UNITARILY INVARIANT NORM INEQUALITIES FOR ELEMENTARY OPERATORS INVOLVING $G_1$ OPERATORS

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Abstract. In this paper, motivated by perturbation theory of operators, we present some upper bounds for $|||f(A)Xg(B)+X|||$ in terms of $|||AXB|||+||X|||$ and $|||f(A)Xg(B)−X|||$ in terms of $|||AX|||+||XB|||$, where $A, B$ are $G_1$ operators, $|||·|||$ is a unitarily invariant norm and $f, g$ are certain analytic functions. Further, we find some new upper bounds for the the Schatten 2-norm of $f(A)X ± Xg(B)$. Several special cases are discussed as well.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ denote the $C^*$-algebra of all bounded linear operators on a separable complex Hilbert space $\mathcal{H}$ equipped with the operator norm $||·||$. If $\dim \mathcal{H} = n$, we can identify $\mathcal{B}(\mathcal{H})$ with the matrix algebra $M_n$ of all $n \times n$ matrices with entries in the complex field $\mathbb{C}$. If $z \in \mathbb{C}$, then we write $z$ instead of $zI$, where $I$ denotes the identity operator on $\mathcal{H}$. We write $A \geq 0$ when $A$ is positive (positive semi-definite for matrices). For any operator $A$ in the algebra $\mathbb{K}(\mathcal{H})$ of all compact operators, we denote by $\{s_j(A)\}$ the sequence of singular values of $A$, i.e. the eigenvalues $\lambda_j(|A|)$, where $|A| = (A^*A)^{1/2}$, in decreasing order and repeated according to multiplicity. If the rank $A$ is $n$, we put $s_k(A) = 0$ for any $k > n$.

In addition to the operator norm $||·||$, which is defined on whole of $\mathcal{B}(\mathcal{H})$, a unitarily invariant norm is a map $|||·||| : \mathbb{K}(\mathcal{H}) \rightarrow [0, \infty]$ given by $|||A||| = g(s_1(A), s_2(A), \cdots)$, where $g$ is a symmetric norming function. The set $\mathcal{C}_{|||·|||} = \{A \in \mathbb{K}(\mathcal{H}) : |||A||| < \infty\}$ is a closed self-adjoint ideal $\mathcal{J}$ of $\mathcal{B}(\mathcal{H})$ containing finite rank operators. It enjoys the properties:

(i) For all $A, B \in \mathcal{B}(\mathcal{H})$ and $X \in \mathcal{J}$,

$$|||AXB||| \leq |||A||| |||X||| ||B||| .$$

(1.1)

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(ii) If $X$ is a rank one operator, then
\[ |||X||| = \|X\| \tag{1.2} \]
Inequality (1.1) implies that $|||UAV||| = |||A|||$ for all unitary matrices $U, V \in \mathbb{B}(\mathcal{H})$ and all $A \in \mathcal{J}$. In addition, employing the polar decomposition of $X = W|X|$ with $W$ a partial isometry and (1.1), we have
\[ |||X||| = |||W|X||| \tag{1.3} \]

The Ky Fan norms as an example of unitarily invariant norms are defined by $\|A\|_{(k)} = \sum_{j=1}^{k} s_j(A)$ for $k = 1, 2, \ldots$. The Ky Fan dominance theorem [3, Théorème IV.2.2] states that $\|A\|_{(k)} \leq \|B\|_{(k)}$ ($k = 1, 2, \ldots$) if and only if $|||A||| \leq |||B|||$ for all unitarily invariant norms $|||\cdot|||$; see [3, 9] for more information on unitarily invariant norms. For the sake of brevity, we will not explicitly mention this norm ideal. Thus, when we consider $|||A|||$, we are assuming that $A$ belongs to the norm ideal associated with $|||\cdot|||$. It is known that the Schatten $p$-norms $\|A\|_p = \left( \sum_{j=1}^{\infty} s_j^p(A) \right)^{1/p}$ are unitarily invariant for $1 \leq p < \infty$; cf. [3, Section IV]. We use the notation $A \oplus B$ for the diagonal block matrix $\text{diag}(A, B)$. Its singular values are $s_1(A), s_1(B), s_2(A), s_2(B), \ldots$. It is evident that
\[ \|A \oplus B\| = \max\{\|A\|, \|B\|\} \quad \text{and} \quad \|A \oplus B\|_p = (\|A\|_p^p + \|B\|_p^p)^{1/p}. \tag{1.4} \]

The inequalities involving unitarily invariant norms have been of special interest; see e.g., [15].

An operator $A \in \mathbb{B}(\mathcal{H})$ is called $G_1$ operator if the growth condition
\[ |||(z - A)^{-1}||| = \frac{1}{\text{dist}(z, \sigma(A))} \tag{1.5} \]
holds for all $z$ not in the spectrum $\sigma(A)$ of $A$. Here $\text{dist}(z, \sigma(A))$ denotes the distance between $z$ and $\sigma(A)$. It is known that hyponormal (in particular, normal) operators are $G_1$ operators (see, e.g., [17]).

Let $A \in \mathbb{B}(\mathcal{H})$ and let $f$ be a function which is analytic on an open neighborhood $\Omega$ of $\sigma(A)$ in the complex plane. Then $f(A)$ denotes the operator defined on $\mathbb{H}$ by the Riesz-Dunford integral as
\[ f(A) = \frac{1}{2\pi i} \int_C f(z)(z - A)^{-1}dz, \tag{1.6} \]
where $C$ is a positively oriented simple closed rectifiable contour surrounding $\sigma(A)$ in $\Omega$ (see, e.g., [8, p. 568]). The spectral mapping theorem asserts that $\sigma(f(A)) = f(\sigma(A))$. Throughout this note, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denotes the
unit disk, $\partial \mathbb{D}$ stands for the boundary of $\mathbb{D}$ and $d_A = \text{dist}(\partial \mathbb{D}, \sigma(A))$. In addition, we adopt the notation

$$\mathcal{H} = \{f : \mathbb{D} \to \mathbb{C} : f \text{ is analytic, } \Re(f) > 0 \text{ and } f(0) = 1\}.$$  

The Sylvester type equations $AXB \pm X = C$ have been investigated in matrix theory; see, e.g. [2]. In addition, operators of the form $R(X) = \sum_{i=1}^{n} A_i X B_i$, in particular $\Delta_{A,B} = AXB - X$, are called elementary and have been studied in various aspects by several people; see, e.g. [6]. Some mathematicians try to find some upper and lower bounds for norms of elementary operators; cf. [18]. Regarding $AXB \pm X$, it is shown in [14] that there is a constant $\gamma_p$ for any $1 < p < \infty$ such that for any $A, B, X \in \mathfrak{B}(\mathcal{H})$ such that $A, B$ are positive, it holds that $\|AXB \pm X\|_p \geq \gamma_p \|X\|_p$.

Several perturbation bounds for the norm of sum or difference of operators have been presented in the literature by employing some integral representations of certain functions; cf. [4, 13]. For some other perturbation results the reader is referred to [10, 12]. In this paper, we present some upper bounds for $\|f(A)Xg(B) \pm X\|$, where $A, B$ are $G_1$ operators, $\|\cdot\|$ is a unitarily invariant norm and $f, g \in \mathcal{H}$. Further, we find some new upper bounds for the the Schatten 2-norm of $f(A)X \pm Xg(B)$. Several applications are presented as well.

2. Upper bounds for $\|f(A)Xg(B) \pm X\|$

In this section, we find some upper bounds for $\|f(A)Xg(B) \pm X\|$ in terms of $\|AXB\| + |X|\|$ and $\|f(A)Xg(B) - X\|$ in terms of $\|AX\| + |XB|\|$, where $A, B$ are $G_1$ operators, $\|\cdot\|$ is a unitarily invariant norm and $f, g \in \mathcal{H}$, and present several consequences.

Our main result of this section reads as follows.

**Theorem 2.1.** If $A, B \in \mathfrak{B}(\mathcal{H})$ are $G_1$ operators with $\sigma(A) \cup \sigma(B) \subset \mathbb{D}$ and $f, g \in \mathcal{H}$, then for every $X \in \mathfrak{B}(\mathcal{H})$ and for every unitarily invariant norm $\|\cdot\|$, the inequalities

$$\|f(A)Xg(B) + X\| \leq \frac{2\sqrt{2}}{d_A d_B} \|AXB\| + |X|\|, \quad (2.1)$$

and

$$\|f(A)Xg(B) - X\| \leq \frac{2\sqrt{2}}{d_A d_B} \|AX\| + |XB|\|, \quad (2.2)$$

hold.
Proof. We prove inequality (2.1), the other inequality can be proved in a similar fashion.

It follows from the Herglotz representation theorem (see, e.g., [7, p. 21]) that $f \in H$ can be represented as

$$f(z) = \int_0^{2\pi} \frac{e^{i\alpha} + z}{e^{i\alpha} - z} d\mu(\alpha) + i3f(0) = \int_0^{2\pi} \frac{e^{i\alpha} + z}{e^{i\alpha} - z} d\mu(\alpha)$$

(2.3)

where $\mu$ is a positive Borel measure on the interval $[0, 2\pi]$ with finite total mass $\int_0^{2\pi} d\mu(\alpha) = f(0) = 1$. Similarly $g(z) = \int_0^{2\pi} \frac{e^{i\alpha} + z}{e^{i\alpha} - z} d\nu(\alpha)$ for some positive Borel measure $\nu$ on the interval $[0, 2\pi]$ with finite total mass 1. We have

$$f(A)Xg(B) + X$$

$$= \int_0^{2\pi} \int_0^{2\pi} \left[ (e^{i\alpha} - A)^{-1} (e^{i\alpha} + A) X (e^{i\beta} + B) (e^{i\beta} - B)^{-1} + X \right] d\mu(\alpha)d\nu(\beta).$$

A simple computation shows that

$$(e^{i\alpha} - A)^{-1} (e^{i\alpha} + A) X (e^{i\beta} + B) (e^{i\beta} - B)^{-1} + X$$

$$= (e^{i\alpha} - A)^{-1} (e^{i\alpha} + A) X (e^{i\beta} + B) (e^{i\beta} - B)^{-1}$$

$$+ (e^{i\alpha} - A)^{-1} (e^{i\alpha} - A) X (e^{i\beta} - B) (e^{i\beta} - B)^{-1}$$

$$= (e^{i\alpha} - A)^{-1} [(e^{i\alpha} + A) X (e^{i\beta} + B) + (e^{i\alpha} - A) X (e^{i\beta} - B)] (e^{i\beta} - B)^{-1}$$

$$= 2 (e^{i\alpha} - A)^{-1} (AXB + e^{i\alpha} X e^{i\beta}) (e^{i\beta} - B)^{-1}.$$ 

Thus, by inequality (1.1), we have

$$|||f(A)Xg(B) + X|||$$

$$\leq \int_0^{2\pi} \int_0^{2\pi} 2 \left\| (e^{i\alpha} - A)^{-1} \right\| \left\| AXB + e^{i\alpha} X e^{i\beta} \right\| \left\| (e^{i\alpha} - B)^{-1} \right\| d\mu(\alpha)d\nu(\beta).$$

(2.4)

Due to $A$ and $B$ are $G_1$ operators, we deduce from (1.5) that

$$\left\| (e^{i\alpha} - A)^{-1} \right\| = \frac{1}{\text{dist}(e^{i\alpha}, \sigma(A))} \leq \frac{1}{\text{dist}(\partial D, \sigma(A))} = \frac{1}{d_A},$$

(2.5)

and similarly

$$\left\| (e^{i\beta} - B)^{-1} \right\| \leq \frac{1}{d_B}.$$ 

(2.6)
In addition, we have
\[
||| (AXB + e^{i\alpha}Xe^{i\beta}) \oplus 0 |||
= ||| (e^{-i\beta}AXB + e^{i\alpha}X) \oplus 0 |||
= \left\| \begin{bmatrix} e^{-i\beta} & e^{i\alpha} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} AXB & 0 \\ X & 0 \end{bmatrix} \right\|
\leq \sqrt{2} \left\| \begin{bmatrix} AXB & 0 \\ X & 0 \end{bmatrix} \right\| 
\leq \sqrt{2} \left\| (|AXB|^2 + |X|^2)^{1/2} \oplus 0 \right\|.
\]
(2.7)

To get the last inequality we applied a result of Ando and Zhan [1, p. 775] to the function \( h(t) = t^{1/2} \). It states that for every positive operators \( C, D \), every non-negative operator monotone function \( h(t) \) on \([0, \infty)\) and every unitarily invariant norm \( \| \cdot \| \) it holds that \( \| |h(A + B)|| \| \leq \| |h(A) + h(B)|| \). Now from the Ky Fan dominance theorem and (2.7) we infer that
\[
||| AXB + e^{i\alpha}Xe^{i\beta} ||| \leq \sqrt{2} \||| AXB \| + |X| \||.
\]
(2.8)

It follows from inequalities (2.4), (2.5), (2.6) and (2.8) that
\[
||| f(A)Xg(B) + X ||| \leq \frac{2\sqrt{2}}{d_A d_B} ||| AXB \| + |X| \|| \int_0^{2\pi} d\mu(\alpha) d\nu(\beta)
= \frac{2\sqrt{2} \mu (\text{d}D) \nu (\text{d}D)}{d_A d_B} \||| AXB \| + |X| \||
\leq \frac{2\sqrt{2}}{d_A d_B} ||| AXB \| + |X| \||
\]
as required.

\[
\square
\]

Remark 2.2. Under the assumptions of Theorem 2.1, we have
\[
|||(f(A)Xg(B) + X) \oplus 0 ||| \leq \frac{4\sqrt{2}}{d_A d_B} ||| AXB \oplus X \||.
\]
To see this, first note that, the Ky Fan dominance theorem and (2.1) yield that
\[
|||(f(A)Xg(B) + X) \oplus 0 ||| \leq \frac{2\sqrt{2}}{d_A d_B} ||| (AXB \| + |X|) \oplus 0 \||.
\]
(2.9)
On the other hand, by inequality (3) in [11], \( s_j((C + D) / 2) \leq s_j(C \oplus D) \) for operators \( C \) and \( D \). Hence \( |||(C + D) \oplus 0||| \leq 2 |||C \oplus D||| \). Utilizing (1.3), we therefore get

\[
|||(|AXB| + |X|) \oplus 0||| \leq 2 |||AXB| \oplus |X|||| = 2 |||AXB \oplus X|||
\]

from which and inequality (2.9), we reach the required inequality.

**Corollary 2.3.** Let \( f, g \in \mathcal{F} \) and \( A \in \mathcal{B}(\mathcal{H}) \) be a \( G_1 \) operator with \( \sigma(A) \subset \mathbb{D} \). Then for every normal operator \( X \in \mathcal{B}(\mathcal{H}) \) commuting with \( A \) and for every unitarily invariant norm \( |||\cdot||| \), the inequalities

\[
|||f(A)Xg(A^*) + X||| \leq \frac{2}{d_A^2} |||A|X|A^* + |X|||,
\]

and

\[
|||f(A)Xg(A^*) - X||| \leq \frac{2}{d_A^2} |||AX| + |XA^*|||
\]

are valid.

**Proof.** First, note that under the assumptions of normality of \( X \) and normality of \( AXB \) and using this fact that \( |||C + D||| \leq |||C| + |D||| \) for any normal operators \( C \) and \( D \) (see [5]), the constant \( \sqrt{2} \) can be reduced to 1 in (2.8).

Second, recall that the Fuglede–Putnam theorem states that if \( A \in \mathcal{B}(\mathcal{H}) \) is an operator, \( X \in \mathcal{B}(\mathcal{H}) \) is normal and \( AX =XA \), then \( AX^* =X^*A \); see [16] and references therein. Thus if \( X \) is a normal operator commuting with a \( G_1 \) operator \( A \), then \( AXA^* \) is normal, \( |AXA^*| = A|X|A^* \) and \( A^* \) is a \( G_1 \) operator with \( d_A^* = d_A \). Hence we get the required inequalities by employing Theorem 2.1. \( \square \)

Next, letting \( A = B \) in (2.2) of Theorem 2.1, we get the following inequality.

**Corollary 2.4.** Let \( f, g \in \mathcal{F} \) and \( A \in \mathcal{B}(\mathcal{H}) \) be a \( G_1 \) operator with \( \sigma(A) \subset \mathbb{D} \). Then

\[
|||f(A)g(A) + I||| \leq \frac{2\sqrt{2}}{d_A d_B} |||AB| + I|||
\]

for every \( X \in \mathcal{B}(\mathcal{H}) \) and for every unitarily invariant norm \( |||\cdot||| \).

Setting \( X = I \) in Theorem 2.1, we obtain the following result.

**Corollary 2.5.** Let \( f, g \in \mathcal{F} \) and \( A, B \in \mathbb{M}_n \) be \( G_1 \) matrices such that \( \sigma(A) \cup \sigma(B) \subset \mathbb{D} \). Then for every unitarily invariant norm \( |||\cdot||| \),

\[
|||f(A)g(B) + I||| \leq \frac{2\sqrt{2}}{d_A d_B} |||AB| + I|||
\]
and
\[ |||f(A)g(B) - I||| \leq \frac{2\sqrt{2}}{d_Ad_B} |||A| + |B||| \].

To achieve our next result, we need the following lemma. Its proof is standard but we provide a proof for the sake of completeness.

**Lemma 2.6.** If \( A \in \mathbb{B}(\mathcal{H}) \) is self-adjoint and \( f \) is a continuous complex function on \( \sigma(A) \), then \( f(UAU^*) = Uf(A)U^* \) for all unitaries \( U \).

**Proof.** By the Stone-Weierstrass theorem, there is a sequence \((p_n)\) of polynomials uniformly converging to \( f \) on \( \sigma(A) \). Hence
\[ f(UAU^*) = \lim_n p_n(UAU^*) = U(\lim_n p_n(A))U^* = Uf(A)U^* . \]
Note that \( \sigma(UAU^*) = \sigma(A) \). \( \square \)

**Proposition 2.7.** Let \( f, g \in \mathcal{H} \) and \( A \in \mathbb{B}(\mathcal{H}) \) be a positive operator with \( \sigma(A) \subset [0, 1) \). Then for every unitarily invariant norm \( |||·||| \) and every unitary operator \( U \in \mathbb{B}(\mathcal{H}) \), it holds that
\[ |||f(A)Ug(A) - U||| \leq \frac{2\sqrt{2}}{d_A^2} |||AU + UA||| . \]

**Proof.**
\[
|||f(A)Ug(A) - U||| = |||f(A)I(Ug(A)U^*) - I|||
= |||f(A)Ig(UAU^*) - I||| \quad \text{(by Lemma 2.6)}
\leq \frac{2\sqrt{2}}{d_A d_{UAU^*}} |||AI| + |IUAU^*||| \quad \text{(by inequality (2.2))}
= \frac{2\sqrt{2}}{d_A^2} |||AU + UA||| ,
\]
since \( d_{UAU^*} = \text{dist}(\partial \mathbb{D}, \sigma(UAU^*)) = \text{dist}(\partial \mathbb{D}, \sigma(A)) = d_A \).

3. Upper bounds for \( |||f(A)X \pm Xg(B)||| \)

In the first result of this section, we find some upper bounds for \( |||f(A)X \pm Xg(B)||| \).

**Theorem 3.1.** Let \( A, B \in \mathbb{B}(\mathcal{H}) \) be \( G_1 \) operators such that \( \sigma(A) \cup \sigma(B) \subset \mathbb{D} \) and \( X \in \mathbb{B}(\mathcal{H}) \). Let \( f, g \in \mathcal{H} \) and \( |||·||| \) be a unitarily invariant norm. Then
\[ |||f(A)X + Xg(B)||| \leq \frac{2\sqrt{2}}{d_Ad_B} |||AXB| + |X||| . \]
and
\[
||| f(A)X - Xg(B) ||| \leq \frac{2\sqrt{2}}{d_A d_B} \left( ||| AX ||| + |X| ||| + ||| XB ||| + |X| ||| + ||| AX ||| + |X| ||| \right).
\]

Proof. Noting that
\[
\int_0^{2\pi} d\mu(\alpha) = \int_0^{2\pi} d\nu(\beta) = 1,
\]
we have
\[
f(A)X + Xg(B)
\]
\[
= \int_0^{2\pi} (e^{i\alpha} - A)^{-1}(e^{i\alpha} + A)X d\mu(\alpha) + \int_0^{2\pi} X(e^{i\beta} + B)(e^{i\beta} - B)^{-1} d\nu(\beta)
\]
\[
= \int_0^{2\pi} \int_0^{2\pi} \left\{ (e^{i\alpha} - A)^{-1}(e^{i\alpha} + A)X(e^{i\beta} - B)(e^{i\beta} - B)^{-1} + (e^{i\alpha} - A)^{-1}(e^{i\alpha} - A)X(e^{i\beta} + B)(e^{i\beta} - B)^{-1} \right\} d\mu(\alpha) d\nu(\beta)
\]
\[
= \int_0^{2\pi} \int_0^{2\pi} (e^{i\alpha} - A)^{-1} [(e^{i\alpha} + A)X(e^{i\beta} - B) + (e^{i\alpha} - A)X(e^{i\beta} + B)] (e^{i\beta} - B)^{-1} d\mu(\alpha) d\nu(\beta)
\]
\[
= 2 \int_0^{2\pi} \int_0^{2\pi} (e^{i\alpha} - A)^{-1}(e^{i\alpha} Xe^{i\beta} - AXB)(e^{i\beta} - B)^{-1} d\mu(\alpha) d\nu(\beta).
\]

Applying the same reasoning as in the proof of Theorem 2.1, we get the required inequality. The proof of the second inequality can be completed similarly. \qed

The next result reads as follows.

**Proposition 3.2.** Under the same assumptions of Theorem 3.1, the inequalities
\[
||| f(A)X + f(A)Xg(B) + Xg(B) |||\]
\[
\leq \frac{\sqrt{2}}{d_A d_B} \left( ||| AXB ||| + |X| ||| + ||| XB ||| + |X| ||| + ||| AX ||| + |X| ||| \right),
\]

and
\[
||| f(A)X + f(A)Xg(B) + Xg(B) |||\]
\[
\leq \frac{2}{d_A d_B} \left( ||| AXB ||| + |AX| + |XB| + 3|X| ||| \right)
\]
hold.

Proof. We prove the first inequality. As before, we have
\[
f(A)X + Xg(B) = 2 \int_0^{2\pi} \int_0^{2\pi} (e^{i\alpha} - A)^{-1}(e^{i\alpha} Xe^{i\beta} - AXB)(e^{i\beta} - B)^{-1} d\mu(\alpha) d\nu(\beta)
\]
(3.1)
and
\[
f(A)Xg(B) = \int_0^{2\pi} \int_0^{2\pi} (e^{i\alpha} - A)^{-1} (e^{i\alpha} + A)X(e^{i\beta} + B)(e^{i\beta} - B)^{-1} d\mu(\alpha)d\nu(\beta).
\]
(3.2)

Adding (3.1) and (3.2), we get
\[
f(A)X + f(A)Xg(B) + Xg(B)
\]
\[
= \int_0^{2\pi} \int_0^{2\pi} (e^{i\alpha} - A)^{-1} \left[ 2(e^{i\alpha}Xe^{i\beta} - AXB) + (e^{i\alpha} + A)X(e^{i\beta} + B) \right] (e^{i\beta} - B)^{-1} d\mu(\alpha)d\nu(\beta)
\]
\[
= \int_0^{2\pi} \int_0^{2\pi} (e^{i\alpha} - A)^{-1} \left[ (e^{i\alpha}Xe^{i\beta} - AXB) + (e^{i\alpha}Xe^{i\beta} + e^{i\alpha}XB) + (e^{i\alpha}Xe^{i\beta} + AXe^{i\beta}) \right] (e^{i\beta} - B)^{-1} d\mu(\alpha)d\nu(\beta).
\]
Consequently,
\[
\|f(A)X + f(A)Xg(B) + Xg(B)\|
\]
\[
\leq \frac{1}{d_A d_B} \left( \|e^{i\alpha}Xe^{i\beta} - AXB\| + \|e^{i\alpha}Xe^{i\beta} + e^{i\alpha}XB\| + \|e^{i\alpha}Xe^{i\beta} + AXe^{i\beta}\| \right).
\]
(3.3)

As before, it can be shown that
\[
\|e^{i\alpha}Xe^{i\beta} - AXB\| \leq \sqrt{2} \|AXB\| + |X| \|.
\]
(3.4)

Moreover,
\[
\|e^{i\alpha}Xe^{i\beta} + e^{i\alpha}XB\| = \|Xe^{i\beta} + XB\|
\]
\[
= \|e^{i\alpha}Xe^{i\beta} + IXB\|
\]
\[
\leq \sqrt{2} \|XB\| + |X| \|.
\]
(3.5)

Similarly,
\[
\|e^{i\alpha}Xe^{i\beta} + AXe^{i\beta}\| \leq \sqrt{2} \|AXB\| + |X| \|.
\]
(3.6)

Now considering (3.4), (3.5) and (3.6) in (3.3), we get the desired inequality. The proof of the second inequality can be completed by an argument similar to that used in the proof of (2.7).

We conclude this article by presenting some inequalities involving the Hilbert-Schmidt norm \(\|\cdot\|_2\).
Theorem 3.3. Let $A, B \in \mathbb{M}_n$ be Hermitian matrices satisfying $\sigma(A) \cup \sigma(B) \subset \mathbb{D}$ and let $f, g \in \mathcal{S}$. Then

$$\|f(A)X \pm Xg(B)\|_2 \leq \left\| \frac{X + |A|X}{d_A} + \frac{X + X|B|}{d_B} \right\|_2,$$

and

$$\|f(A)Xg(B) \pm X\|_2 \leq \left\| \frac{I + |A|}{d_A} X \frac{I + |B|}{d_B} + X \right\|_2.$$

Proof. We prove the first inequality. The second inequality follows similarly. Let $A = UD(\nu_j)U^*$ and $B = VD(\mu_k)V^*$ be the spectral decomposition of $A$ and $B$ and let $Y = U^*XV := [y_{jk}]$. Noting that $|e^{i\alpha} - \lambda_j| \geq d_A$ and $|e^{i\beta} - \mu_k| \geq d_B$, we have

$$\|f(A)X \pm Xg(B)\|_2^2 = \sum_{j,k} |f(\lambda_j) \pm g(\mu_k)|^2 |y_{jk}|^2$$

$$= \sum_{j,k} \left| \int_0^{2\pi} \frac{e^{i\alpha} + \lambda_j}{e^{i\alpha} - \lambda_j} d\mu(\alpha) \pm \int_0^{2\pi} \frac{e^{i\beta} + \mu_k}{e^{i\beta} - \mu_k} d\nu(\beta) \right|^2 |y_{jk}|^2$$

$$\leq \sum_{j,k} \left( \int_0^{2\pi} \frac{|e^{i\alpha} + \lambda_j|}{|e^{i\alpha} - \lambda_j|} d\mu(\alpha) \pm \int_0^{2\pi} \frac{|e^{i\beta} + \mu_k|}{|e^{i\beta} - \mu_k|} d\nu(\beta) \right) |y_{jk}|^2$$

$$\leq \sum_{j,k} \left( \frac{1}{d_A} + \frac{1}{d_B} \right)^2 |y_{jk}|^2$$

$$= \left\| \frac{X + |A|X}{d_A} + \frac{X + X|B|}{d_B} \right\|_2^2,$$

which completes the proof. \hfill \Box

Remark 3.4. The results in Theorem 3.3 can be extended to infinite dimensional separable complex Hilbert spaces. This can be done by using a result in [19], which insures that if $A, B$ are normal operators acting on an infinite dimensional separable Hilbert space, then $A$ and $B$ are Hilbert-Schmidt perturbations of diagonal operators. That is, given $\epsilon > 0$, there exist diagonal operators $\Gamma_1, \Gamma_2$ and unitary operators $U, V$ such that $\|A - UT_1U^*\|_2 < \epsilon$ and $\|B - VT_2V^*\|_2 < \epsilon$.

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