Prediction of Future Failures for Heterogeneous Reliability Field Data

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Abstract

This article introduces methods for constructing prediction bounds or intervals to predict the number of future failures from heterogeneous reliability field data. We focus on within-sample prediction where early data from a failure-time process is used to predict future failures from the same process. Early data from high-reliability products, however, often suffers from limited information due to small sample sizes, censoring, and truncation. We use a Bayesian hierarchical model to jointly model multiple lifetime distributions arising from different sub-populations of similar products. By borrowing information across sub-populations, our method enables stable estimation and the computation of corresponding prediction intervals, even in cases where there are few observed failures. Two applications are provided to illustrate this methodology, and a simulation study is used to validate the coverage performance of the prediction intervals.

Keywords: Bayesian estimation, Censored data, Prior information, Reliability prediction, Stan, Truncated data
1 Introduction

1.1 Motivation and Prediction Problems

Consumers and producers often have to predict future values of random quantities given data from a failure-time process. These random quantities fall into one of two different classes of prediction problems, referred to as new-sample and within-sample prediction. New-sample includes making predictions on a future unit or collection of units from the same population. Within-sample problems predict future events based on early observed data from the failure-time process. Such applications include

- Managers of a fleet of systems want to predict the number of needed replacement components or subsystems for spare part provisioning.

- A company needs to predict the number of future failures for a manufactured product to determine if a recall is required.

- A reliability engineer conducting a life test needs to estimate the number of future failures from a group of products that will occur in a given interval in order to determine when the test will be terminated.

In within-sample prediction, computing prediction intervals or one-sided prediction bounds, however, can be challenging. High-reliability products often lead to heavy right censoring. With few observed failures, relying on large-sample theory for prediction intervals is problematic. Field data can also involve a population of products with heterogeneous sub-populations. These sub-populations may enter service over a period of time (known as staggered entry), exhibit different lifetime distributions, and have varying amounts of time in service. In this scenario it may be possible to make predictions only for sub-populations with many observed failures. Jointly modeling the sub-populations, however, provides prediction intervals for the individual sub-populations or the entire population, providing an appealing alternative.

This paper presents a general framework for making within-sample predictions for the number of future failures based on failure-time data from a potentially heterogeneous population of components or subsystems operating within a larger population or fleet of systems. We take a hierarchical modeling approach in order partially pool sub-populations and reduce the amount of data required to produce stable prediction intervals. Estimation is
performed using Bayesian methods, which allows for prior information about lifetime dis-
tribution parameters to be incorporated into the model. Moreover, the construction of
prediction intervals is relatively straightforward with the Bayesian framework, once the set
of draws from the joint posterior distribution are available.

1.2 Related Work

There are many books and papers on the subject of prediction intervals. Guttman (1970)
describes both Bayesian and non-Bayesian tolerance and prediction intervals. Geisser (1993)
describes Bayesian methods for prediction. Meeker and Escobar (1998, Chapter 12) pro-
vide a general overview of reliability-related applications of prediction intervals. Escobar
and Meeker (1999) developed a simulation-based method to provide prediction intervals
for new and within-sample predictions. Nelson (2000) proposed likelihood ratio based pre-
diction intervals for future within-sample failures using a Weibull distribution. Nordman
and Meeker (2002) extended the work of Nelson and studied the properties of likelihood
to-based prediction intervals.

Hamada et al. (2004) explain the differences between Bayesian tolerance and new-sample
prediction intervals and illustrate these with an example using a hierarchical linear model.
Hong et al. (2009) developed methods to predict the number of future failures from a fleet
of power transformers that exhibited both left truncation and right censoring. Other ap-
plications include Wang and Wang (2009), who predicted the future life of a series system
comprised of exponential lifetime distributions, and studied the frequentest coverage prob-
abilities of their Bayesian intervals. Hong and Meeker (2010) predicted warranty returns
for a product with multiple failure modes. Hong and Meeker (2013) extended this work
by incorporating dynamic covariate information to make field-failure predictions both for
individual units and for a population.

1.3 Overview

The remainder of this paper is organized as follows. Section 2 introduces two motivating
applications for our prediction methodology. Section 3 describes general hierarchical lifetime
models that can include varying types of censoring and truncation commonly encountered in
reliability field data. Section 4 presents methodology for constructing prediction intervals
for the number of future failures from sub-populations, as well as an entire population.
Section 5 applies our methods to the motivating applications. Section 6 describes a small simulation study to study the coverage probability properties of our Bayesian prediction intervals based on weakly informative prior distributions. Section 7 discusses the simulation results. Section 8 contains some concluding remarks and suggestions for future research.

2 Motivating Applications

2.1 Heat Exchanger Tube Data

Nuclear power plants contain steam generators which are large heat exchangers comprised of many stainless steel tubes. Over time, the tubes can crack due to a combination of cyclic stressing and corrosion. Periodic inspection is used to detect such cracks. Once a crack is identified it is plugged and the tube is thus removed from service. The heat exchanger can remain in service until 10 to 20 percent of the tubes have been plugged. The first prediction application we consider has heat exchanger tube data from three different plants with $N = 300$ tubes, of which 11 have failed (4 failures are left censored and 7 are interval censored) at $t_c = 3$ years of service for the oldest plant, as illustrated in Figure 1.
Figure 1: Event plot for heat exchanger tube inspection data in operating time. Solid arrows indicate right censored tubes. Dashed line segments with ? indicate left or interval censored tubes.

Given the observed failures, engineers would like to predict the number of additional tubes, $Y$, that will crack over the next $t_c + \Delta t$ years. Both [Nelson (2000)](#) and [Nordman and Meeker (2002)](#) analyzed similar data using maximum likelihood (ML) estimation, and predicted the number of additional tubes that would fail over the next 10 years. However, due to a limited number of observed failures, data from many plants were pooled together to improve estimation. In addition, the value of the Weibull shape parameter, $\beta$, was assumed to be known and the same in all of the plants, based on information from engineers familiar with heat exchanger tubes.

### 2.2 Backblaze Hard Drive Data

Backblaze is a company that provides cloud backup storage to protect against data loss. Since 2013, it has been collecting daily operational data on all of the hard drives operating...
at its facility. Every quarter the company reports detailed operational data and summary statistics on the different drive-models in operation though their website (https://www.backblaze.com/b2/hard-drive-test-data.html, accessed June 1, 2020). The purpose is to provide consumers and businesses with reliability information on different hard drive-models. The hard drives continuously spin in controlled storage pods. Drives are run until failure or until they are replaced with newer technology drives. When a hard drive fails, it is removed and replaced. In addition, the number of storage pods is increasing as Backblaze expands their business and adds drives to its storage capacity.

As of the first quarter of 2019, Backblaze was collecting and reporting data on 85 different drive-models. Some drive-models have been running since 2013 or before, while others were added at a later date. A subset of these data were analyzed by Mittman, Lewis-Beck, and Meeker (2019). However, their focus was on comparing the reliability of the different drive-model brands whereas our interest is in predicting the number of future failures over a fixed future period of time for a current population of drives.

We use data through the first quarter of 2016 to make predictions, and use data starting the second quarter of 2016 through the first quarter of 2019 for validation. Because Backblaze periodically retires old drive-models and replaces them with newer, larger capacity models, only 28 out of the 44 drive-models are in both the testing and prediction data sets. For model identification, a minimum of three failures was the criterion for inclusion of a drive-model into our testing period. Figure 1 shows a scatterplot of the number of active drive-models (i.e., drives in the risk set) at our data-freeze date, March 31, 2016.
Figure 2: Scatterplot of active drive-models versus total number of units at risk at the data-freeze date. Each number labels the number of failures that were observed before the data-freeze date. The vertical axis is a log scale.

3 Models for Heterogeneous Reliability Field Data

The hierarchical models presented in this paper could employ any of the commonly used lifetime distributions such as the Weibull or lognormal (see Chapter 4 of Meeker et al. 1998 for others). In this article, we use the Weibull distribution as well as a mixture of Weibull distributions since the Weibull distribution has been shown to fit the motivating data sets well (Nelson 2000, Mittman et al. 2019). Also, the Weibull distribution is more conservative in predicting future failures. However, using any other combination of distributions in the log-location-scale family of distributions is straightforward.
3.1 Weibull Distribution and Reparameterization

The Weibull cumulative distribution function (cdf) is

\[ Pr(T \leq t|\alpha, \beta) = F(t|\alpha, \beta) = 1 - \exp\left[-\left(\frac{t}{\alpha}\right)^\beta\right], \quad t > 0, \quad (1) \]

where \( \beta > 0 \) is the Weibull shape parameter and \( \alpha > 0 \) is a scale parameter. Because \( \log(T) \) has a smallest extreme value distribution (a member of the location-scale family of distributions), the Weibull cdf can also be written as

\[ Pr(T \leq t|\mu, \sigma) = F(t|\mu, \sigma) = \Phi_{\text{sev}}\left[\frac{\log(t) - \mu}{\sigma}\right], \quad t > 0, \]

where \( \Phi_{\text{sev}}(z) = 1 - \exp[-\exp(z)] \) is the standard smallest extreme value distribution cdf and \( \mu = \log(\alpha) \) and \( \sigma = 1/\beta \) are, respectively, location and scale parameters for the distribution of \( \log(T) \). Following the suggestions given in [Li and Meeker (2014)] and Section 15.2 of [Meeker et al. (2017)] we use an alternative parameterization where the usual scale parameter \( \alpha \) is replaced by the \( p \) quantile \( t_p = \alpha [-\log(1 - p)]^{1/\sigma} \) (which is also a scale parameter). This parameterization is important for estimation as well as eliciting prior information, which is often easier for a quantile as opposed to the usual scale parameter. For a more general discussion of reparameterization and Bayesian inference, see [Gelman (2004)].

3.2 Hierarchical Failure Population Model

In order to jointly model an entire product population consisting of different but similar sub-populations, and because of the limited amount of data from many of the sub-populations, we model sub-population-specific parameters hierarchically, borrowing strength across sub-populations. We extend the simple Weibull lifetime model as follows,

\[ T_{ig}^{\text{ind.}} \sim \text{Weibull}\left(t_{pg}, \sigma_g\right), \quad (2) \]

where \( g = 1, \ldots, G \) indexes the sub-populations. The likelihood for the Weibull model for all subpopulations is a function of the sets of parameters \( \theta_g = (t_{pg}, \sigma_g) \), one set for each sub-population, \( g \). Assuming the lifetimes of all units are independent within and across sub-populations, and conditional on fixed values of the parameters, the likelihood for the data, \( i = 1, \ldots, n_g \), is given by

\[
L(\Theta) = \prod_{g=1}^{G} \prod_{i=1}^{n_g} \left[f(t_{ig}; \theta_g)\right]^{I_{ig}} \left[1 - F(t_{ig}; \theta_g)\right]^{L_{ig}} \left[F(t_{1ig}; \theta_g)\right]^{C_{ig}} \left[F(t_{0ig}; \theta_g)\right]^{L_{ig}} \left[F(t_{1ig}; \theta_g) - F(t_{0ig}; \theta_g)\right]^{I_{ig}}
\]
where $t_{ig}$ is the observed failure or survival time of unit $i$ in sub-population $g$; $t_{0ig}$ and $t_{1ig}$ ($t_{0ig} < t_{1ig}$) are the lower and upper interval censored times; $C_{ig}$ is an indicator if unit $i$ in sub-population $g$ is right censored; $L_{ig}$ is an indicator if unit $i$ in sub-population $g$ is left censored; and $I_{ig}$ is an indicator if unit $i$ in sub-population $g$ is interval censored.

In a hierarchical model, the parameters are modeled as random variables, varying across different subpopulations. Thus, to complete the model we specify the following distributions,

$$
\sigma_g \sim \text{Lognormal} \left( \eta_{\sigma}, \tau_{\sigma}^2 \right),
$$

$$
t_{pg} = \exp \left( \mu_g + \sigma_g \Phi^{-1}(p) \right) \sim \text{Lognormal} \left( \eta_{tp}, \tau_{tp}^2 \right).
$$

### 3.3 Hierarchical Generalized Limited Failure Population Model

The Generalized Limited Failure Population model (GLFP) from Chan and Meeker (1999) is a generalization of the Weibull model to accommodate lifetime data with two failure modes. The GLFP accounts for early failures due to manufacturing defects (infant mortality) and, after this ‘burn-in’ period, failures due to prolonged use (wearout). The GLFP can be seen as a mixture of two Weibull distributions, $F_1, F_2$, with parameters $(t_{p1}, \sigma_1)$ and $(t_{p2}, \sigma_2)$, respectively. The GFLP model uses the parameter, $\pi$, the population proportion of defective units that are susceptible to both failure modes. When $\pi$ is zero, the GLFP model reduces to a single Weibull distribution. If $T \sim \text{GLFP}(\pi, t_{p1}, \sigma_1, t_{p2}, \sigma_2)$, then the cdf of $T$ is

$$
\Pr(T \leq t) = H(t; \pi, t_{p1}, \sigma_1, t_{p2}, \sigma_2) = 1 - (1 - \pi F_1(t)) (1 - F_2(t)), \quad t > 0, \quad 0 < \pi < 1.
$$

and taking the derivative of the cdf gives us the density for the GLFP, $\partial H(t; \theta) / \partial t = h(t; \pi, t_{p1}, \sigma_1, t_{p2}, \sigma_2)$. Here $F_1(t)$ is the cdf for the early failures and $F_2(t)$ is the cdf for the wearout failures.

We extend the GLFP as follows,

$$
T_{ig} \sim \text{GLFP} \left( \pi_g, t_{p1g}, \sigma_{1g}, t_{p2g}, \sigma_{2g} \right),
$$

where again $g = 1, \ldots, G$ indexes the sub-populations. We now define $\theta_g = (\pi_g, t_{p1g}, \sigma_{1g}, t_{p2g}, \sigma_{2g})$, one set for each sub-population, $g$. Under the same independence assumptions as for the single Weibull distribution, the likelihood for the data, $i = 1, \ldots, n_g$, is given by

$$
L(\Theta) = \prod_{g=1}^{G} \prod_{i=1}^{n_g} \left[ \frac{h(t_{ig}; \theta_g)}{1 - H(t_{ig}; \theta_g)} \right]^{1-C_{ig}} \left[ \frac{1 - H(t_{ig}; \theta_g)}{1 - H(t_{ig}; \theta_g)} \right]^{C_{ig}}.
$$
where $t_{ig}$ is the observed failure or survival time of unit $i$ in sub-population $g$; $t^L_{ig}$ is the left truncation time for unit $i$ in sub-population $g$; and $C_{ig}$ is an indicator if unit $i$ in sub-population $g$ is right censored. As in Section 3.2 including left and interval censored observations into the likelihood is straightforward.

Assuming a distribution, common across the $G$ groups, for early failures provides a meaningful interpretation and comparison of the proportion defective, $\pi_g$, and typically the lifetime distribution of the defective subpopulation is not of high interest in practical applications. Thus, we only specify hierarchical distributions for the proportion defective, as well as the Weibull parameters for the wearout failure mode,

$$\sigma_{2g} \overset{\text{ind.}}{\sim} \text{Lognormal}(\eta_{\sigma_2}, \tau^2_{\sigma_2}) \text{ Tr (0, 1)}$$

$$t_{p2g} = \exp \left( \mu_{2g} + \sigma_{2g} \Phi^{-1}(p_2) \right) \overset{\text{ind.}}{\sim} \text{Lognormal}(\eta_{t_{p2}}, \tau^2_{t_{p2}})$$

$$\pi_g \overset{\text{ind.}}{\sim} \text{Logit-normal}(\eta_\pi, \tau^2_\pi).$$

We truncate the distribution of $\sigma_{2g}$ above 1, which restricts the wearout failure mode to have an increasing hazard function.

### 4 Prediction

We focus on the within-sample prediction problem in two contexts: predicting future failures from a single group or sub-population, and an aggregation over all $G$ groups (i.e, the entire population). The aggregation prediction application is more complicated because the groups of units could differ due to a variety of covariates such as entry time, hazard rates, censoring or number of units at risk.

#### 4.1 Prediction of Future Failures From a Single Group

For single-group prediction, we assume $i = 1 \ldots n$ units are in service and observed until time $t_{ci} > 0$. We define $t_{ci}$ in terms of operating time (e.g., days in service) rather than calendar time. While we may observe units up until a data-freeze date, unless all units are placed into service at the same time, the amount of operating time will vary among surviving units. A parametric lifetime model, $F(t; \theta)$, describes the lifetime distribution and by time $t_{ci}$ we have observed $r > 0$ failures. Thus, the remaining $n - r$ units have not
failed (i.e., are right censored). Let $Y$ be the number of additional failures that will occur between $t_{ci}$ and some future time $t_{wi}$, where $t_{wi} = t_{ci} + \Delta t$ ($\Delta t > 0$). Then, conditional on the $r$ failure times, $Y$ has a Binomial($n - r, \rho$) distribution, where

$$\rho = \frac{F(t_{wi}; \theta) - F(t_{ci}; \theta)}{1 - F(t_{ci}; \theta)}.$$

If $\rho$ were known, the lower and upper prediction bounds for the number of future failures could be obtained from the $\alpha$ and $1 - \alpha$ quantiles of $Y$

$$Y_{\alpha} = \text{qbinom}(\alpha, n - r, \rho), \quad \bar{Y}_{1-\alpha} = \text{qbinom}(1 - \alpha, n - r, \rho).$$

Otherwise, there are several methods to obtain these bounds described in the references in Section 1.2. In this paper, we focus on Bayesian prediction methods, which are discussed in more detail in Section 4.3.

### 4.2 Prediction of Future Failures Across Multiple Groups

Predicting the number of future within-sample failures across multiple groups is more challenging than prediction for a single population. Using the same notation as above, we now add the subscript $g = 1 \ldots G$ to denote the heterogeneous groups or sub-populations. For a given group, $g$, with $n_g$ units, the probability of unit $i$ failing within the interval $(t_{c,ig}, t_{w,ig})$ where $t_{w,ig} = t_{c,ig} + \Delta t$ is,

$$\rho_{ig} = \frac{F(t_{w,ig}; \theta_g) - F(t_{c,ig}; \theta_g)}{1 - F(t_{c,ig}; \theta_g)} \quad i = 1 \ldots n_g.$$

The total number of failures in the population over $\Delta t$ is $Y = \sum_{g=1}^{G} \sum_{i=1}^{n_g} I(T_{ig} \in (t_{c,ig}, t_{w,ig}])$. Conditional on the observed data and censoring times, the distribution of $Y$ is Poisson-binomial. Similar to the single group case if the $\rho_g = (\rho_{1g}, \ldots, \rho_{ng})$ values are known, the prediction interval endpoints are obtained from the $\alpha$ and $1 - \alpha$ quantiles of $Y$ as follows,

$$Y_{\alpha} = \text{qpoisbinom}(\alpha, \rho_g), \quad \bar{Y}_{1-\alpha} = \text{qpoisbinom}(1 - \alpha, \rho_g).$$

Computing the cdf of the Poisson-binomial distribution is computationally challenging. However, [Hong (2013)] provides an algorithm to evaluate the Poisson-binomial distribution exactly or by an approximation. We use the Poisson approximation in this paper. The next section shows how to apply the single and multiple group prediction approaches within a Bayesian framework when the model parameters are not known.
4.3 Bayesian Prediction Methods

Bayesian prediction methods require draws from the joint posterior distribution of the model parameters. Once the draws are obtained, the predictive distribution for the random variable \( Y \) is of the form,

\[
J(y) = \int_{\theta} J(y; \theta) f(\theta | t) d\theta
\]  

(4)

where \( J(y; \theta) \) is the cdf of \( Y \) and \( f(\theta | t) \) is the joint posterior for the model parameters. Given a set of posterior draws, \( \theta^*_j, j = 1 \ldots B \), (4) can be approximated by

\[
J(y) \approx \frac{1}{B} \sum_{j=1}^{B} J(y; \theta^*_j).
\]  

(5)

A 100(1 – \( \alpha \))% Bayesian prediction interval is given by the \( \alpha/2 \) and \( 1 - \alpha/2 \) quantiles of \( J(y) \). A point prediction for the number of failures uses the median (or mean) of the predictive distribution to estimate \( E(Y) \).

In cases where it is difficult to evaluate \( J(y; \theta) \) directly, another option is to simulate new values \( Y^*_j \) from \( J(y; \theta^*_j) \) for \( j = 1 \ldots B \) so that \( J(y) \approx (1/B) \sum_{i=1}^{B} I(Y^*_j \leq y) \). One limitation of this method is it requires extra layer of simulation that can be computationally demanding because the number of draws required to control Monte Carlo error, and provide the same precision as computing \( J(y; \theta) \) directly, can be considerable larger. In our applications, however, because the predictand is discrete, we found this additional simulation step did not require extra draws to obtain the same level of precision. That is, when we compared the two methods with \( B = 1,500 \) in repeated trials the methods usually gave the same interval endpoints. In the exceptional cases, the small proportion of off-by-one deviations was similar in the two methods.

5 Application to Examples

5.1 Heat Exchanger Tubes

We extend the analysis of Nordman and Meeker (2002) by using a Bayesian hierarchical model and estimating the Weibull shape parameter (instead of assuming it is given). Then we predict the number of future failures using the methods outlined in Section 4. In the case of the heat exchanger data we have 3 groups (\( G = 3 \)) corresponding to the 3 nuclear power plants.
To assess whether the Weibull distribution is appropriate, we combined the data from the three plants and estimated a pooled Weibull model. Figure 3 plots the posterior median of the pooled Weibull model with axes on the Weibull probability scale. The plot also contains 90% pointwise credible bands and a nonparametric estimate. The Weibull distribution describes the data quite well, and the credible bands are consistent with the nonparametric point estimates.

Figure 3: The estimated pooled Weibull model (with weakly informative priors) plotted on Weibull probability scales. The solid curve corresponds to the median of the posterior draws as a function of plant age; pointwise 90% credible bounds are indicated by the dashed lines. The solid circles are nonparametric estimates with 90% pointwise standard errors (triangles).
For the hierarchical parameters, we specify the following proper prior distributions. Following the approach used in (Meeker et al., 2017, Section 15.2.2) we will use diagonal braces ([..]) to refer to 95% central probability intervals, rather than the standard model parameters when specifying prior distributions. We specify two different priors for $\eta_\sigma$, corresponding to weak ($\eta_\sigma,\text{weak}$) versus informative prior ($\eta_\sigma,\text{inform}$) information about the Weibull shape parameter. If the failure is due to a wearout mechanism, then it is known that $\sigma < 1$. Thus, for our informative prior, we put the majority of probability mass on values less than 1.

$$
\eta_\sigma,\text{weak} \sim \text{Lognormal}(0.08, 4.0), \quad \eta_\sigma,\text{inform} \sim \text{Lognormal}(0.37, 1.0), \quad \eta_p \sim \text{Lognormal}(0.63, 31.78)
$$

We reparameterize the Weibull distributions by replacing the usual scale parameter with the 0.05 quantile ($p = 0.05$) which can be viewed as an alternative scale parameter. We do this because typically we only have information about failures in the lower tail of the distribution and thus it would be easier (and more sensible) to set a prior distribution on such a parameter. For the standard deviation parameters ($\tau_\sigma, \tau_p$) we use half-$t$ distributions ($\nu = 4$) as suggested by Gelman (2006) for hierarchical standard deviation parameters when the number of groups is small.

Figure 4 provides 95% Bayesian prediction intervals using the Poisson approximation to the Poisson-binomial distribution. The widths of the intervals are roughly the same, but the more informative prior distribution on the Weibull shape parameter results in intervals that are more conservative, i.e., predict more failures. This difference becomes larger as we predict further into the future, which makes sense as the hazard rate increases over time. When a tube is found to be cracked it is filled and thus taken out of service. As mentioned earlier, heat exchangers typically have excess capacity to allow between 5% and 10% of their tubes to be removed from service before requiring the entire heat exchanger to be taken out of service (Nordman and Meeker, 2002). Therefore, we predict out to the first year when the lower bound of our prediction interval exceeds 30 tubes (10%), which occurs at year ten for the model with the informative prior distribution. The widths of the prediction intervals increase as we predict further into the future. In practice, however, such prediction intervals would be updated overtime as additional data is collected from maintenance checks (e.g., annually). This would result in narrower prediction intervals.
Figure 4: Yearly 95% prediction intervals for the cumulative number of cracked heat exchanger tubes for Plant 1 (left), and all 3 plants (right). Circles correspond to the model with an informative prior distribution on the shape parameter. Triangles corresponds to the model with a weakly informative prior distribution on the shape parameter.

5.2 Backblaze Disk Drive

We use the Backblaze data through the first quarter of 2016 to fit the GLFP model (subsequent data are used for validation). There are a total of 44 drive-models ($G = 44$). Some drive-models have only 3 observed failures; others have more, up to 1,707 failures. To complete the GLFP model, we specify the following proper prior distributions. We consider these prior distributions weakly informative because they put probability mass on a wide range of values for all model parameters—much larger than the ranges that would expected from typical applications where Weibull distributions are used to describe reliability. We used

$$\sigma_1 \sim \text{Lognormal}(0.14, 7.1)$$
\[ t_{p_1} = \exp(\mu_1 + \sigma_1 \Phi^{-1}(p_1)) \sim \text{Lognormal}(22, 5.5 \times 10^4) \] (6)
\[ \eta_\pi \sim \text{Normal}(0.007, 0.26) \]
\[ \eta_{\sigma_2} \sim \text{Normal}(0.02, 51.2) \]
\[ \eta_{t_{p_2}} \sim \text{Normal}(159, 4 \times 10^5). \]

We reparameterize the Weibull distributions replacing the usual scale parameter with the 0.50 quantile for the early failure mode \((p_1 = 0.50)\) and the 0.20 quantile for the wearout failure mode \((p_2 = 0.20)\). For the hierarchical standard deviation parameters, we use half-Cauchy prior distributions as we have a larger number of groups and observed failures compared to the heat exchanger example. For more details on the choice of these prior distributions see Mittman et al. (2019).

In Figure 5, we overlay the posterior median of the fitted GLFP model for Drive-Model 14 (the drive-model with the most observed failures) onto an adjusted Kaplan-Meier estimate with axes on the Weibull probability scale (Meeker and Escobar, 1998, Chapter 11). The plot also contains 90% pointwise credible bands associated with each estimate. The GLFP model fits quite well, as it is able to adequately describe the rapid increase in the empirical cdf between 8,000 and 20,000 hours.
Figure 5: The estimated GLFP model for Drive-model 14 plotted on Weibull probability scales. The dashed line corresponds to the median of the posterior draws; pointwise 90% credible bounds are indicated by the smooth region. The solid line is an adjusted Kaplan-Meier estimate with 90% pointwise standard errors.

5.2.1 Dynamic Risk Set

When making within-sample predictions for a large population or fleet of systems there is often a need to adjust to the risk set over time. In addition to units failing, units could be removed temporarily from the risk set for inspection or maintenance. Due to advancements in technology, sub-groups of products are also frequently retired as they are replaced by newer technology. Also, in some applications units that will be added to the population in the future may need to be included at the appropriate time. Accurately adjusting the risk-set to reflect the changing sample size in the prediction population is important to avoid biased predictions for the number of future failures.
5.2.2 Disk Drive Failure Predictions

As mentioned in Section 2.2, Backblaze periodically retires certain sub-populations of drive-models from the population before they fail, and replaces them with newer more efficient disk drives that have larger storage capacity. Backblaze does not disclose their decision-making policy for retiring old technology, but says on their website they regularly migrate older drive-models out for newer more efficient drives. Their data reflect such actions. The bottom row in Figure 6 plots the changing risk set size for Drive-Models 6 and 23 starting on the data-freeze date (Q2 2016). For Drive-Model 6 (bottom left), at week 1 there are over 4,000 active drives in the risk set, but by week 26 there are fewer than 500 drives at risk of failing. In contrast, Drive-Model 23 (bottom right) has drives leaving the within-sample prediction population at a roughly linear rate. To avoid making predictions for drives that are no longer in the sample (either due to failure or removal), we predict using one week intervals (i.e, Δ\(t\) = 1 week) for 26 weeks. Each week we update the risk set by removing drives from the prediction sample that are no longer in service (either because they failed or were taken out of service). We make weekly predictions for the next 26 weeks after the data-freeze date. At the end of 26 weeks, we stop predicting because the majority of the drive-models have few units remaining in the risk population.
Figure 6: Top: weekly 95% prediction intervals and predicted average number of failures (dots) for Drive-Models 6 and 23. Squares are observed number of failures. Bottom: weekly risk set (diamonds) for Drive-Models 6 and 23.

The top two plots in Figure 6 shows the 95% weekly prediction intervals for the number of future failures for Drive-models 6 and 23. The dots are the median of the predictive distribution (a point prediction) and the squares are the observed number of failures. The prediction intervals for both drive-models cover the observed number of failures (see the supplementary material for the prediction intervals for all drive-models). The gradual decrease in the number of predicted failures for Drive-Model 6 (top left) is because Backblaze
started heavily retiring units from that model between weeks 15 and 20 of the prediction period. In contrast, fewer than 100 units were retired from Drive-Model 23, which explains the monotonic increase in the predicted number of failures from week 1 to 26.

6 The Simulation Experiment

6.1 Goals of the Simulation

This section provides an overview of the design of our simulation study to examine the properties of Bayesian prediction intervals when applied to within-sample prediction for heterogeneous lifetime data with weakly-informative prior distributions. For each simulation, we generate data, \( t_{ig} \), from the following Weibull hierarchical model,

\[
\begin{align*}
\sigma_g & \sim \text{Lognormal}(\eta_{\sigma}, \tau_{\sigma}^2) \\
t_{pg} & = \exp\left(\mu_g + \sigma_g \Phi^{-1}(p)\right) \sim \text{Lognormal}(\eta_{tp}, \tau_{tp}^2) \\
T_{tg} & \sim \text{Weibull}(t_{pg}, \sigma_g).
\end{align*}
\]

6.2 Factors

Our simulation has all units entering the population at the same time, and are are observed until a fixed censoring time, \( t_c \) (i.e., type I censoring). The goal is to use the information from the observation period to predict the number of failures within each sub-population \( (r_g) \), as well as the whole population, between \( (t_c, t_w] \). This setup mimics certain real-world applications using product reliability data, where products enter service in a single cohort. The particular factors in the simulation experiment used were

- \( G \): the number of sub-populations
- \( t_{pg}, \sigma_g \): the sub-population specific random parameters
- \( E(r) = E\left(\sum_{g=1}^{G} r_g\right) \): the expected number of failures before \( t_c \)
- \( E(p_f) \): the expected proportion of failures before time \( t_c \)
- \( E(p_{\delta}) \): the expected proportion of failures between times \( t_c \) and \( t_w \)
6.3 Factor Levels

In order to cover a range of situations, we use the following levels of the factors.

- \( G = 5, 10 \)
- \( E(r) = 25, 50, 75, 100, 125 \) for \( G = 5 \)
- \( E(r) = 50, 100, 150, 200, 250 \) for \( G = 10 \)
- \( E(p_f) = 0.10, 0.20 \)
- \( E(p_δ) = 0.10, 0.20 \)
- \( t_{pg} \sim \text{Lognormal}(4, 8), t_{pg} \sim \text{Lognormal}(10, 14) \)
- \( σ_g \sim \text{Lognormal}(0.15, 0.40), σ_g \sim \text{Lognormal}(0.50, 0.75) \).

We used different levels of \( E(r) \) so the expected number of failures per group is the same for \( G = 5 \) and \( G = 10 \). For each factor-level combination we simulated 300 data sets. To compute the censoring times for each factor-level combination we simulated 50,000 sets of Weibull parameters from the hierarchical distributions. We then averaged the Weibull cdfs across all the simulated parameters and numerically calculated \( t_c \) and \( t_w \) to ensure the correct \( E(p_f) \) and \( E(p_δ) \) were observed in each simulated data set.

6.4 Estimation

We again use Bayesian estimation for our simulation study. Performance of our prediction intervals is evaluated in terms of coverage probability. Here coverage probability is interpreted as the proportion of times that prediction intervals (for single and multiple groups) cover the true number of future within-sample failures. We do not want our prior distributions to bias the results: diffuse prior distributions (e.g., a flat prior) put probability mass on extreme values not consistent with the range practical situations we are trying to mimic; priors that are too informative can also affect the posterior by overly constraining the likelihood. Therefore, we specify what Gelman et al. (2017) refer to as ‘weakly informative’ prior distributions. Weakly informative priors are constructed in relation to the likelihood and are designed to put probability mass on reasonable values of the parameters while down weighting nonsensical values that would not be consistent with what would be
expected in the data. In hierarchical models with many parameters and small amounts of data, weakly informative priors are especially useful as they can produce more stable estimates than maximum likelihood estimation or a Bayesian model with vague priors.

For all factor level combinations we specify the following weakly informative prior distributions,

\[ \eta_\sigma \sim \text{Lognormal}(0.08, 4.0), \quad \eta_\tau \sim \text{Lognormal}(2.75, 19.70). \]

As in Section 5.1, the standard deviation parameters (\( \tau_\sigma, \tau_\tau \)) receive half-t distributions (\( \nu = 4 \)).

The models were fit using the \texttt{rstan} \citep{rstan} package in \texttt{R} \citep{r}, which implements a variant of Hamiltonian Monte Carlo (HMC) \citep{Betancourt2015}. Four chains were run, each with 2,500 iterations after 2,500 warmup iterations. The Gelman-Rubin potential scale reduction factor was used to provide a check for adequate mixing of the four chains \citep{Gelman92}.

7 Simulation Experiment Results

7.1 Summarizing the Results

We use the methods outlined in Section 4 to predict the number of future within-sample failures for single and multiple groups. To evaluate these methods we calculated each procedure’s coverage probability separately for lower and upper prediction bounds. For the single group prediction, we calculated the coverage probability for each group and then averaged the probabilities over all groups, \( G \). For the multiple groups case we calculated the unconditional coverage probability. A 100(1 - \( \alpha \))% two-sided prediction interval can be obtained by combining 100(1 - \( \alpha_L \))% lower and 100(1 - \( \alpha_U \))% upper prediction bounds where \( \alpha_L + \alpha_U = \alpha \). Generally, it is desirable to have \( \alpha_L \approx \alpha_U \) so that the endpoints of the interval can be interpreted as one-sided bounds. This is because in most practical prediction applications, the loss from predicting too high is different than the loss from predicting too low. To summarize our simulation results, we compare our one-sided coverage probabilities with nominal values (90% and 95%) to assess the adequacy of the prediction procedure.
7.2 Interpreting the Results

Figures 7 and 8 present a few of the most interesting and informative graphical displays of our simulation results. Both figures show lower ($L$) and upper ($U$) coverage probabilities at the one-sided 90% and 95% level. The dashed line corresponds to the nominal coverage level. Both single group (circle) and multiple group (triangle) predictions are plotted. Plots for the other factor-level combinations are given in the supplementary material. Some observations from the simulation results are:

- The coverage probability for the upper one-sided prediction intervals are closer to the nominal coverage level when compared with that for the lower one-sided prediction bounds. This is because the probability of a unit failing is small (e.g., 0.10 or 0.20), which means the lower coverage interval has a higher chance of missing an observed failure as opposed to an upper coverage bound.

- The multiple-group predictions have over-coverage as opposed to under-coverage, especially for the lower one-side prediction bounds. For single group predictions, especially when there is a small amount of information (see top left of Figures 7 and 8), predictions can perform poorly for a given sub-population as opposed to a prediction for the entire population where there is more information. However, once the number of expected failures increases (around 10 failures per group) performance improves.

- The factor levels for the Weibull parameters do not have a large effect on the coverage probabilities. Increasing $E(p_0)$ from 0.10 to 0.20 does, however, move coverage closer to the nominal level. This is due to predictions being made over a wider time window.

- Due to the hierarchical model, we are able to make predictions for all groups, even the ones that have no failures, by borrowing information from the groups that do have failures.
Figure 7: Coverage probabilities for $G = 5$, $t_{pg} \sim \text{Lognormal}(4, 8)$, and $\sigma_g \sim \text{Lognormal}(0.50, 0.75)$. Top: $E(p_\delta) = 0.10$. Bottom: $E(p_\delta) = 0.20$. The dashed lines correspond to the nominal coverage probability.
8 Conclusions and Areas for Further Research

This article introduces a new approach for making within-sample predictions for a population of products or systems with heterogeneous sub-populations. The methodology is applicable to single log-location-scale families of distributions, as well as mixtures of lifetime distributions, such as the GLFP model. Our hierarchical modeling approach borrows strength across sub-populations with many observed failures to improve predictions for sub-populations with few failures. Estimation was performed using Bayesian methods, which allows analysts to incorporate prior information into the model. As exemplified in the heat exchanger example, informative priors can improve prediction intervals and supplement limited data.

We empirically verified our prediction methods using test data from Backblaze, where hold-out data were available. We also conducted a small simulation study to examine the coverage probabilities of single and multiple-group prediction intervals. The results
show that, even with a small number of observed failures, Bayesian prediction intervals can achieve frequentist nominal coverage levels with weakly informative priors. The intervals perform the best (i.e., closer to nominal level) when constructing upper prediction bounds for multiple groups, and gain precision as the number of observed failures increases. Because lifetime data for high reliability products is often limited by sample size, censoring, and truncation, the performance of Bayesian prediction methods in small-amount-of-information settings is useful for applied researchers. Some possible extensions of our work include:

- We assumed exchangeability of units within a sub-population. This assumption is due in part to ignorance about potentially important covariates. However, there may be batch effects or varying conditions in the Backblaze facility that impact observed failure rates. For example, the predictions for Drive-Model 21 (see supplementary material) significantly underestimate the number of future failures between weeks 1 and 10. Then, following week 10, Backblaze removed the majority of the remaining drives from service. Incorporating drive status information into the model (e.g., Self-Monitoring, Analysis and Reporting Technology or S.M.A.R.T. data that are available from individual drives) or other time-varying covariates could improve prediction intervals \(^{[Allen\ 2004]}\).

- Our paper focused on the prediction of future events. However, if an analyst wanted to calculate the amount of money required for spare parts, it would be important to predict future costs as well. Because the Bayesian approach provides full posterior distributions once a model is fit, estimating a functional, such as a depreciation factor that varies across sub-populations, would be straightforward.

- As seen in Figure 6, Backblaze retires batches of hard drives before they fail. To avoid prediction bias, we updated the risk set weekly when predicting the number of future failures. However, an alternative approach, needed when retirements are not directly observed, is to model product retirement. \(^{[Xu\ et\ al.\ 2015]}\) used two Weibull distributions to jointly model a failure-time and retirement-time distribution. Incorporating a retirement distribution into our methodology would allow us to predict the number of future within-sample failures over longer time intervals and to adapt our methodology to applications that have unobserved retirements.
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