Support Theorems for Horocycles on Hyperbolic Spaces.

Sigurdur Helgason

In Memory of S.S. Chern

1 Introduction

The Radon transform associates to a function on a space $X$ a function $\hat{f}$ on a family $\Xi$ of subsets $\xi \subset X$ with the definition,

$$\hat{f}(\xi) = \int_{\xi} f(x) \, dm(x), \quad \xi \in \Xi,$$

$dm$ being a given measure on each $\xi$. Radon’s original question [9] was whether this mapping $f \to \hat{f}$ was injective, in other words whether $f$ is determined by the integrals (1.1). Along with this injectivity problem, determining the range of the mapping $f \to \hat{f}$ is an interesting question.

A part of this question is the so-called support theorem. While the implication

$$\text{supp} (f) \text{ compact} \Rightarrow \text{supp} (\hat{f}) \text{ compact}$$

(supp denoting support) will usually hold for simple reasons, the converse implication

$$\text{supp} (\hat{f}) \text{ compact} \Rightarrow \text{supp} (f) \text{ compact}$$

is designated the support theorem (usually with extra assumption on $f$). Positive answers for some examples lead to various applications:

(i) An explicit description of the range $D(X)$ where $X$ is a Euclidean space or a symmetric space of the noncompact type ([2], [3]). Here ($D = C_c^{\infty}$). In the first case, $\hat{f}$ in (1.1) is integration over hyperplanes in $X = \mathbb{R}^n$; in the latter case $\hat{f}$ in (1.1) refers to integration over horocycles $\xi$ in the symmetric space $X$.

(ii) Medical application in X-ray reconstruction ([6], p.47).

(iii) Existence theorem for invariant differential equations on a symmetric space $X$ ([3], Lemma 8.1 and Theorem 8.2).

While these results rely on special methods for each case, microlocal analysis has been used e.g. by Quinto [8] for results of more general nature, requiring however stronger a priori assumptions about $f$ and its support.

For a symmetric space $X$ of the noncompact type there are two natural Radon transforms, the X-ray transform and the horocycle transform; in both cases (1.3) holds ([4], [3]). If $X$ has rank one, then a horocycle has codimension one and its interior is well defined ([1]). Thus one can raise the question of a support theorem for the X-ray transform $f \to \hat{f}$ relative to a fixed horocycle. If
is assumed exponentially decreasing, the support theorem does indeed hold (\cite{5}). Specifically, a function on $X$ is said to be exponential decreasing if

$$\sup_x f(x)e^{md(0,x)} < \infty$$

for each $m > 0, 0 \in X$ denoting the origin and $d$ the distance.

For $X$ a hyperbolic space we consider in this note the analogous question for the horocycle transform $f \rightarrow \hat{f}$, relative to a fixed horocycle (Theorem 2.2), extending a result by Lax and Phillips (\cite{7}).

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2 The horocycle transform on $H^n$

For the support question we take the hyperbolic space $H^n$ with the metric

$$ds^2 = \frac{dx_1^2 + \cdots + dx_n^2}{x_n^2}, \quad x_n > 0.$$ \hspace{1cm} (2.1)

In the metric (2.1) the geodesics are the circular arcs perpendicular to the plane $x_n = 0$; among these are the half lines perpendicular to $x_n = 0$. The horocycles perpendicular to these last geodesics are the planes $x_n = \text{const}$. The other horocycles are the Euclidean $(n-1)$-spheres tangential to the boundary.

Let $\xi \subset H^n$ be a horocycle in the half space model. It is a Euclidean sphere with center $(x',r)$ (where $x' = (x_1, \ldots, x_{n-1})$) and radius $r$. We consider the intersection of $\xi$ with the $x_{n-1}x_n$ plane. It is the circle $\gamma : x_{n-1} = r \sin \theta, x_n = r(1 - \cos \theta)$ where $\theta$ is the angle measured from the point of contact of $\xi$ with $x_n = 0$. The plane $x_n = r(1 - \cos \theta)$ intersects $\xi$ in an $(n-2)$-sphere whose points are $x' + r \sin \theta \omega'$ where $\omega' = (\omega_1, \ldots, \omega_{n-1})$ is a point on the unit sphere $S_{n-2}$ in $\mathbb{R}^{n-1}$. Let $d\omega'$ be the surface element on $S_{n-2}$.

**Proposition 2.1.** Let $f$ be exponentially decreasing on $H^n$. Then in the notation above,

$$\hat{f}(\xi) = \int_0^\pi \int_{S_{n-2}} f(x' + r \sin \theta \omega', r(1 - \cos \theta)) d\omega' \left(\frac{\sin \theta}{1 - \cos \theta}\right)^{n-2} \frac{d\theta}{1 - \cos \theta}. \hspace{1cm} (2.2)$$

**Proof:** Since horizontal translations preserve (2.1) and commute with $f \rightarrow \hat{f}$ we may assume $x' = 0$.

The plane $\pi_\theta : x_n = r(1 - \cos \theta)$ has the non-Euclidean metric

$$\frac{dx_1^2 + \cdots + dx_{n-1}^2}{r^2(1 - \cos \theta)^2}$$

and the intersection $\pi_\theta \cap \xi$ is an $(n-2)$-sphere with induced metric

$$\frac{r^2 \sin^2 \theta (d\omega')^2}{r^2(1 - \cos \theta)^2},$$
where \((d\omega')^2\) is the metric on the \((n - 2)\)-dimensional unit sphere in \(\mathbb{R}^{n-1}\). The non-Euclidean volume element on \(\xi \cap \pi_\theta\) is thus

\[
\left( \frac{\sin \theta}{1 - \cos \theta} \right)^{n-2} d\omega'.
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The non-Euclidean arc element on \(\gamma\) is by (2.1) equal to \(d\theta/(1 - \cos \theta)\). Putting these facts together (2.2) follows by integrating over \(\xi\) by slices \(\xi \cap \pi_\theta\).

**Theorem 2.2.** Let \(\xi_0 \subset H^n\) be a fixed horocycle. Let \(f\) be exponentially decreasing and assume

\[
\hat{f}(\xi) = 0
\]

for each horocycle \(\xi\) lying outside \(\xi_0\). Then

\[
f(x) = 0 \text{ for } x \text{ outside } \xi_0.
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**Remark.**
For the case \(n = 3\) this is proved in Lax-Phillips [7]. As we see below, this case is an exception and the general case requires additional methods.

**Proof:** By homogeneity we may take \(\xi_0\) as the plane \(x_n = 1\). Assuming \(\hat{f}(\xi) = 0\) we take the Fourier transform in the \(x'\) variable of the right hand side of (2.2), in other words integrate it against \(e^{-i\langle x', \eta' \rangle}\) where \(\eta' \in \mathbb{R}^{n-1}\).

Then

\[
\pi \int_0^\pi \int_{S_{n-2}} \tilde{f}(\eta', r(1 - \cos \theta)) e^{-ir\sin \theta \langle \eta', \omega' \rangle} d\omega' \left( \frac{\sin \theta}{1 - \cos \theta} \right)^{n-2} \frac{d\theta}{1 - \cos \theta} = 0.
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By rotational invariance the \(\omega'\) integral only depends on the norm \(|\eta'| r \sin \theta\) so we write

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J(r \sin \theta | \eta'|) = \int_{S_{n-2}} e^{-ir\sin \theta \langle \eta', \omega' \rangle} d\omega'.
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and thus

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Here we substitute \(u = r(1 - \cos \theta)\) and obtain

\[
(2.3) \quad \int_0^{2r} \tilde{f}(\eta', u) J((2ur - u^2)^{1/2} | \eta'|) \frac{u}{u^{n-1}} (2ur - u^2)^{(n-3)/2} du = 0.
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Since the distance from the origin \((0, 1)\) to \((x', u)\) satisfies

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d((0, 1), (x', u)) \geq d((0, 1), (0, u)) = \int_u^1 \frac{dx_n}{x_n} = -\log u
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so

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e^{d((0, 1), (x', u))} \geq \frac{1}{u},
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and since
\[ \tilde{f}(\eta', u) = \int_{\mathbb{R}^{n-1}} f(x', u)e^{-i\langle x', \eta' \rangle} \, dx', \]
we see from the exponential decrease of \( f \), that the function \( u \to \tilde{f}(\eta', u)/u^{n-1} \) is continuous down to \( u = 0 \).

**The case** \( n = 3 \). In this simplest case (2.3) takes the form
\[ (2.4) \quad 2r \int_0^{2r} \tilde{f}(\eta', u)u^{-2}J((2ur - u^2)^{1/2}|\eta'|) \, du = 0. \]

We need here standard result for Volterra integral equation (cf. Yosida [10]).

**Proposition 2.3.** Let \( a < b \) and \( f \in C[a,b] \) and \( K(s,t) \) of class \( C^1 \) on \([a,b] \times [a,b]\). Then the integral equation
\[ (2.5) \quad \varphi(s) + \int_a^s K(s,t)\varphi(t) \, dt = f(s) \]
has a unique continuous solution \( \varphi(t) \). In particular, if \( f \equiv 0 \) then \( \varphi \equiv 0 \).

**Corollary 2.4.** Assume \( K(s,s) \neq 0 \) for \( s \in [a,b] \). Then the equation
\[ (2.6) \quad \int_a^s K(s,t)\psi(t) \, dt = 0 \quad \text{implies} \quad \psi \equiv 0. \]

This follows from Prop. 2.3 by differentiation. Using Cor. 2.4 on (2.4) we deduce \( \tilde{f}(\eta', u) = 0 \) for \( u \leq 2r \) with \( 2r \leq 1 \) proving Theorem 2.2 for \( n = 3 \).

**The case** \( n = 2 \). Here (2.3) leads to the generalized Abel integral equation \((0 < \alpha < 1)\).

\[ (2.7) \quad \int_a^s \frac{G(s,t)}{(s-t)^\alpha} \varphi(t) \, dt = f(s). \]

**Theorem 2.5.** With \( f \) continuous, \( G \) of class \( C^1 \) and \( G(s,s) \neq 0 \) for all \( s \in [a,b] \), equation (2.7) has a unique continuous solution \( \varphi \). In particular, \( f \equiv 0 \Rightarrow \varphi \equiv 0 \).

This is proved by integrating the equation against \( 1/(x-s)^{1-\alpha} \) whereby the statement is reduced to Cor. 2.4 (cf. Yosida, loc.cit.).

This proves Theorem 2.2 for \( n = 2 \).

**The general case.** Here the parity of \( n \) makes a difference. For \( n \) odd we just use the following lemma.

**Lemma 2.6.** Assume \( \varphi = C^1([a,b]) \) and that \( K(s,t) \) has all derivatives with respect to \( s \) up to order \( m - 2 \) equal to 0 on the diagonal \((s,s)\). Assume the \((m-1)\)th order derivative is nowhere 0 on the diagonal. Then (2.6) still holds.
In fact, by repeated differentiation of (2.6) one can show that (2.5) holds with a kernel

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\frac{K^{(m)}(s,t)}{\{K^{(m-1)}(s,t)\}_{t=s}}
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and \( f \equiv 0 \).

This lemma proves Theorem 2.2 for \( n \) odd. For \( n \) even we write (2.3) in the general form

\[
\int_0^s F(u)H((su - u^2)^{1/2})(su - u^2)^{1(n-3)} du = 0 \quad n \text{ even } \geq 2,
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where \( H(0) \neq 0 \).

**Theorem 2.7.** Assume \( F \in C([0,1]) \) satisfies (2.8) for \( 0 \leq s \leq 1 \) and \( H \in C^\infty \) arbitrary with \( H(0) \neq 0 \). Then \( F \equiv 0 \) on \([0,1]\).

**Proof:**

We proceed by induction on \( n \), the case \( n = 2 \) being covered by Theorem 2.5. We assume the theorem holds for \( n \) and any function \( H \) satisfying \( H(0) \neq 0 \). We consider (2.8) with \( n \) replaced by \( n + 2 \) and take \( d/ds \). The result is with \( H_1(x) = H'(x)x + (n - 1)H(x) \),

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Since \( H_1(0) \neq 0 \) we conclude \( F \equiv 0 \) by induction. This finishes the proof of Theorem 2.2.

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**Proposition 2.1.** Let \( f \) be exponentially decreasing on \( H^n \). Then in the notation above,
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**Proof:** Since horizontal translations preserve (2.1) and commute with \( f \to \hat{f} \) we may assume \( x' = 0 \).

The plane \( \pi_{\theta} : x_n = r(1 - \cos \theta) \) has the non-Euclidean metric
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\]
Here we substitute \(u = r(1 - \cos \theta)\) and obtain
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(2.3) \quad \int_0^{2r} \hat{f}(\eta', u) J((2ur - u^2)^{1/2} |\eta'|) \frac{r}{u^{n-1}} (2ur - u^2)^{\frac{n-3}{2}} du = 0 .
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\[ \tilde{f}(\eta', u) = \int_{\mathbb{R}^{n-1}} f(x', u)e^{-i\langle x', \eta' \rangle} dx', \]
we see from the exponential decrease of \( f \), that the function \( u \to \tilde{f}(\eta', u)/u^{n-1} \) is continuous down to \( u = 0 \).

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\begin{align*}
(2.4) \quad & \int_{0}^{2r} \tilde{f}(\eta', u)u^{-2}J((2ur - u^2)^{1/2}||\eta'||) du = 0.
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We need here standard result for Volterra integral equation (cf. Yosida [10]).

**Proposition 2.3.** Let \( a < b \) and \( f \in C[a, b] \) and \( K(s, t) \) of class \( C^1 \) on \([a, b] \times [a, b] \). Then the integral equation
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has a unique continuous solution \( \varphi(t) \). In particular, if \( f \equiv 0 \) then \( \varphi \equiv 0 \).

**Corollary 2.4.** Assume \( K(s, s) \neq 0 \) for \( s \in [a, b] \). Then the equation
\begin{align*}
(2.6) \quad & \int_{a}^{s} K(s, t)\psi(t) dt = 0 \quad \text{implies} \quad \psi \equiv 0.
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This follows from Prop. 2.3 by differentiation. Using Cor. 2.4 on (2.4) we deduce \( \tilde{f}(\eta', u) = 0 \) for \( u \leq 2r \) with \( 2r \leq 1 \) proving Theorem 2.2 for \( n = 3 \).

**The case** \( n = 2 \). Here (2.3) leads to the generalized Abel integral equation \((0 < \alpha < 1)\).
\begin{align*}
(2.7) \quad & \int_{a}^{s} \frac{G(s, t)}{(s - t)^{\alpha}}\varphi(t) dt = f(s).
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**Theorem 2.5.** With \( f \) continuous, \( G \) of class \( C^1 \) and \( G(s, s) \neq 0 \) for all \( s \in [a, b] \), equation (2.7) has a unique continuous solution \( \varphi \). In particular, \( f \equiv 0 \Rightarrow \varphi \equiv 0 \).

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and \( f \equiv 0 \).

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(2.8) \[ \int_0^s F(u) H((su - u^2)^{1/2})(su - u^2)^{\frac{1}{2}(n-3)} \, du = 0 \quad n \text{ even} \geq 2, \]

where \( H(0) \neq 0 \).

**Theorem 2.7.** Assume \( F \in C([0,1]) \) satisfies (2.8) for \( 0 \leq s \leq 1 \) and \( H \in \mathcal{C}^\infty \) arbitrary with \( H(0) \neq 0 \). Then \( F \equiv 0 \) on \([0,1]\).

**Proof:**
We proceed by induction on \( n \), the case \( n = 2 \) being covered by Theorem 2.5. We assume the theorem holds for \( n \) and any function \( H \) satisfying \( H(0) \neq 0 \). We consider (2.8) with \( n \) replaced by \( n + 2 \) and take \( d/ds \). The result is with \( H_1(x) = H'(x)x + (n-1)H(x) \),

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Since \( H_1(0) \neq 0 \) we conclude \( F \equiv 0 \) by induction. This finishes the proof of Theorem 2.2.

**References**

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