Exponentially stable solution of mathematical model based on graph theory of agents dynamics on time scales

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Abstract

In this paper an emergence of leader-following model based on graph theory on the arbitrary time scales is investigated. It means that the step size is not necessarily constant but it is a function of time. We propose and prove conditions ensuring a leader-following consensus for any time scales using Grönewall inequality. The presented results are illustrated by examples.

Keywords Time scales, graph theory, leader-following problem, Grönewall inequality, multi-agent systems.

AMS Subject classification 34N05, 34D20, 93C10.

1 Introduction

This paper studies consensus problems of multi-agent systems over an undirected network topology on time scales. Based on the theory of time scales, we discuss continuous-time and discrete-time consensus protocols as well as consensus on time scales consisting of the both kinds of points: right-dense and right-scattered simultaneously. We find that consensus can be realized exponentially if the graininess function of the time scale is bounded. Presented here results cover also the case when graininess function approaches zero. Some existing results of discrete-time consensus are special cases of results presented in this paper.

Investigation of the leader-following problem dates back to the 1970s. In 1974 [1], DeGroot considered explicitly described process leading to the consensus. In 2000, Krause [2, 3] proposed the model of a group of agents who have to make a joint assessment of a certain magnitude. Each of the agents has his own opinion but is open to some extent to revise it when being informed about the opinions of all the other agents. Coordination of groups of mobile autonomous agents using nearest neighbor rules was investigated by Jadbabaie et al. in [4].

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In [3,6], Blondel et al. studied Krause’s model with state-dependent connectivity. Girejko et al. studied Krause’s model of opinion dynamics on discrete time scales [7,8]. In 2007, Cucker and Smale [9,10] published two papers devoted to an emergent behavior in flocks. Cucker-Smale model on isolated time scales is studied by Girejko et al. in [8]. In 2015, Wang et al. [11] studied the leader-following consensus of discrete time linear multi-agent systems with communication noises. Recently, in 2018, Girejko, Machado, Malinowska, and Martins, have published some results for consensus in the Cucker-Smale type model on discrete time scales [12]. Presented here results generalize and improve results obtained by the authors in [13] and [14]. In [14] consensus on different types of discrete time scales is considered under assumption that feedback control gain $\gamma$ is constant.

2 Basis of time scales calculus

A time scale is a model of time [15, 16, 17], where the step size is a function of time. From mathematical point of view it is an arbitrary nonempty closed subset $\mathbb{T}$ of the set $\mathbb{R}$ of real numbers.

The mapping $\sigma: \mathbb{T} \to \mathbb{T}$, defined by $\sigma(t) = \inf \{ s \in \mathbb{T} : s > t \}$ with $\inf \emptyset = \sup \mathbb{T}$, is called the forward jump operator. Similarly, we define the backward jump operator $\rho: \mathbb{T} \to \mathbb{T}$ by $\rho(t) = \sup \{ s \in \mathbb{T} : s < t \}$ with $\sup \emptyset = \inf \mathbb{T}$. The following classification of points is used within the theory: a point $t \in \mathbb{T}$ is called right-dense, right-scattered, left-dense and left-scattered if $\sigma(t) = t$ (for $t < \sup \mathbb{T}$), $\sigma(t) > t$, $\rho(t) = t$ (for $t > \inf \mathbb{T}$) and $\rho(t) < t$, respectively. We say that $t$ is isolated if $\rho(t) < t < \sigma(t)$, and that $t$ is dense if $\rho(t) = t = \sigma(t)$. The function $\mu: \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$ and called the graininess function. The delta (or Hilger) derivative of $f: \mathbb{T} \to \mathbb{R}$ at a point $t \in \mathbb{T}^\kappa$, where

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < \infty, \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \infty \end{cases},$$

is defined in the following way.

**Definition 1 ([16]).** The delta derivative $f^\Delta(t)$ is the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood $U$ of $t$ (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|(f(\sigma(t)) - f(s)) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s| \quad \text{for all } s \in U.$$

The following definitions will be used in the sequel, too.

**Definition 2 ([16]).** A function $f: \mathbb{T} \to \mathbb{R}$ is called regulated provided its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. A function $f: \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-side limits exist (finite) at left-dense points in $\mathbb{T}$.
Definition 3 ([16]). Assume $f: \mathbb{T} \to \mathbb{R}$ is a regulated function. We define the indefinite integral of regulated function $f$ by $\int_a^b f(t) \Delta t = F(b) - F(a)$ for all $a,b \in \mathbb{T}$.

Definition 4 ([16]). We say that a function $p: \mathbb{T} \to \mathbb{R}$ is regressive provided $1 + \mu(t)p(t) \neq 0$ holds for all $t \in \mathbb{T}^\kappa$. The set of all regressive and rd-continuous functions $p: \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$. The set of all positively regressive elements of $\mathcal{R}$, is defined as $\mathcal{R}^+$: $= \{ p \in \mathcal{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T} \}$.

Definition 5 ([16]). An $N \times N$-matrix-valued function $P$ on a time scale $\mathbb{T}$ is called regressive (with respect to $\mathbb{T}$) provided $I + \mu(t)P(t)$ is invertible for all $t \in \mathbb{T}^\kappa$, where by $I$ we denote the $N \times N$ identity matrix.

Similarly to the scalar case, the class of all regressive and rd-continuous matrix-valued functions is denoted by $\mathcal{R}$. Notice that, constant $N \times N$ matrix $P$ is regressive iff the eigenvalues $\lambda_i$ of $P$ are regressive for all $1 \leq i \leq N$.

The Grönwall inequality is used in the proof of the main result.

Lemma 1 ([16]). Let $z$ be rd-continuous, $p \in \mathcal{R}^+$ and $p(t) \geq 0$ for $t \in \mathbb{T}$ and $c \in \mathbb{R}$. Then

$$z(t) \leq c + \int_{T_0}^t p(\tau)z(\tau)\Delta \tau \text{ for all } t \in \mathbb{T}$$

implies

$$z(t) \leq c e_p(t,T_0).$$

Here $z(t) = e_p(t,T_0)$, $T_0 \in \mathbb{T}$, is a solution of the initial value problem

$$z(\Delta)(t) = p(t)z(t), \quad z(T_0) = 1 \text{ on } \mathbb{T}. \quad (1)$$

Through this paper, assume that

$$\inf \mathbb{T} = T_0 \geq 0 \text{ and } \sup \mathbb{T} = \infty.$$ 

It implies that $\mathbb{T}^\kappa = \mathbb{T}$.

3 Mathematical model of agents dynamics

We consider a discrete time multi-agent system consisting of $N$ agents and the leader. The dynamics of each agent labeled $i$, $i = 1, 2, \ldots, N$, is given by the following equation

$$x_i^\Delta(t) = f(t,x_i(t)) + \gamma(t) \sum_{j=1}^N a_{ij}(x_j(t) - x_i(t)) + \gamma(t)d_i(x_0(t) - x_i(t)), \quad (2)$$
where \( t \in \mathbb{T}, x_i : \mathbb{T} \to \mathbb{R} \) and \( \gamma : \mathbb{T} \to \mathbb{R} \) represent the state and the feedback control gain at time \( t \), respectively. Here \( a_{ij} \in \mathbb{R}, \, d_i \in \mathbb{R}, \, i, j = 1, 2, \ldots, N, \) and \( D := \text{diag}[d_1, d_2, \ldots, d_N] \) is a diagonal matrix. Throughout this paper, we assume \( a_{ij} = a_{ji} \). It means matrix \( A = [a_{ij}]_{N \times N} \) is a symmetric matrix. Function \( f : \mathbb{T} \times \mathbb{R} \to \mathbb{R} \) describes nonlinear dynamics. The leader, labeled as \( i = 0 \), for multi-agent system (2) is an isolated agent with trajectory described by

\[
x_0^\Delta(t) = f(t, x_0(t)), \quad t \in \mathbb{T}. \tag{3}
\]

Notice that the control law \( \gamma(t) \sum_{j=1}^{N} a_{ij} (x_j(t) - x_i(t)) + \gamma(t) d_i (x_0(t) - x_i(t)) \) for \( i \)-th agent used in system (2) was studied by many authors including Yu, Jiang and Hu in [18].

Let us denote by \( \varepsilon_i(t) = x_i(t) - x_0(t) \) the distance between the leader and the \( i \)-th agent. From (2)-(3) we obtain

\[
\varepsilon_i^\Delta(t) = f(t, x_i(t)) - f(t, x_0(t)) + \gamma(t) \sum_{j=1}^{N} a_{ij} (\varepsilon_j(t) - \varepsilon_i(t)) - \gamma(t) d_i \varepsilon_i(t)
\]

for \( i = 1, 2, \ldots, N \). Setting

\[
\varepsilon(t) = (\varepsilon_1(t), \varepsilon_2(t), \ldots, \varepsilon_N(t))^T,
\]

\[
x(t) = (x_1(t), x_2(t), \ldots, x_N(t))^T
\]

and

\[
F(t, x(t)) = \left( f(t, x_1(t)), f(t, x_2(t)), \ldots, f(t, x_N(t)) \right)^T,
\]

\[
F(t, x_0(t) \mathds{1}) = \left( f(t, x_0(t)), f(t, x_0(t)), \ldots, f(t, x_0(t)) \right)^T,
\]

system (2)-(3) takes the following form

\[
\varepsilon^\Delta(t) = F(t, x(t)) - F(t, x_0(t) \mathds{1}) - \gamma(t) B \varepsilon(t), \quad \varepsilon(T_0) = \varepsilon_{T_0}, \tag{5}
\]

(for details see [19]). Here \( \mathds{1} \) is the vector \([1, \ldots, 1]^T\). We remind \( B = L + D \) is the symmetric matrix since \( A \) is a symmetric matrix where by \( L \) we mean the Laplacian matrix \( L = [l_{ij}] \) with \( l_{ii} = \sum_{j \neq i} a_{ij} \) and \( l_{ij} = -a_{ij}, i, j = 1, \ldots, N, i \neq j \).

If

\[
(-\gamma(t) B) \in \mathbb{R},
\]

in equation (4), then by \( e^{-\gamma B}(t, T_0) \) we denote a solution of initial value problem

\[
\varepsilon^\Delta(t) = -\gamma(t) B \varepsilon(t), \quad \varepsilon(T_0) = \mathds{1}.
\]

By variation of constants (see [16]), the unique solution of equation (4) is given by

\[
\varepsilon(t) = e^{-\gamma B}(t, T_0) \varepsilon_{T_0} + \int_{T_0}^{t} e^{-\gamma B}(t, \sigma(\tau)) \left( F(\tau, x_0(\tau) \mathds{1}) - F(\tau, x(\tau)) \right) \Delta \tau. \tag{6}
\]
Definition 6. Function $F: \mathbb{T} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ fulfills Lipschitz condition with respect to the second variable if there exists a positive constant $L$ such that
\[ \| F(t, x) - F(t, y) \| \leq L \| x - y \|, \quad t \in \mathbb{T}. \] (7)

Definition 7. We say that equation (5), where $T_0 \geq 0$, $\varepsilon_{T_0} \in \mathbb{R}^N$, is exponentially stable if there exist a positive constants $c$ and $d$ such that for any rd-continuous solution $\varepsilon(t, T_0, \varepsilon_{T_0})$ of equation (5) holds
\[ \lim_{t \to \infty} \| \varepsilon(t, T_0, \varepsilon_{T_0}) \| =: \lim_{t \to \infty} \| \varepsilon(t) \| \leq c \| \varepsilon_{T_0} \| \lim_{t \to \infty} e_d(t, T_0) = 0. \]

For some relevant result for exponential stability in the discrete case see [20] and [21].

Definition 8. The multi-agent system (2)-(3) is said to be achieved the leader-following consensus exponentially if equation (5) is exponentially stable.

In 2005 [22], Peterson and Raffoul investigated the exponential stability of the zero solution to systems of dynamic equations on time scales. The authors defined suitable Lyapunov-type functions and then formulated appropriate inequalities on these functions that guarantee that the zero solution decay to zero exponentially. For the growth of generalized exponential functions on time scales see Bodine and Lutz [23].

4 Main results

Assume that function $F: \mathbb{T} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ defined by (1) satisfies Lipschitz condition with respect to the second variable.

Let $\lambda_i$, $i = 1, 2, \ldots, N$ denote the eigenvalues of matrix $B$. By $\mathbb{T}^s$ and $\mathbb{T}^d$ we mean the set of right-scattered and right-dense points of $\mathbb{T}$, respectively. Notice that, since we assumed $\sup \mathbb{T} = \infty$, at least one of sets $\mathbb{T}^s$ or $\mathbb{T}^d$ must be unbounded.

Next, we rewrite time scale $\mathbb{T}$ in the useful way for estimation of norm of solution of initial value problem (5) on a time scale consisting of right-scattered as well as right-dense points. To avoid confusion we underline that any interval throughout this paper is an interval on the time scale, i.e. any interval contains only points of the time scale. Set
\[ T_1 = \min \{ t: t \in [T_0, \infty) \cap \mathbb{T}^d \text{ and } [T_0, t] \subset \mathbb{T}^s \} \]
\[ T_{2i} = \inf \{ t: t \in [T_{2i-1}, \infty) \cap \mathbb{T}^s \text{ and } [T_{2i-1}, t] \subset \mathbb{T}^d \} \]
\[ T_{2i+1} = \min \{ t: t \in [T_{2i}, \infty) \cap \mathbb{T}^d \text{ and } [T_{2i}, t] \subset \mathbb{T}^s \} \]
for $i = 1, 2, \ldots$. In case of $[T_{2i-1}, \infty) \cap \mathbb{T}^s = \emptyset$ for some $i \in \mathbb{N}$ we take $T_{2i} = \infty$ (see Example 5). If $[T_{2i-1}, t) \cap \mathbb{T}^d = [T_{2i-1}, T_{2i-1}) = \emptyset$ for some $i \in \mathbb{N}$ we also take $T_{2i} = \infty$ (see Example 6). Analogously, if $[T_{2i}, \infty) \cap \mathbb{T}^d = \emptyset$ for some $i \in \mathbb{N}$, then $T_{2i+1} = \infty$. If $T_j = \infty$ for some $j \in \mathbb{N}_0$, then we take $T_{j+i} = T_j$ for $i \in \mathbb{N}$ and $[T_{j+i}, T_{j+i+1}] = (T_{j+i}, T_{j+i+1}) = \emptyset$ (see Example 7). We see if $\sigma(T_0) = T_0$, then $T_1 = T_0$.
Example 1. Let \( T = \{1\} \cup [2, 3] \cup [6, \infty) \). Here \( T_0 = 1, T_1 = 2, T_2 = 3, T_3 = 6 \) and \( T_4 = \infty \).

We underline that \( T_{2i+1} \in T^d \) for any \( i \in \mathbb{N}_0 \) while it is possible \( T_{2i} \notin T^s \) for some \( i \in \mathbb{N}_0 \).

Example 2. Let \( T = \bigcup_{i=1}^{\infty} [2i - 1, 2i] \cup \{4i + \frac{1}{j+1} : i, j \in \mathbb{N}\} \). Here \( T_0 = 1, T_1 = T_0, T_i = i \) for \( i \in \{2, 3, \ldots\} \). We see \( T_1 = T_0 \in T^d, T_{2i+1} \in T^d, T_{4i} \in T^d, T_{4i+2} \in T^s \) for \( i \in \mathbb{N} \).

Example 3. Let \( T = \bigcup_{i=1}^{\infty} [2i - 1, 2i] \cup \{4i + 1 - \frac{1}{j+1} : i, j \in \mathbb{N}\} \). Here \( T_0 = 1, T_i = i \) for \( i \in \mathbb{N} \) and \( T_{2i-1} \in T^d \) and \( T_{2i} \in T^s \) for \( i \in \mathbb{N} \).

We can write \( T = \{T_0\} \cup \bigcup_{j=0}^{\infty} (T_j, T_{j+1}) = \{T_0\} \cup \bigcup_{i=0}^{\infty} (T_{2i}, T_{2i+1}) \cup (T_{2i+1}, T_{2i+2}] \)
wherein \((T_{2i}, T_{2i+1}) \subset T^s \) and \((T_{2i+1}, T_{2i+2}) \subset T^d \).

In the next lemma, for any \( i \in \mathbb{N} \), the estimations of the norm of matrices \( e^{-\gamma B(t, T_2)} \) where \( t \in [T_{2i}, T_{2i+1}] \), and \( e^{-\gamma B(t, T_2+1)} \) where \( t \in [T_{2i+1}, T_{2i+2}] \) are presented.

Lemma 2. If for \( i = 1, 2, \ldots, N \) the following conditions are satisfied
\[
\gamma(t) \lambda_i \in (0, \infty) \quad \text{for} \quad t \in T, \\
0 < \delta \leq \mu(t) \gamma(t) \lambda_i < 1 \quad \text{for} \quad t \in T^s, \quad \text{where} \quad \delta \quad \text{is a constant},
\]
then there exists a positive real number \( M < 1 \) such that
\[
\|e^{-\gamma B}(t, T_2)\| \leq \prod_{s \in [T_{2i}, T_{2i+1})} M \quad \text{for} \quad t \in [T_{2i}, T_{2i+1}), \quad i \in \mathbb{N}_0,
\]
\[
\|e^{-\gamma B}(t, T_{2i+1})\| \leq M^t_{T_{2i+1}} \|\gamma(s)\| ds \quad \text{for} \quad t \in [T_{2i+1}, T_{2i+2}), \quad i \in \mathbb{N}_0,
\]
where \( \| \cdot \| \) denotes the spectral norm of considered matrix at the point \( t \).

Proof. Obviously, \( T^s \cup T^d = T \). We consider two cases:

(i) \( t \in T^s \);

(ii) \( t \in T^d \).

In case (i), notice that since matrix \( B \) is symmetric, then \( I - \mu(t) \gamma(t)B \) is a symmetric matrix at the point \( t \), too. Therefore \( \|I - \mu(t) \gamma(t)B\| \) equals the maximum of the absolute value of eigenvalues of matrix \( I - \mu(t) \gamma(t)B \). It means
\[
\|e^{-\gamma B}(t, T_{2i})\| = \prod_{s \in [T_{2i}, t]} \|I - \mu(s) \gamma(s)B\| = \prod_{s \in (T_{2i}, t]} \left( \max_{i \in \{1, 2, \ldots, N\}} \{|1 - \mu(s) \gamma(s) \lambda_i|\} \right)
\]
for \( t \in [T_{2i}, T_{2i+1}) \). Because of positivity of \( \mu \) on \( T^s \) and condition (3), we have \( |\mu(s) \gamma(s) \lambda_i| = \mu(s) \gamma(s) \lambda_i \). Moreover, by (9), \( \mu(s) \gamma(s) \lambda_i \in (0, 1) \) for \( i \in \{1, 2, \ldots, N\} \). We can conclude
\[
\|e^{-\gamma B}(t, T_{2i})\| = \prod_{s \in (T_{2i}, t]} (1 - \min_{i \in \{1, 2, \ldots, N\}} \{\mu(s) \gamma(s) \lambda_i\}).
\]
Again by (9), we have
\[-1 < -\min_{i \in \{1, 2, \ldots, N\}} \mu(s) \gamma(s) \lambda_i \leq -\delta < 0.\]

From above
\[\|e_{-\gamma B}(t, T_{2i})\| \leq \prod_{s \in [T_{2i}, t]} (1 - \delta) = \prod_{s \in [T_{2i}, t]} \mathcal{M}^* = \prod_{s \in [T_{2i}, t]} \mathcal{M}^* \text{ for } t \in [T_{2i}, T_{2i+1}),\]
where \(\mathcal{M}^* = 1 - \delta \in (0, 1)\).

Case (ii). Condition (8) implies
\[(1') \lambda_i > 0 \text{ for any } i = 1, 2, \ldots, N \text{ and } \gamma(t) > 0 \text{ for any } t \in \mathbb{T}^d \]
or
\[(2') \lambda_i < 0 \text{ for any } i = 1, 2, \ldots, N \text{ and } \gamma(t) < 0 \text{ for any } t \in \mathbb{T}^d.\]

If (1'), then
\[\|e_{-\gamma B}(t, T_{2i+1})\| = \|e_B\| \int_{t_{2i+1}}^{T_{2i+1}} \gamma(s) ds = (\max_{i \in \{1, 2, \ldots, N\}} \{e^{\lambda_i}\})^{-1} \int_{t_{2i+1}}^{T_{2i+1}} \gamma(s) ds\]
for \(t \in [T_{2i+1}, T_{2i+2})\), where \(\mathcal{M}^{**} = (\max_{i \in \{1, 2, \ldots, N\}} \{e^{\lambda_i}\})^{-1} \in (0, 1)\).

If (2'), then
\[\|e_{-\gamma B}(t, T_{2i+1})\| = \|e_B\| \int_{t_{2i+1}}^{T_{2i+1}} \gamma(s) ds = (\max_{i \in \{1, 2, \ldots, N\}} \{e^{\lambda_i}\})^{+1} \int_{t_{2i+1}}^{T_{2i+1}} \gamma(s) ds\]
for \(t \in [T_{2i+1}, T_{2i+2})\), where \(\mathcal{M}^{**} = (\max_{i \in \{1, 2, \ldots, N\}} \{e^{\lambda_i}\})^{+1} \in (0, 1)\).

Set \(\mathcal{M} = \max\{\mathcal{M}^*, \mathcal{M}^{**}\}\). Obviously \(\mathcal{M} \in (0, 1)\).

Hence \(\|e_{-\gamma B}(t, T_{2i})\| \leq \prod_{s \in [T_{2i}, t]} \mathcal{M}^{**} \text{ for } t \in [T_{2i}, T_{2i+1})\)
and \(\|e_{-\gamma B}(t, T_{2i+1})\| \leq \mathcal{M}^{**} \int_{t_{2i+1}}^{T_{2i+1}} \gamma(s) ds \text{ for } t \in [T_{2i+1}, T_{2i+2}).\)

Next, we find the estimations of the norm of matrix \(e_{-\gamma B}(t, T_0)\) in two cases: \(t \in [T_{2i}, T_{2i+1})\) and \(t \in [T_{2i+1}, T_{2i+2})\).

**Lemma 3.** If conditions (8), (9) are satisfied, then
\[\|e_{-\gamma B}(t, T_0)\| \leq (\mathcal{M} \sum_{j=1}^{i} \int_{T_{2j-1}}^{T_{2j-1}} |\gamma(s)| ds) \prod_{s \in [T_0, t) \cap \mathbb{T}^d} \mathcal{M} \tag{10}\]
for \(t \in [T_{2i}, T_{2i+1})\), and
\[\|e_{-\gamma B}(t, T_0)\| \leq (\prod_{s \in [T_0, T_{2i+1} \cap \mathbb{T}^d)} \mathcal{M} \sum_{j=1}^{i} \int_{T_{2j-1}}^{T_{2j}} |\gamma(s)| ds + \int_{T_{2i+1}}^{T_{2i+1}} |\gamma(s)| ds) \tag{11}\]
for \(t \in [T_{2i+1}, T_{2i+2}), \text{ where } i \in \mathbb{N}_0.\)
Proof. Let us rewrite function $e^{-\gamma B}(t, T_0)$ in the following form

$$e^{-\gamma B}(t, T_0) = \left( \prod_{s \in [T_0, T_1]} (I - \mu(s)\gamma(s)B) \right) \left( e^{-B \int_{T_1}^{T_2} \gamma(s)ds} \right)$$

$$\cdot \left( \prod_{s \in [T_2, T_3]} (I - \mu(s)\gamma(s)B) \right) \left( e^{-B \int_{T_3}^{T_4} \gamma(s)ds} \right)$$

$$\cdots$$

$$\cdot \left( \prod_{s \in [T_{2i-2}, T_{2i-1}]} (I - \mu(s)\gamma(s)B) \right) \left( e^{-B \int_{T_{2i-1}}^{T_{2i}} \gamma(s)ds} \right)$$

or

$$e^{-\gamma B}(t, T_0) = \left( \prod_{s \in [T_0, T_1]} (I - \mu(s)\gamma(s)B) \right) \left( e^{-B \int_{T_1}^{T_2} \gamma(s)ds} \right)$$

$$\cdot \left( \prod_{s \in [T_2, T_3]} (I - \mu(s)\gamma(s)B) \right) \left( e^{-B \int_{T_3}^{T_4} \gamma(s)ds} \right)$$

$$\cdots$$

$$\cdot \left( \prod_{s \in [T_{2i-2}, T_{2i-1}]} (I - \mu(s)\gamma(s)B) \right) \left( e^{-B \int_{T_{2i-1}}^{T_{2i}} \gamma(s)ds} \right)$$

$$\cdot \left( \prod_{s \in [T_{2i}, T_{2i+1}]} (I - \mu(s)\gamma(s)B) \right) \left( e^{-B \int_{T_{2i+1}}^{T_{2i+2}} \gamma(s)ds} \right)$$

for $t \in [T_{2i+1}, T_{2i+2})$.

By submultiplicativity of the norm, for $t \in [T_{2i}, T_{2i+1})$ we estimate the norm of matrix $e^{-\gamma B}(t, T_0)$

$$\|e^{-\gamma B}(t, T_0)\| \leq \left( \prod_{s \in [T_0, T_1]} \|I - \mu(s)\gamma(s)B\| \right) \left( \|e^B\| - \int_{T_1}^{T_2} \gamma(s)ds \right)$$

$$\cdot \left( \prod_{s \in [T_2, T_3]} \|I - \mu(s)\gamma(s)B\| \right) \left( \|e^B\| - \int_{T_3}^{T_4} \gamma(s)ds \right)$$

$$\cdots$$

$$\cdot \left( \prod_{s \in [T_{2i-2}, T_{2i-1}]} \|I - \mu(s)\gamma(s)B\| \right) \left( \|e^B\| - \int_{T_{2i-1}}^{T_{2i}} \gamma(s)ds \right)$$

$$\cdot \left( \prod_{s \in [T_{2i}, t]} \|I - \mu(s)\gamma(s)B\| \right)$$

$$\leq \left( \prod_{s \in [T_0, T_1]} M \right) \left( M \int_{T_1}^{T_2} \gamma(s)ds \right) \left( \prod_{s \in [T_2, T_3]} M \right) \left( M \int_{T_3}^{T_4} \gamma(s)ds \right) \cdots .$$
From the above, inequalities (10) and (11) imply

\[\prod_{s \in [T_0, t]} \mathcal{M} \left( \int_{T_{2i-1}}^{T_{2i}} |\gamma(s)| ds \right) \]

\[= \left( \prod_{s \in [T_0, t]} \mathcal{M} \right) \left( \int_{T_{2i-1}}^{T_{2i}} |\gamma(s)| ds \right) \]

Analogously, for \( t \in [T_{2i+1}, T_{2i+2}) \), we get

\[\|e_{-\gamma}(t, T_0)\| \leq \left( \prod_{s \in [T_0, t]} \mathcal{M} \right) \left( \int_{T_{2i-1}}^{T_{2i}} |\gamma(s)| ds \right).\]

Remark 1. If conditions (8) - (9) are satisfied, then

\[\|e_{-\gamma}(t, T_0)\| \leq \left( \prod_{s \in [T_0, t]} \mathcal{M} \right) \text{ for } t \in \mathbb{T}.\]

Proof. Since \( \mathcal{M} \in (0, 1) \) and \( \sum_{j=1}^{j-1} \int_{T_{2j-1}}^{T_{2j}} |\gamma(s)| ds + \int_{T_{2i+1}}^{T_{2i+2}} |\gamma(s)| ds \geq 0 \), thus

\[\mathcal{M} \sum_{j=1}^{j-1} \int_{T_{2j-1}}^{T_{2j}} |\gamma(s)| ds + \int_{T_{2i+1}}^{T_{2i+2}} |\gamma(s)| ds \leq 1 \text{ for } t \in \mathbb{T}.\]

From the above, inequalities (10) and (11) imply

\[\|e_{-\gamma}(t, T_0)\| \leq \left( \prod_{s \in [T_0, t]} \mathcal{M} \right) \text{ for } t \in [T_{2i}, T_{2i+1}),\]

\[\|e_{-\gamma}(t, T_0)\| \leq \left( \prod_{s \in [T_0, T_{2i+1})} \mathcal{M} \right) \text{ for } t \in [T_{2i+1}, T_{2i+2}),\]

where \( i \in \mathbb{N}_0 \). It implies the thesis.

The following result concerns of scalar case of exponential function on arbitrary time scale.

Lemma 4. Let \( e_{\mathcal{LM}^{-1}}(\cdot, T_0) : \mathbb{T} \rightarrow \mathbb{R} \). There hold

\[e_{\mathcal{LM}^{-1}}(t, T_0) = e^{\mathcal{LM}^{-1} \sum_{j=1}^{i} (T_{2j-1} - T_{2j-1})} \prod_{s \in [T_0, t]} (1 + \mu(s)\mathcal{LM}^{-1})\]

for \( t \in [T_{2i}, T_{2i+1}) \), and

\[e_{\mathcal{LM}^{-1}}(t, T_0) = \prod_{s \in [T_0, T_{2i+1})} (1 + \mu(s)\mathcal{LM}^{-1}) \cdot e^{\mathcal{LM}^{-1} \sum_{j=1}^{i} (T_{2j-1} - T_{2j-1} + (T_{2j+1} - T_{2j-1}))}\]

for \( t \in [T_{2i+1}, T_{2i+2}) \), where \( i \in \mathbb{N}_0 \).
We are now in a position to present the main theorem of this paper.

**Theorem 1.** If conditions (7)-(9) are satisfied, and for any \( t \in \mathbb{T} \)

there exists a positive constant \( \mu^* \) such that \( \mu(t) \leq \mu^* \),

\[
\lim_{t \to \infty} e^{LM^{-1} \sum_{i=1}^{T_1} (T_2, T_1) \mathcal{M} \sum_{j=1}^{T_j} \int_{T_{j-1}}^{T_j} |\gamma(s)| ds \prod_{s \in [T_0, t] \cap \mathbb{T}_s} (\mathcal{M} + \mu^* \mathcal{L}) = 0 
\]

and

\[
\lim_{t \to \infty} e^{LM^{-1} (\sum_{i=1}^{T_1} (T_2, T_1) + (t-T_2+1))} \mathcal{M} \sum_{j=1}^{T_j} \int_{T_{j-1}}^{T_j} |\gamma(s)| ds + \int_{T_{2i+1}}^{T_2} |\gamma(s)| ds \prod_{s \in [T_0, T_{2i+1}] \cap \mathbb{T}_s} (\mathcal{M} + \mu^* \mathcal{L}) = 0,
\]

then equation (5) is exponentially stable.

**Proof.** Taking the norm of the both sides of equation (6), we obtain

\[
\|\varepsilon(t)\| = \|e_{-\gamma B}(t, T_0)\varepsilon_{T_0} + \int_{T_0}^{t} e_{-\gamma B}(t, \sigma(\tau)) (F(\tau, x_0(\tau)1) - F(\tau, x(\tau))) \Delta \tau\|. 
\]

Using properties of the norm, we get

\[
\|\varepsilon(t)\| \leq \|e_{-\gamma B}(t, T_0)\| \|\varepsilon_{T_0}\| + \| \int_{T_0}^{t} e_{-\gamma B}(t, \sigma(\tau)) (F(\tau, x_0(\tau)1) - F(\tau, x(\tau))) \Delta \tau\|,
\]

and consequently

\[
\|\varepsilon(t)\| \leq \|\varepsilon_{T_0}\| \|e_{-\gamma B}(t, T_0)\| + \int_{T_0}^{t} \|e_{-\gamma B}(t, \sigma(\tau))\| \| (F(\tau, x_0(\tau)1) - F(\tau, x(\tau))) \| \Delta \tau.
\]

By condition (7), we obtain

\[
\|\varepsilon(t)\| \leq \|\varepsilon_{T_0}\| \|e_{-\gamma B}(t, T_0)\| + \mathcal{L} \int_{T_0}^{t} \|e_{-\gamma B}(t, \sigma(\tau))\| \|\varepsilon(\tau)\| \Delta \tau.
\]

For \( t \in [T_2i, T_{2i+1}) \), using (10), we estimate

\[
\|\varepsilon(t)\| \leq \|\varepsilon_{T_0}\| (\mathcal{M} \sum_{j=1}^{T_{2i}} \int_{T_{j-1}}^{T_j} |\gamma(s)| ds) (\prod_{s \in [T_0, T_2i] \cap \mathbb{T}_s} \mathcal{M}) \|\varepsilon(t)\| \Delta \tau.
\]

\[
+ \mathcal{L} \int_{T_0}^{t} (\mathcal{M} \sum_{j=1}^{T_{2i}} \int_{T_{j-1}}^{T_j} |\gamma(s)| ds) (\prod_{s \in [\sigma(\tau), t] \cap \mathbb{T}_s} \mathcal{M}) \|\varepsilon(t)\| \Delta \tau.
\]
Multiplying the both sides of the above inequality by

\[
(M^{-\sum_{j=1}^{i} T_{2j-1}} |\gamma(s)| ds) \left( \prod_{s \in (T_{0}, t) \cap \mathbb{T}^{n}} M^{-1} \right),
\]

we obtain

\[
\|\varepsilon(t)\| (M^{-\sum_{j=1}^{i} T_{2j-1}} |\gamma(s)| ds) \left( \prod_{s \in (T_{0}, t) \cap \mathbb{T}^{n}} M^{-1} \right)
\]

\[
\leq \|\varepsilon_{T_{0}}\| M + \int_{T_{0}}^{t} L \|\varepsilon(\tau)\| (M^{-\sum_{j=1}^{i} T_{2j-1}} |\gamma(s)| ds + f_{T_{2j-1}}^{\mu(s)} |\gamma(s)| ds) \left( \prod_{s \in (T_{0}, \tau) \cap \mathbb{T}^{n}} M^{-1} \right) \Delta \tau.
\]

Since \(M^{-\sum_{j=1}^{i} T_{2j-1}} |\gamma(s)| ds = 1\),

\[
\|\varepsilon(t)\| (M^{-\sum_{j=1}^{i} T_{2j-1}} |\gamma(s)| ds) \left( \prod_{s \in (T_{0}, t) \cap \mathbb{T}^{n}} M^{-1} \right)
\]

\[
\leq \|\varepsilon_{T_{0}}\| M^{-\sum_{j=1}^{i} T_{2j-1}} |\gamma(s)| ds . M
\]

\[
+ \int_{T_{0}}^{t} L \|\varepsilon(\tau)\| (M^{-\sum_{j=1}^{i} T_{2j-1}} |\gamma(s)| ds + f_{T_{2j-1}}^{\mu(s)} |\gamma(s)| ds) \left( \prod_{s \in (T_{0}, \tau) \cap \mathbb{T}^{n}} M^{-1} \right) \Delta \tau.
\]

Using \(\sigma(\tau) = \tau\), we get

\[
\|\varepsilon(t)\| (M^{-\sum_{j=1}^{i} T_{2j-1}} |\gamma(s)| ds) \left( \prod_{s \in (T_{0}, t) \cap \mathbb{T}^{n}} M^{-1} \right)
\]

\[
\leq \|\varepsilon_{T_{0}}\| M^{-\sum_{j=1}^{i} T_{2j-1}} |\gamma(s)| ds . M
\]

\[
+ \int_{T_{0}}^{t} L \|\varepsilon(\tau)\| (M^{-\sum_{j=1}^{i} T_{2j-1}} |\gamma(s)| ds + f_{T_{2j-1}}^{\mu(s)} |\gamma(s)| ds) \left( \prod_{s \in (T_{0}, \tau) \cap \mathbb{T}^{n}} M^{-1} \right) \Delta \tau.
\]

By Lemma \(\square\) it leads to inequality

\[
\|\varepsilon(t)\| (M^{-\sum_{j=1}^{i} T_{2j-1}} |\gamma(s)| ds) \left( \prod_{s \in (T_{0}, t) \cap \mathbb{T}^{n}} M^{-1} \right) \leq \|\varepsilon_{T_{0}}\| M e_{M^{-1}}(t, T_{0}).
\]

Using Lemma \(\square\)

\[
\|\varepsilon(t)\| (M^{-\sum_{j=1}^{i} T_{2j-1}} |\gamma(s)| ds) \left( \prod_{s \in (T_{0}, t) \cap \mathbb{T}^{n}} M^{-1} \right)
\]

\[
\leq \|\varepsilon_{T_{0}}\| M (e_{M^{-1}} \sum_{j=1}^{i} (T_{2j-1} - T_{2j-2})) \left( \prod_{s \in [T_{0}, t) \cap \mathbb{T}^{n}} (1 + \mu(s) e_{M^{-1}}) \right).
\]
By (12),

\[
\|e(t)\| \leq \|e_{T_0}\| \left( e^{\mathcal{L}M^{-1} \sum_{j=1}^{i}(T_{2j} - T_{2j-1})} \right) \\
\cdot (\mathcal{M} \sum_{j=1}^{i} f_{T_{2j-1}}^{T_{2j}} |\gamma(s)| ds) \left( \prod_{s \in [T_0,t] \cap T^*} (\mathcal{M} + \mu(s) \mathcal{L}) \right).
\]

Hence

\[
\|e(t)\| \leq \|e_{T_0}\| \left( e^{\mathcal{L}M^{-1} \sum_{j=1}^{i}(T_{2j} - T_{2j-1})} \right) \\
\cdot (\mathcal{M} \sum_{j=1}^{i} f_{T_{2j-1}}^{T_{2j}} |\gamma(s)| ds) \left( \prod_{s \in [T_0,t] \cap T^*} (\mathcal{M} + \mu(s) \mathcal{L}) \right).
\]

(13)

(14)

(15)

(16)

By (12),

\[
\|e(t)\| \leq \|e_{T_0}\| \left( e^{\mathcal{L}M^{-1} \sum_{j=1}^{i}(T_{2j} - T_{2j-1})} \right) \\
\cdot (\mathcal{M} \sum_{j=1}^{i} f_{T_{2j-1}}^{T_{2j}} |\gamma(s)| ds) \left( \prod_{s \in [T_0,T_{2i+1}] \cap T^*} (\mathcal{M} + \mu^* \mathcal{L}) \right).
\]

Analogously, for \( t \in (T_{2i+1}, T_{2i+2}] \)

\[
\|e(t)\| \leq \|e_{T_0}\| \left( e^{\mathcal{L}M^{-1} \sum_{j=1}^{i}(T_{2j} - T_{2j-1})} \right) \\
\cdot (\mathcal{M} \sum_{j=1}^{i} f_{T_{2j-1}}^{T_{2j}} |\gamma(s)| ds + f_{T_{2i+1}}^{T_{2i+2}} |\gamma(s)| ds) \left( \prod_{s \in [T_0,T_{2i+1}] \cap T^*} (\mathcal{M} + \mu^* \mathcal{L}) \right).
\]

(17)

(18)

(19)

Corollary 1. If conditions (13) and (14) are satisfied,

for any \( t \in T^* \) there exists \( \tilde{t} \in \mathbb{T}^d \) such that \( \tilde{t} > t \) and

\[
\mathcal{M} + \mu^* \mathcal{L} < 1,
\]

and

\[
\lim_{i \to \infty} e^{\text{sum}(i)} < \infty,
\]

then equation (5) is exponentially stable.
**Figure 1:** The topology of the leader-following multi-agent system under the undirected graph

**Proof.** By (17) we get, \( t \to \infty \) iff \( i \to \infty \). Since \( 0 < M + \mu^* L < 1 \) and \( M \in (0,1) \), by properties of functions \( M^t \) and \( e^t \), condition (19) implies conditions (13) and (14). Hence assumptions of Theorem 1 are satisfied. So, the thesis holds. \( \square \)

**Example 4.** Let 
\[
T = \bigcup_{i=3}^{\infty} \left[ \frac{i-2}{i^2} + \frac{1}{i^3} \right].
\]
Here \( T^d = \bigcup_{i=3}^{\infty} \left[ \frac{i-2}{i^2} + \frac{1}{i^3} \right] \) and \( T^s = \left\{ \frac{i-2}{i^2} : i \in \mathbb{N}, i \geq 3 \right\} \),
\[
T_0 = T_1 = 1.500, \quad T_2 \approx 1.537, \quad T_3 \approx 2.000, \ldots, \mu^* = 0.500.
\]
Moreover, let 
\[
f(t, x) = 0.100 \frac{x}{t^2}, \quad \gamma(t) = 0.500(t-1),
\]
\[
A = \begin{bmatrix}
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0
\end{bmatrix},
\]
\[
D = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]
in equation (13) (see Figure 1).
Hence
\[
L = \begin{bmatrix}
2 & 0 & -1 & -1 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 2 & -1 \\
-1 & 0 & -1 & 2
\end{bmatrix}
\quad \text{and} \quad
B = \begin{bmatrix}
2 & 0 & -1 & -1 \\
0 & 3 & 0 & 0 \\
-1 & 0 & 3 & -1 \\
-1 & 0 & -1 & 3
\end{bmatrix}.
\quad (20)
\]
There is \( L = 0.100, \lambda_1 = 2 - \sqrt{2}, \lambda_2 = 2, \lambda_3 = 3 \) and \( \lambda_4 = 2 + \sqrt{2} \). It follows from these that \( \lambda_{\min} = \min\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \approx 0.585 \) and
\[
\max\{1 - 0.462 \cdot 0.250 \cdot 0.585, e^{-0.585}\} \leq \max\{0.933, 0.557\} = 0.933 =: \mathcal{M}.
\]

13
From the above
\[ \prod_{s \in T} (M + \mu^* L) \approx \prod_{s \in T} 0.983 = 0 \]
and
\[ \lim_{i \to \infty} e^{\text{sum}(i)} \approx \lim_{i \to \infty} e^{\sum_{j=3}^{i} (0.108 j^{-3} - 0.070 - 0.250 (i^{-2} - 2i^{-3} - i^{-6}))} < \infty. \]

All assumptions of Corollary 1 are satisfied, thus equation (5) is exponentially stable. System (2) - (3) achieves consensus exponentially.

In Example 4 there is
\[ \lim_{i \to \infty} e^{L\mathcal{M}^{-1} \sum_{j=1}^{i} (T_{2j} - T_{2j-1})} \approx \lim_{i \to \infty} e^{0.108 \sum_{j=3}^{i} j^{-3}} < \infty, \]
but this condition is not required for exponential stability of (5) (see Example 5).

**Remark 2.** If conditions (7) - (9), (12) and (18) are satisfied, and
\[ \gamma(t) \equiv \gamma \in \mathbb{R}, \quad (21) \]
and
\[ L\mathcal{M}^{-1} + \gamma \ln \mathcal{M} < 0, \quad (22) \]
then equation (5) is exponentially stable.

**Proof.** If condition (21) holds, then
\[
\text{sum}(i) = \sum_{j=1}^{i} \left( L\mathcal{M}^{-1} (T_{2j} - T_{2j-1}) + \gamma (T_{2j} - T_{2j-1}) \ln \mathcal{M} \right) \\
= (L\mathcal{M}^{-1} + \gamma \ln \mathcal{M}) \sum_{j=1}^{i} (T_{2j} - T_{2j-1}).
\]

By (22), we see that \( \text{sum}(i) < 0 \) for any \( i \in \mathbb{N} \), and \( e^{\text{sum}(i)} \) is a positive, decreasing function of variable \( i \in \mathbb{N} \). Here \( \prod_{s \in T} (M + \mu^* L) \) as well as \( e^{\text{sum}(i)} \) for any \( i \in \mathbb{N} \), are bounded. If the cardinality of set \( T^s \) is infinity, then \( \lim_{i \to \infty} \prod_{s \in T} (M + \mu^* L) = 0 \). If the cardinality of set \( T^d \) is infinity, then \( \lim_{i \to \infty} e^{\text{sum}(i)} = 0 \). Thus, by Theorem 1 we obtain the thesis. \( \blacksquare \)

**Example 5.** Let
\[ T = \bigcup_{i=3}^{\infty} \left[ \frac{i}{2}, \frac{i}{2} + \frac{1}{i} \right]. \]

Here \( T^d = \bigcup_{i=3}^{\infty} \left[ \frac{i}{2}, \frac{i}{2} + \frac{1}{i} \right] \) and \( T^s = \{ \frac{i}{2} + \frac{1}{i} : i \in \mathbb{N}, i \geq 3 \}, \)
\[ T_0 = T_1 = 1.500, \ T_2 \approx 1.833, \ T_3 = 2.000, \ldots, \]
\[ \mu(t) = \frac{1}{2} - \frac{t}{2} + \frac{1}{2} \sqrt{t^2 - 2} \text{ for } t \in \mathbb{T}^s, \mu^* = 0.500. \]

Moreover, let
\[ f(t, x) = 0.250 \frac{\sin x}{t^2}, \gamma(t) \equiv 2.000, \]
and matrix \( B \) is given by \([24]\) in equation \([5]\). There is \( \mathcal{L} = 0.250, \lambda_{\min} \approx 0.585, \]
\[ \max\{1 - 0.333 \cdot 2.000 \cdot 0.585, e^{-0.585}\} \leq \max\{0.390, 0.557\} = 0.557 =: \mathcal{M}. \]
Finally,
\[ \mathcal{L}\mathcal{M}^{-1} + \gamma \ln \mathcal{M} \approx 0.449 - 1.170 = -0.721 < 0. \]
All assumptions of Remark 2 hold, thus equation \([5]\) is exponentially stable.

In Example 5 there is
\[ \lim_{i \to \infty} e^{\mathcal{L}M^{-1} \sum_{j=1}^{i}(T_{2j} - T_{2j-1})} \approx \lim_{i \to \infty} e^{0.449 \sum_{j=1}^{i} \frac{1}{j}} = \infty, \]
even that system \([24]-[34]\) achieves consensus exponentially.

**Corollary 2.** If conditions \([7]-[9]\) and \([18]\) are satisfied, and
\[ \sum_{i=0}^{\infty} (T_{2i+2} - T_{2i+1}) < \infty, \quad (23) \]
then equation \([5]\) is exponentially stable.

**Proof.** Since \([23]\) holds,
\[ e^{\text{sum}(i)} = \text{constant}. \]
Hence, reminding cardinality of the set \( \mathbb{T}^s \) is infinity, by \([18]\), we obtain
\[ \lim_{i \to \infty} e^{\text{sum}(i)} \prod_{s \in [T_0, T_{2i}) \cap \mathbb{T}^s} (\mathcal{M} + \mu^* \mathcal{L}) \]
\[ = \lim_{i \to \infty} c^* \left( \prod_{s \in [T_0, T_{2i}) \cap \mathbb{T}^s} (\mathcal{M} + \mu^* \mathcal{L}) \right) \]
\[ = c^* \prod_{s \in \mathbb{T}^s} (\mathcal{M} + \mu^* \mathcal{L}) = 0, \]
where \( c^* = e^{\text{sum}(i)}. \)

For two possible cases of carrying out of assumption \([23]\) see Example 4 and Example 7.

Theorem 1 generalize Theorem 2 \([14]\). The following example present an equation on time scale for which Theorem 2 \([14]\) can not be applicable, but our Corollary 2 of Theorem 1 can be.
Example 6. Let
\[ T = \{ i \in \mathbb{N} \} \cup \{ i + \frac{1}{j+1} : i, j \in \mathbb{N}, j \geq 2 \}. \]

Here \( T_d = \{ i \in \mathbb{N} \} \) and \( T^s = \{ i + \frac{1}{j+1} : i, j \in \mathbb{N}, j \geq 2 \} \).

\[ T_0 = 1, \ T_1 = \infty, \ \mu(t) = \frac{(t-i)^2}{1+t-i} \text{ for } t \in T^s, \ \mu^* = 0.500. \]

Set \( f(t, x) = 0.250x \),
\[ \gamma(t) = \begin{cases} \frac{1}{\mu(t)} & \text{for } t \in T^s \\ 0 & \text{for } t \in T^d, \end{cases} \]
and \( B \) is given by (20) in equation (5). There is \( L = 0.250, \ \lambda_{\text{min}} = 0.585, \)
\[ \max\{1 - 1 \cdot 0.585, e^{-0.585}\} \leq \max\{0.515, 0.557\} = 0.557 =: M. \]

Hence
\[ \prod_{s \in T^s} (M + \mu^* L) \approx \prod_{s \in T^s} 0.682 = 0. \]

All assumptions of Corollary 2 are satisfied, thus equation (5) is exponentially stable.

Since \( \liminf_{t \to \infty} \mu(t) = 0 \) results obtained in [14] can not be applied.

The following examples show two different situations concerning time scale in which condition (23) is satisfied. In the first example, \( T^d \) is a bounded set.

In the second one, set \( T^d \) is unbounded.

Example 7. Let
\[ T = [1, 2] \cup [3, 7] \cup \{ n : n \in \mathbb{N}, n \geq 8 \}. \]

Here \( T^d = [1, 2] \cup [3, 7] \) is bounded and \( T^s = \{ n : n \in \mathbb{N}, n \geq 8 \} \). We see that
\[ T_0 = 1, \ T_1 = T_0 = 1, \ T_2 = 2, \ T_3 = 3, \ T_4 = 7, \ T_5 = 8, \ T_6 = \infty, \]
\[ \mu(t) = 1 \text{ for } t \in T^s, \ \mu^* = 1. \]

Let also
\[ f(t, x) = \frac{1}{4\sqrt{t}} \sin x, \ \gamma(t) = \cos t + 2, \]
and matrix \( B \) is given by (20) in equation (5). There is \( L = 0.250, \ \lambda_{\text{min}} = 0.585, \)
\[ \mathcal{M} = \max\{1 - 1 \cdot 1 \cdot 0.585, e^{-0.585}\} \approx 0.557 < 1. \]

In consequence
\[ \prod_{s \in T^s} (\mathcal{M} + \mu^* L) \approx \prod_{s \in T^s} 0.807 = 0. \]

All assumptions of Corollary 2 are satisfied, thus equation (5) is exponentially stable. It means, the multi-agent system (2)–(3) achieves the leader-following consensus exponentially.
Example 8. Let
\[ T = \bigcup_{i=3}^{\infty} \left[ \frac{i}{2} + \frac{1}{i+1}, \frac{i}{2} + \frac{1}{i} \right]. \]
Here either \( T^d = \bigcup_{i=3}^{\infty} \left[ \frac{i}{2} + \frac{1}{i+1}, \frac{i}{2} + \frac{1}{i} \right] \) or \( T^s = \left\{ \frac{i}{2} + \frac{1}{i} : i \in \mathbb{N}, i \geq 3 \right\} \) are unbounded sets. We see that
\[ T_0 = T_1 = 1.750, \quad T_2 \approx 1.833, \quad T_3 = 2.200, \quad T_4 = 2.250, \ldots \]
\[ \mu(t) = \frac{1}{2} - \frac{2}{(t + \sqrt{t^2 - 2})(2 + t + \sqrt{t^2 - 2})} \text{ for } t \in T^s, \quad \mu^* = 0.500. \]
Moreover
\[ f(t, x) = \frac{x}{4t}, \quad \gamma(t) = \frac{1}{4}t^2, \]
and matrix \( B \) is given by \([20]\) in equation \([5]\). There is \( \mathcal{L} = 0.250, \lambda_{\text{min}} = 0.585, \]
\[ \max\{1 - 0.366 \cdot 0.765 \cdot 0.585, e^{-0.585}\} < 0.836 =: M. \]
Hence
\[ \prod_{s \in T^s} (\mathcal{M} + \mu^* L) \approx \prod_{s \in T^s} 0.961 \approx 0. \]
All assumptions of Corollary \([6]\) are satisfied, thus equation \([5]\) is exponentially stable. System \([2] - [3]\) achieves consensus exponentially.

Notice that in Example \([8]\) there is
\[ \lim_{i \to \infty} e^{\mathcal{M}^{-1} \sum_{j=1}^{i} (T_{2j} - T_{2j-1})} = \lim_{i \to \infty} e^{0.299 \sum_{j=1}^{i} \frac{1}{(1+i+1)}} = e^{0.299} < \infty. \]

**Remark 3.** If conditions \([7] - [9]\) are satisfied,
\[ \sum_{j=1}^{\infty} \int_{T_{2j-1}}^{T_{2j}} |\gamma(s)| ds < \infty, \]
and
\[ \lim_{i \to \infty} e^{\mathcal{M}^{-1} \sum_{j=1}^{i} (T_{2j} - T_{2j-1})}, \quad \prod_{s \in [T_0, T_2) \cap T^s} (\mathcal{M} + \mu^* L) = 0, \]
then equation \([5]\) is exponentially stable.

(See Example \([4]\))

**Remark 4.** Let conditions \([7] - [9]\) be satisfied. If the cardinality of the set \( T^s \) is finite and \( \text{sum}(i) < 0 \) for any \( i \in \mathbb{N} \) then equation \([5]\) is exponentially stable.
Example 9. Let 
\[ T = \{1\} \cup \{11\} \cup [12, \infty) \].

Here \( T^d = [12, \infty) \) is unbounded set whereas \( T^s = \{1\} \cup \{11\} \) is bounded, and 
\[ T_0 = 1, \ T_1 = 12, \ T_2 = \infty, \ \mu(1) = 10, \ \mu(11) = 1, \ \mu^* = 10. \]

Let 
\[ f(t, x) = 0.1x, \ \gamma(t) = 1, \]
and matrix \( B \) is given by (20) in equation (5). There is \( \mathcal{L} = 0.100, \lambda_{\text{min}} \approx 0.585, \max \{1 - 0.585, e^{-0.585}\} < 0.557 =: M. \)

Hence 
\[ \text{sum}(i) \approx 0.180(T_2 - T_1) - 0.585 \int_{T_1}^{T_2} ds = -0.405(T_2 - T_1) = -\infty < 0. \]

All assumptions of Remark 4 are satisfied, thus equation (5) is exponentially stable.

Notice that in Example 9 condition (18) does not hold.

Acknowledgments

The second author was supported by the Polish National Science Center grant on the basis of decision DEC-2014/15/B/ST7/05270.

References

[1] M. H. DeGroot, Reaching a consensus, J. Amer. Statist. Assoc., (1974), 69, 118–121

[2] U. Krause, Gordon and Breach Publ., A discrete nonlinear and non-autonomous model of consensus formation, Communications in Difference Equations, 2000, ICDEA, 1998, Poznan

[3] R. Hegselmann and U. Krause, Opinion dynamics and bounded confidence: models, analysis, and simulation, J. Artificial Societies and Social Simulations, (2002), 5, 1–33

[4] A. Jadabaie and J. Lin and A. S. Morse, Coordination of groups of mobile autonomous agents using nearest neighbor rules, IEEE Transactions on Automatic Control, 2003, 48, 6, 988–1001

[5] V. D. Blondel and J. M. Hendrickx and J. N. Tsitsikli, On Krause’s multi-agent consensus model with state-dependent connectivity, IEEE Transactions on automatics control, (2009), 54, 11, 2586–2597
[6] V. D. Blondel and J. M. Hendrickx and J. N. Tsitsiklis, Continuous-time average-preserving opinion dynamics with opinion-dependent communications, SIAM J. Control Optim., (2010), 18, 8, 5214–5240

[7] E. Girejko and L. Machado and A. B. Malinowska and N. Martins, Krause’s model of opinion dynamics on isolated time scales, Math. Methods Appl. Sci., (2016), 39, 18, 5302–5314

[8] E. Girejko and A.B. Malinowska and E. Schmeidel and M. Zdanowicz, IEEE-Explore, The emergence on isolated time scales, 21st International Conference on Methods and Models in Automation and Robotics (MMAR), 2016

[9] F. Cucker and S. Smale, On the mathematics of emergence, Japan. J. Math., (2007), 2, 1, 197–227

[10] F. Cucker and S. Smale, Emergent behavior in flocks, IEEE Transactions on Automatic Control, (2007), 52, 7, 852–862

[11] Y. Wang and L. Cheng and H. Wang and Z. G. Hou and M. Tan and H. Yu, IEEE, Leader-following consensus of discrete-time linear multi-agent systems with communication noises, Lecture Notes in Electrical Engineering 407, 2015, Control Conference (CCC) 34th, 2015, Hangzhou, China

[12] E. Girejko and L. Machado and A. B. Malinowska and N. Martins, On consensus in the Cucker-Smale type model on isolated time scale, Discrete Contin. Dyn. Syst. Ser. B, (2018), 11, 1, 77–89

[13] U. Ostaszewska and E. Schmeidel and M. Zdanowicz, American Institute of Physics, Leader-following consensus on discrete time scales, AIP Conference Proceedings, 2018, ICNAAM 2017, 2017, Thessaloniki, Greece

[14] U. Ostaszewska, E. Schmeidel, M. Zdanowicz, Emergence of consensus of multi-agents systems on time scales, Miskolc Math. Notes, (to appear)

[15] B. Aulbach and S. Hilger, North-Holland, Amsterdam, A unified approach to continuous and discrete dynamics, Qualitative theory of differential equations, 1990, Colloq. Math. Soc. Jnos Bolyai 53, 1988, Szeged

[16] M. Bohner and A. Peterson, Dynamic equations on time scales, 2001, Birkhäuser, Boston

[17] M. Bohner and A. Peterson, Advances in dynamic equations on time scales, 2003, Birkhäuser, Boston

[18] Z. Yu and H. Jiang and C. Hu, Leader-following consensus of fractional-order multi-agent systems under fixed topology, Neurocomputing, (2015), 149, 613–620

[19] E. Schmeidel, The existence of consensus of a leader-following problem with Caputo fractional derivative, Opuscula Math., (2019), 39, 1, 77–89
[20] L. Berezansky, M. Migda, E. Schmeidel, Some stability conditions for scalar Volterra difference equations, Opuscula Math., (2016), 36, 4, 459–470

[21] S. N. Elaydi, An Introduction to Difference Equations, 2005, Springer, New York

[22] A. Peterson and Y. N. Raffoul, Exponential stability of dynamic equations on time scales, Adv. Differ. Equ., (2005), 1–13

[23] S. Bodine and D. A. Lutz, Exponential functions on time scales: their asymptotic behavior and calculation, Dynam. Systems Appl., (2003), 12, 1–2, 23–43