Explicit universal estimate for polyharmonic equations via Morse index

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Abstract
We investigate universal estimate of finite Morse index solutions to polyharmonic equation in a proper open subset of $\mathbb{R}^n$. Differently to previous works [8, 11, 18, 30], we propose here a direct proof under large superlinear and subcritical growth conditions on the term source where we show that the universal constant evolves as a polynomial function of the Morse index. To do so, we introduce a new interpolation inequality and we make use of Pohozaev’s identity and a delicate bootstrap argument.

Thanks to our interpolation inequality, we improve previous nonexistence results [18, 19] dealing with stable at infinity weak solutions to the $p$-polyharmonic equation in the subcritical range.

Keywords: Interpolation inequality, Universal estimate, $p$-polyharmonic equation, Morse index, Liouville-type theorem, Pohozaev identity, Bootstrap argument.

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1. Introduction
Consider $u \in W^{r,p}_{loc}(\Omega)$ where $\Omega$ is an open subset of $\mathbb{R}^n$, with $n, r \geq 2$, and $pr \geq 2$. For any multi-index $j = (j_1, j_2, ..., j_n)$ of order $|j| \leq r$, the weak $j^{th}$ partial derivative $D^j u = \frac{\partial^j u}{\partial x_1^{j_1}...\partial x_n^{j_n}}$ is defined a.e in $\Omega$. The magnitude of the $s^{th}$ gradient is defined a.e in $\Omega$ by

$$|\nabla^s u| = \left( \sum_{|j|=s} |D^j u|^p \right)^{1/p}, \forall 1 \leq s \leq r.$$  

1.1. Explicit universal estimate
In this paper we investigate explicit universal estimate for finite Morse index solutions to the following polyharmonic problem:

$$(-\Delta)^r u = f(x, u), \text{ in } \Omega,$$  

where $u \in C^2(\Omega)$, $f$ and $f' = \frac{\partial f}{\partial s}$ belong to $C(\Omega \times \mathbb{R})$. The associated quadratic form is defined by

$$Q_u(h) = \int_{\Omega} |D^r h|^2 - \int_{\Omega} f'(u)h^2, \forall h \in C^r_c(\Omega).$$
The Morse index of $u$, denoted by $i(u)$ is defined as the maximal dimension of all subspaces $V$ of $C_i^1(\Omega)$ such that $Q_{2i}(h) < 0$, $\forall h \in V \setminus \{0\}$.

Blow-up technique allows us to derive universal estimate; or $L^\infty$-bounds provided that Liouville-type theorems classifying finite Morse index solutions are available. Even in the autonomous case $f(x, u) = f(u)$, this procedure requires the following asymptotical behavior at infinity:

$(h_0)$: There exist $q > 1$ and a positive constant $c_0$ such that $\lim_{s \to \infty} f'(s) = c_0$.

The reader may consult [3, 8, 11, 18, 30] and also [12, 24, 27, 28, 29] for positive solutions. However, from this approach we cannot exhibit how the universal constant depends on $i(u)$. Motivated by [17, 15, 20, 31], we establish explicit universal estimate where we relax assumption $(h_0)$ into the following large superlinear and subcritical growth conditions:

There exist, $s_0 > 0$, $c_1 > 1$ and $1 < p_1 \leq p_2 < \frac{n + 2r}{n - 2r}$ such that for all $(x, s) \in \Omega \times \mathbb{R} \setminus [-s_0, s_0]$, we have

$(h_1)$ (Super-linearity) $f'(x, s)s^2 \geq p_1 f(x, s)s$;

$(h_2)$ (Subcritical growth) $(p_2 + 1)F(x, s) \geq f(x, s)s$ where $F(x, s) = \int_0^s f(x, t)dt$;

$(h_3)$ $|\nabla_x F(x, s)| \leq c_1(F(x, s) + 1)$, for all $(x, s) \in \Omega \times \mathbb{R}$;

$(h_4)$ $|f(x, s)| \leq c_1$, for all $(x, s) \in \Omega \times [-s_0, s_0]$ and $\pm f(x, \pm s_0) \geq \frac{1}{c_1}$, for all $x \in \Omega$.

When $f(x, s) = f(s)$, the above assumptions are reduced to $(h_1)$-$(h_2)$ with $\pm f(\pm s_0) > 0$, which are weaker than $(h_0)$ if $1 < q < \frac{n + 2r}{n - 2r}$. Moreover, $f(s) = s_i^q - s_i^q$ (where $s_i = \max(s, 0)$ and $s_i = \max(-s, 0)$). Let $K \in C^1(\Omega)$ be a positive function such that $K, |\nabla K| \in L^\infty(\Omega)$, then $f(x, s) = K(x)(s_i^q - s_i^q)$ satisfies $(h_1)$-$(h_4)$. Let $\Omega$ be a proper domain of $\mathbb{R}^n$. For $\alpha \in (0, 1)$ and $y \in \Omega$, denote by $\delta_\gamma = \text{dist}(y, \partial \Omega)$, $d_\gamma = \inf(\alpha, \delta_\gamma)$ and set $\Omega_\alpha = \{y \in \Omega; \delta_\gamma \geq \alpha\}$. We use here $[11]$ with $p = 2$. Our explicit estimate reads as follows:

**Theorem 1.1.** Assume that $f$ satisfies $(h_1)$-$(h_3)$. Then, there exist $\alpha_0 \in (0, 1)$ $\gamma_1 > 0$ $\gamma_2 > 0$ and a positive constant $C = C(n, r, p_1, p_2, s_0, c_1)$ independent of $\Omega$ such that for any finite Morse index solution $u$ of (1.2) and for every $\alpha \in (0, \alpha_0)$, we have

$$\sum_{j=0}^{2r-1} d_\gamma^{2j} |\nabla u(y)| \leq C(1 + i(u))^{2j}d_\gamma^{2j}, \forall y \in \Omega.$$  

Precisely, if $2rp_2 + 1 < n$, then $\gamma_1 = \frac{4r^2(p_1 + 1)p_2}{(p_1 - 1)(2r(p_2 + 1) - n(p_2 - 1))}$ and $\gamma_2 = \gamma_1 + \frac{2r(p_2 + 1)}{2r(p_2 + 1) - n(p_2 - 1)}$.

Consequently, for any $\alpha \in (0, \alpha_0)$, we have

$$\|u\|_{C^{r-1}(-\Omega_\alpha)} \leq C(1 + i(u))^{2j} \text{ where } C = C(n, r, p_1, p_2, s_0, c_1) > 0.$$  

To prove Theorem 1.1, we introduce a new interpolation inequality (see section 2) to provide a local integral estimate on a ring (see (1.3) in section 3). Thus, we establish a variant of the Pohozaev identity [28], to derive the following estimate on a ball:

$$d_\gamma^{2j} \int_{B(y, d_\gamma)} |f(x, u)|^{\frac{q+1}{p_2}} \leq C(1 + i(u)) \left(1 + i(u) \right) \frac{d_\gamma^{\frac{2r(q-1)}{p_2 - 1}}}{d_\gamma^{\frac{2r(q-1)}{p_2 - 1}}}, \forall y \in \Omega.$$  

As $p_2$ is less than the critical exponent, we employ a delicate boot strap argument from local $L^q$-$W^{2q,q}$-regularity to obtain the desired estimate (1.4).
Noticing that we can derive \([1.6]\) if in assumptions \((h_1)-(h_2)\) we assume that \(\frac{n + 2r}{n - 2r} < p_1 \leq p_2\). However, it is not clear which procedure would be helpful to derive explicit universal estimate even for \(r = 1\) and \(\frac{n + 2r}{n - 2r} < p_1 \leq p_2 < p_\mu(n, 2)\). Also, similar estimate of \([1.6]\) holds related to the \(p\)-polyharmonic equation. However, we do not dispose to any \(L^q\)-regularity result to employ a bootstrap procedure except in the subcritical \(p\)-laplacian problem where explicit \(L^\infty\)-bounds of finite Morse index solutions has been established under similar assumptions of \((h_1)-(h_2)\) \([17]\).

This extends the result of \([11]\) dealing with the second order Dirichlet boundary-value problem where it has been shown that the \(L^\infty\) norm evolves less rapidly than a polynomial growth on \(i(u)\). Also, in \([20]\) the authors examined the influence of the type boundary conditions related to the biharmonic and triharmonic problems to provide similar polynomial growth estimate. The general higher order polyharmonic problem is still an open question where some local estimates near the boundary are no longer available for \(r \geq 4\) under any type boundary conditions. In estimate \([1.4]\), the exponents \(\gamma_1\) and \(\gamma_2\) converge to \(\infty\) as \(p_2\) tends to \(\frac{n + 2r}{n - 2r}\), which seems coherent with the fact that the \(L^\infty\) norm has an exponential growth on \(i(u)\) when the nonlinearity \(f\) is close to the critical power as \(f(s) = \frac{|s|^{p^*}}{\ln(s^2 + 2)}\) \([15]\).

1.2. Liouville type theorem

Consider the following problem
\[
\Delta_p u = c_1 u^{q_1} - c_2 u^{q_2} \text{ in } \mathbb{R}^n, \quad (1.7)
\]
where \(p \geq 2, q_1, q_2 < p - 1, c_1, c_2 > 0\) and \(u_\infty = \max(0, u), u_- = \max(0, -u)\). The \(p\)-polyharmonic operator \(\Delta_p\) is defined by
\[
\Delta_p u = -\text{div} \left( \Delta^{p-2} \nabla u \right) \quad \text{if } r = 2j - 1 \quad \text{and} \quad \Delta^j(\Delta^{p-2} u) \quad \text{if } r = 2j.
\]
Define the main \(r\)-order differential operator by \(D_r u = \nabla \Delta^{r-1} u\) if \(r = 2j - 1\) and \(\Delta^j u\) if \(r = 2j\). We denote
\[
D_r u \cdot D_r v = \begin{cases} 
\nabla \Delta^{r-1} u \cdot \nabla \Delta^{r-1} v & \text{if } r = 2j - 1; \\
\Delta^j u \Delta^j v & \text{if } r = 2j; \\
|D_r u|^2 - D_r u \cdot D_r u, & \text{else}.
\end{cases}
\]

We designate by \(B_\lambda\) the ball of radius \(\lambda > 0\) centered at the origin \(\overline{\Omega} = \max(q_1, q_2)\) and \(p^* = \frac{pn}{n - pr}\), which is equal to the Sobolev critical exponent of \(W^{p_\mu}(\mathbb{R}^n)\) if \(n > pr\) and \(\infty\) if \(n \leq pr\). The appropriate Banach space of the variational setting of \((1.7)\) is \(E_1 =: W^{p_\mu}(B_1) \cap L^{q_1,1}(B_1)\) if \(\overline{\Omega} > p^* - 1\) (respectively \(E_1 =: W^{p_\mu}(B_4)\) if \(p - 1 < \overline{\Omega} \leq p^* - 1\)).

The energy functional is defined by
\[
I(v) := \frac{1}{p} \int_{B_1} |D_r v|^p - \int_{B_\lambda} \left( c_1 V^{q_1,1} + c_2 V^{q_2,1} + 1 \right), \quad \forall \ v \in E_1,
\]
which belongs in \(C^2(E_1)\) as \(p \geq 2\). We say that \(u \in W^{p_\mu}(\mathbb{R}^n) \cap L^{q_1,1}(\mathbb{R}^n)\) is a weak solution of \((1.7)\) if for any \(h \in E_3\) and for any \(\lambda > 0\), we have
\[
\int_{B_\lambda} |D_r u|^p - D_r u \cdot D_r h = \int_{B_\lambda} (c_1 u^{q_1} - c_2 u^{q_2}) h, \quad (1.9)
\]

The linearized operator of \((1.7)\) at \(u\) is given by
\[
L_0(g, h) := \int_{B_1} |D_r u|^p - D_r u \cdot D_r g \cdot D_r h + (p - 2) |D_r u|^p - (D_r u \cdot D_r g)(D_r u \cdot D_r h) - (c_1 q_1 u^{q_1-1} + c_2 q_2 u^{q_2-1}) g h, \quad \forall (g, h) \in E^2_3.
\]

1. If \(1 < p < 2\), the energy functional \(I\) is only \(C^1\) functional, so it is not clear which definition of stability would be the natural one.

2. As mentioned above, no regularity result is known for the \(p\)-polyharmonic operator if \(r \geq 2\). So, we deal here with weak solutions.
So, the associated quadratic form is defined by $Q_\lambda(h) := L_\lambda(h, h)$ and as
\[
\int_{B_1} |D_x u|^{p-2} (D_x u \cdot D_x h) \leq \int_{B_1} |D_x u|^{p-2} |D_x h|^2, \quad \forall h \in E_\lambda,
\]
then
\[
Q_\lambda(h) \leq (p-1) \int_{B_1} |D_x u|^{p-2} |D_x h|^2 - \int_{B_1} (c_1 q_1 u^{q_1 - 1}_+ + c_2 q_2 u^{q_2 - 1}_-) |h|^2. \tag{1.11}
\]

**Definition 1.1.**

- We say that $u$ is stable if for all $\lambda > 0$ we have $Q_\lambda(h) \geq 0$, for all $h \in E_\lambda$.
- For $R_0 > 0$, $u$ is said to be stable outside the ball $B_{R_0}$ if $\forall \lambda > R_0, \forall h \in E_\lambda \sup h \in \mathbb{R}^n \setminus B_{R_0}$ we have $Q_\lambda(h) \geq 0$. $u$ is also called stable at infinity solution.

For $c_1 = c_2$, $q_1 = q_2$ and $p = 2$, sharp nonexistence results have been obtained in lower order $r \leq 3$ up to the so-called Joseph-Lundgren exponent $p_{JL}(n, 2r)$ [9, 11, 21]. The reader may also consult [13, 15, 16, 30]. The cases $r = 2, 3$ need more involved analysis using powerful monotonicity formula and a delicate blow-down analysis which are very recently extended to the higher order case for large dimension with a convenient explicit expression of $p_{JL}(n, 2r)$ [23]. The nonhomogeneous higher order polyharmonic problem has been discussed in [13] where one only requires that the term source has a subcritical growth at $0$ to provide the nonexistence of nontrivial stable solutions. If $c_1 > 0$, $c_2 = 0$ and $r = 1$, it is shown that any finite morse index solution is spherically symmetric about some point of $\mathbb{R}^n$ [22]. However, the situation is extremely unclear in the higher order case since the decomposition $u = u^r - u^-$ is no longer available for $r \geq 2$. When $p > 2$ the degenerate nonlinear $p$-polyharmonic operator does not yield any monotonicity formula, which provokes an obstruction to examine the supercritical growth case. In contrast, for $r = 1$, from a relevant Moser’s iteration argument, sharp classification results of stable solutions and radial stable at infinity solutions were obtained in [10]. Regarding the $p$-biharmonic problem, only the subcritical case was achieved in [24], where the main integral estimate needs more involved control. Inspired by [13], this result has been extended to the problem (1.7) with $c_1 = c_2$ and $q_1 = q_2$, for all $r \geq 2$. The key ingredient is borrowed from [18, 27, 13] using an appropriate family of test functions and weighted interpolation inequality.

We provide the nonexistence of nontrivial stable and stable at infinity solutions of (1.7) in the subcritical range which improves the results of [18, 19]. To this end, we establish an interpolation inequality (see Lemma 2.3 in section 2) to extend the best integral estimate derived in lower order $r \leq 3$ [9, 21, 16, 30] and therefore we remove the exponential growth condition imposed on unbounded solutions in [18, 19].

**Proposition 1.1.** Let $r \geq 2$, $p \geq 2$, $q_1$, $q_2 > p - 1$ and $R_0 > 0$. Let $u \in W^{r,p}_{\text{loc}}(\mathbb{R}^n) \cap L^{q_1}_\text{loc}(\mathbb{R}^n)$ be a weak solution of (1.7) which is stable outside the ball $B_{R_0}$. Then, there exist two positive constants $C_0 = C_0(u, R_0, n, r, q_1, q_2, p)$ and $C = C(n, r, q_1, q_2, p)$ such that
\[
\int_{B_{R}} |\nabla' u|^p + \int_{B_{R}} (c_1 u^{q_1 + 1}_+ + c_2 u^{q_2 + 1}_-) \leq C_0 + CR^{n-\frac{n}{q_2-r} - \frac{n}{q_1-r}}, \quad \forall R > \max(1, R_0). \tag{1.12}
\]

Moreover, if $u$ is a stable solution, then (1.12) holds with $C_0 = 0$ for all $R > 1$. We also have
\[
\int_{B_{R}} (c_1 u^{q_1 + 1}_+ + c_2 u^{q_2 + 1}_-) \leq C \left( \int_{B_{R}} |\nabla' u|^p + R^{-p} \int_{B_{R}} |u|^p \right). \tag{1.13}
\]

Let us now state our Liouville type theorem. Set $q = \min(q_1, q_2)$, we have

**Theorem 1.2.**

1. The problem (1.7) has no nontrivial weak stable solution belonging to $W^{r,p}_{\text{loc}}(\mathbb{R}^n)$ if $p - 1 < q_1$, $q_2 \leq p^* - 1$.}

\[\text{Inequality (1.13) will be only used to prove the nonexistence of nontrivial weak stable solution when } p^* = p - 1 \text{ and } n > 2r.\]
2. Let $u \in W^{r+1,p}_{loc}(\mathbb{R}^n)$ be a weak solution of (1.7) which is stable outside the ball $B_{R_0}$ for some $R_0 > 0$. Then, $u \equiv 0$ if $p-1 < q_1$, $q_2 < p^* - 1$.

Moreover, if $n > pr$ and $q_1 = q_2 = p^* - 1$, then

$$\int_{\mathbb{R}^n} |D u|^p = \int_{\mathbb{R}^n} \left( c_1 u_+^{q_1} + c_2 u_+^{q_2} \right).$$

(1.14)

If $p-1 < q < p^* - 1$, then (1.14a) holds with respectively $u \geq 0$ if $q = q_1$ and $u \leq 0$ if $q = q_2$.

This paper is organized as follows: Section 2 is devoted to the interpolation inequalities and some preliminary technical Lemmas. In section 2, we give the proof the main Theorem 1.1. The proofs of Proposition 1.1 and Theorem 1.2 will be done in section 4. In the appendix we revised previous $L^p-W^{k,p}$ estimate stated in [14, 27].

2. Interpolation inequalities and preliminary technical lemmas.

2.1. Appropriate cut-off function and interpolation inequalities.

From the standard interpolation inequality on the unit ball $B_1$ and an obvious dilation argument, we have for every $\varepsilon \in (0,1)$ and for any $v \in W^{r,p}_{loc}(\mathbb{R}^n)$

$$R^{(q-r)} \int_{B_\varepsilon} \|v\|^p \leq \varepsilon \int_{B_\varepsilon} \|v\|^{pr} \int_{B_\varepsilon} |v|^p, \quad \forall 1 \leq q \leq r-1,$$

(2.1)

where $C = C(n, p, r) > 0$. According to (2.1), we may easily derive the following weighted interpolation inequality (see [18, 19, 27])

$$R^{(q-r)} \Phi_q^p(v) \leq s \Phi_q^p(v) + C \varepsilon^{\frac{n}{p-r}} R^{-pr} \int_{B_\varepsilon} |v|^p dx.$$

(2.2)

Where $\Phi_q$ is a family of weighted semi-norms defined by

$$\Phi_q(v) = \left( \sup_{0 < a < 1} (1-a)^q \int_{B_a} |\nabla v|^p \right)^{\frac{1}{q}}, \quad 0 \leq q \leq r.$$

To exploit (2.2), we employ the following family of cut-off functions

$$\psi = \psi_{(R, \alpha)} \in C_c^{\infty}(\mathbb{R}^n), \quad 0 \leq \psi \leq 1, \quad \text{supp}(\psi) \subset B_{\alpha R}, \quad \psi \equiv 1, \quad x \in B_{\alpha R},$$

with $\alpha \in (0,1)$, and $\alpha' = \frac{1 + \alpha}{2}$ where

$$\psi = \exp \left( \left( \frac{|x|}{R} - \frac{\alpha}{\alpha'} \right)^{\alpha+1} \right)$$

with $|\nabla^k \psi| \leq C((1 - \alpha)R)^{-k}$ if $\alpha R < |x| < \alpha' R$, and $1 \leq k \leq r$.

We introduce here a more general cut-off function related to two bounded open subset of $\Omega$, $\omega$ and $\omega'$ such that $\overline{\omega} \subset \omega' \subset \overline{\omega'} \subset \Omega$. Set $d = \text{dist}(\omega, \Omega \setminus \omega')$, we have

Lemma 2.1. There exists $\psi \in C_c^{\infty}(\Omega)$ satisfying

$$\begin{cases}
\text{supp}(\psi) \subset \omega' \text{ and } 0 \leq \psi \leq 1, \\
\psi \equiv 1 \text{ if } x \in \omega, \\
|\nabla^k \psi(x)| \leq Cd^{-kp}, \quad \forall x \in \omega' \setminus \omega \text{ and } k \in \mathbb{N},
\end{cases}$$

(2.3)

where $C$ is a positive constant depending only on $(n, p, k)$. 


The case of stable at infinity weak solutions of (1.7) is more difficult and requires the following standard cut-off function $\psi = \psi_{R,R_0} \in C_c^0(B_{2R})$, $R > 2R_0 > 0$ satisfying

$$
\begin{cases}
\psi \equiv 1 & \text{if } 2R_0 < |x| < R,
\psi \equiv 0 & \text{if } |x| < R_0 \text{ or } |x| > 2R, \\
0 \leq \psi \leq 1 & \text{if } |\nabla \psi| \leq CR^{-k}, \text{ for all } R < |x| < 2R, \text{ and } 1 \leq k \leq r.
\end{cases}
$$

Similarly to Lemma 2.2 we have

**Lemma 2.3.** Let $\psi = \psi_{R,R_0}$ and $u \in W^{r,p}_{\text{loc}}(\Omega)$. Then, for every $0 < \epsilon < 1$ there exist two positive constants $C_\epsilon = C(\epsilon, n, p, r, m) > 0$ and $C_0 = C_0(u, R_0, \epsilon, n, p, r, m)$ such that for all $R > 2R_0$, we have

$$
R^{-pk} \int_{B_{2R}} |\nabla^{k} u|^p |\nabla^{k} \psi^{(m-k)}| \leq C_0 + \epsilon \int_{B_{2R}} |\nabla^p u|^p \psi^m + C_\epsilon R^{-pr} \int_{\mathbb{R}^n} |u|^p \psi^{(m-r)}.
$$

**Proof of Lemma 2.1.**

Set $\omega_d = \{x \in \Omega, \ \text{dist}(x, \omega) < \frac{d}{4}\}$, we have $\omega \subset \omega_d \subset \omega'$. Let $h = \chi_{\omega_d}$ be the indicator function of $\omega_d$ and $g \in C^\infty_c(\mathbb{R}^n)$ a nonnegative function such that supp$(g) \subset B_1$ and $\int_{\mathbb{R}^n} g(x) dx = 1$. Set

$$
g_d(x) = \left(\frac{8}{\delta} \right)^n g \left( \frac{x}{\delta} \right) \text{ and } \psi(x) = \int_{B_\delta} g_d(y) h(x-y) dy.
$$

We have $\text{supp}(\psi) \subset \omega_d + B_\delta \subset \omega'$, and $\omega + B_\delta \subset \omega_d$, then $0 \leq \psi \leq 1$ and $\psi(x) = \int_{B_\delta} g_d(y) dy = 1$ if $x \in \omega$. Since $\psi(x) = \int_{B_\delta} g_d(x-y) h(y) dy$, then $\psi \in C^\infty_c(\mathbb{R}^n)$ and $D^j \psi(x) = \int_{B_\delta} D^j g_d(y) h(x-y) dy$. Therefore,

$$
|D^j \psi(x)| \leq \int_{B_\delta} |D^j g_d(y)| dy \leq \lambda^{-|j|} \int_{B_\delta} |D^j (g(y))| dy \leq C \lambda^{-|j|}, \ \text{where } C = C(n, |j|).
$$

Lemma 2.1 is proved.

By virtue of (2.2), one can derive local $L^p-W^{2r,p}$-regularity [13, 27]. Also, this inequality is essential to provide an integral estimate of stable solutions to the higher order $p$-polyharmonic equations [18, 19] which is weaker than the one established in lower order $r \leq 3$ [11, 30, 21]. We introduce an interpolation inequality composed with the cut-off function $\psi$ which will be more relevant than (2.2) in testing higher order PDE’s against $\psi u$ allowing therefore to improve the integral estimate of stable solutions and also to revise local $L^p-W^{2r,p}$-regularity. It will be also helpful in investigating explicit universal estimate of finite Morse index solutions to higher order polyharmonic equation. Motivated by [11] we will use $\psi^m$, $m > r$ as a cut-off function. Let $(q, k) \in \mathbb{N}^+ \times \mathbb{N}^+$, $q + k = r$, we have

**Lemma 2.2.** For every $0 < \epsilon < 1$ there exist two positive constants $C = C(n, r, p, m)$ and $C_\epsilon = C(\epsilon, n, p, r, m) > 0$ such that for any $u \in W^{r,p}_{\text{loc}}(\Omega)$, we have

$$
\int_{\omega} |\nabla^q u|^p |\nabla^r (\psi^m)|^p \leq C_\epsilon d^{-pk} \int_{\omega} |\nabla^q u|^p \psi^{(m-k)} \leq \epsilon \int_{\omega} |\nabla^q u|^p \psi^m + C_\epsilon d^{-pr} \int_{\omega} |u|^p \psi^{(m-r)}.
$$

Consequently,

$$
\int_{\omega} |\nabla^q u|^p \psi^m \leq 2 \int_{\omega} |\nabla^r (u\psi^m)|^p + C_\epsilon d^{-pr} \int_{\omega} |u|^p \psi^{(m-r)}.
$$

and

$$
\int_{\omega} |\nabla^q u|^p |\nabla^k \psi^{(m-k)}|^p \leq C_\epsilon d^{-pk} \int_{\omega} |\nabla^q u|^p |\nabla^k \psi^{(m-k)}| \leq \epsilon \int_{\omega} |\nabla^r (u\psi^m)|^p + C_\epsilon d^{-pr} \int_{\Omega} |u|^p \psi^{(m-r)}.
$$

The case of stable at infinity weak solutions of (1.7) is more difficult and requires the following standard cut-off function $\psi = \psi_{R,R_0} \in C^\infty_c(B_{2R})$, $R > 2R_0 > 0$ satisfying

$$
\begin{cases}
\psi \equiv 1 & \text{if } 2R_0 < |x| < R, \ \psi \equiv 0 & \text{if } |x| < R_0 \text{ or } |x| > 2R, \\
0 \leq \psi \leq 1 & \text{if } |\nabla \psi| \leq CR^{-k}, \text{ for all } R < |x| < 2R, \text{ and } 1 \leq k \leq r.
\end{cases}
$$

Similarly to Lemma 2.2 we have

**Lemma 2.3.** Let $\psi = \psi_{R,R_0}$ and $u \in W^{r,p}_{\text{loc}}(\Omega)$. Then, for every $0 < \epsilon < 1$ there exist two positive constants $C_\epsilon = C(\epsilon, n, p, r, m) > 0$ and $C_0 = C_0(u, R_0, \epsilon, n, p, r, m)$ such that for all $R > 2R_0$, we have

$$
R^{-pk} \int_{B_{2R}} |\nabla^{k} u|^p |\nabla^{k} \psi^{(m-k)}| \leq C_0 + \epsilon \int_{B_{2R}} |\nabla^p u|^p \psi^m + C_\epsilon R^{-pr} \int_{\mathbb{R}^n} |u|^p \psi^{(m-r)}.
$$
As a consequence, we have

\[
\int_{B_{2R}} |\nabla^i u|^p |\psi|^{pm} \leq C_0 + 3 \int_{\mathbb{R}^n} |\nabla^i (u\psi^m)|^p + CR^{-pk} \int_{B_{2R}} |u|^p |\psi|^{pm-k},
\]

and

\[
R^{-pk} \int_{B_{2R}} |\nabla^q u|^p |\psi|^{pm-k} \leq C_0 + \epsilon \int_{B_{2R}} |\nabla^i (u\psi^m)|^p + C_\epsilon R^{-pk} \int_{B_{2R}} |u|^p |\psi|^{pm-k}.
\]

2.2. Proofs of Lemmas 2.2 and 2.3

As the proofs are closely similar, they will be done in parallel. Working by induction we may verify that

\[
\left\{ \begin{array}{l}
|\nabla^i \psi|^p \leq C d^{-pk} |\psi|^{p(m-k)}, \\
|\nabla^i \psi_{R,R_0}|^p \leq CR^{-pk} |\psi|^{p(m-k)},
\end{array} \right. \quad R < |x| < 2R,
\]

for all \(1 \leq k \leq r\) where \(C\) is a positive constant depending only on \((n, p, r, m)\), (see the Appendix of [18]). Firstly, inequalities (2.6) and (2.10) are immediate consequences of (2.4), (2.8) respectively (2.9) and (2.10). Also, inequalities (2.5) and (2.9) follow respectively from (2.4) and (2.8). Indeed, according to (1.11) we have

\[
|\nabla^i u|^p |\psi|^{pm} = \sum_{|j|=r} |D^{|j|} u|^p |\psi|^{pm}.
\]

We apply the following inequality\(^4\)

\[
b^p \leq 2a^p + C(a - b)^p, \quad \forall a, b > 0 \quad \text{where} \quad C = C(p) > 0,
\]

with \(a = |D^{|i|} (u\psi^m)|\) and \(b = |D^{|j|} (u\psi^m)|\) and using Leibniz’s formula\(^1\), we arrive at

\[
|\nabla^i u|^p |\psi|^{pm} \leq 2|\nabla^i (u\psi^m)|^p + C \sum_{|j|=r} |D^{|i|} (u\psi^m) - D^{|i|} (u\psi^m)|^p \leq 2|\nabla^i (u\psi^m)|^p + C \sum_{q+k=r, q \neq r} |\nabla^q u|^p |\nabla^k \psi|^p.
\]

Integrate the above inequality over \(\Omega\) (respectively over \(B_{2R}\) if \(\psi = \psi_{R,R_0}\), and using (2.11), we obtain

\[
\int_{\Omega} |\nabla^i u|^p |\psi|^{pm} \leq 2 \int_{\Omega} |\nabla^i (u\psi^m)|^p + C \sum_{q+k=r, q \neq r} d^{-pk} \int_{\Omega} |\nabla^q u|^p |\nabla^k \psi|^p,
\]

and

\[
\int_{B_{2R}} |\nabla^i u|^p |\psi|^{pm} \leq C_0 + 2 \int_{B_{2R}} |\nabla^i (u\psi^m)|^p + C \sum_{q+k=r, q \neq r} R^{-pk} \int_{\mathbb{R}^n} |\nabla^q u|^p |\nabla^k \psi|^p,
\]

where

\[
C_0 = \sum_{q+k=r, q \neq r} \int_{A_{R_0}} |\nabla^q u|^p |\nabla^k \psi|^p, \quad A_{R_0} = \{R_0 < |x| < 2R_0\}.
\]

Combine these inequalities with respectively (2.4) and (2.8) and choose \(\epsilon = \frac{1}{rC}\), we derive (2.5) and (2.9).

\(^4\)In fact, we have \((1 - 2+ \frac{1}{p}) b \leq |b - a|\) if \(b \geq 2+ a\).
Proof of (2.4).
In the following, $C$ denotes always generic positive constants depending on $(n, p, r, m)$ only, which could be changed from one line to another. Set $I_q = d^{-rk} \int_{\omega} |\nabla^q u|^{p(m-k)}$. By virtue of (2.11), we have $\int_{\Omega} |\nabla^q u|^p |\nabla^{k}(\psi^m)|^p \leq C_l q$. Hence, inequality (2.4) will be an immediate consequence of the following crucial inequality:

$$I_q \leq C e^{1/q} I_0 + \varepsilon I_{q+1}, \quad \forall 1 \leq q \leq r - 1, \quad \forall \varepsilon \in (0, 1).$$  

(2.13)

We divide the proof of (2.13) in two steps.

**Step 1.** We shall establish the following first-order interpolation inequality:

$$I_q \leq C e^{1/q} I_{q-1} + \varepsilon I_{q+1}, \quad 1 \leq q \leq r - 1.$$  

(2.14)

Denote by $u_{\omega'}$ the restriction of $u$ to $\omega'$. As $u \in W^{p,q}(\Omega)$, we have $u_{\omega'} \in W^{p,q}(\omega')$ and from Meyers-Serrin’s density theorem we may assume that $u_{\omega'} \in C^r(\omega') \cap W^{p,q}(\omega')$. Let $j = (j_1, j_2, ..., j_n)$ be a multi index with $|j| = q$ and $1 \leq q \leq r - 1$. As $|j| \neq 0$, there exists $i_0 \in \{1, 2, ..., n\}$ such that $j_{i_0} \neq 0$. Set $j_1 = (j_1, ..., j_{i_0} - 1, ..., j_n)$, $|j_1| = q - 1$ and $j_2 = (j_1, j_{i_0} + 1, ..., j_n)$, $|j_2| = q + 1$. Taking into account that $u_{\omega'} \in C^r(\omega') \cap W^{p,q}(\omega')$, $1 \leq |j| \leq r - 1$ and $p \geq 2$, we deduce

$$|D^j u|^p D^j u \in C^1(\omega') \quad \text{and} \quad \frac{\partial}{\partial x_{i_0}}(|D^j u|^p D^j u) = (p - 1)|D^j u|^p D^j u.$$  

(2.15)

Recall that $\psi \in C^{\infty}_c(\omega'), |\nabla \psi| \leq C d^{-1}$, then integration by parts with respect the variable $x_{i_0}$, yields

$$d^{-rk} \int_{\omega'} |D^j u|^p \psi^m \leq \int_{\omega'} |D^j u|^p D^j u \frac{\partial}{\partial x_{i_0}} \psi^m = -(p - 1)d^{-rk} \int_{\omega'} |D^j u|^p D^j u \psi^m$$

$$-p(m-k)d^{-rk} \int_{\omega'} |D^j u|^p D^j u \psi^m \frac{\partial}{\partial x_{i_0}} \psi^m$$

$$\leq C d^{-rk} \int_{\omega'} |\nabla^{q-1} u|^{p(m-k)} |\nabla^{q+1} u| \psi^m$$

$$+ C d^{-r(p+1)} \int_{\omega'} |\nabla^{q-1} u|^{p(m-k)} \psi^m,$$  

(2.16)

where $C = C(n, p, r, m) > 0$. As $I_q = \sum_{|j| = q} d^{-rk} \int_{\omega'} |D^j u|^p \psi^m$, with $k + q = r$, then (2.16) implies

$$I_q \leq C(J_1 + J_2),$$  

(2.17)

where

$$J_1 = d^{-rk} \int_{\omega'} |\nabla^{q-1} u|^{p(m-k)} |\nabla^{q+1} u| \psi^m \quad \text{and} \quad J_2 = d^{-r(p+1)} \int_{\omega'} |\nabla^{q-1} u|^{p(m-k)} \psi^m.$$  

We invoke now the following Young’s inequalities: for every $a, b, c > 0$ and $0 < \varepsilon < 1$ we have

$$ab^{\varepsilon} \leq \frac{1}{p} e^{\varepsilon - p} a^p + \frac{p - 2}{p} a b^p + \frac{1}{p} e^p.$$  

(2.18)

We used standard approximation argument and Lebesgue’s dominated convergence theorem to extend (2.13) to $u_{\omega'} \in W^{p,q}(\omega')$. 

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5We used standard approximation argument and Lebesgue’s dominated convergence theorem to extend (2.13) to $u_{\omega'} \in W^{p,q}(\omega')$. 

and

\[ ab^{p-1} \leq \frac{1}{p} c^{1-p} a^p + \frac{p-1}{p} c b^p. \]  (2.19)

Observe that \( pk = (k+1) + (p-2) + (k-1), \) \( p(m-k) = (m-(k+1)) + (p-2)(m-k) + (m-(k-1)) \) and apply (2.18) with

\[ a = d^{-k(p+1)} |\nabla^q u|^{q(m-k+1)}; \quad b^{p-2} = d^{-k(p-2)|\nabla^q u|^{p-2} q^{r-k(p-2)}}, \quad c = d^{-(k-1)|\nabla^q u|^{q(m-k-1)}}, \]

there holds

\[ J_1 \leq \frac{1}{p} c^{1-p} I_{q-1} + \frac{p-2}{p} c I_q + \frac{1}{p} c I_{q+1}. \]  (2.20)

Also, as \( pk+1 = (k+1) + (p-1)k \) and \( p(m-k) - 1 = (m-(k+1)) + (p-1)(m-k) \), according to (2.19) with

\[ a = d^{-(k+1)|\nabla^q u|^{q(m-k+1)}}, \quad b^{p-1} = d^{-(p-1)|\nabla^q u|^{p-1} q^{m-k(p-1)}}, \]

we obtain

\[ J_2 \leq \frac{1}{p} c^{1-p} I_{q-1} + \frac{p-1}{p} c I_q. \]  (2.21)

Combine, (2.17) with (2.20) and (2.21), we arrive at

\[ (1 - 2C\varepsilon) I_q \leq C\varepsilon^{1-p} I_{q-1} + C\varepsilon I_{q+1}. \]

We replace \( \varepsilon \) by \( \varepsilon \frac{\varepsilon}{4(1+C)} \), thus the desired inequality (2.14) follows.

**Step 2. End of the proof of (2.13).**

The case \( r = 2 \) is obvious. The case \( r \geq 3 \) needs involved iteration argument. We first prove that

\[ I_q \leq C\varepsilon^{1-p} I_0 + C\varepsilon I_{q+1}, \quad 1 \leq q \leq r-1. \]  (2.22)

The case \( q = 1 \) follows from (2.14). For \( q \geq 2 \), let \( 2 \leq j \leq q \leq r-1 \). We apply (2.14) where we replace \( q \) by \( j-i \) and \( \varepsilon \) by \( \varepsilon^p \) with \( i = 0, 1, 2, \ldots, j-1 \), we derive

\[ C^i \varepsilon^{-p} I_{j-i} \leq C^{i+1} \varepsilon^{-p} I_{j-i-1} + C I_{j-i+1}, \]

Set \( S_j = \sum_{i=2}^{j} I_i \), we make the sum of the above inequalities from \( i = 0 \) to \( i = j-1 \), we obtain\(^6\)

\[ \varepsilon^{-1} I_j \leq C^i \varepsilon^{-p} I_0 + I_{j+1} + \sum_{i=1}^{j-1} C I_{j-i+1} \leq C\varepsilon^{-p} I_0 + I_{j+1} + CS_j. \]

As \( S_j \leq S_q \) as \( I_q \geq 0 \), we have

\[ I_j \leq C\varepsilon^{1-p} I_0 + C\varepsilon S_q \] if \( 2 \leq j \leq q \).  \( (2.23) \)

\(^6\)Recall that \( 0 < \varepsilon < 1 \).
So, we make the sum of (2.23) from \( j = 2 \) to \( j = q \), we arrive at

\[ S_q \leq C e^{1 - q} I_0 + \varepsilon I_{q+1} + C \varepsilon S_q. \]

We replace \( \varepsilon \) by \( \frac{\varepsilon}{2(C + 1)} \), we get

\[ S_q \leq C e^{1 - q} I_0 + \varepsilon I_{q+1}, \quad \text{for all } 1 \leq q \leq r - 1. \]

We insert the above inequality in (2.23) with \( j = q \), then the desired (2.22) follows. Now, as \( 0 < \varepsilon < 1 \), from (2.22), we have

\[
\begin{aligned}
I_q & \leq C e^{1 - q} I_0 + I_{q+1}, \\
I_{q+1} & \leq +C e^{1 - q} I_{2q} + I_{q+2}, \\
& \vdots \\
I_{q+q} & \leq C e^{1 - q} I_{2q} + I_{q+q}, \\
& \vdots \\
I_{r+1} & \leq C e^{1 - q} I_0 + \varepsilon I_r.
\end{aligned}
\]

We make the sum of these inequalities, we deduce

\[ I_q \leq C e^{1 - q} I_0 + \varepsilon I_r. \]

Thus, the proof of Lemma 2.3 is completed.

Let us now end the proof of inequality (2.8) of Lemma 2.3 which is closely similar to the one of (2.4). We will only explain how the constant \( C_0 = C_0(u, R_0, n, p, r, m) \) appears in (2.8). For \( 1 \leq q \leq r - 1 \) and \( k = r - q \), let \( u \in W^{1,p}_{loc}(\mathbb{R}^n) \) and \( \psi = \psi_{R_0} \). Set \( I_q = R^{-k} \int_{B_{2R}} |D^q u|^p \psi^{p(m-k)} \). We first show that for every \( \varepsilon \in (0, 1) \) there exist two positive constants \( C = C(n, p, r, m) \) and \( C_0 = C(u, R_0, n, p, r, m) \) such that

\[ I_q \leq C_0 + C_0 e^{1-p} I_{q-1} + \varepsilon I_{q+1}, \quad \forall 1 \leq q \leq r - 1. \]

Recall that \( R > 2R_0 \) and \( |\nabla \psi| \leq CR^{-1} \) if \( x \in A_k = \{ R < |x| < 2R \} \) (see (2.7)). As above integration by parts gives

\[
R^{-pk} \int_{B_{2R}} |D^q u|^p \psi^{p(m-k)} = -(p - 1) R^{-pk} \int_{B_{2R}} D^j u |D^j u|^p |D^{j-1} u| \psi^{p(m-k)}
- p(m-k) R^{-pk} \int_{A_k \cap A_{R_0}} |D^j u| |D^j u|^p |D^{j-1} u| \psi^{p(m-k-1)} \frac{\partial \psi}{\partial x_0}
\leq C_0 + (p - 1) R^{-pk} \int_{B_{2R}} |D^j u| |D^j u|^p |D^{j-2} u| \psi^{p(m-k)}
+ p(m-k) R^{-pk-1} \int_{B_{2R}} |D^j u| |D^j u|^p |D^{j-1} u| \psi^{p(m-k-1)},
\]

where \( C_0 = pm(\inf(1, R_0))^{-p} \int_{A_{R_0}} |D^j u| |D^j u|^p - |\nabla | \psi| \). Therefore,

\[ I_q \leq C_0 + C R^{-pk} \int_{B_{2R}} |D^{q-1} u| |D^{q-1} u|^p |D^{q-1} u| \psi^{p(m-k)} + R^{-pk-1} \int_{B_{2R}} |D^{q-1} u| |D^{q-1} u|^p |D^{q-1} u| \psi^{p(m-k-1)} \]

where \( C_0 = pm(\inf(1, R_0))^{-p} \int_{B_{2R}} |D^{q-1} u| |D^{q-1} u|^p - |\nabla | \psi| \). Hence, following the proof of (2.14), we derive (2.25). Note that in inequality in (2.25), the additive constant \( C_0 \) does not provoke any mathematical difficulty to employ as above an iteration argument to derive inequality (2.8):

\[ I_q \leq C_0 + C e^{1-p} I_0 + \varepsilon I_r, \quad \forall 1 \leq q \leq r - 1. \]

The proof of Lemma 2.3 is thereby completed.
2.3. Preliminary technical Lemmas.

Thanks to our interpolation inequalities of Lemmas 2.2 and 2.3, we establish the following technical lemmas which will be respectively essential to prove Proposition 1.1 and Theorem 1.1. Let $\psi = \psi_R, \omega (respectively $\psi = \psi_R(\omega')$ the cut-off function defined in (2.7) (respectively defined in Lemma 2.1 with $\omega = B_R$ and $\omega' = B_{2R}$). For $u \in C^2(\mathbb{R}^n)$ and $m > r$, we have

**Lemma 2.4.**

1. For every $0 < \epsilon < 1$, there exist two positive constants $C_0 = C_0(u, R_0, n, m, r, p, \epsilon)$ and $C_\epsilon = C(n, m, r, p, \epsilon)$ such that

$$\int_{B_{2\epsilon}} |D_i u|^{p-2} \left| \left( D_j (u \psi^{\frac{r}{m}}) \right)^2 - D_j u D_i (u \psi^{\frac{m}{m}}) \right| \leq C_0 + I(u, R, r, p, m, \epsilon);$$  

(2.27)

$$\int_{B_{2\epsilon}} |D_i u|^{p-2} u D_i (u \psi^{\frac{m}{m}}) \leq C_0 + I(u, R, r, p, m, \epsilon);$$  

(2.28)

where

$$I(u, R, r, p, m, \epsilon) = \epsilon \int_{B_{\epsilon}} |\nabla^r u|^{p \psi^{\frac{m}{m}}} + C_\epsilon \sum_{q+k=r, q \neq r} R^{-pk} \int_{B_{\epsilon}} |\nabla^q u|^{p \psi^{(m-r)}}.$$  

(2.29)

2. If $\psi = \psi_R$, inequalities (2.27) and (2.28) hold with $C_0 = 0$.

3. As a consequence of Lemma 2.3, we have

$$I(u, R, r, p, m, \epsilon) \leq \epsilon \int_{B_{\epsilon}} |\nabla^r (u \psi^{\frac{m}{m}})|^{p} + C_\epsilon R^{-pr} \int_{B_{\epsilon}} |u|^{p \psi^{(m-r)}}.$$  

(2.30)

**Proof of Lemma 2.4.**

Let $s > 1$ and $\beta_q \in \mathbb{R}$ with $0 \leq q \leq r$. From Young’s inequality, we can easily derive that for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$\left| \beta_{q} \sum_{q=0}^{r-1} \beta_q \right| \leq \epsilon |\beta|^{s} + C_\epsilon \sum_{q=0}^{r-1} |\beta_q|^{s}.$$  

(2.31)

For $\eta \in C^r(\mathbb{R}^n)$ set $A(\eta, u) = D_j (u \eta) - \eta D_j u$. A simple computation and Leibnitz’s formula [1], imply

$$\begin{align*}
& \left\{ \eta^2 |D_i u|^2 - D_i u \cdot D_j (u \eta) \right\} = -D_i u \cdot A(\eta, u); \\
& \left\{ D_j (u \eta) \right\}^2 = \eta^2 |D_i u|^2 + 2\eta D_i u \cdot A(\eta, u) + |A(\eta, u)|^2; \\
& |A(\eta, u)| \leq C |\nabla^\epsilon \eta||\nabla^\epsilon u|.
\end{align*}$$  

(2.32)

Therefore, we have

$$|D_i u|^{p-2} \left| \left( D_j (u \eta)^2 - D_j u D_i (u \eta^2) \right) \right| \leq C \sum_{q+k=r, q \neq r} \left( |D_i u|^{p-1} |\nabla^q u (\eta^2 \nabla^q \eta) + |\nabla^q (\eta^2)| + |D_i u|^{p-2} |\nabla^q u|^2 |\nabla^q \eta|^2 \right).$$

and

$$|D_i u|^{p-2} \left| \left( D_i u \eta^2 - D_i u D_i (u \eta^2) \right) \right| \leq C_{\epsilon, r}|D_i u|^{p-1} \sum_{q+k=r, q \neq r} |\nabla^q u||\nabla^q (\eta^2)|.$$
We choose now $\eta = \psi^\#$, as $|D_i u| \leq C|\nabla u|$, by virtue of (2.11) we obtain
\[
\int_{B_{2R}} \left| D_i u \right|^{p-2} \left( |D_j (u \psi^\#)|^2 - D_i u D_j (u \psi^\#) \right) \leq C_0 + CS (u, R, p, m, r), \tag{2.33}
\]
and
\[
\int_{B_{2R}} \left| D_i u \right|^{p-2} \left( |D_j u|^2 \psi^{pm} - D_i u D_j (u \psi^\#) \right) \leq C_{r,a} (C_0 + CS (u, R, p, m, r)), \tag{2.34}
\]
where
\[
S (u, R, p, m, r) = \int_{A_k} \left( |D_i u|^{p-1} R^{-k} |\nabla u|^{p^{m-k}} \right) + \int_{A_k} \left( |D_i u|^{p-2} R^{-2k} |\nabla u|^2 \psi^{pm-2k} \right),
\]
and
\[
C_0 = C_0 (u, R_0, p, m, r) = \int_{A_k} \sum_{q+k=q, q \neq r} \left( |D_i u|^{p-1} |\nabla u|^{p^{m-k}} |\psi^{k-\omega} \nabla^k (\psi^\#) + |\nabla^k (\psi^\#)|) + |D_i u|^{p-2} |\nabla u|^2 (|\nabla^k \psi^{\#}|^2) \right).
\]
Observe that $pm - k = (p - 1)m + (m - k)$ (respectively $pm - 2k = (p - 2)m + 2(m - k)$) and apply (2.31) in $S(u, R, p, m, r)$ with $s = \frac{p}{p-1}$, $\beta_r = |\nabla u|^{p-1} \psi^{pm}$ and $\beta_q = R^{-k} |\nabla u|^2 \psi^{pm-k}$ (respectively with $s = \frac{p}{p-2}$ and $\beta_r = |\nabla u|^{p-2} \psi^{pm-2k}$ and $\beta_q = R^{-2k} |\nabla u|^2 \psi^{2m-k}$), we obtain $S (u, R, p, m, r) \leq I(u, R, r, p, m, \varepsilon)$. Hence, inequalities (2.27) and (2.28) follow from the last inequality and respectively (2.33) and (2.34). We are now in a position to use our interpolation inequalities of Lemma 2.3 to prove (2.30). In fact, as $A_k \subset B_{2R}$, (2.30) is an immediate consequence of inequalities (2.3) (respectively of (2.10) if $\psi = \psi_k$). The proof of Lemma 2.4 is completed.

For $y \in \Omega$, we designate by $B(y, \lambda)$ the ball of radius $\lambda > 0$ centered at $y$. Let $\omega$ and $\omega'$ be two bounded open subset of $B(y, \lambda)$ such that $\overline{\omega} \subset \omega' \subset \overline{\omega'} \subset B(y, \lambda)$ and $d = \text{dist}(\omega, \Omega \setminus \omega')$. Let $\psi$ the cut-off function defined in Lemma 2.1. Similarly to the proof of Lemma 2.4 we derive the following

**Lemma 2.5.** For every $0 < \varepsilon < 1$, there exist two positive constants $C = C(n, r) = C_0 = C(n, m, r, \varepsilon)$ such that for any $u \in W_0^{1,p}(\Omega)$ we have
\[
\int_{B_{(y, d)}} \left| D_i (u \psi^m) \right|^2 - D_i u D_j (u \psi^m) \leq C_\varepsilon \int_{B_{(y, d)}} |\nabla u|^2 |\nabla (u \psi^m)|^2 + C_\varepsilon d^{-2r} \int_{B_{(y, d)}} u^2 \psi^{2m-\varepsilon}; \tag{2.35}
\]
and
\[
\int_{B_{(y, d)}} \left| D_i u \right|^2 \psi^{2m} - D_i u D_j (u \psi^m) \leq C_\varepsilon \int_{B_{(y, d)}} |\nabla u|^2 |\nabla (u \psi^m)|^2 + C_\varepsilon d^{-2r} \int_{B_{(y, d)}} u^2 \psi^{2m-\varepsilon}. \tag{2.36}
\]

**Proof of Lemma 2.5.** We only give here the outline of the proof since it is closely identical to the one of Lemma 2.4. Set
\[
I(u, d, r, m, \varepsilon) = \varepsilon \int_{B_{(y, d)}} |\nabla u|^p \psi^{pm} + C_\varepsilon \sum_{q+k=q, q \neq r} R^{-pk} \int_{B_{(y, d)}} |\nabla u|^p |\psi^{pm-k}|,
\]
we appeal to Lemma 2.2 and we insert inequalities (2.5) and (2.6) in the right hand side of $I$, we deduce

---

*Precisely we used (2.11) where one replaces respectively $m$ by $\frac{pm}{2}$ and $pm$.**
\[ I(u, d, r, m, \varepsilon) \leq \varepsilon \int_{B(y, \lambda)} |\nabla^r (ud^m)|^2 + C_{r} d^{-2r} \int_{B(y, \lambda)} u^2 \psi^{2(m-r)}. \]  
\hspace{4.5cm} (2.37)\]

Following now the proof of Lemma 2.4, where we replace \( p \) by 2, \( R \) by \( d \) and both \( B_{2R}, A_{R} \) by \( B(y, \lambda) \), we therefore derive the analogue one of inequalities (2.27) and (2.28) and combining them with (2.37), we obtain (2.35), (2.36).

This ends the proof of Lemma 2.5.

According to the standard Calderon-Zygmund’s inequality [13] and working by induction we derive the higher order analogue one [14]:

\[ \int_{B(y, \lambda)} |\nabla^r v|^{\beta} \leq C \int_{B(y, \lambda)} |\nabla^r v|^{\beta}, \forall v \in W^{0, \beta}_{0}((B(y, \lambda))). \]  
\hspace{4.5cm} (2.38)\]

where \( C \) is a positive constant depending only on \((n, r, p)\)^9.

We end this section by recalling the following elementary identity which will be used to provide a variant of the Pohozaev identity (see the proof in [18], page 1866)^10. Fix \( y \in \mathbb{R}^{n} \) and let \( u \in W^{n+1,2}_{0}(\Omega) \). We have

\[ D_{s}u D_{s}(\nabla u \cdot (x - y)) = \frac{1}{2} \nabla(|\nabla u|^{2}) \cdot (x - y) + r|D_{s}u|^{2}, \ a.e \ in \mathbb{R}^{n}. \]  
\hspace{4.5cm} (3.29)\]

3. **Proof of Theorem 1.1**

In the following lemma, \( C \) denotes always generic positive constant depending only on the parameters \((s_{0}, p_{1}, p_{2})\) and the constant \( c_{1} \) of assumptions \((h_{1})-(h_{4})\). Let \( q > 1 \) and set \( q_{1} = \frac{p_{2} + 1}{p_{2}} \). We have

**Lemma 3.1.** There exists a positive constant \( C \) such that for all \((x, s) \in \Omega \times \mathbb{R}_{+} \), we have

- [1] \( f'(x, s) s^{2} \geq p_{1} f(x, s) - C; \)
- [2] \((p_{2} + 1) F(x, s) \geq f(x, s) s - C; \)
- [3] \(|s|^{p_{1} + 1} \leq C(|f(x, s)| + 1) \) and \(|s|^{q} \leq C(|f(x, s)| + 1); \)
- [4] \(|f(x, s)| \leq f(x, s) s + C \) and \(|F(x, s)| \leq C(|f(x, s)| + 1); \)
- [5] \(|f(x, s)|^{q_{1}} \leq C(|f(x, s)|^{q} + 1) \) and \(|f(x, s)|^{q_{1}} \leq C(|s|^{q} + 1); \)
- [6] For all \( s \in (0, 1), 0 \leq a \leq 1 \) and \( b > 0 \) we have \( a^{s} b \leq C + \varepsilon |f(x, s)| a^{\frac{p_{2} + 1}{p_{2}}} + b^{\frac{q_{1}}{q_{1}}} \).

**Proof of Lemma 3.1**

According to assumption \((h_{4})\), we have

\[ |F(x, s)|, \ |f(x, s)| \leq C, \ \forall (x, s) \in \Omega \times [-s_{0}, s_{0}]. \]  
\hspace{4.5cm} (3.1)\]

Thus, points 1 and 2 are an immediate consequence of assumptions \((h_{1})-(h_{2})\) and points 3, 4 and 5 hold for all \((x, s) \in \Omega \times [-s_{0}, s_{0}]. \) As the nonlinearity \(-f(x, s)\) satisfies also \((h_{1})-(h_{4}), \) then we need only to prove these inequalities for all \((x, s) \in \Omega \times \{s_{0}, \infty\}. \)

**Proof of point 3.**

From \((h_{1})\) we have

\[ f'(x, s) s \geq p_{1} f(y, s), \ \forall (x, s) \in \Omega \times [s_{0}, \infty). \]  
\hspace{4.5cm} (3.2)\]

---

9 In fact, an obvious dilation and translation argument show that \( C \) does not depend on \((y, \lambda). \)
10 In [18], \( y = 0 \) and \( D_{s} \) is denoted by \( D^{r}. \)
It follows that \( \left( \frac{f(x, s)}{s^{p_1}} \right)^r \geq 0 \) and \( f(x, s) = \frac{(f(x, s_0))}{s_0^{p_1}} s^{p_1} \). As \( f(x, s_0) \geq \frac{1}{c_1} \) for all \( x \in \Omega \) (see \( (h_4) \)), then
\[
f(x, s) \geq \frac{s^{p_1}}{c_1 s_0^{p_1}} \text{ and } f(x, s) s \geq \frac{s^{p_1 + 1}}{c_1 s_0^{p_1}}.
\]
(3.3)

Hence, \( s^{p_1} \leq C[f(x, s)] \) and \( s^{p_1 + 1} \leq C[f(x, s)] \(y, s) \in \Omega \times [s_0, \infty) \).

For the second inequality of point 3, let \( q > 1 \), from the above inequality we have \( s^q \leq s^{p_1} + C \leq C|f(x, s)|^q + 1 \).

**Proof of point 4.**

The first inequality follows from (3.3). Integrating (3.2) over \([s_0, s]\) and using \((h_2)\), we derive \( \frac{f(x, s)s}{p_1 + 1} < F(x, s) \leq \frac{f(x, s)s}{p_1 + 1} + C, \forall (x, s) \in \Omega \times [s_0, \infty) \) which achieves the proof of the second inequality.

**Proof of point 5.**

As \( f(x, s_0) \geq \frac{1}{c_1}, (h_2) \) gives \( \frac{F(x, s)}{s^{p_1 + 1}} \leq 0 \). Then, \((h_4)\) implies \( F(x, s) \leq \frac{F(x, s_0)}{s_0^{p_1 + 1}} s^{p_1 + 1} \leq \frac{c_1}{s_0^{p_1 + 1}} s^{p_1 + 1} \forall (x, s) \in \Omega \times [s_0, \infty) \). Using again \((h_2)\), we obtain \( f(x, s) s \leq C[s^{p_1 + 1} \forall (x, s) \in \Omega \times [s_0, \infty) \). From above, we have \( f(x, s) s = |f(x, s)| \) if \( (x, s) \in \Omega \times [s_0, \infty) \). Therefore, for \( q > 1 \), we have
\[
|f(x, s)|^q \leq C|f(x, s)| \text{ and } |f(x, s)|^{q_2} \leq C|s|^q + 1, (x, s) \in \Omega \times [s_0, \infty).
\]

**Proof of point 6.**

We apply the following Young’s inequality \( r^2 \leq e^{nt^{p_i + 1}} + e^{nt^{p_i + 1}} \), with \( t = |s|a^2 \) and \( v = b \), we obtain \( a^2 s \leq e^{s^{p_1 + 1}} a^{p_1 \frac{n}{n-1}} + e^{s^{p_1 + 1}} b^{p_1 \frac{n}{n-1}} \). As \( 0 \leq a \leq 1 \), using point 3 we derive inequality 6. This end the proof of Lemma 3.1.

Let \( \alpha \in (0, 1) \), recall that \( d_0 = \text{in}(\alpha, \delta) \) where \( \delta = \text{dist}(y, \partial \Omega) \) and \( y \in \Omega \). We designate by \( B(x, d_0) \) the ball of radius \( d_0 \) centered at \( y \) and \( A_{d_0} = \{ x \in \mathbb{R}^n : a < |x - y| < b \} \). For \( j = 1, 2, \ldots, i(u) + 1 \), set
\[
A_j := A_{d_j} \quad \text{with} \quad a_j = d_j \frac{2(j + i(u))}{4i(u) + 1}, \quad b_j = d_j \frac{2(j + i(u)) + 1}{4i(u) + 1}.
\]
(3.4)

\[
A_j' := A_{d_j'} \quad \text{with} \quad a_j' = d_j \frac{2(j + i(u)) - 1}{4i(u) + 1}, \quad b_j' = d_j \frac{2(j + i(u)) + 1}{4i(u) + 1}.
\]
(3.5)

From Lemma 2.1, with \( \omega = A_j \) and \( \omega' = A_j' \), there exist \( \psi_j \in C_0(B(y, d_j)) \) satisfying
- \( \psi_j = 1 \) for \( x \in A_j \) and \( 0 \leq \psi_j \leq 1 \) for \( x \in A_j' \);
- \( \text{supp}(\psi_j) \subset A_j' \) and \( |\nabla^k \psi_j| \leq C(\frac{1 + i(u)}{d_j})^k \).

As \( \text{dist}(A_j, B(y, d_j)) A_j' = \frac{d_j}{2(i(u) + 1)} \), from inequality 2.11 and for \( m > r \) we have
\[
|\nabla^k (\psi_j^m)| \leq C \psi_j^{2m - k} \left( \frac{1 + i(u)}{d_j} \right)^{2k}, \forall x \in B(y, d_j).
\]
(3.6)

We have \( \text{supp}(\psi_j^m) \cap \text{supp}(\psi_j^m) = \emptyset \) as \( A_j' \cap A_j' = \emptyset \forall 1 \leq j \leq 1 + i(u) \). Consequently
\[
Q_n \left( \sum_{j=1}^{1+i(u)} \lambda_j \psi_j^m \right) = \sum_{j=1}^{1+i(u)} \lambda_j^2 Q_n(\psi_j^m),
\]
where \( Q_u \) is the quadratic form defined in (3.7). According to the definition of \( i(u) \), there exists \( j_0 \in \{ 1, 2, \ldots, 1 + i(u) \} \) such that \( Q_u(\psi_{j_0}^m) \geq 0 \), therefore point 1 of lemma 3.1 implies
\[
\int_{B(y, d_j)} f(x, u)\psi_{j_0}^{2m} - C d_j^n \leq \int_{B(y, d_j)} |D_u(\psi_{j_0}^m)|^2 \leq \int_{B(y, d_j)} |D_u(\psi_{j_0}^m)|^2. \tag{3.7}
\]

**Step 1.** We first prove the following estimate
\[
\int_{A_{j_0}} |\nabla u|^2 + \int_{A_{j_0}} |f(x, u)u| \leq C d_j^n \left( \frac{1 + i(u)}{d_j} \right)^{\frac{2(m+1)}{p_1}}. \tag{3.8}
\]

Multiplying equation (1.2) by \( (1 + \frac{p_1}{2})u \psi_{j_0}^{2m} \) and integrating by parts, yields
\[
\left( \frac{1 + p_1}{2} \right) \int_{B(y, d_j)} D_i u D_j u (u \psi_{j_0}^{2m}) = \frac{1 + p_1}{2} \int_{B(y, d_j)} f(x, u)u \psi_{j_0}^{2m}. \tag{3.9}
\]

So, the sum of (3.7) with (3.9), gives
\[
\frac{p_1 - 1}{2} \left( \int_{B(y, d_j)} f(x, u)u \psi_{j_0}^{2m} + \int_{B(y, d_j)} |D_u(\psi_{j_0}^m)|^2 \right) \leq C d_j^n + \frac{p_1 + 1}{2} \left( \int_{B(y, d_j)} |D_u(\psi_{j_0}^m)|^2 - D_i u D_j u (u \psi_{j_0}^{2m}) \right). \tag{3.10}
\]

As \( u \psi_{j_0}^m \in \mathcal{W}_{\lambda}^2(B(y, d_j)) \), applying (2.38) in the left-hand side of (3.10) (with \( \lambda = d_j \)). We also invoke Lemma 2.5 with \( \lambda = d_j \), \( \psi = \psi_{j_0} \) and \( d = \text{dist}(A_j, B(y, d_j) \setminus A_j) = \frac{d_j}{2(\iota u + 1)} \), precisely we insert inequality (2.38) in the right-hand side of (3.10), we derive
\[
\int_{B(y, d_j)} |\nabla (u \psi_{j_0}^m)|^2 + \int_{B(y, d_j)} f(x, u)u \psi_{j_0}^{2m} \leq C d_j^n + \varepsilon \int_{B(y, d_j)} |\nabla (u \psi_{j_0}^m)|^2 + C \left( \frac{1 + i(u)}{d_j} \right)^{2\gamma} \int_{B(y, d_j)} u^2 \psi_{j_0}^{2(m-r)}. \tag{3.11}
\]

Choose now \( m = \frac{(p_1 + 1)r}{p_1 - 1} > r \) so that \( \frac{(p_1 + 1)(m - r)}{p_1 - 1} = m \) and we apply point 6 of Lemma 3.1 with \( s = u, \ a = \psi_{j_0}^{2(m-r)} \) and \( b = C \left( \frac{1 + i(u)}{d_j} \right)^{2\gamma} \), there holds
\[
C \left( \frac{1 + i(u)}{d_j} \right)^{2\gamma} \int_{B(y, d_j)} u^2 \psi_{j_0}^{2(m-r)} \leq C d_j^n + \varepsilon \int_{B(y, d_j)} f(x, u)u \psi_{j_0}^{2m} + C \varepsilon d_j^n \left( \frac{1 + i(u)}{d_j} \right)^{\frac{2(m+1)}{p_1}}. \tag{3.12}
\]

The last inequality with (3.11) together with point 4 of Lemma 3.1 imply
\[
\int_{B(y, d_j)} |\nabla u|^2 + \int_{B(y, d_j)} |f(x, u)u|^2 \leq C d_j^n \left( \frac{1 + i(u)}{d_j} \right)^{\frac{2(m+1)}{p_1}}. \tag{3.13}
\]

Thus, the desired inequality (3.8) follows as \( \psi_{j_0}^k(x) = 1 \) in \( A_{j_0} \).

**Step 2.**

\[\text{Observe that } d_j^n \leq d_j^n \left( \frac{1 + i(u)}{d_j} \right)^{\frac{2(m+1)}{p_1}} \text{ as } d_j = \inf \{ \alpha, \beta \} < 1.\]
We will establish a variant of the Pohozaev identity to extend the integral estimate (3.8) to the ball $B(y, \frac{d_y}{2})$. Precisely, we prove that

$$d_y^{n} \int_{B(y, \frac{d_y}{2})} |f(x, u)|^{p_1} \leq C(1 + i(u)) \left(1 + \frac{i(u)}{d_y}\right)^{\frac{2(p_1-1)}{p_1-1}}.$$  \hspace{1cm} (3.12)

Recall that $A_{j_0} = \{a_{j_0} < |x-y| < b_{j_0}\}$ where $a_{j_0}$ and $b_{j_0}$ are defined in (3.5). So, Lemma 2.1 with $\omega = B(y, a_{j_0})$, $\omega' = B(y, b_{j_0})$ and $d = \text{dist}(B(y, a_{j_0}), B(y, b_{j_0})) = \frac{d_y}{2(1 + i(u) + 1)}$ guarantees that there exists $\psi \in C^0_c(B(y, d))$ satisfying

- $\psi \equiv 1$ in $B(y, a_{j_0})$, $\psi(y) \equiv 0$ if $|x-y| \geq b_{j_0}$;
- $|\nabla^k \psi^{2m}| \leq C \left(1 + \frac{i(u)}{d_y}\right)^k \forall y \in A_{j_0}$, $k=1,2,...,r$.

In one hand, multiplying (1.2) by $(\nabla u \cdot (x-y)\psi^{2m})$, $m > r$. According to (2.8-9) with $z = y$ and $\lambda = d_y$, integration by parts gives

$$\frac{2r-n}{2} \int_{B(y,d_y)} |D_x u|^2 \psi^{2m} + n \int_{B(y,d_y)} F(x,u)\psi^{2m} = I - \int_{B(y,d_y)} (\nabla u)^2(x,u) \cdot (x-y)\psi^{2m},$$  \hspace{1cm} (3.13)

where

$$I = \frac{1}{2} \int_{A_{j_0}} |D_x u|^2 (\nabla \psi^{2m} \cdot (x-y)) - \int_{A_{j_0}} F(x,u)(\nabla \psi^{2m} \cdot (x-y)).$$

Observe that $|x-y| \leq d_y \leq \alpha$ and $|\nabla \psi^{2m} \cdot (x-y)| \leq C(1 + i(u))$, $\forall x \in A_{j_0}$. Thus, point 4 of Lemma 3.1 assumption $(h_3)$ and the integral estimate (3.8) of step 1 imply

$$\left\{ \begin{array}{l}
|I| \leq C(1 + i(u)) \left(\frac{1 + i(u)}{d_y}\right) \left(1 + \frac{i(u)}{d_y}\right)^{\frac{2(p_1-1)}{p_1-1}}, \\
\left| \int_{A_{j_0}} [(\nabla u)^2(x,u) \cdot (x-y)] \psi^{2m} \right| \leq C \left(1 + \frac{i(u)}{d_y}\right)^{\frac{2(p_1-1)}{p_1-1}} \int_{A_{j_0}} |F(x,u)||x-y|\psi^{2m} \leq C \left(1 + \frac{i(u)}{d_y}\right)^{\frac{2(p_1-1)}{p_1-1}} \int_{A_{j_0}} |f(x,u)\psi^{2m}|.
\end{array} \right.$$  \hspace{1cm} (3.13)

These inequalities combined with (3.13) and point 2 of Lemma 3.1 imply

$$\left(\frac{2n}{(p_2 + 1)(n-2r)} - C\alpha \right) \int_{B(y,d_y)} f(x,u)\psi^{2m} - \int_{B(y,d_y)} |D_x u|^2 \psi^{2m} \leq C d_y^2 \left(1 + i(u)\right) \left(1 + \frac{i(u)}{d_y}\right)^{\frac{2(p_1-1)}{p_1-1}} \hspace{1cm} (3.14)$$

On the other hand, multiplying equation (1.2) by $u\psi^{2m}$, we deduce

$$\int_{B(y,d_y)} D_x uD_x (u\psi^{2m}) = \int_{B(y,d_y)} f(x,u)u\psi^{2m}.$$  

In view of inequality (2.36) of Lemma 2.5 we derive

$$\int_{B(y,d_y)} |D_x u|^2 \psi^{2m} - \int_{B(y,d_y)} f(x,u)u\psi^{2m} \leq C \int_{B(y,d_y)} |\nabla^r (u\psi^{m})|^2 + C\alpha \left(1 + \frac{i(u)}{d_y}\right)^{\frac{2(p_1-1)}{p_1-1}} \int_{B(y,d_y)} u^2 \psi^{2m(r-1)}.$$
Choose now $\alpha = \alpha_0 \in (0, 1)$ small enough so that $\frac{2n}{(p_2 + 1)(n - 2r)} - C\alpha_0 > 1$. Then the above inequality together with (3.14), imply
\[
\int_{B(y, d_i)} |D_i u|^2 \varphi^{2m} + \int_{B(y, d_i)} f(x, u)|u|^{2m} \leq \varepsilon \int_{B(y, d_i)} |\nabla'(uo)^m|^2 + C_\varepsilon (1 + i(u))^{2} \frac{2(\alpha - 1)}{d_y^2} \int_{B(y, d_i)} u^2 \varphi^{2(m-r)} + C d_i^n (1 + i(u)) \left( \frac{1 + i(u)}{d_y} \right)^{\frac{2(\alpha - 1)}{n-1}}.
\]

Combine (3.3.5) with (3.3.6) and using again (3.3.8), we deduce that
\[
C \int_{B(y, d_i)} |\nabla'(uo)^m|^2 \leq \int_{B(y, d_i)} |D_i (uo)^m|^2 \leq \int_{B(y, d_i)} |D_i u|^2 \varphi^{2m} \leq \varepsilon \int_{B(y, d_i)} |\nabla'(uo)^m|^2 + C_\varepsilon (1 + i(u))^{2} \frac{2(\alpha - 1)}{d_y^2} \int_{B(y, d_i)} u^2 \varphi^{2(m-r)}.
\]

We collect the two last inequalities and we choose $m = \frac{(p_1 + 1)r}{2} > r$, then points 4 and 6 of Lemma 3.1 imply
\[
\int_{B(y, d_i)} |\nabla'(uo)^m|^2 + \int_{B(y, d_i)} |f(x, u)|u|^{2m} \leq C d_i^n (1 + i(u)) \left( \frac{1 + i(u)}{d_y} \right)^{\frac{2(\alpha - 1)}{n-1}}.
\]

As $\varphi \equiv 1$ on $B(y, \frac{d_i}{2}) \subset B(y, a_i)$, it follows that
\[
d_i^\alpha \int_{B(y, \frac{d_i}{2})} |f(x, u)|u| \leq C (1 + i(u)) \left( \frac{1 + i(u)}{d_y} \right)^{\frac{2(\alpha - 1)}{n-1}}.
\]

**Step 3. Boot-strap procedure.**

Set $\lambda = \frac{d_y}{2} < 1$, $u_\lambda(x) = u(y + \lambda x)$ and $g_\lambda(x) = f(y + \lambda x, u(y + \lambda x))$, $x \in B_1$, then $u_\lambda$ satisfies
\[
(-\Delta) u_\lambda = \lambda^2 g_\lambda \text{ in } B_1,
\]

By virtue of (3.12), we have
\[
\int_{B_1} |g_\lambda|^p \leq 2^\lambda d_i^\alpha \int_{B(y, \frac{d_i}{2})} |f(y, u)|u|^{p} \leq C \left( \frac{1 + i(u)}{d_y} \right)^{\frac{2(\alpha - 1)}{n-1}}.
\]

We invoke now local $L^p, W^{2,p}$ estimate (see corollary 6 in [27]); or Corollary 4.1 in the appendix) and Rellich-Kondrachov’s theorem [14]. Precisely, for some $q > 1$, point 3 of Lemma 3.1 implies
\[
||u_\lambda||_{L^q(B_1)} \leq C||u_\lambda||_{W^{2,q}(B_1)} \leq C(||g_\lambda||_{L^q(B_1)} + ||u_\lambda||_{L^q(B_1)}) \leq C(||g_\lambda||_{L^q(B_1)} + 1),
\]

where $q^* = \frac{qn}{n - 2rq}$ if $2rq < n$ and $q^* = \frac{n + 1}{2r}$ if $q = \frac{n}{2r}$;

and
\[
||u_\lambda||_{L^{q+1}(B_1)} \leq C||u_\lambda||_{W^{2,q+1}(B_1)} \leq C(||g_\lambda||_{L^q(B_1)} + ||u_\lambda||_{L^q(B_1)}) \leq C(||g_\lambda||_{L^q(B_1)} + 1), \text{ if } 2rq > n.
\]

12Recall that $\frac{2n}{(p_2 + 1)(n - 2r)} > 1$. 
Also, if \( q > p_2 \), from point 5 of Lemma 3.3 we have \( |g_i|^{\frac{2}{q}} \leq C|\mu_i|^{q} + 1 \), therefore (3.19) gives
\[
\|g_i\|_{L^{\frac{2}{q}}(B_{k_0})} \leq C \left( \|g_i\|_{L^q(B_j)} + 1 \right), \quad \text{if } 2rq \leq n. \tag{3.21}
\]
Hence, if \( 2rq_1 > n \) (respectively \( 2rq_1 = n \)), the main estimate (1.4) of Theorem 1.1 follows from (3.18) and (3.20) with \( q = q_1 \) (respectively from (3.19), with \( q = q_1 \) and (3.20) and (3.21) with \( q = p - \frac{n + 1}{2r} \)). We use now the bootstrap argument to discuss the difficult case \( 2rq_1 < n \). Recall that
\[
\alpha \text{ is a stable solution). As } \\
\text{Iterating now (3.21) } k_0 \text{ times and using (3), we obtain}
\]
\[
\|g_i\|_{L^{\frac{2}{q}}(B_{k_0})} \leq C(\|g_i\|_{L^{q}(B_{j})} + 1)^{\frac{k_0}{q}} \leq C(\|g_i\|_{L^{q}(B_{j})} + 1)^{\frac{p}{q}}.
\]
Set \( \gamma_1 = \frac{(p_1 + 1)\beta}{q_1} = \frac{2r(p_1 + 1)p_2}{2r(p_2 + 1) - n(p_2 - 1)} \) and \( \gamma_2 = \beta + \frac{2r}{p_1 - 1} \). As \( rq_{k+1} > n \), the last inequality with (3.20) and (3.13) imply
\[
\|g_i\|_{C^{\gamma_1}(\overline{B}_{2^{k+1}r})} \leq C(1 + |u_1|^2)\frac{q_1}{r^{\frac{2(r_1 + 1)\beta}{q_1}}}. 
\]
According to the definition of \( u_1 \), we get
\[
\sum_{j=0}^{2^{k+1}} d_j |(\nabla u)(y)| \leq C(1 + |u_1|^2)\frac{q_1}{r^{\frac{2(r_1 + 1)\beta}{q_1}}}. 
\]
So, the desired inequality (1.4) follows. Clearly, inequality (1.5) is an immediate consequence of (1.4). This achieves the proof of Theorem 1.1.

4. Proofs of Proposition 1.1 and Theorem 1.2

4.1. Proof of Proposition 1.1

Proof of 1.1.2

Recall that \( \overline{B} = \max(q_1, q_2) \) and \( q = \min(q_1, q_2) > p - 1 \). Let \( u \in W^{p,q}_{\text{loc}}(\mathbb{R}^n) \cap L^{q+1}_{\text{loc}}(\mathbb{R}^n) \) be a weak solution of (1.7) which is stable outside a ball \( B_{k_0} \). \( k_0 > 0 \). We employ the cut-off function \( \psi = \psi_{(R,R_0)} \) (respectively \( \psi = \psi_{R} \) if \( u \) is a stable solution). As \( u\psi_{\overline{B}} \in W^{p}_{0}(\overline{B}_{2R}) \cap L^{q+1}_{\text{loc}}(\overline{B}_{2R}) \), from (1.9) we have
\[
\int_{\overline{B}_{2R}} |D_{r} u|^{p-2}D_{r} u \cdot D_{r} (u\psi_{\overline{B}}) = \int_{\overline{B}_{2R}} (c_1 u^{\overline{B}}_{r} + c_2 u^{\overline{B}}_{r+1}) \psi_{\overline{B}}. \tag{4.1}
\]
Observe that $\text{supp}(u\psi R) \subset B_{2R} \setminus B_{R}$, then inequality (1.11) implies

$$
\int_{B_{2R}} (c_1 q_1 u_{i+1}^{q_1} + c_2 q_2 u_{i+1}^{q_2}) \psi^m \leq (p - 1) \int_{B_{2R}} |D_i u|^p - 2 |D_i (u\psi R)|^2.
$$

(4.2)

We multiply (4.1) by $\frac{q + p - 1}{2}$ and we add it to (4.2), we obtain

$$
\int_{B_{2R}} \left( \frac{2q_1 - q_2 - p + 1}{2} q_1 u_{i+1}^{q_1} + \frac{2q_2 - q_2 - p + 1}{2} q_2 u_{i+1}^{q_2} \right) \psi^m + \frac{q - p + 1}{2} \int_{B_{2R}} |D_i u|^p - 2 |D_i (u\psi R)|^2 \leq K(u, r, p, q_1, q_2, R),
$$

(4.3)

where

$$
K(u, r, p, q_1, q_2, R) = \frac{q + p - 1}{2} \left( \int_{B_{2R}} |D_i u|^p - 2 |D_i (u\psi R)|^2 - D_i uD_i (u\psi R) \right).
$$

We are now in a position to apply Lemma 2.4, we insert inequality (2.27) in $K(u, r, p, q_1, q_2, R)$. Also, as $u\psi R \in W_0^{\gamma, \beta}(B_{2R})$, we apply (2.43) (with $\gamma = 0$ and $\beta = 2R$) in the left-hand side of (4.3). Therefore, we derive

$$
\int_{B_{2R}} |\nabla (u\psi R)|^p + \int_{B_{2R}} (u_{i+1}^{q_1} + u_{i+1}^{q_2}) \psi^m \leq C_0 + \epsilon \int_{B_{2R}} |\nabla u|^p \psi^m + C_\epsilon \sum_{q_1, q_2, q_3, q_4} R^{-p+k} \int_{B_{2R}} |\nabla u|^p \psi^{p(m-k)}.
$$

(4.4)

where $C_0 = 0$ if $u$ is a stable solution. Using again Lemma 2.4 with $\epsilon = \frac{1}{2}$ and applying inequality (2.30) in the right-hand side of (4.4), there holds

$$
\int_{B_{2R}} |\nabla (u\psi R)|^p + \int_{B_{2R}} (u_{i+1}^{q_1} + u_{i+1}^{q_2}) \psi^m \leq C_0 + CR^{-p} \int_{B_{2R}} |u|^p \psi^{p(m-r)} \leq C_0 + CR^{-p} \int_{B_{2R}} (u_{i+1}^{q_1} + u_{i+1}^{q_2}) \psi^{p(m-r)}.
$$

(4.5)

Let $m = \frac{(q + 1)r}{q + 1 - p} > r$ so that $pm = (q + 1)(m - r) \leq (q + 1)(m - r)$. We apply the following Young’s inequality

$$
d^p b \leq \frac{p}{t + 1} d^{t+1} + \frac{t + 1 - p}{t + 1} b^{t+1},
$$

with $a = u \psi R^{m-r}$, $t = q_1$ (respectively $a = u \psi R^{m-r}$, $t = q_2$) and $b = CR^{-p}$, we obtain

$$
CR^{-p} \int_{B_{2R}} (u_{i+1}^{q_1} + u_{i+1}^{q_2}) \psi^{p(m-r)} \leq \frac{p}{q_1 + 1} \int_{B_{2R}} u_{i+1}^{q_1} \psi^m + \frac{p}{q_2 + 1} \int_{B_{2R}} u_{i+1}^{q_2} \psi^m + C(R^{\frac{q_1(q_1+1)}{q_1^2-1}+\gamma} + R^{\frac{q_2(q_2+1)}{q_2^2-1}+\gamma})
$$

Combining (4.5) with the above inequality and taking into account that $\psi \equiv 1$ on $\{2R_0 < |x| < R\}$ (respectively $\psi_R \equiv 1$ if $|x| < R$), then the main integral estimate (1.12) follows.

**Proof of inequality (1.13).**

We comeback to (4.4) with $C_0 = 0$ (as $u$ is a stable solution), we derive

$$
\int_{\mathbb{R}^n} (u_{i+1}^{q_1} + u_{i-1}^{q_2}) \psi^m \leq C_0 + \epsilon \int_{B_{2R}} |\nabla u|^p \psi^m + C_\epsilon \sum_{q_1, q_2, q_3, q_4} R^{-p+k} \int_{B_{2R}} |\nabla u|^p \psi^{p(m-k)}.
$$

We insert (2.1) in the right-hand side of the above inequality, there holds

$$
\int_{B_{2R}} (u_{i+1}^{q_1} + u_{i-1}^{q_2}) \leq 2 \int_{B_{2R}} |\nabla u|^p + CR^{-p} \int_{B_{2R}} |u|^p.
$$

The proof of Proposition 1.1 is thereby completed.

---

13 Recall that $1 \leq p - 1 < q_1$.

14 As $0 \leq \psi \leq 1$ we have $\psi^{2(m-r)} \leq \psi^m$.

15 In fact, inequality (1.12) holds when one replaces $B_0$ by $A_R = \{R < |x| < 2R\}$. 

4.2. Proof of Theorem 1.2

We recall that \( u \in W_{\text{loc}}^{p}(\mathbb{R}^n) \) is a stable weak solution of (1.7); or \( u \in W_{\text{loc}}^{p+1}(\mathbb{R}^n) \) is stable outside the ball \( B_{R_0} \) and \( p - 1 < \frac{q}{q} \leq p' - 1 \) if \( n > pr \) (respectively \( p - 1 < \frac{q}{q} \leq p' - 1 \) if \( n \leq pr \)). Then \( u \in W_{\text{loc}}^{p}(\mathbb{R}^n) \cup L_{\text{loc}}^{p+1}(\mathbb{R}^n) \) and the main estimate (1.12) holds. In particular if \( u \) is a stable weak solution we have \( C_0 = 0 \) and as \( n - \frac{pr(q+1)}{q+1-p} < 0 \) if \( p - 1 < \frac{q}{q} < p' - 1 \), then \( \int_{\mathbb{R}^n} (u_{+}^{q} + u_{-}^{q+1}) = 0 \), therefore \( u \equiv 0 \). If \( \bar{q} = p' - 1 \) and \( n > pr \); or \( u \) is stable outside the ball \( B_{R_0} \) and \( p - 1 < \frac{q}{q} \leq p' - 1 \), from (1.12) we can easily see that \( u_{+}^{q+1} + u_{-}^{q+1} \in L^1(\mathbb{R}^n) \) and \( \nabla' u \in L^p(\mathbb{R}^n) \). We apply Hölder’s inequality, we have

\[
R^{-pr} \int_{A_R} |u|^p \leq CR^{-pr} \int_{B_{2R}} (u_+^{p} + u_-^{q+1}) \leq CR^{-pr} \left( \int_{A_R} u_+^{q+1} + \int_{A_R} u_-^{q+1} \right)^{\frac{p}{q+1}} \leq C \left( \int_{A_R} u_+^{q+1} \right)^{\frac{p}{q+1}} + C \left( \int_{A_R} u_-^{q+1} \right)^{\frac{p}{q+1}}.
\]

Therefore,

\[
R^{-pr} \int_{A_R} |u|^p = o(1), \quad \int_{A_R} (u_+^{q+1} + u_-^{q+1}) = o(1) \quad \text{and} \quad \int_{A_R} |\nabla' u|^p = o(1) \quad \text{as} \quad R \to \infty.
\]

Consequently, if \( u \) is a stable solution of (1.7) and \( n > pr \), it follows from (1.12) and (4.6) that \( \int_{B_R} (u_+^{q+1} + u_-^{q+1}) = o(1) \), as \( R \to \infty \) and therefore \( u \equiv 0 \). This ends the proof of point 1.

Proof of Point 2.

Set \( \phi = \psi_R \). In one hand, from equality (4.1), we have

\[
\int_{B_R} |D_{u}|^{p-2} D_{u} D_{u} (\psi) = \int_{B_R} (c_1 u_+^{q+1} + c_2 u_-^{q+1}) \phi.
\]

According to (2.28) and (1.13), we obtain \(^{16}\)

\[
\int_{B_{2R}} |D_{u}|^{p} \phi - \int_{B_{2R}} (c_1 u_+^{q+1} + c_2 u_-^{q+1}) \phi \leq C \left( \int_{A_R} |\nabla' u|^p + CR^{-pr} \int_{A_R} |u|^p \right).
\]

It follows from (4.6) that

\[
\int_{B_{2R}} |D_{u}|^{p} \phi - \int_{B_{2R}} (u_+^{q+1} + u_-^{q+1}) \phi = o(1) \quad \text{(4.7)}.
\]

On the other hand, since \( u \in W_{\text{loc}}^{p+1}(\mathbb{R}^n) \), then \( \nabla u \cdot x \in W_{\text{loc}}^{\gamma}(\mathbb{R}^n) \subset L_{\text{loc}}^{q+1}(\mathbb{R}^n) \), therefore \( \nabla u \cdot x \phi \) can be used as a test function in (4.1), there holds

\[
\int_{B_{2R}} |D_{u}|^{p-2} D_{u} D_{u} (\nabla u \cdot x \phi) = \int_{B_{2R}} (c_1 u_+^{q} - c_2 u_-^{q}) \nabla u \cdot x \phi.
\]

Multiplying now (2.28) by \( |D_{u}|^{p-2} D_{u} u \) (with \( \gamma = 0 \)), we derive

\[
|D_{u}|^{p-2} D_{u} D_{u} (\nabla u \cdot x) = \frac{1}{p} \nabla (|D_{u}|^{p}) \cdot x + r|D_{u}|^{p} \quad \text{a.e. in } \mathbb{R}^n.
\]

\(^{16}\)Inequality (2.28) holds (with \( C_0 = 0 \) because we used \( \psi_R \) as a cut-off function.)
Thus, direct integrations by parts yield
\[
\int_{B_{2\varepsilon}} |D_{i}u|^{p-2}D_{i}uD_{i}(\nabla u \cdot x)\phi = \frac{rp-n}{p} \int_{B_{2\varepsilon}} |D_{i}u|^{p} \phi - \frac{1}{p} \int_{B_{2\varepsilon}} |D_{i}u|^{p}(\nabla \phi \cdot x)
\]
and
\[
\int_{B_{2\varepsilon}} (c_{1}u_{x}^{q1+1} - c_{2}u_{x}^{q2+1})\nabla u \cdot x\phi = -n \int_{B_{2\varepsilon}} \left( \frac{c_{1}u_{x}^{q1+1}}{q_{1}+1} + \frac{c_{2}u_{x}^{q2+1}}{q_{2}+1} \right) \phi - \int_{B_{2\varepsilon}} \left( \frac{c_{1}u_{x}^{q1+1}}{q_{1}+1} + \frac{c_{2}u_{x}^{q2+1}}{q_{2}+1} \right) \nabla \phi \cdot x.
\]
By virtue of (4.7) we have
\[
\int_{B_{2\varepsilon}} \left( \frac{c_{1}u_{x}^{q1+1}}{q_{1}+1} + \frac{c_{2}u_{x}^{q2+1}}{q_{2}+1} \right) \nabla \phi \cdot x = o(1) \quad \text{and} \quad \int_{B_{2\varepsilon}} |D_{i}u|^{p}(\nabla \phi \cdot x) = o(1), \quad \text{as } R \to \infty.
\]
Collecting these equalities, we arrive at
\[
\frac{n}{p} \int_{B_{2\varepsilon}} \left( \frac{c_{1}u_{x}^{q1+1}}{q_{1}+1} + \frac{c_{2}u_{x}^{q2+1}}{q_{2}+1} \right) \phi - \frac{1}{p} \int_{B_{2\varepsilon}} |D_{i}u|^{p} \phi = o(1) \quad \text{as } R \to \infty.
\]
Thus, if \( n \leq pr \), then (4.8) implies that \( \int_{B_{2\varepsilon}} \left( \frac{c_{1}u_{x}^{q1+1}}{q_{1}+1} + \frac{c_{2}u_{x}^{q2+1}}{q_{2}+1} \right) \phi = o(1) \), so \( u \equiv 0. \) If \( n > pr \) and \( p - 1 < q \leq \frac{q}{p'} - 1 \), we combine (4.8) with (4.7) we derive
\[
c_{1} \left( \frac{np}{(n-rp)(q_{1}+1) - 1} \right) \int_{B_{2\varepsilon}} u_{x}^{q1+1} \phi + c_{2} \left( \frac{np}{(n-rp)(q_{2}+1) - 1} \right) \int_{B_{2\varepsilon}} u_{x}^{q2+1} \phi = o(1),
\]
which implies that \( u \equiv 0. \)

If \( q = \frac{q}{p'} = p' \), taking into account that \( c_{1}u_{x}^{q1+1} + c_{2}u_{x}^{q2+1} \in L^1(\mathbb{R}^n) \) and \( \nabla u \in L^p(\mathbb{R}^n) \), we apply the dominated convergence theorem in (4.7) we obtain (1.14).

Lastly, if \( p - 1 < q < \frac{q}{p'} = p' \), as above we derive that
\[
c_{2} \left( \frac{np}{(n-rp)(q_{2}+1) - 1} \right) \int_{B_{2\varepsilon}} u_{x}^{q2+1} \phi = o(1).
\]
Consequently, (1.14) holds with \( u \geq 0 \) if \( \frac{q}{p'} = q_{1} \) (respectively \( u \leq 0 \) if \( \frac{q}{p'} = q_{2} \). This achieves the proof of Theorem 1.2.

Appendix: Local \( L^p-W^{2,\beta} \)-regularity revisited.
Let \( \Omega \) be a domain of \( \mathbb{R}^n \). consider the linear higher order elliptic problem of the form
\[
Lu = g \quad \text{in} \quad \Omega.
\]
Here
\[
L = \left( - \sum_{i,k=1}^{n} a_{ik}(x) \frac{\partial^2}{\partial x_i \partial x_k} \right) + \sum_{|j| \geq 2} b_j(x)D^j
\]
is a uniformly elliptic operator with coefficients \( b_j \in L_{\text{loc}}^\infty(\Omega) \) and \( a_{ik} \in C^{2}\lambda(\Omega) \), that is that there exists a constant \( \lambda > 0 \) with \( \lambda^{-1}|x|^2 \leq \sum_{i,k=1}^{n} a_{ik}(x)\xi_i \xi_k \leq \lambda|x|^2 \) for all \( x \in \mathbb{R}^n, \xi \in \mathbb{R}^n \)

For \( p \geq 2 \), thanks to Lemma 2.2 we propose a direct proof of local analogue of the celebrated \( L^p-W^{2,\beta} \) estimate of Agmon-Douglis-Nirenberg [2]. Let \( \omega \) and \( \omega' \) be two bounded open subset of \( \Omega \) such that \( \overline{\omega} \subset \omega' \) and \( \overline{\omega'} \subset \Omega \), we have
Corollary 4.1. Let $g \in L^p_\text{loc}(\Omega)$ for some $p > 1$. Then there exists a constant $C > 0$ depending only on $\|u\|_{C^{2-2}\omega, p}$, $\|b\|_{L^\infty(\omega)}$ and $\lambda, \omega', d, n, p, r$ such that for any $u \in W^{2r, p}_0(\omega)$ a weak solution of (4.9), we have

$$
\|d\|_{W^{2r, p}(\omega)} \leq C \left( \|g\|_{L^p(\omega')} + \|d\|_{L^p(\omega')} \right).
$$

In the following $C$ denotes a generic positive constant which depends on the parameters stated in Corollary 4.1 and $d = \text{dist}(\omega, \Omega(\omega'))$.

**Proof of Corollary 4.1**

We distinguish two cases

**Case 1**, $p \geq 2$.

We truncate the equation (4.9) by $\psi^m$, where $\psi$ is the cut-off function defined in Lemma 2.1 and $m \geq 2r$. Thus, we have

$$
L(u\psi^m) = g\psi^m + uL(\psi^m) + b\omega\psi^m + \sum_{1 \leq |i| + j \leq 2r-1} c_{i,j}D^iD^j(\psi^m), \text{ where } c_{i,j} \in L^\infty(\Omega).
$$

As $\psi \in C^\infty_0(\omega')$ and $u \in W^{2r, p}(\omega')$, then $u\psi^m \in W^{2r, p}(\omega') \cap W^{0, p}_0(\omega')$ with compact support, therefore Agmon-Douglis-Nirenberg’s global estimate \(17\) and (2.11) imply

$$
\sum_{0 \leq i \leq 2r} \int_{\omega'} |\nabla^i (u\psi^m)|^p \leq C \left( \|g\|_{L^p(\omega')}^p + \|d\|_{L^p(\omega')}^p + \sum_{1 \leq |i| + j \leq 2r-1} \sum_{1 \leq s \leq 1} \int_{\omega'} |\nabla^i u|^p |\nabla^{s-2j} \psi^m|^p \right).
$$

Using now our interpolation inequality (2.6) (where one replaces $r$ by $s$ in (2.6)), we obtain

$$
\sum_{1 \leq |i| \leq 2r} \sum_{1 \leq s \leq 1} \int_{\omega'} |\nabla^i u|^p |\nabla^{s-2j} \psi^m|^p \leq \varepsilon \int_{\omega'} |\nabla^i (u\psi^m)|^p + C_{c,d} \int_{\omega'} |u|^p \psi^{p(m-s)}.
$$

Also apply the second inequality of (2.6) with $r = s + 1$ and we replace $\varepsilon$ by $\frac{\varepsilon}{d}$, yields

$$
\int_{\omega'} |\nabla^i u|^p |\nabla^{s-2j} \psi^m|^p \leq \varepsilon \int_{\omega'} |\nabla^i u|^p |\nabla^{s-2j} \psi^{m-1}|^p + C_{c,d} \int_{\omega'} |u|^p \psi^{p(m-s-1)}.
$$

Collecting the two last inequalities, we derive

$$
\sum_{1 \leq |i| \leq 2r} \sum_{1 \leq s \leq 1} \int_{\omega'} |\nabla^i u|^p |\nabla^{s-2j} \psi^m|^p \leq \varepsilon \sum_{0 \leq i \leq 2r} \int_{\omega'} |\nabla^i (u\psi^m)|^p + C_{c,d} \int_{\omega'} |u|^p.
$$

We insert the above inequality in the right-hand side of (4.10), and we choose $\varepsilon = \frac{1}{2C}$, we deduce

$$
\|u\|_{W^{2r, p}(\omega')} \leq C \left( \|g\|^p_{L^p(\omega')} + \|d\|^p_{L^p(\omega')} \right).
$$

Since $\psi(x) = 1$ if $x \in \omega$, we obtain

$$
\|u\|_{W^{2r, p}(\omega)} \leq C \left( \|g\|^p_{L^p(\omega')} + \|d\|^p_{L^p(\omega')} \right).
$$

**Case 2**, $1 < p < 2$.

Let $B(x, r)$ be the ball of radius $r$ and centred at $x$. We apply Corollary 6 of (2.7) with $R = \frac{d}{2}$ and $\sigma = \frac{1}{2}$. Then, for all $x \in \omega$, we have

$$
\|u\|_{W^{2r, p}(B(x, \frac{d}{2}))} \leq C \left( \|g\|^p_{L^p(B(x, \frac{d}{2}))} + \|d\|^p_{L^p(B(x, \frac{d}{2}))} \right) \leq C \left( \|g\|^p_{L^p(\omega')} + \|d\|^p_{L^p(\omega')} \right).
$$

\(^{17}\text{See also Theorem 5 in \(17\).}\)
As $\overline{\omega}$ is a compact set, we can find $x_i \in \omega$, $i = 1, 2, ..., k_0$ such that $\overline{\omega} \subset \bigcup B(x_i, \frac{d}{4}) \subset \omega'$ where $k_0 \in \mathbb{N}^+$ depending only on $d$ and $\omega$. Therefore, we derive

$$
\|u\|_{W^{2,p}(\omega')}^p \leq C k_0 \left( \|u\|_{L^p(\omega')}^p + \|u\|_{L^p(\omega')}^p \right).
$$

This achieves the proof of Corollary 4.1.

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