Surface effects on a mode-III reinforced nano-elliptical hole embedded in one-dimensional hexagonal piezoelectric quasicrystals

Zhina ZHAO¹, Junhong GUO¹,²,†

1. Department of Mechanics, Inner Mongolia University of Technology, Hohhot 010051, China;
2. School of Aeronautics, Inner Mongolia University of Technology, Hohhot 010051, China

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Abstract To effectively reduce the field concentration around a hole or crack, an anti-plane shear problem of a nano-elliptical hole or a nano-crack pasting a reinforcement layer in a one-dimensional (1D) hexagonal piezoelectric quasicrystal (PQC) is investigated subject to remotely mechanical and electrical loadings. The surface effect and dielectric characteristics inside the hole are considered for actuality. By utilizing the technique of conformal mapping and the complex variable method, the phonon stresses, phason stresses, and electric displacements in the matrix and reinforcement layer are exactly derived under both electrically permeable and impermeable boundary conditions. Three size-dependent field intensity factors near the nano-crack tip are further obtained when the nano-elliptical hole is reduced to the nano-crack. Numerical examples are illustrated to show the effects of material properties of the surface layer and reinforced layer, the aspect ratio of the hole, and the thickness of the reinforcing layer on the field concentration of the nano-elliptical hole and the field intensity factors near the nano-crack tip. The results indicate that the properties of the surface layer and reinforcement layer and the electrical boundary conditions have great effects on the field concentration of the nano-hole and nano-crack, which are useful for optimizing and designing the microdevices by PQC nanocomposites in engineering practice.

Key words surface effect, reinforcement layer, exact solution, piezoelectric quasicrystal (PQC), nano-hole/crack

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† Corresponding author, E-mail: jhguo@imut.edu.cn
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1 Introduction

Different from traditional forms of solid state, quasicrystals (QCs) exhibit a conventionally forbidden rotational symmetry, incompatible with translational periodicity. The discovery of QCs since 1984 has changed people’s inherent understanding of solid structures[1]. A lot of work has been done on the structures, optics, and mechanical properties of QCs[2–5]. The research shows that QCs have the characteristics of low friction coefficient, low thermal conductivity, wear resistance, corrosion resistance, and plasticity at high temperature[6–8]. Based on these properties, QCs are widely used in surface coatings, thermal insulation materials, solar energy, industrial thin film materials, and reinforced phase of structural materials[9–10]. It is found experimentally that at room temperature, QCs are so brittle that cracks or holes may appear during the manufacturing process[11]. The presence of defects can affect the physical and mechanical properties of QCs. Therefore, it is of significance to investigate the fracture behavior of QCs with defects.

Based on the classical elasticity theory, a lot of research has been done on the defects of QCs. Wang and Pan[12] studied some typical defect problems in one-dimensional (1D) hexagonal and two-dimensional (2D) octagonal QCs. Li et al.[13] derived a set of three-dimensional (3D) general solutions of 1D hexagonal piezoelectric quasicrystals (PQCs) by introducing two displacement functions and utilizing the rigorous operator theory. Yang and Li[14] investigated the anti-plane shear problem of a circular hole with a straight crack in 1D hexagonal PQCs by utilizing the complex variable function method and the technique of conformal mapping. Furthermore, Bai and Ding[15] presented the analytical solutions of stress distributions and field intensity factors at the crack tip near the regular hexagonal hole in 1D hexagonal PQCs under the electrically impermeable boundary condition. Jiang and Liu[16] considered the interaction between a screw dislocation and a wedge-shaped crack in 1D hexagonal PQCs by the conformal mapping and perturbation techniques. Based on the local Petrov-Galerkin approach, Sladek et al.[17] proposed a meshless method to solve the initial-boundary-value crack problems in decagonal QCs. Wang and Ricoeur[18] developed numerical tools in a finite element environment to compute the fracture quantities of 1D QCs and analyzed the phonon-phason coupling effect on crack paths. Based on the operator theory and the Fourier transform, Fan et al.[19] proposed the extended displacement discontinuity method to analyze cracks in the periodic plane of 1D hexagonal QC with the heat effect. Zhao et al.[20] used this method to consider a 3D interface crack of arbitrary shapes in a 1D hexagonal thermo-electro-elastic QC bi-material. Topholme[21] derived the closed-form solutions of a moving non-constantly loaded anti-plane Griffith-type strip crack embedded in 1D PQCs. Zhou and Li[22] considered a Yoffe-type moving crack in 1D hexagonal PQCs by using the Fourier transform technique. By using the integral transform technique, Hu et al.[23] studied an interface crack between dissimilar 1D hexagonal PQCs under anti-plane shear and in-plane electrical loadings. Li et al.[24] analyzed the interaction between a screw dislocation and an elliptical hole with two asymmetrical cracks in 1D hexagonal PQCs. Loboda et al.[25] presented an analytical approach to an electrically permeable interface crack in 1D PQCs. Also, the elastic field near the tip of an anticrack[26] and the arbitrarily shaped planar crack[27] in 2D PQCs were analyzed.

Although a sufficiently large number of solutions associated with defects have been obtained for QCs based on the classical elasticity theory, very few investigations have been known for nano-QCs. Nano-QC composites normally produced by rapid solidification (melt-spinning or gas atomization) containing icosahedral QC particles with typically size under 500 nm[28] are attractive potential structural materials in the automotive and aeronautical industries due to their high strength at elevated temperatures[29]. Experimental observation showed that decreasing the reinforcement particle size to nanoscale could improve both the mechanical strength and the ductility of metal matrix composites[30]. Defect problems in nano-QCs are more complicated than those in macro-scale due to the introduction of the non-classical boundary
The surface/interface effect should be considered in the analysis of nano-structures due to the increasing ratio of surface to bulk volume. Based on the surface/interface model, some defects in piezoelectric materials were analyzed\cite{31–32}. However, very few studies on defects in nano-QCs and nano-PQCs have been done up to now.

The stress concentration around the hole or crack can be effectively reduced by pasting the reinforcement layer with proper shapes and material characteristics\cite{33–35}. Therefore, this paper focuses on an anti-plane shear problem of a reinforced nano-elliptical hole or nano-crack with surface effects in a 1D hexagonal PQC composite to adjust the phonon stress concentration, phason stress concentration, and electric displacement concentration around the hole or crack, which may provide a theoretical reference for designing new composites in engineering practice.

This paper is organized as follows. In Section 2, we present the problem description and basic equations for an anti-plane shear problem of 1D hexagonal PQC composites with surface effects. In Section 3, the size-dependent phonon stress, phason stress, and electric displacement in the reinforced layer and matrix are derived exactly. Numerical examples are illustrated in Section 4, and conclusions are drawn in Section 5.

## 2 Problem description and basic equations

An anti-plane shear problem of an electrically permeable reinforced nano-elliptical hole in 1D hexagonal PQC’s is considered (see Fig. 1(a)). The regions \( \Omega_L, \Omega_L', \) and \( \Omega_M \) represent the nano-elliptical hole, the reinforcement layer, and the matrix, respectively. It is assumed that the reinforcement layer and matrix have different material properties but they are poled along the \( z \)-direction. According to the Gurtin-Murdoch surface/interface model\cite{36–37}, the material properties of the holed surface without thickness are different from those of the reinforcement layer and matrix. For simplicity, we assume that the two ellipses surrounding the hole and the reinforcement layer are confocal with the common foci \((-c, c), (a_1, b_1)\) and \((a_2, b_2)\) are the major semi-axes and minor semi-axes of the ellipse around the hole and the ellipse around the reinforcement layer, respectively. The interface along the ellipse between the reinforcement layer and the matrix is perfectly bonded.

To solve this problem, we introduce a conformal mapping\cite{31–32} \( z = \omega(\zeta) = \zeta + n/\zeta, \) with \( n = (a_1^2 + b_1^2)/4, \) which maps the region of matrix \( \Omega_M \) in the \( z \)-plane into the exterior region \( \Omega_M' \) of the circle with radius \( \rho^2 = (a_2 + b_2)/2 \) in the \( \zeta \)-plane, the region of the reinforced layer \( \Omega_L \) in the \( z \)-plane into the annular region \( \rho_1 \leq |\zeta| < \rho_2 \) in the \( \zeta \)-plane, where \( \rho_1 = (a_1 + b_1)/2, \) and the internal region \( \Omega_L' \) of the nano-elliptical hole in the \( z \)-plane into the circular ring region \( \Omega_L'' (\rho_0 \leq |\zeta| < \rho_1) \) in the \( \zeta \)-plane, where \( \rho_0 = \sqrt{n}, \) as shown in Fig. 1(b).

For simplicity, the generalized stresses (phonon stress, phason stress, and electric displacement) and the generalized strains (phonon strain, phason strain, and electric field) are denoted by the vectors \( \Sigma_j = (\sigma_{zz}, H_{zz}, D_j)^T \) and \( Z_j = (\varepsilon_{zz}, \omega_{zz}, -E_j)^T, \) respectively, where \( j = x, y, \) and the superscript \( T \) denotes the transpose of the vector or matrix. Using the complex variable function method, an analytic function vector \( f(z) \) is introduced. Then, the phonon displacement \( u_z, \) the phason displacement \( w_z, \) and the electric potential \( \phi \) for the current anti-plane shear problem are only functions of the variables \( x \) and \( y, \) and \( u = (u_z, w_z, \phi)^T = \text{Im}(f(z)). \)

The basic equations for the anti-plane shear problem of 1D hexagonal PQC materials are

\[
\Sigma_y + i\Sigma_x = B(Z_y + iZ_x) = Bf'(z), \quad B\nabla^2 u = 0, \quad (1)
\]

where \( i^2 = -1, \) \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) is a 2D Laplace operator, and \( B \) is the material coefficient matrix, i.e.,

\[
B = \begin{pmatrix}
\epsilon_{44} & R_3 & \epsilon_{15} \\
R_3 & K_2 & d_{15} \\
\epsilon_{15} & d_{15} & -\lambda_{11}
\end{pmatrix}, \quad (2)
\]
Fig. 1 A reinforced nano-elliptical hole in an infinite 1D hexagonal PQC matrix in (a) the $z$-plane and (b) the $\zeta$-plane (color online)

in which $c_{44}$, $K_2$, and $R_3$ are the phonon elastic constant, the phason elastic constant, and the coupling elastic constant between the phonon and phason fields, respectively, $e_{15}$ and $d_{15}$ are the piezoelectric constants, and $\lambda_{11}$ is the dielectric constant.

By utilizing the technique of conformal mapping, the relationship between the generalized stress and the generalized strain in the $\zeta$-plane can be expressed as

$$\mathbf{\Sigma}_\rho + i\mathbf{\Sigma}_\theta = B (\mathbf{Z}_\rho + i\mathbf{Z}_\theta) = e^{i\theta} B \frac{f(z)}{|\omega'(\zeta)|},$$

where $\rho$ and $\theta$ represent the polar coordinates in the $\zeta$-plane.

For the convenience of analysis, the field resultant force $\mathbf{F}$ along the arc $ab$ including the phonon field resultant force $F_\sigma$, the phason field resultant force $F_H$, and the sum of the normal components of the electric displacement $F_D$ is also introduced, which can be expressed as

$$\mathbf{F} = (F_\sigma \quad F_H \quad F_D)^T = \int_a^b \left( \begin{array}{ccc} \sigma_{zx} & H_{zx} & D_x \\ \sigma_{zy} & H_{zy} & D_y \end{array} \right)^T \, dy - \left( \sigma_{zy} & H_{zy} & D_y \right)^T \, dx = -B \text{Re}(f(z))|_a^b. \quad (4)$$

By considering the equilibrium of an element of a general curved interface with the unit normal vector $\mathbf{n}$ between the regions of hole $\Omega_E$ and reinforced layer $\Omega_L$, the equilibrium equation of the interface can be obtained as follows\cite{38-39}:

$$(\mathbf{\sigma}^1 - \mathbf{\sigma}^2) \cdot \mathbf{n} = \nabla^S \cdot \mathbf{\sigma}^S, \quad (5)$$

where $\mathbf{\sigma}^1$ and $\mathbf{\sigma}^2$ are the volume stress tensors in $\Omega_E$ and $\Omega_L$, respectively; the right-hand side of Eq. (5) is the surface divergence of the surface stress tensor $\mathbf{\sigma}^S$. Equation (5) is the
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generalized Young-Laplace equation for solids, which characterizes the interfacial behavior between two different solids with interfacial stresses. For specific geometries of interface, i.e., circular cylindrical coordinates $(\rho, \theta, z)$ in the current problem, Eq. (5) can be simplified as\(^{[40]}\)

$$
\Sigma^E_\rho - \Sigma^L_\rho = \frac{1}{\rho_1} \frac{\partial \Sigma^S_{\theta\theta}}{\partial \theta} - \frac{B^S}{\rho_1} \frac{\partial Z^S_{\theta\theta}}{\partial \theta},
$$

(6)

where superscripts $E$, $L$, and $S$ denote quantities of the nano-elliptical hole, reinforcement layer, and nano-hole surface, respectively, $\Sigma^E_\rho = (0, 0, D^E_T)^T$ is an additional generalized stress due to the dielectric property inside the nano-hole, and $B^S$ is the material matrix of the surface of the nano-hole, which can be determined from atomistic simulations and experimental methods.

By considering the dielectric characteristics inside the nano-elliptical hole as shown in Fig. 1(a), the electric displacement and electric potential induced by the internal electric field can be simplified from Eq. (1) as

$$
-\lambda^E_{11} \nabla^2 \phi^E = 0, \quad D^E_i = -\lambda^E_{11} \frac{\partial \phi^E}{\partial x_i},
$$

(7)

where $\lambda^E_{11}$ is the dielectric constant of the medium inside the nano-hole, and $\phi^E = \text{Im}(\Phi^E)$ in which $\Phi^E$ satisfies

$$
\text{Im}(\Phi^E) = \text{Im}(\Phi^L).
$$

(8)

3 Exact solutions

For the interfaces, the tangential components of the generalized strains, displacements, and resultant forces at the interface are continuous, i.e.,

$$
\begin{align*}
Z^S_\theta & = Z^L_\theta \quad \text{at} \quad \rho = \rho_1, \\
u^L & = u^M, \quad F^L = F^M \quad \text{at} \quad \rho = \rho_2.
\end{align*}
$$

(9)

The following single-value condition is satisfied:

$$
\Phi^E(\sqrt{n}e^{i\theta}) = \Phi^E(\sqrt{n}e^{-i\theta}).
$$

(10)

In the nano-elliptic hole, reinforced layer, and matrix, the analytical functions in the $\zeta$-plane can be expanded into the following Laurent series with finite items\(^{[35]}\):

$$
\begin{align*}
f^L & = a^L \zeta + b^L \zeta^{-1} = a^L \frac{z + \sqrt{z^2 - c^2}}{2} + b^L \frac{z - \sqrt{z^2 - c^2}}{2n}, \\
u^L & = a^M \zeta + b^M \zeta^{-1} = a^M \frac{z + \sqrt{z^2 - c^2}}{2} + b^M \frac{z - \sqrt{z^2 - c^2}}{2n}, \\
\phi^E & = e^E \zeta + f^E \zeta^{-1} = e^E \frac{z + \sqrt{z^2 - c^2}}{2} + f^E \frac{z - \sqrt{z^2 - c^2}}{2n},
\end{align*}
$$

(11)

where $c = \sqrt{a_1^2 - b_1^2} = \sqrt{a^2_E - b^2_E}$.

Substituting Eq. (11) into Eq. (6), the last equation of Eq. (9), and Eq. (10), we obtain the relation between unknown quantities as follows:

$$
\begin{align*}
b^L & = \rho_1^2 \left( \frac{B^S}{\rho_1} + B^L \right)^{-1} \left( \frac{B^S}{\rho_1} - B^L \right) a^L + \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \left( -\lambda^E_{11} (e^E + \rho_1^2 f^E) \right), \\
f^L & = e^L \rho_1^2 - e^E \rho_1^2 + f^E, \quad f^E = ne^E,
\end{align*}
$$

(12)
where $b^L = (b^L, q^L, f^L)^T$, $a^L = (a^L, p^L, e^L)^T$, and $e^E$ and $f^E$ are two unknown constants.

When the PQC materials are subject to far-field electromechanical loads, one easily has

$$a^M = \left( \varepsilon_{xy}^\infty \omega_{xy}^\infty - E_y^\infty \right)^T,$$

(13)

where $\varepsilon_{xy}^\infty$, $\omega_{xy}^\infty$, and $E_y^\infty$ are the given phonon strain, phason strain, and electric field at infinity, respectively.

By substituting the last two equations of Eq. (12) and Eq. (13) into the first equation of Eq. (12), the constant term $e^E$ can be obtained as follows:

$$e^E = \eta_1 a^L + \eta_2 p^L + \eta_3 e^L,$$

(14)

where

$$\eta_1 = \left(2\rho_1^2 \left( \det(B_{12}) \left( \frac{c_{44}^S c_{15}^L - c_{44}^S c_{15}^L}{\rho_1} - \det(B_{14}) \frac{R_{13}^L c_{15}^S - R_{15}^S c_{15}^S}{\rho_1} \right) \right) \right) /

\left( (n - \rho^2_1)(\det(B_{11}) \times \det(B_{12}) - \det(B_{13}) \times \det(B_{14})) - \lambda_{11}^E (n + \rho^2_1) \right)$$

$$\eta_2 = \left( \rho_1^2 \left( \frac{d_{15}^S}{\rho_1} - d_{15}^L \right) \left( \det(B_{12}) \left( \frac{c_{44}^S}{\rho_1} + \frac{c_{14}^L}{\rho_1} - \det(B_{14}) \left( \frac{R_{13}^S}{\rho_1} - R_{15}^L \right) \right) \right) \right) /

\left( (n - \rho^2_1)(\det(B_{11}) \times \det(B_{12}) - \det(B_{13}) \times \det(B_{14})) - \lambda_{11}^E (n + \rho^2_1) \right)$$

$$\eta_3 = \left( -2\rho_1^2 \lambda_{11}^E \left( \det(B_{14}) \left( \frac{R_{13}^S}{\rho_1} + R_{15}^L \right) - \det(B_{12}) \left( \frac{c_{44}^S}{\rho_1} + \frac{c_{14}^L}{\rho_1} \right) \right) \right) +

\left( 2\rho_1^2 \left( \frac{c_{44}^S}{\rho_1} + \frac{c_{14}^L}{\rho_1} \right) \left( \det(B_{12}) - d_{15}^L \det(B_{14}) \right) \right) \right) /

\left( (n - \rho^2_1)(\det(B_{11}) \times \det(B_{12}) - \det(B_{13}) \times \det(B_{14})) - \lambda_{11}^E (n + \rho^2_1) \right)$$

(15)

In Eq. (15), $B_{1k}$ ($k = 1, 2, \cdots, 4$) is the second-order submatrix of matrix $B_1 = B^S / \rho_1 + B^L$, and $\det(B_{1k})$ is the determinant of the following matrix $B_{1k}$:

$$B_{11} = \begin{pmatrix}
\frac{c_{44}^S}{\rho_1} + c_{44}^L & \frac{c_{14}^S}{\rho_1} + c_{14}^L \\
\frac{c_{15}^S}{\rho_1} + c_{15}^L & \lambda_{11}^S - \lambda_{11}^L
\end{pmatrix}, \quad B_{12} = \begin{pmatrix}
\frac{R_{13}^S}{\rho_1} + R_{15}^L & \frac{K_{22}^S}{\rho_1} + \frac{K_{22}^L}{\rho_1} \\
\lambda_{11}^S + \lambda_{11}^L & \frac{d_{15}^S}{\rho_1} + d_{15}^L + d_{15}^L
\end{pmatrix},$$

$$B_{13} = \begin{pmatrix}
\frac{d_{15}^S}{\rho_1} + \frac{d_{15}^L}{\rho_1} & \lambda_{11}^S - \lambda_{11}^L \\
\frac{d_{15}^S}{\rho_1} + d_{15}^L & \lambda_{11}^S + \lambda_{11}^L
\end{pmatrix}, \quad B_{14} = \begin{pmatrix}
\frac{c_{44}^S}{\rho_1} + c_{44}^L & \frac{R_{13}^S}{\rho_1} + \frac{R_{15}^L}{\rho_1} \\
\lambda_{11}^S - \lambda_{11}^L & \frac{d_{15}^S}{\rho_1} + d_{15}^L + d_{15}^L
\end{pmatrix}.$$

(16)

Therefore, the first equation of Eq. (12) can be simplified to

$$b^L = Q_1 a^L,$$

(17)
where

\[ Q_1 = \rho_1^2 \left( \frac{B^S}{\rho_1} + B^L \right)^{-1} \left( \frac{B^S}{\rho_1} - B^L \right) - \lambda_{11}^E (1 + \eta \rho_1^{-2}) T \]  

(18)

with the matrix \( T \)

\[ T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \eta_1 & \eta_2 & \eta_2 \end{pmatrix}. \]  

(19)

Obviously, the matrix \( T \) satisfies the relationship \( Ta^L = (0, 0, e^E)^T \).

According to the last term in Eq. (9), the following relation is obtained:

\[ a^L \rho_2^2 - b^L = a^M \rho_2^2 - b^M, \quad B^L(a^L \rho_2^2 + b^L) = B^M(a^M \rho_2^2 + b^M). \]  

(20)

Thus, all the following constant vectors can be determined by \( a^M \):

\[ a^L = 2Pa^M, \quad b^M = Q_2a^M, \quad b^L = 2Q_1Pa^M, \]  

(21)

where

\[ P = (I + (B^M)^{-1}B^L + \rho_2^{-2}((B^M)^{-1}B^L - I)Q_1)^{-1}, \quad Q_2 = \rho_1^2(I - 2P) + 2Q_1P, \]  

(22)

in which \( I \) is the third-order identity matrix.

Along the \( x \)-axis, the phonon stresses, phason stresses, and electric displacements of the electrically permeable nano-elliptical hole in the reinforcement layer and matrix are finally obtained as

\[
\begin{pmatrix}
\sigma^L_{xy}(x, 0) \\
H^L_{zy}(x, 0) \\
D^L_y(x, 0)
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
\eta_1 \eta_2 \eta_2
\end{pmatrix}.
\]

(23)

Furthermore, the generalized stresses at points \( A \) and \( B \) in Fig. 1(a) of the electrically permeable nano-elliptical hole are expressed as follows:

\[
\begin{pmatrix}
\sigma^A_{xy} \\
H^A_{zy} \\
D^A_y
\end{pmatrix} = \frac{1}{\beta_1} B^L \left( I - \rho_1^{-2} Q_1 \right) \begin{pmatrix}
\sigma^\infty_{xy} \\
H^\infty_{zy} \\
D^\infty_y
\end{pmatrix},
\]

(24)

\[
\begin{pmatrix}
\sigma^B_{xy} \\
H^B_{zy} \\
D^B_y
\end{pmatrix} = \frac{1}{\beta_2} B^M \left( I - \rho_2^{-2} Q_2 \right) \begin{pmatrix}
\sigma^\infty_{xy} \\
H^\infty_{zy} \\
D^\infty_y
\end{pmatrix},
\]

where \( \beta_1 = b_1/a_1 \) and \( \beta_2 = b_2/a_2 \). In the case of an electrically impermeable boundary condition and the dielectric coefficient \( \lambda_{11}^E = 0 \), Eq. (18) can be simplified as

\[ Q_1 = \rho_1^2 \left( \frac{B^S}{\rho_1} + B^L \right)^{-1} \left( \frac{B^S}{\rho_1} - B^L \right). \]  

(25)

As special cases, the present solutions can be reduced to the previous results, such as a nano-elliptical hole in piezoelectric materials. Besides, some new results including the reinforced Griffith nano-crack and the reinforced nano-circular hole in PQC, piezoelectric (PE), and QC materials for two kinds of electrical boundary conditions can also be derived from the present results, which are neglected here.
4 Numerical examples

In this section, the effects of the surface layer properties, the reinforcing layer properties, and the thickness of reinforced layer on the field concentration near the nano-elliptical hole and the field intensity factors near the nano-crack tip in PQCs are analyzed. The material properties of the matrix and surface layer are listed in Tables 1 and 2, respectively.\[13,31\]

![Table 1 Material properties of PQC and PE material (PZT-5) matrix](image)

| Property         | PQC      | PZT-5  | Property | PQC | PZT-5 |
|------------------|----------|--------|----------|-----|-------|
| $c_{44}$/(N·m$^{-2}$) | 50×10$^9$ | 21.1×10$^9$ | $K_2$/(N·m$^{-2}$) | 0.3×10$^9$ | – |
| $\lambda_1$/(C$^2$·N$^{-1}$·m$^{-2}$) | 82.6×10$^{-12}$ | 8.107×10$^{-9}$ | $R_3$/(N·m$^{-2}$) | 1.2×10$^9$ | – |
| $e_{15}$/(C·m$^{-1}$) | –0.138 | 12.3 |
| $\lambda_{11}$/(C$^2$·N$^{-1}$·m$^{-2}$) | –0.16 | – |

![Table 2 Material properties of the surface layer](image)

| Property         | PQC | PZT-5 |
|------------------|-----|-------|
| $c_{33}$/(N·m$^{-1}$) | 0.6 | 0.8 |
| $K_2$/(N·m$^{-1}$) | 0 | 5×10$^{-9}$ |
| $\lambda_1$/(C$^2$·N$^{-1}$·m$^{-2}$) | 0 | 0 |
| $R_3$/(N·m$^{-1}$) | 0.8 | 0 |
| $e_{15}$/(C·m$^{-1}$) | 5×10$^{-9}$ | 0 |

Figure 2 shows the variations of the stress and electric displacement intensity factors near the nano-crack tip induced by remotely mechanical loads in a reinforced piezoelectric material with the crack length $a_1$ under two electrical boundary conditions for $a_2/a_1 = 1$ and $\beta_2 = b_2/a_2 = \sqrt{11}/6$. It can be found from Figs. 2(a) and 2(b) that the stress and electric displacement intensity factors in piezoelectric materials under the electrically impermeable boundary condition agree well with the results obtained by Xiao et al.\[31\] when $L_C = 1$ (without reinforcement).

![Fig. 2 Stress and electric displacement intensity factors induced by mechanical loads at the nano-crack tip in a reinforced piezoelectric material under two electrical boundary conditions](image)
Besides, the stress and electric displacement intensity factors in piezoelectric materials under the electrically permeable boundary condition are the same as those obtained by Guo and Li\cite{32} when \( L_C = 1 \) (see Figs. 2(c) and 2(d)). Moreover, the soft reinforcement \((L_C < 1)\) could decrease the magnitude of the stress and electric displacement intensity factors near the nano-crack tip, while the rigid reinforcement \((L_C > 1)\) could increase the magnitude of the stress and electric displacement intensity factors near the nano-crack tip, indicating that the soft reinforcement in piezoelectric materials can reduce the stress concentration near the crack tip. When the crack size becomes large, the field intensity factors with surface effect approach to the classical results (without surface effect).

Figure 3 shows the effects of the thickness of reinforced layer \(a_2/a_1\) on the phonon stress,
phason stress, and electric displacement intensity factors near the nano-crack tip in a reinforced
PQC composite under two electrical boundary conditions for given $a_1 = 5 \text{ nm}$ and $L_C = 0.5$. It can be observed that an electrically permeable nano-crack with the surface effect displays a different behavior from an electrically impermeable one. In general, three fields including the phonon field, phason field, and electric field are uncoupled in the classical elasticity theory, but they are coupled with the surface effect for the electrically impermeable case.

Figure 4 shows the effects of the material property of reinforcement $L_C$ on the phonon stress, phason stress, and electric displacement intensity factors near the nano-crack tip in a reinforced PQC composite under two electrical boundary conditions for given $a_1 = 5 \text{ nm}$, $a_2/a_1 = 1.2$, and $\beta_2 = b_2/a_2 = \sqrt{11}/6$. It can be seen that for an electrically permeable nano-crack, the field

![Image](image-url)

**Fig. 4** Variations of the phonon stress, phason stress, and electric displacement intensity factors near the nano-crack tip in a reinforced PQC composite with the material property of reinforcement layer $L_C$ (color online)
intensity factors induced by electrical loadings are always zero, but they are greatly dependent on the material property of reinforcement $L_c$ under the mechanical loadings of phonon field. By contrast, the field intensity factors induced by mechanical or electrical loadings are always dependent on $L_c$ for an electrically impermeable nano-crack.

Figure 5 shows the variations of the phonon stress, phason stress, and electric displacement in the reinforced layer (short for RL in Fig. 5) and matrix (short for M in Fig. 5) of 1D hexagonal PQCs along the $x$-axis direction under far-field mechanical and electrical loads for given $\lambda_{11}^E = 8.85 \times 10^{-12} \text{C}^2 \cdot \text{N}^{-1} \cdot \text{m}^2$, $L_c = 0.5$, $a_1 = 5 \text{nm}$, $a_2 = 6 \text{nm}$, $\beta_1 = 0.5$, and $\beta_2 = \sqrt{69}/3$. It can be found that the phonon stress, phason stress, and electric displacement have a big jump across the interface between the reinforced layer and the matrix.

Figure 6 displays the effects of the aspect ratio of the nano-elliptical hole and the reinforced
Fig. 6 Effects of the aspect ratio of the elliptical hole and the reinforced layer on the phonon stress, phason stress, and electric displacement at points A and B under mechanical loadings (color online).

Layer on the phonon stress, phason stress, and electric displacement at points A and B in 1D hexagonal PQCs under far-field mechanical loads for given $a_2/a_1 = 1.2$ and $L_C = 0.5$. It can be observed that the magnitudes of the phonon stress, phason stress, and electrical displacement at points A and B decrease with the increase in the aspect ratio of the elliptical hole and the reinforced layer. The field concentration at the point A is smaller than that at the point B, which is attributed to the soft reinforced layer surrounding the hole.

Figure 7 plots the variations of the phonon stress, phason stress, and electric displacement around the nano-circular hole in 1D hexagonal PQCs under far-field mechanical and electrical loads for given $a_2/a_1 = 1.2$ and $L_C = 0.5$. It is seen that the phonon stress, phason stress, and electric displacement around the circular hole are symmetrical about the y-axis ($\theta = \pi/2$). Furthermore, these three fields are nonzero at $\theta = \pi/2$ with surface effect, which are different
Fig. 7 Variations of the phonon stress, phason stress, and electric displacement around the circular hole under mechanical and electrical loadings (color online)

from the classical results. The phonon stress concentration induced by mechanical loadings (see Fig. 7(a)) and the electric displacement concentration induced by electrical loadings (see Fig. 7(f)) occur at $\theta = 0$ and $\pi$, but the other field concentrations occur at $\theta = \pi/2$. It is interesting to note that the effects of the dielectric constant inside the hole on the phonon stress, phason stress, and electric displacement for an electrically permeable hole induced by electrical loadings are larger than those induced by mechanical loadings as compared with the electrically impermeable case. Besides, under the electrical loadings, the field concentration for an electrically permeable hole is smaller than that for an electrically impermeable one.

5 Conclusions

Based on the surface/interface model, an anti-plane shear problem of a reinforced nano-elliptical hole in 1D hexagonal PQC composites is considered under both electrically permeable
and impermeable boundary conditions. Exact solutions in closed-form of the phonon stresses, phason stresses, and electric displacements in the matrix and reinforced layer are derived by the complex variable method. The phonon stress, phason stress, and electric displacement intensity factors near the nano-crack tip are further obtained when the elliptical hole is reduced to a crack. Some useful features can be found from the numerical examples as follows.

(I) The soft reinforcement could decrease the magnitudes of stress and electric displacement intensity factors near the nano-crack tip, while the rigid reinforcement could increase the magnitudes of stress and electric displacement intensity factors near the nano-crack tip.

(II) The electrically permeable nano-hole or nano-crack with surface effects shows a different behavior from an electrically impermeable one. The phonon field, phason field, and electric field are coupled with surface effects for the electrically impermeable boundary condition, which are also different from the classical elasticity results.

(III) The material property and thickness of the reinforced layer have great effects on the phonon field, phason field, and electric field around the nano-holes and nano-cracks.

(IV) The phonon stress, phason stress, and electric displacement of the nano-elliptical hole are discontinuous at the interface between the matrix and the reinforcement layer. The stress concentration around both the hole and the reinforced layer always decreases with the increasing aspect ratio of the hole and the reinforced layer.

(V) Under electrical loadings, the dielectric characteristics inside holes should be considered. Thus, the electrically permeable boundary conditions should be adopted in the analysis.

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