Universal Spin Structure
in Gauge Gravitation Theory

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Abstract

Building on the universal covering group of the general linear group, we introduce the composite spinor bundle whose subbundles are Lorentz spin structures associated with different gravitational fields. General covariant transformations of this composite spinor bundle are canonically defined.

1 Introduction

Einstein’s General Relativity and metric-affine gravitation theory are formulated on fibre bundles $Y \rightarrow X$ which admit a canonical lift of every diffeomorphism of the base $X$. These are called bundles of geometric objects. The lift obtained is a general covariant transformation of $Y$. The invariance of a gravitational Lagrangian density under these transformations leads to the energy-momentum conservation laws where the gravitational energy-momentum flow is reduced to the generalized Komar superpotential $3, 13, 14, 15, 33, 34, 43$.

Difficulties arise in the gauge gravitation theory because of Dirac’s fermion fields. The corresponding Lorentz spin structure is associated with a certain gravitational field, and it is not preserved under general covariant transformations. To overcome this difficulty, the universal spin structure on a world manifold is here introduced.
Consider the universal two-fold covering group $\tilde{GL}_4$ of the general linear group

$$GL_4 = GL^+(4, \mathbb{R})$$

and the corresponding two-fold covering bundle $\tilde{LX}$ of the bundle of linear frames $LX$ [7, 36, 43]. There is the commutative diagram

$$\begin{array}{ccc}
\tilde{GL}_4 & \longrightarrow & GL_4 \\
\downarrow & & \downarrow \\
L_s & \longrightarrow & L
\end{array}$$

(1.1)

where $L = SO^0(1, 3)$ is the proper Lorentz group and $L_s = SL(2, \mathbb{C})$ its two-fold covering spin group. One can consider the spinor representations of the group $\tilde{GL}_4$ which, however, are infinite-dimensional [22, 31].

Here we pursue a different approach as follows. The total space of the $\tilde{GL}_4$-principal bundle $\tilde{LX} \to X$ is the $L_s$-principal bundle $\tilde{LX} \to \Sigma_T$ over the quotient bundle

$$\Sigma_T := \tilde{LX}/L_s \to X$$

whose sections are tetrad gravitational fields $h$.

Let us consider the Lorentz spinor bundle

$$S = (\tilde{LX} \times V)/L_s$$

associated with the principal bundle $\tilde{LX} \to \Sigma_T$. Given a tetrad field $h$, the restriction of this bundle to $h(X) \subset \Sigma_T$ is a subbundle of the composite spinor bundle

$$S \to \Sigma_T \to X$$

which is exactly the Lorentz spin structure associated with the gravitational field $h$. General covariant transformations of the linear frame bundle $LX$ and, consequently, of the bundles $\tilde{LX}$ and $S$ are canonically defined.

2 Preliminairies

Throughout the paper manifolds are real, finite-dimensional, Hausdorff, second-countable (hence paracompact) and connected. By a world manifold $X$ is meant a 4-dimensional manifold, which is assumed to be non-compact, orientable and parallelizable in order that a pseudo-Riemannian metric, a spin structure and a causal space-time structure can exist on it [12, 48]. An orientation on $X$ is chosen.
Remark 1. In classical field theory, if cosmological models are not discussed, some causality conditions should be satisfied (see [20]). A compact space-time does not possess this property because it has closed time-like curves. For these reasons, we restrict our considerations to non-compact manifolds. Every non-compact manifold admits a non-zero vector field and, as a consequence, a pseudo-Riemannian metric ([4], p.167). A non-compact manifold $X$ has a spin structure iff it is parallelizable (i.e., the tangent bundle $TX \to X$ is trivial) [12]. Moreover, the spin structure on a non-compact parallelizable manifold is unique [2, 12]. The orientability of a world manifold is not needed for a pseudo-Riemannian structure and a spin structure to exist. This requirement and the additional condition of time-orientability seem natural if we are not concerned with cosmological models [4]. Note that the requirement of a manifold to be paracompact also has a physical meaning. A manifold is paracompact iff it admits a Riemannian structure ([30]; [26], p.271).

A linear connection $K$ and a pseudo-Riemannian metric $g$ on a world manifold $X$ are said to be a world connection and a world metric, respectively. We recall the coordinate expressions of a world connection on $TX$ and $T^*X$, respectively,

\begin{align}
K &= dx^\lambda \otimes (\partial_\lambda + K_\lambda^\mu \dot{x}^\mu \frac{\partial}{\partial \dot{x}^\mu}), \\
K^* &= dx^\lambda \otimes (\partial_\lambda - K_\lambda^\mu \dot{x}^\mu \frac{\partial}{\partial \dot{x}^\mu}).
\end{align}

Remark 2. Unless otherwise stated, the coordinate atlas $\Psi_X = \{(U_\zeta, \phi_\zeta)\}$ of $X$, the corresponding holonomic atlas

$\Psi_T = \{(U_\zeta, T\phi_\zeta)\}$

of the tangent bundle $TX$ and the holonomic atlases of its associated fibre bundles are assumed to be fixed.

Let

$\pi_{LX} : LX \to X$

be the principal bundle of oriented linear frames in the tangent spaces to a world manifold $X$ (or simply the frame bundle). Its structure group is

$GL_4 = GL^+(4, \mathbb{R})$. 

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By definition, a world manifold $X$ is parallelizable if the frame bundle $LX \to X$ is trivial and, consequently, it admits global sections (i.e., global frame fields).

Considering the holonomic frames $\{\partial_\mu\}$ in the tangent bundle $TX$ associated with the holonomic atlas (2.3), every element $\{H_a\}$ of the frame bundle $LX$ takes the form

$$H_a = H^\mu_a \partial_\mu,$$

where the matrix $H^\mu_a$ belongs to the group $GL_4$. The frame bundle $LX$ is provided with the bundle coordinates

$$(x^\lambda, H^\mu_a), \quad H^\mu_a = \frac{\partial x^\nu}{\partial x^\lambda} H^\lambda_a.$$

In these coordinates, the canonical action of the structure group $GL_4$ on $LX$ on the right reads

$$R_M : H^\mu_a \mapsto H^\mu_b M^b_a, \quad M \in GL_4.$$

The frame bundle $LX$ is equipped with the canonical $\mathbb{R}^4$-valued 1-form $\theta_{LX}$. Its coordinate expression is

$$\theta_{LX} = H^b_\mu dx^\mu \otimes t_a, \quad (2.4)$$

where $\{t_a\}$ is a fixed basis of $\mathbb{R}^4$ and $H^b_\mu$ is the inverse matrix of $H^\mu_a$.

The important peculiarity of the frame bundle $LX \to X$ is that every diffeomorphism $f$ of $X$ gives rise canonically to the automorphism

$$\tilde{f} : (x^\lambda, H^\lambda_a) \mapsto (f^\lambda(x), \partial_\mu f^\lambda H^\mu_a) \quad (2.5)$$

of $LX$ and to the corresponding automorphisms (denoted by the same symbol)

$$\tilde{f} : T = (LX \times V)/GL_4 \to (\tilde{f}(LX) \times V)/GL_4$$

of any fibre bundle $T$ canonically associated with $LX$. These automorphisms are called general covariant transformations.

**Example 3.** If $T = TX$, the lift $\tilde{f} = Tf$ is the familiar tangent morphism to the diffeomorphism $f$. If $T = T^*X$ is the cotangent bundle, $\tilde{f} = T^* f^{-1}$ is the dual of the linear morphism $Tf^{-1}$. ●

The lift (2.3) provides the canonical horizontal lift $\tilde{\tau}$ of every vector field $\tau$ on $X$ over the principal bundle $LX$ and over the associated fibre bundles. The canonical lift of $\tau$ over $LX$ is defined by the relation

$$\mathbf{L}_\tau \theta_{LX} = 0.$$
We have the corresponding canonical lift

\[ \bar{\tau} = \tau^\mu \partial_\mu + [\partial_\nu \tau^{\alpha_1, \ldots, \alpha_m} \dot{x}_{\beta_1, \ldots, \beta_k} + \ldots - \partial_{\beta_1} \tau^{\nu, \alpha_{1, \ldots, \alpha_m}} \dot{x}_{\nu, \beta_2, \ldots, \beta_k} - \ldots] \frac{\partial}{\partial \dot{x}^\alpha_{\beta_1, \ldots, \beta_k}} \]  

(2.6)

of \( \tau \) over the tensor bundle

\[ T = (\otimes^m TX) \otimes (\otimes^k T^*X). \]  

(2.7)

and, in particular, the lifts

\[ \bar{\tau} = \tau^\mu \partial_\mu + \partial_\nu \tau^\alpha \dot{x}^\nu \partial \]  

(2.8)

over the tangent bundle \( TX \) and

\[ \bar{\tau} = \tau^\mu \partial_\mu - \partial_\beta \tau^\nu \dot{x}_\nu \partial \]  

(2.9)

over the cotangent bundle \( T^*X \).

**Example 4.** A pseudo-Riemannian metric \( g \) on a world manifold \( X \) is represented by a section of the fibre bundle

\[ \Sigma_{\text{PR}} = GLX/O(1,3), \]  

(2.10)

where by \( GLX \) is meant the bundle of all linear frames in \( TX \) and \( O(1,3) \) is the complete Lorentz group. We call \( \Sigma_{\text{PR}} \) the metric bundle. Since \( X \) is oriented, \( \Sigma_{\text{PR}} \) is associated with the principal bundle \( LX \) of oriented frames in \( TX \). Its typical fibre is the quotient

\[ GL(4,\mathbb{R})/O(1,3). \]

This quotient space is homotopic to the Grassman manifold \( G(4,3;\mathbb{R}) \) and is homeomorphic to the topological space \( \mathbb{RP}^3 \times \mathbb{R}^7 \) ([36], p.164), where by \( \mathbb{RP}^3 \) is meant the 3-dimensional real projective space.

For the sake of simplicity, we identify the metric bundle with an open subbundle of the tensor bundle

\[ \Sigma_{\text{PR}} \subset \overset{\vee}{} T X. \]

This is coordinatized by \( (x^\lambda, \sigma^{\mu\nu}) \). As usual, by \( \sigma_{\mu\nu} \) are meant the components of the inverse matrix, and \( \sigma = \det(\sigma_{\mu\nu}) \).
The canonical lift $\bar{\tau}$ onto $\Sigma_{PR}$ of a vector field $\tau$ on $X$ reads
\[ \bar{\tau} = \tau^\lambda \partial_\lambda + \left( \partial_\nu \tau^\alpha \sigma^\nu_\beta + \partial_\nu \tau^\beta \sigma^\nu_\alpha \right) \frac{\partial}{\partial \sigma^\alpha_\beta}. \tag{2.11} \]

\textbf{Example 5.} Since the world connections are induced by the principal connections on the linear frame bundle $LX$, there is the one-to-one correspondence between the world connections and the sections of the quotient fibre bundle
\[ C_K = J^1LX/GL_4, \tag{2.12} \]
where by $J^1LX$ is meant the first order jet manifold of the fibre bundle $LX \to X$. We call (2.12) the bundle of world connections.

With respect to the holonomic frames in $TX$, the bundle $C_K$ is coordinatized by $(x^\lambda, k_\lambda^\nu_\alpha)$ so that, for any section $K$ of $C_K \to X$,
\[ k_\lambda^\nu_\alpha \circ K = K_\lambda^\nu_\alpha \]
are the coefficients of the world connection $K$ (2.1).

The bundle $C_K$ fails to be associated with $LX$, but it is an affine bundle modelled on a vector bundle associated with $LX$. There exists the canonical lift
\[ \bar{\tau} = \tau^\mu \partial_\mu + \left[ \partial_\nu \tau^\alpha k_\mu^\nu_\beta - \partial_\beta \tau^\nu k_\mu^\alpha_\nu - \partial_\mu \tau^\nu k_\nu^\alpha_\beta + \partial_\alpha_\beta \tau^\mu \right] \frac{\partial}{\partial k_\mu^\alpha_\beta}, \tag{2.13} \]
of a vector field $\tau$ on $X$ over $C_K$. ●

In fact, the canonical lift $\bar{\tau}$ (2.6) is the horizontal lift of $\tau$ by means of a symmetric world connection (2.1), for which $\tau$ is a geodesic vector field, i.e.,
\[ \partial_\nu \tau^\alpha = K_\nu^\alpha_\beta \tau^\beta. \]

\textbf{Remark 6.} One can construct the horizontal lift
\[ \tau_K = \tau^\lambda (\partial_\lambda + K_\lambda^\beta_\alpha \frac{\partial}{\partial \hat{x}^\alpha_\beta}) \tag{2.14} \]
of a vector field $\tau$ on $X$ over $TX$ (and other tensor bundles) by means of any world connection $K$. This is the generator of a 1-parameter group of non-holonomic automorphisms of the frame bundle $LX$. We meet non-holonomic automorphisms in the gauge theory of the general linear group $GL_4$ [22]. Note that the lifts (2.8) and (2.14) play the role of generators of the gauge group of translations in the pioneer gauge gravitation theories (see [21, 25] and references therein). ●
3 Dirac spinors

We describe Dirac spinors as follows 3, 25, 38 (see 4, 29 for a general description of the Clifford algebra techniques).

Let $M$ be the Minkowski space equipped with the Minkowski metric

$$\eta = \text{diag}(1, -1, -1, -1)$$

written with respect to a basis $\{e^a\}$ of $M$.

Let $C_{1,3}$ be the complex Clifford algebra generated by elements of $M$. This is the complexified quotient of the tensor algebra $\otimes M$ of $M$ by the two-sided ideal generated by the elements

$$e \otimes e' + e' \otimes e - 2\eta(e, e') \in \otimes M, \quad e, e' \in M.$$

Remark 7. The complex Clifford algebra $C_{1,3}$ is isomorphic to the real Clifford algebra $R_{2,3}$, whose generating space is $\mathbb{R}^5$ with the metric $\text{diag}(1, -1, -1, -1, 1)$. The subalgebra generated by the elements of $M \subset \mathbb{R}^5$ is the real Clifford algebra $R_{1,3}$. ●

A spinor space $V$ is defined to be a minimal left ideal of $C_{1,3}$ on which this algebra acts on the left. We have the representation

$$\gamma : M \otimes V \to V,$$

$$\gamma(e^a) = \gamma^a,$$

of elements of the Minkowski space $M \subset C_{1,3}$ by the Dirac $\gamma$-matrices on $V$.

Remark 8. The explicit form of this representation depends on the choice of the minimal left ideal $V$ of $C_{1,3}$. Different ideals lead to equivalent representations. ●

The spinor space $V$ is provided with the spinor metric

$$a(v, v') = \frac{1}{2}(v^+ \gamma^0 v' + v'^+ \gamma^0 v),$$

where $e^0 \in M$ satisfies the conditions

$$(e^0)^+ = e^0, \quad (e^0 e)^+ = e^0 e, \quad \forall e \in M.$$

By definition, the Clifford group $G_{1,3}$ comprises all invertible elements $l_s$ of the real Clifford algebra $R_{1,3}$ such that the corresponding inner automorphisms keep the Minkowski space $M \subset R_{1,3}$, that is,

$$l_s e l_s^{-1} = l(e), \quad e \in M,$$
where \( l \in O(1,3) \) is a Lorentz transformation of \( M \). The automorphisms (3.3) preserve also the representation (3.1), i.e.,

\[
\gamma(lM \otimes l_s V) = l_s \gamma(M \otimes V). \tag{3.4}
\]

Thereby, we have an epimorphism of the Clifford group \( G_{1,3} \) onto the Lorentz group \( O(1,3) \). Since the action (3.3) of the Clifford group on the Minkowski space \( M \) is not effective, one usually considers its pin and spin subgroups. The subgroup \( \text{Pin}(1,3) \) of \( G_{1,3} \) is generated by elements \( e \in M \) such that \( \eta(e, e) = \pm 1 \).

The even part of \( \text{Pin}(1,3) \) is the spin group \( \text{Spin}(1,3) \). Its component of the unity \( L_s = \text{Spin}^0(1,3) \cong \text{SL}(2, \mathbb{C}) \) is the well-known two-fold universal covering group

\[
z_L : L_s \rightarrow L = L_s/\mathbb{Z}_2, \quad \mathbb{Z}_2 = \{1, -1\}, \tag{3.5}
\]

of the proper Lorentz group \( L = \text{SO}^0(1,3) \). As is well-known ([17], p.27), \( L \) is homeomorphic with \( \mathbb{RP}^3 \times \mathbb{R}^3 \). The spin group \( L_s \) acts on the spinor space \( V \) by the generators

\[
L_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]. \tag{3.6}
\]

By virtue of the relation

\[
L_{ab}^+ \gamma^0 = -\gamma^0 L_{ab},
\]

this action preserves the spinor metric (3.2). Accordingly, the Lorentz group \( L \) acts on the Minkowski space \( M \) by the generators

\[
L_{ab}^c \delta_d^e = \eta_{ad} \delta_b^c - \eta_{bd} \delta_a^c. \tag{3.7}
\]

Remark 9. The generating elements \( e \in M, \eta(e, e) = \pm 1 \), of the group \( \text{Pin}(1,3) \) act on the Minkowski space by the adjoint representation which is the composition

\[
e : v \rightarrow eve^{-1} = -v + 2\frac{\eta(e, v)}{\eta(e, e)} e, \quad e, v \in \mathbb{R}^4,
\]

of the total reflection of \( M \) and the reflection across the hyperplane

\[
e^\perp = \{w \in M : \eta(e, w) = 0\}
\]
which is perpendicular to \( e \) with respect to the metric \( \eta \) in \( M \). By the well-known Cartan–Dieudonné theorem, every element of the pseudo-orthogonal group \( O(p, q) \) can be written as a product of \( r \leq p + q \) reflections across hyperplanes in the vector space \( \mathbb{R}^{p+q} \) ([29], p.17). In particular, the group \( \text{Spin}(1, 3) \) consists of the elements of \( \text{Pin}(1, 3) \) which comprise an even number of reflections. The epimorphism of \( \text{Spin}(1, 3) \) onto the Lorentz group \( \text{SO}(3, 1) \), as like as the epimorphism (3.5), are determined by the fact that the elements \( e^\perp \) and \( (-e)^\perp \) define the same reflection.

Let us now consider a bundle of Clifford algebras \( C_{1,3} \) over a world manifold \( X \) whose structure group is the spin group \( L_s \) of automorphisms of \( C_{1,3} \). It possesses as subbundles the spinor bundle \( S_M \rightarrow X \), associated with a \( L_s \)-principal bundle \( P_s \), and a bundle \( E_M \rightarrow X \) of Minkowski spaces of generating elements of \( C_{3,1} \), which is associated with the \( L \)-principal bundle \( P_s/\mathbb{Z}_2 \).

The bundle \( E_M \rightarrow X \) of Minkowski spaces must be isomorphic to the cotangent bundle \( T^*X \) in order that sections of the spinor bundle \( S_M \) describe Dirac’s fermion fields on a world manifold \( X \). In other words, we must consider a spin structure on the cotangent bundle \( T^*X \) of \( X \) [29].

There are several almost equivalent definitions of a spin structure [2, 11, 12, 23, 29]. One can say that a pseudo-Riemannian spin structure on a world manifold \( X \) is a pair \((P_s, z_s)\) of an \( L_s \)-principal bundle \( P_s \rightarrow X \) and a principal bundle morphism

\[
z_s : P_s \rightarrow LX
\]  

(3.8)

over \( X \) of \( P_s \) to the \( GL_4 \)-principal frame bundle \( LX \rightarrow X \). More generally, one can define a spin structure on any vector bundle \( E \rightarrow X \) (see [29], p.80). Then the precedent definition applies to the particular case in which \( E \) is the cotangent bundle \( T^*X \) and the fibre metric in \( T^*X \) is a pseudo-Riemannian metric.

Since the homomorphism \( L_s \rightarrow GL_4 \) factorizes through the epimorphism (3.3), every bundle morphism (3.8) factorizes through a morphism of \( P_s \) onto some principal sub-bundle of the frame bundle \( LX \) with the proper Lorentz group \( L \) as a structure group. It follows that the necessary condition for the existence of a Lorentz spin structure on \( X \) is that the structure group \( GL_4 \) of \( LX \) is reducible to the proper Lorentz group \( L \).

From the physical viewpoint, it means that the existence of Dirac’s fermion matter implies the existence of a gravitational field.
4 Reduced structure

First, we recall some general notions. Let

$$\pi_{PX}: P \rightarrow X$$

be a principal bundle with a structure group $G$, which acts freely and transitively on the fibres of $P$ on the right:

$$R_g : p \mapsto pg, \quad p \in P, \quad g \in G. \quad (4.1)$$

Let $Y$ be a $P$-associated fibre bundle with a standard fibre $V$ on which the structure group $G$ of $P$ acts on the left. Recall that all associated fibre bundles with the same typical fibre are isomorphic to each other, but the isomorphisms are not canonical in general. Unless otherwise stated, we restrict our considerations to the canonically associated fibre bundles, which are the quotient

$$Y = (P \times V)/G \quad (4.2)$$

with respect to the identification of elements $(p, v)$ and $(pg, g^{-1} v)$, $g \in G$. Let this fibre bundle be coordinatized by $(x^\lambda, y^i)$. Automorphisms of the principal bundle $P$, bundle atlases and principal connections on $P$ define similar objects on the associated fibre bundles $(4.2)$ in a canonical way, in contrast with other associated fibre bundles.

**Remark 10.** Let $[p]$ denote the restriction of the canonical morphism

$$P \times V \rightarrow (P \times V)/G$$

to $p \times V$. For the sake of simplicity, we shall write

$$[p](v) = [p, v]_G, \quad p \in P, \quad v \in V.$$

By a principal automorphism of a principal bundle $P$ is meant an automorphism $\Phi_P$ which is equivariant under the canonical action $(4.1)$, that is, the diagram

$$\begin{array}{ccc}
P & \xrightarrow{R_g} & P \\
\Phi_P \downarrow & & \downarrow \Phi_P \\
P & \xrightarrow{R_g} & P
\end{array}$$
A principal automorphism yields the corresponding automorphism
\[ \Phi_Y : (P \times V)/G \to (\Phi_P(P) \times V)/G \] (4.3)
of every fibre bundle \( Y \) associated with \( P \).

Every principal automorphism of a principal bundle \( P \) is represented as
\[ \Phi_P(p) = pf(p), \quad p \in P, \] (4.4)
where \( f \) is a \( G \)-valued equivariant function on \( P \), i.e.,
\[ f(pg) = g^{-1}f(p)g, \quad \forall g \in G. \] (4.5)

There is a one-to-one correspondence between the functions \( f \) and the global sections \( s \) of the \( P \)-associated group bundle
\[ P^G = (P \times G)/G, \] (4.6)
whose typical fibre is the group \( G \) itself and the action is given by the adjoint representation (\([19]\), p.277). There is defined the canonical fibre-to-fibre action of the group bundle \( P^G \) on any \( P \)-associated bundle \( Y \):
\[ P^G \times X Y \to Y, \] \[ ([p, g]_G, [p, v]_G) \to [p, gv]_G, \quad g \in G, \ v \in V. \] (4.7)

Then, the above mentioned correspondence is given by the relation
\[ pf(p) = s(\pi_{PX}(p))p. \]

Let \( H \) be a closed subgroup of \( G \). We have the composite fibre bundle
\[ P \to P/H \to X, \] (4.8)
where
\[ (i)J \quad \Sigma := P/H \xrightarrow{\pi} X \] (4.9)
is a \( P \)-associated fibre bundle with the typical fibre \( G/H \) on which the structure group \( G \) acts naturally on the left, and
\[ (ii) \quad P_\Sigma := P \xrightarrow{\pi} P/H \] (4.10)
is a principal bundle with the structure group \( H \) (\([26]\), p.57).
One says that the structure group $G$ of a principal bundle $P$ is reducible to the subgroup $H$ if there exists a principal subbundle $P^h$ of $P$ with the structure group $H$. This subbundle is called the reduced $G^H$-structure \[16, 27, 49\].

**Remark 11.** Note that in \[16\] and \[27\] the authors are concerned with reduced structures on the principal frame bundle $LX$. This notion is generalized to arbitrary principal bundle in \[49\]. In \[16\] a reduced structure is regarded as a monomorphism of a given principal bundle $P \to X$ with a structure group $H$ into the principal frame bundle $LX$. Thereby, $\text{GL}(4, \mathbb{R})^H$-structures are defined up to isomorphisms.

Let us recall the following theorems.

**Theorem 1.** A structure group $G$ of a principal bundle $P$ is reducible to a closed subgroup $H$ iff $P$ has an atlas $\Psi$ with $H$-valued transition functions \([26]\), p.53). \[\square\]

Given a reduced subbundle $P^h$ of $P$, such an atlas $\Psi$ is defined by a family of local sections $\{z_{\alpha}\}$ which take their values into $P^h$.

**Theorem 2.** There is a one-to-one correspondence

$$P^h = \pi_{P^\Sigma}^{-1}(h(X))$$

between the reduced $H$-principal subbundles $P^h$ of $P$ and the global sections $h$ of the quotient fibre bundle $P/H \to X$ \([26]\), p.57). \[\square\]

Given such a section $h$, let us consider the restriction $h^*P^\Sigma$ of the $H$-principal bundle $P^\Sigma$ \([14]\) to $h(X)$. This is a $H$-principal bundle over $X$ \([24]\), p.60), which is isomorphic to the reduced subbundle $P^h$ of $P$.

In general, there are topological obstructions to the reduction of a structure group of a principal bundle to a subgroup. In accordance with a well-known theorem (see \[14\], p.53), a structure group $G$ of a principal bundle $P$ is always reducible to a closed subgroup $H$ if the quotient $G/H$ is homeomorphic to an Euclidean space $\mathbb{R}^k$ \([14]\), p.53). In this case, all $H$-principal subbundles of $P$ are isomorphic to each other as $H$-principal bundles \([14]\), p.56).

In particular, a structure group $G$ of a principal bundle is always reducible to its maximal compact subgroup $H$ since the quotient space $G/H$ is homeomorphic to an Euclidean space \([14]\), p.59). It follows that there is a one-to-one correspondence between equivalence classes of $G$-principal bundles and those of $H$-principal bundles if $H$ is a maximal compact subgroup of $G$ \([24][44]\). In particular, this consideration applies
to GL\((n, \mathbb{R})\)- and O\((n)\)-principal bundles as like as GL\(^+\)(n, \mathbb{R})- and SO\((n)\)-principal bundles.

**Proposition 3.** Every vertical principal automorphism \(\Phi\) of the principal bundle \(P \to X\) sends a reduced subbundle \(P^h\) onto an isomorphic \(H\)-principal subbundle \(P^{h'}\).

\(\square\)

**Proof.** Let

\[
\Psi^h = \{(U_\alpha, z^h_\alpha), \rho^h_{\alpha\beta}\}, \quad z^h_\alpha(x) = z^h_\beta(x)\rho^h_{\alpha\beta}(x), \quad x \in U_\alpha \cap U_\beta,
\]

be an atlas of the reduced subbundle \(P^h\), where \(z^h_\alpha\) are local sections of \(P^h \to X\) and \(\rho^h_{\alpha\beta}\) are the transition functions. Given a vertical automorphism \(\Phi\) of \(P\), let us provide the reduced subbundle \(P^{h'} = \Phi(P^h)\) with the atlas

\[
\Psi^{h'} = \{(U_\alpha, z^{h'}_\alpha), \rho^{h'}_{\alpha\beta}\}
\]

determined by the local sections

\[z^{h'}_\alpha = \Phi \circ z^h_\alpha\]

of \(P^{h'} \to X\). Then it is readily observed that

\[
\rho^{h'}_{\alpha\beta}(x) = \rho^h_{\alpha\beta}(x), \quad x \in U_\alpha \cap U_\beta.
\]

\(\blacksquare\)

**Proposition 4.** Conversely, let two reduced subbundles \(P^h\) and \(P^{h'}\) of a principal bundle \(P\) be isomorphic to each other as \(H\)-principal bundles and let \(\Phi : P^h \to P^{h'}\) be an isomorphism. Then \(\Phi\) can be extended to a vertical automorphism of \(P\). \(\square\)

**Proof.** The isomorphism \(\Phi\) determines a \(G\)-valued function \(f\) on \(P^h\) given by the relation

\[pf(p) = \Phi(p), \quad p \in P^h.\]

Obviously, this function is \(H\)-equivariant. Its prolongation to a \(G\)-equivariant function on \(P\) is defined to be

\[f(pg) = g^{-1}f(p)g, \quad p \in P^h, \quad g \in G.\]
In accordance with the relation (4.4), this function defines a principal automorphism of \( P \) whose restriction to \( P^h \) coincides with \( \Phi \).

Given a reduced subbundle \( P^h \) of a principal bundle \( P \), let
\[
Y^h = (P^h \times V)/H
\]
be the canonically associated fibre bundle with a typical fibre \( V \). If \( P^{h'} \) is another reduced subbundle of \( P \) which is isomorphic to \( P^h \), the fibre bundle \( Y^h \) is associated with \( P^{h'} \), but not canonically associated in general. One can write
\[
Y^h \ni [p, v]_H = [pf(p), f(p)^{-1}v]_H \in Y^{h'},
\]
\[
p \in P^h; \quad pf(p) \in P^{h'}, \quad v \in V,
\]
if a typical fibre \( V \) of \( Y^h \) admits representation of the whole group \( G \) only.

5 Tetrad fields

Since a world manifold is assumed to be parallelizable, the structure group \( GL_4 \) of the frame bundle \( LX \) is obviously reducible to the Lorentz group \( L \). The subbundle \( L^hX \) is said to be a Lorentz structure.

In accordance with Theorem 2, there is a one-to-one correspondence between the reduced \( L \)-principal subbundles \( L^hX \) of \( LX \) and the global sections \( h \) of the quotient fibre bundle
\[
\Sigma_T = LX/L,
\]
called the tetrad bundle. This is a \( LX \)-associated fibre bundle with the typical fibre \( GL_4/L \). Since the group \( GL_4 \) is homotopic to its maximal compact subgroup \( SO(4) \) and the proper Lorentz group is homotopic to its maximal compact subgroup \( SO(3) \), \( GL_4/L \) is homotopic to the Stiefel manifold
\[
V(4, 1; \mathbb{R}) = SO(4)/SO(3) = S^3
\]
(\([14]\), p.33), and this is homeomorphic to the topological space \( S^3 \times \mathbb{R}^7 \). The fibre bundle (5.1) is the 2-fold covering of the metric bundle \( \Sigma_{PR} \) (2.10). Its global sections are called tetrad fields.

Since \( X \) is parallelizable, any two Lorentz subbundles \( L^hX \) and \( L^{h'}X \) are isomorphic to each other. It follows that, in virtue of Proposition 4 there exists a vertical
bundle automorphism \( \Phi \) of \( LX \) which sends \( L^hX \) onto \( L^{h'}X \). The associated vertical automorphism \( \Phi_\Sigma \) of the fibre bundle \( \Sigma_T \rightarrow X \) transforms the tetrad field \( h \) to the tetrad field \( h' \).

Every tetrad field \( h \) defines an associated Lorentz atlas \( \Psi^h = \{(U_\zeta, z^h_\zeta)\} \) of \( LX \) such that the corresponding local sections \( z^h_\zeta \) of the frame bundle \( LX \) take their values into its Lorentz subbundle \( L^hX \).

Given a Lorentz atlas \( \Psi^h \), the pull-back
\[
h^a \otimes t_a = z^h_\zeta \theta_{LX} = h^\lambda_\zeta dx^\lambda \otimes t_a
\]
(5.2)
of the canonical form \( \theta_{LX} \) by a local section \( z^h_\zeta \) is said to be a (local) tetrad form. The tetrad form (5.2) determines the tetrad coframes
\[
t^a(x) = h^a_\mu(x) dx^\mu, \quad x \in U_\zeta,
\]
in the cotangent bundle \( T^*X \), which are denoted by the same symbol \( h^a \) for the sake of simplicity. These coframes are associated with the Lorentz atlas \( \Psi^h \).

The coefficients \( h^a_\mu \) of the tetrad forms and the inverse matrix elements
\[
h^\mu_a = H^\mu_a \circ z^h_\zeta.
\]
(5.3)
are called the tetrad functions. Given a Lorentz atlas \( \Psi^h \), the tetrad field \( h \) can be represented by the family of tetrad functions \( \{h^a_\mu\} \). In particular, we have the well-known relation
\[
g = h^a \otimes h^b \eta_{ab},
\]
\[
g_{\mu\nu} = h^a_\mu h^b_\nu \eta^{ab},
\]
between tetrad functions and metric functions of the corresponding pseudo-Riemannian metric \( g : X \rightarrow \Sigma_{PR} \).

**Remark 12.** Since the world manifold \( X \) is assumed to be parallelizable, it admits global tetrad forms (5.2). They, however, are not canonical.

Even if \( X \) were not parallelizable, the existence of a Lorentz structure guarantees that there is a Lorentz atlas such that the temporal tetrad form \( h^0 \) is globally defined. This is a consequence of the fact that \( L \) is reducible to its maximal compact subgroup \( SO(3) \) and, therefore, there exists a \( SO(3) \)-principal subbundle \( L^h_0X \subset L^hX \), called a space-time structure associated with \( h \). The corresponding global section of the quotient fibre bundle \( L^hX/SO(3) \rightarrow X \) with the typical fibre \( \mathbb{R}^3 \) is a 3-dimensional spatial distribution \( FX \subset TX \) on \( X \). Its generating 1-form written relative to a
Lorentz atlas is exactly the global tetrad form $h^0$. There is the corresponding decomposition

$$TX = FX \oplus NF,$$

where $NF$ is the 1-dimensional fibre bundle defined by the tetrad frame $h_0 = h^\mu_0 \partial_\mu$. This decomposition is called a space-time decomposition. In particular, if the generating form $h^0$ is exact, the space-time decomposition obeys the condition of stable causality by Hawking [20].

6 Pseudo-Riemannian spin structure

Given a tetrad field $h$, let $L^hX$ be the corresponding reduced Lorentz subbundle. Since $X$ is non-compact and parallelizable, the principal bundle $L^hX$ is extended uniquely (with accuracy of an automorphism) to a $L_s$-principal bundle $P^h \to X$ [12] and to a fibre bundle of Clifford algebras $C_{1,3}$ [12], p.95. There is the principal bundle morphism

$$z_h : P^h \to L^hX \subset LX$$

(6.1)

over $X$ such that

$$z_h \circ R_g = R_{z_{L}(g)}, \quad \forall g \in L_s.$$

This is a $h$-associated pseudo-Riemannian spin structure on a world manifold. We call $P^h$ the $h$-associated principal spinor bundle. Every Lorentz atlas $\Psi^h = \{z^h_\zeta\}$ of $L^hX$ can be lifted to an atlas of the principal spinor bundle $P^h$.

Let us consider the corresponding $L^hX$-associated fibre bundle $E_M$ of Minkowski spaces

$$E_M = (L^hX \times M)/L = (P^h \times M)/L_s$$

(6.2)

and the $P^h$-associated spinor bundle

$$S^h = (P^h \times V)/L_s,$$

(6.3)

called in sequel the $h$-associated spinor bundle. The fibre bundle $E_M$ (6.2) is isomorphic to the cotangent bundle

$$T^*X = (L^hX \times M)/L$$
as a fibre bundle with the structure Lorentz group $L$. Then there exists the representation
\begin{equation}
\gamma_h : T^*X \otimes S^h = (P^h \times (M \otimes V))/L_s \to (P^h \times \gamma(M \otimes V))/L_s = S^h \tag{6.4}
\end{equation}
of covectors to $X$ by the Dirac $\gamma$-matrices on elements of the spinor bundle $S^h$.

Relative to an atlas $\{z_\zeta\}$ of $P^h$ and to the associated Lorentz atlas $\{z^\zeta = z_h \circ z_\zeta\}$ of $LX$, the representation (6.4) reads
\begin{equation}
y^A(\gamma_h(h^a(x) \otimes v)) = \gamma^a B y^B(v), \quad v \in S^h_x,
\end{equation}
where $y^A$ are the corresponding bundle coordinates of $S^h$ and $h^a$ are the tetrad forms (5.2). As a shorthand, we can write
\begin{equation}
\hat{h}^a = \gamma_h(h^a) = \gamma^a, \\
\hat{dx}^\lambda = \gamma_h(dx^\lambda) = h^\lambda(x)\gamma^a.
\end{equation}

Sections $s_h$ of the $h$-associated spinor bundle $S^h$ (3.3) describe Dirac’s fermion fields in the presence of the tetrad field $h$. Indeed, let $A_h$ be a principal connection on $S^h$ and
\begin{equation}
D : J^1 S^h \to T^*X \otimes S^h, \\
D = (y_A^\lambda - A^a_{\lambda B}y^B)dx^\lambda \otimes \partial_A,
\end{equation}
the corresponding covariant differential. Here, we have used the isomorphism
\begin{equation}
V S^h \simeq S^h_X \times S^h.
\end{equation}
The first order differential Dirac operator is defined on $S^h$ as the composition
\begin{equation}
\mathcal{D}_h = \gamma_h \circ D : J^1 S^h \to T^*X \otimes S^h \to S^h, \tag{6.5}
y^A \circ \mathcal{D}_h = h^\lambda \gamma^a B y^B - \frac{1}{2}A^a_{\lambda B}L_{ab} A^B y^B.
\end{equation}

**Remark 13.** The spinor bundle $S^h$ is a complex fibre bundle with a real structure group over a real manifold. One can regard such a fibre bundle as a real one whose real dimension of the fibres is equal to the doubled complex dimension. In particular, the jet manifold $J^1 S^h$ of $S^h$ is defined as usual. It is coordinatized by $(x^\lambda, y^A, y^A_\lambda)$. •
The $h$-associated spinor bundle $S^h$ is equipped with the fibre spinor metric
\[
a_h : S^h \times S^h \to \mathbb{R},
\]
\[
a_h(v, v') = \frac{1}{2}(v^+ \gamma^0 v' + v'^+ \gamma^0 v), \quad v, v' \in S^h.
\]
Using this metric and the Dirac operator (6.5), one can define the Dirac Lagrangian density on $J^1 S^h$ in the presence of a background tetrad field $h$ and a background connection $A_h$ on $S^h$ as
\[
L_h : J^1 S^h \to \Lambda^4 T^*_X,
\]
\[
L_h = [a_h(iD_h(w), w) - m_h(w, w)] h^0 \wedge \cdots \wedge h^3, \quad w \in J^1 S^h.
\]
Its coordinate expression is
\[
L_h = \{ \frac{i}{2} h_y^\lambda [y_A^+(\gamma^0 \gamma^q)^A_B (y^B_\lambda - \frac{1}{2} A_{\lambda}^{ab} L_{ab}^B C y^C) -
(y^+_{\lambda A} - \frac{1}{2} A_{\lambda}^{ab} y^+_{CB} L_{ab}^C (\gamma^0 \gamma^q)^A_B y^B) - m y_A^+(\gamma^0 \gamma^q)^A_B y^B \} \det(h^\mu_\mu).
\]

7 Spin connections

Note that there is a one-to-one correspondence between the principal (spin) connections on the $h$-associated principal spinor bundle $P^h$ and the principal (Lorentz) connections on the L-principal bundle $L^h X$ as follows.

First, let us recall the following theorem ([26], p.79).

**Theorem 5.** Let $P' \to X$ and $P \to X$ be principle bundles with the structure groups $G'$ and $G$, respectively. If $\Phi : P' \to P$ is a principal bundle morphism over $X$ with the corresponding homomorphism $G' \to G$, there exists a unique principal connection $A$ on $P$ for a given principal connection $A'$ on $P'$ such that $T\Phi$ sends the horizontal subspaces of $A'$ onto horizontal subspaces of $A$. □

It follows that every principal connection
\[
A_h = dx^\lambda \otimes (\partial_\lambda + \frac{1}{2} A_{\lambda}^{ab} e_{ab})
\]
on $P^h$ defines a principal connection on $L^h X$ which is given by the same expression (7.1). Conversely, the pull-back $z^*_h \omega_A$ on $P^h$ of a connection form $\omega_A$ of a Lorentz
connection $A_h$ on $L^hX$ is equivariant under the action of group $L_s$ on $P^h$, and this is a connection form of a spin connection on $P^h$.

In particular, as is well-known, the Levi–Civita connection of a pseudo-Riemannian metric $g$ gives rise to a spin connection with the components

$$A_{\lambda}^{ab} = \eta^{kb}h^a_{\mu}(\partial_{\lambda}h^\mu_k - h^\nu_k\{\lambda^\nu\})$$

(7.2)
on the $g$-associated spinor bundle $S^g$.

In gauge gravitation theory, Lorentz connections are treated as gauge potentials associated with the Lorentz group. At the same time, every world connection $K$ on a world manifold $X$ also defines a spin connection on a $h$-associated principal spinor bundle $P^h$. It follows that the gauge gravitation theory is reduced to metric-affine gravitation theory in the presence of Dirac's fermion fields [15, 43].

Note that, in accordance with Theorem 3, every Lorentz connection $A_h$ (7.1) on a reduced Lorentz subbundle $L^hX$ of $LX$ gives rise to a world connection $K$ (2.1) on $LX$ where

$$K_{\lambda}^{\mu} = h^b_k\partial_{\lambda}h^\mu_k + \eta_{ka}h^\mu_b h^b_k A_{\lambda}^{ab}.$$  

(7.3)

At the same time, every principal connection $K$ on the $GL_4$-principal bundle $LX$ defines a Lorentz principal connection $K_L$ on a $L$-principal subbundle $L^hX$ as follows.

It is readily observed that the Lie algebra of the general linear group $GL_4$ is the direct sum

$$\mathfrak{g}(GL_4) = \mathfrak{g}(L) \oplus \mathfrak{m}$$

of the Lie algebra $\mathfrak{g}(L)$ of the Lorentz group and a subspace $\mathfrak{m} \subset \mathfrak{g}(GL_4)$ such that

$$\text{ad}(l)(\mathfrak{m}) \subset \mathfrak{m}, \quad l \in L$$

where $\text{ad}$ is the adjoint representation. Let $\omega_K$ be a connection form of a world connection $K$ on $LX$. Then, by the well-known theorem (26, p.83), the pull-back onto $L^hX$ of the $\mathfrak{g}(L)$-valued component $\omega_L$ of $\omega_K$ is a connection form of a principal connection $K_h$ on the reduced Lorentz subbundle $L^hX$. To obtain the connection parameters of $K_h$, let us consider the local connection 1-form of the connection $K$ with respect to a Lorentz atlas $\Psi^h$ of $LX$ given by the tetrad forms $h^a$. This reads

$$z^h \omega_K = K_{\lambda}^{b}dx^\lambda \otimes e^k_b,$$

$$K_{\lambda}^{b} = -h^{b}_\mu \partial^{}_{\lambda}h^{\mu}_k + K_{\lambda^\nu}^\mu h^{b}_\nu h^\nu_k,$$
where \( \{e^k_b\} \) is the basis of the right Lie algebra of the group \( GL_4 \). Then, the Lorentz part of this form is accurately the local connection 1-form of the connection \( K_h \) on \( L^hX \). We have

\[
z^{h*}\omega_L = \frac{1}{2} A^{ab}_{\lambda} dx^\lambda \otimes e_{ab},
\]

\[
A^a_{\lambda} = \frac{1}{2}(\eta^{kb}h^a_{\mu} - \eta^{ka}h^b_{\mu})(\partial_{\lambda} h^\mu_k - h^\nu_k K_{\lambda}^\mu^\nu).
\]

(7.4)

If \( K \) is a Lorentz connection \( A_h \), then obviously \( K_h = A_h \).

Accordingly, the connection \( K_h \) on \( L^hX \) given by the local connection 1-form (7.4) defines the corresponding spin connection on \( S^hK_h = dx^\lambda \otimes [\partial_{\lambda} + \frac{1}{4}(\eta^{kb}h^a_{\mu} - \eta^{ka}h^b_{\mu})(\partial_{\lambda} h^\mu_k - h^\nu_k K_{\lambda}^\mu^\nu) L^{AB}_{ab} y^B \partial_A] \),

(7.5)

where \( L^{AB}_{ab} \) are the generators \((3.6)\) [15, 43]. Such a connection has been considered in \([1, 37, 47]\).

A substitution of the spin connection (7.5) into the Dirac operator (6.5) and into the Dirac Lagrangian density (6.7) provides a description of Dirac’s fermion fields in the presence of arbitrary linear connections on a world manifold, not only in the presence of the Lorentz ones.

One can utilize the connection (7.5) for constructing a horizontal lift of a vector field \( \tau \) on \( X \) onto \( S^h \). This lift reads

\[
\tau_{K_h} = \tau^\lambda \partial_{\lambda} + \frac{1}{4}(\eta^{kb}h^a_{\mu} - \eta^{ka}h^b_{\mu})(\partial_{\lambda} h^\mu_k - h^\nu_k K_{\lambda}^\mu^\nu) L^{AB}_{ab} y^B \partial_A.
\]

(7.6)

For every vector field \( \tau \) on \( X \), let us choose a symmetric connection \( K \) which has \( \tau \) as a geodesic vector field. Then we get the canonical horizontal lift

\[
\overline{\tau} = \tau^\lambda \partial_{\lambda} + \frac{1}{4}(\eta^{kb}h^a_{\mu} - \eta^{ka}h^b_{\mu})(\tau^\lambda \partial_{\lambda} h^\mu_k - h^\nu_k \partial_{\nu} \tau^\mu) L^{AB}_{ab} y^B \partial_A
\]

(7.7)

of vector fields \( \tau \) on \( X \) onto the \( h \)-associated spinor bundle \( S^h \) [13].

**Remark 14.** The canonical lift (7.7) is brought into the form

\[
\overline{\tau} = \tau_{\{\}} - \frac{1}{4}(\eta^{kb}h^a_{\mu} - \eta^{ka}h^b_{\mu}) h^\nu_k \nabla_{\nu} \tau^\mu L^{AB}_{ab} y^B \partial_A,
\]

where \( \tau_{\{\}} \) is the horizontal lift (7.6) of \( \tau \) by means of the spin Levi–Civita connection (7.2) of the tetrad field \( h \), and \( \nabla_{\nu} \tau^\mu \) are the covariant derivatives of \( \tau \) relative to the same Levi–Civita connection. This is exactly the Lie derivative of spinor fields described in \([10, 28]\). •
8 Universal spin structure

The canonical lift \((7.7)\) fails to be a generator of general covariant transformations because it does not involve transformations of tetrad fields. To define general covariant transformations of spinor bundles, we should consider spinor structures associated with different tetrad fields. The difficulty arises because, though the principal spinor bundles \(P^h\) and \(P^{h'}\) are isomorphic, the \(h\)-associated spinor bundle \(S^h\) fails to be \(h'\)-associated since it is associated, but not canonically associated with \(P^{h'}\). As a consequence, the representations \(\gamma^h\) and \(\gamma^{h'}\) \((6.4)\) for different tetrad fields \(h\) and \(h'\) are not equivalent \([40, 42]\).

Indeed, let \(L^hX\) and \(L^{h'}X\) be the Lorentz subbundles of the frame bundle \(LX\) and \(\Phi\) an automorphism of \(LX\), characterized by an equivariant function \(f\) on \(LX\), which sends \(L^hX\) onto \(L^{h'}X\). Let

\[ t^* = [p, e]_L = [pf(p), (f^{-1}(p)e)_L], \quad p \in L^hX, \quad e \in \mathbb{R}^4, \]

\[ t^* = t_\mu dx^\mu = t_a h^a = t'_a h'^a, \]

be an element of \(T^*X\). Its representations \(\gamma^h\) and \(\gamma^{h'}\) \((6.4)\) read

\[ \gamma^h(t^*) = \gamma(e) = t_\mu h^a \gamma^a, \]

\[ \gamma^{h'}(t^*) = \gamma(f^{-1}(p)e) = t_\mu h'^a \gamma^a. \]

These representations are not equivalent since no isomorphism \(\Phi_s\) of \(S^h\) onto \(S^{h'}\) can obey the condition

\[ \gamma^{h'}(t^*) = \Phi_s \gamma^h(t^*) \Phi_s^{-1}, \quad \forall t^* \in T^*X. \]

It follows that every Dirac’s fermion field must be described by a pair with a certain tetrad (gravitational) field. Thus we observe the phenomenon of symmetry breaking in gauge gravitation theory which exhibits the physical nature of gravity as a Higgs field \([41]\). The goal is to describe the totality of fermion-gravitation pairs.

**Remark 15.** All spin structures on a manifold \(X\) which are related to the two-fold universal covering groups possess the following two properties \([18]\).

Let \(P \rightarrow X\) be a principal bundle with a structure group \(G\) with the fundamental group \(\pi_1(G) = \mathbb{Z}_2\). Let \(\tilde{G}\) be the universal covering group of \(G\).

1. The topological obstruction for the existence of a \(\tilde{G}\)-principal bundle \(\tilde{P} \rightarrow X\) covering the bundle \(P\) is a non-zero element of the Čech cohomology group \(H^2(X; \mathbb{Z}_2)\) of \(X\) with coefficients in \(\mathbb{Z}_2\).
2. Inequivalent lifts of $G$-principal bundle $P$ to a $\tilde{G}$-principal bundle are classified by elements of the Čech cohomology group $H^1(X;\mathbb{Z}_2)$.

In particular, the well-known topological obstruction in order that a Riemannian spin structure and a pseudo-Riemannian spin structure can exist on $X$ is the non-zero second Stiefel–Whitney class $w_2(X) \in H^2(X;\mathbb{Z}_2)$ of $X$ ([29], p.82). The set of these inequivalent spin structures is in bijective correspondence with the cohomology group $H^1(X;\mathbb{Z}_2)$ ([18, 45]; [29], p.82). In the case of 4-dimensional noncompact manifolds that we consider, all Riemannian and pseudo-Riemannian spin structures are equivalent [2, 12].

**Example 16. Riemannian spin structure.** Let us consider spin structures on Riemannian manifolds. Let $X$ be an arbitrary 4-dimensional oriented manifold. The structure group $GL_4$ of the principal frame bundle $LX$ is reducible to its maximal compact subgroup $SO(4)$ since the quotient $GL_4/SO(4)$ is homeomorphic to the Euclidean space $\mathbb{R}^{10}$. It follows that a Riemannian metric $g_R$, represented by a section of the quotient fibre bundle

$$
\Sigma_R := LX/\text{SO}(4) \to X,
$$

always exists on a manifold $X$. The corresponding $\text{SO}(4)$-principal subbundle $L^gX$ is called a Riemannian structure on a world manifold $X$.

Given two different Riemannian metrics $g_R$ and $g'_R$ on $X$, the corresponding $\text{SO}(4)$-principal subbundles $L^gX$ and $L^{g'}X$ of $LX$ are isomorphic as $\text{SO}(4)$-principal bundles since the group space $GL_4$ is homotopic to $\text{SO}(4)$.

To introduce a Riemannian spin structure, one can consider the complex Clifford algebra $\mathbb{C}_4$ which is generated by elements of the vector space $\mathbb{R}^4$ equipped with the Euclidean metric [1, 29]. The corresponding spinor space $V_E$ is a minimal left ideal of $\mathbb{C}_4$ provided with a Hermitian bilinear form. The spin group $\text{Spin}(4)$ is the two-fold universal covering group of the group $\text{SO}(4)$. This is isomorphic to $\text{SU}(2) \otimes \text{SU}(2)$ ([8], p.430).

Let us assume that the second Stiefel–Whitney class $w_2(X)$ of $X$ vanishes. A Riemannian spin structure on a manifold $X$ is defined to be a pair of a $\text{Spin}(4)$-principal bundle $P_s \to X$ and a principal bundle morphism $z$ of $P_s$ to $LX$. Since such a morphism factorizes through a bundle morphism

$$
z_g : P_s \to L^gX
$$

for some Riemannian metric $g_R$, this spin structure is a $g_R$-associated spin structure. We denote the corresponding $g_R$-associated principal spinor bundle by $P^g$. All these
bundles on a 4-dimensional manifold $X$ are isomorphic [2]. Note that, although spin principal bundles $P^g$ and $P^{g'}$ for different Riemannian metric $g$ and $g'$ are isomorphic, the $g$-associated spinor bundle

$$S^g = (P^g \times V_E)/\text{Spin}(4)$$

is not canonically associated with $P^{g'}$. •

The group $GL_4$ is not simply-connected. Its first homotopy group is

$$\pi_1(GL_4) = \pi_1(SO(4)) = \mathbb{Z}_2$$

([1], p.27). Therefore, $GL_4$ admits the universal two-fold covering group $\widetilde{GL}_4$ such that the diagram

$$\begin{array}{ccc}
\widetilde{GL}_4 & \rightarrow & GL_4 \\
\downarrow & & \downarrow \\
\text{Spin}(4) & \rightarrow & \text{SO}(4)
\end{array}$$

(8.2)

is commutative [22, 29, 36, 45].

A universal spin structure on $X$ is defined to be a pair consisting of a $\widetilde{GL}_4$-principal bundle $\widetilde{L}X \rightarrow X$ and a principal bundle morphism over $X$

$$\tilde{z} : \widetilde{L}X \rightarrow LX$$

(8.3)

[4, 36, 45]. There is the commutative diagram

$$\begin{array}{ccc}
\widetilde{L}X & \xrightarrow{\tilde{z}} & LX \\
\downarrow & & \downarrow \\
P^g & \xrightarrow{z_g} & L^gX
\end{array}$$

(8.4)

for any Riemannian metric $g_R$ [36, 45].

Since the group $\widetilde{GL}_4$ is homotopic to the group $\text{Spin}(4)$, there is a one-to-one correspondence between inequivalent universal spin structures and inequivalent Riemannian spin structures [15]. In our case, all universal spin structures as like as the Riemannian ones are equivalent.

Given a universal spin structure (8.3), one can consider the lift of bundle automorphisms of the frame bundle $LX$ (e.g., general covariant transformations) to automorphisms of the principal bundle $\widetilde{L}X \rightarrow X$ and the associated fibre bundles [4]. Spinor representations of the group $\widetilde{GL}_4$, however, are infinite-dimensional [22, 31]. Elements
of this representation are called world spinors. The corresponding field theory has been already developed (see [22] and references therein).

A different procedure is to consider the commutative diagram

\[
\begin{array}{ccc}
\tilde{L}X & \xrightarrow{\psi} & LX \\
\downarrow & & \downarrow \\
\Sigma_R & & \Sigma_R
\end{array}
\]  

(8.5)

and the composite fibre bundle

\[
\tilde{L}X \to \Sigma_R \to X.
\]

Then the restriction of the Spin(4)-principal bundle \(\tilde{L}X \to \Sigma_R\) to \(g_R(X) \subset \Sigma_R\) is isomorphic to the \(g_R\)-associated principal spinor bundle \(P^g\). This is the reason why the spin structure (8.3) is called the universal spin structure. Accordingly, the universal spin structure (8.3) over the fibre bundle \(\Sigma_R\) (8.1), which is given by the diagram (8.5) is said to be the universal Riemannian spin structure.

Let us consider the composite spinor bundle

\[
S \xrightarrow{\pi_{\Sigma_R}} \Sigma_R \to X,
\]

(8.6)

where \(S \to \Sigma_R\) is the spinor bundle associated with the Spin(4)-principal bundle \(\tilde{L}X \to \Sigma_R\). Then, whenever \(g_R\) is a Riemannian metric on \(X\), sections of the spinor bundle \(S^g\) associated with a principal spinor bundle \(P^g\) as in the commutative diagram (8.4) are in bijective correspondence with the sections \(s\) of the composite spinor bundle (8.6) which are projected onto \(g_R\), that is, \(\pi_{\Sigma_R} \circ s = g_R\).

In a similar way, the universal pseudo-Riemannian spin structure can be introduced.

**Remark 17.** It should be emphasized that the total space \(S\) of the spinor bundle (8.6) has the structure of the fibre bundle which is associated with the \(\tilde{GL}_4\)-principal bundle \(\tilde{L}X \to X\) and whose typical fibre is the quotient

\[
(\tilde{GL}_4 \times V_E)/\text{Spin}(4)
\]

(8.7)

by identification of the elements

\[
(\tilde{g}, v) \simeq (a\tilde{g}, a^{-1}v), \quad \tilde{g} \in \tilde{GL}_4, \ v \in V_E, \ a \in \text{Spin}(4).
\]

Then, every morphism of the quotient \(8.7\) into the spin representation space of the group \(\tilde{GL}_4\) yields the corresponding morphisms of the composite spinor bundle (8.6) into the \(\tilde{GL}_4\)-associated bundle of world spinors.
9 Spontaneous symmetry breaking

The construction above using composite fibre bundles illustrates the standard description of spontaneous symmetry breaking in gauge theories where matter fields admit only exact symmetry transformations [39, 41].

Spontaneous symmetry breaking is a quantum phenomenon. In classical field theory, spontaneous symmetry breaking is modelled by classical Higgs fields. In gauge theory on a principal bundle $P \to X$, the necessary condition for spontaneous symmetry breaking is the reduction of the structure group $G$ of this principal bundle to its closed subgroup $H$ of exact symmetries [25, 32, 46]. Higgs fields are described by global sections $h$ of the quotient fibre bundle $\Sigma$.

In accordance with Theorem 2, the set of Higgs fields $h$ is in bijective correspondence with the set of reduced $H$-principal subbundles $P^h$ of $P$. Given such a subbundle $P^h$, let

$$Y^h = (P^h \times V)/H$$

be the associated fibre bundle with a typical fibre $V$. Its sections describe matter fields in the presence of the Higgs fields $h$.

If $V$ does not admit the action of the whole symmetry group $G$, the fibre bundle $Y^h$ (9.1) is not associated canonically with other $H$-principal subbundles. It follows that $V$-valued matter fields can be represented by pairs with a certain Higgs field only. The goal is to describe the totality of these pairs $(s_h, h)$ for all Higgs fields.

Let us consider the composite fibre bundle (4.8) and the composite fibre bundle

$$Y \xrightarrow{\pi_Y} \Sigma \xrightarrow{\pi} X,$$

where $Y \to \Sigma$ is the fibre bundle

$$Y = (P \times V)/H$$

associated with the principal bundle $P_\Sigma$ (4.10) with the structure group $H$ which acts on the typical fibre $V$ of $Y$ on the left. Given a global section $h$ of the fibre bundle $\Sigma \to X$ (4.10), let

$$Y^h = (P^h \times V)/H$$

be a fibre bundle associated with the reduced $H$-principal subbundle $P^h$ of $P$. There is the canonical isomorphism

$$i_h : Y^h = (P^h \times V)/H \to (h^* P \times V)/H$$
of $Y^h$ to the subbundle of $Y \to X$ which is the restriction
$$h^*Y = (h^* P \times V)/H$$
of the fibre bundle $Y \to \Sigma$ to $h(X) \subset \Sigma$. We have
$$i_h(Y^h) \cong \pi_{Y \Sigma}^{-1}(h(X)). \quad (9.5)$$

Then every global section $s_h$ of the fibre bundle $Y^h$ corresponds to the global section
$i_h \circ s_h$ of the composite fibre bundle (9.2). Conversely, every global section $s$ of the composite fibre bundle (9.2) which is projected onto a section $h = \pi_{Y \Sigma} \circ s$ of the fibre bundle $\Sigma \to X$ takes its values into the subbundle $i_h(Y^h) \subset Y$ in accordance with the relation (9.3). Thus, there is a one-to-one correspondence between the sections of the fibre bundle $Y^h$ (9.4) and the sections of the composite fibre bundle (9.2) which cover the section $h$.

Thus, it is the composite fibre bundle (9.2) whose sections describe the above mentione totality of the pairs $(s_h, h)$ of matter fields and Higgs fields in gauge theory with broken symmetries [39, 42].

The feature of the dynamics of field systems on composite fibre bundles consists in the following.

Let $Y$ (9.2) be a composite fibre bundle coordinatized by $(x^\lambda, \sigma^m, y^i)$, where $(x^\lambda, \sigma^m)$ are bundle coordinates of the fibre bundle $\Sigma \to X$. Let
$$A_{\Sigma} = dx^\lambda \otimes (\partial_\lambda + A^i_\lambda \partial_i) + d\sigma^m \otimes (\partial_m + A^i_m \partial_i) \quad (9.6)$$
be a principal connection on the fibre bundle $Y \to \Sigma$. This connection defines the splitting
$$VY = VY_{\Sigma} \oplus (Y \times V\Sigma)$$
$$\dot{y}^i \partial_i + \dot{\sigma}^m \partial_m = (\dot{y}^i - A^i_m \dot{\sigma}^m) \partial_i + \dot{\sigma}^m (\partial_m + A^i_m \partial_i).$$

Using this splitting, one can construct the first order differential operator
$$\widetilde{D} : J^1Y \to T^*X \otimes VY_{\Sigma},$$
$$\widetilde{D} = dx^\lambda \otimes (y^i_\lambda - A^i_\lambda - A^i_m \partial_m h^m) \partial_i, \quad (9.7)$$
on the composite fibre bundle $Y$.

The operator (9.7) posesess the following property. Given a global section $h$ of $\Sigma$, its restriction
$$\widetilde{D}_h = \widetilde{D} \circ J^1i_h : J^1Y^h \to T^*X \otimes VY^h,$$
$$\widetilde{D}_h = dx^\lambda \otimes (y^i_\lambda - A^i_\lambda - A^i_m \partial_m h^m) \partial_i, \quad (9.8)$$
to \( Y^h \) is exactly the familiar covariant differential relative to the principal connection

\[
A^h = dx^\lambda \otimes \left[ \partial_\lambda + \left( A^i_m \partial_\lambda h^m + A^i_\lambda \right) \partial_i \right]
\]
on the fibre bundle \( Y^h \to X \), which is induced by the principal connection \((9.6)\) on the fibre bundle \( Y \to \Sigma \) by the imbedding \( i_h \) \((20), p.81)\).

Thus, we may utilize \( \widetilde{D} \) in order to construct a Lagrangian density on the jet manifold \( J^1Y \) of a composite fibre bundle which factorizes through \( \widetilde{D} \), that is,

\[
L : J^1Y \to \widetilde{D}T^*X \otimes ^{\pi} \Sigma \to ^{\pi^*} T^*X.
\]

**Remark 18.** The total space of the composite fibre bundle \( Y \to X \) \((9.2)\) can be represented as the quotient of the product \( P \times G \times V \) by identification of the elements

\[
(p, g, v) \simeq (pab, b^{-1}g, a^{-1}v), \quad \forall a \in H, \quad \forall b \in G.
\]

It follows that \( Y \) has the structure of the \( P \)-associated bundle

\[
Y = \left( P \times (G \times V)/H \right)/G
\]

with the structure group \( G \) and the typical fibre \((G \times V)/H\) which is the quotient of the product \( G \times V \) by identification of the elements

\[
(g, v) \simeq (ag, a^{-1}v), \quad \forall a \in H.
\]

In particular, if the typical fibre \( V \) of the composite fibre bundle \( Y \to X \) admits the action of the group \( G \), these two bundle structures on \( Y \) are equivalent. \( \bullet \)

### 10 Universal pseudo-Riemannian spin structure

Let us turn now to fermion fields in gauge gravitation theory. We are based on the following two facts.

**Proposition 6.** The \( L \)-principal bundle

\[
P_L := GL_4 \to GL_4/L \tag{10.1}
\]
is trivial. \( \square \)

**Proof.** In accordance with the classification theorem \((14), p.99\), a \( G \)-principal bundle over an \( n \)-dimensional sphere \( S^n \) is trivial if the homotopy group \( \pi_{n-1}(G) \) is trivial. The
base space \( Z = GL_4/L \) of the principal bundle (10.1) is homeomorphic to \( S^3 \times \mathbb{R}^7 \). Let us consider the inclusion \( f_1 \) of \( S^3 \) into \( Z \), \( f_1(p) = (p,0) \), and the pull-back \( L \)-principal bundle \( f_1^* P_L \to S^3 \). Since \( L \) is homeomorphic to \( \mathbb{RP}^3 \times \mathbb{R}^3 \) and \( \pi_2(L) = 0 \), this bundle is trivial. Let \( f_2 \) be the projection of \( Z \) onto \( S^3 \). Then the pull-back \( L \)-principal bundle \( f_2^*(f_1^* P_L) \to Z \) is also trivial. Since the composition morphism \( f_1 \circ f_2 \) of \( Z \) into \( Z \) is homotopic to the identity morphism of \( Z \), the bundle \( f_2^*(f_1^* P_L) \to Z \) is equivalent to the bundle \( P_L \) ([14], p.53). It follows that the bundle (10.1) is trivial. •

**Proposition 7.** As in (8.2), we have the commutative diagram

\[
\begin{array}{ccc}
\widetilde{GL}_4 & \longrightarrow & GL_4 \\
\downarrow & & \downarrow \\
L_s & \longrightarrow & L \\
\end{array}
\]  

(10.2)

\[\square\]

**Proof.** The restriction of the universal covering group \( \widetilde{GL}_4 \to GL_4 \) to the Lorentz group \( L \subset GL_4 \) is obviously a covering space of \( L \). Let us show that this is the universal covering space. Indeed, any non-contractible cycle in \( GL_4 \) belongs to some subgroup \( SO(3) \subset GL_4 \) and the restriction of the fibre bundle \( \widetilde{GL}_4 \to GL_4 \) to \( SO(3) \subset GL_4 \) is the universal covering of \( SO(3) \). Since the proper Lorentz group is homotopic to its maximal compact subgroup \( SO(3) \), its universal covering space belongs to \( \widetilde{GL}_4 \). •

Let us consider the universal spin structure \( \widetilde{LX} \to X \). This is unique since \( X \) is parallelizable. In virtue of Proposition 4, we have the commutative diagram

\[
\begin{array}{ccc}
\widetilde{LX} & \xrightarrow{z} & LX \\
\downarrow & & \downarrow \\
P^h & \xrightarrow{z_h} & \widetilde{L^hX} \\
\end{array}
\]  

(10.3)

for any tetrad field \( h \). It follows that the quotient \( \widetilde{LX}/L_a \) is exactly the quotient \( \Sigma_T \) (5.1) so that there is the commutative diagram

\[
\begin{array}{ccc}
\widetilde{LX} & \xrightarrow{z} & LX \\
\downarrow & & \downarrow \\
\Sigma_T & \xrightarrow{\mu} & \Sigma_T \\
\end{array}
\]  

(10.4)
By analogy with the diagram (8.3), the diagram (10.4) is said to be the universal pseudo-Riemannian spin structure. We have the composite fibre bundle

\[ \widetilde{LX} \to \Sigma_T \to X, \quad (10.5) \]

where \( \widetilde{LX} \to \Sigma_T \) is the \( L_s \)-principal bundle.

The universal pseudo-Riemannian spin structure 10.4 can be regarded as the \( L_s \)-spin structure on the fibre bundle of Minkowski spaces

\[ E_M = (LX \times M)/L \to \Sigma_T \]

associated with the \( L \)-principal bundle \( LX \to \Sigma_T \). Since the principal bundles \( LX \) and \( P_L \) (10.1) are trivial, the fibre bundle \( E_M \to \Sigma_T \) also is trivial, and this is isomorphic to the pullback

\[ \Sigma_T \times T^* X. \quad (10.6) \]

**Remark 19.** Since the bundle \( \Sigma_T \to X \) is trivial, the fibre bundle \( E_M \) is equivalent to the trivial bundle of Minkoski spaces over the product \( S^3 \times \mathbb{R}^7 \times X \). It follows that the set of inequivalent spin structures on the bundle \( E_M \) is in bijective correspondence with the cohomology group \( H^1(S^3 \times \mathbb{R}^7 \times X; \mathbb{Z}_2) \) (29, p.82). Since the cohomology group \( H^1(S^3; \mathbb{Z}_2) \) is trivial and the spin structure on \( S^3 \) is unique [8], one can show that inequivalent spin structures on \( E_M \) are classified by elements of the cohomology group \( H^1(X; \mathbb{Z}^2) \) and, consequently, by inequivalent spin structures on \( X \). It follows that the spin structure (10.4) on the fibre bundle \( E_M \) is unique. •

Following the general discussion on spontaneous symmetry breaking in the previous Section, let us consider the composite spinor bundle

\[ S \xrightarrow{\pi_{S\Sigma_T}} \Sigma_T \xrightarrow{\pi_{S\Sigma_T}} X, \quad (10.7) \]

where

\[ S = (\widetilde{LX} \times V)/L_s \]

is the spinor bundle \( S \to \Sigma_T \) associated with the \( L_s \)-principal bundle \( \widetilde{LX} \to \Sigma_T \).

Given a tetrad field \( h \), there is the canonical isomorphism

\[ i_h : S^h = (P^h \times V)/L_s \to (h^* \widetilde{LX} \times V)/L_s \]
of the $h$-associated spinor bundle $S^h$ (5.3) to the restriction $h^*S$ of the spinor bundle $S \to \Sigma_T$ to $h(X) \subset \Sigma_T$. Then, every global section $s_h$ of the spinor bundle $S^h$ corresponds to the global section $i_h \circ s_0$ of the composite spinor bundle (10.7). Conversely, every global section $s$ of the composite spinor bundle (10.7) which is projected onto a tetrad field $h$ takes its values into the subbundle $i_h(S^h) \subset S$.

Let the frame bundle $LX \to X$ be provided with a holonomic atlas (2.3) and let the principal bundles $\tilde{L}X \to \Sigma_T$ and $LX \to \Sigma_T$ have the associated atlases $\{z^2, U_2\}$ and $\{z_\epsilon = \tilde{z} \circ z_0^\epsilon, U_\epsilon\}$. With these atlases, the composite spinor bundle $S$ is equipped with the fibre spinor metric

$$a_S(v, v') = \frac{1}{2}(v^+ \gamma^0 v' + v'^+ \gamma^0 v), \quad \pi_{S\Sigma}(v) = \pi_{S\Sigma}(v').$$

Since the fibre bundle of Minkowski spaces $E_M \to \Sigma_T$ is isomorphic to the pull-back bundle (10.6), there exists the representation

$$\gamma_{\Sigma} : T^*X \otimes S = (\tilde{L}X \times (M \otimes V))/L_s \to (\tilde{L}X \times \gamma(M \otimes V))/L_s = S,$$

(10.8)

given by the coordinate expression

$$\hat{dx}^\lambda = \gamma_{\Sigma}(dx^\lambda) = \sigma^a_\lambda \gamma^a.$$  

Restricted to $h(X) \subset \Sigma_T$, this representation recovers the morphism $\gamma_h$ (6.4).

Using this representation, one can construct the total Dirac operator on the composite spinor bundle $S$ as follows.

Since the composite fibre bundle (10.4) is the composition of trivial bundles

$$\tilde{L}X := L_s \times GL_4/L \times X \to GL_4/L \times X \to X,$$

let us consider a principal connection $A_{\Sigma}$ (9.6) on the $L_s$-principal bundle $\tilde{L}X \to \Sigma_T$ given by the local connection form

$$A_{\Sigma} = (A_\lambda^a dx^\lambda + A_{\mu}^{ab} d\sigma_{k}^{\mu}) \otimes L_{ab},$$

(10.9)

where

$$A_\lambda^{ab} = \frac{1}{2}(\eta^{kb} \sigma_{k}^{a} - \eta^{ka} \sigma_{k}^{b})\sigma^{\nu}_{k} K_{\lambda}^{\mu \nu},$$

$$A_{\mu}^{ab} = \frac{1}{2}(\eta^{kb} \sigma_{k}^{a} - \eta^{ka} \sigma_{k}^{b}).$$

(10.10)
and $K$ is a world connection on $X$. We choose this connection because of the following properties.

The principal connection (10.9) defines the associated spin connection

$$A_S = dx^\lambda \otimes (\partial_\lambda + \frac{1}{2} A_{\lambda}^{ab} L_{ab}^A B y^B \partial_A) + d\sigma_k^\mu \otimes (\partial_\mu + \frac{1}{2} A_{\mu}^{kab} L_{ab}^A B y^B \partial_A)$$

(10.11)
on the spinor bundle $S \rightarrow \Sigma_T$. Let $h$ be a global section of $\Sigma_T \rightarrow X$ and $S^h$ the restriction of the bundle $S \rightarrow \Sigma_T$ to $h(X)$. It is readily observed that the restriction of the spin connection (10.11) to $S^h$ is exactly the spin connection (7.7).

The connection (10.11) yields the first order differential operator $\tilde{D}$ (9.7) on the composite spinor bundle $S \rightarrow X$. This reads

$$\tilde{D} : J^1 S \rightarrow T^* X \otimes S,$$

$$\tilde{D} = dx^\lambda \otimes [y_A^A - \frac{1}{2} (A_{\lambda}^{ab} + A_{\mu}^{kab} \sigma_k^\mu) L_{ab}^A B y^B] \partial_A =$$

$$dx^\lambda \otimes [y_A^A - \frac{1}{4} (\eta^{kb} \sigma_k^\mu - \eta^{ka} \sigma_k^\mu) (\sigma_\lambda^\mu - \sigma_\lambda^\nu K^\mu_\nu) L_{ab}^A B y^B] \partial_A.$$ 

(10.12)

The corresponding restriction $\tilde{D}_h$ (9.8) of the operator $\tilde{D}$ (10.12) to $J^1 S^h \subset S^1 S$ recovers the familiar covariant differential on the $h$-associated spinor bundle $S^h \rightarrow X$ relative to the spin connection (7.7).

The composition of the representation (10.8) and the differential (10.12) leads to the first order differential operator

$$\mathcal{D} = \gamma_\Sigma \circ \tilde{D} : J^1 S \rightarrow T^* X \otimes S \rightarrow S,$$

$$y^B \circ \mathcal{D} = \sigma_a^\lambda \gamma_{ab}^A [y_A^A - \frac{1}{4} (\eta^{kb} \sigma_k^\mu - \eta^{ka} \sigma_k^\mu) (\sigma_\lambda^\mu - \sigma_\lambda^\nu K^\mu_\nu) L_{ab}^A B y^B].$$

(10.13)
on the composite spinor bundle $S \rightarrow X$. One can think of $\mathcal{D}$ as being the total Dirac operator on $S$ since, for every tetrad field $h$, the restriction of $\mathcal{D}$ to $J^1 S^h \subset J^1 S$ is exactly the Dirac operator $\mathcal{D}_h$ (9.8) on the $h$-associated spinor bundle $S^h$ in the presence of the background tetrad field $h$ and the spin connection (7.7).

It follows that gauge gravitation theory is reduced to the model of metric-affine gravity and Dirac fermion fields.

The total configuration space of this model is the jet manifold $J^1 Y$ of the bundle product

$$Y = (C_K X \times \Sigma_T) \times S = C_K \times S$$

(10.14)
coordinatized by \( (x^\mu, \sigma^\mu, k_\mu^{\alpha\beta}, y^A) \), where \( C_K \) is the bundle of world connections (2.12).

Let \( J^1_Y \) denotes the first order jet manifold of the fibre bundle \( Y \rightarrow \Sigma_T \). This fibre bundle can be provided with the spin connection

\[
A_Y : Y \rightarrow J^1_Y \xrightarrow{pr} J^1_S,
\]

\[
A_Y = dx^\lambda \otimes (\partial_\lambda + \tilde{A}_\lambda^{ab} L^{A}_{ab} B^B \partial_A) + d\sigma^\mu_k \otimes (\partial^a_k + A^{kab}_\mu L^{A}_{ab} B^B \partial_A), \tag{10.15}
\]

where

\[
\tilde{A}_\lambda^{ab} = -\frac{1}{2} (\eta^{a b} \sigma^a_\mu - \eta^{k a} \sigma^b_\mu)(\sigma^\mu_k - \sigma^\nu_k k_\lambda^{\mu\nu}) L^{A}_{ab} B^B \partial_A
\]

and \( A^{kab}_\mu \) is given by the expression (10.10).

Using the connection (10.15), we get the first order differential operator

\[
\tilde{D}_Y : J^1 Y \rightarrow T^* X \otimes S,
\]

\[
\tilde{D}_Y = dx^\lambda \otimes [y^A_\lambda - \frac{1}{4}(\eta^{a b} \sigma^a_\mu - \eta^{k a} \sigma^b_\mu)(\sigma^\mu_k - \sigma^\nu_k k_\lambda^{\mu\nu}) L^{A}_{ab} B^B \partial_A] \tag{10.16}
\]

and the total Dirac operator

\[
\mathcal{D}_Y = \gamma_\Sigma \circ \tilde{D} : J^1 Y \rightarrow T^* X \otimes S \rightarrow S, \tag{10.17}
\]

\[
y^B \circ \mathcal{D} = \sigma^A_\alpha \gamma^a_\beta [y^A_\lambda - \frac{1}{4}(\eta^{a b} \sigma^a_\mu - \eta^{k a} \sigma^b_\mu)(\sigma^\mu_k - \sigma^\nu_k k_\lambda^{\mu\nu}) L^{A}_{ab} B^B]
\]

on the fibre bundle \( Y \rightarrow X \), where \( \gamma_\Sigma \) denotes the pull-back of the morphism (10.8) onto \( Y \rightarrow (C_K \times \Sigma_T) \).

Given a section \( K : X \rightarrow C_K \), the restrictions of the spin connection \( A_Y \) (10.15), the operator \( \tilde{D}_Y \) (10.16) and the Dirac operator \( \mathcal{D}_Y \) (10.17) to \( K^* Y \) are exactly the spin connection (10.11) and the operators (10.12) and (10.13), respectively.

The total Lagrangian density on the configuration space \( J^1 Y \) of the metric-affine gravity and fermion fields is the sum

\[
L = L_{MA} + L_D \tag{10.18}
\]

of the metric-affine Lagrangian density

\[
L_{MA}(k_\lambda^{\alpha\beta}, \sigma^{\mu\nu}), \quad \sigma^{\mu\nu} = \sigma^a_\mu \sigma^b_\nu \eta^{ab},
\]

and of the Dirac Lagrangian density

\[
L_D = [a_Y (i \mathcal{D}_Y (w), w) - ma_S (w, w)] \sigma^0 \wedge \cdots \wedge \sigma^3, \quad w \in J^1 S,
\]
where $\sigma^a = \sigma^a_\mu dx^\mu$ and $a_Y$ is the pull-back of the fibre spinor metric $a_S$ onto the fibre bundle $Y \to (C_K \times \Sigma_T)$. Its coordinate expression is

$$L_D = \left\{ \frac{i}{2} \sigma^\lambda_q [y^+_A (\gamma^0 \gamma^q)^A_B (y^B \gamma^C) - \frac{1}{4} (\eta^{kb} \sigma^a_\mu - \eta^{ka} \sigma^b_\mu) (\sigma^\lambda_{kb} - \sigma^\lambda_{ka} \sigma^b_\mu) L_{ab}^{\ C} y^C] - (y^+_A - \frac{1}{4} (\eta^{kb} \sigma^a_\mu - \eta^{ka} \sigma^b_\mu) (\sigma^\lambda_{kb} - \sigma^\lambda_{ka} \sigma^b_\mu) y^+_C L^{\ C}_{ab} A (\gamma^0 \gamma^A B y^B) - m y^+_A (\gamma^0 A B y^B) \right\} \sqrt{|\sigma|}, \quad \sigma = \det(\sigma_{\mu\nu}).$$ (10.19)

It is readily observed that

$$\frac{\partial L_\psi}{\partial k^\mu_{\nu\lambda}} + \frac{\partial L_\psi}{\partial k^\mu_{\lambda\nu}} = 0,$$ (10.20)

that is, the Dirac Lagrangian density (10.13) depends on the torsion

$$S_{\mu \nu} = k^\alpha_{\mu \nu} - k^\alpha_{\nu \mu}$$

of a world connection.

11 General covariant transformations

Let us turn now to general covariant transformations.

Since the world manifold $X$ is parallelizable and the universal spin structure is unique, the $\tilde{G}L_4$-principal bundle $\tilde{L}X \to X$, as like as the frame bundle $LX$, admits a canonical lift of any diffeomorphism $f$ of the base $X$. This lift is defined by the commutative diagram

$$\begin{array}{ccc}
\tilde{L}X & \xrightarrow{\Phi} & \tilde{L}X \\
\downarrow & & \downarrow \\
LX & \xrightarrow{\phi} & LX \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & X
\end{array}$$

where $\Phi$ is the holonomic bundle automorphism of $LX$ (2.3) induced by $f$ [7].

The associated morphism of the spinor bundle $S$ (10.7) is given by the relation

$$\tilde{\Phi}_S : [p, v]_{L_s} \to [\tilde{\Phi}(p), v]_{L_s}, \quad p \in \tilde{L}X, \ v \in S.$$ (11.1)

Because $\tilde{\Phi}$ is equivariant, this is a fibre-to-fibre automorphism of the bundle $S \to \Sigma_T$ over the canonical automorphism of the $LX$-associated tetrad bundle $\Sigma_T \to X$ (5.1).
which is projected onto the diffeomorphism \( f \) of \( X \). Thus, we have the commutative diagram of general covariant transformations of the spinor bundle \( S \):

\[
\begin{array}{ccc}
S & \xrightarrow{\tilde{\Phi}_S} & S \\
\downarrow & & \downarrow \\
\Sigma_T & \xrightarrow{\Phi_S} & \Sigma_T \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & X \\
\end{array}
\]

Accordingly, there exists a canonical lift \( \tilde{\tau} \) of every vector field \( \tau \) on \( X \) over \( S \). The goal is to discover its coordinate expression. A difficulty arises because tetrad coordinates \( \sigma^\mu_b \) of \( \Sigma_T \) depend on the atlas of the bundle \( LX \to \Sigma_T \). Therefore, non-canonical vertical components appear in the coordinate expression of \( \tilde{\tau} \).

A comparison with the canonical lift (2.11) of a vector field \( \tau \) over the metric bundle \( \Sigma_{PR} \) shows that the similar canonical lift of \( \tau \) over the tetrad bundle \( \Sigma_T \) coordinatized by \((x^\lambda, \sigma^\mu_a)\) takes the form

\[
\tau_{\Sigma} = \tau^\lambda \partial_{\lambda} + \partial_{\nu} \tau^\mu \sigma^\nu_c \frac{\partial}{\partial \sigma^\mu_c} + Q^\mu_c \frac{\partial}{\partial \sigma^\mu_c},
\]

where the coefficients \( Q^\mu_c \) obey the conditions

\[
(Q^\mu_a \sigma^\nu_b + Q^\nu_a \sigma^\mu_b)\eta^{ab} = 0.
\]

These coefficients \( Q^\mu_a \) represent the above mentioned non-canonical part of the lift (11.2).

Let us consider a horizontal lift \( \tilde{\tau}_S \) of the vector field \( \tau_{\Sigma} \) over the spinor bundle \( S \to \Sigma_T \) by means of the spin connection (10.11). Let \( K \) be a symmetric connection whose geodesic vector field is \( \tau \). We find that the canonical part of the vector field \( \tau_{\Sigma} \) is lifted identically, and \( \tilde{\tau}_S \) reads

\[
\tilde{\tau}_S = \tau^\lambda \partial_{\lambda} + \partial_{\nu} \tau^\mu \sigma^\nu_c \frac{\partial}{\partial \sigma^\mu_c} + \frac{1}{4}Q^\mu_c \frac{\partial}{\partial \sigma^\mu_c} + \frac{1}{4}Q^\mu_k (\eta^{kb} \sigma^a_c - \eta^{ka} \sigma^b_c)(L^A_{ab} y^B \partial_A + L^+_A y^+_B \partial^B).
\]

This can be brought into the form

\[
\tilde{\tau}_S = \tau^\lambda \partial_{\lambda} + \partial_{\nu} \tau^\mu \sigma^\nu_c \frac{\partial}{\partial \sigma^\mu_c} + \frac{1}{4}Q^\mu_k \eta^{kb} \sigma^a_c - \eta^{ka} \sigma^b_c)\left[-L^d_{ab} \sigma^\nu_d \frac{\partial}{\partial \sigma^\nu_c} + L^A_{ab} y^B \partial_A + L^+_A y^+_B \partial^B\right],
\]

34
where $L_{ab}^d c$ are the generators (11.3). The corresponding total vector field on the bundle product $Y$ (10.14) reads

$$
\tilde{\tau}_Y = \tau^\lambda \partial_\lambda + \partial_\nu \tau^\mu \sigma^\nu_c \partial_{\sigma^d_c} +
$$

\[\partial_\nu \tau^\alpha k^\nu_{\mu} \beta - \partial_\beta \tau^\nu k^\alpha_{\nu} - \partial_{\mu} \tau^\nu k^\alpha_{\beta} + \partial_{\mu \beta} \tau^\alpha \]
\[\frac{1}{4} Q^\mu_k [\eta^k b_{\sigma^d_c} - \eta^k a_{\sigma^d_c}] - L^d_{ab} c \sigma^\nu_d \partial_{\sigma^d_c} + L^A B y^B \partial_A + L^A_{ab} B y^A \partial^B].
\]

The Dirac Lagrangian density (10.18), by construction, is invariant separately under transformations of holonomic atlases of the frame bundle $LX$ (passive covariant transformations acting on the Greek indices) and under transformations of atlases of the principal bundles $\tilde{LX} \rightarrow \Sigma_T$ and $LX \rightarrow \Sigma$ (passive spin and Lorentz gauge transformations acting on the Latin indices). It follows that this Lagrangian density is invariant under infinitesimal active gauge transformations whose generators are the vector fields (11.3). Moreover, one can exclude from calculations the last term in these vector fields which leads to the Nöther flow and can, thus, consider only their canonical part

$$
\tilde{\tau} = \tau^\lambda \partial_\lambda + \partial_\nu \tau^\mu \sigma^\nu_c \partial_{\sigma^d_c} +
$$

\[\partial_\nu \tau^\alpha k^\nu_{\mu} \beta - \partial_\beta \tau^\nu k^\alpha_{\nu} - \partial_{\mu} \tau^\nu k^\alpha_{\beta} + \partial_{\mu \beta} \tau^\alpha \]
\[\frac{1}{4} Q^\mu_k [\eta^k b_{\sigma^d_c} - \eta^k a_{\sigma^d_c}] - L^d_{ab} c \sigma^\nu_d \partial_{\sigma^d_c} + L^A B y^B \partial_A + L^A_{ab} B y^A \partial^B].
\]

in order to obtain the energy-momentum conservation laws [13, 15, 43].

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