The geometry of null-like disformal transformations

Iarley P. Lobo
Departamento de Física, Universidade Federal da Paraíba,
C. Postal 5008, João Pessoa, PB 58051-970, Brazil

Gabriel G. Carvalho

We study how some geometric objects are affected by the action of a disformal transformation in the metric tensor in the direction of a null-like vector field. Subsequently, we analyze symmetry properties such as mutual geodesics and mutual Killing vectors, generalized Weyl transformations that leave the disformal relation invariant, and introduce the concept of disformal Killing vector fields. In most cases, we use the Schwarzschild metric, in the Kerr-Schild formulation, to verify our calculations and results. Finally, we revisit the disformal operator using a Newman-Penrose basis to show that, in the null-like case, this operator is not diagonalizable.

I. INTRODUCTION

Effective metrics constitute an important tool for the description of physical phenomena, since they allow researchers to explore intermediary regimes of a given system using standard tools of differential geometry, instead of dealing with different mathematical structures derived from different fundamental theories. Just to mention few examples, we can refer to applications in Analogue Gravity (see [1, 2] and references therein). Another interesting application in quantum gravity can be found in [3–5].

Among these effective metrics, here we consider those generated by the so called disformal transformations. They are, essentially, a generalization of conformal transformations, in which besides a conformal rescaling of a background metric, it is considered a rank-2 tensor field with properties that guarantee the invertibility of the disformal metric. Although one could define disformal transformations in the context of Riemannian geometry, the case of a pseudo-Riemannian geometry with Lorentzian signature is far more rich due to the changes in the causal structure and its applications to physics.

Since its appearance, in the nineties due to Bekenstein’s works [6, 7], disformal transformations have had numerous applications. For instance TeVeS theories for MOND [8], bimetric theories of gravity [9], scalar [10] or scalar-tensor theories [11–14] (including Mimetic [15–18] and Horndeski gravity [19–21]), disformal inflation [22], chiral symmetry breaking [23], anomalous magnetic moment for neutrinos [24], disformal invariance of matter fields [25–27], quantum gravity [28] and more on analog gravity [29, 30].

In Ref. [28], besides demonstrating a link between disformal transformations and rainbow gravity, we further analyzed the mathematical properties of disformal transformations defined by time-like vectors (with respect to the background metric). We defined a disformal operator that acted on the (co-)tangent space, which worked as a decomposition of the disformal transformation on the metric tensor. This way, we managed to, in certain sense, explain the group operation rule of successive disformal transformations and solve some ambiguity issues.

In the present text, we continue to study the properties of these transformations, now using null-like vectors to define them (which we call null-like disformal transformations). We explicitly decompose the connection and the curvature, study invariant geodesics and Killing vectors, generalize Weyl transformations for the geometry induced by null and time-like disformal transformations, define the disformal Killing equation and study the disformal operator using a Newman-Penrose basis.

This text is divided as follows: In section II we fix the notation used throughout the text, revise the definition of a derivative operator and list some formulae concerning conformal transformations. In section III we study the geometry of null-like disformal transformations providing some generalizations of results valid in the conformal frame. In multiple examples we provide a test case to verify our results. In section IV we extend the definition of a conformal Killing vector to the disformal case, showing explicitly a solution when the background metric is flat. Finally, in section V we mention some relevant algebraic differences between the null and time-like disformal operators. In section VI we conclude with some future perspectives.

*Electronic address: iarley_lobo@fisica.ufpb.br
†Electronic address: gbracarvalho@gmail.com
II. PRELIMINARIES: DERIVATIVE OPERATORS, CURVATURE AND CONFORMAL TRANSFORMATIONS

Though we assume the reader to be familiarized with differential geometry, we start revising the definition of a derivative operator showing that any two derivative operators differ by a tensor. We shall use Penrose’s abstract index notation [31], in which tensor equations with Latin indices are true tensor equations (i.e., valid in any coordinate system) and tensor equations with Greek indices represent equations valid on a given coordinate system.

Definition 1 (Derivative operators). A derivative operator $D_a$ on a manifold $\mathcal{M}$ is a map which takes each $C^r$ tensor field of type $(k,l)$ to a tensor field of type $(k,l+1)$ and satisfy

1. Linearity: For all $A,B \in \mathcal{T}(k,l)$ and $\alpha,\beta \in \mathbb{R}$,
   \[ D_a [\alpha A^{b_1...b_k}_{c_1...c_l} + \beta B^{b_1...b_k}_{c_1...c_l}] = \alpha D_a A^{b_1...b_k}_{c_1...c_l} + \beta D_a B^{b_1...b_k}_{c_1...c_l}. \] (1)

2. Leibnitz rule: For all $A \in \mathcal{T}(k,l)$ and $B \in \mathcal{T}(k',l')$,
   \[ D_c \left[ A^{a_1...a_k}_{b_1...b_l} B^{c_1...c_{k'}}_{d_1...d_{l'}} \right] = D_c \left[ A^{a_1...a_k}_{b_1...b_l} B^{c_1...c_{k'}}_{d_1...d_{l'}} \right] + A^{a_1...a_k}_{b_1...b_l} D_c \left[ B^{c_1...c_{k'}}_{d_1...d_{l'}} \right]. \] (2)

3. Commutativity with contraction: For all $A \in \mathcal{T}(k,l)$,
   \[ D_d \left[ A^{a_1...a_k}_{b_1...c_l} \right] = D_d A^{a_1...a_k}_{b_1...c_l}. \] (3)

4. Consistency with the notion of vector fields as directional derivatives on scalar fields: For all $f \in C^r(\mathcal{M})$ and all $t^a \in \Gamma(T\mathcal{M})$,
   \[ t(f) = t^a D_a f. \] (4)

5. Torsion free: For all $f \in C^r(\mathcal{M})$,
   \[ D_a D_b f = D_b D_a f. \] (5)

Lemma 2. Consider a manifold $\mathcal{M}$ endowed with two metric tensors $g$ and $\hat{g}$ and their respective derivative operators $\nabla_a$ and $\hat{\nabla}_a$ (i.e., $\nabla_c g_{ab} = 0$ and $\nabla_c \hat{g}_{ab} = 0$). Then, for any tensor field $T^{b_1...b_k}_{c_1...c_l}$ we have

\[ \hat{\nabla}_a T^{b_1...b_k}_{c_1...c_l} = \nabla_a T^{b_1...b_k}_{c_1...c_l} + \sum_i C^{b_i}_{ad} T^{b_1...d...b_k}_{c_1...c_l} - \sum_j C^{cd}_{ac} T^{b_1...b_k}_{c_1...d...c_l}, \] (6)

where

\[ C^c_{ab} = \frac{1}{2} \hat{g}^{cd} \left( \nabla_a \hat{g}_{bd} + \nabla_b \hat{g}_{ad} - \nabla_d \hat{g}_{ab} \right). \] (7)

Proof. See [31], chapter 3. \qed

The reader should reckon the expression for $C^c_{ab}$ as a generalized Christoffel symbol. In fact, the Christoffel symbols are obtained using the same procedure and choosing $\nabla_a$ and $\partial_a$ as derivative operators. Since it depends on the coordinate system used to define the derivative operator $\partial_a$, the Christoffel symbols are not a true tensor in another coordinate system. Using lemma 2 we have, for any 1–form field $\omega_b$, that

\[ (\hat{\nabla}_a - \nabla_a) \omega_b = \hat{\nabla}_a \omega_b - \nabla_a \omega_b = \partial_a \omega_b - \hat{\Gamma}^c_{ab} \omega_c - (\partial_a \omega_b - \Gamma^c_{ab} \omega_c) = \left( \hat{\Gamma}^c_{ab} - \Gamma^c_{ab} \right) \omega_c. \] (8)
On the other hand, using lemma [2] we have  
\[ \hat{\nabla}_a \omega_b = \nabla_a \omega_b - C^c_{ab} \omega_c, \]
yielding
\[ \hat{\Gamma}_a^c = \Gamma_a^c + C_a^c. \]  
Therefore, once we know \( \Gamma_ab \), the knowledge of \( \hat{\Gamma}_a^c \) determines \( C_a^c \), and vice-versa. Let us see how to relate the curvature tensors associated with the two different affine connections \( \nabla \) and \( \hat{\nabla} \) in the form of the following

**Proposition 3.** The curvature tensor \( \hat{R}_{abc} \) associated with the metric \( \hat{g} \) in terms of the geometry defined by \( g \) is given by
\[ \hat{R}_{abc} = R_{abc} - 2\nabla_a C^d_{bc} + 2C_e^d C^c_{d} + \] 

**Proof.** By definition
\[ \hat{R}_{abc} d \omega_d = \left[ \hat{\nabla}_a, \hat{\nabla}_b \right] \omega_c = \] 
where \( a \leftrightarrow b \) represents the same expression but interchanging the indices \( a \) and \( b \). Replacing \( \nabla_m \omega_n = \nabla_m \omega_n - C^p_{mn} \omega_p \) and performing some index substitutions one gets
\[ \left[ \hat{\nabla}_a, \hat{\nabla}_b \right] \omega_c = R_{abc} d \omega_d + \left[ \nabla_b C^d_{ac} - \nabla_a C^d_{bc} + C_e^d C^c_{d} \right] \omega_d, \] 

therefore concluding the proof. \( \square \)

Note that when \( \hat{\nabla} = \nabla \) the \( C \) symbols are all zero and the geometry is kept the same, of course.

**Corollary 4.** The Ricci and scalar curvature associated with \( \hat{g} \) are, respectively, given by
\[ \hat{R}_{ac} = R_{ac} - 2\nabla_a C^b_{bc} + 2C_e^b C^c_{e} \] 
\[ \hat{R} = g^{ac} \hat{R}_{ac}. \]

**A. The geometry of conformal transformations**

Let \((\mathcal{M}, g_{ab})\) be a spacetime. A conformal transformation of \((\mathcal{M}, g_{ab})\), denoted by \((\mathcal{M}, \tilde{g}_{ab})\), is essentially a local angle-preserving change of scale where the new metric tensor is given by
\[ \tilde{g}_{ab} = \Omega^2 g_{ab}. \]

It is worthwhile to mention that a conformal transformation, as defined here, **is not** a change of coordinates, but an actual change of the geometry. Formally, it should be written \((\mathcal{M}, g_{ab}) \rightarrow (\mathcal{M}, \tilde{g}_{ab})\). However, the spacetime \((\tilde{\mathcal{M}}, \tilde{g}_{ab})\) is a subset of the manifold \(\mathcal{M}\) endowed with another metric tensor defined on it, hence the abuse of notation. It should be clear that conformal transformations are not, in general, associated with a diffeomorphism of \(\mathcal{M}\) [31].

Because now one can consider two metric tensors defined on \(\mathcal{M}\), hence two affine connections, it is of major interest to use conformal transformations to change our dynamical variables: anything that is a function of \(g_{ab}\) can be equally thought as a function of \(\tilde{g}_{ab}\) and \(\Omega\). We say that these quantities are expressed in the **conformal frame**. To resume this section we list some quantities of interest in the conformal frame.

**Proposition 5.** In the conformal frame we have:
\[ C^c_{ab} = 2\delta^c_{(a} \nabla_{b)} \ln \Omega - g_{ab} g^{cd} \nabla_d \ln \Omega \] 
\[ \hat{R}_{abc} \] 
\[ \hat{R}_{ac} = R_{ac} - (n - 2) \nabla_a \nabla_c \ln \Omega - g_{ac} g^{de} \nabla_d \nabla_e \ln \Omega \] 
\[ \hat{R} = \Omega^{-2} \left\{ \hat{R} - 2(n - 1) g^{ac} \nabla_a \nabla_c \ln \Omega - (n - 2) (n - 1) g^{ac} \nabla_a \ln \Omega \nabla_c \ln \Omega \right\} \] 
\[ \hat{\phi} = \Omega^{-2} \hat{\phi} + (n - 2) g^{ab} \Omega^{-3} \nabla_a \Omega \nabla_b \phi, \] 

for any \( C'(r \geq 2) \) scalar field \( \phi \).

**Proof.** Check the appendix D of [31]. \( \square \)
III. THE GEOMETRY OF NULL-LIKE DISFORMAL TRANSFORMATIONS

A. Connection, curvature and the d’Alembertian

Disformal transformations can be seen as a generalization of conformal transformations. As such, they do not represent a change in coordinates, but a local change in the geometry instead. One might think of a conformal transformation as a smooth, isotropic and infinitesimal stretch at a point, whereas a disformal transformation is a smooth, anisotropic and infinitesimal stretch at a point. Given a spacetime \((\mathcal{M}, g_{ab})\), a null-like vector \(V^c\) and two spacetime-dependent scalars \(\alpha\) and \(\beta\) with \(\alpha > 0\), we define a null-like disformal transformation \((\mathcal{M}, g_{ab}, V^c, \alpha, \beta) \rightarrow (\mathcal{M}, \hat{g}_{ab})\) as a change in geometry when the metric tensor changes according to

\[
\hat{g}_{ab} = \alpha g_{ab} + \beta V_a V_b \equiv \alpha g_{ab} + \beta g_{ac} g_{bd} V^c V^d.
\]  

(21)

It is easy to check that the inverse of the disformal metric in that case is given by

\[
\hat{g}^{ab} = \frac{1}{\alpha} g^{ab} - \frac{\beta}{\alpha^2} V^a V^b.
\]  

(22)

Since we are now dealing with a manifold endowed with two metric tensors, it is important to distinguish which metric tensor is being used when raising and lowering indices. One shall deal with this problem by explicitly writing the metric in all formulae in which indices are raised or lowered.

We can now consider some of the dynamical variables in the disformal frame. Using the definition of the tensor \(C^c_{ab}\) in lemma 2 we find:

\[
C^c_{ab} = \frac{1}{2\alpha} \left[ 2\alpha (\delta_c^d) + 2\beta V^c V_{(a;b)} + V^c \beta_{(a} V_{b)} + 2\beta V^c (\alpha V_b) + V^c \beta_{(a} V_{b)} - g_{ab} \delta^c_d - 2\beta V_{(a|d)} g^d_V V_{b)} - \beta_d g^d_V V_{b)} \right]
\]  

\[ - \frac{1}{\alpha^2} \left[ 2\beta \alpha (V_a V_b) V^c - g_{ab} V^c \alpha \beta - 2\beta^2 \alpha (V_a V_b) V^c - \beta \beta V_{(a} V_{b)} V^c \right],
\]  

(23)

\[
\bar{R} = \frac{1}{\alpha} R - \frac{\beta}{\alpha^2} R_{ac} V^a V^c + \frac{1}{4\alpha^2} \left\{ 4(1-n)\square \alpha + 4\beta V^b V^a_{;\,a:b} + 2\beta V^b V^a_{;\,a:b} 
\right\}
\]  

\[- 6\beta V^a g^{bc} V_{abc} + 4\beta (\nabla \cdot V)^2 + 2\beta V^a V^b_{;\,a} - 6\beta g^{bc} V^a_{;\,a:c} + 11\beta (\nabla \cdot V) + \beta \alpha_{a:b} V^a V^b + 4\beta \alpha V^b_{;\,a} + \frac{9 (\dot{\beta})^2}{2\beta} \right\} + \frac{1}{4\alpha^3} \left\{ -(n-1)(n-6)\alpha \ast \alpha + 6(n-2)\beta \ddot{\alpha} 
\right\}
\]  

\[ + (11n-24)\beta \dddot{\alpha} + (3n-4)\beta \dddot{\alpha} a_n + (8n-20)\dddot{\beta} \alpha \nabla \cdot V + 5\beta^2 \dot{V}^a V^a + 4\beta^2 \dot{V}^a V^a_{;\,a} + 4\beta^2 \dot{V}^a V^a_{;\,a} \right\} + \frac{1}{4\alpha^2} (2n^2 - 19n + 30)\beta (\dot{\alpha})^2,
\]  

(24)

and

\[
\Box \Phi = \frac{1}{\alpha} \Box \Phi - \frac{1}{\alpha^2} \beta V^a \dot{\Phi}_a + \frac{1}{\alpha^3} (3-n) \dot{\alpha} \Phi
\]  

\[ - \frac{1}{2\alpha^2} \left\{ (2-n) \Phi^a_\Phi \Phi_a + 3\beta \dot{\Phi} + 2\beta \dot{\Phi} (\nabla \cdot V) + 2\beta V^c \Phi_c \right\},
\]  

(25)

where \(n = \dim \mathcal{M}, \alpha_a = \partial_a \alpha, \alpha^a = V^b \partial_b \alpha, \dot{\alpha} = V^a \partial_a \alpha, \alpha_{;\,a} = \nabla_a V^a, \square \alpha = \nabla_a \nabla^a \alpha, \alpha \ast \alpha = g^{ab} \partial_a \alpha \partial_b \alpha \) and \(V^a = V^b \partial_b V^a\). Setting \(V^a = 0\) and \(\alpha = \Omega^2\) one can recover the geometry in the conformal frame given in proposition 3.

In the examples below, we are going to refer them to a test case. They are used to show the validity of our formulae in an actual example. In order to avoid defining it in every example, the reader should have in mind the following

**Definition 6** (The test case). The manifold \(\mathcal{M}\) is \(\mathbb{R}^4\) with Cartesian coordinates \(\{x^\mu\} = (t, x, y, z)\) and the background metric is the Minkowski one \(g_{ab} = \text{diag}(1, -1, -1, -1)\). The disformal parameters are \(\alpha = 1\) and \(\beta = \frac{2\alpha}{\alpha}\) and the null-like vector \(V^a = (1, -\xi, -\eta, -\zeta)\), and \(r = \sqrt{x^2 + y^2 + z^2}\). This is the Schwarzschild metric in the Kerr-Schild formulation. Of course one could use the Kerr-Newman metric as a test case, but this would complicate significantly the calculations without adding anything new to the problem.
Example 7 (The test case – I). The derivative operator compatible with \( g \) is the coordinate system derivative operator \( \partial_a \) and \( \Gamma^c_{ab} \equiv 0 \) (and therefore The Riemann and Ricci tensors are null everywhere). Using Eq. (9) we have

\[
C^c_{ab} = \Gamma^c_{ab},
\]

which is simply the Eq. (7) when one replaces \( \nabla_a \) by \( \partial_a \), as expected. Using Eq. (10) and using the fact that \( R_{abc}^d = 0, \nabla_a = \partial_a \) and \( C^c_{ab} = \Gamma^c_{ab} \) we have

\[
\hat{\delta}_{abc}^d = R_{abc}^d + \nabla_b C^d_{ac} - \nabla_a C^d_{bc} + C^e_{ac}C^d_{be} - C^e_{bc}C^d_{ae} \\
= \partial_b \Gamma_{ac}^d - \partial_a \Gamma_{bc}^d + \Gamma_{ac}^e \hat{\Gamma}_{bc}^d - \hat{\Gamma}_{bc}^e \Gamma_{ac}^d,
\]

which is the expression for the Riemann curvature tensor associated with \( \hat{g} \) as expected. Substituting \( \alpha = 1, \beta = 2m/r, \) and \( n = 4 \) in the expression for the scalar curvature (24) we obtain \( \hat{R} = 0. \)

B. Mutual geodesics

One might be interested in comparing the geodesics with respect to \( \nabla \) with those with respect to \( \hat{\nabla} \). As is the case for conformal transformations, one cannot expect the geodesics to be preserved. Nonetheless, for conformal transformations null geodesics are indeed preserved, although the geodesic in the conformal frame is not affinely parameterized. This is an extraordinary feature that allows one to study causality up to a conformal transformation (for instance, using the Carter-Penrose conformal diagrams). Since disformal transformations provide a generalization of the conformal ones, we cannot expect the geodesics to be preserved, not even in the null case. Surprisingly, if the tangent vector of a null geodesic satisfies an extra condition, this null geodesic is preserved. This result can be stated as the following

Proposition 8. Let \( (\mathcal{M}, \hat{g}_{ab}, V^c, \alpha, \beta) \rightarrow (\mathcal{M}, \hat{g}_{ab}) \) be a null-like disformal transformation. Let \( w^a \in \Gamma(T, \mathcal{M}) \) be the tangent vector to an affinely parameterized null geodesic \( \gamma \) of the background metric. If \( w^a V^a = w^a V_a = g_{ab}w^a V^b = 0 \), then \( \gamma \) is a null geodesic with respect to \( \hat{\nabla} \).

Proof. We have

\[
w^a \hat{\nabla}_a w^b = w^a \nabla_a w^b + C^b_{ac} w^a w^c \\
= C^b_{ac} w^a w^c,
\]

since \( w^a \nabla_a w^b = 0 \). Using the expression for \( C^b_{ac} \), yields

\[
w^a \hat{\nabla}_a w^b = \left\{ \frac{1}{2\alpha} \left[ 2\alpha (a^b_{a^c}) + 2\beta V^b V_{(a^c)} + V^b \beta (a V_c) + 2\beta V^b \Gamma_{(a}^b V_c) + V^b \beta (a V_c) \\
- g_{ac} (a^b_{d^c}) - 2\beta V_{[a)}^b V_{d^c} - \beta g_{bd} V_{(a^c)} \right] \\
- \frac{1}{\alpha^2} \left[ 2\beta (a V_c) V^b - g_{ac} V^b \delta \beta - 2\beta \hat{\nabla}_{(a} V^b \beta (a V_c) V^b - \beta \hat{\nabla}_{(a} V^b \right] \right\} w^a w^c \\
= (w^a \nabla_a \ln \alpha) w^b.
\]

The parameters of the geodesics are then related by \( \frac{\delta \lambda}{\delta \lambda} = c \alpha \), where \( c \in \mathbb{R} \) is a constant, showing that it is not affinely parameterized. We used that \( 2\beta V_{(c^a)} V^b w^a w^c = 0 \). Indeed:

\[
2\beta V_{(c^a)} V^b w^a w^c = \left\{ \beta V^b \nabla_a V_c + \beta V^b \nabla_c V_a \right\} w^a w^c \\
= \beta V^b w^a w^c \nabla_a V_c + \beta V^b w^a w^c \nabla_c V_a.
\]

Investigating the first term (the second term follows by analogy), we have

\[
\beta V^b w^a \nabla_a (w^c V_c) = 0 = \underbrace{\beta V^b w^a \nabla_a V_c + \beta V^b V_c}_{\text{first term}} \underbrace{w^a \nabla_a w^c}_{=0 \ (\text{geodesic})}.
\]
The proposition above shows that for the case of a null-like disformal transformation the null vector \( w^a \) must satisfy an extra condition, to wit \( V_a w^a = 0 \). This extra condition involves a coupling between \( w^a \), the disformal portion of the transformation. Therefore, since conformal transformation preserve causal relations, the study of the causal structure must rely on \( \beta \) and \( V^c \). As shown in [28], \( \beta \) plays the crucial role in the change of the causal structure.

**Example 9** (The test case – II). The vector defining this disformal transformation, \( V^\mu = (1, -\frac{\xi}{\tau}, -\frac{\eta}{\tau}, -\frac{\zeta}{\tau}) \), satisfy the conditions of the proposition above. Therefore:

\[
V^\mu \nabla_\mu V^\nu = V^\mu \tilde{\nabla}_\mu V^\nu = 0.
\]

(28)

These are precisely the radial null-like geodesic of Minkowski and Schwarzschild spacetimes.

C. Mutual Killing vector fields

Let \((\mathcal{M}, g_{ab})\) be a spacetime. We say that \( \xi^a \in \Gamma(T\mathcal{M}) \) is a Killing vector field of \( g_{ab} \) if

\[
\mathcal{L}_\xi g_{ab} = \xi^c \partial_c g_{ab} + (\partial_a \xi^c) g_{cb} + (\partial_b \xi^c) g_{ca} = 0,
\]

(29)

i.e., the Lie derivative of \( g_{ab} \) in the direction of \( \xi^c \) is zero. The Killing vectors of a metric are associated with the symmetries of that metric. Despite the appearance, equation (29) is known to be covariant.

Needless to say, knowing the Killing vectors is of major importance. The proposition below relates the Killing vectors of the background and disformal metrics. The corollary provides a well-known result when the disformal transformation is purely conformal.

**Proposition 10.** Let \((\mathcal{M}, g_{ab}, V^c, \alpha, \beta) \rightarrow (\mathcal{M}, \hat{g}_{ab})\) be a null-like disformal transformation and let \( \xi^c \) be a Killing vector field of \( \hat{g}_{ab} \). Then, \( \xi^c \) is a Killing vector field of \( g_{ab} \) if, and only if,

\[
(\xi^c \partial_c \alpha)g_{ab} + \xi^c \partial_c (\beta V^a V_b) + 2\beta V_a (\partial_b \xi^c) V_c = 0.
\]

(30)

**Proof.** Using the equation (29) for the metric \( \hat{g}_{ab} \), we have:

\[
\mathcal{L}_\xi \hat{g}_{ab} = \xi^c \partial_c \hat{g}_{ab} + (\partial_a \xi^c) \hat{g}_{cb} + (\partial_b \xi^c) \hat{g}_{ca} = 0.
\]

(31)

Replacing the expression for \( \hat{g}_{ab} \) yields

\[
0 = \mathcal{L}_\xi \hat{g}_{ab} = (\xi^c \partial_c \alpha)g_{ab} + \alpha \mathcal{L}_\xi g_{ab} + \xi^c \partial_c (\beta V^a V_b) + 2\beta V_a (\partial_b \xi^c) V_c.
\]

(32)

Since \( \alpha \neq 0 \), \( \mathcal{L}_\xi g_{ab} = 0 \) if, and only if,

\[
(\xi^c \partial_c \alpha)g_{ab} + \xi^c \partial_c (\beta V^a V_b) + 2\beta V_a (\partial_b \xi^c) V_c = 0.
\]

(33)

**Corollary 11.** For conformal transformations, \( \xi^a \) is at least a conformal Killing vector field of \( g_{ab} \) and will be a true Killing vector field if, and only if, \( \alpha \) is constant along the orbits (integral curves) of \( \xi^a \).

**Example 12** (The test case – III). In that case, \( \alpha \equiv 1 \) and therefore the necessary and sufficient condition for \( \xi^c \) to be a mutual Killing vector field of the Minkowski and Schwarzschild metrics is

\[
\xi^c \partial_c (\beta V^a V_b) + 2\beta V_a (\partial_b \xi^c) V_c = 0.
\]

(34)

The rotational Killing vectors \( R = -y \partial_x + x \partial_y \), \( S = z \partial_z - x \partial_z \), \( T = -z \partial_y + y \partial_z \) and the time-translation Killing vector \( \partial_t \) are shown to satisfy the equation (34). Therefore, they are Killing vector fields for both metrics.

D. Generalized Weyl transformations

Weyl geometry is a generalization of Riemannian geometry, that presents a non-metricity tensor, i.e., the compatibility between the metric of the manifold and the connection is determined by the rule

\[
\nabla_c g_{ab} = \sigma_c g_{ab}.
\]

(35)
where σc is a 1-form field named the Weyl field. By assuming this W-\emph{compatibility condition}, along with the torsionless connection requirement, it is possible to generalize Levi-Civita’s theorem to find the unique connection that satisfies (35), given by

\[
\Gamma^c_{ab} = \frac{1}{2} g^{cd} (\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}) - \frac{1}{2} g^{cd} (\sigma_a g_{bd} + \sigma_b g_{ad} - \sigma_d g_{ab}).
\]  

(36)

Such geometry presents an inherent symmetry property. By performing the simultaneous transformation, named Weyl transformations, \((g_{ab}, \sigma_c) \mapsto (\epsilon^f g_{ab}, \sigma_c + f_c)\), the form of the compatibility condition (35) and the connection coefficients (36) are preserved. Which means that for the simultaneous transformations \(\hat{g}_{ab} = \epsilon^f g_{ab}\) and \(\hat{\sigma}_c = \sigma_c + f_c\), the compatibility condition transforms to

\[
\nabla_c \hat{g}_{ab} = \hat{\sigma}_c \hat{g}_{ab}.
\]  

(37)

and consequently the connection coefficients have the same functional dependence as (36), but depending of the pair \((\hat{g}_{ab}, \hat{\sigma}_c)\). This means that different choices of the scalar function \(f\) defines different \emph{frames} in this geometry.

An interesting possibility consists in defining \(\sigma = d\phi\), i.e., the Weyl field as the differential of a scalar field \(\phi\). This defines the so called integrable Weyl geometry. This way, looking at the Weyl transformations, by choosing \(f = -\phi\), there exists a Riemannian frame

\[
(g_{ab}, \phi) \mapsto \left( g_{ab} = e^{-\phi} g_{ab}, d\hat{\phi} = 0 \right),
\]  

(38)

meaning that the integrable Weyl geometry can be effectively described as a Riemannian geometry with an effective conformal metric. In fact, considering the conformal relation \(\hat{g}_{ab} = e^{-\phi} g_{ab}\), it is straightforward to check that it is invariant under the Weyl transformation, i.e., if we transform \(\hat{g}_{ab} = \epsilon^f g_{ab}\) and \(\hat{\phi} = \phi + f\), we have

\[
\hat{g}_{ab} = e^{-\phi} g_{ab} \mapsto e^{-\hat{\phi}} \hat{g}_{ab}.
\]  

(39)

Since there exists a Riemannian frame, a different form of treating the integrable geometry and deriving the Weyl transformation consists in treating it as a Riemannian geometry in an effective, conformal metric. This way, the Riemannian compatibility condition

\[
\nabla_c (e^{-\phi} g_{ab}) = 0
\]  

(40)

is equivalent to equation (35) for \(\sigma = d\phi\). And even the connection coefficients (36) are the Christoffel symbols of \(e^{-\phi} g_{ab}\). Since the Weyl transformation preserves the form of the effective metric by (39), it also preserves the W-\emph{compatibility condition} induced by the conformal metric

\[
\nabla_c (e^{-\phi} g_{ab}) = 0 \iff \nabla_c g_{ab} = \phi, c g_{ab}.
\]  

(41)

This construction can be generalized to an induced geometry inspired by disformal a transformation [32]. Consider a disformal relation

\[
\hat{g}_{ab} = \alpha g_{ab} + \beta g_{ac} g_{bd} V^c V^d.
\]  

(42)

Performing a disformal transformation \((g_{ab}, V^c, \lambda, \gamma) \mapsto (\hat{g}_{ab})\),

\[
\hat{g}_{ab} = \lambda g_{ab} + \gamma g_{ac} g_{bd} V^c V^d.
\]  

(43)

Inverting this relation we have [35]

\[
g_{ab} = \frac{1}{\lambda} \hat{g}_{ab} - \frac{\gamma}{\lambda^3} \hat{V}_a \hat{V}_b,
\]  

(44)

where \(\hat{V}_a \equiv \hat{g}_{ab} V^b\).

Substituting in Eq. (42), we have

\[
\hat{g}_{ab} = \alpha \hat{g}_{ab} + \frac{1}{\lambda^2} \left( \beta - \alpha \gamma \right) \hat{V}_a \hat{V}_b.
\]  

(45)

If we define

\[
\hat{\alpha} = \frac{\alpha}{\lambda}, \quad \hat{\beta} = \frac{1}{\lambda^2} \left( \beta - \frac{\alpha \gamma}{\lambda} \right),
\]  

(46)
the disformal relation is preserved. Therefore, for the simultaneous transformations

\[
\begin{align*}
\tilde{g}_{ab} &= \lambda g_{ab} + \gamma V_{a}V_{b}, \\
\tilde{V}_{a} &= \lambda V_{a}, \\
\tilde{\alpha} &= \alpha^{}/\lambda, \\
\tilde{\beta} &= (\beta - \alpha\gamma^{}/\lambda^{2})/\lambda^{2},
\end{align*}
\]

(47)

For the case of conformal transformations, as we saw above, the simultaneous transformations that preserve the conformal relation are

\[
\begin{align*}
\tilde{g}_{ab} &= \lambda g_{ab}, \\
\tilde{\alpha} &= \alpha^{}/\lambda.
\end{align*}
\]

(48)

These are the Weyl transformations, and the preservation of the conformal relation is related the invariance of the compatibility condition in integrable Weyl geometry.

E. A brief comment on the time-like case

A similar analysis may be done in the time-like case. Consider a fixed time-like disformal relation

\[
\hat{g}_{ab} = \alpha g_{\alpha\beta} + \beta \frac{V_{a}V_{b}}{g_{cd}V^{c}V^{d}}.
\]

(49)

Performing a disformal transformation on \(g\)

\[
\tilde{g}_{ab} = \lambda g_{ab} + \gamma \frac{V_{a}V_{b}}{g_{cd}V^{c}V^{d}}.
\]

(50)

If we invert this transformation using the rule presented in [28] and substitute the result in Eq. (49), we have

\[
\tilde{g}_{ab} = \tilde{\alpha} \tilde{g}_{ab} + \tilde{\beta} \frac{V_{a}V_{b}}{g_{cd}V^{c}V^{d}},
\]

(51)

where we defined

\[
\tilde{\alpha} = \frac{\alpha}{\lambda}, \quad \tilde{\beta} = \frac{\beta}{\lambda + \gamma} - \alpha \frac{\gamma}{\lambda (\lambda + \gamma)}.
\]

(52)

This way, by performing the simultaneous transformations

\[
\begin{align*}
\tilde{g}_{ab} &= \lambda g_{ab} + \gamma \frac{V_{a}V_{b}}{g_{cd}V^{c}V^{d}}, \\
\tilde{V}_{a} &= (\lambda + \gamma) V_{a}, \\
\tilde{\alpha} &= \alpha^{}/\lambda, \\
\tilde{\beta} &= (\beta (\lambda + \gamma)^{-1} - \alpha\gamma^{}/[\lambda (\lambda + \gamma)]^{-1},
\end{align*}
\]

(53)

Which means that also in the time-like case there are different gauges that preserve the disformal relation (49), as was also observed in [32].

IV. DISFORMAL KILLING EQUATION

In this section we generalize the notion of conformal Killing vectors to the disformal case. It is shown that, under some hypotheses, it is possible to find a solution for this disformal Killing equation. The meaning of this solution is discussed in the end of this section. We begin with the following
Definition 13. A vector field \( X^c \), satisfying the equation
\[
(\mathcal{L}_X g)_{ab} = \alpha g_{ab} + \beta V_a V_b,
\]
is called a null-like disformal Killing vector of the metric \( g \), where \( \alpha \) and \( \beta \) are scalar fields and \( V^c \) is a null-like vector field.

It would be an exercise in futility to define such an object if no solution to equation (54) existed. Let us consider a flat metric \( \eta_{ab} \) and the 1-form field \( \sqrt{\beta} V_a = U_a = \partial_a \phi \) for scalar field \( \phi \). We analyze this particular case following the notation of \([33]\).

From the trace of the equation (54), the null-like disformal Killing equation is
\[
\partial_a X_b + \partial_b X_a = \frac{2}{n} \eta_{ab} \partial_c X^c + U_a U_b
\]
(55)

Defining \( \psi = \frac{2}{n} \partial_c X^c \) and differentiating equation (55) we have
\[
\partial_a \partial_b X^c + \partial_b \partial_a X^c = \eta_{ab} \partial_c \psi + U_b \partial_a U_a + U_a \partial_b U_b,
\]
(56)
\[
\partial_a \partial_b X^c + \partial_b \partial_a X^c = \eta_{bc} \partial_a \psi + U_c \partial_a U_b + U_b \partial_a U_c.
\]
(57)

Subtracting (56) and (57):
\[
\partial_b (\partial_c X_a - \partial_a X_c) = \eta_{ab} \partial_c \psi - \eta_{bc} \partial_a \psi + U_a \partial_b U_b - U_c \partial_a U_c.
\]
(58)

Differentiating again, considering that \( \partial_a \partial_b (\partial_c X_a - \partial_a X_c) = \partial_a \partial_b (\partial_c X_a - \partial_a X_c) \), and contracting the result with \( \eta^{ad} \eta^{bc} \), yields
\[
\eta^{ad} \eta^{bc} (\partial_b \partial_a \phi) (\partial_a \partial_c \phi) - (\square \phi)^2 = 0,
\]
(59)
where \( \square = \eta^{ab} \partial_a \partial_b \). For simplicity, let us assume the case in which \( \partial_a \phi = 2 C_{\alpha} \) in Cartesian coordinates (now we are using greek indices because we are fixing the coordinate system to be the Cartesian one). This way, Eq. (59) is simply
\[
\square \psi = 0.
\]
(60)

We will chose the solution to be
\[
\psi = 2 B + 4 B_{\alpha} x^\alpha,
\]
(61)
where \( B, B_{\alpha} \) are constants. Substituting in (58), and integrating
\[
\partial_\lambda X_\alpha - \partial_\alpha X_\lambda = 4 (B_\lambda x_\alpha - B_{\alpha} x_\lambda) + 2 A_{\alpha \lambda},
\]
(62)
where \( A_{\alpha \lambda} = -A_{\lambda \alpha} \). Summing Eqs. (55) and (62), we have
\[
\partial_\lambda X_\alpha = 2 (B_\lambda x_\alpha - B_{\alpha} x_\lambda) + A_{\alpha \lambda} + \eta_{\alpha \lambda} (B + 2 B_{\beta} x^\beta) + 2 C_{\lambda} C_{\alpha}.
\]
(63)

Finally, integrating this equation we find
\[
X_\alpha = A_\alpha + (A_{\alpha \lambda} + 2 C_{\alpha} C_{\lambda}) x^\lambda + B_{x_\alpha} + 2 B_{\alpha} x_\alpha x^\lambda - B_{\alpha} x_\lambda x^\lambda,
\]
(64)
where \( A_{\alpha} \) are arbitrary constants.

This solution can be decomposed as a linear combination of the generators of the conformal group \([33]\)
\[
p_\alpha = \partial_\alpha, \quad m_{\alpha \beta} = x_\alpha \partial_\beta - x_\beta \partial_\alpha, \quad d = x^\alpha \partial_\alpha, \quad k_\alpha = 2 x_\alpha x^\beta \partial_\alpha - x_\beta x^\beta \partial_\alpha,
\]
and the fixed vector
\[
w = 2 C_{\alpha} C_{\beta} x^\beta \partial_\alpha.
\]
(65)

To illustrate our construction, consider the coordinate transformation induced by \( w \):
\[
x_\alpha' = x_\alpha + 2 C_{\alpha} C_{\beta} x^\beta.
\]
(66)

This way, the light cone transforms as
\[
\eta_{\alpha \beta} x_\alpha' x^\beta = (\eta_{\alpha \beta} + 4 C_{\alpha} C_{\beta}) x^\alpha x^\beta = (\eta_{\alpha \beta} + U_{\alpha \beta}) x^\alpha x^\beta.
\]
(67)

Which means that a coordinate transformation induced by the disformal Killing equation preserves the disformal structure.
V. THE DISFORMAL OPERATOR REVISITED

In Refs. [27, 28] the group structure and disformal operator were studied in detail for the time-like case. However, some algebraic differences occur when one deals with null-like disformal transformations/operators. Remember that time-like disformal transformation is defined as

\[ \hat{g}_{ab} = \alpha g_{ab} + \frac{\beta}{V^2} V_a V_b, \]  

(68)

with inverse given by

\[ \hat{g}^{ab} = \frac{1}{\alpha} g^{ab} - \frac{\beta}{\alpha(\alpha + \beta)} V^a V^b. \]  

(69)

In [28] it was verified that the operator \( \hat{g}^a_b \equiv \hat{g}_{ab}g^{ac} \) is the square of the disformal operator, and that such an operator has a basis of eigenvectors. In fact, one can choose an orthonormal basis (orthonormal with respect to \( g \)) at a point \( p \in \mathcal{M} \) where \( \sqrt{V^2} e^a_{(0)} = V^a \) to show that, in that basis,

\[ \hat{g} = \begin{pmatrix} \alpha + \beta & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & 0 & 0 & -\alpha \end{pmatrix}, \]  

(70)

therefore,

\[ D^2_{\text{time-like}} = \begin{pmatrix} \alpha + \beta & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}. \]  

(71)

The trick consisting of taking an unit vector in the direction of \( V^a \) and completing this set to an orthonormal basis proves that the disformal operator has a basis of eigenvectors.

Intending to extend this idea to the case of a null-like disformal transformation, the overt problem is that \( V^a \) cannot belong to an orthonormal basis. One could, however, do the second best thing: Consider a null-like tetrad basis at an arbitrary point \( p \in \mathcal{M} \) (i.e., a Newman-Penrose basis) where \( V^a = e^a_{(0)} \). The background metric in this basis is given by

\[ g = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \]  

(72)

In this case we have

\[ \hat{g} = \begin{pmatrix} \beta & \alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha \\ 0 & 0 & -\alpha & 0 \end{pmatrix} \implies D^2_{\text{null}} = \begin{pmatrix} \alpha & \beta & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}. \]  

(73)

The null tetrad basis chosen is almost a Jordan basis. Changing \( e^a_{(0)} \rightarrow \beta e^a_{(0)} \) and keeping the other vectors in our basis unchanged, we have our Jordan basis and the disformal operator is given by

\[ D^2_{\text{null}} = \begin{pmatrix} \alpha & 1 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}. \]  

(74)

It is now obvious that \( D^2 \) has only \( \alpha \) as an eigenvalue with algebraic multiplicity 4, geometric multiplicity 3 and the minimal polynomial given by \( p_M(x) = (\alpha - x)^2 \). Hence, no basis of eigenvectors of \( D^2_{\text{null}} \) can exist. Finally, it is easy to verify that the product of matrices of the type (73) is a matrix of the same type, showing that the group structure is also satisfied by null type disformal operators/transformations.
Time-like disformal transformations keep the causal character of the vector used in the disformal transformation. This is also the case for null-like transformations, as the reader can easily verify (the fact to the matter is that the vector used to define disformal transformations is always an eigenvector of the disformal operator, therefore the subspace generated by it is preserved). As a result, in the null case the background and foreground light cones coincide along one null direction (one, and only one, since the last two eigenvectors of \((74)\), although null-like in character, are complex), which explains the narrow relation between null-like disformal transformations and the conformal ones pointed out in this work (see also figure 1 below). More information can be found in an independent study performed in \[34\].

![Image of light cones](image)

FIG. 1: The figure in the left represents the relation between the light cones after a time-like disformal transformation (check that the only coincide at the origin) whereas the figure on the right represents the relation between the light cones after a null-like disformal transformation (check that it is nearly conformal in the vicinity of the shared null direction).

VI. CONCLUDING REMARKS

We revised how some geometric objects change when two affine connections are defined on \(\mathcal{M}\) and applied the results for the case of a null-like disformal transformation. As pointed out throughout the text, similar results can be found for both null-like disformal transformations and conformal transformations. Many of these similarities are due to the geometric fact that the light cones of a null-like disformal metric share a null direction with the background light cones (whereas the conformal light cones share all null directions), and are not expected to be true for time-like disformal transformations. Furthermore, the new results presented here are accompanied with important physical examples.

To motivate further studies in this area, we introduced the concept of a disformal Killing vector field showing an explicit solution in a particular case. We also generalized the concept of Weyl transformations that preserve the null-like disformal relation.

Finally, we revisit the disformal operator and its algebraic properties, showing that it is not diagonalizable in the present case and the suitable basis to study it is a basis consisting of null vectors. Future works containing the relation between spinors and null-like disformal transformations beyond the scope of \[27\] are now under investigations.

Acknowledgements

GGC would like to thank J. A. M. Gondim for the valuable discussion. IPL is supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq-Brazil) by the grant No. 150384/2017-3.
References

[1] C. Barcelo, S. Liberati and M. Visser, “Analog gravity from Bose-Einstein condensates,” Class. Quant. Grav. 18, 1137 (2001) doi:10.1088/0264-9381/18/6/012 [gr-qc/0011026].

[2] C. Barcelo, S. Liberati and M. Visser, “Analog gravity,” Living Rev. Rel. 8, 12 (2005) [Living Rev. Rel. 14, 3 (2011)] doi:10.12942/lrr-2005-12 [gr-qc/0505065].

[3] J. Magueijo and L. Smolin, “Gravity’s rainbow,” Class. Quant. Grav. 21, 1725 (2004) doi:10.1088/0264-9381/21/7/001 [gr-qc/0305055].

[4] M. Assanioussi, A. Dapor and J. Lewandowski, “Rainbow metric from quantum gravity,” Phys. Lett. B 751, 302 (2015) doi:10.1016/j.physletb.2015.10.043 [arXiv:1412.6000 [gr-qc]].

[5] R. G. Torromé, M. Letizia and S. Liberati, “Phenomenology of effective geometries from quantum gravity,” Phys. Rev. D 92, no. 12, 124021 (2015) doi:10.1103/PhysRevD.92.124021 [arXiv:1507.03205 [gr-qc]].

[6] J.D. Bekenstein, “New gravitational theories as alternatives to dark matter”, in Proceedings of the Sixth Marcel Grossmann Meeting on General Relativity, eds. H. Sato and T. Nakamura, World Publishing, Singapore, (1992).

[7] J. D. Bekenstein, “The Relation between physical and gravitational geometry,” Phys. Rev. D 48, 3641 (1993) doi:10.1103/PhysRevD.48.3641 [gr-qc/9211017].

[8] J. D. Bekenstein, “Relativistic gravitation theory for the MOND paradigm,” Phys. Rev. D 70, 083509 (2004) Erratum: [Phys. Rev. D 71, 069901 (2005)] doi:10.1103/PhysRevD.70.083509, 10.1103/PhysRevD.71.069901 [astro-ph/0403694].

[9] T. Clifton, P. G. Ferreira, A. Padilla and C. Skordis, “Modified Gravity and Cosmology,” Phys. Rept. 513, 1 (2012) doi:10.1016/j.physrep.2012.01.001 [arXiv:1106.2176 [astro-ph.CO]].

[10] M. Novello, E. Bittencourt, U. Moschella, E. Goulart, J. M. Salim and J. D. Toniato, “Geometric scalar theory of gravity,” JCAP 1306, 014 (2013) doi:10.1088/1475-7516/2014/01/E01, 10.1088/1475-7516/2013/06/014 [arXiv:1212.0770 [gr-qc]].

[11] T. S. Koivisto, D. F. Mota and M. Zumalacarregui, “Screening Modifications of Gravity through Disformally Coupled Fields,” Phys. Rev. Lett. 109, 241102 (2012) doi:10.1103/PhysRevLett.109.241102 [arXiv:1205.3167 [astro-ph.CO]].

[12] H. Y. Ip, J. Sakstein and F. Schmidt, “Solar System Constraints on Disformal Gravity Theories,” JCAP 1510, 051 (2015) doi:10.1088/1475-7516/2015/10/051 [arXiv:1507.00568 [gr-qc]].

[13] J. Sakstein, “Disformal Theories of Gravity: From the Solar System to Cosmology,” JCAP 1412, 012 (2014) doi:10.1088/1475-7516/2014/12/012 [arXiv:1409.1734 [astro-ph.CO]].

[14] J. Sakstein and S. Verner, “Disformal Gravity Theories: A Jordan Frame Analysis,” Phys. Rev. D 92, no. 12, 123005 (2015) doi:10.1103/PhysRevD.92.123005 [arXiv:1509.05679 [gr-qc]].

[15] N. Deruelle and J. Rua, “Disformal Transformations, Veiled General Relativity and Mimetic Gravity,” JCAP 1409, 002 (2014) doi:10.1088/1475-7516/2014/09/002 [arXiv:1407.0825 [gr-qc]].

[16] F. Arroja, N. Bartolo, P. Karmakar and S. Matarrese, “The two faces of mimetic Horndeski gravity: disformal transformations and Lagrange multiplier,” JCAP 1509, 051 (2015) doi:10.1088/1475-7516/2015/09/051 [arXiv:1506.08575 [gr-qc]].

[17] R. Myrzakulov, L. Sebastiani, S. Vagnozzi and S. Zerbini, “Static spherically symmetric solutions in mimetic gravity: rotation curves and wormholes,” Class. Quant. Grav. 33, no. 12, 125005 (2016) doi:10.1088/0264-9381/33/12/125005 [arXiv:1510.02284 [gr-qc]].

[18] L. Sebastiani, S. Vagnozzi and R. Myrzakulov, “Mimetic gravity: a review of recent developments and applications to cosmology and astrophysics,” Adv. High Energy Phys. 2017, 3156915 (2017) doi:10.1155/2017/3156915 [arXiv:1612.08661 [gr-qc]].

[19] D. Bettoni and S. Liberati, “Disformal invariance of second order scalar-tensor theories: Framing the Horndeski action,” Phys. Rev. D 88, 084020 (2013) doi:10.1103/PhysRevD.88.084020 [arXiv:1306.6724 [gr-qc]].

[20] M. Zumalacárregui and J. García-Bellido, “Transforming gravity: from derivative couplings to matter to second-order scalar-tensor theories beyond the Horndeski Lagrangian,” Phys. Rev. D 89, 064046 (2014) doi:10.1103/PhysRevD.89.064046 [arXiv:1308.4685 [gr-qc]].

[21] J. Gleyzes, D. Langlois, F. Piazza and F. Vernizzi, “Exploring gravitational theories beyond Horndeski,” JCAP 1502, 018 (2015) doi:10.1088/1475-7516/2015/02/018 [arXiv:1406.1952 [astro-ph.CO]].

[22] N. Kaloper, “Disformal inflation,” Phys. Lett. B 583, 1 (2004) doi:10.1016/j.physletb.2004.01.005 [hep-th/0312002].

[23] E. Bittencourt, S. Faci and M. Novello, “Chiral symmetry breaking as a geometrical process,” Int. J. Mod. Phys. A 29, no. 26, 1450145 (2014) doi:10.1142/S0217751X14501450 [arXiv:1406.1909 [hep-th]].

[24] M. Novello and E. Bittencourt, “A geometrical origin of the right-handed neutrino magnetic moment,” Int. J. Mod. Phys. A 29, 1450075 (2014) doi:10.1142/S0217751X14500754 [arXiv:1411.2347 [physics.gen-ph]].

[25] F. T. Falciano and E. Goulart, “A new symmetry of the relativistic wave equation,” Class. Quant. Grav. 29, 085011 (2012) doi:10.1088/0264-9381/29/8/085011 [arXiv:1112.1341 [gr-qc]].

[26] E. Goulart and F. T. Falciano, “Disformal invariance of Maxwell’s field equations,” Class. Quant. Grav. 30, 155020 (2013) doi:10.1088/0264-9381/30/15/155020 [arXiv:1303.4350 [gr-qc]].

[27] E. Bittencourt, I. P. Lobo and G. G. Carvalho, “On the disformal invariance of the Dirac equation,” Class. Quant. Grav. 32, 185016 (2015) doi:10.1088/0264-9381/32/18/185016 [arXiv:1505.03415 [gr-qc]].
[28] G. G. Carvalho, I. P. Lobo and E. Bittencourt, “Extended disformal approach in the scenario of Rainbow Gravity”, Phys. Rev. D 93, no. 4, 044005 (2016) doi:10.1103/PhysRevD.93.044005 [arXiv:1511.00495 [gr-qc]].

[29] M. Novello and E. Bittencourt, “Dragged Metrics,” Gen. Rel. Grav. 45, 1005 (2013) doi:10.1007/s10714-013-1507-z [arXiv:1201.2806 [physics.gen-ph]].

[30] M. Novello and E. Goulart, “Beyond Analog Gravity: The Case of Exceptional Dynamics,” Class. Quant. Grav. 28, 145022 (2011) doi:10.1088/0264-9381/28/14/145022 [arXiv:1102.1913 [gr-qc]].

[31] R. M. Wald, General Relativity (The University of Chicago Press, Chicago, 1984).

[32] F. F. Yuan and P. Huang, “Induced geometry from disformal transformation”, Phys. Lett. B 744, 120 (2015) doi:10.1016/j.physletb.2015.03.031 [arXiv:1501.06135 [gr-qc]].

[33] Y. Choquet-Bruhat, C. DeWitt-Morette and M. Dillard-Bleick, Analysis, Manifolds and Physics, Vol. 1 (North Holland Publishing Company, Amsterdam, 1977).

[34] V. Baccetti, P. Martin-Moruno and M. Visser, “Gordon and Kerr-Schild ansatze in massive and bimetric gravity,” JHEP 1208, 108 (2012) doi:10.1007/JHEP08(2012)108 [arXiv:1206.4720 [gr-qc]].

[35] Unlike the time-like case, in the null case, the inverse transformation is not equal to the transformation of the inverse metric.