Elliptic Linear Problem for Calogero-Inozemtsev Model and Painlevé VI Equation

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Abstract

We introduce $3N \times 3N$ Lax pair with spectral parameter for Calogero-Inozemtsev model. The one degree of freedom case appears to have $2 \times 2$ Lax representation. We derive it from the elliptic Gaudin model via some reduction procedure and prove algebraic integrability. This Lax pair provides elliptic linear problem for the Painlevé VI equation in elliptic form.

1 Introduction

1.1 Lax pair for Calogero-Inozemtsev model

The elliptic Calogero-Mosero model (CM) [1] provides a notable example of integrable many-body systems. It is defined by its Hamiltonian

$$H^{CM} = \frac{1}{2} \sum_{i=1}^{N} v_i^2 + g^2 \sum_{i>j}^{N} \wp(u_i - u_j)$$

(1.1)

An important tool for investigating the integrable systems is the Lax representation with a spectral parameter. For CM model it was suggested by I.Krichever [2]. An extension of Calogero type models for root systems of simple Lie algebras was due to M.Olshanetsky and A.Perelomov [3]. Later the Lax pairs for these models were constructed in [4, 5]. The model of much current interest is the Calogero-Inozemtsev (CI) one [6]. It is described by the Hamiltonian

$$H^{CI} = \frac{1}{2} \sum_{i=1}^{N} v_i^2 + g^2 \sum_{i>j}^{N} (\wp(u_i - u_j) + \wp(u_i + u_j)) + \sum_{i=1}^{N} \sum_{\alpha=0}^{3} \nu_\alpha^2 \wp(u_i + \omega_\alpha)$$

(1.2)

on an elliptic curve $\langle 1, \tau \rangle$, $\omega_\alpha = \{0, \frac{1}{2}, \frac{3}{2}, \frac{1+i\tau}{2}\}$ with five free constants $g, \nu_\alpha$. It generalizes the $BC_N$ type CM model. In original paper by V.Inozemtsev [6] the Lax representation was constructed and a principal fact of existence of the spectral parameter was proved. However, an explicit dependence on the spectral parameter failed to be found. Following above mentioned results we suggest a new $3N \times 3N$ Lax representation for CI model with explicit dependence on the spectral parameter.
1.2 Reduction from elliptic Gaudin model

Another line of research is related to the Hitchin approach [7, 8, 9] to the classical integrable systems. The first concrete examples were constructed by N.N. Nekrasov [10]. The spin generalization of CM model [11] and the elliptic top [12] make up typical systems of this kind. (A natural relationship between the top and the spin CM model was found [13, 14].) However, spinless systems only of $A_N$ type were described in the Hitchin-Nekrasov framework. The problem is to find some reduction procedure which would freeze the spin degrees of freedom in a way which produces root systems of other types for spinless CM models. An example of this kind of reduction is going to be introduced and applied to the $2 \times 2$ elliptic Gaudin model [10] with four points on an elliptic curve (G4). As a result we come to the one degree of freedom CI model described by the Hamiltonian

$$H^{PCI} = \frac{1}{2} v^2 + \sum_{\alpha=0}^{3} \nu^2_\alpha \wp(u + \omega_\alpha),$$

(1.3)

where "P" indicates its relation to Painlevé VI equation (see below). Under the reduction we obtain $2 \times 2$ Lax pair for this system with spectral parameter on elliptic curve. Four constants $\nu_\alpha$ in (1.3) appear from the Casimirs of the orbits corresponding to four marked points in G4 model. A particular case, when $\nu_0 = \nu_1 = \nu_2 = \nu_3$ transforms (1.3) to the CM model with one degree of freedom.

1.3 Spectral curve and algebraic integrability in $2 \times 2$ case

We evaluate explicit expression for the spectral curve for G4 model and find out that it is a 2-fold covering of $\mathbb{CP}^1$ branching at eight points. Its genus equals five. The above mentioned reduction allows us to find a way to decrease the genus from five to one and thus provides the proof of the algebraic integrability in $2 \times 2$ case.

1.4 Elliptic form of Painlevé VI equation

The sixth Painlevé equation (PVI) [15] in the elliptic form [16, 17, 18] is the nonautonomous version of equation of motion for (1.3)

$$\frac{d^2 u}{d\tau^2} = - \sum_{\alpha=0}^{3} \nu^2_\alpha \wp'(u + \omega_\alpha).$$

(1.4)

The standard form of PVI equation can be derived from the Schlesinger systems [20] of the isomonodromic deformations on $\mathbb{CP}^1 \setminus \{x_1, x_2, x_3, x_4\}$ [19, 22]. It is desirable to have Schlesinger type description of PVI equation on a torus. A knowledge of $2 \times 2$ Lax representation for (1.3), is the key to solving this task. We would like to notice that the isomonodromy preserving equations on Riemann surfaces of arbitrary genus and in particular of genus one has been much investigated in [23, 24, 25, 26].

Necessary elliptic function definitions and identities can be found in the Appendix A.

2 Lax Pair for Calogero-Inozemtsev Model

As it was mentioned in the Introduction the CI model was defined by the following Hamiltonian:

$$H = \frac{1}{2} \sum_{i=1}^{N} v_i^2 + g^2 \sum_{i>j}^{N} (\wp(u_i - u_j) + \wp(u_i + u_j)) + \sum_{i=1}^{N} \sum_{\alpha=0}^{3} \nu^2_\alpha \wp(u_i + \omega_\alpha)$$

(2.1)
on an elliptic curve $\langle 1, \tau \rangle$, where $\omega_\alpha = \{0, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \}$ while $g, \nu_\alpha$ are arbitrary constants.

**Proposition 2.1** The Calogero-Inozemtsev model (2.1) admits $3N \times 3N$ Lax representation with a spectral parameter on the elliptic curve $^1$

\[
L = \begin{pmatrix} V + A & B_1 & -C_1 \\ B_2 & -V + A^T & C_2 \\ -C_2 & C_1 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} D + A' & B_1' & -C_1' \\ B_2' & D + A'^T & C_2' \\ -C_2' & C_1' & D+E \end{pmatrix} \tag{2.2}
\]

where all entries are $N \times N$ matrices. The matrices $V, D, C_1$ and $C_2$ are diagonal while all others are offdiagonal:

\[
A_{ij} = g(1-\delta_{ij})\Phi(z,u_i-u_j), \quad E_{ij} = g(1-\delta_{ij})(\varphi(u_i-u_j) - \varphi(u_i+u_j))
\]

\[
V_{ij} = \delta_{ij}v_i, \quad D_{ij} = g\delta_{ij}\sum\limits_{k\neq i} (\varphi(u_k-u_i) + \varphi(u_k+u_i)),
\]

\[
C_{1ij} = \delta_{ij}\sum\limits_{\alpha=0}^3 \nu_\alpha \varphi_\alpha(z,\omega_\alpha+u_i), \quad C_{2ij} = \delta_{ij}\sum\limits_{\alpha=0}^3 \nu_\alpha \varphi_\alpha(z,\omega_\alpha-u_i),
\]

\[
B_{1ij} = g(1-\delta_{ij})\Phi(z,u_i+u_j), \quad B_{2ij} = g(1-\delta_{ij})\Phi(z,-u_i-u_j).
\tag{2.3}
\]

It follows from the above proposition that there exists $3 \times 3$ Lax representation which describes PCI model (1.3). Surprisingly, the following assertion holds:

**Proposition 2.2** The Painlevé-Calogero-Inozemtsev model (1.3) admits $2 \times 2$ Lax representation with a spectral parameter on the elliptic curve

\[
L_{\text{PCI}} = \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix} + \sum\limits_{\alpha=0}^3 L_{\alpha}^{\text{PCI}}, \quad L_{\alpha}^{\text{PCI}} = \begin{pmatrix} 0 & \nu_\alpha \varphi_\alpha(z,\omega_\alpha+u) \\ \nu_\alpha \varphi_\alpha(z,\omega_\alpha-u) & 0 \end{pmatrix} \tag{2.4}
\]

\[
M_{\alpha}^{\text{PCI}} = \sum\limits_{\alpha=0}^3 M_{\alpha}^{\text{PCI}}, \quad M_{\alpha}^{\text{PCI}} = \begin{pmatrix} 0 & \nu_\alpha \varphi_\alpha'(z,\omega_\alpha+u) \\ \nu_\alpha \varphi_\alpha'(z,\omega_\alpha-u) & 0 \end{pmatrix}
\]

The existence of $2 \times 2$ Lax pair (2.4) appears to be explicable on the basis of its relation to $sl(2,\mathbb{C})$ the elliptic Gaudin model [10] with four points on the elliptic curve (G4). It will be discussed in the next section.

The proof of the above propositions is based on the identities given in Appendix A. In particular from (A.29) we easily come to a useful equality:

**Lemma 2.1** For $\alpha \neq \beta$ and matrices $L_{\alpha}^{\text{PCI}}, M_{\alpha}^{\text{PCI}}$ (2.4) the following relation holds:

\[
[L_{\alpha}^{\text{PCI}}, M_{\beta}^{\text{PCI}}] + [L_{\beta}^{\text{PCI}}, M_{\alpha}^{\text{PCI}}] = 0 \tag{2.5}
\]

**Proof**

\[
\begin{align*}
\varphi_\alpha(z,\omega_\alpha+u)\varphi_\beta'(z,\omega_\beta-u) - \varphi_\alpha(z,\omega_\alpha-u)\varphi_\beta'(z,\omega_\beta+u) &+ \\
\varphi_\beta(z,\omega_\beta+u)\varphi_\alpha'(z,\omega_\alpha-u) - \varphi_\beta(z,\omega_\beta-u)\varphi_\alpha'(z,\omega_\alpha+u) &+ \\
\varphi_{\alpha+\beta}(z,\omega_{\alpha+\beta}+\omega_\beta)(\varphi(\omega_\beta-u) - \varphi(\omega_\alpha+u) + \varphi(\omega_\alpha-u) - \varphi(\omega_\beta+u)) = 0
\end{align*}
\tag{2.6}
\]

\(^1\)To have an appropriate sign behind the potential in (1.2) one should replace $g, \nu$ with $\sqrt{-1}g, \sqrt{-1}\nu$
3 Algebraic Integrability in $2 \times 2$ Case

3.1 Elliptic Gaudin model and reduction to PCI model

As indicated earlier, in the case of the single degree of freedom the CI model is defined by the Hamiltonian

$$H^{PCI} = \frac{1}{2} v^2 + \sum_{\alpha=0}^{3} \nu^2_\alpha \varphi(u + \omega_\alpha) \tag{3.1}$$

and equations of motion can be represented in the Lax form with matrices (2.4). The aim of the section is to prove the algebraic integrability of the PCI model. For this purpose let us consider the G4 model [10]. It describes 4 degrees of freedom. One corresponds to a motion of pair of particles in the center of mass frame while three others describe dynamics of a complicated manifold $\{O \times O \times O \times O\}/T$, where $T$ is the Cartan subgroup in $SL(2, \mathbb{C})$. The Lax matrix of the G4 model is $sl(2, \mathbb{C})$-valued function on the torus with appropriate quasiperiodic properties and four simple poles. Residues of the four points are the orbits $O_\alpha$ of the coadjoint action by $SL(2, \mathbb{C})$.

Let us specify the Lax matrix for the G4 model on the doubled torus $(2, 2\tau)$:

$$L^{G4} = \left( \begin{array}{cc} v & 0 \\ 0 & -v \end{array} \right) + \sum_{\alpha=0}^{3} L^{G4}_\alpha, \quad L^{G4}_\alpha = \left( \begin{array}{cc} s_{11}^\alpha E_1(z - 2\omega_\alpha, 2\tau) & \tilde{s}_{12}^\alpha \varphi_\alpha(z, \omega_\alpha + u) \\ \tilde{s}_{21}^\alpha \varphi_\alpha(z, \omega_\alpha - u) & -s_{11}^\alpha E_1(z - 2\omega_\alpha, 2\tau) \end{array} \right) \tag{3.2}$$

In doing so we imply functions to be defined on $(1, \tau)$ if the dependence on $2\tau$ is not given explicitly. The four marked points are $\{2\omega_\alpha\}$ or $\{0, 1, \tau, \tau + 1\}$. Notice that unlike the diagonal elements of residues of (3.2) $s_{11}^\alpha$ the offdiagonal $\tilde{s}_{12}^\alpha$ and $\tilde{s}_{21}^\alpha$ do not correspond to a certain orbit $O_\alpha$ but to some linear combination which can be expressed through the use of the matrix $I$ (A.26):

$$s_{12}^\alpha = \sum_{\alpha=0}^{3} \sum_{\alpha=0}^{3} e(2u \partial_\tau \omega_\alpha) I_{\rho_\alpha s_{12}^\alpha} = \sum_{\alpha=0}^{3} \sum_{\alpha=0}^{3} e(2u \partial_\tau \omega_\alpha - 2(\omega_\alpha + u) \partial_\tau \omega_\rho) \tilde{s}_{12}^\alpha \tag{3.3}$$

$$s_{21}^\alpha = \sum_{\alpha=0}^{3} \sum_{\alpha=0}^{3} e(2u \partial_\tau \omega_\alpha) I_{\rho_\alpha \tilde{s}_{21}^\alpha} = \sum_{\alpha=0}^{3} \sum_{\alpha=0}^{3} e(2(\omega_\alpha - u) \partial_\tau \omega_\rho) \tilde{s}_{21}^\alpha \tag{3.4}$$

The inverse change of variables comes from (A.27):

$$\tilde{s}_{12}^\alpha = \frac{1}{4} \sum_{\alpha=0}^{3} e(2u \partial_\tau \omega_\alpha) I_{\rho_\alpha s_{12}^\alpha}, \quad \tilde{s}_{21}^\alpha = \frac{1}{4} \sum_{\alpha=0}^{3} e(-2u \partial_\tau \omega_\alpha) I_{\rho_\alpha \tilde{s}_{21}^\alpha} \tag{3.5}$$

The spectral curve is defined by the equation

$$\det(\lambda + L^{G4}(z)) = 0 \quad \text{or} \quad \lambda^2 + \det L^{G4}(z) = 0 \tag{3.6}$$

Using (A.15) we have

$$\lambda^2 = v^2 + 2v \sum_s s_{11}^s E_1(z - 2\omega_\alpha, 2\tau) + \left( \sum_s s_{11}^s E_1(z - 2\omega_\alpha, 2\tau) \right)^2 + \sum_{s \neq \beta} s_{12}^s s_{21}^\beta (\varphi(z) - \varphi(u + \omega_\alpha)) + \sum_{s \neq \beta} s_{12}^s s_{21}^\beta \varphi_\alpha(z, \omega_\alpha + u) \varphi_\beta(z, \omega_\beta - u) \tag{3.5}$$

The Hamiltonian appears form the decomposition of function $-\det L^{G4}(z) = \frac{1}{2} Tr \left( (L^{G4}(z))^2 \right)$:

$$-\det L^{G4}(z) = \sum_{s=0}^{3} H_{2, s} E_2(z - 2\omega_\alpha, 2\tau) + \sum_{s=0}^{3} H_{1, s} E_1(z - 2\omega_\alpha, 2\tau) + H_{2, 0} \tag{3.6}$$
where $H_{2,\alpha} = C_{\alpha}$ are the Casimir functions of orbits $s^\alpha$, $H_{1,\alpha}$ are the Hamiltonians linear in the momentum $v$ and $H_{2,0}$ is the quadratic one.

The symmetry which underlies the reduction is generated by the following involution:

$$L^{G_4}(z) \to -\sigma_1 L^{G_4}(-z)\sigma_1,$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The reduction procedure implies that we should choose the eigenvalue of the map (3.7). Choosing it to be $+1$ and keeping in mind (A.17) and (A.19) we arrive to the following constraints:

$$s^{\alpha}_1 = 0, \quad s^{\beta}_{12} = s^{\alpha}_{21} \quad (3.8)$$

It follows from (3.4) that

$$e(-2u\partial_\tau \omega_\alpha)s^{\alpha}_{21} = e(2u\partial_\tau \omega_\alpha)s^{\alpha}_{12} \quad (3.9)$$

If it is recalled that the Casimirs $C_{\alpha} = (s^{\alpha}_{11})^2 + s^{\alpha}_{12}s^{\alpha}_{21}$ look like $C_{\alpha} = s^{\alpha}_{12}s^{\alpha}_{21}$ on shell (3.8) we have

$$s^{\alpha}_{12} = e(-2u\partial_\tau \omega_\alpha)\sqrt{C_{\alpha}}, \quad s^{\alpha}_{21} = e(2u\partial_\tau \omega_\alpha)\sqrt{C_{\alpha}} \quad (3.10)$$

And consequently from (3.4)

$$s^{\beta}_{12} = s^{\beta}_{21} = \frac{1}{4} \sum_{\alpha=0}^3 I_{\rho \alpha} \sqrt{C_{\alpha}} = \nu_\rho. \quad (3.11)$$

Thereby the reduction provides transformation from the Lax matrix of the Gaudin model (3.2) to the one (2.4) of Painlevé-Calogero-Inozemtsev.

The fact that the variables $s^{\alpha}_{12}, s^{\alpha}_{21}$ and $s^{\alpha}_{11}$ commute with the quadratic Hamiltonian with respect to Poisson brackets on shell (3.8) is discussed in Appendix B.

### 3.2 Algebraic integrability

To prove the algebraic integrability of PCI one should show that the genus of its spectral curve equals 1. The spectral curve (3.5) has genus 5. Indeed, the r.h.s. of (3.5) is the doubleperiodic function on the elliptic curve $(2, 2\tau)$ with second order poles at 4 points. Thus it has 8 zeros and may be conceived as a 2-fold covering of the elliptic curve branching at eight points. From the point of view of the spectral curve the reduction by the involution (3.7) implies identification of points $z$ and $-z$. The function $\varphi(z)$ is even and has 8 zeros on the torus $(2, 2\tau)$. Thus it is natural to expect that the unknown spectral curve could be written in terms of $\varphi(z)$.

**Proposition 3.1** The spectral curve of the Painlevé-Calogero-Inozemtsev model is of the form

$$\lambda^2 = R(X), \quad X = \varphi(z)$$

where $R(X)$ is some rational function with four simple poles.

Let us transform the last term from the r.h.s of (3.5). Using (A.31) we have:

$$\sum_{\alpha \neq \beta} s^{\alpha}_{12} s^{\beta}_{21} \varphi_{\alpha+\beta}(z, \omega_\alpha + u)\varphi_{\beta}(z, \omega_\beta - u) =$$

$$\sum_{\alpha \neq \beta} s^{\alpha}_{12} s^{\beta}_{21} \varphi_{\alpha+\beta}(z, \omega_\alpha + \omega_\beta)(E_1(z) + E_1(\omega_\alpha + u) + E_1(\omega_\beta - u) - E_1(z + \omega_\alpha + \omega_\beta)) =$$

$$= \sum_{\alpha \neq \beta} s^{\alpha}_{12} s^{\beta}_{21} \varphi_{\alpha+\beta}(z, \omega_\alpha + \omega_\beta)(E_1(z) + E_1(\omega_\alpha + \omega_\beta) - E_1(z + \omega_\alpha + \omega_\beta)) +$$

$$+ \sum_{\alpha \neq \beta} s^{\alpha}_{12} s^{\beta}_{21} \varphi_{\alpha+\beta}(z, \omega_\alpha + \omega_\beta)(E_1(\omega_\beta - u) + E_1(\omega_\alpha + u) - E_1(\omega_\alpha + \omega_\beta))$$

(3.13)
The last sum vanishes under constraints (3.8) \(^2\) and we come to the following equation for the spectral curve of the PCI model:

\[
\lambda^2 = v^2 - \sum_{\alpha} \nu_{\alpha}^2 \varphi(u + \omega_{\alpha}) + \varphi(z) \sum_{\alpha} \nu_{\alpha}^2 + \sum_{\alpha \neq \beta} \nu_{\alpha} \nu_{\beta} \varphi_{\alpha + \beta}(z, \omega_{\alpha} + \omega_{\beta}) (E_1(z) + E_1(\omega_{\alpha} + \omega_{\beta}) - E_1(z + \omega_{\alpha} + \omega_{\beta})) \tag{3.14}
\]

At this moment we need one more relation:

\[
\varphi_{\alpha}(z, \omega_{\alpha}) (E_1(z) + E_1(\omega_{\alpha}) - E_1(z + \omega_{\alpha})) = \sum_{\rho=0}^{3} I_{\alpha \rho} \varphi(z - 2\omega_{\rho}, 2\tau), \tag{3.15}
\]

where matrix \(I\) is defined in (A.26). The proof of (3.15) is based on comparing the structure of singularities and (A.6). So we have

\[
\lambda^2 = 2H^{PCI} + \varphi(z) \sum_{\alpha} \nu_{\alpha}^2 + \sum_{\alpha \neq \beta} \nu_{\alpha} \nu_{\beta} \sum_{\rho} I_{\mu(\alpha, \beta), \rho} \varphi(z - 2\omega_{\rho}, 2\tau), \tag{3.16}
\]

where the index \(\mu(\alpha, \beta)\) is uniquely defined from

\[
\omega_{\mu(\alpha, \beta)} = \omega_{\alpha} + \omega_{\beta} \mod(1, \tau) \tag{3.17}
\]

Substituting \(\varphi(z, \tau) = \sum_{\alpha} \varphi(z - 2\omega_{\alpha}, 2\tau)\) into (3.16) we come to the final answer:

\[
\lambda^2 = 2H^{PCI} + \sum_{\rho=0}^{3} K_{\rho} \varphi(z - 2\omega_{\rho}, 2\tau), \quad K_{\rho} = \sum_{\alpha, \beta=0}^{3} \nu_{\alpha} \nu_{\beta} I_{\mu(\alpha, \beta), \rho} \tag{3.18}
\]

To finish the proof we need one more step:

\[
\varphi(z + \omega_{\alpha}) = \frac{1}{2} \varphi''(\omega_{\alpha}) + \varphi(\omega_{\alpha}) \tag{3.19}
\]

The r.h.s. of (3.18) is the rational function with four simple poles corresponding to \(z = \{\omega_{\alpha}\}\). Thus the desirable curve of genus 1 appears as the 2-fold covering of \(\mathbb{CP}^1\) branching at four points. In doing so we imply \(z\) as a coordinate on the torus \(\langle 1, \tau \rangle \supset \langle 2, 2\tau \rangle\).

### 4 Elliptic form of Painlevé VI equation

As it was shown by Painlevé [16] himself the Painlevé VI equation (PVI) can be represented in the following form

\[
\frac{d^2 u}{d\tau^2} = -\sum_{\alpha=0}^{3} \nu_{\alpha}^2 \varphi'(u + \omega_{\alpha}) \tag{4.20}
\]

\[
\sum_{\alpha, \beta=0}^{3} \nu_{\alpha} \nu_{\beta} \varphi_{\alpha + \beta}(z, \omega_{\alpha} + \omega_{\beta}) (E_1(\omega_{\beta} - u) + E_1(\omega_{\alpha} + u) - E_1(\omega_{\alpha} + \omega_{\beta})) =
\]

\[
\frac{1}{2} \sum_{\alpha, \beta=0}^{3} \nu_{\alpha} \nu_{\beta} \varphi_{\alpha + \beta}(z, \omega_{\alpha} + \omega_{\beta}) (E_1(\omega_{\alpha} - u) + E_1(\omega_{\alpha} + u) + E_1(\omega_{\beta} - u) + E_1(\omega_{\beta} + u) - 2E_1(\omega_{\alpha}) - 2E_1(\omega_{\beta})) = 0
\]
on the elliptic curve Σ parameterized by \( \langle 1, \tau \rangle \) where \( \omega_\alpha = \{ 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \} \) while \( \nu_\alpha \) are arbitrary constants. Later the result was rediscovered in [17, 18]. Initially B. Gambier [15] found PVI in a more complicated form

\[
\frac{d^2X}{dt^2} = \frac{1}{2} \left( \frac{1}{X} + \frac{1}{X - 1} + \frac{1}{X - t} \right) \left( \frac{dX}{dt} \right)^2 - \left( \frac{1}{t} + \frac{1}{t - 1} + \frac{1}{X - t} \right) \frac{dX}{dt} + \frac{X(X - 1)(X - t)}{t^2(t - 1)^2} \left( \alpha + \beta \frac{t}{X^2} + \gamma \frac{t - 1}{(X - 1)^2} + \delta \frac{t(t - 1)}{(X - t)^2} \right)
\]

which transforms into (4.20) by the following rules:

\[
(u, \tau) \rightarrow \left( X = \frac{\varphi(u) - \varphi(\omega_1)}{\varphi(\omega_2) - \varphi(\omega_1)}, \tau = \frac{\varphi(\omega_3) - \varphi(\omega_1)}{\varphi(\omega_2) - \varphi(\omega_1)} \right)
\]

\[
(\nu_0^2, \nu_1^2, \nu_2^2, \nu_3^2) = 4\pi^2(\alpha, -\beta, \frac{1}{2}\gamma, -\delta)
\]

The derivation of PVI equation as the preserving monodromy condition was given by R. Fuchs [19]. Namely it was shown that the PVI equation can be derived from the Schlesinger system [20, 21]

\[
\frac{\partial A_j}{\partial \lambda_i} = \frac{[A_i, A_j]}{\lambda_i - \lambda_j}, \quad i \neq j, \quad \frac{\partial A_i}{\partial \lambda_i} = -\sum_{j \neq i} \frac{[A_i, A_j]}{\lambda_i - \lambda_j}
\]

on \( \mathbb{CP}^1 \setminus \{ \lambda_1, \lambda_2, \lambda_3, \lambda_4 \} \) where \( A_j \) are \( \text{sl}(2, \mathbb{C}) \)-valued matrices. It arises as the isomonodromy condition

\[
\frac{\partial \Psi}{\partial \lambda_j} = -\frac{A_j}{\lambda - \lambda_j} \Psi
\]

for a matrix-valued function \( \Psi(\lambda) \in \text{SL}(2, \mathbb{C}) \) which satisfies the matrix differential equations

\[
\frac{d\Psi}{d\lambda} = \sum_{j=1}^{4} \frac{A_j}{\lambda - \lambda_j} \Psi
\]

In other words the system of equations (4.24) and (4.25) is a linear problem for the Schlesinger equation (4.23) which appears to be equivalent to the rational form of PVI equation (4.21). Here we come to a natural question. What is the analogue of the Schlesinger equation on an elliptic curve which leads to the elliptic form of PVI (4.20)? In other words, we would like to find out if there exists a pair of matrix-valued functions \( L^{PVI}(z), M^{PVI}(z) \) on an elliptic curve \( \Sigma \) which defines a linear problem

\[
\left( \frac{\partial}{\partial z} + L^{PVI}(z) \right) \Psi(z) = 0, \quad \left( \frac{\partial}{\partial \tau} + M^{PVI}(z) \right) \Psi(z) = 0
\]

with the analogue of the Schlesinger equation

\[
\frac{\partial}{\partial \tau} L^{PVI} - \frac{\partial}{\partial z} M^{PVI} = [L^{PVI}, M^{PVI}]
\]

to be equivalent to (4.20).

**Proposition 4.1** The Painlevé VI equation in elliptic form is equivalent to

\[
\partial_{\tau} L^{PCI} - \partial_z M^{PCI} = [L^{PCI}, M^{PCI}]
\]

with matrices \( L^{PCI}, M^{PCI} \) from (2.4).
The proof is based on the Proposition 2.2 and identity:
\[
\partial_\tau \varphi_\alpha(z, u + \omega_\alpha) = \partial_z \partial_u \varphi_\alpha(z, u + \omega_\alpha) + \partial_\tau u \partial_u \varphi_\alpha(z, u + \omega_\alpha)
\] (4.29)
For the case \(\nu_0 = \nu_1 = \nu_2 = \nu_3\) this statement was discovered in [24].

There is another way to describe the Painlevé VI equation which is closed to the one considered in [23, 26]. One may start from \(sl(2, \mathbb{C})\) elliptic top [12, 14]:

\[
L_{\text{top}}(z) = 3 \sum_{\alpha = 1}^3 S_\alpha \varphi_\alpha(z, \alpha) \sigma_\alpha, \quad M_{\text{top}}(z) = 3 \sum_{\alpha = 1}^3 S_\alpha \varphi'_\alpha(z, \alpha) \sigma_\alpha,
\] (4.30)

where \(\sigma_\alpha\) are the Pauli matrices and \(S_\alpha\) are the dynamical variables. The Hamiltonian and Poisson brackets are:

\[
H = \frac{1}{2} \sum_{\alpha = 1}^3 S_\alpha^2 \varphi(\omega_\alpha), \quad \{S_\alpha, S_\beta\} = \varepsilon_{\alpha\beta\gamma} S_\gamma
\] (4.31)

Now if we consider the top on the doubled torus \(\langle 2, 2\tau \rangle\) and put four orbits \(S^{(\beta)}\) into \(0, 1, \tau, \tau + 1\) we obtain the Lax matrix

\[
L_{\text{top}4} = 3 \sum_{\alpha = 1}^3 \tilde{S}^{(\beta)}_\alpha \varphi_\alpha(z, \alpha) \sigma_\alpha, \quad M_{\text{top}4} = 3 \sum_{\alpha = 1}^3 \tilde{S}^{(\beta)}_\alpha \varphi'_\alpha(z, \alpha) \sigma_\alpha,
\] (4.32)

where as in the previous section

\[
\tilde{S}^{(\beta)}_\alpha = \frac{1}{4} \sum_{\rho = 0}^3 I_{\beta\rho} S^{(\rho)}_\alpha
\]

with matrix \(I\) defined in (A.26). The Hamiltonian and Poisson brackets in this case are:

\[
H = \frac{1}{32} \sum_{\alpha = 1}^3 \left( \sum_{\beta, \rho} I_{\beta\rho} S^{(\rho)}_\alpha \right)^2 \varphi(\omega_\alpha), \quad \{S^{(\gamma)}_\alpha, S^{(\rho)}_\beta\} = \delta^{\gamma\rho} \varepsilon_{\alpha\beta\gamma} S_\gamma
\] (4.33)

It follows from (A.24) that \(L\)-matrix defined in (4.32) is the doubleperiodic function on the torus \(\langle 2, 2\tau \rangle\) and thus the sum of residues equals zero

\[
\sum_{\rho = 0}^3 S^{(\rho)} = 0
\] (4.34)

The \(SL(2, \mathbb{C})\) action saves these constrains. Consequently the phase space of \(top4\) model looks like \(O_0 \times O_1 \times O_2 \times O_3 / SL(2, \mathbb{C})\) and thus coincides with the one of the Schlesinger system (4.23). The pair of matrices (4.32) obviously provides the Painlevé VI equation in a sense of Proposition 4.1.

5 Acknowledgments

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A Elliptic Functions

We summarize the main formulae for elliptic functions, borrowed mainly from [27] and [28]. We assume that $q = \exp 2\pi i \tau$, where $\tau$ is the modular parameter of the elliptic curve $\langle 1, \tau \rangle$, half periods $\omega_\alpha = \{0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \}$ and $e(x) = \exp(2\pi \sqrt{-1}x)$. We also deal with the doubled lattice $\langle 2, 2\tau \rangle$ and writing $f(z, w, 2\tau)$ we mean that the function is defined on it while writing $f(z, w)$ we imply that the function is defined on the initial lattice $\langle 1, \tau \rangle$.

The basic element is the theta function:

$$\vartheta(z|\tau) = q^{\frac{1}{8}} \sum_{n \in \mathbb{Z}} (-1)^n e^{\pi i (n(n+1)\tau + 2nz)} =$$

$$q^{\frac{1}{8}} e^{-\frac{\pi i}{4}} (e^{i\pi z} - e^{-i\pi z}) \prod_{n=1}^{\infty} (1 - q^n)(1 - q^n e^{2i\pi z})(1 - q^n e^{-2i\pi z})$$

The Eisenstein functions

$$E_1(z|\tau) = \partial_z \log \vartheta(z|\tau), \quad E_1(z|\tau) \sim \frac{1}{z} - 2\eta_1 z, \quad \eta_1(\tau) = \zeta(\frac{1}{2})$$

$$E_2(z|\tau) = -\partial_z E_1(z|\tau) = \partial_z^2 \log \vartheta(z|\tau), \quad E_2(z|\tau) \sim \frac{1}{z^2} + 2\eta_1$$

Particular values

$$E_1(\omega_\alpha) = -2\pi \sqrt{-1} \partial_\tau \omega_\alpha,$$

where $\partial_\tau \omega_\alpha = \{0, 0, \frac{1}{2}, \frac{1}{2} \}$ or equivalently

$$E_1\left(\frac{1}{2}\right) = 0, \quad E_1\left(\frac{\tau}{2}\right) = E_1\left(\frac{1+\tau}{2}\right) = -\pi \sqrt{-1}.$$  

$$\sum_{\alpha=1}^{3} \varphi(\omega_\alpha) = 0$$

The next important function is

$$\phi(z, u) = \frac{\vartheta(u + z)\vartheta'(0)}{\vartheta(u)\vartheta(z)}.$$  

It has a pole at $z = 0$ and

$$\phi(z, u) = \frac{1}{z} + E_1(u) + \frac{z}{2}(E_1^2(u) - \varphi(u)) + \ldots,$$

and

$$\phi'(z, u) = \phi(z, u)(E_1(u + z) - E_1(u)).$$

We use prime for the derivation with respect to the second argument, i.e.

$$\phi'(z, w) = \frac{\partial}{\partial y} \phi(z, y)|_{y=w}$$

One of the most important notations is

$$\varphi_\alpha(z, \omega_\beta + u) = e(z \partial_\tau \omega_\alpha) \phi(z, \omega_\beta + u)$$
It should be mentioned that at \( z = 0 \)
\[
\varphi_\alpha(z, \omega_\alpha) = \frac{1}{z} - \frac{z}{2} \varphi(\omega_\alpha) + \ldots
\]  

(A.11)

**Relations to the Weierstrass functions**

\[
\zeta(z|\tau) = E_1(z|\tau) + 2\eta_1(\tau)z,
\]

(A.12)

\[
\varphi(z|\tau) = E_2(z|\tau) - 2\eta_1(\tau),
\]

(A.13)

\[
\phi(z, u) = \exp(-2\eta_1uz)\frac{\sigma(u + z)}{\sigma(u)\sigma(z)}.
\]

(A.14)

\[
\phi(z, u)\phi(-u, z) = \varphi(z) - \varphi(u) = E_2(z) - E_2(u).
\]

(A.15)

**Parity**

\[
\vartheta(-z) = -\vartheta(z)
\]

(A.16)

\[
E_1(-z) = -E_1(z)
\]

(A.17)

\[
E_2(-z) = E_2(z)
\]

(A.18)

\[
\phi(z, u) = \phi(z, u) = -\phi(-u, -z)
\]

(A.19)

**Behavior on the lattice**

\[
\vartheta(z + 1) = -\vartheta(z), \quad \vartheta(z + \tau) = -q^{-\frac{1}{2}} e^{-2\pi \sqrt{-1}z} \vartheta(z),
\]

(A.20)

\[
E_1(z + 2\omega_\alpha) = E_1(z) - 4\pi \sqrt{-1}\partial_\tau \omega_\alpha : \quad E_1(z + 1) = E_1(z), \quad E_1(z + \tau) = E_1(z) - 2\pi \sqrt{-1},
\]

(A.21)

\[
E_2(z + 2\omega_\alpha) = E_2(z) : \quad E_2(z + 1) = E_2(z), \quad E_2(z + \tau) = E_2(z),
\]

(A.22)

\[
\phi(u + 1, z) = \phi(z, u), \quad \phi(u + \tau, z) = e^{-2\pi \sqrt{-1}\zeta}\phi(z, u).
\]

(A.23)

The above matrix is defined by

\[
Res_{z=2\omega_\rho}\varphi_\alpha(z, \omega_\alpha + u) = e(2\omega_\rho \vartheta_\omega_\alpha - 2(\omega_\alpha + u)\vartheta_\omega_\rho)
\]

(A.25)

For the symmetric \( 4 \times 4 \) matrix \( Res_{z=2\omega_\rho}\varphi_\alpha(z, \omega_\alpha) \) we keep notation

\[
I_{\rho\alpha} = Res_{z=2\omega_\rho}\varphi_\alpha(z, \omega_\alpha) = e(2\omega_\rho \vartheta_\omega_\alpha - 2\omega_\alpha \vartheta_\omega_\rho)
\]

(A.26)

or

\[
I = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 \\
\end{pmatrix}
\]
Note that
\[ I^{-1}_{\alpha\beta} = \frac{1}{4} I_{\alpha\beta} \] \hspace{1cm} (A.27)

**Addition formula**
\[ \phi(z, u)\partial_v \phi(z, v) - \phi(z, v)\partial_u \phi(z, u) = (E_2(v) - E_2(u))\phi(z, u + v), \] \hspace{1cm} (A.28)
or
\[ \phi(z, u)\partial_v \phi(z, v) - \phi(z, v)\partial_u \phi(z, u) = (v(v) - \varphi(u))\phi(z, u + v). \] \hspace{1cm} (A.29)

In fact, \( \phi(z, u) \) satisfies more general relation which follows from the Fay three-section formula
\[ \phi(u_1, z_1)\phi(u_2, z_2) - \phi(u_1 + u_2, z_1)\phi(u_2, z_2 - z_1) - \phi(u_1 + u_2, z_2)\phi(u_1, z_1 - z_2) = 0 \] \hspace{1cm} (A.30)
A particular case of this formula is
\[ \phi(u_1, z)\phi(u_2, z) = \phi(u_1 + u_2, z)(E_1(u_1) + E_1(u_2) + E_1(z) - E_1(u_1 + u_2 + z)) \] \hspace{1cm} (A.31)

**B Comments on Reduction Procedure**

Let us have a look at the generating function of the Hamiltonians:
\[
\frac{1}{2} Tr \left( L^{G4}(z) \right)^2 = v^2 + 2v \sum_{\alpha} s_{11}^\alpha E_1(z - 2\omega_\alpha, 2\tau) + \left( \sum_{\alpha} s_{11}^\alpha E_1(z - 2\omega_\alpha, 2\tau) \right)^2 + \\
\sum_{\alpha} \tilde{s}_{12}^{\alpha\beta} (\varphi(z) - \varphi(u + \omega_\alpha)) + \sum_{\alpha \neq \beta} \tilde{s}_{12}^{\alpha\beta} \varphi_\alpha(z, \omega_\alpha + u) \varphi_\beta(z, \omega_\beta - u)
\] \hspace{1cm} (B.1)

Consider infinitesimal deformation from the constraints
\[ s_{11}^\alpha = 0, \quad \tilde{s}_{12}^{\alpha\beta} = \tilde{s}_{21}^{\alpha\beta} \]
in the form
\[ \delta s_{11}^\alpha = \epsilon^\alpha \]
Since the change of variables \( s \to \tilde{s} \) is linear we have
\[ \tilde{s}_{12}^{\alpha\beta} = \nu^{\alpha\beta} + \tilde{\epsilon}^{\alpha\beta} \]
\[ \tilde{s}_{21}^{\alpha\beta} = \nu^{\alpha\beta} - \tilde{\epsilon}^{\alpha\beta}, \] \hspace{1cm} (B.2)
where \( \tilde{\epsilon} \) are linear in \( \epsilon \). We would like to show that the deformation of the quadratic Hamiltonian \( H_{2,0} \) from (3.6) is quadratic in \( \epsilon \) on shell.

Substituting the deformations into (B.1) we find that all terms obviously satisfy our assumption except the last one:
\[ \delta \sum_{\alpha \neq \beta} \tilde{s}_{12}^{\alpha\beta} \varphi_\alpha(z, \omega_\alpha + u) \varphi_\beta(z, \omega_\beta - u) = \\
\delta \sum_{\alpha \neq \beta} \tilde{s}_{12}^{\alpha\beta} \varphi_\alpha(z, \omega_\alpha + \omega_\beta) (E_1(z) + E_1(\omega_\alpha + u) + E_1(\omega_\beta - u)) \\
- E_1(z + \omega_\alpha + \omega_\beta) = \sum_{\alpha \neq \beta} (\tilde{\epsilon}^{\alpha\beta} \nu^{\alpha\beta} - \tilde{\epsilon}^{\beta\alpha} \nu^{\beta\alpha}) \varphi_\alpha(z, \omega_\alpha + \omega_\beta) (E_1(z) + E_1(\omega_\alpha + u) \\
+ E_1(\omega_\beta - u) - E_1(z + \omega_\alpha + \omega_\beta)) = \\
\sum_{\alpha \neq \beta} (\tilde{\epsilon}^{\alpha\beta} \nu^{\alpha\beta} - \tilde{\epsilon}^{\beta\alpha} \nu^{\beta\alpha}) \varphi_\alpha(z, \omega_\alpha + \omega_\beta) (E_1(\omega_\alpha + u) + E_1(\omega_\beta - u)) \] \hspace{1cm} (B.3)

However this expression does not provide dynamics via quadratic Hamiltonian since
\[ \varphi_\alpha(z, \omega_\alpha) \sim \frac{1}{z} - \frac{z}{2} \varphi(\omega_\alpha) \] \hspace{1cm} (B.4)
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