HÖLDER REGULARITY FOR WEAK SOLUTIONS TO NONLOCAL DOUBLE PHASE PROBLEMS

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Abstract. We prove local boundedness and Hölder continuity for weak solutions to nonlocal double phase problems concerning the following fractional energy functional

\[ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - v(y)|^p}{|x - y|^{n + sp}} + a(x, y)|v(x) - v(y)|^q \, dx \, dy, \]

where \( 0 < s \leq t < 1 < p \leq q < \infty \) and \( a(\cdot, \cdot) \geq 0 \). For such regularity results, we identify sharp assumptions on the modulating coefficient \( a(\cdot, \cdot) \) and the powers \( s, t, p, q \) which are analogous to those for local double phase problems.

1. Introduction

In this paper, we study the regularity theory for weak solutions to the following integro-differential equation:

\[ Lu(x) := \text{P.V.} \int_{\mathbb{R}^n} |u(x) - u(y)|^{p-2}(u(x) - u(y))K_{sp}(x, y) \, dy + \text{P.V.} \int_{\mathbb{R}^n} a(x, y)(u(x) - u(y))^{q-2}(u(x) - u(y))K_{tq}(x, y) \, dy = 0 \quad \text{in } \Omega. \quad (1.1) \]

Here, \( \Omega \subset \mathbb{R}^n \ (n \geq 2) \) is a bounded domain, \( K_{sp}, K_{tq} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) are suitable kernels with orders \((s, p)\) and \((t, q)\), respectively, for some \( 0 < s \leq t < 1 < p \leq q < \infty \), and \( a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) is a nonnegative modulating coefficient. We note that when \( a(x, y) \equiv 0 \) and \( K_{sp}(x, y) \equiv |x - y|^{-(n+sp)} \), it reduces to the \((s)\)fractional \(p\)-Laplace equation, \((-\Delta)^s_p u = 0\). Precise assumptions will be described in Section 1.1 below.

The regularity theory for nonlocal problems with fractional orders has been extensively studied for the last two decades. Caffarelli and Silvestre [6] proved Harnack inequality for the fractional Laplace equation, \((-\Delta)^s u := (-\Delta)_{\mathbb{R}^n}^s u = 0\), by using an extension argument. Later, Caffarelli, Chan and Vasseur [5] applied De Giorgi’s approach to linear parabolic equations with fractional orders involving general kernels \( K_{sp} \) with \( p = 2 \), and proved Hölder continuity of weak solutions. We refer to, for instance, [7, 25, 26, 31, 39, 40, 41, 42, 44] for regularity results for nonlocal linear equations with fractional orders. For nonlocal equations of fractional \(p\)-Laplacian type, Di Castro, Kuusi and Palatucci [17, 18] employed a nonlocal version of De Giorgi’s approach to prove local Hölder continuity and Harnack inequality. Cozzi [12] extended these results to non-homogeneous problems with lower order terms, by using fractional De Giorgi classes. Such approaches are further applied to several research areas including obstacle problems [29] and measure data problems [30]. We also refer to [20, 22, 27, 28, 33, 34, 35] and references therein for various regularity results for nonlocal problems of fractional \(p\)-Laplacian type. For a general overview of the history and related topics, we refer to the survey paper [38] and the monographs [3, 32].

Nonlocal problems with nonstandard growth are getting more and more attention in the very recent years. For local problems, there are two typical models of nonstandard growth conditions. One is the variable growth condition concerned with the function \( t^{p(x)} \); the other is the double growth condition concerned with the function \( s \log(1 + |x|) \).

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phase growth condition concerned with the function $t^p + a(x)t^q$. Recently, the second author in this paper [30] proved the local Hölder continuity of weak solutions to nonlocal equations with variable growth by developing the technique used in [18]. We would like to mention a recent paper [8] in which a similar result was obtained with more restrictive assumptions. We also refer to the above papers and references therein for the research on nonlocal problems with variable growth or relevant function spaces.

A prototype of nonlocal double phase problems is the following equation:

$$\text{P.V.} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{n+sp}} \, dy$$

$$+ \text{P.V.} \int_{\mathbb{R}^n} a(x,y) \frac{|u(x) - u(y)|^{q-2}(u(x) - u(y))}{|x-y|^{n+sq}} \, dy = 0 \quad \text{in } \Omega,$$

which is the case when $K_{sp}(x,y) \equiv |x-y|^{-(n+sp)}$ and $K_{sq}(x,y) \equiv |x-y|^{-(n+sq)}$ in (1.1). It is in fact the Euler-Lagrange equation of the functional

$$v \mapsto \int_{C_\Omega} \frac{1}{p} |v(x) - v(y)|^p + a(x,y) \frac{1}{q} |v(x) - v(y)|^q \, dx \, dy,$$

where

$$C_\Omega := (\mathbb{R}^n \times \mathbb{R}^n) \setminus ((\mathbb{R}^n \setminus \Omega) \times (\mathbb{R}^n \setminus \Omega)).$$

The local version corresponding to (1.2) is the double phase equation

$$\text{div} \left( |Du|^{p-2} Du + a(x) |Du|^{q-2} Du \right) = 0 \quad \text{in } \Omega.$$ (1.5)

Starting from [9, 10], the regularity for weak solutions to (1.5) and minimizers of corresponding variational integral has been exhaustively studied, see [2, 4, 11, 14, 15, 37] and references therein. In particular, for local boundedness and Hölder continuity, it is shown that, when $1 < p \leq n$,

$$a(\cdot) \in L^\infty_{\text{loc}}(\Omega), \quad q \leq p^* \quad \Rightarrow \quad u \in L^\infty_{\text{loc}}(\Omega),$$

$$u \in L^\infty_{\text{loc}}(\Omega), \quad a(\cdot) \in C^{0,\gamma}_{\text{loc}}(\Omega), \quad q \leq p + \alpha \quad \Rightarrow \quad u \in C^{0,\gamma}_{\text{loc}}(\Omega),$$ (1.6)

see [1, 9, 13].

Nonlocal equations of double phase type were first treated by De Filippis and Palatucci [16], where they proved Hölder continuity for viscosity solutions. Scott and Mengesha [43] proved nonlocal self-improving property for bounded weak solutions. We also mention the paper [21] by Fang and Zhang concerning Hölder continuity for bounded weak solutions and a relationship between weak and viscosity solutions. Here, we point out that the above papers [16, 21, 43] are restricted to solutions bounded in $\mathbb{R}^n$ and are under the assumption that $s \leq s$, which means the second term in (1.2) is a lower order term. Therefore, they were able to consider bounded, possibly discontinuous modulating coefficient $a(\cdot, \cdot)$.

In this paper we prove the local boundedness and Hölder continuity for weak solutions to the nonlocal equation with double phase growth condition, (1.1). We emphasize that we deal with the case $s \leq t$, which is a main difference from the papers [16, 21, 43]. The case $s \leq t$ is more delicate than the other case, since the second term in (1.3) has a higher order in the sense that $t \geq s$, $q \geq p$. To the best of our knowledge, the results presented in this paper are the first regularity results in this case. When we prove Hölder continuity in this case, we assume that the modulating coefficient $a(\cdot, \cdot)$ is Hölder continuous, which together with a restriction of the range of $q$ allows us to replace $a(\cdot, \cdot)$ with a constant. Note that this argument is exactly the same as the one for the local double phase problem. Therefore, we are able to make the assumptions of $q$ and $a(\cdot, \cdot)$ that are analogous to those in (1.6). Moreover, we only assume that the weak solution is not bounded in $\mathbb{R}^n$, but locally bounded in $\Omega$. Therefore, we need to handle the so-called nonlocal tails. The main difficulty arises in deriving the logarithmic type estimate (see Lemma 5.1). For fractional $p$-Laplacian type problems, an analogue estimate was obtained in [18, Lemma 1.3]. However, we could not apply the same approach directly to our problem (1.1). In order to obtain such an estimate, we first assume that the weak solution is locally bounded, and then take advantage of the Hölder continuity of $a(\cdot, \cdot)$ in order to modify and develop the techniques used in the proof of [18, Lemma 1.3].
1.1. Assumptions and main results. We say that a function \( f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) is symmetric if \( f(x,y) = f(y,x) \) for every \( x, y \in \mathbb{R}^n \).

The kernels \( K_{sp}, K_{tq} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) are measurable, symmetric and satisfy

\[
\frac{\Lambda^{-1}}{|x-y|^{n+sp}} \leq K_{sp}(x,y) \leq \frac{\Lambda}{|x-y|^{n+sp}}, \quad \frac{\Lambda^{-1}}{|x-y|^{n+tq}} \leq K_{tq}(x,y) \leq \frac{\Lambda}{|x-y|^{n+tq}}
\]

for a.e. \((x, y) \in \mathbb{R}^n \times \mathbb{R}^n\), where \( \Lambda > 1 \) and

\[
1 < p < q < \infty, \quad 0 < s \leq t < 1.
\]

The modulating coefficient \( a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) is assumed to be nonnegative, measurable, symmetric and bounded:

\[
0 \leq a(x, y) = a(y, x) \leq \|a\|_{L^\infty}, \quad x, y \in \mathbb{R}^n.
\]

In addition, in Theorem 1.1 and Section 5, we also assume that

\[
|a(x_1, y_1) - a(x_2, y_2)| \leq [a]_\alpha(|x_1 - x_2| + |y_1 - y_2|)^\alpha, \quad \alpha > 0,
\]

for every \((x_1, y_1), (x_2, y_2) \in \mathbb{R}^n \times \mathbb{R}^n\).

With the relevant function spaces including \( \mathcal{A}(\Omega) \) and \( L_{sp}^{q-1}(\mathbb{R}^n) \) to be introduced in the next section, we introduce weak solutions under consideration. We say that \( u \in \mathcal{A}(\Omega) \) is a weak solution to (1.1) if

\[
\int_{\mathcal{C}_\Omega} \left[ |u(x) - u(y)|^{p-2}(u(x) - u(y)) (\varphi(x) - \varphi(y)) K_{sp}(x,y) \
+ a(x,y)|u(x) - u(y)|^{q-2}(u(x) - u(y)) (\varphi(x) - \varphi(y)) K_{tq}(x,y) \right] \, dx \, dy = 0
\]

for every \( \varphi \in \mathcal{A}(\Omega) \) with \( \varphi = 0 \) a.e. in \( \mathbb{R}^n \setminus \Omega \). In addition, we say that \( u \in \mathcal{A}(\Omega) \) is a weak subsolution (resp. supersolution) if (1.11) with “\( = \)” replaced by “\( \leq \)” (resp. “\( \geq \)”)
holds for every \( \varphi \in \mathcal{A}(\Omega) \) satisfying \( \varphi \geq 0 \) a.e. in \( \mathbb{R}^n \) and \( \varphi = 0 \) a.e. in \( \mathbb{R}^n \setminus \Omega \). Existence and uniqueness of weak solutions to (1.1) with a Dirichlet boundary condition will be discussed in Section 3.

Now we state our main results. The first one is the local boundedness of weak solutions.

**Theorem 1.1.** Let \( K_{sp}, K_{tq}, a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) be symmetric and satisfy (1.7)-(1.9). If

\[
\begin{cases}
p \leq q \leq \frac{np}{n-sp} & \text{when } sp < n, \\
p \leq q < \infty & \text{when } sp \geq n,
\end{cases}
\]

then every weak solution \( u \in \mathcal{A}(\Omega) \cap L_{sp}^{q-1}(\mathbb{R}^n) \) to (1.1) is locally bounded in \( \Omega \).

The second one is the local Hölder continuity. Here, we assume that \( a(\cdot, \cdot) \) is Hölder continuous in \( \mathbb{R}^n \times \mathbb{R}^n \) and that \( u \) is locally bounded in \( \Omega \).

**Theorem 1.2.** Let \( K_{sp}, K_{tq}, a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) be symmetric and satisfy (1.7)-(1.9). If \( a(\cdot, \cdot) \) satisfies (1.10) and

\[
tq \leq sp + \alpha,
\]

then every weak solution \( u \in \mathcal{A}(\Omega) \cap L_{sp}^{q-1}(\mathbb{R}^n) \) to (1.1) which is locally bounded in \( \Omega \) is locally Hölder continuous in \( \Omega \). More precisely, for every open subset \( \Omega' \subset \Omega \), there exists \( \gamma \in (0,1) \) depending only on \( n, s, t, p, q, \Lambda, \|a\|_{L^\infty}, [a]_\alpha \) and \( \|u\|_{L^\infty(\Omega')} \) such that \( u \in C_{loc}^{0,\gamma}(\Omega') \).

**Remark 1.3.** In view of Theorem 1.1, we also see that, under the setting in Theorem 1.2, if

\[
\begin{cases}
p \leq q \leq \min\left\{ \frac{np}{n-sp}, \frac{sp+\alpha}{t} \right\} & \text{when } sp < n, \\
p \leq q \leq \frac{sp+\alpha}{t} & \text{when } sp = n,
\end{cases}
\]

then every weak solution \( u \in \mathcal{A}(\Omega) \cap L_{sp}^{q-1}(\mathbb{R}^n) \) to (1.1) is locally Hölder continuous.

This paper is organized as follows: In the next section, we introduce basic notation and function spaces which will be used throughout this paper. In Section 3, we examine the existence of solutions to (1.1). In Section 4 we derive a Caccioppoli type estimate and prove Theorem 1.1. Finally, in Section 5, we prove Theorem 1.2 by obtaining a logarithmic estimate.
2. Preliminaries

2.1. Notation. We denote by \( c \) a generic constant greater than or equal to one, whose value may vary from line to line. We denote its specific dependence in parentheses when needed, using the abbreviation like

\[
\begin{align*}
\text{data} & := (n, s, t, p, q, \alpha, \|a\|_{L^{\infty}}) \\
\text{data}_1 & := (n, s, t, p, q, \alpha, [a]_{\alpha}),
\end{align*}
\]

where \( \|a\|_{L^{\infty}}, \alpha \) and \([a]_{\alpha}\) are given in (1.9) and (1.10).

For any open set \( \mathcal{O} \subseteq \mathbb{R}^n, s \in (0, 1) \) and \( p \geq 1 \), the fractional Sobolev space \( W^{s,p}(\mathcal{O}) \) is the set of all functions \( v \in L^p(\mathcal{O}) \) for which

\[
\|v\|_{W^{s,p}(\mathcal{O})} := \|v\|_{L^p(\mathcal{O})} + [v]_{s,p,\mathcal{O}} := \left( \int_{\mathcal{O}} |v|^p \, dx \right)^{1/p} + \left( \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{|v(x) - v(y)|^p}{|x - y|^{n+sp}} \, dxdy \right)^{1/p} < \infty.
\]

Furthermore, we define \( W^{s,p}_0(\mathcal{O}) \) as the closure of \( C_0^\infty(\mathcal{O}) \) in \( W^{s,p}(\mathcal{O}) \). We denote the \((s-)\)fractional Sobolev conjugate of \( p \) by

\[
p^*_s := \begin{cases} np/(n-sp) & \text{when } sp < n, \\ \text{any number in } (p, \infty) & \text{when } sp \geq n. \end{cases}
\]

In particular, if we consider two exponents \( p \) and \( q \) with \( 1 < p < q \), we set \( p^*_s = q + 1 \) when \( sp = n \).

As usual, \( B_r(x_0) \) is the open ball in \( \mathbb{R}^n \) with center \( x_0 \in \mathbb{R}^n \) and radius \( r > 0 \). We omit the center when it is clear in the context. For a measurable function \( v \), we write \( v_{\pm} := \max\{\pm v, 0\} \). If \( v \) is integrable over a measurable set \( S \) with \( 0 < |S| < \infty \), we denote its integral average over \( S \) by

\[
(v)_S := \frac{\int_S v \, dx}{|S|} = \frac{1}{|S|} \int_S v \, dx.
\]

We always assume that \( s, t, p, \) and \( q \) satisfy (1.8) and that \( K_{sp}, K_{tq}, a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) satisfy (1.7) and (1.9). We denote

\[
H(x, y, \tau) := \frac{\tau^p}{|x - y|^p} a(x, y) \frac{\tau^q}{|x - y|^q}, \quad x, y \in \mathbb{R}^n \quad \text{and} \quad \tau \geq 0,
\]

and

\[
\varrho(v; S) := \int_S \int_S H(x, y, |v(x)|^p - |v(y)|^p) \frac{dxdy}{|x - y|^{n}}
\]

for each measurable set \( S \subseteq \mathbb{R}^n \) and \( v : S \to \mathbb{R} \). Then we define a function space concerned with weak solutions to (1.1) by

\[
\mathcal{A}(\Omega) := \left\{ v : \mathbb{R}^n \to \mathbb{R} \mid v|_{\Omega} \in L^p(\Omega) \quad \text{and} \quad \int_{\mathcal{C}_{\Omega}} H(x, y, |v(x)| - |v(y)|) \frac{dxdy}{|x - y|^{n}} < \infty \right\},
\]

where \( \mathcal{C}_{\Omega} \) is defined in (1.4). Note that \( \varrho(v; \Omega) < \infty \) whenever \( v \in \mathcal{A}(\Omega) \), which in particular implies

\[
\mathcal{A}(\Omega) \subset W^{s,p}(\Omega).
\]

We note that if \( sp > n \), then every function in \( W^{s,p}(\Omega) \) is H"{o}lder continuous by the fractional Sobolev embedding. Thus, in this paper we may assume without loss of generality that

\[
sp \leq n.
\]

Moreover, again by the fractional Sobolev embedding, we have

\[
\mathcal{A}(\Omega) \subset L^q(\Omega) \quad \text{if} \quad \begin{cases} p < q \leq \frac{np}{n-sp} & \text{when } sp < n, \\ p < q < \infty & \text{when } sp \geq n. \end{cases}
\]

This will be used later in the proof of several estimates concerning local boundedness.
We next define the tail space. One of the features in [18] is to consider the notion of nonlocal tails in local estimates, which encodes the nonlocal nature of the problem. We define

$$L_{sp}^{q}(\mathbb{R}^n) := \left\{ v : \mathbb{R}^n \to \mathbb{R} \mid \left( \int_{\mathbb{R}^n} \frac{|v(x)|^{q-1}}{(1 + |x|)^{n+sp}} dx < \infty \right) \right\}.$$  

Let $m \in \{s, t\}$ and $\ell \in \{p, q\}$. Since we have

$$\frac{1 + |x|}{|x - x_0|} \leq \frac{1 + |x - x_0|}{|x - x_0|} \leq 1 + \frac{|x_0|}{r}, \quad \frac{|v(x)|^{q-1}}{(1 + |x|)^{n+sp}} \leq \frac{|v(x)|^{q-1}}{(1 + |x|)^{n+2sp}}$$

for $x \in \mathbb{R}^n \setminus B_r(x_0)$, we see that

$$\int_{\mathbb{R}^n \setminus B_r(x_0)} \frac{|v(x)|^{q-1}}{|x - x_0|^{n+2sp}} dx$$

is finite whenever $v \in L_{sp}^{q}(\mathbb{R}^n)$ and $B_r(x_0) \subset \mathbb{R}^n$. We call such a quantity a nonlocal tail.

**Remark 2.1.** If $v \in L^{q_0}(\mathbb{R}^n)$ for some $q_0 \geq q - 1$, or if $v \in L^{q_1}(B_R(0)) \cap L^{q_2}(\mathbb{R}^n \setminus B_R(0))$ for some $R > 0$, then $v \in L_{sp}^{q}(\mathbb{R}^n)$. Moreover, we have that

$$W^{a,p}(\mathbb{R}^n) \subset L_{sp}^{q}(\mathbb{R}^n) \quad \text{if} \quad q \leq q^* + 1.$$

### 2.2. Inequalities

We first collect several inequalities concerning fractional Sobolev functions. For basic properties of fractional Sobolev spaces, we refer to [19].

The following inequality shows an inclusion relation between fractional Sobolev spaces. We also notice that it fails to hold when $s = t$.

**Lemma 2.2.** Let $1 \leq p \leq q$ and $0 < s < t < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded measurable set. Then, for any $v \in W^{s,a}(\Omega)$ we have

$$\left( \int_\Omega \int_\Omega \frac{|v(x) - v(y)|^p}{|x - y|^{n+sp}} dxdy \right)^{\frac{1}{p}} \leq c|\Omega|^{\frac{q-p}{rt}} (\text{diam}(\Omega))^{t-s} \left( \int_\Omega \int_\Omega \frac{|v(x) - v(y)|^q}{|x - y|^{n+sq}} dxdy \right)^{\frac{1}{q}}$$

for a constant $c \equiv c(n, s, t, p, q)$.

**Proof.** In the case $p < q$, this is a special case of [12, Lemma 4.6]. In particular, the constant $c$ is given by

$$c = \left( \frac{n(q - p)}{(t - s)pq} |B_1| \right)^{\frac{q-p}{rt}},$$

which blows up as $t \searrow s$.

If $p = q$, then we directly have

$$\left( \int_\Omega \int_\Omega \frac{|v(x) - v(y)|^p}{|x - y|^{n+sp}} dxdy \right)^{\frac{1}{p}} = \left( \int_\Omega \int_\Omega \frac{|v(x) - v(y)|^p}{|x - y|^{n+tp}} |x - y|^{(t-s)p} dxdy \right)^{\frac{1}{p}} \leq (\text{diam}(\Omega))^{t-s} \left( \int_\Omega \int_\Omega \frac{|v(x) - v(y)|^p}{|x - y|^{n+tp}} dxdy \right)^{\frac{1}{p}}.$$  

We next recall the following fractional Sobolev-Poincaré type inequalities from [36, Lemma 2.5]. In fact, the second one follows from the first one and Lemma 2.2.

**Lemma 2.3.** Let $s \in (0, 1)$, $p \geq 1$ be such that $sp \leq n$. For any $v \in W^{a,p}(B_r)$ we have

$$\left( \int_{B_r} |v - (v)_{B_r}|^{p_1} dx \right)^{\frac{1}{p_1}} \leq c \int_{B_r} \int_{B_r} \frac{|v(x) - v(y)|^p}{|x - y|^{n+sp}} dydx$$

for $c \equiv c(n, s, p) > 0$. Moreover, if $s \leq t < 1$ and $p \leq q$, we also have

$$\left( \int_{B_r} |v - (v)_{B_r}|^{p_2} dx \right)^{\frac{1}{p_2}} \leq c \int_{B_r} \int_{B_r} \frac{|v(x) - v(y)|^q}{|x - y|^{n+sq}} dydx$$

for $c \equiv c(n, s, t, p, q) > 0$, whenever the right-hand side is finite.
The following two lemmas are simple corollaries of the preceding lemma. They will be used in the proof of Theorems 1.1 and 1.2, respectively.

**Lemma 2.4.** Assume that the constants $s$, $t$, $p$ and $q$ satisfy (1.8) and (1.12). Then for every $f \in W^{s,p}(B_r)$ we have

$$\int_{B_r} \left| \frac{f}{r^s} \right|^p + L_0 \left| \frac{f}{r^t} \right|^q \, dx \leq c L_0^{(s-t)q} \left( \int_{B_r} \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} \, dx \right)^{\frac{q}{p}} + c \left( \frac{\text{supp } f}{|B_r|} \right)^{\frac{q}{p+s}} \int_{B_r} \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} \, dx dy + c \left( \frac{\|f\|_{L^\infty(B_r)}}{|B_r|} \right)^{-1} \int_{B_r} |f|^{q-1} \, dx + L_0 \left| \frac{f}{r^t} \right|^q \, dx$$

for a constant $c \equiv c(n, s, t, p, q)$, where $L_0$ is any positive constant.

**Proof.** Applying Hölder’s inequality and Lemma 2.3, we have

$$\int_{B_r} \left| \frac{f}{r^s} \right|^p \, dx \leq c \left( \int_{B_r} \left| \frac{f - (f)_{B_r}}{r^s} \right|^p \, dx \right)^{\frac{q}{p}} + c \left( \frac{\|f\|_{L^\infty(B_r)}}{|B_r|} \right)^{\frac{q}{p+s}} \int_{B_r} \left| \frac{f(x) - f(y)}{|x - y|^{n+sp}} \right|^p \, dx dy + c \left( \frac{\|f\|_{L^\infty(B_r)}}{|B_r|} \right)^{-1} \int_{B_r} |f|^{q-1} \, dx + c \left( \frac{\|f\|_{L^\infty(B_r)}}{|B_r|} \right)^{-1} \int_{B_r} |f|^{q-2} \, dx$$

Likewise, we obtain

$$\int_{B_r} \left| \frac{f}{r^t} \right|^q \, dx \leq c \left( \frac{\text{supp } f}{|B_r|} \right)^{\frac{q}{p+t}} \int_{B_r} \left| \frac{f - (f)_{B_r}}{r^t} \right|^q \, dx + c \left( \frac{\|f\|_{L^\infty(B_r)}}{|B_r|} \right)^{-1} \int_{B_r} |f|^{q-1} \, dx + c \left( \frac{\|f\|_{L^\infty(B_r)}}{|B_r|} \right)^{-1} \int_{B_r} |f|^{q-2} \, dx$$

We also have

$$\left( \frac{\|f\|_{L^\infty(B_r)}}{|B_r|} \right)^{p} + L_0 \left| \frac{f}{r^t} \right|^q$$

$$\leq r^{-sp} \left( \frac{\text{supp } f}{|B_r|} \right)^{p-1} \int_{B_r} |f|^p \, dx + L_0 r^{-tq} \left( \frac{\|f\|_{L^\infty(B_r)}}{|B_r|} \right)^{p-1} \int_{B_r} |f|^q \, dx$$

$$\leq \left( \frac{\|f\|_{L^\infty(B_r)}}{|B_r|} \right)^{-p} \int_{B_r} \left| \frac{f}{r^s} \right|^p + L_0 \left| \frac{f}{r^t} \right|^q \, dx.$$

We combine the inequalities in the above display to complete the proof.

**Lemma 2.5.** Assume that the constants $s$, $t$, $p$ and $q$ satisfy (1.8) and that the function $a(\cdot, \cdot)$ satisfies (1.10) and (1.13). Let $B_r \subseteq B_R$ be concentric balls with $\frac{1}{2} R \leq r \leq R \leq 1$. Then for any $f \in L^\infty(B_r)$ we have

$$\left( \int_{B_r} \left( \left| \frac{f}{r^s} \right|^p + a_2 \left| \frac{f}{r^t} \right|^q \right)^\kappa \, dx \right)^\frac{1}{\kappa} \leq c \left( 1 + \|f\|_{L^\infty(B_r)}^{q-p} \right) \int_{B_r} \int_{B_r} H(x, y, |f(x) - f(y)|) \frac{\, dx \, dy}{|x - y|^n}$$

$$+ c \left( 1 + \|f\|_{L^\infty(B_r)}^{q-p} \right) \int_{B_r} \left| \frac{f}{r^s} \right|^p + a_1 \left| \frac{f}{r^t} \right|^q \, dx$$

for some $c \equiv c(n, s, t, p, q, [a]_a) > 0$, whenever the right-hand side is finite, where

$$\kappa := \min \left\{ \frac{p}{p}, \frac{q}{q} \right\} > 1, \quad a_1 := \inf_{B_r \times B_r} a(\cdot, \cdot) \quad \text{and} \quad a_2 := \sup_{B_r \times B_r} a(\cdot, \cdot).$$
Proof. Using the assumptions, we estimate
\[
\left[ \int_{B_r} \left( \frac{|f|}{p} \right)^p + a_1 \left( \frac{|f|^q}{r^d} \right)^q \right]^\frac{1}{p} \leq c \left[ \int_{B_r} \left( \frac{|f|}{p} \right)^p + a_1 \left( \frac{|f|^q}{r^d} \right)^q \right]^\frac{1}{p} + \left( r^{\alpha + sp - tq} \|f\|_{L^\infty(B_r)} \right)^\frac{1}{p} \int_{B_r} \left( \frac{|f|^p}{r^d} \right)^p dx \right]^{\frac{1}{p}}
\]
\[
\leq c \left( 1 + \|f\|_{L^\infty(B_r)} \right) \left[ \int_{B_r} \left( \frac{|f|}{p} \right)^p + a_1 \left( \frac{|f|^q}{r^d} \right)^q \right]^{\frac{1}{p}}.
\]
We next apply Lemma 2.3 to see that
\[
\left[ \int_{B_r} \left( \frac{|f|}{p} \right)^p + a_1 \left( \frac{|f|^q}{r^d} \right)^q \right]^\frac{1}{p} \leq c \left[ \int_{B_r} \left( \frac{|f|}{p} \right)^p + a_1 \left( \frac{|f|^q}{r^d} \right)^q \right]^\frac{1}{p} + c \left( \frac{|f|}{p} \right)^p + c a_1 \left( \frac{|f|^q}{r^d} \right)^q
\]
\[
\leq c \int_{B_r} \left( |f(x) - f(y)|^p \right)^p + a_1 \left( |f(x) - f(y)|^q \right)^q dx + c \int_{B_r} \left( \frac{|f|^p}{p} \right)^p + a_1 \left( \frac{|f|^q}{q} \right)^q dx
\]
\[
\leq c \int_{B_r} \left( \int_{B_r} H(x, y, |f(x) - f(y)|) \right) dx + c \int_{B_r} \left( \frac{|f|^p}{p} \right)^p + a_1 \left( \frac{|f|^q}{q} \right)^q dx.
\]
Then the conclusion follows. \[\Box\]

The following numerical inequalities will be frequently used in this paper.

Lemma 2.6. Let \( p \geq 1 \) and \( a, b \geq 0 \). Then
\[
a^p - b^p \leq pa^{p-1}|a - b|
\]
and, for any \( \varepsilon > 0 \),
\[
a^p - b^p \leq \varepsilon b^p + c \varepsilon^{1-p}|a - b|^p
\]
for some \( c \equiv c(p) > 0 \).

Proof. The first one is a direct consequence of Mean Value Theorem; note that we may assume \( a \geq b \), otherwise it is obvious. For the proof of the second one, see [18, Lemma 3.1]. \[\Box\]

We end this section with a standard iteration lemma from [24, Lemma 7.1].

Lemma 2.7. Let \( \{y_i\}_{i=0}^\infty \) be a sequence of nonnegative numbers satisfying
\[
y_{i+1} \leq b_1 b_2 y_i^{1+\beta}, \quad i = 0, 1, 2, \ldots,
\]
for some constants \( b_1, \beta > 0 \) and \( b_2 > 1 \). If
\[
y_0 \leq b_1^{-1/\beta} b_2^{-1/\beta^2},
\]
then \( y_i \to 0 \) as \( i \to \infty \).

3. Existence of weak solutions

In this section we show the existence of weak solutions to (1.1). By a standard argument, such as the one in the proof of [18, Theorem 2.3], we see that \( u \in \mathcal{A}(\Omega) \) is a weak solution to (1.1) if and only if it is a minimizer of the functional
\[
\mathcal{E}(v; \Omega) := \int_{\mathcal{C}_0} \int_p \frac{1}{p} |v(x) - v(y)|^p K_{sp}(x, y) + a(x, y) \frac{1}{q} |v(x) - v(y)|^q K_{sq}(x, y) dx dy.
\]
We say that \( u \in \mathcal{A}(\Omega) \) is a minimizer of (3.1) if
\[
\mathcal{E}(u; \Omega) \leq \mathcal{E}(v; \Omega)
\]
for every \( v \in \mathcal{A}(\Omega) \) with \( v = u \) a.e. in \( \mathbb{R}^n \setminus \Omega \). Therefore, we prove the existence and uniqueness of the minimizer of (3.1) with a Dirichlet boundary condition.
Theorem 3.1. Let $\Omega$ be a bounded domain and $g \in A(\Omega)$ be a given boundary data. Let $K_{sp}, K_{tp}, a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be symmetric and satisfy (1.7)-(1.9). Then there exists a unique minimizer $u \in A(\Omega)$ of (3.1) with $g = u$ a.e. in $\mathbb{R}^n \setminus \Omega$. Moreover, if $g \in A(\Omega) \cap L^{2p}_{sp}(\mathbb{R}^n)$, then $u \in A(\Omega) \cap L^{2p-1}_{sp}(\mathbb{R}^n)$.

Proof. The uniqueness follows directly from the fact that the function $\tau \mapsto \tau^p + a(x, y)\tau^q$ is strictly convex for each fixed $(x, y)$. Now we prove the existence. The admissible set

$$A_g(\Omega) := \{v \in A(\Omega) : v = g\ a.e.\ in\ \mathbb{R}^n \setminus \Omega\}$$

is obviously nonempty, as $g \in A_g(\Omega)$. Let $\{u_k\}_k \subset A_g(\Omega)$ be a minimizing sequence. Then there exists a constant $C$ such that

$$\left[ u_k \right]_{p,\Omega}^p = \int_{\Omega} \int_{\Omega} \frac{|u_k(x) - u_k(y)|^p}{|x-y|^{n+sp}} dxdy \leq C \quad \forall k \in \mathbb{N}.$$ 

In particular, Lemma 2.2 implies that $\{|u_k|_{s_0,\Omega}\}_k$ is bounded for any $s_0 \in (0, s)$. Then we choose a ball $B_R \equiv B_R(x_0) \supset \Omega$ with $R \geq 1$ and fix $s_0 \in (0, s/2)$ with $np/(n+sop) =: p_0 > 1$. Since $u_k - g = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$, the fractional Sobolev embedding [19, Theorem 6.5] implies

$$\left( \int_{B_R} |u_k - g|^p dx \right)^{\frac{1}{p}} \leq c |u_k - g|^p_{s_0, p; \mathbb{R}^n}$$

$$\leq c |u_k - g|^p_{s_0, p; B_R} + c \int_{B_R} |u_k(y) - g(y)|^p \left( \int_{\mathbb{R}^n \setminus B_R} \frac{dx}{|x-y|^{n+sop}} \right) dy$$

$$\leq c |u_k - g|^p_{s_0, p; B_R} + c \int_{B_{2R} \setminus B_R} |u_k(y) - g(y)|^p \left( \int_{B_{2R} \setminus B_R} \frac{dx}{|x-y|^{n+sop}} \right) dy$$

$$\leq c \int_{B_{2R}} \int_{B_R} \frac{|(u_k - g)(x) - (u_k - g)(y)|^p}{|x-y|^{n+sop}} dxdy \leq c |u_k - g|^p_{s_0, p; B_{2R}},$$

where we have used the fact that

$$\int_{\mathbb{R}^n \setminus B_R} \frac{dx}{|x-y|^{n+sop}} \leq c(n) \int_{B_{2R} \setminus B_R} \frac{dx}{|x-y|^{n+sop}} \quad \forall y \in B_R.$$ 

Applying Lemma 2.2 to the right-hand side of (3.2), we have for all $k \in \mathbb{N}$

$$\left( \int_{B_R} |u_k - g|^p dx \right)^{\frac{1}{p}} \leq c |u_k - g|^p_{s_0, p; B_{2R}}$$

$$\leq c R^{sp_0} |u_k - g|^p_{s_0, p; B_{2R}}$$

$$\leq c R^{sp_0} \left( \int_{B_{2R}} \frac{|u_k(x) - u_k(y)|^p}{|x-y|^{n+sp}} dxdy + \int_{B_{2R}} \frac{|g(x) - g(y)|^p}{|x-y|^{n+sp}} dxdy \right)^{\frac{1}{p}}$$

$$\leq c R^{sp_0} \left( C + \int_{B_{2R}} \frac{|g(x) - g(y)|^p}{|x-y|^{n+sp}} dxdy \right)^{\frac{1}{p}}.$$ 

This implies that $\{u_k - g\}_k$ is bounded in $L^p(B_R)$, and hence in $W^{s,p}_0(B_R)$. By the compact embedding theorem for fractional Sobolev spaces [19, Theorem 7.1], there exist a subsequence $\{u_{k_j} - g\}_j$ and $v \in L^p(B_R)$ such that

$$u_{k_j} - g \rightharpoonup v \quad \text{in} \quad L^p(B_R),$$

$$u_{k_j} - g \rightarrow v \quad \text{a.e. in} \quad B_R, \quad \text{as} \quad j \rightarrow \infty.$$ 

We extend $v$ to $\mathbb{R}^n$ by letting $v = 0$ on $\mathbb{R}^n \setminus B_R$ and set $u := v + g$. Then $u_{k_j} \rightarrow u$ a.e. in $\mathbb{R}^n$. Finally, Fatou’s lemma implies

$$\liminf_{j \rightarrow \infty} \frac{1}{p} \int_{B_{C_1}} |u(x) - u(y)|^p K_{sp}(x,y) + a(x, y) \frac{1}{q} |u(x) - u(y)|^q K_{tp}(x,y) dxdy$$

$$\leq \liminf_{j \rightarrow \infty} \frac{1}{p} \int_{B_{C_1}} |u_{k_j}(x) - u_{k_j}(y)|^p K_{sp}(x,y) + a(x, y) \frac{1}{q} |u_{k_j}(x) - u_{k_j}(y)|^q K_{tp}(x,y) dxdy.$$
This means that \( u \in \mathcal{A}_g(\Omega) \) and it is a minimizer of \( E \).

The last assertion is clear since \( u \in L^p(\Omega) \) and \( u = g \) a.e. in \( \mathbb{R}^n \setminus \Omega \).

\( \square \)

**Remark 3.2.** In fact, the above theorem still holds even when \( a(\cdot, \cdot) \geq 0 \) is not bounded above.

4. Caccioppoli estimates and Local boundedness

We start with the following lemma which implies that the multiplication of any function in \( \mathcal{A}(\Omega) \) and a cut-off function is also a function in \( \mathcal{A}(\Omega) \). We recall the notation (2.2).

**Lemma 4.1.** Assume that the constants \( s, t, p \) and \( q \) satisfy (1.8), and \( \eta \in W^{1, \infty}_0(B_r) \). If one of the following two conditions holds:

(i) The inequality (1.12) holds and \( v \in L^p(B_{2r}) \) satisfies \( \varrho(v; B_{2r}) < \infty \);

(ii) \( v \in L^q(B_{2r}) \) satisfies \( \varrho(v; B_{2r}) < \infty \),

then \( \varrho(v\eta; \mathbb{R}^n) < \infty \). In particular, \( v\eta \in \mathcal{A}(\Omega) \) whenever \( \Omega \supset B_{2r} \).

**Proof.** Note that we also have \( v \in L^q(B_{2r}) \) in (i) by Lemma 2.3 and (1.12). We write

\[
\varrho(v\eta; \mathbb{R}^n) = \varrho(v\eta; B_{2r}) + 2 \int_{\mathbb{R}^n \setminus B_{2r}} \int_{B_{2r}} H(x, y, |v(y)\eta(x)|) \frac{dxdy}{|x - y|^n}.
\]

The first term is estimated as

\[
\varrho(v\eta; B_{2r}) \leq c \int_{B_{2r}} \int_{B_{2r}} H(x, y, |(v(x) - v(y))\eta(y)|) \frac{dxdy}{|x - y|^n} + c \int_{B_{2r}} \int_{B_{2r}} H(x, y, |v(x)(\eta(x) - \eta(y))|) \frac{dxdy}{|x - y|^n} \leq c \left( \|v\|_{L^p(B_{2r})} + 1 \right)^q \varrho(v; B_{2r}) + c \|D\eta\|_{L^\infty(B_{2r})} \int_{B_{2r}} |v(x)| \left( \int_{B_{2r}} \frac{dxdy}{|x - y|^{n+(s-1)p}} \right) dx
\]

\[
+ c \|D\eta\|_{L^\infty(B_{2r})}^q \|a\|_{L^\infty} \int_{B_{2r}} |v(x)|^q \left( \int_{B_{2r}} \frac{dy}{|x - y|^{n+(t-1)q}} \right) dx \leq c \left( \|v\|_{L^p(B_{2r})} + 1 \right)^q \varrho(v; B_{2r}) + c \|D\eta\|_{L^\infty(B_{2r})} \int_{B_{2r}} |v(x)| dx + c \|D\eta\|_{L^\infty(B_{2r})}^q \|a\|_{L^\infty} \int_{B_{2r}} |v(x)|^q dx < \infty.
\]

The second term is estimated as

\[
\int_{\mathbb{R}^n \setminus B_{2r}} \int_{B_{2r}} H(x, y, |v(x)\eta(x)|) \frac{dxdy}{|x - y|^n} \leq c \left( \|v\|_{L^p(B_{2r})} + 1 \right)^q \int_{\mathbb{R}^n \setminus B_{2r}} \int_{B_{2r}} \frac{|v(x)|^p}{|x - y|^{n+sp}} + \|a\|_{L^\infty} \int_{\mathbb{R}^n} \frac{|v(x)|^q}{|x - y|^{n+sq}} \frac{dxdy}{|x - y|^n} \leq c \left( \|v\|_{L^p(B_{2r})} + 1 \right)^q \left( r^{-sp} \int_{B_r} |v(x)|^p dx + \|a\|_{L^\infty} r^{-sq} \int_{B_r} |v(x)|^q dx \right) < \infty,
\]

and the conclusion follows. \( \square \)

Next, we prove a nonlocal Caccioppoli type estimate. We again recall (2.2) with (2.1), and further define

\[
h(x, y, \tau) := \frac{\tau^{p-1}}{|x - y|^{sp}} + a(x, y) \frac{\tau^{q-1}}{|x - y|^{sq}}, \quad x, y \in \mathbb{R}^n, \tau \geq 0.
\]
for the case (i), we can test the

\[ I_1 + I_2. \]

We first estimate $I_1$. Assume that $u(x) \geq u(y)$. Then,

\[ \Phi_\ell(u(x) - u(y))(w_+(x)\phi^\ell(x) - w_+(y)\phi^\ell(y)) \]

\[ = (u(x) - u(y))^{\ell-1}(u(x) - k_+\phi^\ell(x) - (u(y) - k_+\phi^\ell(y)) \]

\[ = \begin{cases} 
(u(x) - u(y))^{\ell-1}(w_+(x)\phi^\ell(x) - w_+(y)\phi^\ell(y)), & u(x) \geq u(y) \geq k \\
0, & k \geq u(x) \geq u(y) 
\end{cases} \]

\[ \geq (w_+(x) - w_+(y))^{\ell-1}(w_+(x)\phi^\ell(x) - w_+(y)\phi^\ell(y)) \]

\[ = \Phi_\ell(w_+(x) - w_+(y))(w_+(x)\phi^\ell(x) - w_+(y)\phi^\ell(y)), \]

and hence

\[ I_1 \geq \int_{B_r} \int_{B_r} [\Phi_\ell(w_+(x) - w_+(y))(w_+(x)\phi^\ell(x) - w_+(y)\phi^\ell(y))K_{\phi^\ell}(x,y) \\
+ a(x,y)\Phi_\ell(w_+(x) - w_+(y))(w_+(x)\phi^\ell(x) - w_+(y)\phi^\ell(y))K_{\phi^\ell}(x,y)] \, dx dy. \] (4.4)

Moreover,

\[ w_+(x)\phi^\ell(x) - w_+(y)\phi^\ell(y) = \frac{w_+(x) - w_+(y)}{2}(\phi^\ell(x) + \phi^\ell(y)) + \frac{w_+(x) + w_+(y)}{2}(\phi^\ell(x) - \phi^\ell(y)), \]

which implies

\[ \Phi_\ell(w_+(x) - w_+(y))(w_+(x)\phi^\ell(x) - w_+(y)\phi^\ell(y)) \]

\[ \geq |w_+(x) - w_+(y)|\frac{\phi^\ell(x) + \phi^\ell(y)}{2} - |w_+(x) - w_+(y)|^{\ell-1}\frac{w_+(x) + w_+(y)}{2}|\phi^\ell(x) - \phi^\ell(y)|. \]
Here, we use Lemma 2.6 to see that
\[ |\phi^q(x) - \phi^q(y)| \leq q(\phi^{q-1}(x) + \phi^{q-1}(y))|\phi(x) - \phi(y)| \]
\[ \leq c(q(\phi^q(x) + \phi^q(y)))^{(q-1)/q}|\phi(x) - \phi(y)|. \]

Thus, using Young’s inequality, we get
\[ |w_+(x) - w_+(y)|^{q-1}(w_+(x) + w_+(y))|\phi^q(x) - \phi^q(y)| \]
\[ \leq |w_+(x) - w_+(y)|^{q-1}(w_+(x) + w_+(y))(\phi^q(x) + \phi^q(y))^{\frac{1}{q}} \|\phi(x) - \phi(y)\|^{\frac{q}{q-1}} \frac{1}{q}|\phi(x) - \phi(y)| \]
\[ \leq c|w_+(x) - w_+(y)|^{\ell}(\phi^q(x) + \phi^q(y))^{\frac{1}{q}} + c(\epsilon)(\phi^q(x) + \phi^q(y))^{(q-\ell)/q}|\phi(x) - \phi(y)|^{\ell}(w_+(x) + w_+(y))^\ell. \]

Since 0 ≤ φ ≤ 1 and (q - ℓ)/q ≥ 0, after choosing ε so small, we discover
\[ \Phi_\ell(w_+(x) - w_+(y))(w_+(x)\phi^q(x) - w_+(y)\phi^q(y)) \]
\[ \geq |w_+(x) - w_+(y)|^{\ell}(\phi^q(x) + \phi^q(y))^{\frac{1}{q}} - c|\phi(x) - \phi(y)|^{\ell}(w_+(x) + w_+(y))^\ell. \]

We notice that by the symmetry of the above inequality for x and y, we also have the same inequality when u(x) < u(y). Inserting this into (4.4) and using (1.7), we have
\[
I_1 \geq \frac{1}{4A} \int_{B_r} \int_{B_r} H(x, y, |w_+(x) - w_+(y)|(\phi^q(x) + \phi^q(y)) \left| \frac{dx dy}{|x - y|^n} \right.
\]
\[- c \int_{B_r} \int_{B_r} H(x, y, |\phi(x) - \phi(y)|(w_+(x) + w_+(y))) \left| \frac{dx dy}{|x - y|^n} \right. \]

For I_2, we observe that
\[
\Phi_\ell(u(x) - u(y))w_+(x) \geq -w_+^{q-1}(y)w_+(x)
\]
and use (1.7), to find
\[
I_2 \geq -c \int_{\mathbb{R}^n \setminus B_r} \int_{B_r} h(x, y, w_+(y))w_+(x)\phi^q(x) \left| \frac{dx dy}{|x - y|^n} \right. \]
\[- c \sup_{x \in \text{supp } \phi} \int_{\mathbb{R}^n \setminus B_r} h(x, y, w_+(y)) \left| \frac{dy}{|x - y|^n} \right. \int_{B_r} w_+(x)\phi^q(x) dx. \]

Combining the above estimates with I_1 + I_2 = 0, we obtain (4.2). \qed

**Remark 4.3.** If u is locally bounded, then the assumption (1.12) in Lemma 4.2 can be eliminated, see the case (ii) in Lemma 4.1. Moreover, we can also obtain (4.2) when q > p^*_\infty by using a truncation argument as in [36, Lemma 4.2] provided the right-hand side of (4.2) is finite.

Now, we are ready to prove the local boundedness of weak solutions to (1.1).

**Proof of Theorem 1.1.** For convenience, we define
\[ H_0(\tau) := \tau^p + |a|_{L^\infty} \tau^q, \quad \tau \geq 0. \]

In the following, c means a constant depending only on data.

Let B_r \equiv B_r(x_0) \subset \Omega be a fixed ball with r ≤ 1. For r/2 ≤ \rho < \sigma ≤ r and k > 0, we denote
\[ A^+(k, \rho) := \{x \in B_\rho : u(x) \geq k\} \]
and apply Lemma 2.4 with \( f = (u - k)_+ \) to have

\[
\rho^{-sp} \int_{B_{\rho}} H_0(f) \, dx \leq \int_{B_{\rho}} \left( \frac{f}{\rho^p} \right)^p + \|a\|_{L^\infty} \left( \frac{f}{\rho^q} \right)^q \, dx \\
\leq c\|a\|_{L^\infty} \rho^{(s-t)q} \left( \int_{B_{\rho}} \int_{B_{\rho}} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \right)^{\frac{q}{p}} \\
+ c \left( \frac{|A^+(k, \rho)|}{|B_{\rho}|} \right)^{\frac{p-1}{p}} \int_{B_{\rho}} \left( \frac{f}{\rho^p} \right)^p + \|a\|_{L^\infty} \left( \frac{f}{\rho^q} \right)^q \, dx.
\]

(4.5)

We now fix \( 0 < h < k \) and observe that, for \( x \in A^+(k, \rho) \subset A^+(h, \rho) \),

\[
(u(x) - h)_+ = u(x) - h \geq k - h, \\
(u(x) - h)_+ = u(x) - h \geq u(x) - k = (u(x) - k)_+.
\]

This implies

\[
|A^+(k, \rho)| \leq \int_{A^+(k, \rho)} \frac{(u - h)_+^p}{(k - h)^p} \, dx \leq \frac{1}{(k - h)^p} \int_{A^+(h, \rho)} H_0((u - h)_+) \, dx
\]

(4.6)

and

\[
\int_{B_{\rho}} (u - k)_+ \, dx \leq \int_{B_{\rho}} (u - h)_+ \left( \frac{(u - h)_+}{k - h} \right)^{p-1} \, dx \\
\leq \frac{1}{(k - h)^{p-1}} \int_{B_{\rho}} H_0((u - h)_+) \, dx.
\]

(4.7)

We then choose a cut-off function \( \phi \in C_0^\infty(B_{2r}) \) satisfying \( 0 \leq \phi \leq 1, \phi \equiv 1 \) in \( B_{\rho} \) and \( |D\phi| \leq 4/(\sigma - \rho) \). Denoting the tail by

\[
T(v; r) \equiv \int_{B_{\rho} \setminus B_{\rho}} \frac{|v(x)|^{p-1}}{|x - x_0|^{n+sp}} + \|a\|_{L^\infty} \frac{|v(x)|^{q-1}}{|x - x_0|^{n+sp}} \, dx,
\]

Lemma 4.2 gives

\[
\int_{B_{\rho}} \int_{B_{\rho}} \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \\
\leq \frac{c}{(\sigma - \rho)^p} \int_{B_{\rho}} (u(x) - h)_+^p \int_{B_{\rho}} \frac{1}{|x - y|^{n+(s-1)p}} \, dy \, dx \\
+ \frac{c\|a\|_{L^\infty}}{(\sigma - \rho)^q} \int_{B_{\rho}} (u(x) - h)_+^q \int_{B_{\rho}} \frac{1}{|x - y|^{n+(q-1)q}} \, dy \, dx \\
+ c \left( \sup_{x \in \text{supp} \phi} \int_{\mathbb{R}^n \setminus B_{\rho}} \frac{(u(y) - k)_+^{p-1}}{|x - y|^{n+sp}} + \|a\|_{L^\infty} \frac{(u(y) - k)_+^{q-1}}{|x - y|^{n+sp}} \, dy \right) \int_{B_{\rho}} (u(x) - k)_+ \, dx \\
\leq \frac{c\rho^{(1-s)p}}{(\sigma - \rho)^p} \int_{B_{\rho}} (u - h)_+^p \, dx + \frac{c\|a\|_{L^\infty} \rho^{(1-s)q}}{(\sigma - \rho)^q} \int_{B_{\rho}} (u - h)_+^q \, dx \\
+ \frac{c(\sigma + \rho)}{\sigma - \rho} \left( \int_{B_{\rho} \setminus B_{\rho}} \frac{(u(y) - k)_+^{p-1}}{|y - x_0|^{n+sp}} + \|a\|_{L^\infty} \frac{(u(y) - k)_+^{q-1}}{|y - x_0|^{n+sp}} \, dy \right) \int_{B_{\rho}} (u(x) - k)_+ \, dx \\
\leq \frac{c(\sigma + \rho)}{(\sigma - \rho)^q} \int_{B_{\rho}} H_0((u - h)_+) \, dx + \frac{c(\sigma + \rho)}{\sigma - \rho} \left[ T((u - k)_+; \sigma) \right] \int_{B_{\rho}} (u - k)_+ \, dx,
\]

where \( \sigma = \frac{\rho}{2} \).

As a result, we obtain

\[
\rho^{-sp} \int_{B_{\rho}} H_0(f) \, dx \leq \int_{B_{\rho}} \left( \frac{f}{\rho^p} \right)^p + \|a\|_{L^\infty} \left( \frac{f}{\rho^q} \right)^q \, dx \\
\leq c\|a\|_{L^\infty} \rho^{(s-t)q} \left( \int_{B_{\rho}} \int_{B_{\rho}} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \right)^{\frac{q}{p}} \\
+ c \left( \frac{|A^+(k, \rho)|}{|B_{\rho}|} \right)^{\frac{p-1}{p}} \int_{B_{\rho}} \left( \frac{f}{\rho^p} \right)^p + \|a\|_{L^\infty} \left( \frac{f}{\rho^q} \right)^q \, dx.
\]
where we have used that \(|y-x_0|/|y-x| \leq 1 + \frac{|y-x_0|}{|y-x|} \leq 1 + \frac{\sigma + \rho}{\sigma - \rho} \leq 2\frac{\sigma + \rho}{\sigma - \rho}
\) for \(x \in \text{supp} \phi \) and \(y \in \mathbb{R}^n \setminus B_r\).

Combining this estimate together with \((4.5)-(4.7)\) implies

\[
\rho^{-sp} \int_{B_r} H_0((u - k)_+) \, dx \\
\leq c \rho^{(s-t)q} \rho^{(1-t)q} \left( \frac{\int_{B_r} H_0((u - h)_+) \, dx}{(\sigma - \rho)^q} \right)^{\frac{1}{p}} \left( \frac{1}{k-h)^{q/p}} \right) \left( \frac{\int_{B_r} H_0((u - h)_+) \, dx}{(\sigma - \rho)^q} \right)^{\frac{1}{p}} \\
+ c \rho^0 \left( \frac{\int_{B_r} H_0((u - h)_+) \, dx}{(\sigma - \rho)^q} \right)^{\frac{1}{p}} \left( \frac{1}{k-h)^{q/p}} \right) \left( \frac{\int_{B_r} H_0((u - h)_+) \, dx}{(\sigma - \rho)^q} \right)^{\frac{1}{p}} \\
+ c \rho^0 \left( \frac{\int_{B_r} H_0((u - h)_+) \, dx}{(\sigma - \rho)^q} \right)^{\frac{1}{p}} \left( \frac{1}{k-h)^{q/p}} \right) \left( \frac{\int_{B_r} H_0((u - h)_+) \, dx}{(\sigma - \rho)^q} \right)^{\frac{1}{p}} \\
+ c \rho^0 \left( \frac{\int_{B_r} H_0((u - h)_+) \, dx}{(\sigma - \rho)^q} \right)^{\frac{1}{p}} \left( \frac{1}{k-h)^{q/p}} \right) \left( \frac{\int_{B_r} H_0((u - h)_+) \, dx}{(\sigma - \rho)^q} \right)^{\frac{1}{p}}.
\]

Now, for \(i = 0, 1, 2, \ldots\) and \(k_0 > 1\), we write

\[
\sigma_i := \frac{r}{2}(1 + 2^{-i}), \quad k_i := 2k_0(1 - 2^{-i-1}) \quad \text{and} \quad y_i := \int_{A^+(k_i, \sigma_i)} H_0((u - k_i)_+) \, dx.
\]

Since \(H_0(u) \in L^1(\Omega)\) from the assumption \((1.12)\), we see that

\[
y_0 = \int_{A^+(k_0, r)} H_0((u - k_0)_+) \, dx \rightarrow 0 \quad \text{as} \quad k_0 \rightarrow \infty.
\]

First, we consider \(k_0 > 1\) so large that

\[
y_i \leq y_{i-1} \leq \cdots \leq y_0 \leq 1, \quad i = 1, 2, \ldots.
\]

Then, since

\[
T((u - k_i)_+; \sigma_i) \leq T(u; r/2) < \infty,
\]

we have

\[
y_{i+1} \leq \bar{c} \left( 2^{(q/p' y_i)^p + 2(\sigma(p')^{(q' + 1)} y_i^{(1 + sp)/p'} + 2^{(sp^{2}/n)q + 2^{(sp^{2}/n + q) y_i^{1 + (sp)/n}} + 2^{(sp^{2}/n + p) y_i^{1 + (sp)/n}} + 2^{(sp^{2}/n) y_i^{p}}}) \right) \\
\leq \bar{c} 2^{\theta y_i^{1 + \beta}}
\]

for some constant \(\bar{c} > 0\) depending on \(\text{data}, r\) and \(T(u; r/2)\), where

\[
\theta = \max \left\{ \frac{q^2}{p'}, \frac{q}{p'} + 1, \frac{sp^2}{n} + q, p(p - 1) \right\}, \quad \beta = \min \left\{ \frac{q}{p - 1}, \frac{sp}{n}, p - 1 \right\}.
\]

Finally, we can choose \(k_0\) so large that

\[
y_0 \leq \bar{c}^{-1/\beta} 2^{-\theta/\beta^2}
\]

holds. Then Lemma 2.7 implies

\[
y_\infty = \int_{A^+(2k_0, r/2)} H_0((u - 2k_0)_+) \, dx = 0,
\]

which means that \(u \leq 2k_0\) a.e. in \(B_{r/2}\).

Applying the same argument to \(-u\), we consequently obtain \(u \in L^\infty(B_{r/2})\). \hfill \Box

5. Hölder Continuity

Throughout this section, we assume that the modulating coefficient \(a(\cdot, \cdot)\) satisfies \((1.9)-(1.10)\), and that a weak solution \(u \in \mathcal{A}(\Omega) \cap L^{p,p-1}_{\text{loc}}(\mathbb{R}^n)\) under consideration is locally bounded in \(\Omega\). We fix any \(\Omega' \subseteq \Omega\) and define

\[
M \equiv M(\Omega') := 1 + \|u\|^p_{L^\infty(\Omega')}.
\]

(5.1)
5.1. Logarithmic estimate. We start with obtaining a logarithmic type estimate. This implies Corollary 5.2, which will play a crucial role in the proof of Hölder continuity.

Lemma 5.1. Under the assumptions in Theorem 1.2, let \( u \in A(\Omega) \cap L_{sp}^q(\mathbb{R}^n) \) be a weak supersolution to (1.1) such that \( u \in L^\infty(\Omega') \) and \( u \geq 0 \) in a ball \( B_R \equiv B_R(x_0) \subset \Omega' \) with \( R > 1 \). Then the following estimate holds true for any \( 0 < \rho < R/2 \) and \( d > 0 \):

\[
\int_{B_{\rho}} \int_{B_{\rho}} \log \left( \frac{u(x) + d}{u(y) + d} \right) \frac{dydx}{|x - y|^n} \leq c\tilde{M}^2 \left( \rho^n + \rho^{n+sp}d^{-p} \int_{\mathbb{R}^n \setminus B_R} \frac{u^{q-1}(y) + u^{q-1}(y)}{|y - x_0|^{n+sp}} dy \right. \\
+ \left. \rho^{n+td}d^{-q} \int_{\mathbb{R}^n \setminus B_{\rho}(x_0)} \frac{u^{q-1}(y)}{|y - x_0|^{n+td}} dy \right)
\]

for some \( c \equiv c(\text{data}) \), where \( \tilde{M} \equiv \tilde{M}(\Omega') := 1 + (\|u\|_{L^\infty(\Omega')} + d)^{sp} \).

Proof. We recall (2.1), (4.1) and (4.3), and further denote

\[
\tilde{H}(x, y, \tau) := \frac{\tau^p}{\rho^p} + a(x, y)\frac{\tau^q}{\rho^q},
\]

\[
\tilde{h}(x, y, \tau) := \frac{\tau^{p-1}}{\rho^p} + a(x, y)\frac{\tau^{q-1}}{\rho^q},
\]

\[
G(\tau) := \frac{\tau^p}{\rho^p} + a_1^2\frac{\tau^q}{\rho^q},
\]

\[
g(\tau) := \frac{\tau^{p-1}}{\rho^p} + a_2\frac{\tau^{q-1}}{\rho^q},
\]

where \( \tau \geq 0 \) and

\[
a_2 := \sup_{B_{2\rho} \times B_\rho} a(\cdot, \cdot).
\]

Let \( \phi \in C_C^\infty(B_{\rho}/2) \) be a cut-off function satisfying \( 0 \leq \phi \leq 1, \phi \equiv 1 \) in \( B_{\rho} \) and \( |D\phi| \leq 4/\rho \). Testing (1.11) with \( \varphi(x) = \phi^q(x)/g(u(x) + d) \), we have

\[
0 \leq \int_{B_{\rho}} \int_{B_{\rho}} \left[ \Phi_p(u(x) - u(y))K_sp(x, y) \left( \frac{\phi^q(x)}{g(u(x) + d)} - \frac{\phi^q(y)}{g(u(y) + d)} \right) \\
+ a(x, y)\Phi_q(u(x) - u(y))K_tq(x, y) \left( \frac{\phi^q(x)}{g(u(x) + d)} - \frac{\phi^q(y)}{g(u(y) + d)} \right) \right] dxdy \\
+ 2 \int_{\mathbb{R}^n \setminus B_{\rho}} \int_{B_{\rho}} \left[ \Phi_p(u(x) - u(y))K_sp(x, y) \frac{\phi^q(x)}{g(u(x) + d)} \\
+ a(x, y)\Phi_q(u(x) - u(y))K_tq(x, y) \frac{\phi^q(x)}{g(u(x) + d)} \right] dxdy
\]

\[
=: I_1 + I_2.
\]

Moreover, in \( I_1 \) the integrand with respect to the measure \( \frac{dxdy}{|x - y|^n} \) is denoted by \( F(x, y) \), that is,

\[
I_1 = \int_{B_{\rho}} \int_{B_{\rho}} F(x, y) \frac{dxdy}{|x - y|^n},
\]

\[
F(x, y) := \Phi_p(u(x) - u(y))K_sp(x, y)|x - y|^n \left( \frac{\phi^q(x)}{g(u(x) + d)} - \frac{\phi^q(y)}{g(u(y) + d)} \right) \\
+ a(x, y)\Phi_q(u(x) - u(y))K_tq(x, y)|x - y|^n \left( \frac{\phi^q(x)}{g(u(x) + d)} - \frac{\phi^q(y)}{g(u(y) + d)} \right).
\]

We also denote \( \bar{u}(x) := u(x) + d \). Next, we estimate \( I_1 \) and \( I_2 \) separately. The remaining part of the proof is divided into four steps.

Step 1: Estimate of \( F(x, y) \) when \( \bar{u}(x) \geq \bar{u}(y) \geq \frac{1}{q}\bar{u}(x) \). We first observe that

\[
\frac{\phi^q(x)}{g(\bar{u}(x))} - \frac{\phi^q(y)}{g(\bar{u}(y))} = \frac{\phi^q(x) - \phi^q(y)}{g(\bar{u}(x))} + \phi^q(x) \left( \frac{1}{g(\bar{u}(x))} - \frac{1}{g(\bar{u}(y))} \right)
\]

\[
\leq q\phi^{q-1}(x)|\phi(x) - \phi(y)| + \phi^q(x) \int_0^1 \frac{d}{d\sigma} \left( \frac{1}{g(\sigma \bar{u}(x) + (1 - \sigma)\bar{u}(y))} \right) d\sigma.
\]
To estimate the last integral, we first observe that
\[
\frac{d}{d\sigma} \left( \frac{1}{g(\sigma u(x) + (1 - \sigma) u(y))} \right) = \frac{g'(\sigma u(x) + (1 - \sigma) u(y))}{g^2(\sigma u(x) + (1 - \sigma) u(y))} (\bar{u}(x) - \bar{u}(y)),
\]
where a direct calculation shows
\[
\frac{g'(\tau)}{g^2(\tau)} = (p-1) \frac{\tau^{p-2}}{(\tau^{p} + (q-1)\rho^2\tau^q)} + (q-1)\alpha_2 \frac{\tau^q}{\rho^q}, \quad \text{hence} \quad \frac{p-1}{G(\tau)} \leq \frac{g'(\tau)}{g^2(\tau)} \leq \frac{q-1}{G(\tau)}.
\]

Thus we have
\[
\frac{\phi'(x)}{g(u(x))} \frac{\phi'(y)}{g(u(y))} \leq \frac{g\phi'^{-1}(x)(\bar{u}(x) - \bar{u}(y))}{g(u(x))} - \frac{(p-1)\phi'(x)(u(x) - \bar{u}(y))}{G(u(y))} \leq \frac{g\phi'^{-1}(x)(\bar{u}(x) - \bar{u}(y))}{g(u(y))} - \frac{(p-1)\phi'(x)(u(x) - \bar{u}(y))}{G(u(y))}.
\]

Applying this inequality to \(F(x, y)\) and using (1.7), we have
\[
F(x, y) \leq \Lambda q h(x, y, \bar{u}(x) - \bar{u}(y))\phi'^{-1}(x)(\bar{u}(x) - \bar{u}(y)) \quad \text{where for the last inequality we have used that} \quad x, y \in B_{2p}.
\]

Thus, we have
\[
\frac{h(x, y, \bar{u}(x) - \bar{u}(y))}{G(\bar{u}(y))} \leq \frac{\phi'(x)}{G(\bar{u}(y))} \frac{\phi'(y)}{G(\bar{u}(y))} \leq \frac{\phi'(x)}{G(\bar{u}(y))} \frac{\phi'(y)}{G(\bar{u}(y))} \leq \frac{\phi'(x)}{G(\bar{u}(y))} \frac{\phi'(y)}{G(\bar{u}(y))}.
\]

Let us now estimate the first term in the right-hand side of (5.2). Applying Young’s inequality to the numerator, for any small \(\varepsilon > 0\) we obtain
\[
\left( h(x, y, \bar{u}(x) - \bar{u}(y)) \phi'^{-1}(x)(\bar{u}(x) - \bar{u}(y)) \leq \varepsilon (\bar{u}(x) - \bar{u}(y))^p \phi'(x \phi'(y)) + c(\varepsilon) \phi'(x) \phi'(y) \phi'^{-1}(x)(\bar{u}(x) - \bar{u}(y)) \right.
\]
\[
\leq \varepsilon \phi'(x) H(x, y, \bar{u}(x) - \bar{u}(y)) + c(\varepsilon) \left( \frac{\phi'(x) \phi'(y)}{x - y |x - y|^q} \right).
\]

for any small \(\varepsilon > 0\). Putting this into (5.2) and choosing
\[
\varepsilon = \frac{p-1}{2q+1} \Lambda,
\]

we have
\[
F(x, y) \leq c \frac{|x - y|^{(1-s)p}}{\rho^{(1-s)p}} + c \frac{|x - y|^{(1-t)q}}{\rho^{(1-t)q}} \leq \frac{p-1}{2q+1} \Lambda \frac{\phi'(x) H(x, y, \bar{u}(x) - \bar{u}(y))}{G(\bar{u}(y))}.
\]

In order to estimate the last term in the above display, we note that
\[
a_2 = a_2 - a(x, y) + a(x, y) \leq [a] \rho^s + a(x, y), \quad x, y \in B_{2p},
\]

where for the last inequality we have used that \(x, y \in B_{2p}\).
to discover
\[ G(\bar{u}(y)) = \frac{\bar{u}^p(y)}{\rho x} + a\frac{\bar{u}^q(y)}{\rho^q} \]
\[ \leq \frac{\bar{u}^p(y)}{\rho x} + [a]_{\alpha} \rho^{-t} \| \bar{u} \|_{L^{\infty}(\mathbb{V})}^{q-p} \frac{\bar{u}^p(y)}{\rho x} + a(x, y, \bar{u}^q(y)) \]
\[ \leq (1 + \alpha [\alpha]) (1 + \|u\|_{L^{\infty}(\mathbb{V})} + d^q \rho) \bar{H}(x, y, \bar{u}(y)), \]
where we have used the inequality in (1.13) with \( \rho \leq 1 \). Then it follows that
\[ -\frac{\phi^q(x) \bar{H}(x, y, \bar{u}(x) - \bar{u}(y))}{G(\bar{u}(y))} \leq -\frac{1}{cM} \frac{\phi^q(x) \bar{H}(x, y, \bar{u}(x) - \bar{u}(y))}{\bar{H}(x, y, \bar{u}(y))} \]
and therefore
\[ F(x, y) \leq c \frac{|x - y|^{(1-s)p}}{\rho^{(1-s)p}} + c \frac{|x - y|^{(1-t)q}}{\rho^{(1-t)q}} = \frac{1}{cM} \frac{\phi^q(x) \bar{H}(x, y, \bar{u}(x) - \bar{u}(y))}{\bar{H}(x, y, \bar{u}(y))}. \] (5.4)

We now need to derive an estimate for \( \log \bar{u} \). Observe
\[ \log \bar{u}(x) - \log \bar{u}(y) = \int_0^1 \frac{\bar{u}(x) - \bar{u}(y)}{\sigma \bar{u}(x) + (1 - \sigma) \bar{u}(y)} d\sigma \leq \frac{\bar{u}(x) - \bar{u}(y)}{\bar{u}(y)} \frac{|x - y|^s}{\rho^s} \]
and use the monotonicity of the function
\[ \tau \mapsto \frac{\tau^p + a(x, y, \bar{u}(x) - \bar{u}(y))}{\tau}, \quad \tau \geq 0, \]
to obtain
\[ \log \bar{u}(x) - \log \bar{u}(y) \]
\[ \leq \left[ \frac{\bar{u}(x) - \bar{u}(y)}{\rho x} + a(x, y, \bar{u}(x) - \bar{u}(y)) \right] \frac{1}{|x - y|^{(1-s)p}} + \frac{1}{|x - y|^{(1-t)q}} \]
\[ \leq c \frac{H(x, y, \bar{u}(x) - \bar{u}(y))}{\bar{H}(x, y, \bar{u}(y))} + \frac{|x - y|^s}{\rho^s}. \]

For the last inequality, we have used the fact that \(|x - y| \leq 2 \rho\). Finally, inserting this into (5.4), we obtain
\[ F(x, y) \leq c \frac{|x - y|^{(1-s)p}}{\rho^{(1-s)p}} + c \frac{|x - y|^{(1-t)q}}{\rho^{(1-t)q}} + c \frac{|x - y|^s}{\rho^s} - \frac{\phi^q(x)}{cM} \log \left( \frac{\bar{u}(x)}{\bar{u}(y)} \right). \]

Step 2: Estimate of \( F(x, y) \) when \( \bar{u}(x) \geq 2 \bar{u}(y) \). We first observe from the second inequality in Lemma 2.6 with \( \varepsilon = \frac{2^{q-1} - 1}{2^{p-1}} \) and \( \bar{u}(x) \geq 2 \bar{u}(y) \) that
\[ \frac{\phi^q(x)}{g(\bar{u}(x))} - \frac{\phi^q(y)}{g(\bar{u}(y))} \]
\[ \leq \frac{\phi^q(x) - \phi^q(y)}{g(\bar{u}(x))} + \phi^q(y) \left( \frac{1}{g(\bar{u}(x))} - \frac{1}{g(\bar{u}(y))} \right) \]
\[ \leq \frac{\phi^q(x) - \phi^q(y)}{g(\bar{u}(x))} + \phi^q(y) \left( \frac{1}{g(2\bar{u}(y))} - \frac{1}{g(\bar{u}(y))} \right) \]
\[ \leq \frac{\phi^q(x) - \phi^q(y)}{g(\bar{u}(x))} \frac{1}{\bar{u}(y)} - \left( 1 - \frac{1}{2^{p-1}} \right) \frac{\phi^q(y)}{g(\bar{u}(y))} \]
\[ \leq \frac{\phi^q(x) - \phi^q(y)}{g(\bar{u}(x))} + \left( \frac{1}{2^{p-1}} - \frac{1}{2^{p-1}} \right) \frac{\phi^q(y)}{g(\bar{u}(y))} \]
\[ \leq \frac{\phi^q(x) - \phi^q(y)}{g(\bar{u}(x))} + \left( 2^{p-1} - 1 \right) \frac{\phi^q(y)}{g(\bar{u}(y))}. \]
Furthermore, since we have in this case
\[
F(x, y) \leq c \frac{h(x, y, \bar{u}(x) - \bar{u}(y))|\phi(x) - \phi(y)|^q}{g(\bar{u}(x))} - \frac{1}{c} h(x, y, \bar{u}(x) - \bar{u}(y))\phi(y)
\]
Estimating the right-hand side similarly as in (5.3), we find
\[
F(x, y) \leq c \frac{h(x, y, \bar{u}(x) - \bar{u}(y))|\phi(x) - \phi(y)|^q}{g(\bar{u}(x))} - \frac{1}{c} h(x, y, \bar{u}(x) - \bar{u}(y))
\]
The first term in the right-hand side is estimated as
\[
\frac{h(x, y, \bar{u}(x) - \bar{u}(y))|\phi(x) - \phi(y)|^q}{g(\bar{u}(x))}
\]
\[
= \frac{\rho^{\rho p} |\bar{u}(x) - \bar{u}(y)|^{p-1} + a(x, y) \rho^{\rho q} |\bar{u}(x) - \bar{u}(y)|^{q-1}}{\rho^q} |\phi(x) - \phi(y)|^q
\]
\[
\leq \left( \frac{\rho^{\rho p}}{|x - y|^p p} + \frac{\rho^{\rho q}}{|x - y|^q} \right) |x - y|^q,
\]
hence we have
\[
F(x, y) \leq c \frac{|x - y|^{(1-s)q}}{\rho^{(1-s)q}} + c \frac{|x - y|^{(1-t)q}}{\rho^{(1-t)q}} - \frac{1}{c M} h(x, y, \bar{u}(x) - \bar{u}(y)).
\]
Furthermore, since we have in this case
\[
\log \bar{u}(x) - \log \bar{u}(y) \leq \log \left( \frac{2(\bar{u}(x) - \bar{u}(y))}{\bar{u}(y)} \right) \leq c \frac{2(\bar{u}(x) - \bar{u}(y))}{\bar{u}(y)} - \frac{1}{\rho^{(1-\tau)q}} |x - y|\rho^q,
\]
\[
\leq c \frac{2(\bar{u}(x) - \bar{u}(y))}{\bar{u}(y)} - \frac{1}{\rho^{(1-\tau)q}} |x - y|\rho^q,
\]
We then use the monotonicity of the function
\[
\tau \mapsto \frac{\tau^{p-1} + a(x, y) \tau^{q-1} |x - y|^{(s-1)q}}{\tau^{p-1}}, \quad \tau \geq 0,
\]
and use the fact that $|x - y| \leq 2\rho$, to have
\[
\log \bar{u}(x) - \log \bar{u}(y)
\]
\[
\leq c \left[ \frac{2(\bar{u}(x) - \bar{u}(y))}{|x - y|^s} + a(x, y) \frac{2(\bar{u}(x) - \bar{u}(y))}{|x - y|^s} - 1 \right] \frac{1}{|x - y|^{(s-1)q}} + 1
\]
\[
\leq c \frac{h(x, y, \bar{u}(x) - \bar{u}(y))}{\bar{u}(y)} + c \frac{|x - y|^{s(1-\tau)}}{\rho^{s(1-\tau)}}.
\]
Finally, inserting this into (5.5), we obtain
\[
F(x, y) \leq c \frac{|x - y|^{q-1}}{\rho^{q-1}} + c \frac{|x - y|^{(1-t)q}}{\rho^{(1-t)q}} + c \frac{|x - y|^{s(1-\tau)}}{\rho^{s(1-\tau)}} - \frac{1}{c M} \log \left( \frac{\bar{u}(x)}{\bar{u}(y)} \right).
\]
\textbf{Step 3: Estimate of $I_1$.} From Step 1 and Step 2, we have that
\[
F(x, y) \leq c \frac{|x - y|^{(1-s)p}}{\rho^{(1-s)p}} + c \frac{|x - y|^{(1-t)q}}{\rho^{(1-t)q}} + c \frac{|x - y|^{s}}{\rho^{s}} + c \frac{|x - y|^{s(1-\tau)}}{\rho^{s(1-\tau)}}
\]
\[
- \frac{\left( \min \{\phi(x), \phi(y)\} \right)^q}{c M} \left| \log \left( \frac{\bar{u}(x)}{\bar{u}(y)} \right) \right|,
\]
when \( \bar{u}(x) \geq \check{u}(y) \). Moreover, by the symmetry of the above estimate for \( x \) and \( y \), the same estimate still holds when \( \check{u}(x) < \bar{u}(y) \). Therefore, \( I_1 \) is finally estimated as follows:

\[
I_1 \leq \frac{1}{cM} \int_{B_{20}} \int_{B_0} \left| \log \frac{\bar{u}(x)}{\check{u}(y)} \right| \frac{dydx}{|x-y|^n} + c \int_{B_{20}} \int_{B_{0}(x)} \left( \frac{|x-y|^{(1-s)p}}{\rho^{(1-s)q}} + \frac{|x-y|^{(1-t)q}}{\rho^{(1-t)q}} + \frac{|x-y|^s}{\rho^s} + \frac{|x-y|^{s(p-1)}}{\rho^{s(p-1)}} \right) \frac{dydx}{|x-y|^n} \tag{5.6}
\]

\[
\leq \frac{1}{cM} \int_{B_{20}} \int_{B_0} \left| \log \frac{\bar{u}(x)}{\check{u}(y)} \right| \frac{dydx}{|x-y|^n} + c \rho^n. \tag{5.6}
\]

**Step 4**: Estimate of \( I_2 \) and Conclusion. We start with the following observation:

(i) If \( y \in B_{R} \setminus B_{20} \), then \( u(y) \geq 0 \) and \( u(x) - u(y) \leq u(x) \);

(ii) If \( y \in \mathbb{R}^n \setminus B_{R} \), then \( (u(x) - u(y))_+ \leq (u(x) + u_- (y))_+ = u(x) + u_- (y) \).

Using this and the fact that \( \text{supp} \phi \subset B_{3p/2} \), we write

\[
I_2 \leq 2 \int_{B_{3p/2}} \int_{\mathbb{R}^n \setminus B_{20}} \frac{h(x, y, u(x) + d)}{g(u(x) + d)} \frac{dydx}{|x-y|^n} + 2 \int_{B_{3p/2}} \int_{\mathbb{R}^n \setminus B_{R}} \frac{h(x, y, u_- (y))}{g(u(x) + d)} \frac{dydx}{|x-y|^n}. \tag{5.7}
\]

Since we are considering integrals over the complement of balls, we cannot directly compare \( a_2 \) and \( a(x,y) \) there. In order to overcome this difficulty, we observe that \((1.9)\) and \((1.13)\) imply

\[
a(x, y) \leq a(x, y) - a(x, x) + a_2 \leq (a(x, y) - a(x, x)) \frac{\rho^n}{\rho^n} + a_2 \leq c|x-y|^{q-sp} + a_2, \tag{5.8}
\]

whenever \( x \in B_{20} \) and \( y \in \mathbb{R}^n \).

For the first integral in \((5.7)\), we use \((5.8)\) and the fact that \( |x-y| > \frac{2}{2} \) for \( x \in B_{3p/2} \) and \( y \in \mathbb{R}^n \setminus B_{20} \), to find

\[
\frac{h(x, y, u(x) + d)}{g(u(x) + d)} \leq c \frac{\bar{u}^{p-1}(x) + a(x, y) \check{u}^{q-1}(x)}{|x-y|^{sp}} + \frac{\check{u}^{p-1}(x) + a_2 \check{u}^{q-1}(x)}{|x-y|^{q} + a_2 \check{u}^{q-1}(x)} \leq c \frac{\check{M}}{|x-y|^{sp}} \tag{5.9}
\]

which gives

\[
\int_{B_{3p/2}} \int_{\mathbb{R}^n \setminus B_{20}} \frac{h(x, y, u(x) + d)}{g(u(x) + d)} \frac{dydx}{|x-y|^n} \leq c \check{M} \rho^n. \tag{5.9}
\]

For the second integral in \((5.7)\), we use \((5.8)\) and the fact that \( \frac{|y-x_0|}{|y-x|} \leq 1 + \frac{|y-x_0|}{|y-x|} \leq 1 + \frac{3p/2}{p/2} = 4 \) for \( x \in B_{3p/2} \) and \( y \in \mathbb{R}^n \setminus B_{20} \) to find

\[
\frac{h(x, y, u_- (y))}{g(u(x) + d)} \leq \frac{u_-^{p-1}(y) + a(x, y) u_-^{q-1}(y)}{|x-y|^{sp}} + \frac{u_-^{p-1}(y) + |x-y|^{q-sp} u_-^{q-1}(y) + a_2 u_-^{q-1}(y)}{|x-y|^{q}} \leq c \frac{\rho^{sp} d^{1-p} u_-^{p-1}(y) + u_-^{q-1}(y)}{|y-x_0|^{sp}} + c \rho^{sp} d^{1-q} u_-^{q-1}(y) \tag{5.10}
\]

Consequently, we obtain

\[
\int_{B_{3p/2}} \int_{\mathbb{R}^n \setminus B_{R}} \frac{h(x, y, u_- (y))}{g(u(x) + d)} \frac{dydx}{|x-y|^n} \leq c \rho^{n+sp} d^{1-p} \int_{\mathbb{R}^n \setminus B_{R}} \frac{u_-^{p-1}(y) + u_-^{q-1}(y)}{|y-x_0|^{n+sp}} \frac{dy}{|y-x_0|^{n+sp}} + c \rho^{n+sp} d^{1-q} \int_{\mathbb{R}^n \setminus B_{R}} \frac{u_-^{q-1}(y)}{|y-x_0|^{n+sp}} \frac{dy}{|y-x_0|^{n+sp}}. \tag{5.10}
\]

Combining \((5.6)\), \((5.7)\), \((5.9)\) and \((5.10)\), we finally get the desired estimate. \( \square \)

The preceding lemma directly implies the following corollary.
Corollary 5.2. Under the same assumptions as in Lemma 5.1, let \( d, \zeta > 0, \xi > 1 \) and define
\[ v := \min\{ (\log(\zeta + d) - \log(\nu + d))_+, \log \xi \}. \]
Then we have
\[
\int_{B_r} (v - (v)_{B_r}) \, dx \leq cM^2 \left( 1 + \rho^p \rho d^{1-p} \int_{\mathbb{R}^n \setminus B_r} \frac{u^{p-1}(y) + u^{q-1}(y)}{|y - x_0|^{n+sp}} \, dy 
+ \rho^q d^{1-q} \int_{\mathbb{R}^n \setminus B_r} \frac{u^{q-1}(y)}{|y - x_0|^{n+sp}} \, dy \right)
\]
(5.11)
for some \( c \equiv c(\text{data}) > 0 \), where \( M = 1 + (\|u\|_{L^\infty(\Omega')} + d)^{q-p}. \)

Proof. It suffices to observe that
\[
\int_{B_r} |v - (v)_{B_r}| \, dx \leq \int_{B_r} \int_{B_r} |v(x) - v(y)| \, dy \, dx
\leq c\rho^{-n} \int_{B_r} \int_{B_r} \frac{|u(x) + d - u(y) + d|}{|x - y|^n} \, dy \, dx,
\]
as \( v \) is a truncation of \( \log(\nu + d) \). Now Lemma 5.1 gives the desired result. \( \square \)

5.2. Hölder continuity: Proof of Theorem 1.2. We are now in a position to prove Theorem 1.2. We first recall that \( \Omega' \subset \Omega \) has been fixed in the beginning of the section and the constant \( M \) was defined in (5.1). We then fix a ball \( B_{2r} \equiv B_{2r}(x_0) \subset \Omega' \). Let \( \sigma \in (0,1/4] \) be a constant depending only on \( \text{data}_1 \) and \( \|u\|_{L^\infty(\Omega')} \) that satisfies
\[
\sigma \leq \min \left\{ \frac{1}{4}, 2^{-\frac{p}{2}}, 6^{-\frac{d(q-1)}{sp}}, \exp \left( -\frac{c_\ast M^3}{\nu_\ast} \right) \right\},
\]
where the large constant \( c_\ast \equiv c_\ast(\text{data}_1) > 0 \) and the small one \( \nu_\ast \equiv \nu_\ast(\text{data}_1, \|u\|_{L^\infty(\Omega')}) > 0 \) are to be determined in (5.28) and (5.35), respectively, and then choose \( \gamma \in (0,1) \) depending only on \( \text{data}_1 \) and \( \|u\|_{L^\infty(\Omega')} \) satisfying
\[
\gamma \leq \min \left\{ \log_\sigma \left( \frac{1}{2} \right), \frac{sp}{2(p-1)}, \frac{tq}{2(q-1)}, \log_\sigma \left( 1 - \sigma \frac{M^3}{\nu_\ast} \right) \right\}.
\]
(5.13)
We define
\[
\frac{1}{2} K_0 := \sup_{B_r} |u| + \left[ r^{sp} \int_{\mathbb{R}^n \setminus B_r} \frac{|u(x)|^{p-1} + |u(x)|^{q-1}}{|x - x_0|^{n+sp}} \, dx \right]^{rac{1}{p+q}}
+ \left[ r^{tq} \int_{\mathbb{R}^n \setminus B_r} \frac{|u(x)|^{q-1}}{|x - x_0|^{n+sp+tq}} \, dx \right]^{rac{1}{q+1}}
\]
(5.14)
and, for \( j \in \mathbb{N} \cup \{0\} \), we write
\[
r_j := \sigma^j r, \quad B_j := B_{r_j}(x_0) \quad \text{and} \quad K_j := \sigma^j K_0.
\]
Now, we are going to prove the following oscillation lemma, which implies \( u \in C^{0,\gamma}(B_r) \).

Lemma 5.3. Under the assumptions of Theorem 1.2, let \( u \) be a weak solution to (1.1). Then we have for every \( j \in \mathbb{N} \cup \{0\} \)
\[
\omega(r_j) := \operatorname{osc}_{B_j} u \leq K_j.
\]
(5.15)

Proof. Step 1: Induction. The proof goes by induction on \( j \). For \( j = 0 \) it is obvious from the definition of \( K_0 \). Now we assume that (5.15) holds for all \( i \in \{0, \ldots, j\} \) with some \( j \geq 0 \) and show that it holds also for \( j + 1 \). That is, we will show that
\[
\omega(r_{j+1}) \leq K_{j+1}.
\]
(5.16)
Without loss of generality, we assume that
\[ \omega(r_{j+1}) \geq \frac{1}{2} K_{j+1}. \]
Then, this together with the fact that \( \sigma^\gamma \geq \frac{1}{2} \) from (5.13) implies that
\[ \omega(r_j) \geq \omega(r_{j+1}) \geq \frac{1}{2} K_{j+1} = \frac{1}{2} \sigma^\gamma K_j \geq \frac{1}{4} K_j. \]  
(5.17)

We note that either
\[ \frac{2B_{j+1} \cap \{ u \geq \inf_{B_j} u + \omega(r_j)/2 \}}{|2B_{j+1}|} \geq \frac{1}{2} \]
or
\[ \frac{2B_{j+1} \cap \{ u \leq \inf_{B_j} u + \omega(r_j)/2 \}}{|2B_{j+1}|} \geq \frac{1}{2} \]  
(5.18) (5.19)
must hold. We accordingly define
\[ u_j \begin{cases} 
  u - \inf_{B_j} u & \text{if (5.18) holds}, \\
  \sup_{B_j} u - u & \text{if (5.19) holds}.
\end{cases} \]
Then we have
\[ u_j \geq 0 \text{ in } B_j \quad \text{and} \quad \frac{2B_{j+1} \cap \{ u_j \geq \omega(r_j)/2 \}}{|2B_{j+1}|} \geq \frac{1}{2}. \]  
(5.20)
Moreover, \( u_j \) is a weak solution to (1.1) satisfying
\[ \sup_{B_i} |u_j| \leq \omega(r_i) \leq K_i \quad \forall \ i \in \{0, \ldots, j\}. \]  
(5.21)

**Step 2: Tail estimates.** We first claim that
\[ r_j^p \int_{\mathbb{R}^n \setminus B_j} \frac{|u_j(x)|^{p-1} + |u_j(x)|^{q-1}}{|x - x_0|^{n+sp}} \ dx \leq cM \sigma^{2p} K_j^{p-1} \]  
(5.22)
and
\[ r_j^q \int_{\mathbb{R}^n \setminus B_j} \frac{|u_j(x)|^{q-1}}{|x - x_0|^{n+sq}} \ dx \leq cM \sigma^{2q} K_j^{q-1} \]  
(5.23)
for a constant \( c \equiv c(\text{data}) \). We will only give the proof of (5.22), since (5.23) can be proved in almost the same way with \( s \) and \( p \) replaced by \( t \) and \( q \), respectively. From (5.21), (5.14) and (5.1), we have
\[ r_j^p \int_{\mathbb{R}^n \setminus B_j} \frac{|u_j(x)|^{p-1} + |u_j(x)|^{q-1}}{|x - x_0|^{n+sp}} \ dx \]
\[ = r_j^p \sum_{i=1}^{j} \int_{B_i \setminus B_{i-1}} \frac{|u_j(x)|^{p-1} + |u_j(x)|^{q-1}}{|x - x_0|^{n+sp}} \ dx + r_j^p \int_{\mathbb{R}^n \setminus B_0} \frac{|u_j(x)|^{p-1} + |u_j(x)|^{q-1}}{|x - x_0|^{n+sp}} \ dx \]
\[ \leq \sum_{i=1}^{j} \left( \frac{r_j}{r_{i-1}} \right)^p \left[ \left( \sup_{B_{i-1}} |u_j| \right)^{p-1} + \left( \sup_{B_{i-1}} |u_j| \right)^{q-1} \right] + cM \left( \frac{r_j}{r_1} \right)^p K_0^{p-1} \]
\[ \leq cM \sum_{i=1}^{j} \left( \frac{r_j}{r_{i-1}} \right)^p K_i^{p-1}, \]  
(5.24)
where for the first inequality we have used
\[ r_j^p \int_{\mathbb{R}^n \setminus B_0} \frac{|u_j(x)|^{p-1} + |u_j(x)|^{q-1}}{|x - x_0|^{n+sp}} \ dx \]
\[ \leq c \left( \frac{r_j}{r_0} \right)^p \left[ \left( \sup_{B_0} |u| \right)^{p-1} + \left( \sup_{B_0} |u| \right)^{q-1} \right] + cM \left( \frac{r_j}{r_1} \right)^p K_0^{p-1}. \]
Now the sum appearing in (5.24) is estimated as
\[
\sum_{i=1}^{j} \left( \frac{r_j}{r_i} \right)^{sp} K_{i-1}^{p-1} = K_j^{p-1} \left( \frac{r_j}{r_0} \right)^{(p-1)} \sum_{i=1}^{j} \left( \frac{r_i-1}{r_i} \right)^{\gamma(p-1)} \left( \frac{r_j}{r_i} \right)^{-sp - \gamma(p-1)}
\]
\[
= K_j^{p-1} \sigma^{-\gamma(p-1)} \sum_{i=1}^{j} \sigma^{i(\epsilon sp - \gamma(p-1))}
\]
\[
\leq 2^{p-1} K_j^{p-1} \sum_{i=1}^{j} \sigma^{i \epsilon} \leq 2^{p-1} K_j^{p-1} \frac{\sigma^{\epsilon}}{1 - \sigma^{\epsilon}} \leq 2^p \sigma^{\epsilon} K_j^{p-1},
\]
where we have used the facts that \(\sigma^{-\gamma} \leq 2, \epsilon sp - \gamma(p-1) \geq \frac{\epsilon}{2}p\) and \(\frac{\sigma^{\epsilon}}{1 - \sigma^{\epsilon}} \leq \frac{1}{\epsilon}\) from (5.12) and (5.13).

**Step 3: A density estimate.** We next apply Corollary 5.2 to the function
\[
v := \min \left\{ \left[ \log \left( \frac{\omega(r_j)/2 + d}{u_j + d} \right) \right]_+, k \right\},
\]
where \(k > 0\) is to be chosen and
\[
d \equiv d_j := \varepsilon K_j \quad \text{with} \quad \varepsilon := \sigma^{\epsilon \omega(r_j)}/\max \left\{ \sigma^{\epsilon \omega(r_j - 1)}, \sigma^{\epsilon \omega(r_j + 1)} \right\}.
\]
Note that by (5.17) we see that
\[
d_j \leq 4\omega(r_j) \leq 8\|u\|_{L^{\infty}(\Gamma')}, \quad \text{hence} \quad \widetilde{M} \leq c M.
\]
Combining the resulting estimate (5.11) with (5.22)-(5.23), we obtain
\[
\left\{ \int_{2B_{j+1}} |v - (v)_{2B_{j+1}}| dx \leq c M^2 \left[ 1 + M d_j^{1-p} \sigma^{\frac{\epsilon}{2} K_j^{p-1}} + d_j^{1-q} \sigma^{\frac{\epsilon}{2} K_j^{q-1}} \right]
\right.
\]
\[
\leq c M^3 \left[ 1 + \left( d_j^{-1} \sigma^{\epsilon \omega(r_j)} K_j \right)^{p-1} + \left( d_j^{-1} \sigma^{\epsilon \omega(r_j)} K_j \right)^{q-1} \right]
\]
\[
\leq c M^3
\]
for a constant \(c \equiv c(\text{data}_1) > 0\). In addition, we have from (5.20) that
\[
k = \frac{1}{|2B_{j+1} \setminus \{v = 0\}|} \int_{2B_{j+1} \setminus \{v = 0\}} (k - v) \, dx \leq 2 \int_{2B_{j+1}} (k - v) \, dx = 2(k - (v)_{2B_{j+1}}).
\]
This inequality and (5.27) imply
\[
\frac{|2B_{j+1} \cap \{v = k\}|}{|2B_{j+1}|} \leq \frac{2}{k |2B_{j+1}|} \int_{2B_{j+1} \setminus \{v = k\}} (k - (v)_{2B_{j+1}}) \, dx
\]
\[
\leq \frac{2}{k} \int_{2B_{j+1}} |v - (v)_{2B_{j+1}}| \, dx \leq \frac{c M^3}{k}
\]
At this moment, we choose
\[
k = \log \left( \frac{\omega(r_j)/2 + \varepsilon \omega(r_j)}{3\varepsilon \omega(r_j)} \right) = \log \left( \frac{1/2 + \varepsilon}{3\varepsilon} \right) \geq \log \left( \frac{1}{6\varepsilon} \right) \geq \log \left( \frac{1}{\sqrt{\varepsilon}} \right) = \frac{8}{4q - 1} \log \frac{1}{\sigma},
\]
where we have used the fact that \(\sqrt{\varepsilon} = \sigma^{\epsilon \omega(r_j)}/\epsilon \leq \frac{1}{\epsilon}\) from (5.12), to infer that
\[
\frac{|2B_{j+1} \cap \{u_j \leq d_j\}|}{|2B_{j+1}|} \leq \frac{c M^3}{k} \leq c_* M^3 / \log(1/\sigma).
\]
for a constant \(c_* > 0\) depending only on \(\text{data}_1\).

**Step 4: Iteration.** Now we proceed with an iteration argument. For \(i = 0, 1, 2, \ldots\), we set
\[
\rho_i = (1 + 2^{-i}) r_{j+1}, \quad \tilde{\rho}_i = \frac{\rho_i + \rho_{i+1}}{2}, \quad B^i = B_{\rho_i}, \quad \tilde{B}^i = B_{\tilde{\rho}_i}
\]
and choose corresponding cut-off functions satisfying
\[ \phi_i \in C_0^\infty(\tilde{B}'), \quad 0 \leq \phi \leq 1, \quad \phi_i \equiv 1 \text{ on } B^{i+1}, \quad \text{and } |D\phi_i| \leq 2^{i+1}r_{j+1}^{-1}. \]
Furthermore, we set
\[ k_i = (1 + 2^{-i})d_j, \quad w_j = (k_i - u_j)_+ \]
and
\[ A_i = \frac{|B^i \cap \{ u_j \leq k_i \}|}{|B^i|} = \frac{|B^i \cap \{ w_i \geq 0 \}|}{|B^i|}. \]
Notice that
\[ r_{j+1} < \rho_i < 2r_{j+1} \quad \text{and} \quad d_j < k_{i+1} \leq k_i < 2d_j \quad \text{and} \quad 0 \leq w_i \leq k_i \leq 2d_j. \tag{5.29} \]
We then denote
\[ a_1 := \inf_{B_{2r_{j+1}} \times B_{2r_{j+1}}} a(\cdot, \cdot), \quad a_2 := \sup_{B_{2r_{j+1}} \times B_{2r_{j+1}}} a(\cdot, \cdot) \]
and
\[ G(\tau) := \frac{\tau^p}{\rho_{j+1}^p} + a_2 \frac{\tau^q}{r_{j+1}^q}. \]
Using the first inequality in (5.29) and applying Lemma 2.5 with \( f \equiv w_i \), we obtain
\[ A_i^{1/2}G(k_i - k_{i+1}) = \left[ \frac{1}{|B^{i+1}|} \int_{B^{i+1} \cap \{ u_j \leq k_{i+1} \}} [G(k_i - k_{i+1})]^n dx \right]^{1/n} \]
\[ \leq \left[ \frac{1}{|B^{i+1}|} \int_{B^{i+1}} [G(w_i)]^n dx \right]^{1/n} \tag{5.30} \]
\[ \leq cM \int_{B^{i+1}} \int_{B^{i+1}} H(x, y, |w_i(x) - w_i(y)|) \frac{dx dy}{|x - y|^n} + cMG(d_j)A_i, \]
where for the last inequality we have also used the following estimate:
\[ \int_{B^{i+1}} \int_{B^{i+1}} H(x, y, |w_i(x) - w_i(y)|) \frac{dx dy}{|x - y|^n} \]
\[ \leq c \int_{B^{i+1}} \int_{B^{i+1}} H(x, y, (w_i(x) + w_i(y))|\phi_i(x) - \phi_i(y)|) \frac{dx dy}{|x - y|^n} \]
\[ + c \left( \sup_{y \in B'} \int_{R^n \setminus B'} \frac{w_i^{p-1}(x) + u_i^{q-1}(x)}{|x - y|^{n + sp}} + a_2 \frac{w_i^{q-1}(x)}{|x - y|^{n + tq}} \right) \int_{B'} w_i(x) \phi_i^n(x) dx. \tag{5.31} \]
We estimate the terms in the right-hand side of (5.31) separately. By the definitions of \( w_i \) and \( \phi_i \), we have
\[ \int_{B} \int_{B'} H(x, y, (w_i(x) + w_i(y))|\phi_i(x) - \phi_i(y)|) \frac{dx dy}{|x - y|^n} \]
\[ \leq c2^p \int_{r_{j+1}^{-1}} k_j^q \frac{1}{|B|} \int_{B \cap \{ u_j \leq k_i \}} \int_{B'} \frac{1}{|x - y|^{n + (s-1)p}} \frac{dx dy}{|x - y|^n} \]
\[ + c2^q a_2 r_{j+1}^{-q} k_j^q \frac{1}{|B|} \int_{B \cap \{ u_j \leq k_i \}} \int_{B'} \frac{1}{|x - y|^{n + (t-1)q}} \frac{dx dy}{|x - y|^n} \]
\[ \leq c2^{q} \int_{B \cap \{ u_j \leq k_i \}} \left( r_{j+1}{q, d_j^p} + a_2 r_{j+1}^{-q, d_j^q} \right) \]
\[ = c2^{q}G(d_j)A_i, \tag{5.32} \]
and
\[ \int_{B^c} w_i(x) \phi_i^1(x) \, dx \leq c d_j A_i, \quad \text{(5.33)} \]

As for the tail term, we first observe the following facts: \( \frac{|x-x_0|}{|x-y|} \leq 1 + \frac{|x-x_0|}{|x-y|} \leq 1 + \frac{2r_{i+1}}{2r_{i+1}} = 2^{i+3} \)
for \( x \in \mathbb{R}^n \setminus B^c \) and \( y \in B^c; \) \( w_i \leq k_i \leq 2d_i \) in \( B_{j+1}; \) and \( w_i \leq k_i + |u_j| \leq 2d_j + |u_j| \) in \( \mathbb{R}^n \setminus B_j. \) Using these facts, (5.22), (5.23), (5.25) and (5.26), we see that

\[
\begin{align*}
&\sup_{y \in B^c} \int_{\mathbb{R}^n \setminus B^c} \frac{w_i^{p-1}(x) + w_i^{q-1}(x)}{|x-y|^{n+sp}} + a_2 \frac{w_i^{q-1}(x)}{|x-y|^{n+q}} \, dx \\
&\leq c_2^{2(n+q)} \int_{\mathbb{R}^n \setminus B_{j+1}} \frac{w_i^{p-1}(x) + w_i^{q-1}(x)}{|x-x_0|^{n+sp}} + a_2 \frac{w_i^{q-1}(x)}{|x-x_0|^{n+q}} \, dx \\
&\leq c_2^{2(n+q)} \int_{\mathbb{R}^n \setminus B_{j+1}} \frac{d_j^{p-1} + d_j^{q-1}}{|x-x_0|^{n+sp}} + a_2 \frac{d_j^{q-1}}{|x-x_0|^{n+q}} \, dx \\
&\quad + c_2^{2(n+q)} \int_{\mathbb{R}^n \setminus B_{j+1}} \frac{|u_j(x)|^{p-1} + |u_j(x)|^{q-1}}{|x-x_0|^{n+sp}} + a_2 \frac{|u_j(x)|^{q-1}}{|x-x_0|^{n+q}} \, dx \\
&\leq c_2^{2(n+q)} M \left( \frac{d_j^{p-1}}{r_{j+1}} + a_2 \frac{d_j^{q-1}}{r_{j+1}^{1+q}} \right) + c_2^{2(n+q)} M \left( \frac{\sigma^{p-1} r_{j+1}^{p-1}}{\varepsilon^{p-1} r_{j+1}^{q-1}} + a_2 \frac{\sigma^{q-1} r_{j+1}^{q-1}}{\varepsilon^{q-1} r_{j+1}^{q-1}} \right) \\
&\leq c_2^{2(n+q)} M G(d_j) d_j.
\end{align*}
\]

Therefore, combining (5.30), (5.31), (5.32), (5.33) and (5.34), we arrive at
\[ A_{i+1} G(2^{-i-1} d_j) = A_{i+1}^{1/\kappa} G(k_i - k_{i+1}) \leq c_2^{2(n+q)} M^2 G(d_j) A_i, \]
which implies
\[ A_{i+1} \leq c_0 2^{i(n+q+2q)} M^{2\kappa} A_i \]
for a constant \( c_0 > 0 \) depending only on \( data. \) In order to apply Lemma 2.7, it should be assured that
\[ A_0 \leq (c_0 M^{2\kappa} - 1/(\kappa-1) 2^{-(n+q+2q)/\kappa(\kappa-1)^2}) =: \nu_s. \] (5.35)

This inequality holds by (5.28) and (5.12). More precisely, we have
\[ A_0 = \frac{2B_{j+1} \cap \{ u_j \leq 2d_j \}}{|2B_{j+1}|} \leq \frac{c_0 M^{3}}{\log(1/\sigma)} \leq \nu_s. \]

Hence it follows that \( A_i \to 0 \) as \( i \to \infty, \) which means that
\[ u_j \geq d_j = \varepsilon K_j \quad \text{a.e. in } B_{j+1}. \]

From this with (5.21) and (5.13), we finally obtain (5.16) as follows:
\[ \omega(r_{j+1}) = \sup_{B_{j+1}} u_j - \inf_{B_{j+1}} u_j \leq (1 - \varepsilon) K_j = \left( 1 - \sigma^{\frac{r_{j+1}}{r_j}} \right) \sigma^{-q} K_{j+1} \leq K_{j+1}. \]

\[ \square \]

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