ON LOW-DIMENSIONAL MANIFOLDS WITH ISOMETRIC $SO_0(p, q)$-ACTIONS

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Abstract. Let $G$ be a non-compact simple Lie group with Lie algebra $\mathfrak{g}$. Denote with $m(\mathfrak{g})$ the dimension of the smallest non-trivial $\mathfrak{g}$-module with an invariant non-degenerate symmetric bilinear form. For an irreducible finite volume pseudo-Riemannian analytic manifold $M$ it is observed that $\dim(M) \geq \dim(G) + m(\mathfrak{g})$ when $M$ admits an isometric $G$-action with a dense orbit. The Main Theorem considers the case $G = \tilde{SO}_0(p, q)$ providing an explicit description of $M$ when the bound is achieved. In such case, $M$ is (up to a finite covering) the quotient by a lattice of either $\tilde{SO}_0(p + 1, q)$ or $\tilde{SO}_0(p, q + 1)$.

Introduction

Let $G$ be a connected non-compact simple Lie group acting isometrically on a connected analytic manifold $M$ with a finite volume pseudo-Riemannian metric. Following Zimmer’s program, it has been shown that such actions are rigid in the sense of having distinguished properties that restrict the possibilities for $M$ (see for example [10, 20, 22]). The general belief is that any such action, with some additional non-triviality conditions, must essentially be an algebraic double coset of the form $K \backslash H / \Gamma$. More precisely, such coset is given by some Lie group $H$ together with a homomorphism $G \to H$, a lattice $\Gamma \subset H$ and a compact subgroup $K \subset H$ centralizing the image of $G$ in $H$. The $G$-action is then given by the natural left action on $K \backslash H / \Gamma$. We note that when $H$ is semisimple these $G$-actions are isometric for a metric induced by the Killing form of the Lie algebra of $H$. Some results have already been obtained in [15, 16] proving that suitable geometric conditions imply that such $G$-actions are of the double coset type. We refer to [2, 3, 9] for similar related results.

In this work we observe, that for $M$ complete and weakly irreducible with the $G$-action non-transitive but with a dense orbit, the dimension of $M$ has a bound from below in terms of the representation theoretic properties of $\mathfrak{g}$, the Lie algebra of $G$. More precisely, it is noted in Proposition 1.6 that in this case we have

$$\dim(M) \geq \dim(G) + m(\mathfrak{g}),$$

where $m(\mathfrak{g})$ denotes the dimension of the smallest non-trivial representation of $\mathfrak{g}$ that admits an invariant non-degenerate symmetric bilinear form. Recall that a connected pseudo-Riemannian manifold is weakly irreducible if the tangent space at some (and hence any) point has no proper non-degenerate subspaces invariant under the restricted holonomy group at that point.

1991 Mathematics Subject Classification. 57S20, 53C50, 53C24.

Key words and phrases. Pseudo-Riemannian manifolds, simple Lie groups, rigidity results.

The first author was supported by DMS grants 0402068 and 0801010.

The second author was supported by Conacyt, Concyteg and SNI.
For our main result, we consider with further detail the case $G = \widetilde{\mathrm{SO}}_0(p,q)$, the universal covering group of $\mathrm{SO}_0(p,q)$. For $\mathrm{SO}_0(p,q)$-actions with $p,q \geq 1$ and $p+q \geq 4$, the following result establishes that the lower bound just considered can be achieved only for double coset models. Note that for a $G$-action on a manifold $M$ and $X$ in the Lie algebra of $G$ we denote by $X^*$ the vector field on $M$ whose one-parameter group of diffeomorphisms is given by $(\exp(tX))_t$ through the $G$-action on $M$.

**Main Theorem.** Let $M$ be a connected analytic pseudo-Riemannian manifold. Suppose that $M$ is complete weakly irreducible, has finite volume and admits an analytic and isometric $\mathrm{SO}_0(p,q)$-action with a dense orbit, for some integers $p,q$ such that $p,q \geq 1$ and $n = p + q \geq 5$. In this case we have $m(\mathfrak{so}(p,q)) = n$. If the equality:

$$
\dim(M) = \dim(\widetilde{\mathrm{SO}}_0(p,q)) + m(\mathfrak{so}(p,q)) = \frac{n(n+1)}{2},
$$

holds, then for $H$ either $\widetilde{\mathrm{SO}}_0(p,q+1)$ or $\widetilde{\mathrm{SO}}_0(p+1,q)$ there exist:

1. a lattice $\Gamma \subset H$, and
2. an analytic finite covering map $\varphi : H/\Gamma \to M$,

such that $\varphi$ is $\widetilde{\mathrm{SO}}_0(p,q)$-equivariant, where the $\widetilde{\mathrm{SO}}_0(p,q)$-action on $H/\Gamma$ is induced by some non-trivial homomorphism $\widetilde{\mathrm{SO}}_0(p,q) \to H$. Furthermore, we can rescale the metric on $M$ along the $\widetilde{\mathrm{SO}}_0(p,q)$-orbits and their normal bundle to assume that $\varphi$ is a local isometry for the bi-invariant pseudo-Riemannian metric on $H$ given by the Killing form of its Lie algebra. The result holds for the case $(p,q) = (3,1)$ as well if we further assume that $X^* \perp Y^*$ on $M$ for all $X \in \mathfrak{su}(2)$ and $Y \in \mathfrak{isu}(2)$ under the identification $\mathfrak{so}(3,1) \simeq \mathfrak{sl}(2,\mathbb{C})$.

Note that there is no $\mathbb{R}$-rank restriction, and so this result applies to the groups $\widetilde{\mathrm{SO}}_0(p,1)$ when $p \geq 3$. Thus, the Main Theorem provides a rigidity result for $\mathrm{SO}_0(p,1)$-actions.

The proof of the Main Theorem is based on the application of representation theory to the Killing vector fields centralizing the $G$-action, where the latter are as found in Gromov-Zimmer’s machinery (see [10, 22]). With respect to centralizing Killing fields, our main ingredient is Proposition 1.2 as already found in [7, 10, 16, 22] with varying assumptions on the manifold $M$ acted upon by $G$. Proposition 1.2 ensures the existence of a Lie algebra $\mathfrak{g}(x)$, isomorphic to $\mathfrak{g}$, of Killing fields vanishing at a point $x$ on the universal cover of $M$ and with some additional properties. The Lie algebra $\mathfrak{g}(x)$ provides a $\mathfrak{g}$-module structure to the tangent space $T_xM$ that allows to use representation theory to control the behavior of the normal to the orbits. Such $\mathfrak{g}$-module structure is then related to the Lie algebra $\mathcal{H}$ of Killing vector fields centralizing the $G$-action (see Lemma 1.8), thus again providing control on $\mathcal{H}$. By Gromov-Zimmer’s machinery, the Lie algebra of $\mathcal{H}$ has an open dense orbit in the universal covering space of $M$. Also, the Lie algebra $\mathcal{H}$ yields a Lie group action constructed in Section 1. These elements together allow to obtain a very detailed description of the structure of $\mathcal{H}$, which is provided in Section 2. Here again, the application of representation theory is a key element. Finally, Section 3 completes the proof of the Main Theorem using the Lie group action induced by $\mathcal{H}$ and the weak irreducibility assumption on $M$. Some needed facts about the Lie algebra $\mathfrak{so}(p,q)$ are collected in Appendix A. We will use the notation from the introduction and the Appendix without further comments.
1. Isometric actions of simple Lie groups and Killing fields

In this section, we let $G$ be a connected non-compact simple Lie group with Lie algebra $\mathfrak{g}$ and $M$ a connected finite volume pseudo-Riemannian manifold. Hence, every isometric $G$-action on $M$ with a dense orbit is locally free (see [17, 19]) and so the orbits define a foliation that we will denote with $O$. The bundle $TO$ tangent to the foliation $O$ is a trivial vector bundle isomorphic to $M \times \mathfrak{g}$, under the isomorphism $M \times \mathfrak{g} \rightarrow TO$ given by $(x, X) \mapsto X^*_x$. This also defines an isomorphism of every fiber $T_xO$ with $\mathfrak{g}$. We will refer to it as the natural isomorphism between $T_xO$ and $\mathfrak{g}$. Recall that, as before and in the rest of this article, for $X$ in the Lie algebra of a group acting on a manifold, we denote by $X^*$ the vector field on the manifold whose one-parameter group of diffeomorphisms is given by $(\exp(tX))_t$ through the action on the manifold. On the other hand, we will have the occasion to consider both left and right actions and so our convention is to assume that an action is on the left unless otherwise specified.

For any given pseudo-Riemannian manifold $N$, we will denote by $\text{Kill}(N)$ the globally defined Killing vector fields of $N$. Also, we denote by $\text{Kill}_0(N, x)$ the Lie algebra of globally defined Killing vector fields that vanish at the given point $x$. The following result is an easy application of the Jacobi identity and the fact that Killing vector fields are derivations of the corresponding metric. Also note that, in the rest of this work, for a vector space $W$ with a non-degenerate symmetric bilinear form, we will denote with $\mathfrak{so}(W)$ the Lie algebra of linear maps on $W$ that are skew-symmetric with respect to the bilinear form.

**Lemma 1.1.** Let $N$ be a pseudo-Riemannian manifold and $x \in N$. Then, the map

$$\lambda_x : \text{Kill}_0(N, x) \rightarrow \mathfrak{so}(T_xN)$$

given by $\lambda_x(Z)(v) = [Z, V]_x$, where $V$ is any vector field such that $V_x = v$, is a well defined homomorphism of Lie algebras.

For any given point $x$ of a pseudo-Riemannian manifold, the map $\lambda_x$ will denote from now on the homomorphism from the previous lemma.

Gromov [10] proved that the presence of a geometric structure of finite type (in the sense of Cartan) which is invariant under the action of a simple Lie group yields large spaces of Killing vector fields fixing given points in the manifold being acted upon. We refer to [22] for a detailed description of these techniques. The statement below in the case of germs of Killing fields is essentially contained in Section 9 of [7] (see also Proposition 2.3 in [16]). From this, the result for global Killing vector fields is straightforward since $\tilde{M}$ is analytic and simply connected (see [7, 10, 22]). Observe that, following our notation with $M$, we denote with $O$ the foliation by $\tilde{G}$-orbits in $\tilde{M}$.

**Proposition 1.2.** Let $G$ be a connected non-compact simple Lie group acting isometrically and with a dense orbit on a connected finite volume pseudo-Riemannian manifold $M$. Consider the $G$-action on $\tilde{M}$ lifted from the $G$-action on $M$. Assume that $M$ and the $G$-action on $M$ are both analytic. Then, there exists a conull subset $S \subset \tilde{M}$ such that for every $x \in S$ the following properties are satisfied:

1. There is a homomorphism $\rho_x : \mathfrak{g} \rightarrow \text{Kill}(\tilde{M})$ which is an isomorphism onto its image $\rho_x(\mathfrak{g}) = \mathfrak{g}(x)$.
2. $\mathfrak{g}(x) \subset \text{Kill}_0(\tilde{M}, x)$, i.e. every element of $\mathfrak{g}(x)$ vanishes at $x$.
3. For every $X, Y \in \mathfrak{g}$ we have:

$$[\rho_x(X), Y^*] = [X, Y]^* = -[X^*, Y^*].$$
In particular, the elements in \( g(x) \) and their corresponding local flows preserve both \( O \) and \( TO^\perp \).

(4) The homomorphism of Lie algebras \( \lambda_x \circ \rho_x : g \to so(T_x\tilde{M}) \) induces a \( g \)-module structure on \( T_x\tilde{M} \) for which the subspaces \( T_xO \) and \( T_xO^\perp \) are \( g \)-submodules.

With the above setup, assume that the \( G \)-orbits are non-degenerate which, from now on, is considered with respect to the ambient pseudo-Riemannian metric. In particular, the \( \tilde{G} \)-orbits on \( \tilde{M} \) are non-degenerate as well and we have a direct sum decomposition \( T\tilde{M} = TO \oplus TO^\perp \). Hence, we can consider the \( g \)-valued 1-form \( \omega \) on \( \tilde{M} \) which is given, at every \( x \in \tilde{M} \), by the composition \( T_x\tilde{M} \rightarrow T_xO \cong g \) of the natural projection onto \( T_xO \) and the natural isomorphism of this latter space with \( g \). Also, consider the \( g \)-valued 2-form given by \( \Omega = d\omega|_{T^2O^\perp} \). The following result is elementary and a proof can be found in [16].

**Lemma 1.3.** Let \( G \), \( M \) and \( S \) be as in Proposition 1.2. If we assume that the \( G \)-orbits are non-degenerate, then:

1. For every \( x \in S \), the maps \( \omega_x : T_x\tilde{M} \to g \) and \( \Omega_x : \wedge^2T_xO^\perp \to g \) are both homomorphisms of \( g \)-modules, for the \( g \)-module structures from Proposition 1.2.

2. The normal bundle \( TO^\perp \) is integrable if and only if \( \Omega = 0 \).

The non-degeneracy of the orbits is ensured for low dimensional manifolds by the next result, which appears in [16] as Lemma 2.7.

**Lemma 1.4.** Let \( G \) be a connected non-compact simple Lie group acting isometrically and with a dense orbit on a connected finite volume pseudo-Riemannian manifold \( M \). If \( \dim(M) < 2\dim(G) \), then the bundles \( TO \) and \( TO^\perp \) have fibers that are non-degenerate with respect to the metric on \( M \).

It turns out that for complete manifolds, if the \( G \)-orbits are non-degenerate and the normal bundle to such orbits is integrable, then the universal covering space can be split. Such a claim is the content of the following proposition which is a particular case of Theorem 1.1 of [16]. This result is in the spirit of similar ones found in [5, 6, 10].

**Proposition 1.5.** Let \( G \) be a connected non-compact simple Lie group acting isometrically on a connected complete finite volume pseudo-Riemannian manifold \( M \). If the tangent bundle to the orbits \( TO \) has non-degenerate fibers and the bundle \( TO^\perp \) is integrable, then there is an isometric covering map \( \tilde{G} \times N \rightarrow M \) where the domain has the product metric for a bi-invariant metric on \( \tilde{G} \) and with \( N \) a complete pseudo-Riemannian manifold.

As a consequence, we obtain a lower bound on the dimension of \( M \).

**Proposition 1.6.** Let \( M \) be a connected analytic pseudo-Riemannian manifold and \( G \) a connected non-compact simple Lie group. Suppose that \( M \) is complete weakly irreducible, has finite volume and admits an analytic isometric non-transitive \( G \)-action with a dense orbit. Then:

\[
\dim(M) \geq \dim(G) + m(g),
\]

where \( m(g) \) is the dimension of the smallest non-trivial representation of \( g \) that admits an invariant non-degenerate symmetric bilinear form.
Proof. Suppose that \( \dim(M) < \dim(G) + m(\mathfrak{g}) \). Since \( m(\mathfrak{g}) \leq \dim(G) \) (the Killing form defines an inner product), by Lemma 1.4 the bundle \( TO^\perp \) has non-degenerate fibers with dimension \( < m(\mathfrak{g}) \). Hence, the definition of \( m(\mathfrak{g}) \) implies that \( T_xO^\perp \) is a trivial \( \mathfrak{g} \)-module for the structure defined by Proposition 1.2(4). Hence, Lemma 1.3 yields the integrability of \( TO^\perp \), and Proposition 1.5 contradicts the irreducibility of \( M \).

For a \( G \)-action as in Proposition 1.2 consider \( \tilde{M} \) endowed with the \( \tilde{G} \)-action obtained by lifting the \( G \)-action on \( M \). With such setup, let us denote by \( \mathcal{H} \) the Lie subalgebra of \( \text{Kill}(\tilde{M}) \) consisting of the fields that centralize the \( \tilde{G} \)-action on \( \tilde{M} \). The next result provides an embedding of \( \mathfrak{g} \) into \( \mathcal{H} \) that allows us to apply representation theory to study the structure of \( \mathcal{H} \). We observe that this statement is at the core of Gromov-Zimmer’s machinery on the study of actions preserving geometric structures (see [10, 22]).

Lemma 1.7. Let \( S \) be as in Proposition 1.2. Then, for every \( x \in S \) and for \( \rho_x \) given as in Proposition 1.2, the map \( \hat{\rho}_x : \mathfrak{g} \rightarrow \text{Kill}(\tilde{M}) \) given by:

\[
\hat{\rho}(X) = \rho_x(X) + X^*,
\]

is an injective homomorphism of Lie algebras whose image \( \mathcal{G}(x) \) lies in \( \mathcal{H} \). In particular, \( \hat{\rho}_x \) induces on \( \mathcal{H} \) a \( \mathfrak{g} \)-module structure such that \( \mathcal{G}(x) \) is a submodule isomorphic to \( \mathfrak{g} \).

Proof. First, observe that the identity in Proposition 1.2(3) is easily seen to imply that the image of \( \hat{\rho}_x \) lies in \( \mathcal{H} \).

To prove that \( \hat{\rho}_x \) is a homomorphism of Lie algebras we apply Proposition 1.2(3) as follows for \( X, Y \in \mathfrak{g} \):

\[
[\hat{\rho}_x(X), \hat{\rho}_x(Y)] = [\rho_x(X) + X^*, \rho_x(Y) + Y^*]
= [\rho_x(X), \rho_x(Y)] + [\rho_x(X), Y^*] + [X^*, \rho_x(Y)] + [X^*, Y^*]
= \rho_x([X, Y]) + [X, Y]^* + [X, Y]^* + [X^*, Y^*]
= \rho_x([X, Y]) + [X, Y]^*
= \hat{\rho}_x([X, Y]).
\]

For the injectivity of \( \hat{\rho}_x \) we observe that \( \hat{\rho}_x(X) = 0 \) implies \( X^*_x = (\rho_x(X) + X^*)_x = 0 \), which in turns yields \( X = 0 \) because the \( G \)-action is locally free. The last claim is now clear.

We can now relate the \( \mathfrak{g} \)-module structure of \( \mathcal{H} \) to that of \( T_x\tilde{M} \).

Lemma 1.8. Let \( S \) be as in Proposition 1.2. Consider \( T_x\tilde{M} \) and \( \mathcal{H} \) endowed with the \( \mathfrak{g} \)-module structures given by Proposition 1.2(4) and Lemma 1.7, respectively. Then, for every \( x \in S \), the evaluation map:

\[
ev_x : \mathcal{H} \rightarrow T_x\tilde{M}, \quad Z \mapsto Z_x,
\]

is a homomorphism of \( \mathfrak{g} \)-modules that satisfies \( \ev_x(\mathcal{G}(x)) = T_xO \). Furthermore, for almost every \( x \in S \) we have \( \ev_x(\mathcal{H}) = T_x\tilde{M} \).

Proof. For every \( x \in S \), if we let \( Z \in \mathcal{H} \) and \( X \in \mathfrak{g} \) be given, then:

\[
\ev_x(X \cdot Z) = [\hat{\rho}_x(X), Z]_x = [\rho_x(X) + X^*, Z]_x
= [\rho_x(X), Z]_x = X^* \cdot Z_x = X \cdot \ev_x(Z)
\]
where we have used Lemma 1.1 and the definition of the $g$-module structures involved, thus proving the first part. The last claim follows by an easy adaptation of the proof of Lemma 4.1 of [23], which establishes the transitivity of $\mathcal{H}$ on an open conull dense subset of $\tilde{M}$.

To study in the following sections those $G$-actions for which $T\mathcal{O}^\perp$ is non-integrable we will need to use some known results that relate isometries with Killing fields for complete manifolds. First, we have the following result, which follows from the rigidity (in the sense of [10]) of pseudo-Riemannian metrics and their basic properties.

**Lemma 1.9.** Let $N$ be an analytic pseudo-Riemannian manifold. Then, every Killing vector field of $N$, either local or global, is analytic. In particular, the isometry group $\text{Iso}(N)$ acts analytically on $N$.

By Proposition 30 of Chapter 9 from [13], on a complete pseudo-Riemannian manifold every global Killing vector field is complete. Hence, Proposition 33 of Chapter 9 from [13] has the following immediate consequence.

**Lemma 1.10.** Let $N$ be a complete pseudo-Riemannian manifold and suppose that the action of its isometry group $\text{Iso}(N)$ is considered on the left. If $\text{Iso}(N)$ denotes the Lie algebra of $\text{Iso}(N)$, then the map:

$$\text{Iso}(N) \to \text{Kill}(N), \quad X \mapsto X^*,$$

is an anti-isomorphism of Lie algebras. In particular, $[X,Y]^* = -[X^*,Y^*]$ for every $X,Y \in \mathfrak{so}(N)$.

We now use the above to prove that on a complete manifold every Lie algebra of Killing fields can be realized from an isometric right action.

**Lemma 1.11.** Let $N$ be a complete pseudo-Riemannian manifold and $H$ a simply connected Lie group with Lie algebra $\mathfrak{h}$. If $\psi : \mathfrak{h} \to \text{Kill}(N)$ is a homomorphism of Lie algebras, then there exists an isometric right $H$-action $N \times H \to N$ such that $\psi(X) = X^*$, for every $X \in \mathfrak{h}$. Furthermore, if $N$ is analytic, then the $H$-action is analytic as well.

**Proof.** Consider the map $\alpha : \mathfrak{so}(N) \to \text{Kill}(N)$ given by $\alpha(Y) = -Y^*$, which is an isomorphism of Lie algebras by Lemma 1.10. Let $\Psi : H \to \text{Iso}(N)$ be the homomorphism of Lie groups induced by the homomorphism $\alpha^{-1} \circ \psi : \mathfrak{h} \to \mathfrak{so}(N)$. This yields a smooth isometric right $H$-action given by:

$$N \times H \to N, \quad (n,h) \mapsto nh = \Psi(h^{-1})(n).$$

For $X \in \mathfrak{h}$ there is $Y \in \mathfrak{so}(N)$ such that $\psi(X) = -\alpha(Y) = Y^*$. Hence, for the right $H$-action one computes $X^*$ at every $p \in N$ as follows:

$$X_p^* = \left. \frac{d}{dt} \right|_{t=0} \exp(tX) = \left. \frac{d}{dt} \right|_{t=0} \Psi(\exp(-tX))(p)$$

$$= \left. \frac{d}{dt} \right|_{t=0} \exp(-t(\alpha^{-1} \circ \psi)(X))p = \left. \frac{d}{dt} \right|_{t=0} \exp(tY)p$$

$$= Y_p^* = \psi(X)_p,$$

which proves the first part of the lemma. Note that the first and second to last identities use the definition of $Z^*$ for the right $H$-action and the left $\text{Iso}(N)$-action, respectively. Finally, the last part of our statement follows from the last claim of Lemma 1.9.
2. Structure of the centralizer of isometric $\widetilde{SO}_0(p, q)$-actions for low dimensional $M$ and non-integrable $T\mathcal{O}^\perp$

For $n \in \mathbb{Z}_+$ let $p, q \in \mathbb{Z}_+$ be such that $p + q = n$ and

$$I_{p, q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$  

Then, $\mathfrak{so}(p, q)$ is the Lie algebra of linear transformations of $\mathbb{R}^n$ that are anti-symmetric with respect to the inner product $\langle \cdot, \cdot \rangle_{p, q}$ on $\mathbb{R}^n$ defined by $I_{p, q}$. $\mathfrak{so}_0(p, q)$ denotes the connected group of isometries with respect to $\langle \cdot, \cdot \rangle_{p, q}$ and $\mathfrak{so}_0(p, q)$ its universal covering group. We let $\mathbb{R}^{p-q}$ denote the $\mathfrak{so}(p, q)$-module defined by the natural representation of $\mathfrak{so}(p, q)$ in $\mathbb{R}^n$. Denote by $C^+$ and $C^-$ the $\mathfrak{so}(4, 4)$-modules given by real forms of the two half spin representations of $\mathfrak{so}(8, \mathbb{C})$. We refer to $C^+$ and $C^-$ as the half spin representations of $\mathfrak{so}(4, 4)$.

In preparation for the proof of the Main Theorem, we assume in this section that $M$ is a connected finite volume analytic pseudo-Riemannian manifold with $\dim(M) = n(n + 1)/2$. We assume that $p + q \geq 5$ or $(p, q) = (3, 1)$. We also assume that $\mathfrak{so}_0(p, q)$ acts analytically, isometrically and with a dense orbit on $M$.

In particular, by Lemma 1.3 we have the direct sum $TM = T\mathcal{O} \oplus T\mathcal{O}^\perp$. Also note that $T\mathcal{O}^\perp$ has rank $n$. Finally, we also assume in the rest of this section that the bundle $T\mathcal{O}^\perp$ is non-integrable.

As before, the $\mathfrak{so}_0(p, q)$-action on $M$ can be lifted to $\widetilde{M}$, thus inducing a direct sum decomposition $T\widetilde{M} = T\mathcal{O} \oplus T\mathcal{O}^\perp$ as a consequence of the corresponding property for $M$. Again, we denote with $\mathcal{O}$ the foliation in $\widetilde{M}$ whose leaves are the orbits for the $\mathfrak{so}_0(p, q)$-action on $\widetilde{M}$.

In the rest of this work we will denote with $\mathcal{H}$ the Lie subalgebra of $\text{Kill}(\widetilde{M})$ consisting of the fields that centralize the $\mathfrak{so}_0(p, q)$-action. In particular, there is a set $S$ as in Proposition 1.2 for which Lemmas 1.7 and 1.8 hold. We will now see that our hypotheses allow to provide a precise description of the $\mathfrak{so}(p, q)$-module structures obtained through these results from the previous section.

Lemma 2.1. Let $S$ be as in Proposition 1.2. Consider $T\mathcal{O}^\perp$ endowed with the $\mathfrak{so}(p, q)$-module structure given by Proposition 1.2.4. Then, for almost every $x \in S$:

1. for $(p, q) \neq (4, 4)$, the $\mathfrak{so}(p, q)$-module $T_x\mathcal{O}^\perp$ is isomorphic to $\mathbb{R}^{p-q}$, and
2. for $p = q = 4$, the $\mathfrak{so}(4, 4)$-module $T_x\mathcal{O}^\perp$ is isomorphic to either $\mathbb{R}^{4, 4}$, $C^+$ or $C^-$.

In particular, $\mathfrak{so}(T_x\mathcal{O}^\perp)$ is isomorphic to $\mathfrak{so}(p, q)$ as a Lie algebra and as a $\mathfrak{so}(p, q)$-module, for almost every $x \in S$.

Proof. By Lemma 1.3(2) and since we are assuming that $T\mathcal{O}^\perp$ is non-integrable, the 2-form $\Omega$ considered in its statement is non-zero. This 2-form is clearly analytic and thus it vanishes at a proper analytic subset of $\widetilde{M}$ which is necessarily null. Hence, $\Omega_x \neq 0$ for almost every $x \in S$. Let us choose and fix $x \in S$ such that $\Omega_x \neq 0$; we will prove that the conclusions of the statement hold for such $x$.

Lemma 1.3(1) implies that the map $\Omega_x : \wedge^2 T_x\mathcal{O}^\perp \to \mathfrak{so}(p, q)$ is a homomorphism of $\mathfrak{so}(p, q)$-modules, which is then non-trivial by our choice of $x$. Since $\dim(T_x\mathcal{O}^\perp) = n$ and because $\mathfrak{so}(p, q)$ is an irreducible module, it follows that $\Omega_x$ is an isomorphism. Then, the irreducibility of $\mathfrak{so}(p, q)$ implies that $T_x\mathcal{O}^\perp$ is irreducible as well.
By Lemma 1.8 it follows that $T_x \mathcal{O}^\perp \cong \mathbb{R}^{p,q}$ except for the cases given by the Lie algebras $\mathfrak{so}(3,1)$ and $\mathfrak{so}(4,4)$. For these Lie algebras the other possibilities are $C^2_\mathbb{R}$ for $\mathfrak{so}(3,1)$, and real forms $C^+$ and $C^-$ of the two half spin representations of $\mathfrak{so}(8, \mathbb{C})$, for the case of $\mathfrak{so}(4,4)$.

Let us consider the case of $\mathfrak{so}(3,1)$. If $(\pi, V)$ is a $G$-module and $\chi_V = \text{tr} \circ \pi$ is its character, then for every $g \in G$ we have $\chi_{\pi^* V}(g) = \frac{1}{2}(\chi_V(g)^2 - \chi_V(g^2))$. Hence, for any such element we have $\chi_{\mathfrak{so}(3,1)}(g) = \frac{1}{2}(\chi_V(g)^2 - \chi_V(g^2))$. A simple calculation using

$$
g = \begin{pmatrix}
\cosh(t) & 0 & \sinh(t) \\
0 & I_2 & 0 \\
\sinh(t) & 0 & \cosh(t)
\end{pmatrix}
$$

shows that the above holds only for $V \cong \mathbb{R}^{3,1}$, and so $T_x \mathcal{O}^\perp \cong \mathbb{R}^{3,1}$.

For the final claim, we observe that the representation of $\mathfrak{so}(p,q)$ on $T_x \mathcal{O}^\perp$ defines a non-trivial homomorphism $\mathfrak{so}(p,q) \to \mathfrak{so}(T_x \mathcal{O}^\perp)$. Since $\mathfrak{so}(p,q)$ is simple, the latter is injective and so it is an isomorphism. □

The previous results allow us to obtain the following decomposition of the centralizer $\mathcal{H}$ of the $\text{SO}(p,q)$-action into submodules related to the geometric structure on $\tilde{M}$.

**Lemma 2.2.** Let $S$ be as in Proposition 1.2. Then, for almost every $x \in S$ and for the $\mathfrak{so}(p,q)$-module structure on $\mathcal{H}$ from Lemma 1.7 there is a decomposition into $\mathfrak{so}(p,q)$-submodules $\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x)$, satisfying:

1. $\mathcal{G}(x) = \tilde{\rho}_x(\mathfrak{so}(p,q))$ is a Lie subalgebra of $\mathcal{H}$ isomorphic to $\mathfrak{so}(p,q)$ and $ev_x(\mathcal{G}(x)) = T_x \mathcal{O}$.
2. $\mathcal{H}_0(x) = \ker(ev_x)$, which is either 0 or a Lie subalgebra of $\mathcal{H}$ isomorphic to $\mathfrak{so}(p,q)$.
3. $ev_x(\mathcal{V}(x)) = T_x \mathcal{O}^\perp$ and
   - for $(p,q) \neq (4,4)$, $\mathcal{V}(x)$ is isomorphic to $\mathbb{R}^{p,q}$ as $\mathfrak{so}(p,q)$-module,
   - for $p = q = 4$, $\mathcal{V}(x)$ is isomorphic to either $\mathbb{R}^{4,4}$, $C^+$ or $C^-$ as $\mathfrak{so}(4,4)$-module.

In particular, the evaluation map $ev_x$ defines an isomorphism of $\mathfrak{so}(p,q)$-modules $\mathcal{G}(x) \oplus \mathcal{V}(x) \to T_x \tilde{M} = T_x \mathcal{O} \oplus T_x \mathcal{O}^\perp$ preserving the summands in that order.

**Proof.** Note that the conclusions of both Lemmas 1.8 and 2.1 are satisfied for almost every $x \in S$. Let us choose and fix one such point $x \in S$. By Lemma 1.7 we conclude that $\mathcal{G}(x) = \tilde{\rho}_x(\mathfrak{so}(p,q))$ is indeed a Lie subalgebra isomorphic to $\mathfrak{so}(p,q)$.

Define $\mathcal{H}_0(x) = \ker(ev_x)$. By Lemma 1.8 it follows that $\mathcal{H}_0(x)$ is an $\mathfrak{so}(p,q)$-submodule of $\mathcal{H}$. Moreover, since $\mathcal{H}_0(x) = \mathcal{H} \cap \text{Kill}_0(\tilde{M},x)$ it follows that it is a Lie subalgebra as well.

On the other hand, the elements of $\mathcal{G}(x)$ are of the form $\rho_x(X) + X^*$, with $X \in \mathfrak{so}(p,q)$, where $\rho_x$ is the Lie algebra homomorphism from Proposition 1.2. Hence, for any such element we have $ev_x(\rho_x(X) + X^*) = X^*_x$; in particular, the condition $ev_x(\rho_x(X) + X^*) = 0$ implies $X = 0$. In other words, $\mathcal{G}(x) \cap \mathcal{H}_0(x) = \{0\}$. Hence, there exists an $\mathfrak{so}(p,q)$-submodule $\mathcal{V}(x)$ complementary to $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$ in $\mathcal{H}$. In particular, $ev_x$ restricted to $\mathcal{G}(x) \oplus \mathcal{V}(x)$ is an isomorphism of $\mathfrak{so}(p,q)$-modules onto $T_x \tilde{M}$. Hence, if we choose $\mathcal{V}(x)$ as the inverse image of $T_x \mathcal{O}^\perp$ under
this isomorphism, then we have the required decomposition into $\mathfrak{so}(p, q)$-submodules except for the properties of $\mathcal{H}_0(x)$, which we now proceed to consider.

Let $\text{Kill}_0(\bar{M}, x, \mathcal{O})$ be the Lie algebra of Killing vector fields on $\bar{M}$ that preserve the foliation $\mathcal{O}$ and that vanish at $x$. Hence, every Killing field in $\text{Kill}_0(\bar{M}, x, \mathcal{O})$ leaves invariant the normal bundle $T_x\mathcal{O}$, and so the map $\lambda_x$ from Lemma 1.4 induces a homomorphism of Lie algebras:

$$\lambda_x : \text{Kill}_0(\bar{M}, x, \mathcal{O}) \to \mathfrak{so}(T_x\mathcal{O}^\perp), \quad X \mapsto \lambda_x(X)|_{T_x\mathcal{O}^\perp}.$$

We observe that both Lie algebras $\rho_x(\mathfrak{so}(p, q)) = \mathfrak{so}(p, q)(x)$ and $\mathcal{H}_0(x)$ lie inside of $\text{Kill}_0(\bar{M}, x, \mathcal{O})$. A number of properties for the restriction of $\lambda_x$ to $\mathfrak{so}(p, q)(x)$ and $\mathcal{H}_0(x)$ will imply the needed conditions on $\mathcal{H}_0(x)$.

\textbf{Claim 1:} $\lambda_x$ restricted to $\mathcal{H}_0(x)$ is injective. We recall that a Killing vector field is completely determined by its 1-jet at $x$; this is a consequence of the fact that pseudo-Riemannian metrics are 1-rigid (see [7, 10]). If $Z \in \mathcal{H}_0(x)$ is given, then $Z_x = 0$ and so it is completely determined by the values of $[Z, V]_x$ for $V$ a vector field in a neighborhood of $x$. On the other hand $[Z, X^*_x]_x = 0$ for $X \in \mathfrak{so}(p, q)$ and so $[Z, V]_x = 0$ whenever $V_x \in T_x\mathcal{O}$. We conclude that $Z$ is completely determined by the values of $[Z, V]_x$ for $V$ such that $V_x \in T_x\mathcal{O}^\perp$. In other words, $[Z, V]_x = 0$ for every $V_x \in T_x\mathcal{O}^\perp$ implies $Z = 0$. This yields the injectivity of $\lambda_x$ on $\mathcal{H}_0(x)$.

\textbf{Claim 2:} $\lambda_x(\mathfrak{so}(p, q)(x)) = \mathfrak{so}(T_x\mathcal{O}^\perp)$. By Proposition 1.4, the vector space $T_x\mathcal{O}^\perp$ has a $\mathfrak{so}(p, q)$-module structure induced from the homomorphism $\lambda_x \circ \rho_x$. By our choice of $x$ and Lemma 2, such module structure is in fact non-trivial. Hence, $\lambda_x \circ \rho_x : \mathfrak{so}(p, q) \to \mathfrak{so}(T_x\mathcal{O}^\perp)$ is non-trivial as well and so it is injective. But then it has to be surjective because the domain and target have the same dimensions.

\textbf{Claim 3:} $\lambda_x(\mathcal{H}_0(x))$ is an ideal in $\mathfrak{so}(T_x\mathcal{O}^\perp)$. Let $Z \in \mathcal{H}_0(x)$ and $T \in \mathfrak{so}(T_x\mathcal{O}^\perp)$ be given. Then, by Claim 2, there is some $X \in \mathfrak{so}(p, q)$ such that $T = \lambda_x(\rho_x(X))$. For every local vector field $V$ such that $V_x \in T_x\mathcal{O}^\perp$ we have:

$$[T, \lambda_x(Z)](V_x) = [\lambda_x(\rho_x(X)), \lambda_x(Z)](V_x) = [\rho_x(X), [Z, V]]_x = [Z, [\rho_x(X), V]]_x$$

$$= [\rho_x(X), Z]^*_x V_x = [\rho_x(X), Z] V_x = [[\rho_x(X), Z], V]_x = ([\rho_x(X), Z] V) [V].$$

Since the $\mathfrak{so}(p, q)$-module structure on $\mathcal{H}_0(x)$ is defined by $\tilde{\rho}_x$ and $\mathcal{H}_0(x)$ is a submodule of such structure, we have $[\tilde{\rho}_x(X), Z] \in \mathcal{H}_0(x)$, and so the last formula proves that $[T, \lambda_x(Z)] = \lambda_x([\tilde{\rho}_x(X), Z])$, thus showing the claim.

Claim 1 shows that $\mathcal{H}_0(x)$ is a Lie algebra isomorphic to its image in $\mathfrak{so}(T_x\mathcal{O}^\perp)$ under $\lambda_x$. Such image is by Claim 3 an ideal of $\mathfrak{so}(T_x\mathcal{O}^\perp)$. By our choice of $x$ and Lemma 2, the Lie algebra $\mathfrak{so}(T_x\mathcal{O}^\perp)$ is isomorphic to $\mathfrak{so}(p, q)$, which is simple since $n \geq 4$ and $(p, q) \neq (2, 2)$. This implies that $\mathcal{H}_0(x)$ is either 0 or isomorphic to $\mathfrak{so}(p, q)$ as a Lie subalgebra of $\mathcal{H}$.

On the other hand, for $X \in \mathfrak{so}(p, q)$ and $Z \in \mathcal{H}_0(x)$, considering the definitions of the $\mathfrak{so}(p, q)$-module structures involved we have:

$$\lambda_x(X \cdot Z) = \lambda_x([\tilde{\rho}_x(X), Z]) = \lambda_x([\rho_x(X), Z]) = [\lambda_x(\rho_x(X)), \lambda_x(Z)] = X \cdot \lambda_x(Z),$$

where the second identity holds by the definition of $\tilde{\rho}_x$ in terms of $\rho_x$ and because $\mathcal{H}_0(x)$ centralizes the $\text{SO}_0(p, q)$-action. But this last relation shows that $\lambda_x$ restricted to $\mathcal{H}_0(x)$ is a homomorphism of $\mathfrak{so}(p, q)$-modules. By Lemma 2, we conclude that $\mathcal{H}_0(x)$ is either 0 or isomorphic to $\mathfrak{so}(p, q)$ as a $\mathfrak{so}(p, q)$-module. \qed
Lemma 2.3. Let $S$ be as in Proposition 1.2. With the notation from Lemma 2.2, one of the following conditions is satisfied for almost every $x \in S$:

1. The radical $\text{rad}(\mathcal{H})$ of $\mathcal{H}$ is Abelian and $\text{rad}(\mathcal{H}) = \mathcal{V}(x)$.
2. $\mathcal{H}_0(x) \neq \{0\}$ and $\mathcal{H}$ is the sum of two simple ideals one of them being $\mathcal{H}_0(x) \oplus \mathcal{V}(x)$.
3. $\mathcal{H}_0(x) = \{0\}$ and $\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{V}(x)$ is isomorphic as a Lie algebra to either $\mathfrak{so}(p, q + 1)$ or $\mathfrak{so}(p + 1, q)$.

Proof. For this proof let us choose and fix $x \in S$ satisfying the conclusions of Lemmas 1.7, 1.8, 2.1 and 2.2.

Note that $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$ is a Lie subalgebra with $\mathcal{H}_0(x)$ as an ideal because $[\mathcal{G}(x), \mathcal{H}_0(x)] \subset \mathcal{H}_0(x)$ as a consequence of Lemmas 1.7, 1.8. Since $\mathcal{G}(x), \mathcal{H}_0(x)$ are both isomorphic to $\mathfrak{so}(p, q)$ if $\mathcal{H}_0(x) \neq \{0\}$, we conclude that $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$ is semisimple. Furthermore, this implies that $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$ is isomorphic to either $\mathfrak{so}(p, q)$ or $\mathfrak{so}(p, q) \oplus \mathfrak{so}(p, q)$ according to whether $\mathcal{H}_0(x)$ is zero or not, respectively.

Choose a Levi factor $\mathfrak{s}$ of $\mathcal{H}$ that contains the Lie subalgebra $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$. Since the $\mathfrak{so}(p, q)$-module structure of $\mathcal{H}$ is defined by the Lie subalgebra $\mathcal{G}(x)$ (see Lemma 1.7), it follows that $\mathfrak{s}$ is an $\mathfrak{so}(p, q)$-submodule of $\mathcal{H}$. Let $W$ be an $\mathfrak{so}(p, q)$-submodule of $\mathcal{H}$ such that $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus W$. Since $\text{rad}(\mathcal{H})$ is an ideal, this induces the following decomposition of $\mathcal{H}$ as a direct sum of $\mathfrak{so}(p, q)$-submodules:

$$\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus W \oplus \text{rad}(\mathcal{H}).$$

From this decomposition of $\mathfrak{so}(p, q)$-modules, as well as Lemmas 1.7 and 2.2, we conclude that one of the following holds:

(a) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{H}_0(x)$ and $\text{rad}(\mathcal{H}) = \mathcal{V}(x)$, or
(b) $\text{rad}(\mathcal{H}) = \{0\}$ and so $\mathcal{H}$ is semisimple.

Let us assume that case (a) holds. Then, the Lie brackets of $\mathcal{H}$ restricted to $\wedge^2 \mathcal{V}(x)$ define, by the Jacobi identity and Lemma 1.7, a homomorphism of $\mathfrak{so}(p, q)$-modules $\wedge^2 \mathcal{V}(x) \to \mathcal{V}(x)$. This homomorphism is necessarily trivial since $\mathcal{V}(x)$ is $n$-dimensional, $\wedge^2 \mathcal{V}(x) \simeq \mathfrak{so}(p, q)$ is irreducible and $n \geq 4$. This shows that $\mathcal{V}(x) = \text{rad}(\mathcal{H})$ is Abelian and yields (1) from our statement.

Let us now assume that case (b) holds, and write $\mathcal{H} = \mathfrak{h}_1 \times \cdots \times \mathfrak{h}_k$ a direct product of simple ideals. Since each such ideal is invariant by $\mathcal{G}(x)$, it is an $\mathfrak{so}(p, q)$-submodule, and so it follows that $k \leq 3$ since the decomposition of $\mathcal{H}$ from Lemma 2.2 has at most 3 irreducible summands. Moreover, if $k = 3$ we conclude that, after reindexing the ideas, we have $\mathcal{V}(x) = \mathfrak{h}_1$ and $\mathcal{G}(x) \oplus \mathcal{H}(x) = \mathfrak{h}_2 \times \mathfrak{h}_3$. In particular, $[\mathcal{G}(x), \mathcal{V}(x)] = \{0\}$ which implies that $\mathcal{V}(x)$ is a trivial $\mathfrak{so}(p, q)$-submodule and contradicts Lemma 2.2(3). Hence we can further assume that the number of simple ideals of $\mathcal{H}$ is $k \leq 2$.

First suppose that $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2$, the direct product of two simple ideals. If $\mathcal{H}_0(x) = \{0\}$, then the decomposition of $\mathcal{H}$ from Lemma 2.2 has two irreducible summands and we can reindex the ideals $\mathfrak{h}_1, \mathfrak{h}_2$ to assume that $\mathfrak{h}_1 = \mathcal{G}(x)$ and $\mathfrak{h}_2 = \mathcal{V}(x)$. But this implies that $[\mathcal{G}(x), \mathcal{V}(x)] = \{0\}$, a contradiction. Hence we conclude that $\mathcal{H}_0(x) \neq \{0\}$ in the current case. In particular, $\mathcal{H}$ is the direct sum of three irreducible $\mathfrak{so}(p, q)$-submodules. Hence, after decomposing $\mathfrak{h}_1, \mathfrak{h}_2$ as the direct sum of irreducible $\mathfrak{so}(p, q)$-submodules, and reindexing if necessary, we can assume that $\mathfrak{h}_1$ is an irreducible $\mathfrak{so}(p, q)$-submodule and that $\mathfrak{h}_2 = V_1 \oplus V_2$, where $V_1, V_2$ are
irreducible $\mathfrak{so}(p, q)$-submodules. By comparing the decomposition $\mathcal{H} = h_1 \oplus V_1 \oplus V_2$, with the one from Lemma 2.2, we conclude that $\mathcal{V}(x)$ is either one of $h_1$, $V_1$ or $V_2$. If $\mathcal{V}(x) = h_1$, then $[\mathcal{V}(x), \mathcal{V}(x)] \subset \mathcal{V}(x)$ and an argument used above shows that $\mathcal{V}(x)$ is Abelian, which contradicts the simplicity of $h_1$. Hence, without loss of generality, we can assume that $\mathcal{V}(x) = V_2$, and so that $\mathcal{G}(x) \oplus \mathcal{H}_0(x) = h_1 \oplus V_1$. In particular, since $V_1$ is a subspace of both of the Lie algebras $\mathcal{G}(x) \oplus \mathcal{H}_0(x) = h_1 \oplus V_1$ and $h_1 \oplus V_2$ it follows that $[V_1, V_1] \subset V_1$, thus showing that $V_1$ itself is a Lie algebra. But since $[h_1, V_1] = \{0\}$, this implies that the right-hand side of the sum:

$$\mathcal{G}(x) \oplus \mathcal{H}_0(x) = h_1 \oplus V_1$$

is the decomposition into simple ideals. Since $\mathcal{H}_0(x)$ is an ideal of $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$, it is either $h_1$ or $V_1$. If $\mathcal{H}_0(x) = h_1$, then $[\mathcal{H}_0(x), \mathcal{V}(x)] = \{0\}$, which is in contradiction with Claim 1 in the proof of Lemma 2.2 because $ev_x(\mathcal{V}(x)) = T_xO$. Hence, $\mathcal{H}_0(x) = V_1$ and so $\mathcal{H}_0(x) \oplus \mathcal{V}(x) = h_2$ is a simple ideal of $\mathcal{H}$, thus establishing option (2).

Finally, let us assume that $\mathcal{H}$ is a simple Lie algebra. We will prove that in this case (3) holds.

Let us start by assuming that $\mathcal{H}_0(x) \neq \{0\}$. Hence, from the above remarks, we can write $\mathcal{G}(x) \oplus \mathcal{H}_0(x) = g_1 \oplus g_2$, where $g_1, g_2$ are ideals of $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$ both isomorphic to $\mathfrak{so}(p, q)$. Let $V$ be a $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$-submodule of $\mathcal{H}$ such that:

$$\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus V.$$ 

In particular, $V$ has dimension $n$. Moreover, $V$ is necessarily a non-trivial $g_1$-module, since otherwise $g_1$ would be a proper ideal of $\mathcal{H}$. Then, Lemma [A.1] implies that we can decompose $V = V_0 \oplus V_1$ where $V_0$ is a trivial $g_1$-module and $V_1$ is an irreducible $g_1$-module. Note that this can be done so that $V_0 = \{0\}$, except for the cases given by $\mathfrak{so}(3, 2)$ and $\mathfrak{so}(3, 3)$ for which we can assume $\dim(V_0) = 1$ or 2, respectively. In any case, this yields a decomposition of $\mathcal{H}$ into $g_1$-submodules given by:

$$\mathcal{H} = g_1 \oplus g_2 \oplus V_0 \oplus V_1.$$ 

Since $g_1$ and $g_2$ commute with each other, then for every $X \in g_2$ the map $\text{ad}_H(X)$ defines a $g_1$-module homomorphism of $V$ and so preserves its summands corresponding to given isomorphism classes for the $g_1$-module structure. Hence, $V_0$ and $V_1$ are $g_2$-modules as well, and by Lemma [A.1] it follows that $V_0$ is a trivial $g_2$-module, because $\dim(V_0) \leq 2$. As before, $V_1$ is a non-trivial $g_2$-module, since otherwise $g_2$ would be a proper ideal of $\mathcal{H}$.

The above shows that $\text{ad}_H$ restricted to $\mathcal{G}(x) \oplus \mathcal{H}_0(x) \simeq \mathfrak{so}(p, q) \oplus \mathfrak{so}(p, q)$ leaves invariant $V_1$ and induces a representation:

$$\rho : \mathfrak{so}(p, q) \oplus \mathfrak{so}(p, q) \to \mathfrak{sl}(V_1) \simeq \mathfrak{sl}(k, \mathbb{R}),$$

for some $k \leq n$, which is injective when restricted to each summand. Furthermore, $\rho(\mathfrak{so}(p, q) \oplus \{0\})$ and $\rho(\{0\} \oplus \mathfrak{so}(p, q))$ centralize each other, from which the simplicity of $\mathfrak{so}(p, q)$ implies that $\rho(\mathfrak{so}(p, q) \oplus \{0\}) \cap \rho(\{0\} \oplus \mathfrak{so}(p, q)) = \{0\}$. In particular, $\rho$ realizes $\mathfrak{so}(p, q) \oplus \mathfrak{so}(p, q)$ as a Lie subalgebra of $\mathfrak{sl}(k, \mathbb{R})$. Note that the codimension of $\mathfrak{so}(p, q) \oplus \mathfrak{so}(p, q)$ in $\mathfrak{sl}(k, \mathbb{R})$ is $k^2 - 1 - n(n - 1) \leq n - 1$.

Replacing $\mathcal{H}$ with $\mathfrak{sl}(k, \mathbb{R})$ and repeating these arguments, more than once if necessary, yields a non-trivial representation of $\mathfrak{so}(p, q)$ with dimension strictly smaller than the lower bound obtained in Lemma [A.1] for irreducible non-trivial representations. This contradiction proves that $\mathcal{H}_0(x) = \{0\}$.
With our assumption that \( \mathcal{H} \) is simple we thus obtain \( \mathcal{H} = \mathcal{G}(x) \oplus \mathcal{V}(x) \). Hence, to conclude (3) it remains to show that \( \mathcal{H} \) is isomorphic to either \( \mathfrak{so}(p, q + 1) \) or \( \mathfrak{so}(p + 1, q) \).

Recall from the previous remarks that \( \mathcal{H} = \mathcal{G}(x) \oplus \mathcal{V}(x) \) is the decomposition into irreducible \( \mathfrak{so}(p, q) \)-modules for an isomorphism \( \mathcal{G}(x) \simeq \mathfrak{so}(p, q) \) of Lie algebras. Furthermore, either \( \mathcal{V}(x) \simeq \mathbb{R}^{p,q} \) as \( \mathfrak{so}(p,q) \)-modules or \( p = q = 4 \) and \( \mathcal{V}(x) \) is isomorphic to one of \( C^+ \) or \( C^- \) as \( \mathfrak{so}(4,4) \)-modules. Let us first assume that the latter case holds. Then, \( \mathcal{H} \) is a 36-dimensional simple Lie algebra. Since there are no simple complex Lie algebras of dimension 18, it follows that the complexification \( \mathcal{H}^\mathbb{C} \) is a simple Lie algebra. Hence, a direct inspection of the simple complex Lie algebras shows that \( \mathcal{H}^\mathbb{C} \) is up to isomorphism either \( \mathfrak{so}(9, \mathbb{C}) \) or \( \mathfrak{sp}(4, \mathbb{C}) \). For \( \mathcal{H}^\mathbb{C} \simeq \mathfrak{so}(9, \mathbb{C}) \) we conclude that \( \mathcal{H} \simeq \mathfrak{so}(5, 4) \) since it contains the Lie algebra \( \mathcal{G}(x) \simeq \mathfrak{so}(4, 4) \). For the case \( \mathcal{H}^\mathbb{C} \simeq \mathfrak{sp}(4, \mathbb{C}) \) we obtain a non-trivial homomorphism \( \mathfrak{so}(8, \mathbb{C}) \simeq \mathcal{G}(x)^\mathbb{C} \subset \mathcal{H}^\mathbb{C} \simeq \mathfrak{sp}(4, \mathbb{C}) \), which yields an 8-dimensional \( \mathfrak{so}(8, \mathbb{C}) \)-module, non-trivial and so irreducible, with an invariant non-degenerate skew-symmetric form. The latter is a contradiction since every 8-dimensional irreducible \( \mathfrak{so}(8, \mathbb{C}) \)-module carries a unique (up to a constant) invariant non-degenerate symmetric form (see page 217 of [4]).

Hence, we can assume that \( \mathcal{V}(x) \simeq \mathbb{R}^{p,q} \) as \( \mathfrak{so}(p,q) \)-modules. Also, since the \( \mathfrak{so}(p,q) \)-module structure on \( \mathcal{H} \) is induced by \( \mathcal{G}(x) \), we have \( \{ \mathcal{G}(x), \mathcal{V}(x) \} \subset \mathcal{V}(x) \). On the other hand, the Lie brackets and the projection \( \mathcal{H} \to \mathcal{V}(x) \) define a homomorphism of \( \mathfrak{so}(p,q) \)-modules \( \wedge^2 \mathcal{V}(x) \to \mathcal{V}(x) \), which is thus trivial. This implies that \( \{ \mathcal{V}(x), \mathcal{V}(x) \} \subset \mathcal{G}(x) \). Hence, there exists a linear isomorphism:

\[
\varphi : \mathcal{H} = \mathcal{G}(x) \oplus \mathcal{V}(x) \to \mathfrak{so}(p,q) \oplus \mathbb{R}^{p,q},
\]

that preserves the summands in that order, that restricts to an isomorphism of Lie algebras \( \mathcal{G}(x) \to \mathfrak{so}(p,q) \) and that defines an isomorphism of \( \mathfrak{so}(p,q) \)-modules. Moreover, we also have proved the relations:

\[
[\mathcal{G}(x), \mathcal{G}(x)] \subset \mathcal{G}(x), \quad [\mathcal{G}(x), \mathcal{V}(x)] \subset \mathcal{V}(x), \quad [\mathcal{V}(x), \mathcal{V}(x)] \subset \mathcal{G}(x).
\]

Note that the last relation defines an isomorphism \( T : \wedge^2 \mathcal{V}(x) \to \mathcal{G}(x) \) of \( \mathcal{G}(x) \)-modules; otherwise, \( \mathcal{V}(x) \) would be a non-trivial Abelian ideal of \( \mathcal{H} \). Recall the map \( T_c(u \wedge v) = c(\{u,\cdot\}_{p,q}v - \{v,\cdot\}_{p,q}u) \) from Lemma [A.3] and with respect to the isomorphism \( \varphi \), the map \( T \) is of the form \( T_c \) for some \( c \in \mathbb{R} \setminus \{0\} \), if \( n \geq 5 \). Let us now assume that \( n \geq 5 \). Hence, the Lie algebra structure on \( \mathfrak{so}(p,q) \oplus \mathbb{R}^{p,q} \) induced by \( \varphi \) is given by \( \{\cdot,\cdot\}_c \), as in Lemma [A.5] for some \( c \neq 0 \), and so isomorphic to either \( \mathfrak{so}(p,q+1) \) or \( \mathfrak{so}(p+1,q) \). This shows that (3) holds when \( \mathcal{H} \) is simple and \( (p,q) \neq (3,1) \).

Finally, consider the case \( (p,q) = (3,1) \). Under such assumption, we have \( \dim(\mathcal{H}) = \dim(\mathcal{G}(x)) + \dim(\mathcal{V}(x)) = 10 \). Since there is no simple complex Lie algebra of dimension 5 we conclude that \( \mathcal{H}^\mathbb{C} \) is simple. Note that, up to isomorphism, \( \mathfrak{so}(5, \mathbb{C}) \) is the only simple complex Lie algebra of complex dimension 10. From this we conclude that \( \mathcal{H} \) is isomorphic to either \( \mathfrak{so}(4,1) \) or \( \mathfrak{so}(3,2) \) (the only non-compact real forms of \( \mathfrak{so}(5, \mathbb{C}) \) up to isomorphism) thus showing that (3) holds when \( \mathcal{H} \) is simple and \( (p,q) = (3,1) \). \( \square \)

3. Proof of the Main Theorem

In this section we will assume the hypotheses of the Main Theorem. More precisely, we assume that \( M \) is a connected analytic pseudo-Riemannian manifold which is
complete weakly irreducible and has finite volume. We also assume that $M$ admits an analytic and isometric $\text{SO}_0(p,q)$-action with a dense orbit for some integers $p,q$ such that $p,q \geq 1$ and $n = p+q \geq 5$, or $(p,q) = (3,1)$. In the case $(p,q) = (3,1)$ we assume that $X^* \perp Y^*$ on $M$ for every $X \in \mathfrak{su}(2)$ and $Y \in i\mathfrak{su}(2)$. Finally we are assuming that $\dim(M) = n(n+1)/2$ and we will consider the three cases provided by Lemma 2.3. For this we will use the notation from Section 2. Our first goal is to rule out cases (1) and (2) of Lemma 2.3 which is done in the next two subsections. Then, we obtain in the third subsection the conclusions of the Main Theorem when case (3) of Lemma 2.3 holds.

3.1. Case 1: The radical of $\mathcal{H}$ is non-trivial. We are assuming that the conclusion (1) of Lemma 2.3 is satisfied for some fixed $x_0 \in \tilde{M}$. We will see that this yields a contradiction with our assumptions on $M$.

First note that $\mathcal{H}(x_0) = \mathcal{G}(x_0) \oplus \mathcal{V}(x_0)$ is a Lie subalgebra of $\mathcal{H}$. This holds because $\mathcal{V}(x_0)$ is Abelian and $[\mathcal{G}(x_0),\mathcal{V}(x_0)] \subset \mathcal{V}(x_0)$. Hence, $\mathcal{G}(x_0) \oplus \mathcal{V}(x_0)$ is isomorphic to the semidirect product Lie algebra $\mathfrak{so}(p,q) \ltimes W$, where $W$ is an $n$-dimensional $\mathfrak{so}(p,q)$-module endowed with the Abelian Lie algebra structure. Choose a Lie algebra isomorphism $\psi : \mathfrak{so}(p,q) \ltimes W \to \mathcal{H}(x_0)$ that maps $\mathfrak{so}(p,q)$ onto $\mathcal{G}(x_0)$ and $W$ onto $\mathcal{V}(x_0)$.

Let us denote with $\widetilde{\text{SO}_0}(p,q) \ltimes W$ the Lie group structure on $\widetilde{\text{SO}_0}(p,q) \ltimes W$ with the semidirect product given by:

$$(A,v) \cdot (B,w) = (AB,B^{-1}v + w),$$

where we are considering the representation of $\widetilde{\text{SO}_0}(p,q)$ on $W$ induced by that of $\mathfrak{so}(p,q)$. In particular, the Lie algebra of $\widetilde{\text{SO}_0}(p,q) \ltimes W$ is $\mathfrak{so}(p,q) \ltimes W$. By Lemma 1.1 there exists an analytic isometric right action of $\widetilde{\text{SO}_0}(p,q) \ltimes W$ on $\tilde{M}$ such that $\psi(X) = X^*$ for every $X \in \mathfrak{so}(p,q) \ltimes W$. Since $\mathcal{H}$ centralizes the left $\widetilde{\text{SO}_0}(p,q)$-action, then the right $\widetilde{\text{SO}_0}(p,q) \ltimes W$-action centralizes the left $\widetilde{\text{SO}_0}(p,q)$-action as well and so it preserves both $T\mathcal{O}$ and $T\mathcal{O}^\perp$.

Using the right $(\widetilde{\text{SO}_0}(p,q) \ltimes W)$-action, let us now consider the map:

$$f : \widetilde{\text{SO}_0}(p,q) \ltimes W \to \tilde{M}, \quad h \mapsto x_0h,$$

which is clearly $(\widetilde{\text{SO}_0}(p,q) \ltimes W)$-equivariant for the right action on its domain. In what follows, we will denote with $I$ the identity element in $\text{SO}_0(p,q)$. Then, $d\psi(I,0)$ is the composition:

$$\mathfrak{so}(p,q) \ltimes W \to \mathcal{G}(x_0) \oplus \mathcal{V}(x_0) \to T_{x_0}\tilde{M}$$

$$X \mapsto X^* \mapsto X^*_{x_0}.$$

Hence by the property $\psi(X) = X^* (X \in \mathfrak{so}(p,q) \ltimes W)$ and Lemma 2.2 $d\psi(I,0)$ maps $\mathfrak{so}(p,q)$ onto $T_{x_0}\mathcal{O}$ and $W$ onto $T_{x_0}\mathcal{O}^\perp$. In particular, $f$ is a local diffeomorphism at $(I,0)$.

For every $w \in W$, denote with $R_w$ the transformations on both $\widetilde{\text{SO}_0}(p,q) \ltimes W$ and $\tilde{M}$ given by the assignment $x \mapsto x(I,w)$. In particular, a straightforward computation shows that we have:

$$d(R_w)(I,v) : T_{(I,v)}(\widetilde{\text{SO}_0}(p,q) \ltimes W) \to T_{(I,v+w)}(\widetilde{\text{SO}_0}(p,q) \ltimes W)$$

$$(X,Y) \mapsto (X,Y_{v+w}).$$
Also note that $R_w(I \times W) = I \times W$, since $W$ is a subgroup of $\widetilde{SO}_0(p, q) \times W$.

Let $N = f(I \times W)$, which defines a submanifold of $\widetilde{M}$ in a neighborhood of $x_0 = f(I, 0)$. By the above remarks on $df_{f(I, 0)}$ we have:

$$T_f(I, 0)N = df_{f(I, 0)}(T_{f(I, 0)}(I \times W)) = T_{f(I, 0)}O^\perp.$$

But then, the equivariance of $f$ yields:

$$T_{f(I, w)}N = df_{f(I, w)}(T_{(I, w)}(I \times W)) = df_{f(I, w)}(d(R_w)(T_{f(I, 0)}(I \times W))) = d(R_w)(f_{f(I, 0)})(T_{f(I, 0)}N) = d(R_w)f_{f(I, 0)}(T_{f(I, 0)}O^\perp) = T_{R_w(f(I, 0))}O^\perp = T_{f(I, w)}O^\perp,$$

where we have used in the second to last identity that $R_w$ preserves in $\widetilde{M}$ the bundle $T_\mathcal{O}^\perp$. This proves that $N$ is an integral submanifold of $T\mathcal{O}^\perp$ passing through the point $x_0 = f(I, 0)$.

On the other hand, from the left $\widetilde{SO}_0(p, q)$-action on $\widetilde{M}$ we obtain by restriction to $N$ a map:

$$\varphi : \widetilde{SO}_0(p, q) \times N \to \widetilde{M}, \quad (g, x) \mapsto gx,$$

whose differential at $(I, x_0)$ is given by:

$$X + v \mapsto X^*_{x_0} + v,$$

where $X \in so(p, q)$ and $v \in T_{x_0}N$. The latter is an isomorphism and so the map $\varphi$ is a diffeomorphism from a neighborhood of $(I, x_0)$ onto a neighborhood of $x_0$. Since the left $\widetilde{SO}_0(p, q)$-action preserves both $T\mathcal{O}$ and $T\mathcal{O}^\perp$, we conclude that there is an integral submanifold of $T\mathcal{O}^\perp$ passing through every point in neighborhood of $x_0$ in $\widetilde{M}$. Hence, the tensor $\Omega$ considered in Lemma 1.3 vanishes in a neighborhood of $x_0$. Since all our objects are analytic, this implies that $\Omega$ vanishes everywhere and so Lemma 1.3 implies the integrability of $T\mathcal{O}^\perp$ everywhere in $\widetilde{M}$.

This last conclusion and Proposition 1.5 contradict the weak irreducibility of $M$. This shows that case (1) from Lemma 2.3 cannot occur.

3.2. **Case 2**: $\mathcal{H}_0(x_0) \neq \{0\}$ and $\mathcal{H}$ is the sum of two simple ideals. We now assume that the conclusion (2) of Lemma 2.4 is satisfied for some fixed $x_0 \in \widetilde{M}$. As in the previous case, we will rule out this possibility.

In this case, there exist simple Lie algebras $\mathfrak{h}_1$, $\mathfrak{h}_2$ and an isomorphism of Lie algebras $\psi : \mathfrak{h}_1 \times \mathfrak{h}_2 \to \mathcal{H}$ so that $\psi(\mathfrak{h}_2) = \mathcal{H}_0(x_0) \oplus \mathcal{V}(x_0)$. Let $H_1$ and $H_2$ be simply connected Lie groups with Lie algebras $\mathfrak{h}_1$ and $\mathfrak{h}_2$, respectively. By Lemma 1.11 there is an analytic isometric right action of $H_1 \times H_2$ on $\widetilde{M}$ such that $\psi(X) = X^*$ for every $X \in \mathfrak{h}_1 \times \mathfrak{h}_2$. Note that this right action centralizes the left $\widetilde{SO}_0(p, q)$-action on $\widetilde{M}$ and so it preserves the bundles $T\mathcal{O}$ and $T\mathcal{O}^\perp$.

Let us consider the map:

$$f : H_1 \times H_2 \to \widetilde{M}, \quad h \mapsto x_0h,$$

which is clearly $(H_1 \times H_2)$-equivariant for the right action on its domain. In particular, we have $df_e(X) = X^*_{x_0} = \psi(X)_{x_0}$ which is surjective with $\ker(df_e) = \psi^{-1}(\mathcal{H}_0(x_0))$ by Lemma 2.2. We claim that we also have $df_e(\mathfrak{h}_1) = T_{x_0}\mathcal{O}$ and $df_e(\mathfrak{h}_2) = T_{x_0}\mathcal{O}^\perp$. The latter follows from our choice of $\psi$ and Lemma 2.2. To prove the former, first note that both $\psi(\mathfrak{h}_1)$ and $\mathcal{G}(x_0)$ are complementary $so(p, q)$-modules to $\mathcal{H}_0(x_0) \oplus \mathcal{V}(x_0)$ in $\mathcal{H}$, and so they are isomorphic as $so(p, q)$-modules.
Hence, by Lemma 2.2 the evaluation $ev_{x_0} : \mathcal{H} \to T_{x_0} \tilde{M} = T_{x_0} \mathcal{O} \oplus T_{x_0} \mathcal{O}^{\perp}$ necessarily maps $\psi(h_1)$ onto $T_{x_0} \mathcal{O}$ since $T_{x_0} \mathcal{O} \simeq \mathcal{G}(x_0) \neq T_{x_0} \mathcal{O}^{\perp}$ as $\mathfrak{so}(p, q)$-modules. This proves that $df_c(h_1) = T_{x_0} \mathcal{O}$.

Let us denote with $H$ the connected subgroup of $H_2$ with Lie algebra $\psi^{-1}(\mathcal{H}_0(x_0))$, the latter being isomorphic to $\mathfrak{so}(p, q)$ by Lemma 2.2. Since $H_2$ is simply connected and $\psi^{-1}(\mathcal{H}_0(x_0))$ is simple, it follows that $H$ is a closed subgroup of $H_2$ (see Exercise D.4(ii), Chapter II from [12]). Hence, the map:

$$\tilde{f} : H_1 \times H_2 \to \tilde{M}, \quad (h_1, H h_2) \mapsto x_0(h_1, h_2),$$

is a well-defined $(H_1 \times H_2)$-equivariant analytic map of manifolds. From the properties of $df_c$, it also follows that $\tilde{f}$ is a local diffeomorphism at $(e_1, H e_2)$.

By considering $N = \tilde{f}(I \times H \setminus H_2)$ and using the equivariance of $\tilde{f}$, we can prove with arguments similar to those used in the previous subsection that $TO^{\perp}$ is integrable. This again rules out the current case.

3.3. Case 3: $\mathcal{H}$ is simple. In this case we are now assuming that (3) from Lemma 2.3 holds for some $x_0 \in \tilde{M}$ that we now consider fixed.

**Lemma 3.1.** There is an isomorphism $\psi : \mathfrak{so}(p, q) \oplus \mathbb{R}^{p,q} \to \mathcal{H} = \mathcal{G}(x_0) \oplus \mathcal{V}(x_0)$ of Lie algebras that preserves the summands in that order, where the domain has the Lie algebra structure given by $[\cdot, \cdot]_c$ for some $c \neq 0$ as defined in Lemma A.8. In particular, $\psi$ is an isomorphism of $\mathfrak{so}(p, q)$-modules as well.

**Proof.** The result follows from the arguments in the second to last paragraph in the proof of Lemma 2.3 when $\mathcal{V}(x_0) \simeq \mathbb{R}^{p,q}$ as $\mathfrak{so}(p, q)$-modules and $n \geq 5$. Hence, by Lemma 2.1 we can assume that either $(p, q) = (3, 1)$ or $(p, q) = (4, 4)$ in the rest of the proof.

By Lemma 2.3 there is an isomorphism $\psi : \mathfrak{h} \to \mathcal{H}$, for $\mathfrak{h} = \mathfrak{so}(V)$ where $V$ is either $\mathbb{R}^{4,1}$ or $\mathbb{R}^{3,2}$ for $(p, q) = (3, 1)$, and it is $\mathbb{R}^{5,4}$ for $(p, q) = (4, 4)$. The restriction of this homomorphism to $\psi^{-1}(\mathcal{G}(x_0))$ yields a representation of $\mathcal{G}(x_0) \simeq \mathfrak{so}(p, q)$ on the $(n + 1)$-dimensional space $V$. By the description of the irreducible representations of $\mathfrak{so}(3, 1)$ from previous sections we know that there do not exist 5-dimensional irreducible representations of $\mathfrak{so}(3, 1)$. In particular, there is a line $L \subset V$ that is a $\mathcal{G}(x_0)$-submodule for the case $(p, q) = (3, 1)$. On the other hand, since $\mathfrak{so}(4, 4)$ is split and using Weyl’s dimension formula we find that $\mathfrak{so}(4, 4)$ does not admit 9-dimensional irreducible representations. Hence, for the case $(p, q) = (4, 4)$ we similarly conclude the existence of a line $L \subset V$ which is a $\mathcal{G}(x_0)$-submodule.

Let us now consider our two remaining cases $(p, q) \in \{(3, 1), (4, 4)\}$ together. If $L$ as above is a null line, then $\psi^{-1}(\mathcal{G}(x_0))$ lies in the Lie algebra $\mathfrak{s}$ of the stabilizer of some point in one of the pseudo-conformal spaces $C^{p,q-1}$ or $C^{p-1,q}$. We recall that $C^{r,s}$ is the projectivization of the null cone of $\mathbb{R}^{r+1,s+1}$, is homogeneous under $O(r + 1, s + 1)$ and has dimension $r + s$ (see [14] for further details). In particular, $\mathfrak{s}$ has dimension $n(n - 1)/2 + 1$. This yields $\psi^{-1}(\mathcal{G}(x_0)) \subsetneq \mathfrak{s} \subsetneq \mathfrak{h}$, which contradicts Theorem A.5. We conclude that $L$ is a non-null line.

This yields an orthogonal decomposition $V = L \oplus L^\perp$ into non-degenerate subspaces which is clearly a decomposition into $\mathcal{G}(x_0)$-submodules. Hence, $\psi$ induces an isomorphism $\mathfrak{so}(L^\perp) \to \mathcal{G}(x_0)$ and a rank argument shows that $L^\perp$ has signature $(p, q)$. In particular, $\mathfrak{so}(L^\perp) \simeq \mathfrak{so}(p, q)$ as Lie algebras under $\psi$. With respect to the corresponding $\mathfrak{so}(p, q)$-module structure, it is easily seen that $\mathfrak{so}(L^\perp)$ has a complementary module in $\mathfrak{h}$ isomorphic to $\mathbb{R}^{p,q}$. This provides an isomorphism
\[ \mathfrak{h} \simeq \mathfrak{so}(p, q) \oplus \mathbb{R}^{p,q} \] so that the Lie algebra structure on \( \mathfrak{h} \) corresponds to the one given by \([\cdot, \cdot]_c\) on \( \mathfrak{so}(p, q) \oplus \mathbb{R}^{p,q} \) for some \( c \neq 0 \). Hence, under the identification \( \mathfrak{h} \simeq \mathfrak{so}(p, q) \oplus \mathbb{R}^{p,q} \) of Lie algebras, \( \psi \) is the required isomorphism. \( \square \)

Let us fix an isomorphism of Lie algebras \( \psi : \mathfrak{so}(p, q) \oplus \mathbb{R}^{p,q} \to \mathcal{H} = \mathcal{G}(x_0) \oplus \mathcal{V}(x_0) \) as in Lemma 8.1. We will identify \( \mathfrak{h} = \mathfrak{so}(p, q) \oplus \mathbb{R}^{p,q} \) with either \( \mathfrak{so}(p + 1, q) \) or \( \mathfrak{so}(p, q + 1) \) through the appropriate isomorphism as considered in Lemma A.8. Also, we will denote with \( \mathcal{H} \) either \( \tilde{\mathcal{SO}}_0(p + 1, q) \) or \( \tilde{\mathcal{SO}}_0(p, q + 1) \), chosen so that \( \text{Lie}(\mathcal{H}) = \mathfrak{h} \).

By Lemma 1.11 there is an analytic isometric right \( \mathcal{H} \)-action on \( \tilde{\mathcal{M}} \) such that \( \psi(X) = X^* \) for every \( X \in \mathfrak{h} \). As in the previous subsections, we now consider the orbit map:

\[ f : \mathcal{H} \to \tilde{\mathcal{M}}, \quad h \mapsto x_0 h \]

which satisfies \( df_I(X) = X^*_x = \psi(X) x_0 \) for every \( X \in \mathfrak{h} \). By the choice of \( \psi \) and Lemma 2.2 it follows that \( df \) is an isomorphism that maps \( \mathfrak{so}(p,q) \) onto \( T_{x_0}O \) and \( \mathbb{R}^{p,q} \) onto \( T_{x_0}O^\perp \). Since \( f \) is \( \mathcal{H} \)-equivariant for the right action on its domain, we conclude that it is an analytic local diffeomorphism.

**Lemma 3.2.** Let \( \mathcal{G} \) be the metric on \( \mathcal{H} = \mathfrak{so}(p, q) \oplus \mathbb{R}^{p,q} \) defined as the pullback under \( df \) of the metric \( g_{x_0} \) on \( T_{x_0} \tilde{\mathcal{M}} \). Then, \( \mathcal{G} \) is \( \mathfrak{so}(p, q) \)-invariant.

**Proof.** By the above expression of \( df \) and since \( \psi \) is an isomorphism of Lie algebras with \( \psi(\mathfrak{so}(p,q)) = \mathcal{G}(x_0) \), it is enough to show that the metric on \( \mathcal{H} \) defined as the pullback of \( g_{x_0} \) with respect to the evaluation map:

\[ \mathcal{H} \to T_{x_0} \tilde{\mathcal{M}}, \quad X \mapsto X_{x_0} \]

is \( \mathcal{G}(x_0) \)-invariant. For simplicity, we will denote with \( \mathcal{G} \) such metric on \( \mathcal{H} \). Let \( X, Y, Z \in \mathcal{H} \) be given with \( X \in \mathcal{G}(x_0) \). In particular, there exist \( X_0 \in \mathfrak{so}(p,q) \) such that \( X = \rho_{x_0}(X_0) + X_0^* \), where \( \rho_{x_0} \) is the homomorphism from Proposition 1.2 and \( X_0^* \) is the vector field on \( \tilde{\mathcal{M}} \) induced by \( X_0 \) through the left \( \tilde{\mathcal{SO}}_0(p,q) \)-action. Then, the following proves the required invariance:

\[
\mathcal{G}(X, Y, Z) = g_{x_0}(X, Y |_{x_0}, Z |_{x_0}) = g([X, Y], Z |_{x_0}) = g([\rho_{x_0}(X_0) + X_0^*, Y], Z |_{x_0}) = g([\rho_{x_0}(X_0) + X_0^*, Y], Z |_{x_0}) = g(\rho_{x_0}(X_0)(g(Y, Z)) |_{x_0} - g(Y, [\rho_{x_0}(X_0), Z]) |_{x_0} = -g(Y, [\rho_{x_0}(X_0), X_0^*]) |_{x_0} = g(Y, [X, Z]) |_{x_0} = -g(Y, [X, Z]).
\]

We have used in lines 2 and 4 that \( \mathcal{H} \) centralizes \( X_0^* \). To obtain the third line we used that \( \rho_{x_0}(X_0) \) is a Killing field for the metric \( g \). And the first identity in line 4 uses the fact that \( \rho_{x_0}(X_0) \) vanishes at \( x_0 \). \( \square \)

From the previous result and Lemma A.9 for \( n \geq 5 \) we can rescale the metric along the bundles \( T\mathcal{O} \) and \( T\mathcal{O}^\perp \) in \( \tilde{\mathcal{M}} \) so that the new metric \( \tilde{\mathcal{G}} \) on \( \tilde{\mathcal{M}} \) satisfies \( (df_I)^* (\tilde{\mathcal{G}}_{x_0}) = K \), the Killing form on \( \mathfrak{h} \).

Let us now consider the case \( (p, q) = (3,1) \). From the hypotheses of the Main Theorem, we are now assuming that \( X^* \perp Y^* \) in \( \tilde{\mathcal{M}} \) for every \( X \in \mathfrak{su}(2), Y \in \mathfrak{j}\mathfrak{su}(2) \) and the left \( \text{SL}(2, \mathbb{C}) \)-action on \( \tilde{\mathcal{M}} \). By Proposition 1.2 and Lemma 1.7 we have \( \rho_{x_0}(X)_{x_0} = X^*_{x_0} \) and so \( \rho_{x_0}(X)_{x_0} \perp \rho_{x_0}(Y)_{x_0} \), for every \( X \in \mathfrak{su}(2) \) and \( Y \in \mathfrak{j}\mathfrak{su}(2) \).

Hence, for the compact real form \( \mathcal{U} = \rho_{x_0}(\mathfrak{su}(2)) \) of \( \mathcal{G}(x_0) \) we have \( X_{x_0} \perp Y_{x_0} \) when
\(X \in \mathcal{U}\) and \(Y \in \mathcal{J}\mathcal{U}\). By the definition of \(\mathcal{F}\) and the expression for \(df_1\) given above, it follows that for the metric \(\mathcal{F}\) restricted to \(\mathfrak{so}(3,1) \simeq \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}\) we have \(\mathcal{F}(X,Y) = 0\) for every \(X \in \psi^{-1}(\mathcal{U})\) and \(Y \in J\psi^{-1}(\mathcal{U})\). By the remarks that follow Lemma A.9 we conclude that in this case we can also rescale the metric on \(\tilde{M}\) along \(\mathcal{O}\) and \(\mathcal{T}\mathcal{O}^\perp\) to obtain a new metric \(\hat{g}\) such that \((df_1)^*(\hat{g}_{x_0}) = K\), the Killing form of \(h\).

Note that, for both cases \(n \geq 5\) and \((p,q) = (3,1)\), since the elements of \(\mathcal{H}\) preserve the decomposition \(\mathcal{T}M = \mathcal{T}\mathcal{O} \oplus \mathcal{T}\mathcal{O}^\perp\), then \(\mathcal{H} \subset \text{Kill}(\tilde{M}, \hat{g})\). In other words, the elements of \(\mathcal{H}\) are Killing vector fields for the metric \(\hat{g}\) rescaled as above. In particular, \(\hat{g}\) is invariant under the right \(H\)-action. Similarly, the left \(\text{SO}_0(p,q)\)-action on \(\tilde{M}\), from the hypotheses of the Main Theorem, preserves the rescaled metric \(\hat{g}\). Also note that the metric \(\hat{g}\) is the lift of a correspondingly rescaled metric \(\hat{g}\) in \(M\).

Consider the bi-invariant metric on \(H\) induced by the Killing form \(K\), which we will denote with the same symbol. The previous discussion implies that the local diffeomorphism \(f : (H,K) \rightarrow (\tilde{M}, \hat{g})\) is a local isometry. Then, by Corollary 29 in page 202 of [13], the completeness of \((H,K)\) and the simple connectedness of \(\tilde{M}\) imply that \(f\) is an isometry.

Hence, from the previous discussion we obtain the following result.

**Lemma 3.3.** Let \(M\) be as in the statement of the Main Theorem. If \(\dim(M) = n(n+1)/2\), then for \(H\) either \(\text{SO}_0(p,q+1)\) or \(\text{SO}_0(p+1,q)\), there exists an analytic diffeomorphism \(f : H \rightarrow \tilde{M}\) and an analytic isometric right \(H\)-action on \(\tilde{M}\) such that:

1. on \(\tilde{M}\), the left \(\text{SO}_0(p,q)\)-action and the right \(H\)-action commute with each other,
2. \(f\) is \(H\)-equivariant for the right \(H\)-action on its domain,
3. for a pseudo-Riemannian metric \(\hat{g}\) in \(M\) obtained by rescaling the original one on the summands of \(TM = \mathcal{T}\mathcal{O} \oplus \mathcal{T}\mathcal{O}^\perp\), the map \(f : (H,K) \rightarrow (\tilde{M}, \hat{g})\) is an isometry where \(K\) is the bi-invariant metric on \(H\) induced from the Killing form of its Lie algebra.

If we consider \(H\) endowed with the bi-invariant pseudo-Riemannian metric \(K\) induced by the Killing form of its Lie algebra, then Lemma 3.3 allows to consider \((H,K)\) as the isometric universal covering space of \((\tilde{M}, \hat{g})\). We will use this identification in the rest of the arguments.

The isometry group \(\text{Iso}(H)\) for the pseudo-Riemannian manifold \((H,K)\) has finitely many connected components (see for example Section 4 of [15]). Furthermore, the connected component of the identity is given as \(\text{Iso}_0(H) = L(H)R(H)\), the subgroup generated by \(L(H)\) and \(R(H)\), the left and right translations, respectively.

Let \(\rho : \text{SO}_0(p,q) \rightarrow \text{Iso}_0(H)\) be the homomorphism induced by isometric left \(\text{SO}_0(p,q)\)-action on \(H\). With respect to the natural covering \(H \times H \rightarrow L(H)R(H)\), this yields homomorphisms \(\rho_1, \rho_2 : \text{SO}_0(p,q) \rightarrow H\) such that:

\[\rho(g) = L_{\rho_1(g)} \circ R_{\rho_2(g)^{-1}},\]

for every \(g \in \text{SO}_0(p,q)\). By Lemma 3.3 we have \(\rho(g) \circ R_h = R_h \circ \rho(g)\) for every \(g \in \text{SO}_0(p,q)\) and \(h \in H\). In particular, \(\rho_2(\text{SO}_0(p,q))\) lies in the center \(Z(H)\) and so (being connected) it is trivial. We conclude that \(\rho = L_{\rho_1}\), which implies that the
SO(p, q)-action on H is induced by the homomorphism \( \rho_1 : \text{SO}_0(p, q) \to H \) and the left action of H on itself. Note that \( \rho_1 \) is necessarily non-trivial.

By Lemma 3.3 we have \( \pi_1(M) \subset \text{Iso}(H) \), and from the above remarks \( \Gamma_1 = \pi_1(M) \cap \text{Iso}_0(H) \) is a finite index subgroup of \( \pi_1(M) \). In particular, every \( \gamma \in \Gamma_1 \) can be written as \( \gamma = L_{h_1} \circ R_{h_2} \) for some \( h_1, h_2 \in H \).

On the other hand, since the left \( \text{SO}_0(p, q) \)-action on H is the lift of an action on M, it follows that it commutes with the \( \Gamma_1 \)-action. Applying this property to \( \gamma = L_{h_1} \circ R_{h_2} \) we conclude that \( L_{h_1} \circ L_{\rho_1(g)} = L_{\rho_1(g)} \circ L_{h_1} \), for every \( g \in \text{SO}_0(p, q) \), which implies \( \Gamma_1 \subset L(Z)R(H) \), where Z is the centralizer of \( \rho_1(\text{SO}_0(p, q)) \) in H. By Lemma A.6, the center of \( \text{H} \) is the centralizer of \( \rho_1 \) in \( \text{SO}(p, q) \), which implies that \( R(H) \) has finite index in Z, which implies that \( R(H) \) has finite index in \( L(Z)R(H) \). In particular, \( \Gamma = \Gamma_1 \cap R(H) \) is a finite index subgroup of \( \Gamma_1 \), and so has finite index in \( \pi_1(M) \) as well.

Hence, the natural identification \( R(H) = H \) realizes \( \Gamma \) as a discrete subgroup of \( H \) such that \( H/\Gamma \) is a finite covering space of M. Furthermore, if \( \varphi : H/\Gamma \to M \) is the corresponding covering map, and for the left \( \text{SO}_0(p, q) \)-action on \( H/\Gamma \) given by the homomorphism \( \rho_1 : \text{SO}_0(p, q) \to H \), then the above constructions show that \( \varphi \) is \( \text{SO}_0(p, q) \)-equivariant. We also note that \( \varphi \) is a local isometry for the metric \( \hat{g} \) on M considered in Lemma 3.3.

To complete the proof of the Main Theorem it only remains to show that \( \Gamma \) is a lattice in H. For this it is enough to prove that M has finite volume in the metric \( \hat{g} \). The following result provides proofs of these facts since we are assuming that M has finite volume in its original metric.

**Lemma 3.4.** Let us denote with vol and \( \text{vol}_\hat{g} \) the volume elements on M for the original metric on M and the rescaled metric \( \hat{g} \), respectively. Then, there is some constant \( C > 0 \) such that \( \text{vol}_\hat{g} = C \text{vol} \).

**Proof.** Clearly, it suffices to verify this locally, so we consider some coordinates \((x^1, \ldots, x^m)\) of M in a neighborhood U of a given point such that \((x^1, \ldots, x^r)\) defines a set of coordinates of the leaves of the foliation \( \mathcal{O} \) in such neighborhood. For the original metric \( g \) on M, consider as above the orthogonal bundle \( T\mathcal{O}^\perp \) and a set of 1-forms \( \theta^1, \ldots, \theta^{m-r} \) that define a basis for its dual \((T\mathcal{O}^\perp)^*\) at every point in U. Hence, in U the metric \( g \) has an expression of the form:

\[
g = \sum_{i,j=1}^{r} h_{ij} dx^i \otimes dx^j + \sum_{i,j=1}^{m-r} k_{ij} \theta^i \otimes \theta^j.
\]

From this and the definition of the volume element as an m-form, it is easy to see that:

\[
\text{vol} = \sqrt{|\det(h_{ij}) \det(k_{ij})|} dx^1 \wedge \cdots \wedge dx^r \wedge \theta^1 \wedge \cdots \wedge \theta^{m-r}.
\]

On the other hand, the metric \( \hat{g} \) is obtained by rescaling \( g \) along the bundles \( T\mathcal{O}^\perp \) and \( T\mathcal{O}^\perp \), and so it has an expression of the form:

\[
\hat{g} = \sum_{i,j=1}^{r} c_1 h_{ij} dx^i \otimes dx^j + \sum_{i,j=1}^{m-r} c_2 k_{ij} \theta^i \otimes \theta^j,
\]
for some constants $c_1, c_2 \neq 0$. Hence, the volume element of $\hat{g}$ satisfies:

$$\text{vol}_{\hat{g}} = \sqrt{\det(c_1 h_{ij}) \det(c_2 k_{ij})} |dx^1 \wedge \cdots \wedge dx^r \wedge \theta^1 \wedge \cdots \wedge \theta^{m-r}|$$

$$= \sqrt{|c_1^2 c_2^{m-r}| \text{vol.}}$$

\[ \square \]

**Appendix A. Facts on the Lie algebra $\mathfrak{so}(p, q)$**

In this appendix we collect some facts about the pairs $(\mathfrak{so}(p + 1, q), \mathfrak{so}(p, q))$ and $(\mathfrak{so}(p, q + 1), \mathfrak{so}(p, q))$. We now describe some low-dimensional non-trivial $\mathfrak{so}(p, q)$-modules. The following result is an easy consequence of well known facts of classical Lie groups (see [11, 14]). As mentioned in the introduction we denote with $m(\mathfrak{g})$ the lowest dimension of a non-trivial $\mathfrak{g}$-module with an invariant non-degenerate symmetric bilinear form.

**Lemma A.1.** Let $p, q \geq 1$, $n = p + q \geq 4$ such that $(p, q) \neq (2, 2)$. Then, $m(\mathfrak{so}(p, q)) = n$, i.e. there is no non-trivial $\mathfrak{so}(p, q)$-module with dimension $< n$ and carrying an invariant inner product. Moreover, the only non-trivial irreducible $\mathfrak{so}(p, q)$-module of dimension $\leq n$ is $\mathbb{R}^{p,q}$, except for the Lie algebras $\mathfrak{so}(3, 1)$, $\mathfrak{so}(3, 2)$, $\mathfrak{so}(3, 3)$ and $\mathfrak{so}(4, 4)$. For the latter, there also exist the following irreducible modules:

- $\mathbb{C}^2_{\mathbb{R}}$ corresponding to $\mathfrak{so}(3, 1) \simeq \mathfrak{sl}(2, \mathbb{C})_{\mathbb{R}}$.
- $\mathbb{R}^4$ corresponding to $\mathfrak{so}(3, 2) \simeq \mathfrak{sp}(2, \mathbb{R})$.
- $\mathbb{R}^4$ and $\mathbb{R}^{4*}$ corresponding to $\mathfrak{so}(3, 3) \simeq \mathfrak{sl}(4, \mathbb{R})$.
- $\mathfrak{so}(4, 4)$-invariant real forms of the half spin representations of $\mathfrak{so}(8, \mathbb{C})$, both 8-dimensional.

**Remark A.2.** We note that the previous statement is not correct in lower dimensions. We have that $\mathfrak{so}(1, 1)$ is abelian, so every irreducible module is one-dimensional. In fact $\mathbb{R}^{1,1} = \mathbb{R}^2$ decomposes into irreducible submodules where $t \in \mathbb{R} \simeq \mathfrak{so}(1, 1)$ acts by multiplication by $e^t$ respectively $e^{-t}$.

The Lie algebra $\mathfrak{so}(2, 2) \simeq \mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R})$ is not simple and acts on $\mathbb{R}^2$. But there is again no non-trivial invariant symmetric bilinear form on $\mathbb{R}^2$. The lowest dimensional $\mathfrak{so}(2, 2)$-module with a non-trivial invariant form is $\mathbb{R}^{2,1}$ where one of the factors carries the canonical $\mathfrak{so}(2, 1)$ action, and the other factor acts trivially, thus reducing this to the case of $\mathfrak{so}(2, 1)$. Hence we have $m(\mathfrak{so}(2, 2)) = 3$.

The following statement is an easy to prove exercise.

**Lemma A.3.** Let $p, q \geq 1$ and $n = p + q \geq 3$. Then, for every $c \in \mathbb{R}$, the map $T_c : \wedge^2 \mathbb{R}^{p,q} \to \mathfrak{so}(p, q)$ given by:

$$T_c(u \wedge v) = c \langle u, \cdot \rangle_{p,q} v - c \langle v, \cdot \rangle_{p,q} u,$$

for every $u, v \in \mathbb{R}^{p,q}$, is a well defined homomorphism of $\mathfrak{so}(p, q)$-modules. Also, $T_c$ is an isomorphism of $\mathfrak{so}(p, q)$-modules if and only if $c \neq 0$. If $n \neq 4$, then these maps exhaust all the $\mathfrak{so}(p, q)$-module homomorphisms $\wedge^2 \mathbb{R}^{p,q} \to \mathfrak{so}(p, q)$.

**Remark A.4.** The last conclusion fails for $\mathfrak{so}(2, 2) \simeq \mathfrak{so}(2, 1) \times \mathfrak{so}(2, 1)$ as it is not simple. It also fails for $\mathfrak{so}(3, 1) \simeq \mathfrak{sl}(2, \mathbb{C})$, since the complex structure of $\mathfrak{sl}(2, \mathbb{C})$ defines an isomorphism which is not multiplication by a real scalar.

Next we prove the maximality of $\mathfrak{so}(p, q)$ in both $\mathfrak{so}(p, q + 1)$ and $\mathfrak{so}(p + 1, q)$. 
Theorem A.5. Assume that \( p, q \geq 1 \) and \( n = p + q \geq 3 \), and let \( \mathfrak{g} = \mathfrak{so}(p + 1, q) \) or \( \mathfrak{g} = \mathfrak{so}(p, q + 1) \). Suppose that \( \rho : \mathfrak{so}(p, q) \to \mathfrak{g} \) is an injective Lie algebra homomorphism and let \( \mathfrak{h} = \rho(\mathfrak{so}(p, q)) \). If \( \mathfrak{g}, \mathfrak{h} \not\cong \mathfrak{so}(2, 1) \times \mathfrak{so}(2, 1) \), then \( \mathfrak{h} \) is a maximal subalgebra of \( \mathfrak{g} \).

Proof. This follows from Theorem 1.2 of [8] for \( n \) big and a case by case calculation for the other cases. We give here a simple proof for completeness. The map \( \theta(X) = -X^t \) is a Cartan involution on \( \mathfrak{g} \). As all Cartan involutions are conjugate and every Cartan involution on \( \mathfrak{h} \) extends to a Cartan involution on \( \mathfrak{g} \) we can assume that \( \theta(\mathfrak{h}) = \mathfrak{h} \). Then the form \( \beta(X, Y) = \text{Tr}(XY) \) is non-degenerate on \( \mathfrak{g} \) and \( \mathfrak{h} \). Let \( V \) be the \( \beta \)-orthogonal complement of \( \mathfrak{h} \). Hence \( V \) is an \( n \)-dimensional \( \mathfrak{h} \)-module which is necessarily non-trivial since otherwise \( \mathfrak{h} \) is an ideal. Furthermore, \( \beta|_{V \times V} \) is non-degenerate and \( \mathfrak{h} \)-invariant. It follows by Lemma A.1 that \( V \) is an irreducible \( \mathfrak{h} \)-module if \( n \geq 4 \) and \( (p, q) \neq (2, 2), (3, 2), (3, 3) \). Hence \( \mathfrak{h} \) is maximal in those cases.

Let us now assume that \( \mathfrak{h} = \mathfrak{so}(2, 1) \cong \mathfrak{sl}(2, \mathbb{R}) \), \( \mathfrak{g} = \mathfrak{so}(3, 1) \) and \( V \) is not irreducible. Thus \( V = V_1 \oplus V_2 \) with \( V_1 \cong \mathbb{R} \) and \( V_2 \cong \mathbb{R}^2 \) where the representation on \( V_1 \) is trivial and the representation on \( V_2 \) is isomorphic to the natural representation of \( \mathfrak{sl}(2, \mathbb{R}) \). But then it follows that \( V_2 \) is invariant under \( \theta \) and hence \( \beta|_{V_2 \times V_2} \) is an \( \mathfrak{h} \)-invariant non-degenerate form, contradicting the fact that there is no such form on \( \mathbb{R}^2 \). The remaining cases \( (p, q) \in \{(3, 2), (3, 3)\} \) can be considered similarly using Lemma A.1. □

Lemma A.6. Suppose that \( G \) is a connected Lie group locally isomorphic to either \( \text{SO}_0(p, q + 1) \) or \( \text{SO}_0(p + 1, q) \), where \( p, q \geq 1 \) and \( n = p + q \geq 3 \), and consider \( \rho : \text{SO}_0(p, q) \to G \) a non-trivial homomorphism of Lie groups. Assume that \( \mathfrak{so}(p, q) \), \( \mathfrak{so}(p + 1, q) \) and \( \mathfrak{so}(p, q + 1) \) satisfy the same conditions as in Theorem A.5. Then, the centralizer \( Z_G(\rho(\text{SO}_0(p, q))) \) of \( \rho(\text{SO}_0(p, q)) \) in \( G \) contains \( Z(G) \) (the center of \( G \)) as a finite index subgroup.

Proof. Write \( H = \rho(\text{SO}_0(p, q)) \) and write \( \mathfrak{h} \) for the Lie algebra of \( H \). Then, clearly \( Z(G) \subseteq Z_G(H) \). The Lie algebra of \( Z_G(H) \) is

\[
\mathfrak{z}_G(\mathfrak{h}) = \{ X \in \mathfrak{h} \mid [X, Y] = 0 \text{ for all } Y \in \mathfrak{h} \}.
\]

Clearly \( \mathfrak{z}_G(\mathfrak{h}) + \mathfrak{h} \) is a Lie algebra containing \( \mathfrak{h} \). By Theorem A.5 we conclude that \( \mathfrak{z}_G(\mathfrak{h}) \subseteq \mathfrak{h} \). As \( \mathfrak{h} \) is simple, it follows that \( \mathfrak{z}_G(\mathfrak{h}) = \{0\} \) and \( Z_G(H) \) is discrete. Finally, it follows easily from [18], Lemma 1.1.3.7, that \( Z_G(H) \) is contained in any maximal compact subgroup of \( G \) and hence that it is finite. □

Remark A.7. If \( p = q = 1 \) then \( \mathfrak{h} \) is abelian, hence \( H \subseteq Z_G(H) \). On the other hand Theorem A.5 shows that \( Z_G(H)/H \) is finite. For \( \mathfrak{so}(2, 1) \subseteq \mathfrak{so}(2, 1) \times \mathfrak{so}(2, 1) \) the statement remains true if \( \mathfrak{so}(2, 1) \) is embedded diagonally, but clearly not if \( \mathfrak{so}(2, 1) \cong \mathfrak{so}(2, 1) \) is one of the ideals.

We now provide an elementary but useful description of the Lie algebra structures of \( \mathfrak{so}(p, q + 1) \) and \( \mathfrak{so}(p + 1, q) \) in terms of \( \mathfrak{so}(p, q) \)-modules. In the next result \( \mathfrak{so}(p, q) \rtimes \mathbb{R}^{p,q} \) is considered with the usual semidirect product Lie algebra structure coming from the fact that \( \mathbb{R}^{p,q} \) is an \( \mathfrak{so}(p, q) \)-module. Also, we will denote:

\[
I_{p,q}(c) = \begin{pmatrix} c & 0 \\ 0 & I_{p,q} \end{pmatrix} \text{ if } c > 0 \quad \text{and} \quad I_{p,q}(c) = \begin{pmatrix} I_{p,q} & 0 \\ 0 & c \end{pmatrix} \text{ if } c < 0.
\]
Lemma A.8. For \( p, q \geq 1, n = p + q \geq 3 \) and every \( c \in \mathbb{R} \), let:

\[
[\cdot, \cdot]_c : \mathfrak{so}(p, q) \oplus \mathbb{R}^{p,q} \times \mathfrak{so}(p, q) \oplus \mathbb{R}^{p,q} \to \mathfrak{so}(p, q) \oplus \mathbb{R}^{p,q},
\]

be given by:

- \( [X, Y]_c = XY - YX \) for \( X, Y \in \mathfrak{so}(p, q) \).
- \( [X, u]_c = -[u, X]_c = X(u) \) for \( X \in \mathfrak{so}(p, q) \) and \( u \in \mathbb{R}^{p,q} \).
- \( [u, v]_c = T_c(u \wedge v) \) for \( u, v \in \mathbb{R}^{p,q} \), where \( T_c \) is the map defined in Lemma A.3.

Then, \([\cdot, \cdot]_c\) defines a Lie algebra structure on \( \mathfrak{so}(p, q) \oplus \mathbb{R}^{p,q} \) which satisfies:

1. \( \mathfrak{so}(p, q) \oplus \mathbb{R}^{p,q}, [\cdot, \cdot]_0 \simeq \mathfrak{so}(p, q) \ltimes \mathbb{R}^{p,q} \).
2. \( \mathfrak{so}(p, q) \oplus \mathbb{R}^{p,q}, [\cdot, \cdot]_c \simeq \mathfrak{so}(\mathbb{R}^{n+1}, I_{p,q}(c)) \) for every \( c \neq 0 \).

In particular, \( \mathfrak{so}(p, q) \oplus \mathbb{R}^{p,q}, [\cdot, \cdot]_c \) is isomorphic to \( \mathfrak{so}(p+1, q) \oplus \mathfrak{so}(p+1, q) \) for \( c > 0 \) (\( c < 0 \), respectively) under an isomorphism for which the summand \( \mathfrak{so}(p, q) \) is canonically embedded.

Proof. Using Lemma A.3, it is an easy exercise to prove that \([\cdot, \cdot]_c\) defines a Lie algebra structure. On the other hand, (1) is the definition of the Lie brackets on \( \mathfrak{so}(p, q) \ltimes \mathbb{R}^{p,q} \).

Finally, for (2) an isomorphism is easily seen to be given by:

\[
(X, u) \mapsto \begin{pmatrix} 0 & u^* \\ cu & X \end{pmatrix}, \quad c > 0, \text{ and } (X, u) \mapsto \begin{pmatrix} X & cu \\ u^* & 0 \end{pmatrix}, \quad c < 0
\]

where \( u \in \mathbb{R}^{p,q} \) is considered as a column vector and \( u^* = -u^t I_{p,q} \). For \( c > 0 \) an isomorphism \( \mathfrak{so}(\mathbb{R}^{n+1}, I_{p,q}(c)) \simeq \mathfrak{so}(p+1, q) \) is

\[
\begin{pmatrix} 0 & u^* \\ cu & X \end{pmatrix} \mapsto \begin{pmatrix} 0 & \sqrt{cu}^* \\ \sqrt{cu} & X \end{pmatrix}
\]

and similarly for \( c < 0 \). \(\square\)

Next, we state a uniqueness property for \( \mathfrak{so}(p, q) \)-invariant inner products related to the constructions of Lemma A.8. Its proof follows easily from Schur’s Lemma and the uniqueness (up to a multiple) of the Killing form of complex simple Lie algebras.

Lemma A.9. Assume that \( p, q \geq 1, n = p + q \geq 3 \) and \( n \neq 4 \). Let \( \langle \cdot, \cdot \rangle_1 \) and \( \langle \cdot, \cdot \rangle_2 \) be inner products on \( \mathfrak{so}(p, q) \) and \( \mathbb{R}^{p,q} \), respectively. Assume that \( \langle \cdot, \cdot \rangle_1 \) and \( \langle \cdot, \cdot \rangle_2 \) are \( \mathfrak{so}(p, q) \)-invariant, in other words:

- \( \langle [X, Y], Z \rangle_1 = -\langle Y, [X, Z] \rangle_1 \) for every \( X, Y, Z \in \mathfrak{so}(p, q) \), and
- \( \langle X(u), v \rangle_2 = -\langle u, X(v) \rangle_2 \) for every \( X \in \mathfrak{so}(p, q) \) and \( u, v \in \mathbb{R}^{p,q} \).

If \( c \in \mathbb{R} \setminus \{0\} \) is given, then there exist \( a_1, a_2 \in \mathbb{R} \) such that \( a_1 \langle \cdot, \cdot \rangle_1 + a_2 \langle \cdot, \cdot \rangle_2 \) is the Killing form of \( \mathfrak{so}(p, q) \oplus \mathbb{R}^{p,q}, [\cdot, \cdot]_c \).

Remark A.10. The previous result is not valid for \( n = p + q = 4 \). For \( p = q = 2 \), the invariant bilinear forms on \( \mathfrak{so}(2, 2) \simeq \mathfrak{so}(2, 1) \times \mathfrak{so}(2, 1) \) are the linear combinations of the Killing forms of the factors.

On the other hand, for the realification \( \mathfrak{g}_R \) of a simple complex Lie algebra, the complex structure \( J \) induces the invariant bilinear form \( \hat{K}(X, Y) = K(X, J(Y)) \), where \( K \) is its Killing form considered as a real Lie algebra. Note that \( \hat{K} \) is not a multiple of \( K \). In fact, using Schur’s lemma one can show that the invariant bilinear forms on \( \mathfrak{g}_R \) are given by \( aK + b\hat{K} \), where \( a, b \in \mathbb{R} \).
Furthermore, for $u$ a compact real form of $g$ it is easy to see that a $g_R$-invariant bilinear map of the form $\langle \cdot, \cdot \rangle = aK + b\hat{K}$ has $b = 0$ if and only if $u$ and $Ju$ are perpendicular with respect to $\langle \cdot, \cdot \rangle$. This follows from the fact that $g_R = u \oplus Ju$ is a Cartan decomposition and the properties of such decompositions with respect to the Killing form. From this, it is easy to see that the proof of Lemma A.9 remains valid for $so(3,1) \simeq sl(2, \mathbb{C})_R$ if we further assume that $u$ and $Ju$ are perpendicular with respect to $\langle \cdot, \cdot \rangle_1$ for some compact form $u$ of $sl(2, \mathbb{C})$.

References

[1] M.A. Akivis and V.V. Goldberg, Conformal differential geometry and its generalizations. Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1996.
[2] U. Bader, C. Frances and K. Melnick, An embedding theorem for automorphism groups of Cartan geometries. GAFA 19 (2009), no. 2, 333–355.
[3] U. Bader and A. Nevo, Conformal actions of simple Lie groups on compact pseudo-Riemannian manifolds. J. Differential Geom. 60 (2002), no. 3, 355–387.
[4] N. Bourbaki, Lie groups and Lie algebras. Chapters 7–9. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2005.
[5] G. Cairns, Géométrie globale des feuilletages totalement géodésiques. C. R. Acad. Sci. Paris Sér. I Math. 297 (1983), no. 9, 525–527.
[6] G. Cairns and E. Ghys, Totally geodesic foliations on 4-manifolds. J. Differential Geom. 23 (1986), no. 3, 241–254.
[7] A. Candel and R. Quiroga-Barranco, Gromov’s centralizer theorem, Geom. Dedicata 100 (2003), 123–155.
[8] E. B. Dynkin, Maximal subgroups of the classical groups. (Russian) Trudy Moskov. Mat. Obšč. 1, (1952), 39–166. See also: Translations of the AMS (2) 6, (1957), 245–378.
[9] C. Frances and K. Melnick, Conformal actions of nilpotent groups on pseudo-Riemannian manifolds, Duke Math. J. 153 (2010), no. 3, 511–550.
[10] M. Gromov, Rigid transformations groups, in Géométrie différentielle, Colloque Géométrie et Physique de 1986 en l’honneur de André Lichnerowicz (D. Bernard and Y. Choquet-Bruhat, eds.), Hermann, 1988, 65–139.
[11] R. Goodman and N.R. Wallach, Representations and Invariants of the Classical Groups. Encyclopedia of Mathematics and its Applications, 68. Cambridge University Press, Cambridge, 1998.
[12] S. Helgason, Differential Geometry, Lie Groups and Symmetric Spaces, Pure and Applied Mathematics 80. Academic Press Inc., 1978.
[13] B. O’Neill, Semi-Riemannian Geometry. With Applications to Relativity. Pure and Applied Mathematics, 103. Academic Press Inc., New York, 1983.
[14] A. L. Onishchik, Lectures on Real Semisimple Lie Algebras and their Representations, ESI Lectures in Mathematics and Physics. European Mathematical Society (EMS), Zürich, 2004.
[15] R. Quiroga-Barranco, Isometric actions of simple Lie groups on pseudo-Riemannian manifolds, Ann. of Math. (2) 164 (2006), no. 6, 941–969.
[16] R. Quiroga-Barranco, Isometric actions of simple groups and transverse structures: the integrable normal case, “Geometry, Rigidity and Group Actions”, 229–261, Chicago Lectures in Mathematics Series, The University of Chicago Press.
[17] J. Szaro, Isotropy of semisimple group actions on manifolds with geometric structure, Amer. J. Math. 120 (1998), 129–158.
[18] G. Warner, Harmonic Analysis on Semi-Simple Lie Groups I, Springer, Berlin and New York, 1972.
[19] A. Zeghib, On affine actions of Lie groups, Math. Z. 227 (1998), no. 2, 245–262.
[20] R. J. Zimmer, On the automorphism group of a compact Lorentz manifold and other geometric manifolds. Invent. Math. 83 (1986), no. 3, 411–424.
[21] R. J. Zimmer, Representations of fundamental groups of manifolds with a semisimple transformation group. J. Amer. Math. Soc. 2 (1989), no. 2, 201–213.
[22] R. J. Zimmer, Automorphism groups and fundamental groups of geometric manifolds. Differential geometry: Riemannian geometry (Los Angeles, CA, 1990), 693–710, Proc. Sympos. Pure Math., 54, Part 3, Amer. Math. Soc., Providence, RI, 1993.

[23] R. J. Zimmer, Entropy and arithmetic quotients for simple automorphism groups of geometric manifolds, Geom. Dedicata 107 (2004), 47–56.

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