Yukawa Coupling Contribution to Magnetic Field Induced Dynamical Mass

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Abstract

By solving the gap equation in the quenched, ladder approximation for an Abelian gauge model with Yukawa interaction in the presence of a constant magnetic field, we show that the Yukawa interactions enhance the dynamical generation of fermion mass. The theory is then studied at finite temperature, where we prove that the critical magnetic field, required for the mass generation to be important at temperatures comparable to the electroweak critical temperature, can be substantially decreased due to the Yukawa coupling.

I. INTRODUCTION

In recent years the phenomenon of dynamical symmetry breaking in the presence of an external magnetic field [1] has attracted a great deal of attention [1]-[9]. The essence of this effect lies in the dimensional reduction in the dynamics of fermion pairing in the presence of a magnetic field [1]. Due to such a dimensional reduction, the magnetic field catalyses the dynamical generation of a fermion condensate and a fermion mass, even in the weakest attractive interaction between fermions.

The field-induced dynamical generation of mass (FDGM) has been found in many models of field theories, in 2+1 and in 3+1 dimensions. The universality of this phenomenon makes it of interest for a wide set of applications, including problems in different areas such as
In their original papers, Gusynin, Miransky and Shovkovy suggested that the structure of the electroweak phase transition could be affected by the FDGM. Considering FDGM in QED$_4$ at finite temperature, Lee, Leung and Ng, and independently, Gusynin and Shovkovy, calculated the critical temperature at which the fermion mass (and hence the fermion condensate) evaporates. From their results one might conclude that the dynamical generation of mass induced by a magnetic field should play no role in the electroweak phase transition, because for it to exist at the high temperatures typical of the electroweak scale, it would require a primordial magnetic field too large ($\sim 10^{42}G$) to be realistically attainable. However, as we have argued in a previous paper, it is reasonable to expect an essential modification in the order of the critical field when the FDGM takes place in the context of the electroweak model, since a richer set of interactions enters in scene there. As we will show below, this is indeed the case even in a model much simpler than the electroweak theory. By studying a toy model similar to QED$_4$, but including a Yukawa term, we prove that the Yukawa interactions can substantially enhance the FDGM phenomenon.

Any cosmological application of the FDGM has to assume that primordial magnetic fields could have been present during the early times of universe evolution. Nowadays astronomical observations seems to support this assumption. As it is known, primordial magnetic fields may be needed to explain the large-scale galactic magnetic fields $\sim 10^{-6}G$ observed in our own, as well as in other galaxies. The observed galactic fields have been a source of motivation for many works on primordial-field generating mechanisms. Typically, fields as large as $10^{24}G$ are predicted by these mechanisms during the electroweak phase transition. Moreover, Ambjorn and Olesen have claimed that seed primordial fields even larger, $\sim 10^{33}G$, would be necessary at the electroweak scale to explain the observed galactic fields.

From the above discussion it can be understood that the significance of the FDGM at the electroweak scale needs yet to be elucidated. The present paper is a step in that direction. We study the FDGM in an Abelian gauge model of massless fermions with a Yukawa term.
This model, although simple, retains some of the attributes of the electroweak theory. We explicitly show that Yukawa interactions enhance the generation of mass in the presence of a magnetic field, decreasing the critical field needed for the FDGM to be important at temperatures comparable to the electroweak critical temperature. For a Yukawa coupling of order of the top-quark coupling, the critical field strength is decreased in 10 orders of magnitude as compared to the corresponding field strength in QED.

The plan of the paper is the following. In Sec. II, we derive the Schwinger-Dyson equation for the fermion self-energy in the quenched, ladder approximation, solving the corresponding gap equation in the infrared region. Thermal effects are considered in Sec. III, where we calculate the critical temperature at which the field-induced fermion mass disappears and estimate the order of the critical field required for the FDGM to be significant at the electroweak scale. We discuss the implications of our results and state our conclusions in Sec. IV. The solution of the SD equation for the coefficient $Z_\parallel$ of the mass operator is derived in the Appendix.

II. ABELIAN GAUGE MODEL WITH YUKAWA INTERACTION

Let us consider an Abelian gauge model with a Yukawa interaction described by Lagrangian density

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}\gamma^\mu\partial_\mu\psi - g\bar{\psi}\gamma^\mu\psi A_\mu - \frac{1}{2}\xi(\partial_\mu A^\mu)^2 + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\lambda}{4}\phi^4 - \sqrt{2}\lambda y\phi\bar{\psi}\psi \quad (1)$$

Note that $L$ has a U(1) gauge symmetry and a fermion number global symmetry, but it does not have a continuous chiral symmetry. We consider the present model in the presence of a constant and uniform external magnetic field $H$. Our aim is to investigate the gap equation of the theory. We need to go beyond a perturbative calculation to find a non zero solution of the gap equation. In the small coupling regime of the theory a consistent, non perturbative calculation can be carried out by using the quenched, ladder approximation. This approximation has produced gauge invariant non trivial solutions
of the gap equation in other theories with external fields \[1\], so it is natural to expect that
it will yield similar results in the present case.

The gap equation of the theory described by the Lagrangian density \(\mathbb{L}\) can be obtained
from the Schwinger-Dyson equation for the fermion self-energy

\[
\overline{G}^{-1}(x,y) = G^{-1}(x,y) + ig \int d^4ud^4w \gamma \overline{G}(x,u) \overline{D}(x-w) \Gamma_{\psi A}(u,y,w) \\
+ i\sqrt{2}\lambda_y \int d^4ud^4w \overline{G}(x,u) \overline{S}(x-w) \Gamma_{\psi\phi}(u,y,w)
\]

(2)

Here \(G\) refers to fermion propagators, and \(D\) and \(S\) to gauge and scalar boson propagators
respectively. \(\Gamma_{\psi A}\) and \(\Gamma_{\psi\phi}\) are three-fields vertex functions. The bar indicates full Green
functions. A compact notation where tensorial and spinorial indexes have been suppressed
is understood.

After taking the quenched, ladder approximation (where the fermion propagator in the
presence of an external magnetic field is taken full, while the vertices, as well as the gauge
and scalar boson propagators, are taken bare) of Eq. (2), one obtains the following equation
for the fermion mass operator

\[
M(x,y) = \overline{G}^{-1}(x,y) - G^{-1}(x,y) = -ig^2 \int d^4ud^4w \gamma \overline{G}(x,u) \gamma D(x-w) \\
- i \left(\sqrt{2}\lambda_y\right)^2 \int d^4ud^4w \overline{G}(x,u) S(x-w)
\]

(3)

To solve Eq. (3) we need to transform it to momentum space. However, it is known
that the Fourier transform of the fermion Green function in the presence of a magnetic field
does not yield a diagonal-in-p function. The reason is that in the problem with external
field the fermion asymptotic states are no longer plane waves. A suitable solution to this
technical problem was found many years ago by Schwinger \[16\], who introduced the proper-
time representation of the fermion Green function in the presence of a constant external field.
Here, however, we prefer to adopt another approach due to Ritus \[17\]. Ritus’ method is based
on the use of a representation (known as the \(E_p\) representation) spanned by the solutions \(\psi_p\)
of the eigenvalue equation \((\gamma\Pi)^2\psi_p = p^2\psi_p\). The operator \((\gamma\Pi)^2 = (\gamma^\mu(i\partial_\mu - gA_\mu))^2\) commutes
with the mass operator. Hence, the $\psi_p$ can be used to generate a set of complete and orthonormal eigenfunction-matrices of the mass operator. In the chiral representation of the Dirac matrices, where $\gamma_5$ and $\Sigma_3 = i\gamma_1\gamma_2$ are both diagonal, the $\psi_p$ take the form

$$\psi_p = E_{p\sigma\chi}(x)\omega_{\sigma\chi}$$

(4)

The bispinors $\omega_{\sigma\chi}$ are eigenvectors of $\gamma_5$ and $\Sigma_3$, with eigenvalues $\chi = \pm 1$ and $\sigma = \pm 1$, respectively.

In the case of a purely magnetic field background (crossed field case) directed along the $z$-direction, the $E_{p\sigma\chi}$ functions are

$$E_{p\sigma}(x) = N(n)e^{i(p_0x^0 + p_2x^2 + p_3x^3)}D_n(\rho)$$

(5)

where $D_n(\rho)$ are the parabolic cylinder functions \cite{LS} with argument $\rho = \sqrt{2|gH|}(x_1 - \frac{p_3}{gH})$ and positive integer index

$$n = n(k, \sigma) \equiv k + \frac{gH\sigma}{2|gH|} - \frac{1}{2}, \quad n = 0, 1, 2, ...$$

(6)

$N(n) = (4\pi |gH|)^\frac{1}{4}/\sqrt{n!}$ is a normalization factor. Here $p$ represents the set $(p_0, p_2, p_3, k)$, which determines the eigenvalue $\mathbf{p}^2 = -p_0^2 + p_3^2 + 2|gH|k$ in $(\gamma\Pi)^2\psi_p = \mathbf{p}^2\psi_p$. Note that in this case $E_{p\sigma\chi}$ does not depend on $\chi$.

The $E_p$ representation is obtained forming the eigenfunction-matrices

$$E_p(x) = \sum_\sigma E_{p\sigma}(x)\Delta(\sigma),$$

(7)

where

$$\Delta(\sigma) = \text{diag}(\delta_{\sigma_1}, \delta_{\sigma-1}, \delta_{\sigma_1}, \delta_{\sigma-1}), \quad \sigma = \pm 1,$$

(8)

It is easy to check that the $E_p$ functions are orthonormal

$$\int d^4x E_{p'}(x)E_p(x) = (2\pi)^4\delta^{(4)}(p - p') \equiv (2\pi)^4\delta_{kk'}\delta(p_0 - p_0')\delta(p_2 - p_2')\delta(p_3 - p_3')$$

(9)

as well as complete.
\[
\sum_k \int dp_0 dp_2 dp_3 E_p(x) \mathcal{E}_p(y) = (2\pi)^4 \delta^{(4)}(x - y)
\]  

(10)

Here we have used \( \mathcal{E}_p(x) = \gamma^0 E^\dagger_p \gamma^0 \).

They also satisfy two important relations

\[
\gamma \cdot \Pi E_p(x) = E_p(x) \gamma \cdot \slashed{p}
\]  

(11)

\[
\int d^4x' M(x, x') E_p(x) = E_p(x) \tilde{\Sigma}_A(\slashed{p})
\]  

(12)

where \( \tilde{\Sigma}_A(\slashed{p}) \) is the fermion mass operator in momentum coordinates.

In the \( E_p \) representation the SD equation (3) for the mass operator becomes

\[
(2\pi)^4 \delta_{kk'} \delta(p_0 - p'_0) \delta(p_2 - p'_2) \delta(p_3 - p'_3) \tilde{\Sigma}_A(\slashed{p})
\]

\[
= -ig^2 \int d^4xd^4x' \sum_{k} \int \frac{dp'_0 dp''_2 dp''_3}{(2\pi)^4} (E_p(x) \gamma^\mu E_{p'}(x) \frac{1}{\gamma \cdot \slashed{p}' - \tilde{\Sigma}_A(\slashed{p}'))}
\]

\[
\times E_{p'}(x') \gamma^\nu E_{p'}(x') D_{\mu\nu}(x - x') \} - i2\lambda_y^2 \int d^4xd^4x' \sum_{k} \int \frac{dp''_0 dp''_2 dp''_3}{(2\pi)^4} (E_p(x) E_{p'}(x) \]

\[
\times \frac{1}{\gamma \cdot \slashed{p}' - \tilde{\Sigma}_A(\slashed{p}')} E_{p'}(x') E_{p'}(x') S(x - x') \}
\]  

(13)

with

\[
D_{\mu\nu}(x - x') = -\int \frac{d^4q}{(2\pi)^4} \frac{e^{-iq \cdot (x - x')}}{q^2 - i\epsilon} \left( g_{\mu\nu} - (1 - \xi) \frac{q_{\mu}q_{\nu}}{q^2 - i\epsilon} \right)
\]  

(14)

the bare photon propagator, and

\[
S(x - x') = \int \frac{d^4q}{(2\pi)^4} \frac{e^{-iq \cdot (x - x')}}{q^2 - i\epsilon}
\]  

(15)

the bare scalar propagator.

Using the properties of the parabolic cylinder functions and Eqs. (11)-(12), the integrals in \( x \) and \( x' \) in Eq. (13), as well as the integrals in \( p_0, p_2, \) and \( p_3 \), can be done yielding

\[
\delta_{kk'} \tilde{\Sigma}_A(\slashed{p}) = ig^2 2|gH| \sum_{k'} \sum_{(\sigma)} \int \frac{d^4q}{(2\pi)^4} \frac{e^{isgn(gH)(n - n' + \tilde{n} - \tilde{n}')\phi}}{\sqrt{n!n'!\tilde{n}!\tilde{n}'!}} e^{-\tilde{q}^2} J_{n'n'}(\tilde{q}) \tilde{J}_{n'n'}(\tilde{q}) \frac{1}{\tilde{q}^2}
\]
\[
x \left( g_{\mu
u} - (1 - \xi) \frac{\hat{q}_\mu \hat{q}_\nu}{q^2} \right) \Delta \gamma^\mu \Delta'' \frac{1}{\gamma \cdot \vec{p} - \sum_A(\vec{p}')} \tilde{\Delta}'' \gamma'' \Delta'
\]

\[-i2\lambda_y^2 \langle 2 \mid gH \rangle \sum_{k^*} \sum_{\{\sigma\}} \int \frac{d^4 \hat{q}}{(2\pi)^4} \left\{ \frac{e^{isgn(gH)(n-n+\bar{n}''-n')}\phi}{\sqrt{n!n''!n''''!}} e^{-\frac{\hat{q}_\perp^2}{2}} J_{n''}^* (\hat{q}_\perp) J_{n''''}^* (\hat{q}_\perp) \frac{1}{q^2} \right\}
\]

\[
\times \Delta \Delta' \frac{1}{\gamma \cdot \vec{p} - \sum_A(\vec{p}')} \tilde{\Delta}'' \Delta'
\]

(16)

where the following notation has been used

\[
J_{n_p n_r}^* (\hat{q}_\perp) \equiv \sum_{m=0}^{\min(n_p, n_r)} \frac{n_p! n_r!}{m!(n_p - m)!(n_r - m)!} [isgn(gH)\hat{q}_\perp]^n_p + n_r - 2m
\]

(17)

\[
\vec{p}' \equiv (p_0 - q_0, 0, -sgn(gH)\sqrt{2 \mid gH \mid k''}, p_3 - q_3)
\]

(18)

\[
\sum_{\{\sigma\}} \equiv \sum_{\sigma \sigma' \sigma''}
\]

(19)

and the dimensionless variables

\[
\hat{q}_\mu \equiv \frac{q_\mu \sqrt{2 \mid gH \mid}}{2gH}, \quad \mu = 0, 1, 2, 3
\]

(20)

and polar coordinates for the transverse components of \(\hat{q}_\mu\)

\[
\hat{q}_\perp \equiv \sqrt{\hat{q}_1^2 + \hat{q}_2^2}, \quad \varphi \equiv \arctan(\hat{q}_2/\hat{q}_1)
\]

(21)

have been introduced.

The first term of the right hand side (RHS) in Eq. (16) coincides with the result previously found in references [3] for the pure QED case. Because of the presence of the Yukawa interaction in the present model, we have obtained an additional contribution (the second integral in the RHS of Eq. (16)) whose consequences for the gap equation will not be trivial, as we show below.
To solve equation (16), we need the mass operator structure. In the presence of the external magnetic field the mass operator structure is quite rich. However, as argued in a previous paper, within the present approximation a simpler structure can be used. Thus, we consider

\[ \tilde{\Sigma}_A(\tilde{\varphi}) = Z_\parallel(\tilde{\varphi}) \gamma \cdot \tilde{p}_\parallel + Z_\perp(\tilde{\varphi}) \gamma \cdot \tilde{p}_\perp + m(\tilde{\varphi}) \]  

(22)

Note the separation between transverse and perpendicular variables.

After substituting the above structure for \( \tilde{\Sigma}_A(\tilde{\varphi}) \), Eq. (16) can be written as

\[ \delta_{kk'} \left[ Z_\parallel \gamma \cdot \tilde{p}_\parallel + Z_\perp \gamma \cdot \tilde{p}_\perp + m(\tilde{\varphi}) \right] \]

\[ = ig^2 2 |gH| \sum_{kk'} \sum_\{\sigma\} \int \frac{d^4 q}{(2\pi)^4} e^{i \text{sgn}(gH)(n-n'+\tilde{n}^\prime-n')\varphi} e^{-\tilde{q}_i} J_{nn'}(\tilde{q}_\parallel) J_{n'n'}(\tilde{q}_\perp) \frac{1}{q^2} \]

\[ \times \left\{ (g_{\mu\nu} - (1 - \xi) \frac{\tilde{q}_\mu \tilde{q}_\nu}{q^2}) \Delta \gamma^\mu \Delta' m(\tilde{\varphi}) - (1 + Z_\parallel) \gamma \cdot \tilde{p}_\parallel' - (1 + Z_\perp) \gamma \cdot \tilde{p}_\perp' \right\} \]

\[ - 2i \lambda_\parallel^2 |gH| \sum_{kk'} \sum_\{\sigma\} \int \frac{d^4 q}{(2\pi)^4} e^{i \text{sgn}(gH)(n-n'+\tilde{n}^\prime-n')\varphi} e^{-\tilde{q}_i} J_{nn'}(\tilde{q}_\parallel) J_{n'n'}(\tilde{q}_\perp) \frac{1}{q^2} \]

\[ \times \left\{ \Delta \Delta' m(\tilde{\varphi}) - (1 + Z_\parallel) \gamma \cdot \tilde{p}_\parallel' - (1 + Z_\perp) \gamma \cdot \tilde{p}_\perp' \right\} \]  

(23)

It can be further simplified taking into account that large \( \tilde{q}_\perp \) contributions to Eq. (23) are suppressed by the factor \( e^{-\tilde{q}_i} \), and therefore we can approximate the \( J_{nn'}(\tilde{q}_\perp) \) functions by their infrared behavior

\[ J_{nn'}(\tilde{q}_\perp) \approx n! \delta_{nn'} \]  

(24)

This approximation allows us to eliminate the \( \varphi \) dependence in the integrand to obtain

\[ \delta_{kk'} \left[ Z_\parallel \gamma \cdot \tilde{p}_\parallel + Z_\perp \gamma \cdot \tilde{p}_\perp + m(\tilde{\varphi}) \right] \]

\[ = ig^2 2 |gH| \sum_{kk'} \sum_\{\sigma\} \delta_{nn'} \delta_{n'n'} \int \frac{d^4 q}{(2\pi)^4} e^{-\tilde{q}_i} \left( \frac{1}{(1 + Z_\parallel)^2 \tilde{p}_\parallel^2 + (1 + Z_\perp)^2 \tilde{p}_\perp^2 + m^2(\tilde{\varphi})} \right) \]
\[
\times \{m(p') \left[ \Delta \gamma^\mu \Delta'' \tilde{\Delta}'' \gamma'' \Delta' - \frac{(1 - \xi)}{q^2} \Delta(\gamma \cdot \tilde{q}) \Delta'' \tilde{\Delta}''(\gamma \cdot \tilde{q}) \Delta' \right] \\
- (1 + Z_\parallel) \left[ \Delta \gamma^\mu \Delta'' \left( \gamma \cdot \tilde{p}_\parallel'' \right) \tilde{\Delta}'' \gamma'' \Delta' - \frac{(1 - \xi)}{q^2} \Delta(\gamma \cdot \tilde{q}) \Delta'' \left( \gamma \cdot \tilde{p}_\parallel'' \right) \tilde{\Delta}''(\gamma \cdot \tilde{q}) \Delta' \right] \\
- (1 + Z_\perp) \left[ \Delta \gamma^\mu \Delta'' \left( \gamma \cdot \tilde{p}_\perp'' \right) \tilde{\Delta}'' \gamma'' \Delta' - \frac{(1 - \xi)}{q^2} \Delta(\gamma \cdot \tilde{q}) \Delta'' \left( \gamma \cdot \tilde{p}_\perp'' \right) \tilde{\Delta}''(\gamma \cdot \tilde{q}) \Delta' \right] \}
\]

\[
- i 2 \lambda_s^2 (2 |gH|) \sum_{k'} \sum_{\{\sigma\}} \delta_{nn'} \delta_{\tilde{n}'n'} \int \frac{d^4 \tilde{q}}{(2\pi)^4} \frac{e^{-\tilde{q}_z^2}}{q^2} \\
\times \left( \frac{m(p'') \Delta \Delta'' \Delta'' \Delta' \left( \gamma \cdot \tilde{p}_\parallel'' \right) \tilde{\Delta}'' \gamma'' \Delta' - (1 + Z_\parallel) \Delta \Delta'' \left( \gamma \cdot \tilde{p}_\parallel'' \right) \tilde{\Delta}''(\gamma \cdot \tilde{q}) \Delta'}{(1 + Z_\parallel)^2 \tilde{p}''^2 + (1 + Z_\perp)^2 \tilde{p}_\perp^2 + m^2(p')} \right)
\]

(25)

To perform the summation in the spin indices we can make use of the following relations satisfied by the \( \Delta \) matrices (8):

\[
\Delta(\sigma) \Delta(\sigma') = \delta_{\sigma\sigma'} \Delta(\sigma)
\]

(26)

\[
\Delta(1) + \Delta(-1) = I
\]

\[
\Delta \gamma^\mu_\parallel = \gamma^\mu_\parallel \Delta, \quad \text{with} \quad \gamma^\mu_\parallel = (\gamma^0, 0, 0, \gamma^3)
\]

(27)

\[
\Delta \gamma^\mu_\perp = \gamma^\mu_\perp (1 - \Delta), \quad \text{with} \quad \gamma^\mu_\perp = (0, \gamma^1, \gamma^2, 0)
\]

(28)

to find that

\[
\sum_{\{\sigma\}} \delta_{nn'} \delta_{\tilde{n}'n'} \Delta \Delta'' \tilde{\Delta}'' \Delta' = \delta_{kk'} \delta_{kk''}
\]

(29)

\[
\sum_{\{\sigma\}} \delta_{nn'} \delta_{\tilde{n}'n'} \Delta \gamma^\mu \Delta'' \left( \gamma \cdot \tilde{p}_\perp'' \right) \tilde{\Delta}'' \gamma'' \Delta' = 2 \left( \gamma \cdot \tilde{p}_\perp'' \right) \delta_{kk'} \delta_{kk''}
\]

(30)

\[
\sum_{\{\sigma\}} \delta_{nn'} \delta_{\tilde{n}'n'} \Delta \gamma^\mu \Delta'' \left( \gamma \cdot \tilde{p}_\parallel'' \right) \tilde{\Delta}'' \gamma'' \Delta' = 2 \left( \gamma \cdot \tilde{p}_\parallel'' \right) \delta_{kk'} \left( \delta_{k,k'-\text{sgn}(gH)} \Delta(1) + \delta_{k,k'+\text{sgn}(gH)} \Delta(-1) \right)
\]

(31)
\[
\sum_{\{\sigma\}} \delta_{nn'} \delta_{n'n'} \Delta \left( \gamma \cdot \hat{q} \right) \Delta' \left( \gamma \cdot \vec{\mathcal{P}}_\perp \right) \Delta'' \left( \gamma \cdot \hat{q} \right) \Delta' \simeq \Delta \left( \gamma \cdot \vec{\mathcal{P}}_\perp \right) \Delta' \left( \gamma \cdot \hat{q} \right) \Delta'' \left( \gamma \cdot \vec{\mathcal{P}}_\perp \right) \delta_{kk'} \delta_{kk''}
\]

(32)

\[
\sum_{\{\sigma\}} \delta_{nn'} \delta_{n'n'} \frac{\Delta \left( \gamma \cdot \hat{q} \right) \Delta'' \left( \gamma \cdot \vec{\mathcal{P}}_\perp \right) \Delta' \left( \gamma \cdot \hat{q} \right) \Delta'' \left( \gamma \cdot \vec{\mathcal{P}}_\perp \right)}{\hat{q}^2} \simeq \delta_{kk'} \delta_{kk''} \left( \gamma \cdot \vec{\mathcal{P}}_\perp \right) \left( \gamma \cdot \vec{\mathcal{P}}_\perp \right) \left( \gamma \cdot \hat{q} \right) \left( \gamma \cdot \hat{q} \right)
\]

(33)

\[
\sum_{\{\sigma\}} \delta_{nn'} \delta_{n'n'} \Delta \Delta'' \left( \gamma \cdot \vec{\mathcal{P}}_\perp \right) \Delta' = \delta_{kk'} \delta_{kk''} \left( \gamma \cdot \vec{\mathcal{P}}_\perp \right)
\]

(34)

\[
\sum_{\{\sigma\}} \delta_{nn'} \delta_{n'n'} \Delta \Delta'' \left( \gamma \cdot \vec{\mathcal{P}}_\perp \right) \Delta' = \delta_{kk'} \delta_{kk''} \left( \gamma \cdot \vec{\mathcal{P}}_\perp \right)
\]

(35)

In the above equations we dropped terms proportional to \(\hat{q}_\perp\) taking into account the small \(\hat{q}_\perp\) approximation here considered. The remaining \(\sigma\) sums were obtained in ref. [3]. Note that the \(\delta_{kk''}\) appearing in all terms cancels out with the one of the left hand side (LHS) of Eq. (23). The SD equation is then expressed as

\[
Z_\parallel \gamma \cdot \vec{\mathcal{P}}_\parallel + Z_\perp \gamma \cdot \vec{\mathcal{P}}_\perp + m(\vec{\mathcal{P}}) = i2 \left|gH\right| \sum_{k'} \int \frac{d^4\hat{q}}{(2\pi)^4} e^{-\frac{\hat{q}^2}{2}} \left( \frac{1}{(1 + Z_\parallel)^2 \vec{\mathcal{P}}_\parallel^2 + (1 + Z_\perp)^2 \vec{\mathcal{P}}_\perp^2 + m^2(\vec{\mathcal{P}})} \right)
\]

\[
\times \left\{ g^2 \left\{ m(\vec{\mathcal{P}}) \right\} \left[ (1 - \xi) \delta_{kk''} - 2 \left( \delta_{kk''} + \delta_{kk'' - sgn(gH)} \Delta(1) + \delta_{kk'' + sgn(gH)} \Delta(-1) \right) \right] \right. \\
+ \left(1 + Z_\parallel\right) \left[ (1 - \xi) \delta_{kk''} \left( \gamma \cdot \hat{q}_\parallel \right) \left( \gamma \cdot \vec{\mathcal{P}}_\parallel \right) \left( \gamma \cdot \hat{q}_\parallel \right) - 2 \left( \gamma \cdot \vec{\mathcal{P}}_\parallel \right) \left( \delta_{kk'' - sgn(gH)} \Delta(1) + \delta_{kk'' + sgn(gH)} \Delta(-1) \right) \right] \\
+ \left(1 + Z_\perp\right) \left( \left[ (1 - \xi) \delta_{kk''} \left( \gamma \cdot \vec{\mathcal{P}}_\perp \right) - 2 \delta_{kk''} \left( \gamma \cdot \vec{\mathcal{P}}_\perp \right) \right] \right) \\
- 2g^2 \left\{ m(\vec{\mathcal{P}}) \delta_{kk''} - (1 + Z_\parallel) \delta_{kk''} \left( \gamma \cdot \vec{\mathcal{P}}_\parallel \right) - (1 + Z_\perp) \delta_{kk''} \left( \gamma \cdot \vec{\mathcal{P}}_\perp \right) \right\}
\]

(36)

The \(\Delta'\)’s terms, appearing in the RHS of Eq. (36), but not in the LHS, remind us that the mass operator structure used here, Eq. (23), is not the most general one in the presence of an external magnetic field [13]. However, to study the dynamical generation of mass we can restrict our calculations to certain momentum region on which the simplified structure (11) gives rise to self-consistent results. Let us recall that in the case of QED it has been shown [1], [3], [13] that the dynamical generation of mass in the presence of a magnetic field is governed by the fermion infrared dynamics. It is natural to expect that a similar situation takes place in the model we are considering here. Thereby, we shall restrict the above SD
equation to the lower Landau level (LLL) approximation, on which $p_\perp = k = 0$, which is better justified if we simultaneously assume that the external momentum lies in the infrared region $p^2 << |eH|$. If, in addition, we use the Feynman gauge, and realize that the main contribution to the sum in $k''$ comes from the $k'' = k = 0$ terms, we can rewrite the SD equation as

$$Z_{\parallel} \gamma \cdot \vec{p}_{\parallel} + m(\vec{p}_{\parallel}) \simeq i2|gH| \int \frac{d^4\tilde{q}}{(2\pi)^4} \frac{e^{-\tilde{q}_\perp^2}}{\tilde{q}^2} \left( \frac{1}{(1 + Z_{\parallel})^2 p^\top + m^2(p')^2} \right) \times \left\{ -2(g^2 + \lambda_y^2)m(\vec{p}'') + 2\lambda_y^2(1 + Z_{\parallel})\delta_{kk''}(\gamma \cdot \vec{p}'') \right\} \quad (37)$$

Performing a Wick rotation to Euclidean space, Eq.(37) leads to the following two equations, one for each independent structure,

$$Z_{\parallel} \gamma \cdot \vec{p}_{\parallel} = -4\lambda_y^2|gH| \int \frac{d^4\tilde{q}}{(2\pi)^4} \frac{e^{-\tilde{q}_\perp^2} (1 + Z_{\parallel})\gamma \cdot (\vec{p} - q_{\parallel})}{(1 + Z_{\parallel})^2(\vec{p} - q_{\parallel})^2 + m^2(\vec{p} - q_{\parallel})} \quad (38)$$

$$m(\vec{p}_{\parallel}) = 4|gH|(g^2 + \lambda_y^2) \int \frac{d^4\tilde{q}}{(2\pi)^4} \frac{e^{-\tilde{q}_\perp^2} m(\vec{p} - q_{\parallel})}{(1 + Z_{\parallel})^2(\vec{p} - q_{\parallel})^2 + m^2(\vec{p} - q_{\parallel})} \quad (39)$$

As shown in the Appendix, the first equation has solution $Z_{\parallel} = 0$. The second is the gap equation. Note that it is very similar to the gap equation found by Lee, Leung and Ng [3] (after using $Z_{\parallel} = 0$), except that the coupling factor here is $(g^2 + \lambda_y^2)$ instead of just $g^2$. The main contributions to Eq.(39) come from the infrared region $\tilde{q}_\perp^2 << |gH|$, $q_{\parallel}^2 << |gH|$. A solution to Eq. (39) can be explicitly found in the infrared approximation $\vec{p}_{\parallel} \approx 0$, taking into account that the mass parameter is dominated by the small momenta contributions [3]. Then, we can approximate the mass by its infrared value $m(\vec{p}_{\parallel} - q_{\parallel}) \simeq m(\vec{p}_{\parallel}) \simeq m(0)$, and use that $Z_{\parallel} = 0$ to arrive at the infrared gap equation

$$1 \simeq \frac{|gH|(g^2 + \lambda_y^2)}{4\pi^2} \int_0^\infty d^2\tilde{q}_\perp \int_0^\infty d^2\tilde{q}'_\perp \frac{e^{-\tilde{q}_\perp^2}}{\tilde{q}_\perp^2 + \tilde{q}'_\perp^2} \frac{e^{-\tilde{q}'_\perp^2}}{2|gH|\tilde{q}_\perp^2 + m^2} \quad (40)$$

which can be integrated to obtain the non trivial solution
\[ m \approx \sqrt{|gH|} \exp \left[ -\frac{\pi}{\frac{g^2}{4\pi} + \frac{\lambda_y^2}{4\pi}} \right] \]  

(41)

The consistency of the approximation requires \( m \ll \sqrt{|gH|} \), which is satisfied if \( \frac{g^2}{4\pi} + \frac{\lambda_y^2}{4\pi} \ll 1 \), so the dynamical mass appears in the weak coupling region of the theory. Note that, because of the exponential function in Eq. (41), small changes in the exponent can yield substantial changes in the mass. For instance, for \( \lambda_y \approx 0.7 \), a value comparable to the top-quark Yukawa coupling, the dynamical mass \( \frac{g^2}{4\pi} \) is five orders of magnitude larger than the mass found in QED \[1\]–\[3\], which is given by \( m \approx \sqrt{|gH|} \exp \left[ -\sqrt{\frac{4\pi^2}{g^2}} \right] \).

We should underline that in this model the generation of a dynamical mass cannot be linked to the breaking of a continuous chiral symmetry, because there is no chiral symmetry to begin with. Since our goal is to use the present toy model to get insight of the FDGM phenomenon in the electroweak theory, where there is no chiral symmetry to break, the present model is a good candidate for our purpose. We must point out that, even though the appearance of the dynamical mass in the present case can be easily traced to the existence of a fermion-antifermion condensate \[19\], there is no Goldstone field produced in this theory, because there is no continuous symmetry broken by the fermion condensate. We expect that this should not be the case in the electroweak model. If a fermion condensate is catalyzed by the magnetic field there, it might give rise to a field-dependent vev of the scalar field and hence to a Higgs-like spontaneous gauge symmetry breaking.

Eq. (41) clearly indicates that Yukawa interactions enhance the dynamical generation of the fermion mass in the presence of a magnetic field. To grasp the possible significance of this result in the context of the electroweak phase transition, finite temperature effects has to be considered.

**III. FINITE TEMPERATURE CALCULATIONS**

Finite temperature effects can be incorporated using the well known imaginary-time Matsubara formalism, in which the Euclidean time variable is compactified to a circle of
radius $\beta = \frac{1}{T}$, ($T$ is the absolute temperature), the fourth components of the momenta are consequently discretized according to

$$q_4 = 2n\pi/\beta \quad \text{with} \quad n = 0, \pm 1, \pm 2, \ldots \quad \text{for bosons}$$

$$q_4 = (2n + 1)\pi/\beta \quad \text{with} \quad n = 0, \pm 1, \pm 2, \ldots \quad \text{for fermions}$$

and, in the functional integrals, boson (fermion) fields are periodic (antiperiodic) in time with period $\beta$.

The calculation leading to the gap equation in the above section can be performed at finite temperature in a similar way, taking into account that now the integrals in the fourth components of the momenta must be substituted by sums due to the discrete character of these variables. Hence, the finite temperature gap equation can be expressed as

$$m(\omega_n, p) = \left(\frac{g^2}{4\pi} + \frac{\lambda^2}{4\pi}\right) T \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{\omega_n^2 + k^2 + m^2(\omega_n, k)} \int_0^\infty \frac{dk \exp\left(-\frac{x^2}{2|\mu|}\right)}{(\omega_n - \omega_n')^2 + (k - p)^2 + x}$$

with $\omega_n = (2n + 1)\pi T$. Eq. (42) differs from the corresponding equation for QED (Eq.(10) of reference [4]) only in the factor multiplying the integrals, which in the present case contains the contribution $\frac{\lambda^2}{4\pi}$ due to the Yukawa interaction.

It is natural to expect that at some critical temperature the thermal effects evaporate the $\bar{\psi}\psi$ fermion condensate responsible for the nonzero dynamical mass. For the dynamical mass generation to be of any significance at the electroweak scale, it is needed that the critical temperature at which the mass becomes zero results of the order of the electroweak critical temperature $\sim 10^2 Gev$. Of course, we must remember that there is another parameter in the problem, the magnetic field. Different magnetic field strengths will yield different values of the critical temperature. Therefore, our goal is to determine the magnetic field required to have a critical temperature comparable to the electroweak one. Hence, we need to find the critical temperature as a function of the couplings and the magnetic field, that is, we need to solve first the gap equation (42), and then take the limit $m \to 0$. Since our calculations are very similar to those of ref. [4], we remit the interested lector to that paper, and here
we just give the final result, which in our case contains the modification due to the extra coupling $\lambda_y$. Thus, the critical temperature estimate is

$$T_c \approx \sqrt{|gH|} \exp \left[ - \sqrt{\frac{\pi}{g^2 + \frac{\lambda_y^2}{4\pi}}} \right] \approx m(T = 0)$$

which results comparable to the dynamical mass at zero temperature. Critical temperature estimates of the order of the corresponding zero temperature dynamical masses have been also found in QED and NJL, in (2+1) and (3+1) dimensions [1] [4] [7].

We can now estimate the strength of the magnetic field required to have $T_c \sim 10^2 Gev$. For Yukawa coupling $\lambda_y \simeq 0.7$ and gauge coupling $g = \frac{2}{3}e$ we have

$$T_c \simeq 10^2 Gev \simeq 1.8 \times 10^{-4} \sqrt{|gH|} \simeq 1.2 \times 10^{-12} Gev \sqrt{\frac{H(G)}{10^4 G}}$$

thus the critical field is $H \sim 10^{32} G$. This result represents a decreasing of the field in 10 orders of magnitude as compared to the value required in QED (where $\lambda_y \simeq 0, g = e$) to obtain the same critical temperature of $10^2 Gev$.

In a note added in proof, Lee et. al. concluded that the dynamical generation of mass due to a magnetic field plays no role in the electroweak phase transition, because the field that had to be present in the early universe for the FDGM to be important was too large. Their claim was based in results obtained within QED, where the critical field strength is $\sim 10^{42} G$. Here we have a critical field $\sim 10^{32} G$. Although much smaller, it is still larger than $10^{24} G$, which is the estimate generally predicted by primordial field generating mechanisms [13]. We must point out, however, that the present result has been found within a toy model, so it just indicates the tendency of the theory when new interactions are switched on. We anticipate that in the electroweak model the critical field needed shall be much smaller than the one found here.

**IV. CONCLUSIONS**

The main conclusion of this paper is that Yukawa interactions enhance the dynamical generation of fermion bound states and masses in the presence of external magnetic fields.
It is natural to expect that other interactions will have similar consequences. Therefore, it is worth to extend our results to more realistic theories. In the context of the electroweak model the enhancement of the FDGM should be even more substantial due to the interaction richness of the theory.

The generation of a fermion condensate in the presence of a constant magnetic field provides an example of change in the symmetry properties of the vacuum due to external fields and may have very wide applications. The FDGM was originally discovered as a mechanism of catalysis of chiral symmetry breaking [1] and as so, it has been rediscovered in many different theories, including condensed matter phenomena. We believe that in gauge theories with a richer set of interactions and symmetries the FDGM may be connected to the breaking of a gauge symmetry. In the context of the Standard Model our conjecture poses an interesting question: could primordial magnetic (hypermagnetic) fields induce gauge symmetry breaking in the early universe? If the answer is positive, it may led to several cosmological consequences. Work on this direction is in progress and will be publish elsewhere.

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APPENDIX

Let us find the solution of $Z_\parallel$ that satisfies the equation

$$Z_\parallel \gamma \cdot \overline{\nu_\parallel} = -4\lambda^2_y (|gH|) \int \frac{d^4 \hat{q}}{(2\pi)^4} \frac{e^{-\hat{q}_\parallel^2}}{\hat{q}^2} \frac{(1 + Z_\parallel) \gamma \cdot (\overline{\nu}_\parallel - q_\parallel)}{(1 + Z_\parallel)^2 (\overline{\nu}_\parallel - q_\parallel)^2 + m^2 (\overline{\nu}_\parallel - q_\parallel)}$$  (A1)

Doing the variable change
\[ k_\parallel = (k_4, k_3) = (p_4 - q_4, p_3 - q_3) \quad (A2) \]

the integral in Eq. (A11) can be written as

\[
I = \int \frac{d^4 \hat{q}^*}{(2\pi)^4} e^{-\hat{q}^2} \frac{(1 + Z_\parallel) \gamma \cdot (\bar{p}_\parallel - q_\parallel)}{(1 + Z_\parallel)^2 (\bar{p}_\parallel - q_\parallel)^2 + m^2 (|\bar{p}_\parallel|)}
\]

\[
= \int \frac{d^2 q_\perp d^2 k_\parallel}{(2\pi)^4} e^{-\hat{q}^2_\perp} \frac{(1 + Z_\parallel) \gamma \cdot k_\parallel}{(1 + Z_\parallel)^2 (\bar{p}_\parallel - k_\parallel)^2 + m^2 (|k_\parallel|)} \quad (A3)
\]

We can change \( k_\parallel \) now to polar coordinates \((\kappa_\parallel, \theta_k)\), and use Feynman integral formula

\[
\frac{1}{AB} = \int_0^1 dy \frac{1}{[yA + (1-y)B]^2} \quad (A4)
\]

to write the integral (A3) as

\[
I = \int \frac{d^2 q_\perp}{(2\pi)^2} \int_0^\infty dk_\parallel \int_0^{2\pi} d\kappa_\parallel d\theta_k e^{-\hat{q}^2_\perp} (1 + Z_\parallel)
\]

\[
\times \left\{ yq^2_\perp + \frac{y}{2|gH|} (p^2_\parallel + \kappa^2_\parallel) - \frac{2p^2_\parallel \kappa_\parallel}{|2gH|} \sin (\theta_k - \theta_p) + (1 - y) \left[ (1 + Z_\parallel)^2 (\kappa^2_\parallel) + m^2 \right] \right\}^2 
\]

\[
\quad (A5)
\]

where we have written the external parallel momentum in polar coordinates

\[
\bar{p}_4 = p_4 = p_\parallel \cos \theta_p
\]

\[
\bar{p}_3 = p_3 = p_\parallel \sin \theta_p 
\quad (A6)
\]

Using the following formulas, which are valid for \( a^2 \neq b^2 \),

\[
\int d\theta \frac{\cos \theta}{(a + b \cos \theta)^2} = \frac{a \sin \theta}{(a^2 - b^2) (a + b \cos \theta)} + \frac{1}{a^2 - b^2} \int d\theta \frac{1}{a + b \cos \theta} 
\quad (A7)
\]

\[
\int d\theta \frac{\sin \theta}{(a + b \cos \theta)^2} = \frac{1}{(a^2 - b^2)} - \frac{a}{b (a^2 - b^2)} \ln (a + b \cos \theta) \quad (A8)
\]

\[
\int d\theta \frac{1}{a + b \cos \theta} = \begin{cases} 
\frac{2}{\sqrt{a^2 - b^2}} \arctg \frac{(a-b)tg^2}{\sqrt{a^2 - b^2}}, & \text{if } a^2 > b^2 \\
\frac{1}{\sqrt{b^2 - a^2}} \ln \frac{(a-b)tg^2 - \sqrt{b^2 - a^2}}{(a-b)tg^2 + \sqrt{b^2 - a^2}}, & \text{if } a^2 < b^2 
\end{cases} 
\quad (A9)
\]

and the definitions
\[
a = yq_\perp^2 + \frac{y}{2|gH|} \left( p_\parallel^2 + \kappa_\parallel^2 \right) + (1 - y) \left[ (1 + Z_\parallel)^2 (\kappa_\parallel)^2 + m^2 \right] \tag{A10}
\]

\[
b = -\frac{2p_\parallel \kappa_\parallel}{2|gH|} \cos (\theta_k - \theta_p) \tag{A11}
\]

one can straightforwardly show that all the angle integrals are zero. Therefore, Eq. \((A11)\) reduces to

\[
Z_\parallel \gamma \cdot \mathbf{p}_\parallel = 0 \tag{A12}
\]

leading to the solution \(Z_\parallel = 0\).
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