O(a) perturbative improvement for Wilson fermions

Satchidananda Naik*
Max-Planck-Institut für Physik
Werner-Heisenberg-Institut
Föhringer Ring 6
D-8000 Munich 40

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Abstract

The coefficients of the $O(a)$-improved Sheikholeslami-Wohlert action for Wilson fermions are perturbatively determined at one-loop level and estimated at two-loop level.
1. Introduction

Several groups are investing considerable effort to numerically calculate various hadronic decay constants and the mass spectrum with Wilson fermions [1]. One of the systematic errors, that due to the finite lattice spacing, can in principle be diminished by taking an improved action à la Symanzik [2]. In the weak coupling expansion the pure Yang-Mills theory has no $O(a)$ cutoff effects in the infinite volume or with a finite volume with periodic boundary conditions, however the fermionic part of the Wilson action introduces such terms. To eliminate these effects all independent dimension five operators are added to the action. The procedure drastically simplifies by demanding that only the on-shell physical quantities of the theory are improved [3]. Then only the operator $\bar{\psi}(x)\sigma_{\mu \nu}P_{\mu \nu}(x)\psi(x)$ needs to be added to the action where $P_{\mu \nu}$ is the term with four plaquettes touching at point $x$ in the form of a ‘clover leaf’, an equivalent of $F_{\mu \nu}$ [4]. The fermionic part of the action reads

$$S^W_F = a^4 \sum_{x,\mu} \left[ \bar{\psi}(x) \left[ (r - \gamma_\mu)U_\mu(x)\psi(x + \mu a) + (r + \gamma_\mu)U_\mu^\dagger(x - \mu a)\psi(x - \mu a) \right] ight]$$

$$+ \frac{i}{2} c(g_0^2) \sum_\mu \bar{\psi}(x)\sigma_{\mu \nu}P_{\mu \nu}(x)\psi(x)$$

(1)

with

$$P_{\mu \nu}(x) = \frac{1}{8} [U^P_{\mu,\nu}(x) + U^P_{-\mu,\nu}(x) + U^P_{\mu,-\nu}(x) + U^P_{-\mu,-\nu}(x) - (\mu \rightarrow \nu)]$$

(2)

where $U^P_{\mu,\nu}(x)$ is the untraced plaquette variable. The coefficient $c(g_0^2)$ ought to be fixed perturbatively to all orders of $g_0^2$. This coefficient happens to be quite important for the hadronic decay amplitude and for the hadron spectrum when the short distance effects play a role e.g. for the hyperfine splitting in charmonium. At tree level, the improvement condition fixes this to be ‘r’ the Wilson parameter [4]. The one-loop coefficient was first calculated by Wohlert [5] and also there is a mean field estimate given by Lepage and Mackenzie [6]. In this article we independently check this coefficient at the one loop level and give an approximate value in the two loop level.

2. General Strategy

To fix the coefficient $c(g_0^2)$ to all orders of perturbation theory a suitable on-shell quantity needs to be defined. The most obvious of such quantities which instantly comes to mind is to look for the pole of the fermion two point function. This however

[1] The action given in Ref. [5] is not exactly the clover action now used in numerical simulations. We found terms missing from the fermion-three gluon vertex function and errors in the expression of Clebsch-Gordon coefficients. This created a suspicion of a drastic change of the result since it involves the most dominant tadpole graph. We made an independent check however and found that the number quoted is correct and these errors are probably just misprints.
unfortunately cannot be related to $c(g_0^2)$ due to the Slavnov-Taylor identity

$$-i \frac{\partial}{\partial p_\mu} \Sigma(p) = \Gamma_\mu(p, p, 0), \quad (3)$$

and the fact that the improved part of the vertex function $\Gamma_\mu(p, p, k) \propto \sum \sigma_{\mu\nu} \sin k_\nu$ vanishes for $k = 0$. This clearly shows that to all orders of perturbation $c(g_0^2)$ is not sensitive to the on-shell fermion two point function. Another quantity of interest is

$$\langle \bar{\psi}(y) P e^{ig_0 \int_0^1 A_\mu(\xi) d\xi} \psi(x) \rangle,$$

however does not yield a tree level improvement condition. We were so far unable to find a suitable quantity which needs an improvement at tree level in the infinite volume. However, one possibility is to assume that our lattice has finite extent $L$ in $x_1$ and $x_2$ directions with twisted periodic boundary conditions. Due to these boundary conditions, quarks and gluons get mass by a Kaluza-Klein type of mechanism and some of the eigenvalue modes of the transfer matrix remain stable for small coupling in this twisted world. These states are created from the vacuum by gauge invariant operators such as a Wilson loop winding around the torus. The spectrum of the S-matrix for these gauge invariant modes in the LSZ scheme can be studied unambiguously as has been already done for the pure Yang-Mills theory to fix two necessary coefficients [3]. A more detailed discussion of this procedure is given in Ref.[3].

The twisted periodic boundary condition for the gauge field reads,

$$U(x + L_\nu, \mu) = \Omega_\nu U(x, \mu) \Omega_\nu^{-1}, \quad \nu = 1, 2, \quad (4)$$

where $\Omega_\nu$ are constant SU(N) matrices with

$$\Omega_1 \Omega_2 = e^{\frac{2\pi i}{N}} \Omega_2 \Omega_1. \quad (5)$$

To use the twisted boundary condition for fermions one introduces an extra internal degree of freedom “smell” [7] and a fermion belongs to the $\bar{N}_S \times N_C$ representation. The twisted antiperiodic boundary condition on the torus reads

$$\psi(x + L_\nu) = \Omega_\nu \psi(x) \Omega_\nu^{-1} e^{\frac{2\pi i}{N}} \quad \nu = 1, 2. \quad (6)$$

Thus the transverse momenta in these directions take discrete values as

$$p_\perp = (2n_1 + 1, 2n_2 + 1)m, \quad n_i \in Z, \quad (7)$$

where $m = \frac{2\pi}{LN}$. This offers a mass gap even though we have a zero bare mass fermion to start with. The Fourier decomposition of the gauge field reads

$$A_\mu(x) = \frac{1}{L^2 N} \sum_{k_\perp} \int \frac{d_3 k}{k_0 k_3} \exp(ikx) \Gamma_k \exp \frac{i k_\mu}{2} \tilde{A}_\mu(k) \quad (8)$$

where $k_\perp = 2mn_\nu$ and $\Gamma_k$ plays the role of SU(N) group generator. Since $A_\mu(x)$ is traceless, there exists no zero mode. The gluon propagator reads

$$D_{\mu\nu}(k) = -\frac{1}{2} \chi_k Z(k, k)[\delta_{\mu\nu}k^2 + (\alpha - 1)k_\mu k_\nu] \frac{1}{(k^2)^2}, \quad (9)$$
where
\[ \chi_k = \begin{cases} 0 & \text{if } k_\perp = 0, \text{(modN)} \\ 1 & \text{otherwise,} \end{cases} \]
(10)

\(\alpha\) is the gauge fixing parameter and \(\hat{k}_\mu = 2 \sin \frac{k_\perp}{2}\). Also
\[ Z(k, k') = z^\frac{1}{2}(<k,k'>-(k,k')) \]
(11)

where \(z = e^{2\pi i}, <k, k'> = n_1n_1' + n_2n_2' + (n_1 + n_2)(n_1' + n_2')\) and \((k, k') = n_1n_2' - n_2n_1'\). Following Ref. [3] we study the spectrum of the LSZ scattering process (c.f. Fig.1). The on-shell momenta for these fermions are \(p_1 = (iE, m, m, im)\), \(p_1' = (iE, m, -m, -im)\), \(p_2 = (iE, m, -m, -im)\) and \(p_2' = (iE, m, m, im)\). This kinematical choice of momenta simplifies the calculation drastically since the exchange diagram Fig.1b does not contribute and also the fermion wave function renormalization to \(O(a)\) does not contribute to the S-matrix elements. Following Ref. [3], we look for the residue of the pole of the scattering amplitude
\[ S = T_\mu Z_G^2(k)D_{\mu\nu}(k)T_{\nu}', \]
(12)

where \(T_\mu = \bar{U}(p_1) \Gamma_\mu(p_1, p_1', k) U(p_1')\) and \(Z_G(k)\) is the gluon wavefunction renormalization. Also \(T_{\nu}'\) can be obtained by replacing \(p_1 \rightarrow p_2\), \(p_1' \rightarrow p_2'\) and \(k \rightarrow -k\) in \(T_\mu\). The residue of the pole of this S-matrix element;
\[ \lambda = z(-k, p_1')T_\mu T_{\mu}'. \]
(13)

We assume here a perturbative expansion of this residue \(\lambda = \sum_{i=1}^{\infty} g_0^{2i} \lambda^i\) and also for \(T_\mu = \sum_{i=0}^{\infty} g_0^{2i+1} T_\mu^i\) and \(c(g_0^2) = c_0 + g_0^2c_1 + g_0^4c_2 + \ldots\).

To illustrate the case in the tree level
\[ (T_\mu^0)_{\alpha\beta} = T_\mu^0(1) + T_\mu^0(a) = z(k, p)\bar{U}_\alpha(p)[-\gamma_\mu - \frac{i}{2}(c_0 - r)(p + p')_\mu + 0(a^2)]U_\beta(p'). \]
(14)

We look for the residue in a particular polarization of the gluon say \(\mu = 1\) and in the fixed helicity state of the fermion say \(\alpha = \beta = 1\). For the zero bare mass fermion \(U(p) = (\sinh E)^{-1}(\gamma_0 \sinh E + i\gamma_i \hat{p}_i) U^1\), where \(U^1_\beta = \delta_{1, \beta}\). This gives
\[ T_1^0(1) = 2iz, \quad T_1^0(a) = -2izm(c_0 - r) \]
\[ T_1^0(1)' = -2i, \quad T_1^0(a)' = -2im(c_0 - r) \]
(15)

and
\[ \lambda^0(a) = (T_1^0(1) T_1^0(a) + T_1^0(a) T_1^0(1)) = 8zm(c_0 - r). \]
(16)
To demand the tree level improvement we set $\lambda^0(a) = 0$ and thus get the condition $c_0 = r$.

To all orders of perturbation theory $T_1(a)$ gets contributions from the fermion wave function renormalization, vertex function renormalization and also a term coming from the naive expansion of $c(g_0^2)$. So

$$T_1(a) = T^c_1 + T^{WF}_1 + T^{VF}_1.$$  \hspace{1cm} (17)

However for our kinematical choice of momenta, it can be proved that

$$T^0_1(1) T^{WF}_1(a) + T^{WF}_1(a) T^q_1(1) = 0.$$  \hspace{1cm} (18)

The contribution of the gluon wave function renormalization $Z_G(k)$ does not contribute to $\lambda$ for the $O(a)$ improvement. So it remains to calculate only $T^{WF}_1(a)$ and $T^{WF}_1'(a)$ to all orders of perturbation. The three point function $\bar{U}(p) \Gamma_{\mu}(p, p', k) U(p')$ is expressed as $\bar{U} \sum_i O_i B_i U$ where $O_i$’s are sixteen bilinear invariant Dirac basis matrices and $B_i$’s are their coefficients. It is sufficient to look for the residue in the particular channel which we have chosen, here $\propto \sigma_{12}$ for every diagram.

3. $O(g^2)$ Improvement

To one loop order there are six diagrams (c.f.Fig.2) contributing to $T_1(a)$. Due to the twisted periodic boundary conditions all these loop integrals depend on $L$ and $N$. Then the contributions are compared with their asymptotic expansion

$$I \approx \sum_{i=0}^{\infty} (\alpha_i + \beta_i \ln m)(m)^i + d$$  \hspace{1cm} (19)

where the lattice spacing is set to unity, $m = \frac{\pi}{LN}$ and $d$ is the degree of divergence of the graph. The method of evaluation of these graphs is exactly the same as in Ref.[3]. The analytical expressions for the vertex functions and loop integrands are quite lengthy and beyond the scope of this paper. So we present here only the results.

The imaginary part and the coefficient of the log $m$ for diagram 2.b exactly cancel with that of 2.c. This is also true for the diagram 2.d where this part cancels with the sum of the contributions coming from 2.e and 2.f. The small contributions coming from diagram 2.b-f are not checked by us, however we use here the results of Wohlert [5]. The contribution of each diagram is given in Table I. for $N = 2$ and $N = 3$ in the Feynman gauge. Needless to state that the total contributions of these graphs to the residue of the S-matrix is independent of the choice of the gauge.

**Table 1. Values of the one-loop graphs for $N = 2$ and $N = 3$.**

\[\text{This can be analytically checked in the following way. We take a very small generic external momenta } \epsilon \text{ for these diagrams and then make a Taylor’s expansion in } \epsilon \text{ of these loops to get the same tensor structure of all these loops by using symmetry of the integration of the internal momenta. (In the Feynman gauge expressions are considerably simpler.) Then we look for the logarithmic divergence term for each loop.}\]
Here also one observes that the contribution of 2.b and 2.c nearly cancel with each other so also of 2.d nearly cancel with 2.e. The only leftovers are the contributions of 2.a which is the most dominant one and the contribution of 2.f is nearly 10% of 2.a. This clearly shows the evidence of the tadpole dominance [3].

So the residue

\[ \lambda_1(a) = 4m [0.308 - 2c_1], \]  

for \( N = 2 \) and

\[ \lambda_1(a) = 4m [0.5318 - 2c_1] \]

for \( N = 3 \). This gives \( c_1 = 0.154 \) for \( N = 2 \) and \( c_1 = 0.2659 \) for \( N = 3 \).

4. \( O(g^4) \) Improvement

The numerical simulations for the hadron spectrum or decay amplitudes are performed for \( \beta \approx 6 \) i.e. \( g_0^2 \approx 1 \). So it is worthwhile also to fix the coefficient \( c_2 \). From our observation in the previous section and also the earlier investigation of perturbative computation of lattice graphs [3] there is an indication that the main contribution comes from the tadpole graph. In our case at one loop level all graphs nearly cancel with each other except the tadpole one and also all are less than 10% of the latter. Thus it seems legitimate to take only the perturbative correction from tadpoles as a first step towards a full and much lengthier calculation of the two loop improvement. The dominant two loop graphs are given in Fig.3. and the results are presented in Table II. Here again one also observes the tadpole dominance. The main contribution comes from Fig.3.a and the gauge invariant part of Fig.3c. The contribution of the latter which is like plaquette-plaquette correlation function \( D_{\mu\nu,\mu\nu} \), gives 0.1489, the largest among all these graphs. All other contributions are less than 10% of these two quantities.
Table II. Values of the two-loop graphs for $N = 2$ and $N = 3$.

| Fig. | $N=2$   | $N=3$   |
|------|---------|---------|
| 3.a  | 0.03403 | 0.10465 |
| 3.b  | -0.0259 | 0.00648 |
| 3.(c+d+e+f+g) | 0.05346 | 0.16174 |
| 3.(h+i) | $-n_f$ 0.01 | $-n_f$ 0.02 |

This gives $c_2 = 0.0207$ for $N = 2$ and $c_2 = 0.1164$ for $N = 3$ and $n_f = 2$.

5. Conclusion

A meanfield type of estimate of this coefficient is given by

$$c(g_0^2) = (\frac{1}{N} < trU_P >)^{-\frac{3}{4}}.$$  \hspace{1cm} (22)

The perturbative expansion of this plaquette expectation value reads

$$\frac{1}{N} < trU_P > = 1 - u_1 g_0^2 - u_2 g_0^4 + \ldots$$  \hspace{1cm} (23)

where

$$u_1 = \frac{(N^2 - 1)}{8N}, \quad u_2 = \frac{(N^2 - 1)}{48N} \left\{ \frac{2N^2 - 3}{8N} + NK_1 \right\}$$  \hspace{1cm} (24)

with $K_1 = -0.0042$. This gives $c_1 = 0.25$ and $c_2 = 0.098$ whereas we get $c_1 = 0.266$ and $c_2 = 0.1164$ with two flavors. It is quite gratifying to observe that by two different methods of calculation one gets a fair agreement of the estimate of these coefficients, which can now be used in numerical simulations with more confidence.

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Figure captions

Fig.1. The on-shell fermion fermion scattering

Fig.2. The one-loop graphs

Fig.3. The most dominant two-loop graphs