FAST TRACK COMMUNICATION

Discrete holomorphic parafermions in the Ashkin–Teller model and SLE

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Abstract
We find discrete holomorphic parafermions of the Ashkin–Teller model on the critical line, by mapping appropriate interfaces of the model onto the $O(n = 1)$ model. We give support to the conjecture that the curve created by the insertion of parafermionic operators at two points on the boundary is SLE$(4, \rho, \rho)$, where $\rho$ varies along the critical line.

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1. Introduction

The discovery of the Schramm–Loewner evolution (SLE) by Schramm [1] opened a rigorous way to study conformally invariant systems. Some well-known results from the physics literature were proved [2–4], and new formulae were also discovered [5]. Notably, conformal invariance was proved for percolation clusters [2] and for Ising Fortuin–Kasteleyn and spin clusters [3, 4]. One of the key steps in these proofs is to find an appropriate discrete holomorphic parafermion for the critical interface. Relating the holomorphic parafermion to a specific property of the critical curve, one can conjecture and then prove that the continuum limit of the interface is SLE. Different methods of finding discrete holomorphic parafermions for statistical models and their relation to integrability were investigated by Cardy and collaborators in a series of papers [6–8]. In some cases [6, 8] the relation of the parafermion to a critical interface is known, and so there are conjectures on the SLE corresponding to the continuum limit of the interfaces, but in other cases [7] the corresponding interfaces are not known and so there is no known SLE.

One of the interesting models where neither discrete holomorphic parafermions nor corresponding SLE was known until now is the Ashkin–Teller (AT) model [9, 10] (the phase diagram is described in [11–13]). From the conformal field theory (CFT) point of view, the AT model is interesting because it has a critical line with constant central charge $c = 1$ and changing critical exponents. Although it is possible to define different kinds of critical
interfaces for the AT model [14, 15], it is shown numerically [14] that some of the natural possibilities are not related to simple SLEs. Recently, in [16], the AT model was studied on iso-radial graphs, the critical surfaces on generic iso-radial graphs were found, and discrete holomorphic parafermions were defined at some particular points of the critical line, by using algebraic relations between spin and disorder variables. In this communication, we introduce discrete holomorphic parafermions all over the critical line, without using these relations directly. The idea is to map some particular interfaces onto the O\((n = 1)\) model on the square lattice, and then to exploit some of the results of [8].

Our main conjecture is based on the continuum Gaussian theory for the solid-on-solid (SOS) model associated with the AT model, and on the known relation between this theory and SLE\(_4\) [17]: in the upper half-plane, we expect the curve created by the insertion of a parafermion at the origin and at infinity to have the statistics of SLE\(_4(\sqrt{g} - 1, \sqrt{g} - 1)\), where \(g\) is the coupling constant of the Gaussian theory (see section 4 for details on the definition of \(g\)).

The structure of the communication is as follows. In section 2, we recall the definition of the AT model on the square lattice and its mapping onto the staggered six-vertex model. In section 3, the six-vertex model is mapped onto the O\((n = 1)\) model, and, using this mapping, we find a lattice holomorphic parafermion for the AT model all over the critical integrable surface. In section 4, we formulate conjectures on the relation of our interfaces to SLE. Section 5 is dedicated to bulk critical exponents in the O\((n = 1)\) loop model associated with the AT model. The results are checked numerically by transfer-matrix diagonalization.

2. The Ashkin–Teller model

2.1. Definition and graphical expansion

The Ashkin–Teller model can be defined on any graph, but, for clarity, we will restrict to the square lattice in this communication. At each vertex \(j\) of the lattice, we define two spin variables \(\sigma_j\) and \(\tau_j\), which can take individually the values \(\pm 1\). A spin configuration \(\{\sigma_j, \tau_j\}\) gets the Boltzmann weight:

\[
\prod_{\langle i, j \rangle} W(i, j), \quad \text{where} \quad W(i, j) = \exp[\beta_{ij}(\sigma_i \sigma_j + \sigma_j \tau_i) + \alpha_{ij} \sigma_i \sigma_j \tau_i \tau_j],
\]

(1)

and \(\alpha_{ij}, \beta_{ij} = \alpha_x, \beta_x\) (resp. \(\alpha_y, \beta_y\)) if \(\langle i, j \rangle\) is a horizontal (resp. vertical) bond. We denote by \(\langle \cdot \cdot \cdot \rangle\) the averaging with respect to the normalized Boltzmann weights \(W(i, j)/Z\), where

\[
Z = \sum_{\{\sigma_j, \tau_j\}} \prod_{\langle i, j \rangle} W(i, j).
\]

(2)

Let us recall the graphical expansion of this partition function [18, 19]. Introducing the change of variables \((\sigma_j, \tau_j) \rightarrow (\sigma_j, \tau'_j)\), where \(\tau'_j = \sigma_j \tau_j\), the edge interaction can be written as

\[
W(i, j) = e^{\alpha_j \tau'_i \tau'_j} [\cosh \beta_{ij}(1 + \tau'_i \tau'_j) + \sigma_i \sigma_j \sinh \beta_{ij}(1 + \tau'_i \tau'_j)].
\]

(3)

First, we fix the values of the \(\tau'_j\) spins, which define a domain-wall (DW) configuration on the edges of the dual lattice. Let \((i, j)\) be an edge of the original lattice. If \((i, j)\) is crossed by a DW, then it gets a weight \(e^{-\alpha_j}\). If not, then it gets the weight \(e^{\alpha_j [\cosh 2\beta_{ij} + \sigma_i \sigma_j \sinh 2\beta_{ij}]}\). We depict the first term as an empty edge, and the second one as an occupied edge. Thus, the partition function reads

\[
Z = \text{const} \times \sum_{G/G'} (\tanh 2\beta_{i_k})^{\ell_k(G)} (\tanh 2\beta_{j_k})^{\ell_k(G')} (e^{2\alpha_j} \cosh 2\beta_{i_k})^{-\ell_k(G)} (e^{2\alpha_i} \cosh 2\beta_{j_k})^{-\ell_k(G')},
\]

(4)
where $G$ (resp. $G'$) is a subgraph of the original (resp. dual) lattice, all vertices in $G$ and $G'$ must have even degree, $G$ and $G'$ must not intersect each other, and $\ell_x(G), \ell_y(G)$ are the numbers of horizontal and vertical bonds of $G$. An example configuration is shown in figure 1.

2.2. Mapping onto the six-vertex model

One interest of this graphical expansion is that it maps onto the six-vertex model on the medial lattice. The correspondence is given in figure 2. An example configuration is given in figure 3. The Ashkin–Teller model is critical when the six-vertex model is not staggered. The weights of the six-vertex model with parameter $\Delta = (a^2 + b^2 - c^2)/(2ab) = -\cos \lambda$ can be parameterized as

$$a, b, c = \sin(\lambda - u), \sin u, \sin \lambda.$$  \hspace{1cm} (5)

This corresponds to the weights of the AT model:

$$\tanh 2\beta_x = \frac{\sin u}{\sin \lambda}, \quad \frac{e^{-2\alpha_x}}{\cosh 2\beta_x} = \frac{\sin(\lambda - u)}{\sin \lambda}$$  \hspace{1cm} (6)

$$\tanh 2\beta_y = \frac{\sin(\lambda - u)}{\sin \lambda}, \quad \frac{e^{-2\alpha_y}}{\cosh 2\beta_y} = \frac{\sin u}{\sin \lambda}.$$  \hspace{1cm} (7)

The variable $u$ is a spectral parameter, whereas $\lambda$ determines the universality class. The isotropic point is at $u = \lambda/2$. Some special values are $\lambda_{P} = 0, \lambda_{FZ} = \frac{\pi}{4}, \lambda_{I} = \frac{\pi}{2}$ and $\lambda = \frac{3\pi}{4}$, corresponding to the four-state Potts, $\mathbb{Z}_4$ Fateev–Zamoïlochikov [20], Ising $\times$ Ising and XY models. When $\lambda$ is varied, the central charge remains constant $c = 1$, but the critical exponents change. For example, the correlation exponent is

$$\nu(\lambda) = \frac{2\pi - 2\lambda}{3\pi - 4\lambda}.$$  \hspace{1cm} (8)

3 An earlier construction, described in [12], relates the AT model to a staggered eight-vertex model, without using the $\tau'$ variables. However, in this communication, we focus on the mapping onto the six-vertex model because the latter has a known discrete holomorphic parafermion.
More generally, it was shown in [16] that the critical weights of the AT models on any Baxter lattice are given by (6) and (7), with the spectral parameter $u$ related to the angle of $\theta$ the rhombic faces by $\theta = \pi u / \lambda$. In the following section, we describe a discrete holomorphic parafermion in the critical square-lattice AT model. Note that this parafermion is also present for the AT model on a Baxter lattice, at the critical value of the Boltzmann weights.

3. Discrete holomorphic parafermion

In [16], using the spin variables of the AT model, discrete holomorphic parafermions were found at some particular points of the critical line, namely at the four-state Potts, FZ, Ising and XY points. In this section, we exhibit another discrete holomorphic parafermion all along the critical line of the AT model. This parafermion is defined in the loop formulation of the AT model. The key idea is to use the chain of mappings:

\[
\text{Ashkin–Teller} \rightarrow \text{six-vertex} \rightarrow O(n = 1).
\]
Figure 4. Mapping between the six-vertex model and an O\((n = 1)\) loop model. The correspondence between the arrow and loop vertices shown here is valid on sublattice 1 of the medial lattice. On sublattice 2, the same correspondence holds, with all arrows reversed.

The first mapping is described in section 2, and the second one is given, for example, in [8]. For completeness, we recall it in figure 4. The resulting loop model has seven possible vertices, and each closed loop has a Boltzmann weight \(n = 1\). We will denote this model as the O\((n = 1)\) model for short.

It was shown in [8] that the O\((n = 1)\) model, with weights given by (5), possesses a discrete holomorphic parafermion \(\psi(z)\), associated with the insertion of one path at point \(z\). The spin \(s\) of this parafermion is related to the parameter \(\lambda\) by

\[
s = 1 - \frac{\lambda}{\pi}.
\]

In terms of the AT model, \(\psi\) is defined on the edges of the covering lattice (the union of the original and dual lattices). If \(z\) denotes an edge of this lattice, the operator \(\psi(z)\) inserts a \(\sigma\) variable and a \(\tau'\) DW at the ends of the edge \(z\). The two-point function \(\langle \psi(z_1)\psi(z_2) \rangle\) also contains a non-local phase factor given by the simultaneous winding of the \(\tau'\) DW and \(\sigma\) cluster between \(z_1\) and \(z_2\). We might decompose symbolically \(\psi\) into three factors:

\[
\psi = e^{i\theta} \times \sigma \times \mu_{\tau'},
\]

where \(\theta\) is a contribution to the winding angle (see above) and \(\mu_{\tau'}\) creates a \(\tau'\) domain wall.

Let us interpret the effect of \(\psi\) on the AT model, when inserted at two boundary points \(a\) and \(b\). In this situation, there must be, in the O\((n = 1)\) model, a path \(\gamma\) going from \(a\) to \(b\) (see figure 5). The occupied area adjacent to \(\gamma\) is bounded on one side by a \(\tau'\) domain wall, and on the other side by a \(\sigma\) high-temperature cluster. So \(\langle \psi(a)\psi(b) \rangle\) corresponds to the two-point function \(\langle \sigma(a)\sigma(b) \rangle_{1,ab}\), where \(\langle \cdot \cdot \cdot \rangle_{1,ab}\) denotes the averaging with Boltzmann weights \(W(i,j)/Z\) and the following boundary conditions (denoted BC_1): free boundary conditions for \(\sigma\) and \(\tau' = 1\) on one arc \((ab)\) of the boundary, \(\tau' = -1\) on the other arc.

We can also think of boundary conditions which allow the path \(\gamma\) to end anywhere in a given interval of the boundary. Let \(a, b, c\) be three marked points on the boundary. We consider the two-point function \(\langle \sigma(a)\sigma(b) \rangle_{2,abc}\), where \(\langle \cdot \cdot \cdot \rangle_{2,abc}\) is the Boltzmann average with the following boundary conditions: free BC for \(\sigma\), \(\tau' = +1\) on \((ab)\) and \((bc)\), \(\tau' = -1\) on \((ac)\). In the O\((n = 1)\) model, this forces a path \(\gamma\) to go from \(a\) to a point on \((bc)\), as shown in figure 5. We call these boundary conditions BC_2. If \(\mu_{\tau'}\) is viewed as a boundary-condition changing operator, the correlation function \(\langle \sigma(a)\sigma(b) \rangle_{2}\) can be written as a three-point function, reflecting explicitly the role of point \(c\):

\[
\langle \sigma(a)\sigma(b) \rangle_{2,abc} = \langle \psi(a)\sigma(b)\mu_{\tau'}(c) \rangle.
\]
4. Relation to SLE

The six-vertex model can be transformed into an SOS model by introducing height variables \( \psi \) on the faces of the medial lattice, so that two neighbouring \( \psi \)'s differ by \( \pm \pi/2 \), the highest being on the left of each arrow. The SOS model renormalizes to a Gaussian theory with action

\[
A = \frac{g}{4\pi} \int |\nabla \psi|^2 d^2 x, \quad g = \frac{4(\pi - \lambda)}{\pi}.
\]  

(12)

When the AT model is defined on a system with one or more periodic directions (e.g. a torus or a cylinder), the height \( \psi \) is only well defined up to the identifications \( \psi \equiv \psi + \pi, \psi \equiv -\psi \). The first corresponds to a height defect around one direction of the system, while the second is a ‘twist’ induced by an odd number of DW’s winding around the system [19]. The theory with action (12) and the above identifications is called the \( \mathbb{Z}_2 \)-orbifold of the compact boson [24]. However, in this section, we deal only with simply connected domains, so the SOS configurations are always well defined without any identification, and the continuum limit is simply the Gaussian model (12).

In the case of boundary conditions BC1, the presence of the path \( \gamma \) induces a height gap at \( a \) and \( b \). So we expect the SOS model to be a free field with Dirichlet BC, and boundary values \(-\psi_1\) on one arc, \( \psi_2 \) on the other arc. Since the DW and high-T cluster play the same role at the critical point, we must have \( \psi_1 = \psi_2 \). Schramm and Sheffield have shown [17] that the contour line in such a model is SLE\((4, \rho_1, \rho_2)\), where \( \rho_1 = \psi_1/\psi^* - 1, \rho_2 = \psi_2/\psi^* - 1 \), and \( \psi^* \) is a universal constant. Now, we use the results of [21, 27]: when a consistent normalization is chosen for \( \psi \), we have \( \psi^* = \pi/\sqrt{4g} \) and the operator inserting the height gap \( \delta \psi \) has conformal dimension

\[
h = g \left( \frac{\delta \psi}{2\pi} \right)^2.
\]

Comparing to the spin (9) of the parafermion \( s = g/4 \), we obtain \( \phi_1 = \phi_2 = \pi/2 \), and so we conjecture that the curve \( \gamma \) has the statistics of SLE\((4, \sqrt{\mathbb{R}} - 1, \sqrt{\mathbb{R}} - 1)\).

5. Critical exponents and fractal dimension

In this section, we derive the bulk exponents for watermelon correlation functions in the O\((n = 1)\) model, and give numerical results on some of these exponents. The O\((n = 1)\)
model is a peculiar loop model, where the central charge is fixed, but the critical exponents vary along the critical line.

To calculate bulk exponents, it is easiest to consider the loop model on a cylinder of circumference $L$ sites ($L$ even). In this setting, the model is described by the $\mathbb{Z}_2$-orbifold theory, and part of the conformal spectrum [19] is given by ‘electromagnetic’ exponents

$$X_{em} = \frac{e^2}{2g} + \frac{gm^2}{2}.$$  \hspace{1cm} (13)

In the transfer-matrix formalism, the $k$-leg watermelon exponent corresponds to the dominant eigenvalue in the sector with $k$ strands propagating along the cylinder. For a given $k$,
there are several exponents, according to the parity of the sites where the \( k \) strands sit. Indeed, because of the staggering in the 6V/O\((\eta = 1)\) correspondence, strands sitting on an even (resp. odd) edge are oriented positively (resp. negatively). Each strand contributes a height defect of magnetic charge \( m = \pm 1/2 \), so if we write \( k = k_1 + k_2 \), where \( k_1, k_2 \) are the numbers of strands sitting on even and odd edges, the total magnetic charge is \( m = (k_1 - k_2)/2 \). Hence, when \( k_1 \neq k_2 \), we get the exponent

\[
X_{k_1,k_2} = \frac{g(k_1 - k_2)^2}{8}, \quad k_1 \neq k_2. \tag{14}
\]

In the special case \( k_1 = k_2 \), the total magnetic charge vanishes. Our numerical results allow us to conjecture to the exponent

\[
X_{\ell,\ell} = \frac{\ell^2}{2}. \tag{15}
\]
The fractal dimension of the path \( \gamma \) is related to a two-leg watermelon exponent: 
\[ d_f = 2 - X. \]
For \( k = 2 \), there are two choices: \( X_{2,0} = \frac{g}{2} \) and \( X_{1,1} = \frac{1}{2} \). The choice \( X_{1,1} \) gives a fractal dimension \( d_f = \frac{3}{2} \), which is the correct value for SLE\(_4\). Note that the direct numerical calculations of Picco and Santachiara [23] on the AT model show that the fractal dimension of the boundaries of \( \tau' \) clusters is also \( \frac{3}{2} \).

The central charge and the exponents of the loop model are extracted using the finite-size formulae for the eigenvalues \( \Lambda_j \) of the transfer matrix [22]:

\[
-\log \Lambda_j \simeq L f_\infty + \frac{2\pi}{L} \left( -\frac{c}{12} + X_j \right).
\]

(16)

In figures 6–12, we depicted numerical data for the central charge, the thermal exponent and some watermelon exponents of the \( O(\nu = 1) \) loop model.
6. Conclusion

In this communication, we first found a discrete holomorphic parafermion for the AT model on the whole critical line. There may exist other discrete holomorphic parafermions in this model, but the one we describe is defined in terms of the $O(n = 1)$ loop model, which enables us to relate it to the SLE model. Our conjecture is compatible with our numerical calculations on the $O(n = 1)$ model. Of course, more precise numerical calculation is needed to confirm our conjecture, such as the left–right Schramm’s formula. Unfortunately, no Schramm’s formula for SLE$(\kappa, \rho_1, \rho_2)$ is known, but for $\kappa = 4$ it could be tractable, thanks to the relation with the Gaussian free field. One can also think of Monte–Carlo simulation as it was done for the Ising model [25], and martingale arguments [26].

At the point $\lambda = \pi/2$, the AT model is equivalent to two decoupled Ising models. Oshikawa and Affleck have studied the boundary CFT of this model, in relation to the $\mathbb{Z}_2$ orbifold theory [27]. In particular, our parafermionic observable $\psi$, when inserted in the boundary, corresponds to a jump in Dirichlet boundary conditions for the orbifold ($\varphi_0 = 0 \rightarrow \pi$ in the notations of [27]), together with the insertion of a $\sigma$ operator. An interesting direction would be to try and extend these results to the critical line of the AT model, and analyse the null-vector equations of the boundary operators, to relate them properly to SLE.

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