ON THE ASSOUAD DIMENSION AND
CONVERGENCE OF METRIC SPACES

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ABSTRACT. We introduce the notion of pseudo-cones of metric
spaces as a generalization of both of the tangent cones and the as-
ymptotic cones. We prove that the Assouad dimension of a metric
space is bounded from below by that of any pseudo-cone of it. We
exhibit an example containing all compact metric spaces as pseudo-
cones, and examples containing all proper length spaces as tangent
cones or asymptotic cones.

1. Introduction

Assouad [1, 2, 3] introduced the notion of the so-called Assouad
dimension for metric spaces, and studied the relation between the bi-
Lipschitz embeddability into a Euclidean space for metric spaces and
their Assouad dimensions. In general, it seems to be difficult to esti-
mate the Assouad dimension from below. Mackay and Tyson [20] pro-
vided a lower estimation of the Assouad dimensions of metric spaces by
using their tangent spaces. Namely, they proved that if $W$ is a tangent
space of a metric space $X$, then $\dim_A W \leq \dim_A X$, where $\dim_A$ stands
for the Assouad dimension.

Le Donne and Rajala [19] obtained both-sides estimations of the
Assouad dimensions and the Nagata dimensions by tangent spaces under
a certain assumption (see [19 Theorems 1.2, 1.4]). Dydak and Higes
[8] obtained a lower estimation of the Assouad-Nagata dimensions by
asymptotic cones as ultralimits (see [8, Proposition 4.1]).

In this paper, we introduce the notion of pseudo-cones of metric
spaces, which can be considered as a generalization of tangent cones
and asymptotic cones. For $h \in (0, \infty)$, and for a metric space $X$ with
metric $d_X$, we denote by $hX$ the metric space $(X, hd_X)$.

Definition 1.1 (Pseudo-cone). Let $X$ be a metric space. Let $\{A_i\}_{i \in \mathbb{N}}$
be a sequence of subsets of $X$, and let $\{u_i\}_{i \in \mathbb{N}}$ be a sequence in $(0, \infty)$.
We say that a metric space $P$ is a pseudo-cone of $X$ approximated
by $\{(A_i)_{i \in \mathbb{N}}, \{u_i\}_{i \in \mathbb{N}}\}$ if $\lim_{i \to \infty} d_{\text{GH}}(u_i A_i, P) = 0$, where $d_{\text{GH}}$ is the
Gromov–Hausdorff distance.

Date: January 16, 2020.

2010 Mathematics Subject Classification. Primary 53C23; Secondary 54E40.

Key words and phrases. Assouad dimension, Gromov–Hausdorff convergence.

The author was supported by JSPS KAKENHI Grant Number 18J21300.
For instance, every closed ball centered at a based point of a tangent cone or an asymptotic cone is a pseudo-cone. Indeed, if a pointed metric space \((W, w)\) is a tangent (resp. asymptotic) cone of a metric space \(X\) at \(p\), then for every \(R \in (0, \infty)\) the closed ball \(B(w, R)\) centered at \(w\) with radius \(R\) is a Gromov–Hausdorff limit of \(r_i B(p_i, R/r_i)\), where \(\{p_i\}_{i \in \mathbb{N}}\) is a sequence in \(X\) with \(\lim_{i \to \infty} p_i = p\) and \(\lim_{i \to \infty} r_i = \infty\) (resp. 0); in particular, \(B(w, R)\) is a pseudo-cone of \(X\) approximated by \((\{B(p_i, R/r_i)\}_{i \in \mathbb{N}}, \{r_i\}_{i \in \mathbb{N}})\).

For a metric space \(X\), we denote by \(PC(X)\) the class of all pseudo-cones of \(X\). By using the notion of pseudo-cones, we formulate a generalization of the Mackay–Tyson estimation for the Assouad dimension.

**Theorem 1.1.** Let \(X\) be a metric space. Then for every \(P \in PC(X)\) we have
\[
\dim_A P \leq \dim_A X.
\]

The notion of ultralimits of metric spaces is a method of emulating a limit space of a sequence of metric spaces (see Subsection 2.3). Let \(U\) be a non-principal ultrafilter on \(\mathbb{N}\). For a sequence \(\{(X_i, p_i)\}_{i \in \mathbb{N}}\) of pointed metric spaces, we denote by \(\lim_U \{X_i, p_i\}\) the ultralimit of \(\{(X_i, p_i)\}_{i \in \mathbb{N}}\) with respect to \(U\). The existence of an ultralimit is always guaranteed.

For an ultralimit analogy of pseudo-cones, we obtain the following:

**Theorem 1.2.** Let \(X\) be a metric space. Let \(\{A_i\}_{i \in \mathbb{N}}\) be a sequence of subsets of \(X\), and let \(\{u_i\}_{i \in \mathbb{N}}\) be a sequence in \((0, \infty)\). Take \(a_i \in A_i\) for each \(i \in \mathbb{N}\). Then for every non-principal ultrafilter \(U\) on \(\mathbb{N}\) we have
\[
\dim_A \left(\lim_U (u_i, A_i, a_i)\right) \leq \dim_A X.
\]

The lower Assouad dimension was essentially introduced by Larman [18]. This dimension is used for interpolation of the Assouad dimension. We also obtain the similar estimations as Theorems 1.1 and 1.2 for the lower Assouad dimension (see Theorems 3.5 and 3.6).

In the conformal dimension theory, the conformal Assouad dimension is studied as an invariant of quasi-symmetric maps (see [20]). In other words, by comparing the conformal Assouad dimensions of metric spaces, we can distinguish their quasi-symmetric equivalent classes. In general, it seems to be quite difficult to find the exact value of the conformal Assouad dimension.

For a metric space \(X\), we denote by \(KPC(X)\) the class of all pseudo-cones approximated by a pair of a sequence \(\{A_i\}_{i \in \mathbb{N}}\) of compact sets of \(X\) and a sequence \(\{u_i\}_{i \in \mathbb{N}}\) in \((0, \infty)\). We also obtain the following lower estimation of the conformal Assouad dimensions:

**Theorem 1.3.** Let \(X\) be a metric space. Then for every \(P \in KPC(X)\) we have
\[
\text{Cdim}_A P \leq \text{Cdim}_A X,
\]
where \(\text{Cdim}_A\) stands for the conformal Assouad dimension.
As a consequence of Theorem 1.3, for every metric space \( X \), we can estimate the conformal Assouad dimension of \( X \) by the conformal Assouad dimension of closed balls of an ultralimit constructed by scalings of subsets of \( X \) (see Corollary 3.8).

The points in the proofs of Theorems 1.1, 1.2 and 1.3 for pseudo-cones or ultralimits are to use the stability of the Assouad dimension under scaling of metrics, and to extract the techniques of Mackay and Tyson [20] in their lower estimation for their tangent spaces.

We say that a topological space \( X \) is an \((\omega_0+1)\)-space if \( X \) is homeomorphic to the one-point compactification of the countable discrete topological space. For example, the ordinal space \( \omega_0+1 \) with the order topology is an \((\omega_0+1)\)-space.

We construct an \((\omega_0+1)\)-metric space containing any compact metric space as a pseudo-cone.

**Theorem 1.4.** There exists an \((\omega_0+1)\)-metric space \( X \) such that \( \text{PC}(X) \) contains all compact metric spaces.

A metric space is said to be a length space if the distance of two points in the metric space is equal to the infimum of lengths of arcs joining the two points. A metric space is said to be proper if all bounded closed sets in the metric space are compact.

Similarly to Theorem 1.4, we construct metric spaces containing any proper length space as a tangent cone or an asymptotic cone.

**Theorem 1.5.** There exists an \((\omega_0+1)\)-metric space \( X \) for which every pointed proper length space \((K,p)\) is a tangent cone of \( X \) at its unique accumulation point.

**Theorem 1.6.** There exists a proper countable discrete metric space \( X \) for which every pointed proper length space \((K,p)\) is an asymptotic cone of \( X \) at some point.

The metric spaces mentioned in Theorems 1.4, 1.5 or 1.6 are examples showing that analogies of Theorem 1.1 for the topological dimension, the Hausdorff dimension and the conformal Hausdorff dimension are false (see Proposition 4.11).

The organization of this paper is as follows: In Section 2, we review the definitions and basic properties of the Assouad dimension and the Gromov–Hausdorff distance. In Section 3, we prove Theorems 1.1, 1.2 and 1.3. In Section 4, we construct examples stated in Theorems 1.4, 1.5 and 1.6.

Fraser and Yu [10] studied a characterization of subspaces of full Assouad dimension in the Euclidean space from a viewpoint of number theory. In [10], they observed a variant of the Mackay–Tyson estimation in a Euclidean setting. In the forthcoming paper [16], as an application of Theorem 1.1, the author will generalized the characterization theorem of Fraser and Yu to a metric space setting. In [16], the
The author will introduce the notion of tiling spaces, which form a subclass of metric spaces including the Euclidean spaces, the middle-third Cantor spaces, and various self-similar spaces appeared in fractal geometry. In [10], the author will prove that for every doubling tiling space $X$, a subset $F$ of $X$ satisfies $\dim_A F = \dim_A X$ if and only if some specific subset of $X$, called a tile, is a pseudo-cone of $F$.

Acknowledgements. The author would like to thank Professor Koichi Nagano for his advice and constant encouragement. The author also would like to thank Enrico Le Donne, Tapio Rajala and Jerzy Dydak for their helpful comments on the references.

2. Preliminaries

Let $X$ be a metric space. The symbol $d_X$ stands for the metric of $X$. Let $A$ be a subset of $X$. We denote by $\delta(A)$ the diameter of $A$, and we set $\alpha(A) = \inf \{ d_X(x, y) \mid x, y \in A \text{ and } x \neq y \}$. We denote by $B(x, r)$ (resp. $U(x, r)$) the closed (resp. open) ball centered at $x$ with radius $r$. We also denote by $B(A, r)$ the set $\bigcup_{a \in A} B(a, r)$. To emphasize a metric space under consideration, we often use symbols $\delta_X(A)$, $\alpha_X(A)$, $B(x, r; X)$ and $B(A, r; X)$ instead of $\delta(A)$, $\alpha(A)$, $B(x, r)$ and $B(A, r)$, respectively. A subset $A$ of $X$ is said to be $r$-separated if $\alpha(A) \geq r$. A subset $A$ is separated if it is $r$-separated for some $r$.

In this paper, we denote by $\mathbb{N}$ the set of all non-negative integers.

2.1. Assouad dimension. For a positive integer $N \in \mathbb{N}$, a metric space $X$ is said to be $N$-doubling if for every bounded set $S \subset X$ there exists a subset $F \subset X$ such that $S \subset B(F, \delta(S)/2)$ and $\text{card}(F) \leq N$. Note that if a metric space $X$ is $N$-doubling, then so are all subsets of $X$. A metric space is doubling if it is $N$-doubling for some $N$.

For a bounded set $S \subset X$, we denote by $\mathcal{B}_X(S, r)$ the minimum integer $N$ such that $S$ can be covered by at most $N$ bounded sets with diameter at most $r$. We denote by $\mathcal{A}(X)$ the set of all $\beta \in (0, \infty)$ for which there exists $C \in (0, \infty)$ such that for every bounded set $S \subset X$, and for every $r \in (0, \infty)$, we have $\mathcal{B}_X(S, r) \leq C(\delta(S)/r)^\beta$.

The Assouad dimension $\dim_A X$ of $X$ is defined as $\inf(\mathcal{A}(X))$ if the set $\mathcal{A}(X)$ is non-empty; otherwise, $\dim_A X = \infty$. We denote by $\mathcal{B}(X)$ the set of all $\beta \in (0, \infty)$ for which there exists $C \in (0, \infty)$ such that for every finite subset $A$ of $X$ we have $\text{card}(A) \leq C(\delta(A)/\alpha(A))^{\beta}$, where $\text{card}(A)$ stands for the cardinality of $A$.

By definitions, we obtain the next two propositions.

Proposition 2.1. For every metric space $X$, the following are equivalent:

1. $X$ is doubling;
2. $\mathcal{A}(X)$ is non-empty;
3. $\mathcal{B}(X)$ is non-empty;
(4) \( \dim_A X < \infty \).

**Proposition 2.2.** For every metric space \( X \), we have

\[
\dim_A X = \inf(\mathcal{D}(X)).
\]

The **lower Assouad dimension** \( \dim_{LA} X \) of \( X \) is defined as the supremum of all \( \beta \in (0, \infty) \) for which there exists \( C \in (0, \infty) \) such that for every finite set \( S \) in \( X \) we have \( \text{card}(S) \geq C(\delta(S)/\alpha(S))^{\beta} \).

**Proposition 2.3.** For every metric space \( X \), we have

\[
\dim_{LA} X \leq \dim_A X.
\]

2.2. **Gromov–Hausdorff distance.** For a metric space \( Z \), and for subsets \( S, T \subset Z \), we define the Hausdorff distance \( d_H(S, T; Z) \) between \( S \) and \( T \) in \( Z \) as the infimum \( r \in (0, \infty) \) for which \( S \subset B(T, r) \) and \( T \subset B(S, r) \). For two metric spaces \( X \) and \( Y \), the **Gromov–Hausdorff distance** \( d_{GH}(X, Y) \) between \( X \) and \( Y \) is defined as the infimum of all values \( d_H(i(X), j(Y); Z) \), where \( Z \) is a metric space and \( i : X \to Z \) and \( j : Y \to Z \) are isometric embeddings.

To deal with the Gromov–Hausdorff distance, we use the so-called approximation maps. For \( \epsilon \in (0, \infty) \), and for metric spaces \( X \) and \( Y \), a pair \((f, g)\) with \( f : X \to Y \) and \( g : Y \to X \) is said to be an \( \epsilon \)-approximation if the following conditions hold:

1. for all \( x, y \in X \), we have \( |d_X(x, y) - d_Y(f(x), f(y))| < \epsilon \);
2. for all \( x, y \in Y \), we have \( |d_Y(x, y) - d_X(g(x), g(y))| < \epsilon \);
3. for each \( x \in X \) and for each \( y \in Y \), we have \( d_X(g \circ f(x), x) < \epsilon \) and \( d_Y(f \circ g(x), x) < \epsilon \).

By the definitions of the Gromov–Hausdorff distance, we obtain:

**Proposition 2.4.** Let \( X \) and \( Y \) be metric spaces. Then for every \( h \in (0, \infty) \) we have \( d_{GH}(hX, hY) = hd_{GH}(X, Y) \).

The next two claims can be seen in Sections 7.3 and 7.4 in [6].

**Lemma 2.5.** If metric spaces \( X \) and \( Y \) satisfy \( d_{GH}(X, Y) \leq \epsilon \), then there exists a \( 2\epsilon \)-approximation between them.

**Proposition 2.6.** For all metric spaces \( X \) and \( Y \), we have

\[
|\delta(X) - \delta(Y)| \leq 2d_{GH}(X, Y).
\]

We say that a sequence \( \{(X_i, p_i)\}_{i \in \mathbb{N}} \) of pointed metric spaces converges to \( (Y, q) \) in the pointed Gromov–Hausdorff topology if there exist a sequence \( \{\epsilon_i\}_{i \in \mathbb{N}} \) in \( (0, \infty) \) with \( \lim_{i \to \infty} \epsilon_i = 0 \), and a sequence \( \{(f_i, g_i)\}_{i \in \mathbb{N}} \) of \( \epsilon_i \)-approximation maps between \( X_i \) and \( Y \) with \( f_i(p_i) = q \) and \( g(q) = p_i \).
2.3. Ultraproducts. Let \( \mathcal{U} \) be a non-principal ultrafilter on \( \mathbb{N} \). For a sequence \( \{a_i\}_{i \in \mathbb{N}} \) in \( \mathbb{R} \), a real number \( u \) is a \textit{ultralimit} of \( \{a_i\}_{i \in \mathbb{N}} \) with respect to \( \mathcal{U} \) if for every \( \epsilon \in (0, \infty) \) we have \( \{ i \in \mathbb{N} \mid |a_i - u| < \epsilon \} \in \mathcal{U} \). In this case, we write \( \lim_{\mathcal{U}} a_i = u \). Note that an ultralimit of a bounded sequence in \( \mathbb{R} \) always uniquely exists.

Let \( \{(X_i, p_i)\}_{i \in \mathbb{N}} \) be a sequence of pointed metric spaces. We put
\[
B(\{(X_i, p_i)\}_{i \in \mathbb{N}}) = \left\{ \{x_i\}_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}} X_i \mid \sup_{i \in \mathbb{N}} d(p_i, x_i) < \infty \right\}.
\]
Define an equivalence relation \( R_\mathcal{U} \) on \( B(\{(X_i, p_i)\}_{i \in \mathbb{N}}) \) in such a way that \( \{x_i\}_{i \in \mathbb{N}} R_\mathcal{U} \{y_i\}_{i \in \mathbb{N}} \) if and only if \( \lim_{\mathcal{U}} d_{X_i}(x_i, y_i) = 0 \). We denote by \( [\{x_i\}_{i \in \mathbb{N}}] \) the equivalence class of \( \{x_i\}_{i \in \mathbb{N}} \). We denote by \( \lim_{\mathcal{U}}(X_i, p_i) \) the metric space \( B(\{(X_i, p_i)\}_{i \in \mathbb{N}})/R_\mathcal{U} \) equipped with the metric \( d_{\lim_{\mathcal{U}}(X_i, p_i)} \) defined by
\[
d_{\lim_{\mathcal{U}}(X_i, p_i)}(x, y) = \lim_{\mathcal{U}} d_{X_i}(x, y),
\]
where \( x = [\{x_i\}_{i \in \mathbb{N}}] \) and \( y = [\{y_i\}_{i \in \mathbb{N}}] \). We call \( \lim_{\mathcal{U}}(X_i, p_i) \) the ultralimit of the sequence \( \{(X_i, p_i)\}_{i \in \mathbb{N}} \) with respect to \( \mathcal{U} \).

The following can be seen in [5, 1.5.52] or [17, Proposition 3.2].

**Lemma 2.7.** Let \( \{(X_i, p_i)\}_{i \in \mathbb{N}} \) be a sequence of pointed compact metric spaces. If \( \{(X_i, p_i)\}_{i \in \mathbb{N}} \) converges to a pointed compact metric space \( (X, p) \), then \( \lim_{\mathcal{U}}(X_i, p_i) \) and \( (X, p) \) are isometric to each other.

We say that a sequence \( \{X_i\}_{i \in \mathbb{N}} \) of metric spaces is \textit{uniformly bounded} if there exists \( M \in (0, \infty) \) with \( \sup_{i \in \mathbb{N}} \delta(X_i) \leq M \).

For every uniformly bounded sequence \( \{X_i\}_{i \in \mathbb{N}} \), and for every choice \( \{p_i\}_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}} X_i \) of base points, we have \( B(\{(X_i, p_i)\}_{i \in \mathbb{N}}) = \prod_{i \in \mathbb{N}} X_i \).

Therefore Lemma 2.7 implies:

**Lemma 2.8.** Let \( \{X_i\}_{i \in \mathbb{N}} \) be a uniformly bounded sequence of compact metric spaces. If \( \{X_i\}_{i \in \mathbb{N}} \) converges to a compact metric space \( L \), then for every choice \( \{p_i\}_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}} X_i \) of base points, the metric spaces \( \lim_{\mathcal{U}}(X_i, p_i) \) and \( L \) are isometric to each other.

3. **Pseudo-cones and the Assouad Dimension**

In this section, we prove Theorems 1.1, 1.2 and 1.3.

3.1. **Basic properties of pseudo-cones.** By Proposition 2.4, we have:

**Proposition 3.1.** Let \( X \) be a metric space. If \( A \in \text{PC}(X) \), then for every \( h \in (0, \infty) \) we have \( hA \in \text{PC}(X) \).

We say that a sequence \( \{X_i\}_{i \in \mathbb{N}} \) of metric space is \textit{uniformly pre-compact} if for every \( \epsilon \in (0, \infty) \) there exists \( M \in \mathbb{N} \) such that for every \( i \in \mathbb{N} \), every \( \epsilon \)-separated set in \( X_i \) has at most \( M \) elements. Note that if there exists \( N \in \mathbb{N} \) such that for each \( i \in \mathbb{N} \), the space \( X_i \) is \( N \)-doubling, then \( \{X_i\}_{i \in \mathbb{N}} \) is uniformly bounded.
We recall Gromov’s precompactness theorem (see Section 7.4 in [6]). Namely, if a sequence \( \{X_i\}_{i\in \mathbb{N}} \) of compact metric spaces is uniformly precompact and uniformly bounded, then there exists a subsequence of \( \{X_i\}_{i\in \mathbb{N}} \) which converges to a compact metric space in the sense of Gromov–Hausdorff. This guarantees the existence of pseudo-cones for doubling metric spaces.

**Proposition 3.2.** Let \( X \) be a doubling metric space. If a sequence \( \{A_i\}_{i\in \mathbb{N}} \) consists of compact sets in \( X \), and if \( \{u_iA_i\}_{i\in \mathbb{N}} \) is uniformly bounded for a sequence \( \{u_i\}_{i\in \mathbb{N}} \) in \((0, \infty)\), then there exists a convergent subsequence \( \{u_{\phi(i)}A_{\phi(i)}\}_{i\in \mathbb{N}} \) of \( \{u_iA_i\}_{i\in \mathbb{N}} \) in the sense of Gromov–Hausdorff.

Let \( X \) be a proper metric space, and \( p \in X \). A pointed metric space \((Y, y)\) is said to be a tangent (resp. asymptotic) cone of \( X \) at \( p \) if there exist a sequence \( \{p_i\}_{i\in \mathbb{N}} \) in \( X \) with \( \lim_{i \to \infty} p_i = p \), and a sequence \( \{r_i\} \) in \((0, \infty)\) with \( \lim_{i \to \infty} r_i = 0 \) (resp. \( \infty \)) such that for every \( R \in (0, \infty) \) we have \( (r_iB(p_i, R/r_i), p_i) \to (B(y, R), y) \) as \( i \to \infty \) in the pointed Gromov–Hausdorff topology (see Section 8.1 in [6]).

By the definitions of the tangent cones and the asymptotic cones, we obtain:

**Proposition 3.3.** Let \( X \) be a proper metric space, and let \((Y, y)\) be a tangent cone of \( X \) or asymptotic cone of \( X \). Then for every \( R \in (0, \infty) \) we have \( B(y, R) \in \text{PC}(X) \).

### 3.2. Lower estimations

First we prove Theorem 1.1.

**Proof of Theorem 1.1.** Let \( X \) be a metic space, and \( P \in \text{PC}(X) \). We assume that \( P \) is approximated by \((\{A_i\}_{i\in \mathbb{N}}, \{u_i\}_{i\in \mathbb{N}})\). Suppose that \( \dim_A(X) < \dim_A(P) \). Take \( \beta \in \mathcal{B}(X) \) with \( \dim_A(X) < \beta < \dim_A(P) \). Since \( \beta \in \mathcal{B}(X) \), there exists \( M \in (0, \infty) \) such that for every finite set \( T \) of \( X \) we have \( \text{card}(T) \leq M(\delta_X(T)/\alpha_X(T))^{\beta} \).

Put \( C = 4^\beta(M + 1) \). From \( \beta < \dim_A P \), it follows that \( \beta \not\in \mathcal{B}(P) \).

Thus there exists a finite set \( S \) of \( P \) with \( \text{card}(S) > C(\delta_P(S)/\alpha_P(S))^{\beta} \).

Since \( d_{GH}(u_iA_i, P) \to 0 \) as \( i \to \infty \), we can take \( N \in \mathbb{N} \) such that \( d_{GH}(u_NA_N, P) < \alpha_P(S)/20 \). By Lemma 25 there exists an \((\alpha_P(S)/10)\)-approximation \((f, g)\) between \( u_NA_N \) and \( P \). For each \( x \in S \), take \( t_x \in u_NA_N \) such that \( t_x \in B(g(x), \alpha_P(S)/10) \). Note that if \( x \neq y \), then \( t_x \neq t_y \). Put \( T = \{ t_x \mid x \in S \} \). For all \( x, y \in S \), we obtain

\[
u_N d_X(t_x, t_y) \leq d_P(x, y) + 3\alpha_P(S)/10 \leq 2\delta_P(S),
\]

and

\[
u_N d_X(t_x, t_y) \geq d_P(x, y) - 3\alpha_P(S)/10 \geq 2^{-1}\alpha_P(S).
\]
Then for such $u$ and $i$ for each $S$ and for all $P = \lim_{\mathcal{U}} (u_i, a_i)$, we obtain:

$$4^{-\beta} C (2 u_N^{-1} \delta_P(S) / 2^{-1} u_N^{-1} \alpha_P(S))^{\beta} \geq 4^{-\beta} C (\delta_X(T)/\alpha_X(T))^{\beta}.$$  

On the other hand, we also have $\text{card}(T) \leq M (\delta_X(T)/\alpha_X(T))^{\beta}$. These inequalities imply that $4^{-\beta} C \leq M$. This is a contradiction. □

Since for every metric space $X$ we have $X \in \text{PC}(X)$, by Theorem 3.6 we obtain:

**Corollary 3.4.** Let $X$ and $Y$ be metric spaces. If $d_{GH}(X, Y) = 0$, then $\dim_A X = \dim_A Y$.

This corollary slightly generalizes the fact that the Assouad dimension of any metric space is equal to that of its completion.

By a similar proof to Theorem 1.2, we obtain:

**Theorem 3.5.** Let $X$ be a metric space. Then for every $P \in \text{PC}(X)$ we have

$$\dim_{LA} X \leq \dim_{LA} P.$$  

Next we prove Theorem 3.2.

**Proof of Theorem 3.2**. Let $X$ be a metric space. Let $\{A_i\}_{i \in \mathbb{N}}$ be a sequence of subsets of $X$, and let $\{u_i\}_{i \in \mathbb{N}}$ be a sequence in $(0, \infty)$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. Take $a_i \in A_i$ for each $i \in \mathbb{N}$. Put $P = \lim_{\mathcal{U}} (u_i A_i, a_i)$.

Suppose that $\dim_A X < \dim_A P$. Let $\beta \in B(X)$, $M \in (0, \infty)$, $C = 4^{\beta} (M + 1)$ and $S \subset P$ be the same objects in the proof of Theorem 1.1. Put $S = \{[x_{1,i}], [x_{1,i}], \ldots, [x_{n,i}]\}$. Put $S_i = \{x_{1,i}, \ldots, x_{n,i}\} \subset u_i A_i$ for each $i \in \mathbb{N}$. By the definition of ultralimits, for $\mathcal{U}$-almost all $i \in \mathbb{N}$, and for all $k, l \in \{1, \ldots, n\}$ we have

$$|u_{id_X(x_{k,i}, x_{l,i})} - d_P([x_{k,i}], [x_{l,i}])| < \alpha(S)/2.$$  

Then for such $\mathcal{U}$-almost all $i \in \mathbb{N}$ we have

$$\delta_X(S_i) \leq 2 u_i^{-1} \delta(S)$$  

and

$$\alpha_X(S_i) \geq 2^{-1} u_i^{-1}.$$  

Since $\text{card}(S_i) = \text{card}(S)$, by a similar argument to the proof of Theorem 1.1, we obtain $4^{-\beta} C \leq M$. This is a contradiction. □

By a similar argument to the proof of Theorem 1.2, we obtain:

**Theorem 3.6.** Let $X$ be a metric space. Let $\{A_i\}_{i \in \mathbb{N}}$ be a sequence of subsets of $X$, and let $\{u_i\}_{i \in \mathbb{N}}$ be a sequence in $(0, \infty)$. Take $a_i \in A_i$ for each $i \in \mathbb{N}$. Then for every non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ we have

$$\dim_{LA} X \leq \dim_{LA} \left( \lim_{\mathcal{U}} (u_i A_i, a_i) \right).$$
3.3. Conformal Assouad dimension. Let \( \eta : [0, \infty) \rightarrow [0, \infty) \) be a homeomorphism. A homeomorphism \( f : X \rightarrow Y \) between metric spaces is said to be an \( \eta \)-quasi-symmetric map if the following holds:

(QS) if for \( x, y, z \in X \) and for \( t \in [0, \infty) \) we have \( d_X(x, y) \leq t d_X(x, z) \), then \( d_Y(f(x), f(y)) \leq \eta(t)d_Y(f(x), f(z)) \).

A homeomorphism \( f : X \rightarrow Y \) is quasi-symmetric if it is \( \eta \)-quasi-symmetric for some \( \eta \). Note that the inverse of a quasi-symmetric map is also quasi-symmetric.

For a metric space \( X \), the conformal Assouad dimension \( \text{Cdim}_A X \) of \( X \) is defined as the infimum of all Assouad dimensions of all quasi-symmetric images of \( X \).

In the proof of Theorem 1.3, we use the following theorem due to Tukia and Väisälä (see [21] Theorem 2.21).

**Theorem 3.7.** If a map \( f : X \rightarrow Y \) between metric spaces satisfies the condition (QS), then \( f \) is either a constant map or a quasi-symmetric embedding.

We now prove Theorem 1.3.

**Proof of Theorem 1.3.** Let \( X \) be a metric space, and \( P \in \text{KPC}(X) \). Since a non-doubling space has infinite conformal Assouad dimension, we may assume that \( X \) is doubling. Take a metric space \( Y \) and a \( \eta \)-quasi-symmetric map \( f : X \rightarrow Y \). We may assume that \( P \) is compact and \( P \) has at least two elements. We assume that \( P \) is approximated by \( \{ (A_i, u_i) \}_{i \in \mathbb{N}} \), where \( \{ A_i \}_{i \in \mathbb{N}} \) is a sequence of compact sets in \( X \). By Proposition 2.6, we have \( \sup_i \delta(u_i A_i) < \infty \). For each \( i \in \mathbb{N} \), put \( B_i = f(A_i) \) and \( v_i = (\delta_Y(B_i))^{-1} \). By Gromov’s precompactness theorem, by choosing a suitable subsequence if necessary, we find a limit compact metric space \( Q \in \text{KPC}(Y) \) of \( \{ v_i B_i \}_{i \in \mathbb{N}} \).

Let \( \mathcal{U} \) be a non-principal ultrafilter on \( \mathbb{N} \). By Lemma 2.3, we can consider that \( Q = \lim_\mathcal{U} v_i B_i \) and \( P = \lim_\mathcal{U} u_i A_i \). Since \( f \) is continuous and \( \delta(v_i B_i) = 1 \) for all \( i \in \mathbb{N} \), the map \( f : X \rightarrow Y \) induce a map \( F : P \rightarrow Q \) defined by \( F([\{ x_i \}_{i \in \mathbb{N}}]) = [\{ f(x_i) \}_{i \in \mathbb{N}}] \). Replacing the role of \( f \) with that of \( f^{-1} \), we obtain the inverse of \( F \). Thus \( F \) is bijective.

In order to prove that \( F \) satisfies the condition (QS), we assume \( d_P(x, y) \leq t d_P(x, z) \), where \( x = \{ x_i \}_{i \in \mathbb{N}} \), \( y = \{ y_i \}_{i \in \mathbb{N}} \), \( z = \{ z_i \}_{i \in \mathbb{N}} \). For each \( \epsilon \in (0, \infty) \), we have

\[
d_{u_i A_i}(x_i, y_i) < (t + \epsilon)d_{u_i A_i}(x_i, z_i)
\]

for \( \mathcal{U} \)-almost all \( i \in \mathbb{N} \). Thus, since \( f \) is \( \eta \)-quasi-symmetric, we have

\[
d_{v_i B_i}(f(x_i), f(y_i)) < \eta(t + \epsilon)d_{v_i B_i}(f(x_i), f(z_i))
\]

for \( \mathcal{U} \)-almost all \( i \in \mathbb{N} \). Then we conclude

\[
d_Q(F(x), F(z)) < \eta(t + \epsilon)d_Q(F(x), F(z)).
\]
Letting $\epsilon \to 0$, we obtain $d_Q(F(x), F(z)) \leq \eta(t)d_Q(F(x), F(z))$. Since $F$ is bijective and non-constant, by Theorem 3.7 we conclude that $F$ is an $\eta$-quasi-symmetric map. Thus $\text{Cdim}_A P \leq \text{dim}_A Q$. Theorem 1.1 implies $\text{dim}_A Q \leq \text{dim}_A Y$. In particular, $\text{Cdim}_A P \leq \text{Cdim}_A X$. □

As a corollary of Theorem 1.3 we obtain:

**Corollary 3.8.** Let $X$ be a metric space. Let $\{A_i\}_{i \in \mathbb{N}}$ be a sequence of subsets of $X$, and let $\{u_i\}_{i \in \mathbb{N}}$ be a sequence in $(0, \infty)$. Take $p_i \in A_i$ for each $i \in \mathbb{N}$. Put $Y = \lim U_i(u_i, A_i, p_i)$. Then for every $R \in (0, \infty)$ we have

$$\text{Cdim}_A (B(p, R; Y)) \leq \text{Cdim}_A X,$$

where $p = \{\{p_i\}_{i \in \mathbb{N}}\}$.

**Proof.** Since the Assouad of the completion of $X$ coincides with that of $X$, we may assume that $X$ is doubling and complete. Since $\lim U_i(u_i, A_i)$ is isometric to $\lim U_i(u_i, \text{CL}(A_i))$, we may assume that each $A_i$ is closed. Note that $A_i$ is doubling and complete, and hence it is proper. Put

$$S = \left\{ \{x_i\}_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}} A_i \mid d_{A_i}(p_i, x_i) < 2R \right\} / \text{R}_{U_i},$$

and

$$T = \left\{ \{x_i\}_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}} A_i \mid d_{A_i}(p_i, x_i) \leq 2R \right\} / \text{R}_{U_i}.$$

By the definition of ultralimit, we have $B(p, R; \lim U_i(u_i, A_i)) \subset S$. We also have $T = \lim U_i(B(p_i, 2R), p_i)$. Since $A_i$ is proper, the ball $B(p_i, R)$ is compact. Thus by Theorem 1.3 we obtain

$$\text{Cdim}_A T \leq \text{Cdim}_A X.$$

By the monotonicity of the conformal Assouad dimension, we obtain the corollary. □

## 4. Examples

In this section, we study examples containing a large class of metric space as their pseudo-cones, tangent cones or asymptotic cones.

### 4.1. Telescope construction

Let $\mathcal{X} = \{X_i\}_{i \in \mathbb{N}}$ be a sequence of bounded metric spaces. Assume that $\delta(X_i) \leq 2^{-i}$. Put

$$T(\mathcal{X}) = \{\infty\} \sqcup \prod_{i \in \mathbb{N}} X_i,$$

and define a metric $d_{T(\mathcal{X})}$ on $T(\mathcal{X})$ by

$$d_{T(\mathcal{X})}(x, y) = \begin{cases} d_{X_i}(x, y) & \text{if } x, y \in X_i \text{ for some } i, \\ \max\{2^{-i}, 2^{-j}\} & \text{if } x \in X_i, y \in X_j \text{ for some } i \neq j, \\ 2^{-i} & \text{if } x = \infty, y \in X_i \text{ for some } i, \\ 2^{-i} & \text{if } x \in X_i, y = \infty \text{ for some } i. \end{cases}$$
We call the metric space $T(X)$ the *telescope space of* $X$. This construction is a specific version of the telescope spaces discussed in [15].

### 4.2. Proof of Theorem 1.4

We construct an $(\omega_0 + 1)$-metric space containing all compact metric spaces as its pseudo-cones.

We denote by $\mathcal{S}$ the class of all separable metric spaces. We say that a metric space $X$ is *$\mathcal{S}$-universal* if every metric space $A \in \mathcal{S}$ is isometrically embeddable into $X$.

The Urysohn universal space $U$ (see [22, 13, 11]), and the space $C([0,1])$ equipped with the supremum metric (see [4, 13]) are separable and $\mathcal{S}$-universal. Note that the space $\ell^\infty$ of all bounded sequences equipped with the supremum metric is $\mathcal{S}$-universal, which is known as Fréchet’s embedding theorem (see [9]); however, $\ell^\infty$ is not separable.

By the virtue of the telescope construction, and by the existence of separable $\mathcal{S}$-universal metric space, we can prove Theorem 1.4.

**Proof of Theorem 1.4** Let $U$ be a separable $\mathcal{S}$-universal metric space. Let $Q$ be a countable dense set of $U$, and let $J = \{K_i\}_{i \in \mathbb{N}}$ be the set of all finite set of $Q$. Put $X = T(J)$. The metric space $X$ is an $(\omega_0 + 1)$-metric space, and the point $\infty$ is its unique accumulation point. Let $K$ be any compact metric space. Since $K$ is isometrically embeddable into $U$, there exists a subsequence $\{K_{\phi(i)}\}_{i \in \mathbb{N}}$ of $J$ with $d_H(K_{\phi(i)}, K; U) \to 0$ as $i \to 0$. For each $i \in \mathbb{N}$ we have $(2^i \delta(K_{\phi(i)}))^{-1} K_{\phi(i)} \subset X$. Thus $K \in \mathcal{PC}(X)$. This finishes the proof of Theorem 1.4.

By a similar argument, we also obtain:

**Proposition 4.1.** Let $U$ be a separable $\mathcal{S}$-universal metric space. If $Q$ is a countable dense set of $U$, then $\mathcal{PC}(Q) = \mathcal{S}$.

### 4.3. Proofs of Theorems 1.5 and 1.6

By an argument on arcs in a length space, we obtain the following estimation of the Hausdorff distance between concentric balls.

**Proposition 4.2.** Let $X$ be a length space and $p \in X$. Then for all $r, R \in (0, \infty)$ we have

$$d_H(B(p, r), B(p, R); X) \leq |r - R|.$$

The following proposition is a key of our construction of the desired spaces in Theorems 1.5 and 1.6.

**Proposition 4.3.** Let $U$ be a metric space, and $Q$ a countable dense subset of $U$. Let $K$ be a length metric subspace of $U$, and $p \in K$. For all $i, k \in \mathbb{N}$, put $l_{k,i} = k \cdot 2^{-i}$. Assume that a sequence $\{A_i\}_{i \in \mathbb{N}}$ of subsets of $Q$ satisfies the following for every $i \in \mathbb{N}$:

1. $p \in A_i$;
2. for each $k \in \{0, \ldots, 2^i\}$ we have
   $$d_H(B(p, l_{k,i}; K), B(p, l_{k,i}; A_i); U) \leq 2^{-i}.$$
Then for every $R \in (0, \infty)$, the sequence $\{(B(p, R; A_i), p)\}_{i \in \mathbb{N}}$ converges to $(B(p, R; K), p)$ in the pointed Gromov–Hausdorff topology.

Proof. Take $N \in \mathbb{N}$ with $R < 2^{2N}$. Then for each $i \geq N$, we can take $k \in \{0, \ldots, 2^{2i}\}$ with

$$l_{k,i} \leq R < l_{k+1,i}.$$

By the condition (A2), for $m \in \{k, k + 1\}$,

$$d_H(B(p, l_{m,i}; A_i), B(p, l_{m,i}; K); U) \leq 2^{-i}.$$ 

Thus, we have

(4.1) $B(p, l_{k,i}; K) \subset B(B(p, l_{k,i}; A_i), 2^{-i}; U),$

and

(4.2) $B(p, l_{k+1,i}; A_i) \subset B(B(p, l_{k+1,i}; K), 2^{-i}; U).$

Since for $m \in \{k, k + 1\}$ we have $|R - l_{m,i}| \leq 2^{-i}$, by Proposition 4.2, for $m \in \{k, k + 1\}$ we have

$$d_H(B(p, R; K), B(p, l_{m,i}; K); U) \leq 2^{-i}.$$ 

Thus we have

(4.3) $B(p, R; K) \subset B(B(p, l_{k,i}; K), 2^{-i}; U),$

and

(4.4) $B(p, l_{k+1,i}; K) \subset B(B(p, R; K), 2^{-i}; U).$

Since $B(p, l_{k,i}; A_i) \subset B(p, R; A_i)$, by (4.1) and (4.3), we obtain

(4.5) $B(p, R; K) \subset B(B(p, R; A_i); 2^{-i+1}; U).$

Since $B(p, R; A_i) \subset B(p, l_{k+1,i}; A_i)$, by (4.2) and (4.4), we obtain

(4.6) $B(p, R; A_i) \subset B(B(p, R; K); 2^{-i+1}; U).$

Then, by (4.5) and (4.6), we have

$$d_H(B(p, R; K), B(p, R; A_i); U) \leq 2^{-i+1}.$$

Hence we conclude that the sequence $\{(B(p, R; A_i), p)\}_{i \in \mathbb{N}}$ converges to $(B(p, R; K), p)$ in the pointed Gromov–Hausdorff topology. \qed

A metric space $X$ is said to be homogeneous if for all $x, y \in X$, there exists an isometry $f : X \to X$ such that $f(x) = y$. The spaces $U$ and $C([0, 1])$ are homogeneous.

By the definition of homogeneity, we obtain:

**Proposition 4.4.** Let $U$ be a homogeneous $\mathcal{G}$-universal metric space. Then for every $q \in U$, and for every pointed separable metric space $(X, x)$, there exists an isometry $f : X \to U$ such that $f(x) = q$.

We now prove Theorem 1.5.
Proof of Theorem 1.5. We may assume that \( K \) has at least two elements. Let \( U \) be a separable homogeneous \( \mathcal{G} \)-universal metric space. For instance, we can choose \( C([0,1]) \) or \( \mathbb{U} \) as \( U \). Let \( Q \) be a countable dense subset of \( U \). Put \( I = \{ F_i \}_{i \in \mathbb{N}} \) be a sequence consisting of all finite subset of \( Q \). We impose the condition that for every finite subset \( A \) of \( Q \) there exists infinite many \( n \in \mathbb{N} \) with \( F_n = A \). By Proposition 1.4 we may assume that \( K \subset U \) and \( p \in Q \).

For each \( i \in \mathbb{N} \), set \( r_i = (i + 1)! \cdot \delta(F_i) \). Put \( J = \{(r_i)^{-1}F_i \}_{i \in \mathbb{N}} \). Let \( X = T(J) \). The space \( X \) is an \((\omega_0 + 1)\)-metric space, and \( \infty \) is its unique accumulation point.

Since \( K \) is proper, we can take a sequence \( \{A_i\}_{i \in \mathbb{N}} \) of finite subsets of \( Q \) satisfying the conditions (A1) and (A2) in Proposition 4.3. By the definition of \( I = \{ F_i \}_{i \in \mathbb{N}} \), there exists a strictly increasing map \( \phi : \mathbb{N} \to \mathbb{N} \) such that \( r_{\phi(i)}F_{\phi(i)} \) is isometric to \( A_i \) for each \( i \in \mathbb{N} \). Let \( q_i \in F_{\phi(i)} \) be a corresponding point to \( p \in A_i \). Note that \( r_{\phi(i)} \to \infty \) as \( i \to \infty \).

To prove that \((K,p)\) is a tangent cone of \( X \), we show that for each \( R \in (0,\infty) \), the sequence \( \{(r_iB(q_i,R/r_i;X),q_i)\}_{i \in \mathbb{N}} \) converges to \((B(p,R;K),p)\) in the pointed Gromov–Hausdorff topology. By the definition of \( \{ r_i \}_{i \in \mathbb{N}} \), we can take \( N \in \mathbb{N} \) such that if \( i > N \), then we have \( R < r_{\phi(i)} \cdot 2^{-\phi(i)+1} \). Therefore, by the definition of \( X \), for every \( i > N \), the pointed metric space \((r_{\phi(i)}B(q_i,R/r_{\phi(i)};X),q_i)\) is isometric to \((B(p,R;F_{\phi(i)}),p)\). By Proposition 4.3, \( \{(r_{\phi(i)}B(q_i,R/r_{\phi(i)};X),q_i)\}_{i \in \mathbb{N}} \) converges to \((B(p,R;K),p)\) in the pointed Gromov–Hausdorff topology. Since \( q_i \to \infty \) in \( X \) as \( i \to \infty \), we conclude that \((K,p)\) is a tangent cone of \( X \) at \( \infty \). This completes the proof of Theorem 1.5.

We next prove Theorem 1.6. As a core part to construct a metric space mentioned in Theorem 1.6, we begin with the following elementary lemma on a surjective map between countable sets, which guarantees a polite way of indexing a countable set.

**Lemma 4.5.** There exists a surjective map \( C : \mathbb{N} \to \mathbb{N}^2 \times \mathbb{Z} \) satisfying the following:

(B1) \( C(0) = (0,0,0) \)

(B2) for every \( n \in \mathbb{N} \), two points \( C(n) \) and \( C(n + 1) \) are adjunct in \( \mathbb{N}^2 \times \mathbb{Z} \); namely, for every \( n \in \mathbb{N} \) and for every \( i \in \{1,2,3\} \), we have

\[
|\pi_i(C(n)) - \pi_i(C(n + 1))| \leq 1;
\]

where \( \pi_i \) is the i-th projection.

(B3) for each \((x, y, z) \in \mathbb{N}^2 \times \mathbb{Z} \), there exist infinite many \( n \in \mathbb{N} \) such that \( C(n) = (x, y, z) \).

**Proof.** Take a surjective map \( A : \mathbb{N} \to \mathbb{N}^2 \times \mathbb{Z} \) satisfying the conditions (B1) and (B2). Assume that \( A \) does not satisfy the condition (B3). Take a surjective map \( H : \mathbb{N} \to \mathbb{N} \) satisfying the following:
(1) $H(0) = 0$;
(2) for every $n \in \mathbb{N}$, the set $H^{-1}(n)$ is infinite;
(3) for every $n \in \mathbb{N}$, we have $|H(n) - H(n + 1)| \leq 1$.

For example, if for each $n \in \mathbb{N}$ we put $H(n) = \min_{k \in \mathbb{N}} |n - k^2|$, then the map $H : \mathbb{N} \to \mathbb{N}$ satisfies the conditions mentioned above. Put $C = A \circ H$. Then $C$ satisfies the conditions (B1), (B2) and (B3). □

By the conditions (B1) and (B2), we inductively obtain:

**Lemma 4.6.** If a surjective map $C : \mathbb{N} \to \mathbb{N}^2 \times \mathbb{Z}$ satisfies the conditions (B1) and (B2), then for every $n \in \mathbb{N}$, and for every $i \in \{1, 2, 3\}$, we have

$$|\pi_i(C(n))| \leq n.$$

We now show that the existence of a metric space containing all proper length space as its asymptotic cones. Such a space is constructed as follows: Let $U$ be a separable homogeneous $\mathcal{S}$-universal metric space, and let $Q$ be a countable dense subset of $X$. For each $(j, k) \in \mathbb{N} \times \mathbb{Z}$, let $J_{(j,k)} = \{F_{i, j, k}\}_{i \in \mathbb{N}}$ be a sequence consisting of all finite subsets of $Q$ satisfying the following for every $i \in \mathbb{N}$:

(C1) $q \in F_{i, j, k}$;
(C2) $2^{-k} \leq \delta(F_{i, j, k}) < 2^{k+1}$;
(C3) $2^{-j} \leq \alpha(F_{i, j, k})/\delta(F_{i, j, k}) < 2^{-j+1}$.

Take a surjective map $C : \mathbb{N} \to \mathbb{N}^2 \times \mathbb{Z}$ stated in Lemma 4.5. For each $i \in \mathbb{N}$, define $G_i = F_{C(i)}$. Put $J = \{G_i\}_{i \in \mathbb{N}}$. Then $J$ is a sequence consisting of all finite subsets of $Q$ containing $q$.

For each $i \in \mathbb{N}$, let $a_i = (\alpha(G_i))^{-1} \cdot 2^{i^2}$. Put

$$X = \{q\} \sqcup \bigsqcup_{i \in \mathbb{N}} (G_i \setminus \{q\}),$$

and define a metric $d_X : X \times X \to [0, \infty)$ by

$$d_X(x, y) =
\begin{cases}
    a_i d_{G_i}(x, y) & \text{if } x, y \in F_i \text{ for some } i \in \mathbb{N};
    a_j d_{G_j}(x, q) + a_j d_{G_j}(q, y) & \text{if } x \in F_i \text{ and } y \in F_j \text{ for some } i \neq j.
\end{cases}$$

The metric space $X$ is a proper countable discrete metric space.

We are going to prove that every pointed proper length space is an asymptotic cone of $X$. To simplify our notation, for $R \in (0, \infty)$, and for $i \in \mathbb{N}$, put $B_i(R) = (a_i)^{-1}B(q, a_iR; X)$. By the definition of $d_X$, the space $B_i(R)$ contains an isometric copy of $B(q, R; G_i)$ containing $p$. We denote by $S_i(R)$ that isometry copy. We also put $T_i(R) = B_i(R) \setminus S_i(R)$. Note that $S_i(R) \subset G_i$.

**Lemma 4.7.** Let $R \in (0, \infty)$. If $i \in \mathbb{N}$ satisfies $2^{i+1}\delta(G_i) > R$, then for every $k > i$ we have $B_i(R) \cap G_k = \emptyset$. 
Proof. For every \( x \in G_k \), by the definition of \( d_X \), we have

\[
d_X(q, x) \geq 2^{k^2} \geq 2^{(i+1)^2}.
\]

By Lemma 4.6, we obtain

\[
2^{(i+1)^2}/a_i = 2^{(i+1)^2-i^2}\alpha(G_i) \geq 2^{2i+1+2-\pi_2(C(i))}\delta(G_i) \geq 2^{i+1}\delta(G_i) > R.
\]

Hence \( a_i R < 2^{(i+1)^2} \). This leads to the conclusion. \( \square \)

By Lemma 4.7 and by the definition of \( T_i(R) \), we obtain:

**Corollary 4.8.** For every \( i \geq 1 \) and for every \( R \in (0, \infty) \), we have

\[
T_i(R) \subset \bigcup_{j=0}^{i-1} G_j.
\]

**Lemma 4.9.** For every \( i \geq 1 \), we have

\[
\alpha(G_i)/\alpha(G_{i-1}) < 16.
\]

**Proof.** By the conditions (B2), (C2) and (C3), we obtain

\[
\alpha(G_i)/\alpha(G_{i-1}) < 2^{-\pi_2(C(i)) + 1 + \pi_3(C(i-1))}\delta(G_i)/\delta(G_{i-1})
\leq 4 \cdot \delta(G_i)/\delta(G_{i-1})
< 4 \cdot 2^{-\pi_3(C(i)) + 1 + \pi_3(C(i-1))} \leq 16.
\]

This proves the lemma. \( \square \)

We conclude the following:

**Lemma 4.10.** Let \( i \in \mathbb{N} \) and \( R \in (0, \infty) \). For all \( x \in T_i(R) \), we have \((a_i)^{-1}d_X(q, x) < 32 \cdot 2^{-i} \). In particular, we have

\[
d_H(B_i(R), S_i(R); B_i(R)) < 32 \cdot 2^{-i}.
\]

**Proof.** By Corollary 4.8, we have \((a_i)^{-1}d_X(p, x) \leq a_{i-1}\delta(G_{i-1})/a_i \). Lemmas 4.9 and 4.10 imply

\[
a_{i-1}\delta(G_{i-1})/a_i = 2^{(i-1)^2-i^2}\delta(G_{i-1})(\alpha(G_i)/\alpha(G_{i-1}))
< 16 \cdot 2^{-2i+1+2-\pi_3(C(i-1)) + 1} \leq 32 \cdot 2^{-i}.
\]

This leads to the former part of the lemma. The later part follows from the former one, \( q \in S_i(R) \) and \( S_i(R) \subset B_i(R) \). \( \square \)

We now prove Theorem 1.6.

**Proof of Theorem 1.6.** We first show that the metric space \( X \) constructed above is a desired space. By Proposition 4.4 we may assume that \( K \subset U \) and \( p = q \). Since \( K \) is proper, we can take a sequence \( \{A_i\}_{i \in \mathbb{N}} \) of finite subsets of \( Q \) satisfying the conditions (A1) and (A2) in Proposition 4.3. By the definition of \( J = \{G_i\}_{i \in \mathbb{N}} \), and by the condition (B3), there exists a strictly increasing map \( \phi : \mathbb{N} \to \mathbb{N} \) such that \( G_{\phi(i)} = A_i \) for every \( i \in \mathbb{N} \).
To prove our statement, we next show that for each \( R \in (0, \infty) \),
the sequence \( \{(B_{\phi(i)}(R), q)\}_{i \in \mathbb{N}} \) converges to \( (B(q, R; K), q) \) in the pointed Gromov–Hausdorff topology. Note that, by the conditions \( (C2) \) and \( (C3) \) and Lemma 4.6, we have \( a_i \to \infty \) as \( i \to \infty \).

Since \( \delta(G_{\phi(i)}) \cdot 2^{\phi(i)+1} \to \infty \) as \( i \to \infty \), we can take \( N \in \mathbb{N} \) such that for every \( i \geq N \), we have \( R < \delta(G_{\phi(i)}) \cdot 2^{\phi(i)+1} \).

By Lemma 4.10, we have

\[
d_H(B_{\phi(i)}(R), S_{\phi(i)}(R)) < 32 \cdot 2^{-\phi(i)} \leq 32 \cdot 2^{-i}.
\]

Since \( S_{\phi(i)}(R) \) is isometric to \( B(q, R; A_i) \), by Proposition 4.3, we conclude that \( \{(B_{\phi(i)}(R), q)\}_{i \in \mathbb{N}} \) converges to \( (B(q, R; K), q) \) in the pointed Gromov–Hausdorff topology. Therefore \( (K, p) \) is an asymptotic cone of \( X \). Thus we conclude Theorem 1.6.

\[ \square \]

Remark 4.1. Let \( X \) be a metric space mentioned in Theorems 1.4, 1.5, 1.6 or Proposition 4.1. By Theorem 1.1 and Proposition 3.3, we obtain \( \dim A(X) = \infty \).

All \((\omega_0 + 1)\)-metric spaces and all countable metric spaces have the topological dimension 0, and have the Hausdorff dimension 0. Thus, Theorems 1.4, 1.5, 1.6 or Proposition 4.1 tells us that analogies of Theorem 1.1 for the topological dimension, the Hausdorff dimension and the conformal Hausdorff dimension are false. More precisely, we have the following:

**Proposition 4.11.** There exists a metric space \( X \) such that for some \( P \in \text{PC}(X) \) we have

\[
\dim_T X < \dim_T P, \quad \dim_H X < \dim_H P, \quad \text{Cdim}_H X < \text{Cdim}_H P,
\]

where \( \dim_T \), \( \dim_H \) and \( \text{Cdim}_H \) stand for the topological dimension, the Hausdorff dimension and the conformal Hausdorff dimension, respectively.

**Remark 4.2.** In [7], Chen and Rossi studied a metric space containing a large class of metric spaces as tangent cones of it. They constructed a compact subset \( X \) of \( \mathbb{R}^N \) with \( \dim_H X = 0 \) that contains all similarity classes of compact subsets of \([0, 1]^N\) as tangent cones at countable dense subset of \( X \) (see [7, Corollary 5.2]). The metric space \( X \) is an example failing an analogy of Theorem 1.1 for the Hausdorff dimension and the conformal Hausdorff dimension.

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