Principle of the Helical and Nonhelical Dynamo and the $\alpha$ Effect in a Field Structure Model

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Received 2018 November 22; revised 2019 January 16; accepted 2019 January 16; published 2019 February 19

Abstract

We demonstrate the conversion process of helical (nonhelical) kinetic energy into magnetic energy using a field-structure model based on the magnetic induction equation. This approach aims to explain the generation, transport, and conservation of magnetic helicity dependent on a forcing method such as kinetic or magnetic forcing. When a system is driven by helical kinetic or magnetic energy, two kinds of magnetic helicities with opposite signs are induced. Then, asymmetric competing processes between them determine the dominant magnetic helicity. Also, the model shows that the conservation of magnetic helicity is related to a common current density and antiparallel magnetic fields in the large- and small-scale regimes. In addition to the intuitive method, we suggest an analytical method to find the $\alpha$ and $\beta$ coefficients using temporally evolving large-scale magnetic energy and magnetic helicity. The method implies that the $\alpha$ effect and its quenching are generally consistent with the conventional theory. However, the $\beta$ coefficient implies that the role of kinetic energy in a dynamo may be somewhat different from our conventional understanding. We also show how the kinetic energy near the viscous scale can suppress the dynamo process when the magnetic Prandtl number ($P_{TM}$) is small. We verify this using simulation results. Finally, using the $\alpha^2$ effect and differential rotation effect, we suggest a solar dynamo model that explains the periodic magnetic evolution in the Sun.

Key words: dynamo – magnetic fields – magnetohydrodynamics (MHD) – turbulence

1. Introduction

Various scales of magnetic fields $B$ and conducting fluids (plasma) are often observed in space. The energy in plasmas, which compose many astrophysical systems, can be transferred to the magnetic fields (dynamo process), and the magnetic fields can constrain the evolution of the plasma systems through magnetic back-reaction. The dynamo process and the influence of the magnetic field on the plasma system have been studied for a long time (Moffatt 1978; Krause & Rädler 1980; Brandenburg & Subramanian 2005, and references therein). However, it is not yet clear how the magnetic field is amplified or how it affects the evolving astrophysical systems. For example, in addition to energy transfer, the magnetic field is known to stabilize or destabilize a plasma system (e.g., sausage instability, kink instability, Kruskal–Schwarzschild instability, etc.; Boyd & Sanderson 2003). Furthermore, the magnetic field can control the rate of collapse (Machida et al. 2005) and the formation of an accretion disk through the transport of angular momentum (magneto-rotational instability; Balbus & Hawley 1991).

There are many other important influences of magnetic fields on the evolution of astrophysical systems. However, in this paper, we will mainly discuss the fundamental mechanism of the dynamo process using a physical model based on the fluid equation. Our study covers the mechanism of a helical large-scale dynamo (LSD) and small-scale dynamo (SSD) process according to the direction of magnetic energy transport. In addition to the intuitive approach, we show how to semi-analytically find the coefficients responsible for the growth of a helical large-scale magnetic field. Since many physical turbulent phenomena such as transport of momentum or material are mostly controlled by large-scale motions, the evolution and role of a large-scale magnetic field ($\vec{B}$) in a turbulent plasma system are fundamental and practical problems that are not limited to academic interest. Also, it should be noted that related processes are accelerated by fluctuating motions in the small-scale regime.

LSD theory shows how small-scale magnetic energy with helicity (the $\alpha$ effect; Park & Blackman 2012a, 2012b), differential rotation (the $\Omega$ effect; Balbus & Hawley 1991), or shear current (Rogachevskii & Kleeroin 2003) can be (inversely) cascaded toward $\vec{B}$. Of these, the $\alpha$ effect is indispensable to the self-consistent dynamo process, or inverse cascade of magnetic energy $E_M$ in the helical LSD. Moreover, since the properties of helicity provide a relatively clear mathematical aid in the theoretical description of the LSD phenomenon, many LSD theories aim to represent electromotive force (EMF, $\xi \equiv \langle u \times b \rangle$), 1 which is a source of $\vec{B}$, with (pseudo-) tensors $\alpha$ and $\beta$ and $\vec{B}$.

Analytically, the $\alpha$ effect can be derived with a scale-divided function feedback method, which is also a basic principle of numerical calculation. Representative related theories such as the first-order smoothing approximation (FOSA), second-order correlation approximation (Moffatt 1978; Krause & Rädler 1980; Brandenburg & Subramanian 2005), minimal tau approximation (MTA; Blackman & Field 2002), or the quasi-normalized approximation (QN; Frisch et al. 1975) are actually based on the method in a dynamic or stationary state.

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1 EMF is defined as $\xi \equiv \oint (u \times b) \cdot dl$. So what EMF generates is magnetic flux $\int \vec{B} \cdot d\vec{S}$ rather than $\vec{B}$ itself. However, we use $\langle u \times b \rangle$ as EMF without the integral symbol for simplicity in this paper.
However, since the \( \alpha \) effect is not a strict mathematical concept, some ambiguities are inevitable in its analytical derivation. However, these defects do not undermine the importance of the \( \alpha \) effect in the helical dynamo. To transfer energy to a larger eddy with larger turnover time and moment inertia, a more complex and efficient dynamo process is necessary. The \( \alpha \) and \( \beta \) coefficients in the \( \alpha \) effect can be found numerically by applying an external test magnetic field \( B_{ex} \) to the system (test field system; see Schrinner et al. 2005 and references therein). We will introduce a semi-numerical method to find the \( \alpha \) and \( \beta \) coefficients without imposing the test field.

Conventional dynamo theories show what happens over a statistical number of realizations of a magnetohydrodynamic (MHD) system. Analytical theories give a qualitative and more or less quantitative description of the evolution of magnetic fields in a plasma, but they do not tell us how the actual plasma and magnetic fields interact physically within the system. To explain the physical processes of evolving \( B \), a cartoon model of stretching of a \( B \) field, \( (B \cdot \nabla)u \), has been used in analogy to stretching vorticity \( (\omega \cdot \nabla)u, \omega = \nabla \times u \), neglecting the tilting effect (Zeldovich 1983; Schekochihin et al. 2002). However, \( B \) is essentially not so directly related to \( u \) as \( \omega \) is. Moreover, the concept of “stretching, twist, folding” is not relevant to any important physical law or fluid equation. Furthermore, a model implying the co-stretching of \((\mathbf{u} \times \mathbf{B}) \neq 0\). This large gap between the dynamo model and mechanism makes it more difficult to derive an accurate dynamo theory.

Here, we introduce an improved field structure model (Park 2017b) based on the magnetic induction equation for the physical mechanisms of a helical LSD and nonhelical SSD. This model shows the structure of kinetic velocity \( (u_1) \) and magnetic \((b_1)\) eddies for the most efficient energy transfer between them. The dynamo processes inferred from the model are in line with theory and consistent with simulation results. The model can also be applied to the study of more complex natural dynamo processes in the Sun or stars. After discussing the model, we show how to find the \( \alpha \) and \( \beta \) coefficients in the helical LSD from large-scale magnetic energy \( E_{m}(\langle B^2 \rangle / 2) \) and magnetic helicity \( \mathcal{H}_{m}(\langle \mathbf{A} \cdot \mathbf{B} \rangle) \), which can be measured by observation and simulation.

2. Numerical Method

For the numerical investigation we use the PENCIL CODE, which solves the coupled fluid equations for compressible conducting fluids in a periodic box (Brandenburg 2001):

\[
\frac{D\rho}{Dt} = -\rho \nabla \cdot u, \tag{1}
\]

\[
\frac{DU}{Dt} = -\nabla \ln \rho + \frac{1}{\rho} (\nabla \times \mathbf{B}) \times \mathbf{B} + \nu \left( \nabla^2 U + \frac{1}{3} \nabla \nabla \cdot U \right) \hspace{0.5em} (f_{kin}) \tag{2}
\]

\[
\frac{\partial \mathbf{A}}{\partial t} = \mathbf{U} \times \mathbf{B} - \eta \nabla \times \mathbf{B} \hspace{0.5em} (f_{mag}) \tag{2}
\]

\[
\frac{-\partial B}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}. \tag{3}
\]

Here, \( \rho \) and \( D/\text{Dt} = (\partial/\partial t) + \mathbf{U} \cdot \nabla \) indicate the density and Lagrangian time derivative. \( \nu \) and \( \eta \) are kinematic viscosity and magnetic diffusivity respectively. The velocity is in units of the sound speed, and the magnetic field is normalized by \((\rho_0 \mu_0 c_s)^{1/2} \), where \( \mu_0 \) and \( c_s \) are magnetic permeability and sound speed, respectively. The forcing function \( f(x, t) \) in Fourier space is \( \nu \mathbf{f}(t) \exp [i \mathbf{k}(t) \cdot \mathbf{x} + i\phi(t)] \):

\[
\mathbf{f}_{kin,ormag} = \frac{ik(t) \times (\mathbf{k}(t) \times e) - \lambda \mathbf{k}(t)(\mathbf{k}(t) \times e)}{k(t)^2} \sqrt{1 + \lambda^2} \sqrt{1 - (\mathbf{k}(t) \cdot e)²/k(t)^2}. \tag{4}
\]

Here \( e \) is an arbitrary unit vector, \( \lambda \) denotes the helicity ratio, and \( \phi(t) \) is a random phase \( \phi(t) \leq \pi \). For example if \( \lambda \) is \( \pm 1, \mathbf{i} \cdot \mathbf{k} \times \mathbf{f}_{kin} = \pm k \mathbf{f}_0 \). If \( \lambda = 0, \mathbf{i} \cdot \mathbf{k} \times \mathbf{f}_{kin} \) is not proportional to \( f \). The calculated forcing function in Fourier space at \( k = k_f \) is again inversely Fourier transformed and applied to Equations (2) or (3).

We simulated a helical kinetic forcing dynamo (HKFD), a nonhelical kinetic forcing dynamos (NHKFD), and a helical magnetic forcing dynamo (HMFD) with the isothermal and periodic boundary condition (PBC)\(^3\). For the HKFD, a fully helical kinetic energy \((\nabla \times \mathbf{f}_{kin}) = k_f \mathbf{f}_{kin}) \) with a strength of \( f_{kin} = 0.07 \) is applied to Equation (2). However, for the HMFD, a fully helical magnetic energy \((\nabla \times f_{mag}) = k_f \mathbf{f}_{mag}) \) is applied to Equation (3) with \( f_{mag} = f_{kin}/k_f \). Also for the NHKFD, the forcing method and its strength are the same as those of the HKFD except the helicity ratio: \( \nabla \times f \sim f \) (see Table 1). The forcing scale \((l \sim 2\pi/k_f) \) in nature will be much smaller than the domain system scale. However, here we force a moderately small scale \((k_f = 5) \) whose dissipation effect \( \sim k_f^2 \) is not too large. This is to see the evolution of the large-scale magnetic field and other physical properties more clearly.

3. Theoretical Model

3.1. Amplification of the Nonhelical B Field for SSD

The dynamo process shown in Figure 1(a) is based on the ideal (basic) magnetic induction (Park 2017a, 2017b):

\[
\frac{\partial \mathbf{B}}{\partial t} \sim -\mathbf{U} \cdot \nabla \mathbf{B} + \mathbf{B} \cdot \nabla \mathbf{U}. \tag{4}
\]

If the right-hand side of this equation is positive \((-\mathbf{U} \cdot \nabla \mathbf{B} + \mathbf{B} \cdot \nabla \mathbf{U} > 0) \), the amplification (transport) of the \( \mathbf{B} \) field can be explained without ambiguity. Since \( u_1 \) indicates the mean fluid velocity of single-charged particles, \( u_1 \) and \( b_1 \) influence each other. Their interaction yields an EMF \((\xi \sim (u_1 \times b_1)(-\mathbf{i})) \), which is weakest at \( u_1 \) and \( b_1 \) and strongest at \( u_2 \) and \( b_1 \). This spatially inhomogeneous EMF arouses a nontrivial curl effect, which is physically the growth rate of the magnetic field: \( \nabla \times (u_1 \times b_1) \sim \partial \mathbf{b}_1/\partial t \). The growth rate is weakest at \( u_2 \) and \( b_3 \) and strongest at \( u_1 \) and \( b_1 \). This indicates that the magnetic energy is advectively transferred (induced) from \( \mathbf{b}_1 \) to \( \mathbf{b}_1 \) through \(-\mathbf{u} \cdot \nabla b\). At the same time, another magnetic field is transferred from \( u_2 \) to \( u_1 \) through \( \mathbf{b} \cdot \nabla u \), namely, the “nonlocal (nl) transfer term.” More exactly, the transferred magnetic field from \( \mathbf{b}_3 \) to \( \mathbf{b}_1 \) through \(-\mathbf{u} \cdot \nabla \mathbf{b}\) is represented by \( \mathbf{b}_{loc} = -\int \mathbf{u} \cdot \nabla \mathbf{b} \, d\tau \), which is parallel to the magnetic field. Also, the transferred magnetic field through the nonlocal transfer term is represented by \( \mathbf{b}_{nl} = \int \mathbf{b} \cdot \nabla \mathbf{u} \, d\tau \), parallel to the velocity field. The net magnetic field \( \mathbf{b}_{tot} \) from these two transferred magnetic fields again interacts with \( \mathbf{u} \) at the next dynamo process.
In addition to the strength of \( \mathbf{u} \) and \( \mathbf{b} \), the angle \( \theta \) between \( \mathbf{b}_\text{nl} \) and \( \mathbf{b}_\text{loc} \) plays a crucial role in the EMF. If \( \mathbf{b}_\text{nl} \) grows faster than \( \mathbf{b}_\text{loc} \), \( \theta \) and EMF decrease. Several factors can affect their relative ratio, but the magnetic Prandtl number \( Pr_M = \nu/\eta \) has a paradoxical effect. Both \( \nu \) and \( \eta \) are related to the dissipation of energy, but their roles work in the opposite way. Decreasing \( \eta \) increases \( \mathbf{b}_\text{loc} \) and the EMF. However, with small \( \nu \) more kinetic energy is transported to a smaller eddy. Then, more magnetic energy is transferred to the smaller eddy, leading to the growth of \( \mathbf{b}_\text{nl} \), which disturbs the dynamo (see Figure 1(b)).

Astrophysical systems have a wide \( Pr_M \) distribution. As the equation for \( Pr_M \) implies (Brandenburg & Subramanian 2005),

\[
Pr_M = 1.1 \times 10^{-4} \left( \frac{T}{10^6 \text{ K}} \right)^4 \left( \frac{\rho}{0.1 \text{ g cm}^{-3}} \right)^{-1} \left( \frac{\ln \Lambda}{20} \right)^2
\]

a hot and dilute system has a large \( Pr_M \). In contrast, a dense system with low temperature is characterized by a small \( Pr_M \). For example, \( Pr_M \) of the Galaxy \( \rho \sim 10^{-24} \text{ g cm}^{-3} \), \( T \sim 10^4 \text{ K} \), Coulomb logarithm \( \ln \Lambda \sim 10 \) is inferred to be \(~10^{11} \), and \( Pr_M \) of the Sun with a higher density \( \rho \sim 0.1 \text{ g cm}^{-3} \), \( T \sim 10^6 \text{ K} \) is \(~10^{-4} \). However, in spite of such different \( Pr_M \), both systems show various scales and strengths of magnetic field. Using an analytical approach, Boldyrev & Cattaneo (2004) investigated the possibility of dynamo action (SSD) using the basic magnetic induction equation. They concluded that the dynamo process might always be possible regardless of \( Pr_M \) as long as the magnetic Reynolds number \( Re_M = \Omega l/U \eta \). U and \( l \); characteristic velocity and length, respectively) is very large. However, Thaler & Spruit (2015) could not certify SSD with \( Pr_M \ll 1 \) in their numerical experiment. In fact, strong magnetic fields \( \sim 10^3 \text{ G} \) are observed in the Sun with a very small \( Pr_M \) and a very large \( Re_M \sim 10^9 \). These contradictory results imply additional dynamo processes in addition to the energy transport process shown in Figures 1, 2, and 4. One of the most likely candidates is differential rotation. Seshasayanan et al. (2017) showed that

the critical Reynolds number \( Re_{M,crit} \) for the dynamo process can be significantly reduced under global rotation.

For the eddy scale, \( b_1 \) (or \( u_1 \)) in the model can be considered as a large- or small-scale field. However, as \( \partial \mathbf{b}/\partial t \sim \nabla \times (\mathbf{u} \times \mathbf{b}) \sim l^{-1}(\mathbf{u} \times \mathbf{b}) \) indicates, if \( u_1 \) and \( b_1 \) are small-scale eddies, the magnetic energy transfer from \( b_1 \) to \( b_1 \) or from \( u_2 \) to \( U_1 \) occurs more easily due to the small characteristic length. In contrast, if \( b_1 \) and \( u_1 \) are large-scale eddies, a more complex process is required to overcome the large characteristic length \( l_1 \sim 1/\eta \). In fact, the energy transport is essentially bidirectional, which appears clearly in a decaying MHD system.

3.2. Amplification of the Helical B Field for LSD

Figure 2 shows a dynamo system forced by the right-handed helical kinetic energy (HKF) composed of a toroidal field \( b_\text{tor} \) and poloidal one \( u_\text{pol}(\hat{z}) \) \( (\mathbf{u} \cdot \boldsymbol{\omega}) > 0, \boldsymbol{\omega} = \nabla \times \mathbf{u} \). The interaction between \( u_\text{pol}(\hat{z}) \) and \( b_\text{nl}(\hat{z}) \) induces a current density \( j_\text{nl} = \sigma (u_\text{pol} \times b_\text{nl})(-\hat{y}) \), which generates a toroidal magnetic field \( b_\text{pol} = -\nabla \times j_\text{nl} \) around \( b_\text{pol} \), forming a left-handed magnetic helicity \( (\mathbf{a} \cdot \boldsymbol{b} < 0) \). \( b_\text{tor} \) interacts with \( \mathbf{u} \) to induce another circular current density \( J_\text{circ} \) antiparallel to \( b_\text{tor} \); \( J_\text{circ} \) amplifies \( b_\text{pol} \), which amplifies \( b_\text{tor} \) back through the \( \alpha \) dynamo process (Park 2017a, 2017b). This compound process allows \( b_1 \) to surpass \( b_2 \) and \( b_3 \). As \( b_1 \) grows, dissipation plays a decisive role. So if \( b_1 \) is a small-scale field, its dissipation \( \sim k_B^2 b_1 \) \( (k_B \gg 1) \) becomes larger than that of other eddies. Therefore, \( b_1 \) should be a large-scale field for a more

2 In Figure 2(a) only a representative \( u_\text{pol} \) is considered to synchronize the two-scale dynamo theory, \( j_\text{nl}, b_\text{pol}, \) and the resultant left-handed magnetic helicity exists over the whole scale.

3 Since the vector potential \( a \) depends on the choice of gauge \( \chi \), magnetic helicity is generally \( \int (a + \nabla \chi) \cdot dV = \int (a \cdot \mathbf{b}) \cdot dV + \int (\nabla \cdot \chi) \cdot dV = \int (a \cdot \mathbf{b}) \cdot dV + \int (\mathbf{b} \cdot \hat{u}) \cdot dS \). The gauge term can be neglected with a periodic boundary condition, or in a perfect conducting fluid which is simply connected without an external source. Otherwise, the effect of gauge should be considered. However, in this paper we limit our discussion to the effect of a helical (magnetic) velocity field structure on a usual (velocity) magnetic field \( \nabla \times (\mathbf{a} \times \mathbf{b}) \). We will not discuss the effect of gauge further in this paper.

Also, we omit the volume integral symbol in magnetic helicity for simplicity.
efficient dynamo process. Here, the induced current density is
not used in the analytic derivation of the $\alpha$ effect. However, it
will be shown to be a useful quantity to explain the generation
of a poloidal solar magnetic field from a toroidal one, resulting
in periodic solar magnetic activity.

As $b_1$ increases, the energy diffuses toward $b_3$ through $-\mathbf{u} \cdot \nabla b_1$ (Figures 2(a), (b)). However, the direction of curl is
opposite, so the induced field approaches $-\hat{y}$. $b_3(\hat{y})$ can be inferred to approach zero and regrow to become $b_3'(-\hat{y})$. Also,
the toroidal field $b_{tor}'$ around $b_3'$ is induced due to $\mathbf{J}_{ind}$. The direction of $b_{tor}'$ does not change. Subsequently, $b_3'$ and $b_{tor}'$ give rise to right-handed magnetic helicity in the small-scale regime.

This is supported by the changing sign of $H_M$ at the minimum of $b_3$ (see Figures 3(a), (c)). Simultaneously, $b_3'$ can be suppressed by the interaction between $b_{tor}'$ and $\mathbf{u}_1$. Also, the growth of $b_3$ modifies the curvature radius of the magnetic fields so that the Lorentz force suppresses $\mathbf{u}_1$. All of these effects explain the conservation of magnetic helicity in HKFD.

The left panel in Figure 4(a) shows the dynamo process discussed above. However, the right panel shows a different possibility, i.e., that $\mathbf{u}_1 \times b_0$ can generate $\mathbf{J}_{ind,2}(-\hat{y})$ which is parallel to $\mathbf{b}_{tor,2}(=\mathbf{b}_{ind,1})$. $\mathbf{J}_{ind,2}$ induces $\mathbf{b}_{tor,2}$, and $\mathbf{b}_{tor,2}$ forms a right-handed magnetic helicity with $\mathbf{b}_{ind,2}$. These two dynamo processes seem to result in zero net helicity. However, careful observation reveals an essential difference between them. The amplified $\mathbf{b}_{ind,1}$ due to the enhanced $\mathbf{b}_0$ yields stronger $\mathbf{J}_{ind,1}$, which amplifies $\mathbf{b}_{tor,1}$. This toroidal field fortifies $\mathbf{b}_0$ again, which produces an enhanced $\mathbf{b}_{ind,1}$. In contrast, $\mathbf{b}_{ind,2}$ and $\mathbf{b}_{tor,2}$ do not have such a mutual interaction, but rather evolve passively. This essential difference determines the dominant magnetic helicity in the system.

The principle of HMFID in Figure 4(b) is similar to that of HKFD except for some slight but essential differences. In the system forced by the right-handed (positive) helical magnetic energy, $\mathbf{b}_{ind,3}(\hat{y})$ is generated from $\mathbf{u}_0 \times \mathbf{b}_{tor}$. But, $\mathbf{u}_0$ can induce two different current densities: $\mathbf{J}_{ind,3}(\hat{y})$ from $\mathbf{u}_0 \times \mathbf{b}_{pol}$ and $\mathbf{J}_{ind,4}(\hat{z})$ from $\mathbf{u}_0 \times \mathbf{b}_{pol}$. $\mathbf{b}_{ind,3}$ generates $\mathbf{b}_{tor,3}$ around $\mathbf{b}_{ind,3}$, leading to a right-handed magnetic helicity. But $\mathbf{J}_{ind,4}$ generates $\mathbf{b}_{tor,4}$ around $\mathbf{b}_{ind,4}$ to generate a left-handed magnetic helicity. However, since $\mathbf{b}_{tor,4}$ is antiparallel to $\mathbf{b}_{tor}$, the left-handed magnetic helicity cancels the injected right-handed one. This is why the magnetic helicity generated in HMFID has the same sign of a forcing magnetic helicity.

A field structure model can be applied to the dynamo process in the Sun. The solar magnetic fields are thought to be generated through a few dynamo processes. Differential rotation $d\Omega/dr$ stretches the solar poloidal magnetic field $\mathbf{B}_{pol}$ which initially connects the north (N) and south (S) poles in the Sun. The stretched field forms toroidal magnetic fields $\mathbf{B}_{tor}$ in

![Figure 2](image2.png)

Figure 2. (a) $\mathbf{b}_{ind}$ and $\mathbf{b}_{pol}$ form a left-handed (negative) magnetic helicity, and $\mathbf{u}_{ind}$ and $\mathbf{u}_{pol}$ form a right-handed (positive) kinetic helicity. (b) Magnetic energy at $b_1$ diffuses toward $b_3$ through $-\mathbf{u} \cdot \nabla b_1$. These field structures in (a) and (b) represent the distribution of temporally evolving fields $\mathbf{u}(t)$ and $\mathbf{b}(t)$ in a simulation system driven by a randomly chosen kinetic helicity $\mathbf{u} \cdot \omega$. However, their average is positive ($\langle \mathbf{u} \cdot \omega \rangle > 0$) in the small-scale regime.

![Figure 3](image3.png)

Figure 3. Evolving $E_M$ and $H_M$ at $k = 1, 5$ in a system forced with positive kinetic helicity. As $Pr_M$ increases, the forced eddy $k_2$ is more influenced by kinetic helicity than magnetic diffusion from the large-scale magnetic field.
the northern and southern hemisphere in opposite directions (antisymmetric about the equatorial plane). Since the repeating rotation has the effect of overlapping the toroidal magnetic fields, the strength of $B_{\text{tor}}$ becomes amplified. This explains how the strong magnetic field can arise in a system with such an extremely low $\text{Pr}_M$. However, with only differential rotation, the dynamo process cannot be sustained. These consecutive dynamo processes are described by the magnetic induction equation (Equation (3)) in an axisymmetric system as follows:

$$\frac{\partial A_{\text{pol}}}{\partial t} = -\frac{U_{\text{pol}}}{r} \cdot \nabla (r A_{\text{pol}}) + \eta (\nabla^2 - r^{-2}) A_{\text{pol}} + (\alpha B_{\text{tor}}), \quad (6)$$

$$\frac{\partial B_{\text{tor}}}{\partial t} = -r (U_{\text{pol}} \cdot \nabla) B_{\text{tor}} + r (B_{\text{pol}} \cdot \nabla) U_{\text{tor}} - r \eta (\nabla^2 - r^{-2}) B_{\text{tor}} + (\nabla \times \alpha B_{\text{pol}}). \quad (7)$$

The toroidal magnetic field can be amplified by $r (B_{\text{pol}} \cdot \nabla) U_{\text{tor}}/r$, but the poloidal component cannot be amplified (Cowling 1933). Although some spatially inhomogeneous fields ($\nabla \varphi = 0$) which can break axisymmetry may enable the dynamo process, another asymmetric effect is required for a sustainable dynamo process in an axisymmetric structure. For a self-consistent dynamo process, Parker (1955) suggested the $\alpha$ effect due to the Coriolis force and buoyancy. Physically, a degree of buoyancy elevates ($\beta$ magnetic fields) simultaneously, the Coriolis effect rotates the fluid convective motion. Simultaneously, the Coriolis effect rotates the flux (in opposite directions in the northern and southern hemisphere). These mechanical processes result in a helical magnetic field structure. The statistically merged fields can produce a large-scale poloidal magnetic field connecting the north and south poles. However, a more enhanced rotation effect (e.g., $\pi/2 \rightarrow 3\pi/2$) can generate an oppositely directed large-scale poloidal field. Above all, this model cannot explain the inversion of magnetic fields and stability of the solar dynamo.

Using the $\alpha$ dynamo process and $J_{\text{ind}}$ discussed in Figures 2 and 4, we can remove the ambiguity and explain the periodic solar magnetic activity. The dynamo process starts from the initial $B_{\text{pol}}$ field (N \rightarrow S) and its stretched fields $B_{\text{tor}}$ due to the rotation effect in both hemispheres. In the northern hemisphere, $B_{\text{tor},N}$ flows from west to east (\hat{\phi}), but $B_{\text{tor},S}$ in the southern hemisphere flows from east to west (\hat{\phi}). The buoyancy effect and Coriolis force generate the left-handed negative kinetic helicity (-$(u \cdot \omega)$) in the north and the right-handed positive one (+$(u \cdot \omega)$) in the south. Through the interaction between the helical motions and magnetic fields, the current density $J_{\text{ind},N,0}$ parallel to $B_{\text{tor},N}$ and $J_{\text{ind},S,0}$ anti-parallel to $B_{\text{tor},S}$ can be induced. Then, the current densities generate other circular magnetic fields $b_{\text{circ},N}$ and $b_{\text{circ},S}$ around $B_{\text{tor},N}$ and $B_{\text{tor},S}$. Above the solar surface, $b_{\text{circ},N}$ and $b_{\text{circ},S}$ are consistently heading from N to S to form a new $B_{\text{pol},\text{up}}$. Below the solar surface, these $b_{\text{circ},N,S}$ fields consistently flow from S to N, which also form a new $B_{\text{pol},\text{down}}$. $B_{\text{pol,up}}$ and $B_{\text{pol,down}}$ coexist, but only $B_{\text{pol,up}}$ are affected (stretched) by the differential rotation effect to be new $B_{\text{pol,up}}$ flowing from east to west (i.e., -\hat{\phi}) and $B_{\text{pol,down}}$ from west to east (\hat{\phi}). Then, these newly induced magnetic fields $B_{\text{pol,up}}$, opposite to $B_{\text{tor}}$, are amplified due to the continuous rotation. However, what can be easily observed is $B_{\text{pol,up}}$ (N \rightarrow S). Moreover, since all the induced currents in the northern and southern hemisphere are in the same direction, the attraction between the northern and southern toroidal fields can be expected. This will lead to the migration of magnetic fluxes and their annihilation, resulting in a new counter-solar magnetic field evolution.

This $\alpha^2 \Omega$ solar dynamo model based on the field structure and differential rotation effect is one of the possible scenarios in the solar dynamo model. However, the strong $B_{\text{tor}}$ (0.1–1 T), weak $B_{\text{pol}}$ (~10^{-3} T, Charbonneau 2014) due to the rather inefficient $\alpha^2$ dynamo process, and their periodic conversion in the rotating system with very low $\text{Pr}_M$ support its validity.

### 3.3. Derivation of the $\alpha$ Coefficients

Thus far, we have shown the physical mechanisms of LSD and SSD in the field structure. Being different from SSD, LSD requires an additional amplifying process of the $B$ field, the $\alpha$ effect. The model implies that the growth of the $B$ field ($b_i$) is related to the (helical) motion of $u_i$, $b_i$ at the small scale and in the $B$ field itself. Therefore, if the characteristic length $l$ and timescale $\tau$ of turbulent eddies are smaller than those of $B$, the EMF may be expanded as

$$\langle u \times b \rangle_i = \alpha_i \beta \frac{\partial B_i}{\partial x_m} + \gamma_i \frac{\partial^2 B_i}{\partial x_m \partial x_n} \ldots \quad (8)$$
Moreover, if the minimum value of $Re_M \equiv ul/\eta$ and $S \equiv ur/l$ is much smaller than 1 (min($Re_M$, $S$) $\ll 1$), triple correlation terms and $G \equiv \langle (u \times b) - (u \times b) \rangle$ in the magnetic induction equation in the small-scale regime can be neglected. Then, $\xi$ can be calculated from $u \times \int \partial b / \partial t \, dt$ (Moffatt 1978) or from $\int \partial b / \partial t \, dt \times b$ (Keinigs 1983). However, these anti-commutative FOSAs are not generally valid beside the considerable $G$ in space. In MTA, the third-order moment terms, neglected in FOSA, are replaced by $\xi/\tau$ without further calculation. It starts from the differentiation of a multi-variable function $\xi(u, b)$:

$$\frac{\partial}{\partial t} \nabla \times \langle u \times b \rangle = \nabla \times \left\{ \frac{\partial u}{\partial t} \times b \right\} + \nabla \times \left\{ u \times \frac{\partial b}{\partial t} \right\}.$$  

(9)

After some analytical calculations, we can derive the simple forms of $\alpha = 1/3 \int (j \cdot b) - \langle u \cdot \omega \rangle \, dt$ and $\beta = 1/3 \int (u^2) \, dt$. Mathematically, the quenching effect of $j \cdot b$ in the $\sigma$ coefficient comes from the definition of a vector product and differentiation of a multi-variable function. Physically, it is caused by the interaction between the current density and magnetic field at different scales.

Additional differentiation of Equation (9) produces the fourth-order moments which can be decomposed into a combination of second-order ones: $\langle X_i \rangle \langle X_j \big| X_m \rangle \sim \sum_{ilm} \langle X_i \rangle \langle X_j \big| X_m \rangle$ (QN; Kraichnan & Nagarajan 1967). With the assumption of isotropy without reflection symmetry, the second-order moment can be replaced by

$$\langle X_i(k)X_m(-k) \rangle = P_{im}(k)E(k) + \frac{i}{2} \frac{\kappa_n}{k^2} \epsilon_{imn}H(k),$$  

(10)

where $P_{im}(k) = \epsilon_{imn}k_{jm}/k^2$, $\langle X^2/2 \rangle = \int E(k) \, dk$, and $\langle X \cdot \nabla \times X \rangle = \int H(k) \, dk$. With some calculation, the $\alpha$, $\beta$ coefficients similar to those of MTA can be derived (Frisch et al. 1975).

All these methods are essentially to solve a closure issue in the MHD equations approximately, without the exact solution of anisotropy and energy cascade time $\tau$ affected by the magnetic field. Therefore, if there is a practical method to find the exact $\alpha$, $\beta$ coefficients from observation and simulation, it will be helpful to infer a better closing method, leading to a more exact helical dynamo theory.

3.4. Derivation of the $\alpha$ and $\beta$ Coefficients using the Semi-analytic Method

In principle, the $\alpha$ and $\beta$ coefficients can be found from $\partial B / \partial t \sim \nabla \times \alpha \, B + (\gamma + \beta) \nabla^2 B$. However, this vector equation is not so useful for the practical calculation. The scalar equations for $E_{dt}(\equiv\langle B^2/2 \rangle)$ and $H_{dt}(\equiv\langle A \cdot B \rangle)$ are more useful. From the coupled equations (Park 2017b)

$$\frac{d}{dt} \langle A \cdot B \rangle = 2\alpha \langle B \cdot B \rangle - 2(\beta + \eta) \langle B \cdot \nabla \times B \rangle,$$  

(11)

$$\frac{d}{dt} \langle B^2 \rangle = \alpha \langle B \cdot \nabla \times B \rangle - \beta \langle \nabla \times B \cdot \nabla \times B \rangle - \frac{e}{\sigma} \langle J \cdot \nabla \times B \rangle,$$  

(12)

$H_{dt}(t)$ and $E_{dt}(t)$ can be found as follows:

$$2H_{dt} = (H_{dt}(0) + 2E_{dt}(0))e^{2\int_0^t (\alpha - \beta - \eta) \, dt} + (H_{dt}(0) - 2E_{dt}(0))e^{-2\int_0^t (\alpha + \beta + \eta) \, dt},$$  

(13)

$$4E_{dt} = (H_{dt}(0) + 2E_{dt}(0))e^{2\int_0^t (\alpha - \beta - \eta) \, dt} - (H_{dt}(0) - 2E_{dt}(0))e^{-2\int_0^t (\alpha + \beta + \eta) \, dt}.$$  

(14)

Equations (13) and (14) are consistent with the field structure (Figures 4(a), (b)) and the simulation results (Figures 5(a), (b)). If $\alpha < 0$ (positive HKF or negative HMF), the second terms on the right-hand sides of Equations (13) and (14) become dominant with increasing time. Since $2E_{dt}(0) \geq H_{dt}(0)$, $H_{dt}(t)$ becomes negative. In contrast, if $\alpha > 0$ (negative HKF or positive HMF), $H_{dt}(t)$ becomes positive and converges to $2E_{dt}(t)$ eventually. But $E_{dt}(t)$ is positive in any case. When making these theoretical predictions, we refer to the simplest definitions of the $\alpha$ and $\beta$ coefficients. However, these coefficients can be derived from Equations (13) and (14) as follows:

$$\alpha(t) = -\frac{1}{4} \left( \frac{1}{C(t)} \right) \frac{\partial C(t)}{\partial t} - \frac{1}{4} \left( \frac{1}{D(t)} \right) \frac{\partial D(t)}{\partial t},$$  

(15)

$$\beta(t) = -\frac{1}{4} \left( \frac{1}{C(t)} \right) \frac{\partial C(t)}{\partial t} + \frac{1}{4} \left( \frac{1}{D(t)} \right) \frac{\partial D(t)}{\partial t} - \eta,$$  

(16)

where $C(t) \equiv 2E_{dt}(t) + H_{dt}(t)$ and $D(t) \equiv 2E_{dt}(t) - H_{dt}(t)$.

As the results show, if we know the temporal changes of large-scale magnetic helicity $H_{dt}(t)$ and large-scale magnetic energy $E_{dt}(t)$, the $\alpha(t)$ and $\beta(t)$ coefficients can be found. Considering that other theoretical and numerical methods to find these coefficients require considerable analytic calculations and elaborate simulations, these results provide a more direct method of finding them.

Additional differentiation over time leads to the integrand of each coefficient: $d\alpha(t)/dt = d\int t \langle \gamma \rangle \, dt / dt$, $d\beta(t)/dt = d\int t \langle \gamma \rangle \, dt / dt$. They provide an opportunity to test the classical analytic results of FOSA or MTA, $d\alpha(t)/dt \sim \langle j \cdot b \rangle - \langle u \cdot \omega \rangle$ or $d\beta(t)/dt \sim \langle u^2 \rangle$. For a decaying MHD system, these classical results from Equations (13) and (14) reproduce the simulation results quite well (Park 2017b), but a more numerical test is necessary for a forced system. There may be additional effects which are excluded in MTA and FOSA. Equations (15) and (16) are generally valid with the statistical assumption $\langle u \times b \rangle \sim \alpha B - \beta \nabla \times B$, which implies helical forcing ($\alpha \equiv 0$).

On the other hand, it should be also noted that if the large-scale magnetic field is fully helical ($2E_{dt}(t) \equiv \pm H_{dt}(t)$), Equations (11) and (12) are actually the same. Furthermore, careful examination of the results shows that $\beta = -1/4 \ln C(t) / D(t) = -\eta$ is likely to be negative. This means that the plasma kinetic energy at the small scale can boost the dynamo process. This is due to the fact that analytical derivation with the assumption of isotropy does not consider the effect of lateral velocity correlation $\langle u(r)u(r + l) \rangle \sim \langle d\beta / dt \rangle$, which can be
negative at some correlation distance. Figure 6 shows the role of $\beta$ clearly. The $\alpha$ effect is not as dominant as conventional helical dynamo theory predicts.

Using the numerical test field method, Warnecke et al. (2018) measured $\alpha_{ij}$, $\beta_{ij}$, and other nonlinear coefficients for a rotating MHD system with $Pr_M = 1$. They found that $\alpha_{\phi\phi}$ as well as $\Omega$ dominates in the formation of $B_{\text{pin.}}$. The diagonal components of $\beta_{ij}$ ($\beta_{xx}$, $\beta_{yy}$, $\beta_{\phi\phi}$) are overall positive, but the non-diagonal components of $\beta_{ij}$ are in the mixed states of positive and (mostly) negative values according to the direction and radius. In practice, the definite negative $\beta$ coefficient indicates its contribution to the dynamo process. Warnecke et al. pointed out that the ambiguity might be partially from the definition of the coefficients in Equation (8).

The semi-analytic approach to find $\alpha$ and $\beta$ is essentially based on Equation (8) assuming a homogenous and isotropic system without reflection symmetry. However, non-axisymmetric perturbation effects (e.g., nonlinear magnetic helicity, magnetic buoyancy) can easily break axisymmetry ($d/d\phi \neq 0$) in the system, which leads to the possible amplification of the poloidal field in Equation (6). Pipin & Kosovichev (2015, 2018) discussed non-axisymmetric perturbation effects on the $\alpha$ and $\beta$ coefficients. Indeed, nonlinear effects may be substantial factors modifying the $\alpha$ effect in the near-surface rotational shear layer. It needs more study to explain how non-axisymmetric effects modify the $\alpha$ and $\beta$ coefficients in general. We will continue measuring these nonlinear coefficients using the analytical and numerical mean field methods to understand the effect of plasma motion on the dynamo process.

4. Summary

Thus far, we have seen how the field structure model explains the amplification process of the magnetic field in a plasma. The magnetic field $b_{\text{al}}$ parallel to $u$ is transferred through $b \cdot \nabla u$, and the magnetic field $b_{\text{loc}}$ parallel to $b$ is transferred through $-u \cdot \nabla b$. The net magnetic field $b_{\text{net}}$ from $b_{\text{al}}$ and $b_{\text{loc}}$ is used as a seed magnetic field for the next dynamo step. As the field structure shows, growing $b_{\text{al}}$ parallel to $u$ suppresses the dynamo process, whereas growing $b_{\text{loc}}$ perpendicular to $u$ boosts the dynamo action. This result explains the dependence of the dynamo on the magnetic Prandtl number ($Pr_M \equiv \nu/\eta$). With less magnetic diffusion (decreasing $\eta$) and more kinetic dissipation (increasing $\nu$, or increasing $Pr_M$), the dynamo effect increases. In contrast, decreasing mechanical dissipation decreases the dynamo effect. These relations imply that the saturation of the magnetic field in an ideal system is related to the field structure between $u$ and $b$ (angle $\theta$) rather than dissipation. We also explained the mechanism of the helical dynamo ($\alpha$ effect) using vector field analysis. HKFD and HMFD can generate both positive and negative magnetic helicity in principle. However, as we discussed, only the opposite (same) sign of magnetic helicity is left in a forced HKFD (HMFD) system. Finally, we derived the $\alpha$, $\beta$ coefficients from large-scale magnetic energy and magnetic helicity. The exact coefficients are useful in understanding more accurately the internal dynamo processes in an MHD system, which leads to a more general dynamo theory. At present, a field structure model with various physical conditions such as rotation, shear, or $B_{\text{ex}}$ remains to be developed.
Before undertaking this, we will test the method using simulation results and observational data.

K.W.P. appreciates the support from ERC Advanced Grant STARLIGHT: Formation of the First Stars (339177) and the support from bwForCluster for numerical simulation. The author also gratefully acknowledges the data storage service SDS@hd supported by the Ministry of Science, Research and the Arts Baden-Württemberg (MWK) and the German Research Foundation (DFG) through grant INST 35/1314-1 FUGG.

**Appendix**

**Simulation Table**

The Appendix comprises Table 1.

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**Table 1**

Forcing Strength in HMFD: $1/k_f$ of HKFD

| Figures | Dynamo | $\nu$ | $\eta$ | $Pr_{\Omega}$ | $f_{\nu}$ | $|f_{h}|$ or $|f_{mag}|$ |
|---------|--------|------|-------|-----------|------|-----------------|
| 1(b) NHKFD (SSD) | $8 \times 10^{-4}$ | $6 \times 10^{-4}$ | 1.33 | 0 | 0.07 |
| 4(a) HKFD | $6 \times 10^{-3}$ | $9 \times 10^{-3}$ | 0.67 | 1 | 0.07 |
| 4(b), 6 HKFD | $6 \times 10^{-3}$ | $6 \times 10^{-3}$ | 1 | 1 | 0.07 |
| 4(c) HKFD | $6 \times 10^{-4}$ | $6 \times 10^{-3}$ | 10 | 1 | 0.07 |
| 5(a) HKFD | $6 \times 10^{-3}$ | $6 \times 10^{-3}$ | 1 | 1 | 0.07 |
| 5(b) HMFD | $6 \times 10^{-3}$ | $6 \times 10^{-3}$ | 1 | $-1$ | 0.014 |

Note. In a magnetically driven system, the vector potential induction equation is forced ($k_f = 5$) instead of the magnetic induction equation. $f_{\nu}$ (helicity ratio) is defined as $(u \cdot \omega)/(\Omega_k (u^2))$ (KFD) and $k_f (u \cdot b)/(b^2)$ (MFD).