Sphalerons, instantons, and standing waves on $S^3 \otimes R$.

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Abstract

We consider pure $SU(2)$ Yang-Mills theory when the space is compactified to a 3-dimensional sphere with finite radius. The Euclidean classical self-dual solutions of the equations of motion (the instantons) and the static finite energy solutions (the sphalerons) which have been found earlier are rewritten in handy physical variables with the gauge condition $A_0 = 0$. Stationary solutions to the equations of motion in the Minkowski space-time (the standing waves) are discussed. We briefly discuss also the theory defined in a flat finite spherical box with rigid boundary conditions and present the numerical solution describing the sphaleron.

1 Introduction

Topologically nontrivial nonabelian gauge field configurations characterized by a non-zero Pontryagin number play a very important role in $QCD$. They provide the solution to $U(1)$ problem, are responsible for strong violation of the Zweig rule in the scalar and pseudoscalar sectors and are crucial for understanding the structure of $QCD$ vacuum state [1]. Of all such configurations, the distinguished position belongs to instantons [2] which are self-dual and present the solutions to the classical Euclidean equations of motion. In a lot of papers, the role of instantons in $QCD$ was studied numerically on lattices [3]. However, the lattice approximation of $QCD$ involves two technical parameters: i) a finite ultraviolet cutoff and ii) a finite size of the box where the theory is defined. Both effects lead to modification and distortion

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of instantons, and it is important to understand what these distortions are to keep them under control. In this paper, we address the second problem — the modification of the classical BPST solution due to finite volume effects.

In the recent paper [4], a numerical study of the instantons living on the torus with periodic boundary conditions in spatial directions has been performed. Also the sphaleron — the static finite energy solution to the Yang-Mills equations of motion with one unstable mode has been found. (Such solutions do not exist in the Yang-Mills theory defined in infinite space due to conformal invariance of the classical theory [5]. However, a finite size of spatial torus introduces infrared cutoff which breaks down conformal invariance, and sphalerons appear by the same token as they do in electroweak theory where infrared cutoff is provided by the vacuum Higgs average. [6]).

Cubic geometry is complicated, and little hope exists to find the solutions not only numerically, but analytically. It is instructive therefore to study the same problem with the spherical geometry where the analytical solution can be obtained. It is rather easy to write down the solution for the instanton living on a four-dimensional Euclidean sphere: when the coordinates \( x^\mu \) describing the stereographic mapping \( R^4 \rightarrow S^4 \) are chosen, it just coincides with the standard BPST solution. However, in such a geometry where time is also compactified, the instanton loses its transparent physical meaning of the tunneling trajectory connecting topologically non-equivalent vacua [7]. The geometry where only space is compactified on \( S^3 \) and Euclidean time \( \tau \) extends from \(-\infty \) to \( \infty \) is much more interesting from the physical viewpoint and involves nontrivial finite volume effects.

Analytical solutions for the Yang-Mills equations in such a geometry have been studied earlier in a lot of papers [8]-[12]. Actually, any solution on \( S^3 \otimes R \) corresponds to some solution in the flat space due to conformal invariance of the Yang-Mills equations and the fact that the metrics of \( S^3 \otimes R \) is reduced to the flat metrics by a conformal transformation:

1. The metrics of the Euclidean \( R^4 \) space can be written as [12]

\[
dx^2_\mu = e^{2\tau/R}[d\tau^2 + R^2 d\Omega_3^2]
\]

(1.1)

with

\[
x^2_\mu = R^2 e^{2\tau/R}
\]

(1.2)

(\( d\Omega_3^2 \) is the metrics on the unit 3-dimensional sphere.)
• The metrics of the Minkowski $R^4$ space can be written as

$$dt^2 - d\vec{x}^2 = R^2 \frac{d\eta^2 - [d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)]}{(\cos \eta + \cos \chi)^2}$$

(1.3)

where

$$t + r = R \tan \frac{\eta + \chi}{2}$$
$$t - r = R \tan \frac{\eta - \chi}{2}$$

(1.4)

The solutions describing the standing waves on $S^3 \otimes R$ with real Minkowski time were found first as peculiar non-stationary solutions in the flat Minkowski space-time describing ingoing and outgoing spherical waves [13, 14]. [we see from (1.4) that the relation of flat space variables $t, r$ to the time ($\eta$) and spatial ($\chi$) variables on $S^3 \otimes R$ is rather intricate].

The sphaleron (a static solution with one unstable mode) on $S^3 \otimes R$ has been found first in [8], and it corresponds to the well-known meron flat-space solution [15]:

$$A^a_\mu(x) = \frac{\eta^a_{\mu\nu} x^\nu}{x^a}$$

(1.5)

where $\eta^a_{\mu\nu}$ are the standard 't Hooft notations [16]:

$$\eta^a_{ij} = \epsilon_{aij}, \eta^a_{i0} = -\eta^a_{0i} = \delta_{ai}$$

(1.6)

This solution does not involve a scale parameter and the differential form $A^a_\mu(x)dx_\mu$ does not depend on $|x|$. After performing the conformal transformation (1.1), it translates into the static $\tau$ - independent solution on $S^3 \otimes R$. Its time component [ corresponding to the radial component of Eq.(1.5)] is zero.

Multiplying Eq. (1.5) by $dx_\mu$, we easily get

$$A^a_\mu(\text{meron})(x)dx_\mu \equiv A^a_\nu(\text{sphal})(y)dy_\nu = -\eta^a_{\mu\nu} e_\mu de_\nu$$

(1.7)

where $y_i$ are some coordinates on $S^3$ and $e_\mu = x_\mu/\sqrt{x^2}$ is the unit four-dimensional vector. The RHS of (1.7) is nothing else as the Maurer-Cartan forms $\sigma^a$ on $S^3$, and this is the way the sphaleron solution has been presented in [8, 12].

The instanton solutions in the flat Euclidean space have been translated into $S^3 \otimes R$ language in [12].

There are two raisons d’etre for the present paper. The first one is methodical. We discuss all the solutions on $S^3 \otimes R$ found earlier in a uniform way and from the same angle treating the finite 3-dimensional spatial sphere where the theory is
defined as a particularly convenient from the analytical viewpoint way to regularize
the theory in the infrared — much as the standard compactification on the torus
used in the lattice approach is convenient for purposes of numerical computations.
Thus, we will be interested neither in the gravitational [8]-[11] nor in the flat-space
[13, 14] aspects of the problem. Our approach is similar to that of [12]. However,
Minkowski space solutions were not discussed in that paper and also the Euclidean
solutions for instantons and sphaleron on $S^3 \otimes R$ were written in rather non-standard
variables and were not easily visualizable. A considerable fraction of this paper is
devoted to rewriting these solutions in a physical and transparent form and analyzing
their properties. We work consistently in the gauge $A_0 = 0$ where the physical
interpretation of instantons is most evident.

The next section is devoted to fixing the notations. In Sect. 3, we present an
alternative (compared to Refs. [8]-[12]) derivation for the sphaleron solution on $S^3$.
Sect. 4 is devoted to the instanton of the maximal size where the volume density of
action is the same at all points of $S^3$. This instanton goes over from the topologically
trivial vacuum $A_i = 0$ at $\tau = -\infty$ to a nontrivial one $A_i = i\Omega(\vec{x})\partial_i\Omega(\vec{x})$ at $\tau = \infty$
(where $\Omega(\vec{x})$ is a particular function describing a nontrivial mapping $SU(2) \rightarrow S^3$).
It passes through the sphaleron at $\tau = 0$. In Sect. 5, we write down the solution for
instantons of arbitrary size $\lambda$ in handy terms, find the gauge transformation bringing
it to the Hamiltonian gauge $A_0 = 0$, and explore the limit $\lambda \ll R$ where the solution
is reduced to the standard BPST instanton. In Sect. 6, we discuss some Minkowski
space solutions describing nonlinear standing waves on $S^3$ and get free of charge the
increment of instability for the sphaleron.

Sect. 7 presents the second part of the paper. We discuss a related but different
problem where the system is defined in the flat spherical box with the size $R$ and rigid
boundary conditions $A_i(r = R) = 0$. No analytical solution can be obtained in this
case, but , by solving a simple ordinary differential equation, we find the sphaleron
solution numerically, draw its profile and calculate its energy.

2 Notations.

Different authors use different conventions and, for clarity, we list here our own. The
Yang-Mills action on an Euclidean curved manifold is

$$S = \frac{1}{4g_0^2} \int \sqrt{g} \, d^4x \, G^{\mu\nu,\alpha} G_{\mu\nu}^\alpha$$  \hspace{1cm} (2.8)
where $G_{\mu\nu}^a = \partial_{\mu}A_{\nu}^a - \partial_{\nu}A_{\mu}^a + \epsilon^{abc}A_{\mu}^bA_{\nu}^c$, $g_0$ is the coupling constant, and $g$ is the determinant of metrics. The $SU(2)$ gauge transformation is

$$A_\mu^\Omega = \Omega(A_\mu + i\partial_\mu)\Omega^\dagger$$  \hspace{1cm} (2.9)$$

where $A_\mu = A_\mu^{a\alpha} = A_\mu^a\sigma^a/2$, $\sigma^a$ are the Pauli matrices. The topological charge is

$$\nu = \frac{1}{32\pi^2} \int g \, d^4x \, G^{\mu\nu,a}\tilde{G}_{\mu\nu,a}$$  \hspace{1cm} (2.10)$$

where $\tilde{G}_{\mu\nu}^a = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}G^{\alpha\beta,a}$ and the convention $\epsilon_{1230} = 1$ is chosen. Instanton is the field configuration with the minimal action belonging to the topological class $\nu = 1$. It is self-dual $G_{\mu\nu}^a = \sqrt{g}\tilde{G}_{\mu\nu}^a$ which means

$$E_i^a = -B_i^a$$  \hspace{1cm} (2.11)$$

where $E_i^a = G_{0i}^a$, $B_i^a = \frac{1}{2}\epsilon_{ijk}\sqrt{g}G^{jk,a}$. In the following, we shall be interested with the case when metrics is static : $g_{00} = 1$, $g_{0i} = 0$ and $g_{ij}$ is time-independent.

In the Hamiltonian approach which is the most natural one for static geometry, an important characteristic of a configuration $A_i^a(\vec{x}, \tau)$ is the Chern-Simons number

$$Q = -\frac{\epsilon_{ijk}}{16\pi^2} \int g \, d\vec{x} \left[ \text{Tr}\{G^{ij}A^k\} + \frac{2i}{3}\text{Tr}\{A^iA^jA^k\} \right]$$  \hspace{1cm} (2.12)$$

so that

$$\nu = Q(\tau = \infty) - Q(\tau = -\infty)$$  \hspace{1cm} (2.13)$$

Chern-Simons number is invariant under static topologically trivial gauge transformations.

3 Sphaleron on $S^3$.

We choose the metrics of $S^3$ in the form

$$ds^2 = \frac{d\vec{x}^2}{[1 + (r/2R)^2]^2}$$  \hspace{1cm} (3.1)$$

It is just the stereographic projection so that $r = 0$ corresponds to the southern pole of the sphere, $r = \infty$ to the north pole and $r = 2R$ — to the equator. In most formulae we shall set $R = 1$. The dependence on $R$ can be anytime restored on dimensional grounds.
Let us look for a non-trivial solution of Yang-Mills equations of motion $A_a^i(\vec{x})$ which lives on $S^3$ and has a finite energy. We assume that the solution has spherical symmetry so that one can decompose

$$A_a^i(\vec{x}) = \frac{1 - \phi_2}{r} \epsilon_{aib} n_b + \frac{\phi_1}{r} (\delta_{ai} - n_a n_i) + A n_a n_i$$

, $n_i = x^i/r$ (3.2)

where $\phi_{1,2}$ and $A$ are some functions of $r$. This Anzatz is similar to Witten’s Anzatz [17], only we keep $A_0^a = 0$. The magnetic field is

$$B_a^i = (1 + r^2/4) \left[ \frac{\phi_1' - A \phi_2}{r} \epsilon_{aib} n_b + \frac{\phi_2' + A \phi_1}{r} (\delta_{ai} - n_a n_i) - \frac{n_a n_i}{r^2} (1 - \phi_1^2 - \phi_2^2) \right]$$ (3.3)

(prime means differentiation over $r$). The energy functional is

$$E = \frac{4\pi}{g_0^2} \int_0^\infty dr (1 + r^2/4) \left\{ (\phi_1' - A \phi_2)^2 + (\phi_2' + A \phi_1)^2 + \frac{1}{2r^2} (1 - \phi_1^2 - \phi_2^2)^2 \right\}$$ (3.4)

On this stage, it is very convenient to introduce the radial and angular variables

$$\left\{ \begin{array}{l}
\phi_1 = \rho \sin \alpha \\
\phi_2 = \rho \cos \alpha
\end{array} \right.$$ (3.5)

(but we shall not necessarily assume $\rho$ to be positive). Variation of (3.4) over $A(r)$ gives the equation

$$A = \alpha'$$ (3.6)

If the condition (3.6) is satisfied, the functional (3.4) depends only on $\rho(r)$:

$$E = \frac{4\pi}{g_0^2} \int_0^\infty dr (1 + r^2/4) \left[ (\rho')^2 + \frac{(1 - \rho^2)^2}{2r^2} \right]$$ (3.7)

Variation of (3.7) over $\rho(r)$ gives the second equation of motion

$$\rho'' + \frac{r/2}{1 + r^2/4} \rho' + \frac{\rho(1 - \rho^2)}{r^2} = 0$$ (3.8)

A change of variables $r = 2e^u$ is handy after which Eq. (3.8) acquires the form [14]

$$\rho_{uu} + \rho_u \tanh u + \rho (1 - \rho^2) = 0$$ (3.9)

Were the term $\propto \rho_u$ absent, the equation would describe a conservative motion in the potential

$$V(\rho) = -(1 - \rho^2)^2/4$$ (3.10)
When the term with first derivative is present, the “energy” is not conserved. When $u < 0$, it is pumped into the system and when $u > 0$, it is dissipated.

If we want the physical energy (3.7) of the field to be finite, the function $\rho$ should be equal to 1 or -1 at $r = 0$ and $r = \infty$ (i.e. at $u = \pm \infty$). There are only two solutions to the equations (3.8, 3.9) having this property:

$$
\rho = \mp \tanh u = \pm \frac{1 - r^2/4}{1 + r^2/4}
$$

These solutions are physically identical as the overall change of the sign in $\rho$ can be traded for an overall phase shift $\alpha \to \alpha + \pi$. Let us, for definiteness, choose the upper sign in Eq. (3.11).

Speaking of the function $\alpha(r)$, it is not rigidly fixed. The only requirement is that the differential form $A_i^a dx^i$ which, in contrast to the potential $A_i^a(\vec{x})$, has an invariant meaning is uniquely defined at $r = 0$ and $r = \infty$. Such a condition implies that $A_i^a$ tends to zero at infinity faster than $1/r$. In that case, $\alpha(r)$ can be any smooth function satisfying the boundary conditions

$$
\alpha(0) = 0, \quad \alpha(\infty) = (2n + 1)\pi
$$

with integer $n$. The simplest case $n = 0$ corresponds to the sphaleron configuration. The case $n = -1$ describes the antishpaleron, and other values of $n$ correspond to “multisphaleron” solutions to be discussed a bit later. Whence the asymptotics of $\alpha(r)$ is fixed, the solutions with different $\alpha(r)$ are obtained from each other by a topologically trivial gauge transformation

$$
A_i(\vec{x}) \to \Omega(\vec{x}) (A_i(\vec{x}) + i\partial_i) \Omega^\dagger(\vec{x})
$$

where $\Omega(\vec{x}) = \exp\{i\beta(r)n_a\tau_a\}$ with $\beta(0) = \beta(\infty) = 1$. One can be easily convinced that the transformation (3.13) acts on a general spherically symmetric Anzatz (3.2) as

$$
\begin{align*}
\rho^\Omega(r) &= \rho(r) \\
\alpha^\Omega(r) &= \alpha(r) + 2\beta(r) \\
A^\Omega(r) &= A(r) + 2\beta'(r)
\end{align*}
$$

Naturally, the equations of motion (3.6) and (3.8) are invariant under this transformation.

We shall see in the next section that the especially clever choice of the phase is

$$
\alpha(r) = 2 \arctan \frac{r}{2}
$$
(such \( \alpha(r) \) just coincides with the polar angle on \( S^3 \)). In this particular gauge, the sphaleron solution acquires the form

\[
A^a_i = \frac{1}{(1 + r^2/4)^2} \left[ r \epsilon_{ab} n_b + (1 - r^2/4)(\delta_{ai} - n_a n_i) + (1 + r^2/4)n_a n_i \right] \quad (3.16)
\]

The solution (3.16) could of course be obtained directly from (1.7) if expressing \( e_\mu \) in stereographic coordinates

\[
\begin{align*}
  e_0 & = \frac{1 - r^2/4}{1 + r^2/4} \\
  e_i & = \frac{r n_i}{1 + r^2/4}
\end{align*}
\quad (3.17)
\]

where \( n_i = x^i/r \) is the unit 3-dimensional vector.

Antisphaleron has the same form but the signs in Eq. (3.15) and of the second and the third term in square brackets in Eq. (3.16) are reversed. It corresponds to choosing the same form for the phase function as in Eq.(3.13) but with the opposite sign. The energy of the sphaleron is obtained by substituting the solution (3.11) in (3.7) :

\[
E = \frac{3\pi^2}{g_0^2 R} \quad (3.18)
\]

where we have restored the dimensional factor \( R^{-1} \). The volume energy density of the sphaleron

\[
\epsilon = \frac{1}{4g_0^2} F^a_{ij} F^{ij,a} = \frac{3}{2g_0^2 R^4} \quad (3.19)
\]

is constant on the sphere.

The configuration (3.16) is a saddle point of the energy functional and has an unstable mode. In Sect. 6, we shall find the increment of instability \( \mu \) for this mode. The result is

\[
\mu = \frac{\sqrt{2}}{R} \quad (3.20)
\]

The sphaleron solution (3.16) has 3 zero modes corresponding to a global gauge rotation \( A^a_i \rightarrow O^{ab} A^b_i \) where \( O^{ab} \) is an orthogonal matrix. The rotated field has the same energy and all other physical properties, but it loses an explicit spherical symmetry and cannot be presented in the form (3.2).

Finally, let us calculate the Chern-Simons number (2.12) on the sphaleron configuration. For any \( A^a_i \) presented in the form (3.2),

\[
Q = \frac{1}{2\pi} \int_0^\infty dr \left[ A(1 - \phi_1^2 - \phi_2^2) + \phi_2\phi'_2 - \phi_1\phi'_1 - \phi_1' \right] \quad (3.21)
\]
For the sphaleron (antisphaleron), it is just

\[ Q = \frac{1}{2\pi} \int_0^\infty dr \, \alpha'(r) = \pm \frac{1}{2} \]  

(3.22)

The sphaleron field is obtained from the antisphaleron one by the topologically nontrivial gauge transformation

\[ \Omega_0 = \exp\{2i \arctan \frac{r}{2} n^a \sigma_a\} \]  

(3.23)

Applying this transformation once more, one can get a multisphaleron solution with the Chern-Simons number \( Q = 3/2 \) etc. The function \( \rho(r) \) for all such solutions is the same as for the sphaleron, and the phase function for the solution with Chern-Simons number \( Q = (2n + 1)/2 \) is

\[ \alpha_n(r) = 2(2n + 1) \arctan \frac{r}{2} \]  

(3.24)

, \( n = 0, \pm 1, \ldots \)

\section{4 Instantons of maximal size.}

The instanton solution on \( S^3 \otimes R \) which has the maximal size and is spread homogeneously over the sphere has been written down in Ref. [12] in the following form

\[
\begin{cases}
A_0^a = 0 \\
A_i^a dx^i = -\frac{2e^{2\tau}}{1 + r^2/4} \eta^a_{\mu \nu} e_\mu d\epsilon_\nu
\end{cases}
\]  

(4.1)

Substituting there the expressions (3.17) for \( e_\mu \) in stereographic coordinates, we get

\[ A_i^a = \frac{2e^{2\tau}}{e^{2\tau} + 1 (1 + r^2/4)^2} \left[ r \epsilon_{abi} n_b + (1 - r^2/4) (\delta_{ai} - n_an_i) + (1 + r^2/4) n_an_i \right] \]  

(4.2)

One can be convinced that the instanton (4.2) satisfies the self-duality condition (2.11) and has the topological charge (2.10) \( \nu = 1 \) and the action \( S = 8\pi^2/g_0^2 \).

We see that the Euclidean time dependence comes as a common factor, and the spatial structure is exactly the same as for the sphaleron solution in the gauge (3.17). The instanton (4.2) starts from the trivial vacuum \( A_i^a = 0 \) at \( \tau = -\infty \) and passes through the sphaleron (3.16) at \( \tau = 0 \). At \( \tau \to \infty \), the solution (4.2) tends to a pure gauge \( A_i = i\Omega_0 \partial_i \Omega_0^\dagger \) with the same \( \Omega_0 \) as in Eq.(3.23) and corresponds to a topologically nontrivial vacuum with Chern-Simons number \( Q = 1 \). Certainly, the instanton can be written down in any other gauge by applying a gauge transformation (3.13). After that, however, the dependence on \( \tau \) and \( r \) would not factor out so nicely.
and also, at $\tau = -\infty$, the potential would not be just zero but a pure gauge [with a topologically trivial $\Omega(\vec{x})$].

This solution has 4 obvious zero modes: one of them corresponding to shifting the time variable $\tau \to \tau - \tau_0$ and 3 zero modes corresponding to global gauge rotations. There is also the fifth zero mode corresponding to going over to a solution with not the maximal size (such solutions will be described in the next section), but there are no zero modes corresponding to spatial translations — the volume action density is the same at all points of the sphere, and the solution has no centre.

5 Instantons of arbitrary size.

The solution of the arbitrary size has also been found in [12]. It was presented in the following form (we still customized it a little bit)

$$A_0 = -\frac{s b^a \sigma^a}{1 + s^2 + b^2 + 2 s b_\mu e_\mu}$$

$$A_i dx^i = -\frac{(s^2 + s b_\mu e_\mu) \sigma^a + s e^{a\alpha \beta} \tilde{b}^\alpha \sigma^\beta}{1 + s^2 + b^2 + 2 s b_\mu e_\mu} - \frac{\eta_{\mu \nu} e_\mu d e_\nu}{r}$$

(5.1)

where $b_\mu$ is a four-dimensional vector, $\tilde{b}^a = \eta^a_{\mu \nu} b_\mu e_\nu$, $s = \xi e^\tau$, and $\sigma^a$ are the Pauli matrices.

To write down the solution for the instanton centered at $r = 0$, $\tau = 0$ and expressed in standard variables, we have to do the following: $i)$ choose $b_\mu = \tilde{(0, -b)}$, $ii)$ choose $\xi = \sqrt{1 + b^2}$, and $iii)$ to perform the stereographic projection (3.17). It is convenient also to introduce the variable $\lambda = 1/b$. When $\lambda$ is small, it has the meaning of the physical instanton size. After some calculations, one gets the following result

$$A_0^a = -\frac{r n_a}{C_+ \kappa \cosh(\tau) - C_-}$$

$$A_i^a = \frac{1}{C_+[C_+ \kappa \cosh(\tau) - C_-]} [r e^\tau \kappa \epsilon_{abc} n_c + (e^\tau \kappa C_- - C_+) (\delta_{ai} - n_i n_a) +$$

$$(e^\tau \kappa C_+ - C_-) n_i n_a]$$

(5.2)

where

$$\kappa = \sqrt{1 + \lambda^2}, \quad C_\pm = 1 \pm r^2/4$$

The potential (5.2) is zero at $\tau = -\infty$. At $\tau = \infty$, it has exactly the same asymptotics $A_0 \to 0$, $A_i \to i \Omega_0 \partial_\theta \Omega_0^\dagger$, $\Omega_0(\vec{x})$ being given by Eq.(3.23), as the instanton of maximal size described in the previous section. The field strength is now

$$E_i^a = -B_i^a = \frac{\lambda^2}{[C_+ \kappa \cosh(\tau) - C_-]^2} [r \epsilon_{abc} n_b + C_- (\delta_{ai} - n_i n_a) + C_+ n_a n_i]$$

(5.3)
and the volume density of the action is no longer homogeneous in space.

The instanton (5.1), (5.2) has 8 zero modes as it should. As earlier, there are global gauge rotations and time translations, but now there are four additional collective variables \( b_\mu \) describing the instanton scale and the spatial position. To get the antiinstanton solution, one has to substitute \( \bar{\eta}^a_{\mu\nu} \) for \( \eta^a_{\mu\nu} \) in Eq.(5.1) or, in our language, to change the sign of \( A_0^a \) and of the second and the third tensor structures for \( A_i^a \).

Let us first look what happens in the limit \( \lambda \ll 1 \). It corresponds to the case when the physical size of the instanton is much less that the radius of the sphere. If one also assumes that \( r, |\tau| \ll 1 \) so that the effects due to finite curvature on sphere are not essential (the requirement \( |\tau| \ll 1 \) is also necessary because, at \( |\tau| \sim 1 \), the instanton field or whatever is left of it spreads out over the whole sphere irrespectively of what the value of \( \lambda \) is, and the expansion in \( r \) fails), the solution (5.2) acquires the familiar flat space form

\[
A_\mu^a = \frac{2\eta^a_{\mu\nu}x_\nu}{\tau^2 + r^2 + \lambda^2} \quad (5.4)
\]

It is instructive to transform the solution still further and go over into the Hamiltonian gauge \( A_0 = 0 \). It is achieved via a gauge transformation \( \Omega = \exp\{i\beta(\tau, r)n^a\sigma^a\} \). Zero component of the gauge field is killed provided \( 2\dot{\beta}n^a + A_0^a = 0 \) and hence

\[
\beta(\tau, r) = \frac{r}{\sqrt{r^2 + \lambda^2C_2^+}} \left\{ \frac{e^{\tau}\kappa C_+ - C_-}{\sqrt{r^2 + \lambda^2C_2^+}} + \arctan \frac{C_-}{\sqrt{r^2 + \lambda^2C_2^+}} \right\} \quad (5.5)
\]

The second term in the braces appears due to initial condition \( \beta(\tau = -\infty, r) = 0 \) which is to be imposed if we want to preserve the property \( A_i^a(\tau = -\infty, \vec{x}) = 0 \). In the flat space limit, the expression for \( \beta(\tau, r) \) is simplified

\[
\beta_{flat}(\tau, r) = \frac{r}{\sqrt{r^2 + \lambda^2}} \left( \frac{\tau}{\sqrt{r^2 + \lambda^2}} + \frac{\pi}{2} \right) \quad (5.6)
\]

The transformed field \( A_i^a \) can be easily found using the radial-angular decomposition (3.5) and the recipe (3.14). As the full expression is rather cumbersome, we write it down explicitly only in the flat space limit:

\[
\rho_{flat}(\tau, r) = \frac{\sqrt{[\tau^2 + (\lambda + r)^2][\tau^2 + (\lambda - r)^2]}}{\tau^2 + r^2 + \lambda^2}
\]

\[
\alpha_{flat}(\tau, r) = \frac{\tau}{|\tau|} \arccos \left( \frac{\tau^2 + \lambda^2 - r^2}{\sqrt{[\tau^2 + (\lambda + r)^2][\tau^2 + (\lambda - r)^2]}} \right) + \frac{2r}{\sqrt{r^2 + \lambda^2}} \left( \frac{\tau}{\sqrt{r^2 + \lambda^2}} + \frac{\pi}{2} \right)
\]

\[
A_{flat}(\tau, r) = \frac{2\lambda^2}{r^2 + \lambda^2} \left[ \frac{\tau}{\tau^2 + \lambda^2} + \frac{1}{\sqrt{r^2 + \lambda^2}} \left( \frac{\tau}{\sqrt{r^2 + \lambda^2}} + \frac{\pi}{2} \right) \right] \quad (5.7)
\]
In the limit $\tau \to \infty$,

\[
\rho_{\text{flat}}(\infty, r) = 1 \\
\alpha_{\text{flat}}(\infty, r) = \frac{2\pi r}{\sqrt{r^2 + \lambda^2}}
\]  

(5.8)

The phase rapidly grows from $\alpha = 0$ at $r = 0$ up to $\alpha = 2\pi$ at $r \gg \lambda$. It is no surprise, of course, that the instanton field written in the gauge $A_0 = 0$ describes in the limit $\tau \to \infty$ the twisted vacuum with

\[
Q(\tau = \infty) = \frac{1}{2\pi} [\alpha(\infty, \infty) - \alpha(\infty, 0)] = 1
\]  

(5.9)

It is noteworthy, however, that the required rise of the phase was provided exactly by the gauge transformation (5.6) which killed the nonzero $A_0$ of the BPST solution.

It is instructive also to look at the phase of the full solution on $S^3$ at $\tau = \infty$ in the gauge $A_0 = 0$. It has the form

\[
\alpha_{S^3}(\infty, r) = 4 \arctan \frac{r}{2} + \frac{2r}{\sqrt{r^2 + \lambda^2 C_+^2}} \left( \frac{\pi}{2} + \arctan \frac{C_-}{\sqrt{r^2 + \lambda^2 C_+^2}} \right)
\]  

(5.10)

Again, when the physical size of the instanton $\lambda$ is small, the phase reaches its asymptotic value $\alpha = 2\pi$ at $r \sim \lambda$ and stays there [at small $r$ it just coincides with the flat space expression (5.8)]. Thus in this gauge, the solution has the nice property that the integral (2.12) for the Chern-Simons number is saturated in a localized spatial region $r \sim \lambda$ (it was not the case for the original form (5.2) where the phase $\alpha_\infty(r) = 4 \arctan(r/2)$ reached its asymptotic value only at $r \gg 1$ close to the north pole of the sphere irrespectively of the instanton size).

The final comment concerns the slice $\tau = 0$ of the instanton solution. For the instanton of the maximal size (4.2), it was just the sphaleron. When $b \neq 0$, it is not a solution to the static equations of motion anymore, but it still has (in the gauge $A_0 = 0$!) the Chern-Simons number $Q = 1/2$. The energy of such a configuration is $\propto 1/(g_0^2 \lambda)$ when the size of the instanton is small and tends to the sphaleron energy (3.18) when $b$ tends to zero (and $\lambda$ to infinity). It is still interesting to look at the field configuration at $\tau = 0$ in more details. When the size of the instanton $\lambda$ is small, it has the simple form

\[
\rho_{\text{flat}}(0, r) = \frac{|\lambda^2 - r^2|}{r^2 + \lambda^2} \\
\alpha_{\text{flat}}(0, r) = \pm \pi \theta(r - \lambda) + \frac{\pi r}{\sqrt{r^2 + \lambda^2}} \\
A_{\text{flat}}(0, r) = \frac{\pi \lambda^2}{(r^2 + \lambda^2)^{3/2}}
\]  

(5.11)
where the sign of the first term in the second equation depends on whether the limit \( \tau \to +0 \) or the limit \( \tau \to -0 \) is taken. This expression is singular. The phase \( \alpha_{\text{flat}}(0, r) \) is not uniquely defined at all, and whatever sign is chosen, it is discontinuous at the point \( r = \lambda \). Also the derivative \( \rho'_{\text{flat}}(0, r) \) is discontinuous at that point. But these are not physical singularities and depend on the particular convention \((5.11)\).

The potential \( A_i^a(\vec{x}) \) is a smooth function which is best seen if expressing it in terms of

\[
\tilde{\rho}(0, r) = \frac{\lambda^2 - r^2}{r^2 + \lambda^2},
\]

\[
\tilde{\alpha}(0, r) = \frac{\pi r}{\sqrt{r^2 + \lambda^2}} \quad (5.12)
\]

(cf. Eqs.\((3.11), (3.15)\) for the sphaleron potential on \( S^3 \)).

### 6 Standing waves.

The explicit form of the sphaleron solution \((3.16)\) allows one to find rather easily some classical solutions to the Yang-Mills equations of motion not only in Euclidean but also in Minkowski space-time. Let us search for such solutions in the form

\[
A_i^a(t, \vec{x}) = \frac{h(t)}{C^+} [r \epsilon_{ab} n_b + C_-(\delta_{ai} - n_a n_i) + C_+ n_a n_i] \quad (6.1)
\]

where \( t \) is the Minkowskian physical time. With this Ansatz, the problem is greatly simplified and the lagrangian constrained on the class of fields \((6.1)\) reads

\[
L = \frac{1}{2g_0^2} \int \sqrt{g} d\vec{x}(E_i^a E^ai - B_i^a B^ai) = \frac{3\pi^2}{g_0^2} [\dot{h}^2 - h^2(2 - h)^2] \quad (6.2)
\]

This is just the lagrangian of anharmonic oscillator. The potential

\[
V(h) = \frac{3\pi^2}{g_0^2} h^2(2 - h)^2 \quad (6.3)
\]

has the minima at \( h = 0 \) and \( h = 2 \) which correspond to the classical vacua with \( Q = 0 \) and \( Q = 1 \). It also has the local maximum at \( h = 1 \) which is the sphaleron \((Q = 1/2)\). For general \( h \), the Chern-Simons number of the configuration \((6.1)\) is

\[
Q = \frac{3h^2 - h^3}{4} \quad (6.4)
\]

The equation of motion is

\[
\ddot{h} + 2h(1 - h)(2 - h) = 0 \quad (6.5)
\]
A general solution to this equation is given by elliptic functions \( h(t) = \begin{cases} 
1 \pm \sqrt{c+1} \operatorname{dn} \left( (t-t_0)\sqrt{c+1} \left| \frac{2c}{1+c} \right| \right), & 0 \leq c \leq 1 \\
1 + \sqrt{c+1} \operatorname{cn} \left( (t-t_0)\sqrt{2c} \left| \frac{1+c}{2c} \right| \right), & c > 1
\end{cases} \) (6.6)
in the notations of [19]. The constant \( c \) determines the energy of the solution:

\[
E = \frac{3\pi^2}{g_0^2 R^2} c^2
\] (6.7)

(where we have restored again the factor \( R^{-1} \)).

There are two distinguished physically interesting cases. The first one corresponds to small \( c \) when the solution (6.6) describes linear standing waves with small amplitude \( |h| \ll 1 \) or \( |2-h| \ll 1 \). The frequency of the oscillations is

\[
\omega = \frac{2}{R}
\] (6.8)

For small \( R \ll \Lambda_{QCD} \) (i.e. when \( g_0^2(R) \ll 1 \)), the frequency (6.8) can be interpreted as one of the lowest excitation energies of the quantum Yang-Mills hamiltonian (a “glueball mass” if you will).

Another interesting case corresponds to the “particle” standing on the top of the barrier at \( t = -\infty \) which then starts to roll down, say, to the left, turns back at \( t = 0 \), and approaches the top again at \( t \to \infty \). The explicit form of such \( h(t) \) [corresponding to the choice \( c = 1 \), \( t_0 = 0 \) and the lower sign in Eq. (6.6)] is [10]

\[
h(t) = 1 - \frac{\sqrt{2}}{\cosh(t\sqrt{2}/R)}
\] (6.9)

The value \( \mu = \sqrt{2}/R \) determines the eigenvalue of the unstable mode (or increment of instability) of the sphaleron solution as has already been quoted in Eq.(3.20). The same result has been obtained in [12] in a different, more complicated way.

Note that, when the system starts to roll sown from the sphaleron saddle point, it approaches again the sphaleron configuration at \( t \to \infty \), not the antisphaleron one as one could probably guess in advance knowing that the potential should be periodic in the Chern-Simons number \( Q \).

The potential is periodic, indeed, when all degrees of freedom of the field are taken into account. But the Ansatz (6.1) breaks down the symmetry with respect to Minkowskian Yang-Mills solutions in flat geometry were studied. But the independent variable there was not just time, but a complicated function of \( t \) and \( r \) [see Eq. (1.4)], and the solution was not stationary.

---

\(^2\)The equation (6.3) and its solution (6.6) appeared first in Ref. [13] where Minkowskian Yang-Mills solutions in flat geometry were studied. But the independent variable there was not just time, but a complicated function of \( t \) and \( r \) [see Eq. (1.4)], and the solution was not stationary.
to large gauge transformations, the antisphaleron configuration is now unreachable, and the potential (6.3) is not periodic but has the reflecting walls on the left and on the right. Certainly, we could choose right from the beginning another Ansatz based on the antisphaleron configuration or on any other configuration with the phase (3.24) and the Chern-Simons number $Q = (2n + 1)/2$ which is related to (3.16) by a topologically nontrivial gauge transformation. We would get then corresponding Minkowski space solutions which are the gauge copies of those we found.

One should also understand that, will all probability, the system is not stable with respect to small perturbations of initial data distorting the particular spatial dependence of the field (6.1). If it started to roll down from the sphaleron “mountain pass” not exactly along the steepest descent road (6.9), it would miss the pass on the way back, would be reflected again by the slope of a mountain nearby, and starts to wander randomly in the functional space (cf. [20, 21]).

7 Sphalerons in the ball.

Let us consider now a similar but a different problem where the 3-dimensional metrics is flat, but the field is defined in a finite spherical box $D^3$ with the rigid boundary conditions

$$A_0^a(r = R) = 0$$

(and the gauge condition $A_0(x) = 0$ is assumed).

The boundary conditions (7.1) are similar in spirit to the bag boundary conditions but do not coincide with the latter. Boundary conditions (7.1) imply that the electric field and the normal component of the magnetic field vanish on the boundary whereas the bag boundary conditions are the vanishing of the normal component of electric field and the tangential component of the magnetic field [22]. The bag boundary conditions describe the spherical cavity in a dual superconductor and correspond to the physical picture of individual hadron in confining medium.

For us, however, boundary conditions are just the way to regularize the theory in the infrared and, for physical purposes, the size of the spherical box should be chosen much larger than a characteristic hadron size. As the main point of interest for us here are classical solutions, the bag boundary conditions are not convenient — they do not allow for non-trivial sphaleron solutions. But the boundary conditions (7.1) do.

We assume again the spherically symmetric form (3.2) of the potential and use the radial-angular representation (3.5). One of the equations of motion for the
static sphaleron field retains the form (3.6) while the equation of motion (3.8) for $\rho(r)$ carrying nontrivial dynamic information is modified to
\[ \rho'' + \frac{\rho(1 - \rho^2)}{r^2} = 0 \] (7.2)

In accordance with the general theorem of Ref. [5], this equation does not have nontrivial finite-energy solutions when space has no boundaries. Really, one can change the variable $r = e^u$ after which the equation (7.2) acquires the form
\[ \rho_{uu} - \rho_u + \rho(1 - \rho^2) = 0 \] (7.3)

and can be interpreted as a non-conservative motion in the potential (3.10) where energy is constantly pumped into the system [14]. The motion is unbounded, and if it started a little bit away from the unstable equilibrium points $\rho = \pm 1$, $\rho_u = 0$, it ends up at $\rho(u) = \pm \infty$ at a finite value of $u$ [21].

But in finite box, the sphaleron solutions appear. In the full analogy with the $S^3$ case considered earlier, the boundary conditions $\phi_1(R) = \phi_2(R) = A(R) = 0$ and the condition of smoothness of $A^a_i(\vec{x})$ at origin imply
\[ \rho(0) = \pm 1, \quad \rho(R) = \mp 1 \] (7.4)

Let us choose for definiteness the upper sign and be interested with the sphaleron rather than with antispaheron or multisphalerons. Then the phase $\alpha(r)$ can be an arbitrary function satisfying the conditions
\[ \alpha(0) = 0, \quad \alpha(R) = \pi, \quad \alpha'(R) = 0 \] (7.5)

The function $\rho(r)$ is determined from the equation (7.2) with the boundary conditions (7.4). The equation (7.3) has even a name, it is a special case of the Duffing equation [23], and, for space without bounds, it was studied extensively in [21], but we are not aware of any presentation of its solution into known functions. Thus, we solved it numerically. In Fig. 1, we plotted the solution for $\rho(r)$. In Fig. 2, the radial dependence of the volume energy density
\[ \epsilon(r) = \frac{1}{2g_0^2} B^a_i B^a_i(r) = \frac{1}{g_0^2r^2} \left\{ \left[ \rho'(r) \right]^2 + \left[ 1 - \rho^2(r) \right]^2 / (2r^2) \right\} \] (7.6)

is plotted. The total energy is
\[ E_{sph} = 4\pi \int_0^R \epsilon(r)r^2 dr \approx \frac{6.7\pi^2}{g_0^2 R} \] (7.7)

\[ ^3 \text{As an amusing fact, one can note that in the region } r \sim 0, \text{ the function } \rho(r) \text{ behaves as } 1 - 4.00 r^2. \]
Note that $\epsilon(R) \neq 0$ which means that the field strength $B_i^a$ (in contrast to the potential $A_i^a$) is non-zero at the boundary. This is the reason why we did not try to look for numerical solutions for the self-duality equations $E_i^a = -B_i^a$ with the boundary conditions $A_i^a(r = R) = 0$ which would describe instantons on $D^8 \otimes R$. With all probability, such solutions just do not exist. Really, in the gauge $A_0^a = 0$, the electric field $E_i^a = \dot{A}_i^a$ should be zero on the boundary and cannot coincide with $-B_i^a$ which is nonzero (at least for $t = 0$ where the instanton of maximal size is expected to pass through the sphaleron).

The situation is essentially the same as in the standard electroweak model where the infrared regularization is provided by the Higgs expectation value $v$. Sphalerons do exist there [6], but instantons do not — the (classical) action of the instanton with the size $\rho$:

$$S(\rho) = \frac{8\pi^2}{g^2} + 4\pi^2 \rho^2 v^2$$

depends on $\rho$ [10]. The action takes the minimal value in the limit $\rho \to 0$ which is never achieved for non-singular configurations. Obviously, the same is true in our case. Instantons of small sizes practically do not feel the boundary, and their action is almost $8\pi^2/g^2$. But this limit is never achieved for smooth field configurations.

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Figure captions.

**Fig. 1.** Numerical solution for the function $\rho(r)$ describing the sphaleron in $D^3$. Radius of the ball $R$ is set to 1.

**Fig. 2.** Volume energy density of the sphaleron $\epsilon(r)$. The normalization $R = g_0^2 = 1$ is chosen.
