ON NONEXISTENCE OF SEMI-ORTHOGONAL DECOMPOSITIONS IN ALGEBRAIC GEOMETRY

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Abstract. The nonexistence of semi-orthogonal decompositions in algebraic geometry is known to be governed by the base locus of the canonical bundle. We study another locus, namely the intersection of the base loci of line bundles that are isomorphic to the canonical bundle in the Néron-Severi group, and show that it also governs the nonexistence of semi-orthogonal decompositions. As an application by using algebraically moving techniques, we prove that the bounded derived category of the $i$-th symmetric product of a smooth projective curve $C$ has no nontrivial semi-orthogonal decompositions when the genus $g(C) \geq 2$ and $i \leq g(C) - 1$. We prove indecomposability of derived categories of some examples of elliptic surfaces with $P_g(X) = 0$, and some natural examples of minimal surfaces of general type. Finally, an inequality involving phases of skyscraper sheaves for any Bridgeland stability condition is obtained.

1. Introduction

The bounded derived category of coherent sheaves of a variety encodes rich information of geometry of the variety. Some linear homological invariants can be directly read from the category, for example, the Hochschild homology and cohomology. One way to study derived categories is to decompose in sense of semi-orthogonal decomposition. Naively, we would like to consider orthogonal decomposition, but for the connected varieties, their bounded derived categories don’t admit nontrivial orthogonal decompositions. However, there are many examples for the semi-orthogonal decompositions in algebraic geometry, for a survey of semi-orthogonal decompositions, see [Kuz15]. Semi-orthogonal pieces of the derived categories of coherent sheaves are also main examples of noncommutative motives, and some classical notions in algebraic geometry can be extended to this noncommutative setting. For example, many famous conjectures admit their noncommutative counterparts including Weil conjecture, Grothendieck conjecture of type $C$ and $D$ [Tab19], and even Hodge conjecture, see A. Perry [Per20] firstly for the cases of the admissible subcategories of the bounded derived categories of coherent sheaves, and then [Lin21] for generalization to noncommutative motives.

However, there are many examples of varieties whose derived categories are indecomposable. The indecomposability is closely related to birational geometry, and it is interesting to determine whether for a given variety, the bounded derived category admits no nontrivial semi-orthogonal decompositions. Given a smooth projective variety $X$, and blow up to $Y$, $D(Y)$ can be decomposed as $D(X)$ and several pieces from the centers [Orl93]. It is natural to think that the minimal variety is indecomposable, but it is not true. For example, the Enriques surface has
a nontrivial semi-orthogonal decomposition. However, if we assume that the canonical bundle is effective, there are many examples of minimal varieties whose derived categories are indecomposable, for example, the minimal surface of Kodaira dimension 1, and some examples of minimal surfaces of general type [KaOk18, Section 4]. We can make the following conjecture, which was stated for example in [BiGoLee, Conjecture 1.6] and [BasBelOkRi, Question E].

**Conjecture 1.1.** Let $X$ be a smooth projective variety. Assume the canonical bundle $K_X$ is nef and effective, then $D(X)$ has no non-trivial semi-orthogonal decompositions.

A long-term project in the realm of derived categories in algebraic geometry is to make the semi-orthogonal decompositions of derived categories compatible with the minimal model program. For example, the $DK$ hypothesis. A good survey for this direction is [Kaw17]. Therefore, we can also ask the converse question.

**Conjecture 1.2.** [KaOk18, Conjecture 1.1] Let $X$ be a smooth projective variety. If $D(X)$ has no nontrivial semi-orthogonal decompositions, then $X$ is minimal.

**Remark 1.3.** We should not expect to obtain $P_g(X) \neq 0$ since if $X$ is a bielliptic surface of Kodaira dimension 0, then $P_g(X) = 0$, but $D(X)$ admits no nontrivial semi-orthogonal decompositions [KaOk18 Proposition 4].

In this paper, we provide a general theory for the nonexistence of semi-orthogonal decompositions in Section 3, which should be regarded as a mild generalization of the nonexistence theory in paper [KaOk18].

**Theorem 1.4.** (= Corollary 3.5) Let $X$ be a smooth projective variety, define the new base points of the canonical bundle,

$$Z = \text{POs}\{\omega_X\} := \bigcap_{L \in \text{Pic}^0(X)} \text{Bs}\{\omega_X \otimes L\}.$$

If assume a semi-orthogonal decomposition, $D(X) = \langle A, B \rangle$, then one of the following is true.

1. For any $x \in X \setminus Z$, $k(x) \in A$. In this case, support of all objects in $B$ is contained in $Z$.
2. For any $x \in X \setminus Z$, $k(x) \in B$. In this case, support of all objects in $A$ is contained in $Z$.

**Remark 1.5.**

(1) In the paper [KaOk18], the base locus of $\omega_X$ in $\text{Pic}(X)$ governs the non-existence of nontrivial semi-orthogonal decompositions. $\text{Pic}(X)$ is too large to consider. The theorem tells us that intersection of the base locus of class $\omega_X$ in $\text{NS}(X) = \text{Pic}(X)/\text{Pic}^0(X) = H^{1,1}(X, Z)$ is relevant to indecomposability of derived categories, though of course not easy to compute in general. It provides a bridge from the numerical geometry up to torsion to indecomposability of derived categories in algebraic geometry, namely the numerical equivalence classes of line bundles $\text{Num}(X)$ is the torsion-free part of $\text{NS}(X)$.
(2) We point out that it is impossible to generalize to $\text{Num}(X)$ in a naive way. For example, consider the Enriques surface $X$, then $\omega_X$ is numerically trivial, but $D(X)$ admits a nontrivial semi-orthogonal decomposition.

**Corollary 1.6.** (= Theorem 3.8) Let $X$ be a smooth projective variety. Assume $Z = \text{PBs}|\omega_X|$ is empty or $\dim Z = 0$, then $D(X)$ has no nontrivial semi-orthogonal decompositions.

Using this theory, in Section 4, we study elliptic surfaces of Kodaira dimension 1 such that $P_g(X) = 0$. When the canonical bundle has nontrivial sections, it is proved in [KaOk18, Section 4] that $D(X)$ has no nontrivial semi-orthogonal decompositions. The base locus of the canonical bundle is the whole variety in our case. However, since we consider intersection base locus of line bundles that is the class of canonical bundle instead of base locus of the canonical bundle, we can deal with the case $P_g(X) = 0$.

**Theorem 1.7.** (= Theorem 4.14) Let $\pi : X \to C$ be elliptic fibration, $X$ is minimal of Kodaira dimension 1. Assume $P_g(X) = 0$, $L := R\pi^*\mathcal{O}(X)^{-1} \neq \mathcal{O}_C$, $g(C) = 1$, $q(X) = 1$, and $d = \deg L = 0$, then $D(X)$ has no nontrivial semi-orthogonal decompositions.

We study some natural examples of minimal surfaces of general type.

**Theorem 1.8.** (= Theorem 4.16) Let $X$ be a minimal surface of general type such that $P_g(X) = q(X) = 1$, $K^2 = 2$ or $K^2 = 3$, then $D(X)$ has no nontrivial semi-orthogonal decompositions.

Using the algebraically moving techniques, we also prove the following theorems.

**Theorem 1.9.** (= Theorem 4.5) Let $X = S^i C$, the $i$-th symmetric product of a smooth projective curve $C$. Then $D(S^i C)$ has no nontrivial semi-orthogonal decompositions when $g(C) \geq 2$, and $i \leq g(C) - 1$.

**Remark 1.10.**

(1) Some partial results of this theorem are obtained by several people, see [BasBelOkRi, Corollary D] and [BiGoLee], but no complete answers. Note that $S^i C$ in this theorem satisfies the assumption in Conjecture 1.1, see [BiGoLee].

(2) Note that $D(S^i C)$ considered in this theorem are conjecturally the building blocks of the derived category of moduli spaces of rank two bundles with fixed determinant on $C$ (this was conjectured by Narasimhan and independently in [BelGalMu]). The theorem tells us that these blocks are actually indecomposable.

(3) If $i \geq g$, the proof does not work. The essential reason is that $g - 1 - i < 0$. We actually prove that $\text{PBs}|\omega_{S^i C}| = \emptyset$, this would imply that $D(S^i C)$ has no nontrivial semi-orthogonal decompositions by general theory in the paper. For details, see the proof of Theorem 4.3.

(4) There is a semi-orthogonal decomposition for $S^i C$ if $i = g, g + 1, \ldots, 2g - 2$, see [Tod19, Corollary 5.12], or independently [BK, Theorem D], and [JL19, corollary 3.8].

$$D(S^i C) = \langle J(C), \cdots, J(C), D(S^{2g-2-i} C) \rangle.$$
There are \( i - g + 1 \) pieces of \( J(C) \). As for \( i \geq 2g - 1 \), \( S^iC \) is a projective bundle of \( J(C) \). We complete the picture of indecomposability of bounded derived categories of symmetric product of curves.

**Theorem 1.11.** (=Theorem 4.8) Let \( C \) be curves of genus \( g \geq 2 \), and \( A_k \) be an abelian variety. Let \( X = \prod_i S^{j_i}C_i \times \prod_k A_k \), here \( j_i \leq g_i - 1 \). Then, \( D(X) \) has no nontrivial semi-orthogonal decompositions.

**Remark 1.12.** The type of the varieties \( X \) in the corollary are the conjecturally building blocks of motivic decomposition of moduli space of rank \( r \) bundles with fixed determinant [GoLee, Conjecture 1.8], and then the derived categories in the corollary are conjecturally building blocks of the derived category of rank \( r \) vector bundles with fixed determinant.

Recently D. Pirozhkov proposed a notion called stably semi-orthogonal indecomposable varieties, it is a more general notion than the indecomposibility, and stably semi-orthogonal indecomposability implies indecomposability [Pir21].

**Theorem 1.13.** (= Theorem 4.10) Let \( C \) be a smooth projective curve, \( g(C) \geq 2 \) and \( i \leq g(C) - 1 \), then \( X = S^iC \) is stably semi-orthogonal indecomposable if and only if \( \text{Bs} |\omega | = \emptyset \).

**Remark 1.14.** For \( X = S^iC \), \( g(C) \geq 2 \), \( i \leq g(C) - 1 \), two notions stably semi-orthogonal indecomposability and indecomposability can be distinguished by \( \text{PBs} |\omega_X | \) and \( \text{Bs} |\omega_X | \). It is interesting to find more examples.

In Section 5, we give an interesting application to the Bridgeland stability conditions. It is closely related to a notion called geometric stability conditions, see for example [FuLiZhao, Definition 2.8].

**Theorem 1.15.** (= Theorem 5.1) Let \( Z = \text{PBs} |\omega_X | \). Let \( \sigma \) be a Bridgeland stability condition of \( D(X) \). Take any \( k(x) \), \( x \in X \setminus Z \). Assume \( k(x) \) is not semi-stable, then the phase number of HN factors of \( k(x) : \phi_1 > \phi_2 > \cdots > \phi_n \) satisfies

\[
\phi_i - \phi_{i+1} \leq \dim X - 1.
\]

Lastly, observe that if the \( \text{Pic}^0(X) \) is larger, it is more possible to get that \( D(X) \) admits no nontrivial semi-orthogonal decompositions. We propose a refined version of Conjecture 1.1 that will probably be easier to obtain.

**Conjecture 1.16.** Let \( X \) be a smooth projective variety. Assume \( K_X \) is nef and effective, and \( q(X) = h^1(\mathcal{O}_X) \geq 1 \) (this implies \( \dim \text{Pic}^0(X) \geq 1 \)). Then \( D(X) \) admits no nontrivial semi-orthogonal decompositions.

**Remark 1.17.**

1. \( S^iC \) satisfies the assumptions in the conjecture when \( i \leq g(C) - 1 \) and \( g(C) \geq 2 \).
(2) If \( q(X) = 0 \), then
\[
\text{Bs}|\omega_X| = \text{PBs}|\omega_X|.
\]

The criteria of Corollary 1.6 is the same as the one of Corollary 1.5 in [KaOk18], but there are other techniques to deal with these cases. It is to consider the semi-orthogonal decompositions varied in a family [BasBelOkRi]. For example, the Horikawa surfaces and the Ciliberto double planes are all of irregularity 0, and in some cases, the base loci of the canonical bundles will be curves, but their bounded derived categories don’t admit nontrivial semi-orthogonal decompositions by the well-behaviour of semi-orthogonal decompositions in families [BasBelOkRi, Section 4].

Notation. \( X \) is always assumed to be smooth projective variety over \( \mathbb{C} \). We write \( D(X) \) as the bounded derived category of coherent sheaves of \( X \), and \( D_{\text{qch}}(X) \) as the derived category of \( \mathcal{O}_X \) modules with quasi-coherent cohomology. We write \( P_g(X) = h^0(X, \omega_X) \), \( q(X) = h^1(X, \mathcal{O}_X) \), and \( K^2 = c_1(X)^2 \).

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2. Semi-orthogonal decomposition

2.1. Semi-orthogonal Decomposition.

Definition 2.1. Let \( \mathcal{T} \) be a \( k \)-linear triangulated category. We say a collection of sub-triangulated categories \( \{ A_1, A_2, \cdots, A_n \} \) of \( \mathcal{T} \) is a semi-orthogonal decomposition of \( \mathcal{T} \) if the following holds.

1. For \( i > j \), we have \( \text{Hom}(A_i, A_j) = 0 \).
2. Given any object \( E \in \mathcal{T} \), there is a sequence of morphisms
\[
E_{n+1} = 0 \rightarrow E_n \rightarrow E_{n-1} \rightarrow E_{n-2} \rightarrow \cdots \rightarrow E_2 \rightarrow E_1 = E
\]
such that the cone of \( E_i \rightarrow E_{i-1} \) belongs to \( A_{i-1} \).

We write the semi-orthogonal decomposition as \( \mathcal{T} = \langle A_1, A_2, \cdots, A_n \rangle \).

Remark 2.2. We often call this version of semi-orthogonal decomposition the (weak) semi-orthogonal decomposition. We say strong semi-orthogonal decomposition if \( A_i \) is an admissible subcategory, that is, the embedding into \( \mathcal{T} \) has both left and right adjoint.

Lemma 2.3. Let \( \mathcal{T} = \langle A, B \rangle \) be a semi-orthogonal decomposition. Let \( E \in \mathcal{T} \), consider a triangle from the semi-orthogonal decomposition
\[
b \rightarrow E \rightarrow a \rightarrow b[1].
\]
Suppose there is another triangle
\[ b' \rightarrow E \rightarrow a' \rightarrow b'[1] \]
such that \( b' \in B, a' \in A \), then there exist a unique isomorphism \( g \) from \( b \) to \( b' \), and also a unique isomorphism \( f \) from \( a \) to \( a' \) such that the following diagram is commutative.

\[
\begin{array}{ccc}
b & \xrightarrow{i} & E \\
\downarrow g & & \downarrow L \\
b' & \xrightarrow{i'} & E
\end{array}
\begin{array}{ccc}
a & \xrightarrow{f} & b'[1] \\
\downarrow g[1] & & \downarrow L' \\
a' & \xrightarrow{f'} & b'[1]
\end{array}
\]

Proof. Apply \( \text{Hom}(b, \bullet) \) to the second triangle, we get isomorphism \( \text{Hom}(b, b') \cong \text{Hom}(b, E) \). Take element \( \text{id} \circ i \in \text{Hom}(b, E) \). So we get a unique element \( g \) with diagram commutative. Similar to the morphism \( f \). To show that \( g \) and \( f \) are isomorphism, we apply the inverse direction to obtain morphism \( g' \) and \( f' \). Since eventually, the identity morphism is the unique element which transforms the triangle to the same triangle, \( g' \) and \( f' \) are the inverses of \( g \) and \( f \) respectively.

Lemma 2.4. [KaOk18, Theorem 3.9] Let \( X \) be a projective scheme. Assume \( \text{Perf}(X) = \langle A, B \rangle \), then for any \( L \in \text{Pic}^0(X) \), we have \( A \otimes L = A \).

Remark 2.5. This would also imply \( B \otimes L = B \) for any \( L \in \text{Pic}^0(X) \). The proof is easy: consider any object \( E \otimes L \in B \otimes L \), where \( E \in B \). Consider the triangle with respect to the semi-orthogonal decomposition
\[ b \rightarrow E \otimes L \rightarrow a \rightarrow b[1]. \]
Tensor by line bundle \( L^{-1} \), clearly \( a \otimes L^{-1} \in A \). We get a new triangle
\[ b \otimes L^{-1} \rightarrow E \rightarrow a \otimes L^{-1} \rightarrow b \otimes L^{-1}[1] \]
Since \( \text{Hom}^*(E, a \otimes L^{-1}) = 0 \), therefore \( b \otimes L^{-1} \cong E \oplus a \otimes L^{-1}[-1] \). Since \( \text{Hom}^*(b \otimes L^{-1}, a \otimes L^{-1}) = 0 \), therefore we get \( a \cong 0 \). Thus, \( E \otimes L \cong b \in B \).

Lemma 2.6. Let \( X \) be a smooth projective variety. Assume \( D(X) = \langle A, B \rangle \). Take any skyscraper sheaf \( k(x) \), consider the triangle \( b \rightarrow k(x) \rightarrow a \rightarrow b[1] \) from the semi-orthogonal decomposition. Then for any line bundle \( L \in \text{Pic}^0(X) \), we have \( b \otimes L \cong b, a \otimes L \cong a \).

Proof. Tensor with \( L \), we get another triangle, \( b \otimes L \rightarrow k(x) \rightarrow a \otimes L \rightarrow b \otimes L[1] \). By Lemma 2.4 \( b \otimes L \in B \), and \( a \otimes L \in A \). Hence, by uniqueness of the triangle from the semi-orthogonal decomposition, see Lemma 2.3 \( b \otimes L \cong b, a \otimes L \cong a \) \( \square \)

3. General theory for the nonexistence of semi-orthogonal decompositions

3.1. First generalization.
Proposition 3.1. ([KaOk18, Thm 3.1]) Let $X$ be a smooth projective variety, $Z = Bs|\omega_X|$. If assume a semi-orthogonal decomposition, $D(X) = \langle A, B \rangle$, then one of the following is true.

1. For any $x \in X \setminus Z$, $k(x) \in A$. In this case, support of all objects in $B$ is contained in $Z$.
2. For any $x \in X \setminus Z$, $k(x) \in B$. In this case, support of all objects in $A$ is contained in $Z$.

We give a generalization of the Proposition 3.1 for objects which are not skyscraper sheaves.

Theorem 3.2. Let $X$ be a smooth projective variety, and a semi-orthogonal decomposition, $D(X) = \langle A, B \rangle$. Let $E$ be an object supported in a closed subset $Z$ such that $\text{Hom}(E, E) = k$. Suppose $\exists$ a section $s \in H^0(X, \omega_X)$ such that $Z \cap Z(s) = \emptyset$. Then $E \in A$ or $B$. Note that $k(x)$ with $x \in X \setminus Bs|\omega_X|$ satisfies the assumption.

Proof. Consider a triangle $b \to E \to a \to b[1]$ corresponding to the semi-orthogonal decomposition. By assumption, we can choose a section $s$ such that $Z(s) \cap Z = \emptyset$. Write $U = X \setminus Z(s)$, $j : U \hookrightarrow X$. Then compose the section $s$ with morphism $a \to b[1]$, we get a morphism $a \to b \otimes \omega_X[1]$. But according to the semi-orthogonality, $\text{Hom}_X(a, b \otimes \omega_X[1]) \cong \text{Hom}_X(b, a[\dim X - 1]) = 0$. Hence the morphism $a \to b[1]$ pulling back to $U$ becomes zero because $s|_U$ is invertible. Now pulling back the triangle to $D^b(U)$, we have $j^*b \to j^*E \to j^*a \to j^*b[1]$, hence $j^*E \cong j^*a \oplus j^*b$. By adjunctions, we have $\text{Hom}_U(j^*E, j^*E) \cong \text{Hom}_X(E, Rj_*j^*E)$. Since $Z \cap Z(s) = \emptyset$, the support of $E$ is contained in $U$. There is a triangle in $D_{qch}(X)$,

$$R\Gamma_{Z(s)}E \to E \to Rj_*j^*E \to R\Gamma_{Z(s)}[1].$$

Since $R\Gamma_{Z(s)}E = 0$, hence $E \cong Rj_*j^*E$. Therefore,

$$\text{Hom}_U(j^*E, j^*E) \cong \text{Hom}_X(E, E) \cong k.$$ 

Back to equality $j^*E \cong j^*a \oplus j^*b$, the equation above implies that $j^*a \cong 0$ or $j^*b \cong 0$. Suppose $j^*a \cong 0$, it implies that the support of $a$ is in $Z(s)$. Clearly $R\text{Hom}(E, a) \cong 0$. Applying $R\text{Hom}$ and taking homology, we have $\text{Hom}_X(E, a) \cong 0$, which implies $b \cong a[-1] \oplus E$. Therefore $a \cong 0$, and $b \cong E$. Suppose $j^*b \cong 0$, then $E \cong a$ follows from the same reason. $\square$

3.2. Generalization to the intersection of base loci of line bundles that are isomorphic to the canonical bundle in Néron-severi group.

Theorem 3.3. Let $X$ be a smooth projective variety, $Z = Bs|\omega_X \otimes L|$, $L \in \text{Pic}^0(X)$. If assume a semi-orthogonal decomposition, $D(X) = \langle A, B \rangle$, then one of the following is true.

1. For any $x \in X \setminus Z$, $k(x) \in A$. In this case, support of all objects in $B$ is contained in $Z$.
2. For any $x \in X \setminus Z$, $k(x) \in B$. In this case, support of all objects in $A$ is contained in $Z$.

Proof. Take $x \in X \setminus Z$, we prove that $k(x) \in A$ or $B$. Take a section $s \in \Gamma(X, \omega_X \otimes L)$ such that $s(x) \neq 0$. Consider the triangle $b \to k(x) \to a \to b[1]$ from the semi-orthogonal decomposition. Consider the composition of $a \to b[1]$ and $b[1] \to b[1] \otimes \omega_X \otimes L$ via section $s$. According to Lemma 2.6 $b \otimes L \cong b$, hence the composition morphism $a \to b[1] \otimes \omega_X \otimes L$ is zero by Serre
duality and semi-orthogonality. Define $U = X \setminus Z(s)$, write $j : U \to X$. Then pulling pack to $j$, there is a new triangle

$$j^*b \to k(x) \to j^*a \to j^*b[1].$$

But since section $s$ is invertible in $U$, the morphism $j^*a \to j^*b[1]$ is trivial. Therefore, $k(x) \cong j^*a \oplus j^*b$. Since $k(x)$ is simple, $j^*a = 0$, or $j^*b = 0$. Without loss of generality, we assume $j^*a \cong 0$, which implies that support of $a$ is in $Z(s)$. However, since $x \in U$, the morphism $k(x)[m] \to a[m]$ must be zero. Then, $b \cong a[-1] \oplus k(x)$. Thus $a = 0$.

The support statement follows easily from the statement of the skyscraper sheaves since the skyscraper sheaves can only belong to exactly one component, see proof of Theorem 1.2 [KaOk18, Theorem 1.2].

□

Remark 3.4.

(1) In fact, we don’t need the fact $B \otimes L \subset B$ to prove our theorem. Since $a \otimes L^{-1} \in A$, we have

$$\text{Hom}^*(a, b \otimes \omega_X \otimes L) \cong \text{Hom}^*(a \otimes L^{-1}, b \otimes \omega_X) = 0.$$

We thank Shizhuo Zhang for pointing out this simpler proof to the author.

(2) One may wonder to generalize Theorem 3.2. We need to assume that $E$ is preserved by the $\text{Pic}^0(X)$.

Corollary 3.5. Let $X$ be a smooth projective variety, define the base points of the paracanonical systems,

$$Z = \text{PBs}|\omega_X| := \cap_{L \in \text{Pic}^0(X)} \text{Bs}|\omega_X \otimes L|.$$

If assume a semi-orthogonal decomposition, $D(X) = \langle A, B \rangle$, then one of the following is true.

(1) For any $x \in X \setminus Z$, $k(x) \in A$. In this case, support of all objects in $B$ is contained in $Z$.

(2) For any $x \in X \setminus Z$, $k(x) \in B$. In this case, support of all objects in $A$ is contained in $Z$.

Proof. Let $x \notin Z$, then there is a $L \in \text{Pic}^0(X)$ such that $x \notin \text{Bs}|\omega_X \otimes L|$, according to Theorem 3.3 $k(x)$ belongs to $A$ or $B$, the statements of the Corollary follows. □

Remark 3.6.

(1) The base locus of algebraic class $[\omega_X]$ in Néron-Severi group $\text{Pic}(X)/\text{Pic}^0(X)$ would be counted to the problem of semi-orthogonal decomposition. Thus, if we have good knowledge of Néron-Severi group and the class $[\omega_X]$, it is more possible to determine whether the derived categories have nontrivial semi-orthogonal decomposition.

(2) We point out that it is impossible to generalize to $\text{Num}(X)$ in a naive way. For example, consider the Enriques surface $X$. $\omega_X$ is numerically trivial, but $D(X)$ admits a nontrivial semi-orthogonal decomposition. However, it is possible if we first have the intersection base locus of line bundles of canonical class in $\text{NS}(X)$ is a proper closed subset.
Example 3.7. Let $A$ be an abelian variety. Then $\Pic^0(A)$ is exactly the translation invariant line bundle in $A$. Let $L \in \Pic^0(A)$ which is not trivial, we have $H^0(A, L) = 0$. Hence $B_s|\omega_A| = \emptyset$, but $B_s|L| = A$. The base locus in an algebraic class can change.

Theorem 3.8. Let $X$ be a smooth projective variety, $Z = \PBs|\omega_X|$. Suppose $Z = \emptyset$ or $\dim Z = 0$, then $D(X)$ have no nontrivial semi-orthogonal decompositions.

Proof. Let $D(X) = \langle A, B \rangle$, then according to Corollary 3.5, we can assume that all objects of $A$ support in $Z$. If $Z = \emptyset$, then $A = 0$.

If $\dim Z = 0$, then, $A \otimes \omega_X = A$. By semi-orthogonality and Serre duality, the original semi-orthogonal decomposition is an orthogonal decomposition, hence must be trivial since $X$ is connected.

Theorem 3.9. Let $X_i$ be a finite collection of the varieties with the property that $\PBs|\omega_{X_i}| = \emptyset$. Let $Y = X_1 \times X_2 \times \cdots \times X_n$. Then $D(Y)$ have no nontrivial semi-orthogonal decompositions.

Proof. We prove that $\PBs|\omega_Y| = \emptyset$. Take any closed point $z = (z_1, \cdots, z_n) \in Y$. Since $\PBs|\omega_{X_i}| = \emptyset$, there exist effective divisors $D_i$ such that $z_i \notin D_i$, and $D_i$ is algebraic equivalent to $K_{X_i}$. As an algebraic class, $K_Y = \sum X_1 \times X_2 \times \cdots \times D_i \times \cdots \times X_n$, we are done.

3.3. Induction process. Inspired by proof of Theorem 3.3, we provide an induction process to reduce the base points $\PBs|\omega_X|$. Let $D(X) = \langle A, B \rangle$ be a semi-orthogonal decomposition, $Z = \PBs|\omega_X|$. Let $M$ be a line bundle locally trivial around $Z$, then any skyscraper sheaf $k(x)$ with $x \notin Z' = B_s|\omega_X \otimes M|$ must belong to $A$ or $B$.

Proof. Observe that objects of $A$ or $B$ are supported in $Z$. Without loss of generality, assume objects of $A$ are supported in $Z$. Since $M$ is locally trivial around $Z$, take any object $E \in A$, then $E \otimes M \cong E$. The trick is as follows: we only need to consider the point $x \in Z \setminus Z'$.

Let $b \to k(x) \to a \to b[1]$ be the triangle from the semi-orthogonal decomposition. Tensor by line bundle $M$, there is a new triangle

$$b \otimes M \to k(x) \otimes M \to a \otimes M \to b[1].$$

By assumption, there is an open neighborhood $Z \subset U$ such that $M$ is trivial on $U$. Hence $a \otimes M \cong a \otimes M|_U \cong a$, as support of $a$ is in $Z$.

Since $A \otimes M = A$, using the similar argument in proof of Remark 2.5, we get $B \otimes M = B$. Using the same techniques as in proof of Theorem 3.3, we prove that $k(x)$ belong to $A$ or $B$.

In fact, the trick in Remark 3.3 (1) can also apply directly.

Remark 3.11. Actually, we just need to assume that $M$ restricts to $nZ_{red}$ is trivial for any integer $n$. The proof is the same, we can use that if $E$ is supported in $Z$, then there is an $E_n \in D(nZ_{red})$ and $i_n : nZ_{red} \hookrightarrow X$ such that $i_n \ast E_n = E$. Thus, $E \otimes M \cong E$ by projection formula.
Example 3.12. Let $f : X \rightarrow Y$ be a morphism that contracts $Z$ to a point $y$, where $Z = \text{PBs}|\omega_X|$. Since any line bundle $N$ is trivial around the point $y$, hence $M := f^*N$ is a line bundle that is trivial around $Z$.

If we can choose a line bundle $M$ locally trivial around $Z$ such that $Z_1 := Z \setminus \text{Bs}|\omega_X \otimes M|$ becomes smaller, then apply the same process for $Z_1$. We get an induction process to prove that $D(X)$ have no nontrivial semi-orthogonal decompositions. It is interesting to find such an example to work.

3.4. The singular varieties. Let $X$ to be a variety with Cohen-Macaulay singularities. The Proposition 3.1 was generalized to such $X$ by Dylan Spence [Sp21]. The theorems in this section can be generalized to the cases of varieties with Cohen-Macauly singularities in paper [Sp21]. For completeness, we state the theorem below.

Theorem 3.13. Let $X$ be a Cohen-Macaulay projective variety with dualizing sheaf $\omega_X$. Assume a semi-orthogonal decomposition $\text{Perf}(X) = \langle A, B \rangle$. $Z = \text{PBs}|\omega_X|$. Then one of the following is true.

1. For any $x \in X \setminus Z$, $O_{Z_x} \in A$. In this case, support of all objects in $B$ is contained in $Z$.
2. For any $x \in X \setminus Z$, $O_{Z_x} \in B$. In this case, support of all objects in $A$ is contained in $Z$. $O_{Z_x}$ is the sheaf of Koszul zero cycle, and the base locus of sheaves is also defined in [Sp21].

Proof. Since the Koszul zero cycle $O_{Z_x}$ is a perfect complex that is supported in closed point $x$, therefore $O_{Z_x} \otimes L \cong O_{Z_x}$ for any $L \in \text{Pic}^0(X)$. The remaining proof is the same as the proof of [Sp21, Theorem 3.1]. □

4. Examples

4.1. Derived category of symmetric product of curves.

In this subsection, we prove that the derived category of $i$-th symmetric product of a curve has no nontrivial semi-orthogonal decompositions when genus $g$ of the curve is greater or equal to 2, and $i \leq g - 1$. Furthermore, as a by product, we prove that $D(\prod_i S^{j_i}C_i)$ has no non-trivial semi-orthogonal decompositions, where $j_i \leq g(C_i) - 1$.

To warm-up, we deal with a well-known example, to see how generalization in Section 3 works.

Example 4.1. Let $X = C_1 \times \cdots \times C_n$, suppose the genus of $C_i$ is positive, then there are no nontrivial semi-orthogonal decompositions for $D(X)$.

Clearly $K_X \cong K_{C_1} \boxtimes \cdots \boxtimes K_{C_n}$. As a class in Néron-Severi group,

$$K_X = (2g_1 - 2)p_1 \times C_2 \times \cdots \times C_n + \cdots + (2g_n - 2)C_1 \times \cdots \times C_{n-1} \times p_n,$$

where $p_i$ are choosing fixed points. Thus, consider any closed points $x = (x_1, \ldots , x_n) \in C_1 \times \cdots \times C_n$, we can choose generic $p_i$ to make sure that $x \notin \cup_i C_1 \times \cdots \times p_i \times \cdots \times C_n$. Hence, $\text{PBs}|\omega_X| = \emptyset$. By Corollary 3.5, $D(X)$ have no nontrivial semi-orthogonal decomposition.
Remark 4.2.

(1) Clearly $Bs|\omega_X| = \emptyset$. So the result that $D(X)$ admits no non-trivial semi-orthogonal decompositions is already known to the experts. The new idea here is that every points in a curve are algebraically equivalent, so we can move the points.

(2) If one of the curves is $\mathbb{P}^1$, it is well known that we can decompose $D(X)$ via decomposing $D(\mathbb{P}^1)$.

Theorem 4.3. Let $G$ be an algebraic group, and $H$ be a hypersurface of $G$. Take any closed point $p \in H$, there always exists a generic $g \in G$ such that $p \notin gH$.

Proof. Without loss of generality, we assume $p = e$, the identity element of $G$. Define $Z = H^{-1}$, it is a proper closed subset of $G$. Take $g \notin Z$, then $e \notin gH$. □

Corollary 4.4. Let $D$ be a hypersurface of abelian variety $A$. Fix any point $p \in D$, there exists generic $a \in A$ such that $p \notin a + D$.

Theorem 4.5. Let $C$ be a smooth projective curve of genus $g(C) \geq 2$. $S^iC$ is the $i$ symmetric product of $C$. Then $\text{PBs}|\omega_{S^iC}| = \emptyset$ when $i \leq g(C) - 1$. In particular, $D(S^iC)$ has no nontrivial semi-orthogonal decompositions in these cases.

Proof. We mainly use the concrete description of class $\omega_{S^iC}$ in Néron-Severi group of $S^iC$, namely $\omega_{S^iC} \cong^{alg} (g - i - 1)x_p + \theta$, see [BiGoLeC Lemma 2.1] or [MacL2 (14.5), (14.9)]. Here, $x_p$ is the image of

$$j : S^{i-1}C \hookrightarrow S^iC \quad D \mapsto D + p_0,$$

$p_0$ is a fixed point in $C$. $\theta$ is the pulled pack line bundle $u^*O(\Theta)$.

$$u : S^iC \longrightarrow J(C) \quad D \mapsto O(D - dp_0).$$

$\Theta$ is the theta divisor of $J(C)$. Though we don’t have a canonical choice of $\Theta$, we will fix a choice, for example $\Theta = u(S^{g-1}C) + b$ with a constant $b \in J(C)$ to avoid the confusion of pulling back divisors, but the pulling back of line bundle by $u$ always make sense.

Clearly, when $i \leq g - 1$, we have $g - i - 1 \geq 0$. Now take any closed point $z = z_1 + z_2 + \cdots + z_i \in S^iC$. We can choose a point $p \in C$ such that $p \neq z_i$. Again we have isomorphism $\omega_{S^iC} \cong^{alg} (g - i - 1)x_p + \theta$ since $x_p \cong^{alg} x_p$. Clearly, $z \notin x_p$ by our choice of point $p$. This means that we can move $x_{p_0}$ algebraically to $x_p$ which avoids $z$. The next step is to move $\Theta$ algebraically.

If $u(z) \notin \Theta$, the canonical section $S_{(g - i - 1)x_p + \theta} \in \Gamma(S^iC, O((g - i - 1)x_p) \otimes u^*O(\Theta))$ does not vanish at $z$. Hence, $z \notin \text{PBs}|\omega_{S^iC}|$.

If $u(z) \in \Theta$, we can move $\Theta$ algebraically to some $t^*_a \Theta$ by translation $t_a$ of $J(C)$. In fact, if $A$ is an abelian variety, then $\text{Pic}^0(A)$ is exactly the translation invariant line bundles, and $t^*_a \Theta - \Theta$ is translation invariant, see for example [Huy Section 9]. In particular, $u(z) \notin t^*_a \Theta$ via generic choice of $a \in J(C)$ by Corollary 4.4. Therefore,

$$\omega_{S^iC} =^{alg} (g - i - 1)x_p + u^*t^*_a \Theta,$$
and the canonical section of \((g - i - 1)x_p + u^*t_s^*\Theta\) does not vanish at \(z\). That is, \(z \notin \mathcal{PB}_\omega S_{iC}\).

To sum up, \(\mathcal{PB}_\omega S_{iC} = \emptyset\). According to Theorem 3.8, \(D(S^iC)\) has no nontrivial semi-orthogonal decompositions. \(\square\)

**Remark 4.6.** When \(i \geq g\), the proof does not work. Note that there is a semi-orthogonal decomposition for \(S^iC\) if \(i = g, g + 1, \ldots, 2g - 2\), see \cite{Tod19} Corollary 5.12] or independently \cite{BK}, Theorem D, and \cite{JL19} corollary 3.8.

\[
D(S^iC) = \langle J(C), \cdots, J(C), D(S^{2g-2-i}C) \rangle.
\]

There are \(i - g + 1\) pieces of \(J(C)\). As for \(i \geq 2g - 1\), \(S^iC\) is a projective bundle of \(J(C)\).

**Corollary 4.7.** Let \(C_i\) be curves of genus \(g_i \geq 2\). Let \(X = \prod_i S^{j_i}C_i\), here \(j_i \leq g_i - 1\). Then \(D(X)\) has no nontrivial semi-orthogonal decompositions.

**Proof.** Combining Theorem 4.3 and Theorem 3.9. \(\square\)

**Corollary 4.8.** Let \(C_i\) be curves of genus \(g_i \geq 2\). Let \(X = \prod_i S^{j_i}C_i\), here \(j_i \leq g_i - 1\). \(Y = X \times \prod A_j\), where \(A_j\) is an abelian variety. Then \(D(Y)\) has no nontrivial semi-orthogonal decompositions.

**Proof.** This is clear since \(\mathcal{BS}_\omega A_j = \emptyset\). \(\square\)

**Remark 4.9.** The reason we state the obvious corollary here is that the varieties in the corollary are the conjecturally building blocks of motivic decomposition of moduli space of rank \(r\) bundles with fixed determinant \cite{GoLee} Conjecture 1.8], and then the derived categories in the corollary are conjecturally the building blocks of derived category of rank \(r\) vector bundles with fixed determinant.

There is a notion called stably semi-orthogonal indecomposable, which is more general than the notion of indecomposability, and stably semi-orthogonal indecomposability implies indecomposability \cite{Pir21}. It is natural to ask whether \(S^iC\) is stably semi-orthogonal indecomposable when \(i \leq g(C) - 1\) and \(g(C) \geq 2\).

**Theorem 4.10.** Let \(C\) be a smooth projective curve, \(i \leq g(C) - 1\) and \(g(C) \geq 2\), then \(S^iC\) is stably semi-orthogonal indecomposable if and only if \(\mathcal{BS}_\omega K_{S^iC} = \emptyset\).

**Proof.** According to \cite{BiGoLee} Proposition 3.4], the Albanese map \(u : S^iC \to J(C)\) is finite if and only if \(\mathcal{BS}_\omega K_{S^iC} = \emptyset\). If \(u\) is finite, then according to \cite{Pir21} Theorem 1.4], \(S^iC\) is stably semi-orthogonal indecomposable. If \(u\) is not finite, then \(S^iC\) will contain a fiber which is a projective space \(\mathbb{P}^m\), \(m \geq 1\). Since \(D(\mathbb{P}^m)\) is decomposable, hence according to \cite{Pir21} Corollary 2.5], \(S^iC\) will not be stably semi-orthogonal indecomposable. \(\square\)
4.2. Derived categories of elliptic surfaces of Kodaira dimension 1. Let $X$ be a minimal projective smooth surface with Kodaira dimension 1. It was proved that if $P_g(X) \geq 1$, $D(X)$ have no nontrivial semi-orthogonal decompositions [KaOk18, Theorem 4.2]. We give a further description of $D(X)$ for cases $P_g(X) = 0$ in the sense whether it has nontrivial semi-orthogonal decompositions, which can not be achieved by just considering the base locus of the canonical bundle. Note that $X$ admits a fibration $\pi : X \to C$ with general fiber elliptic curve. $C$ is a smooth projective curve.

Proposition 4.11. [Fr98, Chap 7, Lemma 13, Lemma 14] Let $\pi : X \to C$ be an elliptic fibration, then $L := R\pi^1\mathcal{O}(X)^{-1}$ is a line bundle on $C$. In particular, $d = \deg L \geq 0$, and $\mathcal{X}(\mathcal{O}_X) = \deg(L)$.

1. If $L \neq \mathcal{O}_C$, then $q(X) := h^1(X, \mathcal{O}_X) = g(C)$ and $P_g(X) = d + g(C) - 1$.
2. If $L = \mathcal{O}_C$, then $q(X) = g(C) + 1$ and $P_g(X) = g(C)$.

Remark 4.12. Assume $P_g(X) = 0$. There are three possibilities.

1. $L := R\pi^1\mathcal{O}(X)^{-1} = \mathcal{O}_C$, $g(C) = 0$, and $q(X) = 1$.
2. $L \neq \mathcal{O}_C$, $g(C) = 1$, $q(X) = 1$, and $d = 0$.
3. $L \neq \mathcal{O}_C$, $g(C) = 0$, $q(X) = 0$, and $d = 1$. In this case, $\mathcal{O}_X$ is an exceptional object.

Theorem 4.13. [Fr98, Chap 7, Theorem 15] Let $X$ be minimal, and $\pi : X \to C$ is an elliptic fibration. Let $F_i$ be the multiple fibers of $\pi$ with multiplicity $m_i \geq 2$. We have a formula for the canonical bundle of $X$,

$$\omega_X = \pi^*(K_C \otimes L) \otimes \mathcal{O}_X(\sum_i (m_i - 1)F_i).$$

The following theorem is the case of (2) in Remark 4.12.

Theorem 4.14. Let $\pi : X \to C$ be elliptic fibration, $X$ is minimal of Kodaira dimension 1. Assume $P_g(X) = 0$, $L \neq \mathcal{O}_C$, $g(C) = 1$, $q(X) = 1$, and $d = 0$, then $D(X)$ has no nontrivial semi-orthogonal decompositions.

Proof. According to Theorem 4.13

$$\omega_X = \pi^*L \otimes \mathcal{O}_X(\sum_i (m_i - 1)F_i).$$

Since $\deg L = 0$, as an algebraic class,

$$\omega_X \cong \mathcal{O}_X(\sum_i (m_i - 1)F_i).$$

Clearly, the base locus of $\mathcal{O}_X(\sum_i (m_i - 1)F_i)$ is contained in finitely many fibers $m_iF_i$. The remaining argument is the same as the proof in [KaOk18, Theorem 4.2].
Remark 4.18. Since $K$ is trivial, $P_g(X) = 0$, $g(C) = 0$, and $q(X) = 1$. The canonical bundle formula becomes
\[ \omega_X \cong \pi^*\mathcal{O}_{\mathbb{P}^1}(-2) \otimes \mathcal{O}(\sum_i (m_i - 1)F_i). \]

If we move $\mathcal{O}(-1)$ to each factor $\mathcal{O}_X((m_i - 1)F_i)$, it would become a noneffective divisor. We can not say the general theorem for these cases. Since $q(X) = 1$, it is still possible that the paracanonical systems admits nontrivial sections.

**Question 4.15.** Let $\pi : X \to \mathbb{P}^1$ be an elliptic fibration, $X$ is minimal of Kodaira dimension 1. Suppose $P_g(X) = 0$, and $q(X) = 1$. Is $D(X)$ indecomposable?

**4.3. Minimal surfaces of general type, $P_g(X) = q(X) = 1$, $K^2 = 2$ or $K^2 = 3$.**

Firstly, the new approach presented in this paper may work for more examples of minimal surfaces of general type. It is interesting to investigate in the future. We present some boundary cases of Conjecture 1.16, namely $P_g(X) = q(X) = 1$.

**Theorem 4.16.** Let $X$ be a minimal surface of general type such that $P_g(X) = q(X) = 1$, $K^2 = 2$ or $K^2 = 3$, then $D(X)$ has no nontrivial semi-orthogonal decompositions.

**Proof.** For these types of surfaces, there are complete classifications, in particular, it was known that $\dim \text{PBs} |\omega_X| = 0$ or $\text{PBs} |\omega_X| = \emptyset$ [CC91, Section 5, Lemma 4.10]. Hence $D(X)$ has no nontrivial semi-orthogonal decompositions by Corollary 3.8.

The proof in Theorem 4.16 was used to classify the surfaces. We present a different proof for some explicit constructions of surfaces of general type such that $P_g(X) = q(X) = 1$, $K^2 = 2$ or $K^2 = 3$.

Let $E^{(2)}$ be the double symmetric product of an elliptic curve. Define $\pi : E^{(2)} \to E$ via $(x, y) \mapsto x + y$. We denote $D_p$ the class $(p, z - p)$ of $E^{(2)}$, and $F_x$ the fibre of $\pi$ at $x \in E$.

The following proposition is for the case $K^2 = 2$.

**Proposition 4.17.** Let $B$ be a general member of $| - 6D + 2F|$ in $E^{(2)}$. Let $X'$ be a double cover of $E^{(2)}$ ramified over $B$. Let $X$ be the minimal resolution of $X'$, and $X'$ is isomorphic to the canonical model of $X$. Then $D(X)$ has no nontrivial semi-orthogonal decompositions.

**Proof.** These examples indeed exist, see [CC81, Part III] and [CC91, Section 5]. It is clear that $K_{X'}$ is a line bundle. Since $K_{X'} \cong f^*(K_{E^{(2)}} + B/2)$, and $K_X \cong g^*K_{X'}$, where $g$ is the resolution $g : X \to X'$. Write the composition of $g$ and $f$ as $h : X \to E^{(2)}$. Then
\[ K_X \cong h^*(K_{E^{(2)}} + B/2). \]

Since $K_{E^{(2)}} \cong_{alg} -2D + F$, therefore $K_{E^{(2)}} + B/2 \cong_{alg} D$. Since $\text{PBs} |D| = \emptyset$, we have $\text{PBs} |\omega_X| = \emptyset$. Thus, $D(X)$ has no nontrivial semi-orthogonal decompositions.

**Remark 4.18.** We actually obtain examples of surface $X'$ with Gorenstein singularities such that $\text{Perf}(X')$ has no nontrivial semi-orthogonal decompositions by Theorem 3.13.
Let $E^{(3)}$ be the triple symmetric product of $E$, similar notations for $D$ and $F$. The following proposition is for some examples such that $K^2 = 3$, see [CC91, Theorem 5.8].

**Proposition 4.19.** Let $X'$ be a divisor in $E^{(3)}$. Assume $X'$ is homologous to $|4D - F|$ and has only simple singularities. Let $X$ be a minimal resolution of $X'$ where $X'$ is isomorphic to the canonical model of $X$. Then $D(X)$ has no nontrivial semi-orthogonal decompositions.

**Proof.** These examples indeed exist [CC91, Theorem 5.8]. We prove that $\text{PBs}|K_{X'}| = \emptyset$. By Adjunction,

$$K_{X'} \cong K_{E^{(3)}} + X'|_{X'} \cong m - 3D + F + 4D - F|_{X'} = D|_{X'}.$$

Clearly, $\text{PBs}|O_{X'}(D)| = \emptyset$. □

**Remark 4.20.** It is also interesting to investigate the cases $4 \leq K^2 \leq 9$, which would be more complicated.

## 5. Application to Bridgeland stability condition

**Theorem 5.1.** Let $X$ be a smooth projective variety, and $Z = \text{PBs}|\omega_X|$. Let $\sigma$ be a Bridgeland stability condition of $D(X)$. Take any $k(x)$, $x \in X \setminus Z$. Assume $k(x)$ is not semi-stable, then the phase number of $HN$ factors of $k(x) : \phi_1 > \phi_2 > \cdots > \phi_n$ satisfies

$$\phi_i - \phi_{i+1} \leq \dim X - 1.$$

**Proof.** The line bundle $L \in \text{Pic}^0(X)$ preserves the $HN$ filtration of skyscraper sheaves [Pol07, Corollary 3.5.2]. So it is enough to prove the theorem for the case $Z = \text{Bs}|\omega_X|$. Let $s$ be a section of $\omega_X$ which does not vanish at $x \in X \setminus Z$. If $k(x)$ is semi-stable with respect to the stability condition, then there is nothing to prove. We assume that $k(x)$ is not semi-stable. Let

$$0 = E_0 \rightarrow E_1 \rightarrow E_2 \rightarrow \cdots \rightarrow E_{n-1} \rightarrow E_n = k(x)$$

be the $HN$ filtration of $k(x)$, $x \in U = X \setminus Z(s)$, with $HN$ factors $A_i := \text{Cone}(E_{i-1} \rightarrow E_i)$ such that $A_i \in \mathcal{P}(\phi_i)$,

$$\phi_1 > \phi_2 > \cdots > \phi_n.$$

Define $K_i = \text{Cone}(E_i \rightarrow k(x))$. There is a commutative diagram.

Consider composition $K_i \rightarrow E_i[1] \rightarrow E_i[1] \otimes \omega_X$ with section $s$,

$$\text{Hom}(K_i, E_i[1] \otimes \omega_X) \cong \text{Hom}(E_i, K_i[\dim X - 1])^\vee.$$
If $\phi^{-}(E_{i}) > \phi^{+}(K_{i}) + \dim X - 1$, then $\text{Hom}(K_{i}, E_{i}[1] \otimes \omega_{X}) = 0$. Therefore $k(x)|_{U} \cong E_{i}|_{U} \oplus K_{i}|_{U}$. Since $\text{Hom}(k(x), k(x)) = \mathbb{C}$, $E_{i}|_{U}$ or $K_{i}|_{U}$ must be zero. Suppose $E_{i}|_{U} = 0$, it implies that the support of $E_{i}$ is in $Z(s)$, then $\text{Hom}^{*}(E_{i}, k(x)) = 0$. Hence $K_{i} \cong k(x) \oplus E_{i}[1]$. Since $\text{Hom}(E_{i}[1], K_{i}) = 0$ by the assumption that $\phi^{-}(E_{i}) > \phi^{+}(K_{i}) + \dim X - 1$, we have $E_{i} \cong 0$, a contradiction. Similarly, suppose $K_{i}|_{U} = 0$, then $E_{i} \cong k(x) \oplus K_{i}[-1]$, which implies $K_{i} \cong 0$. So we prove $\phi^{-}(E_{i}) \leq \phi^{+}(K_{i}) + \dim X - 1$. Clearly $\phi^{-}(E_{i}) = \phi_{i}$. We prove by induction that $\phi^{+}(K_{i}) \leq \phi_{i+1}$.

Lemma 5.2. $\phi^{+}K_{i} \leq \phi_{i+1}$, $i = 1, 2, \cdots, n - 1$.

Proof. Consider the triangle $E_{n-1} \rightarrow k(x) \rightarrow K_{n-1}$, then $K_{n-1} = A_{n}$ by definition. Clearly $\phi^{+}K_{n-1} = \phi(A_{n}) = \phi_{n}$. Now assume $\phi^{+}(K_{i+1}) \leq \phi_{i+2}$, we need to prove that $\phi^{+}(K_{i}) \leq \phi_{i+1}$. Let $B_{1}$ be a $HN$ factor of $K_{i}$ such that $\phi(B_{1}) = \phi^{+}(K_{i})$. Suppose $\phi(B_{1}) > \phi_{i+1}$, then $\text{Hom}(B_{1}, A_{i+1}) = 0$, and $\text{Hom}(B_{1}, K_{i+1}) = 0$. Therefore $\text{Hom}(B_{1}, K_{i}) = 0$ by the triangle $A_{i+1} \rightarrow K_{i} \rightarrow K_{i+1} \rightarrow A_{i+1}[1]$. But if we apply $\text{Hom}(B_{1}, \bullet)$ to the $HN$ filtration of $K_{i}$, we get $\text{Hom}(B_{1}, B_{1}) \cong \text{Hom}(B_{1}, K_{i}) = 0$, a contradiction. Thus $\phi^{+}(K_{i}) \leq \phi_{i+1}$. □

According to the lemma above, we have inequality $\phi_{i} - \phi_{i+1} \leq \dim X - 1$. □

Corollary 5.3. Let $X = S^{4}C$, $g(C) \geq 2$, and $i \leq g(C) - 1$, or be a minimal surface of general type with $P_{g}(X) = q(X) = 1$, $K^{2} = 2$ or $K^{2} = 3$. If $\sigma$ is any Bridgeland stability condition of $D(X)$, then for any closed point $x \in X$, the phase number of $HN$ factors of skyscraper sheaf $k(x)$ (if $k(x)$ is not semi-stable): $\phi_{1} > \phi_{2} > \cdots > \phi_{n}$ satisfies $\phi_{i} - \phi_{i+1} \leq \dim X - 1$.

References

[BasBelOkRic] Francesco Bastianelli, Pieter Belmans, Shinnosuke Okawa, and Andrea T. Ricolfi. Indecomposability of derived categories in families. arXiv:2007.00994, 2020.

[BelGalMu] Pieter Belmans, Sergey Galkin, and Swarnava Mukhopadhyay. Semiorthogonal decompositions for moduli of sheaves on curves, Oberwolfach Report No. 24/2018, 9–11, DOI:10.4171/OWR/2018/24.

[BiGoLee] Indranil Biswas, Tomás Gómez, and Kyoung-Seog Lee. Semi-orthogonal decomposition of symmetric products of curves and canonical system. Revista Matemática Iberoamericana, Volume 37, pp 1885-1896; doi:10.4171/rmi/1251.

[BK] Pieter Belmans and Andreas Krug. Derived categories of (nested) Hilbert schemes. arXiv:1909.04321, 2019

[CC81] F. Catanese, “On a class of surfaces of general type”, pp. 269–284 in Algebraic surfaces (Liguori, Naples, 1981), 1981.

[CC91] F. Catanese and C. Ciliberto, “Surfaces with pg = q = 1”, pp. 49–79 in Problems in the theory of surfaces and their classification (Cortona, 1988), Sympos. Math. 32, Academic Press, London, 1991.

[Fr98] R. Friedman, Algebraic Surfaces and Holomorphic Vector Bundles. Springer-Verlag New York, Inc. 1998.
[FuLiZhao] Lie Fu, Chunyi Li, and Xiaolei Zhao. Stability manifolds of varieties with finite Albanese morphisms. arXiv:2103.07728, 2021.

[GoLee] Tomás Gómez and Kyoung-Seog Lee. Motivic decompositions of moduli spaces of vector bundles on curves. arXiv:2007.06067.

[Huy] Daniel Huybrechts. Fourier-Mukai transforms in algebraic geometry. Oxford: Clarendon, 2009.

[JL19] Qingyuan Jiang and Naichung Conan Leung. Derived category of projectivization and flops. arXiv:1811.12525, 2019.

[KaOk18] Kotaro Kawatani and Shinnosuke Okawa. Nonexistence of semiorthogonal decompositions and sections of the canonical bundle. arXiv:1508.00682.

[Kaw17] Yujiro Kawamata. Birational geometry and derived categories. arXiv:1710.07370, 2017.

[Kuz15] Alexander Kuznetsov. Semiorthogonal decompositions in algebraic geometry. Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II, Kyung Moon Sa, Seoul, 2014, pp. 635–660. MR 3728631.

[Lin21] Xun Lin. Noncommutative Hodge conjecture. arXiv:2102.03481, 2021.

[Mac62] I.G. MacDonald. Symmetric products of an algebraic curve. Topology, Volume 1, Issue 4, 1962, Pages 319-343, ISSN 0040-9383, https://doi.org/10.1016/0040-9383(62)90019-8. (https://www.sciencedirect.com/science/article/pii/0040938362900198)

[Orl93] D. Orlov. Projective bundles, monoidal transformations, and derived categories of coherent sheaves, Izv. Akad. Nauk SSSR Ser. Mat., 56 (1992): 852–862; English transl., Russian Acad. Sci. Izv. Math., 41 (1993): 133–141.

[Per20] Alexander Perry. The integral Hodge conjecture for two-dimensional Calabi-Yau categories. arXiv:2004.01363, 2020.

[Pir21] Dmitrii Pirozhkov. Stably semiorthogonally indecomposable varieties. arXiv:2011.12743.

[Pol07] A. Polishchuk. Constant families of t-structures on derived categories of coherent sheaves. Mosc. Math. J., 7(1):109–134, 167, 2007.

[Sp21] Dylan Spence. A note on semiorthogonal indecomposability for some Cohen-Macaulay varieties. arXiv:2104.13331.

[Tab19] Goncalo Tabuada. Noncommutative counterparts of celebrated conjectures. arXiv:1812.08774, 2019.

[Tod19] Yukinobu Toda. Semiorthogonal decompositions of stable pair moduli spaces via d-critical flips. J. Eur. Math. Soc. 23 (2021), 1675-1725. doi: 10.4171/JEMS/1041.

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