Continuous-time Mean-Variance Portfolio Selection with Stochastic Parameters

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Abstract

This paper studies a continuous-time market under stochastic environment where an agent, having specified an investment horizon and a target terminal mean return, seeks to minimize the variance of the return with multiple stocks and a bond. In the considered model firstly proposed by [3], the mean returns of individual assets are explicitly affected by underlying Gaussian economic factors. Using past and present information of the asset prices, a partial-information stochastic optimal control problem with random coefficients is formulated. Here, the partial information is due to the fact that the economic factors can not be directly observed. Via dynamic programming theory, the optimal portfolio strategy can be constructed by solving a deterministic forward Riccati-type ordinary differential equation and two linear deterministic backward ordinary differential equations.

Keywords: mean-variance portfolio selection, partial information, filtering.

1 Introduction

Mean-variance is by far an important investment decision rule in financial portfolio selection, which is first proposed and solved in the single-period setting by Markowitz in his Nobel-Prize-winning works [14, 15]. In these seminal papers, the variance of the final wealth is used as a measure of the risk associated with the portfolio and the agent seeks to minimize the risk of his investment subject to a given mean return. This model becomes the foundation of modern finance theory and inspires hundreds of extension and applications. For example, this leads to the elegant capital asset pricing model naturally [19].

The dynamic extension of the Markowitz model has been established in the subsequent years after the appearance of [14, 15], employing heavily among others the martingale theory, convex duality and stochastic control. The pioneer work for continuous time case about multi-period portfolio management is [16]. In [16], Merton uses dynamic programming and partial differential equation (PDE) theory to derive and analyze the relevant Hamilton-Jacobi-Bellman (HJB) equation, and thus obtains the optimal strategy. Alternatively, to avoid dynamic programming, in [18], the author introduce the so-called risk-neutral (martingale) probability measure in order to reduce the computational difficulties associated with PDEs. In [23], the authors formulate the mean-variance problem with deterministic coefficients to a linear-quadratic (LQ) optimal problem. As there is no running cost in the objective function, this formulation is inherently an indefinite stochastic LQ control problem. As extensions of [23], for example, [9] deals with random coefficients case; while [24] considers regime switching market. For discrete time case, [5] completely solves the multiperiod mean-variance portfolio selection problem. Analytical optimal strategy and an efficient algorithm to find this strategy are proposed. For more about the history of the mean-variance model, [20] and [2] are referred.

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In [3], in order to tackle the computational tractability and the statistical difficulties associated with the estimation of model parameters, Bielecki and Pliska introduce a model such that the underlying economic factors such as accounting ratios, dividend yields, and macroeconomic measures are explicitly incorporated in the model. Exactly, the factors are assumed to follow Gaussian processes and the drifts of the stocks are linear functions of these factors. This model motivates many subsequent researches; see, for example, [17] and [4]. In practice, investors can only observe past and present asset prices to decide his current portfolio strategy; and, random factors cannot be completely observed. Therefore, the underlying problem falls into the category of portfolio selection under partial information. A significant progress in the realm of mean-variance concerning partial information is the work of [21]. In [21], a separation principle is shown to hold in this partial information setting; efficient strategies based on the partial information are derived, which involve the optimal filter of the stock appreciation rate processes; in addition, the particle system representation of the obtained filter is employed to develop analytical and numerical approaches. It is valuable to point out that backward stochastic differential equations (BSDEs) methodology is employed to tackle this problem.

This paper attempts to deal with the mean-variance portfolio selection under partial information based on the model of [3]. By exploiting the properties of the filter process and the wealth process, we tackle this problem directly by the dynamic programming approach. We show that optimal strategy can be constructed by solving a deterministic forward Riccati-type ordinary differential equation (ODE) and a system of linear deterministic backward ODEs. Clearly, by reversing the time, a deterministic backward ODE can be converted to a forward one. Therefore, we can easily derive the analytic solutions of the ODEs, and thus the analytic form of the optimal strategies. This is the main contribution of the note. The proposed procedure is different from that of [21], where BSDEs are employed.

The rest of the paper is organized as follows. In Section 2, we formulate the mean-variance portfolio selection model under partial information, and an auxiliary problem is introduced. Section 3 gives the optimal strategy of the auxiliary problem by dynamic programming method. Section 4 studies the original problem, while Section 5 gives some concluding remarks.

## 2 Mean-Variance Model

Throughout this paper \((\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t \geq 0})\) is a fixed filtered complete probability space on which defined a standard \(\mathcal{F}_t\)-adapted \((n+m)\)-dimensional Brownian motion \(W(t), t \geq 0\) with \(W(t) = (W^1(t), \cdots, W^{n+m}(t))^T\) and \(W(0) = 0\). Let \(T > 0\) be the terminal time of an investment, and \(L^2_\mathcal{F}(0, T; \mathbb{R}^d)\) denote the set of all \(\mathbb{R}^d\)-valued, \(\mathcal{F}_t\)-adapted stochastic processes \(f(t)\) with \(\int_0^T |f(t)|^2 dt < +\infty\), similarly \(L^2_\mathcal{F}(0, T; \mathbb{R}^d)\) can be defined for any functions with domain in \(\mathbb{R}^d\) and filtration \(\mathcal{H}_t\).

There is a capital market containing \(m + 1\) basic securities (or assets) and \(n\) economic factors. The securities consist of a bond and \(m\) stocks. The set of factors may include short-term interests, the rate of inflation, and other economic factors [4]. One of the securities is a risk-free bank account whose value process \(S_0(t)\) is subject to the following ordinary differential equation

\[
dS_0(t) = r(t)S_0(t)dt, \quad t \geq 0, \quad S_0(0) = s_0 > 0,
\]

where \(r(t)\) is the interest rate, a deterministic function of \(t\). The other \(m\) assets are risky stocks whose price processes \(S_1(t), \cdots, S_m(t)\) satisfy the following stochastic differential equations (SDEs)

\[
\begin{align*}
\left\{ \begin{array}{ll}
    dS_i(t) &= S_i(t)\left\{ \mu_i(t)dt + \sum_{j=1}^{n+m} \sigma_{ij}(t)dW^j(t) \right\}, \quad t \geq 0, \\
    S_i(0) &= s_i > 0, \quad i = 1, 2, \cdots, m,
\end{array} \right.
\end{align*}
\]

where \(\mu_i(t), i = 1, \ldots, m\), are the appreciation rates, and \(\sigma_i(t), i = 1, \ldots, m\) are the deterministic volatility or dispersion rate of the stocks. In this paper, we assume that the appreciation rates are affine functions of the mentioned economic factors, and the factors are Gaussian processes. To be precise, denoting
Consider an agent with an initial endowment $x_0 > 0$ and an investment horizon $[0,T]$, whose total wealth at time $t \in [0,T]$ is denoted by $X(t)$. Assuming that the trading of shares is self-financed and takes place continuously, and that transaction cost and consumptions are not considered, then $X(t)$ satisfies (see, e.g., [12])

$$dX(t) = \left\{ r(t)X(t) + \sum_{i=1}^{m} \left[ \mu_i^u(t) - r(t) \right] \pi_i(t) \right\} dt + \sum_{i=1}^{m} \sum_{j=1}^{n+m} \pi_i(t) \sigma_{ij}(t) dW^j(t),$$

(2.3)

where $\pi_i(t), i = 1, 2 \cdots , m$, denote the total market value of the agent’s wealth in the $i$-th stock. We call the process $\pi(t) = (\pi_1(t), \cdots , \pi_m(t))^T, 0 \leq t \leq T$, a portfolio of the agent.

Let

$$S(t) = (S_1(t), \ldots , S_m(t))^T,$$

$$\mathcal{G}_t = \sigma(S_0(u), S(u) : u \leq t), t \geq 0.$$  

As pointed out by [21], practically, the investor can only observe the prices of assets. So, at time $t$, the information available to the investor is the past and present asset prices, equivalently, the filtration $\mathcal{G}_t$. Thus, the investor’s strategy should be based on his/her available information. Therefore, $\pi_t$ should be $\mathcal{G}_t$-measurable. To be exact, we define the following admissible portfolio.

Definition 2.1 A portfolio $\pi(\cdot)$ is said to be admissible if $\pi(\cdot) \in L_2^2(0,T; \mathbb{R}^m)$ and the SDE (2.3) has a unique solution $x(\cdot)$ corresponding to $\pi(\cdot)$. The totality of all admissible portfolios is denoted by $\Pi$.

The agent’s objective is to find an admissible portfolio $\pi(\cdot)$, among all such admissible portfolios that his/her expected terminal wealth $EX(T) = \bar{x}$, where $\bar{x} \geq x_0 e^{\int_0^T r(t) dt}$ is given a priori, so that the risk measured by the variance of the terminal wealth

$$\text{Var} X(T) := E[\{X(T) - \bar{x}\}^2] \equiv E[\{X(T) - \bar{x}\}^2]$$

is minimized. The problem of finding such a portfolio $\pi(\cdot)$ is referred to as the mean-variance portfolio selection problem. Mathematically, we have the following formulation.

Definition 2.2 The mean–variance portfolio selection problem, with respect to the initial wealth $x_0$, is formulated as a constrained stochastic optimization problem parameterized by $\bar{x} \geq x_0 e^{\int_0^T r(t) dt}$:

$$\begin{align*}
\text{minimize} \quad & \text{Var} X(T) = E[\{X(T) - \bar{x}\}^2] = E[\{X(T) - \bar{x}\}^2], \\
\text{subject to} \quad & X(0) = x_0, \quad EX(T) = \bar{x}, \\
& (X(\cdot), \pi(\cdot)) \text{ admissible}. 
\end{align*}$$

(2.5)

The problem is called feasible (with respect to $\bar{x}$) if there is at least one admissible portfolio satisfying $EX(T) = \bar{x}$. An optimal portfolio, if it ever exists, is called an efficient portfolio strategy with respect to $\bar{x}$, and $\text{Var} X(T)$ is called an efficient point. The set of all efficient points is obtained when the parameter $\bar{x}$ varies between $[x_0 e^{\int_0^T r(s) ds}, +\infty)$.

We impose the basic assumptions of this paper.

Assumption (PD). For any $t \geq 0$, $\sigma(t)\sigma^T(t) > 0$. 

\[ 3 \]
Remark 2.1 This assumption is popular in the literatures about portfolio selection; see, for example, [3], [4], [7], [17], [21].

Let

\[ B(t) \triangleq (\mu^y(t))^T - r(t)1, \quad \sigma(t) = (\sigma_{ij}(t))_{m \times (n+m)}, \]

with 1 being a m-dimensional row vector with all its entries being 1. Then, (2.3) can be rewritten as

\[
\begin{cases}
  dX(t) = [r(t)X(t) + B(t)\pi(t)]dt + \pi(t)'\sigma(t)dW(t), \\
  X(0) = x_0.
\end{cases}
\]

(2.6)

By the definition of \( \pi \), our problem falls into the category of stochastic control based on partial information. Here, the partial information means that we cannot know the process \( y(t) \), and thus \( B(t) \). In order to design admissible strategy, we firstly need to derive the optimal estimation of \( y_t \). Let

\[
\begin{align*}
  \Gamma &= \sigma(t)\sigma^T(t) \in \mathbb{R}^{m \times m}, \\
  \Gamma &= (\Gamma_{11}(t), ..., \Gamma_{mn}(t))^T, \\
  \Sigma(t) &= \Gamma_{n+1}^2(t) = \Sigma^T(t), \\
  Y(t) &= (\log S_1(t), ..., \log S_m(t))^T \triangleq \log S(t).
\end{align*}
\]

By Itô’s formula we have

\[
dY(t) = \left( a + Ay(t) - \frac{1}{2}\Gamma(t) \right) dt + \sigma(t)dW(t), \quad Y_0 = \log S(0).
\]

Define

\[
dv(t) = \Sigma^{-1} \left[ dY(t) - \left( a + Ay(t) - \frac{1}{2}\Gamma(t) \right) dt \right],
\]

(2.7)

then \( \{ v(t), t \geq 0 \} \) is a Brownian motion under the original probability measure (Liptser and Shiryaev (2001)). The estimation of \( \hat{y}(t) \) is given by (Theorem 10.3 of [13])

\[
\begin{cases}
  d\hat{y}(t) = (d + D\hat{y}(t))dt + (\Lambda \sigma^T + \beta(t)A^T) \Sigma^{-1}dv(t), \\
  \beta(t) = D\hat{y}(t) + \beta(t)D^T + \Lambda\Lambda^T - (\Lambda \sigma^T + \beta(t)A^T)(\Sigma \Sigma^T)^{-1}(\Lambda \sigma^T + \beta(t)A^T)^T, \\
  \hat{y}(0) = y_0, \quad \beta(0) = 0.
\end{cases}
\]

(2.8)

By (2.7), a simple calculation shows that

\[
\sigma(t)dW(t) = \Sigma(t)dv(t) + A[\hat{y}(t) - y(t)].
\]

(2.9)

Substituting (2.9), we have an equivalent representation of the wealth process

\[
\begin{cases}
  dX(t) = [r(t)X(t) + B(t)\pi(t)]dt + \pi(t)\Sigma(t)dv(t), \\
  X(0) = x_0,
\end{cases}
\]

(2.10)

where

\[
\mathcal{B}(t) = (a + A\hat{y}(t))^T - r(t)1.
\]

(2.11)

This is the separation principle developed by [21], which enables us to solve problem (2.5) as if the appreciation rate \( \mu^y(\cdot) \) were known, and then replace \( \mu^y(\cdot) \) by its optimal estimation. So, (2.5) can be equivalently formulated as

\[
\begin{cases}
  \text{minimize} & E[(X(T) - \bar{x})^2], \\
  & EX(T) = \bar{x}, \\
  \text{subject to} & \pi \in \Pi, \\
  & (X(\cdot), \pi(\cdot)) \text{ satisfy (2.8)(2.10)(2.11)}. \tag{2.12}
\end{cases}
\]
By general convex optimization theory, the constrained optimal problem (2.12) with \( EX(T) = \bar{x} \) can be converted into an unconstrained one by introducing a Lagrange multiplier \( \gamma \). To be concrete, for any fixed \( \gamma \), we consider the following problem

\[
\begin{align*}
\text{minimize} & \quad E[ X(T) - \bar{x}]^2 - 2\gamma E[ X(T) - \bar{x}] = E[ X(T) - \gamma - \bar{x}]^2 - \gamma^2, \\
\text{subject to} & \quad \pi \in \Pi, \\
& \quad (X(\cdot), \pi(\cdot)) \text{ satisfy } (2.8)(2.10)(2.11),
\end{align*}
\]

which is equivalent to the following (denoting \( \bar{x} + \gamma \) by \( \alpha \) for any fixed \( \gamma \))

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} E[ X(T) - \alpha]^2, \\
\text{subject to} & \quad \pi \in \Pi, \\
& \quad (X(\cdot), \pi(\cdot)) \text{ satisfy } (2.8)(2.10)(2.11),
\end{align*}
\]

in the sense that two problems have exactly the same optimal strategy. In the following, we will call problem (2.14) the auxiliary problem of the original problem (2.12).

## 3 Optimal Policy for the Auxiliary Problem

The problem (2.14) can be viewed as an unconstrained special stochastic optimal control problem with random coefficients in system equation and zero integral term in the performance index. Different from existing results using BSDEs methodology, in this section, we intend to derive the optimal portfolio strategy from dynamic programming directly. This enables us to derive the optimal policy by solving just two linear deterministic backward ODEs and a Riccati-type forward deterministic ODE.

### 3.1 Analysis of Hamilton-Jacobi-Bellman equation

Let \( J(t, X, \dot{y}) \) denote the performance of problem (2.14) at time \( t \), with boundary condition \( J(T, X, \dot{y}) = \frac{1}{2} E[ (X(T) - \alpha)^2 ] \). Then, it is evident that the following HJB equation is satisfied

\[
\min_{\pi \in \Pi} (LJ)(t, X, \dot{y}) = 0, \quad J(T, X, \dot{y}) = \frac{1}{2} (X - \alpha)^2,
\]

where \( L \) is the infinitesimal generator operator of the closed system (2.8)(2.10)(2.11), and the independence of \( X \) on policy \( \pi \) is suppressed.

To evaluate \( L \), first of all, by (2.8)(2.10) we have

\[
\begin{align*}
dX(t)d\dot{y}(t) &= \pi^T \Sigma \sigma(t) \cdot (\Lambda \sigma^T + \beta(t)A^T) \Lambda T^{-1} \sigma(t) \Sigma T^{-1} \sigma(t) \pi \\
&= \left( \Lambda \sigma^T + \beta(t)A^T \right)^T \Sigma T^{-1} \sigma(t) \Sigma T^{-1} \sigma(t) \pi \\
&= \left( \Lambda \sigma^T + \beta(t)A^T \right) \pi dt.
\end{align*}
\]

By Itô’s formula, it follows that

\[
LJ = J_t + J_X rX + \nabla B \pi + J_{\dot{y}} \left( d + D \dot{y} \right) + \frac{1}{2} J_{XX} \pi^T \Sigma \Sigma T \pi + J_{\dot{y} \dot{y}} \left( \Lambda \sigma^T + \beta A^T \right) \pi
\]

\[
+ \frac{1}{2} T \left( (\Lambda \sigma^T + \beta A^T) \Sigma T^{-1} (\Lambda \sigma^T + \beta A^T)^T J_{\dot{y} \dot{y}} \right),
\]

where \( J_t \) is the partial derivative of \( J \) with respect to \( t \), \( J_{XX} \) is the second order partial derivative of \( J \) with respect to \( X \), and \( J_X, J_{X \dot{y}}, J_{\dot{y} \dot{y}} \) are similarly defined. On the assumption that \( J_{XX} > 0 \), we get the following optimal strategy

\[
\pi = - (\Sigma T^{-1}) \nabla B \frac{J_X}{J_{XX}} + (\Lambda \sigma^T + \beta A^T)^T \frac{J_{X \dot{y}}}{J_{XX}}.
\]

(3.2)
which makes $LJ$ minimal. Substituting (3.2) into (3.1) leads to

$$
\begin{align*}
J_t + rXJ_x + J_y^T (d + D\hat{y}) + \frac{1}{2} Tr \left[ (\Lambda \sigma^T + \beta A^T) (\Sigma \Sigma^T)^{-1} (\Lambda \sigma^T + \beta A^T)^T J_{\hat{y}} \right] \\
- \frac{1}{2} \left[ B^T \frac{J_{xx}}{x^2} + (\Lambda \sigma^T + \beta A^T)^T J_{x\hat{y}} \right] T (\Sigma \Sigma^T)^{-1} \left[ B^T \frac{J_{xx}}{x^2} + (\Lambda \sigma^T + \beta A^T)^T J_{x\hat{y}} \right] J_{xx} = 0,
\end{align*}
$$

(3.3)

In this and the following PDEs and ODEs, the arguments $t, X, \hat{y}$ are always suppressed to simplify the notations.

Noticing that the terminal condition of $J$ is a nonhomogeneous function of $X$, in order to make (3.3) homogeneous, we set

$$
z(t) = X(t) - \alpha e^{-\int_t^T r(s) \, ds}.
$$

(4.4)

Simple calculation shows

$$
J(t, X, \hat{y}) = J(t, z + \alpha e^{-\int_t^T r(s) \, ds}, \hat{y}) \triangleq H(t, z, \hat{y}),
$$

$$
H_t = J_t - \alpha J_X T e^{-\int_t^T r(s) \, ds},
$$

$$
H_X = H_z z_X = H_z, \ H_{XX} = H_{zz}, \ H_{X\hat{y}} = H_{z\hat{y}}.
$$

Substituting $z$ and the above equalities into (3.3), we obtain that

$$
\begin{align*}
H_t + rzH_z + H_{\hat{y}}^T (d + D\hat{y}) + \frac{1}{2} Tr \left[ (\Lambda \sigma^T + \beta A^T) (\Sigma \Sigma^T)^{-1} (\Lambda \sigma^T + \beta A^T)^T H_{\hat{y}} \right] \\
- \frac{1}{2} \left[ B^T \frac{H_{xx}}{x^2} + (\Lambda \sigma^T + \beta A^T)^T H_{x\hat{y}} \right] T (\Sigma \Sigma^T)^{-1} \left[ B^T \frac{H_{xx}}{x^2} + (\Lambda \sigma^T + \beta A^T)^T H_{x\hat{y}} \right] H_{xx} = 0,
\end{align*}
$$

(3.5)

By the special structure of (3.5), the following separation form of $H(t, z, \hat{y})$ is taken

$$
H(t, z, \hat{y}) = \frac{1}{2} f(t, \hat{y}) z^2, \quad \text{with } f(T, \hat{y}) = 1 \quad \text{for all } \hat{y},
$$

(3.6)

whose reasonableness will be proved in Theorem 3.1. Therefore, the optimal control (3.2) has the following structure

$$
\pi = - (\Sigma \Sigma^T)^{-1} \left[ B^T + (\Lambda \sigma^T + \beta A^T)^T \frac{\partial \ln f}{\partial \hat{y}} \right] z,
$$

which is linear in $z$, and (3.5) is equivalent to

$$
\begin{align*}
\frac{1}{2} \frac{\partial f}{\partial t} z^2 + rfz^2 + \frac{1}{2} \frac{\partial f}{\partial \hat{y}} \left( d + D\hat{y} \right) z^2 + \frac{1}{2} Tr \left[ (\Lambda \sigma^T + \beta A^T) (\Sigma \Sigma^T)^{-1} (\Lambda \sigma^T + \beta A^T)^T \frac{\partial f}{\partial \hat{y}} \right] z^2 \\
- \frac{1}{2} \left[ B^T z + (\Lambda \sigma^T + \beta A^T)^T \frac{\partial \ln f}{\partial \hat{y}} \right] z (\Sigma \Sigma^T)^{-1} \left[ B^T z + (\Lambda \sigma^T + \beta A^T)^T \frac{\partial \ln f}{\partial \hat{y}} \right] f = 0,
\end{align*}
$$

(3.7)

Clearly, if $f(t, \hat{y})$ solves the following PDE

$$
\begin{align*}
\frac{\partial f}{\partial t} + 2rf + \frac{\partial f}{\partial \hat{y}} \left( d + D\hat{y} \right) + Tr \left[ (\Lambda \sigma^T + \beta A^T) (\Sigma \Sigma^T)^{-1} (\Lambda \sigma^T + \beta A^T)^T \frac{\partial f}{\partial \hat{y}} \right] \\
- \left[ B^T + (\Lambda \sigma^T + \beta A^T)^T \frac{\partial \ln f}{\partial \hat{y}} \right] T (\Sigma \Sigma^T)^{-1} \left[ B^T + (\Lambda \sigma^T + \beta A^T)^T \frac{\partial \ln f}{\partial \hat{y}} \right] f = 0,
\end{align*}
$$

(3.8)

then $H(t, z, \hat{y})$ has the explicit form of (3.6).
\subsection{Optimal Policy}

Notice that the left hand side of the first equation in (3.8) is linear in $f, \frac{\partial f}{\partial t}, \tilde{y}, \frac{\partial^2 f}{\partial \tilde{y}^2}$, and quadratic in $\frac{\partial \ln f}{\partial \tilde{y}}$. Therefore, we assume that $f$ has the following expression

$$ f(t, \tilde{y}) = \exp \left\{ p(t) + q^T(t)\tilde{y} + \tilde{y}^T G(t)\tilde{y} \right\}, \quad (3.9) $$

with $p(t) \in \mathbb{R}, q(t) \in \mathbb{R}^n, G(t) \in \mathbb{S}^{n \times n}$ to be specified later. Here, $\mathbb{S}^{n \times n}$ denotes the set of all symmetric $n \times n$ real matrices. The form (3.9) of $f$ enables us to get an equivalent equation that is independent of $f$ and is only a quadratic function of $\tilde{y}$. Fixing the coefficients of the obtained equation to be zero, we can determine $p, q, G$ by solving several equations. Thus, we may prove that $H$ given in (3.6) satisfied the HJB equation (3.1), indeed. Therefore, we have the following theorem.

**Theorem 3.1** For problem (2.14), the optimal strategy is given by

$$ \pi(t) = -\left(\Sigma(t)\Sigma^T(t)\right)^{-1} \left( a^T - r(t)1 + (\Lambda\sigma^T(t) + \beta(t)A^T)^T q(t) + (A + (\Lambda\sigma^T(t) + \beta(t)A^T)^T G(t)) \tilde{y}(t) \right) \left( X(t) - \alpha e^{-fT}r(s)ds \right), \quad (3.10) $$

where $\beta(t), q(t), G(t)$ are the unique solutions to the second equation of (2.8) and following ODEs, respectively,

$$ \begin{cases} \frac{dq}{dt} + [q^T D + d^T G] - 2(\sigma^T - r 1)(\Sigma\Sigma^T)^{-1} A \\ - 2(\sigma^T - r 1)(\Sigma\Sigma^T)^{-1}(\Lambda\sigma^T + \beta A^T)^T G - 2q^T(\Lambda\sigma^T + \beta A^T)(\Sigma\Sigma^T)^{-1} A = 0, \quad (3.11) \\
\frac{dG}{dt} - A^T(\Sigma\Sigma^T)^{-1} A - [A^T(\Sigma\Sigma^T)^{-1}(\Lambda\sigma^T + \beta A^T)^T + D^T] G \\ - G \left[ (\Lambda\sigma^T + \beta A^T)(\Sigma\Sigma^T)^{-1} A + D \right] = 0, \quad (3.12) \end{cases} $$

Proof. Bearing the form (3.9) of $f$ in mind, simple calculation shows that

$$ \begin{align*}
\frac{\partial f}{\partial \tilde{y}} &= f(q + G\tilde{y}), \\
\frac{\partial^2 f}{\partial \tilde{y}^2} &= f(qq^T + \tilde{y}q^T G + G\tilde{y}G^T + G\tilde{y}\tilde{y}^T G + G), \\
\frac{\partial \ln f}{\partial \tilde{y}} &= q + G\tilde{y}, \\
\frac{\partial f}{\partial t} &= f \left( \frac{dp(t)}{dt} + \frac{dq(t)}{dt} \tilde{y} + \tilde{y}^T G(t) \frac{dG(t)}{dt} \tilde{y} \right). 
\end{align*} $$

Therefore, (3.8) is equivalent to

$$ \begin{cases} \frac{dp}{dt} + \frac{dq}{dt} \tilde{y} + \tilde{y}^T \frac{dG}{dt} \tilde{y} + 2r + (q + G\tilde{y})^T(d + D\tilde{y}) \\ + Tr \left[ (\Lambda\sigma^T + \beta A^T)(\Sigma\Sigma^T)^{-1} \left( \Lambda\sigma^T + \beta A^T \right)^T q(t) + \tilde{y}q^T G + G\tilde{y}q^T + G\tilde{y}\tilde{y}^T G + G \right] \\ - \left[ B^T + (\Lambda\sigma^T + \beta A^T)^T q(t) + G\tilde{y} \right]^T (\Sigma\Sigma^T)^{-1} \left[ B^T + (\Lambda\sigma^T + \beta A^T)^T q(t) + G\tilde{y} \right] = 0, \quad (3.13) \\
p(T) + q^T(T)\tilde{y}(T) + \tilde{y}^T(T)G(T)\tilde{y}(T) = 0, 
\end{cases} $$

which is equivalent to

$$ \begin{cases} \frac{dp}{dt} + \frac{dq}{dt} \tilde{y} + \tilde{y}^T \frac{dG}{dt} \tilde{y} + 2r + (q + G\tilde{y})^T(d + D\tilde{y}) \\ + Tr \left[ \left( \Lambda\sigma^T + \beta A^T \right)^T \left( \Sigma\Sigma^T \right)^{-1} \left( \Lambda\sigma^T + \beta A^T \right)^T G \right] \\ - \left[ (a + A\tilde{y}) - r 1 \right]^T \left( \Sigma\Sigma^T \right)^{-1} \left[ (a + A\tilde{y}) - r 1 \right]^T \\ - 2\left[ (a + A\tilde{y})^T - r 1 \right] \left[ \Sigma \Sigma^T \right]^{-1} \left[ (\Lambda\sigma^T + \beta A^T)^T q(t) + G\tilde{y} \right] = 0, \quad (3.14) \\
p(T) + q^T(T)\tilde{y}(T) + \tilde{y}^T(T)G(T)\tilde{y}(T) = 0, 
\end{cases} $$

which is equivalent to
Thus it is admissible. Therefore, (satisfies)

\[ \left( \mathcal{G} \right) \] have unique solution, respectively. In fact, it is known that the second equation of (3.15) is equal to (3.9) exactly solves (3.8). Furthermore, by analysis in above subsection, we can conclude that \( H \) defined in (3.6) solves (3.3). Notice that

\[ J_{XX}(t) = H_{zz}(t) = f(t, \dot{y}) = \exp \left\{ p(t) + q(t)^T \dot{y} + \dot{y}^T G(t) \dot{y} \right\} > 0. \]

Thus, \( H \) defined in (3.6) satisfies HJB equation (3.1). Clearly, (3.2) is equal to (3.10). In the end, we need only to confirm that (3.10) is admissible. By classic filtering theory, \( G_t \) is equal to the \( \sigma \)-algebra generated by innovation process \( \{ v_s, s \leq t \} \) (see for example [11]). Clearly, we have (3.10) is \( \sigma(v_s, s \leq t) \)-adapted, and thus it is admissible. Therefore, (3.10) is the optimal strategy, which make the \( E \left[ X(T) - \alpha \right]^2 \) minimal. This completes the proof.

To this end, we give a brief discussion about the solvability in theory of (2.8)(3.11)(3.12). Clearly, \( \beta(t) \) satisfies

\[ \dot{\beta} = (D - \Lambda \sigma^T \Gamma^{-1} A) \beta + \beta(t) \left( D^T - A^T \Gamma^{-1} \sigma A^T \right) - \beta A^T \Gamma^{-1} A \beta + \Lambda \Lambda^T - \Lambda \sigma^T \Gamma^{-1} \sigma A^T \]
with $\beta(0) = 0$. Let $\bar{\beta}(s) = \beta(t), s = T - t$, then it follows
\[-\frac{d\bar{\beta}}{ds} = (D - \Lambda\sigma^T\Gamma^{-1}A)\bar{\beta} + \bar{\beta}(D^T - A^T\Gamma^{-1}\sigma\Lambda^T) - \bar{\beta}A^T\Gamma^{-1}A\bar{\beta} + \Lambda\Lambda^T - \Lambda\sigma^T\Gamma^{-1}\sigma\Lambda^T\]
with $\bar{\beta}(T) = 0$. By known result (see for example Anderson and Moore (1971)), $\bar{\beta}(s)$ can be represented as
\[\bar{\beta}(s) = K(s)L^{-1}(s),\]
where $K(s), L(s)$ are defined as
\[
\begin{bmatrix}
K \\
L
\end{bmatrix} =
\begin{bmatrix}
D^T - A^T\Gamma^{-1}\sigma\Lambda^T & -A^T\Gamma^{-1}A \\
-\Lambda\Lambda^T + \Lambda\sigma^T\Gamma^{-1}\sigma\Lambda^T & -D + \Lambda\sigma^T\Gamma^{-1}A
\end{bmatrix}
\]
Therefore,
\[\beta(t) = \bar{\beta}(s) = K(T - t)L^{-1}(T - t).\]
Clearly, (3.12) is a Lyapunov differential equation, which is solved by introducing the following operator
\[\text{Vec}(G(t)) = \begin{bmatrix} G(t)^{(1)T}, G(t)^{(2)T}, ..., G(t)^{(n)T} \end{bmatrix}^T \triangleq G(t) \in \mathbb{R}^{n^2},\]
where $G(t)^{(i)T}$ is the transpose of $i$-th column of $G$. Clearly,
\[\dot{G}(t) = P^{G_1}(t)G(t) + P^{G_2}(t), \quad G(T) = 0,\]
where
\[
P^{G}(t) = I \otimes \left( (\Lambda\sigma^T + \beta(t)A^T)(\Sigma(t)\Sigma^T(t))^{-1}A - \frac{1}{2}D \right)^T + \left( (\Lambda\sigma^T(t) + \beta(t)A^T)(\Sigma(t)\Sigma^T(t))^{-1}A - \frac{1}{2}D \right) \otimes I,
\]
\[
P^{G_2}(t) = \text{Vec} \left( A^T(\Sigma^2)^{-1}A \right).
\]
Let $\Pi(s) = G_{T-s}$. Then
\[-\frac{d\Pi}{ds} = -P^{G_1}(T-s)\Pi(s) - P^{G_2}(T-s), \quad \Pi_0 = 0, \quad (3.18)\]
Therefore,
\[G(t) = \Pi(s) = -\int_0^s \Phi(s, \tau)P^{G_2}(T-\tau)d\tau,\]
where $\Phi(\cdot, \cdot)$ is the fundamental matrix of (3.18). Thus $G(t) = \text{Vec}^{-1}(G(t))$. At last, (3.11) and (3.15) can be easily solved by the linearity of the equations.

4 Efficient Frontier

In this section, we proceed to derive the efficient frontier for the original portfolio selection problem under partial information. To begin with, we prove a lemma which shows the feasibility of the original problem.

Lemma 4.1 Problem (2.5) is feasible, and the minimal mean-variance of the terminal wealth process is finite.
Proof. The proof follows directly from results of Section 5 in [21]. In the language of [21], (2.10) can be rewritten as

\[
\begin{align*}
    dX(t) &= [r(t)X(t) + \mathcal{B}(t)\Sigma(t)Z(t)]dt + Z(t)dv_t, \\
    X(T) &= v
\end{align*}
\]

where \( v \) is defined by Theorem 5.4 in [21] satisfying \( Ev = \bar{x} \), and

\[
Z(t) = (\Sigma^T(t))^{-1}\pi(t).
\]

Clearly, \( \mathcal{G}_t \) is equivalent to the \( \sigma \)-algebra generated by innovation process \( \{v_u, u \leq t\} \). By general BSDEs theory, (4.1) has a unique \( \mathcal{G}_t \)-adapted, square integrate solution \( (X(\cdot), Z(\cdot)) \). Therefore, problem (2.5) is feasible because \( \Sigma^T(x)Z(t) \) is a feasible strategy. On the other hand, by theorem 5.6 of [21], we know that the minimal mean-variance at the terminal time point is finite.

Now, we state our main theorem.

**Theorem 4.1** The efficient strategy of Problem (2.5) with the terminal expected wealth constraint \( EX(T) = \bar{x} \) is given by

\[
\pi(t) = - (\Sigma(t)\Sigma^T(t))^{-1} \left[ a^T - r(t)1 + (\Lambda \sigma^T(t) + \beta(t)A^T)q(t) \right. \\
+ \left. (A + (\Lambda \sigma^T(t) + \beta(t)A^T)G(t))\hat{y}(t) \right] \left( X(t) - (\bar{x} + \gamma^*)e^{-\int_0^T r(s)ds} \right).
\]

Here, \( \beta(t), q(t), G(t) \) solve equation (2.8)-(3.11) respectively, and \( \gamma^* \) is given by

\[
\gamma^* = \frac{\bar{x} - x_0e^{\int_0^T r(s)ds}E[e^{2\xi_T}]}{e^{\int_0^T r(s)ds}E[e^{2\xi_T}] - 1}E[e^{2\xi_T}],
\]

where \( \xi_T \) is given by

\[
\xi_T = \int_0^T \left[ r(s) - \mathcal{B}(s)(\Sigma \Sigma^T)^{-1}(V(s) + U(s)\hat{y}(s)) \right] ds + \int_0^T [V(s) + U(s)\hat{y}(s)]^T \Sigma^{-1} dv(s) \\
- \frac{1}{2} \int_0^T \left[ [V(s) + U(s)\hat{y}(s)]^T \Sigma^{-1} \right]^2 ds,
\]

and \( V(t) = a^T - r(t)1 + (\Lambda \sigma^T(t) + \beta(t)A^T)q(t) \), \( U(t) = A + (\Lambda \sigma^T(t) + \beta(t)A^T)G(t) \). Moreover, the efficient frontier is given by

\[
\frac{1}{2} \left[ x_0 - (\bar{x} + \gamma^*)e^{-\int_0^T r(s)ds} \right]^2 E[e^{2\xi_T}] - \frac{1}{2} \gamma^*.
\]

Proof. By Lemma 4.1, we know that the constraint Problem (2.5) is feasible, and its minimal terminal mean-variance \( J^* \) is finite. This means that

\[
J^* = \max_{\gamma \in R} \min_{\pi \in \Pi} \left\{ \frac{1}{2} E|X(T) - \bar{x}|^2 - \gamma[E X(T) - \bar{x}] \right\} < \infty
\]

where the equality is true by general convex constraint optimization theory (See, for example, [10].). By Theorem 3.1, the wealth equation (2.10) evolves as

\[
\begin{align*}
    dX(t) &= \left\{ r(t)X(t) - \mathcal{B}(t)(\Sigma(t)\Sigma^T(t))^{-1} [V(t) + U(t)\hat{y}(t)] \left( X(t) - \alpha e^{-\int_0^T r(s)ds} \right) \right\} dt \\
    &\quad - \left( X(t) - \alpha e^{-\int_0^T r(s)ds} \right) [V(t) + U(t)\hat{y}(t)]^T (\Sigma(t)\Sigma^T(t))^{-1} \Sigma(t)dv(t).
\end{align*}
\]
In terms of $z$, this equation is
\[
\begin{align*}
\{ & dz(t) = \{ r(t) - \overrightarrow{B}(t)(\Sigma^T)^{-1} (V(t) + U(t) \hat{y}(t)) \} z(t) dt \\
& - z(t) [V(t) + U(t) \hat{y}(t)]^T \Sigma^{-1} d\nu(t), \\
& z(0) = x_0 - \alpha e^{-\int_0^T r(s) ds} \triangleq z_0. \}
\end{align*}
\]

Clearly,
\[
z(t) = z_0 \exp \left\{ \int_0^t \left[ r(s) - \overrightarrow{B}(s)(\Sigma^T)^{-1} (V(s) + U(s) \hat{y}(s)) \right] ds \\
+ \int_0^t [V(s) + U(s) \hat{y}(s)]^T \Sigma^{-1} d\nu(s) - \frac{1}{2} \int_0^t \left[ [V(s) + U(s) \hat{y}(s)]^T \Sigma^{-1} \right] ds \right\}.
\]

Thus
\[
E[z(T)]^2 = z_0^2 E \left[ e^{2\xi_T} \right],
\]
where $\xi_T$ is defined in (4.3). Notice that
\[
\frac{1}{2} E[z(T)]^2 = \frac{1}{2} E[X(T) - \bar{x}]^2 - \gamma [EX(T) - \bar{x}] + \frac{1}{2} \gamma^2.
\]

For any fixed $\gamma$,
\[
\min_{\pi \in \Pi} \left\{ \frac{1}{2} E[X(T) - \bar{x}]^2 - \gamma [EX(T) - \bar{x}] \right\} = \frac{1}{2} \left[ x_0 - (\bar{x} + \gamma) e^{-\int_0^T r(s) ds} \right]^2 E \left[ e^{2\xi_T} \right] - \frac{1}{2} \gamma^2 \triangleq \mathbb{J}(\gamma). \tag{4.6}
\]

To obtain the optimal mean-variance value and the optimal portfolio strategy of Problem (2.5), we should maximize (4.6) over $\gamma$ within $R$, and the finiteness is ensured by (4.5). We easily show that (4.6) attains its maximum value $\mathbb{J}(\gamma^*)$ at
\[
\gamma^* = \frac{\bar{x} - x_0 e^{\int_0^T r(s) ds}}{e^{2\int_0^T r(s) ds} E \left[ e^{2\xi_T} \right] - 1} E \left[ e^{2\xi_T} \right]. \tag{4.7}
\]

And we can assert that
\[
e^{2\int_0^T r(s) ds} E \left[ e^{2\xi_T} \right] - 1 \neq 0.
\]

If this is not true, the optimal cost will be infinite, which contradicts (4.5). \hfill \Box

**References**

[1] B.D.O. Anderson, J.B. Moore. (1971). *Linear optimal control*. Prentice-Hall,Inc.,Englewood Cliffs,N.J.

[2] T.R. Bielecki, H. Jin, S.R. Pliska, X.Y. Zhou. (2005). Continuous-Time Mean-Variance Portfolio Selection with Bankruptcy Prohibition. *Mathematical Finance*, 15, 213-244.

[3] T.R. Bielecki, S.R. Pliska. (1999). Risk-sensitive dynamic asset management. *Applied Mathematics and Optimization*, 39, 337-360.

[4] T.R. Bielecki, S.R. Pliska. (2004). Risk-sensitive ICAPM with application to fixed-income management. *IEEE Transactions on Automatic Control*, 49(3), 420-432.

[5] D. Li, W.L. Ng. (2000): Optimal Dynamic Portfolio Selection: Multi-period Mean-Variance Formulation,*Mathematical Finance*, vol.10, 387-406.
[6] X. Li, X.Y. Zhou, A.E.B. Lim. (2001). Dynamic mean-variance portfolio selection with no-shorting constraints. SIAM Journal on Control and Optimization, vol.40, 1540-1555.

[7] X. Li, X.Y. Zhou. (2006). Continuous-time mean-variance efficiency: The 80% rule. Annals of Applied Probability, vol.16, 1751-1763.

[8] A.E.B. Lim. (2004). Quadratic Hedging and Mean–Variance Portfolio Selection with Random Parameters in an Incomplete Market. Mathematics of Operations Research, vol.29, 132-161.

[9] A.E.B. Lim, X.Y. Zhou. (2002). Mean-Variance Portfolio Selection with Random Parameters in a Complete Market, Mathematics of Operations Research, vol.27, 101-120.

[10] D.G. Luenberger. (1968). Optimization by vector space method. John Wiley, New York.

[11] G. Kallianpur. (1980). Stochastic filtering theory. Springer-Verlag New York Inc.

[12] I. Karatzas, S.E. Shreve. (1991). Methods of Mathematical Finance. New York: Springer-Verlag.

[13] R.S. Liptser, A.N. Shiryaev. (2001). Statistics of random processes: I general theory. Springer.

[14] H. Markowitz. (1952). Portfolio Selection. Journal of Finance, vol.7, 77-91.

[15] H. Markowitz. (1959). Portfolio selection: efficient diversification of investment. John Wiley & Sons, New York.

[16] R. Merton. (1971). Optimum consumption and portfolio rules in a continuous time model. Journal of Economic Theory, 3, 373-413.

[17] H. Nagai, S.G. Peng. (2002). Risk-sensitive dynamic portfolio optimization with partial information on infinite time horizon. The Annals of Applied Probability, vol.12, 173-195.

[18] S.R. Pliska. (1986). A stochastic calculus model of continuous trading: optimal portfolios. Mathematics of Operations Research, 11, 371-384.

[19] W.F. Sharpe. (1964). Capital asset prices: a theory of market equilibrium under conditions of risk. J. of Finance, 19, 425-442.

[20] M.C. Steinbach. (2001). Markowitz Revisited: Mean-Variance Models in Financial Portfolio Analysis. SIAM Review, vol.43, 31-85.

[21] J. Xiong, X.Y. Zhou. (2007). Mean-variance portfolio selection under partial information. SIAM Journal on Control and Optimization, vol.46, 156-175.

[22] J.M. Yong, X.Y. Zhou. (1999). Stochastic controls: Hamiltonian systems and HJB equations. New York: Springer.

[23] X.Y. Zhou, D. Li. (2000). Continuous-Time Mean-Variance Portfolio Selection: A Stochastic LQ Framework. Applied Mathematics and Optimization, vol.42, 19-33.

[24] X.Y. Zhou, G. YIN. (2003). Markowitz’s Mean-Variance Portfolio Selection with Regime Switching: A Continuous Time Model. SIAM Journal on Control and Optimization, 42, 1466-1482.