On the theory of multi-pulse vibro-impact mechanisms

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Abstract. This paper presents a mathematical model of a new multi-striker eccentric shock-vibration mechanism with a crank-sliding bar vibration exciter and an arbitrary number of pistons. Analytical solutions for the parameters of the model are obtained to determine the regions of existence of stable periodic motions. Under the assumption of an absolutely inelastic collision of the piston, we derive equations that single out a bifurcational unattainable boundary in the parameter space, which has a countable number of arbitrarily complex stable periodic motions in its neighbourhood. We present results of numerical simulations, which illustrate the existence of periodic and stochastic motions. The methods proposed in this paper for investigating the dynamical characteristics of the new crank-type conrod mechanisms allow practitioners to indicate regions in the parameter space, which allow tuning these mechanisms into the most efficient periodic mode of operation, and to effectively analyze the main changes in their operational regimes when the system parameters are changed.

1. Introduction
Recently, alongside with shock-vibration mechanisms with unbalance vibration exciters, eccentric shock-vibration mechanisms (ESVM) with a crank-sliding bar vibration exciter (CSVE) [1, 2] have come to be widely used in civil engineering. This innovative design was based on the principle of “the inverted vibrator”, with its working unit, unbalanced mass, hinged on an eccentric shaft and balanced in rotation by the unbalanced mass. The loading pulse transmitted onto the surface (ground, piles etc.) is produced both due to a spreader with a shoulder of the eccentric shaft and due to the kinetic energy of the falling of the working unit. The effectiveness of compacting and immersing machines not only depends on the quantity of the energy transmitted to the medium being processed, but, to the larger extent, on the character of transmitting this energy – the pulse “form” which should be varied by rearranging certain dynamic factors of a single loading pulse. It is evident that a dense and, at the same time, strong structure of the ground can be achieved only when specific pressure on the surface of the contact of the working unit and the ground in the process of compaction is increased by degrees, its lower limit being determined by the physical properties of the ground in its initial state (before compaction), and the upper one by the ultimate strength of the ground or by technological conditions.

Thus, parameters of such machines and mechanisms must be defined from the conditions close to quasi-plastic interaction. Besides, the pulse rate in each single cycle must be such that it precludes possible development of elastic aftereffects of the processed medium in the intervals between the pulses. Such multi-pulse loading mode can be realized using multi-striker ESVM with CSVE designed
to easily regulate working modes by changing the geometry of cinematic connections and to solve the problem of compaction of soils in strained conditions of industrial and civil engineering processes.

The variety of problems that require the study of dynamic models with discontinuous or piecewise-continuous non-linearities is enormous. Thus, nonlinear dynamic problems of vibration systems with impact interaction of the elements are currently being analyzed by scientific teams headed by B.I. Van de Vrande, D.N. Van Campen, P.I. Leine, G.W. Luo, V.K. Astashov, V.I. Babitsky, M.I. Feygin et al [3-18]. These works analyzed the complex dynamics of vibration systems with impact interactions of various nature. Periodical motion modes in non-self-supporting systems with finite (mostly with one) and infinite (quasi-plastic) number of impacts during one period of the periodic motion have been studied. Where possible, the mathematical apparatus of the point mapping method [17-28] was used in the studies. The systems and mechanisms used as vibro-impact rams for various media (soils, concrete, loose materials etc.) as well as for driving in or extracting piles, were designed in the form of unbalance mechanisms. The designs of such mechanisms, especially when being used as vibro-impact rams, often could not cope with their tasks for various reasons. One of such reasons is their being unsuited for working in conditions of limited space, as well as being highly power-consuming.

2. Problem setting
The impact-vibration mechanism considered in the paper (figure 1) consists of the frame 1 encasing the eccentric shaft 2 with flywheels at its ends.

On the shaft, there are eccentric mechanisms 3, each consisting of two eccentrics nested into each other with a possibility to change the position of the washers, which makes it possible to regulate the value of eccentricity $r_i$ and phase shift $\phi_i$ between them ($i = 1, 2, ..., N$). Slides-strikers (SS) 4 are hinged at the free ends of the connecting rods. The eccentric mechanisms in combination with the connecting rods and SS convert the rotary motion of the shaft with the constant cyclic rate of rotation of the flywheel into reciprocating motion of the frame relative to stands 5. Each SS strikes its own anvil.

The processed medium (soil, pile, etc.) is represented as an elastically fastened mass $M_1$ with elasticity coefficient $C$. Energy losses in the slide-bars of the frame and the medium is accounted for in the form of viscous friction with equivalent damping coefficients $b$ and $b_1$, respectively.

As the eccentrics can be phase-shifted by angles $\phi_i$ relative to each other and have, generally speaking, different lengths, the body oscillates relative the mechanism with the largest sum of
projections of the geometrical dimensions onto the vertical axis at the moment. The impact on the anvil of one or several SS’s takes place either during the change of SS’s interacting with the anvil, or after the breakage of the body together with the eccentric mechanisms.

Neglecting the masses of SS’s, connecting rods and cranks, the equation of free (without impact) motion of the system for \( y_{pi} > y_c \) can be written in the following form

\[
M \frac{d^2 y}{dt^2} + b \frac{dy}{dt} = -Mg; M_1 \frac{d^2 y_c}{dt^2} + b_1 \frac{dy_c}{dt} + C y_c = -M_1 g, \tag{1}
\]

where \( y \) – coordinate of the mass centre of the body counted off from the equilibrium position of mass \( M_i \) for a non-deformed spring \( C \), \( y_c \) is deviation value of mass \( M_i \) from the same initial point, \( y_{pi} \) are values characterizing deviation of bases of the \( i \)-th SS, \( g \) is free-fall acceleration.

When one of the SS contacts the medium \( y_{pi} = y_c \) there takes place either a momentary impact interaction, such that if \( \dot{y}_{pi} \), \( \ddot{y}_{pi} \) and \( \dot{y}_{pi}^+ \), \( \ddot{y}_{pi}^+ \) are velocities of the \( i \)-th SS and the medium immediately before and after the impact interaction, respectively, then

\[
\begin{align*}
\dot{y}_{pi}^+ &= [(\mu_0 - R)\dot{y}_{pi} + (1 + R)\dot{y}_{pi}^- ] / (1 + \mu_0), \\
\ddot{y}_{pi}^+ &= [(\mu_0(1 + R)\ddot{y}_{pi} + (1 - \mu R)\ddot{y}_{pi}^- ] / (1 + \mu_0),
\end{align*} \tag{2}
\]

or simultaneous \((R = 0)\) motion of the shock-vibration mechanism with the medium described by the following equation:

\[
M \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + M_1 \frac{d^2 y_c}{dt^2} + b_1 \frac{dy_c}{dt} + Cy_c = -(M + M_1) g \tag{3}
\]

In equations (2) the following definitions are used: \( 0 \leq R \leq 1 \) is impact velocity recovery coefficient, \( \mu_0 = M / M_1 \).

The position of the eccentrics of length will be measured by angles \( \theta_i = \omega t - \varphi_i \), counted off from the vertical axis. Then it follows that

\[
y_{pi} = y - s_i + r_i \cos(\omega t - \varphi_i) - \sqrt{l_i^2 - r_i^2 \sin^2(\omega t - \varphi_i)}, \tag{4}
\]

where \( s_i \) is distance from the fixture point of the \( i \)-th connecting rod to the base of the \( i \)-th SS, \( l_i \) is length of the \( i \)-th connecting rod.

When changing in system (1-4) to dimensionless time \( \tau = \omega t \), coordinates \( x = (y - s_2 - l) / l \), \( x_i = y_i / l \) of the body and the medium, parameters

\[
\mu = r / l, \gamma_j = r_j / r_i, \epsilon_j = (s_j - s_2) / l, \quad p = g / \omega^2 l, \lambda^2 = C / M_1 \omega^2, 2h = b / M \omega, 2h_i = b_i / M_i \omega, l_i = l,
\]

as well as introducing function \( f(\tau) = r_i - \mu \gamma_i \cos(\tau - \varphi_i) \), for \( r_i \leq l \) the following equations describing impact-oscillatory motions of the mechanism, accounting for the mobility of the processed medium, are obtained:

\[
\begin{align*}
x - x_i &> f(\tau), \\
\ddot{x} + 2h \dot{x} &= -p, \\
\dddot{x}_i + 2h_i \ddot{x}_i + \lambda^2 x_i &= -p,
\end{align*} \tag{5}
\]
\[
\begin{align*}
\dot{x}^+ & = \left[ (\mu_0 - R)\dot{x}^- + (1 + R)\dot{x}_i^- + (1 + r - \mu_0) \frac{df(\tau)}{d\tau} \right] / (1 + \mu_0), \\
\dot{x}_i^- & = \left[ \mu_0 (1 + R)\dot{x}^- + (1 - R\mu_0)\dot{x}_i^- - \mu_0 (1 + R) \frac{df(\tau)}{d\tau} \right] / (1 + \mu_0), \\
(1 + \mu_0)\dot{x} + 2(h_1 + \mu_0 h)\dot{x} + \lambda^2 x & = -(1 + \mu_0)p + F(\tau), \\
x - x_i = f(\tau),
\end{align*}
\]  

\(\tau\) -periodical function \(F(\tau)\) on equations (6) describing simultaneous motion of the striking mechanism with the medium after the impact of the \(i\)-th SS has the following form:

\[
F(\tau) = e_i + (\lambda^2 - 1)f(\tau) + 2h_i \frac{df(\tau)}{d\tau},
\]

\(f(\tau) = \max \{ f_1(\tau), f_2(\tau), ..., f_N(\tau) \}.\)

3. Solution method

Assuming \(x_i = 0, \ M_i = \infty, \ h = 0\) in equations (5-7), the equations of motion with momentary stops [3] of one or several SS’s can be rewritten as

\[
\begin{align*}
\ddot{x} & = -p, x > f(\tau), \\
\dot{x}^+ & = -R\dot{x}^- + (1 + R) \frac{df(\tau)}{d\tau}, x = f(\tau).
\end{align*}
\]  

The phase space of system (8) in coordinates \(x, \dot{x}, \tau\) is truncated along \(x\), and, besides, \(x \geq f(\tau), \dot{x} < +\infty\) (figure 2).

![Figure 2. The phase space.](image)

All the phase trajectories are situated either along the surface \(x = f(\tau)\) or above it, which represents (generally speaking) \(N\) of intersecting cylindrical surfaces \(x = f_i(\tau)\) \((i = 1, 2, ..., N)\). The case of \(x = f(\tau)\) corresponds to free motion of the falling mechanism, and \(x = f(\tau)\) corresponds to impact interaction of one of the SS’s with the stopper.

In the case of impact interactions of SS with the stopper for \(R \neq 0\), the representing point in the phase space moves as follows: from region \(\Phi_i(x > f(\tau))\), the representing point gets on surface \(\Pi(x = f(\tau))\) into a point \(M_0(\tau_0, x_0 = f(\tau_0), \dot{x}_0)\) (see figure 2), and then, moving along surface \(\Pi\) into point \(M_0(\tau_0, x_0 = f(\tau_0), \dot{x}_0 = -R\dot{x}_0 + (1 + R)\frac{df(\tau_0)}{d\tau}\), leaves it and gets into subspace \(\Phi_i\),
where it stays until it gets once again onto surface $\Pi$ into a point $M_i(t_i > t_k, x'_i = f(t_i), \dot{x'_i})$, etc. If at a time $t = t_k$, when the representing point gets onto surface $\Pi$ into point $M_i(t_k, x'_k = f(t_k), \dot{x'_k})$, the value before the impact velocity is equal to the velocity of change $f(t)$ at that moment, or a limiting case of the value of the velocity recovery coefficient is considered for the impact $R = 0$, then the representing point displaces from $M_i$ to point $M_i^1 \left( t_i, x'_i, \dot{x'_i} = df(t)/d\tau \right)$ and then will slide along surface $\Pi$ until, at time $t = t_{k+1}$, it reaches boundary $\mu \gamma_i \cos(t_i - \varphi_i) + \gamma = 0$ of the sliding portion plate. These time intervals $\Delta_{k} = t_k - t_i$ correspond to prolonged contacts of SS with the stopper.

In what follows, the main attention will be paid to periodical motion modes with successive impacts of each SS during period $\Gamma$. It is evident that such a mode is only possible under the condition of paired intersection of two successive surfaces $f_i(t), f_{k+1}(t)$. For this to be possible, the following relations for the system parameters must hold:

$$e_{k+1} - e_k \leq \mu^2 (\gamma_{k+1}^2 + \gamma_k^2 - 2\gamma_{k+1} \gamma_k \cos(\varphi_{k+1} - \varphi_k))$$

If for any pair $f_i(t), f_{k+1}(t)$ this inequality does not hold (no intersection of the surfaces) then, in the technological process of the motion of the mechanism, the mode with the stroke of the $l$-th or the $(l-1)$-th SS is skipped. To this end, conditions (9) are of paramount importance when adjusting a specific mechanism to the mode with alternating strokes of each of the SS’s $N$.

4. Constructing a point map

It follows from the description of the structure of phase space $\Phi$ in system (8) that the dynamics of the mathematical model considered can be investigated by studying the properties of the point transform $T_{k+1}$ transforming surface points $x = f_{k+1}(t)$ into surface points $x = f_{k+2}(t)$, which, using (8), can be written in the following form ($h = 0$)

$$f_{k+2}(t_{k+1}) = \Delta_{k+1} (\dot{x}_k - p\Delta_{k+1} / 2) + f_{k+1}(t_k),$$
$$\dot{x}_k = R(p\Delta_{k+1} - \dot{x}_k)(1+R) \frac{df(t_{k+1})}{d\tau},$$

(10)

The parameters in (10) have to satisfy relations (9) as well as the conditions of existence of point transforms $T_{k+1}$, namely

$$f_{k+1}(t_k) \geq f_i(t_k), k = 0, 1, ..., N - 1; l = 1, ..., N,$$

$$x(t) > f(t), t_k < \tau < t_{k+1}.$$

(11)

Point transform $T$ of point $M_i \in f_i(t_k)$ into point $M_{n+1} \in f_i(t_{n+1})$ under conditions (11) is defined as

$$T = \prod_{i=0}^{N} T_i.$$

(12)

5. Existence and stability conditions for fixed points at a point map $T$ corresponding to periodic two-impact motion modes

If $S_i$ is the number of strokes of the $i$-th SS against the stopper during one period $f(t)$, and $n$ is multiplicity of period $\Gamma$ of the periodic motion mode to period $f(t)$, then the periodic motion will be
characterized by numbers $S_{i,n}$ $(i=1,2,...,N)$. It means that the problem of analyzing the periodic motion mode with a single stroke of each of the $N$ SS’s can be reduced to analyzing the properties of point mapping $T$ and the conditions of existence of the latter ($C$ - bifurcations), analogous to analyzing systems with discontinuous nonlinearities [18, 19]. Thus, the coordinates of a fixed point $M^{*}(\tau^{*}, \hat{X}^{*})$ corresponding to the periodic motion mode with successive strokes of each of the SS’s are determined from system $2(N+1)$ of equations (8) supplemented by periodicity conditions

$$
\hat{X}_{N+1} = \hat{X}_1 = \hat{X}^{*}, \tau_{N+1} = \tau_1 + n\Gamma = \tau^{*}.
$$

(13)

Using simple transformations, it can be found from system (8) and (13) that

$$
\hat{X}^{*} = \frac{b_{N} - R^{N} \sum_{k=1}^{N} (-1)^{k+1} b_{N-k}}{1 + (-1)^{N+1} R^{N}},
$$

(14)

$$
\hat{X}_{k+1} = R^{k} [(-1)^{k+1} \hat{X}^{*} + \sum_{i=0}^{N} (-1)^{i} b_{k-i}],
$$

where the components of $N$-dimensional vector-function $b(h_1,...,h_N)$ are independent of $\hat{X}^{*}, \hat{X}_{k+1}$ but are functions of $\tau^{*}, \tau_{k+1}$ and the parameters of the system

$$
b_{j} = Rp(\tau_{j+1} - \tau_{j}) + (1 + R) \frac{df_{j+1}(\tau_{j+1})}{d\tau}.
$$

(15)

Times $\tau_{k}, k = 1,2,3,... N$ are determined by solving the system of nonlinear equations of the form

$$
f_{j}(\tau^{*}) + \Delta \tau^{*}_{j+1} \{p \Delta \tau_{j} / 2 - \hat{X}^{*}_{j} \} = f_{j}(\tau_{0}),
$$

$$
f_{j}(\tau^{*}) - f_{j}(\tau_{N}) + (\tau^{*} - \tau_{N}^{*}) \{p (\tau^{*} - \tau_{N}^{*}) / 2 - \hat{X}_{N}^{*} \} = 0,
$$

(16)

$(j = 1,2,...,N-1)$.

Stability in the small of the main of the periodic motion modes depends on the value of the roots of the characteristic equation

$$
\chi(Z) = a_{0}Z^{2} + a_{1}Z + a_{2} = 0,
$$

where coefficients $a_{i}, i = 0,1,2$ are defined, as usual [4], after the linearization of the equations of point transform of $T$ in the vicinity of a fixed point. After some simple transformations one obtains

$$
\chi(Z) = \begin{pmatrix}
A(Z)...B \\
C(Z)...D(Z)
\end{pmatrix} = 0,
$$

where $A(Z), B, C(Z), D(Z)$ are square matrices, with non-zero entries having the following form:

$$
a_{i,j+1} = (-1)^{i} \{p \Delta \tau_{i+1} - \hat{X}_{i}^{*} \} + \frac{df_{i+1}(\tau_{i+1})}{d\tau},
$$

$$
a_{N,1} = Z \left[ p (\tau_{N}^{*} - \tau^{*}) + \hat{X}_{N}^{*} - \frac{df_{j}(\tau^{*})}{d\tau} \right],
$$

$$
c_{i,j+1} = -R (-1)^{i} p + (1 + R) j \frac{d^{2}f_{i+1}(\tau_{i+1})}{d\tau^{2}},
$$
\[ c_{\tau,j} = Z \left[ Rp + (1+R) \frac{d^2 f_i(\tau^*)}{d\tau^2} \right], \]
\[ b_{\tau,i} = \Delta \tau^*, d_{\tau,i} = (j-1)R - j, \quad (17) \]
\[ d_{\tau,j} = -Z, j = 0, 1; i = 1, 2, ..., N. \]

Let us consider in a more detail the dynamics of the mechanism for various values of \( N \) of SS’s.

6. The dynamics of single piston mechanism

\( N = 1 \). Using (14-17), the coordinates of the fixed point are determined through the parameters of the system as

\[ \tau^* = \pi - \arcsin \Omega, X^* = \pi np, \quad (18) \]

where \( \Omega = \pi np(1-R)/\mu(1+R) \leq 1 \).

The regions of existence and stability of periodic single-stroke motion modes are limited by surfaces \( N_+, N_-, N_\phi \), whose equations in the parametric form have, respectively, the following form

\[ \mu_+ = \Omega \mu, \mu_- = p\sqrt{4(1+R^2)^2 + [\pi n(1-R^2)]^2} / (1+R^2), R_\phi = 1 \]

Figure 3 shows subdivision of the parameter plane \( \mu/p, R \) into the planes (between the upper \( N_+ \) and the lower \( N_- \) boundaries) of existence and stability of a periodic single stroke \( n \)-tuple motion mode.

![Figure 3](image)

**Figure 3.** The boundaries of the region of existence and stability.

In Figure 4 a bifurcation diagram for parameter \( p \) is shown for and \( \mu = 0.12 \), wherefrom it follows that a single-stroke motion mode of the mechanism does exist on the interval for the frequency parameter presented in Figure 3. In the region of the parameters that do not belong to this interval, there exist, as was expected, periodic motions with two (Feigenbaum doubling bifurcation [9]), three, etc. strokes of SS for the period of motion.

It is to be noted that for \( N = 1 \), \( f(\tau) \) is an analytical function and, thus, by substituting variable \( y = x - f(\tau) \) in equations (8), a mathematical model of a conventional impact oscillator is obtained. Its dynamics has been investigated in numerous works (see [4, 5] and the references thereof).

\( N=2 \). The phase space of system (8) in coordinates \( x, \dot{x}, \tau \) is truncated along \( x \), and, besides, \( x \geq f(\tau), \dot{x} < +\infty \) (figure 5).

Conditions (9) necessary for the existence of the motion modes with strokes by each of the two SS’s in our case (\( N=2 \)) depict in the parameter plane (figure 6) corresponding regions (shaded parts) for various values of \( \phi_i \).
Figure 4. The bifurcation diagram for parameter.

Figure 5. The phase space of system \((N=2)\).

Figure 6. The regions existence of the motion modes with strokes by each of the two SS’s.

One can see that with increasing the phase shift between eccentricities \(r\) the regions of existence of the motion mode with alternating strokes of SS’s grow in the parameter space. Point transform \(T\) of points

\[
M_0(\tau = r_0, x_0 = f_1(\tau_0), \dot{x} = \dot{x}_0) \rightarrow M_1(\tau = r_1, x_1 = f_2(\tau_1), \dot{x} = \dot{x}_1) \rightarrow M_2(\tau = r_2, x = f_1(\tau_2), \dot{x} = \dot{x}_2)
\]
can be written as

\[-\mu \gamma \cos(\tau_1 - \phi) = -p(\tau_1 - \tau_0)^2 / 2 + \dot{x}_0(\tau_1 - \tau_0) + \varepsilon - \mu \cos \tau_0, \quad (19)\]

\[\dot{x}_1 = R(p(\tau_1 - \tau_0) - \dot{x}_0) + (1 + R)\mu \gamma \sin(\tau_1 - \phi), \quad (20)\]

\[\varepsilon - \mu \cos \tau_2 = -p(\tau_2 - \tau_1)^2 / 2 + \dot{x}_1(\tau_2 - \tau_1) - \mu \gamma \cos(\tau_1 - \phi), \quad (21)\]

\[\dot{x}_2 = R(p(\tau_2 - \tau_1) - \dot{x}_1) + (1 + R)\mu \sin \tau_2. \quad (22)\]

7. Conditions of existence and stability of fixed points corresponding to periodic two-stroke motions modes.

Equations for determining fixed points of transform \(T\) corresponding to two-stroke (with alternating strokes of each SS) periodic motions (the main mode), taking account of the periodicity conditions, can be written in the following way

\[\dot{x}_1 = R(p(\xi - \dot{x}) - \dot{x}_0) + (1 + R)\mu \gamma \sin(\tau_0 + \alpha), \]

\[\dot{x} = R(p(2\pi n - \xi) - \dot{x}_1) + (1 + R)\mu \sin \tau_0, \]

\[-\mu \gamma \cos(\tau_0 + \alpha) = -\frac{p}{2}\xi^2 + \xi \dot{x} + \varepsilon - \mu \cos \tau_0, \quad (23)\]

\[\varepsilon - \mu \cos \tau_0 = -\frac{p}{2}(2\pi n - \xi)^2 + (2\pi n - \xi)\dot{x}_1 - \mu \gamma \cos(\tau_0 + \alpha), \]

\[(\alpha = \xi - \varphi, \xi = \tau_1 - \tau_0).\]

After a number of transformations (23) can be rewritten as

\[\mu = \frac{\sqrt{(Ad - bB)^2 + (Ac - Ba)^2}}{|ad - bc|}, \]

where

\[a = (1 - R)(1 - \gamma \cos \alpha) + R\gamma \xi \sin \alpha, b = \xi(1 - R\gamma \cos \alpha) - (1 - R)\gamma \sin \alpha, \]

\[c = (1 - R)(1 - \gamma \cos \alpha) + (2\pi n - \xi)\gamma \sin \alpha, d = (2\pi n - \xi)(R - \gamma \cos \alpha) - (1 - R)\gamma \sin \alpha \]

\[A = (1 - R)(\varepsilon - \Delta k - \eta) + \frac{p\xi}{(1 + R)} \left(2\pi n R - \frac{\xi}{2}(1 + R) \right) \]

\[B = (1 - R)(\varepsilon - \Delta k - \eta) + \frac{p(2\pi n R - \xi)}{2(1 + R)} \left((2\pi n \xi) + (1 + R) \right) - 2R\xi. \]

Stability of a fixed point is determined, as is known [24-28], by the value of the roots of the characteristic equation \(\chi(z) = 0\), which, in our case, has the following form:

\[\chi(z) = b_1 z^2 + (a_{11} h_{22} + (a_{11} h_{22} + a_{22} h_{11} - a_{21} h_{12})z + a_{11} a_{22} - a_{12} a_{21}). \quad (24)\]

The following notations are used in (24):

\[a_{11} = R(1 + R)\mu \gamma \cos(\tau_0 + \alpha) + R\frac{p^2}{\xi}(\dot{x} - \mu \gamma \sin(\tau_0 + \alpha)), \]

\[a_{12} = \frac{R^2}{\xi}(\mu \sin \tau_0 - \dot{x}). \]
8. The results of the numerical experiments.

Due to the fact that the investigation of complex periodic and stochastic motion modes in the analytical form is a rather complicated and, currently, unworkable problem, the dynamics of such motions was investigated in the present work using numerical experiments on a personal computer with the help of a software complex developed in Borland C++ Builder 6 medium. Figure 7 presents, in plane \((p, R)\), the region of existence and stability (shaded region) of a \(2\pi\)-periodic motion with alternating strokes of piston-strikers for the following values of the parameters 
\[
\mu = 0.09, \ \varepsilon = 0.018, \ \varphi = 3.14, \ \gamma = 3
\]
and \(P(0)\). Boundary \(N_c\) [18] (of the existence of the above described point transform \(T\)) cuts from the stability region, bounded by surfaces \(N_+\), \(N_\pm\), \(N_q\), a small part.

![Figure 7. The region of existence and stability (shaded region) of a - periodic motion.](image)

Figure 8 shows a bifurcation diagram for parameter \(p\), constructed for the same parameter values and for \(R = 0.55\). Left part of figure 8 corresponds to the diagram with the impacts on surface \(x = f_1(\tau)\); right part of figure 8 – to the diagram with the impacts on surface \(x = f_2(\tau)\). It is evident from the figure that for \(0.22 \leq p \leq 0.23\). Doubling the number of strokes; the main motion mode exists in the limits of \(0.23 \leq p \leq 0.25\). For \(0.19 \leq p \leq 0.21\), a chaotic motion mode is observed.

![Figure 8. The bifurcation diagrams.](image)

9. Conclusions
The work describes:

- a model of a new multi-striker eccentric shock-vibration mechanism with a crank-sliding bar vibration exciter and an arbitrary number of pistons;
- the phase space of the mathematical model;
- equations of a general (for a random number of SS) point mapping of a two-dimensional non-analytical intersecting surface into itself;
- equations (in the parametric form) of the coordinates of the fixed points corresponding to periodic motions with alternating strokes of each of the SS’s;
- bifurcation boundaries of existence of the above mentioned equations of point mappings.

The main results achieved:

- a methodology and algorithms of constructing point mapping equations of non-analytical secular surfaces truncated along phase coordinates have been developed;
- the parameter space has been subdivided into regions of stability of periodic motion modes;
- the study of dynamics of a single-piston mechanism undergoing a perfectly inelastic collision has been formulated as a Poincare map of a circle into itself;
- the existence of a denumerable number of stable periodic modes of motion in the neighborhood of unreachable boundaries in the extended phase space has been proven;
- bifurcation values of the parameters have been demonstrated, for which chaotic motion modes exist. The scenario of the appearance of chaotic motion modes is very similar to the scenario of doubling of Feigenbaum period [23-28].

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