Compositionality of the MSO+U Logic

Paweł Parys
Institute of Informatics, University of Warsaw, Poland
parys@mimuw.edu.pl

Abstract

We prove that the MSO+U logic is compositional in the following sense: whether an MSO+U formula holds in a tree T depends only on MSO+U-definable properties of the root of T and of subtrees of T starting directly below the root. Another kind of compositionality follows: every MSO+U formula whose all free variables range only over finite sets of nodes (in particular, whose all free variables are first-order) can be rewritten into an MSO formula having access to properties of subtrees definable by MSO+U sentences (without free variables).

2012 ACM Subject Classification

Theory of computation → Logic and verification

Keywords and phrases
Compositionality, MSO+U logic, boundedness

Funding
Work supported by the National Science Centre, Poland (grant no. 2016/22/E/ST6/00041).

1 Introduction

The MSO+U logic extends the MSO logic by the unbounding quantifier, U [6]. A formula using this quantifier, UX.ϕ, says that ϕ holds for arbitrarily large finite sets X. In this paper, we consider MSO+U formulae evaluated over infinite trees.

The MSO+U logic was shown to be undecidable, already over infinite words [4]. Nevertheless, some of its fragments have decidable properties. Among them there is the weak fragment, WMSO+U, where one can only quantify over finite sets [2, 10, 5]. The weak fragment can be also extended by the “exists a branch” quantifier [3]. Another fragment, decidable over infinite words, corresponds to ωBS-automata [7].

In a previous paper [12], we show that WMSO+U formulae can be evaluated over trees generated by higher-order recursion schemes. As an ingredient, we use there compositionality of the WMSO+U logic. In the current note, we extend the compositionality results to the full MSO+U logic.

Namely, we prove two facts. Firstly, we show that whether an MSO+U formula holds in a tree T depends only on MSO+U-definable properties of the root of T and of subtrees of T starting directly below the root. Secondly, every MSO+U formula whose all free variables range only over finite sets of nodes (in particular, whose all free variables are first-order) can be rewritten into an MSO formula having access to properties of subtrees definable by MSO+U sentences (without free variables).

Analogous results hold for most logics, and were often used to obtain decidability results (some selection: [9, 14, 11, 1, 10, 13]).

2 Preliminaries

The powerset of a set X is denoted P(X). The domain of a function f is denoted dom(f). When f is a function, by f[x ↦ y] we mean the function that maps x to y and every other z ∈ dom(f) to f(z).

Trees. We consider ordered trees of bounded arity. Fix some maximal arity r_{max} ∈ N. A tree domain (a set of tree nodes) is a set D ⊆ {1, . . . , r_{max}}^* such that if u_i ∈ D then u ∈ D, and if u(i + 1) ∈ D then u_i ∈ D (where u ∈ {1, . . . , r_{max}}^*, i ∈ {1, . . . , r_{max}}). A tree over
Compositionality of the MSO+U Logic

an alphabet \( \mathbb{A} \) is a function \( T : D \to \mathbb{A} \), for some tree domain \( D \). The set of trees over an alphabet \( \mathbb{A} \) and with maximal arity \( r_{\text{max}} \) is denoted \( T(A,r_{\text{max}}) \). A node \( w \) is the \( i \)-th child of \( u \) if \( w = u_i \).

**MSO+U.** For technical convenience, we use a syntax in which there are no first-order variables. It is easy to translate a formula from a more standard syntax to ours (at least when the maximal arity of considered trees is fixed). We assume an infinite set \( \mathcal{V} \) of variables, which can be used to quantify over sets of tree nodes. In the syntax of MSO+U we have the following constructions:

\[
\varphi ::= a(X) \mid X \downarrow Y \mid X \subseteq Y \mid \varphi_1 \land \varphi_2 \mid \neg \varphi' \mid \exists X.\varphi' \mid \cup X.\varphi'
\]

where \( a \) is a letter, \( i \in \mathbb{N}_+ \), and \( X,Y \in \mathcal{V} \). Free variables of a formula are defined as usual; in particular \( \cup X.\varphi \) is a quantifier, hence it bounds the variable \( X \). By \( FV(\varphi) \) we denote the set of free variables of a formula \( \varphi \).

The MSO logic is defined likewise, with the exception that the \( U \) quantifier is disallowed.

A valuation in a tree \( T \) is a function \( \nu : \mathcal{V} \to \mathcal{P}(\text{dom}(T)) \) (formally, we assume that \( \nu \) is defined for all variables from \( \mathcal{V} \); nevertheless, its value is meaningful only for free variables of a considered formula).

The semantics of a formula \( \varphi \) in a tree \( T \) under a valuation \( \nu \) is defined as follows:

- \( a(X) \) holds when every node in \( \nu(X) \) is labeled by \( a \),
- \( X \downarrow Y \) holds when both \( \nu(X) \) and \( \nu(Y) \) are singletons, and the unique node in \( \nu(Y) \) is the \( i \)-th child of the unique node in \( \nu(X) \),
- \( X \subseteq Y \) holds when \( \nu(X) \subseteq \nu(Y) \),
- \( \varphi_1 \land \varphi_2 \) holds when both \( \varphi_1 \) and \( \varphi_2 \) hold,
- \( \neg \varphi' \) holds when \( \varphi' \) does not hold,
- \( \exists X.\varphi' \) holds when \( \varphi' \) holds under a valuation \( \nu[X \mapsto X] \) for some set \( X \) of nodes of \( T \), and
- \( \cup X.\varphi' \) holds when for every \( n \in \mathbb{N} \), \( \varphi' \) holds under a valuation \( \nu[X \mapsto X_n] \) for some finite set \( X_n \) of nodes of \( T \) of cardinality at least \( n \).

We write \( T,\nu \models \varphi \) to denote that \( \varphi \) holds in \( T \) under the valuation \( \nu \). When \( \varphi \) is a sentence (i.e., does not have free variables), the valuation \( \nu \) is irrelevant, and we simply write \( T \models \varphi \) instead.

In order to see that our definition of MSO+U is not too poor, let us write a few example formulæ.

- The fact that \( X \) represents an empty set can be expressed as \( \text{empty}(X) \equiv \forall Y.\ X \subseteq Y \).
- The fact that \( X \) represents a set of size at least 2 can be expressed as \( \text{big}(X) \equiv \exists Y.(Y \subseteq X \land \neg(X \subseteq Y) \land \neg\text{empty}(Y)) \).
- The fact that \( X \) represents a singleton can be expressed as \( \text{sing}(X) \equiv \neg\text{empty}(X) \land \neg\text{big}(X) \).
- When we only consider trees of a fixed maximal arity \( r_{\text{max}} \), the fact that \( X \) and \( Y \) represent singletons \( \{x\} \), \( \{y\} \), respectively, such that \( y \) is a child of \( x \) can be expressed as

\[
(X \downarrow_1 Y) \lor \cdots \lor (X \downarrow_{r_{\text{max}}} Y),
\]

where \( \varphi_1 \lor \varphi_2 \) stands for \( \neg(\neg\varphi_1 \land \neg\varphi_2) \).

- Let \( A = \{a_1, \ldots, a_k\} \) be a finite set of letters. The fact every node in the set represented by \( X \) has label in \( A \) can be expressed as

\[
\forall Y.\left((\text{sing}(Y) \land Y \subseteq X) \to (a_1(Y) \lor \cdots \lor a_k(Y))\right),
\]

where \( \forall Y.\varphi \) stands for \( \neg\exists Y.\neg\varphi \), and \( \varphi_1 \to \varphi_2 \) stands for \( \neg(\varphi_1 \land \neg\varphi_2) \).
**Subtrees.** For a tree $T$ and its node $u$, by $T|_u$ we denote the subtree of $T$ starting at $u$, defined in the expected way. Moreover, when $\nu$ is a valuation in $T$, by $\nu|_u$ we denote its restriction to $T|_u$; namely, every variable $X$ is mapped to the set $\{w \mid uw \in \nu(X)\}$.

The root tree of $T$, denoted $\text{root}(T)$, is the tree consisting only of the root of $T$ (i.e., $\text{root}(T)$ consists of a single node labeled by $T(\varepsilon)$). For a valuation $\nu$, by $\text{root}(\nu)$ we denote the appropriate restriction of $\nu$; it maps every variable $X$ to the set $\{\varepsilon\} \cap \nu(X)$.

### 3 Results

Our first theorem says that whether an MSO+U formula holds in a tree $T$ depends only on MSO+U-definable properties of the root of $T$ and of subtrees of $T$ starting directly below the root.

**Theorem 3.1.** Fix a finite alphabet $\mathcal{A}$ and a maximal arity $r_{\text{max}}$. For every MSO+U formula $\varphi$ there exists a finite set $\Omega$ of tuples of MSO+U formulae such that for every tree $T \in \mathcal{T}(\mathcal{A}, r_{\text{max}})$ and for every valuation $\nu$ in $T$, it is equivalent whether

- $T, \nu \models \varphi$, and
- for some tuple $(\varphi_0, \varphi_1, \ldots, \varphi_r) \in \Omega$ the root of $T$ has $r$ children, and $\text{root}(T), \text{root}(\nu) \models \varphi_0$, and $T|_i, \nu|_i \models \varphi_i$ for all $i \in \{1, \ldots, r\}$.

Moreover, for every tuple $(\varphi_0, \varphi_1, \ldots, \varphi_r) \in \Omega$ and every $i \in \{0, \ldots, r\}$ it is the case that $FV(\varphi_i) \subseteq FV(\varphi)$.

Notice that the formulae $\varphi_0$ evaluated in the root are necessarily very simple: they can only read the root’s label, and check which variables among $FV(\varphi)$ are mapped to sets containing the root.

Our second theorem says that every MSO+U formula whose free variables range only over finite sets of nodes can be rewritten into an MSO formula having access to properties of subtrees definable by MSO+U sentences.

We say that a valuation $\nu$ is finitary if it maps every variable to a finite set of nodes.

An MSO+U relabeling is given by a tuple of MSO+U sentences $\Psi = (\psi_b)_{b \in \mathcal{B}}$ for a finite set $\mathcal{B}$. Suppose that we have a tree $T$ such that for every subtree $T|_u$ of $T$ there is exactly one $b \in \mathcal{B}$ for which $T|_u \models \psi_b$. The relabeling applied to such a tree produces a tree $\Psi(T)$ with the same domain as $T$, over the alphabet $\mathcal{B}$, where every node $u \in \text{dom}(T)$ gets labeled by that $b \in \mathcal{B}$ for which $T|_u \models \psi_b$ (for trees $T$ not satisfying the above assumption, $\Psi(T)$ is undefined).

**Theorem 3.2.** Fix a finite alphabet $\mathcal{A}$ and a maximal arity $r_{\text{max}}$. For every MSO+U formula $\varphi$ there exists a formula $\varphi_{\text{MSO}}$ of MSO, and an MSO+U relabeling $\Psi$ such that for every tree $T \in \mathcal{T}(\mathcal{A}, r_{\text{max}})$ and for every finitary valuation $\nu$ in $T$, the tree $\Psi(T)$ is defined, and it is equivalent whether

- $T, \nu \models \varphi$, and
- $\Psi(T), \nu \models \varphi_{\text{MSO}}$.

Moreover, $FV(\varphi_{\text{MSO}}) \subseteq FV(\varphi)$.

**Remark 3.3.** Both theorems above are constructive: knowing $\varphi$, $\mathcal{A}$, and $r_{\text{max}}$ one can compute either $\Omega$, or $\varphi_{\text{MSO}}$, respectively. The algorithm can be read out of our proof of existence, presented in the next sections.

**Remark 3.4.** Colcombet [8] has shown that every formula $\varphi$ of MSO can be rewritten into a formula $\varphi_{\text{FO}}$ of first-order logic referring to an MSO relabeling of a considered tree, analogously to our Theorem 3.2 (in his result, however, formulae used in the relabeling to
relabel a node \( u \) should be able to access the whole tree with the node \( u \) marked, instead of just the subtree rooted at \( u \), as in our definition. Combining our Theorem 3.2 with this result, we can deduce that every MSO+U formula \( \varphi \) can be rewritten into a formula \( \varphi_{\text{FO}} \) of first-order logic referring to an MSO+U relabeling of a considered tree (under the aforementioned extended definition of relabeling).

4 Logical types and Theorem 3.1

In this section we prove our first result, Theorem 3.1. To this end, we introduce logical types (aka. phenotypes). These types contain more information than just the truth value of a formula. In consequence, types are compositional (as stated in Lemma 4.1), unlike truth values of formulae.

In the sequel we assume that a finite alphabet \( A \) and a maximal arity \( r_{\text{max}} \) are fixed. Let \( \varphi \) be a formula of MSO+U, let \( T \) be a tree, and let \( \nu \) be a valuation. We define the \( \varphi \)-type of \( T \) under valuation \( \nu \), denoted \( [T]_{\varphi}^{\nu} \), by induction on the size of \( \varphi \) as follows:

- if \( \varphi \) is of the form \( a(X) \) (for some letter \( a \)) or \( X \subseteq Y \) then \( [T]_{\varphi}^{\nu} \) is the logical value of \( \varphi \) in \( T, \nu \), that is, \( \text{tt} \) if \( T, \nu \models \varphi \) and \( \text{ff} \) otherwise,
- if \( \varphi \) is of the form \( X \downarrow Y \), then \( [T]_{\varphi}^{\nu} \) equals
  - \( \text{tt} \) if \( T, \nu \models \varphi \),
  - \( \text{empty} \) if \( \nu(X) = \nu(Y) = \emptyset \),
  - \( \text{root} \) if \( \nu(X) = \emptyset \) and \( \nu(Y) = \{ \varepsilon \} \), and
  - \( \text{ff} \) otherwise,
- if \( \varphi \equiv (\psi_1 \land \psi_2) \), then \( [T]_{\varphi}^{\nu} = ([T]_{\psi_1}^{\nu}, [T]_{\psi_2}^{\nu}) \),
- if \( \varphi \equiv (\neg \psi) \), then \( [T]_{\varphi}^{\nu} = [T]_{\psi}^{\nu} \),
- if \( \varphi \equiv \exists X. \psi \) or \( \varphi \equiv \forall X. \psi \), then
  \[
  [T]_{\varphi}^{\nu} = \{ \sigma \mid \exists X. [T]_{\varphi}^{\nu} \mid X \asymp X = \sigma \},
  \{ \sigma \mid \forall n \in \mathbb{N}. \exists X. [T]_{\varphi}^{\nu} \mid X \asymp X = \sigma \land n \leq |X| < \infty \}\},
  \]
  
  where \( X \) ranges over sets of nodes of \( T \).

For each \( \varphi \), let \( Pht_{\varphi} \) denote the set of all potential \( \varphi \)-types. Namely, \( Pht_{\varphi} = \{ \text{tt}, \text{ff} \} \) in the first case, \( Pht_{\varphi} = \{ \text{tt}, \emptyset, \text{root}, \text{ff} \} \) in the second case, \( Pht_{\varphi} = Pht_{\psi_1} \times Pht_{\psi_2} \) in the third case, \( Pht_{\varphi} = Pht_{\psi} \) in the fourth case, and \( Pht_{\varphi} = (P(Pht_{\psi}))^2 \) in the fifth case.

The following three propositions can be shown by a straightforward induction on the structure of a considered formula.

\section*{Proposition 4.1.} For every MSO+U formula \( \varphi \) the set \( Pht_{\varphi} \) is finite.

\section*{Proposition 4.2.} For every MSO+U formula \( \varphi \) there is a function \( tv_{\varphi} : Pht_{\varphi} \rightarrow \{ \text{tt}, \text{ff} \} \) such that for every tree \( T \in \mathcal{T}(A, r_{\text{max}}) \) and every valuation \( \nu \) in \( T \), it holds that \( tv_{\varphi}([T]_{\varphi}^{\nu}) = \text{tt} \) if and only if \( T, \nu \models \varphi \).

In other words, the fact whether \( \varphi \) holds in \( T, \nu \) is determined by \( [T]_{\varphi}^{\nu} \). On the other hand, the \( \varphi \)-type can be computed by an MSO+U formula:

\section*{Proposition 4.3.} For every MSO+U formula \( \varphi \) and every \( \tau \in Pht_{\varphi} \) there is an MSO+U formula \( \psi_\tau \) with \( \text{FV}(\psi_\tau) \subseteq \text{FV}(\varphi) \) such that for every tree \( T \in \mathcal{T}(A, r_{\text{max}}) \) and every valuation \( \nu \) in \( T \), it holds that \( [T]_{\varphi}^{\nu} = \tau \) if and only if \( T, \nu \models \psi_\tau \).

Next, we observe that types behave in a compositional way, as formalized below.
Lemma 4.4. For every letter $a$, every $r \in \mathbb{N}$, and every MSO+U formula $\varphi$, one can compute a function $\text{Comp}_{a,r,\varphi}: \mathcal{P}(\text{FV}(\varphi)) \times (\text{Ph}_{\varphi})^r \rightarrow \text{Ph}_{\varphi}$ such that for every tree $T$ whose root has label $a$ and $r$ children, and for every valuation $\nu$,

$$[T]_{\varphi} = \text{Comp}_{a,r,\varphi}([X \in \text{FV}(\varphi) \mid \epsilon \in \nu(X)], [T_1]_{\varphi}^{r_1}, \ldots, [T_r]_{\varphi}^{r_r}) .$$

Proof. We proceed by induction on the size of $\varphi$.

When $\varphi$ is of the form $b(X)$ or $X \subseteq Y$, then we see that $\varphi$ holds in $T, \nu$ if and only if it holds in every subtree $T[i, \nu[i]$, and in the root of $T$. Thus, for $\varphi \equiv b(X)$ as $\text{Comp}_{a,r,\varphi}(R, \tau_1, \ldots, \tau_r)$ we take $tt$ when $\tau_i = tt$ for all $i \in \{1, \ldots, r\}$ and either $a = b$ or $X \notin R$. For $\varphi \equiv (X \subseteq Y)$ the last part of the condition is replaced by “if $X \in R$ then $Y \in R$”.

Next, suppose that $\varphi \equiv (X \downarrow k \ Y)$. Then as $\text{Comp}_{a,r,\varphi}(R, \tau_1, \ldots, \tau_r)$ we take

- $tt$ if $\tau_j = tt$ for some $j \in \{1, \ldots, r\}$, and $\tau_i = \text{empty}$ for all $i \in \{1, \ldots, r\} \setminus \{j\}$, and $X \notin R$, and $Y \notin R$,
- $tt$ also if $\tau_k = \text{root}$, and $\tau_i = \text{empty}$ for all $i \in \{1, \ldots, r\} \setminus \{k\}$, and $X \in R$, and $Y \notin R$,
- $\text{empty}$ if $\tau_i = \text{empty}$ for all $i \in \{1, \ldots, r\}$, and $X \notin R$, and $Y \notin R$,
- $\text{root}$ if $\tau_i = \text{empty}$ for all $i \in \{1, \ldots, r\}$, and $X \notin R$, and $Y \in R$, and
- $\text{ff}$ otherwise.

By comparing this definition with the definition of the type we immediately see that the thesis is satisfied.

When $\varphi \equiv (\neg \psi)$, we simply take $\text{Comp}_{a,r,\varphi} = \text{Comp}_{a,r,\psi}$, and when $\varphi \equiv (\psi_1 \land \psi_2)$, as $\text{Comp}_{a,r,\varphi}(R, (\tau_1^1, \tau_1^2), \ldots, (\tau_r^1, \tau_r^2))$ we take the pair of $\text{Comp}_{a,r,\psi_1}(R \cap \text{FV}(\varphi), (\tau_1^1, \ldots, \tau_1^2))$ for $i \in \{1, 2\}$.

Finally, suppose that $\varphi \equiv \exists X, \psi$ or $\varphi \equiv \forall X, \psi$. The arguments of $\text{Comp}_{a,r,\varphi}$ are pairs $(\tau_1, \rho_1), \ldots, (\tau_r, \rho_r)$. Let $A$ be the set of tuples $(\sigma_1, \ldots, \sigma_r) \in \tau_1 \times \cdots \times \tau_r$, and let $B$ be the set of tuples $(\sigma_1, \ldots, \sigma_r)$ such that $\sigma_j \in \rho_j$ for some $j \in \{1, \ldots, r\}$ and $\sigma_i \in \tau_i$ for all $i \in \{1, \ldots, r\} \setminus \{j\}$. As $\text{Comp}_{a,r,\varphi}(R, (\tau_1, \rho_1), \ldots, (\tau_r, \rho_r))$ we take

$$\{(\text{Comp}_{a,r,\psi}(R \cup \{X\}, \sigma_1, \ldots, \sigma_r), \text{Comp}_{a,r,\psi}(R \setminus \{X\}, \sigma_1, \ldots, \sigma_r) \mid (\sigma_1, \ldots, \sigma_r) \in A),$$
$$\{\text{Comp}_{a,r,\psi}(R \cup \{X\}, \sigma_1, \ldots, \sigma_r), \text{Comp}_{a,r,\psi}(R \setminus \{X\}, \sigma_1, \ldots, \sigma_r) \mid (\sigma_1, \ldots, \sigma_r) \in B\} .$$

The two possibilities, $R \cup \{X\}$ and $R \setminus \{X\}$, correspond to the fact that when quantifying over $X$, the root of $T$ may be either taken to the set represented by $X$ or not. The second coordinate is computed correctly due to the pigeonhole principle: if for every $n$ we have a set $X_n$ of cardinality at least $n$ (satisfying some property), then we can choose an infinite subsequence of these sets such that either the root belongs to all of them or to none of them, and one can choose some $j \in \{1, \ldots, r\}$ such that the sets contain unboundedly many descendants of $j$.

Now Theorem 3.1 follows easily:

Proof of Theorem 3.1. Let $\varphi$ be the MSO+U formula under consideration. We should define a set $\Omega$ of tuples of MSO+U formulas. To this end, consider the set $\Phi \subseteq \text{Ph}_{\varphi}$ containing those $\varphi$-types for which $\varphi$ is true, that is, $\varphi$-types $\tau$ such that $\tau \varphi(\tau) = tt$, where $\tau \varphi$ is the function defined in Proposition 1.2. Next, for every $\varphi$-type $\tau \in \Phi$, for every letter $a \in A$ (root’s label), and for every $r \in \{0, \ldots, r_{\max}\}$ (number of root’s children) consider all tuples $(R, \tau_1, \ldots, \tau_r) \in \mathcal{P}(\text{FV}(\varphi)) \times (\text{Ph}_{\varphi})^r$ such that $\text{Comp}_{a,r,\varphi}(R, \tau_1, \ldots, \tau_r) = \tau$, where $\text{Comp}_{a,r,\varphi}$ is the function defined in Lemma 4.4. For every such a tuple, we add to $\Omega$ a tuple $(\eta_a, \psi_{\tau_1}, \ldots, \psi_{\tau_r})$, where

$$\eta_a, \psi \equiv \forall Y. (a(Y) \land \bigwedge_{X \in R} (\forall Y. Y \subseteq X) \land \bigwedge_{X \in \text{FV}(\varphi) \setminus R} \neg (\forall Y. Y \subseteq X)$$
and where \( \psi_i \) are the formulae corresponding to types \( \tau_i \), as defined in Proposition \ref{prop:types}

Because there are finitely many possibilities for \( a, r \), and \( R \), and finitely many \( \varphi \)-types

(\cf Proposition \ref{prop:finite}), the set \( \Omega \) is finite.

Consider now a particular tree \( T \in \mathcal{T}(A, r_{\max}) \), and a valuation \( \nu \) in \( T \). Let \( a \) be the label of

the root of \( T \), let \( r \) be the number of root’s children, and let \( R = \{ X \in \text{FV}(\varphi) \mid \varepsilon \in \nu(X) \} \).

Moreover, let \( \tau = [T]_{\nu} \), and for \( i \in \{1, \ldots, r \} \) let \( \tau_i = [T_i]_{\nu}^{\rho_i} \). By Lemma \ref{lem:valuation}

we have that \( \tau = \text{Comp}_{a, r, \varphi}(T, \tau_1, \ldots, \tau_r) \).

Suppose first that \( T, \nu \models \varphi \). By Proposition \ref{prop:valuation} this implies that \( \tau \in \Phi \), and hence

\( (\eta_a, R, \psi_{\tau_1}, \ldots, \psi_{\tau_r}) \in \Omega \). We see that \( \text{root}(T), \text{root}(\nu) \models \eta_{a, R} \), and that \( T_i, \nu_i \models \psi_{\tau_i} \) for all \( i \in \{1, \ldots, r \} \) (by Proposition \ref{prop:types}), which gives the thesis.

Conversely, suppose that for some tuple \( (\eta_{a', R'}, \psi_{\tau'_1}, \ldots, \psi_{\tau'_r}) \in \Omega \) it is the case that

\( \text{root}(T), \text{root}(\nu) \models \eta_{a', R'} \) and \( T_i, \nu_i \models \psi_{\tau'_i} \) for all \( i \in \{1, \ldots, r \} \). We see that necessarily

\( a' = a \), \( R' = R \), and \( \tau'_i = \tau_i \) for all \( i \in \{1, \ldots, r \} \) (by Proposition \ref{prop:types}). Thus, actually

\( (\eta_a, R, \psi_{\tau_1}, \ldots, \psi_{\tau_r}) \in \Omega \), which implies that \( \tau \in \Phi \), and in consequence \( T, \nu \models \varphi \), by Proposition \ref{prop:valuation}.

\section{Proof of Theorem \ref{thm:decomposition}}

In this section we prove our second result, Theorem \ref{thm:decomposition}. Recall that our goal is to decompose

a formula \( \varphi \) into an MSO+U relabeling \( \Psi \) and an MSO formula \( \varphi_{\text{MSO}} \), assuming that free

variables of \( \varphi \) are valuated to finite sets.

The idea here is that the finite top part of a tree, where all the set variables are valuated, can be handled by MSO (intuitively: in a finite part nothing can be unbounded, so the \( U \) quantifier is void here, and hence it can be eliminated). The remaining part of the tree consists of subtrees in which all variables are valuated to empty sets; the \( \varphi \)-type of every such a subtree is fixed, so it can be precomputed and written in the label of the root of that subtree.

Again, in this section we assume that \( A \) and \( r_{\max} \) are fixed. Let \( \nu_0 \) be the empty valuation, mapping every variable to the empty set. We need formulae computing \( \varphi \)-types under the assumption that the valuation is empty.

\begin{proposition}
For every MSO+U formula \( \varphi \) and every \( \tau \in \text{Pht}_\varphi \) there is an MSO+U sentence \( \psi_\varphi^\tau \) such that for every tree \( T \in \mathcal{T}(A, r_{\max}) \) it holds that \( [T]_{\nu_0}^\tau = \tau \) if and only if \( T \models \psi_\varphi^\tau \).
\end{proposition}

\begin{proof}
By Proposition \ref{prop:types} we have a formula \( \psi_\tau \) checking that the \( \varphi \)-type is \( \tau \) under a given valuation.
We obtain \( \psi_\varphi^\tau \) from \( \psi_\tau \) by making a conjunction with statements of the form \( \text{empty}(X) \) for all free variables \( X \), and then surrounding the formula by quantifiers \( \exists X \) for all free variables \( X \).
\end{proof}

We now prove Theorem \ref{thm:decomposition}.

\begin{proof}[Proof of Theorem \ref{thm:decomposition}]
Beside of the sentences \( \psi_\varphi^\tau \) from Proposition \ref{prop:types} for every \( a \in A \) we consider a sentence \( \eta_a \) saying that the root of a tree is labeled by \( a \). As the relabeling we take

\[ \Psi = (\eta_a \land \psi_\varphi^\tau)_{(a, \tau) \in A \times \text{Pht}_\varphi}. \]

The formula \( \varphi_{\text{MSO}} \) starts with a sequence of \( [\text{Pht}_\varphi] \) existential quantifiers, quantifying over variables \( X_\tau \) for all \( \tau \in \text{Pht}_\varphi \). The intention is that, in a tree \( T \), every \( X_\tau \) represents the set of nodes \( u \) such that \( [T]_u^{\tau_\varphi} = \rho \). Inside the quantification we say that

\( \tau_\varphi \) the sets represented by these variables are disjoint, and every node belongs to some of them,
Consider now a tree $T$. If a node with label $(a,\tau_0)$ belongs to $X_\tau$, and its children belong to $X_{\tau_1},\ldots,X_{\tau_r}$, respectively (where $r \leq \max \rho$), and $R$ is the set of free variables $\phi$ for which the node belongs to $\nu(Y)$, then $\tau = \text{Comp}_{a,r,\phi}(R,\tau_1,\ldots,\tau_r)$ (there are only finitely many possibilities for $\tau,\tau_0,\tau_1,\ldots,\tau_r \in \text{Ph}_{\phi}$, for $r \in \{0,\ldots,\max \rho\}$, for $a \in A$, and finitely many free variables of $\phi$, thus the constructed formula can be just a big alternative listing all possible cases), and

- if a node with label $(a,\tau_0)$ belongs to $X_\tau$ and none of $\nu(Y)$ for $\phi$ free in $\phi$ contains this node or some its descendant, then $\tau = \tau_0$.

Consider now a tree $T \in T(A,\max \rho)$ and a valuation $\nu$ in this tree. If $[T]_{\phi} = \tau$, then we can show that $\varphi_{\text{MSO}}$ is true by taking for $X_\tau$ the set of nodes $u$ for which $[T]_{\phi} = \tau$ (for every $\tau \in \text{Ph}_{\phi}$). Conversely, suppose that $\varphi_{\text{MSO}}$ is true. Then we can prove that a node $u$ can belong to the set represented by $X_\tau$ (for $\tau \in \text{Ph}_{\phi}$) only when $[T]_{\phi} = \tau$. The proof is by a straightforward induction on the number of descendants of $u$ that belong to $\nu(Y)$ for some $\phi$ free in $\phi$; we use Lemma 4.4 for the induction step.

References

1. Achim Blumensath, Thomas Colcombet, and Christof Löding. Logical theories and compatible operations. In Jörg Flum, Erich Grädel, and Thomas Wilke, editors, *Logic and Automata: History and Perspectives [in Honor of Wolfgang Thomas]*, volume 2 of *Texts in Logic and Games*, pages 73–106. Amsterdam University Press, 2008.
2. Mikołaj Bojańczyk. Weak MSO with the unbounding quantifier. *Theory Comput. Syst.*, 48(3):554–576, 2011. doi:10.1007/s00224-010-9279-2
3. Mikołaj Bojańczyk. Weak MSO+U with path quantifiers over infinite trees. In Javier Esparza, Pierre Fraigniaud, Thore Husfeldt, and Elias Koutsoupias, editors, *Automata, Languages, and Programming - 41st International Colloquium, ICALP 2014, Copenhagen, Denmark, July 8-11, 2014, Proceedings, Part II*, volume 8573 of *Lecture Notes in Computer Science*, pages 38–49. Springer, 2014. doi:10.1007/978-3-662-43951-7_4
4. Mikołaj Bojańczyk, Paweł Parys, and Szymon Toruńczyk. The MSO+U theory of $(\mathbb{N},<)$ is undecidable. In Nicolas Ollinger and Heribert Vollmer, editors, *33rd Symposium on Theoretical Aspects of Computer Science, STACS 2016, February 17-20, 2016, Orléans, France*, volume 47 of *LIPIcs*, pages 21:1–21:8. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2016. doi:10.4230/LIPIcs.STACS.2016.21
5. Mikołaj Bojańczyk and Szymon Toruńczyk. Weak MSO+U over infinite trees. In Christoph Dürr and Thomas Wilke, editors, *29th International Symposium on Theoretical Aspects of Computer Science, STACS 2012, February 29th - March 3rd, 2012, Paris, France*, volume 14 of *LIPIcs*, pages 648–660. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2012. doi:10.4230/LIPIcs.STACS.2012.648
6. Mikołaj Bojańczyk. A bounding quantifier. In Jerzy Marcinkowski and Andrzej Tarlecki, editors, *Computer Science Logic, 18th International Workshop, CSL 2004, 13th Annual Conference of the EACSL, Karpa, Poland, September 20-24, 2004, Proceedings*, volume 3210 of *Lecture Notes in Computer Science*, pages 41–55. Springer, 2004. doi:10.1007/978-3-540-30124-0_7
7. Mikołaj Bojańczyk. Boundedness in languages of infinite words. *Logical Methods in Computer Science*, 13(4), 2017. doi:10.23638/LMCS-13(4:3)2017
8. Thomas Colcombet. A combinatorial theorem for trees. In Lars Arge, Christian Cachin, Tomasz Jurdziński, and Andrzej Tarlecki, editors, *Automata, Languages and Programming, 34th International Colloquium, ICALP 2007, Wroclaw, Poland, July 9-13, 2007, Proceedings*, volume 4596 of *Lecture Notes in Computer Science*, pages 901–912. Springer, 2007. doi:10.1007/978-3-540-73420-8_77
Compositionality of the MSO+U Logic

9 Solomon Feferman and Robert Lawson Vaught. The first order properties of products of algebraic systems. *Fundamenta Mathematicae*, 47(1):57–103, 1959. URL: [http://eudml.org/doc/213526](http://eudml.org/doc/213526).

10 Tobias Ganzow and Łukasz Kaiser. New algorithm for weak monadic second-order logic on inductive structures. In Anuj Dawar and Helmut Veith, editors, *Computer Science Logic, 24th International Workshop, CSL 2010, 19th Annual Conference of the EACSL, Brno, Czech Republic, August 23-27, 2010. Proceedings*, volume 6247 of Lecture Notes in Computer Science, pages 366–380. Springer, 2010. doi:10.1007/978-3-642-15205-4_29.

11 Hans Läuchli. A decision procedure for the weak second order theory of linear order. *Studies in Logic and the Foundations of Mathematics*, 50:189–197, 1968. doi:10.1016/S0049-237X(08)70525-1.

12 Paweł Parys. Recursion schemes and the WMSO+U logic. In Rolf Niedermeier and Brigitte Vallée, editors, *35th Symposium on Theoretical Aspects of Computer Science, STACS 2018, February 28 to March 3, 2018, Caen, France*, volume 96 of LIPIcs, pages 53:1–53:16. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2018. doi:10.4230/LIPIcs.STACS.2018.53.

13 Paweł Parys and Szymon Toruńczyk. Models of lambda-calculus and the weak MSO logic. In Jean-Marc Talbot and Laurent Regnier, editors, *25th EACSL Annual Conference on Computer Science Logic, CSL 2016, August 29 - September 1, 2016, Marseille, France*, volume 62 of LIPIcs, pages 11:1–11:12. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2016. doi:10.4230/LIPIcs.CSL.2016.11.

14 Saharon Shelah. The monadic theory of order. *Annals of Mathematics*, 102(3):379–419, 1975. doi:10.2307/1971037.