SPHERICAL SCHUBERT VARIETIES AND PATTERN AVOIDANCE

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Abstract. A normal variety $X$ is called $H$-spherical for the action of the complex reductive group $H$ if it contains a dense orbit of some Borel subgroup of $H$. We resolve a conjecture of Hodges–Yong by showing that their spherical permutations are characterized by permutation pattern avoidance. Together with results of Gao–Hodges–Yong this implies that the sphericality of a Schubert variety $X_w$ with respect to the largest possible Levi subgroup is characterized by this same pattern avoidance condition.

1. Introduction

1.1. Spherical varieties. Following [3, 11], a normal variety $X$ is called $H$-spherical for the action of the complex reductive group $H$ if it contains a dense orbit of some Borel subgroup of $H$. Important examples of spherical varieties include projective and affine toric varieties, complexifications of symmetric spaces, and flag varieties (see Perrin’s survey [13]). Producing families of examples of and classifying spherical varieties is of significant interest. In this paper we resolve a conjecture of Hodges–Yong [7], thereby classifying (maximally) spherical Schubert varieties by a permutation pattern avoidance condition.

1.2. Schubert varieties and pattern avoidance. Let $G$ be a complex reductive algebraic group and $B$ be a Borel subgroup. The Bruhat decomposition decomposes $G$ as

$$G = \bigsqcup_{w \in W} BwB,$$

where $W$ denotes the Weyl group of $G$. The closures

$$X_w = BwB/B$$

of the images of these strata in the flag variety $G/B$ are the Schubert varieties, of fundamental importance in algebraic geometry and representation theory.

In the case $G = GL_n(\mathbb{C})$, the Weyl group $W$ is the symmetric group $S_n$. Beginning with the groundbreaking result of Lakshmibai–Sandhya [9] characterizing smooth Schubert varieties, it has been found that many important geometric and combinatorial properties (see, for example [8, 16]) of $X_w$ are determined by permutation pattern avoidance conditions on $w$. Let $w = w_1 \ldots w_n \in S_n$ be a permutation written in one-line notation, and let $p \in S_k$ be another permutation. Then $w$ is said to have an occurrence of the pattern $p$ at positions

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1 \leq i_1 < \cdots < i_k \leq n \text{ if } w_{i_1} \ldots w_{i_k} \text{ are in the same relative order as } p_1 \ldots p_k. \text{ If } w \text{ does not contain any occurrences of } p, \text{ then } w \text{ is said to avoid } p.

Under the natural left action of } G \text{ on } G/B, \text{ the stabilizer of } X_w \text{ is the parabolic subgroup } P_{J(w)} \subset G \text{ corresponding to the left descent set } J(w) \text{ of } w. \text{ The parabolic subgroup } P_{J(w)} \text{ is not reductive, but contains the Levi subgroup } L_{J(w)} \text{ as a maximal reductive subgroup. Following [7], we say } X_w \text{ is maximally spherical if it is } L_{J(w)}\text{-spherical for the induced action of } L_{J(w)}. \text{ Since all Schubert varieties are known to be normal by the work of DeConcini–Lakshmibai [4] and Ramanan–Ramanathan [14], this is equivalent to the existence of a dense orbit inside } X_w \text{ of a Borel subgroup of } L_{J(w)}.

1.3. Hodges and Yong’s conjecture. We consider the symmetric group } S_n \text{ as a Coxeter group with simple generating set } I = \{s_1, \ldots, s_{n-1}\}, \text{ where } s_i \text{ is the adjacent transposition } (i i+1), \text{ and we write } J(w) \text{ for the left descent set of } w \in S_n. \text{ See Section 2 for background and basic definitions.}

**Definition 1.1** (Hodges and Yong [7]). A permutation } w \in S_n \text{ is spherical if it has a reduced word } s_{i_1} \cdots s_{i_{\ell(w)}} \text{ such that:}

(S.1) \quad |\{t \mid s_i t = s_j\}| \leq 1 \text{ for } s_j \in I \setminus J(w), \text{ and}

(S.2) \quad |\{t \mid s_i t \in C\}| \leq \ell(w_0(C)) + |C| \text{ for any connected component } C \text{ of the induced subgraph of the Dynkin diagram on } J(w).

**Remark.** Hodges and Yong consider a more general class of spherical elements in finite Coxeter groups. Definition 1.1 is the special case which is relevant to Conjecture 1.3 (}J(w)-spherical elements in } S_n).

Spherical permutations were defined because of Conjecture 1.2, which is proven in forthcoming work [6] of Gao–Hodges–Yong.

**Conjecture 1.2** (Conjectured by Hodges and Yong [7]; proof by Gao–Hodges–Yong [6] in preparation). The Schubert variety } X_w \text{ is maximally spherical if and only if } w \text{ is spherical.}

This geometric property is linked to permutation pattern avoidance by Conjecture 1.3

**Conjecture 1.3** (Hodges and Yong [7]). A permutation } w \text{ is spherical if and only if it avoids the twenty one patterns in } P:

\[
P = \{24531, 25314, 25341, 34512, 34521, 35412, 35421, 42531, 45123, 45213, 45231, 45312, 52314, 52341, 53124, 53142, 53412, 53421, 54123, 54213, 54231\}.
\]

Our main result resolves Conjecture 1.3

**Theorem 1.4.** A permutation } w \text{ is spherical if and only if it avoids the patterns in } P.

Combining this result with Gao–Hodges–Yong’s proof of Conjecture 1.2, we thus obtain a characterization of maximally spherical Schubert varieties in terms of pattern avoidance.

**Corollary 1.5.** The Schubert variety } X_w \text{ is maximally spherical if and only if } w \text{ avoids the patterns from } P.
The following result, an immediate consequence of Theorem \ref{thm:main} was conjectured in \cite{stanley} and proven in \cite{tardos} using probabilistic methods.

**Corollary 1.6.**

\[ \lim_{n \to \infty} \frac{|\{\text{spherical permutations } w \in S_n\}|}{n!} = 0. \]

**Proof.** The Stanley–Wilf Conjecture, now a theorem of Marcus and Tardos \cite{marcus-tardos}, says that the number of permutations in \( S_n \) avoiding any fixed set \( Q \) of patterns is bounded above by \( C^n \) for some constant \( C \). Thus Theorem \ref{thm:main} implies that

\[ |\{\text{spherical permutations } w \in S_n\}| \]

grows at most exponentially. \qed

1.4. **Outline.** Section 2 recalls some basic definitions and facts about Bruhat order as well as a result of Tenner \cite{tenner} characterizing Boolean intervals in Bruhat order. In Section 3 we introduce the notion of divisible pairs of permutations and connect these to Boolean permutations and spherical permutations. Divisible pairs, along with a helpful decomposition of the set \( P \) of patterns, are applied in Section 4 to prove Theorem \ref{thm:main}.

2. **Background**

2.1. **Bruhat order.** For \( i = 1, \ldots, n \), let \( s_i \) denote the adjacent transposition \((ii+1)\) in the symmetric group \( S_n \); the symmetric group is a Coxeter group with respect to the generating set \( s_1, \ldots, s_{n-1} \) (see \cite{coxeter} for background on Coxeter groups). For \( w \in S_n \), and expression

\[ w = s_{i_1} \cdots s_{i_t} \]

of minimum length is a reduced word for \( w \), and in this case \( \ell = \ell(w) \) is the length of \( w \).

The (right) weak order is the partial order \( \leq_R \) on \( S_n \) with cover relations \( w <_R ws_i \) whenever \( \ell(ws_i) = \ell(w) + 1 \). The Bruhat order is the partial order \( \leq \) on \( S_n \) with cover relations \( w < wt \) for \( t \) a 2-cycle such that \( \ell(wt) = \ell(w) + 1 \). Both posets have the identity permutation \( e \) as their unique minimal element.

For a permutation \( w = w_1 \cdots w_n \in S_n \) and integers \( 1 \leq a \leq b \leq n \), we write \( w[a,b] \) for the set \( \{w_a, w_{a+1}, \ldots, w_b\} \). For two \( k \)-subsets \( A, B \) of \([n] := \{1, \ldots, n\}\) write \( A \preceq B \) if \( a_1 \leq b_1, \ldots, a_k \leq b_k \), where \( A = \{a_1, \ldots, a_k\} \) and \( B = \{b_1, \ldots, b_k\} \) with \( a_1 < \cdots < a_k \) and \( b_1 < \cdots < b_k \). The following well-known property of Bruhat order will be useful:

**Proposition 2.1** (Ehresmann \cite{ehresmann}). Let \( v, w \in S_n \), then \( v \leq w \) if and only if

\[ v[1,i] \preceq w[1,i] \]

for all \( i = 1, \ldots, n \).

A generator \( s_i \) is a (left) descent of \( w \) if \( \ell(s_iw) < \ell(w) \) (equivalently, if \( w^{-1}(i+1) < w^{-1}(i) \)). We write \( J(w) \) for the set of descents of \( w \). For any \( J \subseteq \{s_1, \ldots, s_{n-1}\} \), we write \( w_0(J) \) for the unique permutation of maximum length lying in the subgroup of \( S_n \) generated by \( J \). Explicitly, the one-line notation for \( w_0(J) \) is an increasing sequence of consecutive decreasing runs, where decreasing run consists of \( i + d, i + d - 1, \ldots, i \) whenever \( s_i, s_{i+1}, \ldots, s_{i+d-1} \in J \) while \( s_{i-1}, s_{i+d} \notin J \).
2.2. Boolean permutations.

**Theorem 2.2** (Tenner [15]). The following are equivalent for a permutation \( w \in S_n \):

1. The interval \([e, w]\) in Bruhat order is isomorphic to a Boolean lattice,
2. No simple generator \( s_i \) appears more than once in a reduced word for \( w \),
3. \( w \) avoids the patterns 321 and 3412.

We will call a permutation satisfying the equivalent conditions of Theorem 2.2 a Boolean permutation. Theorem 2.3 suggests a connection between Boolean permutations and spherical varieties.

**Theorem 2.3** (Karuppuchamy [8]). The Schubert variety \( X_w \) is a toric variety if and only if \( w \) is a Boolean permutation.

3. Divisible pairs of permutations

**Definition 3.1.** Given a pair \((v, w)\) of permutations from \( S_n \), we say that \((v, w)\) is divisible after position \( i \) if

\[ |v[i, i] \cap w[i, i]| \leq i - 2, \]

and divisible at position \( i \) if \( v_i = w_i \) and

\[ |v[i, i] \cap w[i, i]| \leq i - 1. \]

We say simply that \((v, w)\) is divisible if there exists \( 1 \leq i \leq n \) such that \((v, w)\) is divisible at or after position \( i \).

**Proposition 3.2.** A pair \((v, w)\) of permutations from \( S_n \) is divisible if and only if \( v^{-1}w \) is not Boolean.

*Proof.* It is clear from the definition that \((v, w)\) is divisible if and only if \((wv, uw)\) is divisible for all \( u \in S_n \), so it suffices to prove the case \( v = e \).

Suppose that \( w \) is not Boolean, so that \( w \) contains a pattern \( p \in \{321, 3412\} \) by Theorem 2.2. If \( p = 3412 \) occurs as \( w_{i_1}w_{i_2}w_{i_3}w_{i_4} \) then \((e, w)\) is divisible after position \( i_2 \), since \( w[1, i_2] = \{w_{i_1}, w_{i_2}\} \) while \( e[1, i_2] = \{1, \ldots, i_2\} \) must contain \( w_{i_3} \) and \( w_{i_4} \) if it contains either \( w_{i_1} \) or \( w_{i_2} \). If \( p = 321 \) occurs as \( w_{i_1}w_{i_2}w_{i_3} \), consider three cases: If \( w_{i_2} = i_2 \) then \((e, w)\) is divisible at \( i_2 \), since \( w_{i_1} > i_2 \notin e[1, i_2]\); If \( w_{i_2} < i_2 \), then \((e, w)\) is divisible after \( i_2 - 1 \), since \( w_{i_2}, w_{i_3} \) both lie in \( e[1, i_2 - 1] \) but not in \( w[1, i_2 - 1] \); Similarly, if \( w_{i_2} > i_2 \), then \((e, w)\) is divisible after \( i_2 \), since \( w_{i_1}, w_{i_2} \) both lie in \( w[1, i_2] \) but not in \( e[1, i_2] \).

Conversely suppose that \( w \) is divisible. If \( w \) is divisible after position \( i \), then there are two elements \( a < b \in w[1, i] \) which are not in \( e[1, i] = \{1, \ldots, i\} \) and therefore also two elements \( c, d \in w[i + 1, n] \) with \( c < d \in \{1, \ldots, i\} \). Either \( w^{-1}(a) < w^{-1}(b) \) and \( w^{-1}(c) < w^{-1}(d) \) in which case \( w \) contains 3412 or at least one of these statements fails and \( w \) contains 321; in either case \( w \) is not Boolean. If \( w \) is divisible at position \( i \), then \( w_i = e_i = i \) and there is some \( a > i \) in \( w[1, i - 1] \) and some \( b < i \) in \( w[i + 1, n] \); then the values \( a, i, b \) form a 321 pattern in \( w \), so \( w \) is not Boolean.

**Proposition 3.3** (Gao–Hodges–Yong [9]). A permutation \( w \in S_n \) is spherical if and only if \( w_0(J(w))w \) is a Boolean permutation.
The following characterization of spherical permutations will be convenient for our arguments in Section 4.

Corollary 3.4. A permutation \(w \in S_n\) is spherical if and only if \((w_0(J(w)), w)\) is not divisible.

Proof. By Proposition 3.3, \(w\) is spherical if and only if \(w_0(J(w))w\) is Boolean. Since \(w_0(J(w))\) is an involution, Proposition 3.2 implies that this is equivalent to \((w_0(J(w)), w)\) not being divisible. \(\square\)

4. Proof of Theorem 1.3

The following decomposition of the set \(P\) of twenty one patterns will be crucial to the proof of Theorem 1.3

\[ P^{321} = \{24531, 25314, 25341, 42531, 45231, 45312, 52314, 52341, 53124, 53142, 53412\} \]

and

\[ P^{3412} = \{34512, 34521, 35412, 45231, 45123, 45213, 45231, 53412, 53421, 54213, 54231\} \]

Notice that 45231 and 53412 lie in both \(P^{321}\) and \(P^{3412}\). A simple check shows that \(P^{321}\) and \(P^{3412}\) are also characterized by the following properties:

\[
\begin{align*}
(1) & \quad P^{321} = \{p \in S_5 \mid p^{-1}(5) < p^{-1}(3) < p^{-1}(1), p^{-1}(4) \notin [p^{-1}(5), p^{-1}(3)], \\
& \quad \quad \quad p^{-1}(2) \notin [p^{-1}(3), p^{-1}(1)]\}, \\
(2) & \quad P^{3412} = \{p \in S_5 \mid \max(p^{-1}(4), p^{-1}(5)) < \min(p^{-1}(1), p^{-1}(2)), \\
& \quad \quad \quad p^{-1}(3) \notin [p^{-1}(4), p^{-1}(2)]\}.
\end{align*}
\]

The following proposition is obvious from the definitions, but will be useful to keep in mind throughout the proofs of Lemmas 4.2 and 4.3.

Proposition 4.1. For \(w \in S_n\), let \(v = w_0(J(w))\) and \(1 \leq a < b \leq n\). Then \(v^{-1}(b) < v^{-1}(a)\) if and only if \(w^{-1}(b) < w^{-1}(b-1) < \cdots < w^{-1}(a)\).

Lemma 4.2. If \(w \in S_n\) avoids the patterns from \(P\) then \(w\) is spherical.

Proof. We reformulate using Corollary 3.4 and prove the contrapositive: if \((w_0(J(w)), w)\) is divisible, then \(w\) contains a pattern from \(P\).

Case 1: Write \(v\) for \(w_0(J(w))\) and suppose that \((v, w)\) is divisible after \(i\), and furthermore that \(i\) is the smallest index for which this is true. Then we have:

\[
\begin{align*}
&v[1,i] \setminus w[1,i] = \{a, b\}, \\
&w[1,i] \setminus v[1,i] = \{c, d\},
\end{align*}
\]

where we may assume without loss of generality that \(a < b\) and \(c < d\). We have \(v \leq_R w\), so in particular \(v \leq w\) in Bruhat order; thus by Proposition 2.4 we must have \(a < c\) and
Let $a < c < b < d$. Suppose that $c \leq b + 1$; if $c < b$, then, since $c$ appears after $b$ in $v$, it must be that $b, c$ lie in the same decreasing run of $v$, and by Proposition 4.3 this implies that $b$ appears before $c$ in $w$, a contradiction. If $c = b + 1$, then $s_c$ is a descent of $w$, so $c$ appears before $b$ in $v$, again a contradiction. Thus we have $a < b < b + 1 < c < d$.

We wish to conclude that $w$ contains a pattern from $P$. If any value $x \in \{b+1, b+2, \ldots, c-1\}$ does not lie between $b$ and $c$ in $w$, then we are done, since the values $\{a, b, c, d, x\}$ form a pattern from $P^{3412}$ in $w$. Otherwise, all of these values appear between $b$ and $c$ in $w$. Suppose that they do not appear in decreasing order, so $w^{-1}(b + j) < w^{-1}(b + j + 1)$ for some $j + 1 < c - b$. Then the values $\{c, d, b + j, b + j + 1, a, b\}$ either contain a pattern from $P^{3412}$, or appear in $w$ in the order $c, b + j, d, a, b + j + 1, b$. In this last case $c, b + j, a, b + j + 1, b$ forms an occurrence of the pattern $53142$ from $P^{5321}$. Finally, suppose that $\{b + 1, b + 2, \ldots, c - 1\}$ appear in decreasing order in $w$ between $b$ and $c$; then by Proposition 4.4 $c$ appears before $b$ in $v$, a contradiction. Thus in all cases $w$ contains a pattern from $P$.

Case 2: Write $v$ for $w_0(J(w))$ and suppose that $(v, w)$ is divisible at $i$, and furthermore that $i$ is the smallest index at or after which $(v, w)$ is divisible. Then we have $v_i = w_i$ and

$$\begin{align*}
v[1, i - 1] \setminus w[1, i - 1] &= \{a\}, \\
v[1, i - 1] \setminus v[1, i - 1] &= \{c\},
\end{align*}$$

with $a < c$ by Proposition 2.1. We claim that $a$ is the minimal element in a decreasing run of $v$. Indeed, otherwise $a - 1$ appears immediately after $a$ in $v$, and thus $a - 1$ also appears after $a$ in $w$. But then $a - 1 \in v[1, i] \setminus w[1, i]$, contradicting the minimality of $i$, thus $a$ is the minimal element in a decreasing run, and is smaller than all values appearing after it in $v$. Similarly, $c$ is the maximal element in a decreasing run of $v$ and is larger than all values appearing before it in $v$. Also note that $a < v_i - 1$ and $c > v_i + 1$, for if $a = v_i - 1$ then $w^{-1}(a + 1) < w^{-1}(a)$ but $v^{-1}(a + 1) > v^{-1}(a)$, contradicting Proposition 4.4 and similarly for $c$.

We will now see that $c, v_i, a$ participate in an occurrence in $w$ of some pattern $p \in P$. Suppose first that all values $c - 1, c - 2, \ldots, v_i + 1$ lie in between $c$ and $v_i$ in $w$. If these occur in decreasing order, then $c$ and $v_i$ must occur in the same decreasing run of $v$, but this is not the case since $c$ appears at the beginning of its run, but after $v_i$ in $v$. Thus there is some $j \leq c - v_i - 2$ such that $w^{-1}(c - j - 1) < w^{-1}(c - j)$. In this case $c, c - j - 1, c - j, v_i, a$ form an occurrence in $w$ of the pattern $53421 \in P^{3412}$. Similarly, if all values $v_i - 1, v_i - 2, \ldots, a + 1$ lie in between $v_i$ and $a$ in $w$, then $w$ contains an occurrence in $w$ of the pattern $54231 \in P^{321}$.

In the only remaining case, there is some $x \in \{c - 1, c - 2, \ldots, v_i + 1\}$ not lying between $c$ and $v_i$ in $w$ and some $y \in \{v_i - 1, v_i - 2, \ldots, a + 1\}$ not lying between $v_i$ and $a$ in $w$. Then the values $\{c, v_i, a, x, y\}$ form a pattern $p$ from $P^{321}$ in $w$, with $c, v_i, a$ corresponding to the values $5, 3, 1$ in $p$ respectively and $x, y$ corresponding to 4, 2. □

**Lemma 4.3.** If $w \in S_n$ is spherical then $w$ avoids the patterns from $P$.

**Proof.** We will apply Proposition 3.3 and prove the contrapositive: if $w$ contains a pattern $p$ from $P$, then $w$ is not Boolean, where $v = w_0(J(w))$.

Suppose first that $w$ contains a pattern $p$ from $P^{321}$ and that $w_i, w_j, w_k$ with $i < j < k$ correspond to the values $5, 3, 1$ in $p$ respectively. Since the 2 and 4 in the pattern $p$ do not lie
between \(w_j, w_k\) and \(w_i, w_j\) respectively, Proposition 4.1 implies that \(v(w_i) > v(w_j) > v(w_k)\) since \(v = v^{-1}\). Thus \(vw\) contains the pattern 321 and is not Boolean by Theorem 2.2.

Suppose now that \(w\) contains a pattern \(p\) from \(P^{3412}\). Let \(w_i, w_j, w_k, w_\ell\) with \(i < j < k < \ell\) correspond to the values \(\{1, 2, 4, 5\}\) from \(p\) (thus one of \(w_i, w_j\) corresponds to 4 and the other to 5, while one of \(w_k, w_\ell\) corresponds to 1 and the other 2). Since the 3 in the pattern \(p\) does not lie between the 2 and the 4, Proposition 4.1 implies that \(\min(v(w_i), v(w_j)) > \max(v(w_k), v(w_\ell))\). Thus either \(vw\) contains the pattern 3412 in these positions or contains a 321 pattern in some subset of them. In either case \(vw\) is not Boolean. \(\Box\)

Lemmas 4.2 and 4.3 together yield Theorem 1.4.

References

[1] Anders Björner and Francesco Brenti. Combinatorics of Coxeter groups, volume 231 of Graduate Texts in Mathematics. Springer, New York, 2005.
[2] David Brewster, Reuven Hodges, and Alexander Yong. Proper permutations, Schubert geometry, and randomness. 2020. arXiv:2012.09749 [math.CO].
[3] M. Brion, D. Luna, and Th. Vust. Espaces homogènes sphériques. Invent. Math., 84(3):617–632, 1986.
[4] C. De Concini and V. Lakshmibai. Arithmetic Cohen-Macaulayness and arithmetic normality for Schubert varieties. Amer. J. Math., 103(5):835–850, 1981.
[5] Charles Ehresmann. Sur la topologie de certains espaces homogènes. Ann. of Math. (2), 35(2):396–443, 1934.
[6] Yibo Gao, Reuven Hodges, and Alexander Yong. Classification of Levi-spherical Schubert varieties. (In preparation).
[7] Reuven Hodges and Alexander Yong. Coxeter combinatorics and spherical Schubert geometry. 2020. arXiv:2007.09238 [math.RT].
[8] Paramasamy Karuppuchamy. On Schubert varieties. Comm. Algebra, 41(4):1365–1368, 2013.
[9] V. Lakshmibai and B. Sandhya. Criterion for smoothness of Schubert varieties in \(SL(n)/B\). Proc. Indian Acad. Sci. Math. Sci., 100(1):45–52, 1990.
[10] D. Luna. Variétés sphériques de type \(A\). Publ. Math. Inst. Hautes Études Sci., (94):161–226, 2001.
[11] D. Luna and Th. Vust. Plongements d’espaces homogènes. Comment. Math. Helv., 58(2):186–245, 1983.
[12] Adam Marcus and Gábor Tardos. Excluded permutation matrices and the Stanley-Wilf conjecture. J. Combin. Theory Ser. A, 107(1):153–160, 2004.
[13] Nicolas Perrin. On the geometry of spherical varieties. Transform. Groups, 19(1):171–223, 2014.
[14] S. Ramanan and A. Ramanathan. Projective normality of flag varieties and Schubert varieties. Invent. Math., 79(2):217–224, 1985.
[15] Bridget Eileen Tenner. Pattern avoidance and the Bruhat order. J. Combin. Theory Ser. A, 114(5):888–905, 2007.
[16] Alexander Woo and Alexander Yong. When is a Schubert variety Gorenstein? Adv. Math., 207(1):205–220, 2006.

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