STUDYING DEFORMATIONS OF FUCHSIAN REPRESENTATIONS WITH HIGGS BUNDLES

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These notes are based on a three hour minicourse given by the author at University of Illinois at Chicago in June 2018. The main goal is to explain how many components of the character variety of a closed surface are either deformations spaces of representations into the maximal compact subgroup or deformation spaces of certain Fuchsian representations. This latter family, is of particular interest and is related to the field of Higher Teichmüller theory. Our main tool is the theory of Higgs bundles. In these notes we try to develop the general theory of Higgs bundles for real groups and indicate where subtleties arise. However, the main emphasis is placed on concrete examples which are our motivating objects. In particular, we do not prove any of the foundational theorems, rather we state them and show how

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they can be used to prove interesting statements about components of the character variety. We have also not spent any time developing the tools (harmonic maps) which define the bridge between Higgs bundles and the character variety. For this side of the story we refer the reader to the notes of Q. Li who gave a concurrent minicourse.

1. An introduction to the character variety

Let $S$ be a closed surface of genus $g \geq 2$. Denote the fundamental group of $S$ by $\Gamma$, and recall that $\Gamma$ has the standard presentation

$$\Gamma = \langle a_1, \ldots, a_g, b_1, \ldots, b_g \mid \prod_{j=1}^{g} [a_j, b_j] = 1 \rangle .$$

Fix also a real reductive Lie group $G$. For example $G$ could be one of the following groups

$$\text{GL}(n, \mathbb{R}) , \quad \text{GL}(n, \mathbb{C}) , \quad \text{SL}(n, \mathbb{C}) , \quad \text{Sp}(2n, \mathbb{R}) , \quad \text{PSL}(n, \mathbb{R}) = \text{SL}(n, \mathbb{R})/\pm \text{Id} , \quad \text{U}(n) , \quad \text{SO}(n) , \quad \text{SU}(p, q) , \quad \text{SO}(p, q) , \quad \text{S}(O(p) \times O(q)) ,$$

but $G$ cannot be a group like

$$P = \{ \left( \begin{array}{cc} a & b \\ 0 & c \end{array} \right) \in \text{GL}(2, \mathbb{C}) \} .$$

One main property of a reductive Lie group $G$ is that, up to conjugation, there is a unique maximal compact subgroup $H < G$. We will heavily use this property. In fact there is a homotopy equivalence

$$H \simeq G .$$

A group homomorphism $\rho : \Gamma \rightarrow G$ will be referred to as a representation. Consider the set $\text{Hom}(\Gamma, G)$ of all such representations. This set has a topological structure induced by the inclusion

$$\text{Hom}(\Gamma, G) \xrightarrow{\rho} G^{2g} .$$

Since a representation $\rho$ must satisfy $\prod_{j=1}^{g} [\rho(a_j), \rho(b_j)] = \text{Id}$, the maximal dimension of a connected component of $\text{Hom}(\Gamma, G)$ is

$$\dim(\text{Hom}(\Gamma, G)) = 2g \cdot \dim G - \dim \text{G} .$$

The group $G$ acts on $\text{Hom}(\Gamma, G)$ by conjugation: for $\gamma \in \Gamma$ and $g \in G$,

$$(\rho \cdot g)(\gamma) = g^{-1} \rho(\gamma) g .$$

The quotient space $\text{Hom}(\Gamma, G)/G$ consists of conjugacy classes of representations. Unless $G$ is compact, $\text{Hom}(\Gamma, G)/G$ is not Hausdorff.

Example 1.1. Let $\rho_1(\gamma) = \text{Id}$ for all $\gamma \in \Gamma$ and define $\rho_2$ by $\rho_2(a_j) = \text{Id}$ for all $j$ and

$$\rho_2(b_j) = \begin{cases} \text{Id} & j \neq g \\ \left( \begin{array}{c} 1 \\ 0 \end{array} \right) & j = g . \end{cases}$$
If \( g_t = \left( \begin{smallmatrix} t & 0 \\ 0 & \frac{1}{t} \end{smallmatrix} \right) \), then \( g_t^{-1} \rho_2(a_j) g_t = \text{Id} \) for all \( j \) and
\[
g_t^{-1} \rho_2(b_j) g_t = \begin{cases} \text{Id} & j \neq g \\ \left( \begin{smallmatrix} 1 & t^{-2} \\ 0 & 1 \end{smallmatrix} \right) & j = g . \end{cases}
\]
The \( G \)-orbits through \( \rho_1 \) and \( \rho_2 \) are disjoint in \( \text{Hom}(\Gamma, G) \), but
\[
\rho_1 \in \overline{\rho_2 \cdot G}.
\]
Thus, \([\rho_1]\) and \([\rho_2]\) are not separable in \( \text{Hom}(\Gamma, G)/G \).

To get a Hausdorff quotient we restrict to the subset of \( \text{Hom}(\Gamma, G) \). A representation \( \rho : \Gamma \to G \) is called \textit{reductive} if the composition with the adjoint representation
\[
\Gamma \xrightarrow{\rho} G \xrightarrow{\text{Ad}} \text{GL}(g)
\]
decomposes as a direct sum of irreducible representations. This is equivalent to the \( G \) orbit through \( \rho \) being closed in \( \text{Hom}(\Gamma, G) \). When \( G \) is algebraic, this is also equivalent to the Zariski closure of \( \rho(\Gamma) \) being a reductive subgroup.

Denote the set of reductive representations by \( \text{Hom}^+(\Gamma, G) \). The \textit{G-character variety of} \( \Gamma \) is defined by be the Hausdorff space
\[
\mathcal{X}(\Gamma, G) = \text{Hom}^+(\Gamma, G)/G .
\]

\textbf{Remark 1.2.} When \( G \) is complex algebraic, one gets the same spaces as the GIT-quotient.

1.1. \textbf{Fuchsian representations.} Important examples of points in a character variety come from Fuchsian representations and are related to hyperbolic metrics on \( S \). These examples will play a fundamental role through these lectures.

Let \( X \) be a Riemann surface structure on \( S \). By the uniformization theorem,
\[
X = \mathbb{H}^2 / \rho_X(\Gamma) ,
\]
where \( \mathbb{H}^2 \) is the upper half-plane and \( \rho_X : \Gamma \to \text{Bihol}(\mathbb{H}^2) \cong \text{PSL}(2, \mathbb{R}) \). By classical results of Riemann, there are \( 3g - 3 \) complex parameters for the deformations of the Riemann surface \( X \). Such deformations give rise to \( (6g - 6) \)-real dimensional subset of \( \mathcal{X}(\Gamma, \text{PSL}(2, \mathbb{R})) \). Moreover, such deformations define an open neighborhood of \( \rho_X \in \mathcal{X}(\Gamma, \text{PSL}(2, \mathbb{R})) \) since \( 6g - 6 \) is the maximal dimension of \( \mathcal{X}(\Gamma, \text{PSL}(2, \mathbb{R})) \). This leads us to define the set of \textit{Fuchsian representations}:
\[
\text{Fuch}(\Gamma) = \{ \rho : \Gamma \to \text{PSL}(2, \mathbb{R}) \mid \text{discrete and faithful} \}/\text{PSL}(2, \mathbb{R}) .
\]

Since being discrete and faithful is a closed condition in \( \mathcal{X}(\Gamma, G) \), \( \text{Fuch}(\Gamma) \) is a closed subset. By the above argument, \( \text{Fuch}(\Gamma) \) is also open. Thus, \( \text{Fuch}(\Gamma) \subset \mathcal{X}(\Gamma, \text{PSL}(2, \mathbb{R})) \) is an open and closed subset which is identified with the Teichmüller space of isotopy class of complex structures on \( S \) and \( \bar{S} \)
\[
\text{Fuch}(\Gamma) \cong \text{Teich}(S) \sqcup \text{Teich}(\bar{S}) .
\]
The group \( \text{PSL}(2, \mathbb{R}) \) is also the orientation preserving isometry group of the hyperbolic plane. Thus, \( \text{Fuch}(\Gamma) \) also parameterizes the set of isotopy classes of hyperbolic metrics on the surface \( S \) and \( \bar{S} \).

\textbf{Remark 1.3.} The two connected components of \( \text{Fuch}(\Gamma) \subset \mathcal{X}(\Gamma, \text{PSL}(2, \mathbb{R})) \) arise because elements of \( \text{PSL}(2, \mathbb{R}) \) preserve the orientation of \( \mathbb{H}^2 \). The group \( \text{PGL}(2, \mathbb{R}) \) is the full isometry group of \( \mathbb{H}^2 \). Thus if we consider \( \text{Fuch}(\Gamma) \) as a subset of \( \text{PGL}(2, \mathbb{R}) \), then there is only one component. Similarly, if we consider \( \text{Fuch}(\Gamma) \) as a subset of
PSL(2, \mathbb{C})-character variety, then it is a closed connected subset which is no longer open.

1.2. Connected components. One fundamental problem is to determine how many connected components the character variety has. Surprisingly, this question has not been answered in full generality. There is a topological invariant which helps distinguish connected components.

Denote the set of isomorphism classes of topological principal $G$-bundles on $S$ by $B_G(S)$, this is the set of homotopy classes of maps from $S$ to the classifying space of $G$. For connected groups we have $B_G(S) = H^2(S, \pi_1 G)$. Every representation $\rho : \Gamma \to G$ defines a principal $G$-bundle $E_\rho \to S$

$$E_\rho = (\tilde{S} \times G)/\Gamma,$$

where $\Gamma$ acts on its universal cover $\tilde{S}$ by deck transformations and by multiplication by $\rho(\Gamma)$ on $G$. Thus, we have a map $\text{Hom}(\Gamma, G) \to B_G(S)$. Moreover, this map is continuous and descends to a map

$$\tau : \pi_0(X(\Gamma, G)) \longrightarrow B_G(S).$$

If $X^\omega(\Gamma, G) = \tau^{-1}(\omega)$, then $X(\Gamma, G)$ decomposes as

$$\bigsqcup_{\omega \in B_G(S)} X^\omega(\Gamma, G).$$

**Question 1.4.** When is the map $\tau$ from (1.1) injective? In other words, when does the topological invariant distinguish the connected components of the character variety.

**Remark 1.5.** Note that when $\tau$ is injective, the question of connected components counts is not very interesting. We will mainly be interested in when $\tau$ is not injective and understanding the special features of these components.

This question has been answered for many groups, but is open in general. For compact groups the map $\tau$ is injective, this was proven by Narasimhan-Seshadri [22] for $G = U(n)$ and Ramanathan [23] in general.

**Theorem 1.6.** If $G$ is compact (i.e. $G = H$), then $\tau$ is injective. Furthermore, if $G$ is also semisimple, then $\tau$ is a bijection.

Since $H$ and $G$ are homotopic, we have $B_G(S) = B_H(S)$. Moreover, for each $\omega \in B_H(S)$

$$X^\omega(\Gamma, H) \subset X^\omega(\Gamma, G).$$

For complex groups, the map $\tau$ is also injective. This was proven for by J. Li [19] for semisimple groups and Garcia-Prada and Oliveira [10] in general.

**Theorem 1.7.** If $G$ is complex (i.e. $G = H^C$), then $\tau$ is injective.

We will prove these results using Higgs bundles in Section 6.5. For $G$ a semisimple complex Lie group, the following corollary follows immediately from the two above theorems. It holds in general.

**Corollary 1.8.** For $G$ a complex reductive Lie group, every representation $\rho : \Gamma \to G$ can be continuously deformed to a compact representation $\Gamma \to H \hookrightarrow G$.

The above corollary says that the connected components of the character variety are not interesting. For real groups, the situation more subtle.
Example 1.9. For $G = \text{PSL}(2, \mathbb{R})$, the maximal compact subgroup is $H \cong \text{SO}(2)$. Since a circle bundle on a closed surface is determined by its degree, we have $B_H(S) \cong \mathbb{Z}$. Thus,
\[ \mathcal{X}(\Gamma, \text{PSL}(2, \mathbb{R})) = \prod_{d \in \mathbb{Z}} \mathcal{X}^d(\Gamma, \text{PSL}(2, \mathbb{R})) . \]
However, the space $\mathcal{X}^d(\Gamma, \text{PSL}(2, \mathbb{R}))$ is empty when $|d| > 2g - 2$ [21]. Moreover, when $|d| \leq 2g - 2$, the space $\mathcal{X}^d(\Gamma, \text{PSL}(2, \mathbb{R}))$ is nonempty and connected [12]. We will prove these statements using Higgs bundles in Section 4.

For a $\text{PSL}(2, \mathbb{R})$ representation $\rho$, the integer invariant can be defined as follows. Pick any $\rho$-equivariant map $f_\rho : S \to \text{PSL}(2, \mathbb{R})/H \cong \mathbb{H}^2$. Such maps always exits since $\mathbb{H}^2$ is contractible. We have a principal $H$-bundle $\text{PSL}(2, \mathbb{R}) \to \mathbb{H}^2$, thus define $\tau$ to be minus the degree of the pullback bundle:
\[ \tau(\rho) = -\deg(f_\rho^*\text{PSL}(2, \mathbb{R})) . \]
The bundle $\text{PSL}(2, \mathbb{R}) \to \mathbb{H}^2$ is identified with the unit tangent bundle of $\mathbb{H}^2$. If $\rho$ is a Fuchsian representation, then we may choose $f_\rho$ to be the equivariant diffeomorphism uniformizing the Riemann surface $\mathbb{H}^2/\rho(\Gamma)$. In this case, $\tau$ is given by the degree of the cotangent bundle. Namely, $\tau = 2g - 2$. Thus, we have
\[ \text{Fuch}(\Gamma) \subset \mathcal{X}^{2g-2}(\Gamma, \text{PSL}(2, \mathbb{R})) \sqcup \mathcal{X}^{2g-2}(\Gamma, \text{PSL}(2, \mathbb{R})). \]
In fact, the above inclusion is an equality, thus $\text{Teich}(S) \cong \mathcal{X}^{2g-2}(\Gamma, \text{PSL}(2, \mathbb{R}))$.

Example 1.10. For $G = \text{PSL}(n, \mathbb{R})$ the maximal compact subgroup is $\text{SO}(n)$ if $n$ is odd and $\text{SO}(n)/\pm \text{Id}$ when $n$ is even. In this case,
\[ B_{\text{PSL}(n, \mathbb{R})}(S) \cong H^2(S, \pi_1 H) \cong \begin{cases} \mathbb{Z} & \text{if } n = 2, \\ \mathbb{Z}_2 & \text{if } n = 2k + 1, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } n = 4k, \\ \mathbb{Z}_4 & \text{if } n = 4k + 2. \end{cases} \]
In the case $n = 2k + 1$, the invariant $\omega \in H^2(S, \mathbb{Z}_2)$ is the second Stiefel-Whitney class of the $\text{SO}(n)$ bundle.

Example 1.11. For $G = \text{SO}(p, q)$, the maximal compact subgroup is $\text{SO}(p) \times \text{O}(q)$. We have
\[ B_{\text{SO}(p, q)}(S) \cong \begin{cases} H^1(S, \mathbb{Z}_2) = \mathbb{Z}_2^{2g} & \text{if } p = q = 1 \\ H^1(S, \mathbb{Z}_2) \times H^1(S, \mathbb{Z}_2) = \mathbb{Z}_2^{2g+1} & \text{if } p = 1 \text{ and } 2 < q \\ H^1(S, \mathbb{Z}_2) \times H^2(S, \mathbb{Z}_2) \times H^2(S, \mathbb{Z}_2) = \mathbb{Z}_2^{2g+2} & \text{if } 2 < p \leq q . \end{cases} \]
In the above cases, the element of $H^1(S, \mathbb{Z}_2)$ is the first Stiefel-Whitney class of an orthogonal bundle and each element of $H^2(S, \mathbb{Z}_2)$ is the second Stiefel-Whitney class of an orthogonal bundle. The case of $p = 2$ or $q = 2$ is slightly more complicated.

2. Deforming Fuchsian representations

We have seen that the Teichmüller space of the surface $S$ is identified with the connected component $\mathcal{X}^{2g-2}(\Gamma, \text{PSL}(2, \mathbb{R}))$ of the $\text{PSL}(2, \mathbb{R})$-character variety. As a result, the representations in this component have special geometric significance. Given an embedding $\iota : \text{PSL}(2, \mathbb{R}) \to G$, we have
\[ \iota(\text{Fuch}(\Gamma)) \subset \mathcal{X}(\Gamma, G) . \]
2.1. **Isometry groups of hyperbolic spaces.** There is a natural embedding into $\text{Isom}^+(\mathbb{H}^n) \cong \text{SO}_0(1,n)$ given by

$$\text{PSL}(2,\mathbb{R}) \cong \text{Isom}^+(\mathbb{H}^2) \cong \text{SO}_0(1,2) \xrightarrow{\iota_{1,n}} \text{SO}_0(1,n) \cong \text{Isom}^+(\mathbb{H}^n) .$$

This gives $\iota_{1,n}(\text{Fuch}(\Gamma)) \subset \mathcal{X}(\Gamma,\text{SO}_0(1,n))$, and small deformations of these representations are holonomies of complete hyperbolic $n$-manifolds called quasi-Fuchsian manifolds.

**Definition 2.2.** A representation $\rho : \Gamma \to \text{Isom}(\mathbb{H}^n)$ is called *convex cocompact* if it is discrete and faithful and $\rho(\Gamma)$ acts cocompactly on a convex domain in $\mathbb{H}^n$. The set of quasi-Fuchsian representations $\text{QFuch}(\Gamma) \subset \mathcal{X}(\Gamma,\text{SO}_0(1,n))$ is defined to be

$$\text{QFuch}(\Gamma) = \{ [\rho] \in \mathcal{X}(\Gamma,\text{SO}_0(1,n)) \mid \rho \text{ is convex cocompact} \} .$$

**Remark 2.3.** $[\rho] \in \text{QFuch}(\Gamma)$ implies $[\rho]$ can be deformed to $\iota_{1,n}(\text{Fuch}(\Gamma))$. Moreover, the set of convex cocompact representations is open in $\mathcal{X}(\Gamma,\text{SO}_0(1,n))$. Thus, any sufficiently small deformation of a Fuchsian representation in $\iota_{1,n}(\text{Fuch}(\Gamma))$ is quasi-Fuchsian. However, unlike $\text{Fuch}(\Gamma) \subset \mathcal{X}(\Gamma,\text{SO}_0(1,2))$, the set $\text{QFuch}_n(\Gamma) \subset \mathcal{X}(\Gamma,\text{SO}_0(1,n))$ is *not closed*. Namely, there families of convex cocompact representations whose limit is discrete and faithful, but not convex cocompact.

In fact, we have the following:

**Proposition 2.4.** For $n > 2$, any representation $\rho \in \iota_{1,n}(\text{Fuch}(\Gamma))$ can be continuously deformed to a compact representation.

**Proof.** Note that it suffices to prove the statement for $\text{SO}_0(1,3)$. Recall that there is an isomorphism of Lie groups $\text{SO}_0(1,3) \cong \text{PSL}(2,\mathbb{C})$. The result now follows from Corollary 1.8. □

**Remark 2.5.** Another interesting embedding is given by the isomorphism

$$\text{PSL}(2,\mathbb{R}) \cong \text{PU}(1,1)$$

and the embedding $\text{PU}(1,1) \to \text{PU}(1,n) \cong \text{Isom}(\mathbb{C}\mathbb{H}^n)$ into the isometry group of the complex hyperbolic space. Deformations of $\text{Fuch}(\Gamma) \subset \mathcal{X}(\Gamma,\text{PU}(1,n))$ under this embedding satisfy a rigidity phenomenon [11]. This is a special case of the more general situation of maximal representations into a Hermitian Lie group of non-tube type (see for example [5]). We will not discuss this situation further.

2.2. **Principal embedding.** Recall that for each dimension $n$, there is a unique irreducible representation

$$\iota_{pr} : \text{SL}(2,\mathbb{R}) \to \text{SL}(\mathbb{R}^n)$$

which is given by the $(n-1)^{st}$-symmetric product of the standard representation. Moreover, it is straightforward to check that this induces an embedding

$$\iota_{pr} : \text{PSL}(2,\mathbb{R}) \to \text{PSL}(n,\mathbb{R}) .$$

We will call this embedding the *principal embedding.*
More generally, if $G$ is a split real Lie group of adjoint type, there is a unique preferred (principal) embedding
\[ \iota_{pr} : \text{PSL}(2, \mathbb{R}) \to G. \]
We will not go into the Lie theory necessary to define the principal embedding in general, see [17] for more details on the general setup. We will explicitly describe $\iota_{pr}$ for the classical groups.

**Example 2.6.** When $n = 2p + 1$ the principal embedding $\iota_{pr} : \text{SL}(2, \mathbb{R}) \to \text{SL}(\mathbb{R}^{2p+1})$ preserves a symmetric nondegenerate quadratic form of signature $(p, p+1)$. Thus, we have an embedding
\[ \iota_{pr} : \text{PSL}(2, \mathbb{R}) \to \text{SO}_0(p, p+1) \subset \text{PSL}(2p+1, \mathbb{R}). \]
This is the principal embedding (2.2) for the split group adjoint $G = \text{SO}_0(p, p+1)$.

Similarly, when $n = 2p$ the principal embedding $\iota_{pr} : \text{SL}(2, \mathbb{R}) \to \text{SL}(\mathbb{R}^{2p})$ preserves a nondegenerate symplectic form. Thus, we have an embedding
\[ \iota_{pr} : \text{PSL}(2, \mathbb{R}) \to \text{PSp}(2p, \mathbb{R}) \subset \text{PSL}(2p, \mathbb{R}). \]
This is the principal embedding (2.2) for the split group adjoint $G = \text{Sp}(2p, \mathbb{R})$.

The deformation space of $\iota_{pr}(\text{Fuch}(\Gamma)) \subset X(\Gamma, G)$ is called the **Hitchin component** or Hitchin components.

**Definition 2.7.** Let $G$ be a split real simple adjoint Lie group, a Hitchin component $\text{Hit}(G) \subset X(\Gamma, G)$ is a connected component containing a component of $\iota_{pr}(\text{Fuch}(\Gamma))$.

Unlike the embedding $\iota_{1,n} : \text{PSL}(2, \mathbb{R}) \to \text{SO}_0(1, n)$, representations in $\text{Hit}(G)$ cannot be deformed to compact representations.

**Theorem 2.8.** (Hitchin [15]) If $\rho \in \text{Hit}(G)$ then $\rho$ cannot be deformed to a compact representation. In particular,
\[ |\pi_0(X(\Gamma, G))| \geq 1 + |\pi_0(X(\Gamma, H))|. \]

**Remark 2.9.** Labourie showed that all representations in the Hitchin component satisfy a certain dynamical property called the Anosov property [18] which generalize the notion of convex cocompactness to higher rank Lie groups. As a consequence, every representation in a Hitchin component is discrete and faithful. Moreover, like $\text{Fuch}(\Gamma)$ the representations in the Hitchin component are all holonomies of certain geometric structures on compact manifolds [13]. Since $\text{Hit}(G)$ shares many features with the Teichmüller space of $S$, it has been called a **higher Teichmüller component** (see for example [6] and [26]). We will not discuss this perspective anymore, however the components discussed in these notes which are deformation spaces of Fuchsian representations are intimately related with the field of higher Teichmüller theory.

For the group $\text{PSL}(n, \mathbb{R})$, Hitchin also proved that there are no other components.

**Theorem 2.10.** (Hitchin [15]) For $n > 2$ we have
\[ |\pi_0(X(\Gamma, \text{PSL}(n, \mathbb{R}))| = \begin{cases} 3 & \text{if } n \text{ is odd} \\ 6 & \text{if } n \text{ is even}. \end{cases} \]
Remark 2.11. Recall that $\text{Fuch}(\Gamma) \subset \mathcal{X}(\Gamma, \text{PSL}(2, \mathbb{R}))$ has two connected components, however, these components are isomorphic via an outer automorphism of $\text{PSL}(2, \mathbb{R})$. The number of Hitchin components $\text{Hit}(\mathcal{X}(\Gamma, \text{PSL}(n, \mathbb{R})))$ depends on the parity of $n$. Namely, the map $\iota_{\text{pr}}: \text{Fuch}(\Gamma) \to \mathcal{X}(\Gamma, \text{PSL}(n, \mathbb{R}))$ is $2 : 1$ when $n$ is odd and injective when $n$ is even. There are thus two Hitchin components for $G = \text{PSL}(2n, \mathbb{R})$ and one Hitchin component for $G = \text{PSL}(2n + 1, \mathbb{R})$.

Introductory words about the below corollary

Corollary 2.12. If $\rho \in \mathcal{X}(\Gamma, \text{PSL}(n, \mathbb{R}))$, then there is a dichotomy: either $\rho$ can be deformed to compact representation or $\rho$ can be deformed to a Fuchsian representation in $\iota_{\text{pr}}(\text{Fuch}(\Gamma))$.

Remark 2.13. A generalization of the embedding (2.1) is given by

\[(2.3) \quad \iota_{p,q}: \text{SO}(p, p-1) \to \text{SO}(p, q).\]

The embedding

\[\text{PSL}(2, \mathbb{R}) \xrightarrow{\iota_{\text{pr}}} \text{SO}(p, p-1) \xrightarrow{\iota_{p,q}} \text{SO}(p, q)\]

will play an important role in Theorem 7.13. In fact, when $q = p$ the principal embedding $\iota_{\text{pr}}: \text{PSL}(2, \mathbb{R}) \to \text{SO}(p, p)$ is given by principal into $\text{SO}(p, p-1)$ followed by $\iota_{p,p}$

\[\text{PSL}(2, \mathbb{R}) \xrightarrow{\iota_{\text{pr}}} \text{SO}(p, p-1) \xrightarrow{\iota_{p,p}} \text{SO}(p, p)\]

3. Higgs bundles

We now shift our focus to a moduli space of holomorphic objects on a Riemann surface called Higgs bundles. This theory was developed Hitchin [14, 16] and Simpson [24, 25]. At first look, Higgs bundles and surface group representations seem to have little to do with each other. However, a remarkable theorem, known as the Nonabelian Hodge Correspondence, gives a homeomorphism between the two moduli spaces. Higgs bundles thus provide a powerful tool for addressing certain questions about the topology of the character variety.

Theorem 3.1 (Nonabelian Hodge Correspondence). Let $S$ be a closed orientable surface of genus at least two. For each Riemann surface structure $X$ on $S$, the moduli space of $G$-Higgs bundles on $X$ is homeomorphic to character variety $\mathcal{X}(\pi_1(S), G)$.

One direction of the nonabelian Hodge correspondence asserts that to each polystable $G$-Higgs bundle, there is a special metric from which one builds a flat $G$-connection. For principal bundles a metric is by definition a reduction of structure group to the maximal compact subgroup. The other direction asserts that for each reductive representations and each choice of Riemann surface structure on $S$, there is an equivariant harmonic map $\tilde{X} \to G/H$ from the universal cover to the Riemannian symmetric space. From such a map one constructs a polystable $G$-Higgs bundle. For details on this case we refer the reader to Q. Li’s lectures in this volume.
3.1. Definitions. As before, let $G$ be a reductive Lie group with maximal compact $H$ and Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Complexifying gives an $\text{Ad}_{\mathfrak{h}}$-invariant decomposition $\mathfrak{g}^C = \mathfrak{h}^C \oplus \mathfrak{m}^C$. Fix a compact Riemann surface $X$ with genus $g \geq 2$ and let $K$ denote its holomorphic cotangent bundle.

**Definition 3.2.** A $G$-Higgs bundle on $X$ is a pair $(\mathcal{P}, \varphi)$ where

- $\mathcal{P} \to X$ is a holomorphic principal $H^C$-bundle and
- $\varphi$ is a holomorphic section of the associated bundle $\mathcal{P}[\mathfrak{m}^C] \otimes K$.

The holomorphic section $\varphi$ is called the Higgs field.

**Example 3.3.** If $G$ is compact, then $\mathfrak{g} = \mathfrak{h}$ and $\mathfrak{m} = \{0\}$. In this case, a $G$-Higgs bundle is just a holomorphic principal $G^C$-bundle.

**Example 3.4.** If $G$ is complex, then $\mathfrak{g} = \mathfrak{h}^C$ and $\mathfrak{g}^C = \mathfrak{h}^C \oplus \mathfrak{m}^C \cong \mathfrak{g} \oplus \mathfrak{g}$. In this case, a $G$-Higgs bundles is a pair $(\mathcal{P}, \varphi)$ where $\mathcal{P}$ is a holomorphic $G$-bundle and $\varphi$ is a holomorphic section of the adjoint bundle $\mathcal{P}[\mathfrak{g}]$ twisted by $K$.

Rather than work with principal bundles, we will usually pick a faithful linear representation of $G^C$ and work with vector bundles. A faithful representation $G^C \to \text{GL}(V)$ defines a representation $\beta : H^C \to \text{GL}(V)$ and an embedding $\mathfrak{m}^C \to \text{End}(V)$. With this data fixed, a $G$-Higgs bundle $(\mathcal{P}, \varphi)$ gives rise to a pair $(E, \Phi)$ where $E \to X$ is the holomorphic vector bundle $\mathcal{P}[V]$ and $\Phi \in H^0(\text{End}(E) \otimes K)$ is given by $\varphi$ under the inclusion $\mathcal{P}[\mathfrak{m}^C] \otimes K \hookrightarrow \text{End}(\mathcal{P}[V]) \otimes K$.

**Example 3.5.** When $G = \text{SL}(n, \mathbb{C})$ we take $G \to \text{GL}(\mathbb{C}^n)$ to be the standard representation. An $\text{SL}(n, \mathbb{C})$-Higgs bundle is thus defined as a pair $(E, \Phi)$ where $E \to X$ is a holomorphic rank $n$ vector bundle and $\Phi \in H^0(\text{End}(E) \otimes K)$ satisfies $\text{tr}(\Phi) = 0$. Moreover, the standard volume form on $\mathbb{C}^n$ is preserved by the standard representation of $\text{SL}(n, \mathbb{C})$, and a holomorphic principal $\text{SL}(n, \mathbb{C})$-bundle is equivalent to a holomorphic vector bundle $E$ equipped with a holomorphic volume form $\omega \in H^0(\Lambda^n E)$. Thus, an $\text{SL}(n, \mathbb{C})$-Higgs bundle is equivalent to a triple $(E, \omega, \Phi)$. Note that the holomorphic volume form $\omega$ is equivalent to a holomorphic trivialization of the determinant line bundle $\Lambda^n E$. We will usually suppress $\omega$ from the notation.

**Example 3.6.** For $G = \text{SL}(n, \mathbb{R})$ we have $H = \text{SO}(n)$ and the Cartan decomposition is given by

$$\mathfrak{sl}(n, \mathbb{R}) \cong \mathfrak{so}(n) \oplus \text{Sym}_0(\mathbb{R}^n),$$

where $\text{Sym}_0(\mathbb{R}^n)$ is the vector space of traceless symmetric matrices. Again using the standard representation of $\text{SL}(n, \mathbb{C})$ we see that an $\text{SL}(n, \mathbb{R})$-Higgs bundle gives rise to a triple $(E, \omega, \Phi)$ as in the previous example.

Since $H^C = \text{SO}(n, \mathbb{C})$, the restriction of the standard representation of $\text{SO}(n, \mathbb{C})$ preserves a nondegenerate symmetric complex bilinear form on $\mathbb{C}^n$, a holomorphic principal $\text{SO}(n, \mathbb{C})$-bundle is equivalent to a triple $(E, \omega, Q_E)$ where $Q_E \in H^0(S^2(E) \otimes K)$ is everywhere nondegenerate. Equivalently, $Q_E$ defines a symmetric holomorphic isomorphism $Q_E : E \to E^*$. Since $\mathfrak{m}^C = \text{Sym}_0(\mathbb{C}^n)$, the Higgs field $\Phi$ is symmetric with respect to the quadratic form $Q_E$, i.e.

$$\Phi^T Q_E = Q_E \Phi.$$ 

An $\text{SL}(n, \mathbb{R})$-Higgs bundle is thus equivalent to a tuple $(E, \omega, Q_E, \Phi)$.

\footnote{For $\text{O}(n, \mathbb{C})$ a holomorphic principal bundle is equivalent to a pair $(E, Q_E)$.}
Example 3.7. For $G = SO(p, q)$, we have $H = S(O(p) \times O(q))$ and $\mathfrak{h} = \mathfrak{so}(p) \oplus \mathfrak{so}(q)$. With respect to a splitting $\mathbb{R}^{p+q} = \mathbb{R}^p \oplus \mathbb{R}^q$ we may decompose a matrix $X \in \text{End}(\mathbb{R}^{p+q})$ as $X = ( \begin{smallmatrix} A & B \\ C & D \end{smallmatrix} )$. The Lie algebra of $SO(p, q)$ is given by

$$\mathfrak{so}(p, q) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid (A \ B)^T \begin{pmatrix} \text{Id} & 0 \\ 0 & -\text{Id} \end{pmatrix} + \begin{pmatrix} \text{Id} & 0 \\ 0 & -\text{Id} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 0 \right\}.$$ 

This implies that $A \in \mathfrak{so}(p)$, $D \in \mathfrak{so}(q)$ and $B = -C^T$, thus the Cartan decomposition is given by

$$\mathfrak{so}(p, q) = (\mathfrak{so}(p) \oplus \mathfrak{so}(q)) \oplus \text{Hom}(\mathbb{R}^p, \mathbb{R}^q).$$

Similar to the previous examples, we use the standard representation. Since $H^C = S(O(p, \mathbb{C}) \times O(q, \mathbb{C}))$, the restriction of the standard representation of $SO(p + q, \mathbb{C})$ preserves an orthogonal splitting $\mathbb{C}^{p+q} = \mathbb{C}^p \oplus \mathbb{C}^q$. As in the previous example, a holomorphic principal $SO(p + q, \mathbb{C})$-bundle is equivalent to a triple $(E, \omega, Q_E)$. A holomorphic principal $S(O(p, \mathbb{C}) \times O(q, \mathbb{C}))$-bundle is thus equivalent to a triple $(E, \omega, Q_E)$ which decomposes as

$$(E, \omega, Q_E) = (V \oplus W, \omega, \begin{pmatrix} Q_V & -Q_W \end{pmatrix}),$$

where $V$ and $W$ respectively have rank $p$ and $q$ and quadratic forms $Q_V$ and $Q_W$. Using the Cartan decomposition and the description of the Lie algebra, the Higgs field $\Phi \in H^0(\text{End}(V \oplus W) \otimes K)$ is given by

$$\Phi = \begin{pmatrix} 0 & \eta \end{pmatrix},$$

where $\eta \in H^0(\text{Hom}(V, W) \otimes K)$ and, regarding $Q_V$ and $Q_W$ as isomorphism $V \to V^*$ and $W \to W^*$ respectively, $\eta^\dagger = -Q_V^{-1} \eta^T Q_W$. 

Remark 3.8. Taking the determinant of the isomorphisms $Q_V : V \to V^*$ defines an isomorphism of determinant line bundles $\det(V) \cong \det(V^*)$, or equivalently $\det(V)^2 \cong O$. Thus, the determinant line bundle of an orthogonal bundle $(V, Q_V)$ on $X$ is one of the $2^{2g}$ order two points in the Jacobian of $S$. The above volume form $\omega$ defines an isomorphism $\Lambda^{p+q}(V \oplus W) = \Lambda^p V \otimes \Lambda^q W \to O$. Using the orthogonal structures, this implies that $\Lambda^p V \cong \Lambda^q W^* \cong \Lambda^q W.$

Since $SO(p, q)$-Higgs bundles will be a main object of study we record this in a proposition.

Proposition 3.9. An $SO(p, q)$-Higgs bundle on $X$ is equivalent to the data

- a holomorphic rank $p$ vector bundle $V \to X$,
- a holomorphic symmetric isomorphism $Q_V : V \to V^*$,
- a holomorphic rank $q$ vector bundle $W \to X$,
- a holomorphic symmetric isomorphism $Q_W : W \to W^*$,
- a holomorphic isomorphism $\omega : \Lambda^p V \to \Lambda^q W^*$,
- a holomorphic section $\eta \in H^0(\text{Hom}(V, W) \otimes K)$.

The $SO(p + q, \mathbb{C})$-Higgs bundle associated to a tuple $(V, Q_V, W, Q_W, \omega, \eta)$ is

$$(E, \omega, Q_E, \Phi) = (V \oplus W, \omega, \begin{pmatrix} Q_V & 0 \\ 0 & -Q_W \end{pmatrix}, \begin{pmatrix} 0 & \eta \end{pmatrix}),$$

and the associated $SL(p + q, \mathbb{C})$-Higgs bundle is given by forgetting $Q_E$. 
We will often suppress $Q_V$, $Q_W$ and $\omega$ from the notation and just refer to an $SO(p, q)$-Higgs bundle as a triple $(V, W, \eta)$. We will also denote the associated Higgs bundle schematically by

$$V \overrightarrow{\eta} W,$$

where we have suppressed the twisting by $K$ from the notation.

**Remark 3.10.** Recall that the group $SO(p, q)$ has two connected components, $SO_0(p, q) < SO(p, q)$ denotes the connected component of the identity. The maximal compact subgroups of $SO_0(p, q)$ is $SO(p) \times SO(q)$. Thus, an $SO(p, q)$-Higgs bundle $(V, W, \eta)$ reduces to an $SO_0(p, q)$-Higgs bundle if and only if both $V$ and $W$ have trivial determinant.

### 3.2. Stability and the moduli space.

The moduli space of Higgs bundle parameterizes isomorphism classes of Higgs bundles. The isomorphism group for Higgs bundles is called the gauge group. Just as with the character variety, to get a nice moduli we restrict to a special class of Higgs bundles whose gauge orbits are closed.

Given a smooth principal $\mathbb{H}^c$-bundle $P \to X$, the $\mathbb{G}$-gauge group $\mathcal{G}_{\mathbb{H}^c}$ is group of bundle automorphisms $f : P \to P$. The elements of $\mathcal{G}_{\mathbb{H}^c}$ are given by sections of an associated bundle of groups $P[H^c] = P \times_{\text{Ad}_{\mathbb{H}^c}} \mathbb{H}^c$:

$$\mathcal{G}_{\mathbb{H}^c} = \Omega^0(X, P[H^c]).$$

Recall that for a holomorphic structure on a vector bundle $E$ is equivalent to a Dolbeaut operator. That is a differential operator

$$\bar{\partial}_E : \Omega^0(E) \to \Omega^{0,1}(E)$$

so that $2 \bar{\partial}_E(fs) = \bar{\partial}_f \otimes s + f \bar{\partial}_Es$ for all functions $f \in \Omega^0(C)$ and sections $s \in \Omega^0(E)$. Note that a Dolbeaut operator is equivalent to the $(0, 1)$-part of a connection on $E$. In particular, the space of holomorphic structures on $E$ is an infinite dimensional affine space with underlying vector space $\Omega^{0,1}(\text{End}(E))$.

For principal bundles, an analogous theory holds. Namely a holomorphic structure on a principal $\mathbb{H}^c$-bundle $P \to X$ is equivalent to a section $\partial_P \in \Omega^{0,1}(P, \mathfrak{h}^c)$ which defines the $(0, 1)$-part of a connection. Thus, the space of holomorphic structures on $P$ an infinite dimensional affine space with the space of basic $(0, 1)$-forms as underlying vector space. Equivalently, this vector space is given by sections $\Omega^{0,1}(X, P[\mathfrak{h}^c])$ of the adjoint bundle of $P$.

If we fix a smooth $\mathbb{H}^c$-bundle $P \to X$, the set of all Higgs bundles with underlying bundle $P$ is given by

$$\mathcal{H}(\mathbb{G}) = \{(\bar{\partial}_P, \varphi) | \bar{\partial}_P \varphi = 0\}.$$  

Fixing a holomorphic structure on $P$ defines $\mathcal{H}(\mathbb{G})$ as a quadratic subspace of a vector space:

$$\mathcal{H}(\mathbb{G}) \hookrightarrow \Omega^{0,1}(P[\mathfrak{h}^c]) \oplus \Omega^{1,0}(P[\mathfrak{m}^c]).$$

**Remark 3.11.** Note that when $\mathbb{G}$ is complex $\Omega^{0,1}(P[\mathfrak{h}^c]) \oplus \Omega^{1,0}(P[\mathfrak{m}^c]) \cong \Omega^1 (P[\mathfrak{g}])$ since $\mathfrak{h}^c \oplus \mathfrak{m}^c \cong \mathfrak{g} \oplus \mathfrak{g}$.  

$^2$Dolbeaut operators must also satisfy the integrability condition $\bar{\partial}_E^2 = 0$, but this is automatic on a Riemann surface.
For \((\alpha, \psi) \in \Omega^{0,1}(P[h^C]) \oplus \Omega^{1,0}(P[m^C])\), we have \((\bar{\partial}_P + \alpha, \varphi + \psi) \in \mathcal{H}(G)\) if
\[
\bar{\partial}_P \varphi + \bar{\partial}_P \psi + [\alpha, \varphi] + [\alpha, \psi] = 0.
\]
The tangent space is thus given by sections \((\alpha, \psi)\) satisfying this equation to first order:
\[
(3.2) \quad T_{\bar{\partial}_P \varphi} \mathcal{H}(G) = \{ (\alpha, \psi) \in \Omega^{0,1}(P[h^C]) \oplus \Omega^{1,0}(P[m^C]) \mid \bar{\partial}_P \psi + [\alpha, \varphi] = 0 \}.
\]

**Remark 3.12.** The space of Higgs bundles \(\mathcal{H}(G)\) has a natural complex structure given by
\[
I(\alpha, \psi) = (i\alpha, i\psi).
\]
It also has a natural symplectic structure given by
\[
\omega_I((\alpha_1, \psi_1), (\alpha_2, \psi_2)) = \text{Re}\left( i \int_X \text{tr}(\psi_2 \wedge \alpha_1 - \psi_1 \wedge \alpha_2) \right).
\]
The gauge group \(G_{HC}\) acts on \(\mathcal{H}(G)\) by pullback, namely for \(g \in G_{HC}\)
\[
(\bar{\partial}_P, \varphi) \cdot g = (\text{Ad}_{g^{-1}} \bar{\partial}_P, \text{Ad}_{g^{-1}} \varphi).
\]
The orbits of the gauge group are not closed, and to form a nice moduli space we need a notion of (poly)stability. The moduli space \(\mathcal{M}(G)\) of \(G\)-bundles is then defined to be the set \(G_{HC}\)-orbits of polystable \(G\)-Higgs bundles. In general, the notion of the stability involves considering how all holomorphic structure group reductions of an \(\mathcal{H}C\)-bundle to a parabolic subgroup interact with the Higgs field. Instead of developing this theory in general, we will develop the appropriate stability conditions in the vector bundle situation. For the general theory see [9].

Recall that a \(GL(n, \mathbb{C})\)-Higgs bundle is equivalent to a rank \(n\) holomorphic vector bundle \(E\) a section \(\Phi \in H^0(\text{End}(E) \otimes K)\). For \(SL(n, \mathbb{C})\) the bundle \(E\) is equipped with a trivialization of \(\Lambda^n E\), thus, \(\text{deg}(E) = 0\).

**Definition 3.13.** An \(SL(n, \mathbb{C})\)-Higgs bundle \((E, \Phi)\) is
- stable if for all proper holomorphic subbundles \(F \subset E\) such that \(\Phi(F) \subset F \otimes K\) we have \(\text{deg}(F) < 0\), and
- polystable if \((E, \Phi) = \bigoplus_j (E_j, \Phi_j)\) with \((E_j, \Phi_j)\) stable and \(\text{deg}(E_j) = 0\) for all \(j\).

For general groups \(G\) the notion of stability is functorial in the sense that if \(G\) is a real form of a reductive subgroup of \(SL(n, \mathbb{C})\), then a \(G\)-Higgs bundle is polystable if and only if the associated \(SL(n, \mathbb{C})\)-Higgs bundle is polystable. Let \(\mathcal{H}^{ps}(G) \subset \mathcal{H}(G)\) denote the set of polystable Higgs bundles. The gauge group \(G_{HC}\)-preserves \(\mathcal{H}^{ps}(G)\), and the gauge orbits in \(\mathcal{H}^{ps}(G)\) are closed. We define the moduli space \(\mathcal{M}(G)\) to be the quotient space
\[
\mathcal{M}(G) = \mathcal{H}^{ps}(G)/G_{HC}.
\]
We note that the complex structure \(I\) and the symplectic form \(\omega_I\) from Remark 3.12 are preserved by the gauge group action and thus descend to the moduli space.

For the general notion of stability, it is not the case that a \(G\)-Higgs bundle is stable if and only if the associated \(SL(n, \mathbb{C})\)-Higgs bundle is stable. However, can detect stable \(G\)-Higgs bundles inside of the set of polystable Higgs bundles with the following proposition.

**Proposition 3.14.** Let \(G\) be a real form of a complex semisimple subgroup of \(SL(n, \mathbb{C})\). A \(G\)-Higgs bundle \((P, \varphi)\) is stable if it is polystable and has finite automorphism group. Moreover, the set of stable \(G\)-Higgs bundles is open.
Remark 3.15. Note that if the SL(n, ℂ)-Higgs bundle associated to a G-Higgs bundle is stable, then it is stable as a G-Higgs bundle.

Let \( \mathcal{H}^s(G) \subset \mathcal{H}^{ss}(G) \) be the stable locus, the quotient

\[
\mathcal{H}^s / \mathcal{G}_{\mathcal{H}^s} \subset \mathcal{M}(G)
\]

is an orbifold. At a stable Higgs bundle one can show that the real dimension of the tangent space to \( T_{[\tilde{g},\varphi]} \mathcal{M}(G) \) is \( \dim_R(G)(2g - 2) \) (see Remark 6.2). Thus the real dimension of \( \mathcal{M}(G) \) is given by \( \dim_R(G)(2g - 2) \).

4. SO(1, q)-Higgs bundles especially when \( q = 2 \)

In this section we will describe the moduli space of SO(1, q)-Higgs bundles and SO\(_0\)(1, q)-Higgs bundles. When \( q = 2 \) we have \( \text{SO}_0(1, 2) \cong \text{PSL}(2, \mathbb{R}) \). In this case we will recall Hitchin’s parameterization of most of the components of \( \mathcal{M}(\text{SO}_0(1, 2)) \) especially the Teichmüller component.

Recall from Proposition 3.9 that an SO(1, n)-Higgs bundle consists of a tuple \((V, Q_V, W, Q_W, \omega, \eta)\) where \( \text{rk}(V) = 1 \) and \( \text{rk}(W) = q \). We can take \((V, Q_V) = (\Lambda^n W, \det(Q_W))\) and \( \omega = \det(Q_W) : V \to \Lambda^q W^* \). Thus, such a tuple is determined by the triple \((W, Q_W, \eta)\), where

\[
\eta \in H^0(W \otimes \Lambda^q W)^{-1} \otimes K.
\]

Using the notation from (3.1), the associated SL\((1 + q, \mathbb{C})\)-Higgs bundle is given by

\[
\Lambda^n W \underbrace{\eta}_{\eta} W.
\]

When \( q = 1 \), we have \( \eta \in H^0(K) \) and the first Stiefel-Whitney class \( sw_1(W) \in H^1(X, \mathbb{Z}_2) \) of \( W \) labels the components of \( \mathcal{M}(\text{SO}(1, 1)) \). Namely,

\[
\mathcal{M}(\text{SO}(1, 1)) = \coprod_{sw_1 \in H^1(X, \mathbb{Z}_2)} \mathcal{M}_{sw_1}(\text{SO}(1, 1)),
\]

and each space \( \mathcal{M}_{sw_1}(\text{SO}(1, 1)) \) is parameterized by \( H^0(K) \).

For \( q > 1 \), the first and second Steifel-Whitney classes \( (sw_1, sw_2) \in H^1(X, \mathbb{Z}_2) \times H^2(X, \mathbb{Z}_2) \) of \((W, Q_W)\) give a decomposition of the moduli space

\[
\mathcal{M}(\text{SO}(1, n)) = \coprod_{sw_1, sw_2} \mathcal{M}^{sw_1, sw_2}(\text{SO}(1, n)).
\]

The first Steifel-Whitney class of \( W \) vanishes if and only if the \( O(q, \mathbb{C}) \)-bundle reduces to \( \text{SO}(q, \mathbb{C}) \). Thus, for \( q = 2 \) and \( sw_1 = 0 \) the bundle \( W \) reduces to an \( \text{SO}(2, \mathbb{C}) \)-bundle. Since \( \mathbb{C}^* = \text{SO}(2, \mathbb{C}) \), in this case the degree of the \( \mathbb{C}^* \)-bundle provides a refinement of the second Steifel-Whitney class. More precisely, if \( sw_1(W, Q_W) = 0 \), then there is a line bundle \( L \in \text{Pic}(X) \) such that

\[
(W, Q_W) \cong \left( L \oplus L^{-1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right).
\]

The integer \( \text{deg}(L) \) satisfies \( sw_2(W, Q_W) = \text{deg}(L) \mod 2 \), and the isomorphism switching \( L \) with \( L^{-1} \) preserves the \( O(2, \mathbb{C}) \)-structure. Thus, \(|\text{deg}(L)| \in \mathbb{N}\) which is a well defined invariant of \( O(2, \mathbb{C}) \)-bundles with vanishing \( sw_1 \).
This gives a decomposition of the moduli space as
\[ \coprod_{sw_2 \neq 0, sw_2} \mathcal{M}_{sw_2}^{su_2}(\text{SO}(1, 2)) \uplus \coprod_{d \in \mathbb{N}} \mathcal{M}_d(\text{SO}(1, 2)). \]

For Higgs bundles in \( \mathcal{M}_d(\text{SO}(1, 2)) \) the splitting \( W = L \oplus L^{-1} \) gives a decomposition of the Higgs field \( \eta : \mathcal{O} \to W \otimes K \) as
\[ \eta = \begin{pmatrix} \beta \\ \gamma \end{pmatrix} : \mathcal{O} \to LK \oplus L^{-1}K, \]
where \( \beta \in H^0(LK) \) and \( \gamma \in H^0(L^{-1}K) \). Using \( Q_W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), we can write the associated \( \text{SL}(3, \mathbb{C}) \)-Higgs bundle schematically as
\[ \begin{xy}
  0 \ar@{<->}[r]^-{\beta} & L \ar@{<->}[l]^-{\gamma} \ar@{<->}[r]^-{\beta} & \mathcal{O} \ar@{<->}[l]^-{\gamma} \ar@{<->}[r]^-{\beta} & L^{-1},
\end{xy} \]

where we recall that we suppress the twisting by \( K \) from the notation.

The stability condition limits the objects we are considering.

**Proposition 4.1.** If \( (\mathcal{O}, L \oplus L^{-1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \beta \\ \gamma \end{pmatrix}) \) is a polystable \( \text{SO}(1, 2) \)-Higgs bundle with zero first Stiefel-Whitney class, then \( |\text{deg}(L)| \leq 2g - 2 \). Moreover, if \( \text{deg}(L) \in (0, 2g - 2] \), then \( \gamma \neq 0 \) and if \( \text{deg}(L) \in [2g - 2, 0) \), then \( \beta \neq 0 \).

**Proof.** Consider the associated \( \text{SL}(3, \mathbb{C}) \)-Higgs bundle (4.1). By stability, if \( \text{deg}(L) > 0 \) then \( \gamma \neq 0 \) since otherwise \( L \) would define a positive degree invariant subbundle. But, \( \gamma \in H^0(L^{-1}K) \) so if \( \text{deg}(L) > 2g - 2 \) then \( \gamma = 0 \), contradicting stability. Similarly, if \( \text{deg}(L) < 0 \), then stability forces \( \beta \neq 0 \) and we concluded \( \text{deg}(L) > 2 - 2g \). \( \square \)

For \( d = |\text{deg}(L)| > 0 \), we can parameterize the moduli space \( \mathcal{M}_d(\text{SO}(1, 2)) \), this was done by Hitchin in [14] for the group \( \text{PSL}(2, \mathbb{R}) \).

**Theorem 4.2.** [Hitchin [14]] For \( d > 0 \), the moduli space \( \mathcal{M}_d(\text{SO}(1, 2)) \) is smooth and diffeomorphic to the total space of a rank \( (d + g - 1) \)-complex vector bundle over the \((2g - 2 - d)\)-symmetric product \( \text{Sym}^{2g - 2 - d}(X) \) of the Riemann surface \( X \).

**Proof.** By the above discussion a point in \( \mathcal{M}_d(\text{SO}(1, 2)) \) is determined by a triple \((L, \gamma, \beta)\) where \( L \in \text{Pic}^d(X) \), \( \gamma \in H^0(L^{-1}K) \setminus \{0\} \) and \( \beta \in H^0(LK) \). The \( S(\text{O}(1, \mathbb{C}) \times O(2, \mathbb{C})) \)-bundle is given by
\[ (V, Q_V, W, Q_W) = (\mathcal{O}, (1), L \oplus L^{-1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) \]
and the Higgs field \( \eta = \begin{pmatrix} \beta \\ \gamma \end{pmatrix} : V \to W \otimes K \).

For two triples \((L, \beta, \gamma)\) and \((L', \beta', \gamma')\) to define isomorphic \( \text{SO}(1, 2) \)-Higgs bundles it is necessary that \( |\text{deg}(L)| = |\text{deg}(L')| \). Thus we may assume \( L = L' \) as elements \( \text{Pic}^d(X) \). The remaining holomorphic gauge transformation of the \( S(\text{O}(1, \mathbb{C}) \times O(2, \mathbb{C})) \)-bundle is given by
\[ (gV, gw) = (1, \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}) \].
for $\lambda \in \mathbb{C}^*$. This gauge transformation acts on the Higgs field by

$$g_w^{-1}hg_v = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \beta \\ \gamma \end{pmatrix} (1) = \begin{pmatrix} \lambda^{-1}\beta \\ \lambda\gamma \end{pmatrix}.$$ 

In particular, we note that the automorphism group of such an SO(1, 2)-Higgs bundle is trivial since $\gamma \neq 0$. Thus, the moduli space $\mathcal{M}_d(\text{SO}(1, 2))$ is smooth and given $\mathbb{C}^*$-equivalence $[L, \beta, \gamma]$ where $(L, \beta, \gamma) \sim (L', \beta', \gamma')$ if and only if $L = L' \in \text{Pic}^d(X)$, $\beta = \lambda\beta'$ and $\gamma = \lambda^{-1}\gamma'$ for $\lambda \in \mathbb{C}^*$.

Recall that the space of effective divisors on $X$ of degree $n$ is given by the $n$th-symmetric product $\text{Sym}^d(X)$. Taking the projective class of $\gamma \in H^0(L^{-1}K) \setminus \{0\}$ defines a surjective map to the space of effective degree $2g - 2 - d$ divisors on $X$:

$$\mathcal{M}_d(\text{SO}(1, 2)) \longrightarrow \text{Sym}^{2g-2-d}(X).$$

We claim that the fiber of this map is a vector space of rank $(d + g - 1)$. Denote by $\mathcal{O}(\{\gamma\})$ the line bundle associated to the divisor $[\gamma]$. The line bundle $L$ is given by $L = \mathcal{O}(\{\gamma\})^{-1}K$ and $\beta \in H^0(\mathcal{O}(\{\gamma\})^{-1}K^2)$. Thus $L$ is determined by $[\gamma]$ and $\beta$ can be any element of the $(d + g - 1)$ dimensional vector space $H^0(\mathcal{O}(\{\gamma\})^{-1}K^2)$. □

We now collect many corollaries of the above theorem.

**Corollary 4.3.** For $d > 0$ the moduli space $\mathcal{M}_d(\text{SO}(1, 2))$ is connected and homotopy equivalent to the symmetric product $\text{Sym}^{2g-2-d}(X)$.

The cohomology ring of a symmetric product of a Riemann surface was computed in [20], as a result this computes the cohomology ring of $\mathcal{M}_d(\text{SO}(1, 2))$. When $d = 2g - 2$, the space is contractible and we have the following.

**Corollary 4.4.** The moduli space $\mathcal{M}_{2g-2}(\text{SO}(1, 2))$ is parameterized by the $3g - 3$ dimensional complex vector space $H^0(K^2)$ of holomorphic differentials. Moreover, $\mathcal{M}_{2g-2}(\text{SO}(1, 2))$ is the image of a section of the $\text{SO}(3, \mathbb{C})$-Hitchin fibration.

**Proof.** An SO(1, 2)-Higgs bundle in $\mathcal{M}_{2g-2}(\text{SO}(1, 2))$ is determined by a triple $(L, \beta, \gamma)$ where $\text{deg}(L) = 2g - 2$, $\beta \in H^0(LK)$ and $\gamma \in H^0(L^{-1}K) \setminus \{0\}$. The condition on $\gamma$ implies that $L = K$ and thus $\beta \in H^0(K^2)$. If we normalize $\gamma$ to be $\gamma = 1 \in H^0(\mathcal{O})$, then there is no more gauge freedom, and so $\mathcal{M}_{2g-2}(\text{SO}(1, 2)) \cong H^0(K^2)$.

Consider the $\text{SO}(3, \mathbb{C})$-Hitchin fibration

$$\mathcal{M}(\text{SO}(3, \mathbb{C})) \longrightarrow H^0(K^2).$$

Using the above parameterization of $\mathcal{M}_{2g-2}(\text{SO}(1, 2))$ by $H^0(K^2)$, the $\text{SO}(3, \mathbb{C})$-Higgs bundle associated to $\eta \in H^0(K^2)$ is given by

$$[E, Q, \Phi] = \left(\mathcal{O} \oplus K \oplus K^{-1}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & q_2 \\ q_2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right).$$

For this Higgs bundle $\frac{1}{2} \text{tr}(\Phi^2) = q_2$. □
Translating these statements to the character variety $\mathcal{X}(\Gamma, SO(1, 2))$ via the non-abelian Hodge correspondence gives the following.

**Corollary 4.5.** For each $0 < d \leq 2g - 2$, the character variety $\mathcal{X}(\Gamma, SO(1, 2))$ has a connected component $\mathcal{X}_d(\Gamma, SO(1, 2))$ which is smooth and diffeomorphic to a real rank $2d + 2g - 2$ vector bundle over the symmetric product $\text{Sym}^{2g-2-d}(S)$.

**Corollary 4.6.** Every representation $\rho \in \mathcal{X}_d(\Gamma, SO(1, 2))$ factors through the connected component of the identity $SO_0(1, 2)$ and the Fuchsian representations are given by

$$\mathcal{X}_{2g-2}(\Gamma, SO(1, 2)) \cong \text{Fuch}(\Gamma).$$

**Proof.** We space $\text{Fuch}(\Gamma)$ consists of two connected component of the character variety $\mathcal{X}(\Gamma, PSL(2, \mathbb{R})) = \mathcal{X}(\Gamma, SO_0(1, 2))$ which is identified with the Teichmüller space of $S$. Since the representations in these components are conjugate by an element of $SO(1, 2)$ which is not in $SO_0(1, 2)$, the two components of $\text{Fuch}(\Gamma)$ are identified in $\mathcal{X}(\Gamma, SO(1, 2))$. Since $\text{Fuch}(\Gamma)$ is contractible and the only $d$ for which $\mathcal{X}_d(\Gamma, SO(1, 2))$ is contractible is $d = 2g - 2$ we are done. □

**Remark 4.7.** Since the second Steifel-Whitney class invariant of the Higgs bundles in $\mathcal{M}(SO(1, 2))$ is given by $d$ mod 2, the associated $SO(3, \mathbb{C})$-Higgs bundles lift to $\text{Spin}(3, \mathbb{C})$ if and only if $d$ is even. Recall that the isomorphism $\text{Spin}(3, \mathbb{C}) = \text{SL}(2, \mathbb{C})$ is given by the 2 to 1 map $\text{SL}(2, \mathbb{C}) \to \text{SO}(3, \mathbb{C})$ which is induced by the action on the second symmetric product $S^2(\mathbb{C}^2)$. Here, the volume form on $\mathbb{C}^2$ induces a nondegenerate symmetric form on the second symmetric product.

The $\text{SL}(2, \mathbb{R})$-Higgs bundles which give rise to the Higgs bundles in $\mathcal{M}_{2d}(SO(1, 2))$ are thus given by

$$(E, \omega, Q_E, \Phi) \cong (N \oplus N^{-1}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}),$$

where we view the volume form $\omega \in H^0(\Lambda^2 E)$ is viewed as a skew symmetric homomorphism $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : E^* \to E$. Indeed, taking the second symmetric product gives

$$(S^2 E, S^2 \omega, S^2 \Phi) \cong (O \oplus N^2 \oplus N^{-2}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \gamma & \beta \\ \beta & 0 & 0 \\ \gamma & 0 & 0 \end{pmatrix}).$$

In particular, the $\text{SL}(2, \mathbb{R})$-Higgs bundles which define points in the component $\mathcal{M}_{2g-2}(SO(1, 2))$ are given by

$$(K^{\frac{g}{2}} \oplus K^{-\frac{g}{2}}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & q_2 \\ q_2 & 0 \end{pmatrix})$$

for one of the $2^{2g}$ choices of square root $K^{\frac{g}{2}}$ of $K$. In particular, there are $2^{2g}$-connected components of $\mathcal{M}((2, \mathbb{R}))$ which project to $\mathcal{M}_{2g-2}(PSL(2, \mathbb{R})) \cong \mathcal{M}_{2g-2}(SO(1, 2))$.

5. The Hitchin fibration and Hitchin section

So far we have seen that the character variety $\mathcal{X}(\Gamma, G)$ is homeomorphic to the moduli space of $G$-Higgs bundles. The upshot of this correspondence is that the
Higgs bundle moduli space has a lot of useful structures which the character variety is lacking. In this section we define the Hitchin component and see use this additional structure to construct the Hitchin component from Definition 2.7.

5.1. The Hitchin fibration. Suppose $G$ is a complex simple Lie group. Similar to Chern-Weyl theory, we can apply an invariant polynomial to the Higgs field and obtain a holomorphic differential. Fixing a homogeneous basis $p_1, \ldots, p_{\text{rk}(G)}$ of the $\text{Ad}_C$-invariant polynomials $\mathbb{C}[g]$ with $\deg(p_j) = m_j$ defines a map

$$h : \mathcal{M}(G) \longrightarrow \bigoplus_{j=1}^{\text{rk}(G)} H^0(K^{m_j})$$

called the Hitchin fibration. For example, when $G = \text{SL}(n, \mathbb{C})$ we have $m_j = j + 1$ for $1 \leq j \leq n - 1$, and when $G = \text{SO}(2n + 1, \mathbb{C})$ we have $m_j = 2j$ for $1 \leq j \leq n$.

In general, a computation using the Riemann-Roch theorem shows that the base is half the dimension of the moduli space:

$$\dim_{\mathbb{C}} \left( \bigoplus_{j=1}^{\text{rk}(G)} H^0(K^{m_j}) \right) = \frac{1}{2} \dim_{\mathbb{C}} (\mathcal{M}(G)) = \dim_{\mathbb{C}}(G)(g - 1).$$

In [16], Hitchin showed that for any choice of basis the Hitchin fibration is proper. In fact, the generic fibers of the Hitchin fibration are half dimensional tori and makes $\mathcal{M}(G)$ into a algebraic completely integrable system, we will not make use of this additional structure.

Remark 5.1. Notice that the dimension of the base of the Hitchin fibration is the same as the dimension of the moduli space of $G'$-Higgs bundles for $G' < G$ any real form. For example, the Hitchin base of $\text{SO}(2n + 1, \mathbb{C})$ has the same dimension as $\mathcal{M}(\text{SO}(p, q))$ for all $p$ and $q$ satisfying $p + q = 2n + 1$.

5.2. The Hitchin section. Let $\mathfrak{g}$ be a semisimple complex Lie algebra. For $\mathfrak{s} \subset \mathfrak{g}$ a subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{C})$, consider the decomposition of $\mathfrak{g}$ into irreducible $\mathfrak{sl}(2, \mathbb{C})$-representations

$$\mathfrak{g} = \bigoplus_{j=1}^{N} V_j.$$  

For any such $\mathfrak{s} \subset \mathfrak{g}$ we have $N \geq \text{rk}(\mathfrak{g})$, and when $N = \text{rk}(\mathfrak{g})$ the the three dimensional subalgebra $\mathfrak{s}$ is called principal. Up to the conjugation, there is a unique principal three dimensional subalgebra [17]. In this case we have $\dim(V_j) = 2m_j + 1$ where $1 = m_1 \leq m_2 \leq \cdots \leq m_{\text{rk}(\mathfrak{g})}$ are the exponents of $\mathfrak{g}$. Moreover, when we restrict a principal embedding $\mathfrak{sl}(2, \mathbb{C}) \rightarrow \mathfrak{g}$ to the real subalgebra $\mathfrak{sl}(2, \mathbb{R})$, the image lies in the a split real subalgebra of $\mathfrak{g}$. This defines an embedding

$$t_{pr} : \text{PSL}(2, \mathbb{R}) \rightarrow G^{\text{split}}$$

Theorem 5.2 (Hitchin [15]). Let $G$ be a complex simple Lie group, then the Hitchin fibration (5.1) has a section

$$s_h : \bigoplus_{j=1}^{\text{rk}(G)} H^0(K^{m_j}) \longrightarrow \mathcal{M}(G)$$
which maps onto a component of the split real form $\mathcal{M}(G^{\text{split}})$. Under the non-abelian Hodge correspondence (Theorem 3.1) correspondence, the image of this section defines the Hitchin component $\text{Hit}(G^{\text{split}}) \subset \mathcal{X}(G)$ from Definition 2.7.

We will prove the above theorem for $G = \text{SO}(2p + 1, \mathbb{C})$, namely we will construct the Hitchin section and prove that it maps onto a component for the group $\text{SO}(p, p + 1)$. For $\text{SO}(2p + 1, \mathbb{C})$ the Hitchin fibration is given by

$$\mathcal{M}(\text{SO}(2p + 1, \mathbb{C})) \to \bigoplus_{j=1}^{2p} H^0(K^{2j}) .$$

Consider the rank $p$ holomorphic orthogonal bundle

(5.2)

$$K_p = K^{p-1} \oplus K^{p-3} \oplus \cdots \oplus K^{3-p} \oplus K^{1-p} .$$

Note that $K_p$ has a natural orthogonal structure $Q_p = \left( \begin{smallmatrix} 1 & & \cdots & 1 \\ & & & \\ & & & \\ & & & \\ 1 & & & \end{smallmatrix} \right) : K_p \to K_p^* .

Consider the map

$$\hat{\Psi} : \bigoplus_{j=1}^{2p} H^0(K^{2j}) \longrightarrow \mathcal{H}(\text{SO}(p, p + 1))$$

defined by

(5.3)

$$\hat{\Psi}(q_2, \cdots, q_{2p}) = (K_p, Q_p, K_{p+1}, Q_{p+1}, q_2, q_4, \cdots, q_{2p}, 1, q_2, \cdots, q_{2p-2}, 1) : K_p \to K_{p+1} \otimes K^* .$$

We claim that the image of $\hat{\Psi}$ is contained in the stable Higgs bundles $\mathcal{H}^*(\text{SO}(p, p + 1))$ and that the induced map $\Psi : \bigoplus_{j=1}^{p} H^0(K^{2j}) \to \mathcal{M}(\text{SO}(p, p + 1))$ has an open and closed image.

**Proposition 5.3.** The image of $\Psi$ consists of stable $\text{SO}(p, p + 1)$-Higgs bundles.

**Proof.** Consider the stable $\text{SL}(2, \mathbb{C})$-Higgs bundle

(5.4)

$$(E, \Phi) = \left( K^{\frac{p}{2}} \oplus K^{-\frac{p}{2}}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) .$$

Since the unique irreducible $(2p + 1)$-dimensional representation of $\text{SL}(2, \mathbb{C})$ is given by the $2p$-symmetric product, the $\text{SL}(2p + 1, \mathbb{C})$-Higgs bundle given by

$$(S^{2p+1}E, S^{2p+1}\Phi) = \left( K^p \oplus K^{p-1} \oplus \cdots \oplus K^{1-p} \oplus K^{-p}, \begin{pmatrix} 0 & \cdots & 0 \\ p-1 & 0 & \cdots \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix} \right)$$


is also stable. Moreover this is gauge equivalent to

\begin{equation}
(5.5) \quad \left( K^p \oplus K^{p-1} \oplus \cdots \oplus K^{1-p} \oplus K^{-p}, \begin{pmatrix}
0 & 0 \\
1 & 0 \\
& \ddots \\
& & \ddots \\
& & & 1 & 0
\end{pmatrix} \right).
\end{equation}

After rearranging the summands of $K_p \oplus K_{p+1}$, the $\text{SL}(2p + 1, \mathbb{C})$-Higgs bundle associated to $\hat{\Psi}(0, \cdots, 0)$ is given by (5.5). Thus, $\hat{\Psi}(0, \cdots, 0)$ is a stable $\text{SO}(p, p+1)$-Higgs bundle. Since stability is an open condition, for $q_2, \cdots, q_{2p}$ sufficiently close to zero, the Higgs bundle $\hat{\Psi}(q_2, \cdots, q_{2p})$ is also stable.

The $\mathbb{C}^*$-action does not preserve the image of $\hat{\Psi}$. However, for each $\lambda \in \mathbb{C}^*$, the Higgs bundle $\lambda \cdot \hat{\Psi}(q_2, q_3, \cdots, q_{2p})$ is gauge equivalent to $\hat{\Psi}(\lambda^2 q_2, \lambda^4 q_4, \cdots, \lambda^{2p} q_{2p})$. Since stability is preserved by the $\mathbb{C}^*$-action, we conclude that all Higgs bundles in the image of $\hat{\Psi}$ are stable. □

**Proposition 5.4.** Let $\Phi(q_2, \cdots, q_{2p})$ be the Higgs field of the $\text{SO}(2p + 1, \mathbb{C})$-Higgs bundle associated to the $\text{SO}(p, p+1)$-Higgs bundle $\hat{\Psi}(q_2, \cdots, q_{2p})$. There is a basis $(p_1, \cdots, p_{2p})$ of the invariant polynomials $\mathbb{C}[\mathfrak{so}(2p + 1, \mathbb{C})]^{\text{SO}(2p+1, \mathbb{C})}$ so that for all $j$

$$p_j(\Phi(q_2, \cdots, q_{2p})) = q_{2j}.$$ 

**Proof.** In the general setting of a complex semisimple Lie group the existence of such a basis was proven by Kostant in [17]. For $\text{SO}(2p + 1, \mathbb{C})$ we construct such a basis by direct computation. We explain how this works for $p = 2$ and leave the general case to the reader.

After rearranging the summands, the $\text{SO}(5, \mathbb{C})$-Higgs bundle $(E, Q, \Phi)$ associated to the $\text{SO}(2, 3)$-Higgs bundle $\hat{\Psi}(q_2, q_4)$ is given by

$$
\left( K^2 \oplus K \oplus \mathcal{O} \oplus K^{-1} \oplus K^{-2}, \begin{pmatrix}
1 & -1 \\
1 & 1 \\
1 & 1 \\
0 & 0 
\end{pmatrix}, \begin{pmatrix}
0 & q_2 & 0 & q_4 & 0 \\
1 & 0 & q_2 & 0 & q_4 \\
0 & 1 & 0 & q_2 & 0 \\
0 & 0 & 1 & 0 & q_2 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix} \right).
$$

We have $\text{tr}(\Phi^2) = 8q_2$ and $\text{tr}(\Phi^4) = 14q_2^2 + 5q_4$, thus we choose the basis

$$p_1(\Phi) = \frac{1}{8} \text{tr}(\Phi^2) \quad \text{and} \quad p_2(\Phi) = \frac{1}{5} \text{tr}(\Phi^4) - \frac{1}{14} \text{tr}(\Phi^2)^2.$$ 

□

By the previous two propositions, the map $\hat{\Psi}$ gives rise to a well defined map

$$\Psi : \bigoplus_{j=1}^{p} H^0(K^{2j}) \to \mathcal{M}(\text{SO}(p, p + 1))$$

which is a section of the Hitchin fibration for $\mathcal{M}(\text{SO}(2p + 1, \mathbb{C})) \to \bigoplus_{j=1}^{p} H^0(K^{2j})$.

We now show that the image of $\Psi$ is open and closed.

**Proposition 5.5.** The image of the map $\Psi : \bigoplus_{j=1}^{p} H^0(K^{2j}) \to \mathcal{M}(\text{SO}(p, p + 1))$ is open and closed.
5.4. Properness of the Hitchin fibration We conclude that the sequence \( \Psi(q^i_2, \cdots, q^i_{2p}) \) also diverges in \( \mathcal{M}(\text{SO}(p, p + 1)) \).

To complete the proof we need to show that under the nonabelian Hodge correspondence, the component defined by \( \Psi(\bigoplus_{j=1}^p H^0(K^{2j})) \) is the Hitchin component \( \text{Hit}(\text{SO}(p, p + 1)) \) from Definition 2.7. It suffices to show that the representation associated to \( \Psi(0, \cdots, 0) \) is in \( \text{Hit}(\text{SO}(p, p + 1)) \). By Remark 4.7, the Higgs bundle \( \text{SL}(2, \mathbb{R}) \)-Higgs bundle whose corresponding representation in \( \text{Fuch}(\Gamma) \). From Example 2.6, the principal embedding \( \iota_{pr} : \text{PSL}(2, \mathbb{R}) \rightarrow \text{SO}(p, p + 1) \) is given by taking the \( 2p \)-symmetric product of the standard representation of \( \text{SL}(2, \mathbb{R}) \). Thus, the representation associated to \( \Psi(0, \cdots, 0) \) is contained in \( \text{Hit}(\text{SO}(p, p + 1)) \).

6. Structure of the moduli space

6.1. Tangent space and deformation complex. In this section we will assume for simplicity that \( G \) is the real form of a complex semisimple Lie group. Under this assumption, the automorphism group of a stable \( G \)-Higgs bundle is discrete (see Proposition 3.14). Recall that \( \mathcal{H}(G) \) is the set of pairs \( (\overline{\partial} P, \varphi) \) where \( \overline{\partial} P \) is a Dolbeault operator on smooth \( \mathbb{H}^C \)-bundle \( P \rightarrow X \) and \( \varphi \in \Omega^{1,0}(P[\mathbb{m}^C]) \) such that \( \overline{\partial} P \varphi = 0 \).

Since the space of Dolbeault operators is an affine space with underlying vector space isomorphic \( \Omega^{0,1}(P[\mathbb{h}^C]) \) the tangent space of \( \mathcal{H}^{ps}(G) \) at \( (\overline{\partial} P, \varphi) \) is given by the set of \( (\alpha, \psi) \in \Omega^{0,1}(P[\mathbb{h}^C]) \oplus \Omega^{1,0}(P[\mathbb{m}^C]) \) so that \( \varphi + \psi \) is holomorphic with respect to the Dolbeault operator \( \overline{\partial} + \alpha \) to first order. That is,

\[
T_{(\overline{\partial} P, \varphi)} \mathcal{H}(G) = \{ (\alpha, \psi) \in \Omega^{0,1}(P[\mathbb{h}^C]) \oplus \Omega^{1,0}(P[\mathbb{m}^C]) \mid \overline{\partial} P \psi + \alpha \varphi = 0 \in \Omega^{1,1}(P[\mathbb{h}^C]) \}.
\]

The moduli space of \( G \)-Higgs bundles is a set of gauge equivalence classes:

\[
\mathcal{M}(G) = \mathcal{H}^{ps}(G)/\mathcal{G}_{\mathbb{H}^C},
\]

where \( \mathcal{H}^{ps}(G) \) denotes the set of polystable pairs. At stable points of the moduli space, the tangent space can be interpreted as a quotient of the tangent space to the gauge orbit \( \mathcal{G}_{\mathbb{H}^C} \cdot (\overline{\partial} P, \varphi) \):

\[
T_{[\overline{\partial} P, \varphi]} \mathcal{M}(G) = T_{(\overline{\partial} P, \varphi)} \mathcal{H}(G)/T_{(\overline{\partial} P, \varphi)} \mathcal{G}_{\mathbb{H}^C} \cdot (\overline{\partial} P, \varphi).
\]

This is because, under our assumption on \( G \), the automorphism group of a stable \( G \)-Higgs bundle is discrete, and so the gauge group action is locally free. The tangent space to the gauge orbit of a stable \( G \)-Higgs bundle can thus be identified with tangent space at the identity of the gauge group

\[
T_{e} \mathcal{G}_{\mathbb{H}^C} \cong \Omega^{0}(P[\mathbb{h}^C])
\]

The identification of \( \Omega^{0}(P[\mathbb{h}^C]) \) with the tangent space \( T_{[\overline{\partial} P, \varphi]} \mathcal{G}_{\mathbb{H}^C} \cdot (\overline{\partial} P, \varphi) \) is given by the map

\[
\begin{align*}
\Omega^{0}(P[\mathbb{h}^C]) & \longrightarrow \Omega^{0,1}(P[\mathbb{h}^C]) \oplus \Omega^{1,0}(P[\mathbb{m}^C]) \\
x & \longmapsto (\overline{\partial} P x, [\varphi, x])
\end{align*}
\]
Note that for any \((\bar{\partial}_P x, [\varphi, x]) \in T_{\bar{\partial}_P, \varphi} \mathcal{H}(G)\) since \(\bar{\partial}_P ([\varphi, x]) + [\bar{\partial}_P x, \varphi] = 0\).

The tangent space to \(\mathcal{M}(G)\) at a stable Higgs bundle \([\bar{\partial}_P, \varphi]\) is thus identified with
\[
T_{\bar{\partial}_P, \varphi} \mathcal{M}(G) \cong \frac{\{ (\alpha, \psi) \in \Omega^{0,1}(P[h^C]) \oplus \Omega^{1,0}(P[m^C]) \mid \bar{\partial}_P \psi + [\alpha, \varphi] = 0 \}}{\{ (\bar{\partial}_P x, [\varphi, x]) \in \Omega^{0,1}(P[h^C]) \oplus \Omega^{1,0}(P[m^C]) \mid x \in \Omega^2(P[h^C]) \}.
\]

The tangent space fits into a very useful exact sequence.

**Proposition 6.1.** For a stable \(G\)-Higgs bundle \((\bar{\partial}_P, \varphi)\), we have an exact sequence

\[
0 \longrightarrow H^0(P[h^C]) \xrightarrow{\text{ad}_\varphi} H^0(P[m^C] \otimes K) \xrightarrow{i} T_{\bar{\partial}_P, \varphi} \mathcal{M}(G) \xrightarrow{\pi} H^1(P[h^C]) \xrightarrow{\text{ad}_\varphi} H^1(P[m^C] \otimes K)
\]

where the map \(i\) is induced by the inclusion
\[
H^0(P[m^C] \otimes K) \longrightarrow T_{\bar{\partial}_P, \varphi} \mathcal{H}(G) ,
\]
and the map \(\pi\) is induced by the projection
\[
T_{\bar{\partial}_P, \varphi} \mathcal{H}(G) \longrightarrow \Omega^{0,1}(P[h^C]) .
\]

**Remark 6.2.** In fact, the sequence (6.2) is exact on the right, however we have not developed the techniques to prove this. Using exactness of this sequence, a Riemann-Roch calculation implies that the real dimension of the tangent space at a stable Higgs bundle is given by \(\dim_{\mathbb{R}}(G)(2g - 2)\).

**Proof.** First, the elements in the kernel of the map \(\text{ad}_\varphi\) correspond to tangent vectors of one parameter families of automorphisms of \((\bar{\partial}_P, \varphi)\). Thus stability implies that \(\ker(\text{ad}_\varphi) = 0\).

Next, note that the kernel of the map \(i : H^0(P[m^C] \otimes K) \rightarrow T_{\bar{\partial}_P, \varphi} \mathcal{M}(G)\) is given by the set of \((0, \psi) = (\bar{\partial}_P x, [\varphi, x])\). Thus, the kernel of \(i\) equals the image of the map \(\text{ad}_\varphi : H^0(P[h^C]) \rightarrow H^0(P[m^C] \otimes K)\).

The projection \(T_{\bar{\partial}_P, \varphi} \mathcal{H}(G) \rightarrow \Omega^{0,1}(P[h^C])\) descends to a map
\[
\pi : T_{\bar{\partial}_P, \varphi} \mathcal{M}(G) \rightarrow H^1(P[h^C])
\]
since \(\alpha + \bar{\partial}_P x\) defines the same cohomology class as \(\alpha\). Any representative of an element of the kernel of \(\pi\) is a pair \((\bar{\partial}_P x, \psi)\) such that \(\bar{\partial}_P \psi + [\varphi, x] = 0\). Any such pair is equivalent to \((0, \psi - \text{ad}_\varphi x)\). Thus, the kernel of \(\pi\) is the image of \(i\).

Finally, the condition \(\bar{\partial}_P \psi + [\varphi, \alpha] = 0\) implies that \(\text{ad}_\varphi(\alpha)\) is zero in the cohomology group \(H^1(P[m^C] \otimes K)\). Thus, the image of \(\pi\) is the kernel of \(\text{ad}_\varphi\). \(\square\)

**Remark 6.3.** For strictly polystable Higgs bundles we have an analogous sequence which fails to be exact on the left and may or may not also fail to be exact on the right. On way to describe this is with a deformation complex (see [3]). Namely,
the sheaf map \( \text{ad}_\varphi : P[h^C] \to P[m^C] \otimes K \) defines a long exact sequence in hypercohomology

\[
0 \rightarrow H^0(\tilde{\partial}_P, \varphi) \rightarrow H^0(P[h^C]) \xrightarrow{\text{ad}_\varphi} H^0(P[m^C] \otimes K) \rightarrow H^1(\tilde{\partial}_P, \varphi) \rightarrow H^1(P[h^C]) \rightarrow H^1(P[m^C] \otimes K) \rightarrow H^2(\tilde{\partial}_P, \varphi) \rightarrow 0.
\]

In general, \( H^n(\tilde{\partial}_P, \varphi) \) is the space of infinitesimal automorphisms of \((\tilde{\partial}_P, \varphi)\), and for stable Higgs bundles, the tangent space \( T_{(\tilde{\partial}_P, \varphi)}M(G) \) is identified with \( H^1(\tilde{\partial}_P, \varphi) \).

6.2. The \( \mathbb{C}^* \)-action. There is a natural action of \( \mathbb{C}^* \)-action on the \( G \)-Higgs bundle moduli space given by scaling the Higgs field

\[
\mathbb{C}^* \times M(G) \rightarrow M(G),
\]

\[
(\lambda, [\tilde{\partial}_P, \varphi]) \rightarrow [\tilde{\partial}_P, \lambda \varphi]
\]

Note that Hitchin fibration (5.1) is equivariant with respect to a weighted \( \mathbb{C}^* \)-action:

\[
h([\tilde{\partial}_P, \lambda \cdot \varphi]) = (\lambda^{m_p} p_1(\varphi), \ldots, \lambda^{m_k(c_g)p_k(c_g)}(\varphi))
\]

Thus, the fixed points of the \( \mathbb{C}^* \)-action are contained in the nilpotent cone \( h^{-1}(0) \).

Moreover, the properness of \( h \) implies that \( \lim_{\lambda \to 0} [\tilde{\partial}_P, \lambda \varphi] \) always exists and is a \( \mathbb{C}^* \)-fixed point.

Since we are dealing with isomorphism classes, being a \( \mathbb{C}^* \)-fixed point does not imply \( \varphi = 0 \). Rather, it implies that there is a holomorphic gauge transformation \( g_\lambda \) such that \( \text{Ad}_{g_\lambda} \varphi = \lambda \varphi \) for all \( \lambda \in \mathbb{C}^* \). For \( \text{SL}(n, \mathbb{C}) \), the \( \mathbb{C}^* \)-fixed points are classified by the following proposition.

**Proposition 6.4.** Let \((E, \Phi)\) be a polystable \( \text{SL}(n, \mathbb{C}) \)-Higgs bundle. Then \((E, \Phi)\) is gauge equivalent to \((E, \lambda \Phi)\) for all \( \lambda \in \mathbb{C}^* \) if and only if there is a holomorphic splitting \( E = E_1 \oplus \cdots \oplus E_\ell \) in which the Higgs field is given by

\[
\Phi = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
\varphi_1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \varphi_{\ell-1}
\end{pmatrix},
\]

where \( \varphi_j : E_j \to E_{j+1} \otimes K \) is a holomorphic bundle map.

**Remark 6.5.** We will usually represent such a fixed point schematically as

\[
E_1 \overset{\varphi_1}{\longrightarrow} E_2 \overset{\varphi_2}{\longrightarrow} \cdots \overset{\varphi_{\ell-2}}{\longrightarrow} E_{\ell-1} \overset{\varphi_{\ell-1}}{\longrightarrow} E_\ell,
\]

where we suppress the twisting by \( K \) from the notation. The moduli space of such fixed points are a special case of the moduli of holomorphic chains.

For \( \text{SL}(n, \mathbb{C}) \) we have \( H^C = \text{SL}(n, \mathbb{C}) \) and \( m^C = \mathfrak{sl}(n, \mathbb{C}) \). For \( \text{SL}(n, \mathbb{C}) \)-Higgs bundles fixed by the \( \mathbb{C}^* \)-action the \( H^C \)-bundle has a holomorphic reduction \( E = E_1 \oplus \cdots \oplus E_k \) to a subgroup of block diagonal matrices. Such a reduction gives a \( \mathbb{Z} \)-grading on the bundle \( E[m^C] = \text{End}(E) = \bigoplus_j \text{End}(E)_j \) where

\[
\text{End}(E)_j = \bigoplus_{b-a=j} \text{Hom}(E_a, E_b).
\]
Moreover, with respect to this $\mathbb{Z}$-grading we have $\Phi \in H^0(\text{End}(E)_1 \otimes K)$. The characterization of $G$-Higgs bundles fixed by the $\mathbb{C}^*$-action is given by the following proposition.

**Proposition 6.6.** A polystable $G$-Higgs bundle $(P, \phi)$ defines a fixed point of the $\mathbb{C}^*$-action on $\mathcal{M}(G)$ if and only if
1. There is a $\mathbb{Z}$-grading $g^C = \bigoplus_j h^C_j \oplus m^C_j$.
2. There is a holomorphic reduction $\mathcal{P}_{H_0^C} \subset \mathcal{P}$ to an $H_0^C$-bundle, where $H_0^C < H^C$ is the Lie group with Lie algebra $h_0^C$.
3. With respect to the decomposition $\mathcal{P}_{H_0^C}[m^C] = \bigoplus_j \mathcal{P}_{H_0^C}[m^C_j]$, we have
   $$\phi \in H^0(\mathcal{P}_{H_0^C}[m^C_1] \otimes K) .$$

**Remark 6.7.** In terms of vector bundles, the $G$-Higgs bundle which are $\mathbb{C}^*$-fixed points are given by holomorphic chains with extra symmetries which reflect the symmetries of a $G$-Higgs bundle. For example, the $\text{SL}(p + q, \mathbb{C})$-Higgs bundle associated to an $\text{SO}(p, q)$-Higgs bundle $(V, W, \eta)$ is given by $(V \oplus W, \begin{pmatrix} 0 & \eta^t \\ \eta & 0 \end{pmatrix})$, so the associated fixed points are direct sums of holomorphic chains of the form
   $$V_r \xrightarrow{\eta_i} W_{r-1} \xrightarrow{\eta_{i-2}} \cdots \xrightarrow{\eta_2} W_1 \xrightarrow{\eta_1} V_r$$
and
   $$W_s \xrightarrow{\eta_i} V_{s-1} \xrightarrow{\eta_{i-1}} \cdots \xrightarrow{\eta_2} V_{1-s} \xrightarrow{\eta_1} W_s .$$
Here $r$ and $s$ are half integers and the additional symmetry on the grading comes from the orthogonal structure. Namely, the quadratic forms give isomorphisms $W_{-j} \cong W_j^*$ and $V_{-j} \cong V_j^*$.

**6.3. Critical points of a Morse-Bott function.** So far we have not used the full power of the nonabelian Hodge correspondence. Since we have a special metric associated to each polystable Higgs bundle, we can take the $L^2$-norm of the Higgs field. Namely, consider the nonnegative function $f : \mathcal{M}(G) \to \mathbb{R}$ defined by
   $$f([\bar{\partial}P, \phi]) = \int_X |\phi|^2 ;$$
where the norm $|\phi|$ is taken with respect to the Hermitian metric $(\bar{\partial}P, \phi)$ from the nonabelian Hodge correspondence.

**Remark 6.8.** Note that the function $f(\bar{\partial}P, \phi) = 0$ if and only if $\phi = 0$. Equivalently, the global minima of $f$ are given by points in the moduli of polystable $H^C$-bundles $\mathcal{M}(H) \subset \mathcal{M}(G)$.

The $\mathbb{C}^*$-action does not in preserve the metric from the nonabelian Hodge correspondence; however, the metric is preserved by the restriction of the action to $U(1) \subset \mathbb{C}^*$. Thus, the function $f$ is $U(1)$-invariant. Moreover, in [15, Section 8], Hitchin showed that the $U(1)$-action is Hamiltonian with respect to the symplectic structure $\omega_f$ from Remark 3.12, and that the function $f$ is a moment map for this action. That is,
   $$\text{grad}(f) = IX ,$$
where $X$ is the vector field generating the $U(1)$-action. This implies $f$ is a Morse-Bott function on the smooth locus of $\mathcal{M}(\text{SL}(n, \mathbb{C}))$ and critical submanifolds of
that associated to a polystable action we get a decomposition of the tangent space into weight spaces. Recall from \( M_{6.4} \), can be deformed to a compact representation. \( \phi \) local minima have of local minima defines a component of the moduli space. In particular, if the only inequality \( (X, C) \) of Morse theory to do things like compute the cohomology ring. However, using Uhlenbeck compactness Hitchin showed that the function \( f \) proper \([14]\) even on the singular locus. Hence, \( f \) attains a minima on every closed subset. This implies that on each connected component, \( f \) has a local minima. Thus,

\[
|\pi_0(M(G))| \leq |\pi_0(\{\text{local minima of } f\})|.
\]

There are three manifolds which intersect at a \( C^* \)-fixed point \([\bar{\partial}_P, \varphi]\)

1. \( W^s(\bar{\partial}_P, \varphi) = \{ [\bar{\partial}_P, \partial] \mid \lim_{\lambda \to 0} [\bar{\partial}_P, \lambda \partial] = [\bar{\partial}_P, \varphi] \} \)
2. \( W^u(\bar{\partial}_P, \varphi) = \{ [\bar{\partial}_P, \partial] \mid \lim_{\lambda \to \infty} [\bar{\partial}_P, \lambda \partial] = [\bar{\partial}_P, \varphi] \} \)
3. \( W^0(\bar{\partial}_P, \varphi) \) the connected component of the fixed point locus containing \([\bar{\partial}_P, \varphi]\).

For a stable fixed point \([\bar{\partial}_P, \varphi]\), these are exactly the stable, unstable and critical submanifolds of the Morse-Bott function \( f \) at the critical point \([\bar{\partial}_P, \varphi]\). A fixed point \([\bar{\partial}_P, \varphi]\) is thus a local minima of \( f \) if and only if \( W^u([\bar{\partial}_P, \varphi]) = \{ [\bar{\partial}_P, \varphi] \} \).

**Remark 6.9.** Recall from before that we have a lower bound on \( \pi_0(M(G)) \) given by \( \pi_0(M(H)) \). Moreover, when \( H \) is semisimple we have \( \pi_0(M(H)) \) is in bijective correspondence with topological \( H \)-bundles \( \mathcal{B}_H(X) \). This gives the following inequality

\[
|\pi_0(M(H))| \leq |\pi_0(M(G))| \leq |\pi_0(\{\text{local minima of } f\})|.
\]

We now have a strategy for counting the components of the character variety \( \mathcal{X}(G) \). Namely we should classify local minima of \( f \) and show that every component of local minima defines a component of the moduli space. In particular, if the only local minima have \( \varphi = 0 \), then every Higgs bundle can be reduced to \( H \) and if a component of \( M(G) \) has the property that the Higgs field can never be deformed to zero, then no representation in the associated component of the character variety can be deformed to a compact representation.

### 6.4. Local minima criterion

We first describe how for fixed points of the \( C^* \)-action we get a decomposition of the tangent space into weight spaces. Recall from Proposition 6.6 that associated to a polystable \( G \)-Higgs bundle \((\mathcal{P}, \varphi)\) fixed by the \( C^* \)-action there is a \( \mathbb{Z} \)-grading \( g^C = \bigoplus \mathfrak{h}^C_j \oplus m^C_j \) and a holomorphic structure group reduction \( \mathcal{P}|_{\mathcal{H}^C} \) so that \( \varphi \in H^0(\mathcal{P}|_{\mathcal{H}^C}[m^C_1] \otimes K) \).

For such a fixed point, the map \( \text{ad}_\varphi : \mathcal{P}|_{\mathcal{H}^C}[\mathfrak{h}^C_j] \to \mathcal{P}|_{\mathcal{H}^C}[m^C_j + 1] \otimes K \) defines a map

\[
\text{ad}_\varphi : \mathcal{P}|_{\mathcal{H}^C}[\mathfrak{h}^C_j] \to \mathcal{P}|_{\mathcal{H}^C}[m^C_j + 1] \otimes K.
\]

For stable fixed points this gives a decomposition of the exact sequence (6.2), that is, for all \( j \) we have

\[
\begin{align*}
0 \longrightarrow H^0(\mathcal{P}|_{\mathcal{H}^C}[\mathfrak{h}^C_j]) \overset{\text{ad}_\varphi}{\longrightarrow} H^0(\mathcal{P}|_{\mathcal{H}^C}[m^C_j + 1] \otimes K) \overset{i}{\longrightarrow} T^j_{[\bar{\partial}_P, \varphi]}M(G) \\
\overset{\pi}{\longrightarrow} H^1(\mathcal{P}|_{\mathcal{H}^C}[\mathfrak{h}^C_j]) \overset{\text{ad}_\varphi}{\longrightarrow} H^1(\mathcal{P}|_{\mathcal{H}^C}[m^C_j + 1] \otimes K)
\end{align*}
\]
where, similar to (6.1), $T^j_{[\bar{\partial}_P, \varphi]} \mathcal{M}(G)$ is defined by

$$
T^j_{[\bar{\partial}_P, \varphi]} \mathcal{M}(G) = \{(\alpha, \psi) \in \mathbb{O}^{0,1}(\mathcal{P}_H^\mathbb{C}[\mathfrak{h}_j^\mathbb{C}]) \oplus \Omega^{1,0}(\mathcal{P}_H^\mathbb{C}[m_j^\mathbb{C}]) | [\bar{\partial}_P \psi] + [\alpha, \varphi] = 0 \}.
$$

This following result was proven for $\text{SL}(n, \mathbb{C})$ by Hitchin in [15], the general case follows from arguments analogous to the Morse-Bott function’s index computation of Hitchin in [15, Section 8].

**Theorem 6.10** (Hitchin [15, 14]). Let $f : \mathcal{M}(G) \to \mathbb{R}$ be the Morse-Bott function from (6.4). For a stable $G$-Higgs bundle we have the following:

- $[\bar{\partial}_P, \varphi]$ is a fixed point of the $\mathbb{C}^*$-action if and only if it is a critical point of the function $f$,
- $T_{[\bar{\partial}_P, \varphi]} W^k([\bar{\partial}_P, \varphi]) = \bigoplus_{j > 0} T^j_{[\bar{\partial}_P, \varphi]} \mathcal{M}(G),$
- $T_{[\bar{\partial}_P, \varphi]} W^s([\bar{\partial}_P, \varphi]) = \bigoplus_{j < 0} T^j_{[\bar{\partial}_P, \varphi]} \mathcal{M}(G)$ and
- $T_{[\bar{\partial}_P, \varphi]} W^0([\bar{\partial}_P, \varphi]) = T^0_{[\bar{\partial}_P, \varphi]} \mathcal{M}(G).

**Corollary 6.11.** A stable fixed point $[\bar{\partial}_P, \varphi] \in \mathcal{M}(G)$ is a local minimum of $f$ if and only if $\bigoplus_{j > 0} T^j_{[\bar{\partial}_P, \varphi]} \mathcal{M}(G) = 0$.

Using the sequence (6.2), if $\text{ad}_\varphi : \mathcal{P}[\mathfrak{h}_j^\mathbb{C}] \to \mathcal{P}[m_{j+1}] \otimes K$ is an isomorphism for all $j > 0$, then $T^j_{[\bar{\partial}_P, \varphi]} \mathcal{M}(G) = 0$ for all $j > 0$ and we are at a local minimum of the $f$. In fact the converse holds as well (see [4, Section 3.4]), and we have a classification of stable local minima of the Morse-Bott function $f$.

**Proposition 6.12.** A stable $G$-Higgs bundle $(\bar{\partial}_P, \varphi)$ which is a $\mathbb{C}^*$-fixed point is a local minima of the function $f$ from (6.4) if and only if

$$
\text{ad}_\varphi : \mathcal{P}[\mathfrak{h}_j^\mathbb{C}] \to \mathcal{P}[m_{j+1}] \otimes K,
$$

is an isomorphism for all $j > 0$.

6.5. **Some component results.** We have now developed necessary tools to show the map $\tau : \mathcal{X}(\Gamma, G) \to \mathcal{B}_G(S)$ from (1.1) is injective. In fact, the proof is very simple with the above setup.

**Theorem 6.13** (Garcia-Prada and Oliveira [10]). Let $G$ be a complex reductive Lie group with maximal compact subgroup $H$. Then there is a bijection between the components of the moduli space of polystable $G$-Higgs bundles and the moduli space of polystable $G$-bundles:

$$
\pi_0(\mathcal{M}(G)) = \pi_0(\mathcal{M}(H)) .
$$

**Proof.** By the above discussion and Remark 6.9, it suffices to show that a polystable $G$-Higgs bundle $[\mathcal{P}, \varphi]$ is a local minima of the Morse-Bott function $f$ from (6.4) if and only if $\varphi = 0$. Let $[\mathcal{P}, \varphi]$ be a local minima of $f$. Since $[\mathcal{P}, \varphi]$ is a $\mathbb{C}^*$-fixed point, by Proposition 6.6 there is a $\mathbb{Z}$-grading $\mathfrak{g}^\mathbb{C} = \bigoplus_j \mathfrak{h}_j^\mathbb{C} \oplus m_j^\mathbb{C}$ so that $(\mathcal{P}, \varphi)$ is isomorphic to $(\mathcal{P}_H^\mathbb{C}, \varphi)$ with $\varphi \in H^0(\mathcal{P}_H^\mathbb{C}[\mathfrak{m}_j^\mathbb{C}] \otimes K)$.

First suppose $[\bar{\partial}_P, \varphi]$ is a stable local minima of $f$. Then by Proposition 6.12 we have

$$
\text{ad}_\varphi : \mathcal{P}_H^\mathbb{C}[\mathfrak{h}_j^\mathbb{C}] \to \mathcal{P}_H^\mathbb{C}[m_{j+1}] \otimes K
$$
is an isomorphism for all $j > 0$. But, since $g$ is complex, we have $\mathfrak{h}^C \cong \mathfrak{m}^C$, and the only way $P_{\mathfrak{he}}[h_j^C]$ can be isomorphic to $P_{\mathfrak{he}}[m_j^C + 1] \otimes K$ for all $j > 0$ is for $P_{\mathfrak{he}}[h_j^C] = 0$ for all $j > 0$. In this case we have $\varphi = 0$.

To rule out strictly polystable minima with nonzero Higgs field, we note that a $G$-Higgs bundle which is strictly polystable has a holomorphic reduction to a Levi factor $L$ of a parabolic subgroup of $G$ which is stable as a $L$-Higgs bundle. Now repeat the above argument for the moduli space $\mathcal{M}(L)$.

As a immediate corollary we have the following.

**Corollary 6.14.** If $G$ is a complex reductive Lie group, then every polystable $G$-Higgs bundle $[\hat{\partial}, \varphi]$ can be continuously deformed to a polystable $G$-bundle, i.e., a polystable $G$-Higgs bundle with zero Higgs field.

Using the nonabelian Hodge correspondence, Theorem 1.7 now follows as a corollary direct corollary of the above theorem.

**Corollary 6.15.** If $G$ is a complex reductive Lie group with maximal compact $H$, then the map $\tau : \pi_0(\mathcal{X}(\Gamma, G)) \to B_G(S)$ from (1.1) is injective. In particular, every representation $\rho : \Gamma \to G$ can be deformed to a compact representation $\Gamma \to H \hookrightarrow G$.

Using the methods described above, Hitchin gave a complete component count of $\mathcal{M}(\text{PSL}(n, \mathbb{R}))$. The proof idea is to first classify the stable local minima using Proposition 6.6, then construct explicit deformations of strictly polystable fixed points with nonzero Higgs field which decreases the value of $f$.

**Theorem 6.16** (Hitchin [15]). For $n > 2$, the only local minima of the Morse-Bott function (6.4) on $\mathcal{M}(\text{PSL}(n, \mathbb{R}))$ are $\varphi = 0$ and the image of 0 in the Hitchin section.

We thus have the following corollary.

**Corollary 6.17.**

$$|\pi_0(\mathcal{M}(\text{PSL}(n, \mathbb{R})))| = \begin{cases} 4g - 3 & n = 2, \\ 3 & n\text{-odd}, \\ 6 & n > 2 \text{ and even} \end{cases}$$

**Proof.** For $n = 2$ the component count is the same as $\mathcal{M}(\text{SO}_0(1, 2))$. For $n > 2$ and odd we have $B_{\text{PSL}(n, \mathbb{R})} = \mathbb{Z}_2$ and there is only one Hitchin component. This gives three components. For $n > 2$ and $n$-even we have $B_{\text{PSL}(n, \mathbb{R})}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$ depending on the parity of $\frac{n}{2}$ (see (1.2)). Moreover, there are two Hitchin components by Remark 2.11, this gives six components. By Theorem 6.16 there are no other components. \qed

For the character variety $\mathcal{X}(\Gamma, \text{PSL}(n, \mathbb{R}))$ we of course have the same count.

**Corollary 6.18.**

$$|\pi_0(\mathcal{X}(\Gamma, \text{PSL}(n, \mathbb{R})))| = \begin{cases} 4g - 3 & n = 2, \\ 3 & n\text{-odd}, \\ 6 & n > 2 \text{ and even} \end{cases}$$
Theorem 6.16 also gives a dichotomy for deformations of representations into $\text{PSL}(n, \mathbb{R})$, namely for $n > 2$ the components of $\mathcal{X}(\Gamma, \text{PSL}(n, \mathbb{R}))$ are either deformations spaces of compact representations or deformation spaces of special Fuchsian representations.

**Corollary 6.19.** For each $n > 2$ and each $\rho \in \mathcal{X}(\Gamma, \text{PSL}(n, \mathbb{R}))$, exactly one of the following holds

- $\rho$ can be deformed to a compact representation
  \[ \Gamma \to \text{PSO}(n) \to \text{PSL}(n, \mathbb{R}) \]
- $\rho$ can be deformed to a representation
  \[ \Gamma \overset{\rho_{\text{Fuch}}}{\longrightarrow} \text{PSL}(2, \mathbb{R}) \overset{\iota_{pr}}{\longrightarrow} \text{PSL}(n, \mathbb{R}) \]

where $\rho_{\text{Fuch}} \in \text{Fuch}(\Gamma)$ is a Fuchsian representation and $\iota_{pr}$ is the principal embedding from (2.2).

### 7. $\text{SO}(p,q)$-Higgs bundles

We now apply the techniques of understand the components of the $\text{SO}(p,q)$-character variety $\mathcal{X}(\Gamma, \text{SO}(p,q))$. In her thesis [2], Aparicio-Arroyo discovered that Higgs bundle moduli space $\mathcal{M}(\text{SO}(p,q))$ had stable local minima of the Morse-Bott function (6.4) with nonzero Higgs field and which did not arise from the Hitchin section. This was done by classifying stable $\text{SO}(p,q)$-Higgs bundles which were fixed points of the $\mathbb{C}^*$-action and satisfied Proposition 6.12. Due to the potential singularities, these results were not strong enough to classify the components of the moduli space $\mathcal{M}(\text{SO}(p,q))$. We start by recalling the classification of stable minima.

**Remark 7.1.** The case $\text{SO}(2,q)$ is rather special since $\text{SO}(2,q)$ is a group of Hermitian type. This special type of group has its own very interesting connected component results. Since we have not said much about this situation, we will only discuss the non-Hermitian case of, that is, for $2 < p \leq q$. For the case of $\text{SO}(2,q)$ we refer the reader to [8] and [1].

Recall the notation of $\mathcal{K}_p = K^{p-1} \oplus K^{p-3} \oplus \cdots \oplus K^{3-p} \oplus K^{1-p}$ from (5.2) and the map $\tilde{\Psi}(0, \cdots, 0) : \mathcal{K}_p \to \mathcal{K}_{p+1} \otimes \mathcal{K}$ from (5.3). Denote by $\eta_0$ the following transpose

\[ \eta_0 = \tilde{\Psi}(0, \cdots, 0)^T : \mathcal{K}_p \to \mathcal{K}_{p-1} \otimes \mathcal{K} \]

**Theorem 7.2** (Aparicio-Arroyo [2]). Suppose $2 < p \leq q$. If $[V,W,\eta]$ is a stable $\text{SO}(p,q)$-Higgs bundle, then $(V,W,\eta)$ defines a local minima of the Morse-Bott function (6.4) if and only if one of the following holds

1. $\eta = 0$,
2. there is a stable rank $q-p+1$ orthogonal bundle $W_0$ with determinant bundle $I = \text{det}(W_0)$ such that

\[ (V,W,\eta) = \left( \mathcal{K}_p \otimes I, (\mathcal{K}_{p-1} \otimes I) \oplus W_0, \begin{pmatrix} \eta_0 \\ 0 \end{pmatrix} : V \to W \otimes \mathcal{K} \right) \]
(3) $q = p + 1$ and there is a line bundle $M \in \text{Pic}^d(X)$ with $d \in (0, p(2g - 2)]$ and $\mu \in H^0(M^{-1}K^p) \setminus \{0\}$ so that

$$(V, W, \eta) = \left( K_p, M \oplus K_{p-1} \oplus M^{-1}, \left( \begin{array}{c} 0 \\ \eta_0 \\ \eta_\mu \end{array} \right) : V \to W \otimes K \right).$$

where $\eta_\mu = (0 \cdots 0 \mu) : K_p \to M^{-1}K$.

**Remark 7.3.** Note that in case three of the above theorem when $\deg(M) = p(2g - 2)$ the existence of a nonzero section of $M^{-1}K^p$ implies $M = K^p$. In this case, the minima is the minima in the $\text{SO}(p, p+1)$-Hitchin component defined by $\Psi(0, \cdots, 0)$ from (5.3).

In [1] all of the local minima are classified. The result basically says that the only minima not accounted for in Theorem 7.2 arise from polystable Higgs bundles with zero Higgs field and from by the orthogonal bundle $W_0$ to by strictly polystable.

**Theorem 7.4.** Assume $2 < p \leq q$ and $(V, W, \eta)$ is polystable $\text{SO}(p, q)$-Higgs bundle. Then $(V, W, \eta)$ defines a local minima of the Morse-Bott function (6.4) if and only if

1. $\eta = 0$
2. there is a polystable rank $q - p + 1$ orthogonal bundle $W_0$ with determinant bundle $I = \det(W_0)$ such that

$$(V, W, \eta) = \left( K_p \otimes I, (K_{p-1} \otimes I) \oplus W_0, \left( \begin{array}{c} \eta_0 \\ 0 \end{array} \right) : V \to W \otimes K \right).$$

3. $q = p + 1$ and there is a line bundle $M \in \text{Pic}^d(X)$ with $d \in (0, p(2g - 2)]$ and $\mu \in H^0(M^{-1}K^p) \setminus \{0\}$ so that

$$(V, W, \eta) = \left( K_p, M \oplus K_{p-1} \oplus M^{-1}, \left( \begin{array}{c} 0 \\ \eta_0 \\ \eta_\mu \end{array} \right) : V \to W \otimes K \right).$$

where $\eta_\mu = (0 \cdots 0 \mu) : K_p \to M^{-1}K$.

To show that each of the above local minima define a connected component of moduli space $\mathcal{M}(\text{SO}(p, q))$ we define a map $\Theta_{p,q}$ from a parameter space into $\mathcal{M}(\text{SO}(p, q))$ so that

1. $\Theta_{p,q}$ is a homeomorphism onto its image,
2. the image of $\Theta_{p,q}$ is open and closed,
3. each component of the image of $\Theta_{p,q}$ contains exactly one connected component of the local minima with $\eta \neq 0$ from Theorem 7.4.

The connected components of the local minima of type 2 in the above theorem are determined by the number of components of polystable $\text{O}(q - p + 1, C)$-bundles, that is, the components of $\mathcal{M}(\text{O}(q - p + 1))$. For $q > p + 1$ the group $\text{O}(q - p + 1)$ is simple thus, by Theorem 1.6

$$|\pi_0(\mathcal{M}(\text{O}(q - p + 1)))| = |\mathcal{B}_{\text{O}(q-p+1)}(S)| = 2^{2g+1}.$$  

When $q = p + 1$, we have $\mathcal{M}(\text{O}(q - p + 1)) = \mathcal{M}(\text{O}(2))$ and the number of connected components is $2^{2g+1} - 1$. Combining these with the $p(2g - 2)$ components of local minima of type 3 in the above theorem give $2^{2g+1} - 1 + p(2g - 2)$ connected components of local minima in $\mathcal{M}(\text{SO}(p, p+1))$ with $\eta \neq 0$. Finally, when $q = p$ we
have \( q - p + 1 = 1 \) and thus there are \( 2^{2g} \) components of local minima with \( \eta \neq 0 \). In this case all such minima define Hitchin components.

Combined with the \( 2^{2g+1} \)-components of \( \mathcal{M}(S(O(p) \times O(q))) \), for \( 2 < p \leq q \) the following theorem of \cite{1} establishes the component count of \( \mathcal{M}(SO(p,q)) \).

**Theorem 7.5.** For \( 2 < p \leq q \), we have

\[
|\pi_0(\mathcal{M}(SO(p,q)))| = 2^{2g+1} + \begin{cases}
2^{2g} & q = p \\
2^{2g+1} - 1 + p(2g - 2) & q = p + 1 \\
2^{2g+1} & \text{else}.
\end{cases}
\]

To sketch the idea of the above theorem, we need to slightly generalize our notion of Higgs bundles.

**Definition 7.6.** A \( K^p \)-twisted \( G \)-Higgs bundle is a pair \( (\mathcal{P}, \varphi) \) where

- \( \mathcal{P} \to X \) is a holomorphic \( \mathbb{H}^C \)-bundle
- \( \varphi \) is a holomorphic section of the associated bundle \( \mathcal{P}[m^C] \otimes K^p \).

The notions of stability for \( K^p \)-twisted Higgs bundles are defined completely analogous to the notions of stability for regular Higgs bundles, i.e. for \( K^1 \)-twisted Higgs bundles. We will denote the space of polystable \( K^p \)-twisted \( G \)-Higgs bundles and the resulting moduli space by

\[
\mathcal{H}_{K^p}^{ps}(G) \quad \text{and} \quad \mathcal{M}_{K^p}(G) = \mathcal{H}_{K^p}^{ps}(G)/G_{H^C}.
\]

Recall from Section 4 that an \( SO(1,n) \)-Higgs bundle is given by a triple \( (V,W,\eta) = (\Lambda^n W_0, W_0, \eta) \), where \( \eta \in H^0(\Lambda^n W_0 \otimes W_0 \otimes K) \). The map \( \eta \) can be interpreted as a holomorphic bundle map \( \eta : \Lambda^n W_0 \to W_0 \otimes K \). Similarly, a \( K^p \)-twisted \( SO(1,n) \)-Higgs bundle is given by a triple \( (\Lambda^n W_0, W_0, \eta_p) \) where \( \eta_p \in H^0(\Lambda^n W_0 \otimes W_0 \otimes K) \), which we may interpret at as holomorphic bundle map \( \eta_p : K^{1-p} \to W_0 \otimes K \).

Recall the definition of the map \( \hat{\Psi} : \bigoplus_{j=1}^{p} H^0(K^{2j}) \to \mathcal{H}(SO(p,p+1)) \) from (5.3). Taking analogous transpose of this map defines the \( SO(p+1,p) \)-Hitchin component. For our applications we need the map for defining the \( SO(p,p-1) \)-Hitchin component. We we call this map \( \hat{\Psi} : \bigoplus_{j=1}^{p-1} H^0(K^{2j}) \to \mathcal{H}(SO(p,p-1)) \) as well, explicitly it is given by

\[
\hat{\Psi}(q_2,\ldots,q_{2p-2}) = \left(K_p, Q_p, \mathcal{K}_{p-1}, Q_{p+1}, \begin{pmatrix} 1 & q_2 & \cdots & q_{2p-2} \\ \vdots & \ddots & \ddots & \vdots \\ 1 & q_2 \end{pmatrix} : \mathcal{K}_p \to \mathcal{K}_{p-1} \otimes K \right).
\]

Consider the map

\[
(7.1) \quad \hat{\Theta}_{p,q} : \mathcal{H}_{K^p}^{ps}(SO(1,q-p+q)) \times \bigoplus_{j=1}^{p-1} H^0(K^{2j}) \to \mathcal{H}(SO(p,q))
\]

defined by

\[
\hat{\Theta}_{p,q}(W_0, \eta_p, q_2, \ldots, q_{2p-2}) = \left( I \otimes \mathcal{K}_p, (I \otimes \mathcal{K}_{p-1}) \oplus W_0, \left( \hat{\Psi}(q_2,\ldots,q_{2p-2}) \ \hat{\eta}_p \right) \right),
\]

where \( I = \Lambda^{q-p+1} W_0 \) and

\[
\hat{\eta}_p = (0 \ \cdots \ 0 \ \eta_p) : I \otimes (K^{p-1} \oplus K^{p-3} \oplus \cdots \oplus K^{3-p} \oplus K^{1-p}) \to W_0 \otimes K.
\]
Remark 7.7. Note that \( \Lambda^p(I \otimes K_p) = I^p \) and \( \Lambda^q((I \otimes K^{p-1}) \oplus W_0) = I^p \), so this indeed defines an \( \text{SO}(p, q) \)-Higgs bundle.

Remark 7.8. Note also that for \( W_0 \) a polystable orthogonal bundle of rank \( q-p+1 \), we can take \( \eta_p = 0 \). In this case the image of \( \tilde{\Theta}_{p,q}(W_0, 0, \cdots, 0) \) is given by

\[
\left( \Lambda^{q-p+1}(W_0) \otimes K_p, (\Lambda^{q-p+1}(W_0) \otimes K_{p-1}) \oplus W_0, (\tilde{\Psi}(q_2, \cdots, q_{2p-2}) \ 0) \right).
\]

In particular, the \( \text{SO}(p, q) \)-Higgs bundle is a direct sum of an \( \text{SO}(p, p+1) \)-Higgs bundle in the Hitchin component (twisted by an \( O(1, \mathbb{C}) \)-bundle) with a polystable \( O(q-p+1) \)-Higgs bundle.

Theorem 7.9 ([1]). For \( 2 < p \leq q \), the map \( \tilde{\Theta}_{p,q} \) from (7.1) induces a map

\[
\Theta_{p,q} : \mathcal{M}_{K^p}(SO(1, q-p+1)) \times \bigoplus_{j=1}^{p-1} H^0(K^{2j}) \to \mathcal{M}(SO(p,q)),
\]

which is a homeomorphism onto its image. Moreover, the image of \( \Theta_{p,q} \) is open and closed.

The proof has four steps.

1. Well defined: show the \( \text{SO}(p,q) \)-Higgs bundles in the image of \( \Theta_{p,q} \) are polystable.
2. Injectivity: Every \( S(O(1, \mathbb{C}) \times O(q-p+1, \mathbb{C})) \) gauge transformation of a \( K^p \)-twisted \( \text{SO}(1, q-p+1) \)-Higgs bundle induces a unique \( S(O(p, \mathbb{C}) \times O(q, \mathbb{C})) \) gauge transformation preserving the image of \( \tilde{\Theta}_{p,q} \).
3. Openness of image: This is the most difficult and technical step. We first note that

\[
\dim \left( \mathcal{M}_{K^p}(SO(1, q-p+1)) \times \bigoplus_{j=1}^{p-1} H^0(K^{2j}) \right) = \dim(\mathcal{M}(SO(p,q))
\]

then analyzing the local structure of the singularities of the image of \( \Theta_{p,q} \).
4. Closedness of the image: Use properness of the Hitchin fibration analogous to closedness of the Hitchin section.

To see that the component count of Theorem 7.5 is a corollary of Theorems 7.4 and 7.9 we prove the following proposition.

Proposition 7.10. For all \( p > 1 \) the component count of \( \mathcal{M}_{K^p}(SO(1,n)) \) is given by

\[
\pi_0(\mathcal{M}_{K^p}(SO(1,n))) = \begin{cases} 
2g & n = 1 \\
2^g + 1 + p(2g - 2) & n = 2 \\
2^g + 1 & n > 2
\end{cases}
\]

Remark 7.11. The proof of the \( n = 1 \) and \( n > 2 \) are by direct computation, namely we show that every fixed point of the \( \mathbb{C}^* \)-action can be deformed to on with zero Higgs field. The additional \( p(2g - 2) \) components in the \( n = 2 \) case are analogous to the components in Theorem 4.2. In particular, the Higgs field in these components is never zero and these components are parameterized by certain vector bundles over an appropriate symmetric product of the surface.

As a direct corollary of the component count for \( \mathcal{M}(SO(p,q)) \) we have the following component count of the character variety \( \mathcal{X}(\Gamma, SO(p,q)) \).
Corollary 7.12. For $2 < p \leq q$, the component count of the character variety $\mathcal{X}(\Gamma, SO(p,q))$ is given by

$$|\pi_0(\mathcal{X}(\Gamma, SO(p,q)))| = \begin{cases} 2^{2g} & q = p \\ 2^{2g+1} - 1 + p(2g - 2) & q = p + 1 \\ 2^{2g+1} & \text{else} \end{cases}$$

Combining Remarks 7.8 and 7.11 it follows that if $2 < p < q - 1$, then every Higgs bundle in the image of $\Theta_{p,q}$ can be deformed to the direct sum of a polystable orthogonal bundle $W_0$ and a Higgs bundle in the $SO(p,p-1)$-Hitchin component twisted by the determinant of $W_0$. Apply the nonabelian Hodge correspondence to this statement gives a dichotomy for the character variety $\mathcal{X}(\Gamma, SO(p,q))$ when $q > p + 1$ which is analogous to Corollary 2.12.

Theorem 7.13. Suppose $2 < p < q - 1$. If $\rho \in \mathcal{X}(\Gamma, SO(p,q))$, then there is a dichotomy: either $\rho$ can be deformed to compact representation or $\rho$ can be deformed to a Fuchsian representation of the form

$$\left(\iota_{p,q} \circ \iota_{pr} \circ \rho_{Fuch}\right) \otimes \det(\alpha) \oplus \alpha,$$

where

- $\rho_{Fuch} : \Gamma \to \text{PSL}(2,\mathbb{R})$ is a Fuchsian representation,
- $\iota_{pr} : \text{PSL}(2,\mathbb{R}) \to SO(p,p-1)$ is the principal embedding
- $\iota_{p,q} : SO(p,p-1) \to SO(p,q)$ is the embedding given by (2.3)
- $\alpha : \Gamma \to O(q-p + 1)$ is a compact representation.

In particular, every component of $\mathcal{X}(\Gamma, SO(p,q))$ is either the deformation space of compact representations or the deformation space of certain Fuchsian representations.

Remark 7.14. For the case $SO(p,p+1)$ there is a trichotomy, since their are $p(2g - 2) - 1$ components which cannot be deformed to compact representations and cannot be deformed to Fuchsian representations. In [7], the $SO(p,p+1)$-case is studied in great detail, and it is conjectured that every representation in these $p(2g - 2) - 1$ components is Zariski dense.

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