OPTIMAL DECAY FOR THE COMPRESSIBLE NAVIER-STOKES EQUATIONS WITHOUT ADDITIONAL SMALLNESS ASSUMPTIONS

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Abstract. We investigate the large time behavior of solutions to the barotropic compressible Navier-Stokes equations. Precisely, we show that if the initial density and velocity additionally belong to some Besov space $\dot{B}^{\sigma_1}_{2,\infty}$ with $\sigma_1 \in (1-d/2, 2d/p - d/2]$, then the $L^p$ norm (the slightly stronger $\dot{B}^0_{p,1}$ norm in fact) of the critical global solutions admits the optimal decay $t^{-\frac{d}{2}\left(\frac{1}{2} - \frac{1}{p}\right) - \sigma_1}$ for $t \to +\infty$. It is well-known that the case $p = 2$ for solutions with high Sobolev regularity, was first observed by Matsumura & Nishida [30] and generalized by Ponce [34] under the $L^1$ smallness assumption of initial low frequencies. In this paper, the pure energy argument without the spectral analysis (for the linearized system) is performed, which enables us to remove the usual smallness assumption of low frequencies. Indeed, bounding the evolution of $\dot{B}^{\sigma_1}_{2,\infty}$-norm restricted in low frequencies is the key ingredient, which depends on some new and non classical Besov product estimates. In contrast to the recent time-weighted energy approach in [11], a different energy argument is developed in the critical regularity spaces.

1. Introduction

In this paper, we consider the following compressible Navier-Stokes equations

\begin{equation}
\begin{aligned}
&\partial_t \rho + \text{div} (\rho u) = 0, \\
&\partial_t(\rho u) + \text{div} (\rho u \otimes u) - \text{div} (2\mu \text{D}(u) + \lambda \text{div} u \text{Id}) + \nabla \Pi = 0,
\end{aligned}
\end{equation}

which govern the motion of a general barotropic compressible fluid in the whole space $\mathbb{R}^d$. Here $u = u(t, x) \in \mathbb{R}^d$ (with $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$) and $\rho = \rho(t, x) \in \mathbb{R}_+$ stand for the velocity field and density of the fluid, respectively. The barotropic assumption means that the pressure $\Pi \triangleq P(\rho)$ depends only upon the density of fluid and the function $P$ will be taken suitably smooth in what follows. The notation $\text{D}(u) \triangleq \frac{1}{2}(D_x u + T D_x u)$ stands for the deformation tensor, and div is the divergence operator with respect to the space variable. The density-dependent functions $\lambda$ and $\mu$ (the bulk and shear viscosities) are assumed to be smooth enough and to satisfy

\begin{equation}
\mu > 0 \quad \text{and} \quad \nu \triangleq \lambda + 2\mu > 0.
\end{equation}

System (1.1) is supplemented with the initial data

\begin{equation}
\rho(t, u)|_{t=0} = (\rho_0, u_0).
\end{equation}

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The second author would like to thank Professor A. Matsumura for introducing him to the decay problem for partially parabolic equations when he visited Osaka University. Also, he is very grateful to Professor R. Danchin for addressing the conjecture on the regularity of low frequencies when visiting the LAMA in UPEC.
We investigate the solution \((\varrho, u)\) to the Cauchy problem \((1.1)-(1.3)\) fulfilling the constant far-field behavior
\[(\varrho, u) \to (\varrho_\infty, 0)\]
with \(\varrho_\infty > 0\), as \(|x| \to \infty\).

There is a huge literature on the existence, blow-up and large-time behavior of solutions to the compressible Navier-Stokes equations. The local existence and uniqueness of smooth solutions away from vacuum were proved by Serrin [35] and Nash [32]. The local existence of strong solutions with Sobolev regularities was constructed by Solonnikov [37], Valli [40] and Fiszdon-Zajaczkowski [13]. Matsumura and Nishida [29, 30] established the global strong solutions for small perturbations of a linearly stable constant non-vacuum state \((\varrho_\infty, 0)\) in three dimensions. With the additional \(L^1\) smallness assumption of initial data, the definite decay rate was also available:
\[
\|(\varrho - \varrho_\infty, u)(t)\|_{L^2} \lesssim \langle t \rangle^{-\frac{3}{2}} \quad \text{with} \quad \langle t \rangle \triangleq \sqrt{1 + t^2},
\]
which coincides with that of the heat kernel. Indeed, the decay rate in time reveals the dissipative properties of solutions to \((1.1)-(1.3)\). Subsequently, Ponce [34] obtained more general \(L^r\) decay rates:
\[
\|\nabla^k(\varrho - \varrho_\infty, u)(t)\|_{L^r} \lesssim \langle t \rangle^{-\frac{d}{r}(1 - \frac{1}{r}) - \frac{k}{2}}, \quad 2 \leq r \leq \infty, \quad 0 \leq k \leq 2, \quad d = 2, 3.
\]
Later, Matsumura-Nishida’s results were generalized to more physical situations where the fluid domain is not \(\mathbb{R}^d\): for example, the exterior domains were investigated by Kobayashi [23] and Kobayashi-Shibata [24], the half space by Kagei & Kobayashi [21, 22]. In addition, there are some results available which are connected to the information of wave propagation. Zeng [45] investigated the \(L^1\) convergence to the nonlinear Burgers’ diffusive wave. Hoff and Zumbrun [18] performed the detailed analysis of the Green function for the multi-dimensional case and got the \(L^\infty\) decay rates of diffusion waves. In [28], Liu and Wang exhibited pointwise convergence of solution to diffusion waves with the optimal time-decay rate in odd dimension, where the phenomena of the weaker Huygens’ principle was also shown. Huang, Li and the first author [19] established the global existence of classical solutions that may have large highly oscillations and can contain vacuum states, see also [26] where further results including 2-dimensional case and large time behavior of solutions have been obtained. For the existence of solution with arbitrary data, a breakthrough is due to Lions [27], who obtained the global existence of weak solutions with the finite energy when the adiabatic exponent is suitably large. Subsequently, some improvements were made by Feireisl, Novotny & Petzeltová [12] and Jiang-Zhang [20]. In 1998, the first author [41] found a fact that any smooth solution to the Cauchy problem of full compressible Navier-Stokes system without heat conduction (including the current barotropic case) would blow up in finite time if the initial density contains vacuum. See also [42] for general blow-up results.

As regards global-in-time results, \textit{scaling invariance} plays a fundamental role. It is well known that suitable critical quantities (that is, having the same scaling invariance as the system under consideration) may control the possible blow-up of solutions. This trick is now classic. Let us recall the existence context for the incompressible Navier-Stokes equations and go back to the work [14] by Fujita & Kato (see also results by Kozono & Yamazaki [25] and Cannone [2]). Observe that \((1.1)\) is invariant by the transform
\[
\varrho(t, x) \sim \varrho(l^2 t, lx), \quad u(t, x) \sim lu(l^2 t, lx), \quad l > 0
\]
up to a change of the pressure term \( \Pi \) into \( l^2 \Pi \). Danchin [7] solved (1.1) globally in critical homogeneous Besov spaces of \( L^2 \) type, that is, the initial data belong to 
\((B^{\frac{d}{2},1}_d \cap B^{\frac{d}{2}-1}_d) \times (B^{\frac{d}{2},1}_d)^d\). Subsequently, the result of [7] has been extended to Besov spaces that are not related to \( L^2 \) by Charve-Danchin [4] and Chen-Miao-Zhang [6] independently. Haspot [16] achieved essentially the same result by means of more elementary approach based on using the viscous effective flux as in [17] [39] (see also [10] for the case of density-dependent viscosity coefficients). This eventually leads to the following statement:

**Theorem 1.1.** ([4] [6] [16]) Let \( d \geq 2 \) and \( p \) satisfy
\[ 2 \leq p \leq \min(4,d^*) \quad \text{with} \quad d^* \triangleq 2d/(d-2) \quad \text{and, additionally,} \quad p \neq 4 \quad \text{if} \quad d = 2. \]
Assume that \( P'(\varrho_{\infty}) > 0 \) and that (1.2) is fulfilled. There exists a constant \( c = c(p,d,\lambda,\mu,P,\varrho_{\infty}) \) such that if \( a_0 \triangleq \varrho_0 - \varrho_{\infty} \) is in \( \dot{B}^{\frac{d}{2}}_{p,1} \), if \( u_0 \) is in \( \dot{B}^{\frac{d}{2}-1}_{p,1} \) and if in addition \((a_0^\ell,u_0^\ell) \in \dot{B}^{d/2}_{2,1} \) with
\[ X_{p,0} \triangleq \|(a_0,u_0)\|_{\dot{B}^{\frac{d}{2}-1}_{p,1}}^{\ell} + \|a_0\|^{h}_{\dot{B}^{\frac{d}{2}}_{p,1}} + \|u_0\|^{h}_{\dot{B}^{\frac{d}{2}-1}_{p,1}} \leq c \]
then (1.1) has a unique global-in-time solution \((\varrho,u)\) with \( \varrho = \varrho_{\infty} + a \) and \((a,u)\) in the space \( X_p \) defined by:
\[
(a,u)^\ell \in \mathcal{C}_b(\mathbb{R}_+;\dot{B}^{\frac{d}{2}-1}_{2,1}) \cap L^1(\mathbb{R}_+;\dot{B}^{\frac{d}{2}+1}_{2,1}), \quad a^h \in \mathcal{C}_b(\mathbb{R}_+;\dot{B}^{\frac{d}{2}}_{p,1}) \cap L^1(\mathbb{R}_+;\dot{B}^{\frac{d}{2}}_{p,1}), \quad u^h \in \mathcal{C}_b(\mathbb{R}_+;\dot{B}^{\frac{d}{2}-1}_{p,1}) \cap L^1(\mathbb{R}_+;\dot{B}^{\frac{d}{2}+1}_{p,1}).
\]
Furthermore, there exists some constant \( C = C(p,d,\lambda,\mu,P,\varrho_{\infty}) \) such that
\[ X_p \leq C X_{p,0}, \]
with
\[ \begin{align*}
X_p & \triangleq \|(a,u)\|_{L^\infty(\dot{B}^{\frac{d}{2}-1}_{2,1})}^{\ell} + \|(a,u)\|_{L^1(\dot{B}^{\frac{d}{2}+1}_{2,1})}^{\ell} + \|a\|_{L^\infty(\dot{B}^{\frac{d}{2}}_{p,1}) \cap L^1(\dot{B}^{\frac{d}{2}}_{p,1})}^{h} + \|u\|_{L^\infty(\dot{B}^{\frac{d}{2}-1}_{p,1}) \cap L^1(\dot{B}^{\frac{d}{2}+1}_{p,1})}^{h}.
\end{align*} \]

Next, a natural problem is what is the large time asymptotic behavior of global-in-time solutions constructed above. Although providing an accurate long-time asymptotics description is still out of reach in general, there are a number of works concerning the time-decay rates of \( L^q-L^r \) (same as (1.4)-(1.5)) which are available in the critical regularity framework. Okita [33] established the optimal time-decay estimates to (1.1)-(1.3) by using a smart modification of the method of [7]. However, that result cannot cover the 2D case. In the survey [8], Danchin proposed another description of the time decay which allows one to handle dimension \( d \geq 2 \). Recently, Danchin and the second author [11] further developed the method of [8] so as to get the optimal decay rates in more general \( L^p \) critical spaces. The regularity exponent \( d/p - 1 \) for velocity may become negative in physical dimensions \( d = 2,3 \), that result thus applies to large highly oscillating initial velocities. Inspired by the private communication with Danchin, the

\[ 1 \text{The subspace } \mathcal{C}_b(\mathbb{R}_+;\dot{B}^{\frac{d}{2}}_{q,1}) \text{ of } \mathcal{C}_b(\mathbb{R}_+;\dot{B}^{\frac{d}{2}+1}_{q,1}) \text{ is defined in [33], and the norms } \| \cdot \|_{L^\infty(\dot{B}^{\frac{d}{2}}_{p,1})} \text{ are introduced just below Definition 3.2.} \]
second author [43] developed a general low-frequency condition for optimal decay estimates, where the regularity \( \sigma_1 \) of \( B_{2,\infty}^{-\sigma_1} \) belongs to a whole range \((1 - d/2, \sigma_0)\) with 
\[ \sigma_0 \triangleq \frac{2d}{p} - \frac{d}{2}. \]
The proof mainly depends on the refined time-weighted energy approach in the Fourier semi-group framework. To the best of our knowledge, the smallness assumption of low frequencies is usually needed in both Sobolev spaces (with higher regularity) and critical Besov spaces in all previous studies. In this paper, we intend to remove the smallness of low frequencies and establish the optimal decay for the barotropic compressible Navier-Stokes equations in the \( L^p \) critical framework.

2. Reformulation and main results

To state main results, it is convenient to rewrite System (1.1) as the linearized compressible Navier-Stokes equations about equilibrium \((\varrho_\infty, 0)\), and regard the non-linearities as source terms. Precisely, one has

\[
\begin{cases}
\partial_t a + \text{div} u = f, \\
\partial_t u - A u + \nabla a = g,
\end{cases}
\]

with \( f \triangleq -\text{div}(au), \; A \triangleq \mu_\infty \Delta + (\lambda_\infty + \mu_\infty) \nabla \text{div} \) such that \( \mu_\infty > 0 \) and \( \lambda_\infty + 2\mu_\infty > 0 \),
\[
g \triangleq -u \cdot \nabla u - I(a)Au - k(a)\nabla a + \frac{1}{1+a} \text{div} (2\tilde{\mu}(a)D(u) + \tilde{\lambda}(a)\text{div} u \text{Id}),
\]

where
\[
I(a) \triangleq \frac{a}{1+a}, \; k(a) \triangleq \frac{p'(1+a)}{1+a} - 1, \; \tilde{\mu}(a) \triangleq \mu(1+a) - \mu(1) \; \text{ and } \; \tilde{\lambda}(a) \triangleq \lambda(1+a) - \lambda(1).
\]

For simplicity, one can perform a suitable rescaling in (1.1) so as to normalize, at infinity, the density \( \varrho_\infty \), the sound speed \( c_\infty \triangleq \sqrt{p'(\varrho_\infty)} \) and the total viscosity \( \nu_\infty \triangleq \lambda_\infty + 2\mu_\infty \) (with \( \lambda_\infty \triangleq \lambda(\varrho_\infty) \) and \( \mu_\infty \triangleq \mu(\varrho_\infty) \)) to be one.

Denote \( \Lambda^s f \triangleq \mathcal{F}^{-1}(\|s\|^s \mathcal{F}f) \) for \( s \in \mathbb{R} \). Now, we state main results as follows.

**Theorem 2.1.** Let \( (\varrho, u) \) be the global solution addressed by Theorem 1.1. If in addition \((a_0, u_0)^{\ell} \in B_{2,\infty}^{-\sigma_1} (1 - \frac{d}{2} < \sigma_1 \leq \sigma_0 \triangleq \frac{2d}{p} - \frac{d}{2})\) such that \( \|(a_0, u_0)^{\ell}\|_{B_{2,\infty}^{-\sigma_1}} \) is bounded, then we have

\[
\|\Lambda^\sigma(a, u)\|_{L^p} \lesssim (1 + t)^{-\frac{d}{2} - \frac{1}{p} - \frac{\sigma + \sigma_1}{2} \frac{p - d}{p}} \quad \text{if} \quad -\sigma_1 < \sigma \leq \frac{d}{p} - 1,
\]

for all \( t \geq 0 \), where \( \sigma_1 \triangleq \sigma_1 + d\left(\frac{1}{2} - \frac{1}{p}\right) \).

Furthermore, by using improved Gagliardo-Nirenberg inequalities, we obtain optimal decay estimates of \( B_{2,\infty}^{-\sigma_1} - L^r \) type.

**Corollary 2.1.** Let those assumptions of Theorem 2.1 be fulfilled. Then the corresponding solution \((a, u)\) admits the following decay estimates

\[
\|\Lambda^l(a, u)\|_{L^r} \lesssim (1 + t)^{-\frac{d}{2} - \frac{1}{r} - \frac{\sigma + \sigma_1}{2} \frac{r - d}{r}} \quad \text{if} \quad -\sigma_1 < \sigma \leq \frac{d}{r} - 1,
\]

for \( p \leq r < \infty \), \( l \in \mathbb{R} \) and \(-\sigma_1 < l + d\left(\frac{1}{p} - \frac{1}{r}\right) \leq \frac{d}{p} - 1\).

Some comments are in order.

\[2\text{In our analysis, the exact value of functions } k, \tilde{\lambda}, \tilde{\mu} \text{ and even I will not matter: we only need those functions to be smooth enough and to vanish at 0.} \]
(1) In comparison with classical efforts in \cite{30, 34} and the recent work \cite{11}, the innovative ingredient is that the smallness of low frequencies is no longer needed in Theorem 2.1 and Corollary 2.1. Furthermore, choosing the endpoint regularity for example $\sigma_1 = \sigma_0 = d/2$ (if $p = 2$) allows to go back to the classical decay estimates (1.4)-(1.5) with aid of the Sobolev embedding $L^1 \hookrightarrow \dot{B}_{2,\infty}^{-d/2}$. Clearly, there is some freedom in the choice of $\sigma_1$, which allows to obtain more optimal decay estimates in the $L^p$ framework.

(2) In \cite{11}, there is a little loss on decay rates due to using different Sobolev embeddings at low frequencies and high frequencies. In the case of $\sigma_1 = \sigma_0$, for example, we see that the solution itself decays to equilibrium in $L^p$ norm with the rate of $O\left(t^{-d/(1/p-1/4)}\right)$, which is slowly than $t^{-d/2p}$ for $t \to \infty$. The present results avoid this minor flaw and thus are satisfactory.

(3) In the energy argument with interpolation, the optimality of lower bound of $\sigma_1$ can be confirmed by Besov product estimates in Corollary 5.1 and a couple of interpolation inequalities like (5.41), (6.9) and (6.13).

(4) In physical dimensions $d = 2, 3$, Condition (1.7) allows us to consider the case $p > d$ for which the velocity regularity exponent $d/p - 1$ becomes negative. Our result thus applies to large highly oscillating initial velocities (see \cite{4, 6} for more explanations).

(5) Finally, it may be worth pointing out that our approach here is of independent interest in the $L^p$ critical framework and thus is effective for other models that are encountered in fluid mechanism or mathematical physics.

Let us give some illustration on the major difficulty and our strategy to overcome it. The main objective of \cite{11} is to establish the time-weighted inequality including enough time-decay information, which leads to the optimal decay estimate for strong solutions. The decay framework depends more or less on the spectral analysis and the smallness assumption of low frequencies. In the present paper, we develop a new energy argument without the spectral analysis, which is originated from the idea as in \cite{15, 38}. In Sobolev spaces with higher regularity, they deduced the Lyapunov-type inequality with respect to the time variable $t$ by using a family of scaled energy estimates with minimum derivative counts and interpolations among them, which leads to desired decay estimates for several Boltzmann type equations and Navier-Stokes equations. In the critical regularity framework however, it seems very difficult to apply their approaches directly. As a matter of fact, one cannot afford any loss of regularity for the high frequency part of the solution (and some terms as $u \cdot \nabla a$ induce a loss of one derivative since there is no smoothing for $a$, solution of a transport equation). In the low-frequency regime, observe that the first order terms of (2.1) are predominate and hyperbolic energy methods are thus expected to be more appropriate. As in \cite{7}, we decompose the velocity into the “incompressible part $\omega$” and the “compressible part $v$”. It suffices to study the mixed system between $a$ and $v$, since $\omega$ satisfies, up to nonlinear terms, a mere heat equation. The influence of viscous terms is decisive, which enables us to obtain the parabolic decay for both $a$ and $u$.

In the high-frequency regime, the main difficulty comes from the convection term in the density equation, as it may case a loss of one derivative. To get round the difficulty, we shall employ the energy estimate approach in terms of effective velocity, which was used in the critical framework by Haspot \cite{16}. The interested reader is also referred to
for the use of viscous effective flux. Based on the introduction of $w$ by (4.10), it is observed that both the divergence-free part of $u$ (which is equivalent to the incompressible part) and $w$ satisfy some constant coefficient heat equation, while $a$ satisfies a damped transport equation. Thanks to the decay properties of the heat and damped transport equations, we can get suitable energy estimates avoiding the loss of one derivative. In order to achieve the Lyapunov-type inequality for energy norms, some new aspects are involved in our analysis. On the one hand, bounding the evolution of Besov norm $B_{2,\infty}^{-\sigma_0}$ (restricted in the low-frequency part) of solutions is the main ingredient in the present paper. To this end, we develop some non classical Besov product estimates (see Proposition 5.1) by means of Bony's para-product decomposition. As a matter of fact, Proposition 5.1 investigates the endpoint case of product estimates, which is regarded as the supplement of Proposition 3.5 (see also [11]). Furthermore, by using different Sobolev embeddings and interpolations, we can handle the nonlinear estimate of $\| (f, g) \|_{B_{2,\infty}^{-\sigma_0}}$ in the non oscillation case ($2 \leq p \leq d$) and in the oscillation case ($p > d$), respectively, which leads to Lemma 5.1 by employing nonlinear generalisations of the Gronwall’s inequality (with respect to the Bernoulli equation). On the other hand, in order to derive the Lyapunov-type inequality (6.10) for energy norms, we establish real interpolation inequalities of Gagliardo-Nirenberg type for Besov norms restricted in low frequencies or high frequencies only (see Proposition 6.1 for details).

We end the section with a simple overview of our main results. In comparison with [11], a different decay framework is developed in the critical regularity spaces, which enables us to employ energy methods (without using the spectral analysis) to remove the smallness assumption of low frequencies. Of course, vacuum is ruled out in this paper. What happen, if the vacuum occurs. The corresponding mathematical theory for viscous fluids is still far away well known in critical spaces, which is left in the forthcoming work.

The rest of the paper unfolds as follows: In Section 3, we recall briefly the Littlewood-Paley decomposition, Besov spaces and related analysis tools. In Section 4, we establish the low-frequency and high-frequency estimates of solutions by using the pure energy arguments. Section 5 is devoted to bounding the evolution of negative Besov norms, which plays the key role in deriving the Lyapunov-type inequality for energy norms. In the last section (Section 6), we present the proofs of Theorem 2.1 and Corollary 2.1.

3. Preliminary

Throughout the paper, $C > 0$ stands for a generic harmless “constant”. For brevity, we sometime write $u \lesssim v$ instead of $u \leq Cv$. The notation $u \approx v$ means that $u \lesssim v$ and $v \lesssim u$. For any Banach space $X$ and $u, v \in X$, $\| (u, v) \|_X \triangleq \| u \|_X + \| v \|_X$. For all $T > 0$ and $\rho \in [1, +\infty]$, one denotes by $L^\rho_T (X) \triangleq L^\rho ([0, T] ; X)$ the set of measurable functions $u : [0, T] \to X$ such that $t \mapsto \| u(t) \|_X$ is in $L^\rho (0, T)$.

3.1. Littlewood-Paley decomposition and Besov spaces. Let us recall Littlewood-Paley decomposition and Besov spaces for convenience. More details may be found for example in Chap. 2 and Chap. 3 of [1]. Choose a smooth radial non increasing function $\chi$ with $\text{Supp}\, \chi \subset B \left( 0, \frac{3}{4} \right)$ and $\chi \equiv 1$ on $B \left( 0, \frac{3}{8} \right)$. Set $\varphi (\xi) = \chi (\xi / 2) - \chi (\xi)$. It is not difficult to check that

$$\sum_{j \in \mathbb{Z}} \varphi (2^{-j} \cdot) = 1 \quad \text{in} \quad \mathbb{R}^d \setminus \{0\} \quad \text{and} \quad \text{Supp} \, \varphi \subset \left\{ \xi \in \mathbb{R}^d : 3/4 \leq |\xi| \leq 8/3 \right\}.$$
Then homogeneous dyadic blocks $\dot{\Delta}_j$ ($j \in \mathbb{Z}$) are defined by
\[
\dot{\Delta}_j u \triangleq \varphi(2^{-j} D) u = \mathcal{F}^{-1} (\varphi(2^{-j} \cdot) \mathcal{F} u) = 2^{jd} h(2^j \cdot) \ast u \quad \text{with} \quad h \triangleq \mathcal{F}^{-1} \varphi.
\]
Consequently, we have the unit decomposition for any tempered distribution $u \in S'(\mathbb{R}^d)$
\[
(3.1) \quad u = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u.
\]
As it holds only modulo polynomials, it is convenient to consider the subspace of those tempered distributions $u$ such that
\[
(3.2) \quad \lim_{j \to -\infty} \| \dot{S}_j u \|_{L^\infty} = 0,
\]
where $\dot{S}_j f$ stands for the low frequency cut-off defined by $\dot{S}_j u \triangleq \chi (2^{-j} D) u$. Indeed, if (3.2) is fulfilled, then (3.1) holds in $S'(\mathbb{R}^d)$. Hence, we denote by $S'_0(\mathbb{R}^d)$ the subspace of tempered distributions satisfying (3.2).

Now, Besov spaces are defined as follows.

**Definition 3.1.** For $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, the homogeneous Besov spaces $\dot{B}^s_{p,r}$ are defined by
\[
\dot{B}^s_{p,r} \triangleq \left\{ u \in S'_0 : \| u \|_{\dot{B}^s_{p,r}} < +\infty \right\},
\]
where
\[
(3.3) \quad \| u \|_{\dot{B}^s_{p,r}} \triangleq \left\| (2^{js}\| \dot{\Delta}_j u \|_{L^p}) \right\|_{\ell^r(\mathbb{Z})}.
\]
Moreover, a class of mixed space-time Besov spaces are also used when studying evolutionary PDEs, which was introduced by J.-Y. Chemin and N. Lerner [5] (see also [3] for the particular case of Sobolev spaces).

**Definition 3.2.** For $T > 0$, $s \in \mathbb{R}$, $1 \leq r, \theta \leq \infty$, the homogeneous Chemin-Lerner space $\dot{L}^\theta_t(\dot{B}^s_{p,r})$ is defined by
\[
\dot{L}^\theta_t(\dot{B}^s_{p,r}) \triangleq \left\{ u \in L^\theta(0, T; S'_0) : \| u \|_{\dot{L}^\theta_t(\dot{B}^s_{p,r})} < +\infty \right\},
\]
where
\[
(3.4) \quad \| u \|_{\dot{L}^\theta_t(\dot{B}^s_{p,r})} \triangleq \left\| (2^{js}\| \dot{\Delta}_j u \|_{L^\theta_t(L^p)}) \right\|_{\ell^r(\mathbb{Z})}.
\]
For notational simplicity, index $T$ is omitted if $T = +\infty$. We also use the following functional space:
\[
(3.5) \quad \tilde{C}_0(\mathbb{R}_+; \dot{B}^s_{p,r}) \triangleq \left\{ u \in C(\mathbb{R}_+; \dot{B}^s_{p,r}) \ s.t \ \| u \|_{\tilde{L}^\infty(\dot{B}^s_{p,r})} < +\infty \right\}.
\]
Thanks to Minkowski’s inequality, one has the following topological relation between Chemin-Lerner’s spaces and the standard spaces $\dot{L}^\theta_t(\dot{B}^s_{p,r})$.

**Remark 3.1.** It holds that
\[
\| u \|_{\dot{L}^\theta_t(\dot{B}^s_{p,r})} \leq \| u \|_{\dot{L}^\theta_t(B^s_{p,r})} \quad \text{if} \quad r \geq \theta; \quad \| u \|_{\dot{L}^\theta_t(\dot{B}^s_{p,r})} \geq \| u \|_{\dot{L}^\theta_t(B^s_{p,r})} \quad \text{if} \quad r \leq \theta.
\]
Restricting the above norms (3.3) and (3.4) to the low or high frequencies parts of distributions will be crucial in our approach. For example, let us fix some integer $j_0$ (the value of which will follow from the proof of the main theorem) and set

$$
\|u\|_{\dot{B}^s_{p,1}} \triangleq \sum_{j \leq j_0} 2^{js} \|\hat{\Delta}_j f\|_{L^p} \quad \text{and} \quad \|u\|_{\dot{B}^s_{p,1}'} \triangleq \sum_{j \geq j_0-1} 2^{js} \|\hat{\Delta}_j f\|_{L^p},
$$

$$
\|u\|_{L^\infty_{t,x}(\dot{B}^s_{p,1})} \triangleq \sum_{j \leq j_0} 2^{js} \|\hat{\Delta}_j u\|_{L^\infty_{t,x} L^p} \quad \text{and} \quad \|u\|_{L^\infty_{t,x}(\dot{B}^s_{p,1}')} \triangleq \sum_{j \geq j_0-1} 2^{js} \|\hat{\Delta}_j u\|_{L^\infty_{t,x} L^p}.
$$

3.2. Analysis tools in Besov spaces. Recall the classical Bernstein inequality:

$$
(3.6) \quad \|D^k u\|_{L^q} \leq C^{1+k} \lambda^{k+d(\frac{1}{a}-\frac{1}{b})} \|u\|_{L^a}
$$

that holds for all function $u$ such that $\text{Supp} \mathcal{F} u \subset \{ \xi \in \mathbb{R}^d : |\xi| \leq R\lambda \}$ for some $R > 0$ and $\lambda > 0$, if $k \in \mathbb{N}$ and $1 \leq a \leq b \leq \infty$.

More generally, if $u$ satisfies $\text{Supp} \mathcal{F} u \subset \{ \xi \in \mathbb{R}^d : R_1 \lambda \leq |\xi| \leq R_2 \lambda \}$ for some $0 < R_1 < R_2$ and $\lambda > 0$, then for any smooth homogeneous of degree $m$ function $A$ on $\mathbb{R}^d \setminus \{0\}$ and $1 \leq a \leq \infty$, it holds that (see e.g. Lemma 2.2 in [1]):

$$
(3.7) \quad \|A(D) u\|_{L^a} \lesssim \lambda^m \|u\|_{L^a}.
$$

An obvious consequence of (3.6) and (3.7) is that $\|D^k u\|_{\dot{B}^s_{p,r}} \approx \|u\|_{\dot{B}^{s+k}_{p,r}}$ for all $k \in \mathbb{N}$.

The following nonlinear generalization of (3.7) will be also used (see Lemma 8 in [9]):

Proposition 3.1. If $\text{Supp} \mathcal{F} f \subset \{ \xi \in \mathbb{R}^d : R_1 \lambda \leq |\xi| \leq R_2 \lambda \}$ then there exists $c$ depending only on $d$, $R_1$ and $R_2$ so that for all $1 < p < \infty$,

$$
c \lambda^2 \left( \frac{p-1}{p} \right) \int_{\mathbb{R}^d} |f|^p \, dx \leq (p-1) \int_{\mathbb{R}^d} |\nabla f|^2 |f|^{p-2} \, dx = - \int_{\mathbb{R}^d} \Delta f |f|^{p-2} f \, dx.
$$

The following classical properties are often used (see [1]):

Proposition 3.2.  

- Scaling invariance: For any $s \in \mathbb{R}$ and $(p,r) \in [1,\infty)^2$, there exists a constant $C = C(s,p,r,d)$ such that for all $\lambda > 0$ and $u \in \dot{B}^s_{p,r}$, then

$$
C^{-1} \lambda^{-d} \|u\|_{\dot{B}^s_{p,r}} \leq \|u(\lambda \cdot)\|_{\dot{B}^s_{p,r}} \leq C \lambda^{d} \|u\|_{\dot{B}^s_{p,r}}.
$$

- Completeness: $\dot{B}^s_{p,r}$ is a Banach space whenever $s < \frac{d}{p}$ or $s \leq \frac{d}{p}$ and $r = 1$.

- Action of Fourier multipliers: If $F$ is a smooth homogeneous of degree $m$ function on $\mathbb{R}^d \setminus \{0\}$, then

$$
F(D) : \dot{B}^s_{p,r} \to \dot{B}^{s-m}_{p,r}.
$$

Proposition 3.3. Let $1 \leq p, r, r_1, r_2 \leq \infty$.

- Complex interpolation: if $u \in \dot{B}^s_{p,r_1} \cap \dot{B}^s_{p,r_2}$ and $s \neq \bar{s}$, then $u \in \dot{B}^{\theta s+(1-\theta)\bar{s}}_{p,r}$ for all $\theta \in (0,1)$ and

$$
\|u\|_{\dot{B}^{\theta s+(1-\theta)\bar{s}}_{p,r}} \leq \|u\|_{\dot{B}^s_{p,r_1}}^{\theta} \|u\|_{\dot{B}^s_{p,r_2}}^{1-\theta}
$$

with $\frac{1}{r} = \frac{\theta}{r_1} + \frac{1-\theta}{r_2}$.

Note that for technical reasons, we need a small overlap between low and high frequencies.
Proposition 3.4. (Embedding for Besov spaces on $\mathbb{R}^d$)

- For any $p \in [1, \infty]$ we have the continuous embedding $\dot{B}^0_{p,1} \hookrightarrow L^p \hookrightarrow \dot{B}^0_{p,\infty}$.
- If $\sigma \in \mathbb{R}$, $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$, then $\dot{B}^\sigma_{p_1,r_1} \hookrightarrow \dot{B}^{\sigma-d(1/p_1 - 1/p_2)}_{p_2,r_2}$.
- The space $\dot{B}^0_{p,1}$ is continuously embedded in the set of bounded continuous functions (going to zero at infinity if, additionally, $p < \infty$).

The following product estimates in Besov spaces play a key role in our analysis of the bilinear terms of (2.1) (see [11]).

Proposition 3.5. Let $s > 0$ and $1 \leq p, r \leq \infty$. Then $\dot{B}^s_{p,r} \cap L^\infty$ is an algebra and
\[
\|uv\|_{\dot{B}^s_{p,r}} \lesssim \|u\|_{L^\infty} \|v\|_{\dot{B}^s_{p,r}} + \|v\|_{L^\infty} \|u\|_{\dot{B}^s_{p,r}}.
\]

Let the real numbers $s_1, s_2, p_1$ and $p_2$ be such that
\[
s_1 + s_2 > 0, \quad s_1 \leq \frac{d}{p_1}, \quad s_2 \leq \frac{d}{p_2}, \quad s_1 \geq s_2, \quad \frac{1}{p_1} + \frac{1}{p_2} \leq 1.
\]

Then it holds that
\[
\|uv\|_{\dot{B}^{s_1}_{p_1,1}} \lesssim \|u\|_{\dot{B}^{s_2}_{p_2,1}} \|v\|_{\dot{B}^{s_2}_{p_2,1}} \quad \text{with} \quad \frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{s_1}{d}.
\]

Additionally, for exponents $s > 0$ and $1 \leq p_1, p_2, q \leq \infty$ satisfying
\[
\frac{d}{p_1} + \frac{d}{p_2} - d \leq s \leq \min\left(\frac{d}{p_1}, \frac{d}{p_2}\right) \quad \text{and} \quad \frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{s}{d},
\]

one has
\[
\|uv\|_{\dot{B}^{-s}_{p_1,\infty}} \lesssim \|u\|_{\dot{B}^{s}_{p_1,1}} \|v\|_{\dot{B}^{-s}_{p_2,\infty}}.
\]

System (2.1) also involves compositions of functions (through $I(a), k(x), \tilde{a}(x)$ and $\tilde{\eta}(x)$) and they are bounded according to the following conclusion (see [11]).

Proposition 3.6. Let $F : \mathbb{R} \to \mathbb{R}$ be smooth with $F(0) = 0$. For all $1 \leq p, r \leq \infty$ and $s > 0$, it holds that $F(u) \in \dot{B}^s_{p,r} \cap L^\infty$ for $u \in \dot{B}^s_{p,r} \cap L^\infty$, and
\[
\|F(u)\|_{\dot{B}^s_{p,r}} \leq C \|u\|_{\dot{B}^s_{p,r}}
\]
with $C$ depending only on $\|u\|_{L^\infty}$, $F'$ (and higher derivatives), $s$, $p$ and $d$.

In the case $s > -\min\left(\frac{d}{p_1}, \frac{d}{p_2}\right)$ then $u \in \dot{B}^s_{p,r} \cap \dot{B}^{-d}_{p,1}$ implies that $F(u) \in \dot{B}^s_{p,r} \cap \dot{B}^{-d}_{p,1}$, and
\[
\|F(u)\|_{\dot{B}^s_{p,r}} \leq C(1 + \|u\|_{\dot{B}^{-d}_{p,1}}) \|u\|_{\dot{B}^s_{p,r}}.
\]

The following commutator estimates (see [11]) have been used in the high-frequency estimate of the proof of Theorem 2.1.
Proposition 3.7. Let \( 1 \leq p, p_1 \leq \infty \) and
\[
- \min \left( \frac{d}{p_1}, \frac{d}{p} \right) < s \leq 1 + \min \left( \frac{d}{p'}, \frac{d}{p_1'} \right) \quad \text{with} \quad \frac{1}{p} + \frac{1}{p'} = 1.
\]
There exists a constant \( C > 0 \) depending only on \( s \) such that for all \( j \in \mathbb{Z} \) and \( \ell \in \{1, \cdots, d\} \), it holds that
\[
\| [v \cdot \nabla, \partial_t \Delta_j] a \|_{L^p} \leq C c_j 2^{-j(s-1)} \| \nabla v \|_{\dot{B}^{s-1}_{p_1,1}} \| \nabla a \|_{\dot{B}^{s-1}_{p,1}},
\]
where the commutator \([\cdot, \cdot]\) is defined by \([f, g] = fg - gf\) and \((c_j)_{j \in \mathbb{Z}}\) denotes a sequence such that \(\|(c_j)\|_{\ell^1} \leq 1\).

Finally, we end this section with those inequalities of Gagliardo-Nirenberg type, which can be found by the work of Sohinger and Strain [36] (see also Chap. 2 of [1] or [44]).

Proposition 3.8. The following interpolation inequalities hold true:
\[
\| \Lambda^\ell f \|_{L^r} \lesssim \| \Lambda^m f \|_{L^q}^{1-\theta} \| \Lambda^k f \|_{L^q}^\theta,
\]
whenever \( 0 \leq \theta \leq 1, 1 \leq q \leq r \leq \infty \) and
\[
\ell + d \left( \frac{1}{q} - \frac{1}{r} \right) = m(1-\theta) + k\theta.
\]

4. Low-frequency and high-frequency analysis

We are going to establish a Lyapunov-type inequality for energy norms by using a pure energy argument. For clarity, the proof is divided into several steps. In this section, we first derive the low-frequency and high-frequency estimates.

4.1. Low-frequency estimates. Let \( \Lambda^s z \triangleq \mathcal{F}^{-1}(|\xi|^s \mathcal{F} z) \) \((s \in \mathbb{R})\). Denote by \( \omega = \Lambda^{-1} \text{curl} u \) the incompressible part of \( u \) and by \( v = \Lambda^{-1} \text{div} u \) the compressible part of \( u \).

Therefore, we see that \( \omega \) satisfies the heat equation
\[
\partial_t \omega - \mu_\infty \Delta \omega = G_1, \quad G_1 \triangleq \Lambda^{-1} \text{curl} g.
\]

On the other hand, it is easy to check that \((a, v)\) satisfies
\[
\begin{align*}
\partial_t a + \Lambda v &= f, \\
\partial_t v - \Delta v - \Lambda a &= h, \quad h \triangleq \Lambda^{-1} \text{div} g.
\end{align*}
\]

Lemma 4.1. Let \( k_0 \) be some integer. Then it holds that for all \( t \geq 0 \),
\[
\frac{d}{dt} \| (a, v) \|_{\dot{B}^d_{2,1}}^\ell \lesssim \| (f, g) \|_{\dot{B}^d_{2,1}}^{\ell},
\]
where
\[
\| z \|_{\dot{B}^d_{2,1}}^\ell \triangleq \sum_{k \leq k_0} 2^{ks} \| \Delta_k z \|_{L^2} \quad \text{for} \quad s \in \mathbb{R}.
\]

Proof. Set \( z_k \triangleq \Delta_k z \). One may apply the operator \( \Delta_k \) to (4.2). By using the standard energy argument, we arrive at the following three equalities:
\[
\frac{1}{2} \frac{d}{dt} \| a_k \|_{L^2}^2 + \| v_k \|_{L^2}^2 + \| \Delta v_k \|_{L^2}^2 = (f_k |a_k|) + (h_k |v_k|),
\]
Lemma 4.2. \[ \frac{1}{2} \frac{d}{dt} (v_k | \Lambda a_k) + \| \Lambda a_k \|_{L^2}^2 = - (\Delta v_k | \Lambda a_k) + \| \Lambda v_k \|_{L^2}^2 - (h_k | \Lambda a_k) - (v_k | \Lambda f_k) \]

and

\[ \frac{1}{2} \frac{d}{dt} \| \Lambda a_k \|_{L^2}^2 = (\Delta v_k | \Lambda a_k) + (f_k | \Lambda^2 a_k), \]

from which one can deduce that

\[ \frac{1}{2} \frac{d}{dt} L_k^2(t) + \| (\Lambda a_k, \Lambda v_k) \|_{L^2}^2 \]

\[ = 2 (f_k | a_k) + 2 (h_k | v_k) - (h_k | \Lambda a_k) - (v_k | \Lambda f_k) + (f_k | \Lambda^2 a_k) \]

with \( L_k^2 \triangleq 2 (\| a_k \|_{L^2}^2 + \| v_k \|_{L^2}^2) + \| \Lambda a_k \|_{L^2}^2 - 2 (v_k | \Lambda a_k) \). It follows from Young’s inequality that \( L_k^2 \approx \| (a_k, \Lambda a_k, v_k) \|_{L^2}^2 \approx \| (a_k, v_k) \|_{L^2}^2 \) for \( k \leq k_0 \). Consequently, we get the following inequality

\[ \frac{1}{2} \frac{d}{dt} L_k^2 + 2^{2k} L_k^2 \lesssim \| (f_k, h_k) \|_{L^2}^2 \]

which implies that

\[ \frac{d}{dt} L_k + 2^{2k} L_k \lesssim \| (f_k, h_k) \|_{L^2} \]

for \( k \leq k_0 \). So multiplying both sides by \( 2^{k(d/2-1)} \) and summing up on \( k \leq k_0 \) imply that (1.3).

4.2. High-frequency estimates. In the high-frequency regime, the problem is that the structure of \( f \) would cause a loss of one derivative as there is no smoothing effect for \( a \). To overcome the difficulty, as in [16], we perform the energy argument in terms of the effective velocity

\[ w \triangleq \nabla (-\Delta)^{-1} (a - \text{div} u). \]

Note that if (2.1) is written in terms of \( a \), \( w \) and of the divergence free part \( P u \) of \( u \), then, up to low order terms, \( a \) satisfies a damped transport equation, and both \( w \) and \( P u \) satisfy a heat equation. Thanks to the structure of the system, we can establish the high-frequency estimates.

Lemma 4.2. Let \( k_0 \in \mathbb{Z} \) be chosen suitably large, it holds that for all \( t \geq 0 \),

\[ \frac{d}{dt} \| (\nabla a, u) \|_{B^{\frac{d}{2}-1}_{p,1}} + \| \nabla a \|_{B^{\frac{d}{2}-1}_{p,1}} + \| u \|_{B^{\frac{d}{2}+1}_{p,1}} \]

(4.11)

\[ \lesssim \| f \|_{B^{\frac{d}{2}-1}_{p,1}} + \| g \|_{B^{\frac{d}{2}-1}_{p,1}} + \| \nabla u \|_{B^{\frac{d}{2}+1}_{p,1}} \| a \|_{B^{\frac{d}{2}+1}_{p,1}}, \]

where

\[ \| z \|_{B^{s}_{2,1}} \triangleq \sum_{k \geq k_0 - 1} 2^{ks} \| \hat{\Delta}_k z \|_{L^2} \quad \text{for} \ s \in \mathbb{R}. \]

Proof. Note that \( P u \) satisfies

\[ \partial_t P u - \mu_\infty \Delta P u = P g. \]

Applying \( \hat{\Delta}_k \) to the above equation yields for all \( k \in \mathbb{Z} \),

\[ \partial_t P u_k - \mu_\infty \Delta P u_k = P g_k \quad \text{with} \ u_k \triangleq \Delta_k u \ \text{and} \ g_k \triangleq \hat{\Delta}_k g. \]
Then, multiplying each component of the above equation by $|(P u_k)^i|^p (P u_k)^i$ and integrating over $\mathbb{R}^d$ gives for $i = 1, \cdots, d$,

$$
\frac{1}{p} \frac{d}{dt} \|P u_k^i\|_{L_p}^p - \mu_\infty \int \Delta (P u_k)^i |(P u_k)^i|^p - (P u_k)^i dx = \int |(P u_k)^i|^p - (P u_k)^i P g_k dx.
$$

The key observation is that the second term of the l.h.s., although not spectrally localized, may be bounded from below as if it were (see Proposition 3.1). After summation on $i = 1, \cdots, d$, we end up (for some constant $c_p$ depending only $p$) with

$$
\frac{1}{p} \frac{d}{dt} \|P u_k\|_{L_p}^p + c_p \mu_\infty 2^{2k} \|P u_k\|_{L_p}^p \leq \|P g_k\|_{L_p} \|P u_k\|_{L_p}^{p-1},
$$

which leads to

$$
\frac{d}{dt} \|P u_k\|_{L_p} + c_p \mu_\infty 2^{2k} \|P u_k\|_{L_p} \leq \|P g_k\|_{L_p}.
$$

On the other hand, it is clear that $w$ fulfills

$$
\partial_t w - \Delta w = \nabla (-\Delta)^{-1} (f - \text{div} g) + w - (-\Delta)^{-1} \nabla a.
$$

Hence, arguing exactly as for proving (4.13) shows that for $w_k \triangleq \hat{\Delta} w$:

$$
\frac{1}{p} \frac{d}{dt} \|w_k\|_{L_p}^p + c_p 2^{2k} \|w_k\|_{L_p}^p \leq \left( \|\nabla (-\Delta)^{-1} (f_k - \text{div} g_k)\|_{L_p} + \|w_k - (-\Delta)^{-1} \nabla a_k\|_{L_p} \right) \|w_k\|_{L_p}^{p-1}.
$$

Similarly, we obtain

$$
\frac{d}{dt} \|w_k\|_{L_p} + c_p 2^{2k} \|w_k\|_{L_p} \leq \|\nabla (-\Delta)^{-1} (f_k - \text{div} g_k)\|_{L_p} + \|w_k - (-\Delta)^{-1} \nabla a_k\|_{L_p}.
$$

In terms of $w$, the function $a$ satisfies the following damped transport equation:

$$
\partial_t a + \text{div} (au) + a = -\text{div} w.
$$

Then, applying the operator $\partial_i \hat{\Delta} a$ to (4.17) and denoting $R_k^i \triangleq |u \cdot \nabla, \partial_i \hat{\Delta} a|a$ gives

$$
\partial_t \partial_i a_k + u \cdot \nabla \partial_i a_k + \partial_i a_k = -\partial_i \hat{\Delta} a (\text{div} u) - \partial_i \text{div} w_k + R_k^i, \quad i = 1, \cdots, d.
$$

Multiplying by $|\partial_i a_k|^{p-2} \partial_i a_k$, integrating on $\mathbb{R}^d$, and performing an integration by parts in the second term of (4.18), one can get

$$
\frac{1}{p} \frac{d}{dt} \|\partial_i a_k\|_{L_p}^p + \|\partial_i a_k\|_{L_p}^p = \frac{1}{p} \int \text{div} u |\partial_i a_k|^p dx

+ \int \left( R_k^i - \partial_i \hat{\Delta} a (\text{div} u) - \partial_i \text{div} w_k \right) |\partial_i a_k|^{p-2} \partial_i a_k dx.
$$

Summing up on $i = 1, \cdots, d$ and applying Hölder and Bernstein inequalities give

$$
\frac{1}{p} \frac{d}{dt} \|\nabla a_k\|_{L_p}^p + \|\nabla a_k\|_{L_p}^p \leq \left( \frac{1}{p} \|\text{div} u\|_{L^\infty} \|\nabla a_k\|_{L_p} + \|\nabla \hat{\Delta} a (\text{div} u)\|_{L_p}

+ C 2^{2k} \|w_k\|_{L_p} + \|R_k\|_{L_p} \right) \|\nabla a_k\|_{L_p}^{p-1},
$$
which leads to
\[
\frac{d}{dt} \| \nabla a_k \|_{L^p} + \| \nabla a_k \|_{L^p} \\
\leq \left( \frac{1}{p} \| \text{div} u \|_{L^\infty} \| \nabla a_k \|_{L^p} + \| \nabla \Delta_k (\text{adiv} u) \|_{L^p} + C 2^{2k} \| w_k \|_{L^p} + \| R_k \|_{L^p} \right).
\]

Furthermore, adding up (4.20) (multiplying by \( \gamma c_p \) for some \( \gamma > 0 \)) to (4.13) and (4.16) yields
\[
\frac{d}{dt} \left( \| P u_k \|_{L^p} + \| w_k \|_{L^p} + \gamma c_p \| \nabla a_k \|_{L^p} \right) + c_p 2^{2k} (\mu_\infty \| P u_k \|_{L^p} + \| w_k \|_{L^p}) \\
+ \gamma c_p \| \nabla a_k \|_{L^p} \leq \| P g_k \|_{L^p} + \| \nabla (-\Delta)^{-1} (f_k - \text{div} g_k) \|_{L^p} \\
+ \gamma c_p \left( \frac{1}{p} \| \text{div} u \|_{L^\infty} \| \nabla a_k \|_{L^p} + \| \nabla \Delta_k (\text{adiv} u) \|_{L^p} + \| R_k \|_{L^p} \right) \\
+ C \gamma c_p 2^{2k} \| w_k \|_{L^p} + \| w_k - (-\Delta)^{-1} \nabla a_k \|_{L^p}.
\]

Noticing that \((-\Delta)^{-1}\) is a homogeneous Fourier multiplier of degree \(-2\), we have
\[
\| (-\Delta)^{-1} \nabla a_k \|_{L^p} \lesssim 2^{-2k} \| \nabla a_k \|_{L^p} \lesssim 2^{-2k_0} \| \nabla a_k \|_{L^p} \quad \text{for all } k \geq k_0 - 1.
\]

Choosing \( k_0 \) suitably large and \( \gamma \) sufficiently small, we deduce that there exists a constant \( c_0 \) such that for all \( k \geq k_0 - 1 \),
\[
\frac{d}{dt} \left( \| P u_k \|_{L^p} + \| w_k \|_{L^p} + \| \nabla a_k \|_{L^p} \right) + c_0 (2^{2k} \| P u_k \|_{L^p} + 2^{2k} \| w_k \|_{L^p} + \| \nabla a_k \|_{L^p}) \\
\lesssim 2^{-k} \| f_k \|_{L^p} + \| g_k \|_{L^p} \\
+ \left( \| \text{div} u \|_{L^\infty} \| \nabla a_k \|_{L^p} + \| \nabla \Delta_k (\text{adiv} u) \|_{L^p} + \| R_k \|_{L^p} \right).
\]

Since
\[
(4.21) \quad u = w - \nabla (-\Delta)^{-1} a + P u,
\]
it follows that
\[
\frac{d}{dt} \| (\nabla a_k, u_k) \|_{L^p} + c_0 \| (\nabla a_k, 2^{2k} u_k) \|_{L^p} \\
(4.22) \lesssim \| (2^{-k} f_k, g_k) \|_{L^p} + \| \text{div} u \|_{L^\infty} \| \nabla a_k \|_{L^p} + \| \nabla \Delta_k (\text{adiv} u) \|_{L^p} + \| R_k \|_{L^p}.
\]

Hence, by multiplying both sides (4.22) by \( 2^{k(\frac{1}{p} - 1)} \) and summing up over \( k \geq k_0 - 1 \), we arrive at (4.11).

5. THE EVOLUTION OF NEGATIVE BESOV NORM

In this section, we establish the evolution of the negative Besov norms at low frequencies, which plays the key role in deriving the Lyapunov-type inequality for energy norms. To this end, we need some non classical product estimates as follows.

Proposition 5.1. Let the real numbers \( s_1, s_2, p_1 \) and \( p_2 \) be such that
\[
s_1 + s_2 \geq 0, \quad s_1 \leq \frac{d}{p_1}, \quad s_2 < \min \left( \frac{d}{p_1}, \frac{d}{p_2} \right) \quad \text{and} \quad \frac{1}{p_1} + \frac{1}{p_2} \leq 1.
\]

Then it holds that
\[
\| fg \|_{\dot{B}^{s_2}_{q,\infty}} \lesssim \| f \|_{\dot{B}^{s_1}_{p_1,1}} \| g \|_{\dot{B}^{s_2}_{p_2,\infty}} \quad \text{with} \quad \frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{s_1}{d}.
\]
Proof. Here, we need the so-called Bony decomposition for the product of two tempered distributions $f$ and $g$ (see for example [1]):

$$fg = T_1g + R(f, g) + T_2f,$$

where the paraproduct between $f$ and $g$ is defined by

$$T_1g := \sum_j \hat{S}_{j-1}f \hat{\Delta}g \quad \text{with} \quad \hat{S}_{j-1} := \chi(2^{-(j-1)}D),$$

and the remainder $R(f, g)$ is given by the series:

$$R(f, g) := \sum_j \hat{\Delta}f (\hat{\Delta}_{j-1}g + \hat{\Delta}_jg + \hat{\Delta}_{j+1}g).$$

Case 1: $s_2 \geq 0$. It follows from the definition of $T_2f$ and the spectral cut-off property that

$$\hat{\Delta}f T_2f = \hat{\Delta}f \left( \sum_{j'} \hat{S}_{j'-1}g \hat{\Delta}j'f \right) = \sum_{|j-j'| \leq 4} \hat{\Delta}j (\hat{S}_{j'-1}g \hat{\Delta}j'f),$$

where $\hat{S}_{j'-1}g$ can be given by $\sum_{k \leq j'-2} \hat{\Delta}_kg$. Hence, it follows from the Hölder inequality and Bernstein inequality [5,6] that

$$\|\hat{\Delta}j T_2f\|_{L^p} \lesssim \sum_{|j-j'| \leq 4} \|\hat{S}_{j'-1}g \hat{\Delta}j'f\|_{L^p} \leq \left\{ \begin{array}{ll} \sum_{|j-j'| \leq 4} \sum_{k \leq j'-2} \|\hat{\Delta}_kg\|_{L^m} \|\hat{\Delta}j'f\|_{L^{p_1}}, & \frac{1}{p_2} = \frac{1}{m} + \frac{1}{p_1} (p_2 \leq p_1); \\ \sum_{|j-j'| \leq 4} \sum_{k \leq j'-2} \|\hat{\Delta}_kg\|_{L^\infty} \|\hat{\Delta}j'f\|_{L^{p_2}}, & p_2 > p_1 \end{array} \right.$$}

Furthermore, the right side of (5.4) can be bounded by

$$\sum_{|j-j'| \leq 4} 2^{j'(\frac{d}{m_1} - s_1 - s_2)} 2^{j,s_1} \|\hat{\Delta}j'f\|_{L^{p_1}} \|g\|_{\dot{B}^{m_2}_{2,\infty}},$$

provided $s_2 < \min(d/p_1, d/p_2)$. Consequently, we are led to

$$\|T_2f\|_{\dot{B}^{m_2}_{2,\infty}} \lesssim \|f\|_{\dot{B}^{s_1}_{p_1,1}} \|g\|_{\dot{B}^{m_2}_{2,\infty}},$$

and thus

$$\|T_2f\|_{\dot{B}^{p_2}_{2,\infty}} \lesssim \|f\|_{\dot{B}^{s_1}_{p_1,1}} \|g\|_{\dot{B}^{p_2}_{2,\infty}}.$$

Note that $s_1 \geq -s_2$ by assumption. If $s_1 \geq 0$, then we use the fact that $T$ maps $L^{q_1} \times \dot{B}^{s_1}_{p_2,\infty}$ to $\dot{B}^{s_1}_{q_2,\infty}$, that is,

$$\|T_2g\|_{\dot{B}^{s_1}_{q_2,\infty}} \lesssim \|f\|_{L^{q_1}} \|g\|_{\dot{B}^{s_1}_{p_2,\infty}} \quad \text{with} \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{p_2}.$$

Noting that the embedding $\dot{B}^{s_1}_{p_1,1} \hookrightarrow L^{q_1}$ with $\frac{1}{q_1} = \frac{1}{p_1} - \frac{s_1}{d}$, we arrive at

$$\|T_2g\|_{\dot{B}^{s_1}_{q_1,\infty}} \lesssim \|f\|_{\dot{B}^{s_1}_{p_1,1}} \|g\|_{\dot{B}^{s_1}_{p_2,\infty}}.$$
If \(-s_2 \leq s_1 < 0\), then we use the fact that \(T\) maps \(\dot{B}_{p_1,1}^{s_1} \times \dot{B}_{p_2,\infty}^{s_2}\) to \(\dot{B}_{q,\infty}^{s_1+s_2}\) to get
\[
(5.9) \quad \|Tfg\|_{\dot{B}_{q,\infty}^{s_1+s_2}} \lesssim \|f\|_{\dot{B}_{p_1,1}^{s_1}} \|g\|_{\dot{B}_{p_2,\infty}^{s_2}}.
\]

Case 2: \(s_2 < 0\). In this case, the fact that \(T\) maps \(\dot{B}_{p_2,\infty}^{s_2} \times \dot{B}_{p_1,\infty}^{s_1}\) to \(\dot{B}_{p_1,\infty}^{s_1+s_2}\) enables us to obtain
\[
(5.10) \quad \|Tg_{1}f\|_{\dot{B}_{p_1,\infty}^{s_1+s_2}} \lesssim \|g\|_{\dot{B}_{p_2,\infty}^{s_2}} \|f\|_{\dot{B}_{p_1,\infty}^{s_1}}
\]
with \(1/p = 1/p_1 + 1/p_2\). Noticing that the embedding \(\dot{B}_{p_1,\infty}^{s_1+s_2} \hookrightarrow \dot{B}_{q,\infty}^{s_2}\), we arrive at
\[
(5.11) \quad \|Tg_{1}f\|_{\dot{B}_{q,\infty}^{s_2}} \lesssim \|f\|_{\dot{B}_{p_1,1}^{s_1}} \|g\|_{\dot{B}_{p_2,\infty}^{s_2}}.
\]

In addition, we have \(s_1 \geq -s_2 > 0\), and thus \((5.7), (5.8)\) still remains valid.

To bound the term \(R(f, g)\), one can use that the remainder operator \(R\) maps \(\dot{B}_{p_1,1}^{s_1} \times \dot{B}_{p_2,\infty}^{s_2}\) to \(\dot{B}_{p_1,\infty}^{s_1+s_2}\) with \(\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}\), and the embedding \(\dot{B}_{p_1,\infty}^{s_1+s_2} \hookrightarrow \dot{B}_{q,\infty}^{s_2}\). Therefore, the proof of Proposition \((5.1)\) is finished. \(\square\)

**Corollary 5.1.** Let \(1 - \frac{d}{2} < \sigma_1 \leq \sigma_0\). The following two inequalities hold true:
\[
(5.12) \quad \|fg\|_{\dot{B}_{2,\infty}^{\sigma_1}} \lesssim \|f\|_{\dot{B}_{p,1}^{\sigma_1}} \|g\|_{\dot{B}_{2,\infty}^{\sigma_1}},
\]
and
\[
(5.13) \quad \|fg\|_{\dot{B}_{2,\infty}^{\sigma_2}} \lesssim \|fg\|_{\dot{B}_{q,\infty}^{\sigma_1} + \frac{d}{2} + \frac{s_2}{2} + 1} \lesssim \|f\|_{\dot{B}_{p_1,1}^{\sigma_2}} \|g\|_{\dot{B}_{p_2,\infty}^{\sigma_2}},
\]
with \(1/q = 1/p + 1/d\).

**Proof.** Observe that \(\sigma_1 \leq \sigma_0 \leq \frac{d}{p}\) and \(-\sigma_1 < \frac{d}{2} - 1 \leq \frac{d}{p}\) if \(2 \leq p \leq d^*\). Hence, \((5.12)\) follows from Proposition \((5.1)\) with \(s_1 = \frac{d}{p}, s_2 = -\sigma_1, p_1 = p\) and \(p_2 = 2\). The second inequality in \((5.13)\) follows from Proposition \((5.1)\) with \(s_1 = \frac{d}{p} - 1, s_2 = -\sigma_1 + \frac{d}{2} - \frac{s_2}{2} + 1\) and \(p_1 = p_2 = p\). In addition, the first inequality in \((5.13)\) is achieved by the embedding \(\dot{B}_{2,\infty}^{\sigma_1 + \frac{d}{2} - \frac{s_2}{2} + 1} \hookrightarrow \dot{B}_{2,\infty}^{\sigma_2}\), since the relation \(2 \leq p \leq d^*\) indicates \(1 \leq q \leq 2\). \(\square\)

Now, we begin to bound the evolution of negative Besov norm, which is the main ingredient in the proof of Theorem \((2.1)\).

**Lemma 5.1.** Let \(1 - \frac{d}{2} < \sigma_1 \leq \sigma_0\) and \(p\) satisfy \((1.7)\). It holds that
\[
(5.14) \quad \left(\|(a, u)(t)\|_{\dot{B}_{2,\infty}^{\sigma_1}}\right)^2 \lesssim \left(\|(a_0, u_0)\|_{\dot{B}_{2,\infty}^{\sigma_1}}\right)^2
\]
\[\quad + \int_0^t \left(D_1^p(\tau) + D_2^p(\tau)\right)\left(\|(a, u)(\tau)\|_{\dot{B}_{2,\infty}^{\sigma_1}}\right)^2 d\tau + \int_0^t D_3^p(\tau)\|(a, u)(\tau)\|_{\dot{B}_{2,\infty}^{\sigma_1}} d\tau,
\]
where
\[
D_1^p(t) \triangleq \|(a, u)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|a\|_{\dot{B}_{p_1,1}^{\frac{d}{2}}} + \|u\|_{\dot{B}_{p_1,1}^{\frac{d}{2}+1}},
\]
\[
D_2^p(t) \triangleq \|a\|_{\dot{B}_{p_1,1}^{\frac{d}{2}}},
\]
\[
D_3(t) \triangleq \left(\|(a, u)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|a\|_{\dot{B}_{p_1,1}^{\frac{d}{2}}} + \|u\|_{\dot{B}_{p_1,1}^{\frac{d}{2}+1}}\right)\left(\|a\|_{\dot{B}_{p_1,1}^{\frac{d}{2}}} + \|u\|_{\dot{B}_{p_1,1}^{\frac{d}{2}+1}}\right).
\]
Proof. It follows from (4.4) that
\[
\frac{1}{2} \frac{d}{dt} \| (a_k, v_k) \|_{L^2}^2 + \| \Lambda v_k \|_{L^2}^2 \leq (\| f_k \|_{L^2} + \| h_k \|_{L^2}) \| (a_k, v_k) \|_{L^2}.
\]

By performing a routine procedure, one can arrive at
\[
\left( \| (a, u)(t) \|_{B_{2,1}^{-\sigma_1}}^\ell \right)^2 \lesssim \left( \| (a_0, u_0) \|_{B_{2,1}^{-\sigma_1}}^\ell \right)^2 + \int_0^t \| (f, g) \|_{B_{2,\infty}^{-\sigma_1}}^\ell \| (a, u)(\tau) \|_{B_{2,\infty}^{-\sigma_1}}^\ell d\tau.
\]

In what follows, we focus on the nonlinear norm \( \| (f, g) \|_{B_{2,\infty}^{-\sigma_1}}^\ell \). To this end, it is convenient to decompose \( f \) and \( g \) in terms of low-frequency and high-frequency parts:
\[
f = f^\ell + f^h
\]
with
\[
f^\ell \triangleq -a \text{ div } u^\ell - u \cdot \nabla a^\ell, \quad f^h \triangleq -a \text{ div } u^h - u \cdot \nabla a^h
\]
and
\[
g = g^\ell + g^h
\]
with
\[
g^\ell \triangleq -u \cdot \nabla u^\ell - k(a) \nabla a^\ell + g_3(a, u^\ell) + g_4(a, u^\ell), \quad g^h \triangleq -u \cdot \nabla u^h - k(a) \nabla a^h + g_3(a, u^h) + g_4(a, u^h),
\]
where
\[
g_3(a, v) = \frac{1}{1 + a} \left( 2\tilde{\mu}(a) \text{ div } D(v) + \tilde{\lambda}(a) \nabla \text{ div } v \right) - I(a) \mathcal{A}v, \\
g_4(a, v) = \frac{1}{1 + a} \left( 2\tilde{\mu}'(a) D(v) \cdot \nabla a + \tilde{\lambda}'(a) \text{ div } v \nabla a \right)
\]
and
\[
z^\ell \triangleq \sum_{k \leq k_0} \hat{\Delta}_k z, \quad z^h \triangleq z - z^\ell \quad \text{for } z = a, u.
\]

We use (5.12)-(5.13) to estimate those terms of \( f \) and \( g \) with \( a^\ell \) or \( u^\ell \). For instance, for \( a \text{ div } u^\ell = (a^\ell + a^h) \text{ div } u^\ell \), one can get from (5.12) that
\[
\| a^\ell \text{ div } u^\ell \|_{B_{2,\infty}^{-\sigma_1}}^\ell \lesssim \| \text{ div } u^\ell \|_{B_{\frac{3}{2},2}^{-\sigma_1}}^\ell \| a^\ell \|_{B_{2,1}^{-\sigma_1}}^\ell \lesssim \| u \|_{B_{2,1}^{-\sigma_1}}^\ell \| a \|_{B_{2,\infty}^{-\sigma_1}}^\ell
\]
and
\[
\| a^h \text{ div } u^\ell \|_{B_{2,\infty}^{-\sigma_1}}^\ell \lesssim \| a^h \|_{B_{\frac{3}{2},2}^{-\sigma_1}}^h \| \text{ div } u^\ell \|_{B_{2,1}^{-\sigma_1}}^\ell \lesssim \| a \|_{B_{2,1}^{-\sigma_1}}^h \| u \|_{B_{2,\infty}^{-\sigma_1}}^\ell.
\]
Therefore,
\[
\| a \text{ div } u^\ell \|_{B_{2,\infty}^{-\sigma_1}}^\ell \lesssim \left( \| u \|_{B_{2,1}^{-\sigma_1}}^\ell + \| a \|_{B_{2,1}^{-\sigma_1}}^h \right) \| (a, u) \|_{B_{2,\infty}^{-\sigma_1}}^\ell.
\]

The estimates of \( a \cdot \nabla a^\ell \) and \( u \cdot \nabla u^\ell \) follow from essentially the same procedures as \( a \text{ div } u^\ell \) so that
\[
\| a \cdot \nabla a^\ell \|_{B_{2,\infty}^{-\sigma_1}}^\ell \lesssim \left( \| a \|_{B_{2,1}^{-\sigma_1}}^\ell + \| a \|_{B_{2,1}^{-\sigma_1}}^h \right) \| (a, u) \|_{B_{2,\infty}^{-\sigma_1}}^\ell
\]
\[
\| u \cdot \nabla u^\ell \|_{B_{2,\infty}^{-\sigma_1}}^\ell \lesssim \left( \| u \|_{B_{2,1}^{-\sigma_1}}^\ell + \| u \|_{B_{2,1}^{-\sigma_1}}^h \right) \| u \|_{B_{2,\infty}^{-\sigma_1}}^\ell.
\]
Bounding nonlinear terms involving composition functions is more elaborate. For example, let us take a look at the term $k(a) \nabla a^\ell$. Keeping in mind that $k(0) = 0$, one may write

$$k(a) = k'(0)a + \tilde{k}(a)a$$

for some smooth function $\tilde{k}$ vanishing at 0. Arguing similarly as (5.17) gives

$$\|k'(0)a \nabla a^\ell\|_{B^0_{2,\infty}} \lesssim \left(\|a\|_{B^0_{2,1}} + \|a\|_{H^\infty}^h\right)\|a\|_{B^0_{2,\infty}}.$$  

(5.20)

On the other hand, it follows from Propositions 3.3, 3.6 that

$$\|\tilde{k}(a) \nabla a^\ell\|_{B^0_{2,\infty}} \lesssim \|a\|_{B^0_{2,1}}^2 \|\nabla a^\ell\|_{\dot{B}^{-s_1}_2} \lesssim \|a\|_{B^0_{2,1}}^2 \|a\|_{B^0_{2,\infty}}.$$  

(5.21)

Putting (5.20) and (5.21) together leads to

$$\|k(a) \nabla a^\ell\|_{B^0_{2,\infty}} \lesssim \left(\|a\|_{B^{d+1}_{2,1}} + \|a\|_{H^\infty}^h + \|a\|_{B^0_{2,1}}^2\right)\|a\|_{B^0_{2,\infty}}.$$  

(5.22)

Similarly,

$$\|g_3(a, u^\ell)\|_{B^0_{2,\infty}} \lesssim \left(\|a\|_{B^{d+1}_{2,1}} + \|a\|_{H^\infty}^h + \|a\|_{B^0_{2,1}}^2\right)\|(a, u)\|_{B^0_{2,\infty}}.$$  

(5.23)

Next, we estimate $g_4(a, u^\ell)$. It suffices to estimate the first term in $g_4(a, u^\ell)$, since the second one can be similarly handled. Denote by $J(a)$ the smooth function fulfilling $J'(a) = \frac{2\mu'(a)}{1+a}$ and $J(0) = 0$, so that $\nabla J(a) = \frac{2\mu'(a)}{1+a} \nabla a$. For convenience, we use the decomposition $J(a) = J'(0)a + J(a)$. Then, it follows from (5.12) that

$$\|\nabla a^\ell \cdot D(u^\ell)\|_{B^0_{2,\infty}} \lesssim \|\nabla a^\ell\|_{B^{d+1}_{2,1}} \|D(u^\ell)\|_{B^{-s_1}_2} \lesssim \|a\|_{B^{d+1}_{2,1}} \|u\|_{B^{-s_1}_2}.$$  

(5.24)

Regarding the high frequency part of $a^h$, one can get from (5.13)

$$\|\nabla a^h \cdot D(u^\ell)\|_{B^0_{2,\infty}} \lesssim \|\nabla a^h \cdot D(u^\ell)\|_{B^{-s_1+\frac{d}{p}+\frac{d}{q}+1}_{2,\infty}} \lesssim \|\nabla a^h\|_{B^0_{p,\infty}} \|D(u^\ell)\|_{B^{-s_1+\frac{d}{p}+\frac{d}{q}+1}_{2,\infty}},$$

(5.25)

where $1/q = 1/p + 1/d$. Furthermore, the embedding $\dot{B}^{-s_1+1}_{2,\infty} \hookrightarrow \dot{B}^{-s_1+\frac{d}{p}+\frac{d}{q}+1}_{p,\infty}$ implies

$$\|\nabla a^h \cdot D(u^\ell)\|_{B^0_{2,\infty}} \lesssim \|a\|_{B^0_{p,\infty}} \|u\|_{B^{-s_1}_2}.$$  

(5.26)

Moreover, the remaining term with $\tilde{J}(a)a$ can be estimated as similarly:

$$\|\nabla (\tilde{J}(a)a) \cdot D(u^\ell)\|_{B^0_{2,\infty}} \lesssim \|\nabla (\tilde{J}(a)a) \cdot D(u^\ell)\|_{B^{-s_1+\frac{d}{p}+\frac{d}{q}+1}_{2,\infty}} \lesssim \|\tilde{J}(a)a\|_{B^0_{p,\infty}} \|D(u^\ell)\|_{B^{-s_1+\frac{d}{p}+\frac{d}{q}+1}_{p,\infty}} \lesssim \|a\|_{B^0_{p,\infty}} \|u\|_{B^{-s_1}_2}.$$  

(5.27)

Hence, together with (5.24), (5.26) and (5.27), we can conclude that

$$\|g_4(a, u^\ell)\|_{B^0_{2,\infty}} \lesssim \left(\|a\|_{B^{d+1}_{2,1}} + \|a\|_{B^{d+1}_{p,1}} + \|a\|_{B^0_{p,1}}^2\right)\|u\|_{B^{-s_1}_2}.$$  

(5.28)

In addition, it seems to be difficult to get the suitable bounds for those terms of $f$ and $g$ with $a^h$ or $u^h$ if resorting to Proposition 5.1 only. We will take advantage of the following result, whose proof has been shown by [11].
Proposition 5.2. Let $k_0 \in \mathbb{Z}$, and denote $z^\ell \triangleq \hat{S}_{k_0} z$, $z^h \triangleq z - z^\ell$ and, for any $s \in \mathbb{R}$,
\[
\|z\|_{B^s_{2,\infty}} \triangleq \sup_{k \leq k_0} 2^{ks} \|\hat{\Delta}_k z\|_{L^2}.
\]
There exists a universal integer $N_0$ such that for any $2 \leq p \leq 4$ and $\sigma > 0$, we have
\[
\|fg^h\|_{B^{-\sigma}_{2,0}} \leq C(\|f\|_{B^{\sigma}_{p,1}} + \|\hat{S}_{k_0} f\|_{L^p}) \|g^h\|_{B^{-\sigma}_{p,0}}
\]
(5.29)
\[
\|f^h g\|_{B^{-\sigma}_{2,0}} \leq C(\|f^h\|_{B^{\sigma}_{p,1}} + \|\hat{S}_{k_0} f^h\|_{L^p}) \|g\|_{B^{-\sigma}_{p,0}}
\]
(5.30)
with $\sigma_0 \triangleq \frac{2d}{p} - \frac{d}{2}$ and $\frac{1}{p^*} \triangleq 1 - \frac{1}{p}$, and $C$ depending only on $k_0$, $d$ and $\sigma$.

Let us first consider the case $2 \leq p \leq d$. If $2 \leq p < d$, then (5.29) with $\sigma = \frac{d}{p} - 1$ yields
\[
\|fg^h\|_{B^{-\sigma_{1}}_{2,\infty}} \lesssim \|fg^h\|_{B^{-\sigma_{0}}_{2,0}} \lesssim \left(\|f\|_{B^{\frac{d}{p}-1}_{p,1}} + \|f^h\|_{L^p}\right) \|g^h\|_{B^{1-rac{d}{p}}_{p,1}},
\]
(5.31)
\[
\|f^h g\|_{B^{-\sigma_{1}}_{2,\infty}} \lesssim \|f^h\|_{B^{-\sigma_{0}}_{2,0}} \lesssim \|f^h\|_{L^d} \|g^h\|_{L^d} \lesssim \|f\|_{B^{1}_{d,1}} \|g^h\|_{B^{0}_{d,1}}.
\]
(5.32)
since $\sigma_1 \leq \sigma_0$. In the limit case $p = d$, one can get by the Sobolev embedding that
\[
\|fg^h\|_{B^{-\sigma_{1}}_{2,\infty}} \lesssim \|fg^h\|_{B^{-\sigma_{0}}_{2,0}} \lesssim \|f\|_{L^d} \|g^h\|_{L^d} \lesssim \|f\|_{B^{1}_{d,1}} \|g^h\|_{B^{0}_{d,1}}.
\]
Furthermore, using the embedding $\hat{B}^{d}_{2,1} \hookrightarrow L^p$ and the fact that $\frac{d}{2} - 1 \leq \frac{d}{p}$ and $1 - \frac{d}{p} \leq \frac{d}{p} - 1$, we can obtain
\[
\|fg^h\|_{B^{-\sigma_{1}}_{2,\infty}} \lesssim \left(\|f\|_{B^{\frac{d}{p}-1}_{p,1}} + \|f^h\|_{L^p}\right) \|g^h\|_{B^{1-rac{d}{p}}_{p,1}}.
\]
(5.33)
Therefore, we get the following estimates:
\[
\|adiv u^h\|_{B^{-\sigma_{1}}_{2,\infty}} \lesssim \left(\|a\|_{B^{\frac{d}{p}-1}_{p,1}} + \|a^h\|_{L^p}\right) \|u^h\|_{B^{1-rac{d}{p}}_{p,1}}.
\]
(5.34)
\[
\|u \cdot \nabla a^h\|_{B^{-\sigma_{1}}_{2,\infty}} \lesssim \left(\|u\|_{B^{\frac{d}{p}-1}_{p,1}} + \|u^h\|_{L^p}\right) \|a^h\|_{B^{1-rac{d}{p}}_{p,1}}.
\]
(5.35)
\[
\|u \cdot \nabla u^h\|_{B^{-\sigma_{1}}_{2,\infty}} \lesssim \left(\|u\|_{B^{\frac{d}{p}-1}_{p,1}} + \|u^h\|_{L^p}\right) \|u^h\|_{B^{1-rac{d}{p}}_{p,1}}.
\]
(5.36)

Next, using the composition inequality and the embeddings $\hat{B}^{d}_{2,1} \hookrightarrow L^p$ and $\hat{B}^{\sigma_0}_{p,1} \hookrightarrow L^p$ yields
\[
\|k(a)\|_{L^p} \lesssim \|a\|_{L^p} \lesssim \|a^\ell\|_{B^{d}_{2,1}} + \|a^h\|_{B^{\sigma_0}_{p,1}} \lesssim \|a^\ell\|_{B^{d-1}_{2,1}} + \|a^h\|_{B^{d}_{p,1}}
\]
and
\[
\|k(a)\|_{B^{d-1}_{p,1}} \lesssim \|a\|_{B^{d-1}_{p,1}} \lesssim \|a^\ell\|_{B^{d-1}_{2,1}} + \|a^h\|_{B^{d}_{p,1}}
\]
since $d/p - 1 > -d/p$ ($p < 2d$). Therefore, we arrive at
\[
\|k(a) \nabla a^h\|_{B^{-\sigma_{1}}_{2,\infty}} \lesssim \left(\|a\|_{B^{d-1}_{2,1}} + \|a^h\|_{B^{d}_{p,1}}\right) \|a^h\|_{B^{d}_{p,1}}.
\]
(5.37)
\[
\|g^3(a, u^h)\|_{B^{-\sigma_{1}}_{2,\infty}} \lesssim \left(\|a\|_{B^{d-1}_{2,1}} + \|a^h\|_{B^{d}_{p,1}}\right) \|u^h\|_{B^{d+1}_{p,1}}.
\]
(5.38)
Regarding the nonlinear term \( g_4(a, u^h) \), the calculation is a little bit careful. Let us first deal with the case \( \frac{d}{p} - \frac{d}{2} < \sigma_1 \leq \sigma_0 \) if \( p \leq d \). By taking \( f = \nabla K(a) \) and \( g = \nabla u \), it follows from Hölder inequality that

\[
\|g_4(a, u^h)\|_{B_{2, \infty}^{-\sigma_1}} \lesssim \|g_4(a, u^h)\|_{B_{2, \infty}^{-\sigma_0}} = \|\nabla K(a) \otimes \nabla u^h\|_{B_{2, \infty}^{-\sigma_0}} \lesssim \|\nabla K(a)\|_{L^p} \|\nabla u^h\|_{L^p}.
\]

The embeddings \( \dot{B}_{2,1}^{\frac{d}{p} - \frac{d}{2}} \hookrightarrow L^p \) and \( \dot{B}_{p,1}^{1 - \sigma_0} \hookrightarrow L^p \) yields that

\[
\|\nabla K(a)\|_{L^p} \lesssim \|\nabla a\|_{L^p} \lesssim \|\nabla a\|_{\dot{B}_{2,1}^{\frac{d}{p} - \frac{d}{2}}} + \|\nabla a^h\|_{\dot{B}_{p,1}^{1 - \sigma_0}}.
\]

Owing to \( -\sigma_1 < \frac{d}{2} - \frac{d}{p} \leq \frac{d}{2} - 1 \), the real interpolation in Proposition 3.3 and Young inequalities imply that

\[
\|\nabla a\|_{\dot{B}_{2,1}^{\frac{d}{p} - \frac{d}{2}}} = \|\nabla a\|_{\dot{B}_{2,1}^{1 - \sigma_1}} \|\nabla a\|_{\dot{B}_{2,1}^{1 - \sigma_1}} \lesssim \|a\|_{\dot{B}_{2,1}^{1 - \sigma_1}} + \|a\|_{\dot{B}_{2,1}^{1 - \sigma_1}},
\]

where

\[
\sigma_2 = \frac{\sigma_1 + \frac{d}{2} - \frac{d}{p}}{\sigma_1 + \frac{d}{2} - 1} \in (0, 1].
\]

Inserting (5.40)–(5.41) into (5.39) leads to

\[
\|g_4(a, u^h)\|_{B_{2, \infty}^{-\sigma_1}} \lesssim \left(\|a\|_{\dot{B}_{2,1}^{1 - \sigma_1}} + \|a\|_{\dot{B}_{p,1}^{1 - \sigma_0}}\right) \|u\|_{\dot{B}_{p,1}^{\frac{d}{d^*} + 1}} + \|u\|_{\dot{B}_{p,1}^{\frac{d}{d^*} + 1}}\|a\|_{\dot{B}_{2,1}^{1 - \sigma_1}}.
\]

Next, we turn to the case \( 1 - \frac{d}{2} < \sigma_1 \leq \frac{d}{p} - \frac{d}{2} \leq 0 \) if \( 2 \leq p \leq d \). By Sobolev embedding properties and Hölder inequality, we arrive at

\[
\|g_4(a, u^h)\|_{B_{2, \infty}^{-\sigma_1}} \lesssim \|\nabla K(a) \otimes \nabla u^h\|_{B_{2, \infty}^{0}} \lesssim \|\nabla K(a)\|_{L^{d}} \|\nabla u^h\|_{L^{d^*}},
\]

with \( 1/d + 1/d^* = 1/2 \). It follows from Proposition 3.3 and \( \dot{B}_{p,1}^{d - 1} \hookrightarrow L^d \) that

\[
\|\nabla K(a)\|_{L^d} \lesssim \|\nabla K(a)\|_{\dot{B}_{p,1}^{d - 1}} \lesssim \|a\|_{\dot{B}_{p,1}^{d}}
\]

and

\[
\|\nabla u^h\|_{L^{d^*}} \lesssim \|\nabla u^h\|_{\dot{B}_{p,1}^{d - 1}} \lesssim \|\nabla u^h\|_{\dot{B}_{p,1}^{d^*}} \lesssim \|u\|_{\dot{B}_{p,1}^{\frac{d}{d^*} + 1}} \quad (p \leq d^*).
\]

Therefore, we obtain

\[
\|g_4(a, u^h)\|_{B_{2, \infty}^{-\sigma_1}} \lesssim \left(\|a\|_{\dot{B}_{2,1}^{1 - \sigma_1}} + \|a\|_{\dot{B}_{p,1}^{1 - \sigma_0}}\right) \|u\|_{\dot{B}_{p,1}^{\frac{d}{d^*} + 1}}.
\]

Now, we consider the oscillation case \( p > d \). Again applying (5.29) with \( \sigma = 1 - \frac{d}{p} \) implies that

\[
\|f g^h\|_{B_{2, \infty}^{-\sigma_1}} \lesssim \|f g^h\|_{B_{2, \infty}^{-\sigma_0}} \lesssim \left(\|f\|_{\dot{B}_{p,1}^{1 - \frac{d}{p}}} + \|f\|_{L^{d^*}}\right) \|g^h\|_{\dot{B}_{p,1}^{1 - \sigma_0}}.
\]

By using the embedding \( \dot{B}_{2,1}^{1 - \sigma_0} \hookrightarrow \dot{B}_{p,1}^{1 - \frac{d}{p}} \) and the fact \( \frac{d}{2} - 1 < 1 - \sigma_0 \) owing to \( p > d \), we obtain

\[
\|f g^h\|_{B_{2, \infty}^{-\sigma_1}} \lesssim \left(\|f\|_{\dot{B}_{2,1}^{1 - \frac{d}{p}}} + \|f\|_{\dot{B}_{p,1}^{1 - \frac{d}{p}}}\right) \|g^h\|_{\dot{B}_{p,1}^{\frac{d}{d^*} + 1}}.
\]
Consequently, we can get the same estimates for \(a \div v^h, k(a) \nabla a^h\) and \(g_3(a, u^h)\) as (5.34), (5.37) and (5.38), respectively. In addition,

\[
\|u \cdot \nabla a^h\|_{B^d_{2, \infty}} \lesssim \left(\|u\|_{B^d_{2, 1}}^\ell + \|u\|_{B^d_{p, 1}}^h\right)\|a\|_{B^{d+1}_{2, 1}}^h \\
\lesssim \|u\|_{B^d_{2, 1}}^\ell \|a\|_{B^{d+1}_{p, 1}}^h + \|a\|_{B^{d+1}_{p, 1}}^h \|u\|_{B^{d+1}_{p, 1}}^h.
\]

On the other hand, the interpolation inequality implies that

\[
\|u^h\|_{B^{d+1}_{p, 1}}^2 \lesssim \|u\|_{B^{d+1}_{p, 1}}^h \|u\|_{B^{d+1}_{p, 1}}^h,
\]

which leads to

\[
\|u \cdot \nabla u^h\|_{B^{d, \sigma_0}_{2, \infty}} \lesssim \left(\|u\|_{B^{d, \sigma_0}_{2, 1}}^\ell + \|u\|_{B^{d, \sigma_0}_{p, 1}}^h\right)\|u\|_{B^{d, \sigma_0}_{p, 1}}^h \lesssim \left(\|u\|_{B^{d, \sigma_0}_{2, 1}}^\ell + \|u\|_{B^{d, \sigma_0}_{p, 1}}^h\right)\|u\|_{B^{d, \sigma_0}_{p, 1}}^h.
\]

To estimate \(g_1(a, u^h)\), one can use (5.30) with \(\sigma = 1 - d/p\) to show that for any smooth function \(F\) vanishing at 0,

\[
\|\nabla F(a) \otimes \nabla u^h\|_{B^{d, \sigma_0}_{2, \infty}} \lesssim \left(\|\nabla u^h\|_{B^{d, \sigma_0}_{p, 1}}^\ell + \sum_{k=k_0}^{k_0+N_0-1} \|\Delta_k \nabla u^h\|_{L_\sigma^{p^*}}\right)\|\nabla F(a)\|_{B^{d, \sigma_0}_{p, 1}}.
\]

As \(p^* \geq p\), the Bernstein inequality ensures that \(\|\Delta_k \nabla u^h\|_{L_\sigma^{p^*}} \lesssim \|\Delta_k \nabla u^h\|_{L_p}\) for \(k_0 \leq k < k_0 + N_0\). Hence, thanks to Proposition 3.6 the fact \(\sigma_1 \leq \sigma_0\) and \(1 - \frac{d}{p} < \frac{d}{p}\), we have

\[
\|g_1(a, u^h)\|_{B^{d, \sigma_1}_{2, \infty}} \lesssim \|a\|_{B^{d, \sigma_1}_{p, 1}}^\ell \|\nabla u^h\|_{B^{d, \sigma_1}_{p, 1}} + \|a\|_{B^{d, \sigma_1}_{p, 1}}^h \lesssim \left(\|a\|_{B^{d, \sigma_1}_{p, 1}}^\ell + \|a\|_{B^{d, \sigma_1}_{p, 1}}^h\right)\|u\|_{B^{d, \sigma_1}_{p, 1}}^h.
\]

By inserting above all estimates into (5.16) yields (5.14). \(\square\)

Noticing that the definition of \(X_p(t)\) in Theorem 1.1 it is easy to see that

\[
\int_0^t \left(D_p^1(\tau) + D_p^2(\tau)\right)d\tau \leq X_p + X_p^2 \leq C X_p,0,
\]

since \(X_p,0 \ll 1\). In addition, one has

\[
\|a^\ell\|_{L_t^2(B^d_{p, 1})}^2 \lesssim \|a^\ell\|_{L_t^\infty(B^d_{p, 1})} \|a^\ell\|_{L_t^1(B^d_{p, 1})} \lesssim \|a\|_{L_t^\infty(B^d_{p, 1})} \|a\|_{L_t^1(B^d_{p, 1})}
\]

and

\[
\|a^h\|_{L_t^2(B^d_{p, 1})}^2 \lesssim \|a\|_{L_t^\infty(B^d_{p, 1})} \|a\|_{L_t^1(B^d_{p, 1})}.
\]

Consequently, it is shown that

\[
\int_0^t D_p^2(\tau)d\tau \lesssim X_p^2 \leq C X_p,0.
\]

Finally, combining (5.51)–(5.52), one can employ nonlinear generalisations of the Gronwall’s inequality (see for example, Page 360 of [31]) and get

\[
\|(a, u)(t, \cdot)\|_{B^{d, \sigma_1}_{2, \infty}} \leq C_0
\]

for all \(t \geq 0\), where \(C_0 > 0\) depends on the norm \(\|(a_0, u_0)\|_{B^{d, \sigma_1}_{2, \infty}}\).
6. Proofs of main results

The last section is devoted to the proofs of Theorem 2.1 and Corollary 2.1.

6.1. Proof of Theorem 2.1 It follows from Lemmas 4.1 and 1.2 that

\[
\frac{d}{dt}\left(\|(a, u)\|_{B_{2,1}^{r}} + \|\nabla(a, u)\|_{B_{2,1}^{s}} + \|\|\nabla(a)\|_{B_{2,1}^{r}} + \|u\|_{h_{B_{2,1}^{r}}}
\right)
\]

\[
\lesssim \|(f, g)\|_{B_{2,1}^{r}} + \|f\|_{h_{B_{2,1}^{r}}} + \|g\|_{B_{2,1}^{r}} + \|\nabla u\|_{B_{2,1}^{s}} \|a\|_{h_{B_{2,1}^{s}}}.
\]

Due to the fact \(p \geq 2\), the last term above can be bounded easily by \(\mathcal{X}_p(t)\|u\|_{B_{2,1}^{r}} + \|u\|_{h_{B_{2,1}^{r}}}\). Next, we have

\[
\|f\|_{h_{B_{2,1}^{r}}} \lesssim \|a u\|_{h_{B_{2,1}^{r}}}.
\]

We decompose \(a u = a^\ell u^\ell + a^\ell u^h + a^h u\). Hence, it is shown that

\[
\|a^\ell u^h\|_{h_{B_{2,1}^{r}}} \lesssim \|a^\ell\|_{B_{2,1}^{r}} \|u^h\|_{h_{B_{2,1}^{r}}} \lesssim \mathcal{X}_p(t)\|u\|_{h_{B_{2,1}^{r}}}
\]

and

\[
\|a^h u\|_{h_{B_{2,1}^{r}}} \lesssim \|a^h\|_{B_{2,1}^{r}} \|u\|_{h_{B_{2,1}^{r}}} \lesssim \mathcal{X}_p(t)\|a\|_{B_{2,1}^{r}}.
\]

It follows from Proposition 3.5 and Bernstein inequality that

\[
\|a^\ell u^\ell\|_{h_{B_{2,1}^{r}}} \lesssim \|a^\ell u^\ell\|_{B_{2,1}^{r}} \|u^\ell\|_{h_{B_{2,1}^{r}}} + \|a^\ell\|_{L^\infty} \|u^\ell\|_{h_{B_{2,1}^{r}}} \lesssim \mathcal{X}_p(t)\|(a, u)\|_{B_{2,1}^{r}}.
\]

Consequently, we are led to

\[
\|f\|_{B_{2,1}^{r}} \lesssim \mathcal{X}_p(t)\left(\|a\|_{B_{2,1}^{r}} + \|(a, u)\|_{B_{2,1}^{r}} + \|u\|_{B_{2,1}^{r}}\right).
\]

In addition, it is easy to get

\[
\|g\|_{B_{2,1}^{r}} \lesssim \|u\|_{B_{2,1}^{r}} \|\nabla u\|_{B_{2,1}^{r}} + \|a\|_{B_{2,1}^{r}} \|\nabla u\|_{h_{B_{2,1}^{r}}} + \|a\|_{h_{B_{2,1}^{r}}},
\]

where the last term can be estimated as

\[
\|a^h\|_{B_{2,1}^{r}} \lesssim \mathcal{X}_p(t)\|a\|_{B_{2,1}^{r}},
\]

\[
\|a^\ell\|_{B_{2,1}^{r}} \lesssim \|a^\ell\|_{B_{2,1}^{r}} + \|a^\ell\|_{B_{2,1}^{r}} \lesssim \mathcal{X}_p(t)\|a\|_{B_{2,1}^{r}}
\]

Therefore, we arrive at

\[
\|g\|_{B_{2,1}^{r}} \lesssim \mathcal{X}_p(t)\left(\|a\|_{B_{2,1}^{r}} + \|(a, u)\|_{B_{2,1}^{r}} + \|u\|_{B_{2,1}^{r}}\right).
\]

Bounding \(\|(f, g)\|_{B_{2,1}^{r}}\) is a little bit more complicated. We claim that

\[
\|(f, g)\|_{B_{2,1}^{r}} \lesssim \mathcal{X}_p(t)\left(\|a\|_{B_{2,1}^{r}} + \|(a, u)\|_{B_{2,1}^{r}} + \|u\|_{B_{2,1}^{r}}\right).
\]
Here, we shall follow the similar strategy as in [8] and handle the nonlinear term $I(a)Au$ as example. To this end, let us admit the following two inequalities (see [8] for more details):

\begin{align}
(6.6) \quad \|T_f g\|_{B^{s-1+\frac{d}{p}-\frac{d}{p'}}_{p,1}} \lesssim \|f\|_{B^{s-1}_{p,1}} \|g\|_{B^s_{p,1}} & \quad \text{if } d \geq 2 \text{ and } \frac{d}{d-1} \leq p \leq \min(4,d^*) , \\
(6.7) \quad \|R(f,g)\|_{B^{s-1+\frac{d}{p}-\frac{d}{p'}}_{p,1}} \lesssim \|f\|_{B^{s-1}_{p,1}} \|g\|_{B^s_{p,1}} & \quad \text{if } s > 1 - \min\left(\frac{d}{p}, \frac{d}{p'}\right) \text{ and } 1 \leq p \leq 4 ,
\end{align}

where $1/p + 1/p' = 1$ and $d^* \triangleq \frac{2d}{d-2}$. Now, using Bony’s para-product decomposition, one has that

$$I(a)Au = T_{Au}I(a) + R(I(a),Au) + T_{I(a)}Au + T_{I(a)}Au^h .$$

Thanks to (6.6) and (6.7) with $s = \frac{d}{p}$, we get

$$\|T_{Au}I(a)\|_{B^{s-1}_{2,1}} \lesssim \|Au\|_{B^{s-1}_{p,1}} \|I(a)\|_{B^{s+1}_{p,1}} \lesssim \|a\|_{B^{s+1}_{p,1}} \|\nabla u\|_{B^{s+1}_{p,1}} ,$$

$$\|R(I(a),Au)\|_{B^{s-1}_{2,1}} \lesssim \|a\|_{B^{s+1}_{p,1}} \|\nabla u\|_{B^{s+1}_{p,1}} .$$

Since $T$ maps $L^\infty \times B^{s-1}_{2,1}$ to $B^{s-1}_{2,1}$, thus,

$$\|T_{I(a)}Au\|_{B^{s-1}_{2,1}} \lesssim \|I(a)\|_{L^\infty} \|Au\|_{B^{s-1}_{2,1}} \lesssim \|a\|_{B^{s+1}_{p,1}} \|\nabla u\|_{B^{s+1}_{p,1}} .$$

In order to handle the last term in the decomposition of $I(a)Au$, we observe that owing to the spectral cut-off, there exists a universal integer $N_0$ such that

$$\left( T_{I(a)}Au^h \right) = \hat{S}_{j_0+1} \left( \sum_{|j-j_0| \leq N_0} \hat{S}_{j-1}I(a) \hat{\Delta}_jAu^h \right) .$$

Hence $\|T_{I(a)}Au^h\|_{B^{s-1}_{2,1}} \approx 2^{j_0(\frac{d}{p}-1)} \sum_{|j-j_0| \leq N_0} \|\hat{S}_{j-1}I(a) \hat{\Delta}_jAu^h\|_{L^2}$. If $2 \leq p \leq \min(d,d^*)$ then one may use for $|j-j_0| \leq N_0$

$$2^{j_0(\frac{d}{p}-1)} \|\hat{S}_{j-1}I(a) \hat{\Delta}_jAu^h\|_{L^2} \lesssim \|\hat{S}_{j-1}I(a)\|_{L^p} \left(2^{j_0(\frac{d}{p}-1)} \|\hat{\Delta}_jAu^h\|_{L^{d^*}}\right) \lesssim \|a\|_{B^{s+1}_{p,1}} \|u^h\|_{B^{s+1}_{p,1}} ,$$

and if $d \leq p \leq 4$, then it holds that

$$2^{j_0(\frac{d}{p}-1)} \|\hat{S}_{j-1}I(a) \hat{\Delta}_jAu^h\|_{L^2} \lesssim \left(2^{j_0(\frac{d}{p}-1)} \|\hat{S}_{j-1}I(a)\|_{L^{4}}\right) \left(2^{j_0(\frac{d}{p}-1)} \|\hat{\Delta}_jAu^h\|_{L^{d^*}}\right) \lesssim \|a\|_{B^{s+1}_{p,1}} \|u^h\|_{B^{s+1}_{p,1}} .$$

Bounding other nonlinear terms is similar, so the details are omitted. Hence, the inequality (6.5) is proved.

Inserting above estimates into (6.1) and using the fact that $X_p(t) \lesssim X_{p,0} \ll 1$ for all $t \geq 0$ guaranteed by Theorem 1.1 we conclude that

$$\frac{d}{dt} \left( \| (a,u) \|^\ell_{B^{s-1}_{2,1}} + \| (\nabla a, u) \|^h_{B^{s+1}_{p,1}} \right) + \left( \| (a,u) \|^\ell_{B^{s+1}_{2,1}} + \| a \|^h_{B^{s+1}_{p,1}} + \| u \|^h_{B^{s+1}_{p,1}} \right) \leq 0 .$$
In what follows, we prove the interpolation inequalities that hold for those restricted Besov norms.

**Proposition 6.1.** Suppose that \( m \neq \rho \). Then it holds that

\[
\|f\|_{B^m_{p,1}}^{\ell} \lesssim (\|f\|_{B^{m}_{r,\infty}}^{\ell})^{1-\theta}(\|f\|_{B^{\rho}_{r,\infty}}^{\ell})^{\theta}, \quad \|f\|_{B^h_{p,1}}^{m} \lesssim (\|f\|_{B^{m}_{r,\infty}}^{h})^{1-\theta}(\|f\|_{B^{\rho}_{r,\infty}}^{h})^{\theta},
\]

where \( j + d(\frac{1}{r} - \frac{1}{p}) = m(1 - \theta) + \rho \theta \) for \( 0 < \theta < 1 \) and \( 1 \leq r \leq p \leq \infty \).

Proof. Without loss of generality, one may suppose that \( m < \rho \). For \( R \in \mathbb{R} \) to be 
chosen later, we have

\[
\|f\|_{B^h_{p,1}}^{m} = \sum_{k \leq k_0} 2^{kj} \|\hat{f}_{L^p}\|_{L^p} \lesssim \sum_{k \leq k_0} 2^{kj+k(d(\frac{1}{r} - \frac{1}{p}) - \rho \theta)} \|\hat{f}_{L^p}\|_{L^p}
\]

\[
\lesssim 2^{(j + d(\frac{1}{r} - \frac{1}{p}) - m)k_0} \|f\|_{B^{m}_{r,\infty}}^{\ell} \lesssim 2^{(j + d(\frac{1}{r} - \frac{1}{p}) - m)R} \|f\|_{B^{m}_{r,\infty}}^{\ell}.
\]

Choosing

\[
R = \log_2 \left( \frac{\|f\|_{B^h_{r,\infty}}^{m}}{\|f\|_{B^{m}_{r,\infty}}^{m}} \right)^{-\frac{1}{p-m}}
\]

yields the first inequality. The proof of the second inequality is left to the interested reader.

Thanks to \(-\sigma_1 < \frac{d}{p} - 1 \leq \sigma \leq \frac{d}{p} + 1 \), it follows from Proposition 6.1 that

\[
(6.9) \quad \|a, u\|_{B^{\sigma}_{2,1}}^{\ell} \lesssim \left( \|a, u\|_{B^{\sigma}_{2,\infty}}^{\ell} \right)^{\theta_0} \left( \|a, u\|_{B^{\sigma}_{2,\infty}}^{\ell} \right)^{1-\theta_0},
\]

where \( \theta_0 = \frac{2}{d/2+1+\sigma} \in (0, 1) \).

By virtue of (5.53), one can get

\[
\|a, u\|_{B^{\sigma}_{2,1}}^{\ell} \geq c_0 \left( \|a, u\|_{B^{\sigma}_{2,\infty}}^{\ell} \right)^{\frac{1}{1-\theta_0}},
\]

where \( c_0 = C^{-\frac{1}{1-\theta_0}} C_0^{\theta_0/1-\theta_0} \). In addition, it follows the fact \( \|\nabla a, u\|_{B^{\sigma}_{p,1}}^{h} \leq X_p(t) \lesssim X_{p,0} \ll 1 \) for all \( t \geq 0 \) that

\[
\|a\|_{B^{\sigma}_{p,1}}^{h} \geq \left( \|a\|_{B^{\sigma}_{p,\infty}}^{h} \right)^{\frac{1}{1-\theta_0}}, \quad \|u\|_{B^{\sigma}_{p,1}}^{h} \geq \left( \|u\|_{B^{\sigma}_{p,\infty}}^{h} \right)^{\frac{1}{1-\theta_0}}.
\]

Consequently, there exists a constant \( \tilde{c}_0 > 0 \) such that the following Lyapunov-type inequality holds

\[
(6.10) \quad \frac{d}{dt} \left( \|a, u\|_{B^{\sigma}_{2,1}}^{\ell} + \|\nabla a, u\|_{B^{\sigma}_{p,1}}^{h} \right) + \tilde{c}_0 \left( \|a, u\|_{B^{\sigma}_{2,1}}^{\ell} + \|\nabla a, u\|_{B^{\sigma}_{p,1}}^{h} \right)^{1+\frac{2}{d/2+1+\sigma}} \leq 0.
\]
Solving (6.10) directly gives
\begin{align}
\|(a, u)(t)\|_{\dot{B}^{\frac{d}{2}, 1}_{p, 1}} + \| (\nabla a, u)(t)\|_{\dot{B}^{\frac{d}{2}, 1}_{p, 1}} \leq \left( \Lambda_{p, 0}^{\frac{2}{d/2 - 1 + \sigma_1}} + \frac{2C_0}{d/2 - 1 + \sigma_1} \right)^{-\frac{d/2 - 1 + \sigma_1}{2}} (1 + t)^{-\frac{d/2 - 1 + \sigma_1}{2}}
\end{align}
for all \( t \geq 0 \). According to the embedding properties in Proposition 3.4, we arrive at
\begin{align}
\|(a, u)(t)\|_{\dot{B}^{\frac{d}{2}, 1}_{p, 1}} \lesssim \|(a, u)(t)\|_{\dot{B}^{\frac{d}{2}, 1}_{p, 1}}^\ell + \| (\nabla a, u)(t)\|_{\dot{B}^{\frac{d}{2}, 1}_{p, 1}} \lesssim (1 + t)^{-\frac{d/2 - 1 + \sigma_1}{2}}.
\end{align}
In addition, if \( \sigma \in (-\bar{\sigma}_1, \frac{d}{p} - 1) \) with \( \bar{\sigma}_1 = \sigma + d(\frac{1}{2} - \frac{1}{p}) \), then employing Proposition 6.1 once again implies that
\begin{align}
\|(a, u)\|_{\dot{B}^{\sigma}_{p, 1}} \leq (\|(a, u)\|_{\dot{B}^{\sigma}_{2, \infty}})^{\theta_1} \left( \|(a, u)\|_{\dot{B}^{\sigma}_{2, \infty}} \right)^{1 - \theta_1},
\end{align}
where \( \theta_1 = \frac{d}{2} - 1 - \sigma \in (0, 1) \).
Noticing the fact that
\begin{align}
\|(a, u)(t)\|_{\dot{B}^{\sigma}_{p, 1}} \leq C_0
\end{align}
for all \( t \geq 0 \), with aid of (6.12)-(6.13), we deduce that
\begin{align}
\|(a, u)(t)\|_{\dot{B}^{\sigma}_{p, 1}} \lesssim \left[ (1 + t)^{-\frac{d/2 - 1 + \sigma_1}{2}} \right]^{1 - \theta_1} = (1 + t)^{-\frac{d}{2} + \frac{1}{2} - \frac{1}{p} - \frac{\sigma + \sigma_1}{2}}
\end{align}
for all \( t \geq 0 \), which leads to
\begin{align}
\|(a, u)(t)\|_{\dot{B}^{\sigma}_{p, 1}} \lesssim \|(a, u)(t)\|_{\dot{B}^{\sigma}_{p, 1}}^\ell + \|(a, u)(t)\|_{\dot{B}^{\sigma}_{p, 1}} \lesssim (1 + t)^{-\frac{d}{2} + \frac{1}{2} - \frac{1}{p} - \frac{\sigma + \sigma_1}{2}}
\end{align}
for \( \sigma \in (-\bar{\sigma}_1, \frac{d}{p} - 1) \). Therefore, the proof of Theorem 2.1 is completed by \( \dot{B}^0_{p, 1} \hookrightarrow L^p \).

6.2. Proof of Corollary 2.1. Indeed, Corollary 2.1 can be regarded as the direct consequence of Proposition 3.8. It follows from Proposition 3.8 with \( q = p \), \( m = \frac{d}{p} - 1 \) and \( k = -\bar{\sigma}_1 + \varepsilon \) with \( \varepsilon > 0 \) small enough. Furthermore, if we define \( \theta_2 \) by the relation
\begin{align}
k_2 \theta_2 + m(1 - \theta_2) = l + d\left( \frac{1}{p} - \frac{1}{r} \right),
\end{align}
then one can take \( \varepsilon \) so small as \( \theta_2 > 0 \) to be in \((0, 1)\). Therefore we have
\begin{align}
\| \Lambda^l(a, u) \|_{L^r} \lesssim \| \Lambda^m(a, u) \|_{L^p}^{\theta_2} \| \Lambda^k(a, u) \|_{L^p}^{\theta_2} \\
\lesssim \left\{ (1 + t)^{-\frac{d}{2} + \frac{1}{2} - \frac{1}{p} - \frac{k + \sigma_1}{2}} \right\}^{1 - \theta_2} \left\{ (1 + t)^{-\frac{d}{2} + \frac{1}{2} - \frac{1}{p} - \frac{k + \sigma_1}{2}} \right\}^{\theta_2} \\
= (1 + t)^{-\frac{d}{2} + \frac{1}{2} - \frac{1}{p} - \frac{1 + \sigma_1}{2}}
\end{align}
for \( p \leq r \leq \infty \) and \( l \in \mathbb{R} \) satisfying \( -\bar{\sigma}_1 < l + d\left( \frac{1}{p} - \frac{1}{r} \right) \leq \frac{d}{p} - 1 \).

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