of gauge invariant variables. Besides, the symplectic structures, given by the set Dirac brackets (DB) usually becomes more complicated so that subsequent quantization by elevating DB to quantum commutators may have serious operator ordering ambiguities. Indeed, as we shall see that the symplectic structure in our model in the gauge independent scheme is complicated enough, let alone in the gauge fixed scheme, so that we are forced to restrict our analysis to the classical level only. In sec 2 we describe the model. In sec 3. we perform the Hamiltonian analysis of the model followed by a discussion of the system in sec 4.

2. The model

As explained in the introduction in the long wavelength limit the excitations near ferromagnetic ground state can be described by a non-relativistic NLSM. As mentioned earlier this provides a good model in some quantum Hall systems where the g-factor is small. In this case there are excitations which are the solitons of the NLSM. This model is described by a field \( n(x, y, t) \) satisfying the condition \( n^2 = 1 \). The configuration space \( Q \) is given by the set of maps \( \{ n \} \),

\[
\mathbf{n} : R^2 \rightarrow S^2
\]  

For the finite energy configurations to exist in this model we need to impose the condition that \( n(x, y, t) \xrightarrow{x,y \to \infty} \text{Const} \). With this condition we have a compactified \( R^2 \) which is the
same as an $S^2$ so that the configuration space is given by

$$ n : S^2 \rightarrow S^2 $$

The configuration space splits into disjoint union of path connected spaces $Q_N$ each labelled by an integer $N$ known as soliton number. This easily follows from the fact that

$$ \pi_0(Q) = \pi_2(S^2) = \mathbb{Z} $$

The soliton number $N$ can be obtained through a topological conservation law for the current $j^\mu$, given by

$$ j^\mu = \frac{1}{8\pi} \epsilon^{\mu\nu\lambda} n \cdot \partial_\nu n \times \partial_\lambda n $$

$j_0$ of this current is the soliton density for the map.

$$ N = \int d^2x \ j_0(x) $$

Note that the conservation of the current $j^\mu$ in eq.[4] holds without reference to the equations of motion. Using the conservation of the topological current we can write

$$ j^\mu = \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda $$

where $a$ is one-form obtained by pulling back onto the space-time the magnetic monopole connection of Dirac[16]. If we parametrize $n$ through $\Theta$ and $\Phi$, the polar coordinates for $S^2$ then $j$ can be written as mentioned earlier in eq.[4] as the dual of a two-form $da$. Explicitly, $a_\mu$ is given by:

$$ a_\mu = \frac{1}{2}(\pm 1 - \cos \Theta)\partial_\mu \Phi $$

It is also well known that in all the soliton number[6, 15] sectors the fundamental group is nontrivial. This follows from

$$ \pi_1(Q) = \pi_3(S^2) = \mathbb{Z} $$

This implies that the loops based at any point in the configuration space Q falls into separate homotopy classes labelled by an integer. This integer is obtained by the Hopf-term[3, 13]

$$ \mathcal{H} = \int d^3x j^\mu a_\mu. $$

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where $a$ is defined through eq.\([3]\).

It is well known that the introduction the Hopf-term in the action of the relativistic NLSM has the effect of changing the spin and statistics of the soliton\([6, 17]\). This will be so in the antiferromagnetic system. Here one is studying the effect of this term in the nonrelativistic NLSM. One is interested in studying the effect of adding this term to the ferromagnetic system which leads to nonrelativistic NLSM with Hopf term. As was pointed out specifically in the quantum Hall context it has been shown\([7]\) that this term gets generated with a coefficient determined by the filling fraction. They have also shown in path integral formalism that statistics of skyrmion is determined by the same coefficient.

In this paper we would like to study in Hamiltonian analysis the role of the same term. For this purpose it is advantageous to use $CP^1$ variables to describe the NLSM instead of $S^2$ ones. Firstly the NLSM defined in terms of spin variables $n$ has a Dirac singularity and this will disappear in the $CP^1$ description\([18]\). This will help in defining the generators of the symmetry transformations unambiguously\([8]\). This singularity is related to the the fact that the U(1) principal bundle over $S^2$ is nontrivial and there is no global section\([19]\). Secondly the Hopf-term in the spin variables is a non-local term in the action. This can be avoided by enhancing the degrees of freedom and describing the same as a local action. This can be easily seen by recalling that $CP^1$ manifold is described by $Z = (Z_1 \ Z_2)$ with $Z^\dagger Z = 1$ and the identification $Z \sim e^{i\theta}Z$, where $e^{i\theta}$ is an U(1) element. The one-form $a_\mu$ can be obtained by geometrical considerations\([8]\), to get

$$a_\mu = -iZ^\dagger \partial_\mu Z \quad (10)$$

The topological current in $CP^1$ variables can therefore easily be seen to be given by

$$j^\mu = \frac{-i}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu Z^\dagger \partial_\lambda Z \quad (11)$$

The Hopf-term becomes,

$$\mathcal{H} = -\frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \int d^3x Z^\dagger \partial_\mu Z \partial_\nu Z^\dagger \partial_\lambda Z \quad (12)$$
Clearly this expression\(^{[18]}\) for the Hopf-term is local unlike that in spin variables given in eq.\(^{[9]}\). The locality is achieved through a gauge invariance and it is the same expression whether one is considering nonrelativistic or relativistic NLSM. The action corresponding to the nonrelativistic NLSM can be written as

\[
S = - \int d^3x \left\{ \frac{i}{2} Z^\dagger \partial_0 Z + A_0 (Z^\dagger Z - 1) + |D_i Z|^2 \right\} + \theta \mathcal{H} \tag{13}
\]

Here while considering the action [eq.\(^{[13]}\)] we find it convenient to enlarge the phase space and introduce a new gauge field \(a_\mu\). This \(a_\mu\) gauge field should not be confused with the one introduced earlier even though equations of motion will relate them. We couple this gauge field to the topological current and add only the Chern Simons term for this gauge field\(^{[20]}\).

\[
S = S_0 + \theta \epsilon_{\mu \nu \lambda} \int d^2x \left\{ 2ia_\mu \partial_\nu Z^\dagger \partial_\lambda Z + a_\mu \partial_\nu a_\lambda \right\} \tag{14}
\]

This action in eq.\(^{[14]}\) is same as the previous one in eq.\(^{[13]}\) when we use the equations of motion for the \(a_\mu\).

In this paper we will deal with the action eq.\(^{[14]}\) and carry out Hamiltonian analysis. We will find that the Hopf term dramatically changes the content of the theory by modifying the spin algebra. We will follow the Faddeev-Jackiw \(^{[21]}\) symplectic analysis for the first order Lagrangians. This will ease our constraint analysis and help in obtaining the new Dirac-Brackets.

3. Hamiltonian Analysis

In this section we shall perform Hamiltonian analysis of the model \(^{[14]}\). The Lagrangian corresponding to \(^{[13]}\) is given by

\[
L = \int d^2x \left\{ - \frac{i}{2} Z^\dagger \partial_0 Z - A_0 (Z^\dagger Z - 1) - |D_i Z|^2 + \theta \epsilon^{\mu \nu \lambda} a_\mu (2i \partial_\nu Z^\dagger \partial_\lambda Z + \partial_\nu a_\lambda) \right\} \tag{15}
\]

Here we would like to point out that the covariant derivative \(D\) in eq.\(^{[15]}\) involves only the background gauge-field \(A\) and is given by \(D_i = \partial_i - iA_i\) and does not involve the other gauge field \(a_\mu\). The configuration space variables are \([a_0(x), a_i(x), A_i(x), A_0(x), Z_\alpha(x), Z^*_\alpha(x)]\).
However the role of $A_0$ and $a_0(x)$ are basically that of lagrange multipliers. They enforce the constraints

$$G_1(x) \equiv Z^\dagger(x)Z(x) - 1 \approx 0 \quad (16)$$

$$G_2(x) \equiv \epsilon_{ij}(i\partial_i Z^\dagger(x)\partial_j Z(x) + \partial_i a_j(x)) \approx 0 \quad (17)$$

respectively.

The conjugate momenta to $(A_i(x), a_i(x), Z_\alpha(x), Z_\alpha^*(x))$ are given by

$$\Pi_i(x) \equiv \frac{\delta L}{\delta A_i(x)} = 0 \quad (18)$$

$$\pi_i(x) \equiv \frac{\delta L}{\delta a_i(x)} = \theta \epsilon_{ij} a_j \quad (19)$$

$$P_\alpha(x) \equiv \frac{\delta L}{\delta Z_\alpha(x)} = -\frac{i}{2}Z_\alpha^*(x) + 2i\theta \epsilon_{ij} a_i \partial_j Z_\alpha^*(x) \quad (20)$$

$$P_\alpha^*(x) \equiv \frac{\delta L}{\delta Z_\alpha^*(x)} = \frac{i}{2}Z_\alpha(x) - 2i\theta \epsilon_{ij} a_i(x) \partial_j Z_\alpha(x) \quad (21)$$

These are the set of primary constraints of the model. Since the model is first order in time-derivative the Hamiltonian can be readily obtained as

$$H_c = \int d^2x[|D_i Z|^2 + A_0(x)G_1(x) - 2\theta a_0(x)G_2(x)] \quad (22)$$

By requiring the constancy of primary constraints \textsuperscript{[18]} we get the following secondary constraints:

$$C_i(x) \equiv A_i(x) + i Z^\dagger(x)\partial_i Z(x) \approx 0 \quad (23)$$

In view of this strong equality $A_i$ ceases to be an independent degree of freedom. Clearly this constraint is conjugate to the constraint\textsuperscript{[18]} and can be strongly implemented by the Dirac bracket

$$\{A_i(x), \Pi_j(y)\} = 0 \quad (24)$$

With this $A_i$ becomes,

$$A_i(x) \approx -iZ^\dagger\partial_i Z(x) \quad (25)$$
The CS gauge-field also has interestingly the same form:

\[ a_i \approx -iZ^\dagger \partial_i Z. \]  

(26)

Thus both the gauge fields agree modulo gauge transformations on the constraint surface. However they are not identical and must be treated as distinct.

We are thus left with constraints (19-21). These are the second class constraints of the model as the Lagrangian (15) is linear in “velocities” of the associated variables[21]. The constraints imposed by the lagrange multipliers \( A_0 \) and \( a_0 \) are expected to be gauss-law constraints as in electrodynamics. Since gauss’s law constraint is a gauge transformation generator we expect these to be first-class constraints. Note in our model we have \( U(1) \times U(1) \) gauge symmetry corresponding to the two gauge fields \( A_i(x) \) and \( a_i(x) \). We will now demonstrate the fact that (16) and (17) are indeed first-class by explicit calculation. Thus to begin with, we have to invert the matrix formed by the poisson brackets of the constraints (19), (20) and (21).

\[ \chi_i(x) \equiv \pi_i(x) - \theta \epsilon_{ij} a_j(x) \approx 0 \]  

(27)

\[ C_1^\alpha(x) \equiv \mathcal{P}_\alpha(x) + \frac{i}{2} Z^\ast_\alpha(x) - 2i \theta \epsilon_{ij} a_i(x) \partial_j Z^\ast_\alpha(x) \approx 0 \]  

(28)

\[ C_2^\alpha(x) \equiv (C_1^\alpha(x))^\ast \]  

(29)

In order to obtain Dirac brackets we have to thus invert a \( 6 \times 6 \) matrix of the poisson brackets of constraints. We can simplify the procedure by doing the inversion in parts following Hanson et al[9]. Specifically note that pair of constraints in eq.(27) do not involve fields from matter sector. So we can readily implement this pair of constraints strongly by the Faddeev-Jackiw scheme[12, 13, 21] to yield the Dirac bracket:

\[ \{ a_i(x), a_j(x) \}_{DB} = \frac{\epsilon_{ij}}{2\theta} \delta(x - y) \]  

(30)

However this is not the final DB in the gauge field sector. This, along with the basic Poisson brackets \( \{ Z_\alpha, Z_\beta \} \) and \( \{ Z_\alpha, a_i \} \) will undergo a further modification as we implement the other four constraints (28) and (29) strongly. This is because these constraints involve both
$Z$ and $a_i$ fields. To this end, we calculate the following matrix elements of Poisson bracket matrix (where the use of the Dirac bracket in eq.(30) has been made).

$$C_{\alpha\beta}^{11}(x, y) \equiv \{C_{\alpha}^{1}(x), C_{\beta}^{1}(y)\} = iR_{\alpha\beta}(x)\delta(x - y)$$

$$C_{\alpha\beta}^{22}(x, y) \equiv \{C_{\alpha}^{2}(x), C_{\beta}^{2}(y)\} = iS_{\alpha\beta}(x)\delta(x - y)$$

$$C_{\alpha\beta}^{12}(x, y) \equiv \{C_{\alpha}^{1}(x), C_{\beta}^{2}(y)\} = iT_{\alpha\beta}(x)\delta(x - y)$$

where

$$R_{\alpha\beta} = -S_{\alpha\beta}^* = 2i\theta\epsilon_{\alpha\beta}\nabla Z_1^* \times \nabla Z_2^*$$

$$T_{\alpha\beta} = T_{\beta\alpha}^* = (\delta_{\alpha\beta} - 2i\theta\nabla Z_\alpha^* \times \nabla Z_\beta)$$

The final matrix formed by the constraints (28) and (29) is therefore given by

$$C = \begin{pmatrix} C_{11}(x, y) & C_{12}(x, y) \\ C_{21}(x, y) & C_{22}(x, y) \end{pmatrix}$$

The inverse of the matrix can readily be calculated:

$$C^{-1}(x, y) = -\frac{i}{D} \begin{pmatrix} 0 & -S_{12} & T_{22} & -T_{21} \\ S_{12} & 0 & -T_{12} & T_{11} \\ -T_{22} & T_{12} & 0 & -R_{12} \\ T_{21} & -T_{11} & R_{12} & 0 \end{pmatrix} \delta(x - y)$$

where

$$D = R_{12}S_{12} - detT$$

We can simplify the expression for $D$. At the end we get

$$D = - (1 + 2\theta B)$$

where $B \equiv \epsilon_{ij}\partial_i A_j = -i\nabla Z^\dagger \times \nabla Z$ is the magnetic field corresponding to the gauge field $A_i$. Upto a factor this is also the topological density $j_0$. In order to make sense we have
to impose the condition $\mathcal{D} \neq 0$ or equivalently $B \neq -\frac{1}{\mathcal{D}}$. The final Dirac brackets in the matter sector can be written as:

$$
\begin{align*}
\{Z_1(x), Z_1(y)\} & \{Z_1(x), Z_2(y)\} = -\frac{i}{\mathcal{D}} \begin{pmatrix} 0 & -S_{12} \\ S_{12} & 0 \end{pmatrix} \delta(x - y) \\
\{Z_2(x), Z_1(y)\} & \{Z_2(x), Z_2(y)\} = -\frac{i}{\mathcal{D}} \begin{pmatrix} T_{22} & -T_{21} \\ -T_{12} & T_{11} \end{pmatrix} \delta(x - y)
\end{align*}
$$

(40)

(41)

In addition to the above we have to obtain the Dirac brackets of the matter sector with gauge fields. They can be worked out to be:

$$
\begin{align*}
\{Z_1(x), a_k(y)\} &= \frac{1}{\mathcal{D}} \begin{pmatrix} -T_{22} & T_{21} & 0 & -S_{12} \\ T_{12} & -T_{11} & S_{12} & 0 \\ 0 & R_{12} & -T_{22} & T_{12} \\ -R_{12} & 0 & T_{21} & -T_{11} \end{pmatrix} \begin{pmatrix} \partial_k Z_1 \\ \partial_k Z_2 \\ \partial_k Z_1^* \\ \partial_k Z_2^* \end{pmatrix} \delta(x - y) \\
\{Z_2(x), a_k(y)\} &= \\{Z_1^*(x), a_k(y)\} = \\{Z_2^*(x), a_k(y)\} = 0
\end{align*}
$$

(42)

At this stage one can prove the following useful identities:

$$
T_{11} \partial_k Z_2 - T_{12} \partial_k Z_1 - S_{12} \partial_k Z_1^* = \partial_k Z_2; \quad T_{22} \partial_k Z_1 - T_{21} \partial_k Z_2 + S_{12} \partial_k Z_2^* = \partial_k Z_1
$$

(43)

Using these identities the brackets from the mixed sector eq. (12) can be easily simplified to:

$$
\{a_k(x), Z(y)\} = \frac{1}{\mathcal{D}} \partial_k Z \delta(x - y)
$$

(44)

Now we must go back and rework the new Dirac brackets between the gauge fields $a_i$. This turns out to be

$$
\{a_i(x), a_j(y)\}^{DB} = -\frac{\epsilon_{ij}}{2\theta\mathcal{D}} \delta(x - y)
$$

(45)

We thus have at our disposal the Dirac brackets among all the field variables. They are given by the equations (40), (11), (14) and (15). Armed with these we can check that the constraints $G_1(x)$ and $G_2(x)$ generate the gauge transformations.

Let us consider the constraint $G_1$ first. It is easy to check that

$$
\int d^2y f(y) \{Z_1(x), G_1(y)\} = \frac{i f(x)}{\mathcal{D}} (-T_{22} Z_1 + T_{21} Z_2 + S_{12} Z_2^*)
$$

(46)
where \( f(x) \) is an arbitrary function with finite support. Using the explicit forms of \( T, S \) and \( \mathcal{D} \) we can show that

\[
\delta Z_\alpha(x) = \int d^2y f(y) \{ Z_\alpha(x), G_1(y) \} \approx i f(x) Z_\alpha(x)
\]  
(47)

It is not difficult to show that

\[
\{ a_k(x), G_1(y) \} \approx 0, \quad \text{and} \quad \{ Z_\alpha(x), G_2(y) \} \approx 0
\]  
(48)

This establishes that the constraint \( G_1(x) \) has no effect on the gauge field sector and similarly \( G_2(x) \) does not effect any transformation in the matter sector. We will now establish that it is exactly \( 2\theta G_2(x) \) that generates the second \( U(1) \) gauge transformation. For this purpose note that \( G_2(x) = b(x) - B(x) \) where \( b(x) = \epsilon_{ij} \partial_i a_j \) and \( B(x) = \epsilon_{ij} \partial_i A_j \) are the ‘magnetic fields’ of the gauge potentials \( a \) and \( A \) respectively. Simple algebra leads to

\[
\int d^2y f(y) \{ a_i(x), B(y) \} = - \frac{B}{\mathcal{D}} (\partial_i f); \int d^2y f(y) \{ a_i(x), b(y) \} = \frac{1}{2\theta \mathcal{D}} (\partial_i f)
\]  
(49)

From this and the Dirac brackets of the gauge field sector and eq. (44), we can arrive at the following relations using the explicit form of \( \mathcal{D} \)

\[
\int d^2y f(y) \{ a_i(x), G_2(y) \} \approx \frac{1}{2\theta} \partial_i f(x)
\]  
(50)

establishing our claim. It is easy to prove that these two generators have zero Dirac brackets among themselves.

To proceed further with the analysis we consider the spin variables \( n_a(x) \) mentioned in the section 2. They can be obtained from the \( CP^1 \) variables by using the Hopf map:

\[
n_a(x) = Z^\dagger \sigma_a Z
\]  
(51)

We can obtain using eqs. (10) and (11) that

\[
\{ n_a(x), B(y) \} \approx \frac{1}{\mathcal{D}} \nabla \delta(x - y) \times \nabla n_a(x)
\]  
(52)
Given this we will now consider the modifications for the spin algebra that is obtained by the introduction of the Hopf term in the action.

Using the Hopf map we can write down the components of spin variable $n$ as:

$$n_1 = Z_1 Z_2^* + Z_2^* Z_2; \quad n_2 = i(Z_1 Z_2^* - Z_2^* Z_2); \quad n_3 = |Z_1|^2 - |Z_2|^2.$$  

(53)

For convenience let us consider the Dirac bracket between $n_1$ and $n_2$. This is given by, using eqs. (40) and (41):

$$\{n_1(x), n_2(y)\} = \frac{1}{D} [2n_3(x) - 4i\theta M] \delta(x - y)$$

(54)

where $M$ is given by

$$M = |Z_1|^2 \nabla Z_1^* \times \nabla Z_1 - |Z_2|^2 \nabla Z_2^* \times \nabla Z_2 + (Z_1 Z_2 \nabla Z_1^* \times \nabla Z_2^* - c.c.)$$

(55)

One can easily see that $\theta$ dependent term in the above is gauge invariant and hence we can evaluate it in a particular gauge. Following [8], we choose $Z_1 = Z_1^*$ and find the term vanishes. Similarly we get the same result for the other choice $Z_2 = Z_2^*$. We therefore find that

$$\{n_1(x), n_2(y)\} = \frac{2}{D} n_3(x) \delta(x - y)$$

(56)

It is easy to write this covariantly as:

$$\{n_a(x), n_b(y)\} = \frac{2}{D} \epsilon_{abc} n_c(x) \delta(x - y).$$

(57)

Note that in contrast to the NLSM without Hopf term we have an extra factor $D$ in the denominator. In view of the nonvanishing bracket $\{n, B\}$ the conventional spin algebra cannot be restored even if we redefine a new renormalised spin variable $\tilde{n} = D n$.

4. Global generators and Discussions

In this paper we have considered the Hamiltonian analysis of the non-relativistic non-linear sigma model with the topological Hopf term added. This model as was pointed out is quite relevant when we consider polarised quantum Hall system with non zero $g$-factor. As was pointed out in the literature there are solitons in this model and the introduction of the
Hopf term was expected to alter the statistics of these skyrmions. We have chosen to work with $CP^1$ variables in order to avoid introduction of nonlocal terms which naturally arise when we consider Hopf terms. True to the expectations, the introduction of the term changes the symplectic structure defined by the system. Interestingly a modified ‘spin algebra’ was obtained. In this modification the topological charge density plays an important role. Whenever $D = 0$ the spin algebra breaks down. We have obtained the generators of the gauge transformations. We verified that these generators produce appropriate transformations.

For completeness we can easily work out the other global symmetry generators also. For example, the Hamiltonian as time translations generator is given by

$$H = \int d^2x \{ |D_iZ|^2 + A_0(x)G_1(x) - 2\theta a_0(x)G_2(x) \}$$

(58)

This gives the equations of motion for the $Z_\alpha$ as

$$iD_0Z_\alpha = D_iD_\alpha Z_\alpha - \frac{2i\theta}{D}[\nabla Z^\dagger(D_iD_jZ) + (D_iD_jZ)^\dagger\nabla] \times \nabla Z_\alpha$$

(59)

where $D_0 \equiv \partial_0 - iA_0$. In the limit $\theta \to 0$ we get the the equations given in [8]. The equation for $a_k(x)$ is found to be

$$\partial_t a_k(x) = -\frac{1}{D}[\partial_k Z^\dagger(D_iD_jZ) + (D_iD_jZ)^\dagger\partial_k Z]$$

(60)

The translation generator turns out to be

$$P_i = \int d^2x [\mathcal{P}_a(x)\partial_i z_a(x) + \mathcal{P}_a^*(x)\partial_i z_a^*(x) + \pi_j(x)\partial_j a_j(x)]$$

(61)

This, when using the definitions of momenta reduces to

$$P_i = -\frac{i}{2} \int d^2x Z^\dagger \partial_i Z - 2\theta \int d^2x a_i(x)G_2(x)$$

(62)

One can easily show after a straightforward calculation that

$$\{Z_\alpha(x), P_i\}^{DB} = \partial_i Z_\alpha$$

(63)

Similarly one gets for $a_i$

$$\{a_k(x), P_i\} = \partial_i a_k.$$  

(64)
At this stage it is instructive to calculate the \( \{P_i, P_j\}^{DB} \). We can just proceed as was done for the model without Hopf term to find that

\[
\{P_i, P_j\}^{DB} = 2\pi N \epsilon_{ij}
\]  

(65)

where \( N = \int j_0(x) \) is topological charge given in eq.(5). Comparing the similar equation given in [8] we conclude that presence of the Hopf term does not affect the central charge, viz., the right hand side of eq.(65).

Now let us consider the angular momentum generator. It is given by

\[
J = \int d^2x \left[ \epsilon_{ij} x_ip_j + \pi_i \Sigma_{ij}^{12} a_j \right]
\]  

(66)

where \( p_i \) is the momentum density and is the integrand in (61):

\[
p_i = \mathcal{P}_\alpha(x) \partial_i z_\alpha(x) + \mathcal{P}_\alpha^*(x) \partial_i z_\alpha^*(x) + \theta \epsilon_{jk} a_k(x) \partial_i a_j(x)
\]  

(67)

and

\[
\Sigma_{ij}^{12} = \delta_{i1} \delta_{j2} - \delta_{i2} \delta_{j1}
\]  

(68)

We can write a more simplified form of angular momentum in eq.(66) as

\[
J = \int d^2x [\epsilon_{ij} x_ip_j + \theta a_j a_j]
\]  

(69)

Using this one can indeed show after a straightforward calculation that appropriate spatial rotation is generated by \( J \),

\[
\{Z(x), J\} = \epsilon_{ij} x_i \partial_j Z(x)
\]  

(70)

\[
\{a_k(x), J\} = \epsilon_{ij} x_i \partial_j a_k(x) + \epsilon_{ki} a_i(x)
\]  

(71)

Inspite of the dramatic changes in the symplectic structure all these generators effect the same transformations as the model without the Hopf term. Ofcourse in the limit of the coefficient of the Hopf term vanishing we get back the conventional model offering Landau-Lifshitz description of Ferromagnets. Note the ubiquitous presence of \( \mathcal{D} \) in all the new Dirac brackets. Further analysis can be done by considering quantum fluctuations in a skyrmion.
background. Eventhough the spin algebra could not be reconstructed when $\theta \neq 0$ in a general analysis, in a fixed background of soliton density profile we can obtain the algebra through rescaling. Then one can consider fluctuations about this background. The new algebraic description arising here will play an important role for the quantum fluctuations and will be presented elsewhere.

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