Using the Tilted flat-$\Lambda$CDM and the Untilted Non-flat $\Lambda$CDM Inflation Models to Measure Cosmological Parameters from a Compilation of Observational Data

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Abstract

We use the physically consistent tilted spatially flat and untitled non-flat $\Lambda$CDM inflation models to constrain cosmological parameter values with the Planck 2015 cosmic microwave background (CMB) anisotropy data and recent SNe Ia measurements, baryonic acoustic oscillations (BAO) data, growth rate observations, and Hubble parameter measurements. The most dramatic consequence of including the four non-CMB data sets is the significant strengthening of the evidence for non-flatness in the non-flat $\Lambda$CDM model, from 1.8$\sigma$ for the CMB data alone to 5.1$\sigma$ for the full data combination. The BAO data is the most powerful of the non-CMB data sets in more tightly constraining model-parameter values and in favoring a spatially closed universe in which spatial curvature contributes about a percent to the current cosmological energy budget. The untitled non-flat $\Lambda$CDM model better fits the large-angle CMB temperature anisotropy angular spectrum and is more consistent with the Dark Energy Survey constraints on the current value of the rms amplitude of mass fluctuations ($\sigma_8$) as a function of the current value of the nonrelativistic matter-density parameter ($\Omega_m$) but does not provide as good a fit to the smaller-angle CMB temperature anisotropy data, as does the tilted flat-$\Lambda$CDM model. Some measured cosmological parameter values differ significantly between the two models, including the reionization optical depth and the baryonic matter density parameter, both of whose 2$\sigma$ ranges (in the two models) are disjointed or almost so.

Key words: cosmic background radiation – cosmological parameters – cosmology: observations – inflation – large-scale structure of universe – methods: statistical

1. Introduction

In the standard spatially flat $\Lambda$CDM cosmology (Peebles 1984), the cosmological constant $\Lambda$ dominates the current energy budget, and cold dark matter (CDM) and baryonic matter are currently the second and third biggest contributors to the cosmological energy budget, followed by small contributions from neutrinos and photons. For reviews of this model, see Ratra & Vogeley (2008), Martin (2012), and Huterer & Shafer (2018). Many different observations are largely consistent with the standard picture, including cosmic microwave background (CMB) anisotropies data (Planck Collaboration et al. 2016), baryonic acoustic oscillations (BAO) distance measurements (Alam et al. 2017), Hubble parameter observations (Faroq et al. 2017), and Type Ia supernova (SN Ia) apparent magnitude data (Betoule et al. 2014). However, there still is room for mild dark-energy dynamics or a bit of spatial curvature, among other possibilities.

The standard model is characterized by six cosmological parameters that are conventionally taken to be: $\Omega_b h^2$ and $\Omega_c h^2$, the current values of the baryonic and cold dark matter density parameters multiplied by the square of the Hubble constant $H_0$ (in units of 100 km s$^{-1}$ Mpc$^{-1}$); $\theta_{MC}$, the angular diameter distance as a multiple of the sound horizon at recombination; $\tau$, the reionization optical depth; and $A_s$ and $n_s$, the amplitude and spectral index of the (assumed) power-law primordial scalar energy density inhomogeneity power spectrum (Planck Collaboration et al. 2016). The standard model assumes a flat spatial geometry (Planck Collaboration et al. 2016).

However, using a physically consistent non-flat inflation model power spectrum of energy density inhomogeneities (Ratra & Peebles 1995; Ratra 2017), Ooba et al. (2018a) recently found that Planck 2015 CMB anisotropy measurements (Planck Collaboration et al. 2016) do not require flat spatial geometry in the six-parameter non-flat $\Lambda$CDM model. In the non-flat $\Lambda$CDM model, compared to the flat-$\Lambda$CDM model, there is no simple tilt option, so $n_s$ is no longer a free parameter and is replaced by the current value of the spatial curvature density parameter $\Omega_k$.\textsuperscript{3}

In non-flat models non-zero spatial curvature sets the second, new length scale. This is in addition to the Hubble length scale. Inflation provides the only known way to define a physically consistent non-flat model power spectrum. For open spatial geometry the open-bubble inflation model of Gott (1982) is used to compute the non-power-law power spectrum (Ratra & Peebles 1994, 1995).\textsuperscript{4} For closed spatial geometry, Hawking’s prescription for the quantum state of the universe (Hawking 1984; Ratra 1985) can be used to construct a closed inflation model that can be used to compute the non-power-law power

\textsuperscript{3} The CMB anisotropy data also do not require flat spatial geometry in the seven-parameter non-flat XCDM inflation model (Ooba et al. 2018b; Park & Ratra 2019a). Here, the equation of state relating the pressure and energy density of the dark-energy fluid $p_X = w_X \rho_X$ and $\rho_X$ is the additional, seventh parameter. CDM is often used to model dynamical dark energy but is not a physically consistent model, as it cannot describe the evolution of energy density inhomogeneities. Also, XCDM does not accurately model $\Lambda$CDM (Peebles & Ratra 1988; Ratra & Peebles 1988) dark-energy dynamics (Podariu & Ratra 2001). In the simplest, physically consistent, seven-parameter, non-flat $\phi$CDM inflation model (Pavlov et al. 2013)—in which a scalar field $\phi$ with potential energy density $V(\phi) \propto e^{\alpha \phi}$ is the dynamical dark energy and $\alpha > 0$ is the seventh parameter that governs dark-energy evolution—Ooba et al. (2018c) again found that CMB anisotropy data do not require flat spatial hypersurfaces (also see Park & Ratra 2018). (In both the non-flat XCDM and $\phi$CDM cases, $n_s$ is again replaced by $\Omega_k$).

\textsuperscript{4} For early discussions of observational consequences of the open inflation model, see Kamionkowski et al. (1994) and Górski et al. (1995, 1998).
spectrum of energy density inhomogeneities (Ratra 2017). Both these open and closed inflation models are slow-roll inflation models (Gott 1982; Hawking 1984; Ratra 1985) so the resulting energy density inhomogeneity power spectra are untilted (Ratra & Peebles 1995; Ratra 2017).

Non-CMB observations, even combinations thereof to date, do not rule out non-flat dark-energy models (see, e.g., Farooq et al. 2015; Cai et al. 2016; Chen et al. 2016; Li et al. 2016; Yu & Wang 2016; Farooq et al. 2017; L’Huillier & Shafieloo 2017; Rana et al. 2017; Wei & Wu 2017; Mitra et al. 2018, 2019; Park & Ratra 2019a; Ryan et al. 2018, 2019; Yu et al. 2018). The most restrictive constraints on spatial curvature come from CMB anisotropy measurements, but, as shown by Ooba et al. (2018a), when the correct non-power-law power spectrum for energy density inhomogeneities is used for the CMB anisotropy analyses, a spatial curvature density parameter contributes magnitude a percent or two is still allowed, with the CMB anisotropy data (Planck Collaboration et al. 2016) favoring a mildly closed model. Ooba et al. (2018a) also added a few BAO distance measurements to the mix and found that a mildly closed model was still favored. Moreover, the mildly closed model better fits the observed low-$\ell$ CMB temperature anisotropy multipole number ($\ell$) power spectrum $C_\ell$ and was more consistent with rms fractional energy density inhomogeneity averaged over $8h^{-1}$ Mpc radius spheres, $\sigma_8$, and current values determined from weak-lensing observations, although the flat-$\Lambda$CDM model better fits the observed higher-$\ell$ $C_\ell$’s. In this paper we examine the constraints on the non-flat $\Lambda$CDM inflation model that result from a joint analysis of the Planck 2015 CMB anisotropy data (Planck Collaboration et al. 2016), the Joint Light-curve Analysis (JLA) SN Ia apparent magnitude measurements (Betoule et al. 2014), and all reliable BAO distance, growth factor, and Hubble parameter measurements to date. We also perform a similar analysis for the tilted flat-$\Lambda$CDM inflation model.

The main purposes of our analyses here are, first, to examine the effect that the inclusion of a significant amount of reliable, recent, non-CMB data has on the findings of Ooba et al. (2018a) that the Planck 2015 CMB anisotropy observations and a handful of BAO distance measurements are not inconsistent with the untilted closed-$\Lambda$CDM inflation model, and, second, to use this large new compilation of reliable non-CMB data to examine the consistency between the cosmological constraints of each type of data and to more tightly measure cosmological parameters than has been done to date.

Our main findings here are that our carefully gathered compilation of cosmological observations, the largest to date, does not require flat spatial hypersurfaces, with the untilted non-flat $\Lambda$CDM inflation model in which spatial curvature contributes about a percent to the current cosmological energy budget being more than $5\sigma$ away from flatness; the untilted non-flat model better fits the low-$\ell$ CMB temperature anisotropy $C_\ell$’s, as well as the weak lensing constraints in the $\sigma_8$-$\Omega_m$ plane, while the tilted flat-$\Lambda$CDM model is more consistent with the higher-$\ell$ $C_\ell$’s; $H_0$ is robustly measured in an almost model-independent manner and the value is consistent with most other measurements; and some measured cosmological parameter values, including those of $\Omega_m h^2$, $\tau$, and $\Omega_b h^2$, differ significantly between the two models, so care must be exercised when utilizing cosmological measurements of such parameters.

This paper is organized as follows. In Section 2 we describe the cosmological data sets we use in our analyses. In Section 3 we summarize the methods we use for our analyses here. The observational constraints resulting from these data for the tilted flat-$\Lambda$CDM and the non-flat $\Lambda$CDM inflation models are presented in Section 4. We summarize our results in Section 5.

2. Data

2.1. Planck 2015 CMB Anisotropy Data

We use the Planck 2015 TT + lowP and TT + lowP + lensing CMB anisotropy data (Planck Collaboration et al. 2016). Here, TT represents the low-$\ell$ ($2 \leq \ell \leq 29$) and high-$\ell$ ($30 \leq \ell \leq 2508$; PlikTT) Planck temperature-only $C_\ell^{TT}$ data and lowP denotes low-$\ell$ polarization $C_\ell^{TE}$, $C_\ell^{EE}$, and $C_\ell^{BB}$ power spectra measurements at $2 \leq \ell \leq 29$. The collection of low-$\ell$ temperature and polarization measurements is denoted as lowTEB. For CMB lensing data we use the power spectrum of the lensing potential measured by Planck.

2.2. JLA SN Ia Data

We use the JLA compilation of 740 SN Ia apparent magnitude measurements released by the SDSS-II and SNLS collaborations (Betoule et al. 2014). The JLA data set is composed of several low-redshift SN Ia ($z < 0.1$) and higher-redshift samples from the SDSS-II ($0.05 < z < 0.4$) and SNLS ($0.2 < z < 1$).

2.3. BAO Data

The anisotropy of BAO features in the line of sight and the transverse directions enable us to constrain both the Hubble parameter $H(z)$ and the comoving angular diameter distance

$$D_M(z) = (1 + z)D_A(z),$$

where $D_A$ is the physical angular diameter distance at redshift $z$. The radius of the sound horizon at the drag epoch $z_d$ is

$$r_d = \int_{z_d}^{\infty} \frac{c_s(z)}{H(z)} dz,$$

where $c_s(z)$ is the sound speed of the photon-baryon fluid. Because the size of the sound horizon $r_d$ depends on the cosmological model and the energy contents, the BAO features in the large-scale structure actually constrain $D_M(z)/r_d$ and $H(z)/r_d$.

We use the recent, more reliable BAO distance measurements from the 6dF Galaxy Survey (6dFGS; Beutler et al. 2011), the Sloan Digital Sky Survey (SDSS) Data Release 7 (DR7) main galaxy sample (MGS; Ross et al. 2015), the Baryon Oscillation Spectroscopic Survey (BOSS) DR12 galaxies (Alam et al. 2017), the eBOSS DR14 QSO’s (Ata et al. 2018), and the BOSS DR11 and DR12 Ly$\alpha$ forest (Font-Ribera et al. 2014; Bautista et al. 2017), 15 points in total, which are summarized in Table 1. We call this collection of BAO measurements “NewBAO” to distinguish it from the earlier BAO data compilation (which we call “BAO”) of 6dFGS (Beutler et al. 2011), BOSS LOWZ and CMASS.

5 For the BAO data point of Ata et al. (2018) we use the value presented in Ata et al. (2017). In the revised published version (Ata et al. 2018) they updated the data point to $D_A(r_{d, d}(z_d)) = 3843 \pm 147$ Mpc where $r_{d, d}$ is the value of $r_d$ for the fiducial model used in the analysis.
BOSS DR12

| Data Set       | LSS Tracers | $z_{eff}$ | Observable | Measurement | Reference |
|----------------|-------------|-----------|------------|-------------|-----------|
| BOSS DR12      | galaxies    | 0.38      | $D_M (r_{d, fid}/r_d)$ [Mpc] | 1518 ± 22 | Alam et al. (2017) |
|                |             | 0.51      | $D_M (r_{d, fid}/r_d)$ [Mpc] | 1977 ± 27 | Alam et al. (2017) |
|                |             | 0.61      | $D_M (r_{d, fid}/r_d)$ [Mpc] | 2283 ± 32 | Alam et al. (2017) |
|                |             | 0.38      | $H (r_{d, fid}/r_d)$ [km s$^{-1}$ Mpc$^{-1}$] | 81.5 ± 1.9 | Alam et al. (2017) |
|                |             | 0.51      | $H (r_{d, fid}/r_d)$ [km s$^{-1}$ Mpc$^{-1}$] | 90.4 ± 1.9 | Alam et al. (2017) |
|                |             | 0.61      | $H (r_{d, fid}/r_d)$ [km s$^{-1}$ Mpc$^{-1}$] | 97.3 ± 2.1 | Alam et al. (2017) |
| 6dF            | galaxies    | 0.106     | $r_d/D_V$  | 0.327 ± 0.015 | Beutler et al. (2011) |
| SDSS DR7 MGS   | galaxies    | 0.15      | $D_V (r_{d, fid}/r_d)$ [Mpc] | 664 ± 25 | Ross et al. (2015) |
| eBOSS DR14     | QSOs        | 1.52      | $D_V (r_{d, fid}/r_d)$ [Mpc] | 3855 ± 170 | Ata et al. (2018) |
| BOSS DR12 Lyα forest | Lyα | 2.33      | $D_H (0^3 M/3)/r_d$ | 13.94 ± 0.35 | Bautista et al. (2017) |
| BOSS DR11 Lyα forest | QSO and Lyα | 2.36      | $D_H (r_{dL})$ | 9.0 ± 0.3 | Font-Ribera et al. (2014) |
|                |             | 2.36      | $D_H (r_{dH})$ | 10.8 ± 0.4 | Font-Ribera et al. (2014) |

Note. The sound horizon size of the fiducial model is $r_{d, fid} = 147.78$ Mpc in Alam et al. (2017) and Ata et al. (2018), and $r_{d, fid} = 148.69$ Mpc in Ross et al. (2015).

(Anderson et al. 2014), and SDSS MGS (Ross et al. 2015) BAO distance measurements, used in the analyses of Planck Collaboration et al. (2016) and Ooba et al. (2018a, 2018b, 2018c).

For BAO data provided by Alam et al. (2017), we include the growth rate ($f_{rSD}$) data in our BAO (and not in our growth rate) analyses here, to be able to properly account for the correlations in the Alam et al. (2017) measurements. For the SDSS DR7 MGS (Ross et al. 2015) and BOSS DR11 Lyα forest (Font-Ribera et al. 2014) measurements, we use the probability distributions of the BAO data points, instead of using the Gaussian approximation constraints. Bautista et al. (2017) provide one BAO parameter $D_H (0^3 M/3)/r_d$ at $z = 2.33$ measured from BOSS DR12 Lyα forest observations, where $D_H$ is defined as

$$D_H (z) = c/H (z),$$

where $c$ is the speed of light. Font-Ribera et al. (2014) provide BAO parameters ($D_H/r_d$ and $D_A/r_d$) measured from the cross-correlation between QSO and Lyα forest data. They actually provide the probability distribution of parameters that describe shifts of the BAO peak position with respect to the fiducial cosmology in perpendicular and parallel directions to the line of sight,

$$\alpha_L = \frac{D_M (z) r_{d, fid}}{D_M^{fid} (z) r_d}, \quad \alpha_P = \frac{H^{fid} (z) r_{d, fid}}{H (z) r_d}.$$

The angle-averaged shift and the ratio of the two $\alpha$ parameters can be converted into the angle-averaged version of the distance scale,

$$D_V (z) = [czD_H^2 (z)/H (z)]^{1/3},$$

and the Alcock–Paczynski parameter,

$$F_{AP} (z) = D_M (z) H (z)/c.$$  

For the BAO data of Alam et al. (2017), instead of using $D_M (r_{d, fid}/r_d)$ and $H (r_{d}/r_{d, fid})$, we actually transform these into $D_V (r_{d}/r_d)$ and $F_{AP}$ and also use their growth rate $f_{rSD}$ measurements and account for correlations (data publicly available at the BOSS website).

2.4. Hubble Parameter Data

Hubble parameter measurements can be used to constrain dark-energy parameters, as well as other cosmological parameters, including the spatial curvature of the universe (see, e.g., Farooq et al. 2017). Here we adapt and use a recent Hubble parameter measurement compilation to constrain both the tilted flat-$\Lambda$CDM inflation model and the non-flat $\Lambda$CDM inflation model. Table 2 lists all more reliable recent measurements of the Hubble parameter at various redshifts (with 31 data points in total). See Farooq et al. (2017) and Yu et al. (2018) for discussions about how these data were selected.

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6 Early developments include Samushia & Ratra (2006), Samushia et al. (2007), and Chen & Ratra (2011b); recent work includes Tripathi et al. (2017), Lonappan et al. (2017), Rezaei et al. (2017), Magana et al. (2018), Anagnostopoulos & Basilakos (2018), Yu et al. (2018a), and Cao et al. (2018). We note that there are many different $H(z)$ compilations discussed in the literature. Unfortunately, a significant fraction of these include non-independent or unreliable measurements.

7 Table 2 does not list radial or line-of-sight BAO $H(z)$ measurements; these are instead listed in Table 1.

8 The redshift range over which the Hubble parameter has been measured encompasses the redshift of the cosmological deceleration-acceleration transition in the standard cosmological model. This transition is between the earlier nonrelativistic-matter-powered decelerating cosmological expansion and the more recent dark-energy-driven accelerating cosmological expansion. This transition redshift has recently been measured and is at roughly the value expected in the standard $\Lambda$CDM and other dark-energy models (Farooq & Ratra 2013; Moresco et al. 2016; Farooq et al. 2017).
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Table 2

| z   | H(z) (km s⁻¹ Mpc⁻¹) | σₘ (km s⁻¹ Mpc⁻¹) | Reference |
|-----|----------------------|-------------------|-----------|
| 0.070 | 69                   | 19.6              | Zhang et al. (2014) |
| 0.090 | 69                   | 12                | Simon et al. (2005) |
| 0.120 | 68.6                 | 26.2              | Zhang et al. (2014) |
| 0.170 | 83                   | 8                 | Simon et al. (2005) |
| 0.179 | 75                   | 4                 | Moresco et al. (2012) |
| 0.199 | 75                   | 5                 | Moresco et al. (2012) |
| 0.200 | 72.9                 | 29.6              | Zhang et al. (2014) |
| 0.270 | 77                   | 14                | Simon et al. (2005) |
| 0.280 | 88.8                 | 36.6              | Zhang et al. (2014) |
| 0.352 | 83                   | 14                | Moresco et al. (2012) |
| 0.3802| 83                   | 13.5              | Moresco et al. (2016) |
| 0.400 | 95                   | 17                | Simon et al. (2005) |
| 0.4004| 77                   | 10.2              | Moresco et al. (2016) |
| 0.4247| 87.1                 | 11.2              | Moresco et al. (2016) |
| 0.4497| 92.8                 | 12.9              | Moresco et al. (2016) |
| 0.47 | 89                   | 90                | Ratsimbazafy et al. (2017) |
| 0.4783| 80.9                 | 9                 | Moresco et al. (2016) |
| 0.480 | 97                   | 62                | Stern et al. (2010) |
| 0.593 | 104                  | 13                | Moresco et al. (2012) |
| 0.680 | 92                   | 8                 | Moresco et al. (2012) |
| 0.781 | 105                  | 12                | Moresco et al. (2012) |
| 0.875 | 125                  | 17                | Moresco et al. (2012) |
| 0.880 | 90                   | 40                | Stern et al. (2010) |
| 0.900 | 117                  | 23                | Simon et al. (2005) |
| 1.037 | 154                  | 20                | Moresco et al. (2012) |
| 1.300 | 168                  | 17                | Simon et al. (2005) |
| 1.363 | 160                  | 33.6              | Moresco (2015) |
| 1.430 | 177                  | 18                | Simon et al. (2005) |
| 1.530 | 140                  | 14                | Simon et al. (2005) |
| 1.750 | 202                  | 40                | Simon et al. (2005) |
| 1.965 | 186.5                | 50.4              | Moresco (2015) |

where the subscript 0 indicates the present epoch. In the following we denote the present value σₘ₀ as σₘ for simplicity.

Table 3 lists all more reliable recent measurements of growth rate f(z)σₘ(z) at various redshifts, 10 points in total. As already noted, the three growth rate data points of Alam et al. (2017) are included in the collection of BAO data points in order to properly account for correlations between these BAO and growth rate data points.

3. Methods

3.1. Model Computations

We use the publicly available CAMB/COSMOMC package (version of 2016 November; Challinor & Lasenby 1999; Lewis et al. 2000; Lewis & Bridle 2002) to constrain the tilted flat and the non-flat ΛCDM inflation models with Planck 2015 CMB and other non-CMB data sets. The Boltzmann code CAMB computes the CMB angular power spectra for temperature fluctuations, polarization, and lensing potential, and COSMOMC applies the Markov chain Monte Carlo (MCMC) method to explore and determine model-parameter space that is favored by the data used. We use the COSMOMC settings adopted in the Planck team’s analysis (Planck Collaboration et al. 2016). We set the present CMB temperature to T₀ = 2.7255 K (Fixsen 2009) and the effective number of neutrino species to N_eff = 3.046. We assume the existence of a single species of massive neutrinos with mass mν = 0.06 eV. The primordial helium fraction Y₁(0) is set from the big bang nucleosynthesis prediction. In the parameter estimation the lensed CMB power spectra for each model are compared with observations. When the Planck lensing data are included in the analysis, we also need to consider the nonlinear lensing effect that is important in the lensing potential reconstruction (Planck Collaboration et al. 2014). As needed, we turn on the options for CMB lensing and nonlinear lensing in every case, regardless of whether the Planck lensing data are used or not.

The primordial power spectrum in the spatially flat tilted ΛCDM inflation model (Luminchi & Matarrese 1985; Ratra 1992, 1989) is

\[ P(k) = A_s \left( \frac{k}{k_0} \right)^{n_s}, \]

where k is wavenumber and A_s is the amplitude at the pivot scale k₀ = 0.05 Mpc⁻¹. On the other hand, the primordial power spectrum in the non-flat ΛCDM inflation model (Ratra & Peebles 1995; Ratra 2017) is

\[ P(q) \propto \frac{(q^2 - 4K)^2}{q(q^2 - K)}, \]

which goes over to the n_s = 1 spectrum in the spatially flat limit (K = 0). For scalar perturbations, q = √k² + K is the wave-number where K = -(H₀^2/c^2)Ω_k is the spatial curvature. For the spatially closed model, with negative Ω_k, the normal modes are characterized by the positive integers ν = qK⁻¹/₂ = 3, 4, 5, ..., and the eigenvalue of the spatial Laplacian is -(q² - K)/K ≈ -K²/K. We use P(q) as the initial power
spectrum of perturbations for the non-flat model by normalizing its amplitude at the pivot scale $k_0$ to the value of $A_s$.

The Planck 2015 non-flat model analyses (Planck Collaboration et al. 2016) are not based on either of the above power spectra; instead they assume

$$P_{\text{Planck}}(q) \propto \frac{q^2 - 4K^2}{q(q^2 - K)} \left( \frac{k}{k_0} \right)^{n_s - 1}$$

where in addition to the non-flat space wavenumber $q$, the wavenumber $k$ is also used to define and tilt the non-flat model $P_{\text{Planck}}(q)$. The $k^{n_s - 1}$ tilt factor in $P_{\text{Planck}}(q)$ assumes that tilt in non-flat space works somewhat as it does in flat space, which seems unlikely, as spatial curvature sets an additional length scale in non-flat space (i.e., in addition to the Hubble length). It is not known if the power spectrum of Equation (11) can be the consequence of quantum fluctuations during an early epoch of inflation. This power spectrum is physically sensible if $K = 0$ or if $n_s = 1$, when it reduces to the power spectra in Equations (9) and (10), both of which are consequences of quantum fluctuations during inflation.9

### 3.2. Constraining Model Parameters

We explore the parameter space of the tilted flat-$\Lambda$CDM model with six cosmological parameters ($\Omega_m h^2$, $\Omega_b h^2$, $\sigma_8$, $\Lambda$, $A_s$, and $n_s$).
Figure 2. Likelihood distributions of the tilted flat-$\Lambda$CDM model parameters constrained by Planck TT + lowP, JLA SN Ia, NewBAO, $H(z)$, and $f_{rS}$ data. Two-dimensional marginalized likelihood distributions of all possible combinations of model parameters together with one-dimensional likelihoods are shown for cases when each non-CMB data set is added to the Planck TT + lowP data (left panel) and when the growth rate, JLA SN Ia, Hubble parameter data, and the combination of them, are added to TT + lowP + NewBAO data (right panel). For ease of viewing, the cases of TT + lowP (left) and TT + lowP + NewBAO (right) panel are shown as solid black curves.

Table 4
Mean and 68.3% Confidence Limits of Tilted Flat and Untilted Non-flat $\Lambda$CDM Model Parameters

| Parameter | Tilted flat-$\Lambda$CDM model | Tilted flat-$\Lambda$CDM model | Tilted flat-$\Lambda$CDM model | Tilted flat-$\Lambda$CDM model |
|-----------|--------------------------------|--------------------------------|--------------------------------|--------------------------------|
| $\Omega_{C}^{2}$ | 0.02227 ± 0.00020 | 0.02225 ± 0.00020 | 0.02229 ± 0.00020 | 0.02227 ± 0.00020 |
| $\Omega_{b}^{2}$ | 0.1190 ± 0.0013 | 0.1185 ± 0.0012 | 0.1187 ± 0.0012 | 0.1183 ± 0.0012 |
| $100\theta_{MC}$ | 1.04095 ± 0.00042 | 1.04103 ± 0.00041 | 1.04099 ± 0.00041 | 1.04105 ± 0.00040 |
| $\tau$ | 0.080 ± 0.018 | 0.067 ± 0.013 | 0.079 ± 0.017 | 0.066 ± 0.013 |
| $\ln(10^{10}A_{s})$ | 3.092 ± 0.035 | 3.065 ± 0.024 | 3.088 ± 0.034 | 3.064 ± 0.024 |
| $n_{s}$ | 0.9673 ± 0.0044 | 0.9674 ± 0.0044 | 0.9678 ± 0.0044 | 0.9682 ± 0.0044 |
| $H_{0}$ [km s$^{-1}$ Mpc$^{-1}$] | 67.65 ± 0.57 | 67.81 ± 0.54 | 67.78 ± 0.55 | 67.92 ± 0.54 |
| $\Omega_{m}$ | 0.3102 ± 0.0076 | 0.3077 ± 0.0072 | 0.3083 ± 0.0074 | 0.3063 ± 0.0071 |
| $\sigma_{8}$ | 0.829 ± 0.014 | 0.8158 ± 0.0089 | 0.826 ± 0.014 | 0.8150 ± 0.0089 |

| Parameter | Untilted non-flat $\Lambda$CDM model | Untilted non-flat $\Lambda$CDM model | Untilted non-flat $\Lambda$CDM model | Untilted non-flat $\Lambda$CDM model |
|-----------|--------------------------------|--------------------------------|--------------------------------|--------------------------------|
| $\Omega_{C}^{2}$ | 0.02307 ± 0.00020 | 0.02304 ± 0.00020 | 0.02307 ± 0.00020 | 0.02303 ± 0.00020 |
| $\Omega_{b}^{2}$ | 0.1096 ± 0.0011 | 0.1093 ± 0.0011 | 0.1096 ± 0.0011 | 0.1093 ± 0.0010 |
| $100\theta_{MC}$ | 1.04222 ± 0.00042 | 1.04232 ± 0.00041 | 1.04223 ± 0.00042 | 1.04230 ± 0.00042 |
| $\tau$ | 0.135 ± 0.018 | 0.115 ± 0.011 | 0.132 ± 0.017 | 0.115 ± 0.011 |
| $\ln(10^{10}A_{s})$ | 3.179 ± 0.036 | 3.138 ± 0.022 | 3.174 ± 0.034 | 3.139 ± 0.022 |
| $\Omega_{k}$ | −0.0093 ± 0.0019 | −0.0093 ± 0.0018 | −0.0088 ± 0.0017 | −0.0087 ± 0.0017 |
| $H_{0}$ [km s$^{-1}$ Mpc$^{-1}$] | 67.46 ± 0.72 | 67.56 ± 0.67 | 67.69 ± 0.66 | 67.81 ± 0.66 |
| $\Omega_{m}$ | 0.2931 ± 0.0064 | 0.2914 ± 0.0059 | 0.2910 ± 0.0059 | 0.2893 ± 0.0058 |
| $\sigma_{8}$ | 0.832 ± 0.016 | 0.814 ± 0.010 | 0.830 ± 0.015 | 0.8148 ± 0.0097 |

Note. BAO versus NewBAO.

and the untitled non-flat $\Lambda$CDM model with six parameters ($\Omega_{C}^{2}$, $\Omega_{b}^{2}$, $\Omega_{k}$, $\theta_{MC}$, $\tau$, and $A_{s}$), $\theta_{MC}$ is the approximate angular size of the sound horizon ($r_{s}/D_{A}$) at redshift $z_{s}$ for which the optical depth equals unity (Planck Collaboration et al. 2014). Unresolved extragalactic foregrounds due to point sources, cosmic infrared background, and thermal and kinetic Sunyaev–Zeldovich
components contribute to the temperature power spectrum. Thus, the foreground model parameters are also constrained as nuisance parameters by the MCMC method. We also compute three derived parameters, $H_0$, $\Omega_m$, and $\sigma_8$.

For each model (and set of six-parameter values), we compare the lensed CMB power spectra obtained from the CAMB Boltzmann code with the Planck 2015 TT + lowP data and TT + lowP + lensing data, excluding and including the power spectrum of the lensing potential, respectively. For BAO, SN Ia, and Hubble parameter data, the prediction determined from the spatially homogeneous background evolution equations solution for each set of model parameters...
Figure 5. Same as Figure 4 but now including the Planck CMB lensing data.

| Parameter | TT+lowP | TT+lowP+JLA | TT+lowP+NewBAO |
|-----------|---------|-------------|----------------|
| $\Omega_k h^2$ | 0.02222 ± 0.00023 | 0.02226 ± 0.00022 | 0.02229 ± 0.00020 |
| $\Omega_k h^2$ | 0.1197 ± 0.0022 | 0.1193 ± 0.0020 | 0.1187 ± 0.0012 |
| $100\theta_{MC}$ | 1.04086 ± 0.00048 | 1.04092 ± 0.00047 | 1.04099 ± 0.00041 |
| $\tau$ | 0.078 ± 0.019 | 0.080 ± 0.019 | 0.079 ± 0.017 |
| ln(10^10 A_s) | 3.089 ± 0.037 | 3.092 ± 0.035 | 3.088 ± 0.034 |
| $n_s$ | 0.9655 ± 0.0062 | 0.9666 ± 0.0057 | 0.9678 ± 0.0044 |
| $H_0$ [km s^{-1} Mpc^{-1}] | 67.32 ± 0.99 | 67.52 ± 0.89 | 67.78 ± 0.55 |
| $\Omega_m$ | 0.315 ± 0.014 | 0.312 ± 0.012 | 0.308 ± 0.0074 |
| $\sigma_8$ | 0.829 ± 0.015 | 0.826 ± 0.014 | 0.826 ± 0.014 |

| Parameter | TT+lowP+$H(z)$ | TT+lowP+JLA+NewBAO | TT+lowP+NewBAO+$H(z)$ |
|-----------|----------------|---------------------|------------------------|
| $\Omega_k h^2$ | 0.02225 ± 0.00022 | 0.02230 ± 0.00019 | 0.02231 ± 0.00020 |
| $\Omega_k h^2$ | 0.1195 ± 0.0021 | 0.1186 ± 0.0012 | 0.1185 ± 0.0012 |
| $100\theta_{MC}$ | 1.04091 ± 0.00047 | 1.04101 ± 0.00041 | 1.04103 ± 0.00041 |
| $\tau$ | 0.079 ± 0.019 | 0.078 ± 0.017 | 0.079 ± 0.017 |
| ln(10^10 A_s) | 3.091 ± 0.036 | 3.088 ± 0.034 | 3.089 ± 0.034 |
| $n_s$ | 0.9661 ± 0.0060 | 0.9679 ± 0.0044 | 0.9682 ± 0.0042 |
| $H_0$ [km s^{-1} Mpc^{-1}] | 67.44 ± 0.94 | 67.82 ± 0.55 | 67.85 ± 0.52 |
| $\Omega_m$ | 0.313 ± 0.013 | 0.3078 ± 0.0072 | 0.3074 ± 0.0068 |
| $\sigma_8$ | 0.829 ± 0.014 | 0.826 ± 0.014 | 0.826 ± 0.014 |

| Parameter | TT+lowP+$f_{s8}$ | TT+lowP+NewBAO+$f_{s8}$ | TT+lowP+JLA+NewBAO+$H(z)$+$f_{s8}$ |
|-----------|----------------|-------------------------|----------------------------------|
| $\Omega_k h^2$ | 0.02234 ± 0.00023 | 0.02231 ± 0.00020 | 0.02232 ± 0.00020 |
| $\Omega_k h^2$ | 0.1174 ± 0.0020 | 0.1179 ± 0.0012 | 0.1178 ± 0.0011 |
| $100\theta_{MC}$ | 1.04110 ± 0.00046 | 1.04104 ± 0.00041 | 1.04104 ± 0.00042 |
| $\tau$ | 0.074 ± 0.020 | 0.069 ± 0.016 | 0.069 ± 0.017 |
| ln(10^10 A_s) | 3.076 ± 0.037 | 3.068 ± 0.032 | 3.068 ± 0.033 |
| $n_s$ | 0.9702 ± 0.0061 | 0.9689 ± 0.0042 | 0.9690 ± 0.0043 |
| $H_0$ [km s^{-1} Mpc^{-1}] | 68.31 ± 0.94 | 68.09 ± 0.54 | 68.12 ± 0.52 |
| $\Omega_m$ | 0.301 ± 0.012 | 0.3040 ± 0.0069 | 0.3035 ± 0.0067 |
| $\sigma_8$ | 0.817 ± 0.014 | 0.815 ± 0.013 | 0.815 ± 0.013 |
is compared with the observations. For growth rate data, the matter density parameter \(\sigma_8\) is computed using the CAMB code at the needled redshifts.

We set priors for some parameters. The Hubble constant is restricted to the range \(20 < H_0 < 100\), in units of \(\text{km s}^{-1}\text{Mpc}^{-1}\). The reionization optical depth is explored only in the range \(\tau > 0.005\). The other basic parameters have flat priors that are sufficiently wide such that the final constraints are within the prior ranges. For every model considered here, sufficient MCMC chains are generated in order that the Gelman and Rubin \(R\) statistics satisfy the condition \(R \lesssim 0.01\).

### 4. Observational Constraints

We constrain the spatially flat tilted and the untiilted non-flat \(\Lambda\)CDM inflation models using the Planck 2015 TT + lowP (excluding the CMB lensing) data and other non-CMB data sets.

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**Note**: For parameter estimation using the JLA SN Ia data set we need to consider hidden nuisance parameters, \(\alpha_{JLA}\) and \(\beta_{JLA}\), related to the stretch and color correction of the SN Ia light curves, \(B\)-band absolute magnitude \(M_B\), and the offset of the absolute magnitude due to the environment (host stellar mass) \(\Delta_M\). Thus, the number of degrees of freedom for the JLA data is less than the total number of SN Ia data \((N = 740)\). For example, for the flat-\(\Lambda\)CDM model that fits the matter density parameter \(\Omega_m\), \(\Omega_{\Lambda M}\), \(\beta_{JLA}\), \(M_B\), and \(\Delta_M\), the number of degrees of freedom becomes 735 \((=740-5)\). In our analysis, we assume flat priors for these parameters \((0.01 \leq \alpha_{JLA} \leq 2\) and \(0.9 \leq \beta_{JLA} \leq 4.6)\) during parameter estimation.

We first examine how efficient the new BAO data are in constraining parameters, relative to the old BAO data. Figure 1 compares the likelihood distributions of the model parameters for the old (“BAO”) and new BAO (“NewBAO”) data sets, in conjunction with the CMB observations. The mean and 68.3% confidence limits of model parameters are presented in Table 4. We see that adding CMB lensing data results in a reduction of \(\ln (10^{10} A_s)\) and \(\tau\) in both models and that the NewBAO data improve parameter estimation with slightly narrower parameter constraints (moreso for the cases in which the lensing data are excluded).

The entries in the TT + lowP + BAO and TT + lowP + lensing + BAO columns for the non-flat \(\Lambda\)CDM model in Table 4 agree well with the corresponding entries in Table 2 of Ooba et al. (2018a). Ooba et al. (2018a) used CLASS (Blas et al. 2011) to compute the \(C_i\)'s and Monte Python (Audren et al. 2013) for the MCMC analyses, so it is gratifying and reassuring that our results agree well with those of Ooba et al. (2018a).11

We investigate the effect of including non-CMB data sets, with the Planck 2015 CMB data, on the parameter constraints of the tilted flat and the untiilted non-flat \(\Lambda\)CDM models. The results are presented in Figures 2–5 and Tables 5–8. In the

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11 Note that Ooba et al. (2018a) use CLASS \(\theta\), which is defined as the ratio of comoving sound horizon to the angular diameter distance at decoupling, while here we use \(\text{CMB} \theta_{MPC}\), which is an approximate version of \(\theta\).
triangle plots we omit the likelihood contours for TT + lowP (+ lensing) + JLA + NewBAO data (excluding or including the Planck lensing data) in both the tilted flat and the untitled non-flat ΛCDM models because they are very similar to those for TT + lowP (+ lensing) + NewBAO data.

The entries in the CMB-only TT + lowP column of Table 5 and those in the TT + lowP + lensing column of Table 6 for the tilted flat-ΛCDM model agree well with the corresponding entries in Table 4 of Planck Collaboration et al. (2016). Similarly, the entries in the TT + lowP column of Table 7 and those in the TT + lowP + lensing column of Table 8 for the non-flat ΛCDM model agree well with the corresponding entries in Table 1 of Ooba et al. (2018a).

From Tables 5 and 6 we see that, when added to the Planck 2015 CMB anisotropy data, for the tilted flat-ΛCDM model, the NewBAO measurements prove more restrictive than either the H(z), fσ8, or SN Ia observations. We note, however, that our NewBAO compilation includes radial BAO H(z) measurements as well as the fσ8 measurements of Alam et al. (2017). It is likely that even if these are moved to the H(z) and fσ8 data sets, BAO constraints will still be the most restrictive, for the tilted flat-ΛCDM model, but probably closely followed by H(z) and fσ8 constraints, with SN Ia being the least effective.

The situation in the untitled non-flat ΛCDM case is more interesting. When CMB lensing data are excluded, Table 7, adding NewBAO, or JLA SN Ia, or H(z), or fσ8 data to the CMB data results in roughly similarly restrictive constraints on Ω_ch^2, Ω_bh^2, θMC, τ, ln (10^10A_s), and σ_8, while CMB + NewBAO data provide the tightest constraints on Ω_bh, H_0, and Ω_m. When the CMB lensing data are included (Table 8), CMB data with either JLA SN Ia, or NewBAO, or H(z), or fσ8 data, provide roughly similarly restrictive constraints on Ω_bh^2, Ω_ch^2, and θMC, while CMB + NewBAO data provide the tightest constraints on τ, ln (10^10A_s), Ω_bh, H_0, and σ_8.

If we focus on CMB TT + lowP + lensing data, Figures 3 and 5 and Tables 6 and 8, we see that adding only one of the four non-CMB data sets at a time to the CMB measurements (left triangle plots in the two figures) results in four sets of contours that are quite consistent with each other, as well as with the original CMB alone contours, for both the tilted flat-ΛCDM case and for the untitled non-flat ΛCDM model. The same holds true for the tilted flat-ΛCDM model when the CMB lensing data are excluded (left triangle plot of Figure 2). However, in the untitled non-flat ΛCDM case without the lensing data when any of the four non-CMB data sets are added to the CMB data (left triangle plot of Figure 4), they each pull the results toward a smaller [Ω_bh] (closer to the flat model) and slightly larger τ and ln(10^10A_s) and smaller Ω_bh^2 than is favored by the CMB data alone, although all five sets of constraint contours are largely mutually consistent. It is reassuring that the four non-CMB data sets do not pull the CMB constraints in significantly different directions.
As noted above, adding the NewBAO data to the CMB data typically makes the biggest difference, but the other three non-CMB data sets also contribute. Focusing on the TT + lowP + lensing data, we see from Table 6 for the tilted flat-$\Lambda$CDM case that the NewBAO data tightly constrains model parameters, particularly $\Omega_ch^2$, while the growth rate ($f_{\sigma8}$) data shifts $\Omega_bh^2$ and $n_s$ to larger values and $\Omega_ch^2$ to a smaller value. In this case $\Omega_m$ is the quantity whose error bar is reduced the most by the full combination of data relative to the CMB and NewBAO compilation, followed by the $H_0$ error bar reduction. For the untitled non-flat $\Lambda$CDM model, from Table 8, $\Omega_h^2$ and $\tau$ error bars from the CMB and NewBAO data are not reduced by including the $H(z)$, $f_{\sigma8}$, and JLA SN Ia measurements in the mix. In all cases, adding JLA SN Ia or growth rate ($f_{\sigma8}$) data to the combination of CMB + NewBAO data does not much improve the observational constraints.12

Again concentrating on the TT + lowP + lensing data, Tables 6 and 8, we see that for the tilted flat-$\Lambda$CDM model, adding the four non-CMB data sets to the mix most affects $\Omega_ch^2$ and $\Omega_m$, with both central values moving down about 0.5$\sigma$ of the CMB data alone error bars. The situation in the untitled non-flat $\Lambda$CDM case is a little more dramatic, with $\Omega_k$ moving closer to flatness by about $1\sigma$, $H_0$ and $\Omega_m$ also moving by about $1\sigma$, and the $\sigma_8$, $\ln(10^{10}A_s)$, and $\tau$ central values moving by about $0.5\sigma$.

Perhaps the biggest consequence of including the four non-CMB data sets in the analyses is the significant strengthening of the evidence for non-flatness in the untitled non-flat $\Lambda$CDM case, with it increasing from 1.8$\sigma$ away from flatness for the CMB alone case, to 5.1$\sigma$ away from flatness for the full data combination in Table 8,13 where the NewBAO data plays the most important role among the four non-CMB data sets. This is consistent with, but stronger than, the Ooba et al. (2018a) results. The same situation is also seen when the lensing data are excluded, as shown in Table 7. We also note that combining CMB data with either JLA SN Ia, $H(z)$, or growth rate data do not strongly support non-flatness. When combined with CMB data with lensing, SN Ia, $H(z)$, and $f_{\sigma8}$ data result in $\Omega_k$ being 2.1$\sigma$, 1.8$\sigma$, and 1.2$\sigma$ away from flatness, while CMB and NewBAO data favor $\Omega_k$ being 5.1$\sigma$ away from flatness (Table 8). In the untitled non-flat $\Lambda$CDM case, the effect of growth rate data on the model constraints differs from that of the NewBAO data. The results for the untitled non-flat $\Lambda$CDM model from TT + lowP + $f_{\sigma8}$ observations excluding

12 We did not check what happens when just $H(z)$ data are added to the CMB and NewBAO combination, but suspect a similar conclusion holds for this case also.

13 It is possible to assume that all one-dimensional likelihoods are close to Gaussian, except for $\Omega_k$ estimated using the TT + lowP, TT + lowP + $H(z)$, and TT + lowP + $f_{\sigma8}$ data.

Table 8

| Parameter | TT+lowP+lensing+H(z) | TT+lowP+lensing+NewBAO | TT+lowP+lensing+NewBAO+H(z) |
|-----------|----------------------|-------------------------|-----------------------------|
| $\Omega_k h^2$ | $0.02305 \pm 0.00020$ | $0.02304 \pm 0.00020$ | $0.02303 \pm 0.00019$ |
| $\Omega_m$ | $0.316 \pm 0.033$ | $0.306 \pm 0.023$ | $0.2893 \pm 0.0057$ |
| $\sigma_8$ | $0.799 \pm 0.021$ | $0.805 \pm 0.017$ | $0.8148 \pm 0.0097$ |
| $H_0$ [km s$^{-1}$ Mpc$^{-1}$] | $65.1 \pm 3.3$ | $66.0 \pm 2.5$ | $67.81 \pm 0.66$ |
| $\Omega_m$ | $0.316 \pm 0.033$ | $0.306 \pm 0.023$ | $0.2893 \pm 0.0057$ |
| $\sigma_8$ | $0.799 \pm 0.021$ | $0.805 \pm 0.017$ | $0.8148 \pm 0.0097$ |
| $H_0$ [km s$^{-1}$ Mpc$^{-1}$] | $68.4 \pm 1.9$ | $67.82 \pm 0.66$ | $69.73 \pm 0.63$ |
| $\Omega_m$ | $0.285 \pm 0.016$ | $0.2893 \pm 0.0058$ | $0.2885 \pm 0.0055$ |
| $\sigma_8$ | $0.818 \pm 0.014$ | $0.8148 \pm 0.0098$ | $0.8156 \pm 0.0098$ |
| $H_0$ [km s$^{-1}$ Mpc$^{-1}$] | $68.7 \pm 3.0$ | $67.93 \pm 0.67$ | $68.07 \pm 0.63$ |
| $\Omega_m$ | $0.283 \pm 0.024$ | $0.2877 \pm 0.0058$ | $0.2868 \pm 0.0055$ |
| $\sigma_8$ | $0.815 \pm 0.019$ | $0.8111 \pm 0.0098$ | $0.8124 \pm 0.0095$ |
including the lensing data show that the growth rate measurements favor $\Omega_k$ moving closer to spatial flatness with a deviation of only 1.4$\sigma$ (1.2$\sigma$) from zero spatial curvature.

Adding $f\sigma_8$ data to TT + lowP + lensing measurements—that favor the closed model by 5.2$\sigma$ (5.1$\sigma$)—gives a negative $\Omega_k$ deviating from flatness by 5.1$\sigma$ (5.1$\sigma$). Thus, the negativeness of the curvature parameter persists for the combination of BAO and growth rate data, which also implies that the BAO data most tightly constrains the curvature parameter compared to the other non-CMB data.

For the full data combination, $H_0$ measured in the two models (with lensing data) in Tables 6 and 8, 68.17 ± 0.50 and 68.07 ± 0.63 km s$^{-1}$ Mpc$^{-1}$, are very consistent with each other, agreeing to within 0.12$\sigma$ (of the quadrature sum of the two error bars). These values are consistent with the most recent median statistics estimate $H_0 = 68 \pm 2.8$ km s$^{-1}$ Mpc$^{-1}$ (Chen & Ratra 2011a), which is consistent with earlier median statistics estimates (Gott et al. 2001; Chen et al. 2003). Many recent estimates of $H_0$ are also quite consistent with these measurements (Calabrese et al. 2012; Hinshaw et al. 2013; Sievers et al. 2013; Aubourg et al. 2015; Luković et al. 2016; Planck Collaboration et al. 2016; Chen et al. 2017; DES Collaboration et al. 2018a; Planck Collaboration et al. 2018b; L’Huillier & Shaﬁeloo 2017; Lin & Ishak 2017; Wang et al. 2017; Haridasu et al. 2018; Yu et al. 2018), but, as is well known, they are lower than the local measurement of $H_0 = 73.06 \pm 1.74$ km s$^{-1}$ Mpc$^{-1}$ (Anderson & Riess 2018).

In our analyses here, $H_0$ and $\sigma_8$ (discussed below) are the only cosmological parameters that are determined in a cosmological model (spatial curvature and tilt) independent manner. For instance, $\Omega_m$ determined using the tilted flat-$\Lambda$CDM model differs from that measured in the untilted non-

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Figure 6. Likelihood distributions in the $\Omega_m - \sigma_8$ plane for the tilted flat-$\Lambda$CDM model constrained by Planck CMB TT + lowP (+lensing), JLA SN Ia, NewBAO, $H(z)$, and $f\sigma_8$ data. In each panel the $\Lambda$CDM constraints (68.3% and 95.4% confidence limits) obtained from the first-year Dark Energy Survey (DES Y1 All; DES Collaboration et al. 2018a) are shown as thick solid curves for comparison.

14 Potential systematic errors, ignored here, have been discussed by Addison et al. (2016) and Planck Collaboration et al. (2017).
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A compilation of measured primordial deuterium abundances mildly favors the flat case (Penton et al. 2018).
distributions in the $\Omega_m$–$\sigma_8$ plane obtained by adding each non-CMB data set to the Planck CMB data are consistent with each other. As expected, the NewBAO data or the NewBAO data combined with other non-CMB data sets give tighter constraints in all cases. As shown in Figures 6 and 7, there is tension between both $\Lambda$CDM models constrained by Planck TT + lowP data (dotted and dashed curves in the top panels) and the DES constraints. This tension disappears when the CMB lensing data is included. In the bottom panels, the best-fit point for the non-$\Lambda$CDM model constrained by the Planck CMB data (including lensing) combined with all non-CMB data enters well into the 1 $\sigma$ region of the DES Y1 All constraint contour (Figure 7 lower right panel), unlike the case for the tilted-flat-$\Lambda$CDM model (Figure 6 lower right panel).

Table 9 lists the individual and total $\chi^2$ values for the best-fit tilted flat and untitled non-flat $\Lambda$CDM models. The best-fit position in the parameter space is found with the COSMOMC built-in routine that obtains the minimum $\chi^2$ using Powell’s minimization method. This method searches for the local minimum by differentiating the likelihood distribution and is efficient at finding an accurate location of the minimum $\chi^2$. We present the individual contribution of each data set used to constrain the model parameters. The total $\chi^2$ is the sum of those from the high-$\ell$ TT likelihood ($\chi^2_{\text{TTFlk}}$), the low-$\ell$ CMB

Note. $\Delta \chi^2$ and In B of a non-flat $\Lambda$CDM model estimated for a combination of data sets represent the excess $\chi^2$ value and ratio of Bayesian evidence, respectively, relative to the tilted flat model for the same combination of data sets.

| Data Sets                     | $\chi^2_{\text{TT}}$ | $\chi^2_{\text{lowP}}$ | $\chi^2_{\text{lensing}}$ | $\chi^2_{\text{DES}}$ | $\chi^2_{\text{NewBAO}}$ | $\chi^2_{\text{CMB}}$ | $\chi^2_{\text{prior}}$ | Total $\chi^2$ | $\Delta \chi^2$ | In B   |
|--------------------------------|---------------------|------------------------|---------------------------|-------------------------|--------------------------|------------------------|----------------------|-----------------|-----------------|--------|
| TT+lowP                       | 763.57              | 10494.41               |                           |                         |                          |                       |                      | 1.96            | 11261.93        |        |
| +JLA                          | 763.60              | 10496.48               | 695.32                    |                         |                          |                       |                      | 1.92            | 11957.32        |        |
| +NewBAO                       | 762.50              | 10497.73               | 13.46                     |                         |                          |                       |                      | 2.36            | 11276.05        |        |
| +$H(z)$                       | 763.98              | 10496.36               | 14.89                     |                         |                          |                       |                      | 1.70            | 11276.93        |        |
| +$f\sigma_8$                  | 766.83              | 10494.95               | 12.63                     |                         |                          |                       |                      | 1.87            | 11275.80        |        |
| +NewBAO+$f\sigma_8$           | 766.47              | 10494.93               | 12.45                     |                         |                          |                       |                      | 2.02            | 11288.50        |        |
| +JLA+NewBAO                   | 764.11              | 10496.10               | 695.17                    | 12.91                   |                          |                       |                      | 2.21            | 11970.51        |        |
| +JLA+NewBAO+$H(z)$            | 764.30              | 10496.06               | 695.19                    | 12.94                   | 14.81                    |                       |                      | 1.95            | 11955.25        |        |
| +JLA+NewBAO+$H(z)+f\sigma_8$  | 766.81              | 10498.80               | 695.12                    | 12.73                   | 14.79                    |                      | 12.15               | 2.05            | 11998.43        |        |
| TT+lowP+lensing               | 766.20              | 10494.93               | 9.30                      |                         |                          |                       |                      | 2.00            | 11272.44        |        |
| +JLA                          | 767.15              | 10494.77               | 8.98                      | 695.07                  |                          |                       |                      | 2.11            | 11285.06        |        |
| +NewBAO                       | 766.37              | 10494.86               | 12.59                     |                         |                          |                       |                      | 2.04            | 11287.27        |        |
| +$H(z)$                       | 766.20              | 10494.92               | 9.27                      | 14.83                   |                          |                       |                      | 1.94            | 11284.62        |        |
| +$f\sigma_8$                  | 768.26              | 10494.43               | 8.67                      | 11.31                   | 1.94                     |                       |                      | 2.11            | 11297.33        |        |
| +NewBAO+$f\sigma_8$           | 767.47              | 10494.57               | 8.73                      | 12.66                   | 11.80                    |                      | 2.11               | 11297.33        |        |
| +JLA+NewBAO                   | 766.42              | 10494.85               | 9.16                      | 695.19                  | 12.61                    |                          |                      | 2.01            | 11980.24        |        |
| +JLA+NewBAO+$H(z)$            | 766.57              | 10494.76               | 9.04                      | 695.16                  | 12.59                    | 14.81                 |                      | 2.15            | 11995.08        |        |
| +JLA+NewBAO+$H(z)+f\sigma_8$  | 767.50              | 10494.56               | 8.74                      | 695.12                  | 12.65                    | 14.79                 |                      | 2.07            | 12007.21        |        |

Note. $\Delta \chi^2$ and In B of a non-flat $\Lambda$CDM model estimated for a combination of data sets represent the excess $\chi^2$ value and ratio of Bayesian evidence, respectively, relative to the tilted flat model for the same combination of data sets.
Figure 8. Best-fit power spectra of (a) tilted flat (top five panels) and (b) untilted non-flat ΛCDM models (bottom five panels) constrained by the Planck CMB TT + lowP data (excluding the lensing data) together with JLA SN Ia, NewBAO, $H(z)$, and $f_{\sigma 8}$ data.
Figure 9. Same as Figure 8 but now including the lensing data.
power spectra ($\chi^2_{\text{lowTEB}}$, lensing ($\chi^2_{\text{lensing}}$), JLA SN Ia ($\chi^2_{\text{JLA}}$),
NewBAO ($\chi^2_{\text{NewBAO}}$), $H(z)$ ($\chi^2_{H(z)}$), $f_{\sigma_8}$ data ($\chi^2_{f_{\sigma_8}}$), and the
contribution from the foreground nuisance parameters ($\chi^2_{\text{prior}}$). The nonstandard normalization of the Planck 2015 CMB data likelihoods means that only the difference of $\chi^2$ of one model relative to the other is meaningful for the Planck CMB data. In Table 9, for the untitled non-flat $\Lambda$CDM model, we list $\Delta \chi^2$, the excess $\chi^2$ over the value of the tilted flat-$\Lambda$CDM model constrained with the same combination of data sets. For the non-CMB data sets, the numbers of degrees of freedom are $735, 15, 31, 10$ for JLA SN Ia, NewBAO, $H(z)$, $f_{\sigma_8}$ data sets, respectively, for a total of 791 degrees of freedom. The reduced $\chi^2$'s for the individual non-CMB data sets are $\chi^2/\nu \lesssim 1$. There are 189 points in the $TT + lowP$ Planck 2015 data (binned angular power spectrum) and 197 when the CMB lensing observations are included.

Let us first focus on how the model fits the individual data sets. Compared to the tilted flat-$\Lambda$CDM model, the untitled non-flat $\Lambda$CDM model constrained with the Planck CMB data alone (excluding and including CMB lensing data) does worse at fitting the Planck high-$\ell$ $C_{\ell}$'s, while it fits the low-$\ell$ ones a bit better. Inclusion of the non-CMB data with the CMB data also results in the best-fit untitled non-flat model providing a poorer fit to the high-$\ell$ TT measurements, both with and without the lensing data, compared to the tilted flat-$\Lambda$CDM case. Adding JLA SN Ia or NewBAO data to the Planck $TT + lowP +$ lensing data improves the untitled non-flat model fit to the Planck low-$\ell$ TEB data. There is a tendency for the non-flat models to more poorly fit the NewBAO and $H(z)$ data (with larger values of $\chi^2_{\text{NewBAO}}$ and $\chi^2_{H(z)}$) than the flat models do, while the opposite is true for the case of the growth rate ($f_{\sigma_8}$) measurements.

Comparing results for the $TT + lowP +$ lensing analyses, $\Delta \chi^2 = 21$ for the full data compilation, for the non-flat $\Lambda$CDM case relative to the flat-$\Lambda$CDM model (the second from the last column in the last row of Table 9). Unfortunately, it is unclear how to turn this into a quantitative relative probability, as the two six-parameter models are not nested (and the number of degrees of freedom of the Planck CMB anisotropy data is not available). Rather, the best-fit versions of each six-parameter model provide distinct local likelihood maxima in a larger seven-parameter model space. However, it is clear that the untitled non-flat $\Lambda$CDM model does not do as good a job of fitting the higher-$\ell$ $C_{\ell}$'s as it does of fitting the lower-$\ell$ ones. In this context it might be relevant to note that there has been some discussion about systematic differences between constraints derived using the higher-$\ell$ and the lower-$\ell$ Planck 2015 CMB data (Addison et al. 2016; Planck Collaboration et al. 2017). Additionally, in the flat-$\Lambda$CDM model, there appear to be inconsistencies between the higher-$\ell$ Planck 2015 CMB anisotropy data and the South Pole Telescope CMB anisotropy data (Aylor et al. 2017).

To compare the untitled non-flat $\Lambda$CDM model with the tilted flat one, we may also use the Bayes factor $B = E [<\text{nonflat}]/E[\text{flat}]$ that is defined as a ratio of Bayesian evidence of the non-flat model relative to the flat one for the same combination of data sets. The Bayesian evidence of a model $M$ is given by

$$E = p(x|M) = \int d\theta p(x|\theta) \pi(\theta|M),$$

where $x$ indicates a data set, $\theta$ is a vector containing parameters of the model $M$, and $\pi(\theta|M)$ is the prior on the parameters. We calculate the Bayesian evidence using the algorithm developed by Heavens et al. (2017) in which the posterior for the Bayesian evidence is obtained with the nearest-neighbor distances in parameter space. In Table 9 we list the logarithm of Bayes factor $\ln B$ for each untitled non-flat $\Lambda$CDM model relative to the tilted flat one. In all cases, we find that $\ln B < -5$, which indicates very strong evidence that the untitled non-flat $\Lambda$CDM model is less favored than the tilted flat one (Trotta 2008). However, we again take note of possible systematic differences in the CMB data mentioned at the end of

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18 The energy density inhomogeneity power spectrum for this seven-parameter tilted non-flat $\Lambda$CDM model is not known.
the previous paragraph (Addison et al. 2016; Aylor et al. 2017; Planck Collaboration et al. 2017) which, if real, could alter the Bayesian evidence in either direction. In addition, the Bayesian evidence we have computed here does not account for the fact that the best-fit untitled non-flat model has a lower $\Omega_m$ than the best-fit tilted flat model (when both are fit to the cosmological data compilation we have used in our analyses here), so it is in better agreement with the lower $\Omega_m$ determined from weak-lensing measurements. Figures 8 and 9 show the CMB high-$\ell$ TT, and the low-$\ell$ TT, TE, EE power spectra of the best-fit tilted flat and untitled non-flat ΛCDM models, excluding and including the lensing data, respectively. The non-flat ΛCDM model constrained by adding each non-CMB data set to the Planck 2015 CMB anisotropy observations generally gives a poorer fit to the low-$\ell$ EE power spectrum while it better fits the low-$\ell$ TT power spectrum (see the bottom left panel of Figures 8 and 9). The shape of the best-fit $C_\ell$ power spectra of various models relative to the Planck CMB data points are consistent with the $\chi^2$ values listed in Table 9.

Figure 10 shows the best-fit initial power spectra of scalar-type fractional energy density perturbations for the non-flat ΛCDM model constrained by the Planck TT + lowP (left) and TT + lowP + lensing (right panel) data together with other non-CMB data sets. The reduction in power at low $q$ in the best-fit closed-ΛCDM inflation model power spectra shown in Figure 10 is partially responsible for the low-$\ell$ TT power reduction of the best-fit untitled closed model $C_\ell$’s (shown in the lower panels of Figures 8 and 9) relative to the best-fit tilted flat model $C_\ell$’s. Other effects, including the usual and integrated Sachs–Wolfe effects, also play a role in affecting the shape of the low-$\ell$ $C_\ell$’s. For a detailed discussion of how the interplay among these effects influence the low-$\ell$ shape of the $C_\ell$’s in the open inflation case, see Gorski et al. (1998).

5. Conclusion

We use the tilted flat-ΛCDM and the untitled non-flat ΛCDM inflation models to measure cosmological parameters from a carefully gathered compilation of observational data, the largest such collection utilized to date. Our main results, in summary, are:

1. Using a consistent power spectrum for energy density inhomogeneities in the untitled non-flat model, we confirm, with greater significance, the Ooba et al. (2018a) result that cosmological data does not demand spatially flat hypersurfaces. These data (including CMB lensing measurements) favor a closed universe at more than 5σ significance, with spatial curvature contributing about a percent to the current cosmological energy budget.

2. The best-fit untitled non-flat ΛCDM model provides a better fit to the low-$\ell$ temperature anisotropy $C_\ell$’s and better agrees with the $\sigma_8$–$\Omega_m$ DS constraints, but does worse than the best-fit tilted flat-ΛCDM model in fitting the higher-$\ell$ temperature anisotropy $C_\ell$’s.  

3. $H_0$ measured in both models are almost identical, and consistent with most other measurements of $H_0$. However, as is well known, an estimate of the local expansion rate (Anderson & Riess 2018) is 2.7σ larger.

4. $\sigma_8$ measured in both models are identical and consistent with the recent DES measurement (DES Collaboration et al. 2018a).

5. The measured $\Omega_m$ is more model-dependent than the measured $\sigma_8$ and the $\Omega_m$ value measured using the non-flat ΛCDM model is more consistent with the recent DES measurement (DES Collaboration et al. 2018a).

Overall, the tilted flat-ΛCDM model has a lower $\chi^2$ than the untitled non-flat ΛCDM case and thus is more favored. On the other hand, the untitled non-flat ΛCDM model has other advantages, including having a lower $\Omega_m$. It is possible that a more complete understanding of systematic differences between constraints derived using the lower-$\ell$ and higher-$\ell$ Planck CMB anisotropy data, as well as a more complete understanding of the differences between the Planck and South Pole Telescope CMB anisotropy data, might have some bearing on these issues.

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19 We note that the tilted flat XCDM and ϵCDM models, with dynamical dark energy, provide slightly better fits to the data than does the tilted flat ΛCDM model (Eoba et al. 2018d; Park & Ratra 2018, 2019b; Solà et al. 2019, and references therein).
