Conformal Transformation Properties of the Supercurrent in Four Dimensional Supersymmetric Theories

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Abstract

We investigate the superconformal transformation properties of Green functions with one or more insertions of the supercurrent in $N = 1$ supersymmetric quantum field theories. These Green functions are conveniently obtained by coupling the supercurrent and its trace to a classical supergravity background. We derive flat space superconformal Ward identities from diffeomorphisms and Weyl transformations on curved superspace. For the classification of potential quantum superconformal anomalies in the massless Wess-Zumino model on curved superspace a perturbative approach is pursued, using the BPHZ scheme for renormalisation. By deriving a local Callan-Symanzik equation the usual dilatational anomalies are identified and it is shown that no further superconformal anomalies involving the dynamical fields are present.

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1 Introduction

Conformal symmetry is a topic which is of interest from a multitude of points of view and which therefore has received attention repeatedly throughout the years. Of particular interest is superconformal symmetry which links conformal space-time symmetry to an internal symmetry. Supersymmetric quantum field theories provide many non-trivial examples for four dimensional theories with vanishing beta functions. Such theories are expected to be conformally invariant on flat space. Within perturbation theory, necessary and sufficient conditions for the beta functions vanishing to all orders in supersymmetric gauge theories have been given and used for the construction of finite theories in which the beta functions vanish identically \[1\].

Within the context of electric-magnetic duality, there are examples for \(N = 1\) supersymmetric gauge theories with non-trivial renormalisation group fixed points \[2\]. These theories may be used for testing conjectured extensions of the Zamolodchikov C theorem \[3\] to four dimensions. Recently several examples of \(N = 1\) supersymmetric Yang-Mills theories have been explored \[4\] in which the coefficient of the topological Euler density is larger in the UV than in the IR limit, in agreement with a possible four dimensional analogue of the C theorem.

In this paper we consider supersymmetric theories away from the fixed points, where anomalies involving the dynamical fields may be present. It is appropriate to use the superfield formalism in which supersymmetry is manifest. We study the transformation properties of the supercurrent under superconformal transformations. The supercurrent is an axial vector superfield which has the \(R\) and supersymmetry currents among its components, as well as the energy momentum tensor. Its space-time moments and derivatives yield all the currents of the superconformal group. The superconformal transformation properties of Green functions for elementary fields are well-known \[5\] and require considering Green functions with one insertion of the supercurrent only. Here however, we derive the transformation properties of Green functions with multiple insertions of the supercurrent. In analogy to the original proof of the C theorem in two dimensions, these are expected to be relevant for a possible proof of a C theorem for four dimensional supersymmetric theories.

Multiple insertions of the supercurrent are conveniently generated by coupling the quantum field theory considered to a classical curved superspace background. The constraints on torsion and curvature are chosen such as to obtain minimal supergravity with a real vector superfield prepotential, to which the supercurrent couples, and with a chiral compensator. Superconformal Ward identities are obtained by combining diffeomorphisms and Weyl transformations on curved superspace. For definiteness we consider the massless Wess-Zumino model, which is the simplest example for a classically superconformal theory. We expect to extend our results to gauge theories in the future.

For the quantisation of the dynamical fields we adopt a perturbative approach analogous
to the method employed in [6, 7, 8] in a related study of Green functions with double insertions of the energy momentum tensor in scalar $\phi^4$ theory. This approach ensures an off-shell formulation at every stage of the analysis and yields scheme-independent results which hold to all orders in perturbation theory. Thus all potential anomalies are systematically investigated. At the operator level, the Ward identities for Green functions with multiple insertions of the supercurrent constitute the current algebra for all currents of the superconformal group including its anomalies.

In section 2 of this paper we review the elements of classical supergravity which are necessary for our analysis [9, 10]. Furthermore we discuss the superconformal transformation properties of the supergravity fields. In section 3 we discuss superconformal Ward identities for classical $N = 1$ supersymmetric field theories on curved superspace and apply them in particular to the Wess-Zumino model. We show how the Ward identities derived on flat space in [5] (for a review see [11]) follow naturally from the superconformal transformations on curved superspace discussed in section 2. These Ward identities lead to the definition of the supercurrent and of its trace. In section 4 we quantise the Wess-Zumino model on curved superspace using the BPHZ scheme for renormalisation. Green functions with insertions of the supercurrent are defined. The conditions for the validity of a Callan-Symanzik equation are given. With the help of symmetry consistency conditions [12] we are able to show that symmetry breaking terms of matter-background coupling type may be removed in agreement with renormalisability. Furthermore we derive a local Callan-Symanzik equation which determines the anomaly structure of the full superconformal group. In section 5 we discuss the transformation properties of Green functions with supercurrent insertions which follow from the the local Callan-Symanzik equation. Section 6 contains a discussion as well as some concluding remarks. Some longer formulae are relegated to the appendix.

2 Superconformal Transformations in Curved Superspace

We begin this section by presenting a few well-known results of classical supergravity and by introducing our conventions. Subsequently these facts enable us to discuss diffeomorphisms and Weyl transformations in curved superspace. The combination of these two transformations is shown to reduce to the superconformal transformations when restricting to flat superspace.

We consider the superspace manifold $\mathbb{R}^{4|4}$. At each point $z = (z^M) = (x^m, \theta^\mu, \bar{\theta}^\dot{\mu}) \in \mathbb{R}^{4|4}$, the differential operators $(\partial_M) = (\partial_m, \partial_\mu, \bar{\partial}_{\dot{\mu}})$ span the tangent space. A supervierbein is a set of eight vector fields

\[(\hat{E}_A) = (\hat{E}_a, \hat{E}_\alpha, \hat{E}^{\dot{\alpha}}),\] (2.1)
such that the $\tilde{E}_A(z)$ constitute a basis of the tangent space at each point $z$. The supervierbein can be regarded as the gauge covariant derivative for the gauge group of diffeomorphisms in superspace. We require supervierbeins related to each other by a superlocal Lorentz transformation to be physically equivalent, such that the gauge group is extended to the direct product of the group of diffeomorphisms and the superlocal Lorentz group with generators $M_{\alpha\beta}$, $\bar{M}_{\dot{\alpha}\dot{\beta}}$ acting in tangent space. The extended gauge group has an additional gauge field $\tilde{\omega}^{\alpha\beta}_A$, such that the gauge covariant derivative is given by

$$\tilde{D}_A = \tilde{E}_A + \tilde{\omega}_A. \quad (2.2)$$

Under diffeomorphisms $K$ and Lorentz transformations $M = K^{\alpha\beta}M_{\alpha\beta} + \bar{K}^{\dot{\alpha}\dot{\beta}}\bar{M}_{\dot{\alpha}\dot{\beta}}$ these derivatives change in the following way:

$$\tilde{D}_A \rightarrow \tilde{D}'_A = e^K \tilde{D}_A e^{-K}, \quad \tilde{D}_A \rightarrow \tilde{D}'_A = e^M \tilde{D}_A e^{-M}. \quad (2.3)$$

Torsion and curvature of superspace are obtained from the graded commutator of covariant derivatives,

$$[[\tilde{D}_A, \tilde{D}_B]] = T_{AB}^C \tilde{D}_C + R_{AB}^{\alpha\beta}M_{\alpha\beta} + R_{AB}^{\dot{\alpha}\dot{\beta}}\bar{M}_{\dot{\alpha}\dot{\beta}}. \quad (2.4)$$

So far we have used the operators $\partial_A$ as a basis in tangent space. An alternative basis is given by the flat covariant derivatives

$$(D_A) = (D_a, D_\alpha, \bar{D}_{\dot{\alpha}}), \quad (2.5)$$

$$D_a = \partial_a, \quad D_\alpha = \partial_\alpha - i\sigma^a_{\alpha\dot{\alpha}}\bar{D}_{\dot{\alpha}} + i\theta^a_\alpha\sigma^a_\alpha \partial_a.$$ 

The spinor covariant derivatives satisfy the anticommutation relation

$$\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = 2i\sigma^b_{\alpha\dot{\alpha}} \partial_b. \quad (2.6)$$

We may also define $D_{\alpha\dot{\alpha}} = \frac{1}{2i}(D_\alpha, \bar{D}_{\dot{\alpha}})$ which replaces $D_a$. Using this basis, the vierbein is given by

$$\tilde{E}_A = \tilde{E}_A^M D_M = \tilde{E}_A^\mu \partial_\mu + \tilde{E}_A^\mu D_\mu + \tilde{E}_A^{\dot{\mu}} \bar{D}_{\dot{\mu}}.$$

In order to be able to introduce covariantly chiral scalar superfields, it is necessary to impose constraints on the torsion and curvature components. In this paper we impose the torsion constraints of conformal supergravity and the additional condition $T_\alpha = 0$, $T_\alpha = T_{ab}^{\alpha} - T_{\alpha\beta}^{\beta} - T_{\alpha\dot{\beta}}^{\dot{\beta}}$ with $T_{AB}^{C}$ defined in (2.4), such that the superconformal group is the maximal symmetry group. Some of these constraints determine $\omega$ in terms of the vierbein, while the remaining can be solved by introducing prepotentials $W$ and $F$ such that the covariant derivatives may be expressed by

$$\tilde{D}_A = e^{W}(FD_A) e^{-W} + \tilde{\omega}_A, \quad \tilde{D}_\alpha = e^{W}(F\bar{D}_\alpha) e^{-W} + \tilde{\omega}_\alpha, \quad \tilde{D}_a = -\frac{i}{4}\sigma^a_{\alpha\dot{\alpha}}\{\tilde{D}_\alpha, \bar{D}_{\dot{\alpha}}\}, \quad W = W^A D_A + W^{\alpha\beta}M_{\alpha\beta} + W^{\dot{\alpha}\dot{\beta}}\bar{M}_{\dot{\alpha}\dot{\beta}}. \quad (2.7)$$
where \( W^{\alpha\beta} \) and \( W^{\dot{\alpha}\dot{\beta}} \) are symmetric, and \( W^A, W^{\alpha\beta}, W^{\dot{\alpha}\dot{\beta}} \) and \( F \) are complex superfields. \( F \) is constrained by

\[
\bar{E}_\dot{\alpha} \left( \text{sdet}(\bar{E}_A^M) F^2 (1 \cdot e^{\bar{W}}) \right) = 0. \tag{2.8}
\]

The prepotentials determining the covariant derivatives in (2.7) are not unique. There is some arbitrariness in \( W \) and \( F \) which manifests itself in the so-called \( \Lambda \) group of transformations. This group is the group of transformations \( \Lambda \) which leave the covariant derivatives invariant,

\[
e^\Lambda \tilde{D}_A e^{-\Lambda} = \tilde{D}_A, \quad \Lambda = \Lambda^A D_A + \Lambda^{\alpha\beta} M_{\alpha\beta} + \Lambda^{\dot{\alpha}\dot{\beta}} M_{\dot{\alpha}\dot{\beta}}. \tag{2.9}
\]

Thus the gauge group is enlarged further and consists of diffeomorphisms, superlocal Lorentz transformations and \( \Lambda \) transformations. This situation can be simplified by using the chiral representation

\[
\mathcal{D}_a = e^{-\bar{W}} \bar{D}_a e^W, \quad \bar{\mathcal{D}}_{\dot{\alpha}} = e^{-\bar{W}} \bar{D}_{\dot{\alpha}} e^W, \quad A = e^{-\bar{W}} \bar{A}. \tag{2.10}
\]

The chiral scalar field \( A \) plays the role of the matter field in the following sections. This representation is analogous to the flat superspace chiral representation in which all fields \( \tilde{\Phi} \) are replaced by \( \Phi = e^{i\theta_\sigma \bar{\theta} a} \tilde{\Phi} \). Furthermore we define the operator \( H \) by

\[
e^{2iH} = e^{-W} e^W, \tag{2.11}
\]

where \( H = H^A D_A + H^{\alpha\beta} M_{\alpha\beta} + H^{\dot{\alpha}\dot{\beta}} M_{\dot{\alpha}\dot{\beta}} \) is a real superfield. As in the flat space chiral representation, the complex conjugate of an expression in the chiral representation belongs to the antichiral representation. Whenever quantities in the chiral and antichiral representation are combined into the same expression, a representation changing factor \( e^{2iH} \) has to be included. For example, \( \tilde{A} \bar{A} \) becomes \( A e^{2iH} \bar{A} \) in the chiral representation. It should also be noted that in our conventions \( \bar{D}_{\dot{\alpha}} \) is in the chiral and \( \mathcal{D}_a \) is in the antichiral representation.

In the chiral representation the \( K \) and \( M \) transformations are eliminated and the equations (2.3) are replaced by

\[
\mathcal{D}_A \longrightarrow e^\Lambda \mathcal{D}_A e^{-\Lambda}, \quad e^{2iH} \longrightarrow e^\Lambda e^{2iH} e^{-\Lambda}, \tag{2.12}
\]

where

\[
\Lambda = \Lambda^A D_A + \Lambda^{\alpha\beta} M_{\alpha\beta} + \Lambda^{\dot{\alpha}\dot{\beta}} M_{\dot{\alpha}\dot{\beta}}, \quad \bar{D}_{\dot{\alpha}} \Lambda^{\alpha\dot{\alpha}} = 4i \Lambda^{\alpha\delta} \delta^{\dot{\alpha}}_{\dot{\beta}}, \quad \bar{D}_{\dot{\beta}} \Lambda^{\dot{\alpha}} = 0, \quad \Lambda_{\dot{\alpha}\dot{\beta}} = -\bar{D}_{(\dot{\alpha}} \Lambda_{\dot{\beta})}. \tag{2.13}
\]

In the chiral representation the \( \Lambda \) transformations play thus the role of (restricted) diffeomorphisms and Lorentz transformations. Moreover it should be noted that \( \Lambda \) is complex. Its conjugate \( \bar{\Lambda} \) is constrained by relations which are obtained from (2.13) by complex conjugation.
Some of the $\Lambda$ transformations may be used to restrict $H$ to the simpler form

$$ H = H^{\alpha\dot{\alpha}} D_{\alpha\dot{\alpha}}, \hspace{1cm} (2.14) $$

which imposes the conditions

$$ \Lambda_{\dot{\alpha}} = e^{2iH} \bar{\Lambda}_{\dot{\alpha}}, \hspace{5mm} \Lambda_{\alpha\beta} = e^{2iH} \bar{\Lambda}_{\alpha\beta}. \hspace{1cm} (2.15) $$

The Lorentz generators $M_{\alpha\beta}$ and $\bar{M}_{\dot{\alpha}\dot{\beta}}$ act only on fields which carry spinor indices. Since we are not going to consider such fields, we set them to zero from now on. The $\Lambda$ transformations are then simply

$$ \Lambda = \Lambda^{\alpha\dot{\alpha}} D_{\alpha\dot{\alpha}} + \Lambda^\alpha D_{\alpha} + \Lambda_{\dot{\alpha}} \bar{D}^{\dot{\alpha}}. \hspace{1cm} (2.16) $$

The restrictions (2.13) may be solved in terms of a complex superfield $\Omega^\alpha$,

$$ \Lambda^{\alpha\dot{\alpha}} = i \bar{D}^{\dot{\alpha}} \Omega^\alpha, \hspace{5mm} \Lambda^\alpha = \frac{1}{4} \bar{D}^2 \Omega^\alpha. \hspace{1cm} (2.17) $$

Equation (2.17) defines $\Omega^\alpha$ uniquely up to a chiral field. Similarly $\bar{\Lambda}^{\alpha\dot{\alpha}}, \bar{\Lambda}_{\dot{\alpha}}$ are determined in terms of $\bar{\Omega}^{\dot{\alpha}}$. (2.13) and its conjugate ensure that $\Lambda_{\dot{\alpha}}$ is a function of $\bar{\Omega}_{\dot{\alpha}}$ and $\bar{\Lambda}^\alpha$ is a function of $\Omega^\alpha$.

An important quantity for the construction of invariant actions is the determinant of the supervierbein, $E \equiv \text{sdet}(E_A^M)$. Constraint (2.8) now simply reads

$$ \bar{D}_{\dot{\alpha}} (E\bar{F}^2) = 0. $$

This can easily be solved by setting

$$ E\bar{F}^2 = \phi^{-3}, \hspace{1cm} (2.18) $$

where the chiral compensator $\phi$ satisfies $\bar{D}_{\dot{\alpha}} \phi = 0$.

Furthermore there are super Weyl transformations $\sigma, \bar{\sigma}$. If $\bar{D}_{\dot{\alpha}} \sigma = 0$ and $D_\alpha \bar{\sigma} = 0$, the Weyl transformations leave the (anti-) commutation relations of the covariant derivatives unchanged and respect the constraints on torsion and curvature. For the Weyl transformation properties of the prepotentials we have

$$ H \longrightarrow H, \hspace{5mm} \phi \longrightarrow e^{\sigma} \phi. \hspace{1cm} (2.19) $$

For our subsequent investigations it is crucial to note that $H$ is a Weyl invariant. The transformation laws of the prepotentials and of a chiral matter field $A$ under $\Lambda$ and super Weyl transformations are summarised in appendix A.1.

With the help of the prepotential $\bar{F}$ we may define the supersymmetric curvature scalar $R$ which in the chiral representation is given by

$$ R = \bar{D}^2 \bar{F}^2. \hspace{1cm} (2.20) $$
The transformation properties of $\bar{F}$ and $F$ imply that $R$ and $\bar{R}$ transform homogeneously under $\Lambda$ transformations. The Weyl transformation properties of $R$, $\bar{R}$ are given by

$$
\delta_\sigma R = -2\sigma R, \quad \delta_\sigma \bar{R} = (\bar{D}^2 + R)\bar{\sigma}, \quad \delta_\sigma \bar{R} = -2\bar{\sigma} \bar{R}, \quad \delta_\sigma \bar{R} = (D^2 + \bar{R})\sigma.
$$

Moreover, $R$ and $\bar{R}$ are essential for defining a chiral (antichiral) integration measure by

$$
\int d^8z E^{-1} = \int d^6z \phi^3 (\bar{D}^2 + R) = \int d^6\bar{z} \bar{\phi}^3 (D^2 + \bar{R}) e^{-2iH}.
$$

The factor $e^{-2iH}$ ensures the change from the chiral to the antichiral representation.

We proceed by defining infinitesimal transformations which combine diffeomorphism ($\Lambda$) and Weyl transformations. The infinitesimal form of diffeomorphisms and Weyl transformations is obtained by expanding the expressions of appendix A.1. For the fields $H_{\alpha\dot{\alpha}}$, $\phi$, and $A$, the combined infinitesimal transformations are given by

$$
\delta H_{\alpha\dot{\alpha}} = \delta_{\Lambda} H_{\alpha\dot{\alpha}}, \quad \delta \phi = \delta_{\Lambda} \phi + \delta_\sigma \phi, \quad \delta A = \delta_{\Lambda} A + \delta_\sigma A,
$$

where

$$
2i\delta_{\Lambda} H = \Lambda - \bar{\Lambda} - \frac{1}{2} [2iH, \Lambda + \bar{\Lambda}] + \frac{1}{12} [2iH, [2iH, \Lambda - \bar{\Lambda}]] + O(H^3),
$$

$$
\delta_{\Lambda} \phi = \Lambda \phi + \frac{1}{3} (D^{\alpha\dot{\alpha}} \Lambda_{\alpha\dot{\alpha}} + D^\alpha \Lambda_\alpha) \phi, \quad \delta_{\Lambda} A = \Lambda A,
$$

$$
\delta_\sigma \phi = \sigma \phi, \quad \delta_\sigma A = -\sigma A.
$$

We note that the $\Lambda$ transformation properties of $H$ are analogous to the gauge transformations of a non-abelian gauge field. For later convenience it is useful to define a chiral dimensionless field $J$ by

$$
\phi \equiv e^J = 1 + J + \frac{1}{2} J^2 + \ldots,
$$

such that flat superspace is characterised by vanishing external fields $H = 0$, $J = 0$ rather than by $H = 0$, $\phi = 1$. From its definition and the transformation properties of $\phi$ the infinitesimal transformations of $J$ are given by

$$
\delta J = \Lambda J + \frac{1}{3} (D^{\alpha\dot{\alpha}} \Lambda_{\alpha\dot{\alpha}} + D^\alpha \Lambda_\alpha) + \sigma.
$$

To lowest order in an expansion around flat space, $H$ and $J$ remain invariant if

$$
\Lambda = \bar{\Lambda}, \quad \sigma = -\frac{1}{3} (D^{\alpha\dot{\alpha}} \Lambda_{\alpha\dot{\alpha}} + D^\alpha \Lambda_\alpha) = -\frac{1}{12} \bar{D}^2 D^\alpha \Omega_\alpha.
$$

Together with the constraints (2.13, 2.15), the condition (2.28a) is a necessary and sufficient condition for $\Omega^a$ to be restricted to the superconformal coordinate transformations.
The solutions to (2.28a) are listed in appendix A.2. To obtain the superconformal transformation properties of fields on flat space, it is necessary to combine their infinitesimal diffeomorphism and Weyl transformations and to impose both (2.28a) and (2.28b).

It should be noted that on curved space the use of (2.28a) is not mandatory even if the Weyl transformations are restricted by (2.28b). Throughout this paper (2.28a) is imposed only when the flat space limit is taken. To obtain the combined diffeomorphism and Weyl transformations of the fields $H, J, A$ in an expansion around flat space, it is convenient to impose (2.28b) even on curved space in view of expressing the combined transformations with the superfields $\Omega, \bar{\Omega}$ as their only parameters. In this form the infinitesimal combined transformations are listed in appendix A.3.

3 Classical Theory

3.1 Ward Identities and Supercurrent

The superconformal group is generated by translations, supersymmetry, $R$ and Lorentz transformations, dilatations, special conformal and special supersymmetry transformations. The action of the generators $G$ belonging to these transformations on a chiral field $A$ is given by

$$[iG, A(z)] = \delta^G A(z).$$  \hspace{1cm} (3.1)

The explicit form of the generators and the corresponding infinitesimal transformations belong to the standard results of supersymmetry theory and may be found in textbooks [11]. The generators $R, Q_\alpha, \bar{Q}^\dot{\alpha}$ and $P^a$ of $R$ and supersymmetry transformations and of translations form a subalgebra of the full superconformal algebra. The currents associated with these four generators form a supermultiplet, the supercurrent.

The flat space superconformal transformation properties of a flat space field theory with chiral and antichiral superfields $A, \bar{A}$ are characterised by the local Ward identity

$$W_\Omega \hat{\Gamma} = -i \int d^6z \frac{\delta_\Omega A}{\delta A} \frac{\delta \hat{\Gamma}}{\delta A} - i \int d^8\bar{z} \frac{\delta_{\bar{\Omega}} \bar{A}}{\delta \bar{A}} \frac{\delta \hat{\Gamma}}{\delta \bar{A}},$$  \hspace{1cm} (3.2)

where $\Omega$ is defined in (2.17), $\bar{\Omega}$ is its conjugate field and $\hat{\Gamma}$ the classical action. From the infinitesimal conformal transformation $\delta A$ in (2.23) we have in the flat space case using (2.28b)

$$\delta_\Omega A = \bar{D}^2 \left( \frac{1}{4} \Omega^\alpha D_\alpha A + \frac{1}{12} D^\alpha \Omega_\alpha A \right), \hspace{1cm} \delta_{\bar{\Omega}} \bar{A} = D^2 \left( \frac{1}{4} \bar{\Omega}^\dot{\alpha} \bar{D}^{\dot{\alpha}} \bar{A} + \frac{1}{12} \bar{D}_\dot{\alpha} \bar{\Omega}^{\dot{\alpha}} \bar{A} \right),$$  \hspace{1cm} (3.3)

and thus

$$W_\Omega \hat{\Gamma} = -i \int d^8z \left( \Omega^\alpha w_\alpha + \bar{\Omega}^{\dot{\alpha}} \bar{w}^{\dot{\alpha}} \right) \hat{\Gamma}$$  \hspace{1cm} (3.4)
with
\[ w_\alpha = \frac{1}{4} D_\alpha A \frac{\delta}{\delta A} - \frac{1}{12} D_\alpha \left( A \frac{\delta}{\delta A} \right), \quad \bar{w}_\dot{\alpha} = \frac{1}{4} \bar{D}_\dot{\alpha} \bar{A} \frac{\delta}{\delta A} - \frac{1}{12} \bar{D}_\dot{\alpha} \left( \bar{A} \frac{\delta}{\delta A} \right). \tag{3.5} \]

By comparing with the superconformal transformation law (3.1) we may identify the components of \( \Omega \) which lead to the different superconformal transformations. These are listed in appendix A.2.

From appendix A.2 we read off that for the \( R, Q, \bar{Q} \) and \( P \) transformations we may write
\[ \Omega_\alpha = D_\alpha \omega, \quad \bar{\Omega}_{\dot{\alpha}} = -\bar{D}_{\dot{\alpha}} \omega \tag{3.6} \]
with \( \omega \) an imaginary superfunction given by
\[ \omega = \frac{1}{2} i \theta \sigma^\mu t - \bar{\theta}^2 \theta \cdot q + \theta^2 \bar{\theta} \cdot \bar{q} - \frac{1}{7} i \theta^2 \bar{\theta}^2 r \tag{3.7} \]
\[ = -8 (i r + q^\alpha \partial_\alpha + \bar{q}_{\dot{\alpha}} \bar{\partial}^\dot{\alpha} + \frac{1}{4} i \sigma^a_{\alpha \dot{\alpha}} t_a \bar{\partial}^a \bar{\partial}^\dot{\alpha}) \delta^{(2)}(\theta) \delta^{(2)}(\bar{\theta}), \tag{3.8} \]
with \( \delta^{(2)} \) the delta function. Thus for the \( R, Q, \bar{Q}, P \) transformations we have
\[ -i \int d^8 z \left( \Omega^{X} w_\alpha + \bar{\Omega}^{\bar{X}} \bar{w}_{\dot{\alpha}} \right) \hat{\Gamma} = i \int d^8 z \omega \left( D^\alpha w_\alpha - D_{\dot{\alpha}} \bar{w}_{\dot{\alpha}} \right) \hat{\Gamma} \]
\[ \equiv i \int d^8 z \omega w \hat{\Gamma}, \tag{3.9} \]
for which we find, using the expression (3.8) for \( \omega \), the component decomposition
\[ i \int d^8 z \omega w = r W^R + q^\alpha W^Q_\alpha + \bar{q}_{\dot{\alpha}} W^{\bar{Q}\dot{\alpha}} + t^a W^P_a \tag{3.10} \]
with
\[ W^R \hat{\Gamma} = 8 \int d^4 x w \hat{\Gamma} \bigg|_{\theta=\bar{\theta}=0}, \quad W^Q_\alpha \hat{\Gamma} = 8 i \partial_\alpha \int d^4 x w \hat{\Gamma} \bigg|_{\theta=\bar{\theta}=0}, \]
\[ W^{\bar{Q}\dot{\alpha}} \hat{\Gamma} = 8 i \bar{\partial}^\dot{\alpha} \int d^4 x w \hat{\Gamma} \bigg|_{\theta=\bar{\theta}=0}, \quad W^P_\mu \hat{\Gamma} = 2 \sigma^a_{\alpha \dot{\alpha}} \partial^\mu \bar{\partial}^\dot{\alpha} \int d^4 x w \hat{\Gamma} \bigg|_{\theta=\bar{\theta}=0}. \tag{3.11} \]

Then the operator
\[ \hat{W}^R = 8 \int d^4 x w \tag{3.12} \]
is given by
\[ \hat{W}^R = W^R - i \theta W^Q + i \bar{\theta} W^{\bar{Q}} - 2 \theta \sigma^\mu \bar{\theta} W^P_\mu. \tag{3.13} \]
For a superconformal theory for which
\[ \hat{W}^R \hat{\Gamma} = 0 , \] (3.14)
the integrand is a total divergence
\[ w \hat{\Gamma} = -i\frac{1}{8} \partial^\mu V_\mu , \] (3.15)
with \( V_\mu \) a (axial-)vector superfield. Allowing also for symmetry breaking terms, \( \hat{W} \) may be decomposed into a chiral and an antichiral equation using (3.9),
\[ -16 w_\alpha \hat{\Gamma} = \bar{D} \dot{\alpha} V_\dot{\alpha} + 2 D_\alpha S - B_\alpha , \]
\[ -16 \bar{w}_\dot{\alpha} \hat{\Gamma} = D^\alpha V_\alpha - 2 \bar{D}_a \bar{S} - \bar{B}_a , \] (3.16)
with \( B_\alpha \) satisfying the constraint
\[ D^\alpha B_\alpha - \bar{D} \dot{\alpha} \bar{B} = 0 . \] (3.17)

(3.16) is referred to as the trace identity. It leads to the Ward identity
\[ \partial^\alpha V_\alpha = 8iw \hat{\Gamma} + i(D^2 S - \bar{D}^2 \bar{S}) . \] (3.18)

Among the components of the supercurrent there are the \( R \) and supersymmetry currents as well as the energy momentum tensor. However, as is well known \([5],[11]\), these currents are not conserved if symmetry breaking terms of both S and B type are present simultaneously. The construction of conserved currents differs according to which type of breaking is present.

In massless theories, the S and B formulations are both possible. In massive theories, only the S formulation is consistent since mass terms are of this type. Since we would like to understand massless theories as the well-defined limit of massive theories, we consider only S type breaking terms in the following. The trace equations then read
\[ -16 w_\alpha \hat{\Gamma} = \bar{D}^\alpha V_{\alpha \dot{\alpha}} + 2 D_\alpha S , \]
\[ -16 \bar{w}_\dot{\alpha} \hat{\Gamma} = D^\alpha V_{\alpha \dot{\alpha}} + 2 \bar{D}_a \bar{S} , \] (3.19)
and the identification of \( \theta \)-components of \( V_\alpha \) with component currents is such that the superfields
\[ \hat{R}_a = V_\alpha , \]
\[ \hat{Q}_{\alpha \dot{\alpha}} = i (D_\alpha V_\alpha - (\sigma_a \bar{\sigma}_b)_{\alpha \beta} D_\beta V_b) \]
\[ \hat{T}_{ab} = -\frac{1}{4} (V_{ab} + V_{ba}) + \frac{1}{2} g_{ab} V_\lambda ^\lambda \]
\[ V_{ab} = \frac{1}{4} (D \sigma_a \bar{D} - \bar{D} \sigma_a D) V_b , \] (3.20)
have as lowest \( \theta \)-components just the current whose name they bear. Their respective Ward identities have the form
\[ \partial^\alpha \hat{R}_a = i(D^2 S - \bar{D}^2 \bar{S}) + 8iw \hat{\Gamma} \]
\[ \partial^\alpha \hat{Q}_{\alpha \dot{\alpha}} = 8iw_\alpha \hat{Q} \hat{\Gamma} \]
\[ \partial^\alpha \hat{T}_{ab} = 8iw_b \hat{T} \hat{\Gamma} , \] (3.21)
where (see \cite{13})

\[ w^Q_\alpha = iD_\alpha (D^\beta w_\beta - \bar{D}_\beta \bar{w}^\beta) - 4i\sigma^a_{\alpha\dot{\alpha}} \partial_a \bar{w}^{\dot{\alpha}} \]

\[ w^P_\alpha = -\frac{1}{16} \sigma^a_{\alpha\dot{\alpha}} \left( D^2 \bar{D}_a w_\alpha + \bar{D}^2 D_a \bar{w}^{\dot{\alpha}} + [D_\alpha, \bar{D}_{\dot{\alpha}}](D^\beta w_\beta - \bar{D}_\beta \bar{w}^\beta) \right) \]

\[ + \frac{1}{2} i \partial_b (D^\beta w_\beta + \bar{D}_\beta \bar{w}^\beta). \]  

\hfill (3.22)

For the $R$ current the breakdown of conformal symmetry manifests itself in the non-conservation of the current (3.18) in the presence of the multiplet $S$, whereas for the supersymmetry current and the energy momentum tensor it leads to trace contributions of the form

\[ (\hat{Q} a \sigma^a)_{\dot{\alpha}} = 12 i \bar{D}_a \bar{S} + 96 i \bar{w}_a \hat{\Gamma}, \]

\[ \hat{T}_a^a = -\frac{3}{2} \left( D^2 S + \bar{D}^2 \bar{S} \right) + 12 \bar{w} \hat{\Gamma}. \]  

\hfill (3.23)

The currents corresponding to the remaining symmetries are given by moments of the $\hat{Q}$ and $\hat{T}$ currents \cite{11}.

\[ \delta \hat{\Gamma}[\alpha, \bar{\alpha}] = \frac{1}{16} \int d^8 z A \bar{A} + \frac{1}{48} g \int d^6 z A^3 + \frac{1}{48} g \int d^6 \bar{z} \bar{A}^3 \]

\[ + \frac{1}{8} m \int d^6 z A^2 + \frac{1}{8} m \int d^6 \bar{z} \bar{A}^2 \]  

\hfill (3.24)

the Ward identity (3.19) yields

\[ V_{\alpha\dot{\alpha}} = -\frac{1}{6} (D_\alpha A \bar{D}_{\dot{\alpha}} \bar{A} - AD_\alpha \bar{D}_{\dot{\alpha}} \bar{A} + \bar{A} \bar{D}_{\dot{\alpha}} D_\alpha A), \]  

\[ S = -\frac{1}{12} m A^2 \]  

\hfill (3.25) \hfill (3.26)

for the supercurrent $V_{\alpha\dot{\alpha}}$ and the breaking term $S$. We observe that the massless Wess-Zumino model is invariant under superconformal transformations since the breaking term is proportional to the mass.

For constructing Green functions with multiple insertions of the supercurrent it is convenient to introduce the external supergravity field $H^{\alpha\dot{\alpha}}$ conjugate to the supercurrent $V_{\alpha\dot{\alpha}}$,

\[ \frac{\delta \hat{\Gamma}_{\text{ext}}}{\delta H^{\alpha\dot{\alpha}}} = \frac{1}{8} V_{\alpha\dot{\alpha}}, \]

\hfill (3.27)
such that to lowest order in \( H^{\alpha \dot{\alpha}} \) we have
\[
\Gamma = \hat{\Gamma} + \Gamma_{\text{ext}} = \hat{\Gamma} + \frac{i}{8} \int d^8 z H^{\alpha \dot{\alpha}} V_{\alpha \dot{\alpha}}. \tag{3.28}
\]
According to the Noether procedure the action \( \Gamma \) is given to all order in \( H^{\alpha \dot{\alpha}} \) by the most general diffeomorphism invariant action which on flat space reduces to \( (3.24) \). Using the results of the preceding section we find that this action is given by
\[
\Gamma[A, H^{\alpha \dot{\alpha}}, \phi] = \Gamma_0[A, H^{\alpha \dot{\alpha}}, \phi] + \Gamma_S[A, H^{\alpha \dot{\alpha}}, \phi] \tag{3.29}
\]
\[
\Gamma_0[A, H^{\alpha \dot{\alpha}}, \phi] = \frac{1}{16} \int d^8 z \left( E - \frac{1}{2} A e^{2iH} \bar{A} + \frac{1}{48} g \int d^6 \bar{z} \phi^3 \bar{A}^3 \right)
\]
\[
\Gamma_S[A, H^{\alpha \dot{\alpha}}, \phi] = \frac{1}{8} m \int d^6 z \phi^3 A^2 + \frac{1}{8} m \int d^6 z \phi^3 \bar{A}^2 + \frac{1}{8} \xi \int d^6 z \phi^3 R A^2 + \frac{1}{8} \xi \int d^6 z \phi^3 \bar{R} \bar{A}^2.
\]
This action is invariant under \( \Lambda \) transformations as given in appendix A.1. Infinitesimally this diffeomorphism invariance is expressed by the equation
\[
(w^{(A)}(H) + w^{(A)}(J) + w^{(A)}(A)) \Gamma = 0, \tag{3.30}
\]
where the operators \( w^{(A)}(H) \) and \( w^{(A)}(J) \) are defined by
\[
\int d^8 z \delta H^{\alpha \dot{\alpha}} \frac{\delta \Gamma}{\delta H^{\alpha \dot{\alpha}}} = \int d^8 z \Omega^x w^{(A)}(H) \Gamma + \text{c.c.}, \quad \int d^8 z \delta A J \frac{\delta \Gamma}{\delta J} = \int d^8 z \Omega^x w^{(A)}(J) \Gamma,
\]
their explicit form being given in appendix A.4 to second order in \( H^{\alpha \dot{\alpha}} \). The Ward identity \( (3.30) \) holds order by order in the external fields \( H^{\alpha \dot{\alpha}} \) and \( J \), which can be checked explicitly. For this purpose the expansion of \( \Gamma \) to second order in the external fields is given in appendix A.5.

Similarly for the Weyl transformations
\[
\int d^6 z \left( \delta_\sigma A \frac{\delta \Gamma}{\delta A} + \delta_\sigma \phi \frac{\delta \Gamma}{\delta \phi} \right) = \int d^6 z \sigma w^{(\sigma)}(A, J) \Gamma \tag{3.32}
\]
yields
\[
w^{(\sigma)}(A, \phi) = \phi \frac{\delta}{\delta \phi} - A \frac{\delta}{\delta A} \tag{3.33}
\]
or similarly
\[
w^{(\sigma)}(A, J) = \frac{\delta}{\delta J} - A \frac{\delta}{\delta A}, \tag{3.34}
\]
which when acting on \( \Gamma \) yields the breaking term
\[
w^{(\sigma)}(A, J) \Gamma = -\frac{3}{2} S, \quad S = -\frac{1}{16} \left( m e^{3J} A^2 - \xi e^{3J} R A^2 + \xi e^{3J} (\bar{D}^2 + R) e^{2iH} \bar{A}^2 \right).
\]

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If \( \sigma \) is given in terms of \( \Omega^\alpha \) as in (2.28b), we have

\[
\int \! d^6z \, \sigma \, w^{(\sigma)}(A, J) \, \Gamma = \int \! d^8z \, \Omega^\alpha \, w^{(\sigma)}(A, J) \, \Gamma
\]

with

\[
w^{(\sigma)}(A, J) = \frac{1}{12} D_\alpha w^{(\sigma)}(A, J),
\]

which yields

\[
w^{(\sigma)}(A, J) \, \Gamma = -\frac{1}{8} D_\alpha S
\]

for each order in \( H^{\dot{\alpha}} \) and \( J \). Adding (3.30) and (3.38) we find

\[
w^{(A)}(A, H, J) \, \Gamma \equiv (w^{(A)}(A, H, J) + w^{(\sigma)}(A, J)) \, \Gamma = -\frac{1}{8} D_\alpha S
\]

with \( w^{(A)}(A, H, J) \) the Ward operator for the combined chiral diffeomorphism and Weyl transformations of \( H \), \( J \) and \( A \) whose explicit forms are given in appendix A.4.

By inserting the expressions for \( \Omega^\alpha \) given in appendix A.2 into (3.40), we obtain Ward operators for the corresponding superconformal transformations,

\[
W^X = -i \int \! d^8z \, (\Omega^\alpha w^{(\sigma)}(A, H, J) + \bar{\Omega}^\dot{\alpha} \bar{w}^{(\sigma)}(A, H, J)).
\]

When restricting to flat space, \( H = 0 \), \( J = 0 \), (3.39) yields again the Ward identity

\[
-16 w_\alpha(A) \hat{\Gamma} = \bar{D}^\dot{\alpha} V_\alpha + 2 D_\alpha S
\]

which agrees with (3.19) with \( V \) and \( S \) given by

\[
V_\alpha = -\frac{1}{6} \left( D_\alpha A \bar{D}_\alpha \bar{A} - A D_\alpha \bar{D}_\alpha \bar{A} + \bar{A} \bar{D}_\alpha D_\alpha A \right) - \frac{1}{3} \left( \xi \bar{D}_\dot{\alpha} D_\alpha A^2 - \xi D_\alpha \bar{D}_\dot{\alpha} \bar{A}^2 \right),
\]

\[
S = -\frac{1}{12} m A^2 + \frac{1}{12} \xi \bar{D}^2 \bar{A}^2, \quad \bar{S} = -\frac{1}{12} m A^2 + \frac{1}{12} \xi D^2 A^2.
\]

Clearly for a conformal theory with \( S = \bar{S} = 0 \) we must have \( m = 0 \) and \( \xi = 0 \).

It should be noted that when imposing the equations of motion

\[
\frac{\delta \Gamma}{\delta A} = 0,
\]

we have from (3.33)

\[
\frac{\delta \Gamma}{\delta J} = -\frac{3}{2} S,
\]

such that the breaking term \( S \) is conjugate to the chiral compensator \( J \) in the same way as the supercurrent \( V \) is conjugate to the superspace metric field \( H \) in (3.27).
relation (3.44) is frequently used in the literature [9],[10]. However since here we would like to proceed off-shell we continue to work with the general form (3.33) for the Weyl transformation.

The conformal transformation properties of the supercurrent $V$ may be obtained from the Ward identity

$$\int d^8 z \delta \lambda H^b \frac{\delta \Gamma}{\delta H^b} + \int d^6 z \left( \delta \lambda A \frac{\delta}{\delta A} + \delta \lambda J \frac{\delta}{\delta J} + \delta \sigma A \frac{\delta}{\delta A} + \delta \sigma J \frac{\delta}{\delta J} \right) \Gamma + \text{c.c.}$$

$$= -\frac{1}{12} \int d^8 z \Omega^\alpha D_\alpha S + \text{c.c.} \ . \ (3.45)$$

Varying this equation with respect to $H^a$ we obtain

$$\delta V_{\alpha \dot{\alpha}} = -\int d^8 z \left( \frac{\delta (\delta \Lambda H^{\beta \dot{\beta}}|O(H))}{\delta H^{\alpha \dot{\alpha}}} V_{\beta \dot{\beta}} + \frac{\delta (\delta \Lambda H^{\beta \dot{\beta}}|O(H))}{\delta H^{\alpha \dot{\alpha}}} V_{\beta \dot{\beta}} + \Omega^\beta D_\beta \frac{\delta S}{\delta H^{\alpha \dot{\alpha}}} + \Omega_{\dot{\beta}} \bar{D}_{\dot{\beta}} \frac{\delta \bar{S}}{\delta H^{\alpha \dot{\alpha}}} \right),$$

where the conformal transformation $\delta V_{\alpha \dot{\alpha}}$ is given by

$$\delta V_{\alpha \dot{\alpha}} = \left( \int d^8 z \delta \lambda h^b \frac{\delta}{\delta H^b} + \int d^6 z (\delta \lambda A + \delta \sigma A) \frac{\delta}{\delta A} + \int d^6 z (\delta \lambda J + \delta \sigma J) \frac{\delta}{\delta J} + \text{c.c.} \right) V_{\alpha \dot{\alpha}} . \ (3.46)$$

The subscript $O(H)$ in (3.46) indicates that zeroth order terms are to be omitted. For $H = J = 0$ we find

$$\int d^6 z (\delta \lambda A + \delta \sigma A) \frac{\delta V_{\alpha \dot{\alpha}}}{\delta A} + \text{c.c.}$$

$$= -\int d^8 z \frac{\delta (\delta \Lambda H^{\beta \dot{\beta}}|O(H))}{\delta H^{\alpha \dot{\alpha}}} V_{\beta \dot{\beta}} - \frac{3}{2} \int d^6 z \sigma \frac{\delta S}{\delta H^{\alpha \dot{\alpha}}} + \text{c.c.}$$

$$= \frac{1}{4} \left( \{ D_{\beta}, \bar{D}_{\dot{\beta}} \} (\bar{D}_{\dot{\beta}} \Omega^{\beta} V_{\alpha \dot{\alpha}}) + \{ D_{\alpha}, \bar{D}_{\dot{\alpha}} \} (\bar{D}_{\dot{\alpha}} \Omega^{\beta} V_{\beta \dot{\beta}}) + D^{\beta} (\bar{D}^{\dot{\beta}} \Omega^{\beta} V_{\alpha \dot{\alpha}}) \right)$$

$$- \frac{3}{2} \int d^6 z \sigma \frac{\delta S}{\delta H^{\alpha \dot{\alpha}}} + \text{c.c.} \ . \ (3.47)$$

By inserting the explicit forms for $\Omega$ given in appendix A.2 we may find the expressions appropriate for the different conformal transformations.

It may be checked within a BRS type formalism that the representation for $\delta \Lambda H$ resp. $w^{(A)}(H)$ given by (3.33) is stable in the sense that any other choice for $\delta \Lambda H$ satisfying the superconformal algebra is equivalent to (2.24) up to redefinitions of $H$. For this purpose a constant anticommuting factor is extracted from the $\Lambda$ transformations, such that a corresponding BRS operator $s$ is obtained,

$$s \Lambda = \Lambda^2 = (\Lambda^{\alpha \dot{\alpha}} D_{\alpha \dot{\alpha}} + \Lambda^\alpha D_{\alpha} + \Lambda_{\dot{\alpha}} \bar{D}^{\dot{\alpha}}) (\Lambda^{\beta \dot{\beta}} D_{\beta \dot{\beta}} + \Lambda^{\beta} D_{\beta} + \Lambda_{\dot{\beta}} \bar{D}^{\dot{\beta}}) , \quad (3.49)$$

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where the $\Lambda$ transformations are now fermionic, such that $s^2\Lambda = 0$ is automatically satisfied. With $\Lambda$ given by $\{2.17\}$ we have, noting that $\{s, D_\alpha\} = \{s, \bar{D}\dot{\alpha}\} = 0$,

$$s\Lambda = -\bar{D}\dot{\alpha} s\Omega^\alpha D_{\alpha\dot{\alpha}} + \frac{1}{4} \bar{D}\bar{D} s\Omega^\alpha D_\alpha + \frac{1}{4} DD s\bar{\Omega}_{\dot{\alpha}} \bar{D}\dot{\alpha}. \tag{3.50}$$

Inserting $\{2.17\}$ into $\{3.49\}$ and comparing with $\{3.50\}$ implies

$$s\Omega^\alpha = \frac{1}{2} \bar{D}_\beta \Omega^\beta \{\bar{D}\dot{\beta}, D_\beta\}\Omega^\alpha + \frac{1}{4} \bar{D}\bar{D} \Omega^\beta D_\beta \Omega^\alpha + \chi^\alpha \tag{3.51}$$

for the superfield $\Omega^\alpha$ which is now bosonic. $\chi^\alpha$ is a chiral field which represents the freedom in expressing $\Lambda$ in terms of $\Omega^\alpha$. Using this result we may check that

$$s^2 A = s^2 H_\alpha \dot{\alpha} = s^2 J = 0 \tag{3.52}$$

where $sA, sH_\alpha \dot{\alpha}, sJ$ are obtained from $\delta_\Lambda A, \delta_\Lambda H_\alpha \dot{\alpha}, \delta_\Lambda J$ given by $\{2.24\}$ by inserting $\{2.17\}$ and by extracting a constant anticommuting factor from $\Omega_\alpha$. By complex conjugation a similar result may be obtained for $\bar{\Omega}_{\dot{\alpha}}, \bar{A}, \bar{J}$. To first order in $\bar{H}$ it is straightforward but tedious to check that any supersymmetric and Lorentz covariant modification of $sH$ may be absorbed by a suitable redefinition of $H$. For instance modifications of the schematic form

$$s' H_\alpha \dot{\alpha} = sH_\alpha \dot{\alpha} + (\bar{D}\bar{D} D\Omega H)_{\alpha\dot{\alpha}}, \tag{3.53}$$

where all permutations of the symbols in brackets are permitted, are absorbed by

$$H^{\alpha\dot{\alpha}} = H^{\alpha\dot{\alpha}} + (D\bar{D} HH)_{\alpha\dot{\alpha}}, \tag{3.54}$$

such that

$$s' H^{\alpha\dot{\alpha}} = 0. \tag{3.55}$$

4 Quantised Theory

4.1 Green Functions

We quantise the massless Wess-Zumino model by defining its Green functions by the Gell-Mann-Low formula with suitable subtractions within the BPHZ scheme. The essence of this renormalization scheme is to expand integrands whose integrals are potentially divergent into power series in external momenta, and to subtract terms of this series such that the integrals over the remaining terms are well-defined finite expressions. The reader not familiar with the BPHZ scheme is referred to the literature [14].

For the treatment of massless fields, in which particular care has to be taken as far as potential IR singularities are concerned, we use the formalism of Zimmermann and
The renormalisation operator \( R \) performs the subtactions according to the forest formula with the subtraction operator \( \gamma \) acting on each contribution to the perturbation expansion corresponding to a 1PI diagram. Clearly after performing the subtractions the massless limit is given by \( s = 1 \). The Green functions are given by the perturbative expansion

\[
\langle A(z_1) \ldots A(z_n) \rangle = R \frac{\langle A^{(0)}(z_1) \ldots A^{(0)}(z_n) e^{igR\int(A^{(0)})} \rangle}{\langle e^{igR\int(A^{(0)})} \rangle}, \tag{4.2}
\]

where \( (0) \) stands for the vacuum expectation value of free fields, and the interaction is given by

\[
\hat{\Gamma}_{\text{int}} = \frac{1}{16} (\hat{z} - 1) \int d^8 z \, AA + \frac{1}{48} \hat{g} \left( \int d^6 z \, A^3 + \int d^6 \bar{z} \, \bar{A}^3 \right). \tag{4.3}
\]

The renormalisation operator \( R \) denotes that ultraviolet and infrared subtractions are performed according to the forest formula with the subtraction operator

\[
T_\gamma = \left( 1 - t_{p,s}^{(\rho)} \right) \left( 1 - t_{p,s}^{(\delta)} \right) \tag{4.4}
\]

acting on each contribution to the perturbation expansion corresponding to a 1PI diagram \( \gamma \). \( \rho(\gamma) \) and \( \delta(\gamma) \) are the infrared and ultraviolet subtraction degrees associated with this diagram. \( t_{p,s}^{(\delta)} \) is a Taylor expansion operator which implies Taylor expansion in the external momenta \( p \) and in the mass parameter \( s \) up to and including order \( \delta(\gamma) \). The subtraction degrees are given by

\[
\delta(\gamma) = 4 - N_S(1 + d_S) - N_V(2 + d_V) + \sum V_i(\delta_i - 4) + \frac{1}{2} \omega(\gamma),
\]

\[
\rho(\gamma) = 4 - N_S(1 + d_S) - N_V(2 + d_V) + \sum V_i(\rho_i - 4) + \frac{1}{2} \omega(\gamma), \tag{4.5}
\]

where \( N_S, N_V \) are the number of external (anti-)chiral and vector fields and \( d_S, d_V \) the dimensions of these fields. \( \delta_i \) and \( \rho_i \) are the subtraction degrees of the vertex \( V_i \) given by the insertion \( [Q^i(z)]^{\delta_i}_{\theta_i} \). \( \omega(\gamma) \) is the number of independent differences \( \theta_i - \theta_j, \bar{\theta}_i - \bar{\theta}_j \) in the respective \( \theta, \bar{\theta} \) expansion of the contribution associated with the graph \( \gamma \). The parameters \( \hat{z} = 1 + O(h) \) and \( \hat{g} = g + O(h) \) are fixed by the normalisation conditions on the vertex functions

\[
\hat{\Gamma}_{AA} \bigg|_{p^2 = \mu^2, s = \theta} = \frac{1}{16}, \quad \partial_{\theta_1} \partial_{\theta_2} \hat{\Gamma}_{AAA} \bigg|_{p^2 = q^2 = (p+q)^2 = -\mu^2, s = \theta} = \frac{1}{8} g. \tag{4.6}
\]
Moreover, as a consequence of the subtraction we have
\[
\partial^2 \hat{\Gamma}_{AA} \big|_{p=0,s=1} = 0, \tag{4.7}
\]
which guarantees masslessness of the theory. A fundamental ingredient for the derivation of quantum Ward identities is the action principle, which implies for the operator \(w_\alpha(A)\) acting on the vertex functional \(\hat{\Gamma}\)
\[
-16 w_\alpha(A) \hat{\Gamma} = -16 \left[ w_\alpha(A) \hat{\Gamma}_{\text{eff}} \right]^{7/2} \cdot \hat{\Gamma} = \left[ \bar{D}^\dagger V_{\alpha \dot{\alpha}} + 2 D_\alpha S \right]^{7/2} \cdot \hat{\Gamma}, \tag{4.8}
\]
with
\[
V_{\alpha \dot{\alpha}} = -\frac{1}{6} \hat{z} \left( D_\alpha A \bar{D}_\alpha \bar{A} - A D_\alpha \bar{D}_\alpha \bar{A} + \bar{A} \bar{D}_\alpha D_\alpha A \right),
S = \frac{1}{12} M(s-1)A^2. \tag{4.9}
\]

On curved space, the parity invariant effective action is given by
\[
\Gamma_{\text{eff}} = \frac{1}{16} \hat{z} I_{\text{kin}} - \frac{1}{8} \left( I_M + \bar{I}_M \right) + \frac{1}{48} \hat{g} \left( I_g + \bar{I}_g \right) + \frac{1}{8} \hat{\xi} \left( I_\xi + \bar{I}_\xi \right) + \frac{1}{8} \hat{\lambda}_1 \left( I_1 + \bar{I}_1 \right) + \frac{1}{8} \hat{\lambda}_2 \left( I_2 + \bar{I}_2 \right) \tag{4.10}
\]
with
\[
I_{\text{kin}} = \int d^8 z E^{-1} A e^{2iH} \bar{A}, \quad I_g = \int d^6 z \phi^3 A^3
I_M = \int d^6 z \phi^3 M(s-1)A^2, \quad I_\xi = \int d^6 z \phi^3 RA^2
I_1 = \int d^6 z \phi^3 R^2 A, \quad I_2 = \int d^8 z E^{-1} A e^{2iH} \bar{R}.
\tag{4.11}
\]
The dynamical fields \(A, \bar{A}\) are quantised, whereas the background fields \(H, J, \bar{J}\) are treated as classical, i.e. non-propagating. Possible purely geometrical terms are discarded and will be considered in a separate publication.

The Green functions are given by the Gell-Mann-Low formula \((4.2)\) on curved space as well, with the interaction given by
\[
\Gamma_{\text{int}} = \frac{1}{16} \left( \hat{z} - 1 \right) I_{\text{kin}} + \frac{1}{48} \hat{g} \left( I_g + \bar{I}_g \right) + \frac{1}{8} \hat{\xi} \left( I_\xi + \bar{I}_\xi \right) + \frac{1}{8} \hat{\lambda}_1 \left( I_1 + \bar{I}_1 \right) + \frac{1}{8} \hat{\lambda}_2 \left( I_2 + \bar{I}_2 \right). \tag{4.12}
\]
The external fields yield external, i.e. non-integrated, vertices by construction. For the calculation of the subtraction degrees we note that \(H^{\alpha \dot{\alpha}}\) is a vector superfield with \(d_H = -1\) and \(J, \bar{J}\) are (anti-)chiral superfields with \(d_J = 0\). In the spirit of perturbation theory all exponentials of the external fields in \((4.12)\) have to be understood as series expansions,
such that at first sight it seems to be non-trivial that the renormalised couplings \( \hat{\gamma} \) and \( \hat{\xi} \) as well as the field renormalisation \( \hat{\gamma} \) have the same value to all orders in \( H, J \) and \( \bar{J} \). However this is guaranteed by the requirement of diffeomorphism invariance as expressed by the Ward identity (3.30) which we impose also for the quantum theory. Our subtraction scheme guarantees that diffeomorphism invariance is maintained in the quantised theory. \( \hat{\gamma}(g) \) and \( \hat{\gamma}(g) \) may thus be fixed by the flat space normalisation conditions (4.6), the masslessness of the theory still being ensured by (4.7). For the renormalised coupling \( \hat{\xi}(\xi, g) \), we impose the additional vertex function normalisation condition

\[
\frac{\partial}{\partial p^H_a} \partial_\theta^2 \partial_\theta^3 \Gamma^{a \bar{A} A}_{H A A}(p H; p_2, p_3) \bigg|_{p=p_{sym}, \theta=0, s=1} = -\frac{1}{12} \xi ,
\]

where the vertex functions are defined by

\[
\Gamma^{a \bar{A} A}_{H A A}(z_H; z_2, z_3) = \left. \frac{\delta}{\delta H_a(z_H)} \frac{\delta}{\delta A(z_2)} \frac{\delta}{\delta A(z_3)} \Gamma \right|_{A=H=J=0} \]

In (4.14) all coordinates are superspace coordinates. Insertions of the supercurrent and of the trace are given by

\[
\begin{align*}
\frac{\delta}{\delta H_a(z)} \Gamma &= \left[ \frac{\delta \Gamma_{\text{eff}}}{\delta H_a(z)} \right]^3 \cdot \Gamma = \left[ V^a[A, \bar{A}, H, J, \bar{J}](z) \right]^3 \cdot \Gamma , \\
w^{(\sigma)}(A, J) \Gamma &= \left[ w^{(\sigma)}(A, J) \Gamma_{\text{eff}} \right]^3 \cdot \Gamma \equiv -\frac{3}{2} \left[ S[A, \bar{A}, H, J, \bar{J}](z) \right]^3 \cdot \Gamma , \\
w^{(\bar{\sigma})}(A, J) \Gamma &= \left[ w^{(\bar{\sigma})}((A, J) \Gamma_{\text{eff}} \right]^3 \cdot \Gamma \equiv -\frac{3}{2} \left[ \bar{S}[A, \bar{A}, H, J, \bar{J}](z) \right]^3 \cdot \Gamma ,
\end{align*}
\]

from which 1PI Green functions with insertion may be obtained by virtue of

\[
\begin{align*}
\left. \frac{\delta}{\delta A(z_n)} \ldots \frac{\delta}{\delta A(z_1)} \frac{\delta}{\delta H_a(z)} \Gamma \right|_{H=J=\bar{J}=0, A=\bar{A}=0} &= \left\langle \left[ V^a[A, \bar{A}](z) \right]^3 A(z_1) \ldots A(z_n) \right\rangle_{1\text{PI}} ,
\end{align*}
\]

and similarly for the traces. The trace insertions \([S]^3\) and \([\bar{S}]^3\) contain oversubtracted terms of the form \([M(s-1)\phi^3 A^2]^3\) and \([M(s-1)\bar{\phi}^3 \bar{A}^2]^3\), where the expression \((s-1)\) has to be treated like an external momentum in the subtractions, such that some additional information has to be used before the massless limit \(s=1\) may be taken. This is provided by the Zimmermann identities which express oversubtracted insertions by the corresponding minimally subtracted ones plus a basis of local field polynomial insertions with the original subtraction degree [16]. For our purposes it is convenient to use the Zimmermann identities in their non-integrated form. For the chiral and antichiral mass
terms we obtain

\[
\left[ M(s-1)\phi^3 A^2 \right]_3^3 \cdot \Gamma \bigg|_{s=1} = \left[ u_{\text{kin}} \phi^3 A (\bar{D}^2 + R) \bar{A} + u_g \phi^3 A^3 
+ u_\xi \phi^3 R A^2 + u_\xi' \phi^3 (\bar{D}^2 + R) \bar{A}^2 
+ u_1 \phi^3 R^2 A + u_1' \phi^3 (\bar{D}^2 + R) (\bar{A}R) 
+ u_2 \phi^3 (\bar{D}^2 + R) (\bar{D}^2 + R) \bar{A} 
+ u_2' \phi^3 R (\bar{D}^2 + R) \bar{A} + u_\xi \phi^3 \square A \right]_3^3 \cdot \Gamma \bigg|_{s=1} , \tag{4.17}
\]

\[
\left[ M(s-1)\bar{\phi}^3 \bar{A}^2 \right]_3^3 \cdot \Gamma \bigg|_{s=1} = \left[ u_{\text{kin}} \bar{\phi}^3 \bar{A} (\bar{D}^2 + \bar{R}) A + u_g \bar{\phi}^3 \bar{A}^3 
+ u_\xi \bar{\phi}^3 \bar{R} A^2 + u_\xi' \bar{\phi}^3 (\bar{D}^2 + \bar{R}) A^2 
+ u_1 \bar{\phi}^3 \bar{R}^2 \bar{A} + u_1' \bar{\phi}^3 (\bar{D}^2 + \bar{R}) (AR) 
+ u_2 \bar{\phi}^3 (\bar{D}^2 + \bar{R}) (\bar{D}^2 + \bar{R}) \bar{A} 
+ u_2' \bar{\phi}^3 \bar{R} (\bar{D}^2 + \bar{R}) A + u_\xi \bar{\phi}^3 \square \bar{A} \right]_3^3 \cdot \Gamma \bigg|_{s=1} , \tag{4.18}
\]

where the chiral d’Alembertian is given by

\[
\square A = (\bar{D}^2 + R) D^2 A , \tag{4.19}
\]

and, again, purely geometrical terms have been omitted. In (4.17)–(4.19) all representation changing factors \(e^{\pm 2iH}\) have been suppressed. The Zimmermann identities contain all (anti-)chiral local field polynomials which are diffeomorphism invariant and of dimension 3. Parity invariance requires all coefficients \(u\) to be real.

### 4.2 Callan-Symanzik equation

Conformal transformations are in general anomalous to higher orders. On the level of integrated Ward identities, these anomalies are parametrised by the Callan-Symanzik (CS) functions \(\beta\) and \(\gamma\). These functions have to be identified before proceeding to the investigation of local Ward identities.

#### 4.2.1 Independent couplings

As a first application of the Zimmermann identities (4.17), (4.18) we derive a CS equation in which the coupling \(\xi\) of \(A^2\) to the curvature \(R\) is taken to be independent of the elementary coupling \(g\). In this case the counterterm coefficient \(\hat{\xi} = \hat{\xi}(\xi, g)\) is fixed by the normalisation condition (4.13). Similarly, the coefficients of the linear terms in \(\Gamma_{\text{eff}}\) are treated as independent and have to be fixed by some normalisation conditions. We claim that a CS equation

\[
\left. C\Gamma \right|_{s=1} = 0 \tag{4.20}
\]
holds with

\[ C = m \partial_m + \beta_g \partial_g + \beta_\xi \partial_\xi + \beta_{\lambda_1} \partial_{\lambda_1} + \beta_{\lambda_2} \partial_{\lambda_2} - \gamma N, \]  

(4.21)

where \( m \partial_m \) includes all mass parameters of the theory and

\[ N = \int d^6 z \frac{\delta}{\delta A} + \int d^6 \bar{z} \frac{\delta}{\delta \bar{A}} \]  

(4.22)

is the counting operator for matter legs. For the proof we apply \( C \) to \( \Gamma \), use the action principle and the Zimmermann identity and find that

\[ \mathcal{C} \Gamma|_{s=1} = \left[ \frac{1}{16} I_{\text{kin}} \left( \beta_g \partial_g \hat{z} - 2 \gamma \hat{\gamma} - 4(1 - 2 \gamma) u_{\text{kin}} \right) + \frac{1}{48} (I_g + \bar{I}_g) \left( \beta_g \partial_g \hat{g} - 3 \gamma \hat{g} \right) + \frac{1}{8} (I_\xi + \bar{I}_\xi) \left( (\beta_g \partial_g + \beta_\xi \partial_\xi + \beta_{\lambda_1} \partial_{\lambda_1} + \beta_{\lambda_2} \partial_{\lambda_2}) \hat{\xi} - 2 \gamma \hat{\gamma} - (1 - 2 \gamma) (u_\xi + u'_\xi) \right) + \frac{1}{8} (I_1 + \bar{I}_1) \left( (\beta_g \partial_g + \beta_\xi \partial_\xi + \beta_{\lambda_1} \partial_{\lambda_1} + \beta_{\lambda_2} \partial_{\lambda_2}) \hat{\lambda}_1 - \gamma \hat{\lambda}_1 - (1 - 2 \gamma) (u_1 + u'_1) \right) + \frac{1}{8} (I_2 + \bar{I}_2) \left( (\beta_g \partial_g + \beta_\xi \partial_\xi + \beta_{\lambda_1} \partial_{\lambda_1} + \beta_{\lambda_2} \partial_{\lambda_2}) \hat{\lambda}_2 - \gamma \hat{\lambda}_2 - (1 - 2 \gamma) (u_2 + u'_2) \right) \right]^{4} \cdot \Gamma \bigg|_{s=1}. \]

Hence (4.20) is true if the system

\[ \beta_g \partial_g \hat{z} - 2 \gamma \hat{\gamma} - 4(1 - 2 \gamma) u_{\text{kin}} = 0 \]

\[ \beta_g \partial_g \hat{g} - 3 \gamma \hat{g} = 0 \]

\[ (\beta_g \partial_g + \beta_\xi \partial_\xi + \beta_{\lambda_1} \partial_{\lambda_1} + \beta_{\lambda_2} \partial_{\lambda_2} - 2 \gamma) \hat{\xi} - (1 - 2 \gamma) (u_\xi + u'_\xi) = 0 \]  

(4.24)

\[ (\beta_g \partial_g + \beta_\xi \partial_\xi + \beta_{\lambda_1} \partial_{\lambda_1} + \beta_{\lambda_2} \partial_{\lambda_2} - 2 \gamma) \hat{\lambda}_1 - (1 - 2 \gamma) (u_1 + u'_1) = 0 \]

\[ (\beta_g \partial_g + \beta_\xi \partial_\xi + \beta_{\lambda_1} \partial_{\lambda_1} + \beta_{\lambda_2} \partial_{\lambda_2} - 2 \gamma) \hat{\lambda}_2 - (1 - 2 \gamma) (u_2 + u'_2) = 0 \]

is satisfied. This system has a unique solution for \( \beta \) and \( \gamma \) order by order in perturbation theory, hence (4.20) is proved. It may be shown in general that both the vertex functional \( \Gamma \) as well as the \( \beta \) and \( \gamma \) functions are independent of the auxiliary mass \( M \). [11]

**4.2.2 Dependent couplings and \( R \) invariance**

Inspired by analogous results in \( \phi^4 \) theory where the improvement coefficient is a unique function of the elementary coupling \( \xi \), we wish to understand \( \xi, \lambda_1 \) and \( \lambda_2 \) as functions of
$g$ in the supersymmetric case too. Necessary and sufficient conditions for such a functional
dependence to be consistent with renormalisability are the reduction equations [17]
\[
\beta_g \frac{dp}{dg} = \beta_p, \quad p = \xi, \lambda_1, \lambda_2, \quad p = p(g).
\]

These imply that the equations
\[
\begin{align*}
\beta_g \partial_g \hat{\xi} &= 2 \gamma \hat{\xi} + (1 - 2 \gamma)(u_\xi + u'_\xi) \\
\beta_g \partial_g \hat{\lambda}_1 &= \gamma \hat{\lambda}_1 + (1 - 2 \gamma)(u_1 + u'_1) \\
\beta_g \partial_g \hat{\lambda}_2 &= \gamma \hat{\lambda}_2 + (1 - 2 \gamma)(u_2 + u'_2)
\end{align*}
\]

have to be satisfied by $\hat{\xi}, \hat{\lambda}_{1,2}$. In order to show that this system of equations has a solution
we proceed as follows. We show that there exists a particular choice for $\hat{\xi}, \hat{\lambda}_{1,2}$ such that
the corresponding theory is $\mathcal{R}$ invariant. Consistency of the $\mathcal{R}$ transformation and Callan
Symzanizk operators implies for this particular choice that the reduction equations (4.26)
are satisfied, which in turn implies renormalisability of the theory.

As mentioned above (3.42) the classical theory is $\mathcal{R}$ invariant for
\[
\xi = \hat{\xi}^{(0)} = 0 \quad \lambda_{1,2} = \hat{\lambda}_{1,2}^{(0)} = 0.
\]

To higher orders, $\mathcal{R}$ invariance can be maintained by setting
\[
\begin{align*}
\hat{\xi} &= \frac{1}{2}(u'_\xi - u_\xi) \\
\hat{\lambda}_1 &= \frac{1}{4}(u'_1 - u_1) \\
\hat{\lambda}_2 &= \frac{1}{2}(u_2 - u'_2).
\end{align*}
\]

By studying consistency of the CS operator with $\mathcal{R}$ invariance we now show that the
reduction solution coincides with the $\mathcal{R}$ invariant theory, such that reduction implies a
symmetry (and vice versa). The Ward operator for $\mathcal{R}$ transformation is given by (3.40)
with $\Omega^{R\alpha}$ from appendix A.2, and we have
\[
W^R \Gamma|_{s=1} = 0.
\]

Furthermore the consistency condition
\[
[C, W^R] = 0
\]
holds with $C = m \partial_m + \beta_g \partial_g - \gamma N$. As a consequence
\[
W^R \mathcal{C} \Gamma|_{s=1} = 0.
\]

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We now proceed by induction. At order $\hbar^0$ the system (4.26) is trivial. Assuming that it holds for orders up to and including $n$ we have for $C\Gamma$:

\[
C\Gamma|_{s=1} = \frac{1}{8} \left( (\beta g \partial g - 2\gamma) \hat{\xi} - (1 - 2\gamma)(u_\xi + u'_\xi) \right) (I_\xi + \bar{I}_\xi) \\
+ \frac{1}{8} \left( (\beta g \partial g - \gamma) \hat{\lambda}_1 - (1 - 2\gamma)(u_1 + u'_1) \right) (I_1 + \bar{I}_1) \\
+ \frac{1}{8} \left( (\beta g \partial g - \gamma) \hat{\lambda}_2 - (1 - 2\gamma)(u_2 + u'_2) \right) (I_2 + \bar{I}_2) \\
+ o(\hbar^{n+2}).
\] (4.29)

The l.h.s. is of order $\hbar^{n+1}$ by assumption. To this order the r.h.s. is local since the nonlocal contributions are of order $n+2$. Applying $WR$ and using (4.28) we have to order $n+1$

\[
0 = \frac{1}{8} \left( (\beta g \partial g - 2\gamma) \hat{\xi} - (1 - 2\gamma)(u_\xi + u'_\xi) \right) \cdot \frac{1}{3} (I_\xi - \bar{I}_\xi) \\
+ \frac{1}{8} \left( (\beta g \partial g - \gamma) \hat{\lambda}_1 - (1 - 2\gamma)(u_1 + u'_1) \right) \cdot \frac{1}{3} (I_1 - \bar{I}_1) \\
+ \frac{1}{8} \left( (\beta g \partial g - \gamma) \hat{\lambda}_2 - (1 - 2\gamma)(u_2 + u'_2) \right) \cdot \frac{4}{3} (I_2 - \bar{I}_2). 
\] (4.30)

Hence (4.26) holds to order $n+1$, which completes the proof. Thus the quantum theory with $g$ as the only independent coupling is well defined and renormalisable, and $R$ invariance is realised at $s = 1$.

As a check we calculate $\beta$ to one-loop in appendix A.6 and find

\[
\beta^\xi = (a_{10} + a_{11}\xi) g^2 + (a_{20} + a_{21}\xi) g^4 + (a_{30} + a_{31}\xi) g^6 + \ldots 
\] (4.31)

\[
\beta^\xi_{(0)} = 2\gamma(1)^\xi, 
\]

with $a_{10} = 0$, $a_{11} = \frac{1}{(4\pi)^2}$. With

\[
\xi(\gamma) = \xi_0 + \xi_1 g^2 + \xi_2 g^4 + \ldots
\] (4.32)

the reduction equation (4.26) yields

at $o(h)$: \quad $a_{10} + a_{11}\xi_0 = 0$

at $o(h^2)$: \quad $\xi_1(3\frac{1}{(4\pi)^2} - a_{11}) = a_{20} + a_{21}\xi_0$

at $o(h^{n+1})$: \quad $\xi_n(3n\frac{1}{(4\pi)^2} - a_{11}) = f_n(\xi_i, i < n)$.

Hence $\xi_0 = 0$ and all $\xi_n$ are uniquely determined in accordance with the general arguments above. This completes the analysis of dilatational anomalies at the level of integrated Ward identities.
We proceed to consider local Ward identities in order to analyse the anomalies of the full superconformal group, which is necessary in view of determining the current algebra. For this purpose we introduce a local \([\ldots]\) insertion \(L_{\text{eff}}\) whose integral yields \(\Gamma_{\text{eff}}\),

\[
\Gamma_{\text{eff}} = \int d^6 z L_{\text{eff}} + \int d^6 \bar{z} \bar{L}_{\text{eff}},
\]

\[
L_{\text{eff}} = \frac{1}{32} z L_{\text{kin}} + \frac{1}{48} \hat{g} L_g - \frac{1}{8} L_M + \frac{1}{8} \hat{\xi} L_\xi + \frac{1}{8} \hat{\eta} L_\eta
\]

\[
\Gamma_{\text{eff}} = \int d^6 z L_{\text{kin}} + \int d^6 \bar{z} \bar{L}_{\text{kin}} + \int d^6 z L_g + \int d^6 \bar{z} \bar{L}_g - \frac{1}{8} \hat{\xi} L_\xi + \frac{1}{8} \hat{\eta} L_\eta.
\]

Here we have introduced

\[
L_{\text{kin}} = \phi^3 A (\bar{D}^2 + R) e^{2iH} \bar{A}
\]

\[
L_g = \phi^3 A^3
\]

\[
L_M = \phi^3 M(s - 1) \phi^3 A^2
\]

\[
L_\xi = \phi^3 RA^2
\]

\[
L_1 = \phi^3 R^2 A
\]

\[
L_2 = \phi^3 (\bar{D}^2 + R) e^{2iH} (\bar{D}^2 + \bar{R}) e^{-2iH} A
\]

\[
L'_{\text{kin}} = \phi^3 (\bar{D}^2 + R) e^{2iH} \bar{D} e^{-2iH} A
\]

\[
L'_{\text{g}} = \phi^3 \bar{D}^2 + R) e^{2iH} \bar{A}
\]

\[
L'_{\text{M}} = \phi^3 (\bar{D}^2 + R) e^{2iH} \bar{A} \bar{R} e^{-2iH} A
\]

\[
L'_{\xi} = \phi^3 (\bar{D}^2 + R) e^{2iH} \bar{D} e^{-2iH} \bar{A}
\]

\[
L'_{1} = \phi^3 \bar{D}^2 + R) e^{2iH} \bar{A}
\]

\[
L'_{2} = \phi^3 R (\bar{D}^2 + R) e^{2iH} \bar{A}
\]

\[
L'_{\text{kin}} = \phi^3 (\bar{D}^2 + R) e^{2iH} \bar{D} e^{-2iH} A
\]

\[
L'_{\text{g}} = \phi^3 \bar{D}^2 + R) e^{2iH} \bar{A}
\]

\[
L'_{\text{M}} = \phi^3 (\bar{D}^2 + R) e^{2iH} \bar{A} \bar{R} e^{-2iH} A
\]

\[
L'_{\xi} = \phi^3 (\bar{D}^2 + R) e^{2iH} \bar{D} e^{-2iH} \bar{A}
\]

\[
L'_{1} = \phi^3 \bar{D}^2 + R) e^{2iH} \bar{A}
\]

\[
L'_{2} = \phi^3 R (\bar{D}^2 + R) e^{2iH} \bar{A}
\]

\[
\text{such that}
\]

\[
\int d^6 z L_{\text{kin}} = I_{\text{kin}}
\]

\[
\int d^6 z L_{g,M} = I_{g,M}
\]

\[
\int d^6 z L_{\text{kin}} = I_{\text{kin}}
\]

\[
\int d^6 z L_{g,M} = I_{g,M}
\]

\[
\int d^6 z L_{\xi} = I_{\xi}
\]

\[
\int d^6 z L_{\xi} = I_{\xi}
\]

\[
\int d^6 z L_{1} = I_{1}
\]

\[
\int d^6 z L_{1} = I_{1}
\]

\[
\int d^6 z L_{2} = I_{2}
\]

\[
\int d^6 z L_{2} = I_{2}
\]

\[
\text{The functions } \hat{\epsilon} = \hat{\epsilon}(g), \hat{\eta}_{1,2} = \hat{\eta}_{1,2}(g) \text{ and } \hat{\alpha} = \hat{\alpha}(g) \text{ represent ambiguities in the choice of the local basis for } L_{\text{eff}} \text{ and the corresponding terms cancel when integrated.}
\]

We aim at proving an equation which expresses broken Weyl invariance at the local level, of the form

\[
\left( w^{(\sigma)} - \gamma A \frac{\delta}{\delta A} \right) \Gamma \bigg|_{s=1} + [\beta_g \partial_g L_{\text{eff}}] \cdot \Gamma \bigg|_{s=1} = 0,
\]

as well as a corresponding antichiral equation. We denote these equations by local CS equations.
Analogously to the integrated case we find that (4.37) is satisfied if the following equations hold,

\[ \beta_y \partial_y \hat{\varepsilon} - 2\gamma \hat{\varepsilon} - 4(1 - 2\gamma)u_{\text{kin}} = 0 \quad (4.38a) \]
\[ \beta_y \partial_y \hat{\gamma} - 3\gamma \hat{\gamma} = 0 \quad (4.38b) \]
\[ \beta_y \partial_y (\hat{\xi} - \hat{\varepsilon}) - (1 - 2\gamma)u_\xi - (1 + 2\gamma)\hat{\xi} = 0 \quad (4.39a) \]

\[ \beta_y \partial_y \hat{\xi} - (1 - 2\gamma)u'_\xi + \hat{\xi} = 0 \quad (4.39b) \]
\[ \beta_y \partial_y (\hat{\lambda}_1 - \hat{\eta}_1) - (1 - 2\gamma)u_1 - (2 + \gamma)\hat{\lambda}_1 = 0 \quad (4.40a) \]
\[ \beta_y \partial_y \hat{\eta}_1 - (1 - 2\gamma)u'_1 + 2\hat{\lambda}_1 = 0 \quad (4.40b) \]
\[ \beta_y \partial_y (\hat{\lambda}_2 - \hat{\eta}_2) - (1 - 2\gamma)u_2 + (1 - \gamma)\hat{\lambda}_2 = 0 \quad (4.41a) \]
\[ \beta_y \partial_y \hat{\eta}_2 - (1 - 2\gamma)u'_2 - \hat{\lambda}_2 = 0 \quad (4.41b) \]
\[ \beta_y \partial_y \hat{\alpha} - (1 - 2\gamma)u_\alpha = 0 \quad (4.42) \]

Equations (4.38a), (4.38b) are the usual equations for \( \beta_y \) and \( \gamma \), and hence are satisfied. The sums (4.39a) + (4.39b), (4.40a) + (4.40b) and (4.41a) + (4.41b) constitute the reduction equations (4.26) for \( \xi, \lambda_1 \) and \( \lambda_2 \), which are satisfied by the theory constructed in section 4.2.2. Inserting these into (4.39b), (4.40b) and (4.41b) and using (4.27), which also holds in this theory, yields

\[ \beta_y \partial_y \hat{\varepsilon} = \frac{1}{2} \beta_y \partial_y \hat{\gamma} - 3\gamma \hat{\gamma} \quad (4.43a) \]
\[ \beta_y \partial_y \hat{\eta}_1 = \frac{1}{2} \beta_y \partial_y \hat{\lambda}_1 - \frac{3}{2} \gamma \hat{\lambda}_1 \quad (4.43b) \]
\[ \beta_y \partial_y \hat{\eta}_2 = \frac{1}{2} \beta_y \partial_y \hat{\lambda}_2 + \frac{5}{2} \gamma \hat{\lambda}_2. \quad (4.43c) \]

For establishing the local CS equation (4.37), we have to provide \( \hat{\varepsilon}, \hat{\eta}_1, \hat{\eta}_2, \hat{\alpha} \) such that equations (4.43a), (4.43b), (4.43c) and (4.42) are satisfied.

Inserting the power series expansions

\[ \beta_y = \sum_{n=1}^{\infty} \beta_n h^n g^{2n+1}, \quad \gamma = \sum_{n=1}^{\infty} \gamma_n h^n g^{2n} \quad (4.44) \]

and

\[ \hat{\varepsilon} = \sum_{n=0}^{\infty} h^n \hat{\varepsilon}_n(g), \quad \hat{\xi} = \sum_{n=0}^{\infty} h^n \hat{\xi}_n(g) \quad (4.45) \]

into (4.43a), we have to order \( N \) in \( h \)

\[ \beta_1 g^3 \partial_y \hat{\varepsilon}_{N-1}(g) = \sum_{k=1}^{N} q_k(g) - \sum_{k=1}^{N} \beta_k g^{2k+1} \partial_y \hat{\varepsilon}_{N-k}(g), \quad (4.46) \]

\[ q_k(g) = \frac{1}{2} \beta_k g^{2k+1} \partial_y \hat{\varepsilon}_{N-k}(g) - 3\gamma_k g^{2k} \hat{\xi}_{N-k}(g). \]

We assume that the coefficients \( \hat{\varepsilon}_1, \ldots, \hat{\varepsilon}_{N-2} \) have already been determined. Then, since \( \beta_1 = \frac{3}{2(4\pi)^2} \neq 0 \) and the r.h.s. is a power series starting at least with \( g^3 \), the coefficient \( \hat{\varepsilon}_{N-1} \) can be found by trivial integration, such that \( \hat{\varepsilon}_n \) is a power series in \( g \), once we choose the arbitrary integration constant to be \( \hat{\varepsilon}_0(g) = 0 \). The same argument applies to the other differential equations. Hence we have established the local CS equation (4.37).
5 Transformation properties

In this section we calculate the superconformal transformation properties of the supercurrent in the quantum theory using the local Ward identities (3.39) and (4.37) as a starting point. First let us recall that the supercurrent is defined to all orders in (4.15) with $\hat{\xi}$, $\hat{\lambda}_1$, $\hat{\lambda}_2$ fixed by (4.27). The trace $S$, originally given by (4.15), has been brought into the form (4.37), where the symmetry breaking term is separated into anomalous contact terms of the quantised fields $A$ and into an operator insertion such that the character of an $S$ type conformal breaking is maintained.

Since the quantised theory is diffeomorphism invariant by construction, the symmetry breaking which we look for is entirely determined by the breaking of Weyl symmetry in curved space. In particular it has precisely the form of the tree approximation, hence (3.39) may be rewritten to all orders in terms of insertions,

$$\int d^8z \Omega^\alpha w^{(\gamma)}_\alpha(A, J) \Gamma + c.c. = - \int d^8z \delta\Omega H^{\alpha\hat{\alpha}} \frac{\delta\Gamma}{\delta H^{\alpha\hat{\alpha}}} - \frac{3}{2} \int d^6z \sigma S \cdot \Gamma + c.c. , \quad (5.1)$$

where

$$w^{(\gamma)}_\alpha(A, J) \equiv w^{(A)}_\alpha(A, J) + w^{(\sigma)}_\alpha(A, J) - \frac{1}{12} \gamma D\alpha \left( A \frac{\delta}{\delta A} \right), \quad (5.2)$$

$$S \equiv \frac{2}{3} [\beta g \partial g \mathcal{L}_{\text{eff}}]_3^3, \quad (5.3)$$

and $w^{(\sigma)}_\alpha(A, J)$ is given by (3.37).

Differentiation of (5.1) with respect to $H$ yields the transformation properties of Green functions with supercurrent insertions as required. For the first derivative we obtain, when restricting to flat space,

$$\int d^8z \Omega^\alpha w^{(\gamma)}_\alpha(A) [V_{\alpha\hat{\alpha}}(z')] \cdot \hat{\Gamma} + c.c. = - \int d^8z \frac{\delta(\Omega H^{\beta\hat{\beta}}(z'))}{\delta H_{\alpha\hat{\alpha}}(z')} [V_{\beta\hat{\beta}}] \cdot \hat{\Gamma} - 12 \left( \frac{\delta}{\delta H_{\alpha\hat{\alpha}}(z')} \int d^6z \sigma S \cdot \Gamma \right) \bigg|_{H=0} + c.c. \quad (5.4)$$

Only the linear part of $\delta\Omega H$ contributes since the terms involving the inhomogenous parts cancel for superconformal transformations which satisfy $\bar{D}^\alpha \Omega^\alpha = D^\alpha \Omega^\alpha$ by definition. The hats denote that for the corresponding quantities $H = J = 0$. Equation (5.4) holds to all orders and reduces to (3.46) in the classical approximation, where $S$ vanishes and all contributions are purely local.
The transformation laws for vertex functions with multiple insertions of the supercurrent can be obtained by multiple differentiation of (5.1) with respect to $H$. Such differentiations never involve the mass term hence the massless limit $s = 1$ may be taken without the necessity of using a Zimmermann identity. This means that (5.1) contains all information about the transformation properties of the supercurrent even when multiply inserted into vertex functions. $S$ type breaking of superconformal symmetry ensures that the multiple insertions transform covariantly under translations, Lorentz and supersymmetry transformations. All other transformations ($R$, dilatations, special conformal, special supersymmetry) lead to anomalies. However there are no further anomalies than those already present for Green functions of elementary fields. This is the main result of our analysis.

Supercurrent and trace are in direct relation with those of equation (16.1.15) of [11] for $s = 1$ (S type) in the flat space limit, if we define $L = \frac{\hat{\omega}}{32} \hat{D}^2 (\hat{A} \hat{A}) - \frac{1}{8} M (s - 1) A^2 + \frac{1}{48} A^3$, such that on flat space $L_{\text{eff}}$ given by (1.37) and $L$ differ by the terms involving $\hat{\xi}, \hat{\epsilon}, \hat{\lambda}_1, \hat{\lambda}_2, \hat{\eta}_1, \hat{\eta}_2, \hat{\alpha}$. In [11], the trace is taken to be $S' = \hat{2} \hat{\delta}_y \hat{\partial}_y L$ and the coefficients of the terms present in $L_{\text{eff}} - L$ differ accordingly in the supercurrent when compared to the supercurrent discussed in this paper. Using (1.39a) – (4.42) of the present paper we can show that the difference $V'_{\hat{a} \hat{a}} - V_{\hat{a} \hat{a}}$ is in exact agreement with $S - S'$ such that in both cases

$$\hat{D}^a[V_{\hat{a} \hat{a}}] \cdot \Gamma = -16 w^{(y)} [S] \cdot \Gamma$$

is satisfied. This demonstrates that there is an arbitrariness in the definition of $V_{\hat{a} \hat{a}}$ and $S$ on flat space. However the definitions of this paper have the feature of coinciding with the well-defined flat space limit of canonical supergravity.

6 Discussion and conclusions

In the present paper we have derived the superconformal transformation properties of Green functions with multiple insertions of the supercurrent. We have worked to all orders of perturbation theory within the context of the massless Wess-Zumino model. The main tool of our analysis is the embedding of the model into curved superspace with the usual supergravity constraints and a chiral compensator. Supersymmetry and diffeomorphism invariance can be kept manifest, such that all superconformal transformations can be expressed by Weyl transformations. This applies also to higher orders and permits the exhaustive study of possible anomalies. Since our analysis employs Ward identities, it is scheme independent.

As a conclusion we compare our results with the literature. Close in method and spirit is the analogous investigation [6]–[8] in $\phi^4$ theory. The most remarkable difference between the two models is the fact that the $S$ type supercurrent contains an improved energy-momentum tensor. While in $\phi^4$ theory the improvement has to be built in and
necessitates some technically involved steps in the subsequent analysis, it is automatic in the supersymmetric case. In $\phi^4$ theory it is not possible to separate trace effects and hence to define multiple insertions with controlled transformation behaviour easily (s. [8] sect. 5), even when the couplings are generalised to be local fields. Here there is no difficulty in doing so even without local couplings. This is an important feature which indicates that further structural relations may also be derived more easily in the supersymmetric case than in the non-supersymmetric one.

Complementary to our approach is the investigation performed in [18]. There the general aim is to quantise supersymmetric theories in terms of conventional fields and thus to have a simple transition to conventional non-supersymmetric field theories. This effort is essential if one aims at having contact with phenomenology. Supersymmetry transformations become non-linear and cannot be realised trivially. Supermultiplets have a non-trivial form too. This makes a literal translation between the two approaches difficult. In particular it is not obvious how the two supergravity multiplets are related. In any case, comparison in detail is somewhat premature. Whereas in [18] a general supersymmetric gauge theory is considered and here only a non-gauge model is studied, the present analysis treats already higher orders. We hope to come back to this comparison in the near future.

In order to give an outlook to applications and further investigations we complete the Weyl identity, i.e. the local CS equation, by adding purely geometrical terms $C(H, J, \bar{J})$

$$\left( w^\sigma - \gamma A \frac{\delta}{\delta A} \right) \Gamma - \beta_g [\partial g L_{\text{eff}}] \cdot \Gamma = C(H, J, \bar{J}). \quad (6.1)$$

These have to be Weyl invariant and contain at least the supersymmetric version of the Weyl tensor and the Gauß-Bonnet term. (6.1) may be varied arbitrarily often with respect to $H$. Thus the Ward identity for the superconformal current algebra is obtained within the framework of general Green functions. The terms depending on $\beta$ and $\gamma$ represent the dynamical, i.e. model dependent anomalies of the superconformal currents, whereas the r.h.s. corresponds to geometrical anomalies, in which only the coefficients depend on the model. With the help of these explicit results it should be possible to discuss questions of the superconformal current algebra in general.

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## A Appendix

### A.1 Fields and Transformation Laws

| Prepotentials | $H = H^{\alpha \dot{\alpha}} D_{\alpha \dot{\alpha}}$ real  
|              | $\phi$ chiral  
|              | $\phi = e^{J}, J$ chiral  
| Vierbein determinant | $E = \text{sdet} \ E_A^M$  
| Curvature scalar | $R = \bar{D}^2 \left( E^{-1} \phi^{-3} \right)$  
| Matter field | $A$ chiral  
| $\Lambda$ transformations | $\Lambda = \Lambda^{\alpha \dot{\alpha}} D_{\alpha \dot{\alpha}} + \Lambda^{\alpha} D_{\alpha} + \Lambda_{\dot{\alpha}} \bar{D}^{\dot{\alpha}}$  
|              | $\Lambda_c = \Lambda^{\alpha \dot{\alpha}} D_{\alpha \dot{\alpha}} + \Lambda^{\alpha} D_{\alpha}$  
|              | $\Lambda^{\alpha \dot{\alpha}} = i \bar{D}^{\dot{\alpha}} \Omega^{\alpha}$  
|              | $\Lambda^{\alpha} = \frac{1}{4} \bar{D}^2 \Omega^{\alpha}$  
|              | $\Lambda_{\dot{\alpha}} = e^{2iH} \bar{\Lambda}_{\dot{\alpha}}$  
|              | $e^{2iH'} = e^{\Lambda} e^{2iH} e^{-\bar{\Lambda}}$  
|              | $\phi'^3 = \phi^3 e^{\Lambda_c}$  
|              | $J' = e^{\Lambda} J + \frac{1}{3} \ln \left( 1 \cdot e^{\Lambda_c} \right)$  
|              | $(E')^{-1} = E^{-1} e^{\bar{\Lambda}}$  
|              | $R' = e^{\Lambda} R$  
|              | $A' = e^{\Lambda} A$  
| Super Weyl transformations | $\sigma$ chiral  
|              | $H' = H$  
|              | $\phi' = e^{\sigma} \phi$  
|              | $J' = J + \sigma$  
|              | $(E')^{-1} = E^{-1} e^{\sigma} e^{2iH} e^{\bar{\sigma}}$  
|              | $R' = e^{-2\sigma} \left( \bar{D}^2 + R \right) e^{\bar{\sigma}}$  
|              | $A' = e^{-\sigma} A$  

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### A.2 Parameters of Superconformal Transformations

| Transformation     | Parameter $\Omega^a$                                                                 | Action on chiral field $A$                                                                 |
|-------------------|-------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------|
| General           | $\tilde{D}^\alpha \Omega^a = D^\alpha \tilde{\Omega}^\alpha$, $\sigma = -\frac{1}{12} \tilde{D}^2 D^\alpha \Omega^a$ | $\frac{1}{4} \tilde{D}^2 (\Omega^a D_a A) - \sigma A = \delta_\Omega A$               |
| Translation       | $\Omega^a = -\frac{i}{2} \sigma_a^{\alpha\beta} i\theta^\alpha \theta^\beta$, $\sigma = 0$ | $t^a \partial_a A = t^a \delta_a^P A$                                                     |
| Supersymmetry     | $\Omega^a = -\bar{\theta}^2 q^a$, $\sigma = 0$                                     | $q^a (\partial_\alpha + i \sigma_a^{\alpha\beta} \bar{\theta}^\beta \partial_a) A = q^a \delta^Q_a A$ |
|                   | $\Omega^a = 2 \theta^\alpha \bar{\theta}_a q^a$, $\sigma = 0$                      | $\bar{q}_a (-\bar{\theta}^\alpha - i \theta^a \sigma_b^{\alpha\beta} \partial_b) A = \bar{q}_a \delta^Q_a A$ |
| $R$ Transformation | $\Omega^a = -i \theta^a \bar{\theta}^2 r$, $\sigma = \frac{2}{3} i r$                | $ir \left( -\frac{2}{3} + \theta^a \partial_a + \bar{\theta}_a \bar{\theta}^a \right) A = r \delta^R A$ |
| Dilatation        | $\Omega^a = d \left( \frac{1}{2} \theta^a \bar{\theta}^2 - \frac{i}{2} \bar{\theta}_a \sigma_a^{\alpha\beta} \theta^\alpha \theta^\beta \right)$, $\sigma = -d$ | $d \left( 1 + x^a \partial_a + \frac{1}{2} \theta^a \partial_a - \frac{1}{2} \bar{\theta}_a \bar{\theta}^a \right) A = d \delta^D A$ |
| Lorentz Transformation | $\Omega^a = i \sigma_a^{\alpha\beta} \omega^{ab} \sigma^a_b \bar{\theta}_a$, $\sigma = 0$ | $\omega^{ab} \left( x_a \partial_b - x_b \partial_a - \frac{i}{2} \theta^a (\sigma_{ab})^{\alpha\beta} \theta_a \theta_b \right) + \frac{i}{2} \bar{\theta}_a (\sigma_{ab})^{\beta\alpha} \bar{\theta}_a = \omega^{ab} \delta^L_{ab} A$ |
| S Transformation  | $\Omega^a = 2 s^a \sigma_a^{\beta\gamma} \theta^\beta \bar{\theta}^\gamma$, $\sigma = 4i s^a \theta_a$ | $s^a \left( -x_a \sigma_a^{\alpha\beta} \delta^Q \bar{\theta}^\alpha \delta^R \bar{\theta}^\alpha + 2 \bar{\theta}_a \delta^R - i \theta^2 D_a \right)$, $-2i \left( d - \frac{n}{2} \right) \theta_a A = s^a \delta^S_{\alpha} A$ |
|                   | $\Omega^a = -\bar{s}_a \sigma_a^{\alpha\beta} \bar{\theta}_a^2$, $\sigma = 0$         | $\bar{s}_a \left( -x_a \delta^Q \sigma_a^{\alpha} \bar{\theta}_a^\alpha + 2 \bar{\theta}^\alpha \delta^R \bar{\theta}^\alpha \right)$, $+2i \left( d + \frac{n}{2} \right) \bar{\theta}^\alpha A = \bar{s}_a \delta^S \bar{\theta}^\alpha A$ |
| K Transformation  | $\Omega^a = \left( \frac{i}{2} k_b^a x^2 - i k^a x_a x^b \right) \sigma_b^{\alpha\beta} \bar{\theta}_a$ | $k^a \left( 2x_a \delta^P + 2x^b \delta^L_{ab}^{} - 2x_a x_b \partial_b \right)$, $+x^2 \partial_a + 2 \theta \sigma_a \bar{\theta}^R - \frac{2}{3} i \theta \sigma_a \bar{\theta}$, $+\theta^2 \bar{\theta}_a ^{} A = \delta^K a A$ |
A.3 Combined Diffeomorphism and Weyl Transformations

Below we list the combined infinitesimal $\Lambda$ and $\sigma$ transformations in terms of $\Omega^\alpha$. In $\delta_\Omega H$ we include only terms involving $\Omega$ but not $\bar{\Omega}$,

$$\delta_\Omega A = \bar{D}^2 \left( \frac{1}{4} \Omega^\alpha D_\alpha A - \frac{1}{12} D_\alpha \Omega^\alpha A \right)$$  \hspace{1cm} (A.3.1)

$$\delta_\Omega H^{\alpha\bar{\alpha}} = \frac{1}{2} \bar{D}^{\dot{\alpha}} \Omega^\alpha + \frac{1}{4} \bar{D}^{\dot{\beta}} \Omega^\beta \{ D_\gamma, \bar{D}_\dot{\gamma} \} H^{\alpha\bar{\alpha}} - \frac{1}{8} H^{\gamma\bar{\gamma}} \{ D_\gamma, \bar{D}_\dot{\gamma} \} \bar{D}^{\dot{\alpha}} \Omega^\alpha + \frac{1}{4} \bar{D}^{\dot{\beta}} \Omega^\beta D_\beta H^{\alpha\bar{\alpha}}$$

$$- \frac{1}{16} H^{\gamma\bar{\gamma}} \{ D_\gamma, \bar{D}_\dot{\gamma} \} \left( \bar{D}^{\dot{\alpha}} \Omega^{\beta \gamma} \{ D_\gamma, \bar{D}_\dot{\beta} \} H^{\alpha\bar{\alpha}} \right)$$

$$+ \frac{1}{24} \bar{D}^{\dot{\beta}} \Omega^\beta \{ D_\beta, \bar{D}_\dot{\beta} \} \left( H^{\gamma\bar{\gamma}} \{ D_\gamma, \bar{D}_\dot{\gamma} \} H^{\alpha\bar{\alpha}} \right)$$

$$+ O(H^3)$$  \hspace{1cm} (A.3.2)

$$\delta_\bar{\Omega} J = \frac{1}{4} \bar{D}^2 (\Omega^\alpha D_\alpha J) . $$  \hspace{1cm} (A.3.3)
A.4 Local Superconformal Ward Operators

Diffeomorphisms (Λ Transformations)

\[
\begin{align*}
  w_\alpha^{(A)}(A) &= \frac{1}{4} D_\alpha A \frac{\delta}{\delta A}, \\
  w_\alpha^{(J)}(J) &= \frac{1}{4} D_\alpha J \frac{\delta}{\delta J}, \\
  w_\alpha^{(H)}(H) &= w_\alpha^{(0)}(H) + w_\alpha^{(1)}(H) + w_\alpha^{(2)}(H) + O(H^3), \\
  w_\alpha^{(0)}(H) &= \frac{1}{2} \bar{D} \frac{\delta}{\delta H} \frac{\delta}{\delta H}, \\
  w_\alpha^{(1)}(H) &= \frac{1}{4} \bar{D} \left( \left\{ D_\alpha, \bar{D}_\beta \right\} H^{\beta\gamma} \frac{\delta}{\delta H^{\beta\gamma}} \right) + \frac{1}{4} \left\{ D_\beta, \bar{D}_\gamma \right\} \bar{D} \frac{\delta}{\delta H^{\beta\gamma}}, \\
  w_\alpha^{(2)}(H) &= \frac{1}{8} \bar{D}^2 \left( \left\{ D_\gamma, \bar{D}_\gamma \right\} \left( H^{\gamma\beta} D_\alpha \right) \frac{\delta}{\delta H^{\gamma\beta}} \right) + \frac{1}{24} \bar{D} \left( \left\{ D_\alpha, \bar{D}_\beta \right\} H^{\beta\gamma} \left\{ D_\beta, \bar{D}_\gamma \right\} \left( H^{\gamma\beta} \frac{\delta}{\delta H^{\beta\gamma}} \right) \right) + \frac{1}{12} \left( \left\{ D_\alpha, \bar{D}_\beta \right\} \left( H^{\gamma\beta} \left\{ D_\gamma, \bar{D}_\gamma \right\} H^{\beta\gamma} \right) \frac{\delta}{\delta H^{\beta\gamma}} \right).
\end{align*}
\]

Super Weyl Transformations

\[
\begin{align*}
  w^{(\sigma)}(A) &= -A_{\frac{\delta}{\delta A}}, \\
  w^{(\sigma)}(J) &= \frac{\delta}{\delta J}, \\
  w^{(\sigma)}(H) &= 0,
\end{align*}
\]

Combined Transformations

\[
\begin{align*}
  w_\alpha(A) &= \frac{1}{4} D_\alpha A \frac{\delta}{\delta A} - \frac{1}{12} D_\alpha \left( A \frac{\delta}{\delta A} \right), \\
  w_\alpha(J) &= \frac{1}{4} D_\alpha J \frac{\delta}{\delta J}, \\
  w_\alpha(H) &= w_\alpha^{(A)}(H).
\end{align*}
\]
A.5 Expansion of the Action Functional

\[ E^{-1} = 1 + E^{-1}(1) + E^{-1}(2) + \ldots \]  

\[ E^{-1}(1) = \frac{2}{3} \bar{D}_a D_a H^{\alpha \bar{\alpha}} + \frac{1}{3} D_a \bar{D}_a H^{\alpha \bar{\alpha}} + J + \bar{J} \]  

\[ E^{-1}(2) = \frac{2}{3} (J + \bar{J})^2 D_a D_a H^{\alpha \bar{\alpha}} + \frac{1}{3} (J + \bar{J}) D_a \bar{D}_a H^{\alpha \bar{\alpha}} \]  

\[ \Gamma = \Gamma^{(0)} + \Gamma^{(1)} + \Gamma^{(2)} + \ldots, \]  

\[ \Gamma^{(0)} = \frac{1}{16} \int d^8 z \, A \bar{A} + \frac{9}{48} \int d^6 z \, A^3 + \frac{9}{48} \int d^6 z \, \bar{A}^3 + \frac{m}{8} \int d^6 z \, A^2 + \frac{m}{8} \int d^6 z \, \bar{A}^2, \]  

\[ \Gamma^{(1)} = \frac{1}{8} \int d^8 z \, H^{\alpha \bar{\alpha}} V_{\alpha \bar{\alpha}} + \left[ \frac{9}{16} \int d^6 z \, JA^3 + \frac{3m}{8} \int d^6 z \, JA^2 + c.c. \right] \]  

\[ + \frac{1}{16} \int d^8 z (J + \bar{J}) A \bar{A} + \frac{1}{8} \int d^8 z (\xi J A^2 + \xi J \bar{A}^2) \]  

\[ \Gamma^{(2)} = \frac{1}{16} \int d^8 z \left[ -2AH^{\alpha \bar{\alpha}} \{ D_\alpha, \bar{D}_{\bar{\alpha}} \} (H^{\beta \bar{\beta}} \{ D_{\beta}, \bar{D}_{\bar{\beta}} \}) \bar{A} + E^{-1}(1)AH^{\alpha \bar{\alpha}} \{ D_\alpha, \bar{D}_{\bar{\alpha}} \} \bar{A} \right. \]  

\[ + \left. E^{-1}(2)A \bar{A} \right] \]  

\[ + \frac{1}{8} \int d^8 z \left[ -2\xi H^{\alpha \bar{\alpha}} \{ D_\alpha, \bar{D}_{\bar{\alpha}} \} (H^{\beta \bar{\beta}} \{ D_{\beta}, \bar{D}_{\bar{\beta}} \}) \bar{A}^2 + \xi E^{-1}(1)H^{\alpha \bar{\alpha}} \{ D_\alpha, \bar{D}_{\bar{\alpha}} \} \bar{A}^2 \right. \]  

\[ + \left. E^{-1}(2)(\xi A^2 + \xi \bar{A}^2) \right] \]  

\[ + \frac{9g}{24} \int d^6 z \, J^2 A^3 + \frac{9g}{24} \int d^6 z \, J^2 \bar{A}^3 + \frac{9m}{16} \int d^6 z \, J^2 A^2 + \frac{9m}{16} \int d^6 z \, J^2 \bar{A}^2 \]
A.6 $\beta^\xi$ to order $\hbar$

To first order in $H$, zeroth order in $J, \bar{J}$, the supersymmetric curvature scalar is given by

$$R \sim -\frac{2}{3}i \bar{D}^2 \partial_a H^a,$$  \hfill (A.6.1)

such that to first order in $H$, the $RA^2$ term in the action (4.10) generates the vertex in figure 1 while the kinetic term generates the three vertices in figure 2.

Figure 1: Vertex with coupling $\xi$

Figure 2: Vertices for the kinetic term

Testing the CS equation (4.20) with respect to $H^a, A, A$ yields to first order in $\hbar$

$$\Gamma^{(1)}_{H^a A A} + \left( \beta^\xi (1) \partial_\xi + \beta^\eta (1) \partial_\eta - 2 \gamma^{(1)} \right) \Gamma^{(0)}_{H^a A A} = 0$$  \hfill (A.6.2)

at $s = 1$. The zeroth order vertex is given by

$$\Gamma^{(0)}_{H^a_{\alpha A_1 A_2}} = -\frac{2}{3} \xi p^H \delta^S(1, 3, p) \delta^S(2, 3, p + p^H),$$  \hfill (A.6.3)
with $\delta^S$ the chiral $\delta$ function given by
\[
\delta^S(1, 2, p) = -\frac{1}{4} \theta_1 \theta_2 e^{-\theta_1 \gamma_2 p},
\]
(A.6.4)
such that applying the normalisation condition (4.13) to (A.6.2) we find
\[
\mu \partial_\mu \left[ \frac{\partial}{\partial p^{H a}} \partial_\theta_1^2 \partial_\theta_2^2 \Gamma^{(1)}_H AA(p_H; p_1, p_2) \right]_{p_H^2 = p_{sym}^2 = \mu^2, \theta = \bar{\theta} = 0, s = 1} + \beta(1) - 2\gamma(1) = 0,
\]
(A.6.5)
with $p_1 = p, p_2 = p + p^H$. For the subsequent calculation it is crucial to note that the only terms in $\Gamma_H AA$ contributing to the first term in (A.6.5) are those of the form $f(p^H)\theta_1 \theta_2^2$; with $f$ some function of $p^H$, and no further $\theta, \bar{\theta}$ dependence. The contributions to $\Gamma^{(1)}_H AA$ are given by the graphs in figure 3 for the $\xi$ dependent part and by figure 4 for the $\xi$ independent part. All vertices of figure 2 contribute to the top vertex in both graphs of figure 4.

![Figure 3: $\xi$ dependent contribution to $\hat{\Gamma}_H AA$](image)

![Figure 4: $\xi$ independent contribution to $\hat{\Gamma}_H AA$](image)

We note that according to (4.3), the subtraction degree is $\delta = 1$. For the propagators it is useful to define
\[
G_{1, 2}(k, s) = \frac{i\delta^S(1, 2, k)}{k^2 - M^2(s - 1)^2 + i\varepsilon}.
\]
(A.6.6)
Then performing the subtraction of the integrand according to (4.4) we find for the \( \xi \) dependent part of the first order vertex function, shown in figure 3,

\[
\Gamma^{(1)}_{H^\alpha AA}(p^H, p) = \frac{2}{3} g^2 \int d^4k \, \theta_1^2 \theta_2^2 \left( p^H_b \left[ \partial_{p^H} b F_a(p^H, p, k, s = 0) \right]_{p^H=p=0} \right),
\]

(A.6.7)

where

\[
F_a(p^H, p, k, s) = 4M(s - 1)p^H_a G_{1,2}(p - k, s)D_1^2G_{1,3}(k, s)D_3^2G_{3,2}(p^H + k, s),
\]

(A.6.8)

and the limit \( s = 1 \) is taken at the end of the calculation. In (A.6.7) there is no contribution with a factor of \( p^H \theta_1^2 \theta_2^2 \) without further \( \bar{\theta} \) dependence such that (A.6.7) does not contribute to the first term of (A.6.5).

When calculating the \( \xi \) independent contributions to the first order vertex function as shown in figure 4, we see that there are potential contributions of the form \( f(p^H) \theta_1^2 \theta_2^2 \). However these cancel when symmetrising the \( HAA \) vertex, which corresponds to adding the two graphs in figure 4. We have

\[
\Gamma^{(1)}_{a HAA}(p^H, p) = \frac{1}{3} g^2 \int d^4k \left( p^H_a \left[ \partial_{p^H} a I(p^H, p, k, s = 0) \right]_{p^H=p=0} + p_a \left[ \partial_{p^H} a I(p^H, p, k, s = 0) \right]_{p^H=p=0} \right),
\]

(A.6.9)

with

\[
I(p^H, p, k, s) = M^2(s - 1)^2G_{1,2}(p - k, s) \left( \bar{D}_a D_a G_{1,3}(k, s)D^2G_{3,2}(k + p^H, s) \right.
- D_a G_{1,3}(k, s) \left. \bar{D}_a D^2G_{3,2}(k + p^H, s) - G_{1,3}(k, s)D_a \bar{D}_a D^2G_{3,2}(k + p^H, s) \right.
+ D^2G_{1,3}(k, s) \left. \bar{D}_a D_a G_{3,2}(k + p^H, s) + \bar{D}_a D^2G_{1,3}(k, s)D_a G_{3,2}(k + p^H, s) \right.
\]

(A.6.10)

where all derivatives act on the (3) coordinates and \( s = 1 \) is chosen after performing all subtractions. The first three terms in (A.6.10) correspond to the first graph in figure 4 and the last three to the second. Using \( \left\{ \bar{D}_a, D_a \right\} \delta^{S(1,2,p)} = 2\sigma^{a\bar{a}} p_a \delta^{S(1,2,p)} \) it may be checked that \( \frac{\partial}{\partial p^H} \partial_{\theta_1^2} \partial_{\theta_2^2} \Gamma^{(1)}_{H^\alpha AA} \) consists of a constant, momentum independent term. Thus the first term in (A.6.5) vanishes and we have

\[
\beta^{(1)} - 2\gamma^{(1)} \xi = 0.
\]

(A.6.11)
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