Elliptic Curves and Algebraic Geometry Approach in Gravity Theory III. Uniformization Functions for a Multivariable Cubic Algebraic Equation

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Abstract

The third part of the present paper continues the investigation of the solution of the multivariable cubic algebraic equation for reparametrization invariance of the gravitational Lagrangian. The main result in this paper constitutes the fact that the earlier found parametrization functions of the cubic algebraic equation for reparametrization invariance of the gravitational Lagrangian can be considered also as uniformization functions. These functions are obtained as solutions of first-order nonlinear differential equations, as a result of which they depend only on the complex (uniformization) variable $z$. Further, it has been demonstrated that this uniformization can be extended to two complex variables, which is particularly important for investigating various physical metrics, for example the $ADS$ metric of constant negative curvature (Lobachevsky spaces).

1 INTRODUCTION

In previous papers [1, 2] the general approach for investigation of algebraic equations in gravity theory has been presented. The approach is based essentially on the important distinction between covariant and contravariant metric components in the framework of the gravitational theories with covariant and contravariant metrics and connections ($GTCCMC$), which has been described in the review article [3]. In its essence, this distinction is related to the affine geometry approach [4,5], according to which the four-velocity tangent vector at each point of the observer’s worldline is not normalized and

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equal to one, i.e. \( l_\alpha l^\alpha = l^2 \neq 1 \). Similarly, for a second-rank tensor one would have \( g_{\mu \nu} g^{\nu \alpha} = l_\mu \neq \delta^\alpha_\mu \).

In [2] a solution for the chosen variables \( dX^1, dX^2, dX^3 \) in terms of the elliptic Weierstrass function and its derivative has been found of the earlier proposed [6] cubic algebraic equation of reparametrization invariance of the gravitational Lagrangian

\[
 dX^i dX^l \left( p \Gamma^r_{il} g_{kr} dX^k - \Gamma^r_{ik} g_{lr} d^2 X^k - \Gamma^r_{l(i} g_{r)}, d^2 X^k \right) - dX^i dX^l R_{il} = 0 \quad . (1.1)
\]

Finding the solution enables one to find the dependence of the contravariant metric tensor components on the elliptic Weierstrass function and its derivatives and on variables, related to the covariant tensor and its derivatives.

However, it is much more important to find the dependence of the covariant tensor components. For some concrete cases - the Szafron-Szekeres inhomogeneous cosmological model and its subcase - the FLRW (Friedman - Lemaître - Robertson - Walker) cosmology, such a representation has been found in [7]. The general solution of inhomogeneous relativistic cosmology was given in the form

\[
 ds^2 = dt^2 - \frac{\left( M(z) \right)^2}{\left( \rho(u + \epsilon) - \rho(v_0) \right)^2} e^{2\nu} (dx^2 + dy^2) - h^2(z) (\Phi' + \Phi \nu')^2 dz^2 \quad . (1.2)
\]

The function \( \rho(v_0) \) in [7] is supposed to satisfy the elliptic curve

\[
 \left( \rho'(v_0) \right)^2 = 4\rho^3(v_0) - g_2 \rho(v_0) - g_3 \quad , (1.3)
\]

where \( g_2 \) and \( g_3 \) are assumed to be the functions

\[
 g_2 = \frac{K^2(z)}{12} \quad ; \quad g_3 = \frac{1}{216} K^3(z) - \frac{1}{12} \Lambda M^2(z) \quad . (1.7)
\]

Unfortunately, the presented in [7] solution cannot be claimed to be true due to the following simple reason: If \( g_2 \) and \( g_3 \) are functions and at the same time, they satisfy the elliptic curve equation (1.3), then these functions should be equal to the corresponding Eisenstein series (invariants) \( g_2 = 60 \sum_{\omega \in \Gamma} \frac{1}{\omega} g_3 = 140 \sum_{\omega \in \Gamma} \frac{1}{\omega^6} \). If this is taken into account, then it can easily be checked that the obtained solution (1.2) would not be of that form - for example, there would be no dependence on the \( M(z) \) function in the second term.

Consequently, the problem about finding the correct solutions of the Einstein’s equations in terms of elliptic functions still remains open. But even if solutions for the metric tensor are found in the form \( g_{ij}(z, x) \) (here \( z \) is the complex variable, on which the Weierstrass function depends, \( x \) are the coordinates in the chosen metric), then this dependence on additional complex variable would still complicate the solution.

In this third part of the paper, we shall consider an approach, when it will be possible to find the solution not in terms of the parametrization functions \( g_{ij}(z, x) \), but in terms
of uniformization functions - these are the functions, which depend only on the complex variable $z$ (or, as we shall see, on two complex variables $z$ and $v$) and on no other variables. In fact, it will be shown that after solving a first-order system of nonlinear differential equations with respect to the generalized coordinates $X^i$, their dependence on the complex coordinate $z$ can be found. Thus, an important problem from the point of algebraic geometry will be solved - the uniformization functions $dX^i = dX^i(z)$ for the multivariable cubic algebraic equation (1.1) are found, as a result also the metric tensor $g_{ij}(z, X(z)) = g_{ij}(z)$.

2 COMPLEX COORDINATE DEPENDENCE OF THE METRIC TENSOR COMPONENTS FROM THE UNIFORMIZATION OF A CUBIC ALGEBRAIC SURFACE

In this section it will be shown that the solutions (2.16), (2.21) and (2.28) (see [2] - Part II) of the cubic algebraic equation (1.1) enable us to express not only the contravariant metric tensor components through the Weierstrass function and its derivatives, but the covariant components as well.

Let us write down for convenience the system of equations (2.16), (2.21) and (2.28) for $dX^1$, $dX^2$, and $dX^3$ as ($l = 1, 2, 3$)

$$dX^l(X^1, X^2, X^3) = F_l(g_{ij}(X), \Gamma^k_{ij}(X), \rho(z), \rho'(z)) = F_l(X, z), \quad (2.1)$$

where the appearance of the complex coordinate $z$ is a natural consequence of the uniformization procedure, applied with respect to each one of the cubic equations from the "embedded" sequence of equations.

Yet how the appearance of the additional complex coordinate $z$ on the R. H. S. of (2.1) can be reconciled with the dependence of the differentials on the L. H. S. only on the generalized coordinates $(X^1, X^2, X^3)$ (and on the initial coordinates $x^1, x^2, x^3$ because of the mapping $X^i = X^i(x^1, x^2, x^3)$)? The only reasonable assumption will be that the initial coordinates depend also on the complex coordinate, i.e.

$$X^l \equiv X^l(x^1(z), x^2(z), x^3(z)) = X^l(x, z). \quad (2.2)$$

Taking into account the important initial assumptions ($l = 1, 2, 3$)

$$d^2X^l = 0 = dF_l(X(z), z) = \frac{dF_l}{dz} dz, \quad (2.3)$$
one easily gets the system of three inhomogeneous linear algebraic equations with respect to the functions \( \frac{\partial X^1}{\partial z}, \frac{\partial X^2}{\partial z} \) and \( \frac{\partial X^3}{\partial z} \) (\( l = 1, 2, 3 \)):

\[
\frac{\partial F_l}{\partial X^1} \frac{\partial X^1}{\partial z} + \frac{\partial F_l}{\partial X^2} \frac{\partial X^2}{\partial z} + \frac{\partial F_l}{\partial X^3} \frac{\partial X^3}{\partial z} + \frac{\partial F_l}{\partial z} = 0 ,
\]

(2.4)

The solution of this algebraic system \((i, k, l = 1, 2, 3)\)

\[
\frac{\partial X^i}{\partial z} = G_l \left( \frac{\partial F_i}{\partial X^k} \right) = G_l \left( X^1, X^2, X^3, z \right)
\]

(2.5)

represents a system of three first-order nonlinear differential equations. A solution of this system can always be found in the form

\[
X^1 = X^1(z) \quad ; \quad X^2 = X^2(z) \quad ; \quad X^3 = X^3(z)
\]

(2.6)

and therefore, the metric tensor components will also depend on the complex coordinate \( z \), i.e. \( g_{ij} = g_{ij}(X(z)) \). Note that since the functions \( \frac{\partial F_i}{\partial X^k} \) in the R. H. S. of (2.5) depend on the Weierstrass function and its derivatives, it might seem natural to write that the solution of the above system of nonlinear differential equations \( g_{ij} \) will also depend on the Weierstrass function and its derivatives

\[
g_{ij} = g_{ij}(X^1(\rho(z), \rho'(z), X^2(\rho(z), \rho'(z), X^3(\rho(z), \rho'(z)) = g_{ij}(z)
\]

(2.7)

Note however that for the moment we do not have a theorem that the solution of the system \((2.5)\) will also contain the Weierstrass function. But the dependence on the complex coordinate \( z \) will be retained.

Now let us mention the other equations, which will further be taken into account.

The first set of equations simply means that the differentials \( dF_1, dF_2, dF_3 \), equal to the second differentials \( d^2X^1, d^2X^2, d^2X^3 \) can be taken with respect both to the generalized coordinates \( X^1, X^2, X^3 \) and the initial coordinates \( x^1, x^2, x^3 \) (\( l = 1, 2, 3 \))

\[
d^2X^l = dF_l(X(z), z) = dF_l(x(z), z)
\]

(2.8)

Denoting further \( x^1 \equiv \frac{\partial x^1}{\partial z}, x^2 \equiv \frac{\partial x^2}{\partial z} \) and \( x^3 \equiv \frac{\partial x^3}{\partial z} \), the above equalities result again in a system of three inhomogeneous algebraic equations with respect to \( \dot{X}^1 \equiv \frac{\partial X^1}{\partial z}, \dot{X}^2 \equiv \frac{\partial X^2}{\partial z} \) and \( \dot{X}^3 \equiv \frac{\partial X^3}{\partial z} \)

\[
\frac{\partial F_l}{\partial X^1} \frac{\partial X^1}{\partial z} + \frac{\partial F_l}{\partial X^2} \frac{\partial X^2}{\partial z} + \frac{\partial F_l}{\partial X^3} \frac{\partial X^3}{\partial z} = \frac{\partial F_l}{\partial x^1} \dot{x}^1 + \frac{\partial F_l}{\partial x^2} \dot{x}^2 + \frac{\partial F_l}{\partial x^3} \dot{x}^3.
\]

(2.9)

Assuming for the moment that we know the functions \( x^1, \dot{x}^2 \) and \( \dot{x}^3 \), the solutions of this algebraic system will give again another system of three first-order nonlinear differential equations (\( l = 1, 2, 3 \))

\[
\frac{\partial X^l}{\partial z} = H_l \left( X^1, X^2, X^3, z, \dot{x}^1, \dot{x}^2, \dot{x}^3 \right)
\]

(2.10)
Again, a solution of this system like the one in (2.6) can be obtained but with account of the dependence on the additional variables \(x^1, x^2\) and \(x^3\). Let us also here note that the solution (2.6) of the nonlinear system of equations (2.5) can be assumed to be dependent on some another complex variable \(v\)

\[
X^1 = X^1(z, v) \quad ; \quad X^1 = X^1(z, v) \quad ; \quad X^1 = X^1(z, v) .
\]  

(2.11)

The system of equations (2.8) \((i = 1, 2, 3)\)

\[
d^2 X^i = dF_i(\mathbf{X}(z, v), z) = dF_i(\mathbf{x}(z, v), z) ,
\]

(2.12)

with account of the expressions (2.10) now will be rewritten as

\[
\frac{\partial F_i}{\partial X^1} \frac{\partial X^1}{\partial v} + \frac{\partial F_i}{\partial X^2} \frac{\partial X^2}{\partial v} + \frac{\partial F_i}{\partial X^3} \frac{\partial X^3}{\partial v} = \frac{\partial F_i}{\partial x^1} x^1 + \frac{\partial F_i}{\partial x^2} x^2 + \frac{\partial F_i}{\partial x^3} x^3 + \n
\]

\[
+ \frac{\partial F_i}{\partial x^1} x' + \frac{\partial F_i}{\partial x^2} x'' + \frac{\partial F_i}{\partial x^3} x''' - \frac{\partial F_i}{\partial X^1} H_1 - \frac{\partial F_i}{\partial X^2} H_2 - \frac{\partial F_i}{\partial X^3} H_3 = 0 ,
\]

(2.13)

where \(x', x'', x'''\) denote the derivatives \(\frac{\partial x^1}{\partial z}, \frac{\partial x^2}{\partial z}, \frac{\partial x^3}{\partial z}\). The same notation further will be used with respect to the variables \(\frac{\partial X^1}{\partial v}, \frac{\partial X^2}{\partial v}, \frac{\partial X^3}{\partial v}\). Similarly to (2.10), the algebraic solution of this system of equations can be represented as

\[
\frac{\partial X^i}{\partial v} = K_i \left( \mathbf{X}(z, v), z, \dot{\mathbf{x}}, \ddot{\mathbf{x}} \right) .
\]

(2.14)

Note that instead of (2.12), we could have also written

\[
d^2 X^i = dF_i(\mathbf{X}(z, v), z) = dF_i(\mathbf{x}(z, v), z) .
\]

(2.15)

Further in section 3 it shall be proved why this would be incorrect. The complete analysis of the system of equations, when both system of coordinates depend on the two pair of complex variables \(z\) and \(v\) will be given in the following sections. For the moment we give just the general qualitative motivations.

The other set of equations, which will further be used and which relates the generalized coordinates \(X^i\) to the initial ones \(x^i\) is

\[
d^2 X^i = 0 = \frac{\partial^2 X^i}{\partial x^k \partial x^r} dx^k dx^r + \frac{\partial X^i}{\partial x^k} d^2 x^k .
\]

(2.16)

For the moment we assume that the initial coordinates \(x^k\) depend only on the \(z\) coordinate, and therefore

\[
\frac{\partial^2 X^i}{\partial x^k \partial x^r} = \frac{\ddot{X}^i}{\dot{x}^k \dot{x}^r} - \dot{X}^i \frac{\ddot{x}^r}{(\dot{x}^r)^2} .
\]

(2.17)
Taking this into account, the system (2.16) in the \( n \)-dimensional case can be written as
\[
n^2 \dot{X}^i (dz)^2 - (n - 1) \ddot{X}^i \frac{\dot{x}^r}{x} (dz)^2 + n \dot{X}^i \, d^2 z = 0 \quad .
\] (2.18)

Introducing the notation
\[
y^r = \frac{\partial}{\partial z} (\ln \dot{x}^r) = \frac{\dot{x}^r}{x^r}
\] (2.19)
for the three-dimensional case, the system (2.18) can be written as
\[
2\dot{X}^i (dz)^2 (y^1 + y^2 + y^3) = 9\ddot{X}^i (dz)^2 + 3\dot{X}^i \, d^2 z \quad .
\] (2.20)

Dividing the L. H. S. and the R. H. S. of the \( i \)-th and the \( j \)-th equation of this system, it can easily be obtained
\[
(dz)^2 \left( \dddot{X}^i \dot{X}^j - \ddot{X}^j \dddot{X}^i \right) = 0 \quad ,
\] (2.21)
which can be written as
\[
(dz)^2 \left( \dot{X}^j \right)^2 \frac{\partial}{\partial z} \left( \frac{\dddot{X}^i}{\dot{X}^j} \right) = 0 \quad .
\] (2.22)

Neglecting the case when \( \dot{X}^j = 0 \), the above relation simply means that \( \dot{X}^2 \) and \( \dot{X}^3 \) should be proportional to \( \dot{X}^1 \)
\[
\dot{X}^2 = C_2 \dot{X}^1 \quad ; \quad \dot{X}^3 = C_3 \dot{X}^1 \quad ,
\] (2.23)
where \( C_2 \) and \( C_3 \) are constants. Indeed, it is easily seen that (2.23) holds since
\[
dX^1 = \dot{X}^1 \, dz = F_1 \quad ; \quad dX^2 = \dot{X}^2 \, dz = F_2 \quad ; \quad dX^3 = \dot{X}^3 \, dz = F_3
\] (2.24)
and consequently
\[
C_2 = \frac{F_2}{F_1} \quad ; \quad C_3 = \frac{F_3}{F_1} \quad .
\] (2.25)

3 FIRST-ORDER NONLINEAR DIFFERENTIAL EQUATIONS FOR THE COMPLEX FUNCTIONS \( x = x(z) \) AND \( X = X(z) \)

For the purpose, the two systems of algebraic equations (2.4) and (2.9) will be used. If one substitutes the found expressions (2.23) for \( \dot{X}^2 \) and \( \dot{X}^3 \) into the system (2.4), it may
be treated as an algebraic system of equations with respect to the variables \( X^1, C_2 \) and \( C_3 \). Introducing the notation

\[
\{ F_i, F_j \}_{z, X^k} \equiv \frac{\partial F_i}{\partial z} \frac{\partial F_j}{\partial X^k} - \frac{\partial F_i}{\partial X^k} \frac{\partial F_j}{\partial z} \tag{3.1}
\]

de the "one-dimensional" Poisson bracket \( \{ F_i, F_j \}_{z, X^k} \) of the coordinates \( z, X^k \) and also the notation

\[
\{ F_1, F_2, F_3 \}_{z, [X^i, X^j]} \equiv \{ F_1, F_2 \}_{z, X^i} \{ F_1, F_3 \}_{z, X^j} - \{ F_1, F_2 \}_{z, X^j} \{ F_1, F_3 \}_{z, X^i} \tag{3.2}
\]

one can show that the solution of the system of linear algebraic equations (2.4) with respect to \( X^1, C_2 \) and \( C_3 \)

\[
\frac{\partial F_i}{\partial X^1} \dot{X}^1 + \frac{\partial F_i}{\partial X^2} \dot{X}^2 + \frac{\partial F_i}{\partial X^3} \dot{X}^3 + \frac{\partial F_i}{\partial z} = 0 \tag{3.3}
\]

can be represented in the following compact form

\[
C_2 = \frac{\{ F_1, F_2, F_3 \}_{z, [X^1, X^1]} \{ F_1, F_2, F_3 \}_{z, [X^2, X^2]} \{ F_1, F_2, F_3 \}_{z, [X^3, X^3]}}{\{ F_1, F_2, F_3 \}_{z, [X^1, X^2]} \{ F_1, F_2, F_3 \}_{z, [X^2, X^3]}}, \quad C_3 = \frac{\{ F_1, F_2, F_3 \}_{z, [X^1, X^2]} \{ F_1, F_2, F_3 \}_{z, [X^2, X^3]} \{ F_1, F_2, F_3 \}_{z, [X^3, X^3]}}{\{ F_1, F_2, F_3 \}_{z, [X^1, X^3]} \{ F_1, F_2, F_3 \}_{z, [X^2, X^3]}}, \tag{3.4}
\]

\[
\dot{X}^1 = -\frac{\partial F_1}{\partial z} \frac{\{ F_1, F_2, F_3 \}_{z, [X^2, X^3]} K_1}{K_1} \tag{3.5}
\]

In (3.5) the following notation has been introduced for \( K_i \) \((i = 1, 2, 3)\)

\[
K_i \equiv \frac{\partial F_i}{\partial X^1} \{ F_1, F_2, F_3 \}_{z, [X^2, X^2]} + \frac{\partial F_i}{\partial X^2} \{ F_1, F_2, F_3 \}_{z, [X^3, X^1]} + \frac{\partial F_i}{\partial X^3} \{ F_1, F_2, F_3 \}_{z, [X^1, X^2]} \tag{3.6}
\]

The usefulness of introducing this notation will soon be understood.

Now let us rewrite the system of equations (2.9) in the form

\[
\frac{\partial F_i}{\partial x^1} \dot{x}^1 + \frac{\partial F_i}{\partial x^2} \dot{x}^2 + \frac{\partial F_i}{\partial x^3} \dot{x}^3 = M_i \tag{3.7}
\]

where \( M_i \) will be the notation for

\[
M_i \equiv \frac{\partial F_i}{\partial X^1} X^1 + \frac{\partial F_i}{\partial X^2} X^2 + \frac{\partial F_i}{\partial X^3} X^3 \tag{3.8}
\]

Making use of the above formulae (3.4 - 3.6) and also (2.23), \( M_i \) can be calculated to be

\[
M_i = -\frac{\partial F_1}{\partial z} K_i \tag{3.9}
\]

Further, the solutions of the linear algebraic system of equations (3.7) can be represented in the form

\[
\dot{x}^i = S^i_1 M_1 + S^i_2 M_2 + S^i_3 M_3 \tag{3.10}
\]
where the functions $S_1, S_2$ and $S_3$ depend on $\frac{\partial F}{\partial x} (i, k = 1, 2, 3)$. Since $M_1$, $M_2$, $M_3$ according to (3.9) and (3.6) are proportional to $\frac{\partial F_i}{\partial z} (F, F_1, F_3)_{z, [x^k, x^j]} (where (k, j) = (2, 3), (3, 1) or (1, 2)), the resulting solution (3.10) will be of the kind

$$
x^i = \frac{S_1 (\frac{\partial F_i}{\partial x})^2 + S_2 \frac{\partial F_2}{\partial x} \frac{\partial F_3}{\partial x} + S_3 \frac{\partial F_1}{\partial x} \frac{\partial F_3}{\partial x}}{S_4 (\frac{\partial F_1}{\partial z}) + S_5 \frac{\partial F_2}{\partial z} + S_6 \frac{\partial F_3}{\partial z}}, \quad (3.11)
$$

where the functions $S_1, S_2, ..., S_6$ depend both on $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial z}$ and consequently on all the variables $x^k, X^k$ and $z$. We have used also the following relation, obtained after simple algebra with account of (3.1) and (3.2)

$$\{F_1, F_2, F_3\}_{z, [x^i, x^j]} = \left(\frac{\partial F_i}{\partial z}\right)^2 \{F_2, F_3\}_{x^i, x^j} + \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial z} \{F_3, F_1\}_{x^i, x^j} + \frac{\partial F_1}{\partial z} \frac{\partial F_3}{\partial z} \{F_2, F_1\}_{x^i, x^j}. \quad (3.12)$$

Thus we have obtained the system of first order nonlinear differential equations with respect to the initial coordinates $x^i = x^i(z)$. An analogous system of nonlinear differential equations is obtained for $X^1 = X^1(z)$, $X^2 = X^2(z)$ and $X^3 = X^3(z)$ - for $X^1$ this is equation (3.5), and with account of (2.23) and expressions (3.4) for $C_2$ and $C_3$, the corresponding equations for $X^2(z)$ and $X^3(z)$ are

$$\dot{X}^2 = -\frac{\partial F_1}{\partial z} \{F_1, F_2, F_3\}_{z, [x^3, x^2]} ; \quad \dot{X}^3 = -\frac{\partial F_1}{\partial z} \{F_1, F_2, F_3\}_{z, [x^1, x^2]} \quad (3.13)$$

Therefore, if the generalized coordinates $X^1, X^2, X^3$ are determined as functions of the complex variable $z$ after solving the system (3.5), (3.13), the obtained functions $X^1 = X^1(z)$, $X^2 = X^2(z)$ and $X^3 = X^3(z)$ can be substituted into the R. H. S. of the system (3.11) for $x^1, x^2$ and $x^3$ and the corresponding solutions $x^1 = x^1(z)$, $x^2 = x^2(z)$ and $x^3 = x^3(z)$ can be found. Remember that we started from the assumption that only the generalized coordinates $X^1, X^2, X^3$ satisfy the original cubic algebraic equation and therefore equalities (2.24) are fulfilled. Nevertheless, the corresponding functions $x^i = x^i(z)$ is possible to be determined from the system (3.11), the R. H. S. of which also confirms that $dx^i \neq F_i$.

This conclusion is important since it shows that the two systems of coordinates should not be treated on an equal footing. This refers of course to the case of only one complex coordinate.

**4** IS IT POSSIBLE TO HAVE A TWO COM-
PLEX COORDINATE DEPENDENCE OF THE GENERALIZED COORDINATES $X^i = X^i(x(z), z, v)$?

It will be proved below that such a case should be disregarded since it leads to an impossibility to determine the dependence $X^i$ on the $v$ coordinate.

Under the above assumption $X^i = X^i(x(z), z, v)$, the first set of three equations

$$dX^i = \frac{\partial X^i}{\partial x^1} dx^1 + \frac{\partial X^i}{\partial x^2} dx^2 + \frac{\partial X^i}{\partial x^3} dx^3$$

(4.1)

can be represented as

$$F_i = \dot{X}^i \frac{\partial z^j}{\partial x^1} dz^1 + \dot{X}^i \frac{\partial z^j}{\partial x^2} dz^2 + \dot{X}^i \frac{\partial z^j}{\partial x^3} dz^3 = 3 \dot{X}^j dz^j$$  ,

(4.2)

so again relations (2.23) - (2.25) $\dot{X}^2 = F_1 \dot{X}^1, \dot{X}^3 = F_1 \dot{X}^1$ will hold.

The second set of equations

$$d^2X^i = dF_i(X, z) = dF_i(X(z, v), z) = dF_i(z, v)$$

(4.3)

will express the equality of the differentials, expressed in terms of the two different sets of coordinates $(X, z)$ and $(z, v)$

$$d^2X^i = \frac{\partial F_i}{\partial X^1} dX^1 + \frac{\partial F_i}{\partial X^2} dX^2 + \frac{\partial F_i}{\partial X^3} dX^3 + \frac{\partial F_i}{\partial z} dz =$$

$$= \left[ \frac{\partial F_i}{\partial X^1} \dot{X}^1 + \frac{\partial F_i}{\partial X^2} \dot{X}^2 + \frac{\partial F_i}{\partial X^3} \dot{X}^3 + \frac{\partial F_i}{\partial z} \right] dz +$$

$$+ \left[ \frac{\partial F_i}{\partial X^1} dX^1 + \frac{\partial F_i}{\partial X^2} dX^2 + \frac{\partial F_i}{\partial X^3} dX^3 \right] dv$$  .

(4.4)

Taking into account that according to (2.25) $dX_1 = F_1, dX_2 = F_2$ and $dX_3 = F_3$ and also the expressed from (2.25) differential

$$dz = \frac{1}{3} \frac{F_1}{\dot{X}^1}$$  ,

(4.5)

one can obtain for (4.4)

$$\left[ \frac{\partial F_i}{\partial X^1} dX^1 + \frac{\partial F_i}{\partial X^2} dX^2 + \frac{\partial F_i}{\partial X^3} dX^3 \right] dv =$$

$$= \frac{2}{3} \left[ \frac{\partial F_i}{\partial X^1} F_1 + \frac{\partial F_i}{\partial X^2} F_2 + \frac{\partial F_i}{\partial X^3} F_3 \right]$$  .

(4.6)
Dividing the L. H. S. and the R. H. S. for different values of the indice \( i = 1, 2, 3 \), one can obtain the following system of linear homogeneous algebraic equations with respect to \( X^1, X^2 \) and \( X^3 \) (the indice \( i \) takes values 1, 2, 3, 1, 2…i.e. if \( i = 3 \), then \( i + 1 \) would be 1)

\[
\left( \frac{\partial F_i}{\partial X^1} Q_{i+1} - \frac{\partial F_{i+1}}{\partial X^1} Q_i \right) X^1 + \left( \frac{\partial F_i}{\partial X^2} Q_{i+1} - \frac{\partial F_{i+1}}{\partial X^2} Q_i \right) X^2 + \\
\left( \frac{\partial F_i}{\partial X^3} Q_{i+1} - \frac{\partial F_{i+1}}{\partial X^3} Q_i \right) X^3 = 0 ,
\]

where \( Q_i (i = 1, 2, 3) \) denotes the expression

\[
Q_i \equiv \frac{\partial F_i}{\partial X_1} F_1 + \frac{\partial F_i}{\partial X^2} F_2 + \frac{\partial F_i}{\partial X^3} F_3 .
\]

Note that for the moment we have not yet used the equations \( d^2 X^i = dF_i = 0 \), from where \( Q_i = 0 \). Then the system of equations (4.7) would be identically satisfied for all \( X^1, X^2 \) and \( X^3 \) and it would be impossible to express them as solutions of the system. But even without making use of the equations \( d^2 X^i = dF_i = 0 \), the consistency (or inconsistency) of the system (4.7) is a necessary condition for the consistency (or inconsistency) of the assumption about \( X^i = X^i (x(z), z, v) \).

Making use of the notation (3.1), the determinant of the system can be written as

\[
\begin{vmatrix}
\sum_{l_1 \neq 1} F_{l_1} \{F_1, F_2\}_{1,l_1} & \sum_{l_2 \neq 2} F_{l_2} \{F_1, F_2\}_{2,l_2} & \sum_{l_3 \neq 3} F_{l_3} \{F_1, F_3\}_{3,l_3} \\
\sum_{m_1 \neq 1} F_{m_1} \{F_1, F_3\}_{1,m_1} & \sum_{m_2 \neq 2} F_{m_2} \{F_1, F_3\}_{2,m_2} & \sum_{m_3 \neq 3} F_{m_3} \{F_1, F_3\}_{3,m_3} \\
\sum_{n_1 \neq 1} F_{n_1} \{F_2, F_3\}_{1,n_1} & \sum_{n_2 \neq 2} F_{n_2} \{F_2, F_3\}_{2,n_2} & \sum_{n_3 \neq 3} F_{n_3} \{F_2, F_3\}_{3,n_3}
\end{vmatrix},
\]

where instead of \( \{F_i, F_j\}_{X^k, X^m} \) we have written only \( \{F_i, F_j\}_{k,n} \) and each element in the determinant represents a sum either over \( l_1, l_2 \) or \( l_3 \).

The explicite calculation of the determinant (4.9) gives the non-zero expression

\[
\frac{\partial F_2}{\partial X^1} \{F_1, F_3\}_{2,3} + \frac{\partial F_3}{\partial X^1} \{F_1, F_2\}_{1,2} .
\]

Since the determinant is non-zero, the system of linear homogeneous algebraic equations does not have a solution and consequently the assumption that \( X^i = X^i (x(z), z, v) \) turns out to be incorrect.

\section{5 COMPLEX STRUCTURE \( X^i = X^i (x(z, v), z) \) OF THE GENERALIZED COORDINATES AND OF THE METRIC TENSOR COMPONENTS}

Now it shall be proved that the parametrization (2.1) of the initially given cubic algebraic
curve (surface) can be extended to a parametrization in terms of a pair of complex coordinates \((z, v)\) and thus a complex structure can be introduced. Of particular interest in view of possible physical applications to theories with extra dimensions and relation to \(ADS\) theories, which will be discussed in the conclusion, will be the case of \(v = \overline{z}\), when a pair of holomorphic - antiholomorphic variables can be introduced.

In principle a manifold may admit a complex structure \([8]\), if it can be covered with opened sets \(U_1, V_1, U_2, V_2, \ldots\), such that in any intersection \(U_i \cap V_i\) the associated transformations \(z^k = z^k(z_i, v_i)\) are complex (analytical) functions.

The investigated problem may be formulated as follows. Let (again) the system of equations (2.1) is given, subjected to the additional constraining equation \(d^2 X^i = 0\). Then the parametrization (2.1) of the initially given cubic algebraic surface can be extended to a parametrization by means of a pair of complex coordinates \((z, v)\) in the following way

\[
dX^i(X) = F_i(X(X(z, v)), z) .
\]

Therefore, it should be proved that the same system of equations, investigated in the previous sections, is not contradictable under the assumption \(X^i = X^i(X(z, v), z)\).

The first set of equations to be used is similar to (4.3), but this time expressing the equality of the differentials

\[
dF_i(X(z, v), z) = dF_i(X(z, v), z) ,
\]

written in terms of the coordinates \((X(z), z)\) and \((x, z)\)

\[
\left[ \frac{\partial F_i}{\partial X^1} X^1 + \frac{\partial F_i}{\partial X^2} X^2 + \frac{\partial F_i}{\partial X^3} X^3 + \frac{\partial F_i}{\partial z} \right] dz +
\left[ \frac{\partial F_i}{\partial x^1} x^1 + \frac{\partial F_i}{\partial x^2} x^2 + \frac{\partial F_i}{\partial x^3} x^3 \right] dv =
\left[ \frac{\partial F_i}{\partial x^1} x^1 + \frac{\partial F_i}{\partial x^2} x^2 + \frac{\partial F_i}{\partial x^3} x^3 + \frac{\partial F_i}{\partial z} \right] dz +
\left[ \frac{\partial F_i}{\partial x^1} x^1 + \frac{\partial F_i}{\partial x^2} x^2 + \frac{\partial F_i}{\partial x^3} x^3 \right] dv .
\]

The second set of equations takes into account the fact that the second differential \(d^2 X^i\) is zero, or equivalently

\[
d^2 X^i = dF_i(X(z, v), z) = 0 ,
\]

where \(dF_i(X(z, v), z)\) is given by the L. H. S. of equation (5.3).

The third set of equations is

\[
dx^i = F^i = \frac{\partial X_i}{\partial z} dz + \frac{\partial X_i}{\partial v} dv .
\]

Let us now introduce the notations

\[
M_i(X, z) \equiv \frac{\partial F_i}{\partial X^k} X^k ; \quad M_i(x, z) \equiv \frac{\partial F_i}{\partial x^k} x^k ,
\]

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\[ M_i(X, v) = \frac{\partial F_i}{\partial X^k} X'^k; \quad M_i(x, v) = \frac{\partial F_i}{\partial x^k} x'^k, \quad (5.7) \]

which will allow us to write down the first and the second set of equations (4.3) - (4.4) in the following compact form

\[
[M_i(X, z) - M_i(x, z)] dz + [M_i(X, v) - M_i(x, v)] dv = 0, \quad (5.8)
\]

\[
\left[M_i(X, z) + \frac{\partial F_i}{\partial z}\right] dz + M_i(X, v) dv = 0. \quad (5.9)
\]

Expressing \( \frac{\partial X^i}{\partial v} dv \) from (5.5), it can easily be proved that

\[
M_i(X, v) dv = \frac{\partial F_i}{\partial X^k} X'^k dv = -\frac{\partial F_i}{\partial X^k} X^k dz + \frac{\partial F_i}{\partial X^k} F_k, \quad (5.10)
\]

where the last term is zero due to the fulfillment of the second set of equations (5.5). Consequently, from (5.10) it follows

\[
M_i(X, z) dz + M_i(X, v) dv = dF_i(z, v) = dF_i(X(z, v), z) = 0. \quad (5.11)
\]

Additionally, if (5.11) is subtracted from (5.8) and (5.9), one easily obtains

\[
M_i(x, z) dz + M_i(x, v) dv = dF_i(z, v) = dF_i(x(z, v), z) = 0, \quad (5.12)
\]

\[
\frac{\partial F_i}{\partial z} dz = 0 \Rightarrow \frac{\partial F_i}{\partial v} dv = 0. \quad (5.13)
\]

In other words, if the differential \( dF_i(X(z, v), z) \) is zero in terms of the coordinates \((X, z)\), then it necessarily should be zero in the coordinates \((x, z)\). But in the spirit of the discussion at the end of section 2, this does not mean that if \( dX^i = F_i \), then the same should hold also for the initial coordinates \( x^i \), i.e. \( dx^i = F_i \). Indeed, we can find

\[
M_i(x, z) \equiv \frac{\partial F_i}{\partial x^k} x'^k = \frac{\partial F_i}{\partial X^l} \frac{\partial X^l}{\partial x^k} x'^k = \frac{\partial F_i}{\partial X^l} \left[ \frac{X^l}{x'^k} + \frac{X'^l}{x^k} \right] x'^k = M_i(X, z) + M_i(X, v) \frac{x'^k}{x_k}. \quad (5.14)
\]

Similarly

\[
M_i(x, v) = M_i(X, v) + M_i(X, z) \frac{x'^k}{x_k}. \quad (5.15)
\]

If the above two expressions are substituted into (5.12), and (5.11) is taken into account, one can obtain

\[
M_i(X, v) \frac{x'^k}{x_k} dz + M_i(X, z) \frac{x'^k}{x_k} dv = 0. \quad (5.16)
\]
Additionally, we have

\[ M_i(X, v) = M_i(X, z)^{x_k} \frac{x^k}{x_k} ; \quad dv = -\frac{M_i(x, z)}{M_i(x, v)} dz \quad (5.17) \]

Therefore, the following equation in partial derivatives with respect to \( F_i = F_i(x^i) \) can be derived

\[ \frac{\partial F_i}{\partial x^l} \frac{x^k}{x_k} x^m = \frac{\partial F_i}{\partial x^l} \frac{x^k}{x_k} \frac{x^m}{x_m} = 0 \quad (5.18) \]

A stronger statement may be proved, clearly showing that from \( dF_i(X, z) = dF_i(x, z) = 0 \) and \( dX_i = F_i \) it does not follow that \( dx^i = F_i \). If expression (4.17) for \( M_i(X, v) = M_i(X, z)^{x_k} \) is substituted into (5.11), one obtains

\[ M_i(X, z) \left[ dz + \frac{x^k}{x_k} dv \right] = 0 \quad (5.22) \]

and since \( M_i(X, z) \neq 0 \) (and if \( \frac{\partial F_i}{\partial x^k} \neq 0 \)), it follows

\[ dx^k = \dot{x}^k dz + x^k dv = 0 \quad (5.23) \]

6 ANALYSIS OF THE FOURTH AND THE FIFTH SET OF EQUATIONS FOR THE PREVIOUS CASE \( X^i = X^i(x(z, v), z) \)

The fourth set of equations, which will be considered is

\[ dX^k = \frac{\partial X^k}{\partial x^1} dx^1 + \frac{\partial X^k}{\partial x^2} dx^2 + \frac{\partial X^k}{\partial x^3} dx^3 = \]

\[ = 3X^k dz + X^k \dot{x}^m x_m dz + X^k x^m \frac{x^m}{x_m} dv + X^k dv \quad (6.1) \]

If multiplied by \( \frac{\partial F_i}{\partial x^k} dz dv \) and also relation (5.11) \( M_i(X, z) dz + M_i(X, v) dv = 0 \) is taken into account, the fourth set of equations can be written as

\[ \frac{\partial F_i}{\partial x^k} F_k dv = M_i(X, z) dz \left[ \frac{x^m}{x_m} (dv)^2 - \frac{x^m}{x_m} (dz)^2 \right] \quad (6.2) \]

The fifth set of equations is

\[ d^2 X^k = 0 = \frac{\partial^2 X^k}{\partial x^m \partial x^n} dx^m dx^n + \frac{\partial X^k}{\partial x^m} d^2 x^m \quad (6.3) \]
where the expressions on the R.H.S. can easily be computed in terms of the coordinates $X$ and $x$ and their derivatives.

Our goal further will be to see whether the fifth equation (6.3) constitutes a separate equation or whether it follows from the preceding four ones.

For the purpose, let us multiply both sides of the fifth equation by $\frac{\partial F_i}{\partial X^k}$ and see which are the terms, containing the second differentials $d^2z$ and $d^2v$

$$\left(\frac{\partial F_i}{\partial X^k} \frac{\dot{X}^k}{x^m} + \frac{\partial F_i}{\partial X^k} \frac{X'^k}{x'\m} \right) \left(\dot{x}^m \, d^2z + x'^m \, d^2v\right) =$$

$$= M_i(x, z)d^2z + M_i(x, v)d^2v \quad . \quad (6.4)$$

In (6.4) we have used relations (5.14) and (5.15) for $M_i(x, z)$ and $M_i(x, v)$. But we may note that the obtained term in (6.4) can be found from the relation (5.12)

$$M_i(x, z) dz + M_i(x, v) dv = 0, \text{ if it is differentiated by } z \text{ and } v \text{ and the resulting equations are summed up. Therefore}$$

$$M_i(x, z)d^2z + M_i(x, v)d^2v = -M_i(x, z)(dz)^2 - M_i(x, v)(dv)^2 -$$

$$-(\dot{M}_i(x, v) + M'_i(x, z))dxdv \quad . \quad (6.5)$$

The derivatives $\dot{M}_i(x, v)$ and $M'_i(x, z)$ can be found also from the already used expressions (5.14) and (5.15)

$$\dot{M}_i(x, z) = M'_i(X, z) + \frac{x^m}{x'^m} M'_i(X, v) +$$

$$+ M_i(X, v) \frac{\left[\frac{x'^m}{x'^m} - \frac{x^m}{x'^m}\right]}{\left(\frac{x'^m}{x'^m}\right)^2} \quad , \quad (6.6)$$

$$\dot{M}_i(x, v) = M'_i(X, v) + \frac{x^m}{x'^m} M'_i(X, z) +$$

$$+ M_i(X, z) \frac{\left[\frac{x'^m}{x'^m} - \frac{x^m}{x'^m}\right]}{\left(\frac{x'^m}{x'^m}\right)^2} \quad . \quad (6.7)$$

Making use of all the expressions (6.4) - (6.7), the following expression for the fifth equation (6.3), multiplied by $\frac{\partial F_i}{\partial X^k}$, can be obtained:

$$(dz)^2[-2M_i(X, z) + M_i(X, z)\frac{x^m}{x} + \frac{\partial F_i}{\partial X^k} \frac{\partial^2 X^k}{\partial x^m \partial x^n} x^m x^n] +$$

$$+(dv)^2[-2\frac{x^m}{x'} M_i(X, z) + M_i(X, z)\frac{x'^m}{x'} + \frac{\partial F_i}{\partial X^k} \frac{\partial^2 X^k}{\partial x^m \partial x^n} x^m x'^n] +$$

$$+dxdv[2M_i(X, z)\frac{x^m}{x'} + 2\frac{\partial F_i}{\partial X^k} \frac{\partial^2 X^k}{\partial x^m \partial x^n} x^m x' - 2M'_i(X, z)$$
After some lengthy, but straightforward calculations it can be obtained equations. For the purpose, let us take the differential of (5.2) and use equations (5.10).

The last two terms \( \frac{\partial F_i}{\partial x^k} \frac{\partial^2 X^k}{\partial x^m \partial x^n} \dot{x}^m \dot{x}^n \) and \( \frac{\partial F_i}{\partial x^k} \frac{\partial^2 X^k}{\partial x^m \partial x^n} x^m \dot{x}^n \) in the first two square brackets can be found as follows: First, the derivatives \( M_i(X, v) \) and \( M'_i(X, v) \) can be expressed from the relation (4.17) \( M_i(X, v) = M_i(X, z) \frac{\dot{x}^i}{x} \):

\[
\dot{M}_i(X, v) = \dot{M}_i(X, z) \frac{\dot{x}^m}{x} + M_i(X, z) \frac{\left[ \dot{x}^m \dot{x}^m - \dot{x}^m \dot{x}^m \right]}{\left( \dot{x}^m \right)^2},
\]

\[
M'_i(X, v) = M'_i(X, z) \frac{\dot{x}^m}{x} + M_i(X, z) \frac{\left[ x'^m x'^m - x'^m x'^m \right]}{\left( \dot{x}^m \right)^2}.
\]

But on the other hand, the same derivatives can be found by using the defining expressions (5.6 - 5.7)

\[
\dot{M}_i(X, v) = \frac{\partial^2 F_i}{\partial X^k \partial X^l} \frac{\partial X^k}{\partial x^m} \frac{\partial X^l}{\partial x^n} \dot{x}^m \dot{x}^n + \frac{\partial F_i}{\partial X^k} \frac{\partial^2 X^k}{\partial x^m \partial x^n} \dot{x}^m \dot{x}^n +
\]

\[
+ M_i(X, z) \frac{\dot{x}^m \dot{x}^n}{x} + M_i(X, z) \dot{x}^n, \quad (6.11)
\]

\[
M'_i(X, v) = \frac{\partial^2 F_i}{\partial X^k \partial X^l} \frac{\partial X^k}{\partial x^m} \frac{\partial X^l}{\partial x^n} \dot{x}^m \dot{x}^n + \frac{\partial F_i}{\partial X^k} \frac{\partial^2 X^k}{\partial x^m \partial x^n} \dot{x}^m \dot{x}^n +
\]

\[
+ \frac{\partial F_i}{\partial X^k} \frac{\partial X^k}{\partial x^m} \frac{\dot{x}^m \dot{x}^n}{x} + \frac{\partial F_i}{\partial X^k} \frac{\partial X^k}{\partial x^m} \dot{x}^m \dot{x}^n. \quad (6.12)
\]

Therefore, the desired expressions can be found by setting formulae (6.9) equal to (6.11) and also formulae (6.10) equal to (6.12). It can easily be derived how eq. (6.8) will transform, but unfortunately, this would not result in any simplification of the equation with respect to the generalized coordinates \( X^i \).

It remained only to show whether the fourth equation (5.2) is independent from the preceding ones (and thus can be treated separately) or it follows naturally from these equations. For the purpose, let us take the differential of (5.2) and use equations (5.10). After some lengthy, but straightforward calculations it can be obtained

\[
\frac{(dz)(dv)^2}{2} [\dot{M}_i(X, z) \frac{\dot{x}^m}{x} + 2 \frac{\dot{x}^m}{x} M'_i(X, z) - M_i(X, z) \frac{\left( \dot{x}^m \dot{x}^m - \dot{x}^m \dot{x}^m \right)}{\left( \dot{x}^m \right)^2} +
\]

\[
+ M_i(X, z) \frac{\left( \dot{x}^m \dot{x}^m - \dot{x}^m \dot{x}^m + \dot{x}^m \dot{x}^m - \dot{x}^m \dot{x}^m \right)}{\left( \dot{x}^m \right)^2} ] +
\]

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end of Section 3 that the two sets of coordinates, i.e. on both system of coordinates. Of particular importance is the conclusion at the end of first-order nonlinear differential equations is obtained, for which always a solution exists. In this third part of the paper it has been shown that from the expressions (2.1) a system of differential equations can be treated on an equal footing. This means that if \( dX^i = 0 \) for obtaining the solutions (2.1) allows us to derive a system of nonlinear differential equations also for the initial variables \( x^1, x^2, x^3 \) and thus the corresponding solutions \( x^1 = x^1(z), x^2 = x^2(z), x^3 = x^3(z) \) in principle can be found. This analysis has been performed in section 3. In fact, it can easily be guessed that if we have the solutions \( X^1 = X^1(z), X^2 = X^2(z), X^3 = X^3(z) \) and the additional condition \( d^2X^i = 0 \) (which in fact relates the generalized and the initial sets of coordinates), then the solutions \( x^1 = x^1(z), x^2 = x^2(z), x^3 = x^3(z) \) should also be "coordinated" with the previous ones. Indeed, this is evident from the dependence of the functions \( \overline{S}_1^{(i)}, \overline{S}_2^{(i)}, \ldots, \overline{S}_6^{(i)} \) in the system (3.11) for \( \frac{dx}{dz} \), both on the functions \( \frac{\partial F_i}{\partial x^k} \) and \( \frac{\partial E_i}{\partial x^k} \), i.e. on both system of coordinates. Of particular importance is the conclusion at the end of Section 3 that the two sets of coordinates \( X^1, X^2, X^3 \) and \( x^1, x^2, x^3 \) should not be treated on an equal footing. This means that if \( dX^1, dX^2, dX^3 \) satisfy the originally derived cubic algebraic equation, then it is not necessary to assume this for \( dx^1, dx^2, dx^3 \).

Much more interesting is the other investigated case in Sections 4 - 6, where a pair of complex coordinates \( z, v \) has been introduced and thus through the generalized coordinates \( X^1 = X^1(z, v), X^2 = X^2(z, v), X^3 = X^3(z, v) \) a complex structure of the metric tensor components is introduced. For the investigated case under the assumption \( d^2X^i = 0 \), there is only one way for introducing this complex structure - namely, through the dependence of the initial coordinates on \( z \) and \( v \), i.e. \( X^i = X^i(x(z, v), z) \). Otherwise,

\[
+(dz)^2(dv)[M_1(X, z)\frac{\dot{x}^m}{x^m} + 2\dot{M}_1(X, z) + M_1(X, z)\left(\frac{\ddot{x}^m \dot{x}^m - \dot{x}^m \ddot{x}^m}{(x^m)^2}\right)] + \\
+(dz)^3[\dot{M}_1(X, z)\frac{\dot{x}^m}{x^m} + M_1(X, z)\left(\frac{\dddot{x}^m \dot{x}^m - \dot{x}^m \dddot{x}^m}{(x^m)^2}\right)] + \\
+(dv)^3[M_1(X, z)\frac{\dot{x}^m}{x^m} + M_1(X, z)\left(\frac{\dddot{x}^m \dot{x}^m - \dot{x}^m \dddot{x}^m}{(x^m)^2}\right)] = 0 \quad . \tag{6.13}
\]

This is an equation both for the initial coordinates \( x^i \) and for the generalized ones \( X^i \) and it is different from the fifth equation (6.8).

7 DISCUSSION

In this third part of the paper it has been shown that from the expressions (2.1) a system of first-order nonlinear differential equations is obtained, for which always a solution \( X^1 = X^1(z), X^1 = X^1(z), X^1 = X^1(z) \) exists. Thus the dependence on the generalized coordinates \( X^1, X^2, X^3 \) in the uniformization functions (2.1) disappears and only the dependence on the complex coordinate \( z \) remains, as it should be for uniformization functions.

Moreover, the initial assumption \( dX^i = 0 \) for obtaining the solutions (2.1) allows us to derive a system of nonlinear differential equations also for the initial variables \( x^1, x^2, x^3 \) and thus the corresponding solutions \( x^1 = x^1(z), x^2 = x^2(z), x^3 = x^3(z) \) in principle can be found. This analysis has been performed in section 3. In fact, it can easily be guessed that if we have the solutions \( X^1 = X^1(z), X^2 = X^2(z), X^3 = X^3(z) \) and the additional condition \( d^2X^i = 0 \) (which in fact relates the generalized and the initial sets of coordinates), then the solutions \( x^1 = x^1(z), x^2 = x^2(z), x^3 = x^3(z) \) should also be "coordinated" with the previous ones. Indeed, this is evident from the dependence of the functions \( \overline{S}_1^{(i)}, \overline{S}_2^{(i)}, \ldots, \overline{S}_6^{(i)} \) in the system (3.11) for \( \frac{dx}{dz} \), both on the functions \( \frac{\partial F_i}{\partial x^k} \) and \( \frac{\partial E_i}{\partial x^k} \), i.e. on both system of coordinates. Of particular importance is the conclusion at the end of Section 3 that the two sets of coordinates \( X^1, X^2, X^3 \) and \( x^1, x^2, x^3 \) should not be treated on an equal footing. This means that if \( dX^1, dX^2, dX^3 \) satisfy the originally derived cubic algebraic equation, then it is not necessary to assume this for \( dx^1, dx^2, dx^3 \).

Much more interesting is the other investigated case in Sections 4 - 6, where a pair of complex coordinates \( z, v \) has been introduced and thus through the generalized coordinates \( X^1 = X^1(z, v), X^2 = X^2(z, v), X^3 = X^3(z, v) \) a complex structure of the metric tensor components is introduced. For the investigated case under the assumption \( d^2X^i = 0 \), there is only one way for introducing this complex structure - namely, through the dependence of the initial coordinates on \( z \) and \( v \), i.e. \( X^i = X^i(x(z, v), z) \). Otherwise,
if some other possibility is assumed, for example \( X^i = X^i(x(z), z, v) \), then, as proved in section 4, the obtained system of equations is contradictory. Therefore, it remains to investigate the full system of equations for the only allowed case \( X^i = X^i(x(z, v), z) \), which has been performed in Sections 5 and 6. Remarkably, a nonlinear differential equation is obtained only for the initial coordinates. However, no such an equation only for the generalized coordinates can be obtained - the derived equation depends in a complicated manner on both system of coordinates. Since the existence of these noncontradictory systems of equations confirms that a complex structure can be introduced, one may express the line element \( ds^2 = g_{ij}(X)dX^i dX^j \) as

\[
ds^2 = \tilde{g}_{zz}(z, v)(dz)^2 + \tilde{g}_{zv}(z, v)dzdv + \tilde{g}_{vv}(z, v)(dv)^2 ,
\]

where

\[
\tilde{g}_{zz}(z, v) \equiv g_{ij}(X(z, v))\dot{X}^i \dot{X}^j ; \quad \tilde{g}_{zv}(z, v) \equiv g_{ij}(X(z, v))X'^i X'^j ,
\]

\[
\tilde{g}_{vv}(z, v) \equiv g_{ij}(X(z, v)) \left[ \dot{X}^i \dot{X}^j + X'^i X'^j \right] .
\]

This result will be of particular importance in reference to possible physical applications, which will be considered in another paper. For example, the linear element of a unit surface in the Lobachevsky space with a constant negative curvature \(-\frac{1}{R^2}\) can be represented as \([9]\)

\[
ds^2 = R^2\left( a^2 - w^2 \right)du^2 + 2uwdu dw + \left( a^2 - u^2 - w^2 \right)dw^2 ,
\]

which by means of a suitable coordinate transformation can be brought to the form

\[
ds^2 = d\rho^2 + e^{-\frac{2\rho}{R}}d\sigma^2 .
\]

The metric (7.5) is the standard form of the three-dimensional Lobachevsky metric \([10]\) (\(d\sigma^2\) is a two-dimensional surface element), where the ratio \(\frac{2\rho}{R}\) may or may not be identified with the Lobachevsky constant \(k = \frac{1}{c}\) (\(c\) is a natural unit length element in the Lobachevsky space). This is particularly important to be mentioned in reference to Randall - Sundrum models and theories with extra dimensions, which are based on the "multi-dimensional" analogue of the Lobachevsky metric (7.5). Some physical applications of the algebraic geometry formalism in these models will be considered in a separate paper.

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