Chapter 1

Scaling of von Neumann entropy at the Anderson transition

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Extensive body of work has shown that for the model of a non-interacting electron in a random potential there is a quantum critical point for dimensions greater than two—a metal-insulator transition. This model also plays an important role in the plateau-to-plateau transition in the integer quantum Hall effect, which is also correctly captured by a scaling theory. Yet, in neither of these cases the ground state energy shows any non-analyticity as a function of a suitable tuning parameter, typically considered to be a hallmark of a quantum phase transition, similar to the non-analyticity of the free energy in a classical phase transition. Here we show that von Neumann entropy (entanglement entropy) is non-analytic at these phase transitions and can track the fundamental changes in the internal correlations of the ground state wave function. In particular, it summarizes the spatially wildly fluctuating intensities of the wave function close to the criticality of the Anderson transition. It is likely that all quantum phase transitions can be similarly described.

1.1. Introduction

Ever since Anderson’s paper[1] “Absence of diffusion in certain random lattices”, it has been a theme in condensed matter physics to unravel the quantum phase transition between the itinerant and the localized electronic states[2]. The metal-insulator transition embodies the very basic concept of wave-particle complementarity in quantum mechanics. Itinerant states reflect the wave aspect, while the localized states reflect the particle aspect. In one-particle quantum mechanics without disorder, the wave and the particle descriptions are dual to each other. There is no fundamental distinction between them. Coherent superposition of waves are packets that act like spatially compact lumps of energy and momentum, or particles. In contrast, in a disordered medium the metallic state described by non-normalizable wave functions is separated by a quantum phase transition, the Anderson transition, when it exists, from the insulating state with normalizable wave functions. In the insulating state particles are tied to random spatial centers. These two macroscopic states are fundamentally different and can not be analytically continued into each
other.

If the Fermi energy is situated within the localized states, the system is an insulator. It might be argued that in a real physical situation, the role of electron-electron interaction will become more and more important as the system approaches localization and the notion of Anderson localization will lose its validity. In fact, quite the opposite may sometimes be true. A rigorous, but a simple example of spinless fermions, was recently studied\textsuperscript{3,4} where interactions lead to a broken symmetry in the pure system, generating a gap, hence an insulator. But it was shown that for arbitrary disorder this gap is washed out, and there are gapless localized excitations resembling an “Anderson insulator”. In any case, Anderson transition has proven to be a powerful paradigm for metal-insulator transition.

Because the Anderson transition is a quantum phase transition, it is natural to develop a theoretical framework that comes as close as possible to any other thermodynamical quantum phase transitions. Although there are other theoretical approaches, including powerful numerical simulations of an electron in a random potential\textsuperscript{5} interesting insights can be gained by contrasting and comparing with more conventional models of phase transitions. In order to study Anderson localization I shall focus on the scaling properties of the von Neumann entropy (vNE), which is a fundamental concept in quantum mechanics and quantum information theory.

1.2. Statistical field theory of localization

It is well known that the properties of a Brownian particle can be understood from a free Euclidean field theory. The free fields act as a generating function for the Brownian motion. The Green’s function of interacting fields, on the other hand, reflect particles with suitable constraints.\textsuperscript{6} A particularly pretty example is that of the self avoiding random walks that can be described in terms of the correlation functions of the $O(n)$ spin model in the limit $n \to 0$, even though the partition function is exactly unity in that limit.\textsuperscript{7} The lesson is that the language of statistical field theory and its scaling behavior can provide important insights. Similarly, a replica field theory discussed elsewhere in this volume maps the Anderson problem of a single particle with disorder to a suitable non-linear $\sigma$-model, which depends on the relevant symmetries, with the proviso that the number of replicas $N$ has to be set to zero at the end of all calculations. It is only in the $N \to 0$ that the effect of randomness appear; as long as $N \neq 0$, the model is translationally invariant. In spite of the subtleties of the replica limit, much has been learnt as far as the criticality of the Anderson transition is concerned by drawing analogies with the problem of critical phenomena in statistical mechanics.\textsuperscript{8}

One can also reverse the chain of reasoning and learn about the statistical mechanics of critical phenomena from the Anderson problem. As an example, let us consider the universal conductance fluctuations in a mesoscopic system. It was
shown that if we consider the disorder averaged conductance by \( \langle G \rangle \) and its fluctuation by \( \langle (\delta G)^2 \rangle \) then the latter is independent of scale and is universal for dimension \( D < 4 \). A sample is considered to be mesoscopic if its linear dimension \( L \) is larger than the mean free path but smaller than the scale at which the phase coherence of the electrons is broken. The relative fluctuation \( \langle (\delta G)^2 \rangle / \langle G \rangle^2 \) is proportional to \( L^{4-2D} \) and is independent of scale at \( D = 2 \). In fact, it was shown that this result along with many others can be obtained from a replica field theory of an extended non-linear sigma model defined on a Grassmannian manifold.

This raises the possibility that perhaps a similar result should also hold on a much simpler manifold, namely the coset space of \( O(n)/O(n-1) \). For \( n = 3 \), this is the familiar \( O(3) \) \( \sigma \)-model of classical \( n \)-vector spins of unit length \( \hat{\Omega}^2 = 1 \), which is a faithful description of the long wavelength behavior of the classical Heisenberg model. What could possibly be the analog of the conductance for the Heisenberg model? It was argued that it is the spin stiffness constant, defined by the response of the system with respect to a twist in the boundary condition, which measures the rigidity of the system. By a mesoscopic sample we now mean \( L \) such that it is much larger than the microscopic cutoff of the order of lattice spacing and much smaller than the correlation length \( \xi \) of the Heisenberg model. From a one-loop calculation it is easy to show that the absolute fluctuation of the spin stiffness constant \( \rho_s \) is independent of the scale and its relative fluctuation is given by

\[
\frac{\delta \rho_s^2}{\overline{\rho_s^2}} \propto L^{4-2D},
\]

where the overline now represents the average with respect to the thermal fluctuations. More explicitly in the interesting case of \( D = 2 \) we get, including the logarithmic correction,

\[
\frac{\delta \rho_s^2}{\overline{\rho_s^2}} = \frac{2\pi}{(n-2)[\ln(\xi/L)]^2}.
\]

One can find many more interesting connections between these two disparate systems, which behooves us to take a closer look at the “thermodynamics” of the quantum phase transition in the Anderson model, leading us to a discussion of vNE.

1.3. von Neumann entropy

A set of brief remarks seem to be appropriate to place our subsequent discussion in a more general context. In a landmark paper on black hole entropy, Bekenstein demonstrated the power of the notion of information entropy. The concept also applies to any quantum mechanical ground state. Given a unique ground state, the thermodynamic entropy is of course zero. To distinguish various ground states one usually studies the analyticity of the ground state energy as a function of a tuning
parameter. In most cases, a quantum phase transition is characterized by the non-analyticity of the ground state energy. In some cases, for example in the Anderson transition, and in the integer quantum Hall plateau transitions, the ground state energy is analytic through the transitions and does not provide any indication of their existence. Yet we know that the wave function encodes special correlations internal to its state. How can we quantify such correlations? In particular how do they change across these quantum phase transitions? We shall show that in these cases the non-analyticity of vNE can be used as a fingerprint of these quantum phase transitions.

For a pure state $|\Psi\rangle$, the density matrix is $\rho = |\Psi\rangle\langle\Psi|$. Consider partitioning the system into $A$ and $B$, where $A$ denotes the subsystem of interest and $B$ the environment whose details are of no interest. The reduced density matrix $\rho_A$ is constructed by tracing over the degrees of freedom of $B$, similar to integrating out the microstates corresponding to a set of macroscopic thermodynamic variables. The vNE, $S = -\text{Tr}(\rho_A \ln \rho_A)$, is a measure of the bipartite entanglement and therefore contains information about the quantum correlations present in the ground state. The interesting point is that the reduced density matrix is a mixture if the state $|\Psi\rangle$ is entangled, that is, it cannot be factored into $|\Psi\rangle_A \otimes |\Psi\rangle_B$. Of course, partitioning a mixed state will also lead to a mixed state; there is nothing new here. Since $\rho_A$ is a mixture, we can perform a statistical analysis of it and obtain a non-trivial value of entropy that can summarize the essential features of an entangled state. The result follows from the Schmidt decomposition theorem: for a bipartition of a pure state there exist sets of orthonormal states $\{|i_A\rangle\}$ of $A$ and orthonormal states $\{|i_B\rangle\}$ of $B$ such that

$$|\Psi\rangle = \sum_i \lambda_i |i_A\rangle \otimes |i_B\rangle,$$

(1.3)

where $\lambda_i$ are non-negative real numbers satisfying $\sum_i \lambda_i^2 = 1$. The result that a state can be fully known, yet its subsystem is in a mixed state is a remarkable consequence of entanglement. Unfortunately, there no such theorems if we partition the system into more than two parts, say $A$, $B$, and $C$. Multipartite entanglements are consequently less understood.

As we have argued above, the mapping of the Anderson localization to a problem of a statistical field theory has been quite successful. It leaves us little doubt that the notion of criticality and scaling are correct. We might pursue this argument further and ask does this transition fit into the general framework of a quantum phase transition? If we define such a transition in terms of the non-analyticity of the ground state energy as a function of disorder, the answer to this question is no. Edwards and Thouless\(^{[14]}\) have shown rigorously that the ground state energy, which depends on the average density of states, is smooth through the localization transition. We believe that the closest we can come is the non-analyticity of vNE\(^{[13]}\) which is of great current interest in regard to quantum phase transitions\(^{[15]}\). One expects that vNE must play a role in understanding the correlations that exist on all
length scales at a quantum critical point. But a state can be entangled without being critical—consider, for instance, the singlet state of two spin-$1/2$ particles. It is the special critical scaling property of entanglement that we are interested here. Even more paradoxical, it may sound, is that Anderson localization is a single particle problem, and the conventional notion entanglement of particles does not apply. Clearly, the notion of entanglement will have to be extended, and this extension will be the theory of entanglement defined using the site occupation number basis in the second-quantized Fock space.

As noted above, we shall consider two important models to illustrate our expectation. Our first example is Anderson localization in dimension greater than two, which has been extensively studied and is known to have a quantum critical point. At the critical point the wave function exhibits a fractal character. The second example is the plateau-to-plateau transition in the integer quantum Hall effect in which the Anderson localization plays a crucial role in establishing the very existence of the plateaus. We shall see that vNE is nonanalytic at these transitions and exhibits the correct scaling behavior when compared to other approaches. vNE and its scaling behavior characterize the entanglement associated with these quantum phase transitions. Because they are determined by single-particle properties in the presence of disorder, their vNE are different from those associated with disorder-free interacting systems.

Consider the single-particle probability $|\psi_E(r)|^2$ at energy $E$ and position $r$ for a noninteracting electronic system. In the neighborhood of a critical point governed by disorder, it fluctuates so strongly that it has a broad (non-Gaussian) distribution even in the thermodynamic limit. This non-self-averaging nature of the wave function intensity can be seen in the scaling of its moments. In particular, the moments, $P_\ell$, defined as the generalized inverse participation ratios, obey the finite-size scaling Ansatz,

$$P_\ell(E) = \sum_r |\psi_E(r)|^{2\ell} \sim L^{-\tau_\ell} G_\ell[(E - E_c)L^{1/\nu}],$$

where $L$ is the system size and $\nu$ is the exponent characterizing the divergence of the correlation length at the critical point $E_c$, $\xi_E \sim |E - E_c|^{-\nu}$. The quantity $\tau_\ell$ is the multifractal spectrum, and the overline denotes the disorder average. $G_\ell(x)$ is a scaling function with $G_\ell(x \to 0) \to 1$ as $E \to E_c$. As $E$ deviates from the critical point, the system either tends to an ideal metallic state with $P_\ell(E) \sim L^{-D(\ell-1)}$ or to a localized state with $P_\ell(E)$ that is independent of $L$. In the multifractal state, right at the Anderson transition, the intensity of the wave function has local exponents, defined by its sample-size dependence, which vary from point to point. A beautiful simulation of multifractality of the intensity of the wave function at the 3D Anderson transition is shown in Ref. 21. In contrast, a single non-integer scaling exponent applicable to the entire volume corresponds to the fluctuations of a fractal. The multifractal spectrum uniquely characterizes the wildly complex spatial structure of the wave function. It is quite remarkable that the same multifractal
1.4. von Neumann Entropy in Disordered Noninteracting Electronic Systems

We define entanglement using the site occupation number basis in the second-quantized Fock space. Let us partition a lattice of linear dimension $L$ into two parts, $A$ and $B$. A single particle eigenstate at energy $E$ in the site occupation number basis is

$$|\psi_E⟩ = \sum_{r \in A \cup B} \psi_E(r) |1⟩_r \bigotimes_{r' \neq r} |0⟩_{r'}$$  \hspace{1cm} (1.5)

Here $\psi_E(r)$ is the probability amplitude at the site $r$ and $|n⟩_r$ is the occupation number at site $r$, either 0 or 1. We rewrite the above sum over lattice sites $r$ into mutually orthogonal parts,

$$|\psi_E⟩ = |1⟩_A \otimes |0⟩_B + |0⟩_A \otimes |1⟩_B$$  \hspace{1cm} (1.6)

where

$$|1⟩_A = \sum_{r \in A} \psi_E(r) |1⟩_r \bigotimes_{r' \neq r} |0⟩_{r'}; \quad |0⟩_A = \bigotimes_{r \in A} |0⟩_r$$  \hspace{1cm} (1.7)

similarly for $|1⟩_B$ and $|0⟩_B$. Note that

$$⟨0|0⟩_A = ⟨0|0⟩_B = 1, \quad ⟨1|1⟩_A = p_A, \quad ⟨1|1⟩_B = p_B,$$  \hspace{1cm} (1.8)

where

$$p_A = \sum_{r \in A} |\psi_E(r)|^2;$$  \hspace{1cm} (1.9)

and similarly for $p_B$ with $p_A + p_B = 1$.

The reduced density matrix $ρ_A$ is obtained from $ρ = |\psi_E⟩⟨\psi_E|$, after tracing out the Hilbert space over $B$, is

$$ρ_A = |1⟩_A ⟨1| + (1 - p_A)|0⟩_A ⟨0|.$$  \hspace{1cm} (1.10)

The corresponding vNE is given by

$$S_A = -p_A \ln p_A - (1 - p_A) \ln(1 - p_A).$$  \hspace{1cm} (1.11)

Here, manifestly $S_A = S_B$, and either of them is bounded between 0 and ln2 for any eigenstate. Despite the use of a second-quantized language, we are considering a single particle state rather than a many body correlated state. The entanglement entropy can not grow arbitrarily large as the size of $A$ increases, unlike the entanglement entropy in interacting quantum systems where it can be arbitrarily large close to the critical point.
If the system size becomes very large in comparison to the size of the subsystem $A$, we can restrict $A$ to be a single lattice site and study scaling with respect to $L$. Thus, we consider the single site vN:

$$S(E) = - \sum_{r \in L^d} \left\{ |\psi_E(r)|^2 \ln |\psi_E(r)|^2 + \left[ 1 - |\psi_E(r)|^2 \right] \ln \left[ 1 - |\psi_E(r)|^2 \right] \right\}. \tag{1.12}$$

To study the leading critical behavior, the second term in the curly brackets in the right-hand side of Eq. (1.12) can be ignored since $|\psi_E(r)|^2 \ll 1$ for all $r$ for states close to the critical energy. The disorder averaged entropy $\overline{S}$ can be expressed in terms of the multifractal scaling in Eq. (1.4), giving

$$\overline{S}(E) \approx - \frac{dP_\ell}{dt} \bigg|_{t=1} \approx \frac{d\tau_\ell}{dt} \bigg|_{t=1} \ln L - \frac{\partial G_\ell}{\partial t} \bigg|_{t=1}. \tag{1.13}$$

Although we do not know the analytical form of the scaling function $G_\ell$, its approximate $L$ dependence can be obtained in various limiting cases. Exactly at criticality, $G_\ell \equiv 1$ for all values of $\ell$ and

$$\overline{S}(E) \sim \alpha_1 \ln L, \tag{1.14}$$

where the constant $\alpha_1 = d\tau_\ell/d\ell|_{\ell=1}$. The leading scaling behaviors of $\overline{S}(E)$ in both the metallic and the localized states can now be obtained, following the discussion below Eq. (1.4). The results are

$$\overline{S}_{\text{metal}}(E) \sim D \ln L, \quad \overline{S}_{\text{loc}}(E) \sim \alpha_1 \ln \xi_E. \tag{1.15}$$

We see that in general $\overline{S}(E)$ is of the form

$$\overline{S}(E) \sim Q[(E - E_C)L^{1/\nu}] \ln L, \tag{1.16}$$

where the coefficient function $Q(x)$ is $D$ in the metallic state, decreases to $\alpha_1$ at criticality and then goes to zero for the localized state. We now turn to numerical simulations to see the extent to which this scaling behavior is satisfied.

1.5. von Neumann entropy in the three dimensional Anderson Model

Let us consider the disordered Anderson model on a 3D cubic lattice. The Hamiltonian is

$$H = \sum_i V_i c_i^\dagger c_i - t \sum_{\langle i,j \rangle} (c_i^\dagger c_j + H.c.), \tag{1.17}$$

where $c_i^\dagger$ ($c_i$) is the creation (annihilation) operator for an electron at site $i$ and the $\langle i,j \rangle$ indicates that the second sum is over nearest neighbors. The $V_i$ are random variables uniformly distributed in the range $[-W/2, W/2]$. In what follows, we set $t = 1$. Of course, the model has been extensively studied. Below a critical disorder...
strength \( W_c \), there is a region of extended states at the band center. The recent values of the critical disorder strength \( W_c \) and the localization length exponent are \( W_c = 16.3 \) and \( \nu = 1.57 \pm 0.03 \).

To obtain the energy-averaged entropy, we average Eq. (1.12) over the entire band of energy eigenvalues. From this we construct the vNE,

\[
\mathcal{S}(w, L) = \frac{1}{\mathcal{N}} \sum_E \mathcal{S}(E, w, L),
\]

where \( \mathcal{N} \) counts the total number of states in the band. Near \( w = 0 \), we can show, using Eqs. (1.16) and (1.20), that

\[
\mathcal{S}(w, L) \sim C + L^{-1/\nu} f_{\pm}(wL^{1/\nu}) \ln L,
\]

where \( C \) is a constant independent of \( L \) and \( f_{\pm}(x) \) are two universal functions corresponding to the regimes \( w > 0 \) and \( w < 0 \). We numerically diagonalize Eq. (1.17)

Fig. 1.1. Scaling curve in the 3D Anderson model. With the choice of \( \nu = 1.57 \) and \( C = 12.96 \), all data collapse to a universal function \( f_{\pm}(x) \). The two branches correspond to \( w < 0 \) and \( w > 0 \).

for systems of sizes \( L \times L \times L \) with periodic boundary conditions. The maximum system size was \( L = 13 \), and the results were averaged over 20 disorder realizations. The scaling form of \( \mathcal{S}(w, L) \) is given by Eq. (1.19). Figure 1.1 shows the results of the data collapse with a choice of \( \nu = 1.57 \), and the nonuniversal constant \( C = 12.96 \).
Scaling of von Neumann entropy at the Anderson transition

is determined by a powerful algorithm described in the Appendix C of Ref. [24]. The data collapse is reasonable and is consistent with the nonanalyticity of vNE and the multifractal analysis. Clearly, it would be useful to improve the numerics by increasing both the system sizes and the number of disorder realizations to attain a better data collapse.

We can also study vNE at the band center \( E = 0 \) by sweeping \( W \) across the critical value \( W_c \). In this case, the states at \( E = 0 \) will evolve continuously from metallic to critical and then to localized states. The entanglement entropy will be given similarly by another scaling function

\[
S(E = 0, w, L) \sim C(w L^{1/\nu}) \ln L, \tag{1.20}
\]

where \( w = (W - W_c)/W_c \) is the reduced disorder strength and \( C(x) \) is a scaling function, which as remarked earlier, → \( D \) as \( w \to -1 \) and \( \to 0 \) as \( w \to \infty \), and \( C = \alpha_1 \) when \( w = 0 \). For this purpose we use the transfer matrix method\(^{25}\) to study the energy resolved \( S(E, w, L) \) by considering a quasi-one-dimensional system with a size of \( (mL) \times L \times L \), \( m \gg 1; L \) up to 18, and \( m = 2000 \) were found to be reasonable. To compute vNE, we divide the system into \( m \) cubes labeled by \( I = 1, 2, \ldots, m \), each containing \( L^3 \) sites. The wave function within each cube is normalized and the vNE, \( S_I(E, W, L) \) in the \( I \)th cube is computed. Finally \( S(E, W, L) \) was obtained by averaging over all cubes. The validity of the scaling form in Eq. (1.20) is seen in Fig. 1.2.

In particular, the function \( C(x) \) shows the expected behavior, approaching \( D = 3 \) as \( w \to -1 \), and tending to 0 as \( w \to \infty \).

1.6. von Neumann entropy in the integer quantum Hall system

For the integer quantum Hall system, we use a basis defined by the states \(|n, k\rangle\), where \( n \) is the Landau level index and \( k \) is the wave vector in the \( y \)-direction. The Hamiltonian can be expressed\(^{26}\) in terms of the matrix elements in this basis as

\[
H = \sum_{n,k} \langle n, k | \langle n, k | (n + 1/2) \hbar \omega_c + \sum_{n,k,n',k'} \langle n, k | V | n', k' \rangle \langle n', k' | \tag{1.21}
\]

where \( \omega_c = eB/mc \) is the cyclotron frequency, and \( B \) is the magnetic field. \( V(r) \) is the disorder potential. If we focus on the lowest Landau level, \( n = 0 \), and assume that the distribution of disorder is \( \delta \)-correlated with zero mean, that is, \( V(r) = 0 \) and \( \langle V(r) V(r') \rangle = V_0^2 \delta(r - r') \), the matrix elements, \( \langle 0, k | V | 0, k' \rangle \), are\(^{26}\)

\[
\langle 0, k | V | 0, k' \rangle = V_0 \sqrt{\frac{\pi}{\pi L_y}} \exp \left[ -\frac{1}{4} \rho_B^2 (k - k')^2 \right] \int d\chi e^{-\chi^2} u_0(l_B \chi + \frac{k + k'}{2}, k', k - k) \tag{1.22}
\]

where \( l_B = (\hbar c/eB)^{1/2} \) is the magnetic length, and \( u_0(x, k) \) is the Fourier transform of \( V(x, y) \) along the \( y \) direction,

\[
u_0(x, k) = \frac{1}{\sqrt{L_y}} \int dy V(x, y) e^{iky}. \tag{1.23}
\]
Fig. 1.2. The quantity $C$ in Eq. (1.20). The system sizes are too small to observe the weak $L$ dependence. Inset: $\mathbb{S}(E = 0, W, L)$ as a function of $\ln L$ for three different $W$.

We choose a two-dimensional square with a linear dimension $L = \sqrt{2\pi}Ml_B$, where $M$ is an integer. We impose periodic boundary conditions in both directions and discretize by a mesh of size $\sqrt{\pi l_B}/\sqrt{2}M$. The Hamiltonian matrix is diagonalized and the eigenstates $|\phi_a\rangle = \sum_k \alpha_{k,a}|0,k\rangle$, $a = 1 \ldots M^2$, are obtained along with the corresponding eigenvalues $E_a$. The zero of the energy is at the center of the lowest Landau band and the unit of energy is $\Gamma = 2V_0/\sqrt{2\pi l_B}$. For each eigenstate the wave function in real space is constructed:

$$\phi_a(x, y) = \langle x, y|\phi_a\rangle = \sum_k \alpha_{k,a}\phi_{0,k}(x, y),$$  \hspace{1cm} (1.24)

where $\phi_{0,k}(x, y)$ is the wave function with quantum number $k$ in the lowest Landau level. The dimension of the Hamiltonian matrix increases as $N_k \sim M^2$, making it difficult to diagonalize fully. We circumvent this difficulty by computing only those states $|\phi_a\rangle$ whose energies lie within a window $\Delta$ around a fixed value $E$ thus: $E_a \in [E - \Delta/2, E + \Delta/2]$. We take $\Delta$ to be sufficiently small (0.01), but still large enough such it spans large number of states in the interval $\Delta$ (at least 100 eigenstates).

We now uniformly break up the $L \times L$ square into smaller squares $A_i$ of size $l \times l$, where $l = l_B\sqrt{\pi/2}$, independent of the system size $L$. The $A_i$ do not overlap. For
1.7. A brief note on the single-site von Neumann entropy

The scaling of single-site vNE can lead to some misunderstanding in regard to universality. This can be illustrated by considering Ising chain in a transverse field...
for which the Hamiltonian is

$$\mathcal{H} = -J \lambda \sum_i S^z_i S^z_{i+1} - J \sum_i S^x_i$$  \hspace{1cm} (1.25)

where $S_x$, $S_y$ and $S_z$ are spin-1/2 matrices. The sum is over all sites $N \to \infty$. It is well known that the second derivative of the ground state energy

$$\frac{E_0}{N} = -\frac{2J}{\pi} \left( 1 + \lambda \right) \mathbb{E} \left( \frac{2\sqrt{\lambda}}{1+\lambda} \right)$$  \hspace{1cm} (1.26)

has a logarithmic singularity at $\lambda = 1$, signifying a quantum critical point in the conventional sense of the non-analyticity of the ground state energy. This non-analyticity is symmetric as $\lambda \to 1 \pm \epsilon$.

Here, $\mathbb{E}$ is the complete Elliptic integral of the second kind. It is also simple to calculate the single site vNE. A given site constitutes part $A$ of the system, and part $B$ is the rest of the Ising chain of $N-1$ sites. The reduced density matrix $\rho_A$ is

$$\rho_A = \frac{1}{2} \begin{pmatrix} 1 + \langle \sigma^z_i \rangle & \langle \sigma^z_i \rangle \\ \langle \sigma^z_i \rangle & 1 - \langle \sigma^z_i \rangle \end{pmatrix},$$  \hspace{1cm} (1.27)

where the exact known results are

$$\langle \sigma^z_i \rangle = (1 - 1/\lambda^2)^{1/8}, \quad \lambda > 1; 0, \quad \text{otherwise},$$  \hspace{1cm} (1.28)

$$\langle \sigma^x_i \rangle = \frac{1 - \lambda}{\pi} \mathbb{K} \left( 2\sqrt{\lambda} \frac{1}{1 + \lambda} \right) + \frac{1 + \lambda}{\pi} \mathbb{E} \left( 2\sqrt{\lambda} \frac{1}{1 + \lambda} \right),$$  \hspace{1cm} (1.29)

where $\mathbb{K}$ is the complete elliptic integral of the first kind. The vNE, $S$, can now be easily computed from the $2 \times 2$ reduced density matrix $\rho_A$. The singularities approaching the critical point are

$$\lim_{\lambda \to 1^-} \frac{\partial S}{\partial \lambda} = -\frac{1}{2\pi} \ln \left( \frac{\pi + 2}{\pi - 2} \right) \ln |\lambda - 1|,$$

$$\lim_{\lambda \to 1^+} \frac{\partial S}{\partial \lambda} = -\frac{\pi}{219/4} \ln \left( \frac{\pi + 2}{\pi - 2} \right) (\lambda - 1)^{-3/4}$$

The exponents differ as to how we approach the critical point. Nonetheless, the exponents are pure numbers independent of the coupling constant, as are the amplitudes. This then is a perfectly legitimate case of universality. The reason for the asymmetry at the critical point is clear: the magnetization (a local order parameter) vanishes for $\lambda \leq 1$, while it is non-zero for $\lambda > 1$. For the case of Anderson localization and the integer quantum Hall effect, there are no such local order parameters that vanish at the transition. If we regard the average density of states as an order parameter, it is smooth through both the Anderson transition and the plateau-to-plateau transition for the integer quantum Hall effect. Thus, the single site vNE has a scaling function that is symmetric around the transition as deduced from the multifractal scaling. The moral is that the single site entropy is an important and useful quantity to compute.
1.8. Epilogue

Entropy measures uncertainty in a physical system. It is therefore not surprising that it is a central concept in quantum information theory. That it may turn be an essential concept at a quantum critical point can also be anticipated. At a critical point a system cannot decide in which phase it should be. At the Anderson transition the wave function is a highly complex multifractal, and it is not surprising that vNE exhibits non-analyticity in the infinite volume limit, even though the ground state energy in which the complexity of the wave function is averaged over is smooth through it. The non-analyticity of vNE is perfectly consistent with other measures of entanglement, for example the linear entropy, which is

\[ S_L = 1 - \text{Tr} \rho_L^2 \]

The inverse participation ratio

\[ P^{(2)} = \frac{1}{N} \sum_{r,E} |\psi_r(E)|^4 = 1 - S_L/2, \]

where \( N \) is the total number of states in the band. In the extreme localized case, only one site participates and \( S_L = 0 \). In the opposite limit \( S_L = 2 - 2/N \to 2 \), when \( N \to \infty \). The participation ratio, hence \( S_L \), exhibits scaling at the metal-insulator transition:

\[ P^{(2)} = L^{-\frac{2}{\nu}} g_{\pm}(L^{1/\nu} w) \]

where \( g_{\pm} \) is another universal function. This scaling was also verified with \( x \approx 1.4, \nu \approx 1.35 \), and \( W_c \approx 16.5 \) for the 3D Anderson transition.

A key question now is what happens when we have both disorder and interaction. In recent years there has been much progress in one-dimensional systems, especially from the perspective of vNE in the ground state. The quantum criticalities of these disordered interacting systems belong to universality classes different from their counterparts with interaction but no disorder and are generally described by infinite disorder fixed points. Little is understood for similar higher dimensional systems. The principal difficulty in constructing a universal theory when both interactions and disorder are present is qualitatively clear. When interactions are strong and disorder is absent, the ground state can break many symmetries and organize itself into a variety of phases. Introducing disorder may affect the stability of these many body correlated phases in different ways. Although the symmetry of the order parameters can guide us, the strongly correlated nature of these phases makes theories difficult to control. As mentioned above, in a simple case of spinless fermion, we were able to provide some rigorous answers: no matter how strong the interaction is there appears to be gapless excited states and the broken symmetry is broken. In the opposite limit, we have to examine how weak interaction affects the Anderson problem. Here there has been progress in recent years; see the contribution by Finkelstein in the present volume.

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