On general weighted extropy of ranked set sampling

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ABSTRACT
In the past few years, considerable attention has been given to the extropy measure. The extropy and weighted extropy of ranked set sampling were studied by several authors. The general weighted extropy and some results related to it are introduced in this paper. We provide general weighted extropy of ranked set sampling. We also studied characterization results, stochastic comparison and monotone properties of general weighted extropy.

ARTICLE HISTORY
Received 18 July 2022
Accepted 6 February 2023

KEYWORDS
Extropy; general weighted extropy; ranked set sampling; stochastic order; weighted extropy.

2020 MATHEMATICS SUBJECT CLASSIFICATION
62B10, 62D05

1. Introduction
McIntyre (1952) introduced ranked set sampling (RSS) to estimate mean pasture yields. The RSS is a better sampling strategy for estimating a population mean than simple random sampling (SRS).

Let $X$ have the probability density function (pdf) $f$, the cumulative distribution function (cdf) $F$, and the survival function (sf) $\bar{F} = 1 - F$. Assume that $X_{SRS} = \{X_i : i = 1, \ldots, n\}$ denotes a simple random sample of size $n$ from $X$. Here we first explain the one-cycle RSS: randomly select $n^2$ units from $X$, then these units are randomly allocated into $n$ sets, each of size $n$. Then rank those $n$ units in each set with respect to the variable of interest. From the first set, select the smallest ranked unit; then from the second set select the second smallest ranked unit; continue the process until the $n$th smallest ranked unit is selected from the last set. Now repeat the entire sampling procedure $m$ times, to obtain a sample of size $mn$. The obtained sample is called the RSS from the underlying distribution $F$. We assume $m = 1$ throughout the manuscript without loss of generality. Let $X_{RSS}^{(n)} = \{X_{(i:n)} : i = 1, \ldots, n\}$ denotes the ranked-set sample, where $X_{(i:n)}$ represent the $i$-th order statistics from the $i$-th sample with sample size $n$. For more on RSS see Wolfe (2004), Chen, Bai and Sinha (2004), Al-Nasser (2007), Al-Saleh and Diab (2009), Raqab and Qiu (2019).

Entropy was first introduced by Shannon (1948) and is relevant to many disciplines, including information theory, physics, probability and statistics, economics, communication theory, etc. The average level of uncertainty related to the results of the random experiment is measured by entropy.

Entropy measures the average level of uncertainty related to the results of the random experiment. The differential form of Shannon entropy is defined as:
This measure is used in many different applications in order statistics and record statistics; see, for instance, Baratpour, Ahmadi, and Arghami (2007), Raqab and Awad (2000, 2001), Zarezadeh and Asadi (2010), Abo-Eleneen (2011), Qiu and Jia (2018a, 2018b) and Tahmasebi, Jafari, and Eksanarzadeh (2016).

Recently, an alternative measure of uncertainty called extropy has gained importance. Lad, Sanfilippo, and Agro (2015) defined the compliment dual of the Shannon entropy called extropy as

\[ J(X) = -\frac{1}{2} \int_{-\infty}^{\infty} f^2(x)dx = -\frac{1}{2} E(f(X)). \]  

Qiu (2017) studied the characterization results, lower bounds, monotone properties, and statistical applications concerning the extropy of order statistics and record values. Balakrishnan, Buono, and Longobardi (2020) and Bansal and Gupta (2021) independently introduced the weighted extropy as

\[ J^w(X) = -\frac{1}{2} \int_{-\infty}^{\infty} xf^2(x)dx \]  

Balakrishnan, Buono, and Longobardi (2020) studied characterization results and bounds for weighted versions of extropy, residual extropy, past extropy, bivariate extropy and bivariate weighted extropy, whereas Bansal and Gupta (2021) discussed the results for weighted extropy and weighted residual extropy for order statistics and k-record values. Here we introduce the general weighted extropy (GWE) with weight \( w_1(x) \geq 0 \) as

\[ J^{w_1}(X) = -\frac{1}{2} \int_{-\infty}^{\infty} w_1(x)f^2(x)dx = -\frac{1}{2} E(\Lambda^{w_1}_X(U)), \]  

where \( \Lambda^{w_1}_X(u) = w_1(F^{-1}(u))f(F^{-1}(u)) \) and \( U \) is uniformly distributed random variable on \((0,1)\), that is, \( U \sim \text{Uniform}(0,1) \).

Bansal and Gupta (2021) used the following example in which the extropies of random variables \( X \) and \( Y \) are the same but weighted extropies are different. We find that general weighted extropies \( J^{w_1}(X) \) and \( J^{w_1}(Y) \) are also different when \( w_1(x) = x^m, \ m > 0, \ x > 0 \). Let \( X \) and \( Y \) be two random variables with pdf’s

\[
\begin{align*}
  f_X(x) &= \begin{cases} 
  2x, & 0 < x < 1 \\
  0, & \text{otherwise}
  \end{cases} \\
  f_Y(x) &= \begin{cases} 
  2(1-x), & 0 < x < 1 \\
  0, & \text{otherwise}
  \end{cases}
\end{align*}
\]

We get \( J(X) = J(Y) = -2/3 \) but \( J^X(X) = -1/2 \) and \( J^Y(Y) = -1/6 \). Let us consider \( w_1(x) = x^m, \ m > 0, \ x > 0 \) then

\[
\begin{align*}
  J^{w_1}(X) &= -\frac{2}{m+3} \\
  J^{w_1}(Y) &= -2 \left( \frac{1}{m+1} - \frac{2}{m+2} + \frac{1}{m+3} \right)
\end{align*}
\]

As a result, while extropies in this example are the same, weighted extropies and general weighted extropies are different. Therefore, weighted extropies and general weighted extropies
can be used as uncertainty measures. This shift-dependent measure takes into consideration the values of the random variable, unlike the extropy defined in (1.1).

Qiu and Raqab (2022) have given a representation for the weighted extropy of ranked set sampling in terms of quantile and density-quantile functions. Then provided some related results including monotone properties, stochastic orders, characterizations, and sharp bounds. Moreover, they have shown how the weighted extropy of ranked set sampling compares with its counterpart of simple random sampling.

In this article, we also study the monotone and stochastic properties of general weighted extropy of RSS data. Stochastic comparison results are obtained by taking different weights for the extropy of RSS and SRS data is provided. Some characterization results are obtained. We also study the monotone properties for the general weighted extropy of RSS data. These results generalize some of the results available in the literature.

2. Some results on GWE and related properties

Before providing results, let us review definitions from the literature (see, Shaked and Shanthikumar (2007), Sengupta and Nanda (1999) and Barlow and Proschan (1981)) of some useful terminology.

**Definition 1.** Let a random variable \( X \) have pdf \( f(x) \), cdf \( F(x) \) and sf \( \bar{F}(x) = 1 - F(x) \). Let \( l_X = \inf \{ x \in \mathbb{R} : F(x) > 0 \} \), \( u_X = \sup \{ x \in \mathbb{R} : F(x) < 1 \} \) and \( S_X = (l_X, u_X) \), where \(-\infty \leq l_X \leq u_X \leq \infty\).

(i) \( X \) is said to be log-concave (log-convex) if \( \{ x \in \mathbb{R} : f(x) > 0 \} = S_X \) and \( \ln(f(x)) \) is concave (convex) on \( S_X \).

(ii) \( X \) is said to have increasing (decreasing) failure rate IFR (DFR) if \( \bar{F}(x) \) is log-concave (log-convex) on \( S_X \).

(iii) \( X \) is said to have decreasing (increasing) reverse failure rate DRFR (IRFR) if \( F(x) \) is log-concave (log-convex) on \( S_X \).

(iv) \( X \) is said to have decreasing (increasing) mean residual life DMRL (IMRL) if \( \int_{u_X}^{x} \bar{F}(t)dt \) is log-concave (log-convex) on \( S_X \).

(v) \( X \) is said to have increasing (decreasing) mean inactivity time (IMIT (DMIT)) if \( \int_{l_X}^{x} F(t)dt \) is log-concave (log-convex) on \( S_X \).

**Definition 2.** Let \( X \) be a random variable with pdf \( f(x) \), cdf \( F(x) \) and sf \( \bar{F}(x) = 1 - F(x) \). Let \( l_X = \inf \{ x \in \mathbb{R} : F(x) > 0 \} \), \( u_X = \sup \{ x \in \mathbb{R} : F(x) < 1 \} \) and \( S_X = (l_X, u_X) \). Similarly, let \( Y \) be a random variable with pdf \( g(x) \), cdf \( G(x) \) and sf \( \bar{G}(x) = 1 - G(x) \). Let \( l_Y = \inf \{ x \in \mathbb{R} : G(x) > 0 \} \), \( u_Y = \sup \{ x \in \mathbb{R} : G(x) < 1 \} \) and \( S_Y = (l_Y, u_Y) \). If \( l_X \geq 0 \) and \( l_Y \geq 0 \), then

(i) \( X \) is said to be smaller than \( Y \) in usual stochastic (st) ordering \( X \leq_{st} Y \) if \( \bar{F}(x) \leq \bar{G}(x) \), for every \(-\infty < x < \infty \).

(ii) \( X \) is said to be smaller than \( Y \) in the likelihood ratio (lr) ordering \( X \leq_{lr} Y \) if \( g(x)f(y) \leq f(x)g(y) \), whenever \(-\infty < x < y < \infty \).

(iv) \( X \) is said to be smaller than \( Y \) in the dispersive ordering \( X \leq_{disp} Y \) if \( G^{-1}F(x) - x \) is increasing in \( x \geq 0 \).

(v) \( X \) is said to be smaller than \( Y \) in the hazard rate ordering \( X \leq_{hr} Y \) if \( \frac{\bar{G}(x)}{F(x)} \) is increasing in \( x \in S_X \cap S_Y \).
**Theorem 2.1.** Let $X$ and $Y$ be non negative random variables with pdf’s $f$ and $g$, cdf’s $F$ and $G$, respectively having $u_X = u_Y < \infty$.

(a) If $w_1$ is increasing, $w_1(x) \geq w_2(x)$ and $X \leq_{disp} Y$, then $J^{w_1}(X) \leq J^{w_2}(Y)$.

(b) If $w_1$ is increasing, $w_1(x) \leq w_2(x)$ and $X \geq_{disp} Y$, then $J^{w_1}(X) \geq J^{w_2}(Y)$.

**Proof.** (a) Since $X \leq_{disp} Y$, therefore we have $f(F^{-1}(u)) \geq g(G^{-1}(u))$ for $u \in (0, 1)$. Then using Theorem 3.B.13(b) of Shaked and Shanthikumar (2007), $X \leq_{disp} Y$ implies that $X \geq_{st} Y$. Hence $F^{-1}(u) \geq G^{-1}(u)$ for all $u \in (0, 1)$. Since $w_1$ is increasing and $w_1(x) \geq w_2(x)$, then $w_1(F^{-1}(u)) \geq w_1(G^{-1}(u)) \geq w_2(G^{-1}(u))$. Hence

\[
A^{w_1}_X(u) = w_1(F^{-1}(u))f(F^{-1}(u)) \
\geq w_2(G^{-1}(u))g(G^{-1}(u)) \
= A^{w_2}_Y(u).
\] (2.1)

Now using (2.1),

\[
J^{w_1}(X) = -\frac{1}{2}E(A^{w_1}_X(U)) \
\leq -\frac{1}{2}E(A^{w_2}_Y(U)) \
= J^{w_2}(Y).
\]

(b) Proof is Similar to part (a). \qed

If we take $w_1(x) = w_2(x)$ in the above theorem, then we have the following corollary.

**Corollary 2.1.** Let $X$ and $Y$ be non negative random variables with pdf’s $f$ and $g$, cdf’s $F$ and $G$, respectively having $u_X = u_Y < \infty$. Let $w_1$ is increasing.

(a) If $X \leq_{disp} Y$, then $J^{w_1}(X) \leq J^{w_1}(Y)$.

(b) If $X \geq_{disp} Y$, then $J^{w_1}(X) \geq J^{w_1}(Y)$.

The pdf of $X_{(i:n)}$ is

\[
f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}F^{i-1}(x)\tilde{F}^{n-i}(x)f(x), \quad -\infty < x < \infty.
\]

Let

\[
\phi_{2i-1:2n-1}(u) = \frac{(2n-1)!}{(2i-1)!(2n-2i)!}u^{2i-2}(1-u)^{2n-2i}, \quad 0 < u < 1.
\]

Then $\phi_{2i-1:2n-1}(u)$ represents the pdf of a beta-distributed random variable with parameter $(2i-1)$, and $(2n-2i+1)$, we denote this random variable by $B_{2i-1:2n-1}$.

**Theorem 2.2.** Consider a random sample of size $n$ as $X_1, \ldots, X_n$ from a IRFR distribution $F$.

(a) If $w_1$ is increasing, then $J^{w_1}(X_{i:n})$ is decreasing in $i$ for fixed $n$, $1 \leq i \leq n$.

(b) If $w_1$ is decreasing, then $J^{w_1}(X_{i:n})$ is increasing in $n$ for fixed $i$, $1 \leq i \leq n$. 
Proof. (a) The general weighted extropy of $X_{i,n}$ is

$$J^{w_1}(X_{i,n}) = -\frac{1}{2} \int_{-\infty}^{\infty} w_1(x) f_{i,n}^2(x) dx$$

$$= -\frac{1}{2} \int_{-\infty}^{\infty} w_1(x) \frac{(n!)^2}{(i-1)!(n-i)!^2} F_i^{2i-2}(x) F_i^{2n-2i}(x) f_i^2(x) dx$$

$$= -\frac{n c_{i,n}}{2} \int_{0}^{1} w_1(F^{-1}(u)) f(F^{-1}(u)) \frac{(2n-1)!}{(2i-1)!(2n-2i)!} u^{2i-2}(1-u)^{2n-2i} du$$

$$= -\frac{n c_{i,n}}{2} \int_{0}^{1} \Lambda_{X_i}^{w_1}(u) \phi_{2i-1;2n-1}(u) du$$

$$= -\frac{c_{i,n}(2i-1)}{4} \int_{0}^{1} w_1(F^{-1}(u)) r(F^{-1}(u)) \phi_{2i;2n}(u) du$$

$$= -\frac{c_{i,n}(2i-1)}{4} E(M(B_{2i;2n})).$$

where the third step followed by taking transformation $F(x) = u$ and $dx = \frac{du}{f(F^{-1}(u))}$; also $c_{i,n} = \frac{(2i-1)!(2n-2i)!}{(n-i)!}$, $M(u) = w_1(F^{-1}(u)) r(F^{-1}(u))$ and $r(x) = f(x)/F(x)$ denote the reverse failure rate function of $X$. Now, it follows that

$$\frac{J^{w_1}(X_{i,n})}{J^{w_1}(X_{i+1,n})} = \frac{c_{i,n}}{c_{i+1,n}} \left( \frac{2i-1}{2i+1} \right) \frac{E(M(B_{2i;2n}))}{E(M(B_{2i+2;2n}))}$$

$$= \frac{i(2n-2i-1)}{(2i+1)(n-i)} \frac{E(M(B_{2i;2n}))}{E(M(B_{2i+2;2n}))}$$

$$\leq \frac{E(M(B_{2i;2n}))}{E(M(B_{2i+2;2n}))}.$$

Since $B_{2i;2n} \leq_{hr} B_{2i+2;2n}$, hence $B_{2i;2n} \leq_{st} B_{2i+2;2n}$, hence under the hypothesis of the theorem it follows that

$$E(M(B_{2i;2n})) \leq E(M(B_{2i+2;2n})).$$

which further implies that $J^{w_1}(X_{i,n}) \geq J^{w_1}(X_{i+1,n})$. This completes the proof of part (a).

(b) Proceeding in the same fashion as in part (a), under the assumption of part (b) result follows by observing that

$$\frac{J^{w_1}(X_{i,n})}{J^{w_1}(X_{i,n+1})} = \frac{(2n+1)(n-i+1)}{(n+1)(2n-2i+1)} \left( \frac{2i-1}{2i+1} \right) \frac{E(M(B_{2i;2n}))}{E(M(B_{2i+2;2n}))}$$

$$\geq \frac{E(M(B_{2i;2n}))}{E(M(B_{2i+2;2n}))}.$$

and $B_{2i;2n} \geq_{st} B_{2i;2n+2}$. \qed

3. General weighted extropies of RSS

Let $X$ be a random variable with finite mean $\mu$ and variance $\sigma^2$. For $X_{SRS} = \{X_i, i = 1, \ldots, n\}$, the joint pdf is $\prod_{i=1}^{n} f(x_i)$, as $X_i$'s, $i = 1, \ldots, n$ are independent and identically distributed.
(i.i.d.). Hence the general weighted extropy of $X_{\text{SRS}}^{(n)}$ can be defined as

$$J^{w_1}(X_{\text{SRS}}^{(n)}) = -\frac{1}{2} \prod_{i=1}^{n} \left( \int_{-\infty}^{\infty} w_1(x_i)f^2(x_i)dx_i \right)$$

$$= -\frac{1}{2} \left( -2J^{w_1}(X) \right)^n$$

$$= -\frac{1}{2} \left( E(\Lambda^{w_1}_X(U)) \right)^n$$  \hspace{1cm} (3.1)

Now, we can write the GWE of $X_{\text{RSS}}^{(n)}$ as

$$J^{w_1}(X_{\text{RSS}}^{(n)}) = -\frac{1}{2} \prod_{i=1}^{n} \left( -2J^{w_1}(X_{(i:n)}) \right)$$

$$= -\frac{1}{2} \prod_{i=1}^{n} \int_{-\infty}^{\infty} n^2 \left( \frac{n}{i-1} \right)^2 w_1(x)F^{2i-2}(x)F^{2n-2i}(x)f^2(x)dx$$

$$= -\frac{1}{2} \prod_{i=1}^{n} \int_{0}^{1} n^2 \left( \frac{n}{i-1} \right)^2 \Lambda^{w_1}_X(u)u^{2i-2}(1-u)^{2n-2i}du$$

$$= -\frac{Q_n}{2} \prod_{i=1}^{n} E(\Lambda^{w_1}_X(B_{2i-1:2n-1}))$$  \hspace{1cm} (3.2)

where

$$Q_n = n^n \prod_{i=1}^{n} c_{i,n},$$

$$c_{i,n} = \frac{(2i-2)(2n-2i)}{(2n-1)}$$

and $B_{2i-1:2n-1}$ is a beta distributed random variable with parameters $(2i - 1)$ and $(2n - 2i + 1)$. Equation (3.2) provides an expression in simplified form of the GWE of $X_{\text{RSS}}^{(n)}$. Now we provide some examples to illustrate the Equation (3.2).

**Example 3.1.** Let $U$ be a random variable with power distribution. The pdf and cdf of $U$ are, respectively, $f(x) = \theta x^{\theta-1}$ and $F(x) = x^\theta$, $0 < x < 1$, $\theta > 0$. Let $w_1(x) = x^m$, $x > 0$, $m > 0$, then it follows that

$$\Lambda^{w_1}_U(u) = w_1(F^{-1}(u))f(F^{-1}(u)) = \theta u^{m+\frac{\theta-1}{\theta}}$$

for $w_1(x) = x^m$. Then we have

$$J^{w_1}(U_{\text{RSS}}^{(n)})$$

$$= -\frac{Q_n}{2} \prod_{i=1}^{n} E(\Lambda^{w_1}_U(B_{2i-1:2n-1}))$$

$$= -\frac{Q_n}{2} \prod_{i=1}^{n} \int_{0}^{1} \Lambda^{w_1}_U(u) \frac{(2n-1)!}{(2i-2)! (2n-2i)!} u^{2i-2}(1-u)^{2n-2i}du$$

$$= -\frac{Q_n}{2} \prod_{i=1}^{n} E(\Lambda^{w_1}_U(B_{2i-1:2n-1}))$$

**Example 3.2.** Let $V$ be a random variable with beta distribution. The pdf and cdf of $V$ are, respectively, $f(v) = \frac{1}{B(\alpha, \beta)} v^{\alpha-1}(1-v)^{\beta-1}$ and $F(v) = I(v; \alpha, \beta)$, $0 < v < 1$, $\alpha, \beta > 0$. Let $w_2(v) = v^p$, $v > 0$, $p > 0$, then it follows that

$$\Lambda^{w_2}_V(u) = w_2(F^{-1}(u))f(F^{-1}(u)) = \frac{\Gamma(p+\alpha)}{\Gamma(p+\alpha+\beta)} u^p (1-u)^{\beta-1}$$

for $w_2(v) = v^p$. Then we have

$$J^{w_2}(V_{\text{RSS}}^{(n)})$$

$$= -\frac{Q_n}{2} \prod_{i=1}^{n} E(\Lambda^{w_2}_V(B_{2i-1:2n-1}))$$

$$= -\frac{Q_n}{2} \prod_{i=1}^{n} \int_{0}^{1} \Lambda^{w_2}_V(u) \frac{(2n-1)!}{(2i-2)! (2n-2i)!} u^{2i-2}(1-u)^{2n-2i}du$$

$$= -\frac{Q_n}{2} \prod_{i=1}^{n} E(\Lambda^{w_2}_V(B_{2i-1:2n-1}))$$
Example 3.2. Let $w = \Lambda^{\theta + \frac{1}{2}}\frac{u}{\theta \frac{1}{2}}(\beta_1 + \frac{1}{2}) = -\frac{u}{\theta \frac{1}{2}}(\beta_1 + \frac{1}{2})$. Let $w_1(x) = x^m$, $m > 0$, $x > 0$, then it follows that

$$\Lambda^{w_1}(u) = w_1(F^{-1}(u))f(F^{-1}(u)) = \frac{(-1)^m(1-u)(\ln(1-u))^m}{\lambda^{m-1}}, 0 < u < 1.$$ 

Then we have

$$J^{w_1}(Z^{(n)}_{RSS}) = -\frac{Q_n}{2} \prod_{i=1}^{n} E(\Lambda^{w_1}(B_{2i-1:2n-1}))$$

$$= -\frac{Q_n}{2} \prod_{i=1}^{n} \int_0^1 \Lambda^{w_1}(u) \frac{(2n-1)}{(2i-2)!(2n-2i)!} u^{2i-2}(1-u)^{2n-2i} du$$

$$= -\frac{Q_n}{2} \prod_{i=1}^{n} \int_0^1 \frac{(-1)^m(1-u)(\ln(1-u))^m}{\lambda^{m-1}} \frac{(2n-1)}{(2i-2)!(2n-2i)!} u^{2i-2}(1-u)^{2n-2i} du$$

$$= -\frac{Q_n}{2} \prod_{i=1}^{n} \frac{(-1)^m}{\lambda^{m-1}} \int_0^1 \frac{(-1)^m(1-u)(\ln(1-u))^m}{\lambda^{m-1}} \frac{(2n-1)}{(2i-2)!(2n-2i)!} u^{2i-2}(1-u)^{2n-2i+1} du.$$ 

Taking $u = 1 - e^{-x}$ in the above equation, we get

$$J^{w_1}(Z^{(n)}_{RSS})$$

$$= -\frac{Q_n}{2} \prod_{i=1}^{n} \frac{(-1)^m}{\lambda^{m-1}} \int_0^\infty \frac{(\ln(-x))^m}{\lambda^{m-1}} \frac{(2n-1)}{(2i-2)!(2n-2i)!} (1 - e^{-x})^{2i-2} e^{-x} dx$$

$$= -\frac{Q_n}{2} \prod_{i=1}^{n} \frac{1}{\lambda^{m-1}} \int_0^\infty x^m \frac{(2n-1)}{(2i-2)!(2n-2i)!} (1 - e^{-x})^{2i-2} e^{-x} dx$$

$$= -\frac{Q_n}{2} \frac{1}{\lambda^{m-1}} \prod_{i=1}^{n} \int_0^\infty x^m \frac{(2n-1)}{(2i-2)!(2n-2i)!} (1 - e^{-x})^{2i-2} e^{-x} dx$$

$$= -\frac{Q_n}{2} \frac{1}{\lambda^{m-1}} \prod_{i=1}^{n} \left[ \frac{(2n-2i+1)}{2n} \right] \int_0^\infty x^m \frac{(2n)!}{(2i-2)!(2n-2i+1)!}$$
(1 - e^{-x})^{2i-2} (e^{-x})^{2n-2i+2} dx] \]
\[= - \frac{Q_n(2n - 1)!!}{2^n+1} \frac{1}{\lambda^{n(m-1)}} \prod_{i=1}^{n} E(Z_{2i-1:2n}^m),
\]
where $Z_{2i-1:2n}$ is the $(2i - 1)$-th order statistics of a sample of size $2n$ from exponential distribution having pdf given by

\[\phi_{2i-1:2n} = \frac{(2n)!}{(2i - 2)!(2n - 2i + 1)!} (1 - e^{-x})^{2i-2} (e^{-x})^{2n-2i+2}, \ x \geq 0, \]
and $(2n - 1)!! = \prod_{i=1}^{n} (2n - 2i + 1) = \prod_{i=1}^{n} (2i - 1). \ 
\]

**Example 3.3.** Let $V$ be a Pareto random variable with cdf $F(x) = 1 - x^{-\alpha}, \ \alpha > 0, \ x > 1$. Let $w_1(x) = x^m, \ m > 0, \ x > 0$, then we get

\[\Lambda_V^{w_1}(u) = w_1(F^{-1}(u)) f(F^{-1}(u)) \]
\[= \alpha (1 - u)^{\frac{a-m+1}{a}}.
\]

The weighted extropy of $V^{(n)}_{\text{RSS}}$ is

\[J_{w_1}(V^{(n)}_{\text{RSS}}) = - \frac{Q_n}{2} \prod_{i=1}^{n} E(\Lambda_{V}^{w_1}(B_{2i-1:2n-1})) \]
\[= - \frac{Q_n}{2} \prod_{i=1}^{n} \int_0^1 \Lambda_{V}^{w_1}(u) \frac{(2n - 1)!}{(2i - 2)!(2n - 2i)!} u^{2i-2} (1 - u)^{2n-2i} du \]
\[= - \frac{Q_n}{2} \prod_{i=1}^{n} \int_0^1 \alpha (1 - u)^{\frac{a-m+1}{a}} \frac{(2n - 1)!}{(2i - 2)!(2n - 2i)!} u^{2i-2} (1 - u)^{2n-2i} du \]
\[= - \frac{Q_n}{2} \alpha \prod_{i=1}^{n} \int_0^1 \frac{(2n - 1)!}{(2i - 2)!(2n - 2i)!} u^{2i-2} (1 - u)^{2n-2i+\frac{a-m+1}{a}} du \]
\[= - \frac{Q_n}{2} \alpha \prod_{i=1}^{n} \left[ \frac{(2n - 1)!}{(2i - 2)!(2n - 2i)!} \frac{\Gamma(\frac{2na-2ia+2a-\alpha-m+1}{a})}{\Gamma(\frac{a+2na-\alpha-m+1}{a})} \right].
\]

\]

The following result gives the conditions under which the GWE will increase (decrease).

**Theorem 3.1.** Let $X$ be a non negative absolutely continuous random variable with pdf $f$ and cdf $F$. Assume $\phi(x)$ is an increasing function and \[\frac{w_1(\phi(x))}{\phi'(x)(x)} \leq (\geq) w_1(x) \text{ and } \phi(0) = 0. \text{ If } Z = \phi(X), \text{ then } J_{w_1}(X^{(n)}_{\text{RSS}}) \leq (\geq) J_{w_1}(Z^{(n)}_{\text{RSS}}).
\]

**Proof.** Let pdf and cdf of $Z$ be $h$ and $H$, respectively. Then

\[\Lambda_Z^{w_1}(u) = w_1(H^{-1}(u)) h(H^{-1}(u)) \]
\[= \frac{w_1(\phi(F^{-1}(u)))}{\phi'(F^{-1}(u))} f(F^{-1}(u)) \ \forall \ 0 < u < 1.
\]
Note that \( \phi(x) \geq \phi(0), \forall x \geq 0 \). Hence for \( 0 < u < 1 \), we have
\[
\Lambda_{\mathcal{Z}}^{w_1}(u) = \frac{w_1(\phi(F^{-1}(u)))}{\phi'(F^{-1}(u))} f(F^{-1}(u)) \leq w_1(F^{-1}(u))f(F^{-1}(u)) = \Lambda_X^{w_1}(u).
\]
Therefore, \( J_{w_1}(X_{RSS}^{(n)}) \leq J_{w_1}(Z_{RSS}^{(n)}) \) using Equation (3.2). Proof of other parts can be done in a similar fashion.

**Example 3.4.** Let \( Z \) have an exponential distribution with cdf \( F_Z(z) = 1 - e^{-\lambda z}, \lambda > 0, z \geq 0 \). Let \( w_1(x) = x > 0 \). Consider \( \phi(x) = e^x - 1, x \geq 0 \). Then \( \phi(Z) \) is Pareto distribution (see Qiu and Raqab (2022) Example 2.7) with survival function \( \bar{F}_{\phi(Z)}(x) = 1/(1+x)^2, x \geq 0 \). Note that
\[
\frac{w_1(\phi(x))}{\phi'(x)} = \frac{e^x - 1}{e^x} = 1 - e^{-x} \leq x = w_1(x).
\]
Hence using Theorem 3.1, \( J_{w_1}(Z_{RSS}^{(n)}) \leq J_{w_1}(\phi(Z)_{RSS}^{(n)}) \).

We now give a lower bound for the general weighted extropy of RSS data. This lower bound is dependent on the weighted extropy of the SRS data, as shown by the following result.

**Theorem 3.2.** Let \( X \) be an absolutely continuous random variable with pdf \( f \) and cdf \( F \). Then for \( n \geq 2, \)
\[
\frac{J_{w_1}(X_{RSS}^{(n)})}{J_{w_1}(X_{SRS}^{(n)})} \leq \frac{n^{2n}}{(n-1)^{(2n-1)(n-2)}} \prod_{i=2}^{n-1} \left( \left( \frac{n-1}{i-1} \right)^2 \frac{(i-1)^{2i-2}(n-i)^{2n-2i}}{2} \right).
\]

**Proof.** The proof is on similar lines as the proof of theorem 2.8 of Qiu and Raqab (2022).

**4. Characterization results**

**Theorem 4.1.** Let \( X \) be an absolutely continuous random variable with pdf \( f \) and cdf \( F \); and assume \( w_1(-x) = -w_1(x) \). Then \( X \) is a symmetric distributed random variable with mean 0 if and only if \( J_{w_1}(X_{RSS}^{(n)}) = 0 \) for all odd \( n \geq 1 \).

**Proof.** For sufficiency, suppose \( f(x) = f(-x) \) for all \( x \geq 0 \). Also since \( F^{-1}(u) = F^{-1}(1-u) \), \( f(F^{-1}(u)) = f(F^{-1}(1-u)) \) for all \( 0 < u < 1 \) and \( w_1(-x) = -w_1(x) \), which implies that
\[
\Lambda_X^{w_1}(u) = w_1(F^{-1}(u))f(F^{-1}(u)) = -w_1(F^{-1}(1-u))f(F^{-1}(1-u)) = -\Lambda_X^{w_1}(1-u)
\]
In a similar fashion as in Qiu and Raqab (2022), \( J_{w_1}(X_{RSS}^{(n)}) = -J_{w_1}(X_{RSS}^{(n)}) \). This completes the proof of sufficiency. For the necessity, since equation \( J_{w_1}(X_{RSS}^{(n)}) = 0 \) holds for all odd \( n \geq 1 \). For \( n = 1 \),
\[
J_{w_1}(X_{RSS}^{(1)}) = J_w(X) = 0.
\]
Now,

\[
J^{w_1}(X) = -\frac{1}{2} \int_{-\infty}^{\infty} w_1(x)f^2(x)dx \\
= -\frac{1}{2} \left( \int_{-\infty}^{0} w_1(x)f^2(x)dx + \int_{0}^{\infty} w_1(x)f^2(x)dx \right) \\
= -\frac{1}{2} \left( \int_{-\infty}^{0} w_1(x)f^2(x)dx + \int_{0}^{\infty} w_1(x)f^2(x)dx \right) \\
= -\frac{1}{2} \left( -\int_{0}^{\infty} w_1(x)f^2(-x)dx + \int_{0}^{\infty} w_1(x)f^2(x)dx \right) \\
= -\frac{1}{2} \left( -\int_{0}^{\infty} w_1(x)(f(x) + f(-x))(f(x) - f(-x))dx \right) \\
= 0,
\]

since \(w_1(x) > 0\), and \(f(x) = f(-x)\forall x \geq 0\). This provides proof of necessity. \(\square\)

5. Stochastic comparison

In the following result, we provide the conditions for comparing two RSS schemes under different weights.

**Theorem 5.1.** Let \(X\) and \(Y\) be non negative random variables with pdf’s \(f\) and \(g\), cdf’s \(F\) and \(G\), respectively having \(u_X = u_Y < \infty\).

(a) If \(w_1\) is increasing, \(w_1(x) \geq w_2(x)\) and \(X \leq_{\text{disp}} Y\), then \(J^{w_1}(X^{(n)}_{\text{RSS}}) \leq J^{w_2}(Y^{(n)}_{\text{RSS}})\).

(b) If \(w_1\) is increasing, \(w_1(x) \leq w_2(x)\) and \(X \geq_{\text{disp}} Y\), then \(J^{w_1}(X^{(n)}_{\text{RSS}}) \geq J^{w_2}(Y^{(n)}_{\text{RSS}})\).

**Proof.** (a) Since \(X \leq_{\text{disp}} Y\), therefore we have \(f(F^{-1}(u)) \geq g(G^{-1}(u))\) for all \(u \in (0, 1)\). Then using Theorem 3.B.13(b) of Shaked and Shanthikumar (2007), \(X \leq_{\text{disp}} Y\) implies that \(X \geq_{\text{st}} Y\). Hence \(F^{-1}(u) \geq G^{-1}(u)\forall u \in (0, 1)\). Since \(w_1\) is increasing and \(w_1(x) \geq w_2(x)\), then \(w_1(F^{-1}(u)) \geq w_1(G^{-1}(u)) \geq w_2(G^{-1}(u))\). Hence

\[
\Lambda^{w_1}_X(u) = w_1(F^{-1}(u))f(F^{-1}(u)) \\
\geq w_2(G^{-1}(u))g(G^{-1}(u)) \\
= \Lambda^{w_2}_Y(u) \quad (5.1)
\]

Now using (5.1),

\[
J^{w_1}(X^{(n)}_{\text{RSS}}) = -\frac{Qn}{2} \prod_{i=1}^{n} E(\Lambda^{w_1}_X(B_{2i-1:2n-1})) \\
\leq -\frac{Qn}{2} \prod_{i=1}^{n} E(\Lambda^{w_2}_Y(B_{2i-1:2n-1})) \\
= J^{w_2}(Y^{(n)}_{\text{RSS}})
\]

(b) Proof is similar to part (a). \(\square\)
If we take \( w_1(x) = w_2(x) \) in the above theorem, then The following corollary follows.

**Corollary 5.1.** Let \( X \) and \( Y \) be non negative random variables with pdf’s \( f \) and \( g \), cdf’s \( F \) and \( G \), respectively, having \( u_X = u_Y < \infty \); let \( w_1 \) be increasing. Then

(a) If \( X \leq_{\text{disp}} Y \), then \( J^{w_1}(X) \leq J^{w_1}(Y) \).
(b) If \( X \geq_{\text{disp}} Y \), then \( J^{w_1}(X) \geq J^{w_1}(Y) \).

**Remark 5.1.** For \( w_1(x) = x \), the result in Corollary 5.1 was proved by Qiu and Raqab (2022).

**Lemma 5.1.** [Ahmed et al. (1986); also see Qiu and Raqab (2022), lemma 4.3] Let \( X \) and \( Y \) be non negative random variables with pdf’s \( f \) and \( g \), respectively, satisfying \( f(0) \geq g(0) > 0 \). If \( X \leq_{su} Y \) (or \( X \leq_{s} Y \)), then \( X \leq_{\text{disp}} Y \).

One may refer Shaked and Shanthikumar (2007) for details of convex transform order (\( \leq_c \)), star order (\( \leq_s \)), super additive order (\( \leq_{su} \)), and dispersive order (\( \leq_{\text{disp}} \)). In view of Theorem 5.1 and Lemma 5.1, the following result is obtained.

**Theorem 5.2.** Let \( X \) and \( Y \) be non negative random variables with pdf’s \( f \) and \( g \), cdf’s \( F \) and \( G \), respectively, having \( u_X = u_Y < \infty \).

(a) If \( w_1 \) is increasing, \( w_1(x) \geq w_2(x) \) and \( X \leq_{su} Y \) (or \( X \leq_{s} Y \) or \( X \leq_{c} Y \)), then \( J^{w_1}(X) \leq J^{w_2}(Y) \).
(b) If \( w_1 \) is increasing, \( w_1(x) \leq w_2(x) \) and \( X \geq_{su} Y \) (or \( X \geq_{s} Y \) or \( X \geq_{c} Y \)), then \( J^{w_1}(X) \geq J^{w_2}(Y) \).

If we take \( w_1(x) = w_2(x) \) in the above theorem, then we have the following corollary.

**Corollary 5.2.** Let \( X \) and \( Y \) be non negative random variables with pdf’s \( f \) and \( g \), cdf’s \( F \) and \( G \), respectively, having \( u_X = u_Y < \infty \) and \( w_1 \) is increasing.

(a) If \( X \leq_{su} Y \) (or \( X \leq_{s} Y \) or \( X \leq_{c} Y \)), then \( J^{w_1}(X) \leq J^{w_2}(Y) \).
(b) If \( X \geq_{su} Y \) (or \( X \geq_{s} Y \) or \( X \geq_{c} Y \)), then \( J^{w_1}(X) \geq J^{w_2}(Y) \).

**Remark 5.2.** For \( w_1(x) = x \), the result in Corollary 5.2 was proved by Qiu and Raqab (2022).

One may ask that whether the condition \( X \leq_{\text{disp}} Y \) in Theorem 5.1 may be relaxed by \( J^{w_1}(X) \leq J^{w_2}(Y) \). The following result gives a positive answer to this assertion.

**Theorem 5.3.** Let \( X \) and \( Y \) be non negative random variables with pdf’s \( f \) and \( g \), cdf’s \( F \) and \( G \), respectively. Let \( \Delta(u) = w_1(F^{-1}(u)f(F^{-1}(u)) - w_2(G^{-1}(u)g(G^{-1}(u)) \),

\[ A_1 = \{0 \leq u \leq 1: \Delta(u) > 0\}, \quad A_2 = \{0 \leq u \leq 1: \Delta(u) < 0\} \]

If \( \inf_{A_1} \phi_{2i-1:2n-1}(u) \geq \sup_{A_2} \phi_{2i-1:2n-2}(u) \), and if \( J^{w_1}(X) \leq J^{w_2}(Y) \), then \( J^{w_1}(X) \leq J^{w_2}(Y) \).
Proof. Since \( J^w_1(X) \leq J^w_2(Y) \), we have
\[
\int_0^1 \Delta(u) du = \int_0^1 \left[ w_1(F^{-1}(u)) f(F^{-1}(u)) - w_2(G^{-1}(u)) g(G^{-1}(u)) \right] du \geq 0. \tag{5.2}
\]
Now for \( 1 \leq i \leq n \), we have
\[
\int_0^1 w_1(F^{-1}(u)) f(F^{-1}(u)) \phi_{2i-1:2n-2i}(u) du
- \int_0^1 w_2(G^{-1}(u)) g(G^{-1}(u)) \phi_{2i-1:2n-2i}(u) du
= \int_0^1 \Delta(u) \phi_{2i-1:2n-2i}(u) du
= \int_{A_1} \Delta(u) \phi_{2i-1:2n-2i}(u) du + \int_{A_2} \Delta(u) \phi_{2i-1:2n-2i}(u) du
\geq \inf_{A_1} \phi_{2i-1:2n-2i}(u) \int_{A_1} \Delta(u) du + \sup_{A_2} \phi_{2i-1:2n-2i}(u) \int_{A_2} \Delta(u) du
\geq \sup_{A_2} \phi_{2i-1:2n-2i}(u) \int_{A_1} \Delta(u) du
\geq 0,
\]
where the inequalities follow by using the assumption that
\[
\inf_{A_1} \phi_{2i-1:2n-2i}(u) \geq \sup_{A_2} \phi_{2i-1:2n-2i}(u)
\]
and the Equation (5.2). Now using Equation (3.2) and above inequality result follow. \(\square\)

If we take \( w_1(x) = w_2(x) \) in the above theorem, then we have the following corollary.

**Corollary 5.3.** Let \( X \) and \( Y \) be non negative random variables with pdf’s \( f \) and \( g \), cdf’s \( F \) and \( G \). Let \( \Delta(u) = w_1(F^{-1}(u)) f(F^{-1}(u)) - w_1(G^{-1}(u)) g(G^{-1}(u)) \),
\[
A_1 = \{ 0 \leq u \leq 1 | \Delta(u) > 0 \}, \quad A_2 = \{ 0 \leq u \leq 1 | \Delta(u) < 0 \}.
\]
If \( \inf_{A_1} \phi_{2i-1:2n-2i}(u) \geq \sup_{A_2} \phi_{2i-1:2n-2i}(u) \), and if \( J^w_1(X) \leq J^w_1(Y) \), then \( J^w_1(X_{RSS}^{(n)}) \leq J^w_1(Y_{RSS}^{(n)}) \).

**Remark 5.3.** For \( w_1(x) = x \), the result in Corollary 5.3 was proved by Qiu and Raqab (2022).

**Example 5.1.** Let \( X \) and \( Y \) be non negative random variables with pdf’s \( f \) and \( g \), respectively. Let
\[
f(x) = \begin{cases} 
2x, & 0 \leq x < 1 \\
0, & \text{otherwise}
\end{cases} \quad \text{and} \quad g(x) = \begin{cases} 
2(1 - x), & 0 \leq x < 1 \\
0, & \text{otherwise}.
\end{cases}
\]
As pointed in Qiu and Raqab (2022), Example 4.10, \( X \nless disp \ Y \) and \( \inf_{A_1} \phi_{2i-1:2n-2i}(u) = \sup_{A_2} \phi_{2i-1:2n-2i}(u) \), where \( A_1 = (0, 1) \), \( A_2 = \{0\} \). Let \( w_1(x) = x^2 \) and \( w_2(x) = x \), then \( J^w_1(X) = -2/5 \leq 1/6 = J^w_1(Y) \). Hence using Theorem 5.3, we have \( J^w_1(X_{RSS}^{(n)}) \leq J^w_1(Y_{RSS}^{(n)}) \).
6. Monotone properties

Here, we provide a proposition having the monotone properties between ranked set sampling's elements. The proof of the proposition is similar to Proposition 5.1 given in Qiu and Raqab (2022), hence omitted.

Proposition 6.1. Let $X$ be a random variable with pdf $f$ and cdf $F$. If $\Lambda^w_X$ is decreasing, then
\[ J^w_1(X_{(1:n)}) \leq J^w_1(X_{(2:n)}) \leq \ldots \leq J^w_1(X_{(m:n)}) ; \]
and if $\Lambda^w_X$ is increasing, then
\[ J^w_1(X_{(m:n)}) \leq J^w_1(X_{(m+1:n)}) \leq \ldots \leq J^w_1(X_{(n:n)}) , \]
where \( m = \left\lceil \frac{n}{2} \right\rceil \) the least integer greater than or equal to \( \frac{n}{2} \) and \( \lceil \cdot \rceil \) is the ceiling function such that \( \lfloor x \rfloor \) maps \( x \) to the least integer greater than or equal to \( x \).

If we take $w_1(x) = x$ in the above proposition, we obtain Proposition 5.1 of Qiu and Raqab (2022)

Conflict of interest

No conflicts of interest are disclosed by the authors.

Acknowledgement

The authors are thankful to the referees for their valuable suggestions which significantly improved the paper.

Funding

Santosh Kumar Chaudhary would like to acknowledge financial support from the Council of Scientific and Industrial Research (CSIR) (File Number 09/0081(14002)/2022- EMR-I), Government of India.

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