Introduction. — Robust boundary states are vivid physical manifestations of deep physics. Slightly over a decade ago, they came into the spotlight as topologically protected edge states [1–9]. More recently, much attention has also focused on non-Hermitian skin states, which are robust boundary states arising from net unbalanced gain/loss [10–50]. This work shall introduce a third and fundamentally distinct type of robust boundary state known as “Exceptional Boundary” (EB) states.

EB states are predicted to exist in critical non-Hermitian fermionic systems containing exceptional points (EPs) [51–55]. Unlike ordinary gapless points, EPs are also branch points of the complex energy Riemann surface [56–72], and host unusual quantum critical, fractal and thermodynamic behavior as well as potential applications in sensing, multi-mode laser manipulation and topological energy transfer [47, 60, 61, 73–85]. Exceptional Boundary states around EPs are physically distinguished by their anomalously large occupation probabilities, which can be detected long after their host EPs are quenched. In the quantum entanglement context, EB states also give rise to decreased or even negative entanglement entropy (EE), very different from what is possible in Hermitian systems.

Fundamentally, EB state phenomena arise because the system is not just critical, but also defective around the exceptional point. A generic occupied state cannot be completely expanded in terms of the eigenstates, blurring the distinction between occupied and unoccupied states and resulting in an asymmetrically singular propagator. This gives rise to pronounced EB eigenstate accumulation when an effective boundary is imposed through a quench or entanglement cut. Such intriguing EB states are well-separated from the other eigenstates and possess distinctive spatial profiles, violating the celebrated notion of bulk-boundary correspondence in a unique way. They are completely distinct from other types of robust boundary eigenstates [Fig. 1], being protected by the defectiveness of the eigenspace, which is a geometric rather than dynamical or topological property. Perturbations that increase the effective rank at the EP induce a gradual crossover from negative EE scaling to ordinary critical scaling behavior, allowing for continuous tuning of logarithmic critical scaling coefficients.

Exceptional Boundary states from defectiveness. — To understand why exceptional points give rise to EB states, we consider a simple paradigmatic 2-level 1D Hamiltonian containing an EP [56], indexed by momentum $k$:

$$H(k) = \begin{pmatrix} \gamma(k) & a(k) \\ b(k) & -\gamma(k) \end{pmatrix} \xrightarrow{k \to 0} \begin{pmatrix} \gamma_0 k^p & a_0 \\ b_0 k^p & -\gamma_0 k^p \end{pmatrix}$$  \hspace{1cm} (1)

whose eigenenergies are $\pm \epsilon(k) = \pm \sqrt{a(k)b(k) + \gamma^2(k)}$, centered around the Fermi level at $\epsilon(k) = 0$. At the EP
near the EP, \( \langle c_b^\dagger_{\gamma x}, c_{\gamma x} \rangle \sim \sqrt{\frac{\Delta^0}{b_0}} B^{2-4\Delta x^2}/B \) is short-ranged, quadratically decaying with \( \Delta x = x_1 - x_2 \). However, due to the divergent denominator in \( U(k) \sim \sqrt{a_0/b_0} \),

\[
\langle c_b^\dagger_{\gamma x}, c_{\gamma x} \rangle \mid_{B>4} \sim -\sqrt{\frac{a_0}{b_0}} \left( \frac{L}{\pi} \right)^{B/2-1} \times \left( 2 - \frac{\pi^2 \Delta x^2}{L^2} \right),
\]

which is long-ranged in \( \Delta x = x_1 - x_2 \) and diverges as \( L^{B/2-1} \). For the important cases of \( B = 2 \) and \( B = 4 \), we also have \[88\]

\[
\langle c_b^\dagger_{\gamma x}, c_{\gamma x} \rangle \mid_{B=4} \sim -\sqrt{\frac{a_0}{b_0}} \left( L - 2\Delta x \right),
\]

which diverges logarithmically and linearly with both \( L \) and \( x \). Due to the definiteness of at the EP, even a very small asymmetry \( (0 \neq a_0 \ll b_0) \) at criticality can cause extremely large asymmetry in the hopping probability with sufficiently large \( L \).

Evidently, then, a real-space cutoff will truncate \( \langle c_b^\dagger_{\gamma x}, c_{\gamma x} \rangle \) much more than \( \langle c_b^\dagger_{\gamma x}, c_{\gamma x} \rangle \), leading to uncompensated net accumulation. A distant caricature would be the crossing of radiation across the extremal surface in the context of Hawking radiation \[89\, 90\]. Interestingly, this shall give rise to isolated EB states with special spatial profiles, alongside extensively many ordinary un-accumulated states (unlike the skin effect, which yields extensively many boundary accumulated states).

Consider cutoffs at \( x = 0 \), implemented by an operator \( R \), which projects onto the region \( 0 < x < l \), \( l < L \). To quantify their effect, we introduce \( \Lambda = 4(\tilde{P}^2 - \tilde{P}) = (\vec{I} - 2\tilde{P})^2 - \vec{I} \), where \( \tilde{P} = RPR \) is the \( R \)-truncated occupied band projector. Taking advantage of the purely off-diagonal form of \( 1 - 2\tilde{P} \) [Eq. [3]], we simplify \( \Lambda \) by decoupling it into \( \Lambda = \Lambda_+ \oplus \Lambda_- \), where \( \Lambda_+ = \tilde{U} \tilde{D} - \vec{I} \) and \( \Lambda_- = \tilde{U} \tilde{D} - \vec{I} \). Due to the real-space cutoffs, \( \tilde{U} \) and \( \tilde{D} \) are \( L \times L \) Toeplitz matrices that are not diagonal in \( k \) and, importantly, no longer inverses of each other. In particular, since \( \tilde{U} \) contains long-ranged couplings while \( \tilde{D} \) contains only short-ranged couplings, there must be uncompensated off-diagonal elements in \( \tilde{U} \tilde{D} \) near the cutoffs. As rigorously derived in the Supplement \[88\], \( \Lambda_+ \) is mostly contributed by its boundary row and columns i.e.

\[
\Lambda_+ \approx |c_1\rangle \langle 1| + |\gamma r_1\rangle - |\gamma r_1\rangle |1\rangle |1\rangle,
\]

where \( |1\rangle \) is the site next to the \( \gamma = 0 \) cutoff and \( |c_1\rangle \) and \( \langle r_1| \) are the boundary row and column containing \( |1\rangle \). From \( \Lambda_+ = \tilde{U} \tilde{D} - \vec{I} \), their elements are explicitly given by \( \langle x|c_1\rangle = 4 \sum_{j=0}^L \langle c_1^0, c_{x+j} \rangle |c_j^0\rangle \) which are slowly decaying, and \( \langle r_1| |x\rangle = 4 \sum_{j=0}^L \langle c_1^0, c_{x+j} \rangle |c_j^0\rangle \) which are rapidly decaying (short-ranged). Importantly,
is the occupation probability $\langle n \rangle = \langle \Phi^\dagger \Phi \rangle$ of the EB state $|\phi \rangle = \Phi^\dagger |0 \rangle$ or its biorthogonal conjugate $|\phi \rangle = \tilde{\Phi}^\dagger |0 \rangle$:

$$
\langle n \rangle = \sum_k \langle \psi^L(k) | \Phi^\dagger | \psi^R(k) \rangle = \sum_k \langle \tilde{\phi} | \psi^R(k) \langle \psi^L(k) | \phi \rangle
= \langle \tilde{\phi} | \tilde{P} | \phi \rangle = \langle \tilde{\phi} | \tilde{P} | \phi \rangle = p.
$$

**Unique properties of EB states.** EB states are unusual because: (i) They are isolated eigenstates of $\tilde{P}$ with occupation probabilities that are either negative ($p' < 0$) or potentially much greater than one ($p > 1$); (ii) Unlike conventional isolated mid-gap eigenstates which universally decay exponentially [91], their spatial profiles resemble that of $\langle \psi^L_{x,1} | \psi^R_{x,0} \rangle$ [Eqs. 5 to 7], which can be quadratic, logarithmic or linear. This resemblance, which originates from $\langle x | \phi_\pm \rangle \approx (x | c_1 \rangle \approx 4 (c_{0,-} | x_c \rangle \approx (c_{x,-} | c_{a,-} \rangle$ in Eq. 2a, is evident in Fig. 2a for sufficiently singular momentum profiles $B(k) \sim k^2 B^2$. Anomalously large occupation probabilities $p > 1$ of EB states physically originate from the asymmetric accumulation caused by $\langle \psi^L_{x,1} | \psi^R_{x,0} \rangle \neq \langle \psi^L_{x,1} | \psi^R_{x,2} \rangle$, which also diverge strongly with $B(k) \sim k^2 B^2$. The corresponding negative occupation $p' = 1 - p$ of the EB states on the other boundary, away from the direction of asymmetry, represents a loss that compensates the large gain in particle density across the region. As previously explained, such spectacular non-locality cannot be brought about by non-defective critical points, which only lead to power-law decaying 2-site propagators and eigenstates, not the characteristic EB state behavior in Fig. 2a. This asymmetric inter-sub-lattice accumulation is also fundamentally distinct from the non-Hermitian skin effect [11,13,15], which affects all eigenstates, not just special EB states that are eigensolutions to Eq. 3. Furthermore, a Pauli basis rotation, say, from $\sigma_y$ to $\sigma_z$, may transform this asymmetry into physical gain-loss without reciprocity breaking. Although classified by a number $B$, EB states are distinct from topological boundary states, whose robustness is rooted in topological anomalies rather than geometric EP defectiveness. Besides, topological mid-gap states typically straddle the boundary with occupation $p \approx 1/2$ [92,96], unlike EB states whose $p, p' \in [0,1]$.  

**Persistent elevated EB 2-point functions.** The EB state phenomenon leads to the existence of long-lived, high particle density configurations even after a critical system EP state is quenched. Typically, a post-quench Hamiltonian $\tilde{H}_F$ scrambles the eigenstates of the original Hamiltonian, eroding pre-quench initial states rapidly. However, due to the very high $\langle n \rangle = p$ of EB states in large systems, the 2-point function $\langle \tilde{\phi}(0) \tilde{\varphi}(t) \rangle$ of EB states $| \varphi(0) \rangle$ can persist at elevated values.

We consider a quench where the system evolves according to the EP Hamiltonian (Eq. 4) for times $t < 0$, and a post-quench potential landscape $\tilde{H}_F = \sum_x E(x) |x \rangle \langle x|...
for $t \geq 0$. To demonstrate the significance of the bounded nature of EB states, we choose the spatial energy profile to be a potential well $E(x) = \hbar \omega_F / (1 + e^{x(l-x)})$ with a wall of slope $\alpha$ at $x = l$, the EB state cutoff. $\omega_F^{-1}$ sets the decay timescale expected of ordinary, non-EB states. From $t = 0$ onwards, the 2-point correlation function $\langle \phi(0) \phi(t) \rangle = \langle \phi| e^{i H t / \hbar} | \phi \rangle$ evolves from its initial value of $\langle n \rangle$, as plotted in Fig. [2] for various choices of $| \phi \rangle = | \phi(0) \rangle$ for the same well. Evidently, it remains elevated when $| \phi \rangle$ is an EB state $| \phi \rangle$, scaling exponentially with $B$. In particular, for sufficiently singular EPs with $B > 6$, the 2-point function shall remain robustly large even if the quench evolution maximally scrambles the states, since the dominant EB states take up a fraction of $2/L$ of all the states, but scales like $\langle n \rangle = (1 + \sqrt{T + X})/2 \sim L^{(B-2)/4} > L$. However, the 2-point function of generic non-EB states $| \phi \rangle$ remain in the range of $[0,1]$, as typical of ordinary expectation values in the absence of significant asymmetric gain/loss. This is the case even when $| \phi(x) \rangle$ superficially resembles an EB state but is in reality almost orthogonal i.e. the symmetric $| \phi_+ \rangle \otimes | \phi_+ \rangle$ or antisymmetric $| \phi_+ \rangle [\pm | \phi_- \rangle]$ combinations, which are distinct from the actual EB state $| \phi_+ \rangle \otimes | \psi_+ \rangle$. The 2-point function decays even more substantially before fluctuating around the order of $1/\sqrt{L}$ when $| \phi \rangle$ is completely different from any EB state i.e. when it is spatially uniform or random (see [22]).

**Negative entanglement entropy from EB states.** — EB states also fundamentally modify free-fermion entanglement entropy and spectra. Consider a critical half-filled 1D Fermi gas (restricting ourselves to systems with real spectra) governed by the EP Hamiltonian of Eq. [1]. A density operator consistent with probabilistic interpretations [22] [103] [105] is given by $\rho = | \Psi \rangle \langle \Psi |$, where $| \Psi \rangle$ is the many-body state and $\langle \Psi |$ its biorthogonal conjugate. To probe entanglement properties, we examine the reduced density matrix $\rho_{\mathcal{R}} = \text{Tr}_\bar{\mathcal{R}} | \Psi \rangle \langle \Psi |$ obtained by tracing over the degrees of freedom outside an arbitrarily selected region $\mathcal{R}$, which we take as $0 < x < l$ as before. From Wick’s theorem [106], the EE $S = -\text{Tr}_\mathcal{R} \log \rho_{\mathcal{R}}$ and the Hamiltonian $H_E = -\log \rho_{\mathcal{R}}$ can be expressed in terms of operators in the single-particle Hilbert space: $S = -\text{Tr}(\hat{P} \log \hat{P} + (1 - \hat{P}) \log(1 - \hat{P}))$ and $H_E = \log([1 - \hat{P}] / \hat{P})$, where $\hat{P} = R \hat{P} P$ is the $R$-truncated band projector introduced earlier. Hence an EB eigenstate satisfying $\hat{P} | \phi \rangle = | \phi \rangle$ must provide an additional $\epsilon_\phi = \log[p^{-1} - 1]$ contribution to the entanglement spectrum, which will be complex since $\text{Re} p > 1$. The corresponding EB eigenstate on the opposite boundary of $\mathcal{R}$, with eigenvalue $p' = 1 - p$, gives rise to an opposite entanglement eigenenergy $-\epsilon_\phi$.

Most interestingly, EB eigenstates lead to negative contributions to the EE $S_{\text{EB}} = -p \log p - (1-p) \log(1-p)$. Even though $\log p$ and $\log(1-p)$ are complex for $\text{Re} p > 1$, they always combine to produce a real and negative $S_{\text{EB}}$. [FIG. 3. (a) Negative logarithmic scaling of the EE $S$ due to ideal ($\gamma_0 = 0$) EB states. Numerical results (solid) agree well with predicted asymptotic behavior $S = \frac{\xi}{L} \log L$ (dashed), with $\xi = 3(2 - B)$ for $B > 2$ [Eq. [11]] and $\xi = -2$ for $B = 2$. Inset: Variation of $\xi$ with the tuning of $\gamma_0$ for $H'$ [Eq. [13]] at $w = 2, v = 2.5$. (b) Continuous tuning of the logarithmic scaling coefficient of $S$ as $\gamma_0$ is varied at large $a_0 = 10^9$ and $b_0 = 1$. Dashed straight lines represent $S \sim \log L$ (ordinary critical) and $S \sim \frac{1}{2} \log L$ (EB fits), while the dashed curve represents the qualitative form with $S_{\text{EB}} \approx 0.6 - 2 \log L$ as derived in [22]. (c) Scaling behavior of $S$ for $B = 4$, also at $a_0 = 10^9, b_0 = 1$. Unlike in $B = 2$, $c$ does not change continuously as $\gamma_0$ is varied, but crosses over from $\sim \log L$ (red dashed) to the EB behavior of $\sim -2 \log L$ (blue dashed) at a $\gamma_0$-dependent critical size.]

Since $\lambda \sim L^{B/2-1}$ in $p = (1 + \sqrt{1 + \lambda})/2$ for $B > 2$, $S_{\text{EB}} |_{B > 2} \sim -2 \log L - \log \lambda_0 - B/2 \log L$, the constant $\lambda_0$ computable in terms of the 2-site propagators [22]. In the absence of topological contributions, most of the other eigenvalues of $P$ are extremely close to 0 or 1, and contribute negligibly to $S$. Hence $S \approx S_{\text{EB}}$ for EP Hamiltonians [Eq. [1]], with its negative log $L$ dependence accurately corroborated by numerical computations across various $B$ [Fig. [2]]. [107]. Physically, the negative EE originates from the anomalously large expected EB occupancy $\langle n \rangle = p > 1$ and $1 - \langle n \rangle = p' < 1$ for the opposite EB state), which is encoded in the reduced density matrix as:

$$
\rho_{\mathcal{R}} = \left( p | \phi \rangle \langle \phi | + p' | 0 \rangle \langle 0 | \right) \otimes \left( p' | \phi' \rangle \langle \phi' | + p | 0 \rangle \langle 0 | \right) \otimes \rho_{\mathcal{R}}^{\text{EB}}
$$

(12)

where $\rho_{\mathcal{R}}^{\text{EB}} = \bigotimes_{j \in \text{EB}} (p_j | \psi_j \rangle \langle \psi_j | + (1 - p_j) | 0 \rangle \langle 0 |)$ is spanned by the non-EB states $| \psi_j \rangle$ corresponding to all other $P$ eigenvalues $p_j \in [0,1]$. In particular, in the large $B$ or $L \to \infty$ thermodynamic limit where $p \gg 1$, we have $\rho_{\mathcal{R}} \approx -p^2 \sigma^0_\phi \otimes \sigma^0_\phi \otimes \rho_{\mathcal{R}}^{\text{EB}}$, where $\sigma^0_\phi = | \phi \rangle \langle \phi | - | 0 \rangle \langle 0 |$. Intuitively, the coefficient of $| 0 \rangle \langle 0 |$ indicates how much the state is entangled with degrees of freedom outside of $\mathcal{R}$. By contrast, this coefficient is very close to 0 or 1 for gapped systems with short-ranged entanglement, since the state on different sides of the entanglement cut can be almost completely decoupled. For usual gapless systems, there also exists some coefficients nearer to 1/2 due to long-ranged entanglement. But for EB states,
we not just have long-ranged entanglement, but also negative effective entanglement due to EB states having \( \tilde{P} \) eigenvalues \( p, p' \neq [0, 1] \). This unusual scenario occurs due to the divergent unbalanced 2-site propagators, which can drastically amplify or attenuate a state across an entanglement cut.

**Generalizations and tunable entanglement scaling.** – So far, we have only studied “ideal” EPs with the minimal ingredient necessary for defectiveness i.e. \( \sim k^B \) dispersion with \( \sim a_0 \) asymmetry. But actual non-Hermitian critical systems often contain additional complications, such as nontrivial \( \gamma(k) \) [Eq. 1] that breaks sublattice symmetry (or other symmetries upon basis rotation). Since \( \gamma(k) \) enter the \( \epsilon(k) \) denominator in \( P(k) \) [Eq. 2], \( U(k) \neq (D(k))^{-1} \) and the 2-site propagators \( (c_{x_1}^\dagger c_{x_2}) \) may no longer be distinctively long/short-ranged. This attenuates the first column of \( \Lambda_+ \) [Eq. 8], leading to the erosion of EB states.

For illustration, consider the following non-Hermitian SSH-like model generalized from Ref. [105]:

\[
H'(k) = (v - w \cos k) \sigma_x + \gamma_0 \sin k \sigma_y + i(v - w) \sigma_z. \quad (13)
\]

After swapping \( \sigma_y \) and \( \sigma_z \), it takes the form of Eq. 1 with \( B = 2, a_0 = 2(v - w), b_0 = w/2 \). It straddles the transition between the topologically trivial and gapless phase, and hosts an EP at \( k = 0 \), albeit with a diagonal (\( \sigma_z \)) term \( \gamma(k) = \gamma_0 \sin k \). As \( \gamma_0 \) is increased from 0 (with \( v, w \) kept fixed), the coefficient \( c \) in its EE \( S \sim \frac{\gamma_0}{2} \log L \) increases from \(-2 \) to \( 1 \) [Fig. 2a inset]. The negative \( c = -2 \) coefficient is a consequence of the marginally divergent EB state, and has been previously explained in terms of non-unitary CFT for a related model [105]. Increasing \( \gamma_0 \) erodes the EB state till we finally obtain \( c = 1 \) as expected of an ordinary critical free fermion. In a “stronger” or more asymmetric \( B = 2 \) EP with large \( a_0 \) [Fig. 3], \( c \) can be finely tuned by changing the magnitude of \( \gamma_0 \).

For \( B > 2 \), however, \( S \) crossovers from \( c = 1 \) to the asymptotic value of \( c = -3(B - 2) \) as \( L \) increases, and tuning \( \gamma_0 \) can only changing the critical crossover system size \( L \), not \( c \) itself [Fig. 4]. Various representative models that host EB states are tabulated in [88].

More generally, systems with multiple EPs can possess multiple sets of EB states with their own elevated \( \langle n \rangle \) and 2-point functions. These are distinct from *secondary* EBs for a single EP of sufficiently large \( L \), which exist due to increased hopping non-locality and possess related spatial profiles with \( \lambda \sim L^{B/2-1} \) [88]. Remnants of EB states can even exist for an EP with a small gap \( m \), although \( \lambda \) will scale like \( m^{1-B/2} \) instead of \( L^{B/2-1} \).

**Discussion.** – EB states represent a new type of robust boundary phenomena distinct from topological or non-Hermitian skin states. They fundamentally arise because EP defectiveness causes the propagator to be asymmetrically singular, such that special bounded states can possess anomalously large occupation probabilities which may be detectable in optical lattice setups. Such EB states also contribute negatively to the EE, with the logarithmic scaling coefficient \( c \) depending on the exact EP dispersion. Whether \( c \) can be identified with the central charge of a CFT remains an open question.

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* phylch@nus.edu.sg

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Supplementary Materials

I. FUNDAMENTAL PROPERTIES OF EXCEPTIONAL BOUNDARY (EB) EIGENSTATES

Boundary-truncated Projector

We first show why EB states emerge as the eigenstates of the boundary-restricted two-point function, which is defined as \( P = RPR \), where \( P \) is the biorthogonal occupied band projector of the translation-invariant exceptional system and \( R \) the projector onto the real space region demarcated by the boundary. For general particle-hole symmetric Hamiltonians of the form

\[
H = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} = \sum_k \begin{pmatrix} 0 & a(k) \\ b(k) & 0 \end{pmatrix} |k\rangle\langle k| \tag{S1}
\]

in two-component sublattice space, the biorthogonal projector onto the occupied band \( \{ |\psi^{R}(k)\rangle \} \) takes the form

\[
P = \sum_k |\psi^{R}(k)\rangle\langle \psi^{L}(k)| = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -D \\ -D & I \end{pmatrix} \tag{S2}
\]

where \( \langle \psi^{L}(k) | \) is the biorthogonal dual of \( |\psi^{R}(k)\rangle \), and \( U \neq D^{\dagger} \) for a non-Hermitian system. In a translation invariant setting, we can work in momentum space where we have \( U(k)D(k) = 1 \), and \( P \) only has eigenvalues 0 and 1. This holds because the divergence in \( U(k) \to \sqrt{\frac{2\pi}{a^2}} k^{-B/2} \) is canceled by the vanishing of \( D(k) \to \sqrt{\frac{2\pi}{a^2}} k^{B/2} \) as we approach the EP at \( k = 0 \) (Eq. 3 of the main text).

But when projected onto a bounded region in real space i.e. \( u \to \tilde{U} = RUR \) and \( d \to \tilde{D} = RDR \), we have \( \tilde{U}\tilde{D} \neq \tilde{D}\tilde{U} \). Here \( R \) truncates off contributions to \( \tilde{U} \) or \( \tilde{D} \) from outside the stipulated real space region, such that \( \tilde{U} \) and \( \tilde{D} \) are no longer inverses of each other. The departure of \( \tilde{U}\tilde{D} \) or \( \tilde{D}\tilde{U} \) from unity captures the effect of boundaries or, more generally, spatial inhomogeneities. To study this systematically, we introduce the quantity \( \Lambda = (\mathbb{1} - 2\tilde{P})^2 - \mathbb{1} = 4(\tilde{P}^2 - \tilde{P}) \), which also measures how much \( \tilde{P} \neq \tilde{P}^2 \) fails to be a projector. Since \( \Sigma = R(PRPR - P)R = -RPR'PR \) where \( R^c = 1 - R \) projects onto the complement of region \( R \), \( \Sigma \) can also be understood as a measure of how much the occupied band projector \( P \) “entangles” \( R \) and \( R^C \) (If all band were occupied, \( \Sigma \) would be equal to \( RR^cR = 0 \)). Its relation with EE will be elaborated later on.

Due to the off-diagonal form of \( \mathbb{1} - 2\tilde{P} \), \( \Lambda \) is diagonal, such that its eigenstates \( |\phi\rangle = (|\phi_+\rangle, |\phi_-\rangle)^T \) decouple into \( \pm \) sublattice subspaces:

\[
\Lambda \begin{pmatrix} |\phi_+\rangle \\ |\phi_-\rangle \end{pmatrix} = \begin{pmatrix} \Lambda_+ & 0 \\ 0 & \Lambda_- \end{pmatrix} \begin{pmatrix} |\phi_+\rangle \\ |\phi_-\rangle \end{pmatrix} = \begin{pmatrix} \tilde{U}\tilde{D} - \mathbb{1} & 0 \\ 0 & \tilde{D}\tilde{U} - \mathbb{1} \end{pmatrix} \begin{pmatrix} |\phi_+\rangle \\ |\phi_-\rangle \end{pmatrix} = \lambda \begin{pmatrix} |\phi_+\rangle \\ |\phi_-\rangle \end{pmatrix}. \tag{S3}
\]

In particular, from the definition \( \Lambda = 4(\tilde{P}^2 - \tilde{P}) \), each eigenvalue \( \lambda \) of \( \Lambda \) (or \( \Lambda_\pm \)) is related to an eigenvalue \( p \) of the projector via \( \lambda = 4p(p - 1) \), and can be obtained by working in either subspaces. These eigenvalues would have been trivially zero if not for the projection onto the bounded region, where \( p \) cease to be 0 or 1.

Concretely, \( \tilde{U} \) and \( \tilde{D} \) can be obtained from \( U \) and \( D \) by a real-space truncation. We Fourier expand them into real space matrix elements via

\[
U(k) = \sum_x U_x e^{-ikx} \tag{S4}
\]

\[
D(k) = \sum_x D_x e^{-ikx}, \tag{S5}
\]

such that the two-point function takes the form

\[
\langle x_1 | P | x_2 \rangle = \sum_k P(k) e^{ik(x_1-x_2)} = \frac{1}{2} \left( -\sum_x \delta_{x_1,x_2} D_x^2 \delta_{x_1,x_2} + x - \sum_x U_x \delta_{x_1,x_2} \right) = \frac{1}{2} \left( \delta_{x_1,x_2} D_{x_1-x_2} - \delta_{x_1,x_2} \right). \tag{S6}
\]
Since $\langle x_1 | (1 - 2P) | x_2 \rangle = \begin{pmatrix} 0 & U_{x_1 - x_2} \\ D_{x_1 - x_2} & 0 \end{pmatrix}$, when restricted to finite values $0 \leq x_1, x_2 < L$ i.e. $P \to \bar{P}$, we have, for a single boundary at $x_1, x_2 = 0$,

$$\langle x_1 | \Lambda | x_2 \rangle = \langle x_1 | (1 - 2\bar{P})^2 - \bar{P} | x_2 \rangle = \begin{pmatrix} \sum_{x_4 \leq x_1} U_{x_4 - x_2} D_{x_4} \delta_{x_4 + x_3, x_1 - x_2} - \delta_{x_1, x_2} & 0 \\ 0 & \sum_{x_4 \leq x_2} D_{x_4} U_{x_4} \delta_{x_4 + x_3, x_1 - x_2} - \delta_{x_1, x_2} \end{pmatrix}. \tag{S7}$$

This is the explicit form of $\Lambda$ in Eq. [S3] in terms of real space matrix elements. The above summations keep track of the terms that are truncated in the convolution of the Fourier series of $U(k)$ and $D(k)$. Had no boundaries been present, the convolutions would have evaluated to $\delta_{x_1 - x_2} \mathbb{I}$, since $U(k)D(k) = D(k)U(k) = 1$. In the presence of another boundary at $x_1, x_2 = L$, we will also have analogous upper limits in the summations above.

**Two-site propagators in the singular limit**

If both $U(k)$ and $D(k)$ were smooth and non-divergent, their Fourier components (2-site propagators) would decay rapidly, rendering boundary effects largely negligible beyond small edge corrections. This scenario corresponds to $\langle x_1 | \Lambda | x_2 \rangle = \langle x_1 | [(1 - 2\bar{P})^2 - \bar{P}] | x_2 \rangle \approx 0$. However, if $U(k)$ and $D(k)$ possess divergent behavior, they will affect the 2-site propagators defined as their Fourier transforms:

$$\langle c^\dagger_{x_1, +} c_{x_2, -} \rangle = \langle x_1, + | P | x_2, - \rangle = -\frac{1}{2} U_{x_1 - x_2} - \frac{1}{2} \sum_k e^{ik(x_1 - x_2)} U(k) \tag{S8a}$$

$$\langle c^\dagger_{x_1, -} c_{x_2, +} \rangle = \langle x_1, - | P | x_2, + \rangle = -\frac{1}{2} D_{x_1 - x_2} - \frac{1}{2} \sum_k e^{ik(x_1 - x_2)} D(k) \tag{S8b}$$

$$\langle c^\dagger_{x_1, \mp} c_{x_2, \pm} \rangle = \frac{1}{2}. \tag{S8c}$$

Evidently, only the inter-sublattice 2-site propagators are nontrivial (this is due to the absence of any $\sigma_z$ term in $H$, whose complicating effects are treated in a later section.) As explained shortly, $U(k)$ diverges like $\sqrt{\frac{\Delta k}{\Delta \bar{P}}} \frac{1}{\pi} B^{2/3}$ for $B > 2$ due to the EP defectiveness. For $B = 2$, we shall also show below that $U(k)$ diverges like $\log L$. On the other hand, $D(k)$ remains well-behaved, and will for realistic local systems contain not more than a few non-negligible Fourier coefficients.

**Ansatz for $a(k)$ and $b(k)$**

To be concrete, we specialize to the ansatz

$$b(k) = b_0 (2(1 - \cos k))^{B/2} \tag{S9a}$$

$$a(k) = b(-k) + a_0, \tag{S9b}$$

such that

$$U(k) = \sqrt{\frac{a(k)}{b(k)}} = \sqrt{\frac{b_0 (2(1 - \cos k))^{B/2} + a_0}{b_0 (2(1 - \cos k))^{B/2}}}, \tag{S10}$$

$$D(k) = \sqrt{\frac{b(k)}{a(k)}} = \sqrt{\frac{b_0 (2(1 - \cos k))^{B/2}}{b_0 (2(1 - \cos k))^{B/2} + a_0}}. \tag{S11}$$

Of course, there are many other valid ansatze i.e. $b(k) = b_0 \sin^B k$ or $b_0 (1 - e^{ik})^B$, but Eq. [S9b] possesses the most local Fourier coefficients, as shown in Fig. [S1]. Near the EP at $k = 0$, $a(k)$ is dominated by $a_0$, and $D(k) \approx \sqrt{\frac{b_0}{a_0}} (2(1 - \cos k))^{B/4} = \sqrt{\frac{b_0}{a_0}} \sum_{\Delta x = -B/4}^{B/4} \left(\frac{B/2}{B/4 + \Delta x}\right)(-1)^\Delta x e^{i\Delta x k}$ i.e. with Fourier coefficients $(-1)^\Delta x \sqrt{\frac{b_0}{a_0}} \frac{B/2}{B/4 + \Delta x} \sim (-1)^2 \sqrt{\frac{b_0}{a_0}} B^{-1} e^{-4(\Delta x)^2/B}$ that are rapidly decaying with $\Delta x$. Yet, because $U(k) \sim \sqrt{\frac{b_0}{a_0}} (2(1 - \cos k))^{-B/4}$ is highly singular, its Fourier coefficients are slow-decaying and divergent with $L$. Since the closest momentum point to the EP
is \( k_0 \sim \pi/L \), \( a(k) \) is dominated by \( a_0 \) only when \( (2(1 - \cos k_0))^{B/2} = k_0^B \ll a_0 \) i.e. \( L \gg \pi/a_0^{1/B} \). In this limit, which is easily satisfied for modest system sizes of \( L \sim 10^1 - 10^2 \), we have

\[
U_x = -2 \langle c_{x,+}^\dagger c_{0,-} \rangle \sim \sqrt{\frac{a_0}{b_0}} \sum_k \frac{e^{ikx}}{(2(1 - \cos k))^{B/4}} \\
\simeq \frac{2\pi}{L} \sqrt{\frac{a_0}{b_0}} \frac{2 \cos k_0 x}{(2(1 - \cos k_0))^{B/4}} \\
\simeq \frac{2\pi}{L} \sqrt{\frac{a_0}{b_0}} \frac{2 \cos k_0 x}{k_0^{B/2}} \\
= 2 \sqrt{\frac{a_0}{b_0}} \left( \frac{L}{\pi} \right)^{B/2 - 1} \times \left( 2 - \frac{\pi^2 x^2}{L^2} \right). \tag{S12}
\]

This result holds accurately for \( B > 4 \). To understand why, note that we have kept only the most divergent \( k = k_0 \) term in line 2, which corresponds to a momentum width \( \Delta k = 2\pi/L \). The largest omitted terms scale like \((L/(3\pi))^{B/2} = (L/\pi)^{B/2}3^{B/2}B/2\), and can safely be discarded when \( 3^{B/2} \ll 1 \), i.e. when \( B \approx 4 \) or larger. Indeed, in Fig. S1a, it was seen that \( U_x \) follows an approximately parabolic profile for \( B > 4 \). Cases with larger \( B \) have similar profiles, except that they scale with \( L^{B/2 - 1} \).

FIG. S1. Comparison between the two-point functions \( \langle c_{x,+}^\dagger c_{0,-} \rangle = -U_x/2 \) and \( \langle c_{x,-}^\dagger c_{0,+} \rangle = -D_x/2 \) of various ansatz exceptional points. (a) and (b) respectively depict \( U_x \) and \( D_x \) (both normalized to have a maximum value of 1) for EPs defined by \( b(k) = (1 - e^{ik})^3 \), \( 2i\sin k \) and \( (2(1 - \cos k))^B \), \( B = 2, 4, 6 \), where \( a(k) = b(-k) + a_0 \) with \( a_0 \) set to unity. Due to EP defectiveness, \( U_x \) is evidently more nonlocal than \( D_x \), exhibiting power-law, linear and logarithmic decays (\( B = 3, 6, B = 4 \) and \( B = 2 \) respectively). They scale with \( x \) in an exactly analogous fashion as with \( L \) (Eqs. S13 to S19). (c-d) dependence of \( U_x \) and \( D_x \) on the EP asymmetry \( a_0 \). \( D_x \), being already local, is practically independent of \( a_0 \). But \( U_x \) depends significantly on \( a_0 \) for \( a_0 \sim 1 \) or smaller, when our analytical results become no longer valid.
We next turn to a more rigorous derivation of the scaling of \( U_x \) with \( L \). It holds for generic \( B \), and in particular allows us to characterize the special case of \( B = 2 \), as well as \( B < 2 \) cases. Not making any approximations, except that \( L \) sets the lower IR limit \[111\], we have (defining \( k' = k/2 \))

\[
U_x = -2(c_{x,+c_0,-}^4) = \sqrt{\frac{a_0}{b_0}} \int_{\pi/L}^{\pi} \frac{e^{ikx}dk}{(2(1 - \cos k))^{B/4}} + c.c
\]

\[
= \frac{1}{2^{B/2 - 1}} \sqrt{\frac{a_0}{b_0}} \int_{\pi/L}^{\pi} \frac{\cos kx dk}{((1 - \cos k)/2)^{B/4}}
\]

\[
= \frac{1}{2^{B/2 - 2}} \sqrt{\frac{a_0}{b_0}} \int_{\pi/2L}^{\pi/2} \frac{\cos 2k'x dk'}{\sin^{B/2} k'}
\]

\[
= \frac{1}{2^{B/2 - 2}} \sqrt{\frac{a_0}{b_0}} \int_{\pi/2L}^{\pi/2} \left[ \frac{1}{\sin^{B/2} k'} - 2 \sin^2 k' \right] dk'.
\]

(S13)

In particular, \( x \) only enters the second term in the square parentheses, which is positive and vanishes when \( x = 0 \) i.e. for the on-site inter-sublattice correlator. Hence \( U_x \) is maximal for \( x = 0 \), with its divergence with \( L \) obtainable from the first term alone. In general, its scaling can be analytically derived in terms of a Hypergeometric function:

\[
U_0 = -2(c_{1,+,c_{0,-}^4}^4) \sim -\frac{1}{2^{B/2 - 2}} \sqrt{\frac{a_0}{b_0}} \int \frac{dk'}{\sin^{B/2} k'} |k| = \pi/2L
\]

\[
= 4_2F_1 \left[ 1, 1 - \frac{B}{2}, 1 - \frac{B}{2}, \sin^2 \left( \frac{\pi}{2L} \right) \right] \sqrt{\frac{a_0}{b_0}} \left( 2 \sin \left( \frac{\pi}{2L} \right) \right)^{1-B/2}
\]

\[
= \frac{4}{B - 2} \sqrt{\frac{a_0}{b_0}} \left( \frac{L}{\pi} \right)^{B/2 - 1} \left( 1 + \frac{B(B - 2)}{3(B - 6)} \frac{L^2}{\pi^2} + \frac{B(B - 2)(5B + 4)}{90(B - 10)} \frac{L^4}{\pi^4} + ... \right)
\]

(S14)

whose leading order power-law divergence behavior agrees up to a constant with Eq. S12 which was obtained through hand-wavy arguments. Note that the above expansion converges only for \( B \) (mod 4) = 0. A more careful analysis reveals logarithmic divergence behavior or non-universal constants the other cases, and subleading contributions for all cases:

- **\( B = 1 \):**

\[
U_0\big|_{B=1} \sim \sqrt{\frac{32a_0}{b_0}} F \left( \frac{\pi(L - 1)}{4L}, 2 \right) = \sqrt{\frac{a_0}{b_0}} \left( 7.4163 - 4 \sqrt{\frac{\pi}{L}} + ... \right)
\]

(S15)

- **\( B = 2 \):**

\[
U_0\big|_{B=2} \sim 2 \sqrt{\frac{a_0}{b_0}} \log \cot \frac{\pi}{4L} = 2 \sqrt{\frac{a_0}{b_0}} \left( \log \frac{\pi}{L} + \log L + ... \right)
\]

(S16)

- **\( B = 3 \):**

\[
U_0\big|_{B=3} \sim \sqrt{\frac{8a_0}{b_0}} \left( \frac{\cos \frac{\pi}{2L}}{\sin \frac{\pi}{2L}} - E \left( \frac{\pi(L - 1)}{4L}, 2 \right) \right) = \sqrt{\frac{a_0}{b_0}} \left( 4 \sqrt{\frac{\pi}{L}} - 1.6944 + ... \right)
\]

(S17)

- **\( B = 4 \):**

\[
U_0\big|_{B=4} \sim \sqrt{\frac{a_0}{b_0}} \cot \frac{\pi}{2L} = \sqrt{\frac{a_0}{b_0}} \left( \frac{2L}{\pi} - \frac{\pi}{6L} + ... \right)
\]

(S18)

- **\( B = 6 \):**

\[
U_0\big|_{B=6} \sim \frac{1}{4} \sqrt{\frac{a_0}{b_0}} \left( \log \cot \frac{\pi}{4L} + \frac{\cos \frac{\pi}{2L}}{\sin^2 \frac{\pi}{2L}} \right) = \sqrt{\frac{a_0}{b_0}} \left( \frac{L^2}{\pi^2} - \frac{1}{24} + ... \right)
\]

(S19)
Here $F(\phi, m) = \int_0^\phi d\theta/\sqrt{1 - m \sin^2 \theta}$ and $F(\phi, m) = \int_0^\phi d\theta/\sqrt{1 - m \sin^2 \theta}$ are the Elliptic integrals. As evident from Fig. S1, odd $B$ cases can be realized with the ansatz $b(k) = b_0(1 - e^{ik})^B$, with the caveat that $U_x = 0$ for all even $x$.

In particular, the marginal $B = 2$ case, which can be realized commonly in lattices with nearest neighbor couplings, contains two-pt functions $-U_0/2$ that diverges logarithmically with $L$. The $B = 1$ case, which can be realized with $b(k) = b_0(1 - e^{ik})$ that also contains up to only nearest neighbor couplings, is also noteworthy in that $U_0$ saturates to a constant $4 \sqrt{\pi} \mathcal{F}(\pi/2, 1/2) = 7.4163 \sqrt{\pi} b_0/\pi$ as $L \to \infty$. While this lack of divergence is special among all these cases containing an EP, it is typical among non-EP gapless scenarios, where the on-site inter-sublattice propagator should not grow with $L$.

In general, the 2-site propagator $U_x$ scales with $x$ in the same way as it scales with $L$, since these are the only length scales present, except when $x$ is comparable with $L$. This can be explicitly demonstrated for some simple but important cases in the $L \to \infty$ limit. For $B = 2$, we have from Eq. S13:

$$U_x|_{B=2} = U_0|_{B=2} - \frac{1}{2^{1-2}} \sqrt{\frac{a_0}{b_0}} \int_0^{\pi/2} \frac{2 \sin^2 k' x}{\sin k'} dk'$$

$$= U_0|_{B=2} - 4 \sqrt{\frac{a_0}{b_0}} \sum_{j=1}^x \frac{\cos jk}{2j - 1} |_{k'=\pi/2}$$

$$= U_0|_{B=2} - 4 \sqrt{\frac{a_0}{b_0}} \sum_{j=1}^x \frac{1}{2j - 1}$$

$$\sim U_0|_{B=2} - 2 \sqrt{\frac{a_0}{b_0}} (\log x + \gamma + \log 4)$$

$$= 2 \sqrt{\frac{a_0}{b_0}} \left( \log \frac{1}{4x} \cot \frac{\pi}{4L} \right) - \gamma$$

$$= 2 \sqrt{\frac{a_0}{b_0}} \left( \log \frac{L}{\pi x} \right) - \gamma + ...$$

(S20)

where $\gamma$ is the Euler-Mascheroni constant from the approximation in line 4, and the last line contains subleading terms vanishing as $1/x, 1/L$ or faster.

For the $B = 4$ case, we also have:

$$U_x|_{B=4} = U_0|_{B=4} - \frac{1}{2^{2-2}} \sqrt{\frac{a_0}{b_0}} \int_0^{\pi/2} \frac{2 \sin^2 k' x}{\sin^2 k'} dk'$$

$$= U_0|_{B=4} - 2 \sqrt{\frac{a_0}{b_0}} (k' x + [\sin ...]) |_{k'=\pi/2}$$

$$= U_0|_{B=4} - \pi x$$

$$= 2 \sqrt{\frac{a_0}{b_0}} \left( \frac{L}{\pi} - \pi x \right)$$

(S21)

i.e. that $U_x|_{B=4}$ depends linearly on both $L$ and $x$, in qualitative agreement with Fig. S1.

**Emergence of EB eigenstates**

EB eigenstates emerge when the eigenstate is forced to have compact support, and can arise from even a single boundary. Setting the boundary to be at $x = 0$, we return to Eq. S7 which shows that Fourier coefficients $D_j$ are relatively small and rapidly decaying, but Fourier coefficients $U_j$ are diverging like $\sim L^{B/2}$ and slowly decaying with $j$. These turn out to be the exact conditions needed to support Exceptional Boundary (EB) eigenstates of $P$. These special eigenstates have eigenvalues $p_x = \frac{1}{2}(1 + \sqrt{1 + \lambda})$ that are well-separated from the other eigenvalues which are clustered around 0 and 1. Focusing on the upper + sublattice (the − sublattice eigenequation contains equivalent information), the EB eigenvalue $\lambda$ appears in the eigenequation

$$\sum_{x_1} \langle x_1 | \Lambda_+ | x_2 \rangle \phi_+ (x_1) = \sum_{x_1} \sum_{x_{s_1} \leq x_1, x_{s_2} \leq x_2} [U_{-x_2} D_{x_2} \delta_{x_2 + x_{s_4}, x_1 - x_2} - \delta_{x_1, x_2}] \phi_+ (x_1) = \lambda \phi_+ (x_2).$$

(S22)
More explicitly, making use of the fact that $D(k)U(k) = 1$ i.e. $\sum_x D_x \Delta U_{-x} = \delta_{\Delta,0}$, the $x_1$-th column of the $L \times L$ matrix $\Lambda_+ = U\tilde{D} - \Pi$ takes the form

$$- |c_{x_1}\rangle = D_{x_1} \begin{pmatrix} U_{-1} \\ U_{-2} \\ U_{-3} \\ \vdots \end{pmatrix} + D_{x_1+1} \begin{pmatrix} U_{-2} \\ U_{-3} \\ U_{-4} \\ \vdots \end{pmatrix} + \cdots = \sum_{j=0}^{\infty} D_{x_1+j} \begin{pmatrix} U_{-1-j} \\ U_{-2-j} \\ U_{-3-j} \\ \vdots \end{pmatrix}. \quad \text{(S23)}$$

Analogously, the $x_2$-th row of $\langle x_1|\Lambda_+|x_2\rangle = \langle x_1|(U\tilde{D} - \Pi)|x_2\rangle$ takes the form

$$- \langle r_{x_2} \rangle = U_{-x_2} \begin{pmatrix} D_1 \\ D_2 \\ D_3 \\ \vdots \end{pmatrix}^T + U_{-x_2+1} \begin{pmatrix} D_2 \\ D_3 \\ D_4 \\ \vdots \end{pmatrix}^T + \cdots = \sum_{j=0}^{\infty} U_{-x_2-j} \begin{pmatrix} D_{1+j} \\ D_{2+j} \\ D_{3+j} \\ \vdots \end{pmatrix}^T. \quad \text{(S24)}$$

Importantly, the slow decay of the $U_{-j}$ coefficients, which are already divergently scaling as $\sim L^{B/2}$, ensures that the matrix $\Lambda_+ = U\tilde{D} - \Pi$ is dominated by its first slowly-decaying column. The subsequent columns contain much smaller elements due to the rapid decay of the $D_j$ coefficients. Meanwhile, the situation with the rows is not analogous. Due to the rapid decay of the $D_j$ coefficients, none of the rows will contain large, slowly-decaying elements, perhaps with the exception of the first. With a dominant first column $|c_{x_1}\rangle$ and subdominant first row $\langle r_{x_1} \rangle$, the eigenequation Eq. S22 can be approximated as

$$\Lambda_+ |\phi_+\rangle \approx |c_{x_1}\rangle \langle 1| + |1\rangle \langle r_{x_1}| - \langle 1|c_{x_1}\rangle \langle 1| |1\rangle |\phi_+\rangle = \lambda |\phi_+\rangle. \quad \text{(S25)}$$

The first two terms give the dominant column and subdominant row respectively, while the third term subtracts the $\langle 1|\Lambda_+|1\rangle = \langle 1|c_{x_1}\rangle$ matrix contribution that appears in both $\langle r_{x_1}|$ and $|c_{x_1}\rangle$. Now, because the first term $|c_{x_1}\rangle \langle 1|$ is the dominant contribution, the eigenvector $|\phi_+\rangle$ must be approximately parallel to $|c_{x_1}\rangle$. Substituting $|\phi_+\rangle \approx |c_{x_1}\rangle$ and left multiplying the above by $\langle 1|$, we obtain

$$\langle 1|c_{x_1}\rangle^2 + \langle 1|1\rangle \langle r_{x_1}|c_{x_1}\rangle - \langle 1|c_{x_1}\rangle \langle 1|1\rangle \langle 1|c_{x_1}\rangle \approx \lambda \langle 1|c_{x_1}\rangle$$

i.e.

$$\lambda \approx \frac{\langle r_{x_1}|c_{x_1}\rangle}{\langle 1|c_{x_1}\rangle} = \frac{\langle 1|\Lambda_+^2|1\rangle}{\langle 1|\Lambda_+|1\rangle} = \frac{\langle 1|(U\tilde{D}U\tilde{D} - 2\tilde{U}\tilde{D} + \Pi)|1\rangle}{\langle 1|(U\tilde{D} - \Pi)|1\rangle} = - \sum_{j,j'=0} \frac{D_{1+j} U_{-1-j} D_{2+j'} D_{3+j'} \cdots}{\sum_{j=0} D_{1+j} U_{-1-j}} \begin{pmatrix} U_{-1-j} \\ U_{-2-j} \\ \vdots \end{pmatrix}\begin{pmatrix} D_{1+j} \\ D_{2+j} \\ \vdots \end{pmatrix} \quad \text{(S26)}$$

The first line (Eq. S27) follows from the definition of $\langle 1|, |c_{x_1}\rangle$ as the first row and column of $\Lambda_+$, which can be expressed in terms of the final explicit expressions upon making use of the simplifications used in Eqs. S23 and S24.

Eq. S27 expresses the special EB eigenvalue $\lambda$ of $\Lambda_+$ in terms of the corner matrix elements of $\Lambda_+$ and $\Lambda_+^2$, that can be easily computed without further explicit diagonalization. Although we have made various approximations along the way, Eq. S27 or S28 is highly accurate for a huge range in parameter space, even when the EP is not too singular ($B \geq 2$). Note that this $\lambda$ eigenvalue is unique and the other eigenvalues, which constitute the majority, are still very close to 0.

The fact that the EB eigenstate $|\phi_+\rangle$ and $|c_{x_1}\rangle$, the first column of $\Lambda_+$, are almost parallel implies that the EB eigenstate profile is largely determined by the Fourier components of $D(k)$ and $U(k)$ via Eq. S23. For more singular cases in particular, $\phi_+$ and $|c_{x_1}\rangle$ are almost exactly parallel, and the components of $U_j$ accurately determine the EB state. As plotted in Fig. S2, the profile of the first column $|c_{x_1}\rangle$ resemble the profiles of the EB states in Fig. 2a of the main text. As $B$ increases, the size of the elements in $|c_{x_1}\rangle$ increases strongly, leading to more well-defined EB modes.
FIG. S2. Profiles of the matrix elements of Λ+ for (a) B = 2, (b) B = 4, (c) B = 6 order EPs with b(k) = (2(1 − cos k))B/2 and a0 = 10^3. While most elements are close to 0, similar to the Hermitian case where Λ+ vanishes, the first column matrix elements are anomalously large in our cases. Their characteristic concave, linear and quadratic spatial profiles dominate the effects of Λ+, and in fact determine the profiles of their corresponding EB eigenstates. (d) showcases the profile of an alternative B = 4 EP implementation based on b(k) = (2i sin k)^4, which is similar to (b) except for odd/even effects.

Scaling of EB eigenvalues and entanglement entropy

From previous subsections, we already know that the Fourier coefficients Uj diverge like \( L^{B/2-1} \) for \( B > 2 \), but that the first few Fourier coefficients \( D_j \) remain nonzero but fixed. Hence from Eq. S28 we expect \( \lambda \) to scale for \( B > 2 \) like

\[
\lambda \sim \lambda_0 L^{B/2-1} = -\frac{\sum_{j,j',j''=0} D_{1+j-1-j'} D_{1+j''-1-j''} u_j u_{j'} u_{j''}}{\sum_{j=0} D_{1+j} u_{-1-j}} L^{B/2-1}
\]

(S29)

where \( \lambda_0 \) is a constant determined by the quotient above, and \( u_j = U_j/L^{B/2-1} \) are the Fourier coefficients with \( L^{B/2-1} \) divergence factored out. Likewise, for \( B = 2 \), we have \( \lambda|_{B=2} \sim \lambda_0 \log L \). For \( B < 2 \), the first column of \( \Lambda_+ \) are not dominant and the derivations leading to Eq. S28 are not applicable. As such, a well-defined \( \lambda \) and hence EB states do not exist [Fig. 2a of the main text].

For each boundary, the EE contribution from the EB state is, for \( B > 2 \),

\[
S_{\text{EB}} = -\text{Tr}_{\text{EB}}[\hat{P} \log \hat{P} + (\mathbb{1} - \hat{P}) \log(\mathbb{1} - \hat{P})]
\]

\[
= -\sum_{\pm} \left[ \frac{1 \pm \sqrt{1 + \lambda}}{2} \log \frac{1 \pm \sqrt{1 + \lambda}}{2} + \frac{1 \pm \sqrt{1 + \lambda}}{2} \log \frac{1 \pm \sqrt{1 + \lambda}}{2} \right]
\]

\[
\sim \log 4 - 2 - 2 \log \sqrt{1 + \lambda}
\]

\[
\sim \log 4 - 2 - \log \lambda_0 - \frac{B - 2}{2} \log L.
\]

(S30)

On the first line, the trace is only taken over the possible EB states. The first \( \sim \) refers to the asymptotic expansion of the 2nd line at large \( \lambda \), while the second \( \sim \) pertains to the large \( L \) behavior of \( \lambda \). We see that a divergence of positive power \( B \) always tends to suppress the EE, in fact contributing a quantized negative coefficient to \( \log L \). In the presence of other competing terms i.e. \( \sigma_z \) in the Hamiltonian, the EB modes become less protected by the defectiveness, and varying amounts of positive \( \sim \log L \) contributions will also be present, as presented in a later section. The marginal
FIG. S3. (a-f) Magnitudes of the eigenvalues $p$ of $P$ as a function of $L$, plotted on log-log scales for $B = 2, 4, 8, 10, 12, 14$ respectively for the ansatz model Eq. S9b with $a_0 = 10^3$. While only one pair of $|p| > 1$ eigensolutions (EB states) exist at $B = 2$ and $B = 4$, a secondary set of EB state eigensolutions emerge for $B \geq 8$. At $B = 14$, a tertiary set of EB states also appear. For the $B = 12$ case, the emergence of secondary EB states at $L \approx 30$ (e) corresponds to the crossover of the EE $S$ behavior. Below $L \approx 30$, $S \sim -(B - 2) \log L = -10 \log L$ [Eq. S30] (cyan dashed), but above $L \approx 30$, $S \sim -(B - 2) \log L - (B - 4) \log L = -18 \log L$ (green dashed). Primary (h) and secondary (i) EB states differ markedly in their spatial profiles.

$B = 2$ case has $\lambda$ increasing with $L$ logarithmically slowly, resulting in $S_{EB} \sim -\log \log L + \text{const.}$ at very large $L$. This weak double logarithmic increase can be easily overshadowed by other $\sim \log L$ or larger contributions should any competing term be present. Indeed, it was numerically shown that $S \sim -2/3 \log L$ for $B = 2$ in Fig. 3a of the main text, as also justified by CFT for a related model in Ref. [105].

In generic entanglement geometries, the above expression will be multiplied by the number of entanglement cuts i.e. 1 (or 2) cuts for a single entanglement subsystem in a 1D OBC (or PBC) system.

Secondary EB states and beyond

In the case of very strong divergences i.e. large $B$, $\Lambda$ contains not just one set of strongly nonvanishing eigenvalues, but also secondary, tertiary etc. sets as well. They arise because a very large $B$ leads to very large elements in the first few columns of $\Lambda_{+}$, not just the first column as for the EB states described previously. This can be understood from Eq. S23. Large $B$ leads to divergent values of $U_x$ [Eq. S12], and more importantly more nonlocal $D_{x_{1}+j}$ in

\[
\sum_{i} D_{i_{1}+j} \rightarrow \infty \quad \text{for } B \rightarrow \infty
\]
which leads to more slowly decaying columns in $\Lambda_+$. Generalizing the mechanism involved in the derivation of Eq. S28, we empirically find additional secondary EB states that appear at sufficiently large $L$, scaling as $\sim L^{B/2-2}$ instead of $\sim L^{B/2-1}$.

The emergence of these additional EB states is studied in Fig. S3. At small $B$, i.e. $B=2$, there is only one set of weak EB state with eigenvalue $p \geq 0$. The other states with $p \in [0, 1]$ behave like ordinary truncated projector eigenstates. At $B=4$, the corresponding branch of eigenvalues $1-p$ satisfies $|1-p| > 1$ for larger values of $L$, but still have the same origin as the $p$ eigenvalues. But at larger $B$, i.e. at $B=8$, a new distinct branch of $p > 1$ eigenvalues emerge. These secondary $p$ eigenvalues can become very large too for even larger $B$, albeit scaling like $\sim L^{B/2-2}$ instead of $\sim L^{B/2-1}$. This is verified in Fig. S3g, where different sets of EB states contribute additively to the EE.

Unlike those of ordinary (primary) EB states [Fig. S3h], $B$ instead of $|H|$ emerge. These secondary have the same origin as the formations. This would not have been true if the bra had been $\phi$, and the expectation of an arbitrary operator $\hat{O}$ of weak EB state with eigenvalue $\hat{p}$.

**TEMPORAL EVOLUTION OF THE TWO-POINT FUNCTION**

For a non-Hermitian system, the temporal two-point function with respect to a given operator $\varphi(t)$ is given by

$$\langle \varphi^\dagger(t)\varphi(0) \rangle = \langle \hat{\varphi}(0)|P|\varphi(t)\rangle = \langle \hat{\varphi}|Pe^{-iH_F t/H}|\varphi \rangle,$$

(S31)

where $H_F$ is the Hamiltonian for $t \geq 0$, $P$ is the projector onto the occupied Hilbert space and $|\varphi\rangle = |\varphi(0)\rangle = \varphi^\dagger(0)|0\rangle$.

To be consistent with probabilistic interpretations, the bra state $\langle \hat{\varphi} |$ is taken to be the biorthogonal conjugate of the ket $|\varphi\rangle$. To see why, we briefly review the concepts biorthogonal bases. Suppose that $|\phi_i\rangle$ and $\langle \phi_i |$ are corresponding left and right eigenstates of an operator $H$ i.e.

$$H|\phi_i\rangle = h_i|\phi_i\rangle,$$

(S32a)

$$\langle \phi_i | H = h_i\langle \phi_i |.$$

(S32b)

Under a similarity basis transform $H \rightarrow V HV^{-1}$, which is not necessarily unitary, the above equations give us $|\phi_i\rangle \rightarrow V|\phi_i\rangle$ and $\langle \phi_i | \rightarrow \langle \phi_i | V^{-1}$. Since $VV^{-1} = V^{-1}V = I$, biorthogonal expectations are independent of basis transformations. This would not have been true if the bra had been $|\phi_i\rangle$, the Hermitian conjugate of $|\phi_i\rangle$, since $|\phi_i\rangle \rightarrow |\phi_i|$, and the expectation of an arbitrary operator $\hat{O}$ will have transformed like $\langle \phi_i | \hat{O} | \phi_i \rangle \rightarrow \langle \phi_i | V\hat{O}V^{-1} | \phi_i \rangle$, which is not consistent with the similarity transformation $O \rightarrow V\hat{O}V^{-1}$. In particular, normalization and orthonormality is preserved in the biorthogonal basis only: $\langle \phi_j | \phi_i \rangle \rightarrow \langle \phi_j | V^{-1} V | \phi_i \rangle = \delta_{ij}$ but not $\langle \phi_j | \phi_i \rangle \rightarrow \langle \phi_j | V V^{-1} | \phi_i \rangle$.

Suppose that $|\varphi\rangle = \sum_i \chi_i |\phi_i\rangle$ and $\langle \hat{\varphi} \rangle = \sum_i \tilde{\chi}_i \langle \phi_i |$, where $\chi_i = \langle \phi_i | \varphi \rangle$ and $\tilde{\chi}_i = \langle \phi_i | \hat{\varphi} | \varphi \rangle$. The expectation of $H$ is

$$\langle H \rangle = \langle \hat{\varphi} | H | \varphi \rangle = \sum_{i,j} \chi_i \tilde{\chi}_j \langle \phi_j | H | \phi_i \rangle = \sum_i h_i \tilde{\chi}_i \chi_i.$$

(S33)
If \( \langle \varphi \rangle \) is defined such that \( \bar{\chi}_i = \chi_i^* \), we arrive at the probabilistic expression

\[
\langle H \rangle = \sum_i h_i |\chi_i|^2
\]

(S34)

where \( \sum_i |\chi_i|^2 = 1 \) from biorthogonality. (In the Hermitian context, there is no distinction between usual orthogonality and biorthogonality because only unitary rotations \( V \) are considered i.e. \( V^\dagger V = \mathbb{I} \).)

At \( t = 0 \), \( \langle \varphi(t)\varphi(0) \rangle = \langle \varphi|P|\varphi \rangle \) is the density evolution of the state \( |\varphi \rangle = |\varphi(0) \rangle \). As shown in the main text, this density can be anomalously large when \( |\varphi \rangle = |\phi \rangle \), an EB state. Interestingly, \( \langle \varphi(t)\varphi(0) \rangle \) remains elevated at \( t > 0 \) for \( |\varphi \rangle \) an EB state, as long as the energy profile does not excessively exceed the EB profile. As shown in Fig. S4, the 2-point function generally decays initially before transitioning into oscillatory behavior [Figs. S4a-c]. For the EB state [Fig. S4a], whose density expectation is \( \approx 2 \), the 2-point function remains elevated and frequently exceeds unity. But for other states i.e. uniform profiles \( \langle x|\varphi \rangle = \varphi(x) = \theta(\ell/2 - x) \) [Figs. S4a] or \( \langle x|\varphi \rangle = \varphi(x) = \theta(\ell/4 - x) \) [Figs. S4a], the 2-point function starts at \( \approx 0.4 \), and decays further to smaller values.

The extent of decay depends on the Hamiltonian \( H_F \)’s energy profile, as color coded in Fig. S4l. In this work, we have considered \( H_F \) of the form

\[
H_F = \sum_x E(x) |x\rangle \langle x| = \sum_x \frac{E_0}{1 + e^{\alpha(x_0-x)}} |x\rangle \langle x|
\]

(S35)

such that it creates a potential well with soft walls at \( x = x_0 \), with larger \( \alpha \) leading to a steeper well. In Figs. S3 and S5a, we have respectively investigated the effects of tuning \( \alpha \) and \( x_0 \). In Fig. S3, where \( x_0 \) is fixed at \( \ell/2 \), the 2-point function evolution is compared between (a) a \( B = 4 \) EB state “bounded” at \( x = \ell/2 \), i.e. defined by \( R = \sum_x \theta(\ell/2 - x) |x\rangle \langle x| \), (b) an uniform state \( \varphi(x) = \theta(\ell/2 - x) \) between \( x = 0 \) and \( x = \ell/2 \), and (c) an uniform state \( \varphi(x) = \theta(\ell/4 - x) \) between \( x = 0 \) and \( x = \ell/4 \). The various curves correspond to \( H_F \) profiles labeled in Fig. S4l by the same color. For the EB state (a), the 2-point function fluctuates significantly for very gentle potential wells due to the nonzero tail of the nonzero energy within \([0, \ell/2]\). This nonzero energy tail scrambles the state \( \varphi(x) \), and is more pronounced when the well boundary is softer, as evidenced in (b) and (c). While a relatively hard \( \ell/2 \) boundary with \( \alpha = 100 \) (purple) leads to slow oscillations for (b) \( \varphi(x) = \theta(\ell/2 - x) \) and non-existent fluctuations for (c) \( \varphi(x) = \theta(\ell/4 - x) \), very soft boundaries with \( \alpha = 1 \) (blue) and \( \alpha = 2 \) (orange) lead to rapid initial decay, followed by partially recurrent oscillations. Over time, these oscillations completely lose their coherence and descend into increasingly random oscillations.

The effects of tuning the potential wall position \( x_0 \) are presented in Fig. S5a for \( \varphi(x) \) being the same \( B = 4 \) EB state and potential slope \( \alpha = 10 \). For \( x_0/\ell = 5/12 < 1/2 \) (red), the nonzero energy tail within the support of the EB state \( (0 \leq x \leq \ell/2) \) scrambles the state, leading to oscillations. These oscillations become slower as the tail moves away, as plotted for \( x_0/\ell = 1/2 \) (magenta) and \( x_0/\ell = 7/12 \) (blue).

It is also illuminating to investigate the effects of the scrambling by \( H_F \) on \( \varphi(x) \) being a random state with uncorrelated uniformly distributed amplitudes at each site. Shown in Fig. S5a are the 2-point functions for \( x_0 = \ell/2 \) and different slopes \( \alpha = 1, 2, 10 \) (light green, teal, blue). As the slope increases and the potential well becomes more well-defined, scrambling due to time-evolution becomes confined to the part of the wavefunction outside the well, leading to an approximately two-component effective system. This manifest as regular oscillations in the 2-point function.

**Quantum Entanglement and EB State Phenomena**

General results

In the general many-body context, a system consists of occupied single-body states \( \{|\chi_i\rangle\} \) that collectively form a many-body state \( |\Psi\rangle \). The latter defines the density matrix via \( \rho = |\Psi\rangle \langle \Psi| \), where bras are understood to be the biorthogonal conjugate of the kets with respect to the non-Hermitian Hamiltonian (unlike in the main text, they are not indicated with a tilde for brevity). Entanglement information is contained in the reduced density matrix \( \rho_R = \text{Tr}_W \rho \), which is obtained from \( \rho \) by tracing out degrees of freedom in \( \mathcal{R}^c \), where \( \mathcal{R}^c \) is the complement of a prescribed subregion \( \mathcal{R} \).

To put \( \rho_R \) in a palatable form, we would like to rotate the single-body basis into a new basis \( \{|\chi_i'\rangle\} \) such that it consists of basis states that are either wholly contained in \( \mathcal{R} \) or \( \mathcal{R}^c \). To do so, we first explicitly write down the
FIG. S5. (a) Dependence of 2-point function evolution with potential wall position $x_0 = 25, 30, 35$ (red, magenta, blue) [Eq. S35] in a system with $L = 60$ unit cells, for the $B = 4, a_0 = 1$ EB state with support in $x \in [0, 30]$. Evidently, there is more scrambling and hence faster oscillations when more of the EB state falls outside the potential well. (b) 2-point function evolution for different potential wall slopes $\alpha = 1, 2, 10$ (light green, teal, blue) [Eq. S35] for a random state with uncorrelated uniformly distributed site amplitudes. A more well-defined potential well leads to more regular oscillations.

many-body state as

$$|\psi\rangle = \bigotimes_{\xi \in \text{occ}} |\chi_\xi\rangle = \bigotimes_{\xi} P|\chi_\xi\rangle = \bigotimes_{\xi} (RP|\chi_\xi\rangle + R^c P|\chi_\xi\rangle),$$

(S36)

where $R$ and $R^c = I - R$ are projectors onto regions $R$ and $R^c$ respectively, and the occupied subspace projector $P = \sum_{\xi \in \text{occ}} |\chi_\xi\rangle \langle \chi_\xi|$ ensures that only occupied states are included. Importantly, we would like the two terms $RP|\chi_\xi\rangle$ and $R^c P|\chi_\xi\rangle$ on the right to be each proportional to a basis state of $\{|\chi_\xi\rangle\}$. Suppose $|\chi_\xi\rangle \propto RP|\chi_\xi\rangle$. Then

$$\langle \chi_\mu | \langle \psi | \tilde{R} | \psi \rangle | \chi_\xi \rangle = \langle \chi_\mu | P RP | \chi_\xi \rangle = r_\xi \delta_{\mu\xi}$$

(S37)

if we let $|\chi_\xi\rangle$ be an eigenstate of $\tilde{R} = P RP$ with eigenvalue $r_\xi$, i.e., $\tilde{R}|\chi_\xi\rangle = r_\xi |\chi_\xi\rangle$. In Ref. [105], $\langle \psi_\alpha | \tilde{R} | \psi_\beta \rangle$ was interpreted as the overlap matrix between arbitrary states $|\psi_\alpha\rangle$ and $|\psi_\beta\rangle$. With Eq. S37 we can construct the requisite new orthonormal basis $\{|\chi_\xi'\rangle\}$ in $R$ via

$$|\chi_\xi'\rangle = \frac{1}{\sqrt{r_\xi}} RP|\chi_\xi\rangle.$$  

(S38)

Since an eigenvalue $r_\xi$ always occurs together with $1 - r_\xi$, we also have the following basis states that are wholly contained in $R^c$:

$$|\chi_\xi^{r_\xi}\rangle = \frac{1}{\sqrt{1 - r_\xi}} R^c P|\chi_\xi\rangle.$$  

(S39)

Hence the many-body state is expressed as

$$|\Psi\rangle = \bigotimes_{\xi} (\sqrt{r_\xi}|\chi_\xi'\rangle + \sqrt{1 - r_\xi}|\chi_\xi^{r_\xi}\rangle).$$  

(S40)

In this form, the contributions from $R$ and $R^c$ are separated, and the trace over $R^c$ can be easily done to obtain
\[ \rho_R = \text{Tr}_R \rho : \]
\[ \rho_R = \text{Tr}_R |\Psi \rangle \langle \Psi| = \text{Tr}_R \bigotimes_{\xi, \mu} \left( \sqrt{r_\xi} |\chi_\xi \rangle + \sqrt{1 - r_\xi} |\chi_\xi^\prime \rangle \right) \left( \sqrt{r_\mu} |\chi_\mu \rangle + \sqrt{1 - r_\mu} |\chi_\mu^\prime \rangle \right) \]
\[ = \text{Tr}_R \bigotimes_{\xi, \mu} \left( \sqrt{r_\xi} |\chi_\xi \rangle, 0 \rangle \langle 0| + \sqrt{r_\xi (1 - r_\mu)} |\chi_\xi \rangle, 0 \rangle \langle 0| + \sqrt{r_\mu (1 - r_\xi)} |0, \chi_\mu^\prime \rangle \langle 0, \chi_\mu^\prime | \right) \]
\[ + \sqrt{(1 - r_\xi) (1 - r_\mu)} |0, \chi_\mu^\prime \rangle \langle 0, \chi_\mu^\prime | \right) \]
\[ = \sum_\gamma \sum_\xi |0, \chi_\xi^\prime \rangle \bigotimes_{\xi, \mu} \left( \sqrt{r_\xi} |\chi_\xi \rangle, 0 \rangle \langle 0| + \sqrt{r_\xi (1 - r_\mu)} |\chi_\xi \rangle, 0 \rangle \langle 0| + \sqrt{r_\mu (1 - r_\xi)} |0, \chi_\mu^\prime \rangle \langle 0, \chi_\mu^\prime | \right) \]
\[ + \sqrt{(1 - r_\xi) (1 - r_\mu)} |0, \chi_\mu^\prime \rangle \langle 0, \chi_\mu^\prime | \right) |0, \chi_\mu^\prime \rangle \]
\[ = \bigotimes_\xi (r_\xi |\chi_\xi \rangle \langle \chi_\xi | + (1 - r_\xi) |0 \rangle \langle 0|) . \] (S41)

In the second and third lines above, we have used the notation \(|a, b\rangle = |a\rangle_R |b\rangle_{R^c}\) to indicate whether the state belongs to \(R\) or \(R^c\). Note that \(\text{Tr}_R |a, 0\rangle \langle a, 0| = \sum_b \langle 0, b|a, 0 \rangle \langle a, 0|0, b\rangle = |a\rangle \langle a|\).

The EE for region \(R\) can be obtained via
\[ S_R = -\text{Tr}_R \rho_R \log \rho_R = - \sum_\xi [r_\xi \log r_\xi + (1 - r_\xi) \log(1 - r_\xi)] . \] (S42)

**Relation to the 2-site propagator and EB states**

For the case of free fermions, the anticommutation algebra makes it possible to express the reduced density matrix as \(\rho_R = e^{-H_E} / (I + e^{-H_E}) = (1 + e^{H_E})^{-1}\), where \(H_E\) is the single-particle entanglement Hamiltonian. In this case, \(H_E = \log(\bar{P} - 1)\) where \(\bar{P} = RPR\) is the (single-particle) correlation matrix. The real-space elements of \(\bar{P}\) constitute the 2-site propagator, as explained in the main text. In particular, it was \(\bar{P}\) which hosts EB states as eigenstates, even though it was the eigenstates of \(\hat{R}\) that enter the Schmidt decomposition of the entanglement cut as well as the reduced density matrix [Eq. S41].

Let us now elucidate the relationship between \(\bar{P} = RPR\) and \(\hat{R} = PRP\). Given an arbitrary eigenstate \(|\phi_\xi\rangle\) of \(\hat{P}\), i.e. \(P|\phi_\xi\rangle = p_\xi |\phi_\xi\rangle\), we have
\[ \hat{R}(P|\phi_\xi\rangle) = P\hat{R}P^2|\phi_\xi\rangle = PPR|\phi_\xi\rangle = (PRP)R|\phi_\xi\rangle = p_\xi R|\phi_\xi\rangle \] (S43)
where we have used \(R|\phi_\xi\rangle = |\phi_\xi\rangle\). In other words, \(|\chi_\xi\rangle = P|\phi_\xi\rangle / \sqrt{\langle \phi_\xi |P|\phi_\xi\rangle} = 1 / \sqrt{p_\xi} P|\phi_\xi\rangle\) is a normalized eigenstate of \(\hat{R}\) with identical eigenvalue \(r_\xi = p_\xi\). Immediately, this means that we can also express the EE in terms of the eigenvalues \(p_\xi\) of \(\hat{P}\) i.e.
\[ S_R = - \sum_\xi [p_\xi \log p_\xi + (1 - p_\xi) \log(1 - p_\xi)] \] (S44)
which is identical to Eq. S42 except that \(r_\xi\) is replaced by \(p_\xi\). However, the reduced density matrix takes on a slightly simpler form when expressed in terms of the eigenstates \(|\phi_\xi\rangle\) of \(\bar{P}\). Substituting \(|\chi_\xi\rangle = 1 / \sqrt{p_\xi} P|\phi_\xi\rangle\) into Eq. S38, we have
\[ |\chi_\xi'\rangle = \frac{1}{\sqrt{r_\xi}} R\bar{P}|\chi_\xi\rangle = \frac{1}{\sqrt{r_\xi}} R\bar{P}|\phi_\xi\rangle = \frac{1}{\sqrt{p_\xi}} RPR|\phi_\xi\rangle = p_\xi \frac{1}{\sqrt{p_\xi}} |\phi_\xi\rangle = |\phi_\xi\rangle , \] (S45)
such that
\[ \rho_R = \bigotimes_\xi (p_\xi |\phi_\xi\rangle \langle \phi_\xi | + (1 - p_\xi) |0\rangle \langle 0|) \] (S46)
which is expressed in terms of the unrotated eigenstates of \(\bar{P}\). In particular, for a system containing an EP with \(B \geq 2\), at least one of the \(|\phi_\xi\rangle\) will be an EB eigenstate \(|\phi\rangle\) with eigenvalue \(p_\xi = p \notin [0, 1]\). In the large \(p\) limit, each EB Schmidt factor \((p_\xi |\phi_\xi\rangle \langle \phi_\xi | + (1 - p_\xi) |0\rangle \langle 0|)\) approximates \(p(|\phi\rangle \langle \phi| - |0\rangle \langle 0|)\), which is the enigmatic consequence of the singular asymmetric propagator.
In this last section, we present a table of various EP lattice Hamiltonians with some of their quantities that are relevant to EB phenomena.

| Model                           | $B$ | $a_0$ | $b_0$ | $\gamma(k)$ | $U_x \sim$                  | $\lambda(L) \sim$ | $S(L) \sim$ |
|---------------------------------|-----|-------|-------|--------------|----------------------------|-------------------|-------------|
| $(2(1 - \cos k) + a_0)\sigma_x + a_0\sigma_+$ | 2   | $a_0 > 0$ | 1     | 0            | $2\sqrt{\frac{2\lambda}{b_0}}\log \frac{L}{x}$ | $\log L$         | $-\frac{3}{2}\log L$ |
| $(2(1 - \cos k) + a_0)^2\sigma_x + a_0\sigma_+$ | 4   | $a_0 > 0$ | 1     | 0            | $\frac{2}{\pi}\sqrt{\frac{2\lambda}{b_0}}(L - 2x)$ | $L$               | $-2\log L$ |
| $(2(1 - \cos k) + a_0)^3\sigma_x + a_0\sigma_+$ | 6   | $a_0 > 0$ | 1     | 0            | $(\frac{4}{\pi})^2\sqrt{\frac{2\lambda}{b_0}}(L^2 - \frac{a^2}{2})$ | $L^2$             | $-4\log L$ |
| $(2(1 - \cos k) + a_0)^6\sigma_x + a_0\sigma_+$ | 12  | $a_0 > 0$ | 1     | 0            | $2\sqrt{\frac{2\lambda}{b_0}}\left(\frac{L^2 - \frac{a^2}{2}}{L^2}\right)^5$ | $L^5$             | $-18\log L$ |
| $(v - w \cos k)\sigma_x + \gamma_0 \sin k \sigma_y + i(v - w)\sigma_z$ | 2   | $2(v - w)$ | $\frac{\gamma_0 \sin k}{2}$ | depends on $\gamma_0$ | depends on $\gamma_0$ | $\approx 0$ | $\frac{1}{3}\log L$ |
| $2(1 - \cos k)\sigma_x + \sin k \sigma_y$ | 2   | 0     | $\frac{\gamma_0 \sin k}{2}$ | $1/x$ | $\approx 0$ | $\frac{1}{3}\log L$ |

**TABLE I.** Various critical lattice models and the forms of their $a_0, b_0$ and $\gamma(k)$. $U_x = -2\langle c^{\dagger}_x, c_{0-}\rangle$ is the propagator from the $-\to +$ sublattice across $x$ unit cells. $\lambda(L)$ is the eigenvalue of $\Lambda = 4(\bar{P}^2 - \bar{P})$ corresponding to the EB eigenstate, whose $\lambda(L)$ depends on $L$ and which can be very large. $S(L)$ is the entanglement entropy scaling behavior at half-filling, comprising contributions from EB states as well as all other eigenstates. The first four models are of the prototypical form introduced in the text, with different powers $B$. The fourth model however has an anomalously large negative logarithmic EE scaling coefficient of $-18$, which arises due to the secondary EB states from its large $B$. They cannot be predicted from $\lambda(L)$, which only pertains to the (primary) EB states. The following (fifth) model is introduced in the final section of the main text and has a nontrivial $\gamma(k)$. It possesses complicated $U_x$ and $\lambda(L)$ dependencies, with “central charge” $c'(\gamma_0)$ given in the inset of Fig 3a. The last model is a Hermitian critical point with no EB states to speak of ($\lambda(L) \approx 0$), and exhibits ordinary $\frac{1}{3}\log L$ EE scaling.