Variational Optimization for the Submodular Maximum Coverage Problem

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ABSTRACT

We examine the submodular maximum coverage problem (SMCP), which is related to a wide range of applications. We provide the first variational approximation for this problem based on the Nemhauser divergence, and show that it can be solved efficiently using variational optimization. The algorithm alternates between two steps: (1) an E step that estimates a variational parameter to maximize a parameterized modular lower bound; and (2) an M step that updates the solution by solving the local approximate problem. We provide theoretical analysis on the performance of the proposed approach and its curvature-dependent approximate factor, and empirically evaluate it on a number of public data sets and several application tasks.

1 INTRODUCTION

Submodular optimization lies at the core of many data mining and machine learning problems, ranging from summarizing massive data sets [1, 2], cutting and segmenting images [3–5], monitoring network status [6, 7], diversifying recommendation systems [8, 9], searching neural network architectures [10], interpreting machine learning models [11–14], to asset management and risk allocation in finance [15, 16]. Recent works have studied the optimization of submodular functions in various forms, for example, weighted coverage functions [17], rank functions of matroids [18], facility location functions [19], entropies [20], as well as mutual information [21]. In a typical setting, the optimization is subject to the classical cardinality constraint, where the number of elements selected is required to be under a preset constant limit. It’s been shown that even with this simple constraint, many submodular optimization problems are NP-hard, although under certain conditions the greedy algorithm can provide a good approximate solution [22–24].

The forms of constraints in real applications are often very complex and may be given either analytically or in terms of value oracle models. We, therefore, investigate a more generalized formulation, i.e., the problems of maximizing a submodular function \( g(X) \) subject to a general submodular upper bound constraint \( f(X) \leq b \). This problem is referred to as the submodular maximum coverage problem (SMCP), or submodular maximization with submodular knapsack constraint [25]. The pioneer work [25] first examined this problem and introduced an algorithms with bi-criterion approximation guarantees. The importance of SMCP has been widely recognized as it can be regarded as a meta-problem for a breadth of tasks including training the most accurate classifier subject to process unfairness constraints [26], automatically design convolutional neural networks to maximize accuracy with a given forward time constraint [27], and selecting leaders in a social network for shifting opinions [28], to name a few.

While [25] shows the greedy method with a modular approximation has good performance, we take a step further to build a mathematical connection between the variational modular approximation to a submodular function based on Namhauser divergence and classical variational approximation based on Kullback-Leibler divergence. We take advantage of this framework to iteratively solve SMCP, leading to a novel variational approach. Analogously to the counterpart of variational optimization based on Kullback-Leibler divergence, the proposed method consists of two alternating steps, namely estimation (E step) and maximization (M step) to monotonically improve the performance in an iterative fashion. We provide theoretical analysis on the performance of the proposed variational approach and prove that the E step provides the optimal estimator for the subsequent M step. More importantly, we show that the approximate factor of the EM algorithm is decided by the curvature of the objective function and the marginal gain of the constraint function. We evaluated the proposed framework on a number of public data sets and demonstrated it in several application tasks.

2 PROBLEM DEFINITION

2.1 Formulation

Submodularity is an important property that naturally exists in many real-world scenarios, for example, diminishing returns in economics [29], which refers to the phenomenon that the marginal benefit of any given element tend to decrease as more elements are added. Formally, let \([n] = \{1, 2, \ldots, n\}\) be a finite ground set and the set of all subsets of \([n]\) be \(2^n\). The real-valued discrete set function \( f : 2^n \rightarrow \mathbb{R} \) is submodular on \([n]\) if

\[
f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y)
\]

holds for all \( X, Y \subseteq [n] \) [30]. We denote a singleton set with element \( j \) as \( \{j\} \) and the marginal gain as \( f(j|X) \triangleq f(j \cup X) - f(X) \). The marginal gain is also known as the discrete derivative of \( f \) at \( X \) with respect to \( j \), and we use \( \Delta f \) to denote the maximum marginal gain at \( X = \emptyset \):

\[
\Delta f = \max_{j \in [n]} f(j).
\]

In terms of the marginal gain, the submodularity defined in (1) is equivalent to

\[
f(j|X) \geq f(j|Y), \quad \forall X \subseteq Y \subseteq [n], \quad j \not\in Y.
\]

Intuitively, the monotonicity means \( f \) won’t decrease as \( X \) is expanded. A necessary and sufficient monotone condition for \( f \) is that
where with the number of circle nodes smaller than \( SCKC \). The authors further established the equivalence between the configuration of a CNN (e.g., kernel size at each validation accuracy on a held out set of samples is submodular \[10\].

It's shown that the influence function is submodular, although the formulation of SMCP, where

\[
\max_{X} g(X), \quad \text{s.t. } f(X) \leq b,
\]

(4)

where \( f(X) \) and \( g(X) \) are monotone, and are assumed to be normalized such that \( f(\emptyset) = 0 \) and \( g(\emptyset) = 0 \). Our formulation is general enough with minimal assumptions, the techniques developed in this paper, including the analysis, are applicable to general forms of \( g(X) \) and \( f(X) \), including those with analytical forms or given in terms of a value oracle.\(^1\)

Fig 1 illustrates a concrete example of SMCP, where we are given a bipartite graph consisting of two kinds of nodes, i.e., square nodes and circle nodes; each circle node is associated with a non-negative value, and each square node represents a singleton; the goal is to select a subset, \( X \), of the square nodes, such that the circle nodes being covered have as much total value (denoted by \( g(X) \)) as possible yet the number of circle node selected is within a set limit, i.e., \( f(X) < b \).

2.2 Related Problems

The SMCP problem was first studied in \[25\], where it was also referred to as submodular cost with submodular knapsack constraint (SCCKC). The authors further established the equivalence between SMCP (4) and minimizing \( f(X) \) subject to \( g(X) \geq c \) (called submodular cost with submodular cover constraint or SCSC). A greedy algorithm and an ellipsoidal approximation method were employed to solve SMCP in \[25\].

SMCP is regarded as a meta-problem to many application tasks, of which we introduce a few examples. In \[26\], it was shown that training a classifier with fairness constraints involves solving a variant of SMCP, where \( X \) is the feature subset, and both the objective (i.e., loss function) and constraints are submodular. \[27\] studied automatically designing convolutional neural networks (CNNs) to maximize accuracy within a given forward time constraint, where \( X \) represents the configuration of a CNN (e.g., kernel size at each layer) and \( f(X) \) is the forward time function. It’s shown that the validation accuracy on a held out set of samples is submodular \[10\]. \[31\] investigated influence maximization in social networks and show the influence function is submodular, although the formulation is unconstrained. \[28\] studied French-Degroot opinion dynamics in a social network with two polarizing parties to shift opinions in a social network through leader selection. In their formulation, \( g(X) \) is the influence function and \( f(X) \) is the average opinion of all nodes, both of which are submodular.

Cardinality-constrained submodular maximization is a special case of SMCP since the cardinality \( |X| \) is a modular function. A number of important tasks can be approached by this simpler variant of SMCP, for example, data set summarization \[1,2\], network status monitoring \[6,7\], and interpretable machine learning \[11,13,14\].

3 VARIATIONAL BOUNDS

A submodular function resembles both convex functions and concave functions \[25\], in the sense that it can be bounded both from above and below. In this section, we propose variational SMCP (V-SMCP), a variational approximate for SMCP based on Nemhauser divergence.

3.1 Upper Bound for \( f(X) \)

In the seminal work \[22\], it is demonstrated that the submodularity of \( f(X) \) in (1) is equivalent to the following inequality

\[
f(X) - \sum_{j \in X \setminus Y} f(j|X\setminus j) + \sum_{j \in Y \setminus X} f(j|\Theta) \geq f(Y), \quad \forall X, Y \subseteq [n],
\]

(5)

\[
\Delta_{\Theta}(Y; X) \triangleq f(X; \Theta) - f(Y).
\]

where \( \Theta = X \cap Y \). Following from the above inequality, the Nemhauser divergence \[32\] between two set functions \( \Delta_{\Theta}(Y; X) \) and \( f(Y) \) is defined as

\[
D(\Delta_{\Theta}(Y; X)||f(Y)) = \Delta_{\Theta}(Y; X) - f(Y),
\]

(6)

which satisfies \( D(\Delta_{\Theta}(Y; X)||f(Y)) \geq 0 \). The equality holds when \( X = Y \), which implies \( \Theta = Y \). The Nemhauser divergence measures the distance between two set functions and is not symmetric, which is similar to the Kullback-Leibler divergence that measures the distance between two probability distributions.

Note that \[22\] provides another inequality, which is also equivalent to the submodularity of \( f(X) \), given by

\[
\Delta_{\Pi}(Y; X) \triangleq f(X) + \sum_{j \in Y \setminus X} f(j|X \setminus j) - \sum_{j \in X \setminus Y} f(j|\Psi j) \geq f(Y), \quad \forall X, Y \subseteq [n],
\]

(7)

with \( \Psi = X \cup Y \). We can therefore define the divergence with \( \Delta_{\Pi}(Y; X) \), i.e., \( D(\Delta_{\Pi}(Y; X)||f(Y)) = \Delta_{\Pi}(Y; X) - f(Y) \) for the variational optimization. Yet, as there is no guarantee that which one between these two functions provides a better approximation, we focus on \( D(\Delta_{\Theta}(Y; X)||f(Y)) \) in this paper, and all the algorithms and analyses provided can be adapted to the algorithm based on \( D(\Delta_{\Pi}(Y; X)||f(Y)) \).

\(^1\) For a given set \( X \), one can query an oracle to find its value \( f(X) \) and \( g(X) \), and both \( f(X) \) and \( g(X) \) could be computed by a black box.
3.2 Lower Bound for \( g(X) \)

We define a permutation on the elements of \([n]\), i.e., \( \pi : [n] \to [n] \) that orders the elements in \([n]\) as a sequence \((\pi_1, \pi_2, \ldots, \pi_n)\), which denotes that if \( \pi_i = j \), \( j \) is the \( i \)-th element in this sequence. Particularly, given a subset \( X_0 \subseteq [n] \), we choose a permutation \( \pi \) that places the elements in \( X_1 \) first and then includes the remaining elements in \([n]\) \( \setminus X_1 \), where the subscript \( t \) denotes the iteration number used in the EM algorithm introduced in the next section. We further define the corresponding sequence of subsets of \([n]\) as \( S^\pi_t \) with \( t = 0, \ldots, n \), which is given by

\[
S^\pi_0 = \emptyset, \quad S^\pi_1 = \{\pi_1\}, \ldots, S^\pi_n = \{\pi_1, \ldots, \pi_n\},
\]

which results in \( \emptyset = S^\pi_0 \subseteq S^\pi_1 \subseteq S^\pi_2 \cdots \subseteq S^\pi_n = [n] \). Then a lower bound of \( g(X) \) is given by [25]

\[
\tilde{\gamma}_X^\pi_t(X) = \sum_{j \in X} \tilde{\gamma}_X^\pi_t(j), \quad \forall X \subset [n],
\]

where \( \tilde{\gamma}_X^\pi_t(j) \) with \( j = \pi_i \) is defined by [25]

\[
\tilde{\gamma}_X^\pi_t(j) = \tilde{\gamma}_X^\pi_t(S^\pi_t - S^\pi_{t-1}) = g(S^\pi_t) - g(S^\pi_{t-1}).
\]

Since \( X_t \) and \( \pi \) has a mapping relationship, in the following of the paper, we omit the super script \( \pi \) when no confusing is caused. The lower bound property, i.e., \( \tilde{\gamma}_X^\pi(X) \leq g(X) \) can be easily proved [25] according to the submodularity. Further more, substituting (10) into (9) and considering the permutation given by \( \pi \), it guarantees the tightness at \( X_t \) that

\[
\tilde{\gamma}_X^\pi(X_t) = g(X_t).
\]

3.3 Variational Approximation for SMCP

The SMCP in (4) can be approximated, at any given \( X_t \), by the following problem, which we call \( V\text{-SMCP} \):

\[
\max_X \tilde{\gamma}_X^\pi(X)
\]

s.t. \( \tilde{f}_X^\pi(X; \hat{\Theta}_t) \leq b \),

\[
\hat{\Theta}_t = \arg \min_{\Theta} D(\tilde{f}_X^\pi(X; \Theta)||f(X)),
\]

\( \Theta = X \cap X_t \),

where \( \tilde{\gamma}_X^\pi(X) \) and \( \tilde{f}_X^\pi(X; \hat{\Theta}_t) \) are lower bound and upper bound for \( g(X) \) and \( f(X) \), respectively. V-SMCP is an effective approximation of SMCP as both bounds are tight at \( X_t \), i.e., \( \tilde{\gamma}_X^\pi(X_t) = g(X_t) \) and \( \tilde{f}_X^\pi(X_t; X_t) = f(X_t) \).

4 VARIATIONAL OPTIMIZATION

In this section, we introduce an iterative method to solve an SMCP based on a sequence of V-SMCPs. It alternates between (1) an estimation (E) step that minimizes the Nemhauser divergence by estimating the parametric approximation; and (2) a subsequent maximization (M) step that updates the solution.

Since \( \tilde{f}_X^\pi(X; \Theta) \) is an upper bound of \( f(X) \), maximizing \( \tilde{f}_X^\pi(X; \Theta) \) w.r.t \( \Theta \) will equivalently minimizing \( D_{\Theta}(\tilde{f}_X^\pi(X; \Theta)||f(X)) \). We therefore treat \( \Theta \) as a variational parameter and estimate it in the E step to reduce \( D_{\Theta}(\tilde{f}_X^\pi(X; \Theta)||f(X)) \) as much as possible. Then with \( \hat{\Theta}_t \), we update the solution by solving a V-SMCP in the M step. We name this method estimation-maximization (EM) algorithm.

4.1 E step: Estimate \( \hat{\Theta}_t \)

According to the submodularity definition in (3), we have \( f(j|\Theta_1) \geq f(j|\Theta_2) \) if \( \Theta_1 \subseteq \Theta_2 \). Following from (7), for all \( X \subseteq [n] \), we further obtain

\[
\tilde{f}_{X_t}(X; \Theta_1) \geq \tilde{f}_{X_t}(X; \Theta_2), \quad \forall X \subseteq [n].
\]

This inequality indicates that we can decrease the divergence of \( D(\tilde{f}_{X_t}(X; \Theta)||f(X)) \) by enlarging \( \Theta \). Thus, the largest \( \Theta \) is \( X_t \), according to the Nemhauser divergence defined in (7). To avoid notational clumsiness, we use \( E \backslash j \) to denote a set that excludes \( j \), i.e.,

\[
E \backslash j = \{X_t \cup X, \text{ if } j \notin X_t, \quad X_t, \text{ if } j \in X_t \}.
\]

By substituting (14) to (7), we define a permutation operation \( \epsilon : [n] \to [n] \) that orders the elements in \([n]\) as a new sequence \( \epsilon_1, \epsilon_2, \ldots, \epsilon_n \) such that

\[
\tilde{\gamma}_{X_t}(\epsilon_1) \geq \tilde{\gamma}_{X_t}(\epsilon_2) \geq \cdots \geq \tilde{\gamma}_{X_t}(\epsilon_n).
\]

There must exist a \( \tilde{k} \) such that

\[
\tilde{k} = \arg \max_{k} \sum_{k=1}^{n} f(\epsilon_k|E \backslash \epsilon_k) \leq b.
\]

We then obtain an estimation of \( \Theta \) given by

\[
\hat{\Theta}_t = X_t \cap \tilde{X}_t,
\]

with

\[
\tilde{X}_t = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_{k}\}.
\]

While there is no guarantee that \( \tilde{f}_{X_t}(\tilde{X}_t; \hat{\Theta}_t) \leq b \) is satisfied, the estimator \( \hat{\Theta}_t \) as well as \( \tilde{X}_t \) would lead to a larger feasible space for maximizing \( \tilde{\gamma}_{X_t}(X_t) \) in the subsequent M step, which is analytically proved in Section 5.

4.2 M Step: Compute the Maximizer \( X_{t+1} \)

For notational brief, in the M step, we represent a set without \( j \) given \( X_t \) as

\[
M \backslash j = \{X_t \cup X, \text{ if } j \notin X_t, \quad X_t, \text{ if } j \in X_t \}.
\]

Substituting (18) to (7), we further define a new permutation \( \mu : [n] \to [n] \) that orders the elements in \([n]\) as a new sequence \( \mu_1, \mu_2, \ldots, \mu_n \) such that

\[
\frac{\tilde{\gamma}_{X_t}(\mu_1)}{f(\mu_1|M \backslash \mu_1)} \geq \frac{\tilde{\gamma}_{X_t}(\mu_2)}{f(\mu_2|M \backslash \mu_1)} \geq \cdots \geq \frac{\tilde{\gamma}_{X_t}(\mu_n)}{f(\mu_n|M \backslash \mu_1)}.
\]

By letting \( \tilde{m} \) be the largest index that satisfy the following inequality:

\[
\tilde{m} = \arg \max_{m} \sum_{m=1}^{n} f(\mu_m|M \backslash \mu_m) \leq b,
\]

we finally obtain the optimizer at the \( t \)-th iteration:

\[
X_{t+1} = \{\mu_1, \mu_2, \ldots, \mu_{\tilde{m}}\}.
\]

From (9), the corresponding objective value is

\[
\tilde{\gamma}(X_{t+1}) = \sum_{m=1}^{\tilde{m}} \tilde{\gamma}_{X_t}(\mu_m).
\]
Algorithm 1 EM algorithm

Require: Initialization: $X_0$ {EM Algorithm.}
1: while $\hat{g}_{X_t}(X_t) \leq g_{X_t}(X_{t+1})$ do
2:  
3:  
4:  
5:  
6:  
7: endwhile

Algorithm 2 SEM algorithm

Require: Initialization: $X_0$ {SEM Algorithm.}
1: while $\hat{g}_{X_t}(X_t) \leq g_{X_t}(X_{t+1})$ do
2:  
3:  
4:  
5:  
6:  
7: endwhile

The algorithm terminates once $\hat{g}(X_{t+1}) \leq \hat{g}(X_t)$, which is equivalent to $g(X_{t+1}) \leq g(X_t)$ according to (11). The proposed EM algorithm is summarized in Algorithm 1. Note that in both E step and M step, the permutation $\epsilon$ and $\mu$ can be implemented in $O(n \log n)$ time through any efficient sorting procedure.

Fig. 2 shows how the EM algorithm approximates the solution of P1 in the space of $\mathcal{G}[n] \times \mathbb{R}$. The black curve represents the objective function under constraint in SMCP. At the $t$-th iteration, we construct $\hat{g}_{X_t}(X)$ with tightness guarantee at $X_t$ according to (11). In the E step, we compute $\hat{\Theta}_t$ to enlarge the feasible space, in the subsequent M step, we compute $X_{t+1}$, which is the approximate solution for P2. The corresponding function is shown by the red curve. Then at the $(t+1)$-th, we compute the new lower bound with estimation $\hat{\Theta}_{t+1}$ depicted by the blue color.

A simplified version of EM algorithm can be obtained by setting $\hat{\Theta}_t = \emptyset$ in the EM algorithm. This simplified EM (SEM) method saves the computation cost for the permutation $\epsilon$ in the E step. We summarize the SEM in Algorithm 2. However, it is evident that the E step of EM algorithm leads to larger or equal (when $\hat{\Theta} = \emptyset$ in (16)) feasible space than the SEM algorithm. Therefore, it is guaranteed that the EM algorithm has a no smaller objective value than SEM, which is also verified by experiments in Section 6.

5 THEORETICAL ANALYSIS

In this section, we provide analysis of the proposed EM algorithm. By replacing $X$ and $Y$ in (7) with $X_t$ and $X_{t+1}$, we obtain

$$b \geq f(X_t) - \sum_{j \in X_t \setminus X_{t+1}} f(j|X_t \setminus j) + \sum_{j \in X_{t+1} \setminus X_t} f(j|\emptyset) \geq f(X_{t+1}).$$

$$\hat{f}_{X_t}(X_{t+1}; \emptyset)$$

(23)

The quantities $\emptyset = X_t \cap X$ implies that $\emptyset \subseteq \Theta \subseteq X_t$. According to (13), we have $\hat{f}_{X_t}(X|\emptyset) \geq \hat{f}_{X_t}(X; \emptyset) \geq \hat{f}_{X_t}(X; X_t)$. Thus, $\hat{f}_{X_t}(X; X_t)$ is the tightest bound we can achieve. In spite of this, we cannot simply set $\hat{\Theta}_t = X_t$ since it is unsecured that $X_{t+1}$, which is obtained in the M step, satisfies $X_t \cap X_{t+1} = X_t$. Then there is no warranty that $\hat{f}_{X_t}(X_{t+1}; X_t) \geq f(X_{t+1})$. Consequently, $f(X_{t+1}) < b$ is not guaranteed, and $X_{t+1}$ may not lies in the feasible space, which violates the constraint in (12). In the following theorem, we analytically show the optimality of $\hat{\Theta}_t$ (equation (16) in the proposed E step). Here, an optimal $\hat{\Theta}_t$ implies that it provides the feasible space which is a superset of all the feasible space provided by any other $\Theta_t$’s.

**THEOREM 1 (Optimality).** In the E step of the EM algorithm, $\hat{\Theta}_t$, i.e., equation (16), provides the optimal $\hat{\Theta}_t$ for the optimization problem in the M step at each iteration.

**PROOF.** Because in the E step, we have no idea about the $X_{t+1}$, we need to estimate a $\hat{\Theta}_t$ such that for all possible $X_{t+1}$, the constraint $f(X_{t+1}) \leq \hat{f}_{X_t}(X_t; \emptyset)$ is always satisfied, so that $f(X_{t+1}) \leq b$ is guaranteed. Thus, according to (7), we need to show that $f(j|\emptyset)$ is larger than any $f(j|\emptyset)$. We prove this as follows.

First, according to (16), we have $\emptyset \subseteq \hat{\Theta}_t \subseteq X_t$. Then, (13) shows that setting $\hat{\Theta}_t = X_t$ leads to the smallest $f(e|\emptyset)$ at last. Due to the sorting mechanism in (15), the solution $\hat{X}_t$ is the smallest subset containing elements from all possible $X_{t+1}$, which makes any $\Theta_t$ that satisfies $\emptyset \subseteq \Theta_t \subseteq \hat{X}_t \cap X_t$ is a subset of $X_{t+1} \cap X_t$. Thus, we conclude that $\forall X_{t+1}$, $f(X_{t+1}) \leq \hat{f}_{X_t}(X_{t+1}; \emptyset)$ if $\emptyset \subseteq \Theta_t \subseteq \hat{X}_t \cap X_t$. Hence, in the feasible range, the optimal $\Theta$ is obtained by setting $\hat{\Theta}_t = \hat{X}_t \cap X_t$ as it gives the smallest $\hat{f}_{X_t}(X_{t+1}; \emptyset)$, i.e.,

$$\hat{f}_{X_t}(X_{t+1}; \emptyset) \leq \hat{f}_{X_t}(X_{t+1}; \emptyset)$$

for all the feasible $\Theta_t$. Therefore, it leads to the largest feasible space.

**PROPOSITION 2 (Monotonicity).** The EM algorithm monotonically improves the objective function value, i.e., $g(X)$ in the feasible space of SMCP, i.e., $g(X_0) \leq g(X_1) \leq g(X_2) \leq \ldots$

**PROOF.** According to the tightness property in (11), we have $g(X_t) = \hat{g}_{X_t}(X_t)$. Since the proposed EM algorithm leads to increment of $\hat{g}_{X_t}(X)$ at each iteration, we obtain $\hat{g}_{X_t}(X_t) \leq \hat{g}_{X_t}(X_{t+1})$. Moreover, the lower bound property of $\hat{g}_{X_t}(X)$ results in $\hat{g}_{X_t}(X_{t+1}) \leq \hat{g}_{X_t}(X_{t+1})$.
We first give the definition of curvature and then analytically prove \( \kappa \). Then by induction, we have

\[
\frac{\min_{k \in [n]} g(k|X \setminus k)}{g(k)} \geq (1 - \kappa_g) \forall k \in [n].
\]

Equation (27) implies that

\[
\frac{\sum_{j \in X} \frac{\hat{g}_X(j)}{g(j)}}{\sum_{j \in X} \frac{g(j)}{g(j)}} \geq (1 - \kappa_g) \forall X \subseteq [n].
\]

Next, we extend the above inequality from an arbitrary element \( j \in [n] \) to an arbitrary set \( X \subseteq [n] \) by induction. Equation (27) implies that

\[
\frac{\hat{g}_X(j_1) + \hat{g}_X(j_2)}{g(j_1) + g(j_2)} \geq 1 - \kappa_g, \forall j_1, j_2 \in [n].
\]

Then by induction, we have

\[
\frac{\sum_{j \in X} \hat{g}_X(j)}{\sum_{j \in X} g(j)} \geq 1 - \kappa_g, \forall X \subseteq [n].
\]

Next, we extend the approximation ratio of the proposed EM/SEM algorithm for \( \hat{g}_X(X) \) in a V-SMCP. Let OPT\( \hat{g}_X \) denote the optimizer for (12).

**Proposition 4.** At each iteration, both the EM and SEM algorithms obtain a set \( X_{t+1} \) such that

\[
\hat{g}_X(X_{t+1}) \geq (1 - \frac{2\lambda_f}{b})\hat{g}_X(OPT). \tag{31}
\]

The tedious but straightforward proof for this proposition is provided in the Appendix. Yet, this proposition paves the way to the proof of the approximation ratio of the EM algorithm for \( g(X) \) in V-SMCP. Let OPT denote the optimizer for (4), and with the knowledge of Theorem 3 and Proposition 4 in mind, we have the following result.

**Theorem 5 (Approximate Optimality).** The results of both EM and SEM algorithm, i.e., \( g(X_{t+1}) \) hold the approximation ratio

\[
g(X_{t+1}) \geq (1 - \kappa_g)(1 - \frac{2\lambda_f}{b})g(OPT), \tag{32}
\]

Proof. Since OPT\( \hat{g}_X \) is the optimizer of \( \hat{g}_X(X) \), we have the inequality \( g(OPT) \hat{g}_X(OPT) \geq (1 - \kappa_g)g(OPT) \). Further, due to \( g(x) \geq 0 \) for all \( X \subseteq [n] \), and following from (4), we obtain \( g(OPT) \geq 0 \). We then have \( \hat{g}_X(OPT) \hat{g}_X(OPT) \geq \hat{g}_X(OPT) \hat{g}_X(OPT) \), which results in

\[
\hat{g}_X(OPT) \hat{g}_X(OPT) \leq \frac{g(OPT)}{\hat{g}_X(OPT)} \leq \frac{1}{1 - \kappa_g}.
\]

**6 EXPERIMENTS**

Since first proposed in [25], the SMCP has been identified for a wide range of applications from training the most accurate classifier subject to process unfairness constraints [26], automatically designing convolutional neural networks to maximize accuracy within a given forward time constraint [27] to shifting opinions in a social network through leader selection [28].

In order to understand the mechanism, effectiveness, and application potential of the proposed variational framework and EM algorithm for SMCP, we start on the public data set and demonstrate the performance advantages over existing methods. After that, we test the performance in the production environment, first on decision rule selection for fraud transaction detection, and then go further to train a interpretable classifier that covers truth positive in the feature space well, and control the false positive within a predefined bound due to production requirement.

### 6.1 Performance on Discrete Location Data Sets

To compare the performance of our EM algorithm with that of existing methods, we consider four bipartite graphs from the public discrete location data sets [35] including an instance on perfer codes (PCodes), an instance on chess-board (Chess), an instance on finite projective planes (FPP), and an instance on large duality gap (Gap-A)
Figure 3: Comparison of objective values with public data sets.

Figure 4: Convergence and monotonicity of EM and SEM algorithms with different upper-bound constraints for the Gap-A data set.

Figure 5: $X_i$ updating process of the EM algorithm. In this example, the 61-th element was first selected in the 1-st iteration and later removed in the 3-rd iteration.

with 128, 144, 133, and 100 nodes for each type of the corresponding bipartite graphs, respectively. For more detailed information of these data sets, please refer to [35]. Fig. 1 is a running example of this test. A random value, which is uniformly sampled from 1 to 100, is assigned to each circle node, and our goal is to choose a subset of the square nodes to maximize the total sum-value of the covered circle nodes subject to an upper bound constraint of the total number of the square nodes.

We compare the greedy (Gr) algorithm, which was proposed in the classical work [25] and has been widely applied for different applications. Without an E step, it is analogous to the M step in the EM algorithm with a permutation $\varepsilon$ such that

$$
\bar{\tau}(i) \in \arg\max \left\{ g \left( j \mid \pi_{i-1} \right) \mid j \notin \pi_{i-1}, f \left( \pi_{i-1} \cup \{ j \} \right) \leq b \right\}.
$$

(34)

It was shown that Gr shows best performance in most experiments in [25]. We, therefore, compare the EM algorithm with the Gr as well as the SEM algorithms. The ellipsoidal approximation method in [25] is not applied here due to high computational complexity.

By considering 11 upper bounds in each kind of data set, we thus compare the performances of different algorithms in a total of 44 experiments. As shown in Fig. 3, our EM algorithm outperforms all other methods in all the 44 experiments except the only case when the upper bound is 60 in the FPP data set. Gr algorithms and SEM have overlaps with each other in some settings, yet most of the time Gr outperforms SEM. Interestingly, in the sub-figure (b), we notice that the Gr algorithm’s objective values cannot be increased when the constraint upper bound is increased from 55 to 60 as well as from 65 to 70. SEM also suffers from the same problem when the constraint upper bound is increased from 50 to 55, 60 to 65, and 80 to 85. Similar problems can also be identified for Gr and SEM in other data sets. However, it is rare to happen to EM. Thus, the experiment demonstrates that our EM algorithm, which enlarges the approximate feasible space in the E step, makes a better use of the feasible space of the SMCP.

We further test the convergence rate and the monotonicity of the EM algorithm by fixing the data set to be Gap-A and choosing four different upper bounds, i.e., 65, 70, 85, and 90. Fig. 4 shows the objective value versus EM/SEM iteration number. It demonstrates that our EM algorithm converges quickly within 3-5 iterations, and the objective value increases monotonically, which is consistent with Proposition 2. Note that as the initial values are set to be $\emptyset$, the first updates of EM and SEM are the same and hence the corresponding objective values after first iterations are the same.
Variational Optimization for the Submodular Maximum Coverage Problem

We consider four data sets in four different local areas, where each area has their own detection rules due to different attributes in each area. Each data set consists of transaction index and their labels (fraud or not), and the list of rules that cover each transaction. There are in total of 1200, 10052, 1600, and 2400 transactions in each data set, and the number of rules are 85, 92, 112, and 92 respectively.

As shown in Fig. 6 that our EM algorithm outperforms all other methods consistently for different upper bounds as well as in different data sets. Furthermore, Fig. 7 shows the objective value of as a function of the iteration number for data set 2 with different upper bounds. It is demonstrated that our EM algorithm also converges quickly within 3-5 iterations in the industrial environment.

### 6.3 Application to Interpretable Classifier

Following the same context of fraud detection, we further go beyond the rules selection scenario by modeling the problem as designing an interpretable classifier based on SMCP. From the lens of the classifier, we are interested in maximizing the true positive subject to an upper bounded false negative.

More specifically, given a bunch of features for each transactions, we first apply the efficient F-P algorithm [36] for mining the frequent fraud transaction patterns/rules. We limit the maximum rule length to be 4i to make it more interpretable. Let \([n]\) denote the set of rules obtained. To detect as many fraud value as possible (which is equivalently to maximize the truth positive), we maximize the following objective function:

$$g(X) = v(\bigcup_{i \in X} CR(i)),$$

where \(X \subseteq [n]\), \(CR(r_i)\) is the set of frauds covered by rule \(i\), and \(v(\cdot)\) is the total amount of fraud transaction value covered by \(X\). Moreover, the number of interrupted transactions, i.e., normal transactions but classified mistakenly, can be denoted by

$$f(X) = |\bigcup_{i \in X} C(i) \setminus CR(i)|,$$

where \(C(i)\) denotes all the transactions covered by rule \(i\), either correctly or wrongly. We then can train a classifier that is consist of the rules selected by maximizing \(g(X)\) subject to an constraint that \(f(X) < b\). According to the submodularity definition in (3), both \(f(X)\) and \(g(X)\) are monotonic submodular functions. Consequently, training the classifier is equivalent to solving an SMCP. We therefore apply our EM algorithm to train this classifier.

We summarize the data set in Table 1 and split 75% of the data into a training set and 25% of the data into a testing set. For performance comparison, we choose a decision tree with a maximum depth of 4. We summarize the result in Table 2. It shows that the EM based method achieves performance that covers more fraud amount and also achieves less interruptions. The advantages could come from the formulation that builds the classifier, which exchanges false positive and true negative to identify as many frauds as possible in the feasible space.
Table 1: Summary of transaction data set

| # samples | # fraud | # normal | # features | # categorical features | # continuous features |
|-----------|--------|----------|------------|-----------------------|----------------------|
| 50,357    | 369    | 49,988   | 50         | 26                    | 24                   |

Table 2: Classification performance

| Method | Fraud coverage | Intervention rate |
|--------|----------------|-------------------|
| Decision Tree | 82.16% | 1.05% |
| EM     | 83.73% | 0.96% |

7 CONCLUSIONS

In this paper, we have proposed a novel variational frame based on the Nemhauser divergence for the submodular maximum coverage problem (SMCP). The proposed estimation-and-maximization (EM) method monotonically improves optimization performance in a few iterations. We have further proved a curvature dependent approximate factor for the EM method. Empirical results on both public data sets and industrial problems in production environment have shown evident performance improvement over state-of-the-art algorithms.

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A PROOF OF PROPOSITION 4

For the independence of the appendix, we repeat (19) with the first \( \bar{m} + 1 \) terms as below.

\[
\frac{\mathcal{g}^\pi_{X_1}(\mu_1)}{f(\mu_1|M\setminus\mu_1)} \geq \frac{\mathcal{g}^\pi_{X_2}(\mu_2)}{f(\mu_2|M\setminus\mu_2)} \ldots \geq \frac{\mathcal{g}^\pi_{X_{\bar{m}+1}}(\mu_{\bar{m}+1})}{f(\mu_{\bar{m}+1}|M\setminus\mu_{\bar{m}+1})}. \tag{35}
\]

Then by induction we have

\[
\sum_{k=1}^{\bar{m}+1} \frac{\mathcal{g}^\pi_{X_k}(\mu_k)}{f(\mu_k|M\setminus\mu_k)} \geq \frac{\mathcal{g}^\pi_{X_{\bar{m}+1}}(\mu_{\bar{m}+1})}{f(\mu_{\bar{m}+1}|M\setminus\mu_{\bar{m}+1})}. \tag{36}
\]

According to the definition of \( \bar{m} \) in (20), it is evident that

\[
\sum_{k=1}^{\bar{m}+1} f(\mu_k|M\setminus\mu_k) \geq b.
\]

Substituting the above inequality to (36), it holds that

\[
\mathcal{g}^\pi_{X_{\bar{m}+1}}(\mu_{\bar{m}+1}) \leq \frac{1}{b} f(\mu_{\bar{m}+1}|M\setminus\mu_{\bar{m}+1}) \sum_{k=1}^{\bar{m}+1} \mathcal{g}^\pi_{X_k}(\mu_k)
\leq \frac{1}{b} f(\mu_{\bar{m}+1}|\emptyset) \sum_{k=1}^{\bar{m}+1} \mathcal{g}^\pi_{X_k}(\mu_k)
\leq \frac{\Delta_f}{b} \sum_{k=1}^{\bar{m}+1} \mathcal{g}^\pi_{X_k}(\mu_k). \tag{37}
\]

The second inequality is due to (13), and the third inequality follows from (2). By subtracting \( \frac{\Delta_f}{b} \mathcal{g}^\pi_{X_{\bar{m}+1}}(\mu_{\bar{m}+1}) \) on the left-hand side and then adding \( \sum_{k=1}^{\bar{m}} \mathcal{g}^\pi_{X_k}(\mu_k) \) on both sides, we obtain

\[
\sum_{k=1}^{\bar{m}} \mathcal{g}^\pi_{X_k}(\mu_k) + (1 - \frac{\Delta_f}{b}) \mathcal{g}^\pi_{X_{\bar{m}+1}}(\mu_{\bar{m}+1}) \leq \frac{\Delta_f}{b} \sum_{k=1}^{\bar{m}+1} \mathcal{g}^\pi_{X_k}(\mu_k) + \sum_{k=1}^{\bar{m}} \mathcal{g}^\pi_{X_k}(\mu_k),
\]

which equals

\[
\sum_{k=1}^{\bar{m}} \mathcal{g}^\pi_{X_k}(\mu_k) - \frac{\Delta_f}{b} \mathcal{g}^\pi_{X_{\bar{m}+1}}(\mu_{\bar{m}+1}) \leq \frac{\Delta_f}{b} \sum_{k=1}^{\bar{m}+1} \mathcal{g}^\pi_{X_k}(\mu_k) + \sum_{k=1}^{\bar{m}} \mathcal{g}^\pi_{X_k}(\mu_k). \tag{38}
\]

The above inequality still holds after subtracting a positive \( \sum_{k=1}^{\bar{m}} \mathcal{g}^\pi_{X_k}(\mu_k) \) on the left-hand side:

\[
(1 - \frac{2\Delta_f}{b}) \sum_{k=1}^{\bar{m}+1} \mathcal{g}^\pi_{X_k}(\mu_k) \leq \sum_{k=1}^{\bar{m}} \mathcal{g}^\pi_{X_k}(\mu_k). \tag{40}
\]

From (22) \( \sum_{k=1}^{\bar{m}} \mathcal{g}^\pi_{X_k}(\mu_k) = \mathcal{g}(X_{\bar{m}+1}) \), and considering the fact that \( \sum_{k=1}^{\bar{m}+1} \mathcal{g}^\pi_{X_k}(\mu_k) \geq \OPT \), we finally prove Proposition 4:

\[
\mathcal{g}(X_{\bar{m}+1}) \geq (1 - \frac{2\Delta_f}{b}) \OPT.
\]