SUPERCONDUCTIVITY IN ULTRASMALL GRAINS: 
INTRODUCTION TO RICHARDSON'S EXACT SOLUTION

JAN VON DELFT AND FABIAN BRAUN

Institut für Theoretische Festkörperphysik
Universität Karlsruhe
76128 Karlsruhe, Germany

November 1, 1999

Abstract. Studies of pairing correlations in ultrasmall metallic grains have
commonly been based on a simple reduced BCS-model describing the scat-
tering of pairs of electrons between discrete energy levels that come in time-
reversed pairs. This model has an exact solution, worked out by Richardson
in the context of nuclear physics in the 1960s. Here we give a tutorial intro-
duction to his solution, and use it to check the quality of various previous
treatments of this model.

1. Introduction

Recent experiments by Ralph, Black and Tinkham, involving the observa-
tion of a spectroscopic gap indicative of pairing correlations in ultrasmall
Al grains [1], have inspired a number of theoretical [2]-[11] studies of how
superconducting pairing correlations in such grains are affected by reduc-
ing the grains' size, or equivalently by increasing its mean level spacing
\( d \propto \text{Vol}^{-1} \) until it exceeds the bulk gap \( \Delta \). In the earliest of these, a
grand-canonical (g.c.) BCS approach [2, 3, 4] was applied to a reduced
BCS Hamiltonian for uniformly spaced, spin-degenerate levels; it suggested
that pairing correlations, as measured by the condensation energy \( E_C \), van-
ish abruptly once \( d \) exceeds a critical level spacing \( d_c^\infty \) that depends on the
parity (0 or 1) of the number of electrons on the grain, being smaller for
odd grains \( (d_c^\infty \approx 0.89\Delta) \) than even grains \( (d_c^\infty \approx 3.6\Delta) \). A series of more
sophisticated canonical approaches (summarized in Section 3 below) con-
firmed the parity dependence of pairing correlations, but established [6]-[11]
that the abrupt vanishing of pairing correlations at \( d_c^\infty \) is an artifact of g.c.
treatments: pairing correlations do persist, in the form of so-called fluctuations, to arbitrarily large level spacings, and the crossover between the bulk superconducting (SC) regime \(d \ll \Delta\) and the fluctuation-dominated (FD) regime \(d \gg \Delta\) is completely smooth [10]. Nevertheless, these two regimes are qualitatively very different [9, 10]: the condensation energy, e.g., is an extensive function of volume in the former and almost intensive in the latter, and pairing correlations are quite strongly localized around the Fermi energy \(\varepsilon_F\), or more spread out in energy, respectively.

After the appearance of all these works, we became aware that the reduced BCS Hamiltonian on which they are based actually has an exact solution. It was published by R. W. Richardson in the context of nuclear physics (where it is known as the “picket-fence model”), in a series of papers between 1963 and 1977 [12]-[20] which until very recently seem to have completely escaped the attention of the condensed matter community. In this work, we (i) give a tutorial introduction (with no pretense of rigor) to his solution, and (ii) compare the results of various previously-used approximations against the benchmark set by the exact solution, in order to gauge their reliability for related problems for which no exact solutions exist [21, 22].

2. Richardson’s Exact Solution

2.1. REDUCED BCS MODEL

Ultrasmall superconducting grains are commonly described [2]-[11] by a reduced BCS model,

\[
H = \sum_{j\sigma} \varepsilon_j c_{j\sigma}^\dagger c_{j\sigma} - g \sum_{ij} c_{i+}^\dagger c_{i-} c_{j-} c_{j+},
\]

for a set \(S\) of \(N_S\) pairs of time-reversed states \(|j, \pm\rangle\) labeled by a discrete index \(j = 1, \ldots, N_S\), with energies \(\varepsilon_j\) and coupling \(g = \lambda d\), where \(d\) is the mean level spacing and \(\lambda\) a dimensionless coupling constant. Unbeknownst to the authors that have studied this model recently, Richardson had long ago solved it exactly, for an arbitrary set of levels \(\varepsilon_j\) (degenerate levels are allowed, but are to be distinguished by distinct \(j\)-labels, i.e. they have \(\varepsilon_i = \varepsilon_j\) for \(i \neq j\)).

The first step is to note that singly-occupied levels do not participate in the pairscattering described by \(H\), and by the Pauli principle remain “blocked” [23] to such pairscattering; the labels of such levels are therefore good quantum numbers. A general eigenstate of \(H\) thus has the form

\[
|n, B\rangle = \prod_{i \in B} c_{i\sigma}^\dagger |\Psi_n\rangle_U,
\]
\[ |\Psi_n\rangle_U = \sum_{j_1, \ldots, j_n} n \prod_{\nu=1}^n b_{j_{\nu}=1}^\dagger |0\rangle. \]  

This describes \( N = 2n + b \) electrons, \( b \) of which sit in a set \( B \) of singly-occupied, blocked levels, thereby contributing \( E_B = \sum_{i \in B} \varepsilon_i \) to the eigenenergy, while the remaining \( n \) pairs of electrons, created by the pair operators \( b_j^\dagger = c_{j_+}^\dagger c_{j_-}^\dagger \), are distributed among the remaining set \( U = S \setminus B \) of \( N_U = N_S - b \) unblocked levels, with wave function \( \psi(j_1, \ldots, j_n) \) \( (\sum_j^U \equiv \sum_{j \notin B} \) denotes a sum over all unblocked levels). The dynamics of these pairs is governed by

\[ H_U = \sum_{ij} (2 \varepsilon_j \delta_{ij} - g) b_i^\dagger b_j, \]  

and writing the eigenenergy of \( |n, b\rangle \) as \( E_n + E_b \), the state \( |\Psi_n\rangle_U \) satisfies

\[ H_U |\Psi_n\rangle_U = E_n |\Psi_n\rangle_U, \quad \sum_j^U b_j^\dagger b_j |\Psi_n\rangle_U = n |\Psi_n\rangle_U. \]  

Diagonalizing \( H_U \) would be trivial if the \( b \)’s were true bosons. However, they are not, and in the subspace spanned by the set \( U \) of all non-singly-occupied levels, instead satisfy the “hard-core boson” relations,

\[ b_j^{12} = 0, \quad [b_j, b_{j'}^\dagger] = \delta_{jj'} \left(1 - 2b_j^\dagger b_j\right), \quad [b_j^\dagger b_j, b_{j'}^\dagger] = \delta_{jj'} b_j^\dagger, \]  

which reflect the Pauli principle for the fermions they are constructed from. In particular, \( b_j^{12} = 0 \) implies that only those terms in (3) are non-zero for which the indices \( j_1, \ldots, j_n \) are all distinct.

In his original publications [12, 13, 14], Richardson derived a Schrödinger for \( \psi(j_1, \ldots, j_n) \) and showed that its exact solution was simply a generalization of the form that \( \psi(j_1, \ldots, j_n) \) would have had if the \( b \)’s had been true (not hard-core) bosons. With the benefit of hindsight, we shall here follow an alternative, somewhat shorter root, also due to Richardson[24]: we first consider the related but much simpler case of true bosons and write down the generic form of its eigenstates; we then clarify why this form fails to produce eigenstates of the hard-core boson Hamiltonian; and having identified the reason for the failure, we show that (remarkably) only a slight generalization is needed to repair it and to obtain the sought-after hard-core-boson eigenstates.

2.2. TRUE BOSONS

Let \( \tilde{b}_j \) denote a set of true bosons (i.e. \( [\tilde{b}_j, \tilde{b}_j^\dagger] = \delta_{jj'} \)), governed by a Hamiltonian \( \tilde{H}_U \) of precisely the form (4), with \( b_j \rightarrow \tilde{b}_j \). This problem, being
quadratic, can be solved straightforwardly by any number of methods. The solution is as follows:  \( \tilde{H}_U \) can be written as
\[
\tilde{H}_U = \sum_J \tilde{E}_J \tilde{B}_J^\dagger \tilde{B}_J + \text{const.}
\]
where the new bosons \( \tilde{B}_J^\dagger \) (with normalization constants \( C_J \)) are given by
\[
\tilde{B}_J^\dagger = gC_J \sum_j \frac{\tilde{b}_j^\dagger}{2\varepsilon_j - \tilde{E}_J}, \quad \frac{1}{(gC_J)^2} = \sum_j \frac{1}{(2\varepsilon_j - \tilde{E}_J)^2},
\]
and the boson eigenenergies \( \tilde{E}_J \) are the roots of the eigenvalue equation
\[
1 - \sum_j \frac{g}{2\varepsilon_j - \tilde{E}_J} = 0.
\]
This is an equation of order \( N_U \) in \( \tilde{E}_J \). It thus has \( N_U \) roots, so that the label \( J \) runs from 1 to \( N_U \). As the coupling \( g \) is turned to 0, each \( E_J \) smoothly evolves to one of the bare eigenenergies \( \varepsilon_j \). A general \( n \)-boson eigenstate of \( \tilde{H}_U \) and its eigenenergy \( \tilde{E}_n \) thus have the form
\[
|\tilde{\Psi}_n\rangle_U = \prod_{\nu=1}^n \tilde{B}_{J_{\nu}}^\dagger |0\rangle, \quad \tilde{E}_n = \sum_{\nu=1}^n \tilde{E}_{J_{\nu}},
\]
where the \( n \) indices \( J_1, \ldots, J_n \) that characterize this state need not all be distinct, since the \( \tilde{B}_J^\dagger \) are true bosons.

2.3. COMPLICATIONS ARISING FOR HARD-CORE BOSONS

Let us now return to the hard-core boson Hamiltonian \( H_U \). Its eigenstates will obviously not be identical to the true-boson eigenstates just discussed, since matters are changed considerably by the hard-core properties of \( b_j \). To find out exactly what changes they produce, it is very instructive to take an Ansatz for \( |\Psi_n\rangle_U \) similar to (10) (but suppressing the normalization constants and taking all \( J_{\nu} \) to be distinct), namely
\[
|\Psi_n\rangle_U = \prod_{\nu=1}^n B_{J_{\nu}}^\dagger |0\rangle, \quad \text{with} \quad B_J^\dagger = \sum_j \frac{b_j^\dagger}{2\varepsilon_j - E_J},
\]
and to check explicitly whether or not it could be an eigenstate of \( H_U \), i.e. to check under what conditions \( (H_U - \varepsilon_n)|\Psi_n\rangle_U \) would equal zero, where
\( \mathcal{E}_n = \sum_{\nu} E_{J_{\nu}} \). To this end, we commute \( H_U \) to the right past all the \( B^\dagger_{J_{\nu}} \) operators in \( |\Psi_n\rangle_U \), using

\[
\left[ H_U, \prod_{\nu=1}^n B^\dagger_{J_{\nu}} \right] = \sum_{\nu=1}^n \left\{ \left( \prod_{\eta=1}^{\nu-1} B^\dagger_{J_{\eta}} \right) [H_U, B^\dagger_{J_{\nu}}] \left( \prod_{\mu=\nu+1}^n B^\dagger_{J_{\mu}} \right) \right\} .
\] (12)

To evaluate the commutators appearing here, we write \( H_U \) as

\[
H_U = \sum_j U \varepsilon_j b^\dagger_j b_j - g B^\dagger_0 B_0 ,
\] (13)

and use the following relations:

\[
[b^\dagger_j b_j, B^\dagger_j] = \frac{b^\dagger_j}{2 \varepsilon_j - E_J} , \quad [B_0, B^\dagger_j] = \sum_j \frac{1 - 2b^\dagger_j b_j}{2 \varepsilon_j - E_J} ,
\] (14)

\[
[H_U, B^\dagger_j] = E_J B^\dagger_j + B^\dagger_0 \left[ 1 - g \sum_j \frac{1 - 2b^\dagger_j b_j}{2 \varepsilon_j - E_J} \right] .
\] (15)

Inserting these into (12) and using \( H_U |0\rangle = 0 \) and \( \mathcal{E}_n = \sum_{\nu} E_{J_{\nu}} \), we find

\[
H_U |\Psi_n\rangle_U = \mathcal{E}_n |\Psi_n\rangle_U + \sum_{\nu=1}^n \left[ \frac{1 - \sum_j \frac{g}{2 \varepsilon_j - E_{J_{\nu}}} \left( \prod_{\eta=1(\neq \nu)}^n B^\dagger_{J_{\eta}} \right) B^\dagger_0 \left( \prod_{\mu=\nu+1}^n B^\dagger_{J_{\mu}} \right) }{1 - g \sum_j \frac{1 - 2b^\dagger_j b_j}{2 \varepsilon_j - E_{J_{\nu}}} \right] |0\rangle
\] + \sum_{\nu=1}^n \left\{ \left( \prod_{\eta=1}^{\nu-1} B^\dagger_{J_{\eta}} \right) \left[ \sum_j \frac{2g B^\dagger_0 b^\dagger_j b_j}{2 \varepsilon_j - E_{J_{\nu}}} \right] \left( \prod_{\mu=\nu+1}^n B^\dagger_{J_{\mu}} \right) \right\} |0\rangle .
\] (16)

Now, suppose we do the same calculation for true instead of hard-core bosons (i.e. run through the same steps, but place a \( \tilde{\cdot} \) on \( H_U, b_j, E_J \) and \( \mathcal{E}_n \)). Then the second line of (16) would be absent (because the \( b^\dagger_j b_j \) terms in the second of Eqs. (6) and (14) and in (15) would be absent); and the first line of (16) would imply that \( (\tilde{H}_U - \tilde{\mathcal{E}}_n) |\tilde{\Psi}_n\rangle_U = 0 \) provided that the term in square brackets vanishes, which is nothing but the condition that the \( \tilde{E}_J \) satisfy the the true-boson eigenvalue equation of (9)! In other words, we have just verified explicitly that all true-boson states of the form (10) are indeed eigenstates of \( \tilde{H}_U \), provided that the \( \tilde{E}_J \) satisfy (9). Moreover, we have identified the term in second line of (16) as the extra complication that arises for hard-core bosons.
2.4. THE CURE: A GENERALIZED EIGENVALUE EQUATION

Fortunately, this extra complication is tractable: first, we note that

$$\sum_j \frac{U}{2\varepsilon_j - E_{J_\nu}} b_j^\dagger b_j B_{J_\mu}^\dagger = \frac{U}{2\varepsilon_j - E_{J_\nu}} b_j^\dagger - \frac{U}{2\varepsilon_j - E_{J_\nu}} b_j^\dagger B_{J_\mu}^\dagger \frac{B_{J_\mu}}{E_{J_\nu} - E_{J_\mu}}.$$ (17)

The rightmost expression follows via a partial fraction expansion, and remarkably, contains only $B_{J_\mu}^\dagger$ operators and no more $b_j^\dagger b_j$. This enables us to eliminate the $b_j^\dagger b_j$ from the second line of (16), by rewriting it as follows (we commute its term in square brackets to the right, using a relation similar to (12), but with the commutator (17) instead of $[H, B_{J_\mu}^\dagger]$):

$$\sum_{\nu=1}^n \left( \prod_{\eta=1}^{\nu-1} B_{J_\eta}^\dagger \right) \sum_{\mu=\nu+1}^n \left( \prod_{\eta'=\nu+1}^{\mu-1} B_{J_{\eta'}}^\dagger \right) \left( \frac{2g}{E_{J_\nu} - E_{J_\mu}} \right) \left( \prod_{\eta=1(\neq \mu)}^n B_{J_\eta}^\dagger \right) |0\rangle |0\rangle$$

$$= \sum_{\mu=1}^n \sum_{\nu=1}^{\mu-1} \frac{2g}{E_{J_\nu} - E_{J_\mu}} B_{J_\mu}^\dagger \left( \prod_{\eta=1(\neq \mu)}^n B_{J_\eta}^\dagger \right) |0\rangle$$

$$- \sum_{\nu=1}^n \sum_{\mu=\nu+1}^n \frac{2g}{E_{J_\nu} - E_{J_\mu}} B_{J_\mu}^\dagger \left( \prod_{\eta=1(\neq \nu)}^n B_{J_\eta}^\dagger \right) |0\rangle$$

$$= \sum_{\nu=1}^n \sum_{\mu=1(\neq \nu)}^n \frac{2g}{E_{J_\nu} - E_{J_\mu}} B_{J_\mu}^\dagger \left( \prod_{\eta=1(\neq \nu)}^n B_{J_\eta}^\dagger \right) |0\rangle.$$ (18)

(The last line follows by renaming the dummy indices $\nu \leftrightarrow \mu$ in the second line.) Substituting (18) for the second line of (16), we conclude that $(H_U, B_{J_\nu}^\dagger)$ will equal zero provided that

$$1 - \sum_j \frac{U}{2\varepsilon_j - E_{J_\nu}} + \sum_{\mu=1(\neq \nu)}^n \frac{2g}{E_{J_\mu} - E_{J_\nu}} = 0, \quad \text{for } \nu = 1, \ldots, n.$$ (19)

This constitutes a set of $n$ coupled equations for the $n$ parameters $E_{J_1}, \ldots, E_{J_n}$, which may be thought of as self-consistently-determined pair energies. Eq. (19) can be regarded as a generalization of the true-boson eigenvalue equation (9), and was originally derived by Richardson by solving the Schrödinger equation for the wave-function $\psi(j_1, \ldots, j_n)$ of (3). It is truly remarkable that the exact eigenstates of a complicated many-body problem can be constructed by such a simple generalization of the solution of a quadratic (i.e. non-interacting) true-boson Hamiltonian!

Below we shall always assume the $\varepsilon_j$’s to be all distinct. Then there exists a simple relation between the bare pair energies $2\varepsilon_j$ and the solutions
of (19): as \( g \) is reduced to 0, it follows by inspection that each solution \( \{E_{J_1}, \ldots, E_{J_n}\} \) reduces smoothly to a certain set of \( n \) bare pair energies, say \( \{2\epsilon_{j_1}, \ldots, 2\epsilon_{j_n}\} \). Correspondingly, the state \( |\Psi_n\rangle_U \equiv |J_1, \ldots, J_n\rangle_U \) of (11) reduces smoothly to the state \( |j_1, \ldots, j_n\rangle_U \equiv \prod_{\nu=1}^n b_{j_\nu}^\dagger |0\rangle \) (up to a normalization factor not shown here). Thus there is a one-to-one correspondence between the set of all states \( \{|J_1, \ldots, J_n\rangle_U\} \) and the set of all states \( \{|j_1, \ldots, j_n\rangle_U\} \). Since the latter constitute a complete eigenbasis for the \( n \)-pair Hilbert space defined on the set of unblocked levels \( U \), the former do too.

2.5. GROUND STATE

For a given set of blocked levels \( B \), the lowest-lying of all states \( |n, B\rangle \), say \( |n, B\rangle_G \), is obtained by using that particular solution \( E_{J_1}, \ldots, E_{J_n} \) for which the total “pair energy” \( \mathcal{E}_n \) takes its lowest possible value (as \( g \) is increased, some of the \( E_J \)s become complex; however, they always occur in complex conjugate pairs, so that \( \mathcal{E}_n \) remains real [17]).

The lowest-lying of all eigenstates with \( n \) pairs and \( b \) blocked levels, say \( |n, b\rangle_G \) with energy \( \mathcal{E}_b^G(n) \), is that \( |n, B\rangle_G \) for which the blocked levels in \( B \) are all as close as possible to \( \varepsilon_F \), the Fermi energy of the uncorrelated \( N \)-electron Fermi sea \( |F_N\rangle \). The \( E_{J_\nu} \) for the ground state \( |n, b\rangle_G \) coincide at \( g = 0 \) with the lowest \( n \) energies \( 2\epsilon_j \) \( (j = 1, \ldots, n) \), and smoothly evolve toward lower values as \( g \) is turned on. This fact can be exploited during the numerical solution of (19), which can be simplified by first making some algebraic transformations, discussed in detail in [15], that render the equations less singular.

2.6. GENERAL COMMENTS

Since the exact solution provides us with wave functions, it is in principle straightforward to calculate arbitrary correlation functions. Some such correlators are discussed by Richardson in [16, 17], who showed that they can be expressed in terms of certain determinants that are most conveniently calculated numerically. Moreover, it is natural to ask whether in the bulk limit, the standard BCS results can be extracted from the exact solution. Indeed they can, as Richardson showed in [20], by interpreting the problem of solving (19) for the \( E_{J_\nu} \) as a problem in two-dimensional electrostatics. Exploiting this analogy, he showed that in the bulk limit \( (N_S \rightarrow \infty \text{ at fixed } N_Sd) \), Eqs. (19) reduce to the well-known BCS gap equation and the BCS equation for the chemical potential, and the condensation energy \( \mathcal{E}_0^G(n) \) (defined in Eq. (20) below) to its BCS result, namely \( -\Delta^2/2d \).
3. Comparison with Other Approaches

We now apply the exact solution to check the quality of results previously obtained by various other methods. Most previous works [2, 3, 4, 6, 7, 8, 9, 10] studied a half-filled band with fixed width $2\omega_D$ of uniformly-spaced levels (i.e. $\varepsilon_j = j d$), containing $N = 2n + b$ electrons. Then the level spacing is $d = 2\omega_D/N$ and in the limit $d \to 0$ the bulk gap is $\Delta = \omega_D \sinh(1/\lambda)^{-1}$. Following [9], we take $\lambda = 0.224$ throughout this paper. To study the SC/FD crossover, two types of quantities were typically calculated as functions of increasing $d/\Delta$, which mimics decreasing grain size: the even and odd ($b = 0, 1$) condensation energies

$$E_b^C(n) = \mathcal{E}_b^G(n) - \langle F_N | H | F_N \rangle;$$

and a parity parameter introduced by Matveev and Larkin (ML) [6] to characterize the even-odd ground state energy difference,

$$\Delta_{ML}(n) = \mathcal{E}_1^G(n) - [\mathcal{E}_0^G(n) + \mathcal{E}_0^G(n + 1)]/2.$$

Following the initial g.c. studies [2]-[6], the first canonical study was that of Mastellone, Falci and Fazio (MFF) [7], who used Lanczos exact diagonalization (with $n \leq 12$) and a scaling argument to probe the crossover regime. Berger and Halperin (BH) [8] showed that essentially the same results could be achieved with $n \leq 6$ by first reducing the bandwidth and renormalizing $\lambda$, thus significantly reducing the calculational effort involved. To access larger systems and fully recover the bulk limit, fixed-$n$ projected variational BCS wavefunctions (PBCS) were used in [9] (for $n \leq 600$); significant improvements over the latter results, in particular in the crossover regime, were subsequently achieved in [10] using the density matrix renormalization group (DMRG) (with $n \leq 400$). Finally, Dukelsky and Schuck [11] showed that a self-consistent RPA approach, that in principle can be extended to finite temperatures, describes the f.d. regime rather well (though not as well as the DMRG).

To check the quality of the above methods, we [21, 22] computed $E_b^C(n)$ and $\Delta_{ML}(n)$ using Richardson’s solution (Fig. 1). The exact results (a) quantitatively agree, for $d \to 0$, with the leading $-\Delta^2/2d$ behavior for $E_b^C(n)$ obtained in the g.c. BCS approach [2, 3, 4], which in this sense is exact in the bulk limit, corrections being of order $d^0$; (b) confirm that a completely smooth [10] crossover occurs around the scale $d \simeq \Delta$ at which the g.c. BCS approach breaks down; (c) show that the PBCS crossover [9] is qualitatively correct, but not quantitatively, being somewhat too abrupt; (d) are reproduced remarkably well by the approaches of MFF [7] and BH [8]; (e) are fully reproduced by the DMRG of [10] with a relative error of
Figure 1. (a) The even and odd \((b = 0,1)\) condensation energies \(E_C^b\) of Eq. (20), calculated with BCS, PBCS and exact wave functions, as functions of \(d/\Delta = 2 \sinh(1/\lambda)/(2n + b)\), for \(\lambda = 0.224\). For comparison the dotted line gives the “bulk” result \(E_0^\text{bulk} = -\Delta^2/(2d)\). (b) Comparison of the parity parameters \(\Delta_{\text{ML}}\) of Eq. (21) obtained by various authors: ML’s analytical result (dotted lines) \([\Delta(1 - d/2\Delta)\) for \(d \ll \Delta,\) and \(d/2 \log(ad/\Delta)\) for \(d \gg \Delta,\) with \(a = 1.35\) adjusted to give asymptotic agreement with the exact result\]; grand-canonical BCS approach (dash-dotted line) \([\text{the naive perturbative result } 1/2d\text{ is continued to the origin}];\) PBCS approach (short-dashed line); Richardson’s exact solution (thick solid line); exact diagonalization and scaling by MFF (open circles) and BH (long-dashed line).

< 10^{-4} for \(n \leq 400\); our figures don’t show DMRG curves, since they are indistinguishable from the exact ones and are discussed in detail in [10].

4. Conclusions

The main conclusion we can draw from these comparisons is that the two approaches based on renormalization group ideas work very well: the DMRG is essentially exact for this model, but the band-width rescaling method of BH also gives remarkably (though not quite as) good results
with rather less effort. In contrast, the PBCS approach is rather unreliable in the crossover region.

Acknowledgments: The derivation of the exact solution shown above, which is shorter and perhaps somewhat more direct than the original published derivation, was invented by R. W. Richardson too; we thank him for a private communication suggesting this route. We also thank Moshe Schechter for helpful comments on the manuscript.

References

1. D.C. Ralph, C.T. Black and M. Tinkham, Phys. Rev. Lett. 76, 688 (1996); 78, 4087 (1997).
2. J. von Delft et al., Phys. Rev. Lett. 77, 3189 (1996).
3. F. Braun et al., Phys. Rev. Lett. 79, 921 (1997).
4. F. Braun and J. von Delft, Phys. Rev. B 59, 9527 (1999).
5. R.A. Smith and V. Ambegaokar, Phys. Rev. Lett. 77, 4962 (1996).
6. K.A. Matveev and A.I. Larkin, Phys. Rev. Lett. 78, 3749 (1997).
7. A. Mastellone, G. Falci and R. Fazio, Phys. Rev. Lett. 80, 4542 (1998).
8. S.D. Berger and B.I. Halperin, Phys. Rev. B 58, 5213 (1998).
9. F. Braun and J. von Delft, Phys. Rev. Lett. 81, 4712 (1998).
10. J. Dukelsky and G. Sierra, Phys. Rev. Lett. 83, 172 (1999); and cond-mat/9906166.
11. J. Dukelsky and P. Schuck, to appear in Phys. Lett. B.
12. R. W. Richardson, Phys. Lett. 3, 277 (1963).
13. R. W. Richardson, Phys. Lett. 5, 82 (1963).
14. R. W. Richardson and N. Sherman, Nucl. Phys. 52, 221 (1964).
15. R. W. Richardson, Phys. Lett. 14, 325 (1965).
16. R. W. Richardson, J. Math. Phys. 6, 1034 (1965).
17. R. W. Richardson, Phys. Rev. 141, 949 (1966).
18. R. W. Richardson, Phys. Rev. 144, 874 (1966).
19. R. W. Richardson, Phys. Rev. 159, 792 (1966).
20. R. W. Richardson, J. Math. Phys. 18 (1977) 1802.
21. F. Braun, Ph.D. thesis, Karlsruhe University (1999); F. Braun and J.von Delft, Advances in Solid State Physics, (Ed. B. Kramer), p. 341, Vieweg, Braunschweig (1999).
22. G. Sierra, J. Dukelsky, G. G. Dussel, J. von Delft, and F. Braun, cond-mat/9909015.
23. V. G. Soloviev, Mat. Fys. Skrif. Kong. Dan. Vid. Selsk. 1, 1 (1961).
24. R. W. Richardson, private communication (1999).