SCALING LIMITS AND HOMOGENIZATION OF MIXING HAMILTON-JACOBI EQUATIONS

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Abstract. We study the homogenization of nonlinear, first-order equations with highly oscillatory mixing spatio-temporal dependence. It is shown in a variety of settings that the homogenized equations are stochastic Hamilton-Jacobi equations with deterministic, spatially homogenous Hamiltonians driven by white noise in time. The paper also contains proofs of some general regularity and path stability results for stochastic Hamilton-Jacobi equations, which are needed to prove some of the homogenization results and are of independent interest.

1. Introduction

1.1. The set-up. We study the homogenization of nonlinear, first-order equations with highly oscillatory mixing spatio-temporal dependence, which, for small $\varepsilon > 0$ and fixed $\gamma > 0$, are of the form

\begin{equation}
    u_\varepsilon \left( \varepsilon \gamma \nabla \right) + \frac{1}{\varepsilon \gamma} H \left( D u_\varepsilon, x_\varepsilon, t_\varepsilon, \omega \right) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \times \Omega \text{ and } \quad u_\varepsilon(x, 0, \omega) = u_0(x) \quad \text{in } \mathbb{R}^d \times \Omega.
\end{equation}

Here, $(\Omega, \mathcal{F}, \mathbb{P})$ is a given probability space and $u_0 \in UC(\mathbb{R}^d)$, the space of uniformly continuous functions on $\mathbb{R}^d$. For notational ease, when it does not cause confusion, we suppress the dependence on the parameter $\omega \in \Omega$.

We assume that $H$ has a self-averaging effect in the spatial variable and is mixing in the time variable. In addition, to rule out so-called ballistic behavior, it is assumed that $H$ is centered, that is, for each fixed $(p, y, s) \in \mathbb{R}^d \times \mathbb{R}^d \times [0, \infty),$

\begin{equation}
    \mathbb{E} [H(p, y, s, \cdot)] = 0,
\end{equation}

although we later give an example to indicate that more assumptions are needed in general to avoid blow-up.

We demonstrate, in a variety of settings, that (1.1) approximates in law a stochastic partial differential equation with no random spatial oscillations, that is, there exists $M \in \mathbb{N}$, a deterministic $\overline{H} = (\overline{H}^1, \overline{H}^2, \ldots, \overline{H}^M) \in C(\mathbb{R}^d, \mathbb{R}^M)$, and a Brownian motion $B = (B^1, B^2, \ldots, B^M) : [0, \infty) \times \Omega \to \mathbb{R}^M$ such that, as $\varepsilon \to 0$, $u_\varepsilon$ converges locally uniformly and in distribution to the unique stochastic viscosity solution $\overline{u}$ of

\begin{equation}
    d \overline{u} + \overline{H}(D \overline{u}) \circ dB = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad \overline{u}(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d.
\end{equation}

The Lions-Souganidis theory of stochastic viscosity solutions is discussed in Section 2 below. For more details, see also [24, 25, 26, 27, 28, 38].

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When $\gamma = 1$, $u^\varepsilon$ is given by $u^\varepsilon(x, t) = \varepsilon u(x/\varepsilon, t/\varepsilon^2)$, where
\begin{equation}
(1.4) \quad u_t + H(Du, x, t, \omega) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and } \quad u(\cdot, 0) = \varepsilon^{-1}u_0(\varepsilon \cdot) \quad \text{in } \mathbb{R}^d,
\end{equation}
and, hence, the behavior of (1.1) describes the average long range, long time behavior of solutions of (1.4) with large, slowly-varying initial data (the dependence of $u$ on $\varepsilon$, seen only through the initial data, is suppressed here for convenience of presentation).

It is also of interest to study (1.1) for different values of $\gamma$. For some results, the parameters $\varepsilon$ and $\delta := \varepsilon^\gamma$ can effectively be viewed as independent, while for others, the fixed parameter $\gamma > 0$ must be made sufficiently small to ensure that the rate of mixing in time is mild in relation to the spatial oscillations.

The limiting behavior of the problem is very sensitive to the precise structure of the Hamiltonian. For example, if $H$ is of the form
\begin{equation}
H(p, x, t, \omega) := F(p, x)\xi(t, \omega),
\end{equation}
where $F \in C(\mathbb{R}^d \times \mathbb{T}^d, \mathbb{R})$ and the random field $\xi : [0, \infty) \times \Omega \to \mathbb{R}$ is centered and sufficiently mixing, then (1.1) can exhibit ballistic behavior if $F$ is not convex in the gradient variable, and, if
\begin{equation}
H(p, x, t, \omega) = G(p)\xi_1(t, \omega) + f(x)\xi_2(t, \omega)
\end{equation}
for some $G \in C(\mathbb{R}^d, \mathbb{R})$, $f \in C(\mathbb{T}^d, \mathbb{R})$, and centered mixing fields $\xi_1, \xi_2 : [0, \infty) \times \Omega \to \mathbb{R}$, then the correlation between $\xi_1$ and $\xi_2$, as well as their precise laws, have a nontrivial effect on the structure of the homogenized equation (1.3).

An important example from the standpoint of applications is
\begin{equation}
H(p, x, t, \omega) = A\left(\frac{p}{|p|}, x, t, \omega\right)|p|,
\end{equation}
where $A : S^{d-1} \times \mathbb{R}^d \times [0, \infty) \times \Omega \to \mathbb{R}$ is continuous in the first three variables and $S^{d-1} \subset \mathbb{R}^d$ is the unit sphere. This Hamiltonian arises in the study of the asymptotic behavior of an interfacial flow $(\Gamma^\varepsilon_t)_{t \geq 0} \subset \mathbb{R}^d$ evolving with the oscillatory and fluctuating normal velocity
\begin{equation}
V^\varepsilon = -\frac{1}{\varepsilon^\gamma}A\left(n, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, \omega\right),
\end{equation}
where $n \subset S^{d-1}$ is the outward unit normal vector to the surface $\Gamma^\varepsilon_t$ at the point $x$.

We show, under certain structural conditions on $A$, that, as $\varepsilon \to 0$, $(\Gamma^\varepsilon_t)_{t \geq 0}$ converges in distribution and in the Hausdorff distance to an interfacial motion $(\Gamma_t)_{t \geq 0}$ with the normal velocity given, for some deterministic, continuous $\overline{A} : S^{d-1} \to \mathbb{R}^d$ and Brownian motion $B : [0, \infty) \times \Omega \to \mathbb{R}^d$, by
\begin{equation}
\overline{V} = -\overline{A}(n) \circ dB.
\end{equation}

The level-set formulation of the problem leads to the equation
\begin{equation}
(1.5) \quad u^\varepsilon_t + \frac{1}{\varepsilon^\gamma}A\left(Du^\varepsilon, x, \frac{t}{\varepsilon^2}, \omega\right)|Du^\varepsilon| = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and } \quad u^\varepsilon(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d,
\end{equation}
and the homogenized level-set equation is then a special case of (1.3):
\begin{equation}
(1.6) \quad \overline{u}_t + \overline{A}\left(\frac{\overline{D}\overline{u}}{|\overline{D}\overline{u}|}\right)|\overline{D}\overline{u}| \circ dB = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and } \quad \overline{u}(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d.
\end{equation}

There is an extensive literature on the approximation of stochastic partial differential equations by equations with mixing time dependence. For instance, results of this type for linear and semilinear parabolic partial differential equations were obtained by Bouc and Pardoux [7], Kushner and Huang [19], and Watanabe [40], and partial differential equations with spatial averaging and time fluctuations have been studied by Campillo, Kleptsyna, and Piatnitski [8] and Pardoux and Piatnitski [34].
In contrast to these works, the equations we consider here are of first order and with nonlinear dependence on the gradient. This makes the task of obtaining regularity estimates, and, therefore, tightness of probability measures, much more challenging. The nonlinear nature of the problem gives rise to further difficulties in identifying the limiting equation.

1.2. The main results. We now give an informal summary of the main results of the paper. Precise assumptions and statements can be found later on.

We focus our attention on a specific class of Hamiltonians, one which already reveals the complexity of the problem. These take the form

\begin{equation}
H(p, y, t, \omega) = \sum_{i=1}^{m} H^i(p, y, \omega) \xi^i(t, \omega) \quad \text{for } (p, y, t, \omega) \in \mathbb{R}^d \times \mathbb{R}^d \times [0, \infty) \times \Omega,
\end{equation}

where, for each \( i = 1, 2, \ldots, m \), \( H^i : \mathbb{R}^d \times \mathbb{R}^d \times \Omega \to \mathbb{R} \) is self-averaging in space and \( \xi^i : [0, \infty) \times \Omega \to \mathbb{R} \) is (piecewise) smooth, centered, and mixing, so that, in distribution, as \( \varepsilon \to 0 \), the field

\begin{equation}
(t, \omega) \mapsto \frac{1}{\varepsilon^2} \xi^i \left( \frac{t}{\varepsilon^2}, \omega \right)
\end{equation}

approaches white noise in time. More details and examples of such fields are discussed in subsection 2.1.

We divide the results into two cases, depending on whether \( m = 1 \) (the single-noise case) or \( m > 1 \) (the multiple-noise case).

1.2.1. The single-noise case. The problem of interest, for some convex and coercive \( H : \mathbb{R}^d \times \mathbb{R}^d \times \Omega \to \mathbb{R} \) and a white noise approximation \( \xi : [0, \infty) \times \Omega \to \mathbb{R} \), is

\begin{equation}
u^\varepsilon + \frac{1}{\varepsilon^2} H \left( Du^\varepsilon, \frac{x}{\varepsilon}, \omega \right) \xi \left( \frac{t}{\varepsilon^2}, \omega \right) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad u^\varepsilon(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d.
\end{equation}

Many different assumptions for the dependence of the random Hamiltonian on space are covered by the results in Section 2. The Hamiltonian may even be allowed to depend on the “slow” spatial variable, as in

\begin{equation}
u^\varepsilon + \frac{1}{\varepsilon^2} H \left( Du^\varepsilon, x, \frac{x}{\varepsilon}, \omega \right) \xi \left( \frac{t}{\varepsilon^2}, \omega \right) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad u^\varepsilon(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d.
\end{equation}

The field \( \xi \), meanwhile, is allowed to be any reasonable approximation of white noise, or even true white noise, as for the problem

\begin{equation}
du^\varepsilon + H \left( Du^\varepsilon, \frac{x}{\varepsilon}, \omega \right) \circ dB = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad u^\varepsilon(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d,
\end{equation}

where \( B : \Omega \times [0, \infty) \to \mathbb{R} \) is a standard Brownian motion.

As an example of the types of results available in this setting, we assume here that

\begin{equation}
\begin{cases}
\text{the white noise approximation } \xi \text{ is piecewise smooth,} \\
p \mapsto H(p, x, \omega) \text{ is convex and coercive, uniformly for } (x, \omega) \in \mathbb{R}^d \times \Omega, \text{ and} \\
either x \mapsto H(\cdot, x, \omega) \text{ is deterministic and periodic, or} \\
(x, \omega) \mapsto H(\cdot, x, \omega) \text{ is a random, stationary-ergodic field.}
\end{cases}
\end{equation}

**Theorem 1.1.** Let \( \gamma > 0 \) and \( u_0 \in UC(\mathbb{R}^d) \), and assume that \( H \) and \( \xi \) satisfy (1.10). Then there exists a deterministic \( \overline{H} : \mathbb{R}^d \to \mathbb{R} \) satisfying (2.9), which depends only on \( H \), and a Brownian motion \( B : [0, \infty) \times \Omega \to \mathbb{R} \) such that, as \( \varepsilon \to 0 \), the solution \( u^\varepsilon \) of (1.9) converges in distribution to the unique stochastic viscosity solution \( \overline{u} \) of

\begin{equation} \frac{d\overline{u}}{dt} + \overline{H}(D\overline{u}) \circ dB = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad \overline{u}(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d. \end{equation}
The convergence in distribution in Theorem 1.1, and in the subsequent results below, is understood with the topology of local-uniform convergence.

Theorem 1.1 holds without any restrictions on the positive parameter \( \gamma \), or on the correlation between the random functions \( H \) and \( \xi \). This has to do with regularity and stability estimates for pathwise Hamilton-Jacobi equations with convex and coercive Hamiltonians. These estimates, which are of independent interest, are proved in Appendix A.

### 1.2.2. The multiple-noise case.

We now turn to the study of the initial value problem

\[
(1.12) \quad u_\varepsilon^\gamma + \frac{1}{\varepsilon^\gamma} \sum_{i=1}^{m} H^i \left( D u_\varepsilon^\gamma \right) \xi^i \left( \frac{t}{\varepsilon^{2\gamma}}, \omega \right) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad u_\varepsilon^\gamma(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d,
\]

where \( u_0 \in UC(\mathbb{R}^d) \), \( m > 1 \), and, for each \( i = 1, 2, \ldots, m \), \( H^i \in C(\mathbb{R}^d \times T^d) \) and \( \xi^i : [0, \infty) \times \Omega \to \mathbb{R} \) is a white noise approximation.

We will show that there exist \( M \in \mathbb{N} \) and, for each \( j = 1, 2, \ldots, M \), a continuous, deterministic, effective Hamiltonian \( \overline{H}^j : \mathbb{R}^d \to \mathbb{R} \) and a Brownian motion \( B^j \) such that, as \( \varepsilon \to 0 \), \( u^\gamma \) converges locally uniformly in distribution to the unique stochastic viscosity solution \( \overline{\pi} \) of

\[
(1.13) \quad d\overline{\pi} + \sum_{j=1}^{M} \overline{H}^j(D\overline{\pi}) \circ dB^j = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad \overline{\pi}(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d.
\]

Despite the similarity of this statement with Theorem 1.1, there are some fundamental differences in the nature of the problem. Most importantly, the deterministic effective Hamiltonians, and even their number \( M \), depend on the particular laws of the mixing fields.

This can be seen by considering the initial value problem

\[
(1.14) \quad \overline{u}_\varepsilon^\gamma + \frac{1}{\varepsilon^\gamma} |\overline{x}| \xi^1 \left( \frac{t}{\varepsilon^{2\gamma}}, \omega \right) + \frac{1}{\varepsilon^\gamma} f \left( \frac{x}{\varepsilon} \right) \xi^2 \left( \frac{t}{\varepsilon^{2\gamma}}, \omega \right) = 0 \quad \text{in } \mathbb{R} \times (0, \infty) \quad \text{and} \quad \overline{u}_\varepsilon^\gamma(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}.
\]

**Theorem 1.2.** Assume that \( f \in C^{0,1}(\mathbb{T}) \), \( u_0 \in UC(\mathbb{R}) \), and \( 0 < \gamma < 1/2 \). Then there exist piecewise smooth white noise approximations

\[
\xi = (\xi^1, \xi^2) : [0, \infty) \times \Omega \to \mathbb{R}^2 \quad \text{and} \quad \overline{\xi} = \left( \xi^1, \xi^2 \right) : [0, \infty) \times \Omega \to \mathbb{R}^2,
\]

deterministic functions \( \overline{H} \in C(\mathbb{R}, \mathbb{R}^2) \) and \( \overline{H} \in C(\mathbb{R}, \mathbb{R}^d) \), and Brownian motions \( B : [0, \infty) \times \Omega \to \mathbb{R}^d \) such that, if \( u^\gamma \) and \( \overline{u}_\varepsilon \) are the solutions of \( 1.14 \) with respectively the fields \( \xi \) and \( \overline{\xi} \), then

\[
\lim_{\varepsilon \to 0} u_\varepsilon^\gamma = \overline{\pi} \quad \text{and} \quad \lim_{\varepsilon \to 0} \overline{u}_\varepsilon^\gamma = \overline{\pi} \quad \text{locally uniformly and in distribution,}
\]

where \( \overline{\pi} \) and \( \overline{\pi} \) are the unique stochastic viscosity solutions of respectively

\[
d\overline{\pi} + \overline{H}(\overline{\pi}_x) \circ dB = 0 \quad \text{and} \quad d\overline{\pi} + \overline{H}(\overline{\pi}_x) \circ d\overline{B} = 0 \quad \text{in } \mathbb{R} \times (0, \infty)
\]

with \( \overline{\pi}(\cdot, 0) = \overline{\pi}(\cdot, 0) = u_0 \) in \( \mathbb{R} \). Moreover, as \( C(\mathbb{R} \times [0, \infty)) \)-valued random variables, \( \pi \) and \( \pi \) have different laws for general \( u_0 \in UC(\mathbb{R}^d) \).

The next result demonstrates that nontrivial correlation between the fields \( \xi^1 \) in \( 1.12 \) can create ballistic behavior.

**Theorem 1.3.** For some \( V \in C(\mathbb{T}) \), \( F \in C(\mathbb{R}) \), and independent white noise approximations \( \xi^1, \xi^2 : [0, \infty) \times \Omega \to \mathbb{R} \), the following hold:
(a) There exists a deterministic $\bar{H} \in C(\mathbb{R}, \mathbb{R}^3)$ and a Brownian motion $B : [0, \infty) \times \Omega \to \mathbb{R}^3$ such that, if $0 < \gamma < 1/6$, $u_0 \in UC(\mathbb{R})$, and $u^\varepsilon$ solves

$$
\frac{d}{dt} u^\varepsilon + \frac{1}{\varepsilon^7} F(u_\varepsilon) \xi^1 \left( \frac{t}{\varepsilon^2 \gamma}, \omega \right) + \frac{1}{\varepsilon^7} V \left( \frac{x}{\varepsilon} \right) \xi^2 \left( \frac{t}{\varepsilon^2 \gamma}, \omega \right) = 0 \quad \text{in } \mathbb{R} \times (0, \infty) \quad \text{and} \quad u^\varepsilon(\cdot, 0) = u_0 \quad \text{in } \mathbb{R},
$$

then, as $\varepsilon \to 0$, $u^\varepsilon$ converges locally uniformly and in distribution to the unique stochastic viscosity solution of

$$
d\bar{H} + \bar{H}(\bar{H}) \circ dB = 0 \quad \text{in } \mathbb{R} \times (0, \infty) \quad \text{and} \quad \bar{H}(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}.
$$

(b) There exists $p \in \mathbb{R}$ and a deterministic, nonzero constant $\tau \neq 0$ such that, if $0 < \gamma < 1$ and $\tilde{u}^\varepsilon$ is the solution of

$$
\tilde{u}^\varepsilon + \frac{1}{\varepsilon^7} F(\tilde{u}_x) \xi^1 \left( \frac{t}{\varepsilon^2 \gamma}, \omega \right) + \frac{1}{\varepsilon^7} V \left( \frac{x}{\varepsilon} \right) \xi^1 \left( \frac{t}{\varepsilon^2 \gamma}, \omega \right) = 0 \quad \text{in } \mathbb{R} \times (0, \infty) \quad \text{and} \quad \tilde{u}^\varepsilon(x, 0) = p \cdot x \quad \text{in } \mathbb{R},
$$

then, with probability one, for all $T > 0$,

$$
\lim_{\varepsilon \to 0} \sup_{(x, t) \in \mathbb{R} \times [0, T]} |\varepsilon^\gamma u^\varepsilon(x, t) - \tau t| = 0.
$$

Observe that, if $F : \mathbb{R} \to \mathbb{R}$ is convex and coercive, then the hypotheses in (1.10) are satisfied by the Hamiltonian $H(p, x) := F(p) + V(x)$ and the field $\xi^i$. Hence, the example in Theorem 1.3(b), for which the function $F$ is necessarily non-convex, illustrates that the convexity assumption in Theorem 1.4 is necessary in general.

Finally, we describe a result concerning the first order, level set problem

$$
\frac{d}{dt} u^\varepsilon + \frac{1}{\varepsilon^7} A \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2 \gamma} \right) |Du^\varepsilon| = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad u^\varepsilon(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d,
$$

where

$$
A(y, t, \omega) := \sum_{i=1}^m a^i(y) \xi^i(t, \omega),
$$

(1.16)

$$
\left( \left( \xi^1, \xi^2, \ldots, \xi^m \right) ((k, k + 1), .) \right)_{k=0}^\infty \text{ are independent and uniformly distributed over } \{-1, 1\}^m,
$$

$$
a^i \in C^{0,1}(\mathbb{R}^d) \quad \text{for all } i = 1, 2, \ldots, m, \quad \text{and} \quad \sum_{i=1}^m a^i y^i \neq 0 \text{ whenever } y^i \in \{-1, 1\}.
$$

A more general result, which covers Theorem 1.4 below, will be proved in Section 4.

Theorem 1.4. Assume that $0 < \gamma < 1/6$, $u_0 \in UC(\mathbb{R}^d)$, and (1.16) holds. Then there exists $\bar{A} \in C \left( S^{d-1}, \mathbb{R}^{2m-1} \right)$ and a Brownian motion $B : [0, \infty) \times \Omega \to \mathbb{R}^{2m-1}$ such that, as $\varepsilon \to 0$, the solution $u^\varepsilon$ of (1.15) converges locally uniformly and in distribution to the stochastic viscosity solution $\bar{H}$ of

$$
d\bar{A} + \bar{A} \left( \frac{D\bar{A}}{|D\bar{A}|} \right) |D\bar{A}| \circ dB = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad \bar{H}(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d.
$$

Recall that (1.16) is the level set equation for a hypersurface evolving according to the normal velocity $-\varepsilon^{-\gamma} A(x/\varepsilon, t/\varepsilon^2 \gamma)$. Theorem 1.4 then implies that, as $\varepsilon \to 0$ and in distribution, the level-set flow corresponding to the normal velocity $-\varepsilon^{-\gamma} A(x/\varepsilon, t/\varepsilon^2 \gamma)$ converges in the Hausdorff metric to the level-set flow with the normal velocity $dA(n, t, \omega)$, where

$$
\mathcal{B}(n, t, \omega) = \bar{A}(n) \cdot B(t, \omega).
$$
1.3. Organization of the paper. Section 2 contains an overview of the main tools and theories used throughout the paper. The results from the single-noise and multiple-noise cases are proved in respectively Sections 3 and 4. Finally, the appendix contains the proof of new regularity and stability estimates from the pathwise viscosity solution theory, as well as the computation of a certain effective Hamiltonian.

1.4. Notation. Throughout, integration with respect to the probability measure $\mathbb{P}$ is denoted by $\mathbb{E}$. For a domain $U \subseteq \mathbb{R}^N$, $(B)UC(U)$ is the space of (bounded) uniformly continuous functions on $U$, and $C^2_b(U)$ is the space of $C^2$ functions $f$ whose Hessian $D^2 f$ is uniformly bounded. For $H : \mathbb{R}^d \to \mathbb{R}$, $H^* : \mathbb{R}^d \to \mathbb{R}$ is the Legendre transform of $H$. Given a set $A$, the function $1_A$ is the indicator function of $A$. The identity matrix is denoted by $\text{Id}$. The $(d - 1)$-dimensional unit sphere in $\mathbb{R}^d$ is $S^{d-1}$, and the $d$-dimensional torus is $T^d$.

2. Preliminaries

2.1. Mixing fields and white noise approximations. We call a random field $\xi : [0, \infty) \times \Omega \to \mathbb{R}$ a white noise approximation if

\[
\begin{align*}
&\begin{cases}
  t \mapsto \xi(t, \cdot) \text{ is piecewise continuous with } \mathbb{P}\text{-probability one and} \\
  \xi^\delta \xrightarrow{\delta \to 0} B \text{ in distribution in } C([0, \infty), \mathbb{R}), \text{ where} \\
  \xi^\delta(t, \cdot) := \delta \int_0^{t/\delta} \xi(s, \cdot) \, ds \text{ and } B : [0, \infty) \times \Omega \to \mathbb{R} \text{ is a standard Brownian motion.}
\end{cases}
\end{align*}
\]

A random field $\xi$ satisfies (2.1) if it is centered, stationary, and sufficiently mixing. More precisely, we define the mixing rate $\rho : [0, \infty) \to [0, \infty)$ associated to $\xi$ by

\[
\rho(t) = \sup_{s \geq 0} \sup_{A \in \mathcal{F}_{s+t, \infty}} \sup_{B \in \mathcal{F}_{0,s}} |\mathbb{P}(A | B) - \mathbb{P}(A)| \quad \text{for } t \geq 0,
\]

where, for $0 \leq s \leq t \leq \infty$, $\mathcal{F}_{s,t} \subset \mathcal{F}$ is the $\sigma$-algebra generated by the maps $(\omega \mapsto \xi(r, \omega))_{r \in [s,t]}$. The field $\xi$ can then be shown to satisfy (2.1) if

\[
\begin{align*}
&\begin{cases}
  t \mapsto \xi(t, \omega) \text{ is stationary,} \\
  \mathbb{P} \left( \sup_{t \in [0, \infty)} |\xi(t, \cdot)| \leq M \right) = 1 \quad \text{for some } M > 0, \\
  \lim_{t \to \infty} \rho(t) = 0, \quad \int_0^\infty |\rho(t)|^{1/2} \, dt < \infty, \\
  \mathbb{E}[\xi(0)] = 0, \quad \text{and} \quad 2 \int_0^\infty \mathbb{E}[\xi(0)\xi(t)] \, dt = 1.
\end{cases}
\end{align*}
\]

The stationarity in (2.3) is understood to be with respect to real shifts in time. If the field is instead stationary with respect to only integer shifts, then the last identity must be replaced with

\[
\int_0^1 \int_0^1 \mathbb{E}[\xi(s)\xi(t)] \, ds \, dt + 2 \int_0^1 \int_1^\infty \mathbb{E}[\xi(s)\xi(t)] \, ds \, dt = 1.
\]

The above conditions are by no means the most general that imply (2.1). For more details on white noise approximations of this type, as well as stochastic ordinary differential equations with mixing coefficients, see the works of Cogburn, Hersh, and Kac [10], Khasminskii [17], Papanicolaou and Varadhan [33], and Papanicolaou and Kohler [32].
We mention a specific class of examples of white noise approximations, one which plays an important role later in the paper. These are given by

\begin{equation}
\xi(t, \omega) = \sum_{k=1}^{\infty} X_k(\omega) 1_{[k-1,k)}(t) \text{ for } (t, \omega) \in [0, \infty) \times \Omega,
\end{equation}

where \((X_k)_{k=1}^{\infty} : \Omega \to \mathbb{R}\) is a collection of mutually independent and identically distributed random variables with

\[ \mathbb{E}[X_k] = 0 \quad \text{and} \quad \mathbb{E}[(X_k)^2] = 1 \quad \text{for all } k = 1, 2, \ldots. \]

For such \(\xi\), the path \(\zeta^i\) appearing in (2.1) is a linearly interpolated random walk, and (2.1) follows from Donsker’s invariance principle.

2.2. Pathwise Hamilton-Jacobi equations. We give a brief overview of some facts that are needed in this paper regarding pathwise, or stochastic, viscosity solutions of the initial value problems

\begin{equation}
\frac{du}{dt} = H(Du, x) \cdot d\xi \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad u(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d
\end{equation}

and

\begin{equation}
\frac{du}{dt} = \sum_{i=1}^{m} H^i(Du) \cdot d\zeta^i \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad u(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d,
\end{equation}

where \(H \in C(\mathbb{R}^d \times \mathbb{R}^d)\), \(H^1, H^2, \ldots, H^m \in C(\mathbb{R}^d)\), \(\zeta^1, \zeta^2, \ldots, \zeta^m \in C([0, \infty), \mathbb{R})\), and \(u_0 \in UC(\mathbb{R}^d)\). For more details, including the definitions of stochastic viscosity sub- and super-solutions and proofs of well-posedness, see [24, 25, 26, 27, 28, 15, 36, 37, 38].

Both problems (2.5) and (2.6) fall under the scope of the classical viscosity solution theory if the driving paths are continuously differentiable, or, more generally, have finite total variation. See [11] for details on the former and [18, 22] for the latter. The theory of pathwise viscosity solutions was developed by Lions and Souganidis [26, 27, 38] to study equations like (2.5) and (2.6) when the driving paths are merely continuous.

The pathwise viscosity solution of (2.5) or (2.6) may be identified by extending the solution operator for the equation from smooth to continuous paths. More precisely, for a fixed \(u_0 \in UC(\mathbb{R}^d)\), let \(S_{u_0} : C^1([0, \infty)) \to C(\mathbb{R}^d \times (0, \infty))\) denote either the solution operator for (2.5) or (2.6), both of which, under certain structural conditions on the Hamiltonians, are well-defined with the classical viscosity solution theory.

We then say that (2.5) or (2.6) has a unique extension to continuous paths if

\begin{equation}
\left\{
\begin{array}{l}
S_{u_0} : C^1([0, \infty)) \to C(\mathbb{R}^d \times [0, \infty)) \text{ extends continuously} \\
\quad \text{to } C([0, \infty)) \text{ for any } u_0 \in UC(\mathbb{R}^d).
\end{array}
\right.
\end{equation}

As in the classical viscosity theory, there is also a notion of continuous stochastic viscosity solutions that is defined using semi-continuous sub- and super-solutions, for which a comparison principle has been proved in a variety of settings. The existence of the unique solution can then be proved alternatively through Perron’s method, as by the author in [37]. The notions of pathwise sub- and super-solutions are not used in this work, so we do not focus on them in this section. In view of the stability properties of pathwise stochastic viscosity solutions, it is always the case that the solution of (2.5) or (2.6) obtained by extending the solution operator is a pathwise viscosity sub- and super-solution.

For the spatially homogenous equation (2.6), it is possible to characterize exactly for which Hamiltonians well-posedness holds.

**Theorem 2.1** (Lions, Souganidis [28]). The solution operator for (2.6) extends continuously in the sense of (2.7) if and only if each Hamiltonian \(H^i\) satisfies

\begin{equation}
H = H_1 - H_2 \quad \text{for some convex } H_1, H_2 : \mathbb{R}^d \to \mathbb{R}.
\end{equation}
Moreover, given \( L > 0 \), there exists \( C = C_L > 0 \) such that, for all \( u_0 \in C^{0,1}(\mathbb{R}^d) \) with \( \|Du_0\|_\infty \leq L \) and \( \zeta_1, \zeta_2 \in C([0, \infty), \mathbb{R}^m) \), if \( S_{u_0} : C([0, \infty), \mathbb{R}^m) \to C(\mathbb{R}^d \times [0, \infty)) \) is the solution operator for (2.6), then
\[
\sup_{(x,t) \in \mathbb{R}^d \times [0,T]} |S_{u_0}(\zeta_1)(x,t) - S_{u_0}(\zeta_2)(x,t)| \leq C \max_{t \in [0,T]} |\zeta_1(t) - \zeta_2(t)|.
\]

The nontrivial spatial dependence in (2.5) makes the question of well-posedness more complicated. It has been proved for certain classes of Hamiltonians (see [38, 24, 15, 36]) which can be described by the following representative cases:
\[
H(p,x) = \begin{cases} 
\frac{1}{2}|p|^2 - f(x) & \text{for } f : \mathbb{R}^d \to \mathbb{R} \text{ smooth}, \\
a(x) \left( |p|^2 + 1 \right)^{1/2} & \text{for } a : \mathbb{R}^d \to \mathbb{R} \text{ smooth}, \\
(p \cdot g(x)p)^{q/2} & \text{for } q \geq 1 \text{ and } g : \mathbb{R}^d \to \mathbb{S}^d \text{ smooth and strictly positive}, \text{ and} \\
\frac{p}{|p|} x \cdot p & \text{for } a : S^{d-1} \times \mathbb{R}^d \to \mathbb{R} \text{ smooth}.
\end{cases}
\]

In Appendix A we prove a quantitative form of (2.7) under less stringent regularity and structural requirements, as long as the Hamiltonian is convex and has uniform growth in the gradient variable, that is,
\[
\begin{cases} 
H \in C(\mathbb{R}^d \times \mathbb{R}^d), & p \rightarrow H(p,x) \text{ is convex for all } x \in \mathbb{R}^d, \text{ and } \\
\text{there exist convex, increasing functions } \underline{\nu}, \overline{\nu} : [0, \infty) \rightarrow \mathbb{R} \text{ such that } \\
\underline{\nu}(|p|) \leq H(p,x) \leq \overline{\nu}(|p|) & \text{for all } (p,x) \in \mathbb{R}^d \times \mathbb{R}^d.
\end{cases}
\]

**Theorem 2.2.** Assume that \( H \) satisfies (2.9). Then, given \( L > 0 \), there exists \( C \) depending only on \( L, \underline{\nu}, \overline{\nu}, \) and \( \overline{\nu} \) such that, for all \( u_0 \in C^{0,1}(\mathbb{R}^d) \) with \( \|Du_0\|_\infty \leq L \), if \( S_{u_0} : C([0, \infty), \mathbb{R}) \to C(\mathbb{R}^d \times [0, \infty)) \) is the solution operator for (2.6), then, for all \( \zeta \in C([0, \infty], \mathbb{R}) \),
\[
\sup_{t \in [0,\infty)} \|D_x S_{u_0}(\zeta)\|_\infty \leq C,
\]
and, for all \( \zeta_1, \zeta_2 \in C([0, \infty), \mathbb{R}) \),
\[
\sup_{(x,t) \in \mathbb{R}^d \times [0,T]} |S_{u_0}(\zeta_1)(x,t) - S_{u_0}(\zeta_2)(x,t)| \leq C \max_{t \in [0,T]} |\zeta_1(t) - \zeta_2(t)|.
\]

The global Lipschitz estimate in space presented in Theorem 2.2 is immediate for equations with \( x \)-independent Hamiltonians like (2.6). This is due to the translation invariance of the equation and the comparison principle.

### 2.3. Homogenization of Hamilton-Jacobi equations

We discuss next some facts regarding the initial value problem
\[
u_{\varepsilon} + H \left( D u_{\varepsilon}, \frac{x}{\varepsilon} \right) = 0 \text{ in } \mathbb{R}^d \times (0, \infty) \text{ and } u_{\varepsilon}(\cdot, 0) = u_0 \text{ in } \mathbb{R}^d
\]
and its homogenization, that is, the identification of a deterministic, effective Hamiltonian \( \overline{H} \in C(\mathbb{R}^d) \) and the local uniform convergence of \( u_{\varepsilon} \), as \( \varepsilon \to 0 \), to the solution \( \overline{\pi} \) of
\[
\overline{\pi}_{t} + \overline{H}(D \overline{\pi}) = 0 \text{ in } \mathbb{R}^d \times (0, \infty) \text{ and } \overline{\pi}(\cdot, 0) = u_0 \text{ in } \mathbb{R}^d.
\]

In this subsection, we assume that the continuous Hamiltonian is either positively or negatively coercive in the gradient variable, that is,
\[
H \in C(\mathbb{R}^d \times \mathbb{R}^d) \text{ and either } \lim_{|p| \to +\infty \ y \in \mathbb{R}^d} H(p, y) = +\infty \text{ or } \lim_{|p| \to +\infty \ y \in \mathbb{R}^d} H(p, y) = -\infty.
\]
We also discuss when the solutions of the problem
\[ u^\varepsilon_t - H \left( Du^\varepsilon, \frac{x}{\varepsilon} \right) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad u^\varepsilon(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d \]
converge, as \( \varepsilon \to 0 \), locally uniformly to the solution of
\[ \overline{u}_t - \overline{H}(D\overline{u}) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad \overline{u}(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d, \]
or, more concisely, whether or not
\[ (2.13) \quad \overline{(-H)} = -\overline{H}. \]
It is proved in [36] that, although the identity (2.13) may fail for general coercive Hamiltonians, it holds in both the periodic and stationary-ergodic settings if \( p \mapsto H(p, \cdot) \) is convex.

We now describe in more detail the homogenization literature in the periodic and random settings.

**Periodicity.** Assume, in addition to (2.12), that
\[ (2.14) \quad y \mapsto H(p, y) \text{ is } \mathbb{Z}^d \text{-periodic.} \]
Then there exists a Hamiltonian \( \overline{H} \in C(\mathbb{R}^d) \) such that, for any \( T > 0 \) and \( u_0 \in UC(\mathbb{R}^d) \), the solution of (2.10) converges uniformly in \( \mathbb{R}^d \times [0, T] \), as \( \varepsilon \to 0 \), to the solution of (2.11), which was proved by Lions, Papanicolaou, and Varadhan [21] (see also Evans [12]).

Some more aspects of the problem are summarized next:

**Theorem 2.3.** Assume that \( H \) satisfies (2.12) and (2.14). Then, for every \( p \in \mathbb{R}^d \), there exist unique constants \( \overline{H}(p) \) and \( (-H)(p) \) such that the equations
\[ H(p + D_y v^+, y) = \overline{H}(p) \quad \text{and} \quad -H(p + D_y v^-, y) = (-H)(p) \]
admit periodic viscosity solutions \( v^+ \) and \( v^- \). The functions \( p \mapsto \overline{H}(p) \) and \( p \mapsto (-H)(p) \) are continuous. Moreover, if \( H \) is convex in \( p \), then so is \( \overline{H} \), and (2.13) holds.

We shall make use of the following homogenization error estimates:

**Theorem 2.4.** Assume, in addition to (2.12) and (2.14), that \( H \) is locally Lipschitz. Let \( u^\varepsilon \) and \( \overline{u} \) be the solutions of the initial value problems
\[
\begin{cases}
  u^\varepsilon_t + H \left( Du^\varepsilon, \frac{x}{\varepsilon} \right) = 0 & \text{in } \mathbb{R}^d \times (0, \infty), \\
  \overline{u}_t + \overline{H}(D\overline{u}) = 0 & \text{in } \mathbb{R}^d \times (0, \infty), \\
  u^\varepsilon(\cdot, 0) = \overline{u}(\cdot, 0) = u_0 & \text{in } \mathbb{R}^d.
\end{cases}
\]

(a) (Capuzzo-Dolcetta, Ishii [9]) For all \( L > 0 \), there exists \( C = C_L > 0 \) such that, if \( \|Du_0\|_\infty \leq L \), then for all \( T > 0 \),
\[ (2.15) \quad \sup_{(x, t) \in \mathbb{R}^d \times [0, T]} |u^\varepsilon(x, t) - \overline{u}(x, t)| \leq C(1 + T)\varepsilon^{1/3}. \]
The exponent can be improved from 1/3 to 1 if \( u_0(x) = p \cdot x \) for some fixed \( p \in \mathbb{R}^d \).

(b) (Mitake, Tran, Yu [31]) If, in addition, \( d = 1 \) and \( p \mapsto H(p, \cdot) \) is convex, or if \( d = 2 \) and \( p \mapsto H(p, \cdot) \) is convex and positively homogenous of some degree \( q \geq 1 \), then the exponent 1/3 in (2.15) can be replaced with 1.

**Stationarity and ergodicity.** We now consider Hamiltonians \( H = H(p, x, \omega) \) depending on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\).
The main assumption is that the Hamiltonian is stationary with respect to a group of ergodic, measure-preserving transformations \( \{T_y\}_{y \in \mathbb{R}^d} : \Omega \to \Omega \). That is,

\[
\begin{aligned}
P &= P \circ T_y \text{ for all } y \in \mathbb{R}^d, \\
H(p, x, T_y \omega) &= H(p, x + y, \omega) \text{ for all } (p, x, y, \omega) \in \mathbb{R}^{d} \times \Omega, \text{ and,} \\
\frac{E}{E} \in \mathcal{F} \text{ and } T_y \frac{E}{E} = \frac{E}{E} \text{ for all } y \in \mathbb{R}^{d}, \text{ then } P(E) = 1 \text{ or } P(E) = 0.
\end{aligned}
\]

(2.16)

The homogenization result in the following theorem was proved independently by Souganidis \[39\] and Rezakhanlou and Tarver \[35\], and the statement about the consistency condition (2.13), as already mentioned, was proved in \[36\].

**Theorem 2.5.** Assume that \( H \) satisfies (2.12) and (2.16), and is convex in the gradient variable. Then there exists a deterministic, coercive, and convex \( \overline{H} : \mathbb{R}^d \to \mathbb{R} \) and an event \( \Omega_0 \in \mathcal{F} \) with \( P(\Omega_0) = 1 \) such that, for all \( u_0 \in UC(\mathbb{R}^d) \), \( T > 0 \), and \( \omega \in \Omega_0 \),

\[
\lim_{\varepsilon \to 0} \sup_{(x, t) \in B_T \times [0, T]} |u_\varepsilon(x, t, \omega) - \overline{u}(x, t)| = 0,
\]

where \( u_\varepsilon \) and \( \overline{u} \) are the solutions of respectively (2.10) and (2.11). Moreover, the identity (2.13) holds.

The stochastic homogenization of Hamilton-Jacobi equations has been studied in many other settings. For more results and extensions, see the works of Lions and Souganidis \[29\], Armstrong and Souganidis \[3\], Armstrong, Tran, and Yu \[4,5\], Armstrong and Cardaliaguet \[1\], Ziliotto \[41\], Feldman and Souganidis \[14\], Feldman, Fermanian, and Ziliotto \[13\], and Armstrong, Cardaliaguet, and Souganidis \[2\].

We now state corollaries of the above homogenization results. These are a consequence of the contractive property of equations (2.10) and (2.11), and the compact embedding of \( C^{0,1}(\mathbb{R}^d) \) into \( C(\mathbb{R}^d) \).

Let \( (S^\varepsilon(\tau))_{\tau \geq 0} : UC(\mathbb{R}^d) \to UC(\mathbb{R}^d) \) and \( (\overline{S}(\tau))_{\tau \geq 0} : UC(\mathbb{R}^d) \to UC(\mathbb{R}^d) \) be the solution operators for respectively (2.10) and (2.11).

**Corollary 2.1.** (a) Assume \( H \) satisfies (2.12) and (2.14). Then, for all \( L, T > 0 \),

\[
\lim_{\varepsilon \to 0} \sup_{\|D\phi\|_{\infty} \leq L \times \mathbb{R}^d} \sup_{\tau \in [0, T]} |S^\varepsilon(\tau)\phi(x) - \overline{S}(\tau)\phi(x)| = 0.
\]

(b) Under the same hypotheses as Theorem 2.5, for all \( \omega \in \Omega_0 \) and \( L, T > 0 \),

\[
\lim_{\varepsilon \to 0} \sup_{\|D\phi\|_{\infty} \leq L} \sup_{x \in B_T} \sup_{\tau \in [0, T]} |S^\varepsilon(\tau, \omega)\phi(x) - \overline{S}(\tau, \omega)\phi(x)| = 0.
\]

The results described above, in either setting, extend also to the homogenization of equations with “slow” spatial dependence, that is, for problems of the form

\[
u^\varepsilon_t + H \left( Du^\varepsilon \frac{x}{\varepsilon}, x \right) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad u^\varepsilon(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d.
\]

In this case, for some deterministic \( \overline{H} \in C(\mathbb{R}^d \times \mathbb{R}^d) \), the homogenized equation takes the form

\[
\overline{u} + \overline{H}(Du, x) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad u(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d,
\]

and, as above, if \( p \mapsto H(p, \cdot, \cdot) \) is convex, then so is \( p \mapsto \overline{H}(p, \cdot) \), and the consistency condition (2.13) holds. Moreover, in both settings, the effective Hamiltonian satisfies the growth estimates

\[
\inf_{y \in \mathbb{R}^d} H(p, x, y) \leq \overline{H}(p, x) \leq \sup_{y \in \mathbb{R}^d} H(p, x, y) \quad \text{for all } (p, x) \in \mathbb{R}^d \times \mathbb{R}^d.
\]

(2.17)
2.4. Convergence of probability measures. Throughout the paper, we use certain facts about random variables converging in distribution. More details and proofs can be found in the book of Billingsley [6].

Given a Polish space \( A \), that is, a complete and separable metric space, a sequence of Borel probability measures \( (\mu_n)_{n \geq 1} \) on \( A \) is said to converge weakly to \( \mu \) as \( n \to \infty \) if

\[
\lim_{n \to \infty} \int_A f \, d\mu_n = \int_A f \, d\mu \quad \text{for all } f \in C_b(A).
\]

A sequence of \( A \)-valued random variables \( (X_n)_{n \geq 1} \) (not necessarily defined on the same probability space) is said to converge in distribution to \( X \) in the space \( A \), as \( n \to \infty \), if the sequence of probability laws of the \( X_n \)'s converges weakly to the probability law of \( X \).

The following lemma is a specific equivalence from the classical Portmanteau theorem.

**Lemma 2.1.** Let \( (\mu_n)_{n \geq 1} \) and \( \mu \) be Borel probability measures on \( A \). Then \( \mu_n \) converges weakly to \( \mu \) as \( n \to \infty \) if and only if

\[
\liminf_{n \to \infty} \mu_n(G) \geq \mu(G) \quad \text{for all open sets } G \subset A.
\]

We also make use of the following Mapping Theorem. Recall that, for Polish spaces \( A \) and \( B \), a Borel measurable map \( f : A \to B \), and a Borel measure \( \mu \) on \( A \), the Borel measure \( \nu := f^* \mu \) is defined by

\[
\nu(B) := \mu(f^{-1}(B)) \quad \text{for Borel sets } B \subset B.
\]

**Lemma 2.2.** Let \( A \) and \( B \) be complete separable metric spaces, and assume \( f : A \to B \) is continuous. If \( (\mu_n)_{n \geq 1} \) is a sequence of Borel probability measures on \( A \) that, as \( n \to \infty \), converge weakly to some \( \mu \), then the measures \( \nu_n := f^* \mu_n \) converge weakly, as \( n \to \infty \), to \( \nu := f^* \mu \).

The following result, known as Slutsky’s Theorem, is a useful way to compare two sequences of probability measures converging in distribution.

**Lemma 2.3.** Let \( (X_n, Y_n)_{n \geq 1} \) be two sequences of \( A \)-valued random variables. Assume that, for some \( A \)-valued random variable \( X \), as \( n \to \infty \), \( X_n \) converges in distribution to \( X \) and \( X_n - Y_n \) converges in probability to \( 0 \). Then, as \( n \to \infty \), \( Y_n \) converges in distribution to \( X \).

In this paper, we focus mainly on the two spaces \( C(\mathbb{R}^d \times [0, \infty)) \) and \( C([0, \infty), \mathbb{R}^M) \), which are endowed with the topology of local-uniform convergence. These spaces are metrizable with the metrics

\[
d_s(u, v) := \sum_{k=1}^{\infty} \max_{(x,t) \in B_k \times [0,k]} |u(x,t) - v(x,t)| \cdot 2^{-k}
\]

and

\[
d_p(\eta, \zeta) := \sum_{k=1}^{\infty} \max_{t \in [0,k]} |\eta(t) - \zeta(t)| \cdot 2^{-k}
\]

for \( u, v \in C(\mathbb{R}^d \times [0, \infty)) \) and \( \eta, \zeta \in C([0, \infty), \mathbb{R}^M) \).

For the product space, we use the metric

\[
d(\eta, \zeta) := d_s(u, v) + d_p(\eta, \zeta)
\]

for \( u, v \in C(\mathbb{R}^d \times [0, \infty)) \) and \( \eta, \zeta \in C([0, \infty), \mathbb{R}^M) \).

3. The single-noise case

3.1. A general convergence result. The first result we prove in this section is not directly related to homogenization, and is general enough to be applied to a variety of asymptotic problems. We give more details on such examples, including the ones stated in the introduction, at the end of this section.
For an initial datum \( u_0 \in UC(\mathbb{R}^d) \), paths \((\zeta^\varepsilon)_{\varepsilon \geq 0} : [0, \infty) \times \Omega \to \mathbb{R}\) and Hamiltonians \((H^\varepsilon)_{\varepsilon \geq 0} : \mathbb{R}^d \times \mathbb{R}^d \times \Omega \to \mathbb{R}\), we consider, for \( \varepsilon > 0 \), the problems

\[
du^\varepsilon + H^\varepsilon(Du^\varepsilon, x, \omega) \cdot d\zeta^\varepsilon(t, \omega) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad u^\varepsilon(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d
\]

and

\[
du^0 + H^0(Du^0, x, \omega) \cdot d\zeta^0(t, \omega) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad u^0(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d.
\]

Let \((S^\varepsilon_\pm(t))_{\varepsilon, t \geq 0} : (B)UC(\mathbb{R}^d) \to (B)UC(\mathbb{R}^d)\) denote the solution operators for

\[
U^\varepsilon_\pm(t) \pm H^\varepsilon(DU^\varepsilon_\pm, x, \omega) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty), \quad U^\varepsilon_\pm(\cdot, 0) = \phi \quad \text{in } \mathbb{R}^d,
\]

that is, \(U^\varepsilon_\pm(x, t) = S^\varepsilon_\pm(t)\phi(x)\) for \( \varepsilon \geq 0 \) and \((x, t) \in \mathbb{R}^d \times [0, \infty)\).

We assume that there exists \( \Omega_0 \in \mathcal{F} \) such that \( \mathbb{P}(\Omega_0) = 1 \) and the following hold:

\[
\zeta^\varepsilon(\cdot, \omega) \quad \text{is continuous for all } \varepsilon \geq 0 \text{ and } \omega \in \Omega_0, \text{ and,}
\]

\[
\text{as } \varepsilon \to 0, \quad \zeta^\varepsilon \to \zeta^0 \text{ locally uniformly in distribution;}
\]

and

\[
\text{there exist } \nu, \tau : [0, \infty) \to [0, \infty) \text{ as in (2.9) such that, for all } \varepsilon \geq 0 \text{ and } \omega \in \Omega_0, \quad (H^\varepsilon(\cdot, \cdot, \omega))_{\varepsilon \geq 0} \text{ satisfies (2.10), and, for all } L, T, \delta > 0,
\]

\[
\lim_{\varepsilon \to 0} \mathbb{P} \left( \sup_{\|D\phi\|_\infty \leq L} \max_{t \in [0, T]} \left| S^\varepsilon_\pm(t)\phi(x) - S^0_\pm(t)\phi(x) \right| > \delta \right) = 0.
\]

Because \( H^\varepsilon(\cdot, \cdot, \omega) \) satisfies (2.10) for all \( \varepsilon \geq 0 \) and \( \omega \in \Omega_0 \), it follows that the equations (3.1) and (3.2) admit unique pathwise viscosity solutions by extending the solution operator to continuous paths.

**Theorem 3.1.** Assume (3.3) and (3.4), and let \( u_0 \in UC(\mathbb{R}^d) \). Then, as \( \varepsilon \to 0 \), \((u^\varepsilon, \zeta^\varepsilon)\) converges locally uniformly and in distribution to \((u^0, \zeta^0)\).

The key idea in the proof of Theorem 3.1 is to compare with solutions of intermediate equations driven by more regular paths. The stability estimates of Theorem 2.2 allow for this strategy to be effectively carried out.

Throughout the proofs below, we consider paths \( \eta \) that satisfy

\[
\eta : [0, \infty) \to \mathbb{R} \text{ is piecewise-C}^1 \text{ and, for any } T > 0,
\]

\[\dot{\eta} \text{ changes sign finitely many times on } [0, T].\]

Recall that the metric \( d_\lambda \) below, defined in subsection 2.4, metrizes the space \( C(\mathbb{R}^d \times [0, \infty)) \) with the topology of local uniform convergence.

**Lemma 3.1.** Assume that \( v_0 \in UC(\mathbb{R}^d), \eta : [0, \infty) \times \Omega \to \mathbb{R} \) is such that \( \eta(\cdot, \omega) \) satisfies (3.5) for all \( \omega \in \Omega_0, \) and \((H^\varepsilon)_{\varepsilon \geq 0} \) satisfies (3.3), and let \( v^\varepsilon \) and \( v^0 \) solve

\[
\begin{cases}
\nu^\varepsilon + H^\varepsilon(Dv^\varepsilon, x, \omega)\dot{\eta}(t, \omega) = 0 & \text{in } \mathbb{R}^d \times (0, \infty), \\
v^0 + H(Dv^0, x, \omega)\dot{\eta}(t, \omega) = 0 & \text{in } \mathbb{R}^d \times (0, \infty),
\end{cases}
\]

and

\[v^\varepsilon(\cdot, 0) = v^0(\cdot, 0) = v_0 \quad \text{in } \mathbb{R}^d.
\]

Then, for all \( \delta > 0, \)

\[
\lim_{\varepsilon \to 0} \mathbb{P} \left( d_\lambda(v^\varepsilon, v^0) > \delta \right) = 0.
\]
It is necessary to use the following well-known domain-of-dependence result for viscosity solutions of Hamilton-Jacobi equations. For a proof, see the book of Lions [20].

**Lemma 3.2.** Suppose that $G : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is continuous, let $U$ and $V$ be respectively a sub- and super-solution of

$$U_t = G(DU, x) \quad \text{and} \quad V_t = G(DV, x) \quad \text{in} \quad \mathbb{R}^d \times (-\infty, \infty)$$

such that $\max(\|DU\|_\infty, \|DV\|_\infty) \leq L$, and suppose that

$$\mathcal{L} := \sup_{(p, x) \in B_L \times \mathbb{R}^d} |D_p G(p, x)| < \infty.$$

Then, for all $R > 0$ and $-\infty < s < t < \infty$,

$$\max_{x \in B_R - \mathcal{L}(t-s)} (U(x, t) - V(x, t)) \leq \max_{x \in B_R} (U(x, s) - V(x, s)).$$

The strategy for the proof of Lemma 3.1 is similar to one used by the author in [30]. However, the argument is more complicated here, due to the fact that the Hamiltonians and the path are random.

**Proof of Lemma 3.1.** Observe first that, in view of the contractive property of the equations in (3.6), it suffices to prove the result for $v_0 \in C^{0,1}(\mathbb{R}^d)$ with $\|Dv_0\|_\infty \leq L$ for some fixed $L > 0$. Also, it is enough to prove, for any fixed $\delta > 0$ and $T > 0$, that

$$\lim_{\varepsilon \to 0} \mathbb{P} \left( \max_{(x, t) \in B_T \times [0, T]} |v^\varepsilon(x, t) - v^0(x, t)| > \delta \right) = 0.$$

Fix $\omega \in \Omega_0$, so that there exists a partition $\{0 = t_0 < t_1 < t_2 < \cdots < t_N = T\}$ such that $\eta_\omega$ is monotone on each interval $[t_i, t_{i+1}]$. Fix $(x, t) \in B_T \times [0, T]$, let $i$ be such that $t \in (t_i, t_{i+1})$, and assume without loss of generality that $\eta_\omega$ is decreasing on $[t_i, t_{i+1}]$.

Set $\Delta := \eta_{i+1} - \eta_{i}$. Because $\eta$ is monotone on $[t_i, t_{i+1}]$,

$$v^\varepsilon(\cdot, t) = S^\varepsilon_+(\Delta) v^\varepsilon(\cdot, t_i) \quad \text{and} \quad v^0(\cdot, t) = S^0_+(\Delta) v^0(\cdot, t_i).$$

We then write

$$v^\varepsilon(x, t) - v^0(x, t) = \left( S^\varepsilon_+(\Delta) v^\varepsilon(\cdot, t_i)(x) - S^0_+(\Delta) v^0(\cdot, t_i)(x) \right) + \left( S^\varepsilon_+(\Delta) v^0(\cdot, t_i)(x) - S^0_+(\Delta) v^0(\cdot, t_i)(x) \right).$$

In view of Theorem 2.2, there exists a deterministic constant $C_1 > 0$ depending only on $L$ such that

$$\max(\|Dv^\varepsilon\|_\infty, \|Dv^0\|_\infty) \leq C_1.$$

The convexity and uniform growth of $H^0$ in the gradient variable then imply that, for some deterministic constant $C_2 > 0$ depending only on $L$,

$$\sup_{\varepsilon > 0} \sup_{|p| \leq C_1} |D_p H^\varepsilon(p, x, \omega)| \leq C_2.$$

Lemma 3.2 then implies that, for all $x \in B_T$,

$$|S^\varepsilon_+(\Delta) v^\varepsilon(\cdot, t_i)(x) - S^0_+(\Delta) v^0(\cdot, t_i)(x)| \leq \max_{y \in B_T + C_2 \Delta} |v^\varepsilon(y, t_i) - v^0(y, t_i)|,$$

and so

$$|v^\varepsilon(x, t) - v^0(x, t)| \leq \sum_{i=0}^{N-1} \max_{(y, \tau) \in [0, \Delta_i]} \left| S^\varepsilon_+(\tau) v^0(\cdot, t_i)(y) - S^0_+(\tau) v^0(\cdot, t_i)(y) \right|,$$

where

$$\Delta_i := |\eta(t_{i+1}) - \eta(t_i)| \quad \text{and} \quad R_i := T + C_2 \sum_{k=i}^{N-1} \Delta_k,$$
and the subscripts + and − for the solution operators in (3.7) are chosen depending on whether $\eta$ is respectively decreasing or increasing on $[t_i, t_{i+1}]$.

For $M > 0$, define
\[
A_M := \left\{ \omega \in \Omega_0 : N(\omega) \leq M, \max_{i=0,1,2,\ldots,N-1} \Delta_i(\omega) \leq M, R_{N(\omega)-1}(\omega) \leq M \right\}.
\]

Then, for any $M > 0$,
\[
P \left( \max_{(x,t) \in B_T \times [0,T]} |v^\varepsilon(x,t) - v^0(x,t)| > \delta \right) = P \left( \Omega_0 \cap \left\{ \max_{(x,t) \in B_T \times [0,T]} |v^\varepsilon(x,t) - v^0(x,t)| > \delta \right\} \right)
\leq P \left( \Omega_0 \setminus A_M \right) + P \left( A_M \cap \left\{ \sum_{i=0}^{N-1} \max_{(y,\tau) \in B_{R_i} \times [0,\Delta_i]} |S^\varepsilon_\pm(\tau)v^0(y) - S^0_\pm(\tau)v^0(y)| > \delta \right\} \right)
\leq P \left( \Omega_0 \setminus A_M \right) + \sup_{\|D\phi\| \leq C_1} \left( A_M \cap \left\{ \sup_{\|D\phi\| \leq C_1} \max_{\|D\phi\| \leq C_1} |S^\varepsilon_\pm(\phi(x) - S^0_\pm(\phi(x)| > \frac{\delta}{M} \right\} \right),
\]
and so, in view of (3.24),
\[
\limsup_{\varepsilon \to 0} P \left( \max_{(x,t) \in B_T \times [0,T]} |v^\varepsilon(x,t) - v^0(x,t)| > \delta \right) \leq P \left( \Omega_0 \setminus A_M \right).
\]

Sending $M \to \infty$ yields the result.

Proof of Theorem 3.1. Appealing to Lemma 2.1 it suffices to show that, for any open set $U \subset C([0,\infty), \mathbb{R})$, \[
\liminf_{\varepsilon \to 0} P \left( (u^\varepsilon, \zeta^\varepsilon) \in U \right) \geq P \left( (u^0, \zeta^0) \in U \right).
\]
Recall that we metrize the space $C([0,\infty), \mathbb{R})$ with the metric $d := d_s + d_p$ defined in subsection 2.4. For $\sigma > 0$, define the open set
\[
U_\sigma := \{ (v, \eta) \in U : d((v, \eta), (w, \tau)) > \sigma \text{ for all } (w, \tau) \in U^\varepsilon \}.
\]
As in the proof of Lemma 3.1 it suffices to take $u_0 \in C^{0,1}(\mathbb{R}^d)$ with $\|D\phi\| \leq L$ for some fixed $L > 0$.

Fix $\delta > 0$, and let $\eta : [0,\infty) \times \Omega \to \mathbb{R}$ be such that, for all $\omega \in \Omega_0$, $\eta(\omega)$ satisfies (3.3) and $d_p(\zeta^0(\omega), \eta(\omega)) < \delta$.

For example, $\eta$ could be a piecewise linear interpolation of $\zeta^0$ over an appropriately defined (random) partition.

Let $v^\varepsilon$ and $v^0$ be as in the statement of Lemma 3.1 with the path $\eta$. Theorems 2.1 and 2.2 then yield a constant $C > 0$ depending only on $L$ such that, for all $\omega \in \Omega_0$,
\[
d_s(u^\varepsilon(\omega), v^\varepsilon(\omega)) \leq Cd_p(\zeta^\varepsilon(\omega), \eta(\omega)) \quad \text{and} \quad d_s(u^0(\omega), v^0(\omega)) \leq C\delta.
\]
Lemma 3.1 gives the existence of a deterministic $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0,\varepsilon_0)$, $P(\Omega^\varepsilon) \geq 1 - \delta$, where
\[
\Omega^\varepsilon := \{ \omega \in \Omega_0 : d_s(u^\varepsilon(\omega), v^0(\omega)) < \delta \}.
\]
Then, for all $\varepsilon \in (0,\varepsilon_0)$,
\[
\{(u^\varepsilon, \zeta^\varepsilon) \in U \} \cup (\Omega^\varepsilon)^C \supset \{(v^\varepsilon, \eta, \zeta^\varepsilon) \in U_{(C+1)\delta} \times \mathcal{B}_\delta(\eta)) \} \cup (\Omega^\varepsilon)^C
\supset \{(v^0, \eta, \zeta^\varepsilon) \in U_{(C+2)\delta} \times \mathcal{B}_\delta(\eta)) \} \cup (\Omega^\varepsilon)^C,
\]
where $\mathcal{B}_\delta(\eta) \subset C([0,\infty), \mathbb{R})$ denotes the open ball of radius $\delta$ centered at $\eta$ with respect to the metric $d_p$.

From this it follows that, for all $\varepsilon \in (0,\varepsilon_0)$,
\[
P \left( (u^\varepsilon, \zeta^\varepsilon) \in U \right) \geq P \left( (v^0, \eta, \zeta^\varepsilon) \in U_{(C+2)\delta} \times \mathcal{B}_\delta(\eta)) \right) - \delta.
\]
which, together with\((\ref{3.3})\), yields, after sending \(\varepsilon \to 0\),
\[
\liminf_{\varepsilon \to 0} \mathbb{P}\left( (u^\varepsilon, \zeta^\varepsilon) \in \mathcal{U} \right) \geq \mathbb{P}\left( (u^0, \eta, \zeta) \in \mathcal{U}(C+2\delta) \times B_\delta(\eta) \right) - \delta \geq \mathbb{P}\left( (u^0, \zeta) \in \mathcal{U}(2C+3\delta) \right) - \delta.
\]
The result now follows upon sending \(\delta \to 0\).

\section{Applications of Theorem 3.1} \label{3.2}

The assumptions needed for Theorem 3.1, and in particular, those for the Hamiltonians \(H^\varepsilon\), are general enough to apply to a multitude of settings. For instance, the dependence of \(H^\varepsilon\) on \(x/\varepsilon\) can be periodic, almost periodic, or stationary ergodic. There can also be dependence on multiple scales \(x/\varepsilon^\alpha, x/\varepsilon^\beta\), etc. All that is needed is \((\ref{3.4})\), that is, convergence to some \(H^0\) in the solution-operator sense. Here, to have a simplified presentation, we discuss only the periodic and random settings, with \(H^\varepsilon\) given as a function of \(x/\varepsilon\) and possibly \(x\).

We first prove the result from the introduction concerning the initial value problem
\[
(3.8) \quad u^\varepsilon_t + \frac{1}{\varepsilon^\gamma} H\left(Du^\varepsilon, \frac{x}{\varepsilon}, \omega \right) \xi\left( \frac{t}{\varepsilon^2 \gamma}, \omega \right) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad u^\varepsilon(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d
\]
for a fixed \(\gamma > 0\) and \(u_0 \in UC(\mathbb{R}^d)\), a white noise approximation \(\xi : [0, \infty) \times \Omega \to \mathbb{R}\) in the sense of \((2.9)\), and a Hamiltonian \(H : \mathbb{R}^d \times \mathbb{R}^d \times \Omega \to \mathbb{R}\) for which
\[
(3.9) \begin{align*}
\text{there exists } \Omega_0 \in \mathcal{F} \text{ with } \mathbb{P}(\Omega_0) = 1 \text{ and deterministic } \nu, \tau : [0, \infty) \to [0, \infty) \text{ as in } (2.9) \\
\text{such that } H(\cdot, \cdot, \omega) \text{ satisfies } (2.9) \text{ uniformly over } \omega \in \Omega_0, \text{ and} \\
\text{either } y \to H(\cdot, y) \text{ is deterministic and periodic, or} \\
(y, \omega) \mapsto H(y, \omega) \text{ is a stationary-ergodic random field.}
\end{align*}
\]
For \((t, \omega) \in [0, \infty) \times \Omega \to \mathbb{R}\), define
\[
(3.10) \quad \zeta^\varepsilon(t, \omega) := \varepsilon^\gamma \int_0^t \xi(s, \omega) \, ds = \frac{1}{\varepsilon^\gamma} \int_0^t \xi\left( \frac{s}{\varepsilon^2 \gamma}, \omega \right) \, ds.
\]

\textbf{Theorem 3.2.} Let \(\gamma > 0\) and \(u_0 \in UC(\mathbb{R}^d)\) and assume that \(\xi\) and \(H\) satisfy respectively \((2.9)\) and \((3.9)\). Then there exist a deterministic, convex \(\overline{H} : \mathbb{R}^d \to \mathbb{R}\) satisfying \((2.9)\), which depends only on \(H\), and a Brownian motion \(B : [0, \infty) \times \Omega \to \mathbb{R}\) such that, as \(\varepsilon \to 0\), \((u^\varepsilon, \zeta^\varepsilon)\) converges locally uniformly and in distribution to \((\overline{\nu}, B)\), where \(\overline{\nu}\) is the unique stochastic viscosity solution of
\[
(3.11) \quad d\overline{\nu} + \overline{H}(D\overline{\nu}) \circ dB = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad \overline{\nu}(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d.
\]

Note that equation \((3.11)\) is well-posed in the stochastic viscosity sense, by merit of Theorem 2.1.

The theorem is a direct consequence of Theorem 3.1 with \(\zeta^\varepsilon\) defined as in \((3.10)\) for \(\varepsilon > 0\), \(\zeta^0 = B\),
\[
H^\varepsilon(p, x, \omega) := H\left( p, \frac{x}{\varepsilon}, \omega \right) \quad \text{for } \varepsilon > 0 \text{ and } (p, x, \omega) \in \mathbb{R}^d \times \mathbb{R}^d \times \Omega,
\]
and
\[
H^0(p, x) = \overline{H}(p)
\]
for \((p, x) \in \mathbb{R}^d \times \mathbb{R}^d\), where \(\overline{H}\) is the deterministic, convex, effective Hamiltonian in either the periodic or random homogenization settings discussed in subsection 2.3. The convergence in distribution of \(\zeta^\varepsilon\) to the Brownian motion follows from \((2.1)\), and \(H\) satisfies \((3.4)\) in either the periodic or random homogenization settings in view of Corollary 2.1.

We next consider equations with true white noise in time, that is,
\[
(3.12) \quad du^\varepsilon + \left( Du^\varepsilon, \frac{x}{\varepsilon}, \omega \right) \circ dB = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad u^\varepsilon(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d,
\]
where \(B : \Omega \times [0, \infty) \to \mathbb{R}\) is a standard Brownian motion.
Theorem 3.3. Under the same hypotheses of Theorem 3.2 as \( \varepsilon \to 0 \), the solution \( u^\varepsilon \) of (3.12) converges locally uniformly and in distribution to the solution of (3.11).

The result follows from Theorem 3.1 taking \( (H^\varepsilon)_{\varepsilon \geq 0} \) as before and \( \zeta^\varepsilon = B \) for all \( \varepsilon \geq 0 \).

We now mention some results concerning the initial value problems

\[
\begin{align*}
(3.13) & \quad u_t^\varepsilon + \frac{1}{\varepsilon} H \left( Du^\varepsilon, \frac{x}{\varepsilon}, x, \omega \right) \xi \left( \frac{t}{\varepsilon^2}, \omega \right) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad u^\varepsilon(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d \quad \text{and} \\
(3.14) & \quad du^\varepsilon + H \left( Du^\varepsilon, \frac{x}{\varepsilon}, x, \omega \right) \circ dB = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad u^\varepsilon(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d.
\end{align*}
\]

The following theorem is a consequence of Theorem 3.1 and the remarks at the end of subsection 2.3.

Theorem 3.4. Assume that \( \gamma > 0 \), \( u_0 \in UC(\mathbb{R}^d) \), \( B : [0, \infty) \times \Omega \to \mathbb{R} \) is a Brownian motion, \( H \) is uniformly continuous in \( B_R \times \mathbb{R}^d \times \mathbb{R}^d \) for each \( R > 0 \), and there exist \( \Omega_0 \in \mathcal{F} \) and \( \nu, \pi \) as in (3.9) such that, for each fixed \( x \in \mathbb{R}^d \), \( H(\cdot, \cdot, x) \) satisfies (3.9). Then there exists a deterministic \( \overline{\pi} \in C(\mathbb{R}^d \times \mathbb{R}^d) \) satisfying (2.9) such that the following hold:

(a) For any \( \xi : [0, \infty) \times \Omega \to \mathbb{R} \) satisfying (2.1), if \( u^\varepsilon \) is the solution of (3.13) and \( \zeta^\varepsilon \) is as in (3.11), then, as \( \varepsilon \to 0 \), \( (u^\varepsilon, \zeta^\varepsilon) \) converges locally uniformly and in distribution to \( (\pi, B) \), where \( \pi \) is the stochastic viscosity solution \( \overline{\pi} \) of

\[
(3.15) \quad d\overline{\pi} + \overline{D\pi}(D\pi, x) \circ dB = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad \overline{\pi}(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d.
\]

(b) As \( \varepsilon \to 0 \), the solution \( u^\varepsilon \) of (3.14) converges locally uniformly and in distribution to \( \pi \).

We remark that, in all of the results in this subsection, the fact that the effective Hamiltonian satisfies the bounds in (2.9) is a consequence of (2.11).

We conclude this subsection by explaining how the above results can be applied to equations of level-set type. Indeed, if, for some \( a : \Omega \to C(S^{d-1} \times \mathbb{R}^d \times \mathbb{R}^d) \),

\[
(3.16) \quad H(p, y, x, \omega) = a \left( \frac{p}{|p|}, y, x, \omega \right) |p|,
\]

then (3.13) and (3.14) become level-set equations for certain first-order interfacial motions. For some \( \overline{\pi} \in C(S^{d-1} \times \mathbb{R}^d) \), the effective Hamiltonian then has the form

\[
\overline{H}(p, x) := a \left( \frac{p}{|p|}, x \right) |p| \quad \text{for } p \in \mathbb{R}^d.
\]

The Hamiltonian (3.16) satisfies (3.9) if \( a \) is essentially bounded from above and below uniformly in \( S^{d-1} \times \mathbb{R}^d \times \mathbb{R}^d \times \Omega \), and if

\[
p \mapsto a \left( \frac{p}{|p|}, \cdot, \cdot \right) |p|
\]

is convex.

4. THE MULTIPLE-NOISE CASE

We now turn to the study of the initial value problem

\[
(4.1) \quad u_t^\varepsilon + \frac{1}{\varepsilon} \sum_{i=1}^m H_i \left( Du^\varepsilon, \frac{x}{\varepsilon} \right) \xi_i \left( \frac{t}{\varepsilon^2}, \omega \right) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad u^\varepsilon(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d.
\]
Throughout this section, we will assume that each Hamiltonian is deterministic and periodic in space, and that, for each \(i = 1, 2, \ldots, m\), \(\xi^i\) is a discrete mixing field satisfying (4.1), that is,

\[
\begin{aligned}
\xi^i(t, \omega) &= \sum_{k=1}^{\infty} X^i_k(\omega) 1_{[k-1, k)}(t) \quad \text{for } (t, \omega) \in [0, \infty) \times \Omega, \\
\text{where } (X^i_k)_{k=1}^{\infty} : \Omega \to \mathbb{R} \text{ are independent and identically distributed with } \\
\mathbb{E}[X^i_k] &= 0 \text{ and } \mathbb{E}[(X^i_k)^2] = 1.
\end{aligned}
\]

As in (3.10), we set, for each \(i = 1, 2, \ldots, m\),

\[
\zeta^i_t(\omega, \omega) := \frac{1}{\varepsilon^\gamma} \int_0^{t/\varepsilon^{2\gamma}} \xi^i(s, \omega) \, ds \quad \text{for } (t, \omega) \in [0, \infty) \times \Omega,
\]

so that, in view of Donsker’s invariance principle, for some Brownian motion \(B^i : [0, \infty) \times \Omega \to \mathbb{R}\),

\[
\zeta^i \xrightarrow{\varepsilon \to 0} B^i \quad \text{in } C([0, \infty), \mathbb{R}) \text{ in distribution.}
\]

4.1. **Difficulties.** We begin with a discussion of the general strategy of proof in the multiple noise setting, and the challenges that arise.

We first make the formal assumption, one which we later justify by choosing \(\gamma\) sufficiently small (see Lemma 4.3 below), that \(u^\varepsilon\) is closely approximated by a solution \(\overline{u}^\gamma\) of an equation of the form

\[
u^\gamma + \frac{1}{\varepsilon^\gamma} \overline{\nabla} \left( D\overline{u}^\gamma, \xi \left( \frac{t}{\varepsilon^{2\gamma}}, \omega \right) \right) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad \overline{u}^\gamma(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d,
\]

via the expansion

\[
u^\gamma(x, t) \approx \overline{u}^\gamma(x, t) + \varepsilon v(x/\varepsilon, t) + \cdots
\]

for some \(v : \mathbb{T}^d \times [0, \infty) \to \mathbb{R}\). This yields to the following equation for \(v\), for fixed \(p \in \mathbb{R}^d\) and \(\xi \in \mathbb{R}^m\):

\[
\sum_{i=1}^{m} H^i(D\nu^\gamma + p, y) \xi^i = \overline{\nabla}(p, \xi) \quad \text{in } \mathbb{R}^d.
\]

The fixed parameters \(p\) and \(\xi\) stand in place of respectively the gradient \(D\overline{u}^\gamma(x, t)\) and the mild white noise \(\varepsilon^{-\gamma}(t/\varepsilon^{2\gamma})\).

Note that, in deriving (4.3), we have assumed that \(\xi \mapsto \overline{\nabla}(\cdot, \xi)\) is positively homogenous. Later, we justify this by the fact that, under sufficient conditions on the \(H^i\), (4.3) admits periodic solutions for a unique choice of constant \(\overline{\nabla}(p, \xi)\) on the right hand side. The positive homogeneity can then be seen from multiplying both sides of (4.3) by a positive constant.

If \(u_0(x) = p_0 \cdot x\) for some fixed \(p_0 \in \mathbb{R}\), then the solution of (4.3) is given by

\[
\overline{u}^\gamma(x, t) = p_0 \cdot x - \frac{1}{\varepsilon^\gamma} \int_0^t \overline{\nabla} \left( p_0, \xi \left( \frac{s}{\varepsilon^{2\gamma}} \right) \right) \, ds.
\]

Therefore, it can be proved that

\[
\mathbb{E} \left[ \overline{\nabla}(p_0, X^1_0, X^2_0, \ldots, X^m_0) \right] = 0,
\]

it then follows that \(\overline{u}^\gamma\) converges locally uniformly in distribution, as \(\varepsilon \to 0\), to \(p_0 \cdot x + \sigma(p_0)B(t)\), where \(B\) is a standard Brownian motion and

\[
\sigma(p_0)^2 := \mathbb{E} \left[ \overline{\nabla}(p_0, X^1_0, X^2_0, \ldots, X^m_0)^2 \right].
\]

However, the nonlinear nature of the problem makes it difficult to describe the limit of \(\overline{u}^\gamma\) as \(\varepsilon \to 0\) for general initial data \(u_0 \in UC(\mathbb{R}^d)\). This distinguishes the problem from those studied in [7, 19, 40], where the equations are uniformly parabolic and semilinear.
A further complication arises from the fact that, for two \( \mathbb{R}^m \)-valued random variables \( X_0 \) and \( \tilde{X}_0 \) as in (4.2), the identity

\[
(4.7) \quad \mathbb{E} \left[ H(p, X_0)^2 \right] = \mathbb{E} \left[ H(p, \tilde{X}_0)^2 \right] \quad \text{for all } p \in \mathbb{R}^d
\]

may fail in general, which indicates that the law of the field \( \xi \) in equation (4.1) can have a nontrivial effect on the limiting equation.

As shown above, if (4.7) does hold, then, whenever the initial data has the form \( u_0(x) = p \cdot x \) for some \( p \in \mathbb{R}^d \), the laws of the limiting functions depend only on \( p \), and not on the laws of \( X_0 \) and \( \tilde{X}_0 \). However, it could still be the case that the laws of the limiting functions differ for more general initial data.

As an indication of why this is true, consider, for \( u_0 \in UC(\mathbb{R}) \) and two Brownian motions \( B, \tilde{B} : [0, \infty) \times \Omega \to \mathbb{R} \), the initial value problems

\[
\begin{align*}
du - u_x \circ dB &= 0, \quad d\tilde{u} - |\tilde{u}_x| \circ d\tilde{B} = 0 \quad \text{in } \mathbb{R} \times (0, \infty), \quad \text{and} \\
u(x, 0) &= \tilde{u}(x, 0) = u_0 \quad \text{in } \mathbb{R}.
\end{align*}
\]

If \( u_0(x) = px \) for some fixed \( p \in \mathbb{R} \), then the solutions

\[
u(x, t) = px + pB(t) \quad \text{and} \quad \tilde{u}(x, t) = px + |p|\tilde{B}(t)
\]

have the same law as \( C(\mathbb{R} \times [0, \infty)) \)-valued random variables. However, if \( u_0(x) = |x| \), then a simple calculation yields that

\[
u(x, t) = |x + B(t)|,
\]

while it is shown in [26, 24, 58] that

\[
\tilde{u}(x, t) = \max \left\{ |x| + \tilde{B}(t), \max_{0 \leq s \leq t} \tilde{B}(s) \right\}.
\]

4.2. A general class of examples. We now present a class of Hamiltonians and white noise approximations for which, given any initial data \( u_0 \in UC(\mathbb{R}^d) \), the limit as \( \varepsilon \to 0 \) of the solution \( u^\varepsilon \) of (4.1) can be identified as the unique stochastic viscosity solution of a certain initial value problem.

We assume that the Hamiltonians satisfy

\[
(4.8) \quad \begin{cases} 
H^i \in C^{0,1}(\mathbb{R}^d \times \mathbb{T}^d), \quad \text{and, for each } \xi \in \{-1, 1\}^m, \\
p \mapsto \sum_{i=1}^m H^i(p, \cdot) \xi^i \text{ is either convex or concave and} \\
\lim_{|p| \to +\infty} \inf_{y \in \mathbb{T}^d} \sum_{i=1}^m H^i(p, y) \xi^i = +\infty \quad \text{or} \quad \lim_{|p| \to +\infty} \sup_{y \in \mathbb{T}^d} \sum_{i=1}^m H^i(p, y) \xi^i = -\infty.
\end{cases}
\]

As a consequence of Theorem 2.3, the cell problem (4.5) is solvable for all \( p \in \mathbb{R}^d \) and \( \xi \in \{-1, 1\}^m \), \( p \mapsto \overline{H}(p, \cdot) \) is either convex or concave, and

\[
(4.9) \quad \overline{H}(\cdot, \lambda \xi) = \lambda \overline{H}(\cdot, \xi) \quad \text{for all } \lambda \in \mathbb{R} \text{ and } \xi \in \{-1, 1\}^m.
\]

The mixing fields are assumed to satisfy, for \( i = 1, \ldots, m \),

\[
(4.10) \quad \begin{cases} 
\xi^i(t, \omega) = \sum_{k=0}^{\infty} X^i_k(\omega) 1_{(k, k+1)}(t)(t) \quad \text{for } (t, \omega) \in [0, \infty) \times \Omega, \quad \text{where} \\
(X^i_k)_{i=1,2,\ldots,m, \ k=0,1,\ldots} \text{ are independent Rademacher random variables.}
\end{cases}
\]
Define
\[
A^m := \{ j = (j_1, j_2, \ldots, j_l) : j_i \in \{1, 2, \ldots, m\}, j_1 < j_2 < \cdots < j_l \},
\]
\[
|j| = |(j_1, j_2, \ldots, j_l)| := l, \quad \text{and}
\]
\[
A^m_o := \{ j \in A^m : |j| \text{ is odd} \},
\]
and note that \#\(A^m = 2^m - 1\) and \#\(A^m_o = 2^{m-1}\).

For each \(j = (j_1, j_2, \ldots, j_l) \in A^m\), set
\[
\begin{cases}
\xi^j := \xi^{j_1} \xi^{j_2} \cdots \xi^{j_l} & \text{for} \: \xi = (\xi^1, \xi^2, \ldots, \xi^m) \in \{-1, 1\}^m, \\
\overline{H}^j(p) := \frac{1}{2m} \sum_{\xi \in \{-1, 1\}^m} \overline{H}(p, \xi) \xi^j & \text{for} \: p \in \mathbb{R}^d,
\end{cases}
\]
\[
\begin{align*}
X_k^j(\omega) &:= X_k^{j_1}(\omega)X_k^{j_2}(\omega) \cdots X_k^{j_l}(\omega), \\
\zeta^j(0, \omega) &:= 0, \quad \zeta^j(t, \omega) := \sum_{k=0}^{\infty} X_k^j(\omega) 1_{(k,k+1)}(t), \quad \text{and} \\
\zeta^j(\cdot, \omega) &:= \varepsilon^j \zeta^j(t/\varepsilon^{2\gamma}, \omega) \quad \text{for} \: (t, \omega) \in [0, \infty) \times \Omega.
\end{align*}
\]
(4.11)

Observe that, for each \(j \in A^m_o\), \(\overline{H}^j\) is a difference of convex functions, and that the homogeneity property (4.9) implies that \(\overline{H}^j = 0\) whenever \(|j|\) is even.

**Theorem 4.1**. Assume that \(0 < \gamma < 1/6\), \(u_0 \in UC(\mathbb{R}^d), (4.8), \) and (4.10), and let \(u^\varepsilon\) be the solution of (4.11). Then there exist \(2^{m-1}\) independent Brownian motions \(\{B^j\}_{j \in A^m_o}\), such that, in distribution,
\[
(u^\varepsilon, (\zeta^j)_{j \in A^m_o}) \xrightarrow{\varepsilon \to 0} (\overline{\mu}, (B^j)_{j \in A^m_o}) \quad \text{in} \: C(\mathbb{R}^d \times [0, \infty)) \times C\left([0, \infty), \mathbb{R}^{2^{m-1}}\right),
\]
where \(\overline{\mu}\) is the unique stochastic viscosity solution of
\[
\frac{d\overline{\mu}}{dt} + \sum_{j \in A^m_o} \overline{H}^j(D\overline{\mu}) \circ dB^j = 0 \quad \text{in} \: \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad \overline{\mu}(\cdot, 0) = u_0 \quad \text{in} \: \mathbb{R}^d.
\]
(4.12)

If \(d = 1\), or if \(d = 2\) and \(p \mapsto \overline{H}(p, \cdot)\) is homogenous of degree \(q\) for some \(q \geq 1\), then the result holds for \(0 < \gamma < 1/2\).

The result relies on the fact that the \(\xi^j\) take their values in \(\{-1, 1\}\), and functions defined on \(\{-1, 1\}^m\) take a very particular form.

**Lemma 4.1.** Let \(f : \{-1, 1\}^m \to \mathbb{R}\). Then
\[
f(\xi) = f_0 + \sum_{j \in A^m} f_j \xi^j,
\]
(4.13)
where
\[
f_0 := \frac{1}{2m} \sum_{\xi \in \{-1, 1\}^m} f(\xi) \quad \text{and} \quad f_j := \frac{1}{2m} \sum_{\xi \in \{-1, 1\}^m} f(\xi) \xi^j.
\]
If \(f\) is odd, then \(f_0 = 0\), and the sum in (4.13) is taken over \(j \in A^m_o\).

**Proof.** Let \(F^m\) be the \(2^m\)-dimensional space of real-valued functions on \(\{-1, 1\}^m\). The \(2^m\) functions in the collection \(P^m := \{1, (\xi^j)_{j \in A^m_o}\}\) are linearly independent elements of \(F^m\), and therefore, their span is equal to it.
For \( f, g \in \mathcal{F}^m \), define the inner product
\[
\langle f, g \rangle_{\mathcal{F}^m} := \frac{1}{2^m} \sum_{\xi \in \{-1, 1\}^m} f(\xi)g(\xi).
\]
With respect to \( \langle \cdot, \cdot \rangle_{\mathcal{F}^m} \), \( \mathcal{P}^m \) becomes an orthonormal basis, so that, for any \( f \in \mathcal{F}^m \),
\[
f = \sum_{q \in \mathcal{P}^m} \langle f, q \rangle_{\mathcal{F}^m} q,
\]
which is the desired formula. The statements about odd \( f \) now follow easily. \( \square \)

As a consequence of Lemma 4.1, the effective Hamiltonian \( \overline{H} : \mathbb{R}^d \times \{-1, 1\}^m \) in (4.5) takes the form
\[
\overline{H}(p, \xi) := \sum_{j \in A^m} \overline{H}^j(p)\xi^j,
\]
where the functions \((H^j)_{j \in A^m}\) are defined as in (4.11).

The proof of the following lemma is elementary and thus omitted.

**Lemma 4.2.** Let \( \{X_j\}_{j=1}^m \) be mutually independent and Rademacher. Then the random variables defined by
\[
X^j := X_{j_1}X_{j_2} \cdots X_{j_l} \quad \text{for} \quad j = (j_1, j_2, \ldots, j_l) \in A^m
\]
are pairwise independent and Rademacher.

Now let \( \overline{\sigma}^\varepsilon \) be the viscosity solution of the equation
\[
\left(4.14\right) \quad \overline{\sigma}^\varepsilon_t + \sum_{j \in A^m} \overline{H}^j(D\overline{\sigma}^\varepsilon)\zeta^j_{\varepsilon\gamma}(t, \omega) = 0 \quad \text{in} \quad \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad \overline{\sigma}^\varepsilon(\cdot, 0) = u_0 \quad \text{in} \quad \mathbb{R}^d,
\]
where the \( \overline{H}^j \)'s and \( \zeta^j \)'s are as in (4.11).

**Lemma 4.3.** Assume (4.8) and (4.10), and let \( u^\varepsilon \) and \( \overline{\sigma}^\varepsilon \) be the solutions of respectively (4.1) and (4.14). Then there exists \( C = C_L > 0 \) such that, with probability one, whenever \( \|Du_0\|_\infty \leq L, \varepsilon > 0 \) and \( T > 0 \),
\[
\sup_{(x,t)\in\mathbb{R}^d\times[0,T]} |u^\varepsilon(x,t) - \overline{\sigma}^\varepsilon(x,t)| \leq C(1 + T)^{1/3 - 2\gamma}.
\]
If \( d = 1 \), or if \( d = 2 \) and \( p \mapsto \overline{H}(p, \cdot) \) is homogenous of degree \( q \) for some \( q \geq 1 \), then the exponent can be replaced with \( 1 - 2\gamma \).

We do not give the full details of the proof of Lemma 4.3 as it is a simpler version of Lemma 3.1 (see also Lemma 5.2 from [36]).

The argument follows by applying the rate of convergence from the periodic homogenization of Hamilton-Jacobi equations in Theorem 2.4 on each of the \( O(1/\varepsilon^{2\gamma}) \) intervals on which \( \xi^\varepsilon(t) \) is constant. The effective equation on each of those intervals is given by
\[
\overline{\sigma}^\varepsilon_t + \overline{H}(D\overline{\sigma}^\varepsilon, \varepsilon^{-\gamma}\xi(t/\varepsilon^{2\gamma})) = 0,
\]
which is exactly equation (4.14).

**Proof of Theorem 4.1.** Because the solution operators are contractive in the initial data, it suffices to assume that \( u_0 \in C^{0,1}(\mathbb{R}^d) \).

The choice of \( \gamma \) and Lemma 4.3 imply that, with probability one,
\[
\lim_{\varepsilon \to 0} d_s (u^\varepsilon, \overline{\sigma}^\varepsilon) = 0,
\]
where $d_\ast$ is the metric on $C(\mathbb{R}^d \times [0,\infty))$ defined in subsection 2.4.

In view of Lemma 4.2, the path
\[
\zeta^\varepsilon := (\zeta^j)^{j \in A^m_\ast} \in C \left([0,\infty), \mathbb{R}^{2^m-1}\right)
\]
is a random walk which, as $\varepsilon \to 0$, converges in distribution to a $2^{m-1}$-dimensional Brownian motion $B := (B_l)^{l \in A^m_\ast}$.

For the fixed initial datum $u_0 \in C^{0,1}(\mathbb{R}^d)$, let
\[
S : C \left([0,\infty), \mathbb{R}^{2^m-1}\right) \ni \zeta \mapsto v \in C(\mathbb{R}^d \times [0,\infty))
\]
be the solution operator for the equation
\[
dv + \sum_{j \in A^m_\ast} H^j(Dv) \cdot d\zeta^j = 0 \quad \text{in } \mathbb{R}^d \times (0,\infty) \quad \text{and} \quad v(\cdot,0) = u_0 \quad \text{in } \mathbb{R}^d.
\]
The stability result in Theorem 2.1 implies that $S$ is continuous, and, therefore, so is the graph map
\[
(S, \text{Id}) : C \left([0,\infty), \mathbb{R}^{2^m-1}\right) \ni \zeta \mapsto (v,\zeta) \in C \left([0,\infty), \mathbb{R}^{2^m-1}\right) \times C(\mathbb{R}^d \times [0,\infty)).
\]
It follows from the Mapping Theorem (Lemma 2.2) that, if $0 < \gamma < 1/2$, then, as $\varepsilon \to 0$, $(\bar{\pi},\bar{\zeta}^\varepsilon)$ converges in distribution to $(\bar{\pi},B)$ in $C(\mathbb{R}^d \times [0,\infty)) \times C \left([0,T], \mathbb{R}^{2^m-1}\right)$. The result now follows from Slutsky’s Theorem (Lemma 2.3).

### 4.3. A one-dimensional example.

For $u_0 \in C^{0,1}(\mathbb{R}), \xi^1, \xi^2 : [0,\infty) \times \Omega \to \mathbb{R}$ as in (4.10), and $f \in C^{0,1}(\mathbb{T})$, consider the equation
\[
\begin{align*}
\dot{u}^\varepsilon_t + \frac{1}{\varepsilon^2} |u_\varepsilon^\varepsilon_0|^1 \left(\frac{t}{\varepsilon^2\gamma}, \omega\right) + \frac{1}{\varepsilon^2} f \left(\frac{x}{\varepsilon^2\gamma}, \omega\right) & = 0 \quad \text{in } \mathbb{R} \times (0,\infty) \quad \text{and} \quad u^\varepsilon(\cdot,0) = u_0 \quad \text{in } \mathbb{R}.
\end{align*}
\]

Theorem 1.1 implies that, if $0 < \gamma < 1/2$, then, as $\varepsilon \to 0$, $(u^\varepsilon,\xi^1,\xi^2,\zeta^\varepsilon)$ converges in distribution to $(\bar{\pi},B^1,B^2)$, where $\xi^1,\zeta^\varepsilon$ are as in (4.3), $B^1$ and $B^2$ are independent Brownian motions, and, for some $\bar{H}^1, \bar{H}^2 : \mathbb{R} \to \mathbb{R}$, $\bar{\pi}$ is the unique stochastic viscosity solution of
\[
\begin{align*}
d\bar{\pi} + \bar{H}^1(\bar{\pi}_x) \circ dB^1 + \bar{H}^2(\bar{\pi}_x) \circ dB^2 & = 0 \quad \text{in } \mathbb{R} \times (0,\infty) \quad \text{and} \quad \bar{\pi}(\cdot,0) = u_0 \quad \text{in } \mathbb{R}.
\end{align*}
\]

To compute $\bar{H}^1$ and $\bar{H}^2$, we appeal to the following lemma, whose proof is omitted (see [21] for similar computations). Below, define $\langle V \rangle := \int_0^t V(y) \, dy$ for any $V \in C(\mathbb{T})$.

**Lemma 4.4.** Let $F \in C(\mathbb{T})$. Then, for any $p \in \mathbb{R}$, the equation
\[
|p + v'(y)| + F(y) = \bar{H}(p) \quad \text{in } \mathbb{T}
\]
adopts a viscosity solution $v \in C(\mathbb{T})$ if and only if
\[
\bar{H}(p) = \max \left\{\max_{y \in \mathbb{T}} F(y), |p| + \langle F \rangle\right\}.
\]

Using the formulae in (4.11) and Lemma 4.4 with either $f$ or $-f$ taking the place of $F$, we explicitly compute $\bar{H}^1$ and $\bar{H}^2$, splitting into two cases depending on whether
\[
\langle f \rangle > \frac{\max f - \max f}{2} \quad \text{or} \quad \langle f \rangle < \frac{\max f - \min f}{2},
\]
which we refer to by saying that $f$ skews respectively upwards or downwards.
If \( f \) skews upwards, then \( 0 \leq \max f - \langle f \rangle < \langle f \rangle - \min f \),

\[
\begin{align*}
\overline{H}_j^4(p) &= \begin{cases} 
\frac{\max f - \min f}{2} & \text{if } |p| \leq \max f - \langle f \rangle, \\
\frac{1}{2}|p| + \frac{1}{2}(\langle f \rangle - \min f) & \text{if } \max f - \langle f \rangle < |p| \leq \langle f \rangle - \min f, \\
|p| & \text{if } |p| > \langle f \rangle - \min f,
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
\overline{H}_j^2(p) &= \begin{cases} 
\frac{\max f + \min f}{2} & \text{if } |p| \leq \langle f \rangle - \min f, \\
\frac{1}{2}|p| + \frac{1}{2}(\langle f \rangle + \min f) & \text{if } \langle f \rangle - \min f < |p| \leq \max f - \langle f \rangle, \\
\langle f \rangle & \text{if } |p| > \max f - \langle f \rangle,
\end{cases}
\end{align*}
\]

If \( f \) skews downwards, then \( 0 \leq \langle f \rangle - \min f < \max f - \langle f \rangle \),

\[
\begin{align*}
\overline{H}_j^4(p) &= \begin{cases} 
\frac{\max f - \min f}{2} & \text{if } |p| \leq \langle f \rangle - \min f, \\
\frac{1}{2}|p| + \frac{1}{2}(\max f - \langle f \rangle) & \text{if } \langle f \rangle - \min f < |p| \leq \max f - \langle f \rangle, \\
|p| & \text{if } |p| > \max f - \langle f \rangle,
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
\overline{H}_j^2(p) &= \begin{cases} 
\frac{\max f + \min f}{2} & \text{if } |p| \leq \langle f \rangle - \min f, \\
\frac{1}{2}|p| + \frac{1}{2}(\max f - \langle f \rangle) & \text{if } \langle f \rangle - \min f < |p| \leq \max f - \langle f \rangle, \\
\langle f \rangle & \text{if } |p| > \max f - \langle f \rangle.
\end{cases}
\end{align*}
\]

4.4. **Interfacial motions.** Theorem 4.11 can be used to prove Theorem 1.4 from the Introduction, concerning the first-order, level-set problem

\[
(4.18) \quad u^\varepsilon_t + \frac{1}{\varepsilon^2} A \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, \omega \right) |Du^\varepsilon| = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad u^\varepsilon(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d,
\]

where

\[
\begin{align*}
A(y, t, \omega) &:= \sum_{i=1}^m a^i(y)\xi^i(t, \omega) \quad \text{for } (y, t, \omega) \in \mathbb{T}^d \times [0, \infty) \times \Omega, \\
\xi^i &\text{ satisfies (4.10) and } a^i \in C^0(\mathbb{T}^d) \quad \text{for all } i = 1, 2, \ldots, m, \quad \text{and} \\
\sum_{k=1}^m a^k \xi_k &\neq 0 \text{ on } \mathbb{T}^d \text{ for all } \xi \in \{-1, 1\}^m.
\end{align*}
\]

The Hamiltonians \( H_j^4(p, x) := a^i(x)|p| \) then satisfy (4.13). In this case, the effective Hamiltonian \( \overline{H} \) given by (4.13) is positively homogenous in the gradient variable, and, from the formula in (4.11), so are each of the \( \overline{H}_j^4 \) for \( j \in \mathcal{A}_a^m \). Therefore, each \( \overline{H}_j^4 \) has the form

\[
\overline{H}_j^4(p) := \overline{P}^j \left( \frac{p}{|p|} \right) |p| \quad \text{for some } \overline{P} : S^{d-1} \to \mathbb{R}.
\]

For some independent Brownian motions \( B_j^4 \in \mathcal{A}_a^m \), the limiting equation is then

\[
d\overline{P} + \sum_{j \in \mathcal{A}_a^m} \overline{P} \left( \frac{D\overline{P}}{|D\overline{P}|} \right) |D\overline{P}| \circ dB_j^4 = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad \overline{P}(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d.
\]
4.5. A nonconvex example. We now turn to Theorem 1.3 from the introduction. The relevant objects are defined just as in the work of Luo, Tran, and Yu [30].

Let \( F : \mathbb{R} \to \mathbb{R} \) be a smooth, even function such that

\[
F(0) = 0, \quad F'(0) = \frac{1}{2}, \quad F'(\theta_1) = \frac{1}{3}, \quad \lim_{r \to \infty} F(r) = +\infty, \tag{4.20}
\]

for some \( 0 < \theta_3 < \theta_2 < \theta_1 \).

\( F \) is strictly increasing on \([0, \theta_2] \cup [\theta_1, +\infty)\) and strictly decreasing on \([\theta_2, \theta_1]\),

and, for \( 0 < s < 1 \), define

\[
V_s(x) := \begin{cases} \frac{x}{s} & \text{if } 0 \leq x \leq s \text{ and } \\ \frac{1 - x}{1 - s} & \text{if } s < x \leq 1 \end{cases}
\]

and extend \( V_s \) to be 1-periodic on all of \( \mathbb{R} \).

For \( \xi^1 \) and \( \xi^2 \) as in (4.10), we consider the equation

\[
u_s^\varepsilon + \frac{1}{\varepsilon^2} F(u_s^\varepsilon) \xi^1 \left( \frac{t}{\varepsilon^{2\gamma}}, \omega \right) + \frac{1}{\varepsilon^2} V_s \left( \frac{x}{\varepsilon} \right) \xi^2 \left( \frac{t}{\varepsilon^{2\gamma}}, \omega \right) = 0 \quad \text{in } \mathbb{R} \times (0, \infty) \quad \text{and } \ u^\varepsilon(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}. \tag{4.22}
\]

If \( F \) is replaced with a convex function, then (4.22) falls within the scope of Theorem 4.1 and the limiting equation resembles (4.11). However, the nonconvexity of \( F \) and the “crooked” structure of \( V_s \) for \( s \neq 1/2 \) imply that the effective Hamiltonian \( \overline{H} : \mathbb{R} \times (-1, 1)^2 \to \mathbb{R} \) given by the cell problem

\[
F(p + v'(y)) \xi^1 + V_s(y) \xi^2 = \overline{H}(p, \xi^1, \xi^2) \quad \text{in } \mathbb{R}
\]

is not odd in the \((-1, 1)^2\)-variable. As a result, in the decomposition

\[
\overline{H}(\cdot, \xi^1, \xi^2) = \overline{H}^0 + \overline{H}^1 \xi^1 + \overline{H}^2 \xi^2 + \overline{H}^{(1,2)} \xi^1 \xi^2 \quad \text{for } \xi^1, \xi^2 \in \{-1, 1\}
\]

given by Lemma 4.11, the term \( \overline{H}^{(1,2)} \) does not vanish. However, it is the case, as we show below, that \( \overline{H}^0 = 0 \), so that (4.22) does not exhibit ballistic blow-up as \( \varepsilon \to 0 \).

Let \( \overline{H}_s \) be the effective Hamiltonian associated to the Hamiltonian

\[
H_s(p, x) := F(p) - V_s(x).
\]

In Appendix B we obtain an explicit formula for \( \overline{H}_s \), and deduce, in particular, that \( \overline{H}_s \) satisfies (2.8). Moreover, as was established in [30], we have \( \overline{H}_s \neq \overline{H}_{s'} \) unless \( s = s' \).

Simple manipulations of the cell problem, properties of viscosity solutions, and the symmetry properties

\[
V_s(1 - x) = V_{1-s}(x) \quad \text{and} \quad V_s(x) = 1 - V_{1-s}(x - s) \quad \text{for all } s \in (0, 1), x \in \mathbb{T}
\]

can be used to show that

\[
\begin{align*}
\overline{H}(\cdot, 1, 1) &= \overline{H}_{1-s} + 1, \\
\overline{H}(\cdot, 1, -1) &= \overline{H}_s, \\
\overline{H}(\cdot, -1, 1) &= -\overline{H}_{1-s}, \quad \text{and} \\
\overline{H}(\cdot, -1, -1) &= -\overline{H}_s - 1,
\end{align*}
\]
Finally, the formula for \( u^\xi \) field and \((4.10)\) gives
\[
\begin{align*}
\mathcal{H}^0 &= 0, \\
\mathcal{H}^1 &= \frac{\mathcal{H}_s + \mathcal{H}_{1-s} + 1}{2}, \\
\mathcal{H}^2 &= \frac{1}{2}, \quad \text{and} \\
\mathcal{H}^{(1,2)} &= \frac{\mathcal{H}_{1-s} - \mathcal{H}_s}{2}.
\end{align*}
\]

A similar proof as for Theorem 4.1 then gives the following:

**Theorem 4.2.** Assume \(0 < \gamma < 1/6\), \(u_0 \in UC(\mathbb{R})\), \(F\) and \(V_s\) are as in (4.20) and (4.21), \(\xi^1\) and \(\xi^2\) are as in (4.10), the paths \((\xi^{1,\varepsilon})_{\varepsilon \in \mathbb{A}}\) are defined as in (4.11), and \(u^\varepsilon\) is the solution of (4.22). Then, as \(\varepsilon \to 0\), \((u^\varepsilon, (\xi^{1,\varepsilon})_{\varepsilon \in \mathbb{A}})\) converges locally uniformly in distribution to \((\bar{\pi}, (B^1)_{\varepsilon \in \mathbb{A}})\), where \(\bar{\pi}\) is the unique stochastic viscosity solution of
\[
\begin{align*}
\frac{d\bar{\pi}}{d\varepsilon} + \frac{\mathcal{H}_s(\bar{\pi}_s) + \mathcal{H}_{1-s}(\bar{\pi}_s) + 1}{2} \cdot dB^1 + \frac{1}{2} \cdot dB^2 + \frac{\mathcal{H}_{1-s}(\bar{\pi}_s) - \mathcal{H}_s(\bar{\pi}_s)}{2} \cdot dB^{(1,2)} = 0 \quad \text{in } \mathbb{R} \times (0, \infty) \quad \text{and} \\
\bar{\pi}(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}.
\end{align*}
\]

To finish this discussion and the proof of Theorem 4.3, we mention that the independence of the fields \(\xi^1\) and \(\xi^2\) is used in the above result, in particular, through the application of Lemma 4.2. Indeed, for a single field \(\xi\) satisfying (4.10), consider the equation
\[
(4.23) \quad u^\varepsilon_t + \frac{1}{\varepsilon^\gamma} \left( F(u^\varepsilon_x) - V_x \left( \frac{x}{\varepsilon} \right) \right) \xi^\gamma \left( \frac{t}{\varepsilon^{2\gamma}} \right) = 0 \quad \text{in } \mathbb{R} \times (0, \infty) \quad \text{and} \quad u^\varepsilon(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}.
\]

This equation is not covered by the result in the single-noise case, due to the fact that (2.13) fails if \(s \neq 1/2\):
\[
(-\mathcal{H}_s) = -\mathcal{H}_{1-s} \neq \mathcal{H}_s.
\]

As a consequence, we have the following:

**Theorem 4.3.** Assume \(0 < \gamma < 1\), \(F\) and \(V\) are as in (4.20) and (4.21), \(\xi\) is as in (4.10), and, for some fixed \(p_0 \in \mathbb{R}\), \(u^\varepsilon\) is the solution of (4.23) with \(u_0(x) = p_0 \cdot x\). Then, with probability one, for all \(T > 0\),
\[
\lim_{\varepsilon \to 0} \sup_{(x,t) \in \mathbb{R} \times [0,T]} \left| \varepsilon^\gamma u^\varepsilon(x,t) - \frac{\mathcal{H}_{1-s}(p_0) - \mathcal{H}_s(p_0)}{2} \right| = 0.
\]

**Proof.** The solution \(\bar{\pi}^\varepsilon\) of the initial value problem
\[
\begin{align*}
\bar{\pi}^\varepsilon_t + \frac{1}{\bar{\pi}^\varepsilon} \left( \bar{\pi}^\varepsilon_x, \xi \left( \frac{t}{\bar{\pi}^\varepsilon}, \omega \right), \xi \left( \frac{t}{\bar{\pi}^\varepsilon}, \omega \right) \right) = 0 \quad \text{in } \mathbb{R} \times (0, \infty) \quad \text{and} \quad \bar{\pi}^\varepsilon(\cdot, 0) = p_0 \cdot x \quad \text{in } \mathbb{R},
\end{align*}
\]

takes the form
\[
\bar{\pi}^\varepsilon(x,t) = p_0 \cdot x + \varepsilon^\gamma \int_0^{1/\varepsilon^{2\gamma}} \bar{\pi}(p_0, \xi(s), \xi(s))ds.
\]

A similar argument as for Lemma 4.3 gives, for some constant \(C > 0\),
\[
\sup_{(x,t) \in \mathbb{R} \times [0,T]} \left| \varepsilon^\gamma u^\varepsilon(x,t) - \varepsilon^\gamma \bar{\pi}^\varepsilon(x,t) \right| \leq C(1 + T)^{3-\gamma}.
\]

Note that the exponent is \(1 - \gamma\), rather than \(1/3 - \gamma\), because of the form of the initial datum.

Finally, the formula for \(\bar{\pi}\) and the law of large numbers yield, with probability one,
\[
\lim_{\varepsilon \to 0} \sup_{(x,t) \in \mathbb{R} \times [0,T]} \left| \varepsilon^\gamma \bar{\pi}^\varepsilon(x,t) - \frac{\mathcal{H}_{1-s}(p_0) - \mathcal{H}_s(p_0)}{2} \right| = 0,
\]
which establishes the result. \(\square\)
4.6. Dependence of the limit on the noise approximation. We return to the equation

\begin{equation}
(4.24) \quad u^\varepsilon \varepsilon + \frac{1}{\varepsilon^2} |u^\varepsilon|^1 \left( \frac{t}{\varepsilon^2}, \omega \right) + \frac{1}{\varepsilon^2} f \left( \frac{x}{\varepsilon} \right) \xi^2 \left( \frac{t}{\varepsilon^2}, \omega \right) = 0 \quad \text{in } \mathbb{R} \times (0, \infty) \quad \text{and} \quad u^\varepsilon(\cdot, 0) = u_0 \quad \text{in } \mathbb{R},
\end{equation}

but we define the white noise approximations in such a way that the limiting equation has a different law than \((4.16)\), thus establishing Theorem \(1.12\) from the introduction, together with the computations in subsection \(4.3\).

Let \((X_k, Y_k, Z_k)_{k=0}^\infty\) be a collection of independent, Rademacher random variables, let \(0 < b < a\) be such that

\[ a^2 + b^2 = 2 \quad \text{and} \quad a(\max f - \langle f \rangle) < b(\langle f \rangle - \min f), \]

and set

\[ X^1_k := X_k \quad \text{and} \quad X^2_k := \frac{a + b}{2} Y_k + \frac{a - b}{2} Z_k. \]

Note that \(X^1_k\) and \(X^2_k\) are independent for each \(k\), and

\begin{equation}
(4.25) \quad \mathbb{E} \left[ X^i_k \right] = 0 \quad \text{and} \quad \mathbb{E} \left[ X^i_k \right]^2 = 1.
\end{equation}

For \(i = 1, 2\), define \(\zeta^i(0) = 0\) and

\[ \dot{\zeta}^i(t, \omega) = \xi^i(t, \omega) := \sum_{k=0}^\infty X^i_k(\omega) \mathbf{1}_{(k,k+1)}(t) \quad \text{and} \quad \dot{\zeta}^i,\varepsilon(t, \omega) = \varepsilon^i \zeta^i(t/\varepsilon^{2\gamma}, \omega), \]

and, for \(j \in \{1, 2, 3\}\), define the approximating paths \(\zeta^j,\varepsilon(t) := \varepsilon^j \zeta^j(t/\varepsilon^{2\gamma})\), where

\[ \dot{\zeta}^{(1)}(t, \omega) = \xi^{(1)}(t, \omega) := \sum_{k=0}^\infty Y_k(\omega) \mathbf{1}_{(k,k+1)}(t), \quad \dot{\zeta}^{(2)}(t, \omega) := \sum_{k=0}^\infty Z_k(\omega) \mathbf{1}_{(k,k+1)}(t), \quad \text{and} \]

\[ \dot{\zeta}^{(1,2,3)}(t, \omega) := \sum_{k=0}^\infty X_k(\omega) Y_k(\omega) Z_k(\omega) \mathbf{1}_{(k,k+1)}(t). \]

Equation \((4.24)\) can then be written as

\begin{equation}
(4.26) \quad \left\{ u^\varepsilon + |u^\varepsilon|^1 \xi^1(\cdot, t, \omega) + \frac{a + b}{2} f \left( \frac{x}{\varepsilon} \right) \xi^2(\cdot, t, \omega) + \frac{a - b}{2} f \left( \frac{x}{\varepsilon} \right) \xi^3(\cdot, t, \omega) = 0 \quad \text{in } \mathbb{R} \times (0, \infty) \quad \text{and} \quad u^\varepsilon(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}. \right.
\end{equation}

Applying Theorem \(4.11\) then gives that, if \(0 < \gamma < 1/2\), then, for some independent Brownian motions \(B^j\) with \(j \in \{1, 2, 3\}\),

\[ \left( u^\varepsilon, \zeta^{(1)}, \zeta^{(2)}, \zeta^{(3)}, \zeta^{(1,2,3)} \right) \overset{\varepsilon \to 0}{\longrightarrow} \left( \overline{u}, B^{(1)}, B^{(2)}, B^{(3)} \right) \quad \text{locally uniformly and in distribution}, \]

where \(\overline{u}\) is the stochastic viscosity solution of

\begin{equation}
(4.27) \quad \left\{ \begin{array}{l}
\overline{d u} + \overline{H}^{(1)}(\overline{u}) \circ d B^{(1)} + \overline{H}^{(2)}(\overline{u}) \circ d B^{(2)} + \overline{H}^{(3)}(\overline{u}) \circ d B^{(3)} \\
\overline{u}(\cdot, 0) = u_0 \quad \text{in } \mathbb{R},
\end{array} \right.
\end{equation}

and

\[ \overline{H}^{(1,2,3)}(\overline{u}) \circ d B^{(1,2,3)} = 0 \quad \text{in } \mathbb{R} \times (0, \infty) \quad \text{and} \]
The formulae for the effective Hamiltonians are given below, and, as can be checked, the laws of the solutions of (4.16) and (4.27) differ in general, even when \( u_0(x) := p_0 \cdot x \) for some fixed \( p_0 \in \mathbb{R}^d \):

\[
\mathcal{H}^{(1)}(p) := \begin{cases} 
\frac{a+b}{4}(\max f - \min f) & \text{if } 0 \leq |p| \leq b(\max f - \langle f \rangle), \\
\frac{1}{4}|p| + \frac{a}{4}(\max f - \min f) + \frac{b}{4}(\langle f \rangle - \min f) & \text{if } b(\max f - \langle f \rangle) \leq |p| \leq a(\max f - \langle f \rangle), \\
\frac{1}{2}|p| + \frac{a+b}{4}(\langle f \rangle - \min f) & \text{if } a(\max f - \langle f \rangle) \leq |p| \leq b(\langle f \rangle - \min f), \\
\frac{3}{4}|p| + \frac{a}{4}(\langle f \rangle - \min f) & \text{if } b(\langle f \rangle - \min f) \leq |p| \leq a(\langle f \rangle - \min f), \\
|p| & \text{if } |p| \geq a(\langle f \rangle - \min f),
\end{cases}
\]

\[
\mathcal{H}^{(2)}(p) := \begin{cases} 
\frac{a+b}{4}(\max f + \min f) & \text{if } 0 \leq |p| \leq b(\max f - \langle f \rangle), \\
\frac{1}{4}|p| + \frac{a}{4}(\max f + \min f) + \frac{b}{4}(\langle f \rangle + \min f) & \text{if } b(\max f - \langle f \rangle) \leq |p| \leq a(\max f - \langle f \rangle), \\
\frac{1}{2}|p| + \frac{a+b}{4}(\langle f \rangle + \min f) & \text{if } a(\max f - \langle f \rangle) \leq |p| \leq b(\langle f \rangle - \min f), \\
\frac{1}{4}|p| + \frac{a}{4}(\langle f \rangle + \min f) + \frac{b}{2}\langle f \rangle & \text{if } b(\langle f \rangle - \min f) \leq |p| \leq a(\langle f \rangle - \min f), \\
\frac{a+b}{2}\langle f \rangle & \text{if } |p| \geq a(\langle f \rangle - \min f),
\end{cases}
\]

\[
\mathcal{H}^{(3)}(p) := \begin{cases} 
\frac{a-b}{4}(\max f + \min f) & \text{if } 0 \leq |p| \leq b(\max f - \langle f \rangle), \\
-\frac{1}{4}|p| + \frac{a}{4}(\max f + \min f) - \frac{b}{4}(\langle f \rangle + \min f) & \text{if } b(\max f - \langle f \rangle) \leq |p| \leq a(\max f - \langle f \rangle), \\
\frac{a-b}{4}(\langle f \rangle + \min f) & \text{if } a(\max f - \langle f \rangle) \leq |p| \leq b(\langle f \rangle - \min f), \\
\frac{1}{4}|p| + \frac{a}{4}(\langle f \rangle + \min f) - \frac{b}{2}\langle f \rangle & \text{if } b(\langle f \rangle - \min f) \leq |p| \leq a(\langle f \rangle - \min f), \\
\frac{a-b}{2}\langle f \rangle & \text{if } |p| \geq a(\langle f \rangle - \min f),
\end{cases}
\]

and

\[
\mathcal{H}^{(1,2,3)}(p) := \begin{cases} 
\frac{a-b}{4}(\max f - \min f) & \text{if } 0 \leq |p| \leq b(\max f - \langle f \rangle), \\
-\frac{1}{4}|p| + \frac{a}{4}(\max f - \min f) - \frac{b}{4}(\langle f \rangle - \min f) & \text{if } b(\max f - \langle f \rangle) \leq |p| \leq a(\max f - \langle f \rangle), \\
\frac{a-b}{4}(\langle f \rangle - \min f) & \text{if } a(\max f - \langle f \rangle) \leq |p| \leq b(\langle f \rangle - \min f), \\
-\frac{1}{4}|p| + \frac{a}{4}(\langle f \rangle + \min f) & \text{if } b(\langle f \rangle - \min f) \leq |p| \leq a(\langle f \rangle - \min f), \\
0 & \text{if } |p| \geq a(\langle f \rangle - \min f). 
\end{cases}
\]
We prove the regularity and path-stability results in Theorem 2.2. For convenience, we repeat the assumptions on \( H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \):

\[
\begin{cases}
 H \in C(\mathbb{R}^d \times \mathbb{R}^d), \ p \mapsto H(p, x) \text{ is convex for all } x \in \mathbb{R}^d, \text{ and } \\
 \text{there exist convex, increasing functions } \psi, \nu : [0, \infty) \to \mathbb{R} \text{ such that } \\
 \psi(|p|) \leq H(p, x) \leq \nu(|p|) \text{ for all } (p, x) \in \mathbb{R}^d \times \mathbb{R}^d.
\end{cases}
\]

(A.1)

For two smooth (or piecewise smooth) paths \( \zeta, \eta : [0, \infty) \to \mathbb{R} \) and \( u_0^1, u_0^2 \in C^{0,1}(\mathbb{R}^d) \), consider the viscosity solutions \( u^1 \) and \( u^2 \) of

\[
u^j = H(Du^j, x)\dot{\zeta}^j \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad \text{and} \quad u^j(\cdot, 0) = u_0^j \quad \text{in } \mathbb{R}^d.
\]

(A.2)

Theorem A.1. Set \( L := \max \{ \| Du_0^1 \|_\infty, \| Du_0^2 \|_\infty \} \). Then, for all \( t > 0 \) and for \( j = 1, 2 \),

\[
\| Du^j(\cdot, t) \|_\infty \leq \nu^{-1}(\nu(L)),
\]

and, for all \( T > 0 \),

\[
\max_{(x, t) \in \mathbb{R}^d \times [0, T]} |u^1(x, t) - u^2(x, t)| \leq \max_{x \in \mathbb{R}^d} |u_0^1(x) - u_0^2(x)| + \nu(L) \max_{t \in [0, T]} |\zeta^1(t) - \zeta^2(t)| + \nu(0) - \left( \max_{t \in [0, T]} |\zeta^1(t) - \zeta^2(t)| - (\zeta^1(T) - \zeta^2(T)) \right).
\]

We remark that a similar result was obtained by Gassiat, Gess, Lions, and Souganidis [10] using slightly different methods, as a tool to study some finer properties of solutions, such as the cancellation of oscillations and speed of propagation.

Both results in Theorem A.1 follow from the next proposition. The hypotheses require more regularity for the Hamiltonian than is specified by (A.1). The proof of Theorem A.1 then involves a further regularization of \( H \), and the result will follow upon obtaining estimates that do not depend on the regularization parameter.

The proof below uses similar strategies as those in [15, 30, 23].

Proposition A.1. Assume that \( H \) satisfies (A.1),

\[
H \in C^2_b(B_R \times \mathbb{R}^d) \quad \text{for all } R > 0, \quad \text{and } D^2_p H \text{ is strictly positive.}
\]

(A.3)

For \( u_0, v_0 \in UC(\mathbb{R}^d) \) and \( \zeta, \eta \in C^1([0, \infty)) \) with \( \zeta_0 = \eta_0 \), let \( u \) be a sub-solution of

\[
u = H(Du, x)\dot{\zeta}(t) \quad \text{in } \mathbb{R}^d \times (0, \infty), \quad u(\cdot, 0) = u_0 \quad \text{on } \mathbb{R}^d,
\]

and \( v \) a super-solution of

\[
u = H(Dv, x)\dot{\eta}(t) \quad \text{in } \mathbb{R}^d \times (0, \infty), \quad v(\cdot, 0) = v_0 \quad \text{on } \mathbb{R}^d.
\]

Then, for all \( T > 0 \) and \( 0 < \lambda < (\max_{0 \leq t \leq T} (\zeta_t - \eta_t))^{-1} \),

\[
\sup_{(x,y,t) \in \mathbb{R}^d \times [0, T]} \left( u(x, t) - v(y, t) - \left( \frac{1}{\lambda} + \zeta_t - \eta_t \right) \nu^* \left( \frac{\lambda|x - y|}{1 + \lambda(\zeta_t - \eta_t)} \right) \right) \leq \sup_{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d} \left( u_0(x) - v_0(y) - \frac{1}{\lambda} \nu^* \lambda|x - y| \right).
\]

Equipped with Proposition A.1 we proceed with the
Proof of Theorem A.4. Step 1. Assume first that $H$ satisfies (A.3) in addition to (A.1). Applying Proposition A.1 to the case $u = v = u^1$ and $\zeta = \eta = \zeta^1$ yields, for all $(x, y, t) \in \mathbb{R}^d \times \mathbb{R}^d \times (0, \infty)$,

$$u^1(x, t) - u^1(y, t) \leq \inf_{\lambda > 0} \left\{ \frac{1}{\lambda} \nu^*(\lambda|x-y|) + \sup_{s \geq 0} \left\{ Ls - \frac{1}{\lambda} \nu^*(\lambda s) \right\} \right\} = \inf_{\lambda > 0} \left\{ \frac{1}{\lambda} \nu^*(\lambda|x-y|) + \nu(L) \right\} = \nu^{-1}(\nu(L)) |x - y|.$$ 

Thus $\|Du^1(\cdot, t)\|_\infty \leq \nu^{-1}(\nu(L))$, and similarly for $u^2$.

Now setting $(u, v, \zeta, \eta) := (u^1, u^2, \zeta^1, \zeta^2)$ in Proposition A.1 gives

$$u^1(x, t) - u^2(x, t) \leq \left( \frac{1}{\lambda} - (\zeta^1 - \zeta^2) \right) \nu^*(0) + \max_{x \in \mathbb{R}^d} |u^1_0(x) - u^2_0(x)| + \frac{1}{\lambda} \nu(L).$$

The claim follows upon choosing $\lambda = (\max_{s \in [0, t]} |\zeta^1 - \zeta^2|)^{-1}$ and using the fact that

$$\nu^*(0) = -\min_{r \geq 0} \nu(r) = -\nu(0) \leq \nu(0).$$

Step 2. We now return to the general case, where $H$ satisfies only (A.1). Let $\phi \in C^2(\mathbb{R}^d)$ be nonnegative and supported in $B_1(0)$ with $\int \phi = 1$, and, for $\rho > 0$, define

$$\phi_\rho(z) := \frac{1}{\rho^d} \phi \left( \frac{z}{\rho} \right)$$

and

$$H_\rho(p, x) := \rho|p|^2 + \int_{\mathbb{R}^d \times \mathbb{R}^d} H(q, y) \phi_\rho(p - q) \phi_\rho(x - y) \, dq \, dy.$$ 

It is straightforward to verify that $\lim_{\rho \to 0} H_\rho = H$ locally uniformly, and $H_\rho$ satisfies both (A.1) and (A.3) with the growth functions

$$\nu_\rho(s) := \rho s^2 + \nu(s + \rho) \quad \text{and} \quad \nu_\rho(s) := \rho s^2 + \nu((s - \rho)_+).$$

Let $u^1_\rho$ and $u^2_\rho$ be as in the statement of Theorem A.1 for the Hamiltonian $H_\rho$. As proved above, $u^1_\rho$ and $u^2_\rho$ satisfy the Lipschitz bound and stability estimate for $\nu_\rho$ and $\nu_\rho$. Classical arguments from the theory of viscosity solutions yield the local uniform convergence, as $\rho \to 0$, of $u^1_\rho$ to $u^1$ for $j = 1, 2$, where $u^1$ are as in the statement of Theorem A.1 for the Hamiltonian $H$. Since $\nu_\rho$ converge, as $\rho \to 0$, to $\nu$ and $\nu$, the proof is complete. \hfill \Box

The rest of this section is devoted to the proof of Proposition A.2 in [36].

For $x, y \in \mathbb{R}^d$ and $\tau > 0$, define

$$\mathcal{A}(x, y, \tau) := \{ \gamma \in W^{1, \infty}([0, \tau], \mathbb{R}^d) : \gamma_0 = x, \gamma_\tau = y \}$$

and

$$L(x, y, \tau) := \inf \left\{ \int_0^\tau H^* (-\gamma_s, \gamma_s) \, ds : \gamma \in \mathcal{A}(x, y, \tau) \right\}. \tag{A.4}$$

We summarize the main properties of this distance function in the next lemma. We omit the proof, as it follows more or less in the same way as in Lemma A.1 of [36].

For $R > 0$, define

$$\Delta_R := \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : |x - y| \leq R \}. \tag{A.4}$$

**Lemma A.1.** Assume that $H$ satisfies (A.3). Then the following hold:
(a) \( L \) is a viscosity solution of
\[
\frac{\partial L}{\partial \tau} = H(D_x L, x) \quad \text{and} \quad \frac{\partial L}{\partial \tau} = H(-D_y L, y) \quad \text{in} \ \mathbb{R}^d \times \mathbb{R}^d \times (0, \infty).
\]
(b) For all \( x, y \in \mathbb{R}^d \) and \( \tau > 0 \),
\[
\tau \nu \left( \frac{|x-y|}{\tau} \right) \leq L(x, y, \tau) \leq \tau \nu \left( \frac{|x-y|}{\tau} \right).
\]
Furthermore, there exists \( \gamma \in A(x, y, \tau) \) such that \( L(x, y, \tau) = \int_0^\tau H^*(\gamma_s, \gamma_s) \, ds \), and, for some \( c \geq 1 \) and almost every \( s \in [0, \tau] \),
\[
|\gamma_s| \leq \frac{c|x-y|}{\tau}.
\]
(c) For all \( R > 0 \), there exists a constant \( C = C_R > 0 \) such that
\[
|D_x L| + |D_y L| \leq C \quad \text{and} \quad D^2 L \leq C \text{Id} \quad \text{on} \ \Delta_R \times \left[ \frac{1}{R}, R \right].
\]

The upper bound on \( D^2 L \) means that \( L \) is semiconcave in space. As the next result demonstrates, this allows \( L \) to be used as a test function at an important point in the proof of Proposition 1.1 despite the fact that \( L \) is not in general \( C^1 \).

**Lemma A.2.** Under the same assumptions as Lemma 1.1, assume that \( \phi \in C^2(\mathbb{R}^d \times \mathbb{R}^d) \) and \( L(\cdot, \cdot, \tau_0) - \phi \) attains a local minimum at \((x_0, y_0)\). Then \( L \) is differentiable at \((x_0, y_0, \tau_0)\) with
\[
\begin{align*}
\frac{\partial}{\partial \tau}(x_0, y_0, \tau_0) &= H(D_x L(x_0, y_0, \tau_0), x_0) = H(-D_y L(x_0, y_0, \tau_0), y_0).
\end{align*}
\]

**Proof.** In view of the semiconcavity of \( L(\cdot, \cdot, \tau_0) \) on \( \mathbb{R}^d \times \mathbb{R}^d \), the super-differential of \( L(\cdot, \cdot, \tau_0) \) is nonempty at every point. Meanwhile, \((p_0, q_0) := D\phi(x_0, y_0)\) belongs to the sub-differential of \( L(\cdot, \cdot, \tau_0) \) at \((x_0, y_0)\). This implies that \( L(\cdot, \cdot, \tau_0) \) is differentiable at \((x_0, y_0)\), and the first line above holds.

Choose \( \psi^+, \psi^- \in C^2(\mathbb{R}^d \times \mathbb{R}^d) \) such that
\[
\begin{align*}
\psi^- \leq L(\cdot, \cdot, \tau_0) \leq \psi^+,
\psi^-(x_0, y_0) = L(x_0, y_0, \tau_0) = \psi^+(x_0, y_0), \quad \text{and} \quad D\psi^-(x_0, y_0) = D\psi^+(x_0, y_0) = (p_0, q_0).
\end{align*}
\]

The method of characteristics can then be used to construct, for sufficiently small \( \mu > 0 \), solutions \( \Psi^\pm \in C^2(\mathbb{R}^d \times \mathbb{R}^d \times (\tau_0 - \mu, \tau_0 + \mu)) \) of the equations
\[
\frac{\partial \Psi^\pm}{\partial \tau}(x, y, \tau) = H(D_x \Psi^\pm(x, y, \tau), x) \quad \text{in} \ \mathbb{R}^d \times \mathbb{R}^d \times (\tau_0 - \mu, \tau_0 + \mu).
\]

The comparison principle and Lemma 1.1 then yield
\[
(\text{A.5}) \quad \Psi^-(x, y, \tau) \leq L(x, y, \tau) \leq \Psi^+(x, y, \tau) \quad \text{for all} \ (x, y, \tau) \in \mathbb{R}^d \times \mathbb{R}^d \times (\tau_0 - \mu, \tau_0 + \mu).
\]

Finally, the regularity of \( H \) and the equations for \( \Psi^\pm \) allow for the Taylor expansion
\[
\Psi^\pm(x, y, \tau) = L(x_0, y_0, \tau_0) + p \cdot (x - x_0) + q \cdot (y - y_0) + H(p_0, x_0)(\tau - \tau_0) + O(|x - x_0|^2 + |y - y_0|^2 + |\tau - \tau_0|^2).
\]

Together with (A.4), this shows that \( L \) is differentiable at \((x_0, y_0, \tau_0)\) and
\[
\frac{\partial L}{\partial \tau}(x_0, y_0, \tau_0) = H(D_x L(x_0, y_0, \tau_0), x_0).
\]
A similar argument using the equation \( \frac{\partial \Omega}{\partial t} = H(-D_y \Omega, y) \) gives the final desired equality
\[
\frac{\partial \Omega}{\partial t}(x_0, y_0, \tau_0) = H(-D_y \Omega(x_0, y_0, \tau_0), y_0).
\]

\[\square\]

**Proof of Proposition A.7.** We first note that it suffices to assume that \( u_0 \) and \( v_0 \) are bounded. Because the resulting estimates do not depend on \( \|u_0\|_\infty \) or \( \|v_0\|_\infty \), the general result can be obtained through an approximation procedure and the local uniform stability of the equations with respect to the initial data.

Classical viscosity solution arguments show that \( z(x, y, t) := u(x, t) - v(y, t) \) is a sub-solution of
\[
z_t = H(D_x z, x) \hat{\zeta} - H(-D_y z, y) \hat{\eta} \quad \text{in } \mathbb{R}^d \times \mathbb{R}^d \times (0, \infty).
\]

For \( 0 < \lambda < (\max_{0 \leq t \leq T}(\hat{\zeta}_t - \hat{\eta}_t))^{-1} \), define
\[
\Phi_{\lambda}(x, y, t) := L\left(x, y, \frac{1}{\lambda} \hat{\zeta} - \hat{\eta}\right).
\]

A simple computation and Lemma A.11 reveal that \( \Phi \) satisfies (A.6) at any point \( (x, y, t) \) of differentiability.

Next, for \( 0 < \beta < 1 \) and \( \mu > 0 \), define
\[
\Psi(x, y, t) := u(x, t) - v(y, t) - \Phi_{\lambda}(x, y, t) - \frac{\beta}{2}(|x|^2 + |y|^2) - \mu t.
\]

The comparison principle from the classical viscosity solution theory yields that \( |u(x, t)| \leq M \) and \( |v(x, t)| \leq M \) on \( \mathbb{R}^d \times [0, T] \), where
\[
M = \max \left\{ \|u_0\|_\infty + \max(|\nu(0)|, |\nu(0)|) \max_{0 \leq t \leq T} |\zeta(t)|, \|v_0\|_\infty + \max(|\nu(0)|, |\nu(0)|) \max_{0 \leq t \leq T} |\eta(t)| \right\}.
\]

Therefore, \( \Psi \) attains a maximum on \( \mathbb{R}^d \times \mathbb{R}^d \times [0, T] \) at some \( (\hat{x}, \hat{y}, \hat{t}) \) that depends on \( \beta, \lambda, \) and \( \mu \). Assume for the sake of contradiction that \( \hat{t} > 0 \).

Rearranging terms in the inequality \( \Psi(0, 0, \hat{t}) \leq \Psi(\hat{x}, \hat{y}, \hat{t}) \) gives
\[
\frac{\beta}{2}(|\hat{x}|^2 + |\hat{y}|^2) \leq u(\hat{x}, \hat{t}) - v(\hat{y}, \hat{t}) = (u(0, \hat{t}) - v(0, \hat{t})) \leq 4M.
\]

The inequality \( \Psi(\hat{y}, \hat{y}, \hat{t}) \leq \Psi(\hat{x}, \hat{y}, \hat{t}) \) and Lemma A.11 yield
\[
\frac{1}{\lambda} + \frac{\zeta_t - \eta_t}{\lambda} \leq \frac{\lambda}{1 + (\zeta_t - \eta_t)(\hat{x} - y, \hat{t})} \leq u(\hat{x}, \hat{t}) - u(\hat{y}, \hat{t}) + \frac{\beta}{2}(|\hat{y}|^2 - |\hat{x}|^2) \leq 6M.
\]

Then A.11 and A.8 together imply that, for some \( R > 0 \) depending on \( \lambda, M, \|\zeta\|_{\infty, T}, \) and \( \|\eta\|_{\infty, T}, \) but independent of \( \beta, (\hat{x}, \hat{y}, \hat{t}) \in \Omega_{R, \beta}, \) where
\[
\Omega_{R, \beta} := \Delta_R \cap B_{R^{\beta-1/2}} = \left\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : (|x|^2 + |y|^2)^{1/2} \leq R^{\beta-1/2} \text{ and } |x - y| \leq R \right\}.
\]

In the arguments that follow, the constant \( C \) depends only on \( R, \) and may change from line to line.

For \( 0 < \delta < 1 \), set
\[
\Psi_{\delta}(x, y, z, w, t) := u(x, t) - v(y, t) - \frac{1}{2\delta}(|x - z|^2 + |y - w|^2) - \Phi_{\lambda}(z, w, t) - \frac{\beta}{2}(|z|^2 + |w|^2) - \mu t - \frac{1}{2}(|x - \hat{x}|^2 + |y - \hat{y}|^2 + |t - \hat{t}|^2)
\]
and assume that the maximum of \( \Psi_{\delta} \) on \( \Omega_{R, \beta} \cap \Omega_{R, \beta} \times [0, T] \) is attained at \( (x_{\delta}, y_{\delta}, z_{\delta}, w_{\delta}, t_{\delta}) \). Similar arguments as in the proof of Proposition A.2 from [39] then yield
\[
|x_{\delta} - z_{\delta}| + |y_{\delta} - w_{\delta}| + |x_{\delta} - \hat{x}|^2 + |y_{\delta} - \hat{y}|^2 + |t_{\delta} - \hat{t}|^2 \leq C\delta.
\]
Therefore, for sufficiently small $\delta$, $(x_\delta, y_\delta, z_\delta, w_\delta, t_\delta)$ is a local interior maximum point of $\Psi_\delta$ in $\Omega_{R,\beta} \times \Omega_{R,\beta} \times (0, T)$.

Since
\[
(x, y, t) \mapsto u(x, t) - v(y, t) - \frac{1}{2\delta} (|x - z_\delta|^2 + |y - w_\delta|^2) - \Phi_\lambda(z_\delta, w_\delta, t) - \mu t - \frac{1}{2} \left(|x - \hat{x}|^2 - |y - \hat{y}|^2 - |t - \hat{t}|^2\right)
\]
attains an interior maximum at $(x_\delta, y_\delta, t_\delta)$, the definition of viscosity solutions yields
\[
\mu + t_\delta - \hat{t} + \Phi_{\lambda, t}(z_\delta, w_\delta, t_\delta) \leq H \left(\frac{x_\delta - z_\delta}{\delta} + x_\delta - \hat{x}, x_\delta\right) \hat{\zeta}_t - H \left(-\frac{y_\delta - w_\delta}{\delta} - (y_\delta - \hat{y}), y_\delta\right) \eta_t.
\]

Next, $(z_\delta, w_\delta)$ is a minimum point of
\[
(z, w) \mapsto \Phi_\lambda(z, w, t_\delta) + \frac{1}{2\delta} (|x_\delta - z|^2 + |y_\delta - w|^2) + \frac{\beta}{2} (|z|^2 + |w|^2).
\]

In view of Lemma 1.2, $\Phi_\lambda$ is differentiable at $(z_\delta, w_\delta, t_\delta)$, and so
\[
\begin{align*}
D_x \Phi_\lambda(z_\delta, w_\delta, t_\delta) &= \frac{x_\delta - z_\delta}{\delta} - \beta z_\delta, \\
D_y \Phi_\lambda(z_\delta, w_\delta, t_\delta) &= \frac{y_\delta - w_\delta}{\delta} - \beta w_\delta, \\
\Phi_{\lambda, t}(z_\delta, w_\delta, t_\delta) &= H(D_x \Phi_\lambda(z_\delta, w_\delta, t_\delta), z_\delta) \hat{\zeta}_t - H(-D_y \Phi_\lambda(z_\delta, w_\delta, t_\delta), w_\delta) \eta_t.
\end{align*}
\]

It follows that
\[
\mu + t_\delta - \hat{t} + \Phi_{\lambda, t}(z_\delta, w_\delta, t_\delta) \leq H(D_x \Phi_\lambda(z_\delta, w_\delta, t_\delta) + \beta z_\delta + x_\delta - \hat{x}, x_\delta) \hat{\zeta}_t - H(-D_y \Phi_\lambda(z_\delta, w_\delta, t_\delta) - \beta w_\delta - (y_\delta - \hat{y}), y_\delta) \hat{\zeta}_t.
\]

The bounds for $(\hat{x}, \hat{y}, \hat{t})$ and $(x_\delta, y_\delta, z_\delta, w_\delta, t_\delta)$ and the local Lipschitz regularity of $H$ yield
\[
\mu \leq C(\beta^{1/2} + \delta^{1/2} + \delta) \left(\|\xi\|_{\infty, T} + \|\zeta\|_{\infty, T}\right).
\]

We obtain a contradiction for sufficiently small enough $\delta$ and $\beta$.

Therefore, for all $\mu > 0$ and $t \in [0, T]$,
\[
\lim_{\beta \to 0} \sup_{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d} \left( u(x, t) - v(y, t) - \Phi_\lambda(x, y, t) - \frac{\beta}{2} (|x|^2 + |y|^2) \right) = \sup_{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d} (u(x, t) - v(y, t) - \Phi_\lambda(x, y, t)) \leq \sup_{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d} (u_0(x) - v_0(y) - L(x, y, 1/\lambda)) + \mu t.
\]

The desired inequality is established upon letting $\mu \to 0$ and using the bounds in Lemma 1.3.

**Appendix B. Calculation of an effective Hamiltonian in the nonconvex case**

Let $F : \mathbb{R} \to \mathbb{R}$ be a smooth, even function such that
\[
\begin{align*}
F(0) &= 0, & F(\theta_2) &= \frac{1}{2}, & F(\theta_1) &= F(\theta_3) = \frac{1}{3}, & \lim_{r \to \infty} F(r) &= +\infty, \\
F &\text{ is strictly increasing on } [0, \theta_2] \cup [\theta_1, +\infty) \text{ and strictly decreasing on } [\theta_2, \theta_1],
\end{align*}
\]

for some $0 < \theta_3 < \theta_2 < \theta_1$. 

(B.1)
and, for $0 < s < 1$, define the 1-periodic function $V_s : \mathbb{T} \to \mathbb{T}$ by

$$V_s(x) := \begin{cases} \frac{x}{s} & \text{if } 0 \leq x \leq s \text{ and} \\ \frac{1 - x}{1 - s} & \text{if } s < x \leq 1. \end{cases}$$

The goal of this section is to obtain a formula for the effective Hamiltonian associated to

$$H_s(p, x) = F(p) - V_s(x).$$

Not only do we recover the fact from [30] that $H_s = H_{s'}$ if and only if $s = s'$, but we also show that $H_s$ is Lipschitz and piecewise smooth, and, hence, satisfies (2.8).

As in [30], define the functions

$$
\begin{align*}
\psi_1 &:= (F|_{[\theta_1, \infty)})^{-1} : \left[\frac{1}{3}, +\infty\right) \to [\theta_1, +\infty), \\
\psi_2 &:= (F|_{[\theta_2, \theta_1]})^{-1} : \left[\frac{1}{3}, \frac{1}{2}\right] \to [\theta_2, \theta_1] \\
\psi_3 &:= (F|_{[0, \theta_2]})^{-1} : \left[0, \frac{1}{2}\right] \to [0, \theta_2].
\end{align*}
$$

We identify the following gradients $0 < p_{+, s} < q_{-, s} < p_+$, between which $H_s$ changes its shape:

\begin{equation}
\begin{aligned}
p_{+, s} &:= \int_0^{1/3} \psi_3(y) \, dy + \int_{1/2}^{1} \psi_1(y) \, dy + \int_{1/3}^{1/2} [s\psi_1(y) + (1 - s)\psi_3(y)] \, dy, \\
q_{-s} &:= \int_{1/2}^{4/3} \psi_1(y) \, dy + \int_{1/3}^{1/2} [s\psi_1(y) + (1 - s)\psi_3(y)] \, dy, \quad \text{and} \\
p_+ &:= \int_{1/3}^{4/3} \psi_1(y) \, dy.
\end{aligned}
\end{equation}

**Proposition B.1.** Suppose that $p \geq 0$. The function $\overline{H}_s$ can be characterized as follows:

(a) If $0 \leq p \leq p_{+, s}$, then $\overline{H}_s(p) = 0$.

(b) If $p_{+, s} \leq p \leq q_{-s}$, then $\overline{H}_s(p)$ is the unique constant $\lambda \in [0, 1/3]$ for which

$$p = \int_0^{1/3} \psi_3(y) \, dy + \int_{1/2}^{1+\lambda} \psi_1(y) \, dy + \int_{1/3}^{1/2} [s\psi_1(y) + (1 - s)\psi_3(y)] \, dy.$$

(c) If $q_{-s} \leq p \leq q_+$, then $\overline{H}_s(p) = \frac{1}{3}$.

(d) If $p \geq q_+$, then $\overline{H}_s(p)$ is the unique constant $\lambda \geq \frac{1}{3}$ for which

$$p = \int_{1/3}^{1+\lambda} \psi_1(y) \, dy.$$  

(e) If $p < 0$, then $\overline{H}_s(p) = \overline{H}_{1-s}(-p)$.

Obtaining the formula for $\overline{H}_s(p)$ involves constructing viscosity solutions of the equation

\begin{equation}
F(w'(y)) - V_s(y) = \lambda \quad \text{in } \mathbb{R}
\end{equation}

such that $w(x) - px$ is periodic, which is possible only for the unique constant $\lambda = \overline{H}_s(p)$. We make use of the following lemma, whose proof is a consequence of the definition of viscosity solutions:
Lemma B.1. Assume that $F(f(y)) + V_s(y) = \lambda$ at all points $y \in \mathbb{R}$ at which $f$ is continuous, and, whenever $y_0 \in \mathbb{R}$, $p_1 := f(y_0)$, and $p_2 := f(y_0^+)$, then $F(p_1) = F(p_2) = \lambda + V_s(y_0)$ and

\[ p_1 < p_2 \implies F(p) \geq \lambda + V_s(y_0) \text{ for } p \in [p_1, p_2], \]
\[ p_1 > p_2 \implies F(p) \leq \lambda + V_s(y_0) \text{ for } p \in [p_2, p_1]. \]

Then \( \{ y \mapsto w(y) := \int_0^y f(x) \, dx \} \) is a viscosity solution of (B.3), and

\[ \Pi \left( \int_0^1 f(x) \, dx \right) = \lambda. \]

For the rest of the section, we construct correctors using Lemma B.1 as a blueprint, that is, for each $p \in \mathbb{R}$, we construct $f$ as in the hypotheses of Lemma B.1 for the correct constant $\Pi_s(p)$.

Define the gradients $p_{0,s} < p_4 < p_3 < p_2 < p_1 < p_{+,s}$ by

\[
\begin{align*}
  p_0, s := (2s - 1) & \int_0^{1/3} \psi_3(y) \, dy + (2s - 1) \int_{1/3}^1 \psi_1(y) \, dy, \\
  p_1 := (2s - 1) & \int_0^{1/3} \psi_3(y) \, dy + \int_{1/3}^1 \psi_1(y) \, dy + \int_{1/3}^{1/2} [s\psi_1(y) + (1 - s)\psi_3(y)] \, dy, \\
  p_2 := (2s - 1) & \int_0^{1/3} \psi_3(y) \, dy + \int_{1/3}^1 \psi_1(y) \, dy + \int_{1/3}^{1/2} [s\psi_1(y) - (1 - s)\psi_3(y)] \, dy, \\
  p_3 := (2s - 1) & \int_0^{1/3} \psi_3(y) \, dy + \int_{1/3}^1 \psi_1(y) \, dy + \int_{1/3}^{1/2} [s\psi_1(y) - (1 - s)\psi_2(y)] \, dy, \text{ and} \\
  p_4 := (2s - 1) & \int_0^{1/3} \psi_3(y) \, dy + \int_{1/3}^1 \psi_1(y) \, dy + (2s - 1) \int_{1/3}^{1/2} \psi_1(y) \, dy.
\end{align*}
\]

The formula for $\Pi_s(p)$ will be established for all $p \geq p_{0,s}$, and the formula for the remaining gradients follows because $p_{0,1-s} = -p_{0,s}$ and $\Pi_{1-s}(p) := \Pi_s(-p)$.

**Case 1:** $p_1 \leq p \leq p_{+,s}$ and $\lambda = 0$

\[ f(x) := \begin{cases} 
  \phi_3(V_s(x)) & \text{if } x \in \left(0, \frac{s}{3}\right) \cup \left(\frac{1+s}{2}, 1 - \tau(1-s)\right), \\
  \phi_1(V_s(x)) & \text{if } x \in \left(\frac{s}{3}, \frac{1+s}{2}\right), \\
  -\phi_3(V_s(x)) & \text{if } x \in (1 - \tau(1-s), 1),
\end{cases} \]

where $\tau \in [0, 1/3]$ is given uniquely by

\[ p = (2s - 1) \int_0^\tau \psi_3(y) \, dy + \int_0^{1/3} \psi_3(y) \, dy + \int_{1/3}^{1/2} [s\psi_1(y) + (1 - s)\psi_3(y)] \, dy + \int_{1/2}^1 \psi_1(y) \, dy. \]

**Case 2:** $p_2 \leq p \leq p_1$ and $\lambda = 0$

\[ f(x) := \begin{cases} 
  \phi_3(V_s(x)) & \text{if } x \in \left(0, \frac{s}{3}\right) \cup \left(\frac{1+s}{2}, 1 - \tau(1-s)\right), \\
  \phi_1(V_s(x)) & \text{if } x \in \left(\frac{s}{3}, \frac{1+s}{2}\right), \\
  -\phi_3(V_s(x)) & \text{if } x \in (1 - \tau(1-s), 1),
\end{cases} \]
where $\tau \in [1/3, 1/2]$ is given uniquely by

$$p = (2s - 1) \int_0^{1/3} \psi_3(y) \, dy + \int_{1/3}^{\tau} [s\psi_1(y) - (1 - s)\psi_3(y)] \, dy$$

$$+ \int_{\tau}^{1/2} [s\psi_1(y) - (1 - s)\psi_3(y)] \, dy + \int_{1/2}^{1} \psi_1(y) \, dy.$$

**Case 3:** $p_3 \leq p \leq p_2$ and $\lambda = 0$

$$f(x) := \begin{cases} 
\psi_3(V_s(x)) & \text{if } x \in \left(0, \frac{s}{3}\right), \\
\psi_1(V_s(x)) & \text{if } x \in \left(\frac{s}{3}, \frac{1 + s}{2}\right), \\
-\psi_2(V_s(x)) & \text{if } x \in \left(\frac{1 + s}{2}, 1 - \tau(1 - s)\right), \\
-\psi_3(V_s(x)) & \text{if } x \in (1 - \tau(1 - s), 1), 
\end{cases}$$

where $\tau \in [1/3, 1/2]$ is given uniquely by

$$p := (2s - 1) \int_0^{1/3} \psi_3(y) \, dy + \int_{1/3}^{\tau} [s\psi_1(y) - (1 - s)\psi_3(y)] \, dy$$

$$+ \int_{\tau}^{1/2} [s\psi_1(y) - (1 - s)\psi_2(y)] \, dy + \int_{1/2}^{1} \psi_1(y) \, dy.$$

**Case 4:** $p_4 \leq p \leq p_3$ and $\lambda = 0$

$$f(x) := \begin{cases} 
\psi_3(V_s(x)) & \text{if } x \in \left(0, \frac{s}{3}\right), \\
\psi_1(V_s(x)) & \text{if } x \in \left(\frac{s}{3}, \frac{1 + s}{2}\right), \\
-\psi_2(V_s(x)) & \text{if } x \in \left(\frac{1 + s}{2}, 1 - \tau(1 - s)\right), \\
-\psi_1(V_s(x)) & \text{if } x \in (1 - \tau(1 - s), \frac{2 + s}{3}), \\
-\psi_3(V_s(x)) & \text{if } x \in \left(\frac{2 + s}{3}, 1\right), 
\end{cases}$$

where $\tau \in [1/3, 1/2]$ is given uniquely by

$$p := (2s - 1) \int_0^{1/3} \psi_3(y) \, dy + (2s - 1) \int_{1/3}^{\tau} \psi_1(y) \, dy + \int_{\tau}^{1/2} [s\psi_1(y) - (1 - s)\psi_2(y)] \, dy + \int_{1/2}^{1} \psi_1(y) \, dy.$$

**Case 5:** $p_{0,s} \leq p \leq p_4$ and $\lambda = 0$

$$f(x) := \begin{cases} 
\psi_3(V_s(x)) & \text{if } x \in \left(0, \frac{s}{3}\right), \\
\psi_1(V_s(x)) & \text{if } x \in \left(\frac{s}{3}, 1 - \tau(1 - s)\right), \\
-\psi_2(V_s(x)) & \text{if } x \in \left(\frac{1 + s}{2}, 1 - \tau(1 - s)\right), \\
-\psi_1(V_s(x)) & \text{if } x \in (1 - \tau(1 - s), \frac{2 + s}{3}), \\
-\psi_3(V_s(x)) & \text{if } x \in \left(\frac{2 + s}{3}, 1\right), 
\end{cases}$$
where $\tau \in [1/2, 1]$ is given uniquely by
\[
p := (2s - 1) \int_0^{1/3} \psi_3(y) \, dy + (2s - 1) \int_{1/3}^\tau \psi_1(y) \, dy + \int_{\tau}^1 \psi_1(y) \, dy.
\]

**Case 6:** $p_{+, s} \leq p \leq q_{-, s}$ and $\lambda \in [0, 1/3]$ satisfies
\[
p = \int_\lambda^{1/3} \psi_3(y) \, dy + \int_{1/3}^{1+\lambda} \psi_1(y) \, dy + \int_{1/3}^{1/2} [s\psi_1(y) + (1 - s)\psi_3(y)] \, dy.
\]
\[
f(y) := \begin{cases}
\psi_3(\lambda + V_s(x)) & \text{if } s \in \left(0, \left(1 - 3\lambda\right)\frac{s}{3}\right) \cup \left(\frac{1 + s}{2} + \lambda(1 - s), 1\right), \\
\psi_1(\lambda + V_s(x)) & \text{if } x \in \left(1 - 3\lambda, 1 - \frac{1 + s}{2} + \lambda(1 - s)\right).
\end{cases}
\]

Before moving on to the next case, we define
\[
q_1 := \int_{1/2}^{4/3} \psi_1(y) \, dy + \int_{1/3}^{1/2} [s\psi_1(y) + (1 - s)\psi_2(y)] \, dy.
\]

**Case 7:** $q_{-, s} \leq p \leq q_1$ and $\lambda = 1/3$

There exists a unique $\tau \in [1/3, 1/2]$ such that
\[
p = \int_{1/3}^{\tau} [s\psi_1(y) + (1 - s)\psi_2(y)] \, dy + \int_{\tau}^{1/2} [s\psi_1(y) + (1 - s)\psi_3(y)] \, dy + \int_{1/2}^{4/3} \psi_1(y) \, dy.
\]
Let $\mu \in [(5 + s)/6, 1]$ be defined by
\[
\tau = \frac{1}{3} + \frac{1 - \mu}{1 - s} \in \left[\frac{1}{3}, 1\right],
\]
and define
\[
f(x) := \begin{cases}
\psi_1(1/3 + V_s(x)) & \text{if } x \in \left(0, \frac{5 + s}{6}\right), \\
\psi_3(1/3 + V_s(x)) & \text{if } x \in \left(\frac{5 + s}{6}, \mu\right), \\
\psi_2(1/3 + V_s(x)) & \text{if } x \in (\mu, 1).
\end{cases}
\]

**Case 8:** $q_1 \leq p \leq q_+$ and $\lambda = 1/3$

There exists a unique $\tau \in [1/3, 1/2]$ such that
\[
p = \int_{1/3}^{\tau} [s\psi_1(y) + (1 - s)\psi_2(y)] \, dy + \int_{\tau}^{4/3} \psi_1(y) \, dy.
\]
Let $\mu \in [(5 + s)/6, 1]$ be defined by
\[
\tau = \frac{1}{3} + \frac{1 - \mu}{1 - s} \in [1/3, 1/2],
\]
and define
\[
f(x) := \begin{cases}
\psi_1(1/3 + V_s(x)) & \text{if } x \in (0, \mu), \\
\psi_2(1/3 + V_s(x)) & \text{if } x \in (\mu, 1).
\end{cases}
\]
Case 9: If \( p \geq q_{+s} \) and \( \lambda \in (1/3, \infty) \) satisfies
\[
p = \int_{\lambda}^{\lambda^1+\lambda} \psi_1(y) \, dy,
\]
then define
\[
f(x) := \psi_1(\lambda + V_s(x)).
\]

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