Splitting multidimensional necklaces and measurable colorings of Euclidean spaces

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Abstract

A necklace splitting theorem of Goldberg and West [5] asserts that any \( k \)-colored (continuous) necklace can be fairly split using at most \( k \) cuts. Motivated by the problem of Erdős on strongly nonrepetitive sequences, Alon et al. [3] proved that there is a \((t+3)\)-coloring of the real line in which no necklace has a fair splitting using at most \( t \) cuts. We generalize this result for higher dimensional spaces. More specifically, we prove that there is \( k \)-coloring of \( \mathbb{R}^d \) such that no cube has a fair splitting of size \( t \) (using at most \( t \) hyperplanes orthogonal to each of the axes), provided \( k \geq (t+4)^d - (t+3)^d + (t+2)^d - 2^d + d(t+2) + 3 \). We also consider a discrete variant of the multidimensional necklace splitting problem in the spirit of the theorem of de Longueville and Živaljević [7]. The question how many axes aligned hyperplanes are needed for a fair splitting of a \( d \)-dimensional \( k \)-colored cube remains open.

1 Introduction

In this paper we investigate some questions connected to the necklace splitting problem. Let \( c : \mathbb{R} \to \{1,2,\ldots,k\} \) be a \( k \)-coloring of the real line. We assume that \( c \) is a measurable coloring, that is, the set \( c^{-1}(i) \) of all points in color \( i \) is Lebesgue measurable for every \( i \in \{1,2,\ldots,k\} \). A splitting of size \( t \) of an interval \([a,b]\) is a sequence of points \( a = y_0 < y_1 < \ldots < y_t < y_{t+1} = b \). A splitting is said to be fair if it is possible to partition the resulting collection of intervals \( F = \{[y_i, y_{i+1}] : 0 \leq i \leq t\} \) into two disjoint subcollections \( F_1 \) and \( F_2 \), each capturing exactly half of the total measure of every color. The partition \( F = F_1 \cup F_2 \) will be called a fair partition of \( F \).

Goldberg and West [5] proved that every \( k \)-colored interval has a splitting of size at most \( k \) (see also [2] for a short proof using the Borsuk-Ulam theorem, and [8] for other applications of the Borsuk-Ulam theorem in combinatorics). This result is clearly the best possible, as can be seen in a necklace where colors occupy consecutively full intervals.

In [3] we considered colorings of \( \mathbb{R} \) such that no interval has a splitting of bounded size.

**Theorem 1** (Alon et al. [3]) For every \( t \geq 1 \) there is a \((t+3)\)-coloring of the real line such that no interval has a fair splitting of size at most \( t \).
For $t = 1$ the result asserts that there is a 4-coloring of the real line avoiding (continuous) abelian squares (adjacent intervals with equal measure of every color). The question whether a similar property holds for the integers was posed in 1961 by Erdős [4], and solved in the affirmative by Keränen [6] in 1991. Curiously the number of colors is the same in both versions, though in continuous variant it is not known whether it is optimal.

In this paper we extend the above result in the spirit of the theorem of de Longueville and Živaljević [7]. Let $d$ be a fixed positive integer, and let $c$ be a measurable coloring of $\mathbb{R}^d$. A cube in $\mathbb{R}^d$ is just a Cartesian product of $d$ non-empty intervals (of the same length) lying on distinct coordinate axes. A splitting of a cube is specified by a family of axes-aligned hyperplanes. A splitting of a colored cube is fair if there is a partition of the resulting family of cuboids into two families, each capturing exactly half of the total measure of every color.

**Theorem 2** (de Longueville and Živaljević [7]) Every $k$-colored $d$-dimensional cube has a fair splitting using at most $k$ axes aligned hyperplane cuts. Moreover, one may specify the number of cuts in each direction arbitrarily.

We are interested in colorings avoiding cubes admitting a fair splitting with a bounded number of cuts. The size of the splitting is the maximum number of axes aligned hyperplanes in the same direction. Our main result reads as follows.

**Theorem 3** For every pair of integers $t, d \geq 1$, and $k \geq (t + 4)^d - (t + 3)^d + (t + 2)^d - 2^d + d(t + 2) + 3$, there is a $k$-coloring of $\mathbb{R}^d$ such that no cube has a fair splitting of size at most $t$.

The proof uses Baire category argument applied to the space of all measurable colorings of $\mathbb{R}^d$. The lower bound on the number of colors in the above theorem is almost surely not optimal. In the final section we discuss several open problems and further directions. The most intriguing seems a discrete version of the multidimensional necklace splitting problem: what is the least number of axes aligned hyperplanes needed to a fair splitting of a discrete $k$-colored cuboid in $\mathbb{R}^d$?

### 2 Proof of the main result

Recall that a set in a metric space is nowhere dense if the interior of its closure is empty. A set is said to be of first category if it can be represented as a countable union of nowhere dense sets. In the proof of theorem 3 we apply the Baire category theorem (see [9]).

**Theorem 4** (Baire Category Theorem) If $X$ is a complete metric space and $A$ is a set of first category in $X$, then $X \setminus A$ is dense in $X$ (and in particular is nonempty).

Our plan is to follow a similar reasoning to that of [3]. We will construct a suitable metric space of colorings of $\mathbb{R}^d$, and then we will demonstrate that the subset of “bad colorings” is of first category.
2.1 The setting

Let \( k \) be a fixed positive integer and let \( \{1, 2, \ldots, k\} \) be the set of colors. Let \( f \) and \( g \) be two measurable colorings of \( \mathbb{R}^d \). For a positive integer \( n \) we set

\[
D_n(f, g) = \{ x \in [-n, n]^d : f(x) \neq g(x) \}.
\]

Clearly \( D_n(f, g) \) is Lebesgue measurable so we may define the normalized distance between \( f \) and \( g \) on \([-n, n]^d\) by

\[
d_n(f, g) = \frac{\lambda(D_n(f, g))}{n^d},
\]

where \( \lambda \) is the \( d \)-dimensional Lebesgue measure. Since \( d_n(f, g) \) is bounded from above by \( 2^d \), we may define the distance between two measurable colorings \( f \) and \( g \) by

\[
d(f, g) = \sum_{n=1}^{\infty} \frac{d_n(f, g)}{2^{n+1}}.
\]

Identifying colorings whose distance is zero gives a metric space \( \mathcal{M} \) of equivalence classes of all measurable \( k \)-colorings. Note that the splitting properties are preserved by equivalent colorings.

**Lemma 5** The space \( \mathcal{M} \) is a complete metric.

We omit the proof of this lemma since this is a simple generalization of a result stating that sets of finite measure in any metric space form a complete metric space with symmetric difference as the distance function (see [3], [9]).

Let \( t \geq 1 \) be a fixed integer. Let \( D_t \) be a subspace of \( \mathcal{M} \) consisting of those \( k \)-colorings that avoid intervals having a \( d \)-dimensional fair splitting of size at most \( t \) in each dimension. Denote for future convenience

\[
f(d, t) = (t + 4)^d - (t + 3)^d + (t + 2)^d - 2^d + d(t + 2) + 3.
\]

We will show that \( D_t \) is not empty provided that \( k \geq f(d, t) \). By granularity of a splitting we mean the length of the shortest subinterval \([z_j^i, z_{j+1}^i]\) in the splitting. For \( n \geq 1 \) and \( r_1, \ldots, r_n \), let \( B_n^{(r_i)} \) be the set of those colorings from \( \mathcal{M} \) for which there exists at least one \( d \)-dimensional cube in \([-n, n]^d\) having a \( d \)-dimensional fair splitting of size exactly \( r_i \) in the \( i \)-th dimension for each \( i \) and granularity at least \( 1/n \). Finally let us denote all the bad colorings by

\[
B_n(t) = \bigcup_{r_i \leq t} B_n^{(r_i)}.
\]

Clearly we have

\[
D_t = \mathcal{M} \setminus \bigcup_{n=1}^{\infty} B_n(t).
\]

Now our aim is to apply Baire category theorem to show that the sets \( B_n(t) \) are nowhere dense, provided that \( k \geq (t + 4)^d - (t + 3)^d + (t + 2)^d - 2^d + d(t + 2) + 3 \).
2.2 The sets $B_n(t)$

We show that each set $B_n^{(r_i)}$ is a closed subset of $\mathcal{M}$. Since $B_n(t)$ is a finite union of these sets, it must be closed too.

**Theorem 6** The set $B_n^{(r_i)}$ is a closed subset of $\mathcal{M}$ for every $r_i \geq 1$ and $n \geq 1$.

**Proof.** Let $\{f_m\}$ be a sequence of colorings converging in $\mathcal{M}$ to $f$. For each $m$ let $C_m$ denote a $d$-dimensional cube in $[-n, n]^d$ of granularity $\geq 1/n$ and having a fair splitting into exactly $r_i$ points in the $i$-th dimension. Let us denote by $\phi_m$: $[r_1] \times \ldots \times [r_d] \to \{1, 2\}$ the labeling function defining the two families from the fair splitting of $C_m$. Since $[-n, n]^d$ is compact we may assume that vertices of the sliced cube $C_m$ converge to vertices of some cube $C$ and since there is finite number of labeling functions we may assume that $\phi_m = \phi$ for every $m$. Now it is easy to see that $\phi$ gives a fair splitting for $C$. □

Next we prove that each $B_n(t)$ has empty interior provided the number of colors $k$ satisfies $k > (t + 4)^d - (t + 3)^d + (t + 2)^d - 2^d + d(t + 2) + 2$. For this purpose let us call $f \in \mathcal{M}$ a cube coloring on $[-n, n]^d$ if there is a partition of $[-n, n]^d$ into some number of (half open) $d$-dimensional cubes of equal size in each dimension, each filled with only one color. Let $I_n$ denote the set of all colorings from $\mathcal{M}$ that are cube colorings on $[-n, n]^d$.

**Lemma 7** Let $f \in \mathcal{M}$ be a $k$-coloring. Then for every $\epsilon > 0$ and $n \in \mathbb{N}$ there exists a coloring $g \in I_n$ such that $d(f, g) < \epsilon$.

**Proof.** Let $C_i = f^{-1}(i) \cap [-n, n]^d$ and let $C_i^* \subseteq [-n, n]^d$ be a finite union of intervals such that

$$\lambda((C_i^* \setminus C_i) \cup (C_i \setminus C_i^*)) < \frac{\epsilon}{2k^2}$$

for each $i = 1, 2, \ldots, k$. Define coloring $h$ so that for each $i = 1, 2, \ldots, k$ the set $C_i^* \setminus (C_i^* \cup \ldots \cup C_{i-1}^*)$ is filled with color $i$, the rest of the cube $[-n, n]^d$ is filled with any of these colors. Moreover we set $h$ to be equal $f$ outside $[-n, n]^d$. Note that $d(f, h) < \epsilon/2$ and $h^{-1}(i) \cap [-n, n]^d$ is a finite union of cubes. Let $A_1, A_2, \ldots, A_N$ be the whole family of these cubes. Now split the cube $[-n, n]^d$ into $M^d$ cubes $B_1, \ldots B_{M^d}$ equally spaced in $[-n, n]^d$. We define $g$ to be equal $h$ on $A_i$ whenever $A_i \subseteq B_j$ for some $j$ and $g(A_i)$ is of any color otherwise. Note that $g$ differs from $h$ on a set of $d$-dimensional measure at most $t((2n + 4n/M)^d - (2n)^d)$ so that for sufficiently large $M$ $d(g, h) < \epsilon/2$ and we get $d(f, g) < \epsilon$. □

In order to state the next lemma we will use the following notation:

$$D(d) = \sum_{i=1}^{d} \binom{d}{i} (t + 2)^i (2^{d-i} - 1) = (t + 4)^d + 1 - (t + 3)^d - 2^d.$$

**Lemma 8** If $k > (t + 2)^d + d(t + 2) + 1 + D(d)$ then each $B_n(t)$ has empty interior.

**Proof.** Let $f \in B_n(t)$ be any bad coloring. Let $U(f, \epsilon)$ be the open $\epsilon$-neighborhood of $f$ in the space $\mathcal{M}$. Assume the assertion of the lemma is false: there is some $\epsilon > 0$ for
witch $U(f, \epsilon) \subseteq B_n(t)$. By Lemma 7 there is a coloring $g \in I_n$ such that $d(f, g) < \epsilon/2$, so that $U(g, \epsilon/2) \subseteq B_n(t)$. The idea is to modify slightly the cube coloring $g$ so that the new coloring will still be close to $g$, but there will be no cube in $[-n, n]^d$ possessing a fair splitting of size at most $t$ and granularity at least $1/n$. Without loss of generality we may assume that there are equally spaced cubes $C_{i_1, \ldots, i_d}$ for $i_1, \ldots, i_d \in \{1, 2, 3, \ldots, N\}$ in $[-n, n]^d$ such that $1 > 6n^2/N$ each cube is filled with a unique color in the cube coloring $g$. Let $\delta > 0$ be a real number satisfying

$$\delta < \min \left\{ \sqrt[4]{\frac{\epsilon}{2n}}, \frac{2n}{N^2} \right\}.$$ 

Choose a color (which we will call from now on "white"). Let $W'_{i_1, \ldots, i_d}$ where $i_1, \ldots, i_d \in \{1, 2, \ldots, N\}$ be a cube $[0, 2\delta]^d$ colored as follows: choose a countable set

$$\{m_{i_1, \ldots, i_d}\}_{j=1, \ldots, k; i_1, \ldots, i_d \in \{1, 2, \ldots, N\}}$$

of real numbers linearly independent over $\mathbb{Q}$ such that $0 < m_{i_1, \ldots, i_d}^j < (\delta/k)^d$. We color $W'_{i_1, \ldots, i_d}$ white except for small cubes

$$V'_{i_1, \ldots, i_d} = \left( \frac{2n - 1}{k} \delta, \ldots, \frac{2n - 1}{k} \delta \right) + \prod_{j=1}^{d} \left[ -\sqrt[k]{m_{i_1, \ldots, i_d}^j}, \sqrt[k]{m_{i_1, \ldots, i_d}^j} \right]$$

colored using color $\eta$ for $\eta = 1, 2, \ldots, k$. Note that the $d$-dimensional Lebesgue measure of $V'_{i_1, \ldots, i_d}$ is equal $2^d m_{i_1, \ldots, i_d}^d$. Hence measures of these cubes are linearly independent over $\mathbb{Q}$.

Now modify the coloring $g$ to get a coloring $h$ outside $B_n(t)$. The coloring $h$ is equal to $g$ outside $[-n, n]^d$. Inside $C_{i_1, \ldots, i_d}$ the coloring $h$ is equal $g$ except in

$$W_{i_1, \ldots, i_d} = \left( \left( i_1 - \frac{1}{2} - \delta \right) \frac{2n}{N} - n, \ldots, \left( i_d - \frac{1}{2} - \delta \right) \frac{2n}{N} - n \right) + W'_{i_1, \ldots, i_d}$$

where $h$ is defined by the coloring of $W'_{i_1, \ldots, i_d}$.

Note that $d(g, h) < \epsilon/2$ so that there exists a $d$-dimensional cube $C$ in $[-n, n]^d$ with granularity at least $1/n$ such that there is a fair splitting of size at least $t$. The fair splitting divides $C$ into at most $(t+1)^d$ cubes hence we obtain a $d$-dimensional cell complex in $[-n, n]^d$ (which we will also denote by $C$). Let us denote by $A$ the measure of $C_{i_1, \ldots, i_d} \setminus W_{i_1, \ldots, i_d}$ (note that $A$ does not depend on the set of indexes chosen and we may assume it is linearly independent with the $m_{i_1, \ldots, i_d}^j$ chosen before).

By the determinant of $C_{i_1, \ldots, i_d}$ in $C$ (denoted by det$_C C_{i_1, \ldots, i_d}$) we mean the lowest dimension of cells that intersect $C_{i_1, \ldots, i_d}$ (there is only one cell reaching the minimum – denoted by det$_C(C_{i_1, \ldots, i_d})$). If $C_{i_1, \ldots, i_d}$ lays outside $C$ we set det$_C C_{i_1, \ldots, i_d} = d$. Note that cells of $C$ divide each cube $C_{i_1, \ldots, i_d}$ into $2^{\text{codim} \text{det}_C C_{i_1, \ldots, i_d}}$ cubes of measures

$$\alpha_1(d_C(C_{i_1, \ldots, i_d})), \alpha_2(d_C(C_{i_1, \ldots, i_d})), \ldots, \alpha_{2^{\text{codim} \text{det}_C C_{i_1, \ldots, i_d}}}(d_C(C_{i_1, \ldots, i_d}))$$

and their sum is equal to $A$. In fact (up to indexing) $\alpha_i(d_C(C_{i_1, \ldots, i_d}))$ does not depend on $d_C(C_{i_1, \ldots, i_d})$ but on the det$_C(C_{i_1, \ldots, i_d})$-dimensional subspace of $\mathbb{R}^d$ spanned by it. The
subspace can be identified by a suitable choice of \( \text{codim}(\det C_{i_1,\ldots,i_d}) \) slices (or ends) of \( C \) on some of the dimensions. Hence we get that \( \alpha_i(d_{C(C_{i_1,\ldots,i_d})}) = \alpha_i(t_1,\ldots,t_s) \) for \( t_1,\ldots,t_s \in \{0,1,2,\ldots,t+1\} \) and \( s = 0,1,2,\ldots,d \). Of course \( \alpha_1(\emptyset) = A \).

Note that the dimension of the space spanned by \( \alpha_i(t_1,\ldots,t_s) \) where \( s > 0 \) is no greater than \( D(d) \). Now note that all the vertices of \( C \) are colored at most by \( (t+2)^d \) colors. Moreover cells of dimensions \( d-1 \) of \( C \) intersect at most one of the cubes \( V_{i_1,\ldots,i_d}^n \subseteq C_{i_1,\ldots,i_d} \) and two such cell intersect the cubes of the same color if they span the same subspace of \( \mathbb{R}^d \). Since there are at most \( d(t+2) \) different subspaces of \( \mathbb{R}^d \) obtained in such a way then if \( C \) intersects one of \( V_{i_1,\ldots,i_d}^n \subseteq C_{i_1,\ldots,i_d} \) then \( d-1 \) of \( C \) also does and it has one of \( d(t+2) \) colors.

Summing up, let us consider a color \( c \) different from white and the \( (t+2)^d + d(t+2) \) colors mentioned before. Since our splitting is fair, \( d \)-dimensional cells colored partially by \( c \) can be divided into two families having equal measure of \( c \). Hence the measure satisfies equality of the form:

\[
T(0)A + \sum \epsilon(0)^j_{i_1,\ldots,i_d} 2^d m_j^{i_1,\ldots,i_d} + \sum S(0)^j_{i_1,\ldots,i_d} \alpha_i(t_1,\ldots,t_s) - T(0)A - \sum \epsilon(0)^j_{i_1,\ldots,i_d} 2^d m_j^{i_1,\ldots,i_d} - \sum S(0)^j_{i_1,\ldots,i_d} \alpha_i(t_1,\ldots,t_s) = 0
\]

where \( T(0), T(1) \in \mathbb{N}, \epsilon(0)^j_{i_1,\ldots,i_d}, \epsilon(1)^j_{i_1,\ldots,i_d} \in \{0,1\}, S(0), S(1) \in \mathbb{N} \), and not all \( \epsilon(0)^j_{i_1,\ldots,i_d} \) are equal to 0. Note that for each color the numbers

\[
T(0)A + \sum \epsilon(0)^j_{i_1,\ldots,i_d} 2^d m_j^{i_1,\ldots,i_d} - T(0)A - \sum \epsilon(0)^j_{i_1,\ldots,i_d} 2^d m_j^{i_1,\ldots,i_d}
\]

are independent over \( \mathbb{Q} \). On the other hand, they can be generated over \( \mathbb{Q} \) by \( \alpha_i(t_1,\ldots,t_s) \) so they lie in \( D(d) \)-dimensional space. Since the number of remaining colors is greater than \( D(d) \), we get a contradiction that ends the proof.

3 Open problems

Let \( f(t,d) \) denote the minimum number of colors needed for a coloring of \( \mathbb{R}^d \) such that no cube has a fair splitting of size at most \( t \). Our main result asserts that \( f(t,d) \leq (t+4)^d - (t+3)^d + (t+2)^d - 2^d + d(t+2) + 3 \). We expect naturally that this bound is far from optimal, as even for \( t = d = 1 \) it gives worst result than that obtained in [3].

Problem 9 Is it true that \( f(t,d) \leq t + O(d) \)?

Turning into discrete case we get the following generalizations of the problem of Erdős. Let \( g(t,d) \) denote the least number of colors needed for a coloring of \( \mathbb{Z}^d \) such that no cube has a fair splitting using at most \( t \) axes aligned cuts in total. So, by the result of Keränen we know that \( g(1,1) = 4 \). Curiously we do not even know if \( g(t,1) \) is finite for every \( t \geq 2 \).

Problem 10 Determine \( g(1,2) \) and \( g(2,1) \).
Finally let us formulate a natural discrete version of multidimensional necklace splitting problem in the spirit of the theorem of de Longueville and Živaljević. By a \textit{d-dimensional necklace} we mean a discrete cube in $\mathbb{Z}^d$, that is a $d$-fold Cartesian product of the set \{1, 2, \ldots, n\} with itself. We assume that the necklace is colored so that each color appears an even number of times. As before, in the fair splitting problem we allow only axes aligned cuts.

\textbf{Problem 11} What is the least number of axes aligned cuts needed for a fair splitting of $k$-colored $d$-dimensional necklace?

Let $h(k, d)$ denote the number we asked for in the problem. It is not hard to see that $h(k, d) \leq (2d - 1)k$, as noticed by Lasoń (personal communication). For instance, if $d = 2$ we may string the necklace as shown in Fig. 1, and then apply one dimensional theorem. We get at most $k$ places to cut the stringed necklace. However, to separate the resulting pieces accordingly we need to use zigzags consisting of three line segments—one horizontal, two vertical. Therefore for each cutting place of the string we may need three orthogonal plane cuts. This proves the bound $h(k, 2) \leq 3k$. The argument for higher dimensions is analogous.

Fig. 1

The following construction due to Petecki (personal communication) shows that the upper bound for $h(k, 2)$ is close to the truth. Consider the set of red points depicted in Fig. 2. It can be checked that any fair splitting of this set must use at least three lines. So, taking $k - 1$ copies of this set, each in different color, and lying far one from another (so that there is no
vertical or horizontal line crossing any two of the copies) one gets that $h(k, 2) \geq 3(k - 1) + 1$.

Fig. 2

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