Bose-Einstein condensates in optical lattices: mathematical analysis and analytical approximate formulae

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Abstract

We show that the GPE with cubic nonlinearity, as a model to describe the one dimensional Bose-Einstein condensates loaded into a harmonically confined optical lattice, presents a set of ground states which is orbitally stable for any value of the self-interaction (attractive and repulsive) parameter and laser intensity. We also derive a new formalism which gives explicit expressions for the minimum energy $E_{\text{min}}$ and the associated chemical potential $\mu_0$. Based on these formulas, we generalize the variational method to obtain approximate solutions, at any order of approximation, for $E_{\text{min}}$, $\mu_0$ and the ground state.

Key words: Bose-Einstein condensates, stability of ground states, analytical approximate formulae, repulsive or attractive interatomic interactions.

1. Introduction

The Bose-Einstein condensation (BEC) is a fundamental phenomenon connected to superfluidity in liquid helium \cite{1}. Its achievement in practice leads to the first unambiguous manifestation of the existence of a macroscopic quantum state in a many-particle system. Although the condensates of boson particles was firstly predicted by Einstein \cite{2,3} in 1924, BECs were experimentally realized only in 1995 \cite{4,5}. For these reasons, this phenomenon has attracted the attention of many scientists, in particular, during recent years. Nowadays, one of the most interesting problems in cold matter physics is the study of the BEC in a potential trap loaded in a periodical optical lattice (for a detailed discussion see Ref. \cite{6}). This problem can be described by means of the Gross-Pitaevskii equation (GPE) \cite{7,8,9}, for which the BEC are characterized by its ground state solutions.

The ground state solutions of the GPE in an external potential are not necessarily stable, i.e., small initial perturbations around a ground state could give rise to solutions collapsing in finite time. In order to understand the properties of the BEC, it is important to know how the small-amplitude excitations of the ground state evolve in time. One step in this direction was given by Zhang \cite{10}, who proved the stability of these solutions in the case of a harmonic potential for a nonlinear attractive interaction.

Assuming that the harmonic trapping potential has a strong anisotropy (of “cigar-shaped” type), the 1D limit of the GPE with cubic nonlinearity can be considered as a model to describe the condensate, more precisely, by the equation (see \cite{11})

$$\frac{\hbar^2}{2m} \frac{d^2 \Psi}{dx^2} + \frac{1}{2} m \omega^2 x^2 \Psi + \lambda_1 |\Psi|^2 \Psi - V_0 \cos^2 \left( \frac{2\pi}{d} x \right) \Psi = \mu_0 \Psi, \quad (1.1)$$

where $\omega > 0$ is the oscillator trap frequency, $m > 0$ is the atomic mass, $V_L > 0$ is the laser intensity, $d > 0$ is the wavelength of the laser, $\mu_0 \in \mathbb{R}$ is the chemical potential and $\lambda_1 \in \mathbb{R}$ is the self-interaction parameter ($\lambda_1 < 0$ when interatomic forces are attractive and $\lambda_1 > 0$ for repulsive interatomic forces). In its dimensionless form the equation (1.1) can be written as

$$-\frac{d^2 \psi}{d\xi^2} + \xi^2 \psi + \lambda |\psi|^2 \psi - V_0 \cos^2 (\alpha \xi) \psi = \mu \psi, \quad (1.2)$$

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where, for \( l := \sqrt{\hbar/m\omega} \), we set \( \xi = x/l \), \( \psi(\xi) := \sqrt{l}\Psi(x) \) and

\[
\mu := \frac{2\mu_0}{\hbar\omega}, \quad \lambda := \frac{2\lambda_1 D}{\hbar\omega}, \quad V_0 := \frac{2V_L}{\hbar\omega}, \quad \alpha := \frac{2\pi l}{d}.
\]

The solutions of (1.2) can be viewed as standing waves of the time dependent GPE, namely,

\[
i\frac{\partial u}{\partial \tau} = -\frac{\partial^2 u}{\partial \xi^2} + \xi^2 u + \lambda|u|^2 u - V_0 \cos^2(\alpha \xi) u,
\]

where \( \tau := \omega t/2 \) and \( t \) is the time. By standing waves we mean time periodic solutions of the form

\[
u(\tau, \xi) := e^{-i\mu \tau} \psi(\xi).
\]

In this paper we prove the ground state stability (beyond the Bogoliubov approximation) concerning the solutions of (1.3), in both cases, attractive and repulsive, and for any value of \( V_0 \). This result, together with some qualitative properties of the ground states and the general behavior of the minimal energy as function of \( \lambda \), completes the essential of Section 2. Furthermore, taking into account the importance from the physical standpoint to have explicit formulae for the minimal energy, the corresponding chemical potential and the ground state, we develop in Section 3 a new formalism to obtain explicit approximate expressions for the above mentioned magnitudes. The present formalism leads to a new and more general variational approach. To verify the validity of the method, we compare our solutions with those numerical results reported in [12, 13].

2. Existence and stability of ground states

Although the solutions of (1.3) are in general complex valued functions, we can restrict our analysis of the existence of ground states only for real valued ones, as we can see by the following lemma, where \( H^1(\mathbb{R}) \) denotes the usual Sobolev space.

**Lemma 2.1**: If \( \psi \in H^1(\mathbb{R}) \) is a complex solution of (1.2), then there exists a real function \( U(\xi) \) which is a solution of (1.2) and a real number \( \theta \) such that \( \psi(\xi) = e^{i\theta} U(\xi) \).

**Proof**: Assume that \( \psi \) is a complex solution of (1.2) and consider its real and imaginary parts, i.e., \( \psi = u + iv \), \( u \neq 0 \). Then, it follows that

\[
\begin{align*}
-\frac{d^2 u}{d\xi^2} + \xi^2 u + \lambda(u^2 + v^2)u - V_0 \cos^2(\alpha \xi) u &= \mu u, \\
-\frac{d^2 v}{d\xi^2} + \xi^2 v + \lambda(u^2 + v^2)v - V_0 \cos^2(\alpha \xi) v &= \mu v.
\end{align*}
\]

Now, multiplying the first equation in the above system by \( v \), the second by \( u \) and subtracting, we get

\[
u \frac{d^2 v}{d\xi^2} - v \frac{d^2 u}{d\xi^2} = 0 \quad \Rightarrow \quad u \frac{dv}{d\xi} - v \frac{du}{d\xi} = C,
\]

for some \( C \in \mathbb{R} \). Since the solutions of (1.2) tends to zero as \( \xi \to \pm \infty \) (see the next Theorem), we conclude that \( C = 0 \) and hence

\[
u \frac{dv}{d\xi} - v \frac{du}{d\xi} = 0 \quad \Rightarrow \quad \frac{d}{d\xi} \left( \frac{v}{u} \right) = 0.
\]

Therefore, \( v = \gamma u \), \( \gamma \neq 0 \) and each one of the equations of (2.1) reduces to

\[-\frac{d^2 u}{d\xi^2} + \xi^2 u + \lambda(1 + \gamma^2)u - V_0 \cos^2(\alpha \xi) u = \mu u.
\]
Now, considering \( U(\xi) := (1 + \gamma^2)^{1/2}u(\xi) \), it follows that \( U(\xi) \) is a real valued solution of (1.2) and

\[
\psi = u + iv = \left( \frac{1}{\sqrt{1 + \gamma^2}} + \frac{i\gamma}{\sqrt{1 + \gamma^2}} \right) U = e^{i\theta}U,
\]

where \( \theta := \arctan \gamma \). \( \square \)

An important property of the solutions of (1.2) is their asymptotic decay at infinity, as asserted in the following result.

**Theorem 2.2:** Let \( \mu, \lambda, V_0 \in \mathbb{R} \) be given. If \( \varphi \in H^1(\mathbb{R}) \) is a solution of (1.2), then \( \varphi \in C^2(\mathbb{R}) \) and for \( \delta \in (0, 1) \) there exists \( C(\delta) > 0 \) such that

\[
\forall \xi \in \mathbb{R}, \quad |\varphi(\xi)| \leq C(\delta) \exp[-(1 - \delta)\xi^2/2]. \tag{2.2}
\]

Moreover, if \( \lambda > 0 \) and \( \mu < 1 - |V_0| \), the above inequality holds for \( \delta = 0 \).

**Proof:** We proceed as in [14]. Since \( \varphi \in H^1(\mathbb{R}) \), we have that \( \varphi(\xi) \) is a continuous function satisfying

\[
\lim_{|\xi| \to +\infty} \varphi(\xi) = 0 \tag{2.3}
\]

and it follows directly from the equation that \( \psi'' \in C(\mathbb{R}) \). In order to prove the exponential decay of \( \varphi \), let

\[
a(\xi) := \xi^2 - \mu + \lambda|\varphi(\xi)|^2 - V_0 \cos^2(\alpha \xi).
\]

By Kato’s inequality, if \( z(\xi) := |\varphi(\xi)| \), we have \( z'' \geq \text{sign}(\varphi)\varphi'' \) in the sense of distributions. Therefore, \(-z'' + a(\xi)z \leq 0 \) in the same sense. On the other hand, if we set \( \psi_0(\xi) := C \exp[-(1 - \delta)\xi^2/2] \) for \( 0 \leq \delta < 1 \) and \( C > 0 \), a simple calculation gives

\[
-\psi''_0 + a(\xi)\psi_0 = \delta(2 - \delta)\xi^2 - \mu + 1 - \delta + \lambda|\varphi(\xi)|^2 - V_0 \cos^2(\alpha \xi) \psi_0.
\]

If \( \lambda > 0 \) and \( \mu < 1 - |V_0| \) set \( \delta = 0 \), otherwise assume that \( 0 < \delta < 1 \). Then, for \( R > 0 \) large enough, it follows from (2.3) that \( a(\xi) \geq 1 \) and

\[
\delta(2 - \delta)\xi^2 - \mu + 1 - \delta + \lambda|\varphi(\xi)|^2 - V_0 \cos^2(\alpha \xi) > 0, \quad \forall |\xi| > R,
\]

so that \(-z'' + az \leq -\psi''_0 + a\psi_0 \) for \( |\xi| > R \). Moreover, if we choose \( C > 0 \) such that \( z(\pm R) \leq \psi_0(\pm R) \), then from the maximum principle we infer that

\[
|\varphi(\xi)| = z(\xi) \leq \psi_0(\xi), \quad \forall |\xi| \geq R,
\]

which implies the exponential decay of \( \varphi \), as asserted in (2.2). \( \square \)

**- Existence of ground states**

We introduce the variational problem which allows to prove the existence and stability of ground states for Eq. (1.2). Let

\[
X := \left\{ \psi \in H^1(\mathbb{R}) : \int_{\mathbb{R}} \left( |\psi'(\xi)|^2 + |\xi^2|\psi(\xi)|^2 \right) dx < +\infty \right\}. \tag{2.4}
\]

\( X \) is a real Hilbert space if endowed with the following usual inner product

\[
(\phi|\psi)_X := \int_{\mathbb{R}} \left( \psi'(\xi)\phi'(\xi) + \xi^2\psi(\xi)\phi(\xi) \right) d\xi.
\]

Then, the associated norm is given by

\[
\|\psi\|_X^2 := \int_{\mathbb{R}} \left( |\psi'(\xi)|^2 + \xi^2|\psi(\xi)|^2 \right) d\xi.
\]
Now we define the “energy” $E : X \to \mathbb{R}$ and the “charge” $Q : X \to \mathbb{R}$ respectively by

$$E(\psi) := \int_{\mathbb{R}} |\psi'(\xi)|^2 \, d\xi + \int_{\mathbb{R}} \xi^2 |\psi(\xi)|^2 \, d\xi + \frac{\lambda}{2} \int_{\mathbb{R}} |\psi(\xi)|^4 \, d\xi - V_0 \int_{\mathbb{R}} \cos^2 (\alpha \xi) |\psi(\xi)|^2 \, d\xi,$$

$$Q(\psi) := \int_{\mathbb{R}} |\psi(\xi)|^2 \, d\xi,$$

and the manifold

$$\Sigma_1 := \left\{ \psi \in X : Q(\psi) = 1 \right\}.$$

With these ingredients we look for solutions $\psi$ of Eq. (1.2) that minimizes the energy $E$ among all functions in $\Sigma_1$. More precisely, we look for $\psi_{\min} \in \Sigma_1$ such that

$$E(\psi_{\min}) = \min \left\{ E(\psi) : \psi \in \Sigma_1 \right\}.$$  

**Remark 2.3:** Before proceeding to prove that there exist solutions of the variational problem (2.6), we shall remember the following well known facts. Let $\varphi_0 : \mathbb{R} \to \mathbb{R}$ be the function

$$\varphi_0(\xi) := \frac{1}{\sqrt{\pi}} \exp(-\xi^2/2).$$  

It is easy to see that $\varphi_0 \in \Sigma_1$ and that $-\varphi_0''(\xi) + \xi^2 \varphi_0(\xi) = \varphi_0(\xi)$. This means that $\varphi_0$ is an eigenfunction of the operator $L = -\frac{d^2}{d\xi^2} + \xi^2$ corresponding to the eigenvalue $\lambda_0 = 1$. In fact, $L$ has an infinite sequence of eigenvalues $\lambda_0 < \lambda_1 < \cdots$, where $\lambda_n = (2n + 1), \quad (n = 0, 1, \ldots)$, and the Hermite functions are the corresponding eigenfunctions. It is also known that $\lambda_0$ has the following variational characterization,

$$\lambda_0 = \inf \left\{ \int_{\mathbb{R}} \left[ |\psi'(\xi)|^2 + \xi^2 |\psi(\xi)|^2 \right] \, d\xi : \psi \in \Sigma_1 \right\}$$

and we can easily verify that $\lambda_0 = 1$ and that the above infimum is actually a minimum attained at $\varphi_0$, i.e., $\lambda_0 = \|\varphi_0\|_X^2 = 1$.

We are now in position to prove the existence of ground states of (1.2).

**Theorem 2.4:** Let $\lambda, V_0 \in \mathbb{R}$ be given. Then, there exists $\psi_{\min} \in \Sigma_1$ such that

$$E(\psi_{\min}) = \min \left\{ E(\psi) : \psi \in \Sigma_1 \right\}.$$  

**Proof:** We divide the proof in three steps.

**Step 1:** The energy $E$ is bounded from bellow on $\Sigma_1$.

From Remark 2.3 we know that $\|\psi\|_X^2 \geq 1$ for all $\psi \in \Sigma_1$. Hence, if $\lambda \geq 0$, we have

$$E(\psi) \geq \|\psi\|_X^2 - |V_0| \geq 1 - |V_0|, \quad \forall \psi \in \Sigma_1.$$  

On the other hand, from Gagliardo-Nirenberg inequality, there exists a constant $C_{gn} > 0$ such that

$$\|\psi\|_4^4 \leq C_{gn} \|\psi\|_X^2 \|\psi'\|_2, \quad \forall \psi \in H^1(\mathbb{R}),$$

where $\|\cdot\|_4$ and $\|\cdot\|_2$ are the standard norms of the spaces $L^4(\mathbb{R})$ and $L^2(\mathbb{R})$, respectively. Hence, in the attractive case ($\lambda < 0$), we have for any $\psi \in \Sigma_1$,

$$E(\psi) \geq \|\psi\|_X^2 + \frac{\lambda C_{gn}}{2} \|\psi'\|_2 - |V_0| \geq \|\psi\|_X^2 + \frac{\lambda C_{gn}}{2} \|\psi\|_X - |V_0|,$$

from which we get

$$E(\psi) \geq - \frac{\lambda^2 C_{gn}^2}{16} - |V_0|, \quad \forall \psi \in \Sigma_1.$$  

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Step 2: The variational problem (2.8) has a solution:

Let \( E_{\min} := \inf \{ E(\psi) ; \psi \in \Sigma_1 \} \). From Step 1, it follows that \( E_{\min}, \in \mathbb{R} \) and from the definition of infimum, we conclude that there exists a sequence of minimizing functions \( \{ \psi_n \}_{n \in \mathbb{N}} \) in the manifold \( \Sigma_1 \), i.e.,

\[
\forall n \in \mathbb{N}, \exists \psi_n \in \Sigma_1 \ 	ext{such that} \lim_{n \to +\infty} E(\psi_n) = E_{\min}.
\]

Assuming that \( \lambda \geq 0 \), we obtain easily from (2.9) that \( \{ \psi_n \} \) is a bounded sequence in \( X \). On the other hand, if \( \lambda < 0 \), from (2.11) and the Young’s inequality, we have

\[
\| \psi_n \|^2_X \leq E(\psi_n) - \frac{\lambda C}{2} \| \psi_n \|_X + |V_0| \leq E(\psi_n) + \frac{1}{2} \| \psi_n \|^2_X + \frac{\lambda^2 C^2}{8} \| \psi_n \|^2 + |V_0|.
\]

So, we obtain

\[
\frac{1}{2} \| \psi_n \|^2_X \leq E(\psi_n) + \frac{\lambda^2 C^2}{8} \| \psi_n \|^2 + |V_0|,
\]

from which we conclude that \( \{ \psi_n \} \) is a bounded sequence in \( X \). Therefore, in both cases, it follows from the Banach-Alaoglu Theorem, that there exists a subsequence of \( \psi_n \) that converges to some \( \psi_{\min} \) in the weak topology of \( X \), i.e., \( \psi_{n_k} \rightharpoonup \psi_{\min} \). To simplify the notation, we still write \( \psi_n \) for this subsequence. Since the embedding \( X \subset L^p(\mathbb{R}) \) (see [14,15]) is compact for all \( 2 \leq p < +\infty \), we have

\[
\lim_{n \to +\infty} \int_{\mathbb{R}} \psi_n(\xi)|d\xi| = \int_{\mathbb{R}} \psi_{\min}(\xi)|d\xi|,
\]

\[
\lim_{n \to +\infty} \int_{\mathbb{R}} \psi_n(\xi)^2 d\xi = \int_{\mathbb{R}} \psi_{\min}(\xi)^2 d\xi.
\]

By hypothesis, \( \psi_n \in \Sigma_1 \) for all \( n \in \mathbb{N} \), and the second limit above implies that \( \psi_{\min} \in \Sigma_1 \). Moreover, as the norm \( \| \cdot \|_X \) is semi-continuous for the weak topology of \( X \), we have

\[
\| \psi \|^2_X \leq \liminf_{n \to +\infty} \| \psi_n \|^2_X.
\]

From (2.13) and (2.14), we conclude that \( E(\psi_{\min}) = E_{\min} \) and hence \( \psi_{\min} \) is a solution of (2.8).

Step 3: The function \( \psi_{\min} \) is a solution of (1.2).

This follows directly from the fact that \( E(\psi) \) and \( Q(\psi) \) are differentiable functionals in \( X \). Indeed, from the Lagrange Theorem, there exists \( \mu \in \mathbb{R} \) (a Lagrange multiplier) such that

\[
E'(\psi_{\min}) = \mu Q'(\psi_{\min}),
\]

where \( E'(\psi) \) and \( Q'(\psi) \) are the Fréchet derivatives of \( E \) and \( Q \) at \( \psi \), respectively. Note that this last equation is the same as (1.2). This completes the proof.

If we denote by \( \mathcal{G} \) the set of ground states of (1.2), i.e.,

\[
\mathcal{G} := \{ \psi \in \Sigma_1 ; E(\psi) = E_{\min} \},
\]

it follows easily from (2.9) and (2.11) that \( \mathcal{G} \) is a bounded set of \( X \) and we have the following properties:

Theorem 2.5: Let \( \lambda, V_0 \in \mathbb{R} \).

a) If \( \lambda \geq 0 \), there exists a unique positive symmetric function \( \psi_{\min} \in \Sigma_1 \) such that

\[
\mathcal{G} = \{ e^{i\theta} \psi_{\min} ; \theta \in \mathbb{R} \}.
\]

b) If \( V_0 = 0 \), then there exists a positive symmetric function \( \psi_{\min} \in \mathcal{G} \) such that \( \xi \mapsto \psi_{\min}(\xi) \) is decreasing in the interval \( \xi \geq 0 \). In particular, \( \psi_{\min}(0) = \max \{ \psi_{\min}(\xi) ; \xi \in \mathbb{R} \} \).
Proof: To prove (a) we proceed as in [16]. Suppose that there are two real functions $\psi_0, \psi_1 \in \mathcal{G}$, $\psi_0 \neq \psi_1$. If $|\psi_0| \neq |\psi_1|$, define $\psi_\nu := [\nu \psi_0^2 + (1 - \nu)\psi_1^2]^{1/2}$, where $0 < \nu < 1$. It follows that $\psi_\nu \in \Sigma_1$ and

$$\int_\mathbb{R} \xi^2 |\psi_\nu(\xi)|^2 d\xi = \nu \int_\mathbb{R} \xi^2 |\psi_0(\xi)|^2 d\xi + (1 - \nu) \int_\mathbb{R} \xi^2 |\psi_1(\xi)|^2 d\xi,$$

$$\int_\mathbb{R} \cos^2(\alpha \xi) |\psi_\nu(\xi)|^2 d\xi = \nu \int_\mathbb{R} \cos^2(\alpha \xi) |\psi_0(\xi)|^2 d\xi + (1 - \nu) \int_\mathbb{R} \cos^2(\alpha \xi) |\psi_1(\xi)|^2 d\xi. \tag{2.15}$$

Since $x \mapsto |x|^4$ is strictly convex, we have

$$\int_\mathbb{R} |\psi_\nu(\xi)|^4 d\xi < \nu \int_\mathbb{R} |\psi_0(\xi)|^4 d\xi + (1 - \nu) \int_\mathbb{R} |\psi_1(\xi)|^4 d\xi. \tag{2.16}$$

Moreover, by differentiating both sides of $\psi_\nu^2 = \nu \psi_0^2 + (1 - \nu)\psi_1^2$ and using the Cauchy-Schwarz inequality in $\mathbb{R}^2 (ab + cd \leq \sqrt{a^2 + c^2} \sqrt{b^2 + d^2})$, we get

$$\psi_\nu \psi_\nu' = \nu \psi_0\psi_0' + (1 - \nu)\psi_1\psi_1' \leq \nu \sqrt{\nu(\psi_0^2)}(1 - \nu)(\psi_1^2)^2,$$

from which it follows that

$$|\psi_\nu'(\xi)|^2 \leq \nu |\psi_0'(\xi)|^2 + (1 - \nu)|\psi_1'(\xi)|^2. \tag{2.17}$$

As we are assuming that $\lambda \geq 0$, it follows from (2.15)–(2.17) that $E(\psi_\nu) < \nu E_{\text{min}} + (1 - \nu)E_{\text{min}} = E_{\text{min}}$, which is impossible by the definition of $E_{\text{min}}$.

Since $|\psi_i| \in \Sigma_1$ for $i = 0, 1$, it follows from Kato’s inequality that $E(|\psi_i|) \leq E(\psi_i) = E_{\text{min}}$. So, $|\psi_0|, |\psi_1| \in \mathcal{G}$ and we conclude by the previous arguments that $|\psi_0| = |\psi_1|$. By assuming that $\psi_0(\xi_0) = \psi_1(\xi_0) = 0$ for some $\xi_0 \in \mathbb{R}$ and taking into account that $|\psi_i| \in C^2$ (see Theorem 2.2), it follows that $\psi_0'(\xi_0) = \psi_1'(\xi_0) = 0$, which implies from the uniqueness of solutions of ODEs that $\psi_i \equiv 0$. This is in contradiction from the fact that $\psi_i \in \Sigma_1$. Therefore, we can assume that $\psi_i = -\psi_i$, with $\psi_0(\xi) > 0$ for all $\xi \in \mathbb{R}$. Moreover, since $\tilde{\psi}(\xi): = \psi(-\xi)$ belongs to $\mathcal{G}$ for all $\psi \in \mathcal{G}$, the unique positive ground state $\psi_0$ is necessarily symmetric.

To prove (b), let $\psi \in \mathcal{G}$ be a real function and consider $\psi_*(\xi)$ the symmetric-decreasing rearrangement of $|\psi(\xi)|$. It is well known (see [9, 17]) that $\psi_*$ is positive, symmetric, decreasing in $[0, +\infty)$ and

$$\int_\mathbb{R} |\psi_*(\xi)|^p d\xi = \int_\mathbb{R} |\psi(\xi)|^p d\xi, \quad 1 \leq p \leq \infty,$$

$$\int_\mathbb{R} |\psi'_*(\xi)|^2 d\xi \leq \int_\mathbb{R} |\psi'(\xi)|^2 d\xi. \tag{2.18}$$

Hence, from the first equality of (2.18) with $p = 2$ we get $\psi_* \in \Sigma_1$. On the other hand, we have for any $c > 0$ (see [9, 17]),

$$\int_\mathbb{R} (c - \xi^2)^+ |\psi(\xi)|^2 d\xi \leq \int_\mathbb{R} (c - \xi^2)^+ |\psi_*(\xi)|^2 d\xi,$$

which gives

$$c \int_{-\sqrt{c}}^{\sqrt{c}} (|\psi(\xi)|^2 - |\psi_*(\xi)|^2) d\xi \leq \int_{-\sqrt{c}}^{\sqrt{c}} \xi^2 (|\psi(\xi)|^2 - |\psi_*(\xi)|^2) d\xi. \tag{2.19}$$

Since the symmetric-decreasing rearrangement is order-preserving, it follows that $\psi_*$ also satisfies (2.2). Therefore, using the L’Hospital rule we get, for $f(\xi) := |\psi(\xi)|^2 - |\psi_*(\xi)|^2$,

$$\lim_{c \to +\infty} c \int_{-\sqrt{c}}^{\sqrt{c}} f(\xi) d\xi = \lim_{c \to +\infty} c^2 [f(-\sqrt{c}) - f(\sqrt{c})] = 0$$

and consequently

$$\int_\mathbb{R} \xi^2 |\psi_*(\xi)|^2 d\xi \leq \int_\mathbb{R} \xi^2 |\psi(\xi)|^2 d\xi. \tag{2.20}$$
The first equality of \((2.18)\) with \(p = 4\), together with the second inequality in \((2.18)\) and \((2.20)\) imply that \(E(\psi_\ast) = E_{\text{min}}\), which means that \(\psi_\ast \in \mathcal{G}\). Moreover, for \(p = +\infty\), it follows that \(\psi_\ast(0) = \max \{|\psi(\xi)| : \xi \in \mathbb{R}\}\), and the proof is complete. \[\square\]

**Remark 2.6:** Theorem 2.4 asserts that \(\mathcal{G}\) is not empty. In fact, \(\mathcal{G}\) has infinitely many elements, because if \(\psi \in \mathcal{G}\), then \(e^{i\theta}\psi \in \mathcal{G}\), for all \(\theta \in \mathbb{R}\). Nevertheless, we do not know if \(\mathcal{G}\) has only one real positive valued function in the case \(\lambda < 0\). However, if there are multiple real valued ground states \(\psi\) in \(\mathcal{G}\), one should be aware that the Lagrange multiplier \(\mu\) might depend also on \(\psi\). Indeed, multiplying both sides of Eq. \((1.2)\) by \(\psi\), we get

\[
\mu(\psi) = E_{\text{min}} + \frac{\lambda}{2} \int_{\mathbb{R}} |\psi(\xi)|^4 d\xi. \tag{2.21}
\]

As we are going to see in the study of stability, it is important to notice that the set of Lagrange multipliers is bounded. This is immediate because

\[
|\mu(\psi)| \leq |E_{\text{min}}| + \frac{|\lambda|}{2} \|\psi\|_4^4 \leq |E_{\text{min}}| + \frac{|\lambda|C_{\text{gn}}}{2} \|\psi\|_X
\]

and because \(\mathcal{G}\) is bounded in \(X\).

**Remark 2.7:** Theorem 2.4 states that Eq. \((1.2)\) admits ground state solutions in both cases: attractive \((\lambda < 0)\) and repulsive \((\lambda > 0)\). In the repulsive case, Eq. \((1.2)\) presents a different behavior when compared with the classical NLS. In fact, assume that \(\varphi \in H^1(\mathbb{R})\) is a nontrivial solution of

\[
-\varphi'' + \lambda|\varphi|^2 \varphi = \mu \varphi. \tag{2.22}
\]

If we multiply both sides of \((2.22)\) by \(\varphi(\xi)\) (respectively \(-\xi\varphi'(\xi)\)) and integrate on \(\mathbb{R}\), we get

\[
\mu = \int_{\mathbb{R}} |\varphi'(\xi)|^2 d\xi + \frac{\lambda}{2} \int_{\mathbb{R}} |\varphi(\xi)|^4 d\xi,
\]

\[
\mu = -\int_{\mathbb{R}} |\varphi'(\xi)|^2 d\xi + \frac{\lambda}{2} \int_{\mathbb{R}} |\varphi(\xi)|^4 d\xi.
\]

Hence,

\[
2 \int_{\mathbb{R}} |\varphi'(\xi)|^2 d\xi + \frac{\lambda}{2} \int_{\mathbb{R}} |\varphi(\xi)|^4 d\xi = 0,
\]

which is impossible if \(\lambda \geq 0\) and \(\varphi \not= 0\). Therefore, \((2.22)\) has no nontrivial solution in \(H^1(\mathbb{R})\) if \(\lambda \geq 0\).

- **Stability of ground states**

  In order to prove the stability of ground states of \((1.2)\), let us consider the Cauchy problem

  \[
i \frac{\partial v}{\partial \tau} = \frac{1}{2} E'(v), \quad v(0, \xi) = v_0(\xi), \tag{2.23}
\]

  where \(E'\) is the Fréchet derivative of \(E\) in \(X\), i.e.,

  \[
  \frac{1}{2} E'(v) := -\frac{\partial^2 v}{\partial \xi^2} + \xi^2 v + \lambda|v|^2 v - V_0 \cos^2 (\alpha \xi) v.
  \]

  It is well known [18, 19] that \((2.23)\) has a unique solution that is global in time, i.e., for any \(v_0 \in X\), there exists a unique \(v \in C([0, +\infty), X)\) satisfying \((2.23)\). In particular, if \(\psi \in \mathcal{G}\), the unique solution \(u\) of \((2.23)\) such that \(u(0, \xi) = \psi(\xi)\) is the standing wave given by

  \[
u(\tau, \xi) = e^{-i\mu \tau} \psi(\xi).
  \]

Saying that \(\psi\) is stable means that if the initial datum \(v_0\) of Eq. \((2.23)\) is close enough to \(\psi\), then the trajectories \(v(\tau, \cdot)\) remain close to the set \(\mathcal{G}\), as \(\tau\) varies in \(\mathbb{R}\). More precisely,
Definition 2.8: We will say that $G$ is stable if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $v_0 \in X$ satisfies $\inf_{\psi \in G} \|v_0 - \psi\|_X < \delta$, then the solution of (2.23) satisfies

$$\sup_{\tau \in \mathbb{R}} \inf_{\psi \in G} \|v(\tau, \cdot) - e^{-i\mu\tau} \psi\|_X < \varepsilon.$$  

Before proving the stability of $G$, we state and prove the following lemma about the compactness of the set $G$, which will be needed in the proof of the stability result.

Lemma 2.9: The set $G$ is compact and weakly sequentially closed in $X$. More precisely, if $\{\psi_n\}_{n \in \mathbb{N}}$ is a sequence of $G$, then there exists $\psi \in G$ and a subsequence $\{\psi_{n_k}\}_{k \in \mathbb{N}}$ such that $\psi_{n_k} \rightarrow \psi$ in $X$. Also, if $\{\psi_n\}_{n \in \mathbb{N}}$ is a sequence of $G$ and $\psi_n \rightarrow \psi$ in $X$-weakly, then $\psi_n \rightarrow \psi$ strongly in $X$.

Proof: Let $\{\psi_n\}_{n \in \mathbb{N}}$ be a sequence of $G$. Since $G$ is bounded in $X$, there exists a subsequence $\{\psi_{n_k}\}_{k \in \mathbb{N}}$ and $\psi \in X$ such that $\psi_{n_k} \rightarrow \psi$ in $X$-weak. By the compactness of the embedding $X \subset L^p(X)$ for $2 \leq p < \infty$, we conclude that $\psi \in X$. Now, arguing as in the step 2 of the proof of Theorem 2.4, we obtain that $\psi \in G$. To conclude the proof, it remains to show that $\psi_{n_k} \rightarrow \psi$ in $X$ strongly. To see this, it is enough to remark that $\|\psi_{n_k}\|_X \rightarrow \|\psi\|_X$, but this follows from (2.13) and the fact that

$$\|\psi_{n_k}\|_X^2 = E_{\text{min}} - \frac{\lambda}{2} \int_{\mathbb{R}} |\psi_{n_k}(\xi)|^4 d\xi + V_0 \int_{\mathbb{R}} \cos^2(\alpha \xi)|\psi_{n_k}(\xi)|^2 d\xi. \quad (2.24)$$

This proves that $G$ is compact and the remaining claims follow easily. \(\square\)

In order to prove the stability of $G$, we need the following well known conservation laws that hold for all solutions of (2.23): Assume that $v(\tau, \xi)$ is a solution of (2.23) such that $v(0, \xi) = v_0(\xi)$. Then, for each $\tau \in \mathbb{R}$, we have

$$Q(v(\tau, \cdot)) = Q(v_0) \quad \text{and} \quad E(v(\tau, \cdot)) = E(v_0).$$

Concerning the above identities, the first one is known as the “conservation of charge” and the second one is the “conservation of energy”. The first one can be obtained multiplying Eq. (2.23) by $-iv(\tau, \xi)$ (i.e., the complex conjugate of $iv(\tau, \xi)$). By the same way, one obtains the conservation of energy, but in that case we should multiply the equation by $\frac{d\xi}{d\tau}$.

We are now in position to prove the stability of $G$.

Theorem 2.10: The set $G$ is stable in the sense of the Definition 2.8.

Proof: Arguing by contradiction, we proceed as in Cazenave-Lions [20]. If $G$ is not stable, there exists $\varepsilon_0 > 0$ such that for all integer $n \in \mathbb{N}$, we can find $v_n \in X$ satisfying

$$r_n := \inf_{\psi \in G} \|v_0 - \psi\|_X < \frac{1}{n} \quad (2.25)$$

and

$$\sup_{\tau \in \mathbb{R}} \inf_{\psi \in G} \|v_n(\tau, \cdot) - e^{-i\mu\tau} \psi\|_X \geq \varepsilon_0, \quad (2.26)$$

where $v_n \in C(\mathbb{R}; X)$ is the unique solution of (2.23) with initial datum $v_0$.

Let $\psi_n \in G$ such that $r_n \leq \|v_0 - \psi_n\|_X < r_n + 1/n$. Since $G$ is bounded in $X$ and the sequence $\mu(\psi_n)$ is bounded in $\mathbb{R}$ (see Remark 2.6), there exists $(\psi_\infty, \mu_\infty) \in X \times \mathbb{R}$ and a subsequence still denoted by $(\psi_n, \mu(\psi_n))$ such that $\psi_n$ converges to $\psi_\infty$ weakly in $X$ and strongly in $L^4(\mathbb{R}) \cap L^2(\mathbb{R})$, while $\mu(\psi_n)$ converges to $\mu_\infty$ in $\mathbb{R}$. By Lemma 2.9 we know that $\psi_n \rightarrow \psi_\infty$ in $X$ and $\psi_\infty \in G$. Also, from the fact that $\|v_0 - \psi_n\|_X \rightarrow 0$, we may infer that $v_n$ converges to $\psi_\infty$ in $X$ and in $L^4(\mathbb{R}) \cap L^2(\mathbb{R})$ as well.

On the other hand, one may observe from (2.26) that there exists $r_n \in \mathbb{R}$ such that

$$\inf_{\psi \in G} \|v_n(\tau_n, \cdot) - e^{-i\mu(\psi)\tau_n} \psi\|_X \geq \frac{1}{2}\varepsilon_0. \quad (2.27)$$
Setting $\tilde{\psi}_n := e^{j n \langle \psi_n \rangle_n} v_n (\tau_n, \cdot)$, it follows that the conservation of charge and energy that

$$Q(\tilde{\psi}_n) = Q(v_{\infty}) = 1, \ E(\tilde{\psi}_n) = E(v_{\infty}) = E_{\min}$$

As $n \to \infty$.

All this means that $\{ \tilde{\psi}_n \}_{n \in \mathbb{N}}$ is a bounded sequence in $X$ and there exists $\tilde{\psi}_{\infty} \in X$ and a subsequence (still denoted by $\{ \tilde{\psi}_n \}_{n \in \mathbb{N}}$) converging to $\tilde{\psi}_{\infty}$ weakly in $X$ and strongly in $L^4 (\mathbb{R}) \cap L^2 (\mathbb{R})$. Therefore, $\tilde{\psi}_{\infty} \in \mathcal{G}$ and $E(\tilde{\psi}_n) \to E(\tilde{\psi}_{\infty})$. Again, invoking relation (2.24) we observe that $\tilde{\psi}_n \to \tilde{\psi}_{\infty}$ strongly in $X$ and this is a contradiction with (2.27). This finishes the proof. $\square$

**The minimal energy as function of $\lambda$**

In order to explicit the dependence of the minimal energy relatively to the parameter $\lambda$, let us denote

$$E_{\min} (\lambda) := \min \{ E_\lambda (\psi) : \psi \in \Sigma_1 \},$$

where $E_\lambda$ is the energy functional introduced in (2.5), and $\mathcal{G}_\lambda := \{ \psi \in \Sigma_1 : E_\lambda (\psi) = E_{\min} (\lambda) \}$ the set of corresponding ground-states. Theorem 2.4 assures that $E_{\min}$ is well-defined as function of $\lambda \in \mathbb{R}$ and it is easy to see that it is strictly increasing. Indeed, for $h > 0$ and $\psi \in \mathcal{G}_{\lambda + h}$ we have

$$E_{\min} (\lambda) \leq E_\lambda (\psi) = E_{\min} (\lambda + h) - \frac{h}{2} \| \psi \|^4_4 < E_{\min} (\lambda + h).$$

**Proposition 2.11:** $E_{\min} (\lambda)$ is a strictly increasing and concave function such that

$$\lim_{\lambda \to \pm \infty} E_{\min} (\lambda) = \pm \infty.$$  

The proof of the Proposition relies on the following:

**Lemma 2.12:** For each $a, b \in \mathbb{R}$, $a < b$, the set $\bigcup_{a \leq \lambda \leq b} \mathcal{G}_\lambda$ is bounded in $X$. More precisely, there exists a constant $C_{a,b} > 0$ such that

$$\| \psi \|_X \leq C_{a,b} \quad \forall \psi \in \bigcup_{a \leq \lambda \leq b} \mathcal{G}_\lambda.$$  

**Proof:** It suffices to prove for $a < 0 < b$. Let $\lambda \in [a, b]$ and $\psi \in \mathcal{G}_\lambda$. Then

$$E_{\min} (b) \geq E_{\min} (\lambda) = E_\lambda (\psi) \geq \| \psi \|^4_4 + \frac{\lambda}{2} \| \psi \|^4_4 - |V_0|.$$  

If $\lambda \geq 0$, then $\| \psi \|^2_X \leq E_{\min} (b) + |V_0|$. If $\lambda < 0$, it follows from Gagliardo-Nirenberg inequality (2.10) that

$$E_{\min} (b) \geq \| \psi \|^2_X + \frac{\lambda C g n}{2} \| \psi \|_X - |V_0| \geq \| \psi \|^2_X + \frac{a C g n}{2} \| \psi \|_X - |V_0|$$

and we have

$$\| \psi \|^2_X \leq 2 \left( E_{\min} (b) + \frac{a^2 C^2 g n}{8} + |V_0| \right).$$

**Proof of Proposition 2.11:** It follows from (2.29) that

$$E_{\min} (\lambda) \leq \lim_{h \to 0^{+}} E_{\min} (\lambda + h).$$

Let $\psi \in \mathcal{G}_\lambda$. Then,

$$E_{\min} (\lambda + h) \leq E_{\lambda+h} (\psi) = E_{\min} (\lambda) + \frac{h}{2} \| \psi \|^4_4 \forall h \in \mathbb{R},$$

(2.32)
from which we obtain
\[ \limsup_{h \to 0} E_{\min}(\lambda + h) \leq E_{\min}(\lambda). \] (2.33)

From (2.31) and (2.33) we conclude that
\[ \lim_{h \to 0^+} E_{\min}(\lambda + h) = E_{\min}(\lambda). \] (2.34)

On the other hand, if \( h < 0 \) and \( \psi \in G_{\lambda + h} \), we have
\[ E_{\min}(\lambda) \leq E(\psi) = E_{\min}(\lambda + h) - \frac{h}{2} \| \psi \|^4_4, \] (2.35)
from which, using the Gagliardo-Nirenberg inequality we obtain,
\[ E_{\min}(\lambda) \leq E_{\min}(\lambda + h) - \frac{h C_{gn}}{2} \| \psi \|_{X}. \] (2.36)

By choosing and fixing \( a \in \mathbb{R} \) such that \( a < \lambda + h < \lambda \), it follows from Lemma 2.12 that there exists a constant \( C_{a, \lambda} \) such that
\[ E_{\min}(\lambda) \leq E_{\min}(\lambda + h) - \frac{h C_{a, \lambda}}{2}, \]
from which we get
\[ \liminf_{h \to 0} E_{\min}(\lambda + h) \geq E_{\min}(\lambda). \] (2.37)

So, (2.33) and (2.37) give
\[ \lim_{h \to 0^-} E_{\min}(\lambda + h) = E_{\min}(\lambda) \]
and we conclude from (2.34) that \( E_{\min} \) is a continuous function. Moreover, from (2.32) we have
\[ E_{\min}(\lambda + h) + E_{\min}(\lambda - h) - 2E_{\min}(\lambda) \leq 0 \quad \forall \lambda, h \in \mathbb{R}, \]
which, under continuity, is sufficient to assure the concavity of \( E_{\min} \).

In order to prove (2.30), consider \( \psi_k \in G_k, k \in \mathbb{N} \). Then, using (2.29) and (2.32) with \( h = 1 \) we have
\[ \frac{1}{2} \| \psi_{k+1} \|^4_4 \leq E_{\min}(k + 1) - E_{\min}(k) \leq \frac{1}{2} \| \psi_k \|^4_4, \quad \forall k \in \mathbb{N} \]
from which we obtain for all \( n \in \mathbb{N} \)
\[ \frac{1}{2} \sum_{k=1}^{n+1} \| \psi_k \|^4_4 \leq E_{\min}(n + 1) - E_{\min}(0) \leq \frac{1}{2} \sum_{k=0}^{n} \| \psi_k \|^4_4. \]

Now, arguing by contradiction, assume that there exists a positive constant \( C \) such that \( E_{\min}(\lambda) \leq C \) for all \( \lambda \in \mathbb{R} \). Then, it follows from the first inequality in the above expression that the sequence \( \{ \psi_n \}_n \) converges to zero in \( L^4(\mathbb{R}) \). But, repeating the arguments used in the first step of the proof of Theorem 2.4, we conclude that the sequence \( \{ \psi_k \}_k \) is bounded in \( X \), so that, passing to a subsequence if necessary, we have that \( \psi_k \) converges to \( \psi \) weakly in \( X \). Since the embedding \( X \subset L^p(\mathbb{R}) \) is compact for \( 2 \leq p < \infty \), we conclude, by choosing \( p = 4 \) that \( \psi = 0 \), and by choosing \( p = 2 \) that \( \psi \in \Sigma_1 \), which is absurd. Since the same arguments apply for \( \psi_k \in G_{-k}, k \in \mathbb{N} \), we finish the proof. \( \Box \)

**Remark 2.13:** Concerning the dependence of the chemical potential relatively to \( \lambda \), it follows from the formula (2.21) and the characterization of \( G_\lambda \) given by Theorem 2.5 that we can also define, for \( \lambda \geq 0 \), the function \( \mu_{\min}(\lambda) \). However, as reported in Remark 2.6, it is unclear that \( \mu_{\min}(\lambda) \) be uniquely determined.
if $\lambda < 0$. No matter whether or not it be uniquely determined, the fact that $\lambda(\mu(\psi) - E_{\min}(\lambda)) \geq 0$ for all $\lambda \in \mathbb{R}$ and for all $\psi \in \mathcal{G}_\lambda$, allows to state that

$$\lim_{\lambda \to \pm \infty} \mu(\psi_\lambda) = \pm \infty, \quad \forall \psi_\lambda \in \mathcal{G}_\lambda.$$  

**Corollary 2.14:** Let $\psi_i \in \mathcal{G}_\lambda$, $i = 1, 2$. If $\lambda_1 < \lambda_2$, then $\|\psi_2\|_4 \leq \|\psi_1\|_4$.

**Proof:** Let $h = \lambda_2 - \lambda_1$. Then, from (2.29) and (2.32), it follows that

$$\frac{1}{2}\|\psi_\lambda\|_4^4 \leq \frac{E_{\min}(\lambda_2) - E_{\min}(\lambda_1)}{h} \leq \frac{1}{2}\|\psi_1\|_4^4. \quad \square$$

### 3. A new method to obtain approximations for the minimal energy

The results presented in the previous section, although mathematically rigorous, do not provide sufficiently precise quantitative information for some relevant physical quantities. Since it seems to be not possible to calculate exact explicit solutions of Eq. (1.2) (in particular, the ground state solution $\psi_{\min}$), the exact values of such quantities cannot be expressed in terms of the known parameters. Therefore, it would be useful to obtain some kind of explicit formulae through which one could approximate them. Indeed, these are outcomes mainly interesting from the physical point of view. In this sense, there exist some well known methods that were already applied to the problem we are dealing with (see [12,13,21]). Nevertheless, as we shall see below, a new interesting and powerful approach can be developed.

Let $\{\varphi_\lambda\}_{\lambda \in \mathbb{R}}$ a family of functions in $\Sigma_1$ such that $\lambda \mapsto \varphi_\lambda$ defines a differentiable curve in $X$. Then,

$$\int_X \varphi_\lambda(\xi) \frac{d}{d\lambda} \varphi_\lambda(\xi) \, d\xi = 0 \quad \forall \lambda \in \mathbb{R}. \quad (3.1)$$

Since the energy functional $E_\lambda$ is differentiable in $X$, it follows from the chain rule that $\lambda \mapsto E_\lambda(\varphi_\lambda)$ is also a differentiable function and

$$\frac{d}{d\lambda} E_\lambda(\varphi_\lambda) = \langle E'_\lambda(\varphi_\lambda) \mid \frac{d}{d\lambda} \varphi_\lambda \rangle + \frac{1}{2} \int_X |\varphi_\lambda(\xi)|^4 d\xi, \quad (3.2)$$

where $\langle \cdot \mid \cdot \rangle$ denotes the duality product between $X$ and its dual $X^*$. From now on we assume that, for any $\lambda \in \mathbb{R}$, one can choose a real ground state $\psi_\lambda \in \mathcal{G}_\lambda$ such that $\lambda \mapsto \psi_\lambda$ defines a differentiable curve in $X$. In this case,

$$E_{\min}(\lambda) = \int_X |\psi_\lambda'(\xi)|^2 d\xi + \int_X \xi^2 |\psi_\lambda(\xi)|^2 d\xi + \frac{\lambda}{2} \int_X |\psi_\lambda(\xi)|^4 d\xi - V_0 \int_X \cos^2(\alpha \xi) |\psi_\lambda(\xi)|^2 d\xi$$

is differentiable as a function of $\lambda$ and, as $E'_\lambda(\psi_\lambda) = \mu \psi_\lambda$, we get from (3.1) and (3.2)

$$\frac{d}{d\lambda} E_{\min}(\lambda) = \frac{1}{2} \int_X |\psi_\lambda(\xi)|^4 d\xi = \frac{1}{2}\|\psi_\lambda\|_4^4 \quad (3.3)$$

and we have the formula

$$E_{\min}(\lambda) = E_{\min}(0) + \frac{1}{2} \int_0^\lambda \|\psi_\lambda\|_4^4 \, ds. \quad (3.4)$$

On the other hand, if we denote $\mu_{\min}(\lambda) = \mu(\psi_\lambda)$, it follows from (2.21) that

$$\frac{d}{d\lambda} \mu_{\min}(\lambda) = \|\psi_\lambda\|_4^4 + \frac{\lambda}{2} \frac{d}{d\lambda} \|\psi_\lambda\|_4^4, \quad (3.5)$$

and we get by integration on $\lambda$ the formula

$$\mu_{\min}(\lambda) = \mu_{\min}(0) + \frac{1}{2} \left( \lambda \|\psi_\lambda\|_4^4 + \int_0^\lambda \|\psi_\lambda\|_4^4 \, ds \right). \quad (3.6)$$

Notice that $E_{\min}(0) = \mu_{\min}(0)$ and, in the case $V_0 = 0$, we have from Remark 2.3 that $E_{\min}(0) = \mu_{\min}(0) = 1$. 

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Remark 3.1: Formulae (3.4) and (3.6) express the minimal energy $E_{\text{min}}$ and the corresponding chemical potential $\mu_{\text{min}}$ as functions of $\lambda$, which also depend explicitly on the $L^4$-norm of unknown ground states. However, by eliminating this explicit dependence, we can obtain an exact formula relating these two quantities. More precisely, by using (3.3), we can rewrite (3.5) as

$$\frac{d}{d\lambda}\mu_{\text{min}}(\lambda) = 2\frac{d}{d\lambda}E_{\text{min}}(\lambda) + \lambda\frac{d^2}{d\lambda^2}E_{\text{min}}(\lambda),$$

from which we get easily

$$\frac{d}{d\lambda}\left(E_{\text{min}}(\lambda) - \frac{1}{\lambda}\int_{0}^{\lambda} \mu_{\text{min}}(s)\,ds\right) = 0.$$

Hence, there exists a constant $C$ such that

$$E_{\text{min}}(\lambda) - \frac{1}{\lambda}\int_{0}^{\lambda} \mu_{\text{min}}(s)\,ds = C, \quad \forall \lambda \in \mathbb{R}.$$

Since $E_{\text{min}}(0) = \mu_{\text{min}}(0)$, it follows that $C = 0$ and we have the identity

$$E_{\text{min}}(\lambda) = \frac{1}{\lambda}\int_{0}^{\lambda} \mu_{\text{min}}(s)\,ds, \quad \forall \lambda \in \mathbb{R}. \quad (3.7)$$

The formulæ (3.4) and (3.6) can be used to obtain explicit approximate functions depending on $\lambda$ for the minimal energy $E_{\text{min}}$ and the corresponding chemical potential $\mu_{\text{min}}$, by choosing appropriate trial functions that generate differentiable curves in $\Sigma_1$.

Motivated by inequality (2.2), we consider the trial functions $\phi_\kappa \in \Sigma_1$ defined as

$$\phi_\kappa(\xi) = \sqrt{\frac{2\kappa}{\pi}} \exp(-\kappa\xi^2).$$

A direct calculation shows that the function $\kappa \mapsto E_\lambda(\phi_\kappa)$ is given by

$$E_\lambda(\phi_\kappa) = \kappa + \frac{1}{4\kappa} + \frac{\lambda\sqrt{\kappa}}{2\sqrt{\pi}} - \frac{V_0}{2} \left(1 - \exp\left(-\frac{\alpha^2}{2\kappa}\right)\right).$$

We can show that, for all $\lambda \in \mathbb{R}$, the value of $\kappa$ that minimizes the function $\kappa \mapsto E_\lambda(\phi_\kappa)$ is given by the largest solution (in fact, unique solution if $\alpha^2V_0 < 1$) $\kappa(\lambda)$ of the following transcendental equation

$$\kappa^{3/2} \left(\kappa^{1/2} + \frac{\lambda}{4\sqrt{\pi}}\right) + \frac{V_0\alpha^2}{4} \exp\left(-\frac{\alpha^2}{2\kappa}\right) = \frac{1}{4}. \quad (3.8)$$

In order to obtain explicit approximate formulæ for the minimal energy $E_{\text{min}}(\lambda)$ and the corresponding chemical potential $\mu_{\text{min}}(\lambda)$, we introduce the functions

$$E_{\text{app}}(\lambda) = E_{\text{min}}(0) + \frac{1}{2} \int_{0}^{\lambda} \|\varphi_s\|_4^4\,ds,$$

$$\mu_{\text{app}}(\lambda) = \mu_{\text{min}}(0) + \frac{1}{2} \left(\lambda\|\varphi_\lambda\|_4^4 + \int_{0}^{\lambda} \|\varphi_s\|_4^4\,ds\right),$$

where, in this case,

$$\varphi_\lambda(\xi) = \phi_{\kappa(\lambda)}(\xi) = \left(\frac{2\kappa(\lambda)}{\pi}\right)^{1/4} \exp(-\kappa(\lambda)\xi^2).$$
By a straightforward calculation we get

\[
\begin{align*}
E_{\text{app}}(\lambda) &= E_{\text{min}}(0) + \frac{1}{2\sqrt{\pi}} \int_0^\lambda \sqrt{\kappa(s)} \, ds, \\
\mu_{\text{app}}(\lambda) &= \mu_{\text{min}}(0) + \frac{1}{2\sqrt{\pi}} \left( \lambda \sqrt{\kappa(\lambda)} + \int_0^\lambda \sqrt{\kappa(s)} \, ds \right). \quad (3.9)
\end{align*}
\]

Moreover, by arguing as in Remark 3.1, we can show that

\[E_{\text{app}}(\lambda) = \frac{1}{\lambda} \int_0^\lambda \mu_{\text{app}}(s) \, ds.\]

From the above identity we can easily relate the Taylor coefficients \(E_n\) of \(E_{\text{app}}\) with the ones of \(\mu_{\text{app}}\). In fact, we have \(\mu_n = (n+1)E_n\) for all \(n \in \mathbb{N}\).

A relatively simple situation is the one at which the optical lattice potential is not present \((V_0 = 0)\). In this case, (3.8) has a unique solution and if we define \(\sigma(\lambda) := \sqrt{\kappa(\lambda)}\), the equation (3.8) with \(V_0 = 0\) can be written as

\[\sigma^4 + \frac{\lambda}{4\sqrt{\pi}} \sigma^3 = \frac{1}{4}, \quad \forall \lambda \in \mathbb{R}. \quad (3.10)\]

By differentiating this equation implicitly with respect to \(\lambda\), we get easily the following properties of the function \(\sigma(\lambda)\):

**Lemma 3.2:** The function \(\sigma : \mathbb{R} \to \mathbb{R}\) is \(C^\infty\), positive, strictly decreasing, convex and satisfies the following properties:

\[\sigma(0) = \sqrt{2}/2; \quad \lim_{\lambda \to -\infty} \sigma(\lambda) = +\infty \quad \text{and} \quad \lim_{\lambda \to +\infty} \sigma(\lambda) = 0.\]

More precisely,

\[\sigma(\lambda) \sim -\frac{\lambda}{4\sqrt{\pi}} \quad \text{as} \quad \lambda \to -\infty \quad \text{and} \quad \sigma(\lambda) \sim \left(\frac{\sqrt{\pi}}{\lambda}\right)^{1/3} \quad \text{as} \quad \lambda \to +\infty. \]

As consequence of the previous lemma, we can show that \(E_{\text{app}}(\lambda)\) satisfies the general properties of \(E_{\text{min}}(\lambda)\) as reported in Proposition 2.11. More precisely,

**Corollary 3.3:** The functions \(E_{\text{app}}(\lambda)\) and \(\mu_{\text{app}}(\lambda)\) defined in (3.9) are \(C^\infty\), strictly increasing and concave. Moreover

\[
\begin{align*}
E_{\text{app}}(\lambda) \quad \text{and} \quad \mu_{\text{app}}(\lambda) \quad \text{are} \quad O(-\lambda^2) \quad \text{as} \quad \lambda \to -\infty, \\
E_{\text{app}}(\lambda) \quad \text{and} \quad \mu_{\text{app}}(\lambda) \quad \text{are} \quad O(\lambda^{2/3}) \quad \text{as} \quad \lambda \to +\infty. \quad \square
\end{align*}
\]

By implicit differentiation the Eq. (3.10) on \(\lambda\), we get

\[
\begin{align*}
\sigma(0) &= \frac{\sqrt{2}}{2}, \quad \sigma'(0) = -\frac{1}{16\sqrt{\pi}}, \quad \sigma''(0) = \frac{3}{128\pi\sqrt{2}}, \quad \sigma'''(0) = -\frac{3}{512\pi\sqrt{\pi}}, \quad \sigma^{(4)}(0) = \frac{45}{16384\pi^2\sqrt{2}}, \ldots \quad (3.11)
\end{align*}
\]

Then, for \(\lambda\) small enough, we can consider the approximation

\[\sigma(\lambda) \approx \frac{\sqrt{2}}{2} - \frac{\lambda}{16\sqrt{\pi}} + \frac{3\lambda^2}{256\pi\sqrt{2}} - \frac{\lambda^3}{1024\pi\sqrt{\pi}} + \frac{45\lambda^4}{393216\pi^2\sqrt{2}}.\]

By substituting the above approximation in (3.9) we have, for \(\lambda\) small enough,

\[
\begin{align*}
E_{\text{app}}(\lambda) &\approx 1 + \frac{\lambda}{2\sqrt{2\pi}} - \frac{\lambda^2}{64\pi} + \frac{\lambda^3}{512\pi\sqrt{2\pi}} - \frac{\lambda^4}{8192\pi^2} + \frac{9\lambda^5}{786432\pi^2\sqrt{2}}, \\
\mu_{\text{app}}(\lambda) &\approx 1 + \frac{\lambda}{\sqrt{2\pi}} - \frac{3\lambda^2}{64\pi} + \frac{4\lambda^3}{512\pi\sqrt{2\pi}} - \frac{5\lambda^4}{8192\pi^2} + \frac{54\lambda^5}{786432\pi^2\sqrt{2}}. \quad (3.12)
\end{align*}
\]
Remark 3.4: The above results are valid for attractive ($\lambda < 0$) as well as for repulsive ($\lambda > 0$) interatomic interaction strengths. By using (3.3), (3.5) and (2.7) we can show that $E'_{\min}(0) = E'_{\app}(0) = 1/2\sqrt{2\pi}$, which implies that the formulæ given by (3.12) coincide with $E_{\min}(\lambda)$ and $\mu_{\min}(\lambda)$, respectively, up to first order terms in $\lambda$. Notice that, up to second order terms in $\lambda$, the approximate chemical potential $\mu_{\app}$ can be written as

$$\mu_{\app}(\lambda) \approx 1 + \frac{\lambda}{\sqrt{2\pi}} - \varepsilon \lambda^2, \quad (3.13)$$

where $\varepsilon = \frac{3}{64\pi} \approx 0.0149207$.

It is noteworthy to compare Eq. (3.13) with the one obtained in [12] by a perturbative method. Considering that our dimensionless parameters are in fact twice the ones used there, both formulæ (3.13) and Eq. (31) in [12] coincide, except for the values of $\varepsilon$: $\varepsilon = 0.0149207$ and $\varepsilon = 0.016553$ respectively, which are also very close to each other. Moreover, Fig. 1 displays a comparison between the values of $\mu_{\app}$ from Eq. (3.12) (solid line) and those obtained by perturbation theory (dashed line) [12]. Also and for sake of comparison, the numerical evaluation of Eq. (1.2) is shown by full stars. From the figure is observed that Eq. (3.12) increases the accuracy of the solution with respect to the perturbative method. In the scale of the figure no significant differences are observed between the numerical solutions and the calculated values using Eq. (3.12) for the interval range $|\lambda| < 8$. Nevertheless the perturbation method gives a large error for $|\lambda| > 6$.

![Fig.1: Dimensionless chemical potential $\mu = 2\mu_0/\hbar \omega$ as a function of $\lambda = 2\lambda_{1D}/(\hbar \omega)$. Solid line: calculation following Eq. (3.12). Dashed line: perturbation theory from Ref. [12]. Stars: numerical evaluation of Eq. (1.2). Furthermore, in [13] a closed expression for the order parameter is given by (in the case $V_0 = 0$)

$$\psi(\xi) = \frac{1}{\sqrt{\pi}} \exp\left(\frac{-\xi^2}{2}\right) \left[ 1 + \frac{\lambda}{\sqrt{2\pi}} \int_{-1}^{\sqrt{7/2}} \left[ \exp\left(\frac{-\xi^2 (1 - z^2) z - z^2}{1 - z^2}\right) - z \right] dz \right].$$

In our approach, we propose the function of $\Sigma_1$:

$$\varphi_{\lambda}(\xi) = \left( \frac{2\kappa(\lambda)}{\pi} \right)^{1/4} \exp(-\kappa(\lambda)\xi^2), \quad (3.14)$$

where $\kappa(\lambda)$ is the unique root of the equation (3.8) with $V_0 = 0$. Recalling that $\kappa(\lambda) = \sigma(\lambda)^2$, it follows
from (3.11) that, for λ small enough,

\[
\begin{cases}
\kappa(\lambda) \approx \frac{1}{2} - \frac{1}{16} \sqrt{\frac{2}{\pi}} \lambda + \frac{1}{128\pi} \lambda^2,

\sqrt{\kappa(\lambda)} \approx \frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{32\sqrt{\pi}} \lambda + \frac{\sqrt{2}}{1024\pi \sqrt{2}} \lambda^2,
\end{cases}
\]

and (3.14) can be approximated up to second order terms in λ by

\[
\tilde{\varphi}_{\text{app}}(\xi) = \frac{1}{\sqrt{\pi}} \exp \left( \frac{-\xi^2}{2} \right) \left[ 1 - \frac{\sqrt{2}\lambda}{32\sqrt{\pi}} + \frac{\lambda^2}{1024\pi} \right] \exp \left[ \left( \frac{\lambda}{8\sqrt{2\pi}} - \frac{\lambda^2}{128\pi} \right) \xi^2 \right].
\]  

(3.15)

It is also interesting to notice that, for all λ ∈ ℝ,

\[
\begin{cases}
\frac{1}{2} - \frac{\lambda}{8\sqrt{2\pi}} + \frac{\lambda^2}{128\pi} \geq \frac{1}{4},

1 - \frac{\sqrt{2}\lambda}{32\sqrt{\pi}} + \frac{\lambda^2}{1024\pi} \geq \frac{1}{2},
\end{cases}
\]

which implies that \(\tilde{\varphi}_{\text{app}}\) is a positive function of \(X\) for any \(\lambda \in \mathbb{R}\).

**Remark 3.5:** It follows from the properties stated in Lemma 3.2 and the Lebesgue Theorem that

\[
\lim_{\lambda \to 0} \|\varphi_\lambda - \varphi_0\|_X = \lim_{\lambda \to 0} \|\tilde{\varphi}_\lambda - \varphi_0\|_X = 0,
\]

where (always assuming that \(V_0 = 0\)) \(\varphi_0, \varphi_\lambda\) and \(\tilde{\varphi}_\lambda\) are given by (2.7), (3.14) and (3.15), respectively. Therefore, from Theorem 2.10, for λ small enough and up to a change of phase, the unique solutions of Eq. (1.3) with initial data \(\varphi_\lambda\) and \(\tilde{\varphi}_\lambda\) respectively, remain close (in the sense Definition 2.8) to \(u(\tau, \xi) := e^{-i\tau} \varphi_0(\xi)\) in \(X\), for all time \(\tau \in \mathbb{R}\).

4. Conclusions

Our main results in the first part of this work concern qualitative properties of the minimal energy solutions of 1D GP equation with cubic nonlinearity in a harmonically confined periodical potential. We prove the existence of ground states for any λ and \(V_0\) (Theorem 2.4). Regardless the value of the laser intensity \(V_0\), such ground states have a Gaussian-like exponential asymptotic behavior, as was pointed out in Theorem 2.2. For \(\lambda > 0\) (repulsive interatomic forces), we prove that there exists a unique positive and symmetric ground state \(\psi_{\text{min}}\), which is decreasing for \(\xi > 0\) if \(V_0 = 0\), and that any other solution of the problem differs from it in a phase factor (Theorem 2.5). We also prove that, independently of the value of \(V_0\) and for any \(\lambda \in \mathbb{R}\), the set \(G_\lambda\) of ground states is orbitally stable (Theorem 2.10). An important consequence of this fact is the stability of those physical magnitudes, which are described by operators defined in the Hilbert space \(X\) (see (2.4)). This result is particularly related to the superfluidity properties, among other physical phenomena, of the harmonically confined condensates loaded in optical lattices [22].

In the second part (Section 3) we present a new and simple method to construct formulæ (see (3.4) and (3.6)) that allows to approximate the minimal energy, the corresponding chemical potential as well as the ground state. The functions described by these formulæ (see (3.9)) preserve some global properties of \(E_{\text{min}}(\lambda)\) and \(\mu_{\text{min}}(\lambda)\), as pointed out by Proposition 2.11 and Corollary 3.3. In the case \(V_0 = 0\), we obtain approximations for \(E_{\text{min}}(\lambda)\) and \(\mu_{\text{min}}(\lambda)\) as Taylor polynomials of \(E_{\text{app}}(\lambda)\) and \(\mu_{\text{app}}(\lambda)\) respectively (see (3.12)).
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