Periodic magnetic geodesics on Heisenberg manifolds

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Abstract
We study the dynamics of magnetic flows on Heisenberg groups, investigating the extent to which properties of the underlying Riemannian geometry are reflected in the magnetic flow. Much of the analysis, including a calculation of the Mañé critical value, is carried out for \((2n+1)\)-dimensional Heisenberg groups endowed with any left invariant metric and any exact, left-invariant magnetic field. In the three-dimensional Heisenberg case, we obtain a complete analysis of left-invariant, exact magnetic flows. This is interesting in and of itself, because of the difficulty of determining geodesic information on manifolds in general. We use this analysis to establish two primary results. We first show that the vectors tangent to periodic magnetic geodesics are dense for sufficiently large energy levels and that the lower bound for these energy levels coincides with the Mañé critical value. We then show that the marked magnetic length spectrum of left-invariant magnetic systems on compact quotients of the Heisenberg group determines the Riemannian metric. Both results confirm that this class of magnetic flows carries significant information about the underlying geometry. Finally, we provide an example to show that extending this analysis of magnetic flows to the Heisenberg-type setting is considerably more difficult.

Keywords R rigidity · Magnetic flow · Mañé critical value · Two-step nilmanifold · Periodic magnetic geodesic · Heisenberg group

Mathematics Subject Classification 53C22 · 53C24 · 37C27

1 Introduction

From the perspective of classical mechanics, the geodesics of a Riemannian manifold \((M, g)\) are the possible trajectories of a point mass moving in the absence of any forces and in zero potential. A magnetic field can be introduced by choosing a closed two-form \(\Omega\) on \(M\). A charged particle moving on \(M\) now experiences a Lorentz force, and its trajectory is called a magnetic geodesic. As with Riemannian geodesics, magnetic geodesics can be
handled collectively as a single object called the magnetic geodesic flow on $TM$ or $T^*M$ (see Sect. 2.1 for precise definitions). Many classical questions concerning geodesic flows have corresponding analogs for magnetic flows. Indeed, magnetic flows display a number of remarkable properties. See [4, 6, 19, 29], and [1] for a sampling of results.

A magnetic flow may be interpreted as a perturbation of the underlying geodesic flow. We adopt this perspective and investigate the extent to which properties of the underlying Riemannian geometry are reflected in the magnetic flow. Furthermore, we focus on magnetic flows generated by left-invariant magnetic fields on Riemannian two-step nilmanifolds. These questions have been addressed in related contexts, but not in this more complicated setting.

Much of the prior work on magnetic flows has been done from the perspective of magnetic flows as perturbations. In [6], the authors consider the property of topological entropy of magnetic flows on SOL manifolds, while [13] considers the same property in the setting of magnetic flows on two-step nilmanifolds. In addition to topological entropy, [30, 31] and [5] all consider the Anosov property. In [33], the authors show that at high enough energy levels the magnetic geodesics are quasi-geodesics with respect to the underlying Riemannian structure. Both [7] and [32] present the example of a magnetic flow on the Heisenberg group. The former paper develops useful symplectic techniques, while the latter paper provides an example of a Tonelli Lagrangian system with empty Aubrey set in the universal cover. We build on these lines of inquiry by analyzing magnetic flows on larger classes of nilmanifolds; in addition, we explore other geometric properties.

For a large class of Riemannian two-step nilmanifolds, it is possible to describe precisely the set of smoothly closed geodesics, along with their lengths. See Eberlein [11] for the Heisenberg case and Gornet-Mast [16] for the more general setting of Heisenberg-like manifolds. We extend this investigation to magnetic flows. We note that much of the analysis is carried out for general class of $(2n + 1)$-dimensional Heisenberg groups. In the three-dimensional case, we obtain a complete analysis of left-invariant, exact magnetic flows on three-dimensional Heisenberg groups. We summarize this as follows.

**Theorem** (See Sect. 3, Lemma 8, and Theorem 4) Let $\Gamma$ be any cocompact discrete subgroup of the Heisenberg group $H$. The lengths and initial conditions of closed magnetic geodesics for any left-invariant Riemannian metric and any left-invariant, exact magnetic field on $\Gamma \backslash H$ may be explicitly computed in terms of metric Lie algebra information.

Unlike the Riemannian case, closed magnetic geodesics exist in all nontrivial homotopy classes only for sufficiently large energy. In addition, there exist closed and contractible magnetic geodesics on sufficiently small energy levels. The energy level at which this transition occurs is known as the Mañé critical value. The critical value has been extensively studied in many different contexts. In [7], the authors investigate how the symplectic topology of the energy hypersurfaces in the cotangent bundle changes at the critical value. As an illustration of their results, they analyze an exact left-invariant magnetic system on a compact quotient of the three-dimensional Heisenberg group and compute the corresponding critical value. We compute the Mañé critical value for the more general class of exact left-invariant magnetic systems on two-step nilmanifolds.

We give two applications of our main result. The first concerns the density of tangent vectors to closed magnetic geodesics. Eberlein analyzes this property for Riemannian geodesic flows on two-step nilmanifolds with a left-invariant metric, showing that for certain types of two-step nilpotent Lie groups (including Heisenberg groups), the vectors tangent
to smoothly closed unit speed geodesics in the corresponding nilmanifold are dense in the unit tangent bundle [11]; Mast [24] and Lee-Park [23] broadened this result.

**Theorem** (see Theorem 5) The density property continues to hold for magnetic flows with sufficiently high energy on the Heisenberg group.

The second application is a marked length spectrum rigidity result (see Sect. 4.4 for the definition). It is known that within certain classes of Riemannian manifolds, if two manifolds have the same marked length spectrum, then they are isometric. This is true in the class of negatively curved surfaces (see [8] and [27, 28]) and compact flat manifolds (see [2, 3, 26]). In [20], S. Grognet studies marked length spectrum rigidity of magnetic flows on surfaces with pinched negative curvature.

**Theorem** (see Theorem 6) The marked magnetic length spectrum of left-invariant magnetic systems on compact quotients of the Heisenberg group determines the Riemannian metric.

Although it is a perturbation of geodesic flow, the magnetic flow still carries information about the underlying Riemannian manifold.

This paper is organized as follows. In Sect. 2, we present the necessary preliminaries in order to state and prove the main theorems. The definition and basic properties of magnetic flows are given in Sect. 2.1, and the necessary background on nilmanifolds is given in Sect. 2.4. Next, we show how a left-invariant Hamiltonian system on the cotangent bundle of a Lie group reduces to a so-called Euler flow on the dual to the Lie algebra. Such Hamiltonians are known as collective Hamiltonians, and this process is outlined in Sect. 2.3. Section 2.5 specializes the preceding to the case of exact, left-invariant magnetic flows on two-step nilpotent Lie groups. The Mañé critical value is computed in Sect. 2.6. In Sect. 3, the magnetic geodesic equations on the \((2n + 1)\)-dimensional Heisenberg group are solved. In Sect. 4, we use this analysis to obtain the main results. Many geometric results for the Heisenberg group have been shown to hold for the larger class of Heisenberg-type manifolds. In Sect. 5, we use a specific example to show why our analysis of magnetic flows on Heisenberg-type manifolds is considerably more difficult. Lastly, the so-called \(j\)-maps are a central part of the theory of two-step Riemannian nilmanifolds. In the appendix, we provide an alternative approach to studying the magnetic geodesics using \(j\)-maps instead of collective Hamiltonians.

## 2 Preliminaries

### 2.1 Magnetic flows

A **magnetic structure** on a Riemannian manifold \((M, g)\) is a choice of a closed 2-form \(\Omega\) on \(M\), called the **magnetic 2-form**. The **magnetic flow** of \((M, g, \Omega)\) is the Hamiltonian flow \(\Phi_t\), determined by the symplectic form on the tangent bundle

\[
\vec{\sigma}_{\text{mag}} = \vec{\sigma} + \pi^*\Omega
\]

and the kinetic energy Hamiltonian \(H_0 : TM \to \mathbb{R}\), given by
Here, $\pi : TM \to M$ denotes the canonical projection and $\tilde{\sigma}$ denotes the pullback via the Riemannian metric of the canonical symplectic form on $T^*M$.

The magnetic flow models the motion of a charged particle under the effect of a magnetic field whose Lorentz force $F : TM \to TM$ is the bundle map defined via
$$\Omega_x(u, v) = g_x(F_x u, v)$$
for all $x \in M$ and all $u, v \in T_x M$. The orbits of the magnetic flow have the form $t \mapsto \dot{\sigma}(t)$, where $\sigma$ is a curve in $M$ such that
$$\nabla_{\dot{\sigma}} \dot{\sigma} = F \dot{\sigma}.$$  \hfill (3)

In the case that $\Omega = 0$, the magnetic flow reduces to Riemannian geodesic flow. A curve $\sigma$ that satisfies (3) is called a magnetic geodesic. The physical interpretation of a magnetic geodesic is that it is the path followed by a particle with unit mass and charge under the influence of the magnetic field. Because $F$ is skew-symmetric, the acceleration of the magnetic geodesic is perpendicular to its velocity.

Remark 1 It is straightforward to show that magnetic geodesics have constant speed. In contrast to the Riemannian setting, a unit speed reparametrization of a solution to (3) may no longer be a solution. To see this, let $\sigma(s)$ be a solution that is not unit speed and denote energy $E = |\dot{\sigma}| > 0$. Define $\tau(s) = \sigma(s/E)$, which is unit speed. Then,
$${\nabla_{\dot{\tau}} \dot{\tau}} = \frac{1}{E^2} \nabla_{\dot{\sigma}} \dot{\sigma} = \frac{1}{E^2} F \dot{\sigma} = \frac{1}{E} F \dot{\tau} \neq F \dot{\tau},$$
in general. Therefore, one views a magnetic geodesic as the path, not the parameterized curve. (Observe that $\tau$ is a solution to the magnetic flow determined by the magnetic form $\sqrt{E} \Omega$.)

Recall that the tangent and cotangent bundles of a Riemannian manifold are canonically identified, and the Riemannian metric on $TM \to M$ induces a non-degenerate, symmetric 2-tensor on $T^*M \to M$. We will present most of the theory in the setting of the cotangent bundle, while occasionally indicating how to translate to the tangent bundle. Note that many authors use the tangent bundle approach. See for example [4].

Slightly abusing notation, we now let $\pi$ denote the basepoint map of the cotangent bundle, let $g$ denote the metric on the cotangent bundle, and define $H_0 : T^* M \to \mathbb{R}$ as $H_0(p) = \frac{1}{2} g(p, p) = \frac{1}{2} |p|^2$. Accordingly, the magnetic flow of $(M, g, \Omega)$ is the Hamiltonian flow $\Phi_t$ on the symplectic manifold $(T^*M, \omega + \pi^* \Omega)$ determined by the Hamiltonian $H_0$. Regardless of approach, the projections of the orbits to the base manifold will be the same magnetic geodesics determined by (3).

On the cotangent bundle
$$\omega_{\text{mag}} = \omega + \pi^* \Omega$$  \hfill (4)
defines a symplectic form as long as $\Omega$ is closed; $\Omega$ may be non-exact or exact. In the former case, $\Omega$ is referred to as a monopole. In the latter case, when $\Omega$ is exact, the magnetic flow can be realized either as the Euler–Lagrange flow of an appropriate Lagrangian, or
(via the Legendre transform) as a Hamiltonian flow on $T^*M$ endowed with its canonical symplectic structure. Note that even if two magnetic fields $\Omega_1$ and $\Omega_2$ represent the same cohomology class, i.e., $\Omega_1 - \Omega_2$ is exact, they generally determine distinct magnetic flows.

Suppose that $\Omega = d\theta$ for some 1-form $\theta$. A computation in local coordinates shows that the diffeomorphism $f : T^*M \to T^*M$ defined by $f(x, p) = (x, p - \theta_x)$ conjugates the Hamiltonian flow of $(T^*M, \varpi + \pi^*\Omega, H_0)$ with the Hamiltonian flow of $(T^*M, \varpi, H_1)$ where

$$H_1(x, p) = \frac{1}{2} |p + \theta_x|^2.$$ (5)

### 2.2 Example: magnetic geodesics in the Euclidean plane

Before introducing two-step nilmanifolds in the following subsection, we first provide an example of a left-invariant magnetic system in a simpler context.

Let $M = \mathbb{R}^2$ endowed with the standard Euclidean metric $g$. Let $\Omega = B \ dx \wedge dy$ denote a magnetic 2-form, where $(x, y)$ denote global coordinates and $B$ is a real parameter that can be interpreted as modulating the strength of the magnetic field.

Let $\sigma_v(t) = (x(t), y(t))$ denote the magnetic geodesic through the identity $e = (0, 0)$ with initial velocity $v = (x_0, y_0) = x_0 \frac{\partial}{\partial x} + y_0 \frac{\partial}{\partial y} \neq 0$ and energy $E = \sqrt{x_0^2 + y_0^2}$. The Lorentz force $F$ satisfies $F(1, 0) = B(0, 1)$ and $F(0, 1) = -B(1, 0)$. By (3) $\sigma_v(t)$ satisfies

$$(\dot{x}, \dot{y}) = F(\dot{x}, \dot{y}) = B(\dot{y}, -\dot{x}).$$

The unique solution satisfying $\sigma_v(0) = e$ and $\dot{\sigma}_v(0) = v$ is

$$x(t) = -\frac{y_0}{B} (1 - \cos (tB)) + \frac{x_0}{B} \sin (tB)$$
$$y(t) = \frac{x_0}{B} (1 - \cos (tB)) + \frac{y_0}{B} \sin (tB).$$

Then, $\sigma_v(t)$ is a circle of radius $\frac{E}{|B|}$ and center $\left(-\frac{y_0}{B}, \frac{x_0}{B}\right)$. It is immediate that magnetic geodesics cannot be reparameterized. For if $\sigma_{v'}(t)$ is another magnetic geodesic through the identity with $v'$ parallel to $v$ but with $|v| \neq |v'|$, then $\sigma_{v'}(t)$ will describe a circle of different radius. Furthermore, magnetic geodesics are not even time-reversible. The magnetic geodesic $\sigma_{-v}(t)$ is a circle of radius $\frac{E}{|B|}$ and center $\left(\frac{y_0}{B}, -\frac{x_0}{B}\right)$; in particular, $\sigma_{-v}(t)$ and $\sigma_v(t)$ are both circles of the same radius but trace different paths. Note that every magnetic geodesic in this setting is periodic. This will not be the case for two-step nilmanifolds.

### 2.3 Left-invariant Hamiltonians on Lie groups

Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. On the one hand, $T^*G (\cong G \times \mathfrak{g}^*)$ is a symplectic manifold and each function $H : T^*G \to \mathbb{R}$ generates a Hamiltonian flow with infinitesimal generator $X_H$. On the other hand, $\mathfrak{g}^*$ is a Poisson manifold and each function $f : \mathfrak{g}^* \to \mathbb{R}$ determines a derivation of $C^\infty(\mathfrak{g}^*)$ and hence a vector field $E_f$, called the Euler vector field associated to $f$. When the function $H$ is left-invariant, i.e., $H((L_x)^\alpha) = H(\alpha)$ for all $x \in G$ and all $\alpha \in T^*G$, it induces a function $h : \mathfrak{g}^* \to \mathbb{R}$ and the flow of $X_H$ factors onto the flow of $E_h$. Moreover, the flow of $X_H$ can be reconstructed from $E_h$ and knowledge of the group structure of $G$. Note that this is a special
case of a more general class of Hamiltonians, called collective Hamiltonians. More
details and physical motivation can be found in Sections 28 and 29 of [21]. We outline
below how we will use this approach to study magnetic flows.

A Poisson manifold is a smooth manifold $M$ together with a Lie bracket $\{\cdot,\cdot\}$ on the
algebra $C^\infty(M)$ that also satisfies the property
\[
\{f,gh\} = \{f,g\}h + g\{f,h\}
\]
for all $f,g,h \in C^\infty(M)$. Hence, for a fixed function $h \in C^\infty(M)$, the map $C^\infty(M) \to C^\infty(M)$
defined by $f \mapsto \{f,h\}$ is a derivation of $C^\infty(M)$. Therefore, there is an Euler vector field $E_h$
on $M$ such that $E_h(\cdot) = \{\cdot,h\}$.

An important source of Poisson manifolds is the vector space dual to a Lie algebra. We will make use of the standard identifications $T_p g^* \simeq g^*$ and $T_p^* g^* \simeq (g^*)^* \simeq g$.
and $\langle \cdot,\cdot\rangle$ will denote the natural pairing between $g$ and $g^*$. For a function $f \in C^\infty(g^*)$,
its differential $df_p$ at $p \in g^*$ is identified with an element of the Lie algebra $g$. The Lie bracket structure on $g$
duces the Poisson structure on $g^*$ by
\[
\{f,g\}(p) = -\langle p, [df_p, dg_p]\rangle = -p(\{df_p,dg_p]\}).
\]
Antisymmetry and the Jacobi Identity follow from the properties of the Lie bracket $\{ \cdot, \cdot \}$,
while the derivation property (6) follows from the Leibniz rule for the exterior derivative.

It is useful to express the Euler vector field $E_h$ in terms of $h$ and the representation
$\text{ad}^*: g \to \mathfrak{gl}(g^*)$ dual to the adjoint representation, defined as
\[
\langle \text{ad}^*_h p, Y \rangle = -\langle p, \text{ad}_h Y \rangle.
\]
From the definition of the differential of a function,
\[
\langle E_h(p), df_p \rangle = E_h(f)(p) = \{f,h\}(p) = -\langle p, [df_p, dh_p]\rangle = -\langle \text{ad}^*_h p, df_p \rangle.
\]
From this, we conclude that
\[
E_h(p) = -\text{ad}^*_h p.
\]
Now, consider $T^* G \simeq G \times g^*$ trivialized via left-multiplication. Let $r: G \times g^* \to g^*$ be
projection onto the second factor. If $h: g^* \to \mathbb{R}$ is any smooth function, then $H = h \circ r$ is a
left-invariant Hamiltonian on $T^* G$. Conversely, any left-invariant Hamiltonian $H$ factors as
$H = h \circ r$. Recall that the canonical symplectic structure $\sigma$ on $T^* G \simeq G \times g^*$ is
\[
\sigma_{(x,p)}((U_1,\alpha_1),(U_2,\alpha_2)) = \alpha_2(U_1) - \alpha_1(U_2) + p([U_1,U_2])
\]
where we identify $T_{(x,p)} T^* G \simeq g \times g^*$ (see section 4.3 of [12] for more details). To find an
expression for the Hamiltonian vector field $X_H(x,p) = (X, \lambda)$ of a left-invariant Hamiltonian,
first consider vectors of the form $(0,\alpha)$ in the equation $\sigma(X_H, \cdot) = dH(\cdot)$. We have
\[
\sigma_{(x,p)}((X,\lambda),(0,\alpha)) = dH_{(x,p)}(0,\alpha) = d(h \circ r)_{(x,p)}(0,\alpha),
\]
$\lambda(X) - \lambda(0) + p([X,0]) = dh_p(\alpha)$,
$\alpha(X) = \alpha(dh_p)$.
Since this is true for all choices of $\alpha$, we get $X = dh_p$. Next, consider vectors of the form
$(U,0)$. Since $H$ is left-invariant,
\[ \psi_{(x,p)}((dh_p, \lambda), (U, 0)) = dH_{(x,p)}(U, 0), \]
\[ -\langle \lambda, U \rangle + \langle p, [dh_p, U] \rangle = 0, \]
\[ \langle \lambda, U \rangle = -\langle \text{ad}_{dh_p}^* \lambda, U \rangle. \]

Since this must be true for every \( U \), we have that \( \lambda = -\text{ad}_{dh_p}^* p = E_h(p) \). For a left-invariant Hamiltonian, the equations of motions for its associated Hamiltonian flow are

\[ X_H(x, p) = \begin{cases} \dot{x} = (L_x)_* (dh_p) \\ \dot{p} = E_h(p) = -\text{ad}_{dh_p}^* p \end{cases}. \quad (11) \]

### 2.4 The geometry of two-step nilpotent metric Lie groups

Our objects of study in this paper are simply connected two-step nilpotent Lie groups endowed with a left-invariant metric. For an excellent reference regarding the geometry of these manifolds, see [11].

Let \( \mathfrak{g} \) denote a two-step nilpotent Lie algebra with Lie bracket \([ \cdot, \cdot \]\) and non-trivial center \( \mathfrak{z} \). That is, \( \mathfrak{g} \) is nonabelian and \([X, Y] \in \mathfrak{z} \) for all \( X, Y \in \mathfrak{g} \). Let \( G \) denote the unique, simply connected Lie group with Lie algebra \( \mathfrak{g} \); then, \( G \) is a two-step nilpotent Lie group. The Lie group exponential map \( \exp : \mathfrak{g} \to G \) is a diffeomorphism, with inverse map denoted by \( \log : G \to \mathfrak{g} \). Using the Campbell–Baker–Hausdorff formula, the multiplication law can be expressed as

\[ \exp(X) \exp(Y) = \exp \left( X + Y + \frac{1}{2} [X, Y] \right). \quad (12) \]

For any \( A \in \mathfrak{g} \) and any \( X \in T_A \mathfrak{g} \cong \mathfrak{g} \), the push-forward of the Lie group exponential at \( A \) is

\[ (\exp)_* A(X) = (L_{\exp(A)})_* \left( X + \frac{1}{2} [X, A] \right). \]

Using this, the tangent vector to any smooth path \( \sigma(t) = \exp(U(t)) \) in \( G \) is given by

\[ \sigma'(t) = (L_{\sigma(t)})_* \left( U'(t) + \frac{1}{2} [U'(t), U(t)] \right). \quad (13) \]

When a two-step nilpotent Lie algebra \( \mathfrak{g} \) is endowed with an inner product \( g \), then there is a natural decomposition \( \mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z} \), where \( \mathfrak{z} \) is the center of \( \mathfrak{g} \) and \( \mathfrak{v} \) is the orthogonal complement to \( \mathfrak{z} \) in \( \mathfrak{g} \). Every central vector \( Z \in \mathfrak{z} \) determines a skew-symmetric linear transformation of \( \mathfrak{v} \) (relative to the restriction of \( g \)), denoted \( j(Z) \), as follows:

\[ g(j(Z)V_1, V_2) = g([V_1, V_2], Z) \quad (14) \]

for any vectors \( V_1, V_2 \in \mathfrak{v} \). In fact, this correspondence is a linear map \( j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v}) \). These maps, first introduced by Kaplan [22], capture all of the geometry of a two-step nilpotent metric Lie group. For example, the \( j \)-maps provide a very useful description of the Levi-Civita connection. For \( V_1, V_2 \in \mathfrak{v} \) and \( Z_1, Z_2 \in \mathfrak{z} \),
\[ \nabla_{X_1}X_2 = \frac{1}{2}[X_1,X_2], \]
\[ \nabla_{X_1}Z_1 = \nabla_{Z_1}X_1 = -\frac{1}{2}j(Z)X, \]
\[ \nabla_{Z_1}Z_2 = 0. \]

### 2.5 Exact, left-invariant magnetic forms on simply connected two-step nilpotent Lie groups

We use the formalism of Sect. 2.3 to express the equations of motion for the magnetic flow of an exact, left-invariant magnetic form on a simply connected two-step nilpotent Lie group. Throughout this section, \(g\) denotes a two-step nilpotent Lie algebra with an inner product and \(G\) denotes the simply connected Lie group with Lie algebra \(g\) endowed with the left-invariant Riemannian metric determined by the inner product on \(g\).

As a reminder, angled brackets denote the natural pairing of a vector space and its dual. Recall that any (finite dimensional) vector space \(V\) is naturally identified with \(V^*\) as \(\mathbb{R}\)-vector spaces, so that \(\langle \cdot \rangle\) induces a map \(\langle \cdot \rangle : V^* \to \mathbb{R}\). Using this identification, we can and do view elements of \(V\) simultaneously as elements of \(V^*\).

The inner product on \(g^*\) is specified by a choice of linear map \(# : g^* \to g\) such that (a) \(\langle p, #(q) \rangle > 0\) for all \(p \neq 0\) and (b) \(\langle p, #(q) \rangle = \langle #(p), q \rangle\) for all \(p, q \in g^*\). The inner product of \(p, q \in g^*\) is then given by \(\langle p, #(q) \rangle\). Conversely any inner product on \(g^*\) induces a map \(# : g^* \to g^{**} \cong g\) with the properties (a) and (b). Of course, \(#^{-1} = \flat\) is then the flat map and the inner product of \(X, Y \in g\) can be computed as \(\langle X, \flat(Y) \rangle\).

Let \(g = b \oplus z\) be the decomposition of \(g\) into the center and its orthogonal complement. Let \(g^* = b^* \oplus z^*\) be the corresponding decomposition where \(b^*\) is the set of functionals that vanish on \(z\) and vice versa.

**Lemma 1** If \(\Omega\) is an exact, left-invariant two-form on \(G\), then there exists \(B \in \mathbb{R}\) and \(\zeta_m \in z^*\) such that \(\zeta_m = 1\) and \(\Omega = dB\zeta_m\).

**Proof** By hypothesis, \(\Omega = d\theta\) for some left-invariant 1-form \(\theta\). By left-invariance, \(\theta\) can be expressed as \(\theta = \theta_b + \theta_z\), where \(\theta_b \in b^*\) and \(\theta_z \in z^*\), and \(d\theta_b(X, Y) = -\theta_b([X, Y])\) for any \(X, Y \in g\). Because \([X, Y] \in z\), \(d\theta_b = 0\). Hence,

\[ \theta = d\theta = d(\theta_b + \theta_z) = d\theta_z. \]

Lastly, \(\zeta_m = \theta_z / |\theta_z|\) and \(B = |\theta_z|\).

Given \(B \in \mathbb{R}\) and \(\zeta_m \in g^*\), we define the function \(H : T^*G \to \mathbb{R}\) by

\[ H(x, p) = \frac{1}{2}|p + B\zeta_m|^2. \]

By the previous lemma, we may assume \(\zeta_m\) is a unit element in \(g^*\) that vanishes on \(b\). Because \(\zeta_m\) is left-invariant, \(H\) is left-invariant and factors as \(H = h\circ r\), where \(h : g^* \to \mathbb{R}\) is the function.
\[ h(p) = \frac{1}{2} |p + B\zeta_m|^2. \] (17)

Note that when \( B = 0 \), the Hamiltonian flow of \( H \) is the geodesic flow of the chosen Riemannian metric.

**Lemma 2** The differential of \( h \) is \( dh_p = \sharp(p + B\zeta_m) \).

**Proof** For any \( p \in \mathfrak{g} \) and any \( q \in T_p\mathfrak{g}^* \cong \mathfrak{g}^* \), we compute

\[
\langle q, dh_p \rangle = \frac{d}{dt} \bigg|_{t=0} h(p + tq) = \frac{1}{2} \frac{d}{dt} \bigg|_{t=0} |p + tq + B\zeta_m|^2
\]

\[
= \frac{1}{2} \frac{d}{dt} \bigg|_{t=0} \langle p + B\zeta_m + tq, \sharp(p + B\zeta_m + tq) \rangle
\]

\[
= \frac{1}{2} \frac{d}{dt} \bigg|_{t=0} (|p + B\zeta_m|^2 + 2t\langle p + B\zeta_m, \sharp(q) \rangle + t^2 |q|^2)
\]

\[
= \langle p + B\zeta_m, \sharp(q) \rangle.
\]

Lemma now follows from the properties of \( \sharp \). \( \square \)

We now prove that the Euler vector field on \( \mathfrak{g}^* \) is independent of the choice of exact magnetic field, including the choice \( \Omega = 0 \).

**Lemma 3** Let \( h \in C^\infty(\mathfrak{g}^*) \) be any function of the form (17) and define the function \( h_0 \in C^\infty(M) \) by \( h_0(p) = \frac{1}{2} |p|^2 \). Then, \( E_{h_0} = E_h \).

**Proof** For any \( \zeta \in \mathfrak{z}^* \) and any \( V \in \mathfrak{v} \), \( \langle V, \mathfrak{v}(\sharp(\zeta)) \rangle = \langle V, \zeta \rangle = 0 \) shows that \( \sharp(\mathfrak{z}^*) = \mathfrak{z} \). For any \( X \in \mathfrak{g} \), by the previous lemma,

\[
\langle \text{ad}^*_{dh_p} p, X \rangle = -\langle p, [\sharp(p + B\zeta_m), X] \rangle = -\langle p, [\sharp p, X] \rangle = \langle \text{ad}_p^* \sharp(p), X \rangle.
\]

Hence, \( \text{ad}^*_{dh_p} = \text{ad}^*_{(dh_0)_p} \), and the proof follows from the expression (9) for the Euler vector field. \( \square \)

We now describe the structure of the Euler vector field. Much of this can be gleaned from the results of [11]. However, we include it here for the sake of self-containment. For any \( X \in \mathfrak{g} \) and \( p \in \mathfrak{g}^* \), we write \( X = X_\mathfrak{v} + X_\mathfrak{z} \) and \( p = p_\mathfrak{v} + p_\mathfrak{z} \) for the respective orthogonal decomposition according to \( \mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z} \) and \( \mathfrak{g}^* = \mathfrak{v}^* \oplus \mathfrak{z}^* \).

**Lemma 4** The integral curves of the Euler vector field \( E_p \) are of the form \( p(t) = p_\mathfrak{v}(t) + \zeta_0 \) where \( \zeta_0 \in \mathfrak{z}^* \) and \( p_\mathfrak{v}(t) \in \mathfrak{v}^* \) is a path that satisfies \( p_\mathfrak{v}(t) = A(p_\mathfrak{v}(t)) \) for some skew-symmetric transformation of \( \mathfrak{v}^* \).

**Proof** From (8), the dual adjoint representation clearly has the following properties: \( \text{ad}_Z^* = 0 \) for every \( Z \in \mathfrak{z} \), \( \text{ad}_X^*(\mathfrak{g}^*) \subset \mathfrak{v}^* \) for all \( X \in \mathfrak{g} \), and \( \text{ad}_X^*(\mathfrak{v}^*) = \{0\} \) for every \( X \in \mathfrak{g} \).
From this, if \( p(t) = p_b(t) + p_\zeta(t) \) is an integral curve of \( E_h \), then \( p_\zeta(t) = p_\zeta(0) = \zeta_0 \) is constant, and, using Lemmas 2 and 3, \( p_b(t) \) must satisfy the system

\[
p'_b(t) = E_b(p(t)) = -\text{ad}^{\ast}_{p_b(t)} p(t) = -\text{ad}^{\ast}_{p_\zeta(p_b(t))} p_\zeta(t) = -\text{ad}^{\ast}_{p_\zeta(p_b(t))} \zeta_0.
\]

Since \( A : \mathfrak{b}^* \to \mathfrak{b}^* \) is skew-symmetric with respect to the inner product restricted to \( \mathfrak{b}^* \), this completes Lemma.

Let \((G,g,\Omega)\) be a magnetic symmetric system, where \( G \) is a simply connected two-step nilpotent Lie group, \( g \) is a left-invariant metric, and \( \Omega \) an exact, left-invariant magnetic form. Let \( b : g \to g^* \) and \( \sharp = b^{-1} \) be the associated flat and sharp maps, and let \( \zeta_m \) be as in Lemma 1. The magnetic flow can be found as follows. First, compute the coadjoint representation of \( \text{ad}^{\ast} : g \to \mathfrak{gl}(g^*) \) and integrate the vector field \( E(p) = -\text{ad}^{\ast}_{p_b(t)} p \). It follows that the curves \( \sigma(t) \) satisfying \( \sigma'(t) = dh_{p(t)} \), where \( p(t) \) is an integral curve of \( E \), will be magnetic geodesics. To make this step more explicit, let \( q = b \oplus \zeta \) be the decomposition of \( q \) where \( \zeta \) is the center and \( b \) is its orthogonal complement. Suppose that \( p(t) = p_1(t) + \zeta_0 \) is an integral curve of \( E \), where \( p_1(t) \in \mathfrak{b}^* \) and \( \zeta_0 \in \zeta^* \), and \( \sigma(t) = \exp(\mathbf{X}(t) + \mathbf{Z}(t)) \) is a path in \( G \), where \( \mathbf{X}(t) \in \mathfrak{b} \) and \( \mathbf{Z}(t) \in \zeta \). Using (13), we can decompose the equation \( \sigma'(t) = dh_{p(t)} = \sharp(p(t) + B\zeta_m) \) as

\[
\mathbf{X}'(t) = \sharp(p_1(t)),
\]

\[
\mathbf{Z}'(t) + \frac{1}{2} [\mathbf{X}'(t), \mathbf{X}(t)] = \sharp(\zeta_0 + B\zeta_m).
\]

Assuming that the path satisfies \( \sigma(0) = e \), the first equation can be integrated to find \( \mathbf{X}(t) \), which then allows the second equation to be integrated to find \( \mathbf{Z}(t) \).

**Remark 2** The presence of the magnetic field can be thought of as a perturbation of the geodesic flow of \((G,g)\), modulated by the parameter \( B \). In the procedure outlined here for two-step nilpotent Lie groups, the magnetic field only appears in the final step. The Euler vector field, and hence its integral curve, is unchanged by the magnetic field. In addition, the non-central component of the magnetic geodesics is the same as that of the Riemannian geodesics. The presence of a left-invariant exact magnetic field only perturbs the geodesic flow in central component of the Riemannian geodesics.

**Remark 3** For a magnetic geodesic \( \sigma(t) \), we will call \( |\sigma'(t)| \) its energy. Note that this is a conserved quantity for magnetic flows. Since we are not considering a potential, the total energy of a charged particle in a magnetic system is its kinetic energy \( |\sigma'(t)|^2 / 2 \). Although this would be commonly referred to as the energy in the physics and dynamics literature, we find our convention to be more convenient from our geometric viewpoint.

**Remark 4** Although \( t \to (\sigma(t), p(t)) \) is an integral curve of the Hamiltonian vector field, the Hamiltonian \( h \) is not the kinetic energy, and hence, the energy of the magnetic geodesic is not equal to \( |p(0)| \). Instead, by (18) and (19), the energy squared is

\[
|\sigma'(t)|^2 = |\sharp(p(0)) + B\sharp(\zeta_m)|^2 = |\sharp(p_1(0))|^2 + |\sharp(\zeta_0 + B\zeta_m)|^2.
\]
2.6 The Mañé critical value

Although the energy levels of $T^*M$ are preserved by the magnetic flow, the dynamics on each level depend on the particular value of its energy (unlike the situation for the geodesic flow). There is a distinguished energy level, known as the Mañé critical value, at which the dynamics often change dramatically. The introduction to [32] and Section 5 of [7] (and the references therein) are good references for the Mañé critical value.

First, we recast magnetic flows in the Lagrangian terms. Let $(M, g)$ be a closed Riemannian manifold endowed with an exact magnetic field $\Omega = d\theta$ and define a Lagrangian on $M$ by $L(x, v) = \frac{1}{2}|v|^2 - \theta(v)$. The projection to $M$ of the solutions to the Euler–Lagrange equations coincides with the magnetic geodesics as defined in Sect. 2.1. The Legendre transform $L : TM \to T^*M$ smoothly conjugates the Euler–Lagrange flow of $L$ on $TM$ with the Hamiltonian flow of $(T^*M, \pi, H_1)$, where $H_1$ is the Hamiltonian defined in (5). For any absolutely continuous curve $\sigma : [a, b] \to M$, the action of the Lagrangian $L$ on $\sigma$ is

$$A_L(\sigma) = \int_a^b L(\sigma(t), \sigma'(t))dt.$$

**Definition 1** (Mañé critical value) The Mañé critical value of the Lagrangian $L$ is

$$c(L) = \inf\{k \in \mathbb{R} : A_{L+k}(\sigma) \geq 0 \text{ for any absolutely continuous closed curve } \sigma \text{ defined on any closed interval } [0, T]\}$$

where $L + k$ is the Lagrangian defined by $(L + k)(x, v) = L(x, v) + k$.

We now specialize to the case where the manifold is a compact quotient of a simply connected two-step nilpotent Lie group $G$ endowed with left-invariant metric $g$ and left-invariant exact magnetic field $\Omega = d\theta$. Let $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$ be the decomposition of the Lie algebra of $G$ into its center and the orthogonal complement to the center. Let $\Gamma < G$ denote a cocompact discrete subgroup so that $\pi : G \to \Gamma \backslash G$ is the universal Riemannian covering, and let $g, \Omega, \theta$ denote the corresponding structures induced on the closed manifold $\Gamma \backslash G$. By Lemma 1, we can write $\theta = B\zeta_m$, where $\zeta_m \in \mathfrak{z}$ and $|\zeta_m| = 1$. Therefore, the corresponding magnetic Lagrangian $L : T(\Gamma \backslash G) \simeq \Gamma \backslash G \times \mathbb{R} \to \mathbb{R}$ is

$$L(x, v) = \frac{1}{2}|v|^2 - B\zeta_m(v).$$

Suppose that $\sigma : [0, \epsilon] \to \Gamma \backslash G$ is any curve of the form $\sigma(t) = \pi \circ \exp(tX)$ for some $X \in \mathfrak{z}$. By (13), $\sigma'(t) = X$ and therefore its $L + k$ action is

$$A_{L+k}(\sigma) = \int_0^\epsilon \left(\frac{1}{2}|X|^2 - B\zeta_m(X) + k\right)dt = \epsilon \left(\frac{1}{2}|X|^2 - B\zeta_m(X) + k\right).$$

**Theorem 1** Let $G$ be a simply connected two-step nilpotent Lie group, $\Gamma < G$ a cocompact discrete subgroup, $g$ a left-invariant metric on $G$, and $\Omega = d(B\zeta_m)$ a left-invariant exact 2-form on $G$. The Mañé critical value associated with the magnetic system $(\Gamma \backslash G, g, \Omega)$ is $B^2/2$. 
We consider a sequence of vectors such that $\sigma_i 
rightarrow \text{log}(Z(\Gamma))$ and $\log(\text{Z}(\Gamma))$ is a lattice in the vector space $\mathfrak{z}$. Hence, the set of vectors

$$E := \left\{ \frac{|B|}{|X|} X : X \in \log(\text{Z}(\Gamma)) \right\}$$

is dense in the sphere of radius $|B|$ in $\mathfrak{z}$. Let $\{Y_i\} \subset E$ be a sequence of vectors such that $\ell_i Y_i \in \log(\text{Z}(\Gamma))$, $|Y_i| = |B|$ and $Y_i \to BZ_m$. Writing $Y_i = Y_i^* + c_i Z_m$, where $Y_i^* \in \mathbb{R}^m$, we have that $Y_i^* \to 0$ and $c_i \to B$. Now, let $\sigma_i : [0, \ell_i] \to \Gamma \setminus G$ be the path $\sigma_i(t) = \pi \circ \exp(t Y_i)$. By (23), the $L + k$ action of $\sigma_i$ is

$$A_{L+k}(\sigma_i) = \ell_i \left( \frac{1}{2} |Y_i^*|^2 + \frac{1}{2} c_i^2 - Bc_i + k \right) = \ell_i \left( |Y_i|^2 + (c_i - B)^2 + (2k - B^2) \right)$$

When $k < B^2/2$, then $|Y_i|^2 \to 0$, $c_i \to B$, and $\ell_i > 0$, $A_{L+k}(\sigma_i) < 0$ for sufficiently large $i$. This implies that $c(L) \geq B^2/2$.

To obtain a bound in the other direction, note that for $k > B^2/2$, the function $L + k$ is always positive on $T(\Gamma \setminus G)$. Hence, the action on any absolutely continuous closed curve will be positive, which implies that $c(L) \leq B^2/2$.

**Remark 5** The critical energy level corresponding to the value $c(L) = B^2/2$ is $E^{-1}(B^2/2)$ where $E : TM \to \mathbb{R}$ is the energy function associated with the magnetic Lagrangian, $E(x, v) = \frac{1}{2} |v|^2$. In the Hamiltonian context, the critical energy level is $H_0^{-1}(B^2/2)$. In Remark 3, we explain that we use the term energy to refer to the norm of a vector, not half its norm squared. So in the terminology of this paper, the phrase “critical energy level” will refer to $|B|$.

**Remark 6** Given a cover $\pi : \hat{M} \to M$, let $\hat{L} : T\hat{M} \to \mathbb{R}$ be the lifted Lagrangian defined by $\widehat{L} = L \circ \pi_*$. We can then associate a critical value $c(\hat{L})$ to the lifted Lagrangian as in Definition 1. It is easy to check that $c(\hat{L}) \leq c(L)$. There are two distinguished covers: the universal cover $\pi_u : M_u \to M$ and the abelian cover $\pi_0 : M_0 \to M$. The latter is defined as the cover of $M$ whose fundamental group is the kernel of the Hurewicz homomorphism $\pi_1(M) \to H_1(M; \mathbb{Z})$. Let $c(L_u)$ and $c(L_0)$ be the associated critical values. For any magnetic system as in Theorem 1, we will have $c(L) = c(L_u) = c(L_0)$. The abelian cover will be $\pi_u : Z(\Gamma) \setminus G \to \Gamma \setminus G$. Hence, the same argument as in the proof can be repeated to establish $c(L) = c(L_u)$. The second equality follows from the fact that $c(L_u) = c(L_0)$ whenever the fundamental group $\pi_1(M)$ is amenable. The fundamental group of any nilmanifold will be nilpotent and hence amenable.

### 3 Simply connected $(2n + 1)$-dimensional Heisenberg groups

Let $\mathfrak{h}_n = \text{span}\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\}$ and define a bracket structure on $\mathfrak{h}_n$ by declaring the only nonzero brackets among the basis vectors to be $[X_i, Y_j] = Z$ and extending $[\cdot, \cdot]$ to all of $\mathfrak{h}_n \times \mathfrak{h}_n$ by bilinearity and skew-symmetry. Then, $\mathfrak{h}_n$ is a two-step nilpotent Lie algebra called the Heisenberg Lie algebra of dimension $2n + 1$ and the simply connected Lie group $H_n$ with Lie algebra $\mathfrak{h}_n$ is called the Heisenberg group of dimension $2n + 1$. Let $\{a_1, b_1, \ldots, a_n, b_n, \zeta\}$ be the dual basis of $\mathfrak{h}_n^*$. The following Lemma, proven in Lemma 3.5 of [14], shows that to
consider every inner product on \( \mathfrak{h}_n \), we need only consider inner products on \( \mathfrak{h}_n \) that have a simple relationship to the bracket structure.

**Lemma 5** Let \( g \) be any inner product on \( \mathfrak{h}_n \). There exists \( \varphi \in \text{Aut}(\mathfrak{h}_n) \) such that

\[
\left\{ \frac{X_1}{\sqrt{A_1}}, \ldots, \frac{X_n}{\sqrt{A_n}}, \frac{Y_1}{\sqrt{A_1}}, \ldots, \frac{Y_n}{\sqrt{A_n}}, Z \right\}
\]

is an orthonormal basis relative to \( \varphi^*g \), where \( A_i > 0, i = 1 \ldots n \), are positive real numbers.

**Proof** Consider the linear map defined by

\[
X_i \mapsto \frac{X_i}{\sqrt{|Z|}} \quad Y_i \mapsto \frac{Y_i}{\sqrt{|Z|}} \quad Z \mapsto \frac{Z}{|Z|}.
\]

This is an automorphism of \( \mathfrak{h}_n \) and \( Z \) is a unit vector relative to the pullback of the metric. Hence, we can and will assume that \( |Z| = 1 \).

Let \( \psi_i \) be the linear map defined by \( \psi_i(X_i) = X_i - g(X_i,Z)Z, \psi_i(Y_i) = Y_i - g(Y_i,Z)Z, \) and \( \psi_i(Z) = Z \). Now, \( \psi_i \in \text{Aut}(\mathfrak{h}_n) \) and \( \mathfrak{v} = \text{span}\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\} \) is orthogonal to \( \mathfrak{z} = \text{span}\{Z\} \) relative to \( \psi_i^*g \).

Next, consider the map \( j(Z) \in \text{so}(\mathfrak{h}, \psi_i^*g) \). Because it is skew-symmetric, there exists a \( \psi_i^*g \)-orthonormal basis \( \{\tilde{X}_1, \ldots, \tilde{X}_n, \tilde{Y}_1, \ldots, \tilde{Y}_n\} \) of \( \mathfrak{v} \) such that \( j(Z)\tilde{X}_i = d_i\tilde{Y}_i \) and \( j(Z)\tilde{Y}_i = -d_i\tilde{X}_i \) for some real numbers \( d_i > 0 \). Because

\[
(\psi_i^*g)(Z, [\tilde{X}_i, \tilde{Y}_i]) = (\psi_i^*g)(j(Z)\tilde{X}_i, \tilde{Y}_i) = (\psi_i^*g)(d_i\tilde{Y}_i, \tilde{Y}_i) = d_i
\]

we see that \([\tilde{X}_i, \tilde{Y}_i] = d_iZ\). Define the linear map \( \psi_2 \) by

\[
\psi_2(X_i) = \frac{1}{\sqrt{d_i}}\tilde{X}_i \quad \psi_2(Y_i) = \frac{1}{\sqrt{d_i}}\tilde{Y}_i \quad \psi_2(Z) = Z.
\]

Then, \( \psi_2 \in \text{Aut}(\mathfrak{h}_n) \) because

\[
[\psi_2(X_i), \psi_2(Y_i)] = Z = \psi_2(Z) = \psi_2([X_i, Y_i])
\]

and, setting \( A_i = d_i \), it is clear that the basis (24) is orthonormal relative to \( \psi_2^*(\psi_i^*g) \). Hence, \( \varphi = \psi_1 \circ \psi_2 \) is the desired automorphism of \( \mathfrak{h}_n \). \( \square \)

When (24) is an orthonormal basis of \( \mathfrak{h}_n \), the sharp and flat maps are given by

\[
\begin{align*}
\flat(X_i/\sqrt{A_i}) &= \sqrt{A_i} \alpha_i, & \flat(\sqrt{A_i} \alpha_i) &= X_i/\sqrt{A_i}, \\
\flat(Y_i/\sqrt{A_i}) &= \sqrt{A_i} \beta_i, & \flat(\sqrt{A_i} \beta_i) &= Y_i/\sqrt{A_i}, \\
\flat(Z) &= \zeta, & \flat(\zeta) &= Z.
\end{align*}
\]

Relative to the basis \( \{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\} \), the adjoint representation is

\[
\text{ad}_U = \begin{bmatrix}
0 & \cdots & 0 & 0 \\
\vdots & & \vdots & \vdots \\
0 & \cdots & 0 & 0 \\
-y_1 & \cdots & -y_n & x_1 & \cdots & x_n & 0
\end{bmatrix}
\]
where $U = \sum x_i X_i + \sum b_j y_j + zZ$. Relative to the dual basis, the coadjoint representation is the negative transpose

$$\text{ad}_U^* = -(\text{ad}_U)^T = \begin{bmatrix} 0 & \cdots & 0 & y_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & y_n \\ 0 & \cdots & 0 & -x_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -x_n \\ 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Because the center of $\mathfrak{h}_n$ is one-dimensional, $\zeta_m = \zeta$, where $\zeta_m$ is as specified in Lemma 1. Letting $p = \sum a_i A_i + \sum b_j A_j + c \zeta$ be a point in $\mathfrak{h}_n^*$, the differential of the Hamiltonian is

$$dh_p = \sharp(p + B\zeta) = \sum_i \frac{a_i}{A_i} X_i + \sum_i \frac{b_j}{A_j} Y_i + (c + B)Z$$

and the Euler vector field is

$$E_h(p) = -\text{ad}^*_{dh_p} p = \sum_i -\frac{c b_i}{A_i} A_i + \sum_i \frac{c a_i}{A_i} A_i.$$

To integrate the system $p' = E_h(p)$, note that the central component of the Euler vector field is constant by Lemma 4. Suppose that $p(t) = \sum a_i(t) A_i + \sum b_j(t) A_j + c(t) \zeta$ is a solution that satisfies the initial condition $p(0) = \sum u_i A_i + \sum v_i A_j + z_0 \zeta$. Then, $c(t) = z_0$ and the remaining components form a linear system,

$$a_i'(t) = -\frac{z_0}{A_i} b_i(t) \quad b_i'(t) = \frac{z_0}{A_i} a_i(t)$$

that is directly integrated to find

$$a_i(t) = u_i \cos \left( \frac{z_0 t}{A_i} \right) - v_i \sin \left( \frac{z_0 t}{A_i} \right),$$

$$b_i(t) = u_i \sin \left( \frac{z_0 t}{A_i} \right) + v_i \cos \left( \frac{z_0 t}{A_i} \right).$$

With an expression for the integral curves of the Euler vector field now established, we use equations (18) and (19) to obtain a coordinate expression for the magnetic geodesics through the identity. Let $X(t) = \sum x_i(t) X_i + \sum y_j(t) Y_j$. If $z_0 \neq 0$, a direct integration of (18) together with $X(0) = 0$ yields

$$x_i(t) = \frac{u_i}{z_0} \sin \left( \frac{z_0 t}{A_i} \right) + \frac{v_i}{z_0} \cos \left( \frac{z_0 t}{A_i} \right) - \frac{v_i}{z_0},$$

$$y_j(t) = -\frac{u_i}{z_0} \cos \left( \frac{z_0 t}{A_i} \right) + \frac{v_i}{z_0} \sin \left( \frac{z_0 t}{A_i} \right) + \frac{u_i}{z_0}. $$

If $z_0 = 0$, we obtain
Because the center is one-dimensional, the central component \( Z(t) \) in (19) can be expressed as \( Z(t) = z(t)Z \). To integrate (19) in the case that \( z_0 \neq 0 \), first compute

\[
[X'(t), X(t)] = \sum (x'_i y_i - x_i y'_i) Z = \sum \frac{u_i^2 + v_i^2}{A_i z_0} \left( \cos \left( \frac{z_0 t}{A_i} \right) - 1 \right) Z
\]

so that

\[
Z'(t) = z'(t)Z = \#(z_0 \zeta + B \xi) - \frac{1}{2} [X'(t), X(t)]
\]

\[
= (z_0 + B)Z - \sum \frac{u_i^2 + v_i^2}{2A_i z_0} \left( \cos \left( \frac{z_0 t}{A_i} \right) - 1 \right) Z
\]

\[
= \left( z_0 + B + \sum \frac{u_i^2 + v_i^2}{2A_i z_0} \right) Z - \sum \frac{u_i^2 + v_i^2}{2A_i z_0} \cos \left( \frac{z_0 t}{A_i} \right) Z
\]

and hence,

\[
z(t) = \left( z_0 + B + \sum \frac{u_i^2 + v_i^2}{2A_i z_0} \right) t - \sum \frac{u_i^2 + v_i^2}{2z_0^2} \sin \left( \frac{z_0 t}{A_i} \right).
\]

In summary, when \( z_0 \neq 0 \), every magnetic geodesic \( \sigma(t) = \exp(\sum x_i(t)X_i + \sum y_i(t)Y_i + z(t)Z) \) satisfying \( \sigma(0) = e \) has the form

\[
x_i(t) = \frac{u_i}{z_0} \sin \left( \frac{z_0 t}{A_i} \right) - \frac{v_i}{z_0} \left( 1 - \cos \left( \frac{z_0 t}{A_i} \right) \right), \tag{25}
\]

\[
y_i(t) = \frac{u_i}{z_0} \left( 1 - \cos \left( \frac{z_0 t}{A_i} \right) \right) + \frac{v_i}{z_0} \sin \left( \frac{z_0 t}{A_i} \right), \tag{26}
\]

\[
z(t) = \left( z_0 + B + \sum \frac{u_i^2 + v_i^2}{2A_i z_0} \right) t - \sum \frac{u_i^2 + v_i^2}{2z_0^2} \sin \left( \frac{z_0 t}{A_i} \right). \tag{27}
\]

When \( z_0 = 0 \), we obtain

\[
x_i(t) = \frac{u_i}{A_i} t, \tag{28}
\]

\[
y_i(t) = \frac{v_i}{A_i} t, \tag{29}
\]

\[z_i(t) = Bt. \tag{30}\]
Remark 7 A magnetic geodesic \( \sigma(t) \) will be a one-parameter subgroup if and only if \( z_0 = 0 \) or \( z_0 \neq 0 \) and \( u_i = v_i = 0 \) for all \( i \). We will sometimes call a magnetic geodesic spiraling if it is not a one-parameter subgroup, and non-spiraling if it is. We will also call a magnetic geodesic central if it is of the form \( \sigma(t) \in Z(H_n) \) for all \( t \).

The initial velocity of the magnetic geodesic \( \sigma(t) \) is

\[
\sigma'(0) = \sum \left( \frac{u_i}{A_i} X_i + \frac{v_i}{A_i} Y_i \right) + (z_0 + B)Z.
\]

Because \( |X_i|^2 = |Y_i|^2 = A_i \), we can compute the square of the energy \( E = |\sigma'(t)| = |\sigma'(0)| \) as (see Remark 3)

\[
E^2 = |\sigma'(0)|^2 = \sum \frac{u_i^2 + v_i^2}{A_i} + (z_0 + B)^2
\]

(31)

Note that this expression is valid for all values of \( z_0 \).

Theorem 2 There exist periodic magnetic geodesics with energy \( E \) if and only if \( 0 < E < |B| \). For any \( 0 < E < |B| \), let \( z_0 = -\text{sgn}(B)\sqrt{B^2 - E^2} \) and let \( u_i \) and \( v_i \) be any numbers satisfying (31). Then, the spiraling magnetic geodesics determined by \( u_1, v_1, \ldots, u_n, v_n, z_0 \) will be periodic of energy \( E \). Moreover, the period of such a geodesic is \( \omega = 2\pi A/z_0 \).

Proof Recall that non-spiraling magnetic geodesics cannot be periodic. Inspection of the coordinate functions (25)–(27) of a spiraling magnetic geodesic shows they will yield a periodic magnetic geodesic if and only if the coefficient of \( t \) in (27) is zero. This condition is

\[
0 = z_0 + B + \sum \frac{u_i^2 + v_i^2}{2A_i z_0} = z_0 + B + \frac{1}{2z_0}(E^2 - (z_0 + B)^2)
\]

or

\[
z_0^2 = B^2 - E^2.
\]

It can only be satisfied when \( E < |B| \). To obtain a spiraling magnetic geodesic, we need to require that \((z_0 + B)^2 < E^2\) or, equivalently, \(z_0 \in (-B - E, -B + E)\). Since this interval contains only negative or positive numbers, depending on the sign of \( B \), we must choose \( z_0 = -\text{sgn}(B)\sqrt{B^2 - E^2} \). Finally, to see that \( z_0 \) is indeed contained in this interval, note that \( \sqrt{(B - E)(B + E)} = \sqrt{(-B + E)(-B - E)} \) is the geometric mean of the endpoints of interval. \( \square \)

Example 1 For convenience, we state the component functions of a magnetic geodesic \( \sigma(t) = \exp(x(t)X + y(t)Y + z(t)Z) \) in the three-dimensional Heisenberg group (i.e., \( n = 1 \)) with \( \sigma(0) = e \). To ease notation, we use the dual bases \( \{a, \beta, \zeta\} \) and \( \{X, Y, Z\} \) for \( H^*_1 \) and \( H_1 \), respectively, and we let \( A = A_1 \). Given a point \( p(0) = u_0 a + v_0 \beta + z_0 \zeta, z_0 \neq 0 \), the corresponding magnetic geodesic has component functions
When $z_0 = 0$, we obtain

$$x(t) = \frac{u_0}{z_0} \sin \left( \frac{z_0 t}{A} \right) - \frac{v_0}{z_0} \left( 1 - \cos \left( \frac{z_0 t}{A} \right) \right),$$

$$y(t) = \frac{u_0}{z_0} \left( 1 - \cos \left( \frac{z_0 t}{A} \right) \right) + \frac{v_0}{z_0} \sin \left( \frac{z_0 t}{A} \right),$$

$$z(t) = \left( z_0 + \frac{u_0^2 + v_0^2}{2Az_0} \right) t - \frac{u_0^2 + v_0^2}{2z_0^2} \sin \left( \frac{z_0 t}{A} \right).$$

**Remark 8** It is instructive to compare the magnetic geodesics on $\mathbb{R}^2$ given in Sect. 2.2 and the magnetic geodesics on $H_1$ given in Example 1. In the former, all magnetic geodesics are closed circles with radii that depend on the energy. In the latter, the paths $x(t)X + y(t)Y$ through the complement to the center are also circles whose radii depend on both the energy and $z_0$. It is also worth noting some qualitative differences between Riemannian geodesics and magnetic geodesics on Heisenberg groups. In Riemannian case, one-param-eter subgroups of the form $\exp(t(x_0X + y_0Y))$ are always geodesics. In contrast, the central component $z(t)$ of a magnetic geodesic can never be zero. Finally, note that in the Rie-
mannian setting there are never closed geodesics in $H_n$ (compare with Theorem 2).

### 4 Compact quotients of Heisenberg groups

A geodesic $\sigma : \mathbb{R} \rightarrow M$ in a Riemannian manifold $M$ is called periodic or (smoothly) closed if $\sigma(t + \omega) = \sigma(t)$ for some $\omega \neq 0$ and for all $t \in \mathbb{R}$. A periodic or closed magnetic geodesic is defined similarly, and we now investigate the closed magnetic geodesics on manifolds of the form $\Gamma \backslash H_n$, where $\Gamma$ is a cocompact (i.e., $\Gamma \backslash H_n$ compact), discrete subgroup of the $(2n + 1)$-dimensional simply connected Heisenberg group $H_n$. As is common, we proceed by considering $\gamma$-periodic magnetic geodesics on the universal cover $H_n$. An important distinction between the magnetic and Riemannian settings is that in the former one needs to address each energy level separately because magnetic geodesics cannot be reparameterized.

#### 4.1 $\gamma$-Periodic magnetic geodesics

**Definition 2** Let $N$ be a simply connected nilpotent Lie group with left invariant metric and magnetic form. For any $\gamma \in N$ not equal to the identity, a magnetic geodesic $\sigma(t)$ is called $\gamma$-periodic with period $\omega$ if $\omega \neq 0$ and for all $t \in \mathbb{R}$

$$\gamma \sigma(t) = \sigma(t + \omega).$$

We also say that $\gamma$ translates the magnetic geodesic $\sigma(t)$ by amount $\omega$. The number $\omega$ is called a period of $\gamma$.

When $\Gamma < N$ is a cocompact discrete subgroup, we identify $\Gamma$ with the deck transforma-
tions of the universal cover $N \rightarrow \Gamma \backslash N$ by left-multiplication. The magnetic geodesic condition is local and by left-invariance the deck transformations preserve the magnetic
structure. So, a $\gamma$-periodic magnetic geodesic will project to a smoothly closed magnetic geodesic under the mapping $N \rightarrow \Gamma \backslash N$ and will be contained in the free homotopy class corresponding to $\gamma$. Conversely, every closed magnetic geodesic on $\Gamma \backslash N$ will lift to a magnetic geodesic on the universal cover. Every non-contractible periodic magnetic geodesic on $\Gamma \backslash N$ arises as the image of a $\gamma$-periodic magnetic geodesic on $N$.

**Lemma 6** Let $\gamma = \exp(V_\gamma + z_\gamma Z) \in H_n$ where $|Z| = 1$ and $V_\gamma$ is orthogonal to $Z(\mathfrak{h}_n)$, and let $\sigma(t) = \exp(X(t) + Z(t))$ be a $\gamma$-periodic magnetic geodesic. If $V_\gamma \neq 0$, then $\sigma$ is a non-central one-parameter subgroup (see Remark 7).

**Proof** Repeated use of (32) shows that $\gamma^* \sigma(t) = \sigma(t + k \omega)$. Using the multiplication formula (12) on each side of the equation, the non-central components must satisfy $kV_\gamma + X(t) = X(t + k \omega)$. If $V_\gamma \neq 0$, then the vector-valued function $X(t + \omega) - X(t)$ must be unbounded. Inspection of the magnetic geodesic equation (25)–(30) shows that this can only happen if $z_\gamma = 0$, i.e., $\sigma$ is a one-parameter subgroup. Moreover, $\sigma$ cannot be a central one-parameter subgroup because then the left-hand side of (32) would be noncentral and right-hand side would be central, a contradiction.

**Theorem 3** Let $\gamma = \exp(V_\gamma + z_\gamma Z) \in H_n$, with $V_\gamma \neq 0$. For each $E > |B|$, there exist two $\gamma$-periodic magnetic geodesics $\sigma(t)$ with energy $E$ and periods $\omega = \pm |V_\gamma|/\sqrt{E^2 - B^2}$. There do not exist any $\gamma$-periodic magnetic geodesics with energy $E \leq |B|$.

**Proof** By Lemma 6, we need only consider non-spiraling magnetic geodesics. The energy of any such magnetic geodesics satisfies

$$E^2 = \sum \frac{u_i^2 + v_i^2}{A_i} + B^2 \geq B^2$$

If equality holds, then $\sigma$ is a central one-parameter subgroup, which is excluded by Lemma 6. Hence, $E > |B|$.

Fix $V_0 \in \mathfrak{b}$ such that its magnitude satisfies $|V_0|^2 + B^2 = E^2$ and its direction is parallel to $V_\gamma$, $V_0 = (B/k)V_\gamma$ for some $k \in \mathbb{R}_{\neq 0}$. Define $\gamma^* = \exp(V_\gamma + kZ)$ and $\sigma^*(t) = \exp(t(V_0 + BZ))$. Then,

$$\gamma^* \sigma^*(t) = \exp\left(\frac{k}{B}\left(\frac{B}{k}V_\gamma + BZ\right)\right) \exp(t(V_0 + BZ))$$

$$= \exp\left(\frac{k}{B}(V_0 + BZ)\right) \exp(t(V_0 + BZ))$$

$$= \exp\left((t + \frac{k}{B})(V_0 + BZ)\right)$$

$$= \sigma^*\left(t + \frac{k}{B}\right)$$

shows that $\sigma^*$ is a $\gamma^*$-periodic magnetic geodesic of energy $E$ with period $\omega = k/B$. Using the multiplication formula (12) and the fact that $Z(\mathfrak{h}_n)$ is one-dimensional, it is straightforward to see that $\gamma$ and $\gamma^*$ are conjugate in $H_n$. Thus, there exists $a \in H_n$ such that $a \gamma^* a^{-1} = \gamma$. Now, $\sigma = a \cdot \sigma^*$ is a magnetic geodesic of energy $E$ and

$$\gamma \cdot \sigma(t) = a \gamma^* a^{-1} \sigma(t) = a \gamma^* \sigma^*(t) = a \sigma^*(t + \omega) = \sigma(t + \omega)$$

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shows that it is $\gamma$-periodic of period $\omega$. The expression for $\omega$ follows from $\pm k/B = |V_\gamma|/|V_0|$, and $|V_0| = \sqrt{E^2 - B^2}$.

Having dealt with the periods of a non-central element of $H_n$, we now consider the case when $\gamma = \exp(z_0Z)$ is central. In this case, there exist $\gamma$-periodic magnetic geodesics starting at the identity of energy both greater than and less than $|B|$. For a fixed energy $E > |B|$, there will be finitely many distinct periods associated with $\gamma$-periodic magnetic geodesics, while there will be infinitely many distinct periods when $E < |B|$.

**Lemma 7** Let $\gamma = \exp(z_0Z)$ for some $z_0 \in \mathbb{R}^*$ and suppose that $\sigma(t)$ is a $\gamma$-periodic magnetic geodesic and a one-parameter subgroup. Then, $\sigma(t) = \exp(i z_0 Z)$ for some $z_0 \in \mathbb{H}^*$. Moreover, for every $E > 0$, there exist two $\gamma$-periodic magnetic geodesics of energy $E$, $\sigma(t) = \exp(t(\pm E)Z)$, with period $\omega = z_0/(\pm E)$.

**Proof** Since $\sigma$ is a one-parameter subgroup by hypothesis, $\sigma(t) = \exp(itV_0 + BtZ)$. On the one hand, $\gamma\sigma(t) = \exp(itV_0 + (Bt + z_0)Z)$ and on the other $\sigma(t + \omega) = \exp((t + \omega)V_0 + B(t + \omega)Z)$. Hence, $\omega V_0 = 0$ and since $\omega \neq 0$, we conclude that $V_0 = 0$, showing the first claim.

For each energy $E > 0$, let $z_0 = -B \pm E$ and let $\sigma(t)$ be the magnetic geodesic $\sigma(t) = \exp(t(\pm E)Z)$. Then, $\sigma$ is a magnetic geodesic of energy $E$ and

$$\gamma \sigma(t) = \sigma((z_0 \pm Et)Z) = \exp \left( \pm E \left( \frac{z_0}{\pm E} + t \right) Z \right) = \sigma(t + \omega)$$

shows that it is $\gamma$-periodic of period $\omega$.

Next, suppose that $\sigma(t)$ is a spiraling magnetic geodesic, so that the component functions of $\sigma(t)$ have the form (25)–(27). Comparing the coefficients of $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ in $\gamma \sigma(t)$ and $\sigma(t + \omega)$ gives conditions

$$\sin \left( \frac{z_0}{A_i} (t + \omega) \right) = \sin \left( \frac{z_0}{A_i} t \right), \quad \cos \left( \frac{z_0}{A_i} (t + \omega) \right) = \cos \left( \frac{z_0}{A_i} t \right)$$

(33)

for each $i = 1, \ldots, n$ such that $u_i^2 + v_i^2 \neq 0$.

We now specialize to case of the three-dimensional Heisenberg group and obtain a complete description of the spiraling $\gamma$-periodic magnetic geodesics through the identity. Since the left-invariant metric is determined by one parameter, and a magnetic geodesic through the identity is determined by $z_0$ and only one pair of $u_i, v_i$, we write $A = A_1, u_0 = u_1$ and $v_0 = v_1$ to ease notation. In general, the analysis will depend on the relative size of $E$ and $B$ and hence, breaks up naturally into the three cases $E > |B|$, $E < |B|$ and $E = |B|$. In each case, we first establish the range of permissible integers $\ell$. Next, for each permissible $\ell$, we describe the magnetic geodesics through the identity translated by $\gamma$ along with their respective periods.

In this case, the period $\omega$ and the coordinate $z_0$ must be related by $\omega z_0 = 2\pi A\ell$, where $\ell \in \mathbb{Z}$. (We abuse notation by allowing $\pi$ to denote both vector bundle projection maps and the irrational number. Its meaning will be clear from context.) As the meaning will be clear from context, we comparing the central components in $\gamma \sigma(t)$ and $\sigma(t + \omega)$ gives the condition $z(t) + z_0 = z(t + \omega)$. That is,
This simplifies to
\[ z_\gamma = \left( z_0 + B + \frac{u_0^2 + v_0^2}{2A z_0} \right) \omega, \]  
and using (31) to eliminate the fraction and \( \omega z_0 = 2\pi A \ell' \) to eliminate \( \omega \) this can be written as
\[ z_\gamma = \left( z_0 + B + \frac{1}{2z_0} \left( E^2 - (z_0 + B)^2 \right) \right) \frac{2\pi A \ell'}{z_0}. \]  
If \( E = |B| \), then the above simplifies to \( z_\gamma = \pi A \ell' \). If \( E \neq |B| \), then after clearing denominators and solving for \( z_0 \), we obtain the expression
\[ z_0^2 = \frac{E^2 - B^2}{\pi A \ell' - 1}. \]

**Lemma 8** Let \( \gamma = \exp(z_\gamma Z) \) be a central element of the Heisenberg group. For each nonzero energy level, the range of admissible integers \( \ell' \) and the corresponding choices of \( z_0 \) for which there exists a \( \gamma \)-periodic magnetic geodesic through the identity are given by the following table.

| \( \ell' \) | \( z_0 \) |
|---|---|
| (1a) \( E > |B| \) | 1 < \( \frac{2E}{E+|B|} < \frac{z_\gamma}{\pi A \ell'} \) | \( -\sqrt{\frac{E^2-B^2}{2\pi \ell' - 1}} \) |
| (1b) \( E > |B| \) | 1 < \( \frac{2E}{E-|B|} < \frac{z_\gamma}{\pi A \ell'} \) | \( +\sqrt{\frac{E^2-B^2}{2\pi \ell' - 1}} \) |
| (2a) \( 0 < E < B \) | \( \frac{2E}{E-|B|} < \frac{z_\gamma}{\pi A \ell'} < \frac{2E}{E+|B|} < 1 \) | \( -\sqrt{\frac{E^2-B^2}{2\pi \ell' - 1}} \) |
| (2b) \( B < -E < 0 \) | \( \frac{2E}{E-|B|} < \frac{z_\gamma}{\pi A \ell'} < \frac{2E}{E+|B|} < 1 \) | \( +\sqrt{\frac{E^2-B^2}{2\pi \ell' - 1}} \) |
| (3a) \( E = B \) | \( \ell' = \frac{z_\gamma}{\pi A} \) | \( -2B < z_0 < 0 \) |
| (3b) \( E = -B \) | \( \ell' = \frac{z_\gamma}{\pi A} \) | \( 0 < z_0 < -2B \) |
In all cases, the associated period is $\omega = 2\pi A \ell / z_0$ and one can choose any $u_0$ and $v_0$ such that $u_0^2 + v_0^2 = A(E^2 - (z_0 + B)^2)$.

**Proof** The condition $(z_0 + B)^2 < E^2$ is equivalent to

$$-E - B < \pm \sqrt{\frac{E^2 - B^2}{z_\gamma} - 1} < E - B.$$ (37)

In case (1), $-E - B < 0$ and $E - B > 0$, so this leads to the two inequalities

$$-E - B < \sqrt{\frac{E^2 - B^2}{z_\gamma} - 1} < 0, \quad 0 < \sqrt{\frac{E^2 - B^2}{z_\gamma} - 1} < E - B.$$ After squaring both inequalities and isolating $z_\gamma/(\pi A \ell)$, these become

$$1 < \frac{2E}{E + B} < \frac{z_\gamma}{\pi A \ell}, \quad 1 < \frac{2E}{E - B} < \frac{z_\gamma}{\pi A \ell},$$

yielding cases (1a) and (1b), respectively. Notice that one of these ranges for $\ell$ is a subset of the other. We keep them separate as they affect the choice of sign for $z_0$.

In case (2), either $-B - E < -B + E < 0$ if $B > 0$, or $0 < -B - E < -B + E$ if $B < 0$. A similar computation as above leads to the inequalities

$$\frac{2E}{E - B} < \frac{z_\gamma}{\pi A \ell} < \frac{2E}{E + B} < 1, \quad \frac{2E}{E + B} < \frac{z_\gamma}{\pi A \ell} < \frac{2E}{E - B} < 1.$$ Both of these ranges can be expressed simultaneously in terms of $|B|$ as in Lemma statement. However, in case (2a), when $B > 0$, $z_0$ is chosen according to the negative branch, and vice versa in case (2b).

For case (3), it was noted above (36) that if $E = |B|$, then $z_\gamma = \pi A \ell$. Choose $z_0$ so that $(z_0 + B)^2 < E^2$. When $B > 0$, this inequality is the same as $-2B < z_0 < 0$. Setting $\omega = 2\pi A \ell / z_0 = 2z_\gamma / z_0$, it is straightforward to check that $\frac{z_\gamma}{\pi A} (t + \omega) = \frac{z_\gamma}{\pi A} t$ and that (34) holds. The case when $B < 0$ is handled similarly.

**Remark 9** In case (2), the condition that $E < |B|$ ensures that $E^2 - B^2 < 0$, while the conditions on $\ell$ ensure that $z_\gamma/(\pi A \ell) - 1 < 0$. Hence, the expression under the radical in $z_0$ will be positive.

**Remark 10** In every case, for each admissible $z_0$ there is a one-parameter family of $\gamma$-periodic magnetic geodesics.

**Remark 11** The cases where $E = |B|$ are to be interpreted as follows. When $z_\gamma$ and $A$ are such that $z_\gamma/\pi A \in \mathbb{Z}$, then there exist $\gamma$-periodic magnetic geodesics with energy $E$ and $z_0$ as described in the table. Otherwise, the collection of such magnetic geodesics is empty.

### 4.2 Lengths of closed magnetic geodesics

We are now in a position to compute the lengths of closed magnetic geodesics on $\Gamma \backslash H$ in the free homotopy class of $\gamma \in \Gamma$. If $\Gamma < H$ is a cocompact discrete subgroup, and $\gamma \in \Gamma$, then...
then the length of the corresponding closed magnetic geodesic on the compact quotient $\Gamma \backslash H$ will be

$$\int_0^{[\omega]} |\sigma'(t)| dt = E|\omega|. \quad (38)$$

Previous results concerning the lengths of closed geodesics in the Riemannian case include [11, 14, 16, 17]. Unlike the Riemannian case, magnetic geodesics cannot be reparameterized to have a different energy. So it is more natural to consider the collection of lengths of closed geodesics of a fixed energy. Let $L(\gamma;E)$ denote the set of distinct lengths of closed magnetic geodesics in the free homotopy class of $\gamma$.

**Theorem 4** Let $\Gamma < H$ be a cocompact discrete subgroup of the Heisenberg group $H$ and let $\gamma = \exp(V_\gamma + z_\gamma Z) \in \Gamma$.

- If $\gamma = e$ is the identity ($V_\gamma = 0$ and $z_\gamma = 0$), then

  $$L(e;E) = \begin{cases} \emptyset & \text{if } E \geq |B| \\ \left\{ \frac{2\pi A}{\sqrt{E^2 - 1}} \right\} & \text{if } 0 < E < |B| \end{cases} \quad (39)$$

- If $\gamma$ is not central ($V_\gamma \neq 0$), then

  $$L(\gamma;E) = \begin{cases} \emptyset & \text{if } 0 < E \leq |B| \\ \left\{ \frac{|V_\gamma|}{\sqrt{1 - \frac{E^2}{B^2}}} \right\} & \text{if } E > |B| \end{cases} \quad (40)$$

- If $\gamma$ is central ($V_\gamma = 0$ and $z_\gamma \neq 0$), then

  $$L(\gamma;E) = \begin{cases} \emptyset & \text{if } 0 < E \leq |B| \\ \left\{ \frac{|V_\gamma|}{\sqrt{1 - \frac{E^2}{B^2}}} \right\} & \text{if } E > |B| \end{cases} \quad (41)$$

**Proof** The case when $\gamma = e$ follows from Theorem 2. The lengths of closed magnetic geodesics obtained in that theorem are

$$E|\omega| = E \left| \frac{2\pi A}{-\text{sgn}(B)\sqrt{B^2 - E^2}} \right| = \frac{2\pi AE}{\sqrt{B^2 - E^2}}$$

The case when $\gamma = \exp(V_\gamma + z_\gamma Z)$ is not central follows from Theorem 3. The length of closed magnetic geodesics obtained in that theorem is
The case when $\gamma$ is central follows from Lemmas 7 and 8. In the former case, which applies to every energy, the length of the closed magnetic geodesic is

$$E|\omega| = E \sqrt{\frac{\left| V_\gamma \right|}{E^2 - B^2}} = \frac{E|V_\gamma|}{\sqrt{E^2 - B^2}}.$$  

In the latter case, when $E > |B|$ the lengths are

$$E|\omega| = E \frac{2\pi A\ell}{z_0} = 2\pi AE\ell \pm \sqrt{\frac{\zeta_{\frac{\pi A}{A^2}} - 1}{E^2 - B^2}} = \frac{2E\sqrt{\pi A\ell(z_\gamma - \pi A\ell)}}{\sqrt{E^2 - B^2}}$$

and when $E < |B|$ the lengths are

$$E|\omega| = E \frac{2\pi A\ell}{z_0} = 2\pi AE\ell \pm \sqrt{\frac{1 - \frac{\zeta_{\frac{\pi A}{A^2}}}{B^2}}{B^2 - E^2}} = \frac{2E\sqrt{\pi A\ell(\pi A\ell - z_\gamma)}}{\sqrt{B^2 - E^2}}.$$  

The lengths when $E = |B|$ depend not on $\ell$ (which must be $\ell = z_\gamma/(\pi A)$) but instead on $z_0$ and are given by

$$E|\omega| = E \frac{2\pi A\ell}{z_0} = E \frac{2z_\gamma}{z_0}.$$

\[\square\]

**Remark 12** As $E \rightarrow \infty$ or $B \rightarrow 0$, the denominator $\sqrt{1 - B^2/E^2} \rightarrow 1$. Roughly speaking, the cases $E \leq |B|$ will be eliminated, and the collection of lengths in the case $E > |B|$ will approach the length spectrum in the Riemannian case, which was computed in [16]. This reflects the following physical intuition: when the magnetic field is very weak charged particles will behave more like they would in the absence of any forces, and when a particle is very energetic the magnetic field will have less of an effect on its trajectory.

**Remark 13** The dynamics of the magnetic flow on the various energy levels splits roughly into three regimes:

- For fixed energy levels $E > |B|$, there exist closed magnetic geodesics in every free homotopy class and the set of their lengths is finite.
- For fixed energy levels $E < |B|$, there exist free homotopy classes without any closed magnetic geodesics, and in the case that there are closed magnetic geodesics, the set of their lengths is countably infinite. This reflects the paradigm that the dynamics on high energy levels will resemble that of the underlying geodesic flow.
- Finally, when $E = |B|$, $\gamma$ is central, and $z_\gamma \in \pi AZ$ (i.e., the set of lengths is non-empty), then the infinite set of lengths is not discrete.
The following three lemmas address bounds on the collection of lengths of closed magnetic geodesics in a given central free homotopy class.

**Lemma 9** Consider the case $|B| < E$ in (41). The set

$$\left\{ \frac{\sqrt{4\pi A\ell (z_\gamma - \pi A\ell)}}{\sqrt{1 - \frac{B^2}{E^2}}} : \ell \in \mathbb{Z}, \frac{2E}{E + |B|} < \frac{z_\gamma}{\pi A\ell} \right\}$$

is bounded above by $|z_\gamma|/\sqrt{1 - B^2/E^2}$, which is larger than $|z_\gamma|$. The example below shows that this upper bound is the best possible.

**Proof** Without loss of generality, we assume $z_\gamma > 0$. The condition on $\ell$ implies $0 < \ell < \frac{z_\gamma}{2\pi A} \left(1 + \frac{|B|}{E}\right)$. We define

$$\lambda(\ell) = \frac{4\pi A\ell(z_\gamma - \pi A\ell)}{(1 - \frac{B^2}{E^2})}.$$ 

The parabola $\lambda(\ell)$ opens downward and has zeroes at $\ell = 0$ and $\ell = z_\gamma/\pi A$, hence achieves a maximum of $z_\gamma^2/(1 - B^2/E^2)$ at $\ell = z_\gamma/2\pi A$. See the example below for values of $A, B, z_\gamma$ such that this maximum is achieved. The result follows.

**Example 2** Consider the particular example where $A = 1$, $B = 1$ and $E = 2$. Choose the central element $\gamma = \exp(20\pi Z)$ so that $z_\gamma = 20\pi$. In this case, for each $\ell$ such that $0 < \ell < 15$, there is a closed magnetic geodesic with length given by (41). In particular, when $\ell = 10$ the corresponding length is $(2/\sqrt{3})20\pi > z_\gamma$.

**Remark 14** In the setting of Riemannian two-step nilmanifolds, the maximal length of a closed magnetic geodesic in a central free homotopy class is the length of the central geodesic. In fact, the maximal length spectrum determines the length spectrum for central free homotopy classes (see Proposition 5.15 of [11]). Example 2 shows that in the magnetic setting the central magnetic geodesic need not be the longest one.

**Lemma 10** Consider the case $|B| > E$ in (41). The set

$$\left\{ \frac{\sqrt{4\pi A\ell (\pi A\ell - z_\gamma)}}{\sqrt{\frac{B^2}{E^2} - 1}} : \ell \in \mathbb{Z}, \frac{2E}{E - |B|} < \frac{z_\gamma}{\pi A\ell} < \frac{2E}{E + |B|} \right\}$$

is bounded below by $|z_\gamma|$.

**Proof** Without loss of generality, we assume $z_\gamma > 0$. We define

$$\lambda(\ell) = \frac{4\pi A\ell(\pi A\ell - z_\gamma)}{\left(\frac{B^2}{E^2} - 1\right)}.$$
The parabola $\lambda(\ell')$ opens upward and has zeroes at $\ell' = 0$ and $\ell' = z_\gamma/\pi A$. The condition on $\ell'$ implies $\ell' > \frac{z_\gamma}{2\pi A} \left(1 + \frac{|B|}{E}\right) > \frac{z_\gamma}{\pi A}$ or $\ell' < \frac{z_\gamma}{2\pi A} \left(1 - \frac{|B|}{E}\right) < 0$. A lower bound of the set is thus provided by the minimum of $\sqrt{\lambda\left(\frac{z_\gamma}{2\pi A} \left(1 + \frac{|B|}{E}\right)\right)}$ and $\sqrt{\lambda\left(\frac{z_\gamma}{2\pi A} \left(1 - \frac{|B|}{E}\right)\right)}$. However, both of these evaluate to $z_\gamma$, and the result follows.

Lemma 11 Consider the case $|B| = E$ in (41). If the set
\[
\left\{ \frac{2E|z_\gamma|}{|z_0|} : z_0 \in \mathbb{R}, (z_0 + B)^2 < E^2 \right\}
\]
is nonempty (see Remark 11), then it is unbounded above and has an infimum of $|z_\gamma|$.

Proof If $B > 0$, then $z_0$ can be chosen in the interval $-2B < z_0 < 0$. As $z_0 \to 0^-$, the length diverges to infinity, and as $z_0 \to (-2B)^+$ the lengths converge to $|z_\gamma|$. The case when $B < 0$ is analogous.

4.3 Density of closed magnetic geodesics

Given a Riemannian manifold $M$, define $S^E M = \{ V \in TM : |V| = E \}$ and let $S^B M$ denote the tangent sphere of radius $E$ at the point $\gamma$. Given a vector $V \in TM$, let $\sigma_V$ denote the magnetic geodesic such that $\sigma_V'(0) = V$. We are interested in the size of the set of vectors that determine periodic magnetic geodesics. In the Riemannian case, this set is scale invariant. That is, if $V$ determines a periodic geodesic, then so does $cV$ for any $c \neq 0$. So it is natural in this case to restrict attention to unit vectors. However, this property does not hold for magnetic geodesics. Therefore, in the following definition we include a dependence on the energy of the vectors.

\[
\text{Per}^E(M) := \{ V \in S^E M : \sigma_V \text{ is periodic} \} \subset S^E M.
\]

In the context of Riemannian two-step nilmanifolds, the density of this set was first investigated in [11], and subsequently in [10, 23–25, 9]. The following result shows that for magnetic flows on the Heisenberg group, density persists for all energy levels, except perhaps one.

Theorem 5 For each $E \neq |B|$, $\text{Per}^E(\Gamma \backslash H)$ is dense in $S^E(\Gamma \backslash H)$. When $E = |B|$, $\text{Per}^{|B|}(\Gamma \backslash H) = S^{|B|}(\Gamma \backslash H)$ if $\pi A/\bar{z} \in \mathbb{Q}$ and $\text{Per}^{|B|}(\Gamma \backslash H)$ is not dense in $S^{|B|}(\Gamma \backslash H)$ if $\pi A/\bar{z} \notin \mathbb{Q}$.

Proof We begin with a series of reductions. First, we claim that it suffices to show that the set of $V \in S^E(H)$ such that $\sigma_V$ is $\gamma$-periodic for some $\gamma \in \Gamma$ is dense in $S^E(H)$. For any $V \in S^E(\Gamma \backslash H)$, let $W \in \pi^{-1}(V)$ and let $\{W_i\} \subset S^E(H)$ be such that $\sigma_{W_i}$ is $\gamma_i$-periodic for some $\gamma_i \in \Gamma$ and $W_i \to W$. Then, $\{V_i = \pi(W_i)\} \subset S^E(\Gamma \backslash H)$ is a sequence of tangent vectors such that $\sigma_{V_i}$ is periodic and $V_i \to V$.

Next, we claim that it suffices to show that the set of $W \in S^E(H)$ such that $\sigma_W$ is $\gamma$-periodic for some $\gamma \in Z(\Gamma)$ is dense in $S^E(H)$. If $\sigma_W$ is such a magnetic geodesic and $\phi \in H$ is any element, then $\phi \cdot \sigma_W(t) = (\phi \gamma \phi^{-1})$-periodic magnetic geodesic satisfying $\phi \cdot \sigma_W(0) = \phi$ and $(\phi \cdot \sigma_W)'(0) = L_{\phi}(V)$. Because $\gamma$ is central, $\phi \gamma \phi^{-1} = \gamma$ and $\phi \cdot \sigma_W$ is a $\gamma$-periodic magnetic geodesic. Because $L_{\phi^*} : S^E(H) \to S^E(H)$ is a diffeomorphism, this proves the claim.
Lastly, we claim that it suffices to show that set \( z_0 \in [-B - E, -B + E] \) chosen according to Lemma 8 (for some choice of \( \gamma \in Z(\Gamma) \)) is dense in \([-B - E, -B + E]\). As noted in Lemma 8, for any such \( z_0 \) there is a one parameter family of \( \gamma \)-periodic magnetic geodesics given by any choice of \( u_0, v_0 \) such that \( u_0^2 + v_0^2 = A(E^2 - (z_0 + B)^2) \). Hence, if the resulting \( z_0 \) are dense in \([-B - E, -B + E]\), then there is a dense set of latitudes in the ellipsoid \( E^2 = ((u_0^2 + v_0^2)/A) + (z_0 + B)^2 \subset \mathbb{R}^3 \) such that those vectors yield \( \gamma \)-periodic magnetic geodesics for some \( \gamma \in \Gamma \). The initial conditions \( (u_0, v_0, z_0) \in \mathbb{R}^3 \) determine the magnetic geodesic \( \sigma_v \) where \( V = (u_0/A)X + (v_0/A)Y + (z_0 + B)Z \), showing that the set of \( V \in S_v E^2 \) tangent to \( \gamma \)-periodic magnetic geodesics \( (\gamma \in \Gamma) \) is dense in \( S_v E^2 \).

By Proposition 5.4 of [11], \( \Gamma \cap Z(H) = Z(\Gamma') \) is a lattice in \( Z(H) \). Hence, there exists \( \tilde{z} \in \mathbb{R}^+ \) such that \( \Gamma_0 \cap Z(H) = \{ \exp(h\tilde{z}Z) : h \in \mathbb{Z} \} \). By replacing \( \tilde{z} \) with \(-\tilde{z}\), if necessary, we can assume that \( \tilde{z} > 0 \). Consider the set of numbers

\[
\left\{ \frac{h}{\ell} : h, \ell \in \mathbb{Z}^+ \text{ and } \frac{2\pi AE}{\tilde{z}(E + B)} \ell < h \right\}
\]

This set is dense in the interval \((2\pi AE/\tilde{z}(E + B)), \infty\). Via a sequence of continuous mappings of \( \mathbb{R} \), each of which preserves density,

\[
\left\{ \frac{-\sqrt{E^2 - B^2}}{\frac{h\tilde{z}}{\pi AE} - 1} : h, \ell \in \mathbb{Z}^+ \text{ and } \frac{2E}{E + B} \ell < h \right\}
\]

is dense in the interval \((-E - B, 0)\). These are precisely the values for \( z_0 \) appearing in case (1a) of Lemma 8. Starting instead with the set

\[
\left\{ \frac{h}{\ell} : h, \ell \in \mathbb{Z}^+ \text{ and } \frac{2\pi AE}{\tilde{z}(E - B)} \ell < h \right\}
\]

and using a parallel sequence of transformations shows that

\[
\left\{ \frac{\sqrt{E^2 - B^2}}{\frac{h\tilde{z}}{\pi AE} - 1} : h, \ell \in \mathbb{Z}^+ \text{ and } \frac{2E}{E - B} \ell < h \right\}
\]

is dense in \((0, E - B)\). These numbers are the \( z_0 \) appearing in case (1b) of Lemma 8. This shows the density of permissible \( z_0 \) in the interval \([-E - B, E - B]\) and hence, the density of \( \text{Per}^E(\Gamma \setminus H) \) for \( E > |B| \).

When \( E < |B| \), we start from the observation that the set

\[
\left\{ \frac{h}{\ell} : h, \ell \in \mathbb{Z} \text{ and } \frac{2\pi AE}{\tilde{z}(E - |B|)} < \frac{h}{\ell} < \frac{2\pi AE}{\tilde{z}(E + |B|)} \right\}
\]

is dense in the interval \((2\pi AE/\tilde{z}(E - |B|)), 2\pi AE/(\tilde{z}(E + |B|))\). Using a similar sequence of transformations, the set

\[
\left\{ \frac{E^2 - B^2}{\frac{h}{\ell} - 1} : h, \ell \in \mathbb{Z} \text{ and } \frac{2E}{E - |B|} < \frac{h}{\ell} < \frac{2E}{E + |B|} \right\}
\]
is dense in the interval \(((E - |B|)^2, (-E - |B|)^2)\). Then,

\[
\{-\text{sgn}(B) \sqrt{\frac{E^2 - B^2}{h^2}} : h, \ell \in \mathbb{Z} \text{ and } \frac{2E}{E - |B|} < \frac{h}{\ell} < \frac{2E}{E + |B|}\}
\]

is dense in the interval \((-B - E, -B + E)\). These numbers are the values of \(z_0\) from cases (2a) and (2b) in Lemma 8, showing that \(\text{Per}^\phi(\Gamma\setminus H)\) is dense for \(0 < E < |B|\).

Lastly, we consider the case \(E = |B|\). By Theorem 2, there are no closed contractible magnetic geodesics, and by Theorem 3 there are no \(\gamma\)-periodic magnetic geodesics for \(\gamma \notin Z(1)\). Hence, the only source of closed magnetic geodesics is \(\gamma\)-periodic magnetic geodesics when \(\gamma \in Z(\Gamma)\). If \(\pi A/\pi Z \notin \mathbb{Q}\), then there exist integers \(h, \ell\) such that \(\pi z = \pi A\ell\). Consider \(\gamma = \exp(\pi z) \in Z(\Gamma)\). By case 3 of Lemmas 8 and 7, every choice of \(z_0\) in the interval \([-2B, 0]\) if \(B > 0\) and \([0, -2B]\) if \(B < 0\) yields a \(\gamma\)-periodic magnetic geodesic. Hence, \(\text{Per}^{\pi A}(\Gamma\setminus H) = S^{\pi A}(\Gamma\setminus H)\). If \(\pi A/\pi Z \notin \mathbb{Q}\), then the condition in case 3 of Lemma 8 is not satisfied, and so \(\text{Per}^{\pi A}(\Gamma\setminus H)\) is not dense. In fact, the only \(\gamma\)-periodic magnetic geodesics in this case are the two furnished by Lemma 7.

\[\square\]

### 4.4 Rigidity and the marked magnetic length spectrum

We begin by recalling the notion of marked length spectrum for a compact Riemannian manifold \(M\). For each nontrivial free homotopy class \(C\), there exists at least one smoothly closed Riemannian geodesic. Let \(L(C)\) denote the collection of all lengths of smooth closed geodesics that belong to \(C\). Recall that free homotopy classes of closed curves on \(M\) are in bijection with conjugacy classes of \(\pi_1(M)\). If \(\tilde{M}\) is another compact Riemannian manifold and \(\phi : \pi_1(M) \to \pi_1(\tilde{M})\) is an isomorphism, then \(\phi\) maps conjugacy classes of \(\pi_1(M)\) bijectively onto conjugacy classes of \(\pi_1(\tilde{M})\). Hence, \(\phi\) induces a bijection \(\phi_*\) of the set of free homotopy classes of closed curves on \(M\) onto the set of free homotopy classes of closed curves on \(\tilde{M}\). Two compact Riemannian manifolds \(M\) and \(\tilde{M}\) are said to have the same marked length spectrum if there exists an isomorphism \(\phi : \pi_1(M) \to \pi_1(\tilde{M})\) such that \(L(\phi_*C) = L(C)\) for all nontrivial free homotopy classes of closed curves on \(M\). Specializing to the case at hand, let \(G_1\) and \(G_2\) be two simply connected two-step nilpotent Lie groups and \(\Gamma_1 < G_1\) and \(\Gamma_2 < G_2\) cocompact discrete subgroups. Then, \(\pi_1(\Gamma_i \setminus G_i) \simeq \Gamma_i\). With these identifications, we say the nilmanifolds \(\Gamma_i \setminus G_i\) and \(\Gamma_i \setminus G_i\) have the same marked length spectrum if there is an isomorphism \(\phi : \Gamma_1 \to \Gamma_2\) such that \(L(\phi_*C) = L(C)\) for all nontrivial free homotopy classes of closed curves on \(\Gamma_1 \setminus G_1\). See [11] and [18] for previous results on marked length spectrum rigidity of Riemannian two-step nilmanifolds.

While the above definition could be used in the context of magnetic flows on nilmanifolds, it seems more natural to modify it in light of the dependence of the dynamics on the relative magnitudes of \(E\) and \(|B|\). For a fixed homotopy class \(C\), the collection of lengths of closed magnetic geodesics of any energy could be an infinite open interval. Therefore, let \(L(C; E)\) denote the collection of all lengths of smoothly closed magnetic geodesics that belong to \(C\) and have energy \(E\). Note that, unlike the Riemannian case, for magnetic flows it may occur that \(L(C; E) = \emptyset\) with \(C\) nontrivial.

**Definition 3** Let \(H\) be the simply connected three-dimensional Heisenberg group. Let \(g_1\) and \(g_2\) be two left-invariant Riemannian metrics on \(H\) with parameters \(A_1\) and \(A_2\). Let
\( \Omega_1 \) and \( \Omega_2 \) be two left-invariant magnetic forms on \( H \) with parameters \( B_1 \) and \( B_2 \). Let \( \Gamma_1, \Gamma_2 < H \) be two cocompact discrete subgroups, and \( \phi : \Gamma_1 \to \Gamma_2 \) an isomorphism. The nilmanifolds \( \Gamma_1 \backslash H \) and \( \Gamma_2 \backslash H \) with corresponding magnetic structures are said to have the same marked magnetic length spectrum if \( L(C;E) = L(\phi C;E) \) for each nontrivial free homotopy class \( C \).

Even though the magnetic flow is a perturbation away from the underlying geodesic flow, it reflects enough of the underlying Riemannian geometry to exhibit a degree of geometric rigidity.

**Theorem 6** Let \( H \) be the simply connected, three-dimensional Heisenberg group endowed with left-invariant Riemannian metric \( g \) and left-invariant magnetic form \( \Omega \), with corresponding parameters \( A \) and \( B \), respectively. Let \( \Gamma_1, \Gamma_2 < H \) be two cocompact lattices. Suppose that for some \( E > |B| \), the two manifolds \( \Gamma_1 \backslash H \) and \( \Gamma_2 \backslash H \) have the same marked magnetic length spectrum at energy \( E \). Then, \( \Gamma_1 \backslash H \) and \( \Gamma_2 \backslash H \) are isometric.

The proof of Theorem 6 is similar to the proof of Theorem 5.20 in [11], with one notable exception. The latter uses the maximal marked length spectrum, i.e., only the length longest closed geodesic in each free homotopy class. For Riemannian geodesics in central free homotopy classes (on two-step nilpotent Lie groups), this is always length of the one-parameter subgroup. Example 2 and Remark 14 show that the maximal magnetic marked length spectrum is not so well behaved. To circumvent this, we consider all the lengths of closed magnetic geodesics in central free homotopy classes. This argument is given in the following Lemma.

**Lemma 12** Under the same hypotheses as Theorem 6, let \( \exp(\bar{z}_1 Z) \) and \( \exp(\bar{z}_2 Z) \) be generators for the central lattices \( \Gamma_1 \cap H \) and \( \Gamma_2 \cap H \), respectively. Then, \( |\bar{z}_1| = |\bar{z}_2| \).

**Proof** First, we claim that

\[
\sup_{h \in \mathbb{Z}} \left\{ \frac{\max(L(\exp(h \bar{z}_1 Z)_1);E))}{|h|} \right\} = \sup_{h \in \mathbb{Z}} \left\{ \frac{\max(L(\exp(h \bar{z}_2 Z)_2);E))}{|h|} \right\} \tag{44}
\]

where \( [\gamma] \) denotes the free homotopy class of closed curves on \( \Gamma_i \backslash H \) determined by \( \gamma \in \Gamma_i \). Let \( \phi : \Gamma_1 \to \Gamma_2 \) be an isomorphism. Since \( \phi \) is an isomorphism of \( Z(\Gamma_1) \) onto \( Z(\Gamma_2) \), \( \phi(\exp(h \bar{z}_1 Z)) = \exp(\pm h \bar{z}_2 Z) \), and so \( \phi_{\gamma}[\exp(h \bar{z}_1 Z)]_1 = [\exp(\pm h \bar{z}_2 Z)]_2 \). By hypothesis, the sets of lengths of closed magnetic geodesics in these two classes are equal. Moreover, the positive integer \( |h| \) is the same for both free homotopy classes. Hence, the sets over which the supremums are taken are equal.

Next, we evaluate the supremums in (44). By Lemma 9, the set of lengths of smoothly closed magnetic geodesics in the free homotopy class determined by an element of the form \( \exp(h \bar{z}_1 Z) \) is bounded above by \( |h \bar{z}_1|/\sqrt{1 - B^2/E^2} \). After dividing all the lengths in each set by \( |h| \), respectively, we obtain a uniform upper bound,

\[
\sup_{h \in \mathbb{Z}} \left\{ \frac{\sqrt{4\pi A^2 \ell^2(h \bar{z}_1 - \pi A^2)}^2}{|h|/\sqrt{1 - B^2/E^2}} : \ell \in \mathbb{Z}, \frac{2E}{E + |B|} < \frac{h \bar{z}_1}{\pi A^2} \right\} \leq \frac{|\bar{z}_1|}{\sqrt{1 - B^2/E^2}}. \tag{45}
\]
We now claim that the inequality in (45) is actually an equality. If the quantity \( z_i/(2\pi A) \in \mathbb{Q} \), then for \( h \) large enough \( \ell = (h\tilde{z})/(2\pi A) \) will be an allowable integer value for \( \ell \), and \( \max, \left(\sqrt{4\pi A((h\tilde{z}) - \pi A\ell)}\right) = |h\tilde{z}| \). If the quantity \( z_i/(2\pi A) \notin \mathbb{Q} \), then the numbers \( (h\tilde{z})/(2\pi A) \) will come arbitrarily close to an integer. In either case, the supremum is \(|\tilde{z}_i|/\sqrt{1 - B^2/E^2}\). Lemma now follows.

We now proceed with the proof of Theorem 6.

**Proof** First, extend the marking \( \phi : \Gamma_1 \to \Gamma_2 \) to an automorphism \( \phi : H \to H \). Because \( \phi_* \) is a Lie algebra automorphism of \( \mathfrak{h} = \mathfrak{b} \oplus \mathfrak{z} \), we can decompose it as \( \phi_* = R_1 + R_2 + S \), where \( R_1 : \mathfrak{b} \to \mathfrak{b} \), \( R_2 : \mathfrak{b} \to \mathfrak{z} \), and \( S : \mathfrak{z} \to \mathfrak{z} \) are linear maps. Using Lemma 12,

\[
|S(\tilde{z}_1)| = |\pm \tilde{z}_2| = |\tilde{z}_2| = |\tilde{z}_1| = |\tilde{z}_1|,
\]

shows that \( S \) is an isometry of \( \mathfrak{z} \). Let \( \pi_\mathfrak{b} : \mathfrak{h} \to \mathfrak{b} \) denote the projection. For any \( V \in \pi_\mathfrak{b} \log \Gamma_1 \), there is some \( \xi \in \Gamma_1 \) such that \( \xi = V + Z \) and \( Z \in \mathfrak{z} \). By hypothesis, \( \exp(\xi) \) and

\[
\phi(\exp(V + Z)) = \phi(\exp(\xi)) = \exp(\phi_\xi) = \exp(R_1(V) + R_2(V) + S(Z))
\]

have the same lengths of closed magnetic geodesics. By (40), we have

\[
\frac{|V|}{\sqrt{1 - B^2/E^2}} = \frac{|R_1(V)|}{\sqrt{1 - B^2/E^2}}
\]

and we conclude that \( R_1 \) is an isometry of \( \mathfrak{b} \). It is straightforward to check that \( R_1 + S : \mathfrak{h} \to \mathfrak{h} \) is an isometric Lie algebra isomorphism. Let \( \phi_1 : H \to H \) be the Lie group isomorphism such that \( (\phi_1)_s = R_1 + S \). Define \( T : \mathfrak{h} \to \mathfrak{h} \) by \( T(V + Z) = V + Z + (S^{-1} \circ R_2)(V) \). Once can verify directly that \( T \) is an inner automorphism of \( \mathfrak{h} \) and \( (\phi_1)_s \circ T = \phi_s \). Let \( \phi_2 \) be the inner automorphism of \( H \) such that \( (\phi_2)_s = T \).

Now, we have that \( \phi = \phi_1 \circ \phi_2 \), where \( \phi_1 \) is an isometric automorphism of \( H \) and \( \phi_2 \) is an inner automorphism of \( H \). Because \( \phi_2 \) is an inner automorphism, \( \Gamma_1 \setminus H \) is isometric to \( \phi_2(\Gamma_1) \setminus H \) (via a left-translation), and \( \phi_2(\Gamma_1) \setminus H \) is isometric to \( \phi_1(\phi_2(\Gamma_1)) \setminus H = \phi(\Gamma_1) \setminus H = \Gamma_2 \setminus H \). \( \square \)

**Remark 15** For \( E < |B| \), the marked magnetic length spectrum does not determine the metric. Since the set of lengths in any noncentral free homotopy class is empty (see Theorem 4), the marked magnetic lengths spectrum does not carry any information about the noncentral elements of lattices.

**Remark 16** The isometry \( \Gamma_1 \setminus H \to \Gamma_2 \setminus H \) preserves the magnetic form \( \Omega = d(B\xi) \) up to sign. Since the isometry is realized by \( \phi_1 \circ L_\xi \) for some \( x \in H \),

\[
(\phi_1 \circ L_\xi)^* \xi = L_\xi^*(\phi_1^* \xi) = L_\xi^*(\pm \xi) = \pm \xi.
\]
5 Heisenberg-type manifolds

Heisenberg-type manifolds are Riemannian manifolds that generalize the Heisenberg group endowed with a left-invariant metric. A metric two-step nilpotent Lie algebra \( \mathfrak{h} \) is of \textit{Heisenberg type} if

\[
J(Z)^2 = -|Z|^2 \mathbb{I}
\]

for every \( Z \in \mathfrak{g} \) (see (14)). A simply connected two-step nilpotent Lie group with left-invariant metric is of Heisenberg type if its metric Lie algebra is of Heisenberg type. It is often the case that theorems concerning the Heisenberg group endowed with a left-invariant metric, or their analogous formulations, are also true for Heisenberg-type manifolds. For an example, see the results of [16]. This paradigm does not appear to extend to the setting of Heisenberg-type manifolds endowed with a left-invariant magnetic field. In this section, we show by way of a simple example that the computation of the lengths of closed magnetic geodesics becomes significantly more complex for Heisenberg-type manifolds.

Let \( \mathfrak{h} = \text{span}\{X_1, \ldots, X_4, Z_1, Z_2\} \) and define a bracket structure by

\[
[X_1, X_2] = Z_1 \quad [X_1, X_3] = Z_2 \quad [X_2, X_4] = -Z_2 \quad [X_3, X_4] = Z_1
\]

and extending by bilinearity and skew-symmetry to all of \( \mathfrak{h} \). Define the metric on \( \mathfrak{h} \) by declaring \( \{X_1, X_2, X_3, X_4, Z_1, Z_2\} \) to be an orthonormal basis. It is straightforward to check that \( \mathfrak{h} \) is of Heisenberg type with two-dimensional center \( \mathfrak{z} = \text{span}\{Z_1, Z_2\} \). Let \( \{a_1, \alpha_2, \alpha_3, \alpha_4, \zeta_1, \zeta_2\} \) be the basis of \( \mathfrak{h}^\ast \) dual to \( \{X_1, X_2, X_3, X_4, Z_1, Z_2\} \). Let \( H \) be the simply connected Lie group with left-invariant metric with metric Lie algebra \( \mathfrak{h} \). As in Lemma 1, any exact, left-invariant 2-form is of the form \( \Omega = d(\zeta_m) \) for some \( \zeta_m \in \mathfrak{z}^\ast \). For simplicity, we take as magnetic field \( \Omega = d(B\zeta_1) \). For each \( \gamma \in H \), we wish to understand the \( \gamma \)-periodic geodesics and the associated periods.

Proceeding as in the Heisenberg case in Sect. 4, let \( p(t) = \sum a_i(t) \alpha_i + \sum c_i(t) \zeta_i \) be the integral curve of the Euler vector field on \( \mathfrak{h}^\ast \) with initial condition \( p(0) = \sum u_i \alpha_i + \sum z_i \zeta_i \). Then, the component functions are

\[
\begin{align*}
  a_1(t) &= u_1 \cos(\hat{z}t) + \left( \frac{-\bar{z}_1 u_2 - \bar{z}_2 u_3}{\hat{z}} \right) \sin(\hat{z}t) \\
  a_2(t) &= u_2 \cos(\hat{z}t) + \left( \frac{\bar{z}_1 u_1 + \bar{z}_2 u_4}{\hat{z}} \right) \sin(\hat{z}t) \\
  a_3(t) &= u_3 \cos(\hat{z}t) + \left( \frac{\bar{z}_2 u_1 - \bar{z}_1 u_4}{\hat{z}} \right) \sin(\hat{z}t) \\
  a_4(t) &= u_4 \cos(\hat{z}t) + \left( \frac{-\bar{z}_2 u_2 + \bar{z}_1 u_3}{\hat{z}} \right) \sin(\hat{z}t) \\
  c_1(t) &= z_1 \\
  c_1(t) &= z_2
\end{align*}
\]

where \( \hat{z} = \sqrt{\bar{z}_1^2 + \bar{z}_2^2} \). Next, let \( \sigma(t) \) be the magnetic geodesic through the identity determined by the integral curve \( p(t) \). Hence, \( \sigma(t) \) solves \( \sigma'(t) = dh_{p(t)} \), where \( h : \mathfrak{h}^\ast \to \mathbb{R} \) is the Hamiltonian (see (17)). Writing \( \sigma(t) = \exp(X(t) + Z(t)) \), where \( X(t) = \sum x_i(t)X_i \) and \( Z(t) = \sum z_i(t)Z_i \), we have on the one hand under trivialization by left-multiplication,
\[\sigma'(t) = X'(t) + Z'(t) + \frac{1}{2} [X'(t), X(t)] = \sum x'_i(t)X_i + Z'(t) + \frac{1}{2} [X'(t), X(t)],\]

and on the other hand

\[dh_{\rho(t)} = \pi(\rho(t) + B\zeta) = \sum a_i(t)X_i + (z_1 + B)Z_1 + z_2Z_2.\]

Matching up the non-central components shows that

\[
\begin{align*}
x_1(t) &= \frac{\mu_1}{\zeta} \sin(\zeta t) + \frac{-z_1 u_2 - z_2 u_3}{\xi^2} (1 - \cos(\zeta t)) \\
x_2(t) &= \frac{\mu_2}{\zeta} \sin(\zeta t) + \frac{z_1 u_1 + z_2 u_4}{\xi^2} (1 - \cos(\zeta t)) \\
x_3(t) &= \frac{\mu_3}{\zeta} \sin(\zeta t) + \frac{z_2 u_1 - z_1 u_4}{\xi^2} (1 - \cos(\zeta t)) \\
x_4(t) &= \frac{\mu_4}{\zeta} \sin(\zeta t) + \frac{-z_2 u_2 + z_1 u_3}{\xi^2} (1 - \cos(\zeta t))
\end{align*}
\]

while the central components satisfies

\[Z'(t) = (z_1 + B)Z_1 + z_2Z_2 - \frac{1}{2} [X'(t), X(t)].\]

A tedious computation shows

\[
[X'(t), X(t)] = \left( -\frac{z_1 \hat{u}^2}{\xi^2} + \frac{z_1 \hat{u}^2}{\xi^2} \cos(\zeta t) \right) Z_1 + \left( -\frac{z_2 \hat{u}^2}{\xi^2} + \frac{z_2 \hat{u}^2}{\xi^2} \cos(\zeta t) \right) Z_2
\]

\[
= -\frac{z_1 \hat{u}^2}{\xi^2}(1 - \cos(\zeta t))Z_1 - \frac{z_2 \hat{u}^2}{\xi^2}(1 - \cos(\zeta t))Z_2
\]

\[
= (1 - \cos(\zeta t)) \left( -\frac{z_1 \hat{u}^2}{\xi^2} Z_1 - \frac{z_2 \hat{u}^2}{\xi^2} Z_2 \right)
\]

where \(\hat{u} = \sqrt{u_1^2 + \cdots + u_4^2}\). A final integration now provides the central components:

\[
\begin{align*}
z_1(t) &= \left( z_1 + B + \frac{z_1 \hat{u}^2}{2\xi^2} \right) t - \frac{z_1 \hat{u}^2}{2\xi^3} \sin(\zeta t) \\
z_2(t) &= \left( z_2 + \frac{z_2 \hat{u}^2}{2\xi^2} \right) t - \frac{z_2 \hat{u}^2}{2\xi^3} \sin(\zeta t).
\end{align*}
\]

Let \(\gamma = \exp(\xi_1 Z_1 + \xi_2 Z_2)\) be a central element of \(H\). Comparing the components of \(\gamma \sigma(t) = \sigma(t + \omega)\) shows that \(\omega = 2\pi k/\xi, k \in \mathbb{Z}\). With this choice of \(\omega\), the non-central components are equal, while the central component yields the system

\[
\begin{align*}
\left( z_1 + B + \frac{z_1 \hat{u}^2}{2\xi^2} \right) \frac{2\pi k}{\zeta} &= \xi_1 \\
\left( z_2 + \frac{z_2 \hat{u}^2}{2\xi^2} \right) \frac{2\pi k}{\zeta} &= \xi_2.
\end{align*}
\]

Each choice of \(u_1, \ldots, u_4, z_1, z_2\) satisfying this system, and the energy constraint
will yield a unit speed magnetic geodesic translated by $\gamma$. In the case that the magnetic field and $\gamma$ are “parallel”, i.e., $\xi_2 = 0$, then (47) becomes

$$z_2 \left( 1 + \frac{\tilde{u}^2}{2z_2^2} \right) \frac{2\pi k}{z} = 0.$$

The second and third factor are necessarily nonzero, so $z_2 = 0$. This reduces (46) to an equation that can be solved in the same way as the Heisenberg case, according to the strength of the magnetic field relative to the energy. When $\gamma$ is an arbitrary element of the center, it is much more difficult to completely solve (46) and (47), and hence obtain an explicit description of all the $\gamma$-periodic geodesics.

Appendix: Tangent bundle viewpoint: periodic magnetic geodesics in Heisenberg manifolds

In the presence of a Riemannian metric, the tangent and cotangent bundles are canonically identified. Structures can be defined equivalently on either vector bundle and computations can be carried out in the more convenient of the two settings. For the bulk of the paper, we found it easier to work on the cotangent bundle and this section demonstrates how the computations would proceed on the tangent bundle.

In [22], A. Kaplan introduced so-called $j$-maps (see Sect. 2.4 for the definition) to study Clifford modules. A metric two-step nilpotent Lie algebra is completely characterized by its associated $j$-maps. Since being introduced, they have proven very useful in the study of two-step nilpotent geometry. In this appendix, we show how the magnetic geodesic equations can be characterized in terms of the $j$-maps.

Let $G$ be a two-step nilpotent Lie group endowed with a left-invariant metric $\g$ and an exact, left-invariant magnetic form $\Omega$. Let $g = \mathfrak{g} \oplus \mathfrak{z}$ be the decomposition of the Lie algebra into the center and its orthogonal complement. By Lemma 1, there is $\zeta_m \in \mathfrak{z}^*$ such that $\Omega = d(B\zeta_m)$.

Lemma 13 The Lorentz force associated with the magnetic field $\Omega$ satisfies $F_\mathfrak{b} = j(-B\zeta_m)$ and $F_\mathfrak{z} = 0$, where $Z_m = \sharp(\zeta_m)$.

Proof Let $X \in \mathfrak{g}$, $V \in \mathfrak{b}$ and $Z \in \mathfrak{z}$. Then,

$$g(F(Z), X) = \Omega(Z, X) = d(B\zeta_m)(Z, X) = -B\zeta_m([Z, X]) = 0,$$

and

$$g(F(V), X) = \Omega(V, X) = d(B\zeta_m)(V, X) = -B\zeta_m([V, X])$$

$$= -Bg(\sharp(\zeta_m), [V, X]) = -Bg(j(Z_m)V, X)$$

$$= g(j(-BZ_m)V, X).$$

Because of Lemma 13, we will write $F = j(-BZ_m)$ with the understanding that $F$ vanishes on central vectors and agrees with $j(-BZ_m)$ on vectors in $\mathfrak{b}$. Let $\gamma(t) = \exp(X(t) + Z(t))$.
be a magnetic geodesic on $G$ where $X(t) \in \mathfrak{v}$ and $Z(t) \in \mathfrak{z}$. By (13), we can express the velocity vector of $\gamma$ as $\gamma'(t) = X'(t) + \frac{1}{2} [X'(t), X(t)] + Z'(t)$. The condition for $\gamma$ to be a magnetic geodesic is $\nabla_{\gamma'(t)} \gamma'(t) = F(\gamma'(t))$. Using (15) to expand this condition and imposing the initial conditions $\gamma(0) = e$ and $\gamma'(0) = X_0 + Z_0$, the geodesic equations on $\mathfrak{v}$ and $\mathfrak{z}$ separately are

$$X''(t) = j(Z_0 - BZ_m)X'(t)$$  \hspace{1cm} (48)

$$Z'(t) + \frac{1}{2} [X'(t), X(t)] = Z_0.$$  \hspace{1cm} (49)

We restrict to the three-dimensional Heisenberg case and consider the magnetic geodesics in this context. Following the approach as illustrated in Prop. 3.5 on pp. 625–628 of [11], and reducing to the three-dimensional Heisenberg case, a straightforward calculation gives the following result.

**Corollary 1** If $z_0 - B = 0$, (or if $z_0 - B \neq 0$ and $x_0 = y_0 = 0$), then $\sigma(t)$ is the one parameter subgroup

$$\sigma(t) = \exp (x(t)X + y(t)Y + z(t)Z) = \exp \left( t \left( x_0X + y_0Y + z_0Z \right) \right).$$ \hspace{1cm} (50)

If $z_0 - B \neq 0$, the solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{A}{z_0 - B} \begin{pmatrix} \sin \left( t \left( \frac{z_0 - B}{A} \right) \right) - \left( 1 - \cos \left( t \left( \frac{z_0 - B}{A} \right) \right) \right) \sin \left( t \left( \frac{z_0 - B}{A} \right) \right) \\ 1 - \cos \left( t \left( \frac{z_0 - B}{A} \right) \right) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$ \hspace{1cm} (51)

and

$$z(t) = \left( z_0 + \frac{A(x_0^2 + y_0^2)}{2(z_0 - B)} \right) t - \frac{A^2(x_0^2 + y_0^2)}{2(z_0 - B)^2} \sin \left( t \left( \frac{z_0 - B}{A} \right) \right).$$ \hspace{1cm} (52)

**Remark 17** The coordinate functions (51) and (52) are equivalent the one obtained in (25)–(27) in the following sense. In order to obtain the magnetic geodesic through the origin determined by $(u_0, v_0, z_0)$ as in Sect. 3 take as initial tangent vector in (51) and (52) to be $(x_0, y_0, z_0) = (u_0/A, v_0/A, z_0 + B)$.

We now present some of the main results about the three-dimensional Heisenberg manifold proved in the body of the paper, but expressed using the tangent bundle, rather than the cotangent bundle.

Continuing the notation from the previous sections, we fix energy $E$, magnetic strength $B$, and metric parameter $A$. Let $(H, g_A, \Omega)$ denote a simply connected Heisenberg manifold. The theorems in this section state precisely the set of periods $\omega$ such that there exists an initial velocity $v_p \in TH$ such that $\sigma_{v_p}(t)$ is periodic with period $\omega$. We also precisely state the set of initial velocities $v_p$, hence the set of geodesics, that produce each period $\omega$.

Let $\Gamma$ denote a cocompact discrete subgroup of $H$ and, as above, denote the resulting compact Heisenberg manifold by $(\Gamma\backslash H, g_A, \Omega)$. For all $\gamma \in \Gamma$, we state below precisely the set of periods $\omega$ such that there exists an initial velocity $v_p \in TH$ such that $\sigma_{v_p}(t)$ is $\gamma$-periodic with period $\omega$. We also precisely state the set of initial velocities $v_p$, hence the set of geodesics, that produce each period $\omega$. 

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Periodic magnetic geodesics on the simply connected Heisenberg group

We now consider the existence of periodic geodesics in \((H, g_A, d(B\zeta))\), the three-dimensional Heisenberg Lie group \(H\) with left-invariant metric determined by the orthonormal basis \(\left\{ \frac{1}{\sqrt{A}} X, \frac{1}{\sqrt{A}} Y, Z \right\}\) and magnetic form \(\Omega = -B\alpha \wedge \beta\). Recall that for a vector \(v \in \mathfrak{h}\), \(\sigma_v(t)\) denotes the magnetic geodesic through the identity with initial velocity \(v\). Note that if \(v_p \in T_pH\), then \(\sigma_{v_p}(t)\) denotes the magnetic geodesic through \(p = \sigma_{v_p}(0)\) with initial velocity \(v_p\). Also note that because \(g_A\) and \(\Omega\) are left-invariant, that \(\sigma_{v_p}(t) = L_p\sigma_{v}(t)\), where \(v_p = L_{p*}(v)\); i.e., magnetic geodesics through \(p \in H\) are just left translations of magnetic geodesics through the identity. Clearly, a magnetic geodesic through \(p \in H\) is periodic with period \(\omega\) if and only if its left translation by \(p^{-1}\) is a magnetic geodesic through the identity with period \(\omega\).

**Theorem 7** With notation as above, fix energy \(E\), magnetic strength \(B\), and metric parameter \(A\).

1. If \(B^2 > E^2\), then there exists a one-parameter family of vectors \(v \in \mathfrak{h}, |v| = E\), such that \(\sigma_v(t)\) is periodic. In particular, \(\sigma_v(t)\) is periodic if and only if \(z_0 = B - \text{sgn}(B)\sqrt{B^2 - E^2}\) and
   \[x_0^2 + y_0^2 = -2z_0(z_0 - B)/A.\]
   The set of periods of \(\sigma_v(t)\) is \(\frac{2\pi}{\sqrt{B^2 - E^2}} \mathbb{Z}_{\neq 0}\), and the smallest positive period is
   \[\omega = \left| \frac{2\pi A}{|z_0 - B|} \right| = \frac{2\pi A}{\sqrt{B^2 - E^2}}.\]
2. If \(B^2 \leq E^2\), then there does not exist a vector \(v\) with \(|v| = E\) such that \(\sigma_v(t)\) is periodic.

Periodic geodesics on compact quotients of the Heisenberg group

We ultimately wish to consider closed magnetic geodesics on Heisenberg manifolds of the form \(\Gamma \backslash H\), where \(\Gamma\) is a cocompact discrete subgroup of \(H\). As above, we proceed by considering periodic magnetic geodesics on the cover \(H\).

The purpose of this section is stating more precisely, and proving, the following, which is divided into several cases. See Theorems 9, 10 and 11 below.

**Theorem 8** Consider the three-dimensional Heisenberg Lie group \(H\) with left invariant metric \(g_A\) determined by the orthonormal basis \(\left\{ \frac{1}{\sqrt{A}} X, \frac{1}{\sqrt{A}} Y, Z \right\}\) and magnetic form \(\Omega = d(B\zeta) = -B\alpha \wedge \beta\). Fix \(\gamma \in H\) and fix energy \(E\), magnetic strength \(B\) and metric parameter \(A\). Then, we can state precisely the set of periods \(\omega\) such that there exists an initial velocity \(v_p \in TH\) with \(|v_p| = E\) such that \(\sigma_{v_p}(t)\) is \(\gamma\)-periodic with period \(\omega\). We can also precisely state the set of initial velocities \(v_p\), hence, the set of geodesics, that produce each period \(\omega\).
Noncentral case

Let \( \gamma = \exp (x \gamma X + y \gamma Y + z \gamma Z) \in H \) with \( x^2 + y^2 \neq 0 \). Let \( a = \exp (a \gamma X + a \gamma Y + a \gamma Z) \in H \). From (12), the conjugacy class of \( \gamma \) in \( H \) is \( \exp (x \gamma X + y \gamma Y + \mathbb{R}Z) \).

**Theorem 9** Fix energy \( E \), magnetic strength \( B \) and metric parameter \( A \). Let \( \gamma = \exp (x \gamma X + y \gamma Y + z \gamma Z) \in H \) with \( x^2 + y^2 \neq 0 \).

1. If \( E^2 > B^2 \) (i.e., if \( \mu > 1 \)), then there exists a two-parameter family of elements \( a \in H \) such that \( ay a^{-1} = \exp \left( x \gamma X + y \gamma Y + z_{\gamma}' Z \right) \) where \( z_{\gamma}' = \pm B \sqrt{A (x^2 + y^2) / (E^2 - B^2)} \).

   Letting \( v = B \frac{x\gamma}{z_{\gamma}'} \left( x \gamma X + y \gamma Y + z_{\gamma}' Z \right) \), which satisfies \( |v| = E \), then \( \gamma \) translates the (non-spiraling) magnetic geodesic \( a^{-1} \exp (iv) \) with period \( \omega = \pm \sqrt{A (x^2 + y^2) / (E^2 - B^2)} \).

   These are the only magnetic geodesics with energy \( E \) translated by \( \gamma \).

2. If \( E^2 \leq B^2 \) (i.e., if \( \mu \leq 1 \)), then neither \( \gamma \) nor any of its conjugates in \( H \) translate a magnetic geodesic with energy \( E \).

Central case

Throughout this subsection, we assume that \( x \gamma = y \gamma = 0 \); i.e., that \( \gamma \) lies in \( Z(H) \), the center of three-dimensional Heisenberg group \( H \). Recall that since \( \gamma \) is central, \( \gamma \) translates a magnetic geodesic \( \sigma(t) \) through the identity \( e \in H \) with period \( \omega \) if and only if for all \( a \in H \), \( \gamma \) translates a magnetic geodesic through \( a \) with period \( \omega \). That is, without loss of generality, if \( \gamma \) lies in the center, we may assume that \( \sigma(0) = e \).

Recall that magnetic geodesics in \( H \) are either spiraling or one parameter subgroups. We first consider the case of one-parameter subgroups.

**Theorem 10** Fix energy \( E \), magnetic strength \( B \) and metric parameter \( A \). Let \( \gamma = \exp (z \gamma Z) \in Z(H) \), with \( z \gamma \neq 0 \). The element \( \gamma \) translates the magnetic geodesics \( \sigma(t) = \exp (\pm tEZ) \) with initial velocities \( v = \pm E \) and periods \( \omega = \pm z \gamma / E \). This pair of one-parameter subgroups and their left translates are the only straight magnetic geodesics translated by \( \gamma \).

**Theorem 11** Fix energy \( E \), magnetic strength \( B \), and metric parameter \( A \). Denote \( \mu = \frac{E}{|B|} \). Let \( \gamma = \exp (z \gamma Z) \in Z(H) \), \( z \gamma \neq 0 \). If there exists a vector \( v = x_0X + y_0Y + z_0Z \) and a period \( \omega \neq 0 \) such that the spiraling geodesic \( \sigma(v)(t) \) is \( \gamma \)-periodic with period \( \omega \), then there exists \( \ell \in Z_{\neq 0} \) such that \( \gamma_{\ell} = \frac{z \gamma}{\omega} \) satisfies the conditions relative to \( \mu \) specified in the following six cases and \( A (x_0^2 + y_0^2) / z_0 \) and \( \omega \) are as expressed below. Conversely, for every choice of \( \ell \in Z_{\neq 0} \) such that \( \gamma_{\ell} = \frac{z \gamma}{\omega} \) satisfies the conditions in one of the cases below, there exists at least one vector \( v \) as given below such that \( \sigma(v)(t) \) is \( \gamma \)-periodic (spiraling) geodesic with period \( \omega \) as given below. Note that Case 1 requires \( E^2 < B^2 \). Cases 2 through 5 require \( E^2 > B^2 \), and Case 6 requires \( E^2 = B^2 \). Note that in all cases, \( \zeta_{\ell} \neq 0 \).

1. \( -2\mu / (1 - \mu) < \frac{\zeta_{\ell}}{A} < \frac{2\mu}{1 + \mu} < 1 \),
In Cases 1 through 4, we choose any \( x_0, y_0 \in \mathbb{R} \) so that

\[
A(x_0^2 + y_0^2) = B^2 \left( \frac{\mu^2 - 1}{A} - 1 \left( \frac{\zeta}{A} - 2 \right) + 2 \sqrt{\frac{\mu^2 - 1}{A} - 1} \right)
\]

and let

\[
z_0 = -B \left( -1 + \sqrt{\frac{\mu^2 - 1}{A} - 1} \right)
\]

and

\[
\omega = \frac{2\zeta A}{\zeta (z_0 - B)} = \frac{2\pi \ell'}{\sqrt{E^2 - B^2}}.
\]

In Case 4, we may also choose any \( x_0, y_0 \in \mathbb{R} \) so that

\[
A(x_0^2 + y_0^2) = B^2 \left( \frac{\mu^2 - 1}{A} - 1 \left( \frac{\zeta}{A} - 2 \right) - 2 \sqrt{\frac{\mu^2 - 1}{A} - 1} \right)
\]

and let

\[
z_0 = -B \left( -1 - \sqrt{\frac{\mu^2 - 1}{A} - 1} \right)
\]

and

\[
\omega = \frac{2\zeta A}{\zeta (z_0 - B)} = -\frac{2\pi \ell'}{\sqrt{E^2 - B^2}}.
\]

The conditions on \( \mu, \zeta, x_0, y_0 \) and \( z_0 \) imply \( \frac{\mu^2 - 1}{A} > 0, x_0^2 + y_0^2 > 0, E^2 = A(x_0^2 + y_0^2) + z_0^2 \), and the (spiraling) magnetic geodesic through the identity \( \sigma(t) \) with initial velocity \( v = x_0 X + y_0 Y + z_0 Z \) is \( \gamma = \exp \left( z_0 Z \right) \)-periodic with energy \( E \) and period \( \omega \) as given.

In Case 5, which only occurs if \( \frac{\zeta}{A} \in 2\pi \mathbb{Z} \neq 0 \), we choose any \( x_0, y_0 \in \mathbb{R} \) so that

\[
A(x_0^2 + y_0^2) = 2|B|\sqrt{E^2 - B^2}
\]

and
\[ z_0 = B - \frac{A(x_0^2 + y_0^2)}{2B}. \]

Then, the conditions on \( \mu, \zeta_\ell, x_0, y_0 \) and \( z_0 \) imply that \( E^2 = A(x_0^2 + y_0^2) + z_0^2 \) and the (spiraling) magnetic geodesic \( \sigma_\gamma(t) \) starting at the identity with initial velocity \( v = x_0X + y_0Y + z_0Z \) is \( \gamma \)-periodic with energy \( E \) and period
\[ \omega = -\text{sgn}(B) \frac{z_\gamma}{\sqrt{E^2 - B^2}}. \]

In Case 6, which only occurs if \( \frac{z_\gamma}{A} \in \pi \mathbb{Z} \neq 0 \), we choose any \( x_0, y_0, z_0 \in \mathbb{R} \) so that \( E^2 = B^2 = A(x_0^2 + y_0^2) + z_0^2 \) and \( z_0 \neq \pm B \). The conditions on \( \mu \) and \( \zeta_\ell \) imply that the (spiraling) magnetic geodesic \( \sigma_\gamma(t) \) with initial velocity \( v = x_0X + y_0Y + z_0Z \) will yield a \( \gamma \)-periodic magnetic geodesic with energy \( E \) and period
\[ \omega = \frac{2z_\gamma}{z_0 - B}. \]

Remark 18 In Case 1, there are infinitely many values of \( \ell \) that satisfy the conditions, hence infinitely many distinct periods \( \omega \). In particular, if \( \mu < 1 \) and there exists \( \ell_0 \in \mathbb{Z}_{>0} \) such that \( \zeta_{\ell_0} \in \left( \frac{-2\mu}{1-\mu}, \frac{2\mu}{1+\mu} \right) \), then for all \( \ell > \ell_0 \), \( \zeta_\ell \in \left( \frac{-2\mu}{1-\mu}, \frac{2\mu}{1+\mu} \right) \). Likewise if there exists \( \ell_0 \in \mathbb{Z}_{<0} \) such that \( \zeta_{\ell_0} \in \left( \frac{-2\mu}{1-\mu}, \frac{2\mu}{1+\mu} \right) \), then for all \( \ell < \ell_0 \), \( \zeta_\ell \in \left( \frac{-2\mu}{1-\mu}, \frac{2\mu}{1+\mu} \right) \).

Remark 19 In Case 6, the magnitude of the periods takes all values in the interval \( \left( \frac{|z_\gamma|}{E}, \infty \right) \). The period \( \omega = \frac{|z_\gamma|}{E} \) is achieved when \( v = -BZ \), which implies \( \sigma_\gamma \) is a one-parameter subgroup; i.e., non-spiraling. The magnitude of the period approaches \( \infty \) as \( v \to BZ \). This behavior is in contrast to the Riemannian case; i.e., the case \( B = 0 \). In the Riemannian case, there are finitely many periods associated with each element \( \gamma \). However, if there exists \( \gamma \in \Gamma \) such that \( \log \gamma \in 2\pi \mathbb{Z} \), then \( \Gamma \backslash H \) does not satisfy the Clean Intersection Hypothesis, so the fact that unusual magnetic geodesic behavior occurs in this case is not unprecedented (see [15]).

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