Hopf Bifurcation of Relative Periodic Solutions: Case Study of a Ring of Passively Mode-Locked Lasers

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Abstract
In this paper, we consider an equivariant Hopf bifurcation of relative periodic solutions from relative equilibria in systems of functional differential equations respecting $\Gamma \times S^1$-spatial symmetries. The existence of branches of relative periodic solutions together with their symmetric classification is established using the equivariant twisted $\Gamma \times S^1$-degree with one free parameter. As a case study, we consider a delay differential model of coupled identical passively mode-locked semiconductor lasers with the dihedral symmetry group $\Gamma = D_8$.

Keywords: Functional differential equation, Symmetric coupling, Equivariant Hopf bifurcation, Equivariant degree method, Passively mode-locked laser

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1 Introduction
A natural counterpart of an equilibrium state in dynamical systems with continuous symmetries is a relative equilibrium, i.e., an equilibrium modulo the group action. Similarly, a counterpart of a periodic solution is a relative periodic solution. In particular, in $S^1$-symmetric systems, the $S^1$-equivariant Hopf bifurcation is responsible for branching of relative periodic solutions from a relative equilibrium. This scenario is analogous to the classical Hopf bifurcation of periodic solutions from an equilibrium state in generic systems (without symmetry).

Relative equilibrium states and relative periodic motions are well-known for conservative systems related to rigid bodies [34], deformable bodies [35], molecular vibrations [36], celestial mechanics [6,37] and vortex theory [38] (see also [33,40] and references therein). In addition to $S^1$-symmetry, many of such systems respect a finite group $\Gamma$ of spatial symmetries such as, for example, a symmetry of coupling of atoms in molecules [39]. This naturally leads to the problem of classification of relative equilibria/periodic solutions according to their symmetric properties (spatio-temporal symmetries) represented by a subgroup $H$ of the group $\Gamma \times S^1$ for relative equilibria and the group $\Gamma \times S^1 \times S^1$ for relative periodic solutions, respectively, where the second copy of $S^1$ is associated with time periodicity. Examples include dynamics of a deformable body in an ideal irrotational fluid [33], symmetric celestial motions, for instance, central configurations [13,45], etc. On the other hand, there is a long list of applications described by non-conservative systems of ODEs admitting relative equilibria/relative periodic solutions (see, for example, [40], where the Couette-Taylor experiment is discussed in detail).

In his pioneering work [41] (cf. [42]), M. Krupa proposed a general method for analysis of the bifurcation of relative periodic solutions from a relative equilibrium for systems of ordinary differential equations (in general, non-Hamiltonian). This elegant method reduces the problem to the analysis of a generic (non-symmetric) Hopf bifurcation for an explicit differential equation on the normal slice to the relative equilibrium. An extension of Krupa’s method to the case of more complicated spatial symmetries (including $\Gamma \times S^1$) has been developed in [43] (see also [44] for the moving frames method and [54] for the hierarchy of secondary bifurcations). Essentially, the analysis of the Hopf bifurcation includes two main problems: (i) finding the bifurcation points and establishing the occurrence of the bifurcation (in equivariant setting, this problem

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additionally requires to describe symmetric properties of the bifurcating solutions), and (ii) analysis of stability properties of the bifurcating solutions. Krupa’s method allows one to solve both problems for $S^1$-equivariant (or $\Gamma \times S^1$-equivariant, if an additional group $\Gamma$ of spatial symmetries is involved) ordinary differential systems using the center manifold reduction and the analysis of the normal forms of the bifurcation for the system on the normal slice.

The applicability of Krupa’s method to $S^1$-symmetric functional differential equations (FDEs) appears to be problematic. In particular, it is unclear whether it is possible to reduce the problem in question to the analysis of a non-equivariant FDE on some kind of a normal slice. However, if the problem is limited to establishing the occurrence of the bifurcation (problem (i) above), then alternative methods are available. In the present paper, we address this more modest problem. In this case, one can reduce the analysis to studying a fixed point problem for a $\Gamma \times S^1 \times S^1$-equivariant operator equation in a functional space of periodic functions. For the latter problem, one can adapt either equivariant Lyapunov-Schmidt reduction techniques or $S^1$-equivariant topological methods (which is the case in this paper).

More specifically, we obtain conditions for the occurrence of the Hopf bifurcation of relative periodic solutions (together with their complete symmetric classification) from a relative equilibrium in general $\Gamma \times S^1$-equivariant systems of FDEs using the method based on twisted equivariant degree with one free parameter. For a systematic exposition of this method, we refer to [1–4, 12, 46, 47]. As is well-known, this method is insensitive to violations of genericity assumptions [7, 48] (these assumptions include the simplicity of purely imaginary eigenvalues at the bifurcation point, transversality of the eigenvalue crossing, and non-resonance conditions). Our results are formulated for a general FDE system, which respects a group $\Gamma \times S^1$ of spatial symmetries with an arbitrary finite group $\Gamma$, and include the method of classification of symmetries of the relative periodic solutions based on the linearization of the problem. In the second part of the paper, these results are applied for the analysis of delay differential equations (DDE) describing a system of 8 mode-locked lasers coupled in $D_8$-symmetric fashion. The rate equations of semiconductor lasers are $S^1$-symmetric (see, for example, [49–51]) and can naturally include delays, the classical example being the Lang-Kabayashi model where the light is re-injected into the laser cavity by an external mirror [52, 53].

Semiconductor mode locked lasers are compact reliable low-cost devices that emit short optical pulses at high repetition rates suitable for applications in telecommunications, machining, probing, all optical systems, etc. [15]. We use a delay differential model of a passively mode-locked laser proposed in [16]. This delay differential system has been extensively applied to analyze instabilities [17–20] and hysteresis [21, 22] in mode-locked lasers, optically injected lasers [23, 24], hybrid mode locking [25], noise reduction [26], resonance to delayed feedback [27], and Fourier domain mode locking [28]. An increasing interest in small systems of symmetrically coupled lasers and large synchronized laser arrays motivates the analysis of the corresponding variants of the model [29–32].

In the mode-locking regime, the laser emits a periodic sequence of light pulses with the period close to the cold cavity round trip time, which equals to the delay in the model. A typical bifurcation scenario associated with formation of this regime is the Hopf bifurcation of a relative equilibrium (continuous wave solution) from the “laser off” equilibrium followed by the $S^1$-equivariant Hopf bifurcation of a relative periodic solution from the relative equilibrium with the increase of the bifurcation parameter (pump current). As the bifurcation parameter increases further, the relative periodic solution continuously transforms to acquire a pulsating shape. This transformation is simultaneous with a sequence of secondary Hopf bifurcations from the equilibrium and relative equilibrium solutions.

The paper is organized as follows. In the next short section, some equivariant jargon and notation are introduced. In Section 3, we first classify symmetries of branches of relative equilibria, which bifurcate from a $\Gamma \times S^1$-fixed equilibrium of an equivariant FDE. Then, the main theorem on classification of symmetries of relative periodic solutions bifurcating from the branches of relative equilibria (Theorem 3.10) is presented and proved. In Section 4, the results of Section 3 are applied to delay rate equations of the $D_8 \times S^1$-symmetric laser system. We prove the occurrence of infinitely many branches of relative equilibria with various symmetries from the laser off state. Then the analytic method is combined with numerical computations to analyze symmetric properties of relative periodic solutions that branch from the relative equilibrium states. A short appendix (Section 5) lists a few twisted subgroups, which are used in Section 4 to describe symmetries of solutions.
2 Equivariant Jargon

Consider a compact Lie group $G$. In what follows we will always assume that all considered in this paper subgroups $H \subset G$ are closed and denote by $N(H)$ the normalizer of $H$ in $G$, by $W(H) = N(H)/H$ the Weyl group of $H$ in $G$, and by $(H)$ the conjugacy class of $H$ in $G$. We will denote by $\Phi(G)$ the set of all conjugacy classes of subgroups in $G$. Clearly, $\Phi(G)$ admits a partial order defined by:

$$(H) \leq (K) \iff \exists_{g \in G} \ gHg^{-1} \subset K.$$ 

We also put $\Phi_n(G) := \{(H) \in \Phi(G) : \dim W(H) = n\}$.

Let $X$ be a $G$-space and $x \in X$. We denote by $G_x := \{g \in G : gx = x\}$ the isotropy (or stabilizer) of $x$, by $G(x) := \{gx : g \in G\} \cong G/G_x$ the orbit of $x$, and by $X/G$ the orbit space of $X$. The conjugacy class $(G_x)$ will be called the orbit type of $x$. We will also adopt the following notation:

$$\Phi(G; X) := \{(H) \in \Phi(G) : H = G_x \text{ for some } x \in X\},$$

$$\Phi_n(G; X) := \Phi_n(G) \cap \Phi(G; X),$$

$$X_H := \{x \in X : G_x = H\},$$

$$X^H := \{x \in X : G_x \supset H\},$$

$$X_{(H)} := \{x \in X : (G_x) = (H)\},$$

$$X^{(H)} := \{x \in X : (G_x) \geq (H)\}.$$ 

One can easily verify that $W(H)$ acts on $X^H$ and this action is free on $X_H$.

For two $G$-spaces $X$ and $Y$, a continuous map $f : X \to Y$ is said to be equivariant if $f(gx) = gf(x)$ for all $x \in X$ and $g \in G$. If the $G$-action on $Y$ is trivial, then $f$ is called invariant. Clearly, for any subgroup $H \subset G$ and equivariant map $f : X \to Y$, the map $f^H : X^H \to Y^H$, with $f^H := f|_{X^H}$, is well-defined and $W(H)$-equivariant.

Finally, given two orthogonal $G$-representations $W$ and $V$ and an open bounded subset $\Omega \subset W$, a continuous map $f : W \to V$ is called $\Omega$-admissible if $f(x) \neq 0$ for all $x \in \partial \Omega$ (in such a case we will also say that $(f,\Omega)$ is an admissible pair). Similarly, two $\Omega$-admissible maps $f$ and $g$ are called $\Omega$-admissible homotopic if there exists a continuous map $h : [0,1] \times W \to V$ (called $\Omega$-admissible homotopy between $f$ and $g$) such that: (i) $h(0,x) = f(x), h(1,x) = g(x)$ for all $x \in W$, and (ii) the map $h_t(x) := h(t,x)$ is $\Omega$-admissible for all $t \in [0,1]$. We denote by $\Lambda(W,V)$ the set of all admissible pairs $(f,\Omega)$.

For further details of the equivariant jargon used in this paper, we refer to [3, 5, 10, 11].

3 $\Gamma \times S^1$ -Symmetric Systems of FDEs

3.1 Notation and statement of the problem

Assume that $\Gamma$ is a finite group and let $V := \mathbb{R}^n$ be an orthogonal $\Gamma \times S^1$-representation such that the $S^1$-action on $V$ is given by the homomorphism $T : S^1 \to O(n)$. Assume that $J$ is the infinitesimal operator of the subgroup $T(S^1) \subset O(n)$, i.e.

$$J = \lim_{\tau \to 0} \frac{1}{\tau} [T(e^{i\tau}) - \text{Id}].$$

The action of $S^1$ on $V$ satisfies for all $e^{i\tau} \in S^1$

$$\forall_{v \in V} \ e^{i\tau}v = e^{i\tau}J(v),$$

and we also have $Je^{i\tau} = e^{i\tau}J$.

We will denote $\mathcal{G} := \Gamma \times S^1$ and use the notation

$$\Gamma := \Gamma \times \{1\} \quad \text{and} \quad S := \{e\} \times S^1,$$

where $e \in \Gamma$ is the neutral element.
Let
\[ V = V_0 \oplus V_1 \oplus \cdots \oplus V_m \] (1)
be the \( S \)-isotypical decomposition of \( V \), where \( V_k \) is modeled on the \( S^1 \)-irreducible representation \( V_k \cong \mathbb{C} \) with the \( S^1 \)-action given by \( e^{i\tau}z := e^{ik\tau} \cdot z \), where \( \cdot \) stands for the complex multiplication. Then, each of the components \( V_k \), \( k > 0 \), has a natural complex structure such that for \( v \in V_k \)
\[ e^{i\tau}v = e^{ik\tau} \cdot v, \quad Jv = ik \cdot v. \]
Also, for \( v \in V_0 \), we have \( Jv = 0 \).

Let \( r > 0 \) and denote by \( C_{-r}(V) \) the Banach space
\[ C([-r,0]; V) := \{ x : \text{where } x : [-r,0] \to V \text{ is a continuous function} \}, \]
equipped with the norm \( \|x\|_{\infty} := \sup\{|x(\theta)| : \theta \in [-r,0]\} \). Clearly, \( C_{-r}(V) \) is an isometric \( \Gamma \times S^1 \)-representation, with the action given by
\[ (\gamma,e^{i\tau})x(\theta) = e^{rJ}(\gamma x(\theta)), \quad x \in C_{-r}(V), \quad (\gamma,e^{i\tau}) \in \Gamma \times S^1. \]
In addition, we have the following \( S \)-isotypical decomposition of \( C_{-r}(V) \):
\[ C_{-r}(V) = \bigoplus_{k=0}^{m} C_{-r}(V_k), \]
where each of the components \( C_{-r}(V_k) \) with \( k > 0 \) has a natural complex structure induced from \( V_k \).

For a continuous function \( x : \mathbb{R} \to V \) and \( t \in \mathbb{R} \), let \( x_t : [-r,0] \to V \) be a function defined by
\[ x_t(\theta) := x(t + \theta), \quad \theta \in [-r,0]. \]

We make the following assumption:
(A0) \( f : \mathbb{R} \times C_{-r}(V) \to V \) is a \( C^1 \)-differentiable \( G \)-equivariant functional, i.e. for \( (\gamma,e^{i\tau}) \in G \)
\[ f(\alpha,(\gamma,e^{i\tau})x) = e^{rJ}(\gamma f(\alpha,x)) \quad \text{for all } x \in C_{-r}(V). \] (2)

### 3.2 Symmetric bifurcation of relative equilibria from an equilibrium

Consider the parametrized system of FDEs
\[ \dot{x}(t) = f(\alpha,x_t), \quad x(t) \in V. \] (3)

In what follows, we will study bifurcations of continuous branches of periodic/quasi-periodic solutions to (3) of special type and describe their symmetric properties.

In this subsection, we are interested in periodic solutions to (3) of the type
\[ x(t) = e^{wJt}x \quad \text{for some } x \in V \text{ and } w \in \mathbb{R}. \] (4)

**Relative equilibria.** By substituting (4) into equation (3), we obtain
\[ we^{wtJ}Jx(t) = f(\alpha,e^{w(t+\cdot)J}x). \] (5)

Then, using the equivariance condition (2), we can rewrite (5) as
\[ wJx = f(\alpha,e^{wJ}x), \quad x \in V. \] (6)

Take the orthogonal \( S \)-invariant decomposition (1) of the space \( V \), where \( V_0 = V^S \), and denote
\[ V_s := V_0^\perp = V_1 \oplus \cdots \oplus V_m. \]
For a fixed \( \lambda = u + iw \in \mathbb{C} \), define the linear operator \( \xi(\lambda) : V \rightarrow C_{-r}(V) \) by

\[
(\xi(\lambda)x)(\theta) = e^{u\theta} e^{wJ\theta}x_0 + x_0, \quad \theta \in [-r, 0],
\]

where \( x = x_0 + x_\ast \), \( x_\ast \in V_\ast \) and \( x_0 \in V_0 \), and consider the function \( \tilde{f} : \mathbb{R} \times \mathbb{C} \times V \rightarrow V \) defined by

\[
\tilde{f}(\alpha, \lambda, x) := f(\alpha, \xi(\lambda)x).
\]

With this notation, equation \( (6) \) can be written as

\[
wJx = \tilde{f}(\alpha, iw, x), \quad x \in V.
\]

Furthermore, assumption (A0) implies \( G \)-equivariance of \( \tilde{f} \):

\[
\tilde{f}(\alpha, \lambda, (\gamma, e^{\tau i})x) = e^{rJ\tau} \tilde{f}(\alpha, \lambda, x) \quad \text{for all } x \in V, \lambda \in \mathbb{C}.
\]

Hence, solutions to \( (9) \) come in \( S \)-orbits. It is clear that any \( S \)-orbit, which is a solution to \( (9) \), is flow invariant for \( (3) \) and therefore, satisfies the standard definition of relative equilibrium (see, for example, [8]). In what follows, we refine this concept to the setting relevant to our discussion.

Assume that \( x \in V^S = V_0 \). Then, one has \( \tilde{f}(\alpha, iw, x) = \tilde{f}(\alpha, 0, x) \) for all \( w \). Hence, any solution \( x \in V^S \) of \( (9) \) is an equilibrium for equation \( (3) \) with \( S(x) = \{x\} \). On the other hand, solutions of \( (9) \) that satisfy \( x \notin V^S \) form one-dimensional orbits.

**Definition 3.1.** Suppose that \( (9) \) holds for some \( \alpha_0, w_0 \in \mathbb{R} \) and \( x_0 \notin V^S \). Then, the orbit \( S(x_0) \) is a one-dimensional curve in \( V \) called a relative equilibrium of equation \( (3) \).

(i) For \( w_0 \neq 0 \), this orbit is a trajectory of time-periodic solutions \( x(\cdot) = e^{(w_0 + \tau)J}x_0, e^{ir} \in S^1 \), to equation \( (3) \) called a rotating wave.

(ii) For \( w_0 = 0 \), the relative equilibrium consists of equilibrium points \( e^{rJ}x_0, e^{ir} \in S^1 \), of \( (3) \) (the so-called frozen wave).

**Characteristic quasi-polynomials.** Let \( \alpha_0 \in \mathbb{R} \) be given and let \( x_0 \in V^G \) be an equilibrium for \( (3) \). We will also call the pair \( (\alpha_0, x_0) \) an equilibrium, or a stationary solution, in this case.

Let us consider the bifurcation of relative equilibria from this equilibrium. Let

\[
D_x f(\alpha, x) : \mathbb{R} \times C_{-r}(V) \rightarrow V
\]

(10)

 denote the derivative of the functional \( f \) with respect to \( x \in C_{-r}(V) \). If \( x_0 \in V_0 \), then the Jacobi matrix \( D_x \tilde{f}(\alpha, \lambda, x_0) : V \rightarrow V \), which is given by

\[
D_x \tilde{f}(\alpha, \lambda, x_0) = D_x f(\alpha, \xi(\lambda)x_0)\xi(\lambda)
\]

(11)

is \( S \)-equivariant. Therefore, the subspaces \( V_0 \) and \( V_\ast \) are \( S \)-invariant for this matrix. Consider the restrictions \( D_x \tilde{f}(\alpha, \lambda, x_0)|_{V_0} \) and \( D_x \tilde{f}(\alpha, \lambda, x_0)|_{V_\ast} \) and define the characteristic quasi-polynomials for \( x_0 \in V_0 \) and \( \lambda \in \mathbb{C} \):

\[
P_0(\alpha, \lambda, x_0) := \det_C \left( D_x \tilde{f}(\alpha, \lambda, x_0)|_{V_0} - \lambda \text{Id} \right),
\]

\[
P_\ast(\alpha, \lambda, x_0) := \det_C \left( D_x \tilde{f}(\alpha, \lambda, x_0)|_{V_\ast} - \lambda \text{Id} \right).
\]

(12)

We make the following assumption.

(A1) At the equilibrium point \( (\alpha_0, x_0) \), where \( x_0 \in V^G \), the characteristic quasi-polynomial \( P_0(\alpha_0, \cdot, x_0) \) has no zero roots, i.e. \( P_0(\alpha_0, 0, x_0) \neq 0 \).
By assumption (A1) and implicit function theorem, there exists a $C^1$-function $x : (\alpha_0 - \varepsilon, \alpha_0 + \varepsilon) \to V_0$ (for some $\varepsilon > 0$) such that $x(\alpha_0) = x_o$ and $\{(\alpha, x(\alpha)) : \alpha \in (\alpha_0 - \varepsilon, \alpha_0 + \varepsilon)\}$ is a curve of equilibrium points for (3). By equivariance of $f$, and due to $x_o \in V^G$, it follows that $x(\alpha) \in V^G$, consequently the 2-manifold $M \subset \mathbb{R}^2 \times V^G$ given by

$$M := \{(\alpha, w, x(\alpha)) : \alpha \in (\alpha_0 - \varepsilon, \alpha_0 + \varepsilon), w \in \mathbb{R}\}$$

is composed of solutions to (9), which can be called trivial.

The roots of the the quasi-polynomials are called the characteristic roots for $(\alpha_o, x_o)$. Assume that:

(A2) The quasi-polynomial $P_*(\alpha_o, \cdot, x_o)$ has a characteristic root $\lambda = iw_o$ for some $w_o \in \mathbb{R}$ at the equilibrium point $(\alpha_o, x_o)$, but for any other equilibrium $(\alpha, x)$ from a neighborhood of $(\alpha_o, x_o)$ in $\mathbb{R} \times V_0$, the corresponding characteristic polynomial has no roots of the form $\lambda = iw$, $w \in \mathbb{R}$.

In order to find nontrivial solutions to (9) bifurcating from $M$, consider the equation

$$\Phi(\alpha, w, x) := f(\alpha, iw, x) - wJx = 0$$

as a $G$-symmetric bifurcation problem with two free parameters $\alpha$ and $w$.

By applying the standard terminology (see [3]), if $D_x f(\alpha, iw, x) - wJ : V \to V$ is not an isomorphism, we call the point $(\alpha, w, x) \in V$ an $M$-singular point of $\Phi$. By the implicit function theorem, a necessary condition for a point $(\alpha', w', x') \in M$ to be a bifurcation point for equation (9) is that it is an $M$-singular point. Assumption (A2) implies that the point $(\alpha_o, w_o, x_o)$ satisfies this necessary condition.

**Remark 3.2.** Due to equivariance of $f$, the linearization of the delayed system (3) at the equilibrium point $(\alpha, x(\alpha))$ has two flow invariant subspaces $C_{-r}(V_0)$ and $C_{-r}(V_s)$. Assumption (A1) means that $\lambda = 0$ is not an eigenvalue for the restriction of the linearization to $C_{-r}(V_0)$. Assumption (A2) means that the restriction of the linearization to $C_{-r}(V_s)$ has a pair of eigenvalues $\lambda = \pm iw_o$ for $\alpha = \alpha_o$ and has no eigenvalues of the form $iw$, $w \in \mathbb{R}$, for $\alpha \neq \alpha_o$ sufficiently close to $\alpha_o$.

**Sufficient condition for bifurcation of relative equilibria.** In order to provide a sufficient condition for the bifurcation of relative equilibria from the point $(\alpha_o, w_o, x_o)$ and an equivariant topological classification of the bifurcating branches, we apply the twisted $G$-equivariant degree with one free parameter (for more details, see [3]). To be more precise, consider the $G$-isotypical decomposition of $V$ (see (1)):

$$V = V_0 \oplus V_s = \bigoplus_i V_i^0 \oplus \bigoplus_{j,k} V_{j,k} \quad (i = 0, \ldots, r; \ j = 0, \ldots, s; \ k = 1, \ldots, m),$$

where $V_{j,k}$ is the isotropic component modeled on the irreducible $G$-representation $V_{j,k}$ and $V_i^0$ can be identified with the $G$-representation modeled on an irreducible $G$-representation $V_i$.

**Remark 3.3.** Let $(H_o)$ be a maximal twisted orbit type in $V$. Then, $(H_o)$ is also a maximal twisted orbit type for some $V_{j,k_o}$ in (15), $k_o > 0$. In fact, if $U$ is a direct sum of two $G$-representations $U_1$ and $U_2$, then $G(x,y) = G_x \cap G_y$ for any $(x, y) \in U$, $x \in U_1$, $y \in U_2$.

For any $j = 0, \ldots, s$ and $k = 1, \ldots, m$, put

$$P_{j,k}(\alpha, \lambda) := \text{det}_C \left( D_x f(\alpha, \lambda, x(\alpha))|_{V_{j,k}} - \lambda \text{Id} \right), \quad \lambda \in \mathbb{C}. $$

Notice that the characteristic equation at $(\alpha, x(\alpha))$ can be written as

$$P_*(\alpha, \lambda) := P_*(\alpha, \lambda, x(\alpha)) = \prod_{\lambda > 0} \prod_{j=0}^s P_{j,k}(\alpha, \lambda) = 0. $$

This implies that $\lambda$ is a characteristic root for $(\alpha, x(\alpha))$ if it is a root of $P_{j,k}(\alpha, \lambda) = 0$ for some $k > 0$ and $j \geq 0$.

To formulate our first bifurcation result, we need two additional concepts. Observe that using (A2), one can choose a small neighborhood $Q$ of the point $iw_o$ in the right half-plane $\text{Re}\lambda > 0$ of $C$ and a sufficiently small real $\delta = \delta(Q) > 0$ such that, as $\alpha$ varies over the interval $|\alpha - \alpha_o| \leq \delta$, the roots $\lambda(\alpha)$ of $P_{j,k}(\alpha, \cdot)$ can only leave $Q$ through the ‘exit’ at the point $iw_o$ and only when $\alpha = \alpha_o$.  


Definition 3.4. Define the $V_{j,k}$-isotypical crossing number at $(\alpha_0, w_0)$ by the formula

$$t_{j,k}(\alpha_0, w_0) := t_{j,k}^-/(\alpha_0, w_0) - t_{j,k}^+(\alpha_0, w_0),$$

where $t_{j,k}^\pm/(\alpha_0, w_0)$ is the number of roots $\lambda(\alpha)$ of $P_{j,k}(\alpha, \cdot)$ (counted according to their $V_{j,k}$-isotypical multiplicity) in the set $Q$ for $\alpha < \alpha_0$, and $t_{j,k}^\pm/(\alpha_0, w_0)$ is the number of roots of $P_{j,k}(\alpha, \cdot)$ in $Q$ for $\alpha > \alpha_0$.

Definition 3.5. A set $K$ of solutions $(\alpha, w, x)$ to equation (9) is called a continuous branch of relative equilibria bifurcating from the equilibrium $(\alpha_0, x_0)$ of equation (3) if:

(i) $x \not\in V^S$ for all $(\alpha, w, x) \in K$;

(ii) $\overline{\mathcal{X}}$ contains a connected component $K_o$ such that $K_o \cap M \neq \emptyset$ (cf. (13));

(iii) For any $\varepsilon > 0$ there is a $\delta > 0$ such that if $(\alpha, w, x) \in K \cap K_o$ and $\|x\| < \delta$, then $|\alpha - \alpha_0| < \varepsilon$ and $|w - w_0| < \varepsilon$.

A sufficient condition for the bifurcation of relative equilibria from the equilibrium $(\alpha_0, x_0)$, which provides an estimate for the number of possible branches of relative equilibria with their symmetric properties, can be formulated as follows.

Proposition 3.6. Given system (3), assume conditions (A0)-(A2) are satisfied. Let $(\mathcal{H}_0)$ be a maximal twisted orbit type in $V$. Take decomposition (15) and denote by $\mathcal{M}$ the set of all $G$-isotypical components in which $(\mathcal{H}_0)$ is an orbit type (cf. Remark 3.3). Assume there exists $V_{j,k,0} \in \mathcal{M}$ such that:

(i) $(\mathcal{H}_0)$ is a maximal twisted type in $V_{j,k,0}$;

(ii) $t_{j,k,0}(\alpha_0, w_0) \neq 0$ (cf. (17));

(iii) $t_{j,k}(\alpha_0, w_0) \cdot t_{j,k,0}(\alpha_0, w_0) \geq 0$ for all $V_{j,k,0}, V_{j,k} \in \mathcal{M}$.

Then, there exists at least $|G/\mathcal{H}_0|_S$ continuous branches of relative equilibria of equation (3) bifurcating from the equilibrium $(\alpha_0, x_0)$ with the minimal symmetry $(\mathcal{H}_0)$ (here $| \cdot |_S$ stands for the number of $S$-orbits).

The proof literally follows the argument presented in [3]. For completeness, here we give a brief sketch of the proof. Under extra transversality/genericity conditions, this statement is well-known, see for example [8, 9].

Sketch of the proof of Proposition 3.6. The proof splits into three steps.

(a) Auxiliary function and admissibility. Take $\mathbb{R}^2$ with the trivial $G$-action and define a $G$-invariant neighborhood of the point $(\alpha_0, w_0, x_0)$ in $\mathbb{R}^2 \oplus V$ of the form

$$\Omega := \{(\alpha, w, x) : |\alpha - \alpha_0| < \varepsilon, |w - w_0| < \varepsilon, \|x - x_0\| < \varepsilon, \|x\| < \delta\},$$

where $(\alpha, w, x) = (\alpha, w, \overline{x} + x_0)$, $(\alpha, w, \overline{x}) \in M$, $x_0 \perp \overline{x}$, and a $G$-invariant continuous function $\zeta : \overline{\Omega} \to \mathbb{R}$ satisfies

$$\zeta(\alpha, w, x) < 0 \text{ if } (\alpha, w, x) \in M \cap \overline{\Omega},$$

$$\zeta(\alpha, w, x) > 0 \text{ if } (\alpha, w, x) \in \overline{\Omega} \text{ and } \|x\| = \delta$$

(recall that $\zeta$ is called an auxiliary function). By condition (A0), the map $\Phi_\zeta : \mathbb{R}^2 \times V \to \mathbb{R} \times V$ defined by

$$\Phi_\zeta(\alpha, w, x) = (\zeta(\alpha, w, x), \Phi(\alpha, w, x)),$$

is $G$-equivariant (cf. (14)). Moreover, condition (A2) allows us to choose the parameters $\varepsilon, \delta > 0$ of the set $\Omega$ to be sufficiently small to ensure that the map $\Phi_\zeta$ is $\Omega$-admissible (i.e., $\Phi_\zeta$ does not have zeroes on $\partial\Omega$).

(b) Twisted degree and a sufficient condition for the bifurcation of relative equilibria. Since $\Phi_\zeta$ is $G$-equivariant and $\Omega$-admissible, the twisted degree

$$G\text{-deg}(\Phi_\zeta, \Omega) = \sum_{(\mathcal{H})} n_\mathcal{H}(\mathcal{H})$$

is correctly defined (here, $n_\mathcal{H} \in \mathbb{Z}$ and the summation is going over all twisted orbit types occurring in $V$). The following statement is parallel to Theorem 9.28 from [3].
Proposition 3.7. Given (18), assume that $n_{\mathcal{H}_o} \neq 0$ for some maximal twisted orbit type $(\mathcal{H}_o)$ in $V$. Then, the conclusion of Proposition 3.6 holds.

(c) Twisted degree and crossing numbers. To effectively apply Proposition 3.7 for proving Proposition 3.6, one needs to link the twisted degree (18) to (isotypical) crossing numbers (cf. Definition 3.4). To this end, one can use the following standard computational formula (cf. [3] and (15)):

$$
\mathcal{G}\text{-deg}(\Phi_{\zeta}, \Omega) = \prod_{\mu \in \sigma_{-}} \prod_{i=0}^{r} \left( \deg_{V_i} \right)^{m_i(\mu)} \sum_{j,k} t_{j,k}(\alpha_o, w_o) \deg_{V_j,k},
$$

where $j = 0, 1, \ldots, s$, $k = 1, \ldots, m$; $\sigma_{-}$ denotes the set of all (real) negative roots $\mu$ of the quasi-polynomial $P(\alpha_o, \lambda)$ at $x_o$; $m_i(\mu)$ stands for the $V_i$-isotypical multiplicity of $\mu$; $\deg_{V_i}$ (resp. $\deg_{V_j,k}$) denote the so-called basic degrees related to irreducible $\Gamma$-representations (resp. $\mathcal{G}$-representations); and, “$\bullet$” stands for the multiplication in the Euler ring $U(\mathcal{G})$ (cf. [10]; see also Appendix for more details). Take $(\mathcal{H}_o)$ and $V_{j_o,k_o}$ satisfying (i)–(iii). Then (see conditions (i)–(ii)), $(\mathcal{H}_o)$ appears with non-zero coefficient in $t_{j_o,k_o}(\alpha_o, w_o) \deg_{V_{j_o,k_o}}$. Condition (iii) implies that $(\mathcal{H}_o)$ “survives” in the sum given in the right hand side of (19). Finally, since any $\deg_{V_i}$ is an invertible element of the Burnside ring $A(\mathcal{G}) \subset U(\mathcal{G})$, the result follows.

3.3 Hopf bifurcation from a relative equilibrium

Regular relative equilibria. Suppose that for some $\alpha = \pi$ and $\pi \in V \setminus V^S$, equation (3) has a relative equilibrium $S(\pi)$ (see Definition 3.1). Then, equation (9) is satisfied for some $\pi \in \mathbb{R}$,

$$
\Phi(\pi, \bar{\pi}, \pi) = \bar{f}(\pi, \bar{\pi}, \pi) - \bar{\pi}/J = 0,
$$

where $\pi = (\pi, \bar{\pi}) \in V^S \oplus V^\ast$ with $\pi \neq 0$. Since

$$
\frac{d}{d\tau} \Phi(\pi, \bar{\pi}, e^{\tau J}) \bigg|_{\tau=0} = D_x \Phi(\pi, \bar{\pi}, e^{\tau J}) J e^{\tau J} \bigg|_{\tau=0} = D_x \Phi(\pi, \bar{\pi}, \pi) J \pi,
$$

relation (20) and the $S$-equivariance of $\Phi$ imply that the directional derivative of $\Phi$ at the point $\pi$ in the direction of the orbit $S(\pi)$ is zero:

$$
D_x \Phi(\pi, \bar{\pi}, \pi) J \pi = 0.
$$

That is, the map $D_x \Phi(\pi, \bar{\pi}, \pi)$ has a non-zero kernel.

Definition 3.8. A relative equilibrium $S(\pi)$ will be called regular if the kernel of the map given by the following block-matrix

$$
\begin{bmatrix}
D_w \Phi(\pi, \bar{\pi}, \pi) & D_x \Phi(\pi, \bar{\pi}, \pi)
\end{bmatrix} : \mathbb{R} \times V \to V
$$

is one-dimensional.

We make the following assumption:

(A3) Equation (3) has a regular relative equilibrium $S(\pi)$ for some $\alpha = \pi$, $w = \bar{\pi}$.

The implicit function theorem implies that there exist a neighborhood $\mathcal{W}$ of $\pi$ in $\mathbb{R}$ and the functions $w : \mathcal{W} \to \mathbb{R}$, $w(\pi) = \bar{\pi}$, and $u : \mathcal{W} \to S := \{x \in V : x \cdot J \pi = 0\}$, $u(\pi) = 0$, such that

$$
\Phi(\alpha, w(\alpha), \pi + u(\alpha)) = 0, \quad \alpha \in \mathcal{W}.
$$

Put $x(\alpha) = \pi + u(\alpha)$, then we clearly have that

$$
x_{\alpha}(t) := e^{(w(\alpha)t + \tau)J} x(\alpha)
$$

is a branch of relative equilibria parametrized by $\alpha \in \mathcal{W}$. It will be assumed that this branch has symmetric properties (cf. Proposition 3.6).
(A4) The regular relative equilibrium $S(\pi)$ admits a twisted group symmetry $\mathcal{H} \leq \mathcal{G}$.

Due to the equivariance, the twisted symmetry group $\mathcal{H}$ is the same for all relative equilibria $S(x(\alpha))$, $\alpha \in \mathcal{V}$, of the branch (24). We note that the relation

$$D_x \Phi(\alpha, w(\alpha), x(\alpha)) J x(\alpha) = 0,$$

which is similar to (21), holds, and the relative equilibria $S(x(\alpha))$ are regular for all $\alpha \in \mathcal{V}$.

**Hopf bifurcation of relative periodic solutions.** We are interested in finding solutions to (3) of the form

$$x(t) = e^{(w(\alpha) + \phi)t} (x(\alpha) + y(t)),$$

where $y(t)$ is a non-stationary $p$-periodic function with $p = 2\pi/\beta$ for some $\beta > 0$, and in symmetric properties of these solutions. Here $y(t)$ and $\phi, \beta \in \mathbb{R}$. Periodic and quasi-periodic solutions of type (26) are called relative periodic solutions.

To be more precise, let us define the so-called equivariant Hopf bifurcation of small amplitude relative periodic solutions of type (26) from the family of relative equilibria (24).

The following definition is similar to Definition 3.5.

**Definition 3.9.** A set $K$ of quadruplets $(\alpha, \beta, \phi, x)$, where $x$ is a solution to equation (3) of the form (26), is called a continuous branch of relative periodic solutions bifurcating (via the equivariant Hopf bifurcation) from the relative equilibrium $(\pi, \pi, S(\pi))$ if there exists a $\beta_0 > 0$ such that:

(i) $\overline{K}$ contains a connected component $K_0$ such that $(\pi, \beta_0, 0, \pi) \in K_0$;

(ii) For any $\varepsilon > 0$ there is a $\delta > 0$ such that if $(\alpha, \beta, \phi, x) \in K \cap K_0$ and $\|y\| < \delta$, then $|\alpha - \pi| < \varepsilon$, $|\phi| < \delta$, and $|\beta - \beta_0| < \varepsilon$.

For a given $\alpha \in \mathcal{V}$, consider the derivative $D_x f(\alpha, x) : C_{-r}(V) \to V$ of the functional $f$ with respect to $x \in C_{-r}(V)$ and put

$$B_\alpha := D_x f(\alpha, e^{w(\alpha)J} x(\alpha)) = D_x f(\alpha, \xi(iw(\alpha)) x(\alpha)) : C_{-r}(V) \to V.$$

For $\alpha \in \mathcal{V}$ and $\lambda \in \mathbb{C}$, define the linear map $R_\alpha : V^c \to V^c$ in the complexification $V^c$ of $V$ by the formula

$$R_\alpha(\lambda)y := B_\alpha(e^{(w(\alpha)J + \lambda \text{Id})}y), \quad y \in V^c.$$  

Then,

$$\det_{\mathbb{C}} (R_\alpha(\lambda) - w(\alpha)J - \lambda \text{Id}) = 0$$

is the characteristic equation for the linearization of system (3) on the relative equilibrium $S(x(\alpha))$. Since

$$R_\alpha(0) - w(\alpha)J = D_x \Phi(\alpha, w(\alpha), x(\alpha)),$$

equation (25) implies that the characteristic equation (29) has a zero root $\lambda = 0$ corresponding to the eigenvector $J x(\alpha)$; furthermore, due to the regularity of the relative equilibrium $S(x(\alpha))$, this root is simple.

A necessary condition for the Hopf bifurcation is that characteristic equation (29) has a pair of purely imaginary roots $\lambda = \pm i\beta_0$, $\beta_0 > 0$, for $\alpha = \pi$. We make a stronger assumption:

(A5) Characteristic equation (29) has a pair of purely imaginary roots $\lambda = \pm i\beta_0$, $\beta_0 > 0$, for $\alpha = \pi$, and has no roots of the form $\lambda = i\beta$, $\beta \geq 0$, for $\alpha \neq \pi$, $\alpha \in \mathcal{V}$.

Put $\mathcal{K} := \mathcal{H} \times S^1$ and consider the $\mathcal{K}$-isotypical decomposition of $V^c$:

$$V^c = U_{0,1} \oplus U_{1,1} \oplus \cdots \oplus U_{p,1},$$

where $S^1$-action is given by complex multiplication. Due to the equivariance, each isotypical component $U_{j,1}$ is invariant for the map $R_\alpha(\lambda)$ and for $J$. Therefore, we can introduce the characteristic polynomial

$$\overline{\mathcal{P}}_{j,1}(\alpha, \lambda) := \det_{\mathbb{C}} \left((R_\alpha(\lambda) - w(\alpha)J - \lambda \text{Id})|_{U_{j,1}}\right), \quad \lambda \in \mathbb{C},$$

(32)
associated with each isotypical component \(U_{j,1}\), and define the \(U_{j,1}\)-isotypical crossing numbers
\[
\mathfrak{I}_{j,1}(\alpha, \beta) = \mathfrak{I}_{j,1}^{-1}(\alpha, \beta) - \mathfrak{I}_{j,1}^{+1}(\alpha, \beta)
\] (33)

at the point \((\alpha, \beta)\) in the same way as we did in Subsection 3.2 (cf. (17)).

**Theorem 3.10.** Given system (3), assume conditions (A0) and (A3)-(A5) are satisfied. Take decomposition (31) and let \((L_0)\) be a maximal twisted orbit type in \(V^c\). Denote by \(\mathfrak{R}\) the set of all \(K\)-isotypical components in (31) in which \((L_0)\) is an orbit type. Assume there exists \(U_{j,1} \in \mathfrak{R}\) such that:

(i) \((L_0)\) is a maximal twisted orbit type in \(U_{j,1}\) (cf. Remark 3.3);
(ii) \(t_{j,1}(\alpha, w_0) \neq 0\) (cf. (17));
(iii) \(t_{j,1}(\alpha, w_0) \cdot t_{j,1}(\alpha, w_0) \geq 0\) for all \(U_{j,1}, U_{j',1} \in \mathfrak{R}\).

Then, there exist at least \(|\mathcal{H}/L_0|_{S^1}\) continuous branches of relative periodic solutions (26) bifurcating via the Hopf bifurcation from the relative equilibrium \((\bar{n}, \bar{n}, S(\bar{n}))\) (cf. Definition 3.9) and having the minimal symmetry \((L_0)\).

### 3.4 Proof of Theorem 3.10

For the proof, which splits into several steps, we modify the twisted equivariant degree approach described in Sections 10.1-2 of [3] (see also the sketch of the proof of Proposition 3.6).

(a) **Rescaling time.** Substituting (26) in (3) (see also (23) and (24)) leads to equations
\[
\begin{align*}
\dot{y}(t) &= f(\alpha, \bar{x} + \bar{y}_t) - (w(\alpha) + \phi)J(x(\alpha) + y(t)), \\
y(t) &= y(t + p),
\end{align*}
\] (34)

where \(p > 0\) is an unknown period of \(y\) and
\[
\bar{x}(\theta) := e^{(w(\alpha) + \phi)\theta J} x(\alpha), \quad \bar{y}_t(\theta) := e^{(w(\alpha) + \phi)\theta J} y(t + \theta).
\] (35)

By normalizing the period \(p = 2\pi/\beta\) of \(y\), we obtain the system
\[
\begin{align*}
\dot{y}(t) &= \frac{1}{p} \left( f(\alpha, \bar{x} + \bar{y}_t^\beta) - (w(\alpha) + \phi)J(x(\alpha) + y(t)) \right), \\
y(t) &= y(t + 2\pi)
\end{align*}
\] (36)

with
\[
\bar{y}_t^\beta(\theta) := e^{(w(\alpha) + \phi)\theta J} y(t + \beta \theta).
\] (37)

(b) **Constraint.** This step reflects the specifics of the Hopf bifurcation of relative periodic solutions from a relative equilibrium. Namely, in order to ensure that the unknown function \(y(t)\) is determined “uniquely” (i.e. up to shifting the argument), we will assume that this function satisfies an additional constraint. From assumption (A3) and (21)-(23), it follows that for any \(\alpha \in \mathcal{W}\), the map given by the matrix \(D_{x} \Phi(\alpha, w(\alpha), x(\alpha))\) has the one-dimensional kernel span \(\{Jx(\alpha)\}\). Denote by \(g^\dagger(\alpha)\) the adjoint eigenvector of the transpose matrix \(D_{x} \Phi(\alpha, w(\alpha), x(\alpha))^T\) corresponding to the zero eigenvalue:
\[
D_{x} \Phi(\alpha, w(\alpha), x(\alpha))^T g^\dagger(\alpha) = 0, \quad g^\dagger(\alpha) \bullet Jx(\alpha) = 1, \quad \alpha \in \mathcal{W}.
\]

We will look for a solution to (36) with the \(y\)-component satisfying the constraint
\[
\mathcal{J}_\alpha(y) := g^\dagger(\alpha) \bullet \int_0^{2\pi} y(t) dt = 0.
\] (38)

(c) **Setting system (36) in functional spaces.** Using the standard identification of a \(2\pi\)-periodic \(V\)-valued function with the \(V\)-valued function on \(S^1\), we reformulate system (36) with constraint (38) as a \(K\)-equivariant operator equation in the space \(\mathbb{R}^2_+ \times \mathcal{W}\), where \(K\) acts trivially on \(\mathbb{R}^2_+ := \mathbb{R} \times \mathbb{R}^+\) and \(\mathcal{W} := H^1(S^1; V)\) stands for the first Sobolev space equipped with the \(K\)-action given by
\[
(h, e^{i\tau})(u)(t) := hu(t + \tau) \quad ((h, e^{i\tau}) \in \mathcal{H} \times S^1 =: \mathcal{K}, \ u \in \mathcal{W}).
\] (39)
To this end, denote
\[ v_\alpha := D_w \Phi(\alpha, w(\alpha), x(\alpha)) \in V \] (40)
and observe that
\[ v_\alpha \in V^H. \] (41)
Indeed, the \( \mathcal{H} \)-action on \( V \) induces the \( \mathcal{H} \)-action on \( \mathbb{R} \times V \), where \( \mathcal{H} \) acts trivially on \( \mathbb{R} \). Since the map \( \Phi(\alpha, \cdot, \cdot) : \mathbb{R} \times V \to V \) is \( \mathcal{H} \)-equivariant and \( (w(\alpha), x(\alpha)) \in (\mathbb{R} \oplus V)^\mathcal{H} \), one has that \( D\Phi(\alpha, w(\alpha), x(\alpha)) : \mathbb{R} \times V \to V \) is \( \mathcal{H} \)-equivariant as well, which implies (41).

Next, given an \( \alpha \in \mathcal{W} \), we identify a function \( z \in \mathcal{W} \) with the pair \((y, \phi)\), where \( y \in \mathcal{W} \) satisfies (38) and \( \phi \in \mathbb{R} \), by the relationships
\[ z = \phi v_\alpha + y, \quad \mathcal{J}_\alpha(y) = 0, \] (42)
and define the corresponding projections
\[ \phi = \hat{\pi}_\alpha(z), \quad y = z - \hat{\pi}_\alpha(z)v_\alpha. \] (43)

Let us introduce the following operators:
\[
L : \mathcal{W} \to L^2(S^1; V), \quad L(z) = \dot{z}, \\
j : \mathcal{W} \to C(S^1; V), \quad j(z) = z,
\]
where \( C(S^1; V) \) is the space of continuous functions equipped with the usual sup-norm. Furthermore, define
\[ F : \mathbb{R}^2_+ \times C(S^1; V) \to V \] by
\[
F(\alpha, \beta, z(t)) := \frac{1}{\beta} \left( f(\alpha, \overline{x} + \overline{y}_t^z) - (w(\alpha) + \phi)J(x(\alpha) + y(t)) \right), \quad t \in \mathbb{R},
\] (44)
with \( (\alpha, \beta, z) \in \mathbb{R}^2_+ \), \( z \in C(S^1; V) \), where the function \( y \in C(S^1; V) \) and the scalar \( \phi \) are defined by (43); \( \overline{x}, \overline{y}_t^z \) are defined in (35), (37). Next, denote by \( \mathcal{N}_F : \mathbb{R}^2_+ \times C(S^1, V) \to L^2(S^1; V) \) the Nemytsk operator associated with the map \( F \), i.e.
\[
\left( \mathcal{N}_F(\alpha, \beta, z) \right)(t) := F(\alpha, \beta, z(t)), \quad z \in C(S^1; V).
\] (45)

Since \( Lz = Ly \), system (36) with constraint (38) is equivalent to the following operator equation:
\[
Lz = \mathcal{N}_F(\alpha, \beta, j(z)), \quad (\alpha, \beta) \in \mathbb{R}^2_+, \quad z \in \mathcal{W}.
\] (46)
Using the formulas similar to (39), one can define the \( \mathcal{H} \)-actions on an \( C(S^1, V) \) and \( L^2(S^1; V) \). Clearly, all the operators involved in formula (46) are \( \mathcal{K} \)-equivariant, therefore equation (46) can be transformed to a \( \mathcal{K} \)-equivariant fixed-point problem in \( \mathbb{R}^2_+ \times \mathcal{W} \) as follows. Define the operator \( K : \mathcal{W} \to L^2(S^1; V) \) by
\[
K(z) := \frac{1}{2\pi} \int_0^{2\pi} z(t) \, dt, \quad z \in \mathcal{W},
\] (47)
which is simply a projection on the subspace \( V \) of constant functions. Then, the operator \( L + K : \mathcal{W} \to L^2(S^1; V) \) is an isomorphism. Put
\[
\mathcal{F}(\alpha, \beta, z) := (L + K)^{-1} \left[ \mathcal{N}_F(\alpha, \beta, j(z)) + K(z) \right],
\] (48)
\[
\mathfrak{F}(\alpha, \beta, z) := z - \mathcal{F}(\alpha, \beta, z).
\] (49)
In this way, the following equation is equivalent to (46):
\[
\mathfrak{F}(\alpha, \beta, z) = 0, \quad (\alpha, \beta, z) \in \mathbb{R}^2_+ \times \mathcal{W}.
\] (50)

\textit{(d) Reduction to twisted degree.} Take \( \overline{\alpha}, \mathcal{W} \) and \( S(\alpha(x)) \) provided by condition (A5) (see also (23)–(25)) and \( \beta_0 \) provided by (A5). Put
\[
M := \{ (\alpha, \beta, z) : \alpha \in \mathcal{W}, \beta \in \mathbb{R}_+, z \in S(\alpha(x)) \} \subset \mathbb{R}^2_+ \times \mathcal{W},
\]
where

$$\mathcal{W} = V \oplus \bigoplus_{i=1}^{\infty} \mathcal{W}_i, \quad \mathcal{W}_i = \{ e^{it} : y_t \in V^c \},$$

(51)

and the subspace of constant functions is identified with the space \( V \). For any small \( \varepsilon > 0 \), define a three-dimensional \( \mathcal{K} \)-invariant submanifold

$$M_\varepsilon := \{ (\alpha, \beta, z) \in M : |\alpha - \alpha| < \varepsilon, |\beta - \beta_0| < \varepsilon \} \subset \mathbb{R}_+^2 \times V \subset \mathbb{R}_+^2 \times \mathcal{W}$$

of \( M \). Take a small \( r > 0 \), define a normal \( \mathcal{K} \)-invariant neighborhood of \( M_\varepsilon \) by

$$\mathcal{N}_{\varepsilon, r} := \{ u + v \in \mathbb{R}_+^2 \times \mathcal{W} : u \in M_\varepsilon, v \perp \tau_u(M_\varepsilon), \| v \| < r \}$$

and denote

$$\partial_M^N := \partial(\mathcal{N}_{\varepsilon, r}) \cap M, \quad \partial_r^N := \{ u + v \in \mathcal{N}_{\varepsilon, r} : \| v \| = r \}.$$

By condition (A5), one can choose \( \varepsilon \) and \( r \) to be so small that

$$\hat{\mathfrak{g}}^{-1}(0) \cap \partial(\mathcal{N}_{\varepsilon, r}) \subset \partial_M^N \cup \partial_r^N.$$

Let \( \xi : \mathcal{N}_{\varepsilon, r} \to \mathbb{R} \) be a \( \mathcal{K} \)-invariant Urysohn function which is positive on \( \partial_r^N \) and negative on \( \partial_M^N \). Then, the map \( \hat{\mathfrak{g}} \xi : \mathcal{N}_{\varepsilon, r} \subset \mathbb{R}_+^2 \times \mathcal{W} \to \mathbb{R} \times \mathcal{W} \) given by

$$\hat{\mathfrak{g}} \xi(\alpha, \beta, z) := (\xi(\alpha, \beta, z), \hat{\mathfrak{g}}(\alpha, \beta, z))$$

is \( \mathcal{K} \)-equivariant and \( \mathcal{N}_{\varepsilon, r} \)-admissible, therefore the \( \mathcal{K} \)-equivariant twisted degree

$$\mathcal{K}-\text{deg}(\hat{\mathfrak{g}} \xi, \mathcal{N}_{\varepsilon, r}) = \sum_{(\mathcal{L})} n_{\mathcal{L}}(\mathcal{L})$$

(52)

is correctly defined.

**Proposition 3.11.** Let \( (\mathcal{L}_o) \) and \( (\pi, \pi, S(\pi)) \) be as in Theorem 3.10 and assume that \( n_{\mathcal{L}_o} \neq 0 \) in (52). Then, the conclusion of Theorem 3.10 holds.

**Proof.** Following the same argument as in the proof of Theorem 9.28 from [3], one can establish the existence of a continuous branch of solutions \( (\alpha, \beta, z) \) to equation (50) bifurcating from \( (\pi, \beta_0, \pi) \) with symmetry \( (\mathcal{L}_o) \). For any solution \( (\alpha, \beta, z) \) belonging to this branch, take \( \nu_o \) given by (40) and identify \( \phi \) and \( y \) using (42) and (43). Then, the quadruplets \( (\alpha, \beta, \phi, y) \) constitute a continuous branch required in the conclusion of Theorem 3.10. Symmetric properties of this branch are guaranteed by condition (41) and assumption \( n_{\mathcal{L}_o} \neq 0 \).

(6) **Computation of twisted degree.** To effectively apply Proposition 3.11 to proving Theorem 3.10, one needs to prove that the hypotheses of Theorem 3.10 indeed guarantee a non-zero summand \( n_{\mathcal{L}_o}(\mathcal{L}_o) \) in twisted degree (52). To estimate (52), one can use a computational product formula similar to (19) (cf. [3]). To this end, one needs:

(i) to show that the restriction of \( D_x \hat{\mathfrak{g}}(\alpha, \beta, \pi) \) to \( V \) is invertible;

(ii) to link the restriction of \( D_x \hat{\mathfrak{g}}(\alpha, \beta, \pi) \) to \( \bigoplus_{i=1}^{\infty} \mathcal{W}_i \) to crossing numbers.

Both problems require to evaluate the linearization of \( F \) (cf. (34)–(37) and (44)–(49)). Assuming in (44) \( \phi \) and \( y \) to be small, one obtains for the first summand (up to the terms of higher order):

$$f(\alpha, \beta, \pi) = f(e^{(w(\alpha)+\phi)J}(x(\alpha) + y(t + \beta \theta))) = f(e^{(w(\alpha)\theta)J}(x(\alpha)))
+ D_x f(e^{(w(\alpha)\theta)J}(x(\alpha))) \left[ e^{(w(\alpha)+\phi)J}(x(\alpha) + y(t + \beta \theta)) - e^{w(\alpha)\theta}J(x(\alpha)) \right]$$

(53)

$$+ (t.o.h.o.).$$
The expression in square brackets reads:

$$e^{w(\alpha)\theta J} e^{\phi \theta J} x(\alpha) + e^{(w(\alpha)+\phi)\theta J} y(t + \beta \theta) - e^{w(\alpha)\theta J} x(\alpha)$$

$$= e^{w(\alpha)\theta J} (e^{\phi \theta J} - \text{Id}) x(\alpha) + e^{(w(\alpha)+\phi)\theta J} y(t + \beta \theta)$$

$$= \phi \theta J e^{w(\alpha)\theta J} x(\alpha) + e^{w(\alpha)\theta J} (\text{Id} + \phi \theta J) y(t + \beta \theta)$$

Combining \((53)\) and \((54)\) yields

$$f(\alpha, \bar{x} + \bar{y}_t) = f(e^{w(\alpha)\theta J} x(\alpha))$$

$$+ D_x f(e^{w(\alpha)\theta J} x(\alpha)) (\phi \theta J e^{w(\alpha)\theta J} x(\alpha) + e^{w(\alpha)\theta J} y(t + \beta \theta)) + (t.o.h.o.).$$

The linearization of other summands in \((44)\) gives:

$$-(w(\alpha) + \phi) J(x(\alpha) + y(t)) = -w(\alpha) J y - \phi J x(\alpha) + (t.o.h.o.).$$

Combining now \((55), (56), (44)\) with \((40)\) and \((14)\) yields the following formula for the linearization of \(F\):

$$D_x F(\alpha, \beta, e^{w(\alpha)\theta J} x(\alpha)) = \frac{1}{\beta} \left( \phi e_\alpha + D_x f(\alpha, e^{w(\alpha)\theta J} x(\alpha)) e^{w(\alpha)\theta J} y(t + \beta \theta) - w(\alpha) J y(t) \right),$$

where \(y\) and \(\phi\) are defined by \((43)\). Therefore, \(D_x \mathcal{F}(\alpha, \beta, \mathcal{T})|_V\) has the form

$$D_x \mathcal{F}(\alpha, \beta, \mathcal{T}) z = \phi D_x \Phi(w(\alpha), x(\alpha)) + D_x \Phi(w(\alpha), x(\alpha)) y_0,$$

where \(\phi = \pi_\alpha(z) \in \mathbb{R}\) and \(y_0 = K(z) \in V\) satisfies \(g^l(\alpha) \cdot y_0 = 0\). Due to \((58)\), from assumption \((A3)\) (see \((22)\)), one obtains that \(D_x \mathcal{F}(\alpha, \beta, \mathcal{T})|_V\) is invertible in a neighborhood of the point \(\alpha = \mathcal{T}\). Therefore (cf. Step \((c)\) of the proof of Proposition \(3.6\)), \(D_x \mathcal{F}(\alpha, \beta, \mathcal{T})|_V\) does not affect the existence of maximal twisted orbit types in \((52)\) and, therefore, is of no consequence for the analysis of maximal twisted orbit types of relative periodic solutions.

On the other hand, \(D_x \mathcal{F}(\alpha, \beta, \mathcal{T})|_{\gamma_l}\) acts as follows (cf. \((51)\)):

$$D_x \mathcal{F}(\alpha, \beta, \mathcal{T})|_{\gamma_l} = D_x f(\alpha, e^{w(\alpha)\theta J} x(\alpha)) e^{(w(\alpha)J + i \beta \text{Id})\theta} y_l - (w(\alpha) J + i \beta \text{Id}) y_l.$$

Also, since \(D_x \mathcal{F}(\alpha, \beta, \mathcal{T})\) is \(K\)-equivariant, it preserves \(K\)-isotypical decompositions of \(\gamma_l\) for all \(l\). Take \(l = 1\) and consider decomposition \((31)\). For the restriction \(D_x \mathcal{F}(\alpha, \beta, \mathcal{T})|_{\gamma_{l,1}}\), one has:

$$\Delta_j(\alpha, \beta) := \det_C D_x \mathcal{F}(\alpha, \beta, \mathcal{T})|_{\gamma_{l,1}} = \mathcal{T}_{j,1}(\alpha, i \beta).$$

Therefore, the degree of the planar vector field \(\Delta_j\) equals the crossing number \((33)\).

Applying the same argument as in Step \((c)\) of the proof of Proposition \(3.6\) completes the proof of Theorem \(3.10\).

4 DDE Model of a Symmetric Configuration of Passively Mode-Locked Semiconductor Lasers

4.1 Mathematical model

In [14], a model for a mode-locked semiconductor laser with gain and absorber sections was introduced as a system of the following delay differential equations:

$$\begin{cases}
\dot{g}(t) = g_0 - \gamma g g(t) - \frac{1}{E_g} [e^{-q(t)}(e^{q(t)} - 1)] a(t)^2, \\
\dot{q}(t) = q_0 - \gamma q q(t) - \frac{1}{E_q} (1 - e^{-q(t)}) |a(t)|^2, \\
\dot{a}(t) = -\gamma a(t) + \sqrt{\kappa} \exp \left[ \frac{(1 - i \eta q)(t - T) - (1 - i \eta q)(t - T)}{2} \right] a(t - T).
\end{cases}$$

(60)
The complex-valued function $a(t)$ is the field amplitude at the entrance of the absorber section with $|a(t)|^2$ representing the optical power. The real-valued functions $q(t)$ and $q(t)$ represent saturable gain and losses, respectively, and $\eta_0$, $\eta_q$ are the linewidth enhancement factors corresponding to self-phase modulation. The constants $g_0$ and $g_0$ stand for unsaturated gain and absorption. The constants $\gamma_g$ and $\gamma_q$ are the carrier density relaxation rates in the gain and absorbing sections; $E_g$ and $E_q$ are the saturation energies in these sections; the ratio $s = E_g/E_q$ is important for laser dynamics. Finally, $T$ stands for the cold cavity round-trip time, and $\sqrt{n}$ is the linear non-resonant attenuation factor per pass. The parameter $g_0$ is proportional to the pump current, which is the physical control parameter.

Assume $(g(t), q(t), a(t))^T \in \mathbb{R}^2 \oplus \mathbb{C} \simeq \mathbb{R}^4 := \mathcal{V}$ and equip $\mathcal{V}$ with the natural $S^1$-representation (trivial on $(g, q)$-components and complex multiplication on $a$-component). Clearly, system (60) is $S^1$-equivariant. In what follows, assuming the value $\alpha := g_0$ to be the bifurcation parameter, we will show how Proposition 3.6 (resp. Theorem 3.10) can be used to study bifurcations of relative equilibria (resp. relative periodic solutions) for the network of identical oscillators (60) coupled in a $D_n$-symmetric fashion.

### 4.2 $D_n$-configuration of identical semiconductor lasers

Let $f : \mathbb{R} \times C([-T, 0]; \mathcal{V}) \to \mathcal{V}$ be the map induced by the right-hand side of system (60). Put $V := \mathcal{V}^n$ and define the map $f_o : \mathbb{R} \oplus C([-T, 0]; V) \to V$ by

$$f_o(\alpha, x_t) = \left( f(\alpha, x_t^0), f(\alpha, x_t^1), \ldots, f(\alpha, x_t^{n-1}) \right)^T,$$

where $x = (x^0, x^1, \ldots, x^{n-1})^T \in V$. Take the linear operator $C : \mathcal{V} \to \mathcal{V}$ with the matrix

$$C := \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & e^{i\psi} & \end{bmatrix}$$

and let $\mathcal{C} : V \to V$ be given by the block matrix

$$\mathcal{C} := \begin{bmatrix}
0 & C & 0 & \ldots & 0 & C \\
C & 0 & C & \ldots & 0 & 0 \\
0 & C & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & C \\
C & 0 & 0 & \ldots & C & 0 \\
\end{bmatrix}.$$ (63)

We are interested in solutions to the system

$$\dot{x} = f(\alpha, \eta, x, x_t) := f_o(\alpha, x_t) + \eta \mathcal{C} x, \quad x \in V, \alpha, \eta \in \mathbb{R},$$

where $\eta$ stands for the strength of coupling.

Clearly, the space $V$ is an orthogonal $D_n \times S^1$-representation for $\Gamma := D_n$, where $D_n$-action on $V$ is defined by permutation of the coordinates of the vector $x \in V$. More precisely, $D_n$ stands for the dihedral group being the group of symmetries of a regular $n$-gone, i.e. one can consider it to be a subgroup of the symmetric group $S_n$ of $n$-vertices $\{0, 1, \ldots, n-1\}$ of the regular $n$-gone. This group is generated by the “rotation” $\xi := (0, n-1, n-2, \ldots, 1)$ and the “reflection” $\kappa := (1, n-1)(2, n-2)\ldots(m, n-m)$, where $m = \left\lfloor \frac{n-1}{2} \right\rfloor$. Then, the $D_n \times S^1$-action on $V$ is given by

$$(h, e^{i\tau})x = (e^{i\tau} x^{h(0)}, e^{i\tau} x^{h(1)}, \ldots, e^{i\tau} x^{h(n-1)})^T, \quad e^{i\tau} \in S^1, h \in D_n,$$

where $x = (x^0, x^1, \ldots, x^{n-1})^T \in V$ and $e^{i\tau}$ acts on $x^i \in \mathcal{V} \simeq \mathbb{R}^2 \oplus \mathbb{C}$ trivially on the first two components and by complex multiplication on the $\mathbb{C}$-component ($i = 0, \ldots, n-1$). Obviously, system (64) satisfies condition (A0).
Remark 4.1. Recall, if \( S(\pi) \) is a relative equilibrium for system (64), then symmetries of \( S(\pi) \) are completely determined by a (twisted) isotropy subgroup \( \mathcal{G} \) with respect to the \( \mathcal{G} := D_n \times S^1 \)-action.

Observe that \( x_\alpha(\alpha) := (\frac{\alpha}{\gamma_g}, \frac{\alpha}{\gamma}, 0) \in \mathcal{V} \) is an equilibrium of system (60) for any \( \alpha \), hence

\[
\mathcal{O}(\alpha) := (x_\alpha(\alpha), x_\alpha(\alpha), ..., x_\alpha(\alpha)) \in V
\]

is an equilibrium of system (64) for any \( \alpha \). Also (cf. (10)),

\[
D_x f(\alpha, x_\alpha(\alpha))|_{\mathcal{V}} = \begin{bmatrix}
-\gamma_g & 0 & 0 \\
0 & -\gamma_g & 0 \\
0 & 0 & \left(\sqrt{2} \exp\left(\frac{(1-i\eta_g)\pi}{2} - \frac{(1-i\eta_g)\pi}{2}\right) - 1\right)\gamma
\end{bmatrix}.
\]

4.3 Bifurcation of symmetric relative equilibria

Hereafter, for the sake of simplicity, we will restrict ourselves to the case \( n = 8 \).

Isotypical decomposition and maximal twisted orbit types. To apply Proposition 3.6 for studying relative equilibria bifurcating from the the equilibrium \( \mathcal{O}(\alpha) \), observe that \( V \) admits the isotypical \( D_8 \)-decomposition:

\[
V = \bigoplus_{j=0}^{4} W_j,
\]

where \( W_j \) is modeled on \( \mathcal{V}_j \), \( \mathcal{V}_4 \) is a one-dimensional \( D_8/D_4 \)-representation and \( \{\mathcal{V}_j\}_{j=1}^{3} \) are three two-dimensional non-equivalent irreducible representations with different actions of the rotational generator (see [3] for details). Observe also (see (15)) that decomposition (67) can be refined to the \( D_8 \times S^1 \)-decomposition:

\[
W_j = V^0_j \oplus V_{j,1}, \quad j = 0, ..., 4, \quad \text{where } V^0_j \text{ is modeled on } \mathcal{V}_j \simeq \mathbb{R}^2 \quad \text{and } S^1 \text{ acts trivially, while } V_{j,1} \text{ is modeled on } \mathcal{V}_{j,1} \simeq \mathbb{C}^2 \quad \text{and } S^1 \text{ acts by complex multiplication (see [3]).}
\]

Clearly, \( \text{dim } W_0 = \text{dim } W_4 = 1 \), while \( \text{dim } W_2 = \text{dim } W_3 = 4 \).

Let us now describe maximal twisted orbit types in \( V \). By inspection, for any \( j = 0, 1, 2, 3, 4, \) if \( (\mathcal{H}_\alpha) \) is a maximal orbit type in \( V_{j,1} \), then \( (\mathcal{H}_\alpha) \) is a maximal twisted type in \( V \) (cf. Proposition 3.6, assumption (i)).

In turn, the list of maximal twisted types in any \( V_{j,1} \) is given by

- for \( V_{0,1} \): \((D_8 \times \{1\}) \simeq (D_8)\);
- for \( V_{1,1} \): \((2^{t_1}_{\mathcal{V}_1}), (D_2^{t_1}), (\tilde{D}_2^{t_1})\);
- for \( V_{2,1} \): \((2^{t_2}_{\mathcal{V}_2}), (D_2^{t_2}), (\tilde{D}_2^{t_2})\);
- for \( V_{3,1} \): \((2^{t_3}_{\mathcal{V}_3}), (D_2^{t_3}), (\tilde{D}_2^{t_3})\);
- for \( V_{4,1} \): \((D_4^{t_4})\).

We refer to Subsection 5.1 the Appendix for explicit description of all these subgroups, see also Remark 4.1.

Equivariant spectral reduction and condition (A1). The linearization \( D_x f(\alpha, x) : \mathbb{R} \times C_{-r}(V)|V \rightarrow V \) of system (64) at \( \mathcal{O}(\alpha) \) respects isotypical decomposition (67). To describe its action on isotypical components, define a (real) \( 4 \times 4 \)-matrix \( \xi \) by

\[
\xi := \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & e^{i\frac{\pi}{2}}
\end{bmatrix}
\]

and put

\[
A_j := D_x f(\alpha, x_\alpha(\alpha))|_{\mathcal{V}} + \eta C(\xi^j + \xi^{-j}), \quad j = 0, 1, 2, 3, 4
\]
(cf. (66) and (62); it should be stressed that $A_j$ is considered here as a real $4 \times 4$-matrix). Then,

$$D_{x\lambda}(\alpha, \mathcal{O}(\alpha))|_{W_j} = \begin{cases} A_j & \text{if } j = 0, 4, \\
A_j & \text{if } j = 1, 2, 3. \end{cases} \quad (70)$$

Since the action of $S^1$ on $(g, q)$-components of (60) is trivial, it follows from (66) and (68)-(70) that $\text{det}(D_{x\lambda}(\alpha, \mathcal{O}(\alpha))|_{W^1}) = (\gamma q)^8 \neq 0$, hence (see (12)), $\mathcal{P}_{\alpha}(0, 0, \mathcal{O}(\alpha)) \neq 0$ so that system (64) satisfies condition (A1).

**Characteristic quasi-polynomial.** Next, let us consider the characteristic quasi-polynomial $\mathcal{P}_{\lambda}(\alpha, \lambda, \mathcal{O}(\alpha))$ (see (12)). For any $j = 0, 1, 2, 3, 4$, put

$$\tilde{P}_j := \lambda + \gamma - \gamma \sqrt{k} \exp \left[ \frac{\alpha}{2\gamma g} - \frac{q_0}{2\gamma g} \right] + i \left( \frac{\eta_g q_0}{2\gamma g} - \frac{\eta_\lambda}{2\gamma g} \right) e^{-\lambda T} + 2\eta \cos \frac{2\pi j}{8} e^{i\psi}. \quad (71)$$

Then, the restriction of the characteristic quasi-polynomial to $V_{j,1}$ reads

$$\mathcal{P}_{j,1}(\alpha, \lambda, \mathcal{O}(\alpha)) = \begin{cases} \tilde{P}_j & \text{if } j = 0, 4, \\
(P_j)^2 & \text{if } j = 1, 2, 3, \end{cases} \quad (72)$$

so that

$$\mathcal{P}_{\lambda}(\alpha, \lambda, \mathcal{O}(\alpha)) = \prod_{j=0}^{4} \mathcal{P}_{j,1}(\alpha, \lambda, \mathcal{O}(\alpha)). \quad (73)$$

**Condition (A2): existence of centers.** In order to simplify the notations (cf. (71)), put

$$x(\alpha) := \frac{\alpha}{2\gamma g} - \frac{q_0}{2\gamma g}, \quad y(\alpha) := \frac{\eta_g q_0}{2\gamma g} - \frac{\eta_\lambda}{2\gamma g}. \quad (74)$$

and

$$a_j + ib_j := 2\eta \cos \frac{2\pi j}{8} \quad (75)$$

Let us identify the values of $\alpha$ for which $\mathcal{O}(\alpha)$ is a center, i.e. we are looking for those values of $\alpha$ for which there exists $w > 0$ such that $\mathcal{P}_{\lambda}(\alpha, iw, \mathcal{O}(\alpha)) = 0$. Equivalently (cf. (74)-(75)),

$$iw = -\gamma + \gamma \sqrt{k} \exp(x(\alpha) + iy(\alpha) - wT) + a_j + ib_j, \quad j = 0, 1, 2, 3, 4.$$ 

This complex equation can be reduced to the real equation

$$\tan(y(\alpha) - w(\alpha)T) = \frac{w(\alpha) - b_j}{\gamma - a_j}, \quad j = 0, 1, 2, 3, 4, \quad (76)$$

with

$$w(\alpha) := \gamma \sqrt{k} e^{x(\alpha)} \left[ 1 - \frac{(\gamma - a_j)^2}{\gamma^2 k e^{2x(\alpha)}} + b_j. \right] \quad (77)$$

For $\alpha$ large enough, the right-hand side of (76) is close to $\frac{\gamma \sqrt{k}}{\gamma - a_j} e^{\frac{2\pi j}{8}} - \frac{2\pi j}{8}$ (see (74) and (77)). Combining this with periodicity of the function tangent, one concludes that (76) has infinitely many solutions $\alpha$ together with the corresponding limit frequencies $w(\alpha)$.

**Proposition 4.2.** Suppose $\alpha = \omega^j$ is a root of (76), (77) for some $j = 0, 1, 2, 3, 4$ and

$$\gamma > 2\eta \cos(\psi) \cos \frac{2\pi j}{8} \quad \text{and} \quad \omega(\omega^j) > 2\eta \cos(\psi) \sin \frac{2\pi j}{8}. \quad (78)$$

Then, the following continuous branches of relative equilibria bifurcate from the equilibrium $(\omega^j, \mathcal{O}(\omega^j))$ of equation (64):
for \( j = 0 \), a branch with symmetry \((D_8)\);
for \( j = 1 \), two branches with symmetry \((\mathbb{Z}_4^4)\), four branches with symmetry \((D_4^2)\) and four branches with symmetry \((\bar{D}_4^2)\);
for \( j = 2 \), two branches with symmetry \((\mathbb{Z}_8^2)\), two branches with symmetry \((D_4^2)\) and two branches with symmetry \((\bar{D}_4^2)\);
for \( j = 3 \), two branches with symmetry \((\mathbb{Z}_8^2)\), four branches with symmetry \((D_4^2)\) and four branches with symmetry \((\bar{D}_4^2)\);
for \( j = 4 \), a branch with symmetry \((D_8^4)\).

Proof. Let us show that the center \(\mathcal{O}(\alpha^e)\) is isolated (cf. condition (A2)). Put \(\lambda(\alpha) := r(\alpha) + iw(\alpha)\) and rewrite the characteristic equation as follows (cf. (71)-(75)):

\[
\begin{cases}
\begin{align*}
r(\alpha) &= -\gamma + \gamma \sqrt{\kappa} e^{z(\alpha) - r(\alpha)T} \cos(y(\alpha) - w(\alpha)T) + a_j, \\
w(\alpha) &= \gamma \sqrt{\kappa} e^{z(\alpha) - r(\alpha)T} \sin(y(\alpha) - w(\alpha)T) + b_j,
\end{align*}
\end{cases}
\]

(79)

where \( j = 0, 1, 2, 3, 4 \). Assume that for \( \alpha = \alpha_o \), the equilibrium \(\mathcal{O}(\alpha_o)\) is a center with the limit frequency \(w(\alpha_o) = w_o\) and put

\[
x_o := x(\alpha_o), \quad y_o := y(\alpha_o), \quad x'_o := x'(\alpha_o) = \frac{1}{2\gamma_q}, \quad y'_o := y'(\alpha_o) = -\frac{\gamma_q}{2\gamma_q}.
\]

(80)

Differentiating (79) with respect to \( \alpha \), one obtains

\[
\begin{cases}
\begin{align*}
r'(\alpha_o) &= (\gamma - a_j)(x'_o - r'(\alpha_o)T) - (w_o - b_j)(y'_o - w'(\alpha_o)T), \\
w'(\alpha_o) &= (w_o - b_j)(x'_o - r'(\alpha_o)T) + (\gamma - a_j)(y'_o - w'(\alpha_o)T),
\end{align*}
\end{cases}
\]

which leads to

\[
r'(\alpha_o) = \frac{[(\gamma - a_j)^2 x'_o + (w_o - b_j)^2 y'_o]T + (\gamma - a_j) x'_o - (w_o - b_j) y'_o}{(1 + (\gamma - a_j)T)^2 + (w_o - b_j)^2 T^2}. \quad (81)
\]

Formulas (80), (81) show that \( r'(\alpha_o) > 0 \) provided that relations (78) are satisfied. Hence, relations (78) guarantee that the transversality condition for \(\lambda(\alpha')\) is satisfied at \( \alpha = \alpha_o \), in which case the center \(\mathcal{O}(\alpha_o)\) is isolated. Moreover, relations (78) imply that condition (iii) from Proposition 3.6 is satisfied. Since the other conditions have been verified, the result follows.

Recall that \( \eta \) stands for the coupling strength. In particular, conditions (78) are satisfied for all \( j \) for any relatively weak coupling.

Table 1 illustrates Proposition 4.2. Assume that \( \eta = 2, \alpha_q = 1, \alpha_q = 1, \gamma_q = 10^{-2}, \gamma_q = 1, \gamma = 15, \kappa = \sqrt{0.2}, q_o = 2, E_q = 1, E_q = 0.1, T = 2.5 \). For this set of parameters conditions (78) are fulfilled for all \((\alpha, \omega(\alpha))\) satisfying equations (76), (77). For \( \alpha < 0.036 \), the equilibrium \(\mathcal{O}(\alpha)\) is stable. In Table 1, we localize Hopf bifurcation points along the horizontal direction, and specify isotypical components \( V(j,1) \) along the vertical direction. In each cell, we indicate the number of unstable roots of the corresponding characteristic polynomial \( P_{j,1} \) defined by (72). One can easily see a change of stability as \( \alpha \) increases. An entry of the table is circled to indicate a “jump” in the number of unstable roots and hence a Hopf bifurcation point. In particular, Proposition 4.2 guarantees Hopf bifurcations of branches of relative equilibria as follows:

(i) with symmetry \((D_8)\) for \( \alpha \approx 0.03606 \);
(ii) with symmetries \((\mathbb{Z}_8^4)\), \((D_4^2)\) and \((\bar{D}_4^2)\) for \( \alpha \approx 0.03607 \);
(iii) with symmetries \((\mathbb{Z}_8^2)\), \((D_4^2)\) and \((\bar{D}_4^2)\) for \( \alpha \approx 0.0361 \);
(iv) with symmetries \((\mathbb{Z}_8^2)\), \((D_4^2)\) and \((\bar{D}_4^2)\) for \( \alpha \approx 0.03613 \);
(v) with symmetry \((D_8)\) for \( \alpha \approx 0.03617 \);
(vi) with symmetries \((D_8^4)\) and \((\mathbb{Z}_8^4)\) for \( \alpha \approx 0.0362 \),
to mention a few (see Proposition 4.2 for the number of branches of each type).
Table 1: Number of unstable eigenvalues in each isotypical component for the equilibrium $O(\alpha)$

| Isotypical component $V_i$ | Intervals for values of parameter $\alpha \cdot 10^2$ |
|---------------------------|--------------------------------------------------|
|                           | $[3.6, 3.606]$ | $[3.6065, 3.607]$ | $[3.6075, 3.6095]$ | $[3.61, 3.613]$ | $[3.6135, 3.617]$ | $[3.618, 3.62]$ | $[3.6205, 3.622]$ |
| $V_{0,1}$                 | 0           | 2           | 2           | 2           | 4           | 4           | 8           |
| $V_{1,1}$                 | 0           | 0           | 4           | 4           | 4           | 4           | 8           |
| $V_{2,1}$                 | 0           | 0           | 0           | 0           | 4           | 4           | 4           |
| $V_{3,1}$                 | 0           | 0           | 0           | 0           | 4           | 4           | 4           |
| $V_{4,1}$                 | 0           | 0           | 0           | 0           | 0           | 0           | 2           |
| $\bigoplus_{j=0}^4 V_{j,1}$ | 0           | 2           | 6           | 10          | 14          | 16          | 22          |

4.4 Bifurcation of relative periodic solutions

4.4.1 Application of Theorem 3.10 to the laser system

In this subsection, we show how Theorem 3.10 can be applied to classify symmetries of relative periodic solutions, which bifurcate from branches of relative equilibria of system (64) with $n = 8$. We restrict the presentation to bifurcations from relative equilibria that have 3 particular types of symmetry, $(D_8)$, $(Z_{16})$, and $(\tilde{D}_8)$. These branches are listed under the items (i), (ii), and (vi), respectively, on page 17. Further, an infinite number of Hopf bifurcations of relative periodic solutions occurs along each branch of relative equilibria. To be specific, we consider a few successive Hopf bifurcations at the beginning of each branch of our choice. In contrast to the application of Proposition 3.6 to studying bifurcation of relative equilibria (in which case, all the necessary symbolic computations were explicitly presented), we have to resort to numerical computations for verifying conditions (A3), (A5) and (ii), (iii) of Theorem 3.10.

Based on the numerical evidence, Theorem 3.10 allows us to predict the following bifurcations of branches of relative periodic solutions.

Consider the $(D_8)$-symmetric branch of relative equilibria, which is denoted by (i) on page 17. The following branches of relative periodic solutions bifurcate from this branch (we refer to Section 5 for the notation):

(i) with symmetries $(Z^4_8)$, $(D^4_2)$, $(\tilde{D}^4_2)$ for $\alpha \approx 0.0386$;
(ii) with symmetries $(Z^4_8)$, $(D^4_4)$, $(\tilde{D}^4_4)$ for $\alpha \approx 0.0533$;
(iii) with symmetry $(D_8)$ for $\alpha \approx 0.0602$.

Consider the $(Z^4_8)$-symmetric branch of relative equilibria, which is denoted by (ii) on page 17. The following branches of relative periodic solutions bifurcate from this branch:

(i) with symmetries $(Z^4_8)$ and $(Z^{14}_8)$ for $\alpha \approx 0.0366$;
(ii) with symmetry $(Z^{14}_8)$ for $\alpha \approx 0.0399$;
(iii) with symmetries $(Z^4_8)$ and $(Z^{14}_8)$ for $\alpha \approx 0.0416$;
(iv) with symmetry $(Z^8_8)$ for $\alpha \approx 0.064$;
(v) with symmetry $(Z^{12}_8)$ for $\alpha \approx 0.0641$;
(vi) with symmetry $(Z_8)$ for $\alpha \approx 0.0788$.

Consider the $(D^4_8)$-symmetric branch of relative equilibria, which is denoted by (vi) on page 17. The following branches of relative periodic solutions bifurcate from this branch:

(i) with symmetries $(Z^4_8)$, $(D^4_4)$, $(\tilde{D}^4_4)$ for $\alpha \approx 0.0384$;
(ii) with symmetry \( (D_8) \) for \( \alpha \approx 0.0405; \)

(iii) with symmetries \( (Z_8^{t_1}), (D_4^d), (\tilde{D}_2^d) \) for \( \alpha \approx 0.0539; \)

(iv) with symmetry \( (D_8^d) \) for \( \alpha \approx 0.066; \)

(v) with symmetry \( (D_4^d) \) for \( \alpha \approx 0.0731; \)

(vi) with symmetries \( (Z_8^{t_1}), (D_4^d), (\tilde{D}_2^d) \) for \( \alpha \approx 0.0757. \)

Further bifurcations along these and other branches of relative equilibria can be classified in a similar manner. Note that branches of relative periodic solutions with symmetries \( (Z_8^{t_1}), (D_4^d), (\tilde{D}_2^d) \) come in pairs, while the branches with symmetries \( (D_4^d), (\tilde{D}_2^d) \) appear in quadruples.

In the rest of the paper, we show how the above bifurcations can be deduced from Theorem 3.10. Given a relative equilibrium with symmetry group \( H \), the verification of assumptions of Theorem 3.10 splits into the following steps: (a) finding the isotypical decomposition of \( H \times S^1 \)-representation \( (31) \) and providing a list of maximal orbit types in each component (see Subsections 4.4.2 and 4.4.3); (b) obtaining characteristic quasi-polynomials associated with each isotypical component (see Subsection 4.4.4); and, (c) analyzing roots of the quasi-polynomials and verifying conditions (A3), (A5), (i) and (ii) of Theorem 3.10 (see Subsections 4.4.6). The last step relies on numerical computations. Condition (A3) is reduced to an explicit inequality in Subsection 4.4.5.

4.4.2 Symmetries of relative equilibria

To begin with, below we will describe some of the relative equilibria identified in the previous subsection more explicitly.

Observe first that the group \( D_n \) described in Subsection 4.2 can be identified (for convenience) with \( D_n = \{1, \xi, ..., \xi^{n-1}, \kappa, \xi \kappa, ..., \xi^{n-1} \kappa\} \), where

\[
\xi := e^{i \frac{2 \pi}{n}} = \begin{bmatrix}
\cos\left(\frac{2 \pi}{n}\right) & -\sin\left(\frac{2 \pi}{n}\right) \\
\sin\left(\frac{2 \pi}{n}\right) & \cos\left(\frac{2 \pi}{n}\right)
\end{bmatrix}
\text{ and } \kappa = \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}.
\]

(82)

Let \( S(\pi) \) be a relative equilibrium of system (64) (cf. Remark 4.1). Fix an integer \( l \) satisfying \( 0 \leq l < n \), put \( \zeta := \xi^l \) and assume that

\[
\pi := (\pi^1, \pi^2, \pi^3, ..., \pi^{n-1})^T, \quad \pi^p = (g, q, a) \in \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{C} \sim \mathcal{V}, \quad a \neq 0.
\]

(83)

One can easily verify that in this case, under the \( \mathcal{G} := D_n \times S^1 \)-action, the isotropy of \( \pi \) is completely determined by the relations

\[
(\zeta^k, z) \in \mathcal{G}_\pi \iff z \zeta^{-k} = 1 \iff z = \xi^{-lk},
\]

where \( 0 \leq k \leq n - 1 \), and

\[
(\kappa, z) \in \mathcal{G}_\pi \iff l = 0 \text{ and } z = 1.
\]

By direct verification, \( \pi \) is of the form (83) if and only if

\[
(\mathcal{G}_\pi) := \begin{cases}
(D_n \times \{1\}) \simeq (D_n) & \text{if } l = 0, \\
D_n^d & \text{if } l = \frac{d}{2}, \\
Z_n^d := \{(\zeta^k, \xi^{kl}) \in D_n \times S^1 : k = 0, 1, ..., n - 1 \} & \text{otherwise}.
\end{cases}
\]

(84)

Remark 4.3. In what follows, as in Subsection 4.3, we will restrict ourselves to the case \( n = 8 \). As it was established in Subsection 4.3, twisted subgroups listed in (84) do not exhaust possible symmetries of relative equilibria of system (64). For example, one can easily check that if

\[
\pi = (\pi^1, \pi^2, \pi^3, \pi^4, -\pi^1, -\pi^2, -\pi^3, -\pi^4), \quad \pi^1 \in \mathcal{V},
\]

then \( (\mathcal{G}_\pi) = (D^d_8) \), and if

\[
\pi = (\pi^1, -\pi^1, \pi^2, -\pi^2, \pi^3, -\pi^3, -\pi^4), \quad \pi^1 \in \mathcal{V},
\]

then \( (\mathcal{G}_\pi) = (\tilde{D}^d_2) \). However, in order to keep our paper reasonably simple and of appropriate size, we omit these cases.
4.4.3 $\mathcal{H}$-isotypical decomposition of $V^c$ and maximal twisted orbit types

(a) Identification. In this subsubsection, we describe the $\mathcal{H}$-isotypical decomposition of the space $V^c$, where $\mathcal{H} = D_n \times \{1\}$, $D_n^l$ and $Z_n^l$ with $0 < l < \frac{n}{2}$ (cf. (84)). We will assume that $n > 2$ is an even integer and put $r := \frac{n}{2}$. Notice that $D_n \times \{1\}$ and $D_n^l$ can be identified with $D_n$ while $Z_n^l$ can be identified with $Z_n$.

Complex irreducible $Z_n$-representations $U'_j$ can be easily described: (a) the trivial representation $U'_0 = \mathbb{C}$, (b) $U'_r = \mathbb{C}$ with the natural antipodal action of $Z_2 := Z_n/Z_r$, and (c) $U'_{l,j} = \mathbb{C}$, where the $Z_n$-action is given by

$$\xi z = \xi^{\pm j} \cdot z, \quad z \in \mathbb{C}.$$ 

In the case of the group $D_n$, we have the following irreducible $D_n$-representations: (a) the trivial representation $U_0 = \mathbb{C}$, (b) the representation $U_r = \mathbb{C}$ with the natural antipodal action of $Z_2 := D_n/D_r$, and (c) the representations $U_j = \mathbb{C} \oplus \mathbb{C}$ for $0 < j < r$ with the $D_n$-action given by

$$\xi(z_1, z_2) = (\xi^1 \cdot z_1, \xi^{-j} \cdot z_2), \quad \kappa(z_1, z_2) = (z_2, z_1) \quad (z_1, z_2 \in \mathbb{C}).$$

Notice that $Z_n \subset D_n$, therefore, for $0 < j < r$, we have the decomposition

$$U_j = U'_j \oplus U'_{-j}.$$ 

We do not consider other (one-dimensional) irreducible $D_n$-representations since they are irrelevant for the decomposition of the substitutional $D_n$-representation we are dealing with in what follows.

If $\mathcal{H} \simeq \mathbb{Z}_n$, then the complex $\mathcal{H}$-representation $V^c$ admits the following $\mathbb{Z}_n$-isotypical decomposition

$$V^c = U_0 \oplus U_1^+ \oplus U_1^- \oplus \cdots \oplus U_{r-1}^+ \oplus U_{r-1}^- \oplus U_r,$$

where the components $U_j^\pm$ (resp. $U_0$ and $U_r$) are modeled on the complex irreducible $\mathbb{Z}_n$-representation $U'_{l,j}$ (resp. $U'_0$ and $U'_r$). Furthermore, if $\mathcal{H} \simeq D_n$, then

$$V^c = U_0 \oplus U_1 \oplus \cdots \oplus U_{r-1} \oplus U_r,$$

where $U_j = U_j^+ \oplus U_j^-$ for $0 < j < r$ and the isotypical component $U_j$ is modeled on the irreducible $D_n$-representation $U_j$. Also, $U_0$ and $U_r$ are modeled on $U'_0$ and $U'_r$, respectively.

Remark 4.4. (i) The complexification $\mathcal{V}^c$ of the space $\mathcal{V} := \mathbb{R}^2 \oplus \mathbb{C} = \mathbb{R}^2 \oplus (\mathbb{R} \oplus \mathbb{R})$ can be represented as

$$\mathcal{V}^c = \mathbb{C}^2 \oplus (\mathbb{C} \oplus \mathbb{C}) = \mathbb{C}^4,$$

thus $V^c = (\mathcal{V}^c)^n = (\mathbb{C}^4)^n$ for which decomposition (85) takes place.

(ii) Any complex $\mathcal{H}$-equivariant linear operator $A : V^c \rightarrow V^c$ is also $\mathbb{Z}_n$-equivariant, thus it preserves isotypical decomposition (85).

(iii) Clearly, the space $V^c$ admits a natural $S^1$-action induced by the complex multiplication. Put $\mathcal{K} := \mathcal{H} \times S^1$. Then (cf. (31)), the $S^1$-action converts the (complex) $\mathcal{H}$-isotypical decomposition (86) into a (real) $\mathcal{K}$-isotypical decomposition

$$V^c = U_{0,1} \oplus U_{1,1} \oplus \cdots \oplus U_{r-1,1} \oplus U_{r,1},$$

(iv) By inspection, for $n = 8$ (our case study), if $\mathcal{H}$ is a maximal twisted orbit type in an isotypical component of $V^c$, then $\mathcal{H}$ is a maximal twisted orbit type in $V^c$ itself.

(b) $\mathcal{H} := D_n \times \{1\}$-isotypical decomposition of $V^c$. One can explicitly describe the $\mathcal{H}$-isotypical components of (86) as follows:

$$U_0 = \{(z, z, \ldots, z)^\top : z \in \mathbb{C}^4\},$$

$$U_j = U_j^+ \oplus U_j^-,$$

$$U_j^\pm = \{(z, \xi^{\pm j}z, \ldots, \xi^{\pm j(n-1)}z)^\top : z \in \mathbb{C}^4\} \quad (0 < j < r),$$

$$U_r = \{(z, -z, z, -z, \ldots, z, -z)^\top : z \in \mathbb{C}^4\}.$$
Further, one can easily verify that the coupling matrix $\mathcal{C} : V^c \to V^c$ given by (63) preserves the $\mathcal{H}$-isotypical components. Put

$$\mathcal{C}_j^\pm := \mathcal{C}|_{U_j^\pm} \quad (0 < j < r).$$

(89)

Then,

$$\mathcal{C}_j^\pm = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 2a_j \cos \psi & -2a_j \sin \psi \\
0 & 2a_j \sin \psi & 2a_j \cos \psi
\end{bmatrix}, \quad a_j = \operatorname{Re}(\xi^j) = \cos \frac{2\pi j}{n} \quad (0 < j < r),$$

$$\mathcal{C}_0 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 2 \cos \psi & -2 \sin \psi \\
0 & 2 \sin \psi & 2 \cos \psi
\end{bmatrix}, \quad \mathcal{C}_r = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -2 \cos \psi & 2 \sin \psi \\
0 & -2 \sin \psi & -2 \cos \psi
\end{bmatrix}.$$

Finally, for $n = 8$, the list of maximal twisted types in the isotypical components of the $\mathcal{H} := D_8 \times \{1\}$-representation $V^c$ is as follows (see Appendix, Subsection 5.2, for the exact definition of the related twisted subgroups):

(i) for $U_{0,1}$: $(D_8)$;
(ii) for $U_{1,1}$: $(Z_8^t), (D_2^d), (\tilde{D}_2^d)$;
(iii) for $U_{2,1}$: $(Z_8^{t2}), (D_4^d), (\tilde{D}_4^d)$;
(iv) for $U_{3,1}$: $(Z_8^t), (D_2^d), (\tilde{D}_2^d)$;
(v) for $U_{4,1}$: $(D_8^d)$.

(c) $\mathcal{H} := \mathbb{Z}_n^t$-isotypical decomposition of $V^c$. For this group $\mathcal{H}$, the $\mathcal{H}$-isotypical components of (86) can be described as follows (cf. (87)):

$$U_0 = \mathcal{Y}_0 \oplus \mathcal{Y}_0^r,$$

where

$$\mathcal{Y}_0 := \{(z, z, ..., z)^T : z \in \mathbb{C}^2\}$$

and

$$\mathcal{Y}_0^r := \left\{(\left[\begin{array}{c} z_1 \\ z_2 \\ \xi z_2 \\ \xi^{-1} z_2 \\ \xi^{(n-1)} z_2 \\ \xi^{-(n-1)} z_2 \end{array}\right])^T : \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \mathbb{C} \oplus \mathbb{C}\right\};$$

$$U_j^\pm := \mathcal{Y}_j^\pm \oplus \mathcal{Y}_j^\pm \quad (0 < j < r),$$

where

$$\mathcal{Y}_j^\pm := \{(z, \xi^j z, \xi^j z, ..., \xi^j z)^T : z \in \mathbb{C}^2\}$$

and

$$\mathcal{Y}_j^\pm := \left\{(\left[\begin{array}{c} z_1 \\ z_2 \\ \xi^{j+1} z_1 \\ \xi^{j-1} z_2 \\ \xi^{(n-1)(j+1)} z_1 \\ \xi^{(n-1)(j-1)} z_2 \end{array}\right])^T : \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \mathbb{C} \oplus \mathbb{C}\right\};$$

$$U_r := \mathcal{Y}_r \oplus \mathcal{Y}_r,$$

where

$$\mathcal{Y}_r := \{(z, -z, -z, ..., -z)^T : z \in \mathbb{C}^2\}.$$
and

\[ \mathcal{W}_r := \left\{ \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-1} \\ z_n \end{pmatrix} : \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} \right\} \in \mathbb{C}^n \right\} . \]

Under the same notations as in (89), one has

\[ \mathcal{E}_j^+: = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2a_j \cos \psi & -2a_j \sin \psi \\ 0 & 2a_j \sin \psi & 2a_j \cos \psi \end{bmatrix} , \quad \mathcal{E}_j^- := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2a_j \cos \psi & 2a_j \sin \psi \\ 0 & 2a_j \sin \psi & -2a_j \cos \psi \end{bmatrix} , \quad a_{\pm j} = \cos \frac{2\pi (\pm j - 1)l}{n} \quad (0 < j < r), \]

where \( a_0 := \cos \frac{2\pi j}{n} \) and \( a_r := -\cos \frac{2\pi j}{n} \).

For \( n = 8 \), one obtains the following list of maximal twisted types in the isotypical components of the \( \mathcal{H} := \mathbb{Z}_8^l \times \{1\} \)-representation \( V^c \), \( l = 1, 2, 3 \) (see Appendix, Subsection 5.3, for the definition of the related twisted subgroups):

(i) for \( U_{0,1} \): \( (\mathbb{Z}_8) \);
(ii) for \( U_{1,1} \): \( (\mathbb{Z}_8^2) \);
(iii) for \( U_{3,1} \): \( (\mathbb{Z}_8^2) \);
(iv) for \( U_{3,1} \): \( (\mathbb{Z}_8^2) \);
(v) for \( U_{4,1} \): \( (\mathbb{Z}_8^2) \).

(d) \( \mathcal{H} := D_n^d \)-isotypical decomposition of \( V^c \). In this case, one can explicitly describe the \( \mathcal{H} \)-isotypical components of (86) as follows:

\[ U_0 = \mathcal{W}_0 \oplus \mathcal{W}_0, \]

where

\[ \mathcal{W}_0 := \{(z, z, ..., z)^\top : z \in \mathbb{C}^2\} \]

and

\[ \mathcal{W}_0 := \left\{ \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} : \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} \right\} \in \mathbb{C}^n \right\} ; \]

\[ U_j^\pm := \mathcal{W}_j^\pm \oplus \mathcal{W}_j^\pm \quad (0 < j < r), \]

where

\[ \mathcal{W}_j^\pm := \{(z, \xi^j z, \xi^{2j} z, ..., \xi^{(n-1)j} z)^\top : z \in \mathbb{C}^2\} \]

and

\[ \mathcal{W}_j^\pm := \left\{ \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} : \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} \right\} \in \mathbb{C}^n \right\} ; \]
\[ U_r := \mathcal{U}_r \oplus \mathcal{W}_r, \]  

where  
\[ \mathcal{U}_r := \{ (z, -z, z, -z, \ldots, z, -z) \} : z \in \mathbb{C}^2 \]  
and  
\[ \mathcal{W}_r := \left\{ \left( \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_1 \\ z_2 \end{bmatrix} \right) \in \mathbb{C}^\mathbb{N} : \right\}. \]

Also,  
\[ \mathcal{C}_j^\pm := \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 2a_{\pm j} \cos \psi & -2a_{\pm j} \sin \psi \\
0 & 2a_{\pm j} \sin \psi & 2a_{\pm j} \cos \psi
\end{bmatrix}, \quad a_{\pm j} = -\cos \frac{2\pi j}{n} \quad (0 < j < r), \]

\[ \mathcal{C}_0 := \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -2 \cos \psi & 2 \sin \psi \\
0 & -2 \sin \psi & -2 \cos \psi
\end{bmatrix}, \quad \mathcal{C}_r := \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 2 \cos \psi & -2 \sin \psi \\
0 & 2 \sin \psi & 2 \cos \psi
\end{bmatrix}. \]

Hence, for \( n = 8 \), the list of maximal twisted types in the isotypical components of the \( \mathcal{H} := D^4 \) representation \( V^c \) is (see Appendix, Subsection 5.4, for the definition of the twisted subgroups):  
(i) for \( U_{0,1} \): \( (D_8) \);  
(ii) for \( U_{1,1} \): \((Z_8), (D^d_2), (\bar{D}^d_2)\);  
(iii) for \( U_{2,1} \): \((Z^4_8), (D^4_2), (\bar{D}^4_2)\);  
(iv) for \( U_{3,1} \): \((Z^4_8), (D^4_2), (\bar{D}^4_2)\);  
(v) for \( U_{4,1} \): \((D^d_8)\).

### 4.4.4 Linearization on a relative equilibrium and characteristic quasi-polynomials

For any \( \varpi' = (g, q, a) \in \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{C} \cong \mathcal{Y} \), one has (cf. (7)–(8) and (60)):

\[ \bar{f}(\alpha, i\omega, \varpi') = \begin{bmatrix}
\frac{\alpha - \gamma g - g - \frac{1}{E_g} e^{-q}(e^{g} - 1)}{2} |a|^2 \\
\frac{g_0 - \gamma q - \frac{1}{E_q} (1 - e^{-q})}{2} |a|^2 \\
-\gamma a + \sqrt{\kappa} \exp \left[ \frac{(1-i\eta_0^g)\gamma-(1-i\eta_0^q)}{2} \right] ae^{-i\omega T}
\end{bmatrix}. \]  

(91)

Take \( \lambda \in \mathbb{C} \). Combining (91) with (27), (28) and (30) allows us to define a “linearization operator” \( \mathcal{R}^\varpi_{\alpha}(\lambda) : \mathcal{Y}^c \to \mathcal{Y}^c \) by

\[ \mathcal{R}^\varpi_{\alpha}(\lambda) := \begin{bmatrix}
-\gamma g - \frac{1}{E_g} e^{-q} e^q |a|^2 \\
\frac{1}{E_g} e^{-q} (e^q - 1) |a|^2 \\
-\gamma q - \frac{1}{E_q} e^{-\varpi} (e^{\varpi} - 1) a
\end{bmatrix}, \]

(92)

where

\[ B_{31}(\lambda) = \sqrt{\kappa} \exp \left[ \frac{(1-i\eta_0^g)\gamma-(1-i\eta_0^q)}{2} \right] ae^{-i\omega T} \gamma T; \]

\[ B_{32}(\lambda) = -\gamma + \sqrt{\kappa} \exp \left[ \frac{(1-i\eta_0^g)\gamma-(1-i\eta_0^q)}{2} \right] ae^{-i\omega T} \gamma T; \]

\[ B_{33}(\lambda) = -\gamma + \sqrt{\kappa} \exp \left[ \frac{(1-i\eta_0^g)\gamma-(1-i\eta_0^q)}{2} \right] ae^{-i\omega T} \gamma T. \]
For any \( \overline{x} = (\overline{x}^1, \ldots, \overline{x}^n) \in V \), put
\[
\tilde{f}_0(\alpha, i\omega, \overline{x}) := (\tilde{f}(\alpha, i\omega, \overline{x}^1), \ldots, \tilde{f}(\alpha, i\omega, \overline{x}^n))^\top.
\] (93)

For a given \( \alpha \), \( S(\overline{x}(\alpha)) \) is a relative equilibrium for system (64) corresponding to the frequency \( \omega(\alpha) \) if and only if
\[
\Phi(\alpha, \omega(\alpha), \overline{x}(\alpha)) := \tilde{f}_0(\alpha, i\omega(\alpha), \overline{x}(\alpha)) + \eta \xi \overline{x}(\alpha) - \omega(\alpha) J \overline{x}(\alpha) = 0
\] (94)
(cf. (14)). Assume that \( S(\overline{x}(\alpha)) \) is a relative equilibrium with \( H := \mathcal{G}_{\overline{x}(\alpha)} \) of the form (84) (cf. (83) and condition (A4)). Take \( \mathcal{R}_{\alpha}(\lambda) \) determined by (94) and (27)-(28) and consider decompositions (85)-(86). Then (cf. (92) and (94)), one has:
\[
\mathcal{R}_{\alpha}(\lambda)|_{\mathbb{U}} = \begin{cases} \mathcal{R}_{\alpha}^\ell(\lambda) + \eta \xi_0 & \text{if } \mathbb{U} = U_0; \\ \mathcal{R}_{\alpha}^j(\lambda) + \eta \xi_j^\pm & \text{if } \mathbb{U} = U_j^\pm (0 < j < r); \\ \mathcal{R}_{\alpha}^r(\lambda) + \eta \xi_r & \text{if } \mathbb{U} = U_r. \end{cases}
\] (95)

We refer to Subsection 4.4.3, where explicit formulas for \( \xi_0, \xi_j^\pm \) and \( \xi_r \) are given according to three possible values of \( H \). Combining (95) with (29) and (32), one can define the characteristic quasi-polynomials \( \overline{\mathcal{R}}_j(\alpha, \lambda), j = 0, \pm 1, \ldots, (r-1), r \) and study Hopf bifurcation of relative periodic solutions for different values of \( H = D_n \times \{ 1 \}, D_n', \mathbb{Z}_n' \).

### 4.4.5 Condition (A3)

Suppose that equation (64) with \( n = 8 \) has a relative equilibrium \( \mathcal{S}(\overline{x}), \overline{x} = (g, q, a) \), for some \( \overline{x} \) and \( \overline{x} \). Without loss of generality, assume that \( a \in \mathbb{C} \) is real. Take decomposition (67) and let us describe the restriction of matrix (22) to \( \mathbb{R} \times W_j \). For any \( j = 0, \ldots, 4 \), define the operator \( \mathcal{B}_j = \mathcal{B}_j(\overline{x}, \overline{x}, \overline{x}) : V \rightarrow V \) by
\[
\mathcal{B}_j := \mathcal{R}_{\alpha}^j(0) + \eta (\xi^j + \xi^{-j}) C - \overline{x} J \mathcal{V}.
\] (96)

Here \( \mathcal{R}_{\alpha}^j(0) \) is considered as a real linear operator in \( \mathcal{V} \cong \mathbb{R}^2 \oplus \mathbb{C} \) (cf. (92)) and \( J \mathcal{V} : \mathcal{V} \rightarrow \mathcal{V} \) is given by
\[
J \mathcal{V}(\overline{g}, \overline{q}, \overline{a})^\top = (0, 0, i\overline{a})^\top; \text{ see also (62) and (68)}.
\]
Define a vector \( \mathcal{B} = \mathcal{B}(\overline{x}, \overline{x}, \overline{x}) \in \mathcal{V} \cong \mathbb{R}^2 \oplus \mathbb{C} \) by
\[
\mathcal{B} := \left\{ 0, 0, -iT \gamma \sqrt{\xi} \exp \left[ \frac{(1-i\eta_b)g - (1-i\eta_a)q}{2} \right] a e^{-i\omega T} - i\overline{a} \right\}^\top.
\] (97)

Then (see (22), (91), (96) and (97)),
\[
\left[ D_{\alpha} \Phi(\overline{x}, \overline{x}, \overline{x}) \mid D_{\overline{x}} \Phi(\overline{x}, \overline{x}, \overline{x}) \right]_{\mathbb{R} \times W_j} = \begin{cases} [\mathcal{B} \mid \mathcal{B}_j] & \text{if } j = 0, 4 \\ [\mathcal{B} \mid \mathcal{B}_j] \quad 0 & \text{if } j = 1, 2, 3. \end{cases}
\] (98)

Put \( \mathcal{B} := [\mathcal{B} \mid \mathcal{B}_j] \). It follows from (98) that condition (A3) is satisfied if \( \operatorname{rank}(\mathcal{B}) = 4 \). Note that
\[
(0, 0, i)^\top \in \mathbb{R}^2 \oplus \mathbb{C} \text{ is an eigenvector of } \mathcal{B}_j \text{ corresponding to the zero eigenvalue.}
\]
Denote by \( \mathcal{E} \) the direct sum of generalized eigenspaces corresponding to non-zero eigenvalues of \( \mathcal{B}_j \). Clearly, \( \operatorname{rank}(\mathcal{B}) = 4 \) if
\begin{enumerate}
\item \( \operatorname{rank}(\mathcal{B}_j) = 3 \) (i.e., zero is a simple eigenvalue of \( \mathcal{B}_j \)), and
\item \( \mathcal{B} e \notin \mathcal{E} \), where \( e := (1, 0, 0, 0, 0) \in \mathbb{R}^5 \cong \mathbb{R} \oplus \mathbb{R}_2 \oplus \mathbb{C} \cong \mathbb{R} \oplus \mathcal{V} \).
\end{enumerate}

**Remark 4.5.** Condition (a) can be effectively expressed in terms of the derivative of the characteristic polynomial associated with \( \mathcal{B}_j \). Condition (b) is satisfied if
\[
(b') \quad \text{Im} \left[ -iT \gamma \sqrt{\xi} \exp \left[ \frac{(1-i\eta_b)g - (1-i\eta_a)q}{2} \right] a e^{-i\omega T} - a \right] \neq 0
\]
(recall, \( a \in \mathbb{R} \)).
4.4.6 Isotypical crossing

In order apply Theorem 3.10 to classify symmetries of relative periodic solutions bifurcating from relative equilibria $S(\mathcal{F})$ with $(\mathcal{G}_T)$ given by (84), it remains to analyze the isotypical crossing of the roots of characteristic quasi-polynomials $\mathcal{P}_j(\alpha, \lambda)$, $j = 0, \pm 1, \ldots, \pm 3, 4$ (cf. (95), (29) and (32)), as $\alpha$ crosses some critical value $\alpha_0$. Numerical results illustrating isotypical crossing of characteristic roots through the imaginary axis are described in Table 2 for $(\mathcal{G}_T) = D_6 \times \{1\}$, in Tables 3, 4, 5 for $(\mathcal{G}_T) = \mathbb{Z}_5^d \times \mathbb{Z}_5^d$, respectively, and in Table 6 for $(\mathcal{G}_T) = D_4^d$. All parameters except $\alpha$ are the same as in Section 4.3. In these tables, we follow the same agreement as in Table 1 except that we use a circle to indicate a Hopf bifurcation point and a rectangle to indicate a steady-state bifurcation. In particular, an entry in a given cell indicates the number of unstable roots for the characteristic quasi-polynomial $\mathcal{P}_j(\alpha, \lambda)$ associated with the isotypical component $U_j$ (shown in the left column) for the corresponding interval of $\alpha$-valus (shown in the upper row). The results presented in Subsection 4.4.1 follow from these tables.

Table 2: Number of unstable eigenvalues for each isotypical component along the branch of the relative equilibrium with $(D_6)$ symmetry (see item (i) on page 17)

| Intervals for values of parameter $\alpha \cdot 10^2$ | [3.61, 3.69] | [3.70, 3.85] | [3.86, 4.01] | [4.02, 4.58] | [4.59, 5] | [5.01, 5.32] | [5.33, 6.01] | [6.02, 8.97] |
|-----------------------------------------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| Isotypical component                         |             |             |             |             |             |             |             |             |
| $U_{0,1}$                                    | 0           | 0           | 0           | 0           | 0           | 0           | 0           | 2           |
| $U_{1,1}$                                    | 0           | 2           | 6           | 6           | 6           | 6           | 6           | 6           |
| $U_{2,1}$                                    | 0           | 0           | 0           | $\Box$      | 2           | 2           | 6           | 6           |
| $U_{3,1}$                                    | 0           | 0           | 0           | 0           | $\Box$      | 2           | 2           | 2           |
| $U_{4,1}$                                    | 0           | 0           | 0           | 0           | 0           | $\Box$      | 1           | 1           |
| $\bigoplus_{j=0} U_{j,1}$                    | 0           | 2           | 6           | 8           | 10          | 11          | 15          | 17          |

Table 3: Number of unstable eigenvalues for each isotypical component along the branch of the relative equilibrium with $(\mathbb{Z}_5^d)$ symmetry (see item (ii) on page 17)

| Intervals for values of parameter $\alpha \cdot 10^2$ | [3.61, 3.65] | [3.66, 3.98] | [3.99, 4.15] | [4.16, 4.2] | [4.21, 6.39] | 6.40           | [6.41, 7.87] | [7.88, 10.02] |
|-----------------------------------------------|-------------|-------------|-------------|-------------|-------------|----------------|--------------|--------------|
| Isotypical component                         |             |             |             |             |             |                |              |              |
| $U_{0,1}$                                    | 0           | 0           | 0           | 0           | 0           | 0              | 0            | 2            |
| $U_{1,1}$                                    | 2           | 4           | 4           | 6           | 6           | 6              | 6            | 6            |
| $U_{2,1}$                                    | 0           | $\Box$      | 2           | 2           | 2           | 2              | $\Box$       | 4            |
| $U_{3,1}$                                    | 0           | 0           | $\Box$      | 4           | 4           | 4              | 4            | 4            |
| $U_{4,1}$                                    | 0           | 0           | $\Box$      | 1           | $\Box$      | $\Box$         | 3            | 3            |
| $\bigoplus_{j=0} U_{j,1}$                    | 2           | 6           | 8           | 12          | 13          | 15             | 17           | 19           |

5 Appendix

If $\mathcal{W}$ is a $G$-representation, then for any function $x : S^1 \to \mathcal{W}$, the spatio-temporal symmetry of $x$ is a group $\mathfrak{H} < G \times S^1$ such that $g \cdot x(t - \theta) = x(t)$ for any $t \in \mathbb{R}/2\pi \mathbb{Z} \sim S^1$ and any $(g, e^{i\theta}) \in \mathfrak{H}$. If $x$ is non-constant, then $\mathfrak{H}$ has the structure of a graph of a homomorphism $\varphi : H \to S^1$, where $H$ stands some subgroup of $G$. To emphasize this nature of the group $\mathfrak{H}$, the following notation is commonly used:

$$H^\varphi := \{ \langle h, \varphi(h) : h \in H \rangle \}.$$
Table 4: Number of unstable eigenvalues for each isotypical component along the branch of the relative equilibrium with \( (\mathbb{Z}_8^t) \) symmetry (see item (iii) on page 17)

| Intervals for values of parameter \( \alpha \cdot 10^2 \) | [3.61, 5.48] | [5.49, 6.6] | [6.61, 8.09] | [8.10, 8.47] | [8.48, 13.53] |
|--------------------------------|----------------|----------------|----------------|----------------|----------------|
| Isotypical component          |                |                |                |                |                |
| \( U_{0,1} \)                | 0              | 0              | 0              | 0              | 0              |
| \( U_{1,1} \)                | 2              | 2              | 2              | 4              | 4              |
| \( U_{2,1} \)                | 2              | 2              | 4              | 4              | 6              |
| \( U_{3,1} \)                | 0              | 2              | 2              | 4              | 4              |
| \( U_{4,1} \)                | 0              | 0              | 0              | 0              | 2              |
| \( \bigoplus U_{i,1} \)      | 6              | 8              | 12             | 14             | 18             |

Table 5: Number of unstable eigenvalues for each isotypical component along the branch of the relative equilibrium with \( (\mathbb{Z}_8^3) \) symmetry (see item (iv) on page 17)

| Intervals for values of parameter \( \alpha \cdot 10^2 \) | [3.61, 3.68] | [3.69, 3.84] | [3.85] | [3.86, 5.24] | [5.25, 5.29] | [5.3, 5.52] | [5.53, 6.57] | [6.38, 8.65] | [8.66, 9.25] |
|--------------------------------|----------------|----------------|--------|----------------|----------------|----------------|----------------|----------------|----------------|
| Isotypical component          |                |                |        |                |                |                |                |                |                |
| \( U_{0,1} \)                | 0              | 0              | 0      | 0              | 0              | 0              | 0              | 0              | 0              |
| \( U_{1,1} \)                | 2              | 2              | 2      | 4              | 4              | 6              | 6              | 6              | 6              |
| \( U_{2,1} \)                | 2              | 2              | 4      | 4              | 6              | 6              | 6              | 6              | 6              |
| \( U_{3,1} \)                | 4              | 4              | 6      | 6              | 6              | 6              | 6              | 8              | 8              |
| \( U_{4,1} \)                | 2              | 2              | 2      | 2              | 2              | 2              | 4              | 4              | 4              |
| \( \bigoplus U_{i,1} \)      | 10             | 12             | 14     | 16             | 18             | 20             | 22             | 24             | 26             |

Table 6: Number of unstable eigenvalues for each isotypical component along the branch of the relative equilibrium with \( (D_8^t) \) symmetry (see item (vi) on page 17)

| Intervals for values of parameter \( \alpha \cdot 10^2 \) | [3.62, 3.83] | [3.84, 4.04] | [4.05, 5.38] | [5.39, 6.59] | [6.6, 7.3] | [7.31, 7.56] | [7.57, 13.55] |
|--------------------------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| Isotypical component          |                |                |                |                |                |                |                |
| \( U_{0,1} \)                | 0              | 0              | 0              | 0              | 2              | 2              | 2              |
| \( U_{1,1} \)                | 4              | 4              | 4              | 8              | 4              | 4              | 8              |
| \( U_{2,1} \)                | 4              | 8              | 8              | 4              | 4              | 8              | 8              |
| \( U_{3,1} \)                | 4              | 4              | 4              | 8              | 4              | 4              | 8              |
| \( U_{4,1} \)                | 4              | 4              | 4              | 8              | 4              | 4              | 8              |
| \( \bigoplus U_{i,1} \)      | 16             | 20             | 22             | 26             | 28             | 26             | 30             |

The group \( H^\varphi \) is called a twisted symmetry group with twisting homomorphism \( \varphi \).

Relative periodic solutions of our interest have symmetry groups which are subgroups of \( \Gamma \times S^1 \times S^1 \). Such a subgroup can be characterized by two twisting homomorphisms \( \varphi : K \to S^1 \) and \( \psi : K^\varphi \to S^1 \) for some subgroup \( K < \Gamma \). However, in order to simplify our notations, instead of writing \( K^\varphi, \psi \), we used the bold symbol \( K^\varphi \) to distinguish it from the group \( K^\varphi \) used for twisted symmetries of periodic solutions.
5.1 Notations used for the twisted subgroups of $\mathcal{H} := D_8 \times S^1$

The following symbols are used for the twisted subgroups of $\mathcal{K}$: we put $\xi := e^{\frac{2\pi i}{7}}$ and $\kappa := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and denote

\[
\begin{align*}
D_8 & := \{(\xi^k, 1) : k = 0, 1, \ldots, 7\} \cup \{(\xi^k \kappa, 1) : k = 0, 1, \ldots, 7\}, \\
D_8^d & := \{(\xi^k, (-1)^k) : k = 0, 1, \ldots, 7\} \cup \{(\xi^k \kappa, (-1)^k) : k = 0, 1, \ldots, 7\}, \\
\tilde{D}_4 & := \{(1, 1), (i, -1), (-1, 1), (-i, -1), (\xi \kappa, 1), (\xi \kappa, -1), (-\xi \kappa, 1), (-\xi \kappa, -1)\}, \\
D_4 & := \{(1, 1), (i, -1), (-1, 1), (-i, -1), (\kappa, 1), (i \kappa, 1), (-\kappa, 1), (-i \kappa, 1)\}, \\
D_2 & := \{(1, 1), (-1, -1), (\kappa, 1), (-\kappa, -1)\}, \\
\tilde{D}_2 & := \{(1, 1), (-1, -1), (\xi \kappa, 1), (-\xi \kappa, -1)\}, \\
Z_8^{t_1} & := \{(\xi^k, \xi^k) : k = 0, 1, \ldots, 7\}, \\
Z_8^{t_2} & := \{(\xi^k, \xi^{2k}) : k = 0, 1, \ldots, 7\}, \\
Z_8^{t_3} & := \{(\xi^k, \xi^{3k}) : k = 0, 1, \ldots, 7\}.
\end{align*}
\]

5.2 Notations used for the twisted subgroups of $\mathcal{K} := D_8 \times \{1\} \times S^1$

The following symbols are used for the twisted subgroups of $\mathcal{K}$:

\[
\begin{align*}
D_8 & := D_8 \times \{1\} \times \{1\}, \\
Z_8^{t_1} & := \{(\xi^k, 1, \xi^k) \in \mathcal{K} : k = 0, 1, \ldots, 7\}, \quad \xi := e^{\frac{2\pi i}{7}}, \\
D_2^d & := \{(1, 1, 1), (-1, -1), (\kappa, 1, 1), (-\kappa, 1, -1)\}, \\
\tilde{D}_2^d & := \{(1, 1, 1), (-1, -1), (\xi \kappa, 1, 1), (-\xi \kappa, 1, -1)\}, \\
Z_8^{t_2} & := \{(\xi^k, 1, \xi^{2k}) \in \mathcal{K} : k = 0, 1, \ldots, 7\}, \\
D_4 & := \{(1, 1, 1), (i, 1, -1)(-1, 1, 1), (-i, 1, -1), (\kappa, 1, 1), (i \kappa, 1, -1), (-\kappa, 1, 1), (-i \kappa, 1, 1)\}, \\
\tilde{D}_4^d & := \{(1, 1, 1), (i, 1, -1)(-1, 1, 1), (-i, 1, -1), (\xi \kappa, 1, 1), (i \xi \kappa, 1, -1), (-\xi \kappa, 1, 1), (-i \xi \kappa, 1, -1)\}, \\
Z_8^{t_3} & := \{(\xi^k, 1, \xi^{3k}) \in \mathcal{K} : k = 0, 1, \ldots, 7\}, \\
D_8^d & := \{(\xi^k, 1, (-1)^k) : (\xi^k \kappa, 1, (-1)^k) \in \mathcal{H} : k = 0, 1, \ldots, 7\}.
\end{align*}
\]

5.3 Notations used for the twisted subgroups of $\mathcal{K} := \mathbb{Z}_8^l \times S^1$, $l = 1, 2, 3$

In this case, the following symbols are used for the twisted subgroups of $\mathcal{K}$:

\[
\begin{align*}
Z_8^l & := \mathbb{Z}_8^l \times \{1\}, \\
Z_8^{t_1} & := \{(\xi^k, \xi^{lk}, \xi^k) \in \mathcal{K} : k = 0, 1, \ldots, 7\}, \quad \xi := e^{\frac{2\pi i}{7}}, \\
Z_8^{t_2} & := \{(\xi^k, \xi^{lk}, \xi^{2k}) \in \mathcal{K} : k = 0, 1, \ldots, 7\}, \\
Z_8^{t_3} & := \{(\xi^k, \xi^{lk}, \xi^{3k}) \in \mathcal{K} : k = 0, 1, \ldots, 7\}, \\
Z_8^{c} & := \{(\xi^k, \xi^{lk}, (-1)^k) \in \mathcal{K} : k = 0, 1, \ldots, 7\}.
\end{align*}
\]
5.4 Notations used for the twisted subgroups of $K := D_8^d \times S^1$

For this group, the following symbols are used for the twisted subgroups of $K$:

$$
\begin{align*}
D_8 & := D_8^d \times \{1\}, \\
Z_{8^d}^{t_1} & := \{(\xi^k, (-1)^k, \xi^k) \in K : k = 0, 1, \ldots, 7\}, \quad \xi := e^{\frac{2\pi}{7}}, \\
D_2^d & := \{(1, 1, 1), (-1, 1, -1), (\kappa, 1, 1), (-\kappa, 1, -1)\}, \\
\tilde{D}_2^d & := \{(1, 1, 1), (-1, 1, -1), (\xi \kappa, -1, 1), (-\xi \kappa, -1, -1)\}, \\
Z_{8^d}^{t_2} & := \{(\xi^k, (-1)^k, \xi^{2k}) \in K : k = 0, 1, \ldots, 7\}, \\
D_4^d & := \{(1, 1, 1), (i, 1, -1)(-1, 1, 1), (-i, 1, -1), (\kappa, 1, 1), (i \kappa, 1, -1), \\
& \quad (-\kappa, 1, 1), (-i \kappa, 1, -1)\}, \\
\tilde{D}_4^d & := \{(1, 1, 1), (i, 1, -1)(-1, 1, 1), (-i, 1, -1), (\xi \kappa, -1, 1), (i \xi \kappa, -1, -1), \\
& \quad (-\xi \kappa, -1, 1), (-i \xi \kappa, -1, -1)\}, \\
Z_{8^d}^{t_3} & := \{(\xi^k, (-1)^k, \xi^{3k}) \in K : k = 0, 1, \ldots, 7\}, \\
D_8^d & := \{(\xi^k, (-1)^k, (-1)^k), \{(\xi^k \kappa, (-1)^k, (-1)^k) \in H : k = 0, 1, \ldots, 7\}.
\end{align*}
$$

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