Generalized nil-Coxeter algebras, cocommutative algebras, and the PBW property

Apoorva Khare

Abstract. Poincaré–Birkhoff–Witt (PBW) Theorems have attracted significant attention since the work of Drinfeld (1986), Lusztig (1989), and Etingof–Ginzburg (2002) on deformations of skew group algebras $H \ltimes \text{Sym}(V)$, as well as for other cocommutative Hopf algebras $H$. In this paper we show that such PBW theorems do not require the full Hopf algebra structure, by working in the more general setting of a “cocommutative algebra”, which involves a coproduct but not a counit or antipode. Special cases include infinitesimal Hecke algebras, as well as symplectic reflection algebras, rational Cherednik algebras, and more generally, Drinfeld orbifold algebras. In this generality we identify precise conditions that are equivalent to the PBW property, including a Yetter–Drinfeld type compatibility condition and a Jacobi identity. We also characterize the graded deformations that possess the PBW property. In turn, the PBW property helps identify an analogue of symplectic reflections in general cocommutative bialgebras.

Next, we introduce a family of cocommutative algebras outside the traditionally studied settings: generalized nil-Coxeter algebras. These are necessarily not Hopf algebras, in fact, not even (weak) bialgebras. For the corresponding family of deformed smash product algebras, we compute the center as well as abelianization, and classify all simple modules.

1. Introduction

In the study of deformation algebras, their structure and representations, one commonly begins by understanding their connection to the corresponding associated graded algebras (which are generally better behaved). Such connections of course provide desirable “monomial bases”, but also additional structural and representation-theoretic knowledge.

A first step in understanding these connections involves showing that these filtered algebras satisfy the Poincaré–Birkhoff–Witt (PBW) property, in that they are isomorphic as vector spaces to their associated graded algebras. Such results are known as PBW theorems in the literature. The terminology of course originates with the classical result for the universal enveloping algebra of a Lie algebra.
However, it has gathered renewed attention over the past few decades owing to tremendous interest in the study of orbifold algebras and their generalizations, which we now briefly describe.

In a seminal paper [12], Drinfeld pioneered the study of smash product algebras of the form $kG \ltimes \text{Sym}(V)$, where a group $G$ acts on a $k$-vector space $V$. Drinfeld’s results were rediscovered and extended by Etingof and Ginzburg in their landmark paper [14], which introduced symplectic reflection algebras and furthered our understanding of rational Cherednik algebras. These algebras serve as “non-commutative” coordinate rings of the orbifolds $V/G$; see [32] for a related setting. Subsequently, Etingof, Ginzburg, and Gan replaced the group by algebraic distributions of a reductive Lie group $G$. This led to the study of infinitesimal Hecke algebras in [13] (and several recent papers), where $U \mathfrak{g}$ acts on $\text{Sym}(V)$, with $\mathfrak{g} = \text{Lie}(G)$. These families of deformed algebras continue to be popular and important objects of study, with connections to representation theory, combinatorics, and mathematical physics.

A common theme underlying all of these settings is that a cocommutative Hopf algebra $H$ acts on the vector space $V$ and hence on $\text{Sym}(V)$. The aforementioned families of algebras $\mathcal{H}_{\lambda,\kappa}$ are created by deforming two sets of relations:

- The relations $V \wedge V \mapsto 0$ in the smash product algebra $H \ltimes \text{Sym}(V)$ are deformed using an anti-symmetric bilinear form $\kappa : V \wedge V \to H$, or more generally, $\kappa : V \wedge V \to H \oplus V$. These deformed relations feature in [12]–[14], and follow-up works.
- The relations $g \cdot v = g(v)g$ for grouplike elements $g$ with $H$ a group algebra, were deformed by Lusztig [32] to create graded affine Hecke algebras, using a bilinear form $\lambda : H \otimes V \to H$.

The forms $\lambda, \kappa$ define a filtered algebra, and an important question is to characterize those deformations $\mathcal{H}_{\lambda,\kappa}$ whose associated graded algebra is isomorphic to $\mathcal{H}_{0,0} = H \ltimes \text{Sym}(V)$. Such parameters $\lambda, \kappa$ are said to correspond to PBW deformations, and have been studied in the aforementioned works as well as by Braverman and Gaitsgory [5] among others. More recently, in a series of papers [35]–[37], Shepler and Witherspoon have shown PBW theorems in a wide variety of settings (skew group algebras, Drinfeld orbifold algebras, Drinfeld Hecke algebras, . . . ), that encompass many of the aforementioned cases. We also point the reader to the comprehensive survey [38] for more on the subject. This includes the case of $\text{Sym}(V)$ replaced by a quantum symmetric algebra. Perhaps one of the most general versions in the literature is the recent work [43] by Walton and Witherspoon, in which $H$ is replaced by a Hopf algebra, and $\text{Sym}(V)$ by a Koszul algebra. For completeness, we also mention work in related flavors: [19] studies generalized Koszul algebras, while [144] analyze deformations of Hopf algebra actions on “doubled” pairs of module algebras.

We now point out some of the novel features and extensions in the present paper. First, all of the aforementioned settings involve $H$ being a bialgebra – in fact, a Hopf algebra. In this paper we isolate the structure required to study the PBW property, and show that it includes the coproduct but not the counit or antipode. More precisely, we work in the more general framework of a (cocommutative) algebra with coproduct. This is a strictly weaker setting than that of a bialgebra, as it also includes examples such as the nil-Coxeter (or nil-Hecke) algebra associated to a Weyl group, $NC_W$. Recall that these algebras were originally introduced by
Fomin and Stanley \cite{Fomin2005} as Demazure operators in the study of Schubert polynomials, though they appear implicitly in previous work \cite{Binz1990,Joseph1995} on the cohomology of generalized flag varieties for semisimple and Kac–Moody groups, respectively; see also \cite{Parshall1987}. Nil-Coxeter algebras have subsequently been studied in their own right \cite{Kamnitzer2011,Zelevinsky2012} as well as in the context of categorification \cite{Khovanov1999,Khovanov2000}, among others.

Nil-Coxeter algebras are necessarily not bialgebras (hence not Hopf algebras). Thus, deformations over such cocommutative algebras have not been considered to date in the literature.

Second, we introduce a novel class of Hecke-type algebras, the \emph{generalized nil-Coxeter algebras}, which encompass the usual nil-Coxeter algebras. These algebras have not been studied in the literature. In this paper we will specifically study deformations over generalized nil-Coxeter algebras. Moreover, our results are characteristic-free.

An additional novelty of the present work is that in all of the aforementioned works in the literature, either the bilinear form $\kappa_V : V \wedge V \to V$ is assumed to be identically zero, or $\lambda : H \otimes V \to H$ is identically zero. The present paper addresses this gap by working with algebras for which all three parameters $\lambda, \kappa_V, \kappa_A = \kappa - \kappa_V$ are allowed to be nonzero. (All notation is explained in Definition \ref{def:3.3} below.)

**Organization of the paper.** We now outline the contents of the present paper, which can be thought of as having two parts. In Section \ref{sec:2} we introduce the general notion of a \emph{cocommutative} $k$-\emph{algebra} $A$, i.e., an algebra with a multiplicative coproduct map that is cocommutative (over a unital ground ring $k$). We next state and prove one of our main results: a PBW-type theorem for deformations $H_{\lambda,\kappa}$ of the smash product algebra $H_{0,0} = A \ltimes \text{Sym}(V)$. Here, $A$ acts on tensor powers of $V$ via the coproduct, and on the symmetric algebra because of cocommutativity.

In Section \ref{sec:3} we explain the connection between the PBW theorem and deformation theory. Specifically, we identify the graded $k[t]$-deformations of $H_{0,0}$ whose fiber at $t = 1$ has the PBW property. This extends various results in the literature; see \cite{Rallis2002,Wallach1988}. The first part of the paper concludes in Section \ref{sec:4} by examining well-known notions in the Hopf algebra literature in the broader setting of cocommutative algebras. This includes studying the cases where $A$ is a cocommutative bialgebra or Hopf algebra. We classify the parameters $\lambda, \kappa$ for which $H_{\lambda,\kappa}$ has the same structure, and relate the PBW property to the Yetter–Drinfeld condition, a natural compatibility condition that arises in Hopf-theoretic settings. We also extend the notion of ‘symplectic reflections’ from groups to all cocommutative bialgebras.

In the second part of the paper, we study a specific family of cocommutative algebras that are not yet fully explored in the literature. Thus, in Section \ref{sec:5} we introduce a family of \emph{generalized nil-Coxeter algebras} associated to a Coxeter group $W$; these are closely related to Coxeter groups and their generalizations studied by Coxeter and Shephard–Todd \cite{Coxeter1950,Shephard1953,Shephard1954}.

Generalized nil-Coxeter algebras are necessarily not bialgebras; thus they fall strictly outside the Hopf-theoretic setting. In the remainder of the paper, we study the deformations $H_{\lambda,\kappa}$ over generalized nil-Coxeter algebras. We first study the Jacobi identity in such algebras $H_{\lambda,\kappa}$, and classify all Drinfeld-type deformations $H_{0,\kappa}$ with the PBW property. In the final section of the paper, we study additional
properties of the algebras $\mathcal{H}_{\lambda,\kappa}$, including computing the center and abelianization, and classifying simple modules.

2. Cocommutative algebras, smash products, and the PBW theorem

Global assumptions: Throughout this paper, we work over a ground ring $k$, which is a unital commutative ring. We also fix a cocommutative $k$-algebra $(A, \Delta)$, defined below, and a $k$-free $A$-module $V$.

By $\dim V$ for a free $k$-module $V$, we will mean the (possibly infinite) $k$-rank of $V$. In this paper, all $k$-modules, including all $k$-algebras, are assumed to be $k$-free.

Unless otherwise specified, all (Hopf) algebras, modules, and bases of modules are with respect to $k$, and all tensor products are over $k$.

2.1. Cocommutative algebras and the PBW theorem. We begin by introducing the main construction of interest and the main result of the first part of this paper.

Definition 2.1. Suppose $A$ is a unital associative $k$-algebra.

(1) $A$ is an algebra with coproduct if there exists a $k$-algebra map $\Delta : A \to A \otimes_k A$ called the coproduct, such that $\Delta(1) = 1 \otimes 1$ and $\Delta$ is coassociative, i.e., $(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta : A \to A \otimes A \otimes A$.

(2) An algebra with coproduct is said to be cocommutative if $\Delta = \Delta^{\text{op}}$.

Notice that bialgebras and Hopf algebras (with the usual coproduct) are examples of algebras with coproduct (with $k$ a field). As pointed out to us by Susan Montgomery, one could a priori have considered weak bialgebras (these feature prominently in the theory of fusion categories [15]), but these provide no additional examples, as explained at the end of [4 §2.1]: since $\Delta(1) = 1 \otimes 1$ by assumption, a cocommutative algebra is a bi/Hopf-algebra if and only if it is a weak bi/Hopf-algebra. Additional examples do arise, however, using nil-Coxeter algebras, as explained in Remark 5.3 below. These algebras show that the notion of an algebra with coproduct is strictly weaker than that of a (weak) bialgebra.

We also remark that every unital $k$-algebra $A$ is an algebra with coproduct, if we define $\Delta_L(a) := a \otimes 1$ or $\Delta_R(a) := 1 \otimes a$. (Thus, the definition essentially involves a choice of coproduct.) However, $A$ need not have a cocommutative coproduct in general.

Given $a \in A$, write $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$ and $\Delta^{\text{op}}(a) = \sum a_{(2)} \otimes a_{(1)}$, in the usual Sweedler notation. We now use $\Delta$ to first define tensor and symmetric product $A$-module algebras, as well as undeformed Drinfeld Hecke algebras. Suppose $(A, \Delta)$ acts on a free $k$-module $V$ (not necessarily of finite rank), denoted by $v \mapsto a(v)$. Notice that $TV := T_k V$ has an augmentation ideal $T^+ V := V \cdot T_k V$, and this ideal is an $A$-module algebra via:

$$a(v_1 \otimes \cdots \otimes v_n) := \sum a_{(1)}(v_1) \otimes \cdots \otimes a_{(n)}(v_n), \quad \forall a \in A, \ v_1, \ldots, v_n \in V, \ n \geq 1.$$ 

We do not include the case $n = 0$ here, since $A$ does not have a counit $\varepsilon$.

Definition 2.2. Given a $k$-algebra $A$, let $A^{\text{mult}}$ denote the left $A$-module $A$, under left multiplication. Now given $(A, \Delta)$ and $V$ as above, the smash product of $TV$ and $A$, denoted by $TV \rtimes A^{\text{mult}}$, is defined to be the $k$-algebra $T(V \oplus A^{\text{mult}})$, with
the multiplication relations given by \( a \cdot a' := aa' \) in \( A \), \((v' \otimes a') \cdot (1 \otimes a) = v' \otimes a'a\), and

\[
(v' \otimes a') \cdot (v \otimes a) := \sum (v' \cdot a'_1(v)) \otimes a'_2(a), \quad \forall a, a' \in A, v', v \in T^+ V.
\]

We use \( - \otimes A \) rather than \( A \otimes - \) in this paper. Also note that for \( 1_A \) to commute with \( V \) requires \( \Delta(1) = 1 \otimes 1 \) as above. Now denote by \( \wedge^2 V \subset V \otimes_k V \) the k-span of \( v \wedge v' := v \otimes v' - v' \otimes v \); then \( \wedge^2 V \) is an \( A \)-submodule of \( T^+ V \) because of the cocommutativity assumption on \( A \), which implies that \( a(v_1 \wedge v_2) = \sum a_{(1)}(v_1) \wedge a_{(2)}(v_2) \). Thus, one can quotient \( TV \rtimes A \) by the related two-sided “\( A \)-module ideal”, to define:

\[
(2.1) \quad \mathcal{H}_{0,0}(A, V) = \mathcal{H}_{0,0} = \text{Sym}(V) \rtimes A := \frac{TV \rtimes A}{(TV \cdot \wedge^2 V \cdot TV) \rtimes A}.
\]

The algebra \( \mathcal{H}_{0,0}(A, V) \) will be referred to as the smash product of \( \text{Sym}(V) \) and \( A \).

We are now able to introduce deformations of this smash product algebra.

**Definition 2.3.** Given \((A, \Delta)\) and \( V \) as above, as well as bilinear forms \( \lambda \in \text{Hom}_k(V \otimes A, A) \) and \( \kappa \in \text{Hom}_k(\wedge^2 V, A \otimes V) \), the deformed smash product algebra \( \mathcal{H}_{\lambda, \kappa} = \mathcal{H}_{\lambda, \kappa}(A, V) \) with parameters \( \lambda, \kappa \) is defined to be the quotient of \( T(V \oplus A) \) by the multiplication in \( A \) and by

\[
(2.2) \quad av - \sum a_{(1)}(v)a_{(2)} = \lambda(a, v), \quad vv' - v'v =: [v, v'] = \kappa(v, v'), \quad \forall a \in A, v, v' \in V.
\]

Also define \( \kappa_V \in \text{Hom}_k(V \wedge V, V) \) and \( \kappa_A \in \text{Hom}_k(V \wedge V, A) \) to be the projections of \( \kappa \) to \( V, A \) respectively.

Observe that \( \lambda \) being trivial is equivalent to the \( A \)-action preserving the grading on \( \text{Sym}(V) \). Moreover, we will write \( \mathcal{H}_{\lambda, \kappa} \) instead of \( \mathcal{H}_{\lambda, \kappa}(A, V) \) if \( A, V \) are clear from context.

The deformed smash product algebras \( \mathcal{H}_{\lambda, \kappa} = \mathcal{H}_{\lambda, \kappa}(A, V) \) encompass a very large family of deformations considered in the literature, including universal enveloping algebras, skew group algebras, Drinfeld orbifold algebras, Drinfeld Hecke algebras, symplectic reflection algebras, rational Cherednik algebras, degenerate affine Hecke algebras and graded Hecke algebras, Weyl algebras, infinitesimal Hecke algebras, and many others. This is an area of research that is the focus of tremendous recent activity; see \cite{11, 14, 24, 31, 32, 10, 42}, and subsequent follow-up works in the literature.

**Remark 2.4.** In order to place the work in context, we briefly comment on how our framework compares to other papers in the PBW literature. The paper encompasses other works in two aspects: first, the algebra \((A, \Delta)\) is strictly weaker than a bialgebra. Second, the deformation parameters \( \lambda, \kappa_V, \kappa_A \) can all be nonzero. At the same time, we impose two restrictions that are present in some papers but not in others: first, we work with \( \text{Sym}(V) \) and not a quantum algebra, nor a general Koszul algebra (e.g., a PBW algebra). Second, for ease of exposition we only consider algebras with \( \text{im}(\kappa_V) \) a subset of \( V \) instead of \( V \otimes A \); this is akin to the assumption \( \lambda \equiv 0 \) in \cite{35, 43}, or \( \kappa_V \equiv 0 \) in \cite{37}.

Notice that the algebras \( \mathcal{H}_{\lambda, \kappa} \) are filtered, by assigning \( \deg A = 0, \deg V = 1 \). We say that \( \mathcal{H}_{\lambda, \kappa} \) has the **PBW property** if the surjection from \( \mathcal{H}_{0,0} = \text{Sym}(V) \rtimes A \) to the associated graded algebra of \( \mathcal{H}_{\lambda, \kappa} \) is an isomorphism. Equivalently, the
PBW theorem holds for $\mathcal{H}_{\lambda,\kappa}$ if for any (totally) ordered $k$-basis $\{x_i : i \in I\}$ of the free $k$-module $V$ and $\{a \in J_1\}$ of the $k$-free $k$-algebra $A$, the collection $$\{X \cdot a : X \text{ is a word in the } x_i \text{ in non-decreasing order of subscripts, } a \in J_1\}$$
is a $k$-basis of $\mathcal{H}_{\lambda,\kappa}$. We now state the main result of the first part of the paper, which is a PBW Theorem for the algebras $\mathcal{H}_{\lambda,\kappa}$.

**Theorem 2.5 (PBW Theorem).** Suppose $(A, \Delta)$ is a $k$-free cocommutative $k$-algebra, and $V$ a $k$-free $A$-module. Define $\mathcal{H}_{\lambda,\kappa}$ with $\kappa = \kappa_V \oplus \kappa_A : V \otimes V \to V \oplus A$ as above, and suppose $A = k \cdot 1 \oplus A'$ for a free $k$-submodule $A' \subset A$. Then the following are equivalent:

1. $\mathcal{H}_{\lambda,\kappa}$ has the PBW property (for a $k$-basis of $V$ and a $k$-basis of $A$ containing 1).
2. The natural map $A \otimes (V \otimes A) \to \mathcal{H}_{\lambda,\kappa}$ is an injection.
3. $\lambda : A \otimes V \to A$ and $\kappa : V \otimes V \to V \oplus A$ satisfy the following conditions:
   
   (a) $A$-action on $V$: For all $a, a' \in A$ and $v \in V$, the following equation holds in $A$:

   $$(2.3) \quad \lambda(aa', v) = a\lambda(a', v) + \sum \lambda(a, a'(v))a'(2).$$

   (b) $A$-compatibility of $\lambda, \kappa$: For all $a \in A$ and $v, v' \in V$, the following equations hold in $A$ and $V \otimes A$ respectively:

   $$(2.4) \quad a\kappa_A(v, v') - \sum \kappa_A(a(1)(v), a(2)(v'))a(3) = \lambda(\lambda(a, v), v') - \lambda(\lambda(a, v'), v) - \lambda(a, \kappa_V(v, v')).$$

   $$(2.5) \quad \sum a(1)(\kappa_V(v, v'))a(2) - \sum \kappa_V(a(1)(v), a(2)(v'))a(3) = \sum \lambda(a, v)(1)(v')\lambda(a, v)(2) - \sum \lambda(a, v')(1)(v)\lambda(a, v')(2) + \sum a(1)(v)\lambda(a(2), v') - \sum a(1)(v')\lambda(a(2), v).$$

   (c) Jacobi identities: For all $v_1, v_2, v_3 \in V$, the following cyclic sum vanishes:

   $$(2.6) \quad \sum \kappa(v_1, v_2, v_3) := \kappa(v_1, v_2, v_3) + \kappa(v_2, v_3, v_1) + \kappa(v_3, v_1, v_2) = 0.$$
we point the reader to \[37\] Theorem 3.1 for the analogous result with \( k \) a field, \( A \) a group algebra, and \( \kappa_V \equiv 0 \).

**Remark 2.6.** Notice that the conditions in part (3) of the theorem always hold in \( \mathcal{H}_{A,V} \). In other words, Equations (2.4)–(2.8) hold in the image of the space \( A \oplus (V \otimes A) \) in \( \mathcal{H}_{A,V} \), by considering the equations corresponding to the associativity of the algebra \( \mathcal{H}_{A,V} \):

\[
aa' \cdot v = a \cdot (a' \cdot v), \quad a \cdot (vv' - v'v) = a \cdot \kappa(v, v'), \quad \sum_{\lambda} [\kappa(v_1, v_2), v_3] = 0.
\]

The assertion of Theorem 2.5 is that the PBW property is equivalent to these equations holding in \( A \oplus (V \otimes A) \).

**Remark 2.7.** It is easy to verify that the Jacobi identities (2.4), (2.6) hold in \( A \oplus (V \otimes A) \) if \( \dim_k V \leq 2 \), since in that case the left and right hand sides of both equations vanish. If moreover \( \dim_k V \leq 1 \), then the \( A \)-compatibility conditions (2.4), (2.6) also hold in \( A \oplus (V \otimes A) \), since \( \kappa_V, \kappa_A \equiv 0 \).

### 2.2. Proof of the PBW Theorem

We now prove Theorem 2.5 using the Diamond Lemma \[2\]. As we work with a general cocommutative algebra (which is strictly weaker than a cocommutative bialgebra), and moreover, work with possibly nonzero \( \lambda, \kappa_V \), the proof is written out in some detail. To prove Theorem 2.5, we will require the unit 1 to be one of our \( k \)-basis vectors for \( A \); words involving this basis vector are to be considered “without” the 1.

**Proof of the PBW Theorem 2.5.** Clearly, (1) \( \Rightarrow \) (2), and (2) \( \Rightarrow \) (3) using Remark 2.6. The goal in the remainder of this proof (and this section) is to show that (3) \( \Rightarrow \) (1). We begin by writing down the relations in \( \mathcal{H}_{A,V} \) systematically. Recall that \( A = k \cdot 1_A \oplus A' \); now suppose \( \{ a_j : j \in J \} \) is a \( k \)-basis of the \( k \)-submodule \( A' \). Write

\[
J_1 := \{ a_j : j \in J \} \cup \{ 1_A \}, \quad a_0 := 1_A, \quad J_0 := J \cup \{ 0 \}.
\]

We also fix a total ordering on \( J_1 \) and correspondingly on \( J_0 \), with \( 0 \leq j \) for all \( j \in J_0 \).

Next, fix a totally ordered \( k \)-basis of \( V \), denoted by \( \{ x_i : i \in I \} \). (Thus, \( I \) is also totally ordered.) We then define various structure constants, with the sums running over \( J_0 \) and \( I \), and using Einstein notation throughout. We first define the structure constants from \( A \) and its action on \( V' \):

\[
a_j a_k = u_{jk}^l a_l, \quad a_j (x_k) = s_{jk}^h x_h, \quad \Delta(a_j) = r_{jk}^l a_k \otimes a_l.
\]

In particular, \( u_{j0}^i = u_{0j}^i = \delta_{i,j}, \ s_{0k}^h = \delta_{i,k}, \) and \( r_{0}^{kl} = \delta_{k,0} \delta_{l,0} \). Next, we define the structure constants for the maps \( \lambda, \kappa \):

\[
\kappa_A(x_j, x_k) = v_{jk}^i a_l, \quad \kappa_V(x_j, x_k) = w_{jk}^h x_l, \quad (j > k); \quad \lambda(a_j, x_k) = q_{jk}^l a_l.
\]

It now follows that \( \mathcal{H}_{A,V} \) is a quotient of \( T(V \oplus A) \), with the defining relations:

\[
x_j x_k = x_k x_j + v_{jk}^l a_l + w_{jk}^h x_l \quad (j > k),
\]

\[
a_j a_k = u_{jk}^l a_l, \quad a_j x_k = t_{jk}^{lm} x_m a_l + q_{jk}^l a_l, \quad \text{where} \ t_{jk}^{lm} = r_{jk}^{lm} s_{lk}^{-1}.
\]

Thus, the \( q, r, s, t, u, v, w \) are all structure constants in \( k \), for all choices of indices.
To show (1), we first write down additional consequences of the structure of $A, V$. The following equations encode the associativity, coassociativity, and cocommutativity of $A$:

\[
\begin{align*}
    u_{jm}^l u_{ln}^m &= u_{ij}^l u_{lm}^m \quad \forall i, j, m, n; \\
    r_{kl}^j r_{lm}^m &= r_{ij}^k r_{jm}^m \quad \forall i, j, m, n; \\
    r_{kl}^{jl} &= r_{jk}^{jl} \quad \forall j, k, l \in J_0 = J \cup \{0\}.
\end{align*}
\]

The next condition is that $\Delta$ is multiplicative, which yields:

\[
\begin{align*}
    u_{jk}^l r_{lm}^{mn}(a_m \otimes a_n) &= u_{jk}^l \Delta(a_l) = \Delta(a_j a_k) = \Delta(a_j) \Delta(a_k) \\
    &= r_{ij}^{cd} r_{jk}^{ef} (a_c \otimes a_d)(a_e \otimes a_f) = r_{ij}^{cd} r_{jk}^{ef} u_{ce}^m u_{df}^n (a_m \otimes a_n).
\end{align*}
\]

Equating coefficients in $A \otimes A$, we conclude that

\[
(2.13) \quad u_{jk}^l r_{lm}^{mn} = r_{ij}^{cd} r_{jk}^{ef} u_{ce}^m u_{df}^n.
\]

Finally, $V$ is an $A$-module, which yields:

\[
\begin{align*}
    s_{jm}^n s_{ki}^m x_n &= a_j (s_{km}^n x_m) = a_j (a_k(x_i)) = (a_j a_k)(x_i) = u_{jk}^l a_m(x_i) = u_{jk}^l s_{mi}^n x_n,
\end{align*}
\]

whereby we get

\[
(2.14) \quad s_{jm}^n s_{ki}^m = u_{jk}^l s_{mi}^n.
\]

We now proceed with the proof, using the terminology of [2]. The reduction system $S$ consists of the set of algebra relations (2.11). Then expressions in the left and right hand sides in the equations in (2.11) are what Bergman calls $f_\sigma$ and $W_\sigma$, respectively.

Define $X := \{a_j : j \in J\} \cup \{x_i : i \in I\}$. Then the expressions in the free semigroup $\langle X \rangle$ generated by $X$ that are irreducible (i.e., cannot be reduced via the operations $f_\sigma \mapsto W_\sigma$ via the Equations (2.11)) are precisely the PBW-basis that was claimed earlier, i.e. words $x_{i_1} \cdots x_{i_l} \cdot a_j$, for $j \in J_0$ and $i_1 \leq \cdots \leq i_l$, all in $I$. This also includes the trivial word $1$.

Next, define a semigroup partial ordering $\leq$ on $X$, first on its generators via:

\[
(2.15) \quad 1 < x_i < a_j, \quad \forall j \in J, \ i \in I,
\]

and then extend to a total order on $\langle X \rangle$, as follows: words of length $m$ are strictly smaller than words of length $n$, whenever $m < n$; and words of equal lengths are (totally) ordered lexicographically. It is easy to see that $\leq$ is a semigroup partial order on $\langle X \rangle$, i.e., if $a \leq b$ then $aw \leq wbw'$ for all $w, w' \in \langle X \rangle$. Moreover, $\leq$ is indeed compatible with $S$, in that each $f_\sigma$ reduces to a linear combination $W_\sigma$ of monomials strictly smaller than $f_\sigma$.

We now recall the descending chain condition, which says that given a monomial $B \in \langle X \rangle$, any sequence of reductions applied to $B$ yields an expression that is irreducible in finitely many steps. Now the following result holds.

**Lemma 2.8.** The semigroup partial order $\leq$ on $\langle X \rangle$ satisfies the descending chain condition.

**Proof.** We prove a stronger assertion; namely, we produce an explicit upper bound for the number of reductions successively applicable on a monomial. Given
a word \( w = T_1 \cdots T_n \), with \( T_i \in X \forall i \), define its \textit{misorordering index} \( \text{mis}(w) \) to be \( o + p + pr + q + r^3 \), where

\[
\begin{align*}
o &= o(w) := \# \{ (i, j) : i < j, T_i, T_j \in V, T_i > T_j \}, \\
p &= p(w) := \# \{ (i, j) : i < j, T_i \in A', T_j \in V \}, \\
q &= q(w) := \# \{ i : T_i \in A' \}, \\
r &= r(w) := \# \{ i : T_i \in V \} = n - q.
\end{align*}
\]

We now claim that each reduction strictly reduces the misordering index of each resulting monomial; this claim shows the result. As an illustration of the claim, we present the most involved case: when \( f_\sigma = x_j x_k \) with \( j > k \), and the monomial we consider via the reduction \( f_\sigma \mapsto W_\sigma \) corresponds to \( a_l \) for some \( l \in J \). For this new word \( w' \), notice that \( q \) increases by 1, whereas \( r \) reduces by 2 (so \( r \geq 2 \)), \( o \) reduces by at least 1, and \( p \) may increase by at most the number of \( x \) to the right of the new \( a \), which is at most \( r - 2 \). So, \( o + q \) does not increase, and we now claim that \( p + pr + r^3 \) strictly reduces. Indeed, \( p' \leq p + r - 2 \), \( r' \leq r - 2 \), whence:

\[
\begin{align*}
p'(1 + r') + (r')^3 &\leq (p + r - 2)(1 + (r - 2)) + (r - 2)^3 \\
&\leq p(1 + r) + (r - 2) + (r - 2)^2 + (r - 2)^3 \\
&= p(1 + r) + (r - 2)(r^2 - 3r + 3) < p(1 + r) + r \cdot r^2.
\end{align*}
\]

Hence \( \text{mis}(w') < \text{mis}(w) \) as desired. \( \square \)

The final item utilized in the proof of the PBW theorem, is the notion of ambiguities. It is clear that no \( f_\sigma \) is a subset of \( f_\tau \) for some \( \sigma, \tau \in S \); hence there are no \textit{inclusion ambiguities}. In light of Lemma \( \ref{lem:overlap} \) and the Diamond Lemma \( \ref{thm:diamond} \), it suffices to resolve all \textit{overlap ambiguities} using the given conditions in (3). We begin by writing down these conditions explicitly using the structure constants in \( A \). Explicit computations using these constants and Equations \( \ref{eq:u} \)–\( \ref{eq:u'} \) yield the following five equations, respectively:

\[
\begin{align*}
(2.16) & \quad u_{jk}^l q_{il}^h = q_{kl}^l u_{jl}^h + q_{jk}^l q_{jn}^h u_{lm}, \\
(2.17) & \quad v_{jk}^l u_{il}^h - i_{ij}^m v_{mk}^d v_{nc}^l u_{ld}^h = q_{kl}^l q_{jl}^h - q_{jk}^l q_{jh}^h - w_{jk}^h, \\
(2.18) & \quad w_{jk}^l r_{ij}^m - i_{ij}^m i_{mk}^d w_{ni}^l = q_{lk}^l r_{ik}^d - q_{jk}^l r_{jc}^d + r_{ij}^m i_{mk}^d - r_{ij}^m q_{mk}^d, \\
(2.19) & \quad \sum_{(i,j,k)} v_{ij}^l q_{lk}^h = \sum_{(i,j,k)} w_{jk}^l u_{im}^h, \\
(2.20) & \quad \sum_{(i,j,k)} w_{ij}^l w_{lk}^h \cdot (x_h \otimes a_0) = \sum_{(i,j,k)} v_{jk}^l (x_i \otimes a_m) - \sum_{(i,j,k)} v_{ij}^l r_{ik}^d \cdot (x_c \otimes a_d).
\end{align*}
\]

We now resolve the overlap ambiguities, which are of four types, and correspond to the associativity of the algebra \( A_{\lambda, \kappa} \) (see Remark \( \ref{rem:associativity} \)):

\[
a_i a_j a_k, \; a_j a_k x_i, \; a_k x_i x_j (i > j), \; x_i x_j x_k (i > j > k).
\]

Notice that the first type is resolvable because \( A \) is an associative algebra. We only analyse the second type of ambiguity in what follows; the others involve carrying out similar (and more longwinded) computations, that use the structure constants of the cocommutative algebra \( A \) with coproduct.
Our first goal in this section is to show that the PBW property for the algebras further assuming that $V$ commutative, and $B^{-}$ homogeneous of degree related to that in loc. cit. 

The overlap ambiguity is resolved if these two expressions are shown to be equal.

On the other hand, 

$$a_j(a_k x_i) = t_{jk}^{gf} a_j x_g a_f + q_{jk} a_j a_l = t_{jk}^{gf} y_{lj} x_h a_m + t_{jk}^{hf} a_j a_f + q_{jk} u_{lf} a_h$$

The overlap ambiguity is resolved if these two expressions are shown to be equal.

In light of (2.16), it suffices to show that, after relabelling indices, we have for all $i, j, k, l, h$ (or $h-t$):

$$u_{jk}^{m} s_{f k} = t_{jk}^{gf} u_{hf}^{l}.$$ 

To see why this holds, begin with the right-hand side, expand using the definition of $t$, and then use Equations (2.13), (2.14) above:

$$t_{jk}^{gf} u_{hf}^{l} = r_{jk}^{ga} a_{x_{f g}} u_{gh}^{l} = r_{jk}^{ga} a_{x_{f g}} u_{gh}^{l} \cdot (s_{hi} s_{gi})$$

which is precisely the left-hand side. Thus the ambiguity is resolved. 

3. Characterization via deformation theory

We now explain how PBW deformations can be naturally understood via deformation theory. In this section, suppose $k$ is a field. Given an associative algebra $B$ and an indeterminate, a deformation of $B$ over $k[t]$ is an associative $k[t]$-algebra $(B_t, \ast)$ that is isomorphic to $B[t]$ as a vector space, such that $B_t/tB_t$ is isomorphic to $B$ as a $k$-algebra. In particular, we can write the multiplication of two elements $b_1, b_2 \in B \otimes t^0 \subset B_t$ as:

$$b_1 \ast b_2 = b_1 b_2 + \sum_{j>0} \mu_j(b_1, b_2) t^j,$$

where $\mu_j : B \otimes B \to B$ is $k$-linear and only finitely many terms are nonzero in the above sum.

If moreover $B$ is $Z^{\geq 0}$-graded, then a graded $k[t]$-deformation of $B$ is a deformation of $B$ over $k[t]$ that is graded with $\deg t = 1$, i.e., each $\mu_j : B \otimes B \to B$ is homogeneous of degree $-j$. The map $\mu_j$ is also called the $j$th multiplication map.

Henceforth in this section we will consider the special case of the $Z^{\geq 0}$-graded algebra $B := \mathcal{H}_{0,0} = \text{Sym}(V) \rtimes A$, with $(A, \Delta)$ a cocommutative algebra as above. Our first goal in this section is to show that the PBW property for the algebras $\mathcal{H}_{\lambda, \kappa}$ has a natural reformulation in terms of graded deformations of $\mathcal{H}_{0,0}$ over $k[t]$. Such a result was shown in [37 §6] in the special case of $A$ a group algebra, and further assuming that $\kappa_V \equiv 0$. We now explain how the assumption $\kappa_V \equiv 0$ is related to that in loc. cit. of requiring $V \otimes V \subset \ker \mu_1$, by extending the result to general $\kappa_V : V \otimes V \to V$ and all cocommutative algebras $A$.

Theorem 3.1. Suppose $k$ is a field (of arbitrary characteristic), $(A, \Delta)$ is cocommutative, and $V$ an $A$-module. Consider the following two statements.

1. $\mathcal{H}_{\lambda, \kappa}$ satisfies the PBW Theorem 2.5.
(2) There exists a graded \( k[t] \)-deformation \( B_t \) of \( B := \mathcal{H}_{0,0} = \text{Sym}(V) \ltimes A \), whose multiplication maps \( \mu_1, \mu_2 \) satisfy (for all \( v, v' \in V \) and \( a \in A \)):

\[
\lambda(a, v) = \mu_1(a \otimes v) - \sum \mu_1(a_{(1)}(v) \otimes a_{(2)}) ,
\]

\[
\kappa_v(v, v') = \mu_1(v \otimes v') - \mu_1(v' \otimes v),
\]

\[
\kappa_A(v, v') = \mu_2(v \otimes v') - \mu_2(v' \otimes v).
\]

Then \((1) \implies (2)\), and the converse holds if \( \dim A, \dim V \) are both finite. Moreover, if these statements hold then \( \mathcal{H}_{\lambda, \kappa} \cong B_{t \mid t = 1} \).

Thus, the structure maps \( \lambda, \kappa_v, \kappa_A \) in \( \mathcal{H}_{\lambda, \kappa} \) can be naturally reformulated using the multiplication maps \( \mu_1, \mu_2 \) in a graded deformation of \( \mathcal{H}_{0,0} \), whenever \( \mathcal{H}_{\lambda, \kappa} \) has the PBW property.

**Proof.** We provide a sketch of the proof as it closely resembles the arguments for proving [37] Proposition 6.5 and Theorem 6.11. First suppose (1) holds. Define \((B_t, \ast)\) to be the associative algebra over \( k[t] \) generated by \( A, V \), with the following relations (for all \( a \in A, v, v' \in V \)):

\[
a \ast v = \sum a_{(1)}(v) \ast a_{(2)} + \lambda(a, v)t ,
\]

\[
v \ast v' - v' \ast v = \kappa_v(v, v')t + \kappa_A(v, v')t^2.
\]

This yields a \( \mathbb{Z} \geq 0 \)-graded algebra with \( \deg(t) = \deg(V) = 1 \) and \( \deg(A) = 0 \). Moreover, \( B_t \cong \mathcal{H}_{0,0} \otimes k[t] \) as vector spaces, since \( \mathcal{H}_{\lambda, \kappa} \) has the PBW property. Now verify using the definitions and the relations in the algebra \((B_t, \ast)\), that

\[
\kappa_v(v, v')t + \kappa_A(v, v')t^2 = v \ast v' - v' \ast v = vv' + \sum_{j > 0} \mu_j(v \otimes v')t^j - v'v - \sum_{j > 0} \mu_j(v' \otimes v)t^j.
\]

As this is an equality of polynomials in \( \mathcal{H}_{0,0}[t] \), we equate the linear and quadratic terms in \( t \) on both sides, to obtain the last two equations in (3.1). The first equation in (3.1) follows from a similar computation. This shows (2), and moreover, \( B_{t \mid t = 1} \cong \mathcal{H}_{\lambda, \kappa} \).

Conversely, suppose (2) holds, and \( \dim V, \dim A < \infty \). Define \( F_t := T_{k[t]}(V \oplus A)/(a \cdot a' - aa') \); then we have an algebra map \( f : F_t \to B_t \), which sends monomials \( x_1 \cdots x_k \) (with each \( x_i \in V \oplus A \)) to \( x_1 \ast \cdots \ast x_k \). One shows as in [37] that \( f \) is surjective, and the vectors

\[
av - \sum a_{(1)}(v)a_{(2)} - \lambda(a, v)t = av - \sum a_{(1)}(v)a_{(2)} - \mu_1(a, v)t + \sum \mu_1(a_{(1)}(v) \otimes a_{(2)})t
\]

and

\[
v v' - v'v - \kappa_v(v, v')t - \kappa_A(v, v')t^2 = vv' - v'v - \mu_1(v, v')t + \mu_1(v', v)t - \mu_2(v, v')t^2 + \mu_2(v', v)t^2
\]

lie in \( \ker(f) \). We use here that \( a \ast v = av + \mu_1(a \otimes v)t \) and \( v \ast v' = vv' + \mu_1(v \otimes v')t + \mu_2(v \otimes v')t^2 \), since \( \deg \mu_j = -j \) for all \( j > 0 \).

This analysis implies that \( \mathcal{H}_{\lambda, \kappa, t} \to B_t \) as \( \mathbb{Z} \geq 0 \)-graded \( k \)-algebras, where \( \mathcal{H}_{\lambda, \kappa, t} \) is the quotient of \( F_t \) by the relations

\[
av - \sum a_{(1)}(v)a_{(2)} - \lambda(a, v)t, \quad vv' - v'v - \kappa_v(v, v')t - \kappa_A(v, v')t^2.
\]

Now using that \( A, V \) are finite-dimensional, verify that the graded components of the two algebras satisfy: \( \deg \mathcal{H}_{\lambda, \kappa, t}[m] \leq \deg B_t[m] \). Hence the dimensions agree
for each $m$, whence $\mathcal{H}_{\lambda,\kappa,t} \cong B_t$. It follows that $\mathcal{H}_{\lambda,\kappa} = \mathcal{H}_{\lambda,\kappa,t}|_{t=1} \cong B_t|_{t=1}$ as filtered algebras. Now as explained at the end of the proof of [37] Theorem 6.11, $\mathcal{H}_{\lambda,\kappa}$ has the PBW property. □

4. The case of bialgebras and Hopf algebras

In this section we study a special case of the general framework above, but now requiring that $A$ is a cocommutative bialgebra (with counit $\varepsilon$), or Hopf algebra (with counit $\varepsilon$ and antipode $S$). This is indeed the case in a large number of prominent and well-studied examples in the literature, as discussed after Definition 2.3.

We begin by observing that the cocommutative algebra structure on $A$ automatically extends to $\mathcal{H}_{0,0} = \text{Sym}(V) \rtimes A$, setting $\Delta(v) = v \otimes 1 + 1 \otimes v$ for all $v \in V$. Akin to the usual Hopf-theoretic setting, we now introduce the following notation.

**Definition 4.1.** Given a cocommutative algebra $(A, \Delta)$, an element $a \in A$ is said to be primitive (respectively, grouplike), if $\Delta(a) = 1 \otimes a + a \otimes 1$ (respectively, $\Delta(a) = a \otimes a$).

We now observe that it is possible to classify when the deformed algebra $\mathcal{H}_{\lambda,\kappa}$ is a cocommutative algebra, a bialgebra, or a Hopf algebra, under the assumption that $A$ has the same structure and $V$ is primitive.

**Proposition 4.2.** $(A, \Delta)$ and $V$ as above. Fix $\lambda : A \otimes V \to A$ and $\kappa = \kappa_A \oplus \kappa_V : V \wedge V \to A \oplus V$ as above.

1. Then $\mathcal{H}_{\lambda,\kappa}$ is a cocommutative algebra with (the image of) $V$ primitive, if

\[
\Delta(\lambda(a, v)) = \sum \lambda(a(1), v) \otimes a(2) + \sum a(1) \otimes \lambda(a(2), v), \quad \kappa_A(v, v') \text{ is primitive},
\]

for all $v, v' \in V$, $a \in A$. The converse is true if $\mathcal{H}_{\lambda,\kappa}$ has the PBW property.

2. Suppose $A$ is a cocommutative bialgebra (with counit $\varepsilon$). Then $\mathcal{H}_{\lambda,\kappa}$ is a cocommutative bialgebra with $V$ primitive, if (4.1) holds and $\text{im} \lambda \subset \text{ker} \varepsilon$.

The converse is true if $\mathcal{H}_{\lambda,\kappa}$ has the PBW property.

3. Suppose $A$ is a cocommutative Hopf algebra (with counit $\varepsilon$ and antipode $S$). Then $\mathcal{H}_{\lambda,\kappa}$ is a cocommutative Hopf algebra with $V$ primitive, if (4.1) holds and moreover,

\[
\text{im} \lambda \subset \text{ker} \varepsilon, \quad S(\lambda(a, v)) = \sum \lambda(S(a(1)), a(2)(v)).
\]

The converse is true if $\mathcal{H}_{\lambda,\kappa}$ has the PBW property.

In particular, notice that in all three cases, the structure on $A$ automatically extends to $\mathcal{H}_{0,0} = \text{Sym}(V) \rtimes A$, and more generally, to all $\mathcal{H}_{\lambda,\kappa}$ for which $\text{im} \kappa_A$ is primitive.

**Proof.** To prove the first part, suppose $\mathcal{H}_{\lambda,\kappa}$ has the PBW property. If $V$ is primitive, then we compute in the algebra $\mathcal{H}_{\lambda,\kappa} \otimes \mathcal{H}_{\lambda,\kappa}$:

\[
\Delta(\lambda(a, v)) = \Delta(av) - \sum \Delta(a(1)(v))a(2)
\]

\[
= \Delta(a)\Delta(v) - \sum \Delta(a(1)(v))\Delta(a(2))
\]

\[
= \sum \lambda(a(1), v) \otimes a(2) + \sum a(1) \otimes \lambda(a(2), v),
\]

for each $m$, whence $\mathcal{H}_{\lambda,\kappa,t} \cong B_t$. It follows that $\mathcal{H}_{\lambda,\kappa} = \mathcal{H}_{\lambda,\kappa,t}|_{t=1} \cong B_t|_{t=1}$ as filtered algebras. Now as explained at the end of the proof of [37] Theorem 6.11, $\mathcal{H}_{\lambda,\kappa}$ has the PBW property. □
and similarly,
\[
\Delta(\kappa_A(v, v')) - (1 \otimes \kappa_A(v, v') + \kappa_A(v, v') \otimes 1)
\]
\[
= \Delta(\kappa_A(v, v')) + \Delta(\kappa_V(v, v')) - (1 \otimes \kappa(v, v') + \kappa(v, v') \otimes 1)
\]
\[
= \Delta([v, v']) - (1 \otimes [v, v'] + [v, v'] \otimes 1) = 0.
\]
Since \( \mathcal{H}_{\lambda, \kappa} \) has the PBW property, the above equalities in fact hold inside \( V \otimes A \) and \( A \otimes A \), which inject into \( \mathcal{H}_{\lambda, \kappa} \otimes \mathcal{H}_{\lambda, \kappa} \) by Theorem 2.5. To prove the converse, even when \( \mathcal{H}_{\lambda, \kappa} \) need not have the PBW property, one uses essentially the same computations as above (but slightly rearranged).

This proves the first part. For the second part, that \( \varepsilon(\text{im} \kappa_A) = 0 \) follows from its primitivity, and that \( \varepsilon(\text{im} \lambda) = 0 \) follows from applying \( \varepsilon \) to the defining relations. The third part now follows from the following computation (using that \( S|_V = -\text{id}_V \) as \( V \) is primitive):
\[
S(\lambda(a, v)) = S(a)S(v) - \sum S(a_{(1)}(v))S(a_{(2)})
\]
\[
= (-v)S(a) + \sum S(a_{(2)})a_{(1)}(v))S(a_{(3)}) + \sum \lambda(S(a_{(2)}), a_{(1)}(v)),
\]
and now applying the cocommutativity of \( A \), to cancel the first two expressions. \( \square \)

4.1. Symplectic reflections in bialgebras. Our next goal is to show that the notion of “symplectic reflections” generalizes to arbitrary cocommutative bialgebras. The following result extends to such a setting, its group-theoretic counterparts in \([12][14]\).

**Proposition 4.3.** Suppose \( \mathbb{k} \) is a field, and \( (A, \Delta, \varepsilon) \) is a cocommutative \( \mathbb{k} \)-bialgebra. Suppose \( \kappa_V \) is \( \mathbb{k} \)-cocommutative, and \( \mathcal{H}_{\lambda, \kappa} \) has the PBW property. Given \( 0 \neq a' \in A \), suppose there exists nonzero \( a'' \in A \) and a vector space complement \( U \) to \( \mathbb{k}a'' \) in \( A \) such that
\[
\Delta(\text{im} \kappa_A) \subset \mathbb{k}(a' \otimes a'') \oplus (A \otimes U),
\]
but \( \Delta(\text{im} \kappa_A) \not\subset A \otimes U \). Then \( a' - \varepsilon(a') \in \text{End}_\mathbb{k} V \) has image with dimension at most 2.

In other words, if \( \kappa_A \) is supported on \( a' \otimes a'' \), then \( a' - \varepsilon(a') \) is akin to a symplectic reflection \([14]\). For instance, for symplectic reflection algebras as in \([12][14]\), with \( A = \mathbb{k}W \) a group ring, if \( a' = g \in W \), then choose \( U := \sum_{g' \neq g} \mathbb{k}g' \).

**Proof.** We may assume throughout that \( a' \neq \varepsilon(a') \). By choice of \( a' \), there exist \( x, y \in V \) such that \( \Delta(\kappa_A(x, y)) - r(a' \otimes a'') \in A \otimes U \), for some \( r \in \mathbb{k}^\times \). We now claim that for all \( v \in V \),
\[
(a' - \varepsilon(a'))(v) \in \mathbb{k}v_x + \mathbb{k}v_y, \quad \text{where} \quad v_x := (a' - \varepsilon(a'))(x), \quad v_y := (a' - \varepsilon(a'))(y).
\]
To show the claim, consider the Jacobi identity \((2.7)\) for \( v_1 = x, v_2 = y, v_3 = v \), which yields:
\[
\sum \left( \kappa_A(v_1, v_2)(1) - \varepsilon(\kappa_A(v_1, v_2)(1)) \right) (v_3) \kappa_A(v_1, v_2)(2) = 0.
\]
Denote the summand by \( f(x, y, v) \). Now split the term \( \kappa_A(x, y) \) (and the other two cyclically permuted such terms) into their \( a' \otimes a'' \)-components and \( A \otimes U \)-components. Hence there exist \( r_{xy} = r, r_{yx}, r_{xx} \in \mathbb{k} \) such that by the PBW property,
\[
\varepsilon (a' - \varepsilon(a'))(v) \otimes a'' + r_{yx}(a' - \varepsilon(a'))(x) \otimes a'' + r_{xx}(a' - \varepsilon(a'))(y) \otimes a'' \in V \otimes U.
\]
This shows that the left-hand side vanishes. The claim now follows by the PBW property. □

4.2. Yetter–Drinfeld condition. In the remainder of this section, we work with Hopf algebras. Assume throughout this subsection that $A$ is a $k$-free cocommutative $k$-Hopf algebra, and $V$ is a $k$-free $A$-module. In this case it is easy to verify that the $A$-action on $TV$ (respectively, $\text{Sym}(V)$) agrees with the adjoint action of $A$: $\text{ad} a(x) := \sum a(1)xS(a(2)) = a(x)$, for $x \in TV$ (respectively, $\text{Sym}(V)$).

Our goal is to show that one of the conditions in Theorem 2.5 required for the PBW property to hold is equivalent to a compatibility condition called the Yetter–Drinfeld condition (see e.g. [1] Theorem 3.3). To state the result, we require some preliminaries.

**Proposition 4.4.** Suppose a $k$-Hopf algebra $A$ acts on a free $k$-module $V$, and a $k$-algebra $B$ contains $A, V$.

(1) Then the following relations in $B$ are equivalent for all $v \in V$:
(a) $\sum a(1)vS(a(2)) = a(v)$ for all $a \in A$.
(b) $av = \sum a(1)(v)a(2)$ for all $a \in A$.

If $A$ is cocommutative, then both of these are also equivalent to:
(c) $va = \sum a(1)S(a(2))(v)$ for all $a \in A$.

Now suppose in the remaining parts that the conditions (a),(b) hold.

(2) Suppose $A$ is cocommutative. Then $\tau : A \otimes V \rightarrow V \otimes A$, given by $a \otimes v \mapsto \sum a(1) \otimes a(2)$, as well as $\tau^{op} : V \otimes A \rightarrow A \otimes V$, given by $v \otimes a \mapsto \sum a(1) \otimes S(a(2))(v)$, are $A$-module isomorphisms that are inverse to one another.

(3) Any unital subalgebra $M$ of $B$ that is also an $A$-submodule (via $\text{ad}$), is an $A$-(Hopf) module algebra under the action
$$a(m) := \text{ad} a(m) = \sum a(1)mS(a(2)) \ \forall a \in A, \ m \in M.$$ 

The proof of the following result is standard and is hence omitted. The result may be applied to $B = \mathcal{H}_{\epsilon,k}$. Note as in [36] §4 that the map $\tau$ is an isomorphism of the Yetter–Drinfeld modules $A \otimes V$ and $V \otimes A$, called the “braiding”.

The following preliminary result can (essentially) be found in [28] Lemma 1.3.3. To state the result, recall that given a module $M$ over a Hopf $k$-algebra $A$, the $\epsilon$-weight space $M_{\epsilon}$ is $\{m \in M : a \cdot m = \epsilon(a)m \ \forall a \in A\}$.

**Lemma 4.5.** Given a Hopf algebra $A$ and a $k$-algebra map $\varphi : A \rightarrow B$, the centralizer of $\varphi(A)$ in $B$ is the weight space $B_{\epsilon}$ (where $B$ is an $A$-module via: $a \cdot b := \sum \varphi(a(1))bS(a(2))$).

Consequently, the deformation $\mathcal{H}_{\epsilon,k}$ is commutative if and only if $A = A_{\epsilon}$ under the adjoint action (equivalently, $A$ is commutative), $V = V_{\epsilon}$ (under the given $A$-action), and $k \equiv 0$.

We now discuss the Yetter–Drinfeld condition in detail. In the following result, $\tau^{op} : M \otimes A \rightarrow A \otimes M$ is defined as in Proposition 4.4(2), and $A^{ad}, A^{\text{mult}}$ refer to different $A$-module structures on $A$ (via the adjoint action, and via left multiplication respectively).
PROP 4.6. Suppose $A$ is a Hopf $k$-algebra, $V, M$ are $k$-free $A$-modules, and $\kappa \in \text{Hom}_k(V \otimes V, M)$. Suppose $(B, \mu_B, 1_B)$ is an (associative) $k$-algebra containing $A, M$, with the additional relations $m \cdot a = \mu_B(\tau^\text{op}(m \otimes a))$ in $B$. The following are equivalent in $B$:

1. $\kappa : V \otimes V \rightarrow M$ is $A$-equivariant, or an $A$-module map:
   \[ a(\kappa(v, v')) = \sum \kappa(a_{(1)}(v), a_{(2)}(v')) \forall a \in A, v, v' \in V. \]

2. $\kappa$ satisfies the Yetter–Drinfeld (compatibility) condition, i.e.
   \[ \tau^\text{op} \left( \sum \kappa(a_{(1)}(v), v')a_{(2)} \right) = \sum a_{(1)} \kappa(v, S(a_{(2)})(v')) \forall a \in A, v, v' \in V. \]

3. $\kappa$ is $A$-compatible: $\alpha(\kappa(v, v')) = \sum \kappa(a_{(1)}(v), a_{(2)}(v'))a_{(3)} \forall a, v, v'$.

4. $\kappa$ satisfies: $\kappa(v, v') = \sum a_{(1)} \kappa(S(a_{(2)})(v), S(a_{(3)})(v')) \forall a, v, v'$.

5. $\text{im } \kappa$ commutes (in $B$) with all of $A$.

The proof is a relatively straightforward exercise in computations involving Hopf algebras, and is hence omitted. We remark that the proof uses Proposition 4.4, Lemma 4.5 and that $A$ is cocommutative.

To conclude this section, we point out how the Yetter–Drinfeld condition arises, as in [11 Theorem 3.3]: in the associative algebra $B$ above, compute $v' \cdot a \cdot v$ in two different ways (i.e. using the maps $\tau, \tau^\text{op}, \kappa$). Then,
\[ \sum a_{(1)} \kappa(v, S(a_{(2)})(v')) = \sum a_{(1)}(v)a_{(2)}S(a_{(3)})(v') - v'av = \sum \kappa(a_{(1)}(v), v')a_{(2)}, \]
and this is precisely the Yetter–Drinfeld condition.

5. Generalized nil-Coxeter algebras and grouplike algebras

In the remainder of this paper, we introduce a class of cocommutative algebras that incorporates group algebras as well as nil-Coxeter algebras and their generalizations, which are necessarily not bialgebras or Hopf algebras. We then study the Jacobi identity in detail; this is useful in classifying PBW deformations over nil-Coxeter algebras.

We begin by setting notation concerning unitary/complex reflection groups.

DEF 5.1. A Coxeter matrix is a symmetric matrix $A := (a_{ij})_{i,j \in I}$ indexed by a finite set $I$ and with integer entries, such that $a_{ii} = 1$ and $2 \leq a_{ij} \leq \infty$ for all $i \neq j$. Given a Coxeter matrix $A$, define the corresponding braid group $B_W = B_{W(A)}$ to be the group generated by simple reflections $\{s_i : i \in I\}$, satisfying the braid relations $s_is_js_i \cdots = s_js_is_j \cdots$ for all $i \neq j$, with precisely $a_{ij}$ factors on either side. Finally, define the Coxeter group $W = W(A)$ to be the quotient of the braid group by the additional relations $s_i^2 = 1 \forall i$. More broadly, given an integer tuple $d$ with $d_i \geq 2 \forall i \in I$, define the corresponding generalized Coxeter group $W(d)$ to be the quotient of $B_{W(A)}$ by $s_i^{d_i} = 1 \forall i$.

We now introduce the corresponding families of generalized (nil-)Coxeter groups and algebras. This involves considering the “non-negative part” of the braid group, i.e., the Artin monoid.
Definition 5.2. Given a Coxeter matrix $A$, first define the Artin monoid $B_{W_A}^{20}$ to be the monoid generated by $\{T_i : i \in I\}$ modulo the braid relations. Now given an integer vector $d = (d_i)_{i \in I}$ with each $d_i \geq 2$, define the generalized nil-Coxeter algebra $NC_{W_A}(d)$ as:

\[
NC_{W_A}(d) := \frac{k\langle T_i, i \in I \rangle}{\prod_{a_{ij} \text{ times}} T_i T_j T_i^{-1}, T_i^{d_i} = 0, \forall i \neq j \in I \rangle} = \frac{kB_{W_A}^{20}}{(T_i^{d_i} = 0 \forall i)}.
\]

Remark 5.3. The algebras $NC_W(d)$ provide a large family of examples of cocommutative algebras via $\Delta(T_i) := T_i \otimes T_i$ for all $i \in I$ (and extending $\Delta$ by multiplicativity). Moreover, no algebra $NC_W(d)$ can be a (weak) bialgebra under this coproduct. This is because any counit $\varepsilon$ necessarily maps the nilpotent element $T_i$ to 0; but $T_i$ is grouplike so $\varepsilon(T_i) = 1$.

Generalized nil-Coxeter algebras $NC_W(d)$ include the well-studied case (see the Introduction) of the nil-Coxeter algebra $NC_W$, where $d_i = 2$ $\forall i$. Note that $\dim NC_W(d) \geq NC_W$, as $NC_W(d)$ surjects onto $NC_W$. Moreover, if $W$ is finite, then $\dim NC_W((2, \ldots, 2)) = |W| < \infty$; see e.g. Chapter 7. Notice that there are other finite-dimensional algebras of the form $NC_W(d)$. For instance, $NC_{A_1}(d) \cong k[T_1]/(T_1^d)$ is finite-dimensional; hence, so is the algebra $NC_{A_1}(d_1, \ldots, d_n)$ with all $d_i \geq 2$. This question is completely resolved in related work [25], where we characterize the generalized nil-Coxeter algebras $NC_W(d)$ that are finite-dimensional. We show that apart from the usual nil-Coxeter algebras $NC_W((2, \ldots, 2))$, there is precisely one other family of type-A algebras, $NC_A((2, \ldots, 2, d))$ with $d > 2$, which are finite-dimensional. See [25] Theorems A,C] for further details.

5.1. Grouplike algebras. We begin by unifying the group algebras $kW$ and the algebras $NC_W(d)$ (as well as other algebras considered in the literature) in the following way.

Definition 5.4. A grouplike algebra is a unital $k$-algebra $A$, together with a distinguished $k$-basis $\{T_m : m \in M_A\}$ containing the unit $1_A$, such that the map $\Delta : A \otimes A, \ T_m \mapsto T_m \otimes T_m$ is an algebra map.

Remark 5.5. Observe from the definitions that the grouplike elements $g := \sum_{m \in M_A} c_m T_m$ in a grouplike algebra $A$ can all be easily identified. Indeed, if $g \neq 0$ and $k$ is a domain, then

\[
\sum_{m,m' \in M_A} c_m c_{m'} T_m \otimes T_{m'} = \Delta(g) = \sum_{m \in M_A} c_m T_m \otimes T_m,
\]

from which it follows that the sum is a singleton, with coefficient 1. Thus $g = T_m$ for some $m$. As a consequence, it follows that the set $\{T_m : m \in M_A\} \cup \{0\}$ is closed under multiplication, making it a monoid with both a unit and a zero element. This is formalized presently.

Notice that every grouplike algebra is a cocommutative algebra with coproduct. (Henceforth we will suppress the monoid operation $*$ when it is clear from context.) As we presently show, generalized Coxeter groups and generalized nil-Coxeter algebras are examples of grouplike algebras. First we introduce the following notation.
We now present several examples of (cocommutative) grouplike algebras.

1. Every monoid algebra \( kM \) is a grouplike algebra, using \( T_m := m \) for all \( m \). This includes the group algebra of every (generalized) Coxeter group.

2. Suppose \( M \) contains a zero element \( 0_M \). Then \( k0_M \) is a two-sided ideal in the monoid algebra \( kM \), and so \( kM/k0_M \) is also a grouplike algebra with basis \( \{ T_m : m \in M \setminus 0_M \} \). The previous example is a special case, since to each monoid \( M \) we can formally attach a zero element 0, to create a new monoid with zero element 0.

3. Another special case of the preceding example is a nil-Coxeter algebra \( NC_W \). This corresponds to the monoid \( W \sqcup \{ 0_W \} \), with \( T_w * T_w' := 0_W \) if \( \ell(ww') > \ell(w) + \ell(w') \) in \( W \). More generally, define for \( k \in \mathbb{N} \) the ideal \( I_k \) to be the k-span of \( \{ T_w : \ell(w) \geq k \} \). Then \( NC_W/I_k \) is a grouplike algebra with distinguished basis \( \{ T_w : \ell(w) < k \} \).

4. The generalized nil-Coxeter algebra \( NC_{A^+}((d_1, \ldots, d_n)) \), with \( d_i \geq 2 \) for all \( i \), is yet another example of the above construction. In this case we use the monoid

\[
M := \{ 0 \} \sqcup \times_i \{ 1, \ldots, d_i - 1 \},
\]

with \( (e_i)_i * (e'_i)_i \) equal to \( (e_i + e'_i)_i \) if \( \max_i(e_i + e'_i - d_i) < 0 \), and 0 otherwise.

5. As a final example, recall the 0-Hecke algebra

\[
H_W(0) := \frac{kB_W^{\geq 0}}{(T_i^2 - T_i \forall i \in I)},
\]

where \( B_W^{\geq 0} \) is as in Definition 5.2. This algebra was defined in [33] and has been extensively studied since; see [16, 20, 39] and the references therein. We recall from [21] that \( H_W(0) \) is the monoid algebra of a monoid in bijection with \( W \). As we presently show, it is also a grouplike algebra with distinguished basis \( \{ T_w : w \in W \} \).

Given the profusion of Coxeter-theoretic examples above, it is desirable to consider a subclass of grouplike algebras that incorporates them all in a systematic manner. We now present such a family.

**Definition 5.7.** Given a Coxeter matrix \( A \) and an integer vector \( \mathbf{d} \) with \( 2 \leq d_i \leq \infty \) \( \forall i \), a generic Hecke algebra is any algebra of the form

\[
E_W(\mathbf{d}, \mathbf{p}) := \frac{kB_W^{\geq 0}}{(T_i^{d_i} - p_i(T_i) \forall i \in I)},
\]

where \( W = W_A \), and \( p_i \in k[T] \) has degree at most \( d_i - 1 \) for \( i \in I \).

These algebras are so named after the family of “generic Hecke algebras” studied in [7, 8]; however, unlike loc. cit., we do not require the \( p_i \) to be equal when
Thus, we now study when generic Hecke algebras provide examples of such algebras.

**Proposition 5.8.** Suppose $\mathbb{k}$ is a domain, $W = W_A$ is a Coxeter group, and $d, p$ are as in Equation (5.3).

1. The map $\Delta : T_i \mapsto T_i \otimes T_i$ extends to make $\mathcal{E}_W(d, p)$ a (cocommutative) group-like algebra, if for all $i \in I$, $p_i(T)$ is either zero or equals $T^{e_i}$ for some $0 \leq e_i < d_i$.
2. $\mathcal{E}_W(d, p)$ is a bialgebra if for all $i \in I$, $p_i(T) = T^{e_i}$ for some $0 \leq e_i < d_i$.
3. $\mathcal{E}_W(d, p)$ is a Hopf algebra if $p_i(T) = 1 \, \forall i \in I$.

The converse statements are all true if for all $i$, the vectors $1, T_i, \ldots, T_i^{d_i-1}$ are $\mathbb{k}$-linearly independent in $\mathcal{E}_W(d, p)$.

Notice that the last condition is not always true. For instance, standard arguments as in [29 Introduction] show that the condition fails to hold in a generalized Coxeter group $W$ (or $\mathbb{k}W$ to be precise) whenever $a_{ij}$ is odd, $p_i = 1$ is constant for all $i$, and $d_i \neq d_j$. However, the condition does hold if group algebras, 0-Hecke algebras, and nil-Coxeter algebras corresponding to Coxeter groups.

**Proof.** We begin by showing the first three assertions. Suppose for all $i$ that $p_i(T) = 0$ or $T^{e_i}$ for some $0 \leq e_i < d_i$. Then it is easily verified that $\Delta : T_i \mapsto T_i \otimes T_i$ extends to the tensor algebra over the $T_i$, hence to the Artin monoid $\mathbb{k}B^{>0}_W$, and hence to $\mathcal{E}_W(d, p)$. Similarly one verifies that a counit that sends $T_i$ to 1 for all $i$, can be extended to $\mathcal{E}_W(d, p)$ if $p_i(T) = T^{e_i}$ for all $i$. Finally, an antipode that sends $T_i$ to $T_i^{-1} = T_i^{d_i-1}$ can be extended to $\mathcal{E}_W(d, p)$.

The “converse” statements are slightly harder to show. Suppose $1, T_i, \ldots, T_i^{d_i-1}$ are $\mathbb{k}$-linearly independent in $\mathcal{E}_W(d, p)$. To show (the converse of) (1), notice that every algebra of the form $\mathcal{E}_W(d, p)$ is a quotient of $\mathbb{k}B^{>0}_W$, so it suffices to classify the polynomials $p_i$ such that the ideal generated by all $T_i^{d_i} - p_i(T_i)$ is a coideal. Define $p_i(T) := \sum_{j=0}^{d_i-1} p_{ij} T^j_i$, and compute using the multiplicativity of $\Delta$:

\[
\Delta(T_i^{d_i}) = T_i^{d_i} \otimes T_i^{d_i} = \sum_{j,k=0}^{d_i-1} p_{ij} p_{ik} T_j^i \otimes T_k^i, \quad \sum_{j=0}^{d_i-1} \Delta(p_{ij} T_j^i) = \sum_{j=0}^{d_i-1} p_{ij} T_j^i \otimes T_j^i.
\]

It follows by the assumptions that each nonzero $p_i(T)$ is a monomial $p_{ij} T^j_i$, with $p_{ij}^2 = p_{ij}$ in the domain $\mathbb{k}$. This proves (1). To show (2), it suffices to produce a counit $\varepsilon$ that is compatible with the coproduct. Since $T_i$ is group-like, it follows that $\varepsilon(T_i)$ must equal 1 for all $i$. This is indeed compatible with the relations $T_i^{d_i} = T_i^{e_i}$, which shows one implication. On the other hand, the relation $T_i^{d_i} = 0$ implies $\varepsilon(T_i) = 0$, a contradiction.

Finally, we show (3). If $p_i(T) = 1$ for all $i$ then $\mathcal{E}_W(d, p)$ is a group algebra, hence a Hopf algebra. Conversely, suppose $p_i(T) = T^{e_i}$ for some $0 < e_i < d_i$ and $i \in I$. Then from above, the subalgebra generated by $T_i$ is isomorphic to $\mathbb{k}[T]/(T^{d_i} - T^{e_i})$, which surjects onto the algebra $\mathbb{k}[T]/(T^2 - T)$. This is precisely the 0-Hecke algebra of type $A_1$, in which one knows that $T$ is not invertible, yet $T$ is group-like. Thus $T_i$ is not invertible in $\mathcal{E}_W(d, p)$. \(\square\)
Remark 5.9. Let $A := \mathcal{E}_W(d, p)$. If $p_i(T) = 0 \ \forall i$, and $M := \text{span}_k \{ T_i : i \in I \}$, then $AM = MA = AMA =: \mathfrak{m}$ is a maximal ideal of $A$. This is because $\mathfrak{m}$ is a quotient of the tensor algebra $T_k M$, by relations that strictly lie in the augmentation ideal $T_k^+ M$.

5.2. The Jacobi identity for grouplike algebras. Having defined grouplike algebras and presented examples of them, we specialize the conditions in the PBW Theorem 2.5 to such a setting. For instance, if $\lambda, \kappa$ are identically zero, and $A$ is a group algebra $kG$ as in [12][14], then defining $\kappa_A(v, v') := \sum_{g \in G} \kappa_g(v, v') T_g$, we see easily that the $A$-compatibility of $\kappa_A$ is equivalent to the following condition found in loc. cit.:

$$\kappa_{gh^{-1}}(T_g(v), T_g(v')) = \kappa_g(v, v'), \quad \forall g, h \in G, \ v, v' \in V.$$  

Our goal in the remainder of this section is to study the Jacobi identity (2.7) in the case $\kappa_V \equiv 0$, over a grouplike algebra $A$.

Standing Assumption 5.5. For the remainder of this section, $k$ is a field and $\kappa_V \equiv 0$.

We begin by setting notation. Define the fixed point space of $a \in A$ and its codimension:

$$\text{Fix}(a) := \{ v \in V : a(v) = v \}, \quad d_a := \text{codim}_V \text{Fix}(a).$$

Thus, $d_a = \dim_k \text{im}(\text{id}_V - a)$.

Now suppose we have fixed a $k$-basis $\{ a_j : j \in J_1 \}$ of $A$. Then we will write

$$\kappa(x, y) = \kappa_A(x, y) := \sum_{j \in J_1} \kappa_j(x, y)a_j, \quad \forall x, y \in V.$$  

Thus, $\kappa_j$ is a skew-symmetric bilinear form on $V$. We also define $\text{Rad}(\kappa_j)$ to be the radical of the bilinear form, $\text{Rad}(\kappa_j) := \{ v \in V : \kappa_j(v, V) = 0 \}$. Specifically, this notation will be applied to a grouplike algebra $A$ with a distinguished basis $\{ T_m : m \in M_A \}$ of grouplike elements; see Remark 5.5. In this setting, we will write $\kappa_{T_m} = \kappa_m$ and $d_{T_m} = d_m$.

We now characterize the Jacobi identity in this general setting.

Theorem 5.10. Suppose $\kappa_V \equiv 0$.

1. Suppose $A$ contains a grouplike element $T_m$ and a vector space complement $V_0$ to $kT_m$, such that $\Delta(V_0) \subset V_0 \otimes V_0$. Extend $T_m$ to any basis of $V_0$. Now if the Jacobi identity (2.7) holds in $\mathcal{H}_{\lambda, \kappa}$ (with $\kappa_V \equiv 0$), then one of the following conditions holds:

(a) $\kappa_m \equiv 0$.

(b) $T_m \equiv \text{id}_V$, i.e. $d_m = 0$.

(c) $d_m$ is 1 or 2, and $\text{Rad}(\kappa_m)$ is a subspace of $\text{Fix}(T_m)$, of codimension $2 - d_m$.

2. Conversely, if $A$ is a grouplike algebra with distinguished $k$-basis $\{ T_m : m \in M_A \}$ of grouplike elements, and for each $m \in M_A$ one of the above three conditions holds, then the Jacobi identity (2.7) holds in $\mathcal{H}_{\lambda, \kappa}$ (with $\kappa_V \equiv 0$).

For completeness, we remark that part (1) extends to arbitrary grouplike algebras a result found in [12][14] for $A$ a group algebra; see also [18][37].
PROOF. Write out the Jacobi identity (2.7) using the distinguished \( k \)-basis of 
\( A \), and isolate the \( T_m \)-component to get:
\[
\sum \circ \ k_m(v_2, v_3) = \sum \circ \ k_m(v_2, v_3) T_m(v_1),
\]
or equivalently, for all \( x, y, z \in V \),
\[
\kappa_m(y, x)(id_V - T_m)(z) = \kappa_m(y, z)(id_V - T_m)(x) + \kappa_m(z, x)(id_V - T_m)(y).
\] (5.8)

Before proving the two parts, we make two observations. First, it follows from (5.8) that \( \kappa_m \equiv 0 \) or \( \text{Rad}(\kappa_m) \subset \text{Fix}(T_m) \). Moreover, if \( \text{Rad}(\kappa_m) \subset \text{Fix}(T_m) \) has codimension at most 1, then by the skew-symmetry of \( \kappa_m \) it is clear that \( \text{Fix}(T_m) \) is \( \kappa_m \)-isotropic.

(1) Suppose the Jacobi identity holds. Assume \( \kappa_m \) is not identically zero; thus, choose \( x, y \) so that \( \kappa_m(y, x) \neq 0 \). Then Equation (5.8) implies that
\[
\text{im}(id_V - T_m) \subset \ker(x' + ky', \text{ where } x' := (id_V - T_m)(x) \text{ and } y' := (id_V - T_m)(y)).
\] (This is similar to the proof of Proposition 4.3.) In particular, \( d_m = \dim \text{im}(id_V - T_m) \leq 2 \) if \( \kappa_m \neq 0 \).

If \( d_m = 0 \) then assertion (b) holds, so we may assume now that \( d_m \) is 1 or 2. Also notice by Equation (5.8) that \( \text{Rad}(\kappa_m) \subset \text{Fix}(T_m) \), so it remains to show that the codimension is \( 2 - d_m \).

First suppose \( d_m = 2 \), whence \( x', y' \) are linearly independent. We claim that \( \text{Rad}(\kappa_m) \supset \text{Fix}(T_m) \). Indeed, suppose \( z \in \text{Fix}(T_m) \). Then Equation (5.8) yields:
\[
\kappa_m(y, z)x' + \kappa_m(z, x)y' = 0.
\] (5.9)

Similarly, replacing \( x \) by \( z' \in \ker(id_V - T_m) \) yields: \( \kappa_m(z', s')y' = 0 \). From this and (5.9), it follows that \( \kappa_m(z, -) \) kills \( x, y \) as well as \( \ker(id_V - T_m) = \text{Fix}(T_m) \). Hence it kills their \( k \)-span, which is all of \( V \).

The final case is when \( d_m = 1 \). Fix \( v_1 \notin \text{Fix}(T_m) \); thus \( V = k v_1 \oplus \text{Fix}(T_m) \). We may assume \( v_1 \notin \text{Rad}(\kappa_m) \). Indeed, if instead \( \kappa_m(v_1, V) = 0 \), then \( \kappa_m(v_1, v_0) \neq 0 \) for some \( v_0, v_0' \in \text{Fix}(T_m) \), since \( \kappa_m \neq 0 \). Then \( \kappa_m(v_1 + v_0, v_0) \neq 0 \), so we can replace \( v_1 \) by \( v_1 + v_0' \). Proceeding, notice that \( \kappa_m(v_1, v_0) \neq 0 \) for some \( v_0 \in \text{Fix}(T_m) \). Now define \( V_0 := \{ v \in \text{Fix}(T_m) : \kappa_m(v_1, v) = 0 \} \); then \( \text{Fix}(T_m) = k v_0 \oplus V_0 \), and \( V_0 \supset \text{Rad}(\kappa_m) \) from the observations following (5.8). Finally, applying (5.8) to \( x, y \in \text{Fix}(T_m) \), \( x = v_1 \) shows that \( \text{Fix}(T_m) \) is \( \kappa_m \)-isotropic. Hence \( V_0 = \text{Rad}(\kappa_m) \).

(2) Conversely, suppose \( A \) is grouplike with basis \( \{ T_m : m \in M_A \} \) as given. We are to show that Equation (5.8) holds for all \( m \in M_A \). Certainly this holds if \( \kappa_m \equiv 0 \) or \( T_m = id_V \). Thus we assume henceforth that \( \kappa_m \neq 0 \), and show Equation (5.8) for a fixed \( m \in M_A \), in the two cases \( d_m = 1, 2 \).

First suppose \( d_m = 2 \), and \( x, y \in V \) are linearly independent modulo \( \text{Rad}(\kappa_m) \). Notice that \( \kappa_m(v, v') \) is nonzero only if \( v, v' \) are independent modulo \( \text{Rad}(\kappa_m) \), so it suffices to prove (5.8) with \( x, y \) as above, whence \( z = \alpha v + \beta v + v \) for some \( \alpha, \beta \in k \) and \( v \in \text{Rad}(\kappa_m) = \text{Fix}(T_m) \). In this case it is easily shown that both sides of (5.8) equal \( \kappa_m(y, x) \cdot (id_V - T_m)(\alpha v + \beta v) \).

Finally, suppose \( d_m = 1 \), with \( V \supset \text{Fix}(T_m) \supset \text{Rad}(\kappa_m) \) a chain of codimension one subspaces. Choose \( x \in V \setminus \text{Fix}(T_m) \) and \( y \in \text{Fix}(T_m) \setminus \text{Rad}(\kappa_m) \); once again, if \( \kappa_m(v, v') \) is nonzero we may replace \( v, v' \) by \( x, y \),
and set $z = \alpha x + \beta y + v$ for $v \in \text{Rad}(\kappa_m)$. Now both sides of (5.8) are equal to $\kappa_m(y, x) \cdot (\text{id}_V - T_m)(\alpha x)$.

\[\square\]

Theorem 5.10 is useful in characterizing PBW deformations, via the following consequence.

**Corollary 5.11.** Suppose $A$ contains a grouplike and nilpotent element $T_m$, and a vector space complement $V_0$ to $\mathbb{k}T_m$ such that $\Delta(V_0) \subset V_0 \otimes V_0$. If the Jacobi identity (2.7) holds in $\mathbb{H}_{\lambda, \kappa}$ with $\kappa_V \equiv 0$, then either $\kappa_m \equiv 0$ or $\dim \mathbb{k}V = 2$.

**Proof.** Since $\text{id}_V - T_m$ is invertible, Theorem 5.10(1) implies that either $\kappa_m \equiv 0$, or $d_m = \dim \mathbb{k}V$ and $\text{Rad}(\kappa_m) = \text{Fix}(T_m) = 0$, whence $d_m = 2$. \[\square\]

We conclude this section by specializing to the case of a generalized nil-Coxeter algebra $A = NC_W(d)$. Recall from Remark 2.7 that the condition $\dim \mathbb{k}V = 2$ is sufficient for the Jacobi identities (2.4),(2.7) to hold for $\mathbb{H}_{\lambda, \kappa}$. The following result shows that over $A = NC_W(d)$ and under the original setting of $\lambda, \kappa_V \equiv 0$ considered in (12),(13), either $\kappa_A$ is highly constrained, or else the condition $\dim \mathbb{k}V = 2$ is also necessary.

**Theorem 5.12.** Suppose $A = NC_W(d)$ is such that the maximal ideal $\mathfrak{m}$ generated by $\{T_i : i \in I\}$ is nilpotent. Given an $A$-module $M$, define $\text{Prim}(M) := \{m \in M : T_i m = 0 \forall i\}$.

1. If $\dim \mathbb{k}V \leq 2$, then $\mathbb{H}_{0, \kappa}$ has the PBW property if and only if $\text{im} \kappa_V \subset \text{Prim}(V)$ and $\text{im} \kappa_A \subset \text{Prim}(A^{\text{mult}})$.

2. If $\dim \mathbb{k}V > 2$, and $\lambda, \kappa_V \equiv 0$, then $\mathbb{H}_{0, \kappa}$ has the PBW property if and only if $\kappa_A \equiv 0$.

Thus (using Remark 2.7), if $\mathbb{H}_{0, \kappa}$ satisfies the PBW property for $A = NC_W(d)$ finite-dimensional, then either $\kappa_A \equiv 0$ or $\dim \mathbb{k}V = 2$.

We also provide examples of $\text{Prim}(\cdot)$ for generalized nil-Coxeter algebras. Indeed, $\text{Prim}(A^{\text{mult}})$ equals $\mathbb{k}T_{w_0}$ if $A = NC_W$ is the usual nil-Coxeter algebra over a finite Coxeter group $W$ with unique longest element $w_0$. If $A = NC_{A^\lambda_{\mathbb{A}}}((d_1, \ldots, d_n))$, then $\text{Prim}(A^{\text{mult}}) = \prod T_i^{d_i-1}$. In both of these cases, the maximal ideal $\mathfrak{m}$ is indeed nilpotent, and hence $A$ satisfies the hypotheses of the above theorem for these families of generalized nil-Coxeter algebras.

**Proof.** Suppose $\mathfrak{m}^n = 0 \neq \mathfrak{m}^{n-1}$ for some $n \in \mathbb{N}$. Before proving the result, we consider the following filtration on an $A$-module $V$:

\[V \supset \mathfrak{m}V \supset \mathfrak{m}^2V \supset \cdots \supset \mathfrak{m}^nV = 0.\]

We fix $k \leq n - 1$ such that $\mathfrak{m}^kV = 0 \neq \mathfrak{m}^{k-1}V$.

1. By Remark 2.7 and given that $\lambda \equiv 0$, it suffices to characterize the $A$-compatibilities (2.4),(2.5), assuming further that $\dim V = 2$. Now observe that $\mathfrak{m}^{k-1}V \subset \text{Prim}(V)$. Choose $v_0 \in \mathfrak{m}^{k-1}V$, and $v_1 \notin \mathbb{k}v_0$; thus $V = \mathbb{k}v_0 \oplus \mathbb{k}v_1$. Now notice that $\kappa_{|V\wedge V}$ is completely determined by $\kappa_{|V(v_0, v_1)}$, since $\dim V = 2$. Thus, we compute using the $A$-compatibility (2.4), for any non-trivial grouplike element $1 \neq T_m \in NC_W(d)$:

\[T_m \kappa_A(v_0, v_1) = \kappa_A(T_m(v_0), T_m(v_1))T_m = 0.\]
This equation holds for all non-unital $T_m$, if and only if $\kappa_m \equiv 0$ for $T_m \not\in \text{Prim}(A^{nul})$. Similarly, Equation (2.5) reduces to:

$$T_m(\kappa_V(v_0, v_1)) = \kappa_V(T_m(v_0), T_m(v_1))T_m = 0,$$

which holds if and only if $\kappa_V(v_0, v_1) \in \text{Prim}(V)$, as claimed.

(2) By Corollary 6.11, we see that $\kappa_A \equiv \kappa_1$, since each non-unital grouplike element $T_m$ is nilpotent by assumption. Now as above, Equation (2.4) reduces to:

$$T_m\kappa_A(x, y) = \kappa_A(T_m(x), T_m(y))T_m, \quad \forall m \in M_A,$n

so it follows that $\kappa_A(x, y) = \kappa_A(T_m(x), T_m(y))$ for all non-unital $T_m$ and all $x, y \in V$. Repeated applications of this fact show that $\kappa_A(x, y) = \kappa_A(T_m^k(x), T_m^k(y)) = 0$. Conversely, $\mathcal{H}_{0,0} = \text{Sym}(V) \rtimes A$ has the PBW property.

For completeness, we mention two properties of generalized nil-Coxeter algebras, even though they will not be used in the paper. First, the algebras $NC_W(d)$, and more generally, every generic Hecke algebra $\mathcal{E}_W(d, p)$, is equipped with an anti-involution that fixes every generator $T_i$. This is because the defining relations are preserved by such a map. Such an anti-involution can be used to construct an exact contravariant duality functor on a suitable category of $A$-modules, which preserves the simple object $k = A/m$.

Second, as discussed in [26], for all finite Coxeter groups $W$ the nil-Coxeter algebra is a Frobenius algebra, by defining a trace map to kill all words in the $T_i$ except for the longest word $T_{w_0}$. The same turns out to hold also for the generalized nil-Coxeter algebra $A := NC_{A^\gamma}(d)$, by defining a trace map to kill all words in the $T_i$, except for $\prod_{i=1}^n t_i^{d_i-1}$. Note that these two words $T_{w_0}$ and $\prod_{i=1}^n t_i^{d_i-1}$ span the space $\text{Prim}(A) = \text{Prim}(A^{op})$, as we note after Theorem 6.7 below.

6. Deformations over cocommutative algebras with nilpotent maximal ideals

In this final section, we study the representations of deformed smash product algebras over nil-Coxeter algebras. We will work in somewhat greater generality.

Standing Assumption 6.1. Henceforth, $k$ is a field, and $(A, \Delta)$ is a cocommutative $k$-algebra with coproduct, with a nilpotent maximal ideal $m = \Delta m A \neq 0$ that satisfies:

$$A = m \oplus k \cdot 1_A, \quad \exists f_A \in \mathbb{N} : m^{\ell_A} = 0 \neq m^{\ell_A-1}, \quad \Delta(m) \subset m \otimes m.$$

We will use without further reference the following observations, when required:

- $(A, m)$ is local, since every element in $A \setminus m$ is invertible. From this one can show that $m$ is the Jacobson radical of $A$, and $\text{Ext}_{A-\text{mod}}(k, k) \cong (m^2)^\perp$, where $(m^2)^\perp \subset m^\ast$.
- The assumption $\Delta(m) \subset m \otimes m$ is required if $\text{char} \, k > 0$. Cocommutative algebras not satisfying this assumption exist; for instance, consider $A := (\mathbb{Z}/p\mathbb{Z})[T]/(T^p)$, with $p > 0$ prime and $\Delta(T) = 1 \otimes T + T \otimes 1$. However, we do not need to assume $\Delta(m) \subset m \otimes m$ if $\text{char} \, k = 0$. Indeed, given $a \in m$, let $\Delta(a) \in (1 \otimes 1) \oplus (1 \otimes m) \oplus (m \otimes 1) \oplus (m \otimes m)$, with $c, d, e \in k^\times$. By multiplicativity, $\Delta(a)^n = 0$ for $n \gg 0$, which works out to: $c = d = e = 0$. 

The prototypical example of an algebra satisfying Assumption 6.1 is the nil-Coxeter algebra $NC_V$ for a finite Coxeter group $W$. Another example is the generalized nil-Coxeter algebra $NC_{CW}((d_1, \ldots, d_n)) = \otimes_{i=1}^n \mathbb{k}[T_i]/(T_i^{d_i})$. In both cases, $m$ is the two-sided augmentation ideal generated by the particular, $25$ equality that in related work [25, Theorem C], we characterize the generalized nil-Coxeter algebras $NC_W(d)$ for which the maximal ideal $m$ is nilpotent. This property turns out to be equivalent to the finite-dimensionality of $NC_W(d)$, which was discussed following Remark 5.3.

6.1. Simple $H_{\lambda,\kappa}$-modules. We begin by exploring simple modules over $H_{\lambda,\kappa}$. In order to state our results, some notation is required.

**Definition 6.1.** Suppose $A$ is as in Assumption 6.1 and $M$ is an $A$-module.

1. The level of a nonzero vector $m \in M$ is the integer $k > 0$ such that $m^k m = 0 \neq m^{k-1} m$. Define the level of $0_M$ to be $0$ for convention. The level of the module, denoted by $\ell_M$, is the highest level attained in $M$.
2. For $k \geq 0$, define $\mathcal{L}_{\leq k}(M)$ to be the set of elements of level at most $k$.
3. A vector $m \in M$ is primitive if $m m = 0$. Let $\text{Prim}(M)$ denote all primitive elements.

The following lemma is easily shown.

**Lemma 6.2.** Suppose $M$ is any $A$-module. Then $\mathcal{L}_{\leq k}(M) = \ker_M m^k$; in particular, $\text{Prim}(M) = \mathcal{L}_{\leq 1}(M), \quad M = \mathcal{L}_{\leq \ell_M}(M), \quad \ell_M \leq \ell_{A^\text{mult}} = \ell_A$.

Moreover, $\mathcal{L}_{\leq k}(M)$ is a proper submodule of $\mathcal{L}_{\leq k+1}(M)$ for all $k < \ell_M$.

We now study $H_{\lambda,\kappa}$-modules. Our first result aims to classify all simple $H_{\lambda,\kappa}$-modules in the case when $\kappa_V \equiv 0$.

**Theorem 6.3.** Suppose $A$ satisfies Assumption 6.1 and $V$ is an $A$-module. If $\lambda$ satisfies Equation (2.3) in $A$, then $\lambda(m^k, \mathcal{L}_{\leq k}(V)) \subseteq m^k$ for all $k \geq 0$. If instead we assume $\kappa_V \equiv 0$, then the following are equivalent for $H_{\lambda,\kappa}$:

1. $\lambda(m^k, V) \subseteq m^k$ for all $k \geq 0$, and $\kappa_A : V \wedge V \to m$.
2. $\lambda(m, V) \subseteq m$ and $\kappa_A : V \wedge V \to m$.
3. There exists a one-dimensional $H_{\lambda,\kappa}$-module killed by $m$.
4. There is a bijection from simple $H_{\lambda,\kappa}$-modules to simple $\text{Sym}(V)$-modules, determined uniquely by restriction from $H_{\lambda,\kappa}$ to the image of $V$; moreover, the inverse map is given by restriction to $V$ and inflation to $H_{\lambda,\kappa}$, letting $m$ act trivially.

The condition $\kappa_A : V \wedge V \to m$ is a natural one in characteristic zero, in the sense that it is necessary if $H_{\lambda,\kappa}$ has a finite-dimensional module and $\text{char} \, k = 0$. This is because if $\pi : H_{\lambda,\kappa} \to \text{End}_k M$ is a finite-dimensional representation, then for all $a \in m$, $\pi(a)$ is nilpotent, hence has trace zero. It follows that $\text{im} \, \kappa_A = [V, V] \subseteq m$.

The following result will be useful in proving Theorem 6.3.

**Proposition 6.4.** Suppose $M$ is an $A$-module.

1. $M$ is $A$-semisimple if and only if $m M = 0$. 
(2) Any finite filtration \( M = M_0 \supset M_1 \supset \cdots \supset M_k = 0 \) of \( A \)-modules (such as \( M \supset M \)) can be refined to a possibly longer finite filtration, so that the successive subquotients are \( A \)-semisimple modules. In particular, \( \text{Prim}(M) \neq 0 \) if \( M \neq 0 \).

(3) Every maximal submodule of a nonzero \( A \)-module has codimension one. Thus a \( d \)-dimensional \( A \)-module has a flag of \( A \)-submodules of length \( d+1 \).

(4) \( \text{Prim}(A) \subset m \).

(5) If \( M \) is nonzero, \( mM \) is contained in every maximal proper (i.e. codimension one) submodule of \( M \). In particular, it is a proper submodule of \( M \) if \( M \neq 0 \).

**Proof.**

(1) If \( mM = 0 \) then \( M \) is clearly \( A \)-semisimple. Conversely, if \( M \) is \( A \)-semisimple, notice that \( M = mM \oplus M_1 \) for some \( A \)-semisimple complement \( M_1 \). But then \( M_1 \cong M/mM \) is annihilated by \( m \). Repeat this construction on \( mM \) to produce \( M_2 \), and so on; this process stops after finitely many steps as \( m \) is nilpotent. But then \( M \) is a direct sum of submodules killed by \( m \).

(2) It suffices to prove the result for the filtration \( M \supset 0 \). Define \( M_i := m^iM \) for all \( i > 0 \), and \( M_0 := M \). Now apply the previous part.

(3) This follows from the previous part.

(4) If \( a \in A \setminus m \), then \( a \) is invertible, hence cannot lie in \( \text{Prim}(A) \).

(5) Suppose \( M = \mathbb{km}_0 \oplus M' \) where \( M' \) is a proper submodule. Fix \( a \in m \) such that \( am_0 = rm_0 + m' \), with \( r \in k \) and \( m' \in M' \). Then one shows by induction on \( i \) that

\[
a^i m_0 = r^i m_0 + (r^{i-1}m' + r^{i-2}am' + \cdots + a^{i-1}m')
\]

for all \( i > 0 \). In particular, since \( a^{\ell_a} \in m^{\ell_a} = 0 \), hence \( r^{\ell_a} m_0 \in M' \), whence \( r^{\ell_a} = 0 \). Thus \( r = 0 \), and \( am_0 = m' \in M' \) for all \( a \in m \), whence \( mM \subset M' \) as claimed.

\( \square \)

**Proof of Theorem 6.3.** The first assertion holds because the \( A \)-action (2.3) implies that if \( m^k(v) = 0 \), then (with a slight abuse of notation)

\[
0 = \lambda(m^{\ell_a-k} m_k, v) = m^{\ell_a-k} \lambda(m^k, v) + \lambda(m^{\ell_a-k}, m^k(1)(v))m^k(2) = m^{\ell_a-k} \lambda(m^k, v),
\]

from which it follows that \( \lambda(m^k, v) \subset m^k \).

We now assume \( \kappa \subset V \equiv 0 \), and show that (1) and (2) are equivalent. Clearly \( (1) \implies (2) \); conversely, if (2) holds, then we compute for \( a_1, \ldots, a_k \in m \), by induction on \( k \):

\[
\lambda(a_1 \cdots a_k, v) = a_1 \lambda(a_2 \cdots a_k, v) + \sum \lambda(a_1, ((a_2(1)) \cdots (a_k(1))(v))(a_2(1)) \cdots (a_k(1))((a_1(1)) \cdots (a_k(1)) \subset m \cdot m^{k-1} + m \cdot m^{k-1} = m^k.
\]

Next, given (2), we show (4) as follows: if \( M \) is a simple \( \text{Sym}(V) \)-module then the construction in (4) makes it a simple \( \mathcal{H}_{\lambda, k} \)-module, as the relations in \( \mathcal{H}_{\lambda, k} \) indeed hold in \( \text{End}_k M \) via (2). On the other hand, given any \( \mathcal{H}_{\lambda, k} \)-module \( M \), by Proposition 6.4, \( \ker_M m \neq 0 \). We now claim that if \( \lambda(m, V) \subset m \) and \( M \) is a \( \mathcal{H}_{\lambda, k} \)-module, then \( \ker_M m^k \) is a \( \mathcal{H}_{\lambda, k} \)-submodule of \( M \). Given the claim, if \( M \) is now
a simple $\mathcal{H}_{\lambda,\kappa}$-module, then $0 \neq \ker M$ is a $\mathcal{H}_{\lambda,\kappa}$-submodule, whence $mM = 0$, proving (4).

It remains to show the claim (in order to complete the proof of (2) $\implies$ (4)). Let $M' = \ker M m^k$; then for $a \in m$ and $m' \in M'$, we have

$$m^k(am') \subset m^kA \cdot m' = m^km' = 0,$$

whence $am' \in M'$. Thus $M'$ is an $A$-submodule. It thus remains to show that $vm' \in M'$ for $v \in V$. But if we have $a_1, \ldots, a_k \in m$, then

$$\prod_{i=1}^k a_i \cdot vm' = \sum \left( \prod_{i=1}^k (a_i(1)) \right) (v) \cdot \prod_{i=1}^k (a_i(2)) \cdot m' + \lambda(a_1 \cdots a_k, v)m' ,$$

and this is killed by using Assumption 6.1 and the equivalence of (1) and (2). Hence $vm' \in M'$.

Finally, we show $4) \implies 3) \implies 2)$. If (4) holds, choose any linear functional $\mu \in V^*$ and consider the simple one-dimensional $\text{Sym}(V)$-module

$$M_\mu := \text{Sym}(V)/\text{Sym}(V) \cdot (\text{id}_V - \mu).$$

By (4). $M_\mu$ yields a one-dimensional simple $\mathcal{H}_{\lambda,\kappa}$-module which is killed by $m$, and this shows (3). Next, if (3) holds for $M$ then $V$ acts on $M$ by scalars, i.e., by $\mu \in V^*$. It follows that $\text{im} \kappa_A = [V, V]$ kills $M$, whence $\kappa_A : V \wedge V \to m$. Similarly if $a \in m$, then $\lambda(a, v) \in \text{im} \kappa_A$ also kills $M$, whence $\lambda(m, V) \subset A$.

**Corollary 6.5.** Suppose $k$ is algebraically closed and $V$ is finite-dimensional. If $\lambda(m, V) \subset \kappa_V \equiv 0$, and $\kappa_A : V \wedge V \to m$, then all simple finite-dimensional $\mathcal{H}_{\lambda,\kappa}$-representations are one-dimensional, and in bijection with $V^*$.

**6.2. PBW property.** Our next goal is to prove a result similar to Theorem 5.12 that classifies the PBW deformations $\mathcal{H}_{\lambda,\kappa}$, but in the more general setting of cocommutative algebras $A$ satisfying Assumption 6.1. Thus we do not assume the existence of a grouplike basis as for the nil-Coxeter algebra, and alternate methods are required. In particular, the following provides a second proof of Theorem 5.12.

**Theorem 6.6.** Suppose $A$ satisfies Assumption 6.1, and $V$ is an $A$-module.

1. Suppose $\kappa_V \equiv 0$. Then the Jacobi identity (2.7) holds in $\mathcal{H}_{\lambda,\kappa}$ if and only if $\dim_k V \leq 2$ or $\text{im} \kappa_A \subset k \cdot 1_A$.

2. If $\dim_k V \leq 2$, then $\mathcal{H}_{0,\kappa}$ has the PBW property if and only if $\text{im} \kappa_V \subset \text{Prim}(V)$ and $\text{im} \kappa_A \subset \text{Prim}(A^{\text{mult}})$.

3. If $\dim_k V > 2$, and $\lambda, \kappa_V \equiv 0$, then $\mathcal{H}_{0,\kappa}$ has the PBW property if and only if $\kappa_A \equiv 0$.

**Proof.**

1. By Remark 2.7, and since $\kappa_V \equiv 0$, it suffices to characterize the Jacobi identity (2.7) under the additional assumption that $\dim V > 2$. Now write down the identity:

$$\sum_{\mathcal{O}} [\kappa(v_1, v_2), v_3] = 0, \quad v_1, v_2, v_3 \in V.$$

We may assume without loss of generality that the $v_i$ are linearly independent in $V$. Moreover, the $\kappa_1$-component is killed by commuting with
elements of $V$. (Here, we work with a distinguished $k$-basis of $\mathfrak{m}$, along with $\{1_A\}$.) If we now define $\gamma_{v,v'} := \kappa_A(v,v') - \kappa_1(v,v') \in \mathfrak{m}$, then

$$\sum_{\ell} (v_1 \gamma_{v_2,v_3} - \sum (\gamma_{v_2,v_3})_{(1)}(v_1)(\gamma_{v_2,v_3})_{(2)}) = 0.$$ 

Now assume without loss of generality that $v_1 \in \mathcal{L}_{\ell k+1}(V) \setminus \mathcal{L}_{\ell k}(V)$ for some $k \geq 0$, and $v_1, v_2, v_3 \in \mathcal{L}_{\ell k+1}(V)$. Then $(\gamma_{v_2,v_3})_{(1)}(v_r) \in \mathcal{L}_{\ell k}(V)$ for all $\{p, q, r\} = \{1, 2, 3\}$. Working modulo $\mathcal{L}_{\ell k}(V)$, it follows by the linear independence of the $v_i$ that $\gamma_{v_2, v_3} = 0$, and hence an entire summand in the above cyclic sum vanishes. Repeat the same argument twice to show all summands are zero, and hence, $\kappa_A \equiv \kappa_1$ on $V \wedge V$.

(2) This is similar to the proof of Theorem 6.12(1) and is omitted for brevity.

(3) Clearly $\mathcal{H}_{0,o}$ has the PBW property. Conversely, assume $\mathcal{H}_{0,\kappa}$ has the PBW property. By a previous part, we have $\text{im} \kappa_A \subset k \cdot 1_A$. Suppose $\kappa_A \neq 0$. Then there exists $k \geq 0$ such that $\kappa_A(\mathcal{L}_{\ell k+1}(V), V) \neq 0 = \kappa_A(\mathcal{L}_{\ell k}(V), V)$. Choose nonzero $a \in \mathfrak{m}$, and any $v_0 \in \mathcal{L}_{\ell k+1}(V), v_1 \in V$ such that $\kappa_A(v_0, v_1) \neq 0$. Then by Theorem 2.5,

$$0 \neq a \kappa_A(v_0, v_1) = \sum \kappa_A(a_{(1)}(v_0), a_{(2)}(v_1))a_{(3)}.$$ 

But by assumption $a_{(1)}(v_0) \in \mathcal{L}_{\ell k}(V)$, whence the right hand side vanishes. This contradiction shows that $\kappa_A = 0$. 

\[\square\]

### 6.3. Center and abelianization

We end the paper by computing the center and abelianization of the algebra $\mathcal{H}_{\lambda,\kappa}$, i.e., the zeroth Hochschild (co)homology.

**Theorem 6.7.** Suppose $A$ satisfies Assumption 6.1. $\lambda, \kappa$ are such that $\mathcal{H}_{\lambda,\kappa}$ has the PBW property, and $\text{Prim}(A) = \text{Prim}(A^{op})$. If $\lambda(\mathfrak{m}, V) \subset \mathfrak{m}$, then $\mathcal{H}_{\lambda,\kappa}$ has trivial center, i.e., $H^0(\mathcal{H}_{\lambda,\kappa}, \mathcal{H}_{\lambda,\kappa}) = k$.

Akin to the remarks following Assumption 6.1, the condition $\text{Prim}(A) = \text{Prim}(A^{op})$ is satisfied by all nil-Coxeter algebras $NC_W$ for a finite Coxeter group $W$, as well as by $NC_{A^1}(d)$. The condition $\lambda(\mathfrak{m}, V) \subset \mathfrak{m}$ was discussed in detail in Theorem 6.3.

**Proof.** We first choose a totally ordered basis of $V$ as follows: via Proposition 6.4 fix the filtration $0 = \mathcal{L}_{\leq 0}(V) \subset \mathcal{L}_{\leq 1}(V) \subset \cdots \subset \mathcal{L}_{\leq \ell_V}(V) = V$ according to the level; then choose any $k$-basis $B_k$ of the corresponding vector space complement of $\mathcal{L}_{\leq k-1}(V)$ in $\mathcal{L}_{\leq k}(V)$ for $k = 1, \ldots, \ell_V$. Now index $B_k$ by any totally ordered set $S_k$, and let $S := \bigsqcup_k S_k$ be totally ordered via: $s_i < s_j$ if $i > j$ and $s_i \in S_i, s_j \in S_j$. Thus, every element of $B_1$ is primitive. Now use the PBW property to write any vector in $\mathcal{H}_{\lambda,\kappa}$ as $\sum_I v_I a_I$, where $I$ denotes a word in $S$ whose letters occur in non-increasing order, $a_I \in A$, and $v_I$ denotes the corresponding monomial in $\bigsqcup_k B_k$.

Note that $\mathfrak{m}$ acts on each $v_I$ and yields a linear combination of elements $v_J$ such that $I > J$ in the lexicographic order on words in $S$. More precisely, if we define $\ell(v_I)$ to be the sum of the levels of the letters in the monomial $v_I$ (see Definition 6.1), then $\mathfrak{m}$ strictly reduces $\ell(v_I)$.

We now proceed to the proof. Suppose $0 \neq z = \sum_I v_I a_I$ is central in $\mathcal{H}_{\lambda,\kappa}$, with the $v_I$ linearly independent. We first claim that for each non-empty $I$, the
vector $a_I$ is primitive in $A$. Indeed, choosing $a \in m$ and writing out $az = za$ yields:
\[
\sum_I \left( \sum a_{(1)}(v_I)a_{(2)} + \lambda(a, v_I) \right) a_I = \sum v_I a_I a.
\]
Choosing $I \neq \emptyset$ such that $v_I$ has maximal $\ell$-value, it follows from above that $a_I a = 0$ for all $a \in m$. Hence $a_I \in \text{Prim}(A^p) = \text{Prim}(A)$ by assumption. Now say $v_I = v_{i_1} \cdots v_{i_j}$ for some $i_j \in I$. We notice by induction on $k$ that $av_I a_I = 0$ as well. Indeed,
\[
av_I a_I = \sum a_{(1)}(v_{i_k}) a_{(2)} v_{i_{k-1}} \cdots v_{i_1} a_I + \lambda(a, v_{i_k}) \cdot v_{i_{k-1}} \cdots v_{i_1} a_I,
\]
and both expressions vanish by the induction hypothesis (the base case of $k = 1$ is easy). It follows that $av_I a_I = 0 = v_I a_I a$, where $I \neq \emptyset$ is such that $\ell(v_I)$ is maximal. Now cancel these terms from the above equation and work with $I$ of the next highest $\ell$-value. Repeating the above analysis shows the claim.

Next, let $v \in \text{Prim}(V)$ and consider $zv = vz$ in $H_{\lambda, \kappa}$:
\[
a_0 v + \sum_I v_I a_I v = va_0 + \sum_I vz a_I.
\]
Since $a_I \in \text{Prim}(A) \subset m$ (by Proposition 6.4), hence $a_I v = \lambda(a_I, v)$ for all nonempty $I$. Hence working modulo the filtered degree $\leq 1$ piece and using the PBW property, $a_I = 0$ if $I \neq \emptyset$. In other words, $z = a_0 \in A$. Since $A = k \cdot 1 \oplus m$, we may assume that $z \in m$. Now choose nonzero primitive $v \in V$; then,
\[
vz = zv = \sum z_{(1)}(v)z_{(2)} + \lambda(z, v) = \lambda(z, v),
\]
whence we get that $z = 0$ by the PBW property. Hence $Z(H_{\lambda, \kappa}) = k \cdot 1$ as claimed.

Next, we compute the zeroth Hochschild homology.

**Theorem 6.8.** Suppose $\lambda$ and $\kappa_V$ are identically zero, $\kappa_A : V \wedge V \to m$, and $H_{\lambda, \kappa}^{0, k}$ satisfies the PBW property. If $k$ is an infinite field, then as abelian $k$-algebras, we have
\[
HH_0(H_{\lambda, \kappa}, H_{\lambda, \kappa}) = \frac{H_{\lambda, \kappa}}{H_{\lambda, \kappa} \cdot H_{\lambda, \kappa}^1}
\]
\[
\cong k \cdot 1 + \left( \text{Sym}^+(V) \bigoplus (m/([m, m] + A \cdot (\text{im} \kappa_A) \cdot A)) \right),
\]
where the direct sum indicates that the two factors are ideals and hence multiply to zero.

**Proof.** The proof is in steps. The first step is to show that $[H_{\lambda, \kappa}, H_{\lambda, \kappa}]$ contains the image of $(V \cdot m$, where given a subspace $U \subset V$, $\langle U \rangle := TV \cdot U \cdot TV$ is the two-sided ideal in $TV$ generated by $U$. More precisely, we show by induction on $k$ that $\langle L_{\leq k}(V) \rangle \cdot m \subset [H_{\lambda, \kappa}, H_{\lambda, \kappa}]$. This is clear for $k = 0$, and given the result for $k$, Assumption 6.1 implies that
\[
a(p) \in \langle L_{\leq k}(V) \rangle, \quad \forall a \in m, \quad p \in \langle L_{\leq k+1}(V) \rangle.
\]
It follows by the induction hypothesis that
\[
p \cdot a = [p, a] + a \cdot p
\]
\[
= [p, a] + \sum a_{(1)}(p)a_{(2)} \in [H_{\lambda, \kappa}, H_{\lambda, \kappa}] + \langle L_{\leq k}(V) \rangle m \subset [H_{\lambda, \kappa}, H_{\lambda, \kappa}].
\]
Next, fix a total ordering on a basis of $V$. Given any nonzero sum $v$ of monomial “ordered” words, since $k$ is an infinite field there exists $\mu \in V^*$ such that $\mu(v) \neq 0$. Now since $\lambda \equiv 0$, it follows by Theorem 6.3 that $H_{\lambda, \kappa}$ has a one-dimensional representation $M_\mu$ killed by $m$, and on which $V$ acts by $\mu$. Since $[H_{\lambda, \kappa}, H_{\lambda, \kappa}]$ necessarily kills $M_\mu$, it follows that $v$ has nonzero image in $H_{\lambda, \kappa}/[H_{\lambda, \kappa}, H_{\lambda, \kappa}]$. Hence $V$ generates the symmetric algebra in $H_{\lambda, \kappa}/[H_{\lambda, \kappa}, H_{\lambda, \kappa}]$.

It remains to consider the image of $A$ inside the abelianization. Note that $im\, \kappa_A = [V, V]$ and $[m, m]$ lie in $[H_{\lambda, \kappa}, H_{\lambda, \kappa}]$, and are subspaces of $m$ by assumption. (That this image and $\text{Sym}^+(V)$ are ideals follows from the above analysis.) To complete the proof, it suffices to show the commutator intersects $A$ in $[m, m] + A \cdot (im\, \kappa_A) \cdot A$. Note $H_{\lambda, \kappa} = A \bigoplus (V) \cdot A$ by the PBW property. Now $[A, A] = [m, m]$, while $[(V) \cdot A, A] \subset (V) \cdot A$, which intersects $A$ trivially.

It remains to consider $[(V) \cdot A, (V) \cdot A] \cap A$. By the relations in $H_{\lambda, \kappa}$ as well as the PBW property, the only elements that occur here arise from the relations $[v, v'] = \kappa_A(v, v') \in A$, and hence the intersection is contained in $A \cdot (im\, \kappa_A) \cdot A$. We now show that this containment is an equality, via the claim that $a \kappa_A(v, v')a' \in [H_{\lambda, \kappa}, H_{\lambda, \kappa}]$ for $v, v' \in V$ and $a, a' \in A$. The claim is obvious if $a = a' = 1$. Otherwise we may assume that at least one of $a, a'$ lies in $m$. In this case,

$$[av, v'a'] = avv'a' - v'a'av = a[v, v']a' + av'va' - v'a'va = a[v, v']a' + \sum a(1)(v')a(2)(v)a(3)a' - v' \sum (a'a)(1)(v)(a'a)(2).$$

Since $\Delta(m) \subset m \otimes m$, it follows that all summands of both sums lie in $(V) \cdot m$, hence in $[H_{\lambda, \kappa}, H_{\lambda, \kappa}]$ from above. This proves the claim, and with it, the result. \qed

Acknowledgments

The author would like to thank Sarah Witherspoon for many stimulating and informative conversations regarding this paper. The author also thanks Ivan Marin, Susan Montgomery, and Victor Reiner for useful references and discussions. Finally, the author is grateful to Chelsea Walton for going through a preliminary draft of this work and for her helpful suggestions.

References

1. Y. Bazlov and A. Berenstein, Braided doubles and rational Cherednik algebras, Adv. Math. 220 (2009), no. 5, 1466–1530, DOI 10.1016/j.aim.2008.11.004 MR2493618
2. G. M. Bergman, The diamond lemma for ring theory, Adv. Math. 29 (1978), no. 2, 178–218, DOI 10.1016/0001-8708(78)90010-5 MR500909
3. I. N. Bernstein, I. M. Gel’fand, and S. I. Gel’fand, Schubert cells, and the cohomology of the spaces $G/P$, Russian Math. Surveys 28 (1973), no. 2, 1–26. MR429933
4. G. Böhm, F. Nill, and K. Szlachányi, Weak Hopf algebras. I. Integral theory and $C^*$-structure, J. Algebra 221 (1999), no. 2, 385–438, DOI 10.1006/jabr.1999.7984 MR1726707
5. A. Braverman and D. Gaitsgory, Poincaré–Birkhoff–Witt theorem for quadratic algebras of Koszul type, J. Algebra 181 (1996), no. 2, 315–328, DOI 10.1006/jabr.1996.0122 MR1383469
6. J. Brichard, The center of the nilCoxeter and 0-Hecke algebras, preprint, available at arXiv:0811.2959 (2008).
7. M. Broué, G. Malle, and R. Rouquier, On complex reflection groups and their associated braid groups, Representations of groups (Banff, AB, 1994), CMS Conf. Proc., vol. 16, Amer. Math. Soc., Providence, RI, 1996, pp. 1–13. MR1357192
8. M. Broué, G. Malle, and R. Rouquier, Complex reflection groups, braid groups, Hecke algebras, J. Reine Angew. Math. 500 (1998), 127–190, DOI 10.1515/crll.1998.064 MR1637497
9. H. S. M. Coxeter, Discrete groups generated by reflections, Ann. of Math. (2) 35 (1934), no. 3, 588–621, DOI 10.2307/1968753 MR1503182
[10] H. S. M. Coxeter, *Factor groups of the braid group*, Proceedings of the 4th Canadian Mathematical Congress (Banff, AB, 1957), University of Toronto Press, 1959, pp. 95–122.

[11] W. Crawley-Boevey and M. P. Holland, *Noncommutative deformations of Kleinian singularities*, Duke Math. J. 92 (1998), no. 3, 605–635, DOI 10.1215/S0012-7094-98-09218-3 MR1620538

[12] V. G. Drinfeld, *Degenerate affine Hecke algebras and Yangians*, Funct. Anal. Appl. 20 (1986), no. 1, 58–60, DOI 10.1007/BF01077318 MR831053

[13] P. Etingof, W. L. Gan, and V. Ginzburg, *Continuous Hecke algebras*, Transform. Groups 10 (2005), no. 3-4, 423–447, DOI 10.1007/s00031-005-0404-2 MR2183119

[14] P. Etingof and V. Ginzburg, *Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism*, Invent. Math. 147 (2002), no. 2, 243–348, DOI 10.1007/s002220100171 MR1881922

[15] P. Etingof, D. Nikshych, and V. Ostrik, *On fusion categories*, Ann. of Math. (2) 162 (2005), no. 2, 581–642, DOI 10.4007/annals.2005.162.581 MR2183279

[16] M. Fayers, *0-Hecke algebras of finite Coxeter groups*, J. Pure Appl. Algebra 199 (2005), no. 1-3, 27–41, DOI 10.1016/j.jpaa.2004.12.001. MR2134290

[17] S. Fomin and R. P. Stanley, *Schubert polynomials and the nil-Coxeter algebra*, Adv. Math. 103 (1994), no. 2, 196–207, DOI 10.1006/aima.1994.1009. MR1265793

[18] S. Griffeth, *Towards a combinatorial representation theory for the rational Cherednik algebra of type $G(r, p, n)$*, Proc. Edinb. Math. Soc. (2) 53 (2010), no. 2, 419–445, DOI 10.1017/S0013091508000904 MR2653242

[19] J.-W. He, F. Van Oystaeyen, and Y. Zhang, *PBW deformations of Koszul algebras over a nonsemisimple ring*, Math. Z. 279 (2015), no. 1-2, 185–210, DOI 10.1007/s00209-014-1362-y. MR3299848

[20] X. He, *A subalgebra of 0-Hecke algebra*, J. Algebra 322 (2009), no. 11, 4030–4039, DOI 10.1016/j.jalgebra.2009.04.003. MR2556136

[21] F. Hivert, A. Schilling, and N. M. Thiéry, *Hecke group algebras as quotients of affine Hecke algebras at level 0*, J. Combin. Theory Ser. A 116 (2009), no. 4, 844–863, DOI 10.1016/j.jcta.2008.11.010. MR2513638

[22] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1990. MR1066460

[23] A. Joseph, *Quantum groups and their primitive ideals*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 29, Springer-Verlag, Berlin, 1995. MR1315960

[24] A. Khare, *Category $O$ over a deformation of the symplectic oscillator algebra*, J. Pure Appl. Algebra 195 (2005), no. 2, 131–166, DOI 10.1016/j.jpaa.2004.06.004 MR2108468

[25] A. Khare, *Generalized nil-Coxeter algebras over discrete complex reflection groups*, preprint, available at arXiv:1601.08231 (2016).

[26] M. Khovanov, *Nilcoxeter algebras categorify the Weyl algebra*, Comm. Algebra 29 (2001), no. 11, 5033–5052, DOI 10.1081/AGB-100006800 MR1856929

[27] M. Khovanov and A. D. Lauda, *A diagrammatic approach to categorification of quantum groups. I*, Represent. Theory 13 (2009), 309–347, DOI 10.1090/S1088-4165-09-00346-X MR2525917

[28] B. Kostant and S. Kumar, *The nil Hecke ring and cohomology of G/P for a Kac–Moody group $G$*, Adv. Math. 62 (1986), no. 3, 187–237, DOI 10.1016/0001-8708(86)90101-5 MR866159

[29] D. W. Koster, *COMPLEX REFLECTION GROUPS*, ProQuest LLC, Ann Arbor, MI, 1975. Thesis (Ph.D.)–The University of Wisconsin - Madison. MR2625485

[30] A. Lascoux and M.-P. Schützenberger, *Fonctorialité des polynômes de Schubert*, Invariant theory (Denton, TX, 1986), Contemp. Math., vol. 88, Amer. Math. Soc., Providence, RI, 1989, pp. 585–598, DOI 10.1090/conm/088/1000001 [French, with English summary]. MR1000001

[31] I. Losev and A. Tsymbaliuk, *Infinitesimal Cherednik algebras as $W$-algebras*, Transform. Groups 19 (2014), no. 2, 495–526, DOI 10.1007/s00031-014-9261-1 MR3200433

[32] G. Lusztig, *Affine Hecke algebras and their graded version*, J. Amer. Math. Soc. 2 (1989), no. 3, 593–635, DOI 10.1090/S0894-0347-1989-0991016 MR991016

[33] P. N. Norton, *0-Hecke algebras*, J. Austral. Math. Soc. Ser. A 27 (1979), no. 3, 337–357, DOI 10.1017/S1446788700012453 MR532754

[34] G. C. Shephard and J. A. Todd, *Finite unitary reflection groups*, Canadian J. Math. 6 (1954), 274–304, DOI 10.4153/CJM-1954-028-3 MR0059914
[35] A. V. Shepler and S. Witherspoon, Drinfeld orbifold algebras, Pacific J. Math. 259 (2012), no. 1, 161–193, DOI 10.2140/pjm.2012.259.161 MR2988488

[36] A. V. Shepler and S. Witherspoon, A Poincaré–Birkhoff–Witt theorem for quadratic algebras with group actions, Trans. Amer. Math. Soc. 366 (2014), no. 12, 6483–6506, DOI 10.1090/S0002-9947-2014-06118-7 MR3267016

[37] A. V. Shepler and S. Witherspoon, PBW deformations of skew group algebras in positive characteristic, Algebr. Represent. Theory 18 (2015), no. 1, 257–280, DOI 10.1007/s10468-014-9492-9 MR3317849

[38] A. V. Shepler and S. Witherspoon, Poincaré–Birkhoff–Witt Theorems, Commutative Algebra and Noncommutative Algebraic Geometry, Volume I: Expository Articles, Mathematical Sciences Research Institute Proceedings, vol. 67, Cambridge University Press, 2015, pp. 259–290.

[39] V. V. Tewari and S. J. van Willigenburg, Modules of the 0-Hecke algebra and quasisymmetric Schur functions, Adv. Math. 285 (2015), 1025–1065, DOI 10.1016/j.aim.2015.08.012 MR3406520

[40] A. Tikaradze, On maximal primitive quotients of infinitesimal Cherednik algebras of $\mathfrak{gl}_n$, J. Algebra 355 (2012), 171–175, DOI 10.1016/j.jalgebra.2012.01.013 MR2889538

[41] A. Tikaradze and A. Khare, Center and representations of infinitesimal Hecke algebras of $\mathfrak{sl}_2$, Comm. Algebra 38 (2010), no. 2, 405–439, DOI 10.1080/00927870903448740 MR2598890

[42] A. Tsymbaliuk, Infinitesimal Hecke algebras of $\mathfrak{so}_N$, J. Pure Appl. Algebra 219 (2015), no. 6, 2046–2061, DOI 10.1016/j.jpaa.2014.07.022 MR3299718

[43] C. Walton and S. Witherspoon, Poincaré–Birkhoff–Witt deformations of smash product algebras from Hopf actions on Koszul algebras, Algebra Number Theory 8 (2014), no. 7, 1701–1731, DOI 10.2140/ant.2014.8.1701 MR3272279

[44] C. Walton and S. Witherspoon, PBW deformations of braided products, preprint, available at arXiv:1601.02274 (2016).

[45] G. Yang, Nil-Coxeter algebras and nil-Ariki-Koike algebras, Front. Math. China 10 (2015), no. 6, 1473–1481, DOI 10.1007/s11464-015-0498-3 MR3403207

Departments of Mathematics and Statistics, Stanford University, Stanford, CA - 94305

E-mail address: khare@stanford.edu