Turbulent transport and coherent structures in 3D plasmas

Keith Leitmeyer

University of Virginia, Charlottesville, Virginia, USA

E-mail: kl2ju@virginia.edu

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Abstract
Inertial transport of kinetic and magnetic enstrophy is shown to have a concentrative effect towards smaller scales from a range spanning the integral scale to a Kraichnan-type scale for the 3D magnetohydrodynamic equations. This is an improvement of the result from Bradshaw and Grujić (2013), using redesigned ensemble averages.

Keywords: magnetohydrodynamics, turbulence, solar wind

1. Introduction

Observational and numerical evidence suggests that in magnetohydrodynamic (MHD) turbulence the vorticity \( \omega = \nabla \times u \) and current \( j = \nabla \times b \) (here, \( u \) and \( b \) are the velocity and the magnetic field, respectively) concentrate on coherent quasi low-dimensional structures—predominantly quasi two-dimensional sheets—which become increasingly thin exhibiting morphological dynamics consistent with a process known as turbulent cascade. Understanding this process is important due to its intimate relationship with violent reconnection events in the realm of solar wind turbulence, and is highly relevant to the goals outlined in the Space Studies Board of the National Research Council (NRC) survey, 22A Decadal Strategy for Solar and Space Physics (Heliophysics), completed in 2012, and in particular, to the strategic goal to ’discover and characterize fundamental processes that occur both within the heliosphere and throughout the Universe’ [17].

As a matter of fact—in the last five years—there has been a flurry of activity in the heliophysics community directed at understanding the phenomenon of turbulent dissipation, especially within the range of kinetic scales. One of the most promising theories, supported by extensive computational simulations, is that the coherent structures (most notably current sheets) exhibit a process of turbulent cascade down to the kinetic scales, essentially, all the way to electron scales, where they trigger extremely strong and localized heating of the plasma, dissipating the energy (the so-called heating via current sheets). There is also a sense
of optimism in part of the community that the relevance of this theory to the solar wind could be confirmed via the data to be collected by the upcoming NASA magnetospheric multiscale mission. The zero-step in this theory is an assumption on the geometry of the turbulent plasma at the interface between the continuum (described by the MHD system) and the kinetic (described by the Vlasov–Maxwell–Poisson system)-scale dynamics; shortly, the predominance of the current sheet geometry originating at continuum scales is assumed, and then utilized as the input when descending into the kinetic-scale dynamics. This brings out the question of existence of turbulent cascades—and in particular, the cascades of kinetic and magnetic enstrophies—in the continuum/MHD regime to the forefront of scientific interest.

In particular, statistical properties of current sheets and the relationship to magnetic reconnection were investigated in [23] using a numerical simulation of MHD turbulence. In [14], kinetic simulations are used to study energy dissipation of plasmas.

This main result of this paper will show that the inertial transport of kinetic and magnetic enstrophy is oriented towards smaller scales, for a range of scales. It is emphasized that the Lorentz force, for example, may counteract this effect, so that there is no cascade of enstrophy to smaller scales. Nevertheless, the theorem provides mathematical evidence that current sheets are dynamically generated in MHD turbulence. See [1] for further discussion.

The concentrative effect of the inertial term on kinetic and magnetic enstrophy was shown in [1] using ensemble averages in physical scales. Here the ensemble averages used are redesigned to allow weaker assumptions. The ensemble averages used to state the enstrophy concentration theorem are described first. The following section goes over the conditions under which enstrophy concentration is shown. Then we recall the bounds obtained in [1], and finally the enstrophy concentration theorem is proven.

Related work on cascades and locality in hydrodynamic turbulence can be found in [2–5, 7–13, 15, 16, 18–21].

2. Fluxes and ensemble averages

The MHD equations, which model evolution of the velocity and magnetic fields in an electrically conducting incompressible fluid, read

\[ \frac{\partial}{\partial t} u - \Delta u + (u \cdot \nabla) u - (b \cdot \nabla) b + \nabla P = 0, \]
\[ \frac{\partial}{\partial t} b - \Delta b + (u \cdot \nabla) b - (b \cdot \nabla) u = 0, \]
\[ \nabla \cdot u = \nabla \cdot b = 0. \]

(Here, \( P \) is the total pressure, and the magnetic resistivity and the kinematic viscosity are normalized to 1.)

Taking the curl of equation (1) and using vector calculus identities gives the following equations for vorticity and current

\[ \frac{\partial}{\partial t} \omega - \Delta \omega = - (u \cdot \nabla) \omega + (\omega \cdot \nabla) u + (b \cdot \nabla) j - (j \cdot \nabla) b, \]
\[ \frac{\partial}{\partial t} j - \Delta j = - (u \cdot \nabla) j + (j \cdot \nabla) u + (b \cdot \nabla) \omega - (\omega \cdot \nabla) b + 2 \sum_{i=1}^{3} \nabla b_i \times \nabla u_i. \]

A complete derivation of these equations is given in the appendix.

These equations can be used to study inward kinetic and magnetic enstrophy fluxes. Rather than traditional fluxes through the boundary of a ball \( B = B(x_0, 2R) \),
we use a smooth cutoff function $\psi$ supported on $B(x_0, 2R)$ and equal to 1 on $B(x_0, R)$—with inward pointing gradient—and study fluxes through the spherical layer of thickness $R$,

$$
\int \frac{1}{2} |\omega|^2 (u \cdot \nabla \psi) \, dx = - \int (u \cdot \nabla) \omega \cdot \psi \, dx,
$$

$$
\int \frac{1}{2} |j|^2 (u \cdot \nabla \psi) \, dx = - \int (u \cdot \nabla) j \cdot \psi j \, dx.
$$

The reason is that this form of the flux is more amenable to mathematical analysis, while at the same time preserving the physics. Time-averaged quantities are studied in turbulence, so we take time averages weighted according to a smooth function $h(t)$ which is 0 on $[0, T/3]$ and 1 on $[2T/3, T]$, denoting $\psi(x, t) = \psi(x) h(t)$:

$$
\int_0^T \int \frac{1}{2} |\omega|^2 (u \cdot \nabla \phi) \, dx \, dt = - \int_0^T \int (u \cdot \nabla) \omega \cdot \phi \, dx \, dt,
$$

$$
\int_0^T \int \frac{1}{2} |j|^2 (u \cdot \nabla \phi) \, dx \, dt = - \int_0^T \int (u \cdot \nabla) j \cdot \phi j \, dx \, dt.
$$

Multiplying equation (2) by $\omega$ and $j$ respectively, and integrating over space and time, will yield expressions which can be used to dynamically estimate the fluxes:

$$
\int_0^T \int \frac{1}{2} |\omega|^2 (u \cdot \nabla \phi) \, dx \, dt = \int \frac{1}{2} |\omega(x, T)|^2 \psi(x) \, dx + \int_0^T \int |\nabla \omega|^2 \phi \, dx \, dt
$$

$$
- \int_0^T \int \frac{1}{2} |\omega|^2 (\partial_t \phi + \Delta \phi) \, dx \, dt
$$

$$
- \int_0^T \int (\omega \cdot \nabla) u \cdot (\phi \omega) \, dx \, dt
$$

$$
- \int_0^T \int (b \cdot \nabla) j \cdot (\phi j) \, dx \, dt
$$

$$
+ \int_0^T \int (j \cdot \nabla) b \cdot (\phi j) \, dx \, dt
$$

$$
= \int \frac{1}{2} |\omega(x, T)|^2 \psi(x) \, dx
$$

$$
+ \int_0^T \int |\nabla \omega|^2 \phi \, dx \, dt + H^\omega + N_1^\omega + L^\omega + N_2^\omega
$$

$$
\int_0^T \int \frac{1}{2} |j|^2 (u \cdot \nabla \phi) \, dx \, dt = \int \frac{1}{2} |j(x, T)|^2 \psi(x) \, dx + \int_0^T \int |\nabla j|^2 \phi \, dx \, dt
$$

$$
- \int_0^T \int \frac{1}{2} |j|^2 (\partial_t \phi + \Delta \phi) \, dx \, dt
$$

$$
+ \int_0^T \int (\omega \cdot \nabla) b \cdot (\phi j) \, dx \, dt
$$

$$
- \int_0^T \int (b \cdot \nabla) \omega \cdot (\phi j) \, dx \, dt
$$
\[- \int_0^T \int (j \cdot \nabla) u \cdot (\phi j) \, dx \, dt \]
\[- \int_0^T \int \left( \frac{3}{2} \sum_{i=1}^{2} \nabla u_i \times \nabla b_i \right) \cdot (\phi j) \, dx \, dt \]
\[= \int \frac{1}{2} (j(x, T))^2 \psi(x) \, dx \]
\[+ \int_0^T \int |\nabla j|^2 \phi \, dx \, dt + H^j + N^j_1 + L^j + N^j_2 + X. \tag{7} \]

Enstrophy concentration will be demonstrated locally, over a ball \( B(0, 2R_0) \). Ultimately we want to show that the average of these fluxes over suitable collections of functions \( \psi \) of a particular scale is positive, for a range of scales.

Fix \( C_0 > 1 \) and \( 3/4 < \rho < 1 \). A refined test function at scale \( R \) is any smooth function \( \psi \) supported in a ball of radius \( 2R \) satisfying \( 0 \leq \psi \leq 1 \), \( |\nabla \psi| < \frac{C_0}{R} \psi^\rho \), and \( |\Delta \psi| < \frac{C_0}{R^2} \psi^{2\rho - 1} \).

Now fix a scale \( R_0 \) refined test function \( \psi_0 \) centered at 0. An ensemble at scale \( R \) with global multiplicity \( K_1 \) and local multiplicity \( K_2 \) is a collection of scale \( R \) test functions \( \{ \psi_i \}_{i=1}^{n} \) satisfying the following properties:

1. \( \psi_i \leq \psi_0 \leq \sum \psi_i \)
2. \( (R_0/R)^3 \leq n \leq K_1 (R_0/R)^3 \)
3. No point of \( B(0, 2R_0) \) is contained in more than \( K_2 \) of the supports of \( \psi_i \).

For a function \( f \), denote by \( \langle F \rangle_R \) the ensemble average \( \frac{1}{n} \sum_{i=1}^{n} \frac{1}{R^3} \int f \psi_i \, dx \), and \( F_0 = \frac{1}{R_0^3} \int f \psi_0 \, dx \).

Property 1 above is needed to compare \( \langle F \rangle_R \) to \( F_0 \). Due to property 1, test functions near the boundary of the support of \( \psi_0 \) will have small integrals, effectively skewing the ensemble average towards zero. Larger \( K_1 \) and \( K_2 \) allow ensembles that have higher weight on functions away from the boundary, making the skewing insignificant.

These ensemble averages can be viewed as a way to detect whether a function is significantly negative above some spatial scale. If every ensemble average \( \langle F \rangle_R \) is positive, no matter how one arranges and stacks the test functions, then the function is not significantly negative at scales larger than \( R \). Increasing \( K_1 \) and \( K_2 \) lowers the threshold for a function to be considered significantly negative.

One way to explicitly construct ensembles is to apply lemma 2 to \( \psi_0 \) and varying the multiplicity of the resulting functions (assume \( \psi_0 \) satisfies the stronger \( C_0^\rho \)-bounds to get an ensemble with \( C_0^\rho \)-bounds).

The following lemma states that ensemble averages (at any scale) of positive functions are comparable to the large scale mean. An upper bound holds when powers of the test functions \( \psi_i^\delta \) are used in the averaging instead of just \( \psi_i \). The proof immediately follows from the definitions.

**Lemma 1.** If \( f \geq 0 \) then \( \frac{1}{K_1} F_0 \leq \langle F \rangle_R \leq K_2 F_0 \). Additionally, \( \frac{1}{n} \sum_{i=1}^{n} \frac{1}{R^3} \int f \psi_i^\delta \, dx \leq K_2 \frac{1}{R_0^3} \int f \psi_0^\delta \, dx \) (\( \delta > 0 \)).
Using a refined partition of unity, one can turn larger scale ensembles into smaller scale ensembles.

**Lemma 2.** Any scale $R$ test function satisfying $C_0$ bounds is a sum of $8 \left\lfloor \frac{R}{R'} \right\rfloor^3$ scale $R'$ test functions satisfying $C_0$ bounds (where $R > R'$ and $C_0$ depends only on $C_0$).

Therefore for each $(K_1, K_2, C_0)$-ensemble at scale $R$ and each $R' < R$, there exists a $(64K_1, 8K_2, C_0')$-ensemble at scale $R'$ such that $(F)_R = (F)_{R'}$.

**Proof.** Let $\psi$ be a scale $R$ test function satisfying $C_0$ bounds. Now to construct the partition of unity, take a scale $\frac{R}{2}$ test function $g_0$ (satisfying $C_0$ bounds), centered at zero and equal to 1 on $[-\frac{R}{2}, \frac{R}{2}]$. Define $g_p = g_0(x - 2R_p)$, where $p \in \mathbb{Z}^3$. Then $1 \leq \sum_p g_p \leq 2$ so we may define $h_p = g_p/\sum_p g_p$.

Some calculus shows that $\left| \nabla (\psi h_p) \right| < \frac{6C_0}{R} h_p^2$ and $\left| \Delta (\psi h_p) \right| < \frac{3C_0 + 10C_0^2}{R^2} h_p^{2p-1}$, so $\left| \nabla (\psi h_p) \right| < \frac{7C_0}{R} (\psi h_p)^2$ and $\left| \Delta (\psi h_p) \right| < \frac{4C_0 + 22C_0^2}{R^2} (\psi h_p)^{2p-1}$. Fewer than $8 \left\lfloor \frac{R}{R'} \right\rfloor^3 \leq 64 \left( \frac{R}{R'} \right)^3$ of the functions $\psi h_p$ are nonzero, and for any $x$, $\psi(x) = 0$ for at most 8 functions.

Since $\psi = \sum_p \psi h_p$, the first claim is proven. For the second claim, given an ensemble $\{\psi_i\}$, the new ensemble will be $\{\psi_i h_p\}_{p,0}$.

### 3. Assumptions

Now we come to the conditions on the current and vorticity over $B(0, 2R_0) \times (0, T)$ under which we can show that there is enstrophy concentration.

**3.1. Geometry/smoothness**

Denote by $\theta(u,v)$ the angle between two vectors $u$, $v$. It is required that for some $M$, $C_1 > 0$,

$$|\sin \theta(\omega(x + y, t), \omega(x, t))| \leq C_1 |y|^{1/2}$$

(8)

for every $t \in (0, T)$, every $x \in B(0, 2R_0 + R_0^{2/3})$ with $|\nabla u(x, t)| > M$, and every $y$ such that $|y| < 2(\sigma_0/\beta) + (\sigma_0/\beta)^{2/3}$, and

$$|j(x + y, t) - j(x, t)| \leq |j(x + y, t)||y|^{1/2}$$

(9)

for every $t \in (0, T)$, every $x \in B(0, 2R_0 + R_0^{2/3})$ with $|\nabla b(x, t)| > M$, and every $y$ such that $|y| < 2(\sigma_0/\beta) + (\sigma_0/\beta)^{2/3}$. The quantity $\sigma_0/\beta$ is defined in section 3.2.

Condition 8 depends only on the angle of the vorticity vector $\omega$. This $\frac{1}{2}$-Holder coherence will deplete the vortex stretching term. This condition is needed because condition 9 is insufficient to control the vortex stretching term $N_1^\omega$, which has no explicit dependence on the magnetic field.

Condition 9 is needed for the nonlinear terms that do not have a geometric kernel available. It requires $\frac{1}{2}$-Holder continuity, and more. It would be too restrictive if $|j(x + y, t)|$ were ever small, but $x + y$ is always close to the region where $|\nabla b|$ is large, and roughly, $\nabla b$ and $j$ are large in the same regions.
In [1], condition 9 was used with \( j \) replaced by \( \omega \) with no assumption corresponding to 8. The present formulation is preferred because current appears to be more regular than vorticity in numerical simulations.

3.2. Kraichnan-type scale

Let \( e_0, E_0, \) and \( P_0 \) denote the time-averaged total energy, total enstrophy, and total palinstrophy at the integral scale. Precisely

\[
e_0 = \frac{1}{T} \int_0^T \frac{1}{R_0^2} \int \phi_{0}^2 \left( \frac{|u|^2}{2} + \frac{|b|^2}{2} \right) \, dx \, dt,
\]

\[
E_0 = \frac{1}{T} \int_0^T \frac{1}{R_0^2} \int \phi_{0}^2 \left( |\omega|^2 + |j|^2 \right) \, dx \, dt,
\]

\[
P_0 = \frac{1}{T} \int_0^T \frac{1}{R_0^2} \int \phi_{0} \left( |\nabla \omega|^2 + |\nabla j|^2 \right) \, dx \, dt
\]

\[
+ \frac{1}{TR_0^3} \int \frac{1}{2} \left( |\omega(x, T)|^2 + |j(x, T)|^2 \right) \psi_0 \, dx.
\]

Define the modified Kraichnan-type scale \( \sigma_0 \) by

\[
\sigma_0 = \max \left\{ \left( \frac{E_0}{P_0} \right)^{1/2}, \left( \frac{e_0}{P_0} \right)^{1/4} \right\}.
\]

Assumption 2 is that \( \sigma_0 < \beta R_0 \), where \( \beta \) is a constant \((0 < \beta < 1)\) identified in the proof. If \( \omega \) and \( j \) have large gradients as expected in a turbulent flow, this assumption will be satisfied.

3.3. Localization

Any smooth solution to the equations will have \( \omega \in L^3((0, T) \times B(0, 2R_0 + R_0^{2/3})) \). It is required that the kinetic and magnetic enstrophy is not too highly concentrated around any point:

\[
\int_0^T \int_{B(y, R)} |\omega|^2 + |j|^2 \, dx \, dt < \frac{1}{C_2}
\]

for any \( y \in B(0, 2R_0) \) where \( C_1 \) is a constant and \( R = 2\sigma_0/\beta + (\sigma_0/\beta)^{2/3} \).

In [1], it was required that

\[
\int_0^T \int_{B(0, 2R_0 + R_0^{2/3})} |\omega|^2 \, dx \, dt < \frac{1}{C_3}.
\]

This assumption restricts the extent of the range of scales over which we can show enstrophy concentration. The magnetic enstrophy is in the present assumption because of the geometric/smoothness assumption on the current that is not found in [1].

3.4. Modulation

The assumption imposes a restriction on the time evolution of the integral-scale kinetic and magnetic enstrophies across \((0, T)\) consistent with our choice of the temporal cutoff. Precisely
\[ \int |\omega(x, T)|^2 \psi_0(x) \, dx \geq \frac{1}{2} \sup_t \int |\omega(x, t)|^2 \psi_0(x) \, dx, \]
\[ \int |j(x, T)|^2 \psi_0(x) \, dx \geq \frac{1}{2} \sup_t \int |j(x, t)|^2 \psi_0(x) \, dx, \]

4. Bounds

In [1], the terms of equations (6) and (7) are bounded by quantities which can be related to \( e_0, E_0, \) and \( P_0. \)

Obtaining the desired bounds using assumption 3.1 will use all of the same techniques as in [1] with the major difference being labelling, except for the vortex stretching term which uses geometric depletion as in [6].

Ultimately, we have

\[ H^\omega + H^j + N^\omega + N^j + L^\omega + L^j + X \leq K_F \left( \frac{1}{\alpha} + \frac{1}{2} \sup_{t \in (0, T)} \int \psi(x) \left( |\omega(x, t)|^2 + |j(x, t)|^2 \right) \, dx \right) + \frac{K_E}{R^2} \int_0^T \int \phi^{2\beta-1} \left( |\omega|^2 + |j|^2 \right) \, dx \, dt \]
\[ + \alpha^2 K_e \int_0^T \int \phi^{\beta-1} |u|^2 + |b|^2 \, dx \, dt, \]

where \( K_F, K_E, K_e \) are constants and \( \alpha \) is an interpolation parameter that we may choose. The constants \( K_F, K_E, K_e \) are fixed upon a choice of the parameters \( K_1, K_2, \rho, C_0, C, \) and \( M. \) In the following section \( K_F, K_E, K_e \) are instead the constants obtained corresponding to \( 64K_1, 8K_2, \) and \( C_0. \)

5. Main result

Kinetic and magnetic enstrophy is on average being transported to smaller scales down to the Kraichnan-type scale \( \sigma_0/\beta. \) The following notation will be used in the proof: for a density \( f \) and ensemble \( \{\psi_i\}_{i=1}^n, \) the average over one element of the ensemble is \( F_i = \int f \psi_i \, dx \) and the ensemble average is \( \langle F \rangle_R = \frac{1}{n} \frac{1}{TR} \sum_{i=1}^n F_i. \) Let \( \varphi = -\int_0^T (u \cdot \nabla) \omega + (u \cdot \nabla) j \cdot j \, dt, \) the enstrophy flux density, so that \( \Phi = \int \varphi \psi_i \, dx \) is the kinetic and magnetic enstrophy flux centered at \( x_i \) at scale \( R. \) Referring to equations (6) and (7), suppressing the subscript \( i, \) denote \( H = H^\omega + H^j \) and likewise for \( N \) and \( L. \) Define similarly

\[ e = \int_0^T \int \psi_1^{\alpha-3} \frac{1}{2} \left( |u|^2 + |b|^2 \right) \, dx \, dt, \]
\[ E = \int_0^T \int \psi_2^{\beta-1} \left( |\omega|^2 + |j|^2 \right) \, dx \, dt, \]
\[ P = \int \psi_3(x) \left( |\omega(x, T)|^2 + |j(x, T)|^2 \right) \, dx + \int_0^T \int \psi_4 \left( \left| \nabla \omega \right|^2 + \left| \nabla j \right|^2 \right) \, dx \, dt, \]
and
\[
\bar{P} = \frac{1}{2} \sup_{t \in (0, T)} \int_\Omega \psi(x) \left( |\omega(x, t)|^2 + |j(x, t)|^2 \right) dx + \int_0^T \int_\Omega \phi_1 \left( |\nabla \omega|^2 + |\nabla j|^2 \right) dx \, dt.
\]

**Theorem 1.** Under Assumptions 1–4, for any \( K_1 \) and \( K_2 \) there exists \( K_* \) such that for any \((K_1, K_2)\)-ensemble at scale \( R \) ranging from \( \sigma_0/\beta \) to \( R_0 \), we have \( \frac{1}{K_*} P_0 \leq \langle \Phi \rangle_R \leq K_* P_0 \).

**Proof.** We start by showing that for \( \sigma_0 = R \), for \((64K_1, 8K_2)\)-ensembles satisfying \( C_exec \) bounds, we have \( \frac{1}{K_*} P_0 \leq \langle \Phi \rangle_R \leq K_* P_0 \).

Referring to equations (6) and (7), \( \langle \Phi \rangle_R = \langle P \rangle_R + \langle H + N + L + X \rangle_R \), where \( p \) is the positive density \( \frac{1}{\rho} \). Referring to equations (6) and (7), \( \langle \Phi \rangle_R = \langle P \rangle_R + \langle H + N + L + X \rangle_R \), where \( p \) is the positive density \( \frac{1}{\rho} \).

so that \( \frac{1}{64K_1} P_0 \leq \langle P \rangle_R \leq 8K_2 P_0 \). The rest of the terms are relatively small upon averaging:

\[
|H + N + L + X| \leq K_P \left( \frac{1}{\alpha} + ||\omega||_{L^2(B(x, 2R + R/s) \times (0, T))} \right) \bar{P} + \frac{K_E}{R^2} E + \frac{\alpha^2 K_e}{R^4} e,
\]

so that

\[
|\langle H + N + L + X \rangle_R| \leq \left( K_P \left( \frac{1}{\alpha} + ||\omega|| \right) \bar{P} + \frac{K_E}{R^2} E + \frac{\alpha^2 K_e}{R^4} e \right)_R
\leq 8K_2 K_P \left( \frac{1}{\alpha} + ||\omega|| \right) \bar{P}_0 + 8K_2 K_E \frac{1}{R^2} E_0 + \frac{\alpha^2 8K_2 K_e}{R^4} e_0
\leq \left( 8K_2 K_P \left( \frac{1}{\alpha} + ||\omega|| \right) \bar{P}_0 + 8K_2 K_E \beta^2 + \alpha^2 8K_2 K_e \beta^4 \right) P_0
\leq \frac{3}{4 \cdot 64K_1} P_0.
\]

by taking \( \alpha = 4 \cdot 64K_1 \cdot 8K_2 K_P \), using that \( \beta \) is sufficiently small such that \( 8K_2 K_E \beta^2 + \alpha^2 \cdot 8K_2 K_e \beta^4 \leq \frac{1}{4 \cdot 64K_1} \) and using the assumption that \( ||\omega||_{L^2(B(x, 2R + R/s) \times (0, T))} \leq \frac{1}{C_2} = \frac{1}{4 \cdot 64K_1 \cdot 8K_2 K_P} \).

Therefore \( \frac{1}{4 \cdot 64K_1} P_0 \leq \langle \Phi \rangle_R \leq (8K_2 + \frac{3}{4 \cdot 64K_1}) P_0 \) for any \((64K_1, 8K_2, C_exec')\)-ensemble at scale \( R = \alpha_0/\beta \). For the range of scales from \( \alpha_0/\beta \) up to \( R_0 \), by lemma 2, \( \frac{1}{4 \cdot 64K_1} P_0 \leq \langle \Phi \rangle_R \leq (8K_2 + \frac{3}{4 \cdot 64K_1}) P_0 \) for all scale \( R(K_1, K_2, C_0)\)-ensembles.

The locality of magnetic and kinetic enstrophy flux stated in [11] also holds in our setting of modified assumptions and ensemble averages. The result is stated in terms of the time-averaged enstrophy flux, rather than the time-averaged enstrophy flux per unit mass used above. That is
Corollary 1. Under Assumptions 1–4, for any \( K_1, K_2 \) there exists \( K_* \) such that for any \( r, R \) between \( \sigma_0/\beta \) and \( R_0 \) and any \((K_1, K_2)\) ensembles, enstrophy flux is local:

\[
\frac{1}{K_*^2} \left( \frac{r}{R} \right)^3 \leq \frac{\langle \Psi_r \rangle}{\langle \Psi_r \rangle} \leq K_*^2 \left( \frac{r}{R} \right)^3.
\]

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Appendix

The vorticity and current equations (equation (2)) can be derived from the MHD equations as follows.

For any divergence free vector fields \( A, B \) and scalar field \( \psi \), the following identities hold:

\[
\nabla_b (A \cdot B) = (A \cdot \nabla)B + A \times (\nabla \times B), \tag{12}
\]

\[
\nabla \times (A \times B) = (B \cdot \nabla)A - (A \cdot \nabla)B, \tag{13}
\]

\[
\nabla \times (\psi A) = \psi (\nabla \times A) + \nabla \psi \times A, \tag{14}
\]

\[
\nabla \times (\nabla \psi) = 0, \tag{15}
\]

where equation (13) uses the notation \( \nabla_b (A \cdot B)_k = \sum_{l=1}^{3} A_l \partial_l B_l \).

Taking the curl of the MHD equations yields

\[
\partial_t \omega - \Delta \omega + \nabla \times ((u \cdot \nabla)u) - \nabla \times ((b \cdot \nabla)b) = 0,
\]

\[
\partial_t j - \Delta j + \nabla \times ((u \cdot \nabla)b) - \nabla \times ((b \cdot \nabla)u) = 0. \tag{16}
\]

Now \((u \cdot \nabla)b = \sum_i u_i \nabla b_i - u \times j\) by equation (12). So

\[
\nabla \times ((u \cdot \nabla)b) = \nabla \times \left( \sum_i u_i \nabla b_i \right) - \nabla \times (u \times j).
\]

By equations (14) and (15),

\[
\nabla \times (u_i \nabla b_j) = u_i (\nabla \times \nabla b_j) + \nabla u_i \times \nabla b_j = \nabla u_i \times \nabla b_j.
\]

Next, equation (13) gives that \( \nabla \times (u \times j) = (j \cdot \nabla)u - (u \cdot \nabla)j \). In total, we have that

\[
\nabla \times ((u \cdot \nabla)j) = (u \cdot \nabla)j - (j \cdot \nabla)u + \sum_l \nabla u_l \times \nabla b_l. \tag{17}
\]

Similarly

\[
\nabla \times ((b \cdot \nabla)u) = (b \cdot \nabla)\omega - (\omega \cdot \nabla)b - \sum_l \nabla u_l \times \nabla b_l. \tag{18}
\]
\[ \nabla \times ((u \cdot \nabla)u) = (u \cdot \nabla)\omega - (\omega \cdot \nabla)u + \sum_{i} \nabla u_i \times \nabla u_i = (u \cdot \nabla)\omega - (\omega \cdot \nabla)u, \]

and

\[ \nabla \times ((b \cdot \nabla)b) = (b \cdot \nabla)j - (j \cdot \nabla)b. \]

By using equations (17)–(20) on equation (16), we arrive at the desired equations

\[ \partial_t \omega - \Delta \omega = - (u \cdot \nabla)\omega + (\omega \cdot \nabla)u + (b \cdot \nabla)j - (j \cdot \nabla)b, \]

\[ \partial_t j - \Delta j = - (u \cdot \nabla)j + (j \cdot \nabla)u + (b \cdot \nabla)\omega - (\omega \cdot \nabla)b + 2 \sum_{i=1}^{3} \nabla u_i \times \nabla u_i. \]

References

[1] Bradshaw Z and Grujić Z 2013 On the transport and concentration of enstrophy in 3D magnetohydrodynamic turbulence *Nonlinearity* 26 2373
[2] Cheskidov A, Constantin P, Friedlander S and Shvydkoy R 2008 Energy conservation and Onsager’s conjecture for the Euler equations *Nonlinearity* 21 1233
[3] Constantin P 1990 Navier–Stokes equations and area of interfaces *Commun. Math. Phys.* 129 241
[4] Constantin P 1994 Geometric statistics in turbulence *SIAM Rev.* 36 73–98
[5] Dascaliuc R and Grujić Z 2011 Energy cascades and flux locality in physical scales of the 3D Navier–Stokes equations *Commun. Math. Phys.* 305 199
[6] Dascaliuc R and Grujić Z 2013 Coherent vortex structures and 3D enstrophy cascade *Commun. Math. Phys.* 317 547
[7] Eyink G 2005 Locality of turbulent cascades *Physica D* 207 91–116
[8] Eyink G and Aluie H 2009 Localness of energy cascade in hydrodynamic turbulence: I. Smooth coarse graining *Phys. Fluids* 21 115107
[9] Foias C, Jolly M S, Manley O and Rosa R 2005 Statistical estimates for the Navier–Stokes equations and the Kraichnan theory of 2-D fully developed turbulence *J. Stat. Phys.* 108 591–645
[10] Foias C, Manley O, Rosa R and Temam R 2001 Estimates for the energy cascade in three-dimensional turbulent flows *C. R. Acad. Sci., Paris* I 333 499–504
[11] Grujić Z Vortex stretching and anisotropic diffusion in 3D Navier–Stokes equations *Contemp. Math.* to appear
[12] Grujić Z 2009 Localization and geometric depletion of vortex-stretching in the 3D NSE *Commun. Math. Phys.* 290 861
[13] Jimenez J J, Wray A A, Saffman P G and Rogallo R S 1993 The structure of intense vorticity in isotropic turbulence *J. Fluid Mech.* 255 65–90
[14] Karimabadi H, Roytershteyn V, Wan M, Matthaeus W H and Daughton W 2013 Coherent structures, intermittent turbulence, and dissipation in high-temperature plasmas *Phys. Plasmas* 20 012303
[15] Leitmeyer K 2015 Enstrophy cascade in physical scales for the 3D Navier–Stokes equations *SIAM J. Math. Anal.* at press arXiv: http://arxiv.org/abs/1502.01258
[16] L’vov V and Falkovich G 1992 Countervaled interaction locality of developed hydrodynamic turbulence *Phys. Rev.* A 46 4762
[17] A decadal strategy for solar and space physics (heliophysics), http://sites.nationalacademies.org/ssb/currentprojects/ssb 056864
[18] Rosa R 2002 Some results on the Navier–Stokes equations in connection with the statistical theory of stationary turbulence *Appl. Math.* 47 485
[19] She Z-S, Jackson E and Orszag S 1991 Structure and dynamics of homogeneous turbulence: models and simulations *Proc. R. Soc. A* 434 101–124
(20) Taylor G I 1938 Production and dissipation of vorticity in a turbulent fluid Proc. R. Soc. 164 15–23
(21) Vincent A and Meneguzzi M 1994 The dynamics of vorticity tubes in homogeneous turbulence J. Fluid Mech. 225 245–254
(22) Yoshimatsu K, Okamoto N, Kawahara Y, Schneider K and Farge M 2013 Coherent vorticity and current density simulation of three-dimensional magnetohydrodynamic turbulence using orthogonal wavelets Geophys. Astrophys. Fluid Dyn. 107 73
(23) Zhankin V, Uzdensky D, Perez J and Boldyrev S 2013 Statistical analysis of current sheets in three-dimensional magnetohydrodynamic turbulence Astrophys. J. 771 124