Normal transversality and uniform bounds

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1 Introduction

Let $A$ be a commutative ring. A graded $A$-algebra $U = \oplus_{n \geq 0} U_n$ is a standard $A$-algebra if $U_0 = A$ and $U = A[U_1]$ is generated as an $A$-algebra by the elements of $U_1$. A graded $U$-module $F = \oplus_{n \geq 0} F_n$ is a standard $U$-module if $F$ is generated as an $U$-module by the elements of $F_0$, that is, $F_n = U_n F_0$ for all $n \geq 0$. In particular, $F_n = U_1 F_{n-1}$ for all $n \geq 1$. Given $I$, $J$, two ideals of $A$, we consider the following standard algebras: the Rees algebra of $I$, $R(I) = \oplus_{n \geq 0} I^n t^n = A[It] \subset A[t]$, and the multi-Rees algebra of $I$ and $J$, $R(I, J) = \oplus_{n \geq 0} (\oplus_{p+q=n} I^p J^q u^p v^q) = A[u, Jv] \subset A[u, v]$. Consider the associated graded ring of $I$, $G(I) = R(I) \otimes A/I = \oplus_{n \geq 0} I^n/I^{n+1}$, and the multi-associated graded ring of $I$ and $J$, $G(I, J) = R(I, J) \otimes A/(I + J) = \oplus_{n \geq 0} (\oplus_{p+q=n} I^p J^q/(I + J)^{p+q})$. We can always consider the tensor product of two standard $A$-algebras $U = \oplus_{p \geq 0} U_p$ and $V = \oplus_{q \geq 0} V_q$ as an standard $A$-algebra with the natural grading $U \otimes V = \oplus_{n \geq 0} (\oplus_{p+q=n} U_p \otimes V_q)$. If $M$ is an $A$-module, we have the standard modules: the Rees module of $I$ with respect to $M$, $R(I; M) = \oplus_{n \geq 0} I^n M t^n = M[It] \subset M[t]$ (a standard $R(I)$-module), and the multi-Rees module of $I$ and $J$ with respect to $M$, $R(I, J; M) = \oplus_{n \geq 0} (\oplus_{p+q=n} I^p J^q M u^p v^q) = M[u, Jv] \subset M[u, v]$ (a standard $R(I, J)$-module). Consider the associated graded module of $I$ with respect to $M$, $G(I; M) = R(I; M) \otimes A/I = \oplus_{n \geq 0} I^n M/I^{n+1} M$ (a standard $G(I)$-module), and the multi-associated graded module of $I$ and $J$ with respect to $M$, $G(I, J; M) = R(I, J; M) \otimes A/(I + J) = \oplus_{n \geq 0} (\oplus_{p+q=n} I^p J^q M/(I + J)^{p+q} M)$ (a standard $R(I, J)$-module). If $U, V$ are two standard $A$-algebras and $F$ is a standard $U$-module and $G$ is a standard $V$-module, then $F \otimes G = \oplus_{n \geq 0} (\oplus_{p+q=n} F_p \otimes G_q)$ is a standard $U \otimes V$-module.

Denote by $\pi: R(I) \otimes R(J; M) \to R(I, J; M)$ and $\sigma: R(I, J; M) \to R(I + J; M)$ the natural surjective graded morphisms of standard $R(I) \otimes R(J)$-modules. Let $\varphi: R(I) \otimes R(J; M) \to R(I + J; M)$ be $\sigma \circ \pi$. Denote by $\pi: G(I) \otimes G(J; M) \to G(I, J; M)$ and $\sigma: G(I, J; M) \to G(I + J; M)$ the tensor product of $\pi$ and $\sigma$ by $A/(I + J)$; these are two natural surjective graded morphisms of standard $G(I) \otimes G(J)$-modules. Let $\overline{\varphi}: G(I) \otimes G(J; M) \to G(I + J; M)$ be $\overline{\sigma} \circ \overline{\pi}$. The first purpose of this note is to prove the following theorem:

**Theorem 1** Let $A$ be a noetherian ring, $I$, $J$ two ideals of $A$ and $M$ a finitely generated $A$-module. The following two conditions are equivalent:

(i) $\overline{\varphi}: G(I) \otimes G(J; M) \to G(I + J; M)$ is an isomorphism.

(ii) $\text{Tor}_1(A/I^p, R(J; M)) = 0$ and $\text{Tor}_1(A/I^p, G(J; M)) = 0$ for all integers $p \geq 1$.

In particular, $G(I) \otimes G(J; M) \simeq G(I + J)$ if and only if $\text{Tor}_1(A/I^p, A/J^q) = 0$ and $\text{Tor}_2(A/I^p, A/J^q) = 0$ for all integers $p, q \geq 1$. 




The morphism $\varphi$ has been studied by Hironaka [H], Grothendieck [G] and Hermann, Ikeda and Orbanz [HIO], among others, but assuming always $A$ is normally flat along $I$ (see 21.11 in [HIO]). We will see how Theorem 2 generalizes all this former work.

Let us now recall some definitions in order to state the second purpose of this note. If $U$ is a standard $A$-algebra and $F$ is a graded $U$-module, put $s(F) = \min\{r \geq 1 \mid F_n = 0 \text{ for all } n \geq r + 1\}$, where $s(F)$ may possibly be infinite. If $U_n = \oplus_{n>0} U_n$ and $r \geq 1$, the following three conditions are equivalent: $F$ can be generated by elements of degree at most $r$; $s(F/U_+ F) \leq r$; and $F_n = U_1 F_{n-1}$ for all $n \geq r + 1$. If $\varphi : G \to F$ is a surjective graded morphism of graded $U$-modules, we denote by $E(\varphi)$ the graded $A$-module $E(\varphi) = \ker \varphi / U_+ \ker \varphi = \ker \varphi \oplus (\oplus_{n \geq 1} \ker \varphi_n / U_1 \ker \varphi_{n-1}) = \oplus_{n \geq 0} E(\varphi)_n$. If $F$ is a standard $U$-module, take $S(U_1)$ the symmetric algebra of $U_1$, $\alpha : S(U_1) \to U$ the surjective graded morphism of standard $A$-algebras induced by the identity on $U_1$ and $\gamma : S(U_1) \otimes F_0 \to U \otimes F_0 \to F$ the composition of $\alpha \otimes 1$ with the structural morphism. Since $F$ is a standard $U$-module, $\gamma$ is a surjective graded morphism of graded $S(U_1)$-modules. The module of effective $n$-relations of $F$ is defined to be $E(F)_n = E(\gamma)_n = \ker \gamma_n / U_1 \ker \gamma_{n-1}$ (for $n = 0$, $E(F)_0 = 0$). Put $E(F) = \oplus_{n \geq 1} E(F)_n = \oplus_{n \geq 1} E(\gamma)_n = E(\gamma) = \ker \gamma / S_\alpha(U_1) \ker \gamma$. The relation type of $F$ is defined to be $rt(F) = s(E(F))$, that is, $rt(F)$ is the minimum positive integer $r \geq 1$ such that the effective $n$-relations are zero for all $n \geq r + 1$. A symmetric presentation of a standard $U$-module $F$ is a surjective graded morphism of standard $V$-modules $\varphi : G \to F$, with $\varphi : G = V \otimes M \overset{f \otimes h}{\to} U \otimes F_0 \to F$, where $V$ is a symmetric $A$-algebra, $f : V \to U$ is a surjective graded morphism of standard $A$-algebras, $h : M \to F_0$ is an epimorphism of $A$-modules and $U \otimes F_0 \to F$ is the structural morphism. One can show (see [P2]) that $E(F)_n = E(\varphi)_n$ for all $n \geq 2$ and $s(E(F)) = s(E(\varphi))$. Thus the module of effective $n$-relations and the relation type of a standard $U$-module are independent of the chosen symmetric presentation. Roughly speaking, the relation type of $F$ is the largest degree of any minimal homogeneous system of generators of the submodule defining $F$ as a quotient of a polynomial ring with coefficients in $F_0$. For an ideal $I$ of $A$ and an $A$-module $M$, the module of effective $n$-relations and the relation type of $I$ with respect to $M$ are defined to be $E(I; M)_n = E(\mathcal{R}(I; M))_n$ and $rt(I; M) = rt(\mathcal{R}(I; M))$, respectively. Then:

**Theorem 2** Let $A$ be a commutative ring, $U$ and $V$ two standard $A$-algebras, $F$ a standard $U$-module and $G$ a standard $V$-module. Then $U \otimes V$ is a standard $A$-algebra, $F \otimes G$ is a standard $U \otimes V$-module and $rt(F \otimes G) \leq \max(rt(F), rt(G))$.

As a consequence of Theorems [P1] and [P2] one deduces the existence of an uniform bound for the relation type of all maximal ideals of an excellent ring.

**Theorem 3** Let $A$ be an excellent (or $J - 2$) ring and let $M$ be a finitely generated $A$-module. Then there exists an integer $s \geq 1$ such that, for all maximal ideals $m$ of $A$, the relation type of $m$ with respect to $M$ satisfies $rt(m; M) \leq s$.

In fact, Theorem [P3] could also been deduced from the proof of Theorem 4 of Trivedi in [H]. Finally, and using Theorem 2 of [P2], one can recover the following result of Duncan and O’Carroll.

**Corollary 4** [DO] Let $A$ be an excellent (or $J - 2$) ring and let $N \subseteq M$ be two finitely generated $A$-modules. Then there exists an integer $s \geq 1$ such that, for all integers $n \geq s$ and for all maximal ideals $m$ of $A$, $m^n M \cap N = m^{n-s}(m^s M \cap N)$. 

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2 Normal transversality

Lemma 2.1 Let $A$ be a commutative ring, $I$ an ideal of $A$, $U$ a standard $A$-algebra, $F$ and $G$ two standard $U$-modules and $\varphi : G \to F$ a surjective graded morphism of standard $A$-algebras. If $\overline{A} = A/I$, then $\overline{U} = U \otimes \overline{A}$ is a standard $\overline{A}$-algebra, $\overline{F} = F \otimes \overline{A}$ and $\overline{G} = G \otimes \overline{A}$ are two standard $\overline{U}$-modules and $\overline{\varphi} = \varphi \otimes 1_{\overline{A}} : \overline{G} \to \overline{F}$ is a surjective graded morphism of standard $\overline{U}$-modules. Moreover, $s(E(\overline{\varphi})) \leq s(E(\varphi))$.

Proof. Consider the following commutative diagram of exact rows:

\[
\begin{array}{ccccccccc}
U_1 \otimes \ker \varphi_{n-1} & \longrightarrow & U_1 \otimes G_{n-1} & \longrightarrow & U_1 \otimes F_{n-1} & \longrightarrow & 0 \\
\downarrow & & \downarrow \varphi_{n-1} & & \downarrow \varphi_n & & \\
\ker \varphi_n & \longrightarrow & G_n & \longrightarrow & F_n & \longrightarrow & 0.
\end{array}
\]

By the snake lemma, $\ker \partial_n^G \to \ker \partial_n^F \to E(\varphi)_n \to 0$ is an exact sequence of $A$-modules. If we tensor this sequence by $\overline{A}$, then $(\ker \partial_n^G) \otimes \overline{A} \to (\ker \partial_n^F) \otimes \overline{A} \to E(\varphi)_n \otimes \overline{A} \to 0$ is an exact sequence of $\overline{A}$-modules. On the other hand, we have the following commutative diagram of exact rows:

\[
\begin{array}{ccccccccc}
\overline{U}_1 \otimes \ker \overline{\varphi}_{n-1} & \longrightarrow & \overline{U}_1 \otimes \overline{G}_{n-1} & \longrightarrow & \overline{U}_1 \otimes \overline{F}_{n-1} & \longrightarrow & 0 \\
\downarrow & & \downarrow \overline{\varphi}_{n-1} & & \downarrow \overline{\varphi}_n & & \\
\ker \overline{\varphi}_n & \longrightarrow & \overline{G}_n & \longrightarrow & \overline{F}_n & \longrightarrow & 0.
\end{array}
\]

By the snake lemma, $\ker \partial_n^G \otimes \overline{A} \to \ker \partial_n^F \otimes \overline{A} \to E(\overline{\varphi})_n \to 0$ is an exact sequence of $\overline{A}$-modules. In order to see the relationship between $\ker \partial_n^F$ and $\ker \partial_n^F$, tensor by $\overline{A}$ the exact sequence of $A$-modules $0 \to \ker \partial_n^F \to U_1 \otimes F_{n-1} \to F_n \to 0$ and consider the commutative diagram of exact rows:

\[
\begin{array}{ccccccccc}
(\ker \partial_n^F) \otimes \overline{A} & \longrightarrow & (U_1 \otimes F_{n-1}) \otimes \overline{A} & \longrightarrow & F_n \otimes \overline{A} & \longrightarrow & 0 \\
\downarrow & & \downarrow \sim & & \downarrow \sim & & \\
\ker \partial_n^F & \longrightarrow & U_1 \otimes \overline{F}_{n-1} & \longrightarrow & \overline{F}_n & \longrightarrow & 0.
\end{array}
\]

It induces an epimorphism of $\overline{A}$-modules $(\ker \partial_n^F) \otimes \overline{A} \to \ker \partial_n^F$. Analogously, there exists an epimorphism of $\overline{A}$-modules $(\ker \partial_n^G) \otimes \overline{A} \to \ker \partial_n^F$. Both epimorphisms make commutative the following diagram of exact rows:

\[
\begin{array}{ccccccccc}
(\ker \partial_n^G) \otimes \overline{A} & \longrightarrow & (\ker \partial_n^F) \otimes \overline{A} & \longrightarrow & E(\overline{\varphi})_n \otimes \overline{A} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\ker \partial_n^G & \longrightarrow & \ker \partial_n^F & \longrightarrow & E(\overline{\varphi})_n & \longrightarrow & 0,
\end{array}
\]

from where we deduce an epimorphism $E(\overline{\varphi})_n \otimes \overline{A} \to E(\overline{\varphi})_n$. In particular, $s(E(\overline{\varphi})) \leq s(E(\varphi))$.

Lemma 2.2 Let $A$ be a commutative ring, $I$, $J$ two ideals of $A$ and $M$ an $A$-module. Consider $\sigma : \mathcal{R}(I, J; M) \to \mathcal{R}(I + J; M)$ and $\overline{\sigma} = \mathcal{G}(I, J; M) \to \mathcal{G}(I + J; M)$. Then
(a) \( \text{ker}(\sigma_1) \simeq IM \cap JM \).

(b) \( \text{ker}(\overline{\sigma}_1) = 0 \) if and only if \( IM \cap JM \subset (I+J)M \cap (I+J)JM \).

(c) If \( IpM \cap JqM = IpJqM \) for all integers \( p, q \geq 1 \), then \( s(E(\sigma)) = 1 \) and \( \overline{\sigma} \) is an isomorphism.

Proof. Consider \( 0 \to IM \cap JM \xrightarrow{\rho} IM \oplus JM \xrightarrow{\sigma_1} (I+J)M \to 0 \) where \( \rho(a) = (a, -a) \) and \( \sigma_1(a,b) = a+b \). Clearly it is an exact sequence of \( A \)-modules. Thus \( \text{ker}(\sigma_1) = \rho(IM \cap JM) \simeq IM \cap JM \). If we tensor this exact sequence by \( A/(I+J) \) we get \( (IM \cap JM) \otimes A/(I+J) \xrightarrow{\overline{\sigma_1}} (I+J)M/(I+J)^2M \to 0 \). Then

\[
\text{ker}(\overline{\sigma}_1) = \text{im}\overline{\rho} = \{(\overline{\sigma}, -\overline{\sigma}) \in IM/I(I+J)M \oplus JM/(I+J)JM \mid a \in IM \cap JM \}.
\]

Hence \( \text{ker}(\overline{\sigma}_1) = 0 \) if and only if \( IM \cap JM \subset (I+J)M \cap (I+J)JM \). Now, let us prove (c).

Let \( z \in \text{ker}\sigma_n \subset R(I,J;M)_n = \oplus_{p+q=n} IpJqM u^p v^q \subset M[u,v] \). Thus, \( z = a_0 u^n + a_1 u^{n-1} v + \ldots + a_{n-1} v^{n-1} + a_n v^n \), \( a_i \in I^{n-i}J^iM \), and \( 0 = \sigma_n(z) = (a_0 + a_1 + \ldots + a_{n-1} + a_n)u^n \in R(I+J;M)_n = (I+J)^nM \). So \( a_0 + a_1 + \ldots + a_{n-1} + a_n = 0 \). Let us denote:

\[
\begin{align*}
&b_0 = a_0 \in I^n \cap JM = I^nJM \\
b_1 = a_0 + a_1 \in I^{n-1} \cap J^2M = I^{n-1}J^2M \text{ and } a_1 = b_1 - b_0 \\
b_2 = a_0 + a_1 + a_2 \in I^{n-2} \cap J^3M = I^{n-2}J^3M \text{ and } a_2 = b_2 - b_1 \\
\vdots \\
b_{n-2} = a_0 + \ldots + a_{n-2} \in I^2 \cap J^{n-1}M = I^2J^{n-1}M \text{ and } a_{n-2} = b_{n-2} - b_{n-3} \\
b_{n-1} = a_0 + \ldots + a_{n-1} \in IM \cap J^nM = IJ^nM \text{ and } a_{n-1} = b_{n-1} - b_{n-2} \\
b_n = -b_{n-1} \in IJ^nM \;.
\end{align*}
\]

We can rewrite \( z \) in \( M[u,v] \) in the following manner:

\[
z = a_0 u^n + a_1 u^{n-1} v + \ldots + a_{n-1} v^{n-1} + a_n v^n = \\
= b_0 u^n + (b_1 - b_0) u^{n-1} v + (b_2 - b_1) u^{n-2} v^2 + \ldots + \\
+(b_{n-2} - b_{n-3}) u^{n-3} v^3 + (b_{n-1} - b_{n-2}) u^{n-4} v^4 + \ldots + b_n v^n = \\
= (b_0 u^n + b_1 u^{n-1} v + b_2 u^{n-2} v^2 + \ldots + b_{n-2} u^{n-3} v^3 + b_{n-1} u^{n-4} v^4 + \ldots + b_n v^n)(u - v) := p(u,v)(u - v),
\]

where \( p(u,v) \in A[Iu,Jv]_{n-1} \cdot (IJM) = R(I,J)_{n-1} \cdot (IJM) \). Since by hypothesis \( IM \cap JM = IJM \), then \( \text{ker}(\sigma_1) = (IJM)(u - v) \), \( \text{ker}(\overline{\sigma}_1) = 0 \) and \( z = p(u,v)(u - v) \in R(I,J)_{n-1} \cdot (IJM)(u - v) = R(I,J)_{n-1} \cdot \text{ker}\sigma_1 \). Thus \( \text{ker}\sigma_n = R(I,J)_{n-1} \cdot \text{ker}\sigma_1 \) for all \( n \geq 2 \) and \( s(E(\sigma)) = 1 \). By Lemma 2.1, \( s(E(\overline{\sigma}_1)) \leq s(E(\sigma)) = 1 \). Therefore \( \text{ker}(\overline{\sigma}_n) = G(I,J)_{n-1} \cdot \text{ker}(\overline{\sigma}_1) = 0 \) for all \( n \geq 2 \) and \( \overline{\sigma} \) is an isomorphism.

\[\blacksquare\]

**Proposition 2.3** Let \( A \) be a noetherian ring, \( I, J \) two ideals of \( A \) and \( M \) a finitely generated \( A \)-module. The following two conditions are equivalent:

(i) \( \overline{\sigma} : G(I,J;M) \to G(I+J;M) \) is an isomorphism.

(ii) \( IpM \cap JqM = IpJqM \) for all integers \( p, q \geq 1 \).

**Proof.** Remark that we can suppose \( A \) is local. By Lemma 2.2, (ii) \(\Rightarrow\) (i). Let us see (i) \(\Rightarrow\) (ii), proving by double induction in \( p, q \geq 1 \) that

\[
IpM \cap JqM \subset Ip(I+J)J^{n-1}M \cap (I+J)pJ^nM .
\]
Remark that if $I^p M \cap J^q M \subseteq I^p (I + J) J^{q-1} M$ for all $p, q \geq 1$, then $I^p M \cap J^q M \subseteq I^{p+1} M + I^p J^q M$ and $I^p M \cap J^q M \subseteq I^{p+1} M \cap J^q M + I^p J^q M$. Recursively, and using $A$ is noetherian local and $M$ is finitely generated, $I^p M \cap J^q M \subseteq (\cap_{r \geq 1} I^{p+r} M \cap J^q M) + I^p J^q M \subseteq (\cap_{n \geq 1} I^n M) + I^p J^q M = I^p J^q M$, concluding $(ii)$. Take $q = 1$. Let us prove by induction in $p \geq 1$ that

$$I^p M \cap J^1 M \subseteq I^p (I + J) M \cap (I + J)^p J^1 M.$$  

For $p = 1$, we apply Lemma 2.4 (b), using the hypothesis $\sigma_1$ is an isomorphism. Suppose

$$I^p M \cap J^1 M \subseteq I^p (I + J) M \cap (I + J)^p J^1 M$$

is true and let us prove

$$I^{p+1} M \cap J^1 M \subseteq I^{p+1} (I + J) M \cap (I + J)^{p+1} J^1 M.$$  

Then $I^{p+1} M \cap J^1 M \subseteq I^p M \cap J^1 M \subseteq (I + J)^p J^1 M$. Consider the short complex of $A$-modules:

$$I^p M \cap J^1 M \xrightarrow{\alpha} I^{p+1} M \cap (I + J)^p J^1 M \xrightarrow{\beta} (I + J)^{p+1} M,$$

where $\alpha(a) = (a, -a)$ and $\beta(a, b) = a + b$. Remark that $\beta \circ \alpha = 0$, $\beta$ is surjective and that there exists a natural epimorphism $\gamma$ of $A$-modules such that $\beta \circ \gamma = \sigma_{p+1}$. If we tensor this short complex by $A/(I + J)$ we obtain:

$$(I^{p+1} M \cap J^1 M) \otimes A/(I + J) \xrightarrow{\pi} I^{p+1} M/I^{p+1} (I + J) M \oplus (I + J)^p J^1 M/(I + J)^{p+1} J^1 M$$

$$I^{p+1} M/I^{p+1} (I + J) M \oplus (I + J)^p J^1 M/(I + J)^{p+1} J^1 M \xrightarrow{\pi} (I + J)^{p+1} M/(I + J)^{p+2} M,$$

with $\overline{\beta} \circ \overline{\pi} = 0$. Since $\sigma_{p+1} = \overline{\beta} \circ \overline{\pi}$ is an isomorphism, then $\overline{\beta}$ is an isomorphism, $\overline{\pi} = 0$ and

$$I^{p+1} M \cap J^1 M \subseteq I^{p+1} (I + J) M \cap (I + J)^{p+1} J^1 M.$$  

By the symmetry of the problem, the following inclusion is also true for all $q \geq 1$:

$$I^1 M \cap J^q M \subseteq (I + J) J^q M \cap I (I + J)^q M.$$  

In particular, if $I^p M \cap J^q M \subseteq I^p (I + J) M$ for all $p \geq 1$, then $I^p M \cap J^q M \subseteq I^{p+1} M + I^p J^q M$ and $I^p M \cap J^q M \subseteq I^{p+1} M \cap J^1 M + I^p J^q M$. Recursively, and using $A$ is noetherian local and $M$ is finitely generated, $I^p M \cap J^q M \subseteq (\cap_{r \geq 1} I^{p+r} M \cap J^q M) + I^p J^q M \subseteq (\cap_{n \geq 1} I^n M) + I^p J^q M = I^p J^q M$ concluding $I^p M \cap J^q M = I^p J^q M$ for all $p \geq 1$. Again, by the symmetry of the problem, $I^1 M \cap J^q M = I J^q M$ for all $q \geq 1$. Now, suppose

$$I^p M \cap J^q M \subseteq I^p (I + J) J^{q-1} M \cap (I + J)^p J^q M$$

holds for all $p \geq 1$ and let us prove, by induction in $p \geq 1$, that

$$I^p M \cap J^{q+1} M \subseteq I^p (I + J) J^q M \cap (I + J)^p J^{q+1} M.$$  

Remark that if $I^p M \cap J^q M \subseteq I^p (I + J) J^{q-1} M$ for all $p \geq 1$, then $I^p M \cap J^q M \subseteq I^{p+1} M + I^p J^q M$ and $I^p M \cap J^q M \subseteq I^{p+1} M \cap J^q M + I^p J^q M$. Recursively, and using $A$ is noetherian local and $M$ is finitely generated, $I^p M \cap J^q M \subseteq (\cap_{r \geq 1} I^{p+r} M \cap J^q M) + I^p J^q M \subseteq (\cap_{n \geq 1} I^n M) + I^p J^q M = I^p J^q M$ concluding $I^p M \cap J^q M = I^p J^q M$ for all $p \geq 1$. For $p = 1$, we have to show:

$$I^1 M \cap J^{q+1} M \subseteq I (I + J) J^q M \cap (I + J)^1 J^{q+1} M.$$
We have $IM \cap J^{q+1}M \subset IM \cap J^qM = IJ^qM$. Consider the short complex of $A$-modules:

$$IM \cap J^{q+1}M \xrightarrow{\alpha} I^qM \oplus \ldots \oplus IJ^qM \oplus J^{q+1}M \xrightarrow{\sigma_{q+1}} (I + J)^{q+1}M,$$

where $\alpha(a) = (0, \ldots, 0, a, -a)$. Remark that $\sigma_{q+1} \circ \alpha = 0$. If we tensor this complex by $A/(I + J)$ we obtain $\overline{\sigma}_{q+1} \circ \overline{\alpha} = 0$. Since $\overline{\sigma}_{q+1}$ is an isomorphism, then $\overline{\alpha} = 0$ and

$$IM \cap J^{q+1}M \subset I(I + J)J^qM \cap (I + J)J^{q+1}M.$$

Suppose now true

$$IP^{p}M \cap J^{q+1}M \subset IP^{p}(I + J)J^qM \cap (I + J)^pJ^{q+1}M$$

and let us prove

$$IP^{p+1}M \cap J^{q+1}M \subset IP^{p+1}(I + J)J^qM \cap (I + J)^{p+1}J^{q+1}M.$$

Then $IP^{p+1}M \cap J^{q+1}M \subset IP^{p}M \cap J^{q+1}M \subset (I + J)^{p+1}J^{q+1}M$ and $IP^{p+1}M \cap J^{q+1}M \subset IP^{p+1}M \cap J^qM = IP^{p+1}J^qM$. Consider the short complex of $A$-modules:

$$IP^{p+1}M \cap J^{q+1}M \xrightarrow{\alpha} IP^{p+1}M \oplus \ldots \oplus IP^{p+1}J^qM \oplus (I + J)^{p+1}J^{q+1}M \xrightarrow{\beta} (I + J)^{p+1}J^{q+1}M,$$

where $\alpha(a) = (0, \ldots, 0, a, -a)$ and $\beta(a_1, \ldots, a_{q+2}) = a_1 + \ldots + a_{q+2}$. Remark that $\beta \circ \alpha = 0$, $\beta$ is surjective and that there exists a natural epimorphism $\gamma$ of $A$-modules such that $\beta \circ \gamma = \sigma_{p+q+1}$. If we tensor this complex by $A/(I + J)$ we obtain $\overline{\beta} \circ \overline{\alpha} = 0$. Since $\overline{\sigma}_{p+q+1} = \overline{\beta} \circ \overline{\alpha}$ is an isomorphism, then $\overline{\beta}$ is an isomorphism, $\overline{\alpha} = 0$ and

$$IP^{p+1}M \cap J^{q+1}M \subset IP^{p+1}(I + J)J^qM \cap (I + J)^{p+1}J^{q+1}JM.$$

**Proposition 2.4** Let $A$ be a commutative ring, $I$ an ideal of $A$ and $\lambda : M \otimes N \to P$ an epimorphism of $A$-modules. Consider $f : R(I; M) \otimes N \to R(I; P)$ and $\overline{f} = f \otimes 1_{A/I} : G(I; M) \otimes N \to G(I; P)$ the natural surjective graded morphisms of standard modules. Then, for each integer $n \geq 2$, there exists an exact sequence of $A$-modules $E(f)_n \to E(f)_n \to E(\overline{f})_n \to 0$. In particular, if $A$ is noetherian, $M, N, P$ are finitely generated and $\overline{f}$ is an isomorphism, then $f$ is an isomorphism.

**Proof.** For each integer $n \geq 1$, the natural morphism $\text{Tor}_1(A/I^n, M) \otimes N \to \text{Tor}_1(A/I^n, M \otimes N)$ and $\lambda : M \otimes N \to P$ define the following commutative diagram of exact rows:

$$\begin{array}{cccccc}
\text{Tor}_1(A/I^n, M) \otimes N & \xrightarrow{1 \otimes \lambda} & I^n \otimes M \otimes N & \xrightarrow{1 \otimes \lambda} & I^nM \otimes N & \to 0 \\
0 & \xrightarrow{1 \otimes \lambda} & \text{Tor}_1(A/I^n, P) & \xrightarrow{1 \otimes \lambda} & I^n \otimes P & \to 0 \\
\end{array}$$

We deduce an epimorphism $f_n : I^nM \otimes N \to I^nP$. On the other hand, $R(I; M) \otimes M$ is a standard $R(I)$-module and $f = \oplus_{n \geq 0} f_n : R(I; M) \otimes N \to R(I; P)$ defines a surjective graded morphism of standard $R(I)$-modules. If we tensor $f$ by $A/I$, we get $\overline{f} : G(I; M) \otimes N \to G(I; P)$ a surjective graded morphism of standard $G(I)$-modules.

Let $X$ be an $A$-module. The following is a commutative diagram of exact columns with rows the last three nonzero terms of the complexes $K(R(I; X))_{n+1}, K(R(I; X))_n$ and $K(G(I; X))_n$ (see Proposition 2.6 in [P] for more details):
In other words, \( \ker(\mathcal{R}(I; X))_{n+1} \xrightarrow{u} \ker(\mathcal{R}(I; X))_n \xrightarrow{v} \ker(\mathcal{G}(I; X))_n \to 0 \) is an exact sequence of complexes. It induces the morphisms in homology: \( H_1(\ker(\mathcal{R}(I; X))_{n+1}) \xrightarrow{u} H_1(\ker(\mathcal{R}(I; X))_n) \) and \( H_1(\ker(\mathcal{R}(I; X))_n) \xrightarrow{\cdot} H_1(\ker(\mathcal{G}(I; X))_n) \). By Proposition 2.6 in [P3], \( H_1(\ker(\mathcal{R}(I; X))_n) = E(I; X)_n \) and \( H_1(\ker(\mathcal{G}(I; X))_n) = E(\mathcal{G}(I; X))_n \). Thus we have \( E(I; X)_{n+1} \xrightarrow{u} E(I; X)_n \xrightarrow{v} E(\mathcal{G}(I; X))_n \). Since \( v \circ u = 0 \), then \( v \circ u = 0 \). Since \( u_0 \) is injective, then \( \ker \subset \text{im} u_0 \). Since \( H_0(\ker(\mathcal{R}(I; X))_{n+1}) = 0 \), then \( v \) is surjective. So \( E(I; X)_{n+1} \xrightarrow{v} E(I; X)_n \xrightarrow{v} E(\mathcal{G}(I; X))_n \to 0 \) is an exact sequence of \( A \)-modules. For \( X = P \) we get the exact sequence of \( A \)-modules: \( E(I; P)_{n+1} \xrightarrow{v} E(I; P)_n \xrightarrow{v} E(\mathcal{G}(I; P))_n \to 0 \). Take \( X = M \) in \( \ker(\mathcal{R}(I; X))_{n+1} \xrightarrow{u} \ker(\mathcal{R}(I; X))_n \xrightarrow{v} \ker(\mathcal{G}(I; X))_n \to 0 \) and tensor it by \( N \). Then we get the exact sequence of complexes

\[
\ker(\mathcal{R}(I; M))_{n+1} \otimes N \xrightarrow{\alpha = u \otimes 1} \ker(\mathcal{R}(I; M))_n \otimes N \xrightarrow{\beta = v \otimes 1} \ker(\mathcal{G}(I; M))_n \otimes N \to 0.
\]

That is, we obtain the exact sequence:

\[
\ker(\mathcal{R}(I; M) \otimes N)_{n+1} \xrightarrow{\alpha} \ker(\mathcal{R}(I; M) \otimes N)_n \xrightarrow{\beta} \ker(\mathcal{G}(I; M) \otimes N)_n \to 0,
\]

which induces the morphisms in homology

\[
H_1(\ker(\mathcal{R}(I; M) \otimes N)_{n+1}) \xrightarrow{\alpha} H_1(\ker(\mathcal{R}(I; M) \otimes N)_n) \xrightarrow{\beta} H_1(\ker(\mathcal{G}(I; M) \otimes N)_n).
\]

Again, by Proposition 2.6 in [P3], \( H_1(\ker(\mathcal{R}(I; M) \otimes N)_n) = E(\mathcal{R}(I; M) \otimes N)_n \) and \( H_1(\ker(\mathcal{G}(I; M) \otimes N)_n) = E(\mathcal{G}(I; M) \otimes N)_n \). Moreover, since \( \beta \circ \alpha = 0 \), then \( \beta \circ \alpha = 0 \), and since \( H_0(\ker(\mathcal{R}(I; M) \otimes N)_{n+1}) = 0 \), then \( \beta \) is an epimorphism. Thus we have

\[
E(\mathcal{R}(I; M) \otimes N)_{n+1} \xrightarrow{\alpha} E(\mathcal{R}(I; M) \otimes N)_n \xrightarrow{\beta} E(\mathcal{G}(I; M) \otimes N)_n \to 0
\]

with \( \beta \circ \alpha = 0 \) and \( \beta \) surjective. Remark that since we do not know if \( \alpha_0 = u_0 \otimes 1 \) is injective, we cannot deduce \( \ker \beta \subset \text{im} \alpha \). On the other hand, consider \( g : S(I) \otimes M \otimes N \to \mathcal{R}(I; M) \otimes N \) and \( \overline{g} : S(I/I^2) \otimes M \otimes N \to \mathcal{G}(I; M) \otimes N \) the natural surjective graded morphisms of standard modules, where \( S(I) \), \( S(I/I^2) \) stands for the symmetric algebras of \( I \) and \( I/I^2 \), respectively. By Lemma 2.3 in [P2], for each \( n \geq 2 \), there exists exact sequences of \( A \)-modules \( E(g)_n \to E(f \circ g)_n \to E(f)_n \to 0 \) and \( E(\overline{g})_n \to E(f \circ \overline{g})_n \to E(\overline{f})_n \to 0 \). In other words, we have exact sequences

\[
E(\mathcal{R}(I; M) \otimes N)_n \to E(\mathcal{R}(I; P))_n \to E(f)_n \to 0
\]

and

\[
E(\mathcal{G}(I; M) \otimes N)_n \to E(\mathcal{G}(I; P))_n \to E(\overline{f}) \to 0.
\]

Consider the following commutative diagram of exact columns:
The commutativity induces two morphisms \( \xi : E(f)_{n+1} \to E(f)_n \) and \( \mu : E(f)_n \to E(f)_n \). Since \( v \circ u = 0 \), then \( \mu \circ \xi = 0 \). Since \( v \) is surjective, then \( \mu \) is surjective too. Since \( \beta \) is surjective and the middle row is exact, then \( \ker \mu \subseteq \im \xi \). Therefore,

\[ E(f)_{n+1} \xrightarrow{\xi} E(f)_n \xrightarrow{\mu} E(f)_n \to 0 \]

is an exact sequence of \( A \) modules. Finally, if \( A \) is noetherian and \( M, N \) and \( P \) are finitely generated, then \( E(f)_n = 0 \) for \( n \gg 0 \) big enough.

**Theorem 2.5** Let \( A \) be a noetherian ring, \( I, J \) two ideals of \( A \) and \( M \) a finitely generated \( A \)-module.

The following two conditions are equivalent:

1. \( \varphi : G(I) \otimes G(J; M) \to G(I + J; M) \) is an isomorphism.
2. \( \Tor_1(A/I^p, R(J; M)) = 0 \) and \( \Tor_1(A/I^p, G(J; M)) = 0 \) for all integers \( p \geq 1 \).

In particular, \( G(I) \otimes G(J) \simeq G(I + J) \) if and only if \( \Tor_1(A/I^p, A/J^q) = 0 \) and \( \Tor_2(A/I^p, A/J^q) = 0 \) for all integers \( p, q \geq 1 \).

**Proof.** Remark that \( \Tor_1(A/I^p, J^q M) = \ker(\pi_{p,q} : I^p \otimes J^q M \to I^p J^q M) \). Moreover, under the hypothesis \( \Tor_1(A/I^p, R(J; M)) = 0 \) for all \( p \geq 1 \), then the following two conditions are equivalent:

- \( \Tor_1(A/I^p, G(J; M)) = 0 \) for all \( p \geq 1 \).
- \( I^p M \cap J^q M = I^p J^q M \) for all \( p, q \geq 1 \).

Suppose \((ii)\) holds, i.e., \( \Tor_1(A/I^p, J^q M) = 0 \) and \( I^p M \cap J^q M = I^p J^q M \) for all \( p, q \geq 1 \). Then, \( \pi : R(I) \otimes R(J; M) \to R(I, J; M) \) is an isomorphism and, by Lemma 2.2, \( \varphi : G(I, J; M) \to G(I + J; M) \) is an isomorphism. Thus \( \varphi = \pi \circ \varphi \) is an isomorphism and \((i)\) holds. Let us now prove \((i) \Rightarrow (ii)\).

If \( \varphi = \varphi \circ \varphi \) is an isomorphism, then \( \varphi \) and \( \varphi \) are two isomorphisms. By Proposition 2.3, \( \varphi \) an isomorphism implies \( I^p M \cap J^q M = I^p J^q M \) for all \( p, q \geq 1 \). In particular,

\[ R(I; J^p M/J^q+1 M)_p = \frac{I^p J^q M + J^q+1 M}{J^q+1 M} = \frac{I^p J^q M}{J^q+1 M} = \frac{I^p J^q M}{I^p J^q+1 M} = G(J; I^p M)_q \]

and

\[ G(I; J^q M/J^q+1 M)_p = \frac{I^p J^q M + J^q+1 M}{I^p J^q+1 M} = \frac{I^p J^q M}{I^p J^q+1 M} = \frac{I^p J^q M}{I^p J^q+1 M} = G(I; J^q M)_{p,q}. \]

Fix \( q \geq 1 \). Since \( \varphi_{p,q} : G(I)_p \otimes G(J; M)_q \to G(I, J; M)_{p,q} \) is an isomorphism for all \( p \geq 1 \) and \( G(I, J; M)_{p,q} = G(I, J^q M/J^q+1 M)_{p,q} \), then \( \varphi_{p,q} : G(I) \otimes J^q M/J^q+1 M \to G(I; J^q M/J^q+1 M) \) is an isomorphism for all \( q \geq 1 \). In other words, \( I^p \otimes G(J; M) \to G(J; I^p M) \) is an isomorphism for all \( p \geq 1 \) (since \( R(I; J^q M/J^q+1 M)_p = G(J; I^p M)_q \)). By Proposition 2.4, \( I^p \otimes R(J; M) \to R(J; I^p M) \) is an isomorphism for all \( p \geq 1 \). So \( \pi : R(I) \otimes R(J; M) \to R(I, J; M) \) is an isomorphism and \( \Tor_1(A/I^p, R(J; M)) = 0 \) for all \( p \geq 1 \).
3 Some examples

Example 3.1 Let $A$ be a noetherian local ring, $I$, $J$ two ideals of $A$ and $M$ a finitely generated $A$-module. If $I = (x)$ is principal and $x$ $A$-regular, then $\varphi : \mathcal{G}(I) \otimes \mathcal{G}(J; M) \to \mathcal{G}(I + J; M)$ is an isomorphism if and only if $x$ is a nonzero divisor in $\mathcal{R}(J; M)$ and in $\mathcal{G}(J; M)$. Indeed, let $K(y; N)$ denote the Koszul complex of a sequence of elements $y = y_1, \ldots, y_m$ of $A$ with respect to an $A$-module $N$ and let $H_i(y; N)$ denote its $i$-th Koszul homology group. Then $\text{Tor}_1(A/I, N) = H_1(K(x; A) \otimes N) = H_1(x; M) = 0$ if and only if $x$ is a non-zerodivisor in $N$.

Example 3.2 Let $A$ be a noetherian local ring and let $I = (x)$ and $J = (y)$ be two principal ideals of $A$. If $(0 : x) \subset (y)$ and $(0 : y) \subset (x)$, then $\varphi : \mathcal{G}(I) \otimes \mathcal{G}(J) \to \mathcal{G}(I + J)$ is an isomorphism if and only if $x, y$ is an $A$-regular sequence.

Example 3.3 Let $R$ be a noetherian local ring and let $z, t$ be an $R$-regular sequence. Let $A = R/(zt)$, $x = z + (zt)$, $y = t + (zt)$, $I = (x)$ and $J = (y)$. Then $\varphi : \mathcal{G}(I, J) \to \mathcal{G}(I + J)$ is an isomorphism, but $\varphi : \mathcal{G}(I) \otimes \mathcal{G}(J) \to \mathcal{G}(I, J)$ is not an isomorphism.

An example of a pair of ideals $I$, $J$ with the property $\text{Tor}_1(A/I^p, A/J^q)$ for all integers $p, q \geq 1$ arises from a product of affine varieties (see [V], pages 130 to 136, and specially Proposition 5.5.7). The next result is well known (see, for instance, [HIO]). We give here a proof for the sake of completeness.

Proposition 3.4 Let $A$ be a noetherian local ring, $I$ and $J$ two ideals of $A$ and $M$ a finitely generated $A$-module. Let $x = x_1, \ldots, x_r$ be a system of generators of $I$ and $y = y_1, \ldots, y_r$, $y_i = \overline{x}_i = x_i + J$, a system of generators of the ideal $\overline{I} = I + J/J$ of the quotient ring $\overline{A} = A/J$. If $\mathcal{G}(J)$ and $\mathcal{G}(J; M)$ are free $\overline{A}$-modules and $y$ is an $\overline{A}$-regular sequence in $\overline{I}$, then $x$ is an $A$-regular sequence in $I$ and then $\varphi : \mathcal{G}(I) \otimes \mathcal{G}(J; M) \to \mathcal{G}(I + J; M)$ is an isomorphism.

Proof. Since, for all $q \geq 1$, $J^qM/J^{q+1}M$ is $\overline{A}$-free and $y$ is an $\overline{A}$-regular sequence, then

$$0 = \text{Tor}_1(\overline{A}/I, J^qM/J^{q+1}M) = H_1(K(y; \overline{A}) \otimes J^qM/J^{q+1}M) = H_1(y; J^qM/J^{q+1}M).$$

So $y$ is a $J^qM/J^{q+1}M$-regular sequence in $\overline{I}$ for all $q \geq 1$. In particular, $x$ is a $J^qM/J^{q+1}M$-regular sequence in $I$ and $H_i(x; J^qM/J^{q+1}M) = 0$ for all $q \geq 1$. Using the long exact sequences in homology associated to the short exact sequences of $A$-modules $0 \to J^qM/J^{q+1}M \to M/J^{q+1}M \to M/J^qM \to 0$, we deduce $H_i(x; M/J^qM) = 0$ and $x$ is an $M/J^qM$-regular sequence in $I$ for all $q \geq 1$. In particular, $x$ is an $M$-regular sequence in $I$. Analogously, but using the hypothesis $\mathcal{G}(J)$ is $\overline{A}$-free, we deduce $x$ is an $A$-regular sequence in $I$. Therefore

$$\text{Tor}_1(A/I, M) = H_i(K(x; A) \otimes M) = H_i(K(x; M)) = 0$$

and

$$\text{Tor}_1(A/I, M/J^qM) = H_i(K(x; A) \otimes M/J^qM) = H_i(K(x; M/J^qM)) = 0.$$

Using the long exact sequences in homology associated to the short exact sequences

$$0 \to J^qM \to M \to M/J^qM \to 0$$

and

$$0 \to J^qM/J^{q+1}M \to M/J^{q+1}M \to M/J^qM \to 0,$$

we deduce $\text{Tor}_1(A/I, \mathcal{R}(J; M)) = 0$ and $\text{Tor}_1(A/I, \mathcal{G}(J; M)) = 0$. Since $I^p/I^{p+1}$ is $A/I$-free, then $\text{Tor}_1(I^p/I^{p+1}, \mathcal{R}(J; M)) = \text{Tor}_1(A/I, \mathcal{R}(J; M)) \otimes I^p/I^{p+1} = 0$ and $\text{Tor}_1(I^p/I^{p+1}, \mathcal{G}(J; M)) = \text{Tor}_1(A/I, \mathcal{G}(J; M)) \otimes I^p/I^{p+1} = 0$. Applying the long exact sequences in homology to the short exact sequences $0 \to I^p/I^{p+1} \to A/I^{p+1} \to A/I^p \to 0$, we deduce $\text{Tor}_1(A/I^p, \mathcal{R}(J; M)) = 0$ and $\text{Tor}_1(A/I^p, \mathcal{G}(J; M)) = 0$ for all $p \geq 1$. 





4 Relation type of tensor products

**Lemma 4.1** Let $U$ be a standard $A$-algebra and $F$ a standard $U$-module. If $M$ is an $A$-module, then $F \otimes M$ is a standard $U$-module and $\text{rt}(F \otimes M) \leq \text{rt}(F)$. If $\lambda : M \rightarrow N$ is an epimorphism of $A$-modules, then $1 \otimes \lambda : F \otimes M \rightarrow F \otimes N$ is a surjective graded morphism of standard $U$-modules. Moreover, for each integer $n \geq 1$, $\ker(1_{F_n} \otimes \lambda) = U_1 \cdot \ker(1_{F_{n-1}} \otimes \lambda)$. In particular, for each $n \geq 1$, there exists an epimorphism of $A$-modules $E(F \otimes M)_n \rightarrow E(F \otimes N)_n$ and $\text{rt}(F \otimes N) \leq \text{rt}(F \otimes M)$.

**Proof.** Clearly $F \otimes M$ is a standard $U$-module and $1 \otimes \lambda : F \otimes M \rightarrow F \otimes N$ is a surjective graded morphism of standard $U$-modules. By Proposition 2.6 in [1], for each $n \geq \text{rt}(F) + 1$, the following sequence is exact:

$$\mathbf{A}_2(U_1) \otimes F_{n-2} \rightarrow U_1 \otimes F_{n-1} \rightarrow F_n \rightarrow 0.$$  

If we tensor it by $M$, we obtain the exact sequence

$$\mathbf{A}_2(U_1) \otimes F_{n-2} \otimes M \rightarrow U_1 \otimes F_{n-1} \otimes M \rightarrow F_n \otimes M \rightarrow 0,$$

for all $n \geq \text{rt}(F) + 1$. Thus $E(F \otimes M)_n = 0$ for all $n \geq \text{rt}(F) + 1$ and $\text{rt}(F \otimes M) \leq \text{rt}(F)$. Consider the following commutative diagram of exact columns and rows:

Using a diagram chasing argument, one deduces $\ker(1_{F_n} \otimes \lambda) = U_1 \cdot \ker(1_{F_{n-1}} \otimes \lambda)$ for all $n \geq 1$. If $g : X \rightarrow F \otimes M$ is a symmetric presentation of $F \otimes M$, then, by Lemma 2.3 in [1], there exists an exact sequence of $A$-modules $E(g)_n \rightarrow E((1 \otimes \lambda) \circ g)_n \rightarrow E(1 \otimes \lambda)_n \rightarrow 0$ for all $n \geq 1$. But $E(g)_n = E(F \otimes M)_n$, $E((1 \otimes \lambda) \circ g)_n = E(F \otimes N)_n$ and $E(1 \otimes \lambda)_n = 0$ for all $n \geq 1$. Thus $E(F \otimes M)_n \rightarrow E(F \otimes N)_n$ is surjective for all $n \geq 1$ and $\text{rt}(F \otimes N) \leq \text{rt}(F \otimes M)$.

**Theorem 4.2** Let $A$ be a commutative ring, $U$ and $V$ two standard $A$-algebras and $F$ a standard $U$-module and $G$ a standard $V$-module. Then $U \otimes V$ is a standard $A$-algebra, $F \otimes G$ is a standard $U \otimes V$-module and $\text{rt}(F \otimes G) \leq \max(\text{rt}(F), \text{rt}(G))$.

**Proof.** Clearly $U \otimes V$ is a standard $A$-algebra and $F \otimes G$ is a standard $U \otimes V$-module. Take $\varphi : X \rightarrow F$ and $\psi : Y \rightarrow G$ two symmetric presentations of $F$ and $G$, respectively. Then $\varphi \otimes \psi : X \otimes Y \rightarrow F \otimes G$ is a symmetric presentation of $F \otimes G$. Since $\varphi \otimes \psi = (\varphi \otimes 1_G) \circ (1_X \otimes \psi)$, then, for each integer $n \geq 2$, there exists an exact sequence of $A$-modules

$$E(1_X \otimes \psi)_n \rightarrow E(\varphi \otimes \psi)_n \rightarrow E(\varphi \otimes 1_G)_n \rightarrow 0.$$ 

Since $\psi : Y \rightarrow G$ is a symmetric presentation of $G$, then $1_{X_0} \otimes \psi : X_0 \otimes Y \rightarrow X_0 \otimes G$ is a symmetric presentation of $X_0 \otimes G$ and $E(X_0 \otimes G)_n = E(1_{X_0} \otimes \psi)_n$. Using Lemma 1.4, $\ker(1_{X_1} \otimes \psi_{n-1}) = \cdots$
\[ U_1 \cdot \ker(1_{X_{i-1}} \otimes \psi_{n-i}) \text{ for all } i \geq 1. \]

Then

\[
E(1_X \otimes \psi)_n = \frac{\ker(1_X \otimes \psi)_n}{(U \otimes V)_1 \cdot \ker(1_X \otimes \psi)_{n-1}} \oplus \cdots \oplus \frac{V_1 \cdot \ker(1_{X_{n-1}} \otimes \psi) + V_1 \cdot \ker(1_{X_1} \otimes \psi) - \cdots - \frac{U_1 \cdot \ker(1_{X_{n-2}} \otimes \psi) + V_1 \cdot \ker(1_{X_{n-1}} \otimes \psi)}{U_1 \cdot \ker(1_{X_1} \otimes \psi)}}{\ker(1_{X_0} \otimes \psi)}.
\]

Therefore \( E(1_X \otimes \psi)_n = E(1_{X_0} \otimes \psi)_n = E(X_0 \otimes G)_n \) for all \( n \geq 1 \). Analogously, \( E(\varphi \otimes 1_G)_n = E(\varphi \otimes 1_{G_0})_n = E(F \otimes G_0)_n \) for all \( n \geq 1 \). Hence there exists an exact sequence of \( A \)-modules

\[
E(X_0 \otimes G)_n \rightarrow E(F \otimes G)_n \rightarrow E(F \otimes G_0)_n \rightarrow 0
\]

for all \( n \geq 2 \) and, by Lemma \([4.3]\), \( \text{rt}(F \otimes G) \leq \max(\text{rt}(F \otimes G_0), \text{rt}(X_0 \otimes G)) \leq \max(\text{rt}(F), \text{rt}(G)). \]

**Remark 4.3** Let \( A \) be a commutative ring and let \( U \) and \( V \) be two standard \( A \)-algebras. If \( \text{Tor}^A_1(U, V) = 0 \), then \( E(U \otimes V) = E(U) \oplus E(V) \). This follows from the characterization \( E(U) = H_1(A, U, A) \) (see Remark 2.3 in \([P_3]\)) and Proposition 19.3 in \([A]\).

### 5 Uniform bounds

**Lemma 5.1** Let \((A, m)\) be a noetherian local ring and \( M \) be a finitely generated \( A \)-module. Let \( p \) a prime ideal of \( A \) such that \( A/p \) is regular local and \( G(p) \) and \( G(p; M) \) are free \( A/p \)-modules. Then \( \text{rt}(m; M) \leq \text{rt}(p; M) \).

**Proof.** Since \( A/p \) is regular local, there exists a sequence of elements \( x = x_1, \ldots, x_r \) in \( A \) such that \( y = y_1, \ldots, y_r \), defined by \( y_i = x_i + p \), is a system of generators of \( m/p \) and an \( A \)-regular sequence. Let \( I \) be the ideal of \( A \) generated by \( x \). In particular, \( I + p/p = m/p \) and \( I + p = m \). By Proposition 3.4, \( x \) is an \( A \)-regular sequence and \( \text{Tor}_1(A/p^i, \mathcal{R}(p; M)) = 0 \) and \( \text{Tor}_1(A/p^i, \mathcal{G}(p; M)) = 0 \) for all \( p^i \geq 1 \). By Theorem 2.\( B \), \( \mathcal{G}(I) \otimes \mathcal{G}(p; M) \rightarrow \mathcal{G}(m; M) \) is an isomorphism. By Theorem 4.2, \( \text{rt}(\mathcal{G}(m; M)) \leq \max(\text{rt}(\mathcal{G}(I)), \text{rt}(\mathcal{G}(p; M))) \). By Remark 2.7 in \([P_3]\), \( \text{rt}(\mathcal{G}(J; M)) = \text{rt}(J; M) \) for any ideal \( J \) of \( A \). Since \( I \) is generated by a regular sequence, then \( \text{rt}(I) = 1 \) (see, for instance, \([V]\) page 30). Thus \( \text{rt}(m; M) \leq \text{rt}(p; M) \).

The next result is a slight generalization of a well known Theorem of Duncan and O’Carroll \([DO]\). In fact the proof of our theorem is directly inspired in their. We sketch it here for the sake of completeness.

**Theorem 5.2** Let \( A \) be an excellent (or \( J - 2 \)) ring and let \( M \) be a finitely generated \( A \)-module. Then there exists an integer \( s \geq 1 \) such that, for all maximal ideals \( m \) of \( A \), the relation type of \( m \) with respect to \( M \) satisfies \( \text{rt}(m; M) \leq s \).

**Proof.** For every \( p \in \text{Spec}(A) \), let us construct a non-empty open subset \( U(p) \) of \( V(p) = \{ q \in \text{Spec}(A) \mid q \supseteq p \} \approx \text{Spec}(A/p) \). Remark that \( A/p \) is a noetherian domain, \( G(p) \) is a finitely generated \( A/p \)-algebra and \( G(p; M) \) is a finitely generated \( G(p) \)-module. By Generic Flatness (Theorem 22.A in \([M]\)), there exist \( f, g \in A - p \) such that \( G(p)_f \) is an \((A/p)_f\)-free module and \( G(p; M)_g \) is an \((A/p)_g\)-free
module. Since $A$ is $J - 2$, the set $\text{Reg}(A/p) = \{q \in V(p) \mid (A/p)_q$ is regular local$\}$ is a non-empty open subset of $V(p)$. Define $U(p)$ as the intersection $D(f) \cap D(g) \cap \text{Reg}(A/p) = \{q \in V(p) \mid q \not\subseteq f, q \not\subseteq g, (A/p)_q$ is regular local$\}$, which is a non-empty open subset of $V(p)$. Remark that for all $q \in U(p)$, $(A/p)_q$ is regular local and $G(p)_q$ and $G(p; M)_q$ are free $G(p)_q$-modules. By Lemma 5.1, $rt(qA_q; M_q) \leq rt(pA_q; M_q) \leq rt(p; M)$ for all $q \in U(p)$. In particular, $rt(m; M) \leq rt(p; M)$ for all maximal ideals $m \in U(p)$. For each minimal prime $p_i$ of $A$, let $V(p_i) - U(p_i) = V(p_{i,1}) \cup \ldots \cup V(p_{i,r_i})$ be the decomposition into irreducible closed subsets of the proper closed subset $V(p_i) - U(p_i)$, $p_{i,j} \in \text{Spec}(A)$, $p_{i,j} \not\subseteq p_i$. Since $A$ is noetherian, $\text{Spec}(A)$ can be covered by finitely many locally closed sets of type $U(p)$, i.e., there exists a finite number of prime ideals $q_1, \ldots, q_m$, such that $\text{Spec}(A) = \bigcup_{i=1}^m U(q_i)$. Hence, $rt(m; M) \leq \max\{rt(q_i; M) \mid i = 1, \ldots, m\}$ for any maximal ideal $m$ of $A$.

Using Theorem 2 in [P] we deduce the result of Duncan and O'Carroll in [DO].

**Corollary 5.3** [DO] **Let $A$ be an excellent (or $J - 2$) ring and let $N \subseteq M$ be two finitely generated $A$-modules. Then there exists an integer $s \geq 1$ such that, for all integers $n \geq s$ and for all maximal ideals $m$ of $A$, $m^n M \cap N = m^{n-s}(m^s M \cap N)$.**

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