Recent Developments in Seiberg-Witten Theory and Complex Geometry

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0 Introduction

About two years ago, in October 1994, E. Witten revolutionized the theory of 4-manifolds by introducing the now famous Seiberg-Witten invariants [W]. These invariants are defined by counting gauge equivalence classes of solutions of the Seiberg-Witten monopole equations, a system of non-linear PDE’s which describe the absolute minima of a Yang-Mills-Higgs type functional with an abelian gauge group.

In a very short period of only a few weeks after Witten’s seminal paper became available, several long-standing conjectures were solved, many new and totally unexpected results were found, and much simpler and more conceptional proofs of already established theorems were given.

Among the most spectacular applications in this early period are the solution of the Thom conjecture [KM], new results about Einstein metrics and Riemannian metrics of positive scalar curvature [L1] [L2], a proof of a $10/8$ bound for intersection forms of $\text{Spin}$ manifolds [F], as well as several results about the $C^\infty$-classification of algebraic surfaces [OT1], [OT2], [FM], [Bru]. The latter include Witten’s proof of the $C^\infty$-invariance of the canonical class of a minimal surface of general type with $b_+ \neq 1$ up to sign, and a simple proof of the Van de Ven conjecture by the authors [1].

In two of the earliest papers on the subject, C. Taubes found a deep connection between Seiberg-Witten theory and symplectic geometry in dimension four: He first showed that many aspects of the new theory extend from the case of Kähler surfaces to the more general symplectic case [Ta1], and then he went on to establish a beautiful relation between Seiberg-Witten invariants and Gromov-Witten invariants of symplectic 4-manifolds [Ta2].

A report on some papers of this first period can be found in [D].

\footnote{Combining results in [L1], [L2] with ideas from [OT2], P. Lupaşcu recently obtained the optimal characterization of complex surfaces of Kähler type admitting Riemannian metrics of non-negative scalar curvature [Lu]}
Since the time this report was written, several new developments have taken place:

The original Seiberg-Witten theory, as introduced in [W], has been refined and extended to the case of manifolds with $b_+ = 1$. The structure of the Seiberg-Witten invariants is more complicated in this situation, since the invariants for manifolds with $b_+ = 1$ depend on a chamber structure. The general theory, including the complex-geometric interpretation in the case of Kähler surfaces, is now completely understood [OT6].

At present, three major directions of research have emerged:

- Seiberg-Witten theory and symplectic geometry
- Non-abelian Seiberg-Witten theory and complex geometry
- Seiberg-Witten-Floer theory and contact structures

In this article, which had its origin in the notes for several lectures which we gave in Berkeley, Bucharest, Paris, Rome and Zürich during the past two years, we concentrate mainly on the second of these directions.

The reader will probably notice that the non-abelian theory is a subject of much higher complexity than the original (abelian) Seiberg-Witten theory; the difference is roughly comparable to the difference between Yang-Mills theory and Hodge theory. This complexity accounts for the length of the article. In rewriting our notes, we have tried to describe the essential constructions as simply as possible but without oversimplifying, and we have made an effort to explain the most important ideas and results carefully in a non-technical way; for proofs and technical details precise references are given.

We hope that this presentation of the material will motivate the reader, and we believe that our notes can serve as a comprehensive introduction to an interesting new field of research.

We have divided the article in three chapters. In chapter 1 we give a concise but complete exposition of the basics of abelian Seiberg-Witten theory in its most general form. This includes the definition of refined invariants for manifolds with $b_1 \neq 0$, the construction of invariants for manifolds with $b_+ = 1$, and the universal wall crossing formula in this situation.

Using this formula in connection with vanishing and transversality results, we calculate the Seiberg-Witten invariant for the simplest non-trivial example, the projective plane.

In chapter 2 we introduce non-abelian Seiberg-Witten theories for rather general structure groups $G$. After a careful exposition of $Spin^G$-structures
and $G$-monopoles, and a short description of some important properties of their moduli spaces, we explain one of the main results of the Habilitationsschrift of the second author [T2], [T3]: the fundamental Uhlenbeck type compactification of the moduli spaces of $PU(2)$-monopoles.

Chapter 3 deals with complex-geometric aspects of Seiberg-Witten theory: We show that on Kähler surfaces moduli spaces of $G$-monopoles, for unitary structure groups $G$, admit an interpretation as moduli spaces of purely holomorphic objects. This result is a Kobayashi-Hitchin type correspondence whose proof depends on a careful analysis of the relevant vortex equations. In the abelian case it identifies the moduli spaces of twisted Seiberg-Witten monopoles with certain Douady spaces of curves on the surface [OT1]. In the non-abelian case we obtain an identification between moduli spaces of $PU(2)$-monopoles and moduli spaces of stable oriented pairs [OT5], [T2].

The relevant stability concept is new and makes sense on Kähler manifolds of arbitrary dimensions; it is induced by a natural moment map which is closely related to the projective vortex equation. We clarify the connection between this new equation and the parameter dependent vortex equations which had been studied in the literature [Br]. In the final section we construct moduli spaces of stable oriented pairs on projective varieties of any dimension with GIT methods [OST]. Our moduli spaces are projective varieties which come with a natural $\mathbb{C}^*$-action, and they play the role of master spaces for stable pairs. We end our article with the description of a very general construction principle which we call ”coupling and reduction”. This fundamental principle allows to reduce the calculation of correlation functions associated with vector bundles to a computation on the space of reductions, which is essentially a moduli space of lower rank objects.

Applied to suitable master spaces on curves, our principle yields a conceptional new proof of the Verlinde formulas, and very likely also a proof of the Vafa-Intriligator conjecture. The gauge theoretic version of the same principle can be used to prove Witten’s conjecture, and more generally, it will probably also lead to formulas expressing the Donaldson invariants of arbitrary 4-manifolds in terms of Seiberg-Witten invariants.

1 Seiberg-Witten invariants
1.1 The monopole equations

Let \((X, g)\) be a closed oriented Riemannian 4-manifold. We denote by \(\Lambda^p\) the bundle of \(p\)-forms on \(X\) and by \(A^p := \Lambda^0(\Lambda^p)\) the corresponding space of sections. Recall that the Riemannian metric \(g\) defines a Hodge operator \(* : \Lambda^p \to \Lambda^{4-p}\) with \(*^2 = (-1)^p\). Let \(\Lambda^2 = \Lambda_+^2 \oplus \Lambda_-^2\) be the corresponding eigenspace decomposition.

A \(\text{Spin}^c\)-structure on \((X, g)\) is a triple \(\tau = (\Sigma^\pm, \iota, \gamma)\) consisting of a pair of \(U(2)\)-vector bundles \(\Sigma^\pm\), a unitary isomorphism \(\iota : \det \Sigma^+ \to \det \Sigma^-\) and an orientation-preserving linear isometry \(\gamma : \Lambda^1 \to \mathbb{R}SU(\Sigma^+, \Sigma^-)\). Here \(\mathbb{R}SU(\Sigma^+, \Sigma^-) \subset \text{Hom}_C(\Sigma^+, \Sigma^-)\) is the subbundle of real multiples of (fibre-wise) isometries of determinant 1. The spinor bundles \(\Sigma^\pm\) of \(\tau\) are – up to isomorphism – uniquely determined by their first Chern class \(c := \det \Sigma^\pm\), the Chern class of the \(\text{Spin}^c(4)\)-structure \(\tau\). This class can be any integral lift of the second Stiefel-Whitney class \(w_2(X)\) of \(X\), and, given \(c\), we have

\[
c_2(\Sigma^\pm) = \frac{1}{4}(c^2 - 3\sigma(X) \mp 2e(X)) .
\]

Here \(\sigma(X)\) and \(e(X)\) denote the signature and the Euler characteristic of \(X\).

The map \(\gamma\) is called the Clifford map of the \(\text{Spin}^c\)-structure \(\tau\). We denote by \(\Sigma\) the total spinor bundle \(\Sigma := \Sigma^+ \oplus \Sigma^-\), and we use the same symbol \(\gamma\) also for the induced the map \(\Lambda^1 \to su(\Sigma)\) given by

\[
u \mapsto \begin{pmatrix} 0 & -\gamma(u)^* \\ \gamma(u) & 0 \end{pmatrix}.
\]

Note that the Clifford identity

\[
\gamma(u)\gamma(v) + \gamma(v)\gamma(u) = -2g(u, v)
\]

holds, and that the formula

\[
\Gamma(u \wedge v) := \frac{1}{2} [\gamma(u), \gamma(v)]
\]

defines an embedding \(\Gamma : \Lambda^2 \to su(\Sigma)\) which maps \(\Lambda^2_\pm\) isometrically onto \(su(\Sigma^\pm) \subset su(\Sigma)\).

The second cohomology group \(H^2(X, \mathbb{Z})\) acts on the set of equivalence classes \(c\) of \(\text{Spin}^c(4)\)-structures on \((X, g)\) in a natural way: Given a representative \(\tau = (\Sigma^\pm, \iota, \gamma)\) of \(c\) and a Hermitian line bundle \(M\) representing...
a class \( m \in H^2(X, \mathbb{Z}) \), the tensor product \((\Sigma^\pm \otimes M, \iota \otimes \text{id}_{M^{\otimes 2}}, \gamma \otimes \text{id}_M)\) defines a \( Spin^c \)-structure \( \tau_m \). Endowed with the \( H^2(X, \mathbb{Z}) \)-action given by \((m, [\tau]) \mapsto [\tau_m]\), the set of equivalence classes of \( Spin^c \)-structures on \((X, g)\) becomes a \( H^2(X, \mathbb{Z}) \)-torsor, which is independent of the metric \( g \) up to canonical isomorphism \([OT6]\). We denote this \( H^2(X, \mathbb{Z}) \)-torsor by \( Spin^c(X) \).

Recall that the choice of a \( Spin^c(4) \)-structure \((\Sigma^\pm, \iota, \gamma)\) defines an isomorphism between the affine space \( A(\det \Sigma^+) \) of unitary connections in \( \det \Sigma^+ \) and the affine space of connections in \( \Sigma^\pm \) which lift the Levi-Civita connection in the bundle \( \Lambda^2_\pm \simeq su(\Sigma^\pm) \). We denote by \( \hat{a} \in A(\Sigma) \) the connection corresponding to \( a \in A(\det \Sigma^+) \).

The Dirac operator associated with the connection \( a \in A(\det \Sigma^+) \) is the composition
\[
\mathcal{D}_a : A^0(\Sigma^\pm) \xrightarrow{\nabla_a} A^1(\Sigma^\pm) \xrightarrow{\gamma} A^0(\Sigma^\mp)
\]
of the covariant derivative \( \nabla_a \) in the bundles \( \Sigma^\pm \) and the Clifford multiplication \( \gamma : \Lambda^1 \otimes \Sigma^\pm \rightarrow \Sigma^\mp \).

Note that, in order to define the Dirac operator, one needs a Clifford map, not only a Riemannian metric and a pair of spinor bundles; this will later become important in connection with transversality arguments. The Dirac operator \( \mathcal{D}_a : A^0(\Sigma^\pm) \rightarrow A^0(\Sigma^\mp) \) is an elliptic first order operator with symbol \( \gamma : \Lambda^1 \rightarrow \mathbb{R}SU(\Sigma^\pm, \Sigma^\mp) \). The direct sum-operator \( \mathcal{D}_a : A^0(\Sigma) \rightarrow A^0(\Sigma) \) on the total spinor bundle is selfadjoint and its square has the same symbol as the rough Laplacian \( \nabla^2 \) on \( A^0(\Sigma) \).

The corresponding Weitzenböck formula is
\[
\mathcal{D}_a^2 = \nabla^*_a \nabla_a + \frac{1}{2} \Gamma(F_a) + \frac{s}{4} \text{id}_\Sigma ,
\]
where \( F_a \in iA^2 \) is the curvature of the connection \( a \), and \( s \) denotes the scalar curvature of \((X, g)\) \([LM]\).

To write down the Seiberg-Witten equations, we need the following notations: For a connection \( a \in A(\det \Sigma^+) \) we let \( F^\pm_a \in iA^2_{\pm} \) be the (anti) selfdual components of its curvature. Given a spinor \( \Psi \in A^0(\Sigma^+) \), we denote by \( (\Psi \overline{\Psi})_0 \in A^0(\text{End}_0(\Sigma^\pm)) \) the trace free part of the Hermitian endomorphism \( \Psi \otimes \overline{\Psi} \). Now fix a \( Spin^c(4) \)-structure \( \tau = (\Sigma^\pm, \iota, \gamma) \) for \((X, g)\) and a closed 2-form \( \beta \in A^2 \). The \( \beta \)-twisted monopole equations for a pair \((a, \Psi) \in A(\det \Sigma^+) \times A^0(\Sigma^+)\) are
\[
\begin{cases}
\mathcal{D}_a \Psi = 0 \\
\Gamma(F^+_a + 2\pi i\beta^+) = (\Psi \overline{\Psi})_0.
\end{cases}
\tag{SW^\beta_{\tau}}
\]
These $\beta$-twisted Seiberg-Witten equations should not be regarded as perturbations of the equations $(SW_0)$ since later the cohomology class of $\beta$ will be fixed. The twisted equations arise naturally in connection with non-abelian monopoles (see section 2.2). Using the Weitzenböck formula one gets easily

**Lemma 1.1.1** Let $\beta$ be a closed 2-form and $(a, \Psi) \in \mathcal{A}(\text{det } \Sigma^+) \times \mathcal{A}^0(\Sigma^+)$. Then we have the identity

\[
\| \mathcal{D}_a \Psi \|^2 + \frac{1}{4} \| (F^+ - 2\pi i \beta^+) - (\Psi \bar{\Psi})_0 \|^2 =
\| \nabla A \Psi \| + \frac{1}{4} \| F^+ + 2\pi i \beta^+ \|^2 + \frac{1}{8} \| \Psi \|_{L^4}^4 + \int_X (\mathcal{F}_{(4}) \text{id}_{\Sigma^+} - \Gamma(\pi i \beta^+)) \Psi, \Psi).
\]

**Corollary 1.1.2** [W] On manifolds $(X, g)$ with non-negative scalar curvature $s$ the only solutions of $(SW_0)$ are pairs $(a, 0)$ with $F^+_a = 0$.

### 1.2 Seiberg-Witten invariants for 4-manifolds with $b_+ > 1$

Let $(X, g)$ be a closed oriented Riemannian 4-manifold, and let $c \in \text{Spin}(X)$ be an equivalence class of $\text{Spin}^c$-structures of Chern class $c$, represented by the triple $\tau = (\Sigma^\pm, \iota, \gamma)$. The configuration space for Seiberg-Witten theory is the product $\mathcal{A}(\text{det } \Sigma^+) \times \mathcal{A}^0(\Sigma^+)$ on which the gauge group $\mathcal{G} := \text{C}^\infty(X, S^1)$ acts by

\[ f \cdot (a, \Psi) := (a - 2f^{-1}df, f\Psi). \]

Let $\mathcal{B}(c) := \mathcal{A}(\text{det } \Sigma^+) \times \mathcal{A}^0(\Sigma^+)/\mathcal{G}$ be the orbit space; it depends up to homotopy equivalence only on the Chern class $c$. Since the gauge group acts freely in all points $(a, \Psi)$ with $\Psi \neq 0$, the open subspace

\[ \mathcal{B}(c)^* := \mathcal{A}(\text{det } \Sigma^+) \times \left( \mathcal{A}^0(\Sigma^+) \setminus \{0\} \right)/\mathcal{G} \]

is a classifying space for $\mathcal{G}$. It has the weak homotopy type of a product $K(\mathbb{Z}, 2) \times K(H^1(X, \mathbb{Z}), 1)$ of Eilenberg-Mac Lane spaces and there is a natural isomorphism

\[ \nu : \mathbb{Z}[u] \otimes \Lambda^*(H^1(X, \mathbb{Z})/\text{Tors}) \rightarrow H^*(\mathcal{B}(c)^*, \mathbb{Z}), \]

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where the generator $u$ is of degree 2. The $G$-action on $\mathcal{A}(\det \Sigma^+) \times A^0(\Sigma^+)$ leaves the subset $[\mathcal{A}(\det \Sigma^+) \times A^0(\Sigma^+)]^{SW^\tau}$ of solutions of $(SW^\tau)$ invariant; the orbit space

$$W^\tau := [\mathcal{A}(\det \Sigma^+) \times A^0(\Sigma^+)]^{SW^\tau}_G$$

is the moduli space of $\beta$-twisted monopoles. It depends, up to canonical isomorphism, only on the metric $g$, on the closed 2-form $\beta$, and on the class $c \in Spin^c(X)$ [OT6].

Let $W^\tau_{\beta}^* \subset W^\tau_{\beta}$ be the open subspace of monopoles with non-vanishing spinor-component; it can be described as the zero-locus of a section in a vector-bundle over $\mathcal{B}(c)^*$. The total space of this bundle is

$$[\mathcal{A}(\det \Sigma^+) \times (A^0(\Sigma^+) \setminus \{0\})] \times_G [iA^2_+ \oplus A^0(\Sigma^-)]$$

and the section is induced by the $G$-equivariant map

$$SW^\tau : \mathcal{A}(\det \Sigma^+) \times (A^0(\Sigma^+) \setminus \{0\}) \rightarrow iA^2_+ \oplus A^0(\Sigma^-)$$
given by the equations $(SW^\tau)$.

Completing the configuration space and the gauge group with respect to suitable Sobolev norms, we can identify $W^\tau_{\beta}^*$ with the zero set of a real analytic Fredholm section in the corresponding Hilbert vector bundle on the Sobolev completion of $\mathcal{B}(c)^*$, hence we can endow this moduli space with the structure of a finite dimensional real analytic space. As in the instanton case, one has a Kuranishi description for local models of the moduli space around a given point $[a, \Psi] \in W^\tau_{\beta}$ in terms of the first two cohomology groups of the elliptic complex

$$0 \longrightarrow iA^0 \stackrel{D^0}{\longrightarrow} iA^1 \oplus A^0(\Sigma^+) \stackrel{D^1}{\longrightarrow} iA^2_+ \oplus A^0(\Sigma^-) \longrightarrow 0$$

obtained by linearizing in $p = (a, \Psi)$ the action of the gauge group and the equivariant map $SW^\tau_{\beta}$. The differentials of this complex are

$$D^0_p(f) = (-2df, f\Psi),$$

$$D^2_p(\alpha, \psi) = \left( d^+ \alpha - \Gamma^{-1}[(\Psi \bar{\psi})_0 + (\psi \bar{\Psi})_0]_0, \mathcal{D}_0(\psi) + \gamma(a)(\Psi) \right),$$

and its index $w_c$ depends only on the Chern class $c$ of the $Spin^c$-structure $\tau$ and on the characteristic classes of the base manifold $X$:

$$w_c = \frac{1}{4}(c^2 - 3\sigma(X) - 2\varepsilon(X)).$$
The moduli space $\mathcal{W}_\beta$ is compact. This follows, as in [KM], from the following consequence of the Weitzenböck formula and the maximum principle.

**Proposition 1.2.1** *(A priori $C^0$-bound of the spinor component)* If $(a, \Psi)$ is a solution of $(SW^\tau_\beta)$ then

$$\sup_x |\Psi|^2 \leq \max \left( 0, \sup_X (-s + |4\pi \beta^+|) \right).$$

Moreover, let $\tilde{\mathcal{W}}^\tau$ be the moduli space of triples $(a, \Psi, \beta) \in \mathcal{A}(\det \Sigma^+) \times A^0(\Sigma^+) \times \mathbb{Z}_{DR}^2$ solving the Seiberg-Witten equations above now regarded as equations for $(a, \Psi, \beta)$. Two such triples define the same point in $\tilde{\mathcal{W}}^\tau$ if they are congruent modulo the gauge group $\mathcal{G}$ acting trivially on the third component. Using the proposition above and arguments of [KM], one can easily see that the natural projection $\tilde{\mathcal{W}}^\tau \to \mathbb{Z}_{DR}^2$ is proper. Moreover, one has the following transversality results:

**Lemma 1.2.2** After suitable Sobolev completions the following holds:
1. *[KM]* The open subspace $[\tilde{\mathcal{W}}^\tau]^* \subset \tilde{\mathcal{W}}^\tau$ of points with non-vanishing spinor component is smooth.
2. *[OT6]* For every de Rham cohomology class $b \in H^2_{DR}(X)$ the moduli space $[\tilde{\mathcal{W}}^\tau]^* := [\tilde{\mathcal{W}}^\tau]^* \cap p^{-1}(b)$ is also smooth.

Now let $c \in H^2(X, \mathbb{Z})$ be a characteristic element, i.e. an integral lift of $w_2(X)$. A pair $(g, b) \in \mathcal{M}et(X) \times H^2_{DR}(X)$ consisting of a Riemannian metric $g$ on $X$ and a de Rham cohomology class $b$ is called $c$-good when the $g$-harmonic representant of $c-b$ is not $g$-antiselfdual. This condition guarantees that $\mathcal{W}^\tau_\beta = \mathcal{W}^\tau_\beta^*$ for every $\text{Spin}^c$-structure $\tau$ of Chern class $c$ and every 2-form $\beta$ in $b$. Indeed, if $(a, 0)$ would solve $(SW^\tau_\beta)$, then the $g$-antiselfdual 2-form $\frac{i}{2\pi} F_a - \beta$ would be the $g$-harmonic representant of $c - b$.

In particular, using the transversality results above, one gets the following

**Theorem 1.2.3** *[OT6]* Let $c \in H^2(X, \mathbb{Z})$ be a characteristic element and suppose $(g, b) \in \mathcal{M}et(X) \times H^2_{DR}(X)$ is $c$-good. Let $\tau$ be a $\text{Spin}^c$-structure of Chern class $c$ on $(X, g)$, and $\beta \in b$ a general representant of the cohomology class $b$. Then the moduli space $\mathcal{W}^\tau_\beta = \mathcal{W}^\tau_\beta^*$ is a closed manifold of dimension $w_c = \frac{1}{4}(c^2 - 3\sigma(X) - 2e(X))$. 

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Let us fix a maximal subspace \( H_2^+(X, \mathbb{R}) \) of \( H^2(X, \mathbb{R}) \) on which the intersection form is positive definite. The dimension \( b_+(X) \) of such a subspace is the number of positive eigenvalues of the intersection form. The moduli space \( W_\beta \) can be oriented by the choice of an orientation of the line \( \det H^1(X, \mathbb{R}) \otimes \det H_2^+(X, \mathbb{R})^\vee \).

Let \( [W_\beta]_\sigma \in H_{w_2}(B(c)^*, \mathbb{Z}) \) be the fundamental class associated with the choice of an orientation \( \sigma \) of the line \( \det H^1(X, \mathbb{R}) \otimes \det H_2^+(X) \).

The Seiberg-Witten form associated with the data \((g, b, c, \sigma)\) is the element \( SW_{X, \sigma}^{(g, b)}(c) \in \Lambda^*(H^1(X, \mathbb{Z})) \) defined by

\[
SW_{X, \sigma}^{(g, b)}(c)(l_1 \wedge \ldots \wedge l_r) := \left< \nu(l_1) \cup \ldots \nu(l_r) \cup u\frac{w_2}{2}, [W_\beta]_\sigma \right>
\]

for decomposable elements \( l_1 \wedge \ldots \wedge l_r \) with \( r \equiv w_2 \mod 2 \). Here \( \tau \) is a Spin\(^c\)-structure on \((X, g)\) representing the class \( c \in Spin(X) \), and \( \beta \) is a general form in the class \( b \).

One shows, using again transversality arguments, that the Seiberg-Witten form \( SW_{X, \sigma}^{(g, b)}(c) \) is well defined, independent of the choices of \( \tau \) and \( \beta \). Moreover, if any two \( c \)-good pairs \((g_0, b_0), (g_1, b_1)\) can be joined by a smooth path of \( c \)-good pairs, then \( SW_{X, \sigma}^{(g, b)}(c) \) is also independent of \((g, b)\) [OT6].

Note that the condition \( "(g, b)\) is not \( c \)-good" is of codimension \( b_+(X) \) for a fixed class \( c \). This means that for manifolds with \( b_+(X) > 1 \) we have a well defined map

\[
SW_{X, \sigma} : Spin^c(X) \longrightarrow \Lambda^*H^1(X, \mathbb{Z})
\]

which associates to a class of Spin\(^c\)-structures \( c \) the form \( SW_{X, \sigma}^{(g, b)}(c) \) for any \( b \in H^2_{DR}(X) \) such that \((g, b)\) is \( c \)-good. This map, which is functorial with respect to orientation preserving diffeomorphisms, is the Seiberg-Witten invariant.

Using the identity in Lemma 1.1.1, one can easily prove

**Remark 1.2.4** [W] Let \( X \) be an oriented closed 4-manifold with \( b_+(X) > 1 \). Then the set of classes \( c \in Spin^c(X) \) with nontrivial Seiberg-Witten invariant is finite.

In the special case \( b_+(X) > 1, b_1(X) = 0 \), \( SW_{X, \sigma} \) is simply a function

\[
SW_{X, \sigma} : Spin^c(X) \longrightarrow \mathbb{Z}.
\]
The values $SW_{X,c}(c) \in \mathbb{Z}$ are refinements of the numbers $n^c_c$ defined by Witten [W]. More precisely:

$$n^c_c = \sum_c SW_{X,c}(c),$$

the summation being over all classes of $Spin^c$-structures $c$ of Chern class $c$. It is easy to see that the indexing set is a torsor for the subgroup $\text{Tors}_2 H^2(X, \mathbb{Z})$ of 2-torsion classes in $H^2(X, \mathbb{Z})$.

The structure of the Seiberg-Witten invariants for manifolds with $b_+(X) = 1$ is more complicated and will be described in the next section.
1.3 The case $b_+ = 1$ and the wall crossing formula

Let $X$ be a closed oriented differentiable 4-manifold with $b_+(X) = 1$. In this situation the Seiberg-Witten forms depend on a chamber structure: Recall first that in the case $b_+(X) = 1$ there is a natural map $\mathcal{M}et(X) \rightarrow \mathbb{P}(H^2_{DR}(X))$ which sends a metric $g$ to the line $\mathbb{R}[\omega_+] \subset H^2_{DR}(X)$, where $\omega_+$ is any non-trivial $g$-selfdual harmonic form. Let $\mathcal{H}$ be the hyperbolic space

$$\mathcal{H} := \{ h \in H^2_{DR}(X) \mid h^2 = 1 \} .$$

This space has two connected components, and the choice of one of them orients the lines $\mathbb{H}^2_{+,g}(X)$ for all metrics $g$. Furthermore, having fixed a component $\mathcal{H}_0$ of $\mathcal{H}$, every metric defines a unique $g$-selfdual form $\omega_g$ of length 1 with $[\omega_g] \in \mathcal{H}_0$.

Let $c \in H^2(X, \mathbb{Z})$ be characteristic. The wall associated with $c$ is the hypersurface

$$c^\perp := \{ (h, b) \in \mathcal{H} \times H^2_{DR}(X) \mid (c - b) \cdot h = 0 \} ,$$

and the connected components of $[\mathcal{H} \times H^2_{DR}(X)] \setminus c^\perp$ are called chambers of type $c$. 

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Notice that the walls are non-linear. Each characteristic element $c$ defines precisely four chambers of type $c$, namely

$$C_{H_0,\pm} := \{(h, b) \in H \times H^2_{DR}(X) | \pm (c - b) \cdot h < 0\} , \ H_0 \in \pi_0(H) ,$$

and each of these four chambers contains elements of the form $([\omega_g], b)$ with $g \in \text{Met}(X)$.

Let $\sigma_1$ be an orientation of $H^1(X, \mathbb{R})$. The choice of $\sigma_1$ together with the choice of a component $H_0 \in \pi_0(H)$ defines an orientation $\sigma = (\sigma_1, H_0)$ of $\text{det}(H^1(X, \mathbb{R})) \otimes \text{det}(H^2_+(X, \mathbb{R}))$. Set

$$SW_{X,(\sigma_1,H_0)}^\pm(c) := SW_{X,\sigma}^{(g,b)}(c) ,$$

where $(g, b)$ is a pair such that $([\omega_g], b)$ belongs to the chamber $C_{H_0,\pm}$. The map

$$SW_{X,(\sigma_1,H_0)} : \text{Spin}^c(X) \rightarrow \Lambda^*H^1(X, \mathbb{Z}) \times \Lambda^*H^1(X, \mathbb{Z})$$

which associates to a class $c$ of $\text{Spin}^c$-structures on the oriented manifold $X$ the pair of forms $(SW_{X,(\sigma_1,H_0)}^+(c), SW_{X,(\sigma_1,H_0)}^-(c))$ is the Seiberg-Witten invariant of $X$ with respect to the orientation data $(\sigma_1, H_0)$. This invariant is functorial with respect to orientation-preserving diffeomorphisms and behaves as follows with respect to changes of the orientation data:

$$SW_{X,(-\sigma_1,H_0)}(c) = -SW_{X,(\sigma_1,H_0)}(c) , \ SW_{X,(\sigma_1,-H_0)}^\pm(c) = -SW_{X,(\sigma_1,H_0)}^\mp(c) .$$

More important, however, is the fact that the difference

$$SW_{X,(\sigma_1,H_0)}^+(c) - SW_{X,(\sigma_1,H_0)}^-(c)$$

is a topological invariant of the pair $(X, c)$. To be precise, consider the element $u_c \in \Lambda^2 \left(H_1(X, \mathbb{Z})/\text{Tors}\right)$ defined by

$$u_c(a \wedge b) := \frac{1}{2} \langle a \cup b \cup c, [X]\rangle$$

for elements $a, b \in H^1(X, \mathbb{Z})$. The following universal wall-crossing formula generalizes results of [W], [KM] and [LL].
Theorem 1.3.1 [OT6] (Wall crossing formula) Let $l_{c_1} \in \Lambda^{b_1} H^1(X, \mathbb{Z})$ be the generator defined by the orientation $c_1$, and let $r \geq 0$ with $r \equiv w_c \pmod{2}$. For every $\lambda \in \Lambda^r \left( H_1(X, \mathbb{Z})/\text{Tors} \right)$ we have

$$[SW^+_{X,(c_1,H_0)}(c) - SW^-_{X,(c_1,H_0)}(c)](\lambda) = \left( -1 \right)^{\frac{b_1 - r}{2}} \langle \lambda \wedge u_c, l_{c_1} \rangle$$

when $r \leq \min(b_1, w_c)$, and the difference vanishes otherwise.

We want to illustrate these results with the simplest possible example, the projective plane.

Example: Let $\mathbb{P}^2$ be the complex projective plane, oriented as complex manifold, and denote by $h$ the first Chern class of $\mathcal{O}_{\mathbb{P}^2}(1)$. Since $h^2 = 1$, the hyperbolic space $\mathcal{H}$ consists of two points $\mathcal{H} = \{ \pm h \}$. We choose the component $H_0 := \{ h \}$ to define orientations.

An element $c \in H^2(X, \mathbb{Z})$ is characteristic iff $c \equiv h \pmod{2}$. In the picture below we have drawn (as vertical intervals) the two chambers

$$C_{H_0,\pm} = \{(h,b) \in H_0 \times \mathcal{H}^2_{\text{DR}}(\mathbb{P}^2) | \pm (c - b) \cdot h < 0\}$$

of type $c$, for every $c \equiv h \pmod{2}$. 

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The set $Spin^c(\mathbb{P}^2)$ can be identified with the set $(2\mathbb{Z}+1)h$ of characteristic elements under the map which sends a $Spin^c$-structure $c$ to its Chern class $c$. The corresponding virtual dimension is $w_c = \frac{1}{4}(c^2 - 9)$. Note that, for any metric $g$, the pair $(g, 0)$ is $c$-good for all characteristic elements $c$. Also recall that the Fubini-Study metric $g$ is a metric of positive scalar curvature which can be normalized such that $[\omega_g] = h$. We can now completely determine the Seiberg-Witten invariant $SW_{\mathbb{P}^2, H_0}$ using three simple arguments:

i) for $c = \pm h$ we have $w_c < 0$, hence $SW_{X, H_0}^\pm(c) = 0$, by the transversality results of section 1.2.

ii) Let $c$ be a characteristic element with $w_c \geq 0$. Since the Fubini-Study metric $g$ has positive scalar curvature and $(g, 0)$ is $c$-good, we have $\mathcal{W}_0^r = \mathcal{W}_0^{r*} = 0$ by Corollary 1.1.2. But this moduli space can be used to compute $SW_{\mathbb{P}^2, H_0}^\pm(c)$ for characteristic elements $c$ with $\pm c \cdot h < 0$. Thus we find $SW_{\mathbb{P}^2, H_0}^\pm(c) = 0$ when $w_c \geq 0$ and $\pm c \cdot h < 0$.

iii) The remaining values, $SW_{\mathbb{P}^2, H_0}^\pm(c) = 0$ for classes with $w_c \geq 0$ and $\pm c \cdot h < 0$, are determined by the wall-crossing formula. Altogether we get

$$SW_{\mathbb{P}^2, H_0}^\pm(c) = \begin{cases} 1 & \text{if } c \cdot h \geq 3 \\ 0 & \text{if } c \cdot h < 3 \end{cases}, \quad SW_{\mathbb{P}^2, H_0}^\pm(c) = \begin{cases} -1 & \text{if } c \cdot h \leq -3 \\ 0 & \text{if } c \cdot h > -3 \end{cases}.$$ 

2 Non-abelian Seiberg-Witten theory

2.1 G-monopoles

Let $V$ be a Hermitian vector space, and let $U(V)$ be its group of unitary automorphisms. For any closed subgroup $G \subset U(V)$ which contains the central involution $-\text{id}_V$, we define a new Lie group by

$$Spin^G(n) := Spin(n) \times_{\mathbb{Z}_2} G.$$ 

By construction one has the following exact sequences:

$$1 \rightarrow Spin \rightarrow Spin^G \rightarrow G/\mathbb{Z}_2 \rightarrow 1$$

$$1 \rightarrow G \rightarrow Spin^G \rightarrow SO \rightarrow 1$$
where $Spin$ ($Spin^G$, $SO$) denotes one of the groups $Spin(n)$ ($Spin^G(n)$, $SO(n)$).

Given a $Spin^G$-principal bundle $P^G$ over a topological space, we form the following associated bundles:

$$\delta(P^G) := P^G \times_\delta (G/\mathbb{Z}_2), \quad \mathcal{G}(P^G) := P^G \times_{Ad} G, \quad \mathfrak{g}(P^G) := P^G \times_{ad} \mathfrak{g},$$

where $\mathfrak{g}$ stands for the Lie algebra of $G$. The group $\mathcal{G}$ of sections of the bundle $\mathcal{G}(P^G)$ can be identified with the group of automorphism of $P^G$ over the associated $SO$-bundle $P^G \times_\pi SO$.

Consider now an oriented manifold $(X, g)$, and let $P_g$ be the $SO$-bundle of oriented $g$-orthonormal coframes. A $Spin^G$-structure in $P_g$ is a principal bundle morphism $\sigma : P^G \to P_g$ of type $\pi$ [KN]. An isomorphism of $Spin^G$-structures $\sigma, \sigma'$ in $P_g$ is a bundle isomorphism $f : P^G \to P'^G$ with $\sigma' \circ f = \sigma$.

One shows that the data of a $Spin^G$-structure in $(X, g)$ is equivalent to the data of a linear, orientation-preserving isometry $\gamma : \Lambda^1 \to P^G \times_\pi \mathbb{R}^n$, which we call the Clifford map of the $Spin^G$-structure [T2].

In dimension 4, the spinor group $Spin(4)$ splits as

$$Spin(4) = SU(2)_+ \times SU(2)_- = Sp(1)_+ \times Sp(1)_-.$$

Using the projections

$$p_\pm : Spin(4) \to SU(2)_\pm$$

one defines the adjoint bundles

$$ad_\pm(P^G) := P^G \times_{ad_\pm} su(2).$$

Coupling $p_\pm$ with the natural representation of $G$ in $V$, we obtain representations $\lambda_\pm : Spin^G(4) \to U(\mathbb{H}_\pm \otimes_c V)$ and associated spinor bundles

$$\Sigma^\pm(P^G) := P^G \times_{\lambda_\pm} (\mathbb{H}_\pm \otimes_c V).$$

The Clifford map $\gamma : \Lambda^1 \to P^G \times_\pi \mathbb{R}^4$ of the $Spin^G$-structure yields identifications

$$\Gamma : \Lambda^2_\pm \to ad_\pm(P^G).$$
An interesting special case occurs, when $V$ is a Hermitian vector space over the quaternions, and $G$ is a subgroup of $Sp(V) \subset U(V)$. Then one can define real spinor bundles

$$\Sigma^\pm_\mathbb{R}(P^G) := P^G \times_{\rho_\pm} (\mathbb{H}_\pm \otimes_\mathbb{H} V),$$

associated with the representations

$$\rho_\pm : Spin^G(4) \to SO(\mathbb{H}_\pm \otimes_\mathbb{H} V).$$

**Examples:** Let $(X, g)$ be a closed oriented Riemannian 4-manifold with coframe bundle $P_g$.

1) $G = S^1$: A $Spin^S^1$-structure is just a $Spin^c$-structure as described in Chapter 1 [T2].

2) $G = Sp(1)$: $Spin^{Sp(1)}$-structures have been introduced in [OT5], where they were called $Spin^h$-structures. The map which associates to a $Spin^{Sp(1)}$-structure $\sigma : P^h \to P_g$ the first Pontrjagin class $p_1(\delta(P^h))$ of the associated $SO(3)$-bundle $\delta(P^h)$, induces a bijection between the set of isomorphism classes of $Spin^{Sp(1)}$-structures in $(X, g)$ and the set

$$\{ p \in H^4(X, \mathbb{Z}) \mid p \equiv w_2(X)^2 \mod 4 \}.$$

There is a 1-1 correspondence between isomorphism classes of $Spin^{Sp(1)}$-structures in $(X, g)$ and equivalence classes of triples $(\tau : P^{S^1} \to P_g, E, \iota)$ consisting of a $Spin^{S^1}$-structure $\tau$, a unitary vector bundle $E$ of rank 2, and an unitary isomorphism $\iota : det \Sigma^\pm \to det E$. The equivalence relation is generated by tensorizing with Hermitian line bundles [OT5], [T2]. The associated bundles are – in terms of these data – given by

$$\delta(P^h) = P_E/S^1, \quad \mathcal{G}(P^h) = SU(E), \quad \mathfrak{g}(P^h) = su(E),$$

$$\Sigma^\pm(P^h) = (\Sigma^\pm_\tau)^\vee \otimes E, \quad \Sigma^\pm_\mathbb{R}(P^h) = \mathbb{R}SU(\Sigma^\pm_\tau, E),$$

where $P_E$ denotes the principal $U(2)$-frame bundle of $E$.

3) $G = U(2)$: In this case $G/\mathbb{Z}_2$ splits as $G/\mathbb{Z}_2 = PU(2) \times S^1$, and we write $\delta$ in the form $(\delta, \det)$. The map which associates to a $Spin^{U(2)}$-structure
\[ \sigma : P^u \rightarrow P_g \] the characteristic classes \( p_1(\tilde{\delta}(P^u)), c_1(\det P^u) \) identifies the set of isomorphism classes of \( Spin^{U(2)} \)-structures in \((X,g)\) with the set 

\[ \{(p,c) \in H^4(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \mid p \equiv (w_2(X) + \tilde{c})^2 \pmod{4}\} . \]

There is a 1-1 correspondence between isomorphism classes of \( Spin^{U(2)} \)-structures in \((X,g)\) and equivalence classes of pairs \((\tau : P^{S^1} \rightarrow P_g, E)\) consisting of a \( Spin^{S^1} \)-structure \( \tau \) and a unitary vector bundle of rank 2. Again the equivalence relation is given by tensorizing with Hermitian line bundles \([T2]\). If \( \sigma : P^u \rightarrow P_g \) corresponds to the pair \((\tau : P^{S^1} \rightarrow P_g, E)\), the associated bundles are now

\[ \tilde{\delta}(P^u) = P^E / S^1, \quad \det P^u = \det[\Sigma^\pm]^\vee \otimes \det E , \]

\[ G(P^u) = U(E), \quad \mathfrak{g}(P^u) = u(E), \quad \Sigma^\pm(P^u) = (\Sigma^\pm)^\vee \otimes E . \]

We will later also need the subbundles \( G_0(P^u) := P^u \times_{\text{Ad}} SU(2) \cong SU(E) \) and \( \mathfrak{g}_0 := P^u \times_{\text{ad}} su(2) \cong su(E) \). The group of sections \( \Gamma(X,G_0) \) can be identified with the group of automorphisms of \( P^u \) over \( P_g \times (P^u \times_{\text{det}} S^1) \).

Consider now again a general \( Spin^G \)-structure \( \sigma : P^G \rightarrow P_g \) in the 4-manifold \((X,g)\). The spinor bundle \( \Sigma^\pm(P^G) \) has \( \mathbb{H}^\pm \otimes_C V \) as standard fiber, so that the standard fiber \( su(2)^\pm \otimes \mathfrak{g} \) of the bundle \( \text{ad}^\pm(P^G) \otimes \mathfrak{g}(P^G) \) can be viewed as real subspace of \( \text{End}(\mathbb{H}^\pm \otimes_C V) \). We define a quadratic map

\[ \mu_{0G} : \mathbb{H}^\pm \otimes_C V \longrightarrow su(2)^\pm \otimes \mathfrak{g} \]

by sending \( \psi \in \mathbb{H}^\pm \otimes_C V \) to the orthogonal projection \( pr_{su(2)^\pm \otimes \mathfrak{g}}(\psi \otimes \tilde{\psi}) \) of the Hermitian endomorphism \( (\psi \otimes \tilde{\psi}) \in \text{End}(\mathbb{H}^\pm \otimes_C V) \). One can show that \( -\mu_{0G} \) is the total (hyperkähler) moment map for the \( G \)-action on the space \( \mathbb{H}^\pm \otimes_C V \) endowed with the natural hyperkähler structure given by left multiplication with quaternionic units \([T2]\).

These maps give rise to quadratic bundle maps

\[ \mu_{0G} : \Sigma^\pm(P^G) \longrightarrow \text{ad}^\pm(P^G) \otimes \mathfrak{g}(P^G) . \]

In the case \( G = U(2) \) one can project \( \mu_{0U(2)} \) on \( \text{ad}^\pm(P^G) \otimes \mathfrak{g}_0(P^G) \) and gets a map

\[ \mu_{00} : \Sigma^\pm(P^u) \longrightarrow \text{ad}^\pm(P^u) \otimes \mathfrak{g}_0(P^u) . \]
Note that a fixed Spin$^G$-structure $\sigma : P^G \to P_g$ defines a bijection between connections $A \in \mathcal{A}(\delta(P^G))$ in $\delta(P^G)$ and connections $\hat{A} \in \mathcal{A}(P^G)$ in the Spin$^G$-bundle $P^G$ which lift the Levi-Civita connection in $P_g$ via $\sigma$. This follows immediately from the third exact sequence above. Let

$$\bar{D}_A : A^0(\Sigma^\pm(P^G)) \to A^0(\Sigma^\mp(P^G))$$

be the associated Dirac operator, defined by

$$\bar{D}_A : A^0(\Sigma^\pm(P^G)) \xrightarrow{\nabla_{\hat{A}}} A^1(\Sigma^\pm(P^G)) \xrightarrow{\sigma} A^0(\Sigma^\mp(P^G)).$$

Here $\gamma : \Lambda^1 \otimes \Sigma^\pm(P^G) \to \Sigma^\mp(P^G)$ is the Clifford multiplication corresponding to the embeddings $\gamma : \Lambda^1 \to P^G \times \pi \mathbb{R}^4 \subset \text{Hom}_C(\Sigma^\pm(P^G), \Sigma^\mp(P^G))$.

**Definition 2.1.1** Let $\sigma : P^G \to P_g$ be a Spin$^G$-structure in the Riemannian manifold $(X, g)$. The $G$-monopole equations for a pair $(A, \Psi) \in \mathcal{A}(\delta(P^G)) \times A^0(\Sigma^+(P^G))$ are

$$\begin{cases} 
\bar{D}_A \Psi = 0 \\
\Gamma(F_A^+) = \mu_{0G}(\Psi). 
\end{cases} \quad (SW^g)$$

The solutions of these equations will be called $G$-monopoles. The symmetry group of the $G$-monopole equations is the gauge group $G := \Gamma(X, G(P^G))$. If the Lie algebra of $G$ has a non-trivial center $z(g)$, then one has a family of $G$-equivariant ”twisted” $G$-monopole equations $(SW^g_{\beta})$ parameterized by $iz(g)$-valued 2-forms $\beta \in A^2(iz(g))$:

$$\begin{cases} 
\bar{D}_A \Psi = 0 \\
\Gamma((F_A + 2\pi i\beta)^+) = \mu_{0G}(\Psi). 
\end{cases} \quad (SW^g_{\beta})$$

We denote by $\mathcal{M}^g$, respectively $\mathcal{M}^g_{\beta}$ the corresponding moduli spaces of solutions modulo the gauge group $G$.

Since in the case $G = U(2)$ there exists the splitting

$$U(2)/\mathbb{Z}_2 = PU(2) \times S^1,$$

the data of a connection in $\delta(P^u) = \delta(P^u) \times_X \text{det} P^u$ is equivalent to the data of a pair of connections $(A, a) \in \mathcal{A}(\delta(P^u)) \times \mathcal{A}(\text{det} P^u)$. This can be
used to introduce new important equations, obtained by fixing the abelian connection \( a \in \mathcal{A}(\det P^u) \) in the \( U(2) \)-monopole equations, and regarding it as a parameter. One gets in this way the equations

\[
\begin{align*}
\mathcal{D}_{A,a} \Psi &= 0 \\
\Gamma(F^+_A) &= \mu_{00}(\Psi)
\end{align*}
\]

for a pair \((A, \Psi) \in \mathcal{A}(\mathfrak{h}(P^u)) \times A^0(\Sigma^+(P^u))\), which will be called the \( PU(2) \)-monopole equations. These equations should be regarded as a twisted version of the quaternionic monopole equations introduced in [OT5], which coincide in our present framework with the \( SU(2) \)-monopole equations. Indeed, a \( Spin^{U(2)} \)-structure \( \sigma : P^u \to P_g \) with trivialized determinant line bundle can be regarded as \( Spin^{SU(2)} \)-structure, and the corresponding quaternionic monopole equations are \((SW^\sigma)\), where \( \theta \) is the trivial connection in \( \det P^u \).

The \( PU(2) \)-monopole equations are only invariant under the group \( G_0 := \Gamma(X, G_0) \) of automorphisms of \( P^u \) over \( P_g \times_X \det P^u \). We denote by \( \mathcal{M}_a^\sigma \) the moduli space of \( PU(2) \)-monopoles modulo this gauge group. Note that \( \mathcal{M}_a^\sigma \) comes with a natural \( S^1 \)-action given by the formula \( \zeta \cdot [A, \Psi] := [A, \zeta^\frac{1}{2} \Psi] \).

Comparing with other formalisms:

1. For \( G = S^1 \), \( V = \mathbb{C} \) one recovers the original abelian Seiberg-Witten equations and the twisted abelian Seiberg-Witten equations of [L1], [Bru], [OT6].
2. For \( G = S^1 \), \( V = \mathbb{C}^{\oplus k} \) one gets the so called ”multimonopole equations” studied by J. Bryan and R. Wentworth [BW].
3. In the case \( G = U(2) \), \( V = \mathbb{C}^2 \) one obtains the \( U(2) \)-monopole equations which were studied in [OT1] (see also chapter 3).
4. In the case of a \( Spin \)-manifold \( X \) and \( G = SU(2) \) the corresponding monopole equations have been studied from a physical point of view in [LM].
5. If \( X \) is simply connected, the \( S^1 \)- quotient \( \mathcal{M}_a^\sigma / S^1 \) of a moduli space of \( PU(2) \)-monopoles can be identified with a moduli space of ”non-abelian monopoles” as defined in [PT]. Note that in the general non-simply connected case, one has to use our formalism.

**Remark 2.1.2** Let \( G = Sp(n) \cdot S^1 \subset U(\mathbb{C}^{2n}) \) be the Lie group of transformations of \( \mathbb{H}^{2n} \) generated by left multiplication with quaternionic matri-
ces in $Sp(n)$ and by right multiplication with complex numbers of modulus 1. Then $G/\mathbb{Z}_2$ splits as $PSp(n) \times S^1$. In the same way as in the $PU(2)$-case one defines the $PSp(n)$-monopole equations $(SW_\sigma)$ associated with a $Spin^{Sp(n)-S^1}(4)$-structure $\sigma : P^G \to P_g$ in $(X, g)$ and an abelian connection $a$ in the associated $S^1$-bundle.

The solutions of the (twisted) $G$- and $PU(2)$-monopole equations are the absolute minima of certain gauge invariant functionals on the corresponding configuration spaces $\mathcal{A}(\delta(P^G)) \times A^0(\Sigma^+(P^G))$ and $\mathcal{A}(\delta(P^u)) \times A^0(\Sigma^+(P^G))$. The investigation of these non-abelian Seiberg-Witten functionals and of their associated sets of critical points is the subject of forthcoming thesis by A. M. Teleman [Te].

For simplicity we describe here only the case of non-twisted $G$-monopoles. The Seiberg-Witten functional $SW^\sigma : \mathcal{A}(\delta(P^G)) \times A^0(\Sigma^+(P^G)) \to \mathbb{R}$ associated to a $Spin^G$-structure is defined by

$$SW^\sigma(A, \Psi) := \| \nabla_A \Psi \|^2 + \frac{1}{4} \| F_A \|^2 + \frac{1}{2} \| \mu_{0G}(\Psi) \|^2 + \frac{1}{4} \int_X s|\Psi|^2.$$

The Euler-Lagrange equations describing general critical points are

$$\begin{cases} d_A^* F_A + J(A, \Psi) = 0 \\ \Delta_A \Psi + \mu_{0G}(\Psi)(\Psi) + \frac{1}{4} s \Psi = 0, \end{cases}$$

where the current $J(A, \Psi) \in A^1(g(P^G))$ is given by $\sqrt{32}$ times the orthogonal projection of the $\text{End}(\Sigma^+(P^G))$-valued 1-form $\nabla_A \Psi \otimes \Psi \in A^1(\text{End}(\Sigma^+(P^G)))$ onto $A^1(g(P^G))$.

In the abelian case $G = S^1$, $V = \mathbb{C}$, a closely related functional and the corresponding Euler-Lagrange equations have been investigated in [JPW].

### 2.2 Moduli spaces of $PU(2)$-monopoles

We retain the notations of the previous section. Let $\sigma : P^u \to P_g$ be a $Spin^U(2)$-structure in a closed oriented Riemannian 4-manifold $(X, g)$, and let $a \in \mathcal{A}(\det P^u)$ be a fixed connection. The $PU(2)$-monopole equations

$$\begin{cases} \mathcal{D}_{A,a} \Psi = 0 \\ \Gamma(F_A^+) = \mu_{00}(\Psi) \end{cases} (SW_\sigma)$$

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associated with these data are invariant under the action of the gauge group \( G_0 \), and hence give rise to a closed subspace \( \mathcal{M}_a^\sigma \subset \mathcal{B}(P^u) \) of the orbit space \( \mathcal{B}(P^u) := \mathcal{A}(\bar{\delta}(P^u)) \times A^0(\Sigma^+(P^u))/G_0 \).

The moduli space \( \mathcal{M}_a^\sigma \) can be endowed with the structure of a ringed space with local models constructed by the well-known Kuranishi method [OT1], [OT5], [DK], [LT]. More precisely: The linearization of the \( PU(2) \)-monopole equations in a solution \( p = (A, \Psi) \) defines an elliptic deformation complex

\[
0 \to A^0(\mathfrak{g}_0(P^u)) \xrightarrow{D^0_p} A^1(\mathfrak{g}_0(P^u)) \bigoplus A^0(\Sigma^+(P^u)) \xrightarrow{D^1_p} A^2(\mathfrak{g}_0(P^u)) \bigoplus A^0(\Sigma^-(P^u)) \to 0
\]

whose differentials are given by

\[
D^0_p(f) = (-d_A f, f \Psi)
\]

\[
D^1_p(\alpha, \psi) = (d^+_A \alpha - \Gamma^{-1} [m(\psi, \Psi) + m(\Psi, \psi)], D_{A,a} \psi + \gamma(\alpha) \Psi).
\]

Here \( m \) denotes the sesquilinear map associated with the quadratic map \( \mu_{00} \). Let \( \mathbb{H}^i_p, i = 0, 1, 2 \) denote the harmonic spaces of the elliptic complex above. The stabilizer \( G_0p \) of the point \( p \in \mathcal{A}(\bar{\delta}(P^u)) \times A^0(\Sigma^+(P^u)) \) is a finite dimensional Lie group, isomorphic to a closed subgroup of \( SU(2) \), which acts in a natural way on the spaces \( \mathbb{H}^i_p \).

**Proposition 2.2.1** [OT5], [T2] For every point \( p \in \mathcal{M}_a^\sigma \) there exists a neighborhood \( V_p \subset \mathcal{M}_a^\sigma \), a \( G_0p \)-invariant neighborhood \( U_p \) of \( 0 \in \mathbb{H}^1_p \), an \( G_0p \)-equivariant map \( K_p : U_p \to \mathbb{H}^2_p \) with \( K_p(0) = 0 \), \( dK_p(0) = 0 \), and an isomorphism of ringed spaces

\[
V_p \simeq Z(K_p)/G_0p
\]

sending \( p \) to \([0]\). The local isomorphisms \( V_p \simeq Z(K_p)/G_0p \) define the structure of a smooth manifold on the open subset

\[
\mathcal{M}^\sigma_{a, \text{reg}} := \{ [A, \Psi] \in \mathcal{M}_a^\sigma | G_0 = \{1\}, \mathbb{H}^0_p = \{0\} \}
\]

and a real analytic orbifold structure in the open set of points \( p \in \mathcal{M}_a^\sigma \) with \( G_0p \) finite. The dimension of \( \mathcal{M}^\sigma_{a, \text{reg}} \) coincides with the expected dimension.
of the $PU(2)$-monopole moduli space, which is given by the index $\chi(SW_a^\sigma)$ of the elliptic deformation complex:

$$\chi(SW_a^\sigma) = \frac{1}{2}(-3p_1(\bar{\delta}(P^u)) + c_1(\det P^u)^2) - \frac{1}{2}(3\epsilon(X) + 4\sigma(X)) .$$

Our next goal is to describe the fixed point set of the $S^1$-action on $M_a^\sigma$ introduced above.

First consider the closed subspace $\mathcal{D}(\bar{\delta}(P^u)) \subset M_a^\sigma$ of points of the form $[A,0]$. It can be identified with the Donaldson moduli space of anti-selfdual connections in the $PU(2)$-bundle $\bar{\delta}(P^u)$ modulo the gauge group $G_0$. Note however, that if $H^1(X, \mathbb{Z}_2) \neq \{0\}$, $\mathcal{D}(\bar{\delta}(P^u))$ does not coincide with the usual moduli space of $PU(2)$-instantons in $\bar{\delta}(P^u)$ but is a finite cover of it.

The stabilizer $G_{0_p}$ of a Donaldson point $(A,0)$ contains always $\{\pm \mathrm{id}\}$, hence $M_a^\sigma$ has at least $\mathbb{Z}_2$-orbifold singularities in the points of $\mathcal{D}(\bar{\delta}(P^u))$.

Second consider $S^1$ as a subgroup of $PU(2)$ via the standard embedding $S^1 \ni \zeta \mapsto \begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} \in PU(2)$. Note that any $S^1$-reduction $\rho : P \rightarrow \bar{\delta}(P^u)$ of $\bar{\delta}(P^u)$ defines a reduction $\tau_\rho : P^\rho := P^u \times_{\bar{\delta}(P^u)} P \rightarrow P^u \stackrel{\sigma}{\rightarrow} P_g$ of the $Spin^U(2)$-structure $\sigma$ to a $Spin^c \times S^1$-structure, hence a pair of $Spin^c$-structures $\tau^{\pm}_\rho : P^{\rho_{\pm}} \rightarrow P_g$. One has natural isomorphisms

$$\det P^{\rho_1} \otimes \det P^{\rho_2} = (\det P^u)^{\otimes 2}, \quad \det P^{\rho_1} \otimes (\det P^{\rho_2})^{-1} = P^{\otimes 2} ,$$

and natural embeddings $\Sigma^{\pm}(P^\rho) \rightarrow \Sigma^{\pm}(P^u)$ induced by the bundle morphism $P^{\rho_{\pm}} \rightarrow P^u$. A pair $(A, \Psi)$ will be called abelian if it lies in the image of $A(P) \times A^0(\Sigma^+(P^\rho))$ for a suitable $S^1$-reduction $\rho$ of $\bar{\delta}(P^u)$.

**Proposition 2.2.2** The fixed point set of the $S^1$-action on $M_a^\sigma$ is the union of the Donaldson locus $\mathcal{D}(\bar{\delta}(P^u))$ and the locus of abelian solutions. The latter can be identified with the disjoint union $\bigsqcup_{\rho} \mathcal{W}_{\tau^{\pm}_\rho}^{\sigma_{\bar{\delta}(P^u)}}$, where the union is over all $S^1$-reductions of the $PU(2)$-bundle $\bar{\delta}(P^u)$.

This result suggests to use the $S^1$-quotient of $M_a^\sigma \setminus (M_a^\sigma)^{S^1}$ for the comparison of Donaldson invariants and (twisted) Seiberg-Witten invariants, as explained in [OT5].

Note that only using moduli spaces $M_a^\sigma$ of quaternionic monopoles one gets, by the proposition above, moduli spaces of non-twisted abelian monopoles
in the fixed point locus of the $S^1$-action. This was one of the motivations for studying the quaternionic monopole equations in [OT5]. There it has been shown that one can use the moduli spaces of quaternionic monopoles to relate certain $Spin^c$-polynomials to the original non-twisted Seiberg-Witten invariants.

The remainder of this section is devoted to the description of the Uhlenbeck compactification of the moduli spaces of $PU(2)$-monopoles [T3].

First of all, the Weitzenböck formula and the maximum principle yield a bound on the spinor component, as in the abelian case. More precisely, one has the a priori estimate

$$\sup_X |\Psi|^2 \leq C \cdot \max \left( 0, C \sup (-s^2 + |F_a|) \right)$$

on the space of solutions of $(SW^a)$, where $C$ is a universal positive constant.

The construction of the Uhlenbeck compactification of $M^a$ is based, as in the instanton case, on the following three essential results.

1. A compactness theorem for the subspace of solutions with suitable bounds on the curvature of the connection component.
2. A removable singularities theorem.
3. Controlling bubbling phenomena for an arbitrary sequence of points in the moduli space $M^a$.

1. A compactness result.

**Theorem 2.2.3** There exists a positive number $\delta > 0$ such that for every oriented Riemannian manifold $(\Omega, g)$ endowed with a $Spin^U(2)$-structure $\sigma : P^u \to P_g$ and a fixed connection $a \in A(\det P^u)$, the following holds:

If $(A_n, \Psi_n)$ is a sequence of solutions of $(SW^a)$, such that any point $x \in \Omega$ has a geodesic ball neighborhood $D_x$ with

$$\int_{D_x} |F_{A_n}|^2 < \delta^2$$

for all large enough $n$, then there is a subsequence $(n_m) \subset \mathbb{N}$ and gauge transformations $f_m \in G_0$ such that $f_m^* (A_{m_n}, \Psi_{m_n})$ converges in the $C^\infty$-topology on $\Omega$. 

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2. Removable singularities

Let \( g \) be a metric on the 4-ball \( B \), and let
\[
\sigma : P^u = B \times Spin^{U(2)}(4) \to P_g \simeq B \times SO(4)
\]
be a \( Spin^{U(2)} \)-structure in \((B,g)\). Fix \( a \in iA_B^1 \) and put \( B^\bullet := B \setminus \{0\} \), \( \sigma^\bullet := \sigma|_{B^\bullet} \).

**Theorem 2.2.4** Let \((A_0, \Psi_0)\) be a solution of the equations \((SW_a^\bullet)\) on the punctured ball such that
\[
\| F_{A_0} \|_2^2 < \infty .
\]
Then there exists a solution \((A, \Psi)\) of \((SW_a^\bullet)\) on \( B \) and a gauge transformation \( f \in C^\infty(B^\bullet, SU(2)) \) such that \( f^*(A|_{B^\bullet}, \Psi|_{B^\bullet}) = (A_0, \Psi_0) \).

3. Controlling bubbling phenomena

The main point is that the selfdual components \( F^+_{A_n} \) of the curvatures of a sequence of solutions \([(A_n, \Psi_n)]_{n \in \mathbb{N}}\) in \( \mathcal{M}^\sigma_a \) cannot bubble.

**Definition 2.2.5** Let \( \sigma : P^u \to P_g \) be a \( Spin^{U(2)} \)-structure in \((X,g)\) and fix \( a \in A(\det P^u) \). An **ideal monopole** of type \( (\sigma, a) \) is a pair \([(A', \Psi'), \{x_1, \ldots, x_l\}]\) consisting of a point \([A', \Psi'] \in \mathcal{M}^{\sigma'}_a\), where \( \sigma'_i : P^{nu} \to P_g \) is a \( Spin^{U(2)} \)-structure satisfying
\[
\det P^{nu} = \det P^u , \quad p_1(\delta(P^{nu})) = p_1(\delta(P^u)) + 4l ,
\]
and \( \{x_1, \ldots, x_l\} \in S^l(X) \). The set of ideal monopoles of type \( (\sigma, a) \) is
\[
IM^\sigma_a := \coprod_{l \geq 0} \mathcal{M}^{\sigma'}_a \times S^l X .
\]

**Theorem 2.2.6** There exists a metric topology on \( IM^\sigma_a \) such that the moduli space \( \mathcal{M}^\sigma_a \) becomes an open subspace with compact closure \( \overline{\mathcal{M}^\sigma_a} \).

**Proof:** (sketch) Given a sequence \([(A_n, \Psi_n)]_{n \in \mathbb{N}}\) of points in \( \mathcal{M}^\sigma_a \), one finds a subsequence \([(A_{n_m}, \Psi_{n_m})]_{m \in \mathbb{N}}\), a finite set of points \( S \subset X \), and gauge transformations \( f_m \) such that \((B_m, \Phi_m) := f_m^*(A_{n_m}, \Psi_{n_m})\) converges on \( X \setminus S \)

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in the $C^\infty$-topology to a solution $(A_0, \Psi_0)$. This follows from the compactness theorem above, using the fact that the total volume of the sequence of measures $|F_{A_n}|^2$ is bounded. The set $S$ consists of points in which the measure $|F_{A_{nm}}|^2$ becomes concentrated as $m$ tends to infinity.

By the Removable Singularities theorem, the solution $(A_0, \Psi_0)$ extends after gauge transformation to a solution $(A, \Psi)$ of $(SW_{\sigma'})$ on $X$, for a possibly different $\text{Spin}^{U(2)}$-structure $\sigma'$ with the same determinant line bundle. The curvature of $A$ satisfies

$$|F_A|^2 = \lim_{n \to \infty} |F_{A_{nm}}|^2 - 8\pi^2 \sum_{x \in S} \lambda_x \delta_x$$

where $\delta_x$ is the Dirac measure of the point $x$. Now it remains to show that the $\lambda_x$’s are natural numbers and $\sum_{x \in S} \lambda_x = \frac{1}{4}(p_1(\delta(Pu)) - p_1(\delta(Pu)))$. This follows as in the instanton case, if one uses the fact that the measures $|F_{A_{nm}}|^2$ cannot bubble in the points $x \in S$ as $m \to \infty$ and that the integral of $|F_{A_n}|^2 - |F_{A_n}^+|^2$ is a topological invariant of $\delta(Pu)$. In this way one gets an ideal monopole $m := ([A, \Psi], \{\lambda_1 x_1, \ldots, \lambda_k x_k\})$ of type $(\sigma, a)$. With respect to a suitable topology on the space of ideal monopoles, one has $\lim_{m \to \infty} [A_{nm}, \Psi_{nm}] = m$. ■

3 Seiberg-Witten theory and Kähler geometry

3.1 Monopoles on Kähler surfaces

Let $(X, J, g)$ be an almost Hermitian surface with associated Kähler form $\omega_g$. We denote by $\Lambda^{pq}$ the bundle of $(p, q)$-forms on $X$ and by $A^{pq}$ its space of sections. The Hermitian structure defines an orthogonal decomposition

$$\Lambda^2_+ \otimes \mathbb{C} = \Lambda^{20} \oplus \Lambda^{02} \oplus \Lambda^{00} \omega_g$$

and a canonical $\text{Spin}^c$-structure $\tau$. The spinor bundles of $\tau$ are

$$\Sigma^+ = \Lambda^{00} \oplus \Lambda^{02}, \quad \Sigma^- = \Lambda^{01},$$

and the Chern class of $\tau$ is the first Chern class $c_1(T^1_0) = c_1(K_X^\vee)$ of the complex tangent bundle. The complexification of the canonical Clifford map
γ is the standard isomorphism
\[ \gamma : \Lambda^1 \otimes \mathbb{C} \longrightarrow \text{Hom}(\Lambda^{00} \oplus \Lambda^{02}, \Lambda^{01}), \quad \gamma(u)(\varphi + \alpha) = \sqrt{2}(\varphi u^{01} - i\Lambda_g u^{10} \wedge \alpha), \]
and the induced isomorphism \( \Gamma : \Lambda^{20} \oplus \Lambda^{02} \oplus \Lambda^{00} \omega_g \longrightarrow \text{End}_0(\Lambda^{00} \oplus \Lambda^{02}) \) acts by
\[ (\lambda^{20}, \lambda^{02}, f \omega_g) \Gamma \mapsto 2 \left[ \begin{array}{c} - if \\ \lambda^{02} \wedge \cdot \\ \ast f \end{array} \right] \in \text{End}_0(\Lambda^{00} \oplus \Lambda^{02}). \]

Recall from section 1.1 that the set \( \text{Spin}^c(X) \) of equivalence classes of \( \text{Spin}^c \)-structures in \((X, g)\) is a \( H^2(X, \mathbb{Z}) \)-torsor. Using the class of the canonical \( \text{Spin}^c \)-structure \( c := [\tau] \) as base point, \( \text{Spin}^c(X) \) can be identified with the set of isomorphism classes of \( S^1 \)-bundles: When \( M \) is an \( S^1 \)-bundle with \( c_1(M) = m \), the \( \text{Spin}^c \)-structure \( \tau_m \) has spinor bundles \( \Sigma^\pm \otimes M \) and Chern class \( 2c_1(M) - c_1(K_X) \). Let \( \mathfrak{c}_m \) be the class of \( \tau_m \).

Suppose now that \((X, J, g)\) is Kähler, and let \( k \in A(K_X) \) be the Chern connection in the canonical line bundle. In order to write the (abelian) Seiberg-Witten equations associated with the \( \text{Spin}^c \)-structure \( \tau_m \) in a convenient form, we make the variable substitution \( a = k \otimes e^{\otimes 2} \) for a connection \( e \in A(M) \) in the \( S^1 \)-bundle \( M \), and we write the spinor \( \Psi \) as a sum \( \Psi = \varphi + \alpha \in A^0(M) \oplus A^{02}(M) \).

**Lemma 3.1.1** [W], [OT6] Let \((X, g)\) be a Kähler surface, \( \beta \in A^{11}_R \) a closed real (1,1)-form in the de Rham cohomology class \( b \), and let \( M \) be a \( S^1 \)-bundle with \((2c_1(M) - c_1(K_X) - b) \cdot [\omega_g] < 0 \). The pair \((k \otimes e^{\otimes 2}, \varphi + \alpha) \in A(\text{det}(\Sigma^+ \otimes M)) \times A^0(\Sigma^+ \otimes M)\) solves the equations \((SW_{X, \beta})\) iff \( \alpha = 0 \), \( F^{20}_e = F^{02}_e = 0 \), \( \partial_e \varphi = 0 \), and
\[ i\Lambda_g F_e + \frac{1}{4} \varphi \bar{\varphi} + \left( \frac{s}{2} - \pi \Lambda_g \beta \right) = 0. \] (*)

Note that the conditions \( F^{20}_e = F^{02}_e = 0 \), \( \partial_e \varphi = 0 \) mean that \( e \) is the Chern connection of a holomorphic structure in the Hermitian line bundle \( M \) and that \( \varphi \) is a holomorphic section with respect to this holomorphic structure. Integrating the relation (*), and using the inequality in the hypothesis, one sees that \( \varphi \) cannot vanish.

To interpret the condition (*) consider an arbitrary real valued function \( t : X \longrightarrow \mathbb{R} \), and let
\[ m_t : A(M) \times A^0(M) \longrightarrow iA^0 \]
be the map defined by

\[ m_t(e, \varphi) := \Lambda_g F_e - \frac{i}{4} \varphi \bar{\varphi} + it. \]

It easy to see that (after suitable Sobolev completions) \( \mathcal{A}(M) \times A^0(M) \)
has a natural symplectic structure, and that \( m_t \) is a moment map for the
action of the gauge group \( \mathcal{G} = C^\infty(X, S^1) \). Let \( \mathcal{G}^C = C^\infty(X, \mathbb{C}^*) \) be the
complexification of \( \mathcal{G} \), and let \( \mathcal{H} \subset A(M) \times A^0(M) \) be the closed set
\[ \mathcal{H} := \{(e, \varphi) \in A(M) \times A^0(M) | F_{e}^{02} = 0, \bar{\partial}_e \varphi = 0\} \]
of integrable pairs. For any function \( t \) put
\[ \mathcal{H}_t := \{(e, \varphi) \in \mathcal{H} | \mathcal{G}^C(e, \varphi) \cap m_t^{-1}(0) \neq \emptyset\} . \]
Using a general principle in the theory of symplectic quotients, which also
holds in our infinite dimensional framework, one can prove that the \( \mathcal{G}^C \)-orbit
of a point \( (e, \varphi) \in \mathcal{H}_t \) intersects the zero set \( m_t^{-1}(0) \) of the moment map \( m_t \)
precisely along a \( \mathcal{G} \)-orbit.
In other words, there is a natural bijection of quotients

\[ [m_t^{-1}(0) \cap \mathcal{H}] / G \simeq \mathcal{H} / G^c. \quad (1) \]

Now take \( t := -\left( \frac{s}{2} - \pi \Lambda g \beta \right) \) and suppose again that the assumptions in the proposition hold. We have seen that \( m_t^{-1}(0) \cap \mathcal{H} \) cannot contain pairs of the form \((e, 0)\), hence \( G \cdot (\mathcal{G}^c) \) acts freely in \( m_t^{-1}(0) \cap \mathcal{H} \). Using this fact one can show that \( \mathcal{H} \cdot (G^c) \) acts freely in \( m_t^{-1}(0) \cap \mathcal{H} \). A point in this moduli space can be regarded as an isomorphism class of pairs \((M, \varphi)\) consisting of a holomorphic line bundle \( M \) of topological type \( M \), and a holomorphic section in \( M \). Such a pair defines a point in \( \mathcal{H} / G^c \) if and only if \( M \) admits a Hermitian metric \( h \) satisfying the equation

\[ i \Lambda F_h + \frac{1}{4} \varphi \bar{\varphi}^h = t. \quad (V_t) \]

This equation – for the unknown metric \( h \) – is the vortex equation associated with the function \( t \): it is solvable iff the stability condition

\[ (2c_1(M) - c_1(K_X) - b) \cdot [\omega_g] < 0 \quad (> 0) \]

is fulfilled. Let \( \mathcal{D}ou(m) \) be the Douady space of effective divisors \( D \subset X \) with \( c_1(\mathcal{O}_X(D)) = m \). The map \( Z : \mathcal{H} / G^c \to \mathcal{D}ou(m) \) which associates to an orbit \([e, \varphi]\) the zero-locus \( Z(\varphi) \subset X \) of the holomorphic section \( \varphi \) is an isomorphism of complex spaces.

Putting everything together, we have the following interpretation for the monopole moduli spaces \( \mathcal{W}_{\beta_{\tau_m}} \) on Kähler surfaces.

**Theorem 3.1.2** [OT1], [OT6] Let \((X, g)\) be a compact Kähler surface, and let \( \tau_m \) be the Spin\(^c\)-structure defined by the \( S^1 \)-bundle \( M \). Let \( \beta \in A^1_{\mathbb{R}} \) be a closed 2-form representing the de Rham cohomology class \( b \) such that

\[ (2c_1(M) - c_1(K_X) - b) \cdot [\omega_g] < 0 \quad (> 0) \].
If \( c_1(M) \notin NS(X) \), then \( W^{\tau_m}_\beta = \emptyset \). When \( c_1(M) \in NS(X) \), then there is a natural real analytic isomorphism

\[
W^{\tau_m}_\beta \simeq Dou(m) (Dou(c_1(K_X) - m)) .
\]

A moduli space \( W^{\tau_m}_\beta \neq \emptyset \) is smooth at the point corresponding to \( D \in Dou(m) \) iff \( h^0(O_D(D)) = \dim_D Dou(m) \). This condition is always satisfied when \( b_1(X) = 0 \). If \( W^{\tau_m}_\beta \) is smooth at a point corresponding to \( D \in Dou(m) \), then it has the expected dimension in this point iff \( h^1(O_D(D)) = 0 \).

The natural isomorphisms \( W^{\tau_m}_\beta \simeq Dou(m) \) respects the orientations induced by the complex structure of \( X \) when \((2c_1(M) - c_1(K_X) - b) \cdot [\omega_g] < 0\). If \((2c_1(M) - c_1(K_X) - b) \cdot [\omega_g] > 0\), then the isomorphism \( W^{\tau_m}_\beta \simeq Dou(c_1(K_X) - m) \) multiplies the complex orientations by \((-1)^{\chi(M)} \) [OT6].

Example: Consider again the complex projective plane \( \mathbb{P}^2 \), polarized by \( h = c_1(O_{\mathbb{P}^2}(1)) \). The expected dimension of \( W^{\tau_m}_\beta \) is \( m(m+3h) \). The theorem above yields the following explicit description of the corresponding moduli spaces:

\[
W^{\tau_m}_{\mathbb{P}^2,\beta} \simeq \begin{cases} 
|O_{\mathbb{P}^2}(m)| & \text{if } (2m+3h-\beta) \cdot h < 0 \\
|O_{\mathbb{P}^2}(-(m+3))| & \text{if } (2m+3h-\beta) \cdot h > 0 
\end{cases} .
\]

E. Witten has shown [W] that on Kählerian surfaces \( X \) with geometric genus \( p_g > 0 \) all non-trivial Seiberg-Witten invariants \( SW_{X,c}(\mathfrak{c}) \) satisfy \( w_c = 0 \).

In the case of Kählerian surfaces with \( p_g = 0 \) one has a different situation. Suppose for instance that \( b_1(X) = 0 \). Choose the standard orientation \( o_1 \) of \( H^1(X,\mathbb{R}) = 0 \) and the component \( H_0 \) containing Kähler classes to orient the moduli spaces of monopoles. Then, using the previous theorem and the wall-crossing formula, we get:

**Proposition 3.1.3** Let \( X \) be a Kähler surface with \( p_g = 0 \) and \( b_1 = 0 \). If \( m \in H^2(X,\mathbb{Z}) \) satisfies \( m(m - c_1(K_X)) \geq 0 \), i.e. the expected dimension \( w_{2m-c_1(K_X)} \) is non-negative, then

\[
SW^+_{X,H_0}(\mathfrak{c}_M) = \begin{cases} 
1 & \text{if } Dou(m) \neq \emptyset \\
0 & \text{if } Dou(m) = \emptyset 
\end{cases} ,
\]

\[
SW^-_{X,H_0}(\mathfrak{c}_M) = \begin{cases} 
0 & \text{if } Dou(m) \neq \emptyset \\
-1 & \text{if } Dou(m) = \emptyset 
\end{cases} .
\]
Our next goal is to show that the $PU(2)$-monopole equations on a Kähler surface can be analyzed in a similar way. This analysis yields a complex geometric description of the moduli spaces whose $S^1$-quotients give formulas relating the Donaldson invariants to the Seiberg-Witten invariants. If the base is projective, one also has an algebro-geometric interpretation [OST], which leads to explicitly computable examples of moduli spaces of $PU(2)$-monopoles [T3]. Such examples are important, because they illustrate the general mechanism for proving the relation between the two theories, and help to understand the geometry of the ends of the moduli spaces in the more difficult $C^\infty$-category.

Recall that, since $(X,g)$ comes with a canonical $Spin^c$-structure $\tau$, the data of of a $Spin^U(2)$-structure in $(X,g)$ is equivalent to the data of a Hermitean bundle $E$ of rank 2. The bundles of the corresponding $Spin^U(2)$-structure $\sigma : P^u \to P_g$ are given by $\overline{\delta}(P^u) = P_E/\mathbb{Q}$, $\det P^u = \det E \otimes K_X$, and $\Sigma^\pm(P^u) = \Sigma^\pm \otimes E \otimes K_X$.

Suppose that $\det P^u$ admits an integrable connection $a \in \mathcal{A}(\det P^u)$. Let $k \in \mathcal{A}(K_X)$ be the Chern connection of the canonical bundle, and let $\lambda := a \otimes k^\vee$ be the induced connection in $L := \det E$. We denote by $\mathcal{L} := (L, \overline{\delta})$ the holomorphic structure defined by $\lambda$. Now identify the affine space $A(\overline{\delta}(P^u))$ with the space $A(\lambda \otimes k \otimes 2(E \otimes K_X))$ of connections in $E \otimes K_X$ which induce $\lambda \otimes k^\otimes 2 = a \otimes k$ in $\det(E \otimes K_X)$, and identify $A^0(\Sigma^+(P^u))$ with $A^0(E \otimes K_X) \oplus A^0(E) = A^0(E \otimes K_X) \oplus A^02(E \otimes K_X)$.

**Proposition 3.1.4** Fix an integrable connection $a \in \mathcal{A}(\det E \otimes K_X)$. A pair $(A, \varphi + \alpha) \in A(\lambda \otimes k \otimes 2(E \otimes K_X)) \times [A^0(E \otimes K_X) \oplus A^02(E \otimes K_X)]$ solves the $PU(2)$-monopole equations $(SW'_a)$ if and only if $A$ is integrable and one of the following conditions is satisfied:

I) $\alpha = 0, \quad \overline{\delta} A \varphi = 0 \quad \text{and} \quad i\Lambda g F^0_A + \frac{1}{2}(\varphi \varphi)_0 = 0$

II) $\varphi = 0, \quad \partial_A \alpha = 0 \quad \text{and} \quad i\Lambda g F^0_A - \frac{1}{2} \ast (\alpha \wedge \alpha)_0 = 0$.

Note that solutions $(A, \varphi)$ of type I give rise to holomorphic pairs $(\mathcal{F}_A, \varphi)$, consisting of a holomorphic structure in $F := E \otimes K$ and a holomorphic section $\varphi$ in $\mathcal{F}_A$. The remaining equation $i\Lambda g F^0_A + \frac{1}{2}(\varphi \varphi)_0 = 0$ can again be interpreted as the vanishing condition for a moment map for the $G_0$-action.
in the space of pairs \((A, \varphi) \in A_{1,2}^\lambda \times A_{1,0}^0(F)\). We shall study the corresponding stability condition in the next section.

The analysis of the solutions of type II can be reduced to the investigation of the type I solutions: Indeed, if \(\varphi = 0\) and \(\alpha \in A^{02}(E \otimes K_X)\) satisfies \(\partial_A \alpha = 0\), we see that the section \(\psi := \bar{\alpha} \in A^0(\bar{E})\) must be holomorphic, i.e. it satisfies \(\bar{\partial}_{A_{1,0}[\bar{\alpha}]^\vee} \psi = 0\). On the other hand one has \(-*(\alpha \wedge \bar{\alpha})_0 = *((\bar{\alpha} \wedge \hat{\alpha})_0 = (\bar{\psi} \bar{\psi})_0\).

3.2 Vortex equations and stable oriented pairs

Let \((X, g)\) be a compact Kähler manifold of arbitrary dimension, and let \(E\) be a differentiable vector bundle of rank \(r\), endowed with a fixed holomorphic structure \(L := (L, \bar{\partial}_L)\) in \(L := \text{det } E\).

An oriented pair of type \((E, \mathcal{L})\) is a pair \((\mathcal{E}, \varphi)\), consisting of a holomorphic structure \(\mathcal{E} = (E, \partial_E)\) in \(E\) with \(\bar{\partial}_{\text{det } E} = \bar{\partial}_L\), and a holomorphic section \(\varphi \in H^0(\mathcal{E})\). Two oriented pairs are isomorphic if they are equivalent under the natural action of the group \(SL(E)\) of differentiable automorphisms of \(E\) with determinant 1.

An oriented pair \((\mathcal{E}, \varphi)\) is simple if its stabilizer in \(SL(E)\) is contained in the center \(\mathbb{Z}_r \cdot \text{id}_E\) of \(SL(E)\); it is strongly simple if this stabilizer is trivial.

Proposition 3.2.1 [OT5] There exists a (possibly non-Hausdorff) complex analytic orbifold \(\mathcal{M}^{si}(E, \mathcal{L})\) parameterizing isomorphism classes of simple oriented pairs of type \((E, \mathcal{L})\). The open subset \(\mathcal{M}^{ssi}(E, \mathcal{L}) \subset \mathcal{M}^{si}(E, \mathcal{L})\) of classes of strongly simple pairs is a complex analytic space, and the points \(\mathcal{M}^{si}(E, \mathcal{L}) \setminus \mathcal{M}^{ssi}(E, \mathcal{L})\) have neighborhoods modeled on \(\mathbb{Z}_r\)-quotients.

Now fix a Hermitian background metric \(H\) in \(E\). In this section we use the symbol \((SU(E)) \cup (E)\) for the groups of (special) unitary automorphisms of \((E, H)\), and not for the bundles of (special) unitary automorphisms.

Let \(\lambda\) be the Chern connection associated with the Hermitian holomorphic bundle \((\mathcal{L}, \text{det } H)\). We denote by \(\mathcal{A}_{\partial_L}(E)\) the affine space of semiconnections in \(E\) which induce the semiconnection \(\partial_L = \bar{\partial}_E\) in \(L = \text{det } E\), and we write \(\mathcal{A}_\lambda(E)\) for the space of unitary connections in \((E, H)\) which induce \(\lambda\) in \(L\). The map \(A \mapsto \partial_A\) yields an identification \(\mathcal{A}_\lambda(E) \rightarrow \mathcal{A}_{\bar{\partial}_L}(E)\), which endows the affine space \(\mathcal{A}_\lambda(E)\) with a complex structure. Using this identification and the Hermitian metric \(H\), the product \(\mathcal{A}_\lambda(E) \times A_{1,0}^0(E)\) becomes – after
suitable Sobolev completions – an infinite dimensional Kähler manifold. The map
\[ m : A_{\lambda}(E) \times A^{0}(E) \to A^{0}(su(E)) \]
defined by \( m(A, \varphi) := \Lambda g F_{A}^{0} - \frac{i}{2}(\varphi \bar{\varphi})_{0} \) is a moment map for the \( SU(E) \)-action on the Kähler manifold \( A_{\lambda}(E) \times A^{0}(E) \).

We denote by \( H_{\lambda}(E) := \{(A, \varphi) \in A_{\lambda}(E) \times A^{0}(E) | F_{A}^{02} = 0, \bar{\partial}_{A} \varphi = 0\} \) the space of integrable pairs, and by \( H_{\lambda}^{si}(E) \) the open subspace of pairs \( (A, \varphi) \in H_{\lambda}(E) \) with \( (\bar{\partial}_{A}, \varphi) \) simple. The quotient
\[ V_{\lambda}(E) := H_{\lambda}(E) \cap m^{-1}(0)/SU(E), \quad (V_{\lambda}^{*}(E) := H_{\lambda}^{si}(E) \cap m^{-1}(0)/SU(E)) \]
is called the moduli space of (irreducible) projective vortices. Note that a vortex \((A, \varphi)\) is irreducible iff \( SL(E)_{(A, \varphi)} \subset \mathbb{Z}_{r} id_{E} \). Using again an infinite dimensional version of the theory of symplectic quotients (as in the abelian case), one gets a homeomorphism
\[ j : V_{\lambda}(E) \cong H_{\lambda}^{ps}(E)/SL(E) \]
where \( H_{\lambda}^{ps}(E) \) is the subspace of \( H_{\lambda}(E) \) consisting of pairs whose \( SL(E) \)-orbit meets the vanishing locus of the moment map. \( H_{\lambda}^{ps}(E) \) is in general not open, but \( H_{\lambda}^{*}(E) := H_{\lambda}^{ps}(E) \cap H_{\lambda}^{si}(E) \) is open, and restricting \( j \) to \( V_{\lambda}^{*}(E) \) yields an isomorphism of real analytic orbifolds
\[ V_{\lambda}^{*}(E) \cong H_{\lambda}^{*}(E)/SL(E) \subset M^{*}(E, \mathcal{L}). \]

The image
\[ M^{*}(E, \mathcal{L}) := H_{\lambda}^{*}(E)/SL(E) \]
of this isomorphism can be identified with the set of isomorphism classes of simple oriented holomorphic pairs \((E, \varphi)\) of type \((E, \mathcal{L})\), with the property that \( E \) admits a Hermitian metric with \( \text{det} h = \text{det} H \) which solves the projective vortex equation
\[ i\Lambda g F_{h}^{0} + \frac{1}{2}(\varphi \bar{\varphi}^{h})_{0} = 0. \]

Here \( F_{h} \) is the curvature of the Chern connection of \((E, h)\).
The set $\mathcal{M}^s(\mathcal{E}, \mathcal{L})$ has a purely holomorphic description as the subspace of elements $[\mathcal{E}, \varphi] \in \mathcal{M}^s(E, \mathcal{L})$ which satisfy a suitable stability condition. This condition is rather complicated for bundles $E$ of rank $r > 2$, but it becomes very simple when $r = 2$.

Recall that, for any torsion free coherent sheaf $F \neq 0$ over a $n$-dimensional Kähler manifold $(X, g)$, one defines the \textit{g-slope} of $F$ by

$$\mu_g(F) := \frac{c_1(\det F) \cup [\omega_g]^{n-1}}{\text{rk}(F)}.$$  

A holomorphic bundle $\mathcal{E}$ over $(X, g)$ is called \textit{slope-stable} if $\mu_g(F) < \mu_g(\mathcal{E})$ for all proper coherent subsheaves $F \subset \mathcal{E}$. The bundle $\mathcal{E}$ is \textit{slope-polystable} if it decomposes as a direct sum $\mathcal{E} = \oplus \mathcal{E}_i$ of slope-stable bundles with $\mu_g(\mathcal{E}_i) = \mu_g(\mathcal{E})$.

\textbf{Definition 3.2.2} Let $(\mathcal{E}, \varphi)$ be an oriented pair of type $(\mathcal{E}, \mathcal{L})$ with $\text{rk} E = 2$ over a Kähler manifold $(X, g)$. The pair $(\mathcal{E}, \varphi)$ is \textit{stable} if $\varphi = 0$ and $\mathcal{E}$ is slope-stable, or $\varphi \neq 0$ and the divisorial component $D_\varphi$ of the zero-locus $Z(\varphi) \subset X$ satisfies $\mu_g(\mathcal{O}_X(D)) < \mu_g(E)$. The pair $(\mathcal{E}, \varphi)$ is \textit{polystable} if it is stable or $\varphi = 0$ and $\mathcal{E}$ is slope-polystable.

\textbf{Example:} Let $D \subset X$ be an effective divisor defined by a section $\varphi \in H^0(\mathcal{O}_X(D)) \setminus \{0\}$, and put $\mathcal{E} := \mathcal{O}_X(D) \oplus [\mathcal{L} \otimes \mathcal{O}_X(-D)]$. The pair $(\mathcal{E}, \varphi)$ is stable iff $\mu_g(\mathcal{O}_X(2D)) < \mu_g(\mathcal{L})$.

The following result gives a metric characterization of polystable oriented pairs.

\textbf{Theorem 3.2.3} [OT5] Let $E$ be a differentiable vector bundle of rank 2 over $(X, g)$ endowed with a Hermitian holomorphic structure $(\mathcal{L}, l)$ in $\det E$. An oriented pair of type $(E, \mathcal{L})$ is polystable iff $\mathcal{E}$ admits a Hermitian metric $h$ with $\det h = l$ which solves the projective vortex equation

$$i \Lambda_g F_0^h + \frac{1}{2}(\varphi \bar{\varphi}^h)_0 = 0.$$  

If $(\mathcal{E}, \varphi)$ is stable, then the metric $h$ is unique.
This result identifies the subspace $\mathcal{M}^s(E, \mathcal{L}) \subset \mathcal{M}^{si}(E, \mathcal{L})$ as the subspace of isomorphism classes of stable oriented pairs.

Theorem 3.2.3 can be used to show that the moduli spaces $\mathcal{M}_a^s$ of $PU(2)$-monopoles on a Kähler surface have a natural complex geometric description when the connection $a$ is integrable. Recall from section 3.1. that in this case $\mathcal{M}_a^s$ decomposes as the union of two Zariski-closed subspaces

$$
\mathcal{M}_a^s = (\mathcal{M}_a^s)_I \cup (\mathcal{M}_a^s)_II
$$

according to the two conditions I, II in Proposition 3.1.4. By this proposition, both terms of this union can be identified with moduli spaces of projective vortices. Using again the symbol $\ast$ to denote subsets of points with central stabilizers, one gets the following Kobayashi-Hitchin type description of $(\mathcal{M}_a^s)_I$ in terms of stable oriented pairs.

**Theorem 3.2.4** [OT5], [T2] If $a \in \mathcal{A}(\det P^u)$ is integrable, the moduli space $\mathcal{M}_a^s$ decomposes as an union $\mathcal{M}_a^s = (\mathcal{M}_a^s)_I \cup (\mathcal{M}_a^s)_II$ of two Zariski closed subspaces isomorphic with moduli spaces of projective vortices, which intersect along the Donaldson moduli space $\mathcal{D}(\bar{\delta}(P^u))$. There are natural real analytic isomorphisms

$$(\mathcal{M}_a^s)_I \cong \mathcal{M}^s(E \otimes K_X, \mathcal{L} \otimes K_X^\otimes 2), \quad (\mathcal{M}_a^s)_II \cong \mathcal{M}^s(E^\nu, \mathcal{L}^\nu),$$

where $\mathcal{L}$ denotes the holomorphic structure in $\det E = \det P^u \otimes K_X^\otimes$ defined by $\bar{\partial}_a$ and the canonical holomorphic structure in $K_X$.

**Example:** (R. Plantiko) On $\mathbb{P}^2$, endowed with the standard Fubini-Study metric $g$, we consider the $Spin^{U(2)}(4)$-structure $\sigma : P^u \longrightarrow P_9$ defined by the standard $Spin^c(4)$-structure $\tau : P^c \longrightarrow P_9$ and the $U(2)$-bundle $E$ with $c_1(E) = 7$, $c_2(E) = 13$, and we fix an integrable connection $a \in \mathcal{A}(\det P^u)$. This $Spin^{U(2)}(4)$-structure is characterized by $c_1(\det(P^u)) = 4$, $p_1(\bar{\delta}(P^u)) = -3$, and the bundle $F := E \otimes K_{\mathbb{P}^2}$ has Chern classes $c_1(F) = 1$, $c_2(F) = 1$. It is easy to see that every stable oriented pair $(\mathcal{F}, \varphi)$ of type $(F, \mathcal{O}_{\mathbb{P}^2}(1))$ with $\varphi \neq 0$ fits into an exact sequence of the form

$$0 \longrightarrow \mathcal{O} \xrightarrow{\varphi} \mathcal{F} \longrightarrow J_{Z(\varphi)} \otimes \mathcal{O}_{\mathbb{P}^2}(1) \longrightarrow 0,$$
where \( F = T_{\mathbb{P}^2}(-1) \) and the zero locus \( Z(\varphi) \) of \( \varphi \) consists of a simple point \( z_\varphi \in \mathbb{P}^2 \). Two such pairs \((F, \varphi), (F, \varphi')\) define the same point in the moduli space \( \mathcal{M}^s(F, \mathcal{O}_{\mathbb{P}^2}(1)) \) if and only if \( \varphi' = \pm \varphi \). The resulting identification

\[
\mathcal{M}^s(F, \mathcal{O}_{\mathbb{P}^2}(1)) = H^0(T_{\mathbb{P}^2}(-1))/\{\pm \text{id}\}
\]

is a complex analytic isomorphism.

Since every polystable pair of type \((F, \mathcal{O}_{\mathbb{P}^2}(1))\) is actually stable, and since there are no polystable oriented pairs of type \((\mathcal{E}^\vee, \mathcal{O}_{\mathbb{P}^2}(-7))\), Theorem 3.2.4 yields a real analytic isomorphism

\[
\mathcal{M}^\sigma_a = H^0(T_{\mathbb{P}^2}(-1))/\{\pm \text{id}\}
\]

where the origin corresponds to the unique stable oriented pair of the form \((T_{\mathbb{P}^2}(-1), 0)\). The quotient \( H^0(T_{\mathbb{P}^2}(-1))/\{\pm \text{id}\} \) has a natural algebraic compactification \( \mathcal{C} \), given by the cone over the image of \( \mathbb{P}(H^0(T_{\mathbb{P}^2}(-1))) \) under the Veronese map to \( \mathbb{P}(S^2H^0(T_{\mathbb{P}^2}(-1))) \). This compactification coincides with the Uhlenbeck compactification \( \overline{\mathcal{M}}^\sigma_a \) (see section 2.2, [T3]). More precisely, let \( \sigma' : P^u \rightarrow P_g \) be the \( \text{Spin}^U(2)(4) \)-structure with \( \det P^u = \det P^a \) and \( p_1(P^u) = 1 \). This structure is associated with \( \tau \) and the \( U(2) \)-bundle \( E' \) with Chern classes \( c_1(E') = 7, c_2(E') = 12 \). The moduli space \( \mathcal{M}^\sigma_a' \) consists of one (abelian) point, the class of the abelian solution corresponding to the stable oriented pair \((\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1), \text{id}_{\mathcal{O}_{\mathbb{P}^2}})\) of type \((E' \otimes K_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1))\). \( \mathcal{M}^\sigma_a' \) can be identified with the moduli space \( \mathcal{W}_{\mathbb{P}^2}^{\tau_{1/2\pi F_a}} \) of \( [1/2\pi F_a] \)-twisted abelian Seiberg-Witten monopoles. Under the identification \( \mathcal{C} = \overline{\mathcal{M}}^\sigma_a \), the vertex of the cone corresponds to the unique Donaldson point which is given by the stable oriented pair \((T_{\mathbb{P}^2}(-1), 0)\). The base of the cone corresponds to the space \( \mathcal{M}^\sigma_a' \times \mathbb{P}^2 \) of ideal monopoles concentrated in one point.

We want to close this section by explaining the stability concept which describes the subset \( \mathcal{M}^s_X(E, \mathcal{L}) \subset \mathcal{M}^s(E, \mathcal{L}) \) in the general case \( r \geq 2 \). This stability concept does not depend on the choice of parameter and the corresponding moduli spaces can be interpreted as ”master spaces” for holomorphic pairs (see next section); in the projective framework they admit Gieseker type compactifications [OST].
We shall find this stability concept by relating the $SU(E)$-moment map $m : \mathcal{A}(E) \times A^0(E) \rightarrow A^0(su(E))$ to the universal family of $U(E)$-moment maps $m_t : \mathcal{A}(E) \times A^0(E) \rightarrow A^0(u(E))$ defined by

$$m_t(A, \varphi) := \Lambda g F_A - \frac{i}{2} (\varphi \bar{\varphi}) + \frac{i}{2} t \text{id}_E ,$$

where $t \in A^0$ is an arbitrary real valued function. Given $t$, we consider the following system of equations

$$\begin{cases}
F^0_A = 0 \\
\partial_A \varphi = 0 \\
i \Lambda g F_A + \frac{1}{2} (\varphi \bar{\varphi}) = \frac{t}{2} \text{id}_E
\end{cases} \quad (V_t)$$

for pairs $(A, \varphi) \in A(E) \times A^0(E)$. Put $\rho_t := \frac{1}{4\pi n} \int_X t \omega_g^n$. To explain our first result, we have to recall some classical stability concepts for holomorphic pairs.

For any holomorphic bundle $E$ over $(X, g)$ denote by $S(E)$ the set of reflexive subsheaves $F \subset E$ with $0 < \text{rk}(F) < \text{rk}(E)$, and for a fixed section $\varphi \in H^0(\mathcal{E})$ put

$$S_\varphi(E) := \{ F \in S(E) | \varphi \in H^0(F) \} .$$

Define real numbers $\underline{m}_g(E)$ and $\overline{m}_g(E, \varphi)$ by

$$\underline{m}_g(E) := \max(\mu_g(E), \sup_{F \in S(E)} \mu_g(F')) , \quad \overline{m}_g(E, \varphi) := \inf_{F \in S_\varphi(E)} \mu_g(E/F) .$$

A bundle $E$ is $\varphi$-stable in the sense of S. Bradlow when $\underline{m}_g(E) < \varphi < \overline{m}_g(E, \varphi)$. Let $\rho \in \mathbb{R}$ be any real parameter. A holomorphic pair $(E, \varphi)$ is called $\rho$-stable if $\rho$ satisfies the inequality

$$\underline{m}_g(E) < \rho < \overline{m}_g(E, \varphi) .$$

The pair $(E, \varphi)$ is $\rho$-polystable if it is $\rho$-stable or $E$-splits holomorphically as $E = E' \oplus E''$ such that $\varphi \in H^0(E')$, $(E', \varphi)$ is $\rho$-stable and $E''$ is a slope-polystable vector bundle with $\mu_g(E'') = \rho$ [Br]. Let $GL(E)$ be the group of bundle automorphisms of $E$. With these definitions one proves [OT1] (see [Br] for the case of a constant function $t$):

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Proposition 3.2.5 The complex orbit \((A, \varphi) \cdot GL(E)\) of an integrable pair \((A, \varphi) \in A(E) \times A^0(E)\) contains a solution of \((V_t)\) if and only if the pair \((E_A, \varphi)\) is \(\rho_t\)-polystable.

Let us now fix again a Hermitian metric \(H\) in \(E\) and an integrable connection \(\lambda\) in the Hermitian line bundle \(L := (\det E, \det H)\). Consider the system of equations
\[
\begin{cases}
F_A^0 &= 0 \\
\bar{\partial}_A \varphi &= 0 \\
i\Lambda g F_A^0 + \frac{1}{2}(\varphi \bar{\varphi})_0 &= 0
\end{cases}
\tag{V^0}
\]
for pairs \((A, \varphi) \in A_\lambda(E) \times A^0(E)\). Then one can prove

Proposition 3.2.6 Let \((A, \varphi) \in A_\lambda(E) \times A^0(E)\) be an integrable pair. The following assertions are equivalent:
i) The complex orbit \(SL(E) \cdot (A, \varphi)\) contains a solution of \((V^0)\).
i) There exists a function \(t \in A^0\) such that the \(GL(E)\)-orbit \(GL(E) \cdot (A, \varphi)\) contains a solution of \((V_t)\).
iii) There exists a real number \(\rho\) such that the pair \((E_A, \varphi)\) is \(\rho\)-polystable.

Corollary 3.2.7 The open subspace \(M^s(E, \mathcal{L}) \subset M^{ai}(E, \mathcal{L})\) is the set of isomorphism classes of simple oriented pairs which are \(\rho\)-polystable for some \(\rho \in \mathbb{R}\).

Remark 3.2.8 There exist stable oriented pairs \((E, \varphi)\) whose stabilizer with respect to the \(GL(E)\)-action is of positive dimension. Such pairs cannot be \(\rho\)-stable for any \(\rho \in \mathbb{R}\).

Note that the moduli spaces \(M^s(E, \mathcal{L})\) have a natural \(\mathbb{C}^*\)-action defined by \(z \cdot [E, \varphi] := [E, z^t \varphi]\). This action is well defined since \(r\)-th roots of unity are contained in the complex gauge group \(SL(E)\).

There exists an equivalent definition for stability of oriented pairs, which does not use the parameter dependent stability concepts of [Br]. The fact that it is expressible in terms of \(\rho\)-stability is related to the fact that the moduli spaces \(M^s(E, \mathcal{L})\) are master spaces for moduli spaces of \(\rho\)-stable pairs.
3.3 Master spaces and the coupling principle

Let $X \subset \mathbb{P}^N_C$ be a smooth complex projective variety with hyperplane bundle $\mathcal{O}_X(1)$. All degrees and Hilbert polynomials of coherent sheaves will be computed corresponding to these data.

We fix a torsion-free sheaf $\mathcal{E}_0$ and a holomorphic line bundle $\mathcal{L}_0$ over $X$, and we choose a Hilbert polynomial $P_0$. By $P_F$ we denote the Hilbert polynomial of a coherent sheaf $\mathcal{F}$. Recall that any non-trivial torsion free coherent sheaf $\mathcal{F}$ admits a unique subsheaf $\mathcal{F}'_{\text{max}}$ for which $P_{\mathcal{F}'}\text{rk}\mathcal{F}'$ is maximal and whose rank is maximal among all subsheaves $\mathcal{F}'$ with $P_{\mathcal{F}'}\text{rk}\mathcal{F}'_{\text{max}}$ maximal.

An $\mathcal{L}_0$-oriented pair of type $(P_0, \mathcal{E}_0)$ is a triple $(\mathcal{E}, \varepsilon, \varphi)$ consisting of a torsion free coherent sheaf $\mathcal{E}$ with determinant isomorphic to $\mathcal{L}_0$ and Hilbert polynomial $P_\mathcal{E} = P_0$, a homomorphism $\varepsilon : \det \mathcal{E} \to \mathcal{L}_0$, and a morphism $\varphi : \mathcal{E} \to \mathcal{E}_0$. The homomorphisms $\varepsilon$ and $\varphi$ will be called the orientation and the framing of the oriented pair. There is an obvious equivalence relation for such pairs. When $\ker \varphi \neq 0$, we set

$$\delta_{\varepsilon, \varphi} := P_{\mathcal{E}} - \frac{\text{rk}\mathcal{E}}{\text{rk}|\ker(\varphi)|_{\text{max}}} P_{\text{ker}(\varphi)_{\text{max}}}.$$

An oriented pair $(\mathcal{E}, \varepsilon, \varphi)$ is semistable if one of the following conditions is satisfied:

1. $\varphi$ is injective.
2. $\varepsilon$ is an isomorphism, $\ker \varphi \neq 0$, $\delta_{\varepsilon, \varphi} \geq 0$, and for all non-trivial subsheaves $\mathcal{F} \subset \mathcal{E}$ the following inequality holds

$$\frac{P_{\mathcal{F}}}{\text{rk}\mathcal{F}} - \frac{\delta_{\varepsilon, \varphi}}{\text{rk}\mathcal{F}} \leq \frac{P_{\mathcal{E}}}{\text{rk}\mathcal{E}} - \frac{\delta_{\varepsilon, \varphi}}{\text{rk}\mathcal{E}}.$$

The corresponding stability concept is slightly more complicated [OST]. Note that the (semi)stability definition above does not depend on a parameter. It is, however, possible to express (semi)stability in terms of the parameter dependent Gieseker-type stability concepts of [HL2]. E.g., $(\mathcal{E}, \varepsilon, \varphi)$ is semistable iff $\varphi$ is injective, or $\mathcal{E}$ is Gieseker semistable, or there exists a rational polynomial $\delta$ of degree smaller than $\dim X$ with positive leading coefficient, such that $(\mathcal{E}, \varphi)$ is $\delta$-semistable in the sense of [HL2].

For all stability concepts introduced so far there exist analogous notions of slope-(semi)stability. In the special case when the reference sheaf $\mathcal{E}_0$ is the trivial sheaf $\mathcal{O}_X$, slope stability is the algebro-geometric analog of the stability concept associated with the projective vortex equation.
Theorem 3.3.1 [OST] There exists a projective scheme $M_{ss}(P_0, E_0, L_0)$ whose closed points correspond to gr-equivalence classes of Gieseker semistable $L_0$-oriented pairs of type $(P_0, E_0)$. This scheme contains an open subscheme $M^s(P_0, E_0, L_0)$ which is a coarse moduli space for stable $L_0$-oriented pairs.

It is also possible to construct moduli spaces for stable oriented pairs where the orienting line bundle is allowed to vary [OST]. This generalization is important in connection with Gromov-Witten invariants for Grassmannians [BDW].

Note that $M_{ss}(P_0, E_0, L_0)$ possesses a natural $\mathbb{C}^*$-action, given by

$$z \cdot [\mathcal{E}, \varepsilon, \varphi] := [\mathcal{E}, \varepsilon, z\varphi],$$

whose fixed point set can be explicitly described. The fixed point locus $[M^{ss}(P_0, E_0, L_0)]^{\mathbb{C}^*}$ contains two distinguished subspaces, $M_0$ defined by the equation $\varphi = 0$, and $M_{\infty}$ defined by $\varepsilon = 0$. $M_0$ can be identified with the Gieseker scheme $M^{ss}(P, L_0)$ of equivalence classes of semistable $L_0$-oriented torsion free coherent sheaves with Hilbert polynomial $P_0$. The subspace $M_{\infty}$ is the Grothendieck Quot-scheme $\text{Quot}^P_{E_0/L_0} P_{E_0 - P_0}$ of quotients of $E_0$ with fixed determinant isomorphic with $(\text{det } E_0) \otimes L_0^{-1}$ and Hilbert polynomial $P_{E_0 - P_0}$.

In the terminology of [BS], $M_0$ is the source $M_{source}$ of the $\mathbb{C}^*$-space $M^{ss}(P_0, E_0, L_0)$, and $M_{\infty}$ is its sink when non-empty.

The remaining subspace of the fixed point locus

$$M_R := [M^{ss}(P_0, E_0, L_0)]^{\mathbb{C}^*} \setminus [M_0 \cup M_{\infty}],$$

the so-called space of reductions, consists of objects which are of the same type but essentially of lower rank.

Note that the Quot scheme $M_{\infty}$ is empty if $\text{rk}(E_0)$ is smaller than the rank $r$ of the sheaves $E$ under consideration, in which case the sink of the moduli space is a closed subset of the space of reductions.

Recall from [BS] that the closure of a general $\mathbb{C}^*$-orbit connects a point in $M_{source}$ with a point in $M_{sink}$, whereas closures of special orbits connect points of other parts of the fixed point set.

The flow generated by the $\mathbb{C}^*$-action can therefore be used to relate data associated with $M_0$ to data associated with $M_{\infty}$ and $M_R$.

The technique of computing data on $M_0$ in terms of $M_R$ and $M_{\infty}$ is a very general principle which we call coupling and reduction. This principle
has already been described in a gauge theoretic framework in section 2.2 for relating monopoles and instantons. However, the essential ideas may probably be best understood in an abstract Geometric Invariant Theory setting, where one has a very simple and clear picture.

Let $G$ be a complex reductive group, and consider a linear representation $\rho_A : G \to GL(A)$ in a finite dimensional vector space $A$. The induced action $\bar{\rho}_A \to \text{Aut}(\mathbb{P}(A))$ comes with a natural linearization in $O_{\mathbb{P}(A)}(1)$, hence we have a stability concept, and thus we can form the GIT quotient

$$M_0 := \mathbb{P}(A)^{ss} // G.$$ 

Suppose we want to compute ”correlation functions”

$$\Phi_I := \langle \mu_I, [M_0] \rangle,$$

i.e. we want to evaluate suitable products of canonically defined cohomology classes $\mu_i$ on the fundamental class $[M_0]$ of $M_0$. Usually the $\mu_i$'s are slant products of characteristic classes of a ”universal bundle” $\mathcal{E}_0$ on $M_0 \times X$ with homology classes of $X$. Here $X$ is a compact manifold, and $\mathcal{E}_0$ comes from a tautological bundle $\tilde{\mathcal{E}}_0$ on $A \times X$ by applying Kempf’s Descend Lemma.

The main idea is now to couple the original problem with a simpler one, and to use the $\mathbb{C}^*$-action which occurs naturally in the resulting GIT quotients to express the original correlation functions in terms of simpler data. More precisely, consider another representation $\rho_B : G \to GL(B)$ with GIT quotient $M_\infty := \mathbb{P}(B)^{ss} // G$. The direct sum $\rho := \rho_A \oplus \rho_B$ defines a naturally linearized $G$-action on the projective space $\mathbb{P}(A \oplus B)$. We call the corresponding quotient

$$\mathcal{M} := \mathbb{P}(A \oplus B)^{ss} // G$$

the master space associated with the coupling of $\rho_A$ to $\rho_B$.

The space $\mathcal{M}$ comes with a natural $\mathbb{C}^*$-action, given by

$$z \cdot [a, b] := [a, z \cdot b],$$

and the union $\mathcal{M}_0 \cup \mathcal{M}_\infty$ is a closed subspace of the fixed point locus $\mathcal{M}^{\mathbb{C}^*}$.

Now make the simplifying assumptions that $\mathcal{M}$ is smooth and connected, the $\mathbb{C}^*$-action is free outside $\mathcal{M}^{\mathbb{C}^*}$, and suppose that the cohomology classes
μ_i extend to ℳ. This condition is always satisfied if the μ_i’s were obtained by the procedure described above, and if Kempf’s lemma applies to the pull-back bundle p_δ^*(E_0) and provides a bundle on ℳ × X extending E_0.

Under these assumptions, the complement

\[ ℳ_R := ℳ^{C^*} \setminus (ℳ_0 \cup ℳ_∞) \]

is a closed submanifold of ℳ, disjoint from ℳ_0, and ℳ_∞. We call ℳ_R the manifold of reductions of the master space. Now remove a sufficiently small S^1-invariant tubular neighborhood U of ℳ^{C^*} ⊂ ℳ, and consider the S^1-quotient W := [ℳ \setminus U]/S^1. This is a compact manifold whose boundary is the union of the projectivized normal bundles \( \mathbb{P}(N_{ℳ_0}) \) and \( \mathbb{P}(N_{ℳ_∞}) \), and a differentiable projective fiber space \( P_R \) over ℳ_R. Note that in general \( P_R \) has no natural holomorphic structure. Let \( n_0, n_∞ \) be the complex dimensions of the fibers of \( \mathbb{P}(N_{ℳ_0}), \mathbb{P}(N_{ℳ_∞}) \), and let \( u ∈ H^2(W, \mathbb{Z}) \) be the first Chern class of the S^1-bundle dual to \( ℳ \setminus U \rightarrow W \). Let \( µ_I \) be a class as above. Then, taking into account orientations, we compute:

\[ Φ_I := ⟨µ_I, [ℳ_0]⟩ = ⟨µ_I∪u^{n_0}, [\mathbb{P}(N_{ℳ_0})]⟩ = ⟨µ_I∪u^{n_0}, [\mathbb{P}(N_{ℳ_∞})]⟩ − ⟨µ_I∪u^{n_0}, [P_R]⟩. \]

In this way the coupling principle reduces the calculation of the original correlation functions on ℳ_0 to computations on ℳ_∞ and on the manifold of reductions ℳ_R. A particular important case occurs when the GIT problem given by \( ρ_B \) is trivial, i.e. when \( \mathbb{P}(B)^{ss} = \emptyset \). Under these circumstances the functions \( Φ_I \) are completely determined by data associated with the manifold of reductions ℳ_R.

Of course, in realistic situations, our simplifying assumptions are seldom satisfied, so that one has to modify the basic idea in a suitable way.

One of the realistic situations which we have in mind is the coupling of coherent sheaves with morphisms into a fixed reference sheaf E_0. In this case, the original problem is the classification of stable torsion-free sheaves, and the corresponding Gieseker scheme \( ℳ^{ss}(P_0, ℳ_0) \) of \( ℳ_0 \)-oriented semistable sheaves of Hilbert polynomial \( P_0 \) plays the role of the quotient ℳ_0. The corresponding master spaces are the moduli spaces \( ℳ^{ss}(P_0, ℳ_0, E_0, ℰ_0) \) of semistable \( ℳ_0 \)-oriented pairs of type \((P_0, ℰ_0)\).

Coupling with \( ℰ_0 \)-valued homomorphisms \( ϕ : ℰ \rightarrow ℰ_0 \) leads to two essentially different situations, depending on the rank r of the sheaves ℰ under consideration:
1. When $\text{rk}(E_0) < r$, the framings $\varphi : E \rightarrow E_0$ can never be injective, i.e. there are no semistable homomorphisms. This case corresponds to the GIT situation $\mathcal{M}_\infty = \emptyset$.

2. As soon as $\text{rk}(E_0) \geq r$, the framings $\varphi$ can become injective, and the Grothendieck schemes $\text{Quot}_{P_0, E_0}^L$ appear in the master space $\mathcal{M}^{ss}(P_0, E_0, L_0)$. These Quot schemes are the analogues of the quotients $\mathcal{M}_\infty$ in the GIT situation.

In both cases the spaces of reductions are moduli spaces of objects which are of the same type but essentially of lower rank.

Everything can be made very explicit when the base manifold is a curve $X$ with a trivial reference sheaf $E_0 = \mathcal{O}_X^\oplus k$. In the case $k < r$, the master spaces relate correlation functions of moduli spaces of semistable bundles with fixed determinant to data associated with reductions. When $r = 2$, $k = 1$, the manifold of reductions are symmetric powers of the base curve, and the coupling principle can be used to prove the Verlinde formula, or to compute the volume and the characteristic numbers (in the smooth case) of the moduli spaces of semistable bundles.

The general case $k \geq r$ leads to a method for the computation of Gromov-Witten invariants for Grassmannians. These invariants can be regarded as correlation functions of suitable Quot schemes [BDW], and the coupling principle relates them to data associated with reductions and moduli spaces of semistable bundles. In this case one needs a master space $\mathcal{M}^{ss}(P_0, E_0, L)$ associated with a Poincaré line bundle $L$ on $\text{Pic}(X) \times X$ which set theoretically is the union over $L_0 \in \text{Pic}(X)$ of the master spaces $\mathcal{M}^{ss}(P_0, E_0, L_0)$ [OST]. One could try to prove the Vafa-Intriligator formula along these lines.

Note that the use of master spaces allows us to avoid the sometimes messy investigation of chains of flips, which occur whenever one considers the family of all possible $\mathbb{C}^*$-quotients of the master space [T], [BDV].

The coupling principle has been applied in two further situations.

Using the coupling of vector bundles with twisted endomorphisms, A. Schmitt has recently constructed projective moduli spaces of Hitchin pairs [S]. In the case of curves and twisting with the canonical bundle, his master spaces are natural compactifications of the moduli spaces introduced in [H].

Last but not least, the coupling principle can also be used in certain gauge theoretic situations:

The coupling of instantons on 4-manifolds with Dirac-harmonic spinors has been described in detail in chapter 2. In this case the instanton moduli
spaces are the original moduli spaces \( M_0 \), the Donaldson polynomials are the original correlation functions to compute, and the moduli spaces of \( PU(2) \)-monopoles are master spaces for the coupling with spinors. One is again in the special situation where \( M_\infty = \emptyset \), and the manifold of reductions is a union of moduli spaces of twisted abelian monopoles. In order to compute the contributions of the abelian moduli spaces to the correlation functions, one has to give explicit descriptions of the master space in an \( S^1 \)-invariant neighborhood of the abelian locus.

Finally consider again the Lie group \( G = Sp(n) \cdot S^1 \) and the \( PSp(n) \)-monopole equations \((SW^a_\sigma)\) for a \( Spin^{Sp(n) \cdot S^1 (4)} \)-structure \( \sigma : P^G \to P^g \) in \((X,g)\) and an abelian connection \( a \) in the associated \( S^1 \)-bundle (see Remark 2.1.2). Regarding the compactification of the moduli space \( M^a_\sigma \) as master space associated with the coupling of \( PSp(n) \)-instantons to harmonic spinors, one should get a relation between Donaldson \( PSp(n) \)-theory and Seiberg-Witten type theories.
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