Frequency Theorem for discrete time stochastic system with multiplicative noise

Peter Situmbeko Nalitolela*

Department of Mathematics, University of Dar Es Salaam

Dar Es Salaam, Tanzania

Nikolai Dokuchaev†

Department of Mathematics & Statistics, Curtin University

Perth, Australia

Abstract

In this paper we consider the problem of minimizing a quadratic functional for a discrete-time linear stochastic system with multiplicative noise, on a standard probability space, in infinite time horizon. We show that the necessary and sufficient conditions for the existence of the optimal control can be formulated as matrix inequalities in frequency domain. Furthermore, we show that if the optimal control exists, then certain Lyapunov equations must have a solution. The optimal control is obtained by solving a deterministic linear-quadratic optimal control problem whose functional depends on the solution to the Lyapunov equations. Moreover, we show that under certain conditions, solvability of the Lyapunov equations is guaranteed. We also show that, if the frequency inequalities are strict, then the solution is unique up to equivalence.

Mathematics Subject Classification: 93E20, 49N10

Key words: Stochastic Optimal Control, Frequency Theorem, Kalman–Yakubovich Lemma, Kalman–Szegö Lemma, Lyapunov equations.

*peter.nalitolela@gmail.com
†N.Dokuchaev@curtin.edu.au
1 Introduction

Kalman–Yakubovich Lemma (KY Lemma) was a groundbreaking result that paved way for a solutions to lots of problems in control theory, including optimal control. The first variant of the Lemma was derived by Yakubovich in 1962 (see [22]). The following year, the discrete-time version of that result was derived by Szegö and Kalman (see [20]). It is called sometimes the Kalman–Szegö Lemma (KS Lemma); see [5,13,17] for a comprehensive review of various results in control theory derived from the KY Lemma. Various works such as [1–4] considered problems with quadratic functionals whereas Yakubovich (see [25,26]) derived the KY Lemma for the case in which both the control and state vectors are both Hilbert spaces.

Dokuchaev [6] considered a continuous time stochastic linear-quadratic optimal control problem, with the state evolution described by Itô equations, with state dependent coefficients; a generalization of the Frequency Theorem was obtained. We consider a discrete-time analogy of the problem studied in [6]. We show that the necessary and sufficient conditions for the existence of the optimal control can be formulated as matrix inequalities in frequency domain. Furthermore, we show that if the optimal control exists, then certain Lyapunov equations must have a solution. The optimal control is obtained by solving a deterministic linear-quadratic optimal control problem whose functional depends on the solution to the Lyapunov equations. Moreover, we show that under certain conditions, solvability of the Lyapunov equations is guaranteed. We also show that, if the frequency inequalities are strict, then the solution is unique up to equivalence.

2 Problem Statement

We consider the following optimization problem on a standard probability space, $(\Omega, \mathcal{F}, P)$.

$$\Phi (u) = \sum_{t=0}^{+\infty} \text{Minimize } E \left[ x_t^* G x_t + 2 \text{Re} x_t^* \gamma u_t + u_t^* \Gamma u_t \right]$$  \hspace{1cm} (2.1)

over the set

$$U = \left\{ u_t \in \mathbb{R}^m : \sum_{t=0}^{+\infty} |u_t|^2 < +\infty \right\}$$  \hspace{1cm} (2.2)

subject to

$$x_{t+1} = Ax_t + bu_t + Cx_t \xi_{t+1}, \quad t = 0, 1, 2, \ldots$$  \hspace{1cm} (2.3)

$$x_0 = a.$$  \hspace{1cm} (2.4)
Here $x_t$ is a random $n$-vector of states, $u_t$ is an $m$-vector of controls and $U$ is the set of admissible controls. Matrices $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{n \times n}$, $G = G^\top \in \mathbb{R}^{n \times n}$, $\gamma \in \mathbb{R}^{n \times n}$, and $\Gamma = \Gamma^\top \in m \times m$ are constant. The scalar $\xi_t \in \mathbb{R}$ is the discrete-time white noise adapted to a flow of non-decreasing $\sigma$-algebras $\mathcal{F}_t \subset \mathcal{F}$ such that $\mathbb{E} \xi_t = 0$, $\text{Var} \ (\xi_t) = 1$. The vector $a$ is random, measurable with respect to $\mathcal{F}_0$, independent of $\{\xi_t\}_{t=0}^{\infty}$ and is such that $\mathbb{E} \left| a \right|^2 < +\infty$ and $\mathbb{E} \left| \left| aa^\top \right| \right| < +\infty$; we denote by $\left| . \right|$ the Euclidean norm for vectors and Frobenius norm for matrices.

We assume all the matrices in (2.2) and (2.3) are real and we restrict our considerations to the case when all eigenvalues $\lambda (A)$ of $A$ lie inside the unit disk on the complex plane (that is, the spectral radius of $A$ is $\rho (A) < 1$). Moreover, we assume that the system is stable in mean-square sense for $u_t \equiv 0$. Various sufficient conditions of this stability can be found in [7, 12, 15, 17, 18, 21] and other works.

For random $x_t, y_t \in \mathbb{C}^n$ we denote the inner product $(x, y)$ by $(x, y) = \sum_{t=0}^{+\infty} \mathbb{E} x_t^\top y_t$ and the norm by $\|x\| = \sqrt{(x, x)}$. Furthermore, we write $\|x\|_1 = \sum_{t=0}^{+\infty} \mathbb{E} |x_t|_1$ where $|x_t|$ is the $l_1$-norm $|x_t|_1 = \sum_i |x_{it}|$ of a vector $x$ or an entrywise $l_1$-norm $|x|_1 = \sum_{ij} |x_{ij}|$ of a matrix $x$.

3 Main Results

**Condition 3.1.** There exist symmetric matrices $H$ and $\Theta$ in $\mathbb{C}^{n \times n}$ satisfying

\begin{align}
A^\top HA - H + \Theta &= 0, \\
\Theta - C^\top HC - G &= 0.
\end{align}

Let $\Theta$ be the matrix satisfying Condition 3.1. Consider the hermitian form $\mathcal{F} : \mathbb{C}^n \times \mathbb{C}^m \mapsto \mathbb{R}$ given by

\[ \mathcal{F} (x, u) = x^* \Theta x + 2 \text{Re} x^* \gamma u + u^* \Gamma u. \]  

(3.7)

Let $g : \mathbb{C} \mapsto \mathbb{C}^{n \times n}$ be the matrix-valued function

\[ g (z) = (zI - A)^{-1}, \]  

(3.8)

We denote the unit circle by $\zeta = \{z \in \mathbb{C} : |z| = 1\}$.

The following Theorem establishes necessary and sufficient conditions for the existence of optimal $u^o$ for the problem 2.4.

**Theorem 3.1.** If there exists exists a $u^o \in U$ such that $\Phi (u^o) \leq \Phi (u)$, for all $u \in U$ then
i) it is necessary that
\[ F(g(z)bu, u) \geq 0, \quad (\forall z \in \zeta, \forall u \in \mathbb{C}^m). \] (3.9)

ii) Furthermore, if there exists a \( \delta > 0 \) such that
\[ F(g(z)bu, u) \geq \delta |u|^2, \quad (\forall z \in \zeta, \forall u \in \mathbb{C}^m), \] (3.10)

then \( u^o \) is unique (up to equivalence).

Theorem 3.1 above is an analog of KS Lemma for discrete-time optimal stochastic control problem (2.1)-(2.4). This is a discrete time version of a continuous-time result obtained in [5] for the case when \( \gamma = 0 \) and in Chapter 5 of [10] for the general \( \gamma \).

3.1 Proof of Theorem 3.1

Lemma 3.1. If \( u_t \in U \), then \( \sup_{t \geq 0} \mathbb{E} |x_t|^2 < +\infty \) for the solution of system (2.3)-(2.4).

Proof. Let
\[ \mu_t = \mathbb{E} x_t, \] (3.11)
\[ M_t = \mathbb{E} x_t x_t^\top. \] (3.12)

From (2.3)-(2.4) and (3.11), we have
\[ \mu_{t+1} = A\mu_t + bu_t \quad t = 0, 1, 2, \ldots, \] (3.13)
\[ \mu_0 = \mathbb{E} a. \] (3.14)

Note that \( |\mathbb{E} a|^2 = \sum_{i=1}^n (\mathbb{E} a_i)^2 \leq \sum_{i=1}^n \mathbb{E} a_i^2 = \mathbb{E} |a|^2 < +\infty \). Thus, using the fact that \( u_t \in U \) and \( \rho(A) < 1 \), it follows from (3.13)-(3.14) that \( ||\mu_t|| < +\infty \).

From (2.3)-(2.4) and (3.12), we have
\[ M_{t+1} = AM_t A^\top + A\mu_t u_t^\top b^\top + bu_t \mu_t A^\top + bu_t u_t^\top b^\top + C M_t C^\top, \] (3.15)
\[ M_0 = \mathbb{E} aa^\top. \] (3.16)

Let \( Q_t = A\mu_t u_t^\top b^\top + bu_t \mu_t A^\top + bu_t u_t^\top b^\top \). Let us denote the \( j \)-th column of a matrix \( D \) by \( D^{(j)} \).
We define the vectors \( q_t, m_t \in \mathbb{C}^{n^2} \) as
\[
q_t = \begin{bmatrix}
Q_t^{(1)} \\
Q_t^{(2)} \\
\vdots \\
Q_t^{(n)}
\end{bmatrix}, \quad m_t = \begin{bmatrix}
M_t^{(1)} \\
M_t^{(2)} \\
\vdots \\
M_t^{(n)}
\end{bmatrix}.
\]
(3.17)

The vectors \( q_t \) and \( m_t \) are formed by stacking up the columns of the matrices \( Q_t \) and \( M_t \), respectively. Set \( \mathcal{A} = A \otimes A + C \otimes C \) (where \( \otimes \) denotes the Kronecker product). We can then rewrite (3.15) as
\[
m_{t+1} = \mathcal{A}m_t + q_t.
\]
(3.18)

Note that the system in (3.18) is of dimension \( n^2 \), however, due symmetry, it can be reduced to a system of dimension \( n^2 + n \).

The assumption that the system (2.3)-(2.4) is stable in the mean-square sense for \( u_t = 0 \), is equivalent to \( m_t \) being stable for \( q_t = 0 \), which is true if and only if the spectral radius of \( \mathcal{A} \) is \( \rho(\mathcal{A}) < 1 \). From the solution of (3.18), we can show, using Hölder’s inequality and Young’s theorem, that \( \|m_t\|_1 < +\infty \), therefore \( \sup_{t \geq 0} \mathbb{E}|x_t|^2 < +\infty \). This completes the proof of Lemma 3.1. \( \square \)

It follows from Lemma 3.1 that the Z-transform, \( \hat{x}(z) \), of \( x_t \), exists, and it’s radius of convergence contains the unit circle, \( \zeta \). If we set \( x_t = 0, u_t = 0 \) for all \( t < 0 \) we can then take the Z-transform of the system (2.3)-(2.4) and obtain
\[
\hat{x}(z) = zg(z)a + g(z)b\hat{u}(z) + g(z)C \sum_{t=-\infty}^{\infty} \xi_{t+1} \frac{x_t}{z^t}.
\]
(3.19)

Let \( D \) be an \( n \times n \) real symmetric matrix and let \( T : \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \) be defined by
\[
T(D) = \frac{1}{2\pi i} \oint_{\zeta} C^\top g(z)^\top Dg(z)C \frac{1}{z} dz.
\]
(3.20)

Lemma 3.2. Condition (3.1) is satisfied if and only if \( \Theta \) satisfies
\[
G = \Theta - T(\Theta).
\]
(3.21)

Proof. Suppose there exists a \( \Theta \) such that (3.21) holds. Let
\[
H = \frac{1}{2\pi i} \oint_{\zeta} g(z)^\top \Theta g(z) \frac{1}{z} dz.
\]
(3.22)
It follows from Parseval’s identity that \( H = \sum_{t=0}^{+\infty} (A^T)^t \Theta A^t = \Theta + A^T HA \). It therefore follows from (3.21) that \( G = \Theta - C^T HC \). Hence Condition 3.1 is satisfied.

Conversely, suppose (3.5) holds, then
\[
\sum_{t=0}^{+\infty} (A^T)^t HA^t - \sum_{t=0}^{+\infty} (A^T)^{t+1} HA^{t+1} = \sum_{t=0}^{+\infty} (A^T)^t \Theta A^t.
\]

It follows from Parseval’s identity that
\[
H = \frac{1}{2\pi i} \int g(z)^T \Theta g(z) \frac{1}{z} dz
\]
and it follows from (3.6) and (3.20) that \( G = \Theta - T(\Theta) \). Thus (3.21) holds. This completes the proof of Lemma 3.2.

**Lemma 3.3.** If the system (2.3)-(2.4) is stable in the mean-square sense for \( u_t \equiv 0 \) then Condition 3.1 holds.

**Proof.** Let us denote the \( i \)-th column of a matrix \( D \) by \( D_{\cdot,i} \), let the matrices \( H \) and \( G \) be as in Condition 3.1 and let \( h = [H_{\cdot,1}, \ldots, H_{\cdot,n}] \), \( \theta = [\Theta_{\cdot,1}, \ldots, \Theta_{\cdot,n}] \) and \( g = [G_{\cdot,1}, \ldots, G_{\cdot,n}] \). Let \( A_1 = A^T \otimes A^T - I_{n^2} \) and \( A_2 = -C^T \otimes C^T \), where \( I_{n^2} \) is the \( n^2 \times n^2 \) identity matrix. We can rewrite (3.5)-(3.6) as
\[
\begin{bmatrix}
A_1 & I_{n^2} \\
A_2 & I_{n^2}
\end{bmatrix}
\begin{bmatrix}
h \\
\theta
\end{bmatrix} =
\begin{bmatrix}
0 \\
g
\end{bmatrix}.
\]

Please notice that the system in (3.23) would be degenerate if and only if \( A_1 = A_2 \); however, this would require that \( A = A \otimes A + C \otimes C = I_{n^2} \) which would violate the assumption that the matrix \( A \) from (3.18) satisfies \( \rho(A) < 1 \) (which is equivalent to the requirement that the system (2.3)-(2.4) be stable in the mean-square sense for \( u_t = 0 \)). Therefore, if the mean-square stability is satisfied, we can assume that the system in (3.23) always has a solution, \( [h^T, g^T]^T \).

Hence matrices \( H \) and \( G \) exist (that is, Condition 3.1 holds). This proves Lemma 3.3.

It follows from Lemma 3.2 that \( G = \Theta - T(\Theta) \). Therefore, if we set \( x_t = 0 \) and \( u_t = 0 \) for \( t < 0 \), we can rewrite (2.1) as
\[
\Phi(u_t) = \sum_{t=-\infty}^{+\infty} E F(x_t, u_t) - \sum_{t=-\infty}^{+\infty} E x_t^T T(\Theta) x_t
\]

Let the matrix-valued function \( \Pi : \mathbb{C} \to \mathbb{C}^{m \times m} \) be defined by
\[
\Pi(z) = b^T g(\tau)^T \Theta g(z) b + b^T g(\tau)^T \gamma + \gamma^T g(z) b + \Gamma,
\]
and let
\begin{align}
(\hat{u}(z), R\hat{u}(z)) &= \frac{1}{2\pi i} \oint_{\zeta} \hat{u}(z)\rightP \Pi(z) \hat{u}(z) \frac{1}{z} dz, \quad (3.26) \\
(r, \hat{u}(z)) &= \frac{1}{2\pi i} \oint_{\zeta} \mathbb{E} z a^\top g(\tau)^\top [Gg(z) b + \gamma] \hat{u}(z) \frac{1}{z} dz, \quad (3.27) \\
\rho &= \frac{1}{2\pi i} \oint_{\zeta} \mathbb{E} a^\top g(\bar{\tau})^\top Gg(z) a \frac{1}{z} dz. \quad (3.28)
\end{align}

It follows from (3.26)-(3.28) and Parseval’s identity that we can rewrite (3.24) as
\[
\Phi(u.) = (\hat{u}(z), R\hat{u}(z)) + (r, \hat{u}(z)) + \rho. \quad (3.29)
\]

Thus, \( \Phi(u.) \) is a quadratic form in \( \hat{u}(.) \). Consider the deterministic control problem below.

Minimize
\[
\Phi_1(u.) = \sum_{t=0}^{+\infty} |y_t^\top \Theta x_t + 2\text{Re} y_t^\top \gamma u_t + u_t^\top \Gamma u_t| 
\]
over the set
\[
U = \left\{u_t \in \mathbb{R}^m : \sum_{t=0}^{+\infty} |u_t|^2 < +\infty \right\} \quad (3.30)
\]
subject to
\begin{align}
y_{t+1} &= A y_t + b u_t, \quad t = 0, 1, 2, \ldots \\
y_0 &= \mathbb{E} a. \quad (3.32)
\end{align}

Here \( y_t \) is an \( n \)-vector of states and \( u_t \) is an \( m \)-vector of controls. Let matrices \( G, \gamma, \Gamma, A, \) and \( b \) and the vector \( a \) have the same properties as in the stochastic optimization problem (2.1)-(2.4) above, and let the matrix \( \Theta \) be such that (3.21) is satisfied. Using Parseval’s identity, we can rewrite (3.30) as \( \Phi_1(u.) = (\hat{u}(.), R_1\hat{u}(.) + (r_1, \hat{u}(.)) + \rho_1, \) where
\begin{align}
(\hat{u}(z), R_1\hat{u}(z)) &= \frac{1}{2\pi i} \oint_{\zeta} \hat{u}(z)^\top \Pi(z) \hat{u}(z) \frac{1}{z} dz, \quad (3.34) \\
(r_1, \hat{u}(z)) &= \frac{1}{2\pi i} \oint_{\zeta} \mathbb{E} z a^\top g(\tau)^\top [Gg(z) b + \gamma] \hat{u}(z) \frac{1}{z} dz, \quad (3.35) \\
\rho_1 &= \frac{1}{2\pi i} \oint_{\zeta} \mathbb{E} a^\top g(\bar{\tau})^\top Gg(z) a \frac{1}{z} dz. \quad (3.36)
\end{align}

**Theorem 3.2.** An optimal control \( u_t^* \) for the stochastic optimization problem (2.1)-(2.4) exists if and only if an optimal control for the deterministic optimization problem (3.30)-(3.33) exists. Furthermore, if (3.10) holds then the optimal controls in optimization problems (2.1)-(2.4) and (3.30)-(3.33) are identical and unique to within equivalence.
Proof. Note that, the necessary and sufficient conditions for the existence of optimal $u^o$ that minimizes the quadratic form $(u, Ru) + (r, u) + \rho$ depend on $R$ and $r$. Moreover, the optimal $u^o$, when it exists, is given by the solution to $Ru^o + r = 0$. We can see from (3.26)-(3.28) and (3.31)-(3.36) that $R_1 = R$ and $r_1 = r$ for the functionals $\Phi$ and $\Phi_1$. It therefore follows that the solution to (2.1)-(2.4) exists if and only if the solution to (3.30)-(3.33) exists. Furthermore, if (3.10) holds, it follows from the results from [25, 26], that the optimal control for (3.30)-(3.33) exists and is unique. This completes the proof of Theorem 3.2.

It follows that if the optimization problem (3.30)-(3.33) has an optimal solution then (3.9) must hold. Furthermore if (3.10) holds, it follows that the solution exists and is unique (up to within equivalence). Hence the proof for Theorem 3.1 follows from Theorem 3.2.

Remark 3.1. If $C = 0$ in the problem (2.1)-(2.4), then the requirement that the system (2.3)-(2.4) be stable in the mean-square sense for $u_t = 0$ will be equivalent to requiring that the matrix $A$ satisfy $\rho(A) < 1$. In addition, if we set $\Theta = G$ then (3.21) holds. Therefore Condition 6.1 is satisfied and $F(x, u) = x^*Gx + 2Re x^*g x + u^*G u$, and Theorem 3.1 will be the same as the results from [25, 26] with $x_0 = E a$.

3.2 Numerical Algorithm

In this section we provide a Matlab code that takes matrices $G$, $A$ and $C$ as inputs, then checks if the system is stable in the mean-square sense. If it is stable, the program calculates matrices $\Theta$ and $H$.

function [Theta, H] = numerics1(G, A, C)

% Verify that the inputs are all square matrices of the same dimension
s1=size(G); s2=size(A); s3=size(C);
if((s1(1)==s2(1))&&((s1(1)==s3(1))&&(s2(1)==s3(1)))))
%--------------------------------------------------------------------
%FILE NAME: numerics.m
%DESCRIPTION: Check if the discrete-time Linear Quadratic Control
%Problem is Solvable. That is, we calculate H and Theta
%that satisfy: {A'HA-H+Theta=0, Theta-C'HC-G=0}
%INPUTS: Matrices G, A, C
%OUTPUT: Matirx Theta, H
%--------------------------------------------------------------------
(s1(2)\approx s2(2))|(s1(2)\approx s3(2))|(s2(2)\approx s3(2))|

(s1(1)\approx s1(2))|(s2(1)\approx s2(2))|(s3(1)\approx s3(2))

disp('ERROR! Dimension Mismatch');
return;
end;

%--------------------------------------------------------------------

%Get the symmetric part of G
G=0.5*(G+G');

%--------------------------------------------------------------------

%Verify that the spectral radius of A is less than 1
if(max(abs(eig(A))) >= 1)
    disp('Matrix A is not convergent');
    return;
end;

%--------------------------------------------------------------------

%Verify that the system is Exponential Bounded in the mean-square sense
Big_A = kron(A,A)+kron(C,C);
if(max(abs(eig(Big_A))) >= 1)
    disp('The system is not EMS stable');
    return;
end;

%--------------------------------------------------------------------

%Solve the system, i.e. Calculate matrices H and Theta
A1=kron(A',A')-eye(size(A').^2);
B1=eye(size(A').^2);
A2=-kron(C',C');
B2=eye(size(A').^2);
M=[A1, B1; A2, B2];
v=[zeros(size(G(:)));G(:)];
solution=M\v;
theta=solution(length(solution)/2+1:length(solution));
h=solution(1:length(solution)/2);

%--------------------------------------------------------------------
Theta=reshape(theta,size(A));
H=reshape(h,size(A));

References

[1] V. A. Andreev. The synthesis of optimal controls for inhomogeneous linear systems with a quadratic quality criterion. *Sibirsk. Matem. Zh.*, 13(3):698–702, 1972.

[2] V. A. Andreev, Yu. F. Kazarinov, and V. A. Yakubovich. Synthesis of optimal controls for linear nonhomogeneous systems in problems of minimization of quadratic functionals. *Soviet Mathematics Doklady*, 199(2):257–261, 1971.

[3] V. A. Andreev, Yu. F. Kazarinov, and V. A. Yakubovich. On the synthesis of optimal controls in the problem of minimization of a quadratic functional. *Soviet Mathematics Doklady*, 202(6):1247–1250, 1972.

[4] V. A. Andreev and A. I. Shepelyavyi. Synthesis of optimal controls for pulse–amplitude systems in the problem of minimization of the mean value of a quadratic functional. *Sibirsk. Matem. Zh.*, 14(2):250–276, 1972.

[5] N. E. Barabanov. Kalman–Yakubovich lemma in general finite dimensional case. *Int. J. Robust Nonlinear Control*, 17:369–386, 2007.

[6] N. G. Dokuchaev. A frequency criterion for the existence of an optimal control for Itô equations. *Vestnik Leningrad University. Mathematics*, 16:41–47, 1984.

[7] D. G. Korenevskii. Algebraic criteria and sufficient conditions for asymptotic stability and boundedness with probability 1 of solutions of a system of linear stochastic difference equations. *Ukr. Mat. Zh.*, 38(4):447–452, 1986.

[8] D. G. Korenevskii. Matrix criteria and sufficient conditions for asymptotic stability and boundedness with probability one of solutions of linear stochastic difference equations. *Doklo Akad. Nauk SSSR*, 290(6):1294–1298, 1986.

[9] D. G. Korenevskii. Equivalence of spectral and coefficient criteria for the mean–square asymptotic stability of solutions of systems of linear stochastic differential and difference equations. In *Mathematical Methods for the Investigation of Applied Problems of Dynamics*
of Solids Carrying Liquid, pages 47–52. Institute of Mathematics, Ukrainian Academy of Sciences, Kiev, 1992.

[10] D. G. Korenevskii. Criteria for the mean-square asymptotic stability of solutions of systems of linear stochastic difference equations with continuous time and delay. *Ukrainian Mathematical Journal*, 50(8):1073–1081, 1998.

[11] D. G. Korenevskii. Relationship between spectral and coefficient criteria of mean–square stability for systems of linear stochastic differential and difference equations. *Ukrainian Mathematical Journal*, 52(2):260–266, 2000.

[12] M. V. Levit and V. A. Yakubovich. Algebraic criterion for stochastic stability of linear systems with parametric action of the white noise type. *J. Appl. Math. Mech.*, 36:130–136, 1972. Prikl. Mat. Mekh. 36, 142–148 (1972).

[13] A. L. Likhtarnikov, N. E. Barabanov, G. A. Leonov, A. H. Gelig, A. S. Matveev, V. B. Smirnova, and A. L. Fradkov. Frequency domain theorem (Yakubovich–Kalman lemma) in the control theory. *Automation and Remote Control*, 57(10):3–40, 1996.

[14] A. I. Lur’e. A minimum quality criterion for control systems. *Izv. Akad. Nauk SSSR, Otd. Tekhn. Nauk, Tekhnicheskaya Kibernetika*, 4:140–146, 1963.

[15] T. Morozan. Stability of stochastic discrete systems. *J. Math. Anal. Appl.*, 23(1):1–9, 1968.

[16] P. S. Nalitolela. Frequency Criteria of Optimal Control Existence for Stochastic Models. Master’s thesis, Trent University, Peterborough, Ontario, Canada, January 2010.

[17] P. V. Pakshin and V. A. Ugrinovskii. Stochastic problems of absolute stability. *Automation and Remote Control*, 67(11):1811–1846, 2006. Original Russian Text published in *Avtomatika i Telemekhanika*, No. 11, pp. 122–158, 2006.

[18] B. L. Ryashko and H. Schurz. Mean square stability analysis of some linear stochastic systems. *Dynamics Systems Appl.*, 6(2):165–190, 1996.

[19] Saburou Saitoh, Vu Kim Tuan, , and Mashiro Yamamoto. Convolution inequalities and applications. *Journal of Inequalities in Pure and Applied Mathematics*, 4(3):Article 50, 2003.

[20] G. Szegő and R. E. Kalman. Sur la stabilité absolue d’un système d’équations aux différences finies. *Comptes Rendus del Academie des Sciences, Paris*, 257:388–390, 1963.
[21] J. L. Willems. Mean square stability criteria for linear white noise stochastic systems. 
*Probl. Contr. Inf. Theory*, 2(3–4):199–217, 1973.

[22] V. A. Yakubovich. The solution of certain matrix inequalities encountered in automatic 
control theory. *Soviet Mathematics Doklady*, 143(6):1304–1307, 1962.

[23] V. A. Yakubovich. Solution of one algebraic problem encountered in the control theory. 
*Dokl. Akad. Nauk SSSR*, 193(1):57–60, 1970.

[24] V. A. Yakubovich. Frequency domain theorem in control theory. *Siberian Mathematical 
Journal*, 14(2):265–289, 1973.

[25] V. A. Yakubovich. A frequency theorem for the case in which the state and control vec-
 tors are Hilbert spaces, with an application to some problems in the synthesis of optimal 
controls. *Sibirsk. Math. Zh.*, 15(3):639–668, 1974.

[26] V. A. Yakubovich. A frequency theorem for the case in which the state and control vec-
tors are Hilbert spaces, with an application to some problems in the synthesis of optimal 
controls, II. *Sibirsk. Math. Zh.*, 16(5):1081–1102, 1975.