Colliot-Thélène’s conjecture and finiteness of $u$-invariants

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Introduction

Let $X$ an excellent Noetherian regular scheme of dimension 2 which is projective over an affine Dedekind scheme Spec$(A)$ and $K$ the function field of $X$. Let $l$ be a prime which is a unit on $X$ (i.e. unequal to characteristic of the residue field of any point in $X$). The work of Saltman on the division algebras over surfaces ([S1], [S3]) imply that given an element $\alpha$ in the $l$-torsion of the Brauer group of $K$, there exist elements $f, g, h \in K^*$ such that $\alpha \otimes K(\sqrt[l]{f}, \sqrt[l]{g}, \sqrt[l]{h})$ is unramified on a regular proper model of $K(\sqrt[l]{f}, \sqrt[l]{g})$ over $A$ (cf. [B]). If $K$ is the function field of a curve over a $p$-adic field and $l$ a prime not equal to $p$, the unramified Brauer group on a regular proper model over $\mathbb{Z}_p$ is zero. Hence, for $\alpha \in \text{Br}(K)$, index of $\alpha$ divides $l^2$.

In this paper, we split the ramification of division algebras on surfaces in a more general setting without any assumption on the characteristic of the residue fields of points of the scheme. More precisely, we prove the following (cf. Theorem 2.9):

**Theorem.** Let $X$ be an excellent regular integral scheme of dimension 2 and $K$ its function field. Suppose that char$(K) = 0$ and for every codimension one point $x$ of $X$, if the characteristic of the residue field $\kappa(x)$ at $x$ is $p$, then, $[\kappa(x) : \kappa(x)^p] = p$ (i.e. $p$-dimension of $\kappa(x)$ is 1). If $\alpha$ is an element in the $p$-torsion of the Brauer group of $K$, then there exists $f, g, h \in K^*$ such that $\alpha \otimes K(\sqrt[p]{f}, \sqrt[p]{g}, \sqrt[p]{h})$ is unramified on a regular model $Y$ of $K$ proper over $X$.

The above theorem leads to the following results on splitting ramification of exponent $p$ division algebras over function fields of curves over $p$-adic fields and number fields (cf. Corollaries 2.10 and 2.11).
**Corollary.** If $k$ is a $p$-adic field and $K$ a function field of a curve over $k$, then for every element $\alpha \in p\text{Br}(K)$, $\text{index}(\alpha)$ divides $\text{period}(\alpha)^3$.

**Corollary.** Let $k$ be a number field, $S$ the ring of integers in $k$, and $K$ the function field of a curve over $k$. Let $\alpha \in Br(K)$ be a $p$-torsion element. There is an explicit degree $p^3$ extension $L$ of $K$ such that $\alpha \otimes L$ is the restriction of a class in the Brauer group of a regular proper model of $L$ over $S$.

Thus the study of the period-index problem for such function fields is reduced to a corresponding problem for unramified Brauer classes. Recall the following conjecture of Colliot-Thélène:

**CT-Conjecture**([CT1]): If $Y$ is a smooth projective geometrically connected variety over a global field $k$ then the Brauer-Manin obstruction to the existence of 0-cycles of degree 1 is the only obstruction.

Using a moving lemma for 0-cycles, we extend results of [L2] to show the following (cf. Theorem 3.3).

**Theorem.** Let $k$ be a totally imaginary number field and $K$ the function field of a curve over $k$. Let $X$ be a regular proper model of $K$ over the ring of integers in $k$. If the CT-Conjecture holds then for any $\alpha \in Br(X)$, the $\text{ind}(\alpha)$ divides $\text{per}(\alpha)^2$.

Combining this with the results described above on splitting ramification, we deduce the following corollary (cf. Theorem 4.1).

**Corollary.** Let $K$ be a function field in one variable over a totally imaginary number field. The CT-conjecture implies that for any $\alpha \in Br(K)$, $\text{ind}(\alpha)$ divides $\text{period}(\alpha)^5$.

This together with a result on degree 3 Galois cohomology groups ([Su]) leads to the final result of the paper (cf. Theorem 4.5).

**Theorem.** If the CT-Conjecture holds then the function field of a curve over a totally imaginary number field has finite $u$-invariant.

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1. Brauer group of discretely valued fields

Let \((K, \nu)\) be a complete discrete valued field with ring of integers \(R\), maximal ideal \(m\) and residue field \(\kappa\). Suppose that \(\text{char}(K) = 0\), \(\text{char}(\kappa) = p > 0\) and that \(K\) contains a primitive \(p\)th root of unity \(\zeta\). Write \(N = \nu(p)p/(p - 1)\). Since

\[(p - 1)\nu(\zeta - 1) = \nu(p),\]

\(N\) is a positive integer divisible by \(p\). Finally, for \(i \geq 0\), let

\[U_i = \{u \in R^* \mid x \equiv 1 \mod m^i\}.

The following assumption about the residue field will play a key role throughout this paper.

**Assumption \((\ast)\):** \([\kappa : \kappa^p] = p\).

Given \(a, b \in K^*\), let \((a, b) \in \nu Br(K)\) be the class of the cyclic \(K\)-algebra defined by the relations

\[
x^p = a, \\
y^p = b, \\
xy = \zeta yx.
\]

Let \(br(K)_0 = \nu Br(K)\) and for \(i \geq 1\), let

\[br(K)_i \subset br(K)_0
\]

be the subgroup generated by cyclic algebras \((u, a)\) with \(u \in U_i\) and \(a \in K^*\). Since \(K\) is complete, \(br(K)_n = 0\) for \(n > N\) ([CT2], 4.1.3 and [MS]). In this section we recall a few basic facts about \(\nu Br(K)\) and \(br(K)_n\) due to Kato and Saltman.
Lemma 1.1. (Kato [K], Thm. 2) Let
\[ \alpha \in br(K)_m \setminus br(K)_{m+1} \]
with \(0 \leq m \leq N\). Let \(\pi \in R\) be a parameter and \(f \in R^*\) with \(\overline{f} \notin \kappa^p\).

(a) If \(m = 0\), then
\[ \alpha \equiv (u, \pi) \pmod{br(K)_1} \]
for some \(u \in R^*\) with \(\overline{u} \notin \kappa^p\).

(b) If \(m\) is coprime to \(p\), then there exists \(x \in R^*\) such that
\[ \alpha \equiv (1 + \pi^m x, f) \pmod{br(K)_{m+1}}. \]

(c) If \(0 < m < N\) and \(m\) a multiple of \(p\), then
\[ \alpha \equiv (1 + \pi^m x, \pi) \pmod{br(K)_{m+1}} \]
for some \(x \in R^*\) with \(\overline{x} \notin \kappa^p\).

(d) Let \(b = \zeta - 1\). If \(m = N\) then
\[ \alpha = (1 + xb^p, f) + (1 + x'b^p, \pi) \]
for some \(x, x' \in R\). Further \((1 + xb^p, f)\) is unramified at \(\nu\).

Proof. Let \(K_2(\kappa)\) be the Milnor \(K\)-group and
\[ k_2(\kappa) = K_2(\kappa) / pK_2(\kappa). \]
The group \(k_2(\kappa)\) is isomorphic to a subgroup of \(\Omega_\kappa^2\) (cf. [CT], 3.0). Since (by Assumption (\(*\))) we have
\[ [\kappa : \kappa^p] = p, \]
we know that
\[ \Omega^2_\kappa = 0 \]
and hence \(k_2(\kappa) = 0\). Define a map
\[ \kappa^*/\kappa^{*p} \rightarrow br(K)_0 / br(K)_1 \]
by
\[ \lambda \mapsto (\tilde{\lambda}, \pi), \]
where for any \( \lambda \in \kappa, \tilde{\lambda} \in R \) is a lift. By ([K], Thm. 2, cf. [CT], 4.3.1), this map is an isomorphism. Hence every element in \( \text{br}(K)_0 \setminus \text{br}(K)_1 \) is equivalent to \( (u, \pi) \) modulo \( \text{br}(K)_1 \) for some \( u \in R^* \) with \( \pi \notin \kappa^p \), establishing part (a).

Let us prove part (b). Suppose \( 0 < m < N \) is coprime to \( p \). Define a map
\[ \Omega_{\kappa} \to \text{br}(K)_m/\text{br}(K)_{m+1} \]
by
\[ x \frac{dy}{y} \mapsto (1 + \pi^m \tilde{x}, \tilde{y}), \]
where
\[ x, y \in \kappa^* \]
are arbitrary elements with lifts
\[ \tilde{x}, \tilde{y} \in R^*. \]
By ([K], Thm. 2), this map is an isomorphism. Since \( \overrightarrow{f} \notin \kappa^p \), we have \( \kappa = \kappa^p(\overrightarrow{f}) \). In particular, since the (absolute) differential vanishes on \( \kappa^p \), every element in \( \Omega_{\kappa}^1 \) has the form
\[ \omega = \sum_{i=1}^{p-1} \alpha_i \overrightarrow{f} \frac{df}{\overrightarrow{f}} \tag{1} \]
for \( \alpha_i \in \kappa^p \). Thus every element of \( \text{br}(K)_m/\text{br}(K)_{m+1} \) is the image of an element of the form \( (1 + x\pi^m, f) \) for some \( x \in R^* \).

Now we will prove part (c). Let \( 1 < m < N \) be divisible by \( p \). Define a map
\[ \kappa/\kappa^p \to \text{br}(K)_m/\text{br}(K)_{m+1} \]
by
\[ \lambda + \kappa^p \to (1 + \pi^m \tilde{\lambda}, \pi), \]
where for \( \lambda \in \kappa \) we have chosen a lift \( \tilde{\lambda} \in R \). Using (1) above and the fact that \( d \) is \( \kappa^p \)-linear, we see that every element of \( \Omega_{\kappa}^1 \) is closed. Thus, the first summand in ([K], map (iii) immediately preceding Thm. 2) is trivial. The map described above is just the composition with the remaining summand of ([K], map (iii) before Thm. 2) with the natural map \( u_2^{(n)}/u_2^{(n+1)} \to \text{br}_n(K) \)
(in the notation of [K]). Hence every non-zero element in \( br(K)_m/br(K)_{m+1} \) is the image of an element of the form \((1 + \pi^m x, \pi)\) for some \( x \in R^* \) with \( \overline{\pi} \notin \kappa^p \).

To prove part (d) (the case \( n = N \)), let \( b = \zeta - 1 \). By ([K], Thm. 2) or ([CT2], Thm. 4.3.1(d)) and using (1) above, the element \( \alpha \) can be written in the desired form

\[
\alpha = (1 + xb^p, f) + (1 + x'b^p, \pi).
\]

It remains to show that the first summand is unramified (i.e., split by an unramified extension of \( K \)). If \( x \in \pi R \) then the first summand lies in \( br_{N+1}(K) = 0 \) and is unramified.

Thus, we may assume that \( x \in R^* \), so that \( \nu(x) = 0 \). We will show that in this case the field

\[
L = K(\sqrt[\nu]{1 + xb^p})
\]

is unramified over \( K \); since this splits the algebra \((1 + xb^p, f)\), this will complete the proof.

Let

\[
\theta = \sqrt[\nu]{1 + xb^p}
\]

and

\[
d = \theta - 1.
\]

Let \( \tilde{\nu} \) be the valuation on \( L \) extending \( \nu \) on \( K \). Since \( 1 + xb^p \) is not a \( p \)th power, the minimal polynomial of \( \theta \) is

\[
g(z) = (z + 1)^p - 1 - xb^p = 0.
\]

In particular, the norm of \( \theta \) is \(-xb^p\), so

\[
\tilde{\nu}(\theta) = \frac{1}{p} \nu(-xb^p) = \nu(b)
\]

by our assumption that \( x \in R^* \). This means that \( d = wb \) for some unit \( w \) in the ring of integers in \( L \). It is easy to see that \( \overline{w^p} - \overline{w} = \overline{\pi} \). Hence the \( L/K \) is an unramified extension, as desired.

\[\square\]

**Corollary 1.2.** Every element \( \alpha \in p\text{Br}(K) \) is of the form \((g, f) \cdot (h, \pi)\) for some \( g, h \in R^* \).
Proof. Let $0 \leq n \leq N$. Then, by (1.1), every element in $br(K)_n$ is equivalent to $(g_n, f)(h_n, \pi)$ modulo $br(K)_{n+1}$ for some $g_n, h_n \in R^*$. Since $br(K)_n = 0$ for $n > N$, the corollary follows. \qed

Lemma 1.3. Let $n < N$ be coprime to $p$ and $c \in R^*$. Then there exists $v \in R^*$ such that $(1 + \pi^n c, v \pi) = 1$.

Proof. Since $n$ is coprime to $p$, there exists an integer $m$ such that $nm \equiv 1 \pmod{p}$. We have

$$1 = (1 + \pi^n c, -\pi^n c)^m = (1 + \pi^n c, (-1)^m \pi^n c^m) = (1 + \pi^n c, v \pi)$$

with $v = (-1)^m c^m$. \qed

Lemma 1.4. Let $n = mp < N$ and $i > 0$. Let $u \in R^*$ be such that $\pi \notin \kappa^P$. Suppose $n + i$ is coprime to $p$. Then every element in $br(K)_{n+i} \setminus br(K)_{n+i+1}$ can be represented by $(1 + u \pi^n, u')$ for some $u' \in R^*$.

Proof. Fix

$$\alpha \in br(K)_{n+1} \setminus br(K)_{n+i+1}.$$ 

By (1.1) there is an $x \in R^*$ such that

$$\alpha \equiv (1 + x \pi^{n+i}, u) \pmod{br(K)_{n+i+1}}.$$ 

Let

$$u' = (-1)^{p+1} (u - x \pi^i) (1 + x \pi^{n+i})^{-1}.$$ 

Since $i > 0$, $u \equiv u'$ modulo $\pi$ and hence $u' = vu$ for some $v \in R^*$ with $v \equiv 1 \pmod{\pi}$. Thus, by ([CT], 4.1.1(b)),

$$(1 + x \pi^{n+i}, u) \equiv (1 + x \pi^{n+i}, u') \pmod{br(K)_{n+i+1}}.$$ 

Since $(1 - z^p y, y) = 0$, we have

$$(1 + x \pi^{n+i}, u') = ((1 + x \pi^{n+i})(1 - (-1)^p \pi^m u'), u').$$ 

On the other hand, we know

$$(1 + x \pi^{n+i})(1 - (-1)^p \pi^m u') = 1 + u \pi^n.$$ 

Thus

$$\alpha \equiv (1 + u \pi^n, u') \pmod{br(K)_{n+i+1}},$$

as desired. \qed
Lemma 1.5.  (Saltman [S4]) Let $\alpha \in \text{br}(K)_0 \setminus \text{br}(K)_1$. Then there exist $u, v \in R^*$ such that $\overline{u} \not\in \kappa^p$ and

$$\alpha - (u, v \pi)$$

is unramified.

Proof. First we show by induction that for each $0 \leq i < N$, there exist $u_i, v_i \in R^*$ such that

$$\alpha - (u_i, v_i \pi) \in \text{br}(K)_{i+1}$$

with $\overline{u}_i \not\in \kappa^p$. By (1.1),

$$\alpha - (u_0, \pi) \in \text{br}(K)_1$$

for some $u_0 \in R^*$ with $\overline{u}_0 \not\in \kappa^p$. Suppose there exist $u_i, v_i \in R^*$ such that

$$\alpha - (u_i, v_i \pi) \in \text{br}(K)_{i+1}$$

and $\overline{u}_i \not\in \kappa^p$.

Suppose that $i + 1$ is coprime to $p$. Since $\overline{u}_i \not\in \kappa^p$, by (1.1), there exists $c \in R^*$ such that

$$\alpha - (u_i, v_i \pi) - (u_i, 1 + \pi^{i+1}c) \in \text{br}(K)_{i+2}.$$ 

Thus

$$\alpha - (u_i, v_i \pi(1 + \pi^{i+1}c)) \in \text{br}(K)_{i+2}.$$ 

Suppose that $i + 1$ is divisible by $p$ and $i + 1 < N$. By (1.1), there exists $c \in R^*$ such that

$$\alpha - (u_i, v_i \pi) - (1 + \pi^{i+1}c, v_i \pi) \in \text{br}(K)_{i+2}.$$ 

Then

$$\alpha - (u_i(1 + \pi^{i+1}c), v_i \pi) \in \text{br}(K)_{i+2},$$

and $\overline{u}_i(1 + \pi^{i+1}c) = \overline{u}_i$, so the induction hypothesis is confirmed.

In particular,

$$\alpha - (u_{N-1}, v_{N-1} \pi) \in \text{br}(K)_N.$$ 

By (1.1),

$$\alpha - (u_{N-1}, v_{N-1} \pi) = \alpha' + (u, v_{N-1} \pi)$$

with $\alpha'$ unramified and $u \in 1 + \pi R$. Thus

$$\alpha - (u_{N-1}u, v_{N-1} \pi) = \alpha'$$

is unramified and $\overline{u}_{N-1} \not\in \kappa^p$, as desired. $\square$
Lemma 1.6. (Saltman [S4]) Let $1 \leq n < N$ be coprime to $p$ and $\alpha \in br(K)_n \setminus br(K)_{n+1}$. Then there exist $u, v \in R^*$ such that

$$\alpha - (1 + v\pi^n, u\pi)$$

is unramified.

Proof. First we show by induction that for each $0 \leq i < N - n$, there exist $u_i, v_i \in R^*$ such that

$$\alpha - (1 + v_i\pi^n, u_i) \in br(K)_{n+i+1}$$

with $\overline{u}_i \not\in \kappa^p$. Since $n$ is coprime to $p$, by (1.1),

$$\alpha - (1 + v_0\pi^n, u_0) \in br(K)_{n+1}$$

for some $u_0, v_0 \in R^*$ with $\overline{u}_0 \not\in \kappa^p$. Suppose there exist $u_i, v_i \in R^*$ such that

$$\alpha - (1 + v_i\pi^n, u_i) \in br(K)_{n+i+1} \setminus br(K)_{n+i+2}$$

and $\overline{u}_i \not\in \kappa^p$. We will break the proof into two cases, depending upon the divisibility properties of $n + i + 1$.

Case 1: $n + i + 1$ is coprime to $p$. Since $\overline{u}_i \not\in \kappa^p$, by (1.1), there exists $c \in R^*$ such that

$$\alpha - (1 + v_i\pi^n, u_i) - (1 + \pi^{n+i+1}c, u_i) \in br(K)_{n+i+2}.$$ 

Let

$$v_{i+1} = v_i + \pi^{i+1}c + \pi^{n+i+1}v_ic.$$ 

Then

$$\alpha - (1 + v_{i+1}\pi^n, u_i) \in br(K)_{n+i+2},$$

verifying the inductive hypothesis.

Case 2: $n + i + 1$ is divisible by $p$ and $n + i + 1 < N$. Since $n$ is coprime to $p$, by (1.3) there is a $v \in R^*$ such that

$$(1 + v_i\pi^n, v\pi) = 1.$$ 

In particular,

$$(1 + v_i\pi^n, u_i) = (1 + v_i\pi^n, u_iv\pi).$$
By (1.1), there exists $c \in R^*$ such that
\[
\alpha - (1 + v_i \pi^n, u_i) - (1 + \pi^{n+i}c, u_i v \pi) \in \text{br}(K)_{n+i+2}.
\]
Thus
\[
\alpha - (1 + v_i \pi^n, u_i) - (1 + \pi^{n+i+1}c, u_i v \pi) = \alpha - (1 + v_i \pi^n, u_i v \pi) - (1 + \pi^{n+i+1}c, u_i v \pi)
\]
\[
= \alpha - ((1 + v_i \pi^n)(1 + \pi^{n+i+1}c), u_i v \pi)
\]
\[
\in \text{br}(K)_{n+i+2}.
\]

Let $v_{i+1} = v_i + \pi^{i+1}c + v_i c \pi^{n+i+1}$. Then
\[
\alpha - (1 + v_{i+1} \pi^n, u_i v \pi) \in \text{br}(K)_{n+i+2}.
\]
Once again, by (1.3),
\[
(1 + v_{i+1} \pi^n, v' \pi) = 1
\]
for some $v' \in R^*$. Hence
\[
(1 + v_{i+1} \pi^n, v'^{-1} \pi^{-1}) = 1
\]
and
\[
(1 + v_{i+1} \pi^n, u_i v \pi) = (1 + v_{i+1} \pi^n, u_i v \pi)(1 + v_{i+1} \pi^n, v'^{-1} \pi^{-1})
\]
\[
= (1 + v_{i+1} \pi^n, u_i v' v^{-1}).
\]
Let $u_{i+1} = u_i v' v^{-1}$. Then
\[
\alpha - (1 + v_{i+1} \pi^n, u_{i+1}) \in \text{br}(K)_{n+i+2}.
\]
Suppose that $v_{i+1} \in \kappa^p$. Replacing $u_{i+1}$ by $a^p u_{i+1}$, we assume that $v_{i+1} = 1$. Then, by ([CT], 4.1.1(b)), $v_{i+1} \pi^n, u_{i+1}) \in \text{br}(K)_{n+i+1}$, contradicting the fact that $\alpha \notin \text{br}(K)_{n+i+1}$. Thus $v_{i+1} \notin \kappa^p$, thereby verifying the inductive hypothesis.

In particular, for $i = N - n$, we have
\[
\alpha - (1 + v_{N-1} \pi^n, u_{N-1}) \in \text{br}(K)_N.
\]
Since $n$ is coprime to $p$, by (1.1), we have
\[
(1 + v_{N-1} \pi^n, u_{N-1}) = (1 + v_{N-1} \pi^n, u_{N-1} v \pi)
\]
for some $v \in R^*$. By (1.1(d)),
\[
\alpha - (1 + v_{N-1} \pi^n, u_{N-1} v \pi) = \alpha' + (1 + x' b^p, u_{N-1} v \pi)
\]
for some $\alpha'$ unramified and $x' \in R$. Hence
\[
\alpha - ((1 + v_{N-1} \pi^n)(1 + x' b^p), u_{N-1} v \pi) = \alpha'
\]
is unramified, as desired. \qed
Lemma 1.7. (Saltman [S4]) Let $1 < n < N$ be divisible by $p$ and 
\[ \alpha \in \text{br}(K)_n \setminus \text{br}(K)_{n+1}. \]
Then there exist $u, v \in R^*$ such that 
\[ \alpha - (1 + u\pi^n, v\pi) \]
is unramified.

Proof. First we show by induction that for each $0 \leq i < N - n$, there exist $u_i, v_i \in R^*$ such that 
\[ \alpha - (1 + u_i\pi^n, v_i\pi) \in \text{br}(K)_{n+i+1} \]
and 
\[ \pi_i \not\in \kappa^p. \]
Since $n$ is divisible by $p$, by (1.1) there exists $u_0 \in R^*$ with $\pi_0 \not\in \kappa^p$ such that 
\[ \alpha - (1 + u_0\pi^n, \pi) \in \text{br}(K)_{n+1}. \]
Suppose there exist $u_i, v_i \in R^*$ such that 
\[ \alpha - (1 + u_i\pi^n, v_i\pi) \in \text{br}(K)_{n+i+1} \setminus \text{br}(K)_{n+i+2} \]
and $\pi_i \not\in \kappa^p$. We again break into two cases.

Case 1: $n + i + 1$ is divisible by $p$. By (1.1), there exists $c \in R^*$ such that 
\[ \alpha - (1 + u_i\pi^n, v_i\pi) = \alpha - (1 + (u_i + \pi^{i+1}c + u_i\pi^{n+i+1})\pi^n, v_i\pi) \in \text{br}(K)_{n+i+2}. \]
Thus 
\[ \alpha - ((1 + u_i\pi^n)(1 + \pi^{n+i+1}), v_i\pi) = \alpha - (1 + (u_i + \pi^{i+1}c + u_i\pi^{n+i+1})\pi^n, v_i\pi) \in \text{br}(K)_{n+i+2}. \]
Let 
\[ u_{i+1} = u_i + \pi^{i+1}c + u_i\pi^{n+i+1}. \]
Then $\pi_{i+1} = \pi_i \not\in \kappa^p$ and 
\[ \alpha - (1 + u_{i+1}\pi^n, v_i\pi) \in \text{br}(K)_{n+i+2}, \]
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establishing the inductive hypothesis.

**Case 2:** $n + i + 1$ is coprime to $p$. By (1.1), there are $c, f \in R^*$ with $f \not\in \kappa$ such that

$$\alpha - (1 + u_i \pi^n, v_i \pi) - (1 + c \pi^{n+i+1}, f) \in br(K)_{n+i+2}.$$

By (1.4), there is an $f' \in R^*$ such that

$$(1 + c \pi^{n+i+1}, f) = (1 + v_i \pi^n, f').$$

In particular

$$\alpha - (1 + u_i \pi^n, v_i f' \pi) = \alpha - (1 + u_i \pi^n, v_i \pi) - (1 + u_i \pi^n, f')$$

$$= \alpha - (1 + u_i \pi^n, v_i \pi) - (1 + c \pi^{n+i+1}, f)$$

$$\in br(K)_{n+i+2}.$$

Proceeding as in the proof of (1.5), we conclude that there are $u, v \in R^*$ such that

$$\alpha - (1 + u \pi^n, v \pi)$$

is unramified. \qed

**Proposition 1.8.** (Saltman [S4]) Let $\alpha \in _p \mathrm{Br}(K)$. Then there exists a parameter $\pi'$ such that $\alpha \otimes K(\sqrt[p]{\pi'})$ is unramified.

**Proof.** If $\alpha \in br(K)_n$ for $0 \leq n < N$ $\alpha \in br(K)_N$, then applying (1.5, 1.6, 1.7) we see that there is a uniformizer $\pi$ and an element $a \in K$ such that

$$\alpha - (a, \pi)$$

is unramified. If $\alpha \in br(K)_N$ then by (1.1) we have (in fact) that there is an element $x \in R$ such that

$$\alpha - (1 + x \pi^N, \pi)$$

is unramified for any choice of uniformizer $\pi$. In any case, this implies that for the appropriate choice of $\pi$, the base change

$$\alpha \otimes K(\sqrt[p]{\pi'})$$

is unramified, as desired. \qed
2. The Brauer group of the function field of a surface

Let $R$ be an integral domain with field of fractions $K$. Let $C$ be a central simple algebra over $K$.

Definitions.

- The algebra $C$ is \textit{unramified} on $R$ if there is an Azumaya algebra $C$ over $R$ such that $C \otimes_R K$ is Brauer equivalent to $C$.

- An element in $\text{Br}(K)$ is \textit{unramified} on $R$ if it is represented by central simple algebra over $K$ which is unramified on $R$.

- If an element in $\text{Br}(K)$ is not unramified on $R$, then we say that it is \textit{ramified} on $R$.

- Let $P$ be a prime ideal of $R$. If an element in $\text{Br}(K)$ is unramified (resp. ramified) on $R_P$, then we say that $\alpha$ is unramified (resp. ramified) at $P$.

Let $X$ be a regular integral scheme and $K$ its function field. Let $X^i$ be the set of points of $X$ of codimension $i$. For $x \in X^i$, let $\mathcal{O}_{X,x}$ be the local ring at $x$, $m_x$ be the maximal ideal at $x$ and $\kappa(x)$ be the residue field at $x$. For $x \in X^1$, if a central simple algebra $C$ over $K$ is unramified (resp. ramified) at $\mathcal{O}_{X,x}$, then we say that $C$ is unramified (resp. ramified) at $x$. Let $K_x$ denote the field of fractions of the completion of $\mathcal{O}_{X,x}$ at $m_x$. If $m_x$ is generated by $\pi$, we also denote $K_x$ by $K_\pi$. For $x \in X^1$, let $\nu_x$ be the valuation on $K$ given by $x$. Suppose that $\text{char}(K) = 0$ and $\text{char}(\kappa(x)) = p > 0$. Let $N_x = \nu_x(p)p/(p - 1)$. If $m_x$ is generated by $\pi$, we also denote $N_x$ by $N_\pi$.

We will make (implicit) essential use of the following results throughout the rest of this paper. The following results are true in much more generality for torsors (cf. [CTS], 6.13, [APS], 4.3).

\textbf{Lemma 2.1} Suppose $X$ is a Noetherian two dimensional regular integral scheme with function field $K$. Given $\alpha \in \text{Br}(K)$, the following are equivalent.
1. For every discrete valuation of $K$ with center on $X$ and valuation ring $R$, $\alpha$ is unramified on $R$.

2. For every discrete valuation of $K$ with center of codimension 1 on $X$ and valuation ring $R$, $\alpha$ is unramified on $R$.

3. For any central simple algebra $C$ over $K$ representing $\alpha$, there is a sheaf of Azumaya algebras $A$ on $X$ and an isomorphism $A \otimes_{O_X} K \simto C$ of $K$-algebras.

**Proof.** Fix a central simple algebra $C$ over $K$ representing $\alpha$ and let $A$ be a maximal order in $C$. By [Prop. 1.2, AG] any localization of $A$ at a point of $X$ is a maximal order in $C$ over $O_{X,x}$. Since $X$ is regular and 2-dimensional and $A$ is reflexive as an $R$-module [Thm. 1.5, AG], $A$ is a finite locally free $R$-algebra. Consider the map

$$\mu : A \otimes A^\circ \to \mathcal{E}nd_{O_X}(A)$$

of locally free sheaves of $\mathcal{O}_X$-modules of rank $n^2$ associated to the map of presheaves that sends an elementary tensor $a \otimes b$ in $A(U) \otimes_{O(U)} A(U)$ to the $\mathcal{O}_X(U)$-linear map sending $x$ to $axb$. The maximal order $A$ is Azumaya if and only if $\mu$ is an isomorphism. We therefore assume that $X = \text{Spec}(B)$ with $B$ a regular local ring of dimension 2. In this case $A \otimes_B A^\circ$ and $\text{End}_B(A)$ are free $B$-modules of same rank, since $\mu \otimes_B K$ is an isomorphism. For a choice of a basis of these modules over $B$, $\mu$ is an isomorphism if and only if the determinant of $\mu$ is a unit in $B$. By Krull’s theorem, the determinant is a unit in $B$ if and only if each it is a unit at every codimension one point. This shows that the second two conditions are equivalent.

On the other hand, the first condition certainly implies the second, and hence the third. It remains to show that the third condition implies the first. Suppose $A$ is an Azumaya algebra on $X$ restricting to $C$. If $R$ is a valuation ring with center on $X$ then there is a commutative diagram

$$\begin{array}{ccc}
\text{Spec} K & \longrightarrow & X \\
\text{Spec} R & \longrightarrow & \\
\end{array}$$

in which the two diagonal arrows are the canonical ones. Pulling $A$ back yields an Azumaya algebra on $R$ restricting to $C$ on $K$, showing that $C$ is unramified on $R$, as desired. \[\square\]
Corollary 2.2 Suppose that $R$ is a two dimensional regular Noetherian integral domain with field of fractions $K$. Then a central simple algebra $C$ over $K$ is unramified on $R$ if and only if it is unramified on $R_P$ for every height one prime ideal $P$ of $R$.

Corollary 2.3 Suppose that $X$ is a Noetherian two dimensional regular scheme with fraction field $K$. Then an element $\alpha \in \text{Br}(K)$ is unramified if and only if it is in the image of the injective restriction map

$$\text{Br}(X) \rightarrow \text{Br}(K).$$

Lemma 2.4 Let $R$ be a discrete valuation ring with field of fractions $K$. Let $\hat{R}$ be the completion of $R$ at the discrete valuation and $\hat{K}$ the field of fractions of $\hat{R}$. Then a central simple algebra $C$ over $K$ is unramified at $R$ if and only if $C \otimes_K \hat{K}$ is unramified at $\hat{R}$.

Proof. Let $A$ be an Azumaya algebra over $\hat{R}$ such that there is an isomorphism $\phi : A \otimes_{\hat{R}} \hat{K} \rightarrow C \otimes_K \hat{K}$. Then $C = A \times_{\phi} C = \{(a, b) \in A \times C \mid \phi(a \times 1) = (b, 1)\}$ is an Azumaya $R$-algebra with $C \otimes_R \hat{K} \simeq C$. \hfill \square

Lemma 2.5 Let $R$ be a discrete valuation ring with field of fractions $K$ and residue field $F$. Suppose that $\text{char}(F) \neq p$ and $\alpha \in p\text{Br}(K)$. If $\pi$ is a parameter of $R$, $\alpha \otimes_K K(\sqrt[p]{\pi})$ is unramified on $R[\sqrt[p]{\pi}]$.

Proof. See ([S2], 0.4). \hfill \square

Proposition 2.6. Let $A$ be a regular local ring of dimension two with maximal ideal $m = (\pi, \delta)$. Let $K$ be the field of fractions of $A$ and $k$ the residue field of $A$. Suppose that $\text{char}(K) = 0$, $\text{char}(k) = p > 0$ and $k = k^p$. Let $\alpha \in p\text{Br}(K)$. If $\alpha$ is ramified on $A$ at most at $(\pi)$ and $(\delta)$, then $\alpha \otimes_K K(\sqrt[p]{\pi}, \sqrt[p]{\delta})$ is unramified at any discrete valuation of $K(\sqrt[p]{\pi}, \sqrt[p]{\delta})$ dominating $A$.

Proof. Let $\nu$ be a discrete valuation of $L = K(\sqrt[p]{\pi}, \sqrt[p]{\delta})$ dominating $A$. Let $R$ be the valuation ring at $\nu$. The completion of $A$ at $m$ is contained in the
completion of $R$ at its maximal ideal. By (2.4), to show that $\alpha$ is unramified at $R$, we replace $A$ by its completion at $m$ and assume that $A$ is complete.

Let $B = A[\sqrt[p]{\pi}, \sqrt[p]{\delta}]$. Then $B$ is a complete regular local ring with field of fractions $L$ and $B$ is the integral closure of $A$ in $L$. Hence $B \subset R$. Thus, to show that $\alpha$ is unramified at $\nu$, it is enough to show that $\alpha$ is unramified on $B$. Since $B$ is a regular ring of dimension 2, by (2.2), it is enough to show that $\alpha$ is unramified at every height one prime ideal of $B$.

Let $Q$ be a height one prime ideal of $B$. Since $B$ is integral over $A$, $Q \cap A = P$ is a height one prime ideal of $A$. Suppose $P \neq (\pi)$ and $P \neq (\delta)$. Then $\alpha$ is unramified at $P$ and hence $\alpha$ is unramified at $Q$.

Suppose $P = (\pi)$. Then $L_Q = K_P(\sqrt[p]{\pi}, \sqrt[p]{\delta})$. Suppose that $\text{char}(\kappa(P)) \neq p$. Since $\pi$ is a parameter at $P$, by (2.5), $\alpha$ is unramified over $K_P(\sqrt[p]{\pi}) \subset L_Q$ and hence unramified at $Q$. Suppose that $\text{char}(\kappa(P)) = p$. Since $A$ is a complete regular local ring of dimension 2 with maximal ideal $m = (\pi, \delta)$, we have $A/P \simeq k[[\delta]]$, where $\delta$ is the image of $\delta$ in $A/P$. Since $k = k^p$, $[\kappa(P) : \kappa(P)^p] = p$ and $\delta$ is not a $p^{th}$ power in $\kappa(P)$. Thus, by (1.2), $\alpha$ is split over $K_Q = K_P(\sqrt[p]{\pi}, \sqrt[p]{\delta})$. In particular, $\alpha$ is unramified at $Q$. The case $P = (\delta)$ is similar.

\begin{proof}
Let $B = A[\sqrt[p]{\pi}]$. Then $B$ is a regular local ring with field of fractions $K(\sqrt[p]{\pi})$ and $B$ is integral over $A$. Every discrete valuation of $L = K(\sqrt[p]{\pi})$ dominating $A$ also dominates $B$. Thus, $B$ being regular, it is enough to show that $\alpha$ is unramified on $B$. Let $Q$ be a height one prime ideal of $B$. Since $B$ is integral over $A$, $Q \cap A = P$ is a height one prime ideal. Suppose $P \neq (\pi)$. Since $\alpha$ is unramified at $P$, $\alpha$ is unramified at $Q$. Suppose $P = (\pi)$. Then by the assumption, $\alpha$ is unramified over $L_Q = K_{\pi}(\sqrt[p]{\pi})$ and hence unramified at $Q$ by (2.4).
\end{proof}

\begin{lemma}
Let $A$ be a one-dimensional Noetherian local domain with field of fractions $K$ and residue field $k$. Suppose that $\text{char}(K) = p > 0$ and
\end{lemma}
$[K : K^p] \leq p^{d+1}$ for some $d \geq 0$. Then $[k : k^p] \leq p^d$.

**Proof.** Let $B$ be the integral closure of $A$ in $K$ and $m$ a maximal ideal of $B$. Then $B/m$ is a finite extension of $k$ and hence $[(B/m) : (B/m)^p] = [k : k^p]$. Thus, by replacing $A$ by $B_m$, we assume that $A$ is a discrete valuation ring. Let $\pi \in A$ be a generator of the maximal ideal of $A$. Let $u_1, \cdots, u_r \in A^*$ be such that the images of $u_1, \cdots, u_r$ in $k$ are $p$-independent. Then, it is easy to see that $\pi, u_1, \cdots, u_r$ are $p$-independent in $K$. \hfill $\square$

**Theorem 2.9.** Let $X$ an excellent regular integral scheme of dimension 2 and $K$ the function field of $X$. Let $p$ be a prime number. Suppose that $\text{char}(K) = 0$ and $K$ contains a primitive $p^{th}$ root of unity. Suppose that for every codimension one point $x$ of $X$ with $\text{char}(\kappa(x)) = p$, $[\kappa(x) : \kappa(x)^p] = p$. Let $\alpha \in \pi \text{Br}(K)$. Then there exist $f, g, h \in K^*$ such that $\alpha \otimes K(\sqrt[p]{f}, \sqrt[p]{g}, \sqrt[p]{h})$ is unramified at every discrete valuation of $K(\sqrt[p]{f}, \sqrt[p]{g}, \sqrt[p]{h})$ whose restriction to $K$ is centered at a point of $X$.

**Proof.** Let $P$ be a closed point of $X$. Suppose that $\text{char}(\kappa(P)) = p$. Let $\pi \in \mathcal{O}_{X,P}$ be a prime dividing $p$. Then $A = \mathcal{O}_{X,P}/(\pi)$ is a one-dimensional Noetherian local domain. Let $K(\pi)$ be the field of fractions of $A$. Then $K(\pi)$ is the residue field of a codimension one point of $X$ and $\text{char}(K(\pi)) = p$. Thus, by the assumption, $[K(\pi) : K(\pi)^p] = p$. Hence, by (2.8), $[\kappa(P) : \kappa(P)^p] = 1$. Let $X'$ be a blow-up of $X$ at finitely many closed points. Let $x' \in X'$ be a point of codimension 1. If $x'$ is an exceptional curve, then $\kappa(x') = \kappa(P)(t)$ for some closed point $P$ of $X$ and $t$ a variable. In particular, $[\kappa(x') : \kappa(x')^p] = p$. Thus $X'$ also satisfies the hypothesis of the theorem.

By blowing up $X$ at finitely many closed points ([Li]), we assume that $\text{ram}(\alpha)$ and $\text{Supp}(p)$ are contained in $C + E$, where $C$ and $E$ are regular curves with $C$ intersecting $E$ transversally. Let $D$ be an irreducible component in $C \cup E$. If $\text{char}(\kappa(D)) \neq p$, let $\pi_D$ be any parameter at $D$. Then, by (2.5), $\alpha$ is unramified over $K_D(\sqrt[p]{\pi_D})$. If $\text{char}(\kappa(D)) = p$, then by the assumption, $[\kappa(D) : \kappa(D)^p] = p$ and hence by (1.8) there exists a parameter $\pi_D \in K_D^*$ at $D$ such that $\alpha$ is unramified over $K_D(\sqrt[p]{\pi_D})$. By the weak approximation, choose $f \in K^*$ such that $f = \pi_D$ modulo $K_D^p$ for every irreducible component $D$ in $C \cup E$. Then, we have $K_D(\sqrt[p]{f}) = K_D(\sqrt[p]{\pi_D})$ and

$$\text{div}(f) = C + E + F$$
for some $F$ not containing any component of $C$ and $E$. In particular, $\alpha$ is unramified over $K_D(\sqrt[3]{T})$ for every $D$ in $C \cup E$.

Let $\mathcal{P}$ be the finite set of closed points containing $C \cap E$, $C \cap F$ and $E \cap F$ and at least one point from each component of $C$, $E$, $F$. Let $A$ be the semi-local regular two dimensional ring at the points in $\mathcal{P}$. Let $\pi, \delta \in A$ be such that the divisor of $\pi$ on $A$ is $C$ and the divisor of $\delta$ on $A$ is $E$. For $P \in \mathcal{P}$, let $m_P$ denote the maximal ideal of $A$ at $P$. If $P \in C \cap F$, $P \not\in E$, let $\delta_P \in m_P$ be such that $\delta_P \not\in m_Q$ for all $Q \in \mathcal{P}, Q \neq P$ and $\delta_P \not\in \langle \pi \rangle + m_P^2$. Similarly, choose $\pi_Q$ for each $Q \in E \cap F$, $Q \not\in C$.

Let
\[
g = \pi \prod_{Q \in (E \cap F) \setminus C} \pi_Q \quad \text{and} \quad h = \delta \prod_{P \in (C \cap F) \setminus E} \delta_P.
\]
Let $P \in \mathcal{P}$. By the choice of $g$ and $h$, we have $m_P = (g, h)$. If $P \in C$, then $g$ defines $C$ at $P$ and if $P \in E$, then $h$ defines $E$ at $P$.

We claim that $\alpha$ is unramified over $L = K(\sqrt[3]{T}, \sqrt[3]{g}, \sqrt[3]{h})$. Let $\nu$ be a discrete valuation of $L$. Let $R$ be the discrete valuation ring of $L$ at $\nu$. Then there is a point $x$ of $X$ such that $R$ dominates the local ring $A_x = \mathcal{O}_{X,x}$.

If $x$ is not on $C \cup E$, then $\alpha$ is unramified on $A_x$ and hence unramified at $\nu$. Suppose that $x \in C \cup E$.

Suppose that $x$ is a codimension one point. Then $x$ corresponds to an irreducible component of $C$ or $E$. By the choice of $f$, $\alpha$ is unramified over $K_x(\sqrt[3]{f}) \subset L_\nu$. In particular $\alpha$ is unramified at $\nu$.

Suppose that $x$ is a closed point. Suppose that $x \not\in \mathcal{P}$. Since $x \in C \cup E$, $C$, $E$ are regular curves on $X$, $f \in m_x \setminus m_x^2$ and $\alpha$ is ramified on $\mathcal{O}_{X,x}$ only along $(f)$. By the choice of $f$ and by (2.7), $\alpha$ is unramified at $\nu$ restricted to $K(\sqrt[3]{f})$. Since $K(\sqrt[3]{f}) \subset L$, $\alpha$ is unramified at $\nu$.

Assume that $x \in \mathcal{P}$. Then, by the choice of $g$ and $h$, the maximal ideal $m_x$ of $\mathcal{O}_{X,x}$ is generated by $g$ and $h$. Further $\alpha$ is unramified on $A$ except at $(g)$ and $(h)$. Since $\nu$ is centered on $\mathcal{O}_{X,x}$, its restriction to $K(\sqrt[3]{g}, \sqrt[3]{h})$ is also centered on $\mathcal{O}_{X,x}$. Hence, by (2.6), $\alpha$ is unramified at $\nu$ restricted to $K(\sqrt[3]{g}, \sqrt[3]{h})$. Since $K(\sqrt[3]{g}, \sqrt[3]{h}) \subset L$, $\alpha$ is unramified at $\nu$. \hfill \square

**Corollary 2.10.** Let $K$ be a $p$-adic field and $K$ a function field of a curve over $k$. Then for every central simple algebra $A$ over $K$, index($A$) divides $\text{period}(A))^3$.
Proof. Let $A$ be a central simple algebra over $K$ and $\alpha$ its class in $Br(K)$. Let $n$ be the period of $A$. It is enough to prove the result for a prime $n$ and assuming $K$ contains a primitive $n$th root of unity. If $n$ is coprime to $p$, then by ([S1]), period of $A$ divides the square of the index. Assume that $n = p$. Since $K$ is a field of fractions of a regular proper scheme over the ring of integers in $k$ of dimension 2, by (2.9), there exist $f, g, h \in K^*$ such that $\alpha$ is unramified over $K(\sqrt[3]{f}, \sqrt[3]{g}, \sqrt[3]{h})$. Since the unramified Brauer group of $K(\sqrt[3]{f}, \sqrt[3]{g}, \sqrt[3]{h})$ is zero ([G], 2.15 and 3.1), $\alpha$ is trivial over $K(\sqrt[3]{f}, \sqrt[3]{g}, \sqrt[3]{h})$. Hence the index of $\alpha$ divides $p^3$. 

Corollary 2.11. Let $k$ be a number field and $K$ function field of a curve over $K$. Then for every central simple algebra of period $p$, there exist $f, g, h \in K^*$ such that $\alpha \otimes K(\sqrt[3]{f}, \sqrt[3]{g}, \sqrt[3]{h})$ is unramified at every discrete valuation of $K(\sqrt[3]{f}, \sqrt[3]{g}, \sqrt[3]{h})$. 

Proof. We assume that $K$ contains a primitive $p$th root of unity. Since $K$ is a field of fractions of a regular proper scheme over the ring of integers in $k$ of dimension 2 and every discrete valuation of $K$ is centered at a point of $X$, by (2.9), there exist $f, g, h \in K^*$ such that $\alpha$ is unramified over $K(\sqrt[3]{f}, \sqrt[3]{g}, \sqrt[3]{h})$ at every discrete valuation of $K(\sqrt[3]{f}, \sqrt[3]{g}, \sqrt[3]{h})$. 

3. Period-index bounds

In this section we ameliorate some of the results of [L2] on the period-index problem for Brauer classes on arithmetic surfaces. Let $k$ be a number field, $A$ the ring of adèles of $k$ and $S$ the scheme of integers of $k$. We fix a regular proper surface $X$ with a flat projective generically smooth morphism $X \to S$. We will write $C$ for the generic fiber of $X/S$, and we assume that $C$ is geometrically connected. Unlike in Section 7 of [L2], we do not assume that $C$ admits a $k$-point; instead, we will use the following Lemma to get 0-cycles of degree 1 on appropriate moduli spaces.

Lemma 3.1 Let $\overline{M}$ be a proper smooth connected $k$-scheme. For every nonempty open subscheme $M \subset \overline{M}$, we have that $M$ contains a 0-cycle of degree 1 if and only if $\overline{M}$ contains a 0-cycle of degree 1.
Proof. It is enough to show that if $\overline{M}$ contains a point with residue field a finite extension $L/k$ then $M$ contains a 0-cycle of degree equal to $[L:k]$. Since $\overline{M}$ is smooth, it is easy to see (by choosing generic parameters in the local ring at $Q$ and taking a Zariski closure) that there is a smooth projective curve $D$ over $k$ with an $L$-rational point $R \in D$ and a finite morphism $f : D \to \overline{M}$ such that $f(R) = Q$ and $U = f^{-1}(M) \neq \emptyset$. Since any point of $U$ is ample, it is easy to see that there is a divisor $E \subset U \subset D$ of degree $[L:k]$. The proper pushforward of $E$ to $\overline{M}$ gives a 0-cycle of degree equal to $[L:k]$ contained in $M$, as desired.

Fix a prime number $\ell$ and a $\mu_\ell$-gerbe $\mathcal{X} \to X$. Write $\mathcal{C} \to C$ for the restriction of $\mathcal{X}$ to $C$. Let $\mathcal{M} := \mathcal{M}_C(\ell)$ denote the stack of stable $\mathcal{C}$-twisted sheaves of rank $\ell$ and trivial determinant (as described in [L1]). We know the following facts about this stack.

1. $\mathcal{M}$ is a $\mu_\ell$-gerbe over a smooth quasi-projective variety $M$ admitting a locally factorial compactification $M \subset M^{ss}$ such that
   \[ \text{codim}(M^{ss} \setminus M, M^{ss}) > 2 \]

   and

   \[ \text{Pic}(M^{ss}) = \text{Pic}(M) = \mathbb{Z}. \]

2. the Brauer group $\text{Br}(M \otimes \overline{k})$ is generated by the class associated to $\mathcal{M} \otimes \overline{k}$, this class has period and index $\ell$, and the sequence
   \[ 0 \to \text{Br}(k) \to \text{Br}(M) \to \text{Br}(M \otimes \overline{k}) \to 0 \]

   is exact. Note that since $\mathcal{M}$ is a $\mu_\ell$-gerbe over $M$, its Brauer class has period $\ell$ and thus the universal obstruction provides a canonical splitting of the above exact sequence.

Proposition 3.2 If $k$ is totally imaginary then the Brauer-Manin set $M(\mathbb{A})^{\text{Br}}$ is non-empty.

Proof. We first claim that for every place $v$ of $k$, the category $\mathcal{M}(k_v)$ is non-empty. First, suppose $v$ is finite. Consider the base change $\mathcal{X} \otimes_S \mathcal{O}_{k_v} \to X \otimes_S \mathcal{O}_{k_v}$. This is a $\mu_\ell$-gerbe on a proper curve over a complete discrete valuation ring with finite residue field. Since the Brauer group of
such a scheme is trivial, we have an invertible \( C_k \)-twisted sheaf \( \mathcal{L} \). Writing

\[
\mathcal{M}_{C_k} (\ell, L) \cong \mathcal{M} \otimes k_v.
\]

Since stable vector bundles of rank \( \ell \) and determinant \( L \) exist on any curve over an infinite field, the former has an object over \( k_v \), whence the latter does as well. When \( v \) is infinite, the completion is algebraically closed (as \( k \) is assumed to be totally imaginary), so we can use Tsen's theorem to trivialize the Brauer class and similarly reduce to the existence of stable vector bundles on curves over algebraically closed fields.

Write \((x_v)\) for the system of objects of \( \mathcal{M}(k_v) \). Projecting to \( M \) gives a point \((x_v) \in M(A)\) with the property that the pairing \((x_v) \cdot [\mathcal{M} \to M] = 0\) in \( \mathbb{Q}/\mathbb{Z} \). As noted above, the Leray spectral sequence shows that \( \text{Br}(M)/\text{Br}(k) \) is generated by the class of \( \mathcal{M} \to M \). We conclude that the point \((x_v)\) lies in \( M(\text{Br}) \), as desired.

\[\text{Theorem 3.3}\]

Assuming the CT-conjecture (see page 2), if \( k \) is totally imaginary any class \( \alpha \in \text{Br}(X) \) satisfies \( \text{ind}(\alpha)|\text{per}(\alpha)^2 \).

**Proof.** By standard inductive arguments, we may assume that \( \alpha \) has prime index \( \ell \).

We retain the notation from above. Let \( M \subset \overline{M} \) be any smooth compactification of \( M \); this is possible by Hironaka's theorem, since \( k \) has characteristic 0. By functoriality we have that \( M(A)^\text{Br} \subset \overline{M(A)^\text{Br}} \), hence by Proposition 3.2 \( \overline{M(A)^\text{Br}} \neq \emptyset \). The CT-conjecture applies to show that \( \overline{M} \) has a 0-cycle of degree 1. Applying Lemma 3.1, we have a 0-cycle of degree 1 in \( M \). Put another way, there are two finite étale \( k \)-algebras \( A_1 \) and \( A_2 \) such that \([A_1 : k]\) is relatively prime to \([A_2 : k]\) and such that \( M(A_1) \neq \emptyset \) and \( M(A_2) \neq \emptyset \).

Given a finite étale closed subscheme \( Z \subset M \), the Brauer class of \( \mathcal{M} \times_M Z \) lies in \( \text{Br}(Z)[\ell] \) and thus has index dividing \( \ell \) (as the residue fields of the Artinian scheme \( Z \) are global fields). It follows that there is a finite flat \( Z \)-scheme \( Y \to Z \) of degree \( \ell \) such that \( \mathcal{M}(Y) \neq \emptyset \). Applying this to \( A_1 \) and \( A_2 \) above, we find that for \( i = 1, 2 \) there is a finite flat \( A_i \)-algebra \( B_i \) of degree \( \ell \) and a locally free \( \mathcal{C}_{B_i} \)-twisted sheaf of rank \( \ell \). Putting these together and arguing as in Section 4.1.1 of [L3], we see that the index of \( \alpha \) divides \( \ell^2[A_i : k] \) for \( i = 1, 2 \). Since \([A_1 : k]\) and \([A_2 : k]\) are relatively prime, we see that \( \text{ind}(\alpha)|\ell^2 \), as desired. \(\square\)
4. $u$-invariant of the function field a curve over a number field

**Theorem 4.1.** Assume that the CT-conjecture holds. Let $k$ be a totally imaginary number field and $K$ the function field of a curve over $k$. Then for any central simple algebra $A$ over $K$, $\text{ind}(A)$ divides $\text{period}(A)^5$.

**Proof.** Let $A$ be a central simple algebra over $K$. By a standard inductive argument, it is enough to consider the case $\text{period}(A) = p$ a prime and $K$ contains a primitive $p^{th}$ root of unity. By (2.11), there exist $f, g, h \in K^*$ such that $A \otimes_K K(\sqrt[p]{f}, \sqrt[p]{g}, \sqrt[p]{h})$ is unramified at every discrete valuation of $K(\sqrt[p]{f}, \sqrt[p]{g}, \sqrt[p]{h})$. By (3.3), $\text{index}(A \otimes K(\sqrt[p]{f}, \sqrt[p]{g}, \sqrt[p]{h}))$ divides $p^2$. Hence $\text{index}(A)$ divides $p^5$. $\square$

**Corollary 4.2.** Assume that the CT-conjecture holds. Let $k$ be a number field and $K$ the function field of a curve over $k$. Then for any central simple algebra $A$ over $K$, $\text{ind}(A)$ divides $\text{period}(A)^6$.

**Proof.** By (4.1), $\text{ind}(A \otimes_k k(\sqrt{-1}))$ divides $2^5$. Hence $\text{ind}(A)$ divides $2^6$. $\square$

**Theorem 4.3.** Assume that the CT-conjecture holds. Let $k$ be a number field and $K$ the function field of a curve over $k$. Then there exists an integer $N_2$ (which does not depend on $K$ or $k$) such that every element in $H^2(K, \mu_2)$ is a sum of at most $N_2$ symbols.

**Proof.** Let $\alpha \in H^2(K, \mu_2)$. By (4.2), $\text{index}(\alpha)$ divides $2^6$. Let $\mathcal{A}$ be the generic division algebra of degree $2^6$ with center $Z$ and $X_{\mathcal{A}^2}$ be the Severi-Brauer variety of $\mathcal{A} \otimes \mathcal{A}$ over $Z$. Let $\alpha$ be represented by a central simple algebra $A$ over $K$ of degree $2^6$. Then there is a specialisation from the function field $Z(X_{\mathcal{A}^2})$ to $K$ which specialises $\mathcal{A}$ to $A$. Since $\mathcal{A}$ is 2-torsion element in the Brauer group of $Z(X_{\mathcal{A}^2})$, $\mathcal{A}$ is a product of $N_2$ quaternion algebras over $Z(X_{\mathcal{A}^2})$ ([M]). In particular $A$ is a product of $N_2$ quaternion algebras over $K$. Hence $\alpha$ is a sum of at most $N_2$ symbols. $\square$

**Theorem 4.4.** Assume that the CT-conjecture holds. Let $k$ be a number field and $K$ the function field of a curve over $k$. Then every element in $H^3(K, \mu_2)$ is a sum of $N_2$ symbols.
Proof. Let $\beta \in H^3(K, \mu_2)$. Then by ([Su]), there exists $f \in K^*$ such that $\alpha$ is zero over $K(\sqrt{f})$. Hence $\beta = (f) \cdot \alpha$ for some $\alpha \in H^2(K, \mu_2)$ ([A], 4.6). By (4.3), $\alpha = (a_1) \cdot (b_1) + \cdots + (a_{N_2}) \cdot (b_{N_2})$. Thus $\beta = (f) \cdot \alpha = (f) \cdot (a_1) \cdot (b_1) + \cdots + (f) \cdot (a_{N_2}) \cdot (b_{N_2})$. \hfill $\blacksquare$

**Theorem 4.5** Assume that the CT-conjecture holds. Let $k$ be a totally imaginary number field and $K$ the function field of a curve over $k$. Then $u(K)$ is finite.

Proof. By (4.3), there exists $N_2$ such that every element in $H^2(K, \mu_2)$ is a sum of at most $N_2$ symbols. By (4.4), every element in $H^3(K, \mu_2)$ is a sum of at most $N_2$ symbols. Since $k$ is a totally imaginary number field, $H^4(K, \mu_2) = 0$. Hence $u(K)$ is finite (cf. [PS]). \hfill $\blacksquare$

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