ON LOCALIZING TOPOLOGICAL ALGEBRAS

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Abstract. Through the subsequent discussion we consider a certain particular sort of (topological) algebras, which may substitute the “structure sheaf algebras” in many—in point of fact, in all—the situations of a geometrical character that occur, thus far, in several mathematical disciplines, as for instance, differential and/or algebraic geometry, complex (geometric) analysis etc. It is proved that at the basis of this type of algebras lies the sheaf-theoretic notion of (functional) localization, which, in the particular case of a given topological algebra, refers to the respective “Gel’fand transform algebra” over the spectrum of the initial algebra. As a result, one further considers the so-called “geometric topological algebras”, having special cohomological properties, in terms of their “Gel’fand sheaves”, being also of a particular significance for (abstract) differential-geometric applications; yet, the same class of algebras is still “closed”, with respect to appropriate inductive limits, a fact which thus considerably broadens the sort of the topological algebras involved, hence, as we shall see, their potential applications as well.

0. Introduction

Our aim by the present paper is to present a certain particular type of topological algebras, that seems to be at the basis of what we may call “geometric (topological) algebras”, in the sense that this sort of (topological) algebras are, indeed, fundamental and, in point of fact, determine the inner structure of several important mathematical disciplines of a geometrical character, as, for instance, differential geometry (of smooth manifolds), geometry of complex (analytic) manifolds, through (\(\mathbb{C}\)-, or even, vector-valued) holomorphic functions, algebraic geometry (commutative case), and the like. Our study is
essentially sheaf-theoretic, given that sheaf theory is, as we shall see, the appropriate set-up to formulate and treat structural properties of the topological algebras, under consideration. The present account may also be considered, as a natural continuation and even further improvement and/or extension of our previous study in [12], [13].

We start, by first giving, in the next Section, the necessary preliminary material, as well as, the relevant results, concerning the sheaf-theoretic rudiments for the study of the topological algebras in the title of this paper and their further (geometrical) applications.

1. Functional presheaves and their (functional) localizations

Suppose we are given a topological space $X$, along with a functional presheaf on $X$,

$$A \equiv \{ A(U) : U \subseteq X, \text{ open} \},$$

in the sense that one has (by definition of the concept at issue),

$$A(U) \subseteq \mathcal{F}(U, Y), \text{ for any open } U \subseteq X,$$

the second member of (1.2) denoting the set of maps (not a priori continuous) on $U \subseteq X$ in the set (space) $Y$, which in the sequel is identified, for convenience, with the complex numbers $\mathbb{C}$ (the general case being, however, also of a particular interest; see, for instance, Note 2.1 in the sequel). So $A(U)$ is just a given set of $\mathbb{C}$-valued maps on the open set $U \subseteq X$, hence, the previously applied terminology. In this connection, we first remark that:

$$\text{any given presheaf (of sets) on } X \text{ “is” converted, via its sheafification, into a functional presheaf on } X.$$ 

Concerning the terminology on sheaf theory, applied herewith, see e.g. A. Mallios [14: Vol. I; Chapt. I]. Yet, for the inverted commas in “is”, as above, see also (1.8), (1.11) in the sequel. Thus, expressing (1.3), otherwise, in the natural question, whether “functional presheaves” are so important, one can say that, indeed, they are, in point of fact, the only ones (!), since,
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(1.3) any given presheaf is converted, along the way of its “sheafification” (i.e., “completion”, à la Leray), into a functional one.

Now, according to the general theory (loc. cit.), for any given presheaf (of sets, not necessarily functional), one obtains a sheaf, its sheafification,

\[ S(A) \equiv \mathcal{A} \equiv (\mathcal{A}, \pi, X), \text{ viz. a map } \pi : \mathcal{A} \rightarrow X, \]

such that one has, by definition,

\[ \mathcal{A} := \sum_{x \in X} \mathcal{A}_x, \]

where we still set;

\[ \mathcal{A}_x \equiv \pi^{-1}(x) := A_x := \lim_{U \in \mathcal{B}(x)} A(U), \]

with $U$ in the last “limit” varying over a fundamental system $\mathcal{B}(x)$ of open neighborhoods of the point $x \in X$. Thus, one gets at a map (see (1.1))

\[ \rho \equiv (\rho_U) : \mathcal{A} \equiv \{ A(U) \} \rightarrow \Gamma(\mathcal{A}) \equiv \{ \Gamma(U, \mathcal{A}) \equiv \mathcal{A}(U) \}, \]

such that one has;

\[ \rho_U : A(U) \rightarrow \rho_U(A(U)) \equiv \rho(A(U)) \equiv \rho(A)(U) \equiv A^\sim(U) \subseteq \mathcal{A}(U) \equiv \Gamma(U, \mathcal{A}), \]

where, in particular, we set

\[ \rho_U(s) \equiv \tilde{s} : U \rightarrow \mathcal{A} : x \mapsto \tilde{s}(x) := \rho^x_U(s) \in A_x = \mathcal{A}_x \subseteq \mathcal{A}, \]

for any $s \in A(U)$, while

\[ \rho^x_U : A(U) \rightarrow A_x = A_x \]

is the corresponding to (1.6) canonical map, for any $U \in \mathcal{B}(x)$ (see also ibid., p. 30; (7.9)). Now, a given presheaf (of sets) $A$ on $X$, as above, is said to be a monopresheaf, if each one of the maps $\rho_U$, as in (1.8), is one-to-one, for any open $U \subseteq X$. It is easy to see, by the very definitions, that
the condition of being a given presheaf a monopresheaf is equivalent with the first one of the two classical conditions, à la Leray, of being the given presheaf “complete”, viz. actually a sheaf.

See loc. cit., Chapt. I; p. 46, Section 11, in particular, p. 51ff, proof of Proposition 11.1, yet, p. 54; (11.36’), along with p. 47, Remark 11.1, i). As an immediate consequence of the above, one obtains that:

any functional presheaf (cf. (1.1), (1.2)) is, in effect, a monopresheaf.

As a result of the preceding, we remark that:

the sheafification of a given (abstract) presheaf is just a manner of converting its elements (: elements of the sets \(A(U), U \subseteq X\), cf. (1.1)) into (generalized) functions, viz. sections of the corresponding sheaf (: complete presheaf).

In particular, by considering the so-called Gel’fand presheaf of a given topological algebra, thus, by its very definition, as we shall see right below, a functional presheaf over the spectrum (: Gel’fand space) of the given topological algebra, the obstruction to the above procedure, as in (1.13), that is, to converting the elements of the algebra at issue into sections (of an appropriate sheaf, viz. of the respective Gel’fand sheaf) is measured exactly by the non-localness of the topological algebra, under consideration, a notion of our main concern, which will be examined right away, by the next Section.

However, as a preamble to that study, we first turn our attention to the way one succeeds, in “completing” a given functional presheaf (cf. (1.2)) into a “complete presheaf”, viz. into a sheaf, thus, alias, we are looking, first, at the so-called “functional completion” of the initial (functional) presheaf. As we shall see, right in the sequel, this procedure consists actually in
adding to the given (functional) presheaf all locally defined functions, which further locally belong, as well, to the initial presheaf.

In that sense, we can already understand, in anticipation (see also (1.21) below), the reason that some important classical functional presheaves, as, for instance, those of continuous, differentiable, holomorphic functions, are, in effect, sheaves, that is, complete presheaves; thus, they cannot be further completed, their elements being by their very definition, functions, locally characterizable, hence, already elements of the given presheaves. In other words, we can further state, in anticipation (see thus Section 2 in the sequel), that:

a functional presheaf, consisting of elements, that can be locally characterized, is virtually a complete presheaf; hence, a sheaf (viz., in effect, isomorphic to its sheafification, that is, to the sheaf, it generates, cf. Section 2 below). That is, a functional presheaf, as before, coincides, in effect, with its "localization" being thus a complete presheaf, hence (Leray), a sheaf (see also (1.21) in the sequel, along with subsequent comments therein).

So we first explain, more precisely, the terminology we employed in (1.14): That is, by considering a functional presheaf, as in (1.1), (1.2), "its elements", viz. elements of the particular sets \( A(U) \), with \( U \) open in \( X \), are, by definition, numerical-valued (for convenience, \( \mathbb{C} \)-valued) functions, "locally defined" on the various open sets \( U \subseteq X \). Yet, we also set the following crucial definition. That is, one has.

**Definition 1.1.** Suppose we have a functional presheaf

\[
(1.16) \quad A \equiv \{ A(U) : U \text{ open in } X \}
\]

(see (1.1), (1.2)). Then, a given \( \mathbb{C} \)-valued map \( \alpha : U \to \mathbb{C} \), with \( U \) open in \( X \), locally belongs to \( A \), if, for every point \( x \in U \), there is an open set \( V \subseteq X \),
with $x \in V \subseteq U$, and an element ("local function") $h \in A(V)$, such that;

(1.17) $\alpha = h|_V$.

(: precisely, $\alpha|_V = h$).

Now, the previous definition leads naturally to the important notion for our purpose of the localization of a given (functional) presheaf: That is, suppose we are given a functional presheaf $A$, as in (1.16) (see also (1.2)). Then, by definition,

the localization of $A$, say $\tilde{A}$, consists of those, locally defined $\mathbb{C}$-valued functions on $X$, which locally belongs to $A$ (cf. Definition 1.1).

Thus, technically speaking, given $A$, as above, one considers its localization $\tilde{A}$, viz. the presheaf on $X$, given by

(1.19) $\tilde{A} \equiv \{(\tilde{A}(U) : U \text{ open in } X); \tilde{\rho}_V^U\}$,

where one sets (Definition 1.1);

(1.20) $\tilde{A}(U) := \{\alpha : U \to \mathbb{C} | \forall x \in U \exists V \text{ open in } X, \text{ with } x \in V \subseteq U, \text{ and } h \in A(V), \text{ such that } \alpha|_V = h\}$.

Of course, $\tilde{A}$ is a presheaf on $X$; moreover, by its very definition, it is a functional presheaf (cf. (1.20)), hence, a monopresheaf (see (1.12)). Indeed, our claim is that, in effect, $\tilde{A}$ is a complete presheaf on $X$ (Leray’s definition): That is, given an open covering $(U_i)_{i \in I}$ of an open $U \subseteq X$, viz. $U = \bigcup_i U_i$,

and an element $(\alpha_i) \in \prod_i \tilde{A}(U_i)$, with $\alpha_i = \alpha_j|_{U_i \cap U_j \neq \emptyset}$, one has to prove that these exists $\alpha \in \tilde{A}(U)$, with $\alpha|_{U_i} = \alpha_i$, $i \in I$; indeed, the proof follows straightforwardly from Definition 1.1 and (1.20), along with our hypothesis for the family $(\alpha_i)$. Thus, we have proved, so far, the basic result that:
the localization of a given functional presheaf (as defined by (1.20)) is a complete presheaf; hence (Leray’s definition), a sheaf.

See also, for instance, A. Mallios [14: Chapt. I; Section 11], concerning the terminology applied in (1.21). Thus, in view of the preceding, we fully understand now our previous comments in (1.15), since a functional presheaf, whose elements (: local functions) are locally characterized, coincides, in point of fact, with its localization, according to the very definition of the sets in (1.20) and the term “locally characterized”, (1.20) being, in effect, the precise definition of the latter term (!). In other words, in view of (1.21), the given presheaf is complete.

We turn now, by the next Section 2, to describe (1.19), in terms of “germs” of functions, equivalently, to look at the notion of a sheaf, as a “local homeomorphism” (concerning the map \(\pi\), as in (1.4); Lazard’s definition), and further connecting it with our previous discussion. For technical details, we still refer to A. Mallios [14: Chapt. I; Sections 7, 8].

2. SHEAFIFICATION OF A FUNCTIONAL PRESHEAF

“... to understand what’s what ... is a vital aspect of Mathematics”.
S. Mac Lane, in “Mathematics Forms and Function” (Springer-Verlag (1986), p. 288).

By considering a functional presheaf \(A\) on a topological space \(X\), as above (see (1.1), (1.2)), and then its sheafification (cf. (1.5), (1.6)),

\[
S(A) \equiv A, \tag{2.1}
\]

our main objective in this Section is to prove the relation (: isomorphism of sheaves)

\[
S(A) \equiv A \cong \tilde{A}; \tag{2.2}
\]

in other words, we claim that:
if we fill up a given functional presheaf \( A \) on a topological space \( X \), by all those (local) functions, that locally belong to it (Definition 1.1) (viz., by looking at the localization \( \tilde{A} \) of \( A \), cf. (1.19), (1.20)), then the resulting (functional) presheaf \( \tilde{A} \) is (complete, cf. (1.21), and, in point of fact, isomorphic to) the sheafification of the given presheaf \( A \).

Consequently,

the “completion” of a given functional presheaf \( A \), to become a complete presheaf, hence, to its sheafification, “is” virtually reduced to its localization \( \tilde{A} \). We also speak of \( \tilde{A} \), by extending herewith a relevant terminology of R. Arens [1], as the hull of \( A \), with respect to \( X \).

Thus, in other words,

a given functional presheaf on a topological space \( X \) is complete, viz. it coincides with the (complete presheaf of sections of the) sheaf it generates, if, and only if, it is also “functionally complete”; that is, equivalently, if, and only if, it contains any (local) function that locally belongs to it.

Indeed, the claimed isomorphism (2.2) is actually referred to an isomorphism of the complete presheaves concerned; that is, the complete presheaf of local sections of \( \mathcal{A} \equiv \mathcal{S}(A) \) is isomorphic to \( \tilde{A} \), the latter being also a complete presheaf on \( X \), cf. (1.21)). Namely, one has to prove the following (set-theoretic) bijection:

\[
\tilde{A}(U) = \Gamma(U, \mathcal{A}) \equiv \mathcal{A}(U),
\]

for any open \( U \subseteq X \): Thus, for any \( \alpha \in \tilde{A}(U) \), one defines a map

\[
\mathcal{A}(U) \ni \tilde{\alpha} : U \longrightarrow \mathcal{A} : x \longmapsto \tilde{\alpha}(x) := [s]_x = \tilde{s}(x) \in \mathcal{A}_x \subseteq \mathcal{A}
\]

(see also (1.5)), where, by definition of \( \tilde{\alpha} \) (cf. (1.20)), one has

\[
\alpha|_V = s \in A(V) \subseteq A(V), \quad \text{with } x \in V \subseteq U,
\]
given $x$ in $U$ (see also (1.12)); so (2.7) is well-defined, hence, also the correspondence

$$(2.9) \quad \tilde{A}(U) \xrightarrow{i_U} A(U) : \alpha \mapsto \tilde{\alpha},$$

as given by (2.7). Now, the same map $i_U$ is one-to-one; this follows straightforwardly, in view of (2.7) and (2.8). Finally, $i_U$ is onto. That is, given $s \in A(U)$, one defines a map $t \equiv (t_x) \in \tilde{A}(U)$, according to the relation;

$$(2.10) \quad t(x) := t_x(x), \quad x \in V_x \subseteq U = \bigcup_{x \in U} V_x,$$

such that,

$$(2.11) \quad s(x) = \rho^x_{V_x} (t_x) \in A_x = \lim_{V \in \mathcal{V}(x)} A(V),$$

with $t_x \in A(V_x) \subseteq A(V_x)$, while, by virtue of (2.11), one still obtains;

$$(2.12) \quad t_x = t_y |_{V_x = V_y \cap V_y \neq \emptyset},$$

which also yields, in view of (2.10) and (1.21), that $t \in \tilde{A}(U)$. Furthermore, one also gets at the relation,

$$(2.13) \quad \tilde{t} = s \in A(U),$$

based on (2.7), (2.8), and the preceding last three relations, which proves the assertion, hence, finally (2.6), as well. In sum, one thus obtains the following isomorphism of complete presheaves (see also (2.1)),

$$(2.14) \quad \tilde{A} \equiv (\tilde{A}(U), \rho^U_V) \cong \Gamma(\mathcal{S}(A)) \equiv \Gamma(A) \equiv (A(U), \sigma^U_V),$$

that also explains the slight abuse of notation employed, for convenience, in (2.2).
Note 2.1. – The previous procedure of constructing the sheafification of a given functional presheaf, by just considering its localization, viz. the complete presheaf, containing the initial one, as explained in (2.3), represents, in point of fact, the general case, as well; viz. the sheafification of an arbitrary (not necessarily functional) presheaf: Indeed, this is explained by considering the map (1.7), converting a given presheaf $A$ into a functional one $\rho(A)$, whose elements are \((local)\) sections of $\mathcal{A}$, thus local $A$-valued functions on $X$ (cf. (1.9)), $\mathcal{A}$ being here the final sheaf generated by $A$, and, in effect, by $\rho(A)$ (see A. Mallios [14: Chapt. I, p. 31; (7.12)]); therefore, $\mathcal{A}$ (when also identified with the complete presheaf of its sections, $\Gamma(\mathcal{A})$, cf. (1.7)) is virtually isomorphic, according to the preceding, with the localization of $\rho(A)$, $\tilde{\rho}(A)$ (: localization of the “functionalization”, $\rho(A)$, of $A$). That is, one actually obtains;

\begin{equation}
\tilde{\rho}(A) = \Gamma(\mathcal{A}) \equiv \Gamma(S(A)) = \Gamma(S(\rho(A)));
\end{equation}

within an isomorphism of complete presheaves (see (1.20), (1.21)). In this connection, cf. also loc. cit., Chapt. I; Sections 3, 7, in particular, p. 31, Scholium 7.1.

So the moral here is that;

sheafifying a given presheaf (à la Leray), means, in effect, localize its “functionalization” (: the presheaf obtained, by converting the elements of the initial presheaf into functions (indeed, sections), cf. (1.9)).

Yet, roughly speaking, we may still say that;

\begin{equation}
\text{“sheafifying” means, in point of fact, “localize”}.
\end{equation}
By closing the preceding discussion, we still remark, in passing, that our previous considerations, pertaining to the sheafification of a given functional presheaf, through its localization, apart from its usual application in important classical examples, as already mentioned in the foregoing (see also loc. cit., Chapt. I; p. 17ff, Section 4), one also encounters another interesting justification of the same point of view in some recent applications of the notion of sheaf in Quantum Field Theory, or even in Quantum Relativity; see thus, for instance, R. Haag [7: p. 326], I. Raptis [26], A. Mallios-I. Raptis [19]. To quote e.g. R. Haag (ibid.), one realizes that (italicization below is ours);

\[(2.18)\] \[... \text{the central message of Quantum Field Theory} \text{[is]} \text{that all information characterizing the theory is strictly local} \ldots\]

Now, based on the preceding, we also note that the above “local information” is further improved, through the “functional completion”, as before, that is, by adding thus to any given initial (local) information all the possible “equivalent” ones, a function (: procedure), which, in point of fact, lies at the same basis of implementing (cf. (1.20)) the very notion of sheaf in our arguments (see also (2.3) and/or (2.15)). Accordingly, the sort of application of the foregoing in nowadays quantum field theory, as alluded to above. Yet, we can still conclude, in view of the foregoing (cf., in particular, (2.15), as above), that, in other words,

by considering the sheaf, generated by a given presheaf, means, in point of fact, to

\[(2.19)\]

\[(2.19.1)\]

\begin{equation}
\text{collect together all the local information, we can get, on the basis of that one, which is, already, locally afforded, by the elements of the given presheaf.}
\end{equation}

The above still supports the aforementioned applicability of the notion of sheaf in quantum theory and/or quantum relativity; see also, for instance, H.F. de Groote [6].
3. Localization of topological algebras

“There are many advantages in developing a theory in the most general context possible.”

R. Hartshorne, in “Algebraic Geometry” (Springer-Verlag, New York, 1977). p. 59.

As the title of this Section indicates, our main objective by the ensuing discussion is to apply our previous considerations in the particular case, when one has to deal with (pre)sheaves of topological algebras, in the general framework of which, one can still examine, as already hinted at in the preceding, the sort of topological algebras, we are looking at, by the present study.

We start, by presenting the relevant general set-up: Thus, first, by a topological algebra space, we mean a pair,

\[(A, \mathcal{M}(A)),\]

(3.1)

consisting of a given topological algebra \(A\), together with its (global) spectrum (alias, Gel’fand space), \(\mathcal{M}(A)\); of course, we still posit here, by definition, that the latter (a Hausdorff completely regular) space is not trivial (: empty). Yet, we refer to A. Mallios [11] for the relevant terminology employed herewith.

Now, a topological function algebra, which is naturally associated with a given topological algebra space, as in (3.1), is the algebra

\[\mathcal{C}_c(\mathcal{M}(A)),\]

(3.2)

that is, the set of \(\mathbb{C}\)-valued continuous functions on \(\mathcal{M}(A)\), being a \(\mathbb{C}\)-algebra, with point-wise defined operations, further endowed with the compact-open topology, becoming thus, in particular, a locally m-convex (topological) algebra (ibid.). Yet, an important representation of \(A\) into the latter algebra is achieved, via the so-called Gel’fand representation, and the associated with it Gel’fand map,

\[G_A \equiv \mathcal{G} : A \rightarrow \mathcal{C}(\mathcal{M}(A)),\]

(3.3)
such that,
\[(3.4) \quad G_A(x) \equiv \hat{x} : \mathcal{M}(A) \to \mathbb{C} : f \mapsto \hat{x}(f) := f(x),\]
for every \(f \in \mathcal{M}(A)\) (loc. cit.; we call \(\hat{x}\), the Gelfand transform of \(x \in A\)).

Thus, we are now in the position to turn ourselves to our main objective, viz. to look at the possibility of being a given topological algebra localizable, in the sense in which the latter term will become clear by the subsequent discussion:

So given a topological algebra space, as in (3.1), one can naturally associate with it, by means of the Gelfand representation, as above, a functional (\(\mathbb{C}\)-)algebra presheaf on \(\mathcal{M}(A)\) (: base space of (3.1)), which we call the Gelfand presheaf of the given topological algebra space, or, for short, just, of \(A\), defined by,
\[(3.5) \quad G^\text{presh}(A) \equiv \hat{A} \equiv \{A(U)^\wedge : U \subseteq \mathcal{M}(A), \text{ open}\},\]
where we set (cf. also (3.4), while we still put \(A^\wedge \equiv \text{im} G\), called the Gelfand transform algebra of \(A\)),
\[(3.6) \quad A(U)^\wedge := A^\wedge |_U = \{\hat{x}|_U : x \in A\},\]
for every open \(U \subseteq \mathcal{M}(A)\), along with the corresponding obvious restriction maps,
\[(3.7) \quad \rho^U_V : A^\wedge|_U \to A^\wedge|_V,\]
for any open \(U, V\) in \(\mathcal{M}(A)\), with \(V \subseteq U\). Therefore, by its very definition,
\[(3.8) \quad \text{the Gelfand presheaf of } A \text{ (cf. (3.5)) is a functional presheaf on } \mathcal{M}(A), \text{ hence (see (1.12)), a monopresheaf.}\]

On the other hand, we are actually interested in finding, what we may call the Gelfand sheaf of \(A\), denoted in the sequel by
\[(3.9) \quad A,\]
thus, by its very definition, as we shall presently see, a sheaf of (\(\mathbb{C}\)-)algebras over \(\mathcal{M}(A)\); yet, we still refer to \(A\), as the sheafification of the given topological algebra \(A\).
Thus, by definition, $\mathcal{A}$, viz. the Gel'fand sheaf of $\mathcal{A}$, is the sheaf on $\mathfrak{M}(\mathcal{A})$, generated by the Gel'fand presheaf $\hat{\mathcal{A}}$ of $\mathcal{A}$ (cf. (3.5)). That is, we set;

$$(3.9)' \quad \mathcal{A} := S(\hat{\mathcal{A}}).$$

Therefore, in view of (3.8), and of what has been said in Section 2 (see e.g. (2.3), (2.4) or even (2.17)), one actually obtains the following.

**Theorem 3.1.** Given a topological algebra space $(\mathcal{A}, \mathfrak{M}(\mathcal{A}))$ (see (3.1)), one has;

$$(3.10) \quad \mathcal{A} = \tilde{\mathcal{A}}(\equiv \hat{\mathcal{A}}),$$

within an isomorphism of (complete pre)sheaves, where the second member of (3.10) stands for the localization of the Gel'fand presheaf of $\mathcal{A}$. That is, in other words, the sheaf of germs of the Gel'fand transforms of the elements of $\mathcal{A}$ (or else, the Gel'fand sheaf $\mathcal{A}$ of $\mathcal{A}$, is (isomorphic to) the sheaf (: complete presheaf) $\tilde{\mathcal{A}}$ of germs of $\mathbb{C}$-valued functions, locally belonging to $\mathcal{A}^\wedge$.

For convenience, we only recall the definition of the (complete) presheaf (see (1.21) $\tilde{\mathcal{A}}$, based on (1.20); viz. one has, for any open $U \subseteq \mathfrak{M}(\mathcal{A})$;

$$\tilde{\mathcal{A}}(U) = \{ \alpha : U \to \mathbb{C} | \forall f \in U \exists V \text{ open in } \mathfrak{M}(\mathcal{A}), \text{ with } f \in V \subseteq U, \text{ and } \hat{x}|_V \in \mathcal{A}(V)^\wedge \text{ (hence, actually an element } x \in \mathcal{A}), \text{ such that; } \alpha|_V = \hat{x}|_V \text{ (for short, } \alpha = \hat{x}|_V)\}. \quad (3.12)$$

As a result, one thus infers, according to the very definitions, that;

$$\text{the localization of the Gel'fand presheaf of a given topological algebra } \mathcal{A} \text{ consists of those } \mathbb{C}\text{-valued local functions on } \mathfrak{M}(\mathcal{A}), \text{ which are locally Gel'fand transforms of elements of } \mathcal{A}. \quad (3.13)$$

Of course, in view of (3.10), the same characterization, as above, is still valid for the Gel'fand sheaf itself of $\mathcal{A}$ (when applying Leray’s terminology).
On the other hand, in view of (3.8), one has;

\[ \hat{A} \subseteq \tilde{A} \]  

(3.14)

(within an isomorphism into of the presheaves concerned; see also A. Mallios [13: Chapt. I; Section 6]). Indeed, the previous relation leads us now to the following basic.

**Definition 3.1.** Given a topological algebra space, as in (3.1), we say that the topological algebra \( A \) is *localizable*, whenever one has

\[ \hat{A} = \tilde{A}, \]

viz. whenever the Gel’fand presheaf of \( A \) (cf. (3.5)) is already localized, hence (cf. (1.21)), complete.

Therefore, based on (3.10), in the case of a localizable topological algebra, one gets at the following relation;

\[ \hat{A} = \tilde{A} = A \]

(3.16)

(in the sense of (3.10), concerning the last equality). Thus, as a straightforward application of our terminology in the preceding, one still obtains, in the case of a localizable topological algebra \( A \) (cf. (3.6)),

\[ G(A) \equiv A^\wedge = A^\wedge \big|_{\mathfrak{M}(A)} = \hat{A}(\mathfrak{M}(A))^\wedge = \hat{A}(\mathfrak{M}(A)) \]

\[ = \hat{A}(\mathfrak{M}(A)) = \Gamma(\mathfrak{M}(A), A) = A(\mathfrak{M}(A)), \]

that is, in short, one has, the following isomorphism of \( \mathbb{C} \)-algebras;

\[ A^\wedge = \Gamma(\mathfrak{M}(A), A) \equiv A(\mathfrak{M}(A)), \]

(3.18)

viz., in the case of a localizable (topological) algebra \( A \), the Gel’fand transform algebra of \( A \) may be construed, as the global section algebra of its Gel’fand sheaf over \( \mathfrak{M}(A) \), so that one also speaks then of a sectional representation of (the localizable algebra) \( A \). However, one actually infers, in full generality that;
given a topological algebra space \((A, \mathcal{M}(A))\), then, \(A\) always admits, through its Gel'fand transform algebra, a sectional representation, via the corresponding Gel'fand sheaf over \(\mathcal{M}(A)\).

Indeed, one has, in view of (3.10), (3.14) and (3.17), the relations;

\[
(3.20) \quad A^\wedge = A(\mathcal{M}(A))^\wedge \equiv \hat{A}(\mathcal{M}(A)) \subset \hat{A}(\mathcal{M}(A)) \cong \Gamma(\mathcal{M}(A), A) \equiv \mathcal{A}(\mathcal{M}(A)).
\]

That is, the Gel'fand transforms of the elements of \(A\), can always be viewed as global (continuous) sections over \(\mathcal{M}(A)\) of the Gel'fand sheaf of \(A\). (Thus, one gets at a more intrinsic (direct) sectional representation of \(A\), than that one obtained, anyway, through the Gel'fand transform algebra \(A^\wedge\), as continuous \(\mathbb{C}\)-valued functions on \(\mathcal{M}(A)\), cf. (3.3)).

Now, it may happen that the Gel'fand map of \(A\), as given by (3.4), is one-to-one; equivalently, the points of \(\mathcal{M}(A)\) (: continuous characters of \(A\)) separate the elements of \(A\). We call then \(A\), a (functionally) semisimple topological algebra.

Thus, a topological algebra \(A\), as in (3.1), which is localizable and also semisimple, is just called a local topological algebra. Hence, in that case, one obtains, in view of (3.18), the relation;

\[
(3.21) \quad \mathcal{A} \cong \mathcal{A}^\wedge = \mathcal{A}(\mathcal{M}(A)) \equiv \Gamma(\mathcal{M}(A), A),
\]

within isomorphism of \(\mathbb{C}\)-algebras. [Caution! The term “local”, applied here-with, is to be distinguished from a similar one used in Algebra, pertaining to algebras with just one maximal (2-sided) ideal].

The previous notion lies, indeed, at the basis of what we consider further, as a geometric topological algebra, a notion that mostly occurs in several applications of a geometrical character; see e.g. our previous study in A. Mallios [11; 13]. However, before we proceed to that aspect, in the next Section 4, we first comment a bit more on some particular instances of local topological algebras:
Thus, it is well-known that \textit{not every topological algebra is localizable}; this is simply also the case, even for a unital commutative Banach algebra. Indeed, one has here the well-known \textit{Eva Kalin's counter example} [10]. On the other hand, about the same time R. Blumenthal [2; 3], based on earlier work of S.J. Sidney [27; 28], deciphered Kalin’s example, by supplying an \textit{abstract method of constructing non-local function (Banach) algebras}. Quite recently, R.I. Hadjigeorgiou [8; 9] was able to extend the relevant part of the aforementioned work of Sidney and Blumenthal to the general context of \textit{topological algebra theory}, providing thus, in turn, a corresponding \textit{machinery of constructing}, à la Blumenthal, \textit{non-local topological (non-normed) algebras} [9]. Yet, lately, A. Oukhouya proved in his Thesis [24; 25] that \textit{every regular locally m-convex uniform algebra is localizable}, supplying thus another aspect of a \textit{“local theorem”} for topological algebras, yet, with \textit{non compact spectra}, that was already the case in [12: p. 307, Lemma 2.1]. In this connection, see also previous relevant work of R.M. Brooks [5], where he employs \textit{“partitions of unity”}.

4. \textsc{Geometric topological algebras}

The topological algebras, referred to in the title of this Section, appear, as, of course, their very name indicates, in particular geometrical contexts, that we have already hinted at in the preceding, including, among others, the \textit{“structure algebras”} (another synonym of \textit{“geometric”}) of (geometric) complex (analytic) function theory, and, in particular, in differential geometry, this latter case being also our main motivation for the ensuing discussion.

Thus, for any given \textit{topological algebra space}

\begin{equation}
(A, \mathcal{M}(A)),
\end{equation}

as in (3.1), and under suitable conditions for $A$, one is able to use $A$, or rather its corresponding \textit{Gel’fand sheaf} $\mathcal{A}$ on $\mathcal{M}(A)$ (cf. Theorem 3.1), as the \textit{“sheaf of coefficients”}, for an extended \textit{“differential-geometric”} context on $\mathcal{M}(A)$. This type of applications has already considered in A. Mallios [14: Chapt. XI], along with further potential applications. Now, the aforementioned type of
applications, led us here to associate with the sort of topological algebras, alluded to by the title of this Section, a more restricted version of the same notion, than that one, previously applied in A. Mallios [12]. Namely, we set the following.

**Definition 4.1.** Given a topological algebra space (see (4.1)), we say that \( \mathcal{A} \) is a **geometric topological algebra**, whenever one has;

\[
\mathcal{A} = \Gamma(X, \mathcal{A}) = H^0(X, \mathcal{A}),
\]

within a \( \mathbb{C} \)-algebra isomorphism of the algebras concerned, while we also assume that,

\[
H^p(X, \mathcal{A}) = 0, \quad p \geq 1,
\]

where \( \mathcal{A} \) is a \( \mathbb{C} \)-algebra sheaf on \( X \), the latter being a topological space of the same homotopy type, as \( \mathcal{M}(\mathcal{A}) \).

The “cohomology groups”, in point of fact, \( \mathcal{A}(X) \)-modules appeared in the last relations above, are meant in the sense of sheaf cohomology theory; see, for instance, A. Mallios [14: Chapt. III; yet, in particular, p. 234, Lemma 8.1]. Indeed, the “geometric topological algebras” in the sense of our previous work in [12] (see also [14: Chapt. XI]), are still such, in the point of view of the above Definition 4.1; thus, cf. [14: Chapt. XI; p. 320, (4.19), as well as, Chapt. III; p. 238, (8.24)]. On the other hand, we further remark that the sort of topological algebras that might be geometric, in the sense of the above Definition 4.1, can be associated, in effect, with a larger spectrum of topological algebras, than that one, we might suspect, at first sight, just, by virtue of the following, indeed, quite general result. That is, one gets at the next.

**Theorem 4.1.** Let

\[
(E_\alpha, f_{\beta\alpha})
\]
be an inductive system of unital geometric topological algebras (Definition 4.1), having compact spectra, and let

\[(4.5) \quad E = \lim_{\alpha} E_{\alpha}\]

be the corresponding inductive limit topological algebra (see A. Mallios [11: p. 115, Lemma 2.2]. Then, the topological algebra space

\[(4.6) \quad (E, \mathcal{M}(E) = \lim_{\leftarrow} \mathcal{M}(E_{\alpha}))\]

(see also loc. cit., p. 156, Theorem 3.1), defines \(E\), as a (unital) geometric topological algebras, as well.

**Proof.** As a result of topological algebra theory (loc. cit.), one obtains, based on our hypothesis, the topological algebra space (4.6), having \(\mathcal{M}(E)\), a compact Hausdorff space, such that one has;

\[(4.7) \quad \mathcal{M}(E) = \lim_{\leftarrow} \mathcal{M}(E_{\alpha}),\]

within a homeomorphism of the topological spaces concerned. Thus, by considering the canonical maps,

\[(4.8) \quad t_{\alpha} \equiv \phi_{\alpha} : \mathcal{M}(E) \rightarrow \mathcal{M}(E_{\alpha}), \quad \alpha \in I,\]

(ibid., pp. 152, 156; (3.2), (3.25)), associated, by our hypothesis, with the corresponding herewith, topological algebra spaces,

\[(4.9) \quad (E_{\alpha}, \mathcal{M}(E_{\alpha})), \quad \alpha \in I,\]

one further looks at the resulting \(\mathbb{C}\)-algebra sheaf on \(\mathcal{M}(E)\),

\[(4.10) \quad \mathcal{E} := \lim_{\rightarrow} \phi_{\alpha}^{*}(\mathcal{E}_{\alpha}),\]

inductive limit of the pull-backs on \(\mathcal{M}(E)\) of the corresponding to (4.9) \(\mathbb{C}\)-algebra sheaves \(\mathcal{E}_{\alpha}\) on \(\mathcal{M}(E_{\alpha})\), \(\alpha \in I\). (For the terminology applied herewith, see also, for instance, A. Mallios [14: Chapt. I; p. 79ff, Subsection 14.(b)]). Now, our claim is that;
the \( \mathbb{C} \)-algebra sheaf \( \mathcal{E} \) on \( \mathcal{M}(E) \), as given by (4.10), defines

\[
E = \lim_\rightarrow E_\alpha, \quad \mathcal{M}(E) = \varprojlim \mathcal{M}(E_\alpha)
\]

(see also (4.5), (4.7)), as a geometric topological algebra.

Indeed, the assertion is a straightforward application of the standard “continuity theorem” for (sheaf) cohomology on compact (Hausdorff) spaces: see, for instance, G.E. Bredon [4: p. 102, Theorem 14.4], along with A. Mallios [14: Chapt. III; p. 173ff, Section 4]. That is, one has:

\[
H^p(\mathcal{M}(E), \mathcal{E}) = H^p\left(\varprojlim \mathcal{M}(E_\alpha), \varinjlim \phi_\alpha^*(\mathcal{E}_\alpha)\right)
\]

\[
= \varinjlim H^p(\mathcal{M}(E_\alpha), \mathcal{E}_\alpha), \quad p \geq 0,
\]

that yields the assertion, according to our hypothesis; cf. also (4.1) and (4.2).

**Note 4.1.**— The type of topological algebras, singled out by the previous Theorem 4.1, are thus, in principle, unital ones, with a compact spectrum. Now, the first issue (“unital”) is actually referred to a technical item, pertaining to the non-triviality of the spectrum of \( E \), the inductive limit topological algebra considered, by the aforesaid theorem (cf. (4.5)), as well as, to the validity of (4.7), both conditions being already pointed out in A. Mallios [11: pp. 155, 156; (3.24), (3.29), along with Theorem 3.1 therein]. On the other hand, the hypothesis of compactness of the spectra involved can be relaxed to local compactness, however, by still assuming proper “transition maps” in (4.7), while considering (sheaf) cohomology with compact supports (see G.E. Bredon, loc. cit.).

As already mentioned in the preceding, the previous type of topological algebras is of a particular importance for a vast extension of a differential-geometric character to spaces which are not (smooth, viz. \( C^\infty \)-) manifolds, in the classical sense of the term. Notice that \( C^\infty(X) \), with \( X \) a paracompact (Hausdorff) \( C^\infty \)-manifold, is a topological algebra of the previous type, whose spectrum is (homeomorphic to) \( X \), the same algebra being thus a geometric one,
with respect to $C^\infty_X$, the sheaf of germs of $\mathbb{C}$-valued smooth ($\mathbb{C}^\infty$-) functions on $X$, where the latter is a fine sheaf on $X$; see A. Mallios [14: Chapts. III, XI]. Potential applications of the aforesaid generalized perspective of the classical differential geometry to problems, pertaining, for instance, to quantum gravity, have been discussed, recently, in [19], [20], [21]; in this connection, see also A. Mallios [15], [17], [18], as well as, [22], [23].

**Scholium 4.1.**—Motivated by the situation, that is relevant to the above Theorem 4.1, one is led to single out the following general point of view: Thus, by an *algebra space*, one means a pair,

\[(X, \mathcal{A}),\]

consisting of a topological space $X$ and a ($\mathbb{C}$-)algebra sheaf $\mathcal{A}$ on $X$. In particular, we speak of a *geometric algebra space*, whenever (4.13), as before, satisfies, in addition, the corresponding herewith two conditions (4.2) and (4.3), as above. Yet, we say that (4.13) is a *compact algebra space*, if the topological space $X$, appeared therein, is, in particular, a compact (Hausdorff) space.

In this connection, what we have actually inferred, by the previous Theorem 4.1, is simply that:

\[\text{the inductive limit of compact geometric algebra spaces is a space of the same sort.}\]

Indeed, the assertion is a straightforward application of the proof of the aforesaid result (": continuity theorem" in sheaf cohomology), according to the following data, with an obvious meaning of the notation applied:

\[\lim\downarrow (\mathcal{E}_\alpha, X_\alpha) := (\mathcal{E} \equiv \lim\downarrow \mathcal{E}_\alpha, X \equiv \lim\downarrow X_\alpha).\]
On the other hand, if the previous data are further supplied with the appropriate “differentials”, in the sense of “abstract differential geometry” (ADG), cf. [14], one still concludes that:

\[(4.16) \text{the inductive limit of compact differential-geometric algebra spaces is a space of the same type.}\]

Now, both of the previous two propositions seems to stand in close connection with several recent aspects in quantum relativity; cf., for instance, [20], [18], [16].

References

[1] R. Arens, *The problem of locally-\(A\) functions in a commutative Banach algebra*. A. Trans. Amer. Math. Soc. 104(1962), 24-36.
[2] R.G. Blumenthal, *The geometric structure of the spectrum of a functional algebra*. Ph. D. Thesis, Yale University, 1968.
[3] R.G. Blumenthal, *The spectrum of a function algebras*. Proc. Amer. Math. Soc. 25(1970), 343-346.
[4] G.E. Bredon, *Sheaf Theory* (2nd Edition). Springer-Verlag, New York, 1997.
[5] R.M. Brooks, *Partitions of unity in \(F\)-algebras*. Math. Ann. 177(1968), 265-272.
[6] H.F. de Groote, *Quantum sheaves. An outline of results*. math-ph/0110035 (30 Oct. 2001).
[7] R. Haag, *Local Quantum Physics. Fields, Particles, Algebras* (2nd Ed.). Springer, Berlin, 1996.
[8] R.I. Hadjigeorgiou, *Gluing spectra and boundaries of topological algebras together*. J. Math. Sci. (New York) 95(1999), 2626-2637.
[9] R.I. Hadjigeorgiou, *Spectral geometry of non-local topological algebras* (submitted).
[10] E. Kallin, *A nonlocal function algebra*. Proc. Nat. Acad. Sci. U.S.A. 49(1963), 821-824.
[11] A. Mallios, *Topological Algebras. Selected Topics*. North-Holland, Amsterdam, 1986.
[12] A. Mallios, *On geometric topological algebras*. J. Math. Anal. Appl. 172(1993), 301-322.
[13] A. Mallios, *The de Rham-Kähler complex of the Gel’fand sheaf of a topological algebra*. J. Math. Anal. Appl. 175(1993), 143-168.
[14] A. Mallios, *Geometry of Vector Sheaves. An Axiomatic Approach to Differential Geometry*. Vols I-II. Kluwer, Dordrecht, 1998. [Russian transl.; MIR, Moscow, 2000/2001].
[15] A. Mallios, *Abstract differential geometry, general relativity, and singularities*. In “Unsolved Problems on Mathematics for the 21st Century: A Tribute to Kiyoshi Iséki’s
ON LOCALIZING TOPOLOGICAL ALGEBRAS

80th Birthday”, J.M. Abe & S. Tanaka (Eds.) I0S Press, Amsterdam (2001), pp. 77-100. (invited paper)

[16] A. Mallios, K-Theory of topological algebras and second quantization. Intern. Conf. on “Topological Algebras and Applications”, Oulu (Finland), 2001 (Proceedings, to appear); math-ph/0207033.

[17] A. Mallios, Remarks on “singularities”. gr-qc/0202028 (v2, 27 May 2002).

[18] A. Mallios, Gauge Theories from the Point of View of Abstract Differential Geometry (book, in 2-vols, continuation of [14]; in preparation, 2002).

[19] A. Mallios and I. Raptis, Finitary space-time sheaves of quantum causal sets: Curving quantum causality. Intern. J. Theor. Physics 40 (2001), 1885-1928.

[20] A. Mallios – I. Raptis, Finitary Čech - de Rham cohomology: much ado without \( \mathcal{C}^\infty \)-smoothness. Intern. J. Theor. Physics 41(2002), 1877-1922.

[21] A. Mallios – I. Raptis, Finitary, causal and quantal Einstein gravity (submitted); gr-qc/0209048.

[22] A. Mallios and E.E. Rosinger, Abstract differential geometry, differential algebras of generalized functions, and de Rham cohomology. Acta Appl. Math. 55 (1999), 231-250.

[23] A. Mallios and E.E. Rosinger, Space-time foam dense singularities and de Rham cohomology. Acta Appl. Math. 67 (2001), 59-89.

[24] A. Oukhouya, On local topological algebras. Sc. Math. Japon. 7(2002), 277-281.

[25] A. Oukhouya, Cloture locale des algèbres topologiques (to appear).

[26] I. Raptis, Finitary spacetime sheaves. Intern. J. Theor. Physics 39(2000), 1703-1716.

[27] S.J. Sidney, Powers of maximal ideals in function algebras. Thesis, Harvard Univ., Cambridge, Mass., 1966.

[28] S.J. Sidney, Properties of the sequence of closed powers of a maximal ideal in a sup-norm algebra. Trans. Amer. Math. Soc. 131(1968), 128-148.

[29] J. Wagner, Faisceau structural associé à une algèbre de Banach. In Séminaire P. Lelong (Analyse) 9e année, 1968/69. Lect. Notes in Math. no 116, Springer-Verlag, Berlin, 1970. pp. 164-167.