Quantum walk in terms of quantum Bernoulli noises

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Abstract Quantum Bernoulli noises are the family of annihilation and creation operators acting on Bernoulli functionals, which satisfy a canonical anti-commutation relation in equal time. In this paper, we first present some new results concerning quantum Bernoulli noises, which themselves are interesting. Then, based on these new results, we construct a time-dependent quantum walk with infinitely many degrees of freedom. We prove that the walk has a unitary representation and hence belongs to the category of the so-called unitary quantum walks. We examine its distribution property at the vacuum initial state and some other initial states and find that it has the same limit distribution as the classical random walk, which contrasts sharply with the case of the usual quantum walks with finite degrees of freedom.

Keywords Quantum Bernoulli noise · Quantum walk · Infinitely many degrees of freedom · Limit theorem

Mathematics Subject Classification 81S25 · 60G50 · 81P68

1 Introduction

Quantum walks are quantum analogs of the classical random walk, which have found wide application in quantum information, quantum computing and many other fields [2].
Typically, a quantum walk can be described by an appropriate evolution sequence \( (\Psi_n)_{n \geq 0} \) in a space of form \( \mathcal{L}^2 (\mathbb{Z}^d) \otimes \mathcal{H} \), where \( \mathcal{H} \) is a Hilbert space standing for the internal degrees of freedom of the walk. When the dimension of \( \mathcal{H} \) is finite, the walk is said to have finite degrees of freedom. Otherwise, it is said to have infinitely many degrees of freedom. In the literature, quantum walks with \( k \) degrees of freedom are also known as \( k \)-state quantum walks.

In the past 15 years, quantum walks with finite degrees of freedom (especially those with 2 or 3 degrees of freedom) have been intensively studied and many deep results have been obtained of them (see [1,4,5,7,10] and references therein). One typical result in this aspect is that a quantum walk with finite degrees of freedom usually has a limit distribution with scaling speed \( n \), instead of \( \sqrt{n} \), which is far from Gaussian [5].

From a theoretical point of view, quantum walks with infinitely many degrees of freedom are also of interest. However, little attention has been paid to these walks. In this paper, we consider a type of quantum walk with infinitely many degrees of freedom.

Quantum Bernoulli noises are the family of annihilation and creation operators acting on Bernoulli functionals [8], which can describe a two-level quantum system with infinitely many sites. They satisfy a canonical anti-commutation relation (CAR) in equal time and admit some other interesting properties as well [9]. In this paper, we first present some new results concerning quantum Bernoulli noises, which themselves are also interesting. Then, based on these new results, we construct a time-dependent quantum walk with infinitely many degrees of freedom. We prove that the walk has a unitary representation and hence belongs to the category of the so-called unitary quantum walks. We examine its distribution property at the vacuum initial state and some other initial states and find that it has the same limit distribution as the classical random walk, which contrasts sharply with the case of the usual quantum walks with finite degrees of freedom (e.g., the Hadamard walk [3]).

**Notation and conventions** Throughout, \( \mathbb{Z} \) always denotes the set of all integers, while \( \mathbb{N} \) means the set of all nonnegative integers. We denote by \( \Gamma \) the finite power set of \( \mathbb{N} \), namely

\[
\Gamma = \{ \sigma \mid \sigma \subset \mathbb{N} \text{ and } \# \sigma < \infty \},
\]

where \( \# \sigma \) means the cardinality of \( \sigma \). Unless otherwise stated, letters like \( j, k \) and \( n \) stand for nonnegative integers, namely elements of \( \mathbb{N} \).

## 2 Quantum Bernoulli noises

In this section, we briefly recall main notions and facts about quantum Bernoulli noises. We refer to [8] for details.

Let \( \Omega = \{-1, 1\}^\mathbb{N} \) be the set of all mappings \( \omega: \mathbb{N} \mapsto \{-1, 1\} \), and \( (\zeta_n)_{n \geq 0} \) the sequence of canonical projections on \( \Omega \) given by

\[
\zeta_n (\omega) = \omega(n), \quad \omega \in \Omega.
\]
Denote by \( \mathcal{F} \) the \( \sigma \)-field on \( \Omega \) generated by the sequence \((\zeta_n)_{n \geq 0}\). Let \((p_n)_{n \geq 0}\) be a given sequence of positive numbers with the property that \(0 < p_n < 1\) for all \(n \geq 0\).

It is known [6] that there exists a unique probability measure \(P\) on \(\mathcal{F}\) such that

\[
\mathbb{P} \circ (\zeta_1, \zeta_2, \ldots, \zeta_n)^{-1} \{ (e_1, e_2, \ldots, e_k) \} = \prod_{j=1}^{k} p_j^{1+e_j} q_j^{1-e_j}.
\]  

(2.2)

for \(n_j \in \mathbb{N}, e_j \in \{-1, 1\} (1 \leq j \leq k)\) with \(n_i \neq n_j\) when \(i \neq j\) and \(k \in \mathbb{N}\) with \(k \geq 1\). Thus, we come to a probability measure space \((\Omega, \mathcal{F}, \mathbb{P})\), which is referred to as the Bernoulli space and random variables on it are known as Bernoulli functionals.

Let \(Z = (Z_n)_{n \geq 0}\) be the sequence of random variables defined by

\[
Z_n = \frac{\zeta_n + q_n - p_n}{2 \sqrt{p_n q_n}}, \quad n \geq 0.
\]  

(2.3)

where \(q_n = 1 - p_n\). Clearly \(Z = (Z_n)_{n \geq 0}\) is an independent sequence of random variables on \((\Omega, \mathcal{F}, \mathbb{P})\), and, for each \(n \geq 0\), \(Z_n\) has a distribution

\[
\mathbb{P}\{Z_n = \theta_n\} = p_n, \quad \mathbb{P}\{Z_n = -1/\theta_n\} = q_n, \quad n \geq 0
\]  

(2.4)

with \(\theta_n = \sqrt{q_n/p_n}\).

To be convenient, we set \(\mathcal{F}_{-1} = \emptyset, \Omega\) and \(\mathcal{F}_n = \sigma(Z_k; 0 \leq k \leq n)\), the \(\sigma\)-field generated by \((Z_k)_{0 \leq k \leq n}\), for \(n \geq 0\). By convention \(\mathbb{E}\) will denote the expectation with respect to \(\mathbb{P}\).

Let \(L^2(\Omega)\) be the space of square integrable complex-valued random variables on \((\Omega, \mathcal{F}, \mathbb{P})\), namely

\[
L^2(\Omega) = L^2(\Omega, \mathcal{F}, \mathbb{P}).
\]  

(2.5)

We denote by \(\langle \cdot, \cdot \rangle\) the usual inner product of the space \(L^2(\Omega)\) and by \(\| \cdot \|\) the corresponding norm. It is known [6] that \(Z\) has the chaotic representation property. Thus, \(L^2(\Omega)\) has \(\{Z_\sigma | \sigma \in \Gamma\}\) as its orthonormal basis, where \(Z_\emptyset = 1\) and

\[
Z_\sigma = \prod_{j \in \sigma} Z_j, \quad \sigma \in \Gamma, \sigma \neq \emptyset,
\]  

(2.6)

which shows that \(L^2(\Omega)\) is an infinite dimensional Hilbert space.

Lemma 2.1 [8] For each \(k \in \mathbb{N}\), there exists a bounded operator \(\partial_k\) on \(L^2(\Omega)\) such that

\[
\partial_k Z_\sigma = 1_\sigma(k) Z_\sigma \setminus k, \quad \sigma \in \Gamma,
\]  

(2.7)

where \(\sigma \setminus k = \sigma \setminus \{k\}\) and \(1_\sigma(k)\) the indicator of \(\sigma\) as a subset of \(\mathbb{N}\).

Lemma 2.2 [8] Let \(k \in \mathbb{N}\). Then, \(\partial_k^*\), the adjoint of operator \(\partial_k\), has following property:

\[
\partial_k^* Z_\sigma = [1 - 1_\sigma(k)] Z_{\sigma \cup k}, \quad \sigma \in \Gamma,
\]  

(2.8)

where \(\sigma \cup k = \sigma \cup \{k\}\).
The operator $\partial_k$ and its adjoint $\partial_k^*$ are referred to as the annihilation operator and creation operator at site $k$, respectively.

**Definition 2.1** [8] The family $\{\partial_k, \partial_k^*\}_{k \geq 0}$ of annihilation and creation operators is called quantum Bernoulli noises.

The next lemma shows that quantum Bernoulli noises satisfy the canonical anti-commutation relations (CAR) in equal time.

**Lemma 2.3** [8] Let $k, l \in \mathbb{N}$. Then, it holds true that

$$\partial_k \partial_l = \partial_l \partial_k, \quad \partial_k^* \partial_l^* = \partial_l^* \partial_k^*$$

$k \neq l$ \hspace{1cm} (2.9)

and

$$\partial_k \partial_k^* = \partial_l \partial_l^* = 0, \quad \partial_k \partial_k + \partial_k^* \partial_k = I,$$ \hspace{1cm} (2.10)

where $I$ is the identity operator on $L^2(\Omega)$.

### 3 New results concerning quantum Bernoulli noises

In this section, we prove some new results concerning quantum Bernoulli noises, which will play a fundamental role in constructing our quantum walk.

For brevity, we write $\mathcal{H} = L^2(\Omega)$ and denote by $B(\mathcal{H})$ the Banach algebra of all bounded operators on $\mathcal{H}$.

Recall that $\mathcal{H}$ has an orthonormal basis $\{Z_\sigma \mid \sigma \in \Gamma\}$. Thus, for each $n \geq 0$, we can naturally introduce a subspace of $\mathcal{H}$:

$$\mathcal{H}_n = \text{span}\{Z_\sigma \mid \max \sigma \leq n, \ \sigma \in \Gamma\}, \hspace{1cm} (3.1)$$

namely the one spanned by $\{Z_\sigma \mid \max \sigma \leq n, \ \sigma \in \Gamma\}$. We additionally write $\mathcal{H}_{-1} = \text{span}\{Z_\emptyset\}$, which is actually the same as $\mathbb{C}$. It can be easily checked that the dimension of $\mathcal{H}_n$ is just $2^{n+1}$.

For $n \geq 0$, we also introduce two self-adjoint operators $R_n, L_n \in B(\mathcal{H})$ as follows:

$$R_n = \frac{1}{2}(\partial_n^* + \partial_n + I), \quad L_n = \frac{1}{2}(\partial_n^* + \partial_n - I), \hspace{1cm} (3.2)$$

where $I$ is the identity operator on $\mathcal{H}$.

**Theorem 3.1** For each $n \geq 0$, $R_n + L_n$ is a unitary operator on $\mathcal{H}$ and moreover

$$R_n^2 = R_n, \quad R_n L_n = L_n R_n = 0, \quad L_n^2 = -L_n.$$ \hspace{1cm} (3.3)

**Proof** It is easy to see that $R_n + L_n = \partial_n^* + \partial_n$. Thus by Lemma 2.3, we have \[(R_n + L_n)^* (R_n + L_n) = (R_n + L_n) (R_n + L_n)^* = (\partial_n^* + \partial_n)^2 = I,\]

namely $R_n + L_n$ is a unitary operator. Similarly, we can prove the three formulas in (3.3). \qed
Theorem 3.2 Let \( n \geq 0 \). Then, for all \( \xi \in \mathcal{H}_{n-1} \) one has

\[
\| R_n \xi \|^2 = \| L_n \xi \|^2 = \frac{1}{2} \| \xi \|^2. \tag{3.4}
\]

**Proof** Let \( \xi \in \mathcal{H}_{n-1} \). Then, we have the following expansion

\[
\xi = \sum_{\sigma \in \Gamma_{n-1}} (Z_{\sigma}, \xi) Z_{\sigma},
\]

where \( \Gamma_{n-1} = \{ \sigma \in \Gamma \mid \max \sigma \leq n - 1 \} \). In view of the fact that \( n \notin \sigma \) for all \( \sigma \in \Gamma_{n-1} \), we get

\[
R_n \xi = \frac{1}{2} \left( \partial_n^* + \partial_n + I \right) \xi
\]

\[
= \frac{1}{2} \left[ \sum_{\sigma \in \Gamma_{n-1}} (Z_{\sigma}, \xi) Z_{\sigma \cap n} + \sum_{\sigma \in \Gamma_{n-1}} (Z_{\sigma}, \xi) Z_{\sigma} \right],
\]

which implies that

\[
\| R_n \xi \|^2 = \frac{1}{2} \sum_{\sigma \in \Gamma_{n-1}} |(Z_{\sigma}, \xi)|^2 = \frac{1}{2} \| \xi \|^2.
\]

Similarly, we can verify \( \| L_n \xi \|^2 = \frac{1}{2} \| \xi \|^2 \). \( \square \)

Let \( l^2(\mathbb{Z}, \mathcal{H}) \) be the space of square summable functions defined on \( \mathbb{Z} \) and valued in \( \mathcal{H} \), namely

\[
l^2(\mathbb{Z}, \mathcal{H}) = \left\{ \Phi : \mathbb{Z} \to \mathcal{H} \mid \sum_{x=-\infty}^{\infty} \| \Phi(x) \|^2 < \infty \right\}. \tag{3.5}
\]

It is known that \( l^2(\mathbb{Z}, \mathcal{H}) \) is a complex Hilbert space with the usual inner product induced by that of \( \mathcal{H} \).

**Theorem 3.3** Let \( n \geq 0 \) and \( \Phi \in l^2(\mathbb{Z}, \mathcal{H}) \). If a function \( \Psi : \mathbb{Z} \to \mathcal{H} \) satisfies condition

\[
\Psi(x) = R_n \Phi(x - 1) + L_n \Phi(x + 1), \quad x \in \mathbb{Z}, \tag{3.6}
\]

then \( \Psi \in l^2(\mathbb{Z}, \mathcal{H}) \) and moreover \( \| \Psi \|_{l^2(\mathbb{Z}, \mathcal{H})} = \| \Phi \|_{l^2(\mathbb{Z}, \mathcal{H})} \).
Proof It follows from formula $R_n L_n = L_n R_n = 0$ and the unitary property of operator $R_n + L_n$ that

$$
\sum_{x=-\infty}^{\infty} \| \Psi(x) \|^2 = \sum_{x=-\infty}^{\infty} \| R_n \Phi(x-1) \|^2 + \sum_{x=-\infty}^{\infty} \| L_n \Phi(x+1) \|^2
$$

$$
= \sum_{x=-\infty}^{\infty} \| R_n \Phi(x) \|^2 + \sum_{x=-\infty}^{\infty} \| L_n \Phi(x) \|^2
$$

$$
= \sum_{x=-\infty}^{\infty} \| (R_n + L_n) \Phi(x) \|^2
$$

$$
= \sum_{x=-\infty}^{\infty} \| \Phi(x) \|^2,
$$

which together with the assumption $\Phi \in l^2(\mathbb{Z}, \mathcal{H})$ means that $\Psi \in l^2(\mathbb{Z}, \mathcal{H})$ and $\| \Psi \|_{l^2(\mathbb{Z}, \mathcal{H})} = \| \Phi \|_{l^2(\mathbb{Z}, \mathcal{H})}$ as well. $\square$

**Theorem 3.4** Let $n \geq 0$. Then, there exists a unitary operator $U_n : l^2(\mathbb{Z}, \mathcal{H}) \to l^2(\mathbb{Z}, \mathcal{H})$ such that

$$
[U_n \Phi](x) = R_n \Phi(x-1) + L_n \Phi(x+1), \quad x \in \mathbb{Z}, \quad \Phi \in l^2(\mathbb{Z}, \mathcal{H}). \tag{3.7}
$$

Moreover, the adjoint $U_n^*$ of $U_n$ satisfies

$$
[U_n^* \Phi](x) = R_n \Phi(x+1) + L_n \Phi(x-1), \quad x \in \mathbb{Z}, \quad \Phi \in l^2(\mathbb{Z}, \mathcal{H}). \tag{3.8}
$$

Proof For each $\Phi \in l^2(\mathbb{Z}, \mathcal{H})$, we define the function $\Psi_\Phi : \mathbb{Z} \to \mathcal{H}$ as

$$
\Psi_\Phi(x) = R_n \Phi(x-1) + L_n \Phi(x+1), \quad x \in \mathbb{Z},
$$

which, by Theorem 3.3, still belongs to $l^2(\mathbb{Z}, \mathcal{H})$ and satisfies

$$
\| \Psi_\Phi \|_{l^2(\mathbb{Z}, \mathcal{H})} = \| \Phi \|_{l^2(\mathbb{Z}, \mathcal{H})}.
$$

Thus, we have an isometric operator $U_n : l^2(\mathbb{Z}, \mathcal{H}) \to l^2(\mathbb{Z}, \mathcal{H})$ such that

$$
U_n \Phi = \Psi_\Phi, \quad \Phi \in l^2(\mathbb{Z}, \mathcal{H}),
$$

namely $U_n$ satisfies (3.7).

We now consider the adjoint $U_n^*$ of $U_n$. Let $\Phi \in l^2(\mathbb{Z}, \mathcal{H})$. Then, for any $x \in \mathbb{Z}$ and $\sigma \in \Gamma$, there exists a function $\Phi^{(\sigma)} \in l^2(\mathbb{Z}, \mathcal{H})$ such that

$$
\Phi^{(\sigma)}(y) = \begin{cases} 
Z_{\sigma}, & y = x; \\
0, & y \neq x, \ y \in \mathbb{Z},
\end{cases}
$$

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which gives

\[ \langle U_n^* \Phi, \Phi^{(\sigma)}(\cdot) \rangle_{l^2(\mathbb{Z}, \mathcal{H})} = (\Phi, U_n \Phi^{(\sigma)}(\cdot))_{l^2(\mathbb{Z}, \mathcal{H})} = (\Phi(x + 1), R_n Z_{\sigma}) + (\Phi(x - 1), L_n Z_{\sigma}) = (R_n \Phi(x + 1) + L_n \Phi(x - 1), Z_{\sigma}). \]

Thus, we have (3.8).

Finally, we prove that \( U_n \) is a unitary operator. Since \( U_n \) is an isometric operator, we need only to show that \( U_n U_n^* = I \), where \( I \) denotes the identity operator on \( l^2(\mathbb{Z}, \mathcal{H}) \).

Let \( \Phi^{(0)} \in l^2(\mathbb{Z}, \mathcal{H}) \) and \( \Psi = U_n^* \Phi^{(0)} \). Then, by using (3.7), (3.8) as well as Theorem 3.1, we obtain that

\[ [U_n \Psi](x) = R_n \Psi(x - 1) + L_n \Psi(x + 1) = R_n^2 \Phi(x) + R_n L_n \Psi(x - 2) + L_n R_n \Psi(x + 2) + L_n^2 \Phi(x) = R_n \Phi(x) - L_n \Phi(x) = \Phi(x), \]

where \( x \in \mathbb{Z} \). Thus, \( U_n \Psi = \Phi \), which together with the arbitrariness of \( \Phi \in l^2(\mathbb{Z}, \mathcal{H}) \) implies that \( U_n U_n^* = I \). \( \square \)

4 Quantum walk constructed from quantum Bernoulli noises

In the present section, we introduce our model of quantum walk and examine its distribution property.

We take \( l^2(\mathbb{Z}, \mathcal{H}) \) as the state space of the walk. It is known that \( l^2(\mathbb{Z}) \otimes \mathcal{H} = l^2(\mathbb{Z}, \mathcal{H}) \). Thus, just as in the case of finite degrees of freedom, here \( l^2(\mathbb{Z}) \) stands for the position space of the walk, while \( \mathcal{H} \) stands for the space describing the internal degree of freedom of the walk.

As usual, we call \( \Phi \in l^2(\mathbb{Z}, \mathcal{H}) \) a state if it satisfies the normalized condition \( \| \Phi \|_{l^2(\mathbb{Z}, \mathcal{H})} = 1 \). Define function \( \phi: \mathbb{Z} \to \mathcal{H} \) as

\[ \phi(x) = \begin{cases} Z_\emptyset, & x = 0; \\ 0, & x \neq 0, x \in \mathbb{Z}. \end{cases} \quad (4.1) \]

Obviously \( \phi \in l^2(\mathbb{Z}, \mathcal{H}) \) and \( \| \phi \|_{l^2(\mathbb{Z}, \mathcal{H})} = 1 \), which means that \( \phi \) is a state. We call \( \phi \) the vacuum state in \( l^2(\mathbb{Z}, \mathcal{H}) \).

Let \( \Phi_n \in l^2(\mathbb{Z}, \mathcal{H}) \) be the state of the walk at time \( n \geq 0 \). Then, for \( x \in \mathbb{Z} \), \( \Phi_n(x) \) just means the amplitude of the walker at position \( x \) and time \( n \). We assume that the walk starts form the initial state \( \Phi_0 \) and its time evolution is governed by equation

\[ \Phi_{n+1}(x) = R_n \Phi_n(x - 1) + L_n \Phi_n(x + 1), \quad x \in \mathbb{Z}, \ n \geq 0. \quad (4.2) \]
Note that (4.2) does make sense mathematically since by Theorem 3.3 the function on the right-hand side of (4.2) remains a state in $l^2(\mathbb{Z}, \mathcal{H})$.

Clearly, for each $n \geq 0$, the function $x \mapsto \|\Phi_n(x)\|^2$ makes a probability distribution on $\mathbb{Z}$. Thus, the probability that the quantum walker is at position $x \in \mathbb{Z}$ at time $n \geq 0$ starting from the initial state $\Phi_0$ is naturally defined to be

$$P\{X_n = x\} = \|\Phi_n(x)\|^2. \quad (4.3)$$

Remark 4.1 As shown in Sect. 2, the dimension of $\mathcal{H} = L^2(\Omega)$ is infinite, which means that our quantum walk has infinitely many degrees of freedom. Additionally, evolution Eq. (4.2) implies that the walk is also time-dependent.

The next theorem is an immediate consequence of Theorem 3.4, which shows that the walk belongs to the category of the so-called unitary quantum walks.

**Theorem 4.1** The quantum walk has a unitary representation, more precisely

$$\Phi_n = \left( \prod_{k=0}^{n-1} U_k \right) \Phi_0, \quad n \geq 1, \quad (4.4)$$

where $U_k : l^2(\mathbb{Z}, \mathcal{H}) \to l^2(\mathbb{Z}, \mathcal{H})$ is the unitary operator defined by (3.7).

**Proof** According to (4.2) and (3.7), we have

$$\Phi_n(x) = [U_{n-1} \Phi_{n-1}](x), \quad x \in \mathbb{Z}, \quad n \geq 1,$$

which implies (4.4). \qed

**Theorem 4.2** Let the initial state $\Phi_0 \in l^2(\mathbb{Z}, \mathcal{H})$ be such that $\Phi_0(x) \in \mathcal{H}_{n-1}$ for all $x \in \mathbb{Z}$. Then,

$$\{\Phi_n(x) \mid x \in \mathbb{Z}\} \subset \mathcal{H}_{n-1}, \quad n \geq 1, \quad (4.5)$$

where $\mathcal{H}_{n-1}$ is the subspace as defined by (3.1).

**Proof** We first show that for each $k \geq 0$, $\mathcal{H}_k$ is the invariant subspace of both $R_k$ and $L_k$. In fact, for any $\sigma \in \{\tau \in \Gamma \mid \max \tau \leq k\}$, we have

$$R_k Z_\sigma = \frac{1}{2} (\partial_k^* + \partial_k + 1) Z_\sigma$$

$$= \frac{1}{2} [(1 - 1_\sigma(k)) Z_{\sigma \cup k} + 1_\sigma(k) Z_{\sigma \backslash k} + Z_\sigma],$$

which clearly belongs to $\mathcal{H}_k$. Thus, by the definition of $\mathcal{H}_k$, we find $R_k(\mathcal{H}_k) \subset \mathcal{H}_k$.

In the same way, we can show $L_k(\mathcal{H}_k) \subset \mathcal{H}_k$.

We now verify (4.5). Clearly it holds for $n = 0$. Since $\mathcal{H}_{-1} \subset \mathcal{H}_0$, we have $\{\Phi_0(x) \mid x \in \mathbb{Z}\} \subset \mathcal{H}_0$, which together with the following invariance

$$R_0(\mathcal{H}_0) \subset \mathcal{H}_0, \quad L_0(\mathcal{H}_0) \subset \mathcal{H}_0$$

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implies that

\[ \Phi_1(x) = R_0 \Phi_0(x - 1) + L_0 \Phi_0(x + 1) \in \mathcal{H}_0, \quad x \in \mathbb{Z}. \]

In other words, (4.5) also holds for \( n = 1 \). It then follows from induction that (4.5) holds for all \( n \geq 0 \).

We now turn to examine distribution properties of the walk. Recall that \( \phi \) denotes the vacuum state in \( l^2(\mathbb{Z}, \mathcal{H}) \) [for its definition see (4.1)]. The next theorem offers the probability distribution of the walk at the vacuum initial state.

**Theorem 4.3** Let the initial state \( \Phi_0 \in l^2(\mathbb{Z}, \mathcal{H}) \) be such that \( \Phi_0 = \phi \). Then, for all \( n \geq 1 \) one has

\[
P\{X_n = x\} = \|\Phi_n(x)\|^2 = \begin{cases} \frac{1}{2^n} \binom{n}{j}, & x = 2j - n, 0 \leq j \leq n; \\ 0, & \text{otherwise.} \end{cases} \tag{4.6}
\]

**Proof** We use the method of induction. Clearly \( \Phi_0(x) = \phi(x) \in \mathcal{H}_{-1} \) for all \( x \in \mathbb{Z} \).

When \( n = 1 \), we have \( \Phi_1(x) = R_0 \phi(x - 1) + L_0 \phi(x + 1) \) for all \( x \in \mathbb{Z} \), which gives \( \Phi_1(x) = 0 \) for all \( x \in \mathbb{Z} \setminus \{-1, 1\} \) and

\[ \Phi_1(-1) = L_0 \phi(0) = \frac{1}{2}(Z_0 - Z_\emptyset), \quad \Phi_1(1) = R_0 \phi(0) = \frac{1}{2}(Z_0 + Z_\emptyset), \]

which implies that \( P\{X_1 = x\} = 0 \) for all \( x \in \mathbb{Z} \setminus \{-1, 1\} \) and

\[ P\{X_1 = -1\} = \frac{1}{4}\|Z_0 - Z_\emptyset\|^2 = \frac{1}{2}, \quad P\{X_1 = 1\} = \frac{1}{4}\|Z_0 + Z_\emptyset\|^2 = \frac{1}{2}. \]

Thus, the theorem holds true for \( n = 1 \).

Now assume that the theorem holds true for \( n = k \) with \( k \geq 1 \). Then, for \( n = k + 1 \), by using Theorems 3.1, 3.2 and 4.2, we have

\[
\|\Phi_{k+1}(x)\|^2 = \|R_k \Phi_k(x - 1)\|^2 + \|L_k \Phi_k(x + 1)\|^2 \\
= \frac{1}{2}\|\Phi_k(x - 1)\|^2 + \frac{1}{2}\|\Phi_k(x + 1)\|^2, \quad x \in \mathbb{Z},
\]

which implies that

\[ P\{X_{k+1} = x\} = \frac{1}{2}P\{X_k = x - 1\} + \frac{1}{2}P\{X_k = x + 1\}, \quad x \in \mathbb{Z}. \]
When $x = 2j - (k + 1)$ with $0 \leq j \leq k + 1$, it follows from the assumption of induction that

$$ P\{X_{k+1} = 2j - (k + 1)\} = \frac{1}{2} P\{X_k = 2(j-1) - k\} + \frac{1}{2} P\{X_k = 2j - k\} $$

$$ = \frac{1}{2^{k+1}} \binom{k}{j-1} + \frac{1}{2^{k+1}} \binom{k}{j} $$

$$ = \frac{1}{2^{k+1}} \binom{k+1}{j} . $$

When $x \notin \{2j - (k + 1) \mid 0 \leq j \leq k + 1\}$, we have

$$ x - 1 \notin \{2j - k \mid 0 \leq j \leq k\}, \quad x + 1 \notin \{2j - k \mid 0 \leq j \leq k\}, $$

which together with the assumption of induction gives

$$ P\{X_{k+1} = x\} = \frac{1}{2} P\{X_k = x - 1\} + \frac{1}{2} P\{X_k = x + 1\} = 0. $$

Thus, the theorem also holds true for $n = k + 1$. \hfill \Box

It has been shown that the usual unitary quantum walks with finite degrees of freedom (such as the Hadamard walk) have limit distributions that are quite different from that of the classical random walk. The following theorem, however, shows that our walk behaves at the vacuum initial state much the same as the classical random walk as time goes to infinity.

**Theorem 4.4** Let the initial state $\Phi_0 \in l^2(\mathbb{Z}, \mathcal{H})$ be such that $\Phi_0 = \phi$. Then,

$$ \frac{X_n}{\sqrt{n}} \Rightarrow N(0, 1) , \quad (4.7) $$

namely the probability distribution of $\frac{X_n}{\sqrt{n}}$ converges in law to the standard Gaussian distribution as $n \to \infty$.

**Proof** Write $\xi_n = \frac{X_n}{\sqrt{n}}$. We compute the characteristic function $Ee^{it\xi_n}$ of $\xi_n$, where $i$ denotes the imaginary unit. According to Theorem 4.3, we have

$$ Ee^{it\xi_n} = \frac{1}{2^n} \sum_{j=0}^{n} \binom{n}{j} e^{i\frac{(2j-n)t}{\sqrt{n}}} = \frac{1}{2^n} \left( e^{\frac{it}{\sqrt{n}}} + e^{-\frac{it}{\sqrt{n}}} \right)^n = \cos^n \frac{t}{\sqrt{n}} , \quad t \in \mathbb{R} . $$

It is well known in calculus that $\cos^n \frac{t}{\sqrt{n}} \to e^{-\frac{t^2}{2}}$ as $n \to \infty$. Thus,

$$ \lim_{n \to \infty} Ee^{it\xi_n} = e^{-\frac{t^2}{2}} , \quad t \in \mathbb{R} , $$

\label{eq:4.7}
Quantum walk in terms of quantum Bernoulli noises

which means that the limit distribution of \( \frac{X_n}{\sqrt{n}} \) is just the standard Gaussian distribution \( N(0, 1) \). \( \square \)

**Remark 4.2** It is easy to show that both Theorems 4.3 and 4.4 still hold when \( \Phi_0 = e^{i\theta} \phi \), where \( \theta \in [0, 2\pi) \).

The next theorem shows that the limit distribution of the walk at some other initial states remains Gaussian.

**Theorem 4.5** Let the initial state \( \Phi_0(x) = \begin{cases} \alpha Z_0 + \beta Z_0, & x = 0; \\ 0, & x \neq 0, x \in \mathbb{Z}, \end{cases} \)

where \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha|^2 + |\beta|^2 = 1 \). Then,

\[
\frac{X_n}{\sqrt{n}} \Rightarrow N(0, 1),
\]

namely \( \frac{X_n}{\sqrt{n}} \) converges in law to the standard Gaussian distribution as \( n \to \infty \).

**Proof** In the same way as in the proof of Theorem 4.3, we can obtain that for all \( n \geq 1 \)

\[
P\{X_n = x\} = \|\Phi_n(x)\|^2 = \begin{cases} \frac{1}{2\pi} \left( \binom{n-1}{j} |\alpha - \beta|^2 + \binom{n-1}{j-1} |\alpha + \beta|^2 \right), & x = 2j - n, 0 \leq j \leq n; \\ 0, & \text{otherwise}, \end{cases}
\]

where \( \binom{n-1}{j} = \binom{n}{j} = 0 \) and \( \binom{0}{0} = 1 \). Based on this formula, we can further obtain the characteristic function of \( \xi_n = \frac{X_n}{\sqrt{n}} \) as follows.

\[
Ee^{it\xi_n} = \frac{|\alpha - \beta|^2}{2n} \sum_{j=0}^{n-1} \binom{n-1}{j} e^{it(2j-n)/\sqrt{n}} + \frac{|\alpha + \beta|^2}{2n} \sum_{j=1}^{n} \binom{n-1}{j-1} e^{it(2j-1-n)/\sqrt{n}}
\]

\[
= \frac{|\alpha - \beta|^2}{2} e^{-it} \cos^{n-1} \frac{t}{\sqrt{n}} + \frac{|\alpha + \beta|^2}{2} e^{it} \cos^{n-1} \frac{t}{\sqrt{n}},
\]

where \( n \geq 1, t \in \mathbb{R} \), which implies that

\[
\lim_{n \to \infty} Ee^{it\xi_n} = \frac{|\alpha - \beta|^2 + |\alpha + \beta|^2}{2} e^{-t^2/2} = e^{-t^2/2}, \quad t \in \mathbb{R}.
\]

Thus, \( \frac{X_n}{\sqrt{n}} \Rightarrow N(0, 1) \), namely \( \frac{X_n}{\sqrt{n}} \) also converges in law to the standard Gaussian distribution. \( \square \)
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