LOOP SPACE HAMILTONIAN FOR $c \leq 1$ OPEN STRINGS

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We construct a string field Hamiltonian describing the dynamics of open and closed strings with effective target-space dimension $c \leq 1$. In order to do so, we first derive the Dyson-Schwinger equations for the underlying large $N$ vector+matrix model and formulate them as a set of constraints satisfying decoupled Virasoro and U(1) current algebras. The Hamiltonian consists of a bulk and a boundary term having different scaling dimensions. The time parameters corresponding to the two terms are interpreted from the point of view of the fractal geometry of the world surface.

Submitted for publication to: Physics Letters B

SPhT/95-001

01/95

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1. Introduction

The string field theory seems to be the most complete and geometrically the most
natural formalism for studying strings in interaction. The construction of such a theory
implies a decomposition of the world sheet into propagators and vertices which is a sub-
tle and highly nontrivial problem and can be done in different ways. An unambiguous
prescription for construction of a string field theory stems from the discretization of the
string path integral in which both the world sheet and the target space are discretized [1].
Recently, the strings with discrete target space were reformulated in terms of a transfer
matrix formalism [2][3][4]. One of the interesting aspects of this approach is the inter-
pretation of the time parameter for the string field Hamiltonian as geodesic distance on
the world sheet of the string. If this interpretation is correct, then the dimension of the
time parameter is related to the ”intrinsic fractal dimension” of the world surface. The
computation of the latter in the Liouville theory formalism is a very difficult and still
unsolved problem [6]. Moreover, the finite-time correlation functions can be interpreted as
correlations between points at given geodesic distance on the world sheet.

In this letter we resolve some minor inconsistencies of the transfer matrix approach of
ref. [3] and extend it to the case of interacting closed and open strings. Our motivation is
the hope that such an extension would help to understand the fractal structure of random
surfaces with boundaries. Our starting point will be the soluble large $N$ vector+matrix
model proposed in ref. [7]. We will derive the loop space Dyson-Schwinger equations
for this model and reformulate them as a set of decoupled Virasoro and $U(1)$ current
algebra constraints, one for each point $x$ of the target space. The Virasoro constraints can
be obtained from a bulk Hamiltonian $H$ which generalizes the closed string Hamiltonian
constructed in [3]. Its time parameter $t$ measures the geodesic distance on the world sheet.
The $U(1)$ constraints follow from a boundary Hamiltonian $H_B$ whose evolution parameter
$t_B$ allows an interpretation as the geodesic distance along the edge of the world surface
of the open string. We will evaluate the dimensions of these two time parameters and
interpret their meaning from the point of view of the fractal structure of the world surface.

1 In an equivalent approach based on stochastic quantization [5], the interpretation of this
parameter is as a fictitious stochastic time.
2. Large $N$ field theory for discretized open and closed strings

By a string we understand an oriented chain of particles immersed in the target space $\mathbb{Z}$ and such that the coordinates of the adjacent particles either coincide or differ by 1. The configuration space of strings consists of all closed and open paths $\Gamma = [x_0 x_1 x_2 ... x_n]; x_k \in \mathbb{Z}, x_{k+1} - x_k = 0, \pm 1$. The evolution of the string can be decomposed into elementary moves creating or annihilating a particle ($[...x...x]\leftrightarrow [...x'...x']$) or changing its topology ($[...x...x...x]\leftrightarrow [...x[x...x]x...]$). It is evident that the world surface of such a string will represent a triangulated surface immersed in $\mathbb{Z}$. A string theory with effective dimension $c = 1 - 6/m(m + 1)$ of the target space will be obtained by modifying the vacuum state by means of a background momentum $p_0 = 1/m$.

The discretized string field theory describes the perturbative expansion of a one-dimensional model of vector and matrix fields with the same color structure as the $U(N)$ lattice gauge theory with quarks \cite{7}. The $N$-vector ”quark” field $\psi_x, \psi^*_x$ is associated with the points $x \in \mathbb{Z}$ and creates the ends of the open string. The line elements of the discretized string are represented by an $N \times N$ matrix ”gluon” field $A_{xx'}$ defined for the oriented pairs of points $\{x, x'\}$ such that $|x - x'| \leq 1$. The diagonal matrices $A_{xx}$ are hermitian and the nondiagonal ones satisfy the condition $A_{xx'} = A_{x'x}^\dagger$. The gauge invariant operators creating closed and open strings are the the Wilson lines and loops

$$\Omega_{[x_0 x_1 ... x_n]} = \psi_{x_0}^* A_{x_0 x_1} A_{x_1 x_2} ... A_{x_{n-1} x_n} \psi_{x_n}$$ (2.1)

$$W_{[x_0 x_1 ... x_n x_0]} = \text{tr} \left( A_{x_0 x_1} A_{x_1 x_2} ... A_{x_n x_0} \right).$$ (2.2)

The partition function is defined as

$$Z = \int dA \ d\psi \ d\psi^* e^{W[A, \psi, \psi^*]}$$ (2.3)

$$W[A, \psi, \psi^*] = -\frac{1}{2} \text{tr} \sum_{x, x'} A_{xx'} A_{x'x} + \frac{\lambda}{3\sqrt{N}} \text{tr} \sum_{x, x', x''} A_{xx'} A_{x'x''} A_{x''x}$$

$$+ \sum_{x, x'} \psi_{x'}^* ( - \delta_{xx'} + \frac{\lambda B}{\sqrt{N}} A_{xx'} ) \psi_{x'}. $$ (2.4)
where the $\lambda$ is the bare string tension and $\lambda_B$ is the bare "quark" mass. The components of the $\psi$-field can be considered as commuting or anticommuting variables. The two choices lead to different signs of the string interaction constant.

One can write a collective-field Hamiltonian in terms of the gauge-invariant loop fields (2.2)-(2.1) following the general recipe [8]. It is however possible to find a Hamiltonian acting in a smaller configuration space, namely, the contours associated with a single point $x$. The corresponding Wilson loops and lines involve only the diagonal matrices $A_{xx}$. Indeed, the action (2.4) is quadratic with respect to the non-diagonal matrices $A_{xx'} = A_{x'x}^\dagger$, $x' = x \pm 1$, and the corresponding integration is trivial. Therefore, if we restrict ourselves to contours localized at a single point $x$, the model (2.3)-(2.4) can be formulated only in terms of the "quark" fields $\psi_x, \psi_x^*$ and the hermitian matrix field $M_x$

$$M_x^{ij} = A_{xx}^{ij} - \frac{\sqrt{N}}{2\lambda} \delta^{ij}. \quad (2.5)$$

The reduced Hilbert space is spanned by the operators

$$W_x(L) = \text{tr} \, e^{LM_x}, \quad \Omega_x(L) = \psi_x^* e^{LM_x} \psi_x \quad (2.6)$$

or their Laplace transforms

$$\tilde{W}_x(Z) = \int_0^\infty dL \, e^{-LZ} W_x(L) = \text{tr} \, \frac{1}{Z - M_x} \quad (2.7)$$

$$\tilde{\Omega}_x(Z) = \int_0^\infty dL \, e^{-LZ} \Omega_x(L) = \psi_x^* \frac{1}{Z - M_x} \psi_x. \quad (2.8)$$

The generating functional for the correlation functions of these operators is obtained from (2.3) by introducing the source terms

$$J \cdot W = \sum_x \int_0^\infty dL J_x(L) W_x(L) = \sum_x \oint \frac{dZ}{2\pi i} \tilde{J}_x(Z) \tilde{W}_x(Z) \quad (2.9)$$

$$J^B \cdot \Omega = \sum_x \int_0^\infty dL J_x^B(L) \Omega_x(L) = \sum_x \oint \frac{dZ}{2\pi i} \tilde{J}_x^B(Z) \tilde{\Omega}_x(Z) \quad (2.10)$$
Integrating over the nondiagonal $A$-matrices and adding the source terms, we find the following expression for the generating functional for the loop fields

$$Z[J, J^B] = \int \prod_x d\psi_x d\psi^*_x dM_x \ e^{J \cdot W + J^B \cdot \Omega + A[W, \Omega]}$$

(2.11)

$$A[W, \Omega] = \tfrac{1}{2} \sum_{x,x'} C_{xx'} \int_0^\infty dL \left[ \tfrac{1}{L} W_x(L) W_{x'}(L) + \Omega_x(L) \Omega_{x'}(L) \right]$$

(2.12)

where $C_{xx'}$ denotes the adjacency matrix of the target space

$$C_{xx'} = \delta_{x,x'+1} + \delta_{x,x'-1}. \quad (2.13)$$

We have eliminated the explicit dependence on the couplings $\lambda$ and $\lambda_B$ by shifting the sources and rescaling the $\psi$-fields. The sources now should be considered as small perturbations around the polynomial potentials $V(Z)$ and $V^B(Z)$.

The action (2.4) leads to $V(Z) = \text{[polynomial of third degree]}$ and $V^B = \text{[constant]}$. However one can consider more general polynomial potentials. The multicritical regimes of the closed string are obtained by tuning the potential $V$.

The effective action (2.12) has an evident geometrical interpretation. The integration over the nondiagonal $A$-matrices organizes the elementary triangles that compose the world sheet into rings (the first term) and strips (the second term). A ring can be interpreted as the amplitude for a jump of a closed string with length $L$ from the point $x$ to the adjacent point $x' = x \pm 1$. Similarly, a strip represents a jump $x \to x'$ of an open string with length $L$. In this way the evolution of the string string can be decomposed into elementary processes representing either splittings or joinings, or propagation, but not both. This factorization is an important feature of the strings with discrete target space and gives the clue for the exact solution of the theory.

The sum over the embeddings of the world surface can be thought of as the sum over all possible configurations of the domain walls separating the domains with constant $x$ (see fig.7 of ref. [9]). This is the partition function of a gas of nonintersecting loops and lines.
on the world surface. Each loop or line has two orientations and therefore has to be taken with a factor of 2. The endpoints of the domain lines are arranged along the boundary of the world surface formed by the ends of open strings. In ref. [9] this type of boundary is called ”Neumann boundary” because it is characterized by a free boundary condition. On the other hand, the pieces of the boundary associated with the incoming and outgoing loops are characterized by a constant (Dirichlet) boundary condition. The domain walls do not touch the Dirichlet boundary. Once we have restricted the Hilbert space to the operators of the form (2.6), the ”time” direction on the world sheet is fixed: the lines of constant time go along the domain walls.

The model (2.11)- (2.12) describes strings whose target space is the one-dimensional lattice $\mathbb{Z}$ characterized by the adjacency matrix (2.13). The dimension of the target space, or the central charge of the matter in the language of 2d quantum gravity, can be lowered to $c < 1$ by introducing a background momentum $p_0$ coupled to the intrinsic curvature of the world sheet. This is explained in the context of the loop-gas model in ref. [9]. In our formalism this is achieved by using twisted adjacency matrix (2.13)

$$C_{xx'} \rightarrow \begin{cases} C^{(p_0)}_{xx'} = e^{ip_0} \delta_{x,x+1} + e^{-ip_0} \delta_{x,x'-1}, & \text{for closed strings;} \\ C^{(p_0/2)}_{xx'} = e^{ip_0/2} \delta_{x,x+1} + e^{-ip_0/2} \delta_{x,x'-1}, & \text{for open strings} \end{cases} \quad (2.14)$$

For example, the action (2.12) becomes

$$\mathcal{A}[W,\Omega] = \frac{1}{2} \sum_{x,x'} \int_0^\infty dL \left[ C^{(p_0)}_{xx'} \frac{1}{L} W_x(L) W_{x'}(L) + C^{(p_0/2)}_{xx'} \Omega_x(L) \Omega_{x'}(L) \right]. \quad (2.15)$$

In the loop gas picture the twisted adjacency matrix means that the domain walls are weighed by phase factors depending on their orientation.

The background momentum changes the vacuum of the string theory and the effective dimension of the target space then becomes

$$c = 1 - 6 \frac{(g - 1)^2}{g}, g = p_0 + 1. \quad (2.16)$$

The momentum space is periodic with period 2, but nevertheless it makes sense to consider the parameter $g$ in the interval $0 < g < \infty$. The integral part of $g$ specifies the critical
regime obtained by tuning the potential $V(Z)$. The dense phase of the loop gas is described by the interval $0 < g < 1$, the dilute phase by the interval $1 < g < 2$, and the $m$-critical regime by $m - 1 < g < m$. The half-integer values $g = m - \frac{1}{2}, m = 1, 2, ...$, describe the possible critical regimes of strings without embedding space.

In what follows by adjacency matrix $C_{xx'}$ we will understand the twisted adjacency matrix \(2.14\).

3. Loop equations

The invariance of the integral \(2.11\) under an infinitesimal change of variables

\[ M_x \rightarrow M_x + \frac{\epsilon_x}{Z - M_x}; \quad \psi_x \rightarrow (1 + \frac{\theta_x}{Z - M_x})\psi_x \]  

(3.1)

yields a closed set of Dyson-Schwinger equations for the loop fields \(2.6\)

\[ \langle \int_0^L dL' W_x(L')W_x(L - L') + \int_0^\infty dL'[L'J_x(L') + \sum_{x'} C_{xx'}W_{x'}(L')]W_x(L + L') \]  

\[ + \int_0^\infty dL'L'[J_x^B(L') + \sum_{x'} C_{xx'}\Omega_{x'}(L')]\Omega_x(L + L') \rangle = 0 \]  

(3.2)

\[ \langle W_x(L) + \int_0^\infty dL'[J_x^B(L') + \sum_{x'} C_{xx'}\Omega_{x'}(L')]\Omega_x(L + L') \rangle = 0 \]  

(3.3)

where we denoted by \(\langle \rangle\) the expectation value with the measure \(2.11\). The normalization of the closed string field corresponds to

\[ \langle W_x(0) \rangle = N. \]  

(3.4)

We can alternatively formulate the loop equations in terms of the Laplace transformed loop fields \(2.7\) and \(2.8\). The equations read

\[ \langle \tilde{W}_x^2(Z) + \frac{1}{2\pi i} \int dZ' \frac{\tilde{W}_x(Z')}{Z - Z'} [\partial \tilde{J}_x(Z') + \sum_{x'} C_{xx'}\tilde{W}_{x'}(-Z')] \]  

\[ + \frac{1}{2\pi i} \int dZ' \frac{\tilde{\Omega}_x(Z')}{(z - Z')^2} [\tilde{J}_x^B(Z') + \sum_{x'} C_{xx'}\tilde{\Omega}_{x'}(-Z')] \rangle = 0 \]  

(3.5)
\[ \langle \tilde{W}_x(Z) \rangle + \frac{1}{2\pi i} \oint \frac{dZ'}{Z-Z'} \left[ \tilde{J}_x^B(Z') + \sum_{x'} C_{xx'} \tilde{\Omega}_x'(-Z') \right] = 0. \] (3.6)

The contour of integration encloses the singularities of \( \tilde{W}_x(Z') \) and leaves outside the point \( Z' = Z \) as well as the singularities of \( \tilde{W}_x(-Z') \). It is selfconsistent to assume that in any finite order of the \( 1/N \) expansion the correlation functions of \( W_x(Z) \) and \( \Omega_x(Z) \) are meromorphic functions defined in the \( Z \) plane cut along the same interval \( X < Z < Y \). The location of the endpoints can be determined as usual by the condition

\[ \langle \tilde{W}_x(Z) \rangle = \frac{N}{Z} + \mathcal{O}(1/Z^2), \quad Z \to \infty. \]

The contour integrals in (3.5)-(3.6) makes sense only if the intervals \([X,Y]\) and \([-Y,-X]\) do not overlap, i.e., if \( Y < 0 \). The typical length of strings is \( L \sim |Y|^{-1} \) and the critical point is achieved when \( Y \to Y^* = 0 \).

4. The scaling limit

The typical world surface is characterized by its area \( A \), the length \( L \) of its Dirichlet boundaries and the length \( L_B \) of its Neumann boundaries. Near the critical point \( \lambda = \lambda^*, \lambda_B = \lambda^*_B \) these parameters diverge as

\[ A \sim \frac{1}{\lambda^* - \lambda}, \quad L \sim -\frac{1}{Y}, \quad L_B \sim \frac{1}{\lambda^*_B - \lambda_B}. \] (4.1)

Let us introduce a scale parameter (elementary length) \( a \) by

\[ Y = -Ma. \] (4.2)

and rescaling the variables as

\[ L = \ell/a, \quad Z = za. \] (4.3)

The scaling limit is characterized by the renormalized parameters (the string interaction constant \( \kappa \), the string tension \( \Lambda \) and the "quark" mass \( \mu \)) defined by

\[ \kappa = \frac{1}{a^{1+g} N}, \quad \lambda^* - \lambda \sim a^{2\nu} \Lambda, \quad \lambda^*_B - \lambda_B \sim a^g \mu. \] (4.4)
\[ \nu = \begin{cases} g, & \text{if } g < 1 ; \\ 1, & \text{if } g > 1 \end{cases} \] (4.5)

The parameter \( M \) is a function of \( \Lambda \); in an appropriate normalization it is given by

\[ M^{2\nu} = c^2 \Lambda, \quad e^{\nu/g} = 2 \sin\left(\frac{1}{2} \pi g\right). \] (4.6)

The so called “string susceptibility” exponent is equal to \( \gamma_{\text{str}} = -|g - 1|/\nu \). For given \( n = -2 \cos \pi g \) the branch \( 0 < g < 1 \) describes the dense phase of the loop gas, the branch \( 1 < g < 2 \) describes the dilute phase, and the branches with \( g > 2 \) correspond to an infinite sequence of multicritical phases. The scaling exponents follow from the analysis of the loop equations (3.5) and (3.6) (see [1], Sect. 3).

As mentioned above, we assume polynomial potentials \( \tilde{J}_x(z) = -V(z), \tilde{J}_x^B(z) = -V^B(z) \). By shifting the loop fields by appropriate polynomials in \( z \), the dependence of the l.h.s. of the loop equations on the potential can be eliminated. The r.h.s. of the loop equations then is no more zero, but a polynomial in \( z \). The renormalized loop fields and sources are defined by

\[ \tilde{W}_x(Z) = P(Z) + \frac{1}{\kappa} \tilde{w}_x(z), \quad \tilde{\Omega}_x(Z) = P_B(Z) + \frac{1}{\sqrt{\kappa}} \tilde{\omega}_x(z) \] (4.7)

\[ \tilde{J}_x(Z) = -V(Z) + \kappa \tilde{J}_x^x(z), \quad \tilde{J}_x(Z) = -V^B(Z) + \sqrt{\kappa} \tilde{J}_x^B(z) \] (4.8)

where the polynomials \( P(Z) \) and \( P_B(Z) \) are chosen to cancel the dependence on the potential of the l.h.s. of the loop equations.

The disc amplitudes \( (\kappa = 0) \)

\[ \tilde{w}_c(z) = \langle \tilde{w}_x(z) \rangle, \quad \tilde{\omega}_c(z) = \langle \tilde{\omega}_x(z) \rangle \] (4.9)

do not depend on \( x \) and satisfy the following functional equations [3]

\[ \tilde{w}_c(z)^2 + \tilde{w}_c(-z)^2 - 2 \cos \pi g \cdot \tilde{w}_c(z)\tilde{w}_c(-z) = \Lambda^{g/\nu} \] (4.10)

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\[ 2 \sin \frac{1}{2} \pi g \quad \tilde{\omega}_c(z)\tilde{\omega}_c(-z) + \tilde{w}_c(z) + \tilde{w}_c(-z) = 2 \mu. \quad (4.11) \]

The solution of these equations in the case of a string with massless ends \((\mu = 0)\) is given by simple algebraic functions

\[ \tilde{w}_c(z) = -\frac{(z + \sqrt{z^2 - M^2})^g + (z\sqrt{z^2 - M^2})^g}{2 \cos \pi g/2} \quad (4.12) \]

\[ \tilde{\omega}_c(z) = -\frac{(z + \sqrt{z^2 - M^2})^{g/2} + (z - \sqrt{z^2 - M^2})^{g/2}}{\sqrt{2 \sin \pi g/2}} \quad (4.13) \]

The solution \((4.12)\) corresponds to the normalization \((4.6)\).

The amplitudes for \(c = 1\) are obtained by taking the limit \(g \to 1\) of these expressions; we see that logarithmic singularities occur in this limit. The complete solution of \((4.10)\) is given in ref [9].

The renormalized loop fields satisfy the same loop equations as the bare fields, but a polynomial of \(z\) will appear on the r.h.s. of the equations \((3.5)\) and \((3.6)\). For generic value of the parameter \(g\) only the constant term of this polynomial will have the right dimension, but for half-integer \(g\) higher powers of \(z\) can survive. For example, in the case \(g = 3/2\) the r.h.s. of \((3.5)\) contains terms proportional to \(z\Lambda\) and \(z^3\).

5. The algebra of constraints

The logarithm of the partition function \(Z[J, J^B]\) is the generating functional for the connected loop correlators

\[ \langle w_x(\ell)\ldots\omega_{x'}(\ell')\ldots \rangle = \left( \frac{\delta}{\delta j_x(\ell)} \ldots \frac{\delta}{\delta j^B_{x'}(\ell')} \ldots \ln Z[j, j^B] \right)_{j = j^B = 0}. \quad (5.1) \]

The loop equations \((3.2)\) and \((3.3)\) can be formulated as second-order variational equations for \(Z[j, j^B]\)

\[ T_x(\ell) \quad Z = B_x(\ell) \quad Z = 0 \quad (5.2) \]
where

\[
T_x(\ell) = \int_0^\ell d\ell' \frac{\delta}{\delta j_x(\ell')} \frac{\delta}{\delta j_x(\ell - \ell')} + \int_0^\infty d\ell' \sum_{x'} C_{xx'} \left( \frac{\delta}{\delta j_x(\ell')} \frac{\delta}{\delta j_x(\ell + \ell')} + \kappa \ell' \frac{\delta}{\delta j_x^B(\ell')} \frac{\delta}{\delta j_x^B(\ell + \ell')} \right)
\]

(5.3)

\[
T_x(\ell) = \int_0^\infty d\ell' \ell' \left( \frac{\delta}{\delta j_x(\ell')} \frac{\delta}{\delta j_x(\ell + \ell')} + \sum_{x'} C_{xx'} \frac{\delta}{\delta j_x^B(\ell')} \frac{\delta}{\delta j_x^B(\ell + \ell')} \right)
\]

(5.4)

A consistency condition for the integrability of these equations is that the operators \( T, B \) should close a Lie-algebra. This is indeed the case:

\[
[T_x(\ell), T_{x'}(\ell')] = \kappa^2 (\ell - \ell') \delta_{xx'} T_x(\ell + \ell'); \quad [B_x(\ell), B_{x'}(\ell')] = 0
\]

(5.5)

\[
[B_x(\ell), T_{x'}(\ell')] = \kappa^2 \ell \delta_{xx'} B_x(\ell + \ell')
\]

The constraints form a set of decoupled algebras, one for each point \( x \). Each such algebra is a semi-direct product of a ”continuum” Virasoro and an \( U(1) \) current algebra, truncated to \( \ell > 0 \). The fact that the closed string sector is described by decoupled Virasoro algebras has been already pointed out by Ishibashi and Kawai in \[3\].

6. The loop space Hamiltonian

The loop space Hamiltonian will be constructed following the same logic as in ref.\[3\]. Let us introduce a representation of the fields \( W \) and \( \Omega \) as creation and annihilation operators of the string states. Denote by \( \Psi_x(\ell) \) (\( \Psi_x^\dagger(\ell) \)) the operator that annihilates (creates) an open string state with length \( \ell \) located at the point \( x \). Similarly we define the operators \( \Phi_x^\dagger(\ell) \) creating a closed string with a marked point and the corresponding annihilation operator \( \Phi_x(\ell) \). These operators satisfy the canonical commutation relations

\[
[\Psi_x(\ell), \Psi_{x'}^\dagger(\ell')] = \delta_{xx'} \delta(\ell - \ell'), \quad [\Phi_x(\ell), \Phi_{x'}^\dagger(\ell')] = \delta_{xx'} \delta(\ell - \ell').
\]

(6.1)
The vacuum state is defined by
\[ \langle 0 | \Psi_x^\dagger (\ell) = \langle 0 | \Phi_x^\dagger (\ell) = 0, \quad \Psi_x (\ell) | 0 \rangle = \Phi_x (\ell) | 0 \rangle = 0. \] (6.2)

Let us introduce the shorthand notations
\[ \{ f, g | h \} = \int_0^\infty d\ell d\ell' f(\ell) g(\ell') h(\ell + \ell'), \]
\[ \{ h | f, g \} = \int_0^\infty d\ell d\ell' h(\ell + \ell') f(\ell) g(\ell') \]
\[ f \cdot g = \sum_x \int_0^\infty d\ell f_x(\ell) g_x(\ell) \] (6.3)
and define the two Hamiltonians
\[ H = \sum_x \left( \{ \Phi_x^\dagger , \Phi_x^\dagger | \ell \Phi_x \} + \kappa^2 \{ \Phi_x^\dagger | \ell \Phi_x , \ell \Phi_x \} + \kappa^2 \{ \Psi_x^\dagger | \ell \Psi_x, \ell \Phi_x \} \right) \]
\[ \sum_{x,x'} C_{x,x'} \left( \{ \Phi_x^\dagger \Phi_{x'}^\dagger , \ell \Phi_x \} + \kappa \{ \Psi_x^\dagger | \ell \Psi_{x'}, \ell \Phi_x \} \right) \] (6.4)
\[ H_B = \sum_{x,x'} C_{x,x'} \{ \Psi_x^\dagger , \Psi_{x'} \} + \kappa \sum_x \{ \Psi_x^\dagger | \Psi_x \} + \Phi^\dagger \cdot \Psi \] (6.5)

We will argue that the limit \( t, t_B \to \infty \) of the functional
\[ Z_{t,t_B}[J, J^B] = \langle 0 | e^{w_c \cdot \Phi + \omega_c \cdot \Psi} e^{-tH - t_B H_B} e^{j^B \cdot \Phi^\dagger + j \cdot \Psi^\dagger} | 0 \rangle \] (6.6)
gives a formal solution to the constraints (5.2). First let us note that in the classical limit \( \kappa = 0 \), the functional (6.6) does not depend on the two time parameters \( t \) and \( t_B \) and is given by its value at \( t = t_B = 0 \)
\[ \left( Z_{t,t_B}[j, j^B] \right)_{\kappa = 0} = e^{w_c \cdot j + \omega_c \cdot j^B}. \] (6.7)
This can be checked by taking the derivatives in \( t \) and \( t_B \) and using the fact that \( w_c \) and \( \omega_c \) satisfy the planar loop equations. The solution for \( \kappa \neq 0 \) satisfies the differential equations
\[ \frac{\partial}{\partial t} Z_{t,t_B}[j, j^B] = \sum_x \int_0^\infty d\ell \ell j_x(\ell) T_x(\ell) Z_{t,t_B}[j, j^B] \] (6.8)
\[ \frac{\partial}{\partial t_B} Z_{t,t_B}[J,J^B] = \sum_x \int_0^\infty d\ell \ j_x^B(\ell) \ B_x(\ell) \ Z_{t,t_B}[j,j^B]. \] (6.9)

One can think of the evolution of the string as a sequence of elementary processes of splitting and joining. The operators (6.4) and (6.5) describe the evolution with respect to two time parameters \( t \) and \( t_B \). We can consider only one time \( t \) and introduce a dimensional constant \( R \) such that \( t_B = Rt \). The parameter \( R \) will compensate the difference of the time scales in the bulk and at the boundaries. The bulk Hamiltonian \( \mathcal{H} \) involves processes that can occur at any point of a string (see ref. \[9\] for a description). This is the origin of the \( \ell \)-factors associated with the annihilation operators. On the other hand, the boundary Hamiltonian \( \mathcal{H}_B \) describes elementary processes that are possible only at the endpoints of an open string. After an infinite amount of ”time” the system comes to an equilibrium and does not evolve any more. It follows from (6.8) and (6.9) that the functional

\[ Z[J,J^B] = \lim_{t't_B \to \infty} Z_{t,t_B}[J,J^B] \] (6.10)

satisfies the constraints (5.2).

An important feature of the Hamiltonians (6.4) and (6.5) is that they do not contain tadpole terms, in contrast with the Hamiltonian considered in \[4\] and \[9\]. Such terms can appear only for half-integer values of \( g \) corresponding to the multicritical regimes of a random surface with \( C = 0 \). Of course, the tadpole terms are present far from the critical point but they are multiplied by positive powers of the cutoff \( a \) and disappear in the scaling limit. This means that the dynamics in the scaling limit is played at large distances and the fraction of the strings with infinitesimal length is always small. This fact can be explained with the processes describing propagation of strings, which increase the length of the initial string state. Therefore, in order to obtain a nontrivial expectation value, we multiplied the left vacuum by \( e^{w_c \cdot j + \omega_c \cdot j^B} \). Let us also remark that the bulk and boundary Hamiltonians do not commute. Therefore one can construct more complicated Hamiltonians by adding commutator terms. In this way one can search a connection with the Hamiltonian for open \( C = 0 \) strings recently proposed in \[10\].
The dimensions of the fields and parameters, in terms of units of length \( L_0 \), are

\[
[\Phi_x(\ell)] = L_0^g, \quad [\Phi_x^\dagger(\ell)] = L_0^{-g-1}, \quad [\Psi_x(\ell)] = L_0^{g/2}, \quad [\Psi_x^\dagger(\ell)] = L_0^{-g/2-1}
\]

(6.11)

\[
[\ell] = L_0, \quad [M] = L_0^{-1}, \quad [\kappa] = L_0^{-g-1}, \quad [\Lambda] = L_0^{-2\nu}, \quad [\mu] = L_0^{-g}.
\]

(6.12)

This gives the following values for the dimensions of the two time parameters

\[
[t] = L_0^{g-1}, \quad [t_b] = L_0^{g/2}.
\]

(6.13)

The parameter \( t \) was identified in [2] with the time direction for a special choice of the coordinates on the world sheet, the so called ”temporal gauge”. A peculiarity of this gauge choice is that the string is allowed to suffer unlimited number of splittings before disappearing into the vacuum. This makes a sharp contrast with the conformal gauge. The geometrical meaning of the time parameter \( t \) is the geodesic (minimal) distance on the world sheet. More concretely, the time coordinate of a point close to the time slice \( t_0 \) is \( t_0 + \delta t \), where \( \delta t \) is the minimal distance from this point to the time slice \( t = t_0 \).

In our case, the interpretation of the time \( t \) as geodesic distance on the world sheet makes sense only if a statistical interpretation of the sum of the surfaces is possible, i.e., if all Boltzmann factors are positive. This condition is fulfilled if \( g < 2 \) and \( n = -2 \cos \pi g > 0 \). We also exclude the dense phase \( g < 1 \) in which the area between the domain walls vanishes and the dimension of the time \( t \) is negative. This leads us to the interval \( 1 \leq g \leq 3/2 \). In this interval, which describes string theories with \( 0 \leq c \leq 1 \), the intrinsic fractal dimension \( d_H \) of the world sheet is defined by \( A(r) \sim r^{d_H} \) where \( A(r) \) is the area of a circle with radius \( r \). The dimension if the area is \( [A] = [1/\Lambda] = L_0^{2\nu} \) where \( \nu = 1 \). Hence, \( d_H = 2/(g - 1) \) or, in terms of \( c \),

\[
d_H(c) = \frac{24}{1 - c + \sqrt{(1 - c)(25 - c)}}, \quad 0 \leq c < 1.
\]

(6.14)

In particular, for the unitary series \( c = 1 - 6/m(m+1) \) this formula gives the integer value \( d_H = 2m \) obtained in [4], where a different Hamiltonian was used. We see that the intrinsic
fractal dimension of the world sheet increases with \( c \) and becomes infinite at the critical dimension \( c = 1 \). The fractal dimension \( d_H = 4 \) for strings without embedding \( (C = 0) \) has been predicted both by numerical simulations \[11\] and analytic arguments \[12\]. It is also evident that eq. (6.14) is not true when considered for \( C < 0 \) since the expected behavior of \( d_H \) is \( D = 2 \) in the "classical" limit \( C \to -\infty \). The numerical simulations give the bound \( d_H > 3.5 \) for \( C = -2 \) \[13\].

The parameter \( t_B \) can be interpreted, in a similar manner, as the geodesic distance along the boundary representing a "quark" world line. The typical lengths of the Neumann and Dirichlet boundaries are \( \ell \sim 1/M \) and \( \ell_B \sim 1/\mu \), correspondingly. Note that the lengths measured along the two types of boundary have the same dimension only for \( c = 1 \).

The intrinsic fractal dimension \( d^B_H \) of the Neumann boundary is defined by \( \ell_B(r) \sim r^{d^B_H} \) where \( \ell(r) \) is the length of the set of points of the boundary at geodesic distance \( d < r \) from given point. The interpretation of \( t_B \) as geodesic distance gives the value

\[
d^B_H = 2.
\]

This means that for an observer living on the world surface, the true linear extension of the (Neumann) boundary will be the square root of its length. Therefore, independently of the dimension of the embedding space, the boundary will look from the interior of the surface as a random walk.

As a future development one may think of calculating the correlation functions of local operators at fixed geodesic distance by considering finite time intervals. Such quantities have been considered in \[12\]. Finally, let us remind that the closed strings with \( C = 1 - 6/m(m+1) \) can be obtained from two different Hamiltonians. The Hamiltonian constructed in \[3\] generates \( m - 1 \) decoupled Virasoro algebras while this of ref. \[4\] generates the \( W_m \) algebra of constraints on the partition function. Each of the two approaches has its advantages and it would be helpful to understand better their relation.

The author would like to thank Galen Sotkov and Y. Watabiki for useful discussions and F. David for critical reading of the manuscript.
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