PHOTO-ACOUSTIC INVERSION IN CONVEX DOMAINS

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Abstract. In photo-acoustics one has to reconstruct a function from its averages over spheres around points on the measurement surface. For special surfaces inversion formulas are known. In this paper we derive a formula for surfaces that bound smooth convex domains. It reconstructs the function modulo a smoothing integral operator. For special surfaces the integral operator vanishes, providing exact reconstruction.

1. Introduction. Recently there has been much interest in deriving inversion formulas for photo-acoustic imaging. One has to solve the following inverse problem: Let $p$ be the solution of the initial value problem

\[
\frac{\partial^2 p}{\partial t^2} - \Delta p = 0,
\]

\[
p(x, 0) = p_0(x), \quad \frac{\partial p}{\partial t}(x, 0) = 0.
\]

Let $\Omega$ be a domain in $\mathbb{R}^3$ containing the support of $p_0$ and let $\partial \Omega$ be its boundary. The problem is to determine $p_0$ from the knowledge of $p$ on $\partial \Omega \times \mathbb{R}$. In principle this problem can be solved by back-propagation. However there is much interest in explicit inversion formulas. Various special cases have been treated in the literature. Inversion formulas have been derived for special geometries: $\Omega$ a ball [1], various domains [5], balls, slabs and cylinders [8], cubes [2], ellipsoids [6].

In the present paper we consider the case of a bounded smooth convex domain $\Omega$. We show that in such a domain the “universal back-projection formula” [8] reconstructs $p_0$ up to a smoothing integral operator. This operator disappears for special sets $\Omega$, such as those mentioned above. Our derivation is in the spirit of [4].

2. The general inversion formula. Let us consider the integral

\[
I(x) = \frac{1}{2\pi} \text{div} \int_{\partial \Omega} \nu_y \frac{p(y, |x - y|)}{|x - y|} d\sigma_y
\]

where $\nu$ is the exterior normal and $\sigma$ the surface measure on $\partial \Omega$. This is the inversion integral of [8]. It is shown there that $I(x) = p_0(x)$ for special measurement surfaces, such as spheres, cylinders and planes.

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Let $R$ be the Radon transform

$$(R\chi)(\theta, s) = \int_{x \theta = s} \chi(x) dx, \ \theta \in S^2, \ s \in R^1.$$ 

The derivative of $(R\chi)(\theta, s)$ with respect to $s$ is denoted by $(R\chi)'(\theta, s)$. For $x, z \in \Omega$ let $\theta(z, x) = (z - x)/|z - x|$, $s(z, x) = \frac{1}{2}(|z|^2 - |x|^2)/|z - x|$. $(R\chi)(\theta(z, x), s(z, x))$ is the integral of $\chi$ over the plane of those points which have the same distance to $z$ and $x$.

**Theorem:** Assume that $\Omega$ is a convex bounded and smooth domain in $R^3$. Let $\chi$ be the characteristic function of $\Omega$. Then,

$$I(x) = p_0(x) + \frac{1}{16\pi^2} \int p_0(z) \frac{(R\chi)'''(\theta(z, x), s(z, x))}{|z - x|^2} dz.$$ 

This is the main result of our paper. It states that for $\Omega$ bounded, smooth and convex the inversion formula of [8] is correct up to a smoothing integral operator. As $\Omega$ is bounded, smooth and convex, $(R\chi)(\theta(z, x), s(z, x))$ is a smooth function of $z, x$ on $\Omega$ for $x \neq z$.

Incidentally the Theorem extends the result of [7] to convex measurement geometries.

The proof of the theorem requires some preparations. We first state that according to Kirchhoff’s formula the solution of problem (1,2) is given by

$$p(x, t) = \frac{d}{dt} \left( \frac{1}{4\pi t} \int_{|x - z| = t} p_0(z) dz \right).$$

Thus our problem calls for the reconstruction of $p_0$ from its averages over spheres centered on $\partial \Omega$. Using the language of distributions we can write this as

$$(3) \quad p(x, t) = \frac{1}{4\pi} \int p_0(z) \frac{\delta'(t - |x - z|)}{|x - z|} dz$$

where $\delta$ is the Dirac $\delta$ function in $R^1$. We state some well known facts about distributions. For the Dirac $\delta$ function in $R^3$ we have

$$(4) \quad div \frac{x}{|x|^3} = -div \nabla \frac{1}{|x|} = -\Delta \frac{1}{|x|} = 4\pi \delta(x).$$

By direct calculation one shows that for the $\delta$ function in $R^1$ we have

$$(5) \quad \delta'(t \Phi(t)) = \delta'(t)/\Phi(0)^2$$

where $\Phi \in C^2(R^1)$ satisfies $\Phi(0) > 0$, $\Phi'(0) = 0$. Finally we have

$$(6) \quad \delta(x) = \delta(|x|)/(2\pi|x|^2)$$

with the $\delta$ functions in $R^3$ and $R^1$, respectively, on the left and the right hand side. This is easily seen by introducing polar coordinates.

Let for $x, z \in \Omega$

$$(7) \quad I_\Omega(z, x) = \int_{\Omega} \frac{\delta'(|y - x| - |y - z|)}{|y - x||y - z|} dy.$$ 

The key tool of the proof is the following

**Lemma:** Let $\Omega$ be a bounded, smooth and convex domain in $R^3$. With $\chi$ the characteristic function of $\Omega$ we have for $x, z \in \Omega$, $x \neq z$

$$I_\Omega(z, x) = -\frac{1}{|z - x|^2} (R\chi)'(\theta(z, x), s(z, x)).$$
Proof of the Lemma: In the integral we make the substitution \( y = (z+x)/2 + t(z-x)/2 + w \) where \( t \in R^4 \) and \( w \in \theta(z,x)^{+} \). Then, \( y - z = (t-1)(z-x)/2 + w, \ y - x = (t+1)(z-x)/2+w \), hence \( |y-z| = \sqrt{(t-1)^2v^2 + |w|^2}, \ |y-x| = \sqrt{(t+1)^2v^2 + |w|^2} \) where \( v = |z-x|/2 \). This yields

\[
I_\Omega(z,x) = v \int_{\theta(z,x)^{+}} \frac{\chi((z+x)/2+t(z-x)/2+w)}{(t-1)^2v^2 + |w|^2 \sqrt{(t+1)^2v^2 + |w|^2}} \delta'(t \Phi_w(t)) dt dw
\]

Putting

\[
\Phi_w(t) = \left( \sqrt{(t+1)^2v^2 + |w|^2} - \sqrt{(t-1)^2v^2 + |w|^2} \right) / t
\]

we have \( \Phi'_w(0) = 0 \) and \( \Phi_w(0) = 2v^2/\sqrt{v^2 + |w|^2} > 0 \). Applying (5) to the \( t \)-integral we obtain

\[
I_\Omega(z,x) = v \int_{\theta(z,x)^{+}} \frac{1}{\Phi_w(0)^2} \int_{R^4} \frac{\chi((z+x)/2+t(z-x)/2+w)}{(t-1)^2v^2 + |w|^2 \sqrt{(t+1)^2v^2 + |w|^2}} \delta'(t) dt dw
\]

\[
= -\frac{v}{4v^4} \frac{z-x}{2} \cdot \int_{\theta(z,x)^{+}} \nabla \chi((z+x)/2+w) dw
\]

\[
= -\frac{v}{4v^4} \frac{z-x}{2} \cdot (R \nabla \chi)(\theta(z,x), s(z,x))
\]

Since \( (R \nabla \chi)(\theta, s) = \theta(R \chi)'(\theta, s) \) (see formula (II.1.2) in [3]) we finally obtain the result.

Now we proceed to the

Proof of the Theorem: Applying the Gauss integral theorem we obtain

\[
I(x) = \frac{1}{2\pi} \text{div} \int_{\Omega} \frac{\nabla_y p(y, |x-y|)}{|x-y|} dy.
\]

Carrying out first \( \nabla_y \) and then \( \text{div}_x \) under the integral sign we get

\[
I(x) = \frac{1}{2\pi} \text{div} \int_{\Omega} \left( -p(y, |x-y|) + |x-y| \frac{\partial p}{\partial t}(y, |x-y|) \right) \frac{y-x}{|y-x|^3} dy
\]

\[
+ \frac{1}{2\pi} \text{div} \int_{\Omega} \frac{(\nabla p)(y, |x-y|)}{|x-y|} dy
\]

\[
= \frac{1}{2\pi} \int_{\Omega} \left( -p(y, |x-y|) + |x-y| \frac{\partial p}{\partial t}(y, |x-y|) \right) \text{div}_x \frac{y-x}{|y-x|^3} dy
\]

\[
+ \frac{1}{2\pi} \int_{\Omega} \nabla_x \left( -p(y, |x-y|) + |x-y| \frac{\partial p}{\partial t}(y, |x-y|) \right) \cdot \frac{y-x}{|y-x|^3} dy
\]

\[
+ \frac{1}{2\pi} \text{div} \int_{\Omega} \frac{(\nabla p)(y, |x-y|)}{|x-y|} dy.
\]

Since

\[
\nabla_x \left( -p(y, |x-y|) + |x-y| \frac{\partial p}{\partial t}(y, |x-y|) \right)
\]

\[
= -\frac{\partial p}{\partial t}(y, |x-y|) \frac{x-y}{|x-y|} + \frac{x-y}{|x-y|} \frac{\partial p}{\partial t}(y, |x-y|) + \frac{\partial^2 p}{\partial^2 t}(y, |x-y|)(x-y)
\]

and, by (4),

\[
\text{div}_x \frac{y-x}{|y-x|^3} = -4\pi \delta(x-y)
\]
we conclude that
\[ I(x) = 2p_0(x) - \frac{1}{2\pi} \int_{\Omega} \frac{(\Delta p)(y, |x-y|)}{|x-y|} \, dy \]
\[ \quad + \frac{1}{2\pi} \text{div} \int_{\Omega} \frac{(\nabla p)(y, |x-y|)}{|x-y|} \, dy. \]  
(8)

For the evaluation of the integrals in (8), our lemma comes into play. First we note that due to (3),
\[ \int_{\Omega} \frac{p(y, |x-y|)}{|x-y|} \, dy = \frac{1}{4\pi} \int p_0(z) I_{\Omega}(z, x) \, dz. \]
Since \( \Delta p \) is the solution of (1,2) with \( p_0 \) replaced by \( \Delta p_0 \) we also have
\[ \int_{\Omega} \frac{(\Delta p)(y, |x-y|)}{|x-y|} \, dy = \frac{1}{4\pi} \int (\Delta p_0)(z) I_{\Omega}(z, x) \, dz \]
\[ = \frac{1}{4\pi} \int p_0(z) \Delta_z I_{\Omega}(z, x) \, dz \]
and, correspondingly,
\[ \text{div} \int_{\Omega} \frac{(\nabla p)(y, |x-y|)}{|x-y|} \, dy = \frac{1}{4\pi} \text{div} \int (\nabla p_0)(z) I_{\Omega}(z, x) \, dz \]
\[ = -\frac{1}{4\pi} \int p_0(z) \text{div}_z \nabla_z I_{\Omega}(z, x) \, dz. \]

Hence,
\[ I(x) = 2p_0(x) - \frac{1}{8\pi^2} \int p_0(z) \text{div}_z (\nabla_z + \nabla_x) I_{\Omega}(z, x) \, dz. \]  
(9)

From the lemma we get
\[ (\nabla_z + \nabla_x) I_{\Omega}(z, x) = -\frac{z-x}{|z-x|^3} (R\chi)^{\prime \prime \prime}(\theta(z,x), s(z,x)) \]
and
\[ \text{div}_z (\nabla_z + \nabla_x) I_{\Omega}(z, x) = \]
\[ -\text{div}_z \left( \frac{z-x}{|z-x|^3} \right)(R\chi)^{\prime \prime \prime}(\theta(z,x), s(z,x)) - \frac{z-x}{|z-x|^3} \cdot \nabla_z (R\chi)^{\prime \prime \prime}(\theta(z,x), s(z,x)). \]  
(10)

The second term on the right hand side of (10) is evaluated as follows: We have (see formula (II.1.5) of [3])
\[ \frac{\partial}{\partial z_j} (R\chi)^{\prime \prime \prime}(\theta, s) = -(Rx\chi)^{\prime \prime \prime}(\theta, s) \cdot \frac{\partial \theta}{\partial z_j} + (R\chi)^{\prime \prime \prime}(\theta, s) \cdot \frac{\partial s}{\partial z_j}. \]
Since
\[ (z-x) \cdot \frac{\partial \theta(z,x)}{\partial z_j} = 0, \quad (z-x) \cdot \nabla_z s(z,x) = \frac{1}{2} |z-x| \]
we get
\[ \frac{z-x}{|z-x|^3} \cdot \nabla_z (R\chi)^{\prime \prime \prime}(\theta(z,x), s(z,x)) = \frac{1}{2} \frac{1}{|z-x|^2} (R\chi)^{\prime \prime \prime}(\theta(z,x), s(z,x)). \]
In the first integral we introduce polar coordinates by putting $z$ all these results (10) becomes

$$div_z (\nabla_z + \nabla_z) I_\Omega (z, x) = -4\pi \delta (z - x) (R\chi)''(\theta(z, x), s(z, x))$$

$$-\frac{1}{2|z - x|^2} (R\chi)'''(\theta(z, x), s(z, x)).$$

Therefore (9) assumes the form

$$I(x) = 2p_0(x) + \frac{1}{2\pi} \int p_0(z) \delta (z - x) (R\chi)''(\theta(z, x), s(z, x)) dz$$

$$+ \frac{1}{16\pi^2} \int p_0(z) \frac{(R\chi)'''(\theta(z, x), s(z, x))}{|z - x|^2} dz.$$

In the first integral we introduce polar coordinates by putting $z = x + r\omega, \omega \in S^2$, obtaining

$$\int_0^\infty \int_0^{2\pi} p_0(x + r\omega) \delta (r\omega) (R\chi)''(\omega, x \cdot \omega + r/2) dr d\theta$$

which, because of (6), reduces to $p_0(x) (R^*(R\chi)''(x))/(4\pi)$ where $R^*$ is the adjoint Radon transform

$$(R^* g)(x) = \int_{S^2} g(\omega, x \cdot \omega) d\omega.$$

Using the Radon inversion formula $\chi = -(R^* R\chi)''/(8\pi^2)$ (see e.g. formula (II.2.10) of [3]) we finally arrive at the Theorem.

3. **Special measurement surfaces.** In some cases the integral operator in the Theorem vanishes, yielding an exact inversion formula. The most simple example is a ball of radius 1. We have $\chi(x) = 1$ for $|x| < 1$ and 0 otherwise, hence $(R\chi)(\theta, s) = \pi (1 - s^2)$ for $|s| < 1$ and 0 otherwise. For $z, x \in \Omega$, $s(z, x)$ is always smaller than 1 in absolute value, hence $(R\chi)(\theta(z, x), s(z, x)) = \pi (1 - s(z, x)^2)$. Thus $(R\chi)'''(\theta(z, x), s(z, x)) = 0$, i.e. the integral in (11) vanishes.

Our second example is an ellipsoid $|Ax| \leq 1$ where $A$ is a positive definite $3 \times 3$ matrix. The characteristic function $\chi_A$ is now $\chi_A(x) = \chi(Ax)$ with $\chi$ as in the previous example, and

$$(R\chi_A)(\theta, s) = (R\chi)\left(\frac{A^{-1}\theta}{|A^{-1}\theta|}, \frac{s}{|A^{-1}\theta|}\right).$$

For $z, x \in \Omega$, i.e. $|Az| < 1, |Ax| < 1$ we have

$$\frac{|s(z, x)|}{|A^{-1}\theta(z, x)|} = \frac{1}{2} \frac{|z|^2 - |x|^2}{|A^{-1}(z - x)|} = \frac{1}{2} \frac{|A^{-1}(z - x) \cdot A(z + x)|}{|A^{-1}(z - x)|} \leq \frac{1}{2} |A(z + x)| < 1,$$

hence the kernel of the integral in the Theorem vanishes exactly as in the previous example.

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