Phase Correlations and Topological Measures of Large-scale Structure

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1 Introduction

The process of gravitational instability initiated by small primordial density perturbations is a vital ingredient of cosmological models that attempt to explain how galaxies and large-scale structure formed in the Universe. In the standard picture (the “concordance” model), a period of accelerated expansion (“inflation”) generated density fluctuations with simple statistical properties through quantum processes (Starobinsky 1979, 1980, 1982; Guth 1981; Guth & Pi 1982; Albrecht & Steinhardt 1982; Linde 1982). In this scenario the primordial density field is assumed to form a statistically homogenous and isotropic Gaussian Random Field (GRF). Over years of observational scrutiny this paradigm has strengthened its hold in the minds of cosmologists and has survived many tests, culminating in those furnished by the Wilkinson Microwave Anisotropy Probe (WMAP; Bennett et al. 2003; Hinshaw et al 2003).

Gaussian random fields are the simplest fully-defined stochastic processes (Adler 1981; Bardeen et al. 1986), which makes their analysis relatively straightforward. Robust and powerful statistical descriptors can be constructed that have a firm mathematical underpinning and are relatively simple to implement. Second-order statistics such as the ubiquitous power-spectrum (e.g. Peacock & Dodds 1996) furnish a complete description of Gaussian fields. They have consequently yielded invaluable insights into the behaviour of large-scale structure in the latest generation of redshift surveys, such as the 2dFGRS (Percival et al. 2001). Important though these methods undoubtedly are, the era of precision cosmology we are now entering requires more thought to be given to methods for both detecting and exploiting departures from Gaussian behaviour.

Even if the primordial density fluctuations were indeed Gaussian, the later stages of gravitational clustering must induce some form of non-linearity. One particular way of looking at this issue is to study the behaviour of Fourier modes of the cosmological density field. If the hypothesis of primordial Gaussianity is correct then these modes began with random spatial phases. In the
early stages of evolution, the plane-wave components of the density evolve independently like linear waves on the surface of deep water. As the structures grow in mass, they interact with other in non-linear ways, more like waves breaking in shallow water. These mode-mode interactions lead to the generation of coupled phases. While the Fourier phases of a Gaussian field contain no information (they are random), non-linearity generates non-random phases that contain much information about the spatial pattern of the fluctuations. Although the significance of phase information in cosmology is still not fully understood, there have been a number of attempts to gain quantitative insight into the behaviour of phases in gravitational systems. Ryden & Gramann (1991), Soda & Suto (1992) and Jain & Bertschinger (1998) concentrated on the evolution of phase shifts for individual modes using perturbation theory and numerical simulations. An alternative approach was adopted by Scherrer, Melott & Shandarin (1991), who developed a practical method for measuring the phase coupling in random fields that could be applied to real data. Most recently Chiang & Coles (2000), Coles & Chiang (2000), Chiang (2001) and Chiang, Naselsky & Coles (2002) have explored the evolution of phase information in some detail.

Despite this recent progress, there is still no clear understanding of how the behaviour of the Fourier phases manifests itself in more orthodox statistical descriptors. In particular there is much interest in the usefulness of the simplest possible generalisation of the (second-order) power-spectrum, i.e. the (third-order) bispectrum (Peebles 1980; Scoccimarro et al. 1998; Scoccimarro, Couchman & Frieman 1999; Verde et al. 2000; Verde et al. 2001; Verde et al. 2002). Since the bispectrum is identically zero for a Gaussian random field, it is generally accepted that the bispectrum encodes some form of phase information but it has never been elucidated exactly what form of correlation it measures. Further possible generalisations of the bispectrum are usually called polyspectra; they include the (fourth-order) trispectrum (Verde & Heavens 2001) or a related but simpler statistic called the second-spectrum (Stirling & Peacock 1996). Exploring the connection between polyspectra and non-linearly induced phase association is one of the aims of this paper.

Gravitational instability is expected to generate phase correlations (and non-Gaussianity) even if the primordial fluctuations were Gaussian. The Cosmic Microwave Background (CMB) allows us to probe the fluctuations while they are still in the linear regime and thus test the level of primordial non-Gaussianity without having to worry about non-linear effects. A second aim of this paper is to explain how one can use phase correlations in spherical harmonic expansions of temperature fluctuations in order to detect departures from standard fluctuation statistics.

Finally I discuss the use of topological invariants such as the Euler–Poincaré characteristic of isodensity contours to assess the level of non-Gaussianity in large-scale structure.
2 Basic Statistical Tools

I start by giving some general definitions of concepts which I will later use in relation to the particular case of cosmological density fields. In order to put our results in a clear context, I develop the basic statistical description of cosmological density fields; see also, e.g., Peebles (1980) and Coles & Lucchin (2002).

2.1 Fourier Description

I follow standard practice and consider a region of the Universe having volume $V_u$, for convenience assumed to be a cube of side $L \gg l_s$, where $l_s$ is the maximum scale at which there is significant structure due to the perturbations. The region $V_u$ can be thought of as a “fair sample” of the Universe if this is the case. It is possible to construct, formally, a “realisation” of the Universe by dividing it into cells of volume $V_u$ with periodic boundary conditions at the faces of each cube. This device is often convenient, but in any case one often takes the limit $V_u \to \infty$. Let us denote by $\bar{\rho}$ the mean density in a volume $V_u$ and take $\rho(x)$ to be the density at a point in this region specified by the position vector $x$ with respect to some arbitrary origin. As usual, the fluctuation is defined to be

$$\delta(x) = \frac{\rho(x) - \bar{\rho}}{\bar{\rho}}. \quad (1)$$

We assume this to be expressible as a Fourier series:

$$\delta(x) = \sum_k \delta_k \exp(i k \cdot x) = \sum_k \delta_k^* \exp(-i k \cdot x); \quad (2)$$

the appropriate inverse relationship is of the form

$$\delta_k = \frac{1}{V_u} \int_{V_u} \delta(x) \exp(-i k \cdot x) d x. \quad (3)$$

The Fourier coefficients $\delta_k$ are complex quantities,

$$\delta_k = |\delta_k| \exp(i \phi_k), \quad (4)$$

with amplitude $|\delta_k|$ and phase $\phi_k$. The assumption of periodic boundaries results in a discrete $k$-space representation; the sum is taken from the Nyquist frequency $k_{Ny} = 2\pi/L$, where $V_u = L^3$, to infinity. Note that as $L \to \infty$, $k_{Ny} \to 0$. Conservation of mass in $V_u$ implies $\delta_{k=0} = 0$ and the reality of $\delta(x)$ requires $\delta_k^* = \delta_{-k}$.

If, instead of the volume $V_u$, we had chosen a different volume $V'_u$, the perturbation within the new volume would again be represented by a series of the form (2), but with different coefficients $\delta_k$. Now consider a (large) number $N$ of realisations of our periodic volume and label these realisations by $V_{u1},$
It is meaningful to consider the probability distribution $P(\delta_k)$ of the relevant coefficients $\delta_k$ from realisation to realisation across this ensemble. One typically assumes that the distribution is statistically homogeneous and isotropic, in order to satisfy the Cosmological Principle, and that the real and imaginary parts of $\delta_k$ have a Gaussian distribution and are mutually independent, so that

$$P(w) = \frac{V_u^{1/2}}{(2\pi\alpha_k^2)^{1/2}} \exp\left(-\frac{w^2V_u}{2\alpha_k^2}\right),$$  \hfill (5)

where $w$ stands for either Re $[\delta_k]$ or Im $[\delta_k]$ and $\alpha_k^2 = \sigma_k^2/2$; $\sigma_k^2$ is the spectrum. This is the same as the assumption that the phases $\phi_k$ in equation (5) are mutually independent and randomly distributed over the interval between $\phi = 0$ and $\phi = 2\pi$. In this case the moduli of the Fourier amplitudes have a Rayleigh distribution:

$$P(|\delta_k|, \phi_k)d|\delta_k|d\phi_k = \frac{|\delta_k|V_u}{2\pi\sigma_k^2}\exp\left(-\frac{|\delta_k|^2V_u}{2\sigma_k^2}\right)d|\delta_k|d\phi_k.$$  \hfill (6)

Because of the assumption of statistical homogeneity and isotropy, the quantity $P(\delta_k)$ depends only on the modulus of the wavevector $k$ and not on its direction. It is fairly simple to show that, if the Fourier quantities $|\delta_k|$ have the Rayleigh distribution, then the probability distribution $P(\delta)$ of $\delta = \delta(x)$ in real space is Gaussian, so that:

$$P(\delta)d\delta = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left(-\frac{\delta^2}{2\sigma^2}\right)d\delta,$$  \hfill (7)

where $\sigma^2$ is the variance of the density field $\delta(x)$. This is a strict definition of Gaussianity. However, Gaussian statistics do not always require the distribution (7) for the Fourier component amplitudes. According to its Fourier expansion, $\delta(x)$ is simply a sum over a large number of Fourier modes whose amplitudes are drawn from some distribution. If the phases of each of these modes are random, then the Central Limit Theorem will guarantee that the resulting superposition will be close to a Gaussian if the number of modes is large and the distribution of amplitudes has finite variance. Such fields are called weakly Gaussian.

### 2.2 Covariance Functions & Probability Densities

I now discuss the real-space statistical properties of spatial perturbations in $\rho$. The covariance function is defined in terms of the density fluctuation by

$$\xi(r) = \frac{\langle[\rho(x) - \bar{\rho}][\rho(x+r) - \bar{\rho}]\rangle}{\bar{\rho}^2} = \langle\delta(x)\delta(x+r)\rangle.$$  \hfill (8)
The angle brackets in this expression indicate two levels of averaging: first a volume average over a representative patch of the universe and second an average over different patches within the ensemble, in the manner of §2.1. Applying the Fourier machinery to equation (8) one arrives at the Wiener–Khintchin theorem, relating the covariance to the spectral density function or power spectrum, \( P(k) \):

\[
\xi(r) = \sum_k \langle |\delta_k|^2 \rangle \exp(-i k \cdot r),
\]

(9)

which, in passing to the limit \( V_u \to \infty \), becomes

\[
\xi(r) = \frac{1}{(2\pi)^3} \int P(k) \exp(-i k \cdot r) dk.
\]

(10)

Averaging equation (9) over \( r \) gives

\[
\langle \xi(r) \rangle_r = \frac{1}{V_u} \sum_k \langle |\delta_k|^2 \rangle \int \exp(-i k \cdot r) dr = 0.
\]

(11)

The function \( \xi(r) \) is the two–point covariance function. In an analogous manner it is possible to define spatial covariance functions for \( N > 2 \) points. For example, the three–point covariance function is

\[
\zeta(r,s) = \langle \rho(x) - \bar{\rho} \rangle \langle \rho(x + r) - \bar{\rho} \rangle \langle \rho(x + s) - \bar{\rho} \rangle
\]

(12)

which gives

\[
\zeta(r,s) = \langle \delta(x) \delta(x + r) \delta(x + s) \rangle,
\]

(13)

where the spatial average is taken over all the points \( x \) and over all directions of \( r \) and \( s \) such that \( |r - s| = t \): in other words, over all points defining a triangle with sides \( r, s \) and \( t \). The generalisation of (12) to \( N > 3 \) is obvious.

The covariance functions are related to the moments of the probability distributions of \( \delta(x) \). If the fluctuations form a Gaussian random field then the \( N \)-variate distributions of the set \( \delta_i \equiv \delta(x_i) \) are just multivariate Gaussians of the form

\[
\mathcal{P}_N(\delta_1, \ldots, \delta_N) = \frac{1}{(2\pi)^{N/2} \det C^{1/2}} \exp \left( -\frac{1}{2} \sum_{i,j} \delta_i C^{-1}_{ij} \delta_j \right).
\]

(14)

The correlation matrix \( C_{ij} \) can be expressed in terms of the covariance function

\[
C_{ij} = \langle \delta_i \delta_j \rangle = \xi(r_{ij}).
\]

(15)

It is convenient to go a stage further and define the \( N \)-point connected covariance functions as the part of the average \( \langle \delta_i \ldots \delta_N \rangle \) that is not expressible in terms of lower order functions e.g.
\[\langle \delta_1 \delta_2 \delta_3 \rangle = \langle \delta_1 \rangle_c \langle \delta_2 \delta_3 \rangle_c + \langle \delta_2 \rangle_c \langle \delta_1 \delta_3 \rangle_c + \langle \delta_3 \rangle_c \langle \delta_1 \delta_2 \rangle_c + \langle \delta_1 \rangle_c \langle \delta_2 \rangle_c \langle \delta_3 \rangle_c + \langle \delta_1 \delta_2 \delta_3 \rangle_c,\]

where the connected parts are \(\langle \delta_1 \delta_2 \delta_3 \rangle_c, \langle \delta_1 \delta_2 \rangle_c,\) etc. Since \(\langle \delta \rangle = 0\) by construction, \(\langle \delta_1 \rangle_c = \langle \delta_1 \rangle = 0.\) Moreover, \(\langle \delta_1 \delta_2 \rangle_c = \langle \delta_1 \delta_2 \rangle\) and \(\langle \delta_1 \delta_2 \delta_3 \rangle_c = \langle \delta_1 \delta_2 \delta_3 \rangle.\) The second and third order connected parts are simply the same as the covariance functions. Fourth and higher order quantities are different, however. The connected functions are just the multivariate generalisation of the cumulants \(\kappa_N\) (Kendall & Stewart 1977). One of the most important properties of Gaussian fields is that all of their N-point connected covariances are zero beyond \(N=2,\) so that their statistical properties are fixed once the set of two-point covariances (15) is determined. All large-scale statistical properties are therefore determined by the asymptotic behaviour of \(\xi(r)\) as \(r \to \infty.\)

### 3 Phase Coupling

In §2 we pointed out that a convenient definition of a Gaussian field could be made in terms of its Fourier phases, which should by independent and uniformly distributed on the interval \([0, 2\pi].\) A breakdown of these conditions, such as the correlation of phases of different wavemodes, is a signature that the field has become non-Gaussian. In terms of cosmic large-scale structure formation, non-Gaussian evolution of the density field is symptomatic of the onset of non-linearity in the gravitational collapse process, suggesting that phase evolution and non-linear evolution are closely linked. A relatively simple picture emerges for models where the primordial density fluctuations are Gaussian and the initial phase distribution is uniform. When perturbations remain small evolution proceeds linearly, individual modes grow independently and the original random phase distribution is preserved. However, as perturbations grow large their evolution becomes non-linear and Fourier modes of different wavenumber begin to couple together. This gives rise to phase association and consequently to non-Gaussianity. It is clear that phase associations of this type should be related in some way to the existence of the higher order connected covariance functions, which are traditionally associated with non-linearity and are non-zero only for non-Gaussian fields. In this sections such a relationship is explored in detail using an analytical model for the non-linearly evolving density fluctuation field. Phase correlations of a particular form are identified and their connection to the covariance functions is established.

A graphic demonstration of the importance of phases in patterns generally is given in Figure 1. Since the amplitude of each Fourier mode is unchanged in the phase reshuffling operation, these two pictures have exactly the same power-spectrum, \(P(k) \propto |\tilde{\delta}(k)|^2.\) In fact, they have more than that: they have exactly the same amplitudes for all \(k.\) They also have totally different
morphology. Further demonstrations of the importance of Fourier phases in defining clustering morphology are given by Chiang (2001).

3.1 Quadratic density fields

It is useful at this stage to a particular form of non-Gaussian field that serves both as a kind of phenomenological paradigm and as a reasonably realistic model of non-linear evolution from Gaussian initial conditions. The model involves a field which is generated by a simple quadratic transformation of a Gaussian distribution, hence the term *quadratic* non-linearity. Quadratic fields have been discussed before from a number of contexts (e.g. Coles & Barrow 1987; Moscardini et al. 1991; Falk, Rangarajan & Srednicki 1993; Luo & Schramm 1993; Luo 1994; Gangui et al. 1994; Koyoma, Soda & Taruya 1999; Peebles 1999a,b; Matarrese, Verde & Jimenez 2000; Verde et al. 2000; Verde et al. 2001; Komatsu & Spergel 2001; Shandarin 2002; Bartolo, Matarrese & Riotto 2002); for further discussion see below. The motivation is very similar to that of Coles & Jones (1991), which introduced the lognormal density field as an illustration of some of the consequences of a more extreme form of non-linearity involving an exponential transformation of the linear density field.

3.2 A simple non-linear model

We adopt a simple perturbative expansion of the form

\[ \delta(x) = \delta_1(x) + \epsilon \delta_2(x) \]  

(17)
to mimic the non-linear evolution of the density field. Although the equivalent transformation in formal Eulerian perturbation theory is a good deal more complicated, the kind of phase associations that we will deal with here are precisely the same in either case. In terms of the Fourier modes, in the continuum limit, we have for the first order Gaussian term

$$\delta_1(x) = \int d^3k \, |\delta_k| \exp[i\phi_k] \exp[ik \cdot x] \quad (18)$$

and for the second-order perturbation

$$\delta_2(x) = [\delta_1(x)]^2 = \int d^3k \, d^3k' \, |\delta_k||\delta_{k'}| \exp[i(\phi_k + \phi_{k'})] \exp[i(k + k') \cdot r]. \quad (19)$$

The quadratic field, $\delta_2$, illustrates the idea of mode coupling associated with non-linear evolution. The non-linear field depends on a specific harmonic relationship between the wavenumber and phase of the modes at $k$ and $k'$. This relationship between the phases in the non-linear field, i.e.

$$\phi_k + \phi_{k'} = \phi_{k+k'}, \quad (20)$$

where the RHS represents the phase of the non-linear field, is termed quadratic phase coupling.

### 3.3 The two-point covariance function

The two-point covariance function can be calculated using the definitions of §2, namely

$$\xi(r) = \langle \delta(x)\delta(x+r) \rangle. \quad (21)$$

Substituting the non-linear transform for $\delta(x)$ (equation 17) into this expression gives four terms

$$\xi(r) = \langle \delta_1(x)\delta_1(x+r) \rangle + c \langle \delta_1(x)\delta_2(x+r) \rangle + c \langle \delta_2(x)\delta_1(x+r) \rangle + c^2 \langle \delta_2(x)\delta_2(x+r) \rangle. \quad (22)$$

The first of these terms is the linear contribution to the covariance function whereas the remaining three give the non-linear corrections. We shall focus on the lowest order term for now.

As we outlined in Section 2, the angle brackets $\langle \rangle$ in these expressions are expectation values, formally denoting an average over the probability distribution of $\delta(x)$. Under the fair sample hypothesis we replace the expectation values in equation (21) with averages over a selection of independent volumes so that $\langle \rangle \rightarrow \langle \rangle_{\text{vol, real}}$. The first average is simply a volume integral over a sufficiently large patch of the universe. The second average is over various realisations of the $\delta_k$ and $\phi_k$ in the different patches. Applying these rules to the first term of equation (22) and performing the volume integration gives
\[ \xi_{11}(r) = \int d^3k d^3k' \langle |\delta_k||\delta_{k'}| \exp[i(\phi_k + \phi_{k'})] \rangle_{\text{real}} \delta_D(k + k') \exp[ik' \cdot s], \]  

(23)

where \( \delta_D \) is the Dirac delta function. The above expression is simplified given the reality condition

\[ \delta_k = \delta^*_{-k}, \]  

(24)

from which it is evident that the phases obey

\[ \phi_k + \phi_{-k} = 0 \mod[2\pi]. \]  

(25)

Integrating equation (23) one therefore finds that

\[ \xi_{11}(r) = \int d^3 k \langle |\delta_k|^2 \rangle_{\text{real}} \exp[-i k' \cdot s]. \]  

(26)

so that the final result is independent of the phases. Indeed this is just the Fourier transform relation between the two-point covariance function and the power spectrum we derived in §2.1.

### 3.4 The three-point covariance function

Using the same arguments outlined above it is possible to calculate the 3-point connected covariance function, which is defined as

\[ \zeta(r, s) = \langle \delta(x) \delta(x + r) \delta(x + s) \rangle_c. \]  

(27)

Making the non-linear transform of equation (17) one finds the following contributions

\[ \zeta(r, s) = \langle \delta_1(x) \delta_1(x + r) \delta_1(x + s) \rangle_c + c \langle \delta_1(x) \delta_1(x + r) \delta_2(x + s) \rangle_c \]
\[ + \text{perms}(121, 211) + c^2 \langle \delta_1(x) \delta_2(x + r) \delta_2(x + s) \rangle_c \]
\[ + \text{perms}(212, 221) + c^3 \langle \delta_2(x) \delta_2(x + r) \delta(x + s) \rangle_c. \]  

(28)

Again we consider first the lowest order term. Expanding in terms of the Fourier modes and once again replacing averages as prescribed by the fair sample hypothesis gives

\[ \zeta_{111}(r, s) = \int d^3 k d^3 k' d^3 k'' \langle |\delta_k||\delta_{k'}||\delta_{k''}| \exp[i(\phi_k + \phi_{k'} + \phi_{k''})] \rangle_{\text{real}} \]
\[ \times \delta_D(k + k' + k'') \exp[ik' \cdot r] \exp[ik'' \cdot s]. \]  

(29)

Recall that \( \delta_1 \) is a Gaussian field so that \( \phi_k, \phi_{k'} \) and \( \phi_{k''} \) are independent and uniformly random on the interval \([0, 2\pi]\). Upon integration over one of the wavevectors the phase terms is modified so that its argument contains the sum \((\phi_k + \phi_{k'} + \phi_{-k'-k''})\), or a permutation thereof. Whereas the reality condition of equation (24) implies a relationship between phases of
anti-parallel wavevectors, no such conditions hold for modes linked by the triangular constraint imposed by the Dirac delta function. In other words, except for serendipity,

$$\phi_k + \phi_{k'} + \phi_{-k''} \neq 0.$$  

(30)  

In fact due to the circularity of phases, the resulting sum is still just uniformly random on the interval $[0, 2\pi]$ if the phases are random. Upon averaging over sufficient realisations, the phase term will therefore cancel to zero so that the lowest order contribution to the 3-point function vanishes, i.e. $\zeta_{111}(r, s) = 0$. This is not a new result, but it does explicitly illustrate how the vanishing of the three-point connected covariance function arises in terms of the Fourier phases.

Next consider the first non-linear contribution to the 3-point function given by

$$\zeta_{112}(r, s) = \epsilon \left< \delta_1(x) \delta_1(x + r) \delta_2(x + s) \right>,$$

(31)  

or one of its permutations. In this case one of the arguments in the average is the field $\delta_2(x)$, which exhibits quadratic phase coupling of the form (20). Expanding this term to the point of equation (29) using the definition (19) one obtains

$$\zeta_{112}(r, s) = \int d^3k \, d^3k' \, d^3k'' \, d^3k'''$$



$$\times \left< |\delta_k| |\delta_{k'}| |\delta_{k''}| |\delta_{k'''}| \exp \left[ i(\phi_k + \phi_{k'} + \phi_{k''} + \phi_{k'''})) \right] \right> \delta_D(k + k' + k'' + k''')$$

$$\times \exp [i k' \cdot r] \exp [i (k'' + k''') \cdot s].$$  

(32)  

Once again the Dirac delta function imposes a general constraint upon the configuration of wavevectors. Integrating over one of the $k$ gives $k''' = -k - k' - k''$ for example, so that the wavevectors must form a closed loop. This general constraint however, does not specify a precise shape of loop, instead the remaining integrals run over all of the different possibilities. At this point we may constrain the problem more tightly by noting that most combinations of the $k$ will contribute zero to $\zeta_{112}$. This is because of the circularity property of the phases and equation (30). Indeed, the only nonzero contributions arise where we are able to apply the phase relation obtained from the reality constraint, equation (25). In other words the properties of the phases dictate that the wavevectors must align in anti-parallel pairs: $k = -k'$, $k'' = -k'''$ and so forth.

There is a final constraint that must be imposed upon the $k$ if $\zeta$ is the connected 3-point covariance function. In a graph theoretic sense, the general (unconnected) $N$-point function $\left< \delta_{l_1}(x_1) \delta_{l_2}(x_2) \ldots \delta_{l_N}(x_N) \right>$ can be represented geometrically by a sum of tree diagrams. Each diagram consists of $N$ nodes of order $l_i$, representing the $\delta_{l_i}(x_i)$, and a number of linking lines denoting their correlations; see Fry (1984) or Bernardeau (1992) for more detailed
accounts. Every node is made up of \( l_i \) internal points, which represent a factor \( \delta_k = |\delta_k| \exp(i\phi_k) \) in the Fourier expansion. According to the rules for constructing diagrams, linking lines may join one internal point to a single other, either within the same node or in an external node. The connected covariance functions are represented specifically by the subset of diagrams for which every node is linked to at least one other, leaving none completely isolated. This constraint implies that certain pairings of wavevectors do not contribute to the connected covariance function. For more details, see Watts & Coles (2002).

The above constraints may be inserted into equation (32) by re-writing the Dirac delta function as a product over Delta functions of two arguments, appropriately normalised. There are only two allowed combinations of wavevectors so we have

\[
\delta_D(k+k'+k''+k''') \rightarrow \frac{1}{2V_u} [\delta_D(k+k'')\delta_D(k'''+k''') + \delta_D(k+k''')\delta_D(k'+k'')].
\]  

Integrating over two of the \( k \) and using equation (25) eliminates the phase terms and leaves the final result

\[
\zeta_{112}(r,s) = \frac{1}{V_u} \int d^3k \, d^3k' \langle |\delta_k|^2 |\delta_{k'}|^2 \rangle_{\text{real}} \exp[ik' \cdot r] \exp[-i(k+k') \cdot s].
\]  

The existence of this quantity has therefore been shown to depend on the quadratic phase coupling of Fourier modes. The relationship between modes and the interpretation of the tree diagrams is also dictated by the properties of the phases.

One may apply the same rules to the higher order terms in equation (28). It is immediately clear that the \( \zeta_{122} \) terms are zero because there is no way to eliminate the phase term \( \exp[i(\phi_k + \phi_{k'} + \phi_{k''} + \phi_{k'''} + \phi_{k''''})] \), a consequence of the property equation (30). Diagrammatically this corresponds to an unpaired internal point within one of the nodes of the tree. The final, highest order contribution to the 3-point function is found to be

\[
\zeta_{222}(r,s) = \frac{1}{V_u^2} \int d^3k \, d^3k' \, d^3k'' \langle |\delta_k|^2 |\delta_{k'}|^2 |\delta_{k''}|^2 \rangle_{\text{real}} \times \exp[i(k-k') \cdot r] \exp[i(k'-k'') \cdot s],
\]  

where the phase and geometric constraints allow 12 possible combinations of wavevectors.

### 3.5 Power-spectrum and Bispectrum

The formal development of the relationship between covariance functions and power-spectra developed above suggests the usefulness of higher–order versions of \( P(k) \). It is clear from the above arguments that a more convenient notation for the power-spectrum than that introduced in §2.1 is
\( \langle \delta_k \delta_{k'} \rangle = (2\pi)^3 P(k) \delta_D(k + k'). \) (36)

The connection between phases and higher-order covariance functions obtained above also suggests defining higher-order polyspectra of the form

\[ \langle \delta_k \delta_{k'} \ldots \delta_{k^{(n)}} \rangle = (2\pi)^3 P_n(k, k', \ldots k^{(n)}) \delta_D(k + k' + \ldots k^{(n)}) \] (37)

where the occurrence of the delta-function in this expression arises from a generalisation of the reality constraint given in equation (25); see, e.g., Peebles (1980). Conventionally the version of this with \( n = 3 \) produces the bispectrum, usually called \( B(k, k', k'') \) which has found much effective use in recent studies of large-scale structure (Peebles 1980; Scoccimarro et al. 1998; Scoccimarro, Couchman & Frieman 1999; Verde et al. 2000; Verde et al. 2001; Verde et al. 2002). It is straightforward to show that the bispectrum is the Fourier-transform of the (reduced) three-point covariance function by following similar arguments; see, e.g., Peebles (1980).

Note that the delta-function constraint requires the bispectrum to be zero except for \( k \)-vectors \( (k, k', k'') \) that form a triangle in \( k \)-space. It is clear that the bispectrum can only be non-zero when there is a definite relationship between the phases accompanying the modes whose wave-vectors form a triangle. Moreover the pattern of phase association necessary to produce a real and non-zero bispectrum is precisely that which is generated by quadratic phase association. This shows, in terms of phases, why it is that the leading order contributions to the bispectrum emerge from second-order fluctuations of a Gaussian random field. The bispectrum measures quadratic phase coupling.

Three-point phase correlations have another interesting property. While the bispectrum is usually taken to be an ensemble-averaged quantity, as defined in equation (37), it is interesting to consider products of terms \( \delta_k \delta_{k'} \delta_{k''} \) obtained from an individual realisation. According to the fair sample hypothesis discussed above we would hope appropriate averages of such quantities would yield an estimate of the bispectrum. Note that

\[ \delta_k \delta_{k'} \delta_{k''} = \delta_k \delta_{k'} \delta_{-k-k'} = \delta_k \delta_{k'} \delta_{k+k'} \equiv \beta(k, k'), \] (38)

using the requirement (25), together with the triangular constraint we discussed above. Each \( \beta(k, k') \) will carry its own phase, say \( \phi_{k, k'} \), which obeys

\[ \phi_{k, k'} = \phi_k + \phi_{k'} - \phi_{k+k'}. \] (39)

It is evident from this that it is possible to recover the complete set of phases \( \phi_k \) from the bispectral phases \( \phi_{k, k'} \), up to a constant phase offset corresponding to a global translation of the entire structure (Chiang & Coles 2000). This furnishes a conceptually simple method of recovering missing or contaminated phase information in a consistent way, an idea which has been exploited, for example, in speckle interferometry (Lohmann, Weigelt & Wirnitzer 1983). In the case of quadratic phase coupling, described by equation (20), the left-hand-side of equation (39) is identically zero leading to a particularly simple approach to this problem.
4 Phase Correlations in the CMB

Since the release of the first (preliminary) WMAP data set it has been subjected to a number of detailed independent analyses that have revealed some surprising features. Eriksen et al. (2004) have pointed out the existence of a North-South asymmetry suggesting that the cosmic microwave background (CMB) revealed by the WMAP data is not statistically homogeneous over the celestial sphere. This is consistent with the results of Coles et al. (2004) who found evidence for phase correlations in the WMAP data; see also Hajian & Souradeep (2003) and Hajian, Souradeep & Cornish (2004). The low–order multipoles of the CMB also display some peculiarities (de Oliveira-Costa et al. 2004a; Efstathiou 2004). Vielva et al. (2004) found significant non–Gaussian behaviour in a wavelet analysis of the same data, as did Chiang et al. (2004), Larson & Wandelt (2004) and Park (2004). Other analyses of the statistical properties of the WMAP have yielded results consistent with the standard form of fluctuation statistics (Komatsu et al. 2003; Colley & Gott 2003). These unusual properties may well be generated by residual foreground contamination (Banday et al. 2003; Naselsky et al. 2003; de Oliveira-Costa et al. 2004; Dineen & Coles 2004) or other systematic effects, but may also provide the first hints of physics beyond the standard cosmological model.

In order to tap the rich source of information provided by future CMB maps it is important to devise as many independent statistical methods as possible to detect, isolate and diagnose the various possible causes of departures from standard statistics. One particularly fruitful approach is to look at the behaviour of the complex coefficients that arise in a spherical harmonic analysis of CMB maps. Chiang et al. (2004), Chiang, Naselsky & Coles (2004), and Coles et al. (2004) have focussed on the phases of these coefficients on the grounds that a property of a statistically homogenous and isotropic GRF is that these phases are random. Phases can also be used to test for the presence of primordial magnetic fields (Chen et al. 2004; Naselsky et al. 2004) or evidence of non-trivial topology (Dineen, Rocha & Coles 2004).

4.1 Spherical Harmonics and Gaussian Fluctuations

We can describe the distribution of fluctuations in the microwave background over the celestial sphere using a sum over a set of spherical harmonics:

$$\Delta(\theta, \phi) = \frac{T(\theta, \phi) - \bar{T}}{\bar{T}} = \sum_{l=1}^{\infty} \sum_{m=-l}^{m+l} a_{l,m} Y_{l,m}(\theta, \phi).$$  (40)

Here $\Delta(\theta, \phi)$ is the departure of the temperature from the average at angular position $(\theta, \phi)$ on the celestial sphere in some coordinate system, usually galactic. The $Y_{l,m}(\theta, \phi)$ are spherical harmonic functions which we define in terms of the Legendre polynomials $P_{lm}$ using
\[ Y_{lm}(\theta, \phi) = (-1)^m \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_{lm}(\cos \theta) e^{im\phi}, \] (41)

i.e. we use the Condon-Shortley phase convention. In Equation (1), the \( a_{l,m} \) are complex coefficients which can be written

\[ a_{l,m} = x_{l,m} + iy_{l,m} = |a_{l,m}| \exp[i\phi_{l,m}]. \] (42)

Note that, since \( \Delta \) is real, the definitions (40) & (41) requires the following relations between the real and imaginary parts of the \( a_{l,m} \): if \( m \) is odd then

\[ x_{l,m} = \Re(a_{l,m}) = -\Re(a_{l,-m}) = -x_{l,-m}, \]
\[ y_{l,m} = \Im(a_{l,m}) = \Im(a_{l,-m}) = y_{l,-m}; \] (43)

while if \( m \) is even

\[ x_{l,m} = \Re(a_{l,m}) = \Re(a_{l,-m}) = x_{l,-m}, \]
\[ y_{l,m} = \Im(a_{l,m}) = -\Im(a_{l,-m}) = y_{l,-m}; \] (44)

and if \( m \) is zero then

\[ \Im(a_{l,m}) = y_{l,0} = 0. \] (45)

From this it is clear that the \( m = 0 \) mode always has zero phase, and there are consequently only \( l \) independent phase angles describing the harmonic modes at a given \( l \). Without loss of information we can therefore restrict our analysis to \( m \geq 0 \).

If the primordial density fluctuations form a Gaussian random field in space the temperature variations induced across the sky form a Gaussian random field over the celestial sphere. This means that

\[ \langle a_{l,m}a_{l',m'}^* \rangle = C_l \delta_{ll'}\delta_{mm'}, \] (46)

where \( C_l \) is the angular power spectrum, the subject of much scrutiny in the context of the cosmic microwave background (e.g. Hinshaw et al. 2003), and \( \delta_{xx'} \) is the Kronecker delta function. Since the phases are random, the stochastic properties of a statistically homogeneous and isotropic Gaussian random field are fully specified by the \( C_l \), which determines the variance of the real and imaginary parts of \( a_{l,m} \) both of which are Gaussian:

\[ \sigma^2(x_{l,m}) = \sigma^2(y_{l,m}) = \sigma_l^2 = \frac{1}{2} C_l. \] (47)

4.2 Testing for Phase Correlations

The approach we take is to assume that we have available a set of phases \( \phi_{l,m} \) corresponding to a set of spherical harmonic coefficients \( a_{l,m} \) obtained
from a data set, either real or simulated. We can also form phase differences in according to

$$D_m(l) = \phi_{l,m+1} - \phi_{l,m}. \quad (48)$$

If the orthodox cosmological interpretation of temperature fluctuations is correct, the phases of the $a_{l,m}$ should be random and so should phase differences of the form $\phi_{l,m+1} - \phi_{l,m}$ and $\phi_{l+1,m} - \phi_{l,m}$. Let us assume, therefore, that we have $n$ generic angles, $\theta_1 \ldots \theta_n$. Under the standard statistical assumption these should be random, apart from the constraints described in the previous section. The first thing we need is a way of testing whether a given set of phase angles is consistent with being drawn from uniform distribution on the unit circle. This is not quite as simple as it seems, particularly if one does not want to assume any particular form for actual distribution of angles, such as a bias in a particular direction; see Fisher (1993). Fortunately, however, there is a fully non–parametric method available, based on the theory of order statistics, and known as Kuiper’s statistic (Kuiper 1960).

Kuiper’s method revolves around the construction of a statistic, $V$, obtained from the data via the following prescription. First the angles are sorted into ascending order, to give the set $\{\theta_1, \ldots, \theta_n\}$. It does not matter whether the angles are defined to lie in $[0, 2\pi]$, $[-\pi, +\pi]$ or whatever. Each angle $\theta_i$ is divided by $2\pi$ to give a set of variables $X_i$, where $i = 1 \ldots n$. From the set of $X_i$ we derive two values $S^+_n$ and $S^-_n$ where

$$S^+_n = \max \left\{ \frac{1}{n} - X_1, \frac{2}{n} - X_2, \ldots, 1 - X_n \right\} \quad (49)$$

and

$$S^-_n = \max \left\{ X_1, X_2 - \frac{1}{n}, \ldots, X_n - \frac{n-1}{n} \right\}. \quad (50)$$

Kuiper’s statistic, $V$, is then defined as

$$V = (S^+_n + S^-_n) \cdot \left( \sqrt{n + 0.155 + \frac{0.24}{\sqrt{n}}} \right). \quad (51)$$

Anomalously large values of $V$ indicate a distribution that is more clumped than a uniformly random distribution, while low values mean that angles are more regular. The test statistic is normalized by the number of variates, $n$, in such a way that standard tables can be constructed to determine significance levels for any departure from uniformity; see Fisher (1993). In this context, however, it is more convenient to determine significance levels using Monte Carlo simulations of the “null” hypothesis of random phases. This is partly because of the large number of samples available for test, but also because we can use them to make the test more general.

The first point to mention is that a given set of phases, say belonging to the modes at fixed $l$ is not strictly speaking random anyway, because of the constraints noted in the previous section. One could deal with this by
discarding the conjugate phases, thus reducing the number of data points, but there is no need to do this when one can instead build the required symmetries into the Monte Carlo generator.

In addition, suppose the phases of the temperature field over the celestial sphere were indeed random, but observations were available only over apart of the sky, such as when a galactic cut is applied to remove parts of the map contaminated by foregrounds. In this case the mask may introduce phase correlations into the observations so the correct null hypothesis would be more complicated than simple uniform randomness. As long as any such selection effect were known, it could be built into the Monte Carlo simulation. One would then need to determine whether $V$ from an observed sky is consistent with having been drawn from the set of values of $V$ generated over the Monte Carlo ensemble.

There is also a more fundamental problem in applying this test to spherical harmonic phases. This is that a given set of $a_{l,m}$ depends on the choice of a particular coordinate axis. A given sky could actually generate an infinite number of different sets of $\phi_{l,m}$ because the phase angles are not rotationally invariant. One has to be sure to take different choices of $z$-axis into consideration when assessing significance levels, as a random phase distribution has no preferred axis while systematic artifacts may. A positive detection of non-randomness may result from a chance alignment of features with a particular coordinate axis in the real sky unless this is factored into the Monte Carlo simulations to. For both the real sky and the Monte Carlo skies we therefore need not a single value of $V$ but a distribution of $V$-values obtained by rotating the sky over all possible angles. A similar approach is taken by Hansen, Marinucci & Vittorio (2003). This method may seem somewhat clumsy, but a test is to be sensitive to departures from statistical homogeneity one should not base the test on measures that are rotationally invariant, such as those suggested by Ferreira, Mageuijo & Gorski (1998) as these involve averaging over the very fluctuations one is trying to detect.

4.3 Rotating the $a_{l,m}$

In view of the preceding discussion we need to know how to transform a given set of $a_{l,m}$ into a new set when the coordinate system is rotated into a different orientation. The method is fairly standard, but we outline it here to facilitate implementation of our approach.

Any rotation of the cartesian coordinate system $S\{x, y, z\} \mapsto S'\{x, y, z\}$ can be described using a set of three Euler angles $\alpha, \beta, \gamma$, which define the magnitude of successive rotations about the coordinate axes. In terms of a rotation operator $\hat{D}(\alpha, \beta, \gamma)$, defined so that a field $f(r, \theta, \phi)$ transforms according to

$$\hat{D}(\alpha, \beta, \gamma)f(r, \theta, \phi) = f'(r, \theta, \phi) = f(r, \theta', \phi'),$$

a vector $r$ is transformed as
\[ r' = D(0, 0, \gamma)D(0, \beta, 0)D(\alpha, 0, 0)r \equiv D(\alpha, \beta, \gamma)r. \] (53)

Here \( D \) is a matrix representing the operator \( \hat{D} \), i.e.

\[
D(\alpha, \beta, \gamma) = 
\begin{pmatrix}
\cos \gamma & \sin \gamma & 0 \\
-\sin \gamma & \cos \gamma & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\cos \beta & 0 & -\sin \beta \\
0 & 1 & 0 \\
\sin \beta & 0 & \cos \beta
\end{pmatrix}
\begin{pmatrix}
\cos \alpha & \sin \alpha & 0 \\
-\sin \alpha & \cos \alpha & 0 \\
0 & 0 & 1
\end{pmatrix}.
\] (54)

The Wigner \( D \) functions describe the rotation operator used to realise the transformations of covariant components of tensors with arbitrary rank \( l \). The functions, written as \( D_{m, m'}^l \), transform a tensor from \( S\{x, y, z\} \) to \( S'\{x', y', z'\} \). Consider a tensor \( Y_{l, m}(\theta, \phi) \) defined under the coordinate system \( S \) and apply the rotation operator we get:

\[
\hat{D}(\alpha, \beta, \gamma)Y_{l, m}(\theta, \phi) = Y_{l, m'}(\theta', \phi') = \sum_{m} Y_{l, m}(\theta, \phi)D_{m, m'}^l(\theta, \phi) \] (55)

This means that the transformation of the tensor under the rotation of the coordinate system can be represented as a matrix multiplication. Finding the rotated coefficients therefore requires a simple matrix multiplication once the appropriate \( D \) function is known. To apply this in practice one needs a fast and accurate way of generating the matrix elements \( D_{m, m'}^l \) for the rotation matrix. There are \((2l + 1)^2\) elements needed to describe the rotation of each mode and the value of each element depends upon the particular values of \((\alpha, \beta, \gamma)\) used for that rotation. Details of how to implement this are given in Coles et al. (2004).

In order to apply these ideas to make a test of CMB fluctuations, we first need a temperature map from which we can obtain a measured set of \( a_{l, m} \). Employing the above transformations with some choice of Euler angles yields a rotated set of the \( a_{l, m} \). It is straightforward to choose a set of angles such that random orientations of the coordinate axis can be generated. Once a rotated set has been obtained, Kuiper’s statistic is calculated from the relevant transformed set of phases. For example, Coles et al. (2004) generated 3000 rotated sets of each CMB map using this kind of resampling of the original data, producing 3000 values of \( V_{\text{cmb}} \). The values of the statistic were then binned to form a measured (re-sampled) distribution of \( V_{\text{cmb}} \). The same procedure is applied to the 1000 Monte Carlo sets of \( a_{l, m} \) drawn from a uniformly random distribution, i.e. each set was rotated 3000 times and a distribution of \( V_{\text{MC}} \) under the null hypothesis is produced. These realizations were then binned to created an overall global average distribution under the null hypothesis.

In order to determine whether the distribution of \( V_{\text{cmb}} \) is compatible with a distribution drawn from a sky with random phases, we use a simple \( \chi^2 \) test, using

\[
\chi^2 = \sum_i \frac{(f_i - f_{i0})^2}{f_i} \] (56)
where the summation is over all the bins and $f_i$ is the number expected in the $i$th bin from the overall average distribution. The larger the value of $\chi^2$ the less likely the distribution functions are to be drawn from the same parent distribution. Values of $\chi^2_{MC}$ are calculated for the 1000 Monte Carlo distributions and $\chi^2_{cmb}$ is calculated from the distribution of $V_{cmb}$. If the value of $\chi^2_{cmb}$ is greater than a fraction $p$ of the values of $\chi^2_{MC}$, then the phases depart from a uniform distribution at significance level $p$. We have chosen 95 per cent as an appropriate level for the level at which the data are said to display signatures that are not characteristic of a statistically homogeneous Gaussian random field.

Application of this relatively straightforward method to the WMAP first-year data shows the existence of phase correlations, as demonstrated in Figure 2.

![Figure 2](image)

**Fig. 2.** A reconstruction of the WMAP ILC made using the spherical harmonic mode amplitudes $a_{l,m}$ for $l = 16$ only. Our analysis method (Coles et al. 2004) shows that these modes at different $m$ have correlated phases in harmonic space, and the reconstructed sky shows this is aligned with the Galactic Plane.

### 4.4 Random Walks in Harmonic Space

To begin with, we concentrate on a simple measure based on the distribution of total displacements. Consider a particular value of $l$. The set of values $\{a_{l,m}\}$ can be thought of as steps in a random walk in the complex plane, a structure which can be easily visualized and which has well-known statistical properties.

The simplest statistic one can think of to describe the set $\{a_{l,m}\}$ is the net displacement of a random walk corresponding to the spherical harmonic mode $l$, i.e.
\[ R_l = \sum_{m>0} a_{l,m}, \quad (57) \]

where the vector \( a_{l,m} \equiv (x_{l,m}, y_{l,m}) \) and the random walk has an origin at \( a_{l,0} \) (which is always on the \( x \)-axis). The length of each step \( a_{l,m} = |a_{l,m}| \) is the usual spherical harmonic coefficient described in the previous section and defined by equation (1). If the initial fluctuations are Gaussian then the two components of each displacement are independently normal with zero mean and the same variance (8). Each step then has a Rayleigh distribution so that the probability density for \( a_{l,m} \) to be in the range \((a, a + da)\) is

\[ p(a) = \frac{a}{\sigma_l^2} \exp \left( -\frac{a^2}{2\sigma_l^2} \right). \quad (58) \]

This is a particularly simple example of a random walk (McCrea & Whipple 1940; Chandrasekhar 1943; Hughes 1995). Since the displacements in \( x \) and \( y \) are independently Gaussian the next displacement after \( l \) steps is itself Gaussian with variance \( l\sigma_l^2 \). The probability density of \( |R_l| \) to be in the range \((r, r + dr)\) is then itself a Rayleigh distribution of the form

\[ p_l(r) = \frac{r}{l\sigma_l^2} \exp \left( -\frac{r^2}{2l\sigma_l^2} \right). \quad (59) \]

This requires an estimate of \( \sigma_l^2 \). This can either be made using the same data or by assuming a given form for \( C_l \), in which case the resulting test would be of a composite hypothesis that the fluctuations constitute a Gaussian random field with a particular spectrum. For large \( l \) this is can be done straightforwardly, but for smaller values the sampling distribution of \( R_l \) will differ significantly from (59) because of the uncertainty in population variance from a small sample of \( a_{l,m} \). This is the so-called “cosmic variance” problem.

So far we have concentrated on fixed \( l \) with a random walk as a function of \( m \). We could instead have fixed \( m \) and considered a random walk as a function of \( l \). Or indeed randomly selected \( N \) values of \( l \) and \( m \). In either case the results above still stand except with \( \sigma_l^2 \) replaced by an average over all the modes considered:

\[ \sigma^2 = \frac{1}{N} \sum_{l,m} \sigma_{l,m}^2. \quad (60) \]

We do not consider this case any further in this paper.

The result (59) only obtains if the steps of the random walk are independent and Gaussian. If the distribution of the individual steps is non-Gaussian, but the steps are independent, then the result (59) will be true for large \( l \) by virtue of the Central Limit Theorem. Exact results for finite \( l \) for example non-Gaussian distributions are given by Hughes (1995). In such cases the overall 2D random walk comprises two independent 1D random walks in \( x \).
and $y$. The Gaussianity of the individual step components can be tested using their empirical distributions via a Kolmogorov-Smirnov (K-S) or similar approach. Lack of independence of step size or step direction (i.e. phase correlations) would appear as anisotropy of their joint distribution which could be quantified by direct measures of cross-correlation or by testing the bivariate distribution using an appropriate 2D K-S test. The latter task is harder, especially if the number of modes available is small. Using the net displacement in 1D or 2D corresponds to using the sum of a sample of $n$ variables to test the parent distribution. This is not necessarily powerful, but is robust and has well-defined properties. The true advantage of the random-walk representation is that it encapsulates the behaviour of the set $\{a_{l,m}\}$ in a graphical fashion which is ideal for data exploration.

A slightly different approach is to keep each step length constant. The simplest way of doing this is to define

$$\hat{R}_l = \sum_{m > 0} \frac{a_{l,m}}{|a_{l,m}|},$$

so that each step is of unit length but in a random direction. This is precisely the problem posed in a famous letter by Pearson (1905) and answered one week later by Rayleigh (1905). In the limit of large numbers of steps the result maps into the previous result (59) with $\sigma_l^2 = 1$ by virtue of the Central Limit Theorem. For finite values of $l$ there is also an exact result which can be derived in integral form using a method based on characteristic functions (Hughes 1995). The result is that the probability density for $\hat{R}_l$ to be in the range $r, r + dr$ is

$$q_l(r) = r \int_0^\infty u J_0(ur)[J_0(u)]^2 du.$$

The integral is only convergent for $l > 2$ but for $l = 1$ or $l = 2$ straightforward alternative expressions are available (Hughes 1995). One can use this distribution to test for randomness of the phase angles without regard to the amplitudes.

A simple test of the hypothesis that the fluctuations are drawn from a statistically homogeneous and isotropic Gaussian random field on the sky could be furnished by comparing the empirical distribution of harmonic random flights with the form (59). As we explained above, however, the net displacement of the random walk is a simple but rather crude indication of the properties of the $\{a_{l,m}\}$, as it does not take into account the ordering of the individual steps. The possible non-Gaussian behaviour of the set $\{a_{l,m}\}$ is encoded not so much in the net displacement but in the shape of the random walk. To put this another way, there are many possible paths with the same net displacement, and these will have different shapes depending on the correlations between step size and direction. Long runs of directed steps or regular features in the observed structure could be manifestations of phase correlation (Coles et al. 2004). The graphical representation of the set $\{a_{l,m}\}$
in the form illustrated by Figure 3 provides an elegant way of visualizing the behaviour of the harmonic modes and identifying any oddities. These could be quantified using a variety of statistical shape measures: moment of inertia (Rudnick, Beldjenna & Gaspari 1987), fractal dimension, first-passage statistics, shape statistics (e.g. Kuhn & Uson 1982), or any of the methods use to quantify the shape of minimal spanning trees (Barrow, Bhavsar & Sonoda 1985). Specific examples of correlated random walks are given in Hughes (1995).

**Fig. 3.** The random walk performed by the spherical harmonic coefficients for \( l = 532 \) in the WMAP ILC data, statistically the mode that displays the greatest departure from that expected under the null hypothesis. The outer circles correspond to 99.9, 99 and 95 per cent upper confidence limits (from outer to inner); the inner circles are the corresponding lower limits, though the 99.9 per cent lower limit is too small to see.

In practice the most convenient way to assess the significance of departures from the relevant distribution would be to perform Monte Carlo experiments of the null hypothesis. For statistical measures more complicated than the net displacement, the best way to set up a statistical test is to use Monte-Carlo re-orderings of the individual steps to establish the confidence level of any departure from Gaussianity. This also enables one to incorporate such complications as galactic cuts.

The WMAP team released an Internal Linear Combination (ILC) map that combined five original frequency band maps in such a way to maintain unit response to the CMB whilst minimising foreground contamination. The
construction of this map is described in detail in Bennett et al. (2003). The weighted map is produced by minimizing the variance of the temperature scale such that the weights add to one. To further improve the result, the inner Galactic plane is divided into 11 separate regions and weights determined separately. This takes account of the spatial variations in the foreground properties. Thus, the final combined map does not rely on models of foreground emission and therefore any systematic or calibration errors of other experiments do not enter the problem. The final map covers the full-sky and the idea is that it should represent only the CMB signal. Following the release of the WMAP 1 yr data Tegmark, Oliveira-Costa & Hamilton (2003; TOH) produced a cleaned CMB map. They argued that their version contained less contamination outside the Galactic plane compared with the ILC map produced by the WMAP team.

The ILC map is not intended for statistical analysis but in any case represents a useful “straw man” for testing statistical techniques for robustness. To this end, we analyzed the behaviour of the random-walks representing spherical harmonic from $l = 1$ to $l = 600$ in the WMAP ILC. Similar results are obtained for the TOH map so we do not discuss the TOH map here. For both variable-length (57) and unit-length (61) versions of the random-walk we generated 100000 Monte Carlo skies assuming Gaussian statistics. These were used to form a distribution of $|R_l|$ (or $|\hat{R}_l|$) over the ensemble of randomly-generated skies. A rejection of the null hypothesis (of stationary Gaussianity) at the $\alpha$ per cent level occurs when the measured value of the test statistic lies outside the range occupied by $\alpha$ per cent of the random skies.

Application of this simple test to the WMAP data (Stannard & Coles 2004) does not strongly falsify the null hypothesis, which is not surprising given the simplicity of the measure we have used. The number of modes outside the accepted range is close to that which would be expected if the null hypothesis were true. Notice that slightly more modes show up in the unit length case than in the other, perhaps indicating that the phase correlations that are known to exist in this data (Chiang et al. 2004) are masked if amplitude information is also included. The most discrepant mode turns out to be $l = 532$ in both cases. For interest a plot of the random walk for this case is included as Figure 3.

5 Topological Measures of Large-scale Structure

The application of phase analysis is obviously all performed in harmonic space (whether Fourier-harmonic or spherical harmonic). But what does the presence of phase correlations mean for the morphology of large-scale structure? What is the real-space morphology of a fluctuation field with random phases? In studying morphology, one is typically interested in the question of how the individual filaments, sheets and voids join up and intersect to form
the global pattern shown in Figure 1. Is the pattern cellular, having isolated voids surrounded by high-density sheets, or is it more like a sponge in which under- and over-dense regions interlock?

Looking at ‘slice’ surveys gives the strong visual impression that we are dealing with bubbles; pencil beams (deep galaxy redshift surveys with a narrow field of view, in which the volume sampled therefore resembles a very narrow cone or “pencil”) reinforce this impression by suggesting that a line-of-sight intersects at more-or-less regular intervals with walls of a cellular pattern. One must be careful of such impressions, however, because of elementary topology. Any closed curve in two dimensions must have an inside and an outside, so that a slice through a sponge-like distribution will appear to exhibit isolated voids just like a slice through a cellular pattern. It is important therefore that we quantify this kind of property using well-defined topological descriptors.

In an influential series of papers, Gott and collaborators have developed a method for doing just this (Gott, Melott & Dickinson 1986; Hamilton, Gott & Weinberg 1986; Gott et al. 1989; Gott et al. 1990; Melott 1990; Coles et al. 1996). Briefly, the method makes use of a topological invariant known as the genus, related to the Euler–Poincaré characteristic, of the iso-density surfaces of the distribution. To extract this from a sample, one must first smooth the galaxy distribution with a filter (usually a Gaussian is used; see §14.3) to remove the discrete nature of the distribution and produce a continuous density field. By defining a threshold level on the continuous field, one can construct excursion sets (sets where the field exceeds the threshold level) for various density levels. An excursion set will typically consist of a number of regions, some of which will be simply connected, e.g. a deformed sphere, and others which will be multiply connected, e.g. a deformed torus is doubly connected. If the density threshold is labelled by \( \nu \), the number of standard deviations of the density away from the mean, then one can construct a graph of the genus of the excursion sets at \( \nu \) as a function of \( \nu \): we call this function \( G(\nu) \). The genus can be formally expressed as an integral over the intrinsic curvature \( K \) of the excursion set surfaces, \( S_\nu \), by means of the Gauss–Bonnet theorem.

The general form of this theorem applies to any two-dimensional manifold \( \mathcal{M} \) with any (one-dimensional) boundary \( \partial \mathcal{M} \) which is piecewise smooth. This latter condition implies that there are a finite number \( n \) vertices in the boundary at which points it is not differentiable. The Gauss–Bonnet theorem states that

\[
\sum_{i=1}^{n} (\pi - \alpha_i) + \int_{\partial \mathcal{M}} k_g ds + \int_{\mathcal{M}} k dA = 2 \pi \chi_E(\mathcal{M}),
\]

(63)

where the \( \alpha_i \) are the angle deficits at the vertices (the \( n \) interior angles at points where the boundary is not differentiable), \( k_g \) is the geodesic curvature of the boundary in between the vertices and \( k \) is the Gaussian curvature of the
manifold itself. Clearly $ds$ is an element of length taken along the boundary and $dA$ is an area element within the manifold $\mathcal{M}$. The right-hand side of this equation is the Euler–Poincaré characteristic, $\chi_E$ of the manifold.

This probably seems very abstract but the definition above allows us to construct useful quantities for both two and three-dimensional examples. If we have an excursion set as described above in three-dimensions then its surface can be taken to define such a manifold. The boundary is just where the excursion sets intersect the limits of the survey and it will be taken to be smooth. Ignoring this, we see that the Euler–Poincaré characteristic is just the integral of the Gaussian curvature over the all compact bits of the surface of the excursion set. Hence, in this case,

$$2\pi\chi_E = \int_{S^c} K dS = 4\pi [1 - G(\nu)]. \quad (64)$$

Roughly speaking, the quantity $G$ is the genus, which for a single surface is the number of “handles” the surface possesses; a sphere has no handles and has zero genus, a torus has one and therefore has a genus of one. For technical reasons to do with the effect of boundaries, it has become conventional not to use $G$ but $G_S = G - 1$. In terms of this definition, multiply connected surfaces have $G_S \geq 0$ and simply connected surfaces have $G_S < 0$. One usually divides the total genus $G_S$ by the volume of the sample to produce $g_S$, the genus per unit volume.

One of the great advantages of using the genus measure to study large scale structure, aside from its robustness to errors in the sample, is that all Gaussian density fields have the same form of $g_S(\nu)$:

$$g_S(\nu) = A (1 - \nu^2) \exp\left(\frac{-\nu^2}{2}\right), \quad (65)$$

where $A$ is a spectrum-dependent normalisation constant. This means that, if one smooths the field enough to remove the effect of non–linear displacements of galaxy positions, the genus curve should look Gaussian for any model evolved from Gaussian initial conditions, regardless of the form of the initial power spectrum which only enters through the normalisation factor $A$. This makes it a potentially powerful test of non–Gaussian initial fluctuations, or of models which invoke non–gravitational physics to form large–scale structure. The observations support the interpretation that the initial conditions were Gaussian, although the distribution looks non–Gaussian on smaller scales. The nomenclature for the non–Gaussian distortion one sees is a ‘meatball shift’: non–linear clustering tends to produce an excess of high–density simply–connected regions, compared with the Gaussian curve. The opposite tendency, usually called ‘swiss–cheese’, is to have an excess of low density simply connected regions in a high density background, which is what one might expect to see if cosmic explosions or bubbles formed the large–scale structure. What one would expect to see in the standard picture of gravitational instability from Gaussian initial conditions is a ‘meatball’ topology
when the smoothing scale is small, changing to a sponge as the smoothing
scale is increased. This is indeed what seems to be seen in the observations
so there is no evidence of bubbles; an example is shown in Figure 4.

![Diagram](image-url)

**Fig. 4.** Genus curve for galaxies in the IRAS PSCz survey. The noisy curve is
the smoothed galaxy distribution while the solid line is the best–fitting curve for a
Gaussian field; from Canavezes et al. (1998).

The smoothing required also poses a problem, however, because present
redshift surveys sample space only rather sparsely and one needs to smooth
rather heavily to construct a continuous field. A smoothing on scales much
larger than the scale at which correlations are significant will tend to produce
a Gaussian distribution by virtue of the central limit theorem. The power of
this method is therefore limited by the smoothing required, which, in turn,
depends on the space–density of galaxies. An example is given in the Figure,
which shows the genus curve for the PSCz survey of IRAS galaxies.

Topological information can also be obtained from two–dimensional data
sets, whether these are simply projected galaxy positions on the sky (such
as the Lick map, or the APM survey) or ‘slices’ (such as the various CfA
compilations). Here the excursion sets one deals with are just regions of the
plane where the (surface) density exceeds some threshold. This method can
also be applied to CMB temperature fluctuations where one looks at the
topology of regions bounded by lines of constant temperature (Coles 1988;
Gott et al. 1990; Colley & Gott 2003; Komatsu et al. 2003).

In such case we imagine the manifold referred to in the statement of the
Gauss–Bonnet theorem to be not the surface of the excursion set but the
surface upon which the set is defined (i.e. the sky). For reasonably small angles this can be taken to be a flat plane so that the Gaussian curvature of $\mathcal{M}$ is everywhere zero. (The generalization to large angles is trivial; it just adds a constant curvature term.) The Euler characteristic is then simply an integral of the line curvature of around the boundaries of the excursion set:

$$2\pi \chi_E = \int k_g ds.$$  \hspace{1cm} (66)

In this case the Euler–Poincaré characteristic is simply the number of isolated regions in the excursion set minus the number of holes in such regions.

This is analogous to the genus, but has the interesting property that it is an odd function of $\nu$ for a two–dimensional Gaussian random field, unlike $G(\nu)$ which is even. In fact the mean value of $\chi$ per unit area on the sky takes the form

$$\chi(\nu) = B\nu \exp(-\nu^2/2),$$  \hspace{1cm} (67)

where $B$ is a constant which depends only on the (two–dimensional) power spectrum of the random field. Notice that $\chi < 0$ for $\nu < 0$ and $\chi > 0$ for $\nu > 0$. A curve shifted to the left with respect to this would be a meatball topology, and to the right would be a swiss–cheese.

There are some subtleties with this. Firstly, as discussed above, two–dimensional topology does not really distinguish between ‘sponge’ and ‘swiss–cheese’ alternatives. Indeed, there is no two-dimensional equivalent of a sponge topology: a slice through a sponge is topologically equivalent to a slice through swiss-cheese. Nevertheless, it is possible to assess whether, for example, the mean density level ($\nu = 0$) is dominated by underdense or overdense regions so that one can distinguish swiss–cheese and meatball alternatives to some extent. The most obviously useful application of this method is to look at projected catalogues, the main problem being that, if the catalogue is very deep, each line of sight contains a superposition of many three–dimensional structures. This projection acts to suppress departures from Gaussian statistics by virtue of the central limit theorem. Nevertheless, useful information is obtainable from projected data simply because of the size of the data sets available; as is the case with three–dimensional studies, the analysis reveals a clear meatball shift which is what one expects in the gravitational instability picture. The methods used for the study of two–dimensional galaxy clustering can also be used to analyze the pattern of fluctuations on the sky seen in the cosmic microwave background.

More recently, this approach has been generalized to include not just the Euler–Poincaré distribution but all possible topological invariants. This means all quantities that satisfy the requirement that they be additive, continuous, translation invariant and rotation invariant. For an excursion set defined in $d$ dimensions there are $d + 1$ such quantities that can be regarded as independent. Any characteristic satisfying these invariance properties can
be expressed in terms of linear combinations of these four independent quantities. These are usually called Minkowski functionals. Their use in the analysis of galaxy clustering studies was advocated by Mecke, Buchert & Wagner (1994) and has become widespread since then.

In three dimensions there are four Minkowski functionals. One of these is the integrated Gaussian curvature (equivalent to the genus we discussed above). Another is the mean curvature, $H$ defined by

$$H = \frac{1}{2} \int \left( \frac{1}{R_1} + \frac{1}{R_2} \right) dA. \quad (68)$$

In this expression $R_1$ and $R_2$ are the principal radii of curvature at any point in the surface; the Gaussian curvature is $1/(R_1 R_2)$ in terms of these variables. The other two Minkowski functionals are more straightforward. They are the surface area of the set and its volume. These four quantities give a “complete” topological description of the excursion sets.

6 Discussion

In this paper I have tried to explain how phase correlations, arising from primordial non-Gaussianity, non-linear evolution (or indeed systematic error) can be measured and use to test cosmological models. The use of direct phase information is relatively new in cosmology, so I concentrated on basic properties and explained in some detail how phases relate to more familiar descriptors such as the bispectrum and three-point covariance functions. The magnitude of these statistical descriptors is of course related to the amplitude of the Fourier modes, but the factor that determines whether they are zero or non-zero is the arrangement of the phases of these modes.

The connection between polyspectra and phase information is an important one and it opens up many lines of future research, such as how phase correlations relate to redshift distortion and bias. Using small volumes of course leads to sampling uncertainties which are quite straightforward to deal with in the case of the power-spectra but more problematic for higher-order spectra like the bispectrum. Understanding the fluctuations about ensemble averages in terms of phases could also lead to important insights. On the other hand, the application of phase methods to galaxy clustering studies is complicated by the non-linear evolution of perturbations as they collapse and form bound structures. Structures which are highly localized in real space are highly dispersed in Fourier space, so it is quite difficult to disentangle any primordial phase correlations from artifacts of non-linear evolution.

The CMB is a much more promising arena for the application of these methods. Late-time non-linear effects should be small (at least on large angular scales) so any phase correlations will almost certainly arise from either primordial effects or residual foreground contamination. The preliminary analysis we have performed using the WMAP data shows that there are indeed
phase correlations, but Figure 2 suggests the likely interpretation of this is that it relates to the galaxy. As the constraints on early Universe physics get stronger, the importance of identifying low-amplitude foregrounds becomes all the more important. The next era of CMB physics is likely to be dominated by polarization studies where the effects of foregrounds are likely to be even more complicated. There remains a great deal to learn about how to fully characterize the polarization maps that will soon be obtained. We can be certain, however, that phase information (in one way or another) will help us understand what is going on.

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