1. Introduction

The purpose of this paper is to generalize the coinvariant algebra – a representation whose combinatorics is controlled by permutations – to a new class of graded representations whose combinatorics will be controlled by ordered set partitions. Our new representations will be related to Macdonald polynomial theory in that their graded Frobenius character will be a specialization of the combinatorial side of the Delta Conjecture. It remains an open problem to give a bigraded coinvariant algebra, whose Frobenius image is even conjecturally equal to any of the expressions in the Delta Conjecture; our module \( R_{n,k} \) solves this problem in the specialization \( t = 0 \).

Abstract. The symmetric group \( S_n \) acts on the polynomial ring \( \mathbb{Q}[x_n] = \mathbb{Q}[x_1, \ldots, x_n] \) by variable permutation. The invariant ideal \( I_n \) is the ideal generated by all \( S_n \)-invariant polynomials with vanishing constant term. The quotient \( R_n = \frac{\mathbb{Q}[x_n]}{I_n} \) is called the coinvariant algebra. The coinvariant algebra \( R_n \) has received a great deal of study in algebraic and geometric combinatorics.

We introduce a generalization \( I_{n,k} \subseteq \mathbb{Q}[x_n] \) of the ideal \( I_n \) indexed by two positive integers \( k \leq n \). The corresponding quotient \( R_{n,k} = \frac{\mathbb{Q}[x_n]}{I_{n,k}} \) carries a graded action of \( S_n \) and specializes to \( R_n \) when \( k = n \). We generalize many of the nice properties of \( R_n \) to \( R_{n,k} \). In particular, we describe the Hilbert series of \( R_{n,k} \), give extensions of the Artin and Garsia-Stanton monomial bases of \( R_n \) to \( R_{n,k} \), determine the reduced Gröbner basis for \( I_{n,k} \), describe the graded Frobenius series of \( R_{n,k} \). Just as the combinatorics of \( R_n \) are controlled by permutations in \( S_n \), we will show that the combinatorics of \( R_{n,k} \) are controlled by ordered set partitions of \( \{1, 2, \ldots, n\} \) with \( k \) blocks.

The Delta Conjecture of Haglund, Remmel, and Wilson is a generalization of the Shuffle Conjecture in the theory of diagonal coinvariants. We will show that the graded Frobenius series of \( R_{n,k} \) is (up to a minor twist) the \( t = 0 \) specialization of the combinatorial side of the Delta Conjecture. It remains an open problem to give a bigraded \( S_n \)-module \( V_{n,k} \) whose Frobenius image is even conjecturally equal to any of the expressions in the Delta Conjecture; our module \( R_{n,k} \) solves this problem in the specialization \( t = 0 \).

Key words and phrases. ordered set partition, coinvariant algebra, symmetric function.
The coinvariant algebra $R_n$ is among the most important representations in combinatorics. Let us recall some of its properties, deferring various definitions to Section 2. We will use the usual $q$-analogs of numbers, factorials, and multinomial coefficients:

$$[n]_q := 1 + q + \cdots + q^{n-1}, \quad [n]!_q := [n]_q[n-1]_q \cdots [1]_q,$$

where $\text{Stir}(n, k)$ and we may identify $[n, k]_q$.

Definition 1.1. Given two positive integers $n$ and $k$, let $I_n$ be the ideal $I_n = \langle e_1(x_n), e_2(x_n), \ldots, e_n(x_n) \rangle$.

Let $R_{n,k}$ be the corresponding quotient ring:

$$R_{n,k} := \frac{\mathbb{Q}[x_n]}{I_{n,k}}.$$
Since the ideal \( I_{n,k} \) is homogeneous, the quotient \( R_{n,k} \) is a graded vector space. Moreover, since \( I_{n,k} \) is stable under the action of \( S_n \), the algebra \( R_{n,k} \) carries a graded action of \( S_n \).

When \( k = n \), it can be shown\(^2\) that for any \( 1 \leq i \leq n \), the variable power \( x_i^n \) lies in the invariant ideal \( I_n \), so that \( I_{n,n} = I_n \) and \( R_{n,n} = R_n \).

At the other extreme, when \( k = 1 \) we have \( I_{n,1} = \langle x_1, x_2, \ldots, x_n \rangle \), so that \( R_{n,1} = \frac{\mathbb{Q}[x_n]}{\langle x_1, \ldots, x_n \rangle} \cong \mathbb{Q} \) is the trivial \( S_n \)-module in degree 0.

We will prove that the modules \( R_{n,k} \) extend many of the nice properties of the modules \( R_n \), where one replaces permutations in \( S_n \) with ordered set partitions in \( \mathcal{OP}_{n,k} \). If \( x = (x_1, x_2, \ldots) \) is an infinite set of variables and \( Z[[x]] \) is the ring of formal power series in \( x \) with integer coefficients, let \( \text{rev}_q : Z[[x]][q] \to Z[[x]][q] \) be the operator which reverses polynomials with respect to the variable \( q \). Explicitly, if \( f = a_0 q^d + \cdots + a_1 q + a_0 \) with \( a_i \in Z[[x]] \) and \( a_d \neq 0 \), then \( \text{rev}_q(f) = a_0 q^d + \cdots + a_{d-1} q + a_d \). For example, we have

\[
\text{rev}_q(q^5 + 7q^3 - 8q) = -8q^4 + 7q^2 + 1
\]

and

\[
\text{rev}_q(e_3(x) q^2 + 2e_2(x) q - 3) = -3q^2 + 2e_2(x) q + e_3(x).
\]

- We have \( \dim(R_{n,k}) = |\mathcal{OP}_{n,k}| = k! \cdot \text{Stir}(n,k) \). The Hilbert polynomial \( \text{Hilb}(R_{n,k};q) \) is given by

\[
\text{Hilb}(R_{n,k};q) = \text{rev}_q([k]! q \cdot \text{Stir}(n,k)),
\]

where \( \text{Stir}(n,k) \) is the \( q \)-Stirling number (see Theorem 4.10). There is a generalization \( \mathcal{A}_{n,k} \) of Artin’s basis of \( R_n \) to \( R_{n,k} \) which witnesses this identity (see Theorem 4.13).

- As ungraded \( S_n \)-modules we have

\[
R_{n,k} \cong_{S_n} Q[\mathcal{OP}_{n,k}],
\]

so that \( R_{n,k} \) gives a graded version of the action of \( S_n \) on ordered set partitions (see Theorem 4.11).

- The graded isomorphism type of \( R_{n,k} \) can be described in terms of standard Young tableaux (see Corollary 6.13).

In terms of the dual Hall-Littlewood basis we have (see Theorem 6.14)

\[
\text{grFrob}(R_{n,k};q) = \sum_{T \in \text{SYT}(n)} q^{\text{maj}(T)} \left[ \frac{n - \text{des}(T) - 1}{n - k} \right] s_{\text{shape}(T)}(x).
\]

We also have a combinatorial description of \( \text{grFrob}(R_{n,k};q) \) using Mahonian statistics on ordered multisets.

Our results have connections to the theory of Macdonald polynomials. The Delta Conjecture of Haglund, Remmel, and Wilson [18] is a generalization of the Shuffle Conjecture of Haglund, Haiman, Loehr, Remmel, and Ulyanov [16]. The Delta Conjecture asserts an equality of three quasisymmetric functions; see Section 2 for details. Two of these quasisymmetric functions are defined combinatorially and denoted \( \text{Rise}_{n,k}(x;q,t) \) and \( \text{Val}_{n,k}(x;q,t) \).

\(^1\)It can also be shown that \( e_n(x_n) \) lies in the ideal generated by \( x_1^n, \ldots, x_k^n, c_{n-1}(x_n), \ldots, c_{n-k+1}(x_n) \), so that its presence as a generator of \( I_{n,k} \) is redundant. We include \( e_n(x_n) \) as a generator of \( I_{n,k} \) because it will not be redundant as a generator in a more general family of ideals \( I_{n,k,s} \) introduced in Section 6.

\(^2\)By [5, Sec. 7.2] we have \( x_n^n \in I_n \) and \( I_n \) is \( S_n \)-stable.
Using statistics on ordered multiset partitions, it follows from work of Wilson and Rhoades\cite{33,28} that
\begin{equation}
\text{Rise}_{n,k}(x; q, 0) = \text{Rise}_{n,k}(x; 0, q) = \text{Val}_{n,k}(x; q, 0) = \text{Val}_{n,k}(x; 0, q).
\end{equation}

Let $C_{n,k}(x; q)$ denote this common polynomial, which is known \cite{33,28} to be symmetric. We deform $C_{n,k}(x; q)$ somewhat by setting
\begin{equation}
D_{n,k}(x; q) := (\text{rev}_q \circ \omega)C_{n,k}(x; q),
\end{equation}
where $\omega$ is the usual involution on symmetric functions sending $s_\lambda(x)$ to $s_{\lambda'}(x)$. We will prove that
\begin{equation}
gr\text{Frob}(R_{n,k}; q) = D_{n,k}(x; q).
\end{equation}

In other words, the module $R_{n,k}$ has graded Frobenius image equal to either of the combinatorial expressions in the Delta Conjecture at $t = 0$ (up to a twist).

It is an open problem to determine a bigraded $\mathfrak{S}_n$-module which (even conjecturally) has Frobenius image equal to any of the expressions in the Delta Conjecture. In the case $k = n$, the module of diagonal coinvariants plays this role. Our result is the first theorem in this direction for general $k \leq n$.

Although our new module $R_{n,k}$ generalizes many of the nice combinatorial properties of the classical coinvariant module $R_n$, the proofs of these properties will be substantially different. To see why, recall that a sequence of polynomials $f_1, \ldots, f_n$ is a regular sequence in the ring $\mathbb{Q}[x_n]$ if $f_i$ is not a zero divisor in $\mathbb{Q}[x_n]_{(f_1, \ldots, f_{i-1})}$ for all $i$. The regularity of $f_1, \ldots, f_n$ immediately implies the exactness of
\begin{equation}
0 \to \mathbb{Q}[x_n]_{(f_1, \ldots, f_{i-1})} \to \mathbb{Q}[x_n]_{(f_1, \ldots, f_{i-1})} \to \mathbb{Q}[x_n]_{(f_1, \ldots, f_i)} \to 0
\end{equation}
for all $i$, and in turn (if the $f_i$ are homogeneous)
\begin{equation}
\text{Hilb} \left( \frac{\mathbb{Q}[x_n]}{(f_1, \ldots, f_n)}; q \right) = [\text{deg}(f_1)]q \cdots [\text{deg}(f_n)]q.
\end{equation}

In particular, we have $\dim \left( \frac{\mathbb{Q}[x_n]}{(f_1, \ldots, f_n)} \right) = \text{deg}(f_1) \cdots \text{deg}(f_n)$. Moreover, the Koszul complex, a certain free resolution of the $\mathbb{Q}[x_n]$-module $\frac{\mathbb{Q}[x_n]}{(f_1, \ldots, f_n)}$, is guaranteed to be exact in this case.

It can be shown that the generators $e_1(x_n), e_2(x_n), \ldots, e_n(x_n)$ of $I_n$ form a regular sequence. The regularity of this sequence gives many of the properties of $R_n$ for free. It is immediate that $\dim(R_n) = n!$ and $\text{Hilb}(R_n; q) = [n]_q!$. Since the maps involved in the Koszul complex commute with the action of $\mathfrak{S}_n$, it is readily derived that
\begin{equation}
gr\text{Frob}(R_n; q) = Q'_{(1^n)}(x; q).
\end{equation}

This commutative algebra machinery breaks down in our setting. The ideal $I_{n,k}$ cannot be generated by a regular sequence in $\mathbb{Q}[x_n]$. Indeed, the lack of a nice product formula for the Stirling number $\text{Stir}(n,k)$ makes it impossible to find a homogeneous regular sequence $f_1, \ldots, f_n$ in $\mathbb{Q}[x_n]$ such that $\dim \left( \frac{\mathbb{Q}[x_n]}{(f_1, \ldots, f_n)} \right) = \text{deg}(f_1) \cdots \text{deg}(f_n) = [\mathcal{OP}_{n,k}]$. Trying to modify the above program to determine the Hilbert or Frobenius image of $R_{n,k}$ is therefore hopeless.

To obtain the Hilbert series of $R_{n,k}$, we use the theory of Gröbner bases. For a given monomial order $<$, any ideal $I \subseteq \mathbb{Q}[x_1, \ldots, x_n]$ has a unique reduced Gröbner basis. While the polynomials in this basis can have unpredictable monomials and ugly coefficients, even for nicely presented ideals $I$, in our context a miracle occurs. If we take $<$ to be the lexicographic term ordering, the reduced Gröbner basis for $I_{n,k}$ consists of the variable powers $x_1^k, x_2^k, \ldots, x_n^k$ together with certain (predictable) Demazure characters $\kappa_n(x_n, x_{n-1}, \ldots, x_1) \in \mathbb{Q}[x_n]$ in a reversed variable set (see Theorem 4.14). The polynomials $\kappa_n$ are characters of indecomposable polynomial representations.
of the Borel subgroup $B \subseteq GL_n(\mathbb{C})$ of upper triangular matrices; their appearance as Gröbner basis elements of $I_{n,k}$ is mysterious to the authors.

Our Gröbner basis for $I_{n,k}$ generalizes known results on the reduced Gröbner basis of the classical invariant ideal $I_n$. In particular (see, for example, Sturmfels [32, Thm. 1.2.7] or Bergeron [5, Sec. 7.2]), the reduced Gröbner basis for $I_n$ with respect to lexicographical order is

$$\{h_i(x_i, x_{i+1}, \ldots, x_n) : 1 \leq i \leq n\},$$

where $h_i$ is the homogeneous symmetric function of degree $i$. Moreover, the leading term of $h_i(x_i, x_{i+1}, \ldots, x_n)$ is $x_i^i$, so that the initial ideal of $I_n$ is generated by $x_1, x_2^2, \ldots, x_n^n$. Fomin, Gelfand, and Postnikov [9, Prop. 12.1] obtain a $q$-analog of this result for the quantum cohomology of the complete flag variety. The Demazure characters will reduce to the polynomials $\gamma_i$ these two sequences which preserves the relative orders of the $\gamma_i$s and the $b_i$s. An $(n,k)$-staircase is a shuffle of the two sequences $(0,1,\ldots,k-1)$ and $(k-1,k-1,\ldots,k-1)$, where there are $n-k$ copies of $k-1$. For example, the $(5,3)$-staircases are the shuffles of $(0,1,2)$ and $(2,2)$:

$$(0,1,2,2,2), (0,2,1,2,2), (0,2,2,1,2), (2,0,1,2,2), (2,0,2,1,2), \text{ and } (2,2,0,1,2).$$

We will prove that the set of monomials

$$A_{n,k} = \{x_1^{a_1} \cdots x_n^{a_n} : (a_1, \ldots, a_n) \text{ is componentwise } \leq \text{ some } (n,k)-\text{staircase}\}$$

descends to a basis of $R_{n,k}$. Since the only $(n,n)$-staircase is $(0,1,\ldots,n-1)$, we get $A_{n,n} = A_n$. Although it is not obvious at this point, the number of monomials in $A_{n,k}$ is $|OP_{n,k}|$.

The careful study of the basis $A_{n,k}$ will give us our expression for the Hilbert series of $R_{n,k}$. We will also derive a generalization $GS_{n,k}$ of the Garsia-Stanton basis of $R_n$ whose combinatorics is governed by a major index-like statistic on ordered set partitions [10, 13]. This will also turn out to give the ungraded module isomorphism $R_{n,k} \cong \mathbb{Q}[OP_{n,k}]$.

To determine the graded Frobenius image of $R_{n,k}$, we use a recursive method dating back to the work of Garsia and Procesi on the graded isomorphism type of the cohomology of Springer fibers [12]. In particular, we apply the fact that two symmetric functions with equal constant terms are equal if and only if their images under the operator $e_j(x)^\frac{1}{j}$ coincide for all $j \geq 1$. On the algebraic side, this will involve a generalization $R_{n,k,s}$ of the rings $R_{n,k}$ which satisfy $R_{n,k,k} = R_{n,k}$.

The rest of the paper is organized as follows. In Section 2 we will present definitions related to ordered set partitions, symmetric functions, Demazure characters, and Gröbner bases. In Section 3 we will prove a variety of identities involving polynomials and symmetric functions which will be crucial in the analysis of $R_{n,k}$ in the following sections. In Section 4 we will prove our formula for $\text{Hilb}(R_{n,k}; q)$ and give a generalization of the Artin basis to $R_{n,k}$. We will also describe the reduced Gröbner basis of $I_{n,k}$ with respect to the lexicographic monomial ordering. In Section 5 we will give our extension $GS_{n,k}$ of the Garsia-Stanton monomial basis to $R_{n,k}$. In Section 6 we will derive the graded Frobenius image of $R_{n,k}$. We make concluding remarks in Section 7.

2. Background

2.1. Ordered Set Partitions. Let $\pi = \pi_1 \ldots \pi_n \in \mathcal{S}_n$ be a permutation written in one-line notation. The descent set $\text{Des}(\pi)$ and ascent set $\text{Asc}(\pi)$ are given by

$$\text{Des}(\pi) := \{1 \leq i \leq n-1 : \pi_i > \pi_{i+1}\}, \quad \text{Asc}(\pi) := \{1 \leq i \leq n-1 : \pi_i < \pi_{i+1}\}.$$  

We let $\text{des}(\pi) := |\text{Des}(\pi)|$ and $\text{asc}(\pi) := |\text{Asc}(\pi)|$ be the number of descents or ascents of $\pi$. The major index of $\pi$ is $\text{maj}(\pi) := \sum_{i \in \text{Des}(\pi)} i$. An inversion in $\pi$ is a pair $i < j$ with $\pi_i > \pi_j$; we let $\text{inv}(\pi)$ be the number of inversions in $\pi$. 

An ordered set partition of size \( n \) is a set partition of \([n]\) equipped with a total order on its blocks. For example,
\[
\sigma = \{2, 4\} \prec \{6\} \prec \{1, 3, 5\}
\]
is an ordered set partition of size 6 with 3 blocks.

We will write ordered set partitions in two ways. The first denotes separation between blocks with bars and writes letters in an increasing fashion within blocks, so that the above ordered set partition may be written more succinctly as
\[
\sigma = (24 \mid 6 \mid 135).
\]

We will sometimes use stars to indicate connectives relating elements in an ordered set partition in such a way that letters are increasing within starred segments. Our example ordered set partition can then be expressed
\[
\sigma = 2\ast 4 \ 6 \ 1\ast 3\ast 5.
\]

An ascent starred permutation is a pair \((\pi, S)\) where \(\pi \in \mathfrak{S}_n\) and \(S \subseteq \text{Asc}(\pi)\). Our star notation gives an identification
\[
\mathcal{OP}_{n,k} = \{\text{ascent starred permutations } (\pi, S) : \pi \in \mathfrak{S}_n \text{ and } |S| = n - k\}.
\]

Our example ordered set partition becomes
\[
\sigma = (246135, \{1, 4, 5\}).
\]

Let \(\mathcal{OP}_{n,k}\) denote the collection of ordered set partitions of size \( n \) with \( k \) blocks. We have \(|\mathcal{OP}_{n,k}| = k! \cdot \text{Stir}(n, k)\), where \(\text{Stir}(n, k)\) is the (signless) Stirling number of the second kind counting \(k\)-block set partitions of \([n]\).

For \(\sigma = (B_1 | \cdots | B_k) \in \mathcal{OP}_{n,k}\), an inversion in \(\sigma\) is a pair of letters \(i < j\) such that \(i\) is minimal in \(B_m\), \(j \in B_t\), and \(t < m\). Let \(\text{inv}(\sigma)\) denote the number of inversions of \(\sigma\); this is the usual inversion statistic when \(k = n\) and \(\sigma\) is a permutation. For example, the inversions in \((24 \mid 6 \mid 135)\) are the pairs \((1, 2), (1, 4), (1, 6)\) and \(\text{inv}(24 \mid 6 \mid 135) = 3\).

It will be convenient to consider the statistic on ordered set partitions which is complementary to the inversion statistic. Given \(\sigma \in \mathcal{OP}_{n,k}\), define
\[
(2.1) \quad \text{coinv}(\sigma) := \max\{\text{inv}(\sigma') : \sigma' \in \mathcal{OP}_{n,k}\} - \text{inv}(\sigma)
\]
\[
(2.2) \quad = (n - k)(k - 1) + \binom{k}{2} - \text{inv}(\sigma),
\]
where the second equality follows from \(\text{inv}\) being (uniquely) maximized on
\[
(k(k + 1) \cdots n \mid k - 1 \mid \cdots \mid 2 \mid 1) \in \mathcal{OP}_{n,k}.
\]

In a private communication to the authors, A. Wilson noted that for any \(\sigma \in \mathcal{OP}_{n,k}\), \(\text{coinv}(\sigma)\) is the numbers of pairs \((a, b), 1 < a < b \leq n\), such that:
\[
(2.3) \quad \begin{cases} 
\text{at least one of } a \text{ and } b \text{ is minimal in its block in } \sigma, \\
\text{a and b are in different blocks, and} \\
\text{if a’s block is to the right of b’s block, then only b is minimal in its block.}
\end{cases}
\]

For example, in the ordered set partition \((45 \mid 167 \mid 23)\) the pairs that satisfy the above condition are 12, 13, 34, 46, and 47. We leave it as an exercise for the interested reader to verify that this description of \(\text{coinv}\) is equivalent to our definition \(\text{coinv}(\sigma) = (n - k)(k - 1) + \binom{k}{2} - \text{inv}(\sigma)\).

Given \(\sigma = (\pi, S) \in \mathcal{OP}_{n,k}\) represented as an ascent-starred permutation, we define the major index \(\text{maj}(\sigma)\) as follows. For \(1 \leq i \leq n\), let \(i^c := (n - i + 1)\) and let \(\pi^c \in \mathfrak{S}_n\) be the permutation \(\pi^c = \pi_1^c \cdots \pi_n^c\). Define
\[
(2.4) \quad \text{maj}(\sigma) := \text{maj}(\pi^c) - \sum_{i \in S} |\text{Asc}(\pi) \cap \{i, i + 1, \ldots, n - 1\}|.
\]
An equivalent version of this major index statistic appears in [27, p. 12]. For example, we have
\[ \text{maj}(2, 4, 6, 1, 3, 5) = \text{maj}(531642) - (1 + 2 + 4) = (1 + 2 + 4 + 5) - (1 + 2 + 4) = 5. \]

Just as in the case of \( \text{inv} \), we will consider the complementary statistic to \( \text{maj} \) on \( \mathcal{OP}_{n,k} \). Given \( \sigma \in \mathcal{OP}_{n,k} \), define

\[
\text{comaj}(\sigma) := \max\{\text{maj}(\sigma') : \sigma' \in \mathcal{OP}_{n,k}\} - \text{maj}(\sigma)
\]

\[= (n - k)(k - 1) + \binom{k}{2} - \text{maj}(\sigma),\]

where the second equality comes from the fact that \( \text{maj} \) is (uniquely) maximized on
\[1 2 \ldots (k - 1) k (k + 1) \ldots (n - 1) n \in \mathcal{OP}_{n,k}.\]

The \( q \)-\textit{Stirling numbers} \( \text{Stir}_q(n, k) \) are defined by the recursion

\[
\text{Stir}_q(n, k) = \text{Stir}_q(n - 1, k - 1) + [k]_q \cdot \text{Stir}_q(n - 1, k)
\]

and the initial condition \( \text{Stir}_q(1, k) = \begin{cases} 1 & k = 1 \\ 0 & k > 1 \end{cases} \). Steingrímsson [31] and Remmel-Wilson [27] proved that the product \([k]_q \cdot \text{Stir}_q(n, k)\) is the generating function of \( \text{inv} \) and \( \text{maj} \) on \( \mathcal{OP}_{n,k} \):

\[
\sum_{\sigma \in \mathcal{OP}_{n,k}} q^{\text{inv}(\sigma)} = \sum_{\sigma \in \mathcal{OP}_{n,k}} q^{\text{maj}(\sigma)} = [k]!_q \cdot \text{Stir}_q(n, k).
\]

Any statistic on \( \mathcal{OP}_{n,k} \) which shares this distribution is called \textit{Mahonian}. Reversing in \( q \), we get

\[
\sum_{\sigma \in \mathcal{OP}_{n,k}} q^{\text{coinv}(\sigma)} = \sum_{\sigma \in \mathcal{OP}_{n,k}} q^{\text{comaj}(\sigma)} = \text{rev}_q([k]!_q \cdot \text{Stir}_q(n, k)).
\]

Recall that a \textit{(weak) composition} \( \gamma = (\gamma_1, \ldots, \gamma_n) \) is a sequence of nonnegative integers. We write \(|\gamma| := \gamma_1 + \cdots + \gamma_n\), write \( \gamma = |\gamma| \), and say that \( \gamma \) has \( n \) parts, or \( \ell(\gamma) = n \). We let \( \gamma^* = (\gamma_n, \ldots, \gamma_1) \) be the reverse of \( \gamma \).

An \textit{ordered multiset partition} is a finite sequence \( \mu = (M_1 | \cdots | M_k) \) of nonempty finite sets of positive integers. We say that \( \mu \) has size \( |M_1| + \cdots + |M_k| \) and \( k \) \textit{blocks}. For example,

\[
\mu = (247 | 15 | 145)
\]
is an ordered multiset partition of size 8 with 3 blocks. Note that the elements in an ordered multiset partition are \textit{sets}; we do not allow repeated letters within blocks.

The \textit{content} of an ordered multiset partition \( \mu \) is the composition \( \text{cont}(\mu) = (\text{cont}(\mu)_1, \text{cont}(\mu)_2, \ldots) \), where \( \text{cont}(\mu)_i \) is the multiplicity of \( i \) as a letter in \( \mu \). If \( \mu \) is the ordered multiset partition above, we have \( \text{cont}(\mu) = (2, 1, 0, 2, 2, 0, 1) \). For any composition \( \gamma \), let \( \mathcal{OP}_{\gamma,k} \) be the collection of ordered multiset partitions of content \( \gamma \) with \( k \) blocks. When \( \gamma = (1^n) \), we recover the notion of an ordered set partition.

The definition of \( \text{inv} \) on ordered set partitions extends verbatim to ordered multiset partitions. There is also an extension of the \( \text{maj} \) statistic to ordered multiset partitions (which are viewed in this context as \textit{descent} starred words); see [27, 33]. Remmel-Wilson and Wilson defined two other statistics on ordered multiset partitions called \text{dinv} and \text{minimaj}. We will not use the statistics \( \text{maj}, \text{dinv}, \) and \( \text{minimaj} \) on ordered multiset partitions explicitly in our work; see [27, 33] for their definitions.
2.2. Symmetric functions. Our notation for symmetric functions is standard; see [23].

A partition $\lambda$ of $n$ is a weakly decreasing sequence $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)$ of positive integers with $\sum \lambda_i = n$. We write $\lambda \vdash n$ to mean that $\lambda$ is a partition of $n$ and $\ell(\lambda) = k$ for the number of parts of $\lambda$. We denote the multiplicity of $i$ as a part of $\lambda$ by $m_i(\lambda)$. Given two partitions $\lambda, \mu$ we say $\lambda \leq \mu$ in dominance order if $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$ for all $i$.

Given $\lambda \vdash n$, the (English) Ferrers diagram of $\lambda$ consists of $\lambda_i$ left-justified boxes in row $i$, for all $i$. The Ferrers diagram of $(4,2,2) \vdash 8$ is shown below.

\begin{center}
\begin{tabular}{ccc}
2 & 3 & 4\\
1 & 5 & 6\\
7 & 8 & \\
\end{tabular}
\end{center}

The transpose $\lambda'$ of $\lambda$ is the partition of $n$ whose Ferrers diagram is obtained from the Ferrers diagram of $\lambda$ by reflecting across the line $y = x$. For example, we have $(4,2,2)' = (3,3,1,1)$.

For $\lambda \vdash n$, a standard Young tableau of shape $\lambda$ is a filling of the boxes of $\lambda$ with $1,2,\ldots, n$ which is increasing down columns and across rows. One possible standard Young tableau of shape $(4,2,2)$ is

\begin{center}
\begin{tabular}{ccc}
1 & 2 & 3 & 7\\
4 & 6 & \\
5 & 8 & \\
\end{tabular}
\end{center}

Let SYT($n$) denote the set of standard Young tableaux with $n$ boxes. For $T \in$ SYT($n$), let $\text{shape}(T) \vdash n$ denote the shape of $T$. Given $T \in$ SYT($n$), a descent of $T$ is a letter $i$ which appears in a higher row than $i + 1$ in $T$. Let $\text{des}(T)$ denote the number of descents of $T$ and $\text{maj}(T)$ denote the sum of the descents of $T$. For example, the descents in the tableau $T$ above are 3, 4, and 7, so that $\text{des}(T) = 3$ and $\text{maj}(T) = 3 + 4 + 7 = 14$.

Let $x = (x_1, x_2, \ldots)$ be an infinite set of variables and let $\Lambda$ denote the ring of symmetric functions in $x$ with coefficients in the field $\mathbb{Q}(q,t)$. The ring $\Lambda = \bigoplus_{n \geq 0} \Lambda_n$ is graded by degree and the dimension of the graded piece $\Lambda_n$ equals the number of partitions of $n$.

For $\lambda \vdash n$, let

$$m_\lambda(x), \; e_\lambda(x), \; h_\lambda(x), \; s_\lambda(x), \; \bar{H}_\lambda(x; q, t)$$

be the corresponding monomial, elementary, (complete) homogeneous, Schur, and modified Macdonald symmetric functions. Here we are using $x$ to refer to an infinite set of variables to avoid confusion with the finite variable set $x_n$. Let $\omega$ be the algebra involution on $\Lambda$ defined by $\omega(s_\lambda(x)) = s_\lambda'(x)$.

The Hall-Littlewood polynomials $P_\lambda(x; q)$ are the basis of $\Lambda$ defined implicitly in terms of the Schur basis by the formula

$$s_\lambda(x) = \sum_\mu K_{\lambda, \mu}(q) P_\mu(x; q),$$

where $K_{\lambda, \mu}(q)$ is the Kostka-Foulkes polynomial (i.e., the generating function of charge on semistandard tableaux of shape $\lambda$ and content $\mu$). The dual Hall-Littlewood polynomials $Q'_\mu(x; q)$ are defined by

$$Q'_\mu(x; q) = \sum_\lambda K_{\lambda, \mu}(q) s_\lambda(x).$$

In particular, the polynomials $Q'_1(x; q)$ are Schur positive.

Let $\langle \cdot, \cdot \rangle$ be the Hall inner product on $\Lambda$ with respect to which the Schur basis is orthogonal. For any symmetric function $F(x) \in \Lambda$, let $F(x)^\perp$ denote the linear operator on $\Lambda$ which is adjoint to multiplication by $F(x)$ with respect to the Hall inner product. In other words, we have

$$\langle F(x)^\perp G(x), H(x) \rangle = \langle G(x), F(x) H(x) \rangle$$

for all symmetric functions $G(x), H(x) \in \Lambda$.

The irreducible representations of the symmetric group $\mathfrak{S}_n$ are naturally labeled by partitions of $n$. Given a partition $\lambda \vdash n$, let $S^\lambda$ be the corresponding irreducible representation of $\mathfrak{S}_n$. 

Let $V$ be a finite-dimensional $S_n$-module. Then $V$ is completely decomposable and we may write

$$V \cong \bigoplus_{\lambda \vdash n} (S^\lambda)^{c_{\lambda}}$$

as a direct sum of irreducible representations for some nonnegative integers $c_{\lambda}$. The Frobenius character $\text{Frob}(V) \in \Lambda_n$ is the symmetric function

$$\text{Frob}(V) := \sum_{\lambda \vdash n} c_{\lambda} s_{\lambda}(x).$$

For example, let $\lambda \vdash n$ and consider the Young subgroup

$$S_\lambda := S_{\lambda_1} \times S_{\lambda_2} \times \cdots$$

of the symmetric group $S_n$. The corresponding left coset representation $\text{Ind}_{S_\lambda}^S(\text{triv}) = \mathbb{Q}[S_n/S_\lambda]$ of $S_n$ has Frobenius image $\text{Frob} : \mathbb{Q}[S_n/S_\lambda] \mapsto h_{\lambda}(x)$.

Let $V = \bigoplus_{d \geq 0} V_d$ be a graded vector space in which each graded piece $V_d$ is finite-dimensional. The Hilbert series of $V$ is the power series in $q$ given by

$$(2.11) \quad \text{Hilb}(V; q) := \sum_{d \geq 0} \text{dim}(V_d)q^d.$$

If $V$ carries a graded action of $S_n$, the graded Frobenius character is

$$(2.13) \quad \text{grFrob}(V; q) := \sum_{d \geq 0} \text{Frob}(V_d)q^d.$$

2.3. The Delta Conjecture. Our results are related to the Delta Conjecture arising in the theory of Macdonald polynomials. Let us briefly review this conjecture of Haglund, Remmel, and Wilson [13]. To state the Delta Conjecture, we will need some definitions.

For any symmetric function $F(x)$, we define the linear transformation

$$(2.14) \quad \Delta_F : \Lambda_n \to \Lambda_n$$

to be the Macdonald eigenoperator $\Delta_F : \tilde{H}_{\mu}(x; q, t) \mapsto F[B_{\mu}(q, t) - 1] \cdot \tilde{H}_{\mu}(x; q, t)$. Here we are using the plethystic notation

$$(2.15) \quad F[B_{\mu}(q, t) - 1] = F(\ldots, q^{i-1}t^{j-1}, \ldots),$$

where $(i, j)$ range over all matrix coordinates $\neq (1, 1)$ of cells in the Ferrers diagram of $\mu$ (and all other variables in $F(x)$ are set to 0).

For example, if $\mu = (3, 2) \vdash 5$, we have

$$\Delta_F : \tilde{H}_{\mu}(x; q, t) \mapsto F(q, q^2, t, qt)\tilde{H}_{\mu}(x; q, t),$$

corresponding to the filling

$$\begin{array}{ccc}
\cdot & q & q^2 \\
t & qt & \end{array}$$

A Dyck path of size $n$ is a lattice path from $(0, 0)$ to $(n, n)$ consisting of north and east steps which stays weakly above $y = x$. We number the rows of any Dyck path $D$ as $1, 2, \ldots, n$ from bottom to top. A labeled Dyck path is a Dyck path with (not necessarily unique) positive integers assigned to its north steps such that these labels strictly increase going up columns. We let $D_n$ denote the set of Dyck paths of size $n$ and $LD_n$ denote the collection of labeled Dyck paths of size $n$. Given $P \in LD_n$, let $x^P \in \mathbb{Q}[\{x_i\}]$ be the monomial in which the power of $x_i$ is the multiplicity of $i$ as a label of $P$. Also let $\ell_i(P)$ denote the label of $P$ in row $i$.

For $D \in D_n$ and $1 \leq i \leq n$, let $a_i(D)$ be the number of full squares between $D$ and the line $y = x$ in row $i$. Let $\text{area}(D) := a_1(D) + \cdots + a_n(D)$ be the area of $D$. If $P \in LD_n$ is a labeled Dyck path and $D(P) \in D_n$ is its underlying Dyck path, we set $a_i(P) := a_i(D(P))$ and $\text{area}(P) := \text{area}(D(P))$. 

Let \( P \in \mathcal{LD}_n \) be a labeled Dyck path and \( 1 \leq i \leq n \). Define \( d_i(P) \) by
\[
d_i(P) := |\{i < j \leq n : a_i(P) = a_j(P), \ell_i(P) < \ell_j(P)\}| + |\{i < j \leq n : a_i(P) = a_j(P) + 1, \ell_i(P) > \ell_j(P)\}|
\]
The \( \text{dinv} \) statistic is \( \text{dinv}(P) = d_1(P) + \cdots + d_n(P) \). The contractible valleys of \( P \) are
\[
\text{Val}(P) := \{2 \leq i \leq n : a_i(P) < a_{i-1}(P)\}
\]
\[
\cup \{2 \leq i \leq n : a_i(P) = a_{i-1}(P), \ell_i(P) > \ell_{i-1}(P)\}.
\]
As an example of these concepts, let \( P \in \mathcal{LD}_5 \) be the labeled Dyck path shown below. We have
\[
\text{area}(P) = 2, \text{dinv}(P) = 4, \text{Val}(P) = \{4, 5\}, \text{and } x^P = x_1x_2^2x_3x_6.
\]

\[
\begin{array}{ccc}
 i & a_i & d_i \\
 \hline
 1 & 5 & 0 \\
 2 & 4 & 0 \\
 3 & 1 & 1 \\
 4 & 2 & 2 \\
 5 & 1 & 0 \\
\end{array}
\]

Conjecture 2.1. [18] (The Delta Conjecture) For positive integers \( k \leq n \),
\[
\Delta'_{e_{k-1}} e_n(x) = \left\{ z^{n-k} \right\} \sum_{P \in \mathcal{LD}_n} q^{\text{dinv}(P) \cdot \text{area}(P)} \prod_{i : a_i(P) > a_{i-1}(P)} (1 + \frac{z}{t^{a_i(P)}}) x^P
\]
\[
= \left\{ z^{n-k} \right\} \sum_{P \in \mathcal{LD}_n} q^{\text{dinv}(P) \cdot \text{area}(P)} \prod_{i \in \text{Val}(P)} \left(1 + \frac{z}{q^{d_i(P)+1}}\right) x^P.
\]
Here the operator \( \left\{ z^{n-k} \right\} \) extracts the coefficient of \( z^{n-k} \).

When \( k = n \), we have \( \Delta'_{e_{n-1}} e_n(x) = \nabla e_n(x) \), where \( \nabla \) is the Bergeron-Garsia Macdonald eigenoperator, and the Delta Conjecture reduces to the Shuffle Conjecture of Haglund, Haiman, Loehr, Remmel, and Ulyanov [10]. Haiman proved that \( \nabla e_n(x) \) is the bigraded Frobenius image of the diagonal coinvariant algebra \([20] \). If we set \( t = 0 \), we get that \( \nabla e_n(x)|_{t=0} \) is the graded Frobenius image of the classical coinvariant algebra \( R_n : \text{grFrob}(R_n; q) = \nabla e_n(x)|_{t=0} \).

For general \( k \leq n \), there is not even a conjectural bigraded \( \mathfrak{S}_n \)-module whose bigraded Frobenius image equals any of the expressions in the Delta Conjecture. We will prove that (up to minor modification), our rings \( R_{n,k} \) provide such a module in the specialization \( t = 0 \) of either combinatorial expression in the Delta Conjecture.

Let \( \text{Rise}_{n,k}(x; q, t) \) and \( \text{Val}_{n,k}(x; q, t) \) denote the middle and right sides of the Delta Conjecture, respectively. \(^3\) By the work of Remmel-Wilson, Wilson, and Rhoades [27, 33, 28] we have
\[
\text{Rise}_{n,k}(x; q, 0) = \text{Rise}_{n,k}(x; 0, q) = \text{Val}_{n,k}(x; q, 0) = \text{Val}_{n,k}(x; 0, q).
\]

Let \( C_{n,k}(x; q) \) denote this common symmetric function \(^4\) and set
\[
D_{n,k}(x; q) := \text{rev}_q \circ \omega[C_{n,k}(x; q)].
\]

\(^3\)Our conventions are ‘off by one’ from those in [18] and elsewhere – our \( \text{Rise}_{n,k}(x; q, t) \) is their \( \text{Rise}_{n,k-1}(x; q, t) \), etc.

\(^4\)While this paper was under review, Garsia, Haglund, Remmel, and Yoo [11] proved that we have the additional equalities \( \Delta'_{e_{k-1}} e_n(x)|_{t=0} = \Delta'_{e_{k-1}} e_n(x)|_{q=0, t=q} = C_{n,k}(x; q) \), so we can also define \( C_{n,k}(x; q) \) in terms of the operator \( \Delta'_{e_{k-1}} \).
The symmetric function $C_{n,k}(x; q)$ is related to ordered multiset partitions as follows. If $\gamma$ is any composition, let $x^\gamma = x_1^{\gamma_1} x_2^{\gamma_2} \cdots$. Haglund, Remmel, and Wilson proved the following formulas [18 Prop. 4.1]:

\[
\begin{align*}
\text{Rise}_{n,k}(x; q, 0) &= \sum_{|\gamma|=n} \sum_{\mu \in \mathcal{OP}_{\gamma,k}} q^{\text{inv}(\mu)} x^\gamma \\
\text{Val}_{n,k}(x; q, 0) &= \sum_{|\gamma|=n} \sum_{\mu \in \mathcal{OP}_{\gamma,k}} q^{\text{inv}(\mu)} x^\gamma
\end{align*}
\]

By the standardization of a word $w$ of $n$ positive integers, we mean the unique permutation $\pi \in \mathfrak{S}_n$ which satisfies $w_i < w_j$ iff $\pi_i < \pi_j$. For example, the standardization of 131 is 132. For $\pi \in \mathfrak{S}_n$, let $i\text{Des}(\pi)$ denote the descent set of $\pi^{-1}$.

For $\mu$ an ordered multiset partition, let the reading word $\text{rword}(\mu)$ be the word obtained from $\mu$ by reading along “diagonals” (the $m$th diagonal consists of all elements which are the $m$th largest in their block, left to right), larger $m$ first. For example, if $\mu = (247 \mid 35 \mid 3)$, then $\text{rword}(\mu) = 7452133$. This choice of a reading word guarantees that if $\mu$ is an ordered multiset partition, and the standardization of $\text{rword}(\mu)$ equals $\text{rword}(\sigma)$ for some $\sigma \in \mathcal{OP}_{n,k}$, then a pair of elements in $\mu$ form an inversion pair iff the corresponding pair in $\sigma$ do as well. For example, $\text{rword}(247 \mid 26 \mid 6) = 746226$, which standardizes to the permutation 634125, and this is the reading word of $(136 \mid 24 \mid 5)$. The inversion pairs of $(136 \mid 24 \mid 5)$ are $(5, 6), (2, 6)$ and $(2, 3)$, while those of $(247 \mid 26 \mid 6)$ are the second 6 and the 7, the second 2 and the 7, and the second 2 and the 4.

Let $F_{n,D}(x)$ denote the Gessel fundamental quasisymmetric function corresponding to the descent set $D \subseteq \{1, 2, \ldots, n-1\}$. One way of defining $F_{n,D}(x)$ is the sum of the $x$-weights of all words which standardize to a permutation $\pi$ with $i\text{Des}(\pi) = D$. See [14 pp. 99-101] for background on standardization and Gessel fundamental quasisymmetric functions (there denoted $Q_{n,D}(x)$). Thus an equivalent way of writing the third expression for $C_{n,k}(x, q)$ from (2.19) is

\[
\begin{align*}
\text{Val}_{n,k}(x; q, 0) &= \sum_{\sigma \in \mathcal{OP}_{n,k}} q^{i\text{Des}(\text{rword}(\sigma))} F_{n,i\text{Des}(\text{rword}(\sigma))}(x) \\
G(x) &= \sum_{\pi \in \mathfrak{S}_n} c(\pi) F_{n,i\text{Des}(\pi)}(x)
\end{align*}
\]

where the sum is over the symmetric group and the $c(\pi)$ are independent of $x$. Then

\[
\omega G(x) = \sum_{\pi \in \mathfrak{S}_n} c(\pi) F_{n,i\text{Des}(\text{rword}(\pi))}(x),
\]

where $\text{rword}(\sigma)$ is the word obtained by reversing $\text{rword}(\sigma)$. This follows easily if $G$ is a Schur function, using the well-known decomposition of a Schur function into Gessel fundamentals [30], and hence for general $G$ since the Schur functions form a basis for the ring of symmetric functions. Combining this with (2.20) gives

\[
\begin{align*}
D_{n,k}(x; q) &= \sum_{\sigma \in \mathcal{OP}_{n,k}} q^{i\text{coinv}(\sigma)} F_{n,i\text{Des}(\text{rword}(\sigma))}(x) \\
2.4. \text{ Demazure characters.} \quad \text{To any composition } \gamma = (\gamma_1, \ldots, \gamma_n) \text{ we have a (augmented) skyline diagram consisting of columns of heights } \gamma_1, \ldots, \gamma_n \text{ augmented with the basement which reads, from left to right, } n, n-1, \ldots, 1. \text{ For example, the skyline diagram of (3,0,1,3) is shown below.}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{4} \\
\text{3} \\
\text{2} \\
\text{1}
\end{array}
\end{array}
\]

Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be a composition and let $1 \leq i < j \leq n$ index a pair of columns of $\gamma$. A type A triple is a set of three cells $a, b, c$ of the form $(i, k), (j, k), (i, k - 1)$ of the skyline diagram such that $\gamma_i \geq \gamma_j$. A type B triple is a set of three cells $a, b, c$ of the form $(j, k + 1), (i, k), (j, k)$ where $\gamma_i < \gamma_j$. These two situations are shown schematically below. (Note that basement cells are allowed to be members of triples.)

\[
\begin{array}{ccc}
& a & \\
\Gamma & b & c \\
\end{array}
\quad
\begin{array}{ccc}
& b & \\
\Gamma & a & c \\
\end{array}
\]

type A ($\gamma_i \geq \gamma_j$) \quad type B ($\gamma_i < \gamma_j$)

If $a, b, c$ are positive integers coming from a filling, the corresponding triple is called a coinversion triple if $a \leq b \leq c$. Otherwise the triple is called an inversion triple.

Let $\gamma$ be a composition with $\ell(\gamma) = n$. A semistandard skyline filling (SSK) of shape $\gamma$ is a filling of the skyline diagram of $\gamma$ with positive integers such that

1. The entries decrease weakly up each column (including the basement) and
2. every triple (including those involving basement cells) is an inversion triple.

An example of a SSK of shape $(3, 0, 1, 3)$ is shown below.

\[
\begin{array}{cccc}
3 & 1 & \quad & \\
3 & \quad & 1 & \\
4 & 2 & 1 & \\
\end{array}
\]

Let $\text{SSK}(\gamma)$ be the set of SSK of shape $\gamma$.

Let $\gamma$ be a composition with $\ell(\gamma) = n$. The Demazure character is the polynomial $\kappa_\gamma(x_n) \in \mathbb{Q}[x_n]$ given by

\[
(2.24) \quad \kappa_\gamma(x_n) = \sum_{T \in \text{SSK}(\gamma^*)} x_1^{\# \text{ of } 1s \text{ in } T} x_2^{\# \text{ of } 2s \text{ in } T} \cdots.
\]

Note that the Demazure character labeled by $\gamma$ is the generating function of SSK of shape given by the reverse composition $\gamma^*$. For example, the SSK shown above contributes $x_1^3 x_2^2 x_3^2 x_4^1$ to $\kappa_{(3,1,0,3)}(x_4)$. While the original definition of the Demazure character was not combinatorial, we will take this combinatorial reformulation (due to Mason [24]) as our definition.

Demazure characters are rarely symmetric polynomials. In fact, the collection of all Demazure characters $\{\kappa_\gamma(x_n) : \gamma \text{ a weak composition}\}$ forms a basis of the polynomial ring $\mathbb{Q}[x_n]$. Let $B \subseteq GL_n(\mathbb{C})$ be the subgroup of upper triangular matrices. The Demazure character $\kappa_\gamma(x_n)$ is the trace of the diagonal matrix $\text{diag}(x_1, \ldots, x_n)$ acting on the indecomposable polynomial representation of $B$ indexed by $\gamma$.

It will be convenient to consider Demazure characters in a reversed set of variables. We denote by $x_n^* = (x_n, \ldots, x_1)$ our list of $n$ variables in reverse order. Hence, if $f(x_n) = f(x_1, \ldots, x_n) \in \mathbb{Q}[x_n]$ is any polynomial, we set $f(x_n^*) := f(x_n, \ldots, x_1)$. In particular, for any composition $\gamma$ with $\ell(\gamma) = n$ we have the reverse Demazure character

\[
\kappa_\gamma(x_n^*) = \kappa_\gamma(x_n, \ldots, x_1).
\]

Note that if $f(x_n) \in \mathbb{Q}[x_n]^{|\mathbb{S}_n|}$ is a symmetric polynomial, we have $f(x_n) = f(x_n^*)$.

Let us mention a recursive construction of the Demazure characters. If $\gamma = (\gamma_1 \geq \cdots \geq \gamma_n)$ is a dominant composition, we have $\kappa_\gamma(x_n) = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$. In general, suppose that $\gamma = (\gamma_1, \ldots, \gamma_n)$ is obtained from $\gamma' = (\gamma_1', \ldots, \gamma_n')$ by swapping $\gamma_i' > \gamma_{i+1}'$. Then

\[
\kappa_\gamma(x_n) = \frac{1 - s_i}{x_i - x_{i+1}}[x_i \cdot \kappa_{\gamma'}(x_n)].
\]
Here \( s_i \) acts on polynomials by interchanging \( x_i \) and \( x_{i+1} \), so that \( \frac{1-s_i}{x_i-x_{i+1}} \) is the divided difference operator and \( \frac{1-s_i}{x_i-x_{i+1}} \cdot x_i \) is the isobaric divided difference operator. Although this recursive construction (the Demazure Character Formula) could be used to prove some of our results, Mason’s [21] combinatorial interpretation of Demazure characters in terms of SSK will be crucial in our work.

2.5. Gröbner bases. A total order \(<\) on the monomials in the polynomial ring \( \mathbb{Q}[x_n] \) is called a monomial order if

\begin{enumerate}
\item for any monomial \( m \) we have \( 1 \leq m \), and
\item for any monomials \( m, m', m'' \), we have that \( m < m' \) implies \( m \cdot m'' < m' \cdot m'' \).
\end{enumerate}

The lexicographic monomial order \(<_{\text{lex}}\) is defined by \( x_1^{a_1} \cdots x_n^{a_n} <_{\text{lex}} x_1^{b_1} \cdots x_n^{b_n} \) if there is an index \( i \) such that \( a_1 = b_1, \ldots, a_{i-1} = b_{i-1}, \) and \( a_i < b_i \).

Let \(<\) be a monomial order on \( \mathbb{Q}[x_n] \). For any nonzero polynomial \( f \in \mathbb{Q}[x_n] \), let \( \text{in}_{<}(f) \) be the leading monomial of \( f \) with respect to \(<\). For an ideal \( I \subseteq \mathbb{Q}[x_n] \), let \( \text{in}_{<}(I) = \langle \text{in}_{<}(f) : f \in I - \{0\} \rangle \) be the ideal generated by the leading monomials of nonzero polynomials in \( I \).

A finite collection \( G = \{g_1, \ldots, g_r\} \) of nonzero polynomials in \( I \) is a Gröbner basis if \( \text{in}_{<}(I) = \langle \text{in}_{<}(g_1), \ldots, \text{in}_{<}(g_r) \rangle \); this immediately implies that \( I = \langle g_1, \ldots, g_r \rangle \). The Gröbner basis \( G \) is called reduced if

\begin{enumerate}
\item the coefficient of \( \text{in}_{<}(g_i) \) is 1 for all \( i \), and
\item for \( i \neq j \), no monomial in \( g_j \) is divisible by \( \text{in}_{<}(g_i) \).
\end{enumerate}

For a fixed monomial order, every ideal \( I \subseteq \mathbb{Q}[x_n] \) has a unique reduced Gröbner basis.

Gröbner bases are helpful in constructing linear bases of quotient rings. In particular, let \( I \subseteq \mathbb{Q}[x_n] \) be an ideal and let \( G = \{g_1, \ldots, g_r\} \) be a Gröbner basis of \( I \) with respect to some monomial order \(<\). A monomial \( m \in \mathbb{Q}[x_n] \) is called a standard monomial if \( m \nmid \text{in}_{<}(f) \) for all \( f \in I - \{0\} \). Equivalently, a monomial \( m \) is standard if \( m \nmid \text{in}_{<}(g_i) \) for \( 1 \leq i \leq r \). The collection

\[ \{m + I : m \text{ a standard monomial}\} \]

gives a vector space basis for the quotient ring \( \mathbb{Q}[x_n]/I \); this is called the standard monomial basis.

3. Some polynomial identities

In this section we prove a family of identities involving polynomials and symmetric functions which will serve as lemmata for the analysis of the quotient ring \( R_{nk} \). The first of these is a vanishing property satisfied by certain alternating products of elementary and homogeneous symmetric function evaluations.

**Lemma 3.1.** Let \( k \leq n \), let \( \alpha_1, \ldots, \alpha_k \in \mathbb{Q} \) be distinct rational numbers, and let \( \beta_1, \ldots, \beta_n \in \mathbb{Q} \) be rational numbers with the property that \( \{\beta_1, \ldots, \beta_n\} = \{\alpha_1, \ldots, \alpha_k\} \). For any \( n - k + 1 \leq r \leq n \) we have

\[ \sum_{j=0}^{r} (-1)^j e_{r-j}(\beta_1, \ldots, \beta_n) h_j(\alpha_1, \ldots, \alpha_k) = 0. \tag{3.1} \]

**Proof.** The left hand side of Equation 3.1 is the coefficient of \( t^r \) in the power series

\[ \prod_{i=1}^{n} (1 + t \beta_i) \prod_{i=1}^{k} (1 + t \alpha_j). \tag{3.2} \]

By assumption, every term in the denominator cancels a term in the numerator, so this power series is in fact a polynomial in \( t \) of degree \( n - k \). If \( r > n - k \), the left hand side of Equation 3.1 therefore equals zero. \( \square \)
The next lemma will be used to show that certain reverse Demazure characters \( \kappa_\gamma(x^n) \) lie in the ideal \( I_{n,k} \). In order to state this lemma, we will introduce a family of ‘skip’ objects related to a set \( S \subseteq [n] \).

**Definition 3.2.** Let \( S = \{ s_1 < \cdots < s_m \} \subseteq [n] \) be a set.

1. The skip monomial \( x(S) \in \mathbb{Q}[x_n] \) is the monomial
   \[
   x(S) := x_{s_1}^n x_{s_2}^{n-1} \cdots x_{s_m}^{n-m+1}.
   \]
2. The skip composition \( \gamma(S) = (\gamma_1, \ldots, \gamma_n) \) is the weak composition of length \( n \) defined by
   \[
   \gamma_i := \begin{cases} 
   0 & i \notin S \\
   s_j - j + 1 & i = s_j \in S.
   \end{cases}
   \]

For example, let \( n = 8 \) and take \( S = \{2, 3, 6, 8\} \). We have \( x(S) = x_2^3 x_3^2 x_6 x_8 \in \mathbb{Q}[x_8] \) and \( \gamma(S) = (0, 2, 2, 0, 0, 4, 0, 5) \). In general, the support of the monomial \( x(S) \) is the set \( S \) and \( \gamma(S) \) is the exponent vector of \( x(S) \). The terminology here comes from the fact that the power of the lowest variable in \( x(S) \) is the variable index, and powers of higher variables increase according to how many indices the set \( S \) skips. Skip monomials will be crucial in our study of \( R_{n,k} \).

Let \( \gamma = (\gamma_1, \ldots, \gamma_n) \) be a weak composition and consider a collection \( \rho \) of cells lying immediately above the columns of the skyline diagram of \( \gamma \). The collection \( \rho \) will be called right biased (RB) for \( \gamma \) if

1. there is at most one cell in \( \rho \) on top of any column of \( \gamma \), and
2. among the columns of \( \gamma \) with a fixed height \( h \), the cells of \( \rho \) are right-justified. Note that 0 is a possible value for \( h \).

For example, consider the composition \( \gamma = (0, 4, 0, 3, 3, 0, 2, 0, 0) \). The collection of cells marked with \( \circ \) on the left is RB while the collection of \( \circ \) cells on the right is not RB.

The dual Pieri rule describes how to expand products of the form \( e_d(x) \kappa_\gamma(x) \) in the Schur basis of symmetric functions. The following Demazure version of the dual Pieri rule is due to Haglund, Luoto, Mason, and van Willigenburg. It describes how to expand products of the form \( e_d(x_n) \kappa_\gamma(x_n) \) in the Demazure character basis of polynomials. If \( \rho \) is a RB collection of cells for \( \gamma \), let \( \gamma \cup \rho \) be the composition which is the set theoretic union of the skyline diagram of \( \gamma \) and the collection of cells \( \rho \).

**Theorem 3.3.** [17 Thm. 6.1, \( \lambda = (1^d) \)] Let \( \gamma = (\gamma_1, \ldots, \gamma_n) \) be a weak composition and \( d \geq 0 \).

We have
\[
e_d(x_n) \kappa_\gamma(x_n) = \sum_{|\rho| = d} \kappa_{(\gamma \cup \rho)}(x_n),
\]
where the sum is over all RB collections of cells \( \rho \) of size \( d \).

We will add RB collections to the skyline diagrams of compositions; we now describe collections of cells we will remove. Let \( \gamma = (\gamma_1, \ldots, \gamma_n) \) be a weak composition. A collection \( \lambda \) of non-basement cells in the skyline diagram of \( \gamma \) will be called left leaning (LL) for \( \gamma \) if

1. the cells of \( \lambda \) are top-justified within any column,
2. there is at most one cell of \( \lambda \) in any row, and
3. it is impossible to move a cell of \( \lambda \) to the left in such a way that (1) remains satisfied.
For example, if $\gamma = (0, 4, 0, 3, 3, 0, 2, 0, 0)$ the collection of cells marked with $\bullet$ on the left is LL while the collection of $\bullet$ cells on the right is not LL.

If $\lambda$ is any LL collection of cells for $\gamma$, let $\gamma - \lambda$ be the composition whose skyline diagram is the set theoretic difference of $\gamma$ and $\lambda$.

We will need a little bit more notation to state our lemma. For any composition $\gamma$, let $\gamma^{\downarrow}$ denote the decremented composition obtained by subtracting 1 from every nonzero part of $\gamma$:

$$\gamma_i := \begin{cases} 
\gamma_i - 1 & \gamma_i > 0 \\
0 & \gamma_i = 0.
\end{cases}$$

In particular, if $\gamma(S)$ is a skip composition, we have $|\gamma(S)| = |\gamma(S)| - |S|$. Also, for any collection of cells $\nu$, let $|\nu|$ denote the number of cells in $\nu$.

**Lemma 3.4.** Let $k \leq n$ and let $S \subseteq [n]$ with $|S| = n - k + 1$. Let $\gamma(S)$ be the corresponding skip monomial and consider the reversal $\gamma(S)^*$ and the decremented reversal $\gamma(S)^{\downarrow}$.

We have the identity

$$(3.4) \quad \kappa_{\gamma(S)^*}(x_n) = \sum_{\lambda} (-1)^{|\lambda|} \kappa_{\gamma(S)^{\downarrow} - \lambda}(x_n) e_{n-k+1+|\lambda|}(x_n),$$

where the sum is over all LL collections $\lambda$ for $\gamma(S)^{\downarrow}$.

Reversing variables and applying the symmetry of $e_r(x_n)$ we get

$$\kappa_{\gamma(S)^{\downarrow}}(x_n^*) = \sum_{\lambda} (-1)^{|\lambda|} \kappa_{\gamma(S)^{\downarrow} - \lambda}(x_n^*) e_{n-k+1+|\lambda|}(x_n).$$

The proof of this lemma is involved, but worth it. Equation (3.5) will be our tool for proving that reverse Demazure characters lie in various ideals.

**Proof.** It suffices to prove Equation (3.4). To do this, we will introduce a sign-reversing involution on a set of combinatorial objects called ‘partisan skylines’.

A partisan skyline $\nu$ is a skyline diagram of a composition whose non-basement cells have one of four labels

$$\square \quad \circ \quad \bullet \quad \bigodot$$

satisfying the following conditions. For any collection of labels $L$, we let $\nu(L)$ denote the collection of cells in $\nu$ which have labels in $L$.

1. Reading any column of $\nu$ from bottom to top gives a sequence of labels $\square$, followed by a sequence of labels $\circ$, followed by a sequence of labels $\bullet$, followed by a sequence of labels $\bigodot$.
   Any of these four sequences could be empty. If a column contains the label $\circ$, it is the lone non-$\square$ box in its column.
2. We have $\nu(\square, \bullet, \bigodot) = \gamma(S)^*$.
3. We have $|\nu(\square, \bullet, \bigodot)| = |\gamma(S)^*|$.
4. The collection of cells $\nu(\bullet, \bigodot)$ is LL for $\gamma(S)^*$.
5. The collection of cells $\nu(\bigodot, \bigodot)$ is RB for $\nu(\square)$.

The sign of a partisan skyline $\nu$ is

$$\text{sign}(\nu) = (-1)^{|\nu(\bullet, \bigodot)|}$$
and the weight of $\nu$ is the composition

$$\text{wt}(\nu) = \nu(\square, \circ, \circ).$$

Partisan skylines combinatorially encode the right hand side of Equation (3.4). By Theorem 3.3, a typical term in this expression is obtained by choosing a LL collection $\lambda$ for $\gamma(S)^*$ to remove (at the cost of a sign), and then adding a RB collection $\rho$ of cells to the resulting composition with $|\rho| = n - k + 1 + |\lambda| = |\gamma(S)^*| + |\lambda|$. Cells which are removed are labeled $\bullet$, cells which are added are labeled $\circ$, and cells which are removed and then added are labeled $\square$. Said differently, we have

$$\sum_{\lambda} (-1)^{|\lambda|} \kappa_{\gamma(S)^* - \lambda}(x_\cdot) e_{n-k+1+|\lambda|}(x_n) = \sum_\nu \text{sign}(\nu) \kappa_{\text{wt}(\nu)}(x_n),$$

where the sum is over all partisan skylines $\nu$.

As an example of this construction, consider the case $n = 9$, $k = 5$, and $S = \{1, 3, 5, 6, 9\}$. Then $\gamma(S) = (1, 0, 2, 0, 3, 3, 0, 5), \gamma(S) = (0, 0, 1, 0, 2, 2, 0, 4)$, and the skyline diagram of $\gamma(S)^*$ is shown below.

Three possible partisan skylines are shown in (3.7) below.

(3.7) 9 8 7 6 5 4 3 2 1

From left to right, the weights of these partisan skylines are $(3, 0, 1, 3, 1, 1, 1, 1), (4, 0, 1, 1, 3, 1, 2, 1, 1)$, and $(5, 0, 0, 3, 3, 0, 2, 0, 1) = \gamma(S)^*$. Also from left to right, the signs of these partisan skylines are $(-1)^3 = -1, (-1)^1 = -1$, and $(-1)^0 = 1$. If $\nu$ is the partisan skyline on the left of (3.7), then $\nu(\square, \circ, \circ), \nu(\circ, \circ)$, and $\nu(\circ, \circ)$ are (from left to right):

A column in a partisan skyline $\nu$ is declared frozen if it contains a cell labeled $\circ$. There is a unique partisan skyline in which all nonempty columns are frozen: one simply places a single cell labeled $\circ$ on top of every nonempty column in the skyline diagram of $\gamma(S)^*$, together with the appropriate number of cells labeled $\circ$ in the rightmost empty columns of $\gamma(S)^*$. This is the completely frozen partisan skyline $\nu_0$; an example is shown on the right (3.7). The completely frozen partisan skyline $\nu_0$ always satisfies $\text{sign}(\nu_0) = 1$ and $\text{wt}(\nu_0) = \gamma(S)^*$.

We wish to define a sign-reversing and weight-preserving involution $\iota$ on the set of partisan skylines which are not completely frozen. To do this, we will use

**Claim 1:** If $\nu$ is any partisan skyline, then $\nu$ does not contain a nonempty column consisting only of cells labeled $\bullet$.

**Proof.** (of Claim 1) Write $S = \{s_1 < \cdots < s_{n-k+1}\}$, let $1 \leq i \leq n - k + 1$, and suppose the nonempty column $C$ in position $n - s_i + 1$ of $\nu$ consisted only of cells labeled $\bullet$. Since the column $C$ contains $s_i - i$ cells, there must be at least $(n - k + 1) + (s_i - i)$ cells in $\nu$ labeled $\circ$. Let $e$ be
the number of empty columns strictly to the right of $C$ in $\gamma(S)^\nu$. By considering the number of nonempty columns in $\gamma(S)^\nu$ besides $C$, Conditions (3) and (5) in the definition of a partisan skyline force at least $e + 1$ of these $\circ$ cells to be at height 1. But then Condition (5) forces the bottom box of $C$ to be labeled $\circ$, which is a contradiction. \hfill \Box

We are ready to define our involution $\iota$. Let $\nu$ be a partisan skyline which is not completely frozen. For from left to right, write the unfrozen nonempty columns of $\nu$ as $C_1, \ldots, C_m$. For $1 \leq i \leq m$, let $h_i$ be the number of boxes in $C_i$ labeled $\square$ or $\circ$. Let $h = \min\{h_i : 1 \leq i \leq m\}$ and let $C$ be the leftmost column among $C_1, \ldots, C_m$ with $h$ boxes labeled $\square$ or $\circ$. By Claim 1, we have $h > 0$. The highest cell in $C$ which is not labeled $\bullet$ is labeled either $\circ$ or $\square$. If this cell is labeled $\circ$, change its label to $\square$. If this cell is labeled $\square$, change its label to $\circ$. Let $\iota(\nu)$ be the resulting diagram. As an example of the map $\iota$, the left and middle partisan skylines of $[3, 7]$ are sent to

![Diagrams showing the transformation of partitions under the involution $\iota$](image)

respectively (we have $h = 1$ in either of these examples). To give another example (where $h = 2$ this time) the map $\iota$ interchanges

![Diagrams showing another transformation](image)

**Claim 2:** Let $\nu$ be a partisan skyline which is not completely frozen. Then $\iota(\nu)$ is also a partisan skyline which is not completely frozen. In fact, the sets of not frozen columns of $\nu$ and $\iota(\nu)$ coincide.

**Proof.** (of Claim 2) Let $C$ be the column which differs between $\nu$ and $\iota(\nu)$ and let $(i, j)$ be the coordinates the cell whose label was changed. In the three examples above we have $(i, j) = (7, 1), (4, 1),$ and $(4, 2)$, respectively. Conditions (1), (2), and (3) are obviously satisfied by $\iota(\nu)$.

Condition (4) is clearly preserved if $\iota$ changes the label of the cell $(i, j)$ from $\circ$ to $\square$. The only way Condition (4) could fail is if $\iota$ changes the label of $(i, j)$ from $\square$ to $\circ$, resulting in a collection of cells $\iota(\nu)(\circ, \bullet)$ which is not LL for $\iota(\nu)(\square, \circ, \bullet) = \nu(\square, \circ, \bullet)$. If $\iota(\nu)(\circ, \bullet)$ is not LL for $\iota(\nu)(\square, \circ, \bullet)$ then either

(i) there exists a cell $(i', j - 1)$ in $\nu$ with $i' > i$ containing the label $\circ$ or $\bullet$, or

(ii) there exists a cell $(i'', j + 1)$ in $\nu$ with $i'' < i$ containing the label $\circ$ or $\bullet$.

Suppose there exists a cell $(i', j - 1)$ in $\nu$ with $i' > i$ containing the label $\circ$ or $\bullet$. The column $C'$ containing $i'$ is not frozen and has strictly fewer boxes labeled $\square$ or $\circ$ than $C$. This contradicts the definition of $\iota$.

Suppose there exists a cell $(i'', j + 1)$ in $\nu$ with $i'' < i$ containing the label $\circ$ or $\bullet$. If $(i'', j + 1)$ contains the label $\bullet$, then the column $C''$ containing $(i'', j + 1)$ would have no more boxes labeled $\square$ or $\circ$ than $C$; since $C''$ is to the left of $C$, this contradicts the definition of $\iota$. If $(i'', j + 1)$ contains the label $\circ$, then since $(i, j)$ contains the label $\square$ in $\nu$, the collection $\nu(\circ, \circ)$ is not RB for $\nu(\square)$, contradicting the assumption that $\nu$ is a partisan skyline. We conclude that $\iota(\nu)(\circ, \circ)$ is LL for $\gamma(S)^\nu$, so that Condition (4) holds for $\iota(\nu)$.

Let us consider Condition (5). If $\iota$ changes the label of $(i, j)$ from $\circ$ to $\square$, suppose the collection $\iota(\nu)(\circ, \circ) = \nu(\circ, \circ) - \{(i, j)\}$ were not RB for $\iota(\nu)(\square)$. There must then exist some cell $(i'', j + 1)$ in $\nu$ with $i'' < i$ which is labeled with $\circ$ or $\circ$. Since the cell $(i, j)$ has the label $\circ$ in $\nu$ and no two cells of $\nu$ in the same row can have label $\bullet$ or $\circ$, the cell $(i'', j)$ in $\nu$ must be labeled $\square$. However,
this violates the assumption that $\nu(\bullet, \odot)$ is LL for $\nu(\square, \bullet, \odot)$ (because we could have moved the $\odot$
 in position $(i, j)$ of $\nu$ to the (leftward) position $(i', j)$). We conclude that $\nu(\bullet, \odot)$ is RB for $\nu(\square)$, so that Condition (5) holds for $\nu(\nu)$ in this case.

Finally, consider Condition (5) in the case where $\iota$ changes the label of $(i, j)$ from $\square$ to $\odot$. Suppose the collection $\nu(\bullet, \odot) = \nu(\square, \bullet, \odot) \cup \{(i, j)\}$ were not RB for $\nu(\square)$. Then there must be some $i' > i$ such that $(i', j)$ is not a cell in $\nu(\square, \bullet, \odot)$ but $(i', j - 1)$ is a cell in $\nu(\square)$. In particular, the column $C'$ of $\nu$ containing $(i', j - 1)$ is not frozen. If $j > 1$, then $C'$ is also nonempty, and contains strictly fewer boxes labeled $\square$ or $\odot$ than $C$, contradicting the definition of $\iota$. If $j = 1$, the fact that $\nu$ is partisan forces every coordinate $(i', j)$ with $i' > i$ to be a cell of $\nu(\square, \bullet, \odot)$, again a contradiction.

We have demonstrated that $\nu(\nu)$ is a partisan skyline. It is clear that $\iota$ does not freeze any columns. This completes the proof of Claim 2. \hfill $\Box$

By Claim 2 $\iota$ is a well defined operator on the set of partisan skylines which are not completely frozen. Moreover, the map $\iota$ does not affect the total number of boxes labeled $\square$ or $\odot$ in any column (it merely swaps an $\odot$ for a $\square$ or vice versa). It is also evident that $\iota$ does not freeze any columns. This implies that $\iota$ is an involution. Since $\nu(\square, \bullet, \odot) = \nu(\iota(\nu)(\square, \bullet, \odot))$, the map $\iota$ preserves weight. Since $\iota$ changes the parity of the set $[\nu(\bullet, \odot)]$, the map $\iota$ reverses sign. The right hand side of Equation 3.6 becomes

\[
\sum_{\nu} \text{sign}(\nu)\kappa_{\text{wt}(\nu)}(x_n) = \text{sign}(\nu_0)\kappa_{\text{wt}(\nu_0)}(x_n) = \kappa_{\gamma(S)^*}(x_n)
\]

and the proof is complete. \hfill $\Box$

Our proof of Lemma 3.4 was combinatorial and relied on a sign-reversing involution. Since the polynomials involved in Equation 3.4 are characters of indecomposable representations of the Borel subgroup $B \subseteq GL_n(C)$, Lemma 3.4 suggests the existence of a long exact sequence whose terms are (tensor products of) the corresponding modules. It may be interesting to find a direct algebraic proof of Lemma 3.4.

In order to determine the reduced Gröbner basis of the ideal $I_{n,k}$, we will need to study the monomials appearing in the reverse Demazure characters $\kappa_{\gamma(S)^*}(x_n^*)$. Our lemma in this direction is as follows. It states that the leading terms of these Demazure characters are skip monomials.

**Lemma 3.5.** Let $k \leq n$ and let $S \subseteq [n]$ satisfy $|S| = n - k + 1$. Let $<$ be lexicographic order. We have $\text{in}_< (\kappa_{\gamma(S)^*}(x_n^*)) = x(S)$. Moreover, for any $1 \leq i \leq n$ we have $x_i^{\max(S) - n + k + 1} \triangleright m$ for any monomial $m$ appearing in $\kappa_{\gamma(S)^*}(x_n^*)$. Finally, if $T \subseteq [n]$ satisfies $|T| = n - k + 1$ and $T \neq S$, then $x(S) \triangleright m$ for any monomial $m$ appearing in $\kappa_{\gamma(T)^*}(x_n^*)$.

**Proof.** We use the SSK model for Demazure characters. Given $S$, the big filling of $\gamma(S)$ is obtained by letting the entries in any column equal the entry in the basement of that column. For example, if $(n, k) = (5, 3)$ and $S = \{2, 3, 5\}$, then $\gamma(S) = (0, 2, 2, 0, 3)$ and the big filling of $\gamma(S)$ is shown below.

\[
\begin{array}{cccccc}
4 & 3 & 1 & 1 & 1 \\
4 & 3 & 1 & 1 & 1 \\
5 & 4 & 3 & 2 & 1 \\
\end{array}
\]
It can be observed that the big filling is SSK. The big filling above contributes \(x_2^2 x_3^2 x_4^2\) to \(\kappa_{(0,2,2,0,3)}(x_5)\), and hence \(x_2^2 x_3^2 x_4^2 = x(235)\) to \(\kappa_{(0,2,2,0,3)}(x_5^n)\). In general, the big filling of \(\gamma(S)\) contributes \(x(S)\) to \(\kappa_{\gamma(S)}(x_5^n)\).

The reverse content of any SSK \(F\) is the vector

\[
\text{revcont}(F) := (\# \text{ of n's in } F, \ldots, \# \text{ of 2's in } F, \# \text{ of 1's in } F).
\]

From the condition that SSK are weakly increasing up columns it follows that the big filling of \(\gamma(S)\) is the unique filling of \(\gamma(S)\) with the lexicographically largest reverse content vector. This immediately implies that \(x_5^{\max(S) - n + k + 1} \mid m, \) there would be a SSK \(F\) of shape \(\gamma(S)\) containing at least \((\max(S) - n + k + 1)\) copies of \(i\). By the definition of \(\gamma(S)\), the highest column in the skyline diagram of \(\gamma(S)\) has height \((\max(S) - n + k)\).

This implies that \(F\) contains two copies of \(i\) in the same row. If these two copies of \(i\) occur in columns of the same height, the triple of cells in \(F\)

\[
\begin{array}{c}
i \\
\downarrow \\
\end{array} 
\begin{array}{c}
j \\
\downarrow \\
\end{array} 
\begin{array}{c}
k \\
\end{array}
\]

forms a type A coinversion triple, a contradiction. If these copies occur in columns which are not of the same height, the triple of cells in \(F\)

\[
\begin{array}{c}
i \\
\downarrow \\
\end{array} 
\begin{array}{c}
j \\
\downarrow \\
\end{array} 
\begin{array}{c}
k \\
\end{array}
\]

forms a type B coinversion triple, again a contradiction. We conclude that \(x_5^{\max(S) - n + k + 1} \nmid m\).

Finally, suppose \(T \subseteq [n]\) satisfies \(|T| = n - k + 1\) and \(x(S) \mid m\) for some monomial \(m\) appearing in \(\kappa_{\gamma(T)}(x_5^n)\). Let \(F\) be a SSK of shape \(\gamma(T)\) which contributes \(m\) to \(\kappa_{\gamma(T)}(x_5^n)\). Since \(x(S) \mid m\) and \(F\) decreases weakly up columns, for all \(i\) we have

\[
\# \text{ of cells in columns } 1, 2, \ldots, i \text{ of } \gamma(T) \geq \# \text{ of cells in columns } 1, 2, \ldots, i \text{ of } \gamma(S).
\]

The definition of skip compositions forces \(S = T\).

To determine the graded Frobenius series of \(R_{n,k}\), we will need to carefully study the symmetric function operators \(e_j(x)^\perp\). The basic tool we will use is as follows.

**Lemma 3.6.** Let \(F(x), G(x) \in \Lambda\) be symmetric functions with equal constant terms. We have \(F(x) = G(x)\) if and only if \(e_j(x)^\perp F(x) = e_j(x)^\perp G(x)\) for all \(j \geq 1\).

**Proof.** The forward direction is obvious. For the reverse implication, observe that for any partition \(\lambda\) and \(j \geq 1\) we have

\[
\langle F(x), e_j(x)e_\lambda(x) \rangle = \langle e_j(x)^\perp F(x), e_\lambda(x) \rangle
\]

\[
= \langle e_j(x)^\perp G(x), e_\lambda(x) \rangle
\]

\[
= \langle G(x), e_j(x)e_\lambda(x) \rangle,
\]

so that \(F(x) = G(x)\). \(\square\)

To use Lemma 3.6 to prove \(\text{grFrob}(R_{n,k}; q) = D_{n,k}(x; q)\), we will need to determine the images of \(\text{grFrob}(R_{n,k}; q)\) and \(D_{n,k}(x; q)\) under the operator \(e_j(x)^\perp\) for \(j \geq 1\). The case of \(\text{grFrob}(R_{n,k}; q)\) will be handled in Section 6 after we develop a better understanding of the algebraic combinatorics of \(R_{n,k}\). The case of \(D_{n,k}(x; q)\) can be treated now.

**Lemma 3.7.** We have

\[
e_j(x)^\perp D_{n,k}(x;q) = q^{\binom{k}{j}} \cdot \sum_{m = \max(1,k-j)}^{\min(k,n-j)} q^{(k-m)(n-j-m)} \binom{j}{k-m} D_{n-j,m}(x;q).
\]
Proof. General facts about Gessel fundamental quasisymmetric functions and the superization of symmetric functions \[16, 14\] pp. 99-101 imply that (2.23) is equivalent to the statement that for any (strong) composition \(\beta = (\beta_1, \ldots, \beta_p)\) of \(n\) into positive parts
\[
\langle D_{n,k}(x; q), e_{\beta_1}(x) \cdots e_{\beta_p}(x) \rangle = \sum_{\sigma \in OP_{n,k} \atop \text{revword(}\sigma\text{) is a } \beta\text{-shuffle}} q^{\text{coinv}(\sigma)},
\]
where the sum is over all \(\sigma\) whose reverse reading word \(\text{revword}(\sigma)\) is a shuffle of the decreasing sequences
\[(\beta_1, \ldots, 2, 1), (\beta_1 + \beta_2, \ldots, \beta_1 + 1), \ldots, (n, n - 1, \ldots, \beta_1 + \ldots + \beta_{p-1} + 1).\]
Furthermore, letting \(\beta_1 = j\), this sum also equals \(\langle e_j^+(x) D_{n,k}(x; q), e_{\beta_2}(x) \cdots e_{\beta_p}(x) \rangle\).

We now give a combinatorial interpretation for the right-hand-side of (3.12) (with the parameter \(m\) there replaced by \(k - r\)). Given \(0 \leq r \leq k - 1\) and \(\sigma \in OP_{n-j,k-r}\), let \(T\) be a way of adding the \(j\) elements of the set \(\{n - j + 1, \ldots, n\}\) (hereafter referred to as \(\text{big letters}\)) to the blocks of \(\sigma\) to form an ordered set partition \(\sigma'\) of \(n\) with \(k\) blocks, in such a way that the big letters occur in reverse order \(n, n - 1, \ldots, n - j + 1\). For example, if \(\sigma = (246|15|3)\) and \(j = 3\), one choice for \(\sigma'\) would be \((2467|8|15|9|3)\), since then \(\text{revword}(\sigma) = 391825467\). Note that the conditions on the big letters force each block to have at most one big letter, and that \(T\) must have \(r\) blocks consisting of a single big letter.

Call the letters from \(\{1, 2, \ldots, n-j\}\) \(\text{small letters}\). Furthermore let big letters which are minimal (non-minimal) in their block be denoted by \(\text{minb}\) (\(\text{nminb}\)) letters, respectively, and small letters which are minimal (non-minimal) in their block be denoted by \(\text{mins}\) (\(\text{nmins}\)) letters, respectively. Recalling the conditions (2.3), in \(T\) each \(\text{minb}\) letter forms a coinversion pair with each \(\text{nmins}\) letter, giving \(r(n - j - (k - r))\) coinversions of this type. Each of the \(\text{minb}\) letters also form a coinversion pair with each of the \(\text{nmins}\) letters (giving \(r(j - r)\) pairs), and with each of the \(\text{minb}\) letters to its left (yielding \(j\) pairs). In addition, each \(\text{minb}\) letter forms a coinversion pair with each \(\text{mins}\) letter to its left; as we sum over all ways of interleaving the \(r\) blocks of \(\text{minb}\) letters with the other \(k - r\) blocks, leaving everything else in \(T\) fixed, we generate a factor of \([k]\) from these pairs.

Next note that each \(\text{nmins}\) letter forms a coinversion pair with each \(\text{mins}\) letter to its left. It follows that if we ignore the \(r\) blocks of \(\text{minb}\) letters, and sum over all ways to distribute the \(\text{nmins}\) letters amongst the \(k - r\) blocks with a \(\text{mins}\) letter, we generate a factor of \(q^{(j^r)}\) from these pairs.

Since
\[
\binom{r}{2} + r(j - r) + \binom{j-r}{2} = \binom{j}{2},
\]
we have
\[
\sum_T q^{\text{coinv}(T)} = q^{\text{coinv}(\sigma)} q^{r(n-j-k+r)+(j)} \binom{k}{r} q^{k-r} \binom{j-r}{r} q^j.
\]
If we sum (3.14) over all \(\sigma \in OP_{n-j,k-r}\), multiplying by the appropriate Gessel fundamental, and also sum over \(r\), we get the right hand side of (3.12). Taking the scalar product of the resulting symmetric function with \(e_{\beta_2}(x) \cdots e_{\beta_p}(x)\), again using the results on superization and (3.13), we get the scalar product of the left hand side of (3.12) with \(e_{\beta_2}(x) \cdots e_{\beta_p}(x)\). \(\square\)

4. Hilbert series and generalized Artin basis

4.1. The point sets \(Y_{n,k}\). Our strategy for determining the Hilbert and Frobenius series of \(R_{n,k}\) is inspired from the work of Garsia and Procesi [12]. Garsia and Procesi used this method to
determine the Hilbert and Frobenius series of the cohomology of the Springer fiber $R_\lambda$ for $\lambda \vdash n$.

We recall the method, then apply it to our situation.

Let $Y \subseteq \mathbb{Q}^n$ be any finite set of points and let $I(Y) \subseteq \mathbb{Q}[x_n]$ be the ideal of polynomials which vanish on $Y$:

$$I(Y) = \{ f \in \mathbb{Q}[x_n] : f(y) = 0 \text{ for all } y \in Y \}. \quad (4.1)$$

We can think of the quotient $\mathbb{Q}[Y] = \frac{\mathbb{Q}[x_n]}{I(Y)}$ as the ring of polynomial functions $Y \rightarrow \mathbb{Q}$, and in particular we have

$$\dim \left( \frac{\mathbb{Q}[x_n]}{I(Y)} \right) = |Y|. \quad (4.2)$$

Moreover, if $Y$ is stable under the action of any finite subgroup $G \subseteq \text{GL}_n(\mathbb{Q})$, we have an isomorphism of $G$-modules

$$\frac{\mathbb{Q}[x_n]}{I(Y)} \cong_G \mathbb{Q}[Y]. \quad (4.3)$$

The ideal $I(Y)$ is almost never homogeneous. To produce a homogeneous ideal, let $\tau(f)$ be the top degree component of any nonzero polynomial $f \in \mathbb{Q}[x_n]$. That is, if $f = f_d + f_{d-1} + \cdots + f_0$ with each $f_i$ homogeneous of degree $i$ and $f_d \neq 0$, we define $\tau(f) := f_d$. Let $T(Y) \subseteq \mathbb{Q}[x_n]$ be the ideal generated by the top degree components of all of the polynomials in $I(Y)$:

$$T(Y) := \langle \tau(f) : f \in I(Y) - \{0\} \rangle. \quad (4.4)$$

The ideal $T(Y)$ is homogeneous by definition. It can also be shown (see [12]) that any set of homogeneous polynomials in $\mathbb{Q}[x_n]$ which descends to a basis of $\frac{\mathbb{Q}[x_n]}{I(Y)}$ also descends to a basis of $\frac{\mathbb{Q}[x_n]}{T(Y)}$. In particular, we have

$$\dim \left( \frac{\mathbb{Q}[x_n]}{T(Y)} \right) = \dim \left( \frac{\mathbb{Q}[x_n]}{I(Y)} \right) = |Y|. \quad (4.5)$$

Moreover, if $Y$ is stable under the action of a finite subgroup $G \subseteq \text{GL}_n(\mathbb{Q})$, we have an isomorphism of $G$-modules

$$\frac{\mathbb{Q}[x_n]}{T(Y)} \cong_G \frac{\mathbb{Q}[x_n]}{I(Y)} \cong_G \mathbb{Q}[Y]. \quad (4.6)$$

Our strategy is as follows.

1. Find a finite point set $Y_{n,k} \subseteq \mathbb{Q}^n$ which is stable under the action of $\mathfrak{S}_n$ such that there is a $\mathfrak{S}_n$-equivariant bijection between $Y_{n,k}$ and $\mathcal{OP}_{n,k}$.

2. Prove that $I_{n,k} \subseteq T(Y_{n,k})$ by showing that the generators of $I_{n,k}$ arise as top degree components of polynomials $f \in I(Y_{n,k})$ vanishing on $Y_{n,k}$.

3. Prove that

$$\dim(R_{n,k}) = \dim \left( \frac{\mathbb{Q}[x_n]}{I_{n,k}} \right) \leq |\mathcal{OP}_{n,k}| = \dim \left( \frac{\mathbb{Q}[x_n]}{T(Y_{n,k})} \right)$$

and use the relation $I_{n,k} \subseteq T(Y_{n,k})$ to conclude that $I_{n,k} = T(Y_{n,k})$.

The execution of this strategy will show that $\dim(R_{n,k}) = |\mathcal{OP}_{n,k}|$ and $R_{n,k} \cong Q[\mathcal{OP}_{n,k}]$ as ungraded $\mathfrak{S}_n$-modules.

To achieve Step 1 of our strategy, we introduce the following point sets.

**Definition 4.1.** Fix distinct rational numbers $\alpha_1, \ldots, \alpha_k \in \mathbb{Q}$. Let $Y_{n,k} \subseteq \mathbb{Q}^n$ be the set of points with coordinates occurring in $\{\alpha_1, \ldots, \alpha_k\}$ such that each $\alpha_i$ appears at least once. In other words,

$$Y_{n,k} := \{(y_1, \ldots, y_n) \in \mathbb{Q}^n : \{\alpha_1, \ldots, \alpha_k\} = \{y_1, \ldots, y_n\}\}.$$
There is a bijection from $Y_{n,k}$ and $\mathcal{OP}_{n,k}$ obtained by sending $(y_1, \ldots, y_n)$ to $(B_1 \cdots | B_k)$, where $B_i = \{ j : y_j = \alpha_i \}$. For example, we have

$$(\alpha_3, \alpha_2, \alpha_2, \alpha_1, \alpha_3) \leftrightarrow (4 \mid 23 \mid 15).$$

It is evident that this bijection is $S_n$-equivariant, which implies

(4.7) $Q[Y_{n,k}] \cong S_n \cdot Q[\mathcal{OP}_{n,k}]$.

Lemma 3.1 allows us to achieve Step 2 of our strategy right away.

**Lemma 4.2.** We have $I_{n,k} \subseteq T(Y_{n,k})$.

**Proof.** For $n - k + 1 \leq r \leq n$, Lemma 3.1 guarantees that

(4.8) $\sum_{j=0}^{r} (-1)^j h_j(\alpha_1, \ldots, \alpha_k) e_{r-j}(x_n) \in T(Y_{n,k}).$

Taking the top degree component, we get $e_r(x_n) \in T(Y_{n,k})$. Moreover, since the coordinates of points in $Y_{n,k}$ lie in the set $\{ \alpha_1, \ldots, \alpha_k \}$, for $1 \leq i \leq n$ we have

(4.9) $(x_i - \alpha_1) \cdots (x_i - \alpha_k) \in T(Y_{n,k})$.

Taking the top degree component, we get $x_i^k \in T(Y_{n,k})$. We conclude that $I_{n,k} \subseteq T(Y_{n,k})$. $\square$

Step 3 of our strategy will require more work. We aim to prove the dimension inequality $\text{dim} \left( \frac{\mathbb{Q}[x_n]}{I_{n,k}} \right) \leq |\mathcal{OP}_{n,k}|$. To do so, we will use Gröbner theory.

### 4.2. Skip and nonskip monomials

Let $<$ be the lexicographic monomial order. We know that

(4.10) $\text{dim} \left( \frac{\mathbb{Q}[x_n]}{I_{n,k}} \right) = \text{dim} \left( \frac{\mathbb{Q}[x_n]}{\text{in}_<(I_{n,k})} \right)$.

Moreover, we know that a basis for $\frac{\mathbb{Q}[x_n]}{\text{in}_<(I_{n,k})}$ is given by

(4.11) $\{ m + I_{n,k} : m \in \mathbb{Q}[x_n] \text{ is a monomial and } \text{in}_<(f) \nmid m \text{ for all } f \in I_{n,k} - \{0\} \}$.

Our aim is to bound the size of this set above by $|\mathcal{OP}_{n,k}|$. To do this, we will calculate $\text{in}_<(f)$ for some strategically chosen $f \in I_{n,k}$.

**Lemma 4.3.** Let $k \leq n$. For any $S \subseteq [n]$ of size $|S| = n - k + 1$, the reverse Demazure character $\kappa_{\gamma(S)^*}(x_n^*)$ satisfies $\kappa_{\gamma(S)^*}(x_n^*) \in I_{n,k}$. In particular, we have

$x(S) \in \text{in}_<(I_{n,k})$.

Moreover, for $1 \leq i \leq n$ we have

$$x_i^k \in \text{in}_<(I_{n,k}).$$

**Proof.** Lemma 3.1 (and in particular Equation 3.5) implies that $\kappa_{\gamma(S)^*}(x_n^*) \in I_{n,k}$. By Lemma 3.5 taking the lexicographic leading term shows that

$$\text{in}_<(\kappa_{\gamma(S)^*}(x_n^*)) = x(S) \in \text{in}_<(I_{n,k}).$$

Since $x_i^k \in I_{n,k}$, we have $x_i^k \in \text{in}_<(I_{n,k})$. $\square$

It will turn out that the leading monomials furnished by Lemma 4.3 are all we need. We give the collection of monomials which are not divisible by any of these monomials a name.

**Definition 4.4.** Let $k \leq n$. A monomial $m \in \mathbb{Q}[x_n]$ is $(n,k)$-nonskip if

1. $x(S) \nmid m$ for all $S \subseteq [n]$ with $|S| = n - k + 1$ and
2. $x_i^k \nmid m$ for all $1 \leq i \leq n$.

Let $\mathcal{M}_{n,k}$ denote the set of all $(n,k)$-nonskip monomials in $\mathbb{Q}[x_n]$. 

There is some redundancy in Definition 4.1. If \( n \in S \) and \( |S| = n - k + 1 \), the skip monomial \( \mathbf{x}(S) \) contains the variable power \( x^k \). It is therefore enough to consider those subsets \( S \) with \( |S| = n - k + 1 \) and \( n \notin S \).

We will prove that \( |M_{n,k}| = |OP_{n,k}| \). This will imply that \( R_{n,k} \equiv Q[OP_{n,k}] \). We will also show that the degree statistic on \( M_{n,k} \) is equidistributed with coinv on \( OP_{n,k} \). This will imply that \( \text{Hib}(R_{n,k};q) = \text{rev}_q([k]q \cdot \text{Stir}_q(n,k)) \). Finally, we will show that \( M_{n,k} = A_{n,k} \), proving that generalized Artin monomials descend to a basis of \( R_{n,k} \).

The proofs of the claims in the last paragraph will involve a combinatorial analysis of skip and nonskip monomials. A key observation regarding skip monomials and divisibility is as follows.

**Lemma 4.5.** Let \( m \in Q[\mathbf{x}_n] \) be a monomial and let \( S, T \subseteq [n] \). If \( \mathbf{x}(S) \mid m \) and \( \mathbf{x}(T) \mid m \), then \( \mathbf{x}(S \cup T) \mid m \).

**Proof.** Let \( i \in S \). Then
\[
\text{exponent of } x_i \text{ in } \mathbf{x}(S \cup T) = i - |(S \cup T) \cap \{1, 2, \ldots, i - 1\}|
\leq i - |S \cap \{1, 2, \ldots, i - 1\}|
= \text{exponent of } x_i \text{ in } \mathbf{x}(S).
\]
Similarly, if \( j \in T \) then the exponent of \( x_j \) in \( \mathbf{x}(S \cup T) \) is \( \leq \) the exponent of \( x_j \) in \( \mathbf{x}(T) \). The lemma follows. \( \square \)

An immediate useful consequence of Lemma 4.5 is:

**Lemma 4.6.** Let \( m \in Q[\mathbf{x}_n] \) be a monomial and let \( r \) be a positive integer such that \( \mathbf{x}(S) \mid m \) for some \( S \subseteq [n] \) with \( |S| = r \) but there does not exist \( T \subseteq [n] \) with \( |T| > r \) such that \( \mathbf{x}(T) \mid m \).

Then there exists a unique \( S \subseteq [n] \) with \( |S| = r \) such that \( \mathbf{x}(S) \mid m \).

**Proof.** If there were two such sets \( S \neq S' \), Lemma 4.5 would force \( \mathbf{x}(S \cup S') \mid m \), contrary to the assumption on \( r \). \( \square \)

For any subset \( S = \{i_1, \ldots, i_r\} \subseteq [n] \), let \( \mathbf{m}(S) := x_{i_1} \cdots x_{i_r} \in Q[\mathbf{x}_n] \) be the product of the corresponding variables. For example, we have \( \mathbf{m}(245) = x_2 x_4 x_5 \). The following existence-uniqueness type result will be crucial in bijecting \( M_{n,k} \) with \( OP_{n,k} \).

**Lemma 4.7.** Let \( k \leq n \) and let \( m \in M_{n,k} \). There exists a unique set \( S \subseteq [n] \) with \( |S| = n - k \) such that
\[
(1) \quad \mathbf{x}(S) \mid (\mathbf{m}(S) \cdot m), \text{ and}
(2) \quad \mathbf{x}(U) \nmid (\mathbf{m}(S) \cdot m) \text{ for all } U \subseteq [n] \text{ with } |U| = n - k + 1.
\]

**Proof.** We start with uniqueness. Suppose \( S = \{s_1 < \cdots < s_{n-k}\} \) and \( T = \{t_1 < \cdots < t_{n-k}\} \) were two distinct such sets. Let \( r \) be such that \( s_1 = t_1, \ldots, s_{r-1} = t_{r-1} \), and \( s_r \neq t_r \). Without loss of generality we have \( s_r < t_r \). Let \( U = \{s_1 < \cdots < s_r < t_r < t_{r+1} < \cdots < t_{n-k}\} \). Since \( \mathbf{x}(S) \mid (\mathbf{m}(S) \cdot m) \) and \( \mathbf{x}(T) \nmid (\mathbf{m}(T) \cdot m) \), it follows that \( \mathbf{x}(U) \nmid (\mathbf{m}(S) \cdot m) \) and \( |U| = n - k + 1 \).

To prove existence, consider the collection \( \mathcal{C} \) of sets
\[
C := \{S \subseteq [n] : |S| = n - k \text{ and } \mathbf{x}(S) \mid (\mathbf{m}(S) \cdot m)\}.
\]

The collection \( \mathcal{C} \) is nonempty; indeed, we have \( \{1, 2, \ldots, n - k\} \in \mathcal{C} \). Let \( S_0 \in \mathcal{C} \) be the lexicographically final subset in \( \mathcal{C} \). We argue that \( \mathbf{m}(S_0) \cdot m \) satisfies Condition 2 in the statement of the lemma, thus finishing the proof of the lemma.

Let \( T \subseteq [n] \) have size \( |T| = n - k + 1 \). Working towards a contradiction, suppose \( \mathbf{x}(T) \mid (\mathbf{m}(S_0) \cdot m) \).

If there were an element \( t \in T \) with \( t < \min(S_0) \), then the relation \( \mathbf{x}(S_0) \mid (\mathbf{m}(S_0) \cdot m) \) would imply \( \mathbf{x}(\{t\} \cup S_0) \mid m \), a contradiction. Since \( |T| > |S| \), there exists an element \( t_0 \in T - S_0 \) with
The five possible ways to do this, together with their effect on \( \text{inv} \), are as follows.

(Thus adding a star to the star model of \( \text{OP} \) or \( \text{OP}^+ \).) Whereas performing a bar insertion results in an element of \( \text{OP}_{n+1,k+1} \) (thus adding a bar to the bar model of \( \sigma \)).

For example, consider \( \sigma = (5 \mid 146 \mid 23 \mid 7) \in \text{OP}_{7,4} \). The four possible star insertions of 8, together with their effect on \( \text{inv} \), are as follows.

\[
\begin{align*}
(5 \mid 146 \mid 23 \mid 78) & \quad \text{inv} + 0 \\
(5 \mid 146 \mid 238 \mid 7) & \quad \text{inv} + 1 \\
(5 \mid 1468 \mid 23 \mid 7) & \quad \text{inv} + 2 \\
(58 \mid 146 \mid 23 \mid 7) & \quad \text{inv} + 3
\end{align*}
\]

Rather than \( \text{inv} \) itself, we are interested in the complementary statistic \( \text{coinv} \). In our example, the effect of star insertion on \( \text{coinv} \) is as follows.

\[
\begin{align*}
(5 \mid 146 \mid 23 \mid 78) & \quad \text{coinv} + 3 \\
(5 \mid 146 \mid 238 \mid 7) & \quad \text{coinv} + 2 \\
(5 \mid 1468 \mid 23 \mid 7) & \quad \text{coinv} + 1 \\
(58 \mid 146 \mid 23 \mid 7) & \quad \text{coinv} + 0
\end{align*}
\]

In general, if \( \tilde{\sigma} \) is the result of star inserting \( n \) into the block \( B_i \) of \( \sigma = (B_1 \mid \cdots \mid B_k) \in \text{OP}_{n,k} \), we have

\[
\text{coinv}(\tilde{\sigma}) = \text{coinv}(\sigma) + (i - 1).
\]

On the other hand, consider using bar insertion to insert 8 into \( \sigma = (5 \mid 146 \mid 23 \mid 7) \in \text{OP}_{7,4} \). The five possible ways to do this, together with their effect on \( \text{inv} \), are as follows.
The effect of these bar insertions on \( \text{coinv} \) is shown below.

\[
\begin{array}{c|c|c|c|c|c|c|c}
5 & 146 & 23 & 7 & 8 & 5 & 146 & 23 & 8 & 7 & 5 & 146 & 8 & 23 & 7 & 8 & 5 & 146 & 23 & 7 \\
\text{inv + 0} & \text{inv + 1} & \text{inv + 2} & \text{inv + 3} & \text{inv + 5} \\
\end{array}
\]

In general, if \( \hat{\sigma} \) is the result of bar inserting the singleton block \( \{n\} \) in the space between \( B_{i-1} \) and \( B_i \) of \( \sigma = (B_1 | \cdots | B_k) \in \mathcal{OP}_{n,k} \), we have

\begin{equation}
\text{coinv}(\hat{\sigma}) = \text{coinv}(\sigma) + (n - k) + (i - 1).
\end{equation}

Equations 4.14 and 4.15 imply that the generating function \( P_{n,k}(q) \) of \( \text{coinv} \) on \( \mathcal{OP}_{n,k} \) satisfies the recursion

\begin{equation}
P_{n,k}(q) = [k]_q \cdot (P_{n-1,k}(q) + q^{n-k} P_{n-1,k-1}(q)).
\end{equation}

Together with the initial conditions

\begin{equation}
\begin{cases}
P_{1,1}(q) = 1 \\
P_{n,k}(q) = 0 & n < k,
\end{cases}
\end{equation}

this determines \( P_{n,k}(q) \) completely. The idea for constructing our bijection \( \Psi : \mathcal{OP}_{n,k} \rightarrow \mathcal{M}_{n,k} \) is to show combinatorially that the degree generating function \( M_{n,k}(q) \) on \( \mathcal{M}_{n,k} \) also satisfies the recursion of Equation 4.16.

The base case of our map \( \Psi \) is given by the unique function \( \Psi : \mathcal{OP}_{1,1} \rightarrow \mathcal{M}_{1,1} \) mapping \((1) \mapsto 1\).

In general, suppose \( \sigma = (B_1 | \cdots | B_k) \in \mathcal{OP}_{n,k} \) and let \( \overline{\sigma} \) be the ordered set partition of size \( n - 1 \) obtained by erasing \( n \) from \( \sigma \), as well as the block containing \( n \) if that block is a singleton.

Then \( \overline{\sigma} \in \mathcal{OP}_{n-1,k} \) or \( \overline{\sigma} \in \mathcal{OP}_{n-1,k-1} \) according to whether \( \{n\} \) is a singleton block of \( \sigma \).

We inductively assume that the portions of the function \( \Psi \) given by \( \Psi : \mathcal{OP}_{n-1,k} \rightarrow \mathcal{M}_{n-1,k} \) and \( \Psi : \mathcal{OP}_{n-1,k-1} \rightarrow \mathcal{M}_{n-1,k-1} \) have already been defined (so that in particular \( \Psi(\overline{\sigma}) \) is defined). We define \( \Psi(\sigma) \) as follows, according to whether \( \overline{\sigma} \in \mathcal{OP}_{n-1,k} \) or \( \overline{\sigma} \in \mathcal{OP}_{n-1,k-1} \).

If \( \overline{\sigma} \in \mathcal{OP}_{n-1,k} \), then \( \{n\} \) is not a singleton block of \( \sigma = (B_1 | \cdots | B_k) \) and \( \sigma \) arises from \( \overline{\sigma} \) from star insertion. There exists \( 0 \leq i < k - 1 \) such that \( n \in B_{i+1} \). Define

\begin{equation}
\Psi(\sigma) := \Psi(\overline{\sigma}) \cdot x_i^n.
\end{equation}

If \( \overline{\sigma} \in \mathcal{OP}_{n-1,k-1} \), then \( \{n\} \) is a singleton block of \( \sigma \), so that \( \sigma \) arises from \( \overline{\sigma} \) by bar insertion. We have \( \Psi(\overline{\sigma}) \in \mathcal{M}_{n-1,k-1} \). Invoking Lemma 4.7, let \( S \subseteq [n-1] \) be the unique subset with \(|S| = n - k\) such that \( x(S) \mid (\Psi(\overline{\sigma}) \cdot m(S)) \) but \( x(U) \not\mid (\Psi(\overline{\sigma}) \cdot m(S)) \) for all \( U \subseteq [n-1] \) with \(|U| = n - k + 1\). Since \( \{n\} \) is a singleton block of \( \sigma \), there exists \( 0 \leq i < k - 1 \) such that \( \{n\} = B_{i+1} \). Define

\begin{equation}
\Psi(\sigma) := \Psi(\overline{\sigma}) \cdot m(S) \cdot x_i^n.
\end{equation}

The map \( \Psi \) is best understood with an example. Let \( \sigma \in \mathcal{OP}_{8,5} \) be the ordered set partition \( \sigma = (5 | 146 | 8 | 23 | 7) \). We calculate \( \Psi(\sigma) \) by starting with \( \Psi : (1) \mapsto 1 \in \mathcal{M}_{1,1} \) and repeatedly inserting larger letters to build up to \( \sigma \).
Let \( \sigma \) denote the word \((1 \ 2 \ 3) \ 0 \ 0 \ x \ \sigma \ x \ 23 \ 1 \ \bar{3} \) and assume that we have well-defined functions \( \Psi : \mathcal{OP}_{n-1,k} \to \mathcal{M}_{n-1,k} \) and \( \Psi : \mathcal{OP}_{n-1,k} \to \mathcal{M}_{n-1,k} \). Consider an ordered set partition \( \sigma = (B_1 \ | \cdots \ | B_k) \in \mathcal{OP}_{n,k} \). We need to check that the above procedure defining \( \Psi(\sigma) \) actually yields an element in \( \mathcal{M}_{n,k} \).

**Case 1:** \( \{n\} \) is not a singleton block of \( \sigma \).

In this case \( \Psi(\sigma) = \Psi(\bar{\sigma}) \cdot x_i^n \), where \( n \in B_{i+1} \). In particular, we have \( x^n \not\mid \Psi(\sigma) \). Since \( \Psi(\bar{\sigma}) \in \mathcal{M}_{n-1,k} \), we also know that \( x_j^k \not\mid \Psi(\sigma) \) for all \( 1 \leq j \leq n-1 \).

Let \( T \subseteq [n] \) satisfy \(|T| = n - k + 1 \) and suppose \( x(T) \mid \Psi(\sigma) \). Let \( T' := T \setminus \{\text{max}(T)\} \), so that \( T' \subseteq [n-1] \) and \(|T'| = n - k \). Since \( T' \subseteq [n-1] \) and \( x(T) \mid \Psi(\sigma) \), we have \( x(T') \mid \Psi(\bar{\sigma}) \), contradicting the assumption that \( \Psi(\bar{\sigma}) \in \mathcal{M}_{n-1,k} \). We conclude that \( x(T) \not\mid \Psi(\bar{\sigma}) \), so that \( \Psi(\sigma) \in \mathcal{M}_{n,k} \).

**Case 2:** \( \{n\} \) is a singleton block of \( \sigma \).

In this case \( \Psi(\sigma) = \Psi(\bar{\sigma}) \cdot m(S) \cdot x_i^n \), where \( B_{i+1} = \{n\} \) and \( S \subseteq [n-1] \) is as above. By construction \( x_i^n \not\mid \Psi(\sigma) \). Since \( \Psi(\bar{\sigma}) \in \mathcal{M}_{n-1,k-1} \), we know that \( x_j^{k-1} \not\mid \Psi(\bar{\sigma}) \) for all \( 1 \leq j \leq n-1 \), so that \( x_j^k \not\mid \Psi(\sigma) \).

Let \( T \subseteq [n] \) satisfy \(|T| = n - k + 1 \) and suppose \( x(T) \mid \Psi(\sigma) \). Our choice of \( S \) (and Lemma 4.7) guarantee that \( T \not\subseteq [n-1] \). On the other hand, if \( n \in T \), the variable power \( x^n \) would appear in \( x(T) \), implying the contradiction \( x_i^n \mid \Psi(\sigma) \). We conclude that \( x(T) \not\mid \Psi(\bar{\sigma}) \), so that \( \Psi(\sigma) \in \mathcal{M}_{n,k} \).

The construction of \( \Psi \) suggests the construction of its inverse. Given an \((n,k)\)-nonskip monomial \( m = x_1^{i_1} \cdots x_{n-1}^{i_{n-1}} x_n^i \in \mathcal{M}_{n,k} \), consider the monomial \( m' = x_1^{i_1} \cdots x_{n-1}^{i_{n-1}} \) obtained by erasing the last variable \( x_n \). Either \( m' \) is divisible by some skip monomial \( m(S) \) with \(|S| = n - k \) or it is not. If so, then \( S \) is unique and we have \( m' \not\mid m(S) \in \mathcal{M}_{n-1,k-1} \). If not, then we have \( m' \in \mathcal{M}_{n-1,k-1} \). Either way, we recursively have a size \( n - 1 \) ordered set partition. To get a size \( n \) ordered set partition, use this branching structure to determine whether to insert \( n \) as a singleton block and use the exponent \( i \) of \( x_n \) to determine where to insert \( n \) from left to right.

**Theorem 4.9.** The function \( \Psi : \mathcal{OP}_{n,k} \to \mathcal{M}_{n,k} \) is a bijection with the property that \( \text{coinv}(\sigma) = \deg(\Psi(\sigma)) \) for all \( \sigma \in \mathcal{OP}_{n,k} \).

In particular, we have \( P_{n,k}(q) = M_{n,k}(q) \) and \( |\mathcal{OP}_{n,k}| = |\mathcal{M}_{n,k}| \).

**Proof.** We recursively define the inverse \( \Phi \) to the function \( \Psi \). When \((n,k) = (1,1)\), there is only one choice: let \( \Phi : \mathcal{M}_{n,k} \to \mathcal{OP}_{n,k} \) be the unique assignment \( \Phi : 1 \mapsto (1) \).

In general, fix positive \( k \leq n \) and assume inductively that \( \Phi : \mathcal{M}_{n-1,k} \to \mathcal{OP}_{n-1,k} \) and \( \Phi : \mathcal{M}_{n-1,k-1} \to \mathcal{OP}_{n-1,k-1} \) have already been defined. Let \( m = x_1^{i_1} \cdots x_{n-1}^{i_{n-1}} x_n^i \in \mathcal{M}_{n,k} \) and define \( m' := x_1^{i_1} \cdots x_{n-1}^{i_{n-1}} \). Either \( m' \in \mathcal{M}_{n-1,k} \) or \( m' \not\in \mathcal{M}_{n-1,k} \).

If \( m' \in \mathcal{M}_{n-1,k} \), then \( \Phi(m') = (B_1 \mid \cdots \mid B_k) \) is an ordered set partition of size \( n - 1 \). To define \( \Phi(m) = \Phi(m' \cdot x_n^i) := (B_1 \mid \cdots \mid B_{i+1} \cup \{n\} \mid \cdots \mid B_k) \in \mathcal{OP}_{n,k} \).

(4.20) \[ \Phi(m) = \Phi(m' \cdot x_n^i) := (B_1 \mid \cdots \mid B_{i+1} \cup \{n\} \mid \cdots \mid B_k) \in \mathcal{OP}_{n,k}. \]
If \( m' \notin \mathcal{M}_{n-1,k} \), then there exists a subset \( S \subseteq [n-1] \) such that \( |S| = n-k \) and \( x(S) \mid m' \). Since \( m = m' \cdot x_n^i \in \mathcal{M}_{n,k} \), Lemma 4.6 guarantees that the set \( S \) is unique. Since \( x(S) \mid m' \), we have \( m(S) \mid m' \) and the quotient \( \frac{m'}{m(S)} \) is a monomial.

Claim: \( \frac{m'}{m(S)} \in \mathcal{M}_{n-1,k-1} \).

Since \( m \in \mathcal{M}_{n,k} \), we know that \( x(T) \mid \frac{m'}{m(S)} \) for all \( T \subseteq [n-1] \) with \( |T| = n-k+1 \). Let \( 1 \leq j \leq n-1 \). We need to show \( x_j^{k-1} \mid \frac{m'}{m(S)} \). If \( j \in S \) this is immediate from the fact that \( x_j^k \mid m' \).

If \( j \notin S \) and \( x_j^{k-1} \mid \frac{m'}{m(S)} \), then \( x_j^{k-1} \mid m' \) and \( x(S \cup \{j\}) \mid m' \), a contradiction to the assumption that \( m = m' \cdot x_n^i \in \mathcal{M}_{n,k} \). This finishes the proof of the Claim.

By the Claim, we recursively have an ordered set partition \( \Phi \left( \frac{m'}{m(S)} \right) = (B_1 \mid \cdots \mid B_{k-1}) \) of size \( n-1 \). To define \( \Phi(m) = \Phi(m' x_n^i) \), we bar insert a singleton block \( \{n\} \) to the left of \( B_{i+1} \)

\[
\Phi(m) := (B_1 \mid \cdots \mid B_i \mid n \mid B_{i+1} \mid \cdots \mid B_{k-1}) \in \mathcal{OP}_{n,k}.
\]

As with \( \Psi \), the map \( \Phi \) is best understood with an example. Consider the monomial \( m = x_2^2 x_3^4 x_6 x_7^2 x_8^2 \in \mathcal{M}_{8,5} \). To calculate \( \Phi(m) \in \mathcal{OP}_{8,5} \), we write down the following table. We show entries in the right hand column in bold to indicate that no more letters can be added to their block.

| \( m \) | \( m' \) | \( (n,k) \) | \( m' \in \mathcal{M}_{n-1,k} ? \) | \( S \) | \( \frac{m'}{m(S)} \) | \( i \) | \( \Phi(m) \) |
|---|---|---|---|---|---|---|---|
| \( x_1^2 x_2^4 x_3 x_6 x_7^2 x_8^2 \) | \( x_1^2 x_2^4 x_3 x_6 x_7^2 x_8^2 \) | (8, 5) | no | 123 | \( x_1^2 x_2^4 x_3 x_6 x_7^2 x_8^2 \) | 2 | (\( \cdot \mid \cdot \mid 8 \cdot \mid \cdot \) |
| \( x_1 x_2^4 x_3 x_6 x_7^2 x_8^2 \) | \( x_1 x_2^4 x_3 x_6 x_7^2 x_8^2 \) | (7, 4) | no | 123 | \( x_1^2 x_2^4 x_3 x_6 x_7^2 x_8^2 \) | 3 | (\( \cdot \mid \cdot \mid 8 \cdot \mid \cdot \) |
| \( x_2^2 x_3 x_6 \) | \( x_2^2 x_3 x_6 \) | (6, 3) | yes | 1 | \( \cdot \mid 6 \cdot \mid 8 \cdot \mid \cdot \) |
| \( x_2^2 x_3 \) | \( x_2^2 x_3 \) | (5, 3) | no | 23 | \( x_2 x_3 \) | 0 | (5 | 6 | 8 | \cdot | \cdot) |
| \( x_2 x_3 \) | \( x_2 x_3 \) | (4, 2) | yes | 0 | \( \cdot \mid \cdot \mid 6 \cdot \mid 8 \cdot \mid \cdot \) |
| \( x_2 x_3 \) | \( x_2 \) | (3, 2) | yes | 1 | \( \cdot \mid 6 \cdot \mid 8 \cdot \mid 3 \cdot \mid \cdot \) |
| \( x_2 \) | 1 | (2, 2) | no | \( \emptyset \) | 1 | 1 | (5 | 46 | 8 | 23 | \cdot) |
| 1 | 1 | (1, 1) | no | \( \emptyset \) | 1 | 0 | (5 | 146 | 8 | 23 | 7) |

The rules for proceeding from one row of this table to the next are as follows.

- Define \( m \) to be the monomial \( m' \) from the above row (if the answer to the query in the above row was ‘yes’) or \( \frac{m'}{m(S)} \) from the above row (if the answer to the query in the above row was ‘no’).
- Define the \( (n,k) \)-column entry to be \( (n-1,k) \) from the above row (if the answer to the query in the above row was ‘yes’) or \( (n-1,k-1) \) from the above row (if the answer to the query in the above row was ‘no’).
- Using \( (n,k) \) in the current row, define the monomial \( m' \) by the relation \( m = m' \cdot x_n^i \).
- Record the value of \( i \).
- Check whether \( m' \in \mathcal{M}_{n-1,k} \).
- If \( m' \notin \mathcal{M}_{n-1,k} \), let \( S \subseteq [n-1] \) be the unique set of size \( |S| = n-k \) such that \( x(S) \mid m' \).
- If \( m' \notin \mathcal{M}_{n-1,k} \), calculate \( \frac{m'}{m(S)} \).
- Finally, insert \( n \) into the partial ordered set partition \( \Phi(m) \) from the above row. If the answer to the query in the current row was ‘yes’, insert \( n \) into the \( i^{th} \) unfrozen block from the left. If the answer to the query in the current row was ‘no’, insert \( n \) into the \( i^{th} \) unfrozen block from the left and freeze that block.

The above table shows that \( \Phi : x_1^2 x_2^4 x_3^4 x_6 x_7^2 x_8^2 \mapsto (5 | 146 | 8 | 23 | 7) \).

We leave it to the reader to check that the recursions \( \Phi \) and \( \Psi \) undo each other, so that we have the compositions \( \Phi \circ \Psi = \text{id}_{\mathcal{OP}_{n,k}} \) and \( \Psi \circ \Phi = \text{id}_{\mathcal{M}_{n,k}} \). The definition of coinv makes it clear that \( \text{coinv}(\sigma) = \deg(\Phi(\sigma)) \) for all \( \sigma \in \mathcal{OP}_{n,k} \). \( \square \)
4.4. Gröbner basis and Hilbert series. We exploit the bijection \( \Psi \) of Theorem 4.9. First, we determine \( \text{Hilb}(R_{n,k}; q) \).

**Theorem 4.10.** Let \( k \leq n \) be positive integers. The Hilbert series \( \text{Hilb}(R_{n,k}; q) \) of the graded vector space \( R_{n,k} \) is the generating function for \( \text{coinv} \) on \( \mathcal{O}_n \).

\[
\text{Hilb}(R_{n,k}; q) = P_{n,k}(q) = \text{rev}_q([k]! \cdot \text{Stir}(n,k)).
\]

In particular, we have

\[
\text{dim}(R_{n,k}) = |\mathcal{O}_n| = k! \cdot \text{Stir}(n,k).
\]

**Proof.** Let \( < \) be the lexicographic order on \( \mathbb{Q}[x_n] \) and consider the ideals \( I_{n,k} \) and \( T(Y_{n,k}) \). By Lemma 4.2, we know \( I_{n,k} \subseteq T(Y_{n,k}) \). From this we get the containment of monomial ideals

\[
\text{in}_<(I_{n,k}) \subseteq \text{in}_<(T(Y_{n,k})).
\]

Let \( \mathcal{B}_{n,k}^I \) be the standard monomial basis of \( \mathbb{Q}[x_n]/I_{n,k} \), and let \( \mathcal{B}_{n,k}^T \) be the standard monomial basis of \( \mathbb{Q}[x_n]/T(Y_{n,k}) \). The containment above immediately gives

\[
\mathcal{B}_{n,k}^T \subseteq \mathcal{B}_{n,k}^I.
\]

Lemma 4.3 extends this inclusion to the triple

\[
\mathcal{B}_{n,k}^T \subseteq \mathcal{B}_{n,k}^I \subseteq \mathcal{M}_{n,k}.
\]

Since \( \mathbb{Q}[x_n]/I_{n,k} \cong \mathbb{Q}[x_n]/T(Y_{n,k}) \cong \mathbb{Q}[\mathcal{O}_n] \), we know \( |\mathcal{B}_{n,k}^T| = |\mathcal{O}_n| \). On the other hand, Theorem 4.9 says \( |\mathcal{M}_{n,k}| = |\mathcal{O}_n| \). Therefore, the above inclusions are actually equalities and we have

\[
|\mathcal{B}_{n,k}^T| = |\mathcal{M}_{n,k}| = |\mathcal{O}_n|.
\]

This equality and the containment \( I_{n,k} \subseteq T(Y_{n,k}) \) imply that we have the equality of ideals

\[
I_{n,k} = T(Y_{n,k}).
\]

Consequently, we have

\[
\text{dim}(R_{n,k}) = |\mathcal{B}_{n,k}^T| = |\mathcal{M}_{n,k}| = |\mathcal{O}_n|.
\]

Moreover, we have

\[
\text{Hilb}(R_{n,k}; q) = \text{Hilb}(\mathbb{Q}[x_n]/I_{n,k}; q) = \text{Hilb}(\mathbb{Q}[x_n]/T(Y_{n,k}); q) = \sum_{m \in \mathcal{B}_{n,k}^T} q^{\deg(m)} = \sum_{\sigma \in \mathcal{O}_n} q^{\text{coinv}(\sigma)}
\]

The proof is complete. \( \Box \)

We are also in a position to identify the ungraded Frobenius character of \( R_{n,k} \).
Theorem 4.11. Let $k \leq n$ be positive integers. As ungraded $\mathfrak{S}_n$-modules, we have

$$R_{n,k} \cong \mathfrak{S}_n \mathbb{Q}[OP_{n,k}].$$

Equivalently, we have the ungraded Frobenius character

$$\text{Frob}(R_{n,k}) = \sum_{\lambda \vdash n} \binom{k}{m_1(\lambda), \ldots, m_k(\lambda)} h_\lambda(x),$$

where $\binom{k}{m_1(\lambda), \ldots, m_k(\lambda)}$ is the multinomial coefficient.

Proof. We know that $Q[\mathfrak{x}_n] \cong Q[\mathfrak{S}_n] \cong \mathbb{Q}[OP_{n,k}]$ as ungraded $\mathfrak{S}_n$-representations. The proof of Theorem 4.10 shows $T(Y_{n,k}) = I_{n,k}$, so that $R_{n,k} = \frac{Q[\mathfrak{x}_n]}{I_{n,k}} = \frac{Q[\mathfrak{x}_n]}{Y_{n,k}}$. □

Recall that an $(n,k)$-staircase is a shuffle of the sequences $(0, 1, \ldots, k-1)$ and $(k-1, \ldots, 1)$, where there are $n-k$ copies of $k-1$ in the second sequence.

Definition 4.12. The $(n,k)$-Artin monomials $A_{n,k}$ are those monomials in $\mathbb{Q}[\mathfrak{x}_n]$ whose exponent vectors fit under at least one $(n,k)$-staircase.

In particular, the set $A_{n,n}$ consists of the usual Artin monomials.

Theorem 4.13. Let $k \leq n$ be positive integers. We have $A_{n,k} = M_{n,k}$. Moreover, the set $A_{n,k}$ descends to a monomial basis of $R_{n,k}$.

Proof. The proof of Theorem 4.10 implies that $M_{n,k}$ is the standard monomial basis for $R_{n,k}$ with respect to lexicographic order. Therefore, we need only show that $A_{n,k} \subseteq M_{n,k}$.

If $(a_1, \ldots, a_n)$ is any $(n,k)$-staircase, a direct check shows that the monomial $x_1^{a_1} \cdots x_n^{a_n} \in \mathbb{Q}[\mathfrak{x}_n]$ is $(n,k)$-nonskip. Therefore, we have $A_{n,k} \subseteq M_{n,k}$.

To verify the reverse containment, we show that the bijection $\Psi : OP_{n,k} \to M_{n,k}$ of Theorem 4.10 satisfies $\Psi(\mathbb{Q}[OP_{n,k}]) \subseteq A_{n,k}$. Let $\sigma \in OP_{n,k}$ and let $\overline{\sigma}$ be the ordered set partition of size $n - 1$ obtained by removing $n$ from $\sigma$.

Case 1: $\{n\}$ is not a singleton block of $\sigma$.

In this case $\overline{\sigma} \in OP_{n-1,k}$. We may inductively assume that $\Psi(\overline{\sigma}) \in A_{n-1,k}$. Therefore, there exists an $(n-1,k)$-staircase $(a_1, \ldots, a_{n-1})$ such that $\Psi(\overline{\sigma}) \mid x_1^{a_1} \cdots x_{n-1}^{a_{n-1}}$. We have $\Psi(\sigma) = \Psi(\overline{\sigma}) \cdot x_n^i$ where $0 \leq i \leq k-1$. Since $(a_1, \ldots, a_{n-1}, k-1)$ is an $(n,k)$-staircase and $\Psi(\sigma) \mid x_1^{a_1} \cdots x_n^{a_n} x_n^{k-1}$, we have $\Psi(\sigma) \in A_{n,k}$.

Case 2: $\{n\}$ is a singleton block of $\sigma$.

In this case $\overline{\sigma} \in OP_{n-1,k-1}$. We have $\Psi(\sigma) = \Psi(\overline{\sigma}) \cdot x_n^i$, where $S \subseteq [n-1]$ satisfies $|S| = n - k$, $0 \leq i \leq k-1$, and $x(S) \mid (\Psi(\overline{\sigma}) \cdot x_n(S))$. Consider the $(n,k)$-staircase $(a_1, \ldots, a_n)$ defined by $a_j = k - 1$ if $j \in S$ or $j = n$.

We claim $\Psi(\sigma) \mid x_1^{a_1} \cdots x_n^{a_n}$, so that $\Psi(\sigma) \in A_{n,k}$. To see this, write $\Psi(\sigma) = x_1^{b_1} \cdots x_n^{b_n}$. Since $\Psi(\sigma) \in M_{n,k}$ we know $0 \leq b_j \leq k-1$ for all $j$, so that $b_j \leq a_j$ if $j \in S$ or $j = n$. If $j \in [n-1] - S$ is such that $b_j > a_j$, the fact that $x(S) \mid \Psi(\sigma)$ would imply that $x(S \cup \{j\}) \mid \Psi(\sigma)$, contradicting the fact that $\Psi(\sigma) \in M_{n,k}$. We conclude that $\Psi(\sigma) \mid x_1^{a_1} \cdots x_n^{a_n}$, as desired. □

As an example of Theorem 4.13 consider the case $(n,k) = (3,2)$. The $(3,2)$-staircases are the shuffles of the sequences $(0,1)$ and $(1)$, i.e. the sequences $(0,1,1)$ and $(1,0,1)$. The $(3,2)$-Artin monomials are therefore

$$A_{3,2} = \{x_2x_3, x_1x_3, x_1, x_2, x_3, 1\}.$$
This is precisely the set $\mathcal{M}_{3,2}$ of monomials in $\mathbb{Q}[x_3]$ which are divisible by none of the monomials in the list

$$x_1^2, x_2^2, x_3^2, x(12) = x_1x_2, x(13) = x_1^2x_3^2, x(23) = x_2^2x_3^2.$$  

We conclude that $\mathcal{A}_{3,2} = \mathcal{M}_{3,2}$ descends to a basis of $R_{3,2}$, yielding the Hilbert series

$$\text{Hilb}(R_{3,2}; q) = 2q^2 + 3q + 1.$$  

This agrees with the coinv distribution on $\mathcal{OP}_{3,2}$.

| $\sigma$ | coinv |
|---------|-------|
| (12 | 3) | 2 |
| (1 | 23) | 2 |
| (13 | 2) | 1 |
| (2 | 13) | 1 |
| (3 | 12) | 1 |
| (23 | 1) | 0 |

We can also determine the reduced Gröbner basis of $I_{n,k}$.

**Theorem 4.14.** Let $k \leq n$ be positive integers and let $<$ be the lexicographic monomial order. A Gröbner basis for $I_{n,k}$ with respect to $<$ is given by the variable powers

$$x_1^k, x_2^k, \ldots, x_n^k$$

together with the reverse Demazure characters

$$\kappa_{\gamma}(S)^*(x_n^*)$$

for $S \subseteq [n-1]$ satisfying $|S| = n - k + 1$.

If $k < n$, this Gröbner basis is the reduced Gröbner basis for $I_{n,k}$ with respect to $<$.

When $k = n$, the reduced Gröbner basis for the classical invariant ideal $I_{n,n}$ is $\{h_i(x_i, x_{i+1}, \ldots, x_n) : 1 \leq i \leq n\}$ (see [5] Sec. 7.2). Since $h_i(x_i, x_{i+1}, \ldots, x_n) = \kappa_{(0, \ldots, 0, i, 0, \ldots, 0)}^*(x_n^*)$ (where the $i$ in the vector $(0, \ldots, 0, i, 0, \ldots, 0)$ is in position $n-i$), the last sentence of Theorem 4.14 is almost correct for $k = n$; one simply needs to throw out the variable powers $x_1^n, \ldots, x_{n-1}^n$.

**Proof.** By Lemma 3.4 (and Equation 3.5) we know that the polynomials listed lie in $I_{n,k}$. Lemma 3.5 and Theorem 4.13 tell us that the number of monomials in $\mathbb{Q}[x_n]$ which do not divide the leading terms of any of the polynomials listed in the statement equals $\text{dim}(R_{n,k})$. It follows that the polynomials listed in the statement form a Gröbner basis for $I_{n,k}$.

When $k < n$, Lemma 3.5 implies that, for any distinct polynomials $f, g$ listed in the statement, the monomial $\text{in}_{\leq}(f)$ has coefficient 1 and does not divide any monomial in $g$. This implies the claim about reducedness. \qed

For example, consider the case $(n, k) = (6, 4)$. The reduced Gröbner basis of $I_{6,4} \subseteq \mathbb{Q}[x_6]$ is given by the variable powers

$$x_1^4, x_2^4, x_3^4, x_4^4, x_5^4, x_6^4$$

together with the reverse Demazure characters

$$\kappa_{(0,0,0,1,1,1)}(x_6^*), \kappa_{(0,0,2,0,1,1)}(x_6^*), \kappa_{(0,3,0,0,1,1)}(x_6^*), \kappa_{(0,3,0,2,0,1)}(x_6^*), \kappa_{(0,3,3,0,0,1)}(x_6^*),$$

$$\kappa_{(0,3,3,0,1,1)}(x_6^*), \kappa_{(0,3,3,0,2,0)}(x_6^*), \kappa_{(0,3,3,3,0,0)}(x_6^*).$$

The authors find Theorem 4.14 mysterious and know of no conceptual reason to expect Demazure characters to appear as Gröbner basis elements of $I_{n,k}$.

5. **Generalized Garsia-Stanton basis**

5.1. **The classical Garsia-Stanton basis for $R_n$.** The Lehmer code of a permutation $\pi = \pi_1 \ldots \pi_n \in \mathfrak{S}_n$ is the word $c(\pi) = c(\pi)_1 \ldots c(\pi)_n$, where $c(\pi)_i = |\{j < i : \pi_i < \pi_j\}|$. The map $\pi \mapsto c(\pi)$ provides a bijection from $\mathfrak{S}_n$ to the set of words which are componentwise less than $(0, 1, \ldots, n-1)$. The classical Artin monomial basis of $R_n$ therefore witnesses the fact that

$$\text{Hilb}(R_n; q) = \sum_{\pi \in \mathfrak{S}_n} q^{c(\pi)_1 + \cdots + c(\pi)_n} = \sum_{\omega \in \mathfrak{S}_n} q^{\text{coinv}(\pi)}.$$
Just as the Artin basis for $R_n$ is related to the Lehmer code sum or inv statistic on $S_n$, the Garsia-Stanton basis for $R_n$ is related to the major index statistic maj on $S_n$. Given a permutation $\pi = \pi_1 \cdots \pi_n \in S_n$, the Garsia-Stanton monomial $gs_\pi$ is the product
\begin{equation}
    gs_\pi := \prod_{i \in \text{Des}(\pi)} x_{\pi_1, x_{\pi_2}, \cdots, x_{\pi_i}} \in \mathbb{Q}[x_i].
\end{equation}
For example, if $\pi = 34256187$ we have
\[
gs_{\pi} = (x_3 x_1) \cdot (x_3 x_4 x_2 x_5 x_6) \cdot (x_3 x_4 x_2 x_5 x_6 x_1 x_8) \in \mathbb{Q}[x_i].
\]
It is clear that $\deg(gs_\pi) = \text{maj}(\pi)$ for any permutation $\pi \in S_n$.

Garsia used Stanley-Reisner theory to show that $\{gs_\pi : \pi \in S_n\}$ descends to a basis of $R_n$ [10]; this basis was then studied by Garsia and Stanton in the context of invariant theory [13]. Adin, Brenti, and Roichman later gave a different proof of this fact using a straightening argument [1]. This witnesses the fact that $\text{Hilb}(R_n; q) = \sum_{\pi \in S_n} q^{\text{maj}(\pi)}$. The GS monomials were also studied by E. E. Allen under the name ‘descent monomials’ [2].

5.2. The generalized Garsia-Stanton basis for $R_{n,k}$. Remmel and Wilson proved that inv and maj share the same distribution on $\mathcal{OP}_{n,k}$ [27]. Taking complementary statistics, we get
\[
\sum_{\sigma \in \mathcal{OP}_{n,k}} q^{\text{comaj}(\sigma)} = \sum_{\sigma \in \mathcal{OP}_{n,k}} q^{\text{maj}(\sigma)}.
\]

Theorem 4.13 shows that the $(n, k)$-Artin basis $A_{n,k}$ of $R_{n,k}$ witnesses the fact that the left hand side is $\text{Hilb}(R_{n,k}; q)$. It is natural to ask for a generalized Garsia-Stanton basis which witnesses the fact that the right hand side is also $\text{Hilb}(R_{n,k}; q)$. We will provide such a basis.

To start, we will recast the right hand side in a more illuminating form.

**Lemma 5.1.** We have
\begin{equation}
    \sum_{\sigma \in \mathcal{OP}_{n,k}} q^{\text{comaj}(\sigma)} = \sum_{\pi \in S_n} q^{\text{maj}(\pi)} \cdot \left[\binom{\text{asc}(\pi)}{n-k} q\right].
\end{equation}

**Proof.** We consider the model for ordered set partitions in $\mathcal{OP}_{n,k}$ as ascent starred permutations $(\pi, S)$, where $\pi = \pi_1 \cdots \pi_n \in S_n$ and $S$ is a set of ascents of $w$ with $|S| = n - k$. For a fixed permutation $\pi$, there are $\binom{\text{asc}(\pi)}{n-k}$ choices for the set of stars $S$. For example, if $(n, k) = (6, 4)$ and $\pi = 245136 \in S_6$ we have $n - k = 2$ and the $\binom{6}{2}$ elements of $\mathcal{OP}_{6,4}$ shown below, together with the values of maj and comaj.

| $\sigma$ | 2,4,5 1 3 6 | 2,4 5 1 3,6 | 2,4 5 1 3,6 | 2,4 5 1 3,6 | 2,4 5 1 3,6 |
|-----------|-------------|-------------|-------------|-------------|-------------|
| maj       | 5           | 6           | 7           | 5           | 4           |
| comaj     | 7           | 6           | 8           | 5           | 3           |

We claim that, for $\pi \in S_n$ fixed, we have
\begin{equation}
    \sum_{S \subseteq \text{Asc}(\pi), |S|=n-k} q^{\text{comaj}(\pi, S)} = q^{\text{maj}(\pi)} \cdot \left[\binom{\text{asc}(\pi)}{n-k} q\right].
\end{equation}

Indeed, the standard combinatorial interpretation for $\left[\binom{\text{asc}(\pi)}{n-k} q\right]$ is as the generating function for size on partitions $\lambda$ fitting inside a box of dimensions $(n - k) \times (\text{asc}(\pi) - n + k)$. For $\pi$ fixed, the choice of $\lambda$ corresponds to the choice of $S \subseteq \text{Asc}(\pi)$ in the ordered set partition $(\pi, S)$. The definition of comaj yields the factor of $q^{\text{maj}(\pi)}$.

We introduce the following generalization of the GS monomials.
Definition 5.2. Let \( k \leq n \) be positive integers. The \((n,k)\)-Garsia-Stanton monomials \( \mathcal{GS}_{n,k} \) are given by

\[
\mathcal{GS}_{n,k} := \{ gs_{\pi} \cdot x_{\pi_1}^{i_1} x_{\pi_2}^{i_2} \cdots x_{\pi_{n-k}}^{i_{n-k}} : \pi \in \mathfrak{S}_n, \text{des}(\pi) < k, \text{ and } (k-\text{des}(\pi)) > i_1 \geq i_2 \geq \cdots \geq i_{n-k} \geq 0 \}.
\]

When \( k = n \) we recover the usual GS monomials.

As an example of these monomials, take \((n,k) = (6,3)\) and consider the permutation \( \pi = \lambda \in \mathfrak{S}_6 \). We have \( n-k = 3, k-\text{des}(\pi) = 2 \) and \( gs_{\pi} = x_3 x_5 x_6 \). This gives rise to the following four elements of \( \mathcal{GS}_{6,3} \):

\[
(x_3 x_5 x_6) \cdot 1, \quad (x_3 x_5 x_6) \cdot (x_3), \quad (x_3 x_5 x_6) \cdot (x_3 x_5), \quad (x_3 x_5 x_6) \cdot (x_3 x_5 x_6).
\]

In general, if \( \pi \in \mathfrak{S}_n \) satisfies \( \text{des}(\pi) < k \), we have

\[
(5.4) \quad \sum_{\text{des}(\pi) > i_1 \geq i_2 \geq \cdots \geq i_{n-k} \geq 0} q^{\deg(gs_{\pi} \cdot x_{\pi_1}^{i_1} x_{\pi_2}^{i_2} \cdots x_{\pi_{n-k}}^{i_{n-k}})} = q^{\text{maj}(\pi)} \cdot \left[ (k-\text{des}(\pi) - 1) + (n-k) \right]_{n-k} q
\]

\[
(5.5) \quad = q^{\text{maj}(\pi)} \cdot \left[ n - \text{des}(\pi) - 1 \right]_{n-k} q
\]

\[
(5.6) \quad = q^{\text{maj}(\pi)} \cdot \left[ \text{asc}(\pi) \right]_{n-k} q,
\]

which agrees with the summand corresponding to \( \pi \) in Lemma 5.1. In particular, we have \( \vert \mathcal{GS}_{n,k} \vert \leq \vert \mathcal{OP}_{n,k} \vert \). We cannot yet assert equality because two \((n,k)\)-GS monomials could a priori coincide for different choices of \( \pi \). The following theorem guarantees that this does not actually happen.

Theorem 5.3. Let \( k \leq n \) be positive integers. The set \( \mathcal{GS}_{n,k} \) of \((n,k)\)-Garsia-Stanton monomials descends to a basis for \( R_{n,k} \).

Proof. The strategy for proving \( \mathcal{GS}_{n,k} \) is a basis of \( R_{n,k} \) is to apply the straightening algorithm of Adin, Brenti, and Roichman [1]. To describe this algorithm, we will need a partial order on monomials in \( \mathbb{Q}[x_n] \).

Given any monomial \( m = x_1^{a_1} \cdots x_n^{a_n} \in \mathbb{Q}[x_n] \), let \( \lambda(m) = \text{sort}(a_1, \ldots, a_n) \) be the partition obtained by sorting the exponent sequence of \( m \) into weakly decreasing order. Let \( \pi(m) = \pi_1 \cdots \pi_n \in \mathfrak{S}_n \) be the permutation obtained by listing the indices of variables in decreasing order of exponents in \( m \), breaking ties by choosing smaller indexed variables first. For example, if \( m = x_1^3 x_2^2 x_3^4 x_4^0 x_5^0 x_6^0 x_7^0 \), we have \( \lambda(m) = (4,3,2,2,0,0,0) \) and \( \pi(m) = 2145367 \). If \( m, m' \) are any two monomials in \( \mathbb{Q}[x_n] \), we say \( m \prec m' \) if \( \deg(m) = \deg(m') \) and

\begin{itemize}
  \item \( \lambda(m) < \lambda(m') \) in dominance order, or
  \item \( \lambda(m) = \lambda(m') \) but \( \text{inv}(\pi(m')) > \text{inv}(\pi(m)) \).
\end{itemize}

If \( \pi = \pi_1 \cdots \pi_n \) is any permutation in \( \mathfrak{S}_n \), let \( d(\pi) = (d_1, \ldots, d_n) \) be the length \( n \) vector given by

\[
d_j = |\text{Des}(\pi) \cap \{j, j+1, \ldots, n\}|.
\]

For example, we have \( d(2145367) = (2,1,1,1,0,0,0) \). We have \( d_1 = \text{des}(\pi) \). If \( m \) is a monomial in \( \mathbb{Q}[x_n] \), let \( d(m) := d(\pi(m)) \). It is shown in [1] that the componentwise difference \( \lambda(m) - d(m) \) is an integer partition (i.e., has weakly decreasing components). Let \( \mu(m) \) be the conjugate of this difference: \( \mu(m)' = \lambda(m) - d(m) \). In our example, we have \( \mu(m)' = (2,2,1,1,0,0,0) \) so that \( \mu(m) = (4,2) \).

Adin, Brenti, and Roichman derive the following straightening lemma [1] Lem. 3.5. If \( m \) is any monomial in \( \mathbb{Q}[x_n] \) we may write

\[
m = gs_{\pi(m)} \cdot e_{\mu(m)}(x_n) - \Sigma,
\]

where \( \Sigma \) is a linear combination of monomials \( m' \prec m \). This implies that the classical GS monomials span \( R_{n,n} \). Together with the fact that \( \dim(R_{n,n}) = n! \), we get that the classical GS monomials are a basis for \( R_{n,n} \).
We want to apply the straightening relation \([5.7]\) to prove that \(\mathcal{GS}_{n,k}\) spans \(R_{n,k}\). Since we know \(|\mathcal{GS}_{n,k}| \leq \dim(R_{n,k})\), this would prove that \(\mathcal{GS}_{n,k}\) is a basis of \(R_{n,k}\).

Consider a monomial \(m\) in \(\mathbb{Q}[x_n]\). We argue that the class \(m + I_{n,k}\) lies in the span of \(\mathcal{GS}_{n,k}\). All of the equivalences in the following paragraphs are modulo the ideal \(I_{n,k}\).

If \(m\) is minimal with respect to \(\prec\), then \(m\) must have the form \(m = x_1^0 \cdot x_r^1 \cdot x_{r+1}^1 \cdots x_n^1\) for some \(r\). If \(r = 1\), then \(m = e_n(x_n) \equiv 0\). If \(r > 1\), then \(m = gs_{r(r+1) \cdots n_{12} \cdots (r-1)}\), so that \(m + I_{n,k}\) lies in the span of \(\mathcal{GS}_{n,k}\).

By the last paragraph, we may inductively assume that if \(m'\) is any monomial such that \(m' \prec m\), the coset \(m' + I_{n,k}\) lies in the span of the monomials in \(\mathcal{GS}_{n,k}\). Apply the straightening relation \([5.7]\) to \(m\), yielding
\[
m = gs_{\pi(m)} \cdot e_{\mu(m)}(x_n) - \Sigma.
\]

By induction, the element \(\Sigma + I_{n,k}\) lies in the span of \(\mathcal{GS}_{n,k}\).

For any \(\pi \in \mathfrak{S}_n\), the definition of GS monomials implies that \(\text{des}(\pi)\) is the exponent of \(x_{\pi_1}\) in the GS monomial \(gs_{\pi}\). In particular, if \(\text{des}(\pi(m)) \geq k\), then \(gs_{\pi(m)} \cdot e_{\mu(m)}(x_n) \equiv gs_{\pi(m)} \equiv 0\) and \(m + I_{n,k}\) lies in the span of \(\mathcal{GS}_{n,k}\). If \(\mu(m)_1 > n - k\) then \(gs_{\pi(m)} \cdot e_{\mu(m)}(x_n) \equiv e_{\mu(m)}(x_n) \equiv e_{\mu(m)_1}(x_n) \equiv 0\), so that \(m + I_{n,k}\) lies in the span of \(\mathcal{GS}_{n,k}\).

We are therefore reduced to the situation where \(m\) is a monomial in \(\mathbb{Q}[x_n]\) with \(\text{des}(\pi(m)) < k\) and \(\mu(m)_1 \leq n - k\).

We claim that any such monomial \(m\) actually lies in \(\mathcal{GS}_{n,k}\), or satisfies \(m \equiv 0\). Indeed, write \(\pi(m) = \pi_1 \cdots \pi_n\), so that
\[
m = x_{\pi_1}^{\lambda(m)_1}x_{\pi_2}^{\lambda(m)_2} \cdots x_{\pi_n}^{\lambda(m)_n}.
\]

The GS monomial \(gs_{\pi(m)}\) is equal to
\[
gs_{\pi(m)} = x_{\pi_1}^{d_1(m)}x_{\pi_2}^{d_2(m)} \cdots x_{\pi_n}^{d_n(m)}.
\]

By the definition of \(\mu(m)\) we have
\[
m = gs_{\pi(m)} \cdot x_{\pi_1}^{\mu(m)_1}x_{\pi_2}^{\mu(m)_2} \cdots x_{\pi_n}^{\mu(m)_n}.
\]

Since \(\mu(m)_1 \leq n - k\), we know that
\[
m = gs_{\pi(m)} \cdot x_{\pi_1}^{\mu(m)_1}x_{\pi_2}^{\mu(m)_2} \cdots x_{\pi_{n-k}}^{\mu(m)_{n-k}},
\]
where \(\mu(m)_1 \geq \cdots \geq \mu(m)_{n-k} \geq 0\). As long as \(\mu(m)_1 \leq k - \text{des}(\pi) = k - d_1(m)\), we have \(m \in \mathcal{GS}_{n,k}\). If \(\mu(m)_1 > k - \text{des}(\pi)\), then \(x_{\pi_1}^{d_1(m)} \mid m\), so that \(m \equiv 0\). \(\square\)

The proof of Theorem \([5.3]\) is an extension of the proof of Adin, Brenti, and Roichman that the classical GS monomials form a basis for \(R_n\) \([1]\). It could be interesting to find an alternative proof of Theorem \([5.3]\) which extends the original poset-theoretic proof of Garsia \([10]\).

6. Frobenius series

6.1. The rings \(R_{n,k,s}\). The goal of this section is to prove our formula for the graded Frobenius character of \(R_{n,k}\). To do this, we will need a one-parameter extension \(R_{n,k,s}\) of these rings.

**Definition 6.1.** Let \(s \leq k \leq n\) be positive integers. Define \(I_{n,k,s} \subseteq \mathbb{Q}[x_n]\) to be the ideal
\[
I_{n,k,s} := \langle x_1^k, x_2^k, \ldots, x_n^k, e_n(x_n), e_{n-1}(x_n), \ldots, e_{n-s+1}(x_n) \rangle.
\]

Let \(R_{n,k,s} := \frac{\mathbb{Q}[x_n]}{I_{n,k,s}}\) be the corresponding quotient ring.
We have \( I_{n,k,1} \subseteq I_{n,k,2} \subseteq \cdots \subseteq I_{n,k,k} = I_{n,k} \), so that \( R_{n,k,k} = R_{n,k} \). The ideals \( I_{n,k,s} \) are homogeneous and stable under the action of \( \mathfrak{S}_n \), so that the \( R_{n,k,s} \) are graded \( \mathfrak{S}_n \)-modules.

We want to give a combinatorial model for the rings \( R_{n,k,s} \). To do so, for \( s \leq k \leq n \) define \( \mathcal{O}P_{n,k,s} \) to be the collection of \( k \)-block ordered set partitions \( \sigma = (B_1 | \cdots | B_k) \) of \( [n + (k - s)] \) such that, for \( 1 \leq i \leq k - s \), we have \( n + i \in B_{s+i} \). For example, we have

\[
(46 \mid 1 \mid 237 \mid 8 \mid 59) \in \mathcal{O}P_{6,5,2}.
\]

We will refer to the letters \( n+1, n+2, \ldots, n+k-s \) of \( \sigma \in \mathcal{O}P_{n,k,s} \) as \textit{big}; these are shown in bold above. The other letters \( 1,2,\ldots,n \) will be called \textit{small}.

The symmetric group \( \mathfrak{S}_n \) acts on \( \mathcal{O}P_{n,k,s} \) by permuting the small letters. We will identify this action with \( R_{n,k,s} \) in a manner completely analogous to the analysis in Section \ref{sec:3} we will be more brief this time. Let us model this action with a point set. Fix distinct rational numbers \( \alpha_1, \ldots, \alpha_k \).

**Definition 6.2.** Let \( Y_{n,k,s} \subseteq \mathbb{Q}^{n+k-s} \) be the set of points \( (y_1, \ldots, y_n, y_{n+1}, \ldots, y_{n+k-s}) \) such that

- \( \{y_1, \ldots, y_n, y_{n+1}, \ldots, y_{n+k-s}\} = \{\alpha_1, \ldots, \alpha_k\} \), and
- \( y_{n+i} = \alpha_{s+i} \) for all \( 1 \leq i \leq k - s \).

It is evident that the action of \( \mathfrak{S}_n \) on the small coordinates gives an identification of \( \mathfrak{S}_n \)-modules \( \mathbb{Q}[\mathcal{O}P_{n,k,s}] = \mathbb{Q}[Y_{n,k,s}] \).

Let \( I(Y_{n,k,s}) \) be the ideal of polynomials in \( \mathbb{Q}[x_{n+k-s}] \) which vanish on \( Y_{n,k,s} \) and let \( T(Y_{n,k,s}) \) be the corresponding top component ideal. Since \( x_{n+i} - \alpha_{n+i} \in I(Y_{n,k,s}) \) for all \( 1 \leq i \leq k - s \), we know that \( x_{n+i} \in T(Y_{n,k,s}) \). Let \( \zeta : \mathbb{Q}[x_{n+k-s}] \rightarrow \mathbb{Q}[x_n] \) be the evaluation map which sets \( \zeta(x_{n+i}) = 0 \) for all \( 1 \leq i \leq k - s \). Let \( T_{n,k,s} := \zeta(T(Y_{n,k,s})) \). It follows that \( T_{n,k,s} \subseteq \mathbb{Q}[x_n] \) is an ideal in \( \mathbb{Q}[x_n] \) and we have isomorphisms of \( \mathfrak{S}_n \)-modules

\[
\frac{\mathbb{Q}[\mathcal{O}P_{n,k,s}]}{I(Y_{n,k,s})} \cong \frac{\mathbb{Q}[x_{n+k-s}]}{I(Y_{n,k,s})} \cong \frac{\mathbb{Q}[x_{n+k-s}]}{T(Y_{n,k,s})} \cong \frac{\mathbb{Q}[x_n]}{T_{n,k,s}}.
\]

We prove a generalization of Lemma \ref{lem:4.2} to the ideals \( I_{n,k,s} \).

**Lemma 6.3.** Let \( s \leq k \leq n \). We have \( I_{n,k,s} \subseteq T_{n,k,s} \).

**Proof.** We show that every generator of \( I_{n,k,s} \) lies in \( T_{n,k,s} \).

For \( 1 \leq i \leq n \), we have \( (x_i - \alpha_1) \cdots (x_i - \alpha_k) \in I(Y_{n,k,s}) \), so that \( x_i^k \in T_{n,k,s} \).

Lemma \ref{lem:3.1} implies that \( e_r(x_{n+k-s}) \in T(Y_{n,k,s}) \) for all \( r \geq n - s + 1 \). Applying the evaluation map \( \zeta \) gives \( \zeta : e_r(x_{n+k-s}) \mapsto e_r(x_n) \in T_{n,k,s} \).

Next, we prove that certain reverse Demazure characters are contained in \( I_{n,k,s} \). These polynomials will ultimately be members of a Gröbner basis of \( I_{n,k,s} \).

**Lemma 6.4.** Suppose \( S \subseteq [n] \) satisfies \( |S| = n - s + 1 \). The reverse Demazure character \( \kappa_{\gamma(S)}(x_n) \) lies in \( I_{n,k,s} \).

**Proof.** Apply Lemma \ref{lem:3.4} and Equation \ref{eq:3.5}.

Let \( \mathcal{M}_{n,k,s} \) be the collection of monomials \( m \in \mathbb{Q}[x_n] \) satisfying the following two conditions:

1. We have \( x_i^k \nmid m \) for all \( 1 \leq i \leq n \), and
2. For any \( S \subseteq [n] \) with \( |S| = n - s + 1 \), we have \( x(S) \nmid m \).

The monomials in \( \mathcal{M}_{n,k,s} \) will turn out to be the standard monomial basis for the ring \( R_{n,k,s} \). When \( k = s \), we have \( \mathcal{M}_{n,k,k} = \mathcal{M}_{n,k} \). A more precise relationship between these two families of monomials is as follows.

**Lemma 6.5.** Let \( s \leq k \leq n \). If \( x_1^{a_1} \cdots x_n^{a_n} x_{n+1}^{a_{n+1}} \cdots x_{n+k-s}^{a_{n+k-s}} \in \mathcal{M}_{n+k-s,k} \), then \( x_1^{a_1} \cdots x_n^{a_n} \in \mathcal{M}_{n,k} \).

On the other hand, if \( x_1^{a_1} \cdots x_n^{a_n} \in \mathcal{M}_{n,k,s} \) and \( 0 \leq a_{n+1} < a_{n+2} < \cdots < a_{n+k-s} \leq k - 1 \), then \( x_1^{a_1} \cdots x_n^{a_n} x_{n+1}^{a_{n+1}} \cdots x_{n+k-s}^{a_{n+k-s}} \in \mathcal{M}_{n+k-s,k} \).
Proof. The first statement is clear from the definitions. For the second statement, suppose
\[ x_1^{a_1} \cdots x_n^{a_n} \in M_{n,k,s} \]
and
\[ 0 \leq a_{n+1} < a_{n+2} < \cdots < a_{n+k-s} \leq k-1. \]
We need to show
\[ m := x_1^{a_1} \cdots x_n^{a_n} x_{n+1}^{a_{n+1}} \cdots x_{n+k-s}^{a_{n+k-s}} \in M_{n+k-s,k}. \]
This amounts to showing that \( x(S) \not| m \) for any \( S \subseteq [n+k-s] \) with \( |S| = n+s-1 \). Certainly \( x(S) \not| m \) if \( S \subseteq [n] \). On the other hand, if \( x(S) \not| m \) and \( n+i \in S - [n] \), the exponent \( e_{n+i} \) of \( x_{n+i} \) in the skip monomial \( x(S) \) is \( \geq s+i \). However, the inequalities \( 0 \leq a_{n+1} < a_{n+2} < \cdots < a_{n+k-s} \leq k-1 \) and the divisibility \( x(S) | m \) force \( e_{n+i} \leq a_{n+i} < k - (s-i) \leq s+i \), which contradicts \( e_i \geq s+i \). ∎

We apply our bijection \( \Psi \) from Section 4 to show that the monomials in \( M_{n,k,s} \) are equinumerous with the ordered set partitions in \( OP_{n,k,s} \).

**Lemma 6.6.** For any \( s \leq k \leq n \) we have \( |M_{n,k,s}| = |OP_{n,k,s}| \).

**Proof.** Consider the bijection \( \Psi : OP_{n+k-s,k} \to M_{n+k-s,k} \) of Theorem 4.14. What is the image of \( OP_{n,k,s} \) under \( \Psi \)? Let \( M'_{n,k,s} \) be the set of monomials \( x_1^{a_1} \cdots x_{n+k-s}^{a_{n+k-s}} \) in \( M_{n+k-s,k} \) which satisfy
\[ (a_{n+1}, a_{n+2}, \ldots, a_{n+k-s}) = (s, s+1, \ldots, k-1). \]
It follows from the definition of \( \Psi \) that
\[ \Psi(\mathcal{OP}_{n,k,s}) = M'_{n,k,s}. \]
On the other hand, Lemma 6.5 guarantees that \( |M'_{n,k,s}| = |M_{n,k,s}| \).

We generalize Theorem 4.14 to get a Gröbner basis for the ideals \( I_{n,k,s} \).

**Lemma 6.7.** Let \( s \leq k \leq n \). A Gröbner basis for the ideal \( I_{n,k,s} \) with respect to \( \triangleq_{\text{lex}} \) consists of the variable powers
\[ x_1^k, \ldots, x_n^k \]
together with the reverse Demazure characters
\[ \{ \kappa_{\gamma(S)}(x_n^s) : S \subseteq [n], |S| = n-s+1 \}. \]
If \( s < k \), this is the reduced Gröbner basis for this term ordering.

In particular, we have \( \dim(R_{n,k,s}) = |\mathcal{OP}_{n,k,s}| \).

**Proof.** The polynomials in question lie in the ideal \( I_{n,k,s} \) by Lemma 6.4. By Lemma 6.6, the number of monomials which do not divide any leading terms of the polynomials listed here equals \( |\mathcal{OP}_{n,k,s}| \). By Lemma 6.3, we have \( \dim(R_{n,k,s}) \geq |\mathcal{OP}_{n,k,s}| \). This forces the set of polynomials here to be a Gröbner basis for \( I_{n,k,s} \). If \( s < k \), reducedness follows from an argument similar to the case of Theorem 4.14. □

Let us remark that the number \( \dim(R_{n,k,s}) = |\mathcal{OP}_{n,k,s}| \) has combinatorial significance. The collection \( \mathcal{OP}_{n,k,s} \) bijects with the collection of functions \( f : [n] \to [k] \) whose image contains \( [s] \). The number of such functions is
\[ \sum_{m=0}^n \binom{n}{m} s! \cdot \text{Stir}(m, s) \cdot (k-s)^{n-m}, \]
where \( m \) parametrizes the size of the preimage of \( [s] \) under \( f \). This formula makes sense (and gives the correct value \( k^n \) when \( s = 0 \) and we set \( I_{n,k,0} := (x_1^k, \ldots, x_n^k) \)). For general \( s \), the expression (6.1) is equal to the \( s \)th difference of the sequence \( k^n \). When \( s = k \), we recover the \( k \)th difference \( k! \cdot \text{Stir}(n, k) \). □

\[ \text{The authors thank Dennis Stanton for pointing this out.} \]
We have a surjection
\[
\frac{\mathbb{Q}[x_n]}{T_{n,k,s}} \to R_{n,k,s}
\]
and an identification of the \(S_n\)-module on the left hand side with \(\mathcal{O}P_{n,k,s}\). Lemma 6.7 tells us that these modules have the same dimension, so we have an isomorphism of ungraded \(S_n\)-modules
\[
R_{n,k,s} \cong \mathbb{Q}[\mathcal{O}P_{n,k,s}].
\]

6.2. **Antisymmetrization and \(\epsilon_j(x)^\perp\).** Consider the parabolic subgroup \(S_{n-j} \times S_j\) of \(S_n\) and let \(\epsilon_j\) be the antisymmetrization operator with respect to the last \(j\) variables. In other words, we have that \(\epsilon_j \in \mathbb{Q}[S_n]\) is the group algebra idempotent
\[
(6.3) \quad \epsilon_j := \frac{1}{j!} \sum_{\pi \in S_{n-j+1,\ldots,n}} \text{sign}(\pi) \cdot \pi.
\]

Let us consider the action of \(\epsilon_j\) on \(\mathcal{O}P_{n,k}\). Since \(\epsilon_j\) kills any \(\pi \in \mathcal{O}P_{n,k}\) with any of the \(j\) letters \(n-j+1, n-j+2, \ldots, n\) in the same block, we have
\[
(6.4) \quad \dim(\epsilon_j \mathbb{Q}[\mathcal{O}P_{n,k}]) = \binom{k}{j} \cdot |\mathcal{O}P_{n-j,k,k-j}|.
\]

Applying the isomorphism (6.2) of ungraded \(S_n\)-modules, we have
\[
(6.5) \quad \dim(\epsilon_j R_{n,k}) = \binom{k}{j} \cdot |\mathcal{O}P_{n-j,k,k-j}|.
\]

Our next goal is to bootstrap Equation (6.5) to a statement involving the rings \(R_{n-j,k,k-j}\).

If \(V\) is any \(S_j\)-module, recall that the space of alternants is
\[
(6.6) \quad \{v \in V : \pi.v = \text{sign}(\pi) \cdot v \text{ for all } \pi \in S_j\}.
\]

The symmetric group \(S_j\) acts on the quotient ring \(\mathbb{Q}[x_{n-j+1,\ldots,n}]_{\pi \in \mathcal{S}_j}\) by variable permutation; let \(A_{n,k,j}\) be the space of alternants for this module. The set
\[
(6.7) \quad \{\epsilon_j \cdot (x_{n-j+1}^{a_{n-j+1}} \cdots x_n^{a_n}) : 0 \leq a_{n-j+1} < \cdots < a_n \leq k-1\}
\]
descends to a basis for \(A_{n,k,j}\); it follows that
\[
(6.8) \quad \text{Hilb}(A_{n,k,j}) = q^{(j)} \left[ \begin{array}{c} k \\ j \end{array} \right]_q.
\]

Observe that \(\epsilon_j R_{n,k}\) is a graded \(S_{n-j}\)-module. The group \(S_{n-j}\) also acts on the first component of the tensor product \(R_{n-j,k,k-j} \otimes A_{n,j}\). The next result states that the natural multiplication map induces an isomorphism between these graded modules.

**Lemma 6.8.** As graded \(S_{n-j}\)-modules we have \(\epsilon_j R_{n,k} \cong R_{n-j,k,k-j} \otimes A_{n,k,j}\).

**Proof.** Consider the direct sum decomposition \(Q[z_j] = Q[z_j]^0 \oplus Q[z_j]^1 \oplus Q[z_j]^2 \oplus \cdots\) where \(Q[z_j]^d\) is the vector space of polynomials in \(z_j\) which are homogeneous of degree \(d\). For any \(d \geq 0\), define \(Q[z_j]^d := Q[z_j]^d \oplus Q[z_j]^{d+1} \oplus Q[z_j]^{d+2} \oplus \cdots\) and let \(\epsilon_j Q[z_j]^d\) be the image of \(Q[z_j]^d\) under \(\epsilon_j\).

We will use the spaces \(Q[z_j]^d\) to obtain descending filtrations of our modules of interest. For any \(d \geq 0\), define
\[
(6.9) \quad U_{\geq d} := \text{image of } Q[y_{n-j}] \otimes \epsilon_j Q[z_j]_{\geq d} \text{ in } Q[y_{n-j}] \otimes A_{n,k,j},
\]
\[
(6.10) \quad V_{\geq d} := \text{image of } Q[y_{n-j}] \otimes \epsilon_j Q[z_j]_{\geq d} \text{ in } R_{n-j,k,k-j} \otimes A_{n,k,j},
\]
\[
(6.11) \quad W_{\geq d} := \text{image of } Q[y_{n-j}] \cdot \epsilon_j Q[z_j]_{\geq d} \text{ in } \epsilon_j R_{n,k}.
\]
Here we have suppressed dependence on \(n, k, j\) to reduce notational clutter. Each of the spaces \(U_{\geq d}, V_{\geq d}\), and \(W_{\geq d}\) is closed under the action of \(\mathfrak{S}_{n-j}\) on the \(y\)-variables and is a \(\mathbb{Q}[y_{n-j}]\)-module. For any \(d \geq 0\), set
\[
(6.12) \quad U_d := U_{\geq d}/U_{\geq d+1}, \quad V_d := V_{\geq d}/V_{\geq d+1}, \quad W_d := W_{\geq d}/W_{\geq d+1}.
\]

Fix \(d \geq 0\) and consider the multiplication map \(\tilde{\mu}_d : U_d \to W_d\) induced by \(f(y_{n-j}) \otimes g(z_j) \mapsto f(y_{n-j}) \cdot g(z_j)\). The map \(\tilde{\mu}_d\) is both a graded map of \(\mathfrak{S}_{n-j}\)-modules and a \(\mathbb{Q}[y_{n-j}]\)-module homomorphism.

For any \(r > n - k\) and any \(g(z_j) \in \epsilon_j \mathbb{Q}[z_j]_{\geq d}\) we have
\[
(6.13) \quad \tilde{\mu}_d : e_r(y_{n-j}) \otimes g(z_j) \mapsto e_r(y_{n-j})g(z_j) = \sum_{a+b=r} e_a(y_{n-j})e_b(z_j)g(z_j) = e_r(x_n)g(z_j) \in \epsilon_j I_{n,k}.
\]
The first equality uses the fact that \(W_{\geq d+1} = 0\) inside \(W_d\). For \(1 \leq i \leq n - j\) one also has
\[
(6.14) \quad \tilde{\mu}_d : y_i^k \otimes g(z_j) \mapsto y_i^k g(z_j) \in \epsilon_j I_{n,k}.
\]

Since \(\tilde{\mu}_d\) is a \(\mathbb{Q}[y_{n-j}]\)-module homomorphism we have
\[
(6.15) \quad (I_{n-j,k,k-j} \otimes \epsilon_j A_{n,k,j}) \cap U_d \subseteq \ker(\tilde{\mu}_d).
\]
It follows that \(\tilde{\mu}_d\) induces a map \(\mu_d : V_d \to W_d\).

Since \(\tilde{\mu}_d(V_{d+1}) \subseteq W_{d+1}/W_{d+1} = 0\), the map \(\tilde{\mu}_d\) in turn induces a map \(\mu_d : V_d \to W_d\). Let
\[
(6.16) \quad \mu : \bigoplus_{d \geq 0} V_d \to \bigoplus_{d \geq 0} W_d
\]
be the direct sum of the maps \(\mu_d\).

We claim that \(\mu\) is an isomorphism of graded \(\mathfrak{S}_{n-j}\)-modules. The map \(\mu\) is a graded \(\mathfrak{S}_{n-j}\)-module map because it is a direct sum of such maps, so it suffices to verify that \(\mu\) is bijective. To do this, we demonstrate that \(\mu\) sends a basis to a basis.

Consider the subset of \(\epsilon_j R_{n,k}\) given by the set \(C_{n,k,j}\) of images under \(\epsilon_j\) of monomials \(m(x_n) = m(y_{n-j})m(z_j) \in M_{n,k}\) with the property that the exponent sequence in \(m(z_j)\) is strictly increasing. In other words, we have
\[
(6.17) \quad C_{n,k,j} := \left\{ \epsilon_j m(x_n) : m(x_n) = m(y_{n-j}) \cdot m(z_j) \in M_{n,k} \text{ and } m(z_j) = z_1^{a_1} \cdots z_j^{a_j} \text{ with } a_1 < \cdots < a_j \right\}.
\]

We claim that \(C_{n,k,j}\) is a basis of \(\epsilon_j R_{n,k}\). Since the elements of \(C_{n,k,j}\) are homogeneous in the \(z\)-variables, this will also show that \(C_{n,k,j}\) gives a basis of the codomain \(\bigoplus_{d \geq 0} W_d\) of \(\mu\).

Since \(M_{n,k}\) is a basis of \(R_{n,k}\), it is immediate that \(\{\epsilon_j m(x_n) : m(x_n) \in M_{n,k}\}\) spans \(\epsilon_j R_{n,k}\). Let \(m(x_n) = m(y_{n-j}) \cdot m(z_j) \in M_{n,k}\) with \(m(z_j) = z_1^{a_1} \cdots z_j^{a_j}\). If any of the numbers \((a_1, \ldots, a_j)\) coincide, then \(\epsilon_j m(x_n) = m(y_{n-j}) \cdot \epsilon_j m(z_j) = m(y_{n-j}) \cdot 0 = 0\). Moreover, if \(m(z_j)' = z_1^{a_1'} \cdots z_j^{a_j'}\) and \((a_1', \ldots, a_j')\) is any permutation of \((a_1, \ldots, a_j)\), we have \(\epsilon_j m(x_n) = \pm \epsilon_j (m(y_{n-j}) \cdot m(z_j)')\). It follows that \(C_{n,k,j}\) spans \(\epsilon_j R_{n,k}\).

To show that the set \(C_{n,k,j}\) actually is a basis of \(\epsilon_j R_{n,k}\), we perform a dimension count. In particular, Lemma 6.5 and Proposition 6.7 tell us that
\[
(6.18) \quad |C_{n,k,j}| = \binom{k}{j} \cdot |OP_{n-j,k,k-j}| = \dim(\epsilon_j R_{n,k}).
\]

Therefore, the set \(C_{n,k,j}\) is a basis for \(\epsilon_j R_{n,k}\) and induces a basis of \(\bigoplus_{d \geq 0} W_d\).

In order to get a basis for the domain \(\bigoplus_{d \geq 0} V_d\) of \(\mu\), consider the following set of simple tensors in \(\mathbb{Q}[x_n] = \mathbb{Q}[y_{n-j}] \otimes \mathbb{Q}[z_j]\):
\[
(6.19) \quad D_{n,k,j} := \left\{ m(y_{n-j}) \otimes \epsilon_j m(z_j) : m(y_{n-j}) \in M_{n-j,k,k-j} \text{ and } m(z_j) = z_1^{a_1} \cdots z_j^{a_j} \text{ with } 0 \leq a_1 < \cdots < a_j < k \right\}.
\]
Since $\mathcal{M}_{n-j,k,k-j}$ is the standard monomial basis of $R_{n-j,k,k-j}$ with respect to the lexicographical term ordering, this is a basis for the tensor product $R_{n-j,k,k-j} \otimes A_{n,k,j}$ Since the elements of $\mathcal{D}_{n,k,j}$ are homogeneous with respect to the $z$-variables, the set $\mathcal{D}_{n,k,j}$ yields a basis of $\bigoplus_{d \geq 0} V_d$.

By definition, the multiplication map $\mu$ carries the basis element $m(\mathbf{y}_{n-j}) \otimes \epsilon_j m(\mathbf{z}_j) \in \mathcal{D}_{n,k,j}$ to the corresponding basis element $\epsilon_j m(\mathbf{x}_n) \in \mathcal{C}_{n,k,j}$ where $m(\mathbf{x}_n) = m(\mathbf{y}_{n-j}) \cdot m(\mathbf{z}_j)$. This proves that $\mu : \bigoplus_{d \geq 0} V_d \rightarrow \bigoplus_{d \geq 0} W_d$ is an isomorphism of graded $\mathfrak{S}_{n-j}$-modules. By standard properties of filtrations we also have isomorphism of graded $\mathfrak{S}_{n-j}$-modules $\epsilon_j R_{n,k} \cong \bigoplus_{d \geq 0} V_d$ and $R_{n-j,k,k-j} \otimes A_{n,k,j} \cong \bigoplus_{d \geq 0} W_d$.

Since we ultimately want to determine the graded Frobenius image $\text{grFrob}(R_{n,k}; q)$, we need to relate the antisymmetrization operator $\epsilon_j$ to symmetric function theory. Let $V$ be any $\mathfrak{S}_n$-module. The image $\epsilon_j V$ is a subspace of $V$ and carries an action of the subgroup $\mathfrak{S}_{n-j}$ in the parabolic decomposition $\mathfrak{S}_{n-j} \times \mathfrak{S}_j$. We may therefore speak of the Frobenius image $\text{Frob}(\epsilon_j V) \in \Lambda_{n-j}$. The symmetric functions $\text{Frob}(\epsilon_j V)$ and $\text{Frob}(V)$ are related by

\begin{equation}
\text{Frob}(\epsilon_j V) = e_j(\mathbf{x})^\perp \text{Frob}(V).
\end{equation}

Equation [6.20] was used extensively by Garsia and Procesi in their study of the cohomology of Springer fibers [12]. The fastest way to prove Equation [6.20] is to use Frobenius reciprocity. If $V$ is a graded $\mathfrak{S}_n$-module, we restate Equation [6.20] for emphasis:

\begin{equation}
\text{grFrob}(\epsilon_j V; q) = e_j(\mathbf{x})^\perp \text{grFrob}(V; q).
\end{equation}

We want to prove $\text{grFrob}(R_{n,k}; q) = D_{n,k}(\mathbf{x}; q)$. By Lemma [3.6] it is enough to show $e_j(\mathbf{x}_n)^\perp \text{grFrob}(R_{n,k}; q) = e_j(\mathbf{x}_n)^\perp D_{n,k}(\mathbf{x}; q)$ for all $j \geq 1$. Lemma [3.7] gives an expression for $e_j(\mathbf{x}_n)^\perp D_{n,k}(\mathbf{x}; q)$ in terms of smaller $D$-functions. We know that

\begin{align}
\text{grFrob}(R_{n,k}; q) &= \text{Hilb}(A_{n,k,j}; q) \cdot \text{grFrob}(R_{n-j,k,k-j}; q) \\
&= q^{l(j)} \left[ \begin{array}{c} k \\ j \end{array} \right]_q \cdot \text{grFrob}(R_{n-j,k,k-j}; q). \quad \text{(by Equation [6.21])}
\end{align}

If we want $\text{grFrob}(R_{n,k}; q)$ to satisfy the same recursion as in Lemma [3.7] we must have

\begin{equation}
\text{grFrob}(R_{n-j,k,k-j}; q) = \sum_{m = \max(1,k-j)}^{\min(k,n-j)} q^{(k-m)(n-j-m)} \left[ \begin{array}{c} j \\ k-m \end{array} \right]_q \text{grFrob}(R_{n-j,m}; q).
\end{equation}

6.3. A short exact sequence. The goal of the rest of this section is to prove Equation [6.25]. Its proof will rely on the following short exact sequence of $\mathfrak{S}_n$-modules.

**Lemma 6.9.** Let $s < k \leq n$. There is a short exact sequence

\begin{equation}
0 \rightarrow R_{n,k-1,s} \rightarrow R_{n,k,s} \rightarrow R_{n,k,s+1} \rightarrow 0
\end{equation}

of $\mathfrak{S}_n$-modules, where the first map is homogeneous of degree $n - s$ and the second map is homogeneous of degree 0.

**Equivalently, we have**

\begin{equation}
\text{grFrob}(R_{n,k,s}; q) = \text{grFrob}(R_{n,k,s+1}; q) + q^{n-s} \cdot \text{grFrob}(R_{n,k-1,s}; q).
\end{equation}

**Proof.** As we have an inclusion of ideals $I_{n,k,s} \subseteq I_{n,k,s+1}$, we take the second map to be the canonical projection

\begin{equation}
\pi : R_{n,k,s} \rightarrow R_{n,k,s+1} \rightarrow 0.
\end{equation}
To build the first map, consider the multiplication

$$\tilde{\phi} : \mathbb{Q}[x_n] \to R_{n,k,s}.$$

We claim that $\tilde{\phi}(I_{n,k-1,s}) = 0$. This amounts to showing that $\tilde{\phi}(x_1^{k-1}) = 0$ in $R_{n,k,s}$ for all $1 \leq i \leq n$. To ease notation, we will only handle the case $i = 1$ (the other cases are similar). We have (where all congruences are modulo $I_{n,k,s}$)

$$\tilde{\phi}(x_1^{k-1}) = x_1^{k-1} \cdot e_{n-s}(x_1, \ldots, x_n)$$

$$= x_1^k e_{n-s-1}(x_2, x_3, \ldots, x_n) + x_1^{k-1} e_{n-s}(x_2, x_3, \ldots, x_n)$$

$$\equiv x_1^{k-1} e_{n-s}(x_2, x_3, \ldots, x_n)$$

$$= -x_1^{k-2} e_{n-s+1}(x_1, x_2, \ldots, x_n) - x_1^{k-2} e_{n-s+1}(x_2, x_3, \ldots, x_n)$$

$$\equiv \pm x_1^{-s} e_{n-1}(x_2, \ldots, x_n) = \pm x_1^{k-s-1} e_n(x_1, \ldots, x_n) \equiv 0.$$

Therefore, the map $\tilde{\phi}$ induces a map

$$\phi : R_{n,k-1,s} \to R_{n,k,s}$$

of homogeneous degree $n - s$ which surjects onto the kernel of $\pi$.

By considering whether the final block of an element of $\mathcal{OP}_{n,k,s}$ is the singleton $\{n + k - s\}$, we get the identity

$$|\mathcal{OP}_{n,k,s}| = |\mathcal{OP}_{n,k,s-1}| + |\mathcal{OP}_{n,k-1,s}|.$$

By Lemma 6.7, this implies that

$$\dim(R_{n,k,s}) = \dim(R_{n,k,s-1}) + \dim(R_{n,k-1,s}).$$

This forces the complex

$$0 \to R_{n,k-1,s} \xrightarrow{\phi} R_{n,k,s} \xrightarrow{\pi} R_{n,k,s+1} \to 0$$

to be exact. To finish the proof, simply observe that the maps $\phi$ and $\pi$ commute with the action of $\mathfrak{S}_n$. 

We are ready to prove Equation 6.25. The proof will rely on the short exact sequence of Lemma 6.9 and the $q$-analog of the Pascal’s Triangle recursion.

**Lemma 6.10.** Let $1 \leq j \leq n$. We have

$$\text{grFrob}(R_{n-j,k,k-j}; q) = \sum_{m=\max(1,k-j)}^{\min(k,n-j)} q^{(k-m)-(n-j-m)} \binom{j}{k-m}_q \text{grFrob}(R_{n-j,m}; q).$$

**Proof.** Reindexing our variables, this is equivalent to

$$\text{grFrob}(R_{n,k,s}; q) = \sum_{m=0}^{k-s} q^{m(n-k+m)} \binom{k-s}{m}_q \text{grFrob}(R_{n,k-m}; q),$$

for any $1 \leq s \leq k \leq n$, where we adopt the convention that $\text{grFrob}(R_{n',k'; q}) = 0$ if $k' > n'$. If $k = s$, we have $R_{n,k,s} = R_{n,k,k} = R_{n,k}$ and we are done. To prove the general case, we use the $q$-Pascal recursion.

Suppose $s < k$ and let $E_{n,k,s}$ be the right hand side of Equation 6.40. Then $E_{n,k,s+1} + q^{n-s} \cdot E_{n,k-1,s}$ is the expression
where we changed the dummy variable from $m$ to $m'$ in the second summation. Grouping like terms gives

$$
\sum_{m=0}^{k-s-1} q^{m(n-k+m)} \begin{bmatrix} [k-s-1] \\ m \end{bmatrix}_q \text{grFrob}(R_{n,k-m}; q) + q^{n-s} \sum_{m'=0}^{k-s-1} q^{m'(n-k+m'-1)} \begin{bmatrix} [k-s-1] \\ m' \end{bmatrix}_q \text{grFrob}(R_{n,k-m'-1}; q),
$$

and the third equality used the $q$-Pascal recursion.

This proves that

$$
E_{n,k,s} = E_{n,k,s+1} + q^{n-s} \cdot E_{n,k-1,s}.
$$

Applying Lemma 6.9 we see that the left hand side of Equation 6.40 satisfies the same recursion, finishing the proof.

We are finally ready to prove $\text{grFrob}(R_{n,k}; q) = D_{n,k}(x; q)$. With the machinery we have so far, this is just a chain of lemma invocations.

**Theorem 6.11.** The graded Frobenius character of the module $R_{n,k}$ is given by

$$
\text{grFrob}(R_{n,k}; q) = D_{n,k}(x; q).
$$

**Proof.** Let $j \geq 1$. By Lemma 3.6 it is enough to show $e_j(x)^\perp \text{grFrob}(R_{n,k}; q) = e_j(x)^\perp D_{n,k}(x; q)$. We inductively assume that this identity holds for all $n' < n$.

We have that

$$
(6.46) \quad e_j(x)^\perp \text{grFrob}(R_{n,k}; q) = \text{grFrob}(e_j R_{n,k}; q)
$$

$$
(6.47) \quad = \text{Hilb}(A_{n,k,j}; q) \cdot \text{grFrob}(R_{n-j,k,k-j}; q)
$$

$$
(6.48) \quad = q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q \text{grFrob}(R_{n-j,k,k-j}; q)
$$

$$
(6.49) \quad = q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q \sum_{m=\max(1,k-j)}^{\min(k,n-j)} q^{(k-m)\cdot(n-j-m)} \begin{bmatrix} j \\ k-m \end{bmatrix}_q \text{grFrob}(R_{n-j,m}; q)
$$

$$
(6.50) \quad = q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q \sum_{m=\max(1,k-j)}^{\min(k,n-j)} q^{(k-m)\cdot(n-j-m)} \begin{bmatrix} j \\ k-m \end{bmatrix}_q D_{n-j,m}(x; q)
$$

$$
(6.51) \quad = e_j(x)^\perp D_{n,k}(x; q).
$$
The first equality is the effect of $e_j(x)^{-1}$ on (graded) Frobenius characters. The second equality follows from Lemma 6.8. The third equality follows from Equation 6.54. The fourth equality follows from Lemma 6.10. The fifth equality uses induction. The final equality follows from Lemma 3.7. 

We therefore have the following combinatorial interpretation of $\text{grFrob}(R_{n,k}; q)$ in terms of ordered multiset partition statistics.

**Corollary 6.12.** The graded Frobenius character $\text{grFrob}(R_{n,k})$ is equal to any of the following four expressions, after applying $\text{rev}_q$ and $\omega$:

\begin{equation}
\sum_{|\gamma|=n} \sum_{\mu} q^{\text{inv}(\mu)} x^{\gamma} = \sum_{|\gamma|=n} \sum_{\mu} q^{\text{maj}(\mu)} x^{\gamma} = \sum_{|\gamma|=n} \sum_{\mu} q^{\text{inv}(\mu)} x^{\gamma} = \sum_{|\gamma|=n} \sum_{\mu} q^{\text{minmaj}(\mu)} x^{\gamma}.
\end{equation}

For example, consider $(n, k) = (3, 2)$. We have the inversion counts

| $\mu$ | inv($\mu$) | 1 | 2 | 3 |
|-------|-----------|---|---|---|
| 0     | 1         | 0 | 1 | 1 |

which implies

\begin{align*}
\sum_{|\gamma|=3} \sum_{\mu} q^{\text{inv}(\mu)} x^{\gamma} &= (1 + q)m_{(2,1)}(x) + (2 + 3q + q^2)m_{(1,1,1)}(x) \\
&= (1 + q)s_{(2,1)}(x) + (q + q^2)s_{(1,1,1)}(x).
\end{align*}

Applying $\text{rev}_q$ and $\omega$ we get

\begin{equation}
\text{grFrob}(R_{3,2}; q) = (q + q^2)s_{(2,1)}(x) + (1 + q)s_{(3)}(x).
\end{equation}

An explicit expansion of $\text{grFrob}(R_{n,k}; q)$ in the Schur basis follows from the work of A. T. Wilson [33].

**Corollary 6.13.** We have

\begin{equation}
\text{grFrob}(R_{n,k}; q) = \sum_{T \in \text{SYT}(n)} q^{\text{maj}(T)} \left[ n - \text{des}(T) - 1 \right]_{q} s_{\text{shape}(T)}(x).
\end{equation}

**Proof.** If we let $m = 0$ and take the coefficient of $u^{n-k}$ in [33 Thm. 5.0.1], and then apply $\omega$ and $\text{rev}_q$, we see that

\begin{equation}
\text{grFrob}(R_{n,k}; q) = \text{rev}_q \left[ \sum_{T \in \text{SYT}(n)} q^{\text{maj}(T) + \binom{n-k}{2} - \text{des}(T)} \left[ \text{des}(T) \right]_{n-k} s_{\text{shape}(T)}(x) \right].
\end{equation}

Since the maximum power of $q$ appearing in $\binom{\text{des}(T)}{n-k}$ is $(\text{des}(T) - n + k) \cdot (n - k)$ and the maximum power of $q$ occurring in $\text{grFrob}(R_{n,k}; q)$ is $(n - k) \cdot (k - 1) + \binom{k}{2}$, applying $\text{rev}_q$ to the right hand side of Equation 6.54 gives

\begin{equation}
\text{grFrob}(R_{n,k}; q) = \sum_{T \in \text{SYT}(n)} q^{\binom{n}{2} - \text{maj}(T)} \left[ \text{des}(T) \right]_{n-k} s_{\text{shape}(T)}(x).
\end{equation}

Since $\text{maj}(T') = \binom{n}{2} - \text{maj}(T)$ and $\text{des}(T') = n - \text{des}(T) - 1$, the corollary follows.

For example, consider $(n, k) = (4, 2)$. The tableaux with contribute to the right hand side of Equation 6.53 are those elements in SYT(4) with $\leq 1$ descent:

\begin{align*}
&\begin{array}{cccc}
1 & 2 & 3 & 4 \\
& 1 & 2 & 3 \\
& 4 & & \\
\end{array}, \\
&\begin{array}{cccc}
1 & 2 & 4 & 3 \\
& 1 & 2 & 4 \\
& 3 & & \\
\end{array}, \\
&\begin{array}{cccc}
1 & 3 & 4 & 2 \\
& 1 & 3 & 4 \\
& 2 & & \\
\end{array}, \\
&\begin{array}{cccc}
1 & 2 & & 3 \\
& 1 & 2 & \\
& & 3 & 4 \\
\end{array}.
\end{align*}
We get that \( grFrob(R_{4,2}; q) \) is equal to
\[
q^0 \left[ \frac{3}{2} \right] q \left[ \frac{2}{2} \right] s_{(4)}(x) + q^2 \left[ \frac{2}{2} \right] s_{(3,1)}(x) + q^4 \left[ \frac{2}{2} \right] s_{(2,2)}(x).
\]

It is well known that \( grFrob(R_n; q) = Q^\prime_{(1^n)}(x; q) \), where \( Q^\prime_\lambda(x; q) \) is the dual Hall-Littlewood symmetric function corresponding to \( \lambda \vdash n \). We have the following generalization of this identity to arbitrary \( k \leq n \).

**Theorem 6.14.** We have

\[
(6.56) \quad grFrob(R_{n,k}; q) = rev_q \left[ \sum_{\lambda \vdash n, \ell(\lambda) = k} q^{\sum(i-1) - (\lambda_i - 1)} \left[ \begin{array}{c} k \\ m_1(\lambda), \ldots, m_n(\lambda) \end{array} \right] Q^\prime_\lambda(x; q) \right].
\]

**Proof.** By Corollary 6.12 we have \( grFrob(R_{n,k}; q) = rev_q \omega C_{n,k}(x; q) \), so an equivalent formulation of (6.56) is

\[
(6.57) \quad \omega C_{n,k}(x; q) = \left[ \sum_{\lambda \vdash n, \ell(\lambda) = k} q^{\sum(i-1) - (\lambda_i - 1)} \left[ \begin{array}{c} k \\ m_1(\lambda), \ldots, m_n(\lambda) \end{array} \right] Q^\prime_\lambda(x; q) \right].
\]

Letting \( t = 0 \) in (2.16), we see a labelled Dyck path \( P \in \mathcal{LD}_n \) will contribute to \( C_{n,k}(x; q, 0) \) iff the underlying Dyck path is “balanced”, i.e. has the property that for every \( i \) satisfying \( a_i > 0 \), we also have \( a_{i-1} = a_i - 1 \). Such paths are in bijection with (strong) compositions \( \alpha \) of \( n \): the path \( p(\alpha) \) corresponding to \( \alpha \) consists of \( \alpha_1 \) north steps followed by \( \alpha_1 \) east steps, then \( \alpha_2 \) north steps followed by \( \alpha_2 \) east steps, etc.. Thus

\[
(6.58) \quad \omega C_{n,k}(x; q, 0) = \omega \sum_\alpha \sum_P q^{dinv(P)} x^P,
\]

where the inner sum is over all labellings of the balanced Dyck path \( p(\alpha) \).

By [13] Thm. 6.8, the inner sum from (6.58) can be expressed as

\[
(6.59) \quad \sum_\lambda q^{mindinv(p(\alpha))} K_{\lambda', \mu(\alpha)}(q) s_\lambda(x),
\]

where mindinv is the minimum value taken by dinv over all labellings of \( p(\alpha) \), \( \mu(\alpha) \) is the rearrangement of \( \alpha \) into non-increasing order, and \( K_{\beta, \mu}(q) \) is the coefficient of \( s_\beta \) in \( Q^\prime_\mu(x; q) \). It is noted in [14] p. 98-99 that this theorem follows from a result of Lascoux, Leclerc, and Thibon on LLT polynomials, and also that mindinv(\( \alpha \)) equals the number of triples \( u, v, w \) of north steps of \( \alpha \) where \( v \) is directly below \( u \), and \( w \) is in a row above \( v \) and in the same diagonal (i.e. line of slope 1) as \( v \). If \( i < j \) and \( \alpha_i > \alpha_j \), the squares in the columns corresponding to \( \alpha_i \) and \( \alpha_j \) will therefore contribute \( \min(\alpha_i, \alpha_j) \) to mindinv, while if \( i < j \) and \( \alpha_i \leq \alpha_j \), they will contribute \( \min(\alpha_i, \alpha_j) - 1 \) to mindinv. It follows easily that the contribution to the right-hand-side of (6.58) from the set of all balanced paths \( \alpha \) for which \( \mu(\alpha) \) equals a fixed partition \( \lambda \) is the \( q \)-multinomial coefficient occurring in the sum in (6.57), times \( Q^\prime_\lambda(x; q) \), times \( q^{mindinv(reverse(\lambda))} \). Furthermore

\[
\text{mindinv}(reverse(\lambda)) = \sum_i (i - 1)(\lambda_i - 1).
\]

□
For example, consider \((n, k) = (6, 3)\). The partitions \(\lambda \vdash 6\) with \(\ell(\lambda) = 3\) are \((4, 1, 1), (3, 2, 1),\) and \((2, 2, 2)\). Consequently, the graded Frobenius character of \(R_{6,3}\) is the \(q\)-reversal of

\[
q^0 \begin{bmatrix} 3 \\ 2, 1 \end{bmatrix}_q Q_{(4,1,1)}(\mathbf{x}; q) + q^1 \begin{bmatrix} 3 \\ 1, 1, 1 \end{bmatrix}_q Q_{(3,2,1)}(\mathbf{x}; q) + q^3 \begin{bmatrix} 3 \\ 3 \end{bmatrix}_q Q_{(2,2,2)}(\mathbf{x}; q).
\]

7. Conclusion

In this paper we studied a generalization \(R_{n,k}\) of the coinvariant algebra \(R_n\) attached to \(\mathfrak{S}_n\). The relevant ideal was a deformation \(I_{n,k} = (x_1^k, \ldots, x_{n-k}^k, e_{n-k+1}(\mathbf{x}_n), \ldots, e_n(\mathbf{x}_n))\) of the usual invariant ideal \(\langle Q[\mathbf{x}_n]\rangle = \langle e_1(\mathbf{x}_n), \ldots, e_n(\mathbf{x}_n)\rangle\).

However, the elementary symmetric functions are not the only interesting choice of generators for the ideal \(\langle Q[\mathbf{x}_n]\rangle\). For example, we have

\[
\langle Q[\mathbf{x}_n]\rangle = \langle h_1(\mathbf{x}_n), \ldots, h_n(\mathbf{x}_n)\rangle = \langle p_1(\mathbf{x}_n), \ldots, p_n(\mathbf{x}_n)\rangle,
\]

where \(h_d(\mathbf{x}_n)\) is a homogeneous symmetric function and \(p_d(\mathbf{x}_n) = x_1^d + \cdots + x_n^d\) is a power sum symmetric function. For \(k < n\), one can define the ideals \(I_{n,k}^h\) and \(I_{n,k}^p\) by replacing \(e\)'s in the definition of \(I_{n,k}\) with \(h\)'s or \(p\)'s.

It turns out that none of the ideals \(I_{n,k}, I_{n,k}^h, I_{n,k}^p\) coincide. We do not have a conjecture for the Hilbert series or Frobenius character of the quotient of \(Q[\mathbf{x}_n]\) by \(I_{n,k}^h\) or \(I_{n,k}^p\). Moreover, we do not have a conceptual understanding of why elementary symmetric functions give rise to ordered set partitions, but homogeneous or power sums do not. It would be interesting to recast our construction of \(I_{n,k}\) in such a way that makes the presence of elementary symmetric functions natural.

Our analysis of the quotient rings \(R_{n,k}\) used many similar methods to those found in the work of Garsia and Procesi \([12]\). In particular, Garsia and Procesi studied the quotient \(\frac{Q[\mathbf{x}_n]}{I_\lambda}\) of \(Q[\mathbf{x}_n]\) by the Tanisaki ideal \(I_\lambda\) for \(\lambda \vdash n\). They proved that

\[
\text{grFrob}\left(\frac{Q[\mathbf{x}_n]}{I_\lambda}\right) = Q_\lambda'(\mathbf{x}; q).
\]

If we compare Equations (6.56) and (7.1) we notice that \(R_{n,k}\) is isomorphic to a direct sum of modules of the form \(\frac{Q[\mathbf{x}_n]}{I_\lambda}\) for \(\ell(\lambda) = k\), with appropriate grading shifts.

**Problem 7.1.** Develop a relationship between the Tanisaki ideals \(I_\lambda\) and our ideal \(I_{n,k}\) (perhaps a sort of filtration) which proves Equation (6.56) algebraically.

In our paper we generalized two important bases of the ring \(R_n\) to the ring \(R_{n,k}\): the Artin basis and the Garsia-Stanton basis. The coinvariant algebra \(R_n\) has another important basis: the basis of Schubert polynomials \(\{\mathfrak{S}_\pi : \pi \text{ a permutation of } 1, 2, \ldots, n\}\). Although the Schubert polynomial \(\mathfrak{S}_\pi\) is almost never a monomial, it has a beautiful positive decomposition into monomials via the theory of pipe dreams. The structure constants of the Schubert basis of \(R_n\) are known to coincide with those for multiplication of Schubert classes inside the cohomology ring of the variety of complete flags in \(\mathbb{C}^n\).

**Problem 7.2.** Extend the Schubert basis to obtain a basis \(\{\mathfrak{S}_\sigma : \sigma \in \mathcal{OP}_{n,k}\}\) of \(R_{n,k}\) indexed by ordered set partitions. The polynomials \(\mathfrak{S}_\sigma\) should be a positive integer combination of monomials and they should have positive structure constants inside \(R_{n,k}\). Furthermore, they should represent classes of a basis of the cohomology ring \(H^\bullet(F_{n,k}; \mathbb{Z})\), where \(F_{n,k}\) is some \((n,k)\)-generalization of the flag variety.\(^6\)

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\(^6\)While this paper was under review, Pawlowski and Rhoades [25] solved Problem 7.2.
Let $W$ be a complex reflection group acting on $V = \mathbb{C}^n$. Then $W$ acts on the coordinate ring $\mathbb{C}[V]$ and the ideal $(\mathbb{C}[V]_W)$ of $W$-invariants with vanishing constant term is the $W$-analog of the classical invariant ideal for $\mathfrak{S}_n$. Chevalley proved that we have an identification of $W$-modules $\mathbb{C}[V]/(\mathbb{C}[V]_W) \cong \mathbb{C}[W]$ [8].

**Problem 7.3.** For $0 \leq k \leq n$, develop a $W$-generalization $I_{W,k} \subseteq \mathbb{C}[V]$ of our ideals $I_{n,k}$. When $W$ is well-generated, the action of $W$ on the quotient $\mathbb{C}[V]_{W,k}$ should give a graded version of the action of $W$ on the $k$-dimensional faces of the Coxeter complex $\Delta(W)$ attached to $W$.

When $W = G(r,1,n)$, Chan and Rhoades [7] have a solution to Problem 7.3.

The modules $R_{n,k}$ have a further refinement as follows. For a monomial $m \in \mathbb{Q}[x_n]$, let $\lambda(m)$ be the partition obtained by sorting the exponent sequence of $m$. For a partition $\mu$, define two subspaces $P_{\leq \mu}$ and $P_{<\mu}$ of $\mathbb{Q}[x_n]$ by

$$P_{\leq \mu} = \text{span}\{m \in \mathbb{Q}[x_n] : \lambda(m) \leq \mu\}, \quad P_{<\mu} = \text{span}\{m \in \mathbb{Q}[x_n] : \lambda(m) < \mu\}. $$

Let $Q_{\leq \mu}$ and $Q_{<\mu}$ be the images of these subspaces in $R_{n,k}$. Finally, let $R_{n,k,\mu} := Q_{\leq \mu}/Q_{<\mu}$ be the corresponding quotient. We have a graded decomposition of $\mathfrak{S}_n$-modules

$$R_{n,k} \cong \bigoplus_{\mu} R_{n,k,\mu},$$

where all but finitely many of the modules in the sum are 0.

When $k = n$, Adin, Brenti, and Roichman used the Garsia-Stanton basis of $R_n$ to determine the isomorphism type of $R_{n,n,\mu}$ for all $\mu$ [11]. This suggests the following problem.

**Problem 7.4.** Determine the Frobenius image $\text{Frob}(R_{n,k,\mu})$ for $k \leq n$ and any partition $\mu$.

The solution of Problem 7.3 in the case $k = n$ found in [1] makes heavy use of the fact that $\mathbb{Q}[x_n]$ is a free module over $\mathbb{Q}[x_n]^{\mathfrak{S}_n}$, which in turn comes from the regularity of the sequence $e_1(x_n), e_2(x_n), \ldots, e_n(x_n)$. Since no such regularity holds for the ideal $I_{n,k}$, the solution of Problem 7.3 for general $k \leq n$ will require different methods. [7]

Huang and Rhoades [21] defined an analog of our quotient $R_{n,k}$ which carries an action of the 0-Hecke algebra. Recall that the 0-Hecke algebra (over any field $\mathbb{F}$) is the unital, associative $\mathbb{F}$-algebra with generators $T_1, T_2, \ldots, T_{n-1}$ and relations

$$T_i^2 = T_i, \quad 1 \leq i \leq n - 1$$

$$T_i T_j = T_j T_i, \quad |i - j| > 1$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad 1 \leq i \leq n - 2.$$ 

The algebra $H_n(0)$ acts on the polynomial ring $\mathbb{F}[x_n]$ by the rule $T_i f := \frac{x_{i+1} f - x_i f}{x_{i+1} - x_i}$. Although the ideal $I_{n,k}$ is not closed under this action, the following ideal $J_{n,k}$ is:

$$J_{n,k} := \langle h_k(x_1), h_k(x_1, x_2), \ldots, h_k(x_1, x_2, \ldots, x_n), e_n(x_n), e_{n-1}(x_n), \ldots, e_{n-k+1}(x_n) \rangle.$$

The corresponding quotient $S_{n,k} := \frac{\mathbb{F}[x_n]}{J_{n,k}}$ plays the role of the ring $R_{n,k}$ in [21].

The Hecke algebra $H_n(q)$ (where $q$ is a parameter) interpolates between the symmetric group algebra and the 0-Hecke algebra. It has generators $T_1, T_2, \ldots, T_{n-1}$ and the same relations as above, except that $T_i^2 = q + (1 - q) T_i$.

**Problem 7.5.** Give an analog of the ring $R_{n,k}$ which carries an action of the Hecke algebra $H_n(q)$.

In light of [21], a solution to Problem 7.5 would most likely involve defining a new ideal to play the role of $I_{n,k}$. [8]

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[7] Kyle Meyer (personal communication) has solved Problem 7.3 and is preparing a writeup.

[8] While this paper was under review, Problem 7.5 was solved by Huang, Rhoades, and Scrimshaw [22]. The relevant ideal replaces $x_i^k$ with the Hall-Littlewood $P$-function $P_k(x_1, x_2, \ldots, x_i; q)$. 

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The modules $R_{n,k}$ of this paper were shown to have graded Frobenius image equal to either of the combinatorial expressions in the Delta Conjecture at $t = 0$. This leads to the following problem.

**Problem 7.6.** Find a nice bigraded $S_n$-module whose bigraded Frobenius character equals any of the expressions $\Delta'_{e_{k-1}} e_n(x)$, $\text{Rise}_{n,k}(x; q, t)$, or $\text{Val}_{n,k}(x; q, t)$ in the Delta Conjecture (after application of $\omega$ and an appropriate change of variables).

Problem 7.6 is probably very difficult. However, there are two specializations of Problem 7.6 which could be quite interesting and tractable. The first concerns the specialization $t = 1$. Romero [29] has proven that $\Delta'_{e_{k-1}} e_n(x) |_{q=1, t=q} = \text{Rise}_{n,k}(x; 1, q)$.

**Problem 7.7.** Find a nice (singly) graded $S_n$-module whose graded Frobenius character is given by $\Delta'_{e_{k-1}} e_n(x) |_{q=1, t=q} = \text{Rise}_{n,k}(x; 1, q)$ (after applying $\omega$ and an appropriate change of variables).

In the case $k = n$ Berget and Rhoades [6] proved that the restriction to $S_n$ of a graded $S_{n+1}$-module $V(n)$ defined by Postnikov and Shapiro [26] solves Problem 7.7. The module $V(n)$ is defined in a matroid-theoretic fashion involving subgraphs of the complete graph $K_{n+1}$ whose complements are connected.

Another interesting specialization in Macdonald polynomial theory is $t = 1/q$.

**Problem 7.8.** Find a nice (singly) graded $S_n$-module whose graded Frobenius character is given by any of $\Delta'_{e_{k-1}} e_n(x)$, $\text{Rise}_{n,k}(x; q, t)$, or $\text{Val}_{n,k}(x; q, t)$ after setting $t = 1/q$ (and applying $\omega$ and an appropriate change of variables).

Haglund, Remmel, and Wilson have a plethystic formula for $\Delta'_{e_{k-1}} e_n(x)$ at $t = 1/q$ [18, Thm. 5.1, Eqn. 27]. The case $k = n$ was solved by Haiman [19]; one considers the quotient of $\frac{Q[x_1, \ldots, x_n]}{(x_1 + \cdots + x_n)}$ by a homogeneous system of parameters $\theta_1, \theta_2, \ldots, \theta_{n-1}$ of degree $n + 1$ which carries the dual of the reflection representation of $S_n$. This construction was generalized to all real reflection groups $W$ and $W$-noncrossing parking functions in the Parking Conjecture of Armstrong, Reiner, and Rhoades [3]. A solution to Problem 7.8 could extend to other reflection groups to give a marriage of Parking and Delta.

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