A generating polynomial for the two-bridge knot with Conway’s notation \( C(n, r) \)

Franck Ramaharo
Département de Mathématiques et Informatique
Université d’Antananarivo
101 Antananarivo, Madagascar
franck.ramaharo@gmail.com
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Abstract

We construct an integer polynomial whose coefficients enumerate the Kauffman states of the two-bridge knot with Conway’s notation \( C(n, r) \).

Keywords: generating polynomial, shadow diagram, Kauffman state.

1 Introduction

A state of a knot shadow diagram is a choice of splitting its crossings [2, Section 1]. There are two ways of splitting a crossing:

(A) \( \bigcirc \bigcirc \implies \bigcirc \bigcirc \), (B) \( \bigcirc \bigcirc \implies \bigcirc \bigcirc \)

By state of a crossing we understand either of the split of type (A) or (B). An example for the figure-eight knot is shown in Figure 1.

Let \( K \) be a knot diagram. If \( m \) denotes the initial number of crossings, then the final states form a collection of \( 2^m \) diagrams of nonintersecting curves. We can enumerate those states with respect to the number of their components – called circles – by introducing the sum

\[ K(x) := \sum_S x^{|S|}, \]  

where \( S \) browses the collection of states, and \(|S|\) gives the number of circles in \( S \). Here, \( K(x) \) is an integer polynomial which we referred to as generating polynomial [6, 7] (in fact, it is a simplified formulation of what Kauffman calls “state polynomial” [2, Section 1–2] or
“bracket polynomial” [3]). For instance, if $K$ is the figure-eight knot diagram, then we have $K(x) = 5x + 8x^2 + 3x^3$ (the states are illustrated in Figure 2).

In this note, we establish the generating polynomial for the two-bridge knot with Conway’s notation $C(n, r)$ [4, 5]. We refer to the associated knot diagram as $B_{n,r}$, where $n$ and $r$ denote the number of half-twists. For example, the figure-eight knot has Conway’s notation $C(2, 2)$. Owing to the property of the shadow diagram which we draw on the sphere [1], we can continuously deform the diagram $B_{n,r}$ into $B_{r,n}$ without altering the crossings configuration. We let $B_{n,r} \rightleftharpoons B_{r,n}$ express such transformation (see Figure 3 (a)). Besides, we let $B_{n,0}$ and $B_{n,\infty}$ denote the diagrams in Figure 3 (b) and (c), respectively. Here, “0” and “$\infty$” are symbolic notations – borrowed from tangle theory [2, p. 88] – that express the absence of half-twists. If $r = \infty$ and $n \geq 1$, then $B_{n,\infty}$ represents the diagram of a $(2, n)$-torus knot ($\rightleftharpoons B_{n-1,1}$). Correspondingly, we let $B_{0,r}$ and $B_{\infty,r}$ denote the diagrams pictured in Figure 3 (d) and (e), respectively.

2 Generating polynomial

Let $K$, $K'$ and $\bigcirc$ be knot diagrams, where $\bigcirc$ is the trivial knot, and let $\#$ and $\sqcup$ denote the connected sum and the disjoint union, respectively. The generating polynomial defined in (1) verifies the following basic properties:

(i) $\bigcirc(x) = x$;

(ii) $(K \sqcup K')(x) = K(x)K'(x)$;

(iii) $(K \# K')(x) = \frac{1}{x}K(x)K'(x)$.

Furthermore, if $K \rightleftharpoons K'$, then $K(x) = K'(x)$ [6].

Lemma 1. The generating polynomial for the knots $B_{n,0}$ and $B_{n,\infty}$ are given by

\[ B_{n,0}(x) = x(x + 1)^n \]  

and

\[ B_{n,\infty}(x) = (x + 1)^n + x^2 - 1. \]
The key ingredient for establishing (2) and (3) consists of the states of specific crossings which produce the recurrences

\[ B_{n,0}(x) = (\bigcirc \sqcup B_{n-1,0})(x) + B_{n-1,0}(x) \]

and

\[ B_{n,\infty}(x) = B_{n-1,0}(x) + B_{n-1,\infty}(x), \]
Figure 3: Two-bridge knots with Conway’s notation $C(n, r)$.

respectively, with initial values $B_{0,0}(x) = x$ and $B_{0,\infty}(x) = x^2$ [6]. Note that the lemma still holds if we replace index $n$ by $r$.

**Proposition 2.** The generating polynomial for the two-bridge knot $B_{n,r}$ is given by the recurrence

$$B_{n,r}(x) = B_{n-1,r}(x) + (x+1)^{n-1}B_{\infty,r}(x),$$

and has the following closed form:

$$B_{n,r}(x) = \left(\frac{(x+1)^r + x^2 - 1}{x}\right)(x+1)^n + \left(\frac{(x+1)^r - 1}{x}\right)(x^2 - 1).$$

**Proof.** By Figure 4 we have

$$B_{n,r}(x) = B_{n-1,r}(x) + (B_{n-1,0}\#B_{\infty,r})(x)$$

$$= B_{n-1,r}(x) + (x+1)^{n-1}B_{\infty,r}(x),$$

4
The splits at a crossing allow us to capture $B_{n-1,r}, B_{n-1,0}$ and $B_{\infty,r}$, where the last relation follows from property (iii). Solving the recurrence for $n$ yields

$$B_{n,r}(x) = B_{0,r}(x) + B_{\infty,r}(x) \left( \frac{(x+1)^n - 1}{x} \right).$$

We conclude by the closed forms in Lemma 2.

**Remark 3.** We can write

$$B_{n,0}(x) = x^2 \alpha_n(x) + x$$

and

$$B_{n,\infty}(x) = x \alpha_n(x) + x^2,$$

where $\alpha_n(x) := \frac{(x+1)^n - 1}{x}$, so that identity (5) becomes

$$B_{n,r}(x) = \left( x^2 \alpha_n(x) + x \right) + \left( x^2 \alpha_r(x) + x \alpha_n(x) \alpha_r(x) \right).$$

Since the coefficients of $\alpha_n(x)$ are all nonnegative, it is clear, by (6), that the polynomial $x^2 \alpha_n(x)$ counts the states of $B_{n,0}$ that have at least 2 circles. This is illustrated in Figure 5 (a). Likewise, we have an interpretation of (7) in Figure 5 (b). In Figure 5 and 6, the dashed diagrams represent all possible disjoint union of $\ell - 1$ circles ($\ell = n$ or $r$, depending on the context), counted by $\alpha_\ell(x)$ and eventually empty.

Therefore, for $n, r \notin \{0, \infty\}$, identity (8) means that we can classify the states into 4 subset as shown in Figure 6. In these illustrations, there are $2^n - 1$ and $2^r - 1$ states of (a) and (b) kind, respectively, and $\binom{n}{1} \times \binom{r}{1} + 1$ one-component states of (c) and (d) kind. The remaining states are of (c) kind, bringing the total number of states to $2^{n+r}$.

### 3 Particular values

Let $\sum_{k \geq 0} b(n, r; k) x^k := B_{n,r}(x)$, or $b(n, r; k) := \left[ x^k \right] B_{n,r}(x)$. Then

$$b(n, r; k) = \binom{n + r}{k + 1} + \binom{n}{k - 1} + \binom{r}{k - 1} - \binom{n}{k + 1} - \binom{r}{k + 1} - \delta_{1,k},$$

where $\delta_{1,k}$ is the Kronecker delta function.
Figure 5: Illustrations of $B_{n,0}(x)$ and $B_{n,\infty}(x)$ as functions of $\alpha_n(x)$.

Figure 6: The states of $B_{n,r}$: states in (a) are counted by $x^2\alpha_n(x)$, those in (b) by $x^2\alpha_r(x)$, those in (c) by $x\alpha_n(x)\alpha_r(x)$, and state in (d) is simply counted by $x$.

where $\delta_{1,k}$ is the Kronecker symbol. By (1), we recognize $b(n, r; k)$ as the cardinal of the set $\{|S| = k : S \text{ is a state of } B_{n,r}\}$, i.e., the number of states having $k$ circles. In this section, the coefficients $b(n, r; k)$ are tabulated for some values of $n$, $r$ and $k$. We give as well the corresponding A-numbers in the On-Line Encyclopedia of Integer Sequences [8].

- $b(n, 0; k) = [x^k] x(x + 1)^n$, essentially giving entries in Pascal’s triangle A007318 (see Table 1).

| $n \backslash k$ | 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
|----------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0              | 0   | 1   |     |     |     |     |     |     |     |
| 1              | 0   | 1   | 1   |     |     |     |     |     |     |
| 2              | 0   | 1   | 2   | 1   |     |     |     |     |     |
| 3              | 0   | 1   | 3   | 3   | 1   |     |     |     |     |
| 4              | 0   | 1   | 4   | 6   | 4   | 1   |     |     |     |
| 5              | 0   | 1   | 5   | 10  | 10  | 5   | 1   |     |     |
| 6              | 0   | 1   | 6   | 15  | 20  | 15  | 6   | 1   |     |
| 7              | 0   | 1   | 7   | 21  | 35  | 35  | 21  | 7   | 1   |

Table 1: Values of $b(n, 0; k)$ for $0 \leq n \leq 7$ and $0 \leq k \leq 8$. 
• $b(n; 1; k) = [x^k] \left((x + 1)^{n+1} + x^2 - 1\right)$, generating a subtriangle in A300453 (see Table 2).

| $n \setminus k$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|-----------------|----|----|----|----|----|----|----|----|----|
| 0               | 0  | 1  | 1  |    |    |    |    |    |    |
| 1               | 0  | 2  | 2  |    |    |    |    |    |    |
| 2               | 0  | 3  | 4  | 1  |    |    |    |    |    |
| 3               | 0  | 4  | 7  | 4  | 1  |    |    |    |    |
| 4               | 0  | 5  | 11 | 10 | 5  | 1  |    |    |    |
| 5               | 0  | 6  | 16 | 20 | 15 | 6  | 1  |    |    |
| 6               | 0  | 7  | 22 | 35 | 35 | 21 | 7  | 1  |    |
| 7               | 0  | 8  | 29 | 56 | 70 | 56 | 28 | 8  | 1  |

Table 2: Values of $b(n, 1; k)$ for $0 \leq n \leq 7$ and $0 \leq k \leq 8$.

• $b(n; 2; k) = [x^k] \left((2x + 2)(x + 1)^n + (x^2 - 1)(x + 2)\right)$, giving triangle in A300454 (see Table 3).

| $n \setminus k$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|-----------------|----|----|----|----|----|----|----|----|----|
| 0               | 0  | 1  | 2  | 1  |    |    |    |    |    |
| 1               | 0  | 3  | 4  | 1  |    |    |    |    |    |
| 2               | 0  | 5  | 8  | 3  |    |    |    |    |    |
| 3               | 0  | 7  | 14 | 9  | 2  |    |    |    |    |
| 4               | 0  | 9  | 22 | 21 | 10 | 2  |    |    |    |
| 5               | 0  | 11 | 32 | 41 | 30 | 12 | 2  |    |    |
| 6               | 0  | 13 | 44 | 71 | 70 | 42 | 14 | 2  |    |
| 7               | 0  | 15 | 58 | 113| 140| 112| 56 | 16 | 2  |

Table 3: Values of $b(n, 2; k)$ for $0 \leq n \leq 7$ and $0 \leq k \leq 8$.

• $b(n, n; k) = [x^k] \left((x + 1)^{2n} + (x^2 - 1)(2(x + 1)^n - 1)\right) / x$, giving triangle in A321127 (see Table 4).

• $b(n, r; 1) = nr + 1$, giving A077028, and displayed as square array in Table 5.

In Kauffman’s language, $b(n, r; 1)$ is, for a fixed choice of star region, the number of ways of placing state markers at the crossings of the diagram $B_{n,r}$, i.e., of the forms

\[ \begin{array}{c}
\text{x}, \quad \text{xx}, \quad \text{xx}, \quad \text{xx}
\end{array} \]
Table 4: Values of $b(n, n; k)$ for $0 \leq n \leq 7$ and $0 \leq k \leq 13$.

| $n \setminus k$  | 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  | 13  |
|------------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0                | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   |     |     |     |     |     |     |
| 1                | 0   | 2   | 2   |     |     |     |     |     |     |     |     |     |     |     |
| 2                | 0   | 5   | 8   | 3   |     |     |     |     |     |     |     |     |     |     |
| 3                | 0   | 10  | 24  | 21  | 8   | 1   |     |     |     |     |     |     |     |     |
| 4                | 0   | 17  | 56  | 80  | 30  | 8   | 1   |     |     |     |     |     |     |     |
| 5                | 0   | 26  | 110 | 220 | 122 | 45  | 10  | 1   |     |     |     |     |     |     |
| 6                | 0   | 37  | 192 | 495 | 804 | 497 | 220 | 66  | 12  | 1   |     |     |     |     |
| 7                | 0   | 50  | 308 | 973 | 3059| 3472| 3017| 2004| 1001| 364 | 91  | 14  | 1   | 1   |

Table 5: Values of $b(n, r; 1)$ for $0 \leq n \leq 7$ and $0 \leq r \leq 7$.

| $n \setminus r$  | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  |
|------------------|----|----|----|----|----|----|----|----|
| 0                | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| 1                | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
| 2                | 1  | 3  | 5  | 7  | 9  | 11 | 13 | 15 |
| 3                | 1  | 4  | 7  | 10 | 13 | 16 | 19 | 22 |
| 4                | 1  | 5  | 9  | 13 | 17 | 21 | 25 | 29 |
| 5                | 1  | 6  | 11 | 16 | 21 | 26 | 31 | 36 |
| 6                | 1  | 7  | 13 | 19 | 25 | 31 | 37 | 43 |
| 7                | 1  | 8  | 15 | 22 | 29 | 36 | 43 | 50 |

so that the resulting states are “Jordan trails” [2, Section 1–2]. Note that a state marker is interpreted as an instruction to split a crossing as shown below:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\hline
\end{array}
\end{array}
\Rightarrow \begin{array}{c}
\begin{array}{c}
\hline
\end{array}
\end{array}
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\hline
\end{array}
\end{array}
\Rightarrow \begin{array}{c}
\begin{array}{c}
\hline
\end{array}
\end{array}
\end{array}
\end{array}
\]

The process is illustrated in Figure 7 for the figure-eight knot.

- $b(n, r; 2) = n \left( \binom{r}{2} + 1 \right) + r \left( \binom{n}{2} + 1 \right)$, giving square array in A300401 (see Table 6).

We paid a special attention to the case $k = 2$ because, surprisingly, columns $(b(n, 1; 2))_n$ and $(b(n, 2; 2))_n$ match sequences A000124 and A014206, respectively [6]. The former gives the maximal number of regions into which the plane is divided by $n$ lines, and the latter the maximal number of regions into which the plane is divided by $(n + 1)$ circles.

- $b(n, r; d(n, r))$ = leading coefficient of $B_{n,r}(x)$, giving square array in A321125 (see Table 7). Here, $d(n, r) = \max(n+1, r+1, n+r-1)$ denotes the degree of $B_{n,r}(x)$, and gives entries in A321126. We have Table 8 giving the numbers $d(n, r)$ for $0 \leq n \leq 7$ and $0 \leq r \leq 7$. 8
Figure 7: Illustration of $b(2, 2; 1)$: mark two adjacent regions by stars (*), then assign a state marker at each crossing so that no region of $B_{2,2}$ contains more than one state marker, and regions with stars do not have any.

Table 6: Values of $b(n, r; 2)$ for $0 \leq n \leq 7$ and $0 \leq r \leq 7$.

| $n$ \ $r$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----------|---|---|---|---|---|---|---|---|
| 0         | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 1         | 1 | 1 | 2 | 4 | 7 | 11| 16| 22| 29|
| 2         | 2 | 4 | 8 | 14| 22| 32| 44| 58|
| 3         | 3 | 7 | 14| 24| 37| 53| 72| 94|
| 4         | 4 | 11| 22| 37| 56| 79|106|137|
| 5         | 5 | 16| 32| 53| 79|110|146|187|
| 6         | 6 | 22| 44| 72|106|146|192|244|
| 7         | 7 | 29| 58| 94|137|187|244|308|

Table 7: Leading coefficients of $B_{n,r}(x)$ for $0 \leq n \leq 7$ and $0 \leq r \leq 7$.

| $n$ \ $r$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----------|---|---|---|---|---|---|---|---|
| 0         | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1         | 1 | 2 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2         | 1 | 1 | 3 | 2 | 2 | 2 | 2 | 1 |
| 3         | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 1 |
| 4         | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 1 |
| 5         | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 1 |
| 6         | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 1 |
| 7         | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 1 |

We have the following properties:

- $d(n, r) = d(r, n)$;
- if $r = 0$, then $d(n, r) = n + 1$;
- if $r = \infty$, then sequence $(d(n, r))_n$ begins: $2, 2, 2, 3, 4, 5, 6, 7, 8, \ldots$ ([A233583](http://oeis.org/A233583) with
Table 8: Values of $d(n, r)$ for $0 \leq n \leq 7$ and $0 \leq r \leq 7$.

| $n \setminus r$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|----------------|---|---|---|---|---|---|---|---|
| 0              | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 1              | 2 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 2              | 3 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| 3              | 4 | 4 | 4 | 5 | 6 | 7 | 8 | 9 |
| 4              | 5 | 5 | 5 | 6 | 7 | 8 | 9 | 10|
| 5              | 6 | 6 | 6 | 7 | 8 | 9 | 10| 11|
| 6              | 7 | 7 | 7 | 8 | 9 | 10| 11| 12|
| 7              | 8 | 8 | 8 | 9 | 10| 11| 12| 13|

Diagrammatically, we give the corresponding illustration for some value $s$ of $n$ and $r$ in Figure 8.

![Illustration of the numbers $d(n, r)$](image)

**Figure 8:** Illustration of the numbers $d(n, r)$.

Correspondingly, we have

- $b(n, r; d(n, r)) = b(r, n; d(r, n))$;
- if $r = 0$, then $b(n, r; d(n, r)) = 1$;
- if $r = \infty$, then sequence $\left(b(n, r; d(n, r))\right)_n$ begins: $1, 1, 2, 1, 1, 1, 1, \ldots$ (A294619 with initial term equals to 0).

Remarkable values in Table 7 correspond to knots $B_{1,1}$ (“Hopf link”, see Figure 9), $B_{2,2}$ (figure-eight knot, see Figure 1, 2) and $B_{n,2}$ (“twist knot” [6]) for $n \geq 3$. The latter case can be observed from identity (8) for which the leading coefficient is larger than 1 when $n + 1 = n + r - 1$ is satisfied. Also, consider the identity below:

$$B_{n,2}(x) = B_{n,0}(x) + B_{n,\infty}(x) + B_{n,\infty}(x) + \left( \bigcirc \sqcup B_{n,\infty} \right)(x).$$

We notice that the leading coefficient is deduced from the contribution of the summands $B_{n,0}(x)$ and $\left( \bigcirc \sqcup B_{n,\infty} \right)(x)$ [6].
Figure 9: The states of the knot $B_{1,1}$: $d(1,1) = 2$ and $b(1,1,d(1,1)) = 2$.

References

[1] Daniel Denton and Peter Doyle, Shadow movies not arising from knots, arXiv preprint, 2011, https://arxiv.org/abs/1106.3545.

[2] Louis H. Kauffman, Formal Knot Theory, Princeton University Press, 1983.

[3] Louis H. Kauffman, State models and the Jones polynomial, Topology 26 (1987), 95–107.

[4] Kelsey Lafferty, The three-variable bracket polynomial for reduced, alternating links, Rose-Hulman Undergraduate Mathematics Journal 14 (2013), 98–113.

[5] Matthew Overduin, The three-variable bracket polynomial for two-bridge knots, California State University REU, 2013, https://www.math.csusb.edu/reu/OverduinPaper.pdf.

[6] Franck Ramaharo, Enumerating the states of the twist knot, arXiv preprint, 2017, https://arxiv.org/abs/1712.06543.

[7] Franck Ramaharo, Statistics on some classes of knot shadows, arXiv preprint, 2018, https://arxiv.org/abs/1802.07701.

[8] Neil J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, published electronically at http://oeis.org, accessed 2019.

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(Concerned with sequences A000124, A007318, A014206, A077028, A233583, A294619, A300401, A300453, A300454, A321125, A321126 and A321127.)