Flexible hinges in orthotropic cylindrical shells facilitated by nonlinear elastic deformations

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ABSTRACT

Flexible hinges enable the design of folding structures without using mechanisms by making use of intrinsic structural characteristics in the action of folding. This technology introduces potential benefits including weight reduction, omission of lubrication and potentially better system reliability. To achieve such technology, we exploit a well-known structural instability characteristic of thin-walled structures under bending: the Brazier effect. Composite materials play a key role in this problem since they enable the critical load for folding to be tuned. Moreover, the minimisation of the Brazier moment is material dependent, offering extra degrees of freedom for morphing purposes. The present work considers the minimisation of the Brazier moment providing insights on its material dependency. For this purpose, an analytical solution for cylindrical shells made of unidirectional laminates and an empirical expression useful for design purposes, comprising 4-ply symmetric laminates, are presented with validation accomplished using finite element analysis.

1. Introduction

Significant advantages, including weight reduction and eliminating the need for lubrication, can be introduced into the design of folding structures by replacing conventional hinged mechanisms with flexible hinges embedded into the structure. Flexible hinges provide localised compliance by exploiting a structural instability characteristic of thin-walled structures under bending, known in literature as the Brazier effect. This technology has been notably applied for the deployment of aerospace booms. The foldable antenna of the Mars Advanced Radar for Subsurface and Ionosphere Sounding (MARSIS) on the Mars Express spacecraft used this technology by embedding a built-in tape-spring in its cylindrical shell boom \(^{[1,2]}\). Multiple authors have studied and optimised these structures including Yee and Pellegrino \(^{[3]}\), Mallikarachchi and Pellegrino \(^{[4]}\) and Fernandes et al. \(^{[5]}\).

Fig. 1a shows a cylindrical shell with an embedded flexible hinge from the design of Yee and Pellegrino \(^{[3]}\). Considering a different application, Lachenal et al. \(^{[6]}\) exploited the Brazier effect to fold a wing of a typically-sized unmanned aerial vehicle, see Fig. 1b. Since the Brazier effect has been verified in a wide range of structures including wind turbine blades \(^{[7]}\) and aircraft wings \(^{[8,9]}\), flexible hinges can also be used for many other applications where the main structure is a thin-walled beam of arbitrary cross-sectional shape under bending. Recently, Bowen et al. \(^{[10]}\) showed that some of the knowledge of the nonlinear mechanics related to the Brazier effect experienced by circular thin-walled beams is transferable to thin-walled beams with aerofoil cross-sections. Therefore, detailed knowledge of the Brazier effect on circular cross-sections, like that presented in this work, not only enrich the design knowledge for applications where the structure is a thin-walled beam but also applications where the structure is a thin-walled beam with aerofoil cross-section.

In 1911, von Kármán \(^{[11]}\) studied the effects of nonlinear deformation on initially curved isotropic cylindrical shells. Later Brazier \(^{[12]}\) studied the same problem for initially straight isotropic cylindrical shells. Both of them successfully explained why linear beam theory was not able to correctly predict the behaviour of thin-walled beams under bending. Considering a long beam having a thin-walled circular cross-section, an increase in applied bending moment results in a progressive ovalisation of the cross-section, see Fig. 2. The ovalisation process steadily decreases the second moment of area until a point where the structure is no longer able to sustain a further increase in bending moment and so collapses. The bending moment at which instability occurs is known as the Brazier moment \((M_{Bz})\).

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Whilst extensive research focuses on how to prevent the Brazier phenomenon or determine its contribution to other failure modes, i.e. local buckling and material failure \[7,9,13\]; only one investigation \[14\] explores the optimisation of structures which exploit, rather than avoid, Brazier phenomena. This less explored perspective, has potential to facilitate and tailor the folding performance of flexible hinges.

Considering flexible hinges in deployable booms as an example, shape changes including folding, deployment and latching take place during their operation. Detailed engineering knowledge of each individual morphing phase is an important consideration for the design optimisation of these structures. In the case of folding, the stress induced into the structure during the folding process can be reduced not only by optimising the shape of the cut-out of the hinge \[5,15\] but also by keeping the instability load, which allows folding, to its minimum attainable magnitude. The minimisation of \(M_{Brz}\) for such structures addresses this design objective.

Composite materials offer scope for stiffness tailoring which can be used for minimising \(M_{Brz}\). The analysis of the Brazier effect applied to composite shells with circular cross-section has been carried out by several authors including Kedward \[16\], Harursampath and Hodges \[17\], Stockwell and Cooper \[18\] and Tatting et al. \[19\].

Driven by these premises, the present work considers the minimisation of \(M_{Brz}\) in a fibre-reinforced composite cylindrical shell. In particular, the goal of the minimisation process is to find the optimum composite layup giving the least moment necessary to reach instability for folding a multi-layered composite cylindrical shell with given

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**Nomenclature**

**Symbols**

- \(A_{11}\) Longitudinal stiffness, N/mm
- \(A_{16}, A_{26}\) Extension-shear coupling stiffness, N/mm
- \(C_{1-9}\) Coefficients of empirical expressions
- \(D_{16}, D_{26}\) Bend-twist coupling stiffness, N/mm
- \(E_{11}\) Longitudinal Young’s modulus, N/mm²
- \(E_{22}\) Transverse Young’s modulus, N/mm²
- \(G_{12}\) Shear modulus, N/mm²
- \(K\) Coefficient of Brazier moment analytical solution
- \(L\) Tube length, mm
- \(M_{Brz}\) Brazier moment, Nmm
- \(m_{Brz}\) Normalised Brazier moment
- \(M_{FE}\) Finite element analysis’ instability load, Nmm
- \(n\) Number of plies
- \(r\) Midsurface radius of the tube, mm

- \(t\) Tube thickness, mm
- \(U_1, U_2, U_3\) Material invariants, N/mm²
- \(x_i\) Volume fraction of the i-th laminate layer
- \(\kappa\) Curvature, 1/mm
- \(\lambda_1\) First buckling load, Nmm
- \(\nu_{12}\) Poisson’s ratio
- \(\theta_i\) Fibre orientation angle of the i-th laminate layer, deg
- \(\xi_1, \xi_2, \xi_9, \xi_{10}\) Lamination parameters

**Subscripts and superscripts**

- 4-ply 4-ply symmetric laminate
- equiv equivalent laminate
- min minimum
- opt optimum
- uni unidirectional laminate

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![Fig. 1. Examples of flexible hinges.](image-url)
thickness and material properties. To the best of the authors’ knowledge only one other work [14] explored this perspective and that for a more restricted design space which considered only cross-ply laminates. The present work considers for first time arbitrary symmetric laminates.

To accomplish this goal, firstly, a numerical minimisation of $M_{Brz}$ applied to orthotropic cylindrical shells is performed. The numerical minimisation procedure identifies the presence of material dependency in the problem, which interestingly does not take place in the maximisation of the same quantity [13]. The non-intuitive nature of the problem underlies the value of an analytical solution. The analytical solution for cylindrical shells comprising single ply orientation, unidirectional laminates is provided, expressing the optimum laminate as a function of material properties. However, numerical minimisation shows that further improvement in minimising $M_{brz}$ can be achieved by increasing the number of plies in the laminate. Cylindrical shells comprising 4-ply symmetric laminates represent a good compromise in accuracy between the minimisation results of $M_{brz}$ (without restrictions) and the mathematical complexity of the problem. An empirical expression for cylindrical shells comprising 4-ply symmetric laminates is also offered for practical design purposes. This expression is validated by finite element (FE) analyses which offer insights on the influence of extension-shear and bend-twist couplings into the problem. Finally, conclusions are drawn.

2. Numerical minimisation

The Brazier moment, $M_{brz}$, of an orthotropic cylindrical shell of infinite length under pure bending has been analytically defined by several authors as

$$M_{Brz} = K \cdot r \cdot \sqrt{A_{11}D_{22}}$$  (1)

where $K$ is a constant differing by author, $K = 3.42, 4.22$ or $\pi$ according to Kedward [16], Harursampath and Hodges [17] or Stockwell and Cooper [18], respectively. The term $r$ is the undeformed radius of the tube, while $A_{11}$ and $D_{22}$ are the laminate longitudinal extensional and circumferential bending stiffness, respectively.

Numerical minimisation can identify the optimum stacking sequence of an $n$-ply symmetric laminate giving the minimum Brazier moment, $M_{brz\;\text{min}}$, for a cylindrical shell under pure bending for pre-selected materials. The objective function is $\sqrt{A_{11}D_{22}}$ normalised using the factor $\frac{t}{r^2}$, where $U_1$ is a material invariant (quasi-isotropic modulus) defined by Tsai and Pagano [20] and $t$ is the laminate thickness.

The normalisation factor is the equivalent of $\sqrt{A_{11}D_{22}}$ for a theoretically isotropic material. The normalised objective function is then expressed as

$$m_{brz} = \frac{\sqrt{12}}{\pi U_1} \sqrt{A_{11}D_{22}}.$$  (2)

The optimisation variables are the fibre orientation angles, $\theta_i$, and the volume fractions of the plies, $x_i$. The material properties $E_{11}, E_{22}, G_{12}$ and $\nu_{12}$, as well as, $t$ and the number of plies of the laminate, $n$, are inputs into the minimisation process. The product $A_{11}D_{22}$ is calculated using material invariants and lamination parameters as defined by Tsai and Pagano [20]. The minimisation is performed using the gradient method implemented in Matlab through the function fmincon [21].

Table 2 summarises numerical minimisation results for cylindrical shells comprising $n$-ply symmetric laminates made of a carbon fibre reinforced polymer with properties given in Table 1. These results, interestingly are not the opposite case of the maximisation of the same quantity, which has been reported by Cecchini and Weaver in [13]. They reported that the maximum $M_{brz}$ is found using a [0 0 90] laminate with 62% of 0-deg plies independently of the material used.

Both, the optimum fibre orientation angles and the optimum volume fractions depend on material properties in the minimisation process. A parametric analysis, performed for 4-ply symmetric laminates with stacking sequence $[\theta_1/\theta_3]$, and volume fraction of the inner layers $x_1$, illustrates the material dependency of the results.

The parametric analysis studies the variables $m_{brz\;\text{min}}, \theta_1, \theta_3, \nu_{12}$, and $x_1$ with the change of material ratios $E_{11}/E_{22}, E_{22}/G_{12}$ and $E_{11}/G_{12}$ in the range $6 \leq E_{11}/E_{22} \leq 20, 15 \leq E_{11}/G_{12} \leq 30, 1.5 \leq E_{22}/G_{12} \leq 4.5$. The selection of the range of material ratios was made based on the material properties characteristic of high-stiffness composites. The Poisson’s ratio is considered to be $\nu_{12} = 0.3$ for all materials. This simplification is done since given the value of $\nu_{12}$, its effect in Eq. 1 and Eq. 2 is negligible and $\nu_{12} = 0.3$ is a representative value for the materials under consideration.

Fig. 3 presents the effect of the material ratios on $m_{brz\;\text{min}}$ and on the geometry of the optimum laminate. The ratio $E_{11}/E_{22}$ plays a key role in the minimisation of $m_{brz\;\text{min}}$, increasing $E_{11}/E_{22}$ decreases $m_{brz\;\text{min}}$.

3. Analytical solution for cylindrical shells comprising unidirectional laminates

The material dependency, which became evident from the numerical minimisation given in the previous section, highlights the value of an analytical solution to the problem. The simplest possible analytical problem is the one of cylindrical shells made of unidirectional laminates presented in this section. The product $A_{11}D_{22}$ for the case of unidirectional laminates is

$$\langle A_{11}D_{22}\rangle_{\text{uni}} = \frac{t^4}{12} \left[ U_1 U_3 \cos(4\theta_{\text{uni}}) - U_2^2 \cos^2(2\theta_{\text{uni}}) + U_3^2 \cos^2(2\theta_{\text{uni}}) \right]$$  (3)

where the material invariants $U_1, U_2, U_3$ as defined by Tsai and Pagano [20] represent the quasi-isotropic Young’s modulus, orthotropy properties and shear properties of the material, respectively.

Then, $M_{brz\;\text{uni}}$ is obtained by substituting Eq. 3 in Eq. 1

$$M_{brz\;\text{uni}} = K \cdot r \cdot \frac{t^2}{12} \left[ U_1 + U_3 \cos(4\theta_{\text{uni}}) \right]^2 - U_2^2 \cos^2(2\theta_{\text{uni}}).$$  (4)
Fig. 3. Effect of the material ratios $E_{11}/E_{22}$ and $E_{22}/G_{12}$ on the minimum normalised Brazier moment and the optimum laminate.

Fig. 4. Analytical (A) vs numerical (N) solution for cylindrical shells made of unidirectional laminates from materials varying in ratios $E_{11}/E_{22}$ and $E_{11}/G_{12}$. 
By differentiating Eq. 3 with respect to $\theta_{uni}$ then
\[
\frac{dA_{11}D_{22}}{d\theta_{uni}} = \frac{t^4}{12} \left[ 4U_2^2 \cos(2\theta_{uni}) \sin(2\theta_{uni}) - 8U_1^2 \cos(4\theta_{uni}) \sin(4\theta_{uni}) 
- 8U_1U_3 \sin(4\theta_{uni}) \right].
\]  
(5)

The critical points of $A_{11}D_{22}$ are obtained by equating Eq. 5 to zero and solving for $\theta_{uni}$. Then, the optimum fibre orientation angle, $\theta_{uni opt}$, is expressed as
\[
\theta_{uni opt} = \frac{\pi}{4} \pm \frac{1}{4} \cos \left( \frac{U_1}{U_3} - \frac{U_2}{2U_3} \right)^2
\]  
(6)

and $M_{Brz uni min}$ can be expressed as a function of the material properties of the cylindrical shell by substituting Eq. 6 into Eq. 4
\[
M_{Brz uni min} = \frac{1}{8} K \cdot r \cdot t^2 \left( \frac{U_2}{U_3} \right)^{\frac{1}{3}} \left( 8U_1U_3 - U_2^2 - 8U_3^2 \right). 
\]  
(7)

The analytical solution is validated using a parametric analysis based on the numerical minimisation presented in Section 2. Fig. 4 shows numerical results against the analytical solution for a range of materials varying in ratios $E_{11}/E_{22}$ and $E_{11}/G_{12}$. Close agreement is observed in all cases.

4. Empirical solution for cylindrical shells comprising 4-ply symmetric laminates

A further reduction in $M_{Brz uni min}$ can be achieved by increasing the number of plies in the laminate. However, results from numerical minimisation show that, by increasing the number of plies in the laminate beyond four does not significantly decrease $M_{Brz uni min}$. In fact, Table 2 shows that the percentage difference between $M_{Brz uni min}$ obtained using 10-ply and 4-ply symmetric laminates from the sample material is less than 2%. Therefore, for symmetric laminates comprising more than four plies, the benefits of the presented empirical solution can be utilised by arranging the laminate so that it corresponds to the optimum 4-ply symmetric laminate. Furthermore, the optimum laminate proposed in this section can also be used in the case of specific antisymmetric laminates as is discussed later in the validation of the solution.

The empirical relations presented in this section allow the calculation of the optimum 4-ply symmetric laminate and the respective $M_{Brz uni min}$ to be made from material properties. All empirical relations are defined by using the least squares regression method implemented in Matlab curve fitting tool [21]. The data used for regression comes from numerical minimisation applied to more than 100 materials with $E_{11}/E_{22} > 6, E_{22}/G_{12} > 1.5$ and $E_{11}/G_{12} < 40$.

As Fig. 5 shows, the minimum normalised Brazier moment of 4-ply symmetric laminates, $m_{Brz 4ply uni min}$ can be estimated using a linear function of $m_{Brz uni min}$. The linear approximation is given by
\[
m_{Brz 4ply uni min} = C_1 \cdot m_{Brz uni min} + C_2
\]  
(8)

where $C_1 = 0.83$, $C_2 = 0.062$ and $m_{Brz uni min}$ is obtained from the normalisation of Eq. 7 according to Eq. 2
\[
m_{Brz uni min} = \frac{\sqrt{12}}{8} K \cdot r \cdot t \left( \frac{U_2}{U_3} \right)^{\frac{1}{3}} \left( 8U_1U_3 - U_2^2 - 8U_3^2 \right).
\]  
(9)

Table 2

Results from numerical minimisation for sample material, see Table 1.

| No. of plies | $\theta_{uni opt [deg]}$ | $x_{uni opt}$ | $m_{Brz uni min}$ |
|-------------|-------------------------|---------------|-------------------|
| 1           | 83                      | 0.56          |
| 4           | [8.650.5]               | 0.80          |
| 6           | [7.228.761.3]           | 0.60          |
| 8           | [5.820.838.667.3]       | 0.49          |
| 10          | [4.816.829.145.271.2]   | 0.42          |

Fig. 5. Empirical solution for the normalised minimum Brazier moment.

Results shown in Section 2 and Eq. 1 present a compromise between minimising the stiffnesses $A_{11}$ and $D_{22}$, which are obviously separately minimised by 90 degree and 0 degree ply orientations, respectively. However, the overall minimum is not so simply realised because these same ply orientations maximise the contrasting stiffness property (i.e. 0 degrees maximises $A_{11}$ and 90 degrees $D_{22}$). Therefore, a compromise is required on their relative volume fractions. While the value of $D_{22}$ is most effectively influenced by outer layers (due to partial second moment of area effects), $A_{11}$ is independent of ply position.

As such, 0 degree plies are chosen for the outer layers and 90 degrees for the inner. The proportion of outer to inner layers is characterised by the optimum volume fraction of the inner layers, $x_{uni opt}$, which can be understood as the measure of compromise between minimising $A_{11}$ and $D_{22}$. Note, that placing these fibre angles in the contrasting positions (i.e. the other way around) maximises the Brazier moment, as shown in [13]. A relation between $x_{uni opt}$ and the normalised ratio $\left( \frac{t^2 A_{11}}{12 D_{22}} \right)_{uni opt}$ of the optimum unidirectional laminate, $\left( \frac{t^2 A_{11}}{12 D_{22}} \right)_{uni opt}$, is straightforwardly observed and is approximated by using a fourth-order polynomial which effectively captures the nonlinearity of the identified relation. This function, shown in Fig. 6 is given by
\[
x_{uni opt} = C_3 \left( \frac{t^2 A_{11}}{12 D_{22}} \right)_{uni opt}^4 + C_4 \left( \frac{t^2 A_{11}}{12 D_{22}} \right)_{uni opt}^3 + C_5 \left( \frac{t^2 A_{11}}{12 D_{22}} \right)_{uni opt}^2 + C_6 \left( \frac{t^2 A_{11}}{12 D_{22}} \right)_{uni opt} + C_7
\]  
(10)

where $C_3 = -9.86 \times 10^{-7}, C_4 = 8.82 \times 10^{-6}, C_5 = 1.13 \times 10^{-3}, C_6 = -3.32 \times 10^{-2}$ and $C_7 = 0.51$, and have been evaluated by regression. The ratio $\left( \frac{t^2 A_{11}}{12 D_{22}} \right)_{uni opt}$ is calculated by substituting Eq. 6 into the ratio $\left( \frac{t^2 A_{11}}{12 D_{22}} \right)$ for a unidirectional laminate, to give

Fig. 6. Empirical solution of the optimum volume fraction in inner layers for 4-ply laminates.
Regarding optimum fibre orientation angles in the inner, $\theta_{1\text{opt}}$, and outer layers, $\theta_{2\text{opt}}$, Fig. 7 shows that the difference between them is governed by material orthotropy and shear stiffness properties. Fig. 8 shows that, for materials with higher orthotropy values ($E_{11}/E_{22} > 7$), $\theta_{2\text{opt}}$ can be effectively estimated by equating its value to the optimum fibre orientation angle of a unidirectional laminate, $\theta_{uni\text{opt}}$. The value of $\theta_{uni\text{opt}}$ is given by Eq. 6. The difference between the values of $\theta_{2\text{opt}}$ and $\theta_{uni\text{opt}}$ grows for materials with lower values of orthotropy, see Fig. 7. However, the maximum difference is less than 2.5 degrees.

Since $\theta_{2\text{opt}}$ and $\theta_{uni\text{opt}}$ are almost identically valued, the additional attainable reduction of $M_{Brz\text{min}}$ offered by 4-ply symmetric laminates can be directly attributed to the inner layers. As mentioned previously, the minimisation of $M_{Brz\text{min}}$ represents a compromise in minimising contrasting stiffness properties, i.e. axial and circumferential. Due to this compromise, material orthotropy plays an important role in the minimisation task. In fact, acting as an additional variable to minimise $M_{Brz\text{min}}$, the inner layers depend on material orthotropy, see Fig. 9. The optimum fibre orientation angle in the inner layers, $\theta_{1\text{opt}}$, can be approximated by using a linear function of the orthotropy ratio $E_{22}/E_{11}$, see Fig. 9.

\[
\left(\frac{t^2}{12} \frac{A_{11}}{D_{uni\text{opt}}}ight)_{\text{opt}} = \frac{U_2 + \sqrt{2} \sqrt{U_3^2 + 4U_3^2 - 4U_1U_3}}{U_2 - \sqrt{2} \sqrt{U_3^2 + 4U_3^2 - 4U_1U_3}}
\]

(11)

Regarding optimum fibre orientation angles in the inner, $\theta_{1\text{opt}}$, and outer layers, $\theta_{2\text{opt}}$, Fig. 7 shows that the difference between them is governed by material orthotropy and shear stiffness properties. Fig. 8 shows that, for materials with higher orthotropy values ($E_{11}/E_{22} > 7$), $\theta_{2\text{opt}}$ can be effectively estimated by equating its value to the optimum fibre orientation angle of a unidirectional laminate, $\theta_{uni\text{opt}}$. The value of $\theta_{uni\text{opt}}$ is given by Eq. 6. The difference between the values of $\theta_{2\text{opt}}$ and $\theta_{uni\text{opt}}$ grows for materials with lower values of orthotropy, see Fig. 7. However, the maximum difference is less than 2.5 degrees.

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\[
\theta_{1\text{opt}} = C_8 \frac{E_{22}}{E_{11}} + C_9
\]

(12)

where $C_8 = 89.7$ deg and $C_9 = 45.7$ deg.

4.1. Validation

The results from the empirical solution for 4-ply symmetric laminates were validated using FE analysis. A parametric FE study was performed for a number of cylindrical shells where the instability load, $M_{Brz}$, is studied as a function of layup. $M_{Brz}$ is obtained from geometrically non-linear static analyses using the Riks arc-length algorithm in ABAQUS. Results from minimisation, for a given material, are verified if the optimum layup from minimisation coincides with the layup...
giving the lowest $M_{Brz}$ in the parametric FE analysis. The results from this validation study are presented in detail for the sample material, see Table 1.

For each circular tube two FE analyses were performed, a linear buckling analysis (LBA) and a geometrically non-linear static analysis (GNA) using the arc-length Riks-algorithm. LBAs were performed for perfect circular tubes to extract the first buckling load, $\lambda_1$, and its corresponding mode shape which is applied as an initial imperfection with maximum amplitude equal $t/100$ in the GNA. Little research is available in literature regarding the effect of imperfections for cylindrical shells under bending. An extensive study on imperfection sensitivity by Fajuyitan et al. [22] shows that cylindrical shells under bending are less imperfection sensitive than the classical compression load case. Interestingly, they found that the widely adopted first buckling mode imperfection is not the most deleterious for the bending load case. Moreover, they show that imperfection sensitivity is length-dependent and less severe for cylindrical shells whose length allows the full development of ovalisation, as those considered in this work. Therefore, the imperfections included in the GNA should not decrease the instability load significantly while they still serve the objective of facilitating convergence of the arc-length Riks algorithm. 

Fig. 10 shows the instability load for a cylinder with stacking sequence [0 90], and $x_1 = 0.33$ calculated using GNAs with different imperfection amplitudes. Both analyses (LBA and GNA) share the boundary conditions illustrated in Fig. 11. The circular tube was modelled in its full length without the use of symmetry. Both models use the quadratic shell element S8R5 with an element size of 12 mm and aspect ratio 1. This element size corresponds to the converged solution for $\lambda_1$ of the LBA analysis and the critical load and curvature of the GNA analysis.

It is worth noting that a 38% decrease in element size delivers a maximum 0.5% change of $\lambda_1$, critical curvature and critical load. The displacements and rotations at both ends of the tube are linked to reference points, RP1 and RP2 located at the centre of the cross-section, through kinematic coupling. These reference points have all degrees of freedom constrained except the rotation with respect to the x-axis, $\varphi_x$, and the displacement in z-direction, $u_z$. Then $u_z^{RP1} = u_z^{RP2} = 0$. Bending moments, $M_z$ of same magnitude and opposite sense are applied through the reference points. Two additional boundary conditions are applied to the mid-span of the tube at the furthest points from the neutral plane, i.e. points A and B. The displacement in the z-direction is constrained for these points $u_z^{A,B} = 0$.

Tubes examined have dimensions $r = 100$ mm and $t = 1$ mm. Careful consideration was provided into the choice of the tube length, $L$, to ensure that boundary conditions have a negligible effect on $M_{Brz}$. The length $L = 10$ m was determined following a parametric length FE analysis. Noting that a 50% increase of this length resulted in less than 2% change of the considered variables ($M_{FE}, \lambda_1$ and both curvature and mid-span flattening at moment of instability).

The model was validated using the FE results from Rotter et al. [23] as a benchmark, for a circular tube having dimensions $r = 100$ mm,

\[ L = 4000 \text{ mm}, \quad t = 1 \text{ mm and made from steel, } E = 210 \text{ GPa and } \nu = 0.32. \]

Fig. 12 shows the comparison of equilibrium curves from Rotter et al. [23] and those obtained in the present study. Both moment and curvature are normalised using their respective values of the Brazier limit point $M_{Brz}$ and $\kappa_{Brz}$ for isotropic circular tubes defined as

\[ M_{Brz} = 0.987 \frac{Et^2}{\sqrt{1-\nu^2}} \quad (13) \]

\[ \kappa_{Brz} = 0.314 \frac{t}{r^2/\sqrt{1-\nu^2}} \quad (14) \]

In the following, $M_{FE}$ corresponds to the first decrease in load of the FE equilibrium curve.

Results show that for the sample material the optimum laminate from both optimisation and parametric FE analysis differ. The optimum laminate from parametric FE analysis has stacking sequence [30/85], with $x_1 = 0.3$ while the optimum laminate from the empirical solution has stacking sequence [9/51], with $x_1 = 0.2$ and $M_{FE}$ 31% higher than the optimum laminate from parametric FE analysis. As such, neither the empirical solution nor the numerical optimisation process match FE results.

The discrepancy could be attributed to the objective function used, which is based on the product $A_{11}D_{22}$ and which was identified from analysis involving orthotropic materials but may not be applicable to arbitrary laminates exhibiting anisotropic effects. This hypothesis contradicts the assumptions of the analytical solution of $M_{Brz}$ from Harursampath and Hodges [17]. An implication of this hypothesis is the possibility that extension-shear and bend-twist couplings play a signif-
icant role in the calculation of $M_{Brz}$ for circular tubes with non-negligible stiffness values $A_{11}$, $A_{22}$, $D_{11}$, and $D_{22}$. Then, the objective function would fail to take into account the effect of these couplings in the minimisation process.

The influence of extension-shear and bend-twist couplings was therefore investigated by repeating the parametric FE analyses using equivalent laminates having identical stiffness values $A_{11}$, $A_{22}$, $A_{44}$, $D_{11}$, $D_{12}$, $D_{23}$ and $D_{66}$ but with stiffnesses $A_{16}$, $A_{26}$, $D_{16}$, $D_{26}$ equal to zero. Then for each laminate from the parametric FE analysis, having stacking sequence $[\theta_1/\theta_2]_l$, its corresponding equivalent orthotropic laminate has stacking sequence $[\theta_1/\theta_2/\theta_3/\theta_4]_l$, with ply thickness scaled, respectively. Boundary conditions, loads and applied initial imperfections were identical to those previously described in the FE model description of this section. Therefore, the influence of extension-shear and bend-twist couplings can be confirmed if the optimum equivalent laminate from FE analysis coincides with its respective 4-ply symmetric laminate from the minimisation routine.

Table 3 presents a selection of 10, from the 21, 4-ply symmetric laminates used during the validation process. The geometry of the laminates together with the percentage difference in $M_{Brz}$ between 4-ply symmetric laminates and equivalent laminates is presented. A maximum difference of 63% in $M_{Brz}$ is found. The effect of extension-shear and bend-twist couplings is evidently not negligible for minimisation purposes. Moreover, for the case of equivalent laminates, the optimum laminate from the layup parametric FE analysis is laminate 6 with $M_{Brz}$ 1.5% less than laminate 10, which is the optimum from the empirical solution of 4-ply symmetric laminates. This difference is negligible for practical purposes, therefore, the optimisation delivers reliable results for laminates with negligible extension-shear and bend-twist couplings. The benefits delivered by the minimisation of 4-ply symmetric laminates can be exploited using the stacking sequence of equivalent laminates.

To model laminates with non-negligible extension-shear and bend-twist couplings, a more robust analytical formulation of $M_{Brz}$ should be used as objective function so that the effect of these stiffnesses is considered. To the best of the authors’ knowledge, there is no available solution of $M_{Brz}$ in literature that would effectively consider the effect of these couplings. It should be highlighted that these couplings can be beneficial for minimisation purposes. In fact, the laminate performing best in the parametric FE analysis, for the sample material, exhibits $M_{Brz}$ 31% less than the optimum laminate predicted ignoring the effect of the couplings.

5. Conclusions

This work presents an extensive study of the minimisation of the Brazier moment to enhance the design of orthotropic cylindrical flexible hinges. A numerical minimisation, applied to more than 100 materials, shows that the optimum laminate and the minimum attainable folding load is material dependent, especially on material orthotropy. Numerical results show that materials with larger orthotropy ratios offer smaller minimum attainable folding loads. For the sample carbon fibre reinforced polymer with $E_{11}/E_{22} = 18$, the folding load of a 4-ply symmetric optimum laminate is 47% smaller than that of a cylindrical shell made from an equivalent theoretical isotropic material.

The material dependency of the optimum laminate, which offers additional degrees of freedom for the design of flexible hinges, motivated an analytical solution to be sought for the problem which enables the optimum laminate to be identified based on material properties. The analytical solution for cylindrical shells comprising unidirectional laminates is found. Results showed that the minimum attainable folding load can be reduced by increasing the number of plies in the laminate. However, increasing the number of plies beyond four does not decrease significantly the minimum folding load. Therefore, an empirical solution for the optimum 4-ply symmetric laminate was found based on the analytical solution for cylindrical shells comprising unidirectional laminates and a parametric analysis. FE validation of the empirical solution shows that, contrary to the assumptions of Harursampath and Hodges [17], the calculation of the Brazier moment based on the product of the stiffness properties $A_{11}$ and $D_{22}$ does not apply to arbitrarily oriented symmetric laminates, but only to laminates having negligible extension-shear and bend-twist couplings. Therefore, the design space to which the minimisation process applies should consider laminates with negligible stiffnesses $A_{16}$, $A_{26}$, $D_{16}$, $D_{26}$. This can be achieved by using alternative stacking sequences having stiffnesses $A_{11}$, $A_{22}$, $D_{11}$, and $D_{22}$ identical to those of the optimum 4-ply symmetric laminates, but with $A_{16} = A_{26} = D_{16} = D_{26} = 0$. For example, for the stacking sequence $[\theta_1/\theta_2/\theta_3/\theta_4]_l$, with ply thickness scaled, the additional degrees of freedom for the design of flexible hinges and it will be considered in future work. The material dependency on optimal laminates for minimal Brazier moment and the potential positive influence of extension-shear and bend-twist couplings in the problem highlight the wide-ranging design potential of flexible hinges made from non-conventional laminates.

Data availability

The raw/processed data required to reproduce these findings cannot be shared at this time as the data also forms part of an ongoing study.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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