Abstract. In this overview article we will consider the deliberate restarting of algorithms, a meta technique, in order to improve the algorithm's performance, e.g., convergence rates or approximation guarantees. One of the major advantages is that restarts are relatively black box, not requiring any (significant) changes to the base algorithm that is restarted or the underlying argument, while leading to potentially significant improvements, e.g., from sublinear to linear rates of convergence. Restarts are widely used in different fields and have become a powerful tool to leverage additional information that has not been directly incorporated in the base algorithm or argument. We will review restarts in various settings from continuous optimization, discrete optimization, and submodular function maximization where they have delivered impressive results.

Keywords: restarts · convex optimization · discrete optimization · submodular optimization.

1 Introduction

Restarts are a powerful meta technique to improve the behavior of algorithms. The basic idea is to deliberately restart some base algorithm, often with changed input parameters, to speed-up convergence, improve approximation guarantees, reduce number of calls to expensive subroutines and many more, often leading to provably better guarantees as well as significantly improved real-world computational performance. In actuality this comes down to running a given algorithm with a given set of inputs for some number of iterations, then changing the input parameters usually as a function of the output, and finally restarting the algorithm with new input parameters; rinse and repeat.

One appealing aspect of restarts is that they are relatively black-box, requiring only little to no knowledge of the to-be-restarted base algorithm except for the guarantee of the base algorithm that is then amplified by means of restarts. The reason why restarts often work, i.e., improve the behavior of the base algorithm is that some structural property of the problem under consideration is not explicitly catered for in the base algorithm, e.g., the base algorithm might work for general convex functions, however the function under consideration might be strongly convex or sharp. Restarts cater to this additional problem structure and are in particular useful when we want to incorporate data-dependent parameters. In fact, for several cases of interest the only known way to
incorporate that additional structure is via restarts often pointing out a missing piece in our understanding.

On the downside, restarts often explicitly depend on parameters arising from the additional structure under consideration and obtained guarantees are off by some constant factor or even log factor. The former can be often remedied with adaptive or scheduled restarts (see e.g., [39,23]) albeit with some minor cost. This way we can obtain fully adaptive algorithms that adapt to additional structure without knowing the accompanying parameters explicitly. The latter shortcoming is inherent to restart scheme as due to their black box nature additional structural information might not be incorporated perfectly.

Restarts have been widely used in many areas and fields and we will review some of these applications below to provide context. We would like to stress that references will be incomplete and biased; please refer also to the references cited therein.

**SAT solving and Constraint Programming.** Restarts are ubiquitous in SAT Solving and Constraint Programming to, e.g., explore different parts of the search space. Also after new clauses have been learned, these clauses are often added back to the formulation and then the solver is restarted. This can lead to dramatic overall performance improvements for practical solving; see e.g., [25,7] and references contained therein.

**Global Optimization.** Another important area where restarts are used is global optimization. Often applied to non-convex problems, the hope is that with randomized restarts different local optima can be explored, ideally one of those being a global one; see e.g., [24] and their references.

**Integer Programming.** Modern integer programming solvers use restarts in many different ways, several of which have been inspired by SAT solving and Constraint Programming. In fact, Integer Programming solvers can be quite competitive for pseudo-Boolean problems [6]. A relatively recent approach [4] is clairvoyant restarts based on online tree-size estimation that can significantly improve solving behavior.

Most of the restart techniques mentioned above, while very important, come without strong guarantees. In this article, we are more interested in cases, where provably strong guarantees can be obtained that also translate into real-world computational advantages. In the following, we will restrict the discussion to three examples from convex optimization, discrete optimization, and submodular function maximization. However, before we consider those, we would like to mention a two related areas where restarts have had a great impact not just from a computational point of view but also to establish new theoretical guarantees, but that are unfortunately beyond the scope of this overview.

**Variance Reduction via Restarts.** Usually when we consider stochastic convex optimization problems where the function is given as a general expectation and we would like to use first-order methods for solving the stochastic problem, we cannot expect a convergence rate better than $O(1/\sqrt{t})$ under usual assumptions, where $t$ is the number of stochastic gradient evaluations. However, it turns out that if we consider so-called finite sum problems, a problem class quite common in machine learning, where the
expectation is actually a finite sum and some mild additional assumptions are satisfied, then we can obtain a linear rate of convergence by means of variance reduction. This is an exponential improvement in convergence rate. Variance reduction techniques replace the stochastic gradient which is an unbiased estimator of the true gradient with a different, lower variance, unbiased estimator that is formed with the help of a reference point obtained from an earlier iterate. This reference point is then periodically reset via a restart scheme. Important algorithms here are for example Stochastic Variance Reduced Gradient Descent (SVRG) \cite{ johnson2013accelerating} and its numerous variants, such as e.g., the Stochastic Variance Reduced Frank-Wolfe algorithm (SVRFW) \cite{li2015accelerated}.

**Acceleration in Convex Optimization.** Restarts have been heavily used in convex optimization both for improving convergence behavior via restarts in real-world computations (see e.g., \cite{xiao2013accelerated}) but also as part of formal arguments to establish accelerated convergence rates and design provably faster algorithms. As the literature is particularly deep, we will sample only a few of those works in the context of first-order methods here that we are particularly familiar with; we provide further references in the sections to come. For example restarts have been used in \cite{behmanesh2009accelerated} to provide an alternative explanation of Nesterov’s acceleration as arising from the coupling mirror descent and gradient descent. In \cite{chambolle2016acceleration} it has been shown how restarts can be leveraged to obtain improved rates as the sharpness of the function (roughly speaking how fast the function curves around its minima) increases and these restart schemes have been also successfully carried over to the conditional gradients case in \cite{kale2016dimension}. Restarts have been also used to establish dimension-independent local acceleration for conditional gradients \cite{kale2016dimension} by means of coupling the Away-step Frank-Wolfe algorithm with an accelerated method. As we will see later in the context of submodular maximization, restarts can be also used to reduce the number of calls to expensive oracles. This has been extensively used for lazification of otherwise expensive algorithms in \cite{hu2014lazily, johnson2017lazier} leading to several orders of speed-up in actual computations while maintaining worst-case guarantees identical to those of the original algorithms and in \cite{li2018direct} a so-called optimal method based on lazification has been derived. Very recently, in \cite{xiao2018adaptive} a new adaptive restart scheme has been presented that does not require any knowledge of otherwise inaccessible parameters and its efficacy for saddle point problems has been demonstrated.

**Outline**

In Section 2 we consider restart examples from convex optimization and in Section 3 we consider examples from discrete optimization. Finally we consider submodular function maximization in Section 4. We keep technicalities to a bare minimum, sometimes simplifying arguments for the sake of exposition. We provide references though with the full argument, for the interested reader.

## 2 Smooth Convex Optimization

Our first examples come from smooth convex optimization. As often, the examples here are (arguably) the cleanest ones. We briefly recall some basic notions:
Definition 1 (Convexity). Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. Then $f$ is convex, if for all $x, y$ it holds:

$$f(y) - f(x) \geq \langle \nabla f(x), y - x \rangle.$$ 

In particular, all local minima of $f$ are global minima of $f$.

Definition 2 (Strong Convexity). Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable convex function. Then $f$ is $\mu$-strongly convex (with $\mu > 0$), if for all $x, y$ it holds:

$$f(y) - f(x) \geq \langle \nabla f(x), y - x \rangle + \frac{\mu}{2} \| y - x \|^2.$$ 

Definition 3 (Smoothness). Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. Then $f$ is $L$-smooth (with $L > 0$), if for all $x, y$ it holds:

$$f(y) - f(x) \leq \langle \nabla f(x), y - x \rangle + \frac{L}{2} \| y - x \|^2.$$ 

In the following let $x^* \in X^* = \arg\min f(x)$ denote an optimal solution from the set of optimal solutions $X^*$. Choosing $x = x^*$ and applying the definition of strong convexity (Definition 2) we immediately obtain:

$$f(y) - f(x^*) \geq \langle \nabla f(x^*), y - x^* \rangle + \frac{\mu}{2} \| y - x^* \|^2 \geq \frac{\mu}{2} \| y - x^* \|^2,$$

where the last inequality follows from $\langle \nabla f(x^*), y - x^* \rangle \geq 0$ by first-order optimality of $x^*$ for $\min f(x)$, i.e., the primal gap upper bounds the distance to the optimal solution. This also implies that the optimal solution $x^*$ is unique.

Smooth Convex to Smooth Strongly Convex: the basic case. Let $f : \mathbb{R}^n \to \mathbb{R}$ be an $L$-smooth convex function. Then using gradient descent, updating iterates $x_t$ according to $x_{t+1} \leftarrow x_t - \frac{1}{L} \nabla f(x_t)$, yields the following standard guarantee, see e.g., [33, 35, 21, 29]

Proposition 1 (Convergence of gradient descent: smooth convex case). Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth convex function and $x_0 \in \mathbb{R}^n$ and $x^* \in X^*$. Then gradient descent generates a sequence of iterates satisfying

$$f(x_t) - f(x^*) \leq \frac{L \| x_0 - x^* \|^2}{t}.$$ (2)

Now suppose we additionally know that the function $f$ is $\mu$-strongly convex. Usually, we would expect a linear rate of convergence in this case, i.e., to reach an additive error of $\epsilon$, we would need at most $T \leq \frac{L}{\mu} \log \frac{f(x_0) - f(x^*)}{\epsilon}$ iterations. However, rather than reproving the convergence rate (which is quite straightforward in this case) we want to reuse the guarantee in Proposition 1 as a black box and the $\mu$-strong convexity of $f$. We will use the simple restart scheme given in Algorithm 1: in restart phase $\ell$ we run a given base algorithm $\mathcal{A}$ for a fixed number of iterations $T_{\ell}$ on the iterate $x_{\ell-1}$ output in the previous iteration:
Algorithm 1 Simple restart scheme

**Input:** Initial point $x_0 \in \mathbb{R}^n$, base algorithm $\mathcal{A}$, iteration counts $(T_\ell)$.

**Output:** Iterates $x_1, \ldots, x_K \in \mathbb{R}^n$.

1: for $\ell = 1$ to $K$ do
2: \hspace{1em} $x_\ell \leftarrow \mathcal{A}(f, x_{\ell-1}, T_\ell)$ \hspace{1em} \{run base algorithm for $T_\ell$ iterations\}
3: end for

A priori, it is unclear whether the restart scheme in Algorithm 1 is doing anything useful, in fact even convergence might not be immediate as we in principle could undo work that we did in a preceding restart phase. Also note that when restarting vanilla gradient descent with a fixed step size of $\frac{1}{L}$ as we do here the final restarted algorithm is identical to vanilla gradient descent, i.e., the restarts do not change the base algorithm. This might seem nonsensical and we will get back to this soon; the reader can safely ignore this for now.

In order to analyze our restart scheme we first chain together Inequalities (2) and (1) and obtain:

$$f(x_t) - f(x^*) \leq \frac{L\|x_0 - x^*\|^2}{t} \leq \frac{2L}{\mu} \frac{f(x_0) - f(x^*)}{t}.$$  (3)

This chaining together of two error bounds is at the core of most restart arguments and we will see several variants of this. Next we estimate how long we need to run the base method, using Inequality (3) to halve the primal gap from some given starting point $x_0$ (this will be the point from which we are going to restart the base method), i.e., we want to find $t$ such that

$$f(x_t) - f(x^*) \leq \frac{2L}{\mu} \frac{f(x_0) - f(x^*)}{t} \leq \frac{f(x_0) - f(x^*)}{2},$$

which implies that it suffices to run gradient descent for $T_\ell = \left\lceil \frac{4L}{\mu} \right\rceil$ steps for all $\ell = 1, \ldots, K$ to halve a given primal bound as there is no dependency on the state of the algorithm in this case. Now, in order to reach $f(x_T) - f(x^*) \leq \varepsilon$, we have to halve $f(x_0) - f(x^*)$ at most $K = \left\lceil \log \frac{f(x_0) - f(x^*)}{\varepsilon} \right\rceil$ times and each of the halving can be accomplished in at most $\left\lceil \frac{4L}{\mu} \right\rceil$ gradient descent steps. All in all we obtain that after at most

$$T \geq \sum_{\ell=1}^{K} T_\ell = K \cdot T_1 = \left\lceil \frac{4L}{\mu} \right\rceil \left\lceil \frac{2L}{\mu} \right\rceil \log \frac{f(x_0) - f(x^*)}{\varepsilon}$$  (4)

gradient descent steps we have obtained a solution $f(x_T) - f(x^*) = f(x_K) - f(x^*) \leq \varepsilon$. With this we have obtained the desired convergence rate. Note that the iterate bound in Inequality (4) is optimal for vanilla gradient descent up to a constant factor of 4; see e.g., [33,29,21].

In the particular case from above it is also important to observe that our base algorithm gradient descent is essentially memoryless. In fact, the restarts do not ‘reset’ anything in this particular case and so we have also indirectly proven that gradient descent
without restarts will converge with the rate from Inequality (4). This is particular to this example though and will be different in our next one. Also, note that a direct estimation would have yielded the same rate up to the factor 4 discussed above.

**Smooth Convex to Smooth Strongly Convex: the accelerated case.** While the rate from Inequality (4) is essentially optimal for vanilla gradient descent it is known that (vanilla) gradient descent itself is not optimal for smooth and strongly convex functions and also Proposition 1 is not optimal for smooth and (non-strongly) convex functions. In fact Nesterov showed in [36] that for smooth and (non-strongly) convex functions a quadratic improvement can be obtained; a phenomenon commonly referred to as acceleration:

**Proposition 2 (Convergence of accelerated gradient descent).** Let $f : \mathbb{R}^n \to \mathbb{R}$ be an $L$-smooth convex function and $x_0 \in \mathbb{R}^n$ and $x^* \in X^*$. Then accelerated gradient descent generates a sequence of iterates satisfying

$$f(x_t) - f(x^*) \leq \frac{cL\|x_0 - x^*\|^2}{t^2},$$

for some constant $c > 0$.

Again, we could try to directly prove a better rate via acceleration for the smooth and strongly case (which is non-trivial this time) or, as before, invoke our restart scheme in Algorithm 1 in a black-box fashion, which is what we will do here. As before we will use an analog of Inequality (3) to estimate how long it takes to halve the primal gap, i.e., we want to find $t$ such that

$$f(x_t) - f(x^*) \leq \frac{2cL}{\mu} \frac{f(x_0) - f(x^*)}{t^2} \leq \frac{f(x_0) - f(x^*)}{2},$$

which implies that it suffices to run accelerated gradient descent for $T_\ell \doteq \lceil \sqrt{\frac{4cL}{\mu}} \rceil$ steps for all $\ell = 1, \ldots, K$ to halve a given primal gap. With the same reasoning as above we need to halve the primal gap at most $K \doteq \lceil \frac{\log (f(x_0) - f(x^*))}{\epsilon} \rceil$ times to reach an additive error of $\epsilon$. Putting everything together we obtain that after at most

$$T \geq \sum_{\ell=1}^{K} T_\ell = K \cdot T_1 = \lceil \sqrt{\frac{4cL}{\mu}} \rceil \left\lceil \frac{\log (f(x_0) - f(x^*))}{\epsilon} \right\rceil$$

accelerated gradient descent steps we have obtained a solution $f(x_T) - f(x^*) = f(x^K) - f(x^*) \leq \epsilon$. Note that the iterate bound in Inequality (5) is optimal for strongly convex and smooth functions (up to a constant factor). In contrast to the unaccelerated case, this time the restart actually ‘resets’ the base algorithm as accelerated gradient descent uses a specific step size strategy that is then reset.

**Remark 1.** Sometimes it is also possible to go backwards. Here we recover the optimal base algorithm for the smooth and (non-strongly) convex case from the strongly convex one. The argument is due to [34] (we follow the variant in [42]). Suppose we know an
optimal algorithm for the strongly convex and smooth case that ensures \( f(x_T) - f(x^*) \leq \epsilon \) after \( O\left(\frac{L}{\mu} \log \frac{f(x_0) - f(x^*)}{\epsilon}\right) \) iterations. Now consider a smooth and convex function \( f \) and an initial iterate \( x_0 \) together with some upper bound \( D \) on the distance to some optimal solution, i.e., \( \|x_0 - x^*\| \leq D \). Given an accuracy \( \epsilon > 0 \), we consider the auxiliary function

\[
  f_\epsilon(x) = f(x) + \frac{\epsilon}{2D^2} \|x - x_0\|^2,
\]

which is \((L + \frac{\epsilon}{2D^2})\)-smooth and \(\frac{\epsilon}{2D^2}\)-strongly convex. It can be easily seen that

\[
  f(x) - f(x^*) \leq f_\epsilon(x) - f_\epsilon(x^*) + \frac{\epsilon}{2},
\]

so that finding an \( \epsilon/2 \)-optimal solution to \( \min f_\epsilon \) provides an \( \epsilon \)-optimal solution to \( \min f \). We can now run the purported optimal method on the smooth and strongly convex function \( f_\epsilon \) to compute an \( \epsilon/2 \)-optimal solution to \( \min f_\epsilon \), which we obtain after:

\[
  O\left(\frac{L + \frac{\epsilon}{2D^2}}{\epsilon} \log \frac{f(x_0) - f(x^*)}{\epsilon}\right) = O\left(\frac{2LD^2 + \epsilon}{\epsilon} \log \frac{(L + \epsilon)D^2}{\epsilon}\right),
\]

iterations, where we used \( f_\epsilon(x_0) - f_\epsilon(x^*) \leq \frac{(L + \epsilon)D^2}{2} \). Finally note, ignoring the log factor, \( \frac{2LD^2 + \epsilon}{\epsilon} = T \Leftrightarrow \frac{2LD^2 + \epsilon}{T} \leq \epsilon \), which is the bound from Proposition 2.

The approach used in this section to obtain better rates of convergence under stronger assumptions by means of the simple restart scheme in Algorithm 1 works in much broader settings in convex optimization (including the constrained case). For example it can be used to improve the \( O(1/\sqrt{T}) \)-rate for general non-smooth convex functions via sub-gradient descent into the \( O(1/t) \)-rate for the non-smooth strongly convex case. Here the base rate is \( f(x_t) - f(x^*) \leq \frac{G\|x_0 - x^*\|}{\sqrt{t}} \), where \( G \) is a bound on the norm of the subgradients. We obtain the restart inequality chain (analog to Inequality 3):

\[
  f(x_t) - f(x^*) \leq \frac{G\|x_0 - x^*\|}{\sqrt{t}} \leq \frac{G}{\sqrt{t}} \sqrt{f(x_0) - f(x^*)} \frac{4G^2}{\mu},
\]

and halving the primal gap takes at most \( \frac{4G^2}{\mu f(x_0) - f(x^*)} \) iterations. Following the argumentation from above, we then arrive that the total number of required subgradient descent iterations using Algorithm 1 to ensure \( f(x_t) - f(x^*) \leq \epsilon \) is at most \( t \geq \frac{4G^2}{\mu} \) for the non-smooth but \( \mu \)-strongly convex case, which is optimal up to constant factors.

**Related approaches.** In a similar way we can incorporate additional information obtained e.g., from so-called Hölder(ian) Error Bounds or sharpness (see, e.g., [9,10] and references contained therein for an overview). The careful reader might have observed that the restart scheme in Algorithm 1 requires knowledge of the parameter \( \mu \). While this could be acceptable in the strongly convex case, for more complex schemes to leverage, e.g., sharpness, this is unacceptable as the required parameters are hard to estimate and generally inaccessible. This however, can be remedied in the case of sharpness, at
the cost of an extra $O(\log^2)$-factor in the rates, via scheduled restarts as done in \cite{39} that do not require sharpness parameters as input or when an error bound (of similar convergence rate) is available as in the case of conditional gradients \cite{27}, see also \cite{23} for a very recent adaptive restart scheme using error bounds estimators.

# 3 Discrete Optimization

In this section we consider a prominent example from integer programming: optimization via augmentation, i.e., optimizing by iteratively improving the current solution.

We consider the problem:

$$\max \{ cx \mid x \in P \cap \mathbb{Z}^n \},$$

where $P \subseteq \mathbb{R}^n$ is a polytope and $c \in \mathbb{Z}^n$.

**Algorithm 2 Augmentation**

**Input**: Feasible solution $x^0$ and objective $c \in \mathbb{Z}^n$

**Output**: Optimal solution of $\max \{ cx \mid x \in P \cap \{0,1\}^n \}$

1: $\tilde{x} \leftarrow x^0$
2: repeat
3: compute $x \in P$ integral with $c(x - \tilde{x}) > 0$ and set $\tilde{x} \leftarrow x$ \quad \{improve solution\}
4: until no improving solution exists
5: return $\tilde{x}$ \quad \{return optimal solution\}

To simplify the exposition we assume that $P \subseteq [0,1]^n$ and $c \geq 0$ (the latter is without loss of generality by flipping coordinates), however the arguments here generalize to the general integer programming case. Suppose further that we can compute improving solutions, i.e., given $c$ and a solution $x_0$, we can compute a new solution $x$, so that $c(x - x_0) > 0$ if $x_0$ was not already optimal; such a step (Line 3 in Algorithm 2) is called an augmentation step. Then a trivial and inefficient strategy is Algorithm 2, where we continue improving the solution until we have reached the optimum. It is not too hard to see that Algorithm 2 can take up to $2^n$ steps, essentially enumerating all feasible solutions to reach the optimal solution; simply consider the cube $P = [0,1]^n$ and an objective $c$ with powers of 2 as entries.

**Bit Scaling.** We will now show that we can do significantly better by restarting Algorithm 2 so that we obtain a number of augmentation steps of $O(n \log \|c\|_\infty)$, where $\|c\|_\infty = \max_{i \in [n]} c_i$. This is an exponential improvement over base algorithm and the restart scheme, called bit scaling, is due to \cite{11} (see also \cite{17,20}). It crucially relies on the following insight: Suppose we decompose our objective $c = 2c_1 + c_0$ with $c_0 \in \{0,1\}^n$ (note this decomposition is unique) and we have already obtained some solution $x_0 \in P \cap \{0,1\}^n$ that is optimal for $\max \{ c_1 x \mid x \in P \cap \mathbb{Z}^n \}$, then we have for
all \( x \in P \cap \{0, 1\}^n \):
\[
c(x - x_0) = \underbrace{2c_1(x - x_0) + c_0(x - x_0)}_{\leq 0} \leq n, \tag{7}
\]
by the optimality of \( x_0 \) for \( c_1 \) and \( c_0 \), \( x, x_0 \in \{0, 1\}^n \). Hence starting from \( x_0 \), for objective \( c \), there are at most \( n \) augmentation steps to be performed with Algorithm 2 to reach an optimal solution for \( c \). Equipped with Inequality (7) the following strategy emerges:

slice by the objective \( c \) according to its bit representation and then successively optimize with respect to the starting point from a previous slice. We first present the formal bit scaling restart scheme in Algorithm 3, where \( \mathcal{A} \) denotes Algorithm 2.

**Algorithm 3 Bit Scaling**

**Input:** Feasible solution \( x^0 \)

**Output:** Optimal solution to \( \max \{cx : x \in P \cap \mathbb{Z}^n\} \)

1: \( C \leftarrow \|c\|_{\infty} + 1, \mu \leftarrow 2^{\lceil \log C \rceil}, \tilde{x} \leftarrow x^0, c^\mu \leftarrow \lfloor c/\mu \rfloor \) \quad \text{[initialization]}

2: repeat

3: Call \( \tilde{x} \leftarrow \mathcal{A}(\tilde{x}, c^\mu) \)

4: \( \mu \leftarrow \mu/2, c^\mu \leftarrow \lfloor c/\mu \rfloor \)

5: until \( \mu < 1 \)

6: return \( \tilde{x} \) \quad \text{[return optimal solution]}

Next, we will show that restart scheme from Algorithm 3 requires at most \( O(n \log \|c\|_{\infty}) \) augmentation steps (Line 3 in Algorithm 2) to solve Problem (6). First observe, that by construction and the stopping criterion in Line 5 of Algorithm 3 it is clear that we call \( \mathcal{A} \) in Line 3 at most \( \lceil \log C \rceil \) times. Next, we bound the number of augmentation steps in Line 3 executed within algorithm \( \mathcal{A} \). To this end, let \( \tilde{x} \) and \( \mu \) denote the input to \( \mathcal{A} \). In the first iteration \( c^\mu \in \{0, 1\}^n \), so that \( \mathcal{A} \) can perform at most \( n \) augmentation steps. For later iterations observe that \( \tilde{x} \) was optimal for \( c^{2\mu} = \lfloor c/(2\mu) \rfloor \). Moreover, we have \( c^\mu = \lfloor c/\mu \rfloor = 2c^{2\mu} + c_0 \), where \( c_0 \in \{0, 1\}^n \) as before. Via Inequality (7) we obtain for all feasible solutions \( x \in P \cap \mathbb{Z}^n \):
\[
c^\mu(x - \tilde{x}) = 2c^{2\mu}(x - \tilde{x}) + c_0(x - \tilde{x}) \leq n,
\]
which holds in particular for the optimal solution \( x^* \) to Problem (6). As each augmentation step reduces the primal gap \( c^\mu(x - \tilde{x}) \) by at least 1, we can perform at most \( n \) augmentation steps. This completes the argument.

**Geometric Scaling.** The restart scheme in Algorithm 3 essentially restarted via bit-scaling the objective function, hence the name. We will now present a more versatile restart scheme that is due to [40] (see also [30] for a comparison and worst-case examples), which essentially works by restarting a regularization of our objective \( c \). For comparability we also consider Problem (6) here, however the approach is much more general, e.g., allowing for general integer programming problems and with modifications even convex programming problems over integers.
Again, we will modify the considered objective function \( c \) in each restart. Given the original linear objective \( c \), we will consider:

\[
c^\mu(x, \tilde{x}) = c(x - \tilde{x}) - \mu \|x - \tilde{x}\|_1.
\]

Note that \( c^\mu(x, \tilde{x}) \) is a linear function in \( x \in \{0, 1\}^n \) for a given \( \tilde{x} \in \{0, 1\}^n \). In particular we can call Algorithm 2 with objective \( c^\mu(\cdot, \cdot) \) and starting point \( \tilde{x} \). The restart scheme works as follows: For a given \( \mu \) we call Algorithm 2 with objective \( c^\mu(\cdot, \cdot) \) and starting point \( \tilde{x} \). Then we halve \( \mu \) and repeat.

As in the bit-scaling case, the key is to estimate the number of augmentation steps performed in such a call. To this end let \( x_0 \) be returned by Algorithm 2 for a given \( \mu \) and starting point \( \tilde{x} \). Then

\[
c^\mu(x, x_0) = c(x - x_0) - \mu \|x - x_0\|_1 \leq 0,
\]

holds for all \( x \in P \cap \mathbb{Z}^n \) and in particular for the optimal solution \( x^* \); this is simply the negation of the improvement condition. Now let \( x^* \) be any iterate in the following call to Algorithm 2 for which an augmentation step is performed with objective \( c^{\mu/2}(\cdot, \cdot) \) and starting point \( x_0 \), i.e., there exists \( x^* \) so that

\[
c^{\mu/2}(x^*, x^*) = c(x^* - x^*) - \mu/2 \|x^* - x^*\|_1 > 0.
\]

We can now combine these two inequalities, substituting \( x \leftarrow x^* \), to obtain

\[
\frac{2}{\|x^* - x^*\|_1} > \mu \geq \frac{c(x^* - x_0)}{\|x^* - x_0\|_1},
\]

which implies

\[
c(x^* - x^*) \geq \frac{1}{2} \frac{\|x^* - x^*\|_1}{\|x^* - x_0\|_1} c(x^* - x_0) \geq \frac{1}{2n} c(x^* - x_0),
\]

where \( \|x^* - x^*\|_1 \geq 1 \) as the iterates are not identical and \( \|x^* - x_0\|_1 \leq n \) as \( x^*, x_0 \in P \subseteq [0, 1]^n \). As such each augmentation step recovers at least a \( \frac{1}{2n} \)-fraction of the primal gap \( c(x^* - x_0) \) and therefore we can do at most \( 2n \) such iterations before the condition in Line 3 has to be violated. With this we can formulate the geometric scaling restart scheme in Algorithm 4. The analysis now is basically identical to the one as for Algorithm 3 however this time we have \( O(\log n \|c\|_{\infty}) \) restarts, leading to an overall number of augmentation steps of \( O(n \log n \|c\|_{\infty}) \), which can be further improved to \( O(n \log \|c\|_{\infty}) \), matching that of bit-scaling, with the simple observation in [30].

**Related Approaches.** Chvátal-Gomory cutting planes, introduced by Chvátal in [13], are an important tool in integer programming to approximate the integral hull \( \text{conv}(P \cap \mathbb{Z}^n) \) by means of successively strengthening an initial relaxation \( P \) with \( \text{conv}(P \cap \mathbb{Z}^n) \subseteq P \). This is done by adding new inequalities valid for \( \text{conv}(P \cap \mathbb{Z}^n) \) cutting off chunks of \( P \) in each round. A key question is how many rounds of such strengthenings are needed until we recover \( \text{conv}(P \cap \mathbb{Z}^n) \). In [14] it was shown that in general the number of
Algorithm 4 Geometric Scaling

Input: Feasible solution $x^0$
Output: Optimal solution of max $\{cx | x \in P \cap \mathbb{Z}^n\}$

1: $C \leftarrow \|c\|_{\infty} + 1, \mu \leftarrow nC, \tilde{x} \leftarrow x^0, \phi(y) \oplus c(x - y) - \mu\|x - y\|_1$. \{initialization\}
2: repeat
3: Call $\tilde{x} \leftarrow A(\tilde{x}, c\mu)$
4: $\mu \leftarrow \mu/2$
5: until $\mu < 1$
6: return $\tilde{x}$ \{return optimal solution\}

rounds can be arbitrarily large. It was then shown in [8] via a restart argument that for the important case of polytopes contained in $[0,1]^n$ the number of rounds can be upper bounded by $O(n^3 \log n)$. The key here is to use basic bounds on the number of rounds, e.g., from [13], first for inequalities with some maximum absolute entry $c$, then doubling up to $2c$, and restarting the argument. This bound was further improved in [13] to $O(n^2 \log n)$ by interleaving two restart arguments, one multiplicative (e.g., doubling) over the maximum absolute entry $c$ and one additive (e.g., adding a constant) over the dimension, which matches the lower bound of $\Omega(n^2)$ of [39] up to a log factor; closing this gap remains an open problem. As mentioned in the context of the scheduled restarts of [39], it might be possible that the additional log factor is due to the restart schemes itself and removing it might require a different proof altogether.

Another important application is the approximate Carathéodory problem, where we want to approximate $x^0 \in P$, where $P$ is a polytope, by means of a sparse convex combination $x$ of vertices of $P$, so that $\|x_0 - x\| \leq \varepsilon$ for some norm $\|\cdot\|$ and target accuracy $\varepsilon$. In general it is known that this can be done with a convex combination of $O(1/\varepsilon^2)$ vertices. However, it turns out as shown in [31] that whenever $x_0$ lies deep inside the polytope $P$, i.e., we can fit a ball around $x_0$ with some radius $r$ into $P$ as well, then we can exponentially improve this bound via restarts to $O(1/r^2 \log 1/\varepsilon)$. This restart argument here is particularly nice. We run the original $O(1/\varepsilon^2)$-algorithm down to some fixed accuracy and obtain some approximation $\tilde{x}$, then scale-up the feasible region by a factor of $2$, and restart the $O(1/\varepsilon^2)$-algorithm on the residual $x_0 - \tilde{x}$ and repeat. The argument in [31] relies on mirror descent as underlying optimization routine. More recently, it was shown in [15] that the restarts can be removed and adaptive bounds for more complex cases can be obtained by using conditional gradients as base optimization algorithm, which automatically adapts to sharpness (and optima in the interior) [43,27].

4 Submodular Function Maximization

We now turn our attention to submodular function maximization. Submodularity captures the diminishing returns property and is widely used in optimization and machine learning. In particular, we will consider the basic but important setup of maximizing a monotone, non-negative, submodular function subject to a single cardinality constraint. To this end we will briefly repeat necessary notions. A set function $g : 2^V \rightarrow \mathbb{R}_+$ is submodular if and only if for any $e \in V$ and $A \subseteq B \subseteq V \setminus \{e\}$ we have $g_A(e) \geq g_B(e)$, where
f

g_{A}(e) \triangleq g(A + e) - g(A) \text{ denotes the marginal gain of } e \text{ w.r.t. } A \text{ and } A + e \triangleq A \cup \{e\}, slightly abusing notation. The submodular function \( g \) is monotone if for all \( A \subseteq B \subseteq V \) it holds \( g(A) \leq g(B) \) and non-negative if \( g(A) \geq 0 \) for all \( A \subseteq V \).

Given a monotone, non-negative submodular function \( g \) over ground set \( V \) of size \( n \) and a budget \( k \), we consider the problem

\[
\max_{S \subseteq V, |S| \leq k} g(S) \tag{8}
\]

It is well known that solving Problem (8) exactly is NP-hard under the value oracle model, however the greedy algorithm (Algorithm 5) that in each iteration adds the element that maximizes the marginal gain yields a \((1 - 1/e)\)-approximate solution \( S^* \subseteq V \) with \( |S^*| \leq k \), i.e., \( g(S^*) \geq (1 - 1/e)g(S^*) \), where \( S^* = \arg\max_{S \subseteq V, |S| \leq k} g(S) \subseteq V \) is an optimal solution to Problem (8) and \( e \) denotes the Euler constant (see [32,19]).

**Algorithm 5** Greedy Algorithm

**Input:** Ground set \( V \) of size \( n \), budget \( k \), and monotone, non-negative, submodular function \( g : 2^V \to \mathbb{R}_+ \).

**Output:** feasible set \( S^* \) with \(|S^*| \leq k\).

1: \( S^* \leftarrow \emptyset \)
2: while \(|S^*| \leq k\) do
3: \( e \leftarrow \arg\max_{e \in V \setminus S^*} g_{S^*}(e) \)
4: \( S^* \leftarrow S^* + e \)
5: end while

The proof of the approximation guarantee of \( 1 - 1/e \) is based on the insight that in each iteration it holds:

\[
g(S^*) - g(S^+) \leq k \cdot \max_{e \in V} g_{S^*}(e). \tag{9}
\]

To see that Inequality (9) holds, let \( S^* = \{e_1, \ldots, e_k\} \), then

\[
g(S^*) \leq g(S^* \cup S^+) = g(S^+ \cup e_{i_1}, \ldots, e_{i_k}) (e_i) 
\leq g(S^+) + \sum_{i=1}^{k} g_{S^*}(e_i) \leq g(S^+) + k \max_{e \in V} g_{S^*}(e),
\]

where the first inequality follows from monotonicity, the equation follows from the definition of \( g_{S^*}(v) \), the second inequality from submodularity, and the last inequality from taking the maximizer.

With Inequality (9), the proof of the \((1 - 1/e)\)-approximation is immediate. In each iteration the greedy element we add satisfies \( \max_{e \in V} g_{S^*}(e) \geq \frac{1}{k}(g(S^*) - g(S^+)) \), therefore after \( k \) iterations we have obtained a set \( S^* \) with \(|S^*| = k\), with

\[
g(S^*) - g(S^+) \leq (1 - 1/k)^k (g(S^*) - g(\emptyset)) \leq (1 - 1/k)^k g(S^*) \leq \frac{1}{e} g(S^*),
\]
so that the desired guarantee \((1 - \frac{1}{e})g(S^*) \leq g(S^+)\) follows.

Unfortunately, due to Line 3 in Algorithm 5 computing such a \((1 - 1/e)\)-approximate solution can cost up to \(O(kn)\) evaluations of \(g\) in the value oracle model, where we can only query function values of \(g\). For realistic functions this is often quite prohibitive. We will now see a different application of a restart scheme to reduce the total number of function evaluations of \(g\) by allowing for a small error \(\varepsilon > 0\). We obtain a total number of evaluations of \(g\) of \(O(n \varepsilon \log n / \varepsilon)\), quasi-linear and independent of \(k\), to compute a \((1 - 1/e - \varepsilon)\)-approximate solution. The argument is due to [5] and similar in nature to the argument in Section 3 for geometric scaling. We simplify the argument slightly for exposition; see [5] for details.

The basic idea is rather than computing the actual maximum in Line 3 in Algorithm 5, we collect all elements of marginal gains that are roughly maximal within a \((1 - \varepsilon)\)-factor, then scale down the estimation of the maximum, and then restart. We present the restart scheme, the so-called Threshold Greedy Algorithm in Algorithm 6. This time we present the scheme and the base algorithm directly together. Note that the inner loop in Lines 5 to 7 in Algorithm 6 adds all elements that have approximately maximal marginal gain. The restarts are happening whenever we go back to the beginning of the outer loop starting in Line 2 with a reset value for \(\Phi\).

**Algorithm 6** Threshold Greedy Algorithm

**Input:** Ground set \(V\) of size \(n\), budget \(k\), accuracy \(\varepsilon\), and monotone, non-negative, submodular function \(g : 2^V \rightarrow \mathbb{R}_+\).

**Output:** feasible set \(S^+\) with \(|S^+| \leq k\).

1: \(S^+ \leftarrow \emptyset\), \(\Phi_0 \leftarrow \max_{e \in V} g(e)\), \(\Phi \leftarrow \Phi_0\)
2: while \(\Phi \geq \varepsilon \Phi_0\) do
3: for \(e \in V\) do
4: if \(|S^+| < k\) and \(g_{S^+}(e) \geq \Phi\) then
5: \(S^+ \leftarrow S^+ + e\)
6: end if
7: end for
8: \(\Phi \leftarrow \Phi (1 - \varepsilon)\)
9: end while

We will first show that the gain from any new element \(e \in V\) added in Line 5 of Algorithm 6 is at least

\[ g_{S^+}(e) \geq \frac{1 - \varepsilon}{k} \sum_{x \in S^* \setminus S^+} g_{S^+}(x). \]

To this end suppose we have have chosen element \(e \in V\) to be added. Then \(g_{S^+}(e) \geq \Phi\) by Line 4 and for all \(x \in S^* \setminus (S^+ + e)\) we have \(g_{S^+}(x) \leq \Phi/(1 - \varepsilon)\); otherwise we would have added \(x\) in an earlier restart with a higher value \(\Phi\) already. Combining the two inequalities we obtain

\[ g_{S^+}(e) \geq (1 - \varepsilon)g_{S^+}(x), \]
for all $x \in S^* \setminus (S^* + e)$ and averaging those inequalities leads to
\[
g_{S^*}(e) \geq \frac{1 - e}{|S^* \setminus S^*|} \sum_{x \in S^* \setminus S^*} g_{S^*}(x) \geq \frac{1 - e}{k} \sum_{x \in S^* \setminus S^*} g_{S^*}(x),
\] (10)
which is the desired inequality. From this we immediately recover the (approximate) analog of Inequality (9). We have via submodularity and non-negativity
\[
\sum_{x \in S^* \setminus S^*} g_{S^*}(x) \geq g_{S^*}(S^*) \geq g(S^*) - g(S^+),
\]
and together with Inequality (10)
\[
g_{S^*}(e) \geq \frac{1 - e}{k} (g(S^*) - g(S^+)).
\]
Therefore, as before, after $k$ iterations we obtain a set $S^+$ with $|S^+| = k$, with
\[
g(S^*) - g(S^+) \leq (1 - (1 - e)/k)^k (g(S^*) - g(\emptyset)) \leq (1 - (1 - e)/k)^k g(S^+)
\]
\[
\leq \frac{1}{e^{(1 - e)}} g(S^*) \leq \left( \frac{1}{e} + \epsilon \right) g(S^*),
\]
leading to our guarantee $g(S^+) \geq (1 - \frac{1}{e} - \epsilon) g(S^*)$. If we do fewer than $k$ iterations, the total gain of all remaining elements is less than $e$, establishing the guarantee in that case.

Now for the number of evaluations of $g$, first consider the loop in Line 2 of Algorithm 6. The loops stops after $\ell$ iterations, whenever $(1 - e)^{\ell} \leq \frac{2}{n}$, which is satisfied if $1/(1 - e)^{\ell} \geq (1 + e)^{\ell} \geq \frac{2}{\epsilon}$ and hence $\ell \geq \frac{1}{\epsilon} \log \frac{2}{\epsilon}$. For each such loop iteration we have at most $O(n)$ evaluations of $g$ in Line 4 leading to the overall bound of $O(\frac{2}{\epsilon} \log \frac{2}{\epsilon})$ evaluations of $g$.

Related Approaches. The approach presented here for the basic case with a single cardinality constraint can be applied more widely as already done in [5] for matroid, knapsack, and $p$-system constraints. It can be also used to reduce the number of evaluations in the context of robust submodular function maximization [24].

A similar restart approach has been used to ‘lazify’ conditional gradient algorithms in [28,11,12]. Here is the number of calls to the underlying linear optimization oracle is dramatically reduced by reusing information from previous iterations by solving the linear optimization problem only approximately as done in the case of the Threshold Greedy Algorithm. The algorithm, in a similar vein, is then restarted, whenever the threshold for approximation of the maximum is too large.

Acknowledgement

We would like to thank Gábor Braun and Marc Pfetsch for helpful comments and feedback on an earlier version of this article.
References

1. Allen-Zhu, Z., Orecchia, L.: Linear coupling: An ultimate unification of gradient and mirror descent. arXiv preprint arXiv:1407.1537 (2014)
2. Anari, N., Haghtalab, N., Naor, S., Pokutta, S., Singh, M., Torrico, A.: Structured Robust Submodular Maximization: Offline and Online Algorithms. Proceedings of AISTATS (2019)
3. Anari, N., Haghtalab, N., Naor, S., Pokutta, S., Singh, M., Torrico, A.: Structured Robust Submodular Maximization: Offline and Online Algorithms. To appear in INFORMS Journal on Computing (2020+)
4. Anderson, D., Hendel, G., Le Bodic, P., Viernickel, M.: Clairvoyant restarts in branch-and-bound search using online tree-size estimation. In: Proceedings of the AAAI Conference on Artificial Intelligence. vol. 33, pp. 1427–1434 (2019)
5. Badanidiyuru, A., Vondrák, J.: Fast algorithms for maximizing submodular functions. In: Proceedings of the twenty-fifth annual ACM-SIAM symposium on Discrete algorithms. pp. 1497–1514. SIAM (2014)
6. Berthold, T., Heinz, S., Pfetsch, M.E.: Solving pseudo-Boolean problems with SCIP (2008)
7. Biere, A.: Adaptive restart strategies for conflict driven sat solvers. In: International Conference on Theory and Applications of Satisfiability Testing. pp. 28–33. Springer (2008)
8. Bockmayr, A., Eisenbrand, F., Hartmann, M., Schulz, A.: On the Chvátal rank of polytopes in the 0/1 cube. Discrete Applied Mathematics 98, 21–27 (1999)
9. Bolte, J., Daniilidis, A., Lewis, A.: The lojasiewicz inequality for nonsmooth subanalytic functions with applications to subgradient dynamical systems. SIAM Journal on Optimization 17(4), 1205–1223 (2007)
10. Bolte, J., Nguyen, T.P., Peypouquet, J., Suter, B.W.: From error bounds to the complexity of first-order descent methods for convex functions. Mathematical Programming 165(2), 471–507 (2017)
11. Braun, G., Pokutta, S., Zink, D.: Lazifying Conditional Gradient Algorithms. Proceedings of the International Conference on Machine Learning (ICML) (2017)
12. Braun, G., Pokutta, S., Zink, D.: Lazifying Conditional Gradient Algorithms. Journal of Machine Learning Research (JMLR) 20(71), 1–42 (2019)
13. Chvátal, V.: Edmonds polytopes and a hierarchy of combinatorial problems. Discrete Mathematics 4, 305–337 (1973)
14. Chvátal, V., Cook, W., Hartmann, M.: On cutting-plane proofs in combinatorial optimization. Linear algebra and its applications 114, 455–499 (1989)
15. Combettes, C.W., Pokutta, S.: Revisiting the Approximate Carathéodory Problem via the Frank-Wolfe Algorithm. Preprint (2019)
16. Diakonikolas, J., Carderera, A., Pokutta, S.: Locally accelerated conditional gradients. To appear in Proceedings of AISTATS (arXiv:1906.07867) (2020)
17. Edmonds, J., Karp, R.M.: Theoretical improvements in algorithmic efficiency for network flow problems. Journal of the ACM 19(2), 248–264 (1972)
18. Eisenbrand, F., Schulz, A.: Bounds on the Chvátal rank on polytopes in the 0/1-cube. Combinatorica 23(2), 245–261 (2003)
19. Fisher, M.L., Nemhauser, G.L., Wolsey, L.A.: An analysis of approximations for maximizing submodular set functions—ii. In: Polyhedral combinatorics, pp. 73–87. Springer (1978)
20. Graham, R.L., Grötschel, M., Lovász, L.: Handbook of combinatorics, vol. 1. Elsevier (1995)
21. Hazan, E.: Lecture notes: Optimization for machine learning. arXiv preprint arXiv:1909.03550 (2019)
22. Hazan, E., Luo, H.: Variance-reduced and projection-free stochastic optimization. In: International Conference on Machine Learning. pp. 1263–1271 (2016)
23. Hinder, O., Lubin, M.: A generic adaptive restart scheme with applications to saddle point algorithms. arXiv preprint arXiv:2006.08484 (2020)
24. Hu, X., Shonkwiler, R., Spruill, M.C.: Random restarts in global optimization (2009)
25. Huang, J., et al.: The effect of restarts on the efficiency of clause learning. In: IJCAI. vol. 7, pp. 2318–2323 (2007)
26. Johnson, R., Zhang, T.: Accelerating stochastic gradient descent using predictive variance reduction. In: Advances in neural information processing systems. pp. 315–323 (2013)
27. Kerdreux, T., d’Aspremont, A., Pokutta, S.: Restarting Frank-Wolfe. Proceedings of AISTATS (2019)
28. Lan, G., Pokutta, S., Zhou, Y., Zink, D.: Conditional Accelerated Lazy Stochastic Gradient Descent. Proceedings of the International Conference on Machine Learning (ICML) (2017)
29. Lan, G.: First-order and Stochastic Optimization Methods for Machine Learning. Springer (2020)
30. Le Bodic, P., Pfletsch, M., Pavelka, J., Pokutta, S.: Solving MIPs via Scaling-based Augmentation. Discrete Optimization 27, 1–25 (2018)
31. Mirrokni, V., Paes Leme, R., Vladu, A., Wong, S.C.W.: Tight bounds for approximate Carathéodory and beyond. In: Proceedings of the 34th International Conference on Machine Learning. pp. 2440–2448 (2017)
32. Nemhauser, G.L., Wolsey, L.A., Fisher, M.L.: An analysis of approximations for maximizing submodular set functions—i. Mathematical programming 14(1), 265–294 (1978)
33. Nemirovski, A.: Lectures on modern convex optimization. In: Society for Industrial and Applied Mathematics (SIAM. Citeseer (2001)
34. Nesterov, Y.: How to make the gradients small. Optima. Mathematical Optimization Society Newsletter (88), 10–11 (2012)
35. Nesterov, Y.: Lectures on Convex Optimization. Springer (2018)
36. Nesterov, Y.: A method for solving the convex programming problem with convergence rate $O(1/k^2)$. In: Dokl. akad. nauk Ssr. vol. 269, pp. 543–547 (1983)
37. O’donoghue, B., Candes, E.: Adaptive restart for accelerated gradient schemes. Foundations of computational mathematics 15(3), 715–732 (2015)
38. Rothvöß, T., Sanità, L.: 0/1 polytopes with quadratic chvátal rank. Operations Research 65(1), 212–220 (2017)
39. Roulet, V., d’Aspremont, A.: Sharpness, restart and acceleration. ArXiv preprint arXiv:1702.03828 (2017)
40. Schulz, A.S., Weismantel, R.: The complexity of generic primal algorithms for solving general integer programs. Mathematics of Operations Research 27(4), 681–692 (2002)
41. Schulz, A.S., Weismantel, R., Ziegler, G.M.: 0/1-integer programming: Optimization and augmentation are equivalent. In: Algorithms – ESA ’95, Proceedings. pp. 473–483 (1995)
42. Scieur, D., d’Aspremont, A., Bach, F.: Regularized nonlinear acceleration. In: Advances In Neural Information Processing Systems. pp. 712–720 (2016)
43. Xu, Y., Yang, T.: Frank-Wolfe Method is Automatically Adaptive to Error Bound Condition (2018)