We introduce large vector spaces $M$ of multivariate homogeneous polynomials with a prescribed lower bound for the rank of each non-zero element of $M$.

Mathematics Subject Classification 14N05 · 15A69

1 Introduction

This paper has two stimuli. E. M. Gabidulin introduced the rank metric (instead of the Hamming metric) to define the minimum distance of a linear code [17, 18]. Hence, it is nice to have large linear spaces of matrices or of tensors or of symmetric tensors such that each of its non-zero elements has at least a given rank, $\delta$. In this paper, we consider linear spaces, $W$, of symmetric tensors, but we do not claim that our examples may be used to give nice codes, because in our examples all symmetric tensors $T \in W$ have rank $\geq \delta$ even over the algebraic closure of the base field. Hence, we do not use the Galois structure of finite fields, which should be essential to construct good Rank-Metric codes. The second input came from our previous work [4, 5], in which certain vector spaces of homogeneous polynomials are a key tool (the projective spaces $W(O_1, \ldots, O_k; d)$ defined below are the projectivations of the vector spaces we consider in this paper). To introduce our vector spaces of homogeneous polynomials, we recall the following classical set-up.

For all integers $m \geq 1$ and $d \geq 1$ let $V_d : \mathbb{P}^m \to \mathbb{P}^n$, $n = \binom{m+d}{m} - 1$, denote the order $d$ Veronese embedding of $\mathbb{P}^m$, i.e., the embedding of $\mathbb{P}^m$ induced by the vector space of all homogeneous polynomials of degree $d$ in $m+1$ variables. For each $P \in \mathbb{P}^n$ the rank or the symmetric tensor rank $r_{X_m,d}(P)$ of $P$ is the minimal cardinality of a set $S \subseteq X_{m,d}$ such that $P \in \langle S \rangle$, where $\langle \rangle$ denotes the linear span.

For each $Q \in X_{m,d}$, let $T_QX_{m,d} \subseteq \mathbb{P}^n$ denote the Zariski tangent space of $X_{m,d}$ at $Q$. The set $T_QX_{m,d}$ is a projective space of dimension $m$. For any $k$ distinct points $O_1, \ldots, O_k \in \mathbb{P}^m$ set

$$W(O_1, \ldots, O_k; d) := \langle \bigcup_{i=1}^k T_{V_d(O_i)}X_{m,d} \rangle \subseteq \mathbb{P}^n.$$ 

For each $r \in \mathbb{N}$ set $W(O_1, \ldots, O_k; d)(r) := \{ P \in W(O_1, \ldots, O_k; d) : r_{X_{m,d}}(P) = r \}$ and $W(O_1, \ldots, O_k; d)(\leq r) := \{ P \in W(O_1, \ldots, O_k; d) : r_{X_{m,d}}(P) \leq r \}$.

We prove the following result.

E. Ballico (✉)
Department of Mathematics, University of Trento, 38123 Povo, TN, Italy
E-mail: ballico@science.unitn.it
Theorem 1.1  Fix integers $m, d, k$ such that $m \geq 2$, $d \geq 7$ and $2 \leq k \leq (d^2 - 10d + 17)/8$. Fix general $O_1, \ldots, O_k \in \mathbb{P}^m$ and take any $r \in \{1, \ldots, d-3\}$ and any $P \in W(O_1, \ldots, O_k; d)(r)$. Then, there is a unique set $S \subseteq \{O_1, \ldots, O_k\}$ such that $P \in \langle v_d(S) \rangle$ and $\sharp(S) = r$.

In the set-up of Theorem 1.1, the set $S$ is the only set evincing the rank of $P$ (Proposition 2.4), i.e., $r_{X_m,d}(P) = r$ and $S$ is the only set $A \subseteq \mathbb{P}^m$ with cardinality $r$ such that $P \in \langle v_d(A) \rangle$. See Proposition 2.2 for a stronger statement if $m = 2$.

We recall that a finite set $S \subseteq \mathbb{P}^m$ is said to be in linearly general position if $\dim(B(E)) = \min\{s, \sharp(E) - 1\}$ for all $E \subseteq S$. When $m = 2$, Theorem 1.1 is quite good (Remark 2.3), but when $m \gg d$ it says almost nothing. For any $m \geq 2$, we prove the following result.

Theorem 1.2  Fix integers $m, d, k, r$ such that $m \geq 2$, $d \geq 7$, $1 \leq r \leq d - 3$ and $k < (m(d - 2) + 4 - r)/2$. Fix a set $\{O_1, \ldots, O_k\} \subseteq \mathbb{P}^m$ in linearly general position and take any $P \in W(O_1, \ldots, O_k; d)(r)$. Then, there is a unique set $S \subseteq \{O_1, \ldots, O_k\}$ such that $P \in \langle v_d(S) \rangle$ and $\sharp(S) = r$.

Corollary 1.3  Fix $m, d, k, r$ and $O_1, \ldots, O_k \in \mathbb{P}^m$ either as in Theorem 1.1 or as in Theorem 1.2. Let $M \subseteq W(O_1, \ldots, O_k; d)$ be a general subspace of dimension $(m+1)k - 2 - r$. Then, $r_{X_m,d}(P) > r$ for all $P \in M$.

We work over an algebraically closed base field $\mathbb{K}$ (see Remarks 2.5 and 3.5 for more general fields, Remark 3.6 for a discussion of the positive characteristic case).

We thank the referees whose advice improved the exposition.

2 The set-up of Theorem 1.1

Remark 2.1  For every $O \in \mathbb{P}^n$ and every $Q \in T_{v_d(O)}X_{m,d}$, there is a degree two zero-dimensional scheme $A \subseteq \mathbb{P}^m$ such that $A_{\text{red}} = \{O\}$, $\deg(A) = 2$ and $Q \in \langle v_d(Z) \rangle$. Hence, for each $P \in W(O_1, \ldots, O_k; d)$, there is a zero-dimensional scheme $Z \subseteq \mathbb{P}^m$ such that $Z_{\text{red}} = \{O_1, \ldots, O_k\}$, each connected component of $Z$ has degree two and $P \in \langle v_d(Z) \rangle$.

For each integer $t \geq 1$, the $t$-secant variety $\sigma_t(X_{m,d}) \subseteq \mathbb{P}^m$ of $X_{m,d}$ is the closure inside $\mathbb{P}^m$ of all linear spaces $\langle A \rangle$ with $A \subseteq X_{m,d}$ and $\sharp(A) = t$. The border rank $b_{X_{m,d}}(P)$ of $P \in \mathbb{P}^m$ is the first integer $t$ such that $P \in \sigma_t(X_{m,d})$. When $b_{X_{m,d}}(P) \leq d + 1$ there is a zero-dimensional scheme $Z \subseteq \mathbb{P}^m$ such that $\deg(Z) = b_{X_{m,d}}(P) = P \in \langle v_d(Z) \rangle$ (Remark 1.1, [12], Lemma 2.16). We say that any such $Z$ evinces the border rank of $P$.

We first do the case $m = 2$, because in this case [16] is a very powerful tool (which is also stated and proved in arbitrary characteristic).

Proposition 2.2  Fix integers $d, k$ such that $d \geq 7$ and $2 \leq k \leq (d^2 - 10d + 17)/8$. Fix general $O_1, \ldots, O_k \in \mathbb{P}^2$. For each $i \neq j$, set $L_{i,j} := \langle \{O_i, O_j\} \rangle$.

(a) Fix $x \in \{1, \ldots, d-3\}$ and $P \in W(O_1, \ldots, O_k; d)(x)$. Then, there is $U \subseteq \{O_1, \ldots, O_k\}$ such that $\sharp(U) = x + P \in \langle v_d(U) \rangle$.

(b) Fix $P \in W(O_1, \ldots, O_k; d)(d-2)$. Then, either there is $U \subseteq \{O_1, \ldots, O_k\}$ such that $\sharp(U) = d - 2$ and $P \in \langle v_d(U) \rangle$ or $P \in \langle L_{i,j} \rangle$.

(c) Fix $P \in \langle v_d(L_{i,j}) \rangle$ and assume $r_{X_{m,d}}(P) \geq d - 2$ and $\text{char}(\mathbb{K}) = 0$. Let $v'$ (resp. $v''$) be the degree 2 zero-dimensional subscheme of $L_{i,j}$ with $P_1$ (resp. $P_j$) as its support. Then, either $r_{X_{m,d}}(P) = d - 2$, $b_{X_{m,d}}(P) = 4$ and $v' \cup v''$ evinces the border rank of $P$ or $r_{X_{m,d}}(P) = d - 1$, $b_{X_{m,d}}(P) = 3$ and either $v' \cup \{O_i\}$ or $v'' \cup \{O_j\}$ evince the border rank of $P$ or $r_{X_{m,d}}(P) = d$, $b_{X_{m,d}}(P) = 2$ and either $v' \cup v''$ evince the border rank of $P$.

Proof Fix $P \in W(O_1, \ldots, O_k; d)(d-2)$ and set $r := r_{X_{m,d}}(P)$. Take any $A \subseteq \mathbb{P}^2$ evincing the rank of $P$, i.e., any finite set $A \subseteq \mathbb{P}^2$ such that $\sharp(A) = r_{2,d}(P)$ and $P \in \langle v_d(A) \rangle$. There is a zero-dimensional scheme $Z \subseteq \mathbb{P}^2$ such that $Z_{\text{red}} \subseteq \{O_1, \ldots, O_k\}$, each connected component of $Z$ has degree two and $P \in \langle v_d(Z) \rangle$ (Remark 2.1). Take $W \subseteq Z$ such that $P \in \langle v_d(W) \rangle$ and $P \notin \langle v_d(W') \rangle$ for each $W' \subseteq W$. Set $w := \deg(W)$ and $w' := \sharp(W)$. Since $W \subseteq Z$, we have $w \leq 2w'$, each connected component of $W$ has degree $\leq 2$ and $W_{\text{red}}$ is general in $\mathbb{P}^2$. In particular, $W_{\text{red}}$ has general postulation, i.e., $h^0(I_{W_{\text{red}}}(t)) = \max\{0, \left(\frac{t+2}{2}\right) - w'\}$ for all $t \in \mathbb{N}$. Take any line $L \subseteq \mathbb{P}^2$. We get $\deg(W \cap L) \leq 4$ and that if $\deg(W \cap L) \geq 3$, then $\sharp(L \cap W_{\text{red}}) \geq 2$ (i.e., $L$ is one of the lines $L_{i,j}$ and $W_{\text{red}} \supseteq \{O_i, O_j\}$).
(i) First assume $W = A$. Since $A$ is reduced, $W$ is reduced in this case. We have $A \subseteq \{O_1, \ldots, O_k\}$. We get the existence of $A \subseteq \{O_1, \ldots, O_k\}$ such that $A$ evinces the rank of $P$. Take $U := A$ to prove parts (a) and (b) if $W = A$.

(ii) Now, assume $W \neq A$ and $r \leq d - 2$ [as in parts (a) and (b)]. To prove part (a), we need to find a contradiction if $r \neq d - 2$. To prove part (b), we need to prove that $P \in \langle v_d(L_{i,j}) \rangle$ for some $i, j$ if $r = d - 2$ and $A \neq W$. We have $h^1(I_{A,W}(d)) > 0$ ([16], Lemma 1). Let $t$ be the maximal integer $t$ such that $h^1(I_{A,W}(t)) > 0$. We just proved that $\tau \geq d$. Set $z := \deg(A \cup W)$ and $s := \lfloor \sqrt{z} \rfloor$. We have $z \leq d - 2 + 2k$ and $s \leq z/s$. Since $h^1(I_{A,W}(d)) > 0$, we have $z \geq d + 2$ ([9], Lemma 34). Hence, $s \geq 3$.

Claim 1: $d \geq 2s + 3$.

Proof of Claim 1: Since $s = \lfloor \sqrt{z} \rfloor$, to prove Claim 1 it is sufficient to prove the inequality $4z \leq d^2 - 6d + 9$.

Since $z \leq 2k + d - 2$, it is sufficient to assume $k \leq (d^2 - 10d + 17)/8$.

Since $z < (s + 1)^2$, we have $z/s \leq s + 2$. Claim 1 gives $d \geq 2s - 1 \geq s - 3 + z/s$. Hence, $\tau \geq s - 3 + z/s$.

Hence, we may apply [16], Corollaire 3, and get that either $s = s - 3 + z/s$ and $A \cup W$ is the complete intersection of a curve of degree $s$ and a curve of degree $z/s$ or there is an integer $t \in \{1, \ldots, s - 1\}$ and a curve $T \subset \mathbb{P}^2$ such that $\deg(T) = t$ and $\deg(T \cap (A \cup W)) \geq (t - \tau + 3)$. 

(ii.1) First assume $\tau = s - 3 + z/s$ and that $A \cup W$ is the complete intersection of a curve of degree $s$ and a curve of degree $z/s$. In particular, $W_{\text{red}}$ is contained in a curve of degree $s$. Since $W_{\text{red}}$ has general postulation, we get $w' \leq (s^2 + 3s)/2$. Hence, $w \leq s^2 + 3s$. Since $r \leq d - 2$, we get $z \leq s^2 + 3s + d - 2$. We also have $d \geq \tau \geq s - 3 + z/s \geq s - 3 + s + (d - 2)/s$. Hence, $(s - 1)d \geq 2s^2 - 2$. Since $d \geq 2s + 3$ by Claim 1, we get a contradiction.

(ii.2) Now, assume the existence of an integer $t \in \{1, \ldots, s - 1\}$ and a curve $T \subset \mathbb{P}^2$ such that $\deg(T) = t$ and $\deg(T \cap (A \cup W)) \geq (t - \tau + 3)$. 

(ii.2.1) For each $x \in \mathbb{R}$ set $\psi_\ell(x) := 2x^2 - (x - 1)d - 2$. We have $\psi_\ell(x) = 4x - d$. Hence, $\psi_\ell(x) \leq 0$ if $x \leq d/4$ and $\psi_\ell(x) \geq 0$ if $x \geq d/4$. We have $\psi_\ell(1) = 0$, $\psi_\ell(2) = 2 - d < 0$, $\psi_\ell(d/4) = d^2/8 - 3d^2/4 - 2 < 0$ and $\psi_\ell(2) = (s - 2)/2 - (s - 2)/d < 0$ by Claim 1. Hence, $\psi_\ell(x) < 0$ if $2 \leq x \leq s - 1$.

(ii.2.2) Since $W_{\text{red}}$ has general postulation, we have $\pi(W_{\text{red}} \cap T) \leq (t^2 + 3t)/2$. Hence, $\deg(W \cap T) \leq t^2 + 3t$. Hence, $\deg((W \cup A) \cap T) \leq t^2 + 3t + d - 2$. Hence, $t^2 + 3t + d - 2 \geq \tau(t - \tau + 3)$, i.e., $\psi_\ell(t) \geq 0$. By step (ii.2.1) we have $t = 1$, i.e., there is a line $L \subset \mathbb{P}^2$ such that $\deg(L \cap (W \cup A)) \geq d + 2$. We saw before step (i) that $\deg(L \cap W) \leq 4$ and that if equality holds, then $L = L_{i,j}$ for some $i, j$. If $r \leq d - 3$, we get $\deg(W \cap L) \geq 5$, a contradiction, concluding the proof of part (a). If $r = d - 2$, we get $L = L_{i,j}$ for some $i, j$ and $A \subset L_{i,j}$. Since $P \in \langle v_d(A) \rangle$, we get $P \in \langle v_d(L_{i,j}) \rangle$, concluding the proof of part (b).

(iii) Take the set-up of part (c). By concision either ([11], Sect. 3.1, or [13], Remark 2.3, or [23], Proposition 3.1, or [21], Exercise 3.2.2.2) $r_{X_m,d}(P)$ is the rank of $P$ with respect to the rational normal curve $v_d(L_{i,j}) = X_{i,d}$. We also have $br_{X_m,d}(P) = br_{X_1,d}(P)$ ([11], Sect. 3.1). Hence, by the bivariante case ([9, 15, 22], Theorem 4.1) we know that $br_{X_{i,j}}(P) + r_{X_{i,j},d}(P) = d + 2$ and that there is a zero-dimensional scheme $Z \subset L_{i,j}$ such that $P \in \langle v_d(Z) \rangle$, $P \notin \langle v_d(Z') \rangle$ for any $Z' \subset Z$ and $\deg(Z) = br_{X_{i,j}}(P)$. Since $r_{X_{i,j},d}(P) \geq d - 2$, we have $\deg(Z) \leq 4$. The proof of part (b) also gives $P \in \langle v_d(v' \cup v'') \rangle$. Let $w \leq v' \cup v''$ be a minimal subscheme such that $P \in \langle w \rangle$ of $v' \cup v''$. Since $d \geq 7 \geq \deg(Z) + \deg(w) - 1$, we have $h^1(I_{Z,W}(d)) = 0$. Since $P \in \langle v_d(Z) \cap w \rangle$, [6], Lemma 1, gives $Z \subset w$. Then, we write all subschemes $\gamma$ of $v' \cup v'$ with degree 1, 2, 3, 4. Since $r \geq d - 2 > 2$ we exclude all cases in which $\gamma$ is reduced, i.e., the cases $\gamma = \{O_i\}$, $\gamma = \{O_j\}$ and $\gamma = \{O_i, O_j\}$.

Remark 2.3: Proposition 2.2 is quite strong, because if $d \geq 5$ and $k \geq [(d + 2)(d + 1)/6]$, then $W(O_1, \ldots, O_k) = \mathbb{P}^m$ [1, 2, 10, 14].

Proof of Theorem 1.1: The case $m = 2$ of Theorem 1.1 follows from the proof of Proposition 2.2. Hence, we may assume $m \geq 3$. Fix $r \in \{1, \ldots, d - 3\}$, any $P \in W(O_1, \ldots, O_k; d)(r)$ and any $A \subset \mathbb{P}^m$ evicing the rank of $P$. Remark 2.1 gives the existence of a minimal zero-dimensional scheme $W \subset \mathbb{P}^m$ such that $P \in \langle v_d(W) \rangle$, $W_{\text{red}} \subseteq \{O_1, \ldots, O_k\}$ and each connected component of $W$ has degree $\leq 2$. Set $w := \deg(W)$. We need to prove that $A = W$. Assume that $W \neq A$. Since $P \notin \langle v_d(W') \rangle$ for any $W' \subset W$ and any $W' \subset A$, we have $h^1(I_{A,W}(d)) > 0$ ([6], Lemma 1). Let $M \subset \mathbb{P}^m$ be a general subspace of dimension $m - 3$. Let $\ell : \mathbb{P}^m \setminus M \rightarrow \mathbb{P}^2$ be the linear projection from $M$. Since $M$ is general, we have $M \cap (A \cup \{O_1, \ldots, O_k\}) = \emptyset$. 

[Image 450x45 to 470x65]
$M$ intersects no line spanned by two of the points of $A \cup \{O_1, \ldots, O_k\}$ or by a degree two subscheme of $W$. Hence, $\ell$ is defined in a neighborhood of $A \cup W$, $\ell(A \cup W_{\text{red}})$ is injective and $\ell$ send isomorphically onto its image each connected component of $W$. Hence, $\ell(A \cup W)$ is a scheme isomorphic to $A \cup W$ (as abstract schemes). Set $A' := \ell(A)$ and $W' := \ell(W)$. For general $M$, we may still assume that $W'_\text{red}$ is formed by general points of $\mathbb{P}^2$.

Claim We have $h^1(\mathbb{P}^2, I_{A \cup W}(d)) > 0$.

Proof of the Claim Assume $h^1(\mathbb{P}^2, I_{A \cup W}(d)) = 0$. Since the linear projection from $M$ induces an isomorphism between $A \cup W$ and $A' \cup W'$, we get that $A \cup W$ imposes $\deg(A \cup W)$ independent conditions to the linear subspace of $|O_{\mathbb{P}^m}(d)|$ formed by the degree $d$ cones with vertex containing $M$. Hence, $A \cup W$ imposes $\deg(A \cup W)$ independent conditions to $|O_{\mathbb{P}^m}(d)|$, i.e., $h^1(I_{A \cup W}(d)) = 0$, a contradiction.

By the Claim there is a minimal subscheme $W_1 \subseteq W'$ and a minimal subscheme $A_1 \subseteq A'$ such that $h^1(\mathbb{P}^2, I_{A_1 \cup W_1}(d)) > 0$. We have $\zeta((W_1)_{\text{red}}) \leq k \leq (d^2 - 10d + 17)/8$ and $\zeta(A_1) \leq \zeta(A') = \zeta(A) \leq d - \delta$. Moreover, $(W_1)_{\text{red}}$ is general in $\mathbb{P}^2$. We are in the set-up of part (a) of Proposition 2.2 and we adapt step (ii) of its proof. Let $r$ be the maximal integer $t$ such that $h^1(\mathbb{P}^2, I_{A_1 \cup W_1}(t)) > 0$. Set $z := \deg(A_1 \cup W_1)$ and $s := \lfloor \sqrt{z} \rfloor$. We Claim 1 and parts (ii.2.1) and (ii.2.2) of the proof of Proposition 2.2 give the existence of a line $L \subset \mathbb{P}^2$ such that $\deg(L \cap (A_1 \cup W_1)) > d + 2$. Since $(W_1)_{\text{red}}$ is general and each connected component of $W_1$ has degree $\leq 2$, we get $\deg(L \cap W) \leq 4$. Hence, $\deg(A_1) \geq \deg(A_1 \cap L) \geq d + 2$, a contradiction. \hfill $\Box$

Proposition 2.4 Fix $P$ and $S$ as in the statement of Theorem 1.1. Then, $S$ is the only subset of $\mathbb{P}^m$ evincing the rank of $P$.

Proof Since $P \in \{v_{d}(S)\}$, $\zeta(S) = r$ and $rx_{m,d}(P) = r$, then $S$ is one of the sets evincing the rank of $P$. Assume the existence of a set $A \neq S$ such that $A$ evinces the rank of $P$. By the definition of rank, we have $P \notin \{v_{d}(E)\}$, if $E$ is either a proper subset of $A$ or a proper subset of $S$. Hence, $h^1(I_{A \cup S}(d)) > 0$ (Lemma 1). Since $\deg(A \cup S) < 2r < 2d + 1$, there is a line $L \subset \mathbb{P}^2$ such that $\zeta(L \cap (A \cup S)) \geq d + 2$ (Lemma 34). Since $O_1, \ldots, O_k$ are general in $\mathbb{P}^m$, we have $\zeta(S \cap L) < 2$. Hence, $\zeta(L \cap A) > \zeta(S \cap L)$. The proof of [6], Theorem 2.2 (alternatively, apply [7], Lemma 5.1), gives $A \setminus A \cap L = S \setminus S \cap L$. Hence, $\zeta(A) > \zeta(S)$, a contradiction. \hfill $\Box$
Proof If \( s = 1 \), then the lemma is true, because \( \deg(A \cup W) \leq t + 1 \) in this case. Hence, we may assume \( s \geq 2 \) and use induction on \( s \). Take a hyperplane \( H \subset \mathbb{P}^3 \) spanned by \( s \) of the points of \( S \), with the only restriction that if \( W \) is not reduced then \( H \) contains the support of the only unreduced connected component of \( W \). The inductive assumption gives \( h^1(H, \mathcal{I}_{(A \cup W) \cap H, H}(t)) = 0 \). Look at the residual exact sequence

\[
0 \to \mathcal{I}_{\text{Res}_H(A \cup W)}(t - 1) \to \mathcal{I}_{A \cup W}(t) \to \mathcal{I}_{H \cap (A \cup W), H}(t) \to 0
\]

(1)

The scheme \( \text{Res}_H(W) \) has degree at most 2 and it is reduced. Since \( \text{Res}_H(A \cup W) \) is the union of \( \text{Res}_H(W) \) and \( A \setminus A \cap H \), we have \( \deg(\text{Res}_H(A \cup W)) \leq t \). Hence, \( h^1(\mathcal{I}_{\text{Res}_H(A \cup W)}(t - 1)) = 0 \) ([9], Lemma 34). Hence, (1) gives \( h^1(\mathcal{I}_{A \cup W}(t)) = 0 \).

The following two elementary lemmas are very classical and in characteristic zero stronger results are known (e.g. [8], Lemma 1.8). However, the statements and proofs must be characteristic free to hope any application to codes over a finite field.

Lemma 3.3 Let \( Z \subset \mathbb{P}^d \), \( s \geq 1 \), be a zero-dimensional scheme such that \( S := \text{Z}_{\text{red}} \) is linearly independent in \( \mathbb{P}^d \) and each connected component of \( Z \) has degree \( \leq 2 \). Then, \( h^1(\mathcal{I}_Z(3)) = 0 \).

Proof We have \( \sharp(S) \leq s + 1 \). The lemma is true if \( s = 1 \), because \( \deg(Z) \leq 4 \) if \( s = 1 \). Hence, we may assume \( s \geq 2 \) and that the lemma is true for lower dimensional projective spaces. Let \( H \subset \mathbb{P}^d \) be a hyperplane such that \( \sharp(S \cap H) \) is maximal, i.e., take any \( H \supset S \) if \( \sharp(S) \leq s \) and any \( H \) spanned by \( s \) of the point of \( S \) if \( \sharp(S) = s + 1 \). Look at the residual exact sequence

\[
0 \to \mathcal{I}_{\text{Res}_H(Z)}(2) \to \mathcal{I}_Z(3) \to \mathcal{I}_{Z \cap H, H}(3) \to 0
\]

(2)

The inductive assumption gives \( h^1(H, \mathcal{I}_{Z \cap H, H}(3)) = 0 \). First assume \( \sharp(S) \leq s \). In this case, \( \text{Res}_H(Z) \) is a reduced scheme contained in \( S \). Hence, \( h^1(\mathcal{I}_{\text{Res}_H(Z)}(2)) = 0 \). Now, assume \( \sharp(S) = s + 1 \). In this case, \( \text{Res}_H(Z) \) is a scheme whose reduction, \( B \), is contained in \( S \) and at most one unreduced connected component (the component of \( Z \) not intersecting \( H \)). Lemma 3.1 gives \( h^1(\mathcal{I}_{\text{Res}_H(Z)}(2)) = 0 \). Hence, (2) gives \( h^1(\mathcal{I}_Z(3)) = 0 \).

Lemma 3.4 Fix an integer \( d \geq 3 \), a finite set \( A \subset \mathbb{P}^d \), \( s \geq 1 \), and a zero-dimensional scheme \( W \subset \mathbb{P}^d \) such that \( \sharp(A) \leq d - 3 \), \( W_{\text{red}} \) is linearly independent, and each connected component of \( W \) has degree \( \leq 2 \). Then, \( h^1(\mathcal{I}_{A \cup W}(d)) = 0 \).

Proof The lemma is true if \( s = 1 \), because \( \deg(W \cup A) \leq 4 + d - 3 \) if \( s = 1 \). Hence, we may assume \( s \geq 2 \) and use induction on \( s \). The lemma is true if \( A = \varnothing \) (Lemma 3.3). Hence, we may assume \( A \neq \varnothing \) and in particular \( d \geq 4 \). Taking a scheme \( W_1 \supseteq W \), we reduce to the case \( \sharp(W_{\text{red}}) = s + 1 \). Assume \( h^1(\mathcal{I}_{A \cup W}(d)) > 0 \). Let \( H \subset \mathbb{P}^d \) be any hyperplane containing \( s \) points of \( W_{\text{red}} \). Since \( W_{\text{red}} \cap H \) is linearly independent, the inductive assumption gives \( h^1(H, \mathcal{I}_{Z \cap H, H}(d)) = 0 \). Hence, (1) shows that it is sufficient to prove \( h^1(\mathcal{I}_{\text{Res}_H(A \cup W)}(d - 1)) = 0 \). The scheme \( \text{Res}_H(A \cup W) \) is the union of \( W' := \text{Res}_H(W) \) and the set \( A' := A \setminus A \cap H \). The scheme \( W' \) has as its reduction a subset of \( W_{\text{red}} \) and at most one of its connected components is not reduced (in this case it has degree two if it exists). Hence, there is a hyperplane \( M \subset \mathbb{P}^d \) such that \( \deg(\text{Res}_M(W')) \leq 1 \). We have \( \sharp(A' \cap M) \leq \sharp(A) \leq d - 3 \). Lemma 3.2 gives \( h^1(M, \mathcal{I}_{M \cap (A' \cup W')}(d - 1)) = 0 \). The set \( \text{Res}_M(A' \cup W') \) has cardinality at most \( d - 2 \). Hence, \( h^1(\mathcal{I}_{\text{Res}_M(A' \cup W')}(d - 2)) = 0 \). A residual sequence like (1) for \( t = d - 1 \) gives \( h^1(\mathcal{I}_{\text{Res}_H(A \cup W)}(d - 1)) = 0 \).

Proof of Theorem 1.2 Fix any \( A \subset \mathbb{P}^m \) evincing the rank of \( P \). Remark 2.1 gives the existence of a minimal zero-dimensional scheme \( W \subset \mathbb{P}^m \) such that \( P \in (\nu_d(W)) \), \( W_{\text{red}} \subseteq \{O_1, \ldots, O_k\} \) and each connected component of \( W \) has degree \( \leq 2 \). Set \( w := \deg(W) \). We need to prove that \( A = W \). Assume that \( W \neq A \). Since \( P \notin (\nu_d(W')) \) for any \( W' \subset W \) and any \( W' \subset A \), we have \( h^1(\mathcal{I}_{A \cup W}(d)) = 0 \) ([16], Lemma 1).

Let \( S_0 \) be the set of all hyperplanes \( M \subset \mathbb{P}^m \) containing at least one point of \( A \). Set \( B_0 := A \cup W, A_0 := A \) and \( W_0 := W \). Fix \( H_1 \subset S_0 \) such that \( \deg(H_1 \cap (A \cup W)) \) is maximal among all \( H_1 \in S_0 \). Set \( B_1 := \text{Res}_{H_1}(B_0), W_1 := \text{Res}_{H_1}(W_0) \) and \( A_1 := \text{Res}_{H_1}(A_0) = A \setminus A \cap H_1 \). If \( A_1 = \varnothing \), then let \( S_1 \) be the set of all hyperplanes of \( \mathbb{P}^m \). If \( A_1 \neq \varnothing \), then let \( S_1 \) be the set of all hyperplanes of \( \mathbb{P}^m \) containing at least one point of \( A_1 \). For all integers \( i \geq 2 \) defined recursively the hyperplanes \( H_i \subset \mathbb{P}^m \), the integer \( a_i \), the zero-dimensional schemes \( B_i, W_i, A_i \) and the set \( S_i \) of hyperplanes of \( \mathbb{P}^m \) in the following way. Let \( H_i \in S_{i - 1} \) be such that \( a_i := \deg(H_i \cap B_{i - 1}) \) is maximal among all hyperplanes of \( S_{i - 1} \) and set \( B_i := \text{Res}_{H_i}(B_{i - 1}) \),
$W_i := \text{Res}_{H_i}(W_{i-1})$ and $A_i := \text{Res}_{H_i}(A_{i-1})$. Hence, $B_i = A_i \cup W_i$. If $A_i = \emptyset$, then let $S_i$ be the set of all hyperplanes of $P^m$. If $A_i \neq \emptyset$, then let $S_i$ be the set of all hyperplanes of $P^m$ containing at least one point of $A_i$.

Every zero-dimensional scheme $E \subset P^m$ of degree $\leq m$ is contained in a hyperplane. Hence, if $a_i \leq m - 1$, then $B_i \subseteq H_i$ and $a_{i+1} = 0$.

Notice that if $A_{i-1} \neq \emptyset$, then $A_i \subseteq A_{i-1}$. Since $\sharp(A_0) = r < d - 3$, we get that for each $i > 0$ either $A_i = \emptyset$ or $\sharp(A_i) \leq d - 3 - i$. For all $i \geq 0$, we have the following residual exact sequences

$$0 \to \mathcal{I}_{B_{i+1}}(d - i - 1) \to \mathcal{I}_{B_i}(d - i) \to \mathcal{I}_{B_i \cap H_{i+1}, H_{i+1}}(d - i) \to 0$$

(3)

Since $h^1(\mathcal{I}_{B_0}(d)) > 0$, the exact sequences (3) for $i \geq 0$ give the existence of a minimal integer $e \geq 0$ such that $h^1(\mathcal{I}_{B_{i+1}}(H_{i+1}, H_{i+1}, H_{i+1})(d - e)) > 0$. Since $(W_i)_{\text{red}} \subseteq H_{i+1}$ is linearly independent in $H_{i+1}$ and either $A_i = \emptyset$ or $\sharp(A_i) \leq d - 3 - i$, Lemma 3.4 gives $e \geq d - 2$. Assume for the moment $e \geq d - 1$. We get $2k + r > w + r \geq m(d - 1)$, a contradiction. Now, assume $e = d - 2$. Since $h^1(\mathcal{I}_{B_i}(H_{i+1}, \mathcal{I}_{B_i \cap H_{i+1}}(H_{i+1})(2)) > 0$, we have $\deg(B_{i-2} \cap H_{d-1}) \geq 4$ ([9], Lemma 34). Hence, $2k + r \geq w + r \geq m(d - 2) + 4$, contradicting the assumption $k < (m(d - 2) + 4 - r)/2$.

Proof of Corollary 1.3 Let $M \subset W(O_1, \ldots, O_k)$ be a general linear subspace with codimension at least $r + 1$. Since $M$ is general, we may have $M \cap \langle \nu_{d}(S) \rangle = \emptyset$ for all $S \subset \{O_1, \ldots, O_k\}$ with $\sharp(S) = \min(k, r)$. Hence, to conclude the proof of Corollary 1.3 it is sufficient to prove that $\dim(W(O_1, \ldots, O_k)) = (m + 1)k - 1$. This is true in the set-up of Theorem 1.1 by a weak form of a theorem of Alexander and Hirschowitz ([1,2,10,14]).

Now, we take the set-up of Theorem 1.2. By [14], Lemma 4, it is sufficient to prove $h^1(\mathcal{I}_Z, \mathcal{I}_Z(d)) = 0$ for each zero-dimensional scheme $Z \subset P^m$ such that $Z_{\text{red}} \subseteq \{O_1, \ldots, O_k\}$ and each connected component of $Z$ as degree at most 2. Repeat the proof of Theorem 1.2 taking $A = \emptyset$.

Remark 3.5 Theorem 1.2 is true over any field $K$ for which there are $k$ distinct points $O_1, \ldots, O_k \in P^m(K)$ in linearly general position, just because we defined the rank using subsets of $X_{m,d}(K)$. The existence of $k$ points of $P^m(K)$ in linearly general position is obvious (for arbitrary $k$) if $K$ is infinite. If $K$ is finite, then it is sufficient to assume $k \leq \sharp(K) + 1$ ([19], Theorem 27.5.1 (iv)); if $m = 2$ and $\sharp(S)$ is even we may even allow the case $k = \sharp(K) + 2$ ([19], Eq. (27.2)). Hence, the statement of Theorem 1.2 is true for an arbitrary field $K$, with the only restriction that $m \geq 3$ if $K$ is finite (but it may be an empty statement when $K$ is finite if $k \geq \sharp(K) + 2$).

Remark 3.6 Assume $K$ algebraically closed with characteristic $p > 0$. We fix the degree $d$ of the homogeneous polynomial we are interested in and hence we fix $v_d$. Fix any integer $m > 0$. We take as the definition of rank of $P \in P^n, n := \frac{(m+d)}{m} - 1$, the $X_{m,d}$-rank, i.e., the rank with respect to the Veronese variety $X_{m,d} = v_d(P^n)$. If $p > d$ (but only if $p > d$), we may translate this definition for a homogeneous polynomial $f \neq 0$ as the minimal number of summands of $d$-powers of linear forms needed to obtain $f$. With our definition in terms of $X_{m,d}$-rank if $p > d$, then the case $m = 1$ is true ([20], Theorem 1.44), but the case $p \leq d$ fails (but it fails in a controlled way (3)); for instance if $p = d = 2$, there is a unique point of $P^3$ (the strange point of the smooth conic $X_{1,2} \subset P^3$) with $X_{1,2}$-rank 3). In the set-up of Proposition 2.2 and in many other places, one can characterize the minimal number of summands of $d$-powers of linear forms needed to obtain $f$. With our definition in terms of $X_{m,d}$-rank if $p > d$, then the case $m = 1$ is true ([20], Theorem 1.44), but the case $p \leq d$ fails (but it fails in a controlled way (3)); for instance if $p = d = 2$, there is a unique point of $P^3$ (the strange point of the smooth conic $X_{1,2} \subset P^3$) with $X_{1,2}$-rank 3). In the set-up of Proposition 2.2 and in many other places, one can characterize the minimal number of summands of $d$-powers of linear forms needed to obtain $f$. With our definition in terms of $X_{m,d}$-rank if $p > d$, then the case $m = 1$ is true ([20], Theorem 1.44), but the case $p \leq d$ fails (but it fails in a controlled way (3)); for instance if $p = d = 2$, there is a unique point of $P^3$ (the strange point of the smooth conic $X_{1,2} \subset P^3$) with $X_{1,2}$-rank 3). In the set-up of Proposition 2.2 and in many other places, one can characterize the minimal number of summands of $d$-powers of linear forms needed to obtain $f$.

As in part (c) of Proposition 2.2 take $d \geq 7$ and $P \in \langle v_d(L_{i,j}) \rangle$ with $r_{X_{2,d}}(P) \geq d - 2$. Then, $b_{X_{2,d}}(P) = b_{X_{1,d}}(P), r_{X_{2,d}}(P) = r_{X_{1,d}}(P)$, every subscheme of $P^2$ evincing the rank of $P$ with respect to $X_{2,d}$ is contained in $L_{i,j}$ and every subset of $P^2$ evincing the rank of $P$ with respect to $X_{2,d}$ is contained in $L_{i,j}$.

Acknowledgments The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

Open Access This article is distributed under the terms of the Creative Commons Attribution License which permits any use, distribution, and reproduction in any medium, provided the original author(s) and the source are credited.

References

1. Alexander, J.; Hirschowitz, A.: La méthode d’Horace éclatée: application à l’interpolation en degré quatre. Invent. Math. 107, 585–602 (1992)
2. Alexander, J.; Hirschowitz, A.: Polynomial interpolation in several variables. J. Alg. Geom. 4(2), 201–222 (1995)

3. Ballico, E.: An upper bound for the X-ranks of points of \( \mathbb{P}^n \) in positive characteristic. Albanian J. Math. 5(1), 3–10 (2011)

4. Ballico, E.: An upper bound for the tensor rank. Hindawi Publishing Corporation ISRN Geometry Volume 2013, Article ID 241835, 3 pages

5. Ballico, E.: Symmetric tensor rank and scheme-rank: an upper bound in terms of secant varieties. Volume 2013, Article ID 614195, 3 pages (2013). doi:10.1155/2013/614195

6. Ballico, E.; Bernardi, A.: Decomposition of homogeneous polynomials with low rank. Math. Z. 271, 1141–1149 (2012).

7. Ballico, E.; Bernardi, A.: Stratification of the fourth secant variety of Veronese variety via the symmetric rank. Adv. Pure Appl. Math. 4(2), 215–250 (2013). doi:10.1515/apam-2013-0015

8. Bernardi, A.; Catalisano, M.V.; Gimigliano, A.; Ida M.: Secant varieties to osculating varieties of Veronese embeddings of \( \mathbb{P}^n \). J. Algebra 321(3), 982–1004 (2009)

9. Bernardi, A.; Gimigliano, A.; Ida, M.: Computing symmetric rank for symmetric tensors. J. Symb. Comput. 46(1), 34–53 (2011)

10. Brambilla, M.C.; Ottaviani, G.: On the Alexander–Hirschowitz theorem. J. Pure Appl. Algebra 212(5), 1229–1251 (2008)

11. Buczyński, W.; Buczyński, J.: On differences between the border rank and the smoothable rank of a polynomial arXiv:1305.1726

12. Buczyński, J.; Ginensky, A.; Landsberg, J.M.: Determinantal equations for secant varieties and the Eisenbud–Koh–Stillman conjecture. J. Lond. Math. Soc. 88(2), 1–24 (2013). doi:10.1112/jlms/jds073

13. Carlini, E.; Catalisano, M.V.; Geramita, A.V.: The solution to the waring problem for monomials and the sum of coprime monomials. J. Algebra 370, 5–14 (2012)

14. Chandler, K.: A brief proof of a maximal rank theorem for generic double points in projective space. Trans. Amer. Math. Soc. 353(5), 1907–1920 (2000)

15. Comas, G.; Seiguer, M.: On the rank of a binary form. Found. Comp. Math. 11(1), 65–78 (2011)

16. Ellia, P.; Peskine, C.: Groupes de points de \( \mathbb{P}^2 \) caractè re et position uniforme. Algebraic geometry (L’Aquila, 1988), pp. 111–116. Lecture Notes in Mathematics, 1417. Springer, Berlin (1990)

17. Gabidulin, E.M.: Theory of codes with maximum rank distance. Proble. Inf. Trans. 21(1), 1–12 (1985)

18. Gadouleau, M.; Yan, Z.: MacWilliams identity for codes with the rank metric. Hindawi Publishing Corporation EURASIP Journal on Wireless Communications and Networking Volume 2008, Article ID 754021, 13 pages. doi:10.1155/2008/754021

19. Hirschfeld, J.W.P.; Thas, J.A.: General Galois Geometries. Clarendon Press, Oxford (1991)

20. Iarrobino, A.; Kanev, V.: Power sums, Gorenstein algebras, and determinantal loci. Lecture Notes in Mathematics, vol. 1721. Springer, Berlin, Appendix C by Iarrobino and Steven L. Kleiman (1999)

21. Landsberg, J.M.: Tensors: Geometry and Applications. Graduate Studies in Mathematics, vol. 128. American Mathematical Society Providence (2012)

22. Landsberg, J.M.; Teitler, Z.: On the ranks and border ranks of symmetric tensors. Found. Comput. Math. 10(3), 339–366 (2010)

23. Lim, L.H.; de Silva, V.: Tensor rank and the ill-posedness of the best low-rank approximation problem. SIAM J. Matrix Anal. Appl. 30(3), 1084–1127 (2008)