D I S C R E M I N A T I V E L E A R N I N G F O R P R O B A B I L I S T I C C O N T E X T - F R E E G R A M M A R S B A S E D O N G E N E R A L I Z E D H - C R I T E R IO N

A P R E P R I N T

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A B S T R A C T

We present a formal framework for the development of a family of discriminative learning algorithms for Probabilistic Context-Free Grammars (PCFGs) based on a generalization of criterion-H. First of all, we propose the H-criterion as the objective function and the Growth Transformations as the optimization method, which allows us to develop the final expressions for the estimation of the parameters of the PCFGs. And second, we generalize the H-criterion to take into account the set of reference interpretations and the set of competing interpretations, and we propose a new family of objective functions that allow us to develop the expressions of the estimation transformations for PCFGs.

K e y w o r d s  Discriminative Learning · Probabilistic Context-Free Grammars · H-criterion · Growth Transformations.

1 Introduction

Throughout time, there has been an interest in Probabilistic Context-Free Grammars (PCFGs) for use in different tasks within the framework of Syntactic Pattern Recognition [1, 2, 3] and Computational Linguistics [4, 5]. The reason for this can be found in the capability of PCFGs to model the long-term dependencies established between the different linguistic units of a sentence, and the possibility of incorporating the probabilistic information which allows for adequate modeling of the variability phenomena that are always present in complex problems.

Given a training sample, the problem of learning a PCFG can be stated as an estimation process of the parameters of the PCFG. To tackle this estimation process two aspects have to be considered: proposing an optimization method and defining a certain objective function. The optimization method we have considered is based on the Growth Transformation framework [6, 7], and the classic objective function is based on the Maximum Likelihood Estimation (MLE) criterion.

Discriminative learning is a method that arises to improved recognition accuracy for Natural Language Processing (NLP) problems ([8, 9, 10]). In this learning framework, several objective functions were proposed: Maximum Mutual Information (MMI) or Conditional Maximum Likelihood Estimation, among others.

Some discriminative techniques for parsing ([11, 5]) get the features of their parsers of training set by methods used in the unlexicalized generative parser for parsing treebank. In general, only a parser for each sample is used so that the resulting grammar is tractable.

Previous researches in Pattern Recognition have shown that parameters estimation using discriminative techniques provides better performance than the MLE training criterion. When MLE is used in parsing, the parameters are reestimated to increase the likelihood of the parsers of the training set without taking into count the probability of
the other possible parsers. Whilst discriminative training techniques consider possible competitors for the parser and reduce the probability of incorrect parser.

In this work, we propose a discriminative method for learning PCFGs based on a generalization of the H-criterion. Our formal framework allows us to simultaneously consider multiple reference trees. We used growth transformation as an optimization method to estimate parameters and we noticed that it converges quickly. We build several discriminative algorithms using the formal framework and they can be implemented using well-known algorithms.

The paper is organized as follows. First, we present some related works. Section 3 introduces the notation related to PCFGs. Section 4, the H-criterion is presented as an objective function to solve the estimation problem of PCFGs through the growth transformations method. In Section 5, we propose a new generalization of the H-criterion and we present some discriminative algorithms based on this H-criterion. Finally, we present the conclusions and future work.

2 Related work

Several parsers ([11, 5]), based on discriminative training, use as input a generative component to increase speed and accuracy. In ([11]), they employ for the parser a max-margin principle of support vector machines. They transform the parsing problem to an optimization problem of a quadratic program, which is solved through their dual problem. They train and test on ≤15 word sentences.

The parser presented in ([4]) has no as input a generative component as in ([11]). They use boost decision trees to select compound features incrementally. For this, the parser implements the search using an agenda that stores entire states to build trees of decision. The parser improved training time concerning the parser of ([11]), which uses five days instead of several months.

In ([5]), they showed dynamic programming based feature-rich discriminative parser. They defined a model based on Conditional Random Fields used as a principal optimization method for stochastic gradient descent. They performed their experiments taking into account two types of features: lexicon features, which are over words and tags, and grammar features which obtained information of the component generative. The parser was trained and tested on sentences of length ≤15 and too was trained and tested on sentences of length ≤40.

3 Preliminaries

Before addressing the study of the discriminative estimation of PCFGs using the H-criterion, we first introduce the notation about SCFGs that is used in this work.

A Context-Free Grammar (CFG) G is a four-tuple \((N, \Sigma, S, P)\), where \(N\) is a finite set of non-terminals, \(\Sigma\) is a finite set of terminals \((N \cap \Sigma = \emptyset)\), \(S \in N\) is the initial non-terminal, and \(P\) is a finite set of rules: \(A \rightarrow \alpha, A \in N, \alpha \in (N \cup \Sigma)^+\) (we only consider grammars with no empty rules). A CFG in Chomsky Normal Form (CNF) is a CFG in which the rules are of the form \(A \rightarrow BC\) or \(A \rightarrow a\) \((A, B, C \in N\) and \(a \in \Sigma)\). A left-derivation of \(x \in \Sigma^+\) in \(G\) is a sequence of rules \(d_x = (q_1, q_2, \ldots, q_m)\), \(m \geq 1\), such that: \((S \xrightarrow{q_1} \alpha_1 \xrightarrow{q_2} \alpha_2 \ldots \xrightarrow{q_m} x)\), where \(\alpha_i \in (N \cup \Sigma)^+\), \(1 \leq i \leq m - 1\) and \(q_i\) rewrites the left-most non-terminal of \(\alpha_{i-1}\). The language generated by \(G\) is defined as \(L(G) = \{x \in \Sigma^+ | S \xrightarrow{*} x\}\). A CFG is called ambiguous, if for some \(x \in L(G)\), there exists more than one left-derivation.

A Probabilistic Context-Free Grammar (PCFG) is defined as a pair \(G_p = (G, p)\), where \(G\) is a CFG and \(p : P \rightarrow [0, 1]\) is a probability function of rule application such that \(\forall A \in N: \sum_{i=1}^{n_A} p(A \rightarrow \alpha_i) = 1\); where \(n_A\) is the number of rules associated to \(A\).

Let \(G_p\) be a PCFG. Then, for each \(x \in L(G)\), we denote \(D_x\) as the set of all left-derivations of the string \(x\). The expression \(N(A \rightarrow \alpha, d_x)\) represents the number of times that the rule \(A \rightarrow \alpha\) has been used in the derivation \(d_x\), and \(N(A, d_x)\) is the number of times that the non-terminal \(A\) has been derived in \(d_x\). Obviously, this equation is satisfied: \(N(A, d_x) = \sum_{i=1}^{n_A} N(A \rightarrow \alpha_i, d_x)\).

Then, we define the following expressions:

- Probability of the derivation \(d_x\) of the string \(x\) as
  \[
  P_{G_p}(x, d_x) = \prod_{\forall (A \rightarrow \alpha) \in P} p(A \rightarrow \alpha)^{N(A \rightarrow \alpha, d_x)},
  \]
where \( \Omega = \sum \) The probability of the derivation \( d \) as

\[
P_{G_p}(x) = \sum_{d_x \in D_x} P_{G_p}(x, d_x),
\]

(1)

Probability of the best derivation of the string \( x \) as

\[
\hat{P}_{G_p}(x) = \max_{d_x \in D_x} P_{G_p}(x, d_x),
\]

(2)

Best derivation of the string \( x \) as

\[
\hat{d}_x = \arg \max_{d_x \in D_x} P_{G_p}(x, d_x).
\]

The probability of the derivation \( d_x \) of the string \( x \) can be interpreted as the joint probability of \( x \) and \( d_x \), so as the probability of the string \( x \) can be interpreted as the marginal probability of \( x \).

Given \( \Delta_x \subseteq D_x \), a finite subset of derivations of \( x \), we can also define:

- Probability of \( x \) with respect to \( \Delta_x \),

\[
P_{G_p, \Delta_x}(x) = \sum_{d_x \in \Delta_x} P_{G_p}(x, d_x)
\]

- Probability of the best derivation of \( x \) with respect to \( \Delta_x \),

\[
\hat{P}_{G_p, \Delta_x}(x) = \max_{d_x \in \Delta_x} P_{G_p}(x, d_x)
\]

These expressions respectively coincide with expression (1) and (2) when \( \Delta_x = D_x \).

Finally, the language generated by a PCFG \( G_p \) is defined as: \( L(G_p) = \{ x \in L(G) | P_{G_p}(x) > 0 \} \). A PCFG \( G_p \) is said to be consistent (14) if the language generated by \( G_p \) is a probabilistic language, that is, \( \sum_{x \in L(G_p)} P_{G_p}(x) = 1 \).

Next we tackle the problem of estimating the parameters of a PCFG \( G_p = (G, p) \). This problem can be stated as follows: given a probabilistic language \( L_p = (L, \Phi) \) where \( L \) is a language and \( \Phi \) is a probabilistic distribution over \( L \), and given a training sample \( \Omega \), the estimation process consists in learning the parameters of \( G_p \) in order to represent \( \Phi \) by means of the probability (1). Assuming that \( \Omega \) is a representative sample made up of a multi-set from \( L \) according to \( \Phi \), and assuming that \( \Phi \) can be represented by \( G_p \), the estimation of the parameters \( p \) of \( G_p \) is made by:

\[
\hat{p} = \arg \max_p f_p(\Omega),
\]

where \( f_p(.) \) is an objective function to be optimized. Two issues have to be considered: the optimization method and the selection of an objective function. In this paper we consider an optimization method, based on the growth transformation (GT) framework [6,7], and an objective function derived from a generalization of the H-criterion [15].

4 PCFGs estimation based on growth transformations and H-criterion

In this section we use the restricted H-criteria as objective function for discriminative training of a PCFG adapting it to the most recent notation [16, 17, 9]. We will first present the H-criterion and then we will develop the method of growth transformations, applying the H-criterion, to implement a discriminative learning method for estimating the parameters of a PCFG.

4.1 H-criterion

The H-criterion based learning framework was proposed by Gopalakrishnan, et al. in [15], as a generalization of the estimators of maximum likelihood (ML), maximum mutual information (MMI) and conditional maximum likelihood (CML). It is defined as follows: Let \( a, b, c \) constants with \( a > 0 \). An H-estimator \( \hat{\theta}(a, b, c) \) is obtained by minimizing the H-criterion.

\[
H_{a,b,c}(\theta; \Omega) = -\frac{1}{n} \sum_{i=1}^{n} \log p_{\hat{\theta}}^a(x_i, y_i) p_{\hat{\theta}}^b(x_i) p_{\hat{\theta}}^c(y_i)
\]

(3)

Where \( \Omega = \{(x_i, y_i)\}_{i=1}^{N} \) denotes the training sample, \( x_i \) are the observations and \( y_i \) are the interpretations, and \( \theta \) are the parameters of the model.

Thus the ML estimator is \( \hat{\theta}(1, 0, 0) \), the MMI estimator is \( \hat{\theta}(1, -1, -1) \) and the CML estimator is \( \hat{\theta}(1, 0, -1) \) [15].
4.2 Objective function based on H-criterion

Given a PCFG, $G_p$, a training sample $\Omega$ and a set of derivations $\Delta_x$, for each $x \in \Omega$, the estimation of the probabilities of $G_p$ can be obtained through the H-criterion minimizing the following estimator (see (3)),

$$H_{1,-h,0}(G_p, \Omega) = -\frac{1}{|\Omega|} \sum_{x \in \Omega} \log \frac{P_{G_p}(x, d_x)}{P_{G_p}(x)^h} = -\frac{1}{|\Omega|} \log \prod_{x \in \Omega} \frac{P_{G_p}(x, d_x)}{P_{G_p}(x)^h}$$

(4)

where $0 \leq h < 1$. In practice, the best derivation $\hat{d}_x$ is used, and the total probability $P_{G_p}(x)$ is expanded by marginalizing over all derivations of $x$ and maximizing the objective function,

$$F_h(G_p, \Omega) = \prod_{x \in \Omega} \frac{P_{G_p}(x, \hat{d}_x)}{\left(\sum_{d_x \in \Delta_x} P_{G_p}(x, d_x)^h\right)^h} = \prod_{x \in \Omega} \frac{P_{G_p}(x, \hat{d}_x)}{P_{G_p}(x, \Delta_x)^h}$$

(5)

The sum in the denominator of (5) is the probability of $x$ with respect to $\Delta_x$, where $\Delta_x$ denotes the set of discriminated or competing derivations. In this case, observations are input strings, $x \in \Omega$, and interpretations are the corresponding left-derivation sequences, $d_x$. If $h > 0$ the H-criterion can be viewed as a discriminative training method. The exponent $h$ aims to establish the degree that competing derivations discriminate against the derivation of reference. A optimization of H-criterion attempts simultaneously to maximize the numerator term $P_{G_p}(x, \hat{d}_x)$ and to minimize denominator term $P_{G_p}(x, \Delta_x)^h$ for each string $x \in \Omega$ in the training sample.

Since $F_h(G_p, \Omega)$ is a rational function, the reduction of the case of rational functions to polynomials proposed in [18] can be applied.

$$P_\pi(G_p, \Omega) = \prod_{x \in \Omega} P_{G_p}(x, \hat{d}_x) - \left(F_h(G_p, \Omega)\right)_\pi \prod_{x \in \Omega} P_{G_p}(x, \Delta_x)^h$$

(6)

Where $\left(F_h(G_p, \Omega)\right)_\pi$ is the constant that results from evaluating $F_h(G_p, \Omega)$ at $\pi$ [18]. $\pi$ is a point of the domain (in our case $\pi$ will be the probabilities of the rules of $G_p$).

4.3 Growth transformations for rational functions

The objective function based on the H-criterion, and developed in equations (5) and (6), can be optimized by growth transformations for rational functions [18]. And the following final expression is obtained (see Appendix A),

$$\bar{p}(A \rightarrow \alpha) = \frac{\sum_{x \in \Omega} \left[N(A \rightarrow \alpha, \hat{d}_x) - \frac{h}{P_{G_p}(x, \Delta_x)} \sum_{d_x \in \Delta_x} N(A \rightarrow \alpha, d_x) P_{G_p}(x, d_x)\right] + p(A \rightarrow \alpha) \bar{C}}{\sum_{x \in \Omega} \left[N(A, \hat{d}_x) - \frac{p(A \rightarrow \alpha)}{P_{G_p}(x, \Delta_x)} \sum_{d_x \in \Delta_x} N(A, d_x) P_{G_p}(x, d_x)\right] + \bar{C}}$$

(7)

The value of the constant is $C = \bar{C} \prod_{x \in \Omega} P_{G_p}(x, \hat{d}_x)$. Gopalakrishnan et al. suggested in [18] that to obtain a fast convergence the constant $\bar{C}$ should be calculated by means of the approximation,

$$\bar{C} = \max \left\{ \max_{p(A \rightarrow \alpha)} \left\{ -\frac{1}{p(A \rightarrow \alpha)} \left[ \sum_{x \in \Omega} N(A \rightarrow \alpha, \hat{d}_x) - \frac{h}{P_{G_p}(x, \Delta_x)} \sum_{d_x \in \Delta_x} N(A \rightarrow \alpha, d_x) P_{G_p}(x, d_x) \right] \right\}, 0 \right\} + \epsilon$$

where $\epsilon$ is a small positive constant.

The growth transformation method allows us to easily obtain the estimation of the probabilities of a PCFG $\bar{G}_p = (G, \bar{p})$ from expression (7). An iterative estimation process can be defined from transformation (7). This process is carried out in two steps on an initial PCFG until a local maximum is achieved. In each iteration, first the set $\Delta_x$ is computed for each $x \in \Omega$ according to the selected criterion and then, transformation (7) is applied and a new PCFG is obtained (more details can be found in Appendix A).
5 Generalization of the H-criterion for discriminative estimation of PCFGs

In this section, we will explain the sense in which the H-criteria is restricted, then we present the generalized H-criteria. Finally, we find a growth transformation for generalized H-criteria and we define some discriminative algorithm related.

5.1 Generalized H-criterion

Note that in expression (5) we are considering that each training sample has only a possible reference interpretation (numerator in (5)). Nevertheless, given the extremely ambiguous nature of the models, the strings of training sample presumably have more than one interpretation (parsing). Therefore we will generalize the H-criteria to take into account this event adapting it to unified criterion style introduced by [16]. We define the generalized H-criteria as:

\[ F_h(G_p, \Omega) = \prod_{x \in \Omega} \frac{P^n_{G_p}(x, \Delta^c_x)}{P^c_{G_p}(x, \Delta^c_x)^h} \]  

(8)

where \( 0 < \eta, \ 0 \leq h < 1 \) and \( \Delta^c_x \subset \Delta^c \). The set \( \Delta^c_x \) must contain only derivations of correct parsing of the sentence \( x \) while the set \( \Delta^c_x \) must contain competing derivations of any parsing of the sentences. Furthermore, it is satisfied that \( F_h(G_p, \Omega) = F_h(G_p, \Omega) \) when \( \Delta^c_x = \{ \tilde{d}_x \} \), \( \Delta^c = \Delta_x \) and \( \eta = 1 \), therefore, it have the same set of maximum points. Finally, if \( h > 0 \) we can conclude that the H-criterion can be viewed as a discriminative training method.

5.2 Growth transformations for generalized H-criterion

The new objective function obtained from the generalization of the H-criterion (8) can be optimized by means of growth transformations for rational functions. In a similar way to that used in [6] [18], the rational function \( F_h \) can be reduced to the polynomial function,

\[ Q_x(G_p, \Omega) = \prod_{x \in \Omega} P^n_{G_p}(x, \Delta^c_x) - \left( F_h(G_p, \Omega) \right)_\pi \prod_{x \in \Omega} P^c_{G_p}(x, \Delta^c_x)^h \]  

(9)

Where \( \left( F_h(G_p, \Omega) \right)_\pi \) is the constant that results from evaluating \( F_h(G_p, \Omega) \) at \( \pi [18] \). As in the previous case, \( \pi \) is a point of the domain (for us, \( \pi \) will be the probabilities of the rules of \( G_p \)).

As in the previous case, the new objective function obtained from the generalization of the H-criterion (8) and (9) can be optimized by growth transformations for rational functions [18]. The complete development can be found in Appendix B, and the final expression is as follows,

\[ \tilde{p}(A \rightarrow \alpha) = \frac{D_{A \rightarrow \alpha}(\Delta^c_x) - h D_{A \rightarrow \alpha}(\Delta^c_x) + p(A \rightarrow \alpha) \tilde{C}}{D_A(\Delta^c_x) - h D_A(\Delta^c_x) + \tilde{C}} \]  

(10)

where,

\[ D_{A \rightarrow \alpha}(\Delta_x) = \sum_{x \in \Omega} \frac{1}{P^n_{G_p}(x, \Delta_x)} \sum_{d_x \in \Delta_x} N(A \rightarrow \alpha, d_x) P^n_{G_p}(x, d_x) \]

\[ D_A(\Delta_x) = \sum_{x \in \Omega} \frac{1}{P^n_{G_p}(x, \Delta_x)} \sum_{d_x \in \Delta_x} N(A, d_x) P^n_{G_p}(x, d_x) \]

The value of the constant is \( C = \tilde{C} \eta \prod_{x \in \Omega} P^n_{G_p}(x, \Delta_x) \). Following Gopalakrishnan et al. in [18], and carrying out a similar development to Section 4 to obtain a fast convergence the constant \( \tilde{C} \) should be calculated by means of the approximation,

\[ \tilde{C} = \max \left\{ \max_{p(A \rightarrow \alpha)} \left\{ - \frac{D_{A \rightarrow \alpha}(\Delta^c_x) - h D_{A \rightarrow \alpha}(\Delta^c_x)}{p(A \rightarrow \alpha)} \right\}, 0 \right\} + \epsilon \]

where \( \epsilon \) is a small positive constant.
6 Discriminative algorithms based on generalized H-criterion

From transformation (10), a family of discriminative learning algorithms for PCFGs can be defined depending on how the set of reference derivations \( \Delta_r^c \) and the set of competing derivations \( \Delta_r^c \) are obtained, and the values of the parameters \( \eta \) and \( h \). For the algorithms studied here, we only analyze the effect of \( h \) over the optimization framework and fix \( \eta = 1 \).

The first issue to address is how the set of competing derivations can be obtained. If \( \Delta_r^c \) is the set of all possible derivations, we can calculate it using the well-known Inside algorithm [1]. If \( \Delta_r^c \) is the set of n-best derivations, we can calculate it using a n-best parsing algorithm [19] and [20]. Even, \( \Delta_r^c \) may be the set of competing derivations that are compatible with a bracketed sample; in that case, we can use the bracketed Inside algorithm [21] and [22].

The second issue to consider is how the set of reference derivations \( \Delta_r^c \) is obtained. In any case, it must be satisfied that \( \Delta_r^c \subseteq \Delta_r^c \). The set of reference derivations \( \Delta_r^c \) may be the best derivation, \( \{ \hat{d}_r \} \), and can be calculated with the well-known Viterbi algorithm [2]. \( \Delta_r^c \) may be the n-best derivations, and can be calculated with the n-best parsing algorithm [20]. Or \( \Delta_r^c \) may be the best derivation that is compatible with a bracketed sample, and can be calculated with the bracketed Viterbi algorithm [22].

6.1 Properties of the estimated models

An important issue is the study of the PCFG’s properties estimated by discriminative algorithms based on the generalized H-criterion. More specifically, if the estimated PCFG generates a probabilistic language, that is, if the estimated PCFG is consistent. Then we discuss the role of

\[
\Delta_r^c \subseteq \Delta_r^c
\]

and fix \( h \). For the algorithms studied here, we only analyze the effect of \( h \) over the optimization framework and fix \( \eta = 1 \).

As can be seen in (8), if \( h = 0(\eta = 1) \) and \( \Delta_r^c = \{ \hat{d}_r \} \), the transformation (10) is equivalent to the estimation algorithm based on the Viterbi Score (VS) [2]. It is well known that PCFGs estimated by the VS algorithm are always consistent [23].

Next we explore what happens when \( h = 1 \), and we show that we cannot guarantee the consistency of the estimated models when \( h = 1 \). For this, we consider that the reference derivations is selected as the best derivation, \( \{ \hat{d}_r \} \), and the set of competing derivations is the set of all possible derivation, \( \Delta_r^c = D_x \). And we illustrate this with an example:

Let \( G_p \) be an initial PCFG, where \( N = \{ S \} \); \( \Sigma = \{ a \} \); and \( P = \{(S \rightarrow S S, [q]), (S \rightarrow a, [1 - q])\} \). We know that for values of \( q \) that satisfy, \( 0.5 < q < 1 \), the grammar \( G_p \) is not consistent.

When training sample is \( \{ aa, aaaa \} \), there is only one derivation for aa with probability \( q(1 - p)^2 \) and there are five derivations for aaaa each one of these with probability \( q^3(1 - q)^4 \). Applying the transformation (10) we obtain,

\[
p(S \rightarrow SS) = \frac{(1 - h) + 3 (1 - h) + q C}{3 (1 - h) + 7 (1 - h) + C} = \frac{4 (1 - h) + q C}{10 (1 - h) + C},
\]

\[
p(S \rightarrow a) = \frac{2 (1 - h) + 4 (1 - h) + (1 - q) C}{3 (1 - h) + 7 (1 - h) + C} = \frac{6 (1 - h) + (1 - q) C}{10 (1 - h) + C}.
\]

If \( h = 1 \) then \( \hat{p}(S \rightarrow SS) = q \) and \( \hat{p}(S \rightarrow a) = (1 - q) \) preserving the inconsistency. However, if \( 0 < h < 1 \), the consistency property is satisfied simply by setting \( \epsilon = (1 - h) \).

7 Conclusion

In this paper, we have presented a formal framework for the development of a family of discriminative learning algorithms for Probabilistic Context-Free Grammars (PCFGs) based on a generalization of criterion-H. First of all, we have presented the H-criterion as the objective function and we have developed the final expressions for the estimation of the parameters of the PCFGs. Finally, we have proposed a generalization of the H-criterion to take into account the set of reference interpretations and the set of competing interpretations, and we have defined a new family of objective functions that allow us to develop the expressions of the estimation transformations for PCFGs.

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References

[1] K. Lari and S.J. Young. Applications of stochastic context-free grammars using the inside-outside algorithm. *Computer, Speech and Language*, pages 237–257, 1991.

[2] H. Ney. Stochastic grammars and pattern recognition. In P. Laface and R. De Mori, editors, *Speech Recognition and Understanding: Recent Advances*, pages 319–344. Springer-Verlag, 1992.

[3] Francisco Álvaro, Joan-Andreu Sánchez, and José-Miguel Benedí. An integrated grammar-based approach for mathematical expression recognition. *Pattern Recognition*, 51:135–147, March 2016. ISSN 0031-3203.

[4] Joseph Turian and I. Dan Melamed. Advances in discriminative parsing. In *Proceedings of the 21st International Conference on Computational Linguistics and 44th Annual Meeting of the Association for Computational Linguistics*, pages 873–880, Sydney, Australia, Jul 2006. Association for Computational Linguistics.

[5] Jenny Rose Finkel, Alex Kleeman, and Christopher Manning. Efficient, feature-based, conditional random field parsing. In *Proceedings of ACL-08: HLT*, pages 959–967, Columbus, Ohio, Jun 2008. Association for Computational Linguistics.

[6] L.E. Baum and G.R. Sell. Growth transformation for functions on manifolds. *Pacific J. Mathematics*, 27(2):211–227, 1968.

[7] F. Casacuberta. Growth transformations for probabilistic functions of stochastic grammars. *IJPRAI*, 10(3):183–201, 1996.

[8] L. Wang and P. C. Woodland. Discriminative adaptive training using the mpe criterion. In *2003 IEEE Workshop on Automatic Speech Recognition and Understanding (IEEE Cat. No.03EX721)*, pages 279–284, 2003.

[9] Xiaodong He, Li Deng, and Wu Chou. Discriminative learning in sequential pattern recognition. *IEEE Signal Processing Magazine*, 25(5):14–36, 2008.

[10] R. Hsiao, Y. Tam, and T. Schultz. Generalized baum-welch algorithm for discriminative training on large vocabulary continuous speech recognition system. In *2009 IEEE International Conference on Acoustics, Speech and Signal Processing*, pages 3769–3772, 2009.

[11] Ben Taskar, Dan Klein, Michael Collins, Daphne Koller, and Christopher Manning. Max-margin parsing, 2004.

[12] Stefan Riezler, Tracy H. King, Ronald M. Kaplan, Richard Crouch, John T. Maxwell III, and Mark Johnson. Parsing the Wall Street Journal using a Lexical-Functional Grammar and discriminative estimation techniques. In *Proceedings of the 40th Annual Meeting of the Association for Computational Linguistics*, pages 271–278, Philadelphia, Pennsylvania, USA, Jul 2002. Association for Computational Linguistics.

[13] F. Casacuberta. Maximum mutual information and conditional maximum likelihood estimations of stochastic regular syntax-directed translation schemes. In Laurent Miclet and Colin de la Higuera, editors, *Grammatical Interference: Learning Syntax from Sentences*, pages 282–291, Berlin, Heidelberg, 1996. Springer Berlin Heidelberg.

[14] T.L. Booth and R.A. Thompson. Applying probability measures to abstract languages. *IEEE Transactions on Computers*, C-22(5):442–450, May 1973.

[15] P. Gopalakrishnan, D. Kanevsky, A. Nadas, D. Nahamoo, and M. Picheny. Decoder selection based on cross-entropies. In *ICASSP-88., International Conference on Acoustics, Speech, and Signal Processing*, pages 20,21,22,23, Los Alamitos, CA, USA, Apr 1988. IEEE Computer Society.

[16] Ralf Schlüter, Wolfgang Macherey, Boris Müller, and Hermann Ney. Comparison of discriminative training criteria and optimization methods for speech recognition. *Speech Communication*, 34(3):287–310, 2001.

[17] Philip Woodland and Daniel Povey. Large scale discriminative training of hidden markov models for speech recognition. *Computer Speech and Language*, 16:25–47, 01 2002.

[18] P.S. Gopalakrishnan, D. Kanevsky, A. Nadas, and D. Nahamoo. An inequality for rational functions with applications to some statistical estimation problems. *IEEE Transactions on Information Theory*, 37(1):107–113, 1991.

[19] V.M. Jiménez and A. Marzal. Computation of the n best parse trees for weighted and stochastic context-free grammars. In F.J. Ferri, J.M. Iñesta, A. Amin, and P. Pudil, editors, *Advances in Pattern Recognition*, Lecture Notes in Computer Science, 1876, pages 183–192. Springer-Verlag, 2000.

[20] Ernesto Noya, Joan-Andreu Sánchez, and José-Miguel Benedí. Generation of hypergraphs from the n-best parsing of 2d-probabilistic context-free grammars for mathematical expression recognition. In *25th International Conference on Pattern Recognition (ICPR 2020)*, pages 5696–5703, Milan, Italy, January 10–15, 2021 2014. IEEE Computer Society. ISBN: 978-1-7281-8808-9.
Appendix A

In this appendix, we demonstrate how expression (7) is derived in order to maximize expression (6), and then we explain how the estimation process is carried out. First, the growth transformation optimization framework is presented, and expression (7) is formally derived applying the growth transformations theorem for rational functions [18] to optimize expression (6),

\[
\bar{p}(A \rightarrow \alpha) = \frac{p(A \rightarrow \alpha) \left[ \frac{\partial P_r(G_p, \Omega)}{\partial p(A \rightarrow \alpha)} + C \right]}{\sum_{i=1}^{n_A} p(A \rightarrow \alpha_i) \left[ \frac{\partial P_r(G_p, \Omega)}{\partial p(A \rightarrow \alpha_i)} + C \right]} \tag{11}
\]

where \(n_A\) is the number of rules with the non-terminal \(A\) in the left side of the rule, and \(\pi = (\pi_{A_1}, \pi_{A_2}, \ldots, \pi_{A_N})\). \(A_i \in \mathbb{N}, 1 \leq i \leq |N|\) is a vector defined as follows: \(\pi_{A_i} = (p(A_i \rightarrow \alpha_{i_1}), p(A_i \rightarrow \alpha_{i_2}), \ldots, p(A_i \rightarrow \alpha_{i_{n_{A_i}}})\).

Furthermore, as shown in [18], \(P_r(G_p, \Omega)\) is a polynomial function where \(F_h(G_p, \Omega) = \prod_{x \in \Omega} \frac{P_{G_p}(x, \hat{d}_x)}{P_{G_p}(x, \Delta_x)}\) is a constant which is obtained by evaluating \(F_h(G_p, \Omega)\) in \(\pi\). In [18] is shown that for every point of the domain \(\pi\) there is a constant \(C\) such that the polynomial \(P_{\pi} + C\) has only non-negative coefficients.

Following a similar development to that used in [22], will allow us to explicitly obtain \(\bar{p}(A \rightarrow \alpha)\) of (11). The numerator of this expression can be written as follows:

\[
p(A \rightarrow \alpha) \left( \frac{\partial}{\partial p(A \rightarrow \alpha)} \frac{\prod_{x \in \Omega} P_{G_p}(x, \hat{d}_x)}{\prod_{x \in \Omega} P_{G_p}(x, \Delta_x)} \right) \cdot F_h(G_p, \Omega) \cdot \sum_{x \in \Omega} \frac{p(A \rightarrow \alpha)}{P_{G_p}(x, \hat{d}_x)} \frac{\partial P_{G_p}(x, \hat{d}_x)}{\partial p(A \rightarrow \alpha)}
\]

...
A similar expression can be obtained for the denominator:

\[
\sum_{i=1}^{n_A} p(A \to \alpha_i) \left[ \frac{\partial}{\partial p(A \to \alpha_i)} \prod_{x \in \Omega} P_{G_p}(x, \Delta_x) - F_\pi(G_p, \Omega) \frac{\partial}{\partial p(A \to \alpha_i)} + C \right]_\pi
\]

\[
= \sum_{i=1}^{n_A} \prod_{x \in \Omega} P_{G_p}(x, \Delta_x) \left[ \sum_{x \in \Omega} N(A \to \alpha_i, \Delta_x) - h \sum_{x \in \Omega} \frac{1}{P_{G_p}(x, \Delta_x)} \sum_{d_x \in \Delta_x} N(A \to \alpha_i, d_x) P_{G_p}(x, d_x) + \frac{\sum_{i=1}^{n_A} p(A \to \alpha_i) C}{\prod_{x \in \Omega} P_{G_p}(x, \Delta_x)} \right]
\]

\[
= \prod_{x \in \Omega} P_{G_p}(x, \Delta_x) \left[ \sum_{x \in \Omega} N(A \to \alpha, \Delta_x) - h \sum_{x \in \Omega} \frac{1}{P_{G_p}(x, \Delta_x)} \sum_{d_x \in \Delta_x} N(A, d_x) P_{G_p}(x, d_x) + \frac{C}{\prod_{x \in \Omega} P_{G_p}(x, \Delta_x)} \right]
\]

Thus, expression (11) can be written as:

\[
\bar{p}(A \to \alpha) = \frac{\sum_{x \in \Omega} N(A \to \alpha, \Delta_x) - h \sum_{x \in \Omega} \frac{1}{P_{G_p}(x, \Delta_x)} \sum_{d_x \in \Delta_x} N(A \to \alpha, d_x) P_{G_p}(x, d_x)}{\sum_{x \in \Omega} N(A, \Delta_x) - h \sum_{x \in \Omega} \frac{1}{P_{G_p}(x, \Delta_x)} \sum_{d_x \in \Delta_x} N(A, d_x) P_{G_p}(x, d_x)} + C
\]

which coincides with expression (7).

**Appendix B**

In this appendix, we demonstrate how expression (10) is derived in order to maximize expression (9), and then we explain how the estimation process is carried out. First, the growth transformation is defined as,

\[
\bar{p}(A \to \alpha) = \frac{\sum_{i=1}^{n_A} p(A \to \alpha_i) \left[ \frac{\partial Q_\pi(G_p, \Omega)}{\partial p(A \to \alpha)} + C \right]_\pi}{\sum_{i=1}^{n_A} p(A \to \alpha_i) \left[ \frac{\partial Q_\pi(G_p, \Omega)}{\partial p(A \to \alpha)} + C \right]_\pi}
\]

(12)

where \(n_A\) is the number of rules with the non-terminal \(A\) in the left side of the rule, and \(\pi = (\pi_{A_1}, \pi_{A_2}, \ldots, \pi_{A_{|N|}})\), \(A_i \in N, 1 \leq i \leq |N|\) is a vector defined as follows: \(\pi_{A_i} = (p(A_i \to \alpha_{i1}), p(A_i \to \alpha_{i2}), \ldots, p(A_i \to \alpha_{in_{A_i}}))\). Furthermore, \(Q_\pi(G_p, \Omega)\) (9) is a polynomial function and as demonstrated in (18), for every point of the domain \(\pi\), there is a constant \(C\) such that the polynomial \(P_{\pi} + C\) has only non-negative coefficients. Following a similar development to that used in (22), will allow us to explicitly obtain \(\bar{p}(A \to \alpha)\) of (12).

Let’s define an auxiliary function,

\[
\mathcal{D}_{A \to \alpha}^h(\Delta_x) = p(A \to \alpha) \left[ \frac{\partial}{\partial p(A \to \alpha)} \prod_{x \in \Omega} P_{G_p}(x, \Delta_x)^h \right]_\pi
\]

then the expression (12) can be rewritten as,

\[
\bar{p}(A \to \alpha) = \frac{\mathcal{D}_{A \to \alpha}^h(\Delta_x) - \tilde{F}_h(G_p, \Omega) \mathcal{D}_{A \to \alpha}^h(\Delta_x) + p(A \to \alpha) C}{\sum_{i=1}^{n_A} \mathcal{D}_{A \to \alpha_i}^h(\Delta_x) - \tilde{F}_h(G_p, \Omega) \sum_{i=1}^{n_A} \mathcal{D}_{A \to \alpha_i}^h(\Delta_x) + p(A \to \alpha) C}
\]

(13)
First, the auxiliary function $D_{A \to \alpha}^h(\Delta_x)$ is computed to evaluate $D_{A \to \alpha}^1(\Delta_x^c)$ and $D_{A \to \alpha}^h(\Delta_x^c)$ in the numerator.

$$D_{A \to \alpha}^h(\Delta_x) =$$

$$= p(A \to \alpha) \left[ h \prod_{x \in \Omega} P_{G_p}^n(x, \Delta_x)^{h-1} \frac{\partial \prod_{x \in \Omega} P_{G_p}^n(x, \Delta_x)}{\partial p(A \to \alpha)} \right]$$

$$= h \left[ \prod_{x \in \Omega} P_{G_p}^n(x, \Delta_x)^h \sum_{x \in \Omega} \frac{p(A \to \alpha)}{P_{G_p}^n(x, \Delta_x)} \frac{\partial P_{G_p}^n(x, \Delta_x)}{\partial p(A \to \alpha)} \right]$$

$$= h \eta \left[ \prod_{x \in \Omega} P_{G_p}^n(x, \Delta_x)^h \sum_{x \in \Omega} \frac{1}{P_{G_p}^n(x, \Delta_x)} \sum_{d_x \in \Delta_x} N(A \to \alpha, d_x) P_{G_p}^n(x, d_x) \right] \tag{14}$$

Then, the expression $\sum_{i=1}^{n_A} D_{A \to \alpha_i}^h(\Delta_x)$ is computed to evaluate $\sum_{i=1}^{n_A} D_{A \to \alpha_i}^1(\Delta_x^c)$ and $\sum_{i=1}^{n_A} D_{A \to \alpha_i}^h(\Delta_x^c)$ in the denominator.

$$\sum_{i=1}^{n_A} D_{A \to \alpha_i}^h(\Delta_x) =$$

$$= \sum_{i=1}^{n_A} h \eta \left[ \prod_{x \in \Omega} P_{G_p}^n(x, \Delta_x)^h \sum_{x \in \Omega} \frac{1}{P_{G_p}^n(x, \Delta_x)} \sum_{d_x \in \Delta_x} N(A \to \alpha_i, d_x) P_{G_p}^n(x, d_x) \right]$$

$$= h \eta \left[ \prod_{x \in \Omega} P_{G_p}^n(x, \Delta_x)^h \sum_{x \in \Omega} \frac{1}{P_{G_p}^n(x, \Delta_x)} \sum_{d_x \in \Delta_x} \sum_{i=1}^{n_A} N(A \to \alpha_i, d_x) P_{G_p}^n(x, d_x) \right]$$

$$= h \eta \left[ \prod_{x \in \Omega} P_{G_p}^n(x, \Delta_x)^h \sum_{x \in \Omega} \frac{1}{P_{G_p}^n(x, \Delta_x)} \sum_{d_x \in \Delta_x} \sum_{i=1}^{n_A} N(A, d_x) P_{G_p}^n(x, d_x) \right] \tag{15}$$

Substituting the expressions (14) and (15) in the transformation (13) and simplifying $\eta \prod_{x \in \Omega} P_{G_p}^n(x, \Delta_x)$ in the numerator and denominator, results in:

$$\bar{p}(A \to \alpha) = \frac{D_{A \to \alpha}(\Delta_x^c) - h D_{A \to \alpha}(\Delta_x^c) + p(A \to \alpha)\bar{C}}{D_A(\Delta_x^c) - h D_A(\Delta_x^c) + C}$$

which coincides with expression (10). Where,

$$D_{A \to \alpha}(\Delta_x) = \sum_{x \in \Omega} \frac{1}{P_{G_p}^n(x, \Delta_x)} \sum_{d_x \in \Delta_x} N(A \to \alpha, d_x) P_{G_p}^n(x, d_x)$$

$$D_A(\Delta_x) = \sum_{x \in \Omega} \frac{1}{P_{G_p}^n(x, \Delta_x)} \sum_{d_x \in \Delta_x} N(A, d_x) P_{G_p}^n(x, d_x)$$