Probability Theory/Partial Differential Equations

Random attractors for stochastic porous media equations perturbed by space–time linear multiplicative noise

Attracteurs aléatoires pour des équations aux milieux poreux stochastiques perturbés par un bruit linéaire multiplicatif, distribué dans l'espace et le temps

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1. Porous medium equation driven by rough signals

Let $\mathcal{O} \subseteq \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial \mathcal{O}$ in arbitrary dimension $d \in \mathbb{N}$, $T > 0$ and $\mathcal{O}_T := [0, T) \times \mathcal{O}$. We consider partial differential equation driven by rough signals of the type

\[ \partial_t u - \Delta u + f(u) = \xi \quad \text{in } \Omega \times (0, T), \]

\[ u = 0 \quad \text{on } \partial \Omega \times (0, T), \]

\[ u(0, x) = u_0(x) \quad \text{in } \Omega, \]

where $\xi$ is a rough path and $f$ is a sufficiently regular function.

A more detailed account of the results presented here can be found in (Gess, 2011 [4]).

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\[ \text{d}X_t = \Delta(|X_t|^m \text{sgn}(X_t)) \text{d}t + \sum_{k=1}^{N} f_k X_t \circ \text{d}z_t^{(k)}, \quad \text{on } \mathcal{O}_T \]

\[ X(0) = X_0, \quad \text{on } \mathcal{O} \]

with Dirichlet boundary conditions, \( m > 1 \), driven by signals \( z^{(k)} \in C([0, T]; \mathbb{R}) \) and with \( f_k \in C^\infty(\mathcal{O}) \) (we assume high regularity of \( f_k \) for simplicity only). Giving meaning to Eq. (1), in particular to the occurring stochastic integral, is part of the results. For detailed proofs of the results presented here we refer to [4].

We emphasize that by the spatial dependency of the functions \( f_k \) the noise acts in space as well as in time. For this type of noise even the generation of a continuous RDS by corresponding stochastic partial differential equations (SPDE) with quasilinear drift has been an open problem and is solved in this paper for the first time. In contrast to the case of additive or real (i.e. non-spatially distributed) multiplicative noise, the standard method of transforming the SPDE into a random PDE becomes highly non-trivial, because the space-dependency of the noise destroys the monotonicity structure of the transformed equation. A construction of stochastic flows and invariant manifolds for semilinear SPDE with linear multiplicative space–time noise can be found in [5].

The construction of solutions to (1) for signals of bounded variation proceeds by first transforming the equation into a PDE and then by construction of solutions to this transformed equation. More precisely, let \( \mu_t(\xi) := -\sum_{k=1}^{N} f_k(\xi)z_t^{(k)} \). Then \( Y := e^{\mu t}X \) satisfies the transformed equation

\[ \partial_t Y_t = e^{\mu t} \Delta \left( (e^{-\mu t}Y_t)^m \text{sgn}(e^{-\mu t}Y_t) \right), \quad \text{on } \mathcal{O}_T \]

with Dirichlet boundary conditions and initial condition \( Y_0 \). This transformation will be rigorously justified below. Our results extend [1] where under restrictions on the dimension \( d \) and the order \( m \) unique existence of solutions for (2) with essentially bounded initial conditions has been shown.

Let us define what we mean by a solution to (1) and (2). Defining \( B(x)(z) := \sum_{k=1}^{N} f_k x z^{(k)} \) for \( x \in L^1(\mathcal{O}) \) and \( z \in \mathbb{R}^N \) we can rewrite \( B(X_t) \text{d}z_t = \sum_{k=1}^{N} f_k X_t \text{d}z_t^{(k)} \). Let \( W^{n,p}(\mathcal{O}) \) be the Sobolev space of order \( n \) in \( L^p(\mathcal{O}) \), \( W^{m,n}(\mathcal{O}) \) the subspace of functions vanishing on \( \partial\mathcal{O} \), \( C^{m,n}(\mathcal{O}_T) \subseteq C(\mathcal{O}) \) be the set of all continuous functions on \( \mathcal{O}_T \) having \( m \) continuous derivatives in time and \( n \) continuous derivatives in space and let \( C^{1 \text{-} \text{var}}([0, T]; H) \) be the set of functions of bounded variation. Further, let \( \Phi(r) := |r|^m \text{sgn}(r) \).

**Definition 1.1.**

(i) Let \( Y_0 \in L^1(\mathcal{O}) \). We call \( Y \in L^1(\mathcal{O}_T) \) a (very) weak solution to (2) if \( \Phi(e^{-\mu t}Y) \in L^1([0, T]; W^{1,1}_0(\mathcal{O})) \) (\( \in L^1(\mathcal{O}_T) \) resp.) and

\[ -\int \mathcal{O}_T Y_t \partial_t \eta \text{d}x \text{d}t - \int \mathcal{O} Y_0 \eta_0 \text{d}x = \int \mathcal{O}_T \Phi(e^{-\mu t}Y_t) \Delta(e^{\mu t}\eta_t) \text{d}x \text{d}t, \]

for all \( \eta \in C^1(\mathcal{O}_T) \) (\( \in C^{1,2}(\mathcal{O}_T) \) resp.) with \( \eta = 0 \) on \([0, T] \times \partial\mathcal{O} \) and on \([T] \times \mathcal{O} \).

(ii) Let \( z \in C^{1 \text{-} \text{var}}([0, T]; \mathbb{R}^N) \) and \( X_0 \in L^1(\mathcal{O}) \). A function \( X \in L^1(\mathcal{O}_T) \) such that \( t \mapsto (\int_{\mathcal{O}} B(X_t) \eta_t \text{d}x) \) is continuous is said to be a (very) weak solution to (1) if \( \Phi(X) \in L^1([0, T]; W^{1,1}_0(\mathcal{O})) \) (\( \in L^1(\mathcal{O}_T) \) resp.) and

\[ -\int \mathcal{O}_T X_t \partial_t \eta \text{d}x \text{d}t - \int \mathcal{O} X_0 \eta_0 \text{d}x = \int \mathcal{O}_T \Phi(X_t) \Delta \eta_t \text{d}x \text{d}t + \int_0^T \left( \int_{\mathcal{O}} B(X_t) \eta_t \text{d}x \right) \text{d}z_t, \]

for all \( \eta \in C^1(\mathcal{O}_T) \) (\( \in C^{1,2}(\mathcal{O}_T) \) resp.) with \( \eta = 0 \) on \([0, T] \times \partial\mathcal{O} \) and on \([T] \times \mathcal{O} \).

A rigorous formulation for the transformation of (1) into (2) can be given as following: Let \( X_0 \in L^1(\mathcal{O}), \ z \in C^{1 \text{-} \text{var}}([0, T]; \mathbb{R}^N) \) and \( X \in L^1(\mathcal{O}_T) \) with \( X \in C([0, T]; H) \) or \( X \in C([0, T]; L^1(\mathcal{O})) \). Then \( X \) is a very weak solution to (1) iff \( Y := e^{\mu t}X \) is a very weak solution to (2). We prove the following:

**Theorem 1.2.** Essentially bounded very weak solutions to (2) are unique.

We define \( H^1_0(\mathcal{O}) := W^{1,2}_0(\mathcal{O}) \) and denote its dual by \( H \).

**Theorem 1.3.** Let \( Y_0 \in L^\infty(\mathcal{O}) \) and \( z \in C([0, T]; \mathbb{R}^N) \). There exists a unique weak solution \( Y \in C([0, T]; H) \cap L^\infty(\mathcal{O}_T) \to (2) \) satisfying \( \Phi(e^{-\mu t}Y) \in L^2([0, T]; H^1(\mathcal{O})) \). There is a function \( U : [0, T] \to \mathbb{R} \) (taking the value \( \infty \) at \( t = 0 \)) which is piecewise smooth on \([0, T]\) such that for all \( Y_0 \in L^\infty(\mathcal{O}) \)

\[ Y_t \leq U_t, \quad \text{a.e. in } \mathcal{O}, \ \forall t \in [0, T]. \]
If \( z \in C^{1,\text{var}}([0, T]; \mathbb{R}^N) \) then this yields the existence of a weak solution to (1) given by \( X = e^{-\mu t} Y \). A key point of Theorem 1.3 is that the upper bound \( U_t \) does not depend on the initial condition \( Y_0 \). Solutions to (1) for continuous signals are constructed by an approximation of the driving signal.

**Definition 1.4.** Let \( z \in C([0, T]; \mathbb{R}^N) \). We call \( X \in C([0, T]; H) \) a weak rough solution to (1) if \( X(0) = X_0 \) and for all approximations \( z^{(k)} \in C^{1,\text{var}}([0, T]; \mathbb{R}^N) \) of the driving signal \( z \) with \( z^{(k)} \to z \) in \( C([0, T]; \mathbb{R}^N) \) and corresponding weak solutions \( X^{(k)} \) to (1) driven by \( z^{(k)} \) we have \( X^{(k)} \to X \) in \( H \) for all \( t \in [0, T] \).

**Theorem 1.5.** Let \( X_0 \in L^\infty(O) \) and \( z \in C([0, T]; \mathbb{R}^N) \). Then there exists a unique rough weak solution \( X \) to (1) given by \( X = e^{-\mu t} Y \), where \( Y \) is the corresponding weak solution to (2). \( X \) satisfies \( X_t \leq U_t \) a.e. in \( O \) for all \( t \in [0, T] \), where \( U \) is as in Theorem 1.3.

Proving Lipschitz continuity in the initial condition with respect to the \( L^1(O) \) norm we obtain existence of solutions to (1) for initial conditions in \( L^1(O) \) in a limiting sense. Let \( (\cdot)^+ := \max(0, \cdot) \) and \( C^\alpha([0, T]; H) \) be the space of weakly continuous functions in \( H \).

**Definition 1.6.** Let \( X_0 \in L^1(O) \) and \( z \in C([0, T]; \mathbb{R}^N) \). A function \( X \in C^\alpha([0, T]; L^1(O)) \) is said to be a solution to (1) if \( X(0) = X_0 \) and for all approximations \( X_0^{(k)} \in L^\infty(O) \) with \( X_0^{(k)} \to X_0 \) in \( L^1(O) \) and corresponding weak rough solutions \( X^{(k)} \) to (1) we have \( X^{(k)} \to X \) in \( L^1(O) \) uniformly in time.

**Theorem 1.7.** Let \( z \in C([0, T]; \mathbb{R}^N) \). For each \( X_0 \in L^1(O) \) there is a unique limit solution \( X \) to (1) satisfying \( \Phi(X) \in L^1(O_T) \). For \( X_0^{(i)} \in L^1(O) \), \( i = 1, 2 \), the corresponding limit solutions satisfy
\[
\sup_{t \in [0, T]} \left\| (X_t^{(1)} - X_t^{(2)})^+ \right\|_{L^1(O)} + \left\| (\Phi(X^{(1)}) - \Phi(X^{(2)}))^+ \right\|_{L^1(O_T)} \leq C \left\| (X_0^{(1)} - X_0^{(2)})^+ \right\|_{L^1(O)}.
\]
In addition, \( X_t \leq U_t \) a.e. in \( O \) for all \( t \in [0, T] \), where \( U_t \) is as in Theorem 1.3.

As a special application we obtain a comparison principle: For \( X_0^{(1)}, X_0^{(2)} \in L^1(O) \) with \( X_0^{(1)} \leq X_0^{(2)} \) almost everywhere we have \( X_t^{(1)} \leq X_t^{(2)} \), for all \( t \in [0, T] \), a.e. in \( O \).

We say that a quantity depends only on the data if it is a function of \( d, m, T \). By proving that the regularity results given in [3] may be applied in our situation we obtain:

**Theorem 1.8.** Let \( z \in C([0, T]; \mathbb{R}^N) \), \( X_0 \in L^1(O) \) and \( X \) be the corresponding limit solution. Then

(i) \( X \) is uniformly continuous on every compact set \( K \subseteq [0, T] \times \bar{O} \), with modulus of continuity depending only on the data and \( \text{dist}(K, \partial \bar{O}) \).

(ii) If \( X_0 \in L^\infty(O) \) is continuous on a compact set \( K \subseteq \partial \bar{O} \), then \( X \) is uniformly continuous on \([0, T] \times K \) for every compact set \( K \subseteq \bar{K} \) with \( \text{modulus of continuity depending only on the data, } \text{dist}(K, \partial \bar{O}), \text{dist}(K', \partial \bar{O}), \|X_0\|_{L^\infty(O)} \) and the modulus of continuity of \( X_0 \) over \( K \).

(iii) Assume:

\( (O1) \) There exist \( \theta^* > 0, R_0 > 0 \) such that \( \forall X_0 \in \partial \bar{O} \) and \( \forall R \leq R_0: |O \cap B_R(x_0)| < (1 - \theta^*)|B_R(x_0)| \).

Then for every \( \tau > 0, X \) is uniformly continuous on \([\tau, T] \times \bar{O} \) with modulus of continuity depending only on the data, \( \theta^* \) and \( \tau \).

**Corollary 1.9.** Let \( z \in C([0, T]; \mathbb{R}^N) \), \( X_0 \in L^1(O) \). Then \( X \in C([0, T]; L^1(O)) \cap C([0, T]; L^p(O)) \) for every \( p \in [1, \infty[ \). If \( X_0 \in L^\infty(O) \) then \( X \in C([0, T]; L^p(O)) \) for every \( p \in [1, \infty[ \).

## 2. Stochastic porous medium equation and RDS

We now pass to the case of stochastically perturbed porous media equations. Let \( (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}) \) be a filtered probability space, \((\zeta_t)_{t \in \mathbb{R}}\) be an \( \mathbb{R}^N \)-valued adapted stochastic process and \((\Omega, \mathcal{F}, \mathbb{P}), (\theta_t)_{t \in \mathbb{R}}\) be a metric dynamical system. We assume

(51) (Strictly stationary increments) For all \( t, s \in \mathbb{R}, \omega \in \Omega: \zeta_t(\omega) - \zeta_s(\omega) = \zeta_{t-s}(\theta_t \omega). \)

(52) (Regularity) \( \zeta_t \) has continuous paths.

We have assumed \( z_0 = 0 \) for notational convenience only. In particular, applications include fractional Brownian Motion with arbitrary Hurst parameter. We then consider the SPDE
\begin{equation}
    dX_t = \Delta \Phi(X_t) dt + \sum_{k=1}^{N} f_k X_t \circ d\tilde{z}_t^k, \quad \text{on } \mathcal{O}_T
\end{equation}

\begin{equation}
    X(0) = X_0, \quad \text{on } \mathcal{O}.
\end{equation}

For $x \in L^1(\mathcal{O})$ and $\omega \in \Omega$, let $X(t,s; \omega)x$ denote the solution to (1) with initial value $x$ at time $s$ driven by the continuous signal $z(\omega)$. If the signal $z$ is given by a continuous semimartingale then (4) can be interpreted in the sense of stochastic Stratonovich integration. In this case we show that the limit solution $X$ is a probabilistic solution to (4). Together with the pathwise convergence of the approximants $X^{(\varepsilon)} \to X$ obtained in Theorem 1.5 via approximation by paths of bounded variation this yields a pathwise Wong–Zakai result (cf. e.g. [6]). For the notions of (order-preserving) RDS and random attractors we refer to [2] and references therein.

**Theorem 2.1.** The map $\varphi$ given by

\[ \varphi(t-s, \theta_s \omega)x := X(t,s; \omega)x \quad (t \geq s, \quad \omega \in \Omega, \quad x \in L^1(\mathcal{O})) \]

is a continuous RDS and $\varphi$ is order preserving, i.e. $\varphi(t, \omega)x_1 \leq \varphi(t, \omega)x_2$ a.e. in $\mathcal{O}$ if $x_1, x_2 \in L^1(\mathcal{O})$ with $x_1 \leq x_2$ a.e. in $\mathcal{O}$.

Let $\mathcal{D}$ be the system of all random closed sets. The RDS $\varphi$ satisfies the same regularity and regularizing properties as proved for the pathwise solutions in Theorem 1.8. Using this we prove

**Theorem 2.2.** The RDS $\varphi$ has a $\mathcal{D}$-random attractor $A$ (as an RDS on $L^1(\mathcal{O})$). $A$ is compact in each $L^p(\mathcal{O})$ and attracting in $L^p(\mathcal{O})$-norm, $p \in [1, \infty)$. Moreover, $A(\omega)$ is a bounded set in $L^\infty(\mathcal{O})$ and the functions in $A(\omega)$ restricted to any compact set $K \subseteq \mathcal{O}$ are equicontinuous on $K$. If (O1) is satisfied, then $A(\omega)$ is compact in $C(\mathcal{O})$ and attracting in $L^\infty(\mathcal{O})$-norm.

The random attractor $A$ is unique since it is an invariant, random closed set.

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