Thermal Renormalons in Scalar Field Theory

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Abstract
In the frame of the scalar theory $g\phi^4$, we explore the occurrence of thermal renormalons, i.e. temperature dependent singularities in the Borel plane. The discussion of a particular renormalon type diagram at finite temperature, using Thermofield Dynamics, allows us to establish that these singularities actually get a temperature dependence. This dependence appears in the residues of the poles, remaining their positions unchanged with temperature.
1 Introduction

During the last years, there have been an impressive amount of theoretical work on renormalons in different scenarios. For a recent review see [1]. One of the main motivations behind this effort concerns the non-perturbative structure of Quantum Chromodynamics (QCD). On the other side, finite temperature effects have also called the attention of many authors [2], due to their crucial role in understanding thermal aspects of the hadron dynamics, with special emphasis on the deconfining phase transition and the production of the quark gluon plasma [3]. It seems natural, therefore, to start a systematic study of the occurrence and also the possible phenomenological role of thermal renormalons. Here we discuss, as a first step in this direction, the scalar theory $g\phi^4$. In spite of being a non-realistic approach for phenomenological purposes, this analysis will allow us to gain a first impression on this kind of effects.

The fundamental characteristic of a renormalon type diagram is an insertion of at least a chain of bubbles in a loop diagram, and behaves like $k!$ for large values of $k$, being $k$ the number of bubbles in the chain. These diagrams, in the usual zero temperature situation, induce the existence of certain types of poles in the Borel plane. In this paper we explore in detail the extension to the finite temperature scenario of one kind of renormalon type Feynman diagrams, see Fig. 1, that contribute to the two-point function. The set of diagrams we have chosen is an example of a renormalon type contribution (ultraviolet renormalons, UVR), and gives us a hint about the properties of the thermal Borel plane, i.e. positions and residues of poles as a function of temperature.

The discussion at finite temperature has been done using the machinery of Thermo Field Dynamics (TFD) [4]. The main result from our analysis is that residues can get an explicit dependence on temperature, remaining, nevertheless, the position of the poles in the Borel plane unchanged.

The plan of this paper is as follows. In section 2, the analysis at zero temperature of a particular set of Feynman diagrams, see Fig. 1, that contribute to the two-point function in the $g\phi^4$ theory, allows us to show the presence of a renormalon type singularity in the Borel plane. Section 3 is devoted to a brief discussion of how to handle the chain of bubbles shown in Fig. 2 at finite temperature. This will be done in the deep euclidean region, i.e.
\(-p^2 \gg m^2\), of the momentum \(p\) that circulates through the chain of bubbles. Using these results, in section 4 we calculate our renormalon type diagram (Fig. 1) at finite temperature and proceed to present our conclusions.

2 The zero temperature renormalon

In the present paper the diagram shown in Fig. 1 will be taken as a source for renormalons. Our goal is to explore the influence of temperature on this type of diagrams. Let us call by \(R_k(p)\) the diagram of order \(k\), where \(k\) denotes the number of vertices. In order to establish our notation, we will first review the zero temperature calculation for the renormalon associated to this diagram [6].

\[
R_k(p) = \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\epsilon} \frac{1}{(-ig)^{k-2}} [B(l)]^{k-1},
\]

where the \((-ig)^{k-2}\) factor is due to the double counting of vertices in \(B(l)\). The relevant contribution to the integral comes from the deep euclidean region. Therefore, it is enough to approximate \(B(l)\) in this region.

\[
B(l) = \frac{(-ig)^2}{2} \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\epsilon} \frac{i}{(q + l)^2 - m^2 + i\epsilon}.
\]

\[
\lim_{-l^2 \to \infty} \frac{-ig^2}{32\pi^2} \log(-l^2).
\]

(2)

(the argument of the logarithm is in mass units)
In this way we have

\[ R_k(p) = \frac{-ig^k}{(32\pi^2)^{k-1}} \int \frac{d^4l}{(2\pi)^4} \frac{1}{(p+l)^2 + m^2} (\log(l^2))^{k-1}, \]  

(3)

where a Wick rotation has been performed.

This expression is ultraviolet divergent. Since \( g\phi^4 \) is a renormalizable theory, we can concentrate on the finite part. In order to do this the propagator is expanded in powers of \( 1/l^2 \). The first two ultraviolet divergent terms are subtracted, and we keep only the first convergent term in the expansion. This procedure, which at the end induces one pole in the Borel plane, i.e. one renormalon, will be followed also in the next sections, where the finite temperature corrections will be computed. Due to this expansion, the dependence on \( p \), the external momentum of the two point function, actually disappears. So we find

\[ R_k \propto -i \left( \frac{g}{32\pi^2} \right)^k \int dl \frac{1}{l^3} (\log(l^2))^{k-1}. \]  

(4)

In the last expression we can see that the main contribution to \( R_k \), at large values of \( k \), comes from large values of \( l \), that corresponds to the deep euclidean region, since we have done a Wick rotation. Introducing the new variable \( l = e^t \), we can see that \( R_k \) becomes proportional to the gamma function.

\[ R_k \propto -i \left( \frac{g}{32\pi^2} \right)^k \Gamma(k). \]  

(5)

Let us remain briefly the idea of the Borel transform. If, in general, we have a divergent series in terms of a certain expansion parameter \( a \), of the form:

\[ D[a] = \sum_{n=1}^{\infty} D_n a^n, \]  

(6)

then, one possibility to give a meaning to this series is to make use of the Borel transform method \cite{7}, according to which a new perturbative expansion, in a new expansion parameter \( b \), is considered by dividing each coefficient of the previous series by \( n! \) in the following way.
\[ B[b] = \sum_{n=0}^{\infty} D_{n+1} \frac{b^n}{n!}. \]  

(7)

Formally, we see that the \( D[a] \) corresponds to the Laplace transform of \( B[b] \). Using this idea in our case, we can introduce \( B[b] \) as the Borel transform of \( R_k \) according to

\[ B[b] = \sum \left( \frac{R_k}{g^k} \right) \frac{b^{k-1}}{(k-1)!}, \]

\[ \propto -\frac{i}{1 - b/32\pi^2}. \]

(8)

The \( k! \) behavior of \( R_k \), for large values of \( k \), is responsible for the appearance of a pole in the real axis of the Borel plane, i.e. in the integration range of the Laplace transform. The meaning of this or other poles in the Borel plane corresponds to an ambiguity in the resummation of the series. Physically, this implies the existence of essential errors associated to perturbative expansions of physical magnitudes.

3 Chain of bubbles at finite temperature

In this section we revise the finite temperature calculation for the chain of bubbles shown in Fig. 2, which contribute to the four-point function. The original discussion can be found in [5]. Later we will approximate our result for deep euclidean values of the momentum that goes through the chain. Let us denote by \( I_k \) the sum of all diagrams of the form shown in Fig. 2, with \( k \) bubbles and fixed external vertices of type one, according to the rules of TFD, including a global imaginary factor \( i \). The sum is over all possible combinations of internal type of vertices, which, as we know, can be of first or second type.

As it was shown in [4], and after correcting some missprints, \( I_k \) can be expressed as a function of \( I_1 \) according to

\[ Re I_k = g \frac{1}{2} (\alpha^k + \gamma^k). \]

(9)

\[ Im I_k = \frac{-ig}{2} \left( \frac{e^{\beta|p_0|} + 1}{e^{\beta|p_0|} - 1} \right) (\alpha^k - \gamma^k). \]

(10)
Figure 2: The chain of bubbles present in Fig. 1.

where

$$\alpha(\gamma) = \frac{A \pm iB}{g}. \quad (11)$$

In the previous expression $\alpha$ corresponds to the plus and $\gamma$ to the minus sign, respectively, and $p$ is the momentum that circulates through the chain. $A$ and $B$ are given by

$$A = \text{Re} I_1, \quad (12)$$

$$B = \frac{e^{\beta |p_0|} - 1}{e^{\beta |p_0|} + 1} \text{Im} I_1, \quad (13)$$

So, for $I_1$, the “fish” diagram at finite temperature including the global factor $i$, we have

$$I_1 = i(-ig)^2 \frac{1}{2} \int \frac{d^4l}{(2\pi)^4} \left\{\frac{i}{l^2 - m^2 + i\epsilon} + \frac{2\pi\delta(l^2 - m^2)}{e^{\beta |l_0|} - 1}\right\}$$

$$\times \left\{\frac{i}{(l + p)^2 - m^2 + i\epsilon} + \frac{2\pi\delta((l + p)^2 - m^2)}{e^{\beta |l_0 + p_0|} - 1}\right\}. \quad (14)$$

It is convenient at this point to give the explicit expressions for the real and imaginary parts of $I_{1o}$ and $I_{1\beta}$, zero and finite temperature parts of $I_1$, respectively, such that $I_1 = I_{1o} + I_{1\beta}$.

$$\text{Re} I_{1o} = \frac{g^2}{32\pi^2} \sqrt{\frac{|4m^2 - p^2|}{|p^2|}} \log \left(\frac{\sqrt{|4m^2 - p^2|} + \sqrt{|p^2|}}{\sqrt{|4m^2 - p^2|} - \sqrt{|p^2|}}\right). \quad (15)$$
\[ Im \, I_{10} = \theta(p^2 - 4m^2) \frac{(-g^2)}{32\pi} \sqrt{1 - \frac{4m^2}{p^2}}. \] (16)

\[ Re \, I_{1,\beta} = \frac{g^2}{16\pi|\vec{p}|} \int_0^\infty \frac{dl \, l}{E(e^{\beta E} - 1)} \log \left( \left| \frac{(2p_0E)^2 - (2l|\vec{p}| + p^2)^2}{(2p_0E)^2 - (2l|\vec{p}| - p^2)^2} \right| \right). \] (17)

\[ Im \, I_{1,\beta} = -\frac{g^2}{8} \int \frac{d^3l}{(2\pi)^2E_1E_2} \left( \delta(p_0 + E_1 + E_2) + \delta(p_0 - E_1 - E_2) \right. \\
\left. + \delta(p_0 + E_1 - E_2) + \delta(p_0 - E_1 + E_2) \right) \times \left( \frac{1}{e^{\beta E_1} - 1} + \frac{1}{e^{\beta E_2} - 1} + \frac{2}{(e^{\beta E_1} - 1)(e^{\beta E_2} - 1)} \right). \] (18)

In the previous formulae \( E_1 = \sqrt{l^2 + m^2} \) and \( E_2 = \sqrt{(l + \vec{p})^2 + m^2} \), and the imaginary part has been obtained following [9]. Since \( \alpha \) and \( \gamma \) are conjugated to each other, we can rewrite

\[ Re \, I_k = ga^k \cos(k\theta), \] (19)

\[ Im \, I_k = ga^k \sin(k\theta) \left( \frac{e^{\beta|p_0|} + 1}{e^{\beta|p_0|} - 1} \right), \] (20)

where we have introduced

\[ a = \frac{\sqrt{A^2 + B^2}}{g}. \] (21)

\[ \theta = \arctan \left( \frac{B}{A} \right). \] (22)

From the zero temperature calculation, we know that the renormalon type contribution of the diagram shown in Fig. 1 comes from the deep euclidean region, specifically for \( \sqrt{-p^2} \approx e^{k/2} \), where \( k \) is the number of bubbles in the chain, and \( p \) the momentum that circulates through the chain. In what follows, we will examine the chain of bubbles in the deep euclidean region at finite temperature, assuming \( \sqrt{-p^2} \approx e^{k/2} \). Later, by replacing the explicit expression for our chain of bubbles shown in Fig. 1, we will check if the previous condition, \( \sqrt{-p^2} \approx e^{k/2} \), is actually realized.

First we calculate \( Im \, I_{1,\beta} \) and \( Re \, I_{1,\beta} \) for \( T \neq 0 \). From the four \( \delta \)'s that appear in \( Im \, I_{1,\beta} \), eq. [18], it is easy to see that only those whose arguments include energy differences survive in the deep euclidean limit. So we have
\[
Im I_{1\beta} \approx \frac{-g^2}{8} \int \frac{d^3l}{(2\pi)^2 E_1 E_2} 2 \delta(E_1 - E_2) \left( \frac{2}{e^{\beta E_1} - 1} + \frac{2}{(e^{\beta E_1} - 1)^2} \right). \tag{23}
\]

By integrating and considering that \(e^{\beta \vec{p}/2} \gg 1\), we have finally
\[
Im I_{1\beta} \approx -\frac{g^2}{4\pi} e^{-\beta \vec{p}/2} - \beta |\vec{p}|. \tag{24}
\]

Turning to the real part of \(I_{1\beta}\), eq. \(17\), note that the argument of the logarithm in the deep euclidean limit can be approximated in such a way that the integral can be written as
\[
Re I_{1\beta} \approx \frac{g^2}{16\pi |\vec{p}|} \int_0^q dl \frac{l}{E(e^{\beta E} - 1)} 2 \log \left( 1 - \frac{4l|\vec{p}|}{(-p^2)} \right), \tag{25}
\]
where \(-p^2\) is sufficiently large so that we can reach the main contribution to the integral in the region where the logarithm can be expanded in powers of \(4l|\vec{p}|/(-p^2)\). In the previous expression, \(q\) denotes a certain bound for the integration in \(l\) such that for values of \(l\) bigger than \(q\), the contribution to the integral turns out to be negligible due to the exponential suppression. At the end we can take \(q \to \infty\). The real part, then, can be written as a series in powers of \(1/\sqrt{-p^2}\)
\[
Re I_{1\beta} \approx \frac{g^2}{32\pi^2} \sum_{n=2}^{\infty} f_n(\beta) \left( \frac{1}{-p^2} \right)^{n/2}, \tag{26}
\]
where the coefficients, which in our limit turn out to be essentially independent of the external momentum, are given by
\[
f_n(\beta) \approx -4\pi \int_0^\infty \frac{dl \frac{l}{E(e^{\beta E} - 1)} (4k)^{n-1}}{n - 1}. \tag{27}
\]
Using the previous results in eq. \(22\), we can see that
\[
|k\theta| \ll 1, \tag{28}
\]
where, once again, we have taken the assumption \(\sqrt{-p^2} \approx e^{k/2}\). Using this fact, we can approximate in eqs. \(19\) and \(20\) \(\cos(k\theta) \approx 1\) and \(\sin(k\theta) \approx 0\). This means that \(I_k \approx ga^k\). Since \(B\) in eq. \(21\) can be neglected then \(I_k \approx g(A/g)^k\).
Therefore, using our expressions for the real part of $I_1$, eqs. 15 and 26, finally we get

$$I_k \approx g \left( \frac{g}{32\pi^2} \right)^k \left[ \left( \log(-p^2) \right)^k + k \left( \log(-p^2) \right)^{k-1} \frac{f_2(\beta)}{-p^2} + \cdots \right]. \quad (29)$$

This is the fundamental expression that we will use in the next section for the discussion of our diagram at finite temperature. Note that the leading or first term, which does not depend on temperature, is the same one we found in the zero temperature analysis. The second term is the first thermal contribution. The dots denote higher order terms in the expansion in powers of $1/p^2$.

It is interesting to mention that we can find an approximated expression for the coefficients $f_n(\beta)$ in the low temperature region, where $\beta m \gg 1$

$$f_n(\beta) \approx \frac{-4^n \pi}{2 (n-1)} \int_0^\infty \frac{dx}{\sqrt{x+1}} \left( \frac{2}{\beta} \right)^{n/2} \Gamma\left( \frac{n+1}{2} \right) K_{-n/2}(\beta), \quad (31)$$

and where $K_\nu$ are Bessel type functions.

4 Renormalons at finite temperature

Using eq. 29 for the chain of bubbles, the renormalon diagram shown in Fig. 1 will be calculated in the frame of TFD. At finite temperature, $R_k(T)$ denotes the sum of all diagrams of the shape shown in Fig. 1 with $k$ vertices, being the external vertices of the first type. As it was the case in the chain of bubbles, the sum is over all possible combinations of internal type of vertices. In order to use our expression for $I_k$, this sum must be performed before the integral over the internal momentum that circulates through the chain of bubbles. Here we will obtain an expression for $R_k(T)$, finding the location of the induced poles in the Borel plane and the corresponding residues.

The expression we have to deal with is
\[ R_k(p) = \int \frac{d^4l}{(2\pi)^4} \left( \frac{i}{(p + l)^2 - m^2 + i\epsilon} + \frac{2\pi\delta((p + l)^2 - m^2)}{e^{\beta|p + l|} - 1} \right) I_{k-1}(l) \frac{1}{i}. \] (32)

The zero temperature contribution arises from the zero temperature part of the propagator times the zero temperature part of \( I_{k-1} \). We would like to discuss the following two cases: a) the product of the thermal part of the propagator times the zero temperature part of \( I_{k-1} \), denoted by \( R^a_k \), and b) the product of the usual zero temperature propagator times \( I_{k-1} \) (including the zero temperature part of \( I_{k-1} \)), denoted by \( R^b_k \).

Let us start with case a). We have

\[ R^a_k(p) = \int \frac{d^4l}{(2\pi)^4} \left( \frac{2\pi\delta((p + l)^2 - m^2)}{e^{\beta|p + l|} - 1} \right) \left( \frac{-ig^k}{(32\pi^2)^{k-1}} \log(-l^2)^{k-1} \right). \] (33)

The delta function excludes the deep euclidean region in the momentum \( l \). In spite of this fact, we can exime the limit \(|l| \gg m\), after the integration in \( l^0 \) has been done. It turns out that \( R^a_k(p) \) is proportional to

\[ R^a_k(p) \propto -i \left( \frac{g}{32\pi^2} \right)^k \int dl \frac{l}{e^{\beta l} - 1} \log(l)^{k-1}. \] (34)

The sum over \( k \) in the expression above is Borel summable and, therefore, it does not imply any renormalon.

The second case, case b), corresponds to

\[ R^b_k(p) = \int \frac{d^4l}{(2\pi)^4} \left( \frac{1}{(p + l)^2 - m^2 + i\epsilon} \right) \left( \frac{g^k}{(32\pi^2)^{k-1}} \right) \times \left( \log(-l^2)^{k-1} + (k - 1)(\log(-l^2))^{k-2} \frac{f_2(\beta)}{-l^2} + \cdots \right). \] (35)

After the Wick rotation and substracting the divergents terms, exactly in the way as we did in section 2, we find that the last expression is proportional to

\[ R^b_k \propto -i \left( \frac{g}{32\pi^2} \right)^k \int dt e^{-2t} \left[ (2t)^{k-1} + (k - 1)(2t)^{k-2} e^{-2t} f_2(\beta) + \cdots \right]. \] (36)
where the maxima of the first and second terms in the integral are reached at 
\( t = \log(\sqrt{-p^2}) \approx k/2 \) and \( t \approx k/4 \), respectively. Note that the assumption we 
made in the previous section, \( \sqrt{-p^2} \approx e^{k/2} \), has changed now in the second 
term, due to the factor 2 in the exponential. This fact does not affect our 
approximations.

Doing the integral, we see that \( R^b_k \) is proportional to

\[
R^b_k \propto -i \left( \frac{g}{32\pi^2} \right)^k \Gamma(k) - 2i f_2(\beta) \left( \frac{g}{64\pi^2} \right)^k \Gamma(k) + \cdots
\]  

(37)

We see that those terms proportional to the product of the temperature 
dependent part of the propagator times the whole series of \( I_k \) are Borel 
summable, since the leading term in the series of \( I_k \), case a), is already Borel 
summable. On the contrary, the zero temperature part of the propagator 
times \( I_k \) gives us a series of terms that behave like \( k! \) being, therefore, non 
Borel summable. We have calculated explicitly the first temperature depen-
dent term of this series associated to a \( k! \) behavior.

The Borel transform, \( B[b] \), of the sum \( \sum R_k \), taking into account only 
those non-summable Borel terms, is given by:

\[
B[b] = \sum \left( \frac{R_k}{g^k} \right) \frac{b^{k-1}}{(k-1)!}, \\
\propto -i \left[ \frac{1}{1 - b/32\pi^2} + 2 f_2(\beta) \frac{1}{1 - b/64\pi^2} + \cdots \right].
\]  

(38)

The first term in the last expression is the zero temperature renormalon 
we found in section 2. The second term corresponds to the first thermal 
singularity in the Borel plane, and the dots denote a whole series of poles. 
The structure of singularities in the Borel plane, associated to our renormalon 
type diagram, is shown in Fig. 3. The leading or first pole along the real 
axis is the zero temperature renormalon. The series of new thermal poles, 
i.e. thermal renormalons, to the right side of the first one, are subleading 
contributions to the Borel sum ambiguity.

Summarizing, in this paper we have discussed thermal corrections to a 
particular renormalon type diagram in the theory \( g\phi^4 \). From this analysis, we 
show the existence of thermal singularities in the Borel plane associated to 
this particular diagram. The main properties of these thermal renormalons
are the following: a) Their location in the Borel plane does not depend on temperature. They are situated at the points where the zero temperature renormalons are usually located, i.e. at $n/\beta_0$, being $n$ an integer and $\beta_0$ the first coefficient of the $\beta$ function \[11\]. b) Their residues, on the contrary, have an explicit dependence on temperature, through the factors $f_n(\beta)$ that vanish when $T \to 0$.

The conclusions from the particular diagram we have discussed here, suggest us to conjecture that these will be actually general properties of the Borel plane associated to correlation functions at finite temperature: residues, in general, will depend on temperature being, however, the location of the poles temperature independent. Thermal renormalons in the scalar theory are always subleading in the ambiguity of Borel sum, i.e. the leading pole is temperature independent. This fact is related to the ultraviolet character of the renormalons in this theory. Thermal corrections are associated to long distance correlations in the system, whereas the leading renormalons come from the singular behavior at short distances. Something equivalent happens with the axial anomaly, where it has been shown that the anomaly itself does not depend on temperature \[12\]. In QCD this situation will be probably different due to the existence of infrared renormalons.
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