DISTRIBUTIONS DEFINED BY $q$-SUPERNOMIALS, FUSION PRODUCTS, AND DEMAZURE MODULES

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Abstract. We prove asymptotic normality of the distributions defined by the central string functions and the basic specialization of fusion modules of the current algebra of $\mathfrak{sl}_2$. The limit is taken over linearly scaled fusion powers of a fixed collection of irreducible representations. This includes as special instances all Demazure modules of the associated affine Kac-Moody algebra. Our results are derived from the interpretation of fermionic expressions for the characters of fusion products as mixtures of probability generating functions. In particular, they describe the position of dominating weights and are accompanied by conjectured local central limit theorems giving the first hint towards the asymptotic growth of dominating weight multiplicities in fusion modules found in the literature. Along with an available complementary result on the asymptotic normality of the basic specialization of graded tensors of the type $A$ standard representation, our result is a central limit theorem for a serious class of graded tensors. It therefore serves as an indication towards universal behavior: The central string functions and the basic specialization of fusion and, in particular, Demazure modules behave asymptotically normal, as the number of fusions scale linearly in an asymptotic parameter, $N$ say. We remark on the geometric interpretation in terms of rational points of unipotent partial flag manifolds over finite fields at the end.

1. Introduction

The $q$-supernomials coefficients are $q$-analogues of the coefficients of the monomials $x^a$ in the expansion of $\prod_{i=1}^{m}(1 + x + \ldots + x^j)^{L_j}$. They have a combinatorial interpretation in terms of generalized Durfee dissection partitions in that they are generating functions of so-called $(L_1, \ldots, L_m)$-admissible partitions with exactly $a$ parts [35 §3]. In there most simple form, when $m = 1$, they are the well-known $q$-binomial coefficients which are the generating functions for restricted partitions [2 §3], a result that goes back to Gauss and coins them Gaussian binomial coefficients. Let us mention that, geometrically, the coefficients of $q$-binomials count the number of invariants of binary forms [39 §3.3], [11 Corollary].
The main focus on $q$-supernomials, from our point of view, lies on their appearance as characters of finite-dimensional modules in the representation theory of infinite-dimensional Lie algebras. Precisely, they describe the string functions in fusion modules, i.e. graded tensor products, of the current algebra $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$ \cite{12,13}. Consequently, their coefficients encode the dimensions of weight spaces, i.e. isotropic components with respect to the action of a maximal abelian subalgebra.

The exact determination of those coefficients is certainly possible in any fixed instance of a $q$-supernomial, simply by extrapolation from the definition of $q$-supernomials and their building blocks, the $q$-binomials. In general though, the explicit description of those coefficients remains intractable and one usually is satisfied with concrete expressions for their generating function, the $q$-supernomial. We will examine those coefficients from a qualitative point of view by the interpretation of $q$-supernomials as probability generating functions of discrete distributions. Furthermore, we investigate the generating function of the $q$-supernomials themselves. This generating function equals the so-called basic specialization of the character of $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$–fusion modules, and can be understood as the Hilbert polynomial associated to the parts that are graded by the action of $t$. Our main results are Theorem 4.20 and Theorem 4.24, and can be summarized as follows:

Consider a sequence $(C^1)^{b_1^{(N)}} \ast (C^2)^{L_1^{(N)}} \ast \ldots \ast (C^{m+1})^{L_m^{(N)}}$ of fusion modules of the current algebra $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$. Assume that the exponents grow on a linear scale, i.e. $(L_1^{(N)}, \ldots, L_m^{(N)})/N \to a \neq 0$. Then, the sequences associated to the central string functions and basic specialization of those modules behave asymptotically normal with mean and variance growing quadratically and cubically in $N$, respectively, with explicitly calculable leading terms.

Along with complementary results for the Demazure module $V_{-N\omega_1}(A_0)$ of the higher rank affine Kac-Moody algebra $\widehat{\mathfrak{sl}_r}$ \cite[\S 5.4]{15}, our findings highlight the first serious indication of a central limit theorem concerning the central string functions and the basic specialization of fusion modules. We will conclude this line of thought by a discussion of fusion of symmetric power representations in type $A$ and the following conjecture.

**Conjecture 6.5.** Consider the sequence of type $A$ fusion modules of symmetric power representations

$$
\mathcal{F}_{\mu^{(N)}} = V^{(N)}_{\omega_1} \ast V^{(N)}_{2\omega_1} \ast \ldots \ast V^{(N)}_{\ell \omega_1}
$$

associated to the partition $\mu^{(N)} = (f_1^{(N)}, 2f_2^{(N)}, \ldots, \ell f_\ell^{(N)})$ with $f_1^{(N)}$-many $i$’s. Assume that as $N \to \infty$ we have

$$
\frac{1}{N}(f_1^{(N)}, f_2^{(N)}, \ldots, f_\ell^{(N)}) \to a \neq 0.
$$
Then, the central string functions and the basic specialization of \( \mathcal{F}_\mu(N) \) behaves asymptotically normal as \( N \to \infty \).

Since Demazure modules of \( \widehat{\mathfrak{sl}}_2 \) are special instances of fusion modules we will compare (see \$4.5.2\) our results to previously derived formulæ for the expectation value of the basic specialization of Demazure modules \([7]\). There the expectation value has been derived via different methods, through the detailed analysis of the recursion given by Demazure’s character formula.

Our results are accompanied by conjectured local central limit theorems in \$5\), that give the first hint towards the asymptotic growth of dominating weight multiplicities in fusion modules found in the literature. We furthermore discuss the geometric interpretation in terms of rational points of unipotent partial flag manifolds over finite fields in Remark 6.4.

2. Notation

2.1. \( q \)-supernomial coefficients. We follow \([35]\) and fix a vector \( \mathbf{L} = (L_1, \ldots, L_m) \in \mathbb{Z}_+^m \), and let \( l_m = \sum_{i=1}^m iL_i \). Consider the expansion \([35, (2.5)]\)

\[
\prod_{j=1}^m (1 + x + \ldots + x^j)^{L_j} = \sum_{a=-l_m/2}^{\infty} \binom{\mathbf{L}}{a} x^{a + l_m/2},
\]

and the fermionic representation \([35, (2.8)]\) of the coefficient

\[
\binom{\mathbf{L}}{a} = \sum_{j_1 + \cdots + j_m = a + l_m/2} (L_{m-1} + j_m) \cdots (L_1 + j_2)
\]

as a convex sum of products of usual binomial coefficients. Such an expression as a positive sum of products of binomials is commonly referred to as fermionic. Schilling and Warnaar define the \( q \)-supernomials \([35, (2.9)]\) as a \( q \)-analogue of those coefficients

\[
\left[ \begin{array}{c} \mathbf{L} \\ a \end{array} \right]_q = \sum_{j_1 + \cdots + j_m = a + l_m/2} q^{\sum_{i=2}^m j_i - (L_i + \cdots + L_m - j_i)} \prod_{\ell=1}^m \left[ \frac{L_\ell + j_\ell + 1}{j_\ell} \right]_q.
\]

Here \( j_{m+1} = 0 \), \([M]_k = \frac{[M]a^k}{[k]! ([M-k]! a^k)}\) denotes the \( q \)-binomial coefficient and \([k]_q! = \prod_{i=1}^k \frac{1 - q^i}{1 - q}\) the \( q \)-factorial (see e.g. \([10]\)). Note that Feigin and Feigin describe in \([12\), Theorem 5.1\) the characters of \( \mathfrak{sl}_2 \otimes \mathbb{C}[t] \) fusion modules in terms of a slight modification of \([L]_a \), that is \([35, (3.1)]\)

\[
\tilde{T}(\mathbf{L}, a)(q) = q^{l_m} \left[ \begin{array}{c} \mathbf{L} \\ a - l_m/2 \end{array} \right]_q.
\]
with its explicit form \[ \left[ \frac{N}{k} \right]_{q}^{1} = q^{-k(N-k)} \left[ \frac{N}{k} \right]_{q} \]

For illustration purposes, in the case \( m = 1 \) the definitions (2.1), (2.2) above give

\[
\left[ (L_{1}) \right]_{a} = \sum_{j_{1} = a + \frac{L_{1}}{2}} q^{j_{1}} \left[ \frac{L_{1}}{j_{1}} \right]_{q} = q^{a^{2}} \left[ \frac{L_{1}}{a} \right]_{q},
\]

That is, \( \hat{T} \) is a shifted (here by \( L_{1}/2 \)) and translated (here by \( a^{2} \)) version of \( \left[ (L_{1}) \right]_{a} \).

### 2.2. Lie algebras, Demazure and fusion modules

For general facts about current and affine Kac-Moody algebras and their representation theory we refer the reader to [8, 23]. We denote by \( \mathfrak{sl}_{r} \) the complex-valued \( r \times r \) matrices with trace 0 (the reader might adapt to the case \( r = 2 \) as this will be the setting we will mostly consider). Then, \( \mathfrak{sl}_{r} \otimes \mathbb{C}[t] \) denotes its current algebra and, closely related, \( \hat{\mathfrak{sl}}_{r} \) its associated affine Kac-Moody algebra which can be realized as the extended loop algebra of \( \mathfrak{sl}_{r} \), i.e. \( \hat{\mathfrak{sl}}_{r} = \mathfrak{sl}_{r} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{Cd} \oplus \mathbb{Cc} \) (see [23, 33]). Here, \( d \) denotes the derivation \( \frac{dt}{dt} \) and \( c \) the canonical central element. In view of that realization let \( \mathfrak{h} \supset \mathfrak{h} \) be the Borel and Cartan subalgebra in \( \hat{\mathfrak{sl}}_{r} \) corresponding to their a priori fixed counterparts in \( \mathfrak{sl}_{r} \). We denote the simple roots by \( \alpha_{0}, \alpha_{1}, \ldots, \alpha_{r-1} \in \mathfrak{h}^{\ast} \), the highest root by \( \theta = \alpha_{1} + \ldots + \alpha_{r-1} \), the imaginary root by \( \delta = \alpha_{0} + \theta \), and the simple coroots by \( \alpha_{0}^{\vee}, \ldots, \alpha_{r-1}^{\vee} \in \mathfrak{h} \). Note that we assume \( \alpha_{1}, \ldots, \alpha_{r-1} \) to correspond to the standard embedding of \( \mathfrak{sl}_{r} \). Let \( s_{0}, s_{1}, \ldots, s_{r-1} \) be the simple reflections associated to the simple roots and let the subgroup \( W_{\text{aff}} = \langle s_{0}, s_{1}, \ldots, s_{r-1} \rangle \) of \( \text{GL}(\mathfrak{h}^{\ast}) \) denote the affine Weyl group.

For a dominant integral weight \( \Lambda = m_{1}\Lambda_{0} + m_{2}\Lambda_{1} + \ldots + m_{r-1}\Lambda_{r-1} \) in the affine weight lattice \( \hat{\Lambda} = \mathbb{Z}\Lambda_{0} \oplus \mathbb{Z}\Lambda_{1} \oplus \ldots \oplus \mathbb{Z}\Lambda_{r-1} \subset \mathfrak{h}^{\ast} \) we let \( V(\Lambda) \) be the integrable highest weight representation of weight \( \Lambda \) of \( \mathfrak{sl}_{r} \) and \( \chi(V(\Lambda)) \) its character. All weights \( \mu \) occurring in \( V(\Lambda) \), that is, having non-trivial weight space \( V(\Lambda)_{\mu} \), are elements of the lattice \( \Gamma = \Lambda + \mathbb{Z}\alpha_{0} + \mathbb{Z}\alpha_{1} + \ldots + \mathbb{Z}\alpha_{r-1} \subset \hat{\Lambda} \).
One usually writes the monomials in the characters of such modules as $e^\mu$, the coefficient $k$ in the monomial $e^{-k\alpha_0}$ is referred to as the degree. The $\Lambda_0, \Lambda_1, \ldots, \Lambda_{r-1}$ are called fundamental weights, the $V(\Lambda_i)$ the fundamental representations and $V(\Lambda_0)$ the basic representation.

We will not be directly concerned with the representation theory of $\widehat{\mathfrak{sl}}_r$ but instead with Demazure and fusion modules, i.e. finite-dimensional representations of the Borel subalgebra $\mathfrak{b}$ and the current algebra $\mathfrak{sl}_r \otimes \mathbb{C}[t]$, respectively (see e.g. [9] [13] [14] [19] [24]).

For $w \in W^{\text{aff}}$ and a dominant integral weight $\Lambda = m_1\Lambda_0 + m_2\Lambda_1 + \ldots + m_{r-1}\Lambda_{r-1}$ denote the associated Demazure module by $V_w(\Lambda)$. Such a Demazure module is defined as the finite-dimensional subspace of the integrable highest weight representation $V(\Lambda)$ generated by the action of the universal enveloping algebra $U(\mathfrak{b})$ on a non-zero weight vector $v_w(\Lambda)$ inside the 1-dimensional weight space $V(\Lambda)_{w(\Lambda)}$, in short $V_w(\Lambda) = U(\mathfrak{b}).v_w(\Lambda)$.

Demazure’s character formula [11] [28] [31] allows the computation of the character $\chi$ of $V_w(\Lambda)$ by an iterated application of certain operators on the monomial $e^\Lambda = \chi(V(\Lambda))$, as follows. We introduce the convention that $\sum_{i=0}^{-1} a_i = 0$ and $\sum_{i=0}^{k} a_i = -a_{-1} - \cdots - a_{k+1}$ for $k < -1$. Note that this is natural in the sense that Gauss’s summation formula $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$ extends to all $k \in \mathbb{Z}$, as does the identity $\sum_{i=0}^{k} 1 = k + 1$. With this convention the Demazure operator $D_j$ associated with a simple reflection $s_j$ acts on monomials $e^\mu$ as

$$D_j e^\mu = \sum_{i=0}^{\mu(a_j)} e^{\mu-ia_j}. \tag{2.3}$$

For an arbitrary Weyl group element $w \in W^{\text{aff}}$ we choose a reduced decomposition $w = s_{j_1}s_{j_2}\cdots s_{j_l}$ and set $D_w = D_{j_1}D_{j_2}\cdots D_{j_l}$. Demazure’s character formula now states that the character of $V_w(\Lambda)$ can be computed recursively as $\chi(V_w(\Lambda)) = D_w e^{\Lambda}$.

For a general definition of a fusion module consider irreducible $\mathfrak{sl}_r$-modules $V(\Lambda_1), \ldots, V(\Lambda_n)$ with integral dominant weights $\lambda = (\lambda_1, \ldots, \lambda_n)$. For regular $Z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ let $\mathfrak{sl}_r \otimes \mathbb{C}[t]$ act by

$$(x \otimes t^i).v = z_i^i(x.v), \quad \text{for } x \in \mathfrak{sl}_r, v \in V(\lambda_i).$$

This so-called evaluation module is denoted by $V_z(\lambda_i)$. Then $V_\lambda(Z) = V_{z_1}(\lambda_1) \otimes \cdots \otimes V_{z_n}(\lambda_n)$ is a cyclic module of the universal enveloping algebra $U(\mathfrak{sl}_r \otimes \mathbb{C}[t])$. Hence, there is an induced filtration

$$V_\lambda(Z)_s = \bigoplus_{j=0}^{s} U(\mathfrak{sl}_r \otimes \mathbb{C}[t])_j v$$

The associated graded module is called the fusion module:

$$F_\lambda(Z) = V_{z_1}(\lambda_1) \ast \cdots \ast V_{z_n}(\lambda_n) = \text{gr} V_\lambda(Z) = \bigoplus_{s \geq 0} V_\lambda(Z)_s / V_\lambda(Z)_{s-1}.$$
The additional grading is induced by the powers of \( t \). We obtain a graded tensor product multiplicity \( m^d_\mu \in \mathbb{Z}[q] \) such that
\[
m_\mu = m^d_\mu(1),
\]
where \( m_\mu \) is the usual tensor product multiplicity of \( V(\mu) \) in \( \otimes_i V(\lambda_i) \). For specific \( \lambda = (\lambda_1, \ldots, \lambda_n) \) it is known that the fusion modules \( F_\lambda(Z) \) are independent of the choice of the evaluation parameter \( Z \), see e.g. \cite{9} \S 1.5.1, \cite{18} \cite{19}.

2.3. **Statistical notions.** Standard sources are \cite{3,14,15,16}. All our random variables \( X \) will be discrete and finite. Recall that the expected value of such a random variable is the weighted average \( \mathbf{E}(X) = \sum_x P(X = x)x \).

The covariance of two random variables \( X \) and \( Y \) is \( \text{Cov}(X,Y) = \mathbf{E}((X - \mathbf{E}(X))(Y - \mathbf{E}(Y))) \). They are said to be uncorrelated if \( \text{Cov}(X,Y) = 0 \). The variance of \( X \) is \( \text{Var}(X) = \text{Cov}(X,X) \). Its probability generating function is \( \mathbf{E}(q^X) = \sum_x P(X = x)q^x \), and the associated probability distribution \( \mu_X = \sum_x P(X = x)\delta_x \). Here, \( \delta_x \) denotes the Dirac distribution (point mass) at \( x \). A sequence \( X_N \) converges \( \mathbf{P} \)-almost surely (\( \mathbf{P} - \text{a.s.} \) for short) to \( X \) if \( \mathbf{P}(\lim_{N \to \infty} X_N = X) = 1 \). Convergence and equality in distribution will be denoted by \( \xrightarrow{d} \) and \( \equiv \), respectively. \( \mathcal{N}(\mu, \Sigma) \) will denote the normal distribution with mean \( \mu \) and covariance matrix \( \Sigma \). Note that \( \mathcal{N}(\mu, 0) = \delta_\mu \).

The conditional probability \( P(Y = y|X = x) = P(X = x)^{-1}P(X = x, Y = y) \) is the probability of \( Y \) taking the value \( y \) given the occurrence of the value \( x \) for \( X \). A mixture of distributions \( \mu_{X_i} \) is a finite convex combination thereof, i.e. \( \sum_i w_i \mu_{X_i} \) for some weights \( w_i \geq 0 \) with \( \sum_i w_i = 1 \). The probability generating function of a mixture is \( \sum_i w_i \mathbf{E}(q^{X_i}) \).

3. **Characters and their specializations**

3.1. **Graded characters of fusion and Demazure modules of \( \mathfrak{sl}_2 \).** We focus on \cite{12} \cite{13} which study in detail fusion modules for the current algebra \( \mathfrak{sl}_2 \otimes \mathbb{C}[t] \). Note that irreducible representations of \( \mathfrak{sl}_2 \) are indexed by natural numbers \( \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\} \). Hence, a fusion module \( F_d(Z) \) for \( \mathfrak{sl}_2 \) is given by a dimension vector \( d = (d_1, \ldots, d_n) \in \mathbb{N}_0^n \) and regular \( Z \in \mathbb{C}^n \). For \( \mathfrak{sl}_2 \) it is known that the fusion modules are independent of the choice of \( Z \in \mathbb{C}^n \) \cite{12}, and it is usual to omit the evaluation parameter in the notation. In particular, Feigin and Feigin \cite{12} study the fusion module \( F_d \) with dimension vector \( d = (1^{b_1}, 2^{b_2}, \ldots, (m + 1)^{b_{m+1}}) \), and denote it
\[
F(b_1, b_2, \ldots, b_{m+1}) = (C^1)^{b_1} * (C^2)^{b_2} * \cdots * (C^{m+1})^{b_{m+1}}.
\]

They prove \cite{12} Theorem 5.1 that, written here in terms of the \( q \)-supernomial \( \bar{T}(L, a)(q) \) \cite{22}, its graded character can be described as
\[
\chi(F(b_1, b_2, \ldots, b_{m+1}))(z, q) = \sum_{a \in \mathbb{Z}} z^a \cdot \bar{T}(L, a)(q).
\]
It is well-known that Demazure modules \( V_w(\Lambda) \) associated to \( \mathfrak{sl}_2 \) carry a \( \mathfrak{sl}_2 \otimes C[t] \)-module structure and as such are special instances of fusion modules (see e.g. [9, §1.5.1] or [19, §3.5]). To be precise, we have isomorphisms of \( \mathfrak{sl}_2 \otimes C[t] \)-modules as follows

\[
V_w(m\Lambda_0 + n\Lambda_1) \cong \begin{cases} \mathcal{F}(0, 1_{m+1}^{}, 0, l(w) - 1_{m+n+1}^{}), & w = w's_0, \\ \mathcal{F}(0, 1_{n+1}^{}, 0, l(w) - 1_{m+n+1}^{}), & w = w's_1. \end{cases}
\]

(3.2)

Here, we write \( l(w) \) for the length of a reduced decomposition of \( w \in W_{aff} \). All elements of \( W_{aff} \) have the form

\[
w_{N,0} = \cdots s_0 s_1 s_0 \quad \text{or} \quad w_{N,1} = \cdots s_1 s_0 s_1
\]

for \( N \geq 0 \). Hence, Demazure’s operator \( D_w \) becomes an alternating iteration \( \cdots D_0 D_1 D_0 D_1 \cdots \) of the operators \( D_0, D_1 \) defined in (2.3). Let us summarize the non-trivial elements of the affine Weyl group \( W_{aff} \) of \( \mathfrak{sl}_2 \) as

\[
(s_1 s_0)^N s_1, N \geq 0 \quad \text{and} \quad (s_0 s_1)^N, (s_1 s_0)^N, s_0(s_1 s_0)^{N-1}, N > 0.
\]

Then, one can precisely identify the characters of Demazure modules via those of fusion modules as follows.

**Proposition 3.1.** Consider the Demazure module \( V_w = V_w(m\Lambda_0 + n\Lambda_1) \) of \( \mathfrak{sl}_2 \) of (fixed) highest weight \( \Lambda = m\Lambda_0 + n\Lambda_1 \) and recall the character formula for fusion modules (3.1). The character \( \chi(V_w)(z, q) \), written in the coordinates \( z = e^{-a_1}, q = e^{-a_2}, \) is given by

\[
\begin{align*}
(3.3) & \quad \text{For the trivial element } w = 1 \text{ one has } \chi(V_1)(z, q) = e^\Lambda. \\
(3.4) & \quad \text{For } w = (s_1 s_0)^N s_1, N \geq 0 \text{ one has } \\
& \quad e^{-\Lambda} \chi(V_w)(z, q) = z^{-(n+m)N-n/2} q^{N^2 m+N(N+1)n} \chi(\mathcal{F}(0, L_w)(zq^{-2N-1}, q)) \\
& \quad \text{where } L_w = (L_1, \ldots, L_{m+n}) = (0, 1_{m+n}, 2N). \\
(3.5) & \quad \text{For } w = (s_0 s_1)^N, N \geq 0 \text{ one has } \\
& \quad e^{-\Lambda} \chi(V_w)(z, q) = z^{-(n+m)N-n/2} q^{N^2 m+N(N+1)n} \chi(\mathcal{F}(0, L_w)(zq^{-2N-1}, q)) \\
& \quad \text{where } L_w = (L_1, \ldots, L_{m+n}) = (0, 1, 2N - 1). \\
(3.6) & \quad \text{For } w = (s_1 s_0)^N, N > 0 \text{ one has } \\
& \quad e^{-\Lambda} \chi(V_w)(z, q) = z^{-(n+m)N-n/2} q^{N^2 m+N(N-1)n} \chi(\mathcal{F}(0, L_w)(zq^{-2N}, q)) \\
& \quad \text{where } L_w = (L_1, \ldots, L_{m+n}) = (0, 1_{m+n}, 2N - 1). \\
(3.7) & \quad \text{For } w = s_0(s_1 s_0)^{N-1}, N > 0 \text{ one has }
\end{align*}
\]
The sum of the entries in \( L \) represents the length \( l(w) \) of the Weyl group element \( w \). When either \( n \) or \( m \) equals 0, then \( L_w = (0, l(w)) \).

**Proof.** Feigin [13, (11)] denotes an integrable highest weight representation \( L_{i,k} = \mathcal{U}(\mathfrak{sl}_2)_v \) with highest weight vector \( v_{i,k} \) such that \( c.v_{i,k} = k v_{i,k} \), \( h_0.v_{i,k} = iv_{i,k} \), and \( d.v_{i,k} = 0 \). In our notation, the canonical central element is \( c = \alpha_0^\vee + \alpha_1^\vee \), the coroot is \( h_0 = \alpha_0^\vee \), and the scaling element \( d \) is given by \( \alpha_0(d) = 1 \) and \( \alpha_1(d) = 0 \). Therefore, by comparison of the highest weight vector we have \( L_{i,k} = V((k-i)\Lambda_0 + i\Lambda_1) \). The bigrading is chosen according to the action of \( h_0 \) and \( d \), and consequently, the character is denoted in the monomials \( e^{\alpha_0} = e^{\delta - \alpha_1} \) and \( e^{-\delta} \), respectively. By [13, Corollary 3.1] each such module \( L_{i,k} \) can be constructed as an inductive limit of fusion products, that is \( L_{i,k} = \mathbb{C}^{i+1} \ast (\mathbb{C}^{k+1})^{2\infty} \). Each fusion product can be identified with the corresponding Demazure module \( V_w((k-i)\Lambda_0 + i\Lambda_1) \) by comparing the weights of the extremal weight vectors described in [13, §1]. Now apply the character formula [12, Theorem 5.1], noting that \( e^{\alpha_0} = zq^{-1} \).

Let us illustrate \( q \)-supernomials, characters of Demazure and fusion modules, and their relations by some examples. We consider \( m = 2, n = 0 \) and the vector \( L = (L_1, L_2) = (0, 4) \). Let us compare:

1. The generating function of the \( q \)-supernomials \( \left[ \begin{array}{c} 0,4 \\ a \end{array} \right]_q \)

   \[
   f_L(z, q) = \sum_{a \in \mathbb{Z}} z^a \cdot \left[ \begin{array}{c} 0,4 \\ a \end{array} \right]_q.
   \]

   Note the explicit form (compare (2.1))

   \[
   \left[ \begin{array}{c} 0,4 \\ a \end{array} \right]_q = \sum_{j_1 + j_2 = a+4} q^{j_1(4-j_2)} \left[ \begin{array}{c} 4 \\ j_2 \end{array} \right] \left[ \begin{array}{c} j_2 \\ j_1 \end{array} \right]_q,
   \]

   and see Figure 1 for the actual plot.

2. The character of the fusion module \( \mathcal{F}(0, 0, 4) = (\mathbb{C}^3)^4 \) as depicted in (3.1), i.e.

   \[
   \chi(\mathcal{F}(0, 0, 4))(z, q) = \sum_{a \in \mathbb{Z}} z^a \cdot \tilde{T}((0, 4), a)(q).
   \]

   Note the explicit form (compare (2.2))

   \[
   \tilde{T}((0, 4), a)(q) = \sum_{j_1 + j_2 = a} q^{j_1^2 + j_2^2} \left[ \begin{array}{c} 4 \\ j_2 \end{array} \right] \left[ \begin{array}{c} j_2 \\ j_1 \end{array} \right]_q,
   \]

   and see Figure 2 for the actual plot.
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Figure 1. Plot of the generating function of \( q \)-supernomials \( f_L(z,q) \).

(3) The character of the Demazure module \( V_{(s_1 s_0)^2(2\Lambda_0)} \) written as a Laurent polynomial in the monomials \( z = e^{-\alpha_1}, q = e^{-\delta} \), i.e. the generating function of its weight space dimensions,

\[
\chi(V_{(s_1 s_0)^2(2\Lambda_0)})(z,q) = \sum_{a,i \in \mathbb{Z}} z^a q^i \cdot \dim(V_{(s_1 s_0)^2(2\Lambda_0)} z^a q^i).
\]

This generating function can be compute via Demazure’s (recursive) character formula

\[
\chi(V_{(s_1 s_0)^2(2\Lambda_0)})(z,q) = D_1 D_0 D_1 D_0 e^{2\Lambda_0}.
\]

See Figure 3 for the actual plot.

As to the relations between the different generating functions, one has

\[
(3.8) \quad e^{-\Lambda} \chi(V_{(s_1 s_0)^2(2\Lambda_0)})(z,q) = q^8 \cdot f_L(z,1/q),
\]

\[
(3.9) \quad e^{-\Lambda} \chi(V_{(s_1 s_0)^2(2\Lambda_0)})(z,q) = z^{-4} q^8 \cdot \chi(\mathcal{F}(0,0,4))(zq^{-4},q).
\]

All expressions are equivalent up to translations (multiplication by e.g. \( z^{-4} \)), reflections (evaluation at the reciprocal \( 1/q \)), and rotations (evaluation at mixed monomials \( zq^{-4} \)), respectively. Note that the example (3.9) is an instance of the general formula (3.6).

3.2. **Real characters and the basic specialization.** As already mentioned in §2.2, the specialization of the character \( \chi(\mathcal{F}(b_1,\ldots,b_m)) \) of a fusion module \( \mathcal{F}(b_1,\ldots,b_m) = (C_1)^{b_1} \otimes (C_2)^{b_2} \otimes \cdots \otimes (C_{m+1})^{b_{m+1}} \) at \( q = 1 \) equals the character of the tensor product \( (C_1)^{b_1} \otimes (C_2)^{b_2} \otimes \cdots \otimes (C_{m+1})^{b_{m+1}} \) of irreducible representations of \( \mathfrak{sl}_2 \). Note that frequently this type of specialization is called a real character. Therefore, (3.2) implies

\[
(3.10) \quad \chi(V_w(m\Lambda_0 + n\Lambda_1))(z,1) = \begin{cases} 
\chi(C^{m+1} \otimes (C^{m+n+1})^{\otimes l(w)-1})(z), & w = ws_0, \\
\chi(C^{n+1} \otimes (C^{m+n+1})^{\otimes l(w)-1})(z), & w = ws_1.
\end{cases}
\]
This so-called factorization phenomenon for Demazure modules is known independent of the theory of fusion modules, first proved in [32, 33], and generalized considerably, e.g. [18]. Note, since characters of (non-graded) tensor products multiply, i.e. $\chi(V \otimes W) = \chi(V) \cdot \chi(W)$, the associated distributions convolute and asymptotic considerations, as $l(w) \to \infty$, take the simplest form of a central limit theorem for sums of i.i.d. random variables. Therefore, characters of usual tensor products, and equivalently the above specializations at $q = 1$, are well understood from a statistical point of view and have been analyzed further in great detail, e.g. [42].

Much less studied is the so-called basic specialization of those characters, i.e. their evaluation at $z = 1$. We borrow this terminology from Kac [23 §1.5, 10.8, 12.2] who analyzed this kind of specialization for characters of integrable highest weight modules $V(\Lambda)$, and obtained Macdonald’s identities.
4. Distributions defined by \(q\)-supernomials

Let \(\mathbf{L} := (L_1, \ldots, L_m) \in \mathbb{Z}_+^m\) and consider the \(q\)-supernomial given in (2.2):

\[
\tilde{T}(\mathbf{L}, a)(q) = \sum_{j_1 + \cdots + j_m = a} q^{\sum_{i=1}^{m} j_i} \prod_{\ell=1}^{m} \left[ \frac{L_\ell + j_\ell + 1}{j_\ell} \right]_q.
\]

Figure 3. Plot of the character \(\chi(V^{(s_1 s_0)^2}(2\Lambda_0))\) written in the coordinates \(z = e^{-\alpha_1}\) and \(q = e^{-\delta}\).
Recall that these generating functions were introduced by Schilling and Warnaar [35] who showed that they enumerate $L$-admissible partitions with exactly $a$ parts. In this section we study the average behavior of the distributions defined by $\tilde{T}(L, a)(q)$ and by the generating function

$$
\tilde{T}(L)(q) := \sum_{a=0}^{L_1 + \ldots + L_m} \tilde{T}(L, a)(q)
$$

of the total number of $L$-admissible partitions that equals the specialization $\chi(F(0, L))(1, q)$ by (3.1). We show that the total number and in certain (typical) cases the $a$-restricted number of $L$-admissible partitions are asymptotically normally distributed with asymptotic parameter being a convergent sequence $\frac{1}{N} L^{(N)}$, as $N \to \infty$.

4.1. Preliminaries. The distributions with probability generating function $F_{a,b}(q) := \left[ \begin{array}{c} a+b \\ a \end{array} \right] q^a / (a+b)^a$ were first investigated by Mann and Whitney [30], who showed:

**Theorem 4.1.** Let $Inv_{a,b}$ be a random variable with distribution $F_{a,b}$. Then $Inv_{a,b}$ has expectation $E(Inv_{a,b}) = \frac{1}{2} ab$, variance $Var(Inv_{a,b}) = \frac{1}{12} ab (a+b+1)$, and

$$
\frac{Inv_{a,b} - E(Inv_{a,b})}{\sqrt{Var(Inv_{a,b})}} \overset{d}{\rightarrow} N(0,1),
$$

as $a, b \to \infty$.

A corresponding local limit theorem was proved by Takacs [41].

**Remark 4.2.** The $q$-binomials enumerate different number theoretic [2], geometric [39], and combinatorial [41] objects. We interpret them here as counting inversions [40]. Consider a word (unordered sequence) $w = (w_1, \ldots, w_n)$ of elements from an ordered set. A 4-tuple $(i, j, w_i, w_j)$ with $i < j$ and $w_i > w_j$ is called inversion. It is well known that $\left[ \begin{array}{c} a+b \\ a \end{array} \right] q^a$ is the generating function for inversions in words of $a$ zeroes and $b$ ones.

We call a vector $j = (j_1, \ldots, j_m) \in N_0^m$ compatible to $L \in Z_+^m$ if $j_i \leq L_i + j_{i+1}$ for $i = 1, \ldots, m$, and a probability distribution $L$-compatible if the set of $L$-compatible values has probability one. For $L$-compatible $j$ we let $Inv(L, j)$ denote a random variable with probability generating function $F(L, j)(q) := \prod_{i=1}^m F_{L_i+j_{i+1},j_i}(q)$ (here and in the sequel $j_{m+1} = 0$), and we let

$$
Q(L, j) = \sum_{i=1}^m j_i \left( j_i + \sum_{\ell=1}^{i-1} L_\ell \right).
$$

We view the normalized fermionic expression $g(q) = \tilde{T}(L)(q)/\tilde{T}(L)(1)$, and $g_a(q) = \tilde{T}(L, a)(q)/\tilde{T}(L, a)(1)$, respectively, as mixtures of probability generating functions, i.e.

$$
g(q) = \sum_{j} q^{Q(L,j)} F(L, j)(q) P(J = j),
$$
DISTRIBUTIONS DEFINED BY $Q$-SUPERNOMIALS

weighted by

$$P(J = j) = P(J_1 = j_1, \ldots, J_m = j_m) = \prod_{i=1}^{m} \frac{(L_i + j_i + 1)}{\tilde{T}(L)(1)}.$$ 

All generating functions and random variables considered here depend implicitly on the admission vector $L$, but this dependence will from now on for convenience be suppressed in the notation. We may equivalently write $g(q)$ as follows.

**Proposition 4.3.** With given conditional distribution $P(Y = i | J = j) = P(Inv(L, j) = i)$ we have

$$g(q) = E(q^{Q(L, J) + Y}).$$

Proof. By the definition of the conditional distribution

$$g(q) = \sum_{j} q^{Q(L, j)} F(L, j)(q) P(J = j)$$

$$= \sum_{j} q^{Q(L, j)} E(q^Y | J = j) P(J = j)$$

$$= \sum_{j, y} q^{Q(L, j) + y} P(Y = y | J = j) P(J = j)$$

$$= \sum_{j, y} q^{Q(L, j) + y} P(Y = y, J = j)$$

$$= E(q^{Q(L, J) + Y}).$$

Our interest lies in the distribution of the random variable

$$T = Q(L, J) + Y.$$ 

We write $Y = E(Y | J) + R$, where $R = Y - E(Y | J)$ is (by the properties of conditional expectation) uncorrelated to (any square-integrable function of) $J$. Furthermore, by [Theorem 4.1](#), we have

$$e(L, J) = E(Y | J) = \frac{1}{2} \left( (L_m - J_m)J_m + \sum_{i=1}^{m-1} (L_i + J_{i+1} - J_i)J_i \right).$$

Let us call the distribution of $J$ the *mixing distribution* (for the total number of $L$-admissible partitions). To understand the behavior of mixing distributions and to introduce a convenient way to refer to them, let us describe a simple random experiment. We focus our attention on the unrestricted case first.

4.1.1. The basic probabilistic setup. Consider $m$ mutually independent random sources $S_1, S_2, \ldots, S_m$ emitting words $W(1), \ldots, W(m)$. Each word $W(i) = (X_{1}(i), X_{2}(i), \ldots, X_{L_i}(i))$ is a sequence of $L_i$ many mutually independent letters $X_{k}(i)$, where each $X_{k}(i)$ is uniformly distributed from the alphabet $\{0, \ldots, i\}$. 


For $0 \leq i \leq k$ let $B_i(k) := \sum_{j=1}^{L_i} 1_{\{i\}}(X_j(k))$ denote the random variable the number of appearances of letter $i$ in word $W(k)$. Then, the random vector $B(k) := (B_0(k), \ldots, B_k(k))$ is the occupancy ( statistic) of word $W(k)$, $S(k) := \sum_{i=1}^{L_i} X_i(k) = \sum_{j=0}^{k} j B_j(k)$ is the sum of word $W(k)$, $B_L := (B(1), \ldots, B(m))$ is the total occupancy of $W(1), \ldots, W(m)$, and $S_L := \sum_{i=1}^{m} S(i)$ is called the total sum of words $W(1), \ldots, W(m)$. Clearly, under the assumptions above, $S_L$ is the sum of independent uniformly distributed random variables and we have

$$E(S_L) = \frac{1}{2} \sum_{i=1}^{m} i L_i,$$

(4.2) and

$$\text{Var}(S_L) = \frac{1}{12} \sum_{i=1}^{m} (i + 2) i L_i.$$

(4.3)

Since $f_i(x) = (1 + x + \ldots + x^i)/(i+1)$ is the probability generating function for the uniform distribution on $\{0, \ldots, i\}$, the coefficient of $x^a$ in the product $\prod_{i=1}^{m} (f_i(x))^{L_i}$ gives the probability that $S_L = a$, which we write as

$$[x^a] \prod_{i=1}^{m} (f_i(x))^{L_i} = P(S_L = a).$$

Now, Schilling and Warnaar [35, (1.3)] define supernomial coefficients $\binom{L}{a}$ (for $a + \ell_m/2 \in \mathbb{Z}_+$) by

$$\prod_{j=1}^{m} (1 + x + \ldots + x^j)^{L_j} = \sum_{a=-\ell_m/2}^{\infty} \binom{L}{a} x^{a+\ell_m/2},$$

where $\ell_m = \sum_{i=1}^{m} i L_i = 2E(S_L)$. We may thus interpret the supernomial coefficients as the probabilities of the random variable $S_L$:

$$\binom{L}{a} \prod_{i=1}^{m} (i+1)^{L_i} = P(S_L = E(S_L) = a).$$

Schilling and Warnaar [35, (2.8)] give the representation

$$\binom{L}{a + \ell_m/2} = \sum_{j_1+\ldots+j_m=a} \prod_{i=1}^{m} \binom{L_i + j_i+1}{j_i},$$

which leads to

$$P(S_L = a) = \left( \sum_{j_1+\ldots+j_m=a} \prod_{i=1}^{m} \binom{L_i + j_i+1}{j_i} \right) / \left( \prod_{i=1}^{m} (i+1)^{L_i} \right),$$

and the following result.
Lemma 4.4. For \(\mathbf{j} = (j_1, \ldots, j_m) \in \mathbb{N}_0^m\) (and with the usual convention about binomial coefficients that \(\binom{a}{b} = 0\) unless \(0 \leq b \leq a\) let
\[
P(J_1 = j_1, \ldots, J_m = j_m) = \left( \prod_{i=1}^m \binom{L_i + j_i + 1}{j_i} \right) / \left( \prod_{i=1}^m (i + 1)^{L_i} \right).
\]
Then, these numbers define a \(L\)-compatible probability distribution.

Proof. Indeed, we have a probability distribution since
\[
\sum_{(j_1,\ldots,j_m)} P(J_1 = j_1, \ldots, J_m = j_m) = L_1 + \cdots + L_m \sum_{a=0}^{L_1+\cdots+L_m} P(S_L = a) = 1.
\]
The \(L\)-compatibility is obvious. \(\square\)

We may describe this probability distribution alternatively as follows.

Proposition 4.5. Let \(\mathbf{B}_L = (B(1), \ldots, B(m))\) be as above. That is, the random vectors \(B(i) = (B_0(i), \ldots, B_i(i))\) are independent, where each \(B(i)\) has a (uniform) multinomial distribution with parameters \(L_i\) and \(p_0 = \ldots = p_i = \frac{1}{i+1}\). Let
\[
J_i := \sum_{k=i}^m A_{k-i+1}(k), \text{ where } A_k(i) := \sum_{j=k}^i B_j(i).
\]
(4.4)

Then, the joint distribution of \((J_1, \ldots, J_m)\) is as in Lemma 4.4.

Proof. We use (formal) generating functions. It is clear that the joint generating function of \((J_1, \ldots, J_m)\) as defined in (4.4) is
\[
\mathbb{E}(t_1^{J_1} \cdots t_m^{J_m}) = \left( \prod_{i=1}^m \left( \sum_{j=0}^i \prod_{k=i-j+1}^i t_k \right)^{L_i} \right) / \left( \prod_{i=1}^m (i + 1)^{L_i} \right).
\]
Now extract coefficients to see that this corresponds to the distribution defined in Lemma 4.4. \(\square\)

Remark 4.6. Since \(A_{k-i+1}(k) = B_{k-i+1}(k) + B_{k-i+2}(k) + \cdots + B_k(k)\) counts the number of appearances of the highest \(i\) letters in word \(W(k)\), the \(J_i\) may be described as the total number (overall count) of the \(i\) highest non-zero letters in all words.

Proposition 4.5 shows that the mixing distribution for the total number of \(L\)-admissible partitions may be realized as a simple linear transformation of \(\mathbf{B}_L\). We call \(\mathbf{B}_L\) the underlying occupancy distribution. This representation can be used for explicit calculations, and reduces the asymptotic treatment of \(J\) in the unrestricted case to the well known asymptotics of multinomial distributions. Let us recall the following classical result about the asymptotic normality of multinomial distributions.
Theorem 4.7. Let $B^{(N)}$ have the multinomial distribution with parameters $N$ and $p = (p_0, p_1, \ldots, p_m)$. Then, we have mean $E(B^{(N)}) = Np$, covariance matrix $Cov(B^{(N)}) = N\Sigma$, and

$$\frac{B^{(N)} - Np}{N^{1/2}} \xrightarrow{d} \mathcal{N}(0, \Sigma),$$

where $\Sigma = \text{diag}(p) - p^t p$.

Given these initial observations, we have a straightforward program to treat the asymptotics of the random variable $T = Q(L, J) + Y$:

1. split $T$ into an “occupancy part” $Q(L, J) + E(Y|J)$, dependent only on $J$ and a remaining “rest-inversion part” $R = Y - E(Y|J)$ “orthogonal” to $J$,
2. find the asymptotics of the two parts, using Theorem 4.1 and Theorem 4.7,
3. combine the results.

This program is carried through in §4.2.

4.1.2. The probabilistic setup for the $a$-restricted case. It is clear that the same experiment describes the $a$-restricted case when we consider only the outcomes with total sum $S_L = a$. That is, for the $a$-restricted case the underlying occupancy distribution is $B_L|S_L = a$, i.e. the distribution of $B_L$, conditioned to have $S_L = a$. In order to have a succinct wording for the “most important” restricted cases we make the following definition (Cf. (4.2)).

Definition 4.8. We call the $a$-restricted cases with $a = E(S_L)$ (resp. $a = E(S_L) \pm \frac{1}{2}$) central.

This terminology is justified on the one hand because the distribution of $S_L$ is symmetric around $E(S_L)$, and on the other hand, by the strong law of large numbers, we have $P - a.s.$ as $\frac{1}{N}L^{(N)} \rightarrow (a_1, \ldots, a_m)$:

$$\frac{1}{N}E(S_L^{(N)}) \rightarrow \frac{1}{2} \sum_{k=1}^{m} ka_k.$$

So these cases are of central importance.

4.2. General asymptotic considerations. Throughout this section let $L^{(N)}$ be a sequence of admission vectors, and $J^{(N)}$ a sequence of $L^{(N)}$-compatible mixing distributions. We consider the sequence of inversion statistics

$$Y^{(N)}$$

defined by $P(Y^{(N)} = i | J^{(N)} = j) = P(Inv(L^{(N)}, j) = i)$, and recall the associated definitions from §4.1, in particular (4.1),

$$R^{(N)} = Y^{(N)} - E(Y^{(N)}|J^{(N)}),$$

$$T^{(N)} = Q(L^{(N)}, J^{(N)}) + Y^{(N)},$$

$$e(L^{(N)}, J^{(N)}) = E(Y^{(N)}|J^{(N)}).$$
We first show a result that enables us to treat the occurring quadratic functions of $J^{(N)}$. For quadratic functions of asymptotically normal random vectors $X^{(N)}$ one has in general:

**Proposition 4.9.** Assume there exists $b \in \mathbb{R}^m$ and a positive semidefinite matrix $\Sigma \in \mathbb{R}^{m \times m}$ of positive rank such that

$$\frac{E(X^{(N)})}{N} \rightarrow b \quad \text{and} \quad \frac{X^{(N)} - Nb}{N^{1/2}} \rightarrow \mathcal{N}(0, \Sigma).$$

Let $M \in \mathbb{R}^{m \times m}$, $v \in \mathbb{R}^m$, and consider the quadratic function $q(x, v) = x^t M x + vx^t$. Assume additionally that $v^{(N)}$ is a sequence such that $\frac{1}{N} v^{(N)} \rightarrow a$ and let $w = b(M + M^t) + a$. Then,

$$\frac{q(X^{(N)}, v^{(N)}) - q(E(X^{(N)}), v^{(N)}))}{N^{3/2}} \rightarrow \mathcal{N}(0, w\Sigma w^t).$$

**Proof.** Let

$$q(X^{(N)}, v^{(N)}) - q(E(X^{(N)}), v^{(N)}) = A^{(N)} + B^{(N)},$$

where

$$A^{(N)} = (E(X^{(N)})(M + M^t) + v^{(N)})(X^{(N)} - E(X^{(N)}))^t,$$

$$B^{(N)} = (X^{(N)} - E(X^{(N)}))^t M (X^{(N)} - E(X^{(N)}))^t.$$

Now, clearly $\frac{A^{(N)}}{N^{3/2}} \rightarrow \sum_{i=1}^m w_i \mathcal{N}(0, \Sigma)_i$ and $\frac{B^{(N)}}{N^{3/2}} \rightarrow 0$, and the claim follows.

For the convergence of moments we have here:

**Proposition 4.10.** In the situation of Proposition 4.9 assume that additionally $\frac{1}{N} \text{Cov}(X^{(N)}) \rightarrow \Sigma$. Then,

$$\frac{E(q(x^{(N)}, v^{(N)}))}{N^2} \rightarrow q(b, a)$$

and,

$$\frac{E(q(X^{(N)}, v^{(N)}) - q(E(X^{(N)}), v^{(N)}))}{N} \rightarrow \sum_{i,j} M_{i,j} \Sigma_{i,j}.$$

If furthermore $E \left( X_i^{(N)} - E(X_i^{(N)}) \right)^4 /N^3 \rightarrow 0$ for all $i$, then

$$\frac{\text{Var}(q(X^{(N)}, v^{(N)}))}{N^3} \rightarrow w\Sigma w^t.$$

We omit the elementary proof, and now look at the asymptotic behavior of the conditional distribution of $R^{(N)}$ given that $J^{(N)} = j^{(N)}$.

**Lemma 4.11.** If $\frac{1}{N} L^{(N)} \rightarrow a \neq 0$, and $j^{(N)}$ is a sequence of $L^{(N)}$-compatible vectors such that $\frac{1}{N} j^{(N)} \rightarrow b \neq 0$, then

$$\frac{\text{Inv}(L^{(N)}, j^{(N)}) - E(\text{Inv}(L^{(N)}, j^{(N)}))}{N^{3/2}} \rightarrow \mathcal{N}(0, v(a, b)).$$
where \( v(a, b) = \frac{1}{N^2} \sum_{i=1}^{m} (a_i + b_{i+1} - b_i)(a_i + b_{i+1}) \) (where \( b_{m+1} = 0 \)).

**Proof.** By definition, the random variable

\[
\text{Inv}(L^{(N)}, j^{(N)}) - E(\text{Inv}(L^{(N)}, j^{(N)}))
\]

is distributed like the sum \( \sum_{i=1}^{m} X_i^{(N)} \) of \( m \) independent random variables

\[
X_i^{(N)} = \text{Inv}(L_i^{(N)} + j_i^{(N)} - j_i^{(N)}, j_i^{(N)}) - \frac{1}{2} \left( L_i^{(N)} + j_i^{(N)} - j_i^{(N)} \right).
\]

Let \( a_i^{(N)} = L_i^{(N)} + j_i^{(N)} - j_i^{(N)} \) and \( b_i^{(N)} = j_i^{(N)} \). By assumption the limits \( a_i = \lim \frac{1}{N} a_i^{(N)} \) and \( b_i = \lim \frac{1}{N} b_i^{(N)} \) exist. If \( a_i = 0 \) or \( b_i = 0 \), then \( \frac{1}{N^{3/2}} X_i^{(N)} \xrightarrow{d} 0 \xrightarrow{d} N(0, 0) \). If \( a_i > 0 \) and \( b_i > 0 \) we find that \( \frac{1}{N^{3/2}} \text{Var}(\text{Inv}(a_i^{(N)}, b_i^{(N)})) \xrightarrow{d} \frac{1}{2} a_i b_i (a_i + b_i) := w(a_i, b_i) \), and **Theorem 4.1** gives that

\[
X_i^{(N)} \xrightarrow{d} N(0, w(a_i, b_i)).
\]

Thus under the conditions above \( \frac{1}{N^{3/2}} \sum_{i=1}^{m} X_i^{(N)} \xrightarrow{d} N(0, \sum_{i=1}^{m} w(a_i, b_i)) = N(0, v(a, b)) \). \( \square \)

**Remark 4.12.** The case \( c_d i = 0 \) for all \( i \) (that is \( v(c, d) = 0 \)) is less interesting but not excluded. In this case we interpret \( N(0, 0) := \delta_0 \) as the Dirac-measure (point mass) at 0.

Finally, we turn to the combination of the results above. It turns out that under mild conditions the limiting distributions of the (normalized) random variables \( R^{(N)} \) and \( J^{(N)} \) are asymptotically independent.

**Theorem 4.13.** Let \( R(a, b) \) denote a random variable with distribution \( N(0, v(a, b)) \). If \( \frac{1}{N} L^{(N)} \rightarrow a \neq 0 \), and if there exists \( b \in R_+^m \) and a positive semidefinite matrix \( \Sigma \in R^{m \times m} \) of positive rank such that

\[
\frac{J^{(N)} - Nb}{N^{1/2}} \xrightarrow{d} N(0, \Sigma).
\]

Then, as \( N \rightarrow \infty \),

\[
\left( \frac{R^{(N)}}{N^{3/2}}, \frac{J^{(N)} - Nb}{N^{1/2}} \right) \xrightarrow{d} (R(a, b), N(0, \Sigma)),
\]

where the constituents on the right-hand side are independent.

**Proof.** Let \( A \subset R \) be a Borel set and \( f : R^m \rightarrow R \) be bounded and continuous. We have

\[
E 1_A \left( \frac{R^{(N)}}{N^{3/2}} \right) f \left( \frac{J^{(N)} - Nb}{N^{1/2}} \right)
\]

\[
= E \left( \frac{R^{(N)}}{N^{3/2}} \in A \mid J^{(N)} \right) f \left( \frac{J^{(N)} - Nb}{N^{1/2}} \right)
\]

\[
= E \left( \frac{\text{Inv}(L^{(N)}, J^{(N)}) - e(L^{(N)}, J^{(N)})}{N^{3/2}} \in A \mid \right) f \left( \frac{J^{(N)} - Nb}{N^{1/2}} \right).
\]
By Skorokhod’s representation theorem (see [4, 38]) we may assume that 
\( \frac{J^{(N)}}{N^{1/2}} \rightarrow X, P - a.s. \), where \( X \sim \mathcal{N}(0, \Sigma) \). Then clearly \( \frac{1}{N} J^{(N)} \rightarrow b, P - a.s. \), and by the preceding lemma

\[
P \left( \frac{\text{Inv}(L^{(N)}, J^{(N)}) - c(L^{(N)}, J^{(N)})}{\sqrt{N}} \in A \right) \rightarrow \mathcal{N}(0, v(a, b))(A), P - a.s.
\]

Therefore,

\[
E 1_A \left( \frac{P^{(N)}}{\sqrt{N^3/2}} \right) f \left( \frac{J^{(N)} - Nb}{\sqrt{N^3/2}} \right) \rightarrow \mathcal{N}(0, v(a, b)) \left( A \right) E(f(X)). \quad \square
\]

Let us collect some immediate corollaries.

**Corollary 4.14.** In the situation of Theorem 4.13 assume that additionally \( \frac{1}{N} E(J^{(N)}) \rightarrow b \). Let \( b_0 = b_{m+1} = 0, J^{(N)}_0 = 0, d^{(N)} = \frac{1}{2} \sum_{i=1}^{m} (L^{(N)}_i + E(J^{(N)}_i) - E(J^{(N)}_i)) E(J^{(N)}_i), \) and \( c = c(a, b) \) be the vector with coordinates \( c_i = a_i + b_{i+1} + b_{i-1} - 2b_i \). Then,

\[
e(L^{(N)}, J^{(N)}) - d^{(N)} \quad \text{d} \rightarrow \mathcal{N}(0, \frac{1}{4} \Sigma \Sigma c^t),
\]

\[
y^{(N)} - d^{(N)} \quad \text{d} \rightarrow \mathcal{N}(0, \frac{1}{4} \Sigma \Sigma c^t + v(a, b)).
\]

*Proof.* The first assertion follows directly from Proposition 4.9. For the second assertion observe that by Theorem 4.13 the limiting distribution is the convolution of the normal distributions \( R(a, b) \) and the limiting distribution in (4.5). \( \square 

**Corollary 4.15.** In the situation of Theorem 4.13 assume that additionally \( \frac{1}{N} E(J^{(N)}) \rightarrow b \). Let \( b_0 = b_{m+1} = 0, J^{(N)}_0 = 0, d^{(N)} = \frac{1}{2} \sum_{i=1}^{m} (L^{(N)}_i + E(J^{(N)}_i) - E(J^{(N)}_i)) E(J^{(N)}_i) \), \( e^{(N)} = \sum_{i=1}^{m} E(J^{(N)}_i)(E(J^{(N)}_i) + \sum_{k=1}^{i-1} L_k) \), and \( f = f(a, b) \) be the vector with coordinates \( f_i = a_i + b_{i+1} + b_{i-1} + 2b_i + 2 \sum_{\ell=1}^{i-1} a_\ell \). Then,

\[
T^{(N)} - (d^{(N)} + e^{(N)}) \quad \text{d} \rightarrow \mathcal{N}(0, \frac{1}{4} \Sigma \Sigma f^t + v(a, b)).
\]

Finally, we note for the convergence of the variance of \( R^{(N)} \):

**Lemma 4.16.** In the situation of Theorem 4.13 assume that additionally \( \frac{1}{N} E(J^{(N)}) \rightarrow b \), and that \( \frac{1}{N} J^{(N)} \) is bounded. Then,

\[
\frac{\text{Var}(R^{(N)})}{N^3} \rightarrow v(a, b).
\]

*Proof.* We have

\[
E((R^{(N)})^2 | J^{(N)}) = \text{Var}(\text{Inv}(L^{(N)}, J^{(N)}))
\]

\[
= \frac{1}{12} \sum_{i=1}^{m} (L_i + J^{(N)}_{i+1} - J^{(N)}_i)J^{(N)}_i(L_i + J^{(N)}_{i+1})
\]
and by our assumptions $E(\langle R(N)\rangle^2 | J(N)) / N^3$ converges boundedly to $v(a, b)$. Hence $E(\langle R(N)\rangle^2 | N^3) = E(\langle (R(N))^2 | J(N)\rangle) / N^3 \rightarrow v(a, b)$. \hfill \qed

### 4.3. Unrestricted number of parts.

We first consider the total number $\tilde{T}(L)(q)$ of $L$-admissible partitions, without restrictions on the number of parts. In this case clearly $A_{k-i+1}(k)$ (as defined in Proposition 4.5) has a binomial distribution with parameters $n = L_k$ and $p = \frac{i}{k+1}$, and hence each $J_i$ can be represented as a sum of independent binomial variables. Furthermore, the covariance of $A_{k-i+1}(k)$ and $A_{k-j+1}(k)$ can be computed as

$$\text{Cov}(A_{k-i+1}(k), A_{k-j+1}(k)) = \frac{L_k}{k+1} \left( \min(i, j) - \frac{ij}{k+1} \right).$$

We therefore have

**Lemma 4.17.** Consider $(J_1, \ldots, J_m)$ as defined in Proposition 4.5. Then,

\begin{align*}
\textbf{E}(J_i) &= \frac{i}{k+1} \sum_{k=i}^{m} L_k, \\
\textbf{Var}(J_i) &= \frac{i}{k+1} \sum_{k=i}^{m} \frac{L_k}{(k+1)^2}, \\
\textbf{Cov}(J_i, J_j) &= \min(i, j) \sum_{k=\max(i,j)}^{m} \frac{L_k}{k+1} - ij \sum_{k=\max(i,j)}^{m} \frac{L_k}{(k+1)^2}. 
\end{align*}

Moreover, straightforward computations lead to the exact expectation value of the random variable defined in (4.1). We will need this for comparison to the basic specialization of Demazure modules in §4.5.2. Note that an asymptotic approximation (by Proposition 4.10) to this mean is used in Corollary 4.15.

**Lemma 4.18.** Consider the random variables defined in (4.1), and let $s_i = \sum_{k=i}^{m} \frac{L_k}{k+1}$, $t_i = \sum_{k=1}^{i-1} L_k$. Then, for $E(T) = E(Y) + E(Q(L, J))$ we have

\begin{align*}
E(Y) &= \frac{1}{2} \sum_{i=1}^{m} is_i^2 - \frac{1}{4} \sum_{i=1}^{m} i \frac{L_i}{i+1}, \\
E(Q(L, J)) &= \sum_{i=1}^{m} i^2 s_i^2 + \sum_{i=1}^{m} is_i t_i + \sum_{i=1}^{m} i(i+2) \frac{L_i}{6(i+1)}. 
\end{align*}

**Theorem 4.19.** Let $\frac{1}{N} L^{(N)} \rightarrow a \neq 0$. Then, $\frac{1}{N} E(J^{(N)}_i) \rightarrow i \sum_{k=1}^{m} \frac{a_k}{k+1}$ for each $i$ and

$$\frac{J^{(N)} - E(J^{(N)})}{N^{1/2}} \xrightarrow{d} \mathcal{N}(0, \Sigma),$$

Consider a sequence \( \Sigma \).

Let \( \mathbf{B}_L^{(N)} \) denote the underlying total occupancy statistic. In §4.1 it was shown that the components \( \mathbf{B}(i)^{(N)} \) are independent multinomial distributions with parameters \( L_i^{(N)} \) and \( u(i) \), where \( u(i)_0 = \ldots = u(i)_i = \frac{1}{i+1} \), and covariances \( L_i^{(N)} \Sigma(i) \), \( \Sigma(i) = \text{diag}(u(i)) - u^t(i)u(i) \). By Theorem 4.7, and since \( \frac{1}{N} L_i^{(N)} \rightarrow a_i \), we have

\[
\frac{\mathbf{B}_L^{(N)} - \mathbf{E}(\mathbf{B}_L^{(N)})}{N^{1/2}} \xrightarrow{d} \left( \mathcal{N}(0, a_1 \Sigma(1)), \ldots, \mathcal{N}(0, a_m \Sigma(m)) \right)
\]

where the components on the right-hand side are independent. Since \( \mathbf{J}^{(N)} \) is a linear image of \( \mathbf{B}_L^{(N)} \) it is clear that \( \mathbf{J}^{(N)} \) is asymptotically normal. The assertion about the covariance matrix is obvious. \( \square \)

Let us emphasize explicitly the representation-theoretic implications based on our discussion in §53.

**Theorem 4.20.** Consider a sequence \( \mathcal{F}(b_1^{(N)}, L_1^{(N)}, \ldots, L_m^{(N)}) = (C^1)^{b_1^{(N)}} \ast (C^2)^{L_1^{(N)}} \ast \cdots \ast (C^{m+1})^{L_m^{(N)}} \) of fusion modules of the current algebra \( \mathfrak{sl}_2 \otimes \mathbb{C}[t] \). If \( \frac{1}{N}(L_1^{(N)}, \ldots, L_m^{(N)}) \rightarrow \mathbf{a} \neq 0 \), then the sequence of basic specializations \( \chi(\mathcal{F}(b_1^{(N)}, L_1^{(N)}, \ldots, L_m^{(N)}))(1, q) = \sum_{a \in \mathbb{Z}} \tilde{T}((L_1^{(N)}, \ldots, L_m^{(N)}), a)(q) \) behaves asymptotically normal with individual means

\[
\mu_{(L_1^{(N)}, \ldots, L_m^{(N)})} = \sum_{i=1}^m \left[ \frac{i}{2} + i^2 \left( \sum_{k=i}^m \frac{L_k}{k+1} \right) \right]^2
\]
\[+ i \left( \sum_{k=1}^{m} \frac{L_k}{k+1} \right) \left( \sum_{k=1}^{i-1} L_k \right) + \left( \frac{4i(i+2) - 6i}{24(i+1)} \right) L_i,\]

and variance, as \(N \to \infty\),

\[
\frac{1}{N^3} \sigma^2 (L_1^{(N)}, \ldots, L_m^{(N)}) \to \frac{1}{4} \Sigma f^t + v(a, b).
\]

Here, the vectors \(a, b, f\), the function \(v\), and the matrix \(\Sigma\) are given as

\[
a = (a_1, \ldots, a_m) = \lim_{N \to \infty} \frac{1}{N} (L_1^{(N)}, \ldots, L_m^{(N)}),
\]

\[
b_i = i \sum_{\ell=i}^{m} \frac{a_\ell}{\ell+1},
\]

\[
f_i = a_i + b_{i+1} + b_{i-1} + 2b_i + 2 \sum_{\ell=1}^{i-1} a_\ell,
\]

\[
v(a, b) = \frac{1}{12} \sum_{i=1}^{m} \left( (a_i + b_{i+1} + b_i)(a_i + b_{i+1}) \right) (\text{where } b_{m+1} = 0),
\]

\[
\Sigma_{i,j} = \min(i, j) \sum_{k=\max(i,j)}^{m} \frac{a_k}{k+1} - ij \sum_{k=\max(i,j)}^{m} \frac{a_k}{(k+1)^2}.
\]

Proof. This is simply a re-formulation of the results obtained in Corollary 4.15 and Lemma 4.18. For the convergence of the variance see Proposition 4.10 and Lemma 4.16.

4.4. The central restricted case. We consider the "central region" (see §4.1.2). Let \(s_N = \lfloor E(S_L^{(N)}) \rfloor = \lfloor \frac{1}{2} \sum_{i=1}^{m} iL_i \rfloor\). It is clear from the above that the underlying occupancy distribution \(B_L^{(N)}\) is the conditional distribution

\[
B_L^{(N)} = (Y(1), \ldots, Y(m)) \mid \sum_{k=1}^{m} \sum_{i=0}^{k} Y_i(k) = s_N,
\]

where \(Y(1), \ldots, Y(m)\) are independent random vectors, each \(Y(k)\) is multinomial with parameters \(L_k^{(N)}\) and \(p_0 = \ldots = p_k = \frac{1}{k+1}\). This conditioning has the following effect on the asymptotic distribution.

Theorem 4.21. Let \(m > 1\), \(s_N = [E(S_L^{(N)})]\) and \(u = (u(1), u(2), u(m))\) with \(u(k) = (\frac{1}{k+1}, \ldots, \frac{1}{k+1})\). Assume that \(\frac{1}{N} L^{(N)} \to a \neq 0\) and let \(\sigma^2(a) = \frac{1}{12} \sum_{k=1}^{m} k(k+2)a_k\). Then,

\[
\frac{B^{(N)} - N u}{\sqrt{N}} \to N(0, \Sigma),
\]

where

\[
\Sigma_{i,j}(k, \ell) = a_k \delta_{k,\ell} \left( \frac{1}{k+1} - \frac{1}{(k+1)^2} \right)
\]

(4.15)
Furthermore, let

$$E \left( \prod_{i=0}^{m} t_i^{B_i^{(N)}} \right) = [x^s](t_0 + t_1 x + \ldots + t_m x^m)^N / ((m + 1)^n P(S_N = s_N)).$$

Hence the joint distribution is given by

$$P(B_0^{(N)} = k_0, \ldots, B_m^{(N)} = k_m) = \left( \begin{array}{c} N \\ k_0, \ldots, k_m \end{array} \right) / ((m + 1)^N P(S_N = s_N))$$

with the constraints that \( \sum_{i=0}^{m} k_i = N \) and \( \sum_{i=1}^{n} ik_i = s_N \). Since there are two linearly independent linear constraints on the values of \( B^{(N)} \) we expect a \((m - 1)\)-dimensional limiting distribution. Let \( x_0, \ldots, x_m \) be real numbers with \( \sum_{i=0}^{m} x_i = 0 \) and \( \sum_{i=0}^{m} ix_i = 0 \), and let \( k_i = N \frac{X_i}{\sigma} \). By Stirling’s approximation for the factorials for the numerator and the local limit theorem for lattice distributions for the denominator we see

$$\left( \sqrt{N} \right)^{m-1} P(B^{(N)} = k) \to \frac{1}{\sqrt{(2\pi)^{m-1}}} \sqrt{(m + 1)^{m+1} \sigma^2} e^{-\frac{m+1}{2} (x_0^2 + \ldots + x_m^2)}.$$

A check that the expression on the right-hand side is (considered as a function of \( x_2, \ldots, x_m \), say) the marginal density of \( N(0, \Sigma)_{2,\ldots,m} \) with \( \Sigma \) as in 4.15 concludes the proof. \( \square \)

For the convergence of moments we have here

**Proposition 4.22.** Under the conditions of [Theorem 4.21]

$$E(B_i^{(N)}(k)) \to a_k u(k) \quad \text{and} \quad \text{Cov}(B_i^{(N)}(k), B_j^{(N)}(\ell)) \to \Sigma_{i,j}(k, \ell)$$

Furthermore, \( E(B_i^{(N)}(k)) - E(B_i^{(N)}(k))^4 / N^3 \to 0 \).

**Proof.** Again we restrict the exposition to the one component case and use the same notation as in the proof of [Theorem 4.21]. From the generating function given there we get

$$(4.16) \quad E(B_i^{(N)}) = \frac{N}{m + 1} \frac{P(S_{N-1} = s_{N-i})}{P(S_N = s_N)}$$

$$(4.17) \quad E(B_i^{(N)})^2 = E(B_i^{(N)}) + \frac{N(N-1)}{(m+1)^2} \frac{P(S_{N-2} = s_{N-2i})}{P(S_N = s_N)}$$

$$(4.18) \quad E(B_i^{(N)} B_j^{(N)}) = \frac{N(N-1)}{(m+1)^2} \frac{P(S_{N-2} = s_{N-i-j})}{P(S_N = s_N)}$$
By the local central limit theorem for lattice distributions [17, Corollary VIII.3] we have

\[ P(S_N = k) = \frac{1}{\sqrt{2\pi N}} e^{-\frac{(k-N\mu)^2}{2N\sigma^2}} \left( 1 + O(N^{-1/2}) \right) \]

if \(|\frac{(k-N\mu)^2}{2N\sigma^2}| < C\). Applying this to the numerator and denominator shows that the quotients \( q_r(N) := \frac{P(S_{N-r} = s_{N-r})}{P(S_N = s_N)} \) are asymptotically of the form

\[ q_r(N) = 1 \quad \text{if} \quad |\frac{(s_{N-r} - N\mu)^2}{2N\sigma^2}| < 2r. \]

The asymptotic assertion about the variance/covariance follows from the formulæ above using the asymptotic form of \( q_1(N), q_2(N) \). Concerning the asserted convergence of the central fourth moment, note that the \( r-th \) factorial moment of \( B(N_i) \) is

\[ E(B(N_i)^4) = N(N-1)(N-r+1)P(S_{N-r} = s_{N-r})P(S_N = s_N) \]

After expressing the central fourth moment as a linear combination of factorial moments and plugging in the asymptotical expressions for the \( q_r(N) \), a little algebra yields

\[ E(B_i(N)^4) - E(B_i(N))^4 = N^3(c_4 + 6c_2 - 4c_3 - 4c_1) + O(N^{5/2}) = O(N^{5/2}) \]

since \( c_4 + 6c_2 - 4c_3 - 4c_1 = 0 \).

**Remark 4.23.** Let \( \frac{1}{\sqrt{N}}L(N) \xrightarrow{d} \alpha \neq 0 \). A comparison to the unrestricted case, discussed in [4,3], shows that asymptotically the underlying total occupancy distributions are quite similar. They concentrate around the same expectations. In the unrestricted case the components of the limiting distribution \( \frac{1}{\sqrt{N}}(B_L(N) - \mu) \xrightarrow{d} Z \) are independent normal vectors with

\[ \text{Cov}(Z(k)) = a_k(\text{diag}(\mu(k)) - \mu(k)^t\mu(k)). \]

The components stay normal in the central restricted case, but the restriction causes an additional negative correlation

\[ \text{Cov}(Z(k)_i, Z(\ell)_j) = -\frac{a_ka_\ell(2k-i)(2\ell-j)}{4(k+1)(l+1)a^2(a)} \]

between the components. This in turn forces the elements of the asymptotic covariance \( \Sigma \) of \( J(N) \) to be smaller than in the unrestricted case, we compute

\[ \Sigma_{i,j,\text{restricted}} = \Sigma_{i,j,\text{unrestricted}} - \frac{ij}{2\sigma^4(a)} c(i)c(j) \]

with \( c(i) = \sum_{k=1}^m \frac{k+1-i}{k+1}a_k \).
Since \( J^{(N)} \) is a linear image of \( B^{(N)} \), its distribution is also asymptotically normal and it is clear from Theorem 4.21 and Corollary 4.10 that \( T^{(N)} \) is asymptotically normal and the preceding results show that the expectation resp. variance of \( T^{(N)} \) are of \( N^2 \) resp. \( N^3 \), but the variance in the restricted case will (on the \( N^3 \) scale) be smaller than in the unrestricted case.

Again, let us emphasize the implications for fusion modules of the current algebra \( \mathfrak{sl}_2 \otimes \mathbb{C}[t] \).

**Theorem 4.24.** Consider a sequence \( \mathcal{F}(b_1^{(N)}, L_1^{(N)}, \ldots, L_m^{(N)}) = (C^1) * b_i^{(N)} \ast (C^2) * L_i^{(N)} \ast \ldots \ast (C^{m+1}) * L_m^{(N)} \) of fusion modules of the current algebra \( \mathfrak{sl}_2 \otimes \mathbb{C}[t] \). If \( \frac{1}{N}(L_1^{(N)}, \ldots, L_m^{(N)}) \rightarrow a \neq 0 \), then the sequence of central string functions \( T(L_1^{(N)}, \ldots, L_m^{(N)}) \), i.e. \( s_N = \lfloor \frac{1}{2} \sum_{i=1}^{m} iL_i \rfloor \), behaves asymptotically normal with asymptotic mean

\[
\frac{1}{N^2} \mu_{L_1^{(N)}, \ldots, L_m^{(N)}, s_N} \to \sum_{i=1}^{m} \left[ \left( \frac{i}{2} + \frac{i^2}{2} \right) \left( \sum_{k=i}^{m} \frac{a_k}{k+1} \right)^2 + i \left( \sum_{k=i}^{m} \frac{a_k}{k+1} \right) \left( \sum_{k=1}^{i-1} \frac{a_k}{k} \right) \right],
\]

and asymptotic variance

\[
\frac{1}{N^3} \sigma^2_{L_1^{(N)}, \ldots, L_m^{(N)}, s_N} \to \frac{1}{4} \mathbf{f} \mathbf{f}^\top + v(a, b).
\]

Here, the vectors \( \mathbf{a}, \mathbf{b}, \mathbf{f} \), the function \( v \), and the matrix \( \Sigma \) are given as

\[
\mathbf{a} = (a_1, \ldots, a_m) = \lim_{N \to \infty} \frac{1}{N} (L_1^{(N)}, \ldots, L_m^{(N)});
\]

\[
b_i = i \sum_{\ell=i}^{m} \frac{a_\ell}{\ell+1},
\]

\[
f_i = a_i + b_{i+1} + b_{i-1} + 2b_i + 2 \sum_{\ell=1}^{i-1} a_\ell,
\]

\[
v(a, b) = \frac{1}{12} \sum_{i=1}^{m} (a_i + b_{i+1} - b_i)(a_i + b_{i+1}) \text{ (where } b_{m+1} = 0),
\]

\[
\Sigma_{i,j} = \min(i, j) \sum_{k=\max(i,j)}^{m} \frac{a_k}{k+1} - ij \sum_{k=\max(i,j)}^{m} \frac{a_k}{k+1} + \frac{ij}{2\sigma^2(a)} c(i)c(j),
\]

where \( \sigma^2(a) = \frac{1}{12} \sum_{k=1}^{m} k(k+2)a_k \), and \( c(i) = \sum_{k=i}^{m} \frac{k+1-i}{k+1} a_k \).

**Proof.** This is again a re-formulation of Theorem 4.21. Since \( J^{(N)} \) is a linear image of \( B^{(N)} \), the asserted convergences follow from Proposition 4.10, Lemma 4.16, Proposition 4.22, and Remark 4.23. Note that by Proposition 4.10 the leading term of the expectation values, i.e. the coefficient of
$N^2$, depend only on $a$ and $b$ in the same manner as in the unrestricted case. Therefore, to derive the asymptotic mean given in (4.19) one simply has to replace the $L_k$ in the quadratic terms of (4.13) by their limit values $a_k$. □

Let us explicitly compare the central restricted one component case $L^{(N)} = (L_1^{(N)}, \ldots, L_{m-1}^{(N)}, L_m^{(N)}) = (0, N)$ to the corresponding unrestricted one. We rewrite

$$Q(L^{(N)}, J^{(N)}) = \sum_{i=1}^{m} (J_i^{(N)})^2$$

$$= \sum_{i=1}^{m} (J_i^{(N)} - E(J_i^{(N)}))^2 + 2 \sum_{i=1}^{m} (J_i^{(N)} - E(J_i^{(N)})) E(J_i^{(N)})$$

$$+ \sum_{i=1}^{m} (E(J_i^{(N)}))^2,$$

and let in the sequel

$$G^{(N)} = \sum_{i=1}^{m} (J_i^{(N)} - E(J_i^{(N)}))^2,$$

$$M^{(N)} = 2 \sum_{i=1}^{m} (J_i^{(N)} - E(J_i^{(N)})) E(J_i^{(N)}).$$

It is straightforward to compute that in the case of jointly normal variables $J_i^{(N)}$ we have $\text{Var}(\sum_{i=1}^{m} (J_i^{(N)})^2) = \text{Var}(G^{(N)}) + \text{Var}(M^{(N)}).

Lemma 4.25. Let $B \overset{d}{=} N(Nu, N\Sigma)$ with $\Sigma$ given as in Theorem 4.7. Then, $G^{(N)}$ has expectation

$$E G^{(N)} = \sum_{i=1}^{m} \text{Var}(J_i^{(N)}) = \frac{N}{15} \frac{(m-1)(m^2 + 20m + 6)}{(m+1)(m+2)},$$

and variance

$$\text{Var}(G^{(N)}) = \frac{N^2(m-1)(11m^5 + 99m^4 + 16482m^3 - 2372m^2 + 6696m + 2304)}{6300(m+1)^2(m+2)^2}.$$ 

Furthermore, the variance of $M^{(N)}$ is given by

$$\text{Var}(M^{(N)}) = \frac{N^3 m(m-1)(m^3 + 247m^2 + 136m + 12)}{180 (m+1)^2(m+2)}.$$ 

Thus $M^{(N)}$ is of order of magnitude $N^{3/2}m$ while $G^{(N)}$ is of order of magnitude $mN$. Therefore the positive random variable $G^{(N)}$ will cause a noticeable distortion of the asymptotic normal distribution of $(Q(L^{(N)}, J^{(N)}) -$
\[ \mathbb{E}(Q(L^{(N)}, J^{(N)}))/N^{3/2} \text{ unless } N >> \sqrt{m}. \] Consequently, one has
\[
\mathbb{E} T^{(N)} = N^2 \frac{m(4m + 5)}{12(m + 1)} + \frac{N}{15} m + 1 \frac{m^3 + 19m^2 - 14m - 6}{m + 2} - \frac{N m - 1}{4 m + 1}
\]

**Remark 4.26.** (1) We computed the expectation of \( T^{(N)} \) and the variance of \( Q(L^{(N)}, J^{(N)}) \) here under the simplifying assumption that the occupancy distribution is normal. Note that we know from [Proposition 4.10](#) and [Theorem 4.21](#) that the leading terms in \( N \) are correct.

(2) Comparing these results to (4.10) given at the end of the previous section, we see that the additional restriction has only small effects on the asymptotic distribution of \( T^{(N)} \). Expectation and shape remain essentially unaltered, only the variance has decreased. This was to be anticipated from the probabilistic setup: the conditioning is “in line” with the law of large numbers.

(3) The cases \( \frac{1}{N} S_{L^{(N)}} \rightarrow e' \neq e := \frac{1}{2} \sum_{k=1}^{m} k a_k \) can be treated by standard large deviation techniques. Again we may safely expect to find normality of the asymptotic distributions.

(4) In terms of fusion modules the results in [Lemma 4.25](#) and (4.21) apply to the central string function \( \tilde{T}((0, N), \lfloor Nm/2 \rfloor) \) in \( F(0, N) = (C_m)^{\ast N} \) and can be re-formulated accordingly (see [Theorem 4.24](#)).

**4.5. Comparison to previously known results.** We compare our findings to established results in the literature [5, 6, 7, 22].

**4.5.1. The unrestricted one component case.** Consider first the one component case \( L = (L_1^{(N)}, \ldots, L_{m-1}^{(N)}, L_m^{(N)}) = (0, N) \). In this case the distribution of \( Y^{(N)} \) has been investigated under the heading “generalized Galois numbers” by Bliem and Kousidis [5] and later by Janson [22]. These authors studied the random variables \( Y^{(N)} \) with probability generating function
\[
\mathbb{E}(q^{Y^{(N)}}) = \frac{1}{(m + 1)^N} \sum_{(k_0, \ldots, k_m) \in \mathbb{N}_0^{m+1}} \left[ \begin{array}{c} N \\ k_0, \ldots, k_m \end{array} \right]_q
\]
Note that if we let \( B_0 = L_m - J_m, B_1 = J_m - J_{m-1}, \ldots, B_{m-1} = J_2 - J_1, B_m = J_1 \) it becomes evident that this distribution coincides with the distribution of \( Y^{(N)} \) in the one component case. Now, Bliem and Kousidis showed

**Theorem 4.27 ([5 Theorem 3.5]).** Consider the random variables \( Y^{(N)} \) defined through (4.22). Then,
\[
\mathbb{E}(Y^{(N)}) = \frac{1}{4} \frac{m}{m + 1} N(N - 1),
\]
\[
\text{Var}(Y^{(N)}) = \frac{1}{72} \frac{(m + 1)^2 - 1}{(m + 1)^2} N(N - 1)(2N + 5),
\]
and \( \frac{Y^{(N)} - E(Y^{(N)})}{N^{3/2}} \xrightarrow{d} \mathcal{N}(0, \frac{1}{36} \frac{(m+1)^2 - 1}{(m+1)^2}) \).

Janson derived the same result in a variety of ways, proved a corresponding local central limit theorem, and gave an overview of different interpretations of the distribution of \( Y^{(N)} \). In addition he showed joint convergence of \( Y^{(N)} \) and \( B^{(N)} \):

**Theorem 4.28 ([22, Theorem 2.4]).** Let \( B^{(N)}, Y^{(N)} \) be as above. Then,

\[
\left( \frac{Y^{(N)} - E(Y^{(N)})}{N^{3/2}}, \frac{B^{(N)} - E(B^{(N)})}{N^{1/2}} \right) \xrightarrow{d} \left( \mathcal{N}(0, \frac{1}{36} \frac{(m+1)^2 - 1}{(m+1)^2}), \mathcal{N}(0, \Sigma) \right),
\]

where the constituents on the right hand side are independent, and the matrix \( \Sigma \) is given by \( \Sigma_{i,j} = \frac{1}{m+1} (\delta_{i,j} - \frac{1}{m+1}) \).

Let us compare these results to our findings above. We obtain from Lemma 4.18:

\[ E(Y^{(N)}) = \frac{1}{4} \frac{m}{m+1} N^2 - \frac{1}{4} \frac{m}{m+1} N, \]

which agrees with the expectation given in Theorem 4.27. Further, we have

\[ E(J_i^{(N)}) = i \frac{N}{m+1}, \quad b_i = \frac{i}{m+1}, \quad a_0 = \ldots = a_{m-1} = 0, \quad a_m = 1, \]

and find that

\[ d^{(N)} = \frac{m(m+1)}{4} N^2, \]

and

\[ v(a, b) = \frac{1}{12(m+1)^2} \sum_{i=1}^{m} i(i+1) = \frac{1}{36} \frac{m(m+2)}{(m+1)^2}. \]

Finally, \( c = 0 \), where \( c = c(a, b) \) is as in Corollary 4.14. Thus, by Corollary 4.14 we have

\[ \frac{Y^{(N)} - d^{(N)}}{N^{3/2}} \xrightarrow{d} \mathcal{N}(0, \frac{1}{36} \frac{m(m+2)}{(m+1)^2}), \]

which is equivalent to the weak convergence assertion in Theorem 4.27 and hence establishes an independent proof. Moreover, by Corollary 4.14

\[ \frac{e(L^{(N)}, J^{(N)}) - d^{(N)}}{N^{3/2}} \xrightarrow{d} \mathcal{N}(0, 0) = \delta_0, \]

that together with Theorem 4.1 independently proves Janson’s Theorem 4.28.

Note that Theorem 4.28 as it stands does not generalize to more general distributions. As an example let \( B^{(N)} \) be multinomial with parameters \( N, p \) where \( p \) is not uniform. Here we get from Corollary 4.14 that

\[ \frac{e(L^{(N)}, J^{(N)}) - d^{(N)}}{N^{3/2}} \xrightarrow{d} \mathcal{N}(0, v_1(p)), \]
where \( v_1(p) = \frac{1}{4} (\sum_{i=0}^{m} p_i^3 - (\sum_{i=0}^{m} p_i^2)^2) \). The corresponding joint limiting distribution (on the right hand side of (4.23)) is normal, but the constituents are not independent.

4.5.2. The unrestricted two component case. For the two component case of \( L = (L_1, \ldots, L_m) \), i.e. \( L_m = M, L_k = K \) for a \( k < m \), and all other \( L_i = 0 \), we find

\[
E(T_L) = \frac{1}{12} \frac{m(4m+5)}{m+1} M^2 + \frac{1}{12} \frac{m(2m+1)}{m+1} M + \frac{1}{12} \frac{k(4k+5)}{k+1} K^2
+ \frac{1}{12} \frac{k(2k+1)}{k+1} K + \frac{1}{2} m KM + \frac{1}{6} \frac{k(k+2)}{k+1} KM.
\]

For \( K = 1 \) this simplifies to

\[
E(T_L) = \frac{1}{12} \frac{m(4m+5)}{m+1} M^2 + \frac{1}{12} \frac{m(8m+7)+2k(k+2)}{m+1} M + \frac{k}{2}.
\]

Let us compare this to the only known result for a two component case in the literature, the case of Demazure modules \( V_w(\Lambda) \) associated to the affine Kac-Moody algebra \( \hat{sl}_2 \). Fix the highest weight \( \Lambda = m\Lambda_0 + n\Lambda_1 \). The random variables \( X_w \) having probability generating function the basic specialization of the character \( \chi(V_w(\Lambda)) \) are given due to Proposition 3.1 by translations and rotations (averaging over the random variable \( S_{L_w} \)) as follows

\[
(4.24) \quad \text{Equations (3.4) and (3.5) read as: } X_w = N^2 m + N(N+1)n + (-2N-1) \cdot S_{L_w} + T_{L_w}.
\]

\[
(4.25) \quad \text{Equations (3.6) and (3.7) read as: } X_w = N^2 m + N(N-1)n - 2N \cdot S_{L_w} + T_{L_w}.
\]

The four cases covered here correspond to the four cases in [7, Theorem 4.1]. Let us restrict for simplicity reasons to (4.25) for \( w = (s_1 s_0)^N \), and compare our findings to [7, Theorem 4.1], where the corresponding case is [7, (4.1)] for even \( N \).

Theorem 4.29 ([7, (4.1) in Theorem 4.1]). For \( L \) with entries 0 except \( L_m = 1, L_m+n = 2N - 1 \), and with \( U = 2N - 1, u = m + n \) one has

\[
(4.26) \quad \mathbb{E}(N^2 m + N(N-1)n - 2S_{L}N + T_{L})
= \frac{2U(m+2)+U(U-1)u(u+2)}{12(u+1)} + \frac{U-1}{2} \frac{u}{2} + \frac{m}{2}.
\]

We can re-establish (4.26) by the computation of the left-hand side through the linearity of \( \mathbb{E}(.) \) and the mean of the random variables \( S_{L}, T_{L} \).

Proof. From the above we have in this case

\[
\mathbb{E}(T_L) = \frac{1}{12} \frac{u(4u+5)}{u+1} U^2 + \frac{1}{12} \frac{u(8u+7)+2m(m+2)}{u+1} U + \frac{m}{2}.
\]
The probability distribution of $X$ with probability generating function given by the basic specialization of our Theorem 4.1. Since $\mu$ is equivalent to the asserted weak convergence of $\tilde{\mu}$ and $\tilde{\mu}$ denote the level of the representation, and consider the random variable $d$ these Demazure modules, i.e. $d$ other cases can be derived equivalently. Let $\mu$ the rectangle $[0,1] \times [-1,1]$. Then, as $k \to \infty$,
\[ \tilde{\mu}^{(k)} \overset{w}{\to} \delta_{\left(\frac{c}{3(c_0+1)},0\right)}, \]
where $c = \alpha_0^\vee + \alpha_1^\vee$ denotes the canonical central element.

\textbf{Proof.} We consider only the Demazure modules $V_{(s_1 s_0)^N} (m \Lambda_0 + n \Lambda_1)$ as the other cases can be derived equivalently. Let $d_N$ denote the maximal degree in these Demazure modules, i.e. $d_N = N^2 m + N (N-1) n$. Let $u = m+n = \langle c, \Lambda \rangle$ denote the level of the representation, and consider the random variable with probability generating function given by the basic specialization of our Demazure module, that is

\[ X_N = d_N - 2N S_{L(N)} + T^{(N)} \]
\[ = E(d_N - 2N E(S_{L(N)}) + T^{(N)}) - 2N (S_{L(N)} - E(S_{L(N)})) \]
\[ + (T^{(N)} - E(T^{(N)})). \]

The probability distribution of $X_N$ and $\frac{1}{d_N} X_N$ is the first coordinate of $\mu^{(N)}$ and $\tilde{\mu}^{(N)}$ for the Weyl group element $w^{(N)} = (s_1 s_0)^N$, respectively. Now, equivalent to the asserted weak convergence of $\tilde{\mu}^{(N)}$ we have
\[ \frac{X_N}{d_N} \overset{d}{\to} \frac{u + 2}{3(u + 1)}, \]
since
\[ \frac{E(d_N - 2N S_{L(N)} + T^{(N)})}{d_N} \overset{d}{\to} \frac{1}{3} \frac{4u + 5}{u + 1} - 1 = \frac{u + 2}{3(u + 1)}, \]
and by [4.3] and Corollary 4.15 we have the convergences in distribution
\[ \frac{S_{L(N)} - E(S_{L(N)})}{N} \overset{d}{\to} 0, \quad \text{and} \quad \frac{T^{(N)} - E(T^{(N)})}{N^2} \overset{d}{\to} 0. \]
Since it is already known that the second coordinate of \( \tilde{\mu}^{(N)} \) concentrates in 0 by the factorization phenomenon \((3.10)\), the claim follows. \( \square \)

The basic specialization can be interpreted as the generating function of dimensions of \( \mathfrak{sl}_2 \) representations in \( V_w(\Lambda) \) splitted by the degree, i.e. the coefficient \( k \) in \( e^{-k\alpha_0} \). Therefore, Lemma 4.30 depicts their asymptotic concentration along the degree as a rational expression in the level of the Demazure module.

One can easily compute that the asymptotic concentration in the central restricted two component case (i.e. along a central string function) is the same. This exhibits the dominating weights in Demazure modules. Note that this is not deducible from \([6, 7]\) due to the limitations of their arguments imposed by the non-positivity of Demazure’s character formula. The basic argument that the concentrations coincide is that the expectation values of the random variables in the central restricted two component case have the same leading terms (in \( N \)) as the unrestricted ones.

Let us briefly describe the central string functions in fusion modules whose study would generalize this asymptotic concentration. Let \( A^{-1} \in \mathbb{Z}^{m \times m}_{+} \) be the matrix with \( A^{-1}_{i,j} = \min(i, j) \), and consider the vector \( \ell = A^{-1}L \). In particular, \( \ell_1 = \sum L_i \) and \( \ell_m = \sum iL_i \). Then by (4.2) we have \( E(S_L) = \frac{1}{2}l_m \), and following \([4.1.2]\) the \( q \)-supernomials \( \tilde{T}(L, \frac{1}{2}l_m) \) describe the central string functions in fusion modules (resp. \( \tilde{T}(L, \frac{1}{2}l_m \pm \frac{1}{2}) \)), that carry the dominant weight multiplicities. Due to \([35, \text{Definition 2.3}]\) and \([35, (3.1)]\) the translation

\[
(4.27) \quad q^{-\frac{1}{2}l_1l_m} \tilde{T}(L, \frac{1}{2}l_m) = \sum_{i=0}^{\frac{1}{2}L^tA^{-1}L} c_i q^i
\]

is a polynomial with degree at most \( \frac{1}{4}L^tA^{-1}L \). The reader is referred to Figure 2 for an example in the case \( L = (0, 4) \), where we have \( q^{-8}\tilde{T}((0, 4), 4) = 1 + q + 3q^2 + 3q^3 + 4q^4 + 3q^5 + 2q^6 + q^7 + q^8 \).

5. Local Central Limit Theorems

It should be possible to prove the following local central limit theorem along the lines of \([22, \S 6]\). We pose them here as conjectures.

**Conjecture 5.1.** In the notation of Theorem 4.20 let \( X_{FN} = X_{F(L_1^{(N)}, \ldots, L_m^{(N)})} \) denote a random variable with probability generating function the normalized basic specialization of the fusion module \( F(b_1^{(N)}, L_1^{(N)}, \ldots, L_m^{(N)}) \). Denote its mean \( \mu_N = \mu_{(L_1^{(N)}, \ldots, L_m^{(N)})} \) and variance \( \sigma_N^2 = \sigma_{(L_1^{(N)}, \ldots, L_m^{(N)})}^2 \). Then, uniformly in \( k \) as \( N \to \infty \),

\[
(5.1) \quad \sqrt{2\pi\sigma_N} \cdot P(X_{FN} = k) = e^{-(k-\mu_N)/2\sigma_N^2} + o(1).
\]
Here, $\sigma^2_N$ can be replaced by the explicit expression $N^3(\frac{1}{4} \sum t^4 + v(a, b))$ from (4.20). In particular, the dimension of the $\mathfrak{sl}_2$ submodule in $F_N$ of degree $k$ grows as (5.1).

The analogue for the central restricted case is:

**Conjecture 5.2.** In the notation of Theorem 4.24 and (4.27) let $S_N$ denote a random variable with probability generating function the normalized central string function

$$E(q^{S_N}) = q^{-\frac{1}{2}\ell_1(N)} \tilde{T}(\mathbf{L}(N), s_N).$$

Assume $\frac{1}{N}\mathbf{L}^{(N)} \to a \neq 0$, then $\frac{1}{N}\ell_1(N) \to \sum_{i=1}^{m} a_i$ and $\frac{1}{N}\ell_m(N) \to \sum_{i=1}^{m} ia_i$, and we let

$$\mu = \lim_{N \to \infty} \frac{1}{N^3} \left( \mu_{\ell_1(N), \ldots, \ell_m(N)_N, s_N} - \frac{1}{2} \ell_1(N) \ell_m(N) \right),$$

$$\sigma^2 = \lim_{N \to \infty} \frac{1}{N^3} \left( \sigma^2_{\ell_1(N), \ldots, \ell_m(N)_N, s_N} \right).$$

Then, uniformly in $k$ as $N \to \infty$,

$$\sqrt{2\pi\sigma} \cdot P(S_N = k) = e^{-(k-\mu)/(2\sigma^2) + o(1)}$$

In particular, the dimension of the $\mathfrak{h} \subset \mathfrak{sl}_2$ weight space with coordinates $\frac{1}{2}\ell_m(N)\alpha_1$ and $-k\delta$ grows as (5.2).

Note that those statements are the first hints towards an asymptotic description of the dimensions of fundamental submodules in fusion products available in the literature.

The proofs should work verbatim as the proof of [22, Theorem 1.5] once one establishes an estimate for the generating function $\tilde{T}(\mathbf{L}(q)\mathbf{L}(1)$ analogous to [22, Lemma 6.1], and for the $q$-supernomials $\tilde{T}(\mathbf{L}(a,q)\mathbf{L}(1)$ analogous to [22, (6.2),(6.3)], respectively.

6. **Fusion of symmetric power representations**

The Kostka numbers are the coefficients in the expansion

$$\prod_i h_{\xi_i}(x) = \sum_{\eta} K_{\eta, \xi} \cdot s_{\eta}(x)$$

of the product of complete symmetric functions $h_{\xi_i}$ in terms of the Schur functions $s_{\eta}$. The Kostka polynomials $K_{\eta, \mu}(q)$ generalize the Kostka numbers in the sense that $K_{\eta, \mu}(1) = K_{\eta, \mu}$. They give the transition matrix between the Schur function $s_{\eta}$ and Hall-Littlewood function $P_{\mu}$, i.e.

$$s_{\eta}(x) = \sum_{\mu} K_{\eta, \mu}(q) \cdot P_{\mu}(x, q).$$

A standard reference for the above functions is [29].
Now, the \(q\)-supernomial \(S_{\xi,\mu}(q)\) \cite{21,25,34,35,36} is defined as the combination of (6.1) and (6.2), i.e. as the transition between the above product of complete symmetric functions and Hall-Littlewood functions

\[
S_{\xi,\mu}(q) = \sum_{\eta} K_{\eta,\xi} \cdot K_{\eta,\mu}(q).
\]

An explicit form of \(S_{\xi,\mu}(q)\) is proven in \cite[Proposition 5.1]{21}, where \(\mu = (\mu_1, \ldots, \mu_m)\) is a partition and \(\xi \in \mathbb{Z}_n^+\) a composition such that \(|\mu| = |\xi| = M\), as

\[
S_{\xi,\mu}(q) = \sum_{\nu} q^{\phi(\nu)} \prod_{1 \leq a \leq n-1 \leq i \leq \mu} \left[ \frac{\nu_i^{(a+1)} - \nu_i^{(a)}}{\nu_i^{(a)} - \nu_{i+1}^{(a)}} \right],
\]

with

\[
\phi(\nu) = \sum_{a=0}^{n-1} \sum_{i=1}^{\mu_1} \left( \frac{\nu_i^{(a+1)} - \nu_i^{(a)}}{2} \right),
\]

and where the sum \(\sum_{\nu}\) is indexed over the sequences of Young diagrams \(\nu^{(1)}, \ldots, \nu^{(n-1)}\) such that

\[
\emptyset \subset \nu^{(0)} \subset \nu^{(1)} \subset \cdots \subset \nu^{(n-1)} \subset \nu = \mu^t,
\]

\[|\nu^{(a)}| = \xi_1 + \cdots + \xi_a\] for \(1 \leq a \leq n-1\).

**Remark 6.1.** For \(n = 2\) and arbitrary \(\mu\), (6.3) agrees with the definition of \(q\)-supernomials as given by Schilling and Warnaar \cite{35}. See \cite[§3.1]{25} for a detailed discussion.

We define a slight variant of the above \(q\)-supernomials, which describes the string functions in the fusion product of \(sl_{r+1}\) symmetric power representations \(F_{\mu} = V_{\mu_1 \omega_1} \ast V_{\mu_2 \omega_1} \ast \cdots \ast V_{\mu_m \omega_1}\). That is,

\[
S_{\xi,\mu}^*(q) = q^{n(\mu)} S_{\xi,\mu}(q^{-1}) = q^{n(\mu)} \sum_{\eta} K_{\eta,\xi} \cdot K_{\eta,\mu}(q^{-1}),
\]

where for the partition \(\mu = (\mu_1, \ldots, \mu_m)\) we set \(n(\mu)\) (Cf. \cite[(3.10)]{24}, \cite[§2.1]{26}, \cite[§2.1]{35}) to be the normalization constant

\[
n(\mu) = \sum_{i=1}^{m} (i-1)\mu_i = \sum_{1 \leq i < j \leq m} \min(\mu_i, \mu_j).
\]

Note that this normalization ensures that \(q^{n(\mu)} K_{\eta,\mu}(q^{-1})\) is a polynomial in \(q\).

Then, we have a fermionic formula (a positive sum of products of \(q\)-binomial coefficients) for the graded character of the above fusion product \(F_{\mu}\). That is,
Proposition 6.2. Let $\mu = (\mu_1, \ldots, \mu_m)$ be a partition of $M$. Then,

$$\chi(\mathcal{F}_\mu) = \sum_{\xi \text{ weight}} S_{\xi,\mu}(q) \cdot m_\xi.$$ 

Proof. Let $m_\xi$ denote the monomial symmetric functions. Then, with $\tilde{K}_{\eta,\mu}(q) = q^{n(\mu)} K_{\eta,\mu}(q^{-1})$ where $n(\mu) = \sum (i-1) \mu_i$ as in [24] (3.10) one has [24] Corollary 7.6:

$$\chi(\mathcal{F}_\mu) = \sum_{\eta \vdash M} \chi(\pi_\eta) \cdot \tilde{K}_{\eta,\mu}(q)$$

$$= \sum_{\eta \vdash M} s_\eta \cdot \tilde{K}_{\eta,\mu}(q)$$

$$= \sum_{\eta \vdash M} \left( \sum_{\xi \text{ weight}} K_{\eta,\xi} \cdot m_\xi \right) \cdot \tilde{K}_{\eta,\mu}(q)$$

$$= \sum_{\xi \text{ weight}} \left( \sum_{\eta \vdash M} K_{\eta,\xi} \cdot \tilde{K}_{\eta,\mu}(q) \right) \cdot m_\xi$$

$$= \sum_{\xi \text{ weight}} S_{\xi,\mu}(q) \cdot m_\xi$$

Note that all partitions except $\mu$ have at most $r$ entries, corresponding to the rank of the Lie algebra. □

Remark 6.3. For the graded character of fusion of fundamental representations $*_{j} V(\omega_{i})$, Chari and Loktev prove an equivalent fermionic formula [9, Proposition 2.1.4].

Kirillov [25, 26] is a great source of various combinatorial, geometric and statistical interpretations of $q$-supernomials $S_{\xi,\mu}(q)$. Let us shortly remark on the geometric one.

Remark 6.4 (Cf. [5, 22, 27, 43]). As pointed out by Kirillov [25, §1.4] it has been proven by Shimomura [37] that the $q$-supernomials count the number of rational points $Fl^\mu_\xi(F_q)$ over the finite field $F_q$ of the unipotent partial flag variety $Fl^\mu_\xi$. To be precise, for a composition $\xi \in \mathbb{Z}^r_+$ of $n$, a $\xi$-flag in a $n$-dimensional vector space $V$ is a sequence $V_1 \subset \cdots \subset V_r$ such that $\dim V_i = \xi_1 + \cdots + \xi_i$. The set of all such flags is the partial flag variety $Fl_\xi$. We let $Fl^\mu_\xi \subset Fl_\xi$ be the subset of the partial flag variety $Fl_\xi$ consisting of the set of all $\xi$-flags $F \in Fl_\xi$ fixed by a unipotent endomorphism $u \in \text{Gl}(V)$ of type $\mu$ (a partition of $n$ that describes the Jordan canonical form of $u$). Then, $Fl^\mu_\xi$ is a closed subvariety of $Fl_\xi$, the so-called unipotent partial flag variety. Now, Shimomura [37] proves that the $q$-supernomials count the number of $F_q$-rational points in $Fl^\mu_\xi$. That is, with $n(\mu)$ as in (6.6) one has

(6.7) \[ #Fl^\mu_\xi(F_q) = q^{n(\mu)}S_{\xi,\mu}(q^{-1}) = S^*_{\xi,\mu}(q). \]
In particular, the basic specialization of the fusion module \( F_{\mu} \) gives the number of \( F_q \)-rational points in \( \coprod_{\xi} F_{\xi}^\mu \):

\[
\chi(F_{\mu})(q) = \sum_{\xi} S_{\xi,\mu}(q) = \sum_{\xi} \#F_{\xi}^\mu(F_q).
\]

(6.8)

Our Proposition 6.2 exhibits the objects that have to be analyzed in order to establish a general central limit theorem along the same lines as Theorem 4.20. The explicit expression (6.5) shows that one can interpret the \( q \)-supernomials again as mixtures of probability distributions. For an Ansatz let

\[
f_{\mu,\eta}(q) = \frac{q^{n(\mu)} K_{\eta,\mu}(q^{-1})}{K_{\eta,\mu}},
\]

\[
P(X_{\mu,\xi} = \eta) = \frac{K_{\eta,\xi} K_{\eta,\mu}}{\sum_{\eta} K_{\eta,\xi} K_{\eta,\mu}}.
\]

Here, \( f_{\mu,\eta}(q) \) would inimitate the inversion statistic, and \( X_{\mu,\xi} \) the mixture distribution. It should be straightforward to check the reductions to the distributions investigated in §4 in the case of \( q \)-supernomials as defined by Schilling and Warnaar (see Remark 6.1). We pose a conjecture for further research.

**Conjecture 6.5.** Consider the sequence of type A fusion modules of symmetric power representations

\[
F_{\mu}^{(N)} = V_{\omega_1}^* f_1^{(N)} \ast V_{2\omega_1}^* f_2^{(N)} \ast \cdots \ast V_{\ell \omega_1}^* f_{\ell}^{(N)},
\]

associated to the partition \( \mu^{(N)} = (1^{f_1^{(N)}}, 2^{f_2^{(N)}}, \ldots, \ell^{f_{\ell}^{(N)}}) \) with \( f_i^{(N)} \)-many \( i \)'s. Assume that as \( N \to \infty \) we have

\[
\frac{1}{N}(f_1^{(N)}, f_2^{(N)}, \ldots, f_{\ell}^{(N)}) \to a \neq 0.
\]

Then, the central string functions and the basic specialization of \( F_{\mu}^{(N)} \) behaves asymptotically normal as \( N \to \infty \).

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