ON HURWITZ TRANSFORMATIONS

M. HAGE HASSAN\textsuperscript{1} and M. KIBLER\textsuperscript{2}

\textsuperscript{1}Universit\'e Libanaise
Facult\'e des Sciences, Section 1
Hadath, Beyrouth, Lebanon

\textsuperscript{2}Institut de Physique Nucl\'eaire de Lyon
IN2P3 - CNRS et Universit\'e Claude Bernard
43, Boulevard du 11 Novembre 1918
69622 Villeurbanne Cedex, France

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ABSTRACT

A bibliography on the Hurwitz transformations is given. We deal here, with some details, with two particular Hurwitz transformations, viz, the $\mathbb{R}^4 \rightarrow \mathbb{R}^3$ Kustaanheimo-Stiefel transformation and its $\mathbb{R}^8 \rightarrow \mathbb{R}^5$ compact extension. These transformations are derived in the context of Fock-Bargmann-Schwinger calculus with special emphasis on angular momentum theory.

1. Introduction

The subject of this paper differs from the one (Hurwitz transformations and quantum groups) of the oral presentation given by one of us (M. K.) at the workshop. In fact, although very interesting, the use of non-bijective canonical transformations (like the Hurwitz transformations) in connection with quantum groups is a rather limited subject as far as the Hurwitz factorization problem is concerned. Therefore, the present work is devoted to some non-bijective canonical transformations directly connected to the Hurwitz factorization problem.
A prototype of non-bijective canonical transformations is provided by the (now so-called) Kustaanheimo-Stiefel (KS) transformation [1]. Indeed, this transformation is inherent to the Hopf fibration $S^3/S^1 = S^2$ [2] and is closely related to the theory of Cartan spinors [1,3,4] and, therefore, to the theory of Hamilton quaternions. Since the basic works of Refs. [1,4], the KS transformation and more general non-bijective canonical transformations have been the object of numerous studies [5-23]. In particular, the algebra of (ordinary) quaternions [1,6,7,9,11,12] and, more generally, the Cayley-Dickson algebras of (ordinary and hyperbolic) hypercomplex numbers [10,16] proved to be an appropriate framework for deriving quadratic transformations which extend the Levi-Civita [1] and the KS transformations. In addition, it has been shown [22] that the use of Cayley-Dickson algebras also allows to generate non-quadratic transformations as for example the Fock [24] (stereographic) transformation.

According to Lambert and Kibler [16], an Hurwitz transformation is defined as follows. Let $\mathcal{A}(c)$ be a Cayley-Dickson algebra of dimension $2m$ and $j$ an anti-involution of $\mathcal{A}(c)$. (The abbreviation $(c)$ stands for $(c_1)$, $(c_1, c_2)$, $(c_1, c_2, c_3)$, ... for the algebras of complex numbers, quaternions, octonions, ..., respectively, where the $c_i$’s are $\pm 1$.) The map

$$\mathcal{A}(c) \to \mathcal{A}(c) : u \mapsto x = u j(u)$$ \hspace{1cm} (0a)

defines a (right) Hurwitz transformation which can be described in matrix form by

$$x = A(u) \varepsilon u$$ \hspace{1cm} (0b)

where $\varepsilon$ is a $2m \times 2m$ matrix describing the anti-involution $j$ and $A(u)$ a matrix generalizing the ones encountered in the Hurwitz factorization problem which originally corresponds to the cases $(2m = 2 ; c_1 = -1)$, $(2m = 4 ; c_1 = -1, c_2 = -1)$, $(2m = 8 ; c_1 = -1, c_2 = -1, c_3 = -1)$. The KS transformation is associated to $(2m = 4 ; c_1 = -1, c_2 = -1)$. It is to be noticed that, besides the Hurwitz transformations defined by equation (0), there are other quadratic transformations (the so-called quasi-Hurwitz and pseudo-Hurwitz transformations [16]) which can be obtained by replacing $\varepsilon$ by an euclidean or pseudo-euclidean metric matrix.
Furthermore, non-quadratic transformations can be obtained [22] when replacing \( A(u) \) by \( A(u)^N \) in equation (0); the geometrical interpretation of the latter transformations is an interesting problem. The reader may consult the papers by Lambert and Randriamihamison in these proceedings for other transformations in an enlarged Hurwitz context.

It is the aim of this work to generate the (\( R^4 \rightarrow R^3 \)) KS transformation and its \( R^8 \rightarrow R^5 \) compact extension, corresponding to the Hurwitz transformation with \( (2m = 8; c_1 = -1, c_2 = -1, c_3 = -1) \), in the framework of Fock-Bargmann-Schwinger [25-27] calculus. In this framework, the vector fields associated to the \( R^8 \rightarrow R^5 \) and \( R^4 \rightarrow R^3 \) transformations acquire a nice significance in terms of angular momenta. The material reported here thus sheds light on the Hurwitz transformations in a new direction. In particular, it might be useful for the definition of Hurwitz transformations in the context of quantum matrix groups à la Woronowicz [28]; as a matter of fact, the presentation adopted here makes use of a matrix realization of \( SU(2) \) that might be easily deformed in order to switch to the quantum group \( SU_q(2) \). From a physical point of view, the formalism described in the present paper is clearly of interest for the connection between the hydrogen atom (in \( R^3 \) or \( R^5 \)) and the isotropic harmonic oscillator (in \( R^4 \) or \( R^8 \)) as well as for the separation between vibration and rotation motions. The reader may consult the paper by Campigotto in these proceedings for a recent application of the KS transformation to the hydrogen-oscillator connection.

The paper is organized in the following way. Section 2 is devoted to some preliminaries. In section 3 and 4, we deal with the compact Hurwitz transformations \( R^4 \rightarrow R^3 \) and \( R^8 \rightarrow R^5 \). Appendices 1 and 2 contain some elements on Fock-Bargmann-Schwinger calculus. Finally, some differential aspects of the considered Hurwitz transformations are relegated to appendices 3 and 4.

The present work constitutes a revision and an extension of the one contained in a preliminary note published in Ref. [23].

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2. Preliminaries

A basic ingredient of this work is the generating function $\Psi$ for the rotation matrix elements $D^j(\hat{U})_{mm'}$ of the group $SU(2)$. To establish the notation, let $\Psi$ be the function defined by (cf. Bargmann [26])

$$\Psi(a_1, a_2, b_1, b_2; z_1, z_2) = \exp(\bar{a}Ub) \quad \bar{a} = (a_1 a_2) \quad U = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

(1)

The matrix $\hat{U} = r^{-1/2}U$, with $r = z_1\bar{z}_1 + z_2\bar{z}_2$, belongs to $SU(2)$ and $\Psi$ can be expanded as

$$\Psi(a_1, a_2, b_1, b_2; z_1, z_2) = \sum_{jm'm'} \Phi_{jm}(a_1, a_2) r^j D^j(\hat{U})_{mm'} \Phi_{jm'}(b_1, b_2)$$

(2)

where the function $\Phi_{jm}$ is defined in the Fock-Bargmann space $F_2$ via

$$\Phi_{jm}(\zeta_1, \zeta_2) = \frac{\zeta_1^{j+m}\zeta_2^{-j-m}}{\sqrt{(j+m)!(j-m)!}}$$

(3)

In the main body of this paper we shall use the coordinates $\sqrt{r}, \psi, \theta, \varphi$ of $\mathbb{R}^4$ defined through

$$z_1 = \sqrt{r} \cos \frac{\theta}{2} \exp(i\frac{\psi + \varphi}{2}) \quad z_2 = \sqrt{r} \sin \frac{\theta}{2} \exp(i\frac{\psi - \varphi}{2})$$

(4)

i.e., the Euler-angle (quaternionic) coordinates.

The whole philosophy of our approach can be summarized as follows. Starting from the relation

$$P_q q' = 0$$

(5)

valid for independent coordinates $q$ and $q'$ (with $P_q = \frac{\partial}{\partial q}$), we look for invariants $\{q'\}$ with respect to some group with infinitesimal operators $\{P_q\}$. The coordinates $\{q, q'\}$ chosen are precisely those simple combinations of the quaternionic coordinates of $\mathbb{R}^4$ which are especially adapted to an $SO(4)$ presentation of the theory of angular momentum.
3. The $\mathbb{R}^4 \to \mathbb{R}^3$ case

3.1. The $\mathbb{R}^4 \to \mathbb{R}^3$ map. Let $(\sqrt{r}, \psi, \theta, \varphi)$ be the curvilinear coordinates of $\mathbb{R}^4$. We choose

$$x_1 = r \sin \theta \cos \varphi \quad x_2 = r \sin \theta \sin \varphi \quad x_3 = r \cos \theta$$

(6)

for the coordinates of $\mathbb{R}^3$. Therefore, our starting point is

$$P_\psi x_i = 0 \quad (i = 1, 2, 3)$$

(7)

for the choice of coordinates $(x_1, x_2, x_3, \psi)$ in $\mathbb{R}^4$. Since $P_\psi \mathcal{D}^j(\hat{U})_{mm'} = 0$ for $m = 0$, it is natural to consider the series

$$S_b = \sum_{\ell m'} r^\ell \mathcal{D}^\ell(\hat{U})_{0m'} \Phi_{\ell m'}(b_1, b_2)$$

(8)

especially in view of its importance for generating basis functions of the group $SO(3)$.

The expression $S_b$ can be derived by integrating in the Fock-Bargmann space $\mathcal{F}_2$ the function $\Psi$, in the form given by (2), multiplied by a convenient factor. As a point of fact, it is immediate to show that

$$\int_{\mathbb{C}} \int_{\mathbb{C}} \exp(a_1 a_2) \Psi(a_1, a_2, b_1, b_2; z_1, z_2) d\mu(a_1) d\mu(a_2) = S_b$$

(9)

by using the basic integral of Appendix 1. Now, we pass to the form (1) for the function $\Psi$ and easily obtain

$$\int_{\mathbb{C}} \int_{\mathbb{C}} \exp(\bar{a}_1 \bar{a}_2) \Psi(a_1, a_2, b_1, b_2; z_1, z_2) d\mu(a_1) d\mu(a_2) = \exp[(z_1 b_1 + z_2 b_2) (-\bar{z}_2 b_1 + \bar{z}_1 b_2)]$$

(10)

after repeated use of the composition rule of Appendix 1. As an intermediate result, from equations (8-10) we have

$$\exp[(z_1 b_1 + z_2 b_2) (-\bar{z}_2 b_1 + \bar{z}_1 b_2)] = \sum_{\ell m'} r^\ell \mathcal{D}^\ell(\hat{U})_{0m'} \Phi_{\ell m'}(b_1, b_2)$$

(11)
On another hand, we know that the generating function for the $SO(3) \supset SO(2)$ solid harmonics

$$
y_{\ell m}(\vec{r}) = \sqrt{\frac{2\ell + 1}{4\pi}} r^\ell \mathcal{D}^\ell(\hat{U})_{0m}
$$

is given by (cf. Schwinger [27])

$$
\exp\left( -\frac{x_1 + ix_2 b_1^2 + x_1 - ix_2 b_2^2 + x_3 b_1 b_2}{2} \right) = \sum_{\ell m} \sqrt{\frac{4\pi}{2\ell + 1}} y_{\ell m}(\vec{r}) \Phi_{\ell m}(b_1, b_2)
$$

with $\vec{r} = (x_1, x_2, x_3)$. Introducing (12) into (13) and identifying the so-obtained relation to (11), we finally get

$$
x_1 = z_1 \bar{z}_2 + \bar{z}_1 z_2 \quad x_2 = -i(z_1 \bar{z}_2 - \bar{z}_1 z_2) \quad x_3 = z_1 \bar{z}_1 - z_2 \bar{z}_2
$$

The latter relations are a simple rewriting of three of the relations defining the KS transformation. Indeed, by putting

$$
z_1 = u_1 + iu_2 \quad z_2 = u_3 + iu_4
$$

equation (14) can be rewritten in terms of the Cartesian coordinates $(u_1, u_2, u_3, u_4)$ of $\mathbb{R}^4$ as

$$
x_1 = 2(u_1 u_3 + u_2 u_4) \quad x_2 = 2(-u_1 u_4 + u_2 u_3) \quad x_3 = u_1^2 + u_2^2 - u_3^2 - u_4^2
$$

with the evident property that $r = \left( \sum_{i=1}^{3} x_i^2 \right)^{1/2} = \sum_{\alpha=1}^{4} u_{\alpha}^2$.

The $\mathbb{R}^4 \rightarrow \mathbb{R}^3$ differentiable map defined by (16) corresponds to the transformation originally introduced by Kustaanheimo and Stiefel [1] up to permutations on the $x_i$’s and the $u_{\alpha}$’s. The compatibility between (4), (6) and (14) gives to (16) its specific form owing to (15). In the terminology of Lambert and Kibler [16] and of Kibler and Winternitz [19], the transformation (16) can be identified to an Hurwitz transformation of type $(c')$ associated to a certain anti-involution of the Cayley-Dickson algebra $\mathcal{A}(-1, -1)$, which is isomorphic to the algebra of Hamilton quaternions.

Should we have chosen to work with the series

$$
S_a = \sum_{\ell m} \Phi_{\ell m}(a_1, a_2) r^\ell \mathcal{D}^\ell(\hat{U})_{m0}
$$

$$
= \int_{\mathbb{C}} \int_{\mathbb{C}} \exp(b_1 \bar{b}_2) \Psi(a_1, a_2, b_1, b_2; z_1, z_2) d\mu(b_1) d\mu(b_2)
$$

6
instead of $S_b$, we would have arrived at the transformation

$$
x'_1 = z_1 \bar{z}_2 + \bar{z}_1 z_2 = 2(u_1 u_3 - u_2 u_4)
$$

$$
x'_2 = -i(z_1 z_2 - \bar{z}_1 \bar{z}_2) = 2(u_1 u_4 + u_2 u_3)
$$

$$
x'_3 = z_1 \bar{z}_1 - z_2 \bar{z}_2 = u_1^2 + u_2^2 - u_3^2 - u_4^2
$$

(18)

which corresponds, in the notation of Ref. [16], to the Hurwitz transformation associated to the anti-involution $j_1$ of $A(-1, -1)$ through the relation $x' = A(u) \bar{z}_1 u$, up to a re-labeling in $x'$ and $u$. Equations (4) and (18) can be combined to yield

$$
x'_1 = r \sin \theta \cos \psi 
$$

$$
x'_2 = r \sin \theta \sin \psi 
$$

$$
x'_3 = r \cos \theta
$$

(19)

and thus the transformation (18) is inherent to an approach the starting point of which is

$$
P_{i} x'_i = 0 \quad (i = 1, 2, 3)
$$

(20)

for the choice of coordinates $(x'_1, x'_2, x'_3, \varphi)$ in $\mathbb{R}^4$.

Of course, the transformation (18) is equivalent to the transformation (16). Both transformations are associated to the Hopf fibration on spheres $S^3/S^1 = S^2$ with compact fiber $S^1$. The reader will find in Appendix 3 some differential aspects of the KS transformation defined by (16).

3.2. The $\mathbb{R}^4 \to \mathbb{R}^3$ vector fields. The next step is to examine, still in the framework of Fock-Bargmann-Schwinger calculus, the implications of (16) and (18) in the language of angular momentum theory. The basic relations are (see Refs. [29,30] and Appendix 2)

$$
\hat{K} \Psi = K^\dagger \Psi \quad \hat{L} \Psi = L^\dagger \Psi
$$

(21)

where $\hat{K} = \hat{K}_+, \hat{K}_z$ or $\hat{K}_-$ and $\hat{L} = \hat{L}_+, \hat{L}_z$ or $\hat{L}_-$ are the images in $\mathcal{F}_2$ of $K = K_+, K_z$ or $K_-$ and $L = L_+, L_z$ or $L_-$ in the variables $(a_1, a_2)$ and $(b_1, b_2)$, respectively. The operators $\hat{K} = (K_+, K_z, K_-)$ and $\hat{L} = (L_+, L_z, L_-)$ generate the Lie algebra of the group $SO(4)$. Let us simply recall that $\mathcal{D}^j(\hat{U})_{mm'}$ is an eigenvector of the operators $K_z$ and $L_z$ with the
eigenvalues $m$ and $m'$, respectively. The Fock-Bargmann representations of $\vec{K}$ and $\vec{L}$ are given by

\[
\hat{K}_+ = -a_1 \frac{\partial}{\partial a_2} \quad \hat{K}_z = \frac{1}{2} \left( a_1 \frac{\partial}{\partial a_1} - a_2 \frac{\partial}{\partial a_2} \right) \quad \hat{K}_- = -a_2 \frac{\partial}{\partial a_1}
\]

\[
\hat{L}_+ = +b_1 \frac{\partial}{\partial b_2} \quad \hat{L}_z = \frac{1}{2} \left( b_1 \frac{\partial}{\partial b_1} - b_2 \frac{\partial}{\partial b_2} \right) \quad \hat{L}_- = +b_2 \frac{\partial}{\partial b_1}
\]

and act on the space $\mathcal{F}_2 \otimes \mathcal{F}_2$.

We begin with the image $\hat{K}$ of $K$. From straightforward calculations, we have

\[
\hat{K}_+ \Psi = - \left( \bar{z}_1 \frac{\partial}{\partial z_2} - \bar{z}_2 \frac{\partial}{\partial z_1} \right) \Psi \quad \hat{K}_- \Psi = + \left( z_1 \frac{\partial}{\partial \bar{z}_2} - z_2 \frac{\partial}{\partial \bar{z}_1} \right) \Psi
\]

\[
\hat{K}_z \Psi = \frac{1}{2} \left( z_1 \frac{\partial}{\partial z_1} - \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + z_2 \frac{\partial}{\partial z_2} - \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \right) \Psi
\]

Therefore, from (21) we obtain the three spherical components of $\vec{K}$

\[
K_+ = + \left( z_1 \frac{\partial}{\partial z_2} - z_2 \frac{\partial}{\partial z_1} \right) \quad K_- = - \left( \bar{z}_1 \frac{\partial}{\partial \bar{z}_2} - \bar{z}_2 \frac{\partial}{\partial \bar{z}_1} \right)
\]

\[
K_z = \frac{1}{2} \left( z_1 \frac{\partial}{\partial z_1} - \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + z_2 \frac{\partial}{\partial z_2} - \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \right)
\]

With the help of (15) and of the well-known relations

\[
\frac{\partial}{\partial u_1} = \frac{1}{2} (\partial_1 - i\partial_2) \quad \frac{\partial}{\partial u_2} = \frac{1}{2} (\partial_3 - i\partial_4)
\]

where $\partial_\alpha$ stands for $\frac{\partial}{\partial u_\alpha}$, we can derive from (24) the expressions of $K_+$, $K_z$ and $K_-$ in terms of the coordinates $u_\alpha$ ($\alpha = 1, 2, 3, 4$). Finally, by introducing the Cartesian components

\[
K_1 = \frac{1}{2} (K_+ + K_-) \quad K_2 = \frac{1}{2i} (K_+ - K_-) \quad K_3 = K_z
\]

we get the vector fields

\[
K_1 = -\frac{i}{2} (u_4 \partial_1 + u_3 \partial_2 - u_2 \partial_3 - u_1 \partial_4)
\]

\[
K_2 = -\frac{i}{2} (-u_3 \partial_1 + u_4 \partial_2 + u_1 \partial_3 - u_2 \partial_4)
\]

\[
K_3 = +\frac{i}{2} (u_2 \partial_1 - u_1 \partial_2 + u_4 \partial_3 - u_3 \partial_4)
\]
defined in the real symplectic Lie algebra $sp(8, \mathbb{R})$.

In a similar fashion, starting from $\hat{L}\Psi = L^\dagger\Psi$, we would obtain the following spherical components of $\vec{L}$

$$
\begin{align*}
L_+ &= \left( z_1 \frac{\partial}{\partial z_2} - \bar{z}_2 \frac{\partial}{\partial \bar{z}_1} \right) \\
L_- &= - \left( \bar{z}_1 \frac{\partial}{\partial \bar{z}_2} - z_2 \frac{\partial}{\partial z_1} \right) \\
L_z &= \frac{1}{2} \left( z_1 \frac{\partial}{\partial z_1} - \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} - z_2 \frac{\partial}{\partial z_2} + \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \right)
\end{align*}
$$

so that

$$
\begin{align*}
L_1 &= \frac{i}{2} \left( u_4 \partial_1 - u_3 \partial_2 + u_2 \partial_3 - u_1 \partial_4 \right) \\
L_2 &= \frac{i}{2} \left( u_3 \partial_1 + u_4 \partial_2 - u_1 \partial_3 - u_2 \partial_4 \right) \\
L_3 &= \frac{i}{2} \left( u_2 \partial_1 - u_1 \partial_2 - u_4 \partial_3 + u_3 \partial_4 \right)
\end{align*}
$$

(28)

are the three Cartesian components of $\vec{L}$ expressed in the algebra $sp(8, \mathbb{R})$.

It is easy to verify that the commutation relations

$$
\begin{align*}
[K_k, K_\ell] &= i \epsilon_{k\ell m} K_m \\
[L_k, L_\ell] &= i \epsilon_{k\ell m} L_m \\
[K_k, L_\ell] &= 0
\end{align*}
$$

(30)

hold for $k, \ell, m = 1, 2, 3$. Consequently, the set $\{K_j, L_j : j = 1, 2, 3\}$ spans the Lie algebra $so(4)$ in an $su(2) \oplus su(2)$ basis, as expected from the angular momentum theory developed in an $SO(4)$ presentation (cf. Ref. [31]).

At this stage, it should be noted that the vector operators $\vec{K}$ and $\vec{L}$ are angular momenta associated to the coordinates $(x'_1, x'_2, x'_3)$ and $(x_1, x_2, x_3)$, respectively. More precisely, it is a simple matter of calculation to show that the components of $\vec{K}$ can be written

$$
\begin{align*}
K_1 &= +i \left( x'_2 \frac{\partial}{\partial x'_3} - x'_3 \frac{\partial}{\partial x'_2} \right) = -i \left( + \sin \psi \frac{\partial}{\partial \theta} + \cot \theta \cos \psi \frac{\partial}{\partial \psi} \right) \\
K_2 &= +i \left( x'_3 \frac{\partial}{\partial x'_1} - x'_1 \frac{\partial}{\partial x'_3} \right) = -i \left( - \cos \psi \frac{\partial}{\partial \theta} + \cot \theta \sin \psi \frac{\partial}{\partial \psi} \right) \\
K_3 &= -i \left( x'_1 \frac{\partial}{\partial x'_2} - x'_2 \frac{\partial}{\partial x'_1} \right) = -i \frac{\partial}{\partial \psi}
\end{align*}
$$

(31)
Similarly, the components of $\vec{L}$ satisfy

\[
L_1 = -i \left( x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2} \right) = i \left( +\sin \varphi \frac{\partial}{\partial \theta} + \cot \theta \cos \varphi \frac{\partial}{\partial \varphi} \right)
\]

\[
L_2 = -i \left( x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3} \right) = i \left( -\cos \varphi \frac{\partial}{\partial \theta} + \cot \theta \sin \varphi \frac{\partial}{\partial \varphi} \right)
\]

\[
L_3 = -i \left( x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} \right) = -i \frac{\partial}{\partial \varphi}
\] (32)

and are nothing but the components of the ordinary angular momentum in the coordinates $(x_1, x_2, x_3)$. Further, we note that in order to pass from the operator $\vec{K}$ defined by (24) to the operator $\vec{L}$ defined by (28) it is sufficient to replace $z_2$ by $\bar{z}_2$. This reflects the fact that we can pass from the coordinates $(x'_1, x'_2, x'_3)$ to the coordinates $(x_1, x_2, x_3)$ by changing $z_2$ into $\bar{z}_2$.

Finally, it should be observed that the constraint operator $X$ discussed by many authors (see, for example, Refs. [5,8,16]) is

\[
X = \frac{2}{i} K_3 = -2P_\varphi = u_2 \partial_1 - u_1 \partial_2 + u_4 \partial_3 - u_3 \partial_4
\] (33)

for the transformation (16) and

\[
X = \frac{2}{i} L_3 = -2P_\varphi = u_2 \partial_1 - u_1 \partial_2 - u_4 \partial_3 + u_3 \partial_4
\] (34)

for the transformation (18). In this respect, the constraint relation

\[
X f = 0
\] (35)

holds either for $X = (2/i)K_3$ and $f = f(x_1, x_2, x_3)$ or for $X = (2/i)L_3$ and $f = f(x'_1, x'_2, x'_3)$, where $f$ is a (one-fold) differentiable function. (Equation (35) was the starting point of our derivation of the transformations (16) and (18).) The operator $X$ generates an $so(2)$ subalgebra, with $so(2) = so(2)_K$ for $X = (2/i)K_3$ and $so(2) = so(2)_L$ for $X = (2/i)L_3$, of the symplectic Lie algebra $sp(8, \mathbb{R})$. The corresponding Lie algebra under constraints is

\[
\text{cent}_{sp(8, \mathbb{R})}(so(2))/so(2) = so(4,2) \sim su(2,2)
\] (36)
(see Kibler and Négadi [8] and Kibler and Winternitz [19]). To close this section, note that the one-form $\omega$ (see Ref. [16]) associated to the vector field $X$ can be recovered from the property $\omega\left[\frac{1}{2}\pi X\right] = 1$.

4. The $\mathbb{R}^8 \to \mathbb{R}^5$ case

We are now in a position to handle the $\mathbb{R}^8 \to \mathbb{R}^5$ case. To deal with $\mathbb{R}^8$, we start from two copies of $\mathbb{R}^4$ and choose the coordinates in the following manner. Let us consider two particles, say, 1 and 2. Let $(\psi, \theta, \varphi)$ be, in this section, the angular coordinates of the collective motion and $(x_1, x_2, \ldots, x_5)$ the five remaining coordinates necessary to completely describe, in terms of the $\mathbb{R}^4 \oplus \mathbb{R}^4$ space, the position of the two particles. In this case, equation (5) reads

$$P_\psi x_i = P_\theta x_i = P_\varphi x_i = 0 \quad (i = 1, 2, \ldots, 5)$$

(37)

Equation (37) may be taken in the form

$$K_1 x_i = K_2 x_i = K_3 x_i = 0 \quad (i = 1, 2, \ldots, 5)$$

(38)

where $K_1$, $K_2$ and $K_3$ denote the (Cartesian) coordinates of the total vector operator $\vec{K} = \vec{K}(1) + \vec{K}(2)$ for the system of the two particles. (The operators $\vec{K}(1)$ and $\vec{K}(2)$ correspond to the operator $\vec{K}$ of section 3 for the particles 1 and 2, respectively.) We call $SO(3)_K$ the Lie group whose infinitesimal generators are $K_1$, $K_2$ and $K_3$.

The function

$$\Psi_{uc} = \Psi_1 \Psi_2$$

(39)

where $\Psi_1$ and $\Psi_2$ stand for generating functions of type $\Psi$ (see section 3) associated to the particles 1 and 2, respectively, is simply the uncoupled generating function corresponding to the system of the two particles. More precisely, we take

$$\Psi_1 = \exp(\tilde{a}Ub) = \sum_{j_1m_1m_1'} \Phi_{j_1m_1}(a_1, a_2) r_1^{j_1} D^{j_1}(\hat{U})_{m_1m_1'} \Phi_{j_1m_1'}(b_1, b_2)$$

$$\Psi_2 = \exp(\tilde{c}Vd) = \sum_{j_2m_2m_2'} \Phi_{j_2m_2}(c_1, c_2) r_2^{j_2} D^{j_2}(\hat{V})_{m_2m_2'} \Phi_{j_2m_2'}(d_1, d_2)$$

(40)
where $\tilde{a}$, $U$ and $b$ are defined via equation (1) and similarly

$$\tilde{c} = (c_1 c_2) \quad V = \begin{pmatrix} z_3 & z_4 \\ -\bar{z}_4 & \bar{z}_3 \end{pmatrix} \quad d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$  \hspace{1cm} (41)

Furthermore, the rotational invariants $r_1$ and $r_2$ occurring in (40) are given by

$$r_1 = z_1 \bar{z}_1 + z_2 \bar{z}_2 \quad r_2 = z_3 \bar{z}_3 + z_4 \bar{z}_4$$  \hspace{1cm} (42)

Following Bargmann [26] and Schwinger [27], the coupled generating function $\Psi_c$ for the two-particle system can be taken as

$$\Psi_c = \int_{C^2} \int_{C^2} \exp[\alpha(\tilde{c}_1 \eta - \tilde{c}_2 \bar{\eta}) + \beta(\bar{\xi} \tilde{a}_2 - \eta \tilde{a}_1) + \gamma(\tilde{a}_1 \tilde{c}_2 - \tilde{a}_2 \tilde{c}_1)] \Psi_{uc} \, d\mu_2(a) \, d\mu_2(c)$$ \hspace{1cm} (43)

Now, let us introduce the generating function (see Ref. [27])

$$e^{\alpha(b_1 c_2 - b_2 c_1) + \beta(c_1 a_2 - c_2 a_1) + \gamma(a_1 b_2 - a_2 b_1)} = \sum_{j_1 j_2 j_3} \sum_{m_1 m_2 m_3} N_{j_1 j_2 j_3} \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{array} \right)$$

$$\alpha^{-j_1-j_2-j_3} \beta^{j_1-j_2+j_3} \gamma^{j_1+j_2-j_3} \Phi_{j_1 m_1}(a_1, a_2) \Phi_{j_2 m_2}(b_1, b_2) \Phi_{j_3 m_3}(c_1, c_2)$$

$$N_{j_1 j_2 j_3} = \left[ \frac{(j_1 + j_2 + j_3 + j + 1)!}{(-j_1 + j_2 + j_3)! (j_1 - j_2 + j_3)! (j_1 + j_2 - j_3)!} \right]^\frac{1}{2}$$ \hspace{1cm} (44)

for the $3-jm$ symbols of the group $SU(2)$ in an $SU(2) \supset U(1)$ basis. Then, the exponential in (43) can be transformed owing to (44) and the integration performed by expanding $\Psi_{uc}$ in terms of $\Phi_{jm}$’s and by using the orthonormality property of the functions $\Phi_{jm}$ (see Appendix 1). We thus obtain

$$\Psi_c = \sum_{j_1 j_2 j_3} \sum_{m_1 m_2 m_3} \sum_{m'_1 m'_2} N_{j_1 j_2 j_3} \alpha^{-j_1+j_2+j_3} \beta^{j_1-j_2+j_3} \gamma^{j_1+j_2-j_3} \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{array} \right)$$

$$r_1^{j_1} D^{j_1}(\tilde{U})_{m_1 m'_1} r_2^{j_2} D^{j_2}(\tilde{V})_{m_2 m'_2} \Phi_{j_1 m'_1}(b_1, b_2) \Phi_{j_2 m'_2}(d_1, d_2) \Phi_{j_3 m_3}(\xi, \eta)$$ \hspace{1cm} (45)

The $SO(3)_C$ rotational invariance depicted by (38) is ensured if we assume that $\alpha = \beta = 0$ in (45). Then, by taking $\alpha = \beta = 0$ and by introducing (39) and (40) into (43), the integration can be easily performed after repeated use of the composition rule of Appendix 1. This leads to

$$\Psi_c = \exp \{\gamma[(z_1 b_1 + z_2 b_2)(-\bar{z}_4 d_1 + \bar{z}_3 d_2) - (-\bar{z}_2 b_1 + \bar{z}_1 b_2)(z_3 d_1 + z_4 d_2)]\}$$ \hspace{1cm} (46)
On the other hand, we may introduce $\alpha = \beta = 0$ in (45). This yields

$$\Psi_c = \sum_{JMM'} \gamma^{2J} \Phi_{JM}(-b_2, b_1) (r_1 r_2)^J D^J(\hat{U}^\dagger \hat{V})_{MM'} \Phi_{JM'}(d_1, d_2)$$

(47)

which can be obtained equally well from (43) with $\alpha = \beta = 0$ and $\Psi_{uc}$ expanded in terms of $\Phi_{jm}$'s. Without loss of generality, we can assume that $\gamma = 1$. Therefore, equations (46) and (47) show that $\Psi_c$ is of the form (cf. (1) and (41))

$$\Psi_c = \exp(\tilde{b}' W d) \quad \tilde{b}' = (-b_2 b_1) \quad W = \begin{pmatrix} +s_1 + is_2 & s_3 + is_4 \\ -s_3 + is_4 & s_1 - is_2 \end{pmatrix} \quad d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

(48)

and we have to identify $W$ with $\hat{U}^\dagger \hat{V}$. As a result, we obtain the four non-trivial rotational invariants $x_i = 2s_i$ ($i = 1, 2, 3, 4$) given by

$$\frac{1}{2}(x_1 + ix_2) = \bar{z}_1 z_3 + z_2 \bar{z}_4 \quad \frac{1}{2}(x_3 + ix_4) = \bar{z}_1 z_4 - z_2 \bar{z}_3$$

(49)

(The particular normalization for $s_i = (1/2)x_i$ is chosen in view of having the property $r = (\sum_{i=1}^5 x_i^2)^{1/2} = \sum_{\alpha=1}^8 u_\alpha^2$, see equation (53) below.) The fifth rotational invariant $x_5$ is obtained by looking for an expression “orthogonal” to the trivial invariant

$$r = r_1 + r_2 = z_1 \bar{z}_1 + z_2 \bar{z}_2 + z_3 \bar{z}_3 + z_4 \bar{z}_4$$

(50)

Thereby, we take

$$x_5 = r_1 - r_2 = z_1 \bar{z}_1 + z_2 \bar{z}_2 - z_3 \bar{z}_3 - z_4 \bar{z}_4$$

(51)

Finally, from (49) we are left with

$$x_1 = z_1 \bar{z}_3 + \bar{z}_1 z_3 + z_2 \bar{z}_4 + \bar{z}_2 z_4$$

$$x_2 = i(z_1 \bar{z}_3 - \bar{z}_1 z_3 - z_2 \bar{z}_4 + \bar{z}_2 z_4)$$

$$x_3 = z_1 \bar{z}_4 + \bar{z}_1 z_4 - z_2 \bar{z}_3 - \bar{z}_2 z_3$$

$$x_4 = i(z_1 \bar{z}_4 - \bar{z}_1 z_4 + z_2 \bar{z}_3 - \bar{z}_2 z_3)$$

(52)

Note that the invariant $r$ depends on $x_1, x_2, x_3, x_4$ and $x_5$ since

$$r^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2$$

(53)
The use of the relation (15) for the particle 1 and of the parent relation

\begin{align*}
    z_3 &= u_5 + iu_6 \\
    z_4 &= u_7 + iu_8
\end{align*}

(54)

for the particle 2 allows us to rewrite (51) and (52) as

\begin{align*}
    x_1 &= 2(u_1u_5 + u_2u_6 + u_3u_7 + u_4u_8) \\
    x_2 &= 2(u_1u_6 - u_2u_5 - u_3u_8 + u_4u_7) \\
    x_3 &= 2(u_1u_7 + u_2u_8 - u_3u_5 - u_4u_6) \\
    x_4 &= 2(u_1u_8 - u_2u_7 + u_3u_6 - u_4u_5) \\
    x_5 &= u_1^2 + u_2^2 + u_3^2 + u_4^2 - u_5^2 - u_6^2 - u_7^2 - u_8^2
\end{align*}

(55)

that defines an \( \mathbb{R}^8 \to \mathbb{R}^5 \) map. By putting

\begin{align*}
    \vec{x} &= (x_2, x_3, x_4) & q_{10} &= u_1 & \vec{q}_1 &= (u_2, u_3, u_4) & q_{20} &= u_5 & \vec{q}_2 &= (u_6, u_7, u_8)
\end{align*}

(56)

the relations (55) can be recast in the compact form (see also Cordani [21])

\begin{align*}
    x_1 &= 2(q_{10}q_{20} + \vec{q}_1 \cdot \vec{q}_2) & \vec{x} &= 2(q_{10}\vec{q}_2 - q_{20}\vec{q}_1 - \vec{q}_1 \land \vec{q}_2) & x_5 &= q_{10}^2 + \vec{q}_1^2 - q_{20}^2 - \vec{q}_2^2
\end{align*}

(57)

which obviously reflects the fact that our approach is connected with bi-quaternions.

According to Lambert and Kibler [16], the latter map is associated to a certain Hurwitz transformation. The three corresponding constraint operators (generalizing the operator \( X \) of the KS transformation) are

\[ X_j = \frac{2}{i} K_j = \frac{2}{i} [K_j(1) + K_j(2)] \quad (j = 1, 2, 3) \]

(58)

and thus acquire a significance in terms of generalized angular momentum. In this connection, the Lie algebra of \( SO(3)_K \) identifies to what Kibler and Négadi [8] and Kibler and Winternitz [19] call a constraint Lie algebra. Finally, the one-forms

\begin{align*}
    \omega_1 &= -2(u_4du_1 + u_3du_2 - u_2du_3 - u_1du_4 + u_8du_5 + u_7du_6 - u_6du_7 - u_5du_8) \\
    \omega_2 &= -2(-u_3du_1 + u_4du_2 + u_1du_3 - u_2du_4 - u_7du_5 + u_8du_6 + u_5du_7 - u_6du_8) \\
    \omega_3 &= +2(u_2du_1 - u_1du_2 + u_3du_4 - u_4du_3 - u_6du_5 - u_5du_6 + u_8du_7 - u_7du_8)
\end{align*}

(59)
are the duals of $X_1, X_2, X_3$ in the sense that $\omega_\alpha [\frac{1}{27} X_\beta] = \delta(\alpha, \beta)$ (see Ref. [16]).

Similarly, instead of starting from (38), we could have chosen to look for the coordinates $x'_i \ (i = 1, 2, \ldots, 5)$ invariant under the group $SO(3)_L$ corresponding to the total angular momentum $\vec{L} = \vec{L}(1) + \vec{L}(2)$, where $\vec{L}(1)$ and $\vec{L}(2)$ refer to the angular momenta for the particles 1 and 2, respectively. The basic requirement would have been

$$L_1 x'_i = L_2 x'_i = L_3 x'_i = 0 \quad (i = 1, 2, \ldots, 5) \quad (60)$$

a set of constraints equivalent to

$$P_\varphi x'_i = P_\theta x'_i = P_\psi x'_i = 0 \quad (i = 1, 2, \ldots, 5) \quad (61)$$

Thus, we would have been left with $W' = UV^\dagger$, i.e.

$$x'_1 = \tilde{z}_1 z_3 + z_1 \bar{z}_3 + z_2 \bar{z}_4 + \bar{z}_2 z_4$$
$$x'_2 = \imath(\tilde{z}_1 z_3 - z_1 \bar{z}_3 - z_2 \bar{z}_4 + \bar{z}_2 z_4)$$
$$x'_3 = -\tilde{z}_1 \bar{z}_4 - z_1 z_4 + z_2 z_3 + \bar{z}_2 \bar{z}_3$$
$$x'_4 = \imath(-\tilde{z}_1 \bar{z}_4 + z_1 z_4 - z_2 z_3 + \bar{z}_2 \bar{z}_3)$$
$$x'_5 = z_1 \bar{z}_1 + z_2 \bar{z}_2 - z_3 \bar{z}_3 - z_4 \bar{z}_4 \quad (62)$$

and, therefore, with the $\mathbb{R}^8 \rightarrow \mathbb{R}^5$ map defined by

$$x'_1 = 2(+u_1 u_5 + u_2 u_6 + u_3 u_7 + u_4 u_8)$$
$$x'_2 = 2(-u_1 u_6 + u_2 u_5 - u_3 u_8 + u_4 u_7)$$
$$x'_3 = 2(-u_1 u_7 + u_2 u_8 + u_3 u_5 - u_4 u_6)$$
$$x'_4 = 2(-u_1 u_8 - u_2 u_7 + u_3 u_6 + u_4 u_5)$$
$$x'_5 = u_1^2 + u_2^2 + u_3^2 + u_4^2 - u_5^2 - u_6^2 - u_7^2 - u_8^2 \quad (63)$$

or alternatively (in a quaternionic form) by

$$x'_1 = 2(q_{10} q_{20} + q_1 \cdot \bar{q}_2) \quad \bar{x}' = 2(-q_{10} \bar{q}_2 + q_{20} \bar{q}_1 - \bar{q}_1 \wedge \bar{q}_2) \quad x'_5 = q_{10}^2 + q_1^2 - q_{20}^2 - q_2^2 \quad (64)$$
with \(\vec{x}' = (x'_2, x'_3, x'_4)\). Such a map corresponds, up to a change of notations, to the Hurwitz transformation associated to the anti-involution \(j_1\) of the algebra of octonions \(A(-1, -1, -1)\) through the relation \(x' = A(u) \varepsilon_1 u\) (see Ref. [16]). Here, the three constraint operators are

\[
X_j = \frac{2}{i} \mathcal{L}_j = \frac{2}{i} [L_j(1) + L_j(2)] \quad (j = 1, 2, 3)
\]

and the corresponding one-forms \(\omega_j\) \((j = 1, 2, 3)\) are

\[
\omega_1 = 2(u_4 du_1 - u_3 du_2 + u_2 du_3 - u_1 du_4 + u_8 du_5 - u_7 du_6 + u_6 du_7 - u_5 du_8)
\]

\[
\omega_2 = 2(u_3 du_1 + u_4 du_2 - u_1 du_3 - u_2 du_4 + u_7 du_5 + u_8 du_6 - u_5 du_7 - u_6 du_8) \quad (66)
\]

\[
\omega_3 = 2(u_2 du_1 - u_1 du_2 - u_4 du_3 + u_3 du_4 + u_6 du_5 - u_5 du_6 - u_8 du_7 + u_7 du_8)
\]

with the property that \(\omega_\alpha \left[ \frac{1}{2r} X_\beta \right] = \delta(\alpha, \beta)\).

The \(\mathbb{R}^8 \to \mathbb{R}^5\) maps defined via equations (55) and (58), on one hand, and via equations (63) and (65), on the other hand, are identical up to a re-labeling of the various variables. For both maps, we have

\[
Xf = 0 \quad (67)
\]

with either \(X \in so(3)_K\) and \(f = f(x_1, x_2, \ldots, x_5)\) or \(X \in so(3)_L\) and \(f = f(x'_1, x'_2, \ldots, x'_5)\), where \(f\) is a (one-fold) differentiable function. The constraint Lie algebra \(so(3) = so(3)_K\) or \(so(3)_L\) is clearly a subalgebra of the real symplectic Lie algebra \(sp(16, \mathbb{R})\) and the corresponding Lie algebra under constraints is

\[
\text{cent}_{sp(16, \mathbb{R})}(so(3)) = so(6, 2) \sim so^*(8) \quad (68)
\]

(see Ref. [19]). Indeed, the two maps under consideration correspond to the Hopf fibration on spheres \(S^7/S^3 \to S^4\) with compact fiber \(S^3\). Some further differential aspects of these \(\mathbb{R}^8 \to \mathbb{R}^5\) maps are discussed in Appendix 4.
Appendix 1: Fock-Bargmann-Schwinger calculus

Let \( F_n \) be the Bargmann space of entire analytical functions \( f(\zeta) \), the \( n \)-uple \( \zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \) being a point of the \( n \)-dimensional complex (Euclidean) space \( \mathbb{C}^n \). Such a space can be endowed with the scalar product defined via

\[
(f, f') = \int_{\mathbb{C}^n} \overline{f(\zeta)} f'(\zeta) \, d\mu_n(\zeta)
\]

where the bar indicates complex conjugation and the measure of integration over \( \mathbb{C}^n \) is given by

\[
d\mu_n(\zeta) = \prod_{k=1}^{n} d\mu(\zeta_k) = \frac{1}{\pi} \exp \left( -\bar{\zeta}_k \zeta_k \right) dx_k \, dy_k \quad \zeta_k = x_k + iy_k
\]

As a particular \((f, f')\), we have

\[
\left( \zeta^{(h)}, \zeta^{(h')} \right) = \delta(\langle h' \rangle, \langle h \rangle) \langle h \rangle!
\]

for \( f(\zeta) = \zeta^{(h)} \) and \( f'(\zeta) = \zeta^{(h')} \), where we use the notation

\[
\zeta^{(h)} = \zeta_1^{h_1} \zeta_2^{h_2} \ldots \zeta_n^{h_n} \quad \zeta^{(h')} = \zeta_1^{h_1'} \zeta_2^{h_2'} \ldots \zeta_n^{h_n'}
\]

\[
\delta(\langle h' \rangle, \langle h \rangle) = \delta(h_1', h_1) \delta(h_2', h_2) \ldots \delta(h_n', h_n)
\]

\[
\langle h \rangle! = h_1! \; h_2! \ldots \; h_n! \quad h_k \in \mathbb{N} \quad k = 1, 2, \ldots, n
\]

Repeated use of the latter scalar product leads to the composition rule

\[
(e^{\alpha \zeta_k}, e^{\beta \zeta_l}) = 1 \quad \text{if} \quad k \neq l \quad \text{or} \quad e^{\alpha \beta} \quad \text{if} \quad k = l
\]

for any \( \alpha \) and \( \beta \) in \( \mathbb{C} \).

The case \( n = 1 \). In this trivial case, we obtain the basic integral

\[
\int_{\mathbb{C}} \zeta^h \zeta'^{h'} \, d\mu(\zeta) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (x - iy)^h (x + iy)^{h'} e^{-(x^2+y^2)} \, dx \, dy = \delta(h', h) \; h!
\]
for the scalar product $(ζ^h, ζ^{h'})$ on $F_1$, so that the set
\[
\left\{ \varphi_h(ζ) = \frac{ζ^h}{\sqrt{h!}} = \frac{(x + iy)^h}{\sqrt{h!}} : h \in \mathbb{N} \right\}
\]
constitutes an orthonormal basis for $F_1$. Furthermore, the composition rule
\[
\int_{\mathbb{C}} e^{αζ} e^{βζ} dμ(ζ) = \frac{1}{\pi} \int_{-∞}^{+∞} \int_{-∞}^{+∞} e^{α(x-iy)} e^{β(x+iy)} e^{-(x^2+y^2)} dx dy = e^{αβ}
\]
gives the scalar product $(e^{αζ}, e^{βζ})$ on $F_1$ for any $α$ and $β$ in $\mathbb{C}$.

**The case** $n = 2$. This case is of special interest for the theory of angular momentum. Let us consider the function
\[
Φ_{JM}(ζ_1, ζ_2) = \frac{ζ_1^{J+M} ζ_2^{J-M}}{\sqrt{(J+M)! (J-M)!}} \quad 2J ∈ \mathbb{N} \quad M = -J, -J+1, \ldots, J
\]
It is obvious that
\[
(Φ_{JM}, Φ_{J'M'}) = \int_{\mathbb{C}^2} Φ_{JM}(ζ_1, ζ_2) Φ_{J'M'}(ζ_1, ζ_2) dμ_2(ζ) = δ(J', J) δ(M', M)
\]
and consequently the set
\[
\{ Φ_{JM} : 2J ∈ \mathbb{N}, M = -J, -J+1, \ldots, J \}
\]
spans an orthonormal basis for $F_2$. It is easy to find a realization ($\hat{J}_±$ and $\hat{J}_z$) in $F_2$ of the angular momentum operators $J_± = J_x ± iJ_y$ and $J_z$. It is a matter of simple calculation to verify that the operators (with $h = 1$)
\[
\hat{J}_+ = ζ_1 \frac{∂}{∂ζ_2} \quad \hat{J}_z = \frac{1}{2} \left( ζ_1 \frac{∂}{∂ζ_1} - ζ_2 \frac{∂}{∂ζ_2} \right) \quad \hat{J}_- = ζ_2 \frac{∂}{∂ζ_1}
\]
satisfy the relations
\[
\hat{J}_± Φ_{JM} = √{J(J+1) - M(M±1)} Φ_{JM±1} \quad \hat{J}_z Φ_{JM} = M Φ_{JM}
\]
Operators of the type $\hat{J}_±$ and $\hat{J}_z$ are employed in section 3. First, for $(ζ_1, ζ_2) = (b_1, b_2)$, we take $\hat{J}_± = \hat{L}_±$ and $\hat{J}_z = \hat{L}_z$. Second, for $(ζ_1, ζ_2) = (a_1, a_2)$, we take $\hat{J}_± = -\hat{K}_±$ and
\( \hat{J}_z = \hat{K}_z. \) (Referring to the change of sign when going from \( L_\pm \) to \( K_\pm \), we note that the set \( \{-J_+, J_z, -J_-\} \) spans the same, viz, with the same structure constants, \( su(2) \) Lie algebra as the set \( \{J_+, J_z, J_-\} \).)

The miscellaneous formulas reported in this appendix are at the root of Fock-Bargmann-Schwinger calculus [25-27] and are used for \( n = 2 \) (with \( \zeta = a \) or \( b \)) and \( n = 4 \) (with \( \zeta = a \) and \( c \) or \( b \) and \( d \)) in sections 3 and 4 of the present paper.
Appendix 2: Images of $\vec{L}$ and $\vec{K}$ in $\mathcal{F}_2$

This appendix deals with the derivations of equations (21). The basic relations for the operators $\vec{L}$ and $\vec{K}$ are (in units $\hbar = 1$)

\begin{align*}
L_\pm D^j(\hat{U})_{mm'} &= +\sqrt{j(j+1) - m'(m'\pm1)} \ D^j(\hat{U})_{m,m'\pm1} \\
L_z D^j(\hat{U})_{mm'} &= m' D^j(\hat{U})_{mm'} \\
K_\pm D^j(\hat{U})_{mm'} &= -\sqrt{j(j+1) - m(m\pm1)} \ D^j(\hat{U})_{m\pm1,m'} \\
K_z D^j(\hat{U})_{mm'} &= m D^j(\hat{U})_{mm'}
\end{align*}

We begin with the proof of $\hat{L}_- \Psi = L_+ \Psi$. We have

\begin{align*}
L_+ \Psi &= \sum_{jmm'} \Phi_{jm}(a_1, a_2) \ r^j \ L_+ D^j(\hat{U})_{mm'} \ \Phi_{jm'}(b_1, b_2) \\
&= \sum_{jmm'} \Phi_{jm}(a_1, a_2) \ r^j \ \sqrt{j(j+1) - m'(m' + 1)} \ D^j(\hat{U})_{m,m'+1} \ \Phi_{jm'}(b_1, b_2)
\end{align*}

On the other hand, from Appendix 1 we get

\begin{align*}
\hat{L}_- \Psi &= \sum_{jmm'} \Phi_{jm}(a_1, a_2) \ r^j \ D^j(\hat{U})_{mm'} \ \hat{L}_- \ \Phi_{jm'}(b_1, b_2) \\
&= \sum_{jmm'} \Phi_{jm}(a_1, a_2) \ r^j \ D^j(\hat{U})_{mm'} \ \sqrt{j(j+1) - m'(m' - 1)} \ \Phi_{jm'-1}(b_1, b_2)
\end{align*}

By comparing the expressions for $L_+ \Psi$ and $\hat{L}_- \Psi$, we deduce

$$\hat{L}_- \Psi = L_+ \Psi = L_+^\dagger \Psi$$

which completes the proof. Similarly, we would obtain

$$\hat{L}_+ \Psi = L_- \Psi \quad \hat{L}_z \Psi = L_z \Psi \quad \hat{K}_\pm \Psi = K_\mp \Psi \quad \hat{K}_z \Psi = K_z \Psi$$

so that the images $\hat{L}$ and $\hat{K}$ of $\vec{L}$ and $\vec{K}$ satisfy (21).

Equation (21) can be generalized in the form $A^\dagger \Psi = \hat{A} \Psi$. The latter relation allows to derive the image $\hat{A}$ of $A$ (or the operator $A$ if we know $\hat{A}$) [29,30].
Appendix 3: Laplacians for the KS transformation

This appendix is devoted to the volume elements $dv$, the line elements $ds^2$ and the Laplacians $\Delta$ in $\mathbb{R}^3$ and $\mathbb{R}^4$. It covers results obtained by Kustaanheimo and Stiefel [1], Ikeda and Miyachi [4], Boiteux [5], Kibler and Négadi [8], Polubarinov [10], D’Hoker and Vinet [32], and Lambert and Kibler [16] for the KS transformation. We shall deal in this appendix with the KS transformation in the form corresponding to the $\mathbb{R}^4 \rightarrow \mathbb{R}^3$ map defined by equations (4) and (14-16).

**Volume elements.** The volume elements in $\mathbb{R}^4$ and $\mathbb{R}^3$ are given by

$$dv(\mathbb{R}^4) = du_1 du_2 du_3 du_4 = \frac{1}{16} r \sin \theta dr d\psi d\theta d\varphi$$

$$dv(\mathbb{R}^3) = dx_1 dx_2 dx_3 = r^2 \sin \theta dr d\theta d\varphi$$

Thus, we have

$$dv(\mathbb{R}^4) = \frac{1}{16r} dv(\mathbb{R}^3) d\psi$$

Integration on $\psi$ from 0 to $4\pi$ (i.e., integration on the fiber of the $\mathbb{R}^4 - \{0\} = S^3 \times \mathbb{R}^+ \rightarrow R^3 - \{0\} = S^2 \times R^+\ast$ transformation) leads to

$$\int_{\mathbb{R}^4} \ldots dv(\mathbb{R}^4) = \frac{\pi}{4} \int_{\mathbb{R}^3} \ldots \frac{1}{r} dv(\mathbb{R}^3)$$

The latter equation allows us to formally define a Jacobian for the KS transformation. Indeed, we have

$$dv(\mathbb{R}^3) = |J| dv(\mathbb{R}^4)$$

where

$$J = \det \begin{pmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \frac{\partial x_1}{\partial u_3} & \frac{\partial x_1}{\partial u_4} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \frac{\partial x_2}{\partial u_3} & \frac{\partial x_2}{\partial u_4} \\ \frac{\partial x_3}{\partial u_1} & \frac{\partial x_3}{\partial u_2} & \frac{\partial x_3}{\partial u_3} & \frac{\partial x_3}{\partial u_4} \\ \frac{1}{8\pi u_2} & -\frac{1}{8\pi u_1} & \frac{1}{8\pi u_4} & -\frac{1}{8\pi u_3} \end{pmatrix} = \frac{4}{\pi} \left(u_1^2 + u_2^2 + u_3^2 + u_4^2\right)$$

21
plays the rôle of a Jacobian (the origin of which is not clear) for the considered $\mathbb{R}^4 \to \mathbb{R}^3$ map.

**Line elements.** The line element of $\mathbb{R}^4$

$$ds^2(\mathbb{R}^4) = du_1^2 + du_2^2 + du_3^2 + du_4^2 = dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2$$

is given in the coordinates $(\sqrt{r}, \psi, \theta, \varphi)$ by

$$ds^2(\mathbb{R}^4) = \frac{1}{4r} [dr^2 + r^2 d\theta^2 + r^2 (d\psi^2 + 2 \cos \theta d\psi d\varphi + d\varphi^2)]$$

Similarly, the line element of $\mathbb{R}^3$

$$ds^2(\mathbb{R}^3) = dx_1^2 + dx_2^2 + dx_3^2$$

is known to be

$$ds^2(\mathbb{R}^3) = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2$$

in the coordinates $(r, \theta, \varphi)$. Therefore, the connection between $ds^2(\mathbb{R}^3)$ and $ds^2(\mathbb{R}^4)$ is

$$4r ds^2(\mathbb{R}^4) = ds^2(\mathbb{R}^3) + r^2 (\cos \theta d\varphi + d\psi)^2$$

The constraint

$$\cos \theta d\varphi + d\psi = 0$$

ensures that

$$ds^2(\mathbb{R}^3) = 4r ds^2(\mathbb{R}^4)$$

In other words, the $\mathbb{R}^4 \to \mathbb{R}^3$ map defined by equations (16) is conformal once the one-form $\omega$ given by

$$\omega = 2(u_2 du_1 - u_1 du_2 + u_4 du_3 - u_3 du_4) = -r (\cos \theta d\varphi + d\psi)$$

is taken to vanish (see Kustaanheimo and Stiefel [1]). However, the relation

$$ds^2(\mathbb{R}^3) = 4r ds^2(\mathbb{R}^4) - \omega^2$$
Laplacians. The Laplace-Beltrami operator

$$\Delta = \frac{1}{\sqrt{\|G\|}} \partial a \sqrt{\|G\|} G_{ab}^{ab} \partial b \sqrt{\|G\|} = |\det(G_{ab})|$$

where the metric matrix \((G_{ab}) = (G_{ab})^{-1}\) is defined via

$$ds^2 = G_{ab} d\xi_a d\xi_b$$

can be particularized to \(\mathbb{R}^4\) and \(\mathbb{R}^3\) as follows. In the Euler-angle coordinates \((\sqrt{r}, \psi, \theta, \varphi)\) of \(\mathbb{R}^4\), we get the Laplacian

$$\Delta(\mathbb{R}^4) = 4 \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{4}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{4}{r \sin^2 \theta} \left( \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial \psi^2} \right) - \frac{8 \cos \theta}{r \sin^2 \theta} \frac{\partial^2}{\partial \psi \partial \varphi}$$

while in the spherical coordinates \((r, \theta, \varphi)\) of \(\mathbb{R}^3\) we have the usual Laplacian

$$\Delta(\mathbb{R}^3) = \frac{1}{r} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}$$

Consequently, the two Laplacians are connected through

$$\frac{1}{4r} \Delta(\mathbb{R}^4) = \Delta(\mathbb{R}^3) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \psi} \left( \frac{\partial}{\partial \psi} - 2 \cos \theta \frac{\partial}{\partial \varphi} \right)$$

When acting on (two-fold differentiable) functions of the variables \(x_1, x_2\) and \(x_3\), the latter relation reduces to

$$\Delta(\mathbb{R}^3) = \frac{1}{4} \frac{1}{u_1^2 + u_2^2 + u_3^2 + u_4^2} \Delta(\mathbb{R}^4)$$

in agreement with a well-known result (see Boiteux [5] and Kbler and Négadi [8]). However, when \(\Delta(\mathbb{R}^3)\) acts on (two-fold differentiable) functions of the variables \(u_1, u_2, u_3\) and \(u_4\), we have

$$\Delta(\mathbb{R}^3) = \frac{1}{4r} \Delta(\mathbb{R}^4) + \frac{1}{r^2} K_3^2 + \frac{\cos \theta}{r^2 \sin^2 \theta} K_3 (\cos \theta K_3 - 2L_3)$$

The third term in the right-hand side of the latter general formula does not occur in the corresponding formula of Ref. [8]; the geometrical interpretation of this fact is an interesting
problem, especially in the case of a general Hurwitz transformation. Finally, note that by introducing
\[
\lambda = \frac{1}{2}(r + x_3) \quad \mu = \frac{1}{2}(r - x_3) \quad A = -i(K_3 + L_3) \quad B = -i(K_3 - L_3)
\]
we have
\[
\frac{1}{r^2} K_3^2 + \frac{\cos \theta}{r^2 \sin^2 \theta} K_3(\cos \theta K_3 - 2L_3) = -\frac{1}{16} \frac{1}{\lambda + \mu} \frac{1}{\lambda \mu} [(-\lambda + 3\mu)A^2 + (3\lambda - \mu)B^2 + 2(\lambda + \mu)AB]
\]
so that the difference \( \Delta(\mathbb{R}^3) - (4r)^{-1}\Delta(\mathbb{R}^4) \) can be rewritten (in the general case) in a somewhat symmetric form.

We close this appendix with a remark concerning the canonical character of the KS transformation. In fact, we can verify that
\[
\sum_{i=1}^{3} dx_i \frac{\partial}{\partial x_i} = \sum_{\alpha=1}^{4} du_\alpha \frac{\partial}{\partial u_\alpha}
\]
provided that the constraint
\[
\frac{1}{2r} \omega X \equiv -\frac{i}{r} \omega K_3 = 0
\]
be satisfied. The canonicity of the KS transformation arises from the latter two relations (to be compared with \( \sum p_k dq_k = \sum P_k dQ_k + dF \) characterizing a canonical or contact transformation in classical mechanics).

**Extensions.** The results contained in this appendix can be developed for the non-compact extensions of the \( \mathbb{R}^4 \rightarrow \mathbb{R}^3 \) map defined by (4) and (14-16). These extensions are of the type \( \mathbb{R}^{2,2} \rightarrow \mathbb{R}^{1,2} \) or \( \mathbb{R}^{2,2} \rightarrow \mathbb{R}^{2,1} \) and are associated to fibrations on hyperboloids with compact or non-compact fiber, respectively (see Ref. [16]).
Appendix 4: Differential aspects of the $\mathbb{R}^8 \to \mathbb{R}^5$ transformation

We shall deal in this appendix with the transformation defined through equations (55) and (58). As a preliminary, we note that by taking
\[ u_3 = u_4 = u_7 = u_8 = 0 \]
and by making the replacements
\[ u_1 \to u_2 \quad u_5 \to u_4 \quad u_2 \to u_1 \quad u_6 \to u_3 \]
the $\mathbb{R}^8 \to \mathbb{R}^5$ map defined by (55) reduces to the $\mathbb{R}^4 \to \mathbb{R}^3$ map defined by (16). Therefore, we may think to introduce in an easy way Euler-angle coordinates $(\sqrt{r}, \psi, \theta, \varphi, \theta_1, \theta_2, \theta_3, \theta_4)$ for the $\mathbb{R}^8 \to \mathbb{R}^5$ transformation.

**Euler-angle coordinates.** It is sufficient to work with the inverse transformation, defined up to an $S^3$ sphere, for obtaining the $u_i$'s in terms of, what we call, the Euler-angle coordinates $(\sqrt{r}, \psi, \theta, \varphi, \theta_1, \theta_2, \theta_3, \theta_4)$. In fact, we have
\[
\begin{align*}
 u_1 &= \sqrt{r} \cos \frac{\theta_1}{2} \cos \psi \cos \theta \cos \varphi \\
 u_2 &= \sqrt{r} \cos \frac{\theta_1}{2} \sin \psi \cos \theta \cos \varphi \\
 u_3 &= \sqrt{r} \cos \frac{\theta_1}{2} \sin \theta \cos \varphi \\
 u_4 &= \sqrt{r} \cos \frac{\theta_1}{2} \sin \varphi \\
 u_5 &= \sqrt{r} \sin \frac{\theta_1}{2} (- \sin \theta_2 \sin \theta_3 \cos \theta_4 \sin \theta \cos \varphi - \sin \theta_2 \cos \theta_3 \sin \psi \cos \theta \cos \varphi \\
 &\quad + \cos \theta_2 \cos \psi \cos \theta \cos \varphi - \sin \theta_2 \sin \theta_3 \sin \theta_4 \sin \varphi) \\
 u_6 &= \sqrt{r} \sin \frac{\theta_1}{2} (- \sin \theta_2 \sin \theta_3 \cos \theta_4 \sin \varphi + \sin \theta_2 \cos \theta_3 \sin \psi \cos \theta \cos \varphi \\
 &\quad + \cos \theta_2 \sin \psi \cos \theta \cos \varphi + \sin \theta_2 \sin \theta_3 \sin \theta_4 \sin \theta \cos \varphi) \\
 u_7 &= \sqrt{r} \sin \frac{\theta_1}{2} (+ \sin \theta_2 \sin \theta_3 \cos \theta_4 \cos \psi \cos \theta \cos \varphi + \sin \theta_2 \cos \theta_3 \sin \varphi \\
 &\quad + \cos \theta_2 \sin \theta \cos \varphi - \sin \theta_2 \sin \theta_3 \sin \theta_4 \sin \psi \cos \theta \cos \varphi) \\
 u_8 &= \sqrt{r} \sin \frac{\theta_1}{2} (+ \sin \theta_2 \sin \theta_3 \cos \theta_4 \sin \psi \cos \theta \cos \varphi - \sin \theta_2 \cos \theta_3 \sin \theta \cos \varphi \\
 &\quad + \cos \theta_2 \sin \varphi + \sin \theta_2 \sin \theta_3 \sin \theta_4 \cos \psi \cos \theta \cos \varphi)
\end{align*}
\]
Then, by introducing the latter equations into (55), we obtain

\[
\begin{align*}
    x_5 &= r \cos \theta_1 \\
    x_1 &= r \sin \theta_1 \cos \theta_2 \\
    x_2 &= r \sin \theta_1 \sin \theta_2 \cos \theta_3 \\
    x_3 &= r \sin \theta_1 \sin \theta_2 \sin \theta_3 \cos \theta_4 \\
    x_4 &= r \sin \theta_1 \sin \theta_2 \sin \theta_3 \sin \theta_4
\end{align*}
\]

and \((r, \theta_1, \theta_2, \theta_3, \theta_4)\) are clearly the “spherical” coordinates for \(\mathbb{R}^5\).

**Line elements.** From equation (55), we easily check that the line elements

\[
\begin{align*}
    ds^2(\mathbb{R}^8) &= \sum_{\alpha=1}^{8} du^2_{\alpha} \\
    ds^2(\mathbb{R}^5) &= \sum_{i=1}^{5} dx^2_i
\end{align*}
\]

are connected through

\[
ds^2(\mathbb{R}^5) = 4r \, ds^2(\mathbb{R}^8) - (\omega_1^2 + \omega_2^2 + \omega_3^2)
\]

where the one-forms \(\omega_i\) (\(i = 1, 2, 3\)) are defined in (59).

**Laplacians.** The Laplacians \(\Delta(\mathbb{R}^8)\) and \(\Delta(\mathbb{R}^5)\) can be expressed in the coordinates \((\sqrt{r}, \psi, \theta, \varphi, \theta_1, \theta_2, \theta_3, \theta_4)\). For saving space purposes, the relation between \(\Delta(\mathbb{R}^8)\) and \(\Delta(\mathbb{R}^5)\) is not given here. Note, however, that this relation reduces to

\[
\Delta(\mathbb{R}^5) = \left( 4 \sum_{\alpha=1}^{8} u^2_{\alpha} \right)^{-1} \Delta(\mathbb{R}^8)
\]

when acting on a (two-fold) differentiable function \(f(x_1, x_2, \ldots, x_5)\). The corresponding relation to be used when \(f(x_1, x_2, \ldots, x_5)\) is replaced by a (two-fold) differentiable function \(g(u_1, u_2, \ldots, u_8)\) can be obtained from one of the authors (M. K.).

Let us close this article by noticing that the \(\mathbb{R}^8 \rightarrow \mathbb{R}^5\) transformation (55-59) turns out to be a canonical transformation since we have

\[
\sum_{i=1}^{5} dx_i \frac{\partial}{\partial x_i} = \sum_{\alpha=1}^{8} du_{\alpha} \frac{\partial}{\partial u_{\alpha}}
\]
once the constraint
\[
\frac{1}{2r} \sum_{j=1}^{3} \omega_j X_j \equiv -\frac{i}{r} \sum_{j=1}^{3} \omega_j K_j = 0
\]
is satisfied.

**Extensions.** The results contained in this appendix can be developed for the non-compact extensions of the \( \mathbb{R}^8 \to \mathbb{R}^5 \) map defined by (55) and (58). These extensions are of the type \( \mathbb{R}^{4,4} \to \mathbb{R}^{1,4} \) or \( \mathbb{R}^{4,4} \to \mathbb{R}^{3,2} \) and are associated to fibrations on hyperboloids with compact or non-compact fiber, respectively (see Ref. [16]).
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