BOUNDARY BEHAVIORS AND SCALAR CURVATURE OF COMPACT MANIFOLDS

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ABSTRACT. In this paper, by modifying the arguments in \textsuperscript{16}, we get some rigidity theorems on compact manifolds with nonempty boundary. The results in this paper are similar with those in \textsuperscript{14} and \textsuperscript{16}. Like \textsuperscript{14} and \textsuperscript{16}, we still use quasi-spherical metrics introduced by \textsuperscript{1} to get monotonicity of some quantities.

1. Introduction

In \textsuperscript{14}, the authors proved the following: Let $(\Omega, g)$ be a compact manifold of dimension three with smooth boundary $\Sigma$ which has positive Gaussian curvature and has positive mean curvature. Suppose $\Omega$ has nonnegative scalar curvature, then for each boundary component $\Sigma_i$ of $\Sigma$ satisfies,

\begin{equation}
\int_{\Sigma_i} (H^i_0 - H) \, d\Sigma_i \geq 0
\end{equation}

where $H^i_0$ is the mean curvature of $\Sigma_i$ with respect to the outward normal when it is isometrically embedded in $\mathbb{R}^3$, $d\Sigma_i$ is the volume form on $\Sigma_i$ induced from $g$. Moreover, if equality holds for some $\Sigma_i$ then $\Sigma$ has only one component and $\Omega$ is a domain in $\mathbb{R}^3$.

The result gives restriction on a convex surface $\Sigma$ in $\mathbb{R}^3$ which can bound a compact manifold with nonnegative scalar curvature such that the mean curvature of $\Sigma$ is positive. It is interesting to see what one can say for convex surface $\Sigma$ in $\mathbb{H}^3_{-\kappa^2}$, the hyperbolic space with constant curvature $-\kappa^2$.

The result mentioned above has other interpretation. It implies the quasi-local mass introduced by Brown-York \textsuperscript{3, 4} is positive under the condition that the boundary has positive Gaussian curvature. In \textsuperscript{9},

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Liu and Yau introduced a quasi-local mass in spacetime. This quasi-local mass were also introduced by Epp [6] and Kijoswki [8]. More importantly, Liu and Yau proved its positivity, using [14]. A recent definition of quasi-local mass that relates with these works please see [17].

Motivated by [14, 9, 10], in a recent work [16] Wang and Yau proves the following: Suppose $(\Omega, g)$ is a three dimensional manifold with smooth boundary $\Sigma$ with positive mean curvature $H$, which is a topological sphere. Suppose the scalar curvature $R$ of $\Omega$ satisfies $R \geq -6\kappa^2$ and the Gaussian curvature of $\Sigma$ is larger than $-\kappa^2$, then there is a future directed time-like vector value function $W^0$ on $\Sigma$ such that

$$\int_{\Sigma} (H_0 - H) W^0 d\Sigma$$

is time-like. Here $H_0$ is the mean curvature of $\Sigma$ when isometrically embedded in $H^3_\kappa$, which is in turns isometrically embedded in $\mathbb{R}^{3,1}$, the Minkowski space. In this result, the vector $W^0$ is not very explicit because it is obtained by solving a backward parabolic equation by prescribing data at infinity.

In this work, by modifying the argument in [16], we get similar result by replacing $W^0$ by $W_{\Sigma_0} = (x_1, x_2, x_3, \alpha t)$ for some $\alpha > 1$ depending only on the intrinsic geometry of $\Sigma$. Here $(x_1, x_2, x_3, t)$ is the future directed unit normal vector of $H^3_\kappa$ in $\mathbb{R}^{3,1}$. See Theorem 3.1 for a more precise statement. We believe that the same result should be true with $W_{\Sigma_0} = (x_1, x_2, x_3, t)$, but we cannot prove it for the time being.

As a consequence, if $o$ is a point inside of $\Sigma$ in $H^3_\kappa$, then

$$\int_{\Sigma} (H_0 - H) \cosh \kappa r d\Sigma \geq 0,$$

where $r$ is the distance function from $o$ in $H^3_\kappa$. Moreover, equality holds if and only if $(\Omega, g)$ is a domain in $H^3_\kappa$. The results can be considered as generalization of the results in [14]. In fact, if we let $\kappa \to 0$, we may obtain the inequality (1.1).

The paper is organized as follows. In §2, we list some facts that we need, most of them are from [16]. In §3, we prove our main results. We will also give some examples that $\alpha$ in Theorem 3.1 can be taken to be 1 and also study some properties of $\int_{\Sigma} (H_0 - H) \cosh \kappa r d\Sigma$.

2. Preliminary

Most materials in this section are from Wang and Yau [16]. Let $(\Omega, g)$ be a compact manifold with smooth boundary so that $\Sigma = \partial \Omega$ is a topologically sphere. Let $H$ be the mean curvature with respect
to the outward normal and $K$ be the Gaussian curvature of $\Sigma$ and let $R$ be the scalar curvature of $\Omega$. In our convention, the mean curvature of the unit sphere in $\mathbb{R}^3$ with respect to the outward normal is 2. By [12], we have the following:

**Lemma 2.1.** Suppose the Gaussian curvature $K$ of $\Sigma$ satisfies $K > -\kappa^2$. Then $\Sigma$ can be isometrically embedded into the hyperbolic space $\mathbb{H}^3_{-\kappa^2}$ with constant curvature $-\kappa^2$ as a convex surface which bounds a convex domain $D$ in $\mathbb{H}^3_{-\kappa^2}$. Moreover, the embedding is unique up to an isometry of $\mathbb{H}^3_{-\kappa^2}$.

Since $\Sigma$ is a topological sphere, its image $\Sigma_0$ under the embedding divides $\mathbb{H}^3_{-\kappa^2}$ into two components, the exterior and the interior of $\Sigma_0$. Let $N$ be the unit outward normal of $\Sigma_0$. It is said to be convex if the second fundamental form $h(X,Y) = -\langle \nabla^\kappa_X Y, N \rangle$ is positive definite for $X$ and $Y$ tangent to $\Sigma_0$, where $\nabla^\kappa$ is the covariant derivative of $\mathbb{H}^3_{-\kappa^2}$. The interior $D$ of $\Sigma_0$ being convex means that $D$ is geodesically convex.

The existence and uniqueness of the embedding were proved by Pogorelov [12]. The convexity of $\Sigma_0$ and $D$ were proved by do Carmo and Warner [5].

Further identify $\mathbb{H}^3_{-\kappa^2}$ with
$$\left\{ (x_1, x_2, x_3, t) \in \mathbb{R}^{3,1} \mid x_1^2 + x_2^2 + x_3^2 - t^2 = -\frac{1}{\kappa^2}, \ t > 0 \right\}$$
where $\mathbb{R}^{3,1}$ is the Minkowski space consisting of $X = (x_1, x_2, x_3, t)$ with the Lorentz metric $dx_1^2 + dx_2^2 + dx_3^2 - dt^2$. Position vectors in $\mathbb{R}^{3,1}$ can be parametrized as

$$X(p, \rho) = \frac{1}{\kappa} (\sinh \kappa r \cos \theta, \sinh \kappa r \sin \theta \cos \psi, \sinh \kappa r \sin \theta \sin \psi, \cosh \kappa r).$$

(2.1)

The metric of $\mathbb{H}^3_{-\kappa^2}$ is then

$$dr^2 + \kappa^{-2} \sinh^2 \kappa r d\sigma^2 = dr^2 + \kappa^{-2} \sinh^2 \kappa r (d\theta^2 + \sin^2 \theta d\psi^2).$$

Note that $r$ is the geodesic distance of a point from $(0, 0, 0, 1/\kappa) \in \mathbb{H}^3_{-\kappa^2}$.

Let $\Sigma_\rho$ be the level surface outside $\Sigma_0$ in $\mathbb{H}^3_{-\kappa^2}$ with distance $\rho$ from $\Sigma_0$. Foliate $\mathbb{H}^3_{-\kappa^2} \setminus D$ by $\Sigma_\rho$, $\rho \geq 0$. The hyperbolic metric can be written as $ds^2_{\mathbb{H}^3_{-\kappa^2}} = dr^2 + g_\rho$, where $g_\rho$ is the induced metric on $\Sigma_\rho$.

Suppose $f F : \Sigma \to \mathbb{H}^3_{-\kappa^2}$ is the embedding with unit outward normal $N$. Then $\Sigma_\rho$ as a subset of $\mathbb{R}^{3,1}$ is given by

$$X(p, \rho) = \cosh \kappa \rho \ X(p, 0) + \kappa^{-1} \sinh \kappa \rho \ N(p, 0)$$

(2.2)
Here for simplicity, \((p, \rho)\) denotes a point \(\Sigma_\rho\) which lies on the geodesic from the point \(p \in \Sigma_0\) and \(X(p, 0) = X(F(p))\).

Suppose in addition that the mean curvature of \(H\) of \(\Sigma\) with respect to \((\Omega, g)\) is positive and the scalar curvature \(R\) of \(\Omega\) is greater than or equal to \(-6\kappa^2\). Wang and Yau \cite{16} are able to solve the following parabolic equation

\[
\begin{cases}
2H_0 \frac{\partial u}{\partial \rho} = 2u^2 \Delta \rho u + (u - u^3)(R^\rho + 6\kappa^2), \\
u(p, 0) = \frac{H_0(p, 0)}{H(p)},
\end{cases}
\]

for all \(\rho \geq 0\) with positive and bounded solution \(u\). Here \(\Delta_\rho\) is the Laplacian operator of \(\Sigma_\rho\), \(R^\rho\) is scalar curvature of \(\Sigma_\rho\), and \(H_0\) is the mean curvature of \(\Sigma_\rho\) which is positive. We need the following result which is proved by Wang and Yau, see Theorem 6.1 and Corollary 6.1 in \cite{16}.

**Theorem 2.2.** [Wang-Yau] Let \((\Omega, g)\) be a 3-dimensional compact Riemannian manifold with nonempty smooth boundary which is a topological sphere. Suppose the scalar curvature of \(R\) of \(\Omega\) satisfies \(R \geq -6\kappa^2\), the Gaussian curvature of its boundary \(\Sigma\) satisfies \(K > -\kappa^2\), and the mean curvature \(H\) of the boundary with respect to outward unit norm is positive. Let \(X\) be the position vector of \(H^3_{-\kappa^2}\) in \(\mathbb{R}^3\) then

\[
\lim_{\rho \to \infty} \int_{\Sigma_\rho} (H_0 - H)X \cdot \zeta \leq 0
\]

for any future directed null vector \(\zeta\) in \(\mathbb{R}^3\).

**Corollary 2.3.** With the same assumptions and notations as in Theorem 2.2,

\[
\lim_{\rho \to \infty} \int_{\Sigma_\rho} (H_0 - H) \cosh \kappa r d\Sigma_\rho \geq 0.
\]

where \(r\) is as in \(\mathbb{2.1}\).

We also have the following rigidity result.

**Proposition 2.4.** With the same assumptions and notations as in Theorem 2.2, Suppose

\[
\lim_{\rho \to \infty} \int_{\Sigma_\rho} (H_0 - H)X \cdot \zeta d\Sigma_\rho = 0
\]

for some future directed null vector \(\zeta\) in \(\mathbb{R}^3\), where the inner product is given by the Lorentz metric. Then \(\Omega\) is a domain in \(\mathbb{H}^3_{-\kappa^2}\).
Proof. For simplicity, let us assume that $\kappa = 1$. As in the proof of Theorem 2.2 in \cite{16}, let $ds^2 = u^2d\rho^2 + g_\rho$ be the quasi-spherical metric where $u$ is the solution of (2.3). Let $(M, \check{g})$ be the manifold by gluing $(\Omega, g)$ with $\mathbb{H}^3_{-1} \setminus D$ with metric $ds^2$. By \cite{16}, if (2.4) is true, then the manifold $(M, \check{g})$ has a Killing spinor $\phi$ which is nontrivial, smooth away from $\Sigma$ and is continuous. More precisely, $\phi$ satisfies:

\begin{equation}
\nabla_V \phi + \frac{\sqrt{-1}}{2} c(V) \cdot \phi = 0
\end{equation}

where $c(V)$ is the Clifford multiplication. Hence $M \setminus \Omega$ is Einstein. Since $M$ has dimension three, the sectional curvature is $-1$ in $(M \setminus \Omega, \check{g})$, see \cite{2}, for example. Let $h^0_{ij}$ and $h_{ij}$ be the second fundamental form of $\Sigma_0$ with respect to the metrics $ds^2_{\mathbb{H}^3_{-1}}$ and $ds^2$ respectively. Then $h_{ij} = u^{-1} h^0_{ij}$. By the Gauss equation and the fact that both $ds^2_{\mathbb{H}^3_{-1}}$ and $ds^2$ have constant curvature $-1$, $u \equiv 1$.

On the other hand, $\phi$ is not zero on $\Sigma_0$ and so $\phi$ is a nontrivial Killing spinor in $(\Omega, g)$ satisfying (2.5) and $g$ has constant curvature $-1$ as before.

We claim that the second fundamental forms of $\Sigma_0$ with respect to $g$ and $ds^2_{\mathbb{H}^3_{-1}}$ are equal. If this is true, then by the proof of \cite[Lemma 4.1]{14}, we can conclude that $\check{g}$ is smooth. From this it is easy to see that $(M, \check{g}) = \mathbb{H}^3_{-1}$.

Let us prove the claim. Since $\Sigma_0$ is a topological sphere, for some $a > 0$ a tubular neighborhood of $\Sigma_0 \times (0, a)$ in $(\Omega, g)$ is simply connected. Since it has constant curvature $-1$, $(\Sigma_0 \times (0, a), g)$ can be isometrically embedded in $\mathbb{H}^3_{-1}$, see \cite[p.43]{15}. Denote the embedding by $\iota = (u_1, u_2, u_3)$ where $(u_1, u_2, u_3)$ are global coordinates in $\mathbb{H}^3_{-1}$. Since $\iota$ is an isometry, the normal curvatures of $\Sigma_0 \times \{\tau\}$ for $0 < \tau < a$ with respect to $g$ and the hyperbolic metric are equal. Hence they are uniformly bounded on $\Sigma_0 \times (0, a)$. Note that $\Sigma_0$ is convex in $\mathbb{H}^3_{-1}$, so $\Sigma_0 \times \{\tau\}$ is also convex when embedded in $\mathbb{H}^3_{-1}$, for $0 < \tau < a$ provided $a$ is small. By \cite[VI, \S 3]{13}, for any $k \geq 0$, $|\nabla^\iota u_i|$ are uniformly bounded on $\Sigma_0 \times (0, a)$, where $\nabla^\iota$ is the covariant derivatives of $\Sigma_0 \times \{\tau\}$ with induced metric by $g$. Hence by taking a subsequence of $\tau_j \to 0$, we obtain an isometric embedding of $(\Sigma, g)$. In this embedding the second fundamental form with respect to $g$ and $ds^2_{\mathbb{H}^3_{-1}}$ are equal. Since the embedding of $\Sigma$ is unique up to an isometry of $\mathbb{H}^3_{-1}$, the claim is true.\hfill $\square$
3. Main results

Let \((\Omega, g)\) be as in Theorem 2.2 and let \(\partial \Omega = \Sigma\). With the same notations as in the theorem, suppose \(o = (0, 0, 0, 1/\kappa)\) is in \(D\) which is the interior of \(\Sigma\) in \(\mathbb{H}^3_{-\kappa^2} \subset \mathbb{R}^{3,1}\). Let \(\Sigma_0\) be the image of \(\Sigma\) under the embedding described in §1. \(H_0\) is the mean curvature of \(\Sigma_0\) in \(\mathbb{H}^3_{-\kappa^2}\). We also identify \(\Sigma\) with \(\Sigma_0\). Let \(B_o(R_1)\) and \(B_o(R_2)\) be geodesic balls in \(\mathbb{H}^3_{-\kappa^2}\) such that \(B_o(R_1) \subset D \subset B_o(R_2)\).

We want to prove the following:

**Theorem 3.1.** Let \((\Omega, g)\) be a compact manifold with smooth boundary \(\Sigma\). Assume the following are true

(i) The scalar curvature \(R\) of \((\Omega, g)\) satisfies \(R \geq -6\kappa^2\) for some \(\kappa > 0\).

(ii) \(\Sigma\) is a topological sphere with Gaussian curvature \(K > -\kappa^2\) and with positive mean curvature \(H\).

With the above notations, for any future directed null vector \(\zeta\) in \(\mathbb{R}^{3,1}\),

\[
(3.1) \quad m(\Omega; \zeta) = \int_{\Sigma} (H_0 - H) W_{\Sigma_0} \cdot \zeta d\Sigma \leq 0
\]

where \(W_{\Sigma_0} = (x_1, x_2, x_3, \alpha t)\) with

\[
\alpha = \coth \kappa R_1 + \frac{1}{\sinh \kappa R_1} \left( \frac{\sinh^2 \kappa R_2}{\sinh^2 \kappa R_1} - 1 \right)^{1/2},
\]

\(X = (x_1, x_2, x_3, t)\) being the position vector in \(\mathbb{R}^{3,1}\) and the inner product is given by the Lorentz metric. Moreover, if equality holds in (3.1) for some future directed null vector \(\zeta\), then \((\Omega, g)\) is a domain in \(\mathbb{H}^3_{-\kappa^2}\).

**Remark 3.2.** If \(R_1, R_2 \to \infty\) in such a way that \(R_2 - 2R_1 \to -\infty\), then \(\alpha \to 1\).

In (2.1), the position vector in \(\mathbb{R}^{3,1}\) is given by

\[
(3.2) \quad X = (x_1, x_2, x_3, t) = \frac{1}{\kappa} (\phi_1 \sinh \kappa r, \phi_2 \sinh \kappa r, \phi_3 \sinh \kappa r, \cosh \kappa r)
\]

where \((\phi_1, \phi_2, \phi_3)\) denote position vectors of points of \(\mathbb{S}^2\) in \(\mathbb{R}^3\). Let \(\{\Sigma_\rho\}\) be the foliation of \(\mathbb{H}^3_{-\kappa^2} \setminus D\) described in §1. We need the following:

**Lemma 3.3.** With the same assumptions and notations as in Theorem 3.1 let \(y_1, y_2, y_3 \in \mathbb{R}\) with \(\sum_{i=1}^{3} y_i^2 = 1\), the following are true.

(i) For any \(\rho > 0\),

\[
(3.3) \quad \frac{\partial r}{\partial \rho} \geq \frac{\sinh \kappa R_1}{\sinh \kappa R_2}
\]
(ii) If \( \phi = \sum_{i=1} y_i \phi_i \), then for \( \rho > 0 \)
\[
(\frac{\partial \phi}{\partial \rho})^2 \leq (1 - \phi^2) \kappa^2 \sinh^{-2} \kappa r \left( 1 - (\frac{\partial r}{\partial \rho})^2 \right)
\]
Hence
\[
\mu \cdot \kappa \frac{\partial r}{\partial \rho} \geq \left| \frac{\partial \phi}{\partial \rho} \right|
\]
where
\[
\mu = \frac{1}{\sinh \kappa R_1} \left( \frac{\sinh^2 \kappa R_2}{\sinh^2 \kappa R_1} - 1 \right)^{\frac{1}{2}}.
\]

Proof. (i) Recall that \( o = (0, 0, 0, 1/\kappa) \in D \) and \( r \) is the geodesic distance in \( \mathbb{H}^3_{-\kappa^2} \) from \( o \). \( D \) is geodesically convex by Lemma 2.1. For any \( x, y \in \mathbb{H}^3_{-\kappa^2} \), denote the geodesic from \( x \) to \( y \) parametrized by arc length by \( \overline{xy} \).

Let \( p \in \Sigma \) and let \( \gamma(p) \) be the geodesic through \( p \) so that \( \gamma(p) \) is orthogonal to \( \Sigma \) at \( p \) with arc length parametrization. Moreover, \( \gamma(0) = p \) and \( \gamma(p) \) is outside \( \Sigma \) for \( \rho > 0 \). Let \( q \) be the point on \( \gamma \) such that \( d(o, q) = d(o, \gamma) \). Then \( q = \gamma(p_1) \) with \( p_1 < 0 \) because \( D \) is geodesically convex. Since \( \frac{\partial r}{\partial \rho} = \langle \nabla r, \nabla \rho \rangle \), \( \frac{\partial r}{\partial \rho} \) is nondecreasing on \( \rho > 0 \) along \( \gamma \) and \( \frac{\partial r}{\partial \rho} > 0 \) at \( p \). Let \( \beta = \overline{op} \) and let \( \eta \) be the geodesic from \( p \) on the totally geodesic \( \mathbb{H}^2_{-\kappa^2} \) containing \( \gamma \) and \( \beta \) such that \( \eta \) is tangent to \( \Sigma_0 \). Let \( x, y \) be the intersection of \( \eta \) with \( \partial B_o(R_2) \). Then \( \frac{\partial r}{\partial \rho} \geq \sin \varphi \) where \( \varphi \) is the angle between \( \overline{xo} \) and \( \overline{xp} \). Since \( \eta \) is outside \( B_o(R_1) \) so
\[
\sin \varphi \geq \frac{\sinh \kappa R_1}{\sinh \kappa R_2}.
\]
Hence we have
\[
\frac{\partial r}{\partial \rho} \geq \frac{\sinh \kappa R_1}{\sinh \kappa R_2}
\]
on \( \mathbb{H}^3_{-\kappa^2} \setminus D \).

(ii) Since the inner product in \( \mathbb{R}^3 \) of \( (y_1, y_2, y_3) \) and \( (\phi_1, \phi_2, \phi_3) \) is \( \phi \), we may assume that \( \phi = \cos \theta \) in (2.1). The hyperbolic metric outside \( D \) is given by
\[
ds^2_{\mathbb{H}^3_{-\kappa^2}} = d\rho^2 + g_\rho = dr^2 + \frac{1}{\kappa^2} \sinh^2 \kappa r (d\theta^2 + \sin^2 \theta d\psi^2).
\]
Compute \( ds^2_{\mathbb{H}^3_{-\kappa^2}} \left( \frac{\partial}{\partial \rho}, \frac{\partial}{\partial \rho} \right) \) in the above two forms of \( ds^2_{\mathbb{H}^3_{-\kappa^2}} \), we have
\[
1 = \left( \frac{\partial r}{\partial \rho} \right)^2 + \frac{1}{\kappa^2} \sinh^2 \kappa r \left[ \left( \frac{\partial \theta}{\partial \rho} \right)^2 + \sin^2 \theta \left( \frac{\partial \psi}{\partial \rho} \right)^2 \right] \geq \left( \frac{\partial r}{\partial \rho} \right)^2 + \frac{1}{\kappa^2} \sinh^2 \kappa r \left( \frac{\partial \theta}{\partial \rho} \right)^2.
\]
Since $\phi = \cos \theta$, (ii) follows.

The last assertion follows from (i), (ii), the fact that $|\phi| \leq 1$ and the fact that $r \geq R_1$ for $\rho \geq 0$.

□

Lemma 3.4. With the same assumptions and notations as in Theorem 3.1,

$$H_0 \frac{\partial}{\partial \rho} X + \Delta_\rho X - 2\kappa^2 X = 0$$

in $\mathbb{H}^3_{-\kappa^2} \setminus D$.

Proof. In the representation of $\mathbb{H}^3_{-\kappa^2}$ in (3.2), the Laplacian of $\mathbb{H}^3_{-\kappa^2}$ is given by

$$\Delta = \frac{\partial^2}{\partial \rho^2} + 2\kappa \coth \kappa r \frac{\partial}{\partial r} + \kappa^2 \sinh^{-2} \kappa r \Delta_{S^2}.$$ 

So $\Delta X = 3\kappa^2 X$. In the foliation (2.2), the metric of $\mathbb{H}^3_{-\kappa^2}$ is given by $d\rho^2 + g_\rho$ where $g_\rho$ is the induced metric on level surface $\Sigma_\rho$. The Laplacian on $\mathbb{H}^3_{-\kappa^2}$ is given by

$$\Delta = \frac{\partial^2}{\partial \rho^2} + H_0 \frac{\partial}{\partial \rho} + \Delta_\rho.$$ 

Using (2.2), we have

$$3\kappa^2 X = \frac{\partial^2}{\partial \rho^2} X + H_0 \frac{\partial}{\partial \rho} X + \Delta_\rho X$$

$$= \kappa^2 X + H_0 \frac{\partial}{\partial \rho} X + \Delta_\rho X.$$  

(3.7)

From this the result follows. □

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. By [16] and Lemma 3.4

$$\frac{d}{d\rho} \int_{\Sigma_\rho} (H_0 - H) X d\Sigma_\rho$$

$$= \int_{\Sigma_\rho} -\frac{1}{2} u^{-1} (u - 1)^2 (R^\rho + 2\kappa^2) X + (u - 1) \left( \frac{H_0}{u} \frac{\partial}{\partial \rho} X + \Delta_\rho X - 2\kappa^2 X \right) d\Sigma_\rho$$

$$= -\int_{\Sigma_\rho} u^{-1} (u - 1)^2 \left[ \frac{1}{2} (R^\rho + 2\kappa^2) X + H_0 \frac{\partial}{\partial \rho} X \right] d\Sigma_\rho.$$ 

Let $\lambda_a (p, \rho)$ be the principal curvature of the level surface. Then $\lambda_a = \kappa \tanh \kappa (\mu_a + \rho)$, $\kappa$ or $\kappa \coth \kappa (\mu_a + \rho)$ with $\mu_a > 0$, see [16]. Hence $H_0 = \lambda_1 + \lambda_2$ and $R^\rho + 2\kappa^2 = 2\lambda_1 \lambda_2$. Let $W_0 = \alpha \cosh \kappa r$ and $W =
\[ \phi \sinh \kappa r \text{ where } \phi = \sum_{i=1}^{3} y_i \phi_i, \sum_{i=1}^{3} y_i^2 = 1 \text{ and } \alpha \text{ is the constant in the theorem. Then} \]

\[ \frac{1}{2}(R^\rho + 2\kappa^2)W_0 + H_0(W_0) = \alpha \left[ \lambda_1 \lambda_2 \cosh \kappa r + \kappa(\lambda_1 + \lambda_2) \sinh \kappa r \cdot \frac{\partial r}{\partial \rho} \right], \]

(3.9)

\[ \frac{1}{2}(R^\rho + 2\kappa^2)W + H_0W = \lambda_1 \lambda_2 \phi \sinh \kappa r + \kappa(\lambda_1 + \lambda_2) \left( \phi \cosh \kappa r \cdot \frac{\partial r}{\partial \rho} + \frac{1}{\kappa} \sinh \kappa r \cdot \frac{\partial \phi}{\partial \rho} \right). \]

Combining this with by Lemma 3.3, (3.2), (3.8) and the fact that \( r > R_1 \) in \( \mathbb{H}^3_{-\kappa^2} \setminus D \), we have

\[ \frac{d}{d\rho} \int_{\Sigma_\rho} (H_0 - H)(W - W_0) d\Sigma_\rho \geq 0. \]

(3.11)

By Theorem 2.2 and Corollary 2.3 we conclude that (3.1) is true.

Suppose equality holds in (3.1) for some future directed null vector \( \zeta \), then using Corollary 2.3 we have

\[ \lim_{\rho \to \infty} \int_{\Sigma_\rho} (H_0 - H)X \cdot \zeta d\Sigma_\rho = 0. \]

(3.12)

By Proposition 2.4, \((\Omega, g)\) is a domain in \( \mathbb{H}^3_{-\kappa^2} \).

\[ \text{Remark 3.5.} \quad (3.11) \text{ means that for any future directed null vector } \zeta, \]

\[ \frac{d}{d\rho} \int_{\Sigma_\rho} (H_0 - H)W_{\Sigma_\rho} \cdot \zeta d\Sigma_\rho \geq 0. \]

(3.13)

From the proof of (3.11), it is easy to see that (3.13) is still true if \( \zeta \) is future directed time-like.

Corollary 3.6. With the same assumptions and notations as in Theorem 3.1. Then for any \( \rho \geq 0 \), the following vector is either zero or is future directed non space-like:

\[ \mathbf{m}(\rho) = \int_{\Sigma_\rho} (H_0 - H)W_{\Sigma_\rho} d\Sigma_\rho. \]

In particular, if \((\Omega, g)\) is not a domain in \( \mathbb{H}^3_{-\kappa^2} \), then

\[ \mathbf{m}(\Omega) = \int_{\Sigma} (H_0 - H)W_{\Sigma} d\Sigma \]

is future directed time-like.
Proof. By the proof of Theorem 3.1 and the characterization of future directed non space-like vector [16], we conclude the first part of the corollary is true. If \((\Omega, g)\) is not a domain in \(H^3_{-\kappa^2}\), by the rigidity part of the theorem, this vector cannot be zero and cannot be null. Hence it is future directed time-like. □

**Corollary 3.7.** With the same assumptions and notations as in Theorem 3.1, let

\[
m(\rho) = \int_{\Sigma_\rho} (H_0 - H) W_{\Sigma_0} d\Sigma_\rho.
\]

then \(\frac{d}{d\rho} (|m(\rho)|^2) \geq 0\), where \(|m(\rho)|\) is the Lorentz norm of \(m(\rho)\).

Proof. For any fixed \(\rho_0\), let \(\zeta\) be the vector \(m(\rho_0)\). By Corollary 3.6 as mentioned above, \(\zeta\) is a future directed non space-like, note that

\[
\frac{d}{d\rho} (|m(\rho)|^2) \bigg|_{\rho=\rho_0} = 2 \left( \frac{d}{d\rho} \int_{\Sigma_\rho} (H_0 - H) W_{\Sigma_0} d\Sigma_\rho \right) \bigg|_{\rho=\rho_0} \cdot \zeta
= 2 \frac{d}{d\rho} \int_{\Sigma_\rho} (H_0 - H) W_{\Sigma_0} \cdot \zeta d\Sigma_\rho \bigg|_{\rho=\rho_0}
\]

By Remark 3.5 we have

\[
\frac{d}{d\rho} (|m(\rho)|^2) \geq 0.
\]

□

**Theorem 3.8.** Let \((\Omega, g)\) be as in Theorem 3.1 and let \(\Sigma_0\) be the image of isometric embedding of \(\partial \Omega = \Sigma\) in \(H^3_{-\kappa^2}\) which encloses \(D\). Then for any \(o \in \Sigma_0:\)

\[
\int_{\Sigma_0} (H_0 - H)(y) \cosh \kappa r(o, y) d\Sigma_0(y) \geq 0
\]

Equality holds for some \(o\) if and only if \((\Omega, g)\) is a domain of \(H^3_{-\kappa^2}\). In particular, if \(\Sigma\) is a standard sphere, then

\[
\int_{\Sigma_0} (H_0 - H)(y) d\Sigma_0(y) \geq 0
\]

and equality holds if and only if \((\Omega, g)\) is a domain of \(H^3_{-\kappa^2}\).

Proof. We may embed \(H^3_{-\kappa^2}\) in \(R^{3,1}\) such that \(o = (0, 0, 0, 1/\kappa)\). The result then follows from Theorem 3.1. □

Theorem 3.1 implies a previous result of the authors [14]...
Theorem 3.9. Let $(\Omega, g)$ be a compact three manifold with nonnegative scalar curvature and with smooth boundary $\Sigma$. Suppose $\Sigma$ has positive Gaussian curvature and positive mean curvature $H$, then

$$\int_\Sigma (H_0 - H) \, d\Sigma \geq 0$$

where $H_0$ is the mean curvature of $\Sigma$ when it is isometrically embedded in $\mathbb{R}^3$.

This result follows from Theorem 3.8 and the following lemma.

Lemma 3.10. With the same assumptions and notations as in Theorem 3.9, suppose $H_\kappa$ is the mean curvature of $\Sigma$ when it is isometrically embedded in $\mathbb{H}^3_{-\kappa^2}$, $\kappa > 0$. Then there exists $\kappa_i \to 0$ such that $\lim_{i \to \infty} H_{\kappa_i} = H_0$.

Proof. Let $\mathbb{H}^3_{-\kappa^2}$ be represented by the metric:

$$(3.14) \frac{4}{(1 - \kappa^2|x|^2)^2}(dx_1^2 + dx_2^2 + dx_3^2)$$

defined on $x_1^2 + x_2^2 + x_3^2 = |x|^2 < \kappa^{-2}$. We may assume that $(\Sigma, g)$ is embedded to $\mathbb{H}^3_{-\kappa^2}$ with embedding $\iota_\kappa$ such that $p$ is mapped to the origin, where $p$ is some fixed point. Then it is easy to see that $\Sigma \subset B_\kappa(0, 2d)$ where $d$ the intrinsic diameter of $\Sigma$ and $B_\kappa$ is the geodesic ball with respect to the metric (3.14).

Let $\iota_\kappa = (u_{1,\kappa}, u_{2,\kappa}, u_{3,\kappa})$ in terms of the global coordinates $x_1, x_2, x_3$. Since the Gauss curvature of $\Sigma$ is positive, by [13] VII[2,§3], we conclude that for any $k \geq 0$, for $0 < \kappa \leq 1$ and $1 \leq j \leq 3$, $|\nabla^k u_j|\kappa|$ are uniformly bounded. There exists $\kappa_i \to 0$ such that $\iota_{\kappa_i}$ together its derivatives converge to an embedding of $\Sigma$ in the Euclidean space. Using the fact that the embedding of $\Sigma$ in $\mathbb{R}^3$ is unique, it is easy to see that the lemma is true. \qed

Finally, we would like to give some examples to illustrate that in certain situations, $\alpha$ in Theorem 3.1 can be chosen as 1. The proofs of these examples are direct application of Theorem 3.1, Corollary 3.8 and the representation of $X$ given by (3.2).

Example 3.11. With the same assumptions and notations as in Theorem 3.1, if $\Sigma$ a standard sphere and $H$ is constant, then $m(\Omega, \zeta) \leq 0$ with $W_{\Sigma_0} = (x_1, x_2, x_3, t)$, for all future directed null vector $\zeta$. Here we have assumed that $(0, 0, 0, 1/\kappa)$ is inside $\Sigma_0$, which is the image of $\Sigma$ under the embedding described in §1.
Example 3.12. With the same assumptions and notations as in Theorem 3.1, if $\Sigma$ a standard sphere its mean curvature $H$ is orthogonal to the first eigenfunctions of $SS^2$, then $m(\Omega, \zeta) \leq 0$ with $W_{\Sigma_0} = (x_1, x_2, x_3, t)$, for all future directed null vector $\zeta$. Here we assume that $(0, 0, 0, 1/\kappa)$ is the center of the geodesic ball in $H^3_{-\kappa^2}$ enclosed by $\Sigma_0$, which is the image of $\Sigma$ under the embedding described in $\S 1$.

Remark 3.13. It is easy to see that in Example 3.11 and Example 3.12, if $(\Omega, g)$ is not a domain of $H^3$, then on all of their small perturbations, $m(\Omega, \zeta) \leq 0$ with $W_{\Sigma_0}$. For perturbations of Example 3.11, we only assume that $(0, 0, 0, 1/\kappa)$ is inside $\Sigma_0$. For perturbations of Example 3.12, the inequality is true if $(0, 0, 0, 1/\kappa)$ is at some particular position inside $\Sigma_0$.

Let $f(p) = \int_{\Sigma_0} (H_0 - H)(y) \cosh \kappa r(p, y) d\Sigma_0(y)$, then it is a smooth function on $D \subset H^3_{-\kappa^2}$, and it would be interesting to see some properties of this function. We first have:

Proposition 3.14. Suppose $f$ has a critical point $o$ in the interior of $D$, and let $o = (0, 0, 0, 1/\kappa)$, then for $1 \leq i \leq 3$,

$$\int_{\Sigma_0} (H_0 - H) \sinh \kappa r \cdot \phi_i = 0.$$

Proof. It is easy to see that $\nabla_o r(o, y)$ is the unit tangential vector of geodesic $\overrightarrow{yo}$, hence,

$$\nabla_o r(o, y) = (\cos \theta, \sin \theta \cos \psi, \sin \theta \sin \psi, 0)$$

$$= (\phi_1, \phi_2, \phi_3, 0)$$

Combine this fact and a direct computation, we see that the conclusion is true. \qed

Remark 3.15. Suppose $(\Omega, g)$ satisfying the assumptions in Theorem 3.1 and $o = (0, 0, 0, 1/\kappa)$ be a critical point of $f$, then Theorem 3.1 is true with $\alpha = 1$.

Again by a direct computation we have

Proposition 3.16. Let $f$ be defined as above, then

$$\Delta f = 3\kappa^2 f$$

Remark 3.17. Suppose $(\Omega, g)$ satisfying the assumptions in Theorem 3.1 then by maximal principle, we know that $f$ cannot attain a local maximum inside of $D$; if there is $o \in D$ with $f = 0$, then $f$ is identical to 0 on the whole $D$ which implies $(\Omega, g)$ is a domain of $H^3_{-\kappa^2}$. 
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