Long-time asymptotics for a fourth-order dispersive nonlinear Schrödinger equation with nonzero boundary conditions

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Abstract
In this work, we consider the long-time asymptotics for the Cauchy problem of a fourth-order dispersive nonlinear Schrödinger equation with nonzero boundary conditions at infinity. Firstly, in order to construct the basic Riemann-Hilbert problem associated with nonzero boundary conditions, we analysis direct scattering problem. Then we deform the corresponding matrix Riemann-Hilbert problem to explicitly solving models via using the nonlinear steepest descent method and employing the $g$-function mechanism to eliminate the exponential growths of the jump matrices. Finally, we obtain the asymptotic stage of modulation instability for the fourth-order dispersive nonlinear Schrödinger equation.

Key words: Long-time asymptotics; Fourth-order dispersive nonlinear Schrödinger equation; Riemann-Hilbert problem; Nonlinear steepest descent method.

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1 Introduction

The fourth-order dispersive nonlinear Schrödinger (NLS) equation with nonzero boundary conditions (NZBCs) is

\[ i\dot{q} + q_{xx} + 2(|q|^2 - q_0^2)q + \gamma(q_{xxxx} + 8q_{xx}|q|^2 + 2q_x^2q^2 + 6q_x^2q^* + 4q_x^2 + 6(|q|^4 - q_0^4)) = 0, \]

where \( q_\pm \) are complex constants and independent of \( x, t \) with \( |q_\pm| = q_0 > 0 \). Moreover

\[ q(x, 0) - q_\pm \in L^{1,1}(\mathbb{R}^\pm), \quad L^{1,1}(\mathbb{R}^\pm) = \{ f : \mathbb{R} \to \mathbb{C} | \int_{\mathbb{R}^\pm} (1 + |x|)|f(x)|dx < \infty \}. \] (1.2)

With the Gauge transformation

\[ q(x, t) = u(x, t)e^{-2i(3\gamma q_0^2 + 1)q_0^2 t}, \] (1.3)

Eq. (1.1) can be reduced to the following general fourth-order dispersive NLS equation

\[ iu_t + u_{xx} + 2|u|^2u + \gamma(u_{xxxx} + 8u_{xx}|u|^2 + 2u_x^2u^2 + 6u_x^2u^* + 4u_x^2 + 6u|u|^4) = 0, \]

where \( u \) is a complex function with temporal variable \( t \) and spatial variable \( x \), which denotes the slowly varying envelope of the wave. The parameter \( \gamma \) is a small dimensionless real number. In long distance and high-speed optical fiber transmission systems, the fourth-order dispersive NLS equation plays a leading role in describing the transmission of ultrashort optical pulses \[2,3,4\]. Moreover, the equation can also depict the nonlinear spin excitation in a one-dimensional isotropic biquadratic Heisenberg ferromagnetic spin with octopole-dipole interaction \[5,6\]. So far, there are some works on the study of the fourth-order dispersive NLS equation. Many methods have been used to derive the exact solutions for the fourth-order dispersive NLS equation, such as Darboux transformation method, Hirota bilinear method and inverse scattering transform (IST) method \[7,8\], and the Lax pair, conservation laws, local wave solutions have also been discussed \[9,10,11\]. Recently, the long-time asymptotic behavior for the fourth-order dispersive NLS equation
under zero boundary conditions (ZBCs) was investigated \cite{12,13}. To our known of knowledge, the long-time asymptotic behaviors for the fourth-order dispersive NLS equation with NZBCs have not been analyzed yet.

In fact, the asymptotic behavior of solutions for nonlinear integrable systems has a long history and is always a hot topic. Early studies can be traced back to literatures \cite{14,15,16,17,18,19}. It is worth mentioning that Deift and Zhou, motivated by the pioneering work of Its \cite{19}, proposed the nonlinear steepest descent method to investigate the long-time asymptotic behavior for the Cauchy problem of the mKdV equation with a oscillatory Riemann-Hilbert (RH) problem \cite{20}. Subsequently, this method was further developed in references \cite{21,22,23}. Since the nonlinear steepest descent method been an efficient technique to research the Cauchy problem of integrable equations, the long-time asymptotics for lots of integrable equations as followed have been analyzed \cite{24,25,26,27,28,29,30,31}. Besides, the method have been extended to the long-time asymptotics of the Cauchy problems for nonlinear integrable systems with a variety of non-decaying initial data, such as the time-periodic boundary conditions \cite{32,33}, the shock problem \cite{34}, and the step-like initial data \cite{35,36,37}. Moreover, as a significant development of RH problem, \( \hat{\vartheta} \) generalization of the nonlinear steepest descent method was raised to derive the long-time asymptotic expansion of the solution in different fixed space-time regions \cite{38,39,40,41}. Recent years, the researches about NZBCs at infinity have already been become a focal point. Biondini and his cooperators have studied the soliton solutions and the long-time asymptotics for the focusing NLS equation with NZBCs in \cite{42} and \cite{43}, respectively. After that, long-time asymptotics of the focusing Kundu-Eckhaus equation with NZBCs were studied in \cite{44}, long-time dynamics of the Gerdjikov-Ivanov type derivative nonlinear Schrödinger equation with NZBCs were studied in \cite{45}, long-time dynamics of the Hirota equation with NZBCs were studied in \cite{46}, and long-time dynamics of the modified Landau-Lifshitz equation with NZBCs were studied in \cite{47}. Besides, the long-time asymptotic behavior of nonlocal integrable NLS solutions with NZBCs were studied in \cite{48}.

In this work, motivated by the long-time asymptotic analysis presented in \cite{43}, we consider the long-time asymptotics of Eq. (1.1) with the NZBCs at infinity. To the best knowledge of the authors, the long-time asymptotics for the fourth-order dispersive NLS equation under the NZBCs has never been reported up to now.

The major results of this work is summarized in what follows:

**Theorem 1.1.** As \( t \to \infty \), the asymptotic stage of modulation instability for \( q(x, t) \) is given by

\[
q(x, t) = \frac{q_0}{q_+^*}(\chi_2 + q_0) \Theta \left( \frac{\Omega t + \vartheta + i \ln \left( \frac{q_0^*}{q_0} \right)}{2\pi} - V(\infty) + C \right) \Theta(V(\infty) + C) e^{2i(g(\infty) + G(\infty)t)} + O(t^{-1/2}),
\]

where \( V(\infty) = \int_{q_0}^\infty d\vartheta, and \chi_2, \Omega, G(\infty), \vartheta, g(\infty), C \) are given by Eqs. \cite{3.20,3.32,3.33,3.37,3.40,3.56}.
Organization of this work: In Section 2, we perform the spectral analysis for the Cauchy problem of the fourth-order dispersive NLS equation, and construct the basic RH problem, which is the premise to give out the asymptotic behavior of the fourth-order dispersive NLS equation under NZBCs. In Section 3, the asymptotic stage of modulation instability for Eq. (1.1) is analysed in detail.

2 Reconstructing the basic Riemann-Hilbert problem

We recall some notations that will used in our paper. The classical Pauli matrices are defined as follows

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.1)
\]

For a $2 \times 2$ matrix $A$ and a scalar variable $\alpha$, we define

\[
e^{\alpha \hat{\sigma}_3} A e^{-\alpha \sigma_3}.
\] (2.2)

2.1 Direct scattering problem with NZBCs

The Lax pair of Eq. (1.1) is

\[
\psi_x = X \psi, \quad \psi_t = T \psi,
\] (2.3)

with the vector eigenfunction $\psi = (\psi_1, \psi_2)^T$ being a $2 \times 2$ matrix, where the superscript $T$ represents the transpose of the vector,

\[
X = -ik\sigma_3 + Q, \quad Q = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix},
\]

\[
T = \left[ 3i\gamma |q|^4 + i|q|^2 + i\gamma(q_{xx}q^* + qq_{xx}^*) - 2k\gamma(|q^*_x - q_x|^2) - 2i\gamma^2q_{xx^*} - 2\gamma^2k^2(q^*_x - q_x^*) \right] \sigma_3 - 4i\gamma k^2\sigma_3 Q_x \]

\[-8\gamma k^3 Q + 6i\gamma Q^2 Q_x\sigma_3 + i\sigma_3 Q_x + i\gamma \sigma_3 Q_{xxx} + 2k(Q + \gamma Q_{xx} - 2\gamma Q^3),
\] (2.4)

where $k$ represents the spectrum parameter. For convenience, we set $\gamma = 1$ for the following analysis.

Taking $x \to \pm \infty$ and combining the NZBCs, we turn the Lax pair in Eq. (2.3) into

\[
\psi_{\pm x} = X_{\pm} \psi_{\pm}, \quad \psi_{\pm t} = T_{\pm} \psi_{\pm},
\] (2.5)

where

\[
X_{\pm} = -ik\sigma_3 + Q_{\pm}, \quad T_{\pm} = (-8k^3 + 2k + 4kq_0^2)X_{\pm}(k),
\]

\[
Q_{\pm} = \begin{pmatrix} 0 & q_{\pm} \\ -q_{\pm}^* & 0 \end{pmatrix},
\] (2.6)

and we have defined $Q_{\pm} = \lim_{x \to \pm \infty} Q$. 

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It is not hard to obtain the eigenvalues of the matrix $X \pm$ given by $\pm i\lambda$, and $\lambda = \sqrt{k^2 + q_0^2}$. Obviously, the branch cut of $\lambda$ is $\eta = [-iq_0, iq_0]$ which is oriented upward in Figure 1. Here, we should also define $\eta_+ = [0, iq_0]$ and $\eta_- = [-iq_0, 0]$.

\[
\begin{array}{c}
\eta_+ \\
\text{0} \\
\eta_- \\
-iq_0
\end{array}
\xrightarrow{iq_0} \mathbb{R}
\]

**Figure 1.** (Color online) The contour $\Sigma = \mathbb{R} \cup \eta$ of the basic RH problem.

The asymptotic spectral problem (2.5) can be solved by

\[
\psi \pm = E \pm e^{-i\theta(x,t,k)\sigma_3},
\]

where

\[
\theta = \lambda \left[ x + (-8k^3 + 2k + 4kq_0^2)t \right], \quad E \pm = \begin{pmatrix} 1 & \frac{\lambda - k}{iq_+} \\ \frac{\lambda - k}{iq_-} & 1 \end{pmatrix}.
\]

Supposing that $\Psi \pm(x, t, k)$ are both the Jost solutions of the Lax pair (2.3), we can define $\Psi \pm(x, t, k) = \psi \pm(x, t, k) + o(1)$ as $x \to \infty$. Furthermore, using variable transformation

\[
\mu \pm(x, t, k) = \Psi \pm(x, t, k)e^{i\theta(x,t,k)\sigma_3},
\]

we have

\[
\mu \pm(x, t, k) = E \pm + o(1), \quad x \to \pm \infty,
\]

which arrive the following two Volterra integral equations

\[
\begin{align*}
\mu_-(x, t, k) &= E_- + \int_{-\infty}^{x} E_- e^{-i\lambda(x-y)\sigma_3} \left[ E_-^{-1}(Q_-) \mu_-(y, t, k) \right] dy, \\
\mu_+(x, t, k) &= E_+ - \int_{x}^{+\infty} E_+ e^{-i\lambda(x-y)\sigma_3} \left[ E_+^{-1}(Q_-) \mu_+(y, t, k) \right] dy.
\end{align*}
\]

**Proposition 2.1.** Suppose $q - q_0 \in L^1(\mathbb{R})$, then $\mu_\pm(x, t, k)$ given in Eq. (2.10) uniquely satisfy the Volterra integral equation (2.11) in $\Sigma_0$, and $\mu_\pm(x, t, k)$ admit:

- $\mu_-(x, t, k)$ and $\mu_2(x, t, k)$ is analytical in $\mathbb{C}_+ \setminus \eta_+$ and continuous in $\mathbb{C}_+ \cup \Sigma_0$;
- $\mu_+(x, t, k)$ and $\mu_2(x, t, k)$ is analytical in $\mathbb{C}_- \setminus \eta_-$ and continuous in $\mathbb{C}_- \cup \Sigma_0$;
- $\mu_\pm(x, t, k) \to I$ as $k \to \infty$;
Therefore, the concrete form of scattering matrix $s(k)$ of which the scattering matrix $s$ and $2.2$ Inverse scattering problem and reconstructing the formula for potential

Now, the fundamental matrix-value function is formulated as

\[
m(x, t, k) = \begin{cases} \\
\frac{(\Psi^-_1, \Psi^-_2)e^{i\varrho_2\lambda}}{\lambda_1}, & k \in \mathbb{C}_+ \setminus \eta_+, \\
(\Psi^-_1, \Psi^-_2)e^{i\varrho_2\lambda}, & k \in \mathbb{C}_- \setminus \eta_-, \end{cases}
\]
Then the matrix-value function $m(x, t, k)$ has following jump condition across $\mathbb{R}$:

$$m_+(x, t, k) = m_-(x, t, k) \left( \begin{array}{cc} \frac{1}{2} [1 + \gamma(k)\gamma^*(k)] & \gamma^*(k)e^{-2i\theta(x,t,k)}d(k) \\ \gamma(k)e^{2i\theta(x,t,k)} & \frac{1}{2} [1 + \gamma(k)\gamma^*(k)] \end{array} \right), \quad k \in \mathbb{R}, \quad (2.20)$$

where $m_+(x, t, k)$ denote the boundary values of $m(x, t, k)$ as $k$ approaches the contour from a chosen side, and the reflection coefficient $\gamma(k) = \frac{b(k)}{a(k)}$. In terms of (2.18), (2.19), (2.20), the jump condition of the matrix-value function $m(x, t, k)$ across $\eta_+$ is given by

$$m_+(x, t, k) = m_-(x, t, k) \left( \frac{\lambda-k}{iq_+} \gamma^*(k)e^{-2i\theta(x,t,k)} + \frac{2i\lambda}{q_+} \frac{i\gamma^*(k)}{2\lambda} \right), \quad k \in \eta_+. \quad (2.21)$$

Similarly, the matrix-value function $m^{(0)}(x, t, k)$ has jump condition across $\eta_-$:

$$m_+(x, t, k) = m_-(x, t, k) \left( \frac{i(\lambda+k)}{q_+} \gamma^*(k)e^{-2i\theta(x,t,k)} + \frac{iq_+}{2\lambda} [1 + \gamma(k)\gamma^*(k)] \right), \quad k \in \eta_-.
\quad (2.22)$$

Finally, assuming that the $a \neq 0$ for all $k \in \mathbb{C}_+ \cup \Sigma$, then a matrix RH problem is constructed:

$$\begin{cases}
  m(x, t, k) \text{ is analytic in } \mathbb{C} \setminus \Sigma, \\
  m_+(x, t, k) = m_-(x, t, k)J(x, t, k), \quad k \in \Sigma, \\
  m(x, t, k) \to I, \quad k \to \infty,
\end{cases} \quad (2.23)$$

of which the jump matrix $J(x, t, k) = \{J_i(x, t, k)\}_{i=1}^3$ is (see Figure 1)

- $J_1 = \left( \begin{array}{cc} \frac{1}{2} [1 + \gamma(k)\gamma^*(k)] & \gamma^*(k)e^{-2i\theta(x,t,k)t} \\ \gamma(k)e^{2i\theta(x,t,k)t} & \frac{1}{2} [1 + \gamma(k)\gamma^*(k)] \end{array} \right)$,

- $J_2 = \left( \frac{\lambda-k}{iq_+} \gamma^*(k)e^{-2i\theta(x,t,k)t} + \frac{2i\lambda}{q_+} \frac{i\gamma^*(k)}{2\lambda} \right)$,

- $J_3 = \left( \frac{i(\lambda+k)}{q_+} \gamma^*(k)e^{-2i\theta(x,t,k)t} + \frac{iq_+}{2\lambda} [1 + \gamma(k)\gamma^*(k)] \right)$,

where $f = \lambda \left[ \xi - 8k^3 + 2k + 4kq_0^2 \right], \xi = \frac{\xi}{\eta}$.

In addition, expanding the $M^{(0)}(x, t, k)$ at large $k$ as

$$m(x, t, k) = I + \frac{m_1(x, t)}{k} + \frac{m_2(x, t)}{k^2} + \mathcal{O}(\frac{1}{k^3}), \quad k \to \infty, \quad (2.24)$$

and combining equations (2.3), (2.19), (2.24), we recover the solution $q(x, t)$ of the original initial value problem (1.1) in the following form

$$q(x, t) = 2i \lim_{k \to \infty} km_{12}(x, t, k). \quad (2.25)$$
2.3 The sign structure of Re(if)

To find contour deformations, we firstly need to discuss the sign structure of the quantity Re(if). Through taking

$$\frac{df(\xi, k)}{dk} = \frac{-32k^4 - (16q_0^2 - 4)k^2 + k\xi + 4q_0^4 + 2q_0^2}{\sqrt{k^2 + q_0^2}},$$

(2.26)

for convenience, let $q_0^2 = \frac{1}{2}$, we can get the four stationary phase points (i.e., the points $k_s$ such that $f'_k(k_s) = 0$), given by

$$k_1 = \frac{-\sqrt{2y} \pm \sqrt{-\frac{\xi}{16\sqrt{2y}} - 2y}}{2}, \quad k_2 = \frac{\sqrt{2y} \pm \sqrt{\frac{\xi}{16\sqrt{2y}} - 2y}}{2},$$

(2.27)

where

$$y = \left(\frac{\sqrt{\xi^4 + 128}}{16384} + \frac{\xi^2}{16384}\right)^{\frac{1}{3}} - \left(\frac{\sqrt{\xi^4 + 128}}{16384} - \frac{\xi^2}{16384}\right)^{\frac{1}{3}}.$$

(2.28)

Using Maple symbolic computation, we find there are two real stationary phase points and two complex stationary phase points for arbitrary $\xi \neq 0$, and the corresponding sign structure of Re(if) is presented at what follows.

Figure 2. (Color online) The sign structure of Re(if) in the complex $k$-plane.

3 Asymptotic stage of modulation instability

In order to derive long-time asymptotics of solution for the Eq. (1.1), we carry out similar deformations of the RH problem (2.23) as that in Refs. [34, 43]. As shown in Figure 2, the curves $\text{Im}\theta(k) = 0$ will not intersect the real axis. In order to study the long-time asymptotics of $q(x, t)$ in this case, we construct a $g$-function mechanism and introduce the point $k_0$.  

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3.1 First deformation

To achieve the first deformation, we first decompose the jump matrix \( J_1, J_2, J_3 \) into following form

\[
\begin{align*}
J_1 &= J_0^{(1)} J_1^{(1)}, & \text{on} & \ (k_1, k_0) \cup (k_2, \infty), \\
J_1 &= J_4^{(1)} J_3^{(1)}, & \text{on} & \ (-\infty, k_1) \cup (k_0, k_2), \\
J_2 &= (J_{3-}^{(1)})^{-1} J_0^{(1)} J_{3+}^{(1)}, & \text{on} & \ \eta_+ \text{ cut}, \\
J_3 &= J_4^{(1)} J_{\eta}^{(1)} (J_{4+}^{(1)})^{-1}, & \text{on} & \ \eta_- \text{ cut},
\end{align*}
\]

where

\[
\begin{align*}
J_0^{(1)} &= \begin{pmatrix} 1 + \gamma \gamma^* & 0 \\ 0 & \frac{1}{1 + \gamma \gamma^*} \end{pmatrix}, & J_1^{(1)} &= \begin{pmatrix} d^{-\frac{1}{2}} & \frac{d^2 \gamma e^{-2ift}}{1 + \gamma^2} \\ 0 & d^\frac{1}{2} \end{pmatrix}, & J_2^{(1)} &= \begin{pmatrix} d^{-\frac{1}{2}} & 0 \\ \frac{d^2 \gamma e^{2ift}}{1 + \gamma^2} & d^\frac{1}{2} \end{pmatrix}, \\
J_3^{(1)} &= \begin{pmatrix} d^{-\frac{1}{2}} & 0 \\ \frac{d^2 \gamma e^{2ift}}{1 + \gamma^2} & d^\frac{1}{2} \end{pmatrix}, & J_4^{(1)} &= \begin{pmatrix} d^{-\frac{1}{2}} & \frac{d^2 \gamma e^{-2ift}}{1 + \gamma^2} \\ 0 & d^\frac{1}{2} \end{pmatrix}, & J_{\eta}^{(1)} &= \begin{pmatrix} 0 & \frac{iq_+}{q_0} \\ \frac{iq_-}{q_0} & 0 \end{pmatrix}.
\end{align*}
\]

Therefore, we can transform \( m \) into \( m^{(1)} \) by using

\[
m^{(1)} = mB(k),
\]

where

\[
B(k) = \begin{cases} 
(J_1^{(1)})^{-1} & \text{on} k \in \Omega_1, \\
J_2^{(1)} & \text{on} k \in \Omega_2, \\
(J_3^{(1)})^{-1} & \text{on} k \in \Omega_3 \cup \Omega_5, \\
J_4^{(1)} & \text{on} k \in \Omega_4 \cup \Omega_6, \\
I & \text{on} k \in \text{others},
\end{cases}
\]

then the following RH problem about \( m^{(1)} \) can be given

\[
\begin{align*}
m^{(1)}(x, t, k) & \text{ is analytic in } \mathbb{C} \setminus \Sigma^{(1)}, \\
m_{\pm}^{(1)}(x, t, k) & = m_{\pm}^{(1)}(x, t, k), & k & \in \Sigma^{(1)}, \\
m^{(1)}(x, t, k) \rightarrow I, & k & \rightarrow \infty,
\end{align*}
\]

where the jump matrix \( J^{(1)} \) has been defined in \((3.1)\). The contour \( \Sigma^{(0)} \) in Figure 3 also become the new contour \( \Sigma^{(1)} \) as shown in Figure 4.
3.2 Second deformation

Through introducing a scale RH problem

$$\begin{cases}
\delta(k) \text{ is analytic in } \mathbb{C} \setminus (k_1, k_0) \cup (k_2, \infty), \\
\delta_+(k) = \delta_-(k)[1 + \gamma(k)\gamma^*(k)], \quad k \in (k_1, k_0) \cup (k_2, \infty), \\
\delta(k) \to 1, \quad k \to \infty,
\end{cases} \quad (3.5)$$

we can delete the jump across the cut $(-\infty, k_0) \cup (k_2, \infty)$. The above RH problem can be solved by Plemelj formula, given by

$$\delta(k) = \exp \left\{ \frac{1}{2\pi i} \int_{(k_1,k_0)\cup(k_2,\infty)} \frac{\ln[1 + \gamma(y)\gamma^*(y)]}{y - k} dy \right\}. \quad (3.6)$$

To finish the second deformation, we choose the transformation

$$m^{(2)} = m^{(1)} \delta^{-\sigma_3} \quad (3.7)$$
to get a new matrix-value function $m^{(2)}$, which meets the following RH problem with the contour $\Sigma^{(2)}$ displayed in Fig. 5

$$
\begin{cases}
    m^{(2)}(x, t, k) \text{ is analytic in } \mathbb{C} \setminus \Sigma^{(2)}, \\
    m^{(2)}(x, t, k) = m^{(2)}_-(x, t, k)J^{(2)}(x, t, k), \quad k \in \Sigma^{(2)}, \\
    m^{(2)}(x, t, k) \to I, \quad k \to \infty,
\end{cases}
$$

and with the help of $J^{(2)} = \delta^- \sigma^3 f^{(1)} \delta^- \sigma^3$, we have

$$
\begin{align*}
J^{(2)}_1 &= \begin{pmatrix} d^{-\frac{1}{2}} & \frac{\delta^2}{\delta^2 \gamma^* e^{-2itc}} \\ 0 & \frac{\delta^2}{\delta^2 \gamma^* e^{-2itc}} \end{pmatrix}, \\
J^{(2)}_2 &= \begin{pmatrix} d^{-\frac{1}{2}} & 0 \\ \frac{\delta^2}{\delta^2 \gamma^* e^{-2itc}} & \frac{\delta^2}{\delta^2 \gamma^* e^{-2itc}} \end{pmatrix}, \\
J^{(2)}_3 &= \begin{pmatrix} d^{-\frac{1}{2}} & 0 \\ \frac{\delta^2}{\delta^2 \gamma^* e^{-2itc}} & \frac{\delta^2}{\delta^2 \gamma^* e^{-2itc}} \end{pmatrix}, \\
J^{(2)}_4 &= \begin{pmatrix} d^{-\frac{1}{2}} & \frac{\delta^2}{\delta^2 \gamma^* e^{-2itc}} \\ 0 & \frac{\delta^2}{\delta^2 \gamma^* e^{-2itc}} \end{pmatrix},
\end{align*}
$$

(3.8)

3.3 Third deformation

For the third deformation, we select the following transformation

$$
m^{(3)} = m^{(2)} \hat{B}(k),
$$

(3.10)

with

$$
\hat{B}(k) = \begin{cases}
    d^{\frac{\sigma_3}{2}} \text{ on } k \in \hat{\Sigma}_1, \\
    d^{-\frac{\sigma_3}{2}} \text{ on } k \in \hat{\Sigma}_2, \\
    I \text{ on } k \in \hat{\Sigma}_3 \cup \hat{\Sigma}_4,
\end{cases}
$$

(3.11)

The goal of this transformation is to wipe out the term $\Delta(k)$. Then the following RH problem about $m^{(3)}$ is obtained

$$
\begin{cases}
    m^{(3)}(x, t, k) \text{ is analytic in } \mathbb{C} \setminus \Sigma^{(3)}, \\
    m^{(3)}_-(x, t, k) = m^{(3)}_-(x, t, k)J^{(3)}(x, t, k), \quad k \in \Sigma^{(3)}, \\
    m^{(3)}(x, t, k) \to I, \quad k \to \infty,
\end{cases}
$$

(3.12)
of which the contour $\Sigma^{(3)} = \Sigma^{(2)}$ is shown in Figure 5, and $J^{(3)}$ is

$$J_1^{(3)} = \begin{pmatrix} 1 & 0 \\ \delta^2 \gamma e^{2i\sigma t} & 1 \end{pmatrix}, \quad J_2^{(3)} = \begin{pmatrix} 1 & 0 \\ \delta^{-2} \gamma e^{2i\sigma t} & 1 \end{pmatrix},$$

$$J_3^{(3)} = \begin{pmatrix} 1 & 0 \\ \delta^{-2} \gamma e^{2i\sigma t} & 1 \end{pmatrix}, \quad J_4^{(3)} = \begin{pmatrix} 1 & 0 \\ \delta^2 \gamma e^{-2i\sigma t} & 1 \end{pmatrix}, \quad J_\eta^{(3)} = J_\eta^{(2)}.$$  \hspace{1cm} \text{(3.13)}

### 3.4 Eliminating of the exponential growth

According to the sign structure of $Re(\sigma t)$ shown in Figure 2, we see that the jump matrices $J_3^{(3)}$ and $J_4^{(3)}$ in Eq. (3.13) should be grow exponentially in the segment $[k_0, \chi]$ and $[k_0, \chi^*]$, respectively. Thus, the matrices $J_3^{(3)}$ and $J_4^{(3)}$ must be decomposed (see Figure 6)

$$J_3^{(3)} = J_5^{(3)} J_7^{(3)} J_5^{(3)}, \quad J_4^{(3)} = J_6^{(3)} J_8^{(3)} J_6^{(3)}.$$ \hspace{1cm} \text{(3.14)}

where

$$J_5^{(3)} = \begin{pmatrix} 1 & 0 \\ \delta^2 \gamma^{-1} e^{-2i\sigma t} & 1 \end{pmatrix}, \quad J_6^{(3)} = \begin{pmatrix} 1 & 0 \\ \delta^{-2} (\gamma^*)^{-1} e^{-2i\sigma t} & 1 \end{pmatrix},$$

$$J_7^{(3)} = \begin{pmatrix} 0 & \delta^{-2} \gamma e^{2i\sigma t} \\ \delta^{-2} \gamma e^{2i\sigma t} & 0 \end{pmatrix}, \quad J_8^{(3)} = \begin{pmatrix} 0 & \delta^2 \gamma^* e^{-2i\sigma t} \\ -\delta^{-2} (\gamma^*)^{-1} e^{2i\sigma t} & 0 \end{pmatrix}.$$ \hspace{1cm} \text{(3.15)}

![Figure 6. (Color online) The contour $\hat{\Sigma}^{(3)}$.](image)

Next, we set the transformation $m_{(4)}^{(4)} = m_{(3)}^{(3)} e^{iG(k)\sigma_3}$ via using a time-dependent $G$ function which is analytic off the cuts $\eta \cup \varpi$. Of which $\varpi = \varpi_+ \cup \varpi_- \varpi_+ = [k_0, \chi]$
and $\varpi = [k_0, \chi^*]$. Then the new jump matrices $J^{(4)}$ is calculated as follows

\[
J^{(4)}_1 = \begin{pmatrix}
1 & \delta^2 \gamma^* e^{-2i(f+G)t} \\
0 & 1 + \gamma^* \\
\end{pmatrix},
\]
\[
J^{(4)}_2 = \begin{pmatrix}
1 & 0 \\
\delta^{-2} \gamma e^{2i(f+G)t} & 1 \gamma^* \\
\end{pmatrix},
\]
\[
J^{(4)}_3 = \begin{pmatrix}
1 & 0 \\
\delta^{-2} \gamma e^{2i(f+G)t} & 1 \gamma^* \\
\end{pmatrix},
\]
\[
J^{(4)}_4 = \begin{pmatrix}
1 & 0 \\
\delta^{-2} \gamma e^{2i(f+G)t} & 1 \gamma^* \\
\end{pmatrix},
\]
\[
J^{(4)}_5 = \begin{pmatrix}
1 & \delta^2 \gamma^{-1} e^{-2i(f+G)t} \\
0 & 1 \\
\end{pmatrix},
\]
\[
J^{(4)}_6 = \begin{pmatrix}
1 & 0 \\
\delta^{-2} (\gamma^*)^{-1} e^{-2i(f+G)t} & 1 \\
\end{pmatrix},
\]
\[
J^{(4)}_7 = \begin{pmatrix}
0 & -\delta^2 \gamma^{-1} e^{-2i(2f+G+G_-)t} \\
\delta^{-2} (\gamma^*)^{-1} e^{-2i(2f+G+G_-)t} & 0 \\
\end{pmatrix},
\]
\[
J^{(4)}_8 = \begin{pmatrix}
0 & -\delta^2 \gamma^{-1} e^{-2i(2f+G+G_-)t} \\
\delta^{-2} (\gamma^*)^{-1} e^{-2i(2f+G+G_-)t} & 0 \\
\end{pmatrix},
\] (3.16)

Furthermore, we introduce a new function $\omega$:

\[
\omega(k) = f(k) + G(k),
\] (3.17)

whose properties need to be investigated up to find the parameters $k_0$ and $\chi$. First of all, a function $z$ should be defined as

\[
z(k) = \sqrt{(k^2 + k_0^2)(k - \chi)(k - \chi^*)},
\] (3.18)

which has branch cuts $\eta \cup \varpi$ and satisfies $z(k) = -z_+(k) = z_-(k)$. We implement this algebraic curve as two Riemann surfaces, and the basis $\{L_1, L_2\}$ cycles of this Riemann surface can be described as: the $L_1$-cycle is a simple counterclockwise closed ring around the bifurcation incision $\eta$, which lies on the lower sheet. The $L_2$-cycle starts from the point $\chi$ on the upper sheet, then accesses $-i\alpha_0$ and gets back to the starting point on the lower sheet.

Next, we let $\omega(k)$ meets

\[
\omega(k) = \frac{1}{2} \left( \int_{-i\alpha_0}^{i\alpha_0} + \int_{-i\alpha_0}^{i\alpha_0} \right) d\omega(y),
\] (3.19)

which is a Abelian integral and $d\omega$ is given by

\[
d\omega(k) = -\frac{32}{z(k)} \frac{(k - k_0)(k - k_1)(k - k^*_2)(k - k_2)(k - \chi)(k - \chi^*)}{z(k)} dk.
\] (3.20)

Moreover, the sign signatures of $\text{Im}\omega(k)$ must be same as that ones of $\text{Im}f(k)$ for large $k$, given by

\[
\text{Im}\omega = \text{Im}f + O\left(\frac{1}{k}\right), \quad k \to \infty.
\] (3.21)

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Therefore, we have
\[ \omega(k) = -8k^4 + 2k^2 + \xi k + \omega_0 + O\left(\frac{1}{k}\right), \quad k \to \infty. \] (3.22)

When \( k \to \infty \), the large \( k \) expression of \( z(k) \) become
\[ z(k) = k^2 \left[ 1 - \frac{\chi + \chi^*}{2k} + \frac{4q_0^2 - (\chi - \chi^*)^2}{8k^2} + O\left(\frac{1}{k^3}\right) \right], \quad k \to \infty. \] (3.23)

Taking \( \chi = \chi_1 + \chi_2i \), from (3.20), we easily obtain
\[
\frac{d\omega}{dk} = -32k^3 + 32(\chi_1 + k_0 + k_1 + k_2)k^2 - [32(k_0 + k_1 + k_2)\chi_1 + 32(k_1k_0 + k_2k_0 + k_2k_1)]
- 16q_0^2 + 16\chi_2^2k + (32(k_1k_0 + k_2k_0 + k_2k_1) - 16q_0^2 + 16\chi_2^2)\chi_1 + 32k_0k_2
+ 16(\chi_2^2 - q_0^2)(k_0 + k_1 + k_2) + O\left(\frac{1}{k}\right), \quad k \to \infty.
\] (3.24)

Meanwhile, since \( f = \lambda \left[ \xi - 8k^3 + 2k + 4kq_0^2 \right] \), one has
\[
\frac{df(k)}{dk} = -32k^3 + 4k + \xi + O\left(\frac{1}{k}\right), \quad k \to \infty.
\] (3.25)

Since (3.21) is allowed, and from Eqs. (3.24), (3.25), we can derive
\[
\chi_1 = -k_0 - k_1 - k_2,
\]
\[
\chi_2 = \frac{1}{2} \sqrt{8(k_0^2 + k_1^2 + k_2^2) + 8(k_1k_0 + k_2k_0 + k_2k_1) + 4q_0^2 - 1}, \tag{3.26}
\]
where parameter \( k_0 \) is still need to be derived later. Observing that
\[
-16 \left( \int_{iq_0}^{k} + \int_{-iq_0}^{k} \right) (y^3 - \frac{y}{8} - \frac{\xi}{32})dy = -8k^4 + 2k^2 + \xi k + 8q_0^4 + 2q_0^2, \tag{3.27}
\]
then the expression of \( \omega(k) \) in Eq. (3.10) can been reconstructed as
\[
\omega(k) = -16 \left( \int_{iq_0}^{k} + \int_{-iq_0}^{k} \right) \left[ \frac{(y - k_0)(y - k_1)(y - k_2)(y - \chi)(y - \chi^*)}{z(y)} \right]dy
- \left( y^3 - \frac{y}{8} - \frac{\xi}{32} \right)dy - 8k^4 + 2k^2 + \xi k + 8q_0^4 + 2q_0^2. \tag{3.28}
\]

As \( k \to \infty \) in (3.28), we have
\[
\omega_0 = -16 \left( \int_{iq_0}^{\infty} + \int_{-iq_0}^{\infty} \right) \left[ \frac{(y - k_0)(y - k_1)(y - k_2)(y - \chi)(y - \chi^*)}{z(y)} \right]dy
- \left( y^3 - \frac{y}{8} - \frac{\xi}{32} \right)dy + 8q_0^4 + 2q_0^2. \tag{3.29}
\]

Next, we will devote to reveal the parameter \( k_0 \) on the real line by presenting the asymptotic expansions of \( \omega(k) \) near point \( \chi \). Similar to reference [33], it is not hard to
obtain that

\[
\int_{-i\varrho_0}^{i\varrho_0} \frac{(y - k_0)(y - k_1)(y - k_2)(y - \chi)(y - \chi^*)}{z(y)} \, dy = \int_{-i\varrho_0}^{i\varrho_0} \sqrt{(y - \chi)^2 + \chi^2} \frac{(y - k_0)(y - k_1)(y - k_2)}{y^2 + \varrho_0^2} (y - k_0)(y - k_1) \, dy = 0,
\]

which uniquely gives us the point \(k_0\) is uniquely expressed.

Now, the function \(\omega(k)\) satisfies the following jump condition:

\[
\omega_+(k) + \omega_-(k) = 0, \quad k \in \eta,
\]

\[
\omega_+(k) + \omega_-(k) = \Omega, \quad k \in \varpi,
\]

where \(\Omega\) is real constant given by

\[
\Omega = -32 \left( \int_{i\varrho_0}^{\chi} + \int_{-i\varrho_0}^{\chi^*} \frac{(k - k_0)(k - k_1)(k - k_2)(k - \chi)(k - \chi^*)}{z(k)} \, dk \right).
\]

Besides, since function \(\omega(k)\) is defined in Eq. (3.17), one easily obtains

\[
G(\infty) = \omega_0 - 3q_0^4 - q_0^2, \quad k \to \infty,
\]

and we also have \(m^{(4)} \to e^{iG(\infty)\sigma_3} \) as \(k \to \infty\). Finally, we can derive the RH problem for \(m^{(4)}\), whose jump matrices \(J^{(4)}\) are

\[
J^{(4)}_1 = \begin{pmatrix} \delta^2 \gamma^* e^{-2i\omega t} & 0 \\ 0 & 1 \end{pmatrix}, \quad J^{(4)}_2 = \begin{pmatrix} 1 & 0 \\ \delta - 2 \gamma e^{2i\omega t} & 1 \end{pmatrix},
\]

\[
J^{(4)}_3 = \begin{pmatrix} 1 & 0 \\ \delta - 2 \gamma e^{2i\omega t} & 1 \end{pmatrix}, \quad J^{(4)}_4 = \begin{pmatrix} 1 & \delta^2 \gamma e^{-2i\omega t} \\ 0 & 1 \end{pmatrix},
\]

\[
J^{(4)}_5 = \begin{pmatrix} 1 & \delta^2 \gamma^* e^{-2i\omega t} \\ 0 & \delta - 2 \gamma e^{2i\omega t} \end{pmatrix}, \quad J^{(4)}_6 = \begin{pmatrix} 1 & 0 \\ \delta - 2 \gamma e^{2i\omega t} & 1 \end{pmatrix},
\]

\[
J^{(4)}_7 = \begin{pmatrix} 1 & \delta^2 \gamma^* e^{-i\Omega t} \\ 0 & \delta - 2 \gamma e^{i\Omega t} \end{pmatrix}, \quad J^{(4)}_8 = \begin{pmatrix} 1 & 0 \\ \delta - 2 \gamma e^{i\Omega t} & \delta^2 \gamma^* e^{-i\Omega t} \end{pmatrix}.
\]

The sign signature of \(\text{Im}(\omega)(k)\) ensures that the jump matrices \(J^{(4)}_i (i = 1, 2, 3, 4, 5, 6)\) are all exponentially decaying in the associated branch cuts.

### 3.5 Further deformation

In order to delete the variable \(k\) from the jump matrices \(J^{(4)}_7, J^{(4)}_8\), we need to introduce the \(g\)-function mechanism again. In the same way, we select following transformation

\[
m^{(5)} = m^{(4)(k)} g^{(3)},
\]
where the function \( g(k) \), which is analytic in \( \mathbb{C} \setminus (\eta \cup \varpi) \), satisfies
\[
g_+(k)g_-(k) = \begin{cases} 
\delta^2 & \text{on } k \in \eta, \\
e^{i\vartheta} \frac{\delta^2}{\gamma} & \text{on } k \in \varpi_+, \\
e^{i\vartheta} \delta^2 \gamma^* & \text{on } k \in \varpi_-, 
\end{cases}
\]  
(3.36)
of which \( \vartheta \) is a real constant and given by
\[
\vartheta = \frac{\int_{\eta} \frac{2 \ln \delta}{z} ds + \int_{[k_0, \chi]} \frac{\ln \delta^2}{z} ds + \int_{[k_0, \chi^*]} \frac{\ln \delta^2 \gamma^*}{z} ds}{\int_{[k_0, \chi] \cup [k_0, \chi^*]} \frac{1}{z} ds}.
\]  
(3.37)
Applying the Plemelj’s formula, the \( g(k) \) function can be solved by the following integral representation
\[
g(k) = \exp\{-\frac{z}{2\pi i} \int_{\eta} \frac{2 \ln \delta}{z(s-k)} ds + \int_{[k_0, \chi]} \frac{i\vartheta + \ln \delta^2}{z(s-k)} ds + \int_{[k_0, \chi^*]} \frac{i\vartheta + \ln \delta^2 \gamma^*}{z(s-k)} ds\},
\]  
(3.38)which implies that \( g(k) \) has the following behavior for the large \( k \):
\[
g(k) = e^{ig(\infty)} + o\left(\frac{1}{k}\right), \quad k \to \infty,
\]  
(3.39)where the \( g(\infty) \) is a real constant, given by
\[
g(\infty) = -\frac{1}{2\pi i} \int_{\eta} \frac{2 \ln \delta}{z} ds + \int_{[k_0, \chi]} \frac{i\vartheta + \ln \delta^2}{z} ds + \int_{[k_0, \chi^*]} \frac{i\vartheta + \ln \delta^2 \gamma^*}{z} ds) \]  
(3.40)
Finally, we get the following RH problem for \( m^{(5)} \)
\[
\begin{cases}
m^{(5)}(x, t, k) \text{ is analytic in } \mathbb{C} \setminus \Sigma^{(5)}, \\
m^{(5)}(x, t, k) = m^{(5)}_-(x, t, k)J^{(5)}(x, t, k), \quad k \in \Sigma^{(5)}, \\
m^{(5)}(x, t, k) \to e^{i(g(\infty)+G(\infty)i)\sigma_3}, \quad k \to \infty,
\end{cases}
\]  
(3.41)where the contour \( \Sigma^{(5)} = \tilde{\Sigma}^{(5)} \) and the jump matrices \( J^{(5)} \) become
\[
J^{(5)}_1 = \begin{pmatrix} 1 & \delta^2 -2 e^{-2i\omega t} g^2 \\ 0 & 1 + \gamma^* \end{pmatrix}, \quad J^{(5)}_2 = \begin{pmatrix} 1 & 0 \\ \delta -2 e^{2i\omega t} \gamma^* g^2 & 1 \end{pmatrix},
\]
\[
J^{(5)}_3 = \begin{pmatrix} 1 & 0 \\ \delta -2 e^{2i\omega t} \gamma^* g^2 & 1 \end{pmatrix}, \quad J^{(5)}_4 = \begin{pmatrix} 1 & \delta^2 -2 e^{-2i\omega t} g^2 \\ 0 & 1 + \gamma^* \end{pmatrix},
\]
\[
J^{(5)}_5 = \begin{pmatrix} 1 & \delta^2 -2 e^{-2i\omega t} g^2 \\ 0 & 1 + \gamma^* \end{pmatrix}, \quad J^{(5)}_6 = \begin{pmatrix} 1 & 0 \\ \delta -2 e^{2i\omega t} \gamma^* g^2 & 1 \end{pmatrix},
\]
\[
J^{(5)}_7 = \begin{pmatrix} 0 & -e^{-i(\Omega t+\vartheta)} \\ e^{i(\Omega t+\vartheta)} & 0 \end{pmatrix}, \quad J^{(5)}_8 = \begin{pmatrix} 0 & -e^{-i(\Omega t+\vartheta)} \\ e^{i(\Omega t+\vartheta)} & 0 \end{pmatrix},
\]  
(3.42)
3.6 Model problem and the results

Since the jump matrices $J_i^5 (i = 1, 2, 3, 4, 5, 6)$ are all exponentially decaying to the identity away from the points $k_0, k_1, k_2, \chi$ and $\chi^*$ as $t \to \infty$, we can obtain a model problem to determine the leading term of the solution, given by

\[
\begin{align*}
\begin{cases}
m^{\text{mod}}(x, t, k) \text{ is analytic in } \mathbb{C} \setminus (\eta \cup \varpi_+ \cup (-\varpi_-)), \\
m^+(x, t, k) = m^{\text{mod}}(x, t, k) J^{\text{mod}}(x, t, k), \\
m^{\text{mod}}(x, t, k) \to e^{i(g(\infty)+G(\infty)t)} \sigma_3, \quad k \to \infty,
\end{cases}
\end{align*}
\]

(3.43)

where

\[
J^{\text{mod}}_\eta = J^5_\eta = \begin{pmatrix}
0 & \frac{iq^*}{q_0} \\
\frac{iq}{q_0} & 0
\end{pmatrix},
\]

\[
J^{\text{mod}}_{\varpi_+(\varpi_-)} = \begin{pmatrix}
0 & -e^{-i(\Omega t+\vartheta)} \\
e^{i(\Omega t+\vartheta)} & 0
\end{pmatrix},
\]

(3.44)

and $-\varpi_-$ means the negative direction of cut $\varpi_-$. For large $k$, introducing the factorization $m^5 = m^{\text{err}} m^{\text{mod}}$ and taking the Laurent series for matrices $m^{\text{err}}, m^{\text{mod}}$ as

\[
m^{\text{err}} = I + \frac{m^{\text{err}}_1(x, t)}{k} + \frac{m^{\text{err}}_2(x, t)}{k^2} + \mathcal{O}(\frac{1}{k^3}), \quad k \to \infty,
\]

\[
m^{\text{mod}} = e^{i(g(\infty)+G(\infty)t)} \sigma_3 + \frac{m^{\text{mod}}_1(x, t)}{k} + \frac{m^{\text{mod}}_2(x, t)}{k^2} + \mathcal{O}(\frac{1}{k^3}), \quad k \to \infty,
\]

(3.45)

we can represent the solution $q(x, t)$ for the Eq.(1.1) via the solution of model problem

\[
q(x, t) = 2i \left( m^{\text{mod}}_1(x, t) e^{i(g(\infty)+G(\infty)t)} + m^{\text{err}}_1(x, t) \right)_{12}.
\]

(3.46)

Similar to reference [43], we get $|m^{\text{err}}| = \mathcal{O}(t^{-\frac{3}{2}})$.

To solve the model RH problem (3.43), we first define the Abelian differential in what follows

\[
d\vartheta = \frac{\vartheta_0}{z(k)} dk, \quad \vartheta_0 = \left( \oint_{L_1} \frac{1}{z(k)} dk \right)^{-1},
\]

(3.47)

which is normalized at the case of $\oint_{L_1} d\vartheta = 1$. At the same time, the above Abelian differential (3.47) admits following Riemann period $\tau$

\[
\tau = \oint_{L_2} d\vartheta,
\]

(3.48)

which is purely imaginary when $i\tau < 0[49]$. Therefore, the theta function can be written into

\[
\Theta(k) = \sum_{\vartheta \in \mathbb{Z}} e^{2\pi i\vartheta k + \pi i\vartheta^2},
\]

(3.49)
which yields the properties

$$\Theta(k + n) = \Theta(k), \quad \Theta(k + n\tau) = e^{-(2\pi i k + \pi i \tau n^2)} \Theta(k), \quad n \in \mathbb{Z}. \quad (3.50)$$

According to the Abelian map

$$V(k) = \int_{i q_0}^{k} d\theta, \quad (3.51)$$

it arrives at

$$V_+(k) + V_-(k) = n - \tau, \quad n \in \mathbb{Z}, \quad k \in \varpi_+ \cup (-\varpi_-),$$

$$V_+(k) + V_-(k) = n, \quad n \in \mathbb{Z}, \quad k \in \eta. \quad (3.52)$$

Finally, a $2 \times 2$ matrix-valued function $M(k) = M(x, t, k)$ are constructed to solve the mod problem (3.43), whose elements are

$$M_{11}(k) = \frac{1}{2} [r(k) + r^{-1}(k)] \frac{\Theta \left( \frac{\Omega t + \vartheta + i \ln \left( \frac{i q^*}{q_0} \right)}{2\pi} + V(k) + C \right)}{\sqrt{\frac{2\pi}{i q_0}} \Theta \left( V(k) + C \right)},$$

$$M_{12}(k) = \frac{i}{2} [r(k) - r^{-1}(k)] \frac{\Theta \left( \frac{\Omega t + \vartheta + i \ln \left( \frac{i q^*}{q_0} \right)}{2\pi} - V(k) + C \right)}{\sqrt{\frac{2\pi}{i q_0}} \Theta \left( -V(k) + C \right)},$$

$$M_{21}(k) = \frac{-i}{2} [r(k) - r^{-1}(k)] \frac{\Theta \left( \frac{\Omega t + \vartheta + i \ln \left( \frac{i q^*}{q_0} \right)}{2\pi} + V(k) - C \right)}{\sqrt{\frac{2\pi}{i q_0}} \Theta \left( V(k) - C \right)},$$

$$M_{22}(k) = \frac{1}{2} [r(k) + r^{-1}(k)] \frac{\Theta \left( \frac{\Omega t + \vartheta + i \ln \left( \frac{i q^*}{q_0} \right)}{2\pi} - V(k) - C \right)}{\sqrt{\frac{2\pi}{i q_0}} \Theta \left( -V(k) - C \right)}; \quad (3.53)$$

of which the function $r(k)$ is

$$r(k) = \left( \frac{(k - \chi)(k - i q_0)}{(k - \chi^*)(k + iq_0)} \right)^{1/4}, \quad (3.54)$$

which has the identical jump discontinuity across $\eta$ and $\varpi_+ \cup (-\varpi_-)$, as well as $r_+(k) = \text{ir}_-(k)$, and it’s large-$k$ asymptotic is

$$r(k) = 1 - \frac{i(\chi_2 + q_0)}{2k} + O \left( \frac{1}{k^2} \right), \quad k \to \infty,$$

$$r(k) - r^{-1}(k) = -\frac{i(\chi_2 + q_0)}{k} + O \left( \frac{1}{k^2} \right), \quad k \to \infty. \quad (3.55)$$
Besides, we also have
\[ C = V(\hat{k}) + \frac{1}{2}(1 + \tau), \quad \hat{k} = \frac{q_0 \chi_1}{q_0 + \chi_2}. \]  
(3.56)

Then the model RH problem (3.43) is solved as
\[ m_{\text{mod}}(x, t, k) = e^{i\left(g(\infty)+G(\infty)t\right)\sigma_3}M^{-1}(\infty, C)M(k, C), \]  
(3.57)

further we obtain
\[ (m_{1\text{mod}})_{12} = \frac{q_0(\chi_2 + q_0)\Theta\left(\frac{\Omega t + \vartheta + i \ln\left(\frac{iq^*_0}{q_0}\right)}{2\pi} - V(\infty) + C\right)\Theta(V(\infty) + C)}{2iq^*_0\Theta\left(\frac{\Omega t + \vartheta + i \ln\left(\frac{iq^*_0}{q_0}\right)}{2\pi} + V(\infty) + C\right)\Theta(-V(\infty) + C)e^{-i\left(g(\infty)+G(\infty)t\right)}}. \]  
(3.58)

which implies the long-time asymptotics of solution \( q(x, t) \) for the fourth-order dispersive NLS equation (1.1) is
\[ q(x, t) = \frac{q_0}{q^*_0}(\chi_2 + q_0)\Theta\left(\frac{\Omega t + \vartheta + i \ln\left(\frac{iq^*_0}{q_0}\right)}{2\pi} - V(\infty) + C\right)\Theta(V(\infty) + C) \\
\frac{e^{2i\left(g(\infty)+G(\infty)t\right)} + O(t^{-\frac{1}{2}})}{\Theta\left(\frac{\Omega t + \vartheta + i \ln\left(\frac{iq^*_0}{q_0}\right)}{2\pi} + V(\infty) + C\right)\Theta(-V(\infty) + C)}. \]  
(3.59)

where \( V(\infty) = \int_{q_0}^{\infty} d\vartheta, \) and \( \chi_2, \Omega, G(\infty), \vartheta, g(\infty), C \) are presented in Eqs. (3.26), (3.32), (3.33), (3.37), (3.40), (3.56), respectively.

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