ON CONSTRUCTION OF RECURSION OPERATORS FROM LAX REPRESENTATION

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Abstract

In this work we develop a general procedure for constructing the recursion operators for non-linear integrable equations admitting Lax representation. Several new examples are given. In particular we find the recursion operators for some KdV-type systems of integrable equations.
1 Introduction

It is well known that most of the integrable nonlinear partial differential equations

\[ u_t = F(t, x, u, u_x, \cdots, u_{nx}), \]  

admit Lax representation

\[ L_t = [A, L], \]  

so that inverse scattering method is applicable. The generalized symmetries \([1]\) of (1) have also Lax representations with the same \(L\)-operator

\[ L_{tn} = [A_n, L], \quad n \geq 1. \]  

The recursion operator \(R\), satisfying the equation (see \([2]\))

\[ R_t + [D_F, R] = 0, \]  

where \(D_F\) is the Frechét derivative of the function \(F\), generates symmetries of (1) starting from the simplest ones. In general \(R\) is a nonlocal operator (a pseudo-differential operator).

The construction of the recursion operator of a given integrable system (1) is not an easy task. Several works are devoted to this subject. Among these works most of the authors use (4) for the construction of the recursion operator \([3]-[8]\). There are several difficulties in this direct approach. The main problems are the choices of the order of \(R\) and the structure of the nonlocal terms. This is an approach, having no relation with the Lax representation (2).

On the other hand some of the authors used Lax representation for this purpose. Most of these works are related to the squared eigenfunctions of the Lax operator \([9]-[13]\) and are based on finding eigenvalue equation for the squared eigenfunctions of the Lax operator. The operator corresponding to this eigenvalue equation turns out to be the adjoint of the recursion operator.

There is an alternative use of the Lax representation to construct recursion operators. This approach is based on the explicit construction of the \(A_n\)-operators (3). It was first used by Symes \([14]\), Adler \([15]\) (see also Dorfman-Fokas \([16]\), Fokas-Gel’fand \([17]\) and Antonowicz-Fordy \([18],[19]\). Although
these authors use the Lax representation in different ways their approach is basically the same. Symes and Adler use the Gel’fand-Dikii construction of the $A_n$-operators. On the other hand Antonowicz-Fordy determines these operators from integrability condition (3) and by using an ansatz for $A_n$. Their basic aim is to determine the Hamiltonian operators $\theta_1$ and $\theta_2$ of the equations under consideration. The recursion operator is simply given by $R = \theta_2 \theta_1^{-1}$. Their approach is based on some explicit formulas for coefficients of $A_n$-operator. This is necessary to find the Hamiltonian operators $\theta_1$ and $\theta_2$ and it seems that this approach is quite effective to determine the bi-Hamiltonian structure for the simple cases but it becomes more complicated when $L$-operator has a sophisticated structure.

If one is interested only in the determination of the recursion operator $R$, we shall show in this work that it is possible to succeed this without any concrete information of the coefficients of $A_n$-operators. We use only an ansatz $\hat{A} = \mathcal{P} A + R$ that relates $A_n$-operators for different $n$. Here $\mathcal{P}$ is some operator which commutes with the $L$-operator and $R$ is the remainder.

We follow this basic idea, partially used by Symes [14], Adler [15], Shabat and Sokolov [22], and establish an extremely simple, effective and algorithmic method for the construction of recursion operators when the Lax representation (2) is given.

In the next section we consider the case where $L$ is a scalar operator. We first consider the case where $L$ is a differential operator and then the case where it is a pseudo-differential operator. In each case we present our method, discuss the reductions and give examples for illustrations. In section 3 we consider Lax operator taking values in a Lie algebra. We give our method both for the general case and also for the reductions. We give one example for each case in the text. Several additional examples are given in the Appendices A, B, and C corresponding to all different cases.

## 2 Scalar Lax representations.

First we consider equations with the scalar Lax representations of the form

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1In 1980 Shabat and Sokolov independently found the recursion operator for the Sawada-Kotera equation. This result was published in [23]. In [22] Sokolov found the recursion operator for the Krichever-Novikov equation.
where $L_t = [A, L]$, \hspace{1cm} \hspace{1cm} (5)

where $L$ is, in general, a pseudo-differential operator of order $m$ and $A$ is a differential operator whose coefficients are functions of $x$ and $t$.

The different choice of operators $A$ for a given $L$ leads to a hierarchy of nonlinear systems (3). It is well known that one can define operators $A_n$ by the following formula \hspace{1cm} \hspace{1cm} (6)

where $L^{n/m}$ is a pseudo-differential series of the form $L^{n/m} = \sum_{n=0}^{\infty} v_i D^i$ and $(L^{n/m})_+ = \sum_{i=0}^{n} v_i D^i$. Here $v_i$ are some concrete functions depending on the coefficients of $L$ and $D$ is the total derivative with respect to $x$.

In [24],[25] the relationships between the Kac-Moody algebras and special types of scalar differential and pseudo-differential operators $L$ were established. All corresponding integrable systems are Hamiltonian ones. For most of them a second Hamiltonian structure is not known up to now.

In this section and Appendices A,B, and C we consider the simplest systems from [24] and [25] as examples and find their recursion operators. In the sequel these examples will be referred as Drinfeld-Sokolov (DS) systems. It is interesting to note that in all these examples the order of the recursion operator is equal to the Coexeter number of the corresponding Kac-Moody algebra.

2.1 Gel’fand-Dikii Systems.

In this section we shall consider the case where $L$ is a differential operator,

\[ L = D^m + u_{m-2}D^{m-2} + \cdots + u_0, \]  

where $u_i, i = 0, 1, \ldots, m-2$ are functions of $x, t$. In the framework of [24] this corresponds to the Kac-Moody algebras of the type $A^{(1)}_{m-1}$.

To show that (3) is equivalent to a system of $(m-1)$ evolution equations with respect to $u_i$ one can use the following standard reasoning. Set

\[ L^\pm = (L^\pm)_+ + (L^\pm)_-, \]  

3
where \((L^n_m)^+\) is the differential part of the series \(L^n_m\) and \((L^n_m)^-\) is a series of order \(\leq -1\). Since \([L, L^n_m] = 0\) we have

\[
[(L^n_m)^+, \ L] = [L, (L^n_m)^-].
\] (9)

The left hand side of (9) is a differential operator, but the right side is a series of order \(\leq n - 2\). Thus both side of (3) are differential operators of order \(\leq n - 2\) and it is equivalent to a system of evolution equations for the dependent variables \(u_i, i = 0, 1, ..., m - 2\). This system can be obtained by comparing the coefficients of \(D^i\) where \(0, \cdots m - 2\) in (3).

Since \(L_{n+m} = L L^n_m\) then we have

\[
A_{m+n} = (L L^n_m)^+ = L (L^n_m)^+ + (L (L^n_m)^-)^+,
\] (10)

which leads directly to

\[
L t_{n+m} = [A_{n+m}, L] = L L t_n + [(L (L^n_m)^-)^+, L].
\] (11)

The above equation (11) has been given also by Symes [14] (see also Adler’s paper [15]). In his work Symes expressed the coefficients of the both parts of (11), in a rather complicated way, in terms of some finite set of coefficients of the resolvent for \(L\)- operator. That allows him to express \(L t_{n+m}\) in terms of \(L t_n\). This relation gives directly the recursion operator. He gave explicit formulas for the cases \(m = 2\) and \(m = 3\).

In this section we shall show that in order to construct the recursion operator it suffices to know only that

\[
L t_{n+m} = L L t_n + [R_n, L].
\] (12)

Obviously, it follows from the following

**Proposition 1.** For any \(n\)

\[
A_{n+m} = L A_n + R_n,
\] (13)

where \(R_n\) is a differential operator of order \(\leq m - 1\).

**Proof:** The relation (13) coincides with (10) if we put

\[
R_n = (L (L^n_m)^-)_+.
\] (14)

Since \((L^n_m)^-\) is a series of order \(\leq -1\), then \(ord(R_n) \leq m - 1\).
Remark 1: It follows from the formula

\[ A_{n+m} = (L^m_n L)_+ = (L^m_n)_+ L + ((L^m_n)_- L)_+, \quad (15) \]

that

\[ A_{n+m} = A_n L + \bar{R}_n, \quad (16) \]

and

\[ L_{t_{n+m}} = L_{t_n} L + [L, \bar{R}_n], \quad (17) \]

where \( \bar{R}_n \) is a differential operator of order \( \leq m - 1 \).

To find the recursion operator we can simply equate the coefficients of different powers of \( D \) in (12). It is easy to see that in this comparison of the coefficients of \( D^i, \ i = 2m - 2, \ldots, m - 1 \) we determine \( R_n \) in terms of the coefficients of operators \( L \) and \( L_{t_n} \). It is important that the resulting formulas turn out to be linear in the coefficients of \( L_{t_n} \). The remaining coefficients of \( D^i, \ i = m - 2, \ldots, 0 \) in (12) give us the relation

\[
\begin{pmatrix}
  u_0 \\
  \cdot \\
  \cdot \\
  u_{m-2}
\end{pmatrix}_{t_{n+m}} = \mathcal{R}
\begin{pmatrix}
  u_0 \\
  \cdot \\
  \cdot \\
  u_{m-2}
\end{pmatrix}_{t_n},
\]

where \( \mathcal{R} \) is a recursion operator. Instead of (12) one can use (17). The corresponding recursion operators coincide.

Example 1. KdV Equation. The KdV equation

\[ u_t = \frac{1}{4}(u_{3x} + 6uu_x), \quad (19) \]

has a Lax representation with

\[ L = D^2 + u, \quad A = (L^{3/2})^+_+. \]

Since in this case \( L_{t_{n+2}} = u_{t_{n+2}} \equiv u_{n+2} \) and \( L_{t_n} = u_{t_n} \equiv u_n \) the main relation (12) takes the form
\[ u_{n+2} = (D^2 + u) \cdot u_n + [R_n, L], \]  
(21)

with \( R_n = a_n D + b_n \).

Now if we equate successively to zero the coefficients of \( D^2, D \) and \( D^0 \) in above equation, we obtain

\[ a_n = \frac{1}{2} D^{-1} (u_n), \quad b_n = \frac{3}{4} u_n, \]

and

\[ u_{n+2} = \left( \frac{1}{4} D^2 + u + \frac{1}{2} u_x D^{-1} \right) u_n, \]

that gives the standard recursion operator for KdV equation

\[ \mathcal{R} = \frac{1}{4} D^2 + u + \frac{1}{2} u_x D^{-1}. \]

(22)

In the same way one can find a recursion operator for the Boussinesq equation (see Appendix A).

### 2.2 Symmetric and skew-symmetric reductions of differential Lax operator.

The standard reductions of the Gel’fand-Dikii systems are given by the conditions \( L^* = L \) or \( L^* = -L \). Here \( * \) denotes the adjoint operation defined as follows. Let \( L \) be a differential operator, \( L = \sum a_i D^i \). Its adjoint \( L^* \) is given by \( L^* = \sum (-D)^i \cdot a_i \). It is easy to see that if \( L^* = L \) then \( m = \text{ord} (L) \) must be an even integer. For the case \( L^* = -L \) it must be an odd integer.

It is well known that for both reductions all possible \( A_n \) are defined by (6), where \( n \) takes odd integer values. This condition provides that \( (A_n)^* = -A_n \) which is necessary for (3) to be compatible.

If \( L^* = L \), the formula \( A_{n+m} = (L L^m)_+ = (L^{m+n})_+ \) gives a correct \( A_n \)-operator since \( n + m \) is an odd integer. Thus, in this case Proposition 1 remains valid and the recursion operator can be found from (22) or (17).

On the other hand, if \( L^* = -L \) then both integers \( m \) and \( n \) are odd and hence their sum \( m + n \) is an even integer. This means that \( (L^{m+n})_+ \) can not be taken as an \( A_n \)-operator. In this (skew adjoint) case we must take...
\[ A_{n+2m} = (L_{-m}^{n+2m})_+ = (L^2 L_m^m)_+, \]
to find the recursion operator. Following the proof of Proposition 1 we obtain

**Proposition 2.** If \( L^* = -L \) then

\[ A_{n+2m} = L^2 A_n + R_n, \quad (23) \]

where \( \text{ord}(R_n) < 2 \text{ord}(L) \). It follows from (23) that

\[ L_{tn+2m} = L^2 L_{tn} + [R_n, L]. \quad (24) \]

**Remark 2:** Instead of (23) we can use the ansatz

\[ A_{n+2m} = L A_n L + \tilde{R}_n, \quad (25) \]
or

\[ A_{n+2m} = A_n L^2 + \tilde{R}_n. \quad (26) \]

The recursion operators obtained by the utility of (23), (25), and (26) all coincide.

In the works [24], [25] more general reductions \( L^\dagger = \pm L \) were also consid-
ered. Here \( L^\dagger = KL*K^{-1} \) where \( K \) is a given differential operator, such that
\( LK^{-1} \) is a differential operator. In this general reductions, as well, possible
\( A_n \)-operators are given by (12) with \( n \) being an odd integer. The Propositions
1 and 2 are valid for this general symmetric and skew-symmetric cases and
hence one can use the equations (12), (24) accordingly to obtain the recursion
operators.

**Example 2. Kupershmidt equation.** This equation

\[ u_t = u_{5x} + 10uu_{3x} + 25u_xu_{2x} + 20u^2u_x \quad (27) \]
has the Lax pair

\[ L = D^3 + 2u D + u_x, \quad A = (L^\dagger_3)_+. \quad (28) \]

In this case \( L^* = -L \) therefore we use the equation (24) with
\[ \tilde{R}_n = a_n D^5 + b_n D^4 + c_n D^3 + d_n D^2 + e_n D + f_n. \]  
(29)

By equating the coefficients of powers of \( D \) in (24) we obtain

\[
a_n = \frac{2}{3} D^{-1}(u_n), \quad b_n = \frac{11}{3} u_n, \quad c_n = \frac{1}{9}(20uD^{-1}(u_n) + 73u_n, x),
\]
\[
d_n = \frac{1}{3}(10u_x D^{-1}(u_n) + 22uu_n + 27u_{n,2x}),
\]
\[
e_n = \frac{1}{27}(70u_{2x} D^{-1}(u_n) - 2D^{-1}(u_{2x}u_n) + 40u^2 D^{-1}(u_n) - 8D^{-1}(u^2 u_n)
\]
\[\quad + 134u_{n,3x} + 212uu_{n,x} + 184u_x u_n,
\]
\[
f_{n,x} = \frac{1}{27}(20u_{4x} D^{-1}(u_n) + 74u_{3x} u_n + 126u_{2x} u_{n,x} + 40uu_{2x} D^{-1}(u_n)
\]
\[\quad + 40u_x^2 D^{-1}(u_n) + 136u_x u_{n,2x} + 27uu_{x} u_n + 28u_{n,5x} + 64uu_{n,3x}
\]
\[\quad + 16u^2 u_{n,x}),
\]

and the recursion operator for the Kupershmidt equation:

\[
\mathcal{R} = D^6 + 12u D^4 + 36u_x D^3 + (49u_{2x} + 36u^2) D^2
\]
\[\quad + 5(7u_{3x} + 24uu_x) D + 13u_{4x} + 82uu_{2x} + 69u_x^2 + 32u^3 +
\]
\[\quad 2u_x D^{-1}(u_{2x} + 4u^2) + 2(u_{5x} + 10uu_{3x} + 25u_x u_{2x} + 20u^2 u_x) D^{-1}, \quad (30)
\]

### 2.3 Pseudo-differential Lax operator.

In this section we generalize our scheme to the case of pseudo-differential Lax operators. The only difference is that in formulas like (13) and (23) the \( R_n \) operator becomes also a pseudo-differential operator.

It follows from these formulas that the structure of the nonlocal terms in \( R_n \) is, in general, similar to the nonlocal terms in \( L \) since \( A_{n+m} \) and \( A_n \) are differential operators.

For skew-symmetric case, \( A_n \) may be defined by either (23) or (25), or (26). In the pseudo-differential case they are not equivalent in the sense that the nonlocal part of \( R_n \) depends on which ansatz we choose. For illustration, let us consider the case \( L = MD^{-1} \), where \( M \) is a differential operator. The following lemma shows that if \( L^\dagger = L \) or \( L^\dagger = -L \), where
then the formulas (13) and (25) are much suitable then (16), (23), and (26).

Lemma. Let \( L^\dagger = \varepsilon L \), where \( \varepsilon = \pm 1 \). Then

\[
R_n = D^{m-1} + \cdots + a_0, \quad \text{for } \varepsilon = 1,
\]
(32)

where \( R_n \) is defined by (13), and

\[
\tilde{R}_n = D^{2m-1} + \cdots + a_{-1} D^{-1}, \quad \text{for } \varepsilon = -1,
\]
(33)

where \( \tilde{R}_n \) is defined by (25).

Proof: If \( L = MD^{-1} \) then \( L^\dagger = \varepsilon L \) implies \( M^* = -\varepsilon M \). It is easy to show that \( (L_n^\dagger)^\dagger = -L_n^\dagger \). Hence \( (L_n^\dagger)^\dagger = -L_n^\dagger \) for an odd integer \( n \). Define now a series \( K_n \) by

\[
L_n^\dagger = D K_n.
\]

It is easy to prove that \( K_n^* = K_n \). Since \( K_n = (K_n)_+ + (K_n)_- \) and \( (K_n)^* = K_n \), we have

\[
(K_n)_+^* = (K_n)_+, \quad (K_n)_-^* = (K_n)_-.
\]

From the last formula it follows that \( \text{ord}(K_n)_- \leq -2 \) that leads to an important result

\[
A_n = (L_n^\dagger)_+ = D (K_n)_+.
\]

This implies that

\[
LA_n = M(K_n)_+,
\]
(34)

is a differential operator. Now using (34) in (13) and (25) for the cases \( \varepsilon = 1 \) and \( \varepsilon = -1 \) respectively we find the ansatz for \( A_n \) given by (32) and (33).

Example 3 (\( \varepsilon = -1 \)). It is known that the KdV equation has, besides the standard Lax representation, the following Lax pair:

\[
L = (D^2 + u) D^{-1}, \quad A = (L^3)_+.
\]
(35)
The $L$-operator satisfies the reduction $L^\dagger = -L$. According to the formula (33) we have

$$\tilde{R}_n = a_n D + b_n + c_n D^{-1}.$$ 

It follows from (25) that

$$a_n = D^{-1}(u_n), \quad b_n = u_n, \quad c_n = -u_{n,x} - u D^{-1}(u_n).$$

The remaining equation in (25) gives the recursion operator

$$\mathcal{R} = D^2 + 4u + 2u_x D^{-1}. \quad (36)$$

**Example 4** ($\epsilon = 1$). **DSIII system.** The DSIII system [24] [25] is given by

$$u_t = -u_{3x} + 6uu_x + 6v_x,$$
$$v_t = 2v_{3x} - 6uv_x. \quad (37)$$

The nonlocal Lax representation for this system is

$$L = (D^5 - 2u D^3 - 2D^3 u - 2D w - 2w D)D^{-1}, \quad (38)$$
$$A = (L^\frac{5}{2})_+,$$

where $w = v - u_{2x}$. Since $L^\dagger = L$ we can use (32) which gives us

$$R_n = a_n D^3 + b_n D^2 + c_n D + d_n. \quad (39)$$

By equating the coefficients of the powers of $D$ in (23) we obtain

$$a_n = D^{-1}(u_n), \quad b_n = 4u_n,$$
$$c_n = \frac{1}{2}(-6 u D^{-1}(u_n) + 11 u_{n,x} + 2D^{-1}(u u_n) + 2 D^{-1}(v_n)), $$
$$d_{n,x} = -\frac{1}{2}(6 u_{2x} D^{-1}(u_n) + 10 u_x u_n - 5 u_{n,3x} + 4 u u_{n,x} - 6 v_{n,x}).$$

The recursion operator of the DSIII is found as
\[ \mathcal{R} = \begin{pmatrix} \mathcal{R}_0^0 & \mathcal{R}_0^1 \\ \mathcal{R}_1^0 & \mathcal{R}_1^1 \end{pmatrix} \]  

with

\[ \begin{align*}
\mathcal{R}_0^0 &= D^4 - 8uD^2 - 12u_xD - 8w_2 + 16u^2 + 16v + (-2u_{3x} + 12uu_x + 12v_x)D^{-1} + 4u_xD^{-1}u, \\
\mathcal{R}_0^1 &= -10D^2 + 8u + 4u_xD^{-1}, \\
\mathcal{R}_1^0 &= 10v_xD + 12v_{2x} + (4v_{3x} - 12uv_x)D^{-1} + 4v_xD^{-1}u, \\
\mathcal{R}_1^1 &= -4D^4 + 16uD^2 + 8uxD + 16v + 4v_xD^{-1}.
\end{align*} \]  

This recursion operator has recently been given in [6].

### 3 Matrix L-operator of the first order.

In this section we demonstrate how our approach, given in the previous sections, can be generalized to the case where \( L \) is a matrix operator of the form

\[ L = D_x + \lambda a + q(x,t). \]  

#### 3.1 General Case

Let us consider the Lax operator (42), where \( q \) and \( a \) belong to a Lie algebra \( g \) and \( \lambda \) is the spectral parameter. The constant element \( a \) is supposed to be such that

\[ g = \text{Ker} (ad_a) \oplus \text{Im} (ad_a). \]  

First let us recall the procedure [24] of constructing the \( A \)-operators for the Lax operator (42).

**Proposition 3.** There exist unique series

\[ \begin{align*}
u &= u_{-1} \lambda^{-1} + u_{-2} \lambda^{-2} + \cdots, & u_i &\in \text{Im} (ad_a), \\
h &= h_0 + h_{-1} \lambda^{-1} + h_{-2} \lambda^{-2} + \cdots, & h_i &\in \text{Ker} (ad_a),
\end{align*} \]
such that
\[ e^{ad_u}(L) = L + [u, L] + \frac{1}{2} [u, [u, L]] + \cdots = D_x + a\lambda + h. \] (46)

Let \( b \) be a constant element of \( g \) such that \( [b, Ker(ad_a)] = \{0\} \). It follows from (43) that \( [b \lambda^n, D_x + a\lambda + h] = 0 \). Hence \( [\Phi_{b,n}, L] = 0 \), where
\[ \Phi_{b,n} = e^{-ad_u}(b\lambda^n). \] (47)

Then the corresponding \( A \)-operator of the form
\[ A_{b,n} = b\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0, \] (48)
is defined by the formula
\[ A_{b,n} = (\Phi_{b,n})_+. \] (49)

where
\[ (\sum_{-\infty}^{n} \alpha_i \lambda^i)_+ = \sum_{0}^{n} \alpha_i \lambda^i. \] (50)

According to (47)
\[ \Phi_{b,n+1} = \lambda \Phi_{b,n}. \] (51)

Hence
\[ A_{b,n+1} = (\lambda \Phi_{b,n})_+ = \lambda (\Phi_{b,n})_+ + (\lambda (\Phi_{b,n})_-)_+. \] (52)

The last formula shows that
\[ A_{b,n+1} = \lambda A_{b,n} + R_n, \quad R_n \in g, \] (53)
where \( R_n \) does not depend on \( \lambda \). Substituting (53) into the Lax equation \( L_{t_{n+1}} = [A_{b,n+1}, L] \) we get
\[ L_{t_{n+1}} = \lambda L_{t_n} + [R_n, L]. \] (54)

Using the ansatz (54) one can easily find the corresponding recursion operator.

**Example 5.** The system
\begin{align*}
  u_t &= -\frac{1}{2} u_{xx} + u^2 v, \\
  v_t &= \frac{1}{2} v_{xx} - v^2 u,
\end{align*}

is equivalent to the nonlinear Schrödinger equation, has a Lax operator

\begin{equation}
  L = D + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda + \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}.
\end{equation}

The Lie algebra $g$ in this example coincides with $sl(2)$. Using (54) with $R_n = \begin{pmatrix} a_n & b_n \\ c_n & -a_n \end{pmatrix}$ we find that

\begin{align*}
  a_n &= \frac{1}{2} D^{-1} (vu_n + uv_n), \\
  b_n &= \frac{1}{2} u_n, \\
  c_n &= -\frac{1}{2} v_n,
\end{align*}

and the recursion operator of the system (55) is given by

\begin{equation}
  \mathcal{R} = \begin{pmatrix} -\frac{1}{2} D + u D^{-1} v & u D^{-1} u \\ -v D^{-1} v & \frac{1}{2} D - v D^{-1} u \end{pmatrix}.
\end{equation}

### 3.2 Reductions in matrix case

In the general case considered in the previous section the $A_n$-operators belong to Lie algebra

\begin{equation}
  a_+ = \{ \Sigma_{i=0}^\kappa a_i \lambda^i, \ a_i \in g, \ \kappa \in \mathbb{Z}_+ \},
\end{equation}

that is a sub-algebra of the Lie algebra

\begin{equation}
  a = \{ \Sigma_{-\infty}^\kappa a_i \lambda^i, \ a_i \in g, \ \kappa \in \mathbb{Z} \}.
\end{equation}

A standard $\sigma$-reduction is defined by any automorphism $\sigma$ of the Lie algebra $g$ of finite order $\kappa$. Because $\sigma^\kappa = Id$, the eigenvalues of $\sigma$ are $\varepsilon^i, i = 0, \cdots, \kappa - 1$ where $\varepsilon$ is primitive $\kappa$-root of unity.
Let $g_i$ be an eigenspace corresponding to eigenvalue $\varepsilon^i$. Then the following reduction $a_j \in g_i$, where $i = j \mod \kappa$ in (58) and (59) is compatible with the equations (58). Note that according to this definition $a \in g_1$ and the potential $q(x, t)$ in (12) belongs to $g_0$ or, the same, satisfies $\sigma(q) = q$.

It is easy to see that, to satisfy such a reduction, we must use the ansatz

$$A_{b,n+\kappa} = \lambda^\kappa A_{b,n} + R_n,$$

where

$$R_n = r_{\kappa-1} \lambda^\kappa + \cdots + r_0, \quad r_i \in g_i. \tag{61}$$

Further generalizations are associated with modifications of sign " + " in (50) which corresponds to the simplest decomposition of algebra $a$ into direct sum of two sub-algebras

$$a = a_+ \oplus a_-; \tag{62}$$

where $a_+$ is given by (58) and

$$a_- = \{ \Sigma^{-1}_{-\infty} a_i \lambda_i, \quad a_i \in g \}. \tag{63}$$

The sign " + " in (50) is the projection of onto $a_+$ parallel to $a_-$. If we have a different decomposition (62) then the construction from Proposition 3 is also valid but we have the following condition

$$R_n \in a_+ \cap \lambda a_-, \tag{64}$$

instead of $R_n \in g$. If we also have the $\sigma$-reduction we must use the most general ansatz (61) where

$$R_n \in a_+ \cap \lambda^\kappa a_-.$$

**Example 6.** Let us consider the following equation

$$u_t = \frac{1}{4} u_{xxx} - \frac{3}{8} u_{xx} u + \frac{3}{8} u u_{xx} - \frac{3}{8} u u_x u, \tag{66}$$

where $u$ is a square matrix of arbitrary size, or more generally, $u$ belongs to arbitrary associative algebra $\mathcal{K}$. This equation has a Lax representation with
\[ L = D + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \lambda + \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix}. \] (67)

Here 1 is the unity of \( \mathcal{K} \). The reduction (67) can be described as follows (see [26]). The Lie algebra \( g \) is the algebra of all \( 2 \times 2 \) matrices with entries belonging to \( \mathcal{K} \). The automorphism \( \sigma \) is defined by

\[ \sigma(X) = T X T^{-1}, \] (68)

where \( T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \). Obviously \( \sigma^2 = Id \) and eigenvalues of \( \sigma \) are 1 and -1. The corresponding eigenspaces are

\[ g_0 = \{ * 0 0 \}, \quad g_1 = \{ 0 * \}, \] (69)

and therefore the coefficients \( a_i \) in (59) have the following structure

\[ a_{2j} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \quad a_{2j+1} = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}. \] (70)

The sub-algebra \( a_+ \) is given by (58), where the coefficients have the structure (70) and, additionally \( a_0 = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix} \). The sub-algebra \( a_- \) has the following form

\[ a_- = \sum_{-\infty}^0 a_i \lambda^i, \] (71)

where \( a_0 \) is of the form \( a_0 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \quad \alpha \in \mathcal{K} \).

The \( A \)-operator for (66) is given by formula \( A = (\Phi_{a,3})_+ \) (see (49)), where \( a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) and " + " means the projection onto \( a_+ \) parallel to \( a_- \).

According to (65), \( R_n \) is of the form

\[ R_n = \begin{pmatrix} a_n & 0 \\ 0 & a_n \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & b_n \\ c_n & 0 \end{pmatrix} \lambda + \begin{pmatrix} d_n & 0 \\ 0 & 0 \end{pmatrix}. \] (72)

It follows from

\[ L_{t_{n+2}} = \lambda^2 L_{t_n} + [R_n, L], \] (73)
that
\begin{align*}
    u_n - a_{n,x} + [a_n, u] + b_n - c_n &= 0, \\
    c_n - b_n - a_{n,x} &= 0, \\
    d_n - b_{n,x} - u b_n &= 0, \\
    d_n + c_{n,x} - c_n u &= 0, \\
    u_{n+2} &= -d_{n,x} + [d_n, u].
\end{align*}

Finding $a_n, b_n, c_n$, and $d_n$ from this system we obtain the following recursion operator

$$
\mathcal{R} = -(D + ad_u) (2D + R_u) (2D + ad_u)^{-1} (2D + L_u) D (2D + ad_u)^{-1},
$$

(74)

where $R_u$ and $L_u$ are the operators of right and left multiplications by $u$, respectively.

Note that in the commutative case (66) coincides with the modified KdV equation. It is easy to verify that (74) becomes the standard recursion operator of modified KdV equation. All factors in (74) have to be regarded as operators acting on (non-commutative) polynomial depending on $u, u_x, u_{xx}, \cdots$.

4 Conclusion

In this work we devoted ourselves in the construction of recursion operators when the Lax representation is given. We have shown that our approach can be easily generalized to all cases where $L$-operator is a polynomial of $\lambda$. It would be interesting to generalize it for the cases of more complicated $\lambda$ dependence of $L$ as well as for the cases 2 + 1 dimensional equations, Toda-type lattices and ordinary differential equations.

Acknowledgments

We would like to thank Dr. Jing Ping Wang for reading the manuscript and pointing out some misprints. This work is partially supported by the Scientific and Technical Research Council of Turkey (TUBITAK) and Turkish Academy of Sciences (TUBA). V.S is supported by RFFR - Grant 99-01-00294 and INTAS.
5 APPENDIX A

In this Appendix we give an additional example to Section 2.1.

A. Boussinesq System.

The Boussinesq equation

\[ u_{tt} = -\frac{1}{3}(u_{4x} + 2(u^2)_{2x}) \]  

(75)

can be expressed in the form of a pair of first-order evolution equations

\[
\begin{align*}
u_t &= v_x, \\
v_t &= -\frac{1}{3}(u_{3x} + 8u u_x).
\end{align*}
\]

(76)

This system has a Lax pair

\[ L = D^3 + 2uD + u_x + v, \quad A = (L^2)_. \]  

(77)

To construct the recursion operator for this system we use the equation (12) with the differential operator

\[ R_n = a_n D^2 + b_n D + c_n. \]

By equating the coefficients of the powers of \( D \) in (12) we find

\[
\begin{align*}
a_n &= \frac{2}{3} D^{-1}(u_n), \\
b_n &= \frac{1}{3} (5u_n + D^{-1}(v_n)), \\
c_n &= \frac{1}{9} (6v_n + 8u D^{-1}(u_n) + 10u_{n,x}),
\end{align*}
\]

and after that we obtain the recursion operator of the form (40) for (76) with

\[
\begin{align*}
\mathcal{R}_0^0 &= 3v + 2v_x D^{-1}, \\
\mathcal{R}_1^0 &= D^2 + 2u + u_x D^{-1}, \\
\mathcal{R}_0^1 &= -(\frac{1}{3} D^4 + \frac{10}{3} u D^2 + 5u_x D + 3u_{2x} + \frac{16}{3} u^2 + (\frac{2}{3} u_{3x} + \frac{16}{3} u u_x) D^{-1}), \\
\mathcal{R}_1^1 &= 3v + v_x D^{-1}.
\end{align*}
\]

(78)
6 APPENDIX B

In this Appendix we give additional examples to the Section 2.2.

B.1. Sawada-Kotera Equation.

The Lax pair for the Sawada-Kotera equation \[27\]

\[ u_t = u_{5x} + 5uu_{3x} + 5u_xu_{2x} + 5u^2u_x, \quad (79) \]

is given by

\[ L = D^3 + uD, \quad A = (L^3)_+. \quad (80) \]

In this example, \( L^\dagger = -L \), where \( L^\dagger = D^{-1}L^*D \) and \( L \) is skew-symmetric, then we use (24). The operator \( \tilde{R}_n \) has the same form as (29) with the coefficients given by

\[
\begin{align*}
    a_n &= \frac{1}{3} D^{-1}(u_n), \quad b_n = \frac{5}{3} u_n, \quad c_n = \frac{1}{9} (5uD^{-1}(u_n) + 29u_{n,x}), \\
    d_n &= \frac{1}{9} (5u_x D^{-1}(u_n) + 14uu_n + 26u_{n,2x}), \\
    e_n &= \frac{1}{27} (10u_{2x} D^{-1}(u_n) - 2D^{-1}(u_{2x}u_n) - D^{-1}(u^2u_n) + 5u^2D^{-1}(u_n) \\
    & \quad + 28u_{n,3x} + 32uu_{n,x} - 32u_xu_n), \quad f_n = 0.
\end{align*}
\]

The recursion operator is given as

\[
\mathcal{R} = D^6 + 6uD^4 + 9u_xD^3 + (9u^2 + 11u_{2x})D^2 + (10u_{3x} + 21uu_x)D \\
+ 5u_{4x} + 16uu_{2x} + 6u_x^2 + 4u^3 + (u_{5x} + 5uu_{3x} + 5u_xu_{2x} + 5u^2u_x)D^{-1} + u_xD^{-1}(u^2 + 2u_{2x}). \quad (81)
\]

B.2. DSI System.

The DSI system \[24\], \[25\]

where

\[ u_t = 3v v_x, \]
\[ v_t = 2v_{3x} + 2uv_x + vu_x, \]  

(82)

has a Lax representation with

\[
L = [D^3 + (u + v) D + \frac{1}{2} (u + v)_x] [D^3 + (u - v) D + \frac{1}{2} (u - v)_x], \tag{83}
\]

\[ A = (L^\frac{1}{2})_+. \]

Here \( R_n \) is a differential operator of order 5 and since \( L \) is symmetric we again use the equation (12). The expressions for the coefficients of the operator \( R_n \) are very long and complicated. Hence we do not display them here.

We find that the recursion operator \( R \) of this system is of the form (10), where

\[
\begin{align*}
R_0^0 &= -4D^6 - 24uD^4 - 72u_xD^3 + 2(-49u_{2x} - 18u^2 + 42v^2)D^2 \\
&+ 10(-7u_{3x} - 12uu_x + 30vv_x)D \\
&- 26u_{4x} - 82uu_{2x} - 69u_x^2 + 222vv_x + 141v_x^2 - 16u^3 + 48v^2u \\
&+ 2(-2u_{5x} - 10uu_{3x} - 25u_xu_{2x} - 10u^2u_x + 15v^2u_x + 30vv_{3x} \\
&+ 45v_x^2v_{2x} + 30vvu_x)D^{-1} + 2u_xD^{-1}(3v^2 - 2u^2 - u_{2x}), \\
R_0^1 &= 168vD^4 + 204vu_x^3 + 6(21v_{2x} + 32uv)D^2 + 6(40vu_x + 7v_{3x} + 22uv_x)D \\
&+ 6(13uvu_{2x} + 10u_xv_x + v_{4x} + 5uv_{2x} + 4vu^2 + 12v^3) \\
&+ 108v_{2x}D^{-1}v + 2u_xD^{-1}(6uv + 9v_{2x}), \\
R_1^0 &= 56vD^4 + 268vu_x^3 + 2(243v_{2x} + 32uv)D^2 \\
&+ 2(36vu_x + 219v_{3x} + 106uv_x)D + 2(27vu_{2x} + 92u_xv_x + 99v_{4x} + 99uv_{2x} \\
&+ 4vu^2 + 12v^3) + 2(10vu_{3x} + 35u_xv_{2x} + 45u_x^2v_x + 10uvu_x + 18v_{5x} \\
&+ 30uv_{3x} + 10u^2v_x + 15v^2v_x)D^{-1} + 2v_xD^{-1}(3v^2 - u^2 - u_{2x}), \\
R_1^1 &= 108D^6 + 216uD^4 + 432u_xD^3 + 6(81u_{2x} + 18u^2 + 22v^2)D^2 \\
&+ 6(45u_{3x} + 36uu_x + 70vv_x)D \\
&+ 3(18u_{4x} + 18uv_{2x} + 9u_x^2 + 98v_{2x} + 67v_{3x}^2 + 32uv^2) \\
&+ 36(2v_{3x} + 2v_xu + vu_x)D^{-1}v + 2v_xD^{-1}(6uv + 9v_{2x}). \tag{84}
\end{align*}
\]
B.3. DSII System.

The DSII system \cite{24}, \cite{25}

\[ u_t = 3v_x, \quad v_t = -2(v_3x + uv_x + vu_x), \quad (85) \]

has a Lax representation with

\[ L = (D^6 + uD^3 + D^3u + (v + \frac{1}{2}u^2)D + D(v + \frac{1}{2}u^2), \quad (86) \]

\[ A = (L^2)_+. \]

Since \( L \) is symmetric we again use the equation \((12)\). In this case the operator \( R_n \) is given as follows

\[ R_n = a_n D^5 + b_n D^4 + c_n D^3 + d_n D^2 + e_n D, \quad (87) \]

where

\[ a_n = \frac{1}{3} D^{-1}(u_n), \quad b_n = \frac{5}{3} u_n, \]

\[ c_n = \frac{1}{9} [5uD^{-1}(u_n) + 3D^{-1}(v_n) + 29u_{n,x}], \]

\[ d_n = \frac{1}{9} [5u_x D^{-1}(u_n) + 26u_{n,2x} + 14uu_n + 12v_n], \]

\[ e_n = \frac{1}{27} [5(2u_{2x} + u^2 + 3v)D^{-1}(u_n) - 3D^{-1}(uv_n + w_n) + 9uD^{-1}(v_n) - 2D^{-1}(u_{2x}u_n + \frac{1}{2}u^2u_n) + 54u_xu_n + 28u_{n,3x} + 32(uu_{n,x} - u_nu_x) + 42v_{n,x}]. \]

The recursion operator \((10)\) for the system can be found as

\[ \mathcal{R}_0^0 = -D^6 - 6uD^4 - 9u_xD^3 - (11u_{2x} + 9u^2 + 42v)D^2 \]
\[ + (-10u_{3x} - 21uu_x - 84v_x)D - 5u_{4x} - 16uu_{2x} - 6u_x^2 - 60v_{2x} - 4u^3 - 24uv \]
\[ + (-u_{5x} - 5uu_{3x} - 5u_xu_{2x} - 5u_x^2u_x - 15vu_{x} - 15v_{3x} - 15uv_x)D^{-1} \]
\begin{align*}
- u_x D^{-1} (2u_{2x} + u^2 + 3v), \\
\mathcal{R}_0^1 &= -42D^4 - 48uD^2 - 87u_x D - 6(7u_{2x} + u^2 - 6v) + 27v_x D^{-1} - 3u_x D^{-1} u, \\
\mathcal{R}_0^1 &= 28vD^4 + 106v_x D^3 + (165v_{2x} + 32uv) D^2 \\
&+ (54uv_x + 132v_{3x} + 74v_x u) D \\
&+ 30uv_{2x} + 79u_x v_x + 54v_{4x} + 57uv_{2x} + 4u^2 v - 24v^2 \\
&+ (10uv_{3x} + 25v_x u_{2x} + 30u_x v_{2x} + 10uvu_x + 9v_{5x} + 15uv_{3x} \\
&+ 5u^2 v_x - 15vv_x) D^{-1} - v_x D^{-1} (3v + u^2 + 2u_{2x}), \\
\mathcal{R}_1^0 &= 27D^6 + 54uD^4 + 135u_x D^3 + 3(54u_{2x} + 9u^2 - 22v) D^2 \\
&+ 3(36u_{3x} + 27uu_x - 28v_x) D + 3(9u_{4x} + 9uu_{2x} + 9u_x^2 - 21v_{2x} - 16vu) \\
&- 18(v_{3x} + u_x v + v_x u) D^{-1} - 3v_x D^{-1} u. \\
\end{align*}

**B.4. DSIV System.**

The DSIV System \cite{24}, \cite{25} which is also known as the Hirota-Satsuma system \cite{28}, \cite{29}

\begin{align*}
  u_t &= \frac{1}{2} u_{3x} + 3uu_x - 6vv_x, \\
  v_t &= -v_{3x} - 3uv_x, \\
\end{align*}

(89)

has Lax representation with

\[
L = (D^2 + u + v)(D^2 + u - v), \quad A = (L^2)^+. \\
\]

(90)

Since the operator \(L\) is symmetric we use the equation (12). In this case the operator \(R_n\) has the same form as (39) with coefficients given by

\[
\begin{align*}
  a_n &= \frac{1}{2} D^{-1} (u_n), \quad b_n = \frac{7}{4} u_n - \frac{1}{2} v_n, \\
  c_n &= \frac{1}{8} [6uD^{-1}(u_n) + 2D^{-1}(uu_n) - 4D^{-1}(vv_n) + 17u_{n,x} - 12v_{n,x}], \\
  d_{n,x} &= \frac{1}{16} [6u_{2x} D^{-1}(u_n) - 12v_{2x} D^{-1}(u_n) + 30u_x u_n - 8u_x v_n \\
  &+ 24uu_{n,x} + 15u_{n,3x} - 12v_x v_n - 8w_{n,x} - 20vw_{n,x} - 28v_{n,3x}].
\end{align*}
\]

The recursion operator (40) for the given system is
\[ R_0^0 = \frac{1}{4} D^4 + 2u D^2 + 3u_x D + 2u_{2x} + 4(u^2 - v^2) \]
\[ + (3uu_x - 6uv_x + \frac{1}{2}u_{3x})D^{-1} + u_x D^{-1}u, \]
\[ R_1^0 = -5v D^2 - 4v_x D - v_{2x} - 4uv - 2u_x D^{-1}v, \]
\[ R_0^1 = -\frac{5}{2} v_x D - 3v_{2x} - (v_{3x} + 3uv_x)D^{-1} + v_x D^{-1}u, \]
\[ R_1^1 = -D^4 - 4u D^2 - 2u_x D - 4v^2 - 2v_x D^{-1}v. \] (91)

**B.5. N=3 Hirota-Satsuma System.**

This system is given by \[28\]
\[ u_t = \frac{1}{4} u_{3x} + 3uu_x + 3(-v^2 + w)_x, \]
\[ v_t = -\frac{1}{2} v_{3x} - 3uv_x, \]
\[ w_t = -\frac{1}{2} w_{3x} - 3uw_x. \] (92)

This is an example for \( N = 3 \) system which covers some other \( N = 2 \) systems as special cases. For instance letting \( w = 0 \) we get DSIV and \( v = 0 \) we get DSIII systems.

The corresponding Lax pair is
\[ L = (D^2 + 2u - 2v)(D^2 + 2u + 2v) + 4w, \quad A = (L^\frac{2}{3})_+. \] (93)

In this case the operator \( L \) is symmetric and hence \( R_n \) has the same form as \[19\] with the coefficients
\[ a_n = D^{-1}(u_n), \quad b_n = \frac{7}{2}u_n + v_n, \]
\[ c_n = \frac{1}{4}[12u D^{-1}(u_n) + 4D^{-1}(uu_n + w_n - 2vv_n)] + 17u_{n,x} + 12v_{n,x}, \]
\[ d_{n,x} = \frac{1}{8}[12u_{2x} D^{-1}(u_n) + 24v_{2x} D^{-1}(u_n)] + 60u_x u_n + 16u_x v_n + 15u_{n,3x} \]
\[ + 48uu_{n,x} + 24v_x u_n - 40v_x v_n + 20v_{n,3x} + 16vv_{n,x} + 20w_{n,x}. \]
The recursion operator is given by
\[
\mathcal{R} = 
\begin{pmatrix}
\mathcal{R}_0^0 & \mathcal{R}_0^1 & \mathcal{R}_0^2 \\
\mathcal{R}_1^0 & \mathcal{R}_1^1 & \mathcal{R}_1^2 \\
\mathcal{R}_2^0 & \mathcal{R}_2^1 & \mathcal{R}_2^2 
\end{pmatrix},
\] (94)

where
\[
\begin{align*}
\mathcal{R}_0^0 &= \frac{1}{4}D^4 + 4uD^2 + 6uxD + 4(u_{2x} + 4u^2 - 4v^2 + 4w) \\
&\quad + 4\left(\frac{1}{4}u_{3x} + 3uu_x - 6uv_x + 3w_x\right)D^{-1} + 4uxD^{-1}u, \\
\mathcal{R}_0^1 &= -2(5D^2 + 4uxD + v_{2x} + 8uv + 4uxD^{-1}v), \\
\mathcal{R}_0^2 &= 5D^2 + 8u + 4uxD^{-1}, \\
\mathcal{R}_1^0 &= -5v_xD - 6v_{2x} - 2(v_{3x} + 6v_xu)D^{-1} + 4v_xD^{-1}u, \\
\mathcal{R}_1^1 &= -D^4 - 8uD^2 - 4uxD + 8(8w - 2v^2) - 8v_xD^{-1}v - 8D^{-1}w_x, \\
\mathcal{R}_1^2 &= 4(v_xD^{-1} + 2D^{-1}v_x), \\
\mathcal{R}_2^0 &= -5v_xD - 6v_{2x} - 2(v_{3x} + 6v_xu)D^{-1} + 4w_xD^{-1}u, \\
\mathcal{R}_2^1 &= -16D^{-1}w_x - 8w_xD^{-1}v, \\
\mathcal{R}_2^2 &= -D^4 - 8uD^2 - 4uxD + 16(w - v^2) + 4w_xD^{-1} + 16vD^{-1}v_x. \tag{95}
\end{align*}
\]

APPENDIX C

In this Appendix we give an additional example to Section 3.

C.1. Non-Abelian Schrödinger equation.

This is the system given by
\[
\begin{align*}
u_t &= -\frac{1}{2}u_{xx} + uu, \\
v_t &= \frac{1}{2}v_{xx} - uv, \tag{96}
\end{align*}
\]

where \(u\) and \(v\) belong to \(\mathcal{K}\) (see Example 6 for the notations)). The Lax operator of (96) is given by
The corresponding formula (54) reduces to

\[
\begin{pmatrix}
0 & u_{n+1} \\
v_{n+1} & 0
\end{pmatrix} = \lambda \begin{pmatrix}
0 & u_n \\
v_n & 0
\end{pmatrix} + [R_n, L],
\]  
(98)

where

\[
R_n = \begin{pmatrix}
a_n & b_n \\
c_n & -a_n
\end{pmatrix}.
\]  
(99)

The formula (98) gives us both \(a_n, b_n, c_n\) and the recursion operator \(\mathcal{R}\). They are given by

\[
a_n = \frac{1}{2} D^{-1} (u_n v + u v_n), \quad b_n = \frac{1}{2} u_n, \quad c_n = -\frac{1}{2} u_n,
\]  
(100)

\[
\mathcal{R} = \frac{1}{2} \begin{pmatrix}
-D + R_u D^{-1} R_v + L_u D^{-1} L_v & R_u D^{-1} L_u + L_u D^{-1} R_u \\
-L_v D^{-1} R_u - R_v D^{-1} L_v & D - R_v D^{-1} R_u - L_v D^{-1} L_u
\end{pmatrix}.
\]  
(101)

C.2. Non-Abelian Modified KdV Equation.

The standard non-Abelian Modified KdV equation is given by

\[
u_t = \frac{1}{4} u_{xxx} - \frac{3}{4} u_x u^2 - \frac{3}{4} u^2 u_x.
\]  
(102)

The Lax representation of this equation is given

\[
L = D + \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \lambda + \begin{pmatrix}
0 & u \\
0 & -u
\end{pmatrix}.
\]  
(103)

The recursion operator \(\mathcal{R}\) can be found from (60) and (61). In our case the automorphism \(\sigma\) is the same as in Example 6 and formulas (60) and (61) give us

\[
\begin{pmatrix}
0 & u_{n+1} \\
v_{n+1} & 0
\end{pmatrix} = \lambda^2 \begin{pmatrix}
0 & u_n \\
v_n & 0
\end{pmatrix} + [R_n, L],
\]  
(104)
where

\[ R_n &= \begin{pmatrix} 0 & a_n \\ b_n & 0 \end{pmatrix} \lambda + \begin{pmatrix} c_n & 0 \\ 0 & d_n \end{pmatrix}. \tag{105} \]

Using (104) we find \( a_n, b_n, c_n, d_n \) from the following:

\[
\begin{align*}
  b_n - a_n &= u_n, \\
  -a_{n,x} - a_n u - u a_n + c_n - d_n &= 0, \\
  -b_{n,x} + b_n u + u b_n + d_n - c_n &= 0, \\
  d_{n,x} + c_{n,x} &= [c_n - d_n, u], \\
  u_{n+1} &= d_{n,x} + [d_n, u].
\end{align*}
\]

The resulting recursion operator is given by

\[
\mathcal{R} = \frac{1}{4} (D - a_d u \cdot D^{-1} \cdot a_d) (D - (L_u + R_u) D^{-1} (L_u + R_u)). \tag{106}
\]

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