Generalized Wintgen inequality for statistical submanifolds in statistical manifolds of constant curvature

M. Evren Aydin1 · Adela Mihai2 · Ion Mihai3

Abstract The Wintgen inequality (1979) is a sharp geometric inequality for surfaces in the 4-dimensional Euclidean space involving the Gauss curvature (intrinsic invariant) and the normal curvature and squared mean curvature (extrinsic invariants), respectively. De Smet et al. (Arch. Math. (Brno) 35:115–128, 1999) conjectured a generalized Wintgen inequality for submanifolds of arbitrary dimension and codimension in Riemannian space forms. This conjecture was proved by Lu (J. Funct. Anal. 261:1284–1308, 2011) and by Ge and Tang (Pac. J. Math. 237:87–95, 2008), independently. In the present paper we establish a generalized Wintgen inequality for statistical submanifolds in statistical manifolds of constant curvature.

Keywords Wintgen inequality · Statistical manifold · Statistical submanifold · Dual connections

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1 Introduction

For surfaces $M^2$ of the Euclidean space $\mathbb{E}^3$, the Euler inequality $G \leq \|H\|^2$ is fulfilled, where $G$ is the (intrinsic) Gauss curvature of $M^2$ and $\|H\|^2$ is the (extrinsic) squared mean curvature of $M^2$.

Furthermore, $G = \|H\|^2$ everywhere on $M^2$ if and only if $M^2$ is totally umbilical, or still, by a theorem of Meusnier, if and only if $M^2$ is (a part of) a plane $\mathbb{E}^2$ or, it is (a part of) a round sphere $S^2$ in $\mathbb{E}^3$.

In 1979, Wintgen [25] proved that the Gauss curvature $G$, the squared mean curvature $\|H\|^2$ and the normal curvature $G^\perp$ of any surface $M^2$ in $\mathbb{E}^4$ always satisfy the inequality

$$G \leq \|H\|^2 - |G^\perp|;$$

the equality holds if and only if the ellipse of curvature of $M^2$ in $\mathbb{E}^4$ is a circle.

The Whitney 2-sphere satisfies the equality case of the Wintgen inequality identically.

A survey containing recent results on surfaces satisfying identically the equality case of Wintgen inequality can be read in [5].

Later, the Wintgen inequality was extended by Rouxel [20] and by Guadalupe and Rodriguez [10] independently, for surfaces $M^2$ of arbitrary codimension $m$ in real space forms $\tilde{M}^{2+m}(c)$; namely

$$G \leq \|H\|^2 - |G^\perp| + c.$$

The equality case was also investigated.

A corresponding inequality for totally real surfaces in $n$-dimensional complex space forms was obtained in [13]. The equality case was studied and a non-trivial example of a totally real surface satisfying the equality case identically was given.

In 1999, De Smet et al. [7] formulated the conjecture on Wintgen inequality for submanifolds of arbitrary dimension $n \geq 2$ and codimension 2 in real space forms $\tilde{M}^{n+2}(c)$ of constant sectional curvature $c$.

Recently, the DDVV conjecture was finally settled for the general case by Lu [12] and independently by Ge and Tang [9].

One of the present authors obtained generalized Wintgen inequalities for Lagrangian submanifolds in complex space forms [14] and Legendrian submanifolds in Sasakian space forms [15], respectively. Moreover, two of the present authors established in [3] a version of the Euler inequality and the Wintgen inequality for statistical surfaces in statistical manifolds of constant curvature.

In this paper, using the sectional curvature defined in [19], we derive a generalized Wintgen inequality for statistical submanifolds in statistical manifolds of constant curvature.
2 Statistical manifolds and their submanifolds

A statistical manifold is a Riemannian manifold \((\tilde{M}^{n+k}, \tilde{g})\) of dimension \((n + k)\), endowed with a pair of torsion-free affine connections \(\tilde{\nabla}\) and \(\tilde{\nabla}^*\) satisfying

\[
Z \tilde{g}(X, Y) = \tilde{g}(\tilde{\nabla}_Z X, Y) + \tilde{g}(X, \tilde{\nabla}_Z^* Y),
\]

(2.1)

for any \(X, Y, Z \in \Gamma(T \tilde{M})\). The connections \(\tilde{\nabla}\) and \(\tilde{\nabla}^*\) are called dual connections (see [1,17,22]), and it is easily shown that \((\tilde{\nabla}^*)^* = \tilde{\nabla}\). The pair \((\tilde{\nabla}, \tilde{g})\) is said to be a statistical structure. If \((\tilde{\nabla}, \tilde{g})\) is a statistical structure on \(\tilde{M}^{n+k}\), so is \((\tilde{\nabla}^*, \tilde{g})\) [1,24].

On the other hand, any torsion-free affine connection \(\tilde{\nabla}\) always has a dual connection given by

\[
\tilde{\nabla} + \tilde{\nabla}^* = 2\tilde{\nabla}^0,
\]

(2.2)

where \(\tilde{\nabla}^0\) is Levi-Civita connection on \(\tilde{M}^{n+k}\).

Denote by \(\tilde{R}\) and \(\tilde{R}^*\) the curvature tensor fields of \(\tilde{\nabla}\) and \(\tilde{\nabla}^*\), respectively. A statistical structure \((\tilde{\nabla}, \tilde{g})\) is said to be of constant curvature \(c \in \mathbb{R}\) if

\[
\tilde{R}(X, Y)Z = c[\tilde{g}(Y, Z)X - \tilde{g}(X, Z)Y].
\]

(2.3)

A statistical structure \((\tilde{\nabla}, \tilde{g})\) of constant curvature 0 is called a Hessian structure.

The curvature tensor fields \(\tilde{R}\) and \(\tilde{R}^*\) of dual connections satisfy

\[
\tilde{g}(\tilde{R}^*(X, Y)Z, W) = -\tilde{g}(Z, \tilde{R}(X, Y)W).
\]

(2.4)

From (2.4) it follows immediately that if \((\tilde{\nabla}, \tilde{g})\) is a statistical structure of constant curvature \(c\), then \((\tilde{\nabla}^*, \tilde{g})\) is also a statistical structure of constant curvature \(c\). In particular, if \((\tilde{\nabla}, \tilde{g})\) is Hessian, so is \((\tilde{\nabla}^*, \tilde{g})\) [8].

On a Hessian manifold \((M^{n+k}, \tilde{\nabla})\), let \(\gamma = \tilde{\nabla}^0 - \tilde{\nabla}\). The tensor field \(Q\) of type (1,3) defined by the covariant differential \(Q = \tilde{\nabla}\gamma\) of \(\gamma\) is said to be the Hessian curvature tensor for \(\tilde{\nabla}\) (see [21]).

By using the Hessian curvature tensor \(Q\), a Hessian sectional curvature can be defined on a Hessian manifold.

A Hessian manifold has constant Hessian sectional curvature \(\tilde{c}\) if and only if (see [21])

\[
Q(X, Y, Z, W) = \frac{\tilde{c}}{2}[g(X, Y)g(Z, W) + g(X, W)g(Y, Z)],
\]

for all vector fields on \(\tilde{M}^{n+k}\).

If \((\tilde{M}^{n+k}, \tilde{g})\) is a statistical manifold and \(M^n\) a submanifold of dimension \(n\) of \(\tilde{M}^{n+k}\), then \((M^n, g)\) is also a statistical manifold with the induced connection by \(\tilde{\nabla}\) and induced metric \(g\). In the case that \((\tilde{M}^{n+k}, \tilde{g})\) is a semi-Riemannian manifold, the induced metric \(g\) has to be non-degenerate. For details, see [23,24].
In the geometry of Riemannian submanifolds (see [4]), the fundamental equations are the Gauss and Weingarten formulas and the equations of Gauss, Codazzi and Ricci.

Let denote the set of the sections of the normal bundle to \( M^n \) by \( \Gamma(TM^n) \). In our case, for any \( X, Y \in \Gamma(TM^n) \), according to [24], the corresponding Gauss formulas are

\[
\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.5}
\]

\[
\tilde{\nabla}^*_X Y = \nabla^*_X Y + h^*(X, Y), \tag{2.6}
\]

where \( h, h^* : \Gamma(TM^n) \times \Gamma(TM^n) \rightarrow \Gamma(TM^n) \) are symmetric and bilinear, called the imbedding curvature tensor of \( M^n \) in \( \tilde{M}^{n+k} \) for \( \tilde{\nabla} \) and the imbedding curvature tensor of \( M^n \) in \( \tilde{M}^{n+k} \) for \( \tilde{\nabla}^* \), respectively.

In [24], it is also proved that \((\nabla, g)\) and \((\nabla^*, g)\) are dual statistical structures on \( M^n \).

Since \( h \) and \( h^* \) are bilinear, we have the linear transformations \( A_\xi \) and \( A^*_\xi \) on \( TM^n \) defined by

\[
g(A_\xi X, Y) = \tilde{g}(h(X, Y), \xi), \tag{2.7}
\]

\[
g(A^*_\xi X, Y) = \tilde{g}(h^*(X, Y), \xi), \tag{2.8}
\]

for any \( \xi \in \Gamma(TM^n) \) and \( X, Y \in \Gamma(TM^n) \). Further, see [24], the corresponding Weingarten formulas are

\[
\tilde{\nabla}_X \xi = -A^*_\xi X + \nabla^{\perp}_X \xi, \tag{2.9}
\]

\[
\tilde{\nabla}^*_X \xi = -A_\xi X + \nabla^{*\perp}_X \xi, \tag{2.10}
\]

for any \( \xi \in \Gamma(TM^n) \) and \( X \in \Gamma(TM^n) \). The connections \( \nabla^{\perp}_X \) and \( \nabla^{*\perp}_X \) given by (2.9) and (2.10) are Riemannian dual connections with respect to induced metric on \( \Gamma(TM^n) \).

Let \( \{e_1, \ldots, e_n\} \) and \( \{\xi_1, \ldots, \xi_k\} \) be orthonormal tangent and normal frames, respectively, on \( M^n \). Then the mean curvature vector fields are defined by

\[
H = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i) = \frac{1}{n} \sum_{\alpha=1}^{k} \left( \sum_{i=1}^{n} h^\alpha_{ii} \right) \xi_\alpha, \quad h^\alpha_{ij} = \tilde{g}(h(e_i, e_j), \xi_\alpha), \tag{2.11}
\]

and

\[
H^* = \frac{1}{n} \sum_{i=1}^{n} h^*(e_i, e_i) = \frac{1}{n} \sum_{\alpha=1}^{k} \left( \sum_{i=1}^{n} h^*_{\alpha ii} \right) \xi_\alpha, \quad h^*_{ij} = \tilde{g}(h^*(e_i, e_j), \xi_\alpha), \tag{2.12}
\]

for \( 1 \leq i, j \leq n \) and \( 1 \leq \alpha \leq k \) (see also [6]).

The corresponding Gauss, Codazzi and Ricci equations are given by the following result.
Proposition 2.1 [24] Let \( \tilde{\nabla} \) and \( \tilde{\nabla}^* \) be dual connections on \( \tilde{M}^{n+k} \) and \( \nabla \) the induced connection by \( \tilde{\nabla} \) on \( M^n \). Let \( \tilde{R} \) and \( R \) be the Riemannian curvature tensors for \( \tilde{\nabla} \) and \( \nabla \), respectively. Then,

\[
\tilde{g}(\tilde{R}(X, Y)Z, W) = g(R(X, Y)Z, W) + \tilde{g}(h(X, Z), h^*(Y, W))
- \tilde{g}(h^*(X, W), h(Y, Z)),
\]

\[
(\tilde{R}(X, Y)Z) = \nabla^\perp_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)
- \{\nabla^\perp_Y h(X, Z) - h(\nabla_Y X, Z) - h(X, \nabla_Y Z)\},
\]

\[
\tilde{g}(\tilde{R}^\perp(X, Y)\xi, \eta) = \tilde{g}(\tilde{R}(X, Y)\xi, \eta) + g([A^*_\xi, A^*_\eta]X, Y),
\]

where \( R^\perp \) is the Riemannian curvature tensor of \( \nabla^\perp \) on \( TM^{n,\perp} \), \( \xi, \eta \in \Gamma(TM^{n,\perp}) \) and \( [A^*_\xi, A^*_\eta] = A^*_\xi A^*_\eta - A^*_\eta A^*_\xi \).

For the equations of Gauss, Codazzi and Ricci with respect to the connection \( \tilde{\nabla}^* \) on \( M^n \), we have

Proposition 2.2 [24] Let \( \tilde{\nabla} \) and \( \tilde{\nabla}^* \) be dual connections on \( \tilde{M}^{n+k} \) and \( \nabla^* \) the induced connection by \( \tilde{\nabla}^* \) on \( M^n \). Let \( \tilde{R}^* \) and \( R^* \) be the Riemannian curvature tensors for \( \tilde{\nabla}^* \) and \( \nabla^* \), respectively. Then,

\[
\tilde{g}(\tilde{R}^*(X, Y)Z, W) = g(R^*(X, Y)Z, W) + \tilde{g}(h^*(X, Z), h(X, W))
- \tilde{g}(h^*(X, W), h^*(Y, Z)),
\]

\[
(\tilde{R}^*(X, Y)Z)^\perp = \nabla^\perp_X h^*(Y, Z) - h^*(\nabla_X Y, Z) - h^*(Y, \nabla_X Z)
- \{\nabla^\perp_Y h^*(X, Z) - h^*(\nabla_Y X, Z) - h^*(X, \nabla_Y Z)\},
\]

\[
\tilde{g}(\tilde{R}^{\perp*}(X, Y)\xi, \eta) = \tilde{g}(\tilde{R}^*(X, Y)\xi, \eta) + g([A^*_\xi, A^*_\eta]X, Y),
\]

where \( R^{\perp*} \) is the Riemannian curvature tensor of \( \nabla^{\perp*} \) on \( TM^{n,\perp} \), \( \xi, \eta \in \Gamma(TM^{n,\perp}) \) and \( [A^*_\xi, A^*_\eta] = A^*_\xi A^*_\eta - A^*_\eta A^*_\xi \).

Geometric inequalities for statistical submanifolds in statistical manifolds with constant curvature were obtained in [2].

3 Statistical surfaces in statistical manifolds of constant curvature

Let \( (\tilde{M}^3, \tilde{g}) \) be a 3-dimensional statistical manifold of constant curvature \( c \) and \( M^2 \) a surface of \( \tilde{M} \). Denote the Gauss curvature, the mean curvature and the dual mean curvature of \( M \), by \( G \), \( H \) and \( H^* \), respectively. In [3], a version of the Euler inequality for statistical surfaces was given.

Proposition 3.1 [3] Let \( M^2 \) be a surface in a 3-dimensional statistical manifold of constant curvature \( c \). Then its Gauss curvature satisfies:

\[
G \leq 2\|H\| \cdot \|H^*\| - c.
\]

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Some examples of statistical surfaces satisfying the equality case of the above Euler inequality can be provided by the following.

**Example 1** (A trivial example) Recall Lemma 5.3 of Furuhata [8].

Let $(\mathbb{H}, \tilde{\nabla}, \tilde{g})$ be a Hessian manifold of constant Hessian sectional curvature $\tilde{c} \neq 0$, $(M, \nabla, g)$ a trivial Hessian manifold and $f : M \rightarrow \mathbb{H}$ a statistical immersion of codimension one. Then one has:

$$A^* = 0, \quad h^* = 0, \quad \|H^*\| = 0.$$

Thus, if $\dim M = 2$, the immersion $f$ of codimension one satisfies the equality case of the statistical version of the Euler inequality given by Proposition 3.1.

**Example 2** Let $(\mathbb{H}^3, \tilde{g})$ be the upper half space of constant sectional curvature $-1$, i.e.,

$$\mathbb{H}^3 = \{y = (y^1, y^2, y^3) \in \mathbb{R}^3 : y^3 > 0\}, \quad \tilde{g} = (y^3)^{-2} \sum_{k=1}^{3} dy^k dy^k.$$

An affine connection $\tilde{\nabla}$ on $\mathbb{H}^3$ is given by

$$\tilde{\nabla}_{\partial/\partial y^3} \frac{\partial}{\partial y^3} = (y^3)^{-1} \frac{\partial}{\partial y^3}, \quad \tilde{\nabla}_{\partial/\partial y^i} \frac{\partial}{\partial y^j} = 2 \delta_{ij} (y^3)^{-1} \frac{\partial}{\partial y^3}, \quad \tilde{\nabla}_{\partial/\partial y^3} \frac{\partial}{\partial y^3} = \tilde{\nabla}_{\partial/\partial y^3} \frac{\partial}{\partial y^j} = 0,$$

where $i, j = 1, 2$. The curvature tensor field $\tilde{R}$ of $\tilde{\nabla}$ is identically zero, i.e., $c = 0$. Thus $(\mathbb{H}^3, \tilde{\nabla}, \tilde{g})$ is a Hessian manifold of constant Hessian sectional curvature 4 (see [21]).

Now let consider a horosphere $M^2$ in $\mathbb{H}^3$ having null Gauss curvature, i.e., $G \equiv 0$ (for details, see [11]). If $f : M^2 \rightarrow \mathbb{H}^3$ is a statistical immersion of codimension one, then, by using Lemma 4.1 of [16], we deduce $A^* = 0$, and then $H^* = 0$. This implies that the horosphere $M^2$ satisfies the equality case of the statistical version of the Euler inequality given by Proposition 3.1.

More generally, let consider a 4-dimensional statistical manifold of constant curvature $c$, i.e. $(\tilde{M}^4, c)$, and a surface $M^2$ of $\tilde{M}^4$. We respectively denote the Gauss curvature, the normal curvature and the Gauss curvature with respect to the Levi-Civita connection by $G$, $G^\perp$ and $G^0$. Similarly, we respectively denote the mean vector field, the dual mean curvature and the sectional curvature with respect to the Levi-Civita connection by $H$, $H^*$ and $\tilde{K}^0$. We have the following Wintgen inequalities.

**Theorem 3.2** [3] Let $M^2$ be a statistical surface in a 4-dimensional statistical manifold $(\tilde{M}^4, c)$ of constant curvature $c$. Then

$$G + |G^\perp| + 2G^0 \leq \frac{1}{2}(\|H\|^2 + \|H^*\|^2) - c + 2\tilde{K}^0(e_1 \wedge e_2).$$

In particular, for $c = 0$ we derive the following.
Corollary 3.3 [3] Let $M^2$ be a statistical surface of a Hessian 4-dimensional statistical manifold $M^4$ of Hessian curvature 0. Then:

$$G + |G^\perp| + 2G^0 \leq \frac{1}{2}(\|H\|^2 + \|H^*\|^2).$$

4 Wintgen inequality for statistical submanifolds

Let $M^n$ be an $n$-dimensional statistical submanifold of a $(n+m)$-dimensional statistical manifold $(\tilde{M}^{n+m}, c)$ of constant curvature $c$.

The sectional curvature $K$ on $M^n$ is defined by [3] (see also [18, 19])

$$K(X \wedge Y) = \frac{1}{2}[g(R(X, Y)X, Y) + g(R^*(X, Y)X, Y)],$$

for any orthonormal vectors $X, Y \in T_pM^n$, $p \in M^n$.

In the case of the Levi-Civita connection, the above definition coincides (up to the sign) to the standard definition of the sectional curvature.

Let $p \in M^n$ and $\{e_1, e_2, \ldots, e_n\}$ an orthonormal basis of $T_pM^n$. Then the normalized scalar curvature $\rho$ is defined by (see [7]):

$$\rho = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j)$$

$$= \frac{1}{n(n-1)} \sum_{1 \leq i < j \leq n} [g(R(e_i, e_j)e_i, e_j) + g(R^*(e_i, e_j)e_i, e_j)].$$

By using the Gauss equations for the dual connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$, respectively, we obtain

$$\rho = \frac{1}{n(n-1)} \sum_{1 \leq i < j \leq n} \left[-c - g(h(e_i, e_i), h^*(e_j, e_j)) + g(h^*(e_i, e_j), h(e_i, e_j))
- c - g(h^*(e_i, e_j), h(e_j, e_j)) + g(h(e_i, e_j), h^*(e_i, e_j))\right].$$

Denoting as usual by

$$h^r_{ij} = g(h(e_i, e_j), \xi_r), \quad h^r_{ij} = g(h^*(e_i, e_j), \xi_r),$$

$\forall i, j = 1, \ldots, n$ and $r = 1, \ldots, m$,

the above equation becomes

$$\rho = -c + \frac{1}{n(n-1)} \sum_{r=1}^{m} \sum_{1 \leq i < j \leq n} \left(2h^r_{ij}h^r_{ij} - h^r_{ii}h^r_{jj} - h^r_{ij}h^r_{jj}\right). \quad (4.1)$$
On the other hand, the normalized normal scalar curvature $\rho^\perp$ is defined by (see also [3]):

$$
\rho^\perp = \frac{1}{n(n-1)} \left\{ \sum_{1 \leq r < s \leq m} \sum_{1 \leq i < j \leq n} \left[ g \left( R^\perp (e_i, e_j) \xi_r, \xi_s \right) + g \left( R^{*\perp} (e_i, e_j) \xi_r, \xi_s \right) \right]^2 \right\}^{1/2}.
$$

The Ricci equations for the dual connections $\tilde{\nabla}$, and $\tilde{\nabla}^*$, respectively, imply

$$
\rho^\perp = \frac{1}{n(n-1)} \left\{ \sum_{1 \leq r < s \leq m} \sum_{1 \leq i < j \leq n} \left[ g \left( A^r_{\xi_i} A^s_{\xi_j} e_i, e_j \right) + g \left( A^r_{\xi_i} A^{*s}_{\xi_j} e_i, e_j \right) \right]^2 \right\}^{1/2}
$$
or equivalently,

$$
\rho^\perp = \frac{1}{n(n-1)} \left\{ \sum_{1 \leq r < s \leq m} \sum_{1 \leq i < j \leq n} \left[ \sum_{k=1}^n \left( h^s_{ik} h^{*r}_{jk} - h^r_{ik} h^s_{jk} + h^s_{ik} h^r_{jk} - h^r_{ik} h^{*s}_{jk} \right) \right]^2 \right\}^{1/2}.
$$

It follows that

$$
\rho^\perp = \frac{1}{n(n-1)} \left\{ \sum_{1 \leq r < s \leq m} \sum_{1 \leq i < j \leq n} \left[ \sum_{k=1}^n \left( h^s_{ik} h^{*r}_{jk} + h^r_{ik} h^s_{jk} - h^r_{ik} h^s_{jk} - h^s_{ik} h^r_{jk} \right)^2 \right] \right\}^{1/2}.
$$

It is known that the components of the second fundamental form $h^0$ of $M^n$ with respect to the Levi-Civita connection $\tilde{\nabla}^0$ are given by $2h^0_{ik} = h^s_{ik} + h^{*s}_{ik}$, $\forall i, k = 1, \ldots, n, r = 1, \ldots, m$. Then we can write

$$
\rho^\perp = \frac{1}{n(n-1)} \left\{ \sum_{1 \leq r < s \leq m} \sum_{1 \leq i < j \leq n} \left[ \sum_{k=1}^n \left( 4 \left( h^0_{ik} h^0_{jk} - h^s_{ik} h^0_{jk} \right) + \left( h^r_{ik} h^s_{jk} - h^s_{ik} h^r_{jk} \right) \right)^2 \right] \right\}^{1/2}.
$$

We shall use the algebraic inequality

$$(a + b + c)^2 \leq 3(a^2 + b^2 + c^2), \quad \forall a, b, c \in \mathbb{R}.$$
Therefore
\[
\rho^\perp \leq \frac{3}{n(n-1)} \left\{ \sum_{1 \leq r < s \leq m} \sum_{1 \leq i < j \leq n} \left( 16 \sum_{k=1}^{n} \left( h_{ik}^0 h_{jk}^0 - h_{ik}^0 h_{jk}^0 \right) \right)^2 \right. \\
+ \left[ \sum_{k=1}^{n} \left( h_{ik}^s h_{jk}^s - h_{ik}^s h_{jk}^s \right)^2 \right] + \left[ \sum_{k=1}^{n} \left( h_{ik}^s h_{jk}^s - h_{ik}^s h_{jk}^s \right)^2 \right] \right\}^{\frac{1}{2}}. \tag{4.3}
\]

Recall an inequality from [12] (see also [14])
\[
\sum_{r=1}^{m} \sum_{1 \leq i < j \leq n} \left( h_{ii}^r - h_{jj}^r \right)^2 + 2n \sum_{r=1}^{m} \sum_{1 \leq i < j \leq n} \left( h_{ij}^r \right)^2 \\
\geq 2n \left[ \sum_{1 \leq i < j \leq n} \sum_{1 \leq r \leq s \leq m} \left( \sum_{k=1}^{n} \left( h_{jk}^r h_{ik}^r - h_{ik}^r h_{jk}^r \right) \right) \right]^{\frac{1}{2}}.
\]

Similarly, we have
\[
\sum_{r=1}^{m} \sum_{1 \leq i < j \leq n} \left( h_{ii}^s - h_{jj}^s \right)^2 + 2n \sum_{r=1}^{m} \sum_{1 \leq i < j \leq n} \left( h_{ij}^s \right)^2 \\
\geq 2n \left[ \sum_{1 \leq i < j \leq n} \sum_{1 \leq r \leq s \leq m} \left( \sum_{k=1}^{n} \left( h_{jk}^s h_{ik}^s - h_{ik}^s h_{jk}^s \right) \right) \right]^{\frac{1}{2}}
\]

and
\[
\sum_{r=1}^{m} \sum_{1 \leq i < j \leq n} \left( h_{ii}^0 - h_{jj}^0 \right)^2 + 2n \sum_{r=1}^{m} \sum_{1 \leq i < j \leq n} \left( h_{ij}^0 \right)^2 \\
\geq 2n \left[ \sum_{1 \leq i < j \leq n} \sum_{1 \leq r \leq s \leq m} \left( \sum_{k=1}^{n} \left( h_{jk}^0 h_{ik}^0 - h_{ik}^0 h_{jk}^0 \right) \right) \right]^{\frac{1}{2}}.
\]

Summing up the above three inequalities, from (4.3) we obtain
\[
\rho^\perp \leq \frac{3}{2n^2(n-1)} \sum_{r=1}^{m} \sum_{1 \leq i < j \leq n} \left( h_{ii}^r - h_{jj}^r \right)^2 + \left( h_{ii}^s - h_{jj}^s \right)^2 + 16 \left( h_{ii}^0 - h_{jj}^0 \right)^2 \\
+ \frac{3}{n(n-1)} \sum_{r=1}^{m} \sum_{1 \leq i < j \leq n} \left( h_{ij}^r \right)^2 + \left( h_{ij}^s \right)^2 + 16 \left( h_{ij}^0 \right)^2. \tag{4.4}
\]
Also, we can write
\[
\begin{align*}
n^2 \| H \|^2 &= \sum_{r=1}^{m} \left( \sum_{i=1}^{n} h_{ii}^{r} \right)^2 = \frac{1}{n-1} \sum_{r=1}^{m} \sum_{1 \leq i < j \leq n} (h_{ii}^{r} - h_{jj}^{r})^2 + \frac{2n}{n-1} \sum_{r=1}^{m} \sum_{1 \leq i < j \leq n} h_{ii}^{r} h_{jj}^{r}
\end{align*}
\]
and similarly,
\[
\begin{align*}
n^2 \| H^* \|^2 &= \frac{1}{n-1} \sum_{r=1}^{m} \sum_{1 \leq i < j \leq n} (h_{ii}^{sr} - h_{jj}^{sr})^2 + \frac{2n}{n-1} \sum_{r=1}^{m} \sum_{1 \leq i < j \leq n} h_{ii}^{sr} h_{jj}^{sr}
\end{align*}
\]
and
\[
\begin{align*}
n^2 \| H^0 \|^2 &= \frac{1}{n-1} \sum_{r=1}^{m} \sum_{1 \leq i < j \leq n} (h_{ii}^{0r} - h_{jj}^{0r})^2 + \frac{2n}{n-1} \sum_{r=1}^{m} \sum_{1 \leq i < j \leq n} h_{ii}^{0r} h_{jj}^{0r}.
\end{align*}
\]
Substituting in (4.4), we get
\[
\begin{align*}
\rho^\perp &\leq \frac{3}{2} \| H \|^2 + \frac{3}{2} \| H^* \|^2 + 24 \| H^0 \|^2 \\
&\quad - \frac{3}{n(n-1)} \sum_{r=1}^{m} \sum_{1 \leq i < j \leq n} (h_{ii}^{r} h_{jj}^{r} + h_{ii}^{sr} h_{jj}^{sr} + 16 h_{ii}^{0r} h_{jj}^{0r}) \\
&\quad + \frac{3}{n(n-1)} \sum_{r=1}^{m} \sum_{1 \leq i < j \leq n} \left[ (h_{ij}^{r})^2 + (h_{ij}^{sr})^2 + 16 (h_{ij}^{0r})^2 \right] \\
&= \frac{3}{2} \| H \|^2 + \frac{3}{2} \| H^* \|^2 + 24 \| H^0 \|^2 \\
&\quad - \frac{3}{n(n-1)} \sum_{r=1}^{m} \sum_{1 \leq i < j \leq n} \left[ (h_{ii}^{r} + h_{ii}^{sr}) (h_{jj}^{r} + h_{jj}^{sr}) - h_{ii}^{sr} h_{jj}^{r} - h_{ii}^{0r} h_{jj}^{0r} + 16 h_{ii}^{0r} h_{jj}^{0r} \right] \\
&\quad + \frac{3}{n(n-1)} \sum_{r=1}^{m} \sum_{1 \leq i < j \leq n} \left[ (h_{ij}^{r} + h_{ij}^{sr})^2 - 2 h_{ij}^{sr} h_{ij}^{r} + 16 (h_{ij}^{0r})^2 \right].
\end{align*}
\]
Using again \(2h_{ij}^{0r} = h_{ij}^{r} + h_{ij}^{sr} , \forall i, j = 1, \ldots, n, r = 1, \ldots, m\), we obtain
\[
\begin{align*}
\rho^\perp &\leq \frac{3}{2} \| H \|^2 + \frac{3}{2} \| H^* \|^2 + 24 \| H^0 \|^2 \\
&\quad - \frac{3}{n(n-1)} \sum_{r=1}^{m} \sum_{1 \leq i < j \leq n} \left[ 20 h_{ii}^{0r} h_{jj}^{0r} - h_{ii}^{sr} h_{jj}^{r} - h_{ii}^{sr} h_{jj}^{r} - 20(h_{ij}^{0r})^2 + 2h_{ij}^{sr} h_{ij}^{r} \right].
\end{align*}
\]
(4.5)
Substituting (4.1) in (4.5), one leads to
\[
\rho^\perp \leq \frac{3}{2} \|H\|^2 + \frac{3}{2} \|H^*\|^2 + 24 \left\| H^0 \right\|^2 - 3\rho - 3c \\
- \frac{60}{n(n-1)} \sum_{r=1}^{m} \sum_{1 \leq i < j \leq n} \left[ h^{0r}_{ii} h^{0r}_{jj} - \left( h^{0r}_{ij} \right)^2 \right].
\] (4.6)

If we denote by
\[
\tilde{\rho}^0 = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} \tilde{R}^0(e_i, e_j, e_i, e_j),
\]
the Gauss equation for the Levi-Civita connection \(\tilde{\nabla}^0\) gives
\[
\tilde{\rho}^0 = \rho^0 - \frac{2}{n(n-1)} \sum_{r=1}^{m} \sum_{1 \leq i < j \leq n} \left[ h^{0r}_{ij} h^{0r}_{ij} - \left( h^{0r}_{ij} \right)^2 \right].
\] (4.7)

From (4.6) and (4.7) we obtain
\[
\rho^\perp \leq \frac{3}{2} \|H\|^2 + \frac{3}{2} \|H^*\|^2 + 24 \left\| H^0 \right\|^2 - 3\rho - 3c + 30 \left( \tilde{\rho}^0 - \rho^0 \right).
\]

Summarizing, we proved the following generalized Wintgen inequality.

**Theorem 4.1** Let \(M^n\) be a submanifold in a statistical manifold \((\tilde{M}^{n+m}, c)\) of constant curvature \(c\). Then
\[
\rho^\perp + 3\rho \leq \frac{15}{2} \|H\|^2 + \frac{15}{2} \|H^*\|^2 + 12 g(H, H^*) - 3c + 30 \left( \tilde{\rho}^0 - \rho^0 \right).
\]

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