Hom-Lie groups of a class of Hom-Lie algebra

Zhen Xiong
Department of Mathematics and Computer, Yichun University, Jiangxi, 336000, China

Abstract: In this paper, the definition of Hom-Lie groups is given and one connected component of Lie group $GL(V)$, which is not a subgroup of $GL(V)$, is a Hom-Lie group. More, we proved that there is a one-to-one relationship between Hom-Lie groups and Hom-Lie algebras $(\mathfrak{gl}(V), [\cdot, \cdot]_\beta, \text{Ad}_\beta)$. Next, we also proved that if there is a Hom-Lie group homomorphism, then, there is a morphism between their Hom-Lie algebras. Last, as an application, we use these results on Toda lattice equation.

Keyword: Hom-Lie groups; Hom-Lie algebras; homomorphism; Toda hierarchy.

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1 Introduction

The notion of Hom-Lie algebras was introduced by Hartwig, Larsson, and Silvestrov in [1] as part of a study of deformations of the Witt and the Virasoro algebras. Some $q$-deformations of the Witt and the Virasoro algebras have the structure of a Hom-Lie algebra [1, 2]. Because of close relation to discrete and deformed vector fields and differential calculus [1, 3, 4], more people pay special attention to this algebraic structure [5, 6, 7, 8, 11]. Its geometric generalization is given in [9] and [10].

In [7], the authors give a Hom-Lie algebra $(\mathfrak{gl}(V), [\cdot, \cdot]_\beta, \text{Ad}_\beta)$. This Hom-Lie algebra play an important role of studying structures of Hom-Lie algebras. Base on Hom-Lie algebra $(\mathfrak{gl}(V), [\cdot, \cdot]_\beta, \text{Ad}_\beta)$, there are many results are given: Omni-Hom-Lie algebra is given in [7]; Hom-big brackets are given in [8]; a Hom-Lie algebroid structure on $\varphi^\ast TM$ is given in [9], and have the following results: there is a purely Hom-Poisson algebra structure on $C^\infty(M)$; a Hom-Lie algebra structure on the set of $(\sigma, \sigma)$- derivations of an associative algebra is given in [11], and so on.

As we know, from a Lie group, we get a Lie algebra; on the other hands, from a Lie algebra, we can have a Lie group. Hence, are there Hom-Lie groups, have similar results like Lie groups? In this paper, first, we study some properties of Hom-Lie algebra $(\mathfrak{gl}(V), [\cdot, \cdot]_\beta, \text{Ad}_\beta)$, then, give the definition of Hom-Lie groups, similar to Hom-Lie algebras, a Hom-Lie group twisted by a diffeomorphism is a Lie group. Next, on matrix space, we give some examples of Hom-Lie groups, and proved that one connected component of $GL(V)$, which is not a subgroup, is a Hom-Lie group. More, we proved that there is a one-to-one relationship a Hom-Lie group and a Hom-Lie algebra $(\mathfrak{gl}(V), [\cdot, \cdot]_\beta, \text{Ad}_\beta)$. Then, we give definitions of homomorphism on Hom-Lie groups, and
proved that a homomorphism of Hom-Lie groups induce a morphism of Hom-Lie algebras. As an application, we study a deformation of Toda lattice equation.

The paper is organized as follow. In Section 2, we recall some necessary background knowledge, including Hom-Lie algebras and morphism, Hom-Lie algebra $\mathfrak{gl}(V), [\cdot, \cdot]_\beta, \text{Ad}_\beta$). In Section 3, we study cohomology of Hom-Lie algebra $\mathfrak{gl}(V), [\cdot, \cdot]_\beta, \text{Ad}_\beta$) and have: cohomologies of Hom-Lie algebra $\mathfrak{gl}(V), [\cdot, \cdot]_\beta, \text{Ad}_\beta$ and Lie algebra $\mathfrak{gl}(V)$ are isomorphic. In Section 4, we give definitions of Hom-Lie groups, homomorphism of Hom-Lie groups, and have mainly results: Theorem [4,7] and Theorem [4,13]. In Section 5, for a class of deformations of Toda lattice equation, it is integrable.

2 Preliminaries

The notion of a Hom-Lie algebra was introduced in [1], see also [6] for more information.

**Definition 2.1.** (1.) A Hom-Lie algebra is a triple $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ consisting of a vector space $\mathfrak{g}$, a skew-symmetric bilinear map (bracket) $[\cdot, \cdot] : \wedge^2 \mathfrak{g} \rightarrow \mathfrak{g}$ and a linear transformation $\alpha : \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying $\alpha(x, y) = [\alpha(x), \alpha(y)]$, and the following hom-Jacobi identity:

$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0, \quad \forall x, y, z \in \mathfrak{g}.$$  

A Hom-Lie algebra is called a regular Hom-Lie algebra if $\alpha$ is a linear automorphism. When $\alpha = \text{id}$, Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ is just Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$.

(2.) A subspace $\mathfrak{h} \subset \mathfrak{g}$ is a Hom-Lie sub-algebra of $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ if $\alpha(\mathfrak{h}) \subset \mathfrak{h}$ and $\mathfrak{h}$ is closed under the bracket operation $[\cdot, \cdot]$, i.e. for all $x, y \in \mathfrak{h}$, $[x, y] \in \mathfrak{h}$.

(3.) A morphism from the Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ to the Hom-Lie algebra $(\mathfrak{h}, [\cdot, \cdot], \delta)$ is a linear map $\psi : \mathfrak{g} \rightarrow \mathfrak{h}$ such that $\psi([x, y]) = [\psi(x), \psi(y)]$ and $\psi \circ \alpha = \delta \circ \psi$. When $\psi$ is invertible, then $\psi$ is an isomorphism.

The set of $k$-cochains on Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ with values in $V$, which we denote by $C^k(\mathfrak{g}; V)$, is the set of skewsymmetric $k$-linear maps from $\mathfrak{g} \times \cdots \times \mathfrak{g}(k$-times) to $V$:

$$C^k(\mathfrak{g}; V) := \{ \eta : \wedge^k \mathfrak{g} \rightarrow V \text{ is a linear map} \}.$$  

Associated to the trivial representation, the set of $k$-cochains is $\wedge^k \mathfrak{g}^*$. The corresponding coboundary operator $d : \wedge^k \mathfrak{g}^* \rightarrow \wedge^{k+1} \mathfrak{g}^*$ is given by (see [7,12])

$$d\xi(x_1, \cdots, x_{k+1}) = \sum_{i<j} (-1)^{i+j} \xi([x_i, x_j], \alpha(x_1), \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, \alpha(x_{k+1})).$$

**Theorem 2.2.** [7] Let $V$ be a vector space, and $\beta \in \text{GL}(V)$. Define a skew-symmetric bilinear bracket operation $[\cdot, \cdot] : \text{gl}(V) \times \text{gl}(V) \rightarrow \text{gl}(V)$ by

$$[A, B]_\beta = \beta A \beta^{-1} B \beta^{-1} - \beta B \beta^{-1} A \beta^{-1}, \quad \forall A, B \in \text{gl}(V),$$

where $\beta^{-1}$ is the inverse of $\beta$. Denote by $\text{Ad}_{\beta} : \text{gl}(V) \rightarrow \text{gl}(V)$ the adjoint action on $\text{gl}(V)$, i.e. $\text{Ad}_{\beta}(A) = \beta A \beta^{-1}$. Then $(\text{gl}(V), [\cdot, \cdot]_\beta, \text{Ad}_\beta)$ is a regular Hom-Lie algebra.

Obviously, when $\beta = \text{id}$, $(\text{gl}(V), [\cdot, \cdot]_\beta, \text{Ad}_\beta)$ is just Lie algebra $(\text{gl}(V), [\cdot, \cdot])$. 

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3 Properties of Hom-Lie algebra \((\mathfrak{gl}(V), [\cdot, \cdot]_\beta, \text{Ad}_\beta)\)

In this paper, we just study \(\beta \circ \beta = \text{id}\), i.e. \(\beta^{-1} = \beta\).

**Proposition 3.1.** For any \(C \in \mathbb{GL}(V)\), let \(\gamma = C\beta C^{-1}\), then there is an isomorphism from \((\mathfrak{gl}(V), [\cdot, \cdot]_\beta, \text{Ad}_\beta)\) to \((\mathfrak{gl}(V), [\cdot, \cdot]_\gamma, \text{Ad}_\gamma)\).

**Proof.** Define map \(F: (\mathfrak{gl}(V), [\cdot, \cdot]_\beta, \text{Ad}_\beta) \rightarrow (\mathfrak{gl}(V), [\cdot, \cdot]_\gamma, \text{Ad}_\gamma)\) by \(F(x) = CxC^{-1}\), then \(F\) is an isomorphism. \(\blacksquare\)

**Remark 3.2.** When \(\beta \neq \pm \text{id}\), there is not an isomorphism from \((\mathfrak{gl}(V), [\cdot, \cdot]_\beta, \text{Ad}_\beta)\) to \((\mathfrak{gl}(V), [\cdot, \cdot]_\beta, \text{Ad}_\beta)\).

For Hom-Lie algebra \((\mathfrak{gl}(V), [\cdot, \cdot]_\beta, \text{Ad}_\beta)\), coboundary operator \(d: \wedge^k \mathfrak{gl}(V)^* \rightarrow \wedge^{k+1} \mathfrak{gl}(V)^*\) is given by

\[
d\xi(x_1, \cdots, x_{k+1}) = \sum_{i<j} (-1)^{i+j} \xi([x_i, x_j]_\beta, \text{Ad}_\beta(x_1), \cdots, \overline{x_{i,j}}, \cdots, \text{Ad}_\beta(x_{k+1})).
\]

For Lie algebra \((\mathfrak{gl}(V), [\cdot, \cdot]_\beta)\), coboundary operator \(\hat{d}: \wedge^k \mathfrak{gl}(V)^* \rightarrow \wedge^{k+1} \mathfrak{gl}(V)^*\) is given by

\[
\hat{d}\xi(x_1, \cdots, x_{k+1}) = \sum_{i<j} (-1)^{i+j} \xi([x_i, x_j], x_1, \cdots, \overline{x_{i,j}}, \cdots, x_{k+1}).
\]

Then, we have two cohomology complexes: \((\oplus_k \wedge^k \mathfrak{gl}(V)^*, d)\) of Hom-Lie algebra \((\mathfrak{gl}(V), [\cdot, \cdot]_\beta, \text{Ad}_\beta)\) and \((\oplus_k \wedge^k \mathfrak{gl}(V)^*\), \(\hat{d})\) of Lie algebra \((\mathfrak{gl}(V), [\cdot, \cdot]_\beta)\). \(\beta\) induce map \(\beta^r: \wedge^k \mathfrak{gl}(V)^* \rightarrow \wedge^k \mathfrak{gl}(V)^*\) by

\[
\beta^r(\xi)(x_1, \cdots, x_k) = \xi(x_1 \beta, \cdots, x_k \beta).
\]

And \(\beta\) also induce map \(\beta^l: \wedge^k \mathfrak{gl}(V)^* \rightarrow \wedge^k \mathfrak{gl}(V)^*\) by

\[
\beta^l(\xi)(x_1, \cdots, x_k) = \xi(\beta x_1, \cdots, \beta x_k).
\]

**Proposition 3.3.** For \(\xi \in \wedge^k \mathfrak{gl}(V)^*\), we have:

\[
\begin{align*}
\beta^r \circ d\xi &= \hat{d}\beta^l(\xi); \\
\hat{d}\beta^r(\xi) &= \beta^l \circ \hat{d}\xi.
\end{align*}
\]

**Proof.** For \(\xi \in \wedge^k \mathfrak{gl}(V)^*\),

\[
\begin{align*}
\beta^r \circ d\xi(x_1, \cdots, x_{k+1}) &= d\xi(x_1 \beta, \cdots, x_{k+1} \beta) \\
&= \sum_{i<j} (-1)^{i+j} \xi([x_i, x_j - \beta x_j x_i, \beta x_1, \cdots, \overline{x_{i,j}}, \cdots, \beta x_{k+1}]) \\
&= \sum_{i<j} (-1)^{i+j} \beta^l(\xi)(x_i x_j - x_j x_i, x_1, \cdots, \overline{x_{i,j}}, \cdots, x_{k+1}) \\
&= \hat{d}\beta^l(\xi)(x_1, \cdots, x_{k+1})
\end{align*}
\]

The proof of the rest of the conclusions is similar. \(\blacksquare\)

**Remark 3.4.** In fact, we also have: \(\beta^r \circ \hat{d} \neq \hat{d} \circ \beta^l\), \(\beta^r \circ d \neq d \circ \beta^r\).
Denote the set of closed $k$-cochains of complex $(\oplus_k \wedge^k \mathfrak{gl}(V)^*, d)$ by $Z^k(\mathcal{H}L; \mathbb{R})$ and the set of exact $k$-cochains of complex $(\oplus_k \wedge^k \mathfrak{gl}(V)^*, d)$ by $B^k(\mathcal{H}L; \mathbb{R})$. Denote the corresponding cohomology by

$$H^k(\mathcal{H}L; \mathbb{R}) = Z^k(\mathcal{H}L; \mathbb{R})/B^k(\mathcal{H}L; \mathbb{R}).$$

Similarly, denote the set of closed $k$-cochains of complex $(\oplus_k \wedge^k \mathfrak{gl}(V)^*, \hat{d})$ by $Z^k(L; \mathbb{R})$ and the set of exact $k$-cochains of complex $(\oplus_k \wedge^k \mathfrak{gl}(V)^*, \hat{d})$ by $B^k(L; \mathbb{R})$. Denote the corresponding cohomology by

$$H^k(L; \mathbb{R}) = Z^k(L; \mathbb{R})/B^k(L; \mathbb{R}).$$

**Theorem 3.5.** With the above notations, we have:

$$H^k(\mathcal{H}L; \mathbb{R}) = H^k(L; \mathbb{R}).$$

**Proof.** For $\xi_1 \in Z^k(\mathcal{H}L; \mathbb{R})$, by $0 = \beta^r \circ d\xi_1 = \hat{d}\beta^l(\xi_1)$, we have: $\beta^l(\xi_1) \in Z^k(L; \mathbb{R})$. For $\xi_2 \in Z^k(L; \mathbb{R})$, by $\beta^r \circ d\beta^l(\xi_2) = d\xi_2 = 0$, we have: $\beta^l(\xi_2) \in Z^k(\mathcal{H}L; \mathbb{R})$. So, we have: $Z^k(\mathcal{H}L; \mathbb{R}) = Z^k(L; \mathbb{R})$.

For $\eta_1 \in B^k(\mathcal{H}L; \mathbb{R})$, there is a $\eta_2 \in \wedge^{k-1} \mathfrak{gl}(V)^*$, and such that $d\eta_2 = \eta_1$. By $\hat{d}\beta^l(\eta_2) = \beta^r \circ d\eta_2 = \beta^r(\eta_1)$, we have: $\beta^r(\eta_1) \in B^k(L; \mathbb{R})$. On the other hand, for $\eta \in B^k(L; \mathbb{R})$, there is a $\eta_3 \in \wedge^{k-1} \mathfrak{gl}(V)^*$, and such that $d\eta_3 = \eta$. By $d\beta^r(\eta_3) = \beta^l d\eta_3 = \beta^l(\eta)$, then, $\beta^l(\eta) \in B^k(\mathcal{H}L; \mathbb{R})$. So, we have: $B^k(\mathcal{H}L; \mathbb{R}) = B^k(L; \mathbb{R})$.}$

4 Hom-Lie groups

**Definition 4.1.** Let $G$ is a Lie group, $S$ is a submanifold of $G$. If there is a diffeomorphism $F : G \to G$, such that $F(S)$ is a subgroup of $G$. Then $(S, F)$ is called a Hom-Lie group. For $\forall x, y \in S$, if $F(y)$ is the inverse of $F(x)$ in Lie group $F(S)$, then we called $y$ is the Hom-inverse of $x$ in Hom-Lie group $(S, F)$. When $F = \text{id}$, Hom-Lie group $(S, F)$ is the Lie group $S$.

**Example 4.2.** Matrix Lie group $\text{GL}(n; \mathbb{R})$, let $S_1 = \{ A \in \text{GL}(n; \mathbb{R}) \mid |A| < 0 \}$, $S_2 = \{ A \in \text{GL}(n; \mathbb{R}) \mid |A| > 0 \}$, $S_1$ and $S_2$ are connected components of $\text{GL}(n; \mathbb{R})$, $S_2$ is a subgroup of $\text{GL}(n; \mathbb{R})$ and $S_1$ is not a subgroup of $\text{GL}(n; \mathbb{R})$. We define the map $F : \text{GL}(n; \mathbb{R}) \to \text{GL}(n; \mathbb{R})$ by $F(A) = AP_{i,j}$, where $P_{i,j}$ is given by exchanging line $i$ and line $j$ of id, $P^{2}_{i,j} = \text{id}$. $F$ is a diffeomorphism and $F^2 = \text{id}$. So $(S_1, F)$ is a Hom-Lie group. $\forall A \in S_1$, $P_{i,j}A^{-1}P_{i,j}$ is the Hom-inverse of $A$.

**Example 4.3.** Matrix Lie group $O(n)$, let $S_3 = \{ A \in O(n) \mid |A| = -1 \}$, $S_4 = \{ A \in O(n) \mid |A| = 1 \}$. $S_3$ and $S_4$ are connected components of $O(n)$, $S_4$ is a subgroup of $O(n)$ and $S_3$ is not a subgroup of $O(n)$. We define the map $F : O(n) \to O(n)$ by $F(A) = AP_{i,j}$, where $P_{i,j}$ is given by exchanging line $i$ and line $j$ of id, $P^{2}_{i,j} = \text{id}$. $F$ is a diffeomorphism and $F^2 = \text{id}$. So $(S_3, F)$ is a Hom-Lie group. $\forall A \in S_1$, $P_{i,j}A^{-1}P_{i,j}$ is the Hom-inverse of $A$.

**Definition 4.4.** A function $p : \mathbb{R} \to (S, F)$ is called a one-parameter subgroup of Hom-Lie group $(S, F)$ if

1. $p$ is continuous,
2. $F(p(0)) = \text{id}$,
3. $F(p(t + s)) = F(p(t))F(p(s))$. 


Example 4.5. For matrix Lie group $O(1;1)$, let

$$S_1 = \{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} | t \in \mathbb{R} \}; \quad S_2 = \{ \begin{pmatrix} -\cosh t & \sinh t \\ \sinh t & -\cosh t \end{pmatrix} | t \in \mathbb{R} \};$$

$$S_3 = \{ \begin{pmatrix} \cosh t & -\sinh t \\ -\sinh t & \cosh t \end{pmatrix} | t \in \mathbb{R} \}; \quad S_4 = \{ \begin{pmatrix} -\cosh t & -\sinh t \\ -\sinh t & \cosh t \end{pmatrix} | t \in \mathbb{R} \}.$$ 

$S_1, S_2, S_3, S_4$ are connected components of $O(1;1)$. $S_2, S_3, S_4$ are not subgroups of $O(1;1)$. $S_1$ is a subgroup of $O(1;1)$, hence $S_1$ is a Lie group.

We define the map $F_2 : O(1;1) \rightarrow O(1;1)$ by $F_2(A) = A(-\text{id})$, then $(S_2, F_2)$ is a Hom-Lie group and $F_2^2 = \text{id}$.

The map $F_3 : O(1;1) \rightarrow O(1;1)$ is given by $F_3(A) = A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then $(S_3, F_3)$ is a Hom-Lie group and $F_3^2 = \text{id}$.

We define the map $F_4 : O(1;1) \rightarrow O(1;1)$ by $F_4(A) = A \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, then $(S_4, F_4)$ is a Hom-Lie group and $F_4^2 = \text{id}$. Element of $S_i$ is a one-parameter subgroup of $(S_i, F_i), i = 2, 3, 4$.

$GL(V)$ is a matrix Lie group, $(\mathfrak{gl}(V), [\cdot, \cdot])$ its Lie algebra, there is a map $\exp : \mathfrak{gl}(V) \rightarrow GL(V)$, for any $X \in \mathfrak{gl}(V)$, $\exp(X) = e^X$. More about the map $\exp$, please see [13].

For $\beta \in GL(V)$, let

$$M_{\beta} = \{ e^{\beta X} | X \in \mathfrak{gl}(V) \},$$

then $M_{\beta} \subset GL(V)$, for $X \in \mathfrak{gl}(V)$, let $p(t) = e^{t\beta X}$, then $p(t) \subset M_{\beta}$.

Proposition 4.6. For $\beta \in GL(V)$, $\beta^2 = \text{id}$, let $M_{\beta} = \{ e^{\beta X} | X \in \mathfrak{gl}(V) \}$, then $(M_{\beta}, R_{\beta})$ is a Hom-Lie group.

Proof. When $\beta = \text{id}$, $M_{\beta} = GL(V)$ is a Lie group.

When $\beta \neq \text{id}$, if $X_0 \in \mathfrak{gl}(V)$ and such that $e^{\beta X_0} = \beta$, we have $e^{\beta X_0} = \beta$, and $e^{2\beta X_0} = \beta^2 = \text{id}$, then $X_0 = 0$, we have $\beta = \text{id}$, but $\beta \neq \text{id}$. So, $M_{\beta}$ is not a subgroup of $GL(V)$.

By $p(t) \subset M_{\beta} \subset \bigcup_{X \in \mathfrak{gl}(V)} p(t)$, we have $M_{\beta} = \bigcup_{X \in \mathfrak{gl}(V)} p(t)$. Because of $\beta \in \bigcap p(t)$, $M_{\beta}$ is a connected component of $GL(V)$. Then $M_{\beta}$ is a submanifold of $GL(V)$.

We define the map $R_{\beta} : GL(V) \rightarrow GL(V)$ by $R_{\beta}(A) = A\beta$, $R_{\beta}$ is a diffeomorphism. Then $R_{\beta}(M_{\beta}) = \{ e^{\beta X} | X \in \mathfrak{gl}(V) \}$. So$(M_{\beta}, R_{\beta})$ is a Hom-Lie group, and Hom-inverse of $e^{\beta X} \beta$ is $e^{-\beta X} \beta$.

For $X \in \mathfrak{gl}(V)$, let $p(t) = e^{t\beta X}$, then $p(t)$ is a one-parameter subgroup of Hom-Lie group $(M_{\beta}, R_{\beta})$. For $Y \in \mathfrak{gl}(V)$, we have

$$\frac{d}{dt} (e^{t\beta X}Ye^{-t\beta X})_{t=0} = \beta X \beta Y e^0 - e^0 \beta Y \beta X \beta = \beta X \beta Y - \beta Y \beta X \beta = [X, Y]_{\beta}.$$ 

Actually, we proved the following results.

Theorem 4.7. For $\beta \in GL(V)$, $\beta^2 = \text{id}$, there is a one-to-one relationship between Hom-Lie group $(M_{\beta}, R_{\beta})$ and regular Hom-Lie algebra $(\mathfrak{gl}(V), [\cdot, \cdot], \text{Ad}_{\beta})$. When $\beta = \text{id}$, Hom-Lie group $(M_{\beta}, R_{\beta})$ is the Lie group $GL(V)$ and Hom-Lie algebra $(\mathfrak{gl}(V), [\cdot, \cdot], \text{Ad}_{\beta})$ is the Lie algebra $\mathfrak{gl}(V)$. 

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Example 4.8. Let $M$ be a manifold and $\varphi : M \longrightarrow M$ a diffeomorphism. Define $\text{Ad}_{\varphi^*} : \Gamma(\varphi^*TM) \longrightarrow \Gamma(\varphi^*TM)$ by

$$\text{Ad}_{\varphi^*}(X) = \varphi^* \circ X \circ (\varphi^*)^{-1}, \forall X \in \Gamma(\varphi^*TM).$$

Define a skew-symmetric bilinear operation $[\cdot, \cdot]_{\varphi^*} : \wedge^2\Gamma(\varphi^*TM) \longrightarrow \Gamma(\varphi^*TM)$ by

$$[X, Y]_{\varphi^*} = \varphi^* \circ X \circ (\varphi^*)^{-1} \circ Y \circ (\varphi^*)^{-1} - \varphi^* \circ Y \circ (\varphi^*)^{-1} \circ X \circ (\varphi^*)^{-1}.$$

Then, $(\varphi^*TM, [\cdot, \cdot]_{\varphi^*}, \text{Ad}_{\varphi^*}, \text{id})$ is a Hom-Lie algebroid and is called tangent Hom-Lie algebroid ([9]).

In this example, now, we suppose $\varphi^2 = \text{id}$ and $\varphi \neq \text{id}$, at a point $m \in M$, $x$ is a vector field on $M$, if $\theta_t(m)$ is a one parameter Lie subgroup with respect to $x_m$, i.e., for any $f \in C^\infty(M)$,

$$df = x_m f = \lim_{t \to 0} \frac{1}{t} [f(\theta_t(m)) - f(m)].$$

For tangent Hom-Lie algebroid, $X_m \in \Gamma(\varphi^*TM)$ and $X_m = x_{\varphi(m)}$, by representation of Hom-Lie algebroids on $C^\infty(M)$ ([7]), for any $f \in C^\infty(M)$

$$d^2f(X_m) = (\varphi^*)^2 \circ X_m \circ \varphi^*(f) = X_m \circ (f \circ \varphi) = x_{\varphi(m)}(f \circ \varphi) = \lim_{t \to 0} \frac{f(\varphi \circ \theta_t(\varphi(m))) - f(m)}{t}.$$

Hence, $X_m$ with respect to one parameter Hom-Lie subgroup is $\varphi \circ \theta_t(\varphi(m))$.

On the other hand, $x_m \in T_mM$,

$$\frac{d}{dt} \varphi^* \circ X_m \circ \varphi^* \circ f|_{t=0} = \varphi^* \circ X_m \circ \varphi^* \circ f = x_{\varphi(m)}(f \circ \varphi).$$

So, we got the same result through different paths.

Actually, we can get the following results:

Theorem 4.9. Let $M$ be a manifold, $\varphi : M \longrightarrow M$ is a diffeomorphism and $\varphi^2 = \text{id}$, $\varphi \neq \text{id}$. If $x$ is a vector field on $M$, for any $m \in M$, then

$$D_m\varphi(x_m) \neq x_{\varphi(m)}.$$
Definition 4.10. Let $G_1$ and $G_2$ are Lie groups, $S_i$ is a submanifold of $G_i$, $i = 1, 2$. $(S_1, F_1)$ and $(S_2, F_2)$ are Hom-Lie groups. $\Phi : G_1 \to G_2$ is a Lie group homomorphism. We called $\Phi$ a Hom-Lie group homomorphism from $(S_1, F_1)$ to $(S_2, F_2)$, if

1.) $\Phi(S_1) \subset S_2$;

2.) $\forall x, y \in S_1$, $F_2 \circ \Phi(xy) = \Phi \circ F_1(xy)$.

Example 4.11. In Example 4.2 and Example 4.3, we define map $\mathbb{I} : O(n) \to GL(n; \mathbb{R})$ by $\mathbb{I}(A) = A$, i.e., $\mathbb{I}$ is an inclusion. $\mathbb{I}$ is a homomorphism from $(S_3, F)$ to $(S_1, F)$.

Example 4.12. The map $g : GL(n; \mathbb{R}) \to \mathbb{R}$ is given by $g(A) = |A|$, then $g$ is a Lie group homomorphism. In Example 4.2, $(S_1, F)$ is a Hom-Lie group, let $S_2 = \{g \in \mathbb{R} | y < 0\}$ and define the map $F_2 : \mathbb{R} \to \mathbb{R}$ by $F_2(y) = -y$, then $(S_2, F_2)$ is a Hom-Lie group. We have: $g(S_1) \subset S_2$ and $\forall A, B \in S_1, g \circ F(AB) = g(ABP_{i,j}) = |A||B||P_{i,j}| = -|AB| = F_2 \circ g(AB)$, so $g$ is a Hom-Lie group homomorphism.

Theorem 4.13. Let $\gamma \neq \beta$, $\gamma \neq id$, $\beta \neq id$, $\gamma^2 = id = \beta^2$, $(M_{\gamma}, R_{\gamma})$ and $(M_{\beta}, R_{\beta})$ are Hom-Lie groups, with Hom-Lie algebras $(gl(V), [\cdot, \cdot], \text{Ad}_{\gamma})$ and $(gl(V), [\cdot, \cdot], \text{Ad}_{\beta})$, respectively. Suppose that $\Phi : (M_{\gamma}, R_{\gamma}) \to (M_{\beta}, R_{\beta})$ is a Hom-Lie group homomorphism. Then, there exists a real linear map $\phi : (gl(V), [\cdot, \cdot], \text{Ad}_{\gamma}) \to (gl(V), [\cdot, \cdot], \text{Ad}_{\beta})$ such that

$$\Phi(e^{\gamma X}) = e^{\beta \phi(X)}$$

for all $X \in gl(V)$. The map $\phi$ has following additional properties:

1.) $\phi \circ \text{Ad}_{\gamma}(X) = \text{Ad}_{\beta} \circ \phi(X)$, for all $X \in gl(V)$;

2.) $\phi(e^{\gamma XY}e^{-\gamma X}) = \Phi(e^{\gamma X})\phi(Y)\Phi(e^{-\gamma X})$, for all $X, Y \in gl(V)$;

3.) $\phi([X,Y]) = [\phi(X),\phi(Y)]_{\beta}$, for all $X, Y \in gl(V)$;

4.) $\beta \phi(X) = \frac{d}{dt}\Phi(e^{\gamma X})|_{t=0}$, for all $X \in gl(V)$.

Proof. Since $\Phi$ is a Lie group homomorphism, then $R_{\beta} \circ \Phi(e^{\gamma X}) = \Phi \circ R_{\gamma}(e^{\gamma X})$, we have $\Phi(\gamma) = \beta$. And $\Phi$ is a continuous group homomorphism, for each $X \in gl(V)$, so $\Phi(e^{\gamma X})$ will be a one-parameter subgroup of Lie group $GL(V)$. Thus, there is a unique $Z \in gl(V)$ such that

$$\Phi(e^{\gamma X}) = e^{\beta Z}.$$  

(4)

We define $\phi(X) = Z$ and check in several steps that $\phi$ has the required properties.

Step 1, $\Phi(e^{\gamma X}) = e^{\beta \phi(X)}$.
This follows from (4) and our definition of $\phi$, by putting $t = 1$.

Step 2, $\phi(sX) = s\phi(X)$, for all $s \in \mathbb{R}$.
This is obviously. Step 3, $\phi(X + Y) = \phi(X) + \phi(Y)$.
By Steps 1 and 2,

$$e^{\beta \phi(X+Y)} = e^{\beta \phi(t(X+Y))} = \Phi(e^{\gamma (X+Y)}).$$

By the Lie product formula and the fact that $\Phi$ is a continuous homomorphism, we have

$$e^{\beta \phi(X+Y)} = \Phi(\lim_{m \to \infty} (e^{\gamma X}me^{\gamma Y}/m)^m) = \lim_{m \to \infty} \left(\Phi(e^{\gamma X}/m)\Phi(e^{\gamma Y}/m)\right)^m.$$
Then, we have
\[ e^{t\beta\phi(X + Y)} = \lim_{m \to \infty} \left( e^{t\beta\phi(X)/m} e^{t\beta\phi(Y)/m} \right)^m = e^{t\beta(\phi(X) + \phi(Y))}. \]

Differentiating this result at \( t = 0 \) gives the desired result.

Step 4, \( \phi \circ \text{Ad}_\gamma(X) = \text{Ad}_\beta \circ \phi(X) \).

By Step 1,
\[ \Phi(e^{\gamma \text{Ad}_\gamma(X)} e^{\gamma}) = \Phi(e^{\gamma \text{Ad}_\gamma(X)}) \Phi(\gamma) = e^{\beta \phi(\text{Ad}_\gamma(X)) \beta}. \] (5)

And \( e^{\gamma \text{Ad}_\gamma(X)} e^{\gamma} = e^X e^{\gamma} = e^{X\gamma} \), then
\[ \Phi(e^{\gamma \text{Ad}_\gamma(X)} e^{\gamma}) = \Phi(e^{X\gamma}) = \beta \Phi(e^{X}) = \beta e^{\beta \phi(X)}. \] (6)

By (5) and (6), we have:
\[ e^{\beta \phi(\text{Ad}_\gamma(X))} = \beta e^{\beta \phi(X) \beta} = e^{\phi(X) \beta}. \]

Hence, we have
\[ \phi(e^{\gamma X\gamma} Y e^{-\gamma X\gamma}) = \Phi(e^{\gamma X\gamma}) \phi(Y) \Phi(e^{-\gamma X\gamma}). \]

By putting \( t = 1 \), we have the desired result.

Step 6, \( \phi([X, Y]_\gamma) = [\phi(X), \phi(Y)]_\beta \).

By Step 5, we have
\[ \beta \phi(e^{t\gamma X\gamma} Y e^{-t\gamma X\gamma}) = \beta \phi(e^{t\gamma X\gamma}) \beta \phi(Y) e^{-t\gamma X\gamma} \beta \]
\[ = e^{t\beta \phi(X) \beta} \beta \phi(Y) e^{-t\beta \phi(X) \beta}. \]

On the other hands,
\[ \phi([X, Y]_\gamma) = \phi \left( \frac{d}{dt} e^{t\gamma X\gamma} Y e^{-t\gamma X\gamma} \big|_{t=0} \right) \]
\[ = \frac{d}{dt} \phi(e^{t\gamma X\gamma} Y e^{-t\gamma X\gamma}) \big|_{t=0} \]
\[ = \frac{d}{dt} e^{t\beta \phi(X) \beta} \beta \phi(Y) e^{-t\beta \phi(X) \beta} \big|_{t=0} \]
\[ = [\phi(X), \phi(Y)]_\beta. \]
Step 6. $\beta \phi(X) = \frac{d}{dt} \phi(e^\gamma X)|_{t=0}$.
This follows our definition of $\phi$. □

5 Applications

Consider the following equation

$$\frac{d}{dt}L = BL - LB = [B, L],$$

(7)

where $L$ is an $n \times n$ symmetric real tridiagonal matrix, and $B$ is the skew symmetric matrix obtained from $L$ by

$$B = L_{>0} - L_{<0},$$

where $L_{>0(<0)}$ denotes the strictly upper (lower) triangular part of $L$. Based on Lie algebras, Kodama, Y. and Ye, J. ([15, 16]) study equation (7), and give an explicit formula for the solution to the initial value problem. Now we consider the following system:

$$\frac{d}{dt}L = \beta B L \beta - \beta L B \beta = [B, L]_{\beta},$$

(8)

where $\beta^2 = \text{id}$, then, from results of Section 4, we know that equation (8) is also integrable.

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