A NEW CLASS OF OPTIMAL WIDE-GAP ONE-COINCIDENCE FREQUENCY-HOPPING SEQUENCE SETS

WENLI REN*
School of Mathematics and Big Data, Dezhou University
Dezhou, 253023, China

FENG WANG
School of Mathematics and Big Data, Dezhou University
Dezhou, 253023, China

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Abstract. In this paper, we propose a new class of optimal one-coincidence FHS (OC-FHS) sets with respect to the Peng-Fan bounds, including prime sequence sets and HMC sequence sets as special cases. Thereafter, through investigating their properties, we determine all of the FHS distances in the OC-FHS set. Finally, for a given positive integer, we also propose a new class of wide-gap one-coincidence FHS (WG-OC-FHS) sets where the FHS gap is larger than the given positive integer. Moreover, such a WG-OC-FHS set is optimal with respect to the WG-Lempel-Greenberger bound and the WG-Peng-Fan bounds simultaneously.

1. Introduction

Frequency-hopping code-division multiple access (FH-CDMA) spread communication system [26] has become one of the competitive technologies for wireless communication systems such as BlueTooth, UWB communications, radar, sonar echolocation, and so on. In this system, each user is assigned a frequency-hopping sequence (FHS) to transmit signals in a chosen frequency band within a given time slot. However, the simultaneous transmission of any two or more users over the same frequency band may cause mutual interferences of signals. Hence, it is very necessary to eliminate or minimize these interferences caused by simultaneous transmission over the same frequency band. To be precise, the Hamming correlation of FHS exactly reflects the interference degree of signals in transmission, and the FHS in which Hamming correlation kept as low as possible is extensively employed to reduce the interferences.

Traditionally, it is interesting to seek constructions of FHS sets with low Hamming correlation, large size, long length, and small frequency alphabet simultaneously. Usually, these parameters are not independent of each other, but are limited by some theoretical bounds. For instance, the Lempel-Greenberger bound [17] used
to measure the optimality of a single FHS, the Peng-Fan bounds [21] used to measure the optimality of FHS sets, or coding theoretical bounds [6], and so on. Therefore, during these four decades, numerous optimal constructions of FHSs meeting these bounds have been proposed, and there are many algebraic and combinatorial methods for the construction of the optimal FHS or FHS set in the literature (see [17, 6, 3, 4, 5, 7, 8, 10, 15, 27, 22, 11], and the references therein).

Note the fact that the mutual interference between distinct users cannot be completely eliminated. Among FHS sets, one of the best choices is the one-coincidence FHS (OC-FHS) set where Hamming cross-correlation of any pair of different non-repeating FHS does not exceed one. In the early 1980s, Shaar et al. ([24, 25]) first introduced the concept of the OC sequence set and proposed a prime sequence set which is exactly an OC-FHS set. Then, Li [18] derived a theoretical bound of an OC-FHS set on the number of sequences, the length of sequences, and the frequency gap. Cao et al. [1] made a subsequent contribution by discussing the combinatorial characterizations of OC-FHS sets in 2006. Some new results have emerged in recent years. Ren et al. [23] used an interleaving technology to extend an OC-FHS set of prime length to that of composite length. Based on prime sequence sets, Fukshan-sky and Shaar [9] proposed another class of OC-FHS sets, called HMC sequence sets. Later on, Lee et al. [16] presented another construction of OC-FHS sets by a primitive element of the prime field. In addition, Niu et al. [20] obtained an OC-FHS set with more flexible parameters by a designed direct product.

On the other hand, along with the challenges of limited spectrum resources, continuous upgrading of complex interference, and complicated variable communication link environment, the traditional FHS cannot fully satisfy the requirements for high efficiency and reliable signal transmission in FH-CDMA communication system. Especially in the Joint Tactical Information Distribution System (JTIDS), if the adjacent frequencies equipped with a small gap, the received signals would be liable to interfere. Conversely, when the gap is large enough, even if a certain signal transmitted over an interfered frequency band, as long as the adjacent signals do not interfere, the interfered signals could be recovered by encoding. It is the basic idea of the wide-gap FHS (WG-FHS) [2] where the gap between any two adjacent frequencies is greater than a given value. Very recently, Li et al. [19] further established the theoretical bounds of the WG-FHS and WG-FHS sets, namely the WG-Lempel-Greenberger bound and the WG-Peng-Fan bounds, and pointed out the existence of optimal WG-FHS sets through a computer search.

Compared to the traditional FHS, the WG-FHS can improve the immunity to narrowband interference, wideband jamming, track interference, capturing interference, frequency blocking interference, multipath fading, and so on. Hence, it is desirable to find WG-FHS sets with good Hamming correlation. However, almost all of the known documented WG-FHS sets are generated by mapping algorithms such as frequency allocation algorithms, frequency revision algorithms, and classified pseudorandom perturbations, rather than generally constructed by algebraic methods (please refer to [28, 12, 13, 30, 14] for example), let alone the constructions of optimal WG-FHS sets with respect to the WG-Lempel-Greenberger bound or the WG-Peng-Fan bounds. These motivate us to provide some optimal algebraic constructions of WG-FHS sets to meet the requirements of FH-CDMA systems.

This paper is organized as follows. In Section II, we will review some preliminaries and notations. In Section III, we will give a new class of OC-FHS sets that is optimal with respect to the Peng-Fan bounds. In Section IV, we will determine all of FHS
distances and give a new class of WG-OC-FHS sets that is optimal with respect to the WG-Lempel-Greenberger bound and the WG-Peng-Fan bounds. Moreover, some examples of optimal OC-FHS sets and optimal WG-OC-FHS sets are given in Tables I and II. Finally, the conclusion is drawn in Section V.

2. Preliminaries

The following notations will be used throughout this paper.

• $p$ is an odd prime;
• $\langle x \rangle$ denotes the least nonnegative residue of $x$ modulo $p$ for an integer $x$;
• $\lfloor x \rfloor$ denotes the largest integer less than or equal to $x$;
• $\lceil x \rceil$ denotes the least integer greater than or equal to $x$.

2.1. Hamming correlation function.

Let $\ell$ be a positive integer, $\mathcal{F} = \{f_0, f_1, \cdots, f_{\ell-1}\}$ be an alphabet of $\ell$ available frequencies. A sequence $X = (X(i))_{i=0}^{L-1}$ is called a frequency-hopping sequence (FHS) of length $L$ over $\mathcal{F}$ if $X(i) \in \mathcal{F}$ for all $0 \leq i \leq L - 1$. For any two FHSs $X$ and $Y$, the Hamming correlation function of $X$ and $Y$ at relative time delay $\tau$ is defined by

$$H_{X,Y}(\tau) = \sum_{i=0}^{L-1} h[X(i), Y(i + \tau)], \quad 0 \leq \tau < L,$$

where $h[x, y] = 1$ if $x = y$ and 0 otherwise, and all the addition operations among the subscript are performed modulo $L$. In particular, $H_{X,X}(\tau)$ is called the Hamming autocorrelation if $X = Y$, denoted by $H_X(\tau)$ for simplicity.

Let $\mathcal{S}$ be an FHS set with $M$ sequences of length $L$ over $\mathcal{F}$. For any two distinct FHS $X, Y \in \mathcal{S}$, the following two measures are defined by

$$H(X) = \max_{1 \leq \tau < L} \{H_X(\tau)\},$$

$$H(X, Y) = \max_{0 \leq \tau < L} \{H_{X,Y}(\tau)\}.$$

The maximum Hamming autocorrelation $H_a(\mathcal{S})$, the maximum Hamming cross-correlation $H_c(\mathcal{S})$ and the maximum nontrivial Hamming correlation $H_m(\mathcal{S})$ of $\mathcal{S}$ are defined as follows, respectively:

$$H_a(\mathcal{S}) = \max\{H(X) | X \in \mathcal{S}\},$$

$$H_c(\mathcal{S}) = \max\{H(X, Y) | X \neq Y, X, Y \in \mathcal{S}\},$$

$$H_m(\mathcal{S}) = \max\{H_a(\mathcal{S}), H_c(\mathcal{S})\}.$$

2.2. One-coincidence FHS set and wide-gap FHS set.

Now we recall some useful definitions including a one-coincidence FHS (OC-FHS) set and a wide-gap FHS (WG-FHS) set.

Definition 2.1 ([24]). If all elements of each FHS $X \in \mathcal{S}$ are distinct, then $\mathcal{S}$ is called a non-repeating FHS set.

Definition 2.2 ([24]). An FHS set $\mathcal{S}$ is called a one-coincidence FHS set if it satisfies the following conditions:

1. Each FHS $X \in \mathcal{S}$ is non-repeating;
2. For any two distinct FHSs $X$ and $Y$ with length $L$ of $\mathcal{S}$, the maximum number of hits between $X$ and $Y$ for any time shift $\tau$ ($0 \leq \tau \leq L - 1$) is equal to one, that is, $H_{X,Y}(\tau) \leq 1$. 
Note that the low interference between distinct users can be achieved by using an OC-FHS set due to the maximal Hamming cross-correlation equals one for any pair of sequences and any time delay.

**Definition 2.3** ([2]). For any FHS $X \in \mathcal{S}$, and a given positive integer $D$, if
\[ |X(i+1) - X(i)| > D, \quad i \geq 0, \]
then $\mathcal{S}$ is called a (broad-sense) wide-gap FHS set with the FHS gap $D$.

### 2.3. Some known bounds of FHS sets

Throughout this paper, an FHS $X$ with parameters $(L, \lambda, \ell)$ denotes the FHS $X$ of length $L$ over $\mathcal{F}$ with $H(X) = \lambda$, and an FHS set $\mathcal{S}$ with parameters $(L, M, \lambda, \ell)$ denotes the FHS set $\mathcal{S}$ with $M$ sequences of length $L$ over $\mathcal{F}$ and $H_m(\mathcal{S}) = \lambda$. In particular, an OC-FHS set $\mathcal{S}$ with parameters $(L, M, \ell)$ denotes the FHS set $\mathcal{S}$ with $M$ sequences of length $L$ over $\mathcal{F}$ and $H_c(\mathcal{S}) = 1$.

In 2004, Peng and Fan developed the following bounds on $H_m(\mathcal{S})$ for an FHS set $\mathcal{S}$.

**Lemma 2.4** (Peng-Fan bounds, [21]). Let $\mathcal{S}$ be an $(L, M, \lambda, \ell)$-FHS set, and define $I = \left\lfloor \frac{LM}{\ell} \right\rfloor$. Then
\[
\lambda \geq \left\lfloor \frac{(LM - \ell)L}{(LM - 1)\ell} \right\rfloor \tag{1}
\]
and
\[
\lambda \geq \left\lfloor \frac{2ILM - (I + 1)L\ell}{(LM - 1)M} \right\rfloor. \tag{2}
\]

In [29], the authors pointed out that the above two bounds are the same, and the parameters of an FHS set either achieve the two Peng-Fan bounds simultaneously, or do not achieve any of them.

Naturally, we give the definition of an optimal FHS set as follows.

**Definition 2.5.** An FHS set $\mathcal{S}$ is said to be optimal if one of bounds in Lemma 2.4 is met with equality.

In 2019, Li et al. established the WG-Lempel-Greenberger bound of a single WG-FHS and the WG-Peng-Fan bounds of a WG-FHS set as follows.

**Lemma 2.6** ([19]). Let $X$ be an $(L, \lambda, \ell)$-WG-FHS. Then
\[
\lambda \geq \left\lfloor \frac{(L - \epsilon)(L + \epsilon - \ell)}{\ell(L - 3)} \right\rfloor, \tag{3}
\]
where $L = \left\lfloor \frac{L}{\ell} \right\rfloor + \epsilon$ with $0 \leq \epsilon < \ell$.

**Lemma 2.7** ([19]). Let $\mathcal{S}$ be an $(L, M, \lambda, \ell)$-WG-FHS set, and define $I = \left\lfloor \frac{LM}{\ell} \right\rfloor$. Then
\[
\lambda \geq \left\lfloor \frac{(LM - \ell)L}{(LM - 3)\ell} \right\rfloor \tag{4}
\]
and
\[
\lambda \geq \left\lfloor \frac{2ILM - (I + 1)L\ell}{(LM - 3)M} \right\rfloor. \tag{5}
\]
In fact, Lemmas 2.6 and 2.7 show that the above bounds are independent of the FHS gap $D$.

Similarly, we give the definition of an optimal WG-FHS set as follows.

**Definition 2.8.** A WG-FHS set $\mathcal{S}$ is said to be optimal if one of bounds in Lemma 2.7 is met with equality.

### 3. A new class of optimal OC-FHS sets

In this section, we will give a new class of OC-FHS sets, which include prime sequence sets and HMC sequence sets as special cases.

**Theorem 3.1.** For a positive integer $w$ with $1 \leq w \leq p - 1$, the FHS set $\mathcal{B}^w = \{B^w_1, B^w_2, \cdots, B^w_{p-1}\}$ is an optimal $(p, p-1, w(p-w) + 1)$-OC-FHS set, where

$$B^w_k = (b_k(0), b_k(1), \cdots, b_k(p-1)), \; 1 \leq k \leq p-1$$

and

$$b_k(i) = \sum_{j=i}^{i+w-1} \langle jk \rangle, \; 0 \leq i \leq p-1.$$

**Proof.** We are going to prove this theorem in four steps:

- Firstly, we will show that every FHS $B^w_k$ is non-repeating.

  Suppose not, then there exist two different integers $0 \leq i_1 \neq i_2 \leq p-1$ such that

  $$\sum_{j=i_1}^{i_1+w-1} \langle jk \rangle = \sum_{j=i_2}^{i_2+w-1} \langle jk \rangle.$$  

  Obviously, Eq. (6) also holds for modulo $p$, then we have

  $$wk_i1 + \frac{w(w-1)k}{2} \equiv wki_2 + \frac{w(w-1)k}{2} \pmod{p},$$

  which deduces that $i_1 \equiv i_2 \pmod{p}$.

  Since $0 \leq i_1, i_2 \leq p-1$, it is easily obtained that $i_1 = i_2$ which contradicts the hypothesis. Therefore, every FHS $B^w_k$ is non-repeating.

- Then, we will show that $\mathcal{B}^w$ is a $(p, p-1, w(p-w) + 1)$ FHS set.

  Obviously, $\mathcal{B}^w$ has $p-1$ FHSs of length $p$. Now, we consider the size $|\mathcal{F}|$ of the available frequency alphabet $\mathcal{F}$.

  For $0 \leq j_1 \neq j_2 \leq p-1$ and $1 \leq k \leq p-1$, it is clear that $\langle j_1k \rangle \neq \langle j_2k \rangle$. Then, for $0 \leq i \leq p-1$, the element $b_k(i)$ of $B^w_k$ satisfies the following inequation,

  $$\sum_{j=0}^{w-1} j \leq \sum_{j=i}^{i+w-1} \langle jk \rangle \leq \sum_{j=p-w}^{p-1} j,$$

  which implies that

  $$\frac{w(w-1)}{2} \leq \sum_{j=i}^{i+w-1} \langle jk \rangle \leq wp - \frac{w(w+1)}{2}.$$ 

  Obviously, one has

  $$|\mathcal{F}| = wp - \frac{w(w+1)}{2} - \frac{w(w-1)}{2} + 1 = w(p-w) + 1.$$  

- Next, we will show that $\mathcal{B}^w$ is an OC-FHS set.
For 1 ≤ k ≤ p − 1, B_k is non-repeating. On the other hand, for 1 ≤ k ≠ l ≤ p − 1, assume that b_k(i) = b_l(i + τ), one gets
\[ \sum_{j=i}^{i+w-1} \langle jk \rangle = \sum_{j=i+\tau}^{i+\tau+w-1} \langle jl \rangle. \]
Performing modulo p and simplifying the above equation, it yields
\[ (k - l)i \equiv \tau l + \frac{w - 1}{2}(l - k) \pmod{p}. \]
Note that 1 ≤ k ≠ l ≤ p − 1, and by the knowledge of linear congruences, Eq. (7) has only one solution. Therefore, \( H_{B_k^w,B_l^w}(\tau) = 1 \) for all 0 ≤ τ ≤ p − 1.

Finally, we will show that \( B_w^w \) is an optimal OC-FHS set with respect to the Peng-Fan bounds.

Substituting the parameters \((p, p - 1, w(p - w) + 1)\) into Eq. (1), the right side of the Peng-Fan bounds is
\[ \left\lceil \frac{(p(p - 1) - w(p - w) - 1)p}{(p(p - 1) - 1)(w(p - w) + 1)} \right\rceil = 1. \]
This implies the desired result.

**Remark 1.** The FHS set \( B_w^w \) is a generalization of results in [25] and [9].
- When w = 1, \( B_w^w \) is the prime sequence set.
- When w = 2, \( B_w^w \) is the HMC sequence set.
- When 2 < w ≤ p − 1, \( B_w^w \) is reported for the first time.

### 4. A new class of optimal WG-OC-FHS sets with a given FHS gap

In this section, we will construct a new class of WG-OC-FHS sets with the FHS gap \( D \), which is optimal with respect to the WG-Lempel-Greenberger bound and the WG-Peng-Fan bounds [19].

Before doing this, we give the definition of the FHS distance \( d(B_w^w) \) of an FHS \( B_w^w \) as follows
\[ d(B_w^w) = \min_{0 ≤ i ≤ p} \{|b_k(i + 1) - b_k(i)|\}. \]
Now, we will describe an interesting property of the FHS distance \( d(B_w^w) \), and we decide to show the proof here for the convenience of readers.

For 1 ≤ k ≤ p − 1, let
\[ B_{k,1}^w = (b_k(0), b_k(1), \cdots, b_k(p - w + 1)) \]
and
\[ B_{k,2}^w = (b_k(p - w + 2), b_k(p - w + 3), \cdots, b_k(p - 1)), \]
then the cascade connection of \( B_{w,1}^w \) and \( B_{w,2}^w \) is \( B_k^w = (B_{k,1}^w, B_{k,2}^w) \). In addition, the reverse of \( B_{k,i}^w, (i = 1, 2) \) is denoted by
\[ B_{k,1}^{w,r} = (b_k(p - w + 1), b_k(p - w), \cdots, b_k(0)) \]
and
\[ B_{k,2}^{w,r} = ((b_k(p - 1), b_k(p - 2), \cdots, b_k(p - w + 2)). \]
Similar to Lemma 3.5 in [9], we get the following lemma.
Lemma 4.1. For \(1 \leq k \leq p-1\), let \(B^w_k = (B^w_{k,1}, B^w_{k,2})\) and \(B^w_{p-k} = (B^w_{p-k,1}, B^w_{p-k,2})\). Then we have
\[
B^w_{p-k} = (B^w_{k,1} \setminus B^w_{k,2}),
\]
and all elements of \(B^w_{p-k}\) can be expressed as
\[
b_{p-k}(p - w + 1 + i) = \begin{cases} b_k(-i), & w - 1 - p \leq i \leq 0, \\ b_k(p - i), & 1 \leq i \leq w - 2. \end{cases}
\]

Proof. For \(w - 1 - p \leq i \leq 0\), we have
\[
b_{p-k}(p - w + 1 + i) = \langle (w - 1 - i)k \rangle + \langle (w - 2 - i)k \rangle + \cdots + \langle (-i)k \rangle = b_k(-i).
\]
For \(1 \leq i \leq w - 2\), we get
\[
b_{p-k}(p - w + 1 + i) = \langle (p - w + 1 + i)(p - k) \rangle + \langle (p - w + 2 + i)(p - k) \rangle + \cdots + \langle (p + i)(p - k) \rangle
\]
\[
= \langle (w - 1 - i)k \rangle + \langle (w - 2 - i)k \rangle + \cdots + \langle (-i)k \rangle
\]
\[
= b_k(p - i).
\]
This completes the proof. \(\Box\)

In order to obtain a WG-OC-FHS set based on \(B^w\), it is natural to discuss the following lemma which determines all those FHS distances \(d(B^w_k)\) for \(1 \leq k \leq p - 1\).

Lemma 4.2. For \(1 \leq k \leq p - 1\), we have
\[
d(B^w_k) = \min\{\langle wk \rangle, p - \langle wk \rangle\} \in \{1, 2, \cdots, \frac{p-1}{2}\}.
\]

Proof. By Lemma 4.1, for \(1 \leq k \leq p - 1\), it is easily seen that \(d(B^w_k) = d(B^w_{p-k})\). Herein, we consider the difference of the consecutive elements \(b_k(i + 1)\) and \(b_k(i)\) of \(B^w_k\). Thus, for \(0 \leq i \leq p - 2\), we have
\[
(9) b_k(i + 1) - b_k(i) = \langle (i + w)k \rangle - \langle ik \rangle = \begin{cases} \langle wk \rangle, & \langle ik \rangle + \langle wk \rangle < p \\ \langle wk \rangle - p, & \langle ik \rangle + \langle wk \rangle \geq p. \end{cases}
\]
It means that \(|b_k(i + 1) - b_k(i)| = \min\{\langle wk \rangle, p - \langle wk \rangle\}\).

In particular,
\[
b_k(p - 1) - b_k(0) = \langle (p - 1)k \rangle - \langle (w - 1)k \rangle
\]
\[
= \begin{cases} \langle (p - w)k \rangle, & \langle (w - 1)k \rangle + \langle (p - w)k \rangle < p \\ \langle (p - w)k \rangle - p, & \langle (w - 1)k \rangle + \langle (p - w)k \rangle \geq p. \end{cases}
\]
It means that \(|b_k(0) - b_k(p - 1)| = \min\{p - \langle wk \rangle, \langle wk \rangle\}\). Combining Eq. (9) and Eq. (10) yields
\[
d(B^w_k) = \min\{\langle wk \rangle, p - \langle wk \rangle\} \leq \frac{p-1}{2}.
\]
Moreover, it is easily seen that \(d(B^w_k) \neq d(B^w_l)\) for any \(1 \leq k \neq l \leq \frac{p-1}{2}\). In short, one gets
\[
\{d(B^w_k) | 1 \leq k \leq p - 1\} = \{1, 2, \cdots, \frac{p-1}{2}\}.
\]
\(\Box\)
A natural question one would ask is whether we can get a class of WG-OC-FHS sets from OC-FHS sets $\mathcal{B}^w$. The following theorem gives a positive answer to this question.

**Theorem 4.3.** For a given integer $D$ with $0 < D \leq \frac{p-1}{2}$, the FHS set $\mathcal{C}^w$ is an optimal $(p, p-1-2D, w(p-w)+1)$-WG-OC-FHS set with the FHS gap $D$, where

$$\mathcal{C}^w = \{ B_k^w \in \mathcal{B}^w | d(B_k^w) > D \}$$

and $|\mathcal{C}^w| = p - 1 - 2D$.

**Proof.** Firstly, we will show that $\mathcal{C}^w = \{ B_k^w \in \mathcal{B}^w | d(B_k^w) > D \}$ and $|\mathcal{C}^w| = p - 1 - 2D$. By Lemma 4.2, $d(B_k^w) = d(B_{k-p}^w)$ and $\{d(B_k^w) | 1 \leq k \leq p-1\} = \{1, 2, \ldots, \frac{p-1}{2} \}$, it is obvious that there are $2D$ FHSs in $\mathcal{B}^w$ where

$$\{d(B_k^w) | d(B_k^w) \leq D\} = \{1, 2, \ldots, D\}.$$  

Then, by removing such $2D$ FHSs from $\mathcal{B}^w$, we can obtain a WG-OC-FHS set $\mathcal{C}^w$ with the FHS gap $D$. Obviously, $\mathcal{C}^w$ is a $(p, p-1-2D, w(p-w)+1)$-WG-OC-FHS set with the FHS gap $D$.

Next, we will show that $\mathcal{C}^w$ is optimal with respect to the bounds (3) and (4). Substituting $L = p, \ell = w(p-w)+1$ (resp. $L = p, M = p-2D-1, \ell = w(p-w)+1$) into the right side of the bound (3) (resp. (4)), one gets

$$\left[ \frac{(p-\epsilon)(p+\epsilon-w(p-w)-1)}{(p-3)(w(p-w)+1)} \right] = 0$$

and

$$\left[ \frac{p(p-1-2D)-(w(p-w)+1)|p}{p(p-1-2D)-3|w(p-w)+1} \right] = 1.$$  

This finishes the proof. \qed

The following example explains exactly how to get our main results in the paper.

**Example 1.** Let $p = 17$ and $w = 7$, the OC-FHS set $\mathcal{B}^7 = \{ B_k^7, 1 \leq k \leq 16 \}$ with parameters $(17, 16, 71)$ is given in Table I.

Verified by a MATLAB program, we have $H_m(\mathcal{B}^7) = 1$. Thus, $\mathcal{B}^w$ is optimal with respect to the Peng-Fan bounds (1).

In particular, if we take the FHS gap $D = 3$, note that

$$d(B_1^7) = d(B_{15}^7) = 3, \quad d(B_2^7) = d(B_{10}^7) = 2, \quad d(B_3^7) = d(B_{12}^7) = 1.$$  

Similar to [9], by removing the “bad” sequences

$$\{ B_2^7, B_5^7, B_7^7, B_{10}^7, B_{12}^7, B_{15}^7 \},$$

we get a WG-OC-FHS set $\mathcal{C}^7$ as shown in Table II.

It is clear that

$$\mathcal{C}^7 = \{ B_1^7, B_3^7, B_4^7, B_6^7, B_8^7, B_9^7, B_{11}^7, B_{13}^7, B_{14}^7, B_{16}^7 \}$$

is a $(17, 10, 71)$-WG-OC-FHS set with the FHS gap $D = 3$, and it is optimal with respect to the WG-Lempel-Greenberger bound (3) and the WG-Peng-Fan bound (4) as well.
Table 1. The optimal OC-FHS set $B^w$ with FHS distances $d(B^w_k)$

| $B^w$  | Frequencies | $d(B^w_k)$ |
|--------|-------------|------------|
| $B^1_1$ | (21, 28, 35, 42, 49, 56, 63, 70, 77, 84, 91, 81, 71, 61, 51, 41, 31) | 7 |
| $B^2_2$ | (42, 56, 70, 67, 64, 61, 58, 55, 52, 49, 63, 60, 57, 54, 51, 48, 45) | 3 |
| $B^3_3$ | (46, 50, 54, 58, 62, 66, 53, 57, 61, 65, 69, 56, 43, 47, 51, 55, 59) | 4 |
| $B^4_4$ | (50, 61, 72, 66, 60, 54, 65, 59, 53, 47, 58, 52, 46, 40, 51, 62, 56) | 6 |
| $B^5_5$ | (54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 48, 49, 50, 51, 52, 53) | 1 |
| $B^6_6$ | (41, 49, 57, 48, 56, 64, 55, 63, 71, 62, 70, 61, 52, 60, 51, 42, 50) | 8 |
| $B^7_7$ | (45, 60, 58, 56, 54, 52, 67, 65, 63, 61, 59, 57, 55, 53, 51, 49, 47) | 2 |
| $B^8_8$ | (66, 71, 76, 64, 69, 57, 62, 50, 55, 43, 48, 36, 41, 46, 51, 56, 61) | 5 |
| $B^9_9$ | (36, 48, 43, 55, 50, 62, 57, 69, 64, 76, 71, 66, 61, 56, 51, 46, 41) | 5 |
| $B^{10}_{10}$ | (57, 59, 61, 63, 65, 67, 52, 54, 56, 58, 60, 45, 47, 49, 51, 53, 55) | 2 |
| $B^{11}_{11}$ | (61, 70, 62, 71, 63, 55, 64, 56, 48, 57, 49, 41, 50, 42, 51, 60, 52) | 8 |
| $B^{12}_{12}$ | (48, 64, 63, 62, 61, 60, 59, 58, 57, 56, 55, 54, 53, 52, 51, 50, 49) | 1 |
| $B^{13}_{13}$ | (52, 58, 47, 53, 59, 65, 54, 60, 66, 72, 61, 50, 56, 62, 51, 40, 46) | 6 |
| $B^{14}_{14}$ | (56, 69, 65, 61, 57, 53, 66, 62, 58, 54, 50, 46, 59, 55, 51, 47, 43) | 4 |
| $B^{15}_{15}$ | (60, 63, 49, 52, 55, 58, 61, 64, 67, 70, 56, 42, 45, 48, 51, 54, 57) | 3 |
| $B^{16}_{16}$ | (81, 91, 84, 77, 70, 63, 56, 49, 42, 35, 28, 21, 31, 41, 51, 61, 71) | 7 |

Table 2. The optimal WG-OC-FHS set $C^7$ with the FHS gap $D = 3$

| $C^7$  | Frequencies | $d(B^w_k)$ |
|--------|-------------|------------|
| $B^1_1$ | (21, 28, 35, 42, 49, 56, 63, 70, 77, 84, 91, 81, 71, 61, 51, 41, 31) | 7 |
| $B^2_2$ | (46, 50, 54, 58, 62, 66, 53, 57, 61, 65, 69, 56, 43, 47, 51, 55, 59) | 4 |
| $B^3_3$ | (50, 61, 72, 66, 60, 54, 65, 59, 53, 47, 58, 52, 46, 40, 51, 62, 56) | 6 |
| $B^4_4$ | (41, 49, 57, 48, 56, 64, 55, 63, 71, 62, 70, 61, 52, 60, 51, 42, 50) | 8 |
| $B^5_5$ | (66, 71, 76, 64, 69, 57, 62, 50, 55, 43, 48, 36, 41, 46, 51, 56, 61) | 5 |
| $B^6_6$ | (36, 48, 43, 55, 50, 62, 57, 69, 64, 76, 71, 66, 61, 56, 51, 46, 41) | 5 |
| $B^7_7$ | (61, 70, 62, 71, 63, 55, 64, 56, 48, 57, 49, 41, 50, 42, 51, 60, 52) | 8 |
| $B^8_8$ | (52, 58, 47, 53, 59, 65, 54, 60, 66, 72, 61, 50, 56, 62, 51, 40, 46) | 6 |
| $B^9_9$ | (56, 69, 65, 61, 57, 53, 66, 62, 58, 54, 50, 46, 59, 55, 51, 47, 43) | 4 |
| $B^{10}_{10}$ | (81, 91, 84, 77, 70, 63, 56, 49, 42, 35, 28, 21, 31, 41, 51, 61, 71) | 7 |

5. Concluding remarks

In this paper, we presented a new class of optimal OC-FHS sets, which includes prime sequence sets and HMC sequence sets as special cases. Moreover, we determined the FHS distances of the proposed OC-FHS set. For a given FHS gap $D$, by removing some FHS with a small distance from the OC-FHS set, we also
obtained a new class of WG-OC-FHS sets with the FHS gap $D$, which is optimal with respect to the WG-Lempel-Greenberger bound and the WG-Peng-Fan bounds simultaneously.

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E-mail address: renwenli80@163.com
E-mail address: flys990126.com