How well do domain wall fermions realize chiral symmetry?

George R. Fleming *

*Physics Dept., Columbia University, New York NY 10027

In the domain wall fermion formulation, chiral symmetry breaking in full QCD is expected to fall exponentially with the length of the extra dimension. We measure the chiral symmetry breaking due to a finite extra dimension in two ways, which can be affected differently by finite volume and explicit fermion mass. For quenched QCD the two methods generally agree, except for the largest extent of the extra dimension, which makes the limit uncertain. We have less data for full QCD, but see exponential suppression for the method where we have data.

1. Introduction

Lattice QCD with massless domain wall fermions (including fermion loop effects) is expected to have the $SU_L(N_f) \otimes SU_R(N_f)$ chiral symmetry of the continuum when the extent of the extra dimension, $L_s$, becomes infinite. For simulations, where the volume is finite and particles are not strictly massless, reliable techniques are needed to quantify the symmetry breaking for finite $L_s$. Such techniques are needed to see the expected $\exp(-\alpha L_s)$ dependence of chiral breaking for full QCD and determine if this is also the case for the quenched theory.

Here we report results from two techniques for measuring chiral symmetry breaking due to finite $L_s$; the first uses the pion mass and the second the axial Ward identity. At zero temperature, the pion mass is governed by the axial Ward identity. However, in simulations with finite volume and with finite quark masses, it is important to check the agreement between these approaches.

The axial Ward identity is the origin of the Gell-Mann-Oakes-Renner (GMOR) relation, discussed previously for domain wall fermions in [3]. The fermion action of [2] is used, with the modifications of [4]. Some details on the numerical methods are in [5]. See [6] for a general review on domain wall fermions and references.

2. $m_{\text{res}}$ for domain wall fermions

If the dominant effect of finite $L_s$ is to produce an extra contribution, $m_{\text{res}}$, to the total quark mass, then one would expect

\[ m_{\pi}^2 = c_0(V) + c_1(V) \cdot (m_f + m_{\text{res}}) + \cdots \] (1)

where $V$ is the space-time volume and it is expected that $c_0(V) \to 0$ as $V \to \infty$. For finite volume, a result we call $m_{\text{res}}^{(m_f = 0)}$, can be found from $m_{\pi}^2(m_f = 0)/c_1(V)$, which is $m_{\text{res}}$ when $c_0(V) = 0$.

Using the flavor non-singlet axial current in [2] we integrate the axial Ward–Takahashi identity to get

\[ \langle \bar{q}_0 q_0 \rangle = m_f \chi_{\pi} + \Delta J_5 \] (2)

where the pseudoscalar susceptibility is (no sum on $a$)

\[ \chi_{\pi} \equiv 2 \sum_x \left\langle \bar{q}_x \gamma_5 \frac{\lambda^a}{2} q_x \bar{q}_0 \gamma_5 \frac{\lambda^a}{2} q_0 \right\rangle, \] (3)

the additional contribution from chiral mixing due to finite $L_s$ is

\[ \Delta J_5 \equiv 2 \sum_x \left\langle j^5_a (x, L_s/2) \bar{q}_0 \gamma_5 \frac{\lambda^a}{2} q_0 \right\rangle, \] (4)

and $q_x$ are four-dimensional fermion fields defined from the appropriate right- and left-handed fields at the boundaries of the extra dimension.

For large volumes in the chirally broken phase, the pseudoscalar susceptibility is expected to behave as

\[ \chi_{\pi} = a_{-1} / (m_f + m_{\text{res}}) + a_0 + O(m_f + m_{\text{res}}). \] (5)
The first term again says that, for large volumes, the pion is massless at \( m_f = -a_{\text{res}} \), while \( a_0 \) gives the contribution due to the massive modes. Clearly, the pion pole contribution only dominates for large enough volumes and small enough \( m_f + m_{\text{res}} \).

\[ j_0^2 (x, L_s/2), \] a pseudoscalar density located midway between the domain walls, also has a pole contribution, whose coefficient is suppressed by propagation from \( L_s/2 \) to the boundaries. Since \( \chi_\pi \) and \( \Delta J_5 \) both have a pole at \( m_f = -a_{\text{res}} \) and when the pole terms dominate \( \langle \bar{q}q \rangle \) is finite, we can write in general

\[ \Delta J_5 = m_{\text{res}} \chi_\pi + b_0 + \mathcal{O}(m_f + m_{\text{res}}). \quad (6) \]

We define \( m_{\text{res}}^{\text{GMOR}} \) by simultaneously fitting to the form

\[ \langle \bar{q}q \rangle = (m_f + m_{\text{res}}) \chi_\pi + b_0, \quad (7) \]

and \( \chi_\pi \) as given in (4). For a given \( L_s \), this is a four parameter fit for \( a_{-1}, a_0, b_0 \) and \( m_{\text{res}}^{\text{GMOR}} \). Note that only if \( b_0/\chi_\pi \) is small, can we get a reliable estimate for \( m_{\text{res}}^{\text{GMOR}} \) from \( \langle \bar{q}q \rangle \) \( / \chi_\pi - m_f \). For full QCD, both \( m_{\text{res}}^{\text{GMOR}} \) and \( b_0 \) should approach zero exponentially in \( L_s \), since both involve propagation from \( L_s/2 \) to the walls.

3. \( m_{\text{res}} \) in quenched QCD

We first find \( m_{\text{res}}^{(m^2)} \) for the the \( \beta = 5.7 \), \( m_0 = 1.65 \), \( 8^3 \times 32 \) quenched domain wall spectrum study we reported last year [2]. For quenched QCD, the observed zero mode contribution to \( \langle \bar{q}q \rangle \) is small for \( m_f \geq 0.02 \), so we restrict our attention this mass range here. Figure 1 shows \( m_{\text{res}}^{(m^2)} \) for our \( \beta = 5.7 \), \( m_0 = 1.65 \), \( 8^3 \times 32 \) simulations from a correlated, linear fit of \( m_{\pi}^2 \) to valence masses 0.02, 0.06, 0.10, with errors from jacknifing. The \( L_s = 32 \) and 48 values are the same within errors, making the large \( L_s \) limit seem non-zero. For \( L_s = 24 \) the result for a \( 16^3 \times 32 \) lattice is also shown, revealing that finite volume effects are noticeable.

Figure 1 also shows \( m_{\text{res}}^{\text{GMOR}} \) from a correlated fit to (4) and (5). The \( 16^3 \) GMOR point is on top of the \( 8^3 \) point in the plot. The fits \( m_{\text{res}} = 0.059(14) \exp[-0.0035(11) L_s] \) and \( b_0 = -0.051(13) L_s \) (not shown) are consistent with \( \Delta J_5 \) vanishing in the \( L_s \to \infty \) limit.

The agreement between the two methods for \( L_s \leq 32 \) is reasonable and only occurs since \( b_0 \) is included in the fits. \( m_{\text{res}}^{\text{GMOR}} \) is volume independent for \( L_s = 24 \), while \( m_{\text{res}}^{(m^2)} \) is not. The discrepancy for \( L_s = 48 \) may be due to lattice volume, but needs further study.

4. \( m_{\text{res}} \) in \( N_f = 2 \) QCD

For full QCD, we have done extensive simulations with the Wilson gauge action and do-
main wall fermions on $8^3 \times 4$ volumes for several values of $L_s$ with $\beta = 5.2$, $m_0 = 1.9$ and $m_f = 0.02$. For these lattices, which are in the low temperature phase, we show $\langle \bar{q}q \rangle / \chi_\pi$ in Figure 2. An exponential fit, yielding $\langle \bar{q}q \rangle / \chi_\pi = 0.02 + 0.082(3) \exp[-0.027(2)L_s]$ for $16 \leq L_s \leq 40$ with $\chi^2/N_{\text{dof}} = 2.76/2$, is also shown.

We have also done uncorrelated fits, using valence masses between 0.02 and 0.14, to extract $m_{\text{res}}^{(\text{GMOR})}$ and $b_0$, since for the dynamical simulations there is not enough data to resolve the covariance matrix. (For the quenched case there was little difference between the correlated and uncorrelated fits.) All fits have $N_{\text{dof}} = 4$ and $\chi^2/N_{\text{dof}} \lesssim 1$. $m_{\text{res}}^{(\text{GMOR})}$ is also shown in Figure 2 and the dashed line fit is $m_{\text{res}}^{(\text{GMOR})} = 0.17(2) \exp[-0.026(6)L_s]$ for $10 \leq L_s \leq 40$ with $\chi^2/N_{\text{dof}} = 0.35/4$, $b_0$ is not shown, but also fits the exponential form $b_0 = -0.0100(16) \exp[-0.0147(67)L_s]$ with $\chi^2/N_{\text{dof}} = 0.20/4$ over the same range in $L_s$.

In Table 1 we compare the two methods of extracting $m_{\text{res}}$ using valence spectrum data from $N_f = 2, 8^3 \times 32$ scale setting calculations. All data was fit for $0.02 \leq m_f^{(\text{val})} \leq 0.1$. For the GMOR fit, $N_{\text{dof}} = 2$ and $\chi^2/N_{\text{dof}} \lesssim 1$ for all fits. We note that the two methods agree within statistics.

We can also calculate $m_{\text{res}}$ both ways but with data as a function of the dynamical mass. Since there are only two dynamical masses, both methods are unconstrained so the errors quoted come from naive extrapolation. The results are summarized in Table 2.

### Table 1

| $L_s$ | $m_f$ | $m_{\text{res}}^{(\text{val})}$ | $m_{\text{res}}^{(\text{GMOR})}$ | $-b_0$ |
|------|-------|-------------------------------|-------------------------------|-------|
|      |       |                               |                               |       |
| Wilson gauge action, $\beta = 5.325, m_0 = 1.9$ |       |                               |                               |       |
| 24   | 0.02  | 0.0622(9)                     | 0.058(2)                      | 0.0047(3) |
|      | 0.06  | 0.0645(6)                     | 0.059(2)                      | 0.0046(2) |
| Iwasaki gauge action, $\beta = 1.9, m_0 = 1.9$ |       |                               |                               |       |
| 24   | 0.02  | 0.0401(5)                     | 0.038(2)                      | 0.0028(2) |
| Iwasaki gauge action, $\beta = 2.0, m_0 = 1.9$ |       |                               |                               |       |
| 24   | 0.02  | 0.0158(9)                     | 0.015(2)                      | 0.0013(3) |
|      | 0.06  | 0.019(1)                      | 0.017(3)                      | 0.0015(4) |
| 48   | 0.02  | 0.0073(16)                    | 0.011(2)                      | 0.0010(2) |

### Table 2

| $\beta$ | $m_{\text{res}}^{(\text{val})}$ | $m_{\text{res}}^{(\text{GMOR})}$ | $-b_0$ |
|---------|---------------------------------|---------------------------------|-------|
|         |                                 |                                 |       |
| 5.325   | 0.059(2)                        | 0.053(7)                        | 0.004(1) |
| 2.0     | 0.013(2)                        | 0.014(5)                        | 0.0011(9) |

### 5. Conclusions

We have gotten good agreement between $m_{\text{res}}^{(\text{val})}$ and $m_{\text{res}}^{(\text{GMOR})}$ for a wide range of quenched and dynamical simulations by including the non-pole terms in the susceptibilities. From our current data, $m_{\text{res}}^{(\text{GMOR})}$ appears less volume dependent. For the quenched simulations at $L_s = 48$, the two methods do not agree, possibly as a result of finite volume effects. This case warrants further study.

Whether chiral symmetry is fully restored for quenched simulations in the $L_s \to \infty$ limit of domain wall fermions is still an open question. For $N_f = 2$ QCD, $m_{\text{res}}^{(\text{GMOR})}$ falls exponentially, even at quite strong coupling. The rate of chiral symmetry restoration is very slow leaving much room for improvement.

All numerical calculations were performed on the 400 Gflops QCDSP machine at Columbia and the 6 Gflops QCDSP machine at Ohio State.

### REFERENCES

1. D.B. Kaplan, Phys. Lett. B288, 342 (1992).
2. V. Furman and Y. Shamir, Nucl. Phys. B439, 54 (1995).
3. G.R. Fleming et al., Nucl. Phys. Proc. Suppl. 73, 207 (1999).
4. P.M. Vranas, Phys. Rev. D57, 1415 (1998).
5. P.M. Vranas et al., Nucl. Phys. Proc. Suppl. 73, 456 (1999).
6. T. Blum, Nucl. Phys. Proc. Suppl. 73, 167 (1999).
7. R. Mawhinney et al., Nucl. Phys. Proc. Suppl. 73, 204 (1999).
8. L. Wu, these proceedings.
9. P.M. Vranas, these proceedings.