Unitary units in modular group algebras

Victor Bovdi and L. G. Kovács

Let p be a prime, K a field of characteristic p, G a locally finite p-group, KG the group algebra, and V the group of the units of KG with augmentation 1. The anti-automorphism $g \mapsto g^{-1}$ of G extends linearly to KG; this extension leaves V setwise invariant, and its restriction to V followed by $v \mapsto v^{-1}$ gives an automorphism of V. The elements of V fixed by this automorphism are called unitary; they form a subgroup. Our first theorem describes the $K$ and $G$ for which this subgroup is normal in V.

For each element $g$ in G, let $\bar{g}$ denote the sum (in KG) of the distinct powers of $g$. The elements $1 + (g - 1)h\bar{g}$ with $g, h \in G$ are the bicyclic units of KG. Our second theorem describes the $K$ and $G$ for which all bicyclic units are unitary.

1. Introduction

Let KG be the group algebra of a group G over a commutative ring $K$ (with 1) and $V(KG)$ the group of normalized units (that is, of the units with augmentation 1) in KG. The anti-automorphism $g \mapsto g^{-1}$ extends linearly to an anti-automorphism $a \mapsto a^*$ of KG; this extension leaves $V(KG)$ setwise invariant, and its restriction to $V(KG)$ followed by $v \mapsto v^{-1}$ gives an automorphism of $V(KG)$. The elements of $V(KG)$ fixed by this automorphism are the unitary normalized units of KG; they form a subgroup which we denote by $V_*(KG)$.

(Interest in unitary units arose in algebraic topology, and a more general definition, involving an 'orientation homomorphism', is also current; the special case we use here arises when the orientation homomorphism is trivial.)

The first question considered here is to find the pairs $K, G$ for which $V_*(KG)$ is normal in $V(KG)$. (Since each unit of a group algebra is a scalar multiple of a normalized unit, if $V_*(KG)$ is normal in $V(KG)$ then it is normal also in the group of all units of KG.) For $K = \mathbb{Z}$, this question was discussed

Research partly supported by the Hungarian National Foundation for Scientific Research grant no. T4265.

The second author is indebted to the 'Universitas' Foundation and the Lajos Kossuth University of Debrecen, Hungary, for warm hospitality and generous support during the period when this work began.
Theorem 1.1. Let $K$ be a field of prime characteristic $p$ and let $G$ be a nonabelian locally finite $p$-group. The subgroup $V_e(KG)$ is normal in $V(KG)$ if and only if $p = 2$ and $G$ is the direct product of an elementary abelian group with a group $H$ for which one of the following holds:

(i) $H$ has no direct factor of order 2, but it is a semidirect product of a group $\langle h \rangle$ of order 2 and an abelian 2-group $A$, with $h^{-1}ah = a^{-1}$ for all $a$ in $A$;

(ii) $H$ is an extraspecial 2-group, or the central product of such a group with a cyclic group of order 4.

We work with the definition that a $p$-group is extraspecial if its centre, commutator subgroup and Frattini subgroup are equal and have order $p$: we do not require the group itself to be finite.

The proof of this theorem will be given in Section 2. The reason we take the $p$-group $G$ locally finite is that, as is well known, this ensures that each non-unit of $KG$ lies in the augmentation ideal.

Every group $G$ may be written (see Lemma 2.3) as a direct product of an elementary abelian 2-group $E$ and a group $H$ which has no direct factor of order 2 (we do not exclude $E = 1$ or $H = 1$). The isomorphism type of $G$ determines the isomorphism types of $E$ and $H$, and vice versa.

It is easy to verify that if $H$ satisfies (i) then $A = \{a \in H \mid a^2 \neq 1\}$ and $A$ has no direct factor of order 2. Conversely, if $A$ is a nontrivial abelian 2-group without a direct factor of order 2 and $H$ is formed as the semidirect product indicated, then (i) holds. The classification of the groups $H$ of this kind is thus reduced to the classification of abelian 2-groups, a problem whose solution in terms of Ulm invariants is well known in the finite or countably infinite case but is beyond reach in general.

As to case (ii), the classification of finite extraspecial groups is well known. Equally conclusive results were obtained for extraspecial groups of countably infinite order by M. F. Newman in [9]; he also showed there that no such results can be expected for extraspecial groups of arbitrary order.

The only group $H$ which satisfies both conditions (i) and (ii) is the dihedral group of order 8.

The second part of the paper concerns the bicyclic units introduced in Ritter and Sehgal [11]. For $K$ a commutative ring and $g$ an element of finite order $|g|$ in a group $G$, let $\bar{g}$ denote the sum (in $KG$) of the distinct powers of $g$:

$$\bar{g} = \sum_{i=0}^{\lfloor |g|^{-1} \rfloor} g^i.$$ 

If also $h \in G$, put

$$u_{g,h} = 1 + (g - 1)h\bar{g}.$$