On the Parallel Complexity of Group Isomorphism via Weisfeiler–Leman∗

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Abstract

In this paper, we show that the constant-dimensional Weisfeiler–Leman algorithm for groups (Brachter & Schweitzer, LICS 2020) can be fruitfully used to improve parallel complexity upper bounds on isomorphism testing for several families of groups. In particular, we show:

• Groups with an Abelian normal Hall subgroup whose complement is $O(1)$-generated are identified by constant-dimensional Weisfeiler–Leman using only a constant number of rounds. This places isomorphism testing for this family of groups into $L$; the previous upper bound for isomorphism testing was $P$ (Qiao, Sarma, & Tang, STACS 2011).

• We use the individualize-and-refine paradigm to obtain a quasi $\text{SAC}^1$ isomorphism test for groups without Abelian normal subgroups, previously only known to be in $P$ (Babai, Codenotti, & Qiao, ICALP 2012).

• We extend a result of Brachter & Schweitzer (ESA, 2022) on direct products of groups to the parallel setting. Namely, we also show that Weisfeiler–Leman can identify direct products in parallel, provided it can identify each of the indecomposable direct factors in parallel. They previously showed the analogous result for $P$.

We finally consider the count-free Weisfeiler–Leman algorithm, where we show that count-free WL is unable to even distinguish Abelian groups in polynomial-time. Nonetheless, we use count-free WL in tandem with bounded non-determinism and limited counting to obtain a new upper bound of $\beta_1(M\text{AC}^0(\text{FOLL})$ for isomorphism testing of Abelian groups. This improves upon the previous $\text{TC}^0(\text{FOLL})$ upper bound due to Chattopadhyay, Torán, & Wagner (ACM Trans. Comput. Theory, 2013).

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1 Introduction

The Group Isomorphism problem (GPI) takes as input two finite groups $G$ and $H$, and asks if there exists an isomorphism $\varphi : G \to H$. When the groups are given by their multiplication tables, it is known that $\text{GPI}$ belongs to $\text{NP} \cap \text{coAM}$. The generator-enumerator algorithm, attributed to Tarjan in 1978 [Mil78], has time complexity $n \log p + O(1)$, where $n$ is the order of the group and $p$ is the smallest prime dividing $n$. In more than 40 years, this bound has escaped largely unscathed: Rosenbaum [Ros13] (see [LGR16, Sec. 2.2]) improved this to $n(\log n)^{O(1)}$. And even the impressive body of work on practical algorithms for this problem, led by Eick, Holt, Leedham-Green and O’Brien (e.g., [BEO02, ELGO02, BE99, CH03]) still results in an $n^6$-time algorithm in the general case (see [Wil19, Page 2]).

In the past several years, there have been significant advances on algorithms with worst-case guarantees. Babai’s breakthrough result that $\text{GI}$ reduces to $\text{AC}^0$-intermediate problems, that is, problems that belong to $\text{NP}$, but are neither in $\text{P}$ nor $\text{NP}$-complete [Lad75]. There is considerable evidence suggesting that $\text{GI}$ is not $\text{NP}$-complete [Bab16]. As a consequence, we also make advances in the descriptive complexity theory of finite groups.

Despite GPI in the Cayley table model being strictly easier than GI under $\text{AC}^0$-reductions, there are several key approaches in the GI literature such as parallelization and individualization that have received comparatively little attention in the setting of GPI—see the discussion of Related Work on Page 5. In this paper, using Weisfeiler–Leman for groups [BS20] as our main tool, we begin to bring both of these techniques to bear on GPI. As a consequence, we also make advances in the descriptive complexity theory of finite groups.

Main Results. In this paper, we show that Weisfeiler–Leman serves as a key subroutine in developing efficient parallel isomorphism tests.

Brachter & Schweitzer [BS20] actually introduced three different versions of WL for groups. While they are equivalent in terms of pebble complexity up to constant factors, their round complexity may differ by at most an additive $O(\log n)$ (see Thm. 2.7), and their parallel complexities differ (see Section 2.4). Because of these differences we are careful to specify which version of WL for groups each result uses.

As we are interested in both the Weisfeiler–Leman dimension and the number of rounds, we introduce the following notation.

Definition 1.1. Let $k \geq 2$ and $r \geq 1$ be integers, and let $J \in \{I, II, III\}$. The $(k, r)$-WL Version $J$ algorithm for groups is obtained by running $k$-WL Version $J$ for $r$ rounds. Here, the initial coloring counts as the first round.

We first examine coprime extensions of the form $H \times N$ where $N$ is Abelian. When either $H$ is elementary Abelian or $H$ is $O(1)$-generated, Qiao, Sarma, & Tang [QST11] gave a polynomial-time isomorphism test for these families of groups, using some nontrivial representation theory. Here, as a proof of concept that WL can successfully use and parallelize some representation theory (which was not yet considered in [BS20, BS22]), we use WL to improve their result’s parallel complexity in the case that $H$ is $O(1)$-generated. We remark
below about the difficulties in extending WL to handle the case that $H$ is Abelian (without restricting the number of generators).

**Theorem 1.2.** Groups of the form $H \times N$, where $N$ is Abelian, $H$ is $O(1)$-generated, and $|H|$ and $|N|$ are coprime are identified by $(O(1), O(1))$-WL Version II. Consequently, isomorphism between a group of the above form and arbitrary groups can be decided in L.

**Remark 1.3.** Despite Qiao, Sarma, and Tang giving a polynomial-time algorithm for case where $H$ and $N$ are coprime, $N$ is arbitrary Abelian, and $H$ is elementary Abelian (no restriction on number of generators for $H$ or $N$), we remark here on some of the difficulties we encountered in getting WL to extend beyond the case of $H$ being $O(1)$-generated. When $H$ is $O(1)$-generated, we may start by pebbling the generators of $H$. After this, by Taunt’s Lemma (reproduced as Lemma 3.3 below), all that is left is to identify the multiset of $H$-modules appearing in $N$. In contrast, when $H$ is not $O(1)$-generated, this strategy fails quite badly: if only a small subset of $H$’s generators are pebbled, then it leaves open automorphisms of $H$ that could translate one $H$-module structure to another. But the latter translation-under-automorphism problem is equivalent to the entire problem in this family of groups (see, e.g., [QST11, Theorem 1.2]).

This same difficulty is encountered even when using the more powerful second Ehrenfeucht–Fraïssé pebble game in Hella’s [He89, He96] hierarchy, in which Spoiler may pebble two elements per turn instead of just one. This second game in Hella’s hierarchy is already quite powerful: it identifies semisimple groups using only $O(1)$ pebbles and $O(1)$ rounds [GL22]. It seems plausible to us that with only $O(1)$ pebbles, neither ordinary WL nor this second game in Hella’s hierarchy identifies coprime extensions where both $H, N$ are Abelian with no restriction on the number of generators.

We next parallelize a result of Brachter & Schweitzer [BS22], who showed that Weisfeiler–Leman can identify direct products in polynomial-time provided it can also identify the indecomposable direct factors in polynomial-time. Specifically, we show:

**Theorem 1.4.** For all $G = G_1 \times \cdots \times G_d$ with the $G_i$ directly indecomposable, and all $k \geq 5$, if $(k, O(\log^c n))$-WL Version II identifies each $G_i$ for some $c \geq 1$, then $(k + 1, O(\log^c n))$-WL identifies $G$.

More specifically, we show that for $k \geq 5$ and $r(n) \in \Omega(\log n)$, if a direct product $G$ is not distinguished from some group $H$ by $(k, r)$-WL Version II, then $H$ is a direct product, and there is some direct factor of $H$ that is not distinguished from some direct factor of $G$ by $(k - 1, r)$-WL.

Prior to Thm. 1.4, the best-known upper bound on computing direct product decompositions was P [Will12, KN09]. While Weisfeiler–Leman does not return explicit direct factors, it can implicitly compute a direct product decomposition in $O(\log n)$ rounds, which is sufficient for parallel isomorphism testing. In light of the parallel WL implementation due to Grohe & Verbitsky, our result effectively provides that WL can decompose direct products in TC$^1$.

We next consider groups without Abelian normal subgroups. Using the individualize-and-refine paradigm, we obtain a new upper bound of quasiSAC$^1$ for not only deciding isomorphisms, but also listing isomorphisms. While this does not improve upon the upper bound of P for isomorphism testing [BCQ12], this does parallelize the previous bound of $n^{\Theta(\log \log n)}$ runtime for listing isomorphisms [BCGQ11].

**Theorem 1.5.** Let $G$ be a group without Abelian normal subgroups, and let $H$ be arbitrary. We can test isomorphism between $G$ and $H$ using an SAC circuit of depth $O(\log n)$ and size $n^{\Theta(\log \log n)}$. Furthermore, all such isomorphisms can be listed in this bound.

**Remark 1.6.** The key idea in proving Thm. 1.5 is to prescribe an isomorphism between Soc$(G)$ and Soc$(H)$ (as in [BCGQ11]), and then use Weisfeiler–Leman to test in L whether the given isomorphism of Soc$(G) \cong$ Soc$(H)$ extends to an isomorphism of $G \cong H$. The procedure from [BCGQ11] for choosing all possible isomorphisms between socles is easily seen to parallelize; our key improvement is in the parallel complexity of testing whether such an isomorphism of socles extends to the whole groups.

Previously, this latter step was shown to be polynomial-time computable [BCGQ11, Proposition 3.1] via membership checking in the setting of permutation groups. Now, although membership checking in permutation groups is in NC [BLSS7], the proof there uses several different group-theoretic techniques, and relies on the Classification of Finite Simple Groups (see the end of the introduction of [BLSS7] for a discussion). Furthermore, there is no explicit upper bound on which level of the NC hierarchy these problems
are in, just that it is $O(1)$. Thus, it does not appear that membership testing in the setting of permutation groups is known to be even $\text{AC}^1$-computable. So already, our quasi-$\text{SAC}^1$ bound is new (the quasi-polynomial size comes only from parallelizing the first step). Furthermore, Weisfeiler–Leman provides a much simpler algorithm; indeed, although we also rely on the fact that all finite simple groups are 2-generated (a result only known via CFSG), this is the only consequence of CFSG that we use, and it is only used in the proof of correctness, not in the algorithm itself. We note, however, that although WL improves the parallel complexity of these particular instances of membership testing, it requires access to the multiplication table for the underlying group, so this technique cannot be leveraged for more general membership testing in permutation groups.

In the case of serial complexity, if the number of simple direct factors of $\text{Soc}(G)$ is just slightly less than maximal, even listing isomorphism can be done in $\text{FP}$ [BCGQ11]. Under the same restriction, we get an improvement in the parallel complexity to $\text{FL}$:

**Corollary 1.7** (Cf. [BCGQ11] Corollary 4.4). Let $G$ be a group without Abelian normal subgroups, and let $H$ be arbitrary. Suppose that the number of non-Abelian simple direct factors of $\text{Soc}(G)$ is $O(\log n/\log \log n)$. Then we can decide isomorphism between $G$ and $H$, as well as list all such isomorphisms, in $\text{FL}$.

It remains open as to whether isomorphism testing of groups without Abelian normal subgroups is even in $\text{NC}$. This would follow if such groups were identified by WL with $O(1)$ pebbles in $(\log n)^{O(1)}$ rounds; while we do not yet know whether this is the case, there is a higher-arity version of WL which identifies such groups with $O(1)$ pebbles in $O(1)$ rounds [GL22], but that higher-arity version is not yet known to be implementable by efficient algorithms.

Given the lack of lower bounds on $\text{GI}$, and Grohe & Verbitsky’s parallel WL algorithm, it is natural to wonder whether our parallel bounds could be improved. One natural approach to this is via the count-free WL algorithm, which compares the set rather than the multiset of colors at each iteration. We show unconditionally that this algorithm fails to serve as a polynomial-time isomorphism test for even Abelian groups.

**Theorem 1.8.** There exists an infinite family $(G_n, H_n)_{n \geq 1}$ where $G_n \neq H_n$ are Abelian groups of the same order and count-free WL requires dimension $\Omega(\log |G_n|)$ to distinguish $G_n$ from $H_n$.

**Remark 1.9.** Even prior to [CFI92], it was well-known that the count-free variant of Weisfeiler–Leman failed to place GI into $\text{P}$ [IL90]. In fact, count-free WL fails to distinguish almost all graphs [Fag76, Imm82], while two iterations of the standard counting 1-WL almost surely assign a unique label to each vertex [BK79, BES80]. In light of the equivalence between count-free WL and the logic FO (first-order logic without counting quantifiers), this rules out FO as a viable logic to capture P on unordered graphs. Finding such a logic is a central open problem in Descriptive Complexity Theory. On ordered structures such a logic was given by Immnerman [Imm86] and Vardi [Var82].

Thm. [L3] establishes the analogous result, ruling out FO as a candidate logic to capture P on unordered groups. This suggests that some counting may indeed be necessary to place GI into P. As DET is the best known lower bound for GI [Lor04], counting is indeed necessary to place GI into P. There are no such lower bound known for GI. Furthermore, the work of [CTW13] shows that GI is not hard (under $\text{AC}^0$-reductions) for any complexity class that can compute PARITY, such as DET. Determining which families of groups can (not) be identified by count-free WL remains an intriguing open question.

While count-free WL is not sufficiently powerful to compare the multiset of colors, it turns out that $O(\log \log n)$-rounds of count-free $O(1)$-WL Version III will distinguish two elements of different orders. Thus, the multiset of colors computed by the count-free $(O(1), O(\log \log n))$-WL Version III for non-isomorphic Abelian groups $G$ and $H$ will be different. We may use $O(\log n)$ non-deterministic bits to guess the color class where $G$ and $H$ have different multiplicities, and then an $\text{MAC}^0$ circuit to compare said color class. This yields the following.

**Theorem 1.10.** Abelian Group Isomorphism is in $\beta_1 \text{MAC}^0(\text{FOLL})$.

**Remark 1.11.** We note that this and Thm. [L3] illustrate uses of WL for groups as a subroutine in isomorphism testing, which is how it is so frequently used in the case of graphs. To the best of our knowledge, the
only previous uses of WL as a subroutine for GPI were in [LQ17, BGL+19]. In particular, Thm. 1.10 motivated follow-up work by Collins & Levet [CL22, Co23], who leveraged count-free WL Version I in a similar manner to obtain novel parallel complexity bounds for isomorphism testing of several families of groups. Most notably, they improved the complexity of isomorphism testing for the CFI groups from $TC^1$ to $\beta_1 MAC^0(FOLL)$. The CFI groups are highly non-trivial, arising via Mekler’s construction [Mek81] from the CFI graphs [CFI92].

**Remark 1.12.** The previous best upper bounds for isomorphism testing of Abelian groups are linear time [Kav07, Vik90, Sav80] and $L \cap TC^0(FOLL)$ [CTW13]. As $\beta_1 MAC^0(FOLL) \subseteq TC^0(FOLL)$, Thm. 1.10 improves the upper bound for isomorphism testing of Abelian groups.

**Methods.** We find the comparison of methods at least as interesting as the comparison of complexity. Here discuss at a high level the methods we use for our main theorems above, and compare them to the methods of their predecessor results.

For Thm. 1.2 its predecessor in Qiao–Sarma–Tang [QST11] leveraged a result of Le Gall [LG09] on testing conjugacy of elements in the automorphism group of an Abelian group. (By further delving into the representation theory of Abelian groups, they were also able to solve the case where $H$ and $N$ are coprime and both are Abelian without any restriction on number of generators; we leave that as an open question in the setting of WL.) Here, we use the pebbling game. Our approach is to first pebble generators for the complement $H$, which fixes an isomorphism of $H$ and its image. For groups that decompose as a coprime extension of $H$ and $N$, the isomorphism type is completely determined by the multiplicities of the indecomposable $H$-module direct summands (Lem. 3.3). So far, this is the same group-theoretic structure leveraged by Qiao, Sarma, and Tang [QST11]. However, we then use the representation-theoretic fact that, since $|N|$ and $|H|$ are coprime, each indecomposable $H$-module is generated by a single element (Lem. 3.4); this is crucial in our setting, as it allows Spoiler to pebble that one element in the WL pebbling game. Then, as the isomorphism of $H$ is fixed, we show that any subsequent bijection that Duplicator selects must restrict to $H$-module isomorphisms on each indecomposable $H$-submodule of $N$ that is a direct summand.

For Thm. 1.5 solving isomorphism of semisimple groups took a series of two papers [BCGQ11, BCQ12]. Our result is really only a parallel improvement on the first of these (we leave the second as an open question). In Babai et al. [BCGQ11], they used Code EQUIVALENCE techniques to identify semisimple groups where the minimal normal subgroups have a bounded number of non-Abelian simple direct factors, and to identify general semisimple groups in time $n^{O(\log \log n)}$. In contrast, WL—along with individualize-and-refine in the second case—provides a single, combinatorial algorithm that is able to detect the same group-theoretic structures leveraged in previous works to solve isomorphism in these families. In parallelizing Brachter & Schweitzer’s direct product result in Thm. 1.4 we use two techniques. The first is simply carefully analyzing the number of rounds used in many of the proofs. In several cases, a careful analysis of the rounds used was not sufficient to get a strong parallel result. In those cases, we use the notion of rank, which may be of independent interest and have further uses.

Given a subset $C$ of group elements, the $C$-rank of $g \in G$ is the minimal word-length over $C$ required to generate $g$. If $C$ is easily identified by Weisfeiler–Leman, then WL can identify $\langle C \rangle$ in $O(\log n)$ rounds. This is made precise (and slightly stronger) with our Rank Lemma:

**Lemma 1.13 (Rank lemma).** If $C \subseteq G$ is distinguished by $(k, r)$-WL, then any bijection $f$ chosen by Duplicator must respect $C$-rank, in the sense that $rk_C(g) = rk_{f(C)}(f(g))$ for all $g \in G$, or Spoiler can win with $k + 1$ pebbles and max$r \log d + O(1)$ rounds, where $d = \text{diam}(\text{Cay}(C, C)) \leq |C| \leq |G|$.

One application of our Rank Lemma is that WL identifies verbal subgroups where the words are easily identified. Given a set of words $w_1(x_1, \ldots, x_n), \ldots, w_m(\vec{x})$, the corresponding verbal subgroup is the subgroup generated by $\langle w_i(g_1, \ldots, g_n) : i = 1, \ldots, m, g_j \in G \rangle$. One example that we use in our results is the commutator subgroup. If Duplicator chooses a bijection $f : G \rightarrow H$ such that $f([x, y])$ is not a commutator in $H$, then Spoiler pebbles $[x, y] \mapsto f([x, y])$ and wins in two additional rounds. Thus, by our Rank Lemma, if Spoiler does not map the commutator subgroup $[G, G]$ to the commutator subgroup $[H, H]$, then Duplicator wins with 1 additional pebble and $O(\log n)$ additional rounds. Brachter & Schweitzer [BS22] obtained a similar result about verbal subgroups using different techniques. Namely, they showed that if WL assigns a distinct coloring to certain subsets $S_1, \ldots, S_t$, then WL assigns a
unique coloring to the set of group elements satisfying systems of equations over $S_1, \ldots, S_l$. They analyzed the WL colorings directly. As a result, it is not clear how to compose their result with the pebble game. For instance, while their result implies that if Duplicator does not map $f([G, G]) = [H, H]$ then Spoiler wins, it is not clear how Spoiler wins nor how quickly Spoiler can win. Our result addresses these latter two points more directly. Recall that the number of rounds is the crucial parameter affecting both the parallel complexity and quantifier depth.

Related Work. There has been considerable work on efficient parallel ($NC$) isomorphism tests for graphs \cite{Lin92, JKM03, KV08, Wag11, ES17, GV06, GK21, DLN09, DNTW09, ADKK12}. In contrast with the work on serial runtime complexity, the literature on the space and parallel complexity for $GpI$ is quite minimal. Around the same time as Tarjan’s $T(n, \log^2(n)) + O(1)$-time algorithm for $GpI$ \cite{Mil78}, Lipton, Snyder, and Zalcstein showed that $GpI \in SPACE(\log^2(n))$ \cite{LSZ77}. This bound has been improved to $\Omega(\log^2(n))$ \cite{LQ17, BGL+19, BH20, BS22}, yielding efficient isomorphism tests for several families \cite{GV06, KPS19, GK21, BS20, G09}. In contrast with \cite{Lin92, JKM03, KV08, Wag11, ES17, GV06, GK21, DLN09, DNTW09, ADKK12}, we show that $GpI \in L \cap TC^0(FOLL)$ \cite{CTW13, Tan13}. In the case of Abelian groups, Chattopadhyay, Torán, and Wagner showed that $GpI \in L \cap TC^1(FOLL)$ \cite{CTW13, Tan13}. Tang showed that isomorphism testing for groups with a bounded number of generators can also be done in $L$ \cite{Tan13}. Since composition factors of permutation groups can be identified in $NC$ \cite{BLS87} (see also \cite{Bab07} for a CFSG-free proof), isomorphism testing between two permutation groups that are both direct products of simple groups (Abelian or non-Abelian) can be done in $NC$, using the regular representation, though this does not allow one to test isomorphism of such a group against an arbitrary permutation group. Finding direct factors in $NC$ is a consequence of our Thm. \ref{thm:direct_factors}. To the best of our knowledge, no other specific family of groups is known to admit an $NC$-computable isomorphism test prior to our paper.

Combinatorial techniques, such as individualization with Weisfeiler–Leman refinement, have also been incredibly successful in $GI$, yielding efficient isomorphism tests for several families \cite{GV06, KPS19, GK21, GK19, GN21, BW13, CST13}. Weisfeiler–Leman is also a key subroutine in Babai’s quasipolynomial-time isomorphism test \cite{Bab16}. Despite the successes of such combinatorial techniques, they are known to be insufficient to place $GI$ into $P$ \cite{CTFD12, NS18}. In contrast, the use of combinatorial techniques for $GpI$ is relatively new \cite{LQ17, BGL+19, BS20, BS22}, and it is a central open problem as to whether such techniques are sufficient to improve even the long-standing upper-bound of $n^{\Theta(\log n)}$ runtime.

Examining the distinguishing power of the counting logic $C_k$ serves as a measure of descriptive complexity for groups. In the setting of graphs, the descriptive complexity has been extensively studied, with Gro17 serving as a key reference in this area. There has been recent work relating first order logics and groups \cite{NT17}, as well as work examining the descriptive complexity of finite abelian groups \cite{Gro17}. However, the work on the descriptive complexity of groups is scant compared to the algorithmic literature on $GpI$.

Ehrenfeucht–Fraïssé games \cite{Ehr61, Fra54}, also known as pebbling games, serve as another tool in proving the inexpressibility of certain properties in first-order logics. Two finite structures are said to be elementary equivalent if they satisfy the same first-order sentences. In such games, we have two players who analyze two given structures. Spoiler seeks to prove the structures are not elementary equivalent, while Duplicator seeks to show that the structures are indeed elementary equivalent. Spoiler begins by selecting an element from one structure, and Duplicator responds by picking a similar element from the other structure. Spoiler wins if and only if the eventual substructures are not isomorphic. Pebbling games have served as an important tool in analyzing graph properties like reachability \cite{AF90, AF97}, designing parallel algorithms for graph isomorphism \cite{GV06}, and isomorphism testing of random graphs \cite{Ros09}.

2 Preliminaries

2.1 Groups

Unless stated otherwise, all groups are assumed to be finite and represented by their Cayley tables. For a group of order $n$, the Cayley table has $n^2$ entries, each represented by a binary string of size $\lceil \log_2(n) \rceil$. For an element $g$ in the group $G$, we denote the order of $g$ as $|g|$. We use $d(G)$ to denote the minimum size of a generating set for the group $G$.

The socle of a group $G$, denoted $\text{Soc}(G)$, is the subgroup generated by the minimal normal subgroups of $G$. If $G$ has no Abelian normal subgroups, then $\text{Soc}(G)$ decomposes as the direct product of non-Abelian
simple factors. The normal closure of a subset \( S \subseteq G \), denoted \( \text{ncl}(S) \), is the smallest normal subgroup of \( G \) that contains \( S \).

We say that a normal subgroup \( N \leq G \) splits in \( G \) if there exists a subgroup \( H \leq G \) such that \( H \cap N = \{1\} \) and \( G = HN \). The conjugation action of \( H \) on \( N \) allows us to express multiplication of \( G \) in terms of pairs \((h, n) \in H \times N \). We note that the conjugation action of \( H \) on \( N \) induces a group homomorphism \( \theta : H \to \text{Aut}(N) \) mapping \( h \mapsto \theta_h \), where \( \theta_h : N \to N \) sends \( \theta_h(n) = hnh^{-1} \). So given \((H, N, \theta)\), we may define the group \( H \ltimes N \) on the set \( \{(h, n) : h \in H, n \in N\} \) with the product \((h_1, n_1)(h_2, n_2) = (h_1h_2, \theta_{h_2^{-1}}(n_1)n_2)\). We refer to the decomposition \( G = H \ltimes N \) as a semidirect product demoposition. When the action \( \theta \) is understood, we simply write \( G = H \ltimes N \).

We are particularly interested in semidirect products when \( N \) is a normal Hall subgroup. To this end, we recall the Schur–Zassenhaus Theorem \([\text{Rob82, (9.1.2)}]\).

**Theorem 2.1** (Schur–Zassenhaus). Let \( G \) be a finite group of order \( n \), and let \( N \) be a normal Hall subgroup. Then there exists a complement \( H \leq G \), such that \( \gcd(|H|, |N|) = 1 \) and \( G = H \ltimes N \). Furthermore, if \( H \) and \( K \) are complements of \( N \), then \( H \) and \( K \) are conjugate.

We will use the following standard observation a few times:

**Fact 2.2.** Let \( G = \langle g_1, \ldots, g_d \rangle \). Then every element of \( G \) can be written as a word in the \( g_i \) of length at most \(|G|\).

**Proof.** Consider the Cayley graph of \( G \) with generating set \( g_1, \ldots, g_d \). Words correspond to walks in this graph. We need only consider simple walks—those which never visit any vertex more than once—since if a walk visits a group element \( g \) more than once, then the part of that walk starting and ending at \( g \) is a word that equals the identity element, so it can be omitted. But the longest simple walk is at most the number of vertices, which is \(|G|\). \( \square \)

### 2.2 Weisfeiler–Leman

We begin by recalling the Weisfeiler–Leman algorithm for graphs, which computes an isomorphism-invariant coloring. Let \( \Gamma \) be a graph, and let \( k \geq 2 \) be an integer. The \( k \)-dimension Weisfeiler–Leman, or \( k \)-WL, algorithm begins by constructing an initial coloring \( \chi_0 : V(\Gamma)^k \to \mathcal{K} \), where \( K \) is our set of colors, by assigning each \( k \)-tuple a color based on its isomorphism type. That is, two \( k \)-tuples \((v_1, \ldots, v_k)\) and \((u_1, \ldots, u_k)\) receive the same color under \( \chi_0 \) iff the map \( v_i \to u_i \) (for all \( i \in [k] \)) is an isomorphism of the induced subgraphs \( \Gamma[\{v_1, \ldots, v_k\}] \) and \( \Gamma[\{u_1, \ldots, u_k\}] \) and for all \( i, j \), \( v_i = v_j \iff u_i = u_j \).

For \( r \geq 0 \), the coloring computed at the \( r \)th iteration of Weisfeiler–Leman is refined as follows. For a \( k \)-tuple \( \overline{v} = (v_1, \ldots, v_k) \) and a vertex \( x \in V(\Gamma) \), define

\[
\overline{v}(v_i/x) = (v_1, \ldots, v_{i-1}, x, v_{i+1}, \ldots, v_k).
\]

The coloring computed at the \((r+1)\)st iteration, denoted \( \chi_{r+1} \), stores the color of the given \( k \)-tuple \( \overline{v} \) at the \( r \)th iteration, as well as the colors under \( \chi_r \) of the \( k \)-tuples obtained by substituting a single vertex in \( \overline{v} \) for another vertex \( x \). We examine this multiset of colors over all such vertices \( x \). This is formalized as follows:

\[
\chi_{r+1}(\overline{v}) = (\chi_r(\overline{v}) \cup \{ (\chi_r(\overline{v})(v_1/x), \ldots, \chi_r(\overline{v}(v_k/x)) \mid x \in V(\Gamma) \})
\]

where \( \{\cdot\} \) denotes a multiset.

Note that the coloring \( \chi_r \) computed at iteration \( r \) induces a partition of \( V(\Gamma)^k \) into color classes. The Weisfeiler–Leman algorithm terminates when this partition is not refined, that is, when the partition induced by \( \chi_{r+1} \) is identical to that induced by \( \chi_r \). The final coloring is referred to as the stable coloring, which we denote \( \chi_\infty := \chi_r \).

Brachter & Schweitzer introduced three variants of WL for groups. WL Versions I and II are both executed directly on the Cayley tables, where \( k \)-tuples of group elements are initially colored. For WL Version I, two \( k \)-tuples \((g_1, \ldots, g_k)\) and \((h_1, \ldots, h_k)\) receive the same initial color iff (a) for all \( i, j, \ell \in [k] \), \( g_\ell g_j = g_\ell \iff h_\ell h_j = h_\ell \), and (b) for all \( i, j \in [k], g_i = g_j \iff h_i = h_j \). For WL Version II, \((g_1, \ldots, g_k)\) and
Prior to the start of the game, each pebble pair \((h_1, \ldots, h_k)\) receive the same initial color iff the map \(g_i \mapsto h_i\) for all \(i \in [k]\) extends to an isomorphism of the generated subgroups \(\langle g_1, \ldots, g_k \rangle\) and \(\langle h_1, \ldots, h_k \rangle\). For both WL Versions I and II, refinement is performed in the classical manner as for graphs. Namely, for a given \(k\)-tuple \(\mathbf{g}\) of group elements,

\[
\chi_{r+1}(\mathbf{g}) = (\chi_r(\mathbf{g})), (\chi_r(\mathbf{g}(g_1/x)), \ldots, \chi_r(\mathbf{g}(g_k/x)) | x \in G
\]

WL Version III works as follows. Given the Cayley table for a group \(G\), we first apply a reduction from GI to GI in the setting of simple, undirected graphs. We then apply the standard \(k\)-WL for graphs and pull back to a coloring on \(G^k\).

Given the Cayley table for a group \(G\), we construct the graph \(\Gamma_G\) as follows. We begin with a set of isolated vertices, corresponding to the elements of \(G\). For each pair of elements \((g, h) \in G^2\), we add a multiplication gadget \(M(g, h)\), which is constructed as follows (see Figure 1).

- We add vertices \(a_{gh}, b_{gh}, c_{gh}, d_{gh}\).
- We add the edges:
  \[\{g, a_{gh}\}, \{h, b_{gh}\}, \{b_{gh}, c_{gh}\}, \{c_{gh}, d_{gh}\}, \{gh, d_{gh}\}\].

Observe that \(\Gamma_G\) has \(\Theta(|G|^2)\) vertices. By consideration of vertex degrees, we also note that \(G\) forms a canonical (isomorphism-invariant) subset of \(V(\Gamma_G)\).

**Remark 2.3.** We note that the construction of \(\Gamma_G\) can be done in \(\text{AC}^0\). We may store \(\Gamma_G\) as an adjacency matrix. Adding a single edge can be done using an \(\text{AC}^0\) circuit. Each edge can be added independently in parallel; and thus, without increasing the depth of the circuit. As \(|\Gamma_G| \in \Theta(|G|^2)\), we need only a polynomial number of gates to add the appropriate edges. So \(\Gamma_G\) can be constructed using an \(\text{AC}^0\) circuit.

### 2.3 Pebbling Game

We recall the bijective pebble game introduced by [Hel89, Hel96] for WL on graphs. This game is often used to show that two graphs \(X\) and \(Y\) cannot be distinguished by \(k\)-WL. The game is an Ehrenfeucht–Fraïssé game (c.f., [EFT94, Lib04]), with two players: Spoiler and Duplicator. We begin with \(k + 1\) pairs of pebbles. Prior to the start of the game, each pebble pair \((p_i, p'_i)\) is initially placed either beside the graphs or on a given pair of vertices \(v_i \mapsto v'_i\) (where \(v_i \in V(X), v'_i \in V(Y)\)). Each round \(r\) proceeds as follows.

1. Spoiler picks up a pair of pebbles \((p_i, p'_i)\).
2. We check the winning condition, which will be formalized later.
3. Duplicator chooses a bijection \(f_r : V(X) \rightarrow V(Y)\) (where here, we emphasize that the bijection chosen depends on the round- and, implicitly, the pebbling configuration at the start of said round).
4. Spoiler places \(p_i\) on some vertex \(v \in V(X)\). Then \(p'_i\) is placed on \(f(v)\).

Let \(v_1, \ldots, v_m\) be the vertices of \(X\) pebbled at the end of step 1 at round \(r\) of the game, and let \(v'_1, \ldots, v'_m\) be the corresponding pebbled vertices of \(Y\). Spoiler wins precisely if the map \(v_i \mapsto v'_i\) does not extend to an isomorphism of the induced subgraphs \(X[[v_1, \ldots, v_m]]\) and \(Y[[v'_1, \ldots, v'_m]]\). Duplicator wins otherwise. Spoiler wins, by definition, at round 0 if \(X\) and \(Y\) do not have the same number of vertices. We note that
\[ \pi \in X^k \text{ and } \overline{\pi} \in Y^k \text{ are not distinguished by the first } r \text{ rounds of } k\text{-WL if and only if Duplicator wins the first } r \text{ rounds of the } (k + 1)\text{-pebble game starting from the configuration } \pi \mapsto \overline{\pi}. \] [Hel89, Hel96, CFI92]

For groups instead of graphs, Versions I and II of the pebble game are defined analogously, where Spoiler pebbles group elements on the Cayley tables. Precisely, for groups \( G \) and \( H \), each round proceeds as follows.

1. **Spoiler picks up a pair of pebbles** \((p_i, p'_i)\).
2. We check the winning condition, which will be formalized later.
3. **Duplicator chooses a bijection** \( f_r : G \to H \).
4. **Spoiler places** \( p_i \) on some vertex \( g \in G \). Then \( p'_i \) is placed on \( f(g) \).

Suppose that \((g_1, \ldots, g_\ell) \mapsto (h_1, \ldots, h_\ell)\) have been pebbled. In Version I, Duplicator wins at the given round if this map satisfies the initial coloring condition of WL Version I: (a) for all \( i, j \in [\ell], g_i g_j = g_m \iff h_i h_j = h_m \), and (b) for all \( i, j \in [\ell], g_i = g_j \iff h_i = h_j \). In Version II, Duplicator wins at the given round if the map \((g_1, \ldots, g_\ell) \mapsto (h_1, \ldots, h_\ell)\) extends to an isomorphism of the generated subgroups \( \langle g_1, \ldots, g_\ell \rangle \) and \( \langle h_1, \ldots, h_\ell \rangle \). Brachter & Schweitzer showed that Duplicator need not respect the subgroup structure of the implicitly pebbled elements, as well as the subgroups induced by the (implicitly) pebbled group elements [BS20].

**Remark 2.4.** In our work, we explicitly control for both pebbles and rounds. In our theorem statements, we state explicitly the number of pebbles on the board at the end of the given round. So if Spoiler can win with \( k \) pebbles on the board, then we are playing in the \((k + 1)\)-pebble game. Note that \( k\text{-WL} \) corresponds to \( k\)-pebbles on the board.

The pebble game for graphs is the same pebble game used to analyze the \( k\text{-WL} \) Version III algorithms for groups. Note that placing a pebble pair on vertices corresponding to the multiplication gadgets \( M(g_1, g_2) \) and \( M(h_1, h_2) \) (but not on the vertices corresponding to the group elements) induces a pairing of group elements. Here, we adopt the convention from Brachter & Schweitzer [BS20] in saying that \((g_1, g_2) \mapsto (h_1, h_2)\) are **implicitly pebbled** if a pebble is placed on a non-group-element vertex of the multiplication gadget \( M(g_1, g_2) \).

Brachter & Schweitzer showed that Duplicator must respect the multiplication structure of the implicitly pebbled elements, as well as the subgroups induced by the (implicitly) pebbled group elements [BS20].

**Lemma 2.5** (Brachter & Schweitzer [BS20], Lemma 3.10). Fix \( m \geq 2 \). Consider the \( k\text{-pebble game} \) on graphs \( \Gamma_G \) and \( \Gamma_H \). If \( k \geq 4 \) and one of the following happens:

(a) Duplicator chooses a bijection on \( f : \Gamma_G \to \Gamma_H \) with \( f(G) \neq H \),

(b) after choosing a bijection \( f : \Gamma_G \to \Gamma_H \), there is a pebble pair \((p, p')\) for which pebble \( p \) is on some vertex of \( M(g_1, g_2) \) that is not a group element and \( p' \) is on some vertex of \( M(h_1, h_2) \) that is not a group element, but
\[
(f(g_1), f(g_2), f(g_1g_2)) \neq (h_1, h_2, h_1h_2),
\]

or

(c) the map induced by the group elements pebbled or implicitly pebbled by \( k-2 \) pebbles does not extend to an isomorphism between the corresponding generated subgroups,

then Spoiler can win with at most two additional pebbles and \( \log_2(n) + O(1) \) additional rounds.

In light of Lem. 2.5 (a), even in Version III, we may simply consider bijections that Duplicator selects on the group elements. That is, we may without loss of generality consider bijections \( f : G \to H \). We stress that condition (b) applies only when there is already a pebble on some element of a multiplication gadget. A priori, Duplicator need not select bijections that respect multiplication gadgets. That is, if there is no pebble on a multiplication gadget \( M(g_1, g_2) \), then Duplicator need not map \( f(M(g_1, g_2)) = M(f(g_1), f(g_2)) \).

This is the same subtlety mentioned above.

We also note that Lem. 2.5 (c) provides that if Duplicator does not respect the subgroup structure of the (implicitly) pebbled group elements, then Spoiler can quickly win.
Remark 2.6. The original statement of [BS20, Lemma 3.10] did not specify the number of additional rounds. However, we are able to modify the proof of [BS20, Lemma 3.10(c)] to show that the same result holds with \( \log_2(n) + O(1) \) additional rounds. We defer the proof to Appendix A.

Brachter & Schweitzer [BS20, Theorem 3.9] also previously showed that WL Version I, II, and III are equivalent up to a factor of 2 in the dimension, though they did not control for rounds. We strengthen their analysis to explicitly control for rounds.

**Theorem 2.7.** Let \( G \) and \( H \) be groups of order \( n \). Let \( k \geq 2, r \geq 1 \). We have the following.

(a) If \((k, r)\)-WL Version I distinguishes \( G \) and \( H \), then \((k, r)\)-WL Version II distinguishes \( G \) and \( H \).

(b) If \((k, r)\)-WL Version II distinguishes \( G \) and \( H \), then \( ([k/2] + 2, 3r + O(\log n))\)-WL Version III distinguishes \( G \) and \( H \).

(c) If \((k, r)\)-WL Version III distinguishes \( G \) and \( H \), then \((2k + 1, 2r)\)-WL Version I distinguishes \( G \) and \( H \).

**Proof.** See Appendix A. \( \square \)

### 2.4 Weisfeiler–Leman as a Parallel Algorithm

Grohe & Verbitsky [GV06] previously showed that for fixed \( k \), the classical \( k \)-dimensional Weisfeiler–Leman algorithm for graphs can be effectively parallelized. More precisely, each iteration (including the initial coloring) can be implemented using a logspace uniform \( TC^0 \) circuit. As they mention [GV06, Remark 3.4], their implementation works for any first-order structure, including groups. However, because here we have three different versions of WL, we explicitly list out the resulting parallel complexities, which differ slightly between the versions.

- **WL Version I:** Let \((g_1, \ldots, g_k)\) and \((h_1, \ldots, h_k)\) be two \( k \)-tuples of group elements. We may test in \( AC^0 \) whether (a) for all \( i, j, m \in [k] \), \( g_ig_j = g_m \iff h_ih_j = h_m \), and (b) \( g_i = g_j \iff h_i = h_j \). So we may decide if two \( k \)-tuples receive the same initial color in \( AC^0 \). Comparing the multiset of colors at the end of each iteration (including after the initial coloring), as well as the refinement steps, proceed identically as in [GV06]. Thus, for fixed \( k \), each iteration of \( k \)-WL Version I can be implemented using a logspace uniform \( TC^0 \).

- **WL Version II:** Let \((g_1, \ldots, g_k)\) and \((h_1, \ldots, h_k)\) be two \( k \)-tuples of group elements. We may use the marked isomorphism test of Tang [Tan13] to test in \( L \) whether the map sending \( g_i \mapsto h_i \) for all \( i \in [k] \) extends to an isomorphism of the generated subgroups \( \langle g_1, \ldots, g_k \rangle \) and \( \langle h_1, \ldots, h_k \rangle \). So we may decide whether two \( k \)-tuples receive the same initial color in \( L \). Comparing the multiset of colors at the end of each iteration (including after the initial coloring), as well as the refinement steps, proceed identically as in [GV06]. Thus, for fixed \( k \), the initial coloring of \( k \)-WL Version II is \( L \)-computable, and each refinement step is \( TC^0 \)-computable.

- **WL Version III:** From Remark 2.3 we have that constructing the graph from the Cayley table is \( AC^0 \)-computable. We now appeal directly to the parallel WL implementation for graphs due to Grohe & Verbitsky [GV06]. Thus, each iteration of WL Version III can be implemented with a logspace uniform \( TC^0 \)-circuit, and constructing the graph does not further increase the asymptotic complexity.

### 2.5 Colored Graphs

Let \( k \in \mathbb{N} \), and let \( \Gamma \) be a graph. A k-coloring\(^1\) over \( \Gamma \) is a map \( \gamma : V(\Gamma)^k \rightarrow \mathcal{K} \), where \( \mathcal{K} \) is our finite set of colors. A k-coloring partitions \( V(\Gamma)^k \) into color classes. When \( k = 1 \), we refer to the coloring as an element coloring. For another natural number \( m < k \), a k-coloring \( \gamma^{(k)} : V(G)^k \rightarrow \mathcal{K} \) induces an m-coloring \( \gamma^{(m)} : V(G)^k \rightarrow \mathcal{K} \) via:

\[
\gamma^{(m)}((g_1, \ldots, g_m)) := \gamma^{(k)}((g_1, \ldots, g_m, g_m, \ldots, g_m)).
\]

\(^1\)Not to be confused with the usual “proper k-coloring” of a graph, that is, an assignment of one out of \( k \) colors to each vertex such that no two adjacent vertices receive the same color. Despite this terminological overloading, we stick with this terminology for consistency with [BS22].
Definition 2.8. A colored graph is a pair \((\Gamma, \gamma)\), where \(\Gamma\) is a graph and \(\gamma : V(\Gamma) \rightarrow K\) is an element-coloring. We say that two colored graphs \((\Gamma_1, \gamma_1), (\Gamma_2, \gamma_2)\) are isomorphic if there is a graph isomorphism \(\varphi : V(\Gamma_1) \rightarrow V(\Gamma_2)\) that respects the colorings; that is, \(\gamma_2 \circ \varphi = \gamma_1\). We define \(\text{Aut}_\gamma(\Gamma) = \{\varphi \in \text{Aut}(\Gamma) : \gamma \circ \varphi = \gamma\}\).

For \(k \geq 2\), there is a natural analogue of \(k\)-WL that starts with a colored graph \((\Gamma, \gamma)\). The only difference is in constructing the initial coloring. Two \(k\)-tuples of vertices \(\overline{u} := (u_1, \ldots, u_k)\) and \(\overline{v} := (v_1, \ldots, v_k)\) receive the same initial color precisely if the following three conditions hold:

(a) \(\gamma(u_i) = \gamma(v_i)\) for all \(i \in [k]\),

(b) The map \(u_i \mapsto v_i\) for all \(i \in [k]\) is an isomorphism of the induced subgraphs \(\Gamma[\overline{u}]\) and \(\Gamma[\overline{v}]\), and

(c) For all \(i, j \in [k]\), \(u_i = u_j \iff v_i = v_j\).

The refinement step is performed in the classical way, where the color at round \(r + 1\) assigned to \(\overline{u}\) is given by:

\[
\chi_{r+1}(\overline{u}) = (\chi_r(\overline{v}), \{ \chi_r(\overline{v}(x_1)), \ldots, \chi_r(\overline{v}(x_m)) \mid x \in V(\Gamma) \}).
\]

It is possible to define colored analogues of groups by simply replacing the term graph with group in Def. 2.8.

In the setting of Weisfeiler–Leman Version III [BS20], we may instead color the corresponding graph in the following way. Let \(G\) be a graph, and let \(\gamma : G \rightarrow \mathcal{K}\) be a coloring. Let \(\Gamma_G\) be the graph produced from \(G\), using the reduction in the classical WL Version III algorithm. We obtain a colored graph \(\Gamma_G\) by constructing a coloring \(\gamma' : V(\Gamma_G) \rightarrow \mathcal{K}\), where \(\gamma'(g) = \gamma(g)\) for all \(g \in G\). We then apply the \(k\)-WL for colored graphs to \((\Gamma_G, \gamma')\) and pull back the stable coloring to \(G^k\).

There is an analogous \((k + 1)\)-pebble game for colored graphs. Let \((X, \gamma_X), (Y, \gamma_Y)\) be colored graphs. Each round proceeds as follows.

1. Spoiler picks up a pair of pebbles \((p_i, p'_i)\).
2. We check the winning condition, which will be formalized later.
3. Duplicator chooses a bijection \(f : V(X) \rightarrow V(Y)\).
4. Spoiler places \(p_i\) on some vertex \(v \in V(X)\). Then \(p'_i\) is placed on \(f(v)\).

Let \(v_1, \ldots, v_m\) be the vertices of \(X\) pebbled at the end of step 1, and let \(v'_1, \ldots, v'_m\) be the corresponding pebbled vertices of \(Y\). Spoiler wins precisely if the map \(v_i \mapsto v'_i\) does not extend to a color isomorphism of the induced subgraphs \(X[\{v_1, \ldots, v_m\}]\) and \(Y[\{v'_1, \ldots, v'_m\}]\). Duplicator wins otherwise. Spoiler wins, by definition, at round 0 if there is no color-preserving bijection \(\varphi : V(X)^k \rightarrow V(Y)^k\) that respects such that \(\gamma_Y \circ \varphi = \gamma_X\). We note that \(X\) and \(Y\) are not distinguished by the first \(r\) rounds of \(k\)-WL if and only if Duplicator wins the first \(r\) rounds of the \((k + 1)\)-pebble game. The proof is analogous to the case of uncolored graphs [Hel89, Hel96, CFI92]. We note that the colored pebble game on graphs is the corresponding pebble game for the colored analogue of WL Version III [CFI92, IL90, BS20].

To the best of our knowledge, the notion of colored group isomorphism was first introduced by Le Gall & Rosenbaum [LCR16]. Brachter & Schweitzer [BS22] subsequently introduced a notion of a colored group in close analogue to that of a colored graph.

2.6 Complexity Classes

We assume familiarity with the complexity classes \(P, \text{NP}, \text{L}, \text{NL}, \text{NC}^k, \text{AC}^k, \text{and \ TC}^k\)- we defer the reader to standard references [Zoo, AB09]. The complexity class \(\text{SAC}^k\) is defined analogously to \(\text{AC}^k\), except that the \(\text{AND}\) gates have bounded fan-in (while the \(\text{OR}\) gates may still have unbounded fan-in). The complexity class \(\text{FOLL}\) is the set of languages decidable by uniform circuit families with \(\text{AND}, \text{OR}, \text{and \ NOT}\) gates of depth \(O(\log \log n)\), polynomial size, and unbounded fan-in. It is known that \(\text{AC}^0 \subseteq \text{FOLL} \subseteq \text{AC}^1\), and it is open as to whether \(\text{FOLL}\) is contained in \(\text{NL}\) [BKLM01].

The complexity class \(\text{MAC}^0\) is the set of languages decidable by constant-depth uniform circuit families with a polynomial number of \(\text{AND},\ \text{OR},\ \text{and \ NOT}\) gates, and at most one \(\text{Majority}\) gate. The class \(\text{MAC}^0\)
was introduced (but not so named) in \textcite{ABFR91}, where it was shown that MAC$^0 \subsetneq$ TC$^0$. This class was subsequently given the name MAC$^0$ in \textcite{JKS02}.

For a complexity class $C$, we define $\beta \cdot C$ to be the set of languages $L$ such that there exists an $L' \in C$ such that $x \in L$ if and only if there exists $y$ of length at most $O(|x|)$ such that $(x, y) \in L'$. For any $i, c \geq 0$, $\beta^i \cdot \text{FO}$ cannot compute Parity \textcite{CTW13}.

For complexity classes $C_1, C_2$, the complexity class $C_1(C_2)$ is the class of $h \circ g$, where $f$ is $C_1$-computable and $g$ is $C_2$-computable. For instance, $\beta_1 \cdot \text{MAC}^0(\text{FOLL})$ is the set of functions $h = g \circ f$, where $f$ is FOLL-computable and $g$ is $\beta_1 \cdot \text{MAC}^0$-computable.

The function class FP is the class of polynomial-time computable functions and FL is the class of logspace-computable functions.

## 3 Weisfeiler–Leman for Coprime Extensions

In this section, we consider groups that admit a Schur–Zassenhaus decomposition of the form $G = H \rtimes N$, where $N$ is Abelian, and $H$ is $O(1)$-generated and $|H|$ and $|N|$ are coprime. Qiao, Sarma, and Tang \textcite{QST11} previously exhibited a polynomial-time isomorphism test for this family of groups, as well as the family where $H$ and $N$ are arbitrary Abelian groups. This was extended by Babai & Qiao \textcite{BQ12} to groups where $|H|$ and $|N|$ are coprime, $N$ is Abelian, and $H$ is an arbitrary group given by generators in any class of groups for which isomorphism can be solved efficiently. Among the class of such coprime extensions where $H$ is $O(1)$-generated and $N$ is Abelian, we are able to improve the parallel complexity to $L$ via WL Version II.

### 3.1 Additional preliminaries for groups with Abelian normal Hall subgroup

Here we recall additional preliminaries needed for our algorithm in the next section. None of the results in this section are new, though in some cases we have rephrased the known results in a form more useful for our analysis.

A Hall subgroup of a group $G$ is a subgroup $N$ such that $|N|$ is coprime to $|G/N|$. When a Hall subgroup is normal, we refer to the group as a coprime extension. Coprime extensions are determined entirely by their actions:

\begin{lemma}[Taunt \textcite{Taunt55}]
Let $G = H \rtimes_\theta N$ and $\hat{G} = \hat{H} \rtimes_\hat{\theta} \hat{N}$. If $\alpha : H \to \hat{H}$ and $\beta : N \to \hat{N}$ are isomorphisms such that for all $h \in H$ and all $n \in N$,
\[
\hat{\theta}_{\alpha(h)}(n) = (\beta \circ \theta_h \circ \beta^{-1})(n),
\]
then the map $(h, n) \mapsto (\alpha(h), \beta(n))$ is an isomorphism of $G \cong \hat{G}$. Conversely, if $G$ and $\hat{G}$ are isomorphic and $|H|$ and $|N|$ are coprime, then there exists an isomorphism of this form.
\end{lemma}

\begin{remark}
Lem. \textcite{Taunt55} can be significantly generalized to arbitrary extensions where the subgroup is characteristic. When the characteristic subgroup is Abelian, this is standard in group theory, and has been useful in practical isomorphism testing (see, e.g., \textcite{HEO05}). In general, the equivalence of group extensions deals with both \textit{Action Compatibility} and \textit{Cohomology Class Isomorphism}. Generalizations of cohomology to non-Abelian coefficient groups was done Dedcker in the 1960s (e.g. \textcite{Dec64}) and Inassaridze at the turn of the 21st century \textcite{Ina97}. Unaware of this prior work on non-Abelian cohomology at the time, Grochow & Qiao re-derived some of it in the special case of $H^2$—the cohomology most immediately relevant to group extensions and the isomorphism problem—and showed how it could be applied to isomorphism testing \textcite[Lemma 2.3]{GQ17}, generalizing Taunt’s Lemma. In the setting of coprime extensions, the Schur–Zassenhaus Theorem provides that the cohomology is trivial. Thus, in our setting we need only consider \textit{Action Compatibility}.

A $ZH$-module is an abelian group $N$ together with an action of $H$ on $N$, given by a group homomorphism $\theta : H \to \text{Aut}(N)$. Sometimes we colloquially refer to these as “$H$-modules.” A submodule of an $H$-module $N$ is a subgroup $N' \leq N$ such that the action of $H$ on $N'$ sends $N'$ into itself, and thus the restriction of the action of $H$ to $N'$ gives $N'$ the structure of an $H$-module compatible with that on $N$. Given a subset
S ⊆ N, the smallest H-submodule containing S is denoted ℜ(\langle S \rangle)_H, and is referred to as the H-submodule generated by S. An H-module generated by a single element is called cyclic. Note that a cyclic H-module N need not be a cyclic Abelian group.

Two H-modules N, N′ are isomorphic (as H-modules), denoted N ≃_H N′, if there is a group isomorphism ϕ: N → N′ that is H-equivariant, in the sense that ϕ(θ(h)(n)) = θ′(h)(ϕ(n)) for all h ∈ H, n ∈ N. An H-module N is decomposable if N ≃_H N_1 ⊕ N_2 where N_1, N_2 are nonzero H-modules (and the direct sum can be thought of as a direct sum of Abelian groups); otherwise N is indecomposable. An equivalent characterization of N being decomposable is that there are nonzero H-submodules N_1, N_2 such that N = N_1 ⊕ N_2 as Abelian groups (that is, N is generated as a group by N_1 and N_2, and N_1 ∩ N_2 = 0). The Remak–Krull–Schmidt Theorem says that every H-module decomposes as a direct sum of indecomposable modules, and that the multiset of H-module isomorphism types of the indecomposable modules appearing is independent of the choice of decomposition, that is, it depends only on the H-module isomorphism type of N. We may thus write

\[ N ≃_H N_1^{⊕m_1} ⊕ N_2^{⊕m_2} ⊕ ⋯ ⊕ N_k^{⊕m_k} \]

unambiguously, where the N_i are pairwise non-isomorphic indecomposable H-modules. When we refer to the multiplicity of an indecomposable H-module as a direct summand in N, we mean the corresponding m_i.\(^2\)

The version of Taunt’s Lemma that will be most directly useful for us is:

**Lemma 3.3.** Suppose that G_i = H ⋊_θ_i N for i = 1, 2 are two semi-direct products with |H| coprime to |N|. Then G_1 ≃ G_2 if and only if there is an automorphism α ∈ Aut(H) such that each indecomposable ZH-module appears as a direct summand in (N, θ_1) and in (N, θ_2 ◦ α) with the same multiplicity.

The lemma and its proof are standard but we include it for completeness.

**Proof.** If there is an automorphism α ∈ Aut(H) such that the multiplicity of each indecomposable ZH-module as a direct summand of (N, θ_1) and (N, θ_2 ◦ α) are the same, then there is a ZH-module isomorphism β: (N, θ_1) → (N, θ_2 ◦ α) (in particular, β is an automorphism of N as a group). Then it is readily verified that the map \((h, n) ↦ (α(h), β(n))\) is an isomorphism of the two groups.

Conversely, suppose that ϕ: G_1 → G_2 is an isomorphism. Since |H| and |N| are coprime, N is characteristic in G_i, so we have ϕ(N) = N. And by order considerations ϕ(H) is a complement to N in G_2. We have θ_1(h)(n) = hnh^{-1}. Since ϕ is an isomorphism, we have ϕ(θ_1(h)(n)) = ϕ(hnh^{-1}) = ϕ(h)ϕ(n)ϕ(h)^{-1} = θ_2(ϕ(h))(ϕ(n)). Thus θ_1(h)(n) = ϕ^{-1}(θ_2(ϕ(h))(ϕ(n))). So we may let α = ϕ|_H, and then we have that \((N, θ_1)\) is isomorphic to \((N, θ_2 ◦ ϕ|_H)\), where the isomorphism of H-modules is given by ϕ|_N. The Remak–Krull–Schmidt Theorem then gives the desired equality of multiplicities.

The following lemma is needed for the case when N is Abelian, but not elementary Abelian. A (Z/p^kZ)[H]-module is a ZH-module N where the exponent of N (the LCM of the orders of the elements of N) divides p^k.

**Lemma 3.4** (see, e.g., Thevénaz [The81]). Let H be a finite group. If p is coprime to |H|, then any indecomposable (Z/p^kZ)[H]-module is generated (as an H-module) by a single element.

**Proof.** Thevénaz [The81 Cor. 1.2] shows that there are cyclic (Z/p^kZ)[H]-modules M_1, \ldots, M_n, each with underlying group of the form (Z/p^kZ)^d_i for some d_i, such that each indecomposable (Z/p^kZ)[H]-module is of the form M_i/p^iM_i for some i, and for distinct pairs (i, j) we get non-isomorphic modules.

### 3.2 Coprime Extensions with an O(1)-Generated Complement

Our approach is to first pebble generators for the complement H, which fixes an isomorphism of H. As the isomorphism of H is then fixed, we show that any subsequent bijection that Duplicator selects must restrict to H-module isomorphisms on each indecomposable H-submodule of N that is a direct summand. For groups

\(^2\)For readers familiar with (semisimple) representations over fields, we note that the multiplicity is often equivalently defined as dimy Hom_H(N_i, N). However, when we allow N to be an Abelian group that is not elementary Abelian, we are working with (Z/p^kZ)[H]-modules, and the characterization in terms of hom sets is more complicated, because one indecomposable module can be a submodule of another, which does not happen with semisimple representations.
that decompose as a coprime extension of $H$ and $N$, the isomorphism type is completely determined by the multiplicities of the indecomposable $H$-module direct summands (Lem. 3.3). We then leverage the fact that, in the coprime case, indecomposable $H$-modules are generated by single elements (Lem. 3.4), making it easy for Spoiler to pebble.

**Lemma 3.5.** Let $G = H \rtimes N$, where $N$ is Abelian, $H$ is $O(1)$-generated, and gcd$(|H|,|N|) = 1$. Let $K$ be an arbitrary group of order $|G|$. If $K$ does not decompose as $H \rtimes N$ (for any action), then $(O(1), O(1))$-WL Version II will distinguish $G$ and $K$.

**Proof.** Let $f : G \rightarrow K$ be the bijection that Duplicator selects. As $N \leq G$, as a subset, is uniquely determined by its orders—it is precisely the set of all elements in $G$ whose orders divide $|N|$—we may assume that $K$ has a normal Hall subgroup of size $|N|$. For first, if for some $n \in N$, $|n| \neq |f(n)|$, Spoiler can pebble $n \mapsto f(n)$ and win immediately. By reversing the roles of $K$ and $G$, it follows that $K$ must have precisely $|N|$ elements whose orders divide $|N|$. Second, suppose that those $|N|$ elements do not form a subgroup. Then there are two elements $x, y \in f(N)$ such that $xy \notin f(N)$. At the first round, Spoiler pebbles $a := f^{-1}(x) \mapsto x$. Let $f' : G \rightarrow K$ be the bijection Duplicator selects at the next round. As $K$ has precisely $|N|$ elements of order dividing $|N|$, we may assume that $f'(N) = f(N)$ (setwise). Let $b \in N$ s.t. $f'(b) = y$. Spoiler pebbles $b \mapsto y$. Now as $N$ is a group, $ab \in N$. However, as $f(a)f'(b) \notin f(N)$, $|ab| \neq |f(a)f'(b)|$. So the map $(a, b) \mapsto (x, y)$ does not extend to an isomorphism. Spoiler now wins.

Now we have that $f(N)$ is a subgroup of $K$, and because it is the set of all elements of these orders, it is characteristic and thus normal. Suppose that $f(N) \not\cong N$. We have two cases: either $f(N)$ is not Abelian, or $f(N)$ is Abelian but $N \not\cong f(N)$. Suppose first that $f(N)$ is not Abelian. Let $x \in f(N)$ such that $x \notin Z(f(N))$, and let $g := f^{-1}(x) \in N$. Spoiler pebbles $g \mapsto x$. Let $f' : G \rightarrow K$ be the bijection that Duplicator selects at the next round. We may again assume that $f'(N) = f(N)$ (setwise), or Spoiler wins with two additional pebbles and two additional rounds. Now let $y \in f(N)$ such that $[x, y] \neq 1$. Let $h \in G$ such that $f'(h) = y$. Spoiler pebbles $h \mapsto y$. Now the map $(g, h) \mapsto (x, y)$ does not extend to an isomorphism, so Spoiler wins. Suppose instead that $f(N)$ is Abelian. As Abelian groups are determined by their orders, we have by the discussion in the first paragraph that Spoiler wins with 2 pebbles and 2 rounds.

So now suppose that $N \cong f(N) \leq K$ is a normal Abelian Hall subgroup, but that $f(N)$ does not have a complement isomorphic to $H$. We note that if $K$ contains a subgroup $K'$ that is isomorphic to $H$, then by order considerations, $H'$ and $f(N)$ would intersect trivially in $K$ and we would have that $K = H' \cdot f(N)$. That is, $K$ would decompose as $K = H \rtimes f(N)$. So as $f(N)$ does not have a complement in $K$ that is isomorphic to $H$, by assumption we have that $K$ does not contain any subgroup isomorphic to $H$. In this case, Spoiler pebbles the $O(1)$ generators of $H$ in $G$. As $K$ has no subgroup isomorphic to $H$, Spoiler immediately wins after the generators for $H \leq G$ have been pebbled. The result follows.

**Theorem 3.6.** Let $G = H \rtimes N$, where $N$ is Abelian, $H$ is $O(1)$-generated, and gcd$(|H|,|N|) = 1$. We have that $(O(1), O(1))$-WL Version II identifies $G$.

**Proof.** Let $K \not\cong G$ be an arbitrary group of order $|G|$. By Lem. 3.5, we may assume that $K = H \rtimes N$; otherwise, Spoiler wins in at most 2 rounds. Furthermore, from the proof of Lem. 3.5 we may assume that Duplicator selects bijections $f : G \rightarrow K$ mapping $N \cong f(N)$ (though $f|_N$ need not be an isomorphism), or Spoiler wins with a single round.

Spoiler uses the first $k := d(H)$ rounds to pebble generators $(g_1, \ldots, g_k) \mapsto (h_1, \ldots, h_k)$ for $H$. As $K = H \rtimes N$, we may assume that the map $(g_1, \ldots, g_k) \mapsto (h_1, \ldots, h_k)$ induces an isomorphism with a copy of $H \leq K$; otherwise, Spoiler immediately wins. Let $f : G \rightarrow K$ be the bijection that Duplicator selects. As $G, K$ are non-isomorphic groups of the form $H \rtimes N$, they differ only in their actions. Now the actions are determined by the multisets of indecomposable $H$-modules in $N$. As $|H|$, $|N|$ are coprime, we have by Lem. 3.4 that the indecomposable $H$-modules are cyclic. As $G \not\cong K$, we have by Lem. 3.3 that there exists $n \in N \leq G$ such that $(n)_H$ is indecomposable, and $(f(n))_H$ is inequivalent $H$-modules. Spoiler now pebbles $n \mapsto f(n)$. Thus, the following map

$$(g_1, \ldots, g_k, n) \mapsto (h_1, \ldots, h_k, f(n))$$

does not extend to an isomorphism. So Spoiler wins.

**Remark 3.7.** We see that the main places we used coprimality were: (1) that $N$ was characteristic, and (2) that all indecomposable $H$-modules (in particular, those appearing in $N$) were cyclic.
4 A “rank” lemma

**Definition 4.1.** Let $C \subseteq G$ be a subset of a group $G$ that is closed under taking inverses. We define the $C$-rank of $g \in G$, denoted $\operatorname{rk}_C(g)$, as the minimum $m$ such that $g$ can be written as a word of length $m$ in the elements of $C$. If $g$ cannot be so written, we define $\operatorname{rk}_C(g) = \infty$.

Our definition and results actually extend to subsets that aren’t closed under taking inverses, but we won’t have any need for that case, and it would only serve to make the wording less clear.

**Remark 4.2.** Our terminology is closely related to the usage of “$X$-rank” in algebra and geometry, which generalizes the notions of matrix rank and tensor rank: if $X \subseteq V$ is a subset of an $F$-vector space, then the $X$-rank of a point $v \in V$ is the smallest number of elements of $x \in X$ such that $v$ lies in their linear span. If we replace $X$ by the union $F^*X$ of its nonzero scaled versions (which is unnecessary in the most common case, in which $X$ is the cone over a projective variety), then the $X$-rank in the sense of algebraic geometry would be the $\mathbb{F}X$-rank in our terminology above. For example, matrix rank is $X$-rank inside the space of $n \times m$ matrices under addition, where $X$ is the set of rank-1 matrices (which is already closed under nonzero scaling).

**Lemma 4.3** (Rank lemma). If $C \subseteq G$ is distinguished by $(k, r)$-WL, then any bijection $f$ chosen by Duplicator must respect $C$-rank, in the sense that $\operatorname{rk}_C(g) = \operatorname{rk}_C(f(g))$ for all $g \in G$, or Spoiler can win with $k + 1$ pebbles and $\max\{r, \log d + O(1)\}$ rounds, where $d = \operatorname{diam}(\operatorname{Cay}(C, C)) \leq |C| \leq |G|$.

Our primary uses of this lemma in this paper are to show that if $C$ is distinguished by $(k, r)$-WL, then $\langle C \rangle$ is distinguished by $(k + O(1), r + \log n)$-WL. However, the preservation of $C$-rank itself, rather than merely the subgroup generated by $C$, seems potentially useful for future applications. In particular, Lem. 4.3 shows that WL can identify verbal subgroups in $O(\log n)$ rounds, provided WL can readily identify each word.

**Proof.** Note that since $C$ is detectable by $(k, r)$-WL, there is a set $C' \subseteq H$ such that for any bijection $f$ chosen by Duplicator, $f(C) = C'$, otherwise Spoiler can win with $k$ pebbles in $r$ rounds. We will thus use $\operatorname{rk}(x)$ to denote $\operatorname{rk}_C(x)$ if $x \in G$, and $\operatorname{rk}_C(x)$ if $x \in H$. We proceed by induction on the rank.

By assumption, rank-1 elements must be sent to rank-1 elements, since $C = \{g \in G : \operatorname{rk}(g) = 1\}$, or Spoiler can win with $k$ pebbles in $r$ rounds.

Let $r \geq 1$, and suppose for all $1 \leq j \leq r$, if $\operatorname{rk}(x) = j$, then $\operatorname{rk}(f(x)) = \operatorname{rk}(x)$. Suppose $x \in G$ is such that $\operatorname{rk}(x) = r + 1$ but $\operatorname{rk}(f(x)) \neq \operatorname{rk}(x)$.

Since $f$ is a bijection on elements of smaller rank, the only possibility is $\operatorname{rk}(f(x)) > r + 1$. Spoiler begins by pebbling $x$ to $f(x)$.

Let $f' : G \to H$ be the bijection that Duplicator selects at the next round. Write $x = x_1 \cdots x_{r+1}$, where for each $i$, $x_i \in C$. For $1 \leq i \leq j \leq r + 1$, write $x[i, \ldots , j] := x_i \cdots x_j$. We consider the following cases.

- **Case 1:** Suppose first that $\operatorname{rk}(y) = \operatorname{rk}(f'(y))$ for all $y \in G$ with $\operatorname{rk}(y) \leq r$. In this case, Spoiler pebbles $x[2, \ldots , r + 1] \mapsto f'(x[2, \ldots , r + 1])$. Let $f'' : G \to H$ be the bijection that Duplicator selects at the next round. If $\operatorname{rk}(x_1) = \operatorname{rk}(f''(x_1)) = 1$, then $f''(x_1) \neq f'(x[2, \ldots , r + 1])$, so their product has rank at most $r + 1 < \operatorname{rk}(f(x))$. In this case, Spoiler pebbles $x_1$ and wins immediately since $f''$ does not extend to a bijection on the pebbled elements $x_1, x[2, \ldots , r + 1]$.

- **Case 2:** Suppose instead that the hypothesis of Case 1 is not satisfied. Then $\operatorname{rk}(y) \neq \operatorname{rk}(f'(y))$ for some $y \in \langle C \rangle$ with $\operatorname{rk}(y) \leq r$. In the next two rounds, Spoiler pebbles $x[1, \ldots , [(r + 1)/2]]$ and $x[[r + 1)/2] + 1, \ldots , r + 1]$. Let $f'' : G \to H$ be the next bijection that Duplicator selects. If $f(x) \neq f''(x[1, \ldots , [(r + 1)/2]]) \cdot f''(x[[r + 1)/2] + 1, \ldots , r + 1])$, then Spoiler immediately wins. If $f(x) = f''(x[1, \ldots , [(r + 1)/2]]) \cdot f''(x[[r + 1)/2] + 1, \ldots , r + 1])$, then...
then either
\[
\text{rk}(x[1, \ldots, [(r+1)/2]]) < \text{rk}(f''(x[1, \ldots, [(r+1)/2]])) \quad \text{or} \\
\text{rk}(x([(r+1)/2] + 1, \ldots, r+1]) < \text{rk}(f''(x([(r+1)/2] + 1, \ldots, r+1))),
\]

since \( \text{rk}(f(x)) > \text{rk}(x) = \text{rk}(x[1, \ldots, [(r+1)/2]]) + \text{rk}(x([(r+1)/2] + 1, \ldots, r+1]). \)

Without loss of generality, suppose that \( \text{rk}(x([(r+1)/2]+1, \ldots, r+1]) < \text{rk}(f''(x([(r+1)/2]+1, \ldots, r+1])) \).
As Spoiler has already pebbled \( x([(r+1)/2]+1, \ldots, r+1] \to f''(x([(r+1)/2]+1, \ldots, r+1]), \)
we may iterate on this argument at most \( \log(r+1) + 1 \) times until we hit the case when
\( \text{rk}(x([(r+1)/2]+1, \ldots, r+1]) = 1 < \text{rk}(f''(x([(r+1)/2]+1, \ldots, r+1))). \)
By assumption, every element of \(<C>\) can be written as a word of length at most \( d \) in the elements of \( C \), so we have \( r \leq d \), and thus only \( O(\log d) \) rounds are required. (That \( d \leq |<C>| \) follows from Fact 2.2.)

We claim that the preceding procedure can be implemented with at most \( k + 2 \) pebbles. After Spoiler pebbles \( x([(r+1)/2]+1, \ldots, r+1] \), they may reuse the pebbles on \( x \) and \( x[1, \ldots, [(r+1)/2]] \). We eventually reach a case in which there exists \( g \in C \) such that Duplicator maps \( g \) to some element outside of \(<C>\). In this case, Spoiler pebbles \( g \), using the pebble on \( x \). Now Spoiler uses the pebbles on \( x[1, \ldots, [(r+1)/2]], x([(r+1)/2]+1, \ldots, r+1], \) and \( k-2 \) additional pebbles to win. In total, Spoiler has used \( k+1 \) pebbles. In a similar manner as Case 1, at most \( k-1 \) additional pebbles are required to identify \( C \).

Finally, we must handle the case of infinite rank. By symmetry, it suffices to show that Spoiler can win in the case when \( \text{rk}(x) = r+1 \), but \( \text{rk}(f(x)) = \infty \). In this case, the same argument starting from the third paragraph works \textit{mutatis mutandis}, as \( \text{rk}(f(x)) = \infty > r+1. \)

\[ \square \]

5 Direct Products

Brachter & Schweitzer previously showed that Weisfeiler–Leman Version II can detect direct product decompositions in polynomial-time. Precisely, they showed that the WL dimension of a group \( G \) is at most one more than the WL dimensions of the direct factors of \( G \). We strengthen the result to control for rounds, effectively showing that WL Version II can compute direct product decompositions using \( O(\log n) \) rounds.

In this section, we establish the following.

Theorem 5.1. Let \( G = G_1 \times \cdots \times G_d \) be a decomposition into indecomposable direct factors, let \( k \geq 5 \), and let \( r := r(n) \in \Omega(\log n) \). If \( G \) and \( H \) are not distinguished by \((k, r)\)-WL Version II, then there exist direct factors \( H_i \leq H \) such that \( H = H_1 \times \cdots \times H_d \) such that for all \( i \in [d] \), \( G_i \) and \( H_i \) are not distinguished by \((k-1, r)\)-WL Version II.

The structure and definitions in this section closely follow those of [BS22 Sec. 6] for ease of comparison.

5.1 Preliminaries

We begin by introducing some additional preliminaries.

Definition 5.2. Let \( G_1, G_2 \) be groups, and let \( Z_i \leq Z(G_i) \) be central subgroups. Given an isomorphism \( \varphi : Z_1 \to Z_2 \), the \textit{central product} of \( G_1 \) and \( G_2 \) with respect to \( \varphi \) is:
\[
G_1 \times_\varphi G_2 = (G_1 \times G_2)/\{(g, \varphi(g^{-1})) : g \in Z_1\}.
\]

A group \( G \) is the \textit{(internal) central product} of subgroups \( G_1, G_2 \leq G \), provided that \( G = G_1G_2 \) and \([G_1, G_2] = \{1\} \).

Remark 5.3. In general, a group may have several inherently different central decompositions. On the other hand, indecomposable direct decompositions are unique in the following sense.

Lemma 5.4 (See, e.g., [Rob82 3.8.3]). Let \( G = G_1 \times \cdots \times G_m = H_1 \times \cdots \times H_n \) be two direct decompositions of \( G \) into directly indecomposable factors. Then \( n = m \), and there exists a permutation \( \sigma \in \text{Sym}(n) \) such that for all \( i \), \( G_i \cong H_{\sigma(i)} \) and \( G_iZ(G) \cong H_{\sigma(i)}Z(G) \).
By the preceding lemma, the multiset of subgroups \( \{ \{ G_i, Z(G) \} \} \) is invariant under automorphism.

**Definition 5.5** ([BS22, Def. 6.3]). We say that a central decomposition \( \{ H_1, H_2 \} \) of \( G = H_1H_2 \) is directly induced if there exist subgroups \( K_i \leq H_i \) (\( i = 1, 2 \)) such that \( G = K_1 \times K_2 \) and \( H_i = K_iZ(G) \).

**Lemma 5.6** ([BS22, Lemma 6.4]). Let \( k \geq 4, r \geq 1 \). Let \( G_1, G_2, H_1, H_2 \) be groups such \( G_i \) and \( H_i \) are not distinguished by \( (k, r) \)-WL. Then \( G_1 \times G_2 \) and \( H_1 \times H_2 \) are not distinguished by \( (k, r) \)-WL.

**Remark 5.7.** The statement of [BS22, Lemma 6.4] does not mention rounds; however, the proof holds when considering rounds.

### 5.2 Abelian and Semi-Abelian Case

**Definition 5.8** ([BS22, Def. 6.5]). Let \( G \) be a group. We say that \( x \in G \) splits from \( G \) if there exists a complement \( H \subseteq G \) such that \( G = \langle x \rangle \times H \).

We recall the following technical lemma [BS22, Lemma 6.6] that characterizes the elements that split from an Abelian \( p \)-group.

**Lemma 5.9** ([BS22, Lemma 6.6]). Let \( A \) be a finite Abelian \( p \)-group, and let \( A = A_1 \times \cdots \times A_m \) be an arbitrary cyclic decomposition. Then \( a = (a_1, \ldots, a_m) \in A \) splits from \( A \) if and only if there exists some \( i \in [m] \) such that \( |a_i| = |a| \) and \( a_i \in A_i \setminus \langle A_i \rangle p \). In particular, \( x \) splits from \( A \) if and only if there exists a \( y \in A \) such that \( |xy^p| < |x| \).

We utilize this lemma to show that WL can detect elements that split from \( A \).

**Lemma 5.10.** Let \( A, B \) be Abelian \( p \)-groups of order \( n \), and let \( f : A \rightarrow B \) be the bijection Duplicator selects. If \( x \in A \) splits from \( A \), but \( f(x) \) does not split from \( B \), then Spoiler can win with 2 pebbles and 2 rounds.

**Proof.** Spoiler begins by pebbling \( x \mapsto f(x) \). Let \( f' : A \rightarrow B \) be the bijection that Duplicator selects at the next round. Since \( f(x) \) does not split from \( B \), there exists \( z \in B \) such that \( |f(x) \cdot z^p| < |f(x)| \). Let \( y = (f')^{-1}(z) \in A \). Spoiler pebbles \( y \mapsto f'(y) = z \). Now \( |xy| \neq |f(x) \cdot z| \). So Spoiler immediately wins.

**Remark 5.11.** To characterize when an element splits in a general Abelian group \( A \), we begin by considering the decomposition of \( A \) into its Sylow subgroups: \( A = P_1 \times \cdots \times P_m \). Now \( x = (x_1, \ldots, x_m) \in A \) splits from \( A \) if and only if for each \( i \in [m] \), \( x_i \) is either trivial or splits from \( P_i \). See, e.g., [BS22, Lemma 6.8].

**Lemma 5.12.** Let \( A, B \) be Abelian groups. Let \( A = P_1 \times \cdots \times P_m \) and \( B = Q_1 \times \cdots \times Q_m \), where the \( P_i \) are the Sylow subgroups of \( A \) and the \( Q_i \) are the Sylow subgroups of \( B \) (for each \( i \), \( P_i \) and \( Q_i \) are \( p_i \)-subgroups for the same prime \( p_i \)). Let \( f : A \rightarrow B \) be the bijection that Duplicator selects. Let \( x = (x_1, \ldots, x_m) \) be the decomposition of \( x \), where \( x_i \in P_i \), and let \( f(x) = (y_1, \ldots, y_m) \), where \( y_i \in Q_i \). Suppose that Spoiler pebbles \( x \mapsto f(x) \). Let \( f' : A \rightarrow B \) be the bijection that Duplicator selects at the next round.

(a) If \( f'(x_i) \neq y_i \), then Spoiler can win with 1 additional pebbles and 1 additional round.

(b) If \( x \in A \) splits from \( A \), but \( f(x) \) does not split from \( B \), then Spoiler can win with 2 pebbles and 2 rounds.

**Proof.** We proceed as follows.

(a) Suppose there exists an \( i \in [m] \) such that \( f'(x_i) \neq y_i \). Spoiler pebbles \( x_i \mapsto f'(x_i) \). Suppose that \( P_i, Q_i \) are Sylow \( p \)-subgroups of \( A, B \) respectively. As \( x_i \in P_i \), we have that \( \langle x \cdot x_i^{-1} \rangle \) has order coprime to \( p \). However, as \( f(x_i) \neq y_i \), \( f(x) \cdot f(x_i)^{-1} \) has order divisible by \( p \). So \( |x \cdot x_i^{-1}| 
eq |f(x) \cdot f(x_i)^{-1}| \). Thus, Spoiler wins at the end of this round.

(b) We recall that nilpotent groups are direct products of their Sylow subgroups. Furthermore, for a given prime divisor \( p \), the Sylow \( p \)-subgroup of a nilpotent group is unique and contains all the elements whose order is a power of \( p \). Thus, each Sylow subgroup of a nilpotent group is characteristic as a set. So now by (a), we may assume that \( f'(x_i) = y_i \). Let \( i \in [m] \) such that \( x_i \) splits from \( P_i \), but \( f'(x_i) = y_i \) does not split from \( Q_i \). Spoiler pebbles \( x_i \mapsto f'(x_i) = y_i \). Now by Lem. 5.10 applied to \( x_i \mapsto f'(x_i) \), Spoiler wins with 2 additional pebbles and 2 additional rounds.
We now show that Duplicator must select bijections that preserve both the center and commutator subgroups setwise. Here is our first application of the Rank Lemma [4.3] which was not present in [BS22]. We begin with the following standard definition.

**Definition 5.13.** For a group $G$ and $g \in G$, the commutator width of $g$, denoted $cw(g)$, is the $\{[g, h] : g, h \in G\}$-rank (see Definition [4.1]). The commutator width of $G$, denoted $cw(G)$, is the maximum commutator width of any element of $[G, G]$.

**Lemma 5.14.** Let $G, H$ be finite groups of order $n$. Let $f : G \to H$ be the bijection that Duplicator selects.

(a) If $f(Z(G)) \neq Z(H)$, then Spoiler can win with 2 pebbles and 2 rounds.

(b) If there exist $x, y \in G$ such that $f([x, y])$ is not a commutator $[h, h']$ for any $h, h' \in H$, then Spoiler can win with 3 pebbles and 3 rounds.

(c) If there exists $g \in G$ such that $cw(g) \neq cw(f(g))$, then Spoiler can win with 4 pebbles and $O(\log cw(G)) \leq O(\log n)$ rounds.

Brachter & Schweitzer previously showed that 2-WL Version II identifies $Z(G)$, and 3-WL Version II identifies the commutator $[G, G]$ [BS22]. Here, using our Rank Lemma [4.3] for commutator width, we obtain that 4-WL identifies the commutator in $O(\log n)$ rounds.

**Proof of Lem. 5.14.**

(a) Let $x \in Z(G)$ such that $f(x) \not\in Z(H)$. Spoiler begins by pebbling $x \mapsto f(x)$. Let $f' : G \to H$ be the bijection that Duplicator selects at the next round. Let $y \in H$ such that $f'(x)$ and $y$ do not commute. Let $a := (f')^{-1}(y) \in G$. Spoiler pebbles $a \mapsto f'(a) = y$ and wins.

(b) Spoiler pebbles $[x, y] \mapsto f([x, y])$. At the next two rounds, Spoiler pebbles $x, y$. Regardless of Duplicator’s choices, Spoiler wins.

(c) Apply the Rank Lemma [4.3] to the set of commutators. By part (b), this set is identified by $(4, O(\log n))$-WL, so the Rank Lemma gives part (c).

By Lem. 5.14 Duplicator must select bijections that preserve the center and commutator subgroups setwise (or Spoiler can win). A priori, these bijections need not restrict to isomorphisms on the center or commutator. We note, however, that we may easily decide whether two groups have isomorphic centers, as the center is Abelian. Precisely, by [BS22 Corollary 5.3], $(2, 1)$-WL identifies Abelian groups. Note that we need an extra round to handle the case in which Duplicator maps an element of $Z(G)$ to some element not in $Z(H)$. So $(2, 2)$-WL identifies both the set of elements in $Z(G)$ and its isomorphism type.

We now turn to detecting elements that split from arbitrary groups. To this end, we recall the following lemma from [BS22].

**Lemma 5.15 ([BS22 Lemma 6.9]).** Let $G$ be a finite group and $z \in Z(G)$. Then $z$ splits from $G$ if and only if $z[G, G]$ splits from $G/[G, G]$ and $z \not\in [G, G]$.

We apply this lemma to show that WL can detect the set of elements that split from an arbitrary finite group, improving [BS22 Corollary 6.10] to control for rounds:

**Lemma 5.16 (Compare rounds cf. [BS22 Corollary 6.10]).** Let $G, H$ be finite groups. Let $f : G \to H$ be the bijection that Duplicator selects. Suppose that $x$ splits from $G$, but $f(x)$ does not split from $H$. Then Spoiler can win with 4 pebbles and $O(\log n)$ rounds.
Proof. By Lem. 5.14 we have that if \( x \notin [G, G] \) but \( f(x) \in [H, H] \), then Spoiler can win with 4 pebbles and \( O(\log n) \) rounds. So suppose that \( x \notin [G, G] \) and \( f(x) \notin [H, H] \). It suffices to check whether \( x[G, G] \) splits from \( G/[G, G] \), but \( f(x)[H, H] \) does not split from \( H/[H, H] \). By [BS22, Lemma 4.11], it suffices to consider the pebble game on \( (G/[G, G], H/[H, H]) \). To this end, we apply Lem. 5.12 to \( G/[G, G] \) and \( H/[H, H] \).

As \( f([G, G]) = [H, H] \), \( f \) induces a bijection \( \tilde{f} : G/[G, G] \to H/[H, H] \) on the cosets. As \( f \) was an arbitrary bijection that preserves \([G, G]\), we may assume WLOG that Duplicator selected \( \tilde{f} \) in the quotient game on \((G/[G, G], H/[H, H])\). The result now follows.

We now consider splitting in two special cases.

**Lemma 5.17 ([BS22 Lemma 6.11]).** Let \( U \subseteq G \) and \( x \in Z(G) \cap U \). If \( x \) splits from \( G \), then \( x \) splits from \( U \).

**Lemma 5.18 ([BS22 Lemma 6.12]).** Let \( G = G_1 \times G_2 \), and let \( z := (z_1, z_2) \in Z(G) \) be an element of order \( p^k \) for some prime \( p \). Then \( z \) splits from \( G \) if and only if there exists an \( i \in \{1, 2\} \) such that \( z_i \) splits from \( G_i \) and \( |z_i| = |z| \).

We now consider the semi-Abelian case. Here our groups are of the form \( H \times A \), where \( H \) has no Abelian direct factors and \( A \) is Abelian.

**Theorem 5.19 (Compare rounds cf. [BS22 Lemma 6.13]).** Let \( G_1 = H \times A \), with a maximal Abelian direct factor \( A \). Then the isomorphism class of \( A \) is identified by \((4, O(1))-\text{WL Version II}\). That is, if \((4, O(1))-\text{WL}\) fails to distinguish \( G \) and \( \tilde{G} \), then \( \tilde{G} \) has a maximal Abelian direct factor isomorphic to \( A \).

Proof. We adapt the proof of [BS22, Lemma 6.13] to control for rounds. Let \( \tilde{G} \) be a group such that \((4, O(1))-\text{WL}\) fails to distinguish \( G \) and \( \tilde{G} \). By Lem. 5.14 and the subsequent discussion, we may assume that \( Z(G) \cong Z(\tilde{G}) \) using \((2, 2))-\text{WL Version II}\). As Abelian groups are direct products of their Sylow subgroups, it follows that \( Z(G) \) and \( Z(\tilde{G}) \) have isomorphic Sylow subgroups. Write \( \tilde{G} = \tilde{H} \times \tilde{A} \), where \( \tilde{A} \) is the maximal Abelian direct factor. As \( Z(G) \cong Z(\tilde{G}) \), we write \( Z \) for the Sylow \( p \)-subgroup of \( Z(G) \cong Z(\tilde{G}) \). Consider the primary decomposition of \( Z \):

\[
Z := Z_1 \times \ldots \times Z_m,
\]

where \( Z_i \cong (Z/p^e_i Z)^{e_i} \), for \( e_i \geq 0 \). For each \( i \in [m] \), there exist subgroups \( H_i \leq Z(H) \) and \( A_i \leq A \) such that \( Z_i \cong H_i \times A_i \). Similarly, there exist \( \tilde{H}_i \leq \tilde{H} \) and \( \tilde{A}_i \leq \tilde{A} \) such that \( Z_i \cong \tilde{H}_i \times \tilde{A}_i \). As \( Z(G) \cong Z(\tilde{G}) \), we have that \( H_i \times A_i \cong \tilde{H}_i \times \tilde{A}_i \). It suffices to show that each \( A_i \cong \tilde{A}_i \). As \( H \) does not have any Abelian direct factors, we have by [BS22, Lemma 6.12] (reproduced as Lem. 5.18 above) that a central element \( x \) of order \( p^k \) splits from \( G \) if and only if the projection of \( x \) onto \( A_i \), denoted \( A_i(x) \), has order \( p^i \). The same holds for \( \tilde{G} \) and the \( \tilde{A}_i \) factors. By Lem. 5.12, we may assume that Duplicator selects bijections \( f : G \to H \) such that if \( x \in Z(G) \) splits from \( Z_i \), then \( f(x) \) splits from \( f(Z_i) \). The result follows.

We now recall the definition of a component-wise filtration, introduced by Brachter & Schweitzer [BS22] to control the non-Abelian part of a direct product.

**Definition 5.20 ([BS22 Def. 6.14]).** Let \( G = L \times R \), and let \( \pi_L(U_i) \) (resp. \( \pi_R(U_i) \)) be the natural projection onto \( L \) with kernel \( R \) (resp., the natural projection onto \( R \) with kernel \( L \)). A component-wise filtration of \( U \subseteq G \) with respect to \( L \) and \( R \) is a chain of subgroups \( \{1\} = U_0 \leq \cdots \leq U_r = U \), such that for all \( i \in [r] \), we have that \( U_{i+1} \leq \pi_L(U_i) \) or \( U_{i+1} \leq \pi_R(U_i) \). For \( J \in \{I, II, III\} \), the filtration is \( k \)-WL\(_J\)-detectable, provided all subgroups in the chain are \( k \)-WL\(_J\)-detectable.

Brachter & Schweitzer previously showed [BS22, Lemma 6.15] that there exists a component-wise filtration of \( Z(G) \) with respect to \( H \) and \( A \) that is \( 4 \)-WL\(_J\)-detectable. We extend this result to control for rounds. The proof that such a filtration exists is identical to that of [BS22, Lemma 6.15]: we get a bound on the rounds using our Lem. 5.16 which is a round-controlled version of their Corollary 6.10. For completeness, we indicated the needed changes here.

**Lemma 5.21.** Let \( G = H \times A \), with maximal Abelian direct factor \( A \). The component-wise filtration of \( Z(G) \) with respect to \( H \) and \( A \) from [BS22, Lemma 6.15] (reproduced above) is \((4, O(\log n))-\text{WL}_{\text{II}}\)-detectable.
Proof. Their proof that the filtration is 4-WL detectable uses only two parts: the fact that central $e$-th powers are detectable, and their Corollary 6.10. Using our Lem. 5.16 in place of their Corollary 6.10, we get 4 pebbles and $O(\log n)$ rounds, so all that is left to handle is central $e$-th powers. Suppose Duplicator selects a bijection $f : G \to H$ where $g = x^e$ for some $x \in Z(G)$ and $f(g)$ is not a central $e$-th power. We have already seen that Duplicator must map the center to the center, so we need only handle the condition of being an $e$-th power. At the first round, Spoiler pebbles $g \mapsto f(g)$. At the next round, Spoiler pebbles $x$ and wins. Thus Spoiler can win with 2 pebbles in 2 rounds. \qed

We now show that in the semi-Abelian case $G = H \times A$, with maximal Abelian direct factor $A$, the WL dimension of $G$ depends on the WL dimension of $H$.

**Lemma 5.22** (Compare rounds to [BS22, Lemma 6.16]). Let $G = H \times A$ and $\tilde{G} = \tilde{H} \times \tilde{A}$, where $A$ and $\tilde{A}$ are maximal Abelian direct factors. Let $k \geq 5$ and $r \in \Omega(\log n)$. If $(k-r)$-WL fails to distinguish $G$ and $\tilde{G}$, then $(k-r)$-WL fails to distinguish $H$ and $\tilde{H}$.

Proof. By Thm. 5.19 we may assume that $A \cong \tilde{A}$. Consider the component-wise filtrations from the proof of [BS22, Lemma 6.15], $\{1\} = U_0 \leq \cdots \leq U_r = Z(G)$ with respect to the decomposition $G = H \times A$ and $\{1\} = \tilde{U}_0 \leq \cdots \leq \tilde{U}_r = Z(\tilde{G})$ with respect to the decomposition $\tilde{G} = \tilde{H} \times \tilde{A}$.

Let $V_i, W_i, \tilde{V}_i, \tilde{W}_i$ be as defined in the proof of [BS22, Lemma 6.15] and recalled above. We showed in the proof of Lem. 5.21 that for any bijection $f : G \to \tilde{G}$ Duplicator selects, $f(V_i) = \tilde{V}_i$ and $f(W_i) = \tilde{W}_i$, or Spoiler may win with 4 pebbles and $O(\log n)$ rounds.

In the proof of [BS22, Lemma 6.16], Brachter & Schweitzer established that for all $1 \neq x \in Z(H) \times \{1\}$ and all $1 \neq y \in \{1\} \times A$, $\min\{i : x \in U_i\} \neq \min\{i : y \in U_i\}$. Furthermore, by [BS22, Lemma 4.14], we may assume that Duplicator selects bijections at each round that respect the subgroup chains and their respective cosets, without altering the number of rounds (their proof is round-by-round). It follows that whenever $g_1g_2^{-1} \in Z(H) \times \{1\}$, we have that $f(g_1)f(g_2)^{-1} \notin \{1\} \times A$.

Furthermore, Brachter & Schweitzer also showed in the proof of [BS22, Lemma 6.16] that Duplicator must map $H \times \{1\}$ to a system of representatives modulo $\{1\} \times \tilde{A}$. Thus, Spoiler can restrict the game to $H \times \{1\}$. Now if $(k,r)$-WL Version II distinguishes $H$ and $\tilde{H}$, then Spoiler can ultimately reach a configuration $((h_1,1),\ldots,(h_k,1)) \mapsto (x_1,a_1),\ldots,(x_k,a_k)$ such that the induced configuration over $(G/\{1\} \times A),\tilde{G}/\{1\} \times \tilde{A}\}$ fulfills the winning condition for Spoiler. That is, considered as elements of $G/\{1\} \times A$ (resp., $\tilde{G}/\{1\} \times \tilde{A}\}$, the map $(h_1,\ldots,h_k) \mapsto (x_1,\ldots,x_k)$ does not extend to an isomorphism. This implies that the pebbled map $((h_1,1),\ldots,(h_k,1)) \mapsto (x_1,a_1),\ldots,(x_k,a_k)$ in the original groups (rather than their quotients) does not extend to an isomorphism. For suppose $f$ is any bijection extending the pebbled map. By the above, without loss of generality, $f$ maps $H \times \{1\}$ to a system of coset representatives of $\{1\} \times \tilde{A}$, that is, if Duplicator can win, Duplicator can win with such a map. Let $\tilde{f}$ be the induced bijection on the quotients $G/\{1\} \times A \to \tilde{G}/\{1\} \times \tilde{A}$. Since the pebbled map on the quotients does not extend to an isomorphism, there is a word $w$ such that $\tilde{f}(w(h_1,\ldots,h_k)) \neq w(x_1,\ldots,x_k)$. But then when we consider $f$ restricted to $H \times \{1\}$, we find that $f(w(h_1,1),\ldots,(h_k,1))) = f((w(h_1,\ldots,h_k),1)) \neq (w(x_1,\ldots,x_k),w(a_1,\ldots,a_k))$, because their $H$ coordinates are different. \qed

Lem. 5.22 yields the following corollary.

**Corollary 5.23.** Let $G = H \times A$, where $H$ is identified by $(O(1),O(\log n))$-WL and does not have an Abelian direct factor, and $A$ is Abelian. Then $(O(1),O(\log n))$-WL identifies $G$. In particular, isomorphism testing of $G$ and an arbitrary group $\tilde{G}$ is in $\text{TC}^1$.

Proof. By Lem. 5.22 as $H$ is identified by $(O(1),O(1))$-WL, we have that $G$ is identified by $(O(1),O(\log n))$-WL. As only $O(1)$ rounds are required, we apply the parallel WL implementation due to Grohe & Verbitsky [GV06] to obtain the bound of $\text{TC}^1$ for isomorphism testing. \qed

**Remark 5.24.** Das & Sharma [DS19] previously exhibited a nearly-linear time algorithm for groups of the form $H \times A$, where $H$ has size $O(1)$ and $A$ is Abelian. Cor. 5.23 generalizes this to the setting where $H$ is $O(1)$-generated. While Cor. 5.23 does not improve the runtime, it does establish a new parallel upper bound for isomorphism testing.
5.3 General Case

Following the strategy in [BS22], we reduce the general case to the semi-Abelian case. Consider a direct decomposition $G = G_1 \times \ldots \times G_d$, where each $G_i$ is directly indecomposable. The multiset of subgroups $\{\langle G, Z(G) \rangle \}$ is independent of the choice of decomposition. We first show that Weisfeiler–Leman detects $\bigcup_i G_iZ(G)$. Next, we utilize the fact that the connected components of the non-commuting graph on $G$, restricted to $\bigcup_i G_iZ(G)$, correspond to the subgroups $G_iZ(G)$.

**Definition 5.25.** For a group $G$, the non-commuting graph $X_G$ has vertex set $G$, and an edge $\{g, h\}$ precisely when $[g, h] \neq 1$.

**Proposition 5.26 (AAM06 Proposition 2.1).** If $G$ is non-Abelian, then $X_G|G \setminus Z(G)$ is connected.

Our goal now is to first construct a canonical central decomposition of $G$ that is detectable by WL. This decomposition will serve to approximate $\bigcup_i G_iZ(G)$ from below.

**Definition 5.27 (BS22 Definition 6.19).** Let $G$ be a finite, non-Abelian group. Let $M_1$ be the set of non-central elements $g$ whose centralizers $C_G(g)$ have maximal order among all non-central elements. For $i \geq 1$, define $M_{i+1}$ to be the union of $M_i$ and the set of elements $g \in G \setminus \langle M_i \rangle$ that have maximal centralizer order $|C_G(g)|$ amongst the elements in $G \setminus \langle M_i \rangle$. Let $M := M_\infty$ be the stable set resulting from this procedure.

Consider the subgraph $X_G|M$, and let $X_1, \ldots, X_m$ be the connected components. Set $N_i := \langle X_i \rangle$. We refer to $N_1, \ldots, N_m$ as the non-Abelian components of $G$.

Brachter & Schweitzer previously established the following [BS22].

**Lemma 5.28 ([BS22 Lemma 6.20]).** In the notation of Definition 5.27, we have the following.

(a) $M$ is 3-WL$_H$-detectable.

(b) $G = N_1 \cdots N_m$ is a central decomposition of $G$. For all $i$, $Z(G) \leq N_i$ and $N_i$ is non-Abelian. In particular, $M$ generates $G$.

(c) If $G = G_1 \times \ldots \times G_d$ is an arbitrary direct decomposition, then for each $i \in [m]$, there exists a unique $j \in [d]$ such that $N_i \subseteq G_jZ(G)$. Collect all such $i$ for one fixed $j$ in an index set $I_j$. Then

$$N_{i_1}N_{i_2}\cdots N_{i_\ell} = G_jZ(G),$$

where $I_j = \{j_1, \ldots, j_\ell\}$.

We note that Lem. 5.28 (b)-(c) are purely group theoretic statements. For our purposes, it is necessary, however, to adapt Lem. 5.28 (a) to control for rounds. This is our second use of our Rank Lemma 4.3. This time applied to the set $M_i$ from Definition 5.27.

**Lemma 5.29.** Let $G$ and $H$ be finite non-Abelian groups, let $M_{i,G}$ (resp., $M_{i,H}$) denote the sets from Definition 5.27 for $G$ (resp., $H$). Let $f : G \to H$ be a bijection that Duplicator selects. If for some $i$, $\text{rk}_{M_{i,G}}(g) \neq \text{rk}_{M_{i,H}}(f(g))$, then Spoiler can win with 3 pebbles and $O(\log n)$ rounds.

**Proof.** Let $M_{i,G}, M_{i,H}$ be the sets in $G$ as in Definition 5.27, and let $M_{i,H}, M_H$ be the corresponding sets in $H$. We show that $f(M_{i,G}) = M_{i,H}$ and $f(\langle M_{i,G} \rangle) = \langle M_{i,H} \rangle$. These statements imply that $f(M_{i,G}) = M_{i,H}$. The proof proceeds by induction over $i$, and within each $i$, we use the Rank Lemma 4.3 applied to $M_i$-rank. Note that each $M_i$ is closed under taking inverses, since $C_G(g) = C_G(g^{-1})$.

We first note that if $|C_G(g)| \neq |C_H(f(g))|$, then Duplicator may win with 2 pebbles and 2 rounds. Without loss of generality, suppose that $|C_G(g)| > |C_H(f(g))|$. Spoiler pebbles $g \to f(g)$. Let $f' : G \to H$ be the bijection that Duplicator selects at the next round. Now there exists $x \in C_G(g)$ such that $f'(x) \notin C_H(f(g))$. Spoiler pebbles $x \to f'(x)$ and wins immediately.

Thus $M_{i,G}$ is identified by $(2, 2)$-WL. By the Rank Lemma 4.3 applied to $M_i$-rank, we get that $f(\langle M_{i,G} \rangle) = \langle M_{i,H} \rangle$ or Spoiler can win with 3 pebbles in $O(\log n)$ rounds.

As Duplicator must select bijections $f : G \to H$ where $f(\langle M_{i,G} \rangle) = \langle M_{i,H} \rangle$, we may iterate on the above argument replacing 1 with $i$, to obtain that $f(M_{i,G}) = M_{i,H}$ and $f(\langle M_{i,G} \rangle) = \langle M_{i,H} \rangle$. The result now follows by induction. \(\square\)
Definition 5.30. Let $G = N_1 \cdots N_m$ be the decomposition into non-Abelian components, and let $G = G_1 \times \cdots \times G_d$ be an arbitrary direct decomposition. We say that $x \in G$ is full for $(G_j, \ldots, G_j)$, if

$$\{i \in [m] : [x, N_i] \neq 1\} = \bigcup_{\ell=1}^r I_{\ell x},$$

where the $I_{\ell x}$ are as in Lem. 5.28 (c). For all $x \in G$, define $C_x := \prod_{[x, N_i] = 1} N_i$ and $N_x = \prod_{[x, N_i] \neq 1} N_i$.

We now recall some technical lemmas from [BS22].

Remark 5.31 ([BS22 Observation 6.22]). For an arbitrary collection of indices $J \subseteq [m]$, the group elements $x \in G$ that have $C_x = \prod_{i \in J} N_i$ are exactly those elements of the form $x = z \prod_{i \in J} n_i$ with $z \in Z(G)$ and $n_i \in N_i \setminus Z(G)$. In particular, full elements exist for every collection of non-Abelian direct factors and any direct decomposition, and they are exactly given by products over non-central elements of the corresponding non-Abelian components.

Lemma 5.32 ([BS22 Lemma 6.23]). Let $G$ be non-Abelian, and let $G = G_1 \times \cdots \times G_d$ be an indecomposable direct decomposition. For all $x \in G$, we have a central decomposition $G = C_x N_x$, with $Z(G) \leq C_x \cap N_x$. The decomposition is directly induced if and only if $x$ is full for a collection of direct factors of $G$.

Lemma 5.33 (Compare rounds cf. [BS22 Lemma 6.24]). Let $G = G_1 \times G_2$. For $k \geq 4, r \in \Omega(\log n)$, assume that $(k, r)$-WL Version II detects $G_1Z(G)$ and $G_2Z(G)$. Let $H$ be a group such that $(k, r)$-WL Version II does not distinguish $G$ and $H$. Then for $i \in \{1, 2\}$, there exist subgroups $H_i \leq H$ such that $H = H_1 \times H_2$ and $(k, r)$-WL does not distinguish $G_iZ(G)$ and $H_iZ(H)$.

Proof. The proof is largely identical to that of [BS22 Lemma 6.24]. We adapt their proof to control for rounds.

As $(k, r)$-WL detects $G_1Z(G)$ and $G_2Z(G)$, we have that for any two bijections $f, f' : G \to H$ that $f(G_iZ(G)) = f'(G_iZ(G))$ for $i \in \{1, 2\}$. It follows that there exist subgroups of $H_i \leq H$ such that $f(G_iZ(G)) = \tilde{H}_i$. As $Z(G) \leq G_iZ(G)$, we have necessarily that $Z(H) \subseteq \tilde{H}_i$. Consider the decompositions $Z(G) = Z(G_1) \times Z(G_2)$ and $G_iZ(G) = G_i \times Z(G_{i+1 \mod 2})$. By Lem. 5.18 we have that if $x$ splits from $Z(G)$, then $x$ also splits from $G_1Z(G)$ or $G_2Z(G)$.

Write $\tilde{H}_i = R_i \times B_i$, where $B_i$ is a maximal Abelian direct factor of $\tilde{H}_i$.

Claim 1: For all choices of $R_i, B_i$, it holds that $R_1 \cap R_2 = \{1\}$. Otherwise, Spoiler can win with 2 additional pebbles and 2 additional rounds.

Proof of Claim 1. By assumption, $\tilde{H}_1 \cap \tilde{H}_2 = Z(H)$. So $R_1 \cap R_2 \leq Z(H)$. Suppose to the contrary that there exists $z \in R_1 \cap R_2$ such that $|z| = p$ for some prime $p$. Then there also exists a central $p$-element $w$ that splits from $Z(H)$ and where $z \in \langle w \rangle$ (for instance, we may take $w$ to be a root of $z^pN$, where $N$ is the largest $p$-power order in the Abelian group $Z(H)$). Write $w = (r_i, b_i)$ with respect to the chosen direct decomposition for $\tilde{H}_i$. As $z \in \langle w \rangle$, we have that $w^m = z \in R_i \cap R_2$. So $w^m \neq 1$. Furthermore, we may write $w^m = (r_1^m, 1) = (r_2^m, 1)$. As $w$ has $p$-power order, we have as well that $|b_i| < |r_i|$ for each $i \in \{1, 2\}$. Now $w$ does not split from $\tilde{H}_1$; otherwise, by Lem. 5.18 we would have that $r_i$ splits from $R_i$. However, neither $R_1$ nor $R_2$ admit Abelian direct factors.

It follows that $w$ splits from $Z(H)$, but not from $\tilde{H}_1$ or $\tilde{H}_2$. Such elements do not exist in $G_1Z(G)$ or $G_2Z(G)$. Thus, in this case, we have by Lem. 5.16 that Spoiler can win with 2 additional pebbles and 2 additional rounds.

We next consider maximal Abelian direct factors $A \leq G$ and $B \leq H$. Write $H = R \times B$. By Thm. 5.19 we may assume that $A \cong B$. We now argue that $R_1$ and $R_2$ can be chosen such that $R_1R_2 \cap B = \{1\}$. For $i \in \{1, 2\}$, we may write:

$$\tilde{H}_i = (r_1, b_1), \ldots, (r_1, b_i) \leq R \times B.$$
As $B \leq \tilde{H}_i$, we have that:

$$\tilde{H}_i = ((r_1, 1), (1, b_1), \ldots, (r_t, 1), (b_t, 1)) = ((r_1, 1), \ldots, (r_t, 1)) \times B.$$ 

It follows that we may choose $R_1 R_2 \leq R$. By Claim 1, we have that $R_1 \cap R_2 = \{1\}$. So $R_1 R_2 B = R_1 \times R_2 \times B \leq H$. As $(k, r)$-WL fails to distinguish $G$ and $H$, we have necessarily that $|R_1| \cdot |R_2| \cdot |B| = |H|$. So in fact, $H = R_1 \times R_2 \times B$, which we may write as $(R_1 \times B_1) \times (R_2 \times B_2)$, where $B_i \leq H_i$ are chosen such that $B = B_1 \times B_2$ and $B_i$ is isomorphic to a maximal Abelian direct factor of $G_i$. Furthermore, we have that $R_i Z(H) = H_i$, by construction. The result follows.

**Lemma 5.34.** Let $G = N_1 \cdots N_m$ and $H = Q_1 \cdots Q_m$ be the decompositions of $G$ and $H$ into non-Abelian components. Let $G = G_1 \times \ldots \times G_d$ be a decomposition into indecomposable direct factors. Let $f : G \to H$ be the bijection that Duplicator selects. By Cor. 5.35, we may assume that Duplicator preserves the full components. Let $f$ be arbitrary. Let $H = G_1 \times \ldots \times G_d$ be a decomposition into indecomposable direct factors. If $f$ preserves the full components, then $f$ is isomorphic to a maximal Abelian direct factor of $G$. Furthermore, we have that $R_i Z(H) = H_i$, by construction. The result follows.

**Lemma 5.35.** Let $G = G_1 \times \ldots \times G_d$ be a decomposition of $G$ into directly indecomposable factors. Let $H$ be arbitrary. Let $F_G$ be the set of full elements for $G$, and define $F_H$ analogously. If Duplicator does not select a bijection $f : G \to H$ satisfying:

$$f \left( \bigcup_{g \in F_G} N_g \right) = \bigcup_{h \in F_H} N_h,$$

then Spoiler can win using 5 pebbles and $O(\log n)$ rounds.

**Proof.** By Lem. 5.34, we may assume that $f(F_G) = F_H$ (or Spoiler wins with 4 pebbles and $O(\log n)$ rounds). Now suppose that for some $g \in G$ that there exists an $x \in N_g$ such that $f(x) \notin N_h$ for any $h \in F_H$. Spoiler pebbles $x \rightarrow f(x)$. Let $f' : G \to H$ be the bijection Duplicator selects the next round. Again, we may assume that $f(F_G) = F_H$ (or Spoiler wins). Spoiler now pebbles $g \rightarrow f'(g)$. Now on any subsequent bijection, Duplicator cannot map $N_g \rightarrow N_{f'(g)}$. So by Lem. 5.34, Spoiler wins with 4 additional pebbles and $O(\log n)$ rounds.

**Theorem 5.36.** Let $k \geq 5, r \in \Omega(\log n)$. Let $G$ be a non-Abelian group, and let $G = G_1 \times \ldots \times G_d$ be a decomposition into indecomposable direct factors. If $(k, r)$-WL Version II fails to distinguish $G$ and $H$, then there exist indecomposable direct factors $H_i \leq H$ such that $H = H_1 \times \ldots \times H_d$ and $(k-1, r)$-WL Version II fails to distinguish $G$ and $H$ for all $i \in [d]$. Furthermore, $G$ and $H$ have isomorphic maximal Abelian direct factors and $(k, r)$-WL fails to distinguish $G, Z(G)$ from $H, Z(H)$.

**Proof.** Without loss of generality, we may assume that $H$ is non-Abelian as well. Let $f : G \to H$ be the bijection that Duplicator selects. By Cor. 5.35, we may assume that Duplicator preserves the full elements (or Spoiler wins with 4 pebbles and $O(\log n)$ rounds); that is, $f(F_G) = F_H$. It follows that $H$ must admit a decomposition $H = H_1 \times \ldots \times H_d$, where the $H_i$ factors are directly indecomposable and $F_H = \bigcup_{i=1}^d H_i Z(H) \subseteq H$, which we again note is indistinguishable from $F_G$. Let $X_G$ be the non-commuting graph of $G$, and let $X_H$ be the non-commuting graph of $H$. Recall from [AAM06, Proposition 2.1] that as $G, H$ are non-Abelian, $X_G$ and $X_H$ are connected.

As different direct factors centralize each other, we obtain that for each non-singleton connected component in $K$ of $X_G | F_G$ that there exists a unique indecomposable direct factor $G_i$ such that $K = G_i Z(G) \setminus Z(G)$. 

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Thus, \( G_iZ(G) = \langle K \rangle \). Again by [AAM06] Proposition 2.1, all such non-Abelian direct factors appear in this way.

We note that the claims in the preceding paragraph applies to \( H \) as well. So if \((k, r)\)-WL Version II does not distinguish \( G \) and \( H \), there must exist a bijection between the connected components of \( X_G[F_G] \) and \( X_H[F_H] \). Namely, we may assume that \( G \) and \( H \) admit a decomposition into \( \ell = d \) directly indecomposable factors, and that these subgroups are indistinguishable by \((k, r)\)-WL. In particular, we have a correspondence (after an appropriate reordering of the factors) between \( G_iZ(H) \) and \( H_iZ(H) \), where \( G_iZ(H) \) and \( H_iZ(H) \) are not distinguished by \((k, r)\)-WL. By Lem. 5.22 we have that \((k, r)\)-WL Version II does not distinguish \( G_i \) from \( H_i \). By Thm. 5.19 \( G \) and \( H \) must have isomorphic maximal Abelian direct factors. So when \( G_i, H_i \) are Abelian, we even have \( G_i \cong H_i \).

### 6 Weisfeiler–Leman for Semisimple Groups

In this section, we show that Weisfeiler–Leman can be fruitfully used as a tool to improve the parallel complexity of isomorphism testing of groups with no Abelian normal subgroups, also known as semisimple or Fitting-free groups. The main result of this section is:

**Theorem 6.1.** Let \( G \) be a semisimple group, and let \( H \) be arbitrary. We can test isomorphism between \( G \) and \( H \) using an SAC circuit of depth \( O(\log n) \) and size \( n^{\Theta(\log \log n)} \). Furthermore, all such isomorphisms can be listed in this bound.

The previous best complexity upper bounds were \( P \) for testing isomorphism [BCQ12] and \( \text{DTIME}(n^{O(\log \log n)}) \) for listing isomorphisms [BCGQ11].

We start with what we can observe from known results about direct products of simple groups. Brachter & Schweitzer previously showed that 3-WL Version II identifies direct products of finite simple groups. A closer analysis of their proofs [BS22] Lemmas 5.20 & 5.21 show that only \( O(1) \) rounds are required. Thus, we obtain the following.

**Corollary 6.2 (cf. Brachter & Schweitzer [BS22] Lemmas 5.20 & 5.21).** Isomorphism between a direct product of non-Abelian simple groups and an arbitrary group can be decided in \( L \).

Our parallel machinery also immediately lets us extend a similar result to direct products of almost simple groups (a group \( G \) is almost simple if there is a non-Abelian simple group \( S \) such that \( \text{Inn}(S) \leq G \leq \text{Aut}(S) \); equivalently, if \( \text{Soc}(G) \) is non-Abelian simple).

**Corollary 6.3.** Isomorphism between a direct product of almost simple groups and an arbitrary group can be decided in \( TC^1 \).

**Proof.** Because almost simple groups are 3-generated [DVL95], they are identified by \((O(1), O(1))\)-WL. By Thm. 5.1 direct products of almost simple groups are thus identified by \((O(1), O(\log n))\)-WL.

### 6.1 Preliminaries

We recall some facts about semisimple groups from [BCGQ11]. As a semisimple group \( G \) has no Abelian normal subgroups, we have that \( \text{Soc}(G) \) is the direct product of non-Abelian simple groups. The conjugation action of \( G \) on \( \text{Soc}(G) \) permutes the direct factors of \( \text{Soc}(G) \). So there exists a faithful permutation representation \( \alpha : G \to G^* \leq \text{Aut}(\text{Soc}(G)) \). \( G \) is determined by \( \text{Soc}(G) \) and the action \( \alpha \). Let \( H \) be a semisimple group with the associated action \( \beta : H \to \text{Aut}(\text{Soc}(H)) \). We have that \( G \cong H \) precisely if \( \text{Soc}(G) \cong \text{Soc}(H) \) via an isomorphism that makes \( \alpha \) equivalent to \( \beta \).

We now introduce the notion of permutational isomorphism, which is our notion of equivalence for \( \alpha \) and \( \beta \). Let \( A \) and \( B \) be finite sets, and let \( \pi : A \to B \) be a bijection. For \( \sigma \in \text{Sym}(A) \), let \( \sigma^{\pi} \in \text{Sym}(B) \) be defined by \( \sigma^{\pi} := \pi^{-1} \sigma \pi \). For a set \( \Sigma \subseteq \text{Sym}(A) \), denote \( \Sigma^{\pi} := \{ \sigma^{\pi} : \sigma \in \Sigma \} \). Let \( K \leq \text{Sym}(A) \) and \( L \leq \text{Sym}(B) \) be permutation groups. A bijection \( \pi : A \to B \) is a permutational isomorphism \( K \to L \) if \( K^{\pi} = L \).

The following lemma, applied with \( R = \text{Soc}(G) \) and \( S = \text{Soc}(H) \), precisely characterizes semisimple groups [BCGQ11].
Lemma 6.4 ([BCGQ11 Lemma 3.1]). Let $G$ and $H$ be groups, with $R 	riangleleft G$ and $S \triangleleft H$ groups with trivial centralizers. Let $\alpha : G \to G^* \leq \text{Aut}(R)$ and $\beta : H \to H^* \leq \text{Aut}(S)$ be faithful permutation representations of $G$ and $H$ via the conjugation action on $R$ and $S$, respectively. Let $f : R \to S$ be an isomorphism. Then $f$ extends to an isomorphism $\hat{f} : G \to H$ if and only if $f$ is a permutational isomorphism between $G^*$ and $H^*$; and if so, $\hat{f} = \alpha f^* \beta^{-1}$, where $f^* : G^* \to H^*$ is the isomorphism induced by $f$.

We also need the following standard group-theoretic lemmas. The first provides a key condition for identifying whether a non-Abelian simple group belongs in the socle. Namely, if $S_1 \cong S_2$ are non-Abelian simple groups where $S_1$ is in the socle and $S_2$ is not in the socle, then the normal closures of $S_1$ and $S_2$ are non-isomorphic. In particular, the normal closure of $S_1$ is a direct product of non-Abelian simple groups, while the normal closure of $S_2$ is not a direct product of non-Abelian simple groups. We will apply this condition later when $S_1$ is a simple direct factor of $\text{Soc}(G)$; in which case, the normal closure of $S_1$ is of the form $S_1^*$. We include the proofs of these two lemmas for completeness.

Lemma 6.5. Let $G$ be a finite semisimple group. A subgroup $S \leq G$ is contained in $\text{Soc}(G)$ if and only if the normal closure of $S$ is a direct product of nonabelian simple groups.

Proof. Let $N$ be the normal closure of $S$. Since the socle is normal in $G$ and $N$ is the smallest normal subgroup containing $S$, we have that $S$ is contained in $\text{Soc}(G)$ if and only if $N$ is.

Suppose first that $S$ is contained in the socle. Since $\text{Soc}(G)$ is normal and contains $S$, by the definition of $N$ we have that $N \leq \text{Soc}(G)$. As $N$ is a normal subgroup of $G$, contained in $\text{Soc}(G)$, it is a direct product of minimal normal subgroups of $G$, each of which is a direct product of non-Abelian simple groups.

Conversely, suppose $N$ is a direct product of nonabelian simple groups. We proceed by induction on the size of $N$. If $N$ is minimal normal in $G$, then $N$ is contained in the socle by definition. If $N$ is not minimal normal, then it contains a proper subgroup $M \leq N$ such that $M$ is normal in $G$, hence also $M \leq N$. However, as $N$ is a direct product of nonabelian simple groups $T_1, \ldots, T_k$, the only subgroups of $N$ that are normal in $N$ are direct products of subsets of $\{T_1, \ldots, T_k\}$, and all such normal subgroups have direct complements. Thus we may write $N = L \times M$ where both $L, M$ are nontrivial, hence strictly smaller than $N$, and both $L$ and $M$ are direct product of nonabelian simple groups.

We now argue that $L$ must also be normal in $G$. Since conjugating $N$ by $g \in G$ is an automorphism of $N$, we have that $N = gLg^{-1} \times gMg^{-1}$. Since $M$ is normal in $G$, the second factor here is just $M$, so we have $N = gLg^{-1} \times M$. But since the direct complement of $M$ in $N$ is unique (since $N$ is a direct product of non-Abelian simple groups), we must have $gLg^{-1} = L$. Thus $L$ is normal in $G$.

By induction, both $L$ and $M$ are contained in $\text{Soc}(G)$, and thus so is $N$. We conclude since $N \leq N$.

Corollary 6.6. Let $G$ be a finite semisimple group. A nonabelian simple subgroup $S \leq G$ is a direct factor of $\text{Soc}(G)$ if and only if its normal closure $N = ncl_G(S)$ is isomorphic to $S^k$ for some $k \geq 1$ and $S \leq N$.

Proof. Let $S$ be a nonabelian simple subgroup of $G$. If $S$ is a direct factor of $\text{Soc}(G)$, then $\text{Soc}(G) \cong S^k \times T$ for some $k \geq 1$ and some $T$; choose $T$ such that $k$ is maximal. Then the normal closure of $S$ is a minimal normal subgroup of $\text{Soc}(G)$ which contains $S$ as a normal subgroup. Since the normal subgroups of a direct product of nonabelian simple groups are precisely direct products of subsets of the factors, the normal closure of $S$ is some $S^k$ for $1 \leq k \leq k$.

Conversely, suppose the normal closure $N$ of $S$ is isomorphic to $S^k$ for some $k \geq 1$ and $S \leq N$. By Lem. 6.5 $S$ is in $\text{Soc}(G)$, and thus so is $N$ (being the normal closure of a subgroup of the socle). Furthermore, as a normal subgroup of $G$ contained in $\text{Soc}(G)$, $N$ is a direct product of minimal normal subgroups and a direct factor of $\text{Soc}(G)$ (in fact it is minimal normal itself, but we haven’t established that yet, nor will we need to). Since $S$ is a normal subgroup of $N$, and $S$ is a direct product of non-Abelian simple groups, $S$ is a direct factor of $N$. Since $N$ is a direct factor of $\text{Soc}(G)$, and $S$ is a direct factor of $N$, $S$ is a direct factor of $\text{Soc}(G)$. This completes the proof.

Lemma 6.7. Let $S_1, \ldots, S_k \leq G$ be nonabelian simple subgroups such that for all distinct $i, j \in [k]$ we have $[S_i, S_j] = 1$. Then $\langle S_1, \ldots, S_k \rangle = S_1S_2 \cdots S_k = S_1 \times \cdots \times S_k$.

Proof. By induction on $k$. The base case $k = 1$ is vacuously true. Suppose $k \geq 2$ and that the result holds for $k - 1$. Then $T := S_1S_2 \cdots S_{k-1} = S_1 \times \cdots \times S_{k-1}$. Now, since $S_k$ commutes with each $S_i$, and they generate $T$, we have that $[S_k, T] = 1$. Hence $T$ is contained in the normalizer (or even the centralizer) of
Proposition 6.8. Here we establish Thm. 6.1. We begin with the following.

6.2 Groups without Abelian Normal Subgroups in Parallel

Let $H$ be a semisimple group of order $n$, and let $G$ be an arbitrary group of order $n$. If $H$ is not semisimple, then $3$-WL will distinguish $G$ and $H$ in at most $3$ rounds.

Proof. Recall that a group is semisimple if and only if it contains no Abelian normal subgroups. As $H$ is not semisimple, $\text{Soc}(H) = T \times S$, where $T$ is the direct product of elementary Abelian groups and $S$ is a direct product of non-Abelian simple groups. We show that Duplicator can win using at most $4$ pebbles and $5$ rounds.

Let $f : G \to H$ be the bijection that Duplicator selects. Let $a \in A$. So $\text{ncl}_H(a) \leq A$. Let $b := f^{-1}(a) \in G$, and let $B := \text{ncl}_G(b)$. As $G$ is semisimple, we have that $B$ is not Abelian. Spoiler begins by pebbling $b \mapsto a$.

So there exist $g_1, g_2 \in G$ such that $g_1bg_2^{-1}$ and $g_2bg_2^{-1}$ do not commute (for $B$ is generated by $\{gbg^{-1} : g \in G\}$, and if they all commuted then $B$ would be Abelian). Let $f', f'' : G \to H$ be the bijections that Duplicator selects at the next two rounds. Spoiler pebbles $g_1 \mapsto f'(g_1)$ and $g_2 \mapsto f''(g_2)$ at the next two rounds. As $\text{ncl}(a) = A$ is Abelian, $f'(g_1)f(b)f'(g_1)^{-1}$ and $f''(g_2)f(b)f''(g_2)^{-1}$ commute. Spoiler now wins.

We now apply Lemma 6.3 to show that Duplicator must map the direct factors of $\text{Soc}(G)$ to isomorphic direct factors of $\text{Soc}(H)$.

Lemma 6.9. Let $G, H$ be finite semisimple groups of order $n$. Let $\text{Fac}((\text{Soc}(G))$ denote the set of simple direct factors of $\text{Soc}(G)$. Let $S \in \text{Fac}(\text{Soc}(G))$ be a non-Abelian simple group. Let $a \in S$, and let $f : G \to H$ be the bijection that Duplicator selects.

(a) If $f(a)$ does not belong to some element of $\text{Fac}((\text{Soc}(H))$, or

(b) If there exists some $T \in \text{Fac}(\text{Soc}(H))$ such that $f(a) \in T$, but $S \not\cong T$,

then Spoiler wins with at most $4$ pebbles and $5$ rounds.

Proof. Spoiler begins by pebbling $a \mapsto f(a)$. At the next two rounds, Spoiler pebbles generators $x, y$ for $S$. Let $f' : G \to H$ be the bijection Duplicator selects at the next round. Denote $T := \langle f'(x), f'(y) \rangle$. We note that if $T \not\cong S$ or $f(a) \notin T$, then Spoiler wins.

So suppose that $f(a) \in T$ and $T \cong S$. We have two cases.

• Case 1: Suppose first that $T$ does not belong to $\text{Soc}(H)$. As $S \triangleleft \text{Soc}(G)$, the normal closure $\text{ncl}(S)$ is minimal normal in $G$ [ Isa08, Exercise 2.A.7]. As $T$ is not even contained in $\text{Soc}(H)$, we have by Lemma 6.3 that $\text{ncl}(T)$ is not a direct product of non-Abelian simple groups, so $\text{ncl}(S) \not\cong \text{ncl}(T)$. We note that $\text{ncl}(S) = \langle \{gSg^{-1} : g \in G \} \rangle$.

As $\text{ncl}(T)$ is not isomorphic to a direct power of $S$, there is some conjugate $gSg^{-1} \not\in S$ such that $f'(g)Tf'(g)^{-1}$ does not commute with $T$, by Lemma 6.7 Yet since $S \triangleleft \text{Soc}(G)$, $gSg^{-1}$ and $S$ do commute. Spoiler moves the pebble pair from $a \mapsto f(a)$ and pebbles $g$ with $f'(g)$. Since Spoiler has now pebbled $x, y, g$ which generate $\langle S, gSg^{-1} \rangle = S \times gSg^{-1} \cong S \times S$ but the image is not isomorphic to $S \times S$, the map $(x, y, g) \mapsto (f'(x), f'(y), f'(g))$ does not extend to an isomorphism of $S \times T$. Spoiler now wins. In total, Spoiler used $3$ pebbles and $4$ rounds.

• Case 2: Suppose instead that $T \subseteq \text{Soc}(H)$, but that $T$ is not normal in $\text{Soc}(H)$. As $T$ is not normal in $\text{Soc}(H)$, there exists $Q = \langle q_1, q_2 \rangle \in \text{Fac}(\text{Soc}(H))$ such that $Q$ does not normalize $T$. At the next two rounds, Spoiler pebbles $q_1, q_2$, and their respective preimages, which we label $r_1, r_2$. When pebbling $r_1 \mapsto q_1$, we may assume that Spoiler moves the pebble placed on $a \mapsto f(a)$. By Case 1, we may assume that $r_1, r_2 \in \text{Soc}(G)$, or Spoiler wins with an additional $1$ pebble and $1$ round. Now as $S \triangleleft \text{Soc}(G)$, $\langle r_1, r_2 \rangle$ normalizes $S$. However, $Q$ does not normalize $T$. So the pebbled map $(x, y, r_1, r_2) \mapsto (f'(x), f'(y), q_1, q_2)$ does not extend to an isomorphism. Thus, Spoiler used $4$ pebbles and $5$ rounds.

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Lemma 6.10. Let $G$ be a semisimple group. There is a logspace algorithm that decides, given $g_1, g_2 \in G$, whether $(g_1, g_2) \in \text{Fac}(\text{Soc}(G))$.

Proof. Using a membership test [BM91] [Tan13], we may enumerate the elements of $S := (g_1, g_2)$ by a logspace transducer. We first check whether $S$ is simple. For each $g \in S$, we check whether $ncl_S(g) = S$. This check is $L$-computable [Vij08 Thm. 7.3.3].

It remains to check whether $S \in \text{Fac}(\text{Soc}(G))$. By Cor. 6.6, $S \in \text{Fac}(\text{Soc}(G))$ if and only if $N := ncl_2(S) = S^k$ for some $k$ and $S \leq N$. As $S$ is simple, it suffices to check that each conjugate of $S$ is either (1) equal to $S$ or (2) intersects trivially with $S$ and commutes with $S$. For a given $g \in G$ and each $h \in S$, we may check whether $S \in gSg^{-1}$. If there exist non-trivial $h_1, h_2 \in S$ such that $h_1 \in gSg^{-1}$ and $h_2 \notin gSg^{-1}$, we return that $S \notin \text{Fac}(\text{Soc}(G))$. Otherwise, we know that all conjugates of $S$ are either equal to $S$ or intersect $S$ trivially. Next we check that those conjugates that intersect $S$ trivially commute with $S$. For each $g \in G, h_1, h_2 \in S$ we check whether $gh_1g^{-1} \in S$; if not, we check that $[gh_1g^{-1}, h_2] = 1$. If not, then we return that $S \notin \text{Fac}(\text{Soc}(G))$. If all these tests pass, then $S$ is a direct factor of the socle. For both of these procedures, we only need to iterate over 3- and 4-tuples of elements of $G$ or $S$, so this entire procedure is $L$-computable. The result follows.

Lemma 6.11. Let $G$ be a semisimple group. We can compute the direct factors of $\text{Soc}(G)$ using a logspace transducer.

Proof. Using Lem. 6.10 we may identify in $L$ the ordered pairs that generate direct factors of $\text{Soc}(G)$. Now for $x \in G$ and a pair $(g_1, g_2)$ that generates a direct factor of $\text{Soc}(G)$, define an indicator $Y(x, g_1, g_2) = 1$ if and only if $x \in \langle g_1, g_2 \rangle$. We may use a membership test [BM91] [Tan13] to decide in $L$ whether $x \in \langle g_1, g_2 \rangle$. Thus, we are able to write down the direct factors of $\text{Soc}(G)$ and their elements in $L$.

We now prove Thm. 6.7.

Proof of Thm. 6.7. We first note that, by Lem. 6.9 if $\text{Soc}(G) \neq \text{Soc}(H)$, then $(4, (1))$-WL Version II will distinguish $G$ from $H$. For in this case, there is some simple normal factor $S \in \text{Fac}(\text{Soc}(G))$ such that there are more copies of $S$ in $\text{Fac}(\text{Soc}(G))$ than in $\text{Fac}(\text{Soc}(H))$. Thus under any bijection Duplicator selects, some element of $S$ must get mapped into a simple direct factor of $\text{Soc}(H)$ that is not isomorphic to $S$, and thus by Lem. 6.9 Spoiler can win with 4 pebbles and 5 rounds.

So suppose $\text{Soc}(G) \cong \text{Soc}(H)$. By Lem. 6.11 in $L$ we may enumerate the non-Abelian simple direct factors of $\text{Soc}(G)$ and $\text{Soc}(H)$. Furthermore, we may decide in $L$—and therefore, $\text{SAC}^1$—with a membership test [BM91] [Tan13] whether two non-Abelian simple direct factors of the socle are conjugate. Thus, in $\text{SAC}^1$, we may compute a decomposition $\text{Soc}(G)$ and $\text{Soc}(H)$ of the form $T_1^n \times \cdots \times T_k^n$, where each $T_i$ is non-Abelian simple and each $T_i^n$ is minimal normal.

There are $O(|S|^2)$ automorphisms of each simple factor $|S|$, and so there are at most $n^2k! \prod_{i=1}^{k} t_i!$ isomorphisms between $\text{Soc}(G)$ and $\text{Soc}(H)$ that could extend to isomorphisms $G \cong H$. From [BCCQ11], we note that this quantity is bounded by $n^{O(\log \log n)}$. (This is bound is tight, as in the case of the groups $A_5^k$.) Given a bijection $\psi : \text{Fac}(\text{Soc}(G)) \rightarrow \text{Fac}(\text{Soc}(H))$, we may in $L$ enumerate the $O(n^2)$ isomorphisms between $\text{Fac}(\text{Soc}(G))$ and $\text{Fac}(\text{Soc}(H))$ respecting $\psi$ by fixing generators of each element $S \in \text{Fac}(\text{Soc}(G))$ and enumerating their possible images in $\psi(S)$.

We now turn to testing isomorphism of $G$ and $H$. To do so, we use the individualize and refine strategy. We individualize in $G$ arbitrary generators for each element of $\text{Fac}(\text{Soc}(G))$ (2 for each factor). In parallel, we try each of the $\leq k! \prod_{i=1}^{k} t_i! \leq n^{O(\log \log n)}$ possible bijections $\psi : \text{Fac}(\text{Soc}(G)) \rightarrow \text{Fac}(\text{Soc}(H))$ (this is the one place responsible for the quasi-polynomial, rather than polynomial, size of our resulting circuits). Then for each configuration of generators for the elements of $\text{Fac}(\text{Soc}(H))$, we individualize those in such a way that respects $\psi$. Precisely, if $\psi(S) = T$ and $(g_1, g_2)$ are individualized in $S$, then for the desired generators $(h_1, h_2)$ of $T$, we individualize $h_i$ to receive the same color as $g_i$. Note that, although we are individualizing
up to $2 \log |G|$ elements here, we are not choosing them from all possible $\binom{|G|}{2 \log |G|}$ choices (which would be worse than the trivial upper bound!); the algorithm only considers at most $\prod_{S \in \text{Fac}(\text{Soc}(H))} \binom{|S|}{2} \leq O(|G|^2)$ many choices for which tuples to individualize.

Observe that in two more rounds, no two elements of Soc$(G)$ have the same color. Similarly, in two more rounds, no two elements of Soc$(H)$ have the same color. However, an element of Soc$(G)$ and an element of Soc$(H)$ may share the same color.

Suppose now that $G \not\cong H$. Let $f : G \to H$ be the bijection that Duplicator selects. As $G \neq H$, there exists $g \in G$ and $s \in \text{Soc}(G)$ such that $f(\text{gs}^{-1}) \neq f(g)f(s)f(g^{-1})$. Spoiler pebbles $g$. Let $f' : G \to H$ be the bijection Duplicator selects at the next round. As no two elements of Soc$(G)$ have the same color and no two elements of Soc$(H)$ have the same color, we have that $f'(s) = f(s)$. Spoiler pebbles $s$ and wins.

So after the individualization step, (2, 4)-WL Version II will decide whether the given map extends to an isomorphism of $G \cong H$. Now (2, 4)-WL Version II is L-computable, and so SAC$^1$ computable. As we have to test at most $n^{O(\log \log n)}$ isomorphisms of Soc$(G) \cong \text{Soc}(H)$, our circuit has size $n^{O(\log \log n)}$. The result now follows.

\begin{remark}
\begin{proof} We also note that there is at most one such way of extending the given isomorphism between Soc$(G)$ and Soc$(H)$ to that of $G$ and $H$ \cite[Lemma 3.1]{BCGQ11}. So in particular, after individualizing the generators for the non-Abelian simple direct factors of the socles, from the last paragraph in the proof we see that WL will assign a unique color to each element of the group.

We also obtain the following corollary, which improves upon \cite[Corollary 4.4]{BCGQ11} in the direction of parallel complexity.
\end{proof}
\end{remark}

\begin{corollary}
Let $G$ and $H$ be semisimple with \text{Soc}(G) \cong \text{Soc}(H)$. If Soc$(G) \cong \text{Soc}(H)$ have $O(\log n/\log \log n)$ non-Abelian simple direct factors, then we can decide isomorphism between $G$ and $H$, and list all the isomorphisms between $G$ and $H$ in FL.
\end{corollary}

\section{Count-Free Weisfeiler–Leman}

In this section, we examine consequence for parallel complexity of the \textit{count-free} WL algorithm. Our first main result here is to show a $\Omega(\log |G|)$ lower bound (optimal and maximal, up to the constant factor) on count-free WL-dimension for identifying Abelian groups (Thm. 7.10). Despite this result showing that count-free WL on its own is not useful for testing isomorphism of Abelian groups, we nonetheless use count-free WL for Abelian groups, in combination with a few other ideas, to get improved upper bounds on the parallel complexity of testing isomorphism (Thm. 7.15) of Abelian groups.

We begin by defining analogous pebble games and logics for the three count-free WL versions. Furthermore, we establish the equivalence of the three count-free WL versions up to $O(\log n)$ rounds. These results extend \cite[Section 3]{BS20} to the count-free setting.

\subsection{Equivalence Between Count-Free WL, Pebble Games, and Logics}

We define analogous pebble games for count-free WL Versions I-III. The count-free $(k + 1)$-pebble game consists of two players: Spoiler and Duplicator, as well as (k + 1) pebble pairs $(p, p')$. In Versions I and II, Spoiler wishes to show that the two groups $G$ and $H$ are not isomorphic; and in Version III, Spoiler wishes to show that the corresponding graphs $\Gamma_G, \Gamma_H$ are not isomorphic. Duplicator wishes to show that the two graphs (Versions I and II) or two graphs (Version III) are isomorphic. Each round of the game proceeds as follows.

\begin{enumerate}
\item Spoiler picks up a pebble pair $(p_i, p'_i)$.
\item The winning condition is checked. This will be formalized later.
\item In Versions I and II, Spoiler places one of the pebbles on some group element (either $p_i$ on some element of $G$ or $p'_i$ on some element of $H$). In Version III, Spoiler places one of the pebbles on some vertex of one of the graphs (either $p_i$ on some vertex of $\Gamma_G$ or $p'_i$ on some element of $\Gamma_H$).
\end{enumerate}
4. Duplicator places the other pebble on some element of the other group (Versions I and II) or some vertex of the other graph (Version III).

Let \( v_1, \ldots, v_m \) be the pebbled elements of \( G \) (resp., \( \Gamma_G \)) at the end of step 1, and let \( v'_1, \ldots, v'_m \) be the corresponding pebbled vertices of \( H \) (resp., \( \Gamma_H \)). Spoiler wins precisely if the map \( v_r \rightarrow v'_r \) does not extend to a marked equivalence in the appropriate version of WL. Duplicator wins otherwise. Spoiler wins, by definition, at round 0 if \( G \) and \( H \) do not have the same number of elements. We note that \( G \) and \( H \) (resp., \( \Gamma_G, \Gamma_H \)) are not distinguished by the first \( r \) rounds of \( k \)-WL if and only if Duplicator wins the first \( r \) rounds of the \((k + 1)\)-pebble game.

The count-free \( r \)-round, \( k \)-WL algorithm for graphs is equivalent to the \( r \)-round, \((k + 1)\)-pebble count-free pebble game [CF92]. Thus, the count-free \( r \)-round, \( k \)-WL Version III algorithm for groups introduced in Brachter & Schweitzer [BS20] is equivalent to the \( r \)-round, \((k + 1)\)-pebble count-free pebble game on the graphs \( \Gamma_G, \Gamma_H \) associated to the groups \( G, H \). We establish the same equivalence for the count-free WL Versions I and II.

**Lemma 7.1.** Let \( \overline{g} := (g_1, \ldots, g_k) \in G^k \) and \( \overline{h} := (h_1, \ldots, h_k) \in H^k \). If the count-free \((k, r)\)-WL distinguishes \( \overline{g} \) and \( \overline{h} \), then Spoiler can win in the count-free \((k + 1)\)-pebble game within \( r \) moves on the initial configuration \((\overline{g}, \overline{h})\). (We use the same version of WL and the pebble game).

**Proof.**

- **Version I:** For \( r = 0 \), then \( \overline{g} \) and \( \overline{h} \) differ with respect to the Version I marked equivalence type. Fix \( r > 0 \). Suppose that \( \chi_r(\overline{g}) \neq \chi_r(\overline{h}) \). We have two cases. Suppose first that \( \chi_{r - 1}(\overline{g}) \neq \chi_{r - 1}(\overline{h}) \). Then by the inductive hypothesis, Spoiler can win in the \((k + 1)\)-pebble game using at most \( r - 1 \) moves.
  
  Suppose instead that \( \chi_{r - 1}(\overline{g}) = \chi_{r - 1}(\overline{h}) \). So without loss of generality, there exists an \( x \in G \) such that the color configuration \( (\chi_{r - 1}(\overline{g}(g_1/x)), \ldots, \chi_{r - 1}(\overline{g}(g_k/x))) \) does not appear amongst the colored \( k \)-tuples of \( H \). Thus, for some \( j \in [k] \) and all \( y \in H \), \( \chi_{r - 1}(\overline{g}(g_j/x)) \neq \chi_{r - 1}(\overline{h}(h_j/y)) \). Spoiler moves pebble \( p_j \) to \( x \). By the inductive hypothesis, Spoiler wins with \( r - 1 \) additional moves.

- **Version II:** We modify the Version I argument above to use the Version II marked equivalence type. Otherwise, the argument is identical.

We now prove the converse.

**Lemma 7.2.** Let \( \overline{g} := (g_1, \ldots, g_k) \in G^k \) and \( \overline{h} := (h_1, \ldots, h_k) \in H^k \). Suppose that Spoiler can win in the count-free \((k + 1)\)-pebble game within \( r \) moves on the initial configuration \((\overline{g}, \overline{h})\). Then the count-free \((k, r)\)-WL distinguishes \( \overline{g} \) and \( \overline{h} \). (We use the same version of WL and the pebble game).

**Proof.**

- **Version I:** If \( r = 0 \), then the initial configuration is already a winning one for Spoiler. By definition, \( \overline{g}, \overline{h} \) receive different colorings at the initial round of WL. Let \( r > 0 \), and suppose that Spoiler wins at round \( r > 1 \) of the pebble game. Suppose that at round \( r \), Spoiler moved the \( j \)th pebble from \( g_j \) to \( x \). Suppose Duplicator responded by moving the corresponding pebble from \( h_j \) to \( y \). Then the map \( (g_1, \ldots, g_{j - 1}, x, g_{j + 1}, \ldots, g_k) \rightarrow (h_1, \ldots, h_{j - 1}, y, h_{j + 1}, \ldots, h_k) \) is not a marked equivalence. By the inductive hypothesis, \( \overline{g}(g_j/x) \) and \( \overline{h}(h_j/y) \) receive different colors at round \( r - 1 \) of \( k \)-WL. As Spoiler had a winning strategy by moving the \( j \)th pebble from \( g_j \rightarrow x \), we have that for any \( y \in H \), \( \chi_{r - 1}(\overline{g}(g_j/x)) \neq \chi_{r - 1}(\overline{h}(h_j/y)) \). By the definition of the WL refinement, it follows that \( \chi_r(\overline{g}) \neq \chi_r(\overline{h}) \). The result follows.

- **Version II:** We modify the Version I argument above to use the Version II marked equivalence type. Otherwise, the argument is identical.

**Lemma 7.3.** Let \( G \) and \( H \) be groups of order \( n \). Consider the count-free \( k \)-pebble game on the graphs \( \Gamma_G \) and \( \Gamma_H \). If \( k \geq 6 \) and one of the following happens:
(a) Spoiler places a pebble $p$ on a vertex corresponding to a group element $g \in G$ and Duplicator places the corresponding pebble $p'$ on a vertex $v$ that does not correspond to a group element of $H$.

(b) Suppose that there is a pebble pair $(p, p')$ for which pebble $p$ is on some vertex of $M(g_1, g_2)$ that is not a group element and $p'$ is on some vertex of $M(h_1, h_2)$ that is not a group element. Write $g_3 := g_1 g_2$ and $h_3 := h_1 h_2$. If Spoiler places a pebble on $g_i$ ($i = 1, 2, 3$) and Duplicator does not respond by pebbling $h_i$ (or vice-versa),

(c) the map induced by the group elements pebbled or implicitly pebbled by $k - 2$ pebbles does not extend to an isomorphism between the corresponding generated subgroups,

then Spoiler can win with 2 additional pebbles and $O(\log n)$ additional rounds.

Proof. We have the following.

(a) The vertices that do not correspond to group elements have degree at most 3. So Spoiler can win with 4 additional pebbles by pebbling the neighbors of the vertex corresponding to the group element.

(b) We note that if pebble $p$ is not on the same type of vertex (i.e., type $a, b, c$, or $d$, as in Figure 1) as pebble $p'$, then Spoiler wins in $O(1)$ more rounds with at most 4 more pebbles and 4 more rounds, as either the vertices or their neighbors have different degrees.

So suppose now that $p$ and $p'$ are on the same type of vertex. Now without loss of generality, suppose that pebble $q$ is placed on $g_i$ for some $i = 1, 2, 3$ and the corresponding pebble $q'$ is not placed on $h_i$. Observe that for each non-group-element vertex in a gadget $M(g_1, g_2)$, the distances to the three group-element vertices of that gadget are all distinct, and any path from a vertex not in $M(g_1, g_2)$ to a non-group-element vertex in $M(g_1, g_2)$ must go through one of the group element vertices $g_1, g_2, g_1 g_2$.

Thus, there is some $k$ such that the vertex pebbled by $p$ is connected to $g_i$ by a path made up of exactly $k$ non-group element vertices, but the same is not true for any path from the vertex pebbled by $p'$ to that pebbled by $q'$. Using a third pebble pair, Spoiler can explore the path from $p$ to $g_i$ and win, using at most 7 additional rounds (as a multiplication gadget has 7 vertices).

(c) Suppose that the map $f : g_i \mapsto h_i$ for all $i \in [k-2]$ does not extend to an isomorphism of $\langle g_1, \ldots, g_{k-2} \rangle$ and $\langle h_1, \ldots, h_{k-2} \rangle$. Let $\tilde{f} : \langle g_1, \ldots, g_{k-2} \rangle \mapsto \langle h_1, \ldots, h_{k-2} \rangle$ be an extension of $f$. As $\tilde{f}$ is not an isomorphism, there exists a smallest word $\omega = g_{i_1} \cdots g_{i_j}$ over $g_1, \ldots, g_{k-2}$ such that $\tilde{f}(\omega) \neq \tilde{f}(g_{i_1}) \cdots \tilde{f}(g_{i_j})$. By minimality, we have that

$$\tilde{f}(g_{i_1} \cdots g_{i_j}) = \tilde{f}(g_{i_1}) \cdots \tilde{f}(g_{i_j}) \neq f(\omega).$$

Spoiler pebbles $\omega$ in $\Gamma_G$, and Duplicator responds by pebbling $\omega'$ in $\Gamma_H$. Denote $\omega[x, \ldots, y] := g_{x} \cdots g_{y}$. At the next round, Spoiler implicitly pebbles $(\omega[1, \ldots, \lfloor j/2 \rfloor], \omega[\lceil j/2 \rceil + 1, \ldots, j])$. Duplicator responds by pebbling a pair $(c, d)$. By part (b), we may assume that $cd = \omega'$; otherwise, Spoiler can win by reusing the pebble pair on $\omega, \omega'$ and exploring the multiplication gadget $M(\omega[1, \ldots, \lfloor j/2 \rfloor], \omega[\lceil j/2 \rceil + 1, \ldots, j])$. So now either:

$$c \neq h_{i_1} \cdots h_{i_{\lfloor j/2 \rfloor}}, \text{ or }$$

$$d \neq h_{i_{\lceil j/2 \rceil + 1}} \cdots h_{i_j}.$$  

Without loss of generality, suppose that:

$$c \neq h_{i_1} \cdots h_{i_{\lfloor j/2 \rfloor}}.$$

Spoiler iterates on the above strategy, starting from $c$ rather than $\omega$. We eventually reach the case of part (b), for a total of $\log_2 j + O(1) \leq \log_2 n + O(1)$ rounds. To see that two additional pebbles are required, after implicitly pebbling the multiplication gadget, Spoiler may move reuse the pebble from the previous round. The result now follows.
7.2 Logics

We recall the central aspects of first-order logic. We have a countable set of variables \( \{x_1, x_2, \ldots \} \). Formulas are defined inductively. As our basis, \( x_i = x_j \) is a formula for all pairs of variables. Now if \( \varphi \) is a formula, then so are the following: \( \varphi \land \varphi, \varphi \lor \varphi, \neg \varphi, \exists x_i \varphi, \) and \( \forall x_i \varphi \). Variables can be reused within nested quantifiers.

In order to define logics on groups, it is necessary to define a relation that relates the group multiplication. We recall the two different logics introduced by Brachter & Schweitzer [BS20].

- **Version I:** We add a ternary relation \( R \) where \( R(x_i, x_j, x_l) = 1 \) if and only if \( x_i x_j = x_l \) in the group. In keeping with the conventions of [CFI92], we refer to the first-order logic with relation \( R \) as \( L_I \) and its \( k \)-variable fragment as \( \mathcal{L}_I^k \). We refer to the logic \( C_I \) as the logic obtained by adding counting quantifiers \( \exists n x_i \varphi \) and \( \exists n \varphi \) as \( C_I \) and its \( k \)-variable fragment as \( \mathcal{C}_I^k \).

- **Version II:** We add a relation \( R \), defined as follows. Let \( w = (\{x_1, \ldots, x_n\} \cup \{x_1^{-1}, \ldots, x_n^{-1}\})^* \).

  We have that \( R(x_i, \ldots, x_i; w) = 1 \) if and only if multiplying the group elements according to \( w \) yields the identity. For instance, \( R(a, b; [a, b]) \) holds precisely if \( a, b \) commute. Again, in keeping with the conventions of [CFI92], we refer to the first-order logic with relation \( R \) as \( L_{II} \) and its \( k \)-variable fragment as \( \mathcal{L}_{II}^k \). We refer to the logic \( C_{II} \) as the logic obtained by adding counting quantifiers \( \exists n x_i \varphi \) and \( \exists n \varphi \) as \( C_{II} \) and its \( k \)-variable fragment as \( \mathcal{C}_{II}^k \).

Remark 7.4. Brachter & Schweitzer [BS20] refer to \( L_I \) and \( L_{II} \) as the logics with counting quantifiers. We instead adhere to the conventions in [CFI92].

Brachter & Schweitzer [BS20] Lemma 3.6] showed that for \( J \in \{ I, II \} \) two \( k \)-tuples \( \overline{g}, \overline{h} \) receive a different initial color under \( k \)-WL Version \( J \) if and only if there is a quantifier-free formula in \( C_J \) that distinguishes \( \overline{g}, \overline{h} \). As such formulas do not use any quantifiers, \( \overline{g}, \overline{h} \) receive a different initial color under \( k \)-WL Version \( J \) if and only if there is a quantifier-free formula in \( \mathcal{L}_J \) that distinguishes \( \overline{g}, \overline{h} \). Now the equivalence between the \((k + 1)\)-pebble, \( r \)-round Version \( J \) count-free pebble game and the \((k + 1)\)-variable, quantifier-depth \( r \) fragment of \( \mathcal{L}_J \) follows identically from the argument as in the case of graphs [CFI92]. We record this with the following theorem.

Theorem 7.5. Let \( G \) and \( H \) be groups of order \( n \), and let \( J \in \{ I, II \} \). We have that the count-free \((k, r)\)-WL Version \( J \) distinguishes \( G \) from \( H \) if and only if there exists a sentence \( \varphi \in \mathcal{L}_J \) that uses at most \( k + 1 \) variables and quantifier depth \( r \), such that \( \varphi \) holds on one group but not the other.

7.3 Equivalence of Count-Free WL Versions

We show that the three count-free WL Versions are equivalent, up to a factor of 2 in the dimension and up to a tradeoff of \( O(\log n) \) additional rounds.

Definition 7.6. Let \( k, k' \geq 2, r, r' \geq 1, \) and \( J, J' \in \{ I, II, III \} \). We say that \((k, r)\)-WL Version \( J \) distinguishes groups \( G \) and \( H \) if whenever \((k, r)\)-WL Version \( J \) distinguishes groups \( G \) and \( H \), then \((k', r')\)-WL Version \( J' \) also distinguishes \( G \) and \( H \).

Theorem 7.7. Fix \( k \geq 2 \) and \( r \geq 1 \). In the count-free setting, we have the following:

(a) \((k, r)\)-WL Version \( I \) \( \leq \) \((k, r)\)-WL Version \( II \),

(b) \((k, r)\)-WL Version \( II \) \( \leq \) \((k, r)\)-WL Version \( III \),

(c) \((k, r)\)-WL Version \( III \) \( \leq \) \((k, r)\)-WL Version \( I \).

We first note that count-free \((k, r)\)-WL Version \( I \) can simulate each step of \((k, r)\)-WL Version \( II \). Thus, it remains to prove Thm. 7.7 (b)-(c). We do so with a series of lemmas.

Lemma 7.8. Let \( G \) and \( H \) be groups of order \( n \). Suppose that the count-free \((k, r)\)-WL Version \( II \) distinguishes \( G \) and \( H \). Then the count-free \((k, r)\)-WL Version \( III \) algorithm distinguishes \( G \) and \( H \).
Proof. We adapt the strategy of [BS20 Lemma 3.11] to the count-free setting and control for rounds. Suppose that Spoiler has a winning strategy in the \( r \)-round Version II \((k+1)\)-pebble game. Let \( g_1, \ldots, g_r \) be the sequence of group elements that Spoiler pebbles. Suppose that at round \( 1 \leq i \leq r \) of the Version II game that Spoiler introduces a new pebble. In the Version III game, if there are an even number of group elements pebbled then, Spoiler pebbles the group element vertex \( g_i \); if instead there are an odd number of group elements pebbled; then in the Version III game, Spoiler implicitly pebbles \((g_i, g_{i+1})\) and reuses the pebble on \( g_i \) at the next round.

Suppose that Spoiler instead moves a pebble at round \( i \) of the Version II game. If the corresponding pebble in the Version III game is on a group element vertex, then this is treated identically as in the case when a new pebble is introduced. Suppose instead the corresponding pebble in the Version III game is on a multiplication gadget vertex \( M(a, b) \) and in the Version II game, Spoiler moves the pebble from \( b \). In this case in the Version III game, Spoiler introduces a new pebble onto \( g_i \), and then moves the pebble from \( M(a, b) \) to a non-group element vertex of \( M(a, g_i) \). At the next round, Spoiler reuses the pebble on \( g_i \).

We now argue that, without loss of generality, we may assume that the configuration of pebbled group elements at the end of at most \( 3r \) rounds in the Version III game is the same configuration at the end of round \( r \) of the Version II game. Suppose that at the end of round \( r \) of the Version II game that Duplicator has pebbled \((h_1, \ldots, h_k)\), and suppose that at the end of round \( 3r \) of the count-free pebble game that Duplicator has (implicitly) pebbled \((h'_1, \ldots, h'_k)\). By Lem. 7.3, if a pebble \( p_i \) belongs to a group element (respectively, multiplication gadget) vertex and \( p_i' \) does not belong to a group element (respectively, multiplication gadget) vertex, then Spoiler can win with 2 pebbles and \( O(1) \) rounds. So we may assume at the end of round \( 3r \) of the Version III game that Duplicator has (implicitly) pebbled \( k \) group elements.

Now suppose for a contradiction that Duplicator wins with this strategy in the Version III game, even with 2 additional pebbles and \( O(\log n) \) additional rounds. Then by Lem. 2.5 (c), the map \((g_1, \ldots, g_k) \mapsto (h'_1, \ldots, h'_k)\) extends to an isomorphism of the subgroups \((g_1, \ldots, g_k)\) and \((h'_1, \ldots, h'_k)\). So in the Version II game, Duplicator could have won by pebbling \((h'_1, \ldots, h'_k)\) rather than \((h_1, \ldots, h_k)\), contradicting the assumption that Spoiler wins at round \( r \) of the Version II pebble game. Thus, we may assume at the end of round \( r \) of the Version III game that \((h_1, \ldots, h_k) = (h'_1, \ldots, h'_k)\).

As Spoiler wins at the end of round \( r \) in the Version II game, we have that the induced map on the configurations does not extend to an isomorphism. So by Lem. 7.3 (c), Spoiler wins in the Version III game with 2 additional pebbles and \( O(\log n) \) additional rounds, as desired.

Lemma 7.9. Let \( G \) and \( H \) be groups of order \( n \). Suppose that the count-free \((k, r)\)-WL Version III distinguishes \( G \) and \( H \). Then the count-free \( 2k+1 \)-WL Version I algorithm distinguishes \( G \) and \( H \) in at most \( 2r \) rounds.

Proof. We adapt the strategy of [BS20 Lemma 3.12] to the count-free setting and control for rounds. Suppose that at round \( 0 \leq i \leq r \) of the Version III pebble game, that Spoiler pebbles the group element vertex \( g_i \). Then in the Version I game, Spoiler may pebble \( g_i \). Suppose instead in the Version III game that Spoiler implicitly pebbles \( M(x_1, x_2) \), and Duplicator responds by implicitly pebbling \( M(y_1, y_2) \). We simulate this step in two rounds of the Version I game. At the first stage, Spoiler pebbles \( x_1 \). Duplicator responds by pebbling some group element \( y'_1 \). At the next stage, Spoiler pebbles \( x_2 \), and Duplicator responds by pebbling \( y'_2 \). Observe that at most 2 rounds of the Version I game are required to simulate 1 round of the Version III game.

Now suppose at round \( r \) of the Version III game that Duplicator has pebbled \((h_1, \ldots, h_d)\), where due to implicit pebbling, \( d \leq 2k \). Suppose that at round \( 2r \) of the Version I game that Duplicator has pebbled \((h'_1, \ldots, h'_d)\). Now suppose for a contradiction that Duplicator wins at round \( 2r \) of the Version I game. Let \( \Gamma'_G \) be the induced subgraph \( \Gamma_G([g_1, \ldots, g_d]) \) together with the multiplication gadgets \( M(g_i, g_j) \) for all \( i, j \in [d] \) where \( g_i g_j \in [g_1, \ldots, g_d] \). Define \( \Gamma'_H \) analogously for \((h'_1, \ldots, h'_d)\). Consider the map \( \varphi : V(\Gamma'_G) \rightarrow V(\Gamma'_H) \) induced by \((g_1, \ldots, g_d) \mapsto (h'_1, \ldots, h'_d) \). By the definition of the Version I winning condition, the map \((g_1, \ldots, g_d) \mapsto (h'_1, \ldots, h'_d) \) respects multiplication. Thus, \( \varphi \) will be a graph isomorphism, as multiplicativity can be expressed equivalently in terms of mapping the multiplication gadgets accordingly. It follows that Duplicator could have won in the Version III pebble game by (implicitly) pebbling \((h'_1, \ldots, h'_d)\) instead of \((h_1, \ldots, h_d)\), contradicting the assumption that Spoiler wins in the count-free \((k+1)\)-pebble, \( r \)-round Version III game. The result now follows.
### 7.4 Count-Free WL and Abelian Groups

We now turn to showing that the count-free WL Version II algorithm fails to yield a polynomial-time isomorphism test for even Abelian groups.

**Theorem 7.10.** For \( n \geq 5 \), let \( G_n := (\mathbb{Z}/2\mathbb{Z})^n \times (\mathbb{Z}/4\mathbb{Z})^n \) and \( H_n := (\mathbb{Z}/2\mathbb{Z})^{n-2} \times (\mathbb{Z}/4\mathbb{Z})^{n+1} \). The \( n/4 \)-dimensional count-free WL Version II algorithm does not distinguish \( G_n \) from \( H_n \).

**Proof.** The proof is by induction on the number of pebbles. For our first pebble, Spoiler may pebble one element in \( G_n \), which generate one of the following subgroups: \{1\}, \( \mathbb{Z}/2\mathbb{Z} \), or \( \mathbb{Z}/4\mathbb{Z} \). For each of these options, Duplicator may respond in kind.

Now fix 1 \( \leq k \leq n/4 \). Suppose that Duplicator has a winning strategy with \( k \) pebbles. In particular, we suppose that pebble pairs \((p_1, p'_1), \ldots, (p_k, p'_k)\) have been placed on the board, and that the map \( p_i \mapsto p'_i \) for all \( i \in [k] \) extends to a marked isomorphism on a subgroup of the form \((\mathbb{Z}/2\mathbb{Z})^a \times (\mathbb{Z}/4\mathbb{Z})^b\). Furthermore, suppose that \( 0 \leq a_1 \leq a \) of the \( \mathbb{Z}/2\mathbb{Z} \) direct factors of \((p_1, \ldots, p_k)\) are contained in copies of \( \mathbb{Z}/2\mathbb{Z} \) in \( G_n \). Duplicator will maintain the invariant that the same number of copies of \( \mathbb{Z}/2\mathbb{Z} \) direct factors of \((p_1, \ldots, p_k)\) are contained in copies of \( \mathbb{Z}/4\mathbb{Z} \) in \( H_n \).

As Duplicator had a winning strategy in the \( k \)-pebble game, it is not to Spoiler’s advantage to move pebbles \( p_1, \ldots, p_k \). Thus, Spoiler picks up a new pebble \( p_{k+1} \). Spoiler may pebble one additional element \( G_n \). Now if the element Spoiler pebbles belongs to \((p_1, \ldots, p_k)\); then as the map \( p_i \mapsto p'_i \) for all \( i \in [k] \) extends to a marked isomorphism, Duplicator may respond by pebbling the corresponding element in \((p'_1, \ldots, p'_k)\).

We may now assume that Duplicator does not pebble any element in \((p_1, \ldots, p_k)\). Spoiler may pebble one additional element. We have the following cases.

- **Case 1:** If \( k \leq n/4 \), Spoiler pebbles an element \( g_1 \) generating \( \mathbb{Z}/2\mathbb{Z} \), then Duplicator may respond in kind. We note that any copy of \( \mathbb{Z}/2\mathbb{Z} \) that is pebbled and does not belong to \((p_1, \ldots, p_k)\) is a direct complement to \((p_1, \ldots, p_k)\) (in the sense that they commute and intersect trivially—they will not generate the whole group). Similarly, any copy of \( \mathbb{Z}/2\mathbb{Z} \) that is pebbled and does not belong to \((p'_1, \ldots, p'_k)\) is a direct complement to \((p'_1, \ldots, p'_k)\). Furthermore, as \( k \leq n/4 \), we may assume that Duplicator pebbles a copy of \( \mathbb{Z}/2\mathbb{Z} \) that is contained within a copy of \( \mathbb{Z}/4\mathbb{Z} \) in \( H_n \) if and only if Spoiler pebbles a copy of \( \mathbb{Z}/2\mathbb{Z} \) that is contained within a copy of \( \mathbb{Z}/4\mathbb{Z} \) in \( G_n \).

- **Case 2:** We now consider the case in which Spoiler pebbles an element \( g_1 \) generating a copy of \( \mathbb{Z}/4\mathbb{Z} \). We consider the following subcases.
  - **Subcase 2(a):** As \( k \leq n/4 \), there exist copies of \( \mathbb{Z}/4\mathbb{Z} \) in \( G_n \) that intersect trivially (and thus, are direct complements) with \((p_1, \ldots, p_k)\); and similarly, there exist copies of \( \mathbb{Z}/4\mathbb{Z} \) that intersect trivially (and thus, are direct complements) with \((p'_1, \ldots, p'_k)\). So if \((g_1)\) forms a direct complement with \((p_1, \ldots, p_k)\), then Duplicator may respond by pebbling an element \( h_1 \) that generates a copy of \( \mathbb{Z}/4\mathbb{Z} \) which is a direct complement with \((p'_1, \ldots, p'_k)\).
  - **Subcase 2(b):** Suppose instead that \((g_1) \cong \mathbb{Z}/4\mathbb{Z}\) intersects properly and non-trivially with \((p_1, \ldots, p_k)\). In this case, \((g_1)\) shares a copy of \( \mathbb{Z}/2\mathbb{Z} \) with \((p_1, \ldots, p_k)\). This yields two additional subcases.
    - **Subcase 2(b).i:** Suppose first that this copy of \( \mathbb{Z}/2\mathbb{Z} \) is contained within a copy of \( \mathbb{Z}/4\mathbb{Z} \leq (p_1, \ldots, p_k) \). By the inductive hypothesis, both \((p_1, \ldots, p_k)\) and \((p'_1, \ldots, p'_k)\) have \( a_1 \) copies of \( \mathbb{Z}/2\mathbb{Z} \) that are contained in copies of \( \mathbb{Z}/4\mathbb{Z} \) within \( G_n \) and \( H_n \) respectively. Using this fact, together with the fact that \( k \leq n/4 \), we have that Duplicator may respond by pebbling an element \( h_1 \) generating a copy of \( \mathbb{Z}/4\mathbb{Z} \), where \((h_1) \cap (p'_1, \ldots, p'_k)\) is a copy of \( \mathbb{Z}/2\mathbb{Z} \) that is contained within a copy of \( \mathbb{Z}/4\mathbb{Z} \leq (p'_1, \ldots, p'_k) \).
    - **Subcase 2(b).ii:** Suppose instead that this copy of \( \mathbb{Z}/2\mathbb{Z} \) is contained within a copy of \( \mathbb{Z}/4\mathbb{Z} \nless (p_1, \ldots, p_k) \). By the inductive hypothesis, both \((p_1, \ldots, p_k)\) and \((p'_1, \ldots, p'_k)\) have \( a_1 \) copies of \( \mathbb{Z}/2\mathbb{Z} \) that are contained in copies of \( \mathbb{Z}/4\mathbb{Z} \) within \( G_n \) and \( H_n \) respectively. Using this fact, together with the fact that \( k \leq n/4 \), we have that Duplicator may respond by pebbling an element \( h_1 \) generating a copy of \( \mathbb{Z}/4\mathbb{Z} \), where \((h_1) \cap (p'_1, \ldots, p'_k)\) is a copy of \( \mathbb{Z}/2\mathbb{Z} \) that is not contained within a copy of \( \mathbb{Z}/4\mathbb{Z} \leq (p'_1, \ldots, p'_k) \).
Thus, in all cases, Duplicator has a winning strategy at round $k + 1$. The result now follows by induction.

**Remark 7.11.** Thm. 7.10 shows that count-free WL fails to serve as a polynomial-time (or even $|G|^{o(\log |G|)}$) isomorphism test for Abelian groups. In particular, by Thm. 7.7 our lower-bound holds (up to a constant factor in the number of pebbles) for all three versions of WL. As the $n$-dimensional count-free WL algorithm fails to distinguish $G_n$ and $H_n$, we also obtain an $\Omega(\log(\|G_n\|))$ lower bound on the quantifier rank of any FO formula identifying $G_n$. In particular, this suggests that $\text{GPL}$ is not in $\text{FO}(\text{poly log} \log n)$, even for Abelian groups. As $\text{FO}(\text{poly log log} n)$ cannot compute $\text{Parity}$ [SmoSt], this suggests that counting is necessary to solve $\text{GPL}$. This is particularly interesting, as $\text{Parity}$ is not $\text{AC}^0$-reducible to $\text{GPL}$ [CTW13].

While count-free WL is unable to distinguish Abelian groups, the multiset of colors computed actually provides enough information to do so. That is, after count-free WL terminates, the color classes present and their multiplicities provide enough information to identify Abelian groups. The technical difficulty lies in parsing this information; we will show how to do so with a single $\text{Majority}$ gate and $O(\log n)$ non-deterministic bits.

To this end, we first consider the order-finding problem. Barrington, Kadou, Lange, & McKenzie [BKLM01] previously showed that order-finding is $\text{FOLL}$-computable. Our next result (Prop. 7.13) shows that the count-free Weisfeiler–Leman effectively implements this strategy.

**Lemma 7.12.** Let $G, H$ be groups of order $n$. Suppose in the count-free WL-III game, pebbles have already been placed on $g \to h$ and $g^j \to x$ with $x \neq h^i$. ThenSpoiler can win with $O(1)$ additional pebbles in $O(\log \log i)$ rounds.

**Proof.** By induction on $i$. If $i = 0$ the result follows from the fact that the identity is the unique element such that the gadget $M(1, 1)$ has all three of its group element vertices the same and Lem. 7.3 (b). If $i = 1$, the result follows immediately from the winning condition of the game. So we now suppose $i > 1$, and that the result is true for all smaller exponents, say in $\leq c \log \log i'$ rounds for all $i' < i$.

The structure of the argument is as follows. If $i$ is not a power of 2, we show how to cut the number of $1$s in the binary expansion of $i$ by half using $O(1)$ rounds and only $O(1)$ pebbles that may be reused. Since the number of $1$s in the binary expansion of $i$ is at most $\log_2 i$, and we cut this number in half each time, this takes only $O(\log \log i)$ rounds (and $O(1)$ pebbles) before $i$ has just one in its binary expansion, that is, $i$ is a power of 2. Once $i$ is a power of 2, we will show how to cut $\log_2 i$ in half using $O(1)$ rounds and $O(1)$ pebbles that may be reused. This takes only $O(\log \log i)$ rounds (and $O(1)$ pebbles) before getting down to the base case above. Concatenating these two strategies uses only $O(1)$ pebbles and $O(\log \log i)$ rounds. Now to the details.

If $i$ is not a power of 2, we will show how cut the number of $1$s in the binary expansion of $i$ in half. Write $i = j + k$ where $j, k$ each have at most as many $1$s in their binary expansion as $i$ does (rounded up). (Examine the binary expansion $i_{ij}i_{i-1} \cdots i_0$ and finding an index $j$ such that half the ones are on either side of $j$. Then let $j$ have binary expansion $i_{i}i_{i-1} \cdots i_00 \ldots 0$ and let $k$ have binary expansion $i_{i-1}i_{i-2} \cdots i_0$.)

Spoiler implicitly pebbles $(g^i, g^k)$. Duplicator responds by implicitly pebbling a pair $(a, b)$. If $ab \neq x$, then we have a pebble on a non-group-element vertex of $M(g^j, g^k)$ as well as its group element vertex $g^i + k = g^j$. But the corresponding pebbles are on $M(a, b)$ and $x$ which differs from $ab$, so Spoiler wins by Lem. 7.3 (b). Thus we may now assume $ab = x$.

Since $x \neq h^j$, we necessarily have $\{a, b\} \neq \{h^j, h^k\}$. Without loss of generality, suppose $a \notin \{h^j, h^k\}$. Spooner now picks the pebble on $g^j$ and places it on $g^i$. Because of the implicit pebble mapping $M(g^j, g^k) \to M(a, b)$, Duplicator must respond by placing the pebble on $a$ or Spoiler can win by Lem. 7.3 (b). At this point, Spoiler can reuse the implicit pebble on $M(g^j, g^k)$ and the pebble on $g^i$, and we are now in a situation where $g \to h$ and $g^j \to a \neq h^j$ are pebbled, and $j$ has at most half as many $1$s in its binary expansion as $i$ did. (So there are only two pebbles that can’t be re-used, which is precisely the number we started with.) The cost to get here was $O(1)$ rounds and no non-reusable pebbles.

After that has been iterated $\log \log i$ times, we come to the case where $i$ is a power of 2. We will show how to reduce to a case where $\log_2 i$ has been cut in half. Write $i = 2^k$ with $k$ powers of 2 such that $\log_2 j, \log_2 k \leq \floor{\log_2 j}$ (if $i = 2^k$, let $j = 2^{i/2}, k = 2^{i/2} \ldots 2^{j/2}$). Note that we have $g^i = (g^j)^{2^k}$. Spoiler now pebbles $g^i$, and Duplicator responds by pebbling some $a$. If $a \neq h^j$, then Spoiler can re-use the pebble from
Proposition 7.13 (Order finding in WL-III). Let $G$ be a group. Let $g, h \in G$ such that $|g| \neq |h|$. The count-free $(O(1), O(\log \log n))$-WL Version III distinguishes $g$ and $h$.

Proof. We use the pebble game characterization, starting from the initial configuration $((g), (h))$. We first note that if $g = 1$ and $h \neq 1$, that Spoiler implicitly pebbles the multiplication gadget $M(1, 1)$. This is the unique multiplication gadget where all three group element vertices are the same. Regardless of what Duplicator pebbles, we have by Lem. 7.3 that Spoiler can win with $O(1)$ additional pebbles and $O(1)$ additional rounds.

Without loss of generality suppose $|g| < |h|$. Note that $g \mapsto h$ has already been pebbled by assumption. Spoiler now pebbles 1. By the same argument as above, Duplicator must respond by pebbling 1. But we have $g^i = 1$ and by assumption $h^i \neq 1$. Thus, by Lem. 7.12 Spoiler can now win with $O(1)$ pebbles in $O(\log \log |g|) \leq O(\log \log n)$ rounds. □

As finite simple groups are uniquely identified amongst all groups by their order and the set of orders of their elements, we obtain the following immediate corollary.

Corollary 7.14. If $G$ is a finite simple group, then $G$ is identified by the count-free $(O(1), O(\log \log n))$-WL. Consequently, isomorphism testing between a finite simple group $G$ and an arbitrary group $H$ is in FOLL.

We also obtain an improved upper bound on the parallel complexity of Abelian Group Isomorphism:

Theorem 7.15. GP1 for Abelian groups is in $\beta_1 MAC^0(FOLL)$.

Here, $\beta_1 MAC^0(FOLL)$ denotes the class of languages decidable by a (uniform) family of circuits that have $O(\log n)$ nondeterministic input bits, are of depth $O(\log \log n)$, have gates of unbounded fan-in, and the only gate that is not an And, Or, or Not gate is the output gate, which is a Majority gate of unbounded fan-in.

Note that, by simulating the poly($n$) possibilities for the nondeterministic bits, $\beta_1 MAC^0(FOLL)$ is contained in $TC^0(FOLL)$, at the expense of using poly($n$) Majority gates. Thus, our result improves on the prior upper bound of $TC^0(FOLL)$ [CTW13].

Thm. 7.15 is an example of the strategy of using count-free WL, followed by a limited amount of counting afterwards. (We contrast this with the parallel implementation of the classical (counting) WL algorithm, which—for fixed $k$—uses a polynomial number of Majority gates at each iteration [TV00].) After the fact, we realized this same bound could be achieved by existing techniques: we include both proofs to highlight an example of how WL was used in the discovery process.

Proof using Weisfeiler–Leman. Let $G$ be Abelian, and let $H$ be an arbitrary group such that $G \neq H$. Suppose first that $H$ is not Abelian. We show that count-free $(O(1), O(1))$-WL can distinguish $G$ from $H$. Suppose first that $H$ is not Abelian. Spoiler implicitly pebbles a pair of elements $(x, y)$ in $H$ that do not commute. $H$ responds by pebbling $(u, v) \in G$. At the next round, Spoiler implicitly pebbles $(v, u)$. Regardless of what Duplicator pebbles, we have by Lem. 7.3(b) that Spoiler wins with $O(1)$ additional pebbles and $O(1)$ additional rounds.

Suppose now that $H$ is Abelian. We run the count-free $(O(1), O(\log \log n))$-WL using the parallel WL implementation due to Grohe & Verbitsky. As $G$ and $H$ are non-isomorphic Abelian groups, they have different order multisets. In particular, there exists a color class of greater multiplicity in $G$ than in $H$. By Prop. 7.13 two elements with different orders receive different colors. We use a $\beta_1 MAC^0$ circuit to distinguish $G$ from $H$. Using $O(\log n)$ non-deterministic bits, we guess the color class $C$ where the multiplicity differs. At each iteration, the parallel WL implementation due to Grohe & Verbitsky records indicators as to whether two $k$-tuples receive the same color. As we have already run the count-free WL algorithm, we may in $AC^0$ decide whether two $k$-tuples have the same color. For each $k$-tuple of $V(\Gamma_G)^k$ having color class $C$, we feed a 1 to the Majority gate. For each $k$-tuple of $V(\Gamma_H)^k$ having color class $C$, we feed a 0 to the Majority gate. The Majority gate outputs a 1 if and only if there are strictly more 1’s than 0’s. The result now follows. □
Alternative proof using prior techniques, that we only realized after discovering the WL proof. This proof follows the strategy of Chattopadhyay, Torán, & Wagner [CTW13], realizing that their use of many threshold gates can be replaced by $O(\log n)$ nondeterministic bits and a single threshold gate.

Compute the multiset of orders in $\text{FOLL}$ [BKL01, Prop. 3.1], guess the order $k$ such that $G$ has more elements of order $k$ than $H$ does. Use a single $\text{Majority}$ gate to compare those counts.

8 Conclusion

We combined the parallel WL implementation of Grohe & Verbitsky [GV06] with the WL for groups algorithms due to Brachter & Schweitzer [BS20] to obtain an efficient parallel canonization procedure for several families of groups, including: (i) coprime extensions $H \ltimes N$ where $N$ is Abelian and $H$ is $O(1)$-generated, and (ii) direct products, where WL can efficiently identify the indecomposable direct factors.

We also showed that the individualize-and-refine paradigm allows us to list all isomorphisms of semisimple groups with an $\text{SAC}$ circuit of depth $O(\log n)$ and size $n^{O(\log \log n)}$. Prior to our paper, no parallel bound was known. And in light of the fact that multiplying permutations is $\text{FL}$-complete [CM87], it is not clear that the techniques of Babai, Luks, & Seress [BLS87] can yield circuit depth $o(\log^2 n)$.

Finally, we showed that $O(\log(n))$-dimensional count-free WL is required to identify Abelian groups. It follows that count-free WL fails to serve as a polynomial-time isomorphism test even for Abelian groups. Nonetheless, count-free WL distinguishes group elements of different orders. We leveraged this fact to obtain a new $\beta_1\text{MAC}^0(\text{FOLL})$ upper bound on isomorphism testing of Abelian groups.

Our work leaves several directions for further research that we believe are approachable and interesting.

**Question 8.1.** Show that coprime extensions of the form $H \ltimes N$ with both $H, N$ Abelian have constant WL-dimension (the WL analogue of [QST11]). More generally, a WL analogue of Babai–Qiao [BQ12] would be to show that when $|H|, |N|$ are coprime and $N$ is Abelian, the WL dimension of $H \ltimes N$ is no more than that of $H$ (or the maximum of that of $H$ and a constant independent of $N, H$).

**Question 8.2.** Is the WL dimension of semisimple groups bounded?

It would be of interest to address this question even in the non-permuting case when $G = \text{Pker}(G)$. Alternatively, establish an upper bound of $O(\log\log n)$ for the WL dimension of semisimple groups. These questions would form the basis of a WL analogue of [BCGQ11], without needing individualize-and-refine. In a higher-arity version of WL, which is not known to admit efficient algorithms, the analogous question for semisimple groups has a positive answer [GL22].

It is often the case that if an uncolored class of graphs is identified by WL, then so is the corresponding class of colored graphs. So if constant-dimensional WL identifies a class of graphs, it can often readily be extended to an efficient canonization procedure (c.f., [GN21]). In the case of groups, it is not clear whether WL easily identifies colored variants. To this end, we ask the following.

**Question 8.3.** Does constant-dimensional Weisfeiler–Leman identify every colored Abelian group?

In instances where running constant-dimensional WL for a polylogarithmic number of rounds suffices to identify a class of colored groups, we may combine the parallel WL implementation of Grohe & Verbitsky [GV06], the WL canonization procedure for graphs (c.f., [GN21]), and the generator enumeration procedure to obtain efficient parallel canonization. For full details, see Appendix E.

For the classes of groups we have studied, when we have been able to give an $O(1)$ bound on their WL-dimension, we also get an $O(\log n)$ bound on the number of rounds needed. The dimension bound alone puts the problem into $\text{P}$, while the bound on rounds puts it into $\text{TC}^1$. A priori, these two should be distinct. For example, in the case of graphs, Kiefer & McKay [KM20] have shown that there are graphs for which color refinement takes $n - 1$ rounds to stabilize.

**Question 8.4.** Is there a family of groups identified by $O(1)$-WL but requiring $\omega(\log n)$ rounds?

We also wish to highlight a question that essentially goes back to [CTW13], who showed that $\text{GPI}$ cannot be hard under $\text{AC}^0$ reductions for any class containing $\text{PARITY}$. In Theorem 7.10 we showed that count-free WL requires dimension $\geq \Omega(\log(n))$ to identify even Abelian groups. This shows that this particular, natural
method does not put \( \text{GpI} \) into \( \text{FO}(\text{poly log log } n) \), though it does not actually prove \( \text{GpI} \not\in \text{FO}(\text{poly log log } n) \), since we cannot rule out clever bit manipulations of the Cayley (multiplication) tables. While we think the latter lower bound would be of significant interest, we think even the following question is interesting:

**Question 8.5.** Show that \( \text{GpI} \) does not belong to (uniform) \( \text{AC}^0 \).

### A Parallel Equivalence Between WL Versions

In this section, we show that WL Version I-III are equivalent up to a tradeoff of \( O(\log n) \) rounds.

**Theorem A.1** (Thm. 2.7). Let \( G \) and \( H \) be groups of order \( n \). Let \( k \geq 2, r \geq 1 \). We have the following.

(a) If \((k, r)\)-WL Version I distinguishes \( G \) and \( H \), then \((k, r)\)-WL Version II distinguishes \( G \) and \( H \).

(b) If \((k, r)\)-WL Version II distinguishes \( G \) and \( H \), then \( [(k/2) + 2, 3r + O(\log n)]\)-WL Version III distinguishes \( G \) and \( H \).

(c) If \((k, r)\)-WL Version III distinguishes \( G \) and \( H \), then \( (2k + 1, 2r)\)-WL Version I distinguishes \( G \) and \( H \).

We begin by strengthening Lem. 2.5 (c).

**Proof of Lem. 2.5 (c).** By assumption, there are at most \( m := 2(k - 2) \) implicitly pebbled pairs of group elements corresponding to at most \( k - 2 \) pebbles currently on the board. Without loss of generality, we may assume there are exactly \( m \) such elements pebbled: \( (g_1, \ldots, g_m) \mapsto (h_1, \ldots, h_m) \). Let \( f : G \rightarrow H \) by the bijection Duplicator selects. By Lem. 2.5 (b), Duplicator must select bijections respecting the pairing induced by implicitly pebbled group elements. By assumption, this correspondence does not extend to an isomorphism of \( (g_1, \ldots, g_m) \) and \( (h_1, \ldots, h_m) \). Let \( w := g_1 \cdots g_m \) be a minimal word such that

\[
f(w) \neq f(g_i_1) \cdots f(g_i_r).
\]

Spoiler pebbles \( w \mapsto f(w) \). Let \( f' : G \rightarrow H \) be the bijection Duplicator selects on the next round. Suppose that \( f'(g_1 \cdots g_m) = f(g_1 \cdots g_m) \). By the minimality of \( w \), we have that:

\[
f(g_1 \cdots g_m) = f(g_{i_1}) \cdots f(g_{i_r}).
\]

So in this case, Spoiler pebbles \( g_1 \cdots g_m \mapsto f'(g_1 \cdots g_m) = f(g_1 \cdots g_m) \) and wins with 2 additional pebbles and \( O(1) \) additional rounds by Lem. 2.5 (b).

Suppose instead that \( f'(g_1 \cdots g_m) \neq f(g_1 \cdots g_m) \). Then at least one of the following must hold:

\[
f'(g_1 \cdots g_m) \neq f'(g_{i_1} \cdots g_{i_{\lceil i_1/2 \rceil}}) \cdot f'(g_{i_{\lceil i_1/2 \rceil}+1} \cdots g_m),
\]

\[
f'(g_1 \cdots g_{i_{\lceil i_1/2 \rceil}}) \neq f'(g_{i_{\lceil i_1/2 \rceil}} \cdots g_m),
\]

or

\[
f'(g_{i_{\lceil i_1/2 \rceil}+1} \cdots g_m) \neq f'(g_{i_{\lceil i_1/2 \rceil}+1} \cdots g_m).
\]

We now consider the following cases.

- **Case 1:** Suppose that

\[
f'(g_{i_{\lceil i_1/2 \rceil}}) \neq f'(g_{i_{\lceil i_1/2 \rceil}}) \text{ or } f'(g_{i_{\lceil i_1/2 \rceil}+1} \cdots g_m) \neq f'(g_{i_{\lceil i_1/2 \rceil}+1} \cdots g_m).
\]

In this alternative, Spoiler pebbles \( g_{i_1} \cdots g_{i_{\lceil i_1/2 \rceil}} \mapsto f'(g_{i_1} \cdots g_{i_{\lceil i_1/2 \rceil}}) \). The other alternative, namely:

\[
f'(g_{i_{\lceil i_1/2 \rceil}+1} \cdots g_m) \neq f'(g_{i_{\lceil i_1/2 \rceil}+1} \cdots g_m)
\]

is handled identically.

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\begin{itemize}
\item **Case 2:** Suppose that Case 1 does not occur. Then we necessarily have that:
\[ f'(g_{i_1} \cdots g_{i_t}) \neq f'(g_{i_1} \cdots g_{i_{(t/2)}}) \cdot f'(g_{i_{(t/2)+1}} \cdots g_{i_t}). \]

In this case, Spoiler begins by pebbling \( g_{i_1} \cdots g_{i_{(t/2)}} \mapsto f'(g_{i_1} \cdots g_{i_{(t/2)}}) \). At the next round, Duplicator must select a bijection \( f'' : G \rightarrow H \) mapping \( f''(g_{i_{(t/2)+1}} \cdots g_{i_t}) \) so that
\[ f'(g_{i_1} \cdots g_{i_t}) = f'(g_{i_1} \cdots g_{i_{(t/2)}}) \cdot f''(g_{i_{(t/2)+1}} \cdots g_{i_t}). \]

Otherwise, by Lem. 2.5 (b), Spoiler wins with one additional pebble and \( O(1) \) additional rounds. But then
\[ f''(g_{i_{(t/2)+1}} \cdots g_{i_t}) \neq f''(g_{i_{(t/2)+1}} \cdots g_{i_t}). \]

Spoiler pebbles \( g_{i_{(t/2)+1}} \cdots g_{i_t} \mapsto f''(g_{i_{(t/2)+1}} \cdots g_{i_t}). \)

After at most two rounds, we have pebbled a word \( w' \) such that \( |w'| < |w|/2 \). Thus, at most \( 2 \log_2(t) + 1 \) rounds are required until we can reduce to Lem. 2.5 (b). By Fact 2.2 we have \( t \leq n \). So only \( O(\log n) \) rounds are required. To see that only two additional pebbles are required, at the round after Spoiler pebbles \( w' \), Spoiler picks up the pebble pair on \( w \mapsto f(w) \). The result now follows.
\end{itemize}

To prove Thm. 2.7 we follow the strategy of [BS20] Section 3.5.

**Proof of Thm. 2.7.**

(a) Suppose that Spoiler wins with \( k \) pebbles on the board at round \( r \) of Version I of the pebble game. Suppose that \( (g_1, \ldots, g_k) \mapsto (h_1, \ldots, h_k) \) have been pebbled. As Spoiler wins, there exist \( i, j, m \in [k] \) such that WLOG, \( g_i g_j = g_m \) but \( h_i h_j \neq h_m \). So the map \( (g_1, \ldots, g_k) \mapsto (h_1, \ldots, h_k) \) does not extend to an isomorphism of \( (g_1, \ldots, g_k) \) and \( (h_1, \ldots, h_k) \). So Spoiler may use the same strategy in Version II of the pebble game and win with \( k \) pebbles and \( r \) rounds.

(b) Suppose that Spoiler wins with \( k \) pebbles on the board at round \( r \) of Version II of the pebble game. Suppose that \( (g_1, \ldots, g_k) \mapsto (h_1, \ldots, h_k) \) have been pebbled. As Spoiler wins, the map \( (g_1, \ldots, g_k) \mapsto (h_1, \ldots, h_k) \) does not extend to an isomorphism of \( (g_1, \ldots, g_k) \) and \( (h_1, \ldots, h_k) \).

Brachter & Schweitzer [BS20] Lemma 3.11 previously established that, using \( \lceil k/2 \rceil \) pebbles and 3\( r \) rounds in the Version III pebble game, that Spoiler can obtain a configuration \( (g'_1, \ldots, g'_k) \mapsto (h'_1, \ldots, h'_k) \) that does not extend to an isomorphism of \( (g'_1, \ldots, g'_k) \) and \( (h'_1, \ldots, h'_k) \).

Now by Lem. 2.5 (a), we may assume in the Version III pebble game that Duplicator selects bijections \( f : V(\Gamma_G) \rightarrow V(\Gamma_H) \) that restrict to bijections \( f|_G : G \rightarrow H \); otherwise, Spoiler may win with 2 pebbles and 2 rounds. Thus, without loss of generality, we may assume that if Duplicator selects \( f : V(\Gamma_G) \rightarrow V(\Gamma_H) \) at round \( 1 \leq i \leq r \) of the Version III pebble game that Duplicator selects \( f|_G \) at round \( i \) of the Version II pebble game. So without loss of generality, we may assume that
\[ (g'_1, \ldots, g'_k) = (g_1, \ldots, g_k), \text{ and } (h'_1, \ldots, h'_k) = (h_1, \ldots, h_k). \]

Now recall that as the map \( (g_1, \ldots, g_k) \mapsto (h_1, \ldots, h_k) \) does not extend to an isomorphism of \( (g_1, \ldots, g_k) \) and \( (h_1, \ldots, h_k) \), Spoiler wins at round \( r \) of the Version II pebble game. By Lem. 2.5 (c), Spoiler can win in the Version III pebble game using 2 additional pebbles and \( O(\log n) \) additional rounds. The result now follows.

(c) Brachter & Schweitzer [BS20] Lemma 3.12 showed that \((k', r')\)-WL Version III can be simulated by \((2k' + 1, 3r')\)-WL Version I. Now take \( k' = \lceil k/2 \rceil + 2 \) and \( r' = 3r + O(\log n) \). The result follows.
B  Canozaing Groups in Parallel via Weisfeiler–Leman

One approach to isomorphism testing is to canonize the input structures. Precisely, the goal is to compute a standard representation of the input structure that depends only on the isomorphism type and not on the representation of the object. In the setting of graphs, we may define a canonical form as follows.

**Definition B.1.** A graph canonization for a graph class $C$ is a function $\kappa : C \to C$ such that:

(a) $\kappa(G) \cong G$ for all $G \in C$, and

(b) $\kappa(G) = \kappa(H)$ whenever $G \cong H$.

A group canonization may be defined similarly. The isomorphism problem for a class $C$ of either graphs or groups reduces to computing a corresponding canonization. It is open both in the settings of graphs and groups, as to whether a reduction exists from canonization to isomorphism testing. However, combinatorial approaches to isomorphism testing, such as Weisfeiler–Leman, can easily be adapted to canonization procedures. We refer to Fortnow & Grochow [FG11] for a general study on the power of canonization, complete invariants, and polynomial-time algorithms for equivalence relations.

**Theorem B.2** (Folklore). Let $C$ be a class of graphs, and suppose that $k$-WL identifies all colored graphs in $C$. Then there exists a graph canonization for $C$ that can be computed in time $O(n^{k+3}\log n)$.

**Remark B.3.** While Thm. [B.2] is well-known to those who work on Weisfeiler–Leman, an originating reference appears to be unknown. We defer to Grohe & Neuen for a proof of Thm. [B.2] [GN21, Appendix A].

### B.1  Canonizing in Parallel via Weisfeiler–Leman

In this section, we show how to use Weisfeiler–Leman to obtain a parallel canonization procedure for groups.

**Definition B.1.** A graph canonization for a graph class $C$ is a function $\kappa : C \to C$ such that:

(a) $\kappa(G) \cong G$ for all $G \in C$, and

(b) $\kappa(G) = \kappa(H)$ whenever $G \cong H$.

**Theorem B.2** (Folklore). Let $C$ be a class of graphs, and suppose that $k$-WL identifies all colored graphs in $C$. Then there exists a graph canonization for $C$ that can be computed in time $O(n^{k+3}\log n)$.

**Remark B.3.** While Thm. [B.2] is well-known to those who work on Weisfeiler–Leman, an originating reference appears to be unknown. We defer to Grohe & Neuen for a proof of Thm. [B.2] [GN21, Appendix A].
Theorem B.4 ([GN21 Theorem A.1]). Let \( \mathcal{G} \) be a class of graphs such that \( k \)-WL identifies all (colored) graphs \( G \in \mathcal{G} \). Then \( (k+1) \)-WL determines orbits for all graphs \( G \in \mathcal{C} \).

Remark B.5. The original statement of [GN21 Theorem A.1] did not control for rounds, but the proof holds when we consider rounds. The proof also holds when we consider the count-free WL algorithm, as well as when we consider colored groups rather than graphs. For completeness, we provide a proof below.

Theorem B.6. We have the following.

(a) Let \( J \in \{I, II\} \), and suppose that \( \mathcal{G} \) be a class of groups such that \( (k,r) \)-WL Version \( J \) identifies all colored groups \( G \in \mathcal{G} \). Then \( (k+1,r) \) WL Version \( J \) determines orbits for all groups \( G \in \mathcal{C} \).

(b) Let \( J \in \{I, II\} \), and suppose that \( \mathcal{G} \) be a class of groups such that the count-free \( (k,r) \)-WL Version \( J \) identifies all colored groups \( G \in \mathcal{G} \). Then \( (k+1,r) \) WL Version \( J \) determines orbits for all groups \( G \in \mathcal{C} \).

(c) Let \( \mathcal{G} \) be a class of (colored) graphs such that the classical counting \( (k,r) \)-WL algorithm for graphs identifies all colored graphs \( G \in \mathcal{G} \). Then \( (k+1,r) \)WL determines orbits for all graphs \( G \in \mathcal{C} \).

(d) Let \( \mathcal{G} \) be a class of (colored) graphs such that the count-free \( (k,r) \)-WL algorithm for graphs identifies all (colored) graphs \( G \in \mathcal{G} \). Then count-free \( (k+1,r) \)-WL determines orbits for all graphs \( G \in \mathcal{C} \).

Proof. We proceed as follows.

(a) Let \( G \in \mathcal{G} \), and let \( v, w \in G \) such that the coloring \( \chi_{G,k+1}^r(v, \ldots, v) = \chi_{G,k+1}^r(w, \ldots, w) \). Then \( (k,r) \)-WL fails to distinguish the colored group \( (G, \chi_{G}^r(v)) \) from \( (G, \chi_{G}^r(w)) \), where \( (G, \chi_{G}^r(v)) \) is the colored group where we have individualized \( v \) to receive the color \( \chi_{G,k+1}^r(v, \ldots, v) \). As \( (k,r) \)-WL identifies all groups in \( \mathcal{G} \), we have that \( (G, \chi_{G}^r(v)) \cong (G, \chi_{G}^r(w)) \). So there is an automorphism \( \varphi \in \text{Aut}(G) \) such that \( \varphi(v) = w \).

(b) We modify the proof of (a) to use count-free \( (k,r) \)-WL Version \( J \) rather than the standard counting \( (k,r) \)-WL. The proof now goes through \( \text{mutatis mutandis} \).
Let \( G \in \mathcal{G} \), and let \( v, w \in V(G) \) such that the coloring \( \chi_r^{G,k+1}(v, \ldots, v) = \chi_r^{G,k+1}(w, \ldots, w) \). Then \((k, r)-\text{WL}\) fails to distinguish the colored graph \((G, \chi_r^{(v)})\) from \((G, \chi_r^{(w)})\), where \((G, \chi_r^{(v)})\) is the colored graph where we have individualized \( v \) to receive the color \( \chi_r^{G,k+1}(v, \ldots, v) \). As \((k, r)-\text{WL}\) identifies all graphs in \( \mathcal{G} \), we have that \((G, \chi_r^{(v)}) \cong (G, \chi_r^{(w)})\). So there is an automorphism \( \varphi \in \text{Aut}(G) \) such that \( \varphi(v) = w \).

We now establish the correctness of Algorithm \[\text{Algorithm 1}\] we correctly returns a canonical form for any group \( G \in \mathcal{G} \).

**Proof.** Let \( G \in \mathcal{C} \) be our input group, and let \( \kappa(G) \) be the result of Algorithm \[\text{Algorithm 1}\]. We show that \( \kappa \) canonizes \( \mathcal{C} \). By construction, the map \( i \mapsto \psi(i) \) is an isomorphism of \( G \cong \kappa(G) \).

Now let \( H \in \mathcal{C} \) be a second group such that \( G \cong H \). Let \( g_1, \ldots, g_k \in G \) be the sequence of group elements added to Gens by Algorithm \[\text{Algorithm 1}\] and let \( h_1, \ldots, h_k \) be the corresponding sequence for \( H \). We show by induction on \( 0 \leq i \leq k \) that there is an isomorphism \( \varphi : G \cong H \) that restricts to an isomorphism of \( \langle g_1, \ldots, g_i \rangle \cong \langle h_1, \ldots, h_i \rangle \). The base case when \( i = 0 \) is precisely the assumption that \( G \cong H \). Now fix \( i \geq 0 \) and let \( \varphi : G \cong H \) such that \( \varphi(g_j) = h_j \) for all \( j \leq i - 1 \). As \( \varphi \) is an isomorphism mapping \( \varphi(g_j) = h_j \) for all \( j \leq i - 1 \), it follows that \( \varphi \) restricts to the isomorphism of \( \langle g_1, \ldots, g_i \rangle \cong \langle h_1, \ldots, h_i \rangle \) induced by the map \( \varphi(g_j) = h_j \) for all \( j \leq i - 1 \). So we have that \((G, \chi_{G,i}) \cong (H, \chi_{H,i})\).

As \( g_i, h_i \) were selected by the algorithm at line 12, we have that \( \chi_{G,i}(g) = \chi_{H,i}(\varphi(g)) \). As \((k + 1, r)-\text{WL}\) determines orbits for all colored groups \( G \in \mathcal{C} \), it follows that there is a color-preserving isomorphism \( \varphi : (G, \chi_{G,i}) \rightarrow (H, \chi_{H,i}) \). As \( g_j, h_j \) belong to their own color class for \( j \leq i \), \( \varphi \) also has to map \( \varphi(g_j) = h_j \). Necessarily, \( \varphi \) must also restrict to the isomorphism of \( \langle g_1, \ldots, g_i \rangle \cong \langle h_1, \ldots, h_i \rangle \) induced by the map \( g_j \mapsto h_j \) for \( 1 \leq j \leq i \). The result now follows by induction.

**Theorem B.7.** Let \( k \geq 2 \) be a constant, and let \( r := r(n) \) be a function, where \( n \) denotes the order of the input groups. Let \( J \in \{I, II\} \). Let \( \mathcal{C} \) be a class of groups such that \((k, r)-\text{WL Version } J \) identifies all colored groups in \( \mathcal{C} \). For any \( G \in \mathcal{C} \), Algorithm \[\text{Algorithm 1}\] correctly returns a canonical form for \( G \).

**Proof.** By Thm. B.7, we have that Algorithm \[\text{Algorithm 1}\] correctly computes a canonical form for any group \( G \in \mathcal{C} \). It remains to establish the complexity bounds. The key work lies in the \textbf{while} loop. At line 9, we compute \((k + 1, r)-\text{WL}\) using the parallel WL implementation due to Grohe & Verbitsky \cite{GV06}, adapted for \textbf{WL Version II}. We first observe that the initial coloring of WL Version II is \( \mathcal{L} \)-computable. Deciding whether two \( k \)-tuples of group elements are marked isomorphic is \( \mathcal{L} \)-computable \cite{Tan13}. Using a logspace transducer, we can write down for all \( \binom{2^n}{k} \) pairs \{\( \pi, \tau \)\} of \( k \)-tuples whether \( \pi, \tau \) are marked isomorphic.

Each refinement step in the counting WL Version II is \( \mathcal{T}^{\mathcal{L}} \)-computable, and each refinement step in the count-free WL Version II is \( \mathcal{A}^{\mathcal{L}} \)-computable. We thus have the following.

* For (a), both the counting and count-free variants of \((k + 1, O(1))-\text{WL Version II}\) are \( \mathcal{L} \)-computable.
• For (b), (k + 1, r)-WL Version II can be implemented with a logspace uniform TC circuit of depth $O((r(n) + \log(n)) \log n)$ and size $O(r \cdot n^{O(k)})$.

• For (c), the count-free (k + 1, r)-WL Version II can be implemented with a logspace uniform AC circuit of depth $O((r(n) + \log(n)) \log n)$ and size $O(r \cdot n^{O(k)})$.

We now observe that the remaining steps of the while loop are $AC^0$-computable. To see that Line 12 is $AC^0$-computable, we appeal to the characterization that $AC^0 = FO$ [MIS90]. We may write down a first-order formula for the minimum element, and so finding the minimum color class is $AC^0$-computable. Furthermore, identifying the members of a given color class is $AC^0$-computable. Thus, computing the argmin is $AC^0$-computable.

Finally, we note that at line 12, we select $g_i \in G \setminus \langle \text{Gens} \rangle$. Thus, the size of $\langle \text{Gens} \rangle$ is at least doubled at each iteration. At the start of line 9 of the iteration of the while loop after $g_i+1$ is added to Gens, each element $g \in \langle \text{Gens} \rangle$ has a unique color class under $(k+1, r)$-WL (in particular, this is handled by a single refinement step of WL Version II). So the number of iterations $k$ of the while loop is at most $\log n + 1$. Thus, we have the following.

• For (a), we have a logspace uniform family of SAC circuits with depth $O(\log^2 n)$ and size $O(r \cdot n^{O(k)})$.

• For (b), we have a logspace uniform family of TC circuits with depth $O((r(n) + \log(n)) \log n)$ and size $O(r \cdot n^{O(k)})$.

• For (c), we have a logspace uniform family of AC circuits with depth $O((r(n) + \log(n)) \log n)$ and size $O(r \cdot n^{O(k)})$.

The result follows.

For groups that are $O(1)$-generated, using a different strategy, it is possible to canonize such groups using only one call to Weisfeiler–Leman.

**Proposition B.9.** For groups that are $O(1)$-generated, we may compute canonical forms in L.

**Proof.** Let $d := d(G)$. We run the count-free $(d, 1)$-WL Version II. Now for each color class $K \in \text{Im}(\chi)$ where there exists a $d$-tuple $(g_1, \ldots, g_d)$ such that $g_1, \ldots, g_d$ are all distinct (and in such case, the elements of any $d$-tuple in $K$ are all distinct), we may use Tang’s marked isomorphism procedure [Tan13] to test whether the given $k$-tuple generates the group. To obtain a canonical form, we take the coloring $\chi(g_1, \ldots, g_d)$ obtained by individualizing $(g_1, \ldots, g_d)$ from the smallest color class under $\chi$, where $G = \langle g_1, \ldots, g_d \rangle$.

**Remark B.10.** The strategy here is different, in that we don’t need all colored $d$-generated groups to be identified by WL. A $d$-generated subgroup using more than $d$ colors may be harder to identify.

The complexity results in Thm. B.8 also hold if we use WL Version I for canonization. However, the analysis is slightly different. The key idea is that the logspace computations get pushed to computing $\langle \text{Gens} \rangle$ at line 11, rather than at the initial coloring as in WL Version II.

**Theorem B.11.** Let $k \geq 2$ be a constant, and let $r := r(n)$ be a function, where $n$ denotes the order of the input groups.

(a) Let $\mathcal{C}$ be a family of groups. Suppose that $r \geq 2$ is a constant. If $(k, r)$-WL Version I (counting or count-free) identifies all colored groups belonging to $\mathcal{C}$, then we may compute canonical forms using a logspace uniform family of SAC circuits of depth $O(\log^2 n)$ and size $O(r \cdot n^{O(k)})$.

(b) Let $\mathcal{C}$ be a family of groups, and suppose that $r(n) \in \omega(1)$. If $(k, r(n))$-WL Version I identifies all colored groups belonging to $\mathcal{C}$, then we may compute canonical forms using a logspace uniform family of TC circuits of depth $O((r(n) + \log n) \log n)$ and size $O(r \cdot n^{O(k)})$.

(c) Let $\mathcal{C}$ be a family of groups, and suppose that $r(n) \in \omega(1)$. If the count-free $(k, r(n))$-WL Version I identifies all colored groups belonging to $\mathcal{C}$, then we may compute canonical forms using a logspace uniform family of AC circuits of depth $O((r(n) + \log n) \log n)$ and size $O(r \cdot n^{O(k)})$. 

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Theorem B.12. Let \( k \geq 2 \) be a constant, and let \( r := r(n) \) be a function, where \( n \) denotes the order of the input groups. Let \( \mathcal{C} \) be a class of groups such that \((k, r)-\text{WL Version III}\) identifies all colored groups in \( \mathcal{C} \). For any \( G \in \mathcal{C} \), Algorithm \( 4 \) correctly returns a canonical form for \( G \).
Proof. Let $G \in \mathcal{C}$ be our input group, and let $\kappa(G)$ be the result of Algorithm 2. We show that $\kappa$ canonizes $\mathcal{C}$. By construction, the map $i \mapsto \psi(i)$ is an isomorphism of $G \cong \kappa(G)$.

Now let $H \in \mathcal{C}$ be a second group such that $G \cong H$. Let $g_1, \ldots, g_k \in G$ be the sequence of vertices added to Gens by Algorithm 2, and let $h_1, \ldots, h_k$ be the corresponding sequence for $H$ (we will show later that $k \leq \log n + 1$). We show by induction on $0 \leq i \leq k$ that there is an isomorphism $\varphi : G \cong H$ that restricts to an isomorphism of $(g_1, \ldots, g_i) \cong (h_1, \ldots, h_i)$. The base case when $i = 0$ is precisely the assumption that $G \cong H$. Now fix $i \geq 0$ and let $\varphi : G \cong H$ such that $\varphi(g_j) = h_j$ for all $j \leq i - 1$. As $\varphi$ is an isomorphism mapping $\varphi(g_j) = h_j$ for all $j \leq i - 1$, it follows that $\varphi$ restricts to the isomorphism of $(g_1, \ldots, g_{i-1}) \cong (h_1, \ldots, h_{i-1})$ induced by the map $\varphi(g_j) = h_j$ for all $j \leq i - 1$. Then, using the fact that the map $G \mapsto \Gamma_G$ is a many-one reduction from GI to GI, we have that $(\Gamma_G, \chi_{G,i}) \cong (\Gamma_H, \chi_{H,i})$. As $g_i, h_i$ were selected by the algorithm at line 12, we have that $\chi_{G,i}(g) = \chi_{H,i}(\varphi(g))$. As $(k+1, r)$-WL determines orbits for all graphs $\Gamma_G$ arising from groups $G \in \mathcal{C}$, it follows that there is a color-preserving isomorphism $\varphi : (\Gamma_G, \chi_{G,i}) \cong (\Gamma_H, \chi_{H,i})$ such that $\varphi(g_i) = h_i$. As $g_j, h_j$ belong to their own color class for $j \leq i$, $\varphi$ also has to map $\varphi(g_j) = h_j$. Necessarily, $\varphi$ must also restrict to the isomorphism of $(g_1, \ldots, g_i) \cong (h_1, \ldots, h_i)$ induced by the map $g_j \mapsto h_j$ for $1 \leq j \leq i$. The result now follows by induction. \hfill \Box

Theorem B.13. Let $k \geq 2$ be a constant, and let $r := r(n)$ be a function, where $n$ denotes the order of the input groups.

(a) Let $\mathcal{C}$ be a family of groups. If $(k, O(1))$-WL Version III (counting or count-free) identifies all colored groups belonging to $\mathcal{C}$, then we can compute canonical forms using a logspace uniform family of SAC circuits of depth $O(\log^2 n)$ and size $O(n^{O(k)})$.

(b) Let $\mathcal{C}$ be a family of groups, and suppose that $r(n) \in o(1)$. If $(k, r(n))$-WL Version III identifies all colored groups belonging to $\mathcal{C}$, then we can compute canonical forms using a logspace uniform family of TC circuits of depth $O((r(n) + \log n) \log n)$ and size $O(n^{O(k)})$.

(c) Let $\mathcal{C}$ be a family of groups, and suppose that $r(n) \in o(1)$. If the count-free $(k, r(n))$-WL Version III identifies all colored groups belonging to $\mathcal{C}$, then we can compute canonical forms using a logspace uniform family of AC circuits of depth $O((r(n) + \log n) \log n)$ and size $O(n^{O(k)})$.

Proof. The proof is identical to Thm. B.1, replacing WL Version I with WL Version III. We also note that constructing the graph $\Gamma_G$ at line 3 is $\mathcal{AC}$-computable; see Rmk. 2.3. \hfill \Box

As only $\log n + 1$ calls to $(k + 1)$-WL are required in the setting of groups, we obtain the following improvement to the serial runtime, compared to Thm. B.2.

Corollary B.14. Let $\mathcal{C}$ be a class of groups.

(a) If $k$-WL Versions I or II identify all colored groups in $\mathcal{C}$, then there exists a group canonization for $\mathcal{C}$ that can be computed in time $O(n^{k+2} \log^2 n)$.

(b) If $k$-WL Version III identifies all colored graphs in $\Gamma_G = \{ \Gamma_G : G \in \mathcal{C} \}$, then there exists a group canonization for $\mathcal{C}$ that can be computed in time $O(|G|^{2k+3} \log^3 |\mathcal{C}|)$.

Remark B.15. We note that for WL Version III, Thm. B.2 yields a runtime of $O(n^{2k+4} \log n)$.

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