LIFTABILITY OF THE FROBENIUS MORPHISM
AND IMAGES OF TORIC VARIETIES
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Abstract. We formulate a conjecture characterizing smooth projective varieties in positive characteristic whose Frobenius morphism can be lifted modulo $p^2$ — we expect that such varieties, after a finite étale cover, admit a Zariski-locally trivial fibration with toric fibers over an ordinary abelian variety. We prove that this assertion implies a conjecture of Occhetta and Wiśniewski, which states that a smooth image of a projective toric variety is a toric variety. In order to deal with an important special case, we develop a logarithmic variant of the characterization of ordinary varieties with trivial tangent bundle due to Mehta and Srinivas. Furthermore, we verify our conjecture for surfaces, Fano threefolds, and homogeneous spaces (answering a question posed by Buch–Thomsen–Lauritzen–Mehta). Our proofs are based on a comprehensive theory of Frobenius liftings together with a variety of other techniques including deformation theory of rational curves and Frobenius splittings.

1. Introduction

1.1. Liftings of Frobenius. One of the salient features of algebraic geometry in positive characteristic is the existence of the Frobenius morphism $F^p_X : X \to X$ for every $\mathbb{F}_p$-scheme $X$, defined as the $p$-th power map $f \mapsto f^p$ on $O_X$. At a philosophical level, this paper argues that this type of structure becomes extremely rare as we move towards characteristic zero. This idea is not new: for example, in Borger’s point of view on $\mathbb{F}_1$-geometry [Bor09], liftings of the Frobenius (compatible for all $p$) are seen as ‘descent data to $\mathbb{F}_1$.’ In more classical algebraic geometry, it is expected [Ame97, Fak03] that projective varieties in characteristic zero admitting a polarized endomorphism are very scarce.

More precisely, we will be mostly concerned with the following question:

Which smooth projective varieties in characteristic $p$ lift modulo $p^2$ together with Frobenius?

For brevity, we will call such varieties $F$-liftable. This question also has a long history: in their very influential paper [DI87], Deligne and Illusie gave an algebraic proof of Kodaira–Akizuki–Nakano vanishing

$$H^j(X, \Omega^i_X \otimes L) = 0 \quad (L \text{ ample, } i + j > \dim X)$$

for smooth complex varieties by reducing the setting to characteristic $p > 0$. The key ingredient was the analysis of local liftings of Frobenius modulo $p^2$ and their relation to the de Rham complex, namely that such a lifting $\tilde{F}$ induces an injective homomorphism

$$\xi = \frac{d\tilde{F}}{p} : F^*X \Omega^1_X \to \Omega^1_X$$

whose adjoint $\Omega^1_X \to F^*X \Omega^1_X$ induces the inverse of the Cartier operator. In [BTLM97], the authors study smooth projective varieties in characteristic $p > 0$ admitting a global lifting modulo $p^2$ together with Frobenius, and show in particular that such varieties satisfy a much stronger form of Kodaira vanishing called Bott vanishing

$$H^j(X, \Omega^i_X \otimes L) = 0 \quad (L \text{ ample, } j > 0).$$

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This type of vanishing is extremely restrictive, and hence so is $F$-liftable. In fact, all known examples are in some sense built from toric varieties (where the lifting of Frobenius extends the multiplication by $p$ map on the torus) and ordinary abelian varieties (by the Serre–Tate theory). Note that the existence of an injective map $\xi$ as above implies that $(1 - p) K_X$ is effective, so in particular $X$ has non-positive Kodaira dimension; in fact, the section $\det(\xi) \in H^0(X, \omega_X^{1-p})$ corresponds to a Frobenius splitting of $X$.

The main goal of this paper is to provide evidence for the following conjectural answer to the above question, and to relate it to some open problems in characteristic zero.

**Conjecture 1.** Let $X$ be a smooth projective variety over an algebraically closed field $k$ of characteristic $p > 0$. If $X$ is $F$-liftable, then there exists a finite étale Galois cover $f: Y \to X$ such that the Albanese morphism of $Y$

$$a_Y: Y \to \text{Alb}(Y)$$

is a toric fibration. In particular, if $X$ is simply connected (for example, if $X$ is separably rationally connected), then $X$ is a toric variety.

See §2.1 for the definition of a toric fibration. Conjecture 1 almost characterizes $F$-liftable smooth projective varieties, see Remark 3.1.7 for a discussion of the converse.

**1.2. Special cases of Conjecture 1.** A few important special cases were already known for some time. In the paper [MS87], to which our work owes a great deal, Mehta and Srinivas prove (among other things) that if $X$ is $F$-liftable and the canonical bundle $\omega_X$ is numerically trivial, then $X$ admits a finite étale Galois cover by an ordinary abelian variety. In a different direction, the case of homogeneous spaces was considered in the aforementioned paper [BTLM97], where it was shown that most rational homogeneous spaces are not $F$-liftable, and conjectured that the only $F$-liftable ones are products of projective spaces. Recently, the case of minimal surfaces was considered in [Xin16].

Our first contribution confirms the aforementioned expectation of [BTLM97].

**Theorem 1** (see Theorem 6.4.5). Conjecture 1 is true if $X$ is a homogeneous space. More precisely, if $X$ is a smooth projective $F$-liftable variety whose automorphism group acts transitively, then $X$ is isomorphic to a product of projective spaces and an ordinary abelian variety.

Our method of proof is based on the study of rational curves on $X$. In the crucial special case of rational homogeneous spaces of Picard rank one, we analyze the geometry of an étale covering of $X$ induced by the map (1.1.1), and its restrictions to rational curves belonging to a carefully chosen covering family. In the final step, we apply Mori’s characterization of the projective space. In fact, this method shows that Conjecture 1 holds if $T_X$ is nef and $X$ is a Fano variety of Picard rank one (see Proposition 6.3.2).

We also verify Conjecture 1 in low dimensions.

**Theorem 2.** Conjecture 1 is true in the following cases.

(a) If $\dim X \leq 2$ (see Theorem 7.1.9).

(b) If $X$ is a Fano threefold from the Mori–Mukai classification (see Theorem 7.2.9).

Let us say a few words about the proofs. The case of surfaces is quickly reduced to rational and ruled surfaces, in which cases the question is still not completely trivial. Note that for ruled surfaces our results do not entirely agree with [Xin16], see Remark 7.1.10. The case of Fano threefolds relies on the Mori–Mukai classification with about 100 families; see the first paragraph of §7.2 for a discussion. The case which caused us the most trouble initially was the blow-up of $\mathbb{P}^3_k$ along a twisted cubic.

**1.3. Relation to other problems.** Let us mention three problems (in arbitrary characteristic) to which Conjecture 1 is related.
Images of toric varieties. Consider a problem of the following type: given a projective variety $Z$, determine all smooth projective varieties $X$ for which there exists a surjective morphism

$$\varphi : Z \rightarrow X.$$  

One of the first applications of Mori theory was Lazarsfeld’s solution [Laz84] to a problem of Remmert and van de Ven [RvdV61]: if $Z \cong \mathbb{P}^n$, then $X \cong \mathbb{P}^n$ (or $X$ is a point). Subsequently, more general problems of this kind were considered: for example, if $Z$ is an abelian variety, then $X$ admits a finite étale cover by a product of an abelian variety and projective spaces [Deb89, HM01, DHP08]. In [OW02], Occhetta and Wiśniewski proved that if $Z$ is a toric variety and $X$ has Picard rank one, then $X \cong \mathbb{P}^n$ and were led to pose the following conjecture:

**Conjecture 2.** A smooth complex projective variety $X$ admitting a surjective map $\varphi : Z \rightarrow X$ from a complete toric variety $Z$ is a toric variety.

This project gained momentum upon our discovery of a link between Conjecture 1 and Conjecture 2.

**Theorem 3** (see Theorem 4.4.1). Conjecture 1 for simply connected varieties implies Conjecture 2 in characteristic zero.

The key step in the proof uses the functoriality of obstruction classes to lifting Frobenius, an idea due to the third author [Zda17]: after reducing a given surjective map $\varphi : Z \rightarrow X$ modulo $p^2$, one can relate the obstruction classes to lifting Frobenius on $X$ and $Z$. Since $Z$ is a toric variety, its reduction modulo $p$ is $F$-liftable, and so is the reduction of $X$. By the assumed case of Conjecture 1, the reduction of $X$ is a toric variety. Finally, one uses a strengthening of Jaczewski’s characterization of toric varieties [Jac94] from [KW15] to conclude that $X$ must be a toric variety. We obtain the following result as a by-product.

**Theorem 4.** Let $\varphi : Z \rightarrow X$ be a surjective morphism from a complete toric variety $Z$ to a smooth projective variety $X$ defined over a field $k$ of characteristic zero. Then $X$ satisfies Bott vanishing (1.1.2).

We do not know of an example of a non-toric smooth projective rationally connected variety in characteristic zero which satisfies Bott vanishing. It might well be that there are none.

Varieties with trivial log tangent bundle. Let $X$ be a smooth projective variety over an algebraically closed field $k$, and let $D \subseteq X$ be a divisor with normal crossings. Suppose that the sheaf $\Omega_X^1(\log D)$ of differentials with log poles along $D$ is free. If $k$ has characteristic zero, then by a result of Winkelmann [Win04], $X$ admits an action of a semi-abelian variety (an extension of an abelian variety by a torus) which is transitive on $X \setminus D$; in particular, if $D = 0$, then $X$ is an abelian variety. However, in positive characteristic $X$ might not be an abelian variety itself when $D = 0$, but then the main result of [MS87] states that if $X$ is ordinary, then it admits an abelian variety as a finite étale Galois cover. It is not known if the ordinarity assumption is necessary.

The link with Frobenius liftability was first observed in [MS87] in relation to the above result: if $X$ has a trivial tangent bundle and is ordinary, then it is $F$-liftable. Conversely, if $\omega_X$ is numerically trivial and $X$ is $F$-liftable, then the map (1.1.1) is an isomorphism, and it follows that $\Omega_X^1$ becomes trivial on a finite étale cover of $X$.

We shall say that the pair $(X, D)$ is $F$-liftable if there exists a lifting $(\tilde{X}, \tilde{D})$ of $(X, D)$ modulo $p^2$ such that $\tilde{D}$ has relative normal crossings, and a lifting $\tilde{F}$ of Frobenius to $\tilde{X}$ which is compatible with $\tilde{D}$ in the sense that

$$\tilde{F}^* \tilde{D} = p \tilde{D}.$$  

**Theorem 5** (see Theorem 5.1.1). Let $(X, D)$ be a projective nc pair over an algebraically closed field $k$ of positive characteristic. The following conditions are equivalent.
(i) \((X, D)\) is \(F\)-liftable and \(\omega_X(D)\) is numerically trivial,
(ii) \(X\) is \(F\)-split and \(\Omega_X^1(\log D)\) becomes trivial on a finite étale cover of \(X\),
(iii) \(X\) admits a finite étale cover \(f: Y \to X\) whose Albanese map \(a: Y \to \text{Alb}(Y)\) is a toric fibration over an ordinary abelian variety with toric boundary \(f^{-1}(D)\).

The proof is not less complicated than the one in [MS87]. We closely follow their strategy with a few differences. The case \(H^1(X, \mathcal{O}_X) = 0\) is handled by lifting to characteristic zero, applying the aforementioned theorem of Winkelmann [Win04], and reducing the setting back to positive characteristic. Interestingly, our results show that Winkelmann’s theorem is false in characteristic \(p\), while Winkelmann’s theorem indicates that a natural analogue of Theorem 5(iii) \(\Rightarrow\) (ii) is false in characteristic zero (see Remark 5.1.4). But the most important difference comes from the fact that [MS87] used Yau’s work on the Calabi conjecture for a lifting of \(X\) to characteristic zero in one of the key steps. In the log setting, such results are unfortunately unavailable, and instead we reduce to the case of a finite ground field, then lift to characteristic zero together with Frobenius, and use results of [NZ10] and [GKP16] on varieties with a polarized endomorphism. This brings us to the next array of open problems.

**Polarized endomorphisms.** A polarized endomorphism of a projective variety \(X\) is by definition one which extends to an endomorphism of an ambient projective space. It seems that there are many similarities between Frobenius liftings and polarized endomorphisms. Over a finite field, a power of the Frobenius morphism is a polarized endomorphism. In characteristic zero, the toric Frobenius is an example of such, as is the Serre–Tate canonical lifting of Frobenius on an ordinary abelian variety. As in the case of Frobenius liftable, varieties admitting a polarized endomorphism satisfy \(\kappa(X) \leq 0\) (see [NZ10]).

If \(\rho(X) = 1\), then every endomorphism is polarized. In this case, a conjecture by Amerik [Ame97] states that if a smooth rationally connected \(X\) with \(\rho(X) = 1\) admits a polarized endomorphism, then \(X \simeq \mathbb{P}^n\). This has been proved in certain cases, for example, when \(\dim X \leq 3\) [ARValV99], \(X\) is a hypersurface [Bea01], or \(X\) is a rational homogeneous space [PS89]. Moreover, Zhang showed that a smooth Fano threefold with a polarized endomorphism is rational [Zha10]. In general, it has been asked if every smooth rationally connected variety with a polarized endomorphism is toric (see, for instance, [Fak03]).

**1.4. Methods and further results.** The proofs of the results mentioned above rely on a variety of tools: deformation theory, toric fibrations, and most importantly the study of Frobenius liftings. Let us mention here some of our observations which might be of independent interest. Some of these results are valid for arbitrary (non-necessarily smooth) \(k\)-schemes.

The Swiss Army knife in our toolbox is the technique of descending Frobenius liftability (see Theorem 3.3.6): if \(Y\) is \(F\)-liftable and \(\pi: Y \to X\) is a morphism, then under certain assumptions on \(\pi\), one can deduce the \(F\)-liftable of \(X\). In the special case of a blow-up along a smooth center \(Z \subseteq X\), there is an induced Frobenius lifting on \(X\) which is suitably compatible with \(Z\) (Lemma 3.4.4).

In [Zda17], the third author provided an explicit functorial construction of a lifting modulo \(p^2\) associated to a Frobenius splitting (see Theorem 2.5.2). Using this construction and some properties of the Witt vector ring \(\mathbb{W}_2(X)\) we prove that there is at most one lifting of \(X\) modulo \(p^2\) to which the Frobenius morphism lifts (Theorem 3.6.5). Surprisingly, this lifting is isomorphic to any lifting induced by a Frobenius splitting. This observation allows us to descend Frobenius liftability along good quotients by actions of linearly reductive groups (Theorem 3.3.6(c)). We also employ the Witt vector ring \(\mathbb{W}_2(X)\) to show a curious base change property of Frobenius liftings (Proposition 3.5.2 and Corollary 3.5.4).

We provide a few results on toric varieties in families. First, if a general fiber of a smooth projective family is toric, then the generic fiber must be toric as well (Corollary 4.1.5). To this end, we need an auxiliary result (Proposition 4.1.2) on the constructibility of the locus where a given vector bundle splits. Second, given a smooth projective family together with
a relative normal crossings divisor over a connected base, if a single geometric fiber is a toric pair, then the entire family has to be a toric fibration (Proposition 4.3.1). In other words, using the terminology of Hwang and Mok, toric pairs are globally rigid.

1.5. Outline of the paper. In Section 2, we gather some preliminary results and notions: toric varieties over an arbitrary base (i.e., toric fibrations) in §2.1, deformation theory including the crucial technique of descending deformations in §2.2, some basic facts about normal crossing pairs in §2.3, a review of the Cartier operator in §2.4, and Frobenius splittings (especially in the relative setting) in §2.5. Section 3 is a standalone mini-treatise on Frobenius liftability. We hoped to provide the reader with a good reference, and so some statements are much more general than what we need in the sequel. Many of the results are new (to the best of our knowledge). In Section 4, we deal with the abstract problem of showing that the generic fiber of a family is toric if a general one is (§4.1) and with the opposite problem of proving that if the generic fiber of a degeneration is a toric pair, then so is the special one (§4.2). These results are then applied to show a general ‘global rigidity’ result for toric pairs (§4.3) and Theorem 3 (§4.4). Sections 5, 6, and 7 deal with Theorems 5, 1, and 2, respectively.

1.6. Notation and conventions. Let \( p \) be a prime number. If \( X \) is an \( \mathbf{F}_p \)-scheme, we denote by \( F_X : X \to X \) its absolute Frobenius morphism (i.e., the identity on the underlying topological space and the \( p \)-th power map on the structure sheaf). If \( f : X \to S \) is a morphism of \( \mathbf{F}_p \)-schemes, the relative Frobenius \( F_{X/S} : X \to X' \) is the unique morphism making the following diagram commute

\[
\begin{array}{c}
\xymatrix{ & \bigcirc \ar[dl]_{F_X} \ar[d]^{f} \ar[dr]^{X'} & \\
\bigcirc \ar[dr]_{F_{X/S}} & X & X \ar[d]^{F_{X/S}} \\
 & S & S. \\
}\end{array}
\]

The \( S \)-scheme \( X' \) is called the Frobenius twist of \( X \) relative to \( S \). Note that if \( S = \text{Spec} \, k \) for a perfect field \( k \), then \( W_{X/S} \) is an isomorphism of schemes.

Most of the time we shall be working with a fixed perfect field \( k \) of characteristic \( p > 0 \). We denote by \( W_n(k) \) its ring of Witt vectors of length \( n \), that is, the unique (up to a unique isomorphism) flat \( \mathbf{Z}/p^n\mathbf{Z} \)-algebra with an isomorphism \( W_n(k)/pW_n(k) \cong k \), and by \( W(k) = \lim_{\longrightarrow} W_n(k) \) the full ring of Witt vectors. The unique endomorphism restricting to the Frobenius \( F_k \) modulo \( p \) is denoted by \( \sigma : W_n(k) \to W_n(k) \) or \( \sigma : W(k) \to W(k) \).

We often set \( S = \text{Spec} \, k \) and \( \tilde{S} = \text{Spec} \, W_2(k) \). If \( X \) is an \( S \)-scheme, a lifting of \( X \) (modulo \( p^2 \)) is a flat \( \tilde{S} \)-scheme \( \tilde{X} \) with an isomorphism \( \tilde{X} \times_{\tilde{S}} S \cong X \). A lifting of Frobenius on \( X \) to \( \tilde{X} \) is a morphism \( \tilde{F}_X : \tilde{X} \to \tilde{X} \) restricting to \( F_X \) modulo \( p \). We shall give a thorough discussion of these in §3.

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2. Preliminaries

2.1. Toric varieties and toric fibrations. A toric variety (see e.g. [Ful93]) is by definition a normal algebraic variety $X$ over an algebraically closed field $k$ together with an effective action of a torus $T \simeq \mathbb{G}_m^n$ with a dense orbit. Such varieties admit a completely combinatorial description in terms of rational polyhedral fans $\Sigma$ in $N_\mathbb{R} = \text{Hom}(\mathbb{G}_m, T) \otimes \mathbb{Z} \mathbb{R}$. This description is independent of the field $k$, and in particular every toric variety has a natural model $X(\Sigma)$ over $\mathbb{Z}$. We denote by $D(\Sigma) \subseteq X(\Sigma)$ the toric boundary of $X(\Sigma)$, that is, the complement of the open orbit. Sometimes we shall abuse the terminology and say that a variety $X$ is a toric variety meaning that it admits the structure of a toric variety.
In what follows, we will have to deal with families of toric varieties over more general bases (such as the rings $W_2(k)$ and $W(k)$ or the Albanese variety in the statement of Conjecture 1). There is more than one sensible definition of a ‘toric variety over a base scheme $S$,’ and we decided to settle on the following.

**Definition 2.1.1.** Let $S$ be a scheme.

(a) A torus over $S$ is an $S$-group scheme $T$ which is \( \text{étale-locally isomorphic to } G_m^n \) for some \( n \geq 0 \).

(b) A toric fibration over $S$ is a flat $S$-scheme $X$ together with an action of a torus $T$ such that \( \text{étale-locally on } S \) there exist isomorphisms $T \cong G_m^n$ and $X \cong X(\Sigma)_S$ for some rational polyhedral fan $\Sigma \subseteq \mathbb{R}^n$.

In particular, we do not require the torus $T$ to be split, i.e. that $T \cong G_m^n$. In general, $\text{Aut}_S(G_m^n) \cong GL(n, \mathbb{Z})_S$, and hence tori of dimension $n$ over $S$ are parametrized by the \( \text{étale} \) non-abelian cohomology $H^1(S, GL(n, \mathbb{Z}))$; if $S$ is connected and normal, this is $\text{Hom}(\pi_1(S, \bar{z}), GL(n, \mathbb{Z}))$. Since $\pi_1(S, \bar{z})$ is pro-finite and $GL(n, \mathbb{Z})$ is discrete, every such homomorphism has finite image, and we see that every torus over a normal $S$ becomes split on a finite \( \text{étale} \) cover of $S$. If $T$ is split, then we will say that $X$ is a split toric fibration. If $X \rightarrow S$ is a toric fibration for a torus $T$, the dense open orbit $U \subseteq X$ is a $T$-torsor over $S$. If this torsor is trivial (i.e., $U \rightarrow S$ admits a section), we shall say that $X$ is a trivial toric fibration. If $X$ is a split toric fibration under a torus $T$, then the usual description of toric varieties over algebraically closed fields generalizes as follows: there exists a fan $\Sigma$ in $N_\mathbb{R}$ and a $T$-torsor $U \rightarrow S$ such that

$$X \cong U \times^T X(\Sigma)_S$$

as $S$-schemes with a $T$-action. In particular, if $X$ is split and trivial, then $X \cong X(\Sigma)_S$. In any case, such an isomorphism exists \( \text{étale} \) locally on $S$, and the toric boundaries $D(\Sigma)_S$ glue together to give a global toric boundary $D$, which is a closed subscheme supported on $X \setminus U$.

If $X$ is smooth over $S$, then $D$ has normal crossings relative to $S$, and $\Omega^1_{X/S}(\log D)$ is trivial if moreover $T$ is split. We call a pair $(X, D)$ of a smooth $S$-scheme $X$ and a divisor with relative normal crossings $D \subseteq X$ a toric pair if it arises via the above construction. In §4.3, we will show that the torus $T$ and its action on $X$ are (essentially) uniquely defined by the pair $(X, D)$, at least if $X$ is projective over $S$.

**Example 2.1.2.**

1. The projectivization $P_S(E) \rightarrow S$ of a vector bundle $E$ on $S$ has toric fibers, but it admits the structure of a split toric fibration only when $E$ is a direct sum of line bundles.

2. Let $S = A^1_k$ and $X = A^2_k \setminus \{(0,0)\}$, treated as an $S$-scheme via the first projection. Then $X \rightarrow S$ has a natural action of $G_m^n$, its fibers are toric varieties, but it is not a toric fibration.

3. Let $S = \text{Spec } \mathbb{Q}$ and let $T = \{x^2 + y^2 = 1\} \subseteq A^2_\mathbb{Q}$ be the circle group. Let $X = \{x^2 + y^2 = z^2\} \subseteq P^2_\mathbb{Q}$ be the projective closure of $T$. The action of $T$ on itself extends to $X$, and $D = X \setminus T$ is a point of degree two. The map $X \rightarrow S$ is a toric fibration under $T$ which becomes split and trivial after base change to $\mathbb{Q}(\sqrt{-1})$.

2.2. Deformation theory. We use freely the standard results of deformation theory, for which we refer to [Har10, Ill71]. If $k$ is a perfect field of characteristic $p > 0$, we denote by $\text{Art}_{W(k)}(k)$ the category of Artinian local $W(k)$-algebras with residue field $k$. If $X$ is a $k$-scheme, we denote by

$$\text{Def}_X : \text{Art}_{W(k)}(k) \rightarrow \text{Set}, \quad A \mapsto \left\{ \text{isom. classes of flat } \bar{X}/\text{Spec } A \right\}$$

its deformation functor. Similarly, if $Z \subseteq X$ is a closed subscheme, we denote by $\text{Def}_{X,Z}$ the functor of deformations $ar{X}$ of $X$ together with an embedded deformation $\bar{Z} \subseteq \bar{X}$ of $Z$. 
Lemma 2.2.1 ([LS14, Proposition 2.1]). Let \( \pi: Y \to X \) be a morphism of \( k \)-schemes such that \( \mathcal{O}_X \xrightarrow{\sim} \pi_* \mathcal{O}_Y \) and \( R^1 \pi_* \mathcal{O}_Y = 0 \). Then there exists a natural transformation of deformation functors

\[
\pi_*: \text{Def}_Y \to \text{Def}_X,
\]

associating to every lifting \( \bar{Y} \in \text{Def}_Y(A) \) a lifting \( \bar{X} = \pi_*(\bar{Y}) \in \text{Def}_X(A) \) together with an \( A \)-morphism \( \pi: \bar{Y} \to \bar{X} \) lifting \( \pi \). More precisely, the structure sheaf of \( \bar{X} \) is defined by the formula \( \mathcal{O}_{\bar{X}} = \pi_* \mathcal{O}_{\bar{Y}} \), and consequently the pair \( (\bar{X}, \bar{\pi}) \) is unique up to a unique isomorphism inducing the identity on \( X \) and \( \bar{Y} \).

In the case of blow-ups of smooth varieties in smooth centers, one can say a bit more. Let \( X \) be a smooth scheme over \( k \), let \( Z \subseteq X \) be a smooth closed subscheme of codimension \( > 1 \), let \( \pi: Y \to X \) be the blowing-up of \( X \) along \( Z \), and let \( E = \pi^{-1}(Z) \) be the exceptional divisor.

Lemma 2.2.2 ([LS14, Proposition 2.2]). In the above situation, the natural transformations of deformation functors

\[
\text{Def}_{X,Y} \to \text{Def}_{Y,E} \to \text{Def}_Y
\]

are isomorphisms. The composition \( \text{Def}_Y \simeq \text{Def}_{X,Y} \to \text{Def}_X \) equals \( \pi_* \).

2.3. Normal crossing pairs. We recall some basics on normal crossings (nc) pairs over general base schemes.

Definition 2.3.1. Let \( S \) be a scheme. An nc pair over \( S \) is a pair \((X, D)\) of a smooth scheme \( X \) over \( S \) and a divisor \( D \subseteq X \) with normal crossings relative to \( S \) (see [SGA 1, Exp. XIII §2.1], [Kat70, §4]), that is, such that étale-locally on \( X \) there exists an étale map \( h: X \to \mathbb{A}^n_S \) with \( D = h^*(\{x_1 \cdot \ldots \cdot x_n = 0\}) \).

Remark 2.3.2. It is often more natural and convenient to work in the framework of log geometry than with nc pairs. We decided not to do so, as we will need only a small subset of the theory which is easily handled by the classical results on nc pairs. A reader familiar with the language of log geometry will notice that everything we do with nc pairs can easily be phrased using log schemes instead, and that many of our results have natural logarithmic analogues. More precisely, the dictionary is as follows.

An nc log scheme over a scheme \( S \) is an fs log scheme \((X, \mathcal{M}_X)\) over \((S, \mathcal{O}_S^\times)\) which is log smooth and such that the underlying map of schemes \( X \to S \) is smooth. Equivalently, an fs log scheme \((X, \mathcal{M}_X)\) over \((S, \mathcal{O}_S^\times)\) is nc if and only if étale locally on \( X \), there exists a strict \( h: (X, \mathcal{M}_X) \to \mathbb{A}^n_S \) \( = \text{Spec}(\mathbb{N}^n \to \mathcal{O}_S[x_1, \ldots, x_n]) \). Thus, there is a natural equivalence of groupoids, compatible with base change

\[
(\text{nc log schemes}/(S, \mathcal{O}_S^\times)) \xrightarrow{\sim} (\text{nc pairs}/S).
\]

Given an nc pair \((X, D)\) over \( S \), one defines the module of differentials \( \Omega^1_{X/S}(\log D) \), see [Kat70, §4]. It is locally free and its determinant is isomorphic to \( \omega_{X/S}(D) \). Classical deformation theory for smooth schemes has a natural analogue for nc pairs, and in particular is controlled by the groups \( \text{Ext}^i(\Omega^1_{X/S}(\log D), \mathcal{O}_X) \) for \( i = 0, 1, 2 \), see [Kat96]. Moreover, if \( D \) is the union of smooth divisors \( D_1, \ldots, D_r \), there is a short exact sequence

\[
0 \to \Omega^1_{X/S} \to \Omega^1_{X/S}(\log D) \to \bigoplus_{i=1}^r \mathcal{O}_{D_i} \to 0.
\]

Lemma 2.3.3. Let \((X, D)\) be an nc pair over a scheme \( S \), let \( Y \) be a smooth scheme over \( S \), and let \( f: X \to Y \) be a morphism over \( S \). Then \((X, D)\) is an nc pair over \( Y \) if and only if the morphism

\[
f^* \Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/S}(\log D)
\]

is injective and its cokernel \( \Omega^1_{X/Y}(\log D) \) is locally free.
Proof. If \((X, D)\) is an nc pair over \(Y\), then the assertions are clearly satisfied. For the proof in the other direction, we use the language of log geometry. Accordingly, we treat \((X, D) \to Y\) as a morphism of log schemes. The sheaf \(\Omega^1_{X/Y}(\log D)\) is isomorphic to the sheaf of relative log differentials of \((X, D) \to Y\) and therefore, since it is locally free, we may apply [Kat89, Proposition 3.12] to see that the morphism \((X, D) \to Y\) is log smooth. The scheme \(X\) is smooth over \(Y\) and hence \((X, D) \to Y\) is an nc log scheme over \(Y\). We conclude using the equivalence between nc log schemes and nc pairs (Remark 2.3.2). \(\square\)

2.4. The Cartier isomorphism. Here, we present basic properties of the Cartier isomorphism and its logarithmic variant. Let \(X \to S\) be a smooth morphism of schemes over \(k\), and let \(\Omega^\bullet_{X/S}\) be its de Rham complex. Moreover, let \(B^i_{X/S}\) (resp. \(Z^i_{X/S}\)) be the \(i\)th coboundaries (resp. cocycles) in the \(\partial\)-linear \(X/S\) complex \(F_{X/S}^\bullet \Omega^\bullet_{X/S}\), where \(X'\) is the Frobenius twist of \(X\) relative to \(S\).

More generally, if \((X, D)\) is an nc pair over \(S\), then we denote by \(B^i_{X/S}(\log D)\) (resp. \(Z^i_{X/S}(\log D)\)) the \(i\)th coboundaries (resp. cocycles) in the \(\partial\)-linear log de Rham complex \(F^\bullet_{X/S} \Omega_{X/S}^\bullet(\log D)\). In analogy with the holomorphic Poincaré lemma, the following result describes the cohomology of the de Rham complex in characteristic \(p\).

**Theorem 2.4.1 ([Kat70, Theorem 7.2]).** Let \(X \to S\) be a smooth morphism of schemes over \(k\). Then there exists a unique system of isomorphisms of \(\partial\)-modules

\[
\mathcal{C}^{-1}_{X/S} : \Omega^1_{X'/S} \cong H^1(F_{X/S}^\bullet, \Omega^\bullet_{X/S})
\]

satisfying the conditions

(i) \(\mathcal{C}^{-1}_{X/S}(1) = 1\),

(ii) \(\mathcal{C}^{-1}_{X/S}(\omega) \wedge C^{-1}_{X/S}(\eta) = C^{-1}_{X/S}(\omega \wedge \eta)\) for local sections \(\omega \in \Omega^1_{X'/S}\) and \(\eta \in \Omega^1_{X'/S}\),

(iii) \(\mathcal{C}^{-1}_{X/S} : \Omega^1_{X'/S} \to H^1(F_{X/S}^\bullet, \Omega^\bullet_{X/S})\) is defined by \(d(g \otimes 1) \mapsto [g^{p-1}dg]\).

The inverse isomorphisms give rise to short exact sequences

\[
0 \to B^1_{X/S} \to Z^1_{X/S} \to \Omega^1_{X'/S} \to 0,
\]

inducing Cartier morphisms \(C_{X/S} : Z^1_{X/S} \to \Omega^1_{X'/S}\).

**Variant 2.4.2.** In the logarithmic setting, there exists a system of isomorphisms

\[
\Omega^1_{X'/S}(\log D') \cong H^1(F_{X/S}^\bullet, \Omega^\bullet_{X/S}(\log D)),
\]

where \(D'\) is the preimage of \(D\) in \(X'\). Moreover, we have short exact sequences

\[
0 \to B^1_{X/S}(\log D) \to Z^1_{X/S}(\log D) \to \Omega^1_{X'/S}(\log D') \to 0,
\]

inducing logarithmic Cartier morphisms \(C_{X/S} : Z^1_{X/S}(\log D) \to \Omega^1_{X'/S}(\log D')\).

**Remark 2.4.3.** The top Cartier morphism

\[
C_{X/S} : F_{X/S} \omega_{X/S} = Z^1_{X/S} \to \omega_{X'/S}
\]

for \(r = \dim(X/S)\), is often denoted by \(\text{Tr}_{X/S}\) and called the Frobenius trace map. Its importance stems from the fact that the natural pairing

\[
F_{X/S} \mathcal{H}om(\mathcal{M}, \omega_{X/S}) \otimes F_{X/S} \mathcal{M} \to \omega_{X'/S}
\]

sending \(f \otimes m\) to \(\text{Tr}_{X/S}(f(m))\) induces the Grothendieck duality

\[
\mathcal{H}om(F_{X/S} \mathcal{M}, \omega_{X'/S}) \simeq F_{X/S} \mathcal{H}om(\mathcal{M}, \omega_{X/S})
\]

which, as we shall see in §2.5 and §3.2, is closely related to Frobenius splittings and Frobenius liftings.
2.5. Frobenius splittings. The standard reference for general facts about Frobenius splittings is [BK05, Chapter I]. Let $k$ be a perfect field of characteristic $p > 0$.

Definition 2.5.1. Let $X$ be a $k$-scheme.

(a) A Frobenius splitting on $X$ is an $\mathcal{O}_X$-linear splitting $\sigma: F_X, \mathcal{O}_X \to \mathcal{O}_X$ of the map $F_X^*: \mathcal{O}_X \to F_X, \mathcal{O}_X$.

(b) An $F$-split scheme is a pair $(X, \sigma_X)$ of a $k$-scheme $X$ and a Frobenius splitting $\sigma_X$ on $X$; we shall also say that a $k$-scheme $X$ is $F$-split if it admits a Frobenius splitting.

(c) A morphism of $F$-split schemes $f: (X, \sigma_X) \to (Y, \sigma_Y)$ is a morphism $f: X \to Y$ for which the diagram

$$
\begin{array}{ccc}
F_Y f_* \mathcal{O}_X & \xrightarrow{f_*(\sigma_X)} & f_* \mathcal{O}_X \\
\downarrow & & \downarrow \\
F_Y \mathcal{O}_Y & \xrightarrow{\sigma_Y} & \mathcal{O}_Y.
\end{array}
$$

commutes. This defines the category $\textbf{F-split}$ of $F$-split schemes.

Theorem 2.5.2 ([Zda17, Theorem 3.6]). There exists a functor $(X, \sigma) \mapsto \tilde{X}(\sigma): \textbf{F-split} \to (\text{flat schemes}/W_2(k))$

\[\sigma_X \} \text{ together with a functorial identification } X \simeq \tilde{X}(\sigma) \otimes_{W_2(k)} k.\]

For a group $G$ acting on $Y$, we say that a morphism $\pi: Y \to X$ is a good quotient by $G$, if it is $G$-invariant, affine, and $\mathcal{O}_X = (\pi_* \mathcal{O}_Y)^G$. The following lemma will be applied for $G$ being a torus or a finite group of order prime to $p$.

Lemma 2.5.3. Let $Y$ be an $F$-split scheme of finite type over $k$, let $G$ be a linearly reductive algebraic group acting on $Y$, and let $\pi: Y \to X$ be a good quotient. Then there exist splittings $\sigma_Y: F_Y, \mathcal{O}_Y \to \mathcal{O}_Y$ and $\sigma_X: F_X, \mathcal{O}_X \to \mathcal{O}_X$ such that $\pi: (Y, \sigma_Y) \to (X, \sigma_X)$ is a morphism of $F$-split schemes. In particular, $\pi$ admits a lifting $\tilde{\pi}: \tilde{Y} \to \tilde{X}$.

Proof. The relative Frobenius $F_{G/k}: G \to G'$ is a group homomorphism and the relative Frobenius $F_{Y/k}: Y \to Y'$ is $F_{G/k}$-equivariant. It follows that there is a natural linear $G$-action on $\text{Hom}((F_{Y/k})_* \mathcal{O}_Y, \mathcal{O}_Y)$. Furthermore, the ‘evaluation at one’ map

$$
\varepsilon: \text{Hom}((F_{Y/k})_* \mathcal{O}_Y, \mathcal{O}_Y) \to H^0(Y', \mathcal{O}_Y)
$$

is a homomorphism of $G$-representations.

If $Y$ is integral and proper, then this is a map from a finite-dimensional $G$-representation to $k$, and since $Y$ is $F$-split, this map is surjective. By the linear reductivity of $G$, this map admits a $G$-equivariant splitting, and this way we obtain a $G$-invariant Frobenius splitting of $Y$.

In general, let $V = \varepsilon^{-1}(k)$, which is the increasing union of $G$-representations $V_i$ of finite dimension over $k$, and is endowed with a $G$-invariant map $\varepsilon: V \to k$. Moreover, the map $\varepsilon$ is surjective since $Y$ is Frobenius split. For some $i$, the restriction $\varepsilon|_{V_i}: V_i \to k$ is surjective, and since $G$ is linearly reductive, there exists a $G$-equivariant splitting $s: k \to V_i$ of $\varepsilon|_{V_i}$.

In particular, $Y$ admits a $G$-invariant Frobenius splitting $\sigma_Y = s(1)$. This Frobenius splitting preserves $G$-invariant sections of $\mathcal{O}_Y$, and hence it descends to a Frobenius splitting on $X$. Thus $\pi: (Y, \sigma_Y) \to (X, \sigma_X)$ is a map in the category $\textbf{F-split}$. Applying the canonical lifting functor from Theorem 2.5.2 yields the desired lifting $\tilde{\pi}: \tilde{Y}(\sigma_Y) \to \tilde{X}(\sigma_X)$ of $\pi$. □

The method of proof of Lemma 2.5.3 yields the following interesting result which we will not need in the sequel.

Lemma 2.5.4. Let $Y$ be an $F$-split scheme over $k$ and let $G$ be a finite group of order prime to $p$ acting on $Y$. Then there exists a lifting $\tilde{Y}$ of $Y$ together with a lifting of the $G$-action.
We shall need the following lemma describing basic properties of $F$-split schemes.

**Lemma 2.5.5.** Let $X$ be a proper $F$-split scheme over an algebraically closed field $k$. Then the following hold.

(a) The map $F^* X : H^i(X, \mathcal{O}_X) \to H^i(X, \mathcal{O}_X)$ is bijective for all $i \geq 0$.
(b) The cohomology groups $H^i(B^1_X)$ vanish for all $i \geq 0$, where $B^1_X = F^*_X \mathcal{O}_X / \mathcal{O}_X$.
(c) If $X$ is smooth, then the Albanese variety $\text{Alb} X$ is ordinary.
(d) Every étale scheme $Y$ over $X$ is $F$-split.

**Proof.** For (a) and (b), we use the long exact sequence of cohomology associated with $0 \to \mathcal{O}_X \to F^*_X \mathcal{O}_X \to B^1_X \to 0$, and the fact that an injective $p^{-1}$-linear endomorphism of a finite dimensional vector space over a perfect field is bijective.

To prove (c), we reason as follows. By [MS87, Lemma 1.3] we see that $H^1(\text{Alb} X, \mathcal{O}_{\text{Alb} X})$ injects into $H^1(X, \mathcal{O}_X)$. By (a) the Frobenius action on the latter group is bijective, and hence it is also bijective on the former. This implies that $\text{Alb} X$ is ordinary.

For (d), let $f : Y \to X$ be an étale morphism. Then the square

$$
\begin{array}{ccc}
Y & \xrightarrow{F_Y} & Y \\
\downarrow f & & \downarrow f \\
X & \xrightarrow{F_X} & X
\end{array}
$$

is cartesian by [SGA 5, XIV–XV §1 n°2, Pr. 2(c)], and hence $f^* F^*_X \mathcal{O}_X \simeq F^*_Y \mathcal{O}_Y$ by flat base change. Thus given a splitting $\sigma : F^*_X \mathcal{O}_X \to \mathcal{O}_X$, applying $f^*$ we obtain a morphism

$$
f^*(\sigma) : F^*_Y \mathcal{O}_Y \simeq f^* F^*_X \mathcal{O}_X \to \mathcal{O}_Y
$$

which is a Frobenius splitting on $Y$. \hfill $\square$

We now turn to the study of Frobenius splittings in a relative situation.

**Definition 2.5.6.** Let $f : X \to S$ be a morphism of $k$-schemes. An $F$-splitting relative to $f$ is an $\mathcal{O}_X$-linear splitting $\sigma_{X/S}$ of $F^*_X : \mathcal{O}_{X'} \to F^*_X \mathcal{O}_X$, where $X'$ is the Frobenius twist of $X$ relative to $S$.

The Frobenius trace map $\text{Tr}_{X/S} : F^*_X \omega_{X/S} \to \omega_{X'/S}$ plays a fundamental role in the theory of $F$-splittings, as indicated by the following well known result.

**Proposition 2.5.7.** Let $X \to S$ be a smooth morphism of $k$-schemes. Then there is a one-to-one correspondence between splittings $\delta_{X/S}$ of $\text{Tr}_{X/S}$ and $F$-splittings $\sigma_{X/S}$ relative to $f$.

**Proof.** Let $\sigma_{X/S}$ be an $F$-splitting relative to $f$. By applying $\text{Hom}(-, \omega_{X'/S})$ to the splitting

$$
\mathcal{O}_{X'} \xrightarrow{\sigma_{X/S}} F^*_X \mathcal{O}_X \xrightarrow{\text{Tr}_{X/S}} \omega_{X'/S}
$$

we get by the Grothendieck duality (see Remark 2.4.3) a splitting

$$
\omega_{X'/S} \xrightarrow{\delta_{X/S}} F^*_X \omega_{X/S} \xrightarrow{\text{Tr}_{X/S}} \omega_{X'/S}.
$$

The correspondence in the other direction is constructed analogously. \hfill $\square$

The following auxiliary base change result shows in particular that the fibers of a smooth relatively $F$-split morphism are $F$-split.
Lemma 2.5.8 (cf. [PSZ13, Lemma 2.18]). Consider a Cartesian diagram of $k$-schemes

$$
W \xrightarrow{\psi} X \\
\downarrow f' \quad \downarrow f \\
Z \xrightarrow{f} S,
$$

with $f : X \to S$ smooth, and let $\delta_{X/S}$ be a splitting of $\text{Tr}_{X/S}$. Then there exists a splitting $\delta_{W/Z}$ of $\text{Tr}_{W/Z}$ induced by the following commutative diagram.

$$
\psi^* \omega_{X'/S} \sim \omega_{W'/Z} \\
\psi^* \delta_{X/S} \quad \delta_{W/Z} \\
\psi^* (F_{X/S})_* \omega_{X/S} \sim (F_{W/Z})_* \omega_{W/Z},
$$

where $X'$ and $W'$ are the Frobenius twists of $X$ and $W$ relative to $S$ and $Z$, respectively.

Proof. The arrow $\psi^* (F_{X/S})_* \omega_{X/S} \to (F_{W/Z})_* \omega_{W/Z}$ in the above diagram comes from the cohomological base change for the diagram

$$
W \xrightarrow{F_{W/Z}} X \\
\downarrow F_{X/S} \\
W' \xrightarrow{X'}.
$$

To conclude the proof of the lemma it is enough to show that this arrow is an isomorphism, and this is clear because $F_{X/S}^* \omega_{X/S}$ is a vector bundle ($f$ is smooth, so $F_{X/S}$ is finite and flat) and the above diagram is Cartesian. □

The following result shows that to every $F$-splitting we can associate a $\mathbb{Q}$-divisor.

Proposition 2.5.9 ([Sch09] and [PSZ13]). Let $f : X \to S$ be a smooth morphism of $k$-schemes. Then to every splitting $\delta_{X/S}$ of $\text{Tr}_{X/S}$ we can canonically associate an effective $\mathbb{Q}$-divisor $\Delta_{\delta_{X/S}}$ on $X$ such that

(a) $\Delta_{\delta_{X/S}} \sim_{\mathbb{Q}} -K_{X/S}$;

(b) if $f$ has connected fibers, then $\Delta_{\delta_{X/S}}$ is horizontal, i.e., it does not contain any fiber.

Proof. By adjunction, $\delta_{X/S}$ induces a morphism $F_{X/S}^* \omega_{X'/S} \to \omega_{X/S}$ which is equivalent to $\delta_X \to \omega_{X/S}^{1-p}$, and so we get a divisor

$$
D_{\delta_{X/S}} \sim (1-p)K_{X/S}.
$$

Set $\Delta_{\delta_{X/S}} = \frac{1}{p-1} D_{\delta_{X/S}}$. The restriction of this $\mathbb{Q}$-divisor to a fiber of $f$ is non-zero as it corresponds to a splitting of the Frobenius trace map on this fiber (see Lemma 2.5.8). □

For a smooth $k$-variety $X$ and a splitting $\delta_X$ of $\text{Tr}_{X/k}$, we denote the corresponding $\mathbb{Q}$-divisor by $\Delta_{\delta_X}$.

Remark 2.5.10. In the setting of Lemma 2.5.8, we have $\psi^* \Delta_{\delta_{X/S}} = \Delta_{\delta_{W/Z}}$ (see the commutative diagram in the statement of this lemma).

Remark 2.5.11. It is easy to see that the coefficients of $\Delta_{\delta_{X/S}}$ are at most one. When $S = \text{Spec } k$ (which is the only case in which we will apply this observation), this follows from [HW02, Theorem 3.3] (cf. [SS10, Theorem 4.4]).

Remark 2.5.12. It is not necessary to assume that $f : X \to S$ is smooth in order to be able to associate a $\mathbb{Q}$-divisor to a relative $F$-splitting. A far more general setting is described in [PSZ13]. When $S = \text{Spec } k$, then a $\mathbb{Q}$-divisor can be associated to a Frobenius splitting on every normal variety $X$ by an extension from the smooth locus.
3. Frobenius liftability

Throughout this section we fix a perfect field \( k \) of characteristic \( p > 0 \) (sometimes assumed algebraically closed). First, set \( S = \text{Spec } k \) and \( \bar{S} = \text{Spec } W_2(k) \) (see §1.6). If \( X \) is a scheme over \( S \) and \( \bar{X} \) is a lifting of \( X \) to \( \bar{S} \), a lifting of Frobenius on \( X \) to \( \bar{X} \) is a morphism \( \bar{F}_X : \bar{X} \to \bar{X} \) restricting to \( F_X \) on \( X \). It automatically commutes with the map \( \sigma : \bar{S} \to S \) induced by the Witt vector Frobenius on \( W_2(k) \) (see §3.5), and hence it fits into the commutative diagram

![Diagram](https://example.com/diagram.png)

We call a pair \((\bar{X}, \bar{F}_X)\), where \( \bar{X} \) and \( \bar{F}_X \) are as above, a Frobenius lifting of \( X \). If such a pair exists, we say that \( X \) is \( F \)-liftable. In this situation, we can form a diagram lifting the relative Frobenius diagram (1.6.1):

![Diagram](https://example.com/diagram.png)

Given a lifting \( \bar{X} \), the existence of \( \bar{F}_X \) is thus clearly equivalent to the existence of a lifting \( \bar{F}_{X/S} : \bar{X} \to \bar{X}' \) of the relative Frobenius \( F_{X/S} : X \to X' \).

More generally, the above definitions work in the relative setting, that is when \( S \) is an arbitrary \( k \)-scheme endowed with a Frobenius lifting \((\bar{S}, \bar{F}_S)\). If \( \pi : X \to S \) is a scheme over \( S \) and if \( \bar{\pi} : \bar{X} \to \bar{S} \) is a lifting of \( \pi \), then providing a lifting \( \bar{F}_X \) of \( F_X \) to \( \bar{X} \) such that \( \bar{F}_S \circ \bar{\pi} = \bar{\pi} \circ \bar{F}_X \) is equivalent to providing a morphism \( \bar{F}_{X/S} : \bar{X} \to \bar{X}' \) lifting the relative Frobenius \( F_{X/S} \), where \( \bar{X}' \) is the base change of \( \bar{X} \) along \( \bar{F}_S \).

If \((X, D)\) is an nc pair over \( S \), a Frobenius lifting of \((X, D)\) is a triple \((\bar{X}, \bar{D}, \bar{F}_X)\) where \((\bar{X}, \bar{D})\) is an nc pair lifting \((X, D)\) and \( \bar{F}_X \) is a lifting of \( F_X \) to \( X \) satisfying \( \bar{F}_X(\bar{D}) = p\bar{D} \).

We shall study these in more detail in §3.4. In view of Remark 2.3.2, Frobenius liftings of \((X, D)\) correspond to liftings of the associated log scheme \((X, \mathcal{M}_X)\).

3.1. Examples of \( F \)-liftable schemes. We shall be mostly concerned with smooth schemes in this paper. Examples of \( F \)-liftable singularities were studied in [Zda17].

**Example 3.1.1** (Smooth affines). Every smooth affine \( k \)-scheme is \( F \)-liftable. Indeed, the obstruction class to lifting \( X \) together with \( F_X \) lies in \( \text{Ext}^1(\Omega^1_X, B^1_X) = H^1(X, T_X \otimes B^1_X) \) (see [MS87, Appendix], cf. Proposition 3.3.1(b)), which is zero.

**Example 3.1.2** (Toric varieties). Every toric variety \( X = X(\Sigma)_k \) over \( k \) is \( F \)-liftable [BTLM97]. More precisely, let \( \Sigma \) be a fan in \( \mathbb{N}_R \) for a lattice \( N \) and let \( X(\Sigma) \) be the associated toric variety over \( \text{Spec } \mathbb{Z} \). The multiplication by \( p \) map \( N \to N \) preserves \( \Sigma \) and
hence it induces a morphism \( \tilde{F} : X(\Sigma) \to X(\Sigma) \). Its restriction to \( X(\Sigma)_{\mathbb{F}_p} \) is the absolute Frobenius.

Of course not every Frobenius lifting on a toric variety has to be of this type, e.g. any collection of homogeneous polynomials \( f_0, \ldots, f_n \in k[x_0, \ldots, x_n] \) of degree \( p \) defines a lifting of Frobenius on \( \mathbb{P}^n_{W_2(k)} \) by
\[
\tilde{F}(x_0 : \ldots : x_n) = (x_0^p + pf_0(x_0, \ldots, x_n) : \ldots : x_n^p + pf_n(x_0, \ldots, x_n)).
\]
The existence of such non-standard liftings is one of the main difficulties in Conjecture 1, and provides a contrast between its two extreme (toric and abelian) cases.

**Example 3.1.3** (Ordinary abelian varieties). An abelian variety \( A \) over \( k \) is \( F \)-liftable if and only if it is ordinary, in which case there exists a unique Frobenius lifting \( (\tilde{A}, \tilde{F}_{\tilde{A}}) \), called the Serre–Tate canonical lifting of \( A \) (see [MS87, Appendix]). It has the remarkable property that for every line bundle \( L \) on \( A \), there exists a unique up to isomorphism line bundle \( \tilde{L} \) on \( \tilde{A} \) such that \( \tilde{F}_{\tilde{A}}^* \tilde{L} \simeq \tilde{L}^{\otimes p} \).

**Example 3.1.4** (Étale quotients of ordinary abelian varieties). Let \( A \) be an ordinary abelian variety and let \( f : A \to Y \) be a finite étale surjective morphism. Replacing \( f \) by its Galois closure, we can assume that it is Galois. Then \( Y \) is \( F \)-liftable by [MS87, Theorem 2], but it need not be an abelian variety.

**Example 3.1.5** (Toric fibrations over ordinary abelian varieties). In order to combine Examples 3.1.2 and 3.1.3, let \( A \) be an abelian variety, and let \( f : X \to A \) be a toric fibration under a torus \( T \) (see §2.1). Then \( X \) is \( F \)-liftable if and only if \( A \) is ordinary (see Theorem 5.1.1).

**Remark 3.1.6.** In fact, in each of the above examples, the Frobenius liftings exist over \( W(k) \) and not just over \( W_2(k) \). We do not know an example of a smooth \( k \)-scheme admitting a Frobenius lifting only over \( W_2(k) \). However, it is important to work over \( W_2(k) \) as this allows for descending Frobenius liftability under appropriate finite morphisms (see Theorem 3.3.6).

**Remark 3.1.7** (Partial converse to Conjecture 1). Consider the situation of the assertion of Conjecture 1: let \( A \) be an ordinary abelian variety, let \( Y \to A \) be a smooth toric fibration with toric boundary \( D \subseteq Y \), and let \( Y \to X \) be a finite étale map. Then \( (Y, D) \) is \( F \)-liftable by Example 3.1.5, and \( X \) is \( F \)-liftable if one of the following conditions holds.

1. If \( X = Y/G \) where \( G \) is of order prime to \( p \) (see Theorem 3.3.6(c)). Note that since \( \pi_1(Y) \to \pi_1(A) \), passing to the Galois closure we can always assume that \( X = Y/G \) for a free action of a finite group \( G \) on \( Y \).
2. If the toric boundary \( D \subseteq Y \) is a pull-back from \( X \) (Theorem 5.1.1(iii)⇒(i)), for example if \( Y = A \) (Example 3.1.4).

However, \( X \) is not \( F \)-liftable in general: if \( Y = \mathbb{P}^1 \times C \) where \( C \) is an ordinary elliptic curve, and \( G = \mathbb{Z}/p\mathbb{Z} \) acts on \( \mathbb{P}^1 \) by \((x : y) \mapsto (x + ay : y)\) and on \( C \) by translation, then the diagonal action of \( G \) on \( Y \) is free, and \( X = Y/G \) is not \( F \)-liftable (see Lemma 7.1.8 and Proposition 7.1.6). It would be interesting to find an ‘if and only if’ criterion for the \( F \)-liftable of quotients \( X = Y/G \) as above with \( G \) an arbitrary finite group, or even an abelian \( p \)-group.

### 3.2. Consequences of \( F \)-liftable.

As mentioned in the introduction, the existence of a Frobenius lifting has strong consequences for *smooth* schemes. First, let us recall the construction of the map \( \xi \) (see (1.1.1), cf. [DI87, Proof of Théorème 2.1, (b)]).

**Construction of the map \( \xi \).** Let \((\tilde{S}, \tilde{F}_S)\) be a Frobenius lifting of a \( k \)-scheme \( S \), let \((\tilde{X}, \tilde{F}_X)\) be a Frobenius lifting of a \( k \)-scheme \( X \), and let \( \tilde{\pi} : \tilde{X} \to \tilde{S} \) be a smooth morphism of \( W_2(k) \)-schemes such that \( \tilde{F}_S \circ \tilde{\pi} = \tilde{\pi} \circ \tilde{F}_X \). Let \( \tilde{F}_{X/S} \) be the induced lifting of the relative Frobenius. By flatness of \( \tilde{X} \to \tilde{S} \), the differential \( d\tilde{F}_{X/S} \) fits into the following diagram with...
exact rows

\[
\begin{array}{cccccc}
0 & \longrightarrow & F_{X/S}^* \Omega^1_{X'/S} & \longrightarrow & F_{X/S}^* \Omega^1_{X'/S} & \longrightarrow & 0 \\
\downarrow dF_{X/S} & & \downarrow dF_{X/S} & & \downarrow dF_{X/S} & & \\
0 & \longrightarrow & \Omega^1_{X/S} & \longrightarrow & \Omega^1_{X/S} & \longrightarrow & 0
\end{array}
\]

Using the snake lemma, we obtain a mapping

\[\xi = \frac{d\tilde{F}_{X/S}}{p} : F_{X/S}^* \Omega^1_{X'/S} \to \Omega^1_{X/S}.\]

In simple terms, \(\xi(\omega) = \frac{1}{p} d\tilde{F}_X(\omega)\), where \(\omega\) is any lifting of a local form \(\omega \in F_{X/S}^* \Omega^1_{X'/S}\). In particular, if \(S = \text{Spec } k\), then \(\xi(df) = f^{p-1}df + dg\) where \(\tilde{F}_X(f) = \tilde{f}^p + pg\).

Since \(F_{X/S}^* \Omega^1_{X'/S}\) is isomorphic to \(F_X^* \Omega^1_{X/S}\), we may interpret \(\xi\) as an \(\Theta_X\)-linear morphism \(F_X^* \Omega^1_{X/S} \to \Omega^1_{X/S}\).

**Proposition 3.2.1** ([DI87, Proof of Théorème 2.1 and §4.1], [BTLM97, Theorem 2]). The mapping \(\xi : F_{X/S}^* \Omega^1_{X'/S} \to \Omega^1_{X/S}\) satisfies the following properties:

(a) The adjoint morphism \(\xi^{\text{ad}} : \Omega^1_{X/S} \to F_{X/S}, \Omega^1_{X/S}\) has image in the subsheaf \(\mathcal{Z}^1_{X/S}\) of closed forms and provides a splitting of the short exact sequence

\[0 \longrightarrow B^1_{X/S} \longrightarrow Z^1_{X/S} \longrightarrow \Omega^1_{X/S} \longrightarrow 0.\]

(b) By taking exterior powers, the morphism \(\xi^{\text{ad}}\) induces splittings of the short exact sequences

\[0 \longrightarrow B^i_{X/S} \longrightarrow Z^i_{X/S} \longrightarrow \Omega^i_{X/S} \longrightarrow 0,\]

as well as a quasi-isomorphism of differential graded algebras

\[\wedge^* \xi^{\text{ad}} : \bigoplus_{i \geq 0} \Omega^i_{X'/S}[-i] \xrightarrow{\sim} (F_{X/S})_* \Omega^*_{X/S}\]

where the maps \(\wedge^i \xi^{\text{ad}} : \Omega^i_{X'/S} \to (F_{X/S})_* \Omega^i_{X/S}\) are split injections. On the level of cohomology, this map induces the Cartier isomorphism.

(c) The determinant

\[\det(\xi) : F_{X/S}^* \omega^1_{X'/S} \longrightarrow \omega^1_{X/S}\]

corresponds to a Frobenius splitting \(\sigma\) of \(X\) relative to \(S\) (see §2.5). In particular, the homomorphism \(\xi\) is injective.

**Variant 3.2.2** (Logarithmic variant of Proposition 3.2.1). If \((X, D)\) is an nc pair over \(S\), and if \((\tilde{X}, \tilde{D}, \tilde{F}_X)\) is a Frobenius lifting of \((X, D)\), we get a morphism

\[\xi : F_X^* \Omega^1_{X/S}(\log D) \to \Omega^1_{X/S}(\log D)\]

and the assertions of Proposition 3.2.1 hold in this case (cf. [DI87, §4.2]).

**Corollary 3.2.3.** Let \(X\) be an \(F\)-liftable smooth and proper scheme over \(k\). Then \(X\) is ordinary in the sense of Bloch and Kato, i.e., \(H^i(X, B^i_X) = 0\) for all \(i, j \geq 0\).

**Proof.** This follows from the proof of [MS87, Lemma 1.1].
Bott vanishing. By Proposition 3.2.1(b), we have $H^j(X, \Omega^i_X \otimes \mathcal{L}) \subseteq H^j(X, \Omega^i_X \otimes \mathcal{L}^p)$ for every line bundle $\mathcal{L}$. This explains the following result.

**Theorem 3.2.4** (Bott vanishing, [BTLM97, Theorem 3]). Let $X$ be a smooth projective $F$-liftable scheme over $k$, and let $\mathcal{L}$ be an ample line bundle on $X$. Then

$$H^j(X, \Omega^i_X \otimes \mathcal{L}) = 0 \quad \text{for} \ j > 0, \ i \geq 0.$$

Moreover, if $(X, D)$ is an $F$-liftable nc pair, then

$$H^j(X, \Omega^i_X(\log D) \otimes \mathcal{L}) = 0 \quad \text{for} \ j > 0, \ i \geq 0.$$

**Corollary 3.2.5.** Let $X$ be a smooth projective $F$-liftable scheme over $k$. Suppose that $X$ is Fano (i.e., $\omega_X^{-1}$ is ample). Then $H^i(X, T_X) = 0$ for $i > 0$. In particular, $X$ is rigid and has unobstructed deformations.

**Proof.** Since $T_X \cong \Omega^{-1}_X \otimes \omega_X^{-1}$, Bott vanishing implies that

$$H^i(X, T_X) = H^i(X, \Omega^{-1}_X \otimes \omega_X^{-1}) = 0.$$ \hfill $\square$

The above results suggest that $F$-liftable property is a rare property. For instance, we obtain the following examples of non-liftable varieties.

**Example 3.2.6.** A smooth hypersurface $X \subseteq \mathbb{P}^n$ of degree $d > 1$, where $n > 1$, is not $F$-liftable as long as $n + d > 5$. Indeed, we may reason as follows.

1. If $d > n + 1$, then $X$ has positive Kodaira dimension, which contradicts Proposition 3.2.1(c).
2. If $d = n + 1$, then $\omega_X$ is trivial and $X$ is simply connected, contradicting [MS87, Theorem 2].
3. If $2 < d < n + 1$, then $X$ is Fano but not rigid, contradicting Corollary 3.2.5.
4. If $d = 2$ and $n > 3$, then $X$ is not $F$-liftable because $H^1(X, \Omega^{n-2}_X(n-3)) \neq 0$, contradicting Bott vanishing (see [BTLM97, §4.1]). See also [Zda17, Theorem 4.15].

**The sheaf of $\xi$-invariant forms.** Let $(\bar{X}, \bar{F}_X)$ be a Frobenius lifting of a smooth $k$-scheme $X$. We can regard the induced map

$$\xi = \frac{1}{p} d\bar{F}_X : \bar{F}_X \Omega^1_X \rightarrow \Omega^1_X$$

as a Frobenius-linear endomorphism of $\Omega^1_X$, that is

$$\xi(f \cdot \omega) = f^p \cdot \xi(\omega) \quad \text{for} \ f \in \mathcal{O}_X, \ \omega \in \Omega^1_X.$$

As observed in [MS87], if $\omega_X$ is numerically trivial, then $\xi$ is an isomorphism, and therefore $\Omega^1_X$ becomes trivial on a finite étale cover $Y$ of $X$ (see [LS77, Satz 1.4]).

A related geometric idea, which we put to good use in §6, is to look at the (étale) subsheaf $(\Omega^1_X)^\xi$ of $\xi$-invariant forms in $\Omega^1_X$. By the very definition, $(\Omega^1_X)^\xi$ is the étale sheaf of sections of the fixed point locus $(T^*X)^\xi$ inside the cotangent bundle.

**Lemma 3.2.7.** The subscheme $(T^*X)^\xi \subseteq T^*X$ is étale over $X$ and the étale sheaf $(\Omega^1_X)^\xi$ is a constructible sheaf of $F_p$-vector spaces on $X$. Over the dense open subset $U \subseteq X$ where $\xi$ is an isomorphism, $(T^*X)^\xi$ is finite over $X$ of degree $p \dim X$ and $(\Omega^1_X)^\xi$ is an $F_p$-locally constant sheaf of rank $p \dim X$.

**Proof.** The assertions are local on $X$, so we may assume that there exists an isomorphism $\mathcal{O}_{\bar{X}}^{\dim X} \cong \Omega^1_X$. Since $\xi$ as an endomorphism of $\Omega^1_X$ is Frobenius-linear, its fixed point locus is described by the following system of equations in variables $[f_1, \ldots, f_n]$:

$$[f_1, \ldots, f_n] - [f_1^p, \ldots, f_n^p] \cdot A = 0,$$

for some $A \in M_{n \times n}(\Gamma(X, \mathcal{O}_X))$. We immediately see that the Jacobian of the above system of equations is the identity matrix, and thus $(T^*X)^\xi$ is étale over $X$. The other assertions are clear in view of the fact that for a vector space $V$ of dimension $r$ endowed with a Frobenius-linear isomorphism $\xi : F^r V \rightarrow V$, the locus of fixed points $V^\xi$ is isomorphic to $F_p^r$. \hfill $\square$
3.3. Deformation theory of the Frobenius morphism. In this subsection, we first discuss obstruction classes to lifting Frobenius, as originally defined by Nori and Srinivas. Then we apply them, together with other tools, to show in Theorem 3.3.6 that under certain conditions on a morphism \( \pi: Y \to X \), if \( Y \) is \( F \)-liftable, then so is \( X \). We also discuss \( F \)-liftables of products, toric fibrations, and divisorial rings.

Obstruction classes and their functoriality. Since we are dealing with deformations of schemes and their Frobenius along the first order thickening \( S \to \tilde{S} \), it is natural to seek an obstruction theory for lifting the Frobenius. For smooth schemes over a perfect field \( k \), such a theory was developed by Nori and Srinivas in the appendix to [MS87]. The usual deformation theory of morphisms already tells us that

- given a morphism \( f: X \to Y \) and a lifting \( \tilde{Y} \), the obstruction to the existence of \( f: \tilde{X} \to \tilde{Y} \) is a class in \( \text{Ext}^2(L_{X/Y}, \mathcal{O}_X) \) (\( \simeq H^2(X, T_{X/Y}) \)) if \( f \) is smooth,
- if we also fix \( \tilde{X} \) a priori, then we have an obstruction lying in \( \text{Ext}^1(Lf^*L_Y, \mathcal{O}_X) \) (\( \simeq H^1(X, f^*T_Y) \)) if \( Y \) is smooth) to lifting \( f \).

The first obstruction theory does not apply in our context of lifting \( \sigma \) of Frobenius on \( \mathcal{O}_X \) to \( W_n(k) \). In this case, the lifting \( \sigma \) of Frobenius on \( W_n(k) \) is unique. As in their situation, in §5 we shall actually need to lift not only to \( W_2(k) \) but all the way to \( W(k) \) to apply characteristic zero methods.

Their results are as follows.

**Proposition 3.3.1** ([MS87, Appendix]). Let \( X \) be a smooth scheme over \( k \), and let \( (X_n, F_{X_n}) \) be a Frobenius lifting of \( X \) over \( W_n(k) \). Then the following holds.

(a) For every lifting \( X_{n+1} \) of \( X_n \) over \( W_{n+1}(k) \) there exists an obstruction class

\[
o_{X_{n+1}}^F \in \text{Ext}^1(\Omega^1_{X}, F_{X_n} \mathcal{O}_X)
\]

whose vanishing is sufficient and necessary for the existence of a lifting \( F_{X_{n+1}} \) of \( F_{X_n} \) to \( X_{n+1} \). If the obstruction vanishes, then the space of such liftings is a torsor under \( \text{Hom}(\Omega^1_{X}, F_{X_n} \mathcal{O}_X) \).

(b) There exists an obstruction class

\[
o_{X_n} \in \text{Ext}^1(\Omega^1_{X}, B^1_X)
\]

whose vanishing is sufficient and necessary for the existence of a lifting \( (X_{n+1}, F_{X_{n+1}}) \) of \( (X_n, F_{X_n}) \) over \( W_{n+1}(k) \). If the obstruction vanishes, then the space of such liftings is a torsor under \( \text{Hom}(\Omega^1_{X}, B^1_X) \).

(c) The obstruction class \( o_X \in \text{Ext}^1(\Omega^1_{X}, B^1_X) \) to lifting \( X \) over \( W_2(k) \) compatibly with the Frobenius morphism equals the class of the extension

\[
0 \to B^1_X \to Z^1_X \xrightarrow{c_{2X}} \Omega^1_X \to 0.
\]

(d) Let \( (X_{n+1}, F_{X_{n+1}}) \) be a lifting of \( (X_n, F_{X_n}) \) over \( W_{n+1}(k) \) and suppose that the Frobenius action on \( H^i(X, \mathcal{O}_X) \) is bijective for \( i = 1, 2 \). Then for every \( \mathcal{L}_n \in \text{Pic} X_n \) such that \( F^*_{X_{n+1}} \mathcal{L}_n = \mathcal{L}_{n+1} \), there exists a unique \( \mathcal{L}_{n+1} \in \text{Pic} X_{n+1} \) such that \( \mathcal{L}_{n+1}|X_n \simeq \mathcal{L}_n \) and \( F^*_{X_{n+1}} \mathcal{L}_{n+1} = \mathcal{L}_{n+1} \).

**Variant 3.3.2** (Logarithmic variant of Proposition 3.3.1). The above proposition can be repeated word for word for \( nc \) pairs \( (X, D) \). In this case, the sheaves of Kähler differentials are replaced by logarithmic differentials \( \Omega^1_X(\log D) \). The sheaf \( B^1_X \) is left without change,
since the 1-boundaries $B^1_X(\log D)$ of the logarithmic de Rham complex coincide with the 1-boundaries $B^1_X$ of the standard de Rham complex. We do not give the full proof, but remark about the main ingredients of the logarithmic version of (b) and (c).

(1) As in standard deformation theory, for an nc pair $(X_n, D_n)$ over $W_0(k)$ we notice that any two liftings over $W_{n+1}(k)$ (as nc pairs) are Zariski-locally isomorphic and the infinitesimal automorphisms are parametrized by sections of $\mathcal{H}om(\Omega^1_X(\log D), \mathcal{O}_X)$ (see [EV92, Proposition 8.22]).

(2) The logarithmic analogue of Proposition 3.3.1(a) holds, and in particular for every Frobenius lifting $F_{X_n} : (X_n, D_n) \to (X_n, D_n)$ local liftings of the Frobenius morphism over $W_{n+1}(k)$ exist and are torsors under sections of $\mathcal{H}om(\Omega^1_X(\log D), F_{X_n} \mathcal{O}_X)$ (see [EV92, Proposition 9.3]).

Now, to construct the obstruction class for lifting $(X_n, D_n, F_{X_n})$ over $W_{n+1}(k)$ we reason as follows. We take an open affine covering $\{(U_{i+n}^i, D_{i+n}^i)\}$ of $(X_n, D_n)$, and then lift $(U_{i+n}^i, D_{i+n}^i)$ to some nc pairs $(U_{i+n+1}^{i+1}, D_{i+n+1}^{i+1})$ over $W_{n+1}(k)$. By (2) the nc pairs $(U_{i+n+1}^{i+1}, D_{i+n+1}^{i+1})$ admit Frobenius liftings $F_{U_{i+n+1}^{i+1}} : (U_{i+n+1}^{i+1}, D_{i+n+1}^{i+1}) \to (U_{i+n}^i, D_{i+n}^i)$ extending $F_{X_n}|_{U_n^i}$. We set $U_{i+1}^{i+1} = U_{i+n+1}^{i+1} \cap U_{i+n}^i$.

By (1) we may fix isomorphisms

$$\varphi_{ij} : (U_{i+n+1}^{i+1}, D_{i+n+1}^{i+1})|_{U_{i+1}^{i+1}} \cong (U_{i+n}^i, D_{i+n}^i)|_{U_{i+1}^{i+1}} \quad \text{satisfying} \quad \varphi_{ji} = \varphi_{ij}^{-1}.$$ 

The morphisms $F_{U_{i+n+1}^{i+1}}$ and $\varphi_{ij} \circ F_{U_{i+n}^i} \circ \varphi_{ji}$ are two liftings of $F_{X_n}|_{U_{i+1}^{i+1}}$ and hence they give rise to local section $\tau_{ij}$ of the sheaf $\mathcal{H}om(\Omega^1_X(\log D), F_{X_n} \mathcal{O}_X)$. The images of $\tau_{ij}$ under the homomorphism $\mathcal{H}om(\Omega^1_X(\log D), F_{X_n} \mathcal{O}_X) \to \mathcal{H}om(\Omega^1_X(\log D), B^1_X)$ does not depend on the choices made, and give a cocycle whose cohomology class we claim is the required obstruction. For the rest of the proof, we may repeat the reasoning of [MS87, Appendix, Proposition 1 (vi)] word for word. For the proof of (c), we observe that the difference of the morphisms $\xi$ induced by the liftings $F_{\tau_i}$ and $\varphi_{ij} \circ F_{U_{i+1}^j} \circ \varphi_{ji}$ divided by $p$ is a map $h_{ij} : \Omega^1_{U_{i+1}^j}(\log D) \to B^1_{U_{i+1}^j}$, and that $\{h_{ij}\}$ is a cocycle representing the log Cartier sequence.

The obstruction classes satisfy the following functoriality properties.

**Lemma 3.3.3.** Let $\pi : Y \to X$ be a morphism of smooth schemes over $k$.

(a) Let $\pi : \tilde{Y} \to \tilde{X}$ be a lifting of $\pi$ over $W_2(k)$. Then the obstruction classes $\alpha_{\tilde{X}}$ and $\alpha_{\tilde{Y}}$, treated as morphisms in the appropriate derived categories, fit into the following commutative diagrams

\[
\begin{array}{ccc}
\pi^* F^*_{\tilde{X}} \Omega^1_{\tilde{X}} & \xrightarrow{\pi^* \alpha_{\tilde{X}}} & \pi^* \mathcal{O}_X[1] \\
F^*_{\tilde{Y}} \to & & \downarrow \\
F^*_{\tilde{Y}} \Omega^1_{\tilde{Y}} & \xrightarrow{\alpha_{\tilde{Y}}} & \mathcal{O}_Y[1]
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
F^*_{\tilde{Y}} \Omega^1_{\tilde{Y}} & \xrightarrow{\alpha_{\tilde{Y}}} & \mathcal{O}_Y[1] \\
R\pi_* F^*_{\tilde{Y}} \Omega^1_{\tilde{Y}} & \xrightarrow{R\pi_* \alpha_{\tilde{Y}}} & R\pi_* \mathcal{O}_Y[1].
\end{array}
\]

(b) Suppose that $\pi$ is smooth. Then the obstruction classes $\alpha_X$ and $\alpha_Y$ fit into the following commutative diagram

\[
\begin{array}{ccc}
\pi^* \Omega^1_X & \xrightarrow{\pi^* \alpha_X} & \pi^* B^1_X[1] \\
\downarrow & & \downarrow \\
\Omega^1_Y & \xrightarrow{\alpha_Y} & B^1_Y[1].
\end{array}
\]
Proof. For part (a), see [Zda17, Lemma 4.1]. For part (b), use Proposition 3.3.1(c) and the commutativity of the diagram
\[
\begin{array}{cccccc}
0 & \longrightarrow & \pi^* B^1_X & \longrightarrow & \pi^* Z^1_X & \longrightarrow & \pi^* \Omega^1_X & \longrightarrow & 0 \\
& & d\pi & & d\pi & & d\pi & & \\
0 & \longrightarrow & B^1_Y & \longrightarrow & Z^1_Y & \longrightarrow & \Omega^1_Y & \longrightarrow & 0.
\end{array}
\]

Remark 3.3.4. In the case of singular schemes, the cotangent bundle $\Omega^1_X$ can be substituted with its derived variant $L_{X/k} \in D_{\text{Coh}}(\mathcal{O}_X)$ (see [Ill71]). The assertions of Proposition 3.3.1(a) and Lemma 3.3.3(a) remain valid with the standard differential replaced by the derived one.

**Descendent and lifting Frobenius liftings.** With the abstract deformation theory at hand, we can now easily relate $F$-liftable schemes $X$ and $Y$ in the presence of a suitable morphism $\pi: Y \to X$. In the subsequent sections, we shall frequently use the following results.

**Lemma 3.3.5.** Let $\pi: Y \to X$ be an étale morphism of $k$-schemes. For every Frobenius lifting $(\bar{X}, F_{\bar{X}})$ of $X$, there exists a unique Frobenius lifting $(\bar{Y}, F_{\bar{Y}})$ of $Y$ and a lifting $\bar{\pi}: \bar{Y} \to \bar{X}$ of $\pi$ such that $\bar{F}_{\bar{X}} \circ \bar{\pi} = \bar{\pi} \circ F_{\bar{Y}}$.

Proof. This follows from the equivalence of categories between étale schemes over $X$ and over $\bar{X}$.

**Theorem 3.3.6** (Descending Frobenius liftability). Let $\pi: Y \to X$ be a morphism of schemes (essentially) of finite type over $k$ and let $(\bar{Y}, F_{\bar{Y}})$ be a Frobenius lifting of $Y$.

(a) Suppose that $\pi$ admits a lifting $\bar{\pi}: \bar{Y} \to \bar{X}$, and that one of the following conditions is satisfied:
\begin{enumerate}
  \item $\pi^*: \mathcal{O}_X \to R\pi_* \mathcal{O}_Y$ is a split monomorphism in the derived category,
  \item $\pi$ is finite flat of degree prime to $p$, \item $Y$ satisfies condition $S_2$ and $\pi$ is an open immersion such that $X \setminus Y$ has codimension $> 1$ in $X$.
\end{enumerate}
Then $F_X$ lifts to $\bar{X}$.

(b) Suppose that one of the following conditions is satisfied:
\begin{enumerate}
  \item $\mathcal{O}_X \simeq \pi_* \mathcal{O}_Y$ and $R^1 \pi_* \mathcal{O}_Y = 0$,
  \item $X$ and $Y$ are smooth and $\pi$ is proper and birational,
  \item $Y$ satisfies condition $S_3$ and $\pi$ is an open immersion such that $X \setminus Y$ has codimension $> 2$ in $X$.
\end{enumerate}
Then there exists a unique pair of a Frobenius lifting $(\bar{X}, F_{\bar{X}})$ of $X$ and a lifting $\bar{\pi}: \bar{Y} \to \bar{X}$ of $\pi$ such that $\bar{F}_{\bar{X}} \circ \bar{\pi} = \bar{\pi} \circ F_{\bar{Y}}$.

(c) Suppose that $Y$ is normal and that $\pi: Y \to X = Y/G$ is a good quotient by an action of a linearly reductive group $G$ on $Y$. Then there exists a lifting $\bar{\pi}: \bar{Y} \to \bar{X}$ of $\pi$ and a lifting $\bar{F}_{\bar{X}}$ of $F_X$ to $\bar{X}$.

In fact, conditions (a.ii) and (b.ii) imply (a.i) and (b.i), respectively. We do not expect $\bar{F}_{\bar{X}} \circ \bar{\pi} = \bar{\pi} \circ F_{\bar{Y}}$ to hold in general in situations (a) and (c).

**Proof.** (a) Under condition (i), the right arrow of the right diagram (3.3.1) is a split injection by assumption. Thus $\mathcal{O}^0_{\bar{X}} = 0$ if $\mathcal{O}^0_{\bar{Y}} = 0$. Condition (ii) implies (i), as $R\pi_* \mathcal{O}_Y = \pi_* \mathcal{O}_Y$ and $1/\deg(\pi)$ times the trace map yields a splitting. For (iii), we argue as in [Zda17, Corollary 4.3]. Let $K$ be the fiber of the right arrow in the right diagram (3.3.1), fitting into an exact triangle
\[
(3.3.2) \quad K \longrightarrow \mathcal{O}_X[1] \longrightarrow R\pi_* \mathcal{O}_Y[1] \longrightarrow K[1].
\]
Since the bottom map in the right diagram (3.3.1) is zero by assumption, the top map has to factor through $K$. It is therefore enough to show that $\text{Hom}(F^*_X \mathcal{O}_X, K) = 0$. Note that
\[ K = R\Gamma_Z(O_X)[1] \] is the shift by one of the local cohomology complex with supports on \( Z = X \setminus Y \) (see [Har67] or [Sta14, Tag 0A39]). Considering the spectral sequence
\[
E_2^{pq} = \text{Hom}(F_X^q \Omega^1_{\mathcal{X}}, \mathcal{H}^p(\mathcal{O}_Z(O_X))[p]) \Rightarrow \text{Hom}(F_X^q \Omega^1_{\mathcal{X}}, R\Gamma_Z(O_X)[p+q]),
\]
for \( p + q = 1 \), we see that it suffices to show that the local cohomology vanishes up to degree one, which is implied by the Serre’s condition \( S_2 \) and [HK04, Proposition 3.3]. Analogous reasoning works if \( X \) and \( Y \) are not smooth. In this case, as mentioned in Remark 3.3.4, we substitute the sheaf of Kähler differentials with the cotangent complex.

(b) Under condition (i), Lemma 2.2.1 provides a lifting \( \hat{\pi} : \hat{Y} \to \hat{X} \) defined by the assignment \( \mathcal{O}_{\hat{X}} = \pi_* \mathcal{O}_Y \). To obtain a Frobenius lifting on \( \hat{X} \) we just take \( F_{\hat{X}} = \pi_* F_Y \). For (ii), we observe that, since \( X \) and \( Y \) are smooth, [CR15, Theorem 1.1] implies that \( R^i \pi_* \mathcal{O}_Y = 0 \) for \( i > 0 \), and hence we may use (i) to conclude. For (iii), we reason similarly as in (a.iii). More precisely, we consider the long exact sequence of cohomology for (3.3.2) to see that \( R^1 \pi_* \mathcal{O}_Y \) is isomorphic to the second local cohomology supported in \( Z = X \setminus U \), which vanishes by condition \( S_2 \) and [HK04, Proposition 3.3].

(c) Since \( Y \) is normal and \( F \)-liftable, it is \( F \)-split. Lemma 2.5.3 provides compatible Frobenius splittings \( \sigma_Y \) and \( \sigma_X \) and a lifting \( \hat{\pi} : \hat{Y} \to \hat{X} \) of \( \sigma_Y \) for \( \pi \). By Theorem 3.6.5, \( \hat{Y}(\sigma_Y) \cong \hat{Y} \), and we set \( \hat{X} = \hat{X}(\sigma_X) \), obtaining the required lifting of \( \pi \). By the definition of a good quotient, \( \mathcal{O}_{\hat{X}} = (\pi_* \mathcal{O}_Y)^G \) and \( \pi \) is affine, in particular \( R^i \pi_* \mathcal{O}_Y = 0 \) for \( i > 0 \). Since \( G \) is linearly reductive, \( \mathcal{O}_{\hat{X}} = (\pi_* \mathcal{O}_Y)^G \to \pi_* \mathcal{O}_Y \) splits, and hence assumption (a.i) is satisfied. We apply part (a) to conclude. \( \square \)

**Corollary 3.3.7.** Let \( X \) and \( Y \) be smooth and proper schemes over \( k \). Then \( X \times Y \) is \( F \)-liftable if and only if \( X \) and \( Y \) are. Moreover, every Frobenius lifting of \( X \times Y \) arises as a product of Frobenius liftings of the factors.

**Proof.** If \((\hat{X}, \hat{F}_X)\) and \((\hat{Y}, \hat{F}_Y)\) are Frobenius liftings of \( X \) and \( Y \), respectively, then \((\hat{X} \times \hat{Y}, \hat{F}_X \times \hat{F}_Y)\) is a Frobenius lifting of \( X \times Y \). For the converse, we first use the arguments of [BTLM97, Lemma 1]. The sheaf \( \Omega^1_{X \times Y} \) decomposes as a direct sum \( \pi_X^* \Omega^1_X \oplus \pi_Y^* \Omega^1_Y \) and therefore the morphism of de Rham complexes \( \mathcal{O}^*_{\hat{X}} \to \pi_X^* \mathcal{O}^*_{\hat{X} \times Y} \) induced by the differential \( d\pi_X : \Omega^1_{X \times Y} \to \Omega^1_{X \times Y} \) has a natural splitting. This leads to a splitting \( s \) of the morphism of short exact sequences
\[
0 \to B^1_{X \times Y} \to Z^1_{X \times Y} \to \Omega^1_{X \times Y} \to 0
\]
Consequently, we see that the upper row is split if the lower row is. This completes the proof of the first assertion.

For the second part, we take a Frobenius lifting \((\hat{X} \times \hat{Y}, \hat{F}_{X \times Y})\). We already know that \( X \) and \( Y \) admit Frobenius liftings \((\hat{X}, \hat{F}_X)\) and \((\hat{Y}, \hat{F}_Y)\), and their product \((\hat{X} \times \hat{Y}, \hat{F}_X \times \hat{F}_Y)\) is another Frobenius lifting of \( X \times Y \). But by Corollary 3.6.6, we must have
\[
\hat{X} \times \hat{Y} \cong \hat{X} \times \hat{Y}.
\]
Using Proposition 3.3.1 we see that the space of Frobenius liftings on \( \hat{X} \times \hat{Y} \) is a torsor under \( H^0(X \times Y, F^*T_{X \times Y}) \). The last group equals \( H^0(X, F^*T_X) \oplus H^0(Y, F^*T_Y) \) which can be identified with the space of Frobenius liftings of the factors \( X \) and \( Y \). This finishes the proof.

**Toric fibrations.** The goal of this part of the section is to show that split toric fibrations over an \( F \)-liftable base are \( F \)-liftable (see Example 3.1.5). We remove the assumption that the fiber is split in Theorem 5.1.1.
Let $X$ be a normal $k$-scheme, and let $L_1, \ldots, L_n$ be line bundles on $X$. Consider the graded $\mathcal{O}_X$-algebra

\begin{equation}
\mathcal{R} = \bigoplus_{\lambda \in \mathbb{Z}^n} \mathcal{R}_\lambda, \quad \mathcal{R}_\lambda = L_1^{\lambda_1} \otimes \cdots \otimes L_n^{\lambda_n},
\end{equation}

with multiplication given by the tensor product, and set $U = \text{Spec}_X \mathcal{R}$. The natural map $U \to X$ is a torsor under the split torus $T = \mathbb{G}_m^n$. Let $Z = X(\Sigma)_k$ be a toric variety under the torus $T$ and let $Y = U \times^T Z$. The projection $\pi : Y \to X$ is a split toric fibration with fiber $Z$, and conversely every split toric fibration over $X$ arises via this construction.

**Lemma 3.3.8.** In the above situation, let $(\tilde{X}, \tilde{F}_X)$ be a Frobenius lifting of $X$. Then there exists a Frobenius lifting $(\tilde{Y}, \tilde{F}_Y)$ and a lifting $\tilde{\pi} : \tilde{Y} \to \tilde{X}$ of $\pi$ such that $\tilde{F}_X \circ \tilde{\pi} = \tilde{\pi} \circ \tilde{F}_Y$.

**Proof.** Let $\tilde{L}_i$ be the unique liftings of $L_i$ to $\tilde{X}$ satisfying $\tilde{F}_X^\ast \tilde{L}_i = \tilde{L}_i^p$, which exist by Proposition 3.3.1(d) and Lemma 2.5.5(a). We start with the case $Z = \mathbb{G}_m^n$, so $Y = U$. Consider the graded $\mathcal{O}_X$-algebra

\begin{equation}
\mathcal{R} = \bigoplus_{\lambda \in \mathbb{Z}^n} \mathcal{R}_\lambda, \quad \mathcal{R}_\lambda = \tilde{L}_1^{\lambda_1} \otimes \cdots \otimes \tilde{L}_n^{\lambda_n},
\end{equation}

and set $\tilde{U} = \text{Spec}_X \mathcal{R}$. The natural map $\tilde{\pi} : \tilde{U} \to \tilde{X}$ is a $\mathbb{G}_m^n$-torsor lifting $\pi : U \to X$. Moreover, the map

\begin{equation}
\tilde{F}_X^\ast \mathcal{R} = \bigoplus_{\lambda \in \mathbb{Z}^n} \tilde{F}_X^\ast \mathcal{R}_\lambda \simeq \bigoplus_{\lambda \in \mathbb{Z}^n} (\mathcal{R}_\lambda)^{\otimes p} = \bigoplus_{\lambda \in \mathbb{Z}^n} \mathcal{R}_\lambda \hookrightarrow \bigoplus_{\lambda \in \mathbb{Z}^n} \mathcal{R}_\lambda = \mathcal{R}
\end{equation}

induces a map $\tilde{F}_{U/X} : \tilde{U} \to \tilde{U}'$, which lifts the relative Frobenius $F_{U/X} : U \to U'$, where $\tilde{U}'$ is the base change of $\tilde{U}$ along $\tilde{F}_X$ and $U'$ is its reduction modulo $p$.

For the general case, we set

$\tilde{Y} = \tilde{U} \times^T X(\Sigma)_{\mathbb{Z}_p} \xrightarrow{\tilde{\pi}} \tilde{X}, \quad \text{where } \tilde{T} = \mathbb{G}_m^n \times \mathbb{Z}_p$,

the toric fibration with fiber $X(\Sigma)$ associated to $\tilde{U}$. This is a lifting of $Y$, and the lifting of Frobenius on $\tilde{U}$ extends to $\tilde{Y}$.

**Divisorial rings.** Our final application of the functoriality of the obstruction classes concerns $F$-liftability of divisorial rings with a view toward the study of Cox rings. A standard reference for the following construction is the book [ADHL15]. Consider a diagram of the form

$Y = \text{Spec } R \xrightarrow{j} U \xrightarrow{\pi} T = \mathbb{G}_m^n$

where $X$ is a normal projective $k$-scheme, $\pi : U \to X$ is a torsor under a torus $T = \mathbb{G}_m^n$, and $j : U \to Y$ is an open immersion into a normal affine scheme $Y = \text{Spec } R$ of finite type over $k$ such that the complement $Y \setminus U$ has codimension $> 1$. In the previous paragraph we related the $F$-liftability of $X$ and of $U$, and now we wish to relate the $F$-liftability of $X$ and of $Y$.

The action of $T$ yields a grading $\mathcal{R} = \bigoplus_{\lambda \in \mathbb{Z}^n} \mathcal{R}_\lambda$ on the quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{R} = \pi^\ast \mathcal{O}_U$ whose graded pieces $\mathcal{R}_\lambda$ are line bundles on $X$. By the assumptions on $j$, we have

$R = H^0(U, \mathcal{O}_U) = H^0(X, \mathcal{R}) = \bigoplus_{\lambda \in \mathbb{Z}^n} H^0(X, \mathcal{R}_\lambda)$.

Conversely, any choice of line bundles $L_1, \ldots, L_n$ on a normal projective variety $X$ yields a diagram as above by setting

$\mathcal{R}_\lambda = L_1^{\lambda_1} \otimes \cdots \otimes L_n^{\lambda_n}$,

$\mathcal{R} = \bigoplus_{\lambda \in M} \mathcal{R}_\lambda$ with multiplication given by the tensor product, $U = \text{Spec}_X \mathcal{R}$, and $Y = \text{Spec } \Gamma(X, \mathcal{R})$, provided that the following assumptions are satisfied:
Proposition 3.3.9. In the above situation, let $\hat{X}$ be a lifting of $X$ to $\hat{S} = \text{Spec } W_2(k)$, and let $\hat{L}_1, \ldots, \hat{L}_n$ be liftings of $L_1, \ldots, L_n$ to $\hat{X}$. Set
$$\mathcal{R} = \bigoplus_{\lambda \in M} \hat{L}_1^\lambda \otimes \cdots \otimes \hat{L}_n^\lambda$$
and $\hat{R} = \Gamma(\hat{X}, \mathcal{R})$. Suppose that $\hat{R}$ is a flat $W_2(k)$-algebra. Then $\hat{Y} = \text{Spec } \hat{R}$ is a lifting of $Y$. If $\bar{F}_X$ is a lift of $F_X$ to $\hat{X}$ such that $\bar{F}_X \hat{L}_i \simeq \hat{L}_i^p$ (see Lemma 3.3.8), then $F_Y$ lifts to $\hat{Y}$. The liftings of Frobenius on $\hat{X}, \hat{U}$, and $\hat{Y}$ commute with the natural maps.

Proof. By the flatness assumption, $\hat{Y}$ is a lifting of $Y$. Then the lifting of Frobenius on $\hat{U}$ constructed in Lemma 3.3.8 induces a lifting of Frobenius on $\text{Spec } \Gamma(\hat{U}, \mathcal{O}_U) = Y$. \hfill $\square$

The flatness assumption on $\hat{R}$ is satisfied if, for instance, $H^1(X, \mathcal{R}) = 0$, that is,
$$H^1(X, L_1^\lambda \otimes \cdots \otimes L_n^\lambda) = 0 \quad \text{for every } \lambda_1, \ldots, \lambda_n \in \mathbb{Z}.$$ Indeed, the cohomology exact sequence associated to the short exact sequence
$$0 \rightarrow \mathcal{R} \rightarrow \mathcal{R} \rightarrow \mathcal{R} \rightarrow 0$$
shows that $0 \rightarrow R \rightarrow \hat{R} \rightarrow R \rightarrow 0$ is exact as well.

3.4. Frobenius liftings compatible with a divisor or a closed subscheme. From the point of view of applications, it is necessary to consider Frobenius liftings of nc pairs $(X, D)$ (see §2.3). A lifting of $(X, D)$ is a pair $(\hat{X}, \hat{D})$ over $W_2(k)$ where $\hat{X}$ is a lifting of $X$ and $\hat{D} \subseteq \hat{X}$ is an embedded deformation of $D$. Note that the requirement that $\hat{D}$ has normal crossings relative to $W_2(k)$ is not vacuous. Recall that a Frobenius lifting $(\hat{X}, \hat{D}, \bar{F}_X)$ of $(X, D)$ consists of a lifting $(\hat{X}, \hat{D})$ of $(X, D)$ and a lifting $\bar{F}_X$ of $F_X$ to $\hat{X}$ satisfying $\bar{F}_X^p \hat{D} = p \hat{D}$. In this case, we shall say that $\bar{F}_X$ is compatible with $\hat{D}$.

Lemma 3.4.1. Let $(X, D)$ be an nc pair over $k$, and let $(\hat{X}, \hat{D}, \bar{F}_X)$ be a Frobenius lifting of $(X, D)$. Let $D_1, \ldots, D_r$ be the irreducible components of $D$. Then for every $i = 1, \ldots, r$, the Frobenius lifting $\bar{F}_X$ induces a Frobenius lifting $F_{D_i}$ on $\hat{D}_i$ which is compatible with the divisor $(\bigcup_{j \neq i} D_j) \cap D_i \subseteq D_i$.

Proof. The question is local so we can assume that $X = \text{Spec } R$ and $\hat{X} = \text{Spec } \hat{R}$. Moreover, $D_j$ is the zero locus of $f_j \in R$, and $\hat{D}_j$ is the zero locus of $\hat{f}_j \in \hat{R}$ where $1 \leq j \leq r$.

Since $\bar{F}_X(\hat{f}_j) = \hat{u}_j \hat{f}_j^p$ for every $1 \leq j \leq r$ and some $\hat{u}_j \in \hat{R}^*$, we get an induced morphism
$$\bar{F}_{D_i} : \hat{D}_i \rightarrow \hat{D}_i,$$
where $\hat{D}_i = \text{Spec } \hat{R}/\hat{f}_i$, such that $\bar{F}_{D_i}(\hat{f}_j) = \hat{u}_j \hat{f}_j^p$ for $j \neq i$. \hfill $\square$

Let $o_{X, D} \in \text{Ext}^1(\Omega_X^1(\log D), B_X^1)$ be the obstruction to the existence of a Frobenius lifting $(\hat{X}, \hat{D}, \bar{F}_X)$ of a simple normal crossing pair $(X, D)$ (see Variant 3.3.2). Consider the short exact sequence
$$0 \rightarrow \Omega_X^1 \rightarrow \Omega_X^1(\log D) \rightarrow \bigoplus_i \mathcal{O}_{D_i} \rightarrow 0,$$
where $D = \sum D_i$ is a decomposition into irreducible components. Analyzing the construction in Variant 3.3.2 we see that the induced natural morphism
$$\text{Ext}^1(\Omega_X^1(\log D), B_X^1) \rightarrow \text{Ext}^1(\Omega_X^1, B_X^1)$$
maps $o_{X, D}$ into the obstruction $o_X$ to the existence of a Frobenius lifting of $X$. \hfill (3.4.1)
Moreover, if \( o_{X,D} = 0 \), then, after fixing a Frobenius lifting of \((X, D)\), we can identify \( \text{Hom} (\Omega^1_X, B^1_X) \) with the space of Frobenius liftings \((\bar{X}, \bar{F}_X)\) (see Proposition 3.3.1). With that, the natural morphism
\[
(3.4.2) \quad \text{Hom} (\Omega^1_X(\log D), B^1_X) \rightarrow \text{Hom} (\Omega^1_X, B^1_X)
\]
maps Frobenius liftings \((\bar{X}, \bar{D}, \bar{F}_X)\) to Frobenius liftings \((\bar{X}, \bar{F}_X)\).

As a corollary of the above we obtain the following lemma which describes the behaviour of divisors with a negative normal bundle with respect to Frobenius liftable.

**Lemma 3.4.2.** Let \((X, D)\) be a simple normal crossing pair over \( k \) such that
\[
\text{Hom}^0(D_i, \mathcal{O}_{D_i}(mD_i)) = 0 \quad \text{for } 1 \leq m \leq p
\]
and all irreducible components \( D_i \) of \( D \). Let \((\bar{X}, \bar{F}_X)\) be a Frobenius lifting of \( X \). Then there exists a lifting \( \bar{D} \) of \( D \) with which \( \bar{F}_X \) is compatible.

In particular, if \( Y = \text{Bl}_Z X \) is a blow-up of a smooth \( k \)-scheme \( X \) in a smooth center \( Z \subset X \), then the unique embedded deformation \( \bar{E} \subset \bar{Y} \) of the exceptional divisor \( E \) is preserved by every Frobenius lifting of \( \bar{Y} \).

**Proof.** Let \( o_X \) and \( o_{X,D} \) be the obstruction classes as above. Since \( o_X = 0 \) and \( o_{X,D} \) is mapped to it by \((3.4.1)\), we get that \( o_{X,D} \) is the image of an element of
\[
\text{Ext}^1 \left( \bigoplus_{i=1}^r \mathcal{O}_{D_i}, B^1_X \right).
\]
Moreover, if this cohomology group vanishes, then \((3.4.2)\) is surjective. Hence, in order to prove the lemma, it is enough to show that this cohomology group is zero.

To this end, we apply the local to global spectral sequence to see that
\[
\text{Ext}^1 \left( \bigoplus_{i=1}^r \mathcal{O}_{D_i}, B^1_X \right) \cong H^0 \left( X, \mathcal{E}xt^1 \left( \bigoplus_{i=1}^r \mathcal{O}_{D_i}, B^1_X \right) \right).
\]
Using the short exact sequence
\[
0 \rightarrow \mathcal{O}_X(-D_i) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{D_i} \rightarrow 0
\]
we compute \( \mathcal{E}xt^1 \left( \bigoplus_{i=1}^r \mathcal{O}_{D_i}, B^1_X \right) \) as the cokernel of the mapping:
\[
\bigoplus_{i=1}^r \text{Hom} \left( \mathcal{O}_X, B^1_X \right) \rightarrow \bigoplus_{i=1}^r \text{Hom} \left( \mathcal{O}_X(-D_i), B^1_X \right),
\]
which is equal to \( \bigoplus_{i=1}^r B^1_X(D_i)|_{D_i} \), because \( B^1_X \) is locally free. Since \( X \) is \( F \)-split (see Proposition 3.2.1), we have
\[
B^1_X(D_i)|_{D_i} \subseteq (\mathcal{O}_X(D_i) \otimes F^* \mathcal{O}_X)|_{D_i} = F^* (\mathcal{O}_{pD_i}(pD_i)).
\]
Here we used that \( (F^* \mathcal{O}_X)|_{D_i} \) is the cokernel of
\[
F^* (\mathcal{O}_X(-pD_i)) = \mathcal{O}_X(-D_i) \otimes F^* \mathcal{O}_X \rightarrow F^* \mathcal{O}_X,
\]
and so it is equal to \( F^* \mathcal{O}_{pD_i} \).

Hence, it is enough to show that \( H^0(pD_i, \mathcal{O}_{pD_i}(pD_i)) = 0 \). This follows by inductively looking at the global sections in the short exact sequences
\[
0 \rightarrow \mathcal{O}_{D_i}((p - m + 1)D_i) \rightarrow \mathcal{O}_{mD_i}(pD_i) \rightarrow \mathcal{O}_{(m-1)D_i}(pD_i) \rightarrow 0
\]
where \( 2 \leq m \leq p \). \( \square \)
Definition 3.4.3. Let $X$ be a $k$-scheme, let $Z \subseteq X$ be a closed subscheme, let $(\tilde{X}, \tilde{F}_X)$ be a Frobenius lifting of $X$, and let $\tilde{Z} \subseteq \tilde{X}$ be an embedded deformation of $Z$. We say that $\tilde{F}_X$ is compatible with $\tilde{Z}$ if

$$\tilde{F}_X^*(\mathcal{O}_{\tilde{Z}}) \subseteq \mathcal{O}_{\tilde{Z}}^p,$$

or, in other words, if the image of $\tilde{F}_X^*(\mathcal{O}_{\tilde{Z}}) \to \mathcal{O}_{\tilde{Z}}$ is contained in $\mathcal{O}_{\tilde{Z}}^p$.

In particular, if $X$ is smooth and $D \subseteq X$ is a divisor with normal crossings, then $\tilde{F}_X$ is compatible with a lifting $\tilde{D}$ with relative normal crossings if and only if it is compatible with $\tilde{D}$ as a closed subscheme in the sense of the above definition. In fact, if $\tilde{D} \subseteq \tilde{X}$ is an embedded deformation of $D$ with which $\tilde{F}_X$ is compatible, then $\tilde{D}$ automatically has relative normal crossings, see Corollary 3.6.6.

$F$-Liftability and Blowing up. Let $X$ be a smooth scheme over $k$, let $Z \subseteq X$ be a smooth closed subscheme of codimension $> 1$, let $\pi: Y \to X$ be the blowing-up of $X$ along $Z$, and let $E = \text{Exc}(\pi)$ be the exceptional divisor. Further, let $\tilde{Y}$ be a lifting of $Y$ over $\tilde{S}$. By the results of §2.2, there exist unique liftings of $\tilde{E}$, $\tilde{X}$, and $\tilde{Z}$ fitting inside a commutative square

$$\begin{array}{ccc}
\tilde{E} & \to & \tilde{Y} \\
\downarrow & & \downarrow \pi \\
\tilde{Z} & \to & \tilde{X}.
\end{array}$$

Moreover, by Theorem 3.3.6(b), the map $\pi$ induces an injection

$$\{\text{liftings of } F_Y \text{ to } \tilde{Y} \} \to \{\text{liftings of } F_X \text{ to } \tilde{X} \}.$$ 

Lemma 3.4.4. A lifting $\tilde{F}_X$ of $F_X$ to $\tilde{X}$ is in the image of the above map (that is, $\tilde{F}_X$ extends to $\tilde{Y}$) if and only if it is compatible with $\tilde{Z}$ in the sense of Definition 3.4.3.

Proof. Let $\tilde{F}_X$ be a lifting of $F_X$ to $\tilde{X}$. Suppose that $\tilde{F}_X$ is compatible with $\tilde{Z}$. If $\mathcal{I}$ (resp. $\tilde{\mathcal{I}}$) is the ideal sheaf of $Z$ (resp. $\tilde{Z}$), then $Y = \text{Proj} \bigoplus_{n \geq 0} \mathcal{I}^n = \text{Proj} \bigoplus_{n \geq 0} \tilde{\mathcal{I}}^{np}$ and $\tilde{Y} = \text{Proj} \bigoplus_{n \geq 0} \tilde{\mathcal{I}}^n = \text{Proj} \bigoplus_{n \geq 0} \tilde{\mathcal{I}}^{np}$. The relative Frobenius $F_{Y/X}$ corresponds to the map of graded $\mathcal{O}_X$-algebras

$$(3.4.3) \quad F_{Y/X}^*: F_X^* \left( \bigoplus_{n \geq 0} \mathcal{I}^n \right) \to \bigoplus_{n \geq 0} (\mathcal{I}^n)^{[p]} \to \bigoplus_{n \geq 0} \tilde{\mathcal{I}}^{np}$$

induced by the inclusion. Since by assumption $\tilde{F}_X^* \mathcal{I}$ maps into $\tilde{\mathcal{I}}^p$, we see that $\tilde{F}_X^* \mathcal{I}^n$ maps into $\tilde{\mathcal{I}}^{np}$ for all $n \geq 0$, and we can define a map of graded $\mathcal{O}_X$-algebras

$$(3.4.4) \quad \tilde{F}_{Y/X}^*: \tilde{F}_X^* \left( \bigoplus_{n \geq 0} \mathcal{I}^n \right) = \bigoplus_{n \geq 0} F_X^*(\mathcal{O}_{\tilde{Z}}^n) \to \bigoplus_{n \geq 0} \tilde{\mathcal{I}}^{np}$$

again induced by the inclusion. We claim that the rational map induced by $\tilde{F}_{Y/X}^*$ is defined everywhere, and is the identity on the complement of $\tilde{Z}$. It suffices to show that there are no relevant homogeneous prime ideals whose preimage under (3.4.4) becomes irrelevant. To see this, we first observe that this property holds for the inclusion (3.4.3), since it induces a well-defined morphism $F_{Y/X}$. The inclusion (3.4.4) is only a nilpotent extension of (3.4.3), and hence its action on homogeneous ideals is the same. This finishes the proof of the claim. The composition of $\tilde{F}_{Y/X}$ with the projection $\tilde{Y}' \to \tilde{Y}$ gives the desired extension of $\tilde{F}_X$ to $\tilde{Y}$. We note that in this part of the proof the smoothness assumptions were not needed.

Now suppose that $\tilde{F}_X$ extends to $\tilde{Y}$. The question whether $\tilde{F}_X$ is compatible with $\tilde{Z}$ is local on $\tilde{X}$, so we can assume that $\tilde{X} = \text{Spec } \tilde{A}$ and $\mathcal{I} = (\tilde{x}_1, \ldots, \tilde{x}_c)$ for some $\tilde{x}_i \in A$ such that $(p, \tilde{x}_1, \ldots, \tilde{x}_c)$ is a regular sequence and $c > 1$. Write $\tilde{F}_X^* \tilde{x}_i = \tilde{x}_i^p + pf_i$ for $f_i \in A = \tilde{A} \otimes k$. 
The condition that \( \tilde{F}_X^* \) extends to the open subset \( \tilde{Y}_i = \text{Spec} \tilde{A}[\tilde{Y}/\tilde{x}_i] \subseteq \tilde{Y} \) is equivalent to the condition
\[
\tilde{F}_X^*(\tilde{x}_j/\tilde{x}_i) = \frac{\tilde{x}_j^p + pf_j}{\tilde{x}_i^p + pf_i} = \frac{(\tilde{x}_j^p + pf_j)(\tilde{x}_i^p - pf_i)}{\tilde{x}_i^{2p}} \in \tilde{A}[\tilde{Y}/\tilde{x}_i]
\]
which amounts to saying that
\[
x_i^pf_j - x_j^pf_i \in \mathcal{O}^{2p} \quad \text{for all } j \neq i,
\]
where \( x_i \in A \) are the images of \( \tilde{x}_i \). These equations imply that
\[
f_i \in (\mathcal{O}^{2p} + (x_i^p) : (x_j^p)) = \mathcal{O}^p \quad \text{for } i \neq j.
\]

In the following proposition, we call a morphism of \( k \)-schemes \( f : X \to Y \) separable if it is a composition of a generically smooth morphism and a closed immersion.

**Proposition 3.4.5.** Let \((\tilde{X}, \tilde{F}_X)\) and \((\tilde{Y}, \tilde{F}_Y)\) be Frobenius liftings of smooth and proper \( k \)-schemes \( X \) and \( Y \). Let \( V \subseteq X \times Y \) be an integral subscheme such that one of the projections \( \pi_X \) or \( \pi_Y \) is separable when restricted to \( V \). Suppose that the lifting of Frobenius \( \tilde{F}_X \times \tilde{F}_Y \) on \( \tilde{X} \times \tilde{Y} \) is compatible with a lifting of \( V \). Then \( V = V_X \times V_Y \) for some integral subschemes \( V_X \subseteq X \) and \( V_Y \subseteq Y \).

**Proof.** We set \( V_X \) (resp. \( V_Y \)) to be the image of \( V \) under the projection \( \pi_X : X \times Y \to X \) (resp. \( \pi_Y : X \times Y \to Y \)), and assume without loss of generality that \( \pi_Y \) is separable when restricted to \( V \). We claim that the closed immersion \( V \hookrightarrow V_X \times V_Y \) is an isomorphism. To see this, we apply Corollary 3.5.4(c) to the projection
\[
\pi_Y : (\tilde{X} \times \tilde{Y}, \tilde{F}_X \times \tilde{F}_Y) \to (\tilde{Y}, \tilde{F}_Y)
\]
compatible with the respective Frobenius liftings, and observe that for every \( y \in V_Y \) the Frobenius lifting \( \tilde{F}_X \) is compatible with a lifting of the subscheme \( V_y = V \cap (X \times \{y\}) \), when interpreted as a subscheme of \( X \). By the assumptions the projection \( \pi_Y \) is separable and therefore for a general \( y \) the subscheme \( V_y \) is a union of integral subschemes. Using Corollary 3.6.4 we now notice that there are only finitely many choices for \( V_y \), and therefore they are all isomorphic since \( V_Y \) is connected. This implies that the immersion \( V \hookrightarrow V_X \times V_Y \) is an isomorphism on the fibers of the projection to \( V_Y \) and hence is an isomorphism. \( \square \)

### 3.5. Base change of a lifting of Frobenius.
In this subsection, we show that a morphism from a \( W_2(k) \)-liftable scheme to an \( F \)-liftable scheme lifts to \( W_2(k) \) after composing with the Frobenius. Moreover, if the source is endowed with a lifting of Frobenius, this lifting commutes with the lifting of Frobenius. Although we do not use it much in the sequel, we regard the result as essential for a good understanding of Frobenius liftings.

**Witt vector schemes.** Since our construction uses the Witt vector schemes \( W_2(X) \), let us briefly recall what they are. For an \( \mathbf{F}_p \)-scheme \( X \) and an integer \( n \), we denote by \( W_n(X) \) the scheme of Witt vectors of length \( n \) of \( X \); it has the same underlying topological space as \( X \) and \( W_n(\mathcal{O}_X) \) as its structure sheaf. In particular, if \( X = \text{Spec} \ A \), then \( W_n(X) = \text{Spec} W_n(A) \).

We denote by \( \sigma_n \) the Frobenius map \( W_n(X) \to W_n(X) \) induced by the Witt vector Frobenius on \( \mathcal{O}_X \). The construction \( X \to (W_n(X), \sigma_n) \) is functorial in \( X \).

We do not know of an easy universal property characterizing \( W_n(X) \) except for the case of perfect schemes.

**Lemma 3.5.1** ([GR03, Lemma 6.5.13 i]), [Sch12, Proposition 5.13(i) and Remark 5.14]). Suppose that \( S \) is a perfect \( \mathbf{F}_p \)-scheme (i.e., \( F_S \) is an isomorphism), and let \( n \geq 1 \). Then for every flat \( \mathbf{Z}/p^n\mathbf{Z} \)-scheme \( X \), the map induced by restriction
\[
\text{Hom}(X, W_n(S)) \to \text{Hom}(X \otimes \mathbf{F}_p, S)
\]
is bijective.
Proof. We first claim that \( L_{S/F} = 0 \). Indeed, since \( S \) is perfect, its Frobenius morphism is an isomorphism and therefore its differential \( dF: F^*L_{S/F} \to L_{S/F} \) is a quasi-isomorphism. This quasi-isomorphism is also zero and hence the claim is proven. Now the assertion follows by \([Ill71, \text{Proposition 2.2.4}]\). \( \square \)

**Proposition 3.5.2.** Let \((\tilde{Y}, \tilde{F}_Y)\) be a Frobenius lifting of a \( k \)-scheme \( Y \). Let \( \varphi: Z \to Y \) be a morphism of \( k \)-schemes and let \( \tilde{Z} \) be a lifting of \( Z \) over \( W_2(k) \). Then there exists a morphism \( \psi: \tilde{Z} \to \tilde{Y} \) (canonically defined by (3.5.3) below) such that \( \psi|_Z = F_Y \circ \varphi \):

\[
\begin{array}{ccc}
\tilde{Z} & \xrightarrow{\psi} & \tilde{Y} \\
\downarrow \varphi & & \downarrow \tilde{F}_Y \\
Z & \xrightarrow{\varphi} & Y \\
\end{array}
\]

If \( \tilde{F}_Z \) is a lifting of \( F_Z \) to \( \tilde{Z} \), then \( \psi \circ \tilde{F}_Z = \tilde{F}_Y \circ \psi \). Further, if \( \varphi: \tilde{Z} \to \tilde{Y} \) is any lifting of \( \varphi \), then \( \psi = \tilde{F}_Y \circ \varphi \).

Recall that if \( \tilde{X} \) is any \( W_2(k) \)-scheme and \( X = \tilde{X} \otimes_{W_2(k)} k \), then there is a canonical affine morphism \( \theta_\tilde{X}: \tilde{X} \to W_2(X) \) defined on functions by the formula

\[
\theta_\tilde{X}^*(f_0, f_1) = (f_0)^p + p\cdot f_1
\]

where \( f_0, f_1 \in \mathcal{O}_\tilde{X} \) are any liftings of \( f_0, f_1 \in \mathcal{O}_X \).

Moreover, if \( \tilde{X} \) is flat over \( W_2(k) \), then a lifting \( \tilde{F}_X: \tilde{X} \to \tilde{X} \) of the absolute Frobenius \( F_X: X \to X \) induces an affine morphism \( \nu_{\tilde{X}, \tilde{F}_X}: W_2(X) \to \tilde{X} \) in the opposite direction, defined on functions by

\[
(3.5.1) \quad \nu_{\tilde{X}, \tilde{F}_X}^*(f) = (f, \delta(f)) \quad \text{for} \quad \tilde{f} \in \mathcal{O}_\tilde{X},
\]

where \( f \) the image of \( \tilde{f} \) in \( \mathcal{O}_X \) and \( \delta(f) \) is the unique element such that \( \tilde{F}_X^*(\tilde{f}) = \tilde{f}^p + p \cdot \delta(f) \).

We can recover \( \tilde{F}_X \) from \( \nu_{\tilde{X}, \tilde{F}_X} \) by the formula \( \tilde{F}_X = \nu_{\tilde{X}, \tilde{F}_X} \circ \theta_\tilde{X} \), while the other composition \( \theta_X \circ \nu_{\tilde{X}, \tilde{F}_X} \) coincides with the Witt vector Frobenius \( \sigma_X: W_2(X) \to W_2(X) \), given by \( \sigma_X^*(f_0, f_1) = (f_0^p, f_1^p) \).

\[
(3.5.2) \quad W_2(X) \xleftarrow{\nu_{\tilde{X}, \tilde{F}_X}} \tilde{X} \xrightarrow{\theta_\tilde{X}} W_2(X) \xrightarrow{\nu_{\tilde{X}, \tilde{F}_X}} \tilde{X}.
\]

**Proof of Proposition 3.5.2.** We define the desired lifting \( \psi: \tilde{Z} \to \tilde{Y} \) as the composition

\[
(3.5.3) \quad \tilde{Z} \xrightarrow{\theta_Z} W_2(Z) \xrightarrow{W_2(\varphi)} W_2(Y) \xrightarrow{\nu_{\tilde{Y}, \tilde{F}_Y}} \tilde{Y}.
\]

Explicitly, on functions, \( \psi \) takes the form

\[
\psi^*(\tilde{f}) = \varphi^*(f)^p + p \cdot \varphi^*(\delta_{F_Y}(\tilde{f}))
\]

where \( \varphi^*(f) \in \mathcal{O}_Z \) is a local section mapping to \( \varphi^*(f) \in \mathcal{O}_Z \). It is thus clear that \( \psi|_Z = F_Y \circ \varphi \).
We now check that $\psi \circ \tilde{F}_Z = \tilde{F}_Y \circ \psi$ when $Z$ is flat over $W_2(k)$ and endowed with a Frobenius lifting $\tilde{F}_Z$. This follows from the commutativity of the following diagram:

\[ \begin{array}{ccc}
\tilde{Z} & \xrightarrow{\theta_Z} & W_2(Z) \\
\downarrow{\theta_Z} & & \downarrow{\theta_Z} \\
\tilde{Y} & \xrightarrow{\theta_Y} & W_2(Y)
\end{array} \]

Indeed, by (3.5.2) the composition $\tilde{Z} \rightarrow W_2(Z) \rightarrow \tilde{Z}$ (resp. $\tilde{Y} \rightarrow W_2(Y) \rightarrow \tilde{Y}$) equals $\tilde{F}_Z$ (resp. $\tilde{F}_Y$). To show the commutativity of the diagram, we note that the compositions $W_2(Z) \rightarrow \tilde{Z} \rightarrow W_2(Z)$ and $W_2(Y) \rightarrow \tilde{Y} \rightarrow W_2(Y)$ are the Witt vector Frobenius morphisms $\sigma_Z$ and $\sigma_Y$. These are functorial, which shows that the middle square (and hence the whole diagram) commutes.

The final assertion follows from the commutativity of the following diagram:

\[ \begin{array}{ccc}
\tilde{Z} & \xrightarrow{\theta_Z} & W_2(Z) \\
\downarrow{\varphi} & & \downarrow{W_2(\varphi)} \\
\tilde{Y} & \xrightarrow{\theta_Y} & W_2(Y) \xrightarrow{\nu_Y} \tilde{Y}
\end{array} \]

Here the square commutes by functoriality of the maps $\theta$.

**Corollary 3.5.3.** Let $f : X \rightarrow Y$ and $\varphi : Z \rightarrow Y$ be morphisms of $k$-schemes, and let $(\tilde{X}, \tilde{F}_X)$, $(\tilde{Y}, \tilde{F}_Y)$, and $(\tilde{Z}, \tilde{F}_Z)$ be Frobenius liftings of $X$, $Y$, and $Z$, respectively. Let $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ be a lifting of $f$ commuting with the Frobenius liftings, and let $\psi : \tilde{Z} \rightarrow \tilde{Y}$ be the lifting of $F_Y \circ \varphi$ given by Proposition 3.5.2. Form the cartesian diagram

\[ (\tilde{X}, \tilde{F}_X), (\tilde{Y}, \tilde{F}_Y), (\tilde{Z}, \tilde{F}_Z) \]

Then $\tilde{W}$ admits a Frobenius lifting $\tilde{F}_W$ such that the maps $\tilde{W} \rightarrow \tilde{Z}$ and $\tilde{W} \rightarrow \tilde{X}$ commute with the Frobenius liftings. Moreover, for every subscheme $\tilde{V} \subseteq \tilde{X}$ compatible with $\tilde{F}_X$, its preimage under $\tilde{W} \rightarrow \tilde{X}$ is compatible with $\tilde{F}_W$.

Setting $Z = \text{Spec } k$ (in which case $F_Z$ is an isomorphism), we obtain the following.

**Corollary 3.5.4.** Let $(\tilde{Y}, \tilde{F}_Y)$ be a Frobenius lifting of a $k$-scheme $Y$.

(a) The construction of Proposition 3.5.2 with $Z = \text{Spec } k$ yields a section of the specialization map

\[ \tilde{Y}(W_2(k)) \rightarrow Y(k). \]

In particular, every $k$-point of $Y$ lifts (canonically) to a point of $\tilde{Y}$.

(b) Let $(\tilde{X}, \tilde{F}_X)$ be a Frobenius lifting of a $k$-scheme $X$, and let $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ be a map commuting with the Frobenius liftings. Then for every $y \in Y(k)$, the fiber $X_y = f^{-1}(y)$ is $F$-liftable.
(c) In the situation of (b), for every subscheme \( Z \subseteq X \) such that there exists a lifting \( \tilde{Z} \subseteq \tilde{X} \) compatible with \( \tilde{F}_X \), the Frobenius lifting of \( X_y \) is compatible with the induced lifting of the intersection \( f^{-1}(y) \cap Z \).

**Example 3.5.5.** The scheme \( \tilde{Y} = \{xy = p\} \subseteq \mathbb{A}^2_{W_2(k)} \) does not admit a lifting of Frobenius (not even locally), because the \( k \)-point \((0,0) \in \tilde{Y} = \{xy = 0\} \) does not admit a lifting modulo \( p^2 \). Of course \( Y \) admits a Frobenius lifting 
\[
\tilde{Y}' = \{xy = 0\} \subseteq \mathbb{A}^2_{W_2(k)}, \quad \tilde{F}_Y'(x) = x^p, \quad \tilde{F}_Y'(y) = y^p.
\]
In fact, we will see in Corollary 3.6.6, that since \( Y \) is \( F \)-split, it admits at most one lifting to which \( F_Y \) lifts. This phenomenon seems to be related to some aspects of the log crystalline cohomology of Hyodo and Kato [HK94, §4], where it is more natural to consider the ‘hollow’ log structure on the base \( \text{Spec} W(k) \) rather than the standard one.

### 3.6. Frobenius liftings and Frobenius splittings.

To every Frobenius lifting \((\tilde{X}, \tilde{F}_X)\) of a smooth (or just normal) \( k \)-scheme \( X \) we can associate a corresponding \( F \)-splitting \( \sigma_{\tilde{F}_X} \) on \( X \) and to every smooth morphism \( \tilde{f}: \tilde{Y} \to \tilde{X} \) commuting with the liftings of Frobenius on \( Y \) and \( X \) we can associate a relative \( F \)-splitting \( \sigma_{F_{Y/X}} \) of \( Y/X \) (see Proposition 3.2.1). Here \( F_{Y/X} \) denotes the induced lifting of the relative Frobenius.

This provides us with \( \mathcal{Q} \)-divisors \( \Delta_{F_Y}, \Delta_{F_X}, \) and \( \Delta_{F_{Y/X}} \) as in Proposition 2.5.9. Let \( f: Y \to X \) be the reduction of \( \tilde{f} \) modulo \( p \).

**Lemma 3.6.1.** Let \( X \) and \( Y \) be smooth \( k \)-schemes. Then \( \Delta_{F_Y} = \Delta_{F_{Y/X}} + f^*\Delta_{F_X} \), and if \( f \) has connected fibers, then \( \Delta_{F_{Y/X}} \) (resp. \( f^*\Delta_{F_X} \)) is horizontal (resp. vertical).

**Proof.** Let \( Y' \) be the base change of \( Y \) along \( F_X \). By the construction of the map \( \xi \), we get the following commutative diagram
\[
\begin{array}{c}
0 \to f^*F_X^*\Omega^1_X \to F_Y^*\Omega^1_Y \to F_{Y/X}^*\Omega^1_{Y'/X} \to 0 \\
\text{Vertical} \downarrow \quad \text{Horizontal} \downarrow \quad \xi_Y \downarrow \quad \xi_{Y/X} \\
0 \to f^*\Omega^1_X \to \Omega^1_Y \to \Omega^1_{Y/X} \to 0,
\end{array}
\]
where \( F_{Y/X}^*\Omega^1_{Y'/X} \cong F_Y^*\Omega^1_{Y/X} \) and \( f^*F_X^*\Omega^1_X \cong F_Y^*f^*\Omega^1_X \). The first part of the lemma follows since the considered \( \mathcal{Q} \)-divisors multiplied by \( p - 1 \) are equal to \( \text{div} (\det \xi_Y) \), \( \text{div} (\det \xi_{Y/X}) \), and \( \text{div} (\det \xi_X) \), respectively. If \( f \) has connected fibers, then \( \Delta_{F_{Y/X}} \) is horizontal by Proposition 2.5.9. \( \square \)

The following corollary lists all the properties of \( \Delta_{F_X} \) we need in this article. Given a flat morphism \( f: Y \to X \) of normal varieties such that \( f_*\mathcal{O}_Y = \mathcal{O}_X \) and a \( \mathcal{Q} \)-divisor \( D \) on \( Y \), we denote by \( D^h \) and \( D^v \) the horizontal and the vertical part, respectively.

**Corollary 3.6.2.** Let \((\tilde{Y}, \tilde{F}_Y)\) be a Frobenius lifting of a smooth \( k \)-scheme \( Y \).

(a) If \( D \subseteq Y \) is a smooth irreducible divisor such that \( H^0(D, \mathcal{O}_D (mD)) = 0 \) for \( 1 \leq m \leq p \), then \( D \leq \Delta_{F_Y} \).

(b) In the situation of Theorem 3.3.6(b.ii), we have \( \Delta_{F_X} = \pi_*\Delta_{F_Y} \).

(c) In the situation of Theorem 3.3.6(b.ii), assume that \( \pi: Y \to X \) is smooth and let \( \tilde{F}_Y/X \) be the induced lifting of the relative Frobenius. Then \( \Delta_{F_Y}^h \) is the \( \mathcal{Q} \)-divisor associated to the relative \( F \)-splitting \( \sigma_{\tilde{F}_Y/X} \). In particular,
\[
\Delta_{F_Y}^h \sim_{\mathcal{Q}} -K_{Y/X}, \quad \text{and} \quad \Delta_{F_Y}^v = \pi^*\Delta_{F_X} \sim_{\mathcal{Q}} -\pi^*K_X.
\]
Proof. Statement (a) follows from Lemma 3.4.2. Indeed, we have the morphism
\[ \xi_{(X,D)}: F_X^{*} \Omega_{X/k}^1(\log D') \longrightarrow \Omega_X^1(\log D) \]
(see Variant 3.2.2) such that \( \text{div}(\det \xi_X) = \text{div}(\det \xi_{(X,D)}) + (p - 1)D \).
Statement (b) is clear by the construction of \( \Delta_{F_Y} \) since \( g(\text{Exc} \, g) \) has codimension at least two. Statement (c) follows from Lemma 3.6.1. \qed

We now relate certain conditions on the compatibility of subschemes for Frobenius liftings and Frobenius splittings.

**Lemma 3.6.3.** Let \((\bar{X}, \bar{F}_X)\) be a Frobenius lifting of a quasi-projective smooth \(k\)-scheme \(X\). Suppose that \(\bar{F}_X\) is compatible with a lifting of an integral subscheme \(Z \subset X\). Then the associated Frobenius splitting is compatible with \(Z\).

Proof. Let \(\sigma_X: F_X, \mathcal{O}_X \rightarrow \mathcal{O}_X\) be the Frobenius splitting associated with \(\bar{F}_X\). First, we consider the case when \(Z\) is smooth. By Lemma 3.4.4 we see that \(Y = \text{Bl}_Z X\) admits a Frobenius lifting \((\bar{Y}, \bar{F}_Y)\) compatible with the unique lifting of the exceptional divisor \(E\) and equipped with a lifting \(\pi: (\bar{Y}, \bar{F}_Y) \rightarrow (\bar{X}, \bar{F}_X)\) of the contraction morphism \(\pi: Y \rightarrow X\).
By Variant 3.2.2 we see that \(E\) is compatible with the Frobenius splitting \(\sigma_Y: F_Y, \mathcal{O}_Y \rightarrow \mathcal{O}_Y\) of \(Y\) induced by \(\bar{F}_Y\). By [BK05, Lemma 1.1.8(ii)] we therefore see that \(Z = \pi(E)\) is compatible with the push-forward of \(\sigma_Y\) under \(\pi\), which is equal to \(\sigma_X\). This finishes the proof for \(Z\) smooth. For an arbitrary integral \(Z\), we just observe that the condition of being compatibly split can be checked at the generic point. \qed

**Corollary 3.6.4.** Let \((\bar{X}, \bar{F}_X)\) be a Frobenius lifting of a quasi-projective smooth \(k\)-scheme \(X\). Then there are only finitely integral subschemes \(Z\) such that \(\bar{F}_X\) is compatible with a lifting of \(Z\).

Proof. By Lemma 3.6.3 we observe that every subscheme compatible with \(\bar{F}_X\) is compatible with the associated \(F\)-splitting. Then we conclude by [Sch09, Theorem 5.8], which states that there are only finitely many subschemes compatible with a given Frobenius splitting. \qed

**Uniqueness of liftings admitting a lifting of Frobenius.** So far we have discussed Frobenius splittings arising from Frobenius liftings, but recall from §2.5 that a Frobenius splitting gives rise to a canonical lifting to \(W_2(k)\). It is natural to ask whether the lifting induced by the Frobenius splitting associated to a Frobenius lifting \((\bar{X}, \bar{F}_X)\) of a normal \(k\)-scheme \(X\) is canonically isomorphic to \(\bar{X}\). In fact much more is true, as indicated by the following simple but surprising result.

**Theorem 3.6.5.** Let \(X\) be a scheme over \(k\), let \(\sigma\) be a Frobenius splitting on \(X\), and let \((\bar{X}, \bar{F}_X)\) be a Frobenius lifting of \(X\). Let \(\bar{X}(\sigma)\) be the canonical lifting of \(X\) induced by \(\sigma\) (see Theorem 2.5.2). Then one has a canonical isomorphism \(\bar{X}(\sigma) \cong \bar{X}\) (depending on \(\bar{F}_X\) and \(\sigma\) of liftings of \(X\)).

Proof. The canonical lifting \(\bar{X}(\sigma)\) admits the following description (see [Zda17, §3.5]). Recall that the ring \(W_2(\mathcal{O}_X)\) is a square zero extension of \(\mathcal{O}_X\) by \(F_2, \mathcal{O}_X\). Then \(I(\sigma) = \ker(\sigma: F_2, \mathcal{O}_X \rightarrow \mathcal{O}_X) \subseteq W_2(\mathcal{O}_X)\) is an ideal, and \(\mathcal{O}_{\bar{X}(\sigma)} = W_2(\mathcal{O}_X)/I(\sigma)\), sitting inside the pushout diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & F_2, \mathcal{O}_X & \longrightarrow & W_2(\mathcal{O}_X) & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
\sigma & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{\bar{X}(\sigma)} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0.
\end{array}
\]
The upshot is that we have a natural closed immersion \( i_\sigma : \tilde{X}(\sigma) \to W_2(X) \) making the following triangle commute

\[
\begin{array}{ccc}
X & \xrightarrow{i_\sigma} & W_2(X) \\
\downarrow & & \downarrow \\
\tilde{X}(\sigma) & \xrightarrow{i_\sigma} & W_2(X) \\
\end{array}
\]

On the other hand, the Frobenius lifting \((\tilde{X}, \tilde{F}_X)\) induces the map (3.5.1)

\[
\nu_{\tilde{X}, \tilde{F}_X} : W_2(X) \longrightarrow \tilde{X}, \quad \nu^*(\tilde{f}) = (\tilde{f} \bmod p, \delta(\tilde{f}))
\]

where \(\tilde{F}_X(\tilde{f}) = \tilde{f}^p + p \cdot \delta(\tilde{f})\). The composition

\[
\nu_{\tilde{X}, \tilde{F}_X} \circ i_\sigma : \tilde{X}(\sigma) \longrightarrow \tilde{X}
\]

restricts to the identity on \(X\), and hence is an isomorphism of liftings of \(\tilde{X}\). This is the required map. \(\square\)

The proof shows that \(\nu_{\tilde{X}, \tilde{F}_X} \circ i_\sigma : \tilde{X}(\sigma) \to \tilde{X}\) fits into the commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & \mathcal{O}_X \\
\downarrow & \downarrow & \downarrow \\
0 & \longrightarrow & \mathcal{O}_{\tilde{X}} \\
\downarrow & \downarrow & \downarrow \\
0 & \longrightarrow & \mathcal{O}_X \\
\end{array}
\]

where the left composition \(\sigma \circ F_X^*\) is the identity.

**Corollary 3.6.6.** Let \(X\) be a normal \(k\)-scheme, and let \((\tilde{X}^{(i)}, \tilde{F}_X^{(i)})\) for \(i = 1, 2\) be two Frobenius liftings of \(X\). Then \(\tilde{X}^{(1)} \simeq \tilde{X}^{(2)}\).

**Remark 3.6.7.** If \(X\) is smooth, then Corollary 3.6.6 can be seen more directly as follows. The application of \(\text{Hom}(\Omega^1_X, -)\) to the short exact sequence

\[
\begin{array}{ccc}
0 & \longrightarrow & \mathcal{O}_X \\
\downarrow & \downarrow & \downarrow \\
F_X & \to & \mathcal{O}_{\tilde{X}} \\
\downarrow & \downarrow & \downarrow \\
0 & \longrightarrow & \mathcal{O}_X \\
\end{array}
\]

yields a connecting homomorphism \(\delta : \text{Hom}(\Omega^1_X, B^1_X) \to \text{Ext}^1(\Omega^1_X, \mathcal{O}_X)\). The forgetful map \((\tilde{X}, \tilde{F}_X) \to \tilde{X} : \{\text{Frobenius liftings of } X\} / \text{isom.} \longrightarrow \{\text{liftings of } X\} / \text{isom.}\) is a map from a torsor under \(\text{Hom}(\Omega^1_X, B^1_X)\) to a torsor under \(\text{Ext}^1(\Omega^1_X, \mathcal{O}_X)\) which is equivariant with respect to the map \(\delta\). If \(X\) is \(F\)-liftable, it is \(F\)-split, and hence \(\delta = 0\). Thus (3.6.1) is constant.

4. **Toric varieties in families**

In this section, we address the following three questions:

- **Generalization:** Given a family \(f : X \to S\) such that \(X_s\) is a toric variety for a dense set of \(s \in S\), can we deduce that the generic fiber is a toric variety? (§4.1)
- **Specialization:** Given a family \(f : X \to S\) whose generic fiber is a toric variety and \(S = \text{Spec } R\) for a discrete valuation ring \(R\), when can we deduce that the special fiber is a toric variety as well? (§4.2)
- **Global rigidity:** Given a proper nc pair \((X,D)\) over a connected scheme \(S\), if one geometric fiber \((X,D)_s\) is a toric pair, must \((X,D)\) globally come from a toric fibration? (§4.3)

The result of §4.1 will be used in the subsequent §4.4 to show that Conjecture 2 follows from (a special case of) Conjecture 1. The results of §4.2–§4.3 will be needed in Section 5.
4.1. Generalization. We start with a somewhat lengthy proof of the following fact which should be well-known but for which we were unable to find a reference.

**Lemma 4.1.1.** Let $S$ be a geometrically unibranch [EGA IV$_1$, 6.15.1] noetherian scheme, and let $\pi: X \to S$ be a smooth proper morphism whose geometric fibers are connected and satisfy

$$H^1(X_s, \mathcal{O}_{X_s}) = H^2(X_s, \mathcal{O}_{X_s}) = 0.$$  

Then there exists a finite étale surjective morphism $S' \to S$ such that $\text{Pic}_{X'/S'}$ is the constant sheaf associated to a finitely generated group on the big étale site of $S'$, where $X'$ is the base change of $X$ to $S'$.

**Proof.** By [Kle05, Theorem 9.4.8], $\text{Pic}_{X/S}$ is representable by the disjoint union of quasi-projective schemes over $S$. By the deformation theory of line bundles, $H^2(X_s, \mathcal{O}_{X_s})$ is the obstruction space and $H^1(X_s, \mathcal{O}_{X_s})$ is the tangent space of the deformation functor of a line bundle on $X_s$, and hence (4.1.1) shows that line bundles deform uniquely over thickenings of $X_s$. Consequently, $\text{Pic}_{X/S}$ is formally étale, and hence étale, over $S$. Moreover, $\text{Pic}_{X/S} \to S$ satisfies the valuative criterion of properness. Thus if $P$ is a connected component of $\text{Pic}_{X/S}$, then $P \to S$ is a connected étale covering of $S$, and since $S$ is geometrically unibranch, $P$ is finite over $S$. We conclude that $\text{Pic}_{X/S}$ is the disjoint union of connected finite étale coverings of $S$.

We assume without loss of generality that $S$ is connected, and pick a geometric point $\bar{s}$ of $S$. Let $M = \text{Pic}_{X/S}(\bar{s}) = \text{Pic}(X_{\bar{s}})$, which is a finitely generated abelian group. Pick a finite set of generators $p_1, \ldots, p_k \in M$, and for each $i = 1, \ldots, k$, let $P_i$ be the connected component of $\text{Pic}_{X/S}$ containing the corresponding geometric point $\bar{p}_i \to \text{Pic}_{X/S}$. Let $S' \to S$ be a finite étale cover after pullback to which each $P_i \to S$ becomes constant. Replacing $S$ by $S'$, we can assume that each $P_i$ maps isomorphically onto $S$, that is, each $p_i \in \text{Pic}_{X/S}(\bar{s})$ is the restriction of a (unique) global section $\bar{q}_i \in \text{Pic}_{X/S}(S)$. We claim that in this case $\text{Pic}_{X/S}$ is actually constant. Let $P = M \times S$ be the constant group scheme over $S$ associated to $M$; we will construct an isomorphism $\text{Pic}_{X/S} \cong P$. The sections $p_i$ and $q_i$ define surjective morphisms of group schemes over $S$:

$$\alpha: \mathbb{Z}^k \times S \longrightarrow \text{Pic}_{X/S} \quad \text{and} \quad \beta: \mathbb{Z}^k \times S \longrightarrow P.$$  

Moreover if the $p_i$ satisfy a relation $\sum a_ip_i = 0$, then so do the sections $q_i$, and hence there is a surjective morphism $\gamma: \text{Pic}_{X/S} \to P$. Its kernel $K$ is a closed subscheme of $\text{Pic}_{X/S}$ which is flat, and hence étale, over $S$. Moreover, $K_\bar{s} = \{1\}$ by construction, and hence $K \cong S$. Thus $\gamma$ is an isomorphism. \hfill $\Box$

The above assertion need not be true when $H^2(X_\bar{s}, \mathcal{O}_{X_\bar{s}}) \neq 0$, for example for non-isotrivial families of K3 surfaces over complete complex curves [BKPS98].

**Proposition 4.1.2.** Let $S$ be a noetherian excellent scheme and let $\pi: X \to S$ be a smooth projective morphism whose geometric fibers are connected and satisfy (4.1.1). Let $\mathcal{E}$ be a locally free sheaf of rank $r$ on $X$. Then the set

$$\{ s \in S | \mathcal{E}_s \text{ is a direct sum of line bundles on } X_s \text{ for every geom. pt. } \bar{s} \text{ over } s \} \subseteq S$$

is a constructible subset of $S$.

**Proof.** Stratifying $S$, we may assume that $S$ is connected and regular. By Lemma 4.1.1, after replacing $S$ with some finite étale cover $S' \to S$, we have $\text{Pic}_{X/S} \cong M \times S$ for a finitely generated group $M$, and hence $\text{Pic}(X_s) = \text{Pic}(X_{\bar{s}}) = M$ for all $s \in S$. Given $H, D \in M$, there is a well-defined intersection number $H^{d-1} \cdot D \in \mathbb{Z}$, the same for all fibers, where $d$ is the relative dimension of $f$.

We can assume that there exist $H_1, \ldots, H_s \in M$ which give ample line bundles on every fiber and which span $M \otimes \mathbb{Q}$. If $H$ is one of these, then by Noetherian induction there is a natural number $n_H$ such that $\mathcal{E}(n_HH)$ is globally generated on all fibers. If $L \in M$ is
a direct summand of \( \mathcal{E}_s \), it follows that \( L + n_H H \) is a direct summand of a globally generated sheaf, and hence is effective on \( X_s \), so \((L + n_H H) \cdot H^{d-1} \geq 0 \) (independent of \( s \)). Thus \( L \cdot H^{d-1} \geq -n_H H^d \). Applying the same argument to \( \mathcal{E}^\vee \), we get a natural number \( m_H \) such that \(-L \cdot H^{d-1} \geq -m_H H^d \), i.e. \( L \cdot H^{d-1} \leq m_H H^d \). We conclude that there exists a natural number \( B \) such that

\[
|L \cdot H_i^{d-1}| < B \quad \text{for} \quad i = 1, \ldots, s
\]

whenever \( L \) is a direct summand of some \( \mathcal{E}_s \).

Because \( H_i \) span \( M \otimes \mathbb{Q} \), the set \( K \) of all \( L \in M \) satisfying the inequalities (4.1.2) is finite. For \( v = (L_1, \ldots, L_r) \in K^r \), consider the functor

\[
F_v: (\text{Schemes}/S) \longrightarrow \text{Sets}, \quad T \mapsto \text{Isom}_{X \times S T}((\oplus_{i=1}^r L_i)_T, \mathcal{E}_T).
\]

Stratifying \( S \) further, we may assume that \( \pi_* \mathcal{H}om((\oplus_{i=1}^r L_i), \mathcal{E}) \) is locally free and its formation commutes with base change. This implies that \( F_v \) is representable by an open subscheme of the total space of the vector bundle \( \pi_* \mathcal{H}om((\oplus_{i=1}^r L_i), \mathcal{E}) \), and hence is a scheme of finite type over \( S \). Let \( W \) be the disjoint union of these schemes, which is still of finite type as \( K^r \) is a finite set. The locus where \( \mathcal{E} \) splits is the set-theoretic image of \( W \to S \), which is constructible by Chevalley’s theorem.

We shall now apply this to the study of toric varieties in families. As observed by Jaczewski and developed further by Kędzierski and Wiśniewski, smooth toric varieties admit a generalization of the Euler sequence, and this property in fact characterizes smooth projective toric varieties, at least in characteristic zero.

**Definition 4.1.3** (cf. [Jac94, Definition 2.1] and [KW15, §1.1]). Let \( \pi: X \to S \) be a smooth projective morphism of schemes. Suppose that \( S \) is affine, and that the coherent sheaf \( \mathcal{H} = R^1 \pi_* \mathbb{O}^{-1}_{X/S} \) is locally free. Then the image of \( \text{id}_\mathcal{H} \) under the natural identification

\[
\text{Hom}_S(\mathcal{H}, \mathcal{H}) \cong H^0(S, R^1 \pi_* (\pi^* \mathcal{H}^\vee \otimes \mathbb{O}_{X/S}^{-1}))
\]

\[
\cong H^0(S, \mathcal{H} \mathcal{O}_{X/S}^1(\pi^* \mathcal{H}, \mathbb{O}_{X/S}^{-1}) \cong \text{Ext}^1(\pi^* \mathcal{H}, \mathbb{O}_{X/S}^{-1})
\]

gives rise to an extension

\[
(4.1.3) \quad 0 \longrightarrow \mathbb{O}_{X/S}^1 \longrightarrow R_{X/S} \longrightarrow \mathcal{H} \longrightarrow 0.
\]

The locally free sheaf \( R_{X/S} \) is called the potential sheaf of \( X \) over \( S \).

**Theorem 4.1.4** ([KW15, Theorem 1.1 and Corollary 2.9]). Let \( K \) be an algebraically closed field, and let \( X \) be a smooth projective integral scheme over \( K \). If \( X \) is a toric variety, then the sheaf \( R_{X/K} \) splits into the direct sum of line bundles. Conversely, if \( K \) has characteristic zero, the potential sheaf \( R_{X/K} \) splits into the direct sum of line bundles, and we have the vanishing

\[
H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0,
\]

then \( X \) is a toric variety.

**Corollary 4.1.5.** Let \( S \) be an integral noetherian scheme with generic point \( \eta \) of characteristic zero, and let \( \pi: X \to S \) be a smooth and projective morphism. Suppose that the geometric fiber \( X_\eta \) is a toric variety for a dense set of closed points \( s \in S \). Then the geometric generic fiber \( X_\eta \) is a toric variety.

**Proof.** Shrinking \( S \), we may assume that \( S \) is affine and that \( R^1 \pi_* \mathbb{O}_{X/S}^{-1} \) is locally free, in which case the potential sheaf \( R_{X/S} \) is defined. Moreover, since \( H^0(Y, \mathcal{O}_Y) = k \) and \( H^1(Y, \mathcal{O}_Y) = 0 \) for any \( i > 0 \) and a toric variety \( Y \) over a field \( k \), we see that the assumption on \( H^1(X_\eta, \mathcal{O}_{X_\eta}) \) in Proposition 4.1.2 is satisfied. We apply this result to the potential sheaf \( R_{X/S} \). Since \( R_{Y/k} \) splits for a toric variety \( Y \) by the first part of Theorem 4.1.4, we deduce that \( R_{X_\eta/\eta} = (R_{X/S})_\eta \) must be split as well. By the other direction of Theorem 4.1.4, this implies that \( X_\eta \) is a toric variety.

\( \square \)
4.2. **Specialization.** Let $R$ be a discrete valuation ring with residue field $k$ and fraction field $K$. We set

\[ S = \text{Spec} R, \quad s = \text{Spec} k, \quad \bar{s} = \text{Spec} \bar{k}, \quad \eta = \text{Spec} K, \quad \bar{\eta} = \text{Spec} \bar{K}. \]

Suppose that $X$ is a smooth projective scheme over $S$ whose general fiber $X_\eta$ is a toric variety. It is not true in general that the geometric special fiber $X_{\bar{s}}$ is a toric variety, as the following basic example shows.

**Example 4.2.1.** Let $s_0, s_1, s_2 \in \mathbb{P}^2(R)$ be three sections over $S$ which give a triple of distinct collinear points in the special fiber $\mathbb{P}^2(k)$, but which are not collinear in $\mathbb{P}^2(K)$. Let $X$ be the blow-up of $\mathbb{P}^2_S$ along the union of these three sections. Then $X$ is a smooth projective surface over $S$, and $X_\eta$ is a toric variety but $X_{\bar{s}}$ is not.

The above phenomenon cannot happen if we consider deformations of toric varieties together with their toric boundaries. The goal of this subsection is to show Proposition 4.2.4 which says that toric pairs can only degenerate to toric pairs.

**Lemma 4.2.2 ([Ful98, Lemma, p. 66]).** Let $X$ be a toric variety over a field $k$ and let $D_1, \ldots, D_r$ be all of its toric prime divisors. Let $\mathcal{L}$ be a line bundle on $X$ and set

\[ V_\mathcal{L} = \left\{ (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}_{\geq 0}^r \mid \mathcal{L} \cong \mathcal{O}_X(\lambda_1 D_1 + \ldots + \lambda_r D_r) \right\}. \]

For $\lambda \in V_\mathcal{L}$, let $s_\lambda \in H^0(X, \mathcal{L})$ be the (well-defined up to scaling) section whose divisor of zeros is $\lambda_1 D_1 + \ldots + \lambda_r D_r$. Then

\[ H^0(X, \mathcal{L}) = \bigoplus_{\lambda \in V_\mathcal{L}} k \cdot s_\lambda. \]

**Remark 4.2.3.** Choose $1 \leq i \leq r$ and consider the standard exact sequence:

\[ 0 \to H^0(X, \mathcal{L}(-D_i)) \to H^0(X, \mathcal{L}) \to H^0(D_i, \mathcal{L}|_{D_i}) \]

The above lemma shows that

\[ H^0(X, \mathcal{L}(-D_i)) = \bigoplus_{\lambda \in V_\mathcal{L}, \lambda_i = 0} k \cdot s_\lambda, \]

and hence $\bigoplus_{\lambda \in V_\mathcal{L}, \lambda_i = 0} k \cdot s_\lambda$ injects into $H^0(D_i, \mathcal{L}|_{D_i})$.

**Proposition 4.2.4.** Let $X$ be a smooth projective scheme over $S$, and let $D \subseteq X$ be a divisor with normal crossings relative to $S$. If $(X_\eta, D_\eta)$ (resp. $(X_{\bar{\eta}}, D_{\bar{\eta}})$) is a toric pair (see §2.1), then so is $(X_s, D_s)$ (resp. $(X_{\bar{s}}, D_{\bar{s}})$).

**Proof.** Suppose that $(X_\eta, D_\eta)$ is a toric pair. Let $D = D_1 \cup \ldots \cup D_r$ be the decomposition of $D$ into irreducible components. By induction, we can assume that $(D_i, (D - D_i)|_{D_i})$ is a toric pair for every $1 \leq i \leq r$.

Our goal is to show that the assertion of Lemma 4.2.2 holds over $R$. To this end, for an arbitrary line bundle $\mathcal{L}$ on $X$ we define

\[ R_{\mathcal{L}}(X, D) = \bigoplus_{\lambda \in V_\mathcal{L}} R \cdot \bar{s}_\lambda, \]

where

\[ V_\mathcal{L} = \left\{ (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}_{\geq 0}^r \mid \mathcal{L}_\eta \cong \mathcal{O}_X(\lambda_1(D_1)_\eta + \ldots + \lambda_r(D_r)_\eta) \right\}, \]

and $\bar{s}_\lambda$ is a formal symbol identifying the $\lambda$-th component. There is a natural map

\[ R_{\mathcal{L}}(X, D) \to H^0(X, \mathcal{L}), \]

sending $\bar{s}_\lambda$ to the unique (up to scaling) section corresponding to the divisor $\lambda_1 D_1 + \ldots + \lambda_r D_r$.

Note that $\text{Pic} X = \text{Pic} X_\eta$. We claim that

\[ R_{\mathcal{L}}(X, D) \otimes K \cong H^0(X_\eta, \mathcal{L}_\eta), \]

and

\[ R_{\mathcal{L}}(X, D) \otimes k \to H^0(X_s, \mathcal{L}_s) \]

is injective,
In particular,\(\Omega\) irreducible. Consider the scheme
\[
\mathcal{L}.
\]
Proof. Implications (i)\(\Rightarrow\)(ii)\(\Rightarrow\)(iii) are obvious. For (iii)\(\Rightarrow\)(ii), it is enough to consider \(S\) irreducible. Consider the scheme\(I = \text{Isom}_S((X,D), (X(\Sigma), D(\Sigma))_S)\), which is locally of finite type over \(S\). If \(i\) is a geometric point of \(I\) and \(\tilde{s}\) is its image in \(S\), then \(\Omega^1_{X(\Sigma)}(\log D) \simeq \Omega^1_{X(\Sigma)}(\log D(\Sigma))_S\) is trivial and \(H^1(X(\Sigma), \mathcal{O}_{X}) \simeq H^1(X(\Sigma), \mathcal{O}_{X(\Sigma)\tilde{s}}) = 0\).
for $i > 0$. Deformation theory shows that $I \to S$ is formally smooth at $\bar{i}$. We conclude that $I \to S$ is smooth, and in particular its image is an open subset of $S$.

We shall now prove that the assertion of (iii) holds for every geometric point of $S$, or in other words, that $I \to S$ is surjective. By assumption, $I$ is non-empty, and hence the image of $I \to S$ is a non-empty open subset of $S$. It is also dense in $S$ (as $I$ is irreducible), so it only remains to show that it is closed under specialization. By Lemma 4.3.4, it is enough to consider $S = \text{Spec} V$ for a discrete valuation ring $V$. In this case, Proposition 4.2.4 implies the required assertion. Now since $I \to S$ is smooth and surjective, it admits sections étale locally on $S$, which shows (ii).

Finally, assume (ii), and let $T = \text{Aut}^0_S(X, D)$ be the connected component of $\text{Aut}_S(X, D)$. Then by Lemma 4.3.2, étale-locally on $S$ there exists an isomorphism $T \simeq G^n_m$ and an equivariant isomorphism $(X, D) \simeq (X(\Sigma), D(\Sigma))$. Thus $(X, D)$ is a toric fibration over $S$. □

**Lemma 4.3.2.** Let $\Sigma \subseteq \mathbb{R}^n$ be a smooth complete fan, and let $S$ be a scheme. Then the group scheme $A = \text{Aut}_S((X(\Sigma), D(\Sigma))_S)$ is an extension

$$1 \to G^n_{m,S} \to A \to \text{Aut}(\Sigma)_S \to 1$$

where $\text{Aut}(\Sigma)$ is the (finite) group of automorphisms of the fan $\Sigma$.

**Proof.** Let $\sigma(1)$ denote the set of rays (one-dimensional cones) of $\Sigma$. Let $D(\Sigma) = \bigcup_{\sigma \in \sigma(1)} D_{\sigma}$ be a decomposition of $D(\Sigma)$ into irreducible components. Then every automorphism of the pair $(X(\Sigma), D(\Sigma))$ over $S$ has to permute the divisors $D_{\sigma}$, which yields a homomorphism $A \to \text{Aut}(\Sigma(1))$. Let $A^0$ denote its kernel and let $\sigma \in \Sigma$ be a top-dimensional cone, corresponding to an open immersion

$$(A^0_{\sigma}, \{x_i = 0\}_{i=1,\ldots,n}) \to (X(\Sigma), D(\Sigma))_S$$

whose image is preserved by $A^0$. A direct calculation shows that $\text{Aut}_S(A^n_{\sigma}, \{x_i = 0\}_{i=1,\ldots,n})$ can be naturally identified with $G^n_{m,S}$. □

**Remark 4.3.3.** In the proof of Lemma 4.3.2, the completeness assumption was only used to find a top-dimensional cone in $\Sigma$. On the other hand, $\text{Aut}_S(G_{m,S})$ (where $G_{m,S}$ is treated as an $S$-scheme) might be bigger than the semidirect product of $G_{m,S}$ and $\mathbb{Z} \times 2\mathbb{Z}$ (and non-representable) if $S$ is non-reduced. Smoothness is probably not necessary.

**Lemma 4.3.4.** Let $S$ be a connected noetherian scheme and let $U \subseteq S$ be an open subset. Suppose that for every discrete valuation ring $V$, and for every morphism $h: \text{Spec} V \to S$ mapping the generic point into $U$, we have $h(\text{Spec} V) \subseteq U$. Then $U = S$.

**Proof.** The valuative criterion of properness shows that $U \to S$ is proper, and hence $U$ is also closed. □

**Remark 4.3.5.** Smoothness is probably not necessary for Proposition 4.3.1. Properness seems essential: for example, the pair $(A^2, G_m \times \{0\} + \{0\} \times G_m)$ has non-trivial first-order deformations. Projectivity seems to be only an artifact of the proof of Proposition 4.2.4.

4.4. Images of toric varieties. We shall now combine the technique of descending $F$-liftness (Theorem 3.3.6) with the main result of §4.1 to show the following.

**Theorem 4.4.1.** Suppose that Conjecture 1 is true for simply connected (e.g. separably rationally connected) varieties, that is, that every smooth projective simply connected $F$-liftable variety over an algebraically closed field of characteristic $p > 0$ is a toric variety. Then Conjecture 2 is true, that is, a smooth projective image of a complete toric variety in characteristic zero is a toric variety.

**Proof.** Let $\varphi_K: Z_K \to X_K$ be a surjective morphism from a complete toric variety $Z_K$ to a smooth projective variety $X_K$ defined over an algebraically closed field $K$ of characteristic zero. Reasoning as in [OW02], we can assume that $\varphi_K$ is finite, in which case it is also flat by “Miracle Flatness” [EGA IVIII, 15.4.2], because $X_K$ is smooth and $Z_K$ is Cohen–Macaulay.
There exists a finitely generated subring $R \subseteq K$ and a finite flat surjective map $\varphi: Z \to X$ of schemes over $S = \text{Spec} \ R$, satisfying the following properties:

1. $\varphi_K$ is the base change of $\varphi$ to $K$,
2. $S$ is smooth over $\text{Spec} Z$,
3. $Z$ is a proper constant toric fibration over $S$ (see §2.1),
4. $X$ is smooth and projective over $S$,
5. $d = \deg(\varphi_K)$ is invertible on $S$.

If $\bar{s}: \text{Spec} k \to S$ is a geometric point of $S$, then $\varphi_{\bar{s}}: Z_{\bar{s}} \to X_{\bar{s}}$ is a finite surjective map from a toric variety $Z_{\bar{s}}$ to a smooth projective variety $X_{\bar{s}}$ over $\bar{s}$, of degree invertible in $k$. Let $\bar{s}: \text{Spec} W_2(k) \to S$ be a lifting of $\bar{s}$ mod $p^2$ (such a lifting exists thanks to condition (2)). Then $Z_{\bar{s}}$ is a constant toric fibration over $\bar{s}$, and hence $F_{Z_{\bar{s}}}$ lifts to $Z_{\bar{s}}$. By Theorem 3.3.6(a) applied to $\varphi_{\bar{s}}: Z_{\bar{s}} \to X_{\bar{s}}$, we get that $F_{X_{\bar{s}}}$ lifts to $X_{\bar{s}}$. Moreover, since $X_K$ is simply connected, so is $X_{\bar{s}}$, as the specialization map $\pi_1(X_K) \to \pi_1(X_{\bar{s}})$ is surjective [SGA 1, Exp. X, Corollaire 2.3]. The assumed case of Conjecture 1 implies that $X_{\bar{s}}$ is a toric variety. Thus $X_K$ is toric, by Corollary 4.1.5.

**Remark 4.4.2.** If we believe in Conjecture 1 in its full strength, then the connection between Conjecture 2 and a special case of Conjecture 1 suggests that in characteristic zero, images of abelian varieties should be also of this type, up to a finite étale cover. This would be a common generalization of Conjecture 2 and the results of [Deb89, HM01, DHP08] on the images of abelian varieties.

The first obvious obstacle in deducing such a statement from Conjecture 1 is the ordinary reduction conjecture for abelian varieties. But even assuming that, the method of proof of Theorem 4.4.1 does not apply in this case, as the assumptions of Theorem 3.3.6(a) may no longer be satisfied. More precisely, if $A \to S$ is an abelian scheme over a base $S$ which is of finite type over $Z$, if $\bar{s} = \text{Spec} k \to S$ is a closed geometric point, and $\bar{s} = \text{Spec} W_2(k) \to S$ is a lifting of $\bar{s}$, then to apply the argument from the proof of Theorem 4.4.1 we need not only $A_{\bar{s}}$ to be ordinary, but also $A_{\bar{s}}$ to be its Serre–Tate canonical lifting. We do not know whether one should expect such $\bar{s}$ to exist, even if $A$ is an elliptic curve over $S$.

Using Theorem 3.3.6(c) we can show the following.

*Let $K$ be an algebraically closed field of characteristic zero and let $Z$ be a smooth projective variety over $K$ whose Albanese morphism $Z \to A$ is a toric fibration. Let $G$ be a finite group acting on $Z$. Suppose that $X = Z/G$ is smooth, and that the abelian variety $A$ satisfies the ordinary reduction conjecture. Assume that Conjecture 1 is valid. Then $X$ admits a finite étale cover by a variety whose Albanese morphism is a Zariski-locally trivial fibration with toric fibers.*

Since the proof is long and technical, we refrained from including it in this article.

Using Galois closures and Albanese mappings it is not difficult to show that if $f: A \to X$ is a finite morphism from an abelian variety $A$ to a smooth variety $X$ defined over an algebraically closed field $k$ of characteristic zero, then $X$ is a quotient of some (possibly different) abelian variety by a finite group. Therefore, the above result partially recovers the classification of smooth images of abelian varieties under finite morphisms mentioned above contingent upon the validity of Conjecture 1.

5. Structure of $F$-liftable nc pairs with trivial canonical bundle

5.1. Statement of the main result and some preliminaries. In this section, we provide a logarithmic generalization of [MS87, Theorem 2], settling a special case of Conjecture 1. More precisely, we prove the following theorem which characterizes projective $F$-liftable nc pairs $(X, D)$ with $\omega_X(D)$ numerically trivial.
**Theorem 5.1.1.** Let \((X, D)\) be a projective nc pair over an algebraically closed field \(k\) of positive characteristic. The following conditions are equivalent:

(i) \((X, D)\) is \(F\)-liftable and \(\omega_X(D)\) is numerically trivial,

(ii) \(X\) is \(F\)-split and \(\Omega_X^1(\log D)\) becomes trivial on a finite étale cover of \(X\),

(iii) \(X\) admits a finite étale cover \(f: Y \rightarrow X\) whose Albanese map \(\alpha: Y \rightarrow A\) is a toric fibration over an ordinary abelian variety with toric boundary \(f^{-1}(D)\).

**Proof.** We start by showing (i)⇒(ii). If \(X\) is \(F\)-liftable, it is \(F\)-split, by Proposition 3.2.1(c). Using Variant 3.2.2, we observe that there exists an injective morphism

\[\xi: F^*\Omega_X^1(\log D) \rightarrow \Omega_X^1(\log D)\]

The determinant of \(\xi\) gives rise to a non-zero section of the \((p-1)\)-st power of the numerically trivial bundle \(\omega_X(D)\), and hence it is an isomorphism. Therefore, we see that \(F^*\Omega_X^1(\log D) \simeq \Omega_X^1(\log D)\) and thus by [LS77, Satz 1.4] the bundle \(\Omega_X^1(\log D)\) becomes trivial on a finite étale cover \(\pi: Y \rightarrow X\).

Now we show (iii)⇒(i). Replacing \(A\) with a finite étale cover and \(Y\) with its base change, we may assume that \(Y \rightarrow A\) is a split toric fibration. By Lemma 3.3.8, there exists a natural Frobenius lifting of \(\alpha: Y \rightarrow A\) over the Serre–Tate canonical lifting \((\tilde{A}, \tilde{F}_A)\) of \(A\), and consequently \(Y\) is \(F\)-liftable. Moreover, this Frobenius lifting is compatible with the toric boundary \(f^{-1}D\) of \(Y \rightarrow A\). Further, since \(Y \rightarrow A\) is a split toric fibration with toric boundary \(f^{-1}(D)\), the bundle \(\Omega_{Y/A}^1(\log f^{-1}D)\) is trivial, and hence \(\omega_Y(f^{-1}(D))\) is trivial as well. Thus \((Y, f^{-1}D)\) satisfies (i), and hence by what we already proved it also satisfies (ii). Replacing \(Y\) by a further finite étale cover, we can thus assume that \(\Omega_X^1(\log f^{-1}D)\) is trivial and that \(Y \rightarrow X\) is Galois under a finite group \(G\).

To finish the argument, we argue as in [MS87, Proof of Theorem 2, (i)⇒(ii)]. By Variant 3.3.2, Frobenius liftings of \((Y, f^{-1}D)\) correspond to splittings of the short exact sequence

\[0 \rightarrow B_{Y'}^1 \rightarrow \mathcal{Z}_Y^1(\log f^{-1}D) \rightarrow \Omega_{Y'}^1(\log f^{-1}D') \rightarrow 0\]

where \(Y'\) is the Frobenius twist of \(Y\). Since \(\Omega_Y^1(\log f^{-1}D)\) is trivial and \(H^0(Y, B_{Y'}^1) = 0\) (as \(Y\) is \(F\)-split), the above extension admits a unique splitting. In particular, this splitting is \(G\)-invariant, and hence descends to a splitting of the corresponding sequence on \(X\). Thus \((X, D)\) is Frobenius liftable. Since \(f^*\omega_X(D) = \omega_Y(f^{-1}D) \simeq \mathcal{O}_Y\), we see that \(\omega_X(D)\) is numerically trivial.

Finally, (ii)⇒(iii) follows from Theorem 5.2.1, whose proof occupies the subsequent subsection. \(\square\)

The proof of the implication (ii)⇒(iii) in the case \(D = 0\) in [MS87] relied on lifting to characteristic zero and using Yau’s work on the Calabi conjecture. Instead, we use the following two characteristic zero results.

**Theorem 5.1.2 ([Win04, Corollary 1]).** Let \((X, D)\) be a projective nc pair defined over \(\mathbb{C}\). Then the following conditions are equivalent:

(i) the log cotangent bundle \(\Omega_X^1(\log D)\) is trivial,

(ii) there exists a semi-abelian group variety \(T\) (an extension of an abelian variety by a torus) acting on \(X\) with an open dense orbit \(X \setminus D\).

If \(\dim \text{Alb} X = 0\), then \((X, D)\) is a toric pair.

We say that a group \(G\) is virtually abelian if it contains a finitely generated abelian subgroup of finite index.

**Theorem 5.1.3 ([GKP16, Theorem 10.1], cf. [NZ10]).** Let \(X\) be a normal projective variety over \(\mathbb{C}\) which admits a polarized endomorphism of degree greater than one. Then the topological fundamental group \(\pi_1(X(\mathbb{C}))\) is virtually abelian. The same holds for \(\pi_1^{\text{et}}(X)\) since it is the profinite completion of \(\pi_1(X(\mathbb{C}))\).
Remark 5.1.4. Since [MS87] was motivated by the corresponding result in characteristic zero, we must warn the reader that the following statements are false.

"Theorem" A (characteristic zero analogue of Theorem 5.1.1). Let \((X, D)\) be a projective nc pair over an algebraically closed field \(k\) of characteristic zero. The following conditions are equivalent:
(i) \(\Omega^1_X(\log D)\) becomes trivial on a finite étale cover of \(X\),
(ii) \(X\) admits a finite étale cover \(f: Y \to X\) whose Albanese map \(a: Y \to A\) is a toric fibration over an ordinary abelian variety with toric boundary \(f^{-1}(D)\).

"Theorem" B. The assertion of Winkelmann’s theorem (Theorem 5.1.2) holds over algebraically closed fields of positive characteristic.

"Theorem" C. Let \(X\) be a smooth projective scheme over an open subset \(S \subseteq \text{Spec} \, K\) where \(K\) is the ring of integers in a number field \(K\). Suppose that for infinitely many closed points \(s \in S\), the reduction \(X_s\) has a finite étale cover whose Albanese map is a toric fibration. Then the same holds for \(X_{\overline{K}}\).

In “Theorem” A, we have (i) \(\Rightarrow\) (ii) by Theorem 5.1.2, but (ii) does not imply (i). A basic counterexample is as follows. Let \(C\) be an elliptic curve, let \(L\) be a line bundle of non-zero degree on \(C\), and let \(E = \mathcal{O}_C \oplus L\). Let \(X = \mathbb{P}_C(E)\) and let \(D\) be the sum of the two sections \(C \to X\) corresponding to the projections \(E \to \mathcal{O}_C\) and \(E \to L\). Then condition (ii) is satisfied with \(Y = X\). Moreover, every finite étale cover of \(X\) is of the same type, so if (ii) \(\Rightarrow\) (i) were to hold, we could find such an \(X\) with \(\Omega^1_X(\log D)\) trivial. So suppose that \(\Omega^1_X(\log D)\) is trivial. Again by Theorem 5.1.2, the open subset \(U = X \setminus D\) (which is the \(\mathbb{G}_m\)-torsor corresponding to \(L\)) admits a group structure making \(U \to C\) a group homomorphism. But this is only possible if \(\deg L = 0\) by the “Barsotti–Weil formula”

\[
\text{Pic}^0(C) \simeq \text{Ext}^1(C, \mathbb{G}_m) \quad \text{(see [Ser59, VII.16, Théorème 6])}.
\]

Note that in the two extreme cases \(A = Y\) and \(A = 0\), the implication (ii) \(\Rightarrow\) (i) does indeed hold. This construction also provides a counterexample to “Theorem” B (i) \(\Rightarrow\) (ii).

For a counterexample to “Theorem” C, we take an elliptic curve \(C\) over \(\mathbb{Z}[1/N]\) for some \(N\), and the non-split extension

\[
0 \longrightarrow \mathcal{O}_C \longrightarrow E \longrightarrow \mathcal{O}_C \longrightarrow 0.
\]

We again set \(X = \mathbb{P}_C(E)\). If \(p\) is a prime of ordinary reduction of \(E\) and \(k = \overline{\mathbb{F}}_p\), then \(X_k\) satisfies the required property by Remark 3.1.7 and Lemma 7.1.8. On the other hand, the above extension does not become split over any finite étale cover of \(C_{\overline{\mathbb{Q}}}\), and consequently no finite étale cover of \(X_{\overline{\mathbb{Q}}}\) is a toric fibration over an abelian variety.

5.2. \(F\)-split nc pairs with trivial cotangent bundle. In this section, we prove the following result generalizing [MS87, Theorem 1] and yielding the implication (ii) \(\Rightarrow\) (iii) needed above.

Theorem 5.2.1 (Log version of [MS87, Theorem 1]). Let \((X, D)\) be a projective nc pair over an algebraically closed field \(k\) such that \(\Omega^1_X(\log D)\) is trivial and \(X\) is \(F\)-split. Then after an étale covering \(X\) admits a structure of a toric fibration over an ordinary abelian variety.

Even though the proof closely follows the ideas of Mehta and Srinivas, there are many important details that need to be figured out in the logarithmic setting. For this reason, we precede the proof with a sequence of lemmas generalizing their results.

Lemma 5.2.2 (Log version of [MS87, Lemma 1.2]). Let \((X, D)\) be an nc pair satisfying the hypotheses of Theorem 5.2.1, and let \(\pi: Y \to X\) be a finite étale covering. Then \((Y, \pi^{-1}D)\) also satisfies the hypotheses of Theorem 5.2.1.
Lemma 5.2.4. \(\text{Let } \Omega^1_X(\log \pi^{-1}D) \text{ is isomorphic to } \pi^*\Omega^1_X(\log D) \text{ and is therefore trivial. To prove that } Y \text{ is } F\text{-split, we use Lemma 2.5.5(d).} \)

Proof. Since \(\pi\) is étale, we see that \(\Omega^1_X(\log \pi^{-1}D)\) is isomorphic to \(\pi^*\Omega^1_X(\log D)\) and is therefore trivial. To prove that \(Y\) is \(F\)-split, we use Lemma 2.5.5(d).

Lemma 5.2.3 (Log version of [MS87, Lemma 1.4]). \(\text{Let } (X,D) \text{ be an nc pair satisfying the hypotheses of Theorem 5.2.1. Then the Albanese map } f: X \rightarrow \text{Alb}X \text{ is smooth, surjective, and has connected fibers. Moreover, } (X,D) \text{ is an nc pair over } \text{Alb}X \text{ and } \Omega^1_{X/\text{Alb}X}(\log D) \text{ is trivial.} \)

Proof. First, we observe that the logarithmic differential \(f^*\Omega^1_{\text{Alb}X} \rightarrow \Omega^1_{X}(\log D)\) decomposes as

\[ f^*\Omega^1_{\text{Alb}X} \hookrightarrow \Omega^1_{X} \rightarrow \Omega^1_{X}(\log D), \]

where the first morphism is the differential and the second is an inclusion (see §2.3). Since \(H^0(\text{Alb}X, \Omega^1_{\text{Alb}X}) = H^0(X, f^*\Omega^1_{\text{Alb}X})\), by [MS87, Lemma 1.3] we see that the induced morphism \(H^0(X, f^*\Omega^1_{\text{Alb}X}) \rightarrow H^0(X, \Omega^1_X(\log D))\) is injective and hence

\[ (5.2.1) \quad H^0(X, f^*\Omega^1_{\text{Alb}X}) \hookrightarrow H^0(X, \Omega^1_X(\log D)) \]

is injective as well. As \(f^*\Omega^1_{\text{Alb}X}\) and \(\Omega^1_X(\log D)\) are trivial, the map \(f^*\Omega^1_{\text{Alb}X} \rightarrow \Omega^1_X(\log D)\) is uniquely determined by (5.2.1) and is therefore injective. We conclude that \(f\) is separable and dominant and that there exists a short exact sequence of trivial bundles

\[ 0 \rightarrow f^*\Omega^1_{\text{Alb}X} \rightarrow \Omega^1_X(\log D) \rightarrow \Omega^1_{X/\text{Alb}X}(\log D) \rightarrow 0. \]

Consequently, the sheaf of relative log differentials \(\Omega^1_{X/\text{Alb}X}(\log D)\) is locally free and therefore \(f: (X,D) \rightarrow \text{Alb}X\) is an nc pair over \(\text{Alb}X\) by Lemma 2.3.3.

To see that the fibers of \(f\) are connected, that is, we have \(f_*\mathcal{O}_X = \mathcal{O}_{\text{Alb}X}\), we consider the Stein factorization \(X \rightarrow Z \rightarrow \text{Alb}X\). Since \(X \rightarrow \text{Alb}X\) is smooth, the morphism \(Z \rightarrow \text{Alb}X\) is étale (by [SGA 1, Exp. X, Prop. 1.2] or [EGA II, 7.8.10 (i)]) and therefore \(Z\) is an abelian variety. By the universal property of the Albanese morphism, \(Z \rightarrow \text{Alb}X\) is an isomorphism, which finishes the proof.

Lemma 5.2.4. \(\text{Let } (X,D) \text{ be an nc pair satisfying the hypothesis of Theorem 5.2.1. Then there exists a projective lifting } (\mathcal{E}, \mathcal{D}) \text{ of } (X,D) \text{ over } W(k) \text{ together with a lifting of the Frobenius morphism } F_{\mathcal{E}}: (\mathcal{E}, \mathcal{D}) \rightarrow (\mathcal{E}, \mathcal{D}) \text{. Moreover, for every line bundle } L \text{ on } X \text{ there exists a line bundle } \mathcal{L} \text{ on } \mathcal{E} \text{ such that } L \simeq \mathcal{L}|_X \text{ and } F_{\mathcal{E}}^*\mathcal{L} \simeq \mathcal{L}^{\otimes p}. \)

Proof. We apply Variant 3.3.2 to see that the obstruction classes to lifting the nc pair \((X,D)\) together with the Frobenius morphism over consecutive Witt rings \(W_n(k)\) lie in

\[ \text{Ext}^1(\Omega^1_X(\log D), B^1_X). \]

Since \(\Omega^1_X(\log D)\) is trivial and \(X\) is \(F\)-split, we see by Lemma 2.5.5 that the latter group satisfies

\[ \text{Ext}^1(\Omega^1_X(\log D), B^1_X) = \text{Ext}^1(\mathcal{O}_X^{\otimes n}, B^1_X) = H^1(X, B^1_X)^{\otimes n} = 0 \]

and therefore \((X,D)\) deforms to a formal nc pair \(\{(X_n, D_n)\}_{n \geq 1}\) over the formal spectrum of \(W(k)\) together with a compatible lifting of the Frobenius morphism

\[ \{F_{X_n}: (X_n, D_n) \rightarrow (X_n, D_n)\}_{n \geq 1}. \]

Since \(X\) is \(F\)-split, by Lemma 2.5.5 we see that the Frobenius action on \(H^1(X, \mathcal{O}_X)\) is bijective and hence by Proposition 3.3.1(d) we see that every line bundle \(L\) on \(X\) admits a formal lifting \(\{L_n \in \text{Pic}(X_n)\}_{n \geq 1}\) such that \(F_{X_n}^*L_n \simeq L_n^{\otimes p}\).

To finish the proof we need to show that the given inductive systems are algebraizable. For this purpose, since every ample line bundle deforms to the formal nc pair \(\{(X_n, D_n)\}_{n \geq 1}\), we may just apply Grothendieck’s algebraization theorem (see [EGA III1, Section 3.4] or [Sta14, Tag 089A]).

Remark 5.2.5. The lifting of the Frobenius morphism we exhibit above is not a \(W(k)\)-linear endomorphism, it is only Frobenius-linear. However, if \(X\) is defined over a finite field \(k = \mathbf{F}_p^d\), then the \(e\) th iterate of \(F_{\mathcal{E}}\) is in fact a polarized \(W(k)\)-endomorphism of \(\mathcal{E}\).
The following lemma is the essential part of our argument. It differs substantially from its counterpart in [MS87], and is based on the two theorems stated in §5.1.

**Lemma 5.2.6.** Let \((X,D)\) be an nc pair satisfying the hypotheses of Theorem 5.2.1. Then either \((X,D)\) is a toric pair (see §2.1) or there exists a finite étale covering \(\text{Alb} Y \neq 0\).

**Proof.** First, we assume that there exists an étale covering \(Y \rightarrow X\) such that \(H^1(Y, \mathcal{O}_Y) = 0\). In this case, we claim that \((X,D)\) is in fact a toric pair. In order to see this, we use Lemma 5.2.4 to obtain a \(W(k)\)-lifting \((\mathcal{F}, \mathcal{O})\) of \((Y,E)\), where \(E\) is the preimage of \(D\). The induced deformation of the log cotangent bundle \(\Omega_{Y}^1(\log E)\) is trivial because the tangent space of its deformation functor is isomorphic to

\[
H^1 \left( Y, \mathcal{E}nd(\Omega_{Y}^1(\log E)) \right) = H^1 \left( Y, \mathcal{O}_Y \right) \oplus \mathbb{N}^2 = 0.
\]

This implies that the log cotangent bundle of the generic fiber \(\mathcal{F}_q\) is trivial. By semicontinuity of cohomology we also see that \(H^1(\mathcal{F}_q, \mathcal{O}_{\mathcal{F}_q}) = 0\) and therefore \((\mathcal{F}_q, \mathcal{O}_{\mathcal{F}_q})\) is a toric pair by Theorem 5.1.2. By Proposition 4.2.4 this means that the special fiber \((Y,E)\) is a toric pair as well. To finish the proof of the claim, we show that \(Y \rightarrow X\) is an isomorphism. For this purpose, we observe using the Hirzebruch–Riemann–Roch theorem (see [Ful98, Corollary 15.2.1]) that

\[
1 = \chi(Y, \mathcal{O}_Y) = d \cdot \chi(X, \mathcal{O}_X),
\]

where \(d\) is the degree of the finite map \(Y \rightarrow X\). This clearly implies that \(d = 1\) and hence \(Y \rightarrow X\) is an isomorphism.

Now we proceed to the second case, where \(H^1(Y, \mathcal{O}_Y) \neq 0\) for every étale covering \(Y \rightarrow X\). We follow the strategy of [MS87, Lemma 1.6] substituting the application of the Calabi conjecture with Theorem 5.1.3 describing the algebraic fundamental groups of varieties admitting a polarized endomorphism. We claim that there exists an étale covering \(Y \rightarrow X\) such that \(\text{Alb} Y \neq 0\).

For the proof, we first use the spreading out technique to reduce to the case of nc pairs defined over finite fields. Let \((\mathcal{S}, \mathcal{D})\) be an nc pair over a spectrum of a finitely generated local \(\mathbb{F}_p\)-algebra \(R\) such that the geometric generic fiber is isomorphic to \((X,D)\) and the residue field is finite with \(q = p^e\) elements. Spreading out the trivialization of the log cotangent bundle and the Frobenius splitting (which can interpreted as a morphism of vector bundles on the Frobenius twist of \(X\)), we may assume that \((\mathcal{S}, \mathcal{D})\) satisfies the assumptions of Theorem 5.2.1. Assume now that there exists a finite étale covering

\[
\pi_{\mathcal{F}_q}: \mathcal{F}_q \rightarrow \mathcal{D}_{\mathcal{F}_q}
\]

of the geometric special fiber such that \(\text{Alb} \mathcal{F}_q \neq 0\). By [SGA 4 1/2], Arcata, IV, Proposition 2.2] (or [Sta14, Tag 0BQC]) we see that such a covering extends to a covering of \(\mathcal{S} \times_{\text{Spec} R} \text{Spec} R^{\text{sh}}\), where \(R^{\text{sh}}\) is the strict henselization of \(R\). Since \(R^{\text{sh}}\) is the colimit of étale extensions of \(R\) we observe that, possibly after taking an étale covering of \(\text{Spec} R\), the morphism \(\pi_{\mathcal{F}_q}\) arises as a special fiber of a covering \(\pi: \mathcal{S} \rightarrow \mathcal{S}^{\text{sh}}\). The geometric generic fiber of \(\pi\) yields an étale covering of \(Y \rightarrow X\) which satisfies \(H^1(\text{Alb} Y = H^1_{\text{et}}(Y, \mathbb{Q}_l) \neq 0\) by smooth base change for étale cohomology (we take \(l \neq p\)). This finishes the reduction step.

For \((X,D)\) defined over a finite field \(k = \mathbb{F}_q\), we reason in two steps which we describe briefly. First, we lift the pair \((X,D)\) to characteristic zero and apply Theorem 5.1.3 to prove that \(\pi_{\mathcal{F}_q}(X)\) is virtually abelian. Consequently, using the approach described in [MS87, Lemma 1.6], we prove that for an étale cover \(Y \rightarrow X\) the crystalline cohomology group \(H^1_{\text{crys}}(Y/K)\) over the field \(K = \text{Frac} W(k)\) is non-zero, and hence \(\dim \text{Alb} Y = \dim_k H^1_{\text{crys}}(Y/K)\) is non-zero as well.

Now, we present the details of the first step. Using Lemma 5.2.4 we construct a \(W(k)\)-lifting \(((\mathcal{S}, \mathcal{D}), F_{\mathcal{D}})\) of \(((X,D), F_X)\). We set \(\eta\) to be the generic point of \(W(k)\). Then, applying Remark 5.2.5, we observe that \(F^e_X: X \rightarrow X\) is in fact a \(k\)-linear endomorphism for some \(e > 0)
and therefore the geometric generic fiber $F^{\eta}_{\eta}$ of the lifting of $F^X$ is a polarized endomorphism of $\mathcal{X}_{\eta}$. By Theorem 5.1.3 the geometric fundamental group $\pi^\ell_1(\mathcal{X}_{\bar{\eta}})$ is virtually abelian. Using the surjectivity of the specialization morphism $\pi^\ell_1(\mathcal{X}_{\bar{\eta}}) \to \pi^\ell_1(X)$ we see that the same holds for $\pi^\ell_1(X)$ and therefore after taking an étale covering of $X$ we may assume $\pi^\ell_1(X)$ is abelian.

We proceed to the second step. We assume that $\pi^\ell_1(X)$ is abelian and follow [MS87, Lemma 1.6] closely. First, by [Ill79, Chapitre II, Théorème 5.2], to prove that $\dim Alb(X) = \dim_X H^1_{\text{crys}}(X/K)$ is zero it suffices to show that $H^1_{\text{ét}}(X, \mathbb{Z}_p)$ is non-torsion. We have

$$H^1_{\text{ét}}(X, \mathbb{Z}_p) \cong \text{Hom}_\mathbb{Z}(\pi^\ell_1(X)^{\wedge p}, \mathbb{Z}_p),$$

where $\pi^\ell_1(X)^{\wedge p} = \varprojlim \pi^\ell_1(X) \otimes \mathbb{Z}/p^n$. The $\mathbb{Z}_p$-module $\pi^\ell_1(X)^{\wedge p}$ is finitely generated and therefore it is torsion if and only of it is finite. If $\pi^\ell_1(X)^{\wedge p}$ is finite, then there would exist an étale covering $X' \to X$ such that $H^1_{\text{ét}}(X', \mathbb{F}_p) = 0$. This gives a contradiction with $H^1(X', \mathcal{O}_{X'}) \neq 0$. Indeed, using the Artin–Schreier sequence of étale sheaves

$$0 \to \mathbb{F}_p \to \mathcal{O}_X \xrightarrow{1-\pi} \mathcal{O}_X \to 0$$

we see that $H^1(X', \mathbb{F}_p)$ is the $\mathbb{F}_p$-vector space of elements in $H^1(X', \mathcal{O}_{X'})$ fixed by the Frobenius morphism. This is non-zero because Frobenius is bijective on $H^1(X', \mathcal{O}_{X'})$ for an $F$-split scheme by Lemma 2.5.5(a).

**Lemma 5.2.7** (Log version of [MS87, Lemma 1.7]). Let $(X, D)$ be an nc pair satisfying the hypothesis of Theorem 5.2.1, and let $Y \to X$ be a Galois étale cover with $\text{Alb}Y \neq 0$. Then there exists an intermediate Galois étale cover $Z \to X$ of degree $p^m$, for some $m \geq 0$, such that $Y \to Z$ induces an isogeny on Albanese varieties, in particular $\text{Alb}Z \neq 0$.

**Proof.** We apply the argument given in the proof of [MS87, Lemma 1.7] to the logarithmic cotangent bundle instead of cotangent bundle. \hfill \Box

**Lemma 5.2.8** (Log version of [MS87, Lemma 1.8]). Let $(X, D)$ be an nc pair satisfying the hypothesis of Theorem 5.2.1, and let $\pi : X \to \text{Alb}X$ be the Albanese mapping. Then

(a) all geometric fibers of $\pi$ are $F$-split,

(b) for each $i \geq 0$, $R^i\pi_*\mathcal{O}_X$ is a locally free $\mathcal{O}_{\text{Alb}X}$-module which becomes free on a finite étale cover of $\text{Alb}X$,

(c) $\text{Alb}X$ is an $F$-split abelian variety.

**Proof.** This follows by the same proof with the caveat that the Cartier isomorphism and the Grothendieck duality need to be replaced with their logarithmic versions. For the convenience of the reader, we present a slightly simplified argument below.

For the proof of (a) it is sufficient to show that $\pi$ is relatively $F$-split, which follows from [Eji17, Theorem 1.2]. To get (b), we reason as in [MS87, Lemma 1.8]. More precisely, let $A = \text{Alb}X$, let $X'$ be the base change of $X$ along $F_A$, and let $\pi' : X' \to A$ denote the induced projection. By flat base change we obtain an isomorphism $F^\mathcal{A}_A R^i\pi_*\mathcal{O}_X \cong R^i\pi'_*\mathcal{O}_{X'}$. Since the fibers of $\pi'$ are $F$-split we observe that the natural homomorphisms $R^i\pi_*\mathcal{O}_X \to R^i\pi'_*F^1_{X/A}\mathcal{O}_{X'}$ induced from the long exact sequence of $R^i\pi'_*$ for

$$0 \to \mathcal{O}_{X'} \to F_{X/A}\mathcal{O}_X \to B^1_{X/A} \to 0$$

are isomorphisms and therefore $F^\mathcal{A}_A R^i\pi_*\mathcal{O}_X \cong R^i\pi_*\mathcal{O}_X$. By [MN84, Lemma 1.4] we see that $R^i\pi_*\mathcal{O}_X$ is locally free, and [LS77, Satz 1.4] implies that it is étale trivializable. Part (c) follows from Lemma 2.5.5(c). \hfill \Box

**Lemma 5.2.9** (Log version of [MS87, Lemma 1.9]). Let $(X, D)$ be an nc pair satisfying the hypothesis of Theorem 5.2.1, and let $\pi : X \to \text{Alb}X$ be the Albanese mapping. Let $s \in \text{Alb}X$ be a closed point. Then every finite étale covering of $X_s$ of degree $p^m$, for any $m \geq 0$, is induced by a covering of $X$. 

Proof. Given Lemma 5.2.8(b) the proof of [MS87, Lemma 1.9] applies without change. □

Lemma 5.2.10 (Generalization of [MS87, Remark below Lemma 1.10]). Let \( Y \) be a smooth projective variety and let \( \pi : X \to Y \) be a proper smooth morphism whose geometric fibers are \( F \)-split. Then \( R^1\pi_*Q_\ell \) becomes trivial on a finite étale cover of \( Y \).

Remark about the proof. Once we know that the Albanese varieties of smooth projective \( F \)-split varieties are ordinary (see Lemma 2.5.5), the proof given in [MS87, Remark below Lemma 1.10] can be repeated word for word. □

Equipped with the above we proceed to:

Proof of Theorem 5.2.1. Let \( (X, D) \) be a nc pair satisfying the hypotheses of Theorem 5.2.1. We want to prove that there exists an étale covering admitting a structure of a toric fibration over an ordinary abelian variety with a toric divisor given by the preimage of \( D \). First, following [DHP08, Proof of Theorem 1.1] we consider an étale covering \( Y \to X \) such that \( \dim \text{Alb} Y \) is maximal (by Lemma 5.2.3 we know that \( \dim \text{Alb} Y \leq \dim Y = \dim X \)). Using Lemma 5.2.2 we see that for \( E \) defined as the preimage of \( D \) the nc pair \( (Y, E) \) also satisfies the hypotheses of Theorem 5.2.1.

We claim that the Albanese morphism of \( Y \) is a toric fibration with toric boundary \( E \). Indeed, using Lemma 5.2.3 and Lemma 5.2.8 we see that the fibers of \( Y \to \text{Alb} Y \) satisfy the assumptions of Lemma 5.2.6 and therefore they are either toric, and hence the proof is finished by Proposition 4.3.1, or for some fiber \( Y_s \) there exists an étale covering \( g_s : \tilde{Y}_s \to Y_s \) such that \( \text{Alb} \tilde{Y}_s \neq 0 \). In the latter case, using Lemma 5.2.7, we may assume that \( \deg g_s = p^m \), for some \( m \geq 0 \), and therefore by Lemma 5.2.9 the morphism \( g_s \) is induced by a covering \( \tilde{Y} \to Y \). We consider the composition \( \pi : \tilde{Y} \to \text{Alb} Y \) of the covering and the Albanese morphism of \( Y \). By Lemma 5.2.2 the fibers of \( \pi \) are \( F \)-split and hence we may apply Lemma 5.2.10 to see that \( R^1\pi_*Q_\ell \) is non-zero and étale trivializable. This means that after an étale covering \( \eta : A \to \text{Alb} Y \) we have an isomorphism

\[
\eta^*R^1\pi_*Q_\ell \simeq Q_\ell^d
\]

for some \( d > 0 \). Let \( \tilde{Y} = \tilde{Y} \times_{\text{Alb} Y} A \) be the covering of \( \tilde{Y} \) induced by \( \eta \), and let \( \tilde{\pi} : \tilde{Y} \to A \) be the projection in the cartesian diagram

\[
\begin{array}{ccc}
\tilde{Y} & \xrightarrow{\tilde{\pi}} & \tilde{Y} \\
\downarrow \pi & & \downarrow \pi \\
A & \xrightarrow{\eta} & \text{Alb} Y.
\end{array}
\]

We claim that \( \dim \text{Alb} \tilde{Y} > \dim \text{Alb} Y \). Indeed, as in [MS87, Proof of Theorem 1] we consider the Leray spectral sequence \( H^j_{\text{ét}}(A, R^i\tilde{\pi}_*Q_\ell) \Rightarrow H^j_{\text{ét}}(\tilde{Y}, Q_\ell) \) to obtain the exact sequence

\[
0 \to H^1_{\text{ét}}(A, Q_\ell) \to H^1_{\text{ét}}(\tilde{Y}, Q_\ell) \to H^0(A, R^1\tilde{\pi}_*Q_\ell) \to H^2_{\text{ét}}(A, Q_\ell) \to H^2_{\text{ét}}(\tilde{Y}, Q_\ell).
\]

Since \( \tilde{\pi} : \tilde{Y} \to A \) admits a multi-section the morphism \( H^2_{\text{ét}}(A, Q_\ell) \to H^2_{\text{ét}}(\tilde{Y}, Q_\ell) \) is injective, and therefore we have

\[
\dim \text{Alb} \tilde{Y} = \frac{1}{2}\dim Q_\ell H^1(\tilde{Y}, Q_\ell) = \frac{1}{2}\left( \dim Q_\ell H^1(A, Q_\ell) + \dim Q_\ell H^0(A, R^1\tilde{\pi}_*Q_\ell) \right)
\]

\[
= \dim A + \frac{1}{2}\dim Q_\ell H^0(A, \eta^*R^1\pi_*Q_\ell)
\]

\[
= \dim A + \frac{1}{2}\dim Q_\ell H^0(A, Q_\ell^d)
\]

\[
= \dim \text{Alb} Y + d > \dim \text{Alb} Y.
\]

This gives a contradiction with the choice of \( Y \) and hence finishes the proof. □
6. Homogeneous spaces

In [BTLM97, §4], the authors study Frobenius liftability of rational homogeneous spaces. In many cases, they are able to show that such a variety $X$ is not $F$-liftable because Bott vanishing

$$H^j(X, \Omega^d_X \otimes L) = 0 \quad (j > 0, \ L \text{ ample})$$

does not hold. As in this paper (although using a different argument), they reduce the question to the case of Picard number one. But even under this assumption, finding $i, j,$ and $L$ as above for which $H^i(X, \Omega^d_X \otimes L) \neq 0$ is a difficult task. To this end, the authors use involved results of D. Snow [Sno88, Sno86] on the cohomology of flag varieties of Hermitian symmetric type in characteristic zero. They ask (see [BTLM97, Remark 2]) whether the only $F$-liftable rational homogeneous spaces are products of projective spaces. As these are precisely the toric ones, this is a special case of our Conjecture 1, one which we are actually able to settle.

Our proof in the Picard number one case does not rely on the classification of homogeneous spaces or Bott vanishing. In fact, we only need to assume that $X$ is Fano and that the tangent bundle $T_X$ is nef (note that the Campana–Peternell conjecture ([CP91], [Kol96, V, Conjecture 3.10]) predicts that these two conditions should in fact imply that $X$ is a rational homogeneous space). The main ingredients of our proof are Mori’s characterization of the projective space in terms of rational curves (Theorem 6.1.4) and a careful analysis of the restrictions of the sheaf of $\xi$-invariant forms (§6.2) to some special families of rational curves (Proposition 6.3.2). For the general case, we unfortunately have to look at the classification of homogeneous spaces, but only to check for which vertices of which Dynkin diagrams the corresponding homogeneous space of Picard rank one is a projective space (Lemma 6.4.1). A result of Lauritzen and Mehta [LM97] allows us to exclude the possibility of non-reduced stabilizers. We believe that our ideas could be useful in tackling the Picard rank one case of our Conjecture 1 with the assumption that $T_X$ is nef dropped.

In this section we work over an algebraically closed field $k$ of positive characteristic.

6.1. Families of rational curves. We start by recalling basic definitions pertaining to rational curves. The main reference for this subsection is [Kol96, II §2–3]. In what follows we assume that $X$ is a smooth projective $k$-scheme.

**Definition 6.1.1.** Let $\varphi: \mathbb{P}^1 \to X$ be a non-constant morphism. We say that $\varphi$ is free if $\varphi^*T_X$ is nef, and very free if $\varphi^*T_X$ is ample.

Given a rational curve $C \subseteq X$ we shall say that it is free (resp. very free) if the normalization $\varphi: \mathbb{P}^1 \to C \to X$ is free (resp. very free). Further, for a free $\varphi: \mathbb{P}^1 \to X$ we can write

$$\varphi^*T_X \simeq \oplus_{i=1}^{n-r} \mathcal{O}_{\mathbb{P}^1}(a_i), \quad a_i > 0.$$

With that, $\varphi$ is very free if and only if $r = 0$.

Fix an ample divisor $H$ on $X$. If $X$ is Fano, we always take $H = -K_X$. For an integer $d$, we denote by $\text{Hom}_d(\mathbb{P}^1, X) \subseteq \text{Hilb}(\mathbb{P}^1 \times X)$ the scheme parametrizing morphisms $\varphi: \mathbb{P}^1 \to X$ such that $\deg \varphi^*H = d$. We denote by $\text{Hom}_d^{\text{free}}(\mathbb{P}^1, X) \subseteq \text{Hom}_d(\mathbb{P}^1, X)$ the subscheme parametrizing free $\varphi: \mathbb{P}^1 \to X$ which are generically injective. We drop the subscript $d$ whenever we do not wish to specify the degree, so $\text{Hom}(\mathbb{P}^1, X) = \bigsqcup_d \text{Hom}_d(\mathbb{P}^1, X)$. By [Kol96, II, Corollary 3.5.4], the natural morphism

$$\mathbb{P}^1 \times \text{Hom}_d^{\text{free}}(\mathbb{P}^1, X) \to X$$

is smooth, and so $\text{Hom}_d^{\text{free}}(\mathbb{P}^1, X)$ is smooth as well.

Let $\text{RatCurves}_d^{\text{free}}(X)$ be the quotient $\text{Hom}_d^{\text{free}}(\mathbb{P}^1, X)/\text{Aut}(\mathbb{P}^1)$, which exists by [Kol96, II, Comment 2.7] as $\text{Hom}_d^{\text{free}}(\mathbb{P}^1, X)$ is a smooth open subscheme of $\text{Hom}_d(\mathbb{P}^1, X)$, invariant
under the $\text{Aut}(\mathbb{P}^1)$-action (see [Kol96, II, Corollary 3.5.4]). We denote by $\text{Univ}_d(X)$ the universal $\mathbb{P}^1$-bundle over $\text{RatCurves}_d^{\text{free}}(X)$ so that we have a diagram

$$
\begin{array}{ccc}
\text{Univ}_d(X) & \xrightarrow{\varphi} & X \\
\pi \downarrow & & \downarrow \\
\text{RatCurves}_d^{\text{free}}(X)
\end{array}
$$

It is constructed as a quotient of $\mathbb{P}^1 \times \text{Hom}_d^{\text{free}}(\mathbb{P}^1, X)$ by $\text{Aut}(\mathbb{P}^1)$. We have a factorization

$$
\mathbb{P}^1 \times \text{Hom}_d^{\text{free}}(\mathbb{P}^1, X) \longrightarrow \text{Univ}_d(X) \xrightarrow{\varphi} X
$$

where the left arrow is smooth and surjective (cf. [Kol96, II, Theorem 2.15]), and so $\varphi$ is smooth. In particular, both $\text{Univ}_d(X)$ and $\text{RatCurves}_d^{\text{free}}(X)$ are smooth [Sta14, Tag 02K5].

**Definition 6.1.2.** Let $\varphi: \mathbb{P}^1 \to X$ be a generically injective morphism which is free and such that $d = \deg \varphi^*H$ is minimal among all $\varphi' \in \text{Hom}^{\text{free}}(\mathbb{P}^1, X)$. We call such $\varphi$ a *minimal free rational curve*.

The study of deformations of rational curves of minimal degree plays a vital role in the theory of rationally connected varieties by means of the Mori theory (and *varieties of minimal rational tangents*, see e.g. [Keb02, HM04]). The picture becomes slightly simpler if we assume that $T_X$ is nef. As we have already noted, conjecturally this condition should be equivalent to $X$ being a homogeneous space. When $T_X$ is nef, we denote $\text{RatCurves}_d^{\text{free}}(X)$ by $\text{RatCurves}_d(X)$, as all rational curves are free.

**Lemma 6.1.3 ([Kol96, II, Proposition 2.14.1]).** With notation as above, suppose that $T_X$ is nef and that minimal free rational curves are of degree $d$. Then $\text{RatCurves}_d(X)$ is proper.

**Proof.** By [Kol96, II, Theorem 2.15], the above definition of $\text{RatCurves}_d(X)$ coincides with [Kol96, II, Definition – Proposition 2.11] under the assumption that $T_X$ is nef, and hence $\text{RatCurves}_d(X)$ is proper by [Kol96, II, Proposition 2.14.1].

Finally, let us recall the celebrated result of Mori, which we present here in a slightly different from than that of [Kol96, V, Theorem 3.2] – for the proof of Proposition 6.3.2 we need a variant for minimal free rational curves (cf. [OW02, Families of curves, Theorem]).

**Theorem 6.1.4.** Let $X$ be a smooth projective Fano variety of dimension $n$ defined over an algebraically closed field $k$ and let $x \in X$ be a general point. Suppose that every rational curve through $x$ is free, and that every minimal free rational curve through $x$ is very free. Then $X \simeq \mathbb{P}^n$.

Every rational curve through $x$ is free provided that $k$ is of characteristic zero and $x$ is very general [Kol96, II, Theorem 3.11], or $X$ is $F$-liftable and $x \in X$ is general (Lemma 6.2.1(a)). Note that there exists a minimal free rational curve through a general point $x$ by [Kol96, II, Corollary 3.5.4.2].

**Proof.** A very free rational curve must be of degree at least $n+1$ (see [Kol96, V, Lemma 3.7.2]). Moreover, there exists a rational curve of degree at most $n+1$ through $x$ by bend-and-break (see [Kol96, V, Theorem 1.6.1]), hence minimal free rational curves are of degree $n+1$. Since they are very free by assumption, the rest of the proof follows [Kol96, V, Theorem 3.2] word for word. \qed

### 6.2. The sheaf of $\xi$-invariant forms and rational curves

In this subsection we study positivity conditions imposed on the tangent bundle by the existence of a Frobenius lifting. This is the main component of the proof of Theorem 1 in the Picard rank one case.

Let us fix a Frobenius lifting $(\pi, \tilde{F}_X)$ of a smooth $k$-scheme $X$ and consider the induced morphism $\xi: F^*\Omega^1_X \to \Omega^1_X$. Recall that it is injective, and hence generically an isomorphism, by Proposition 3.2.1. Let $U \subseteq X$ to be the maximal open subset where $\xi$ is an isomorphism.
The following simple lemma allows for the study of families of rational curves by using the sheaf of $\xi$-invariant forms denoted by $(\Omega_X^1)^{\xi}$.

**Lemma 6.2.1.** Let $\varphi: \mathbb{P}^1 \to X$ be a non-constant morphism such that $\varphi(\mathbb{P}^1) \cap U \neq \emptyset$. Then,

(a) $\varphi$ is free (cf. [Xin16, Proposition 5]),

(b) $\varphi^*(\Omega_X^1)^{\xi}$ contains a locally constant $\mathbb{F}_p$-subsheaf of rank $r = h^0(\mathbb{P}^1, \varphi^*(\Omega_X^1)^{\xi})$.

**Proof.** Since $\varphi^*(\Omega_X^1)^{\xi}$ is locally free and $\varphi^*(\xi): F^r \varphi^*(\Omega_X^1) \to \varphi^*(\Omega_X^1)^{\xi}$ is generically an isomorphism, $\varphi^*(\xi)$ must be an injection.

Write $\varphi^*(\Omega_X^1)^{\xi} = \bigoplus_{i=1}^n \mathcal{O}_\mathbb{P}^1(a_i)$, where $a_1 \geq a_2 \geq \cdots \geq a_n$. Since $\varphi^*(\xi)$ is an injection, it induces a non-zero morphism $\mathcal{O}_\mathbb{P}^1(pa_1) \to \mathcal{O}_\mathbb{P}^1(a_j)$ for some $1 \leq j \leq n$. This is only possible if $a_1 \leq 0$, and so $\varphi^*T_X$ is nef, concluding (a).

The morphism $\varphi^*(\xi)$ induces a Frobenius-linear automorphism of the space of global sections $H^0(\mathbb{P}^1, \varphi^*(\Omega_X^1)^{\xi})$, and so by taking $\varphi^*(\xi)$-fixed points

$\mathcal{G}_\varphi = H^0(\mathbb{P}^1, \varphi^*(\Omega_X^1)^{\xi})^{\varphi^*(\xi)} \otimes_{\mathbb{F}_p} (\mathbb{F}_p)_\mathbb{P}^1$

we get a constant $\mathbb{F}_p$-subsheaf of $\varphi^*(\Omega_X^1)^{\xi}$ of rank $r$, where $(\mathbb{F}_p)_{\mathbb{P}^1}$ is the constant sheaf with value $\mathbb{F}_p$ on $\mathbb{P}^1$. \hfill $\square$

As mentioned before, for any free rational curve $\varphi: \mathbb{P}^1 \to X$ we can write $\varphi^*(\Omega_X^1)^{\xi} = \bigoplus_{i=1}^n \mathcal{O}_\mathbb{P}^1(a_i) \oplus \mathcal{O}_\mathbb{P}^1^{nr}$, where $a_i < 0$. In general, the value of $r$ is upper-semicontinuous under deformations of $\varphi$. Here, we show that $r$ is invariant under deformations provided that $X$ is $F$-liftable and $\varphi(\mathbb{P}^1)$ intersects $U$.

**Proposition 6.2.2.** With notation as above, let $\varphi_i: \mathbb{P}^1 \to X$ for $i \in \{1, 2\}$ be two rational curves intersecting $U \subseteq X$ and lying in the same irreducible component $\mathcal{M} \subseteq \text{Hom}(\mathbb{P}^1, X)$. Then $h^0(\mathbb{P}^1, \varphi_1^*(\Omega_X^1)^{\xi}) = h^0(\mathbb{P}^1, \varphi_2^*(\Omega_X^1)^{\xi})$.

**Proof.** We have the following diagram

\[
\begin{array}{ccc}
\mathcal{M} \times \mathbb{P}^1 & \xrightarrow{\varphi} & X \\
\pi \downarrow & & \\
\mathcal{M} & & \\
\end{array}
\]

Replacing $\mathcal{M}$ by an open subset, we can assume that $\varphi((\{m\} \times \mathbb{P}^1))$ intersects $U$ for every closed point $m \in \mathcal{M}$. Pick any closed point $m \in \mathcal{M}$. Then, as in Lemma 6.2.1 we have

\[
h^0(\mathbb{P}^1, \varphi^*(\Omega_X^1)^{\xi}) = \lim_{\text{proj. lim}} h^0(\mathbb{P}^1, \varphi^*(\Omega_X^1)^{\xi}|_{\pi^{-1}(m)}) = \lim_{\text{proj. lim}} \dim_{\mathbb{F}_p} (\pi_* \varphi^*(\Omega_X^1)^{\xi}|_{\pi^{-1}(m)}),
\]

which is lower semi-continuous with respect to $m$ by Lemma 6.2.3. The last equality holds by the proper base change theorem. On the other hand, $h^0(\mathbb{P}^1, \varphi^*(\Omega_X^1)^{\xi})$ is upper semi-continuous with respect to $m$ by the semi-continuity theorem ([Har77, III, Theorem 12.8]). Hence, it is constant over $\mathcal{M}$. \hfill $\square$

We needed the following corollary of the proper base change theorem in the above proof.

**Lemma 6.2.3.** Let $f: X \to S$ be a proper morphism of schemes, let $g: U \to X$ be a separated étale morphism of finite type, and let $\mathcal{F}$ be the étale sheaf of sections of $g$. Then the function $\varphi: S \to \mathbb{Z}$ defined as

\[
\varphi(s) = \|(f_* \mathcal{F})_\varphi\|
\]

for any geometric point $\varphi$ over $s$ is lower semi-continuous.

**Proof.** By [SGA 4iii, Exposé IX, Corollaire 2.7.1] or [Sta14, Tag 03S8], $\mathcal{F}$ is constructible, and hence $f_* \mathcal{F}$ is constructible as well [SGA 4iii, Exposé XIV]. Therefore $\varphi$ is a constructible function on $S$, and hence to prove the assertion it is enough to show that if $s, \mu \in S$ are two points such that $s$ lies in the closure of $\mu$, then $\varphi(s) \leq \varphi(\mu)$.
Let \( \tilde{s} \) be a geometric point lying over \( s \) and let \( \overline{\pi} \) be a geometric point of the localization \( S_\langle \rangle \) lying over \( \mu \). Passing to stalks, we obtain the cospecialization map

\[ c_{\overline{\pi} \to \tilde{s}} : (f, \mathcal{F})_{\overline{\pi}} \to (f, \mathcal{F})_{\tilde{s}}. \]

It remains to show that \( c_{\overline{\pi} \to \tilde{s}} \) is injective. The proper base change theorem implies that \( (f, \mathcal{F})_{\overline{\pi}} \) is the set of sections of \( U \to X \) over \( X \times S \langle \rangle \). Take any two such sections \( u_1, u_2 : X \times S \langle \rangle \to U \), and suppose they are equal after restricting to \( X \times S \langle \overline{\pi} \rangle \to U \). Then \( u_1 = u_2 \), as \( U \to X \) is separated and \( S \langle \overline{\pi} \rangle \to S \langle \rangle \) has dense image. □

### 6.3. The Picard rank one case

In this subsection we prove Theorem 1 in the Picard rank one case (Proposition 6.3.2). Before proceeding with the proof we need the following result.

**Lemma 6.3.1.** Let \( X \) be a smooth projective \( F \)-liftable Fano variety over \( k \). Then \( X \) is simply connected and \( H^0(X, \Omega^1_X) = 0 \).

**Proof.** In order to prove that \( X \) is simply connected, let us consider an étale cover \( f : Y \to X \) of degree \( d \). By the Hirzebruch–Riemann–Roch theorem (see [Ful98, Corollary 15.2.1]), we have \( \chi(Y, \mathcal{O}_Y) = d \cdot \chi(X, \mathcal{O}_X) \). Since \( X \) is \( F \)-liftable, so is \( Y \) (Lemma 3.3.5), and hence Kodaira vanishing holds on both \( X \) and \( Y \) (see Theorem 3.2.4). Thus \( \chi(Y, \mathcal{O}_Y) = \chi(X, \mathcal{O}_X) = 1 \), which shows that \( d = 1 \) and \( X \) is simply connected.

Now, we show that \( H^0(X, \Omega^1_X) = 0 \). Since \( X \) is \( F \)-liftable, all of its global one-forms are closed by Proposition 3.2.1, and

\[ Z^1_X \simeq B^1_X \oplus \Omega^1_X, \]

where \( Z^1_X = \ker(d : F_r \Omega^1_X \to F_{r+1} \Omega^1_X) \) and \( B^1_X = F_r \mathcal{O}_X/\mathcal{O}_X \). In particular, the Cartier operator induces an isomorphism \( H^0(X, Z^1_X) \simeq H^0(X, \Omega^1_X) \). Therefore, the assumptions of [vdGK03, Proposition 4.3] are satisfied and we obtain

\[ H^0(X, \Omega^1_X) \simeq \text{Pic}(X)[p] \otimes \mathbb{Z} \ 	ext{k}. \]

We claim that \( \text{Pic}(X) \) is torsion free. Indeed, if \( L \) is a numerically trivial line bundle on \( X \), then by Kodaira vanishing we have

\[ H^i(X, L) = H^i(X, \omega_X \otimes \omega_X^{-1} \otimes L) = 0 \quad \text{for } i > 0, \]

and therefore \( h^0(X, L) = \chi(X, L) = \chi(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X) = 1 \). □

**Proposition 6.3.2.** Let \( X \) be a smooth projective Fano variety of dimension \( n \) defined over an algebraically closed field of positive characteristic. Suppose that \( T_X \) is nef, \( \rho(X) = 1 \), and \( X \) is \( F \)-liftable. Then \( X \simeq \mathbb{P}^n \).

The idea of the proof is the following. If \( X \not\cong \mathbb{P}^n \), then by Mori theory there exists a rational curve \( C \subseteq X \) such that \( T_X|_C \) is not very ample. In particular, \( h^0(C, \Omega^1_X|_C) > 0 \). Using the global lifting of Frobenius we show that the sections of \( \Omega^1_X|_C \), where \( C_t \) are deformations of \( C \), glue to global sections of \( \Omega^1_X \). This contradicts the fact that \( X \) is Fano (see Lemma 6.3.1).

**Proof.** Choose a general point \( x \in X \). If every minimal free rational curve \( \rho : \mathbb{P}^1 \to X \) passing through \( x \) is very free, then \( X \cong \mathbb{P}^n \) by Theorem 6.1.4. Suppose by contradiction that there exists such a non-very-free minimal \( \rho \) of degree \( d \). Then \( h^0(\mathbb{P}^1, \rho^0 \Omega^1_{\mathbb{P}^1}) = r \geq 1 \).

Let \( \mathcal{M} \) be the irreducible component of \( \text{RatCurves}_d(X) \) containing \( \rho \) (see §6.1 for the notation). By Lemma 6.1.3, we know that \( \mathcal{M} \) is proper and there exist morphisms

\[
\begin{array}{ccc}
\text{Univ} & \xrightarrow{\varphi} & X \\
\pi \downarrow & & \\
\mathcal{M} & & \\
\end{array}
\]

with \( \pi : \text{Univ} \to \mathcal{M} \) being a \( \mathbb{P}^1 \)-fibration, and \( \varphi \) being smooth. Since \( X \) is simply connected (see Lemma 6.3.1), we get that \( \varphi \) has connected fibers. Indeed, let \( \text{Univ} \to Y \to X \) be the
Stein factorization of \( \phi \). Since \( \phi \) is smooth and proper, the morphism \( Y \to X \) is finite étale (see [SGA 1, Exp. X, Prop. 1.2] or [EGA III], 7.8.10 (i)), and so it must be an isomorphism.

We fix a Frobenius lifting \((\tilde{X}, \tilde{F}_X)\) of \( X \) and consider the induced morphism \( \xi: F^*\Omega_X^1 \to \Omega_X^1 \). Let \((\Omega_X^1)^\xi\) be the sheaf of \( \xi \)-invariant forms and let \( U \subseteq X \) be the maximal open subset for which \( \xi|_U: F^*\Omega_U^1 \to \Omega_U^1 \) is an isomorphism.

Set \( \mathcal{M}^o = \pi(\varphi^{-1}(U)) \) to be the locus of all \( m: \mathbb{P}^1 \to X \) in \( \mathcal{M} \) whose image intersects \( U \). It is open as \( \pi \) is smooth. Take \( \text{Univ}^o = \pi^{-1}(\mathcal{M}^o) \) and let \( j: \text{Univ}^o \to \text{Univ} \) denote the inclusion. We define the following subsheaf of \( \varphi^*(\Omega_X^1)^\xi|_{\text{Univ}^o} \)
\[ \mathcal{G} = j^*\pi^*\varphi^*(\Omega_X^1)^\xi \]

Given \( m: \mathbb{P}^1 \to X \) in \( \mathcal{M}^o \), the proper base change theorem implies that \( \mathcal{G}|_{\pi^{-1}(m)} \) is isomorphic to the constant \( \mathbf{F}_p \)-sheaf with value \( H^0(\mathbb{P}^1, m^*(\Omega_X^1)^\xi) \). Lemma 6.2.1(b) and Proposition 6.2.2 show that \( \mathcal{G} \) is a locally constant \( \mathbf{F}_p \)-sheaf of rank \( r \). Indeed, \( (\pi_*\varphi^*(\Omega_X^1))_{|\mathcal{M}^o} \) is a locally free sheaf of rank \( r \) by [Har77, III, Corollary 12.9] and
\[ \mathcal{G} = \pi^*(\pi_*\varphi^*(\Omega_X^1))_{|\mathcal{M}^o} \]

We claim that codim \( \text{Univ} \setminus \text{Univ}^o \geq 2 \). In order to prove this it is enough to show that for an irreducible divisor \( D \subseteq X \setminus U \), the codimension of \( \varphi^{-1}(D) \setminus \text{Univ}^o \) in \( \text{Univ} \) is at least two. Since \( \varphi \) is smooth and has connected fibers, \( \varphi^{-1}(D) \) is irreducible, and hence it is enough to show that \( \varphi^{-1}(D) \cap \text{Univ}^o \neq \emptyset \). This follows from the fact that \( \rho(X) = 1 \). Indeed, for a general closed point \( m \in \mathcal{M} \), the curve \( \varphi(\pi^{-1}(m)) \) intersects \( D \), and so \( \pi^{-1}(m) \subseteq \text{Univ}^o \) intersects \( \varphi^{-1}(D) \).

By Zariski–Nagata purity, the sheaf \( j_*\mathcal{G} \) is a locally constant \( \mathbf{F}_p \)-sheaf of rank \( r \). Since \( \text{Univ} \) is normal and the complement of \( \text{Univ}^o \) is of codimension at least two, we have \( j_*j^*\varphi^*(\Omega_X^1) \simeq \varphi^*(\Omega_X^1) \). In particular,
\[ j_*\mathcal{G} \subseteq j_*j^*\varphi^*(\Omega_X^1)^\xi = (j_*j^*\varphi^*(\Omega_X^1))^{\varphi^*} \simeq \varphi^*(\Omega_X^1)^\xi. \]

Let \( T \) be a fiber of \( \varphi: \text{Univ} \to X \). Since \( j_*\mathcal{G}|_T \) is a locally constant subsheaf of the constant sheaf \( \varphi^*(\Omega_X^1)^\xi|_T \), we see that \( j_*\mathcal{G}|_T \) is constant. In particular, the proper base change theorem implies that \( \varphi_*j_*\mathcal{G} \) is a nonzero locally constant subsheaf of \( (\Omega_X^1)^\xi \). As \( X \) is simply connected, \( \varphi_*j_*\mathcal{G} \) is constant, and hence \( H^0(X, (\Omega_X^1)^\xi) \neq 0 \), contradicting Lemma 6.3.1.

\[ \square \]

### 6.4. The general case.
We recall some facts about homogeneous spaces over an algebraically closed field \( k \). A projective variety \( X \) over \( k \) is called homogeneous if its automorphism group \( \text{Aut}(X) \) acts transitively on \( X \). Note that since \( X \) is projective, \( \text{Aut}(X) \) is actually (the set of \( k \)-points of) a group scheme locally of finite type over \( k \), and in particular its connected component \( \text{Aut}^0(X) \) is a group scheme of finite type acting transitively on \( X \).

The decomposition theorem of Borel and Remmert [BR62] (see [SdS03, §5–6] for its extension to positive characteristic) states that \( X \) decomposes into a product
\[ (6.4.1) \quad X \simeq A \times G_1/P_1 \times \cdots \times G_r/P_r, \]
where \( A \) is an abelian variety, \( G_i \) are simple linear algebraic groups of adjoint type, and \( P_i \subseteq G_i \) are parabolic subgroup schemes (i.e., each \( P_i \) contains a Borel subgroup of \( G_i \)).

Let \( G \) be a simple linear algebraic group over \( k \). A choice of a Borel subgroup \( B \subseteq G \) and a maximal torus \( T \subseteq B \) gives a set \( D \) of simple roots of \( G \), which is the set of nodes of the Dynkin diagram of \( G \). Following the conventions of [Hum78, p. 58], we number them \( 1, \ldots, n \) where \( n \) is the rank of \( G \). Reduced parabolic subgroups of \( G \) containing \( B \) are in an inclusion-preserving bijection with subsets of \( D \). We can thus represent (possibly ambiguously) rational homogeneous spaces with reduced stabilizers by marked Dynkin diagrams, i.e., Dynkin diagrams with a chosen set of nodes. If \( \alpha \in D \) is a simple root, we denote by \( P(\alpha) \) the maximal parabolic subgroup corresponding to \( D \setminus \{\alpha\} \).
Lemma 6.4.1. Let $G$ be as above, and let $\alpha \in D = \{1, \ldots, n\}$ be a simple root. Suppose that $G/P(\alpha) \simeq P^r_k$ for some $r > 0$. Then one of the following holds:

(i) $r = n$, the group $G$ is of type $A_n$, and $\alpha = 1$ or $n$,
(ii) $r = 2n - 1 \geq 3$, the group $G$ is of type $C_n$, and $\alpha = 1$.

In other words, the Dynkin diagram and the node $\alpha$ are as shown below.

\[
\begin{array}{ccccccc}
\includegraphics[width=0.2\textwidth]{dynkin-diagram.png}
\end{array}
\]

Proof. Demazure [Dem77] showed that for most pairs $(G, P)$ of a simple group $G$ of adjoint type and a reduced parabolic subgroup $P$, the natural morphism $G \to \text{Aut}^0(G/P)$ is an isomorphism, and classified the exceptions. The only exception with $\text{Aut}^0(G/P) \simeq PGL_{r+1}(k)$ is (see op.cit., case (a) on p. 181) $G = PSp_r(k)$ and $P$ the stabilizer of a line, in which case $G/P \simeq P^r_k$ ($r$ has to be odd in this case). This is our case (ii). If (ii) does not hold, then $r = n$ and $G \simeq PGL_{n+1}(k)$, and hence $G/P(\alpha) \simeq \text{Gr}(\alpha, n+1)$. So $\dim G/P(\alpha) = (n+1-\alpha) = n$, and hence $\alpha = 1$ or $\alpha = n$.

Below, we denote by $F_{1,n}$ the incidence variety parametrizing partial flags $W_1 \subseteq W_n \subseteq k^{n+1}$ where $\dim W_1 = 1$ and $\dim W_n = n$. It is a hypersurface of degree $(1, 1)$ in $P^r_k \times P^n_k$, and as a homogeneous space it corresponds to the Dynkin diagram of type $A_n$ with all nodes except for the two endpoints $1, n$ marked.

Lemma 6.4.2. Let $G$ be a simple algebraic group over $k$ and let $P \subseteq G$ be a reduced parabolic subgroup. Suppose that for every maximal reduced parabolic $Q \subseteq G$ containing $P$, the homogeneous variety $G/Q$ is isomorphic to $P^r_k$ for some $r$. Then one of the following conditions holds:

(i) $G/P \simeq P^n_k$ for some $n$,
(ii) $G/P \simeq F_{1,n}$ for some $n$.

Proof. This follows from Lemma 6.4.1 and the classification of rational homogeneous spaces with reduced stabilizers.

Lemma 6.4.3. The incidence variety $F_{1,n}$ is not $F$-liftable for $n > 1$.

Proof. This is shown in [BTLM97, §4.2] by using Bott vanishing. Just for fun, we give an alternative proof, the idea of which is the following. The multi-homogeneous polynomial defining $X$ can be treated as a homogeneous polynomial in the usual sense, thus giving rise to a smooth homogeneous quadric hypersurface $\overline{X} \subseteq \mathbb{A}^{2n+2}$. We show that if $X$ is $F$-liftable, then so is $\overline{X}$, which contradicts [Zda17, Theorem 4.15].

Formally, we notice that by the short exact sequence for restrictions, we have

\[
H^0(X, \mathcal{O}_X(a, b)) = H^0(P^n \times P^n, \mathcal{O}_{P^n \times P^n}(a, b))|_X,
\]

and

\[
H^1(X, \mathcal{O}_X(a, b)) = 0,
\]

for $a, b \geq 0$. Moreover, $H^0(X, \mathcal{O}_X(a, b)) = 0$ if $a < 0$ or $b < 0$. Thus, by Proposition 3.3.9, if $X$ is $F$-liftable, then the affine cone $\overline{X} \subseteq \mathbb{A}^{n+1} \times \mathbb{A}^{n+1}$ of $X$ given by

\[
\overline{X} = \text{Spec} \bigoplus_{a,b \geq 0} H^0(X, \mathcal{O}_X(a, b)) \cong \text{Spec} k[x_0, \ldots, x_n, y_0, \ldots, y_n]/(\sum_{i=0}^{n} x_i y_i)
\]

is $F$-liftable as well, contradicting [Zda17, Theorem 4.15].

Theorem 6.4.4 ([LM97]). Let $G$ be a simple algebraic group over $k$, let $P \subseteq G$ be a parabolic subgroup scheme, let $X = G/P$, and let $Y = G/P_\text{red}$. If $X$ is Frobenius split, then $X \simeq Y$ as varieties.

We are ready to show the main theorem of this section.
Theorem 6.4.5. Let $X$ be a projective variety over $k$ whose automorphism group acts transitively on $X(k)$. Then the following are equivalent:

(i) $X$ is $F$-liftable,

(ii) $X \simeq A \times \mathbb{P}^1_k \times \ldots \times \mathbb{P}^n_k$ for some $n_1, \ldots, n_r$, where $A$ is an ordinary abelian variety,

(iii) the Albanese variety $A = \text{Alb}(X)$ is ordinary and the fibers of the Albanese map $\alpha_Y: X \to A$ are toric varieties.

Proof. Consider the decomposition (6.4.1). By Corollary 3.3.7, $X$ is $F$-liftable if and only if $A$ and $G_1/P_1, \ldots, G_r/P_r$ are. Moreover, $A$ is $F$-liftable if and only if it is ordinary, by Example 3.1.3. The Albanese map of $X$ is simply the projection to the first factor of the decomposition. Since the only homogeneous toric varieties are products of projective spaces, we see that (ii) and (iii) are equivalent and imply (i).

It remains to prove that (i) implies (ii). For this, we can assume that $X = G/P$ where $G$ is a simple linear algebraic group of adjoint type and $P \subseteq G$ is a parabolic subgroup scheme. Since $X$ is Frobenius split (Proposition 3.2.1(c)), Theorem 6.4.4 implies that $X \simeq G/P_{\text{red}}$, so we can assume that $P$ is reduced.

If $Q \subseteq G$ is a reduced maximal parabolic subgroup containing $P$, then $Z = G/Q$ inherits a Frobenius lifting from $X$ via the map $X = G/P \to G/Q = Z$ by Theorem 3.3.6(b), as the fibers are isomorphic to the rational homogeneous space $Q/P$ and $H^1(Q/P, \mathcal{O}_{Q/P}) = 0$ for $i > 0$. Now $Z$ is Fano, Pic $Z \simeq \mathbb{Z}$, and $T_Z$ is nef. By Proposition 6.3.2, $Z \simeq \mathbb{P}_k^n$ for some $n$. By Lemma 6.4.2, this implies that $X \simeq \mathbb{P}_k^n$ or $X$ is isomorphic to the incidence variety $F_{1,n}$. But $F_{1,n}$ is not $F$-liftable (Lemma 6.4.3), so $X \simeq \mathbb{P}_k^n$. $\square$

Remark 6.4.6. By Theorem 6.4.5 and the proof of Theorem 4.4.1, we get that Conjecture 2 holds if the target variety is a homogeneous space. This observation is not very interesting because this fact almost follows from [OW02]. What is missing is a direct proof that the incidence variety $F_{1,n}$ is not an image of a toric variety.

7. $F$-Liftability in Low Dimensions

This section pertains to the study of $F$-liftability of low dimensional smooth varieties. By means of the theory developed in Section 3, we classify $F$-liftable surfaces and $F$-liftable Fano threefolds from the Mori–Mukai list, confirming Conjecture 1 for these classes of varieties.

We work over an algebraically closed field $k$ of characteristic $p > 0$. Let $(\tilde{X}, \tilde{F}_X)$ be a Frobenius lifting of a smooth $k$-scheme $X$. As in §3.6, we have the associated effective $\mathbb{Q}$-divisor

$$\Delta_{\tilde{F}_X} = \frac{1}{p-1} \text{div}(\det(\xi_{\tilde{F}_X})) \sim_{\mathbb{Q}} -K_X.$$ 

A careful analysis of $\Delta_{\tilde{F}_X}$ plays a vital role in the proofs of the results of this section.

7.1. Surfaces. The goal of this subsection is to show Conjecture 1 for smooth surfaces. With notation as above, we define $D_{\tilde{F}_X} = |\Delta_{\tilde{F}_X}|$. Note that $D_{\tilde{F}_X}$ is reduced (see Remark 2.5.11). If $X$ is a toric variety with its standard Frobenius lifting (see Example 3.1.2), then $\Delta_{\tilde{F}_X} = D_{\tilde{F}_X}$ is the toric boundary (the complement of the open torus orbit) of $X$.

First, we tackle the case of rational surfaces.

Lemma 7.1.1. Let $X$ be a smooth surface over $k$, and let

$$\pi: Y = \text{Bl}_x X \to X$$

be the blow-up of $X$ at a closed point $x \in X$. Let $(\tilde{X}, \tilde{F}_X)$ and $(\tilde{Y}, \tilde{F}_Y)$ be Frobenius liftings of $X$ and $Y$, and let $\tilde{\pi}: \tilde{X} \to \tilde{Y}$ be a lifting of $\pi$ satisfying $\tilde{F}_X = \tilde{F}_Y$. Suppose that $\text{Supp} \Delta_{\tilde{F}_X}$ has simple normal crossings at $x$. Then $x \in \text{Sing} D_{\tilde{F}_X}$.

Readers familiar with the language of birational geometry may notice that this is a direct consequence of the fact that $x$ is a log canonical center of $(X, \Delta_{\tilde{F}_X})$ by Corollary 3.6.2(a)
(in fact, this shows that the above result is valid in higher dimensions for blow-ups along arbitrary smooth centers). We provide a more elementary explanation below.

**Proof.** For the exceptional divisor \( E = \text{Exc}(\pi) \), we have \( E \leq \Delta_{F_Y} \) by Corollary 3.6.2(a).

Since \( \pi_*\Delta_{F_Y} = \Delta_{F_X} \) (see Corollary 3.6.2(b)), the support of \( \pi^*\Delta_{F_X} - \Delta_{F_Y} \) is exceptional.

By definition, this \( \mathbb{Q} \)-divisor is linearly equivalent to \( K_{Y/X} \), which is equal to \( E \) by [Har77, Proposition V.3.3], and so

\[
\pi^*\Delta_{F_X} - \Delta_{F_Y} = E.
\]

In particular, \( \pi^*\Delta_{F_X} = 2E + \Delta' \), where \( E \not\subseteq \text{Supp} \Delta' \). As \( \text{Supp} \Delta_{F_X} \) has simple normal crossings at \( x \) and the coefficients of \( \Delta_{F_X} \) are at most one (see Remark 2.5.11), this is only possible if \( x \in \text{Sing} D_{F_X} \).

\[\square\]

**Remark 7.1.2.** Let \( f: (Y, D_Y) \to (X, D_X) \) be a toric morphism between two-dimensional toric pairs, which on the level of schemes is a blowing-up of a smooth point \( x \in X \). Then \( x \in \text{Sing}(D_X) \) and \( D_Y = f^{-1}D_X + \text{Exc}(f) \). Moreover, the converse is also true, that is the blowing-up of a smooth toric surface at toric points is a toric morphism.

Let us call a pair \( (X, D) \) of a normal variety \( X \) and a reduced effective divisor \( D \) on \( X \) sub-toric if \( X \) admits the structure of a toric variety such that \( D \) is invariant under the torus action. If \( D \) is the maximal invariant divisor, then we call \( (X, D) \) toric, see §2.1.

**Lemma 7.1.3.** Let \( F_n = P_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(n)) \) be the \( n \)-th Hirzebruch surface for \( n \geq 0 \). Then for every Frobenius lifting \( (\tilde{F}_n, \tilde{F}) \) of \( F_n \), the pair \( (\tilde{F}_n, \tilde{F}) \) is sub-toric. Moreover, if \( n = 0 \), then \( \text{Supp} \Delta_{\tilde{F}} \) has simple normal crossings, and if \( n > 0 \), then \( \text{Supp} \Delta_{\tilde{F}} \) has simple normal crossings at every point of the negative section \( C \subseteq F_n \).

Write \( F_n = P_{\mathbb{P}^1}(E) \) for \( E = \mathcal{O} \oplus \mathcal{O}(n) \). A choice of the splitting \( E \simeq \mathcal{O} \oplus \mathcal{O}(n) \) provides \( F_n \) with a natural toric structure for which the natural morphisms \( P_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(n)) \to F_n \) are toric. Thus \( (F_n, D) \) is a toric pair if and only if

- for \( n = 0 \), we have \( D = G_1 + G_1' + G_2 + G_2' \), where \( G_1, G_1', G_2, G_2' \) are distinct fibers of the two projections \( \pi_1, \pi_2: F_0 \to \mathbb{P}^1 \);
- for \( n > 0 \), we have \( D = C + C' + G_1 + G_2 \), where \( C \) is the unique negative section (corresponding to \( P_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(n)) \to F_n \), \( C' \) is a section disjoint from \( C \), and \( G_1, G_2 \) are two distinct fibers of the projection \( \pi: F_n \to \mathbb{P}^1 \).

**Proof.** Let us fix a Frobenius lifting \( (\tilde{F}_n, \tilde{F}) \). If \( n = 0 \), then, by Corollary 3.3.7, we have \( \Delta_{\tilde{F}} = \pi_1^*\Delta_1 + \pi_2^*\Delta_2 \), where \( \Delta_1 \) and \( \Delta_2 \) are effective \( \mathbb{Q} \)-divisors on \( \mathbb{P}^1 \), and so \( \text{Supp} \Delta_{\tilde{F}} \) is simple normal crossing. Since \( \omega_{\mathbb{P}^1} \simeq \mathcal{O}(\omega_{\mathbb{P}^1}(2, -2)) \), both \( [\Delta_1] \) and \( [\Delta_2] \) consist of at most two irreducible divisors, which concludes the proof.

If \( n > 0 \), then by Theorem 3.3.6(b.i) we get a compatible Frobenius lifting of \( \mathbb{P}^1 \) and so Corollary 3.6.2(c) implies that

\[
\begin{align*}
\Delta^h &\sim_{\mathbb{Q}} -K_{F_n/\mathbb{P}^1}, \\
\Delta^v &\sim_{\mathbb{Q}} -\pi^*K_{\mathbb{P}^1},
\end{align*}
\]

where \( \Delta^h \) and \( \Delta^v \) are the horizontal and the vertical part of \( \Delta_{\tilde{F}} \), respectively. Moreover, Corollary 3.6.2(a) gives \( \Delta^h = \Delta' + C \), where \( \Delta' \) is an effective \( \mathbb{Q} \)-divisor such that \( C \not\subseteq \text{Supp} \Delta' \).

As \( K_{F_n} + \Delta^v + \Delta' + C \sim_{\mathbb{Q}} 0 \), we have

\[
\Delta' \cdot C = -(K_{F_n} + C + \Delta^v) \cdot C = 2 - \Delta^v \cdot C = 0
\]

by adjunction, and so \( \Delta' \) is disjoint from \( C \). Given that \( \text{Supp} (\Delta^v + C) \) is simple normal crossing, so is \( \text{Supp} \Delta_{\tilde{F}} \) along \( C \). Moreover, for a fiber \( G \) of \( \pi \)

\[
\Delta' \cdot G = -(K_{F_n} + \Delta^v + C) \cdot G = -(K_{F_n} + G) \cdot G - (\Delta^v + C) \cdot G = 1,
\]

by adjunction as \( G^2 = 0 \), and so \( [\Delta'] \) is zero or is a single section disjoint from \( C \). Since \( [\Delta^v] \) consists of at most two distinct fibers, this concludes the proof of the lemma.

\[\square\]
Proposition 7.1.4. Let \( Y \) be a smooth projective rational surface. If \( Y \) is \( F \)-liftable, then it is toric.

Proof. Since \( \mathbb{P}^2 \) is toric, we can assume that \( Y \not\cong \mathbb{P}^2 \). Every smooth rational surface which is not isomorphic to \( \mathbb{P}^2 \) admits a birational morphism to a Hirzebruch surface \( \pi: Y \to \mathbb{F}_n \) for some \( n \geq 0 \), which factors into a sequence of monoidal transformations

\[
Y = X_m \xrightarrow{\pi_{m-1}} X_{m-1} \to \cdots \to X_1 \xrightarrow{\pi_0} X_0 = \mathbb{F}_n, \quad X_{i+1} = \text{Bl}_{x_i} X_i.
\]

We assume that \( n \) is minimal among such. It follows that if \( n > 0 \), then \( \pi(\text{Exc}(\pi)) \subseteq C \), where \( C \subseteq \mathbb{F}_n \) is the negative section. Indeed, the blow-up \( \text{Bl}_x \mathbb{F}_n \) at any \( x \notin C \) admits a morphism to \( \mathbb{F}_{n-1} \) constructed by contracting the strict transform of the fiber through \( x \) of the natural projection \( \mathbb{F}_n \to \mathbb{P}^1 \).

Let \( (\tilde{X}, \tilde{F}_Y) \) be a Frobenius lifting of \( Y = X_m \). By Theorem 3.3.6(b), there exist Frobenius liftings \( (\tilde{X}_i, \tilde{F}_i) \) of \( X_i \) for \( 0 \leq i \leq m \), and liftings \( \tilde{\pi}_i: \tilde{X}_{i+1} \to \tilde{X}_i \) such that \( \tilde{F}_i \circ \tilde{\pi}_i = \tilde{\pi}_i \circ \tilde{F}_{i+1} \).

By Lemma 7.1.3 we know that \( (X_0, D_{\tilde{F}_0}) \) is sub-toric and \( \Delta_{\tilde{F}_0} \) has simple normal crossings at \( \pi(\text{Exc}(\pi)) \). Therefore, for every \( i > 0 \) the \( \mathbb{Q} \)-divisor \( \Delta_{\tilde{F}_i} \) has simple normal crossings at \( x_i \in X_i \) (it is contained in the union of the exceptional locus of \( \pi \) and the support of the strict transform of \( \Delta_{\tilde{F}_0} \) by Corollary 3.6.2(b)).

By induction we can show that \( (X_i, D_{\tilde{F}_i}) \) is sub-toric for every \( 0 \leq i \leq m \). Indeed, if \( (X_{i-1}, D_{\tilde{F}_{i-1}}) \) is sub-toric, then by Remark 7.1.2 it is enough to show that \( x_{i-1} \in \text{Sing} D_{\tilde{F}_{i-1}} \), which follows by Lemma 7.1.1. This concludes the proof. \( \square \)

Remark 7.1.5. In the course of the proof, we showed that if \( (\tilde{X}, \tilde{F}_X) \) is a Frobenius lifting of a smooth projective rational surface \( X \not\cong \mathbb{P}^2 \), then \( (X, |\Delta_{\tilde{F}_X}|) \) is sub-toric. This is false for some liftings of Frobenius on \( \mathbb{P}^2 \).

We now turn our attention to ruled surfaces. We say that a rank two vector bundle \( E \) on a curve \( C \) is normalized if \( H^0(C, E) \neq 0 \) and \( H^0(C, E \otimes \mathcal{L}) = 0 \) for every line bundle \( \mathcal{L} \) such that \( \deg \mathcal{L} < 0 \). Given a ruled surface \( X = \mathbb{P}_C(E) \) we can assume that \( E \) is normalized by replacing \( E \) with \( E \otimes \mathcal{L} \) for some line bundle \( \mathcal{L} \).

Proposition 7.1.6. Let \( X = \mathbb{P}_C(E) \) be a smooth projective ruled surface for a normalized rank two vector bundle \( E \) on an ordinary elliptic curve \( C \). Then \( X \) is \( F \)-liftable if and only if \( E \) is not a non-split extension of \( \mathcal{O}_C \) with itself.

Proof. If \( E \) is decomposable (that is, a direct sum of two line bundles), then \( X \) is \( F \)-liftable by Example 3.1.5. Hence, we can assume that \( E \) is indecomposable. By [Har77, Theorem V.2.15], there are only two such ruled surfaces corresponding to \( E \) being a non-split extension of \( \mathcal{O}_C \) with \( \mathcal{O}_C(c) \) where \( c \in C \), and \( E \) being a non-split extension of \( \mathcal{O}_C \) with itself.

In the former case, \( X \cong \text{Sym}^2(C) \) (see for instance [Gar06, Section 6]), and it is easy to see that it is \( F \)-liftable. Indeed, let \( (\tilde{C}, \tilde{F}_C) \) be the canonical Frobenius lifting of \( C \) (see Example 3.1.3), where \( \tilde{F}_C: \tilde{C} \to \tilde{C} \). Then \( \tilde{F}_C \times \tilde{F}_C: \tilde{C} \times \tilde{C} \to \tilde{C} \times \tilde{C} \) is equivariant under the natural \( \mathbb{Z}/2\mathbb{Z} \) action by swapping the coordinates, and so it descends to \( \text{Sym}^2(\tilde{C}) \to \text{Sym}^2(C) \).

Therefore, we are left to show that \( X \) is not \( F \)-liftable when \( E \) is a non-split extension of \( \mathcal{O}_C \) with itself. By contradiction assume that it does admit a Frobenius lifting \( (X, \tilde{F}_X) \). By Lemma 7.1.8, we know that \( X \) is a quotient of \( Y = C' \times \mathbb{P}_p \) by \( \mathbb{F}_p \) acting independently on \( C' \) and \( \mathbb{P}_p \), where \( C' \) is the Frobenius twist of \( C \), and the action on \( (x : y) \in \mathbb{P}_p(k) \) is defined as \( (x : y) \mapsto (x + ly : y) \) for a fixed \( l \in \mathbb{F}_p \). This is illustrated by the following diagram:

\[
\begin{array}{ccc}
\mathbb{P}_1 & \xrightarrow{\rho} & Y \\
\downarrow{\pi'} & & \downarrow{\pi} \\
C' & \xrightarrow{\nu} & C.
\end{array}
\]
Since $V'$ is étale, Lemma 3.3.5 implies that $Y$ admits a Frobenius lifting $(\tilde{Y}, \tilde{F}_Y)$ such that $\Delta_{\tilde{F}_Y} = V''\Delta_{\tilde{F}_X}$, where $\Delta_{\tilde{F}_Y}$ and $\Delta_{\tilde{F}_X}$ are the $\mathbb{Q}$-divisors associated to $\tilde{F}_Y$ and $\tilde{F}_X$, respectively. In particular, $\Delta_{\tilde{F}_Y}$ is $F_p$-invariant.

As $\Delta_{\tilde{F}_Y} \sim \mathbb{Q} - K_{C' \times \mathbb{P}^1} = -\rho^*K_{\mathbb{P}^1}$, we have $\Delta_{\tilde{F}_Y} = \rho^*T$ for some $\mathbb{Q}$-divisor $T$ on $\mathbb{P}^1$. Let $G$ be a fiber of $\pi$ over a general point $c \in C$ and $\Delta = \Delta_{\tilde{F}_X}|_G$. Since $\Delta_{\tilde{F}_Y}$ is $F_p$-invariant and it is a pullback from $\mathbb{P}^1$, we get that $\Delta$ is invariant under the action of $F_p$ on $G \simeq \mathbb{P}^1$.

By Corollary 3.6.2(c) and Remark 2.5.10 applied to $\pi'$, we get that $\Delta$ is the associated $\mathbb{Q}$-divisor of an $F$-splitting of $\mathbb{P}^1$. This contradicts Lemma 7.1.7. □

We needed the following two lemmas in the proof of the above proposition.

**Lemma 7.1.7.** There does not exist an $F_p$-invariant $F$-splitting of $\mathbb{P}^1$, where $F_p$ acts on $\mathbb{P}^1$ via translations, that is $(x : y) \mapsto (x + ly : y)$ for $l \in F_p$.\n
Note that an $F$-splitting is invariant under an action of a group if and only if the corresponding $\mathbb{Q}$-divisor is invariant.

**Proof.** Assume by contradiction that such an $F$-splitting exists and let $\Delta$ be the corresponding $\mathbb{Q}$-divisor (see Proposition 2.5.9). By the definition of $\Delta$, we get that $(p - 1)\Delta$ is an effective integral divisor and $\deg \Delta = 2$. Furthermore, the coefficients of $\Delta$ are at most one (see Remark 2.5.11). Since each orbit of the action of $F_p$ on $\mathbb{P}^1$ is of length $p$ except for the one of the fixed point $\infty \in \mathbb{P}^1$, the only $\mathbb{Q}$-divisor satisfying the aforementioned properties is

$$\Delta = \left(1 - \frac{1}{p - 1}\right)(\infty) + \sum_{i=0}^{p-1} \frac{1}{p - 1}(x_i),$$

where $x_i = x_0 + i \in A^1(k)$. Up to an action of an automorphism we can assume that $x_0 = 0$. This yields a contradiction, because $\mathbb{P}^1$ cannot be $F$-split with such an associated $\mathbb{Q}$-divisor $\Delta$ by [CGS16, Example 3.4] and [SS10, Proposition 5.3 (2)]. One can check that the trace of an $F$-splitting of $\mathbb{P}^1$ cannot be equal to $(p - 1)\Delta$ directly by noticing that $x(x - y) \cdots (x - (p-1)y)p^{p-2}$ has coefficient zero at $xp^{p-1}y^{p-1}$ (see [BK05, Theorem 1.3.8]). □

**Lemma 7.1.8.** Let $C$ be an ordinary elliptic curve, and let $C'$ be the Frobenius twist of $C$. Consider the action of $F_p$ on $C' \times \mathbb{P}^1$ with $l \in F_p$ acting as

$$(c, (x : y)) \mapsto (c + \alpha l, (x + ly : y)),$$

for $c \in C'$ and $(x : y) \in \mathbb{P}^1$, where $\alpha \in C'[p]$ is a fixed $p$-torsion point. Let $X$ be the quotient $(C' \times \mathbb{P}^1)/F_p$. Then $X \simeq P_C(E)$ for $E$ being a non-split extension of $\mathcal{O}_C$ with itself.

**Proof.** By definition we have the following Cartesian diagram

\[
\begin{array}{ccc}
C' \times \mathbb{P}^1 & \xrightarrow{V'} & X \\
\pi' \downarrow & & \downarrow \pi \\
C' & \xrightarrow{V} & C,
\end{array}
\]

where $V$ is the quotient by $C'[p]$. Note that $C' \times \mathbb{P}^1 = P_{C'}(E')$, where $E'$ is a vector bundle sitting inside a split (but not $F_p$-equivariantly so) extension of $F_p$-equivariant sheaves

$$0 \rightarrow \mathcal{O}_{C'} \rightarrow E' \rightarrow \mathcal{O}_{C'} \rightarrow 0,$$

with $E' \rightarrow \mathcal{O}_{C'}$ corresponding to the $F_p$-equivariant section $C' \times (1 : 0)$ of $\pi'$. Therefore, $X = P_C(E)$ for a vector bundle $E$ on $C$ such that $V^*E$ is $F_p$-equivariantly isomorphic to $E'$, and the above short exact sequence descends to $C$ making $E$ a non-split extension of $\mathcal{O}_C$ by itself. □

Having handled rational and ruled surfaces, we are ready to proceed to the general case.
Theorem 7.1.9. Let $X$ be a smooth projective surface over $k$. Then $X$ is $F$-liftable if and only if $X$ is

(i) an ordinary abelian surface,
(ii) a hyperelliptic surface being a quotient of a product of two ordinary elliptic curves,
(iii) a ruled surface $P_C(E)$ for a normalized rank two vector bundle $E$ over an ordinary elliptic curve $C$ except when $E$ is a non-trivial extension of $O_C$ with itself, or
(iv) a toric surface.

In particular, Conjecture 1 is true for surfaces.

Proof. If $X$ is $F$-liftable, then it is $F$-split and $\omega_X^{-p}$ is effective (see Proposition 3.2.1). In particular, we only need to consider the case of $\kappa(X) \leq 0$. If $\kappa(X) = 0$, then $K_X$ is $\mathbb{Q}$-effective, and hence $\omega_X^{-1}$ is trivial and $X$ is minimal. In this case, the theorem follows from [Xin16, Theorem 1], but for the convenience of the reader we present a simplified argument.

When $K_X$ is torsion, $X$ is $F$-liftable if and only if it is a quotient of an ordinary abelian surface by a free action of a finite group (see [MS87, Theorem 2] and Proposition 3.3.1(c)). By Hirzebruch–Riemann–Roch, such quotients satisfy $\chi(X, \mathcal{O}_X) = 0$, and so by the classification of surfaces they are abelian, hyperelliptic, or quasi-hyperelliptic. We can exclude the surfaces of the latter type as they contain rational curves (see [Lie13, §7]): indeed, if $\varphi: P^1 \to X$ is non-constant, then there exists an injection $TP^1 \to \varphi^*T_X$, so $T_X$ cannot be étale trivializable (cf. [Xin16, Proposition 5]). Hyperelliptic surfaces are étale quotients of products of elliptic curves $E_1 \times E_0$, except for case a3) which is an étale quotient of an abelian surface $A = E_1 \times E_0/\mu_2$ (see [BM77, p. 37] for the notation). Therefore they are $F$-liftable if and only if $E_1$ and $E_0$ are ordinary, by [MS87, Theorem 2] and Lemma 3.3.5. Note that $A$ is ordinary if and only if $E_1 \times E_0$ is, as they are isogeneous to each other. This concludes our analysis of Kodaira dimension zero.

As of now, we assume $\kappa(X) = -\infty$. If $X$ is of type (iii) or (iv), then it is $F$-liftable by Example 3.1.2 and Proposition 7.1.6. To conclude, we assume that $X$ is $F$-liftable and show that $X$ is of type (iii) or (iv). If $X$ is rational, then it is toric by Proposition 7.1.4. Thus, we can assume that there exists a birational morphism $f: X \to P_C(E)$, where $P_C(E)$ is a ruled surface over $C \neq P^1$. By Theorem 3.3.6(b), the curve $C$ is $F$-liftable, and so it is an ordinary elliptic curve.

We claim that $X \simeq P_C(E)$. Otherwise, $f$ factors through a monoidal transformation $\pi: \text{Bl}_z P_C(E) \to P_C(E)$, where $z \in P_C(E)$. In particular, Theorem 3.3.6(b) implies that $Y = \text{Bl}_z P_C(E)$ is $F$-liftable, which is impossible. Indeed, if $Y$ is $F$-liftable, then so is $(Y, \text{Exc}\, \pi)$ by Lemma 3.4.2, and hence $Y$ is $F$-split compatibly with $\text{Exc}\, \pi$ by Variant 3.2.2. Therefore, we get an $F$-splitting on $C$ compatible with a point (see [BK05, Lemma 1.1.8(ii)]), which is impossible as $C$ is an elliptic curve.

Since we know that $X$ is a ruled surfaces over an ordinary elliptic curve, the theorem follows from Proposition 7.1.6.

Remark 7.1.10. The $F$-liftability of minimal surfaces has been considered in [Xin16]. Note that our results do not agree when $\kappa(X) = -\infty$, as [Xin16, Theorem 1 (2b)] claims that all ruled surfaces over ordinary elliptic curves are $F$-liftable. It seems to us that the gluing argument used at the end of the proof of Proposition 9 in op.cit. is incomplete, as it is unclear why $h$ extends to a regular function over $V$.

7.2. Fano threefolds. In this subsection, we show that Conjecture 1 is valid for smooth Fano threefolds from the Mori–Mukai list defined over an algebraically closed field $k$ of characteristic $p > 0$. Note that we do not show that the conjecture holds for all smooth Fano threefolds over $k$. Indeed, the full Mori–Mukai classification is only known to be true in characteristic zero. However, $F$-liftable smooth Fano varieties in positive characteristic admit a unique lifting to characteristic zero by Bott vanishing (see Theorem 3.2.4 and Corollary 3.2.5). Given that, we believe that our results could be extended to all smooth Fano threefolds over $k$ with some additional work.
Although a proof based on the classification might seem tedious and not very useful, we embarked on this task for two reasons. First, the study conducted in this section confirms our belief that Frobenius liftability is very rare among smooth projective varieties, and provides substantial empirical evidence for the validity of Conjecture 1. But what is even more important is that the examples analyzed in this section were a source of inspiration in our study of general properties of $F$-liftable smooth projective varieties.

The main reference for the Mori-Mukai classification used in this section is [Sha99, Tables §12.3-§12.6]. We denote a Fano threefold of Picard rank $\rho$ and whose number in the tables is $n$ as $M-M \rho.n$ (for example, $M-M$ 2.12).

**Remark 7.2.1.** Let $f: Y \to X$ be a proper morphism between smooth varieties such that $Rf_*\mathcal{O}_Y = \mathcal{O}_X$. By Theorem 3.3.6(b), if $Y$ is $F$-liftable, then so is $X$. This observation significantly reduces the number of cases of smooth Fano threefolds which need to be considered. Note that $Rf_*\mathcal{O}_Y = \mathcal{O}_X$ when $f$ is birational, by [CR15, Theorem 1.1].

We start by giving main examples of techniques we apply to deal with $F$-liftableability of Fano threefolds.

**Rigidity and Bott vanishing.** By Theorem 3.2.4 and Corollary 3.2.5, smooth $F$-liftable Fano varieties satisfy Bott vanishing, and hence they are rigid. Note that the majority of non-toric smooth Fano threefolds are not rigid (see the proof of Theorem 7.2.9).

The following lemma shows that Bott vanishing does not hold for a certain Fano threefold.

**Lemma 7.2.2** (M–M 1.16). Let $X = \text{Gr}(2,5) \cap \mathbb{P}^6$, where $\text{Gr}(2,5) \subseteq \mathbb{P}^6$ is the Grassmannian of planes in $\mathbb{P}^5$ embedded in $\mathbb{P}^6$ by the Plücker embedding and $\mathbb{P}^6 \subseteq \mathbb{P}^9$ is a general linear subspace. Then $H^1(X, \Omega^2_X(1)) \neq 0$, where $\mathcal{O}_X(1)$ is the restriction of $\mathcal{O}_{\mathbb{P}^9}(1)$ to $X$.

**Proof.** The setting admits a natural lifting to characteristic zero, and by the semicontinuity theorem, it is enough to show that the required non-vanishing holds for this lifting. Therefore, we can assume that $X$ is defined over $\mathbb{C}$.

We have $\omega_{\text{Gr}(2,5)} \simeq \mathcal{O}_{\text{Gr}(2,5)}(-5)$, hence $\omega_X \simeq \mathcal{O}_X(-2)$ and therefore $H^1(X, \Omega^2_X(1)) = H^1(X, T_X(-1))$. First, we prove that $H^i(X, T_{\text{Gr}(2,5)}(-1)|_X) = 0$ for all $i$. Consider the Koszul resolution

$$0 \to \mathcal{O}_{\text{Gr}(2,5)}(-3) \to \mathcal{O}_{\text{Gr}(2,5)}(-2)^{\oplus 3} \to \mathcal{O}_{\text{Gr}(2,5)}(-1)^{\oplus 3} \to \mathcal{O}_{\text{Gr}(2,5)} \to \mathcal{O}_X \to 0,$$

tensor it by $T_{\text{Gr}(2,5)}(-1)$ to get

$$0 \to T_{\text{Gr}(2,5)}(-4) \to T_{\text{Gr}(2,5)}(-3)^{\oplus 3} \to T_{\text{Gr}(2,5)}(-2)^{\oplus 3} \to T_{\text{Gr}(2,5)}(-1) \to T_{\text{Gr}(2,5)}(-1)|_X \to 0.$$

By [Sno86, Theorem, p.171, (3)], all the cohomology groups of $T_{\text{Gr}(2,5)}(-k) \simeq \Omega^2_{\text{Gr}(2,5)}(5-k)$ vanish for $1 \leq k \leq 4$, and so the above exact sequence shows that the same holds for $T_{\text{Gr}(2,5)}(-1)|_X$.

The dual of the conormal exact sequence tensored by $\mathcal{O}_X(-1)$ is

$$0 \to T_X(-1) \to T_{\text{Gr}(2,5)}(-1)|_X \to \mathcal{O}_X^{\oplus 3} \to 0.$$

Since the cohomology groups of the middle sheaf vanish, we get $H^1(T_X(-1)) = H^0(X, \mathcal{O}_X^{\oplus 3}) \neq 0$.

**Blow-ups.** We show that some Fano threefolds constructed as blow-ups are not $F$-liftable. From the viewpoint of our research, smooth Fano threefold 2.27 (see Lemma 7.2.7) was the most intriguing. The study of this rigid Fano variety motivated many results in Section 3.

If a blow-up of a smooth variety $X$ along a smooth locus $V \subset X$ is $F$-liftable, so is $X$ by Remark 7.2.1. Moreover, $V$ must be $F$-liftable as well by results of §3.4.

**Example 7.2.3** (M–M 2.12). Let $X$ be the blow-up of $\mathbb{P}^3$ along a smooth curve $C$ of degree six and genus three. Then $X$ is not $F$-liftable, because otherwise $(\mathbb{P}^3, C)$ would be $F$-liftable by Lemma 3.4.2 and Lemma 3.4.4. In particular, $C$ would be $F$-liftable, which is impossible
as it is of general type. One can also deduce non-$F$-liftability of $X$ by noticing that it is not rigid.

In many cases, a Frobenius lifting of a product of varieties must be a product of Frobenius liftings (see Corollary 3.3.7).

**Example 7.2.4** (M–M 4.6). Let $X$ be the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along the tridiagonal curve

$$C = \{ (x, x, x) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \mid x \in \mathbb{P}^1 \}.$$ 

The variety $X$ cannot be $F$-liftable, because then $(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, C)$ would admit a Frobenius lifting by Lemma 3.4.2 and Lemma 3.4.4, which is impossible by Proposition 3.4.5.

In order to be able to deal with the next example, we need the following lemma.

**Lemma 7.2.5.** Let $D \subseteq \mathbb{P}^2$ be a smooth conic. Then $(\mathbb{P}^2, D)$ is not $F$-liftable.

**Proof.** Assume by contradiction that $(\mathbb{P}^2, D)$ is $F$-liftable and fix a Frobenius lifting. Let $\xi : F^* \Omega_{\mathbb{P}^2}(\log D) \to \Omega_{\mathbb{P}^2}(\log D)$ be the associated morphism constructed in Section 3.2 (see Variant 3.2.2). Choose a general point $x \in D$ and take $C \subseteq \mathbb{P}^2$ to be the line tangent to $D$ at $x$. Since $x$ is general and $\xi$ is generically an isomorphism (see Proposition 3.2.1), we get that $\xi|_C$ is injective. In particular, if $\Omega_{\mathbb{P}^2}(\log D)|_C = \mathcal{O}_C(a) \oplus \mathcal{O}_C(b)$ for some $a, b \in \mathbb{Z}$, then $a, b \leq 0$ (see the proof of Lemma 6.2.1(a)).

We will show that $\Omega_{\mathbb{P}^2}(\log D)|_C = \mathcal{O}_C(-2) \oplus \mathcal{O}_C(1)$ yielding a contradiction (cf. [Xin16, Lemma 4]). To this end, choose a standard affine chart on $\mathbb{A}^2 \subseteq \mathbb{P}^2$ in which $C$ is described as $y = 0$, and $D$ as $y - x^2 = 0$. In the chart $\mathbb{A}^2 \cap C$, we get that $\Omega^1_{\mathbb{P}^2}(\log D)|_C$ is generated by

$$dx \quad \text{and} \quad \frac{dy - x^2}{y - x^2} = -\frac{dy}{x^2} + \frac{2dx}{x}.$$ 

Now, we take the chart of $\mathbb{P}^2$ having coordinates $\frac{1}{x}$ and $\frac{y}{x}$. In this chart $C$ and $D$ are disjoint, and so in the restriction of this chart to $C$ the vector bundle $\Omega^1_{\mathbb{P}^2}(\log D)|_C$ is generated by

$$d\left(\frac{1}{x}\right) = -\frac{1}{x^2}dx \quad \text{and} \quad d\left(\frac{y}{x}\right) = \frac{dy}{x} = 2dx + x \left(\frac{dy}{x^2} - \frac{2dx}{x}\right).$$ 

Therewith, the coordinate-change matrix is:

$$\begin{bmatrix} -\frac{1}{x^2} & 0 \\ 2 & -x \end{bmatrix},$$

and so $\Omega^1_{\mathbb{P}^2}(\log D)|_C = \mathcal{O}_C(-2) \oplus \mathcal{O}_C(1)$.

**Example 7.2.6** (M–M 2.30). Let $X = \text{Bl}_C \mathbb{P}^3$ be a blow-up of $\mathbb{P}^3$ along a conic $C$ contained in a hyperplane $H \subseteq \mathbb{P}^3$. We show that $X$ is not $F$-liftable by assuming otherwise and inferring a contradiction.

Let $\overline{H} \simeq \mathbb{P}^2$ be the strict transform of $H$ on $X$, and let $E$ be the exceptional divisor of the blow-up. By Lemma 3.4.2, we get that $(X, \overline{H} + E)$ admits a Frobenius lifting, and so $(\overline{H}, \overline{H} \cap E)$ is $F$-liftable by Lemma 3.4.1. Since $C \simeq \overline{H} \cap E$ is a conic, this contradicts Lemma 7.2.5.

A surprising property of $F$-liftability is that if $X$ is $F$-liftable and $f : Y \to X$ is a smooth morphism such that $Rf_* \mathcal{O}_Y = \mathcal{O}_X$, then $f$ is relatively $F$-split. This is not true in general, if we assume that $X$ is only $F$-split.

**Lemma 7.2.7** (M–M 2.27). Let $C \subseteq \mathbb{P}^3$ be the twisted cubic, that is, the image of $\mathbb{P}^1$ under the embedding given by the full linear system $|\mathcal{O}_{\mathbb{P}^1}(3)|$. Then the blow-up $Y = \text{Bl}_C \mathbb{P}^3$ is not $F$-liftable.
Proof. Let \( E \) be the exceptional divisor of the blow-up \( \pi : Y \to \mathbb{P}^3 \). Firstly, we observe that by [SW90, Application 1, page 299] \( Y \) is isomorphic to a projective bundle \( \mathbb{P}(\mathcal{E}) \) over \( \mathbb{P}^2 \) for some non-split rank two vector bundle \( \mathcal{E} \). The morphism \( f : Y \to \mathbb{P}^2 \) is given by a pencil of quadrics in \( \mathbb{P}^3 \) containing \( C \).

If \( Y \) is \( F \)-liftable, then Corollary 3.6.2(a) and (c) imply that \(- (K_{Y/P^2} + E)\) is \( \mathbb{Q} \)-linearly equivalent to an effective \( \mathbb{Q} \)-divisor. In what follows, we show that this is not true. By the construction of \( f \), we have \( Q = f^{-1}(L) \) for some line \( L \subseteq \mathbb{P}^2 \), where \( Q \) is the strict transform of a quadric in \( \mathbb{P}^3 \) containing \( C \). Since \( \omega_Y \cong \mathcal{O}_Y(E) \otimes \pi^* \mathcal{O}_{\mathbb{P}^3}(-4) \), we have \( K_Y \sim -2Q - E \), and so:

\[
K_{Y/P^2} + E \sim -2Q + 3f^{-1}(L) \sim Q,
\]

which is not \( \mathbb{Q} \)-linearly equivalent to an anti-effective \( \mathbb{Q} \)-divisor. \( \square \)

Remark 7.2.8. Using Corollary 3.5.3 and a more intricate version of the above argument one can show that a rank two vector bundle \( \mathcal{E} \) on \( \mathbb{P}^n \) is decomposable if and only if \( \mathbb{P}(\mathcal{E}) \) is \( F \)-liftable. This seems intriguing from the viewpoint of Hartshorne’s conjecture predicting that all such vector bundles are decomposable when \( n \geq 7 \).

\( F \)-liftable smooth Fano threefolds are toric. Summing up the above results, we can show the following.

**Theorem 7.2.9.** Let \( X \) be a smooth projective Fano threefold from the Mori–Mukai list, defined over an algebraically closed field \( k \) of characteristic \( p > 0 \). Then \( X \) is \( F \)-liftable if and only if \( X \) is toric.

Proof. We denote a Fano threefold of Picard rank \( \rho \) and whose number in [Sha99, Tables §12.2-§12.6] is \( n \) as \( \rho.n \). We recommend the reader to look at the tables in order to be able to follow the proof.

First, smooth Fano threefold 2.27 is not \( F \)-liftable by Lemma 7.2.7, and Fano threefold 2.32 is not \( F \)-liftable by Lemma 6.4.3.

| Fano threefolds                        | Mori–Mukai list |
|---------------------------------------|-----------------|
| Non-rigid                             | blow-up of a non-rigid curve | 2.28 |
|                                       | negative virtual dimension | 1.1–1.14, 2.1–2.25, 3.1–3.12, 4.1, 4.2 |
| Blow-up on a product                  | 3.17, 3.21, 4.3, 4.6, 4.8 |
| Violating Bott vanishing              | 1.15, 1.16 |
| Inducing a lifting of Frobenius on \( \mathbb{P}^2 \) compatible with a conic | 2.30, 3.22 |
| Mapping to a Frobenius non-liftable variety | 2.26, 2.29, 2.31, 3.13–3.16, 3.18–3.20, 3.23, 3.24, 4.4, 4.5, 4.7, 5.1, 5.4–5.8 |
| Toric                                | 2.33–2.36, 3.25–3.31, 4.9–4.12, 5.2, 5.3 |
| Other                                | 2.27, 2.32 |

**Figure 1.** Fano threefolds.
Non-rigid. The Hirzebruch–Riemann–Roch formula for threefolds reads
\[ \chi(X, E) = \frac{1}{24} \text{rk}(E)c_1(T_X)c_2(T_X) + \frac{1}{12}c_1(E)\left(c_1(T_X)^2 + c_2(T_X)\right) \\
+ \frac{1}{4}c_1(T_X)\left(c_1(E)^2 - 2c_2(E)\right) + \frac{1}{6}(c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E)) . \]
In particular, \( \chi(X, \mathcal{O}_X) = c_1(T_X)c_2(T_X)/24 \); on the other hand, \( \chi(X, \mathcal{O}_X) = 1 \) since \( X \) satisfies Kodaira vanishing, so \( c_1(T_X)c_2(T_X) = 24 \). The number \( c_3(T_X) \) equals the topological Euler characteristic, \( 2 + 2\rho(X) - b_3(X) \). We deduce the following formula
\[ \chi(X, T_X) = \frac{1}{2}(-K_X)^3 - 18 + \rho(X) - \frac{1}{2}b_3(X) . \]
If \( X \) is an \( F \)-liftable Fano threefold, then by Bott vanishing, we have \( H^i(X, T_X) = 0 \) for \( i > 0 \), and hence \( \chi(X, T_X) \geq 0 \). Since \( F \)-liftable Fano threefolds admit a unique lifting to characteristic zero (Theorem 3.2.4), and as Betti numbers are invariant under deformations we can read them off from [Sha99, Tables §12.2–§12.6] (note that \( \rho = 0 \), and hence \( \chi \sim 0 \), and hence \( \chi(X, T_X) \geq 0 \)). This allows us to calculate that smooth Fano threefolds 1.1–14, 2.1–25, 3.1–12, and 4.1–2 have \( \chi(X, T_X) < 0 \). In particular, they are not \( F \)-liftable (Corollary 3.2.5).

Smooth Fano threefold 2.28 is the blow-up of \( \mathbf{P}^3_k \) along a plane cubic \( C \subseteq \mathbf{P}^3_k \). By Lemma 2.2.2, \( \text{Def}_{\text{Bl}_C} \mathbf{P}^3_k \cong \text{Def}_{\mathbf{P}_k^3} C \). Further, the forgetful transformation \( \text{Def}_{\mathbf{P}_k^3} C \to \text{Def} C \) is non-constant. We conclude that \( \text{Bl}_C \mathbf{P}^3_k \) is not rigid.

Blow-ups on a product. Fano threefolds 3.17, 3.21, 4.3, 4.6, and 4.8 are not \( F \)-liftable by the same argument as in Example 7.2.4. Note that Fano threefold 3.17 is the blow-up of \( \mathbf{P}^1 \times \mathbf{P}^2 \) along the graph of the Segre embedding \( \mathbf{P}^1 \to \mathbf{P}^2 \) of degree two.

Violating Bott vanishing. Fano threefolds 1.15 and 1.16 do not satisfy Bott vanishing (Example 3.2.6 and Lemma 7.2.2), and so they are not \( F \)-liftable (Theorem 3.2.4).

Inducing a lifting of Frobenius on \( \mathbf{P}^2 \) compatible with a conic. Fano threefolds 2.30 and 3.22 are not \( F \)-liftable by the same argument as in Example 7.2.6.

Mapping to a Frobenius non-liftable variety. Smooth Fano threefolds 2.26, 2.29, 2.31, 3.13–3.16, 3.18–3.20, 3.23–24, 4.4–5, 4.7, and 5.1 are not \( F \)-liftable, as they admit a birational morphism to another smooth Fano threefold which is not \( F \)-liftable (see Remark 7.2.1).

Fano threefolds 5.4–8 are not \( F \)-liftable, as they admit a morphism \( f : X \to S \) such that \( Rf_*\mathcal{O}_X = \mathcal{O}_S \), where \( S \) is a non-toric rational surface (see Theorem 3.3.6 and Theorem 7.1.9).

Toric. Fano threefolds 2.33–2.36, 2.35–3.31, 4.9–4.12, 5.2, 5.3 are toric (cf. [Sha99, Ch. 12, Remarks (i), p. 216], note that toric varieties 3.25 and 4.12 are missing from this list). □

Combined with Theorem 4.4.1, Theorems 7.1.9 and 7.2.9 show that Conjecture 2 holds if the target \( X \) is a surface or a Fano threefold.

References

[ADHL15]  Ivan Arzhantsev, Ulrich Derenthal, Jürgen Hausen, and Antonio Laface, Cox rings, Cambridge Studies in Advanced Mathematics, vol. 144, Cambridge University Press, Cambridge, 2015. MR 3307753

[Ame97]  Ekaterina Amerik, Maps onto certain Fano threefolds, Doc. Math. 2 (1997), 195–211. MR 1407127

[ARVdV99]  Ekaterina Amerik, Marat Rovinsky, and Antonius Van de Ven, A boundedness theorem for morphisms between threefolds, Ann. Inst. Fourier (Grenoble) 49 (1999), no. 2, 405–415. MR 1697369

[Bea01]  Arnaud Beauville, Endomorphisms of hypersurfaces and other manifolds, Internat. Math. Res. Notices (2001), no. 1, 53–58. MR 1809497

[BK05]  Michel Brion and Shrawan Kumar, Frobenius splitting methods in geometry and representation theory, Progress in Mathematics, vol. 231, Birkhäuser Boston, Inc., Boston, MA, 2005. MR 2107324

[BKPSB98]  Richard E. Borcherds, Ludmil Katzarkov, Tony Pantev, and Nicholas I. Shepherd-Barron, Families of K3 surfaces, J. Algebraic Geom. 7 (1998), no. 1, 183–193. MR 1620702
Ross. Akad. Nauk, Vseross. Inst. Nauchn. i Tekhn. Inform., Moscow, Translation edited by A. N. Parshin and I. R. Shafarevich. MR 1668575

[Sno86] Dennis M. Snow, Cohomology of twisted holomorphic forms on Grassmann manifolds and quadric hypersurfaces, Math. Ann. 276 (1986), no. 1, 159–176. MR 863714

[Sno88] ———; Vanishing theorems on compact Hermitian symmetric spaces, Math. Z. 198 (1988), no. 1, 1–20. MR 938025

[SSI0] Karl Schwede and Karen E. Smith, Globally F-regular and log Fano varieties, Adv. Math. 224 (2010), no. 3, 863–894. MR 2628797

[Sta14] The Stacks Project Authors, Stacks Project, http://stacks.math.columbia.edu, 2014.

[SW90] Michal Szurek and Jarosław A. Wiśniewski, Fano bundles of rank 2 on surfaces, Compositio Math. 76 (1990), no. 1-2, 295–305, Algebraic geometry (Berlin, 1988). MR 1078868

[vdGK03] Gerard van der Geer and Toshiyuki Katsura, On the height of Calabi-Yau varieties in positive characteristic, Doc. Math. 8 (2003), 97–113. MR 2029163

[Win04] Jörg Winkelmann, On manifolds with trivial logarithmic tangent bundle, Osaka J. Math. 41 (2004), no. 2, 473–484. MR 2069097

[Xin16] He Xin, On $W_2$-lifting of Frobenius of algebraic surfaces, Collect. Math. 67 (2016), no. 1, 69–83. MR 3439841

[Zda17] Maciej Zdanowicz, Liftability of singularities and their Frobenius morphism modulo $p^2$, Int. Math. Res. Not. IMRN (2017), https://doi.org/10.1093/imrn/rnw297.

[Zha10] De-Qi Zhang, Polarized endomorphisms of uniruled varieties, Compos. Math. 146 (2010), no. 1, 145–168, With an appendix by Y. Fujimoto and N. Nakayama. MR 2581245

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