Dilatation operator in (super-)Yang-Mills theories on the light-cone

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Abstract

The gauge/string correspondence hints that the dilatation operator in gauge theories with the superconformal $SU(2,2|\mathcal{N})$ symmetry should possess universal integrability properties for different $\mathcal{N}$. We provide further support for this conjecture by computing a one-loop dilatation operator in all (super)symmetric Yang-Mills theories on the light-cone ranging from gluodynamics all the way to the maximally supersymmetric $\mathcal{N} = 4$ theory. We demonstrate that the dilatation operator takes a remarkably simple form when realized in the space spanned by single-trace products of superfields separated by light-like distances. The latter operators serve as generating functions for Wilson operators of the maximal Lorentz spin and the scale dependence of the two are in the one-to-one correspondence with each other. In the maximally supersymmetric, $\mathcal{N} = 4$ theory all nonlocal light-cone operators are built from a single CPT self-conjugated superfield while for $\mathcal{N} = 0, 1, 2$ one has to deal with two distinct superfields and distinguish three different types of such operators. We find that for the light-cone operators built from only one species of superfields, the one-loop dilatation operator takes the same, universal form in all SYM theories and it can be mapped in the multi-color limit into a Hamiltonian of the $SL(2|\mathcal{N})$ Heisenberg (super)spin chain of length equal to the number of superfields involved. For “mixed” light-cone operators involving both superfields the dilatation operator for $\mathcal{N} \leq 2$ receives an additional contribution from the exchange interaction between superfields on the light-cone which breaks its integrability symmetry and creates a mass gap in the spectrum of anomalous dimensions.

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1. Introduction

Four-dimensional gauge theories are expected to admit, at least in the multi-color limit, a complementary description via yet to be identified string theories [1]. The latter operate in terms of collective degrees of freedom (Faraday lines) which are more appropriate to tackle the strong-coupling dynamics of Yang-Mills theories. The most prominent and thoroughly verified to date example of the gauge/string correspondence is the maximally supersymmetric \( \mathcal{N} = 4 \) Yang-Mills (SYM) theory [2] and its dual description in terms of a critical string theory with \( \text{AdS}_5 \times \text{S}^5 \) target space [3, 4, 5]. Recently it has been conjectured [6] that noncritical sigma models, possessing the \( \kappa \)-symmetry and having \( \text{AdS}_5 \) geometry as a factor of the target space, are dual to yet unknown (non)supersymmetric gauge theories which exhibit conformal \( \text{SU}(2, 2|\mathcal{N}) \) invariance with \( \mathcal{N} = 0, 1, 2 \) at finite values of the coupling constant. Both critical and noncritical sigma models on the anti-de Sitter space turn out to be completely integrable [7] and it is believed that this property must manifest itself in hidden symmetries of the corresponding Yang-Mills theory.

Indeed, it has been known for some time that four-dimensional Yang-Mills theories exhibit a remarkable phenomenon of integrability. It has been first discovered in the context of QCD, i.e., \( \mathcal{N} = 0 \) Yang-Mills theory with fundamental matter, in the studies of the Regge asymptotics of scattering amplitudes [8, 9] and anomalous dimensions of high-twist Wilson operators in multi-color limit [10, 11, 12, 13]. In the former case, high-energy asymptotics of the scattering amplitudes is driven by the contribution of multi-gluonic color-singlet states which can be identified as eigenstates of the Heisenberg \( \text{SL}(2, \mathbb{C}) \) spin magnet. In the latter case, the one-loop dilatation operator for a special class of maximal-helicity high-twist operators can be mapped into a Hamiltonian of a completely integrable Heisenberg \( \text{SL}(2, \mathbb{R}) \) spin magnet. The number of sites in this spin chain equals the number of fundamental fields involved and the symmetry group is a collinear subgroup of the \( \text{SO}(4, 2) \sim \text{SU}(2, 2) \) conformal group. Although conformal symmetry of QCD is broken at the quantum level, symmetry breaking effects arise starting from two loops only [14] (for a review, see [15]). This implies that to one-loop accuracy, QCD is not distinct from a conformal field theory. Obviously, the same holds true in supersymmetric \( \mathcal{N} = 1 \) and \( \mathcal{N} = 2 \) SYM theories whereas the \( \mathcal{N} = 4 \) theory remains conformal to all orders of perturbation theory.

In the present paper, we shall study integrability properties of the one-loop dilatation operator in \( \mathcal{N} \)-extended SYM theories. As was already mentioned, in the \( \mathcal{N} = 0 \) theory the integrability phenomenon has been observed in the sector of maximal-helicity operators. The integrability gets extended to a larger class of operators as one goes over from the nonsupersymmetric (\( \mathcal{N} = 0 \)) to the maximally supersymmetric (\( \mathcal{N} = 4 \)) gauge theory. In particular, in the \( \mathcal{N} = 4 \) model, the integrability was found in the sector of local scalar operators. In the multi-color limit, the one-loop mixing matrix for such operators can be mapped into a Hamiltonian of the \( \text{SO}(6) \) Heisenberg magnet with the symmetry group reflecting the \( R \)-symmetry of the model [16]. Eventually, the \( \text{SL}(2, \mathbb{R}) \) and \( \text{SO}(6) \) sectors can be unified together into a \( \text{PSU}(2, 2|4) \) Heisenberg magnet [17, 18]. The gauge/string correspondence hints that the dilatation operator in gauge theories with the \( \text{SU}(2, 2|\mathcal{N}) \) symmetry should possess universal integrability properties for different \( \mathcal{N} \) [6]. This suggests that integrable structures found previously in the \( \mathcal{N} = 0 \) and \( \mathcal{N} = 4 \) SYM should be different facets of the same phenomenon. To address this issue we need an approach that would allow us to treat simultaneously the operator mixing in various \( \mathcal{N} \)-extended SYM theories.

The conventional covariant approach based on calculation of the mixing matrix for local Wilson operators is not particularly suited for these purposes as it has the following shortcomings.
The form of the mixing matrix depends on the sector under consideration. For example, it is given by a finite-dimensional matrix for local composite operators built from fundamental fields without covariant derivatives and by an infinite-dimensional matrix for operators with an arbitrary number of derivatives. In addition, due to different particle content of $N$–extended SYM theories, the number of possible Wilson operators vary with $N$ and, therefore, one would not expect any connection among the mixing matrices for different $N$. Last but not least, hidden integrability symmetry is identifiable only in simplest sectors of Wilson operators but it is not manifest in the most generic case. As was demonstrated in [19], these drawbacks can be circumvented by studying the mixing of Wilson operators within a light-cone superspace formalism [20, 21]. Recently this formalism has been applied to calculating anomalous dimensions of Wilson operators in the $N = 1$ SYM theory within an effective action approach [22]. The interaction vertices in the effective action are manifestly invariant under superconformal transformations and can be mapped into four-point correlation functions.

The Wilson operators in the $N$–extended SYM theories are local composite gauge-invariant operators built as products of an arbitrary number of fundamental fields and an arbitrary number of covariant derivatives acting on them. They can be classified according to representations of the superconformal $SU(2,2|N)$ group. In what follows we shall consider single-trace Wilson operators possessing the maximal Lorentz spin and minimal twist for a given number of constituent fields. They belong to the $SL(2|N)$ subgroup of the full superconformal group and are known in QCD as quasipartonic operators [23]. In this paper, we demonstrate that the one-loop dilatation operator in the $N$–extended SYM theory acting on the space spanned by the quasipartonic operators has a universal form in the multi-color limit and is intrinsically related to a completely integrable $SL(2|N)$ Heisenberg magnet. For $N = 0$ one recovers the $SL(2)$ magnet in the sector of maximal helicity Wilson operators [10, 11, 12, 13], while for $N = 4$ the $SL(2|4)$ magnet forms an autonomous subsector of a bigger $PSU(2,2|4)$ magnet [17, 18].

To study the scale dependence of quasipartonic operators, it is convenient to switch from local Wilson operators to nonlocal light-cone operators. The latter are generating functions for the quasipartonic operators and are defined as

$$\mathcal{O}_{i_1...i_L}(z_1, \ldots, z_L) = \text{tr}\{X_{i_1}(nz_1) \cdots X_{i_L}(nz_L)\},$$  \hspace{1cm} (1.1)

where $X_i = \{\lambda, \bar{\lambda}, n^\mu F_{\mu+}, \phi\}$ is a unified notation for “good” components of fundamental fields in the underlying $N$–extended SYM (fermions, field strength tensor, scalars) given by $N_c \times N_c$ matrices $X_i = X_{i}^a t^a$ with $t^a$ being generators of the fundamental representation of the $SU(N_c)$ group. The fields in (1.1) are located along a light-cone direction defined by a light-like vector $n_\mu$ such that $n_\mu = 0$ and their position on the light-cone is specified by the coordinates $z_1, \ldots, z_L$. It is tacitly assumed that the gauge invariance in (1.1) is restored by inserting Wilson lines $P \exp (ig \int z_{k+1} ds n^\mu A_\mu(ns))$ that run along the light-cone between two adjacent fields. Later, we shall adopt the light-cone axial gauge $n^\mu A_\mu(x) = A_+(x) = 0$ in which these Wilson lines are reduced to a unity matrix. Expanding (1.1) around $z_1 = \ldots = z_L = 0$ one generates the quasipartonic operators

$$\mathcal{O}_{i_1...i_L}(z_1, \ldots, z_L) = \sum_{k_1,\ldots,k_L \geq 0} \frac{z_{k_1}}{k_1!} \cdots \frac{z_{k_L}}{k_L!} \text{tr}\{D_{+}^{k_1} X_{i_1}(0) \cdots D_{+}^{k_L} X_{i_L}(0)\},$$  \hspace{1cm} (1.2)

where $D_{+} = n^\mu D_\mu$ is a projection of the covariant derivative on the light-cone. The operators $\text{tr}\{D_{+}^{k_1} X_{i_1}(0) \cdots D_{+}^{k_L} X_{i_L}(0)\}$ have the maximal possible Lorentz spin, $k_1 + \ldots + k_L$, and their twist
equals the number of constituents $L$. Among them there are the operators with no derivatives \(\text{tr}\{X_{i_1}(0)\ldots X_{i_L}(0)\}\) as well as operators involving an arbitrary number of covariant derivatives. These operators mix under renormalization and the corresponding mixing matrix can be deduced from the scale dependence of nonlocal light-cone operators (1.1) with a help of (1.2).

A very convenient framework for discussing the scale dependence of the quasipartonic operators in the SYM theories is provided by the light-cone superspace formalism \[20, 21\]. In this approach, the SYM theory is quantized on the light-cone and its Lagrangian is built from two distinct chiral superfields $\Phi(x, \theta^A)$ and $\Psi(x, \theta^A)$ (with $A = 1, \ldots, N$) which comprise all “good” components of the fundamental fields $X_i(x)$ describing dynamically independent propagating modes. Both superfields realize a representation of the superconformal $SU(2|2N)$ group and carry a definite value of the conformal spin. While the chiral superfield $\Psi$ has the conformal spin $j_\Psi = (3 - N)/2$ which depends on the number of supercharges $N$, the one of $\Phi$ equals $j_\Phi = -1/2$. For $N \leq 2$ the two superfields are independent on each other whereas in the maximally supersymmetric, $N = 4$ gauge theory they are not independent, $\Phi \sim \Psi$.

By definition, the propagating fields $X_i(z\mu)$ are the coefficients in the Taylor expansion of the superfields $\Phi(z\mu, \theta^A)$ and $\Psi(z\mu, \theta^A)$ in powers of the odd coordinates $\theta^A$. This suggests to generalize further (1.1) and consider composite single-trace operators constructed from an arbitrary number of superfields located on the light-cone $x^\mu = z\mu$. Let us denote $\Phi(z\mu, \theta^A) \equiv \Phi(Z)$ and $\Psi(z\mu, \theta^A) \equiv \Psi(Z)$ and identify $Z = (z, \theta^A)$ as a point in the $(N + 1)$-dimensional light-cone superspace. In general, one can distinguish three types of single-trace operators:

(i) operators built only from $\Phi$–superfields:

\[
\mathcal{O}_{\Phi}(Z_1, \ldots, Z_L) = \text{tr}\{\Phi(Z_1) \ldots \Phi(Z_L)\}, \tag{1.3}
\]

(ii) operators built only from $\Psi$–superfields:

\[
\mathcal{O}_{\Psi}(Z_1, \ldots, Z_L) = \text{tr}\{\Psi(Z_1) \ldots \Psi(Z_L)\}, \tag{1.4}
\]

(iii) operators built from both $\Phi$– and $\Psi$–superfields:

\[
\mathcal{O}(Z_1, \ldots, Z_L) = \text{tr}\{\Phi(Z_1) \ldots \Psi(Z_L)\}, \tag{1.5}
\]

In the $N = 4$ SYM theory all three sectors coincide since $\Psi \sim \Phi$. For $N \leq 2$ each sector has to be considered separately. Expanding the operators (1.3) – (1.5) in $\theta^{A_1} \ldots \theta^{A_L}$, one generates all nonlocal light-cone operators (1.1), symbolically,

\[
\mathcal{O}(Z_1, \ldots, Z_L) = \sum_{\{i_1, \ldots, i_L\}} \theta^{A_{i_1}} \ldots \theta^{A_{i_L}} \mathcal{O}_{i_1 \ldots i_L}(z_1, \ldots, z_L), \tag{1.6}
\]

where $A_i = 1, \ldots, N$, so that the total number of $\theta$–variables in this expansion varies between 0 and $N^L$.

Combining together (1.1), (1.2) and (1.6) one finds that the problem of finding the scale dependence of (an infinite number of) Wilson, quasipartonic operators $\text{tr}\{D_{+1}X_{i_1}(0) \ldots D_{+L}X_{i_L}(0)\}$ can be mapped into the problem of constructing the dilatation operator on the space spanned by nonlocal (super-)light-cone operators (1.3) – (1.5) (see Eq. (1.7) below). As we will show below, to one-loop accuracy in the multi-color limit, the operators (1.6) mix under renormalization with single-trace light-cone operators built from the same number of $\Phi$– and $\Psi$–superfields but
ordered differently inside the trace. This allows one to realize the one-loop dilatation operator for the operators (1.3) as a quantum-mechanical Hamiltonian $H$ acting on $L$ superfields. The resulting one-loop Callan-Symanzik equation for the nonlocal operators (1.6) reads

$$
\left\{ \mu \frac{\partial}{\partial \mu} + \beta_N(g) \frac{\partial}{\partial g} + L \gamma_N(g) \right\} \mathcal{O}(Z_1, Z_2, \ldots, Z_L) = -\frac{g^2 N_c}{8\pi^2} [H \cdot \mathcal{O}](Z_1, Z_2, \ldots, Z_L),
$$

(1.7)

where $\beta_N(g)$ is the beta-function in the SYM theory and $\gamma_N(g) = \beta_N(g)/g$ is the anomalous dimension of the superfields in the light-like axial gauge $A_+(x) = 0$. The superconformal invariance of the SYM theory imposes restrictions on the possible form of the one-loop dilatation operator and allows one to fix $H$ up to a scalar function. We will determine this function performing an explicit calculation of Feynman supergraphs in an $\mathcal{N}$-extended SYM theory.

To one-loop order the dilatation operator $H$ has a two-particle structure. In addition, in the multi-color limit the interaction can happen only between two neighboring superfields and, therefore, $H$ is given by the sum over the nearest neighbors

$$
H = H_{12} + \ldots + H_{L,L-1} + H_{L,1}.
$$

(1.8)

Here the two-particle kernel $H_{k,k+1}$ acts locally on the superfields with the coordinates $Z_k$ and $Z_{k+1}$ and leaves the remaining superfields intact. The explicit form of $H_{k,k+1}$ depends on the superfields involved. For $N = 4$ the light-cone operators (1.6) are built from the superfields $\Phi$ only, Eq. (1.3), and, therefore, $H_{k,k+1}$ coincides with the dilatation operator in the $\Phi\Phi$-sector, $H_{k,k+1} = H_{\Phi\Phi}$. For $N \leq 2$ one has to distinguish four different operators $H_{\Phi\Phi}$, $H_{\Phi\Psi}$, $H_{\Psi\Phi}$ and $H_{\Psi\Psi}$. They define the two-particle evolution kernel $H_{k,k+1}$ in the $\Phi\Phi$-, $\Phi\Psi$-, $\Psi\Phi$- and $\Psi\Psi$-sectors, respectively. Later we shall often use a unifying notation for the superfields, $\Phi_{j\Phi} = \Phi$ and $\Phi_{j\Psi} = \Psi$, and combine these operators into a $2 \times 2$ matrix $H_{ab}$ (with $a, b = \Phi, \Psi$).

The outcome of our consideration can be summarized in a few equations for the two-particle dilatation operators $H_{ab}$. These operators admit the following representation

$$
H_{ab} = [\mathcal{V}^{(j_a,j_b)} - \mathcal{V}^{(j_a,j_b)}_{\text{ex}}] (1 - \Pi_{ab}) \quad (a, b = \Phi, \Psi),
$$

(1.9)

where $j_a$ is the conformal spin of the corresponding superfield ($j_\Phi = -1/2$ and $j_\Psi = (3 - N)/2$) and the operators $\mathcal{V}^{(j_a,j_b)}$, $\mathcal{V}^{(j_a,j_b)}_{\text{ex}}$ and $\Pi_{ab}$ are defined as follows. The kernel $\mathcal{V}^{(j_a,j_b)}$ describes the “diagonal” transition $\Phi_{j_a} \Phi_{j_b} \to \Phi_{j_a} \Phi_{j_b}$. It is given by the following integral operator

$$
\mathcal{V}^{(j_a,j_b)}(Z) = \int_0^1 \frac{d\alpha}{\alpha} \left\{ 2\Phi_{j_a}(Z_1)\Phi_{j_b}(Z_2) - (1 - \alpha)^{2j_a - 1}\Phi_{j_a}((1 - \alpha)Z_1 + \alpha Z_2)\Phi_{j_b}(Z_2) - (1 - \alpha)^{2j_b - 1}\Phi_{j_a}(Z_1)\Phi_{j_b}((1 - \alpha)Z_2 + \alpha Z_1) \right\},
$$

(1.10)

which displaces the superfields in the direction of each other,

$$
\Phi_{j_a}((1 - \alpha)Z_1 + \alpha Z_2) \equiv \Phi_{j_a}((1 - \alpha)z_1 + \alpha z_2, (1 - \alpha)\theta_1^A + \alpha \theta_2^A).
$$

The term with $\mathcal{V}^{(j_a,j_b)}_{\text{ex}}$ arises in (1.9) only for $j_a \neq j_b$, that is, $\mathcal{V}^{(j_a,j_b)}_{\text{ex}} = \mathcal{V}^{(j_b,j_a)}_{\text{ex}} = 0$. The kernel $\mathcal{V}^{(j_b,j_\Psi)}_{\text{ex}}$ describes the exchange transition $\Phi \Psi \to \Psi \Phi$

$$
\mathcal{V}^{(j_\Phi,j_\Psi)}_{\text{ex}}(Z_1)\Psi(Z_2) = \int_0^1 d\alpha \frac{\alpha^{3-N}}{(1 - \alpha)^2} \Psi((1 - \alpha)Z_1 + \alpha Z_2)\Phi(Z_2).
$$

(1.11)
The evolution kernel $V^{(j\Phi,j\Phi)}_{ex}$ describes the transition $\Psi\Phi \rightarrow \Phi\Psi$ and is given by the same expression with the superfields $\Phi$ and $\Psi$ interchanged in both sides of (1.11).

The integral in (1.11) is divergent for $\alpha \rightarrow 1$. The same problem arises in (1.10) if at least one of the superfields carries a negative conformal spin, $j_\Phi = -1/2$. In Eq. (1.9), divergences are removed by the operator $(1 - \Pi_{ab})$, which is a projector. For $N \leq 2$, in the $\Psi\Psi$-sector, the projector is not required, $\Pi_{\Psi\Psi} = 0$, since the superfield $\Psi$ has a positive conformal spin $j_\Psi = (3 - N)/2 > 0$. The expressions for the projectors $\Pi_{\Phi\Phi}, \Pi_{\Psi\Psi}$ and $\Pi_{\Phi\Psi}$ will be given below (see Eqs. (3.44), (3.48) and (3.50)).

There exist nontrivial relations between the two-particle dilatations operators, Eqs. (1.9) – (1.11), for different $N$. Namely, the one-loop dilatation operator in SYM theories with $N \leq 2$ supersymmetries can be obtained from the dilatation operator in the maximally supersymmetric, $N = 4$ theory through a “method of truncation” [21]. It amounts to reducing the number of “odd” dimensions in the light-cone superspace from $N = 4$ down to $N = 0$. In this way one finds that two seemingly different expressions for the evolution kernels (1.10) and (1.11) for $N \leq 2$ follow from the kernel $V^{(-1/2,-1/2)}$ in the $N = 4$ theory. Similar relation between the evolution kernels also at work in the opposite direction. Namely, the expressions for the kernel $V^{(j_{\Phi,\Phi})}$ and $V^{(j_{\Psi,\Phi})}$ in the $N = 0$ theory can be generalized to arbitrary $N$ by simply extending the one-dimensional light-cone direction to the $(N + 1)$-dimensional superspace, $z \rightarrow Z = (z, \theta^4)$.

The two-particle evolution kernels, Eqs. (1.9) – (1.11), allow us to construct a one-loop dilatation operator (1.8). Its eigenvalues determine the spectrum of anomalous dimensions of all quasipartonic operators in SYM theories with $N = 0, 1, 2, 4$ supercharges. Notice that the two-particle kernel in the $\Phi\Phi$-sector, $H_{\Phi\Phi}$, does not depend on the number of supercharges and, therefore, the one-loop dilatation operator (1.8) acting on the light-cone operators (1.3) has a universal form in the SYM theories with $0 \leq N \leq 4$. For the light-cone operators (1.4) and (1.5) the dilatation operator depends on $N$ through the dependence of two-particle kernels $H_{\Psi\Psi}, H_{\Psi\Phi}$ and $H_{\Phi\Phi}$, Eq. (1.9), on the conformal spin of the superfield $\Psi$, $j_\Psi = (3 - N)/2$.

It turns out, the one-loop dilatation operator defined in Eqs. (1.8) – (1.11) has a hidden integrability symmetry: the two-particle kernel $V^{(j_{\Phi,\Phi})}$ can be identified as a Hamiltonian of the Heisenberg $SL(2|\mathcal{N})$ spin chain consisting of two sites [25, 26, 27]. As a consequence, for the light-cone operators (1.3) and (1.4) the one-loop dilatation operator coincides in the multi-color limit with a Hamiltonian of a completely integrable $SL(2|\mathcal{N})$ spin chain of the length equal to the number of superfields $L$. For $N \leq 2$, the dilatation operator acting on the “mixed” light-cone operators (1.5) receives an additional contribution from the exchange interaction $V_{ex}^{(j_{\Psi,\Phi})}$. This interaction breaks integrability symmetry of the dilatation operator and leads to appearance of a mass gap in the spectrum of the anomalous dimensions of the operators (1.5) [11, 12].

Some of the results were reported in an earlier Letter [19]. In this paper we provide a detailed account on our approach and present new results. The paper is organized as follows. In Sect. 2, we review the Brink-Lindgren-Nilsson and Mandelstam approaches to light-cone SYM theories. In Sect. 3 we discuss the superconformal symmetry of the SYM theories on the light-cone and conjecture the form of the one-loop dilatation operator on the basis of symmetry consideration alone. To verify the conjecture, we perform in Sect. 4 the one-loop calculation of renormalization group kernels in the $\mathcal{N}$-extended SYM theories and establish the relation between the one-loop dilatation operators for different $N$. In Sect. 5, we apply the obtained expressions for the dilatation operator to evaluate the one-loop anomalous dimensions of Wilson operators and demonstrate their agreement with the known results. In Sect. 6, we reveal a hidden symmetry of the one-loop dilatation operator in the SYM theory on the light-cone and discuss its relation
to Heisenberg (super)spin chains. Our conclusions are summarized in Sect. 6. Four Appendices contain a detailed derivation of the results formulated in the body of the paper.

2. Super-Yang-Mills theories on the light-cone

To calculate a one-loop dilatation operator in (super) Yang-Mills theories we shall apply the “light-cone formalism” \[28, 20, 21\]. In this formalism one integrates out non-propagating components of fields and formulates the (super) Yang-Mills action in terms of “physical” degrees of freedom. Although the resulting action is not manifestly covariant under the Poincaré transformations, the main advantage of the light-cone formalism for SYM theories is that the \(\mathcal{N}\)-extended supersymmetric algebra is closed off-shell for the propagating fields and there is no need to introduce auxiliary fields. This allows one to design a unifying light-cone superspace formulation of various \(\mathcal{N}\)-extended SYM, including the \(\mathcal{N} = 4\) theory for which a covariant superspace formulation does not exist.

Following \[28, 20, 21\], we split four components of the gauge field \(A_\mu(x) = A_\mu^a(x) t^a\), with \(t^a\) being generators of the fundamental representation of the \(SU(N_c)\), into two longitudinal, \(A_\pm(x)\), and two transverse holomorphic and antiholomorphic components, \(A(x)\) and \(\bar{A}(x)\), respectively,

\[
A_\pm \equiv \frac{1}{\sqrt{2}}(A_0 \pm A_3), \quad A \equiv \frac{1}{\sqrt{2}}(A_1 + iA_2), \quad \bar{A} \equiv A^* = \frac{1}{\sqrt{2}}(A_1 - iA_2). \tag{2.1}
\]

In the light-cone formalism, one quantizes the SYM theory in a noncovariant, light-cone gauge \(A_\pm(x) = 0\). Making a similar decomposition of (Majorana) fermion fields \(\psi(x) = \psi^a(x) t^a\) into the so-called “bad” and “good” components with a help of projectors \(\Pi_\pm = \frac{1}{2} \gamma_\pm \gamma_\mp\) (\(\Pi_\pm^2 = \Pi_\pm\) and \(\Pi_+ \Pi_- = 0\))

\[
\psi = \Pi_+ \psi + \Pi_- \psi \equiv \psi_+ + \psi_- , \tag{2.2}
\]

one finds that the fields \(\psi_-(x)\) and \(A_-(x)\) can be integrated out in this gauge. The resulting action of the SYM theory is expressed in terms of “physical” fields: complex gauge field, \(A(x)\), “good” components of fermion fields \(\psi_+(x)\) and, in general, complex scalar fields \(\phi(x)\). Finally, one combines these fields into superfields and rewrites the SYM action on the light-cone as an integral over the superspace.

At present there exist two different superspace formulations of the SYM theory on the light-cone. In the Brink-Lidgren-Nilsson formulation \[20\], the superspace has \(2\mathcal{N}\) odd directions, \(\theta^A\) and \(\bar{\theta}_A\) with \(A = 1, \ldots, \mathcal{N}\), and the light-cone action is build from chiral and antichiral superfields. In the Mandelstam formulation \[21\], the superspace has only \(\mathcal{N}\) odd directions, \(\theta^A\) with \(A = 1, \ldots, \mathcal{N}\), and the light-cone action involves two distinct chiral superfields. In this Section, we shall review both formulations and demonstrate their equivalence.

2.1. Brink-Lindgren-Nilsson formalism

Let us start from the \(\mathcal{N} = 4\) SYM and reduce step-by-step the number of supersymmetries descending down to \(\mathcal{N} = 0\) SYM (pure gluodynamics).

2.1.1. \(\mathcal{N} = 4\) theory

In the \(\mathcal{N} = 4\) model, the propagating modes consist of the complex field \(A(x)\) describing transverse components of the gauge field, complex Grassmann fields \(\lambda^A(x)\) defining “good” components of four Majorana fermions (see Eqs. \[A.5\] and \[A.10\]) and a matrix of complex scalar fields...
\( \phi^{AB}(x) \) (with \( A,B = 1, \ldots, 4 \)) satisfying \( \phi^{AB} = -\phi^{BA} \). Fields conjugated to them are \( \bar{A}(x), \bar{\lambda}_A(x) \) and \( \bar{\phi}_{AB} = (\phi^{AB})^* = \frac{1}{2} \varepsilon_{ABCD} \phi^{CD} \), respectively.

In the light-cone formalism, all propagating modes can be combined into a single complex scalar \( \mathcal{N} = 4 \) superfield

\[
\Phi(x, \theta^A, \bar{\theta}_A) = e^{i \frac{1}{2} \theta \cdot \partial_\perp} \left\{ \partial_+^{-1} A(x) + \theta^A \partial_+^{-1} \bar{\lambda}_A(x) + \frac{i}{2!} \theta^A \theta^B \bar{\phi}_{AB}(x) \right. \\
+ \left. \frac{1}{3!} \varepsilon_{ABCD} \theta^A \theta^B \theta^C \lambda_D(x) - \frac{1}{4!} \varepsilon_{ABCD} \theta^A \theta^B \theta^C \theta^D \partial_+ \bar{A}(x) \right\}. \tag{2.3}
\]

Here \( \bar{\theta} \cdot \theta \equiv \bar{\theta}_A \theta^A \) and the nonlocal operator \( \partial_+^{-1} \) is defined using the Mandelstam-Leibbrandt prescription \( [21] \) (see Eq. \( (A.3) \)). It is tacitly assumed that \( \Phi = \Phi^a t^a \) with \( t^a \) being the generators of the fundamental representation of the \( SU(N_c) \) group.

The light-cone action of the \( \mathcal{N} = 4 \) SYM reads \( [20] \)

\[
S_{\mathcal{N}=4} = \int d^4x \, d^4 \theta \, d^4 \bar{\theta} \left\{ -\frac{1}{2} \bar{\Phi}^a \Box \Phi^a - \frac{2}{3} g f^{abc} \left( \frac{1}{\partial_+} \bar{\Phi}^a \Phi^b \partial \Phi^c + \frac{1}{\partial_+} \Phi^a \bar{\Phi}^b \partial \bar{\Phi}^c \right) \right. \\
- \frac{1}{2} g^2 f^{abc} f^{fde} \left( \frac{1}{\partial_+} (\Phi^b \partial_+ \Phi^c) \frac{1}{\partial_+} (\Phi^d \partial_+ \Phi^e) + \frac{1}{2} \Phi^b \Phi^d \Phi^e \right) \right\}, \tag{2.4}
\]

where \( \bar{\Phi} = (\Phi(x, \theta^A, \bar{\theta}_A))^* \) is a conjugated superfield\(^1\), \( f^{abc} \) are the structure constants of the \( SU(N_c) \) group, the light-cone derivatives \( \partial_+ \), \( \partial_\perp \) and \( \partial \) are defined in \( [A.2] \) and the integration measure over Grassmann variables is normalized as in \( (A.18) \). The Green’s functions computed from \( (2.4) \) do not contain ultraviolet divergences to all orders of perturbation theory and, therefore, the \( \mathcal{N} = 4 \) light-cone action \( (2.4) \) defines an ultraviolet (UV) finite quantum field theory \( [29, 20, 30, 21] \).

The \( \mathcal{N} = 4 \) light-cone superfield \( (2.3) \) has the following unique properties. It comprises all propagating fields of the model, and expansion in \( \theta^A \) can be viewed as an expansion in different helicity components: +1 for \( A(x) \), 1/2 for \( \bar{\lambda}_A(x) \), 0 for \( \bar{\phi}_{AB} \), –1/2 for \( \lambda^A(x) \) and –1 for \( \bar{A}(x) \). As a consequence, the conjugated superfield is not independent and is related to \( \Phi(x, \theta^A, \bar{\theta}_A) \) as

\[
\bar{\Phi}(x, \theta^A, \bar{\theta}_A) = -\frac{1}{4!} \partial_+^{-2} \varepsilon^{ABCD} D_A D_B D_C D_D \Phi(x, \theta^A, \bar{\theta}_A). \tag{2.5}
\]

Here the notation was introduced for the covariant derivatives in the superspace

\[
D_A = \partial_{\theta^A} - \frac{1}{2} \bar{\theta}_A \partial_+ , \quad \bar{D}_A = \partial_{\bar{\theta}^A} - \frac{1}{2} \theta^A \partial_+ , \tag{2.6}
\]

satisfying \( \{ D_A, D_B \} = \{ \bar{D}_A, \bar{D}_B \} = 0 \) and \( \{ D_A, \bar{D}_B \} = -\delta^B_A \partial_+ \). The superfields \( (2.3) \) and \( (2.5) \) obey the chirality conditions

\[
\bar{D}_B \Phi(x, \theta^A, \bar{\theta}_A) = D_B \bar{\Phi}(x, \theta^A, \bar{\theta}_A) = 0. \tag{2.7}
\]

As usual, they imply that the dependence of the chiral superfield \( \Phi(x, \theta^A, \bar{\theta}_A) \) on \( \bar{\theta}_A \) and antichiral superfield \( \bar{\Phi}(x, \theta^A, \bar{\theta}_A) \) on \( \theta_A \) can be absorbed into a redefinition of the space-time coordinate \( x_\mu \).

Notice that the lowest two components of the superfield \( (2.3) \) are nonlocal fields. As a consequence, the expansion of the light-cone operators \( (1.3) \) around the origin in the superspace yields both Wilson operators and “spurious” operators involving the fields \( \partial_+^{-1} A(0), \lambda^A(0) \) and \( \partial_+^{-1} \bar{\lambda}_A(0) \). The latter operators do not have a clear physical meaning and their appearance is an artefact of the light-cone superspace formalism. We shall return to this issue in Sect. 3.4.

\(^1\)Complex conjugation for Grassmann variables is specified in \( (A.16) \) and \( (A.17) \).
2.1.2. $\mathcal{N} = 2$ theory

The light-cone formulation of Yang-Mills theories with less supersymmetry can be obtained from $\mathcal{N} = 4$ SYM through a “method of truncation” \cite{23}. It is based on the following identity:

$$
\int d^4 x \, d^N \theta \, d^N \bar{\theta} \, \mathcal{L}(\Phi, \bar{\Phi}) = (-1)^N \int d^4 x \, d^{N-1} \theta \, d^{N-1} \bar{\theta} \, \left[ \bar{D}^N D^N \mathcal{L}(\Phi, \bar{\Phi}) \right] \bigg|_{\theta^N = \bar{\theta}^N = 0},
$$

(2.8)

with the covariant derivatives $D^N$ and $\bar{D}^N$ defined in \eqref{2.6}. Subsequently applying \eqref{2.6}, one can rewrite the action of the $\mathcal{N} = 4$ model in terms of the $\mathcal{N} = 2$ light-cone Yang-Mills chiral superfield $\Phi^{(2)}(x, \theta^A, \bar{\theta}_A)$ coupled to the $\mathcal{N} = 2$ Wess-Zumino chiral superfield $\Psi^{(2)}_{WZ}(x, \theta^A, \bar{\theta}_A)$

$$
\Phi^{(2)} = \Phi^{(4)}(x, \theta^A, \bar{\theta}_A) \bigg|_{\theta^3 = \bar{\theta}_3 = 0}, \quad \Psi^{(2)}_{WZ} = D^3 \Phi^{(4)}(x, \theta^A, \bar{\theta}_A) \bigg|_{\theta^3 = \bar{\theta}_3 = 0}.
$$

(2.9)

Here the superscript refers to the underlying $\mathcal{N}$-extended SYM and $\Phi^{(4)}$ is given by \eqref{2.3}. The conjugated (antichiral) superfields are

$$
\Phi^{(2)} = \Phi^{(4)}(x, \theta^A, \bar{\theta}_A) \bigg|_{\theta^3 = \bar{\theta}_3 = 0}, \quad \Psi^{(2)}_{WZ} = D^3 \Phi^{(4)}(x, \theta^A, \bar{\theta}_A) \bigg|_{\theta^3 = \bar{\theta}_3 = 0}.
$$

(2.10)

Expansion of the $\mathcal{N} = 4$ chiral superfield \eqref{2.3} over the $\mathcal{N} = 2$ chiral ($\Phi^{(2)}$, $\Psi^{(2)}_{WZ}$) and antichiral ($\bar{\Phi}^{(2)}$, $\bar{\Psi}_{WZ}^{(2)}$) superfields looks as follows

$$
\Phi^{(4)}(x, \theta^A, \bar{\theta}_A) = e^{\frac{1}{2}(\bar{\theta}_3 \theta^3 + \bar{\theta}_4 \theta^4)} \partial_i \left\{ \Phi^{(2)} + \theta^3 \Psi^{(2)}_{WZ} - \theta^4 (\partial_+^3 \bar{D}^1 \bar{D}^2 \bar{\Phi}^{(2)} - \theta^3 \theta^4 \bar{D}^1 \bar{D}^2 \bar{\Phi}^{(2)}) \right\}.
$$

(2.11)

Substitution of this relation into \eqref{2.8} yields the action \eqref{2.4} rewritten as the $\mathcal{N} = 2$ SYM theory coupled to the $\mathcal{N} = 2$ Wess-Zumino multiplet. To obtain the light-cone formulation of the $\mathcal{N} = 2$ SYM theory it suffices to put $\Psi^{(2)}_{WZ} = \bar{\Psi}^{(2)}_{WZ} = 0$. In this way, one finds the $\mathcal{N} = 2$ action as \eqref{2.4}

$$
S_{\mathcal{N}=2} = \int d^4 x \, d^2 \theta \, d^2 \bar{\theta} \left\{ - \Phi^a \square \Phi^a + 2g f^{abc} (\partial_+ \Phi^a \bar{\Phi}^b \partial \Phi^c + \partial_+ \phi^a \bar{\Phi}^b \partial \Phi^c) - 2g^2 f^{abc} f^{ade} \frac{1}{\partial_+} (\partial_+ \Phi^b \bar{D}^1 \bar{D}^2 \bar{\Phi}^e) \frac{1}{\partial_+} (\partial_+ \Phi^d \bar{D}_1 \bar{D}_2 \Phi^c) \right\},
$$

(2.12)

where $\Phi \equiv \Phi^{(2)}(x, \theta^A, \bar{\theta}_A)$ is a complex chiral $\mathcal{N} = 2$ superfield and $\Phi$ is a conjugated antichiral superfield. Substituting \eqref{2.3} into \eqref{2.9} one gets

$$
\Phi(x, \theta^A, \bar{\theta}_A) = e^{\frac{i}{2} \bar{\theta} \theta \partial_i} \left\{ \partial_+^{1} A(x) + \theta^A \partial_+^{1} \lambda_A(x) + \frac{i}{2} \varepsilon_{AB} \theta^A \theta^B \phi(x) \right\},
$$

(2.13)

with $\phi \equiv \phi_{12}(x)$ and $A, B = 1, 2$. The antichiral superfield $\bar{\Phi}(x, \theta^A, \bar{\theta}_A)$ involves the fields $\bar{A}(x)$, $\lambda^A$ and $\phi$, and in distinction with the $\mathcal{N} = 4$ model, it is independent on the chiral superfield $\Phi^{(2)}(x, \theta^A, \bar{\theta}_A)$.

The propagating fields in the $\mathcal{N} = 2$ theory \eqref{2.12} are the complex gauge field $A(x)$, one complex scalar field $\phi(x)$ and two complex Grassmann fields $\lambda^A(x)$ ($A = 1, 2$) describing “good” components of two Majorana fermions. By construction, the $\mathcal{N} = 2$ SYM action differs from the $\mathcal{N} = 4$ SYM action by the contribution of the Wess-Zumino superfield $\Psi^{(2)}_{WZ}(x, \theta^A, \bar{\theta}_A)$. Had we retained this superfield, the two theories would be equivalent. For $\Psi^{(2)}_{WZ} = 0$, the properties of the theory are changed drastically: the $\mathcal{N} = 2$ SYM acquires a nonvanishing $\beta$–function and its conformal symmetry is broken on the quantum level.
2.1.3. \( \mathcal{N} = 1 \) theory

As a next step, one applies (2.8) to truncate the \( \mathcal{N} = 2 \) down to \( \mathcal{N} = 1 \) SYM. Similar to the previous case, one defines two chiral superfields

\[
\Phi^{(1)} = \Phi^{(2)}(x, \theta^A, \bar{\theta}_A)|_{\theta^2 = \bar{\theta}^2 = 0}, \quad \Psi_{\text{WZ}}^{(1)} = D_2 \Phi^{(2)}(x, \theta^A, \bar{\theta}_A)|_{\theta^2 = \bar{\theta}^2 = 0}
\]

and puts \( \Psi_{\text{WZ}}^{(1)} = 0 \) to retain the contribution of the \( \mathcal{N} = 1 \) SYM superfield only. This leads to

\[
S_{\mathcal{N}=1} = \int d^4 x \, d\theta \, d\bar{\theta} \left\{ \Phi^a \Box \partial_+ \Phi^a + 2g f^{abc} (\partial_+ \Phi^a \partial_+ \bar{\Phi}^b \partial_+ \Phi^c - \partial_+ \bar{\Phi}^a \partial_+ \Phi^b \partial_+ \Phi^c) \\
+ 2g^2 f^{abc} f^{ade} \frac{1}{\partial_+} (\partial_+ \Phi^b \partial^1 \partial_+ \Phi^c) \frac{1}{\partial_+} (\partial_+ \Phi^d \partial^1 \partial_+ \Phi^e) \right\},
\]

where the \( \mathcal{N} = 1 \) light-cone chiral superfield \( \Phi \equiv \Phi^{(1)}(x, \theta, \bar{\theta}) \) is given by

\[
\Phi(x, \theta, \bar{\theta}) = e^\frac{1}{2} \bar{\theta} \partial^{-1} A(x) + \theta \partial^{-1} \lambda(x).
\]

Here \( \bar{\lambda} = \bar{\lambda}_1(x) \), and \( \bar{\Phi} = (\Phi(x, \theta, \bar{\theta}))^* \) is a conjugated, antichiral \( \mathcal{N} = 1 \) superfield. In the \( \mathcal{N} = 1 \) light-cone action (2.15), the propagating fields are the complex gauge field \( A(x) \) and one complex Grassmann field \( \lambda(x) \) describing the “good” component of Majorana fermion.

2.1.4. \( \mathcal{N} = 0 \) theory

Finally, we use (2.8) to truncate the \( \mathcal{N} = 1 \) theory down to \( \mathcal{N} = 0 \) Yang-Mills theory. The resulting light-cone action takes the form

\[
S_{\mathcal{N}=0} = \int d^4 x \left\{ \Phi^a \Box \partial_+^2 \Phi^a - 2g f^{abc} (\partial_+ \Phi^a \partial_+^2 \bar{\Phi}^b \partial_+ \Phi^c + \partial_+ \bar{\Phi}^a \partial_+^2 \Phi^b \partial_+ \Phi^c) \\
- 2g^2 f^{abc} f^{ade} \frac{1}{\partial_+} (\partial_+ \Phi^b \partial^1 \Phi^c) \frac{1}{\partial_+} (\partial_+ \Phi^d \partial^1 \Phi^e) \right\},
\]

where the \( \mathcal{N} = 0 \) field is given by

\[
\Phi(x) = \Phi^{(1)}(x, \theta, \bar{\theta})|_{\theta = \bar{\theta} = 0} = \partial^{-1}_+ A(x),
\]

and \( \bar{\Phi}(x) = \partial^{-1}_+ \bar{A}(x) \). Eq. (2.17) coincides with the well-known expression for the light-cone action of \( SU(N_c) \) gluodynamics [31].

2.2. Mandelstam formalism

In the Brink-Lindgren-Nilsson formalism, the light-cone action \( S_{\mathcal{N}} = \int d^4 x \, d^N \theta \, d^N \bar{\theta} \mathcal{L}_{BLN}(\Phi, \bar{\Phi}) \) involves both chiral and antichiral superfields. The same action can be rewritten in terms of chiral superfields only, without any reference to the conjugated \( \bar{\theta}_A \)-variables. To this end, one trades the antichiral superfield \( \bar{\Phi} \) for yet another chiral superfield

\[
\bar{\Phi}(x, \theta^A, \bar{\theta}_A) = (-1)^{N-1} \partial^{-2}_+ D_1 \ldots D_N \Psi(x, \theta^A, \bar{\theta}_A).
\]
The inverse relation looks as
\[
\Psi(x, \theta^A, \bar{\theta}_A) = -\partial_{+}^{-N} D^{N} \ldots D^{1} \Phi(x, \theta^A, \bar{\theta}_A),
\] (2.20)
so that \( \bar{D} B \Psi(x, \theta^A, \bar{\theta}_A) = 0 \). Comparing (2.19) with (2.24) one finds that for \( N = 4 \), \( \Psi(x, \theta^A, \bar{\theta}_A) = \Phi(x, \theta^A, \bar{\theta}_A) \). For \( N \leq 2 \) the chiral superfields \( \Phi(x, \theta^A, \bar{\theta}_A) \) and \( \Psi(x, \theta^A, \bar{\theta}_A) \) are independent on each other. Their explicit expressions are given below (see Eqs. (2.27) – (2.30)).

Making use of (2.19) one can rewrite the light-cone SYM actions defined in the previous section in terms of chiral superfields \( \Phi \) and \( \Psi \). In general, a chiral field satisfies the relation
\[
\Phi(x, \theta, \bar{\theta}) = 0.
\]
In a similar manner, one redefines the covariant derivatives acting on new superfields
\[
D_{M,A} = e^{-\frac{1}{2} \bar{\theta} \cdot \theta} D_{Ae} e^{\frac{1}{2} \bar{\theta} \cdot \theta} = \partial_{\theta A} - \bar{\theta}_A \partial_+,
\]
so that \( \bar{D}_B \Phi_M(x, \theta^A) = \bar{D}_B \Psi_M(x, \theta^A) = 0 \). Then, one performs integration over the \( \bar{\theta}_A \)–variables inside \( S_N \) and obtains the light-cone SYM action in the Mandelstam formulation
\[
S_N = \int d^4 x \ d^N \theta \ L_M(\Phi_M, \Psi_M), \quad L_M = \int d^N \bar{\theta} \ L_{BLN} = \partial_{\bar{\theta}_N} \ldots \partial_{\bar{\theta}_1} L_{BLN}.
\] (2.23)

It depends on the chiral superfields \( \Phi_M \) and \( \Psi_M \) and involves only “half” of odd variables. From now on, we will suppress the subscript “m” on the superfields and use only the Mandelstam fields throughout our subsequent presentation. This will not lead to a confusion anyway, since the Lagrangian \( L_M \) is evaluated for \( \bar{\theta}_A = 0 \), so that Eq. (2.21) is at work.

Combining together (2.19), (2.21) and (2.23) we find from (2.4), (2.12), (2.15) and (2.17) that the resulting expression for the light-cone action in the Mandelstam formalism can be written in the following form
\[
S_{N=0,1,2} = -\sigma_N \int d^4 x \ d^N \theta \left( \Psi^a \square \Phi^a + 2g f^{abc} \partial_{+} \Phi^a \bar{\Phi}_d \Psi^c + 2g f^{abc} \partial_{+}^{-N} \Phi^a \partial_{+}^{-2N} \psi^b \partial_{+}^{-1} \Psi^e \right)
\]
\[
-2(-1)^N g^2 f^{abc} f^{ade} \partial_{+}^{-2} \left( \partial_{+} \Phi^a \Psi^b \Psi^c \right) \left[ \partial_{+}^{-1} \psi^d, \partial_{+}^{-2N} \Phi^e \right],
\] (2.24)

and
\[
S_{N=4} = - \int d^4 x \ d^4 \theta \left( \frac{1}{2} \Phi^a \square \Phi^a + \frac{2}{3} g f^{abc} \partial_{+} \Phi^a \bar{\Phi}_d \Psi^c + \frac{2}{3} g f^{abc} \partial_{+}^{-1} \Phi^a \left[ \partial_{+}^{-2} \Phi^d, \partial_{+}^{-2} \Phi^e \right] \right) (2.25)
\]
\[
- \frac{1}{2} g^2 f^{abc} f^{ade} \left( \partial_{+}^{-2} \left( \Phi^a \Psi^b \Psi^c \right) \left[ \partial_{+}^{-2} \Phi^d, \partial_{+}^{-1} \Phi^e \right] - \frac{1}{2} \Phi^a \Phi^d \left[ \partial_{+}^{-2} \Phi^e, \partial_{+}^{-2} \Phi^f \right] \right).
\]
Here \( \sigma_N = (-1)^{N(N+1)/2} \) is the signature factor and the notation was introduced for a “square bracket”. For two arbitrary superfields \( \Phi_1(x, \theta^A) \) and \( \Phi_2(x, \theta^A) \), it is defined as (for \( N \geq 1 \))

\[
[\Phi_1, \Phi_2] = \prod_{A=1}^{N} \left( \partial_{\theta^A}^{(1)} \partial_{\bar{\theta}}^{(2)} - \partial_{\theta^A}^{(2)} \partial_{\bar{\theta}}^{(1)} \right) \Phi_1 \Phi_2 ,
\]

(2.26)

where the ordering of fermion derivatives is from the left to right, i.e., \( \prod_{A=1}^{N} \partial_{\theta^A} = \partial_{\theta^1} \ldots \partial_{\theta^N} \), and the superscript indicates the field to which the derivative is applied. For \( N = 0 \) one has \([\Phi_1, \Phi_2] = \Phi_1 \Phi_2 \).

Substituting (2.3), (2.13), (2.16) and (2.18) into (2.20) and (2.21), one finds the explicit expressions for the chiral superfields \( \Phi(x, \theta^A) \) and \( \Psi(x, \theta^A) \) in the Mandelstam formulation

\[
\Phi(x) = \partial_+^{-1} A(x) , \quad \Psi(x) = -\partial_+ \bar{A}(x)
\]

(2.27)

for \( N = 0 \),

\[
\Phi(x, \theta) = \partial_+^{-1} A(x) + \theta \partial_+^{-1} \bar{\lambda}(x) \\
\Psi(x, \theta) = -\lambda(x) + \theta \partial_+ \bar{A}(x)
\]

(2.28)

for \( N = 1 \),

\[
\Phi(x, \theta^A) = \partial_+^{-1} A(x) + \theta^A \partial_+^{-1} \bar{\lambda}_A(x) + \frac{i}{2!} \varepsilon_{AB} \theta^A \theta^B \phi(x) , \\
\Psi(x, \theta^A) = i \phi(x) - \varepsilon_{AB} \theta^A \lambda^B(x) + \frac{1}{2} \varepsilon_{AB} \theta^A \theta^B \partial_+ \bar{A}(x)
\]

(2.29)

for \( N = 2 \), and

\[
\Phi(x, \theta^A) = \Psi(x, \theta^A) = \partial_+^{-1} A(x) + \theta^A \partial_+^{-1} \bar{\lambda}_A(x) + \frac{i}{2!} \theta^A \theta^B \phi_{AB}(x) \\
+ \frac{1}{3!} \varepsilon_{ABCD} \theta^A \theta^B \theta^C \lambda^D(x) - \frac{1}{4!} \varepsilon_{ABCD} \theta^A \theta^B \theta^C \theta^D \partial_+ \bar{A}(x)
\]

(2.30)

for \( N = 4 \). The following comments are in order.

As we demonstrated in this section, the Brink-Lidgren-Nilsson and Mandelstam formulations of the SYM theory on the light-cone are equivalent.\(^2\) In what follows we shall rely on Eqs. (2.24) and (2.25) since they are more suitable for our purposes.

In the Mandelstam formalism, for \( N \leq 2 \) chiral superfields \( \Phi(x, \theta^A) \) and \( \Psi(x, \theta^A) \) describe a half of the propagating fields each. Notice that the superfield \( \Psi(x, \theta^A) \) is bosonic for \( N = 0, 2, 4 \) and fermionic for \( N = 1 \). The important difference between the superfields \( \Phi(x, \theta^A) \) and \( \Psi(x, \theta^A) \) is that the former involves nonlocal fields, \( \partial_+^{-1} A \) and \( \partial_+^{-1} \bar{\lambda} \), whereas the latter contains only local primary fields: scalars, \( \phi \), fermions, \( \lambda^A \), and gauge strength tensor projected onto the light-cone, \( n^\mu F_{\mu \perp} = (\partial_+ A, \partial_+ \bar{A}) \) in the axial gauge \( (n \cdot A) \equiv A_+(x) = 0 \). For \( N \leq 2 \), one could have avoided nonlocal operators from the very beginning if the SYM theory were reformulated in terms of two superfields, chiral \( \Psi(x, \theta^A) \) and antichiral \( \bar{\Psi}(x, \bar{\theta}^A) = (\Psi(x, \theta^A))^\dagger \), by making use of the relation

\[
\Phi(x, \theta^A) = -(i)^N \partial_+^{-2} \partial_{\theta_1} \ldots \partial_{\theta_N} \Psi(x - \theta \cdot \bar{\theta}, \bar{\theta}^A)
\]

(2.31)

\(^2\)Although the expressions for the light-cone action, Eqs. (2.24) and (2.25), differ from those proposed by Mandelstam in Ref. [21], we demonstrate their equivalence in Appendix A4.
which follows from (2.19), (2.21) and (2.22). The reason why we prefer to deal with the superfields $\Phi(x, \theta^A)$ and $\Psi(x, \theta^A)$ is that substitution of (2.31) into (2.24) will break invariance of the light-cone action under translations in the superspace and, as a consequence, the resulting expression for the dilatation operator acting on the light-cone operators involving antichiral superfield $\bar{\Psi}(x, \bar{\theta}^A)$ is more complicated.

3. Superconformal invariance on the light-cone

The $\mathcal{N}$–extended SYM theory is invariant on the classical level under superconformal $SU(2, 2|\mathcal{N})$ transformations. They include

- Conformal $SO(4, 2)$ symmetry generated by translations $P_\mu$, Lorentz transformations $M_{\mu\nu}$, dilatations $D$ and special conformal transformations $K_\mu$;
- Poincaré supersymmetry generated by the supercharges $Q_{\alpha A}$ and their conjugates $\bar{Q}^{\dot{\alpha} A}$;
- Conformal supersymmetry generated by the supercharges $S^A_{\alpha}$ and their conjugates $\bar{S}^{\dot{\alpha} A}$;
- $R$–symmetry generated by the bosonic chiral charge $R$, and, in case of extended $\mathcal{N} \geq 2$ supersymmetry, isotopic $SU(\mathcal{N})$ symmetry generated by charges $T^A_{\dot{A}}$ satisfying the $SU(\mathcal{N})$ commutation relations.

The odd charges $Q_{\alpha A}$, $\bar{Q}^{\dot{\alpha} A}$, $S^A_{\alpha}$ and $\bar{S}^{\dot{\alpha} A}$ are two-dimensional Weyl spinors ($\alpha = 1, 2$ and $\dot{\alpha} = 1, \ldots, \mathcal{N}$). On the quantum level, the superconformal symmetry is broken in $\mathcal{N} = 0$, $\mathcal{N} = 1$ and $\mathcal{N} = 2$ SYM. In the $\mathcal{N} = 4$ SYM theory, it survives to all loops but is reduced due to a $U_R(1)$–anomaly down to the $PSU(2, 2|4)$ group. The symmetry breaking effects manifest themselves starting from two-loops and, therefore, the one-loop dilatation operator in the $\mathcal{N}$–extended SYM enjoys the full $SU(2, 2|\mathcal{N})$ symmetry.

The superfield $\Phi(x, \theta^A)$ (and $\Psi(x, \theta^A)$) realizes a representation of the superconformal algebra. Its infinitesimal variations under the $SU(2, 2|\mathcal{N})$ transformations look as

$$\delta_G \Phi(x, \theta^A) = i[\Phi(x, \theta^A), G] = -G \Phi(x, \theta^A), \quad (3.1)$$

where $G = \varepsilon^\mu P_\mu, \varepsilon^{\mu\nu} M_{\mu\nu}, \ldots$ and $G = \xi^{\alpha A} Q_{\alpha A}, \chi^{\dot{\alpha} A} S^A_{\dot{\alpha}}, \ldots$ for odd generators with $\xi^{\alpha A}, \chi^{\dot{\alpha} A}$ being constant Grassman-valued Weyl spinors. In (3.1), the quantum-field operator $G$ is represented by an operator $G$ acting on the superfield. In the light-cone formalism, the $SU(2, 2|\mathcal{N})$ charges can be split into “kinematical” and “dynamical” charges. For the former, the operator $G$ is given by linear differential operators acting on even and odd coordinates of the superfield, while for the latter it is realized nonlinearly and, in general, does not preserve the number of superfields $^{3}$

3.1. Collinear supergroup

In this paper, we shall calculate the one-loop dilatation operator in the $\mathcal{N}$–extended SYM theory, acting on single-trace operators built from chiral superfields $\Phi(zn^\mu, \theta^A)$ and $\Psi(zn^\mu, \theta^A)$ both located on the light-cone along the $n$–direction ($n^2_\mu = 0$)

$$\mathcal{O}(Z_1, Z_2, \ldots, Z_L) = \text{tr}\{\Phi(Z_1)\Psi(Z_2)\ldots\Phi(Z_L)\}.$$  

$^{3}$Since nonlinear terms are accompanied by powers of the coupling constant, they do not intervene to the lowest order.
Hereafter, we shall use a short-hand notation for the arguments of superfields on the light-cone, \( \Phi(z_n, \theta^A) \equiv \Phi(Z_k) \) where \( Z_k = (z_k, \theta^A_k) \) specifies the position of the \( k \)th superfield in the superspace. We recall that in the \( \mathcal{N} = 4 \) SYM theory we have only one operator \( (3.3) \), while for \( \mathcal{N} \leq 2 \) one has to distinguish three different sets, Eqs. \((3.3) - (3.5)\). The operators \((3.2)\) are generating functions of local composite operators in the underlying \( \mathcal{N} \) extended SYM theory. The latter operators can be obtained from \((3.2)\) by substituting the superfields \( \Phi_j = \{ \Phi, \Psi \} \) by their expansion around the origin in the superspace

\[
\Phi_j(Z) = \Phi_j(0) + Z \cdot \partial Z \Phi_j(0) + \frac{1}{2} (Z \cdot \partial Z)^2 \Phi_j(0) + \ldots , \tag{3.3}
\]

where \( Z \cdot \partial Z = z \partial_z + \theta A \partial_{\theta A} \).

The superfields in \((3.4)\) are located on the light-cone along the ‘+’-direction defined by the light-like vector \( n_{\mu} \). To work out the restrictions on \( \mathbb{H} \) due to superconformal invariance, we have to restrict ourselves to the superconformal transformations \((3.1)\) that map the light-cone operators \((3.1)\) into itself. It is well-known that in nonsupersymmetric Yang-Mills theories such transformations correspond to the so-called collinear \( SL(2,\mathbb{R}) \) subgroup of the conformal \( SO(4,2) \) group (see the review \[15\]). They are generated by the charges \( P_+, M_{-+}, D \) and \( K_+ \) which form the \( SL(2) \) algebra.

Supersymmetry enlarges the \( SL(2) \) subgroup. Examining the \( SU(2,2|\mathcal{N}) \) commutation relations one finds that the resulting collinear superalgebra involves the additional charges: the \( U(1) \) chiral charge \( R \), the \( SU(\mathcal{N}) \) charges \( T_{AB} \), helicity operator \( M_{12} \) and the “odd” charges \( \bar{Q}_A, \bar{S}^A \) and \( \bar{S}_{-A} \).\(^4\) In the light-cone formalism, such one-component spinors are described by a complex Grassmann field without any Lorentz index. Introducing linear combinations of the charges

\[
\begin{align*}
L^- &= -iP_+ , \\
L^+ &= \frac{i}{2}K_+ , \\
L^0 &= \frac{i}{2}(D + M_{-+}) , \\
E &= i(D - M_{-+}) , \\
V_A^- &= \frac{i\bar{\rho}}{2}Q_A , \\
W^A_- &= -\frac{\bar{\rho}}{2} \bar{Q}_A , \\
W^{A,+} &= -\frac{i}{2\bar{\rho}} \bar{S}^A , \\
V_A^+ &= \frac{i}{2\bar{\rho}} \bar{S}_{-A} , \\
B &= \frac{1}{4}(1 - \frac{\bar{\rho}}{\mathcal{N}})R + \frac{1}{2}M_{12} ,
\end{align*}
\tag{3.4}
\]

one finds that together with \( T_{AB} \) they satisfy the \( SL(2|\mathcal{N}) \) (graded) commutation relations. In Eq. \((3.4)\) the normalization factor \( \bar{\rho} = 2^{1/4} \) was introduced to bring these relations to their canonical form \[32\]. To save space we do not display them here.

Using the technique of induced representations \[33, 34\], one can obtain representation of the generators of the collinear superalgebra \((3.4)\) for a general chiral superfield \( \Phi(z_{\mu}, \theta^A) \). The relevant center elements of the superalgebra are

\[
\begin{align*}
[M_{-+}, \Phi(0,0)] &= -is \Phi(0,0) , \\
[D, \Phi(0,0)] &= -i\ell \Phi(0,0) , \\
[M_{12}, \Phi(0,0)] &= h \Phi(0,0) , \\
[R, \Phi(0,0)] &= r \Phi(0,0) ,
\end{align*}
\tag{3.5}
\]

where \( \ell, s, h \) and \( n \) are correspondingly the canonical dimension of the superfield, projection of its spin on the ‘+’- direction, its helicity and its \( R \)-charge. This leads to

\[
\begin{align*}
[L^0, \Phi(0,0)] &= j \Phi(0,0) , \\
[E, \Phi(0,0)] &= t \Phi(0,0) , \\
[B, \Phi(0,0)] &= b \Phi(0,0)
\end{align*}
\tag{3.6}
\]

\(^4\)Here the \(+/-\) subscript indicates “good”/“bad” components of the corresponding Weyl spinors, \( Q_{\alpha A}, \bar{Q}^{\alpha A}, S^A \) and \( \bar{S}_A \) (see Appendix A2 for the definition).
where \( j = \frac{1}{2} (s + \ell) \) is the conformal spin, \( t = \ell - s \) is the twist and \( b = \frac{1}{2} (1 - \frac{1}{N}) r + \frac{1}{2} h \) is the \( B \)-charge of the superfield \([15, 34]\). For the chiral fields the charges \( b \) and \( j \) are related as \([34]\)

\[
b = -j. \tag{3.7}
\]

The parameters \( j \) and \( t \) define the so-called “atypical” representation of the collinear \( SL(2|\mathcal{N}) \) supergroup that we shall denote as \( \mathcal{V}_j \). In this representation, the charges \([34]\) are realized as differential operators acting on the light-cone coordinates of the chiral superfield \( \Phi(zn_\mu, \theta^A) \)

\[
L^- = -\partial_z, \quad L^+ = 2z \partial_z + z (\theta \cdot \partial_\theta), \quad L^0 = j + z \partial_z + \frac{1}{2} (\theta \cdot \partial_\theta), \quad E = t,
\]

\[
W^{A,-} = \theta^A \partial_z, \quad W^{A,+} = \theta^A [2j + z \partial_z + (\theta \cdot \partial_\theta)], \quad V_A^- = \partial_{\theta^A}, \quad V_A^+ = z \partial_{\theta^A},
\]

\[
T_B^A = \theta^A \partial_{\theta^B} - \frac{1}{N} \delta^A_B (\theta \cdot \partial_\theta), \quad B = -j - \frac{1}{2} (1 - \frac{2}{N}) (\theta \cdot \partial_\theta), \tag{3.8}
\]

where \( \partial_z \equiv \partial/\partial z \) and \( \theta \cdot \partial_\theta \equiv \theta^A \partial/\partial \theta^A \).

Let us identify the values of the conformal spin, \( j \), and twist, \( t \), for the chiral superfields \( \Phi(x, \theta^A) \) and \( \Psi(x, \theta^A) \) in Mandelstam formulation, Eqs. (2.27) – (2.30). According to (3.6), they are determined by the properties of the lowest component of the superfields, \( \Phi(0,0) = \partial_+ A(0) \) and \( \Psi(0,0) = -\partial_+ A(0), -\lambda(0), i\phi(0) \) for \( \mathcal{N} = 0, 1, 2 \), respectively. Therefore, for the scalar chiral superfield \( \Phi(x, \theta^A) \) one has \( \ell = r = 0 \), \( s = -1 \) and \( h = 1 \) leading to

\[
j_\Phi = -\frac{1}{2}, \quad t_\Phi = 1. \tag{3.9}
\]

Similarly, for the chiral superfield \( \Psi(x, \theta^A) \) one gets \( \ell = 2 - \mathcal{N}/2 \), \( s = -h = 1 - \mathcal{N}/2 \) and \( r = \mathcal{N} \) leading to

\[
j_\Psi = \frac{3 - \mathcal{N}}{2}, \quad t_\Psi = 1. \tag{3.10}
\]

We see that the chiral superfields \( \Phi(x, \theta^A) \) and \( \Psi(x, \theta^A) \) have the same twist \( t = 1 \) but different conformal spins. Notice that \( j_\Psi \geq 1/2 \) for \( \mathcal{N} \leq 2 \) while \( j_\Phi \) is negative for all \( \mathcal{N} \). As we will show below, this difference has important consequences for the properties of the dilatation operator.

For the nonlocal light-cone operators \([32]\), the generators of the superconformal \( SL(2|\mathcal{N}) \) transformations act on the tensor product of the atypical representations \( \mathcal{V}_{j_\Psi} \) and \( \mathcal{V}_{j_\Phi} \) corresponding to constituent superfields

\[
\mathcal{V}_L = \mathcal{V}_{j_\Phi} \otimes \mathcal{V}_{j_\Psi} \otimes \cdots \otimes \mathcal{V}_{j_\Psi}. \tag{3.11}
\]

They are given by the sum of differential operators \([3,8]\) acting on the coordinates of the superfields, \( Z_k = (z_k, \theta^A_k) \) with \( j = j_\Psi \) or \( j = j_\Phi \) depending on the superfield. Since the twist generator \( E \) in \([3,8]\) is a c-number, the twist of the nonlocal operator \([3,2]\) is equal to the sum of twists of the superfields leading to \( t_G = L \). Obviously, the local composite operators generated by \( \mathcal{O}(Z_1, \ldots, Z_L) \) have the same twist. Such operators are known as quasipartonic operators. In a general classification of local operators, they carry a maximal Lorentz spin and have a minimal possible twist.

The superconformal invariance implies that the evolution equation \([1,7]\) has to be invariant under the \( SL(2|\mathcal{N}) \) transformations of the superfields. For a general light-cone superfield \( \Phi_j(Z) \equiv \)
\( \Phi_j(z, \theta^A) \) with the conformal spin \( j \), these transformations are generated by the operators \( (3.8) \): The operators \( L^-, L^+ \) and \( L^0 \) generate projective, \( SL(2) \) transformations on the light-cone

\[
e^{\epsilon L^-} \Phi_j(Z) = \Phi_j(z - \epsilon, \theta^A),
\]

\[
e^{\epsilon L^0} \Phi_j(Z) = \epsilon^j \Phi_j(e^{i \epsilon z}, e^{i \epsilon/2} \theta^A),
\]

\[
e^{\epsilon L^+} \Phi_j(Z) = (1 - \epsilon z)^{-2j} \Phi_j \left( \frac{z}{1 - \epsilon z}, \frac{\theta^A}{1 - \epsilon z} \right).
\]

(3.12)

The operators \( W^{A,-} \) and \( V_A^- \) generate translations in the superspace and correspond to supersymmetric transformations of the components of the superfield

\[
e^{\xi V^+} \Phi_j(Z) = \Phi_j(z, \theta^A + \xi^A),
\]

\[
e^{\xi W^+} \Phi_j(Z) = \Phi_j(z + \xi \cdot \theta, \theta^A).
\]

(3.13)

The operators \( W^{A,+} \) and \( V_A^+ \) generate conformal transformations in the superspace

\[
e^{\xi V^+} \Phi_j(Z) = \Phi_j(z, \theta^A + z \xi^A),
\]

\[
e^{\xi W^+} \Phi_j(Z) = (1 - \xi \cdot \theta)^{-2j} \Phi_j \left( \frac{z}{1 - \xi \cdot \theta}, \frac{\theta^A}{1 - \xi \cdot \theta} \right).
\]

(3.14)

Then, the evolution equation \( (1.7) \) is invariant under supersymmetric transformations of the superfields \( \Phi(Z) \) and \( \Psi(Z) \) provided that the Hamiltonian \( \mathbb{H} \) commutes with the \( SL(2|N) \) generators

\[
\mathbb{H} \cdot G \circ (Z_1, \ldots, Z_L) = G \cdot \mathbb{H} \circ (Z_1, \ldots, Z_L)
\]

(3.15)

where \( G = \{ L^0, L^+, V_A^+, W^{A, \pm}, T_B^A, B \} \) are the \( SL(2|N) \) generators acting on the tensor product \( (3.11) \), that is \( G = \sum_{k=1}^L G_k \) with \( G_k \) given by the differential operators \( (3.8) \) acting on the coordinates of the \( k \)th superfield. Substituting \( (3.8) \) into \( (3.15) \) one finds that the two-particle kernel \( \mathbb{H}_{k,k+1} \) has to be an \( SL(2|N) \) invariant operator

\[
[\mathbb{H}_{k,k+1}, G_k + G_{k+1}] = 0.
\]

(3.16)

In the next section, we present a general expression for the operator \( \mathbb{H}_{k,k+1} \) satisfying \( (3.16) \).

3.2. The \( SL(2|N) \) invariant operators

By definition, the two-particle kernel \( \mathbb{H}_{12} \) governs the scale dependence of the product of two chiral superfields \( \Phi_{j_1}(Z_1) \Phi_{j_2}(Z_2) \) carrying the conformal spins \( j_1 \) and \( j_2 \). As before, \( \Phi_j(Z) \) stands for the superfields in an \( \mathcal{N} \)-extended SYM theory, \( \Phi(Z) \) and \( \Psi(Z) \), with the conformal spins \( j_\Phi = -1/2 \) and \( j_\Psi = (3 - \mathcal{N})/2 \), respectively.

As we will demonstrate in Sect. 4, to one-loop order the operator \( \mathbb{H}_{12} \) does not change the number of superfields and can be realized as a quantum mechanical Hamiltonian acting on the coordinates of the superfields, \( Z_1 \) and \( Z_2 \). In addition, if the superfields are not identical, \( \mathbb{H}_{12} \) may exchange the superfields inside the single trace, Eq. \( (3.2) \). Therefore, defining the two-particle kernel \( \mathbb{H}_{12} \) one has to distinguish two different channels \( \Phi_{j_1} \Phi_{j_2} \rightarrow \Phi_{j_1} \Phi_{j_2} \) and \( \Phi_{j_1} \Phi_{j_2} \rightarrow \Phi_{j_2} \Phi_{j_1} \). Let us denote the corresponding evolution kernels as \( \Psi^{(j_1j_2)} \) and \( \Psi_{ex}^{(j_1j_2)} \), respectively. By definition, they act on the tensor product of two \( SL(2|N) \) chiral (or atypical) representations as

\[
\Psi^{(j_1j_2)} : \mathcal{V}_{j_1} \otimes \mathcal{V}_{j_2} \rightarrow \mathcal{V}_{j_1} \otimes \mathcal{V}_{j_2}, \quad \Psi_{ex}^{(j_1j_2)} : \mathcal{V}_{j_1} \otimes \mathcal{V}_{j_2} \rightarrow \mathcal{V}_{j_2} \otimes \mathcal{V}_{j_1}.
\]

(3.17)
We shall argue below that the $SL(2|\mathcal{N})$ invariance fixes these operators up to a scalar function. To this end, we will make use of the $SL(2)$ subgroup of the full superconformal group generated by the operators $L^0$, $L^+$ and $L^-$, Eqs. (3.14) to construct the $SL(2)$ invariant operators $\Psi^{(j_1,j_2)}$ and $\Psi^{(j_1,j_2)}_{\text{ex}}$ and, then, generalize them to ensure invariance under the $SL(2|\mathcal{N})$ transformations.

Additional complication arises due to the fact that the $SL(2|\mathcal{N})$ representation $\Psi_j$ is reducible for $j = -1/2$, that is, for the superfield $\Phi(z)$, Eq. (3.9). We shall assume for the moment that the representations $\Psi_{j_1}$ and $\Psi_{j_2}$ are irreducible in (3.17) and extend analysis to the spin $j = -1/2$ representations in Sect. 3.3.

### 3.2.1. The $SL(2)$ invariant operators

Let us consider a nonlocal light-cone operator built from two chiral superfields $\Phi_j(z, \theta^A = 0)$ “living” along the $z$–axis in the superspace

$$\mathcal{O}_{j_1,j_2}(z_1, z_2) = \Phi_{j_1}(z_1, 0)\Phi_{j_2}(z_2, 0).$$

According to (3.12), the superfield $\Phi_j(z, 0)$ is transformed under the $SL(2; \mathbb{R})$ transformations as

$$z \rightarrow \frac{az + b}{cz + d}, \quad \Phi_j(z, 0) \rightarrow (cz + d)^{-2j}\Phi_j\left(\frac{az + b}{cz + d}, 0\right)$$

with $ad − bc = 1$. The generators of these transformations are

$$l_j^- = -\partial_z, \quad l_j^+ = 2j z + z^2\partial_z, \quad l_j^0 = j + z\partial_z.$$ (3.20)

They are obtained from the generators $L^0$, $L^+$ and $L^-$, Eq. (3.8), by neglecting terms involving $\theta$–variables. The $SL(2|\mathcal{N})$ invariant kernels $\Psi^{(j_1,j_2)}$ and $\Psi^{(j_1,j_2)}_{\text{ex}}$ acting on the product of superfields (3.18) should be invariant under the $SL(2)$ transformations (3.19).

According to (3.19), $\Phi_j(z, 0)$ realizes the spin–$j$ representation of the $SL(2, \mathbb{R})$ group, $\Psi_j^{\text{SL}(2)}$. Indeed, as follows from Eqs. (2.27) and (2.30), the superfield $\Phi_j(z, 0)$ is given by its lowest component which in turn is a primary of the $SL(2, \mathbb{R})$ group with the conformal spin $j$.

The light-cone operators (3.18) belong to the tensor product of two $SL(2, \mathbb{R})$ representations $\Psi_j^{\text{SL}(2)} \otimes \Psi_j^{\text{SL}(2)}$ labelled by the spins $j_1$ and $j_2$. The operators $\Psi^{(j_1,j_2)}$ and $\Psi^{(j_1,j_2)}_{\text{ex}}$, Eq. (3.17), act on this product as

$$\Psi^{(j_1,j_2)} : \Psi_j^{\text{SL}(2)} \otimes \Psi_j^{\text{SL}(2)} \rightarrow \Psi_j^{\text{SL}(2)} \otimes \Psi_j^{\text{SL}(2)},$$

$$\Psi^{(j_1,j_2)}_{\text{ex}} : \Psi_j^{\text{SL}(2)} \otimes \Psi_j^{\text{SL}(2)} \rightarrow \Psi_j^{\text{SL}(2)} \otimes \Psi_j^{\text{SL}(2)}.$$ (3.21)

Such operators have been studied thoroughly in the context of QCD conformal operators. As was shown in Ref. [35], the $SL(2)$ invariant operators $\Psi^{(j_1,j_2)}$ and $\Psi^{(j_1,j_2)}_{\text{ex}}$ defined in this way have the following general form

$$\Psi^{(j_1,j_2)} : \mathcal{O}_{j_1,j_2}(z_1, z_2) = e^{i\pi(j_1+j_2)} \int [\mathcal{D}w_1]_{j_1} \int [\mathcal{D}w_2]_{j_2} \mathcal{O}_{j_1,j_2}(w_1, w_2) \times (z_1 - \bar{w}_1)^{-2j_1} (z_2 - \bar{w}_2)^{-2j_2} f(\xi),$$

$$\Psi^{(j_1,j_2)}_{\text{ex}} : \mathcal{O}_{j_1,j_2}(z_1, z_2) = e^{i\pi(j_1+j_2)} \int [\mathcal{D}w_1]_{j_1} \int [\mathcal{D}w_2]_{j_2} \mathcal{O}_{j_1,j_2}(w_1, w_2) \times (z_1 - \bar{w}_2)^{-2j_1} (z_2 - \bar{w}_1)^{-2j_2} f_{\text{ex}}(\xi).$$ (3.22)
Here a notation was introduced for the $SL(2)$ invariant measure
\[ [Dw]_j = \frac{2j - 1}{\pi} d^2 w (\text{Im } w)^{2j-2} \theta(\text{Im } w), \] (3.24)
with the integration region extended over the upper half-plane in the complex $w-$plane, $\bar{w}_k = w_k^*$. Also, $f(\xi)$ and $f_{\text{ex}}(\xi)$ are arbitrary functions of the harmonic ratio
\[ \xi = \frac{(z_1 - \bar{w}_2)(z_2 - \bar{w}_1)}{(z_1 - \bar{w}_1)(z_2 - \bar{w}_2)}. \] (3.25)

It is straightforward to verify that the operators $V^{(j_1 j_2)}$ and $V_{\text{ex}}^{(j_1 j_2)}$ are invariant under the $SL(2)$ transformations (3.19).

We would like to stress that the explicit form of the functions $f(\xi)$ and $f_{\text{ex}}(\xi)$ is not fixed by the $SL(2)$ invariance. These functions determine the dilatation operator in the $\mathcal{N}-$extended SYM theory and one might expect that they should depend on $\mathcal{N}$. Nevertheless, as we will show in Sect. 4 by an explicit calculation of the one-loop dilatation operator, the functions $f(\xi)$ and $f_{\text{ex}}(\xi)$ have the same, universal form in all SYM theories
\[ f(\xi) = \ln \xi + \psi(2j_1) + \psi(2j_2) - 2\psi(1), \] (3.26)
\[ f_{\text{ex}}(\xi) = \xi^{2j_1} \theta(j_2 - j_1) + \xi^{2j_2} \theta(j_1 - j_2). \]

Substituting this ansatz into (3.27) and (3.28), one performs the integration with a help of the identity (3.18) and obtains the following expression for the $SL(2)$ invariant operators acting on the product of two superfields (3.18)
\[ V^{(j_1 j_2)}_{\text{ex}}(z_1, z_2) = \theta(j_2 - j_1) \int_0^1 d\alpha \bar{\alpha}^{2j_1 - 1} \alpha^{2j_2 - j_1} \theta j_1 (\bar{\alpha} z_1 + \alpha z_2, z_2) \] (3.27)
\[ \theta(j_1 - j_2) \int_0^1 d\alpha \bar{\alpha}^{2j_2 - 1} \alpha^{2j_1 - j_2} \theta j_2 (\bar{\alpha} z_1 + \alpha z_2, z_2) \] (3.28)
where $\bar{\alpha} \equiv 1 - \alpha$. These operators have a simple interpretation: they displace the superfields along the light-cone in the direction of each other with the weight functions depending on their conformal spins.

The operators (3.27) and (3.28) commute with the $SL(2; \mathbb{R})$ generators (3.20) acting on the tensor product $V^{(j_1 j_2)}_{\text{ex}} \otimes V^{(j_1 j_2)}_{\text{ex}}$ and, therefore, they are functions of the two-particle conformal spin $\mathbb{J}_{12}$ defined through the $SL(2)$ Casimir operator
\[ l^2 = (l^0)^2 - l^0 + l^+ l^- = \mathbb{J}_{12}(\mathbb{J}_{12} - 1), \] (3.29)
with $l^0 = l^0_{j_1} \otimes l^0_{j_2}$ for $\alpha = 0, \pm$ and the $SL(2)$ generators $l^\alpha_j$ given by (3.20). To establish the explicit form of the dependence of $V^{(j_1 j_2)}$ and $V_{\text{ex}}^{(j_1 j_2)}$ on the conformal spin $\mathbb{J}_{12}$, it suffices to compare their action on the space of test functions which belong to the tensor product $V^{(j_1 j_2)}_{\text{ex}} \otimes$
\( \mathcal{V}_{j_2}^{SL(2)} \). This space is spanned by homogeneous polynomials of two variables \( z_1 \) and \( z_2 \) and it possesses the highest weights \( O^{(n)}_{j_1j_2}(z_1, z_2) = (z_1 - z_2)^n \). These states satisfy the relations

\[
l - O^{(n)}_{j_1j_2}(z_1, z_2) = 0, \quad \mathbb{J}_{12} O^{(n)}_{j_1j_2}(z_1, z_2) = (n + j_1 + j_2) O^{(n)}_{j_1j_2}(z_1, z_2) \tag{3.30}
\]

and, most importantly, they diagonalize the kernels (3.27) and (3.28). One replaces \( O_{j_1j_2}(z_1, z_2) \) in (3.27) and (3.28) by the highest weights \( \mathbb{S}_L \) and, most importantly, they diagonalize the kernels (3.27) and (3.28) by the highest weights

\[
O_{j_1j_2}(z_1, z_2) \to O^{(n)}_{j_1j_2}(z_1, z_2) = (z_1 - z_2)^n, \tag{3.31}
\]

calculates the corresponding eigenvalues of the operators \( \mathbb{V}^{(j_1j_2)} \) and \( \mathbb{V}_{ex}^{(j_1j_2)} \) and casts them into an operator form with a help of (3.30) to get

\[
\mathbb{V}^{(j_1j_2)} = \psi (\mathbb{J}_{12} + j_1 - j_2) + \psi (\mathbb{J}_{12} - j_1 + j_2) - 2\psi(1), \tag{3.32}
\]
\[
\mathbb{V}_{ex}^{(j_1j_2)} = \mathbb{P}_{12} \Gamma(\mathbb{J}_{12} - |j_1 - j_2|) / \Gamma(\mathbb{J}_{12} + |j_1 - j_2|), \tag{3.33}
\]

where \( \mathbb{P}_{12} \) is a permutation operator, \( \mathbb{P}_{12} O_{j_1j_2}(z_1, z_2) = O_{j_2j_1}(z_2, z_1) \), and \( \psi(x) = d \ln \Gamma(x)/dx \) is the Euler \( \psi \)-function. Since \( |j_1 - j_2| \) takes (half)integer values, the operator \( \mathbb{V}_{ex}^{(j_1j_2)} \) is a rational function of the conformal spin \( \mathbb{J}_{12} \). The operator \( \mathbb{V}^{(j_1j_2)} \) is well-known in the theory of lattice integrable models. It can be identified as a two-particle Hamiltonian of a completely integrable Heisenberg \( SL(2; \mathbb{R}) \) spin chain. As was mentioned in the Introduction, it is this property that is responsible for remarkable integrability symmetry of the one-loop dilatation operator in the SYM theory on the light-cone [19].

3.2.2. From the light-cone to the superspace

As a next step, we have to restore the dependence of the superfields in (3.18) on the odd coordinates \( \theta_1^A \) and \( \theta_2^A \) and “lift” the relations (3.27) and (3.28) from the light-cone to the superspace, \( z \to Z = (z, \theta) \). One possibility could be to generalize the relations (3.27) and (3.28) and write down expressions for an \( SL(2|\mathcal{N}) \) invariant operators as integrals over the representation space of the \( SL(2|\mathcal{N}) \) group. We shall choose however another route which is much simpler and leads immediately to the same final expressions.

Let us apply the superconformal transformations generated by the \( SL(2|\mathcal{N}) \) charges \( \mathbb{V}_A^- \) and \( \mathbb{V}_A^+ \) to Eq. (3.18). Taking into account (3.13) and (3.14), we obtain that these generators displace the chiral superfields along odd directions in the superspace and do not alter their positions on the light-cone

\[
e^{\xi^V - \epsilon^V + \epsilon^V} O_{j_1j_2}(z_1, z_2) = \Phi_{j_1}(z_1, \xi^A + z_1\epsilon^A) \Phi_{j_2}(z_2, \xi^A + z_2\epsilon^A) \equiv O_{j_1j_2}(Z_1, Z_2). \tag{3.33}
\]

Denoting \( \theta_1^A = \xi^A + z_1\epsilon^A \) and \( \theta_2^A = \xi^A + z_2\epsilon^A \), one finds that for \( z_1 \neq z_2 \) and \( z_1, z_2 \neq 0 \) the superfields in (3.33) are located in two different points of the superspace, \( Z_1 = (z_1, \theta_1^A) \) and \( Z_2 = (z_2, \theta_2^A) \). Since the \( SL(2|\mathcal{N}) \) invariant operators \( \mathbb{V}_{12} \) and \( \mathbb{V}^{(ex)}_{12} \) have to commute with the generators \( \mathbb{V}_A^\pm \), one gets

\[
\mathbb{V}^{(j_1j_2)} O_{j_1j_2}(Z_1, Z_2) = e^{\xi^V - \epsilon^V} \mathbb{V}^{(j_1j_2)} O_{j_1j_2}(z_1, z_2) \tag{3.34}
\]
and similar for $\mathbf{v}_{ex}^{(j_1,j_2)}$. This relation allows one to reconstruct the operators $\mathbf{V}^{(j_1,j_2)}$ and $\mathbf{V}_{ex}^{(j_1,j_2)}$ acting on the superfields $O_{j_1,j_2}(Z_1,Z_2) = \Phi_j(Z_1)\Phi_{j'}(Z_2)$ from their expressions on the light-cone, Eqs. (3.27) and (3.28).

The transformations (3.33) and (3.34) amount to replacing the arguments of the superfields

$$\Phi_j(\alpha z_1 + \bar{\alpha} z_2, 0) \rightarrow \Phi_j(\alpha Z_1 + \bar{\alpha} Z_2),$$

with $Z_k = (z_k, \theta_k^A)$. As a consequence, the $SL(2)$ invariant operators (3.27) and (3.28) are transformed into

$$\mathbf{V}^{(j_1,j_2)} O_{j_1,j_2}(Z_1,Z_2) = \int_0^1 \frac{d\alpha}{\alpha} \left\{ 2\mathcal{O}_{j_1,j_2}(Z_1,Z_2) - \bar{\alpha}^{2j_1-1}\mathcal{O}_{j_1,j_2}(\alpha Z_1 + \alpha Z_2, Z_2) - \bar{\alpha}^{2j_2-1}\mathcal{O}_{j_2,j_1}(\bar{\alpha} Z_1 + \bar{\alpha} Z_2 + \alpha Z_1) \right\}.$$}

$$\mathbf{V}_{ex}^{(j_1,j_2)} O_{j_1,j_2}(Z_1,Z_2) = \theta(j_2 - j_1) \int_0^1 d\alpha \bar{\alpha}^{2j_1-1}\alpha^{2j_2-2j_1-1}\mathcal{O}_{j_2,j_1} (\bar{\alpha} Z_1 + \alpha Z_2, Z_2)$$

$$+ \theta(j_1 - j_2) \int_0^1 \bar{\alpha}^{2j_2-1}\alpha^{2j_1-2j_2-1}\mathcal{O}_{j_1,j_2}(Z_1, \bar{\alpha} Z_2 + \alpha Z_1).$$}

The only difference with the previous case, Eqs. (3.27) and (3.28), is that $Z = (z, \theta^A)$ has nonvanishing odd coordinates and displacement of the superfields takes place along the line in the superspace connecting two points $Z_1$ and $Z_2$. Obviously, for $\theta_1^A = \theta_2^A = 0$ one recovers the $SL(2)$ expressions. One can verify that (3.36) and (3.37) are invariant under the $SL(2|\mathcal{N})$ transformations, Eqs. (3.12) – (3.14).

Eqs. (3.36) and (3.37) define the $SL(2|\mathcal{N})$ invariant operators $\mathbf{V}^{(j_1,j_2)}$ and $\mathbf{V}_{ex}^{(j_1,j_2)}$ acting on the product of two chiral superfields, $\Phi_j(Z_1)\Phi_{j'}(Z_2)$. In the next section, we shall apply (3.36) and (3.37) to construct an ansatz for the one-loop dilatation operator acting on the space spanned by the light-cone operators (1.3) – (1.5) built from the chiral superfields $\Phi(Z)$ and $\Psi(Z)$.

### 3.3. Ansatz for the dilatation operator

In the $\mathcal{N}$-extended SYM theory, the conformal spins of the chiral superfields, $\Phi$ and $\Psi$, take the values $j_\Phi = -1/2$ and $j_\Psi = (3 - \mathcal{N})/2$, respectively. Substituting $j_1 = j_\Phi = -1/2$ into (1.10) one encounters a problem: due to the presence of the factor $\bar{\alpha}^{2j_1-1}$ the integral over $\alpha$ is divergent for $\alpha \rightarrow 1$ and, therefore, the corresponding operator $\mathbf{V}^{(j_1,j_2)}$ is not well-defined. The problem arises every time the operators $\mathbf{V}^{(j_1,j_2)}$ and $\mathbf{V}_{ex}^{(j_1,j_2)}$ are applied to the product of superfields with at least one of them carrying the conformal spin $j_1 = -1/2$, that is, in the $\Phi\Phi-$, $\Phi\Psi-$ and $\Psi\Phi-$sectors. As we will see later in this section, this divergence is ultimately related to the fact that the $SL(2|\mathcal{N})$ representation defined by the superfield $\Phi_j(Z)$ is reducible for $j = -1/2$, that is, the corresponding representation space $\mathcal{V}_j$ contains an invariant subspace. The above mentioned divergences originate from the states belonging to this subspace.

For $j_1 = -1/2$ the divergences in (1.10) originate from the first two terms of the expansion of nonlocal operator $O_{-1/2,j_2}(\bar{\alpha} Z_1 + \alpha Z_2, Z_2)$ around $\alpha = 1$

$$O_{-1/2,j_2}(\bar{\alpha} Z_1 + \alpha Z_2, Z_2) = \Phi_{-1/2}(Z_2)\Phi_{j_2}(Z_2) + \bar{\alpha} Z_{12} \cdot \partial Z_{\bar{\alpha}} \Phi_{-1/2}(Z_2)\Phi_{j_2}(Z_2) + O(\alpha^2).$$

Here the expansion coefficients are local operators defined at the point $Z_2$. These operators belong to the same $SL(2|\mathcal{N})$ module as the operators $\Phi_{-1/2}(0)\Phi_{j_2}(0)$ and $\partial Z \Phi_{-1/2}(0)\Phi_{j_2}(0)$, since they
are obtained from the latter through translations in the superspace \((3.13)\). According to \((2.27) - (2.30)\)
\[
\Phi_{-1/2}(0) = \partial_+^{-1} A(0), \quad \partial_z \Phi_{-1/2}(0) = A(0), \quad \partial_{\mu A} \Phi_{-1/2}(0) = \partial_+^{-1} \tilde{\lambda}_A(0),
\]
with \(\partial_+ A(0) = n\mu F_{\mu\Sigma}(0)\) in the light-like gauge \(A_+ = 0\). Notice that all fields in \((3.39)\) are \textit{nonlocal}, spurious operators. Their definition involves the inverse derivative \(\partial_+^{-1}\), which is not a well-defined integral operator. In the momentum representation, it induces a spurious pole at \(k_+ = 0\) and the properties of the fields \((3.39)\) depend on the prescription adopted to regularize the pole. Throughout this paper we shall define the operator \(\partial_+^{-1}\) using the Mandelstam-Leibbrandt prescription (see Eq. (A.3)).

One of the advantages of this prescription is that \(1/k_+\)–factors do not induce additional singularities inside Feynman integrals and calculating superficial divergence index of diagrams one can treat them on equal footing with the conventional Feynman propagators. This property plays a crucial role in establishing a UV finiteness of the \(N = 4\) SYM theory \([29, 20, 21, 30]\). In the present case, it also has important consequences for renormalization properties of composite operators involving spurious fields \((3.39)\). As we will show in Sect. 4, the additional \(1/k_+\)–factors improve convergence properties of Feynman integrals in a SYM theory, and, as a consequence, the one-loop corrections to certain operators involving the nonlocal fields \((3.39)\) are ultraviolet finite. The UV finite spurious operators include \(\Phi(0)\Phi(0), \Phi(0)\partial_z\Phi(0), \partial_z\Phi(0)\Phi(0), \partial_z\Phi(0)\Phi(0), \partial_z\Phi(0)\Phi(0), \ldots\) with \(\partial_z\Phi = (\partial_\Phi, \partial_{\mu A}\Phi)\). Notice that this set does not comprise all operators involving the fields \((3.39)\). For instance, the operators like \(\Phi(0)\partial^n\Phi(0)\) (with \(n \geq 3\)) mix under renormalization with “physical” operators \(\partial^n\Phi(0)\partial^{-n-m}\Phi(0)\) and acquire a nontrivial anomalous dimension.

Let us consider separately UV finite spurious operators in the \(\Phi\Phi\) sector, they are given by a bilinear product of the fields \((3.39)\), like \(\Phi(0)\Phi(0), \Phi(0)\partial_z\Phi(0), \partial_{\mu A}\Phi(0)\partial_z\Phi(0), \partial_{\mu A}\Phi(0)\partial_z\Phi(0), \ldots\) with \(\partial_z\Phi = (\partial_\Phi, \partial_{\mu A}\Phi)\) and their \(SL(2|N)\) descendants. For our purposes it is convenient to introduce a “spurious” superfield

\[
\Phi_{sp}(Z) = \Phi(0) + Z \cdot \partial Z \Phi(0) .
\]

and treat the product \(\Phi_{sp}(Z_1)\Phi_{sp}(Z_2)\) as a generating function for such operators. As was just mentioned, \(\Phi_{sp}(Z_1)\Phi_{sp}(Z_2)\) does not acquire anomalous dimension and, therefore, it has to be annihilated by the one-loop dilatation operator

\[
\mathbb{H}_{\Phi\Phi} \Phi_{sp}(Z_1)\Phi_{sp}(Z_2) = 0 .
\]

Let us confront \((3.41)\) with properties of the \(SL(2|N)\) invariant operator, Eq. \((1.10)\). One applies \(V(-1/2,-1/2)\) to the product of two superfields \(\Phi_{sp}(Z_1)\Phi_{sp}(Z_2)\) and, instead of getting zero, arrives at a divergent integral over the \(\alpha\)–parameter. To remove divergencies and, at the same time, to reproduce \((3.41)\), it suffices to introduce a projection operator, \(\Pi_{\Phi\Phi}^2 = \Phi_{\Phi\Phi}\), such that

\[
(1 - \Pi_{\Phi\Phi}) \Phi_{sp}(Z_1)\Phi_{sp}(Z_2) = 0 .
\]

Making use of \(\Pi_{\Phi\Phi}\) one can construct an integral operator

\[
\mathbb{H}_{\Phi\Phi}^{(ansatz)} = V(-1/2,-1/2)(1 - \Pi_{\Phi\Phi}) .
\]

It verifies \((3.41)\) and coincides with \((1.10)\) on the subspace of light-cone operators annihilated by \(\Pi_{\Phi\Phi}\). To preserve the superconformal symmetry one requires that the projector \(\Pi_{\Phi\Phi}\) has to be an \(SL(2|N)\) invariant operator. It acts on the tensor product \(V_{-1/2} \otimes V_{-1/2}\), Eq. \((3.17)\), and has a
The corresponding scalar function $\varphi$ is uniquely fixed by the condition \eqref{3.42}. Going over through the calculation (see Appendix D) one finds that $\varphi(\xi) = c_1 \, \delta(1 - \xi) + c_2 \, \delta'(1 - \xi)$ with $c_1$ and $c_2$ being some coefficients. In this way, we obtain the following expression for the projector

$$
\Pi_{\Phi\Phi} \, \mathcal{O}(Z_1, Z_2) = \frac{1}{2} \left( 1 + Z_{12} \cdot \partial_Z \right) \mathcal{O}(Z, Z_2) \bigg|_{Z=Z_2} + \frac{1}{2} \left( 1 + Z_{21} \cdot \partial_Z \right) \mathcal{O}(Z_1, Z) \bigg|_{Z=Z_1}, \quad \text{(3.44)}
$$

where $(Z_{12} \cdot \partial_Z) \equiv (z_1 - z_2) \partial_z + (\theta_1^+ - \theta_2^+) \partial_{\theta^+}$. One verifies that the operator $\Pi_{\Phi\Phi}$, defined in this way, indeed satisfies \eqref{3.42}.

Let us now examine composite operators in the $\Phi \Psi$--sector. We remind that the superfield $\Psi(Z)$ involves only physical fields and the UV finite spurious operators in this sector are $\Phi(0) \Psi(0)$, $\partial_{Za} \Phi(0) \Psi(0)$ and their $\text{SL}(2|\mathcal{N})$ descendants like $\partial^n \partial_{Za} \Phi(0) \Psi(0)$ with $n$ positive. Similar to the previous case, these operators have to be annihilated by the one-loop dilatation operator in the $\Phi \Psi$--sector

$$
\mathbb{H}_{\Phi\Psi} \, \Phi_{sp}(Z_1) \Psi(0) = 0, \quad \text{(3.45)}
$$

with the auxiliary superfield $\Phi_{sp}$ defined in \eqref{3.40}. In general, $\mathbb{H}_{\Phi\Psi}$ is given by a linear combination of the operators $\mathcal{V}^{(-1/2,j_{\Psi})}$ and $\mathcal{V}_\text{ex}^{(-1/2,j_{\Psi})}$ defined in \eqref{1.10} and \eqref{1.11}, respectively. As before, one can fulfill \eqref{3.45} at an expense of introducing yet another projection operator

$$
\mathbb{H}_{\Phi\Psi}^{(\text{ansatz})} = \left[ \mathcal{V}^{(-1/2,j_{\Psi})} + c \mathcal{V}_\text{ex}^{(-1/2,j_{\Psi})} \right] (1 - \Pi_{\Phi\Psi}), \quad \text{(3.46)}
$$

with a constant $c$. Its value $c = -1$ will be fixed in Sect. 4. The projector $\Pi_{\Phi\Psi}$ acts on the tensor product $\mathcal{V}_{-1/2} \otimes \mathcal{V}_{j_{\Psi}}$ and satisfies

$$
(1 - \Pi_{\Phi\Psi}) \, \Phi_{sp}(Z_1) \Psi(0) = 0. \quad \text{(3.47)}
$$

Looking for $\Pi_{\Phi\Psi}$ in the form of a general $\text{SL}(2|\mathcal{N})$ invariant operator, Eq. \eqref{3.46}, one uses \eqref{3.47} to fix the corresponding scalar function $\varphi$ and obtains (see Appendix B for details)

$$
\Pi_{\Phi\Psi} \, \mathcal{O}(Z_1, Z_2) = \mathcal{O}(Z_2, Z_2) + Z_{12} \cdot \partial_Z \mathcal{O}(Z, Z_2) \bigg|_{Z=Z_2}. \quad \text{(3.48)}
$$

One verifies that $(1 - \Pi_{\Phi\Psi})$ annihilates the operators $\mathcal{O}(Z_1, Z_2)$ linear in $Z_1$ and, therefore, \eqref{3.47} is automatically satisfied.

Finally, one examines UV finite spurious operators in the $\Psi \Phi$--sector. The only difference with the previous case is that the $\Psi$-- and $\Phi$--superfields have to be interchanged inside the trace, so that a generalization of \eqref{3.40} is straightforward

$$
\mathbb{H}_{\Psi\Phi}^{(\text{ansatz})} = \left[ \mathcal{V}_{j_{\Psi} \cdot -1/2} - \mathcal{V}_\text{ex}^{(j_{\Psi} \cdot -1/2)} \right] (1 - \Pi_{\Psi\Phi}), \quad \text{(3.49)}
$$

where the projector is defined as

$$
\Pi_{\Psi\Phi} \, \mathcal{O}(Z_1, Z_2) = \mathcal{O}(Z_1, Z_1) - Z_{12} \cdot \partial_Z \mathcal{O}(Z_1, Z) \bigg|_{Z=Z_1}. \quad \text{(3.50)}
$$

After having inserted the projectors into the expression for the dilatation operator, Eq. \eqref{3.43}, \eqref{3.49} and \eqref{3.46}, we achieved two goals simultaneously. Firstly, the dilatation operator annihilates UV finite spurious operators built from the fields \eqref{3.39}. Secondly, the resulting integrals
over the $\alpha$–parameter are convergent and the corresponding integral operators are well-defined. The projectors are not necessary in the $\Psi\Psi$–sector since the $\Psi$–superfield only involves physical fields and spurious operator do not appear
\[ \mathbb{H}^{\text{ansatz}}_{\Psi\Psi} = \nabla(j, j). \] (3.51)

To summarize, the one-loop dilatation operator in the $\mathcal{N}$–extended SYM theory is given in the multi-color limit by (1.8) with the $SL(2|\mathcal{N})$ invariant two-particle kernel $\mathbb{H}_{k,k+1}$ having a different form for $\mathcal{N} = 4$ and $\mathcal{N} \leq 2$:

- For $\mathcal{N} = 4$ one finds
\[ \mathbb{H}_{k,k+1} \bigg|_{\mathcal{N} = 4} = \mathbb{H}_{\Phi \Phi} = \nabla(-1/2,-1/2)(1 - \Pi_{\Phi \Phi}), \] (3.52)
where the operators $\nabla(-1/2,-1/2)$ and $\Pi_{\Phi \Phi}$, Eqs. (1.10) and (3.44), act on the superfields with the coordinates $Z_k$ and $Z_{k+1}$.

- For $\mathcal{N} \leq 2$ the two-particle kernel has a different form in the $\Phi\Phi$, $\Phi\Psi$, $\Psi\Phi$–sectors and can be represented as a $2 \times 2$ matrix
\[ \left[ \mathbb{H}_{k,k+1} \right]_{ab} = \mathbb{H}_{ab} = \left[ \nabla(j_a, j_b) - \nabla_{\text{ex}}(j_a, j_b) \right] (1 - \Pi_{ab}), \] (3.53)
where $a, b = \Phi, \Psi$. Here $\nabla(j, j) = \nabla_{\text{ex}}(j, j) = 0$ and the projectors $\Pi_{ab}$ were defined in (3.44), (3.48), (3.50) with $\Pi_{\Phi \Psi} = 0$.

The eigenvalues of the dilatation operator defined in this way determine the anomalous dimensions of all quasipartonic operators in the SYM theories with $0 \leq \mathcal{N} \leq 4$. We will demonstrate in Sect. 5 that Eqs. (3.52) and (3.53) lead to the expressions for the anomalous dimensions which are in agreement with the known results in the $\mathcal{N} = 0$ $[36, 23]$, $\mathcal{N} = 1$ $[37, 38, 22]$ and $\mathcal{N} = 4$ $[16, 39, 18]$ theories.

### 3.4. $SL(2|\mathcal{N})$ invariant form of the dilatation operator

By construction, the two-particle evolution kernels $\mathbb{H}^{\text{ansatz}}_{ab}$ (with $a, b = \Phi, \Psi$), Eqs. (3.43), (3.49) and (3.51), commute with the $SL(2|\mathcal{N})$ generators (3.38) acting on the tensor product $V_{j_a} \otimes V_{j_b}$. As in the $SL(2)$ case, Eq. (3.32), one can express the kernels $\mathbb{H}^{\text{ansatz}}_{ab}$ as functions of the two-particle superconformal spin $J_{ab}$. It is defined through the two-particle $SL(2|\mathcal{N})$ Casimir operator
\[ L_{ab}^2 = (L^0)^2 + L^+L^- + (\mathcal{N} - 1)L^0 + \frac{\mathcal{N}}{\mathcal{N} - 2}B^2 - V_A^+W_A^- - W_A^+V_A^- - \frac{1}{2}T_A^B T_B^A. \] (3.54)

where $G = \{L^0, L^\pm, B, V_A^\pm, W_A^\pm, T_A^B \}$ are the $SL(2|\mathcal{N})$ generators acting on the tensor product $V_{j_a} \otimes V_{j_b}$, that is, $G = G_{j_a} + G_{j_b}$ with $G_j$ given by (3.8). Then, the two-particle superconformal spin $J_{12}$ is defined as
\[ L_{ab}^2 = J_{ab}(J_{ab} - 1) + C_{ab} \] (3.55)
where $C_{ab} = \mathcal{N}(j_a + j_b)[1 + (j_a + j_b)/(\mathcal{N} - 2)]$ is a $\mathbb{C}$-valued constant introduced for the latter convenience (see Eq. (3.38)). For $\mathcal{N} = 0$, the relation (3.54) coincides with the $SL(2)$ Casimir operator
The contribution of the $B$–charge to (3.54) is divergent for $\mathcal{N} = 2$. This singularity is spurious since the $B$–charge, Eq. (3.58), is reduced for $\mathcal{N} = 2$ to a c-number, $B = -j$, and, therefore, it can be removed by subtracting constant $C_{ab}$ from the right-hand of (3.54) and (3.56).

As before, to find the explicit form of the dependence of $H_{ab}^{\text{(ansatz)}}$ on $\mathcal{J}_{ab}$, we shall examine the action of both operators on the highest weights in $\mathcal{V}_{j_a} \otimes \mathcal{V}_{j_b}$ that we denote as $O_{j_a j_b}^{(n)}(Z_1, Z_2)$. By definition, these states are annihilated by “lowering” operators $L^-$, $W^A -$ and $V^+_A$ defined in (3.29). The contribution of the $B$ is spurious since the $\mathcal{V}_{j_a \otimes j_b}$ expressions (3.32) to the case of the $\Phi \Phi$ sector, Eq. (3.43), involves the projector $\Pi_{\Phi \Phi}$.

Let us now substitute $O_{j_a j_b} \rightarrow O_{j_a j_b}^{(l)}$ in (3.36) and (3.37). One verifies that both operators become diagonal and the corresponding eigenvalues look as

$$O_{j_1 j_2}^{(l)} = 1, \quad O_{j_1 j_2}^{(k)} = \theta_{12}^{A_1} \cdots \theta_{12}^{A_k}, \quad O_{j_1 j_2}^{(n+N)} = \varepsilon_{A_1 \cdots A_N} \theta_{12}^{A_1} \cdots \theta_{12}^{A_N} z_{12}^{n},$$

where $\theta_{12}^{A_2} = \theta_{1}^{A_2} - \theta_{2}^{A_2}$ and $z_{12} = z_1 - z_2$. These states diagonalize the two-particle Casimir operator (3.32) and carry a definite value of the superconformal spin

$$(\mathcal{J}_{12}^2 - C_{12}) O^{(l)}_{j_1 j_2} = (l + j_1 + j_2)(l + j_1 + j_2 - 1) O^{(l)}_{j_1 j_2} = J_{12}(J_{12} - 1) O^{(l)}_{j_1 j_2},$$

where $J_{12} = l + j_1 + j_2$ is the eigenvalue of the two-particle spin $\mathcal{J}_{12}$, Eq. (3.55).

Let us now substitute $O_{j_1 j_2} \rightarrow O_{j_1 j_2}^{(l)}$ in (3.36) and (3.37). One verifies that both operators become diagonal and the corresponding eigenvalues look as

$$V^{(j_1 j_2)} O^{(l)}_{j_1 j_2} = [\psi (J_{12} + j_1 - j_2) + \psi (J_{12} - j_1 + j_2) - 2\psi(1)] O^{(l)}_{j_1 j_2},$$

where $J_{12} = l + j_1 + j_2$ with $l \geq 0$ and $j_1 \neq j_2$ in the second relation. Eq. (3.59) generalizes the $SL(2)$ expressions (3.32) to the case of the $SL(2|\mathcal{N})$ invariant operators.

Let us set in (3.59) $j_1 = j_2 = j_\Psi = (3 - \mathcal{N})/2$. According to (3.51), the resulting expression for $V^{(j_\Psi, j_\Psi)}$ gives the two-particle kernel in the $\Psi \Psi$–sector (for $\mathcal{N} = 0, 1, 2$)

$$H_{\Psi \Psi}^{\text{(ansatz)}} = 2 [\psi(\mathcal{J}_{\Psi \Psi}) - \psi(1)],$$

with $\mathcal{J}_{\Psi \Psi}$ having the eigenvalues $\mathcal{J}_{\Psi \Psi} = 3 - \mathcal{N} + l$. Then, one puts $j_1 = j_2 = -1/2$ in (3.59) that corresponds to going over to the $\Phi \Phi$–sector. We find that $J_{\Phi \Phi} = -1 + l$ and, as a consequence, the eigenvalues of $V^{(-1/2, -1/2)}$ take infinite values for $l = 0, 1$. It is this divergence that we encountered at the beginning of Sect. 3.3. The expression for the one-loop dilatation operator in the $\Phi \Phi$–sector, Eq. (3.43), involves the projector $\Pi_{\Phi \Phi}$. As follows from its definition (3.44), the operator $(1 - \Pi_{\Phi \Phi})$ annihilates two highest weights with $l = 0, 1$ leading to

$$(1 - \Pi_{\Phi \Phi}) O^{(l)}_{\Phi \Phi} = O^{(l)}_{\Phi \Phi} \theta(l - 1),$$

where the $\theta$–function is defined in such a way that $\theta(n) = 0$ for $n \leq 0$ and $\theta(n) = 1$ for $n > 0$. Combining this relation together with (3.59), we get from (3.43) the following expression for the two-particle evolution kernel in the $\Phi \Phi$–sector

$$H_{\Phi \Phi}^{\text{(ansatz)}} = 2 [\psi(\mathcal{J}_{\Phi \Phi}) - \psi(1)] \theta(\mathcal{J}_{\Phi \Phi}).$$
Similarly, the projector \((1 - \Pi_{\Phi\Psi})\) entering the expression for the kernel in the \(\Phi\Psi\)–sector, Eq. (3.46), annihilates the highest weights with \(l = 0, 1\), so that \((1 - \Pi_{\Phi\Psi})O_{\Phi\Psi}^{(l)} = \theta(l - 1)O_{\Phi\Psi}^{(l)}\). As a result, substituting \(j_1 = -1/2\) and \(j_2 = (3 - N)/2\) into (3.59) we find from (3.46)

\[
H_{\Phi\Psi}^{(\text{ansatz})} = \left[\psi(J_{\Phi\Psi} + c_N) + \psi(J_{\Phi\Psi} - c_N) - 2\psi(1)\right. \\
\left. - P_{\Phi\Psi} \frac{\Gamma(J_{\Phi\Psi} - c_N)}{\Gamma(J_{\Phi\Psi} + c_N)} \theta(J_{\Phi\Psi} - c_N)\right],
\]

where \(c_N = 2 - N/2\) and \(P_{\Phi\Psi}\) is a permutation operator, \(P_{\Phi\Psi}O_{\Phi\Psi}(Z_1, Z_2) = O_{\Phi\Psi}(Z_2, Z_1)\). In this case, the two-particle spin takes the eigenvalues \(J_{\Phi\Psi} = c_N + 1\), that is integer for \(N = 0, 2\) and half-integer for \(N = 1\). Finally, the two-particle kernel in the \(\Psi\Phi\)–sector, Eq. (3.49), is given by the same expression (3.63) modulo substitution \(J_{\Phi\Psi} \rightarrow J_{\Psi\Phi}\) and \(P_{\Phi\Psi} \rightarrow P_{\Psi\Phi}\)

\[
H_{\Psi\Phi}^{(\text{ansatz})} = P_{\Psi\Phi} H_{\Phi\Psi}^{(\text{ansatz})} P_{\Psi\Phi},
\]

where the permutation operator acts as \(P_{\Phi\Psi}O_{\Phi\Psi}(Z_1, Z_2) = O_{\Phi\Psi}(Z_2, Z_1)\). In Eq. (3.63), the term involving the permutation operator describes the exchange interaction between the superfields. The \(\theta\)–functions in (3.64) and (3.65) are induced by the projectors \((1 - \Pi_{ab})\) in Eq. (3.52). They assign zero anomalous dimensions to the spurious operators involving nonlocal fields (3.39).

### 3.5. Wilson operators

The two-particle evolution kernels, Eqs. (3.52) and (3.53), involve the additional projection operators due to the presence of nonlocal fields \(\partial_{\pm}^{-1}A(0), A(0)\) and \(\partial_{\pm}^{-1}\lambda^4(0)\) in the expansion of the superfield \(\Phi(Z)\) around \(Z = 0\). One can avoid the spurious operators from the start by subtracting from \(\Phi(Z)\) the first two terms of its expansion around \(Z = 0\)

\[
\Phi_w(Z) = \Phi(Z) - \Phi(0) - Z \cdot \partial Z \Phi(0) = \Phi(Z) - \Phi_{sp}(Z).
\]

and introducing a nonlocal light-cone operator \(O_w\) built from the superfields \(\Psi(Z)\) and \(\Phi_w(Z)\)

\[
O_w(Z_1, Z_2, \ldots, Z_L) = \text{tr}\{\Phi_w(Z_1) \Psi(Z_2) \ldots \Phi_w(Z_L)\} = \Pi_w \cdot O(Z_1, Z_2, \ldots, Z_L).
\]

By construction, the expansion of \(O_w(Z_1, \ldots, Z_L)\) around \(Z_k = 0\) generates only “physical”, Wilson operators. Here a notation was introduced for the operator \(\Pi_w\) which removes “spurious” operators from the light-cone operator. It is easy to verify that \(\Pi_w\) is a projector, \((\Pi_w)^2 = \Pi_w\). The chiral superfields \(\Phi_w(Z)\) and \(\Psi(Z)\) span all propagating fields in the SYM theory, Eqs. (2.27) – (2.30). For \(Z = (z, \theta^4) = 0\), the derivatives of these superfields along the “odd” directions in the superspace generate all field components, while the derivatives along the “even” direction induce light-cone derivatives. In this way, Eq. (3.66) generates an infinite set of quasipartonic operators.

The Wilson operators mix under renormalization among themselves and form a closed sector with respect to the action of the dilatation operator \(H\). Applying the projector \(\Pi_w\) to both sides of the evolution equation (1.7) one finds that in order to ensure this property one has to require that \(\Pi_w H O(Z_1, \ldots, Z_L) = \Pi_w H O_w(Z_1, \ldots, Z_L)\), or equivalently

\[
\Pi_w H (1 - \Pi_w) = 0.
\]
Let us examine the difference between two light-cone operators

\[ O_{\text{sp}} = O(Z_1, \ldots, Z_L) - O_W(Z_1, \ldots, Z_L) = (1 - \Pi_W)O(Z_1, \ldots, Z_L). \]  

(3.68)

According to (3.65), it involves at least one spurious superfield (3.40). The operators \( O_{\text{sp}} \) mix under renormalization among themselves and with the Wilson operators \( O_W \). The corresponding evolution kernels are given by \( \mathbb{H}_{\text{sp}} = (1 - \Pi_W)\mathbb{H}(1 - \Pi_W) \) and \( \mathbb{H}_{\text{sp} \to W} = (1 - \Pi_W)\mathbb{H}\Pi_W \), respectively.

It is convenient to treat \( O_W \) and \( O_{\text{sp}} \) as two components of the same vector and represent the dilatation operator \( \mathbb{H} \) as a triangular 2 \( \times \) 2 matrix

\[ O(Z_1, \ldots, Z_L) = \begin{pmatrix} O_W \\ O_{\text{sp}} \end{pmatrix}, \quad \mathbb{H} = \begin{pmatrix} \mathbb{H}_W & 0 \\ \mathbb{H}_{\text{sp} \to W} & \mathbb{H}_{\text{sp}} \end{pmatrix}, \]  

(3.69)

where the integral operator \( \mathbb{H}_W \) maps the Wilson operators into themselves

\[ \mathbb{H}_W \equiv \Pi_W\mathbb{H} = \Pi_W\mathbb{H}\Pi_W. \]  

(3.70)

The zero entry in (3.69) reflects the fact that the Wilson operators can not mix with the spurious operators whereas the opposite is possible.

The dilatation operator \( \mathbb{H}_W \) governs the scale dependence of the operators \( O_W(Z_1, \ldots, Z_L) \). As was shown in Sect. 3.4, the two-particle kernels are functions of the two-particle superconformal spin, \( \mathbb{H}_{k,k+1} = h(J_{k,k+1}) \), Eqs. (3.60) – (3.64). It follows from (3.70) and (1.8) that the two-particle kernel \( \mathbb{H}^k_{k+1} = \Pi_W h(J_{k,k+1}) \Pi_W \) is given by the same function with \( J_{k,k+1} \) replaced by a “projected” superconformal spin \( J^W_{k,k+1} = \Pi_W J_{k,k+1} \Pi_W \)

\[ \mathbb{H}_{k,k+1} = h(J_{k,k+1}) \quad \longrightarrow \quad \mathbb{H}^W_{k,k+1} = h(J^W_{k,k+1}). \]  

(3.71)

Thus, the two-particle kernels \( \mathbb{H}^W_{k,k+1} \) have the same eigenvalues as the operators (3.60) – (3.64).

4. One-loop dilatation operator

The one-loop dilatation operator acting on single-trace nonlocal light-cone operators, Eqs. (1.3)–(1.5), is given in the multi-color limit by the sum over the two-particle evolution kernels (1.8). The superconformal invariance of the SYM theory on the light-cone allows one to determine a general form of the two-particle kernels in various sectors, Eqs. (3.43), (3.46) and (3.49), but the obtained expressions (3.36) and (3.37) involve some unknown scalar functions \( f \) and \( f_{\text{ex}} \). Based on previous QCD calculations, we conjectured that these functions should be given by (3.26) leading to the expressions for the one-loop dilatation operator summarized in the Introduction, Eqs. (1.9) – (1.11). In this section, we shall confirm these assertions by calculating the one-loop corrections to the nonlocal light-cone operators, Eqs. (1.3) – (1.5), and matching their divergent part into a general expression for the one-loop dilatation operator.

We remind that the \( N = 4 \) SYM theory involves only one chiral light-cone superfield and, in order to identify the two-particle kernel \( \mathbb{H}_{12} \) entering (1.8), one has to calculate one-loop corrections to the operator \( \Phi(Z_1)\Phi(Z_2) \). For \( N \leq 2 \), the SYM theories are formulated in terms of two independent chiral superfields and, therefore, there are three additional sectors \( \Psi(Z_1)\Psi(Z_2) \), \( \Phi(Z_1)\Psi(Z_2) \) and \( \Psi(Z_1)\Phi(Z_2) \). In what follows we shall denote the corresponding two-particle kernels as \( \mathbb{H}_{\Phi\Phi} \), \( \mathbb{H}_{\Psi\Psi} \), \( \mathbb{H}_{\Phi\Psi} \) and \( \mathbb{H}_{\Psi\Phi} \). The first two kernels will be calculated in Sect. 4.1 and the remaining two in Sect. 4.2.
To calculate the anomalous dimension of the light-cone operators $\mathcal{O}(Z_1, \ldots, Z_L)$ we apply an approach well-known in perturbative QCD [10, 11]. Let us consider the matrix element of this operator between the vacuum and a reference state, $\langle 0 | \mathcal{O}(Z_1, \ldots, Z_L) | P \rangle$. Since the anomalous dimension of the operator does not depend on the choice of the state $| P \rangle$, one can choose it at will, from convenience considerations alone. To this end, we apply the Fourier transformation and expand the superfield over the plane waves in the superspace

$$
\Phi(x, \theta^A) = \int \frac{d^4p}{(2\pi)^4} \int d^N \pi e^{i p \cdot x + \pi \cdot \theta^A} \tilde{\Phi}(p, \pi_A),
$$

(4.1)

where $\pi_A$ is the Grassmann valued momentum conjugated to the odd coordinates $\theta^A$ and $p_\mu$ defines the momentum of the field components entering into expansion of the superfield. Similar expansion holds for the superfield $\Psi(x, \theta^A)$. Let us define $| P \rangle$ to be a state describing $L$ particles with (super)momenta $P_k = (p_{k,\mu}, \pi_{k,A})$

$$
| P \rangle = \left( \prod_{k=1}^L \frac{i \sigma \cdot \pi^2}{p_k^2} \right)^{-1} \text{tr}\{\tilde{\Psi}(P_1) \ldots \tilde{\Phi}(P_L)\}|0\rangle
$$

(4.2)

The total (super)momentum of the state is $P = \sum_{k=1}^L P_k$. In addition, we choose four-dimensional momenta of all particles, $p_{k,\mu}$, to be aligned along the same direction in Minkowski space-time, close to the “$-$” direction on the light-cone

$$
p_{k,\perp} = 0, \quad p_{k,+} \ll p_{k,-}, \quad p_k^2 = 2p_{k,+}p_{k,-}.
$$

(4.3)

Then, in the Born approximation, the matrix element $\langle 0 | \mathcal{O}(Z_1, \ldots, Z_L) | P \rangle$ is given by the product of plane waves accompanied by the propagators (see Eq. (C.5), Appendix C). The latter are cancelled against the prefactor in the right-hand side of (4.2) leading in the multi-color limit to

$$
\langle 0 | \mathcal{O}^{(0)}(Z_1, \ldots, Z_L) | P \rangle = \prod_{k=1}^L e^{-i p_{k,+} Z_k} = \begin{array}{c|c|c|c|c}
\Phi & \psi & \Phi & \ldots & \psi
\end{array}
$$

(4.4)

Here $Z_k = (z_k, \theta_{k}^A)$ defines the position of the $k-$th superfield in the superspace and we used the notation for a scalar product in the superspace $iP \cdot Z = ip_{+,z} + \pi_A \theta^A$ with $p_+ = (p \cdot n)$. The superscript $(0)$ indicates that the matrix element is evaluated at the Born level. For $N \leq 2$, to distinguish the superfields $\Phi(Z)$ and $\Psi(Z)$, we shall denote them by lines with the incoming and outgoing arrows, respectively. In particular, in our notations the right-hand side of (4.4) corresponds to the following operator $\mathcal{O}(Z_1, \ldots, Z_L) = \text{tr}\{\Phi(Z_1)\Psi(Z_2)\Phi(Z_3) \ldots \Psi(Z_L)\}$. For $N = 4$ we shall denote the superfield $\Phi(Z)$ by a line without an arrow.

Substituting (4.1) into the light-cone SYM actions (2.24) and (2.25), it is straightforward to work out the Feynman diagram technique for calculating perturbative corrections to (4.4). The Feynman rules involve three elements: propagators of the superfields, triple and quartic interaction vertices. For $N \leq 2$, the interaction vertices are $\Phi \Psi \Psi$, $\Psi \Phi \Psi$ and $\Phi \Phi \Psi$, whereas for $N = 4$ they are $\Phi \Phi \Phi$ and $\Phi \Phi \Phi \Phi$. Their explicit expressions are given in Appendix C.

---

5If all particles entering $| P \rangle$ are identical, the right-hand side of (4.4) is given in the multi-color limit by a sum over cyclic permutations of their momenta.
\[ N = 0 \] similar technique has been worked out in Ref. [31]. As was demonstrated there, the use of the light-cone action simplifies significantly the calculation of evolution kernels as compared to a conventional “covariant” approach based on the full Yang-Mills action.

Calculating one-loop corrections to the matrix element \( \langle 0 | \mathcal{O}(1)(Z_1, \ldots, Z_L) | P \rangle \), we shall apply the dimensional regularization and evaluate the momentum integrals in \( D = 4 - 2\varepsilon \) dimensions

\[
\int \frac{d^4p}{(2\pi)^4} \rightarrow \mu^{4-D} \int \frac{d^Dp}{(2\pi)^D}
\]

with the scale \( \mu \) playing the role of a UV cut-off. According to the evolution equation (1.7), the one-loop dilatation operator \( H \) is related to the coefficient in front of a pole \( 1/\varepsilon \) in the expression for the matrix element of the nonlocal light-cone operator \( \mathcal{O}(Z_1, \ldots, Z_L) \)

\[
\langle 0 | \mathcal{O}(1) | P \rangle = -\frac{g^2 N_c \mu^{2\varepsilon}}{(4\pi)^2} \varepsilon \langle 0 | \mathbb{H} \cdot \mathcal{O}(0) + L\mathcal{O}(0)\rangle | P \rangle + \ldots ,
\]

where ellipses denote terms regular for \( \varepsilon \rightarrow 0 \) and \( \mathcal{\gamma}_N^{(1)} \) defines the one-loop correction to the anomalous dimension of the superfield, \( \mathcal{\gamma}_N = \frac{g^2 N_c}{(4\pi)^2} \mathcal{\gamma}_N^{(0)} + O(g^4) \). Note that in the SYM theory on the light-cone cone, the anomalous dimensions of the superfields \( \Phi(Z) \) and \( \Psi(Z) \) are equal to each other and are proportional to the \( \beta \)-function, \( \mathcal{\gamma}_N = \beta_N(g) / g \) (see Appendix D1). The reason why we split the right-hand side of (4.6) into the sum of two terms is that the second term containing \( \mathcal{\gamma}_N^{(1)} \) comes entirely from diagrams containing self-energy corrections and can be separated from the very beginning. In what follows, we will not display this term and tacitly imply that it should be added to the final expression for \( \langle 0 | \mathcal{O}(1) | P \rangle \).

### 4.1. Diagonal sector

Let us calculate one-loop corrections to the matrix elements of single-trace operators involving the products \( \Phi(Z_1)\Phi(Z_2) \) and \( \Psi(Z_1)\Psi(Z_2) \) and use them to determine the two-particle evolution kernels \( \mathbb{H}_{\Phi\Phi} \) and \( \mathbb{H}_{\Psi\Psi} \), respectively.

#### 4.1.1. \( N \leq 2 \)

We start with the \( \Psi\Psi \)-sector. For \( N \leq 2 \), the one-loop Feynman diagrams contributing to \( \langle 0 | \text{tr}\{\Psi(Z_1)\Psi(Z_2)\ldots\} | P \rangle \) are shown in Figure 1. Let us examine the diagrams one after another.

The diagram Fig. 1(c) describes the self-energy correction to the superfield and contributes to the one-loop anomalous dimension \( \gamma_N(g) \), Eqs. (1.7) and (4.6). Its calculation can be found in Appendix D1. For the annihilation diagram, Fig. 1(c), one applies the Feynman rules (see Appendix C) and finds that it gives rise to an integral proportional to the holomorphic component of the loop momenta, \( k = (k_1 + ik_2)/\sqrt{2} \) (see Eqs. (A.22) and (4.3))

\[
(p_1 - k, p_2 + k) = -(p_1 + p_2)_+ k .
\]

As a result, it equals zero upon integration over the transverse momenta \( \int d^2k_\perp \equiv \int dk_1dk_2 \). For the sum of the remaining three diagrams, Figs. 1(a), (b) and (d), one gets the following Feynman...
integral (the details can be found in Appendix D2)

\[
\langle 0| O_{\Psi\Psi}^{(1)}(Z_1, Z_2, ...)| P \rangle = -ig^2 N_c \mu^{4-D} \int \frac{d^D k}{(2\pi)^D} e^{-iz_1(p_1-k)-iz_2(p_2+k)} \int d^N \pi e^{-(\pi_1-\pi_A)\theta^A-(\pi_2+\pi)\theta^A} \times \left[ \frac{\delta^{(N)}(\pi - \pi_1 \frac{k_+}{p_1+}) p_1+ \left( \frac{p_1-\frac{k_+}{p_1+}}{p_1+} \right)^{2-N} - \frac{\delta^{(N)}(\pi - \pi_2 \frac{k_+}{p_2+}) p_2+ \left( \frac{p_2+\frac{k_+}{p_2+}}{p_2+} \right)^{2-N}}{k^2(p_2+k)^2 k_+} \right],
\]

(4.8)

where the poles at \( k_+ = 0 \) are regularized using the Mandelstam-Leibbrandt prescription, Eq. (A.4). Here \( \delta(\pi) \) is the Dirac \( \delta \)-function for odd coordinates defined in (A.21)

\[
\int d^N \pi \delta^{(N)}(\pi - \pi') e^{\pi A \theta^A} = e^{\pi A \theta^A}.
\]

(4.9)

For \( N = 0 \) in (4.8), the integral over the odd momenta \( \pi \) is absent and the odd \( \delta \)-functions are replaced by 1. For \( N \geq 1 \), the \( \pi \)-integral in (4.8) is trivial due to (1.9), while integration over the loop momentum \( k_\mu \) can be easily performed with a help of the identity (D.5) (for \( n = 1 \)). In this way, one can express the divergent (for \( \varepsilon \to 0 \)) part of (4.8) as a sum of plane waves integrated over a scalar variable \( \alpha \) which has the meaning of the momentum fraction \( k_+ = \alpha p_+ \)

\[
\langle 0| O_{\Psi\Psi}^{(1)}(Z_1, Z_2, ...)| P \rangle = -\frac{g^2 N_c \mu^{2\varepsilon}}{(4\pi)^2} \varepsilon \int_0^1 \frac{d\alpha}{\alpha} \left\{ 2 e^{-iP_1.Z_1-iP_2.Z_2} \right. - (1-\alpha)^{2-N} \left[ e^{-iP_1.((1-\alpha)Z_1+\alpha Z_2)-iP_2.Z_2} + e^{-iP_1.Z_1-iP_2.((1-\alpha)Z_2+\alpha Z_1)} \right] \right\}.
\]

(4.10)

Here the notation was introduced for the scalar product in the superspace between the vectors \( Z_k = (z_k, \theta^A_k) \) and \( P = (p_{k+}, \pi_{k,A}) \) (with \( k = 1, 2 \))

\[
i(P \cdot Z) \equiv ip_+z + \pi_A \theta^A
\]

(4.11)

Hereafter, to simplicity formulae, we do not display the factors \( e^{-iz_n p_n-\pi_n A \theta^A} \) corresponding to noninteracting superfields with the coordinates \( Z_n = (z_n, \theta^A_n) \) with \( n = 3, \ldots, L \).
Making use of (4.13), one can rewrite the right-hand side of (4.10) in terms of the Born level matrix element leading to

$$\mathcal{O}^{(1)}_{\Psi\Psi}(Z_1, Z_2, \ldots) = \frac{g^2 N_c \mu^{2\epsilon}}{(4\pi)^2 \varepsilon} \int_0^1 \frac{d\alpha}{\alpha} \left\{ 2 \mathcal{O}^{(0)}_{\Psi\Psi}(Z_1, Z_2, \ldots) - (1 - \alpha)^{2N} \left[ \mathcal{O}^{(0)}_{\Psi\Psi}((1 - \alpha)Z_1 + \alpha Z_2, Z_2, \ldots) + \mathcal{O}^{(0)}_{\Psi\Psi}(Z_1, (1 - \alpha)Z_2 + \alpha Z_1, \ldots) \right] \right\}.$$  (4.12)

Matching this expression into (4.6) and keeping in mind that the term involving $\gamma^{(1)}_{\alpha}$ in (4.6) comes from the self-energy diagram, one identifies the two-particle evolution kernel $\mathcal{H}_{\Psi\Psi}$ governing the scale dependence of $\text{tr} \{ \Psi(Z_1)\Psi(Z_2) \ldots \}$ in the $N-$extended SYM theory

$$\mathcal{H}_{\Psi\Psi} \Psi(Z_1)\Psi(Z_2) \bigg|_{N=0,1,2} = \int_0^1 \frac{d\alpha}{\alpha} \left\{ 2\Psi(Z_1)\Psi(Z_2) - (1 - \alpha)^{2N} \left[ \Psi((1 - \alpha)Z_1 + \alpha Z_2)\Psi(Z_2) + \Psi(z_1)\Psi((1 - \alpha)Z_2 + \alpha Z_1) \right] \right\}.$$  (4.13)

The integrand has a pole at $\alpha = 0$ but the linear combination of the superfields vanishes for $\alpha \to 0$ so that the integral is convergent.

We remind that (4.13) is valid in $N = 0$, $N = 1$ and $N = 2$ SYM theories. In a perfect agreement with our expectations, (4.13) coincides with the expression for the $SL(2|N)$ invariant operator (1.10) evaluated for $j_1 = j_2 = (3 - N)/2$ corresponding to the conformal spin of the $\Psi-$superfield, Eq. (3.10),

$$\mathcal{H}_{\Psi\Psi} = \mathcal{V}^{(j_\Psi,j_\Psi)}.$$  (4.14)

Since the $\Psi-$superfield does not contain nonlocal fields, this kernel acts on the subspace of Wilson operators only, $\mathcal{H}_{\Psi\Psi}^{\Psi} = \Pi^\Psi \mathcal{H}_{\Psi\Psi} = \mathcal{H}_{\Psi\Psi}.$

Let us repeat a similar calculation in the $\Phi\Phi-$sector and obtain the one-loop expression for the two-particle kernel $\mathcal{H}_{\Phi\Phi}$. As before, our starting point is the matrix element of the light-cone operator $\langle 0 | \text{tr} \{ \Phi(Z_1)\Phi(Z_2)\ldots \} | P \rangle$. It receives one-loop corrections from the Feynman diagrams similar to those shown in Fig. 1. The only difference is that the direction of the arrow for incoming and outgoing lines should be flipped. As in the previous case, the annihilation diagram (Fig. 1b) vanishes, Eq. (4.17), and the diagram with the self-energy produces the one-loop anomalous dimension of the $\Phi-$field. For the sum of three remaining diagrams, one gets (see Appendix D3 for details)

$$\langle 0 | \mathcal{O}^{(1)}_{\Phi\Phi}(Z_1, Z_2, \ldots) | P \rangle = -i g^2 N_c \mu^{4-D} \int \frac{d^Dk}{(2\pi)^D} e^{-iz_1(p_1-k) - iz_2(p_2+k)} \int d^{N'} \pi e^{-(\pi_1-\pi_2)A\theta^4_1 - (\pi_2+\pi)A\theta^4_2} \left[ \delta^{(N)} \left( \frac{\pi - \pi_2}{p_2+p} \right) \frac{p_1}{k^2(p_1-k)^2} \frac{p_2}{k^2(p_2+k)^2} \right] \left[ \delta^{(N)} \left( \frac{\pi_1}{p_1} \right) \frac{p_1}{k^2(p_1-k)^2} \right],$$  (4.15)

where the poles in $k_+$ are regularized using the Mandelstam-Leibbrandt prescription (A.4). In comparison with (4.8), the momenta of the two incoming lines get interchanged inside the odd $\delta-$functions and the factor $(\ldots)^{2-N}$ is modified. This makes the calculation much more involved.
Indeed, we expect from (3.43) that the two-particle kernel $\mathbb{H}_{\Phi\Phi}$ should have a more complicated form as compared with $\mathbb{H}_{\Psi\Psi}$.

The expression inside the square brackets in (4.14) can be rewritten after some algebra in the following form

$$
\left[\cdots\right]_{N} = \frac{\delta^{(N)}}{k^{2}(p_{1} - k)^{2}} \frac{p_{1}^{3}}{k_{+}(p_{1} + k_{+})^{2}} \delta^{(N)} \left( \pi - \pi_{1} \right)_{p_{1}^{2} + k_{+}^{2}} - \frac{\delta^{(N)}}{k^{2}(p_{2} + k)^{2}} \frac{p_{2}^{3}}{k_{+}(p_{2} + k_{+})^{2}} \delta^{(N)} \left( \pi - \pi_{2} \right)_{p_{2}^{2} + k_{+}^{2}}
$$

where the notation was introduced for

$$
\varpi_{A} = \frac{\pi_{1A} p_{2} + k_{+}}{p_{1} + p_{2}} - \frac{\pi_{2} p_{1} - k_{+}}{p_{1} + p_{2}},
$$

$$
\mathcal{X}_{N=1} = \frac{\pi_{1} p_{1}^{2} - \pi_{2} p_{2}^{2}}{(p_{1} + k_{+})(p_{2} + k_{+})(p_{1} + p_{2})},
$$

$$
\mathcal{X}_{N=2} = \frac{\varepsilon^{AB}(p_{A} - \varpi_{A})(\pi_{1} p_{2} + \pi_{2} p_{1})}{(p_{1} + k_{+})(p_{2} + k_{+})(p_{1} + p_{2})}.
$$

(4.17)

For $N = 0$, one has $\mathcal{X}_{N=0} = 0$ and the odd $\delta$–functions are replaced by 1 in (4.16). The first two terms in the right-hand side of (4.16) depend on a single “external” momentum, $P_{1} = (p_{1}, \pi_{1A})$ and $P_{2} = (p_{2}, \pi_{2A})$, respectively. This allows one to perform the $k$–integration in (4.16) by making use of the identity (D.5). Similarly, one replaces the integration variable $k_{+} = p_{2} + k_{+}$ in the third term in (4.16) and performs the $k'$–integration with a help of the identity (D.5).

The details of the calculation can be found in Appendix D3.

The resulting expression for the Feynman integral in (4.15) is similar to (4.10) and (4.12). Namely, the divergent part of $\langle 0|\mathcal{O}^{(1)}_{\Phi\Phi}(Z_{1}, Z_{2}, \ldots)|P \rangle$ has the form of the $\alpha$–integral with the integrand given by a rather lengthy expression. Remarkably enough, it can be cast into the following form

$$
\mathcal{O}^{(1)}_{\Phi\Phi}(Z_{1}, Z_{2}, \ldots) = -\frac{g^{2} N_{c}}{(4\pi)^{2}} \frac{\mu^{2\varepsilon}}{\varepsilon} \left\{ \mathcal{V}_{\Phi\Phi}(1 - \Pi_{\Phi\Phi}) + \Delta_{\Phi\Phi} \right\} \mathcal{O}^{(0)}_{\Phi\Phi}(Z_{1}, Z_{2}, \ldots),
$$

(4.18)

where the operators $\mathcal{V}_{\Phi\Phi}$, $\Pi_{\Phi\Phi}$ and $\Delta_{\Phi\Phi}$ act on the superfields with the coordinates $Z_{1}$ and $Z_{2}$. They have the same, universal form for $N = 0$, $N = 1$ and $N = 2$. The projector $\Pi_{\Phi\Phi}$ was already defined in (3.44). The operator $\mathcal{V}_{\Phi\Phi}$ is given by

$$
\mathcal{V}_{\Phi\Phi} \mathcal{O}(Z_{1}, Z_{2}, \ldots) = \int_{0}^{1} \frac{d\alpha}{\alpha} \left\{ 2 \mathcal{O}(Z_{1}, Z_{2}, \ldots) - (1 - \alpha)^{-2} \left[ \mathcal{O}((1 - \alpha)Z_{1} + \alpha Z_{2}, Z_{2}, \ldots) + \mathcal{O}(Z_{1}, (1 - \alpha)Z_{2} + \alpha Z_{1}, \ldots) \right] \right\}.
$$

(4.19)

One verifies that it coincides with the $SL(2|N)$ invariant operator (1.10), $\mathcal{V}_{\Phi\Phi} = \mathcal{V}^{(-1/2, -1/2)}$. Eq. (3.9). The integral in (4.19) diverges for $\alpha \to 1$ and, therefore, the operator $\mathcal{V}_{\Phi\Phi}$ is well-defined only for the operators $\mathcal{O}(Z_{1}, Z_{2}, \ldots)$ which vanish sufficiently fast as $Z_{1} \to Z_{2}$. It is easy to verify using (3.43) that

$$
(1 - \Pi_{\Phi\Phi}) \mathcal{O}((1 - \alpha)Z_{1} + \alpha Z_{2}, Z_{2}, \ldots) \sim (1 - \alpha)^{2}
$$

(4.20)
as \( \alpha \to 1 \) and, as a consequence, \( \mathcal{V}_{\Phi}\phi (1 - \Pi_{\Phi}\phi) \) is a well-defined integral operator. Finally, the operator \( \Delta_{\Phi}\phi \) is defined as

\[
\Delta_{\Phi}\phi \mathcal{O}(Z_1, Z_2, \ldots) = \left( 1 - \frac{1}{2} Z_{21} \partial Z_2 \right) \left[ \frac{\partial_{1+} - \partial_{2+}}{\partial_{1+} + \partial_{2+}} \mathcal{O}(Z_1, Z_2, \ldots) \right] \bigg|_{Z_1 = Z_2} + \left( 1 - \frac{1}{2} Z_{12} \partial Z_1 \right) \left[ \frac{\partial_{2+} - \partial_{1+}}{\partial_{1+} + \partial_{2+}} \mathcal{O}(Z_1, Z_2, \ldots) \right] \bigg|_{Z_2 = Z_1},
\]

(4.21)

where \( \partial_{k+} = \partial/\partial z_k \) denotes the derivative with respect to the light-cone coordinate, \( Z_k = (z_k, \theta^A_k) \) and the notation was introduced for \( Z_{jk} = (z_j - z_k, \theta^A_j - \theta^A_k) \) and \( Z_{jk} \cdot \partial Z_j \equiv (z_j - z_k) \partial z_j + (\theta^A_j - \theta^A_k) \partial \theta^A \) with \( j, k = 1, 2 \). In Eq. (4.21), one first evaluates the expressions inside the square brackets for \( Z_1 = Z_2 \) (or \( Z_2 = Z_1 \)) and applies the external derivative afterwards.

Matching (4.18) into (4.6) we conclude that the two-particle evolution kernel in the \( \Phi\Phi \)-sector is given by

\[
\mathbb{H}_{\Phi\Phi} \Phi(Z_1) \Phi(Z_2) \bigg|_{\mathcal{N} = 0, 1, 2} = \left\{ \mathcal{V}(-1/2,-1/2)(1 - \Pi_{\Phi\Phi}) + \Delta_{\Phi\Phi} \right\} \Phi(Z_1) \Phi(Z_2),
\]

(4.22)

where the operators \( \mathcal{V}(-1/2,-1/2) \), \( \Pi_{\Phi\Phi} \) and \( \Delta_{\Phi\Phi} \) were given in Eqs. (1.10), (3.44) and (4.21), respectively. Notice that the \( \mathcal{N} \)-dependence enters (4.22) only through the dimension of the superspace \( Z = (z, \theta^A) \), with \( \lambda = 1, \ldots, \mathcal{N} \).

Comparing (4.22) with our ansatz for the two-particle kernel in the \( \Phi\Phi \)-sector, Eq. (3.43), we find that (4.22) contains the additional operator \( \Delta_{\Phi\Phi} \). To understand its origin, we recall that \( \Phi(Z_1) \Phi(Z_2) \) is a generating function for both Wilson operators and composite operators involving spurious fields. As was explained in Sect. 3.2, the latter operators can be eliminated by applying the projector \( \Pi_{\psi} \) to both sides of (4.22). According to its definition, Eqs. (3.66) and (3.65), the operator \( \Pi_{\psi} \) annihilates the states \( \mathcal{O}(Z_1, Z_2, ...) \) which either do not depend on at least one of the superspace coordinate \( Z_k \) or are linear in \( Z_k \). It is easy to see that each term in the right-hand side of (4.21) verifies these conditions and, therefore,

\[
\Pi_{\psi} \Delta_{\Phi\Phi} \mathcal{O}(Z_1, Z_2, ...) = 0.
\]

(4.23)

This means that the operator \( \Delta_{\Phi\Phi} \) does not affect Wilson operators and only contributes to the scale dependence of spurious operators. Projecting both sides of (4.22) onto the subspace of Wilson operators according to (3.70), we find that the “physical” dilatation operator in the \( \Phi\Phi \)-sector is given by

\[
\mathbb{H}_{\Phi\Phi}^{\psi} \equiv \Pi_{\psi} \mathbb{H}_{\Phi\Phi} = \Pi_{\psi} \mathcal{V}(-1/2,-1/2)(1 - \Pi_{\Phi\Phi}).
\]

(4.24)

One verifies that \( \mathbb{H}_{\Phi\Phi}^{\psi} \) satisfies (3.67) and coincides with \( \Pi_{\psi} \mathbb{H}_{\Phi\Phi}^{\text{(ansatz)}} \), Eq. (3.43). Thus, the evolution kernels \( \mathbb{H}_{\Phi\Phi} \) and \( \mathbb{H}_{\Phi\Phi}^{\text{(ansatz)}} \) are identical on the subspace of Wilson operators.

### 4.1.2. \( \mathcal{N} = 4 \)

In the \( \mathcal{N} = 4 \) SYM theory, there is only the \( \Phi\Phi \)-sector. To calculate the corresponding evolution kernel \( \mathbb{H}_{\Phi\Phi} \) one has to evaluate the one-loop corrections to \( \langle 0 \left| \text{tr} \{ \Phi(Z_1) \Phi(Z_2) \} \right| P \rangle \). They are given by the same Feynman diagrams in Fig. II as before. The only difference is that for \( \mathcal{N} = 4 \)
the lines do not have arrows. In this case, the diagrams in Fig. 1(a) and (b) are identical and only one of them has to be taken into account. The divergent part of the self-energy diagram in Fig. 1(e) is proportional to the $\beta-$function in the $N = 4$ SYM and it vanishes [29, 20, 21, 30] (see Appendix D1). The annihilation diagram in Fig. 1(c) does not contribute by the same reason as before: it is proportional to the holomorphic component of the loop momentum $k = (k_1 + ik_2)/\sqrt{2}$ and vanishes upon integration over $\int dk_1 dk_2$, Eq. (4.17).

Applying the $N = 4$ Feynman rules (see Appendix C) one finds that the sum of the remaining two diagrams is given by the following lengthy expression

$$
\langle 0|O^{(1)}_{\Phi}(Z_1, Z_2, ...)|P \rangle = \frac{i}{4} q^2 N_c \mu^{4-D} \int \frac{d^D k}{(2\pi)^D} \frac{e^{-i z(p_1 + k) - i z(p_2 - k)}}{(p_1 + k)^2 (p_2 - k)^2} \int d^4 \pi e^{-(\pi_1 + \pi_2 A \theta^4 - (\pi_2 - \pi_1) A \theta^4} 
$$

$$
\times \left\{ \left( \frac{[p_1 + k, p_2 - k]}{[p_1, p_2] + (p_1 + p_2)^2} \right) \left( \frac{(p_1 - p_2 + 2k) (p_1 - p_2)}{(p_1 + p_2)^2 + 1} \right) \right. 
$$

$$
- \left( \frac{[p_2, k]}{[p_2 + p_1 + k]^2} \right) \left( \frac{(2p_1 + k) (2p_2 - k) + 1 + 4k^2 \theta^4}{k^2 (p_2 + k)^2} \right) 
$$

$$
+ 2 \left( \frac{[p_2, k]}{[p_2 + p_1 + k]^2} \right) \left( \frac{[p_1, k]}{[p_1 + p_2 + k]^2} \right) 
$$

$$
- \left( \frac{[p_2, p_1 + k]}{[p_2 + p_1 + k]^2} \right) \left( \frac{[p_2, p_1 - k]}{[p_2 + p_1 - k]^2} \right) \right\}.
$$

(4.25)

Here the term involving transverse components of the loop momentum, $k^2_\perp$, comes from the diagram with triple coupling shown in Fig. 1(a) and the rest—from the diagram with quartic coupling, Fig. 1(d).

Eq. (4.25) involves the square bracket between two (super)momenta defined in (A.22). Using its properties, the expression inside the curly brackets in (A.22) can be simplified as described in Appendix D4. Remarkably enough, it can be brought to the very same form as in Eq. (4.16). Namely, it is given by $[\cdots]_{N=4}$ with

$$
\chi_{N=4} = \frac{1}{3!} \epsilon^{ABCD} \left( \pi - \varpi \right)_A \left( \pi - \varphi \right)_B \left( \pi - \varphi \right)_C \left( \pi_1, Dp_2 + \pi_2, Dp_1 \right) 
$$

$$
\left( p_1 + k_+ (p_2 + k_+) (p_1 + p_2) \right),
$$

(4.26)

where the odd momentum $\varpi_A$ was defined in (4.17). Eq. (4.26) generalizes the expression for $\chi_{N=2}$, Eq. (4.17), for $N = 4$. This suggests that (4.25) can be obtained from the similar matrix element for $N = 2$, Eq. (4.18), by simply extending the formulae to $N = 4$. We confirm this by an explicit calculation of (4.25) in Appendix D4. Thus, the one-loop evolution kernel in the $N = 4$ SYM theory is given by

$$
\mathbb{H}_{\Phi \Phi} \Phi(Z_1) \Phi(Z_2) \bigg|_{N=4} = \left\{ \Psi^{(-1/2,-1/2)} (1 - \Pi_{\Phi \Phi}) + \Delta_{\Phi \Phi} \right\} \Phi(Z_1) \Phi(Z_2),
$$

(4.27)

where $Z = (z, \theta^A)$ with $A = 1, \ldots, 4$ and the operators $\Psi^{(-1/2,-1/2)}$, $\Pi_{\Phi \Phi}$ and $\Delta_{\Phi \Phi}$ were introduced in Eqs. (1.10), (3.41) and (4.21), respectively.

The operator $\mathbb{H}_{\Phi \Phi}$, Eq. (4.27), has the same form as the evolution kernel for $N = 2$ in the $\Phi \Phi-$sector, Eq. (1.22). In fact, the two operators would coincide if one formally put $N = 4$ in (1.22). As we will show in Sect 4.3, this property is not accidental and is one of the consequences of a general relation between the evolution kernels in the $N = 4$ and $N = 2$ SYM theories. Finally, projecting (4.27) onto the subspace of Wilson operators (3.70) we obtain the same expression for $\mathbb{H}^W_{\Phi \Phi}$ as before, Eq. (4.24).
4.2. Mixed sector

The two-particle kernel allows one to construct the one-loop dilatation operator in the $\mathcal{N} = 4$ SYM theory. For $\mathcal{N} \leq 2$ the two-particle kernel is given by a $2 \times 2$ matrix. Its diagonal entries, $H_{\phi\psi}$ and $H_{\psi\phi}$, are given by $\epsilon(4.24)$ and $\epsilon(4.23)$. In this section, we calculate the two-particle kernels in the $\Phi\Psi$ and $\Psi\Phi$ sectors, $H_{\phi\psi}$ and $H_{\psi\phi}$, respectively.

To start with, we examine one-loop corrections to the matrix element $\langle 0| \mathrm{tr}\{\Phi(Z_1)\Psi(Z_2)\} | P \rangle$ defined by the Feynman diagrams shown in Fig. 2. As before, the self-energy diagram in Fig. 2(f) gives rise to the anomalous dimension of the superfield while the annihilation diagram in Fig. 2(d) vanishes after integration over the transverse components of the loop momentum. The diagrams in Fig. 2(a) and (b) describe the transition $\Phi\Psi \rightarrow \Phi\Psi$, the diagram in Fig. 2(c) describes the transition $\Phi\Psi \rightarrow \Psi\Phi$ and the diagram in Fig. 2(c) contributes to both.

For the sake of simplicity, we first consider the $\mathcal{N} = 0$ theory. In this case, the superspace does not have “odd” directions and coincides with the light-cone, $Z = z$. Calculating the Feynman diagrams shown in Fig. 2(a), (b), (c) and (e), one finds that the one-loop correction to the matrix element $\langle 0| \mathrm{tr}\{\Phi(Z_1)\Psi(Z_2)\} | P \rangle$ can be split into a sum of two terms corresponding to the $\Phi\Psi \rightarrow \Phi\Psi$ and $\Phi\Psi \rightarrow \Psi\Phi$ transitions. The details of calculations can be found in Appendix D5. The final result for the one-loop correction to $\langle O_{\phi\psi} \rangle$ in the channel $\Phi\Psi \rightarrow \Phi\Psi$ is given by

$$
\langle O_{\Phi\Psi}^{(1)}(Z_1, Z_2, ...) \rangle_{\Phi\Psi \rightarrow \Phi\Psi} = -\frac{g^2 N_c \mu^2 \varepsilon}{(4\pi)^2} \left\{ \mathcal{V}_{\Phi\Psi} (1 - \Pi_{\Phi\Psi}) \langle O_{\Phi\Psi}^{(0)}(Z_1, Z_2, ...) \rangle + \Delta^{(N=0)}_{\phi\psi} \langle O_{\phi\psi}^{(0)}(Z_1, Z_2, ...) \rangle \right\},
$$

and in the channel $\Phi\Psi \rightarrow \Psi\Phi$

$$
\langle O_{\phi\psi}^{(1)}(Z_1, Z_2, ...) \rangle_{\Phi\Psi \rightarrow \Psi\Phi} = -\frac{g^2 N_c \mu^2 \varepsilon}{(4\pi)^2} \left\{ \mathcal{W}_{\phi\psi} (1 - \Pi_{\phi\psi}) \langle O_{\phi\psi}^{(0)}(Z_1, Z_2, ...) \rangle - \Delta^{(N=0)}_{\phi\psi} \langle O_{\phi\psi}^{(0)}(Z_2, Z_1, ...) \rangle \right\}.
$$

Here $\langle O_{\phi\psi}^{(0)}(Z_2, Z_1, ...) \rangle \equiv \langle 0| \mathrm{tr}\{\Psi(Z_2)\Phi(Z_1)\} | P \rangle$ and the superscript $(0)$ indicates the Born level approximation, Eq. (1.14), that is, the product of the plane waves. In Eq. (4.29), the notation was introduced for the integral operators $\mathcal{V}_{\phi\psi}$ and $\mathcal{W}_{\phi\psi}$

$$
\mathcal{V}_{\phi\psi} O_{\phi\psi}(Z_1, Z_2, ...) = \int_0^1 d\alpha \left[ 2 O_{\phi\psi}(Z_1, Z_2, ...) - (1 - \alpha)^2 O_{\phi\psi}(Z_1, \alpha Z_1 + (1 - \alpha)Z_2, ...) - (1 - \alpha)^{-2} O_{\phi\psi}((1 - \alpha)Z_1 + \alpha Z_2, Z_2, ...) \right].
$$

$$
\mathcal{W}_{\phi\psi} O_{\phi\psi}(Z_1, Z_2, ...) = -\int_0^1 d\alpha \left[ \frac{\alpha^2}{(1 - \alpha)^2} O_{\phi\psi}((1 - \alpha)Z_1 + \alpha Z_2, Z_2, ...) \right].
$$

As before, they only act on the first two arguments of a test function $O(Z_1, Z_2, ...)$. Notice that the operator $\mathcal{W}_{\phi\psi}$ interchanges the superfields inside the trace. Comparison with (1.10) and (1.11) allows one to identify these operators as $\mathcal{V}_{\phi\psi} = \mathcal{V}^{-1/2,j_{\psi}}$ and $\mathcal{W}_{\phi\psi} = -\mathcal{V}^{-1/2,j_{\phi}}$ with $j_{\psi} = 3/2$ for $\mathcal{N} = 0$. The operator $\Pi_{\phi\psi}$ is the projector defined in Eq. (3.34). Finally, $\Delta^{(N=0)}_{\phi\psi}$ is the following operator

$$
\Delta^{(N=0)}_{\phi\psi} O(Z_1, Z_2) = (2 - Z_{12} \partial Z_2) \left[ \frac{\partial_{1+}}{\partial_{1+} + \partial_{2+}} O(Z_1, Z_2) \right]_{Z_1 = Z_2}.
$$
where $\partial_{k+} = \partial/\partial z_k$ is the light-cone derivative and the operator $Z_{12}\partial_{Z_2}$ is applied to the square bracket evaluated for $Z_1 = Z_2$. Notice that in Eq. (1.29) the operator $\Delta^{(N=0)}_{\Phi\Psi}$ is applied to the matrix element with the arguments $Z_1$ and $Z_2$ interchanged, that is $O(Z_1, Z_2) = \langle O^{(0)}_{\Psi\Phi}(Z_2, Z_1, ...) \rangle$.

The total one-loop correction to $\langle O^{(1)}_{\Psi\Phi}(Z_1, Z_2, ...) \rangle$ is a sum of the two expressions, Eqs. (1.28) and (1.29). Its matching into (4.6) yields the two-particle evolution kernel in the $\Phi\Psi$-sector in the $N = 0$ theory

$$
\mathcal{H}_{\Phi\Psi} \Phi(Z_1)\Psi(Z_2) = (\nabla^{(-1/2,j_\Psi)} - \nabla^{(-1/2,j_\Psi)}_{\text{ex}})(1 - \Pi_{\Phi\Psi}) \Phi(Z_1)\Psi(Z_2)
+ \Delta_{\Phi\Psi} (\Phi(Z_1)\Psi(Z_2) - \Psi(Z_2)\Phi(Z_1)).
$$

(4.33)

This relation follows from the explicit evaluation of the Feynman diagrams in the $N = 0$ theory shown in Fig. 2. Going over through the calculation of the same diagrams in the $N = 2$ theories one finds that the evolution kernel $\mathcal{H}_{\Phi\Psi}$ is given by the same expression (4.33) with $j_\Psi$ taking the value $j_\Psi = (3 - N)/2$ which depends on the number of supercharges. Also, the superspace acquires extra "odd" dimensions, $Z = (z, \theta^A)$ with $A = 1, \ldots, N$, and the operator $\Delta_{\Phi\Psi}$ is given for an arbitrary $N$ by

$$
\Delta_{\Phi\Psi} O(Z_1, Z_2) = (2 - N - Z_{12}\partial_{Z_2}) \left[ \frac{\partial_{1+}}{\partial_{1+} + \partial_{2+}} O(Z_1, Z_2) \right] \bigg|_{Z_1=Z_2}.
$$

(4.34)

This operator has the same meaning as the operator $\Delta_{\Phi\Psi}$, Eq. (1.21). It contributes to the scale dependence of composite operators involving nonlocal fields and has a vanishing projection onto the subspace of Wilson operators $\Pi_{W}\Delta_{\Phi\Psi} O(Z_1, Z_2, ...) = 0$. Therefore, in agreement with our expectations (3.53), the one-loop evolution kernel for Wilson operators in the $\Phi\Psi$-sector is given by

$$
\mathcal{H}_{\Phi\Psi}^W = \Pi_{W}\mathcal{H}_{\Phi\Psi} = \Pi_{W} \left[ \nabla^{(-1/2,j_\Psi)} - \nabla^{(-1/2,j_\Psi)}_{\text{ex}} \right] (1 - \Pi_{\Phi\Psi}).
$$

(4.35)

To identify the evolution kernel in the $\Psi\Phi$-sector one has to calculate one-loop corrections to the matrix element $\langle 0 | tr \{ \Psi(Z_1)\Phi(Z_2) \} | P \rangle$. The only difference with the previous case is that one has to interchange the two superfields inside the trace. Denoting the corresponding permutation operator as $\mathbb{P}_{\Phi\Psi}$

$$
\Psi(Z_1)\Phi(Z_2) = \mathbb{P}_{\Phi\Psi} \Phi(Z_1)\Psi(Z_2),
$$

(4.36)

one finds that the evolution kernel in the $\Psi\Phi$-sector is related to (4.35) as

$$
\mathcal{H}_{\Psi\Phi}^W = \mathbb{P}_{\Phi\Psi} \mathcal{H}_{\Phi\Psi}^W \mathbb{P}_{\Phi\Psi}^\dagger = \Pi_{W} \left[ \nabla^{(j_\Psi, -1/2)} - \nabla^{(j_\Psi, -1/2)}_{\text{ex}} \right] (1 - \Pi_{\Psi\Phi}),
$$

(4.37)
where the operators $\mathcal{V}(j_{\psi},-1/2)$, $\mathcal{V}_{ex}(j_{\psi},-1/2)$ and $\Pi_{\Psi}$ are given by Eqs. (4.10), (3.37) and (3.50), respectively, with $j_{\psi} = (3 - N)/2$.

### 4.3. Relation between $N = 4$ and $N \leq 2$

According to (4.22) and (4.27), the one-loop evolution kernel in the $\Phi\Phi$–sector has the same, universal form in the SYM theories with $N = 4$ and $N \leq 2$. To understand the origin of this property we remind that the SYM theories with different number of supercharges $N$ are related to each other via the reduction procedure described in Sect. 2.1.

In the Mandelstam formulation, the decomposition of the $N = 4$ superfield over the $N = 2$ superfields looks as follows (see Eqs. (2.11) and (2.11))

$$
\Phi^{(4)}(z, \mu, \theta^A, \theta^3, \theta^4) = \Phi^{(2)}(Z) + \theta^3 \Psi_{WZ}^{(2)}(Z) - \theta^4 \theta_{\mu}^{-1} D_1 D_2 \overline{\Psi}_{WZ}^{(2)}(Z) - \theta^3 \theta^4 \Psi^{(2)}(Z).
$$

where $Z = (z, \theta^A)$ with $A = 1, 2$. We would like to stress that as long as one retains in (4.38) the contribution of the Wess-Zumino superfields, $\Psi_{WZ}^{(2)}$ and $\overline{\Psi}_{WZ}^{(2)}$, the dilatation operators in the $N = 4$ SYM theory and the $N = 2$ SYM theory coupled to the Wess-Zumino superfields are identical. In particular, substituting (4.38) into (4.27) and comparing the coefficients in front of $\theta^3$ and $\theta^4$ in both sides of (4.27), one can identify the two-particle kernels in the various sectors including $\Phi^{(2)} \Phi^{(2)}$, $\Psi^{(2)} \Psi^{(2)}$, $\Psi^{(2)} \Phi^{(2)}$ and $\Phi^{(2)} \Psi^{(2)}$-sectors. In general, these kernels should be different from the same kernels in the $N = 2$ SYM theory since the former receive a nontrivial contribution from the Wess-Zumino superfields. Therefore, in order to derive the evolution kernels in the $N = 2$ theory from the one in the $N = 4$, Eq. (4.27), via the truncation procedure one has to eliminate from the latter kernel the contribution of the superfields $\Psi_{WZ}^{(2)}$ and $\overline{\Psi}_{WZ}^{(2)}$.

For this purpose, it is not sufficient to put $\Psi_{WZ}^{(2)} = \overline{\Psi}_{WZ}^{(2)} = 0$ in (4.27) and (4.38), since the Wess-Zumino superfields could propagate along the internal line in Fig. 1(a)–(c) and inside the loop in Fig. 1(e). In the latter case, the Wess-Zumino superfields contribute to the self-energy and their elimination affects the $\beta$–function of the SYM theory (see Eq. (4.11)). In the former case, since the superfields $\Psi_{WZ}^{(2)}$ and $\overline{\Psi}_{WZ}^{(2)}$ are fermionic, they could couple to bosonic superfields only in pairs and, therefore, can contribute starting from the two-loop level. Thus, going over from the $N = 4$ to $N = 2$ SYM theory, one can safely put $\Psi_{WZ}^{(2)} = \overline{\Psi}_{WZ}^{(2)} = 0$ in (4.38), adjust the value of the $\beta$–function and apply (4.27) to evaluate the one-loop dilatation operator in the $N = 2$ SYM.

Let us apply the reduction procedure to reproduce the $N = 2$ two-particle evolution kernels in different sectors. To obtain the $N = 2$ kernel in the $\Phi\Phi$–sector, one puts $\theta^3_k = \theta^4_k = 0$ (with $k = 1, 2$) in both sides of (4.27). According to (4.38) the product of the superfields reduces to $\Phi^{(2)}(Z_1)\Phi^{(2)}(Z_2)$ leading to

$$
\mathbb{H}^{(4)}_{\Phi\Phi} [\Phi^{(2)}(Z_1)\Phi^{(2)}(Z_2)] = \mathbb{H}^{(2)}_{\Phi\Phi} [\Phi^{(2)}(Z_1)\Phi^{(2)}(Z_2)].
$$

The operator $\mathbb{H}^{(2)}_{\Phi\Phi}$ defined in this way takes the same form as before, Eq. (4.27), but the number of odd dimensions in the superspace is reduced from $N = 4$ to $N = 2$, $Z_k = (z_k, \theta^1_k, \theta^2_k)$. As a result, $\mathbb{H}^{(2)}_{\Phi\Phi}$ coincides with the one-loop $N = 2$ evolution kernel in the $\Phi\Phi$–sector, Eq. (4.22).

In a similar manner, to obtain the $N = 2$ evolution kernel in the $\Psi\Psi$–sector one has to retain in (4.38) the contribution of the $\Psi$–superfield. One substitutes $\Phi(Z_k) = -(\theta \cdot \theta)_k \Psi^{(2)}(Z_k)$ with
Let us demonstrate the relation of our approach based on non-local light-cone operators with the conventional one that deals with local Wilson operators. To this end, we will show how the obtained expressions for the one-loop dilatation operator allow one to evaluate anomalous dimensions of various Wilson operators in $\mathcal{N}$–extended SYM theories.

5. **Anomalous dimensions of Wilson operators**

Let us demonstrate the relation of our approach based on non-local light-cone operators with the conventional one that deals with local Wilson operators. To this end, we will show how the obtained expressions for the one-loop dilatation operator allow one to evaluate anomalous dimensions of various Wilson operators in $\mathcal{N}$–extended SYM theories.
We recall that the Wilson operators of the maximal Lorentz spin, or simply quasiparmonic operators, are local gauge-invariant single-trace operators built from transverse components of the strength tensor \( F_{\mu\nu} \), “good” components of fermions, \( \psi_+^A \) and \( \bar{\psi}_{+A} \), scalar fields, \( \phi^{AB} \) and \( \bar{\phi}_{AB} \), and covariant derivatives \( D_+ \equiv \partial_+ n^\mu \) acting on these fields. In the light-cone formalism, in the light-like gauge \( A_+(x) = 0 \), the same operators are constructed from gauge fields, \( \partial_+ A \) and \( \partial_+ \bar{A} \), Grassman fields, \( \lambda^A \) and \( \bar{\lambda}_A \), complex scalars, \( \phi^{AB} \) and \( \bar{\phi}_{AB} \), and light-cone derivatives \( n^\mu D_\mu \equiv \partial_\perp \). The relation between the two sets of fields looks as follows. The gauge strength tensor \( F_{\mu\nu} = (F_{+x}, F_{+y}) \) and its dual \( \tilde{F}_{\mu\nu} = \frac{i}{\sqrt{2}} \varepsilon_{\mu\nu\rho\lambda} F^{\rho\lambda} \) are expressed in terms of the helicity \( \pm 1 \) gauge fields (2.1):

\[
F_{+x} = \tilde{F}_{+y} = \frac{1}{\sqrt{2}} (\partial_+ A + \partial_+ \bar{A}), \quad F_{+y} = -\tilde{F}_{+x} = -\frac{i}{\sqrt{2}} (\partial_+ A - \partial_+ \bar{A}).
\]

The “good” components of Majorana fermions, \( \psi_+^A \) and \( \bar{\psi}_{+A} \), are expressed in terms of helicity \( \pm 1/2 \) Grassmann fields (see Eqs. (A.5), (A.10) and (A.11))

\[
\psi_+^A = \sqrt{2} \begin{pmatrix} \lambda^A \\ 0 \\ 0 \\ i\bar{\lambda}_A \end{pmatrix}, \quad \bar{\psi}_{+A} = -\sqrt{2} \begin{pmatrix} 0, \lambda^A, i\bar{\lambda}_A, 0 \end{pmatrix}.
\]

Using these relations, one can establish the correspondence between the Wilson operators in the covariant and light-cone formulations. As an example, we present expressions for a few twist-two operators in the \( \mathcal{N} = 4 \) theory: parity even/odd fermion operators

\[
O^q_N(0) = \text{tr} \{ \bar{\psi}_A \gamma^+ (iD_+)^{N-1} \psi^A \} = 2i^{N-2} \text{tr} \{ \bar{\lambda}_A \partial_+^{N-1} \lambda^A + \lambda^A \partial_+^{N-1} \bar{\lambda}_A \},
\]

\[
\tilde{O}^q_N(0) = \text{tr} \{ \bar{\psi}_A \gamma^5 (iD_+)^{N-1} \psi^A \} = 2i^{N-2} \text{tr} \{ \bar{\lambda}_A \partial_+^{N-1} \lambda^A - \lambda^A \partial_+^{N-1} \bar{\lambda}_A \},
\]

parity even/odd gauge field operators

\[
O^g_N(0) = \text{tr} \{ F_{+\nu} (iD_+)^{N-2} F_{+\nu} \} = i^{N-2} \text{tr} \{ \partial_+ A \partial_+^{N-1} \bar{A} + \partial_+ \bar{A} \partial_+^{N-1} A \},
\]

\[
\tilde{O}^g_N(0) = \text{tr} \{ F_{+\nu} (iD_+)^{N-2} i\tilde{F}_{+\nu} \} = i^{N-2} \text{tr} \{ \partial_+ A \partial_+^{N-1} \bar{A} - \partial_+ \bar{A} \partial_+^{N-1} A \},
\]

and scalar operators

\[
O^s_N(0) = \text{tr} \{ \bar{\phi}_{AB} (i\partial_+)^N \phi^{AB} \}.
\]

In the light-cone formalism, one obtains the Wilson operators by expanding the nonlocal light-cone operators (1.3)–(1.5) in powers of even, \( z_k \), and odd variables, \( \theta^A_k \), Eqs. (1.2) and (1.6). The light-cone operators satisfy the evolution equation (1.7) with the one-loop dilatation operator given in the multi-color limit by (1.8) and (1.9). To reconstruct the mixing matrix for the Wilson operators, one has to substitute (1.2) and (1.6) into the evolution equation (1.7) and equate the coefficients in front of different powers of \( z \)'s and \( \theta \)'s in both sides of (1.7). We illustrate below this procedure by calculating the mixing matrices for various Wilson operators in the SYM theories with \( \mathcal{N} = 0, 1, 2, 4 \).

### 5.1. Wilson operators in \( \mathcal{N} = 0 \) theory

In the \( \mathcal{N} = 0 \) theory, that is, pure gluodynamics with the \( SU(N_c) \) gauge group, the light-cone fields are given by (see Eq. (2.27))

\[
\Phi(z) = \partial_+^{-1} A(z), \quad \Psi(z) = -\partial_+ \bar{A}(z), \quad (5.6)
\]
with $A(z)$ and $\tilde{A}(z) = A^*(z)$ being the gauge fields of helicity $+1$ and $-1$, respectively. The conventional local Wilson operators arise from the Taylor expansion of the light-cone operators $O(z_1, \ldots, z_L)$ in the light-cone separations. For the light-cone operators built only from $\Psi-$ or $\Phi-$fields, Eq. (1.3) and (1.4), the corresponding Wilson operators belong to the sector of the aligned-helicity gluon operators.

**$\Psi\Psi-$sector**

The product of two $\Psi-$fields can be expanded as

$$\Psi(z_1)\Psi(z_2) = \sum_{j_1,j_2\geq 0} \frac{z_{j_1}^1 z_{j_2}^2}{j_1! j_2!} O_{j_1j_2}(0) = \sum_{j=0}^{\infty} \frac{1}{j!} \sum_{k=0}^{j} \binom{j}{k} z_{j-k}^1 z_{-k}^2 O_{k,j-k}(0) , \tag{5.7}$$

where the notation was introduced for the aligned-helicity Wilson operators

$$O_{j_1j_2}(0) = \partial^{j_1+1}_+ \tilde{A}(0) \partial^{j_2+1}_+ \tilde{A}(0) = \partial^{j_1}_1 \partial^{j_2}_2 \Psi(z_1)\Psi(z_2) \bigg|_{z_i=0} \tag{5.8}$$

with $\tilde{A}(0) = \tilde{A}^a(0)t^a$ being a matrix of dimension $N_c$.

To one-loop order, $O_{j_1j_2}(0)$ mix under renormalization with the operators $O_{k_1k_2}(0)$ carrying the same Lorentz spin $j = j_1 + j_2 = k_1 + k_2$. The corresponding mixing matrix $V_{j_1j_2}^{k_1k_2}$ is related to the two-particle dilatation operator in the $\Psi\Psi-$sector, Eq. (4.13), as

$$\mathbb{H}_{\Psi\Psi} O_{j_1j_2}(0) = \sum_{k_1+k_2=j_1+j_2} V_{j_1j_2}^{k_1k_2} O_{k_1k_2}(0) . \tag{5.9}$$

For a given $j = j_1 + j_2$, there are $(j+1)$ operators (5.8) so that the mixing matrix has dimension $(j+1)$. Since the Lorentz spin takes the values $0 \leq j < \infty$, the matrix $V_{j_1j_2}^{k_1k_2}$ may have an arbitrary large size. To find this matrix, one substitutes the expansion (5.7) into the expression for the one-loop dilatation operator $\mathbb{H}_{\Psi\Psi}$ at $N = 0$, Eq. (4.13), and equates the coefficients in front of $z_{j_1}^1 z_{j_2}^2$

$$\mathbb{H}_{\Psi\Psi} O_{j_1j_2}(0) = \int_0^1 \frac{d\alpha}{\alpha} [2\partial^{j_1}_1 \partial^{j_2}_2 - \tilde{\alpha}^{j_1+2}\alpha \partial^{j_1}_1 + \tilde{\alpha}^{j_2+2}\alpha \partial^{j_2}_2] \Psi(z_1)\Psi(z_2) \bigg|_{z_i=0} , \tag{5.10}$$

where $\tilde{\alpha} \equiv 1 - \alpha$. This relation establishes the correspondence between the mixing matrix for local Wilson operators and the evolution kernel for nonlocal light-cone operators. Matching (5.10) into (5.9) with a help of (5.8), one calculates the mixing matrix

$$V_{j_1j_2}^{k_1k_2} = \delta_{j_1k_1} \delta_{j_2k_2} [\psi(j_1 + 3) + \psi(j_2 + 3) - 2\psi(1)] - \delta_{j_1+j_2,k_1+k_2} \left[ \frac{\theta_{j_2,k_2} j_2! (j_1+2)!}{j_2-k_2(k_1+2)!} + \frac{\theta_{j_1,k_1} j_1! (j_2+2)!}{j_1-k_1(k_2+2)!} \right] . \tag{5.11}$$

The eigenvalues of this matrix determine the spectrum of anomalous dimensions of the aligned-helicity Wilson operators (5.8).

According to (5.9), the matrix $V_{j_1j_2}^{k_1k_2}$ defines a representation of the dilatation operator on the space spanned by the Wilson operators (5.8). The choice of the basis of local operators in this space is not unique. In order to reveal symmetries of the mixing matrix imposed by the $SL(2)$
invariance of the dilatation operator, one switches to the basis of conformal operators \[15\]. The conformal gauge operators are linear combinations of the operators (5.8)

\[
O_j(0) = \sum_{k=0}^{j} c_{jk} O_{j-k,k} = \partial_+ \bar{A}(0) \left( \partial_+ + \bar{\partial}_+ \right)^j C_j^{5/2} \left( \frac{\partial_+ - \bar{\partial}_+}{\partial_+ + \bar{\partial}_+} \right) \partial_+ \bar{A}(0),
\]

(5.12)

which are expressed in terms of the Gegenbauer polynomials. The expansion coefficients \(c_{jk}\) are fixed from the condition that \(O_j(0)\) have to be the lowest-weight vectors in the tensor product of two \(SL(2)\) moduli. In the conformal basis, the expansion of nonlocal light-cone operator (5.7) looks as follows

\[
\Psi(z_1) \Psi(z_2) = \sum_{j=0}^{\infty} c_j z_{21}^j \int_0^1 d\alpha (\alpha \bar{\alpha})^{j+2} O_j(\alpha z_1 + \alpha z_2),
\]

(5.13)

where \(c_j = 12(2j + 5)/\Gamma(j + 3)\) and \(z_{21} = z_2 - z_1\). A unique feature of the conformal operators is that the mixing matrix (5.9) is diagonal in this basis

\[
\mathbb{H}_{\Psi \Psi} O_j(0) = \gamma_{\Psi \Psi}(j + 3) O_j(0).
\]

(5.14)

The expansion coefficients \(c_{jk}\) entering (5.12) define (left) eigenstates of the mixing matrix and the corresponding anomalous dimension \(\gamma_{\Psi \Psi}(j)\), given below in Eq. (5.19), can be calculated using (5.11).  

7Strictly speaking, \(\gamma_{\Psi \Psi}(j)\) is the eigenvalue of the two-particle dilatation operator rather than anomalous dimension. The two are related to each other, see Eqs. (6.3) and (6.1) below.

There exist a much simpler way of calculating \(\gamma_{\Psi \Psi}(j)\). The conformal operators (5.12) can be written in the following form

\[
O_j(0) = a_j \partial_+ \bar{A} \partial_+^{j+1} \bar{A}(0) + b_j \partial_+ (\partial_+ \bar{A} \partial_+^{j} \bar{A}(0)) + \cdots,
\]

(5.15)

where the expansion goes over local operators involving total derivatives and \(a_j, b_j, \ldots\) are some coefficients. The conformal invariance allows one to reconstruct the whole sum out of the first term only. One can neglect all operators with the total derivatives by going over to the so-called forward limit. It amounts to taking the forward matrix element of the conformal operators with respect to some reference state

\[
\langle P | O_j(0) | P \rangle = a_j \langle P | \partial_+ \bar{A} \partial_+^{j+1} \bar{A}(0) | P \rangle,
\]

(5.16)

since \(\langle P | \partial_+ (\ldots) | P \rangle = 0\). This does not affect the anomalous dimension (5.14), but allows one to substitute the conformal operator inside the forward matrix element by a simple operator \(\partial_+ \bar{A} \partial_+^{j+1} \bar{A}(0)\). We will accept this strategy in the remainder of this section.

The expansion (5.13) looks in the forward limit as

\[
\Psi(z_1) \Psi(z_2) \overset{\text{fw}}{=} \sum_{j=0}^{\infty} \frac{z_{21}^j}{j!} \partial_+ \bar{A} \partial_+^{j+1} \bar{A}(0).
\]

(5.17)

Hereafter \(\overset{\text{fw}}{=}\) means that the relation is only valid upon taking the forward matrix element, that is, up to contribution of Wilson operators involving total derivatives. Substituting (5.17) into (4.13) one finds after a simple calculation

\[
\mathbb{H}_{\Psi \Psi} \Psi(z_1) \Psi(z_2) \overset{\text{fw}}{=} \sum_{j=0}^{\infty} \frac{z_{21}^j}{j!} \gamma_{\Psi \Psi}(j + 3) \partial_+ \bar{A} \partial_+^{j+1} \bar{A}(0),
\]

(5.18)

\[\]
where the anomalous dimension is given by
\[
\gamma_{\Phi\Psi}(j + 3) = 2 \int_0^1 \frac{d\alpha}{\alpha} (1 - \alpha^{2+j}) = 2 \left[ \psi(j + 3) - \psi(1) \right].
\] (5.19)

Here \( j + 3 \) is the two-particle \( SL(2) \) conformal spin \( \bar{J}_{12} = j_1 + j_2 \) for \( j_1 = j_2 = j_{\Psi} = 3/2 \). Eqs. (5.14) and (5.19) are agreement with (3.60) for \( N = 0 \). Comparing the coefficients in front of \( z_{21}^j \) in both sides of (5.18) we conclude that
\[
\left[ \mathbb{H}_{\Phi\Psi} - \gamma_{\Phi\Psi}(j + 3) \right] \partial_+ \bar{A} \partial_+^{j+1} \bar{A}(0) \equiv 0.
\] (5.20)

As was already mentioned, the operator \( \mathbb{H}_{\Phi\Psi} \) can be mapped into a two-particle Hamiltonian of the \( SL(2) \) Heisenberg magnet of spin \( j_{\Psi} = 3/2 \) \([10, 11, 12]\).

\( \Phi\Phi \)–sector

The scale dependence of the operator \( \Phi(z_1)\Phi(z_2) \) is driven to one-loop order by the dilatation operator \( \mathbb{H}_{\Phi\Phi} \), Eq. (4.22). In distinction with the previous case, the first two terms of the expansion of the field \( \Phi(z) \) around \( z = 0 \) involve nonlocal, spurious field components
\[
\Phi(z) = \sum_{k=0,1} z^k \partial_+^{k-1} A(0) + \sum_{k=2}^\infty z^k \partial_+^{k-1} A(0) = \Phi_{sp}(z) + \Phi_w(z),
\] (5.21)
with \( (\Phi_w(z))^* = -z^2 \Psi(z) \). Substituting \( \Phi = \Phi_{sp} + \Phi_w \) into (4.22), one can find anomalous dimensions for different components arising in the product \( \Phi(z_1)\Phi(z_2) \). Going over to the forward limit one gets
\[
\Phi(z_1)\Phi(z_2) \equiv \sum_{j=0}^\infty \frac{z_{21}^j}{j!} \partial_+ A \partial_+^{j-3} A(0),
\] (5.22)
where the terms with \( j \leq 3 \) and \( j > 3 \) correspond to spurious and Wilson operators, respectively.

The one-loop dilatation operator (4.22) involves the projector \( \Pi_{\Phi\Phi} \), Eq. (3.44). The action of the operator \( (1 - \Pi_{\Phi\Phi}) \) on the product \( \Phi(z_1)\Phi(z_2) \) amounts to subtracting the first two terms in the expansion (5.22)
\[
(1 - \Pi_{\Phi\Phi})\Phi(z_1)\Phi(z_2) \equiv \sum_{j=2}^\infty \frac{z_{21}^j}{j!} \partial_+ A \partial_+^{j-3} A(0).
\] (5.23)

In addition, one finds that the expansion of the addendum \( \Delta_{\Phi\Phi} \Phi(z_1)\Phi(z_2) \), Eq. (4.21), around \( z_{12} = 0 \) only involves operators with total derivatives and, therefore, it vanishes in the forward limit
\[
\Delta_{\Phi\Phi} \Phi(z_1)\Phi(z_2) \equiv 0.
\] (5.24)

Substituting (5.22) into (4.22) and taking into account the last two relations we find
\[
\mathbb{H}_{\Phi\Phi} \Phi(z_1)\Phi(z_2) \equiv \sum_{j=2}^\infty \frac{z_{21}^j}{j!} \gamma_{\Phi\Psi}(j - 1) \partial_+ A \partial_+^{j-3} A(0),
\] (5.25)
with \( \gamma_{\Psi\Phi}(j) \) defined in (5.19). Comparing the coefficients in front of \( z_{21}^{j} \) in both sides of this relation, we conclude that

\[
\left[ H_{\Phi\Phi} - \gamma_{\Psi\Phi}(j-1)\theta(j-1) \right] \partial_{+}A \partial_{+}^{j-3}A(0) \overset{fw}{=} 0. \tag{5.26}
\]

We recall that the two-particle SL(2) conformal spin in the \( \Phi\Phi \) sector, Eq. (3.29), equals \( J_{12} = j + 2j_{\Phi} \) with \( j_{\Phi} = -1/2 \). One observes that (5.26) is in an agreement with (3.62).

It follows from (5.26) and (5.20) that the anomalous dimensions of the Wilson operators \( \partial_{+}A \partial_{+}^{j-3}A(0) \) (with \( j \geq 4 \)) and complex conjugated operators \( \partial_{+}\bar{A} \partial_{+}^{j-3}\bar{A}(0) \) coincide, as it should be. Nonlocal operators \( \partial_{+}A \partial_{+}^{j-3}A(0) \) with \( j = 0, 1, 2 \) have vanishing anomalous dimensions, while for \( j = 3 \) the anomalous dimension of the operator \( \partial_{+}A A(0) \) equals \( \gamma(2) = 2 \).

**\( \Phi\Psi \) and \( \Psi\Phi \) sectors**

Let us turn to the mixed sector \( \Phi\Psi \) and go right away to the forward limit

\[
\Phi(z_{1})\Psi(z_{2}) \overset{fw}{=} -\sum_{j=0}^{\infty} \frac{z_{21}^{j}}{j!} \partial_{+}A \partial_{+}^{j-1}\bar{A}(0), \tag{5.27}
\]

with \( z_{21} = z_{2} - z_{1} \). This expansion involves spurious \( (j = 0, 1) \) and Wilson operators \( (j \geq 2) \) both built from the gauge fields of opposite helicity. The one-loop dilatation operator for \( \Phi(z_{1})\Psi(z_{2}) \) is given by (4.33). It involves the projector \( \Pi_{\Phi\Psi} \), Eq. (3.48), which eliminates spurious operators in the right-hand side of (5.27)

\[
(1 - \Pi_{\Phi\Psi})\Phi(z_{1})\Psi(z_{2}) \overset{fw}{=} -\sum_{j \geq 2} \frac{z_{21}^{j}}{j!} \partial_{+}A \partial_{+}^{j-1}\bar{A}(0). \tag{5.28}
\]

One substitutes (5.27) into Eqs. (1.9) and (5.64) and takes into account (5.28) to get

\[
\Psi^{(-1/2,3/2)}(1 - \Pi_{\Phi\Psi})\Phi(z_{1})\Psi(z_{2}) \overset{fw}{=} -\sum_{j \geq 2} \frac{z_{21}^{j}}{j!} \gamma_{\Psi\Phi}(j + 1)\partial_{+}A \partial_{+}^{j-1}\bar{A}(0), \tag{5.29}
\]

and analogously for the exchanged kernel, Eqs. (1.9) and (1.11),

\[
\Psi_{\text{ex}}^{(-1/2,3/2)}(1 - \Pi_{\Phi\Psi})\Phi(z_{1})\Psi(z_{2}) \overset{fw}{=} -\sum_{j \geq 2} \frac{z_{21}^{j}}{j!} \gamma_{\Psi\Phi}^{(\text{ex})}(j + 1)\partial_{+}\bar{A} \partial_{+}^{j-1}A(0). \tag{5.30}
\]

The anomalous dimensions entering into these relations are given by

\[
\begin{align*}
\gamma_{\Psi\Phi}(j) &= \psi(j+2) + \psi(j+1) - 2\psi(1), \\
\gamma_{\Psi\Phi}^{(\text{ex})}(j) &= \frac{6}{(j-2)(j-1)j(j+1)} = \frac{\Gamma(j-2)}{\Gamma(j+2)}\Gamma(4).
\end{align*}
\tag{5.31}
\]

Combining together (5.29) and (5.30), we obtain from (1.9) for \( j \geq 2 \)

\[
H_{\Phi\Psi} \partial_{+}A \partial_{+}^{j-1}\bar{A}(0) \overset{fw}{=} \gamma_{\Psi\Phi}(j + 1)\partial_{+}A \partial_{+}^{j-1}\bar{A}(0) - \gamma_{\Psi\Phi}^{(\text{ex})}(j + 1)\partial_{+}\bar{A} \partial_{+}^{j-1}A(0). \tag{5.32}
\]
As in the previous case, Eq. (5.24), the operator \( \Delta_{\Phi} \), Eq. (4.32), does not contribute to anomalous dimensions of the Wilson operators.

Repeating a similar analysis in the \( \Psi\Phi \)–sector, one finds that in virtue of (4.37), \( \mathbb{H}_{\Phi\Psi}\partial_+ \tilde{A} \partial_+^{-1} A \) is given by the same expression with the fields \( A \) and \( \tilde{A} \) interchanged in the right-hand side of (5.32). Let us rewrite (5.32) in the following form (for \( j \geq 2 \))

\[
\left[ \mathbb{H}_{\Phi\Psi} - \left( \gamma_{\Phi\Psi}(j + 1) - \gamma_{\Phi\Psi}(j + 1)\mathbb{P}_{\Phi\Psi} \right) \right] \partial_+ A \partial_+^{-1} \tilde{A}(0) \overset{\text{fw}}{=} 0, \tag{5.33}
\]

where the permutation operator \( \mathbb{P}_{\Phi\Psi} \) interchanges the gauge fields, \( \mathbb{P}_{\Phi\Psi}\partial_+ A \partial_+^{-1} \tilde{A} = \partial_+ \tilde{A} \partial_+^{-1} A \). The operators \( \partial_+ A \partial_+^{-1} \tilde{A}(0) \) carry the conformal spin \( \mathbb{J}_{12} = j + j_\Psi + j_\Phi \) with \( j_\Phi = -1/2 \) and \( j_\Psi = 3/2 \). Setting \( j + 1 = \mathbb{J}_{12} \) in (5.33) one recovers (3.63) for \( \mathcal{N} = 0 \).

According to (5.32), the Wilson operators \( \partial_+ A \partial_+^{-1} \tilde{A}(0) \) (with \( j \geq 2 \)) mix under renormalization with the operators \( \partial_+ \tilde{A} \partial_+^{-1} A(0) \). One can resolve the mixing by considering their linear combinations \( \partial_+ \tilde{A} \partial_+^{-1} A(0) \pm \partial_+ A \partial_+^{-1} \tilde{A}(0) \), which diagonalize the permutation operator \( \mathbb{P}_{\Phi\Psi} \).

In the special case of the twist-two operators, Eq. (5.4), one finds from (5.33)

\[
\left[ \mathbb{H}_{\Phi\Psi} - \gamma_{\mathcal{N}=0}(j) \right] \text{tr}\{\partial_+ A \partial_+^{-1} \tilde{A}(0)\} \overset{\text{fw}}{=} 0, \tag{5.34}
\]

with the anomalous dimension

\[
\gamma_{\mathcal{N}=0}(j) = \psi(j + 3) + \psi(j - 1) - 2\psi(1) - \frac{6(-1)^j}{(j + 2)(j + 1)j(j - 1)}. \tag{5.35}
\]

For even/odd \( j \), Eq. (5.34) defines the anomalous dimensions of the parity even/odd operators \( \gamma^{(0)(j)} \) and \( \gamma^{(1)(j)} \), respectively, which are in agreement with the known results [36] [23]. Eq. (5.35) can be obtained from the general relation (3.63) by taking into account that \( c_{\mathcal{N}} = 2 \) for \( \mathcal{N} = 0 \) if \( \mathbb{J}_{\Phi\Psi} = c_{\mathcal{N}} - 1 + j = j + 1 \) and \( \mathbb{P}_{\Phi\Psi} = (-1)^j \).

### 5.2. Wilson operators in \( \mathcal{N} = 1 \) and \( \mathcal{N} = 2 \) theories

Let us now extend the analysis to supersymmetric gauge theories and start with a simplest supersymmetric extension of gluodynamics, the \( \mathcal{N} = 1 \) SYM theory. The light-cone superfields are

\[
\Phi(Z_1) = \partial_+^{-1} A(z_1) + \theta_1 \partial_+^{-1} \tilde{A}(z_1), \quad \Psi(Z_2) = -\lambda(z_2) + \theta_2 \partial_+ \tilde{A}(z_2),
\]

with \( Z_k = (z_k, \theta_k) \) (for \( k = 1, 2 \)). The product of superfields can be expanded in both \( z \)'s and \( \theta \)'s. Expansion in powers of the light-cone variables \( z_1 \) and \( z_2 \) generates Wilson operators of arbitrary Lorentz spin while the expansion in powers of \( \theta_1 \) and \( \theta_2 \) produces operators of various partonic content. Supersymmetry imposes restrictions on the mixing matrices of these operators. As before, to simplify analysis, we shall take the forward limit and neglect operators involving total light-cone derivatives.
In the $\Psi\Psi$–sector, the product of two superfields admits the following expansion in terms of local operators in the forward limit

$$
\Psi(Z_1)\Psi(Z_2)_{\text{fw}} = \sum_{j=0}^{\infty} \frac{z^j}{j!} \left\{ \lambda \partial^j_+ \lambda(0) + \theta_2 \cdot \lambda \partial^{j+1}_+ \bar{A}(0) - \theta_1 \cdot \partial_+ \bar{A} \partial^j_+ \lambda(0) + \theta_1 \theta_2 \cdot \partial_+ \bar{A} \partial^{j+1}_+ \bar{A}(0) \right\}.
$$

The scale dependence of this product is driven to one-loop order by the dilatation operator $H_{\Psi\Psi}$, Eq. (4.13). Substitution of (5.36) into (4.13) yields

$$
H_{\Psi\Psi} \Psi(Z_1)\Psi(Z_2)_{\text{fw}} = \sum_{j=0}^{\infty} \frac{z^j}{j!} \left\{ \gamma_{qq}(j) \lambda \partial^j_+ \lambda + \theta_2 \left( \gamma_{qq}(j) \lambda \partial^{j+1}_+ \bar{A} + \gamma^{(\text{ex})}_{qq}(j) \partial_+ \bar{A} \partial^j_+ \lambda \right) - \theta_1 \left( \gamma_{qq}(j) \partial_+ \bar{A} \partial^j_+ \lambda + \gamma^{(\text{ex})}_{qq}(j) \lambda \partial^{j+1}_+ \bar{A} \right) + \theta_1 \theta_2 \gamma_{gg}(j) \partial_+ \bar{A} \partial^{j+1}_+ \bar{A} \right\},
$$

where the anomalous dimensions are

$$
\gamma_{qq}(j) = 2\psi(j + 2) - 2\psi(1), \quad \gamma_{gg}(j) = 2\psi(j + 3) - 2\psi(1),
$$

$$
\gamma_{qq}(j) = \psi(j + 3) + \psi(j + 2) - 2\psi(1), \quad \gamma^{(\text{ex})}_{qq}(j) = \frac{1}{j + 2}.
$$

Equating the coefficients in front of an even number of $\theta$’s in both sides of (5.37), one gets the expressions for anomalous dimensions of the maximal-helicity gauge field operators

$$
[H_{\Psi\Psi} - \gamma_{gg}(j)] \partial_+ \bar{A} \partial^{j+1}_+ \bar{A}(0)_{\text{fw}} = 0,
$$

and maximal-helicity gaugino operators

$$
[H_{\Psi\Psi} - \gamma_{qq}(j)] \lambda \partial^j_+ \lambda(0)_{\text{fw}} = 0.
$$

These relations are in agreement with the known results [23,38,22]. Notice that the anomalous dimension of the operator $\partial_+ \bar{A} \partial^{j+1}_+ A(0)$ is the same as in the $\mathcal{N} = 0$ theory, Eq. (5.14).

Comparing the terms linear in $\theta$’s in both sides of (5.37), one identifies the mixing matrix for the operators $\partial_+ \bar{A} \partial^j_+ \lambda$ and $\lambda \partial^{j+1}_+ \bar{A}$. Its diagonalization reveals that the operators

$$
\lambda \partial^{j+1}_+ \bar{A} - \partial_+ \bar{A} \partial^j_+ \lambda, \quad \lambda \partial^{j+1}_+ \bar{A} + \partial_+ \bar{A} \partial^j_+ \lambda
$$

have an autonomous scale dependence in the forward limit and possess the eigenvalues

$$
\gamma_{qq}(j) - \gamma^{(\text{ex})}_{qq}(j) = \gamma_{qq}(j) = 2\psi(j + 2) - 2\psi(1)
$$

$$
\gamma_{qq}(j) + \gamma^{(\text{ex})}_{qq}(j) = \gamma_{gg}(j) = 2\psi(j + 3) - 2\psi(1),
$$

respectively (see, e.g., [23,38]).

In the $\mathcal{N} = 2$ SYM theory, the analysis goes along the same lines but it is slightly lengthier due to the presence of an extra fermionic direction in the superspace, $Z = (z, \theta^A)$ (with $A = 1, 2$).
The $\mathcal{N} = 2$ light-cone superfields are given by (2.29) and involve an additional complex scalar field $\phi$. For the product of two superfields, we find in the forward limit

$$
\Psi(Z_1)\Psi(Z_2) \overset{fw}{=}_{\Psi} \sum_{j=0}^{\infty} \frac{2j}{j!} \left\{ -\phi \partial_+^{j+1} \phi - \tilde{\theta}_{1A} \tilde{\theta}_{2B} \lambda^A \partial_+ \lambda^B + (\theta_1 \cdot \theta_1)(\theta_2 \cdot \theta_2) \partial_+ \tilde{A} \partial_+ \tilde{A} \right\}
$$

where ellipses stand for fermionic operators built from the gaugino and scalar fields. Here $(AB) = \frac{1}{2} (AB + BA)$ denotes the symmetrization with respect to the $SU(2)$ indices and notations were introduced for $(\theta' \cdot \theta') \equiv \frac{1}{2} \varepsilon_{AB} \theta^A \theta^B$ and $\tilde{\theta}_A \equiv \varepsilon_{AB} \theta^B$.

As before, we substitute (5.43) into (4.13) for $\mathcal{N} = 2$ and evaluate $\mathbb{H}_{\Psi \Psi} \Psi(Z_1)\Psi(Z_2)$ in the forward limit. Matching the coefficients in front of powers of $\theta$’s, we evaluate the anomalous dimensions of various Wilson operators in the $\mathcal{N} = 2$ SYM theory $[37]$. In this manner, one finds that the anomalous dimensions of the gauge field operators, $\tilde{A} \partial_+^{j+2} \tilde{A}$, and gaugino operators in the triplet $SU(2)$ representation, $\lambda^A \partial_+ \lambda^B$, are the same as in the $\mathcal{N} = 1$ theory, Eqs. (5.38) and (5.39), respectively. For the operators built from two scalars one finds

$$
\mathbb{H}_{\Psi \Psi} \phi \partial_+^{j+1} \phi(0) \overset{fw}{=}_{\Psi} 2 \left[ \psi(j + 1) - \psi(1) \right] \phi \partial_+^{j+1} \phi(0).
$$

The remaining three operators, $\phi \partial_+^{j+1} \tilde{A}(0)$, $\partial_+ \tilde{A} \partial_+ \phi(0)$ and $\varepsilon_{AB} \lambda^A \partial_+ \lambda^B(0)$, mix under renormalization. For instance,

$$
\mathbb{H}_{\Psi \Psi} \varepsilon_{AB} \lambda^A \partial_+ \lambda^B(0) \overset{fw}{=}_{\Psi} 2 \left[ \psi(j + 2) - \psi(1) \right] \varepsilon_{AB} \lambda^A \partial_+ \lambda^B + \frac{2i}{j + 2} \left( \phi \partial_+^{j+1} \tilde{A} + \partial_+ \tilde{A} \partial_+ \phi \right).
$$

Diagonalizing the corresponding $3 \times 3$ mixing matrix one constructs three operators

$$
O_j^{(1)}(0) = \phi \partial_+^{j+1} \tilde{A}(0) + \partial_+ \tilde{A} \partial_+ \phi(0) + i \varepsilon_{AB} \lambda^A \partial_+ \lambda^B(0),
$$

$$
O_j^{(2)}(0) = \phi \partial_+^{j+1} \tilde{A}(0) - \partial_+ \tilde{A} \partial_+ \phi(0),
$$

$$
O_j^{(3)}(0) = \phi \partial_+^{j+1} \tilde{A}(0) + \partial_+ \tilde{A} \partial_+ \phi(0) - \frac{j + 2}{j + 1} i \varepsilon_{AB} \lambda^A \partial_+ \lambda^B(0).
$$

They have an autonomous scale dependence in the forward limit

$$
\mathbb{H}_{\Psi \Psi} O_j^{(n)}(0) \overset{fw}{=}_{\Psi} 2 \left[ \psi(j + n) - \psi(1) \right] O_j^{(n)}(0), \quad (n = 1, 2, 3).
$$

Thus, in agreement with our expectations, Eq. (3.60), the anomalous dimensions of Wilson operators in the $\Psi \Psi$–sector in the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ SYM theories are given by the same universal function $\gamma_{\Psi \Psi}(j)$, Eq. (3.19), with the argument determined by the conformal spin of the Wilson operator.

It is straightforward to extend the above analysis to the $\Phi \Phi$–sector. The Wilson operators in this sector can be obtained from those in the $\Psi \Psi$–sector by substituting gauge field, gaugino and scalar by complex conjugated fields. This does not affect their anomalous dimensions and leads to (3.62).

We would like to stress that the operators (5.46) have an autonomous scale dependence only in the forward limit. Beyond this limit they mix under renormalization with Wilson operators.
involving total derivatives. The corresponding mixing matrix takes a triangular form and its non-diagonal elements are fixed by the $SL(2)$ invariance. As in the $\mathcal{N} = 0$ case, Eqs. (5.15) and (5.12), taking the mixing into account amounts to replacing Wilson operators in the right-hand side of (5.46) by conformal operators \[15\]. The resulting operators are primaries of the $SL(2|\mathcal{N})$ group and we shall refer to them as superconformal operators.

**ΦΨ-sector**

Let us consider Wilson operators in the mixed $\Phi\Psi-$ sector in $\mathcal{N} = 1$ SYM theory. The scale dependence of $\Phi(Z_1)\Psi(Z_2)$ is governed to one-loop order by the dilatation operator $\mathbb{H}_{\Phi\Psi}$ defined in (1.9). For the product of two light-cone superfields one gets in the forward limit

$$
\Phi(Z_1)\Psi(Z_2) \overset{fw}{=} \sum_{j=0}^{\infty} \frac{z_{21}^j}{j!} \{ -\partial_+ A \partial_+^{-2} \lambda + \theta_2 \partial_+ A \partial_+^{i-1} \bar{A} + \theta_1 \bar{\lambda} \partial_+^{i-1} \lambda + \theta_1 \theta_2 \bar{\lambda} \partial^i \bar{A} \}.
$$

(5.48)

In this expansion, odd (even) powers of $\theta$'s are accompanied by bosonic (fermionic) operators. In Eq. (5.48), the first few terms with $j \leq 2$ involve spurious operators. The latter are eliminated by the projector $\Pi_{\Phi\Psi}$, Eq. (5.48)

$$
(1 - \Pi_{\Phi\Psi})\Phi(Z_1)\Psi(Z_2) \overset{fw}{=} \frac{\gamma_{\Phi}(j + 1) \partial_+ A \partial_+^{-1} \bar{A}(0) - \gamma_{\Phi}(ex)(j + 1) \partial_+ \bar{A} \partial_+^{i-1} A(0)}{2 \lambda \partial_+^{i-1} \bar{\lambda}(0)} - \frac{\lambda \partial_+^{i-1} \lambda(0)}{j - 1},
$$

(5.49)

where ellipses denote the contribution of fermionic operators $\partial_+ A \partial_+^{i-2} \lambda$ and $\bar{\lambda} \partial_+^{i+2} \bar{A}$. In a similar manner one obtains

$$
(1 - \Pi_{\Phi\Psi})\Phi(Z_1)\Psi(Z_2) \overset{fw}{=} \frac{\gamma_{\Phi}(j + 1) \partial_+ A \partial_+^{-1} \bar{A}(0) - \gamma_{\Phi}(ex)(j + 1) \partial_+ \bar{A} \partial_+^{i-1} A(0)}{2 \lambda \partial_+^{i-1} \bar{\lambda}(0)} + \theta_2 \sum_{j=2}^{\infty} \frac{z_{21}^j}{j!} \lambda \partial_+^{i-1} \bar{\lambda} + \ldots.
$$

(5.50)

The subsequent analysis goes through the same steps as in Sect. 5.1. Namely, one substitutes (5.48) into (1.9) and (5.61), takes into account (5.49) and (5.50) and evaluates $\mathbb{H}_{\Phi\Psi}\Phi(Z_1)\Psi(Z_2)$. One extracts the scale dependence of Wilson operators by comparing the coefficients in front of $\theta_1$ and $\theta_2$.

In this way, one obtains for the gauge field Wilson operators (for $j \geq 2$)

$$
\mathbb{H}_{\Phi\Psi} \partial_+ A \partial_+^{i-1} \bar{A}(0) \overset{fw}{=} \gamma_{\Phi\Psi}(j + 1) \partial_+ A \partial_+^{i-1} \bar{A}(0) - \gamma_{\Phi\Psi}(ex)(j + 1) \partial_+ \bar{A} \partial_+^{i-1} A(0)
$$

(5.51)

where $\gamma_{\Phi\Psi}(j + 1)$ and $\gamma_{\Phi\Psi}(ex)(j + 1)$ the same as in the $\mathcal{N} = 0$ theory, Eq. (5.32), and for gaugino operators (for $j \geq 1$)

$$
\mathbb{H}_{\Phi\Psi} \bar{\lambda} \partial_+^{i-1} \lambda(0) \overset{fw}{=} [\psi(j + 2) + \psi(j) - 2\psi(1)] \bar{\lambda} \partial_+^{i-1} \lambda(0) + \delta_{j1} \frac{1}{3} (\bar{\lambda} \lambda(0) + \bar{\lambda} \lambda(0))
$$

(5.52)

$$
- \theta_{j2} \left( \frac{1}{j + 2} \partial_+ A \partial_+^{i-1} \bar{A}(0) + \frac{2}{j(j + 1)(j + 2)} \partial_+ \bar{A} \partial_+^{i-1} A(0) \right).
$$

45
In Eq. (5.52), the $\theta-$function in front of the gauge field operators ensures that spurious operators $A \partial_+ A(0)$ and $\bar{A} \partial_+ A(0)$ (for $j = 1$) do not mix with the Wilson gaugino operators. The contribution $\sim \delta_{j,1}$ is generated by the terms $z_{21}\lambda\lambda$ and $z_{21}\lambda\lambda\lambda$ in the expression for the projector, Eqs. (5.49) and (5.50). In the conventional approach it comes from the Feynman (annihilation) diagram in which gaugino and antigaugino are first annihilated into a gluon and, then, are produced back, $\lambda\lambda + \lambda\lambda = it^a f^{abc} \lambda^b \lambda^c$.

For gaugino operators with no derivatives one gets from (5.52)

$$\mathbb{H}_{\Phi \Psi} \bar{\lambda}(0) = \frac{11}{6} \lambda(0) + \frac{1}{3} \lambda(0).$$

(5.53)

For $j \geq 2$ the gaugino operators $\partial_+^{j-1} \lambda\lambda(0)$ mix with the gauge field operators, Eq. (5.52). Diagonalizing the $2 \times 2$ mixing matrix one finds that the following operators have autonomous evolution in the forward limit

$$O_j^{(1)} = \text{tr}\{\partial_+ A \partial_+^{j-1} A\} + \text{tr}\{\bar{\lambda} \partial_+^{j-1} \lambda\},$$

$$O_j^{(2)} = \text{tr}\{\partial_+ A \partial_+^{j-1} A\} - \frac{j+2}{j-1} \text{tr}\{\bar{\lambda} \partial_+^{j-1} \lambda\}.$$  

(5.54)

For even/odd $j$, these operators are expressed in terms of the parity even/odd twist-two operators (5.5) and (5.4). The superconformal operators (5.54) diagonalize the $\mathcal{N} = 1$ dilatation operator

$$\left[\mathbb{H}_{\Phi \Psi} - \gamma_{\mathcal{N}=1}(j + n - 1)\right] O_j^{(n)}(0) \overset{fw}{=} 0,$$

(5.55)

with $n = 1, 2$ and their anomalous dimension is given by

$$\gamma_{\mathcal{N}=1}(j) = \psi(j + 2) + \psi(j - 1) - 2\psi(1) - \frac{2(-1)^j}{(j + 1) j (j - 1)}.$$  

(5.56)

Eq. (5.55) is in a perfect agreement with the general relation (3.56). To reproduce (3.56) one has to take into account that $c_{\mathcal{N}} = 3/2$ for $\mathcal{N} = 1$, $\mathbb{J}_{\Phi \Psi} = c_{\mathcal{N}} - 1 + j = j + 1/2$ and $\mathbb{P}_{\Phi \Psi} = (-1)^j$.

It is straightforward to extend the above analysis to the $\mathcal{N} = 2$ SYM theory. The expressions for the mixing matrices of Wilson operators in the $\Phi\Psi-$sector become more involved due to a larger number of operators involved. To save space, we only present explicit expressions for the renormalization group evolution of the scalar operators $\bar{\phi} \partial_+^j \phi(0)$. For $j = 0$, scalar operators with no derivatives form a closed sector

$$\mathbb{H}_{\Phi \Psi} \bar{\phi}(0) = \frac{3}{2} \bar{\phi}(0) - \frac{1}{2} \bar{\phi}(0).$$

(5.57)

For $j = 1$ they only mix with the gaugino operators with no derivatives. For $j \geq 2$ they mix both with the gauge field operators and the $SU(2)$ singlet gaugino operators

$$\mathbb{H}_{\Phi \Psi} \bar{\phi} \partial_+^j \phi(0) \overset{fw}{=} 2 \left[\psi(j + 1) - \psi(1)\right] \bar{\phi} \partial_+^j \phi + \frac{\partial_+ A \partial_+^{j-1} \bar{\lambda} + \partial_+ \bar{A} \partial_+^{j-1} A}{(j + 1)(j + 2)} + \frac{\bar{\lambda} \partial_+^{j-1} \lambda^4}{j + 1}.$$  

(5.58)

Diagonalizing the $3 \times 3$ mixing matrix, one finds that the following operators have an autonomous evolution in the forward limit

$$O_j^{(1)} = \text{tr}\{\partial_+ A \partial_+^{j-1} A\} + \text{tr}\{\bar{\lambda} \partial_+^{j-1} \lambda\} - \text{tr}\{\bar{\phi} \partial_+^j \phi\},$$

$$O_j^{(2)} = \text{tr}\{\partial_+ A \partial_+^{j-1} A\} - \frac{1}{j - 1} \text{tr}\{\bar{\lambda} \partial_+^{j-1} \lambda\} + \frac{j + 1}{j - 1} \text{tr}\{\bar{\phi} \partial_+^j \phi\},$$

$$O_j^{(3)} = \text{tr}\{\partial_+ A \partial_+^{j-1} A\} - \frac{j + 2}{j - 1} \text{tr}\{\bar{\lambda} \partial_+^{j-1} \lambda\} - \frac{(j + 1)(j + 2)}{(j - 1)j} \text{tr}\{\bar{\phi} \partial_+^j \phi\}.$$  

(5.59)
The superconformal operators $O_j^{(n)}$ (with $n = 1, 2, 3$) diagonalize the dilatation operator $H_{\Phi\Psi}$, Eq. (5.63), and satisfy the same relation as before, Eq. (5.54), with the corresponding anomalous
dimension $\gamma_{N=2}(j + n - 1)$ defined as
\[
\gamma_{N=2}(j) = \psi(j + 1) + \psi(j - 1) - 2\psi(1) - \frac{(-1)^j}{j(j-1)}. \tag{5.60}
\]
To match (5.63) one has to take into account that $c_N = 1$ for $N = 2$, $\gamma_{\Phi} = c_N - 1 + j = j$ and $\mathbb{P}_{\Phi} = (-1)^j$.

### 5.3. Wilson operators in $\mathcal{N} = 4$ theory

In the $\mathcal{N} = 4$ SYM theory, all two-particle quasipartonic operators belong to the $\Phi\Phi$-sector. As before they can be obtained as coefficients in the expansion of the product of two light-cone superfields $\Phi(Z_1)\Phi(Z_2)$ in powers of $z$'s and $\theta$'s. The scale dependence of $\Phi(Z_1)\Phi(Z_2)$ is driven to one-loop by the dilatation operator (3.52). This operator has a much simpler form for $N = 4$ as compared with the SYM theories with less supersymmetries, Eq. (5.53). This simplicity gets lost as soon as one replaces the light-cone superfield $\Phi(Z)$ by its explicit expression (2.30) and switches from nonlocal light-cone operators $\Phi(Z_1)\Phi(Z_2)$ to local Wilson operators built from gaugino, scalar and gauge fields. A complete one-loop dilatation operator in the $\mathcal{N} = 4$ theory acting on the space spanned by Wilson operators has been constructed in Ref. [18]. Going over from nonlocal light-cone operators to local Wilson operators, one finds that the obtained expressions for the $\mathcal{N} = 4$ dilatation operator, Eqs. (5.52) and (5.53) (see also (6.4) below), agree with the results of Ref. [18].

As we have seen in Sect. 4.3, the dilatation operator for $N \leq 2$ can be derived from the $\mathcal{N} = 4$ dilatation operator through the truncation procedure. According to (4.39) and (4.40), the anomalous dimensions of Wilson operators in the $\Phi\Phi$- and $\Psi\Psi$-sectors for $N \leq 2$ coincide with anomalous dimensions of the same operators in the $\mathcal{N} = 4$ theory. In particular, this is the case for the maximal helicity gauge field operators $\partial_+ A \partial_+^{-1} A(0)$ and $\partial_+ \bar{A} \partial_+^{-1} \bar{A}(0)$. In the $\mathcal{N} = 4$ theory these operators have the same anomalous dimension as in the SYM theories with $N = 0, 1, 2$, Eqs. (5.26) and (5.39).

Let us examine Wilson operators built from scalar fields $\phi^{AB}$ and $\bar{\phi}_{AB}$ with no derivatives. Such operators have a minimal possible scaling dimension and could only mix to one-loop order with themselves. It is convenient to switch from the complex fields $\phi^{AB}$ and $\bar{\phi}_{AB} = (\phi^{AB})^* = \frac{1}{2} \varepsilon_{ABCD} \phi^{CD}$ to six real scalars $\phi_j$ (with $j = 1, \ldots, 6$) defined as
\[
\phi_j(z) = \frac{1}{2\sqrt{2}} \Sigma_j^{AB} \bar{\phi}_{AB}(z), \tag{5.61}
\]
where $\Sigma_j^{AB}$ is a chiral block of six-dimensional Euclidean Dirac matrices, expressed by means of 't Hooft symbols [12]. According to (2.30), the scalar field is related to the $\mathcal{N} = 4$ superfield as
\[
\phi_j(z) = \frac{i}{2\sqrt{2}} \Sigma_j^{AB} \partial_{\theta^A} \partial_{\bar{\theta}^B} \Phi(z, \theta^A, 0)|_{\theta^A = 0}. \tag{5.62}
\]
This allows one to deduce the scale dependence of the scalar operator $\phi_{j_1}(0)\phi_{j_2}(0)$ from the scale dependence of the nonlocal light-cone operator $\Phi(Z_1)\Phi(Z_2)$
\[
H_{\Phi\Phi}[\phi_{j_1}(0)\phi_{j_2}(0)] = -\frac{1}{8} \left( \Sigma_{j_1}^{AB} \partial_{\bar{\theta}^A} \partial_{\theta^B} \right) \left( \Sigma_{j_2}^{CD} \partial_{\bar{\theta}^C} \partial_{\theta^D} \right) H_{\Phi\Phi}(\Phi(Z_1)\Phi(Z_2))_{Z_1 = Z_2 = 0} = \sum_{k_1 k_2} v_{j_1 j_2}^{k_1 k_2} \phi_{k_1} \phi_{k_2}(0). \tag{5.63}
\]
Replacing $H_{ΦΦ} Φ(Z_1)Φ(Z_2)$ by its expression \(4.27\) and calculating Grassmann derivatives one finds after some algebra
\[
V_{j_1j_2}^{k_1k_2} = δ_{j_1}^{k_1} δ_{j_2}^{k_2} + \frac{1}{2} δ_{j_1j_2} δ_{j_1}^{k_1} δ_{j_2}^{k_2} - δ_{j_1}^{k_2} δ_{j_2}^{k_1}. \tag{5.64}
\]
This relation defines the one-loop mixing matrix the scalar operators $Φ_{j_1}(0)Φ_{j_2}(0)$ in the multi-color limit \(16\). As was shown in \(16\), the mixing matrix \(5.64\) possesses a hidden symmetry—it can be mapped into a Hamiltonian of a completely integrable Heisenberg $SO(6) \sim SU(4)$ spin chain.

As a last example, we examine the scale dependence of the gauge field operator $∂_+ A \partial_{+}^{j-1} A(0)$. To this end one relates the gauge fields with the scalar operators
\[
∂_+ A(z) = \partial_z^2 Φ(z, 0), \quad ∂_+ A(z) = -d_θ Φ(z, θ^A)|_{θ^A=0}, \tag{5.65}
\]
where $d_θ \equiv \frac{1}{d_4} ε^{ABCD} ∂_θ^A ∂_θ^B ∂_θ^C ∂_θ^D$. Then, the evolution equation for the gauge field operator $∂_+ A \partial_{+}^{j-1} A(0)$ can be derived from \(4.27\) as
\[
H_{ΦΦ}[∂_+ A \partial_{+}^{j-1} A(0)] = -\left(∂_z^2 ∂_{+}^{j-2} d_θ\right) H_{ΦΦ} Φ(Z_1)Φ(Z_2)|_{Z_1=Z_2=0}. \tag{5.66}
\]
Calculating the derivatives one obtains in the forward limit (for $j ≥ 2$)
\[
H_{ΦΦ} ∂_+ A \partial_{+}^{j-1} A(0) \equiv γ_{ΦΦ}(j + 1) ∂_+ A \partial_{+}^{j-1} A - γ_{ΦΦ}^{ex}(j + 1) ∂_+ A ∂_{+}^{j-1} A \tag{5.67}
\]
\[
- \frac{\bar{λ}_A ∂_{+}^{j-1} Φ^A}{j - 1} - \frac{2 λ^A ∂_{+}^{j-1} Φ^A}{(j - 1)(j + 1)} - \frac{Φ_{AB} Φ^A}{2j(j - 1)},
\]
where the anomalous dimensions in front of gauge field operators are the same as in \(5.32\). Let us compare \(5.67\) with similar relations in the $N = 0$ and $N = 1$ theories, Eqs. \(5.32\) and \(5.51\), respectively. We observe that \(5.67\) stays intact as one goes over from $N = 4$ down to $N = 0$. The only difference is that the contribution of “unwanted” fields has to be removed in the right-hand side of \(5.67\). This property is yet another manifestation of the truncation procedure described in Sect. 4.3.

The aforementioned scalar and gauge field operators, $Φ_{j_1}(0)Φ_{j_2}(0)$ and $∂_+ A \partial_{+}^{j-1} A(0)$, respectively, represent two special examples of two-particle (twist-two) Wilson operators in the $N = 4$ theory \(39\). A complete classification of such operators has been worked out in Ref. \(42\). Diagonalizing their mixing matrix one can construct two-particle superconformal Wilson operators in the $N = 4$ SYM theory with autonomous scale dependence. It is a straightforward but tedious exercise to verify that using the obtained expression for the dilatation operator \(4.27\) one reproduces the results of Ref. \(42\). One finds from \(5.32\) that, in agreement with Ref. \(18\), the one-loop anomalous dimensions of all superconformal quasipartonic operators in the $N = 4$ theory are given by the same universal function $2[ψ(\mathbb{J}_{ΦΦ}) - ψ(1)]$ with the superconformal spin $\mathbb{J}_{ΦΦ}$ depending on the operator under consideration.

### 6. Hidden symmetries of the dilatation operator

So far we discussed mostly the two-particle dilatation operators in different sectors in the SYM theories, Eqs. \(5.53\) and \(5.52\). These operators govern the scale dependence of the product of two light-cone superfields, $Φ(Z_1)Φ(Z_2)$, $Φ(Z_1)Ψ(Z_2)$, $Ψ(Z_1)Φ(Z_2)$, and allow...
us to construct the one-loop dilatation operator \( H \) acting on arbitrary multi-particle operators, Eqs. (1.5) – (1.3), in the multi-color limit.

The nonlocal light-cone operators \( \mathcal{O}(Z_1, \ldots, Z_L) \), Eqs. (1.5) – (1.3), satisfy the evolution equation (1.7). To solve it, one has to diagonalize the integral operator \( H \), Eq. (1.8),

\[
H \Psi_q(Z_1, \ldots, Z_L) = (H_{12} + \ldots + H_{L1}) \Psi_q(Z_1, \ldots, Z_L) = E_q \Psi_q(Z_1, \ldots, Z_L),
\]

where quantum numbers \( q \) parameterize all possible solutions. Then, a general solution to (1.7) takes the form

\[
\mathcal{O}(Z_1, \ldots, Z_L) = \sum_q \Psi_q(Z_1, \ldots, Z_L) O_q(0),
\]

with the expansion coefficients \( O_q(0) \) being local composite operators. It follows from (1.7) that to one-loop order, the operators \( O_q(0) \) have an autonomous scale dependence and satisfy the evolution equation

\[
\mu \frac{d}{d\mu} O_q(0) = -\gamma_q O_q(0), \quad \gamma_q = \frac{g^2 N_c}{8\pi^2} \left( E_q + L \gamma_N^{(0)} \right),
\]

where the one-loop anomalous dimension of the superfields \( \gamma_N^{(0)} = \beta_0/(2N_c) \) is proportional to the one-loop beta-function, \( \gamma_N^{(0)} = -11/6, -3/2, -1, 0 \) for \( N = 0, 1, 2, 4 \), respectively.

Equation (6.3) determines the spectrum of anomalous dimensions of Wilson operators in the SYM theory built from \( L \) fundamental fields. For \( L = 2 \) one has \( H = 2H_{12} \) and, therefore, \( E_q \) is twice the eigenvalue of two-particle dilatation operators, Eqs. (3.60) – (3.64). It turns out that for some operators the dilatation operator \( H \) can be mapped into a Hamiltonian of integrable lattice models so that their anomalous dimensions are in the one-to-one correspondence with the energy spectrum of these models.

### 6.1. XXX Heisenberg (super)spin chain

For the light-cone operators built from \( \Phi^- \)superfields, Eq. (1.3) the dilatation operator takes the form (1.8) with the two-particle kernel given by the dilatation operator in the \( \Phi\Phi^- \)sector, \( \mathbb{H}_{\Phi\Phi} \), Eqs. (3.43) and (3.62)

\[
\mathbb{H} \mathcal{O}_{\Phi^-\Phi^-}(Z_1, \ldots, Z_L) = \sum_{k=1}^{L} 2 \left[ \psi(J_{k,k+1}) - \psi(1) \right] \theta(J_{k,k+1}) \mathcal{O}_{\Phi^-\Phi^-}(Z_1, \ldots, Z_L).
\]

Here \( J_{k,k+1} \) is the \( SL(2|N) \) superconformal spin in the sector \( \Phi(Z_k)\Phi(Z_{k+1}) \) defined in Eqs. (3.55) and (3.54) and the periodic boundary conditions are imposed, \( J_{L,L+1} = J_{1,2} \). We recall that in the \( N = 4 \) theory all quasipartonic operators reside inside the nonlocal light-cone operators \( \mathcal{O}_{\Phi^-\Phi^-}(Z_1, \ldots, Z_L) \) and, therefore, (6.4) defines a complete one-loop \( N = 4 \) dilatation operator in the multi-color limit.

For \( N \leq 2 \) one has to consider in addition the light-cone operators involving \( \Psi^- \)superfields. For the light-cone operators built from \( \Psi^- \)superfields only, Eq. (1.4), the one-loop dilatation operator is given in the multi-color limit by the sum over two-particle dilatation operators in the \( \Psi\Psi^- \)sector, \( \mathbb{H}_{\Psi\Psi^-} \), Eqs. (3.51) and (3.60),

\[
\mathbb{H} \mathcal{O}_{\Psi^-\Psi^-}(Z_1, \ldots, Z_L) = \sum_{k=1}^{L} 2 \left[ \psi(J_{k,k+1}) - \psi(1) \right] \mathcal{O}_{\Psi^-\Psi^-}(Z_1, \ldots, Z_L),
\]
where $J_{k,k+1}$ is the $SL(2|\mathcal{N})$ superconformal spin in the sector $\Psi(Z_k)\Psi(Z_{k+1})$ and $J_{L,L+1} = J_{L,1}$.

The dilatation operators (6.4) and (6.5) have a hidden symmetry [19]. In both cases, the operator $\mathbb{H}$ can be identified as a Hamiltonian of a completely integrable XXX Heisenberg spin magnet with the $SL(2|\mathcal{N})$ symmetry [25, 26, 27]. The length of the spin chain equals $L$ and the (super)spin in each site is defined by the superconformal spin of the corresponding superfield $j_{\Phi} = -1/2$ and $j_{\Psi} = (3 - \mathcal{N})/2$. As a result, the Schrödinger equation (6.1) for the Hamiltonians (6.4) and (6.5) can be solved exactly by the Quantum Inverse Scattering Method [43] and the spectrum of the anomalous dimensions of the light-cone operators (1.3) and (1.4) can be calculated by the Bethe Ansatz technique.

For the “mixed” light-cone operators (1.5) built from both $\Phi$– and $\Psi$–superfields in the $\mathcal{N} \leq 2$ theory, the dilatation operator has a more complicated form as compared with (6.4) and (6.5). The reason for this is that the operators (1.5) could mix with other light-cone operators containing the same number of $\Phi$– and $\Psi$–superfields but ordered differently inside the trace. In the expression for the two-particle evolution kernels (1.5) such mixing is described by the “exchange” interaction, Eq. (1.11). Its impact on the properties of the dilatation operator has been thoroughly studied in context of the $\mathcal{N} = 0$ theory in Refs. [11, 12]. It was found that the exchange interaction breaks integrability symmetry of the one-loop dilatation operator and modifies the scaling properties of the operators (1.5) in a very peculiar way—it lifts the degeneracy in the “energy” levels of $\mathbb{H}$ and generates a finite mass gap in the spectrum of anomalous dimensions of Wilson operators with large conformal spin.

Another manifestation of the same phenomenon comes from the analysis of the dilatation operator in the sector of Wilson operators with the minimal scaling dimension. As we show in the next section, “nonintegrable” addenda to the dilatation operator in the $\mathcal{N}$–extended SYM theory modify the mixing matrix for these operators in such a way that it can be mapped into a Hamiltonian of the XXZ Heisenberg magnet with the anisotropy parameter depending on $\mathcal{N}$. For $\mathcal{N} = 2$ a similar observation has been made in Ref. [14]. It is interesting to note that XXZ spin chains have previously emerged in QCD in the studies of high-energy (Regge) asymptotics of multi-gluonic scattering amplitudes in the double-logarithmic approximation [45, 31].

6.2. XXZ Heisenberg spin chain

The Wilson operators with minimal scaling dimension are built from the fundamental fields with the lowest dimension, that is, from the gauge field strength $\partial_+ A$ and $\partial_+ \bar{A}$ for $\mathcal{N} = 0$, from the gaugino fields $\lambda$ and $\bar{\lambda}$ for $\mathcal{N} = 1$, from the complex scalar fields $\phi$ and $\bar{\phi}$ for $\mathcal{N} = 2$ and, finally, from the real scalars $\phi_j$ for $\mathcal{N} = 4$. They do not contain additional light-cone derivatives and define a closed sector at one-loop order.

We recall that in the $\mathcal{N} = 4$ theory the two-particle mixing matrix for the scalar operators with no derivatives is given by (5.64) and it can be mapped into the XXX Heisenberg spin chain with the $SO(6)$ symmetry. It turns out that in the SYM theories with $\mathcal{N} \leq 2$, the mixing matrix for the Wilson operators with a minimal scaling dimension built from the fields mentioned above can be mapped into an XXZ Heisenberg spin chain.

Notice that for $\mathcal{N} \leq 2$ the fundamental fields with the lowest dimension are the lowest components of the light-cone superfield $\Psi(Z)$ and the highest components of $\Phi(Z)$, Eqs. (2.27) – (2.29). To write down the mixing matrix it is convenient to associate with them two spin$-1/2$ states

$$|\uparrow\rangle = \{\partial_+ \bar{A}(0), \lambda(0), \phi(0)\}, \quad |\downarrow\rangle = \{\partial_+ A(0), \bar{\lambda}(0), \bar{\phi}(0)\},$$

(6.6)
where the three entries inside the curly brackets correspond to \( \mathcal{N} = 0, 1, 2 \), respectively. Then, multi-particle single-trace Wilson operators of minimal dimension can be mapped into the spin states. For example, the state \(|\uparrow\downarrow \ldots \uparrow\rangle\) gives rise to the operators \( \text{tr}\{\partial_+ \bar{A} \partial_+ \bar{A} \ldots \partial_+ \bar{A}\} \) for \( \mathcal{N} = 0 \), \( \text{tr}\{\lambda \bar{\lambda} \ldots \lambda\} \) for \( \mathcal{N} = 1 \) and \( \text{tr}\{\phi \bar{\phi} \ldots \phi\} \) for \( \mathcal{N} = 2 \).

Let us examine the action of the one-loop dilatation operator on the two-particle Wilson operators defined by the states \(|\uparrow\uparrow\rangle\), \(|\uparrow\downarrow\rangle\), \(|\downarrow\uparrow\rangle\) and \(|\downarrow\downarrow\rangle\). By definition, the operators \(|\uparrow\uparrow\rangle = \{\partial_+ \bar{A} \partial_+ \bar{A}, \lambda \bar{\lambda}, \phi \bar{\phi}\}\) belong to the \( \Psi \Psi \) sector and have the lowest conformal spin possible in this sector, \( J_{\Psi \Psi} = 3 - \mathcal{N} \). Their anomalous dimension is given by (3.60) for \( J_{\Psi \Psi} = 3 - \mathcal{N} \). Eq. (3.63). Therefore, one finds from (3.63)

\[
H_{12} |\uparrow\uparrow \ldots \rangle = \left[ \psi(5 - \mathcal{N}) - \psi(1) \right] |\uparrow\uparrow \ldots \rangle - \left( -1 \right)^N \frac{4}{4 - \mathcal{N}} |\downarrow\uparrow \ldots \rangle,
\]

(6.7)

where the subscript indicates that the dilatation operator acts on the first two spin states. One verifies that for \( \mathcal{N} = 0, 1, 2 \) this relation is in agreement with (5.20), (5.40) and (5.44), respectively. The complex conjugated operators \(|\downarrow\downarrow\rangle = \{\partial_+ \bar{A} \partial_+ \bar{A}, \bar{\lambda} \lambda, \bar{\phi} \phi\}\) have the same anomalous dimension as \(|\uparrow\uparrow\rangle\) and, therefore,

\[
H_{12} |\downarrow\downarrow \ldots \rangle = 2\left[ \psi(5 - \mathcal{N}) - \psi(1) \right] |\downarrow\downarrow \ldots \rangle.
\]

(6.8)

In a similar manner, the operators \(|\uparrow\downarrow\rangle = \{\partial_+ \bar{A} \partial_+ \bar{A}, \bar{\lambda} \lambda, \bar{\phi} \phi\}\) belong to the \( \Psi \Phi \) sector and have the lowest conformal spin possible in this sector, \( J_{\Psi \Phi} = 3 - \mathcal{N} / 2 \). Eq. (3.63). Therefore, one finds from (3.63)

\[
H_{12} |\uparrow\downarrow \ldots \rangle = \left[ \psi(5 - \mathcal{N}) - \psi(1) \right] |\uparrow\downarrow \ldots \rangle - \left( -1 \right)^N \frac{4}{4 - \mathcal{N}} |\downarrow\uparrow \ldots \rangle.
\]

(6.9)

where \( \Phi_{\Psi\Phi} |\uparrow\downarrow\rangle = (-1)^N |\downarrow\uparrow\rangle \) since for \( \mathcal{N} = 1 \) the corresponding (gaugino) fields have Fermi statistics. For \( \mathcal{N} = 0, 1, 2 \) Eq. (6.9) is in agreement with (5.32), (5.53) and (5.57), respectively. For complex conjugated operators \(|\downarrow\uparrow\rangle = \{\partial_+ \bar{A} \partial_+ \bar{A}, \bar{\lambda} \lambda, \bar{\phi} \phi\}\) one gets

\[
H_{12} |\downarrow\uparrow \ldots \rangle = \left[ \psi(5 - \mathcal{N}) - \psi(1) \right] |\downarrow\uparrow \ldots \rangle - \left( -1 \right)^N \frac{4}{4 - \mathcal{N}} |\uparrow\downarrow \ldots \rangle.
\]

(6.10)

Combining together (6.7) – (6.10) one can write the operator \( H_{12} \) as the Hamiltonian of a XXZ Heisenberg spin–1/2 magnet

\[
H_{12} = H_x \sigma^x \otimes \sigma^x + H_x \sigma^y \otimes \sigma^y + H_z \sigma^z \otimes \sigma^z + H_0 \mathbb{I} \otimes \mathbb{1}
\]

(6.11)

where the Pauli matrices act on the spin states as

\[
\sigma^- |\uparrow\rangle = |\downarrow\rangle, \quad \sigma^+ |\uparrow\rangle = 0, \quad \sigma^- |\uparrow\rangle = |\uparrow\rangle, \quad \sigma^+ |\downarrow\rangle = |\uparrow\rangle, \quad \sigma^- |\downarrow\rangle = 0, \quad \sigma^z |\downarrow\rangle = -|\downarrow\rangle
\]

(6.12)

with \( \sigma^\pm = (\sigma^x \pm i \sigma^y) / 2 \). Matching (6.11) into (6.7) – (6.10) one gets

\[
H_x = -\left( -1 \right)^N \frac{4}{2(4 - \mathcal{N})},
\]

\[
H_x = \psi(3 - \mathcal{N}) - \frac{1}{2} \psi(5 - \mathcal{N}) - \frac{1}{2} \psi(1),
\]

\[
H_0 = \psi(3 - \mathcal{N}) + \frac{1}{2} \psi(5 - \mathcal{N}) - \frac{3}{2} \psi(1).
\]

(6.13)
Using these relations one finds the anisotropy parameter $\Delta = H_z/H_x$

$$
\Delta_{N=0} = -\frac{11}{3}, \quad \Delta_{N=1} = \frac{1}{2}, \quad \Delta_{N=2} = 3.
$$

(6.14)

For $\mathcal{N} = 2$ these expressions are in agreement with the results of Ref. [44].

For Wilson operators with the minimal scaling dimension built from $L$ fundamental field, the one-loop dilatation operator is given in the multi-color limit by (1.8) with the two-particle kernels $H_{k,k+1}$ defined in (6.11). It coincides with the Hamiltonian of the XXZ Heisenberg spin $-1/2$ chain of the length $L$ and the anisotropy parameter $\Delta_N$ given by (6.14). According to (6.3) and (6.1), the ground state of the magnet corresponds the Wilson operator with the minimal anomalous dimension. It is well known [43] that in the thermodynamical limit $L \to \infty$ its properties depend on the value of the anisotropy parameter $\Delta_N$:

- For $\Delta_N \geq 1$, or equivalently $\mathcal{N} = 2$, the ground state is ferromagnetic and it is separated from the rest of the spectrum by a mass gap;
- For $-1 \leq \Delta_N < 1$, or equivalently $\mathcal{N} = 1$, the ground state is antiferromagnetic, and the spectrum is gapless;
- For $\Delta_N < -1$, or equivalently $\mathcal{N} = 0$, the ground state is antiferromagnetic, and there is a mass gap.

We should mention that appearance of the XXZ Heisenberg magnet in the sector of Wilson operators with the minimal scaling dimension is not a unique feature of gauge theories [46]. The same structure will appear in a field theory containing complex fields, say $\varphi$ and $\bar{\varphi}$, provided that the two-particle transitions are $\varphi \varphi \to \varphi \varphi$, $\bar{\varphi} \bar{\varphi} \to \bar{\varphi} \bar{\varphi}$ and $(\varphi \bar{\varphi}, \bar{\varphi} \varphi) \to (\varphi \bar{\varphi}, \bar{\varphi} \varphi)$ and the corresponding mixing matrix has real entries.

7. Conclusions

In this paper we employed the light-cone formalism to construct the one-loop dilatation operator, which governs the scale dependence of Wilson operators of the maximal Lorentz spin, in all $\mathcal{N}$-extended SYM theories. The advantage of this formalism is that it provides a unifying superfield description of SYM theories with different number of supercharges $\mathcal{N} = 0, 1, 2, 4$. The $\mathcal{N} = 4$ SYM theory is formulated in terms of a single chiral superfield $\Phi(x, \theta^A)$ which describes all propagating modes in the model, while in SYM theories with less supersymmetry $\mathcal{N} \leq 2$ there are two chiral superfields, $\Phi(x, \theta^A)$ and $\Psi(x, \theta^A)$, each describing half of the propagating modes.

We demonstrated that the one-loop dilatation operator takes a remarkably simple form when realized on the space spanned by single-trace products of the superfields separated by light-like distances. The latter operators serve as generating functions for Wilson operators of the maximal Lorentz spin and the scale dependence of the two are in the one-to-one correspondence with each other. In the maximally supersymmetric, $\mathcal{N} = 4$ theory all nonlocal light-cone operators are built from a single superfield, Eq. (1.3), while for $\mathcal{N} = 0, 1, 2$ one has to distinguish three different types of such operators, Eqs. (1.3) – (1.5). For the nonlocal light-cone operators, the full superconformal $SU(2,2|\mathcal{N})$ group is reduced to its “collinear” $SL(2|\mathcal{N})$ subgroup. The superconformal invariance allowed us to determine the one-loop dilatation operator up to some scalar functions. We deduced their form from previous QCD calculations of anomalous dimensions of
the maximal helicity Wilson operators and confirmed the resulting expressions by explicit superspace calculation of one-loop kernels entering the evolution (Callan-Symanzik) equation for nonlocal light-cone operators.

The superspace formalism allowed us to establish an intricate relation between the one-loop dilatation operators in the SYM theories with $\mathcal{N} \leq 2$ supercharges and the maximally supersymmetric $\mathcal{N} = 4$ theory: all of the former can be deduced from the latter by merely truncating the number of fermionic directions in superspace. In the light-cone approach, the $\mathcal{N} = 4$ theory can be reformulated as a SYM theory with $\mathcal{N} \leq 2$ supercharges coupled to additional Wess-Zumino supermultiplets. The above relation between the dilatation operators is a consequence of vanishing contribution of the Wess-Zumino supermultiplets to two-particle connected Feynman diagrams contributing to the one-loop evolution kernels. They do contribute however to the self-energy diagrams resulting into distinct beta-functions in SYM theories with different $\mathcal{N}$.

We found that the dilatation operator in the sector of light-cone operators built only form the $\Phi-$superfields, Eq. (1.3), has the same, universal form in all SYM theories. It can be identified in the multi-color limit as a Hamiltonian of the $SL(2|\mathcal{N})$ Heisenberg spin chain of length equal to the number of superfields involved [19]. For $\mathcal{N} = 4$ this implies that, in agreement with the findings of Ref. [18], the one-loop dilatation operator is completely integrable. For $\mathcal{N} = 0,1,2$ the one-loop dilatation operator possesses the $SL(2|\mathcal{N})$ integrability in the sector of light-cone operators built from $\Phi-$ and $\Psi-$superfields only, Eqs. (1.3) and (1.4), respectively. At the same time, for “mixed” light-cone operators built from both superfields, Eq. (1.5), the dilatation operator receives an additional contribution from the exchange interaction between the $\Phi-$ and $\Psi-$superfields which breaks its integrability. Thus, in distinction with the $\mathcal{N} = 4$ theory, the dilatation operator in the SYM theories with $\mathcal{N} \leq 2$ supercharges is integrable only for the light-cone operators (1.3) and (1.4). To understand the reason for this, we notice that the mixing matrices for Wilson operators at $\mathcal{N} = 4$ and $\mathcal{N} \leq 2$ are related to each other through the truncation procedure: the mixing matrix for $\mathcal{N} = 0$ is a minor of the same matrix for $\mathcal{N} = 1$ which in its turn is a minor of the $\mathcal{N} = 2$ matrix and so on. Going over from $\mathcal{N} = 4$ down to $\mathcal{N} = 0$ one replaces some of its entries by 0 and, therefore, breaks integrability of the whole matrix. Still, integrability survives in its blocks corresponding to the Wilson operator generated by nonlocal light-cone operators (1.3) and (1.4).

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A Some useful formulae

In this Appendix we specify the conventions used throughout the paper.

A1. Light-cone coordinates

For our purposes, it is convenient to split four-dimensional vectors $x_\mu = (x_0, x_1, x_2, x_3)$ into longitudinal light-cone components $x_\pm = (x_0 \pm x_3)/\sqrt{2}$ and (anti)holomorphic transverse components
\( x = (x_1 + ix_2)/\sqrt{2} \) and \( \bar{x} = x^* \). In these notations, the scalar product looks as

\[
x^\mu y_\mu = x_+ y_- + x_- y_+ - x\bar{y} - \bar{x}y.
\]

We also define the derivatives

\[
\partial_+ \equiv \frac{\partial}{\partial x_-} = \frac{1}{\sqrt{2}}(\partial_{x_0} - \partial_{x_3}), \quad \partial \equiv \frac{\partial}{\partial x} = \frac{1}{\sqrt{2}}(\partial_{x_1} + i\partial_{x_2}), \quad \bar{\partial} = (\partial)^*,
\]

so that \( \partial_+ x_- = \partial_+ \bar{x} = \bar{\partial} x = 1 \) and \( \partial_+ x_+ = \partial_+ \bar{x} = \bar{\partial} \bar{x} = 0 \). The action of the SYM theory on the light-cone, Eqs. (2.24) – (2.25), involves a nonlocal operator \( \partial_+^{-1} \). It is defined in the momentum representation using the Mandelstam-Leibbrandt prescription [21]

\[
\partial_+^{-1} f(x) = i \int \frac{d^4k}{(2\pi)^4} \frac{\hat{f}(k)}{[k_+]_{\text{ML}}},
\]

with the causal prescription for the pole in the momentum space

\[
\frac{1}{[k_+]_{\text{ML}}} \equiv \frac{1}{k_+ + i0 \cdot k_-} = \frac{k_-}{k_+ k_- + i0}.
\]

**A2. Light-cone spinors**

The four-component Majorana spinors (both the gaugino fields and the odd generators of the superconformal group) are composed from two Weyl spinors

\[
\psi = \begin{pmatrix} \lambda_\alpha \\ \bar{\lambda}^\dot{\alpha} \end{pmatrix}, \quad \bar{\psi} = \psi^\dagger \gamma^0 = (\lambda^\alpha, \bar{\lambda}_{\dot{\alpha}}),
\]

where the Weyl indices are lowered/raised according to the rules

\[
\lambda^\alpha = \varepsilon^{\alpha\beta} \lambda_\beta, \quad \bar{\lambda}_{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}} \bar{\lambda}^{\dot{\beta}}, \quad \lambda_\alpha = \lambda_\beta \varepsilon^{\beta\alpha}, \quad \bar{\lambda}^{\dot{\alpha}} = \bar{\lambda}^{\dot{\beta}} \varepsilon^{\dot{\beta}\dot{\alpha}},
\]

with the Levi-Civita tensor normalized as \( \varepsilon^{12} = \varepsilon_{12} = -\varepsilon^{\dot{1}\dot{2}} = -\varepsilon_{\dot{1}\dot{2}} = 1 \). Complex conjugation acts on the covariant Weyl spinors as

\[
(\bar{\lambda}^{\dot{\alpha}})^* = \lambda^\alpha, \quad (\lambda_\alpha)^* = \bar{\lambda}_{\dot{\alpha}},
\]

and the product of two spinors obeys

\[
(\lambda_{1\alpha} \lambda_{2\alpha}^* )^* = (\lambda_{1\alpha}^* )^* (\lambda_{1\alpha})^* = \bar{\lambda}_{\dot{1}\dot{\alpha}} \bar{\lambda}_{\dot{1}\dot{\alpha}}.
\]

In the Weyl basis \([A.5] \), the Dirac matrices admit the representation

\[
\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

where \( \sigma^\mu = (1, \sigma) \) and \( \bar{\sigma}^\mu = (1, -\sigma) \) involve the conventional vector of Pauli \( \sigma \)–matrices.

In the light-cone formalism, one splits Majorana spinors into the “good” and “bad” components using \([2.2] \). In the Weyl representation \([A.5] \), the former is given by

\[
\lambda_{+\alpha} = \frac{1}{2} \bar{\sigma}^-_{\alpha\beta} \sigma^+_{\beta\gamma} \lambda_{\gamma} = \begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix}, \quad \bar{\lambda}^{\dot{\alpha}+}_+ = \frac{1}{2} \sigma^-_{\dot{\alpha}\dot{\beta}} \sigma^+_{\dot{\beta}\dot{\gamma}} \bar{\lambda}^{\dot{\gamma}} = \begin{pmatrix} 0 \\ -\bar{\lambda}_1 \end{pmatrix},
\]
with $\tilde{\lambda}_1 = -\bar{\lambda}^2$. The remaining components of the Weyl spinors $\lambda_\alpha$ and $\bar{\lambda}_{\dot{\alpha}}$ define the “bad” spinors. Thus, the “good” and “bad” spinors, $\lambda_{+\alpha}$ and $\bar{\lambda}_{\dot{\alpha}}$, can be described by a single complex Grassmann number without specifying Lorentz indices. To simplify formulae for the components of superfields $\Phi$ and $\Psi$ building up the light-cone actions in Sect. 2.1 and 2.2, it is convenient to rescale the covariant components of the gaugino field as

$$
\lambda^A \equiv \sqrt{2} \lambda^A, \quad \bar{\lambda} \equiv -i \sqrt{2} \bar{\lambda}_A.
$$

(A.11)

With such a convention the infinitesimal supersymmetric variations of the fields $X = (A, \lambda^A, \phi^{AB})$

$$
\delta_{\xi} X = [\xi^A Q_A, X] + [\bar{\xi}_{\dot{A}} \bar{Q}^{\dot{A}}, X],
$$

(A.12)

with rescaled generators

$$
Q_{1A} \equiv -i \sqrt{8} Q_A, \quad \bar{Q}^A \equiv -\sqrt{8} \bar{Q}^A
$$

(A.13)

and Grassmann transformation parameters, $\xi^{1A} = -\xi^A / \sqrt{8}$ and $\bar{\xi}_{\dot{A}} = i \bar{\xi}_{\dot{A}} / \sqrt{8}$, result in a relatively simple formulae, say for $\mathcal{N} = 4$ SYM (see Ref. [42]),

$$
\begin{align*}
\delta_{\xi} Q_A &= \xi^A \bar{\lambda}_A, \\
\delta_{\xi} \lambda^A &= -\left( \partial^A \bar{A} \right) \xi^A - i \left( \partial^A \phi^{AB} \right) \bar{\xi}_{\dot{B}}, \\
\delta_{\xi} \phi^{AB} &= i \left( \xi^A \lambda^B - \xi^B \lambda^A - \varepsilon^{ABCD} \bar{\xi}_C \bar{\lambda}_D \right). 
\end{align*}
$$

(A.14)

Finally, one introduces rescaled fermionic parameters in light-cone superspace in terms of components of covariant Weyl coordinates,

$$
\theta^A \equiv \sqrt{8} \theta^{1A}, \quad \bar{\theta}_{\dot{A}} \equiv i \sqrt{8} \bar{\theta}_{\dot{A}},
$$

(A.15)

so that the realization of superconformal generators in superspace has a concise form, Eq. (3.8). Due to the presence of additional factors in the right-hand side of (A.11) and (A.15), complex conjugation acts on the odd variables $\chi = (\lambda^A, \theta^A)$ and $\bar{\chi} = (\bar{\lambda}_A, \bar{\theta}_{\dot{A}})$ as

$$
\chi^* = -i \bar{\chi}, \quad \bar{\chi}^* = -i \chi,
$$

(A.16)

while for their product one has

$$
(\chi_1 \chi_2)^* = \chi_2^* \chi_1^* = -\bar{\chi}_2 \bar{\chi}_1 = \bar{\chi}_1 \bar{\chi}_2, \quad (\bar{\chi}_1 \bar{\chi}_2)^* = \chi_2^* \chi_1^* = -\bar{\chi}_2 \bar{\chi}_1 = \chi_1 \chi_2.
$$

(A.17)

**A3. Grassmann integration**

The integration measure over Grassmann variables is normalized as

$$
\int d^N \theta \, \theta^1 \ldots \theta^N = \int d^N \bar{\theta} \, \bar{\theta}_1 \ldots \bar{\theta}_N = 1.
$$

(A.18)

Performing calculation of Feynman diagrams in the momentum representation, we apply the Fourier transformation to the superfield, Eq. (4.11). The inverse Fourier transformation is defined as

$$
\tilde{\Phi}(p, \pi_A) = \int d^D x \int d^N \theta \, e^{-ip \cdot x - \pi_A \theta^A} \Phi(x, \theta^A),
$$

(A.19)
where $\pi_A$ is the momentum conjugated to the odd coordinates $\theta^A$. To establish the normalization of the integration measure over Grassmann valued momenta $\pi_A$, one computes sequentially the Fourier transform and its inverse, i.e., $\Phi(\theta^A) = \int d^N\pi \int d^N\theta_1 \Phi(\theta_1^A) \exp \pi \cdot (\theta - \theta_1)$. Taking into account (A.18), one finds that

$$\int d^N\pi \pi_1 \ldots \pi_N = (-1)^{N(N-1)/2}.$$  \hspace{1cm} (A.20)

For the odd variables the delta-functions in the coordinate and momentum space are defined as

$$\delta^{(N)}(\theta) \equiv \int d^N\pi e^{-\pi^A\theta^A} = \theta_1 \ldots \theta_N,$$

$$\delta^{(N)}(\pi) \equiv \int d^N\theta e^{\pi^A\theta^A} = \pi_1 \ldots \pi_N.$$  \hspace{1cm} (A.21)

They satisfy (4.9) together with \( \int d^N\theta \delta^{(N)}(\theta - \theta_1)\Phi(x, \theta^A) = \Phi(x, \theta_1^A) \).

Going over to the momentum representation in (2.26) we find that the products of differential operators in the right-hand side of (2.26) are replaced by products of momenta \( p_1 \) and \( \pi_A \) which we denote as

\[ (p_1, p_2) = p_1 p_2^+ - p_2 p_1^+ , \quad (p_1, p_2)^* = \bar{p}_1 p_2^+ - \bar{p}_2 p_1^+ , \]

\[ [p_1, p_2] = \prod_{A=1}^N (\pi_{1,A} p_{2}^+ - \pi_{2,A} p_{1}^+) \equiv (\pi_{1,1} p_2^+ - \pi_{2,1} p_1^+) \ldots (\pi_{1,N} p_2^+ - \pi_{2,N} p_1^+) , \]  \hspace{1cm} (A.22)

where the ordering of fermionic momenta is such that the factors with larger \( \lambda \) appear to the right. These brackets have the following properties

\[ (k + xp_{1}, p_{1}) = (k, p_{1}) = (p_{1}, k) , \]

\[ [k + xp_{1}, p_{1}] = [k, p_{1}] = (-1)^{N}[p_{1}, k] \]  \hspace{1cm} (A.23)

with \( x \) arbitrary c-number. The square bracket can be expressed in terms of momentum delta-function, Eq. (A.21),

\[ [p_{1}, p_{2}] = (-1)^{N(N-1)/2}(p_{2}^+)^N \delta^{(N)} \left( \frac{\pi_{1} - \pi_{2} p_{1}^+}{p_{1}^+} \right) \]

\[ = (-1)^{N(N+1)/2}(p_{2}^+)^N \delta^{(N)} \left( \frac{\pi_{2} - \pi_{1} p_{2}^+}{p_{2}^+} \right) , \]  \hspace{1cm} (A.24)

where the delta-function in momentum $\pi-$space is defined in (A.21).

\section{A4. Mandelstam’s approach}

To establish the relation between the expressions for the light-cone SYM actions, Eq. (2.24) and (2.25), with those proposed by Mandelstam in Ref. [21], one introduces the operator

\[ D_A = \partial_{\theta^A} - \theta^A \partial_+ , \quad \{ D_A , D_B \} = -2 \partial_+ \delta_{AB} . \]  \hspace{1cm} (A.25)

In distinction with $D_{M,A}$, Eq. (2.22), this operator is not covariant under the $SU(\mathcal{N})$ rotations of $\theta-$coordinates, generated by the charges $T_{B,A}$, Eq. (3.8). It is straightforward to verify that for
arbitrary light-cone scalar superfield $\Phi(x, \theta^A)$ the following relation holds true (no summation over $A$)

$$D_A \left( D_A \Phi_1 D_A \Phi_2 \right) = -\partial_+ \Phi_1 \partial_\theta \Phi_2 + \partial_\theta \Phi_1 \partial_+ \Phi_2 .$$

Then, comparison with (2.26) yields

$$[\Phi_1, \Phi_2] = (-1)^{N(N-1)/2} D_+(D \Phi_1 D \Phi_2) ,$$

where $D \equiv D_1 \ldots D_N$. The light-cone action (2.24) involves two superfields $\Psi(x, \theta^A)$ and $\Phi(x, \theta^A)$. Let us use the following ansatz for the former field

$$\Psi(x, \theta^A) = \partial_+^{2-N} D \bar{\Phi}(x, \theta^A) ,$$

or equivalently $\bar{\Phi}(x, \theta^A) = (-1)^{N(N+1)/2} \partial_+^{2} D \Psi(x, \theta^A)$. Substituting (A.28) into (2.24) and taking into account (A.27), one obtains the light-cone action of the SYM theory in terms of the superfields $\Phi(x, \theta^A)$ and $\bar{\Phi}(x, \theta^A)$ which coincides with the light-cone actions proposed in Refs. [21, 47].

### B Projectors

To define the one-loop dilatation operator we introduced the $SL(2|N)$ invariant projection operators, Eqs. (3.42) and (3.47). As in Sect. 3.2.2, we shall restrict ourselves to the $SL(2)$ case and make use of the $SL(2|N)$ invariance to “lift” the resulting expressions from the light-cone to the superspace, Eq. (3.35).

To start with one considers a nonlocal light-cone operator $O_{j_1 j_2}(z_1, z_2)$. It belongs to the tensor product of two $SL(2)$ moduli labelled by the conformal spins $j_1$ and $j_2$ which can be decomposed over the irreducible components as $V_{j_1}^{SL(2)} \otimes V_{j_2}^{SL(2)} = \bigoplus_{j_2} V_{j_1 j_2}^{SL(2)}$ with the conformal spin $j_{12} = j_1 + j_2 + n$ for $n = 0, 1, \ldots$. Let us introduce the operator $\Pi^{j_1 j_2}_m$ that projects $O_{j_1 j_2}(z_1, z_2)$ onto the $SL(2)$ moduli $V_{j_1 j_2}^{SL(2)}$ with $j_{12} = j_1 + j_2 + m$.

By the definition, $\Pi^{j_1 j_2}_m$ is the $SL(2)$ invariant operator satisfying $\Pi^{j_1 j_2}_m \Pi^{j_1 j_2}_n = \delta_{mn} \Pi^{j_1 j_2}_m$. The $SL(2)$ invariance fixes the form of integral operator $\Pi^{j_1 j_2}_m$ up to some scalar function $f(\xi)$, Eq. (3.22). The $SL(2)$ integrals in (3.22) can be simplified with a help of the identity [35]

$$\int [Dw]_j (z_1 - \bar{w})^{-2j+x} (z_2 - \bar{w})^{-x} \Phi(w) = \frac{\Gamma(2j)}{\Gamma(2j-x)} \int_0^1 d\alpha \alpha^{-1} \Phi(\bar{\alpha}z_1 + \alpha z_2)$$

where $\bar{\alpha} = 1 - \alpha$. To this end, one uses the integral representation $f(\xi) = \int_C \frac{dx}{2\pi i} \xi^{-x} \tilde{f}(x)$ interchanges the order of integration in (3.22) and applies (B.1) consequently. In this way, one obtains from (3.22)

$$\Pi^{j_1 j_2}_m O_{j_1 j_2}(z_1, z_2) = \int_0^1 d\alpha \int_0^1 d\beta \alpha^{2j_1-2} \beta^{2j_2-2} \varphi_m \frac{\alpha \beta}{\bar{\alpha} \bar{\beta}} O_{j_1 j_2}(\bar{\alpha}z_1 + \alpha z_2, \bar{\beta}z_2 + \beta z_1) ,$$

where $\bar{\alpha} = 1 - \alpha, \bar{\beta} = 1 - \beta$ and the function $\varphi_m$ is related to the function of anharmonic ratio entering (3.22) as

$$f(\xi) = \int_0^1 d\alpha \int_0^1 d\beta \alpha^{2j_1-2} \beta^{2j_2-2} (\bar{\alpha} \alpha + \xi_\alpha)^{-2j_1} \varphi_m \frac{\alpha \beta}{\bar{\alpha} \bar{\beta}} .$$
Important difference between the functions $f(\xi)$ and $\varphi_m$ is that the latter is a distribution.

To determine the function $\varphi_m$ entering (B.2) it is sufficient to examine the action of $\Pi_m^{(j_1,j_2)}$ on the lowest weight in the $SL(2)$ module $V_{j_{12}}^{SL(2)}$ which are given by $(z_1 - z_2)^m \equiv z_{12}^m$. Then, replacing $\mathcal{O}(z_1, z_2)$ in (B.2) by $O^{(n)}(z_1, z_2) = z_{12}^n$ and taking into account that $\Pi_m^{(j_1,j_2)}O^{(n)} = \delta_{mn}O^{(n)}$ one obtains

$$[\Pi_m^{(j_1,j_2)}O^{(n)}](z_1, z_2) = z_{12}^n \int_0^1 d\alpha \int_0^1 d\beta \alpha^{2j_1-2} \beta^{2j_2-2} \varphi_m(\zeta)(1 - \alpha - \beta)^n = \delta_{mn}z_{12}^n. \quad (B.4)$$

where $\zeta \equiv \alpha\beta/(\bar{\alpha}\bar{\beta})$. Solving this relation for $m = 0$ and $m = 1$ one gets

$$\varphi_0(\zeta) = \frac{\Gamma(2j_1 + 2j_2)}{\Gamma(2j_1)\Gamma(2j_2)} \delta(\zeta - 1), \quad (B.5)$$

$$\varphi_1(\zeta) = \frac{\Gamma(2j_1 + 2j_2 + 2)}{\Gamma(2j_1 + 1)\Gamma(2j_2 + 1)} \frac{d}{d\zeta} \delta(\zeta - 1) - (2j_1 + 2j_2 + 1) \frac{\Gamma(2j_1 + 2j_2)}{\Gamma(2j_1)\Gamma(2j_2)} \delta(\zeta - 1).$$

Let us define the following operator

$$\tilde{\mathcal{O}}_{j_1,j_2}(z_1, z_2) = (1 - \Pi_0^{(j_1,j_2)} - \Pi_1^{(j_1,j_2)})\mathcal{O}_{j_1,j_2}(z_1, z_2). \quad (B.6)$$

By the construction, it receives contribution from the $SL(2)$ moduli $V_{j_{12}}^{SL(2)}$ with the conformal spins $j_{12} = j_1 + j_2 + m$ and $m \geq 2$. A distinguished property of the states belonging to these moduli is that for $z_{12} \to 0$ they have the same asymptotic behavior as the lowest weight $(z_1 - z_2)^m$ leading to

$$\tilde{\mathcal{O}}_{j_1,j_2}(z_1, z_2) \sim (z_1 - z_2)^2 \quad (B.7)$$

for $z_1 - z_2 \to 0$. Let us substitute (B.5) into (B.2) and examine the explicit expressions for the projectors $\Pi_m^{(j_1,j_2)}$ for special values of the conformal spins:

- $(j_1 = j_2 = -1/2)$: To define the action of projectors (B.5) on an arbitrary function $\mathcal{O}(z_1, z_2)$ one regularizes the integrand in (B.2) by setting $j_1 = j_2 = -1/2 + \varepsilon$ and takes the limit $\varepsilon \to 0$. Then

$$[\Pi_0^{(-1/2, -1/2)}\mathcal{O}](z_1, z_2) = \lim_{\varepsilon \to 0} \frac{\Gamma(4\varepsilon - 2)}{\Gamma^2(2\varepsilon - 1)} \int_0^1 d\alpha \ (\alpha\bar{\alpha})^{2\varepsilon - 2} \mathcal{O}(\alpha z_1 + \bar{\alpha}z_2, \alpha z_1 + \bar{\alpha}z_2)$$

$$= \frac{1}{2} \left( 1 - \frac{1}{2} z_{12} \partial z_1 \right) \mathcal{O}(z_1, z_1) + \frac{1}{2} \left( 1 - \frac{1}{2} z_{21} \partial z_2 \right) \mathcal{O}(z_2, z_2). \quad (B.8)$$

In the similar manner, one finds for the projector $\Pi_1^{(-1/2, -1/2)}$

$$[\Pi_1^{(-1/2, -1/2)}\mathcal{O}](z_1, z_2) = -\frac{1}{4} z_{12} \partial\mathcal{O}(z_2, z)|_{z=z_2} - \frac{1}{4} z_{21} \partial\mathcal{O}(z, z_2)|_{z=z_2}$$

$$-\frac{1}{4} z_{12} \partial\mathcal{O}(z_1, z)|_{z=z_1} - \frac{1}{4} z_{21} \partial\mathcal{O}(z, z_1)|_{z=z_1}. \quad (B.9)$$

Combining together the last two expressions we define the $SL(2)$ projector

$$\Pi_{\Phi \Phi} \equiv \Pi_0^{(-1/2, -1/2)} + \Pi_1^{(-1/2, -1/2)}, \quad (B.10)$$

which enters into (B.6) and leads to (B.7). Going over from the light-cone to the superspace we apply (B.10) and replace simultaneously the light-cone derivatives by derivatives in the superspace
z_{12} \partial_z \to Z_{12} \cdot \partial_z. In this way, we arrive at the expression for the SL(2|\mathcal{N}) invariant projector \( \Pi_{\Phi \bar{\Psi}} \) given in (3.44).

- \((j_1 = -1/2, j_2 \geq 1/2)\): To be specific we first choose \( j_2 = 3/2 \) which corresponds to the conformal spin of the \( \Psi -(\text{super}) \text{field in the } \mathcal{N} = 0 \text{ theory. As before, we regularize the integrand in (B.2) by setting } j_1 = -1/2 + \varepsilon \text{ for } \varepsilon \to 0 \). Making use of (B.5), one evaluates the projectors

\[
\Pi_0^{(-1,2,3/2)}(z_1, z_2) = \left( 1 - \frac{1}{2} z_{12} \partial_{z_2} \right) \mathcal{O}(z_2, z_2), \quad (B.11)
\]

\[
\Pi_1^{(-1,2,3/2)}(z_1, z_2) = \frac{1}{2} z_{12} \partial_z \mathcal{O}(z_2, z) \big|_{z = z_2} + \frac{3}{2} z_{12} \partial_z \mathcal{O}(z, z_2) \big|_{z = z_2}.
\]

Their sum \( \Pi_{\Phi \bar{\Psi}} = \Pi_0^{(-1,2,3/2)} + \Pi_1^{(-1,2,3/2)} \) is given by

\[
\Pi_{\Phi \bar{\Psi}}(z_1, z_2) = \mathcal{O}(z_2, z_2) + z_{12} \partial_z \mathcal{O}(z, z) \big|_{z = z_2}. \quad (B.12)
\]

Repeating the calculation for other (half)integer positive \( j_2 \) one can verify that the projector \( \Pi_{\Phi \bar{\Psi}} \) does not depend on \( j_2 \) and is given by (B.12). Going over from the light-cone to the superspace one arrives at (3.48).

C Feynman diagram technique

To develop the Feynman diagram technique, we introduce a generating functional in the SYM theory (for \( \mathcal{N} \leq 2 \))

\[
e^{W[J, \bar{J}]} = \int \mathcal{D} \Phi \mathcal{D} \bar{\Psi} e^{i S_N + i \int d^4 x \int d^4 \theta (J_a(x, \theta^4) \Phi^a(x, \theta^4) + \bar{J}_a(x, \theta^4) \bar{\Psi}^a(x, \theta^4))}.
\] (C.1)

For \( \mathcal{N} = 4 \), the theory is formulated in terms of a single superfield, so that there is no integration over \( \Psi \) and corresponding sources \( \bar{J} \) are absent. Connected correlation functions can be calculated from \( W[J, \bar{J}] \) as

\[
\frac{\delta}{\delta J^{a_1}(x_1, \theta^4_1)} \frac{\delta}{\delta J^{a_2}(x_2, \theta^4_2)} \ldots \frac{\delta}{\delta J^{a_N}(x_N, \theta^4_N)} W[J, \bar{J}] = \langle \Phi^{a_1}(x_1, \theta^4_1) \Phi^{a_2}(x_2, \theta^4_2) \ldots \Phi^{a_N}(x_N, \theta^4_N) \rangle,
\]

where the functional derivatives are defined as

\[
\frac{\delta J^a(x, \theta^4)}{\delta J^b(x', \theta^4)} = \delta^{ab} \delta^{(4)}(x - x') \delta^{(\mathcal{N})}(\theta - \theta').
\] (C.2)

It is proves convenient to perform calculation of the Feynman diagrams in the momentum representation. To this end, we apply the Fourier transformation in the superspace (C.1) and switch from the superfields \( \Phi^a(x_\mu, \theta^4) \) and \( \bar{\Psi}^a(x_\mu, \theta^4) \) to the conjugated fields \( \Phi^\dagger(p_\mu, \pi_\lambda) \) and \( \bar{\Psi}^\dagger(p_\mu, \pi_\lambda) \), respectively.

In order to distinguish the lines corresponding to two species of chiral superfields, i.e., \( \tilde{\Phi} \) and \( \tilde{\Psi} \), we will denote them as incoming and outgoing lines, respectively, and indicate the momentum flow by a small arrow

\[
\tilde{\Phi}(p, \pi) = \quad \text{ , } \quad \tilde{\Psi}(p, \pi) = \quad
\]

(C.4)
In what follows we shall use the convention that the external momenta flow \textit{into} vertices in the Feynman diagrams.

Since the Lagrangian of the SYM theory in the Mandelstam formalism is invariant under translations in $x_{\mu}$ and $\theta_{A}$, the correlation functions in the momentum representations are proportional to the product of delta-functions in even and odd momenta. In particular, the free propagator of the superfield can be easily found from the generating functional \textbf{(C.1)} as

$$
\langle \bar{\Phi}^a(p_1, \pi_1) \bar{\Psi}^b(p_2, \pi_2) \rangle = \sigma_N \frac{i \delta^{ab}}{p_1^2 + i0} (2\pi)^4 \delta^{(4)}(p_1 + p_2) \delta^{(N)}(\pi_1 + \pi_2),
$$

where the notation was introduced for the signature factor

$$
\sigma_N = (-1)^{N(N+1)/2},
$$

so that $\sigma_0 = 1$ and $\sigma_1 = \sigma_2 = -1$. The interaction vertices $\Gamma$ are identified as amputated Green functions, Eq. \textbf{(C.2)}, transformed into the momentum space,

$$
\langle \bar{\Phi}^a_1(\pi_1) \bar{\Psi}^b_2(\pi_2) \ldots \bar{\Phi}^a_L(\pi_L) \rangle = (2\pi)^4 \delta^{(4)} \left( \sum_{k=1}^{L} p_k \right) \delta^{(N)} \left( \sum_{k=1}^{L} \pi_k \right) \times \left( \prod_{j=1}^{L} \frac{i \sigma_N}{p_j^2} \right) \Gamma^{a_1 \ldots a_L}(p_1, \pi_1; p_2, \pi_2; \ldots; p_L, \pi_L).
$$

**Feynman rules for $N = 0, 1, 2$:**

\begin{verbatim}
\textbf{Rule 1:}
\begin{align*}
\gamma^b \quad \Phi^a &\quad \rightarrow \quad \frac{-\sigma_N i \delta^{ab}}{p^2 + i0} \\
\end{align*}
\end{verbatim}

\begin{verbatim}
\textbf{Rule 2:}
\begin{align*}
\gamma^1 \quad \Phi^a &\quad \rightarrow \quad -2i g \sigma_N f^{abc}(p_1, p_2)^* \\
\gamma^2 \quad \Phi^a &\quad \rightarrow \quad -2i g \sigma_N f^{abc}(p_1, p_2) \frac{[p_1, p_2]}{(p_1 + p_2)^2} (p_3)^{2-N} \\
\end{align*}
\end{verbatim}

\begin{verbatim}
\textbf{Rule 3:}
\begin{align*}
\gamma^3 \quad \Phi^a &\quad \rightarrow \quad -2i g^2 \sigma_N \left\{ f^{ace} f^{bde} \frac{[p_4, p_2][p_1, (p_2 + p_3)^2]}{p_{4+}(p_1 + p_3)^2 + p_{3+}(p_2 + p_4)^2} + f^{ade} f^{bce} \frac{[p_2, p_3][p_1, (p_2 + p_3)^2]}{p_{3+}(p_1 + p_4)^2 + p_{4+}(p_2 + p_3)^2} \right\} \\
\end{align*}
\end{verbatim}
Feynman rules for $\mathcal{N} = 4$:

\[
\Phi^b \rightarrow_p \Phi^a = \frac{i\delta^{ab}}{p^2 + i0}
\]

\[
= -2igf^{abc}\left\{ (p_1, p_2)^* + \frac{(p_1 + p_2)[p_1, p_2]}{(p_1 + p_2 + p_3 + p_4)^2} \right\}
\]

Here $f^{abc}$ are the $SU(N_c)$ structure constants. The signature factor $\sigma_{N}$ was defined in (C.6) ($\sigma_0 = 1$ and $\sigma_1 = \sigma_2 = -1$). The brackets $(p_1, p_2)$ and $[p_1, p_2]$ were introduced in (A.22). Notice that $[p_1, p_2] = 1$ for $N = 0$.

\section*{D Computation of one-loop dilatation operator}

To calculate one-loop correction to the dilatation operator in the light-cone formalism, we employ the dimensional regularization with $D = 4 - 2\varepsilon$ and extract divergent $\sim 1/\varepsilon$ part of the corresponding Feynman diagrams. The essential steps in the calculation of the Feynman diagrams are:

- Simplify the color factors using the $SU(N_\mathcal{C})$ identities

\[
f^{abc}f^{ab'c'} = N_c\delta^{cc'}, \quad \epsilon^{a'b'b'}f^{ab'c'}f^{abc} = \frac{N_c}{2} \epsilon^{a'b'b'}\epsilon^{a'b',ab}, \quad \epsilon^{a'b'b'}f^{ab'ac}f^{b'bc} = \frac{N_c}{2} \epsilon^{a'b'b'} + \frac{1}{4} \delta^{ab}. \quad (D.1)
\]

In the multi-color limit, we shall drop the term $\sim \delta^{ab}$ in the last relation.
- Simplify the momentum integral using the identities

\[
(p_1, k)(p_2, k)^* = p_1 + p_2 + k\bar{k} + \ldots = \frac{1}{2}p_1 + p_2 + k_\perp^2 + \ldots \quad (D.2)
\]

\[
[p_j, k] = \sigma_0 (p_{j+})^{N} \delta^{(N)} \left( \pi_k - \frac{k_+}{p_{j+}} \right), \quad (j = 1, 2). \quad (D.3)
\]

Hereafter ellipses denote terms which do not produce divergent contribution.
• Get rid of transverse components of the loop momentum

\[ k_1^2 = -k^2 \frac{(p - k)_+}{p_+} - (p - k)^2 \frac{k_+}{p_+} + \ldots, \quad (D.4) \]

with \( p^\mu \) being either \( p_1^\mu \) or \( -p_2^\mu \), and perform integration over the \( k^- \) momentum by making use of the following relation [48]

\[
\int \frac{d^D k}{(2\pi)^D} \frac{e^{-ik \cdot z}}{k^2 (p - k)^2} \frac{p^\mu}{[k^+]_{ML}} = \frac{i}{(4\pi)^2 \varepsilon} \int_0^1 \frac{d\alpha}{\alpha^n} \left( e^{-i\alpha p_+ z} - \sum_{l=0}^{n-1} (-i\alpha p_+ z)^l \right) + O(\varepsilon^0), \quad (D.5)
\]

where the \( \alpha \)-variable has the meaning of the momentum fraction \( k_+ = \alpha p_+ \). Here the pole at \( k_+ = 0 \) is integrated using the Mandelstam-Leibbrandt prescription \([A.4] \).

## D1. Self-energy

Self-energy corrections renormalize the superfields. In the light-cone formalism, in the light-like axial gauge the renormalization constants for the SYM coupling \( g \) and the superfields are equal to each other

\[
\Phi^{(0)}(0) = \sqrt{Z} \Phi, \quad \Psi^{(0)}(0) = \sqrt{Z} \Psi, \quad g^{(0)}(0) = \frac{\mu^2 g}{\sqrt{Z}}, \quad (D.6)
\]

where the superscript \((0)\) stands for the bare field/coupling. Because of this, the beta-function and the anomalous dimensions of the superfields coincide

\[
\frac{\beta(g)}{g} = \frac{\gamma(g)}{g} = \frac{g^2}{16\pi^2} \beta_0 + O(g^4). \quad (D.7)
\]

In the \( \mathcal{N} = 2 \) SYM theory the exact beta-function is given by the one-loop result while in the \( \mathcal{N} = 4 \) SYM theory it equals zero to all loops.

The renormalization factor and the anomalous dimension are computed from the self-energy insertions

\[
\gamma(g) \equiv \frac{d \ln Z}{d \ln \mu^2}, \quad Z = 1 + (-1)^{\mathcal{N}(\mathcal{N}+1)/2} \frac{i}{p^2} \sum_{\text{div}} (p^2), \quad (D.8)
\]

where \( \sum_{\text{div}} (p^2) \) denotes divergent part of the self-energy. To one-loop level, \( \Sigma(p^2) \) receives contribution from the Feynman diagrams shown in Fig. 3. Their calculation in the \( \mathcal{N} \)–extended SYM theory leads to the following result

\[
\Sigma_{\mathcal{N}=0}(p) = 2g^2 N_c \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 (p - k)^2} \left\{ \frac{(k, p)(k, p)^*}{(p - k)^2} \left( \frac{p_+^2}{k_+^2} + 2 \frac{k_+^2}{p_+^2} \right) + 2k_2 p_+ (p - k)_+ \right\}
\]

\[= -i p^2 \frac{11}{3} N_c \frac{g^2 \mu^{2\varepsilon}}{(4\pi)^2}. \quad (D.9)\]
for $\mathcal{N} = 0$, 

$$
\Sigma_{\mathcal{N}=1}(p) = -2g^2N_c \int \frac{d^Dk}{(2\pi)^D} \int d^2\pi \frac{[k,p]}{k^2(p-k)^2} \left\{ \frac{(k,p)(k,p)^*}{(p-k)^2_+} \left( \frac{p_+}{k_+} + 2 \frac{k_+^2}{p_+^2} \right) + 2k^2 \frac{(2p-k)_+}{k_+^2} \right\}
$$

$$
\equiv \frac{i\rho^2}{\varepsilon} \{2 - 1 + 2\} N_c \frac{g^2\mu^{2z}}{(4\pi)^2} = \frac{i\rho^2}{\varepsilon} 3N_c \frac{g^2\mu^{2z}}{(4\pi)^2},
$$

(D.10)

for $\mathcal{N} = 1$, 

$$
\Sigma_{\mathcal{N}=2}(p) = 2g^2N_c \int \frac{d^Dk}{(2\pi)^D} \int d^2\pi \frac{[k,p]}{k^2(p-k)^2} \left\{ \frac{(k,p)(k,p)^*}{(p-k)^2_+} \left( \frac{1}{k_+^2} + 2 \frac{1}{p_+^2} \right) + 2k^2 \frac{1}{k_+^2} \right\}
$$

$$
\equiv \frac{i\rho^2}{\varepsilon} \{0 + 2 + 0\} N_c \frac{g^2\mu^{2z}}{(4\pi)^2} = \frac{i\rho^2}{\varepsilon} 2N_c \frac{g^2\mu^{2z}}{(4\pi)^2},
$$

(D.11)

for $\mathcal{N} = 2$. For $\mathcal{N} = 4$, $\Sigma_{\mathcal{N}=4}(p)$ remains finite for $\varepsilon \to 0$. In Eqs. (D.10) and (D.11), the terms/numbers in the curly brackets correspond to diagrams in Fig. 3. We recover from this calculation the well-known results for the one-loop beta-function, $\beta_0 = -\frac{4}{3}N_c, -3N_c, -2N_c$ and 0 for $\mathcal{N} = 0, 1, 2$ and 4, respectively.

**D2. Dilatation operator in the $\Psi\Psi$–sector**

The one-loop dilatation operator in this sector receives nonvanishing contribution from Feynman diagrams shown in Fig. 1(a), (b) and (d). Using the Feynman rules for $\mathcal{N} \leq 2$, their sum can be written as

$$
\langle \mathcal{O}_{\Psi\Psi}^{(1)}(Z_1, Z_2, \ldots) \rangle = -ig^2N_c \sigma_\mathcal{N} \mu^{4-D} \int \frac{d^Dk}{(2\pi)^D} \frac{e^{-i(p_1-k)\cdot z_1-i(p_2+k)\cdot z_2}}{(p_1-k)^2(p_2+k)^2} \int d^{D-2} \pi \sum_{\mathcal{N} \geq 2} e^{-(\pi_1-\pi)\cdot \theta_1^A-(\pi_2+\pi)\cdot \theta_2^A}
$$

$$
\times \left\{ \frac{2(k,p_1)(k,p_2)^*}{k^2} \frac{[p_1,k]}{(k+p_1)^2} \left[ p_2,k \right] + \frac{2(k,p_1)^*(k,p_2)}{k^2} \frac{(p_2+k)^2_+}{(k+p_2)^2_+} \frac{[p_2,k]}{[p_1,k]} + \frac{(p_1-k)^2_+}{p_1+(k+p_2)^2} \frac{[p_1,k]}{[p_2,k]} \right\}.
$$

(D.12)

Here the first two terms correspond to the diagrams in Fig. 1(a) and (b) containing triple vertices and the last two terms correspond to the diagram in Fig. 1(d) with quartic vertex. The color factor accompanying these diagrams can be easily calculated in the multi-color limit with a help of the identities (D.1). For example, the color factor for the diagram in Fig. 1(a) and (b) is

$$
t^a_{\mu b} f^{abc} f^{\dot{a}d\epsilon} \bar{\Psi}_1^d \tilde{\bar{\Psi}}_2^\epsilon = \frac{N_c}{2} \tilde{\bar{\Psi}}_1 \tilde{\bar{\Psi}}_2
$$

(D.13)

with $\tilde{\bar{\Psi}}_k \equiv \tilde{\bar{\Psi}}^a(p_k, \pi_k)t^a$. One examines the coefficient in front of $[p_1,k]$ in (D.12) and simplifies it with a help of (D.2) and (D.4) as

$$
\frac{2(k,p_1)(k,p_2)^*}{p_1+(k+p_2)^2} + (p_2+k)_+ = \frac{(p_2+k)^2}{k^2} k_+.
$$

(D.14)

Similar relation holds for the coefficient in front of $[p_2,k]$ 

$$
\frac{2(k,p_1)^*(k,p_2)}{p_2+(k+p_2)^2} + (p_1-k)_+ = -\frac{(p_1-k)^2}{k^2} k_+.
$$

(D.15)
Replacing \([p_1, k]\) and \([p_2, k]\) by their expressions in terms of delta-functions, Eq. \((D.3)\), one arrives at Eq. \((1.18)\). The calculation of the momentum integral in \((1.8)\) is straightforward by making use of the identity \((D.5)\). In this way, one gets \((1.10)\) and casts it into the operator form \((1.12)\).

**D3. Dilatation operator in the \(\Phi\Phi\)–sector for \(\mathcal{N} \leq 2\)**

The Feynman diagrams defining one-loop dilatation operator in the \(\Phi\Phi\)–sector for \(\mathcal{N} \leq 2\) are similar to those in the \(\Psi\Psi\)–sector, Fig. III. The only difference is that the direction of all lines has to be flipped. The self-energy diagram is calculated in Sect. D1 and the annihilation diagram vanishes as before, Eq. \((4.7)\). Applying the Feynman rules given in Appendix B, one finds for the sum of the remaining three diagrams

\[
\langle \Phi^{(1)}(Z_1, Z_2, \ldots) \rangle = -ig^2 N_c \sigma \mathcal{N} \mu^{1-D} \int \frac{d^D k}{(2\pi)^D} \frac{e^{-i(p_1 - k) z_1 - i(p_2 + k) z_2}}{(p_1 - k)^2(p_2 + k)^2} \int d^N \pi \epsilon^{-(\pi_1 - \pi_\mathcal{N})_A \epsilon^{(1)}}_{\mathcal{N}}(\pi_2 + \pi_\mathcal{N})_A \epsilon^{(4)}_{\mathcal{N}}
\]

\[
\times \left\{ \frac{2(k,p_1)^{\star}(k,p_2)}{k^2} \left[ \frac{(p_1 + k)_{\mathcal{N}}}{(p_1 + k)_{\mathcal{N}} + (k^2)^2} \right] [p_1,k] + \frac{2(k,p_1)(k,p_2)^{\star}}{k^2} \left[ \frac{(p_2 + k)_{\mathcal{N}}}{(p_2 + k)_{\mathcal{N}} + (k^2)^2} \right] [p_2,k] \right. \\
+ \left. \frac{(p_1 + k)^{\star}}{(p_1 - k)_{\mathcal{N}} + (k^2)^2} [p_1,k] + \frac{(p_2 + k)^{\star}}{(p_2 + k)_{\mathcal{N}} + (k^2)^2} [p_2,k] \right\}.
\]

One repeats the same steps as in the previous case, applies the identities Eqs. \((D.14)\), \((D.15)\) and arrives at \((1.15)\).

The expression inside square brackets in \((4.15)\) can be rewritten as \((4.16)\). To see this, one examines the difference between \([\cdots]_{\mathcal{N}}\) and the first two terms in the right-hand side of \((4.16)\)

\[
-\frac{1}{k_{\mathcal{N}}^2} \left[ \frac{p_{1+}}{(p_1 - k)^2} + \frac{p_{2+}}{(p_2 + k)^2} \right] \left[ \frac{p_{1+}^2}{(p_1 - k)^2} \delta^{(\mathcal{N})} \left( \pi - \pi_1, k_{\mathcal{N}}^{1+} \right) - \frac{p_{2+}^2}{(p_2 + k)^2} \delta^{(\mathcal{N})} \left( \pi - \pi_2, k_{\mathcal{N}}^{1+} \right) \right].
\]

The delta-functions in this relation can expressed in terms of \(\delta^{(\mathcal{N})}(\pi - \omega)\) with the momentum \(\omega_A\) introduced in Eq. \((4.17)\). To this end, one rewrites the momentum \(\omega_A\) as

\[
\omega_A = \pi_1 A_{1A} + \pi_2 A_{2A} + \pi_{1A} A_{1A} + \pi_{2A} A_{2A} = \pi_1 A_{1A} + \pi_2 A_{2A} + \pi_{1A} A_{1A} + \pi_{2A} A_{2A}
\]

\[(D.18)\]

and simplifies the expression for \(\delta^{(\mathcal{N})}(\pi - \omega) - \delta^{(\mathcal{N})}(\pi - \pi_j k_{\mathcal{N}}^{1+}/p_{j+})\) by taking into account that the odd \(\delta\)–function is polynomial in its argument, Eq. \((A.2)\),

\[
\delta^{(\mathcal{N})}(\pi + \pi') - \delta^{(\mathcal{N})}(\pi) = \frac{1}{(\mathcal{N}-1)!} e^{A_{1A}} A_{1A} e^{A_{2A}} A_{2A} \pi_{1A} \ldots \pi_{(\mathcal{N}-1)A} \pi_{(\mathcal{N})A} + \ldots + \delta^{(\mathcal{N})}(\pi')
\]

\[(D.19)\]

Here the expansion goes in powers of \(\pi'\). Substituting \(\pi \rightarrow \pi - \omega\) and \(\pi' \rightarrow \omega - \pi_j k_{\mathcal{N}}^{1+}/p_{j+}\) into \((D.19)\) and applying the resulting identities in \((D.17)\), one recovers after some algebra the last term in the right-hand side of \((1.16)\). The resulting expression for the one-loop matrix element \((D.16)\) involves the Feynman integral which is defined below in \((D.23)\). As described in Appendix D4, it calculation leads to desired expression for the dilatation operator in the \(\Phi\Phi\)–sector, Eq. \((1.22)\).
D4. Dilatation operator in the $\Phi\Phi$–sector for $\mathcal{N} = 4$

For $\mathcal{N} = 4$ the one-loop dilatation operator receives contribution from the Feynman diagrams similar to those shown in Fig. 1(b)–(e). The only difference is that the lines should not have arrows. For $\mathcal{N} = 4$ both self-energy and annihilation diagrams vanish. One applies the $\mathcal{N} = 4$ Feynman rules listed in Appendix C and obtains for the sum of the remaining diagrams the expression given in (4.25).

To begin with, one eliminates $k_{\perp}^2$ from (4.25) with a help of (D.4) and rewrites the second term in the curly bracket in (4.25) as

$$-4 \left( \frac{p_1 + k}{k} \right)^2 \left( \frac{p_2}{p_2 + (k - p)^+} \right)^2 \frac{[p_2, k]}{p_2 + (k - p)^+} + 4 \left( \frac{p_2}{k} \right)^2 \left( \frac{p_1 + k}{k} \right)^2 \frac{[p_1, k]}{p_1 + (p_1 + k)^+}$$

$$+ 2 \left( \frac{p_1 + p_2}{k} \right)^2 \left( \frac{[p_2, k]}{p_2 + (p_2 - k)^+} - \frac{1}{p_2 + (p_1 + k)^+} \right)$$

$$= k^4 \left( \frac{p_1 + p_2}{k} \right)^2 \frac{[p_2, k]}{p_2 + (k - p)^+} + 4 \left( \frac{p_2}{k} \right)^2 \left( \frac{p_1 + k}{k} \right)^2 \frac{[p_1, k]}{p_1 + (p_1 + k)^+}$$

$$+ 2 \left( \frac{p_1 + p_2}{k} \right)^2 \left( \frac{[p_2, k]}{p_2 + (p_2 - k)^+} - \frac{1}{p_2 + (p_1 + k)^+} \right)$$

(D.20)

After substitution into (4.25) the first two terms match (4.16) for $\mathcal{N} = 4$ and can be easily integrated with a help of the identity (D.5). According to (D.3) the last term in (D.20) contains the same combination of delta-functions as (D.17). As before, one applies (D.18) and (D.19) for $\mathcal{N} = 4$ and expands the delta-functions in powers of $\pi_A - \omega_A$. The same set of transformations is applied to the remaining two terms inside the curly brackets in (4.25). Namely, one rewrites them in terms of delta-functions by taking into account Eqs. (D.3) and (A.24) together with the identities

$$[p_1 + k, p_2 - k] = (p_1 + p_2)^4 \delta^{(4)} (\pi - \omega),$$

$$[p_2, p_1 + k] = p_2^4 \delta^{(4)} (\pi + p_1 - p_2),$$

$$[p_1, p_2 - k] = p_1^4 \delta^{(4)} (\pi - p_2)$$

and, then, expands the delta-functions in powers of $\pi_A - \omega_A$ with a help of (D.19). For instance the explicit form of the expansion for $[k, p_1]$ is

$$[k, p_1] = \left( \frac{p_1}{p_1 + p_2} \right)^4 [k - p_1, k + p_2] + \left( \frac{p_1}{p_1 + p_2} \right)^4 [p_1, p_2]$$

$$+ \frac{1}{3!} [p_1 + p_2]^4 \left( \frac{p_1}{p_1 + p_2} \right)^2 \varepsilon^{ABC} \delta^{(4)} (\pi - \omega)_{ABC} \left( \frac{\pi_1}{p_1} - \frac{\pi_2}{p_2} \right)_{D}$$

$$+ \frac{1}{3!} [p_1 + p_2]^4 \left( \frac{p_1}{p_1 + p_2} \right)^2 \varepsilon^{ABC} \delta^{(4)} (\pi - \omega)_{A} \left( \frac{\pi_1}{p_1} - \frac{\pi_2}{p_2} \right)_{BCD}$$

$$+ \frac{1}{4} [p_1 + p_2]^2 \left( \frac{p_1}{p_1 + p_2} \right)^2 \varepsilon^{ABC} \delta^{(4)} (\pi - \omega)_{AB} \left( \frac{\pi_1}{p_1} - \frac{\pi_2}{p_2} \right)_{CD},$$

(D.22)

where the notation was introduced for $(\pi)_{A-CD} = \pi_A \ldots \pi_C$. One substitutes this and analogous expression for $[k, p_2]$ into the momentum integral (4.25) and discovers that the terms containing $\pi_A - \omega_A$ in a power smaller than three cancel against each other. The remaining terms give rise to the last term in (4.16) with $\mathcal{X}_{\mathcal{N}=4}$ given by (4.26).
In this way, the $\mathcal{N} = 4$ Feynman integral entering (4.25) can be written as

$$
\int \frac{d^dk}{(2\pi)^D} \int d^N\pi e^{-i(p_1 + k - p_1)} \frac{[k + p_2, k - p_1]}{(p_1 + k)^2 (p_1 - k)^2} \frac{p_1+(p_2 + k)^+ + p_2+(p_1 - k)^+}{(p_2 + k)^2 (p_1 - k)^2} \equiv \frac{1}{(4\pi)^2 \varepsilon} \sum_{i=1}^{4} J_i ,
$$

where $J_i$ stands for the contribution of $i$th term inside curly brackets. Here we displayed the $\mathcal{N}$—dependence because calculation of the one-loop matrix element in the $\Phi\Phi$—sector for $\mathcal{N} \leq 2$ (see (4.15) and Appendix D3) leads to the same Feynman integral (D.23).

The calculation of (D.23) is straightforward and relies on the identities (D.3), (D.21) and (D.5). To represent the result in a concise manner, one introduces plane waves in the superspace

$$
\phi(Z_1, Z_2) \equiv e^{-iP_1 \cdot Z_1 - iP_2 \cdot Z_2}.
$$

Then the momentum integration in (D.23) leads to

$$
J_1 = \int_0^{1} \frac{d\alpha}{\alpha} \left\{ \left( 1 - i\alpha \frac{p_1}{(p_1 + p_2)^+} (P_1 + P_2) \cdot Z_{12} \right) \left[ \phi(Z_2, Z_2) - \phi(\alpha Z_1 + \alpha Z_2, \alpha Z_1 + \alpha Z_2) \right] 
+ \left( 1 - i\alpha \frac{p_2}{(p_1 + p_2)^+} (P_1 + P_2) \cdot Z_{21} \right) \left[ \phi(Z_1, Z_1) - \phi(\alpha Z_1 + \alpha Z_2, \alpha Z_1 + \alpha Z_2) \right] \right\} ,
$$

$$
J_3 = \int_0^{1} \frac{d\alpha}{\alpha \alpha^2} \left\{ \phi(\alpha Z_1 + \alpha Z_2, Z_2) - (1 - \alpha^2 - i\alpha \bar{\alpha} P_1 \cdot Z_{12}) \phi(Z_2, Z_2) - \bar{\alpha}^2 \phi(Z_1, Z_2) \right\} ,
$$

where $\bar{\alpha} = 1 - \alpha$. Note that $J_1 = J_3 |_{P_1 \leftrightarrow P_2, Z_1 \leftrightarrow Z_2}$. To calculate the integral $J_2$, one applies the identity

$$
\int d^N\pi e^{-i\theta(\pi - \varepsilon)_{1\ldots N-1}} = -(1)^{N(N-1)/2} e^{-\varepsilon \theta N} ,
$$

changes the integration variable $k'=k+p_1$ and takes into account (D.5). This leads to

$$
J_2 = \frac{\theta_{12}^A (\pi_1 A P_2 + \pi_2 A P_1)}{(p_1 + p_2)^+} \int_0^{1} \frac{d\alpha}{\alpha} \left\{ \phi(Z_1, Z_1) + \phi(Z_2, Z_2) 
- \phi(\alpha Z_1 + \alpha Z_2, \alpha Z_1 + \alpha Z_2) - \phi(\alpha Z_2 + \alpha Z_1, \alpha Z_2 + \alpha Z_1) \right\} .
$$

The integrals $J_1, \ldots, J_4$ assume the same form for the SYM theories with $\mathcal{N} = 0, 1, 2, 4$. Their sum reduces to the desired form (4.22) and (4.27)

$$
\text{Eq. (D.23)} = \frac{1}{(4\pi)^2 \varepsilon} \left\{ \bar{\gamma}^{(-1/2, -1/2)} (1 - \Pi_{\Phi\Phi}) \phi(Z_1, Z_2) + \Delta_{\Phi\Phi} \phi(Z_1, Z_2) \right\} .
$$
where after some algebra the remnant $\Delta_{\Phi \Phi}$ can be cast into the form

$$
\Delta_{\Phi \Phi}(Z_1, Z_2) = -\frac{i}{2} \frac{(p_1 - p_2)_+}{(p_1 + p_2)_+} (P_1 + P_2) \cdot Z_{12} \tag{D.27}
$$

$$
\times \int_0^1 d\alpha \left\{ \phi(Z_1, Z_1) + \phi(Z_2, Z_2) - 2\phi(\tilde{\alpha} Z_1 + \alpha Z_2, \alpha Z_1 + \alpha Z_2) \right\}
$$

$$
= \left( \frac{p_1 - p_2}{(p_1 + p_2)_+} \right)' \left( 1 - \frac{1}{2} Z_{21} \partial_{Z_2} \right) \phi(Z_2, Z_2) + \left( \frac{p_2 - p_1}{(p_1 + p_2)_+} \right)' \left( 1 - \frac{1}{2} Z_{12} \partial_{Z_1} \right) \phi(Z_1, Z_1),
$$

with $Z_{jk} \partial_{Z_j} \equiv z_{jk} \partial_{z_j} + \theta^A_{jk} \partial_{\theta^A}$.

### D5. Dilatation operator in the $\Phi \Psi$–sector

The one-loop dilatation operator in the $\Phi \Psi$–sector is given by Feynman diagrams shown in Fig. 2. Since the superfields are different, one encounters a new diagram, Fig. 2(c), in which two superfields are interchanged on the light-cone. As compared with the diagonal $\Phi \Phi$– and $\Psi \Psi$–sectors, the one-loop dilatation operator is given by the sum of two terms corresponding to the transitions $\Phi \Psi \rightarrow \Phi \Psi$ and $\Phi \Psi \rightarrow \Psi \Phi$. For $N = 0$, they are defined in Eqs. (D.28) and (D.29), respectively.

Let us consider these two contributions separately. Applying the $N = 0$ Feynman rules one finds for the sum of Feynman diagrams in Figs. 2(a), (b) and (e)

$$
\langle O^{(1)}_{\Phi \Psi} \rangle_{\Phi \Psi} \equiv \langle O^{(1)}_{\Phi \Psi} \rangle_{\Phi \Psi} = -ig^2 N_c \epsilon^{4-D} \int \frac{d^D k}{(2\pi)^D} \frac{e^{-i(p_1-k)z_1-i(p_2+k)z_2}}{(p_1-k)^2(p_2+k)^2} \frac{p_1^2}{(p_1-k)^2} \left\{ \begin{array}{c}
\frac{k^2}{k^2(k_+^2)} \\
\frac{k^2}{k^2(p_1+p_2)^2} + \frac{(p_1-k)_+(p_2+k)_+}{(p_1+p_2)_+} + \frac{(p_2-k)_+(p_1+k)_+}{(p_1+p_2)_+} \end{array} \right\}, \tag{D.28}
$$

and for the sum of Feynman diagrams in Figs. 2(c) and (e)

$$
\langle O^{(1)}_{\Phi \Psi} \rangle_{\Phi \Psi} \equiv \langle O^{(1)}_{\Phi \Psi} \rangle_{\Phi \Psi} = -ig^2 N_c \epsilon^{4-D} \int \frac{d^D k}{(2\pi)^D} \frac{e^{-i(p_1-k)z_2-i(p_2+k)z_1}}{(p_1-k)^2(p_2+k)^2} \frac{p_1^2}{(p_1-k)^2} \left\{ \begin{array}{c}
\frac{k^2}{k^2(p_1+p_2)^2} \\
\frac{k^2}{k^2(p_1+p_2)^2} + \frac{(p_1-k)_+(p_2+k)_+}{(p_1+p_2)_+} + \frac{(p_2-k)_+(p_1+k)_+}{(p_1+p_2)_+} \end{array} \right\}. \tag{D.29}
$$

One applies the identity (D.31) to get rid of $k^2_-$ and performs integration with a help of (D.5). The calculation gives

$$
\langle O^{(1)}_{\Phi \Psi} \rangle_{\Phi \Psi} = \frac{g^2 N_c}{(4\pi)^2 \epsilon} \left\{ \int_0^1 \frac{d\alpha}{(1-\alpha)^2} \left[ \alpha^4 \phi_1(z_1, (1-\alpha)z_1 + \alpha z_2) \right. \right.
$$

$$
+ \left. \left( 1 - \alpha \right)^2 \phi_1(\alpha z_1 + (1-\alpha)z_2, z_2) \right] - \phi_3(z_1, z_2) \right\}, \tag{D.30}
$$

$$
\langle O^{(1)}_{\Phi \Psi} \rangle_{\Phi \Psi} = \frac{g^2 N_c}{(4\pi)^2 \epsilon} \left\{ \int_0^1 \frac{d\alpha}{(1-\alpha)^2} \left[ \alpha^3 \phi_2(\alpha z_1 + (1-\alpha)z_2, z_2) + \phi_3(z_1, z_2) \right] \right\}. \tag{D.31}
$$
Here the notation was introduced for the functions
\[
\phi_1(z_1, z_2) = (1 - \Pi_{\Phi \Psi}) e^{-ip_1 + z_1 - ip_2 + z_2},
\]
\[
\phi_2(z_1, z_2) = (1 - \Pi_{\Psi \Phi}) e^{-ip_2 + z_1 - ip_1 + z_2},
\]
\[
\phi_3(z_1, z_2) = -\frac{p_1+}{(p_1 + p_2)_+} (2 - z_{12} \partial z_2) e^{-i(p_1 + p_2)_+ z_2},
\]
 equation (D.32)

and the projectors $\Pi_{\Phi \Psi}$ and $\Pi_{\Psi \Phi}$ were defined in (3.48) and (3.50), respectively. Identifying the plane waves as matrix elements of the light-cone operator, equation (4.4), one arrives at Eqs. (4.28) and (4.29).

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