Abstract. We determine which faithful irreducible representations $V$ of a simple linear algebraic group $G$ are generically free for $\text{Lie}(G)$, i.e., which $V$ have an open subset consisting of vectors whose stabilizer in $\text{Lie}(G)$ is zero. This relies on bounds on $\dim V$ obtained in prior work (part I), which reduce the problem to a finite number of possibilities for $G$ and highest weights for $V$, but still infinitely many characteristics. The remaining cases are handled individually, some by computer calculation. These results were previously known for fields of characteristic zero, although new phenomena appear in prime characteristic; we provide a shorter proof that gives the result with very mild hypotheses on the characteristic. (The few characteristics not treated here are settled in part III.) These results are related to questions about invariants and the existence of a stabilizer in general position.

Introduction

Let $G$ be a simple linear algebraic group over a field $k$ acting faithfully on a vector space $V$. In the special case $k = \mathbb{C}$, there is a striking dichotomy between the properties of irreducible representations $V$ whose dimension is small (say, $\leq \dim G$) versus those whose dimension is large, see [AnVE], [E], [Po], etc., for original results and [PoV, §8.7] for a survey and bibliography. For example, if $\dim V < \dim G$, then (trivially) the stabilizer $G_v$ of a vector $v \in V$ is nontrivial. On the other hand (and nontrivially), for $\dim V$ hardly bigger than $\dim G$, the stabilizer $G_v$ for generic $v \in V$ is trivial, i.e., 1; in this case one says that $V$ is generically free or $G$ acts generically freely on $V$. This property has taken on increased importance recently due to applications in Galois cohomology and essential dimension, see [Re] and [Mer] for the theory and [BrRV], [GaGu 17], [K], [LöMMR], [Lö], etc. for specific applications.

With applications in mind, it is desirable to extend the results on generically free representations to all fields. In that setting, [GuL] has shown that, if $V$ is irreducible and $\dim V$ is large enough, then $G(k)$, the group of $k$-points of $G$, acts
generically freely. Equivalently (when \( k \) is algebraically closed), the stabilizer \( G_v \) of a generic \( v \in V \) is an infinitesimal group scheme. For applications, one would like to say that \( G_v \) is not just infinitesimal but is the trivial group scheme, for which one needs to know that the Lie algebra \( \mathfrak{g} \) of \( G \) acts generically freely on \( V \), i.e., \( \mathfrak{g}_v = 0 \). The two conditions are related in that \( \dim G_v \leq \dim \mathfrak{g}_v \), so in particular if \( \mathfrak{g}_v = 0 \), then \( G_v \) is finite. On the other hand, if \( G_v(k) = 1 \), \( \mathfrak{g}_v \) can be nontrivial (i.e., \( G_v \) may be a nontrivial infinitesimal group scheme), see Example 4.2.

In a previous paper, part I [GaGu I], we proved that, roughly speaking, if \( \dim V \) is large enough (where the bound grows like \( \left( \text{rank } G \right)^2 \)) and \( \text{char } k \) is not special, then \( V \) is a generically free \( \mathfrak{g} \)-module. In this paper, we restrict our focus to irreducible modules and settle the question of whether or not \( \mathfrak{g} \) acts generically freely when \( \text{char } k \) is not special.

**Theorem A.** Let \( G \) be a simple linear algebraic group over a field \( k \) and let \( \rho : G \to \text{GL}(V) \) be a faithful and irreducible representation of \( G \). Then \( V \) is generically free for \( \mathfrak{g} \) if and only if \( \dim V > \dim G \) and \( (G, \text{char } k, V) \) is not in Table 1.

**Table 1.** Irreducible and faithful representations \( V \) of simple \( G \) with \( \dim V > \dim G \) that are not generically free for \( \mathfrak{g} \), up to graph automorphism. For each, the stabilizer \( \mathfrak{g}_v \) of a generic \( v \in V \) is a toral subalgebra. The weights are numbered as in Table 2.

| \( G \)      | char \( k \) | rep’n or hst. wt. | \( \dim V \) | \( \dim \mathfrak{g}_v \) |
|-------------|-------------|------------------|-------------|------------------|
| \( \text{SL}_8 / \mu_4 \) | 2          | \( \wedge^4 \)  | 70          | 3                |
| \( \text{SL}_9 / \mu_3 \) | 3          | \( \wedge^3 \)  | 84          | 2                |
| \( \text{Spin}_{16} / \mu_2 \) | 2          | half-spin        | 128         | 4                |
| \( \text{Sp}_4 \) | 5          | 11               | 12          | 1                |
| \( \text{Sp}_8 \) | 3          | 0100             | 40          | 2                |
| \( \text{SL}_4 \) | \( p \) odd | 01p^e, \( e \geq 1 \) | 24          | 1                |
| \( \text{SL}_4 / \mu_2 \) | 2          | 012^e, \( e \geq 2 \) | 24          | 1                |

We say that \( G \) acts faithfully on \( V \) or \( \rho \) is faithful if \( \ker \rho \) is the trivial group scheme. Regardless, there is an induced map \( G/\ker \rho \to \text{GL}(V) \) that is a faithful representation of \( G/\ker \rho \).

We say that char \( k \) is special for \( G \) if char \( k = p \neq 0 \) and the Dynkin diagram of \( G \) has a \( p \)-valent bond, i.e., if char \( k = 2 \) and \( G \) has type \( B_n \) or \( C_n \) for \( n \geq 2 \) or type \( F_4 \), or if char \( k = 3 \) and \( G \) has type \( G_2 \). Equivalently, these are the cases where \( G \) has a very special isogeny. This definition of special is as in [S 63, §10]; in an alternative history, these primes might have been called “extremely bad” because they are a subset of the very bad primes — the lone difference is that for \( G \) of type \( G_2 \), the prime 2 is very bad but not special. In this paper, we prove Theorem A when char \( k \) is not special, and we typically assume that char \( k \) is not special in the rest of this paper. (We do consider some examples where char \( k \) is special, such as for type \( B \) in characteristic 2 in §5.) The case where char \( k \) is special has a different flavor and will be handled in a separate paper, [GaGu III, part III].

We remark that the exceptions in Theorem A, listed in Table 1, can be divided into types. In the first three rows are “\( \theta \)-group” representations, which arise from embedding \( \mathfrak{g} \) in some larger Lie algebra with a finite grading, and the generic
stabilizer $G_v$ is a non-étale, non-infinitesimal finite group scheme. (Premet’s appendix in [GaGu 17] gives a detailed study of the half-spin representation of $D_8$ in characteristic 2. For the other two representations in the left column, see Remark 7.3, [Auld, 4.8.2, 4.9.2], or [GuL, 3.1].) The remaining rows contain two representations where the generic stabilizer is a nonzero infinitesimal group scheme, see Examples 4.2 and 5.2, and two that decompose as tensor products from Example 10.4.

The proof of Theorem A relies heavily on the results of part I, [GaGu I], which include the case of type $A_1$ (ibid., Examples 1.8 and 3.3) and the case where $\dim V$ is large (ibid., Th. A). We also use Magma [BoCP] to verify that certain specific representations in specific characteristics are generically free, a process described in Section 3; the key point is that it suffices to find any vector in the representations whose stabilizer is zero. In Section 4, we prove a criterion by which Magma can verify that a representation is not generically free for $g$. To handle representations where we do not specify the characteristic of the base field, we recall a means to transfer results from characteristic zero (Section 5). Three sections treating specific classes of representations (Sections 6–8) lead to the proof of Theorem A when $\text{char } k$ is not special. This proof occupies Sections 9–11. The first of these treats the case where the highest weight is restricted. The second handles, roughly speaking, the case of a tensor decomposable representation. The third and final section treats the few remaining cases, which are tensor decomposable as representations of the simply connected cover but not necessarily for $G$ itself.

**Notation**

For convenience of exposition, we will assume in most of the rest of the paper that $k$ is algebraically closed of characteristic $p \neq 0$. This is only for convenience, as our results for $p$ prime immediately imply the corresponding results for characteristic zero: simply lift the representation from characteristic 0 to $\mathbb{Z}$ and reduce modulo a sufficiently large prime.

Let $G$ be an affine group scheme of finite type over $k$. If $G$ is additionally smooth, then we say that $G$ is an algebraic group. An algebraic group $G$ is simple if its radical is trivial (i.e., it is semisimple), it is $\neq 1$, and its root system is irreducible. For example, $\text{SL}_n$ is simple for every $n \geq 2$.

If $G$ acts on a variety $X$, the stabilizer $G_x$ of an element $x \in X(k)$ is a subgroup-scheme of $G$ with $R$-points

$$G_x(R) = \{ g \in G(R) \mid gx = x \}$$

for every $k$-algebra $R$. A statement “for generic $x$” means that there is a dense open subset $U$ of $X$ such that the property holds for all $x \in U$.

If $\text{Lie}(G) = 0$ then $G$ is finite and étale. If additionally $G(k) = 1$, then $G$ is the trivial group scheme $\text{Spec } k$.

We write $\mathfrak{g}$ for $\text{Lie}(G)$ and similarly $\mathfrak{spin}_n$ for $\text{Lie}(\text{Spin}_n)$, etc. We put $\mathfrak{z}(\mathfrak{g})$ for the center of $\mathfrak{g}$; it is the Lie algebra of the (scheme-theoretic) center of $G$. As $\text{char } k = p$, the Frobenius automorphism of $k$ induces a “$p$-mapping” $x \mapsto x^{[p]}$ on $\mathfrak{g}$. When $G$ is a sub-group-scheme of $\text{GL}_n$ and $x \in \mathfrak{g}$, the element $x^{[p]}$ is the $p$th power of $x$ with respect to the typical, associative multiplication for $n$-by-$n$
matrices, see [DGa, §II.7, p. 274]. An element \( x \in \mathfrak{g} \) is nilpotent if \( x^{[p]^n} = 0 \) for some \( n > 0 \), toral if \( x^{[p]} = x \), and semisimple if \( x \) is contained in the Lie \( p \)-subalgebra of \( \mathfrak{g} \) generated by \( x^{[p]} \), cf. [SF, §2.3].

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1. Dominant weights and faithful representations

Let \( G \) be a reductive group over the (algebraically closed) field \( k \), and fix a pinning for \( G \) which includes a maximal torus \( T \), a Borel subgroup containing \( T \), and generators for the root subalgebras of \( \mathfrak{g} \).

Irreducible representations of \( G \), up to equivalence, are in one-to-one correspondence with dominant weights (relative to the fixed pinning), where the correspondence is given by sending a representation to its highest weight. We write \( L(\lambda) \) for the irreducible representation with highest weight \( \lambda \), imitating the notation in [J].

We number dominant weights as in Table 2, imitating [Lü] to make it convenient to refer to that paper. We write \( c_1 c_2 c_3 \cdots c_\ell \) as shorthand for the dominant weight \( \sum c_i \varpi_i \), where \( \varpi_i \) is the fundamental dominant weight corresponding to the vertex \( i \) in Table 2. The weight \( \sum c_i \varpi_i \) is restricted if \( \text{char } k = 0 \), or if \( \text{char } k \neq 0 \) and \( 0 < c_i < \text{char } k \) for all \( i \).

Table 2. Dynkin diagrams of simple root systems of classical type, with simple roots numbered as in [Lü].

\[
\begin{align*}
(A_\ell) & : \quad 1 \quad 2 \quad 3 \quad \cdots \quad \ell-2 \quad \ell-1 \quad \ell \\
(B_\ell) & : \quad 1 \quad \ell \quad 2 \quad 3 \quad \cdots \quad \ell-2 \quad \ell-1 \quad \ell \\
(C_\ell) & : \quad 1 \quad 2 \quad 3 \quad \cdots \quad \ell-2 \quad \ell-1 \quad \ell \\
(D_\ell) & : \quad 1 \quad 2 \quad 3 \quad \cdots \quad \ell-2 \quad \ell-1 \quad \ell
\end{align*}
\]

The paper [Lü] studies \( L(\lambda) \) when \( \lambda \) is restricted. When \( \lambda \) is not restricted, we have the following statement: If \( \lambda = \lambda_0 + p\lambda_1 \) for \( \lambda_0 \in T^* \) dominant and restricted, \( \lambda_1 \in T^* \) dominant, and \( p = \text{char } k \), then \( L(\lambda) \cong L(\lambda_0) \otimes L(\lambda_1)^{[p]} \), see [J, II.3.16]. Here \( L(\lambda_1)^{[p]} \) denotes the Frobenius twist of the representation \( L(\lambda_1) \) as in [J, I.9.10]; \( \mathfrak{g} \) acts trivially on it.

Slightly more delicate analysis is required to handle the case where \( \lambda_0, \lambda_1 \) are not assumed to belong to \( T^* \), see Example 10.4 for an illustration.

Lemma 1.1. Let \( G \) be a simple algebraic group over \( k \) of characteristic \( p \neq 0 \). Let \( \lambda = \lambda_0 + p\lambda_1 \) for \( \lambda_0, \lambda_1 \) dominant weights and \( \lambda_0 \) restricted.

(1) If \( \mathfrak{g} \) acts faithfully on \( L(\lambda) \), then \( \lambda_0 \neq 0 \).

(2) If \( \lambda_0 \) and \( \lambda_1 \) are both nonzero, then \( \dim L(\lambda) > \dim G \).
Proof. We write \( \tilde{G} \) for the simply connected cover of \( G \).

For (1), suppose \( \lambda_0 = 0 \). Then \( L(\lambda) \) is isomorphic to a Frobenius twist \( L(\lambda_1)^{[p]} \) as a representation of \( \tilde{G} \), and the composition \( \tilde{g} \to g \to gl(L(\lambda)) \) is zero. As \( \tilde{g} \to g \) is not itself the zero map, \( g \) does not act faithfully on \( L(\lambda) \).

For (2), we may assume that \( G \) is simply connected, so that \( L(\lambda)^{[p]} = L(0) \) and \( \dim L(\lambda) = \dim L(0) \cdot \dim L(1) \). This, in turn, is at least \( m(G)^2 \) for \( m(G) \) the dimension of the smallest nontrivial irreducible representation of \( G \). We list these values in Table 3, obtained from \([\text{L} u]\). We note that in each case 

\[
m(G)^2 > \dim G, \tag{1.2}
\]

proving (2). \( \Box \)

\begin{table}
\begin{tabular}{|c|c|c|c|c|c|c|c|c|}
\hline
\( G \) & \( A_\ell \) & \( B_\ell (\ell \geq 3) \) & \( C_\ell (\ell \geq 2) \) & \( D_\ell (\ell \geq 4) \) & \( E_6 \) & \( E_7 \) & \( E_8 \) & \( F_4 \) & \( G_2 \) \\
\hline
\( m(G) \) & \( \ell + 1 \) & \( 2\ell + 1 - \varepsilon \) & \( 2\ell \) & \( 2\ell \) & 27 & 56 & 248 & 26 - \varepsilon & 7 - \varepsilon \\
\hline
\( \dim G \) & \( \ell^2 + 2\ell \) & \( 2\ell^2 + \ell \) & \( 2\ell^2 + \ell \) & \( 2\ell^2 - \ell \) & 78 & 133 & 248 & 56 & 14 \\
\hline
\end{tabular}
\end{table}

2. Results from part I

For a representation \( V \) of simple \( G \) with \( V^{[g,g]} = 0 \) (such as in Theorem A in this paper), we showed in part I ([\(\text{GaGu I}, \text{Thm. A}\]): If \( \dim V > b(G) \) for \( b(G) \) as in Table 4 and \( \text{char } k \) is not special, then \( g \) acts virtually freely on \( V \). Here, virtually free means that the stabilizer \( g_v \) for a generic \( v \in V \) equals \( \ker[g \to gl(V)] \), i.e., \( g_v \) is as small as possible. It is the natural notion that generalizes “generically free” to allow for the case where the kernel is not zero.

\begin{table}
\begin{tabular}{|c|c|c|c|}
\hline
\text{type of } G & \text{char } k & b(G) & \text{type of } G & \text{char } k & b(G) \\
\hline
A_\ell & \neq 2 & 2.25(\ell + 1)^2 & G_2 & \neq 3 & 48 \\
A_\ell & = 2 & 2\ell^2 + 4\ell & F_4 & \neq 2 & 240 \\
B_\ell & \neq 2 & 8\ell^2 & E_6 & \text{any} & 360 \\
C_\ell & \neq 2 & 6\ell^2 & E_7 & \text{any} & 630 \\
D_\ell & \neq 2 & 2(2\ell - 1)^2 & E_8 & \text{any} & 1200 \\
D_\ell & = 2 & 4\ell^2 & & & \\
\hline
\end{tabular}
\end{table}

Recall that \( g := \text{Lie}(G) \). For \( x \in g \), put

\[
V^x := \{ v \in V \mid d\rho(x)v = 0 \}
\]

and \( x^G \) for the \( G \)-conjugacy class \( \text{Ad}(G)x \) of \( x \). We are going to verify the inequality

\[
\dim x^G + \dim V^x < \dim V \tag{2.1}
\]
for various $x \in \mathfrak{g}$. The following lemma is Lemma 1.6 in part I; it resembles [AnP, Lem. 4], [Gue, §3.3], and [GaGu 17, Lem. 2.6].

**Lemma 2.2.** Suppose $G$ is semisimple over an algebraically closed field $k$ of characteristic $p > 0$, and let $\mathfrak{h}$ be a subspace of $\mathfrak{g}$.

1. If inequality (2.1) holds for every toral or nilpotent $x \in \mathfrak{g} \setminus \mathfrak{h}$, then $\mathfrak{g}_v \subseteq \mathfrak{h}$ for generic $v \in V$.

2. If $\mathfrak{h}$ consists of semisimple elements and (2.1) holds for every $x \in \mathfrak{g} \setminus \mathfrak{h}$ with $x^{[p]} \in \{0, x\}$, then $\mathfrak{g}_v \subseteq \mathfrak{h}$ for generic $v$ in $\mathfrak{g}$.

Taking $\mathfrak{h} = \mathfrak{z}(\mathfrak{g})$ in Lemma 2.2, we see that verifying (2.1) for nonzero nilpotent and noncentral toral $x \in \mathfrak{g}$ implies that $\mathfrak{g}_v \subseteq \mathfrak{z}(\mathfrak{g})$ for generic $v \in V$. This in turn implies that the action is virtually free since $Z(G)$ is a diagonalizable group scheme for the groups $G$ we consider here (so $\mathfrak{g}_v \cap \mathfrak{z}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})_v = \ker d\rho$).

Theorem 12.2 in part I proved a somewhat stronger result than the one stated at the start of this section: If $V$ is a representation of a simple group $G$ such that char $k$ is not special for $G$, $V^{[\ell]} = 0$, and dim $V > b(G)$, then (2.1) holds for all noncentral $x \in \mathfrak{g}$ with $x^{[p]} \in \{0, x\}$.

3. Constructing representations in Magma

In order to prove Theorem A, the results of part I reduce us to considering a finite list of irreducible representations, each of which we will consider. Some of these will be dealt with by invoking calculations done on a personal computer using Magma [BoCP], which we now explain. (Code and output are available at github.com/skipgaribaldi/genfree-code.)

The Magma instructions

\[
\begin{align*}
\mathbb{R} & := \text{IrreducibleRootDatum}(T, \ell); \\
\mathfrak{g} & := \text{LieAlgebra}(\mathbb{R}, \text{GF}(q));
\end{align*}
\]

create a Lie algebra $\mathfrak{g}$ of the split reductive group over the finite field $\mathbb{F}_q$ with root datum $\mathbb{R}$ of type $T_\ell$ (by default simply connected). For a given highest weight $\lambda$,

\[
\text{HighestWeightRepresentation}(\mathfrak{g}, \lambda);
\]

gives a homomorphism $\rho$ from $\mathfrak{g}$ to matrices, and one can identify the space of row vectors $v$ where the action by $\mathfrak{g}$ is $v \mapsto v\rho(x)$ for $x \in \mathfrak{g}$ with the representation $H^0(\lambda)$ of $G$ in the notation of [J]; it is induced from the 1-dimensional representation $\lambda$ of the Borel subgroup. The vector $(1, 0, \ldots, 0)$, the first basis vector in the Magma ordering, is a highest weight vector and it generates a submodule $V$ that is irreducible with highest weight $\lambda$. (Untrusting readers can verify that the submodule generated by this vector has the same dimension as the irreducible representation with the same highest weight as recorded in the literature, and therefore the submodule is the desired irreducible representation.)
For any row vector $v$, it is then a matter of linear algebra to compute the stabilizer $g_v$, i.e., the subspace of $x \in g$ such that $\nu \rho(x) = 0$. It is determined by $\text{Kernel}(\text{VerticalJoin}([\nu \rho(y) : y \text{ in } \text{Basis}(g)]))$.

To verify that a particular $V$ is virtually free, we use $\text{Random}(V)$ to generate random vectors $v \in V$. For each, we compute $\dim g_v$. By upper semicontinuity of dimension, $\dim g_v$ is at least as big as $\dim g_w$ for $w$ generic in $V$. Therefore, if we find any $v \in V$ with $\dim g_v = \dim \ker d\rho$, we have verified that the representation is virtually free.

Remark. Suppose $q: \tilde{G} \to G$ is a central isogeny. Its differential $dq: \tilde{g} \to g$ need not be surjective, i.e., $\ker q$ need not be étale. Nonetheless, if $g$ acts virtually freely on $V$, then so does $\tilde{g}$. Therefore, in the computer calculations described above we work with $g$, the Lie algebra of the group $G$ that acts faithfully on $V$. In Magma, this can be done by invoking the optional argument $\text{Isogeny}$ for $\text{IrreducibleRootDatum}$.

(If we instead assume that $\tilde{g}$ acts virtually freely on $V$, it may occur that $g$ does not. For example, that is the case when char $k = 2$ and (a) $\tilde{G} = \text{SL}_4$, $G = \text{PGL}_4$, and $V$ has highest weight $\omega_2 + 2\omega_3$ as in Example 10.4 or (b) $\tilde{G} = \text{Sp}_8$, $G = \text{PSp}_8$, and $V$ is the 16-dimensional irreducible “spin” representation as in [GaGu III, §8].)

4. Examples where $g$ does not act virtually freely

Lemma 4.1. Let $V$ be a representation of a reductive algebraic group $G$, and suppose that Cartan subalgebras in $g$ are maximal toral subalgebras\(^2\). If there is a $v \in V$ such that

1. $h := g_v$ is a toral subalgebra;
2. $\dim h = \text{rank } G$; and
3. $\dim G - \text{rank } G = \dim V - \dim V^h$,

then there is an open subset $U$ of $V$ containing $v$ such that $g_u$ is a $G$-conjugate of $h$ for every $u \in U$ and there is a maximal torus $T$ such that $G_u$ is $G$-conjugate to a closed sub-group-scheme of $N_G(T)$ for every $u \in U$.

Proof. Because $G$ is reductive and $h$ is toral, there exists a maximal torus $T$ in $G$ whose Lie algebra $t$ contains $h$, see [H, Thm. 13.3, Rem. 13.4]. Since $h = g_v$, the two are equal. In particular, $T$ normalizes $V^h$. Moreover, any element of $G$ that normalizes $h$ also normalizes $3(h) = t$, so $N_G(h) \subseteq N_G(t) = N_G(T)$ (where the latter equality is by the hypothesis on $g$ [DGr, XIII.6.1b]) and $N_G(h)^o = T$.

Put $\tilde{U}$ for the set of $v' \in V^b$ such that $\dim g_{v'}$ is minimal; it is open in $V^b$. On the one hand, $h \subseteq g_{v'}$, and on the other hand, $v \in V^b$, so $\dim g_{v'} \leq \dim h$, whence $g_{v'} = h$ for all $v' \in \tilde{U}$ and $v$ is in $\tilde{U}$. It follows that, if $g \in G(k)$ satisfies $gv' = v$, then $g$ normalizes $h$.

Define $\psi: G \times V^b \to V$ by $\psi(g, w) = gw$. By the preceding paragraph, for generic $w \in V^b$, $\psi^{-1}(w) = \{(g, g^{-1}w) \mid g \in N_G(h)\}$. That is,

$\dim \text{im } \psi = \dim G + \dim V^b - \dim N_G(h),$

\(^2\)This condition is equivalent to condition (2) in Lemma 7.1 by [DGr, XIII.6.1d].
which is \( \dim V \) by (3). Thus \( \psi \) is dominant and there is an open subset \( U \) of \( V \) consisting of elements whose stabilizer in \( g \) is conjugate to \( h \). 

In the language of [PoV, §2.8], the proof shows that \( V^h \) is a “relative section” for the action of \( G \) on \( V \).

The hypotheses of Lemma 4.1 are easy to verify with a computer. For example, to check that \( g \) is toral, one checks that it is abelian (Magma’s \texttt{IsAbelian}) and that a basis consists of semisimple elements (by checking, for each basis vector \( x \), that \( x \) belongs to the subspace spanned by \( x^{[p]} \) for \( i \geq 1 \)).

**Example 4.2 (\( \mathbb{C}_4, 0100, p = 3 \)).** Consider now \( G = \text{Sp}_8 \) over a field \( k \) of characteristic 3. (See Prop. 8.3(3) for the case \( \text{char } k \neq 2, 3 \).) It has a unique irreducible representation \( V \) with \( \dim V = 40 \) [Lu], which occurs as a quotient of the Weyl module of dimension 48 contained in \( \wedge^3(k^8) \) (with \( k^8 \) as the other composition factor), compare [PrS] or Proposition 8.3. Using Magma, one can construct \( V \) (say, with \( k = \mathbb{F}_3 \)) as in the preceding section and verify that for a random \( v \in V \), in the notation of Lemma 4.1, \( \dim h = 2 \) and \( \dim V^h = 8 \). It follows that \( g \) does not act virtually freely on \( V \). On the other hand, \( G_v(k) = 1 \) for generic \( v \in V \) by [GuL], so this is an example of a representation where the scheme-theoretic generic stabilizer \( G_v \) is a nontrivial and infinitesimal group scheme.

Lemma 4.1 shows also that the representation of \( \text{Sp}_4 \) in Table 1 is not virtually free, see Example 5.2.

5. Representations defined over a localization of the integers

Recall that \( G \) is defined over an algebraically closed field \( k \) of characteristic \( p \), and in particular is split. Let now \( R \) be a subring of \( \mathbb{Q} \) with homomorphisms to \( \mathbb{F}_p \) and to a field \( K \) containing a primitive \( p \)th root of unity \( \zeta \) (e.g., take \( R = \mathbb{Z} \) and \( K = \mathbb{C} \)). There exists a smooth affine group scheme \( G_R \) over \( R \) which is split and such that \( G_R \times k \) is isomorphic to \( G \).

**Lemma 5.1.** Let \( \rho: G_R \rightarrow \text{GL}(V) \) be a homomorphism of group schemes over \( R \) for some free \( R \)-module \( V \). Then the following are equivalent:

1. \( \dim x^G + \dim(V_k)^x < \dim V \) for all noncentral \( x \in g \) such that \( x^{[p]} = x \).
2. \( \dim g^{G_K} + \dim(V_K)^g < \dim V \) for all noncentral \( g \in G_R(K) \) such that \( g^p = 1 \).

Here and below we use the shorthand \( X_F \) for \( X_R \times F \), where \( X_R \) is an \( R \)-scheme and there is an implicit homomorphism \( R \rightarrow F \).

**Proof.** This is essentially §3.4 in [Auld], which we reproduce here for the convenience of the reader. Pick a split maximal torus \( T_R \) in \( G_R \) and a basis \( \tau_1, \ldots, \tau_\ell \) of the lattice of cocharacters \( \mathbb{G}_m \rightarrow T_R \). Identifying the Lie algebra of \( \mathbb{G}_m \) with \( k \), the elements \( h_j := \partial \tau_j(1) \) make up a basis of the Lie algebra \( t \) of \( T_R \times \mathbb{F}_p \) such that \( h_j^{[p]} = h_j \). This gives a bijection of toral elements in \( t \) with elements of order \( p \) in \( T_k \) via
\[ \psi: \sum c_j h_j \mapsto \prod \tau_j(\zeta^{c_j}) \quad \text{for } c_j \in \mathbb{F}_p. \]

There is a basis \( \chi_1, \ldots, \chi_\ell \) of the lattice of characters \( T_R \to \mathbb{G}_m \) such that \( \chi_i \circ \tau_j: \mathbb{G}_m \to \mathbb{G}_m \) is the identity for \( i = j \) and trivial for \( i \neq j \), hence \( d\chi_i(h_j) = \delta_{ij} \) for all \( i, j \). Writing a character \( \chi \) as \( \sum d_i \chi_i \) for \( d_i \in \mathbb{Z} \), we find

\[ \chi(\psi(\sum c_i h_i)) = \prod_i \zeta^{c_i d_i} = \zeta^{\sum c_i d_i} = \zeta^{d(\sum c_i h_i)} \]

for \( c_i \in \mathbb{F}_p \). That is, for toral \( x \in \mathfrak{t} \), \( d\chi(x) = 0 \) in \( \mathbb{F}_p \) if and only if \( \chi(\psi(x)) = 1 \). Decomposing \( V \) as a sum of weight spaces relative to \( T_R \) (using that \( R \) is an integral domain), we find that \( \dim(V_k)^x = \dim(V_K)^{\psi(x)} \).

The centralizer in \( \mathfrak{g} \) of \( x \) and the centralizer in \( G_K \) of \( \psi(x) \) contain \( \text{Lie}(T_k) \) and \( T_K \), so their identity components are generated by that and the root subalgebras or subgroups corresponding to roots vanishing on \( x \) or \( \psi(x) \) respectively. As in the preceding paragraph, we find that the centralizers of \( x \) and \( \psi(x) \) have the same dimension, hence (a) \( x \) is central in \( \mathfrak{g} \) if and only if \( \psi(x) \) is central in \( G_K \) and (b) \( \dim x^G = \dim \psi(x)^{G_K} \). The equivalence of (1) and (2) follows. \( \square \)

We now consider five examples and show, in most cases, that inequality (2.1) holds. We use Lemma 5.1 to handle the elements with \( x^{[p]} = x \). In the cases where the characteristic \( p \) module is the reduction of a characteristic 0 module, it suffices to prove the inequality for elements of order \( p \) in the group over \( \mathbb{C} \). In all the examples below, this has been confirmed in [GuL, 2.5.10, 2.5.17, 2.5.18, 2.5.24, 2.6.10]. It is also straightforward to use Magma to compute this in all the examples below as the modules have small dimension. One can also use closure arguments to reduce to the case of nilpotent elements. Thus, it suffices to consider elements \( x \) with \( x^{[p]} = 0 \).

**Example 5.2** (B\(_2\), 11). Let \( G = \text{Spin}_5 \cong \text{Sp}_4 \) and take \( V \) to be the irreducible representation of dimension 12 (if \( \text{char } k = 5 \)) or 16 (if \( \text{char } k \neq 5 \)). It occurs as a composition factor of the tensor product \( X \) of the two fundamental irreducible representations.

In case \( \text{char } k = 5 \), we apply Lemma 4.1. One finds \( \dim \mathfrak{h} = 1 \) and \( \dim V^\mathfrak{h} = 4 \), so \( V \) is not virtually free. We remark that in this case again \( G_v(k) = 1 \), so \( G_v \) is a nonzero infinitesimal group scheme.

In case \( \text{char } k = 2 \), we verify that \( V \) is generically free for \( \mathfrak{g} \) using Magma as in §3.

So assume \( \text{char } k \neq 2, 5 \). As \( X \) is self-dual, it is a direct sum of \( V \) and \( X/V \), the natural representation of \( \text{Sp}_4 \). In this case we argue that \( V \) is virtually free by verifying (2.1).

A long root element \( x \) has a single Jordan block of size 2 on the natural module and 2 Jordan blocks of size 2 on the 5-dimensional module. Since \( \text{char } k \neq 2 \), \( x \) has partition \( (3^2, 2^5, 1^4) \) on \( X \), so \( \dim X^x = 11 \). Since \( X/V \) is the 4-dimensional symplectic module, \( \dim(X/V)^x = 3 \), so \( \dim V^x = 8 \). As \( \dim x^G = 4 \), (2.1) is verified.

For any other nilpotent class, the closure of \( x^G \) in \( \mathfrak{sp}_4 \) contains a nilpotent element with partition \( (2, 2) \), so \( \dim V^x \leq 6 \); as \( \dim x^G \leq 8 \), the inequality is verified.
Example 5.3 (B₃, 101). Let \( G = \text{Spin}_7 \) and take \( V \) to be the irreducible representation of dimension 40 (if \( \text{char } k = 7 \)) or 48 (if \( \text{char } k \neq 7 \)). It occurs as a composition factor of the tensor product \( X \) of the natural and spin representations. In case \( \text{char } k = 2 \) or 7, we construct the representation in Magma as in §3 and observe that it is generically free.

So suppose \( \text{char } k \neq 2, 7 \). Then \( X \) is self-dual so it is a direct sum of \( V \) and \( X/V \), the spin representation. As in the preceding example, we argue that \( V \) is virtually free by verifying (2.1). Suppose that \( x \) is nilpotent. For \( x \) with partition \((3^2, 1)\) on the natural representation, \( \dim V^x \leq 22 \) and \( \dim x^G = 14 \). If \( x \) has partition \((7) \) or \((5, 1^2)\), then \( \dim x^G \leq 18 \) and \( \dim V^x \leq 22 \) (by specialization). A long root element \( x \) (partition \((2^2, 1^3)\)), has \( \dim x^G = 8 \) and \( \dim V^x = 34 \). The remaining possibilities for \( x \) have partition \((3, 2^2) \) or \((3, 1^4)\), which have \( \dim x^G = 12 \) or 10 and by specialization \( \dim V^x \leq 34 \).

Example 5.4 (D₄, 1001). Consider the representation \( V \) of \( G = \text{Spin}_8 / \mu_2 \) with highest weight 1001. In case \( \text{char } k = 2 \), \( \dim V = 48 \) and we verify with Magma that \( V \) is generically free for \( \mathfrak{g} \).

So suppose \( \text{char } k \neq 2 \), in which case \( \dim V = 56 \). Writing \( V_i \) with \( i = 1, 2, 3 \) for the three inequivalent irreducible 8-dimensional representations, we find \( X := V_1 \otimes V_2 \cong V \oplus V_3 \).

Suppose that \( x \) is nonzero nilpotent with \( \dim x^G < 22 \). Certainly \( \dim V^x \leq \dim V^y \) for a root element \( y \). Such a \( y \) has two Jordan blocks of size 2 on the \( V_i \)'s, and so \( y \) acts on \( X \) with partition \((3^4, 2^16, 1^{20})\). Thus \( \dim X^y = 40 \) and \( \dim V^y = \dim X^y - \dim V_3^y = 34 \), and the inequality is verified for \( x \).

We now divide into cases based on the partition of \( x \) on one of the \( V_i \)'s. If \( x \) only has Jordan blocks of size at most 3, then \( \dim x^G < 21 \) and we are done by the previous paragraph.

If \( x \) has two Jordan blocks of size 4, then \( \dim V^x < 16 \). If \( x \) has a Jordan block of size \( \geq 5 \), then \( \dim V^x < 20 \). In either case, as \( \dim x^G \leq 24 \), the inequality is verified. In summary, \( V \) is generically free for \( \mathfrak{g} \).

Example 5.5 (D₅, 20000, char \( k \neq 2 \)). Consider the representation \( V \) of the group \( G = \text{SO}_{10} \) with highest weight 20000 of dimension 126 over a field \( k \) of characteristic different from 2. For one of the half-spin representations \( X \), the second symmetric power \( S^2 X \) is a direct sum of \( V \) and the natural 10-dimensional module.

A root element \( x \in \mathfrak{g} \) has a 12-dimensional fixed space on \( X \) and so has 4 nontrivial Jordan blocks. On \( S^2 X \), it has a fixed space of dimension 84 hence \( \dim V^x = 76 \). Therefore, for every nonzero nilpotent \( x \in \mathfrak{g} \), we have \( \dim V^x \leq 76 \) and of course \( \dim x^G \leq \dim G - \text{rank } G = 40 \), verifying the inequality, so \( V \) is generically free for \( \mathfrak{g} \).

Example 5.6 (C₅, 10000, char \( k \neq 2 \)). Let now \( V \) be the irreducible representation of \( G = \text{Sp}_{10} \) with highest weight 10000. In case \( \text{char } k = 3 \), \( \dim V = 122 \) and one checks using Magma that a generic vector has trivial stabilizer. So assume \( \text{char } k > 3 \), in which case \( \dim V = 132 \).

As above it is enough to verify the inequality for nilpotent elements of \( \text{sp}_{10} \). Restricting to the Levi subgroup \( \text{Sp}_8 \), the representation decomposes as a direct
sum of irreducibles \(X \oplus Y \oplus Y\) where \(\dim X = 48\) and \(\dim Y = 42\). Since \(\text{char} \, k > 3\), \(X\) is a submodule of \(\Lambda^3 k^8\) with quotient \(k^8\). The restriction of \(Y\) to the Levi \(\text{Sp}_6\) in \(\text{Sp}_8\) is a direct sum of irreducibles \(Y' \oplus Y' \oplus Y''\) where \(\dim Y' = \dim Y'' = 14\), \(Y'\) is a submodule of \(\Lambda^3 k^6\) with quotient \(k^6\) and \(Y''\) is a submodule of \(\Lambda^2 k^6\) with quotient \(k\). Using these decompositions, we find that a long root \(x \in \mathfrak{sp}_6 \subset \mathfrak{sp}_{10}\) has \(\dim V^x = 90\) and nilpotent \(y \in \mathfrak{sp}_6 \subset \mathfrak{sp}_{10}\) with partition \((4, 1^6)\) has \(\dim V^y = 19\). In view of the former, it suffices to consider nilpotent \(z \in \mathfrak{g}\) such that \(\dim C_{\text{Sp}_{10}}(z) \leq 13\). Such a \(z\) has a Jordan block of size at least 4 and so specializes to \(y\). Then \(\dim z^{\text{Sp}_{10}} + \dim V^z \leq 50 + 19\), verifying the inequality.

6. Example: symmetric squares and wedge squares

Recall that \(k\) is assumed algebraically closed of characteristic \(p \geq 0\). Put \(\mathfrak{gl}_n\) for the Lie algebra of \(n\)-by-\(n\) matrices with entries in \(k\). We first note that, for \(x \in \mathfrak{gl}_n\), \(\mathbb{Z}_{\mathbb{GL}_n}(x)\) is the group of units in the associative \(k\)-algebra with underlying vector space \(\mathfrak{z}_{\mathfrak{gl}_n}(x)\). Therefore, \(\dim x^{\mathbb{GL}_n} = \dim [\mathfrak{gl}_n, x]\) and we have the following well-known result.

**Lemma 6.1.** For \(x \in \mathfrak{gl}_n\) we have: \(\dim x^{\mathbb{GL}_n} + \dim \mathfrak{z}_{\mathfrak{gl}_n}(x) = n^2. \quad \square\)

Suppose that \(x \in \mathfrak{gl}_n = \mathfrak{gl}(V)\) is nilpotent. It is well known that \(\dim x^{\mathbb{GL}_n}\), and therefore also \(\dim \mathfrak{z}_{\mathfrak{gl}_n}(x)\), depends only on the Jordan form of \(x\) and not on \(k\).

**Lemma 6.2.** Let \(x \in \mathfrak{gl}_n = \mathfrak{gl}(V)\) be nilpotent and assume that \(p \neq 2\). Then \(\dim (S^2 V)^x\) and \(\dim (\Lambda^2 V)^x\) are independent of the characteristic. In particular, if \(x \in \mathfrak{so}_n\), then \(\dim x^{\mathbb{SO}_n} + \dim (\Lambda^2 V)^x = \dim \mathfrak{so}_n\).

**Proof.** Since \(\text{char} \, k \neq 2\), as \(x\)-modules we have \(\mathfrak{gl}_n \cong V \otimes V^* \cong V \otimes V \cong S^2 V \oplus \Lambda^2 V\). Since the dimension of the fixed space of \(x\) can only increase when reducing modulo a prime (\(x\) acting on \(V\) is defined over the integers), the first claim follows.

For the second, \(\Lambda^2 V\) is the adjoint module for \(\mathbb{SO}_n\), so the equality holds in characteristic 0. Since \(\dim (\Lambda^2 V)^x\) depends only on the Jordan form of \(x\) and not on \(k\), and \(\dim x^{\mathbb{SO}_n}\) also does not (as \(p \neq 2\)), the equality also holds over \(k\). \(\square\)

**Lemma 6.3.** Let \(x \in \mathfrak{gl}_n = \mathfrak{gl}(V)\) with \(x\) a regular nilpotent element.

1. The number of Jordan blocks of \(x\) on \(\mathfrak{gl}(V)\) and \(V \otimes V\) is \(n\).

If furthermore \(\text{char} \, k \neq 2\), then

2. the number of Jordan blocks of \(x\) on \(S^2 V\) is \(n/2\) if \(n\) is even and \((n + 1)/2\) if \(n\) is odd; and
3. the number of Jordan blocks of \(x\) on \(\Lambda^2 V\) is \(n/2\) if \(n\) is even and \((n - 1)/2\) if \(n\) is odd.

**Proof.** As \(x\) is nilpotent, \(V\) and \(V^*\) are equivalent \(k[x]\)-modules, hence the number of Jordan blocks on \(V \otimes V\) and \(\mathfrak{gl}(V)\) is the same and is also independent of the characteristic. By Lemma 6.2, we may assume that \(k\) has characteristic 0.

In characteristic 0, we view \(V\) as a module under a principal \(\mathbb{SL}_2\) and see that \(V \otimes V \cong L(n - 1) \otimes L(n - 1) \cong L(2n - 2) \oplus L(2n - 4) \oplus \cdots \oplus L(0)\), proving (1). Examining the weights shows that \(\Lambda^2 V \cong L(2n - 4) \oplus L(2n - 8) \oplus \cdots\), proving (3), from which (2) follows. \(\square\)
Lemma 6.4. Let \( x \in \mathfrak{gl}_n = \mathfrak{gl}(V) \) with \( \text{char } k \neq 2 \). Assume that \( x \) has \( r \) Jordan blocks of odd size. Let \( s \) be the number of Jordan blocks of \( x \) on \( S^2 V \) and \( a \) the number of Jordan blocks on \( \wedge^2 V \). Then \( s - a = r \).

Proof. Write \( V = V_1 \oplus \cdots \oplus V_m \) where \( x \) on \( V_i \) is a single Jordan block. Then (as an \( x \)-module), \( S^2 V = (\bigoplus_{i<j} V_i \otimes V_j) \oplus (\bigoplus_i S^2 V_i) \) and \( \wedge^2 V = (\bigoplus_{i<j} V_i \otimes V_j) \oplus (\bigoplus_i \wedge^2 V_i) \). Thus, the difference in the number of Jordan blocks on \( S^2 V \) and \( \wedge^2 V \) is just the sum of the differences on \( S^2 V_i \) and \( \wedge^2 V_i \) and the result follows by the previous lemma. \( \square \)

Put \( \lambda \) for the highest weight of the natural module of \( \mathfrak{so}_n \), i.e., \( \lambda = \varpi_{n/2} \) as in Table 2. We can now show that \( \mathfrak{so}_n \) acts generically freely on \( W := L(2\lambda) \) in characteristic not 2 by proving that our standard inequality (2.1) holds. (See [GaGu 15, Example 10.7] or [GuL, §3.1] for another proof that the generic stabilizer is an elementary abelian 2-group as a group scheme.) If \( \text{char } k \) does not divide \( n \), then \( W \) is a summand of the natural representation \( V \) with a trivial 1-dimensional complement. If \( \text{char } k \) divides \( n \), then \( S^2 V \) is a uniserial module with trivial head and socle and \( W \) the unique nontrivial composition factor.

Lemma 6.5. Let \( \mathfrak{g} = \mathfrak{so}_n = \mathfrak{so}(V) \) with \( n \geq 5 \) and \( \text{char } k \neq 2 \). Set \( W = L(2\lambda) \). If \( x \in \mathfrak{g} \) is nonzero nilpotent or noncentral semisimple, then \( \dim x^G + \dim W^x < \dim W \).

Proof. If \( x \) is semisimple, by considering weights on \( V \), \( S^2 V \) and \( \wedge^2 V \), we see that \( \dim(S^2 V)^x - \dim(\wedge^2 V)^x = \dim V^x \), using that \( \text{char } k \neq 2 \). Since \( \dim x^G + \dim(\wedge^2 V)^x = \dim G \), we see that

\[
\dim x^G + \dim(S^2 V)^x = \dim G + \dim V^x = \dim S^2 V - (\dim V - \dim V^x),
\]

which is at most \( \dim S^2 V - 2 \), because the fixed space of \( x \) has codimension at least 2. Since \( L(2\lambda) \) is a summand of \( S^2 V \) as an \( x \)-module and \( x \) is trivial on a complement, the result follows. (Note that if \( \text{char } k \) divides \( n \), then \( L(2\lambda) \) is not a summand of \( S^2 V \) for \( G \).

If \( x \) is nilpotent, we argue similarly using the previous lemma. Note that, by Lemma 6.2, \( \dim x^G + \dim(\wedge^2 V)^x = \dim G \). Thus by Lemma 6.4,

\[
\dim x^G + \dim(S^2 V)^x = \dim G + r = \dim S^2 V - (n - r) \leq \dim S^2 V - 2.
\]

Assume \( \text{char } k \) divides \( n \), for otherwise the result follows. Note that \( \dim W^x \leq \dim(S^2 V)^x \) and the result follows unless \( r = n - 2 \) and \( \dim W^x = \dim(S^2 V)^x \). The first condition implies that \( x \) has one nontrivial Jordan block which must be of size 3. In this case, a trivial calculation gives \( \dim W^x = \dim(S^2 V)^x - 2 \) and the result follows. \( \square \)

7. Example: Vinberg representations

Let \( G \) be an algebraic group over a field \( k \) and suppose \( \theta \in \text{Aut}(G)(k) \) has finite order \( m \) not divisible by \( \text{char } k \). Choosing a primitive \( m \)-th root of unity \( \zeta \in k^\times \) gives a \( \mathbb{Z}/m \)-grading \( \mathfrak{g} = \bigoplus_{i \in \mathbb{Z}/m} \mathfrak{g}_i \), where \( \mathfrak{g}_i = \{ x \in \mathfrak{g} \mid \theta(x) = \zeta^i x \} \). The sub-scheme
$G_0$ of fixed points is smooth, see, for example, [Co, Exercise 2.4.10]. In this section we will assume furthermore that $G$ is semisimple simply connected, in which case $G_0$ is connected reductive [CoGP, A.8.12] and can be described explicitly using the recipe in [S 68, §8]. Representations $(G_0, g_1)$ arising in this way are sometimes called Vinberg representations or $\theta$-groups.

**Lemma 7.1.** Let $T$ be a maximal torus in a simple algebraic group $G$ over a field $k$. Then, (1) $G = \text{Sp}_{2n}$ for some $n \geq 1$ and char $k = 2$ or (2) for a generic $t \in \text{Lie}(T)$, the transporter ${\{ x \in \text{Lie}(G) \mid [x, t] \in \text{Lie}(T) \}}$ equals $\text{Lie}(T)$.

**Proof.** Write $x$ as a sum of an element $x_0 \in \text{Lie}(T)$ and a generator $x_\alpha$ in the root subalgebra for each root $\alpha$. Choose $t \in \text{Lie}(T)$ generic and suppose $[x, t] \in \text{Lie}(T)$, i.e., $\alpha(t)x_\alpha = [x_\alpha, t] = 0$ for all $\alpha$. If (1) fails, then an exercise with roots as in [ChR, Lem. 2.13] shows that $\alpha(t) \neq 0$ for every root $\alpha$, whence the claim. □

**Example 7.2** ($m = 2$). Suppose $\theta \in \text{Aut}(G)(k)$ has order 2 and acts on a maximal torus $T$ via $\theta(t) = t^{-1}$ for $t \in T$, so $\text{Lie}(T)$ is contained in $g_1$. As char $k \neq 2$, the centralizer in $\text{Lie}(G)$ of a generic element in $\text{Lie}(T)$ is just $\text{Lie}(T)$ which misses $g_0$, whence $g_0$ acts virtually freely on $g_1$. More precisely, as a group scheme, the stabilizer in $G_0$ of a generic element in $\text{Lie}(T)$ is the 2-torsion subgroup of $T$. In this way, if we pick a subgroup $H$ of $G_0$, we conclude that $h$ acts generically freely on $g_1$. We now consider examples where this applies; in each case a generic element of $g_1$ is a regular semisimple element of $g$, see [RLYG, §7] or [GuL, §3.1].

(1): Take $G$ to have type $E_6$ and $\theta$ to be an outer automorphism so that $G_0$ is the adjoint group $\text{PSp}_8$ of type $C_4$, compare, for example, [GaPT, §5]. In that case, $g_0 = \text{sp}_8$ and $g_1$ is the Weyl module with highest weight 1000 (the “spin” representation). If char $k \neq 3$ (and $\neq 2$), then the representation $g_1$ is irreducible of dimension 42.

If char $k = 3$, $g_1$ has head the irreducible representation of dimension 41 and radical $k = 3(g)$. Let $v$ be a regular semisimple element of $\text{Lie}(T) \subset g_1$. The stabilizer in $g_0 = \text{sp}_8$ of the image of $v$ in $g_1/k$ transports $v$ into $3(g)$, and therefore belongs to $\text{Lie}(T) \cap g_0 = 0$ by Lemma 7.1. In particular, $\text{sp}_8$ acts generically freely on the irreducible representation $g_1/k$.

(2): Take $G$ to be $E_8$ and $\theta$ to be such that $G_0$ has type $D_8$. In this case, $G_0$ is a half-spin group $\text{Spin}_{16}/\mu_2$ and $g_1$ is the 128-dimensional half-spin representation. We conclude that $g_0$ acts generically freely when char $k \neq 2$. (Regardless of char $k$, the generic stabilizer in $G_0$ is $(Z/2)^4 \times (\mu_2)^4$ as a group scheme, see [GaGu 17, Thm. 1.2].)

(3): Take $G$ to be $E_7$ and $\theta$ to be such that $G_0 = \text{SL}_8/\mu_4$. In this case, $g_1$ is the representation $\wedge^4 k^8$, which is generically free for char $k \neq 2$. (We provide a stronger result in Prop. 8.1(1).)

(4): Take $G$ to be $\text{SL}_n$ with $\theta(g) = g^{-T}$, so $G_0 = \text{SO}_n$ and $g_1$ is the Weyl module with head $L(2\lambda)$ as in Lemma 6.5.

The representation $\wedge^3 k^9$ of $G_0 = \text{SL}_9/\mu_3$ arises also in this way when $G = E_8$ and $m = 3$, see [ViE] for a detailed analysis of the orbits in the case char $k = 0$. A generic element of $g_1$ is regular semisimple as an element of $g$ as in the references in Example 7.2 ([GuL] produces an explicit regular nilpotent element), and we find
that \( \mathfrak{sl}_n \) acts generically freely on \( \wedge^3 k^9 \). We will provide a stronger result below in Proposition 8.1(1).

Remark 7.3. The setup above can be generalized to accommodate the case where \( \text{char } k \) divides \( m \). Instead of an element \( \theta \in \text{Aut}(G)(k) \), one picks a homomorphism of group schemes \( \mu_m \to \text{Aut}(G) \) defined over \( k \). Again one obtains a \( \mathbb{Z}/m \)-grading on \( \mathfrak{g} \) and an action of \( \mu_m \) on \( G \) such that \( G_0 \) is smooth. Some statements about the representation \( \mathfrak{g}_1 \) of \( G_0 \) from [Vi] or [L] do not hold in this generality. For example, the representations from Example 7.2(2) and (3) with \( \text{char } k = 2 \) and the representation \( \wedge^3 k^9 \) of \( \text{SL}_9 / \mu_3 \) with \( \text{char } k = 3 \), are not virtually free for \( \mathfrak{g}_0 \). This can be seen by computationally verifying that Lemma 4.1 applies; in each of these three cases the stabilizer of a generic vector is a toral subalgebra whose dimension we list in Table 1. Alternatively, for \( x \in \mathfrak{g}_1 \), \( x^{[p]} \) is in \( \mathfrak{g}_0 (x) \), so finding any \( x \) with \( x^{[p]} \) not in the kernel of the representation (as is done in [Auld, Prop. 4.8.2, 4.9.2]) suffices to show that the representation is not virtually free.

For the spin representation of \( \text{Sp}_8 \), 2 is a special prime so is treated in [GaGu III].

8. Example: 3rd and 4th exterior powers

We now consider the representation \( \wedge^e (k^n) \) of \( \text{SL}_n \) and its analogues for \( \text{SO}_n \) and \( \text{Sp}_n \). Whether or not such representations are virtually free has previously been considered in [Auld] and elsewhere. We will check here the stronger condition of whether or not inequality (2.1) holds for \( x \in \mathfrak{sl}_n \).

Proposition 8.1. For the representation \( V := \wedge^e (k^n) \) of \( \text{SL}_n \) and noncentral \( x \in \mathfrak{sl}_n \) with \( x^{[p]} \in \{0, x\} \), we have:

(1) If (a) \( e = 3 \) and \( n \geq 10 \); (b) \( e = 3 \), \( n = 9 \), and \( \text{char } k \neq 2, 3 \); (c) \( e = 4 \) and \( n \geq 9 \); or (d) \( e = 4 \), \( n = 8 \), and \( \text{char } k \neq 2 \), then \( \dim x^{\text{SL}_n + \dim V^x} < \dim V \) and \( \text{Lie}(\text{SL}_n / \mu_{\gcd(e, n)}) \) acts generically freely on \( V \).

(2) If (a) \( e = 3 \), \( n = 9 \), and \( \text{char } k = 2, 3 \) or (b) \( e = 4 \), \( n = 8 \), and \( \text{char } k = 2 \), then \( \dim x^{\text{SL}_n + \dim V^x} \leq \dim V \).

Proof. Suppose \( x^{[p]} = 0 \). The case where \( x^{[p]} = x \) follows from it by Lemma 4.2 in part I.

Put \( n_0 = 16 \) if \( e = 3 \) and \( n_0 = 10 \) if \( e = 4 \). If \( n > n_0 \), then \( \dim V = \binom{n}{e} > 2.25n^2 geq b(\text{SL}_n) \), and (2.1) holds by the main result of part I.

So suppose \( n \leq n_0 \). We calculate \( \dim x^{\text{SL}_n} \), which does not depend on \( \text{char } k \), using the well-known formulas from, for example, [LiS, p. 39]. For the other term in (2.1), \( \dim V^x \), we view \( V \) as a representation of \( \text{SL}_2 \) where a nilpotent element acts as \( x \) on \( V \). Arguing as in [McN, §3.4], we find that if \( \text{char } k > en \), then the Jordan form of \( x \) acting on \( V \) is the same as in characteristic zero. Therefore, it suffices to check the inequality over \( \mathbb{F}_p \) for \( 2 \leq p < en \) and for some \( p \) larger than \( en \). This is quickly done via computer. For the convenience of the reader, Table 5 lists the partitions corresponding to nilpotent \( x \) for which we have equality in (2). In case \( \gcd(e, n) = 1 \), this shows that \( \mathfrak{sl}_n \) acts generically freely on \( \wedge^e k^n \). For each \( n \leq n_0 \) with \( \gcd(e, n) > 1 \), we verify that \( \text{Lie}(\text{SL}_n / \mu_{\gcd(e, n)}) \) acts generically freely using Magma. \( \square \)
Table 5. Complete list of nilpotent elements \( x \) from Proposition 8.1(2) where equality holds.

| representation | char \( k \) | partition of \( x \) | \( \dim x^G \) | \( \dim V^x \) |
|---------------|-----------|----------------|-------------|-------------|
| \( \wedge^3 \mathfrak{sl}_9 \) | 2         | (2^4, 1)       | 40          | 44          |
| \( \wedge^3 \mathfrak{sl}_9 \) | 3         | (9)            | 72          | 12          |
|              |           | (3^3)          | 54          | 30          |
| \( \wedge^3 \mathfrak{sl}_8 \) | 2         | (8)            | 56          | 14          |
|              |           | (2^4)          | 48          | 22          |
|              |           | (2^4)          | 32          | 38          |

Trivectors and \( \text{SO}_n \)

Consider now \( \text{SO}_n \) with \( n \geq 9 \). The representation \( \wedge^3 (k^n) \) is a fundamental Weyl module and is irreducible if \( \text{char} \ k \neq 2 \), see for example [J, II.8.21] and [McN, Rem. 3.4].

Proposition 8.2. For \( \text{SO}_n \) with \( n \geq 9 \) (over any field \( k \)) and \( V := \wedge^3 (k^n) \), the inequality (2.1) holds for all nonzero \( x \in \mathfrak{so}_n \) with \( x^p \in \{0, x\} \), and \( \mathfrak{g} \) acts generically freely on \( V \).

Proof. Under the tautological inclusion \( \text{SO}_n \hookrightarrow \text{SL}_n \), suppose the inequality holds for \( x \) viewed as an element of \( \mathfrak{sl}_n \). Then as \( \dim x^\text{SO}_n \leq \dim x^\text{SL}_n \), the inequality holds also for \( x \) as an element of \( \mathfrak{so}_n \), completing the proof in case \( n \geq 10 \), or \( n = 9 \) and \( \text{char} \ k \neq 2, 3 \) (Proposition 8.1).

So suppose \( n = 9 \) and \( \text{char} \ k = 2 \) or 3. Write \( y \) for the image of \( x \) in \( \mathfrak{sl}_n \) if \( x \) is nilpotent, and for the image of the nilpotent specialization of \( x \) as in [GaGu I, Lem. 4.2] if \( x \) is toral. As in the previous paragraph, we are done if the inequality holds for \( y \), and therefore we may assume that \( y \) has partition \( (2^4, 1) \) or \( (3^3) \) as in Table 5. In either of these cases, we have \( \dim x^\text{SO}_9 + \dim V^x \leq 32 + \dim V^y \leq 76 < \dim V \), completing the proof. \( \square \)

Trivectors and \( \text{Sp}_{2\ell} \)

The natural representation of \( \text{Sp}_{2\ell} \) has an invariant alternating bilinear form \( b \). The subspace \( V(\varpi_{\ell-2}) \) of \( \wedge^3 k^{2\ell} \) spanned by those \( v_i \wedge v_j \wedge v_k \) with \( b(v_i, v_j) = 0 \) for all \( i, j \) is a submodule of dimension \( \binom{2\ell}{3} - 2\ell \); it is the Weyl module with highest weight \( \varpi_{\ell-2} \), see [GoK, §1]. It is irreducible, i.e., \( V(\varpi_{\ell-2}) = L(\varpi_{\ell-2}) \), if and only if \( \ell - 1 \) is nonzero in \( k \); otherwise \( V(\varpi_{\ell-2}) \) has socle the natural module \( k^{2\ell} \) and head \( L(\varpi_{\ell-2}) \) [PrS, Thm. 2(i)].

Proposition 8.3. Continue the notation of the preceding paragraph and suppose that \( p := \text{char} \ k > 2 \). If

1. \( \ell \geq 7 \) or \( \ell = 5 \); or
2. \( \ell = 6 \) and \( p \neq 5 \); or
3. \( \ell = 4 \) and \( p \neq 3 \),

then for \( V := V(\varpi_{\ell-2}) \) or \( L(\varpi_{\ell-2}) \), inequality (2.1) holds for all nonzero \( x \in \mathfrak{sp}_{2\ell} \) with \( x^p \in \{0, x\} \). In these cases, and also when \( (\ell, p) = (6, 5) \), \( \mathfrak{sp}_{2\ell} \) acts generically freely on \( V \).
In the case $\ell = 4$ and char $k = 3$, $\text{sp}_{8}$ does not act generically freely on $V$, see Example 4.2.

Proof. If $\ell > 6$, then $\dim V(\varpi_{\ell-2}) \geq \dim L(\varpi_{\ell-2}) \geq \binom{2\ell}{3} - 4\ell > b(\text{Sp}_{2\ell})$, and the conclusion holds by [GaGu I, Thm. 12.2]. So suppose $\ell = 4, 5, \text{or } 6$. In particular, $\ell - 1$ is not zero in $k$ and $V = V(\varpi_{\ell-2}) = L(\varpi_{\ell-2})$.

First suppose that $x \in \text{sp}_{2\ell}$ has $x^{[\ell]} = 0$. If $\ell = 5$ or 6, we have $\dim L(\varpi_{\ell-2})^{x} \leq \dim V(\varpi_{\ell-2})^{x} \leq \dim(\Lambda^{3}k^{2\ell})^{x}$ and one checks that $\dim x^{\text{sp}_{2\ell}} + \dim(\Lambda^{3}k^{2\ell})^{x} < \dim L(\varpi_{\ell-2})^{x}$, which need only be done for small characteristics as in the proof of Proposition 8.1 and therefore amounts to a computer calculation. If $\ell = 4$ (and char $k > 3$), then $\Lambda^{3}k^{8}$ is a direct sum of $V$ and $k^{8}$, so $\dim V^{x} = \dim(\Lambda^{3}k^{8})^{x} - \dim(k^{8})^{x}$ and the same computer calculations verify (2.1).

For $x$ toral, we appeal to Lemma 5.1.

Lemma 2.1 gives that $\text{sp}_{2\ell}$ acts generically freely, except in the case $\ell = 6$ and char $k = 5$ which we verify using Magma. \hfill \Box

9. Theorem A for restricted highest weights

In this section, we will prove the following by deducing it from what has come before.

Proposition 9.1. Theorem A holds if char $k$ is not special for $G$ and the highest weight of $\rho$ is restricted.

In addition to the case of type $A_{1}$, many other cases were handled in part I. Corollary B in ibid. reduces us to considering the following:

1. $G$ has type $A_{\ell}$ for $2 \leq \ell \leq 15$;
2. $G$ has type $B_{\ell}$ or $C_{\ell}$ with $2 \leq \ell \leq 11$; or
3. $G$ has type $D_{\ell}$ for $4 \leq \ell \leq 11$.

Theorem A in ibid. allows us to further assume that $\dim V \leq b(G)$ for $b(G)$ as in Table 4. Therefore, $V$ appears in tables A.6–A.48 in [Lü]. (Note that the search space remains infinite: while there are only finitely many possibilities for $G$ and for the highest weight of $V$, we have not exhibited any upper bound on char $k$.)

If $\dim V < \dim G - \dim \mathfrak{g}$, then certainly $\mathfrak{g}$ cannot act virtually freely on $V$. If $\dim G \geq \dim V \geq \dim G - \dim \mathfrak{g}$, then examining the tables shows that $V$ is the irreducible representation with highest weight the highest root, which is not virtually free as in [GaGu I, Example 3.4]. Therefore we assume for the rest of this section that $\dim G < \dim V \leq b(G)$. We check, for each such $V$, that $\mathfrak{g}$ acts generically freely or that $(G, \text{char } k, V)$ appears in Table 1.

Type A

For $A_{\ell}$, we consider $2 \leq \ell \leq 15$.

Proposition 8.1 treats the representations $\Lambda^{3}k^{\ell+1}$ and $\Lambda^{4}k^{\ell+1}$ of $G$ of type $A_{\ell}$ apart from a few cases. For $\Lambda^{3}k^{9}$ when char $k = 2$, $\text{sl}_{9}$ acts generically freely on $V$ by [Auld, Prop. 4.8.3], by reasoning as in §7, or as can be checked in Magma, despite the failure of inequality (2.1). The representations $\Lambda^{3}k^{9}$ of $G = \text{SL}_{9}/\mu_{3}$ when char $k = 3$ or $\Lambda^{4}k^{8}$ of $G = \text{SL}_{8}/\mu_{4}$ when char $k = 2$ are not virtually free, see Remark 7.3.
We refer to [Gue, Thm. 4.3.2] for the representation of $A_\ell$ ($2 \leq \ell \leq 9$) with highest weight $0 \cdots 03$ (and char $k > 3$ so it is restricted); of $A_3$ with highest weight 004 and dimension 35; of $A_3$ with highest weight 102 and char $k \neq 5$; of $A_2$ with highest weight 04 and dimension 15; and of $A_2$ with highest weight 13 or 22 with char $k = 5$.

For the representation of $A_\ell$ ($3 \leq \ell \leq 9$) with highest weight $0 \cdots 011$ with char $k = 3$, we verify using Magma with $G = SL_{\ell+1}/\mu_{\gcd(\ell+1,3)}$. (Guerreiro checked that $SL_{\ell+1}$ acts virtually freely, see Claim 12 on p. 97 of [Gue].)

We refer to [Auld] to see that the following are virtually free: the representation of $A_\ell$ ($\ell = 3, 4, 5$) and char $k \neq 3$ with highest weight $0 \cdots 011$ ($\S$4.5); the representation of $A_2$ with highest weight 12 and dimension 15 when char $k \neq 2$ ($\S$4.1); the representation of $A_3$ with highest weight 102 and dimension 32 when char $k = 5$ ($\S$4.2); the representation of $A_4$ with highest weight 0101 and dimension 40 or 45 ($\S$4.6); the representation of $A_4$ with highest weight 0200 and dimension 45 or 50 when char $k \neq 2$ ($\S$4.7); the representation of $A_4$ with highest weight 0110 and dimension 51 when char $k = 3$ ($\S$4.4); and the representation of $A_5$ with highest weight 01001 and dimension 78 when char $k = 5$ ($\S$4.6).

The representation of $A_3$ with highest weight 020 and char $k \neq 2$ is virtually free by Lemma 6.5.

Types B and D

For $G$ of type $D_\ell$ with $4 \leq \ell \leq 11$ or $B_\ell$ with $2 \leq \ell \leq 11$ and char $k \neq 2$, the representation with highest weight $0 \cdots 02$ is handled by Lemma 6.5.

The (half) spin representations of $B_\ell$ for $7 \leq \ell \leq 11$ and $D_\ell$ for $\ell = 9, 10, 11$ (for $G = \text{Spin}_n$ for $n = 15, 17, 18, 19, 21, 22, 23$ and $G = \text{Spin}_n/\mu_2$ when $n = 20$) are generically free. For $G = \text{Spin}_{16}/\mu_2$, the half-spin representation has generic stabilizer $(\mathbb{Z}/2)^4 \times \mu_2^2$ as a group scheme, so it is a generically free representation of $g$ when char $k \neq 2$ and is not generically free when char $k = 2$. For these results, see [GaGu 17].

The representation $\wedge^3(k^n)$ of $SO_n$ with $n = 9, \ldots, 13$ (i.e., $B_4, B_5, B_6, D_5, D_6$) with char $k \neq 2$ is generically free for $so_n$ by Proposition 8.2. When char $k = 2$ (and $G$ has type $D$), $\wedge^3(k^n)$ is reducible with irreducible quotient $L(\varpi_{2}^{-2})$. For $SO_{10}$, we verify with Magma that $L(\varpi_3)$ is generically free. For $n = 12, 14, \ldots$, dim $L(\varpi_{2}^{-2}) > b(so_n)$.

The representations of $B_2$ with highest weight 11 and $B_3$ with highest weight 101 are handled in Examples 5.2 and 5.3.

The representation of $B_3$ ($so_7$) with highest weight 200 and dimension 35 when char $k \neq 2$ appears as a summand in $S^2 X$ for $X$ the (8-dimensional) spin representation; we have $S^2 X \cong V \oplus k$. The action on $V$ factors through the action of $so_8$ as in Lemma 6.5, whence we have the inequality for $V$. Similarly the representation of $so_9$ with highest weight 2000 and dimension 126 is generically free because it factors through the generically free representation of $D_5$ as in Example 5.5.

We refer to [Gue, Thm. 4.3.3] for the representations of $B_4$ (spin$_9$) with highest weight 1001 and dimension 112 or 128; of $B_3$ with highest weight 011 and dimension 63 and char $k = 3$; of $B_3$ with highest weight 110 and dimension 64 and char $k = 5$; and of $B_2$ with highest weights 30, 12, 03, or 21.
The representation of $D_4$ with highest weight $1001$ has dimension greater than $\dim D_4$ and is generically free as in Example 5.4.

For the representation of $D_5$ with highest weight $10001$ of dimension $144$ with $\text{char } k = 2$, [Gue, Thm. 4.3.5] proves it is generically free. If $\text{char } k = 5$, one checks with Magma that a random vector has zero-dimensional stabilizer.

The representation of $D_5$ with highest weight $20000$ of dimension $126$ is generically free by Example 5.5.

Type C

Type C is similar to types B and D. We consider $3 \leq \ell \leq 11$. Excluding those $V$ with $\dim V > b(G)$ reduces us further to $3 \leq \ell \leq 6$.

The only case for which we refer to [Gue] is type $C_3$ with highest weight $011$ of dimension $50$ with $\text{char } k = 3$ (Th. 4.3.4), which can also be checked using Magma.

The representation of $C_5$ with highest weight $10000$ is generically free by Example 5.6.

We use Magma to verify that a random vector has trivial stabilizer when $\text{char } k = 3$ for $C_5$ with highest weight $01000$ and dimension $121$.

The representation $V$ of $C_4$ with highest weight $1000$ was treated in Example 7.2(1). It has $\dim V > \dim C_4$ and $V$ is generically free.

The representation of $\text{Sp}_{2\ell}$ with highest weight $00 \cdots 0100$ with $\ell = 4,5,6$ is generically free by Proposition 8.3, except for $C_4$ in characteristic $3$, see Example 4.2.

This completes the proof of Proposition 9.1.

10. Theorem A for some tensor decomposable representations

Next we treat a family of irreducible but tensor decomposable representations. Implicitly, we fix a pinning of $G$, which includes a choice of maximal torus $T$; the lattice $T^*$ of characters $T \to \mathbb{G}_m$ is contained in the weight lattice $P$ (and $T^*$ is identified with $P$ when $G$ is simply connected). In this section we will prove:

**Proposition 10.1.** Theorem A holds if $\text{char } k$ is not special and the highest weight $\lambda$ of $\rho$ satisfies $\lambda = \lambda_0 + p\lambda_1$ where $p = \text{char } k \neq 0$, $\lambda_0$ and $\lambda_1$ belong to $T^*$, and $\lambda_0$ is restricted.

**Lemma 10.2.** Let $G$ be a semisimple algebraic group. For every representation $W$ of $G$, $\mathfrak{g}$ acts virtually freely on $W \otimes W^{[p]^i}$ and $W \otimes (W^*)^{[p]^i}$ for all $i > 0$.

**Proof.** Put $V := W \otimes W^{[p]^i}$ or $W \otimes (W^*)^{[p]^i}$.

Suppose first that $G = \text{SL}_n$ and $W$ is the natural module. The representation of $\mathfrak{sl}_n$ on $V$ is equivalent to a direct sum of $\dim W$ copies of the natural module, i.e., is equivalent to $\mathfrak{sl}_n$ acting on $n$-by-$n$ matrices by left multiplication. A generic matrix $v$ is invertible, so the generic stabilizer $(\mathfrak{sl}_n)_v$ is zero. (We remark that the group $\text{SL}_n$ has finitely many orbits on $\mathbb{P}(V)$ [GuLMS, Lem. 2.6].)

Otherwise, the representation $G \to \text{GL}(V)$ factors through $\text{SL}(W) \to \text{GL}(V)$, because $G$ is semisimple, and the previous paragraph shows that $\mathfrak{sl}(W)$ acts virtually freely.
Note that, in the lemma, the inequality (2.1) need not hold. Specifically, a root element \( x \in \mathfrak{s}l_n \) has \( \dim x_{SL_n} = 2(n - 1) \) and kernel of dimension \( n - 1 \) on the natural module, so we find \( \dim x_{SL_n} + \dim V_x = \dim V + n - 2 \) for \( V \) a sum of \( n \) copies of the natural module.

**Example 10.3.** Consider now \( SO_n \) for \( n \geq 3 \) and suppose that \( \text{char } k \neq 2 \) or \( n \) is even. Take \( V_c \) to be a direct sum of \( c \) copies of the natural module \( V_1 \) for some \( 1 \leq c \leq n \). Let \( v \in V_c \) be generic. In particular, the \( SO_n \)-invariant quadratic form \( q \) is nonzero on each component of \( v \), and the \( c \) components of \( v \) generate a \( c \)-dimensional subspace \( U \) of \( V_1 \) on which the bilinearization of \( q \) is nondegenerate if \( \text{char } k \neq 2 \) or \( c \) is even, or has a 1-dimensional radical on which \( q \) does not vanish if \( \text{char } k = 2 \) and \( c \) is odd.

Therefore, if \( c = n - 1 \), an element of \( \mathfrak{so}_n \) that annihilates \( U \) is zero on \( V_1 \), i.e., \( \mathfrak{so}_n \) acts generically freely on \( V_{n-1} \). If \( c = n - 2 \), then an element of \( \mathfrak{so}_n \) that annihilates \( U \) belongs to \( \mathfrak{so}(U^\perp) \) for \( U^\perp \) the 2-dimensional subspace of \( V_1 \) orthogonal to \( U \) with respect to the bilinear form, i.e., the stabilizer of a generic \( v \in V_{n-2} \) is a rank 1 toral subalgebra of \( \mathfrak{so}_n \). (In case \( \text{char } k = 0 \), Table 2 of [E] summarizes this and many similar examples. See also [BuGuS] for more general arguments in a similar vein.)

Finer results can be proved. For example, suppose \( \text{char } k = 2 \) and \( n \geq 8 \) is even. We have already observed that \( \mathfrak{so}_n \) acts generically freely on \( V_{n-1} \), but more is true: the inequality (2.1) holds for noncentral \( x \in \mathfrak{so}_n \) such that \( x^{[2]} \in \{0, x\} \).

If \( x^{[2]} = 0 \), then, as a linear transformation on \( V_1 \), \( x \) has even rank \( r \leq n \) and \( \dim x_{SO_n} \leq (n-r)(n-1+r) \) as noted in Example 10.5 in part I, so

\[
\dim V_{n-1}^x + \dim x_{SO_n} \leq (n-r)(n-1+r).
\]

This is less than \( \dim V_{n-1} \) since \( r \geq 2 \). In case \( x^{[2]} = x \), the inequality is verified by arguing as in the proof of [GaGuI, Cor. 10.6].

**Example 10.4 (A_3).** Suppose \( G \) has type \( A_3, p := \text{char } k \neq 0 \), and consider the irreducible representation \( V \) with highest weight \( \lambda = \omega_2 + p^e \omega_1 \) for \( e \geq 1 \); we take \( G \) to be the member of the isogeny class that acts faithfully on \( V \). The composition \( SL_4 \to G \to GL(V) \) is, as a representation of \( SL_4 \), \( L(\omega_1)[p^e] \otimes L(\omega_2) = (k^4)[p^e] \otimes \wedge^2(k^4) \). As a representation of the Lie algebra \( \mathfrak{sl}_4 \), \( V \) is a sum of 4 copies of \( \wedge^2(k^4) \).

If \( p \) is odd, then \( G \) is \( SL_4 \). The differential of the isogeny \( SL_4 \to SL_4 / \mu_2 = SO_6 \) identifies \( \mathfrak{sl}_4 \) with \( \mathfrak{so}_6 \), and the action of \( \mathfrak{so}_6 \) on a sum of 4 copies of its natural representation is not generically free by Example 10.3, so \( V \) is not generically free for \( \mathfrak{sl}_4 \).

If \( p = 2 \) and \( e \geq 2 \), then \( G \) is \( SO_6 \), and the same argument shows that \( V \) is not generically free for \( \mathfrak{so}_6 \). Nonetheless, \( \mathfrak{sl}_4 \) does act virtually freely. This can be seen by noting that semisimple elements of \( \mathfrak{so}_6 \) have eigenvalues that come in pairs (say, \( \lambda_1, \lambda_2, \lambda_3 \) each occurring twice), the image of \( \mathfrak{sl}_4 \) in \( \mathfrak{so}_6 \) only contains those with \( \lambda_1 + \lambda_2 + \lambda_3 = 0 \) (because the sum on the left side is \( SL_4 \)-invariant [GaGu15, Example 8.5] and the image of \( \mathfrak{sl}_4 \) is the unique codimension-1 \( SL_4 \)-submodule of \( \mathfrak{so}_6 \)), and the elements of the generic stabilizer \( \mathfrak{so}_2 \) in \( \mathfrak{so}_6 \) have two of \( \lambda_1, \lambda_2, \lambda_3 \) equal to zero. Alternatively, Magma verifies that \( \mathfrak{sl}_4 \) acts virtually freely.
Finally, if $p = 2$ and $e = 1$, then $G = \text{PGL}_4$ and we claim that $\mathfrak{pgl}_4$ acts generically freely on $V$. Write down the map $\mathfrak{pgl}_4 \to \mathfrak{gl}(V)$ explicitly as follows. Fixing a pinning for $\text{SL}_4$ and bases for $k^4$ and $\bigwedge^2 k^4$ consisting of weight vectors, we can write down the image in $\mathfrak{gl}(V)$ of the generator of each of the root subalgebras of $\mathfrak{sl}_4$. Now, the image of $\mathfrak{sl}_4$ in $\mathfrak{pgl}_4$ has codimension 1, corresponding to the statement that the weight lattice for $A_3$ is generated by the root lattice and the fundamental weight $\omega_1$, so $\mathfrak{pgl}_4$ is generated by the image of $\mathfrak{sl}_4$ and a semisimple element $h$ corresponding to $\omega_1$ in the sense that $h \delta \omega_5 = \langle \delta, \omega_1 \rangle \omega_5$ for every weight vector $\omega_5$ of weight $\delta$ in every representation of $\text{PGL}_4$; this describes the image of $h$ in $\mathfrak{gl}(V)$. From this, Magma verifies that $\mathfrak{pgl}_4$ acts generically freely on $V$.

Proof of Proposition 10.1. By hypothesis, $V \cong L(\lambda_0) \otimes L(\lambda_1)^{[p]}$ [J, I.3.16] and $\lambda_0 \neq 0$ (Lemma 1.1(1)). If $\lambda_1 = 0$ then $\lambda$ is restricted and we are done by Proposition 9.1, so assume that $\lambda_1 \neq 0$. By Lemma 1.1(2), $\dim V > \dim G$; our task is to show that $V$ is generically free if and only if $(G, \text{char } k, V)$ does not appear in Table 1.

As $\mathfrak{g}$ acts trivially on $L(\lambda_1)^{[p]}$, the representation $V$ of $\mathfrak{g}$ is the same as a sum of $\dim L(\lambda_1)$ copies of $L(\lambda_0)$. Let $m(G)$ be the dimension of the smallest nonzero irreducible representation of $G$ with restricted highest weight as in Table 3. If $\dim L(\lambda_0) > b(G)/m(G)$ for $b(G)$ as in Table 4, then $\dim V > b(G)$ and $\mathfrak{g}$ acts virtually freely on $V$ by part I. In particular, if $m(G)^2 > b(G)$ — as is true for $G$ exceptional — we are done.

If $\dim L(\lambda_0) = m(G)$, then $V$ (considered as a $\mathfrak{g}$-module) contains $L(\lambda_0) \otimes L(\lambda_0)^{[p]}$ as a summand, and we are done by Lemma 10.2. Therefore, it remains to inspect $\oplus^{m(G)} L(\lambda_0)$ for those nonzero restricted dominant weights $\lambda_0$ with

$$m(G) < \dim L(\lambda_0) \leq b(G)/m(G).$$

(10.5)

We proceed case by case, where the possibilities for $\lambda_0$ are enumerated in [Lü]. We find very few possibilities, reflecting the fact that the bounds in (10.5) both grow linearly in the rank of $G$.

Type B. For $G$ of type $B_\ell$ with $\ell \geq 3$, the constraint (10.5) reduces us to consider the case where $G$ has type $B_3$ and $L(\lambda_0)$ is the 8-dimensional spin representation. Then $L(\lambda_0)$ factors through $\text{Spin}_8$ as a vector representation, and we apply Example 10.3 to see that the generic stabilizer in $\mathfrak{so}_8$ is trivial and therefore the same is true for $\mathfrak{spin}_7$.

Type C. For $G$ of type $C_\ell$ with $\ell \geq 2$, the dimension bounds reduce us to considering $C_2$ where $L(\lambda_0)$ is the 5-dimensional fundamental irreducible representation, i.e., $\mathfrak{g} = \mathfrak{sp}_4 = \mathfrak{so}_5$ acting on a sum of four copies of the 5-dimensional module. This action is generically free by Example 10.3.

Type D. For $G$ of type $D_\ell$ with $\ell \geq 4$, the unique dominant weight $\lambda_0$ that must be considered is for type $D_5$ with $\text{char } k \neq 2$ and $L(\lambda_0)$ a half-spin representation, so $G = \text{Spin}_{10}$ and we may take $V = \oplus^{10} L(\lambda_0)$. We verify inequality (2.1) for nonzero $x \in \mathfrak{g}$ with $x^{[p]} \in \{0, x\}$.

Consider first a a nilpotent element $y$ in the Levi of type $\mathfrak{gl}_5$ with two Jordan blocks of size 5 in the natural representation of $\mathfrak{so}_{10}$. On the half-spin module, $\mathfrak{sl}_5$ has three composition factors: one trivial submodule, the natural representation
$k^5$ (or its dual), and $\Lambda^2(k^5)^*$ (or its dual). As $y$ has 1, 1, and 2 Jordan blocks on these representations (see Lemma 6.3(3)), we find that $\dim L(\lambda_0)^y \leq 4$.

Now $\dim L(\lambda_0)^x \leq 12$ by [GaGu 17, Prop. 2.1(i)] and (2.1) holds unless $\dim x^G = 40$ and $x$ is regular. Thus, it suffices to show that $\dim L(\lambda_0)^x < 12$ for $x$ regular. By passing to closures it suffices to take $x$ regular nilpotent. Since the element $y$ in the preceding paragraph is in the closure of $x^G$, we have $\dim L(\lambda_0)^x \leq 4$, hence $V$ is generically free for $\mathfrak{g}$.

**Type A.** For type $A_\ell$ with $\ell \geq 2$, the dimension bounds (10.5) reduce us to the following cases:

(i) $A_2$, where $\text{char } k \neq 2$ and $L(\lambda_0) = S^2(k^3)$;

(ii) $A_3$, where $L(\lambda_0) = \Lambda^2(k^3)$; and

(iii) $A_4$, where $\text{char } k \neq 2$ and $L(\lambda_0) = \Lambda^2(k^5)$.

Case (ii) can be viewed as $\mathfrak{so}_6$ acting on four copies of its natural representation, which is handled in Example 10.3.

For cases (i) and (iii), $G = \text{SL}_{\ell+1}$. We verify inequality (2.1) for nonzero $x \in \mathfrak{g}$ with $x^{[p]} = 0$. This will verify it also for noncentral toral $x \in \mathfrak{g}$ [GaGu 1, Lem. 4.2], whence $\mathfrak{g}$ acts generically freely on $V$.

Consider case (iii). For $x \in \mathfrak{sl}_5$ a root element, i.e., nilpotent with partition $(2,1^3)$, we have $\dim x^G = 8$ and $\dim L(\lambda_0)^x = 7$, and $8 + 5 \cdot 7 = 43 < 50$. For $x$ nilpotent with partition $(2^2,1)$, we have $\dim x^G = 12$ and $\dim L(\lambda_0)^x = 6$, and $12 + 5 \cdot 6 = 42 < 50$.

For $x$ nilpotent with partition $(3,1^2)$, we have $\dim L(\lambda_0)^x = 4$ and $\dim x^G \leq \dim G - \text{rank } G = 20$. Consequently, for every nilpotent $y \in \mathfrak{sl}_5$ such that $x \in \overline{y^G}$, we have:

$$\dim y^G + \dim V^y \leq (\dim G - \text{rank } G) + m(G) \cdot \dim L(\lambda_0)^x < \dim V.$$ 

Thus we have verified the inequality (2.1) for every nonzero nilpotent in $\mathfrak{sl}_5$.

Finally consider case (i). There are two classes of nilpotent elements. If $x$ is a root element, then $\dim L(\lambda_0)^x = 3$ and $\dim x^G = 4$. If $x$ is regular, then $\dim L(\lambda_0)^x = 2$ and $\dim x^G = 6$. In both cases, the inequality (2.1) holds. \qed

**11. Conclusion of proof of Theorem A**

We now complete the proof of Theorem A, assuming $\text{char } k$ is not special. Write the highest weight $\lambda$ of $V$ as $\lambda = \lambda_0 + p\lambda_1$ for $\lambda_0, \lambda_1$ dominant weights (not necessarily in $T^*$) and $\lambda_0$ restricted.

Put $\pi : \tilde{G} \to G$ for the simply connected cover. If $G$ is itself simply connected, then we are done by Proposition 10.1. Thus we are also done if $d\pi$ is surjective (i.e., if $\ker d\pi = 0$), and we may assume that the finite group scheme $\ker \pi$ is not smooth and has exponent divisible by $p$, reducing us to the following cases: $G = \text{SL}_n/\mu_m$ where $p \mid m$ (and $n \geq 3$), $G$ has type $D_\ell$ and $p = 2$, $G$ is adjoint of type $E_6$ and $p = 3$, or $G$ is adjoint of type $E_7$ and $p = 2$.

Suppose that $\lambda_0 = 0$. The composition $d\rho_0 d\pi$ is the representation $L(\lambda_1)^{[p]}$ of $\tilde{G}$, whence $\tilde{\mathfrak{g}}$ acts trivially on $V$, so $V$ is not faithful.
On the other hand, the case where \( \lambda_1 = 0 \) is done by Prop. 9.1, so we may assume that \( \lambda_0 \) and \( \lambda_1 \) are both nonzero.

Now \( \lambda \) vanishes on \( \ker \pi \) (because \( \lambda \in T^* \)) and \( p\lambda_1 \) vanishes on the \( p \)-torsion in \( \ker \pi \), so it follows that \( \lambda_0 \) vanishes on the \( p \)-torsion in \( \ker \pi \). Put \( m_p(G) \) for the minimum of \( \dim L(\mu) \) as \( \mu \) ranges over nonzero restricted dominant weights such that \( \ker \mu \) has exponent divisible by \( p \); the value of \( m_p(G) \) is listed in Table 6. The pullback \( \rho \pi \) of \( \rho \) is the representation \( L(\lambda_0) \otimes L(\lambda_1)^{[p]} \) of \( \tilde{G} \), so \( \dim V \geq m_p(G)m(G) \).

| type \( \mathcal{G} \) | \( p \) | \( m_p(G) \) | \( m(G) \) |
|----------------------|------------------|------------------|
| \( \mathcal{A}_\ell \) (odd \( \ell \geq 3 \)) | 2 | \( \left(\ell + 1\right) \) | \( \ell + 1 \) |
| \( \mathcal{A}_\ell \) (\( \ell \geq 2 \)) | odd \( p \mid \ell + 1 \) | \( \left(\ell + 1\right)^2 - 2 \) | \( \ell + 1 \) |
| \( \mathcal{D}_\ell \) (\( \ell \geq 4 \)) | 2 | 2\( \ell \) | 2\( \ell \) |
| \( \mathcal{E}_6 \) | 3 | 77 | 27 |
| \( \mathcal{E}_7 \) | 2 | 132 | 56 |

In particular, if \( m_p(G)m(G) \) is greater than \( b(G) \), we are done by the main result of part I. This handles the cases where \( G \) has type \( \mathcal{E}_6 \) or \( \mathcal{E}_7 \), or type \( \mathcal{A}_\ell \) when \( p \) is odd.

**Lemma 11.1.** Consider representations \( V \) and \( W \) of a Lie algebra \( L \). For nilpotent \( x \in L \), \( \dim(V \otimes W)^x \leq (\dim V^x)(\dim W) \).

**Proof.** Put \( \psi : L \to \mathfrak{gl}(V) \) and \( \zeta : L \to \mathfrak{gl}(W) \) for the two actions. For each \( t \in k \), \( t\zeta \) is a representation of the Lie algebra \( kx \); since \( x \) is nilpotent the ones with \( t \neq 0 \) are all equivalent. Therefore, writing \( U_t \) for the representation \( \psi \otimes (t\zeta) \) — so \( U_1 = V \otimes W \) — the dimension of \((U_t)^x\) is constant for \( t \neq 0 \). Now \( U_0 \) is a direct sum of \( \dim W \) copies of \((V, \psi)\), so \( \dim(U_0)^x = (\dim V^x)(\dim W) \). On the other hand, by upper semicontinuity of dimension, \( \dim(U_0)^x \geq \dim(U_t)^x \) for \( t \neq 0 \). \( \square \)

**Type \( \mathcal{D}_\ell \).** Suppose now that \( G \) has type \( \mathcal{D}_\ell \) and char \( k = 2 \), in which case we have \( m_p(G)m(G) = 4\ell^2 = b(G) \) and we are done unless \( \dim L(\lambda_0) = \dim L(\lambda_1) = 2\ell \) as representations of \( \tilde{G} \).

If \( \ell > 4 \), the only restricted irreducible representation of \( \tilde{G} \) with restricted highest weight and of dimension \( 2\ell \) is the vector representation \( \text{Spin}_{2\ell} \to \text{SO}_{2\ell} \) with highest weight \( \varpi_\ell \) and a Frobenius twist of it, hence \( \lambda \) is an odd multiple of \( \varpi_\ell \). In particular, \( \varpi_\ell \in T^* \), so \( \lambda_0, \lambda_1 \in T^* \) and we are done by Proposition 10.1.

For \( \ell = 4 \), the representations \( L(\varpi_i) \) with \( i = 1, 3, 4 \) of \( \text{Spin}_8 \) all have dimension 8, and up to graph automorphism we are left with considering the case \( \lambda = 2^e \varpi_1 + \varpi_4 \) for some \( e \geq 1 \). Thus we may view \( G \) as \( \text{SO}_8 \), and the pullback to \( \text{Spin}_8 \) of \( V \) is the natural representation \( k^8 \) of \( \text{SO}_8 \) (with highest weight \( \varpi_1 \)) tensored with a Frobenius twist of a half-spin representation; as a representation of \( \text{SO}_8 \) we find \( k^8 \otimes L(2^e \varpi_1) \).

Arguing as in Example 10.3, a square-zero \( x \in \text{so}_8 \) has even rank \( r \leq 4 \), \( \dim x = r(8 - r) \) and \( \dim(k^8)^x = 8 - r \), so \( \dim(k^8 \otimes L(2^e \varpi_1))^x \leq 8(8 - r) \).
(Lemma 11.1) and \( \dim x^{SO_n} + \dim V_x \leq 64 - r^2 < \dim V \), verifying (2.1). From this, we deduce (2.1) also for noncentral toral \( x \in so_8 \) as in Example 10.3 and it follows that \( so_8 \) acts generically freely on \( V \).

Type \( A_k \). Suppose now that \( G = SL_n / \mu_m \) with \( \text{char } k = 2 \), so \( m \) is even and \( n \geq 4 \). As \( m_p(G) = \dim L(\varpi_2) = \binom{n}{2} \), we have

\[
m_p(G) m(G) - b(G) = \frac{1}{2} \left( n^3 - 5m^2 + 4 \right) .
\]

which is positive for \( n \geq 5 \). So suppose further that \( n = 4 \), in which case \( b(G) = 30 \) and Table A.7 in [Lü] says that the smallest nontrivial restricted irreducible representations of \( SL_4 \) have dimension 4 (the natural representation \( k^4 \) or its dual) or 6 (\( \wedge^2 k^6 \) with highest weight \( \varpi_2 \)), so \( \lambda_0 = \varpi_2 \). As \( SL_4 \) does not act faithfully, up to graph automorphism \( \lambda = 2\varpi_1 + \varpi_2 \) for some \( e \geq 1 \). These representations were handled in Example 10.4. This completes the proof of Theorem A when \( \text{char } k \) is not special. \( \square \)

The proof of Theorem A in the remaining cases, when \( \text{char } k \) is special, will appear in part III, [GaGu III].

References

[AnP] E. M. Andreev, V. L. Popov, О стационарных подгруппах точек общего положения в пространстве представления полупростой группы Ли, Функции, анализ и его приложения. 5 (1971), вып. 4, 1–8. Engl. transl.: E. M. Andreev, V. L. Popov, Stationary subgroups of points of general position in the representation space of a semisimple Lie group, Functional Anal. Appl. 5 (1971), no. 4, 265–271.

[AnVE] E. M. Andreev, Э. Б. Винберг, А. Г. Эlashвili, Орбиты наивысшей размерности полупростых линейных групп Ли, Функции, анализ и его приложения. 1 (1967), вып. 4, 3–7. Engl. transl.: E. M. Andreev, E. B. Vinberg, A. G. Elashvili, Orbits of greatest dimension in semi-simple linear Lie groups, Funct. Anal. Appl. 1 (1968), 257–261.

[Auld] A. Auld, Exceptional modular representations of special linear Lie algebras, PhD thesis, University of Manchester, February 2001.

[BoCP] W. Bosma, J. Cannon, C. Playoust, The Magma algebra system I: the user language, J. Symbolic Computation 9 (1990), 677–698.

[BrRV] P. Brosnan, Z. Reichstein, A. Vistoli, Essential dimension, spinor groups, and quadratic forms, Ann. of Math. (2) 171 (2010), no. 1, 533–544.

[BuGuS] T. Burness, R. Guralnick, J. Saxl, On base sizes for algebraic groups, J. Eur. Math. Soc. (JEMS) 19 (2017), no. 8, 2269–2341.

[ChR] P.-E. Chaput, M. Romagny, On the adjoint quotient of Chevalley groups over arbitrary base schemes, J. Inst. Math. Jussieu 9 (2010), no. 4, 673–704.

[Co] B. Conrad, Reductive group schemes, Autour des Schémas en Groupes I, Panoramas et synthèses, vol. 42–43, Société Mathématique de France, 2014.

[CoGP] B. Conrad, O. Gabber, G. Prasad, Pseudo-reductive Groups, 2nd ed., Cambridge University Press, 2015.
[DGa] M. Demazure, P. Gabriel, Groupes Algébriques. Tome I: Géométrie Algébrique, Généralités, Groupes Commutatifs, Masson, Paris, 1970.

[DGr] M. Demazure, A. Grothendieck, Schémas en groupes II: Groupes de type multiplicatif, et structure des schémas en groupes généraux, Lecture Notes in Mathematics, Vol. 152, Springer, Berlin, 1970.

[È] А. Г. Элашвили, Канонический вид и стационарные подалгебры точек общего положения для простых линейных групп Ли, Функц. анализ и его прил. 6 (1972), вып. 1, 51–62. Engl. transl.: A. G. Èlašvili, Canonical form and stationary subalgebras of points in general position for simple linear Lie groups, Funct. Anal. Appl. 6 (1972), no. 1, 44–53.

[GaGu 15] S. Garibaldi, R. M. Guralnick, Simple groups stabilizing polynomials, Forum of Mathematics: Pi 3 (2015), e3 (41 pages).

[GaGu 17] S. Garibaldi, R. M. Guralnick, Spinors and essential dimension, Compositio Math. 153 (2017), 535–556, with an appendix by A. Premet.

[GaGu I] S. Garibaldi, R. M. Guralnick, Generically free representations I: Large representations, to appear in Algebra & Number Theory, arxiv:1711.05502v3 (2017).

[GaGu III] S. Garibaldi, R. M. Guralnick, Generically free representations III: Extremely bad characteristic, to appear in Transformation Groups.

[GaPT] S. Garibaldi, R. Parimala, J.-P. Tignol, Discriminant of symplectic involutions, Pure Applied Math. Quarterly 5 (2009), no. 1, 349–374.

[GoK] R. Gow, A. Kleshchev, Connections between representations of the symmetric group and the symplectic group in characteristic 2, J. Algebra 221 (1999), 60–89.

[Gue] M. Guerreiro, Exceptional representations of simple algebraic groups in prime characteristic, PhD thesis, University of Manchester, 1997, arxiv:1210.6919 (2012).

[GuL] R. M. Guralnick, R. Lawther, Generic stabilizers in actions of simple algebraic groups I: modules and first Grassmannian varieties, arxiv:1904.13375v2 (2019).

[GuLMS] R. M. Guralnick, M. W. Liebeck, D. Macpherson, G.M. Seitz, Modules for algebraic groups with finitely many orbits on subspaces, J. Algebra 196 (1997), 211–250.

[H] J. E. Humphreys, Algebraic Groups and Modular Lie Algebras, Memoirs of the American Mathematical Society, No. 71, American Mathematical Society, Providence, R.I., 1967.

[J] J. C. Jantzen, Representations of Algebraic Groups, 2nd ed., Mathematical Surveys and Monographs, Vol. 107, American Mathematical Society, Providence, RI, 2003.

[K] N. Karpenko, Canonical dimension, Proceedings of the International Congress of Mathematicians 2010, World Scientific, 2010.

[L] P. Levy, Vinberg’s θ-groups in positive characteristic and Kostant–Weierstrass slices, Transform. Groups 14 (2009), no. 2, 417–461.

[LiS] M. Liebeck, G. Seitz, Unipotent and Nilpotent Classes in Simple Algebraic Groups and Lie Algebras, Mathematical Surveys and Monographs, no. 180, American Mathematical Society, Providence, RI, 2012.
[Lö]  R. Lötscher, *A fiber dimension theorem for essential and canonical dimension*, Compositio Math. **149** (2013), no. 1, 148–174.

[LöMMR]  R. Lötscher, M. MacDonald, A. Meyer, Z. Reichstein, *Essential dimension of algebraic tori*, J. Reine Angew. Math. **677** (2013), 1–13.

[Lü]  F. Lübeck, *Small degree representations of finite Chevalley groups in defining characteristic*, LMS J. Comput. Math. **4** (2001), 135–169.

[McN]  G. J. McNinch, *Semisimplicity of exterior powers of semisimple representations of groups*, J. Algebra **225** (2000), 646–666.

[Mer]  A. Merkurjev, *Essential dimension: a survey*, Transform. Groups **18** (2013), no. 2, 415–481.

[Po]  A. M. Popov, *Конечные стационарные подгруппы общего положения простых линейных групп Ли*, Тр. ММО **48** (1985), 7–59. Engl. transl.: A. M. Popov, *Finite isotropy subgroups in general position of simple linear Lie groups*, Trans. Moscow Math. Soc. **1986** (1988), 3–63.

[PoV]  Э. Б. Винберг, В. Л. Попов, *Теория инвариантов, Алгебраическая геометрия–4*, Итоги науки и техн., сер. Соврем. пробл. мат., Фундам. направл., т. 55, ВИНИТИ, М., 1989, 137–309. Engl. transl.: V. L. Popov, E. B. Vinberg, *Invariant theory*, Encyclopaedia of Mathematical Sciences, Vol. 55, *Algebraic Geometry IV*, Springer-Verlag, Berlin, 1994, pp. 123–284.

[PrS]  A. A. Premet, I. D. Suprunenko, *The Weyl modules and the irreducible representations of the symplectic group with the fundamental highest weights*, Comm. Algebra **11** (1983), no. 12, 1309–1342.

[RLYG]  M. Reeder, P. Levy, J.-K. Yu, B. H. Gross, *Gradings of positive rank on simple Lie algebras*, Transform. Groups **17** (2012), no. 4, 1123–1190.

[Re]  Z. Reichstein, *Essential dimension*, Proceedings of the International Congress of Mathematicians 2010, World Scientific, 2010.

[S63]  R. Steinberg, *Representations of algebraic groups*, Nagoya Math. J. **22** (1963), 33–56.

[S68]  R. Steinberg, *Endomorphisms of Linear Algebraic Groups*, Mem. Amer. Math. Soc. Vol. 80, 1968.

[SF]  H. Strade, R. Farnsteiner, *Modular Lie Algebras and Their Representations*, Monographs and Textbooks in Pure and Applied Math., Vol. 116, Marcel Dekker, New York, 1988.

[Vi]  Э. Б. Винберг, *Группа Вейля градуированной алгебры Ли*, Изв. АН СССР. Сер. матем. **40** (1976), вып. 3, 488–526. Engl. transl.: E. B. Vinberg, *The Weyl group of a graded Lie algebra*, Math. USSR Izv. **10** (1976), 463–495.

[ViE]  Э. Б. Винберг, А. Г. Эlashвили, *Классификация тривекторов девятимерного пространства*, Труды семинара по вектор. и тенз. анализу **18** (1978), 197–233. Engl. transl.: E. B. Vinberg, A. G. Elashvili, *Classification of trivectors of a 9-dimensional space*, Sel. Math. Sov. **7** (1988), no. 1, 63–98.

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