An Extension of Cui-Kano’s Characterization Problem on Graph Factors

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Abstract

Let $G$ be a graph with vertex set $V(G)$ and let $H : V(G) \to 2^N$ be a set function associating with $G$. An $H$-factor of graph $G$ is a spanning subgraphs $F$ such that

$$d_F(v) \in H(v) \quad \text{for every } v \in V(G).$$

Let $f : V(G) \to N$ be an even integer-valued function such that $f \geq 4$ and let $H_f(v) = \{1, 3, \ldots, f(v) - 1, f(v)\}$ for $v \in V(G)$. In this paper, we investigate $H_f$-factors of graphs $G$ by using Lovász’s structural descriptions. Let $o(G)$ denote the number of odd components of $G$. We show that if one of the following conditions holds, then $G$ contains an $H_f$-factor.

(i) $o(G - S) \leq f(S)$ for all $S \subseteq V(G)$;
(ii) $|V(G)|$ is odd, $d_G(v) \geq f(v) - 1$ for all $v \in V(G)$ and $o(G - S) \leq f(S)$ for all $\emptyset \neq S \subseteq V(G)$.

As a corollary, we show that if a graph $G$ with odd order and minimum degree $2n - 1$ satisfies

$$o(G - S) \leq 2n|S| \quad \text{for all } \emptyset \neq S \subseteq V(G),$$

then $G$ contains an $H_n$-factor. In particular, we make progress on the characterization problem for a special family of graphs proposed by Akiyama and Kano.

1 Introduction

All graphs in this paper are simple. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. We denote the degree of $v$ in $G$ by $d_G(v)$. The minimum degree in graph $G$ will be denoted by $\delta(G)$ and the maximum degree by $\Delta(G)$. The subgraph induced by the set $S$

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is denoted by $G[S]$. The number of components of graph $G$ is denoted by $\omega(G)$ and the number of odd components of $G$ by $o(G)$. Let $E_G(S, T)$ denote the set of edges of graph $G$ with one end in $S$ and the other end in $T$ and $e_G(S, T) = |E_G(S, T)|$. The join $G = G_1 + G_2$, is the graph obtained from two vertex disjoint graphs $G_1$ and $G_2$ by joining each vertex in $G_1$ to every vertex in $G_2$.

Let $H$ be a function associating a subset of $\mathbb{Z}$ to each vertex of $G$. A spanning subgraph $F$ of graph $G$ is called an $H$-factor of $G$ if

$$d_F(x) \in H(x) \quad \text{for every vertex } x \in V(G).$$

By specifying $H(x)$ to be an interval or a special set, an $H$-factor becomes an $f$-factor, an $[a, b]$-factor or a $(g, f)$-factor, respectively.

Let $F$ be a spanning subgraph of $G$. Following Lovász [8], one may measure the “deviation” of $F$ from the condition (1) by

$$\nabla_H(F) = \sum_{v \in V(G)} \min \{|d_F(v) - h| : h \in H(v)\}.$$  \hspace{1cm} (2)

Moreover, the “solvability” of (1) can be characterized by

$$\nabla(H) = \min \{\nabla_H(F) : F \text{ is a spanning subgraph of } G\}.$$

The subgraph $F$ is said to be $H$-optimal if $\nabla_H(F) = \nabla(H)$. It is clear that $F$ is an $H$-factor if and only if $\nabla_H(F) = 0$, and any $H$-factor (if exists) is $H$-optimal. Let

$$Q = \{h_1, h_2, \ldots, h_m\},$$

where $h_1 < h_2 < \cdots < h_m$. Then $Q$ is called an allowed set if each of the gaps of $Q$ has at most one integer, i.e.,

$$h_{i+1} - h_i \leq 2 \quad \text{for all } 1 \leq i \leq m - 1.$$

A set function $H$ associating with $G$ is called an allowed set function (following [8]) if $H(v)$ is an allowed set for all $v \in V(G)$.

Lovász [8] showed that if $H$ is not an allowed set, then the decision problem of determining whether a graph has an $H$-factor is known to be $NP$-complete. Cornuéjols [3] provided the first polynomial algorithm for the problem with $H$ allowed.

A special case of $H$-factor problem is the so-called $(1, h)$-odd factor problem, i.e., the problem with

$$H(v) = \{1, 3, \ldots, h(v) - 2, h(v)\},$$

$$H(v) = \{1, 3, \ldots, h(v) - 2, h(v)\},$$
where $h : V(G) \to N$ be an odd function. For a constant odd integer $n \geq 1$, if $h(x) = n$ for all $x \in V(G)$, then $(1, h)$-odd factor is called $(1, n)$-odd factor. The first investigation of the $(1, n)$-odd factor problem is due to Amahashi [2], who gave a Tutte type characterization for graphs having a global odd factor.

**Theorem 1.1 (Amahashi)** Let $n$ be an odd integer. A graph $G$ has an $(1, n)$-odd factor if and only if

$$\omega(G - S) \leq n |S| \quad \text{for all subsets } S \subset V(G).$$

(3)

For general odd value functions $h$, Cui and Kano [4] established a Tutte type theorem.

**Theorem 1.2 (Cui and Kano, [4])** Let $h : V(G) \to N$ be odd value function. A graph $G$ has an $(1, h)$-odd factor if and only if

$$\omega(G - S) \leq h(S) \quad \text{for all subsets } S \subset V(G).$$

(4)

Noticing the form of the condition (4), they asked the question of characterizing graphs $G$ in terms of graph factors such that

$$\omega(G - S) \leq 2n |S| \quad \text{for all subsets } S \subset V(G).$$

(5)

Motivated by Cui-Kano’s problem, Lu and Wang [9] consider the degree prescribed subgraph problem for the special prescription

$$H_n = \{1, 3, \ldots, 2n - 1, 2n\}.$$  

(6)

**Theorem 1.3 (Lu and Wang, [9])** Let $G$ be a connected graph. If

$$\omega(G - S) \leq 2n |S| \quad \text{for all } S \subset V(G),$$

(7)

then $G$ contains an $H_n$-factor.

The condition of Theorem 1.3 implies that $|V(G)|$ is even. Let $H_n^* = H_n \cup \{-1\}$. For odd order graph, they obtained the following result (for convenience, the definition of $H_n^*$-critical graph will be introduced in Section 2).

**Theorem 1.4 (Lu and Wang, [9])** Let $G$ be a connected graph of odd order. Suppose that

$$\omega(G - S) \leq 2n|S| \quad \text{for all } \emptyset \neq S \subset V(G).$$

(8)

Then either $G$ contains an $H_n$-factor, or $G$ is $H_n^*$-critical.
The condition (4) implies that the graph is even order. For odd order graph, Akiyama and Kano propose the following problem (see also [1, Problem (6.14)]).

**Problem 1.5 (Akiyama and Kano, [1])** Let $G$ be a connected graph and $h : V(G) \to \mathbb{N}$ be an even integer-valued function. If $G$ satisfies

$$o(G - S) \leq h(S) \quad \text{for all } \emptyset \neq S \subset V(G),$$

what factor or property does $G$ have?

Let $f \geq 4$ be an even integer-value function and let $H_f : V(G) \to 2^\mathbb{N}$ be an set function such that $H_f(v) = \{1, 3, \ldots, f(v) - 1, f(v)\}$ for $v \in V(G)$. Motivated by Akiyama-Kano’s problem, we investigate the structure of graphs without $H_f$-factor by using Lovász’s $H$-factor structure theory [8]. We obtain the following result, which is an extension of Theorem 1.3.

**Theorem 1.6** Let $G$ be a graph with even order. If

$$o(G - S) \leq f(S) \quad \text{for all } S \subset V(G),$$

then $G$ contains an $H_f$-factor.

The inequality (10) also implies that $|V(G)|$ is even. For odd order graph, we solve Problem 1.5 and obtain a stronger result than Theorem 1.4.

**Theorem 1.7** Let $G$ be a connected graph with odd order. Suppose that $d_G(v) \geq f(v) - 1$ for all $v \in V(G)$. If

$$o(G - S) \leq f(S) \quad \text{for all } \emptyset \neq S \subset V(G),$$

then $G$ contains an $H_f$-factor.

**Corollary 1.8** Let $n \geq 2$ be an integer and let $G$ be a connected graph with odd order and minimum degree $2n - 1$. If

$$o(G - S) \leq 2n|S| \quad \text{for all } \emptyset \neq S \subset V(G),$$

then $G$ contains an $H_n$-factor.

**Remark 1:** In Corollary 1.8, the conditions “$\delta(G) \geq 2n - 1$” is sharp. Let $K_{2n-1}$ denote the complete graph of order $2n - 1$. Take $2n - 2$ disjoint copies of $K_{2n-1}$. Add a new
vertices \( v \) and connect two vertices in each copy of \( K_{2n-1} \) to the new vertex \( v \). This results in a connected graph \( G \) with odd order \( (2n - 2)(2n - 1) + 1 \) and minimum degree \( 2n - 2 \). It is easy to show that

\[
o(G - S) \leq 2n|S| \quad \text{for all } \emptyset \neq S \subset V(G).
\]

Now we show that \( G \) contains no \( H_n \)-factor. Otherwise, suppose that \( G \) contains an \( H_n \)-factor \( F \). By parity, \( K_{2n-1} \) contains no \( H_n \)-factors and so \( F \) contains exactly an edge from a copy of \( K_{2n-1} \). Then we have \( d_F(v) = 2n - 1 \notin H_n \), a contradiction.

**Remark 2:** In Corollary 1.8, the condition (12) is not necessary for the existence of an \( H_n \)-factor in a graph. Let \( m \geq 2n + 2 \) be an even integer. Consider the graph \( G = K_1 + mK_{2n+1} \) obtained by linking a vertex \( v \) to all vertices in \( 2n + 1 \) copies of the complete graph \( K_{2n+1} \). Clearly, \( G \) is a graph with odd order and minimum degree \( 2n + 1 \). It is easy to verify that \( G \) contains an \( H_n \)-factor. However, taking the subset \( S \) to be the single vertex \( v \), we see that the condition (12) does not hold for \( G \).

2 On \( H \)-critical Graphs

In this section, we study \( H \)-factors of graphs based on Lovász’s structural description to the degree prescribed subgraph problem. Denote by \( I_H(v) \) the set of vertex degrees in all \( H \)-optimal subgraphs of graph \( G \), i.e.,

\[
I_H(v) = \{d_F(v) : \text{all } H \text{-optimal subgraphs } F\}.
\]

Comparing the set \( I_H(v) \) with \( H \), one may partition the vertex set \( V(G) \) into four classes:

\[
C_H = \{v \in V(G) : I_H(v) \subseteq H(v)\},
A_H = \{v \in V(G) - C_H : \min I_H(v) \geq \max H(v)\},
B_H = \{v \in V(G) - C_H : \max I_H(v) \leq \min H(v)\},
D_H = V(G) - A_H - B_H - C_H.
\]

It is clear that the 4-tuple \((A_H, B_H, C_H, D_H)\) is a pairwise disjoint partition of \( V(G) \). We call it the \( H \)-decomposition of \( G \). In fact, the four subsets can be distinguished according to the contributions of their members to the deviation \( 2 \). A graph \( G \) is said to be \( H \)-critical if it is connected and \( D_H = V(G) \). For non-consecutive allowed set function, the only
necessary condition of $H$-critical graph is given by Lovász [8]. In this paper, we obtain a sufficient condition for $H$-critical graph.

We write $MH(x) = \max_{H(x)}$ and $mH(x) = \min_{H(x)}$ for $x \in V(G)$. For $S \subseteq V(G)$, let $MH(S) = \sum_{x \in S} MH(x)$ and $mH(S) = \sum_{x \in S} mH(x)$. By the definition of $A_H, B_H, C_H, D_H$, the following holds:

(I) for every $x \in B_H$, there exists an $H$-optimal graph $F$ such that $d_F(x) < mH(x)$;

(II) for every $x \in A_H$, there exists an $H$-optimal graph $F$ such that $d_F(x) > MH(x)$;

(III) for every $x \in D_H$, there exists an $H$-optimal graph $F$ such that $d_F(x) < MH(x)$ and other $H$-optimal graph $F'$ such that $d_F(x) > mH(x)$.

Lovász [8] gave the following properties.

**Lemma 2.1 (Lovász, [8])** If $G$ is a simple graph, then $I_H(v)$ is an interval for all $v \in D_H$.

**Lemma 2.2 (Lovász, [8])** The intersection $I_H(v) \cap H(v)$ contains no consecutive integers for any vertex $v \in D_H$.

Given an integer set $P$ and an integer $a$, we write $P - a = \{p - a \mid p \in P\}$. Let $C$ be a connected induced subgraph of $G$ and $T \subseteq V(G) - V(C)$. Let $H_{C,T} : V(C) \to 2^N$ be a set function such that $H_{C,T}(x) = H(x) - e_G(x, T)$ for all $x \in V(C)$.

**Lemma 2.3 (Lovász, [8])** Every component $R$ of $G[D_H]$ is $H_R,B_H$-critical and if $F$ is $H$-optimal, then $F[V(R)]$ is $H_{R,B_H}$-optimal.

**Lemma 2.4 (Lovász, [8])** If $G$ is $H$-critical, then $\nabla(H) = 1$.

**Lemma 2.5 (Lovász, [8])** For any $H$-optimal graph $F$, $E_G(B_H, B_H \cup C_H) \subseteq E(F)$, and $E_G(A_H, C_H \cup A_H) \cap E(F) = \emptyset$.

**Theorem 2.6 (Lovász, [8])** $\nabla(H) = \omega(G[D_H]) + \sum_{v \in B_H} (mH(v) - d_{G,A_H}(v)) - \sum_{v \in A_H} MH(v)$.

In the proof of main theorems, we need the following two technical lemmas.

**Lemma 2.7** Let $F$ be an $H$-optimal subgraph. For every component $R$ of $G[D_H]$, $F$ misses at most an edge of $E_G(V(R), B_H)$. 
Proof. Let $F$ be an $H$-optimal subgraph of $G$. We write $\tau_H = \omega(G[D_H])$ and $G[D_H] = C_1 \cup \cdots \cup C_{\tau_H}$. Since $C_i$ is $H_{C_i,B_H}$-critical, then $C_i$ contains no $H_{C_i,B_H}$-factors. So if $d_{f_F}(C_i) = 0$, then $F$ either misses at least an edge of $E(C_i,B_H)$ or contains at least an edges of $E(C_i,A_H)$. Let $\tau_B$ denote the number of components of $G[D_H]$ such that $F$ misses at least an edge of $E(C_i,B_H)$ and $\tau_A$ denote the number of the components of $G[D]$ such that $F$ contains at least an edge of $E(C_i,A_H)$. Let $\tau_c$ denote the number of components $C_i$ of $G[D_H]$ such that $F$ contains at least one edge of $E(C_i,A_H)$ and misses at least one edge $E(C_i,B_H)$. Then we have

$$\nabla_H(F) \geq \tau_H - \tau_A - \tau_B + \tau_c + \sum_{x \in A_H \cup B_H} \min\{|r - d_F(x)| \mid r \in H(x)\}$$

$$\geq \tau_H - \tau_A - \tau_B + \tau_c + \sum_{x \in A_H} (d_F(x) - MH(x)) + \sum_{x \in B_H} (mH(x) - d_F(x))$$

$$\geq \tau_H - \tau_A - \tau_B + \tau_c + (e_F(A_H,B_H) - MH(A_H)) + \sum_{x \in B_H} (mH(x) - d_F(x))$$

$$= \tau_H - \tau_B + \tau_c = (e_F(A_H,B_H) - MH(B_H)) + \sum_{x \in B_H} (mH(x) - d_F(x))$$

$$\geq \tau_H - \tau_B + \tau_c + (e_F(A_H,B_H) - MH(A_H)) + (mH(B_H) - (e_F(A_H,B_H) + \sum_{x \in B_H} d_{G-A_H}(x) - \tau_B))$$

$$= \tau_H(A_H,B_H) + \tau_c + mH(B_H) - MH(A_H) - \sum_{x \in B_H} d_{G-A_H}(x) \geq \nabla(H).$$

Since $\nabla_H(F) = \nabla(H)$, then we obtain $\tau_c = 0$ and

$$\sum_{x \in B_H} d_F(x) = e_F(A_H,B_H) + \sum_{x \in B_H} d_{G-A_H}(x) - \tau_B,$$

which implies that $F$ misses at most an edge from $C_i$ to $B$. This completes the proof for $1 \leq i \leq \tau_H$. $\square$

Lemma 2.8 Let $G$ be a graph and let $H : V(G)$ be an allowed set function. If $MH(v) - 1 \in H(v)$ and $d_G(v) \geq MH(v) - 1$ for all $v \in V(G)$, then $G$ is not $H$-critical.

Proof. By contradiction, we firstly assume that $G$ is $H$-critical. Let $F$ be an $H$-optimal subgraph of $G$ such that $E(F)$ is maximal.

Since $G$ is $H$-critical and $F$ is $H$-optimal, then by Lemma 2.4 we have $d_F(v) \leq MH(v) + 1$ for all $v \in V(G)$. We claim that there exists a vertex $x \in V(G)$ such that $d_F(x) = MH(x) + 1$. Otherwise, suppose that $d_F(v) \leq MH(v)$ for all $v \in V(G)$. Then there exists a vertex $v \in V(G)$ such that $d_F(v) \notin H(v)$ and so $0 \leq d_F(v) \leq f(v) - 2$. Hence there exists an edge $e \in E(G) - E(F)$, which is incident with vertex $v$. Then $F \cup \{e\}$ is also $H$-optimal, contradicting to the maximality of $F$. Thus there exists a vertex $x \in V(G)$ such
that \(d_F(x) = MH(x) + 1\). Since \(I_H(x)\) is an interval and \(I_H(x) \cap H(x)\) contains no two consecutive integers, then we have \(\min I_H(x) \geq MH(x)\), contradicting to \(x \in D_H\).

This completes the proof. \(\square\)

**Corollary 2.9** Let \(n \geq 2\) be an integer and let \(G\) be a graph. If \(\delta(G) \geq 2n - 1\), then \(G\) is not \(H_n\)-critical.

### 3 The Proof of Theorems 1.6 and 1.7

In this section, we assume that \(f : V(G) \to Z^+\) be an even integer-valued function such that \(f \geq 4\) and \(H_f(v) = \{1, 3, \ldots, f(v) - 1, f(v)\}\) for all \(v \in V(G)\).

**Theorem 3.1** Let \(G\) be a graph and let \(A_{H_f}, B_{H_f}, C_{H_f}\) and \(D_{H_f}\) be defined as above. Then

(a) \(E_G(B_{H_f}, B_{H_f} \cup C_{H_f}) = \emptyset\);

(b) For every component \(R\) of \(G[D_{H_f}]\), \(|V(R)| + |E_G(V(R), B_{H_f})| \equiv 1 \pmod{2}\);

(c) every component \(R\) of \(G[D_{H_f} \cup B_{H_f}]\) is odd.

**Proof.** Firstly, we prove (a) by contradiction. Suppose that there exists an edge \(e \in E_G(B_{H_f}, B_{H_f} \cup C_{H_f})\). Without loss of generality, we assume that \(e = uv\) and \(u \in B_{H_f}\). For any \(H\)-optimal graph \(F\), by Lemma 2.5 \(e \in E(F)\) and so \(d_F(v) \geq 1\), contradicting to the definition of \(B_{H_f}\). This completes the proof of (a).

Secondly, we prove (b). By Lemma 2.3 \(R\) is \(H_{R,B_{H_f}}\)-critical. For simplicity, we write \(H_R = H_R,B_{H_f}\). We claim that \(f(u) - e_G(u, B_{H_f}) \notin I_{H_R}(u)\) for all \(u \in V(R)\). Otherwise, suppose that there exists a vertex \(x \in V(R)\) such that \(f(x) - e_G(x, B_{H_f}) \notin I_{H_R}(x)\). By Lemma 2.1 \(I_{H_R}(x)\) is an interval and so we have \(\min I_H(x) \geq MH_R(x)\), contradicting to the definition of \(H\)-critical graphs. Hence \(I_{H_R}(u) \subseteq [0, f(u) - 1 - e_G(u, B_{H_f})]\). Let \(F\) be an \(H_f\)-optimal graph and \(F^* = F[V(R)]\). By Lemma 2.3 \(F^*\) is an \(H_{R}\)-optimal subgraph of graph \(R\). Furthermore, by Lemma 2.3 \(R\) is \(H_{R}\)-critical and so there exists a vertex \(x \in V(R)\) such that \(d_{F^*}(x) \notin H_{R}(x)\) and \(d_{F^*}(y) \in H_{R}(y)\) for all \(y \in V(R) - x\).
Hence for every vertex $y \in V(R) - x$, $d_{F^*}(y) \equiv f(y) - 1 - e_G(y, B)$ (mod 2) and $d_{F^*}(x) \equiv f(x) - 2 - e_G(x, B_{H_f})$ (mod 2). Then
\[
\sum_{v \in V(R)} d_{F^*}(v) \equiv \sum_{y \in V(R) - x} (f(y) - 1 - e_G(y, B_{H_f})) + f(x) - 2 - e_G(x, B_{H_f}) \pmod{2}
\]
\[
\equiv \sum_{y \in V(R)} e_G(y, B_{H_f}) + |V(R)| - 1,
\]
which implies
\[
\sum_{y \in V(R)} e_G(y, B_{H_f}) + |V(R)| \equiv 1 \pmod{2}.
\]
This completes the proof of (b).

Finally, we prove (c). We write $B_{H_f} = \{v_1, \ldots, v_{|B_{H_f}|}\}$ and $G[D_{H_f}] = C_1 \cup \cdots \cup C_\tau$. For $1 \leq i \leq |B_{H_f}|$ and $1 \leq j \leq \tau$, we claim $e_G(v_i, V(C_j)) \leq 1$. Otherwise, suppose that there exists $v \in B_{H_f}$ and a component $C_i$ of $G[D_{H_f}]$ such that $e_G(v, V(C_i)) \geq 2$. For arbitrary $H_f$-optimal graph $F$, by Lemma 2.7, then we have $d_F(v) \geq 1$, contradicting $v \in B_{H_f}$.

Let $R$ be an arbitrary connected component of $G[D_{H_f}]$. Without loss of generality, we write $V(R) = C_1 \cup \cdots \cup C_k \cup B_1$, where $B_1 = \{y_1, \ldots, y_r\}$. Now we construct a graph $R^*$ obtained from $R$ by contracting $C_i$ to a vertex $x_i$ for $1 \leq i \leq k$. By (a), $R^*$ is a bipartite graph.

**Claim 1.** $R^*$ is a tree.

Since $R$ is connected, then $R^*$ is connected. Now we show that $R^*$ contains no cycles. Conversely, suppose that $R^*$ contains a cycle $x_1y_1, \ldots, x_my_m,x_1$. We write $W = C_{i_1} \cup \cdots \cup C_{i_m}$ and $B_2 = \{y_1, \ldots, y_m\}$. By Lemma 2.7, for any $H$-optimal graph $F$, $F$ contains at least $m$ edges from $W$ to $B_2$. Now we claim that $d_F(v) = 1$ for all $v \in B_2$, otherwise, there exists a vertex $v \in B_2$ such that $d_F(v) \geq 2$ contradicting to $v \in B_2 \subseteq B_{H_f}$. Since $F$ is an arbitrary $H_f$-optimal graph and $d_F(v) = 1$ for all $v \in B_2$, then we have $B_2 \subseteq C_{H_f}$, a contradiction again. This completes Claim 1.

Let $W^* = V(C_1) \cup \cdots \cup V(C_k)$. By Claim 1, $R^*$ is a tree, which implies that $e_G(W^*, B_1) = k + r - 1$. By (b), we have
\[
k \equiv \sum_{i=1}^{k} ([V(C_i)] + e_G(V(C_i), B_{H_f})) \pmod{2}
\]
\[
= \sum_{i=1}^{k} |V(C_i)| + e_G(W^*, B_1)
\]
\[
= \sum_{i=1}^{k} |V(C_i)| + k + r - 1,
\]
which implies
\[ \sum_{i=1}^{k} |V(C_i)| + r \equiv 1 \pmod{2}. \]

Hence \(|V(R)|\) is odd. This completes the proof. \(\square\)

By Theorems 2.6 and 3.1, we obtain the following result.

**Corollary 3.2** If graph \(G\) contains no \(H_f\)-factors, then there exists two disjoint subsets \(S, T\) of \(V(G)\) such that
\[ f(S) - |T| + \sum_{x \in T} d_{G-S}(x) - q(S,T) < 0, \]
where \(q(S,T)\) denote the number of components \(C\) of \(G - S - T\) such that \(|V(C)| + e_G(V(C),T) \equiv 1 \pmod{2}\).

**Proof of Theorem 1.6.** Since \(|V(G)|\) is even, by Theorem 3.1 (b), \(G\) is not \(H_f\)-critical. By Lemma 3.1 (a) and (c), \(E_G(B_{H_f}, C_{H_f}) = \emptyset\) and every component of \(G[D_{H_f} \cup B_{H_f}]\) is an odd component. We write \(\omega(G[D_{H_f} \cup B_{H_f}]) = k\) and \(G[D_{H_f} \cup B_{H_f}] = R_1 \cup \cdots \cup R_k\). Without loss of generality, suppose that \(V(R_i) = V(C_{i1}) \cup \cdots \cup V(C_{ir_i}) \cup B_i\) for \(1 \leq i \leq k\), where \(C_{ij}\) is a component of \(G[D_{H_f}]\) for \(1 \leq j \leq r_i\) and \(B_i \subseteq B_{H_f}\). By Theorem 2.6, Theorem 3.1 (c) and Claim 1 of Theorem 3.1,
\[ 0 < \nabla(H_f) = \omega(G[D_{H_f}]) + |B_{H_f}| - \sum_{x \in B_{H_f}} d_{G-A_{H_f}}(x) - f(A_{H_f}) \]
\[ = \sum_{i=1}^{k} (|B_i| + r_i - \sum_{x \in B_i} d_{G-A_{H_f}}(x)) - f(A_{H_f}) \]
\[ = \sum_{i=1}^{k} (|B_i| + r_i - e_G(B_i, V(R_i) - B_i)) - f(A_{H_f}) \]
\[ = k - f(A_{H_f}) \]
\[ = o(G[D_{H_f} \cup B_{H_f}]) - f(A_{H_f}) \]
\[ \leq o(G - A_{H_f}) - f(A_{H_f}), \]
a contradiction. This completes the proof. \(\square\)

**Proof of Theorem 1.7.** By Lemma 2.8 \(G\) is not \(H_f\)-critical. Then we have \(A_{H_f} \cup B_{H_f} \neq \emptyset\). For every component \(C_i\) of \(G[D_{H_f}]\) and every vertex \(v\), we claim \(E_G(C_i, v) \leq 1\). Otherwise, suppose that \(e_G(C_i, v) \geq 2\). By Lemma 2.7 we have \(d_F(v) \geq 1\) for any \(H_f\)-optimal graph \(F\), contradicting to \(v \in B_{H_f}\).
Claim 1. \( A_{H_f} \neq \emptyset \).

Otherwise, suppose that \( A_{H_f} = \emptyset \). Then we have \( B_{H_f} \neq \emptyset \). Let \( D_{H_f} = C_1 \cup \cdots \cup C_k \) and \( B_{H_f} = \{v_1, \ldots, v_r\} \). Since \( G \) is connected, then we have \( C_{H_f} = \emptyset \) and \( e_G(C_i, B_{H_f}) \geq 1 \) for \( 1 \leq i \leq k \). Now we show that there exists a component \( C_i \) of \( G[D_{H_f}] \) such that \( e_G(C_i, B_{H_f}) = 1 \). Otherwise, assume that \( e_G(C_i, B_{H_f}) \geq 2 \) for \( 1 \leq i \leq k \). Note that \( d_G(v) \geq f(v) - 1 \geq 3 \) for all \( v \in B_{H_f} \). Then we have \( e_G(D_{H_f}, B_{H_f}) \geq (2k + 3r)/2 > k + r \). For any \( H_f \)-optimal subgraph \( F \), by Lemma 2.7, it misses at most \( k \) edges of \( E_G(B_{H_f}, D_{H_f}) \) and so it contains at least \( r + 1 \) edges of \( E_G(B_{H_f}, D_{H_f}) \). Hence there exists a vertex \( v \in B_{H_f} \), such that \( d_{F}(v) \geq 2 \), contradicting to \( v \in B_{H_f} \). Without loss of generality, suppose that \( e_G(C_1, v_1) = 1 \) and \( u_1v_1 \in E(G) \), where \( u_1 \in V(C_1) \) and \( v_1 \in B_{H_f} \). By Lemma 2.8 \( C_1 \) is \( H'_f \)-critical, where \( H'_f : V(C_1) \to 2^N \) be a set function such that \( H'_f = \{0, 2, \ldots, f(v_1) - 2, f(v_1) - 1\} \) and \( H'_f(u) = H_f(u) \) for all \( u \in V(C_1) - v_1 \). Note that \( d_{C_1}(v_1) \geq f(v_1) - 2 \) and \( d_{C_1}(u) \geq f(u) - 1 \) for all \( u \in V(C_1) - v_1 \), a contradiction by Lemma 2.8. This complete Claim 1.

Let \( m \) denote the number of components and \( G[D_{H_f} \cup B_{H_f}] = R_1 \cup \cdots \cup R_m \). Suppose that \( V(R_i) \cap B_{H_f} = B_i \) and \( R_i \) contains \( r_i \) connected components of \( G[D_{H_f}] \). Then by Theorem 3.1 and Claim 1 of Theorem 3.1 there exists nonempty \( A_{H_f} \), such that
\[
0 < \nabla(H_f) = \omega(G[D_{H_f}]) + |B_{H_f}| - \sum_{x \in B_{H_f}} d_{G-A_{H_f}}(x) - f(A_{H_f}) \\
= \sum_{i=1}^{m} (|B_i| + r_i - \sum_{x \in B_i} d_{G-A_{H_f}}(x)) - f(A_{H_f}) \\
= m - f(A_{H_f}) \\
= o(G[D_{H_f} \cup B_{H_f}]) - f(A_{H_f}) \\
\leq o(G - A_{H_f}) - f(A_{H_f}),
\]
a contradiction. This completes the proof. \( \square \)

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