Necessary and Sufficient Stability Condition for Time-Delay Systems Arising From Legendre Approximation

Mathieu Bajodek, Graduate Student Member, IEEE, Frédéric Gouaisbaut, and Alexandre Seuret

Abstract—Recently, necessary conditions of stability for time-delay systems based on the handling of the Lyapunov–Krasovskii functional have been studied in the literature giving rise to a new paradigm. Interestingly, the necessary condition for stability developed by Gomez et al. has been proven to be sufficient. It is presented as a simple positivity test of a matrix issued from the Lyapunov matrix. This article proposes an extension of this result, where the uniform discretization of the state has been replaced by projections on the first Legendre polynomials. Like in Gomez et al., the stability is guaranteed regarding the sign of the eigenvalues of a matrix, whose size is given analytically from convergence arguments. Compared to them, by relying on the supergeometric convergence rate of the Legendre approximation, the required order to ensure stability can be remarkably reduced. Thanks to this significant modification, it is possible to find an outer estimate of the stability regions, which converges to the expected stability regions with respect to the number of projections, as illustrated in the example section.

Index Terms—Approximation theory, Lyapunov–Krasovskii functionals, stability analysis, time-delay systems.

I. INTRODUCTION

Delays appear unavoidably as soon as time processing or analog-to-digital converters interfere in the communication between interconnected dynamical systems. Numerous numerical methods have been deployed to consider this latency and to analyze the stability of time-delay systems [25]. First of all, the D-partition [26] issued from the modulus-argument calculation is simple to implement and indicates the exact stability properties. Furthermore, stability areas can be inferred using quasipolynomials approaches and the set-up of Mihàkov diagrams [31]. Then, approximated models derived from pseudo-spectral techniques such as collocation [4], or tau [22] methods have also been prevalent. Besides stability sets obtained using bifurcation analysis, the root locus is outlined. In the Laplace domain, frequency-sweeping delay-dependent tests have also been developed to avoid case-by-case studies. The analysis provides accurate stability results [24] and even results in the design of controllers [13], [29]. Lastly, in the time domain, it is well-known that the existence of Lyapunov–Krasovskii functionals leads to a necessary and sufficient condition of stability [23] even though the sufficiency is usually not numerically tractable. The implementation has only been recently made feasible by discretizing the Lyapunov–Krasovskii functional [10]. Henceforth, tractable necessary and sufficient conditions can be formulated as a positive definiteness test of a certain matrix. This method has been applied to various classes of delay systems with single [11], integral [5], neutral [15], or multiple [1, 18] delay types. This article focuses on this last feature.

For linear finite-dimensional systems with state matrix A, stability is equivalent to the positive definiteness of a symmetric matrix P solution of the so-called Lyapunov equation PA + A'P = −I. Concurrently, for linear infinite-dimensional systems with operator A, stability is equivalent to find a positive hermitian operator P solution of the Lyapunov equation PA + A'P = −I (see [8]). Certified implementation techniques need to be developed [11], [27] to use such a theoretical necessary and sufficient condition. From one side, the necessity is directly obtained by the construction of an approximated Lyapunov–Krasovskii functional [12]. On the other side, the sufficiency is obtained asymptotically for sufficiently large approximated orders [9]. In practice, the approximation is realized by discretizing the Lyapunov matrix appearing in the operator P. The interpolated functions are selected on each evenly-spaced subinterval as polynomials (see piece-wise linear or splines schema [20], [27], [28]) or exponential kernels (see [9], [11], [12]). The latter technique makes it possible to elegantly end up with point-wise evaluations of the Lyapunov matrix U. Then, a necessary and sufficient condition of stability is expressed as the positive definiteness of a matrix, approximating P, of size n* [17]. The estimation of the order n* to assess stability has also been given in [19]. Nevertheless, this estimated order seems extremely large, pessimistic, and limited by the discretization schema, which leads us to the following questions. Is it possible to extend the methodology to other approximation techniques and to other support basis? Can the numerical complexity of the numerical test be reduced? In that direction, we propose here another way to approximate the Lyapunov–Krasovskii functional following the idea of projection on a Legendre polynomial basis [32]. The selection of Legendre polynomials is already meaningful insofar tau-Legendre models are very efficient to perform convergent simulations [30] or convergent stability estimates in the linear matrix inequality framework [2]. By taking the benefits of Legendre approximation, especially its supergeometric convergence rate, new necessary and sufficient criterion of stability is derived and the estimated order n* is notably reduced compared to [19].

The rest of this article is organized as follows. Section II presents the complete Lyapunov–Krasovskii functional and recalls the necessity and sufficiency of the converse Lyapunov theorem. Section III is dedicated to the supergeometric convergence occurring when performing Legendre approximations. Then, our novel necessary and sufficient numerical condition of stability is exposed in Section IV. The last section deals with computational issues and performances evaluations of our stability test.

Notations: Throughout this article, N and \( \mathbb{R}^{m \times p} \) and \( S^m \) denote the set of natural numbers, real matrices of size \( m \times p \) and symmetric matrices of size \( m \), respectively. For any square matrix \( M \in \mathbb{R}^{m \times m} \), \( M^\top \) denotes the transpose of \( M \) and \( H(M) \) stands for \( M + M^\top \). For any matrix \( M \in S^m \), \( M \succ 0 \) means that \( M \) is positive definite (i.e., the eigenvalues of \( M \) are strictly positive). Furthermore, for any
matrix \( M \) in \( \mathbb{R}^{m \times p} \), the 2-norm of \( M \) is \( |M| = \sqrt{\sigma(M^T M)} \), where \( \sigma \) defines the maximal eigenvalue. The vector \( u = \text{vec}(M) \) in \( \mathbb{R}^{m \times p} \) \( \text{col} \)locates the columns of \( M \) and the inverse operation is denoted \( \text{vec}^{-1} \) \( \text{and verifies} \ \text{vec}^{-1}(\text{vec}(M)) = M \). Moreover, \( I_m \) is the identity matrix of size \( m \), \( \delta \), and denotes the Kronecker delta, symbol \( \otimes \) represents the Kronecker product, matrix \( \left[ \begin{array}{cc} M_1 & M_2 \\ M_3 & M_4 \end{array} \right] \) stands for \( \left[ \begin{array}{cc} M_1 & M_3 \\ M_2 & M_4 \end{array} \right] \) and \( \text{diag}(d_1, \ldots, d_n) \) is the diagonal matrix with diagonal coefficients \( d_1, \ldots, d_n \). We also declare functions \( e^t \) and \( t \) as the exponential and ceiling part of the real number \( r \), respectively. For functions \( f_1, f_2 \) from \( \mathbb{R} \) to \( \mathbb{R} \), \( f_1(n) \sim f_2(n) \) means that \( f_1(n) \) is finite as \( n \) tends to infinity. The set of piece-wise continuous functions from \([-h, 0]\) to \( \mathbb{R} \) is denoted \( C_{pu}(-h, 0; \mathbb{R}^m) \). For any function \( \varphi \) in this set, the induced norm is \( ||\varphi|| = \sup_{r \in [-h,0]} |\varphi(r)| \). Denote also \( C_{\infty}(-h, 0; \mathbb{R}^m) \), the set of smooth functions from \([-h, 0]\) to \( \mathbb{R}^m \). Finally, the Shimano notation \( x_t : \{-h, 0\} \rightarrow \mathbb{R}^n \), \( t \mapsto x_t(t) = x(t + \tau) \) will be used all along this article.

II. LYAPUNOV NECESSARY AND SUFFICIENT STABILITY CONDITION FOR TIME-DELAY SYSTEMS

A. Time-Delay System and Lyapunov–Krasovskii Functional

Consider a linear time invariant time-delay system given by

\[
\dot{x}(t) = Ax(t) + Ax(t - h) \quad \forall t \geq 0
\]

where \( h > 0 \) and matrices \( A, A_d \) in \( \mathbb{R}^{m \times m} \) are constant and known. Such a system is initialized by \( x_0 = \varphi \in C_{pu}(-h, 0; \mathbb{R}^m) \) \( \text{and, for any} \ t \geq 0, x_t \in C_{pu}(-h, 0; \mathbb{R}^m) \) denotes the state of \( \tau \).

Definition 1: The trivial solution of system (1) is said to be exponentially stable if there exist \( \kappa > 1 \) and \( \alpha > 0 \) such that, for all \( t \geq 0 \) and \( x_0 \in C_{pu}(-h, 0; \mathbb{R}^m) \), \(|x_t| \leq \kappa e^{-\alpha t}\|x_0\|\) holds.

In order to study the stability of system (1), recall the Lyapunov–Krasovskii functional introduced in [23]

\[
V(\varphi) = \int_{-h}^{0} \phi(\tau) \Pi(\tau_1, \tau_2) \phi(\tau_2) d\tau_1 d\tau_2
\]

for any \( \varphi \in C_{pu}(-h, 0; \mathbb{R}^m) \), where matrix II is given by

\[
\Pi(\tau_1, \tau_2) = \left[ \begin{array}{c} U(0) \hfill \\ \tau \hfill \end{array} \right] \left[ \begin{array}{c} U(\tau_1 + \tau_2) A_d \\ \tau \hfill \end{array} \right]
\]

and where the Lyapunov matrix \( U \) in \( \mathbb{R}^{m \times m} \) is given by \( U = \text{vec}^{-1}(U) \) \text{where} \( U = \text{vec}(U) \) is given analytically by

\[
U(\tau) = \left[ \begin{array}{c} M_{1,2} \hfill \\ 0 \hfill \end{array} \right] e^{\tau M} N^{-1} \left[ \begin{array}{c} -\text{vec}(I_m) \\ 0 \hfill \end{array} \right] \text{if} \ \tau \geq 0
\]

\[
= \left[ \begin{array}{c} 0 \hfill \\ 0 \hfill \end{array} \right] e^{(\tau + h)M} N^{-1} \left[ \begin{array}{c} -\text{vec}(I_m) \\ 0 \hfill \end{array} \right] \text{if} \ \tau < 0
\]

with

\[
M = \left[ \begin{array}{cc} A^T \otimes I_m & A_d^T \otimes I_m \\ -I_m \otimes A_d^T & -I_m \otimes A_d \end{array} \right] \\
N = \left[ \begin{array}{cc} A^T \otimes I_m + I_m \otimes A^T & A_d^T \otimes I_m + I_m \otimes A_d^T \\ I_m \otimes A_d^T + I_m \otimes A_d^T & 0 \hfill \end{array} \right] e^{h M}.
\]

The authors of [9] showed that it is the unique functional that satisfies

\[
\dot{V}(x_t) = -\{x(t - h)\}^2
\]

along the trajectories \( x_t \) of system (1).

Remark 1: The Lyapunov matrix \( U \) ensuring (6) is unique if and only if matrix \( N \) is nonsingular. As explained in [23], such a limitation is the Lyapunov condition and excludes all systems with eigenvalues \( s_1, s_2 \) satisfying \( |s_1 + s_2| = 0 \).

B. Necessary and Sufficient Stability Condition

Under the Lyapunov condition, the authors of [9], [19], [23], [28] provide sufficient and necessary conditions for the exponential stability of system (1), which are recalled as follows.

Lemma 1: If system (1) is exponentially stable, then there exists \( \eta > 0 \) such that

\[
V(\varphi) \leq \eta (|\varphi(0)|^2 + \frac{1}{h} \int_{-h}^{0} |\varphi(\tau)|^2 d\tau) \quad \forall \varphi \in C_{pu}(-h, 0; \mathbb{R}^m)
\]

where \( V \) is the functional defined by (2) satisfying (6).

Lemma 2: Assume that system (1) has an eigenvalue with a strictly positive real part. Then,

\[
\exists \varphi \in S, \quad V(\varphi) \leq -\eta_0 = -\frac{e^{-2r h}}{4r} \cos^2(b_0) < 0
\]

where the functional \( V \) is defined by (2) and (6) and \( S \) stands for the compact set given by

\[
S = \{ \varphi \in C_{\infty}(-h, 0; \mathbb{R}^m); \ ||\varphi(0)|| \leq r^k \quad \forall k \in \mathbb{N} \}
\]

with a system dependent parameter \( r \) given by

\[
r = |A| + |A_d|
\]

and with scalar \( b_0 \) the unique root on \([0, \frac{\pi}{2}]\) of the function \( g(b) := \sin^4(b)/(h^2) + b^2 - (h^2) \).

Proof: The proofs of both lemmas are postponed to Appendix A.

III. PRELIMINARIES ON LEGENDRE POLYNOMIALS

A. Legendre Approximation

Legendre polynomials considered on \([-h, 0]\) are defined by

\[
\forall k \in \mathbb{N}, \ l_k(\tau) = \left( \begin{array}{c} \left( \begin{array}{c} k \\ j \end{array} \right) (k + j) \left( \frac{\tau + h}{h} \right)^j \end{array} \right)
\]

where \( \left( \begin{array}{c} k \\ j \end{array} \right) \) stands for the binomial coefficient [14]. These polynomials \( \{l_k\}_{k \in \mathbb{N}} \) form an orthogonal sequence of functions, which spans the space of square-integrable functions [14].

For the sake of simplicity, introduce matrix \( \ell_k \) in \( \mathbb{R}^{m \times m} \) given by

\[
\ell_k(\theta) = \left[ l_0(\theta) I_m \quad l_1(\theta) I_m \ldots l_{n-1}(\theta) I_m \right]^T \quad \forall \theta \in [-h, 0].
\]

For any function \( \varphi \in C_{pu}(-h, 0; \mathbb{R}^m) \) and any approximation order \( n \in \mathbb{N}^* \), let us decompose

\[
\forall \tau \in [-h, 0], \quad \varphi(\tau) = \ell_n(\tau) \Phi_n + \tilde{\varphi}(\tau)
\]

where \( \varphi_n(\tau) = \ell_n^T(\tau) \Phi_n \) is the polynomial approximation and \( \tilde{\varphi}(\tau) = \varphi(\tau) - \varphi_n(\tau) \) is the residual error. The vector \( \Phi_n \) represents the normalized \( n \) first polynomial coefficients of the function \( \varphi \) and is
defined by
\[ \Phi_n = I_n \int_{-h}^{0} \ell_n(\tau) \varphi(\tau) d\tau \in \mathbb{R}^{n,m} \] (14)
where \( I_n \) is the diagonal Gram–Schmidt normalization matrix.

In the sequel, the objective is to prove that the Legendre approximation \( \varphi_n \) converges uniformly toward \( \varphi \) with respect to \( \tau \) and also to quantify its convergence rate on \( S \).

**B. Convergence of the Legendre Remainder**

In light of the polynomial approximation theory [6], it results in an important convergence lemma.

**Lemma 3:** For any function \( \varphi \) in \( S \), the approximation error \( \tilde{\varphi}_n \) in (13) verifies, for any \( \varepsilon > 0 \)
\[ ||\tilde{\varphi}_n|| \leq \varepsilon \quad \forall n \geq N(\varepsilon) \] (15)
where \( N(\varepsilon) \) is given by
\[ N(\varepsilon) = \max \left( 4, \left[ 3 + \mu e^{1+\mathcal{W}(\log(\mu\varepsilon))} \right] \right) \] (16)
where
\[ \mu = \frac{hr}{2}, \quad \rho = \frac{2[\mu]}{\pi^2} \left( \frac{\mu e^{1+\mathcal{W}(\log(\mu\varepsilon))}}{[\mu] + \frac{1}{2}} \right) \] (17)
and where the Lambert function \( \mathcal{W} \) is uniquely defined by the relation \( y e^y = z \).

**Proof:** According to [33, Th. 2.5], an upper bound of the Legendre approximation error \( ||\tilde{\varphi}_n|| = \sup_{[-h,0]} |\tilde{\varphi}_n(\tau)| \) is given by
\[ ||\tilde{\varphi}_n|| \leq \sqrt{\frac{\pi^2}{2[\mu]} (n - \frac{3}{2}) \cdots (1 + [\mu] + \frac{1}{2})} \quad \forall n \geq 1 \] (18)
using Legendre polynomials properties and \( n - 1 - [\mu] \) successive integrations by parts. Applying the logarithm to (18) leads to
\[ \log \left( \sqrt{\frac{2[\mu]}{\pi^2} ||\tilde{\varphi}_n||} \right) \leq (n - [\mu]) \log(\mu) - \sum_{k=1+[\mu]}^{n-2} \log \left( k + \frac{1}{2} \right) . \] (19)

Since the \( \log \) function is monotonically increasing, we obtain
\[ \sum_{k=1+\mu}^{n-2} \log \left( k + \frac{1}{2} \right) \geq \int_{x=\mu}^{x=\mu} \log \left( x + \frac{1}{2} \right) dx \]
where \( c \) denotes the exponential of 1. Reordering the terms and introducing \( \rho \) in (17), inequality (19) becomes
\[ \log(\rho ||\tilde{\varphi}_n||) \leq - \left( n - \frac{3}{2} \right) \log \left( \frac{n - \frac{3}{2}}{\mu \varepsilon} \right) . \] (20)

Denoting \( y_n := \log \left( \frac{n - \frac{3}{2}}{\mu \varepsilon} \right) \), we look for the orders \( n \) such that the upper bound given by (20) is bounded by \( \log(\lambda \varepsilon) \). Then, the following inequality need to be satisfied
\[ y_n e^{y_n} \leq \frac{\log(\rho \varepsilon)}{\mu \varepsilon} . \] (21)

**IV. NEW NECESSARY AND SUFFICIENT STABILITY CONDITION FOR TIME-DELAY SYSTEMS**

**A. Approximated Lyapunov–Krasovskii Functional**

In this section, in order to construct an approximated Lyapunov–Krasovskii functional, the complete Lyapunov–Krasovskii functional given by (2) is regarded for particular functions \( \varphi \), taken from subsets of \( C_{pw}(-h,0;\mathbb{R}^m) \). For instance, we consider here the space spanned by the first Legendre polynomial, and we take support on the first Legendre coefficients of \( \varphi \) denoted \( \Phi_n \) and expressed in (14).

Let the approximated Lyapunov–Krasovskii functional at order \( n \)
\[ V_n(\varphi) = \left[ \varphi(0) \right]^T P_n \left[ \varphi(0) \Phi_n \right] \] (24)
for any \( \varphi \in C_{pw}(-h,0;\mathbb{R}^m) \), with matrix
\[ P_n = \left[ U(0) \right]^{\dagger} Q_n \left[ T_n + I_n \right] . \] (25)

In the previous expression, we have
\[ Q_n = \int_{-h}^{0} U^T(h + \tau) A_d E_n^T(\tau) d\tau \]
\[ T_n = \int_{-h}^{0} \int_{-h}^{0} \epsilon_n(\tau_1) A_d E_n^T(\tau_2) d\tau_1 d\tau_2 . \] (26)

**Remark 2:** Note that \( V_n \) does not involve the Legendre remainder \( \tilde{\varphi}_n \). Functional \( V_n \) is an approximation of the Lyapunov–Krasovskii functional \( V \) defined by (2).

Based on the previous section on polynomial approximation, the convergence of this approximated functional towards the complete Lyapunov–Krasovskii functional given by (2) will be established in the following section.
B. Convergence of the Approximated Lyapunov–Krasovskii Functional

Define the Lyapunov–Krasovskii functional remainder as
\[ \tilde{V}_n(\varphi) = V(\varphi) - V_n(\varphi) \quad \forall \varphi \in C_{pw}(-h, 0; \mathbb{R}^m). \] (27)
Applying expansion (13), this remainder is rewritten as
\[ \tilde{V}_n(\varphi) = \int_{-h}^{0} \int_{-h}^{0} \begin{bmatrix} \varphi(0) \\ \varphi_n(\tau_1) \\ \varphi_n(\tau_2) \end{bmatrix}^\top \Delta_n(\tau_1, \tau_2) \tilde{\varphi}_n(\tau_2) \, d\tau_1 \, d\tau_2 \] (28)
where
\[ \Delta_n(\tau_1, \tau_2) = \begin{bmatrix} \frac{2}{h} U^T(h + \tau_2) A_d \\ 2A_U(h + \tau_2) A_d \\ A_U^T(h + \tau_2) A_d \\ \frac{1}{h} I_m \end{bmatrix}. \] (29)

The main idea is now to prove, at least in the compact subset \( S \) of \( C_{pw}(-h, 0; \mathbb{R}^m) \) given by (9), that the approximated Lyapunov–Krasovskii functional \( V_n \) given by (24) converges toward the complete Lyapunov–Krasovskii functional \( V \) given by (2) with a guaranteed and quantified convergence rate.

**Lemma 4:** For any \( \varphi \in S \) and \( \eta > 0 \), we have
\[ |\tilde{V}_n(\varphi)| \leq \eta \quad \forall n \geq N(\mathcal{E}(\eta)) \] (30)
where the Lyapunov–Krasovskii remainder \( \tilde{V}_n \) is given by (28), where the order \( N(\varepsilon) \) is described in (16) and where
\[ \mathcal{E}(\eta) = \frac{\kappa_1 + \kappa_2}{\kappa_2 + 1} + \sqrt{\frac{(\kappa_1 + \kappa_2)^2}{\kappa_2 + 1}} + \eta \frac{h}{(\kappa_2 + 1)^2} \] (31)
with scalars \( \kappa_1 \) and \( \kappa_2 \) given by
\[ \kappa_1 = \max_{[0, h]} |U(\tau) A_d|, \quad \kappa_2 = \max_{[0, h]} |A_U^T(\tau) A_d|. \] (32)

**Proof:** An upper bound of \( |\tilde{V}_n(\varphi)| \) is obtained by
\[ |\tilde{V}_n(\varphi)| \leq 2 \int_{-h}^{0} \int_{-h}^{0} |\varphi(0)| |\varphi_n(\tau_1)| |\tilde{\varphi}_n(\tau_2)| \, d\tau_1 \, d\tau_2 \]
\[ + (\kappa_2 + 1) \int_{-h}^{0} |\tilde{\varphi}_n(\tau_1)|^2 \, d\tau_1. \] (33)
Hence, having \( \varphi \in S, \|\varphi\| = \sup_{[-h, 0]} |\varphi(\tau)| \leq 1 \) so that
\[ \frac{1}{h} |\tilde{V}_n(\varphi)| \leq 2(\kappa_1 + \kappa_2) \|\tilde{\varphi}_n\| + (\kappa_2 + 1) \|\tilde{\varphi}_n\|^2. \] (34)
We obtain \( |\tilde{V}_n(\varphi)| \leq \eta \) under the following quadratic constraint:
\[ -\frac{\eta}{h(\kappa_2 + 1)} + 2(\kappa_1 + \kappa_2) \|\tilde{\varphi}_n\| + (\kappa_2 + 1) \|\tilde{\varphi}_n\|^2 \leq 0 \] (35)
which is satisfied for \( \|\tilde{\varphi}_n\| \leq \mathcal{E}(\eta) \). The conclusion is finally drawn thanks to Lemma 3, which states that \( \|\tilde{\varphi}_n\| \leq \mathcal{E}(\eta) \) holds for any order \( n \) greater than \( N(\mathcal{E}(\eta)) \).

**Remark 3:** Note that the maximal values \( \kappa_1 \) and \( \kappa_2 \) can easily be computed by grid search with an equally spaced grid.

Contrary to previous works based on discretization procedures with exponential kernels [9], [19], or splines [20], [28], which were limited to algebraic convergence rates, we take the benefits of Legendre polynomial approximation to obtain a supergeometric convergence rate on the remainder \( |\tilde{V}_n(\varphi)| \). Therefore, the proposed convergence property of the remainder will be the key to design a stability test for time-delay systems that extends and enhances existing results.

C. Necessary and Sufficient Stability Test

Lemmas 1 and 2 applied to the approximated Lyapunov–Krasovskii functional \( V_n \) defined by (24) provide a new necessary and sufficient condition of stability for system (1).

**Theorem 1:** System (1) is exponentially stable if and only if matrix \( P_{\mathcal{N}(\mathcal{E}(\eta_0))} \) in (25) is positive definite where \( \mathcal{N}, \mathcal{E} \) are defined in (16), (31), respectively.

**Proof:** Assume that system (1) is exponentially stable. For any vector \( [x] \) in \( \mathbb{R}^{(n+1)m} \) and \( n \in \mathbb{N} \), define function \( \varphi \) as
\[ \varphi(\tau) = \int_{0}^{\tau} \Phi_n(x), \quad \forall \tau \in [-h, 0) \] (36)
Applying Lemma 1 with \( \varphi \) given above, there exists \( \eta > 0 \) such that
\[ V(\varphi) = V_n(\varphi) = \frac{[x] \, P_n \, [x]}{[\Phi_n] \, [\Phi_n]} \geq \eta \frac{[x] \, [x]^\top}{[\Phi_n] \, [\Phi_n]} \] which yields \( P_n > 0 \) for all \( n \in \mathbb{N} \). Finally, the convergence presented in Lemma 4 with \( \eta = \eta_0 \) leads to
\[ V_n(\varphi) = \frac{[x] \, P_n \, [x]}{[\Phi_n]^\top \, [\Phi_n]} \leq 0 \quad \forall n \geq N(\mathcal{E}(\eta_0)) \] (38)
which contradicts \( P_{\mathcal{N}(\mathcal{E}(\eta_0))} > 0 \).

The proposed theorem provides a numerical test to guarantee stability or instability of time-delay systems, which follows the following sequence.
1) Compute \( n^* = N(\mathcal{E}(\eta_0)) \) with \( \eta_0 \) given by (8).
2) Evaluate each element of matrix \( P_n \).
3) Test the positivity of matrix \( P_n \) to state the stability.

Note that this necessary and sufficient stability condition is formulated as shown in [19] on the positivity of matrix \( P_n \) for a given order \( n \).

As a background result, a hierarchical sufficient condition for instability of system (1) is also formulated below.

**Corollary 1:** If there exists \( n \in \mathbb{N} \) such that matrix \( P_n \) given by (25) is not definite positive then system (1) is not exponentially stable. Moreover, if this statement holds at an order \( n \), then it also holds at the order \( n + 1 \).

**Proof:** Relying on the necessity part of the proof of Theorem 1, the sufficient condition for instability is trivial. The hierarchy can then be proven because the upper left part of matrix \( P_{n+1} \) corresponds to \( P_n \). If \( P_n \) is not positive definite then \( P_{n+1} \) cannot be positive definite. Interestingly, Corollary 1 suggests an algorithm to solve the instability test. It consists in testing \( P_n > 0 \) from \( n = 1 \) to \( n = n^* \). If \( P_n \) is not definite positive, then the system is unstable. Once order \( n^* = N(\mathcal{E}(\eta_0)) \) is reached, the system is necessarily stable.
It remains to solve the important problem of the numerical computation of matrix $P_n$, which is necessarily to implement the algorithm. This is detailed in the following section.

V. COMPUTATIONAL ISSUES

A. Numerical Issues

To perform the numerical test presented above, each coefficient of matrix $P_n$ given by (25) needs to be evaluated numerically. It is worth noticing that this problem is not encountered in [19] since the matrix to be evaluated contains point-wise evaluations of the Lyapunov matrix $U$. Here, the situation is more complicated since the matrix $P_n$ given by (25) is equal to eq. (39) shown at the bottom of this page, where $Q_k, T_{jk},$ and $T^o_{jk}$ are the Legendre coefficients of the Lyapunov matrix $U$ given in the vector form as follows:

$$
\begin{align*}
Q_k &= \text{vec}(Q_k) = [I_{m^2} 0] \Gamma_k N^{-1} \left(-\text{vec}(I_m) \right) \\
T_{jk} &= \text{vec}(T_{jk}) = [I_{m^2} 0] \Gamma_{jk} N^{-1} \left(-\text{vec}(I_m) \right) \\
T^o_{jk} &= \text{vec}(T^o_{jk}) = [I_{m^2} 0] \Gamma^o_{jk} N^{-1} \left(-\text{vec}(I_m) \right)
\end{align*}
$$

and where $\Gamma_k, \Gamma_{jk},$ and $\Gamma^o_{jk}$ are defined by

$$
\begin{align*}
\Gamma_k &= \int_{-h}^{0} e^{(h+\tau)M} l_k(\tau) d\tau \quad \forall k \in \{0, \ldots, n-1\} \\
\Gamma_{jk} &= \int_{-h}^{0} \int_{-h}^{\tau_1} e^{(\tau_1-\tau_2)M} l_k(\tau_2) d\tau_2 l_j(\tau_1) d\tau_1 \\
\Gamma^o_{jk} &= \int_{-h}^{0} \int_{-h}^{\tau_1} e^{(\tau_2-\tau_1)M} l_k(\tau_2) d\tau_2 l_j(\tau_1) d\tau_1 \\
&\quad \forall (j, k) \in \{0, \ldots, n-1\}^2.
\end{align*}
$$

The question of the numerical implementation of these integral terms in a reasonable time is then raised. Such computations can be done analytically by computer algebra systems but may turn out to be a tough task, especially for large $n$ or $m$. For instance, for $m = 4$ and $n = 100$, the exact calculation of $P_n$ can take days on a basic computer. An alternative computation through inductive relations is proposed to face the problem and make our results tractable numerically.

B. Iterative Calculation of Legendre Exponential Coefficients

The numerical issue resumes to the calculation of Legendre polynomials coefficients of exponential matrices (41)-(43). To perform this computation recursively, the following relations can be used.

Proposition 1: If $M$ is a nonsingular matrix, then matrices $\Gamma_k$ in (41) can be computed by the recursive relation

$$
\Gamma_k = \Gamma_{k-2} - \frac{2(2k-1)}{h} M^{-1} \Gamma_{k-1} \quad \forall k \geq 2
$$

initialized with

$$
\begin{bmatrix}
\Gamma_0 \\
\Gamma_1
\end{bmatrix} =
\begin{bmatrix}
M^{-1}(e^{hM} - I_{m^2}) \\
M^{-1}(e^{hM} + I_{m^2}) - \frac{2h}{M^{-1}} \Gamma_0
\end{bmatrix}.
$$

Proposition 2: For any matrix $M$ and for matrices $\tilde{\Gamma}_{jk}, \tilde{\Gamma}_{jk}^o$ expressed in (42) and (43), the following equality holds:

$$
\tilde{\Gamma}_{jk} = (-1)^{j+k} \tilde{\Gamma}_{jk}^o \quad \forall (j, k) \in \{0, \ldots, n-1\}^2.
$$

Proposition 3: If $M$ is a nonsingular matrix, then matrices $\bar{\Gamma}_{jk}$ in (42) can be computed by the following relations:

$$
\bar{\Gamma}_{jk} = \begin{cases}
(-1)^{j+k} \tilde{\Gamma}_{kj}, & j < k \\
\Gamma_{jk-2} + M^{-1} \frac{2(2k-1)}{h} \bar{\Gamma}_{jk-1}, & j = k \\
\Gamma_{jk-1} - M^{-1} \frac{h}{2j+1} (\delta_{jk} - \delta_{jk-2}), & j > k
\end{cases}
$$

initialized with

$$
\begin{bmatrix}
\Gamma_{00} \\
\Gamma_{01} \\
\Gamma_{11}
\end{bmatrix} =
\begin{bmatrix}
M^{-1}(\Gamma_0 - hI_{m^2}) \\
- M^{-1} \Gamma_1 \\
- M^{-1} \left( \frac{2h}{M^{-1}} + I_{m^2} \right) (\Gamma_1 - hI_{m^2})
\end{bmatrix}.
$$

Proof: The proofs of these propositions are, respectively, postponed to Appendix A2.

Remark 4: Note that the case where $M$ is singular has not been discussed in this article. It could, for instance, be treated using the Jordan canonical form.

These propositions allow to propose a numerical solution to compute the integral terms (41)-(43) by induction. The induction requires a number of operations in the range of $n^5$ and $m^5$. The necessary and sufficient condition of stability for time-delay systems presented in this article becomes numerically tractable in a reasonable time. In the last section, our numerical test for stability is performed on two examples.

VI. APPLICATION TO NUMERICAL EXAMPLES

A. Presentation of the Examples

Example 1: Consider (1) with $A_1 = 1$ and $A_d = -2$.

Example 2: Consider (1) with

$$
A = \begin{bmatrix}
0 & 0 & 0 & 0 \\
-10 & K & 10 & 0 \\
5 & -15 & 0 & -0.25
\end{bmatrix}
$$

and $A_d = \begin{bmatrix}
0 & 0 & 0 \\
K & 0 & 0 \\
0 & 0 & 0
\end{bmatrix};$

for any $K > 0$.

B. Numerical Results on Stability Analysis

Theorem 1 is applied for point-wise values of delays to evaluate the stability of these systems with respect to the delay. The results are gathered in Tables I and II for Example 1 and 2, respectively. For both systems, one can see that the maximal allowable delay $h$, which guarantees the stability, can be given with a precision 0.001.

$$
P_n = \begin{bmatrix}
U(0) & Q^0_{1} A_d & \cdots & Q^0_{n-1} A_d \\
A_d^T T_{00} + T^o_{00} A_d + h I_m & \cdots & A_d^T T_{0n-1} + T^o_{0n-1} A_d \\
* & A_d^T T_{1n-1} + T^o_{1n-1} A_d + h I_m & \cdots \\
* & & \ddots & \ddots
\end{bmatrix}
$$

(39)
TABLE I
EVALUATION AND COMPARISON OF OUR NECESSARY AND SUFFICIENT TEST FOR STABILITY FOR EXAMPLE 1 WITH SEVERAL DELAYS

| Delay h | Result | Order n* | CPU time | Order n* [19] |
|---------|--------|----------|----------|---------------|
| 0.1     | Stable | 4        | 0.2s     | 36            |
| 0.604   | Stable | 13       | 0.9s     | 10^8          |
| 0.605   | Unstable | 13     | 0.9s     | 10^8          |
| 2       | Unstable | 24      | 2.5s     | 10^{12}       |

TABLE II
EVALUATION OF OUR NECESSARY AND SUFFICIENT TEST FOR STABILITY FOR EXAMPLE 2 FOR K = 10 WITH SEVERAL DELAYS h

| Parameters | Result | Order n* | CPU time |
|------------|--------|----------|----------|
| K = 10, h = 0.552 | Stable | 65       | 150s     |
| K = 10, h = 0.553 | Unstable | 65     | 150s     |

Fig. 2. Example 2: Required orders with respect to (K, h).

The estimated order n* = N(ε(θ0)) for our necessary and sufficient test of stability is reported on Tables I and II. For Example 2, a map of orders in the (K, h) plane is depicted in Fig. 2. One can see that order n* increases as parameters h and K increase. The exact limit between stable (on the right) and unstable (on the left) regions obtained using D-partition is superposed. As emphasized in Remark 1, this line is excluded to our criterion.

The CPU time spent to compute n*, evaluate the components of matrix Pn, by induction and test the positivity of Pn, is also reported on both Tables. The computational load is obviously increasing as the order increases. Actually, the processing time is not only impacted by the positivity test but also by the calculation of the integrals terms in matrix Pn. By using the iterative relation given by Propositions 1–3, this time grows with the square of the order n*.

Other positivity tests of finite size n* for stability of time-delay systems have also been developed by Gomez et al. and Zhabko et al. A brief comparison with [19] is presented in Table I. The approximation of the complete Lyapunov–Krasovskii functional is realized by Legendre polynomial approximation here, whereas a discretization with exponential kernels is used in [19]. From the supergeometric convergence rate of Legendre approximation, our estimation of the minimal order n* = N(ε) satisfies ε ∼ e^{-n* log(n*)}. In [19], this estimation is limited to ε ∼ \frac{1}{n}. On Table I, this order of magnitude significant difference is highlighted. Note that this advantage has to be balanced with the shape of the matrix Pn and the ease of computing its components.

Finally, recalling the sufficient criterion of instability mentioned in Corollary 1. It permits to detect some unstable systems by testing Pn > 0 for a limited number of order. For Example 2, for low orders n = \{1, ..., 5\}, the corresponding unstable areas with respect to parameters (K, h), denoted \{U1, ..., Un\}, are drawn on Fig. 3. These tests are not time consuming and already get an accurate information on the unstable regions of Example 2. Note that U1 already spans the main areas of instability. The hierarchical structure U1 ⊂ ⋯ ⊂ Un is also verified. More interestingly, the hard-to-reach areas are located when the eigenvalues crosses the imaginary axis from the left-half plane to the right-half plane (see red lines).

VII. CONCLUSION

This article has been devoted to the formulation of a necessary and sufficient stability condition of time-delays systems, extending the result of paper [19]. It derives from the positivity of the complete Lyapunov–Krasovskii functional, where an approximation of the Lyapunov matrix has been considered. Stability can then be linked with the positive definiteness of a certain matrix of finite size n*, which depends on systems parameters. The originality of our work relies on the approximation techniques which employs polynomial coefficients instead of discretized elements. The supergeometric convergence rate satisfied by Legendre approximation demonstrates that our condition requires smaller orders n* than in [19]. Based on recurrence relations satisfied by Legendre polynomials, our stability criterion can be easily implemented by induction.

Our approach still to be generalized to distributed [9] and neutral [15] time-delay systems. New tracks of research would be to cover other classes of delay systems and more complex infinite-dimensional systems by this methodology. The Legendre polynomial approximation technique could also be deployed to problems of controller or observer synthesis for infinite-dimensional systems.

APPENDIX

A. Proofs of the Necessary and Sufficient Stability Lemmas

Proof: [Lemma 1] The proof can be found in [9, Th. 3] with W = I_n and follows arguments given in [21, Th. 5.19]. First, we introduce the functional

\[ W(\varphi) = V(\varphi) - \int_{-h}^{0} \left( \varphi(t) \right)^{\top} \begin{bmatrix} \frac{\eta_{0}}{2} I_{m} & 0 & 0 \\ 0 & \frac{\eta_{0}}{2} I_{m} & 0 \\ 0 & 0 & \frac{\eta_{0}}{2} A_{d} \end{bmatrix} \varphi(t) \, dt \] (49)

Functional V is built so that the time derivative along the trajectories x_\tau of system (1) gives (6). Then, the time derivative of the functional W gives

\[ \dot{W}(x_\tau) = - \begin{bmatrix} x(t) \\ x(t - h) \end{bmatrix}^{\top} \begin{bmatrix} \eta_{0} H(A) + \frac{1}{2} I_{m} & \eta_{0} A_{d} \\ \frac{1}{2} I_{m} & \frac{1}{2} I_{m} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t - h) \end{bmatrix} \] (50)
for which there exists a sufficiently small $\eta_0 > 0$ such that $\bar{W}(x_1) \leq 0$. Then, integrating from 0 to $\infty$ and assuming the exponential stability of (1), we yield $W(x_1) \geq 0$, for any initial conditions $x_0 = \phi$ in $C_{x_0}(-h; 0; \mathbb{R}^m)$. Thus, (7) holds with $\eta = \min(\eta_0, \frac{\alpha}{2})$.

**Proof:** [Lemma 2] The following proof is close to [19, Appendix I] and to [28, Lemma 5]. The only difference relies on the definition of $S$ that has been extended to $C_0$, instead of $C_1$. Let us denote $\sigma_0 = \alpha + i\beta$ be the eigenvalue with positive real part of system (1). According to [16], there exists a vector $C = C_1 + iC_2$ such that $|C_2| \leq |C_1| = 1$, $C_2^2 = 0$, and that

$$ w_{01} = -A - Ad e^{-\eta_0}C = 0.$$

Consequently, we have $\alpha \leq |\sigma_0| \leq |A| + |Ad| = \tau$, and

$$\bar{e}(t) = e^{\alpha t} (\cos(\beta t)C_1 - \sin(\beta t)C_2) \forall t \in \mathbb{R}$$

is a solution to system (1). Then, thanks to (6), the derivatives of $V(\bar{e}t)$ along the trajectories of (1) yields to

$$V(\bar{e}t) = -|\bar{e}(t-h)|^2. \quad \text{(51)}$$

After some calculations developed in [19], we also obtain that

$$V(\bar{e}t) \leq -\frac{e^{-2\alpha h}}{4\alpha} \cos^2(\theta_0) - \frac{e^{-2\alpha h}}{4\alpha} \cos^2(\theta_0) \leq -\eta_0. \quad \text{(52)}$$

Let $\varphi = \bar{e}_0$, which belongs to $C_{\varphi}(-h, 0; \mathbb{R}^m)$ and satisfies $|\varphi(0)| = |x(0)| = 1$. We finally prove by induction that $|\varphi(k)(t)| \leq r^k e^{\alpha t}$, for any $t \in [h, 0]$. Initially, $|\varphi(t)| \leq e^{\alpha t}$ holds on $[h, 0]$. Then, assuming $|\varphi(k)(t)| \leq r^{k-1} e^{\alpha t}$, since $\bar{e}$ satisfies (1) and is infinitely differentiable, we obtain

$$|\varphi(k+1)(t)| \leq |A||\varphi(k)(t)| + |Ad||\varphi(k)(t-h)| \leq r^{k+1} e^{\alpha t}.$$ 

Therefore, $|\varphi(k)| = \sup_{[h,0]} |\varphi(k)(t)| \leq r^k$ for any $k \in \mathbb{N}$.

**B. Proofs of the Recursive Relations in Propositions 1–3**

**Proof:** [Proposition 1] The proof is based on the relation

$$l_k' - l_{k-1} = \frac{2(2k-1)}{h} l_{k-1} \quad \forall k \geq 2 \quad \text{(53)}$$

satisfied by Legendre polynomials [14]. To compute $G_k$, an integration by parts leads to

$$\Gamma_k - \Gamma_{k-2} = \int_{-h}^{0} e^{(h+\tau)M}(l_k - l_{k-2})(\tau)d\tau = \int_{-h}^{0} e^{(h+\tau)M}l_{k-1}(\tau)d\tau + M^{-1} e^{(h+\tau)M}(l_k - l_{k-2})(0)$$

and knowing that $l_k(-h) = l_{k-2}(-h) = (-1)^k h$ and $l_k(0) = l_{k-2}(0) = 1$, the last term vanishes. Moreover, for $k \in (0, 1)$, we directly obtain

$$\Gamma_0 = -M^{-1}(e^{hM} - I_{2m}), \quad \Gamma_1 = -M^{-1}(e^{hM} + I_{2m}) - \frac{2}{h} M^{-1} \Gamma_0$$

which concludes the proof.

**Proof:** [Proposition 2] The successive changes of variables $\tau_2 = -(\tau_2 + h)$ and $\tau_1 = -(\tau_1 + h)$ directly lead to

$$\Gamma_{jk} = \int_{-h}^{0} \left(\int_{0}^{-(\tau_1+h)} e^{(\tau_1+\tau_2+h)M} l_1(\tau_1) d\tau_1\right) l_j(\tau_1) d\tau_1$$

following the parity properties of Legendre polynomials, i.e., $l_k(-\tau - h) = (-1)^j l_k(\tau)$ for all $\tau \in [h, 0]$.

**Proof:** [Proposition 3] As for the proof of Proposition 1, an integration by parts and (53) ensure that $G_k$ satisfies the following recursive relation:

$$\Gamma_{jk} = \frac{2(2k-1)}{h} M^{-1} \int_{-h}^{0} \left(\int_{-h}^{0} e^{(\tau_1-\tau_2)M} l_k(\tau_1) d\tau_1\right) l_j(\tau_1) d\tau_1$$

which means that, using the changing of coordinates $\tau_1 = \tau_2$ and $\tau_2 = \tau_1$, the following equations hold:

$$\Gamma_{jk} = \int_{-h}^{0} \int_{-h}^{0} e^{(\tau_1-\tau_2)M} l_j(\tau_1) l_k(\tau_2) d\tau_2 d\tau_1$$

An integration by parts yields the initial values.

**References**

[1] I. V. Alexandrova and A. P. Zhakhko, “Lyapunov-Krasovskii functionals for homogeneous systems with multiple delays,” Vestnik Saint Petersburg Univ., vol. 17, no. 10, pp. 136–149, 2011.

[2] M. Bajdov, A. Seurat, and F. Guaiict, “On the necessity of sufficient LMI conditions for time-delay systems arising from Legendre approximations,” arXiv:2207.0423.

[3] J. Boyd, Chebyshev and Fourier Spectral Methods (Dover Books on Mathematics), New York, NY, USA: Dover, 2001.

[4] D. Breda, S. Maset, and R. Vermiglio, “Pseudospectral differencing methods for characteristic roots of delay differential equations,” SIAM J. Sci. Comput., vol. 27 no. 2, pp. 482–495, 2005.

[5] E. R. Campos, S. Mondé, and M. Di Loretto, “Necessary stability conditions for linear difference equations in continuous time,” IEEE Trans. Autom. Control, vol. 63, no. 12, pp. 4405–4412, Dec. 2018.

[6] E. W. Cheney, Introduction to Approximation Theory, Providence, RI, USA: American Math. Society, 1982.

[7] R. M. Corless, G. H. Gonnet, D. E. Hare, J. D. Jeffrey, and D. E. Knuth, “On the Lambert W function,” Adv. Comput. Math., vol. 5, pp. 329–359, 1996.

[8] R. Datko, “Extending a theorem of A. M. Liapunov to Hilbert space,” J. Math. Anal. Appl., vol. 32, no. 3, pp. 610–616, 1970.

[9] A. V. Egorov, C. Curis, and S. Mondé, “Necessary and sufficient stability conditions for linear systems with pointwise and distributed delays,” Automatica, vol. 80, no. 6, pp. 118–224, 2017.
[10] A. V. Egorov and V. L. Kharitonov, “Approximation of delay Lyapunov matrices,” Int. J. Control, vol. 91, no. 11, pp. 2588–2596, 2017.
[11] A. V. Egorov and S. Mondié, “A stability criterion for the single delay equation in terms of the Lyapunov matrix,” Vestnik Saint Petersburg Univ., vol. 10, no. 1, pp. 106–115, 2013.
[12] A. V. Egorov and S. Mondié, “Necessary stability conditions for linear delay systems,” Automatica, vol. 50, no. 12, pp. 3204–3208, 2014.
[13] E. Fridman, U. Shaked, and V. Suplin, “Input/output delay approach to robust sampled-data $H_\infty$ control,” Syst. Control Lett., vol. 54, no. 3, pp. 271–282, 2005.
[14] W. Gautschi, “Orthogonal polynomials, quadrature, and approximation: Computational methods and software (in Matlab),” in Orthogonal Polynomials and Special Functions (Lecture Notes in Mathematics 1883). Berlin, Germany: Springer, 2006, pp. 1–77.
[15] M. A. Gomez, A. V. Egorov, and S. Mondié, “Necessary stability conditions for neutral type systems with a single delay,” IEEE Trans. Autom. Control, vol. 62, no. 9, pp. 4691–4697, Sep. 2017.
[16] M. A. Gomez, A. V. Egorov, and S. Mondié, “A Lyapunov matrix based stability criterion for a class of time-delay systems,” Vestnik Saint Petersburg Univ. Appl. Math. Comput. Sci. Control Processes, vol. 13, no. 4, pp. 407–416, 2017.
[17] M. A. Gomez, A. V. Egorov, and S. Mondié, “Lyapunov matrix based necessity and sufficient stability condition by finite number of mathematical operations for retarded type systems,” Automatica, vol. 108, 2019, Art. no. 108475.
[18] M. A. Gomez, A. V. Egorov, and S. Mondié, “Necessary stability conditions for neutral-type systems with multiple commensurate delays,” Int. J. Control, vol. 92, no. 5, pp. 1155–1166, 2019.
[19] M. A. Gomez, A. V. Egorov, and S. Mondié, “Necessary and sufficient stability condition by finite number of mathematical operations for time-delay systems of neutral type,” IEEE Trans. Autom. Control, vol. 66, no. 6, pp. 2802–2808, Jun. 2021.
[20] K. Gu, “Complete quadratic Lyapunov-Krasovskii functional: Limitations, computational efficiency, and convergence,” in Advances in Analysis and Control of Time-Delayed Dynamical Systems, J. Q. Sun and Q. Ding, Eds. Singapore: World Scientific, 2013, pp. 1–19.
[21] K. Gu, V. Kharitonov, and J. Chen, Stability of Time-Delay Systems. Boston, MA, USA: Birkhäuser, 2003.
[22] K. Ito, H. T. Tran, and A. Manitius, “A fully-discrete spectral method for delay-differential equations,” SIAM J. Numer. Anal., vol. 28, no. 4, pp. 1121–1140, 1991.
[23] V. L. Kharitonov, Time-Delay Systems: Lyapunov Functionals and Matrices (Control Engineering), 1 ed. Boston, MA, USA: Birkhäuser, 2013.
[24] C. R. Knespe and M. Roodzehani, “Stability of linear systems with interval time delays excluding zero,” IEEE Trans. Autom. Control, vol. 51, no. 8, pp. 1271–1288, Aug. 2006.
[25] V. B. Kolmanovskii and V. R. Nosov. Stability of Functional Differential Equations, vol. 180. Amsterdam, The Netherlands: Elsevier, 1986.
[26] J. McKay, “The D-partition method applied to systems with dead time and distributed lag,” Meas. Control, vol. 3, no. 10, pp. 293–294, 1970.
[27] I. V. Medvedeva and A. P. Zhabko, “Synthesis of Razumikhin and Lyapunov-Krasovskii approaches to stability analysis of time-delay systems,” Automatica, vol. 51, pp. 372–377, 2015.
[28] I. V. Medvedeva and A. P. Zhabko, “Stability of neutral type delay systems: A joint Lyapunov-Krasovskii and Razumikhin approach,” Automatica, vol. 106, pp. 83–90, 2019.
[29] W. Min, H. Yong, and J. H. She, Stability Analysis and Robust Control of Time-Delay Systems. Beijing, China: Springer, 2010.
[30] P. Mokhtary and F. Ghoreishi, “The $L^2$-convergence of the Legendre spectral Tau matrix formulation for nonlinear fractional integro-differential equations,” Numer. Algorithms, vol. 58, pp. 475–496, 2011.
[31] S. Mondie, J. Santos, and V. L. Kharitonov, “Robust stability of quasi-polynomials and the finite inclusions theorem,” IEEE Trans. Autom. Control, vol. 50, no. 11, pp. 1826–1831, Nov. 2005.
[32] A. Seuret and F. Gouaisbaut, “Hierarchy of LMI conditions for the stability analysis of time-delay systems,” Syst. Control Lett., vol. 81, pp. 1–7, 2015.
[33] H. Wang and S. Xiang, “On the convergence rates of Legendre approximation,” Math. Comput., vol. 81, 2012, Art. no. 278.