THREE SOLUTIONS FOR A FRACTIONAL ELLIPTIC PROBLEMS WITH CRITICAL AND SUPERCRITICAL GROWTH

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Abstract. In this paper, we deal with the existence and multiplicity of solutions for the fractional elliptic problems involving critical and supercritical Sobolev exponent via variational arguments. By means of the truncation combining with the Moser iteration, we prove that the problems has at least three solutions.

1. Introduction and main result

2. Introduction

In this paper, we consider the existence and multiplicity of solutions for the fractional elliptic problem

\[
\begin{aligned}
(-\Delta)^s u &= \lambda f(x, u) + \mu |u|^{p-2} u, & \text{in } \Omega, \\
    u &= 0, & \text{on } \partial \Omega,
\end{aligned}
\]

(2.1)

where \( \Omega \subset \mathbb{R}^N, N \geq 2, \) is a smooth bounded domain, \((-\Delta)^s\) stands for the fractional Laplacian, \( p \geq 2^*_s = \frac{2N}{N-2s}, \) \( \mu \) and \( \lambda \) are nonnegative constants and \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function.

The fractional Laplacian appears in diverse areas including physics, biological modeling, mathematical finance and especially partial differential equations involving the fractional Laplacian have been attracted by researchers. An important feature of the fractional Laplacian is its nonlocal property, which makes it difficult to handle. Recently, Caffarelli and Silvestre \cite{1} developed a local interpretation of the fractional Laplacian through Dirichlet-Neumann maps. This is commonly used in the recent literature since it allows to write nonlocal problems in a local way and this permits to use the variational methods for these kinds of problems.

Based on these extensions, many authors studied nonlinear problem of the form \((-\Delta)^s = f(x, u)\) for a certain function \( f : \mathbb{R}^N \to \mathbb{R} \). Among others, it is worthwhile to mention the work by Cabré-Tan \cite{2} and Tan \cite{3} when \( s = \frac{1}{2} \). They established the existence of positive solutions for equations having the subcritical growth, their regularity and symmetry properties. Recently, and for the subcritical case, Choi, Kim and Lee \cite{4} developed a nonlocal analog of the results by Han \cite{5} and Rey \cite{6}.

In this paper, we study the existence and multiplicity of solutions for the problem with critical and supercritical growth. For our problem, the first difficulty lies in that the fractional Laplacian operator \((-\Delta)^s\) is nonlocal, the nonlocal property of \((-\Delta)^s\) makes some calculations difficult. To overcome this difficulty, we do not...
work on the space $H^s_0(\Omega)$ directly, we transform the nonlocal problem into a local problem by the extension introduced by Caffarelli and Silvestre in [1]. After this extension, the problem (2.1) can be reduced to the problem

$$
\begin{align*}
\begin{cases}
\text{div}(y^{1-2s}\nabla w) = 0, & \text{in } C, \\
w = 0, & \text{on } \partial C, \\
\partial^s_\nu w = \lambda f(x, w) + \mu |w|^{p-2}w, & \text{in } \Omega \times \{0\},
\end{cases}
\end{align*}
$$

where $\nu$ is the outward unit normal vector to $C$ on $\Omega \times \{0\}$ and

$$
\partial^s_\nu w(x,0) := -\lim_{y \to 0^+} y^{1-2s} \frac{\partial w}{\partial y}(x,y), \quad \forall x \in \Omega.
$$

Obviously, the equation (2.2) is a local problem.

The second difficult lies in that problem (2.1) is a supercritical problem. Hence, we can not use directly the variational techniques because the corresponding energy functional is not well-defined on the Sobolev space $H^s_0(\Omega)$. To overcome this difficult, one usually uses the truncation and the Moser iteration. This spirit has been widely used in the supercritical Laplacian equation in the past few decades, see [7, 8, 9, 10, 11, 13] and references therein.

The aim of this paper is to study the problem (2.1) when $p \geq 2^*_s$. During this study we develop some nonlocal techniques which also have their own interests. In order to state our main results, we formulate the following assumptions:

1. **(f1)** $\lim_{|t| \to +\infty} \frac{f(x,t)}{|t|} = 0$ uniformly in $x \in \Omega$;

2. **(f2)** $\lim_{|t| \to 0} \frac{f(x,t)}{|t|} = 0$ uniformly in $x \in \Omega$;

3. **(f3)** $\sup_{u \in H^s_0(\Omega)} \int_{\Omega} F(x,u)dx > 0$, and for every $M > 0$, $f(x,u) \in L^\infty(\Omega)$ for each $|u| \leq M$, where $F(x,u) = \int_0^u f(x,t)dt$.

Set

$$
\theta := \frac{1}{2} \inf \left\{ \int_{\Omega} \frac{|(-\Delta)^{\frac{s}{2}} u|^2 dx}{\int_{\Omega} F(x,u)dx} : u \in H^s_0(\Omega), \int_{\Omega} F(x,u)dx > 0 \right\}.
$$

The main results are as follows.

**Theorem 2.1.** Assume that $(f_1) - (f_3)$ hold. Then there exists $\delta > 0$ such that for any $\mu \in [0, \delta]$, there exist an compact interval $[a, b] \subset (\frac{1}{2}, +\infty)$ and a constant $\gamma > 0$ such that for each $\lambda \in [a, b]$, the problem (2.1) has at least three solutions in $H^s_0(\Omega)$, whose norms are less than $\gamma$.

For the general problem

$$
\begin{align*}
\begin{cases}
(-\Delta)^s u = \lambda f(x,u) + \mu g(x,u), & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\end{align*}
$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, and

$$
|g(x,u)| \leq C(1 + |u|^{p-1}), \quad \text{where } p \geq 2^*_s = \frac{2N}{N-2s}, C > 0.
$$

If $f$ satisfies the conditions $(f_1) - (f_3)$, we also have the following result similar to Theorem 2.1.
Theorem 2.2. Let \( f \) satisfy \((f_1) - (f_4)\) and \( g \) satisfy \((g)\). Then there exists \( \delta > 0 \) such that for any \( \mu_0 \in [0, \delta) \), there exist an compact interval \([a, b]\) \( \subset (\frac{1}{2}, +\infty) \) and a constant \( \gamma > 0 \) such that for each \( \lambda \in [a, b] \), the problem \((2.3)\) has at least three solutions in \( H^s_0(\Omega) \), whose \( H^s_0(\Omega) \)-norms are less than \( \gamma \).

The paper is organized as follows. In Section 2, we introduce a variational setting of the problem and present some preliminary results. In Section 3, some properties of the fractional operator are discussed, and apply the truncation and the Moser iteration to obtain the proof of Theorem \((2.1)\) and Theorem \((2.2)\).

For convenience we fix some notations. \( L^p(\Omega) \) \((1 < p \leq \infty)\) denotes the usual Sobolev space with norm \( \| \cdot \|_{L^p} \); \( C_0(\Omega) \) denotes the space of continuous real functions in \( \Omega \) vanishing on the boundary \( \partial \Omega \); \( C \) or \( C_i \) \((i = 1, 2, \ldots)\) denote any positive constant.

3. Preliminaries and functional setting

In this section we recall some basic properties of the fractional Laplacian. In the entire space, the operator \(-\Delta^s\) in \( \mathbb{R}^N \), \(0 < s < 1\), is defined through Fourier transform \( \mathcal{F} \), by
\[
\mathcal{F}[-\Delta^s u](\xi) = |\xi|^{2s} \mathcal{F}[u](\xi).
\]
on a bounded domain \( \Omega \), we define \(-\Delta^s\) through the spectral decomposition of \(-\Delta\) in \( H^1(\Omega)\):
\[
(-\Delta)^s u = \sum_{i=1}^{\infty} \mu_i^s u_i \varphi_i,
\]
where \( u = \sum_{i=1}^{\infty} u_i \varphi_i \), \( u_i = \int_{\Omega} u \varphi_i dx \) and \((\mu_i, \varphi_i)\) are the eigenvalues and corresponding eigenfunctions of \(-\Delta\) on \( H^1(\Omega)\). The fractional Laplacian is well defined in the fractional Sobolev space \( H^s_0(\Omega)\),
\[
H^s_0(\Omega) = \{ u = \sum a_j \varphi_j \in L^2(\Omega) : \| u \|_{H^s_0} = (\sum a_j^2 \lambda^s)^{\frac{1}{2}} < \infty \},
\]
which is a Hilbert space endowed with the following inner product
\[
(\sum_{i=1}^{\infty} a_i \varphi_i, \sum_{i=1}^{\infty} b_i \varphi_i) = \sum_{i=1}^{\infty} a_i b_i \mu_i^s,
\]
and we have the following expression for this inner product
\[
\langle u, v \rangle = \int_{\Omega} (-\Delta)^s u \cdot (-\Delta)^s v dx = \int_{\Omega} (-\Delta)^s u v dx, \quad \forall u, v \in H^s_0(\Omega).
\]

We will often work with an equivalent definition based on an appropriate extension problem introduced by Caffarelli and Silvestre. Let \( \Omega \) be a bounded smooth domain in \( \mathbb{R}^N \), the half cylinder with base \( \Omega \) denote by \( \mathcal{C} = \Omega \times (0, \infty) \subset \mathbb{R}^{N+1}_+ \) and its lateral boundary given that \( \partial_t \mathcal{C} = \partial \Omega \times [0, \infty) \), where
\[
\mathbb{R}^{N+1}_+ = \{ (x, y) = (x_1, x_2, \ldots, x_n, y) \in \mathbb{R}^{N+1} : y > 0 \}.
\]
The space \( H^s_{0,L}(\mathcal{C}) \) is defined as the completion of
\[
C^s_{0,L}(\mathcal{C}) = \{ w \in C^\infty(\mathcal{C}) : w = 0 \text{ on } \partial_L \mathcal{C} \}.
\]
with respect to the norm
\[ \|w\|_{H^s_0(C)} = \left( \int_{\mathcal{C}} y^{1-2s} |\nabla w|^2 \, dxdy \right)^{\frac{1}{2}}. \]

This is a Hilbert space endowed with the following inner product
\[ \langle w, v \rangle = \int_{\mathcal{C}} y^{1-2s} \nabla w \nabla v \, dxdy, \quad \forall w, v \in H^s_0(C). \]

**Definition 3.1.** We say that \( u \in H^s_0(\Omega) \) is a solution of Equation (2.1) such that for every function \( \varphi \in H^s_0(\Omega) \), it holds
\[ \int_{\Omega} (\Delta)^{\frac{s}{2}} u (\Delta)^{\frac{s}{2}} \varphi \, dx = \lambda \int_{\Omega} f(x, u) \varphi \, dx + \mu \int_{\Omega} |u|^{p-2} u \varphi \, dx. \]

Associated with problem (2.1) we consider the energy functional
\[ I(u) = \frac{1}{2} \int_{\Omega} |(\Delta)^{\frac{s}{2}} u|^2 \, dx - \lambda \int_{\Omega} F(x, u) \, dx - \frac{\mu}{p} \int_{\Omega} |u|^p \, dx. \]

We now conclude the main ingredients of a recently developed technique which can deal with fractional power of the Laplacian. To treat the nonlocal problem (2.1), we will study a corresponding extension problem, so that we can investigate problem (2.1) by studying a local problem via classical nonlinear variational methods.

We first define the extension operator and fractional Laplacian for functions in \( H^s_0(\Omega) \).

**Definition 3.2.** Given a function \( u \in H^s_0(\Omega) \), we define its \( s \)-harmonic extension \( w = E_s(u) \) to the cylinder \( \mathcal{C} \) as a solution of the problem
\[
\begin{align*}
div(y^{1-2s} \nabla w) &= 0, \quad \text{in } \mathcal{C}, \\
w &= 0, \quad \text{on } \partial \mathcal{C}, \\
w &= u, \quad \text{on } \Omega \times \{0\}.
\end{align*}
\]

Following [1], we can define the fractional Laplacian operator by the Dirichlet to Neumann map as follows.

**Definition 3.3.** For any regular function \( u(x) \), the fractional Laplacian \( (\Delta)^s \) acting on \( u \) is defined by
\[ (\Delta)^s u(x) = - \lim_{y \to 0^+} y^{1-2s} \frac{\partial w}{\partial y}(x, y), \quad \forall x \in \Omega, \quad y \in (0, \infty), \]
where \( w = E_s(u) \).

From [1] and [12], the map \( E_s(\cdot) \) is an isometry between \( H^s_0(\Omega) \) and \( H^s_{0,L}(\mathcal{C}) \). Furthermore, we have
\begin{enumerate}
\item [(i)] \( \| (\Delta)^s u \|_{H^{-s}(\Omega)} = \| u \|_{H^s_0(\Omega)} = \| E_s(u) \|_{H^s_{0,L}(\mathcal{C})} \), where \( H^{-s}(\Omega) \) denotes the dual space of \( H^s_0(\Omega) \);
\item [(ii)] For any \( w \in H^s_{0,L}(\mathcal{C}) \), there exists a constant \( C \) independent of \( w \) such that
\[ \| \text{tr}_\Omega w \|_{L^r(\Omega)} \leq C \| w \|_{H^s_{0,L}(\mathcal{C})} \]
holds for every \( r \in [2, \frac{2N}{N-2s}] \). Moreover, \( H^s_{0,L}(\mathcal{C}) \) is compactly embedded into \( L^r(\Omega) \) for \( r \in [2, \frac{2N}{N-2s}] \).
In the following lemma, we will list some inequalities.

**Lemma 3.4.** For every $1 \leq r \leq \frac{2N}{N-2s}$ and every $w \in H^s_{0,L}(C)$, it holds

$$\left( \int_{\Omega \times \{0\}} |w|^r \, dx \right)^{\frac{1}{r}} \leq C \int_{\mathbb{R}^N} y^{1-2s} |\nabla w|^2 \, dxdy,$$

where constant $C$ depends on $r$, $s$, $N$, $|\Omega|$.

**Lemma 3.5.** For every $w \in H^s(\mathbb{R}^{N+1}_+)$ the sharp fractional Sobolev inequality for $N > 2s$ and $s > 0$

$$\left( \int_{\mathbb{R}^N} |u(x)|^{2^*} \, dx \right)^{\frac{1}{2^*}} \leq S \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla w(x,y)|^2 \, dxdy,$$

which holds with the constant

$$S = \frac{2^{-1} \pi^{-s} \Gamma(s) \Gamma \left( \frac{N-2s}{2} \right) (\Gamma(N))^\frac{2^*}{2^*}}{\Gamma \left( \frac{N+2s}{2} \right) (\Gamma \left( \frac{N}{2} \right))^\frac{2^*}{2^*}},$$

where $u = tr_{\Omega}w$.

Theorem 2.1 and 2.2 will be proved by using a recent result on the existence of at least three critical points by Ricceri [14, 15]. For the reader’s convenience, we describe it as follows.

If $X$ is a real Banach space, we can denote by $X^*$ the class of all function $\phi : X \to \mathbb{R}$ possessing the following property: if $\{u_n\} \subset X$ is a sequence converging weakly to $u \in X$ and $\lim \inf_{n \to \infty} \phi(u_n) \leq \phi(u)$, then $\{u_n\}$ has a subsequence converging strongly to $u$.

**Theorem 3.6.** Let $X$ be a separable and reflexive real Banach space; $I \subseteq \mathbb{R}$ an interval; $\Phi : X \to \mathbb{R}$ a sequentially weakly lower semi-continuous $C^1$ functional, belonging to $X^*$; $J : X \to \mathbb{R}$ a $C^1$ functional with compact derivative. Assume that, for each $\lambda \in I$, the functional $\Phi - \lambda J$ is coercive and has a strict local, not global minimum, say $\hat{u}_\lambda$. Then, for each compact interval $[a,b] \subseteq I$ for which

$$\sup_{\lambda \in [a,b]} (\Phi(\hat{u}_\lambda) - \lambda J(\hat{u}_\lambda)) < +\infty,$$

there exists $\gamma > 0$ with the following property: for every $\lambda \in [a,b]$ and every $C^1$ functional $\Psi : X \to \mathbb{R}$ with compact derivative, there exists $\delta_0 > 0$ such that, for each $\mu \in [0, \delta_0]$, the equation

$$\Phi'(u) = \lambda J'(u) + \mu \Psi'(u)$$

has at least three solutions whose norm are less than $\gamma$.

4. **Proof of the main results**

Let

$$\Psi(u) = \frac{1}{2} \int_{\mathbb{R}^N} y^{1-2s} |\nabla w|^2 \, dxdy, \quad J(w) = \int_{\Omega \times \{0\}} F(x,w) \, dx.$$ 

Obviously, the condition $(f_3)$ implies

$$\theta = \sup_{\Psi(w) \neq 0} \frac{J(w)}{\Psi(w)} > 0.$$
Lemma 4.1. Let \( f \) satisfy \((f_1) - (f_3)\). Then for every \( \lambda \in (0, \infty) \), the functional \( \Phi - \lambda J \) is sequentially weakly lower semi-continuous and coercive on \( H_{0,L}^s(\mathcal{C}) \), and has a global minimizer \( w_\lambda \).

Proof. By \((f_1)\) and \((f_3)\), for any \( \varepsilon > 0 \), there exist \( M_0 > 0 \) and \( C_1 > 0 \) such that, for all \( s \in H_{0,L}^s(\mathcal{C}) \),

\[
|f(x,s)| \leq \varepsilon |s|, \quad \forall |s| \geq M_0,
\]

and

\[
|f(x,s)| \leq C_1, \quad \forall |s| \leq M_0 + 1.
\]

So, for any \( w \in H_{0,L}^s(\mathcal{C}) \), we have

\[
|F(x,w)| \leq C_1 |w| + \frac{\varepsilon}{2} |w|^2,
\]

which implied that

\[
|F(x,w)| \leq C_1 |w| + \frac{\varepsilon}{2} |w|^2.
\]

Thus, for all \( w \in H_{0,L}^s(\mathcal{C}) \), we obtain

\[
\Phi(w) - \lambda J(w) = \frac{1}{2} \int_\mathcal{C} y^{1-2s} |\nabla w|^2 dxdy - \lambda \int_{\Omega \times \{0\}} F(x,w)dx
\]

\[
\geq \frac{1}{2} \int_\mathcal{C} y^{1-2s} |\nabla w|^2 dxdy - \lambda \int_{\Omega \times \{0\}} (C_1 |w| + \frac{\varepsilon}{2} |w|^2)dx
\]

\[
= \frac{1}{2} \|w\|_{H_{0,L}^s(\mathcal{C})}^2 - \lambda \frac{\varepsilon}{2} \int_{\Omega \times \{0\}} |w|^2 dx - \lambda C_1 \int_{\Omega \times \{0\}} |w| dx
\]

\[
\geq \left( \frac{1}{2} - \lambda \frac{\varepsilon C_2}{2} \right) \|w\|_{H_{0,L}^s(\mathcal{C})}^2 - \lambda C_1 C_3 \|w\|_{H_{0,L}^s(\mathcal{C})}^2
\]

where constants \( C_2 > 0, C_3 > 0 \). Let \( \varepsilon > 0 \) small enough such that \( \frac{1}{2} - \lambda \frac{\varepsilon C_2}{2} > 0 \), then we have

\[
\Phi(w) - \lambda J(w) \to +\infty \quad \text{as} \quad \|w\|_{H_{0,L}^s(\mathcal{C})} \to \infty.
\]

Hence, \( \Phi - \lambda J \) is coercive.

Moreover, from the embedding \( H_{0,L}^s(\mathcal{C}) \hookrightarrow L^r(\Omega) \) \((1 \leq r < 2^*_s)\) is compact and \( \Phi \) is weakly continuous. Obviously,

\[
\Phi(u) = \frac{1}{2} \int_\mathcal{C} y^{1-2s} |\nabla w|^2 dxdy = \frac{1}{2} \|w\|_{H_{0,L}^s(\mathcal{C})}^2
\]

is weakly lower semi-continuous on \( H_{0,L}^s(\mathcal{C}) \). We can deduce that \( \Phi - \lambda J \) is a sequentially weakly lower semi-continuous. So, \( \Phi - \lambda J \) has a global minimizer \( w_\lambda \in H_{0,L}^s(\mathcal{C}) \). The proof is completed. \( \square \)

Next, we will show that \( \Phi - \lambda J \) has a strictly local, not global minimizer for some \( \lambda \), when \( f \) satisfies \((f_1) - (f_3)\).

Lemma 4.2. Let \( f \) satisfy \((f_1) - (f_3)\). Then

(i) \( 0 \) is a strict local minimizer of the functional \( \Phi - \lambda J \) for \( \lambda \in (0, +\infty) \).

(ii) \( w_\lambda \neq 0 \), i.e., \( 0 \) is not the global minimizer \( w_\lambda \) for \( \lambda \in \left( \frac{1}{2}, +\infty \right) \), where \( w_\lambda \) is given by Lemma 4.1.
Proof. Firstly, we prove that
\[ \lim_{\|w\|_{H^s_{0,L}(\Omega)} \to 0} \frac{J(w)}{\Phi(w)} = 0, \quad \forall w \in H^s_{0,L}(\Omega). \]

In fact, by (f2), for any \( \varepsilon > 0 \), there exists a \( \delta > 0 \), such that
\[ |f(x,w)| \leq \varepsilon |w|, \quad |w| < \delta. \quad (4.2) \]
Considering the inequality (4.2), (f1) and (f3), there exists \( r \in (1, 2^*_s - 1) \) such that
\[ |f(x,w)| \leq \varepsilon |w| + |w|^r. \quad (4.3) \]
Then from Lemma [3-4] there exist \( C_4, C_5 > 0 \), such that
\[ |J(w)| \leq \varepsilon C_4 \|w\|^2_{H^s_{0,L}(\Omega)} + C_5 \|w\|^{r+1}_{H^s_{0,L}(\Omega)}. \]
This implies
\[ \lim_{\|w\|_{H^s_{0,L}(\Omega)} \to 0} \frac{J(w)}{\Phi(w)} = 0. \]

Next, we will prove (i) and (ii).

(i) For \( \lambda \in (0, +\infty) \), since
\[ \lim_{\|w\|_{H^s_{0,L}(\Omega)} \to 0} \frac{J(w)}{\Phi(w)} = 0 < \frac{1}{\lambda} \quad \text{and} \quad \Phi(w) > 0 \quad \text{for each} \quad w \neq 0 \quad \text{in some neighborhood} \ U \quad \text{of} \ 0, \]
there exists a neighborhood \( V \subseteq U \) of 0 such that
\[ \Phi(w) - \lambda J(w) > 0, \quad \forall \ w \in V \setminus \{0\}. \]
Hence, 0 is a strict local minimum of \( \Phi - \lambda J \).

(ii) For \( \lambda \in (\frac{1}{\delta}, +\infty) \), from the definition of \( \theta \), there exists \( \hat{w} \in H^s_{0,L}(\Omega) \) such that \( \Phi(\hat{w}) > 0, J(\hat{w}) > 0 \) and \( \frac{J(\hat{w})}{\Phi(\hat{w})} > \frac{1}{\lambda} \). So we have
\[ \Phi(\hat{w}) - \lambda J(\hat{w}) < 0 = \Phi(0) - \lambda J(0). \]
This yields 0 is not a global minimum of \( \Phi - \lambda J \).

This completes the proof. \( \square \)

Let \( K > 0 \) be a real number, whose value will be fixed latter. Define the truncation of \( |w|^{p-2}w \) with \( p > 2^*_s \), be given by
\[ g_K(w) = \begin{cases} |w|^{p-2}w, & \text{if} \ 0 \leq |w| \leq K, \\ K^{p-q}|w|^{q-2}w, & \text{if} \ |w| > K, \end{cases} \]
where \( q \in (2, 2^*_s) \). Then \( g_K(w) \) satisfies
\[ |g_K(w)| \leq K^{p-q}|w|^{q-1}, \]
for \( K \) large enough. Then, we study the truncated problem
\[ \begin{cases} \text{div}(y^{1-2s}\nabla w) = 0, & \text{in} \ C, \\ w = 0, & \text{on} \ \partial L C, \\ \partial^\nu w = \lambda f(x,w) + \mu g_K(w), & \text{in} \ \Omega \times \{0\}, \end{cases} \quad (4.4) \]
We say that \( w \in H^s_{0,L}(\Omega) \) is a weak solution of the problem (4.4) if
\[ \int_C y^{1-2s}\nabla w \cdot \nabla \varphi dxdy = \lambda \int_{\Omega \times \{0\}} f(x,w)\varphi dx + \mu \int_{\Omega \times \{0\}} g_K(w)\varphi dx \]
for every $\varphi \in H^s_{0,1}(C)$.

Let

$$\Psi(u) = \int_{\Omega \times \{0\}} G_K(w) \, dx,$$

where $G_K(w) = \int_0^w g_K(t) \, dt$. So from $|g_K(w)| \leq |K|^{p-1} |w|^{q-1}$, $2 < q < 2^*$, we get that $g_K(w)$ is a super-linear function with subcritical growth, then $\Psi(u)$ has a compact derivative in $H^s_{0,1}(C)$. Moreover, for each compact interval $[a, b] \subset (\frac{1}{2^*}, +\infty)$, $\lambda \in [a, b]$. From (4.3), we have then $J(w)$ has a compact derivative in $H^s_{0,1}(C)$ too. Therefore, it is easy to see that the functional

$$\mathcal{E}(w) = \Phi(w) - \lambda J(w) - \mu \Psi(w) \quad \forall \ w \in H^s_{0,1}(C)$$

is $C^1$ and its derivative is given by

$$\langle \mathcal{E}'(w), \varphi \rangle = \int_{\Omega \times \{0\}} y^{1-2s} \nabla w \nabla \varphi \, dx \, dy - \lambda \int_{\Omega \times \{0\}} f(x, w) \varphi \, dx - \mu \int_{\Omega \times \{0\}} g_K(w) \varphi \, dx,$$

for all $\varphi \in H^s_{0,1}(C)$.

By Lemma 4.1 and Lemma 4.2, all the hypotheses of Theorem 3.6 are satisfied. So there exists $\gamma > 0$ with the following property: for every $\lambda \in [a, b] \subset (\frac{1}{2^*}, +\infty)$, there exists $\delta_0 > 0$, such that for $\mu \in [0, \delta_0]$, the problem (4.4) has at least three solutions $w_0, w_1$ and $w_2$ in $H^s_{0,1}(C)$ and

$$\|w_k\|_{H^s_{0,1}(C)} \leq \gamma, \quad k = 0, 1, 2,$$

where $\gamma$ depends on $\lambda$, but does depend on $\mu$ or $K$.

If the three solutions $w_k$, $k = 0, 1, 2$, satisfy

$$|w_k| \leq K, \quad \text{a.e. } (x, y) \in \Omega \times (0, \infty), \quad k = 0, 1, 2. \quad (4.6)$$

Then in the view of the definition $g_K$, we have $g_K(x, w) = |w|^{p-2} w$ and therefore $w_k$, $k = 0, 1, 2$, are also solutions of the original problem (2.2). Thus, in order to prove Theorem 2.1 it suffices to show that exists $\delta_0 > 0$, such that for $\mu \in [0, \delta_0]$, the solutions obtained by Theorem 3.6 satisfy the inequality (4.6).

**Proof of theorem 2.1.** Our aim is to show that exits $\delta_0 > 0$, such that for $\mu \in [0, \delta_0]$, the solution $w_k$, $k = 0, 1, 2$, satisfy the inequality (4.6). To save notation, we will denote $w := w_k$, $k = 0, 1, 2$.

Set $w_+ = \max\{w, 0\}$, $w_- = - \min\{w, 0\}$. Then $|w| = w_+ + w_-$. We can argue with the positive and negation part of $w$ separately.

We first deal with $w_+$. For each $L > 0$, we define the following functions

$$w_L = \begin{cases} 
  w_+, & \text{if } w_+ \leq L, \\
  L, & \text{if } w_+ > L.
\end{cases} \quad (4.5)$$

For $\beta > 1$ to be determined, we choose in (4.5) that

$$\varphi = w_L^{2(\beta - 1)} w_+, \quad (4.5)$$

and since

$$\nabla \varphi = w_L^{2(\beta - 1)} \nabla w_+ + 2(\beta - 1) w_L^{2(\beta - 1) - 1} w_+ \nabla w_L,$$
we obtain
\[ \int_{\mathcal{C}} y^{1-2s} \nabla w \nabla \varphi \, dx \, dy \]
\[ = \int_{\mathcal{C}} y^{1-2s} (\nabla (w_+ - w_-)) \nabla (w_L^{2(\beta-1)} w_+) \, dx \, dy \]
\[ = \int_{\mathcal{C}} y^{1-2s} (\nabla w_+ - \nabla w_-) (w_L^{2(\beta-1)} \nabla w_+ + 2(\beta - 1) w_L^{2(\beta-1)-1} w_+ \nabla w_L) \, dx \, dy \]
\[ = \int_{\mathcal{C}} y^{1-2s} \left( \nabla w_+^2 w_L^{2(\beta-1)} + 2(\beta - 1) w_L^{2(\beta-1)-1} w_+ \nabla w_L \nabla w_+ \right) \, dx \, dy \]
\[ = \int_{\mathcal{C}} y^{1-2s} w_L^{2(\beta-1)} |\nabla w_+|^2 \, dx \, dy + 2(\beta - 1) \int_{\mathcal{C}} y^{1-2s} w_L^{2(\beta-1)-1} w_+ \nabla w_L \nabla w_+ \, dx \, dy. \]

(4.7)

From the definition of \( w_L \), we have
\[ 2(\beta - 1) \int_{\mathcal{C}} y^{1-2s} w_L^{2(\beta-1)-1} w_+ \nabla w_L \nabla w_+ \, dx \, dy \]
\[ = 2(\beta - 1) \int_{\{w_+ < L\}} y^{1-2s} w_L^{2(\beta-1)-1} w_+ \nabla w_L \nabla w_+ \, dx \, dy \]
\[ = 2(\beta - 1) \int_{\{w_+ < L\}} y^{1-2s} w_L^{2(\beta-1)-1} |\nabla w_+|^2 \, dx \, dy \]
\[ \geq 0. \]

Set
\[ h_K(x, w) = \lambda f(x, w) + \mu g_K(x, w), \quad \forall w \in H^s_{0,L}(\mathcal{C}). \]

From (4.3) and \(|g_K(x, w)| \leq K^{p-q} |w|^{q-1}\), we can choose constant \( C_6 > 0 \) such that
\[ |h_K(x, w)| \leq C_6 |w| + \mu K^{p-q} |w|^{q-1}. \]

(4.9)

We deduce from (4.5), (4.7), (4.8) and (1.9) for \( \beta > 1 \) that
\[ \int_{\mathcal{C}} y^{1-2s} w_L^{2(\beta-1)} |\nabla w_+|^2 \, dx \, dy = \int_{\Omega \times \{0\}} h_K(x, w) \varphi \, dx \leq \int_{\Omega \times \{0\}} |h_K(x, w)\varphi| \, dx \]
\[ \leq \int_{\Omega \times \{0\}} \left( C_6 |w| + \mu K^{p-q} |w|^{q-1} \right) w_L^{2(\beta-1)} w_+ \, dx \]
\[ = \int_{\Omega \times \{0\}} \left( C_6 (w_+ + w_-) + \mu K^{p-q} (w_+ + w_-)^{q-1} \right) w_L^{2(\beta-1)} w_+ \, dx \]
\[ = \int_{\Omega \times \{0\}} \left( C_6 w_+^2 w_L^{2(\beta-1)} + \mu K^{p-q} w_+^q w_L^{2(\beta-1)} \right) \, dx. \]

(4.10)

Let \( \hat{w}_L = w_+ w_L^{\beta-1} \), we have
\[ \nabla \hat{w}_L = w_+ w_L^{\beta-1} \nabla w_+ + (\beta - 1) w_+ w_L^{\beta-2} \nabla w_L. \]
By the Sobolev embedding theorem,

\[ \left( \int_{\Omega \times \{0\}} |\hat{w}|^{2^*} dx \right)^{\frac{1}{2^*}} \leq S \int_{\mathbb{C}} y^{1-2s} |\nabla \hat{w}|^2 dx dy \]

\[ = C \int_{\mathbb{C}} y^{1-2s} |w_{\beta - 1} w_+ + (\beta - 1)w_+ w_{\beta - 2} \nabla w_L|^2 dx dy \]

\[ \leq 2S \int_{\mathbb{C}} y^{1-2s} (\beta - 1)^2 |w_L|^{2(\beta - 1)} |\nabla w_+|^2 dx dy + \int_{\mathbb{C}} y^{1-2s} |w_{\beta - 1} \nabla w_+|^2 dx dy \]

\[ \leq 2S \left( (\beta - 1)^2 + 1 \right) \int_{\mathbb{C}} y^{1-2s} w_L^{2(\beta - 1)} |\nabla w_+|^2 dx dy \]

\[ = 2S \beta^2 \left( \left( \frac{\beta - 1}{\beta} \right)^2 + \frac{1}{\beta^2} \right) \int_{\mathbb{C}} y^{1-2s} w_L^{2(\beta - 1)} |\nabla w_+|^2 dx dy, \tag{4.11} \]

where \( S > 0 \) is the Sobolev embedding constant.

Since \( \beta > 1 \), we have \( \frac{1}{\beta^2} < 1 \) and \( \left( \frac{\beta - 1}{\beta^2} \right)^2 < 1 \). From (4.10) and (4.11), we get

\[ 2S \beta^2 \left( \left( \frac{\beta - 1}{\beta} \right)^2 + \frac{1}{\beta^2} \right) \int_{\mathbb{C}} y^{1-2s} w_L^{2(\beta - 1)} |\nabla w_+|^2 dx dy \]

\[ < 4S \beta^2 \int_{\mathbb{C}} y^{1-2s} w_L^{2(\beta - 1)} |\nabla w_+|^2 dx dy \]

\[ \leq 4S \beta^2 \int_{\Omega \times \{0\}} (C_6 w_L^{2(\beta - 1)} + \mu K^{p-q} u_+^q w_L^{2(\beta - 1)}) dx. \tag{4.12} \]

From the Sobolev embedding \( H^s_{0,L}(\mathbb{C}) \hookrightarrow L^{2^*}(\Omega) \) and \( \|w_+\|_{H^s_{0,L}(\mathbb{C})} \leq \gamma \), we have

\[ \left( \int_{\Omega \times \{0\}} |w_+|^{2^*} dx \right)^{\frac{1}{2^*}} \leq S \int_{\mathbb{C}} y^{1-2s} |\nabla w_+|^2 dx dy \leq S \gamma. \tag{4.13} \]
Let $t = \frac{2^{2\gamma}}{2^{\gamma} - q}$. Since $w_+^q w_L^{2(\beta-1)} = w_+^q w_+^2 w_L^{-2} = w_+^{q-2} w_L^2$ and $w_L = w_L^2 w_L^{2(\beta-1)}$, we can use the Hölder’s inequality, (4.11), (4.12) and (4.13) to conclude that, whenever $\hat{w}_L(\cdot, 0) \in L^t(\Omega)$, it holds

$$
\left( \int_{\Omega \times \{0\}} |\hat{w}_L|^2 \, dx \right)^{\frac{2}{\gamma}} \leq 4S \beta^2 \left( \int_{\Omega \times \{0\}} (C w_+^2 w_L^{2(\beta-1)} + \mu K^{p-q} w_+^{q/2} w_L^{2(\beta-1)}) \, dx \right),
$$

$$
= 4S \beta^2 \left( \int_{\Omega \times \{0\}} \hat{w}_L^2 \, dx + \mu K^{p-q} \int_{\Omega \times \{0\}} w_+^{q-2} \hat{w}_L^2 \, dx \right),
$$

$$
\leq 4S \beta^2 \left[ \left( \int_{\Omega \times \{0\}} \hat{w}_L^2 \, dx \right)^{\frac{2}{\gamma}} + \mu K^{p-q} \left( \int_{\Omega \times \{0\}} |w_+|^{2^*} \, dx \right)^{\frac{2}{2^*}} \left( \int_{\Omega \times \{0\}} \hat{w}_L^2 \, dx \right)^{\frac{2}{\gamma}} \right],
$$

$$
\leq 4S \beta^2 \left( |\Omega|^{\frac{2^*}{2^*}} + \mu K^{p-q} (S \gamma)^{\frac{2-2^*}{2^*}} \right) \left( \int_{\Omega \times \{0\}} |w_+|^{-1} \, w_+^t \, dx \right)^{\frac{2}{t}}.
$$

Set $\beta := \frac{2^*}{t} = 1 + \frac{2^*-q}{2} > 1$. By the definition of $w_L$, we have $w_L \leq w_+$, then we conclude that $\hat{w}_L(\cdot, 0) \in L^t(\Omega)$, whenever $(w_+)(\cdot, 0) \in L^t(\Omega)$. If this is the case, it follow from the above inequality that

$$
\left( \int_{\Omega \times \{0\}} |\hat{w}_L|^2 \, dx \right)^{\frac{2}{\gamma}} = \left( \int_{\Omega \times \{0\}} w_+^{2^*(\beta-1)} w_+^{2^*} \, dx \right)^{\frac{2}{\gamma}} \leq 4S \beta^2 \left( |\Omega|^{\frac{2^*}{2^*}} + \mu K^{p-q} (S \gamma)^{\frac{2-2^*}{2^*}} \right) \left( \int_{\Omega \times \{0\}} |w_+|^{-1} \, w_+^t \, dx \right)^{\frac{2}{t}}.
$$

By Fatou’s Lemma in the variable $L$, we get

$$
\left( \int_{\Omega \times \{0\}} w_+^{2^* \beta} \, dx \right)^{\frac{2}{2^* \beta}} \leq 4S \beta^2 C_{\mu,K} \left( \int_{\Omega \times \{0\}} |w_+|^{t \beta} \, dx \right)^{\frac{2}{t \beta}},
$$

i.e.,

$$
\left( \int_{\Omega \times \{0\}} w_+^{2^* \beta} \, dx \right)^{\frac{1}{2^* \beta}} \leq \left( 4S C_{\mu,K} \right)^{\frac{1}{\beta}} \left( \int_{\Omega \times \{0\}} |w_+|^{t \beta} \, dx \right)^{\frac{1}{t \beta}}, \quad(4.14)
$$

where $C_{\mu,K} = |\Omega|^{\frac{2^*-2}{2^*}} + \mu K^{p-q} (S \gamma)^{\frac{2-2^*}{2^*}}$.

Since $\beta = \frac{2^*}{t} > 1$ and $w_+(\cdot, 0) \in L^2(\Omega)$, the inequality (4.14) holds for this choice of $\beta$. Therefore, from $\beta^2 t = \beta 2^*$, we have that the inequality (4.14) also
holds with $\beta$ replaced by $\beta^2$. Hence
\[
\left( \int_{\Omega \times \{0\}} w_+^{2^*\beta^2} \, dx \right)^{\frac{1}{2^*\beta^2}} \leq \left( 4S C_{\mu, K} \right)^{\frac{1}{2^*\beta^2}} \left( \int_{\Omega \times \{0\}} \left| w_+ \right|^{\beta^2} \, dx \right)^{\frac{1}{\beta^2}}
\]
\[
= \left( 4S C_{\mu, K} \right)^{\frac{1}{2^*\beta^2}} \left( \int_{\Omega \times \{0\}} \left| w_+ \right|^{2^*\beta^2} \, dx \right)^{\frac{1}{\beta^2}}
\]
\[
\leq \left( 4S C_{\mu, K} \right)^{\frac{1}{2^*\beta^2}} \left( 4S C_{\mu, K} \right)^{\frac{1}{2^*\beta^2}} \beta^{\frac{1}{\beta^2}} \left( \int_{\Omega \times \{0\}} \left| w_+ \right|^{\beta} \, dx \right)^{\frac{1}{\beta}}
\]
\[
= \left( 4S C_{\mu, K} \right)^{\frac{1}{2^*\beta^2} + \frac{1}{\beta^2}} \beta^{\frac{1}{\beta^2}} \left( \int_{\Omega \times \{0\}} \left| w_+ \right|^{2^*\beta^2} \, dx \right)^{\frac{1}{\beta^2}}.
\]
By iterating this process and $\beta t = 2^*_t$, we obtain
\[
\left( \int_{\Omega \times \{0\}} w_+^{2^*_t\beta^m} \, dx \right)^{\frac{1}{2^*_t\beta^m}} \leq \left( 4S C_{\mu, K} \right)^{\frac{1}{2^*_t\beta^m}} \left( \int_{\Omega \times \{0\}} \left| w_+ \right|^{\beta^m} \, dx \right)^{\frac{1}{\beta^m}}.
\]
\[\tag{4.15}\]
Taking the limit as $m \to \infty$ in (4.15), we have
\[
\|w_+\|_{L^\infty} \leq \left( 4S C_{\mu, K} \right)^{\theta_1} \|w_+\|_{L^{2^*_1}} \leq \left( 4S C_{\mu, K} \right)^{\theta_1} \beta^{\theta_2}(S \gamma)^{\frac{1}{2}},
\]
where $\theta_1 = \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{\beta^m}$, $\theta_2 = \sum_{m=1}^{\infty} \frac{m}{\beta^m}$ and $\beta > 1$.

Next, we will find some suitable value of $K$ and $\mu$, such that the inequality
\[
\left( 4S C_{\mu, K} \right)^{\theta_1} \beta^{\theta_2}(S \gamma)^{\frac{1}{2}} \leq \frac{K}{2}
\]
\[\tag{4.16}\]
holds. From (4.10), we get
\[
C_{\mu, K} = |\Omega|^{\frac{1}{2^*\beta^2}} + \mu K^{p-q}(S \gamma)^{\frac{2^*}{2^*}} \leq \frac{1}{4S} \left( \frac{K}{2(S \gamma)^{\frac{1}{2}}} \right)^{\frac{1}{\beta^2}}.
\]
Then, choose $K$ to satisfy the inequality
\[
\frac{1}{4S} \left( \frac{K}{2(S \gamma)^{\frac{1}{2}}} \right)^{\frac{1}{\beta^2}} - |\Omega|^{\frac{2^*}{2^*}} = 0,
\]
and fix $\mu_0$ such that
\[
0 < \mu_0 < \mu' := \frac{1}{K^{p-q}(S \gamma)^{\frac{2^*}{2^*}}} \left( \frac{1}{4S} \left( \frac{K}{2(S \gamma)^{\frac{1}{2}}} \right)^{\frac{1}{\beta^2}} - |\Omega|^{\frac{2^*}{2^*}} \right).
\]
Thus, we obtain (4.15) for $\mu \in [0, \mu_0]$, i.e.,
\[
\|w_+\|_{L^\infty} \leq \frac{K}{2}, \quad \text{for } \mu \in [0, \mu_0]. \tag{4.17}
\]
Similarly, we can also have the estimate for the $w_-$, i.e.,
\[
\|w_-\|_{L^\infty} \leq \frac{K}{2}, \quad \text{for } \mu \in [0, \mu_0]. \tag{4.18}
\]
Now, let $\delta = \min \{\delta_0, \mu_0\}$. For each $\mu \in [0, \delta]$, from (4.17), (4.18) and $|w| = w_+ + w_-$, we have
\[
\|w\|_{L^\infty} \leq K, \quad \text{for } \mu \in [0, \mu_0].
\]
Considering this fact and $w := w_k, k = 1, 2, 3$ we get

$$
\|w_k\|_{L^\infty} \leq K, \quad k = 0, 1, 2, \quad \text{for } \mu \in [0, \delta].
$$

Therefore, we obtain the inequality (4.10). The proof is completed.

**Proof of theorem 2.2.** In fact, the truncation of $g_K(x, s)$ can be given by

$$
g_K(x, s) = \begin{cases}
g(x, s) & \text{if } |s| \leq K \\
\min\{g(x, s), C_0(1 + K^{p-q}|s|^{q-2})\} & \text{if } |s| > K
\end{cases}
$$

where $q \in (2, 2^*)$, Then $g_K$ satisfies

$$
|g_K(x, s)| \leq C_0(1 + K^{p-q}|s|^{q-2}), \quad \forall s \in \mathbb{R}.
$$

Let $h_K(x, w) = \lambda f(x, w) + \mu g_K(x, w)$, $\forall w \in H^s_{0,L}(\mathcal{C})$. The truncated problems associated to $h_K$

$$
\begin{align*}
\text{div}(y^{1-2s}\nabla w) &= 0, & \text{in } \mathcal{C}, \\
w &= 0, & \text{on } \partial L \mathcal{C}, \\
\partial^s_w w &= h_K(x, w), & \text{in } \Omega \times \{0\}.
\end{align*}
$$

(4.19)

Similar the proof of the Theorem 2.1, using Theorem 3.6 we can prove that there exists $\delta > 0$ such that the solutions $w$ for the truncated problems (4.19) satisfy $\|w\|_{L^\infty} \leq K$ for $\mu \in [0, \delta]$; and in view of the definition $g_K$, we have

$$
h_K(x, w) = \lambda f(x, w) + \mu g(x, w).
$$

Therefore $w := w_k, k = 0, 1, 2$, are also solutions of the original problem (2.3).

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