Angle-restricted sets
and zero-free regions for the permanent

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1. Introduction

A subset \( S \subset \mathbb{C}^* := \mathbb{C} \setminus \{0\} \) is called a zero-free region for the permanent if the permanent of a square matrix (of any size \( n \)) with entries in \( S \) is necessarily nonzero. The motivation for studying such regions comes from the work of A. Barvinok ([B1]), where he shows that the logarithm of the permanent of such a matrix can be computed within relative error \( \varepsilon \) in quasi-polynomial time \( n^{O \left( \log n - \log \varepsilon \right)} \) (while the problem of efficient computation of general permanents is hopelessly hard). Namely, it is shown in [B1] that the disk \( |z - 1| \leq 1/2 \) and a certain family of rectangles are zero-free regions, which enables efficient approximate computation of permanents of matrices with entries from these regions.\(^1\)

The goal of this note is to give a systematic method of constructing zero-free regions for the permanent. We do so by refining the approach of [B1] using the clever observation that a certain restriction on a set \( S \) involving angles implies zero-freeness ([B1]); we call sets satisfying this requirement angle-restricted. This allows us to reduce the question to a low-dimensional geometry problem (notably, independent of the size of the matrix!), which can then be solved more or less explicitly. We give a number of examples, improving some results of [B1]. This technique also applies to more general problems of a similar kind, discussed in [B2].

**Acknowledgements.** This note was inspired by the Simons lectures of A. Barvinok at MIT in April 2019; namely, it is a (partial) solution of a “homework problem” given in one of these lectures. I am very grateful to A. Barvinok for useful discussions, suggestions and encouragement. I am also very indebted to two anonymous referees for thorough reading of the paper and very useful comments and corrections.

2. Definition and basic properties of angle-restricted sets

For \( u, v \in \mathbb{C}^* \) let \( \alpha(u, v) \in [0, \pi] \) be the angle between \( u \) and \( v \). Let \( \theta, \phi \in (0, 2\pi/3) \). Note that if \( u_1, \ldots, u_n \in \mathbb{C}^* \) are such that \( \alpha(u_i, u_j) \leq \theta \)

\(^1\)We note that a (randomized) efficient algorithm for computing the permanent of a matrix with nonnegative entries was proposed earlier in [JSV].
then there exists \( \lambda \in \mathbb{C}^* \) such that \( |\arg(\lambda u_i)| \leq \theta/2 \) for all \( i \) (where we agree that \( \arg(z) \) takes values in \( (-\pi, \pi] \)).

**Definition 2.1.** (i) We denote by \( A_{\theta,\phi} \) the set of subsets \( S \subseteq \mathbb{C}^* \) such that for any \( u_1, \ldots, u_n \in \mathbb{C}^* \) with \( \alpha(u_i, u_j) \leq \theta \) for all \( i, j \) and any \( a_1, \ldots, a_n, b_1, \ldots, b_n \in S \), the numbers \( v = \sum_i a_i u_i \) and \( w = \sum_i b_i u_i \) are nonzero and \( \alpha(v, w) \leq \phi \). In other words, if \( u_i \) belong to the angle \( |\arg(z)| \leq \theta/2 \) then there exists \( \mu \in \mathbb{C}^* \) such that \( \mu v, \mu w \) belong to the angle \( |\arg(u)| \leq \phi/2 \). We say that a set \( S \subseteq \mathbb{C}^* \) is \((\theta, \phi)\)-angle restricted if \( S \in A_{\theta,\phi} \). If \( \theta = \phi \) then we denote \( A_{\theta,\phi} \) by \( A_\theta \).

(ii) We denote by \( A^2_{\theta,\phi} \) the set of subsets \( S \subseteq \mathbb{C}^* \) such that for any \( a, b, c, d \in S \) the map \( z \mapsto \frac{az+b}{cz+d} \) maps the angle \( \{ z \in \mathbb{C}^* : |\arg(z)| \leq \theta \} \) into the angle \( \{ u \in \mathbb{C}^* : |\arg(u)| \leq \phi \} \). In other words, \( S \in A^2_{\theta,\phi} \) if and only if any \( a, b, c, d \in S \) satisfy the condition of (i) for \( n = 2 \). We denote \( A^2_{\theta,\phi} \) by \( A^2_\theta \).

(iii) We denote by \( B^2_{\theta,\phi} \) the set of subsets \( S \subseteq \mathbb{C}^* \) such that for any \( a, b \in S \) the map \( z \mapsto \frac{az+b}{z+1} \) maps the angle \( \{ z \in \mathbb{C}^* : |\arg(z)| \leq \theta \} \) into the angle \( \{ u \in \mathbb{C}^* : |\arg(u)| \leq \phi/2 \} \). We denote \( B^2_{\theta,\phi} \) by \( B^2_\theta \).

**Remark.** Condition (i) for \( n = 2 \) says that for any \( u_1, u_2 \in \mathbb{C}^* \) with \( \alpha(u_i, u_j) \leq \theta \) and \( a, b, c, d \in S \) we have \( \alpha(au_1 + bu_2, cu_1 + du_2) \leq \phi \). This can be written as \( |\arg(\frac{az+b}{cz+d})| \leq \phi \), where \( z := \frac{u_1}{u_2} \), which implies that the two definitions of \( A^2_{\theta,\phi} \) in (ii) are equivalent.

It is clear that \( A_{\theta,\phi} \subseteq A^2_{\theta,\phi} \) and \( B^2_{\theta,\phi} \subseteq A^2_{\theta,\phi} \) (as \( \frac{az+b}{cz+d} = \frac{az+b}{z+1} \cdot \frac{z+1}{cz+d} \)), and that \( A_{\theta,\phi}, A^2_{\theta,\phi} \) are invariant under rescaling by a nonzero complex number, while \( B^2_{\theta,\phi} \) is invariant under rescaling by a positive real number. Also it is obvious that if \( S \) belongs to any of these sets then so do all subsets of \( S \). Finally, it is clear that any ray emanating from 0 is in \( A_\theta \), so we will mostly be interested in sets \( S \) that are not contained in a line.

The motivation for studying these notions comes from the following result of A. Barvinok ([B1]).

**Theorem 2.2.** (i) If \( S \in A_{\pi/2} \) then any square matrix with entries from \( S \) has nonzero permanent.

(ii) The disk \( |z - 1| \leq 1/2 \) is in \( A_{\pi/2} \).

This implies that any square matrix with entries \( a_{ij} \) such that \( |a_{ij} - 1| \leq 1/2 \) has nonzero permanent. This allowed A. Barvinok to give in [B1] an algorithm for efficient approximate computation of (logarithms of) permanents of such matrices with good precision.

The sets \( A_{\theta,\phi} \) for more general \( \theta \) and \( \phi \), also studied by A. Barvinok, have similar properties and applications (see [B1, B2]). Namely,
as explained in [B2], the condition that $S \in A_{\theta,\phi}$ for suitable $\theta$ and $\phi$ guarantees that some quite general combinatorially defined multivariate polynomials $P(z_1, \ldots, z_n)$, such as the graph homomorphism partition function, are necessarily non-zero whenever $z_1, \ldots, z_n \in U$, and can be efficiently approximated there.

The sets $A^2_{\theta,\phi}, B^2_{\theta,\phi}$ introduced here play an auxiliary role, but they are fairly easy to study (as their definition involves a small number of parameters), and yet we will show that a convex set belonging to $A^2_{\theta,\phi}$ must belong to $A_{\theta,\phi}$.

**Proposition 2.3.** (i) If $S \in A^2_{\theta,\phi}$ and $a, b \in S$ then $\alpha(a, b) < \pi - \theta$ and $\alpha(a, b) \leq \phi$.

(ii) If $S \in A^2_{\theta,\phi}$ and $a_1, \ldots, a_n \in S$ then for any $u_1, \ldots, u_n \in \mathbb{C}^*$ with $\alpha(u_i, u_j) \leq \theta$ for all $i, j$ we have $\sum_j a_j u_j \neq 0$.

**Proof.** (i) If $a, b \in S$ then $au_1 + bu_2$ does not vanish if $\alpha(u_1, u_2) \leq \theta$. Suppose $b/a = re^{i\psi}$ where $0 \leq \psi \leq \pi$ (this can always be achieved by switching $a, b$ if needed). Then $\psi < \pi - \theta$, since otherwise we may take $u_2 = 1$, $u_1 = -b/a$ (so that $\alpha(u_1, u_2) \leq \theta$) and $au_1 + bu_2 = 0$, a contradiction. Also $\psi \leq \phi$, since otherwise $\alpha(au_1 + bu_2, a(u_1 + u_2))$ for $u_1 = 1$ and $u_2 = N \gg 1$ will exceed $\phi$.

(ii) By (i) we have $\alpha(a_i, a_j) < \pi - \theta$ and $\alpha(a_i, a_j) \leq \phi < 2\pi/3$. Thus after rescaling by a complex scalar we may assume that

$$|\text{arg}(a_j)| < \frac{1}{2}(\pi - \theta)$$

for all $j$. Let $u_1, \ldots, u_n \in \mathbb{C}^*$ with pairwise angles $\leq \theta$. By rescaling by a complex scalar we may make sure that $|\text{arg}(u_j)| \leq \theta/2$. Then $|\text{arg}(a_j u_j)| < \pi/2$, so $\text{Re}(a_j u_j) > 0$ for all $j$. Thus $\sum_j a_j u_j \neq 0$. \qed

**Proposition 2.4.** Let $\phi \leq \pi/2$. Then a set $S \subset \mathbb{C}^*$ is in $A^2_{\theta,\phi}$ if and only if the map $z \mapsto \frac{az + b}{cz + d}$ maps the angle $\{z \in \mathbb{C}^* : |\text{arg}(z)| \leq \theta\}$ into $\{u \in \mathbb{C}^* : |\text{arg}(u)| \leq \phi\} \cup \{0, \infty\}$.

**Proof.** Only the “if” direction requires proof. It suffices to show that for $a, b \in S$ and $z \in \mathbb{C}^*$ with $|\text{arg}(z)| \leq \theta$ one has $az + b \neq 0$. Assume the contrary. For any $c \in S$, the map $w \mapsto \frac{aw + b}{cw + d}$ must map the angle $|\text{arg}(z)| \leq \theta$ to the set $\{u \in \mathbb{C}^* : |\text{arg}(u)| \leq \phi\} \cup \{0, \infty\}$, while mapping $z$ to 0. Considering these maps for $c = a, b$ near $w = z$ and using that $\phi \leq \pi/2$, we get that $b/a > 0$, i.e., $z < 0$, a contradiction. \qed
3. Convexity and reduction to $n=2$

The following theorem reduces checking that a convex set is $(\theta, \phi)$-angle restricted to checking that it is in $A_{\theta, \phi}^2$, which is just a low-dimensional geometry problem.

**Theorem 3.1.** (i) If $S \in A_{\theta, \phi}$ then so is the convex hull of $S$.

(ii) If $S \in A_{\theta, \phi}^2$ is convex then $S \in A_{\theta, \phi}$.

**Proof.** (i) Let $CH(S)$ be the convex hull of $S$. Assume $S \in A_{\theta, \phi}$. Let $a_1, ..., a_n, b_1, ..., b_n \in CH(S)$. Then $a_i = \sum_j r_{ij} a_{ij}$ where $a_{ij} \in S$, $r_{ij} > 0$ and $\sum_j r_{ij} = 1$. Similarly, $b_i = \sum_k s_{ik} b_{ik}$ where $b_{ik} \in S$, $s_{ik} > 0$ and $\sum_k s_{ik} = 1$. Let $u_1, ..., u_n \in \mathbb{C}^*$ with angle between each two $\leq \theta$. Let $u_{ijk} = r_{ij} s_{ik} u_i$. Consider

$$v := \sum_{i,j,k} a_{ij} u_{ijk} = \sum_{i,j,k} a_{ij} r_{ij} s_{ik} u_i = \sum_i a_i s_{ik} u_i = \sum_i a_i u_i$$

and

$$w := \sum_{i,j,k} b_{ik} u_{ijk} = \sum_{i,j,k} b_{ik} r_{ij} s_{ik} u_i = \sum_i b_i r_{ij} u_i = \sum_i b_i u_i.$$

Since $a_{ij}, b_{ik} \in S$, we have that $v, w \neq 0$ and the angle between them does not exceed $\phi$. Thus $CH(S) \in A_{\theta, \phi}$.

(ii) Denote by $R_{n, \theta} \subset \mathbb{CP}^{n-1}$ the set of points $u = (u_1, ..., u_n)$ such that the pairwise angles between $u_i$ and $u_j$ (when both are nonzero) are at most $\theta$. It is clear that $R_{n, \theta}$ is closed (hence compact). By Proposition 2.3(ii) for any $a_1, ..., a_n \in S$ we have $\sum_j a_j u_j \neq 0$. Now fix $a_1, ..., a_n, b_1, ..., b_n \in S$ and consider the function

$$f(u_1, ..., u_n) = \text{Im} \log \frac{\sum_j a_j u_j}{\sum_j b_j u_j}$$

(we choose a single-valued branch of this function). The function $f$ is harmonic on $R_{n, \theta}$ in each variable. Let $u \in R_{n, \theta}$ be a global maximum or minimum point of $f$. By the maximum principle we may choose $u = (u_1, ..., u_n)$ so that each $u_i$ is zero or has argument $\pm \theta/2$. By reducing $n$ if needed and relabeling, we may assume that all $u_j$ are nonzero and that $u_j = r_j e^{i\theta/2}$ for $j = 1, ..., m$ and $u_j = r_j e^{-i\theta/2}$ for $j = m+1, ..., n$, where $r_j > 0$ for all $j$. By rescaling by a positive real

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Note that using the coordinates $v_i := \frac{\sum_{j=1}^n u_j}{\sum_{j=1}^n u_j}$, $1 \leq i \leq n-1$, we may identify $R_{n, \theta}$ with a closed region in $\mathbb{C}^{n-1}$. Thus we may apply the maximum principle for harmonic functions on subsets of a Euclidean space.
number, we may assume that \( \sum_{j=1}^{m} r_j = r \) and \( \sum_{j=m+1}^{n} r_j = 1 \). Thus we have
\[
v = \sum_{j} a_j u_j = a r e^{i\theta/2} + b e^{-i\theta/2}, \quad w = \sum_{j} b_j u_j = c r e^{i\theta/2} + d e^{-i\theta/2},
\]
where
\[
a = \sum_{j=1}^{m} a_j r_j / r, \quad b = \sum_{j=m+1}^{n} a_j r_j, \quad c = \sum_{j=1}^{m} b_j r_j / r, \quad d = \sum_{j=m+1}^{n} b_j r_j.
\]

Since \( S \) is convex and \( a, b, c, d \) are convex linear combinations of the numbers \( \{a_j, j \leq m\}, \{a_j, j > m\}, \{b_j, j \leq m\}, \{b_j, j > m\} \) respectively, we get that \( a, b, c, d \in S \). Thus, using that \( S \in A^2_{\theta, \phi} \), and setting \( z = r e^{i\theta} \), we see that the angle between \( v \) and \( w \) does not exceed \( \phi \), as claimed. \( \square \)

**Lemma 3.2.** Let \( S \in A^2_{\theta, \pi/2} \), and \( a, b \in S \) with \( b/a = x + iy \), \( x, y \in \mathbb{R} \). Then we have \( x \geq 0 \) and
\[
|y| \leq \frac{2 \sqrt{x} + (x + 1) \cos \theta}{\sin \theta},
\]
and if \( \theta > \pi/2 \) then
\[
(x + \frac{1}{\cos \theta})^2 + y^2 \leq \tan^2 \theta.
\]

In particular, if \( \theta > \pi/2 \) then
\[
\frac{1 - \sin \theta}{|\cos \theta|} \leq x \leq \frac{1 + \sin \theta}{|\cos \theta|},
\]
i.e., \( b/a \) is separated from the imaginary axis and from infinity (so any \( S \in A^2_{\theta, \pi/2} \) is bounded). Moreover, conditions [1], [2], together with condition [1] with \( a \) and \( b \) switched are also sufficient for the set \( \{a, b\} \) to be in \( A^2_{\theta, \pi/2} \).

**Proof.** Let \( a, b \in S \) with \( b/a = x + iy \). Pick \( u_1 = re^{i\theta} \), \( u_2 = 1 \). The angle between \( au_1 + bu_2 \) and \( au_1 + au_2 \) does not exceed \( \pi/2 \). Hence the real part of \( \frac{au_1 + bu_2}{au_1 + au_2} \) is non-negative. Thus, we have
\[
\text{Re} \left( \frac{re^{i\theta} + x + iy}{re^{i\theta} + 1} \right) \geq 0, \ \forall r > 0.
\]
This yields
\[
\text{Re} \left( (re^{i\theta} + x + iy)(re^{i\theta} + 1) \right) \geq 0, \ \forall r > 0,
\]
i.e.,
\[
r^2 + ((x + 1) \cos \theta \pm y \sin \theta)r + x \geq 0, \ \forall r > 0.
\]
This implies that \( x \geq 0 \), and minimizing with respect to \( r \), we get

\[(x + 1) \cos \theta \pm y \sin \theta \geq -2\sqrt{x},\]

which yields

\[|y| \leq \frac{2\sqrt{x} + (x + 1) \cos \theta}{\sin \theta},\]

as claimed.

Similarly, the real part of \( \frac{a_1}{b_1} + b_2 \) is non-negative. Thus, we have

\[\text{Re} \left( \frac{re^{\pm i\theta} + x + iy}{(x + iy)(re^{\mp i\theta} + 1)} \right) \geq 0, \forall r > 0.\]

This yields

\[\text{Re}((re^{\pm i\theta} + x + iy)((x - iy)(re^{-\mp i\theta} + 1))) \geq 0, \forall r > 0,\]

i.e.

\[xx^2 + (x^2 + y^2 + 1)r \cos \theta + x \geq 0, \forall r > 0.\]

This is satisfied automatically if \( \theta \leq \pi/2 \), but if \( \theta > \pi/2 \) then minimizing the left hand side with respect to \( r \) gives the condition

\[(x^2 + y^2 + 1) \cos \theta + 2x \geq 0, \forall r > 0,\]

which is equivalent to (2).

Finally, to check that \{a, b\} \( \in A_{\theta, \pi/2} \), it suffices to check that for any \( u_1, u_2 \in \mathbb{C}^* \) that are within angle \( \theta \) of each other, the angles

\[\alpha(au_1 + bu_2, au_1 + au_2), \alpha(au_1 + bu_2, bu_1 + au_2), \alpha(au_1 + bu_2, bu_1 + bu_2)\]

do not exceed \( \pi/2 \). Clearly, it suffices to choose \( u_1 = re^{\pm i\theta} \) and \( u_2 = 1 \). Thus, conditions (1), (2), together with condition (1) with \( a \) and \( b \) switched are sufficient for the set \{a, b\} to be in \( A_{\theta, \pi/2} \), as claimed. \( \square \)

Thus we see that the region for \( b/a \) is bounded by two parabolas given by (1) and their inversions under the circle \(|z| = 1\), as well as the circle given by (2) if \( \theta > \pi/2 \) (note that this circle is stable under inversion).

**Proposition 3.3.** Suppose that \( \phi \leq \pi/2 \). Then

(i) if \( S \in A_{\theta, \phi}^2 \) then the closure \( \overline{S} \) of \( S \) in \( \mathbb{C}^* \) belongs to \( A_{\theta, \phi}^2 \); (ii) if \( S \in A_{\theta, \phi}^2 \) then the convex hull \( CH(S) \) of \( S \) belongs to \( A_{\theta, \phi}^2 \).

**Proof.** (i) follows by continuity from Proposition 2.4, since the set \( \{u \in \mathbb{C}^*: \text{arg}(u) \leq \phi \} \cup \{0, \infty\} \) is closed in the Riemann sphere.

(ii) Let \( a, b, b', c, d \in \mathbb{C}^* \) be such that the maps \( z \mapsto \frac{az + b}{cz + d} \) and \( z \mapsto \frac{az + b'}{cz + d} \) satisfy the condition of Proposition 2.4, \( r \in [0, 1] \) and \( b'' := rb + (1 - r)b' \). We claim that the map \( z \mapsto \frac{az + b'}{cz + d} \) also satisfies the
condition of Proposition \[2.4\]. It suffices to show this for \( z \neq -d/c \). We have
\[
\frac{az + b''}{cz + d} = r \frac{az + b}{cz + d} + (1 - r) \frac{az + b'}{cz + d},
\]
and \( \frac{az + b}{cz + d}, \frac{az + b'}{cz + d} \) belong to the set \( \{ u \in \mathbb{C}^* : |\arg(u)| \leq \phi \} \cup \{0\} \), which is convex since \( \phi \leq \pi/2 \). Hence \( \frac{az + b''}{cz + d} \) also belongs to this set, as claimed.

Also note that the condition of Proposition \[2.4\] is invariant under the transpositions \((a, b, c, d) \mapsto (b, a, d, c)\) and \((a, b, c, d) \mapsto (c, d, a, b)\), which generate a group \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) acting transitively on \( a, b, c, d \). Now (ii) follows by using this symmetry and applying the above claim four times (to each of the four variables \( a, b, c, d \)). \[\square\]

This proposition gives a simple method of constructing convex polygons which are in \( A_{\theta, \pi}^2 \) for \( \phi \leq \pi/2 \) by doing a finite check on the vertices. We will see examples of this below.

4. The sets \( A_{\pi/2}^2 \) and \( B_{\pi/2}^2 \)

From now on we focus on the case \( \theta = \phi = \pi/2 \) relevant for zero-free regions for the permanent. The general case can be treated by similar methods.

4.1. Explicit characterization. Let us give a more explicit characterization of the sets \( A_\theta^2 \) and \( B_\theta^2 \) for \( \theta = \pi/2 \). Let
\[
F(a, b, c, d) = (\text{Im}(a\bar{d} - b\bar{c}))^2 - 4\text{Re}(a\bar{c})\text{Re}(b\bar{d}),
\]
and
\[
G_1(a, b) = (a_2 - b_2)^2 - 4a_1b_1, \quad G_2(a, b) = (a_1 - b_1)^2 - 4a_2b_2,
\]
where \( a_1 + ia_2 = e^{i\pi/4}a, b_1 + ib_2 = e^{i\pi/4}b, a_j, b_j \in \mathbb{R} \). Note that
\[
F(a, b, c, d) = F(b, a, d, c) = F(c, d, a, b) = F(d, c, b, a).
\]

**Lemma 4.1.** (i) \( S \in A_{\pi/2}^2 \) if and only if for any \( a, b, c, d \in S \) we have \( F(a, b, c, d) \leq 0 \).

(ii) \( S \in B_{\pi/2}^2 \) if and only if \( |\arg(a)| \leq \pi/4 \) for \( a \in S \), and for any \( a, b \in S \) we have \( G_1(a, b) \leq 0 \), \( G_2(a, b) \leq 0 \).

**Proof.** (i) Suppose that \( F(a, b, c, d) \leq 0 \) for all \( a, b, c, d \in S \). Then \( \text{Re}(a\bar{c}) \geq 0 \) for all \( a, c \in S \) (as we can take \( b = d \)). Therefore, \( \frac{a + b}{c + d} \neq 0 \) when \( \text{Re}(z) \geq 0 \). Indeed, otherwise, we must have \( \text{Re}(b/a) = |a|^{-2}\text{Re}(b\bar{a}) \leq 0 \), so \( \text{Re}(b/a) = 0 \) and \( b/a = it \) for some real \( t \neq 0 \). But then \( F(a, b, a, a) = t^2|a|^4 > 0 \), a contradiction.
Thus by the definition of $A^2_{\pi/2}$, it suffices to show that for $a, b, c, d \in S$ one has $\Re \frac{az + b}{cz + d} \geq 0$ whenever $z = it, t \in \mathbb{R}$. We have

$$\frac{ait + b}{cit + d} = \frac{(ait + b)(-\bar{c}it + \bar{d})}{|cit + d|^2}$$

and

$$\Re \left( (ait + b)(-\bar{c}it + \bar{d}) \right) = \Re(a\bar{c})t^2 - \Im(a\bar{d} - b\bar{c})t + \Re(b\bar{d}).$$

Since $\Re(a\bar{c}), \Re(b\bar{d}) \geq 0$, the condition for this to be $\geq 0$ is that the discriminant of this quadratic function is $\leq 0$, which gives the result.

Conversely, if $S \in A^2_{\pi/2}$ then the above calculation shows that $F(a,b,c,d) \leq 0$ for all $a,b,c,d \in S$.

(ii) Let $a' = a_1 + ia_2, b' = b_1 + ib_2$. The condition on $a', b'$ is that for $t \in \mathbb{R}$ we have $\Re \frac{a'it + b'}{it + 1} \geq 0$ and $\Im \frac{a'it + b'}{it + 1} \geq 0$. We have

$$\frac{a'it + b'}{it + 1} = \frac{(a'it + b')(it + 1)}{t^2 + 1},$$

and

$$(a'it + b')(it + 1) = a't^2 + (a' - b')it + b' = (a_1t^2 - (a_2 - b_2)t + b_1) + i(a_2t^2 + (a_1 - b_1)t + b_2).$$

Since $a_1, a_2, b_1, b_2 \geq 0$ (as seen by setting $t = 0$ and $t = \infty$), the condition is that the discriminants of these two quadratic functions must be $\leq 0$, which gives the result. \hfill \Box

4.2. Examples.

Example 4.2. Lemma 4.1(ii) implies that the interval $[a, b] \subset \mathbb{R}$ for $0 < a \leq b$ is in $B^2_{\pi/2}$ iff $b/a \leq 3 + 2\sqrt{2}$.

Example 4.3. Let $a = 1/2, b = 1 + i/2, c = 1 - i/2$ and $d = 3/2 + t$. Let us find the largest $t > 0$ for which $\{a, b, c, d\}$ is in $B^2_{\pi/2}$ (hence in $A^2_{\pi/2}$). Since $a, b, c$ belong to the disk $|z - 1| \leq 1/2$, which was shown by A. Barvinok in [31] to belong to $B^2_{\pi/2}$, it suffices to check when $G_i(a,d) \leq 0, G_i(b,d) \leq 0, G_i(c,d) \leq 0$. The first condition gives the inequality of Example 4.2, which is $3 + 2t \leq 3 + 2\sqrt{2}$, i.e. $t \leq \sqrt{2}$. The second (or, equivalently, third) condition gives the inequalities $t^2 \leq 2t + 3, (1 + t)^2 \leq 3(2t + 3)$ which hold for $0 \leq t \leq \sqrt{2}$. Thus we find that the optimal value is $t = \sqrt{2}$ and the quadrilateral with vertices $1/2, 1 \pm i/2$ and $3/2 + \sqrt{2}$.
is in $A_{\pi/2}^2$, hence in $A_{\pi/2}$, thus it is a zero-free region for the permanent.

**Example 4.4.** Let us find the values of $t > 1/2$ for which the union of the disk $|z-1| \leq 1/2$ and the point $1+t$ belongs to $B_{\pi/2}^2$ (hence to $A_{\pi/2}^2$). Such $t$ are determined by the condition that $G_1(1+\frac{1}{2}e^{i(\phi-\pi/4)}, 1+t) \leq 0$ for all $\phi$ (the condition involving $G_2$ is the same due to axial symmetry).

This can be written as

$$(t + \frac{1}{\sqrt{2}} \cos \phi)^2 \leq 4(1 + t)(1 + \frac{1}{\sqrt{2}} \sin \phi)$$

for all $\phi$. This gives

$$t \leq 1 + \sqrt{2} \sin \phi - \frac{\sqrt{2}}{2} \cos \phi + \sqrt{6\sqrt{2} \sin \phi - 2\sqrt{2} \cos \phi - \sin 2\phi - \cos 2\phi + 9},$$

and minimizing this function (numerically), we get the answer

$$t \leq t_* = 1.64......$$

Thus the ice cream cone, which is the convex hull of the disk $|z-1| \leq 1/2$ and the point $1 + t_*$ (significantly larger than the disk):

belongs to $A_{\pi/2}^2$, hence to $A_{\pi/2}$, and thus is a zero-free region for the permanent.
Example 4.5. Let \( S = \{ a, b \} \), and \( b/a = x + iy \). Let us compute when \( S \in A_{\pi/2}^2 \). By Lemma 3.2 the conditions for this are
\[
y^2 \leq 4x, \quad y^2 \leq 4(x^2 + y^2).
\]
This gives
\[
(3) \quad |y| \leq 2\sqrt{x}; \quad \text{and} \quad |y| \leq \frac{2x^{3/2}}{\sqrt{1-4x}}, \quad x < 1/4.
\]
So we get a region which is bounded by a parabola and its inversion with respect to the circle \( |z| = 1 \), which is a cissoid of Diocles:

By Proposition 3.3 this is also the necessary and sufficient condition for the segment \([a, b] \subset \mathbb{C}^{*}\) to be in \( A_{\pi/2}^2 \).

Example 4.6. Consider now a 3-element set \( S = \{ 1, a, b \} \) and let us give a necessary condition for it to be in \( A_{\pi/2}^2 \).

Proposition 4.7. Assume \( a \notin \mathbb{R} \). Then one has
\[
a_1 \frac{(|1+a| - 1 - a_1)^2}{a_2^2} \leq b_1 \leq a_1 \frac{(|1+a| + 1 + a_1)^2}{a_2^2},
\]
where \( a = a_1 + ia_2, \ b = b_1 + ib_2 \) and \( a_1, a_2, b_1, b_2 \in \mathbb{R} \). In other words, one has \( K^{-1} \leq \frac{b_1}{a_1} \leq K \), where \( K := \frac{(1+a)^2}{a_2^2} \). Thus any \( S \in A_{\pi/2}^2 \) which is not contained in a line is bounded and separated from the origin.

Proof. We have the inequalities \( F(a, 1, 1, b) \leq 0 \) and \( F(a, b, 1, 1) \leq 0 \), which yields
\[
(a_1b_2 - a_2b_1)^2 \leq 4a_1b_1, \quad (a_2 - b_2)^2 \leq 4a_1b_1,
\]
or, equivalently,
\[
|a_1b_2 - a_2b_1| \leq 2\sqrt{a_1b_1}, \quad |a_2 - b_2| \leq 2\sqrt{a_1b_1}.
\]
From the second inequality
\[ |b_2| \leq 2\sqrt{a_1b_1} + |a_2|. \]
Hence
\[ |a_1b_2| \leq (2\sqrt{a_1b_1} + |a_2|)a_1. \]
Thus
\[ |a_2b_1| \leq 2\sqrt{a_1b_1} + |a_1b_2| \leq 2(1 + a_1)\sqrt{a_1b_1} + |a_2|a_1. \]
Hence
\[ b_1 \leq \frac{2(1 + a_1)\sqrt{a_1b_1}}{|a_2|} + a_1 \]
This yields
\[ b_1 \leq a_1\frac{(1 + a_1) + (1 + a_1)^2}{a_2^2}, \]
as claimed. From this we also have
\[ |a_2| \leq 2\sqrt{a_1b_1} + |b_2| \leq 2\sqrt{a_1b_1} + \frac{2\sqrt{a_1b_1} + |a_2|b_1}{a_1}, \]
which yields
\[ b_1 \geq a_1\frac{(1 + a_1 - 1 - a_1)^2}{a_2^2}, \]
again as claimed.

\[ \square \]

4.3. **Rectangular and trapezoidal regions.** Let us now try to characterize rectangular and trapezoidal regions which are in \( A_{\pi/2} \) (hence in \( A_{\pi/2} \)).

**Proposition 4.8.** (i) Let \( R(M, L, N) \) be the rectangle \( M \leq x \leq M + L, \ |y| \leq N \). Then \( R(M, L, N) \in A_{\pi/2} \) if
\[ N \leq \frac{2M^{3/2}}{\sqrt{L + 24M}}. \]

(ii) Let \( T(M, L, t) \) be the trapezoid \( M \leq x \leq L, |y| \leq tx \). Then \( T(M, L, t) \in A_{\pi/2} \) if \( t < \sqrt{2} - 1 \) and
\[ L \leq M \left( \frac{t^2 + t^{-2} - 4 + (t^{-1} - t)\sqrt{t^2 + t^{-2} - 6}}{2} \right)^{1/2} \]
\[ = Mt^{-1}(1 + o(t)) \text{ as } t \to 0. \]
Proof. In coordinates the desired basic inequality looks like

\[(a_2d_1 - a_1d_2 - b_2c_1 + b_1c_2)^2 \leq 4(a_1c_1 + a_2c_2)(b_1d_1 + b_2d_2),\]

where the subscript 1 denotes the real part and the subscript 2 the imaginary part.

(i) Since the absolute values of \(a_2, b_2, c_2, d_2\) don’t exceed \(N\), the basic inequality would follow from the inequality

\[N^2(a_1 + b_1 + c_1 + d_1)^2 \leq 4(a_1c_1 - N^2)(b_1d_1 - N^2) = 4a_1c_1b_1d_1 - 4N^2(a_1c_1 + b_1d_1) + N^4.\]

(as long as \(N \leq M\), which follows from the inequality in (i)). This, in turn, would follow from the inequality

\[N^2((a_1 + b_1 + c_1 + d_1)^2 + 4(a_1c_1 + b_1d_1)) \leq 4a_1c_1b_1d_1.\]

Let \(q\) be the largest of \(a_1, b_1, c_1, d_1\) and \(p\) the second largest. Then the latter inequality would follow from the inequality

\[N^2((a_1 + b_1 + c_1 + d_1)^2 + 4(a_1c_1 + b_1d_1)) \leq 4M^2pq.\]

Now observe that on the left hand side we have 24 quadratic monomials in \(a_1, b_1, c_1, d_1\), which are all \(\leq pq\) except one, which is \(q^2 \leq (M + L)q\). So the last inequality would follow from the inequality

\[N^2(23p + M + L) \leq 4M^2p,\]

or

\[N^2(M + L) \leq p(4M^2 - 23N^2).\]

This, in turn, follows from the inequality

\[N^2(M + L) \leq M(4M^2 - 23N^2),\]

or

\[N^2(L + 24M) \leq 4M^3,\]

giving

\[N \leq \frac{2M^{3/2}}{\sqrt{L + 24M}},\]

as claimed.

(ii) Since \(|a_2| \leq ta_1, |b_2| \leq tb_1, |c_2| \leq tc_1, |d_2| \leq td_1\), the basic inequality would follow from the inequality

\[4t^2(a_1d_1 + b_1c_1)^2 \leq 4(1 - t^2)^2a_1c_1b_1d_1,\]

which is equivalent to the inequality

\[t^2(a_1^2d_1^2 + b_1^2c_1^2) \leq (1 - 4t^2 + t^4)a_1b_1c_1d_1,\]
or \( \mu + \frac{1}{\mu} \leq t^{-2} - 4 + t^2 \), where \( \mu = \frac{a \cdot d}{b \cdot c} \). The largest value of this ratio is \( L^2 / M^2 \), so it sufficient to require that

\[
\frac{L^2}{M^2} + \frac{M^2}{L^2} \leq t^2 - 4 + t^{-2} := T.
\]

This is satisfied whenever

\[
L \leq M \left( \frac{T + \sqrt{T^2 - 4}}{2} \right)^{1/2},
\]

as claimed.

In particular, if \( L = 1 \) and \( M \) is small then for the rectangle we have \( N = 2M^{3/2}(1 + o(M)) \). Comparing this to the bound \([3]\), we see that this is sharp up to a factor \( 1 + o(M) \). This also relaxes the bound \( N \leq CM^2 \) from \([B1]\).

Also for the trapezoid we have \( M \geq t(1 + o(t)) \), so its short side has half-length \( N = tM \), so the largest possible \( N \) is \( \sim M^2 \).

4.4. Maximal angle-restricted sets. From now on we will only consider closed convex sets \( S \), since we have seen that if \( S \in A_{\pi/2}^2 \) then so do its closure and its convex hull, and a convex set is in \( A_{\pi/2} \) iff it is in \( A_{\pi/2}^2 \).

It is clear from Zorn’s lemma that any \((\pi/2, \pi/2)\)-angle restricted set is contained in a maximal one, which is necessarily closed and convex. The problem of finding and classifying maximal \((\pi/2, \pi/2)\)-angle-restricted sets is a special case of a more general problem of optimal control theory – to find maximal regions \( R \) with the property that a given function \( F(z_1, ..., z_n) \) is \( \leq 0 \) when all \( z_i \in R \); one of the simplest and best known problems from this family is to describe curves of constant width \( \ell \) (in this case \( F(z_1, z_2) = |z_1 - z_2|^2 - \ell^2 \)). As is typical for such problems, the problem of describing maximal regions in \( A_{\pi/2} \) is rather nontrivial; presumably, it can be treated by the methods of the book \([BCGGG]\).

Maximal regions can also be constructed as limits of nested sequences \( \Pi_n \) of convex \( n \)-gons, each obtained from the previous one by “pushing out” a point on one of the sides as far as it can go while still preserving the property of being in \( A_{\pi/2} \). This approach should be good for numerical computation of maximal regions, since the verification that the region is in \( A_{\pi/2}^2 \) (equivalently, in \( A_{\pi/2} \)) is just a finite check on the vertices of the polygon.

Here we will not delve into this theory and will restrict ourselves to proving the following result. Let \( \mu_S(a) := \max_{b, c, d \in S} F(a, b, c, d) \). We have seen that \( S \in A_{\pi/2} \) iff \( \mu_S(a) \leq 0 \ \forall a \in S \).
Proposition 4.9. A closed convex set \( S \in A_{\pi/2} \) (not contained in a line) is maximal iff \( \mu_S(a) = 0 \) for all \( a \in \partial S \).

Proof. Note that \( S \) is bounded by Proposition 4.7, hence compact. Suppose \( S \in A_{\pi/2} \) is maximal and \( a \in \partial S \) is such that there are no \( b,c,d \in S \) with \( F(a,b,c,d) = 0 \). Then \( \mu_S(a) = -\varepsilon < 0 \). Now take sufficiently small \( \delta \) and let \( S' = S \cup \{|z-a| \leq \delta\} \), which is strictly larger than \( S \) as \( a \in \partial S \). Let us maximize \( F(x,b,c,d) \) over \( x,b,c,d \in S' \). If these points are further than \( \delta \) from \( a \) then they are in \( S \) so \( F(x,b,c,d) \leq 0 \). Otherwise, if one of them is \( \delta \)-close to \( a \), say, \( x \) (it does not matter which one because of the permutation symmetry of \( F \)), then \( F(x,b,c,d) \leq F(a,b,c,d) + \varepsilon \leq 0 \) (a number \( \delta \) with this property exists due to uniform continuity of \( F \) on \( S \)). So \( S' \) and its convex hull are in \( A_{\pi/2} \), contradicting the assumption that \( S \) is maximal.

Conversely, suppose \( \mu_S(a) = 0 \) on \( \partial S \), let \( S' \supset S \) be a larger convex region. Then there exists \( a \in \partial S \) which is an interior point of \( S' \). Also there exist \( b,c,d \in S \) with \( F(a,b,c,d) = 0 \). But for fixed \( b,c,d \) the function \( F(z,b,c,d) \) is inhomogeneous quadratic in \( z, \bar{z} \) with nonnegative degree 2 part, which implies that there is a point \( a' \) arbitrarily close to \( a \) with \( F(a',b,c,d) > 0 \). Hence \( S' \notin A_{\pi/2} \) and \( S \) is maximal. \( \square \)

Thus, we see that if \( S \in A_{\pi/2} \) and \( a \in \partial S \) with \( \mu_S(a) < 0 \) then \( S \) can be enlarged near \( a \) (e.g. by adding a point \( a' \notin S \) close to \( a \) and taking the convex hull of \( S \) and \( a' \)), so that the larger set \( S' \) is still in \( A_{\pi/2} \). Otherwise, if \( \mu_S(a) = 0 \), then \( a \) must be on the boundary of any \( S' \in A_{\pi/2} \) containing \( S \). We will say that \( S \) is maximal at \( a \) if \( \mu_S(a) = 0 \) and non-maximal at \( a \) if \( \mu_S(a) < 0 \).

Example 4.10. Let \( S \) be the disk \( |z - 1| \leq 1/2 \). Then \( S \) is maximal at the three points \( a = 1/2, 1 \pm i/2 \) and not maximal at any other points of the boundary circle. The proof is by a direct computation. Namely, if \( a \neq 1/2, 1 \pm i/2 \) but \( |a-1| = 1/2 \), then it can be shown that for any \( b \) with \( |b-1| \leq 1/2 \) one has \( G_1(a,b) < 0 \) and \( G_2(a,b) < 0 \).

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