Collective coherence in planar semiconductor microcavities

J Keeling¹, F M Marchetti², M H Szymańska³ and P B Littlewood¹

¹ Cavendish Laboratory, University of Cambridge, Madingley Road, Cambridge CB3 0HE, UK
² Rudolf Peierls Centre for Theoretical Physics, University of Oxford, 1 Keble Road, Oxford OX1 3NP, UK
³ Clarendon Laboratory, Department of Physics, University of Oxford, Parks Road, Oxford OX1 3PU, UK

Received 28 January 2005, in final form 22 March 2007
Published 23 April 2007
Online at stacks.iop.org/SST/22/R1

Abstract
Semiconductor microcavities, in which strong coupling of excitons to confined photon modes leads to the formation of exciton–polariton modes, have increasingly become a focus for the study of spontaneous coherence, lasing and condensation in solid state systems. This review discusses the significant experimental progress to date, the phenomena associated with coherence which have been observed and also discusses in some detail the different theoretical models that have been used to study such systems. We consider both the case of non-resonant pumping, in which coherence may spontaneously arise, and the related topics of resonant pumping, and the optical parametric oscillator.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Semiconductor microcavities have been designed to greatly enhance the matter–light interaction strength by confining light. The confined light couples to excitonic resonances in the medium inside the microcavity; when this exciton–photon coupling exceeds the exciton and photon damping rates one finds spectrally separated normal modes, microcavity exciton–polaritons [1, 2]. Unlike many other examples of strong coupling studied in quantum optics, this review will focus on planar semiconductor microcavities that confine photons in only one direction, thus leading to a continuum of strongly coupled modes. Due to their dual matter–light nature, exciton–polaritons can be manipulated and studied through their light component, and have an effective interaction through their matter component and the nonlinearity of light–matter coupling. Due to the continuum of modes, the behaviour of exciton–polaritons may be related to the statistical mechanics of interacting bosons. Thus, semiconductor microcavities provide an ideal system in which to study the interface between quantum optics, strong coupling, spontaneous coherence and quantum condensation.

A closely related area of research, although one we will not address in this review, is strong coupling to single excitonic resonances, i.e. excitons in quantum dots in semiconductor microcavities. As well as experiments on quantum dots in planar microcavities [3], experiments with confinement of photons in three spatial directions, producing 0D microcavities, have also been performed. Realizations of strong coupling to single quantum-dot excitons include: photonic crystals [4, 5], in which in-plane confinement results from localization on a defect in the photonic crystal, and vertical confinement from total internal reflection; micropillars [6], where Bragg mirrors provide vertical confinement, and total internal reflection provides in-plane confinement; and microdisks [7], where both vertical and in-plane confinement result from total internal reflection, but the in-plane confinement is significantly increased by using dots coupled to the whispering gallery modes [8] of the microdisk. As well as coupling between quantum dots and confined photon modes, there have also been experiments studying quantum well excitons coupled to photon modes confined in two spatial dimensions using wire structures fabricated by chemical etching [9]. Lying between confinement in...
one and three spatial directions are experiments in patterned microcavities, where schematically a variation in the width of the microcavity provides a shallow in-plane trap for photon modes [10–12]. This results in coexistence of 0D and 2D polariton states, separated in energies: a clear polariton spectrum has been seen along with the quantization induced by a box-like confinement. Polariton localization due to the intrinsic photonic disorder has also been observed [13].

Our review will concentrate on macroscopic collective phenomena arising from the interaction between these special bosonic particles. While some of these issues have been addressed in other contexts—such as quantum condensation in dilute atomic gases [14, 15], and coherent quantum optics of lasers [16]—the combination of effects seen in microcavity polaritons calls for new approaches. In part, the new theoretical challenges arise from description of features of semiconductor microcavities, such as the disorder and decoherence that arise in solids, the spin structure of polaritons, and the effect of pumping and decay. In addition, polariton systems provide new opportunities for experimental probes and observations that differ from those possible in other systems.

The recent experimental and theoretical research in this field can be divided into two main directions: firstly, experiments which use resonant (coherent) pumping have been motivated by the search for all-optical ultrafast switches and amplifiers. The other direction is that of non-resonantly pumped microcavities, where experiments have pursued the search for Bose–Einstein condensation (BEC), polariton lasing and macroscopic phase coherence phenomena.

A number of reviews have been already written on the subject of microcavity polaritons: in [17, 18], linear properties [19] of microcavities have been analysed in great detail. References [20, 21] review problems of nonlinear optics and the theoretical framework necessary to study resonant pumping, parametric amplification and oscillation. In addition, two books [22, 23] and two special issues [24, 25] have also been published. In this review, we will focus attention on those issues of modelling microcavity polaritons that are of particular importance in understanding spontaneous coherence and condensation in such systems. We will address the relation between the various theoretical approaches that have been used, and discuss the limits in which they become equivalent. We will also discuss in some detail the relation between coherence, condensation, lasing and superfluidity in experimental systems which are finite, two-dimensional, decaying and interacting, and thus differ from the Bose–Einstein condensation of ideal three-dimensional bosons. Finally, we will discuss the relation of these features both to the resonantly pumped polariton system, and also to other experimental systems in which similar issues of coherence and condensation in complex systems are addressed.

The review is divided overall into non-resonant pumping in section 2 and resonant pumping in section 3. Within section 2, we first present the experimental development of the subject in section 2.1, and then discuss the theoretical approach, dividing our discussion into the question of choice of model in section 2.2, choice of treatment (i.e. thermal equilibrium, rate equations, etc) in section 2.3 and the phenomena predicted (i.e. experimental signatures, conditions for condensation) in section 2.4. Within section 3 we again divide into a summary of experimental progress in section 3.1, and a discussion of the additional theoretical issues relevant only to the resonantly pumped case in section 3.2. Section 4 finally draws comparisons to phenomena seen in other experimental systems, and briefly summarizes our discussion.

1.1. Introduction to microcavity polaritons

Before discussing experiments and theories of coherence in microcavity polaritons, we provide here a brief introduction to the systems considered, and to microcavity polariton modes. Fuller introductions can be found elsewhere [22, 23, 26]. The semiconductor microcavities we discuss are constructed from distributed Bragg reflectors, containing alternating quarter wavelength thick layers of dielectrics with differing refractive indices. Due to these Bragg reflectors, the cavity contains a standing wave pattern of confined radiation. As illustrated in figure 1, quantum wells (QWs) are placed at the antinodes of this standing wave, thus maximizing the coupling between photons and excitons confined to the quantum wells.

Because the photon modes are confined to the cavity, the volume associated with the radiation mode is small, and so the exciton–photon coupling is strong. This strong coupling means that rather than considering the exciton–photon coupling as leading to radiative decay of the excitons, the exciton and photon modes are instead mixed, to form new normal modes: lower and upper polaritons. At the simplest level, one can write the exciton–photon Hamiltonian in terms of operators \( \psi_k \) creating photons and \( D_k \) creating excitons, with \( k \) labelling the 2D in-plane momentum. Thus:

\[
H = \left( \psi_k^\dagger D_k \right) \begin{pmatrix}
\omega_k & \Omega_k/2 \\
\Omega_k/2 & \varepsilon_k
\end{pmatrix} \begin{pmatrix}
\psi_k \\
D_k
\end{pmatrix}.
\]

(1)

(We have set \( \hbar = 1 \) here and throughout.) Here, \( \omega_k \) is the energy of the photon mode confined in the cavity of width \( L_w \),

\[
\omega_k = (c/n)\sqrt{k^2 + (2\pi N/L_w)^2},
\]

with \( n \) the refractive index, and \( N \) the index of the transverse mode in the cavity. For the situation in figure 1, \( N = 2 \). For small \( k \), the energy can be written as \( \omega_k = \omega_0 + k^2/2m \), where \( m \) is an effective photon mass \( m = (n/c)(2\pi N/L_w)^2 \).

In the absence of disorder, the exciton energy in the QW is \( \varepsilon_k = \varepsilon_0 + k^2/2M \), where \( M \) is the total exciton mass, and \( \varepsilon_0 = E_{cv} - R_{ex} \) comes from the conduction-valence band gap \( E_{cv} \) including QW...

---

**Figure 1.** Schematic diagram of a microcavity, formed by a pair of distributed Bragg reflector stacks, with quantum wells at the antinodes of the cavity photon mode.
confinement and the exciton binding energy (Rydberg) \( R_\text{ex} \).

For convenience, we define the bottom of the exciton band, \( \varepsilon_0 \) as the zero of energies; and denote the detuning between exciton and photon bands as \( \delta = \varepsilon_0 - \varepsilon_0 \). Finally, the off-diagonal term \( \Omega_R/2 \) describes the exciton–photon coupling, where \( \Omega_R \) is the Rabi frequency. Then, diagonalizing the quadratic form in equation (1) gives the polariton spectrum:

\[
E_{k}^{\text{LP,UP}} = \frac{1}{2} \left[ \delta + \frac{k^2}{2M} - \frac{\varepsilon_0^2}{2m} + \frac{1}{2} \sqrt{\left( \delta + \frac{k^2}{2M} - \frac{\varepsilon_0^2}{2m} \right)^2 + \Omega_R^2} \right]. \tag{2}
\]

This spectrum is illustrated in figure 2. It is shown there both as a function of momentum \( k \), and also as a function of angle. The angle corresponds to the angle of emission of a photon out of the cavity; since the in-plane momentum and photon frequency are both conserved as photons escape through the Bragg mirrors, one may write \( (\varepsilon_0 + E_k^{\text{LP}}) \sin(\theta) = c k \), which (since typical values of \( k \) satisfy \( \varepsilon_0 \), \( \varepsilon_0 \gg c k \) can be approximated as \( \omega_0 \sin(\theta) = c k \).

## 2. Non-resonant pumping

### 2.1. Summary of experiments

Optical properties of semiconductor microcavities have been the subject of extensive experimental research since the first observation of the strong coupling regime by Weisbuch et al [2]. Much of the experimental research has been on III–V materials, mainly GaAs/AlGaAs structures, or on II–VI materials, such as CdTe/CdMnTe/CdMgTe structures. The main aim of the experiments described here nonresonant pumping has been to start with incoherently injected polaritons, and observe spontaneous coherent processes emerging from incoherent injection of polaritons: polariton degeneracy, final state stimulation and ultimately polariton BEC.

The authors of the earliest report [27] of nonlinear emission in the presence of strong coupling in GaAs microcavities, suggesting final state stimulation characteristic for bosonic particles, later withdrew those conclusions [28] as further experiments showed that the threshold for nonlinear emission occurred in that case after the crossover to the weak-coupling regime; thus the nonlinear emission should have been attributed to photon lasing. The first unambiguous observation of polariton bosonic stimulation was in CdTe microcavities [29] consisting of 16 quantum wells with a Rabi splitting of around 23 meV. Two distinct stimulation thresholds were observed with increasing intensity of continuous wave pumping, as shown in figure 3. As the pumping intensity was increased above the first threshold, nonlinear emission at energies close to the bottom of the lower polariton branch was clearly seen. The second threshold, reported for much higher intensities, was connected with a weak coupling electron–hole lasing mechanism. Further investigation [30, 31] showed that this nonlinearity had emission varying as the square of pumping intensity, and a threshold that occurred for occupation factors much less than one, and so this nonlinearity was associated with increase of exciton–exciton scattering, rather than final state stimulation. Final state stimulation in III–V materials was demonstrated by pump-probe experiments in [33, 34].

The stimulated scattering to the ground state, and nonlinear build-up of lower polariton population was the first step towards demonstration of spontaneous coherence.
and thermalization—characteristic of quantum condensation. However, the big challenge to realizing a condensed polariton phase was the finite (though very large) quality of the cavity mirrors, and the resultant short polariton lifetime, of the order of picoseconds. In addition, due to the ‘bottleneck effect’ [35], the relaxation of polaritons to the zero momentum state was delayed, hindering the creation of a thermal population in the lower energy states. The first investigation of the coherence properties of emitted light above the threshold for nonlinear emission in strong coupling was based on the measurement [36, 37] of the second order coherence function, \( g_2(t = 0) \), which would take a value of \( g_2(0) = 2 \) for a thermal state, and \( g_2(0) = 1 \) for a coherent state [38]. A decrease of \( g_2(0) \) from 1.8 to 1.4 as pumping power was increased from threshold to 20 times threshold power was seen in a system of 12 GaAs quantum wells placed at the antinodes of light in a GaAs/AlGaAs microcavity, giving 14.9 meV Rabi splitting. This was followed by a report of a characteristic change in the momentum space distribution above threshold [39], as shown in figure 4, and a blueshift of the polariton dispersion [40]. Reference [39] also reported measurements of the polarization of the emission, which both depended on the polarization of the pump, and also varied with the intensity of the pump; such results are well described by models of coherent spin dynamics, as discussed in section 3.2.2.

Experiments on GaAs [36, 41] showed that the delay time between the laser pulse, and emission from the \( k \approx 0 \) states decreased as one increased the pumping power; this implies an increase in scattering rate, and so suggests thermalization rate may also increase with increasing density. Time-resolved photoluminescence measurements were also performed for a CdTe microcavity with 10.5 meV Rabi splitting [42] under non-resonant pulsed excitation, which were able to monitor the buildup of a large polariton population in the \( k = 0 \) state. Analysis of the time dependence showed that, below threshold, the dynamics of the \( k = 0 \) polaritons follows closely the population of cold reservoir excitons; the relaxation from high energy exciton states resonant with the pump to these reservoir exciton states had a characteristic relaxation time of 30 ps. (Note that this time is significantly shorter than the 150 ps observed in similar experiments [41] with GaAs based microcavities.) Above the nonlinear threshold, the population of the reservoir excitons was found to be clamped, and the polariton relaxation dynamics became faster, with the maximum of polariton emission at 70 ps delay after the initial pulse. This delay was further decreased for higher excitation powers. Together these provide evidence of stimulated exciton–exciton scattering to the lower polariton states.

The first clear demonstration of spontaneous first-order coherence in an incoherently pumped microcavity was seen in a 16 QW CdTe microcavity with 26 meV Rabi splitting [43] under non-resonant pulsed pumping. An interesting feature of this particular experiment was that the nonlinear emission was at \( k \neq 0 \) and so resulted in an emission ring at an angle of around 17°; this was associated with the small size of the excitation spot (3 \( \mu m \)). (Note that in later experiments with larger excitation spots on the same sample condensation was at \( k = 0 \) [44].) The first-order coherence was investigated by spectroscopic imaging of the far-field emission. Two momentum space images were superimposed giving fringes (as a function of momentum \( k \)) with over 75% contrast above threshold and up to 35% below threshold. In a later publication of the same group [45], experiments on a 4 QW CdTe microcavity characterized by 13.2 meV Rabi splitting showed macroscopic occupation of the \( k = 0 \) state characterized by narrowing above threshold of the polariton emission line to a linewidth below that of the cavity photon mode. Near-field images showed modulation of the polariton spatial distribution, revealing the effect of photonic disorder. The next challenge in the search for spontaneous condensation was to see similar effects, but accompanied by a thermal distribution of \( k \neq 0 \) polaritons.

Due to the short polariton lifetime and the ‘bottleneck effect’ [35], the realization of equilibrium population has proven to be challenging. Progress came from observing [44, 46] that thermalization processes due to particle–particle scattering can be dramatically increased both by increasing the value of the (non-resonant) pump power, and also by positively detuning the cavity energy above the excitonic energy. Large positive detuning makes polaritons more excitonic and increases their scattering rate. Time- and angle-resolved spectroscopy on a sample consisting of 12 GaAs QCWs characterized by 14.4 meV Rabi splitting [46] showed that for positively detuned cases, where the thermalization time increases while decay time decreases, the thermalization time can reach around one tenth of the polariton lifetime and lower polaritons remain in thermal equilibrium with the phonon bath for a period of about 20 ps. In that paper, fitting of the momentum distribution to a Bose–Einstein distribution was used to extract the temperature and chemical potential. One should note however that for condensation of an interacting gas, the density of states of the low energy modes changes, which makes it hard to extract reliable estimates of the chemical potential from such data.

Finally, a comprehensive set of experiments showing clear evidence for condensation of cavity polaritons was performed in a CdTe [44] structure consisting of 16 quantum wells giving 26 meV Rabi splitting. Above the threshold pumping density they observed: a massive occupation of the \( k \approx 0 \) mode developing from a polariton gas in thermal equilibrium at 19 K (shown in figure 5); an increase of temporal coherence from

---

**Figure 4.** Momentum space distribution of lower polaritons above (main figure) and below (inset) the nonlinear threshold. (Reprinted from [39]. Copyright 2003 National Academy of Sciences.)
1.5 ps below threshold to up to 6 ps above threshold; the build-up of long-range spatial coherence over the whole system size with contrast of interference fringes from less than 5% below threshold to 45% above threshold; linear polarization of the emission. Linear polarization had been predicted to appear in the condensed state [47], and its appearance gives evidence for the single state nature of the condensate, though since the direction of this polarization was pinned to a crystallographic direction the polarization direction symmetry was not spontaneously broken. Evidence that the polarization of light is pinned to one of the crystallographic axes, independently of the excitation polarization, was also independently observed in experiments on both CdTe [48] and InGaAs microcavities [49]. These results were ascribed to birefringence in the mirrors and cavity. The polarization arising above threshold in the presence of a symmetry breaking term, and the possibility of controlling this by application of an electric field was discussed theoretically in [50]. The polarization observed in [44] has been investigated further in [51]. Shortly following the work in [44], a similar nonlinear build-up accompanied by a linear polarization was seen [52] in GaAs structures in stress-induced traps [53]. Very recent work [54] in GaAs structures has also studied the spatial coherence of the system, using a two-slit experiment to measure $g_2(t)$ as a function of the separation of slits. It is expected, and supported by data, that the degree of photonic disorder in GaAs is smaller than in CdTe structures, giving a cleaner demonstration of the decay of spatial coherence with increasing separation. Returning to the CdTe structures studied in [44], the second-order coherence function, $g_2(t = 0)$, has also been measured [55], in contrast to observations reported in GaAs in [36], $g_2(0)$ in CdTe was found to be around 1 at threshold and then increased up to around 1.4 at powers 10 times threshold. This effect has been attributed to the phase diffusion due to interactions.

Wide-band-gap semiconductor structures based on group-III nitrides, such as GaN based cavities have recently attracted considerable interest (see, e.g., [56–65]). The main advantage of these structures over II–VI and other III–V materials lies in the large exciton binding energy (around 26 meV for bulk structures, and over 40 meV for narrow quantum wells) and the large coupling to the photon field, which makes them ideal systems for the realization of functional devices operating at room temperature. Although the study of cavity polaritons in group-III nitrides microcavities is still in its infancy, strong exciton–photon coupling in a bulk GaN cavity [62, 64, 65] and in a quantum well cavity [61] have been reported. In both cases, the substantial inhomogeneous broadening of the excitonic and photonic lines play a key role in establishing the conditions for reaching strong coupling. For similar reasons, ZnO has also been proposed as another possible candidate material [66]. Very recently, a nonlinear build-up of polariton emission accompanied by an increase in the first-order temporal coherence, and a spontaneously chosen linear polarization (independent of the apparatus and different between measurements) has been reported to occur at room temperature in GaN bulk microcavities [67].

2.2. Theoretical models

In this section, we will discuss the different models that have been used to describe microcavity polaritons and study their condensation. We wish to separate clearly two aspects of theoretical description of polaritons; the first aspect is the choice of model, the subject of this section, the second aspect is how that model is treated, which will instead be covered in section 2.3. After having addressed these points, we then in section 2.4 discuss the various theoretical predictions of conditions for condensation, and of possible signatures. Readers who are not interested in the details of how the system is theoretically modelled should jump to section 2.4. A model starting from electrons and holes, taking into account their Coulomb interaction to form bound excitons, their coupling to light, and the effects of disorder would describe polariton systems exactly, but is too complicated to allow any clear understanding of the important features associated with condensation to be gained. Therefore, it is appropriate to use simplified models, that exaggerate some features of the real system, and neglect others. In judging which model is appropriate to address a given problem, it is important to understand how the model relates to the underlying microscopic model of electrons and holes, and so we shall start by discussing this microscopic model.

2.2.1. Microscopic electron–hole Hamiltonian. In this section, we discuss the underlying description of microcavity polaritons formed from photons confined to a two-dimensional cavity, interacting with electrons and holes in two-dimensional quantum wells [26, 68].

$$H = H_{\text{eh}} + H_{\text{coaid}} + H_{\text{disorder}} + H_{\text{photon}} + H_{\text{dipole}}. \quad (3)$$

Consider first the electrons and holes; we have

$$H_{\text{eh}} = \sum_k \varepsilon_k c_k^\dagger c_k + \varepsilon_k^0 v_k^\dagger v_k \quad (4)$$

$$H_{\text{coaid}} = \frac{1}{2\lambda} \sum_q e^2 |q| [\rho_q^\dagger \rho_q^\dagger + \rho_q^\dagger \rho_q^\dagger - 2 \rho_q^\dagger \rho_q^\dagger] \quad (5)$$

$$H_{\text{disorder}} = \int \! dr \!\left[ W_q(\mathbf{r}) c^\dagger(\mathbf{r}) c(\mathbf{r}) - W_q(\mathbf{r}) u(\mathbf{r}) v(\mathbf{r}) \right]. \quad (6)$$
Here $c_{k}^{\dagger} (\nu_{k})$ create electrons in the conduction (valence) bands, which have dispersions, $\epsilon_{c}^{\dagger} (\epsilon_{v})$. Since the 'empty' state is a filled valence band, it is more convenient to describe the valence band via the operator $\nu_{k}$ which creates a hole—i.e. a missing electron. The density of electrons (holes) is given by $\rho_{c}^{\dagger} = \sum_{k} c_{k}^{\dagger} c_{k}^{\dagger}$ ($\rho_{v}^{\dagger} = \sum_{k} h_{k}^{\dagger} h_{k}^{\dagger}$). The factor $1/A$, where $A$ is the quantization area of the cavity, appears explicitly because the Hamiltonian has been written as a sum over momentum labels; this factor plays no role in any final answer, and is absorbed in the definition of $dk$ if summation is replaced by integration. Note also that in general there should be a dependence on the electron and hole spin degrees of freedom, that we neglect here. The last term, equation (6), describes the disorder potential acting on electrons and holes, e.g. due to well-width fluctuations and alloy disorder. In general, disorder can act differently on electrons and holes; in practice for the materials used, the energy scale of disorder is less than the binding energy, so disorder does not dissociate excitons [69]. If the exciton binding energy is significantly larger than a characteristic energy scale of disorder, then as described in [69] one can factorize the wavefunction into a centre-of-mass wavefunction, and a wavefunction of relative electron–hole separation. Then, in the equation for the centre of mass wavefunction, one has an effective disorder potential that is the result of convolving the original disorder with the wavefunction for relative electron–hole separation. As a result of this convolution, the effective disorder potential as seen by the exciton centre-of-mass wavefunction is smoothed over the scale of the exciton Bohr radius [70].

Turning now to the interaction with the photons,

$$H_{\text{photon}} = \sum_{q} \omega_{q} \psi_{q}^{\dagger} \psi_{q} + \int \text{d}r W_{\text{ph}}(\mathbf{r}) \psi_{\mathbf{r}}^{\dagger}(\mathbf{r}) \psi(\mathbf{r})$$

(7)

$$H_{\text{dipole}} = \frac{1}{\sqrt{A}} \sum_{q,k} \epsilon_{\mu_{cv}} \frac{\epsilon_{q}}{2\epsilon_{q} L_{w}} (\psi_{q}^{\dagger} \psi_{k}^{\dagger} + \text{H.c.})$$

(8)

In equation (8), the quantization volume for the electromagnetic field has been factored into $AL_{w}$, where $L_{w}$ is the width of the cavity, and $A$ the quantization area as discussed above. The term $\mu_{cv}$ is the inter-band dipole matrix element, which can be calculated given the Bloch wavefunctions of the two bands. The term $W_{\text{ph}}(\mathbf{r})$ in equation (7) describes photonic disorder, which can arise due to roughness of the Bragg mirrors—i.e. due to layer width fluctuations (monolayer mismatch), or crystal dislocations [13, 71]. The effects of this photonic disorder, and of the exciton disorder introduced above, can be quite different. The photonic disorder is generally on large length scales (typically of the order of a micrometre), comparable to the size of the excitation spot, and so it is primarily associated with the spatial inhomogeneity of polaritons seen in experiment [45]. In contrast, as discussed in section 2.2.3, excitonic disorder is on much shorter length scales (typically of the order of 10 nm for CdTe), and thus does not affect the spatial polariton density profile; however, excitonic disorder does have a significant impact on the distribution of excitonic oscillator strengths. Although the excitons are localized, in the absence of photonic disorder the polaritons formed consist of a superposition of many different localized excitons and extended photon states, and thus one may form delocalized polaritons from localized excitons [72]. We will not explicitly discuss the effects of photonic disorder further, however the discussion of condensation in a trap in section 2.3.3 can apply also to trapping in disorder, as well as any deliberately engineered trapping.

The Hamiltonian in equation (3) already contains a number of important approximations, which should be discussed. The interaction of photons with electrons and holes makes use of both the dipole approximation, and the rotating wave approximation [26, chapter 10]. The interaction strength here is written in the dipole (length) gauge. The choice between the dipole (length) gauge and the Coulomb (velocity) gauge is not arbitrary, as the terms assigned as describing free particles (without interaction with radiation) are different in each gauge [73–75]. This point is worth stressing, as the electromagnetic interaction between excitons is split between the direct Coulomb term, and a photon mediated term. Thus the choice of gauge affects also the Coulomb interaction (equation (5)), controlling which parts of it are absorbed into the definition of exciton states, which parts are associated with the 'photon' operators—in the Dipole gauge, the fields $\psi_{q}^{\dagger}$ are quantized modes of the electric displacement—and which should be written as some effective exciton–exciton interaction [75]. The relation between Coulomb interaction and photon mediated interaction is complicated here because the resonant photons are confined by the DBR (distributed Bragg reflector) mirrors, while the static Coulomb term is modified much less strongly by the mirrors. When one comes to exciton states, it is therefore important to be aware that the choice of gauge affects both the exciton–photon coupling strength, and the form of the inter-exciton Coulomb interaction, and that these two are not separate.

In the next two sections, we will discuss the main two classes of effective Hamiltonians, derived from this full Hamiltonian, used to study microcavity polaritons. The differences between these effective Hamiltonians can be seen as the result of regarding different terms as important; i.e. which terms are treated exactly, and which perturbatively. In both cases, the first step involves changing from electrons and holes to bound excitons—i.e. solving the wavefunction for the relative coordinates. The differences then arise from considering in one case next the effect of disorder, giving localized states, and then approximating the inter-exciton Coulomb term by exclusion—this leads to the boson–fermion model discussed in section 2.2.3—or alternatively, treating the Coulomb term via a quartic exciton–exciton interaction term, then coupling to light, and then treating disorder perturbatively or not at all—this leads to the weakly interacting boson model, discussed in section 2.2.2.

As will be discussed further below, in the low density limit, many features of these models are similar. However, the different models emphasize different features: the boson model can effectively describe the case where the dominant interactions are exciton–exciton Coulomb interactions, while the boson–fermion model instead has the saturation of the exciton–photon coupling as the dominant interaction. As such, these different models may be appropriate in different contexts. For example, to describe the lower polariton blue-shift, and comparable upper polariton red-shift seen, e.g. in
2.2. Weakly interacting boson models. A weakly interacting Bose gas model of polaritons can be achieved by making an Usui transformation \[77, 78\], choosing the bosonic operators to represent bound exciton states, and then truncating the interaction terms at fourth order \[79\]. This results in an effective Hamiltonian describing bosonic excitons coupled to photon modes:

\[
H = \sum_k \left[ \omega_k \psi_k \psi_k^\dagger + \varepsilon_k D_k^\dagger D_k + \frac{\Omega_R}{2} \left( D_k^\dagger \psi_k + \psi_k^\dagger D_k \right) \right] - \frac{\Omega_R}{2 \rho_{sat}} \sum_{k,k'q} \left[ D_{k-q}^\dagger D_{k'q}^\dagger \psi_k \psi_{k'q} + \psi_k^\dagger D_{k'q} \psi_{k-q} + \psi^\dagger_{k-q} D_k \psi_{k'q} \right] + \sum_{k,k'q} \frac{U_{k-k'q}}{2} D_{k'q}^\dagger D_{k-q} D_k D_{k'}.
\]

(9)

Here \( D_k \) creates a bound exciton of energy \( \varepsilon_k \), \( \psi_k^\dagger \) creates a cavity photon of energy \( \omega_k \) and \( \Omega_R \) is the effective exciton–photon coupling strength, or Rabi splitting. By measuring energies from the bottom of the exciton dispersion, we may write \( \varepsilon_k = \omega_k^2/2M + \delta \), with \( \delta \) the exciton–photon detuning. The quartic terms in equation (9) are divided into exciton–exciton interactions, \( U_{k-k'q} \), the strength of which can also be found by calculation of the Coulomb exchange term \[80, 81\] in the Born approximation, and a ‘saturation term’ (second line), which decreases the exciton–photon coupling at large exciton densities due to the fermionic character of the excitons \[79\]. These quartic terms arising from the Usui transformation can be seen as an expansion of the underlying fermionic operators in powers of bosonic operators; this expansion is controlled by the small parameter of the number of excitons per Bohr radius. Note that in general these terms depend also on the spin degrees of freedom of the constituent electron and holes. For a derivation of the dependence on spin of the Coulomb terms, see, e.g., \[82, 83\].

This approach takes into account the intra-exciton Coulomb term, in forming bound excitons, and the inter-exciton Coulomb terms as an effective quartic interaction. The Hamiltonian in equation (9) however neglects disorder acting on the exciton states, and as a result finds that each exciton state couples to a single photon state, with conserved momentum. However, as discussed below in section 2.2.3, and in \[84, 85\], exciton disorder will modify this picture. Including exciton disorder will modify this picture. Including effects of disorder the density at which these saturation effects become important can be much lower than the Mott density. Thus, a quartic description of saturation may become inadequate at modest densities, close to those already studied experimentally. In addition, most bosonic models of polaritons further simplify equation (9), replacing the momentum dependent interaction \( U_{k-k'q} \) with its strength at \( k = k' \), \( q = 0 \). This strength is the interaction between two excitons in the same single particle momentum eigenstate. If exciton eigenstates are localized, it is not obvious that replacing all exciton–exciton Coulomb interactions with an average strength (calculated from delocalized exciton wavefunctions) is appropriate. In addition, the dominant Coulomb interaction between localized, and therefore non-overlapping, exciton states may well be due to the direct dipole–dipole interaction, rather than exchange terms (as it is in the clean case). The boson–fermion model discussed in section 2.2.3 handles this interaction differently—it includes strong on-site repulsion, and neglects inter-site repulsion; this limit is clearly also an exaggeration, and the true effects of Coulomb will be between these two extremes.

It is worth noting parenthetically that a constraint on exciton density \( \sum_k \{ D_k^\dagger D_k \} < \rho_{sat} \) is required to make the Hamiltonian in equation (9) stable. Without such a constraint the free energy is unbounded from below, i.e. for \( |\Psi\rangle = \exp (\lambda \psi_0^\dagger + \beta D_0^\dagger)|0\rangle \), the free energy \( F = \langle H - \mu N \rangle \) corresponding to equation (9) is

\[
F = (\delta - \mu)|\lambda|^2 - \mu |\beta|^2 + \Omega_R \Re(\lambda \beta^*) \left( 1 - \frac{|\beta|^2}{\rho_{sat}} \right) + \frac{U_0}{2} |\beta|^4.
\]

(10)

The minimum free energy can be found for real \( \lambda, \beta \), and so re-parameterizing these as \( \lambda = x \sin(\chi), \beta = x \cos(\chi) \), the quartic term in equation (10) goes like:

\[
F_4 = \frac{U_0}{4} x^4 \cos^2 \chi \left[ 1 + \cos(2\chi) - \frac{2 \Omega_R}{U_0 \rho_{sat}} \sin(2\chi) \right].
\]

(11)

For any non-vanishing \( \Omega_R \), there is a value of \( \chi \) for which this is negative and so unstable. Physically this instability is cured by restoring higher order contributions of the saturation interaction which prevent \( \sum_k \{ D_k^\dagger D_k \} > \rho_{sat} \). Practically the above instability can be avoided if one diagonalizes the quadratic part of equation (9), and then projects onto the basis of lower polariton states \[79\]. By writing

\[
\begin{pmatrix}
\psi_k^\dagger \\
D_k^\dagger
\end{pmatrix} = \begin{pmatrix}
\cos \delta_k & -\sin \delta_k \\
\sin \delta_k & \cos \delta_k
\end{pmatrix} \begin{pmatrix}
U_k^\dagger \\
L_k^\dagger
\end{pmatrix}
\]

(12)

here \( L_k^\dagger, U_k^\dagger \) create lower and upper polaritons respectively, and \( \cos \delta_k, \sin \delta_k \) are the standard Hopfield coefficients \[26, 68\]. In order to diagonalize the quadratic part of equation (9), one must choose

\[
\tan(2\delta_k) = \frac{\Omega_R}{\omega_k - \varepsilon_k},
\]

(13)
with $\omega_k$ and $\epsilon_k$ as defined following equation (9). Having diagonalized the quadratic part, one may project onto the lower polariton basis for the quartic part, giving the effective lower polariton Hamiltonian:

$$H_{LP} = \sum_{k} E_{k}^{LP} L_{k}\dot{L}_{k} + \sum_{k,k',q} V_{kk',q}^{ eff} L_{k}^\dagger L_{k'}^\dagger L_{k'} L_{k}$$

(14)

$$E_{k}^{LP} = \frac{1}{2} \left[ (\omega_k + \epsilon_k) - \sqrt{(\omega_k - \epsilon_k)^2 + \Omega_k^2} \right]$$

(15)

$$V_{kk',q}^{ eff} = \frac{\Omega_k}{2\rho_{sat}} \cos \theta_{kq} \cos \theta_k$$

$$\times \left[ \cos \theta_{k'} q \sin \theta_{k'} + \sin \theta_{k'} q \cos \theta_{k'} \right]$$

(16)

$$+ \frac{U}{2} \cos \theta_{kq} \cos \theta_k \cos \theta_{k'-q} \cos \theta_{k'}.$$

Note that in order for the neglect of upper polaritons to be valid, one must be at temperatures significantly smaller than the Rabi splitting. This requirement of temperature can be translated to a requirement of low densities if one is interested in phase transitions: the density must be low enough that the Bose condensation temperature at that density is much less than the Rabi splitting. It can be shown [86] that this latter requirement means one should have fewer than one polariton per wavelength of light, as discussed further in section 2.2.4; such a density is already exceeded in current experiments.

The Hamiltonian (14) has an effective $k$ dependent interaction strength due to the change of Hopfield coefficient along the lower polariton branch—i.e. Coulomb interaction strength due to the change of Hopfield coefficient per wavelength of light, as discussed further in section 2.2.4; a constraint on total fermion occupancy, eliminates the unphysical states.

This formalism is easy to use, as one may show [101] that $\alpha$ represents a two-level system, where $|\downarrow\alpha⟩$ is the ground state—and $|\uparrow\alpha⟩$ indicates the presence of an exciton on site $\alpha$. Such a model has also been studied in the related context of spontaneous superradiance [97–99]; it was however later shown [100] that including higher order terms beyond the dipole approximation prevents the superradiant transition of the vacuum state of such a model. No such problem however occurs when one considers the system in contact with a reservoir that fixes particle density—the effects discussed in [100] apply to the stability of the vacuum state, i.e. with chemical potential going to negative infinity.

By considering first the effects of disorder acting on the excitons, one finds that in 2D systems the effect of disorder is particularly profound and that formally any arbitrarily small amount of disorder leads to localization [87, 88]. However, the character of the states changes significantly with energy. At high energies states may be described as a random superposition of plane waves with the same modulus of momentum, and localization effects are weak. At very low energies, well below the band edge, the Lifshitz tail states [89–91] have a nodeless form, localized in deep minima. The changing nature of the exciton states with energy also changes their oscillator strength [92, 93], and the exciton states that couple most strongly to the long wavelength radiation modes are those just below the band edge, for which localization effects are important. As a result, those exciton states which contribute most to the relevant (thermally populated) polariton states are effectively localized exciton states [84, 85].

This localization may also be expected to modify details of the inter-exciton Coulomb interaction term compared to the clean picture [79, 80]. Considering strongly localized exciton states, since exchange requires wavefunction overlap, one expects a difference between the strength of on-site Coulomb repulsion—i.e. interaction of excitons localized in the same potential fluctuation—as compared to inter-site interactions. Taking the extreme form of this difference—i.e. on-site exclusion and neglect (or perturbative treatment) of the inter-site interaction—leads one to a generalization [84, 85] of the Dicke model [94–96], describing two-level systems coupled to a bosonic field:

$$\hat{H} = \sum_{\alpha} \varepsilon_{\alpha} \hat{S}_{\alpha}^z + \sum_{p} \alpha_{p} \hat{\psi}_{p}^{\dagger} \hat{\psi}_{p} + \frac{1}{\sqrt{\mathcal{A}}} \sum_{\alpha} \sum_{p} \left( \gamma_{\alpha,p} \hat{\psi}_{p}^{\dagger} \hat{S}_{\alpha}^+ + \text{H.c.} \right).$$

(17)

Here $\hat{S}_{\alpha}^z$ is a spin $1/2$, representing a two-level system, where $|\downarrow\alpha⟩$ is the ground state—and $|\uparrow\alpha⟩$ indicates the presence of an exciton on site $\alpha$. Such a model has also been studied in the related context of spontaneous superradiance [97–99]; it was however later shown [100] that including higher order terms beyond the dipole approximation prevents the superradiant transition of the vacuum state of such a model. No such problem however occurs when one considers the system in contact with a reservoir that fixes particle density—the effects discussed in [100] apply to the stability of the vacuum state, i.e. with chemical potential going to negative infinity.

It is often convenient to represent the two-level systems as two fermionic states so that the ground state is $|\downarrow\alpha⟩ = a_{\alpha}^\dagger |0⟩$, and the excitonic state $|\uparrow\alpha⟩ = b_{\alpha}^\dagger |0⟩ = b_{\alpha}^\dagger a_{\alpha} |\text{g.s.}⟩$. Imposing a constraint on total fermion occupancy, $b_{\alpha}^\dagger b_{\alpha} + a_{\alpha}^\dagger a_{\alpha} = 1$, eliminates the unphysical states $|0⟩$ and $a_{\alpha} b_{\alpha}^\dagger |0⟩$, thus giving the Hamiltonian

$$\hat{H} = \sum_{\alpha} \frac{\omega_{\alpha}}{2} (b_{\alpha}^\dagger b_{\alpha} + a_{\alpha}^\dagger a_{\alpha}) + \sum_{p} \alpha_{p} \hat{\psi}_{p}^{\dagger} \hat{\psi}_{p} + \frac{1}{\sqrt{\mathcal{A}}} \sum_{\alpha} \sum_{p} \left[ \gamma_{\alpha,p} \hat{\psi}_{p}^{\dagger} b_{\alpha}^\dagger a_{\alpha} + \text{H.c.} \right].$$

(18)

This formalism is easy to use, as one may show [101] that the constraint preventing double occupation can be easily...
incorporated in the imaginary time path integral formalism by shifting the fermionic Matsubara frequencies according to

$$\epsilon_n = (2n + 1)\pi/\beta \mapsto \epsilon_n = (2n + 3/2)\pi/\beta. \quad (19)$$

It is important not to confuse these fermionic states (which represent the two levels of a two-level system) with the conduction and valence band states in equation (4). While \(b_\alpha^\dagger a_\alpha\) creates an exciton, one should not think of \(b_\alpha^\dagger (a_\alpha)\) as creating an electron (hole)—i.e. one cannot write \(b_\alpha (a_\alpha)\), as this was neglected in equation (17), and the exciton energies are set by localized states in a disorder potential. This energy shift is important as it relates the observed lower polariton blue-shift to the polariton density, and so is important in the interpretation of experiments. This question of comparing the models is therefore not so simple, including long-range Coulomb interaction has been studied.

Topical Review

A relationship between the model of this section, and that of the previous section, may be established in the limit of low densities, by considering a Holstein–Primakoff transformation of the Hamiltonian in equation (17); i.e.

$$S^+ = D^+_u D^-_u - \frac{1}{2}, \quad S^z = D^+_u \sqrt{1 - D^+_u D^-_u}, \quad S^- = (S^z)^T. \quad (22)$$

Then, assuming the occupation of excitons to be small (i.e. \(\langle D^+_u D^-_u \rangle < 1\), one may expand equation (17) to get

$$R = \sum_\alpha g_{\alpha p} D^+_\alpha D^-_\alpha + \sum_\alpha \psi_\alpha^\dagger \psi_p \langle D^+\psi^\dagger \psi_p (1 - \frac{1}{2} D^+_\alpha D^-_\alpha) + \text{H.c.} \rangle. \quad (23)$$

Comparing this to equation (9) shows that a bosonic model derived in this way has certain differences to the standard bosonic model; it obviously neglects the inter-exciton Coulomb term, as this was neglected in equation (17), and the exciton energies are set by localized states in a disorder potential \(\varepsilon_a\), rather than \(k^2/2M\). Less obviously, but more importantly, the saturation interaction term is significantly stronger than that would be suggested by equation (9); in that case, the mean-field energy shift at polariton density \(n\) is of the order of

$$\delta E_{\text{LP}}^{\text{sat}} \sim \Omega_{\text{R}} n a^2, \quad (24)$$

In contrast, the term in equation (23) is of the order of

$$\delta E_{\text{sat}}^{\text{sat}} \sim \Omega_{\text{R}} h n \xi_{\text{L}}^2, \quad (25)$$

where \(\xi_{\text{L}} \sim (MW_\rho)^{-1/2}\) is a characteristic length scale of the disorder potential. This energy shift is important as it relates the observed lower polariton blue-shift to the polariton density, and so is important in the interpretation of experiments. This result is valid at low temperatures; at higher temperatures one can show [85] that \(\xi_{\text{L}}\) should be replaced by \(\xi_{\text{F}} \sim (Mk_B T)^{-1/2}\). The appearance of this temperature-dependent length scale would not arise from a model that included only bosonic lower polaritons. Comparison of the equilibrium transition temperatures of the two models is discussed later, in section 2.4.1.

2.2.4. Comparison of models. As is clear from the above discussion, the Bose–Fermi model in some sense encompasses a bosonic model. However, its derivation led naturally to the inclusion of saturation interaction, but as yet no generalization including long-range Coulomb interaction has been studied. The question of comparing the models is therefore not so much whether one model is right or wrong, but whether interaction effects beyond a quartic boson–boson interaction are important, and so whether a description like that of equation (17) is necessary. At low enough densities and temperatures (i.e. temperatures a small fraction of the Rabi splitting) it is clear such a description is not necessary. However, the definition of ‘low enough’ that is derived from studying when the (equilibrium) phase boundary of equation (17) is reproduced by a bosonic theory suggests that low enough means exciton separation of the order of the Bohr radius; and temperatures of the order of tenths of the Rabi splitting. This density scale is that at
which the degeneracy temperature is of the order of the Rabi splitting: \( \Omega_k \approx k_B T_{\text{Hub}} \approx \rho_{\text{crossover}} / m \). One may rewrite \( \rho_{\text{crossover}} \approx m \Omega_k \) in terms of microscopic length scales by using expressions for the photon mass \( m = (h/c)(2\pi/\lambda_0) \), and relating the Rabi splitting to the oscillator strength defined in equation (21), via \( \Omega_k^2 \approx \sum_n |g_n|^{2}/A \). One thus finds [86] that the crossover scale is set (up to a numerical factor) by \( 1/L^2 \). Of course, the exact density depends on details, and the transition temperature that results from the full dispersion of the lower polariton, rather than just a quadratic dispersion, will also start to become depressed at a similar temperature scale. However, the estimate of \( \rho_{\text{crossover}} \approx 1/L^2 \) describes a density beyond which one can no longer \textit{a priori} ignore such effects.

As an alternative way to resolve the question of which approximate Hamiltonian, equation (17) or equation (9), is most appropriate for a given physical system, one can propose the following clear, but technically challenging approach. From both Hamiltonians, one can construct an approximate ground state, which can then be rewritten in terms of electrons, holes and photons. In both cases, we consider generalizations of the coherent state, which for a simple structureless boson field \( L^1 \) would be written as \( \exp(\lambda L^1)|0\rangle \). This leads to two different trial wavefunctions for the electron–hole–photon system. While this is not a simple exercise—and would in fact require extensive numerical computation—it is a useful gedanken comparison to highlight the distinctions. Let us consider first the trial wavefunction appropriate to the Hamiltonian of equation (14). Taking \( 0 \) as the filled valence band, we have

\[
|\Psi_{\text{Bose}}\rangle = e^{iL^1}|0\rangle;
\]

\[
L^1_0 = \cos(\xi_0)\psi^\dagger_0 + \sin(\xi_0) \sum_q \phi(q)\epsilon^q_{\text{v}}v_{\text{q}} - \text{q}.
\]

At low densities this wavefunction has a simple interpretation; \( \phi(q) \) is the bound exciton wavefunction, and the \( \xi_0 \) controls the exciton and photon fractions of the lower polariton; i.e. the term in brackets is the lower polariton creation operator, and this is a coherent state of lower polaritons. Note however that \( (\epsilon^q_{\text{v}}v_{\text{q}} - \text{q})^2 = 0 \), as \( \epsilon^q_{\text{v}}v_{\text{q}} - \text{q} \) are fermionic operators, thus this wavefunction can be also written as

\[
|\Psi_{\text{Bose}}\rangle = \exp(\lambda \cos(\xi_0)\psi^\dagger_0) \prod_q (1 + \lambda \sin(\xi_0)\phi(q)\epsilon^q_{\text{v}}v_{\text{q}} - \text{q})|0\rangle.
\]

Thus, if \( \phi(q) \) has a step-like form, this can also describe a BCS-like state [104, 105]. More generally, the parameters \( \lambda, \xi_0 \) and the function \( \phi(q) \) can be taken as variational parameters, and used to minimize the energy.

Starting instead from the Hamiltonian of equation (17) one is instead led to write

\[
|\Psi_{\text{TLS}}\rangle = e^{i\psi_0} \prod_a (\cos(\theta_a) + \sin(\theta_a)D^1_\alpha)|0\rangle_{\text{TLS}}
\]

\[
D^1_\alpha = \sum_{k,q} \tilde{\Phi}_\alpha(k)\tilde{\phi}(q)e^\dagger_{m,k/(m,\text{v})}v_{m,k/(m,\text{v})}-\text{q}.
\]

where we have now introduced \( \tilde{\Phi}_\alpha(k) \) as the localized centre-of-mass wavefunction. Note that the operator, \( D^1_\alpha \), describing a localized exciton does not square to zero. It is thus not possible to rewrite the BCS-like product in equation (28) as an exponential; there is a qualitative difference between the states in equations (27) and (28). Although \( (D^1_\alpha)^2 \neq 0 \), the product in equation (28) only allows each operator \( D^1_\alpha \) to occur at most once, so for a given single-particle state labelled by \( \alpha \), only zero or one excitons may occupy it, and thus prevents multiple occupation. By including the disorder-localized centre-of-mass wavefunctions, equation (28) describes single occupation of a set of localized exciton wavefunctions, while in comparison, equation (26) describes only the single, lowest energy, delocalized exciton mode. As above, we may take the parameters \( \lambda, \theta_a \) and the functions \( \tilde{\phi}_\alpha(k) \), \( \tilde{\phi}(q) \) as variational.

Unfortunately, direct evaluation of the expectation of equation (3) with these trial wavefunctions is challenging. At low enough densities, no multiple occupation occurs, so in this limit equations (26) and (28) become comparable: expanding equation (28) for small \( \theta_a \), the terms in the product can be rewritten approximately as

\[
\prod_a \left( 1 + \theta_a D^1_\alpha + \mathcal{O}(\theta_a^2) \right) \simeq \exp \left( \sum_a \theta_a D^1_\alpha \right) + \mathcal{O}(\theta_a^2).
\]

This would be equivalent to equation (26) except that equation (26) macroscopically occupies the \( k = 0 \) exciton state, whereas equation (30) occupies a collection of disorder-localized states. Although not identical, a superposition of many localized states distributed across the sample can (at low enough densities) behave similarly to the translationally invariant \( k = 0 \) state. Thus depending on the relative importance of disorder localization, and on the difference of Coulomb interaction between different single-particle exciton states versus interaction for multiple occupation of the same single-particle state, one may find which of equation (26) or equation (28) has lower energy.

Furthermore, both of the above wavefunctions are mean-field approximations of the ground state, and in both cases, energy could be lowered by constructing the Nozières–Bogoliubov state. To discuss this, let us consider the simpler case of structureless bosons, \( L^1 \). One can then understand this state in two ways, either as a variational ansatz, as in [106]:

\[
|\Lambda\rangle = \exp \left( \lambda L^1 + \sum_k \lambda_k L^1_{k^\dagger k} \right) |0\rangle,
\]

and then find \( \lambda, \lambda_k \) by minimization. Alternatively, the same state can be described if one considers fluctuation corrections to the mean-field theory. As is well known, in the presence of a condensate, the quasi-particles are the Bogoliubov modes [15], i.e.

\[
B^1_k = \cos(\phi_k) L^1_{k^\dagger} + \sin(\phi_k) L^1_{-k},
\]

with \( \phi_k \) the Bogoliubov rotation angle. Thus, given the Bogoliubov spectrum, the lowest energy state is the Bogoliubov vacuum, \( |\Omega_{\text{Bog}}\rangle \), defined such that it is annihilated by all \( B^1_k \), i.e.

\[
[\cos(\phi_k) L^1_k + \sin(\phi_k) L^1_{-k}]|\Omega_{\text{Bog}}\rangle = 0 \forall k,
\]

which is clearly solved by

\[
|\Omega_{\text{Bog}}\rangle = \exp \left( -\sum_k \tanh(\phi_k) L^1_{k^\dagger k} L^1_{-k} \right) |0\rangle.
\]

Two comments are in order about the significance of this state; firstly, the physical reason this state is of lower energy is the
quartic interaction, in particular terms like $L^1_k L^1_k L^0_0 L^0_0 + \text{H.c.}$, which favour states which are not eigenstates of the number of $k = 0$ particles. Secondly, even when projected to an overall number state, one may retain features of this state, by writing a superposition of terms with different division of the number of particles between the condensate mode and other states.

2.3. Theoretical treatments—effects of the environment

Having discussed various models of the polariton system, we now turn to how these models, and the effects of the environment, may be treated. We first briefly outline the thermal equilibrium case, and compare mean-field theories of the two models discussed above. We then discuss some of the various approaches that one may use to describe the effects of the environment, focusing mainly on non-thermal steady states. Finally, we try to separate and clarify the concepts of the environment, focusing mainly on non-thermal steady states.

2.3.1. Thermal equilibrium.

The simplest approximation for the environment is to consider the system in thermal and chemical equilibrium with a bath. While it is clear that the current experiments involve substantial pumping and decay, which will be discussed next, there are compelling reasons to deal with the equilibrium case. Firstly, the properties of a given model in the equilibrium case are instructive when considering the range of behaviour it can show; while the equilibrium properties of weakly interacting dilute Bose gas are well studied [15, 107, 108], the properties of models like equation (17), with distributions of oscillator strengths and energies [84, 85] are less known. Even within the weakly interacting Bose gas picture, interesting features can arise from considering non-quadratic dispersion [109–111], or the effects of anisotropic spin interactions [47, 112]. The second reason is that with improvements in the quality of mirrors, and refinement to the design of microcavities and the conditions of pumping, experiments have been able to increase the thermalization rate to be comparable to or faster than polariton decay rates [44, 46], and so for these, or future, experiments, the correct description may become increasingly close to equilibrium.

The treatment of both equations (9) and (17) in equilibrium can be put in a similar form by considering their saddle point, or minimum action equations. Formally, these can be derived by writing the imaginary time path integral for the partition function [113], and then considering the configurations that minimize the imaginary time action. Thus, for the bosonic case within the effective lower polariton model, equation (14), the saddle-point solutions satisfy a Gross–Pitaevskii equation:

$$[ -i\partial + E_{k=0}^p + \frac{\nabla^2}{2} + O(\nabla^4) + V_{0,0,0}^\text{eff}|L_0|^2 ] L_0 \simeq 0. $$

(35)

Here, as the dispersion $E_{k=0}^p$ is not quadratic, we have expanded it to quadratic order to find the coefficient of $\nabla^2$. Note that by considering solutions of the form $L_0(t) = L_0 e^{-i\omega t}$, one can introduce the chemical potential, and thus recover the expected static Gross–Pitaevskii equation.

For the fermionic model, more care is required; since there has been no projection onto lower polaritons, the saddle point condition leads to coupled equation for the photon field and two-level systems. Using the spin notation of equation (17) one has

$$ [-i\partial + \omega_0 - \frac{\nabla^2}{2m}] \psi_0 = \sum_u \frac{g_{u,0}}{\sqrt{A}} \delta(r - r_u), $$

(36)

and

$$ \partial_t S_{u} = - B_u \times S_u, \quad B_u = \left( \frac{g_{u,0} \langle \psi_0 + \psi_0^\dagger \rangle}{\epsilon_{r_u}} \right). $$

(37)

Here $r_u$ is the localization site of the two-level system $S_u$. In the case where the only variation is $\psi_0(t) = \psi_0 e^{-i\omega t}$ and the polarization has the same time variation, one can eliminate the time variation by a gauge transformation.

The sum over exciton energy levels can also be simplified if one makes two assumptions: firstly that the excitations are occupied according to a thermal distribution, and secondly that we can average over many realizations of excitonic disorder. This second assumption, that $\psi_0(r)$ varies slowly compared to the distance between excitons, or equivalently that the photon couples to many localized exciton modes, allows one to replace the sum over exciton energy levels with a sum over the statistical distribution of energies and the excitonic coupling strengths. This then yields

$$ [ \omega_0 - \mu - \frac{\nabla^2}{2m} + O(\nabla^4) - \sum_u \frac{g_{u,0}^2}{A} \tanh(\beta E_0) ] \psi_0 \simeq 0, $$

(38)

where the energy $E_u^0 = (\epsilon_{r_u} - \mu)^2 + g_{u,0}^2 |\psi_0(r)|^2$ depends on the local value of the slowly varying $\psi_0(r)$. Note that in this way the exciton disorder does not lead to spatial inhomogeneity of the polariton condensate, and so in the absence of photonic disorder one would expect polariton condensation in the $k = 0$ mode. This has a clear similarity to equation (35), but in this case, the nonlinear interaction term is more complicated than it was in the bosonic case, $V_{0,0,0}^\text{eff}|L_0|^2$, and the polariton–polariton interaction is due to the nonlinearity of the susceptibility arising from the saturable nature of the excitons. For a uniform and static condensed solution $(\nabla \psi_0 = 0 = \partial_t \psi_0)$, the Gross–Pitaevskii equation (38) is also analogous to the gap equation (self-consistency condition) of the BCS theory [113].

Despite their similarity, there is an important distinction between equations (35) and (38): equation (35) is temperature independent, while the nonlinear susceptibility in equation (38) decreases at high temperature, and is eventually incapable of supporting condensation. Thus one can crudely say that equation (35) can support mean-field condensation at any temperature, and fluctuations [107] must be considered to find a transition temperature. For the bosonic model, going beyond mean-field theory, one can also produce a temperature dependent equation by using a Hartree–Fock wavefunction, and thus including the effect of interactions between the condensate and non-condensed particles (see e.g. [15] for further details). In distinction equation (38) contains a finite mean-field transition temperature, and so fluctuations are only important when they significantly decrease this transition.
temperature [86, 102]. Thus, including fluctuations one finds a crossover from a fluctuation dominated phase boundary at low densities, to a phase boundary that is well described by mean-field theory in the high density limit, where long-range interactions dominate. The temperature dependence that appears in equation (38) can also be understood by noting that it was necessary to integrate out the excitonic degrees of freedom in equation (36) to produce an effective action for a single photon field. Thus, the saddle-point density for the Bose–Fermi model contains both the condensate, and a contribution from incoherent excitons.

By considering fluctuation corrections to the saddle-point density, one may in three dimensions, and at low densities, recover the non-interacting transition temperature of a weakly interacting Bose gas. That is, considering the transition from the normal side, a mean-field condensate density appears when \( \mu \rightarrow 0 \), and so the critical temperature is given by \( \rho_{\text{total}} = \rho_{\text{vac}}(T_c, \mu = 0) \). However, for a two-dimensional system, when the transition is of the Berezinskii–Kosterlitz–Thouless (BKT) class [114–116], it is necessary to consider fluctuations in the presence of a quasi-condensate, as the BKT transition, where free vortices proliferate, occurs below the mean-field transition temperature.

In order to calculate the total density in the presence of a condensate, it is important to note that the fluctuation corrections can deplete the condensate population, as well as increase the population of other modes [86]. The condensate density that comes from a mean-field calculation (i.e. from the uniform static solutions to equation (35)) is \( \rho = |L_0|^2 = \mu/V^{\text{eff}} \). One compact way of finding how fluctuations deplete the condensate density is by using the Hugenholtz–Pines relation as discussed in e.g. [107]. Let us briefly summarize here how this argument shows that the condensate density is smaller than the mean-field estimate. To discuss this one must introduce the self-energy of the condensate \( \Sigma \). If we define the matrix of Green’s functions,

\[
G(\omega, k) = \int dt e^{-i\omega t} \left( \begin{array}{cc} L_k^\dagger(t) & L_k(t) \\ L_k(t) & L_0 \\ \end{array} \right),
\]

and introduce \( G_0(\omega, k) \) as the free Green’s function (i.e. in the absence of interactions), then the matrix of self-energies is defined by \( \Sigma(\omega, k) = G_0^{-1}(\omega, k) - G^{-1}(\omega, k) \). The Hugenholtz–Pines relation is the condition required of this self-energy in order that there might be a gapless mode, as one expects for a Bose-condensed system. The condition can be written as \( \Sigma_{11} - \Sigma_{12} = \mu \). By writing the self-energies in terms of the densities of condensate and normal state particles, one may use this identity to write the condensate density in terms of \( \mu \) and the normal state density. At leading order in interaction strength, it can be shown [107] that the self-energy is given by

\[
\Sigma_{11} = 2V^{\text{eff}}(\rho_0 + \rho_1), \quad \Sigma_{12} = V^{\text{eff}}(\rho_0 - \rho_1).
\]

In this expression \( \rho_0 \) is the condensate density, \( \rho_1 \) is the density of particles in other states and \( \rho_1 \) is the anomalous density, \( \sum_k (L_k^\dagger L_k^\dagger) \). One thus finds: \( \rho_0 = (\mu/V^{\text{eff}}) - (2\rho_1 + \rho_1) \), i.e. fluctuations reduce the condensate density below its mean-field value.

To then extract the BKT temperature, one needs to find the condition for free vortices to proliferate [116–118].

As described in those works, this requires one to know the fugacity of a vortex, and the effective vortex–vortex interaction strength, both of which depend on the superfluid stiffness \( \rho_s \), which may be found from the difference between transverse and longitudinal current response functions (see equation (43)). The result is a transition which occurs at \( k_B T_{\text{BKT}} = (2/\pi)(\rho_s/m) \). In the case of bosons with quadratic dispersion, and in the limit of weak quartic interaction strength \( V^{\text{eff}}, i.e. m V^{\text{eff}} \ll 1 \), one may extract an asymptotic relation between the superfluid density and the total density, giving \( k_B T_{\text{BKT}} = (2/\pi) \left( \ln(B/m V^{\text{eff}}) \right)(\rho/m) \), where quantum Monte Carlo calculations [119] give \( B = 380 \pm 3 \). The phase boundary calculated according to the fermionic model, i.e. using equation (38), and the boundary for the BKT transition in a bosonic model following [109–111], but with the effective inclusion of disorder, are shown in figure 7. These boundaries are discussed further in section 2.4.1.

2.3.2. Pumping, decay and non-equilibrium treatments. A more realistic discussion of the experimental environment must consider that polaritons may escape, and so continuous pumping is required to produce a steady state. In addition, if one is to describe pulsed experiments, or the transient behaviour after the pump is switched on, a dynamical approach is required to describe the time dependence of population [112, 120–122]. Considering for the moment steady-state situations—i.e. c.w. (continuous wave) pumping—one may highlight two important features of the difference between the pumped, decaying system and thermal equilibrium. The first is that the distribution function; i.e. the population of each
energy level, may be far from thermal, and set instead by the balance of pumping, decay and thermalization rates [35, 59, 122–126]. The second class of effects is that incoherent pumping and decay introduce dephasing, and can change the excitation spectrum of the system, the additional inclusion of these effects are discussed in [126, 127] (see also the discussion in section 3.2.1). There are a wide variety of approaches that may be applied to study one or either of these features; in the following we discuss briefly how some of these various approaches are related, and what limitations they may have. For a more general discussion see e.g. [38, 128, 129].

In order to describe the properties of the pumped, decaying system, one requires a method to calculate various correlation functions. Given an expression for single particle correlation functions, one may then find many properties of interest, for example the occupation of modes, the luminescence and absorption spectra, and the first-order coherence properties. The most general information about one particle correlations can be written in terms of the two correlation functions \( G^<(t, r) = \langle \psi(t, r) \psi(0, 0) \rangle \), \( G^>(t, r) = \langle \psi^\dagger(t, r) \psi^\dagger(0, 0) \rangle \), which with \( \psi \) describing the photon field correspond directly to luminescence and absorption probabilities. These encode information both about the form of the spectrum and thus the density of states, and also about the population of those states. For example, the density of states is given by \( \text{Im}(\theta \langle \psi^\dagger \psi \rangle) = \frac{1}{\pi} \text{Im}(\theta \langle \psi^\dagger \psi \rangle) \), where the retarded Green’s function can be written as: \( G^R(t, r) = \theta(t) [G^+(t, r) - G^-(t, r)] \). In equilibrium, these two Green’s functions can be related in terms of the thermal distribution function, but out of equilibrium no such simplification is possible.

Let us now discuss different methods to calculate these Green’s functions. The first method is to find and solve the operator equations of motion for \( \psi, \psi^\dagger \), and thus to evaluate the correlation functions directly. In order to describe pumping and decay, one considers coupling the system to baths, which either pump particles and energy into the system, or provide modes into which particles may decay. These baths are assumed to be large, so their properties (e.g. distribution functions) are fixed, and not affected by the system. Since the bath and system are coupled, the equations of motion for the system operators will also include bath operators. If one considers the initial state of the bath to be drawn from some fixed (e.g. thermal) distribution, then the expectation of bath operators will be random quantities, with statistical properties set by the bath’s distribution [38, 120]; thus such coupling to baths introduces noise, giving quantum Langevin equations [38]. The second method is to write equations of motion for Green’s functions, which will now involve correlation functions of bath operators, which again can be found if one assumes a fixed distribution for the baths. These coupled equations are the content of the Keldysh formalism [130], in which it turns out to be simpler to write equations for various linear combinations of the Green’s functions, allowing one to combine the Green’s functions in a compact matrix notation (see e.g. [128]). The equations for the Green’s functions can also be derived in a path integral formulation [129]. The path integral formalism allows one to make a close connection with the methods discussed in section 2.3.1, and can describe changes to both the occupation and the spectrum induced by pumping and decay [126, 127]. As such, it allows one to introduce a complex self-consistency condition (i.e. a complex equivalent of the Gross–Pitaevskii equation); this interpolates between the laser, in which self-consistency requires balancing of pumping and decay, and the equilibrium condensate, where self-consistency instead relates real self-energy shifts. This point will be discussed in more detail in section 3.2.1.

The above two approaches allow one to find self-consistently the population and the spectrum in the presence of pumping and decay. In certain cases—for example in the normal state, with weak pumping and decay—the changes to the spectrum may be small, and one is interested primarily in changes to the population. By assuming a known form for the retarded Green’s function (i.e. for the spectrum of the system) it is possible to extract equations for the population from the Green’s function formalism (see [128, chapter 9] for details). These will lead to the Boltzmann equation [131], which may also be derived phenomenologically, by considering various rates of transfer between different energies. Thus, by neglecting the ‘kinetic’ effects, that change the spectrum, but retaining ‘dynamic’ effects, that change the populations, one may investigate how pumping and decay change the occupation of the spectrum [35, 59, 121–125]. Even without pumping and decay, the spectrum changes due to interactions in the condensed state [15]; and further calculations retaining ‘kinetic’ effects suggest the pumping and decay further modify the spectrum (see [126, 127] and also [132] in a different context). Because changes to the spectrum modify the density of states, they can be expected to in turn affect the population dynamics; it is therefore not clear how valid it is to consider population dynamics in the condensed system without self-consistent treatment of the changes to the spectrum.

A different approach to treating the coupling to baths is to consider the density matrix, which allows calculation of any single-time expectation of operators, as well as certain multi-time correlations, discussed further below. Calculating the full \( N \)-body density matrix allows calculation of correlation functions of arbitrary numbers of particles, rather than just the single particle correlations discussed above. Density matrix methods, and Green’s functions methods can be related as the single-particle density matrix (i.e. tracing over coordinates of all but one particle) is equivalent to the equal time part of the one particle Green’s function.

To numerically calculate evolution of the density matrix, due both to the system Hamiltonian, and to the effect of coupling to baths, one can choose an appropriate basis and write the density matrix in terms of a distribution function over this basis, i.e. \( \tilde{\rho} = \sum |a\rangle \langle b| P(a, b) \). The time evolution of the density matrix then corresponds to the time evolution of this distribution function. Under certain conditions [38, 133], it is possible to interpret the evolution of this distribution as describing evolution of a quasi-probability distribution. In such a case, the equation of motion for \( P(a, b) \) is a Fokker–Planck equation, and can be rewritten in terms of Langevin equations for classical variables. If one chooses to resolve the density matrix onto a basis of coherent states, a variety of ways of doing this exist, among which we mention two important choices: the positive P distribution, and the Wigner distribution. (See [38] for other possible choices, and further details.) The positive P distribution formally has the desired properties, and gives the desired Langevin...
where the eigenvalue, \( \lim_{N \to \infty} \), to which this eigenvalue corresponds will be a state localized to a crossover to a state with a macroscopic eigenvalue (i.e. of the Wigner representation \([133, 134]\) — which allows one to find appropriate Langevin equations, and is numerically stable. The Wigner distribution allows one to find the expectation of symmetrized products of operators; at equal times it is trivial to extend this to find general products of operators, by making use of equal time commutation relations. It is also possible to find multi-time correlation functions from such an approach, however since the unequal time commutation relations are not \( a \, p r i o r i \) known, one cannot generally find expectations of other orders; in this sense the truncated Wigner approach does not allow \( \hat{G}^c + \hat{G}^\tau \), but only the symmetrized combination \( \hat{G}^c \), and hence cannot separate the density of states from its occupation.

2.3.3. Lasing, condensation superfluidity. Condensation, coherence and superfluidity are often connected, however when dealing with two-dimensional systems of composite particles, where finite-size and non-equilibrium effects may be important, it is important to separate and clarify these concepts. The discussion below compares condensation, coherence and superfluidity in equilibrium infinite systems to the case including finite-size and non-equilibrium effects. For simplicity we discuss these ideas in terms of a general bosonic field \( \dot{\psi} \), rather than any specific field appearing in the microcavity polariton system. The first concept is macroscopic occupation of single-particle wave-function; rigorously this can be defined as the existence of a macroscopic eigenvalue of the reduced one-particle density matrix \([135]\):

\[
\rho_{1}(\mathbf{r}, \mathbf{r}') = \langle \dot{\psi}^\dagger(\mathbf{r}) \dot{\psi}(\mathbf{r}') \rangle = \sum_\mathbf{r} n_i \dot{\psi}_i^\dagger(\mathbf{r}) \dot{\psi}_i(\mathbf{r}'),
\]

where \( n_i \) is the occupation of the single-particle mode \( \dot{\psi}_i \). A macroscopic eigenvalue exists if \( \lim_{N \to \infty} n_0 / N \neq 0 \) where \( N = \sum_i n_i \) is the total number of particles. In an infinite system, if the macroscopically occupied state is an extended state, then there is \( o f f - d i a g o n a l \) long range order — i.e. if in the position representation \( \lim_{N \to \infty} \dot{\psi}_0(\mathbf{r}) \neq 0 \), then there remain extensive terms far from the diagonal \([136]\). Thus, in such a case, the correlation function \( \lim_{N \to \infty} \langle \dot{\psi}(\mathbf{r}) \dot{\psi}(0) \rangle = n_0 \dot{\psi}_0^\dagger(\mathbf{r}) \dot{\psi}_0(0) \neq 0 \). In a non-interacting two-dimensional trapped (and thus finite) gas of bosons there can be a sharp crossover\(^4\) to a state with a macroscopic eigenvalue (i.e. of the order of the number of particles) of the one-particle density matrix \([137, 138]\). However, the single-particle state \( \dot{\psi}_0(\mathbf{r}) \) to which this eigenvalue corresponds will be a state localized in the trap, and so despite the existence of a macroscopic eigenvalue, \( \lim_{N \to \infty} \langle \dot{\psi}(\mathbf{r}) \dot{\psi}(0) \rangle = 0 \).

The visibility of interference fringes is directly related to the first-order coherence function \( g_1(t = 0; \mathbf{r}) \), where

\[
g_1(t; \mathbf{r}) = \frac{\langle \dot{\psi}(\mathbf{r}, t) \dot{\psi}(0, 0) \rangle}{\sqrt{\langle \dot{\psi}^\dagger(\mathbf{0}, 0) \dot{\psi}(0, 0) \rangle \langle \dot{\psi}^\dagger(\mathbf{r}, t) \dot{\psi}(\mathbf{r}, t) \rangle}};
\]

coherence can be defined by the properties of this function. As just discussed, if one defines coherence by the limit of \( g_1(0, \mathbf{r} \to \infty) \), then in a trapped system, this function vanishes. However, coherence will exist across the size of the trap. In such inhomogeneous and complicated cases, a binary classification of coherent/incoherent is less useful than a description of how the coherence varies as a function of separation in time and space. Both the size of this variation, and its functional form will depend on the interplay of finite size (and form of trapping potential), temperature, interactions and decay rates \([126]\).

Superfluidity can meanwhile be defined separately as the difference of longitudinal and transverse response functions at vanishing wave-vector \([15, 107, 139]\):

\[
\chi_{ij}(\omega = 0, \mathbf{q}) = 2 \int_0^\beta d \tau \langle J_i(\mathbf{q}, \tau) J_j(-\mathbf{q}, 0) \rangle = \chi_T(q) \left( \delta_{ij} - \frac{q_i q_j}{q^2} \right) + \chi_L(q) \frac{q_i q_j}{q^2};
\]

and \( \rho_s \propto \lim_{q \to 0} \left[ \chi_L(q) - \chi_T(q) \right] \). Since superfluidity results from a change of the response functions, it occurs only in an interacting system; without interaction bosons do not become superfluid. In a two-dimensional infinite interacting system, below the BKT transition \([114–116]\), coherence decays as a power law rather than an exponential — low energy phase fluctuations prevent true long range order \([140]\) — and superfluidity exists, but no macroscopic occupation of a single mode.

When one considers a more realistic system, which is both interacting, but also of finite extent, one cannot ignore \( a \, p r i o r i \) the physics of the BKT transition, nor can one ignore \( a \, p r i o r i \) the physics of the trap. At low enough temperatures it is clear there will be macroscopic occupation of a single mode, and full coherence across the trap. How this state is approached as temperature is reduced, or as density is increased differs depending on whether interactions or finite size effects are dominant. If described as a non-interacting gas, the coherence at all distances increases uniformly as a single mode is increasingly occupied \([137, 138]\). In the BKT scenario, power-law correlations develop on intermediate scales (between some short range thermal length and the trap size); then as temperature decreases, the thermal length increases and the power with which correlations decay decreases, again restoring full coherence as \( T \to 0 \) \([141, 142]\), as shown in figure 8.

Adding non-equilibrium effects, the nature of decay of correlations in an infinite system is significantly altered \([126, 127]\). The long wavelength phase modes, responsible for decay of correlations become diffusive. That is the poles of the Green’s functions, which in the equilibrium case have the form \( \omega \simeq \pm \epsilon k \), take instead the form \( \omega \simeq \epsilon i \pm \sqrt{\epsilon} k^2 - \epsilon^2 \) in the pumped and decaying case. This is illustrated in figure 9. Combining the effects of phase diffusion, and discrete level spacing \([126]\) the properties of coherence are further modified, and one approaches the laser limit: temporal coherence of laser emission comes from one, or at most a few modes (resonant with the laser cavity), and so phase diffusion of a single mode \([16]\) leads always to exponential decay of temporal correlations \([144–146]\). As well as this distinction of forms of temporal coherence, it is worth mentioning here

\(^4\) In the limit of vanishing trap curvature and infinite number of particles the crossover becomes a phase transition, but only if trap curvature vanishes as the correct power of the number of particles \([137, 138]\).

\(^5\) Above the BKT transition, in addition unbound vortices are present, and these lead to exponential decay of correlations.
a few other important distinctions between lasing and the generalized concept of condensation discussed here, as there are evidently also similarities [147]. Most obviously, polariton condensation is seen at the polariton resonance, which is significantly (of the order of the Rabi splitting $\Omega_R$) below the lowest cavity photon mode; as such, nonlinear emission coexisting with strong coupling is a signal that one should consider polaritons, and not just photon lasing. A more fundamental difference is that lasing requires inversion, while condensation does not; this is a consequence of the standard laser systems possessing little coherence in the gain medium, while excitons, being part of a coherent polaritons, are coherent [148–152]. This distinction can also be seen by comparing the critical lasing condition to the Gross–Pitaevskii equations of equations (35) and (38). In the presence of pumping and decay, the susceptibility (describing the nonlinear response of excitons) becomes complex [126, 127]; the real part of the susceptibility gives the nonlinearity in the Gross–Pitaevskii equation. In contrast, the imaginary part of the susceptibility describes absorption or gain, and leads to the lasing condition, that round trip gain and loss balance. A treatment of a model system with pumping and decay elegantly shows how these conditions can be combined, giving an expression in terms of the total susceptibility [126, 127].

Starting from the strong-coupling regime, when crossing over to the weak-coupling regime (see, e.g., [28, 153, 154]), the polariton splitting collapses, and so the lasing mode no longer has any excitonic character, and becomes the standard photon laser. A major advantage of an exciton–polariton laser over standard lasers is that it can operate without the inversion of the electronic population [155], and therefore it has a much smaller threshold pump power. It is interesting to note that wide band gap semiconductors, such as GaN and ZnO, would be particularly suitable as in these cases excitons are stable at higher temperatures and densities, and therefore they could operate at room temperature. Electronic population inversion is not necessary because in the exciton–polariton laser, both photon field and excitons are coherent. In addition, the involvement of the excitonic field leads to strong nonlinear effects compared to conventional lasers, due to exciton–exciton interactions.

2.4. Phenomena

The previous two sections discussed the models, and the treatments of the environment, that have been used to theoretically model polariton condensates. This section in contrast will review a few of the phenomena that have been predicted as possible signatures and properties of a polariton condensate.

2.4.1. $T_c$ and the phase boundary. Within a given model, and effective description of the environment, it is natural to first ask under what conditions a condensate can exist. Within an equilibrium model of the lower polariton branch as weakly interacting bosons, phase diagrams for the physical parameters of various possible materials are shown in [109, 110] (however, see [111] for a discussion of the effects of non-quadratic dispersion on the BKT transition temperature). By considering a simplified version of the Bose–Fermi model, equation (18), where the energies $E_\alpha$ are described by a Gaussian distribution,
while all excitons display a fixed coupling to light, the mean-field phase boundary was first calculated in [95, 96]. The effect of fluctuations, restoring the bosonic limit at low densities was instead considered in [86, 102]. Since the content of the boson–fermion model at small densities and temperatures is equivalent to a bosonic model, and the low momentum part of the polariton dispersion is controlled by the photon mass, it is not surprising that it is possible to recover the standard BKT transition temperature of a weakly interacting Bose gas from the boson–fermion model in the low density limit. A calculation of the mean-field boundary which instead takes into account a realistic description of the quantum well disorder and the full distribution of oscillator strengths (see figure 6) has been performed in [84, 85] (see figure 7).

Owing to finite size effects the experimental systems do not have a sharp phase transition marking the onset of a broken symmetry. All the observed transitions are rounded, and in order to extract a phase boundary from experiment, some criterion has to be chosen. One commonly used criterion is the nonlinear threshold; i.e. the point at which the relation between emission at \( k = 0 \) and input pump power becomes nonlinear. Such a criterion is somewhat problematic. A second-order phase transition can be expected to be accompanied by a region with large susceptibilities, and thus such nonlinearity extends over a significant range of parameters, and so identification of a strict phase boundary from it is hard. Only in a mean-field theory does the onset of nonlinearity occur at the transition. However, because of the long-range nature of interactions, mean-field theory can be an adequate description for lasers, and for polariton condensates except at very low densities (as can be seen from figure 7). The following sections discuss other phenomena that may demonstrate or describe condensation and coherence in microcavity polariton systems, and may thus provide alternative, or corroborating experimental criteria to find the phase boundary.

### 2.4.2. Energy-resolved luminescence, resonant Rayleigh scattering.

In both the weakly interacting boson model, and the Bose–Fermi model, condensation leads to changes in the spectrum of polariton modes—most significantly the appearance of the phase modes [15, 107]. In addition, for a model starting from localized exciton states, with a distribution of energies and oscillator strengths, there is weak emission from sub-radiant exciton states between the upper and lower polaritons [156]. This emission is also modified by condensation [84]: in the presence of a condensate these sub-radiant exciton states have energies \( E_\omega \) as defined following equation (38), and so the density of states is changed by the coupling to the coherent field. This change to the emission is discussed further in detail [85]. In practice, it is however hard to observe the incoherent luminescence of thermally excited modes in the presence of a strong signal from the coherent condensate. One suggestion to overcome this problem is to probe these excited modes via resonant Rayleigh scattering; by using a phase sensitive measurement one may be able to identify a small coherent scattering signal even in the presence of emission from the condensate [84, 85]. Figure 10 shows the Rayleigh scattering signal expected both above and below the phase transition; note in the condensed case, one sees linear modes both above and below the chemical potential; this is as

![Figure 10. Contourplot of the disorder averaged RRS intensity (\( I_{|p|}(\omega) \)) for \(|p| = |q|\) as a function of the dimensionless momentum \(|p|\rho \), and rescaled energy \( 2(\omega - \mu)/|\Omega_0| \), for zero detuning, Rabi splitting \( \Omega_0 = 26 \text{ meV} \), temperature \( k_B T = 20 \text{ K} \), and a disorder strength characterized by an inverse scattering time \( 1/\tau = 1.16 \text{ meV} \). (a) Non-condensed (density \( \rho \approx 0 \)); (b) condensed regime (\( \rho \approx 3.6 \times 10^5 \text{ cm}^{-2} \)). Green lines mark chemical potential: if non-condensed RRS emission is present only above the chemical potential, but when condensed, it is present both above and below. (Adapted with permission from data presented in [84]. Copyright 2006 by the American Physical Society.)](image)

one expects from the Bogoliubov spectrum, where the normal modes are superpositions of particle creation and annihilation.

#### 2.4.3. Momentum distribution of radiation.

By integrating the luminescence at a given wave-vector, one may consider the momentum distribution of polaritons—this has also been discussed in the context of exciton condensation [142, 157]. In a weakly interacting, two-dimensional, infinite system, the density of states would be constant, and this would just show the Bose–Einstein distribution. In experiments, the distribution at small momentum is close to, but can deviate from the Bose–Einstein distribution [39, 44]. However, such deviations are expected; both due to interactions, which modify the spectrum and so modify the density of states [86, 102], and also due to finite size, which cuts off components at small wave-vectors. A naive calculation shows how the change of spectrum causes a change of density of states: for a weakly interacting Bose gas with dispersion \( \epsilon_k \), and mean-field equation \( \mu = g \rho_0 \), we have the Bogoliubov spectrum \( E_k = \pm \sqrt{\epsilon_k^2 + 2g \rho_0} \), and simple algebra gives the density of states:

\[
\nu_\omega(\omega, \mathbf{k}) = \text{Im} [\mathcal{G}^R(\omega + i0^+ , \mathbf{k})] = \frac{\text{Im} \left[ \frac{\omega + \epsilon_k + g \rho_0}{\omega^2 + E_k^2} \right]}{2E_k} \delta(\omega - E_k) + \frac{E_k - \epsilon_k - g \rho_0}{2E_k} \delta(\omega + E_k). \tag{44}
\]

This suggests that as \( k \to 0 \), the density of states goes like \( g \rho_0 / E_k \propto 1/k \), however this neglects the fact that the low
energy modes are phase modes, and phase fluctuations may grow without bounds, as only their gradient costs energy; a full calculation, e.g. [86, 151], shows that this term becomes $1/k^2 - 9$ with $\eta \simeq 2T/T_{BKT}$. In order to include the effects of finite size, one approach is to start with the zero-temperature Thomas–Fermi spatial profile, for which coherence across the whole cloud leads to sharp angular peaks, and then consider how phase fluctuations destroy the long range coherence, and thus soften the peaks [142]. There have also been calculations of this momentum distribution for non-equilibrium situations, where details of the dynamics of polariton relaxation lead to a maximum of the distribution at non-zero $k$, e.g. [35, 125, 121, 122].

2.4.4. Linewidth, first- and second-order coherence, polarization. The line-shape of emission is controlled by the Fourier transform of the first-order coherence function as a function of time, i.e. $g_1(t; r = 0)$, using $g_1(t; r)$ defined in equation (42). A perfectly coherent single mode source would have $g_1(t; 0)$ constant, and thus infinitely sharp lines. Although at any finite temperature, population of slow phase modes will lead to some decay of $g_1(t; 0)$, the transition to a condensed phase will lead to a slower decay of $g_1(t; 0)$, as has been seen [55]. Similarly, as discussed in section 2.3.3, one can also consider the spatial decay of coherence [44]; which has also been much studied in cold atomic gases (see, theoretical discussion in [141, 158, 159] and experiments in [160–162]).

The calculation of coherence has already been discussed briefly in section 2.3.3. As was stressed there, a distinction exists between coherence of a few mode laser [144], applied to the polariton problem in [145, 146]; and coherence in a continuum of modes [120, 126, 127]. In both cases increasing temporal coherence in the system should lead to a narrowing of linewidth as the phase transition is approached, however the behaviour far above the transition may differ. This is clear from the fact that for a single mode with a quartic interaction, the equilibrium state is a number state, and so starting from a coherent state, one has phase diffusion due to self-phase modulation even in the absence of any pumping or decay [146]. For a many-mode system, the ground state is neither a coherent nor a number state, and is better described by the Bogoliubov–Nozières state [106], thus it is not clear that the same effect—i.e. self-phase modulation causing larger broadening—should persist.

A related measurement is the second-order coherence function (see e.g. [38]):

$$g_2(t) = \frac{\langle \psi^\dagger(r, 0)\psi^\dagger(r, t)\psi(r, t)\psi(r, 0) \rangle}{\langle \psi^\dagger(r, 0)\psi^\dagger(r, 0) \rangle \langle \psi(r, t)\psi(r, t) \rangle}. \tag{45}$$

As mentioned above, for a thermal state, $g_2(t = 0) = 2$, while for a coherent state $g_2(t = 0) = 1$. Experimental measurement of $g_2(t = 0)$ is restricted by finite detector integration times, and since $g_2(t) = 1$ at times when $g_1(t; 0) \to 0$, it is hard to distinguish the value of $g_2(t = 0)$ [36, 55]. The dynamical behaviour of $g_2(t = 0)$ as a function of time following switching on the pump laser [163–165], or from an equilibrium picture with separate coherent and incoherent contributions [47] has been considered. Within the Bose–Fermi model, the second-order coherence has been studied for a finite number of excitons coupled to a single coherent field, where the finite number of states replaces the phase transition with a smooth crossover [166].

Considering the differences of interaction strength between polaritons with parallel and anti-parallel spin polarizations, Laussy et al [47] have shown that condensation should be associated with spontaneous development of a linear polarization. That a condensate should choose some definite polarization state is a much more general result, i.e. that repulsive interactions prevent fragmentation of a condensate, even when there are two single-particle states with identical energies [105]. Considering the specific form of the interaction, Laussy et al showed that this specific polarization should be a linear state (as opposed to circular or elliptical). The effects of this polarization on the spectrum of excitations, and its propagation in real space has also been considered [167]. Note, however, that the pinning of the polarization to one of the crystallographic axes observed in [44, 48–51] cannot be an effect of spontaneous symmetry breaking, but it is instead likely due to some optical anisotropy of the microcavities. In atomic gases, investigation of ‘spinor condensates’ is hard, as it requires the use of all optical trapping, as a magnetic field would Zeeman split the atomic spin states; further the populations of spin species are effectively fixed at the start of the experiment, so rather than polarization, phase separation into spin domains has been observed there [168].

3. Resonant pumping

Since its first realization in 2000 [169], the possibility of reaching the stimulated scattering regime for polaritons by resonant pumping has attracted considerable interest [34, 170–181], and has initiated the search for polariton lasers and a new generation of ultrafast optical amplifiers and switches. In resonant excitation experiments, polaritons are optically pumped at an energy and momentum which allows coherent polariton–polariton scattering directly into the ground state. In pulsed, ultrafast pump-probe experiments, a second pump is used to initiate the stimulated scattering process to the ground state, while for c.w. (continuous wave) excitation the stimulated regime can be self-induced above some pump threshold. As in non-resonant excitation experiments, the direct mapping between in-plane momentum and angle of bulk photons is crucial. It is this direct mapping that allows one to directly pump at a given momentum, and thus to perform the experiments described here.

The macroscopic occupation of the ground state in these experiments differs in some ways from condensation and spontaneous coherence arising from incoherent pumping. In experiments without a probe pulse, there is a free phase between the pump and signal modes, however as discussed in section 3.2.1, the existence of a free phase alone is not sufficient to ensure superfluidity in such driven systems. Further, since the coherence of the signal beam is directly coupled to the pump beam, higher order correlations, and the linewidth of the signal may be inherited from the pump. Experiments with
an additional probe beam at the signal frequency differ yet further; in such experiments the phase of the signal is fixed by the probe, and so in this case all excitations ought to be gapped.

It is interesting to note that the idea of parametric scattering discussed here for polaritons has been recently applied in a different field; that of dilute gases of atoms confined in an optical lattice [182, 183]. We will discuss this example in more detail later in section 4.1.

3.1. Summary of experiments

The main idea of resonant pumping experiments [34, 169–181] is that of the coherent scattering of two polaritons from the resonantly pumped mode (pump) into the ground state (signal) and a high energy state (idler) (see figure 11(a)). Energy and momentum conservation in this scattering requires one to have \( \{k_p, k_p\} \leftrightarrow \{0, 2k_p\} \), where

\[
E^{LP}_{k=0} + E^{LP}_{3k_p} = 2E^{LP}_{k_p},
\]

which uniquely selects the momentum (i.e. angle) of pump, signal and idler. If one instead relaxes the condition \( k = 0 \) for the signal mode, then for a fixed pump angle \( k_p \), it can be shown [179, 184] that those final states which satisfy energy and momentum conservation describe a figure-of-eight in momentum space (see figure 11(b)). The non-parabolic dispersion of polaritons is crucial in order to obtain such resonant scattering processes, and so the analogous process for excitons is forbidden. Resonant experiments can be divided into two types: in the first type the scattering is stimulated by a weak probe field (parametric amplification), while in the second type there is no probe and the seed to initiate stimulation is provided by the pump itself, if above a certain power threshold (parametric oscillation).

In their pioneering work [169], Savvidis and collaborators realized the stimulated scattering regime for the first time in an InGaAs/GaAs/AlGaAs microcavity with a Rabi splitting of \( \Omega_R = 7 \) meV. By pumping polaritons at the ‘magic’ angle \( \theta \approx 17^\circ \) (for zero exciton–photon detuning) close to the inflection point of the LP dispersion curve, and using a second pump at zero angle (probe) to initiate the process, a substantial gain of up to 70 was observed. The large observed signal required both the ‘magic’ angle pumping, and the probe at zero angle, and was absent either if no probe was applied [170] (see figure 12), or when pumping at different angles even in the presence of the zero angle probe. These results provide strong evidence for a polariton scattering process stimulated by the probe. The bosonic scattering rate is enhanced by a factor \( N + 1 \) where \( N \) is the population of bosons in the final state [185], in this case the ground state. Moreover, stimulated scattering was confirmed in this experiment by the observed exponential dependence of the gain on pump power. Note however that the obtained stimulated regime is more correctly described as a ‘parametric’ amplifier rather than a laser, i.e. the population of the lowest mode is amplified by a phase coherent parametric process [174].

Much experimental work has followed this first result [34, 170–181]. Evidence of stimulated scattering has been obtained in [34], with a three-beam pulsed experiment, where two pumps excite states at large and opposite angles, \( \theta_p \sim \pm 45^\circ \), and a third beam is used as a probe at normal incidence. Here, the pump creates a quasi-thermal exciton reservoir at large momentum; then, scattering of polariton from \( k_p \) and \(-k_p\) to the \( k = 0 \) upper and lower polariton states can occur, and it is stimulated by the occupation (due to the probe) of the final state. Note that, at zero detuning, such a scattering process conserves energy \( 2E^{\text{ex}}_{k} = E^{LP}_{\text{pump}} + E^{\text{UP}}_{\text{pump}} \) and momentum. Recent experiments in the original two-beam pump probe configuration, but with time resolved measurements as a function of different pump and probe angles have begun the investigation of the hydrodynamic properties of the injected ‘polariton fluid’ [186, 187].

**Figure 11.** (a) Lower polariton dispersion; (b) contour plot of \( |E^{LP}_{k} + E^{LP}_{3k} - 2E^{LP}_{k}| \) as a function of \( k \) and the zero value contour (white dashed). A pair of final states (signal and idler) can be found by intersecting the white dashed curve with a straight line passing through \( k_p \) (indicated as pump). (Reprinted with permission from [184]. Copyright 2001 by the American Physical Society.)
In another series of experiments, making use of resonant c.w. excitation [171, 173] rather than ultrafast pulsed excitation as in [34, 169, 170], the stimulated scattering regime has been reached even without the probe beam (parametric oscillation). Here, the stimulated scattering is self-initiated when the pump power is strong enough that the final state population becomes close to one. It is interesting to note that, as in the non-resonant excitation experiments, a minimum pump power is required (with threshold-like behaviour) in order to overcome the ‘bottleneck’ effect and thus provide the occupation of the ground state required to stimulate scattering. In these particular experiments, occupation of the ground state was estimated to be close to 1 at threshold and around 300 at the highest pump powers. A maximum blue-shift of the lower polariton of the order of 0.5 meV was observed, which is much less than the Rabi splitting of ∼6 meV, and so confirms that the experiment is always in the strong coupling regime. Because of the stimulated scattering, pumping of the ground state mode is efficient, and so these systems have a low power threshold, typically five times smaller than for a high quality vertical cavity surfaces emitting laser (VCSEL) [171].

While the experiments described above were at temperatures of the order of 5 K, Saba and collaborators have reported pump-probe parametric amplification of polaritons with an extraordinary gain up to 120 K in GaAlAs-based microcavities, and up to 220 K in CdTe-based microcavities [175]. The highest possible operating temperatures for observing amplification have been shown to be determined by the exciton binding energy (25 meV for the used CdTe wells and 13.5 meV for GaAs wells), rather than the polariton Rabi splitting, which in these experiments varies from 25 meV for the 24 quantum-well (QW) CdTe microcavity to 15 meV and 20 meV for GaAlAs microcavities with respectively 12 QWs and 36QWs.

Time-resolved measurements, obtained by controlling the delay between pump and probe, show ultrafast dynamics of the parametric gain, promising future applications in high-repetition-rate optical switches and amplifiers.

As an alternative to the finite angle resonant configuration, a few experiments [174, 180] have concentrated on the double energy and momentum resonance, where pump, signal and idler are all at normal incidence, k_p = 0. In this geometry it is possible to investigate the dependence of the phase of the signal on the phase of the pump, and thus to show [174] that in this degenerate configuration the parametric scattering is a coherent process. In addition, in most of the early resonant experiments, such as [170, 171, 173, 175], spectral narrowing of the signal was observed when amplification occurs, also suggesting that the signal is coherent [177]. Direct evidence for the coherent nature of the signal emission has come only recently, in [181], where first-order temporal and spatial coherence were investigated. In that work, in a resonant c.w. pump, no probe, configuration, two spots separated by 70 µm, coming from the same laser excited region of the sample (100 µm FWHM of Gaussian profile), are overlapped in momentum space, showing interference fringes. In addition, by making use of noise measurements, the emitted signal is shown to be in a single-mode quantum state, rather than in a multi-mode state.

Recently, the pair correlation of the emitted signal-idler polaritons has been demonstrated by showing that polaritons in two distinct idler modes can interfere if and only if they share the same signal mode [188].

3.2. Theories

Predictions about threshold conditions, spectral properties and efficiency of the amplification in resonant pump-probe experiments can be easily obtained by making use of the effective lower polariton Hamiltonian \( \hat{H}_{LP} \) (14) described in section 2.2.2, to which one must add the coupling to the external radiation pump \( \hat{\Omega}_{\text{pump}}(t) \) and probe \( \hat{\Omega}_{\text{probe}}(t) \) fields [189]:

\[
H_{\text{ext}} = c \sum_k \left\{ \delta_{k_p k} \hat{\Omega}_{\text{pump}}(t) + \delta_{k_p 0} \hat{\Omega}_{\text{probe}}(t) \right\} \cos \theta_k \hat{a}_k^\dagger + \text{h.c.} \right\}.
\]

(47)

A closed set of equations of motion for the expectation values of probe (signal) \( \langle L_0(t) \rangle \), pump \( \langle L_{k_p}(t) \rangle \) and idler \( \langle L_{k}(t) \rangle \) modes can be obtained by factorizing field expectation values, and neglecting higher order correlations. By solving such equations, both in the steady-state regime and numerically for the pulsed excitation, Ciuti and collaborators [189] have obtained the conditions for gain threshold, showing that the efficiency of the amplifier depends very strongly on the polariton linewidth: the larger the linewidth, the higher the threshold and the lower the maximum gain. Similar results, such as the blue-shift of the signal with increasing pump power, have been obtained by Whittaker [190] in a classical nonlinear optics treatment. In Whittaker’s paper, a phenomenological model—where a nonlinear excitonic oscillator is coupled to the cavity mode, driven by external fields—can describe both parametric amplification and oscillation. This treatment, considering classical fields, and retaining only the frequencies corresponding to pump, signal and idler modes, is equivalent
to factorizing the field expectation values and considering the equation of motion for \( \langle L_0(t) \rangle, \langle L_{k_p}(t) \rangle \) and \( \langle L_{2k_p}(t) \rangle \).

Unsurprisingly, these models of parametric processes in microcavity polaritons are closely related to the more generalized treatment of parametric processes for coupled radiation modes, discussed by Louisell et al. [191].

By considering the case of a c.w. pump without a probe, and expanding up to the second order in the field expectation values, it can be also shown [21, 184] that, below threshold for parametric amplification, the polariton photoluminescence,

\[
\text{PL}(k, t, \omega) \approx \cos^2 \theta \langle \hat{L}_p \rangle \int_0^\infty \text{d}t \ e^{-i(a-\omega)t} \langle L_k^\dagger(t + \tau)L_k(t) \rangle
\]

has a blue-shifted asymmetric emission distribution, as was observed in [170], and shown in figure 12. The lower polariton blue-shift here, \( E_{k_p}^{LP} \rightarrow E_{k_p}^{LP} + \sqrt{\langle \hat{L}_{k_p} \rangle} \langle \hat{L}_{k_p} \rangle \), is due to the interaction term of equation (16). Note that the above formalism is valid only below the threshold for parametric emission, as above threshold the equation of motion for the pump mode, describing pump depletion should also be included.

Theoretical work [192, 193], including the effects of polariton blue-shifts on the parametric oscillator equations, has shown that in the c.w. configuration the ‘magic’ angle is not necessary, as it is in the ultrafast pump-probe case, and that, under suitable pump conditions, the parametric oscillator can in general be observed for pump angles \( \theta \gtrsim 10^\circ \). Such calculations seem to explain the experimental results obtained in [178]. In that experiment, with c.w. pumping, stimulation was achieved over a wide range of pump angles, from 10° to 24° and with the signal always at \( k = 0 \). Including the polariton blue-shift also allows one to distinguish parametric oscillation, where output intensity grows continuously, and bistability, where output suddenly jumps at some threshold pump power, and shows hysteresis if power is then reduced. Bistability, both as a function of pump power, and of varying exciton–photon detuning, has been experimentally observed in [194]. The conditions under which the above two options, bistability or parametric oscillation, occur have been investigated theoretically as a function of pump angle for resonant pumping [193], and also allowing for mismatch between pumping frequency and the polariton energy at the pumping angle [195].

There has also been a proposal to study the hydrodynamic properties of the injected polariton fluid by studying its coherent scattering by disorder [196]; i.e. resonant Rayleigh scattering. In the proposed experiment, a strong pump beam creates a large coherent population of polaritons, and also provides the source of polaritons which may be coherently scattered: in the presence of disorder, the polaritons can resonantly scatter to states with different momentum, but the same energy. This can be observed by looking at the angular distribution of photons escaping the cavity which are resonant with the pump beam. At low pump power, this emission pattern will be a ring, at the pump angle—i.e. those states with the same energy have the same modulus of momentum. However, since a large coherent population of polaritons modifies the polariton dispersion relation, at higher pump powers, both the shape and intensity variation across the pattern of resonantly scattered photons reveal information about the polaritons in the cavity.

3.2.1. Phase degree of freedom and low energy modes.

The laser and an equilibrium polariton condensate form extreme ends of the spectrum of systems in which coherent emission results from a symmetry breaking transition; the resonantly pumped polariton laser falls somewhere in between. Although, as discussed in section 3.1, the coherence of the signal is inherited from the pump, in the parametric oscillator configuration, without a seed signal beam there is in principle a free phase \( \Delta \phi \) between the signal and pump modes. That is, the equations of motion are invariant under the transformation \( L_0 \rightarrow L_0 e^{i\Delta \phi}, \hat{L}_{2k_p} \rightarrow \hat{L}_{2k_p} e^{-i\Delta \phi} \), using the notation introduced in section 3.2. In an equilibrium condensate, the invariance of the energy under a global rotation of phase implies the existence of a soft phase mode. However, the existence of a free phase does not necessarily lead to the same consequences [126, 127, 132, 143], as we will discuss next.

Amongst several distinctions between a laser and an equilibrium condensate, one important difference is that for a laser, the threshold condition is the balance of gain and decay [16, 147, 185], while for a condensate, the Gross–Pitaevskii equation (gap equation, self-consistency condition) involves the real parts of the self-energy. This difference implies a difference in the spectrum of fluctuations. Both cases have a pole in the fluctuation spectrum at \( \omega = 0 \) and zero wave-vector, corresponding to global phase rotations. However, for small, but non-zero wave-vector, the spectrum first acquires a real part for a true condensate, describing linearly dispersing modes [107, 139] (in an interacting system); if instead one must balance gain and decay, the spectrum instead first acquires an imaginary part, describing diffusive modes [126, 127, 132, 143] (see figure 9). Although both the laser and strongly pumped polariton systems share this diffusive structure, there is an important difference in the properties of the phase mode between a conventional laser and the strongly-pumped polariton system. The difference is that, as discussed in section 2.3.3, a conventional laser has a discrete spectrum of wave-vectors (all other modes rapidly decay), while the microcavity polariton system has a continuum of in-plane wave-vectors with comparable decay rates. The continuum of diffusive phase modes should thus lead to differences between the coherence properties of a laser and a resonantly pumped polariton system [132, 143]. The difference between diffusive and dispersive low energy degrees of freedom may also have implications for pattern formation in nonlinear systems [197, 198].

3.2.2. Polarization and spin relaxation.

By considering the spin of polaritons, with resonant pumping, one can consider the coupled dynamics of the polarizations of the pump, signal and idler modes. This dynamics leads to a rich variety of physical effects, due to the interplay between spin dependent stimulated scattering, and precession induced both by momentum dependent TE–TM (transverse-electric, transverse-magnetic) splitting [199], and other energy splittings due to polariton–polariton interactions. The following is a brief summary of the theories that have been applied to explain these features. Interest in the subject began with experiments, in both c.w. [200] and pulsed [201] experiments, that could not be explained by regarding each spin species as acting
independently. In [200], signal intensity as a function of pump ellipticity (from linear to circular) was studied, and a maximum found at an intermediate value. In [201], a large output signal was seen for a linearly polarized pump, but a circularly polarized probe, and was explained there in terms of stimulated spin scattering. In addition, the direction of the linear component of polarization of the signal was observed to vary as a function of the degree of ellipticity of the pump beam. This large rotation was discussed in [202] as a giant Faraday effect, with the cavity amplifying the effect of spin splitting of the exciton energy levels, with the spin splitting being due to unequal spin populations of the exciton states. In contrast, reference [203] showed that parts of the above results could be explained by introducing coupling between the two circular polarizations. Such coupling provides two new terms; parametric scattering of cross polarizations, and processes where pump and signal, or pump and idler polaritons exchange polarization. These terms lead to a threshold power that depends on ellipticity of the pump, and can under some conditions show lowest threshold for an elliptic polarization.

In a different experiment, Martín et al [204] investigated the dependence of the circular polarization of the nonlinear emission on the detuning between the cavity photon modes and excitons. This result, showing oscillations of the polarization, has been attributed to the dependence of the TE–TM splitting on the detuning [205]. This TE–TM splitting, which depends on wave-vector, can lead to time-dependent oscillations of the polarization of the signal and idler modes, as discussed in [206]. In order to combine the precession due to such a splitting with a spin-dependent stimulated scattering to the signal state, one is led to write a spin dependent Boltzmann equation [206]. This formalism was sufficient to reproduce the results of the experiment described in [39, 207]. In that experiment the circular polarization component of the signal was studied as a function of pumping strength, for both linear and circular polarization. It was found that, for a linear pump, there is a maximum of the degree of circular signal polarization near the nonlinear threshold. As discussed in [207], this effect can be understood as a competition between two effects; self-induced Larmor precession, which rotates the pseudospin describing the polarization from linear to circular polarization, and stimulated scattering to the ground state: far above threshold, the rate of scattering to the ground state is too high to allow time for any change of polarization in the pump mode.

An ingredient missing from the works listed so far was stimulated scattering due to polariton–polariton interactions (as opposed to polariton–phonon scattering, stimulated by final state polariton population). Shelykh et al considered the dominant, parallel spin, interaction in [208]. However it was later realized [209, 210] that the scattering of anti-parallel spin states, though small, is important, as it leads (at high densities) to a 90° rotation of linear polarization direction between pump and signal. In a recent experiment [76], the polarization of the output signal was studied for a linearly polarized pump, as a function of pump power and angle of the linear polarization. This experiment showed that, just above threshold, the signal was elliptical, with the major axis (the direction of the linear component) rotated with respect to the linear polarization of the pump—i.e. the direction of major axis depended on the direction of polarization of the pump, but the angle between the two directions was not constant. In addition, the degree of linear polarization of the signal decreased far above threshold (as well as the degree of circular polarization as observed in [207])). This rotation was reproduced in a model combining precession due to static and self-induced (Larmor) splittings (including an extra in-plane splitting, as discussed in [76]) as well as spin rotation in stimulated polariton–polariton scattering. The reduction of linear polarization far above threshold was explained by rapid self-induced Larmor precession, which rotates the linear polarization direction so rapidly that it averages to zero.

4. Connection to other systems, conclusions

4.1. Atomic gases and Feshbach resonances

Microcavity polaritons and their condensation are related to the physics of two-component atomic Fermi gases near Feshbach resonances. In particular, the crossover from a fluctuation dominated phase boundary to a mean-field phase boundary with increasing polariton density is closely related to the BEC–BCS crossover recently studied in these atomic gases [211, 212]. For atomic gases, in contrast to polariton experiments, the density of particles is typically kept fixed, while the interaction strength is varied via magnetically tunable Feshbach resonances, allowing one to go from a BEC of bound molecules to a BCS state of fermionic pairs. The interaction strength is tuned by changing the detuning between the zero of energy for pairs of atoms in their original spin states, and a closed channel resonance level of atoms in some higher energy spin configuration. (See [213] and references therein for more details.) The Bose–Fermi model used for polaritons and described in section 2.2.3 is very similar to the model initially proposed for the description of the BEC–BCS crossover in atomic Fermi gases [214]. The fermionic operators $b^\dagger, a^\dagger$ in the polariton model, introduced in equation (18), are analogous to the two spin species of atoms in the Feshbach resonance model: the photon is analogous to the closed channel resonance level; the dipole interaction relates to the hyperfine interaction; and the polariton to the Feshbach molecule. There are, however, a number of important differences between the two models; most notably the absence of a direct four-fermion interaction in the polariton model, and the existence of an energy dependence, and a distribution, of exciton–photon coupling strengths. Secondy there is a marked difference of mass ratios: for polaritons, the photon mass is typically a factor of 10^5 times smaller than the exciton mass; in the Feshbach case the closed channel resonance has a dispersion controlled by a mass which is twice the atomic mass. Note also that there is a difference in interpretation between the photon and the closed channel resonance level in Feshbach resonance. Although inside the microcavity a photon is not an eigenstate—the upper and lower polaritons are instead eigenstates—outside the cavity it is possible to physically separate the photons from excitons. Similarly, the closed channel resonance level is not an eigenstate; however, in distinction from the photon, it cannot be physically separated from the other two-body states to which it is coupled: in general one cannot have a given molecular resonance level in isolation from other molecular
Another difference between the systems comes in how the gap equation (i.e. equation (38)), and constraint on the density can be used to find chemical potential and temperature: in the mean-field (BCS) limit of the atomic case, the closed channel resonance level lies at high energies, the chemical potential lies well within the band of fermionic states, and so the density depends only on chemical potential, and not on temperature. In the polariton model, the chemical potential remains below the bottom of the band of fermionic states, and so both temperature and chemical potential influence the density. Rather, the fluctuation dominated (BEC) to mean-field (BCS) crossover of the polariton model is in the nature of which excitations are responsible for depopulating the condensate at finite temperatures.

Another issue worth considering is that further analysis has questioned the need to use the Bose–Fermi model in applications to experimentally relevant Feshbach resonances. It has been shown [215, 216] that the molecules created in the vicinity of a Feshbach resonance are halo dimers extending over large distances in which the closed channel admixture is tiny. Thus, the resonance level acts to enhance the effective over large distances in which the closed channel admixture is tiny. Thus, the resonance level acts to enhance the effective interaction between fermionic pairs, but the crossover to loosely bound molecules does not rely on the macroscopic occupation of this level (as was initially suggested). Since the resonance level is not significantly occupied, it was shown, by using realistic atomic potentials [217], that the BCS–BEC crossover in atomic Fermi gases near Feshbach resonances is of the same nature as originally considered by Leggett [218]. That is, it is based on the smooth crossover of the pair size and not on the macroscopic occupation of the resonance level. Thus, microcavity polariton experiments in the normal operating conditions of large photon field occupation present the first experimental realization of the BEC–BCS crossover which differs substantially from the original scenario [86, 102, 218].

Finally, it is interesting to note that the same idea of parametric scattering and amplification in a resonant pumping configuration as is described in section 3 has been recently applied to an ultracold dilute gas of bosonic atoms confined in an optical lattice [183] (see also the related experiments on dynamic instabilities [219, 220], and four wave mixing of matter waves [182]). There, a $^87$Rb Bose–Einstein condensate was loaded into a moving one-dimensional optical lattice. The optical lattice causes the atomic dispersion to deviate from quadratic, and allows parametric scattering: atom pairs with initial momentum $k_p$ inherited from the moving lattice scatter elastically into two final states; $k_p - k$ and $k_p + k$. By generating a seed of atoms with momentum $k_p - k$, parametric amplification of both the seed and the conjugate momentum has been observed, with a gain determined by the atomic scattering length.

### 4.2. Excitons, quantum hall bilayers, triplons

The high quantum degeneracy temperatures and the high degree of control obtained by laser photo-excitation suggest that excitonic systems should also provide excellent environments in which to study macroscopic coherence phenomena. Much interest over the last two decades has been attracted by excitons in bulk Cu$_2$O crystals (see, e.g., [221] and references therein). Excitons in Cu$_2$O have a large binding energy (∼150 meV) and the fact that the direct dipole transition from the exciton ground state is forbidden guarantees low radiative recombination rates. In Cu$_2$O crystals however non-radiative recombination processes such as the Auger effect cause loss and heating, and represent the main obstacle to the observation of quantum degeneracy in these structures.

Low radiative recombination rates together with high cooling rates can be obtained for spatially indirect excitons in coupled quantum wells. Here, an electric field is applied along the growth direction, in such a way that electrons and holes separate into different wells. In contrast to the case of direct excitons, lifetimes up to few microseconds can be achieved, while the high cooling rate gives a much shorter thermalization time, typically in the nanosecond range. As a consequence, in coupled quantum well structures, thermal equilibrium with the lattice can be relatively easily obtained by either waiting a few nanoseconds after photo-excitation or allowing the excitons to travel away from the excitation spot. While much experimental work has been done on these structures in the last few years [222–225] (for a review see [226, 227]), unambiguous evidence for Bose–Einstein condensation of indirect excitons in coupled quantum wells is still missing and is the subject of intensive on-going studies. Recently, gases of indirect excitons have been trapped and equilibrated using in-plane potentials, either by applying localized stress to change the local band energies [228] or by means of optical [229] or electrostatic [230] traps. The confinement of indirect excitons prevents the fast reduction of initial density which occurs in the absence of trapping due to fast expansion driven by their strong dipole–dipole repulsive interaction and their relatively high mobility.

It is interesting to note that, for an untrapped system, while the formation of an external excitonic ring has been explained as an in-plane charge separation, at low temperatures it has been shown that, such a ring can separate into a periodic array of beads, and that the light emitted by each of these beads is coherent [231]. The origin of such a phenomenon is still unknown.

Recently, experimental evidence suggests that, under appropriate conditions, an electron–electron semiconductor bilayer system in the quantum Hall regime can condense into a superfluid state which might be interpreted as an excitonic-like superfluid [232]. In a bilayer 2D electron system with total filling factor $\nu_T = 1$, excitonic pairs can be thought of as formed by filled electron states in one layer and empty electron states in the second layer. By changing the ratio between intra- and inter-layer Coulomb interactions, signatures of the transition to a condensed excitonic phase have been shown by a dramatic increase in the tunnelling rate between the two layers at zero interlayer voltage [233], in Coulomb drag measurements [234], and in counterflow measurements [235].

Another condensed matter system in which a phase transition can be described as condensation of an excitonic mode is that of magnetic ‘triplon’ excitations. This has been seen in a variety of compounds including TiCuCl$_3$ [236, 237], BaCu$_2$Si$_2$O$_6$ [238] and Pb$_x$V$_{1-x}$O [239], where by changing the applied magnetic field, there is a crossing between spin singlet and spin triplet excitations. The resultant magnetic phase transition can be described as condensation of the...
triplet magnon mode. Very recently such a transition was seen at room temperature, using parametric laser pumping to create a non-equilibrium density of triplons in the compound Y$_3$Fe$_2$(FeO$_4$)$_3$ [240].

4.3. Conclusions

In this article we reviewed the experimental results to date demonstrating coherence in microcavities, and discussed the variety of theoretical models and techniques that have been used to describe it. We discussed experiments with both non-resonant pumping, in which coherence may spontaneously arise from an initially incoherent source of polaritons, and the optical parametric amplifier and optical parametric oscillator: both kinds of experiments allow one to explore the interplay of strong-coupling, coherence, lasing and condensation. In our discussion of theoretical descriptions, we highlighted those aspects of these solid state systems which can introduce new questions about coherence—disorder, decoherence, particle flux, potentially non-thermal distributions.

Of course, there are subjects connected to coherence in microcavities that we have not had space to discuss, or have only discussed briefly. For example, we have only briefly mentioned here questions about the dynamics of condensate formation, and of the polariton response following short pump pulses. The last years have seen rapid experimental progress in this field, with the first convincing evidence of coherence developing from incoherently injected polaritons in a variety of systems [44, 52, 67]. This experimental progress both gives hope for the possibility of future experiments, and applications on coherent microcavity polaritons, as well as focusing attention on those areas in which further theoretical work is necessary. There are areas in which our discussion has been brief because some questions have only been partially addressed to date: questions about hydrodynamics in such partially coherent, pumped decaying systems. In order to address questions about the generic behaviour of such systems, it is important to understand how the variety of models used to describe polaritons relate, and what features each can explain. Finally, let us mention that there remains an interesting topic which can be the subject of much future research: what new experiments are possible in these light–matter systems that were not possible in either lasers or in atomic gases.

Acknowledgments

We are grateful to L S Dang, P R Eastham, J Kasprzak, B D Simons for helpful discussions, and to I Carusotto, A Kavokin, T Köhler, D Sarchi, V Savona, L Viña, Y Yamamoto and R Zimmermann for critical reading of, and helpful suggestions on the manuscript. FMM and MHS would like to thank B D Simons for helpful discussions, and to I Carusotto, A Kavokin, T Köhler, D Sarchi, V Savona, L Viña, Y Yamamoto and R Zimmermann for critical reading of, and helpful suggestions on the manuscript. FMM and MHS would like to acknowledge financial support from EPSRC. JK would like to acknowledge financial support from the Lindemann Trust and Pembroke College Cambridge. This work is supported by the EU Network “Photon mediated phenomena in semiconductor nanostructures” HPRN-CT-2002-00292.

References

[1] Björk G, Machida S, Yamamoto Y and Igeta K 1991 Phys. Rev. A 44 669
[2] Weisbuch C, Nishioka M, Ishikawa A and Arakawa Y 1992 Phys. Rev. Lett. 69 3314
[3] Ramon G, Mizzahli U, Akopian N, Braithart S, Gershoni D, Reinecke T L, Gerardot B D and Petroff P M 2006 Phys. Rev. B 73 205330
[4] Yoshie T, Schere A, Hendrickson J, Khirotova G, Gibbs H M, Rupper G, Eil C, Shchekin O B and Deppe D G 2004 Nature 432 200
[5] Badolato A, Hennessy K, Attari M, Dreisner J, Hu E, Petroff P M and Imamoglu A 2005 Science 308 1158
[6] Reithmaler J P, Sk G, Löffler A, Hoffmann C, Kuhn S, Reitzenstein S, Keldysh L, Kulakovskii V D, Reinecke T L and Forchel A 2004 Nature 432 197
[7] Peter E, Senellart P, Martrou D, Lamaitre A, Hours J, Gérard J M and Bloch J 2005 Phys. Rev. Lett. 95 067401
[8] McCull S L, Levi A F J, Slusher R E, Pearton S J and Logan R A 1991 Appl. Phys. Lett. 60 289
[9] Dasbach G, Diederichs C, Tignon J, Ciuti C, Roussignol P, Delalande C, Bayer M and Forchel A 2005 Phys. Rev. B 71 161308
[10] Kaitouni R I et al 2006 Phys. Rev. B 74 155311
[11] Diﬁ O E, Baas A, Guillot T, Brantut J P, Kaitouni R I, Morier-Genoud F and Deveaud B 2006 Appl. Phys. Lett. 88 061105
[12] Baas A et al 2006 Phys. Status Solidi b 243 2311
[13] Langbein W and Hvm J M 2002 Phys. Rev. Lett. 88 047401
[14] Leggett A J 2001 Rev. Mod. Phys. 73 307
[15] Pitavevski L I and Stringari S 2003 Bose-Einstein Condensation (Oxford: Clarendon)
[16] Haken 1970 Quantum Optics ed S M Kay and A Maitland (London: Academic) p 201
[17] Skolnick M S, Fisher T A and M W D 1998 Semicond. Sci. Technol. 13 645
[18] Savona V, Piermarocchi C, Quattropani A, Schwendimann P and Tassone F 1999 Phase Transit. 68 169
[19] Savona V, Andreani L C, Schwendimann P and Quattropani A 1999 Solid State Commun. 93 733
[20] Khirotova G, Gibbs H M, Jahnke F, Kairn M and Koch S W 1999 Rev. Mod. Phys. 71 1591
[21] Ciuti C, Schwendimann P and Quattropani A 2003 Semicond. Sci. Technol. 18 S279
[22] Yamamoto Y, Tassone F and Cao H 2000 Semiconductor Cavity Quantum Electrodynamics (Springer Tracts in Modern Physics vol 167) (Berlin: Springer)
[23] Kavokin A and Malpuech G 2003 Cavity Polaritons (Thin Films and Nanostructures vol 32) (New York: Elsevier)
[24] Baumbreg J and Viña L (ed) 2003 Special issue on microcavities Semiconductor Sci. Technol. 18 (10) S279–S434
[25] Deveaud B (ed) 2005 Special issue on physics of semiconductor microcavities Phys. Status Solidi b 242 (11) 2147–356
[26] Yamamoto Y and Imamoglu A 1999 Mesoscopic Quantum Optics (New York: Wiley)
[27] Pau S, Cao H, Jacobson J, Björk G, Yamamoto Y and Imamoglu A 1996 Phys. Rev. A 54 R1789
[28] Cao H, Pau S, Jacobson J M, Björk G, Yamamoto Y and Imamoglu A 1997 Phys. Rev. A 55 4632
[29] Dang L S, Heger D, André R, Beruf J and Romestain R 1998 Phys. Rev. Lett. 81 3920
[30] Bœuf F, André R, Romestain R, Si Dang L, Pérone E, Lampin J F, Hulin D and Alexandrou A 2000 Phys. Rev. B 62 R2279
[31] Alexandrou A, Bianchi G, Pérone E, Hallé B, Bœuf F, André R, Romestain R and Si Dang L 2001 Phys. Rev. B 64 233318
[32] Senellart P and Bloch J 1999 Phys. Rev. Lett. 82 1233
[33] Senellart P, Bloch J, Sermage B and Marzin J Y 2000 Phys. Rev. B 62 R16263
[34] Huang R, Tassone F and Yamamoto Y 2000 Phys. Rev. B 61 R7854
