Uncertainty principle of discrete Fourier transform and quantum incompatibility

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Discrete Fourier transform (DFT) has many applications both in classical and quantum physics. In this work, we prove an uncertainty principle for DFT which generalizes the well-known uncertainty principle proved by Terence Tao in 2005. To do this, we relate DFT to the quantum incompatibility of two Von Neumann measurements, and introduce the notions of incompatibility order and the index of rank deficiency of transition matrix. The quantification of incompatibility is also discussed.

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I. INTRODUCTION

In quantum physics, Heisenberg’s uncertainty principle says
\[ \Delta x \Delta p \geq \frac{\hbar}{2}, \tag{1} \]
where \( x \) is the position of a particle, \( p \) is its momentum, \( \hbar \) is the reduced Planck constant. This uncertainty principle states that we cannot know both the position and speed of a particle with perfect accuracy; the more we nail down the position, the less we know about its speed, and vice versa. After Heisenberg’s uncertainty principle, many uncertainty principles and a plethora of applications in diverse scenarios have been found. Some recent progress in this area see for examples [1–11]. An uncertainty principle, in quantum or classical physics, is an inequality asserting a fundamental limit to the accuracy with which the values for certain pairs of quantities can be predicted from initial conditions.

Discrete Fourier transform (DFT) is one of the most useful tools in both classical and quantum information processing. A DFT on a \( d \)-dimensional complex Hilbert space \( H \) is expressed by a DFT matrix \( F \) with elements
\[ F_{jk} = \frac{1}{\sqrt{d}} e^{\frac{2\pi i}{d} jk}, \]
for the input nonzero function \( f \) defined on the index set \( \{j\}_{j=0}^{d-1} \), define \( \hat{f}(k) = \sum_{j=0}^{d-1} F_{jk} f(j) \) as its DFT. In 1989, Donoho and Stark [12] established an uncertainty principle for DFT as
\[ |\text{supp} f| |\text{supp} \hat{f}| \geq d, \tag{2} \]
where the support of \( f \), is defined as \( \text{supp} f := \{j | j \in [0, d - 1], f(j) \neq 0\} \), \([0, d - 1]\) denotes the set \( \{j\}_{j=0}^{d-1} \) with the elements of consecutive integers. This bound is sharp, for example the equality holds when \( f \) is a one-point function.

In 2005, Tao [13] proved a stronger uncertainty principle of DFT for \( d = p \) with \( p \) a prime, as
\[ |\text{supp} f| + |\text{supp} \hat{f}| \geq p + 1. \tag{3} \]

It is a long-standing problem that what is the sharp lower bound of \( |\text{supp} f| + |\text{supp} \hat{f}| \) for DFT with general \( d \). In this work, we will show that for DFT with general \( d \), the sharp lower bound of \( |\text{supp} f| + |\text{supp} \hat{f}| \) has the following uncertainty principle
\[ |\text{supp} f| + |\text{supp} \hat{f}| \geq d' + \frac{d}{d'}, \tag{4} \]
\[ d' := \max\{d_1 | 1 \leq d_1 \leq \sqrt{d}, d_1 | d\}, \tag{5} \]
where \( d_1 | d \) means \( d_1 \) divides \( d \).

To prove this result, we regard the DFT matrix \( F \) as a special transition matrix of two orthonormal bases \( A = \{|a_j\rangle\}_{j=1}^d \) and \( B = \{|b_k\rangle\}_{k=1}^d \) of Hilbert space \( H \). The transition matrix \( U^{AB} \) of \( A \) and \( B \) characterizes the incompatibility of two rank-1 projective measurements (Von Neumann measurements) \( \mathcal{A} = \{|a_j\rangle\langle a_j|\}_{j=1}^d \) (corresponding to \( A \)) and \( \mathcal{B} = \{|b_k\rangle\langle b_k|\}_{k=1}^d \) (corresponding to \( B \)). Inspired by the notion of “complete incompatibility” [14], we propose the notion of incompatibility order, denoted by \( \chi^{AB} \), and the index of rank deficiency of the transition matrix \( U^{AB} \), denoted by \( \tau^{AB} \). Having this definitions, we establish a relation \( \chi^{AB} + \tau^{AB} = d \) which reveals some properties of incompatibility of two rank-1 projective measurements. When focusing to the DFT matrix, with the help of this relation we will prove the uncertainty principle in Eq. (4). In addition, we will see that \( \chi^{AB} \) (or \( \tau^{AB} \)) implies only a classification of incompatibility, not a quantification. So we also discuss the topic of quantification of incompatibility.

The concept “incompatible” in the literature usually refers to the meaning that two positive operator-valued measures (POVMs) are not jointly measurable, such as in Refs. [15–21]. A POVM \( D \) can be expressed by a set of positive semidefinite operators \( D = \{D_j\}_{j=1}^m \) which summing to unity. Two POVMs \( D = \{D_j\}_{j=1}^m \) and \( E = \{E_k\}_{k=1}^n \) are called compatible iff (if and only if) there exists a POVM \( G = \{G_{jk}\}_{j=1}^m \) such that
\[ \sum_{j=1}^m G_{jk} = E_k \]
for any \( k \), and \( \sum_{k=1}^n G_{jk} = D_j \) for any \( j \). As a special case, when two measurements are two rank-1 projective measurements \( (\mathcal{A}, \mathcal{B}) \) above, we can check that \( (\mathcal{A}, \mathcal{B}) \) are jointly measurable iff \( \mathcal{A} = \mathcal{B} \). In this work, we only consider the incompatibility of two rank-1 projective measurements \( (\mathcal{A}, \mathcal{B}) \).

We provide a setting of consecutive measurements to
understand this notion of incompatibility. Two persons, Alice and Bob, Alice has the orthonormal basis $A = \{a_j\}_{j=1}^d$ (or say, the measurement $A = \{a_j\}_{j=1}^d$) and Bob has the orthonormal basis $B = \{b_k\}_{k=1}^d$. These mean that, for example, Alice has an apparatus which can measure the energy of the quantum system and $A = \{a_j\}_{j=1}^d$ is the eigenvectors of energy. Bob has an apparatus which can measure the angular momentum of the quantum system and $B = \{b_k\}_{k=1}^d$ is the eigenvectors of angular momentum. We assume that the eigenvalues of energy and angular momentum are all nondegenerate. Now Alice uses $A$ to measure the system which is in a pure state $|\psi\rangle$, suppose the resulting state is $|\psi\rangle = \sum_j c_j |a_j\rangle$ with the corresponding probability $|c_j|^2$. Next Bob uses $B$ to measure the state $|\psi\rangle$ and yields the state $|\psi\rangle$ with probability $|c_j|^2$. We see that, if $A = B$, Bob can infer with certainty that Alice’s previous state is $|a_j\rangle = |b_k\rangle$. However, if $A \neq B$, when Bob finds his state is $|b_k\rangle$, he can not surely infer what the Alice’s previous state is. This is an interpretation of why we can regard $A = B$ as “compatible”. Evidently, $A = B$ is equivalent to that two operators of energy and angular momentum commute.

In Ref. [14], De Bièvre introduced the notion of complete incompatibility. Two orthonormal bases $A = \{a_j\}_{j=1}^d$, $B = \{b_k\}_{k=1}^d$ are completely incompatibel, if for any nonempty subsets $\emptyset \not\subseteq S_A \subseteq A$, $\emptyset \not\subseteq S_B \subseteq B$, $|S_A| + |S_B| \leq d$, it holds that $\text{span}(S_A) \cap \text{span}(S_B) = \{0\}$. Here $|S_A|$ stands for the number of elements in $S_A$, $\text{span}(S_A)$ is the subspace spanned by $S_A$ over the complex field $C$. Although the definition of complete incompatibility is purely algebraic, it possesses the physical interpretation in terms of selective projective measurements [22–24]. It is shown that complete incompatibility closely links with the minimal support uncertainty [14], and also, it is useful to characterize the Kirkwood-Dirac nonclassicality [14].

**II. $s$-ORDER INCOMPATIBILITY AND MINIMAL SUPPORT UNCERTAINTY**

We give the definition of $s$-order incompatibility, and establish a relation between it and the minimal support uncertainty.

**Definition 1.** $s$-order incompatibility. Suppose the integer $s$ satisfies $s \in [2, d+1]$, $A = \{a_j\}_{j=1}^d$ and $B = \{b_k\}_{k=1}^d$ are two orthonormal bases of $d$-dimensional complex Hilbert space $H$. We say $A$ and $B$ are $s$-order incompatible if the following (1.1) and (1.2) hold.

1. For any $\emptyset \not\subseteq S_A \subseteq A$ and $\emptyset \not\subseteq S_B \subseteq B$, if $|S_A| + |S_B| < s$, then $\text{span}(S_A) \cap \text{span}(S_B) = \{0\}$.
2. There exist $\emptyset \not\subseteq S_A \subseteq A$ and $\emptyset \not\subseteq S_B \subseteq B$, such that $|S_A| + |S_B| = s$ and $\text{span}(S_A) \cap \text{span}(S_B) \neq \{0\}$.

We use $\chi_{AB}$ to denote the incompatibility order of $A$ and $B$. When $\chi_{AB} = d + 1$, the $(d+1)$-order incompatibility just coincides with the complete incompatibility introduced in Ref. [14].

We establish a link between $s$-order incompatibility and the minimal support uncertainty. For a pure state $|\psi\rangle$, we express it in the orthonormal bases $A = \{a_j\}_{j=1}^d$ and $B = \{b_k\}_{k=1}^d$ as $|\psi\rangle = \sum_j c_j |a_j\rangle$ and $|\psi\rangle = \sum_k d_k |b_k\rangle$. We use $n_{AB}(\psi)$ to denote the number of nonzero elements in $\{a_j\}_{j=1}^d$, use $n_B(\psi)$ to denote the number of nonzero elements in $\{b_k\}_{k=1}^d$, and let

$$
n_{AB}(\psi) = n_A(\psi) + n_B(\psi),
$$

$$
n_{AB}^\text{min} = \min_{|\psi\rangle \neq 0} n_{AB}(\psi).
$$

$n_{AB}(\psi)$ is called the support uncertainty of $|\psi\rangle$ with respect to $A$ and $B$, and $n_{AB}^\text{min}$ is called the minimal support uncertainty of $|\psi\rangle$ with respect to $A$ and $B$. The support uncertainty $n_{AB}(\psi)$ has many applications in different situations [12, 13, 25–27]. Obviously, $n_{AB}^\text{min} \in [2, d+1]$. It is shown that $\chi_{AB} = d+1$ iff $n_{AB}^\text{min} = d+1$ [14]. We now prove a more general result in Theorem 2.

**Theorem 2.** Suppose $A = \{a_j\}_{j=1}^d$ and $B = \{b_k\}_{k=1}^d$ are two orthonormal bases of $d$-dimensional complex Hilbert space $H$. The incompatibility order $\chi_{AB}$ and minimal support uncertainty $n_{AB}^\text{min}$ are defined in Definition 1 and Eq. (7), then it holds that

$$\chi_{AB} = n_{AB}^\text{min}.\tag{8}$$

**Proof.** By the definition of $n_{AB}^\text{min}$, if $n_{AB}^\text{min} = s$, then there exists pure state $|\phi\rangle$ such that $n_{AB}(\phi) = n_A(\phi) + n_B(\phi) = s$ and there does not exist pure state $|\phi'\rangle$ such that $n_{AB}(\phi') = n_A(\phi') + n_B(\phi') < s$. For such $|\phi\rangle$, there exist $\emptyset \not\subseteq S_A \subseteq A$ and $\emptyset \not\subseteq S_B \subseteq B$, such that $|S_A| = n_A(\phi)$, $|S_B| = n_B(\phi)$, and $|\phi\rangle \in \text{span}(S_A) \cap \text{span}(S_B)$. The nonexistence of such $|\phi'\rangle$ implies that there does not exist $\emptyset \not\subseteq S_A \subseteq A$ and $\emptyset \not\subseteq S_B \subseteq B$, such that $|S_A| = n_A(\phi)$, $|S_B| = n_B(\phi)$, and $|\phi\rangle \in \text{span}(S_A) \cap \text{span}(S_B)$. These two conditions just coincide with (1.1) and (1.2) in Definition 1. Then the claim follows.

Again, when $s = d + 1$. Theorem 1 returns to the corresponding result in Ref. [14].

**III. $s$-ORDER INCOMPATIBILITY AND THE TRANSITION MATRIX**

In this section, we introduce a quantity $\tau_{AB}$. We also establish a link between $\chi_{AB}$ ($n_{AB}^\text{min}$) and $\tau_{AB}$, then $\chi_{AB}$ can be determined via $\tau_{AB}$.

For two orthonormal bases $A = \{a_j\}_{j=1}^d$ and $B = \{b_k\}_{k=1}^d$, the transition matrix $U_{AB} = (U_{jk})_{j,k=1}^d$ is defined as $U_{jk} = \langle a_j | b_k \rangle$. We want to characterize $s$-order incompatibility via the transition matrix $U_{AB}$. To do this, we introduce the definition of $t$-order rank deficiency of $U_{AB}$.

**Definition 3.** $t$-order rank deficiency of $U_{AB}$. For the transition matrix $U_{AB}$ and the integer $t \in [0, d-1]$, we
define the \( t \)-order rank deficiency of \( U^{AB} \), denoted by 
\[ R_t(U^{AB}) \]

\[
R_{t,r}(U^{AB}) = \max_{1 \leq m \leq d-t} \{ m - \text{rank} \left( j_1, j_2, \ldots, j_m; k_1, k_2, \ldots, k_{m+t} \right) \},
\]

\[
R_{t,c}(U^{AB}) = \max_{1 \leq m \leq d-t} \{ m - \text{rank} \left( k_1, k_2, \ldots, k_{m+t}; j_1, j_2, \ldots, j_m \right) \},
\]

\[
R_t(U^{AB}) = \max[R_{t,r}(U^{AB}), R_{t,c}(U^{AB})].
\]

Where \( \left( j_1, j_2, \ldots, j_m \right) \) denotes the submatrix obtained by the \( j_1, j_2, \ldots, j_m \) rows and \( k_1, k_2, \ldots, k_{m+t} \) columns of \( U^{AB} \), for example \( J_{1,3,4} = \left( a_{12} a_{13} a_{14} a_{23} a_{24} a_{34} \right) \).

Clearly, the definitions of \( R_{t,r}(U^{AB}), R_{t,c}(U^{AB}) \), and \( R_t(U^{AB}) \) above can be similarly defined for general matrices, not only the unitary matrices. Note that a similar definition of rank-deficient submatrices was proposed in Ref. [28].

**Proposition 4.** Suppose integers \( \{t, t_1, t_2\} \subseteq [0, d-1] \), then the following (4.1)-(4.4) hold.

(4.1) \( R_t(U^{AB}) \geq 0 \).

(4.2) If \( t_1 < t_2 \) then \( R_{t_1}(U^{AB}) \geq R_{t_2}(U^{AB}) \).

(4.3) \( R_d(U^{AB}) = 0 \).

(4.4) If \( R_0(U^{AB}) = 0 \) then \( R_t(U^{AB}) = 0 \) for any \( t \in [0, d-1] \).

**Proof.** Recall that the matrix rank is defined as the rank of row vectors and which also equals the rank of column vectors, then \( R_t(U^{AB}) \geq 0 \) evidently holds since \( m \geq \text{rank}(j_1, j_2, \ldots, j_m) \).

For \( t_2 \), according to Definition 3, there exist \( 1 \leq m \leq d-t_2 \) and \( j_1, j_2, \ldots, j_m \) such that \( R_{t_2}(U^{AB}) = m - \text{rank}(j_1, j_2, \ldots, j_m) \), or there exist \( 1 \leq n \leq d-t_2 \) and \( k_1, k_2, \ldots, k_{m+t_2} \) such that \( R_{t_2}(U^{AB}) = n - \text{rank}(k_1, k_2, \ldots, k_{m+t_2}) \). We consider the former case, the latter can be discussed similarly. For the former case, we see that

\[
R_{t_2}(U^{AB}) = m - \text{rank}(j_1, j_2, \ldots, j_m; k_1, k_2, \ldots, k_{m+t_2}) \leq (m + t_2 - t_1) - \text{rank}(l_1, l_2, \ldots, l_m; k_1, k_2, \ldots, k_{m+t_2}; l_{m+1}, \ldots, l_{m+t_2-t_1}) \leq R_{t_1}(U^{AB}),
\]

where \( 0 < l_1 < l_2 < \ldots < l_m < l_{m+1} < \ldots < l_{m+t_2-t_1} \leq d \) and \( \{j_1, j_2, \ldots, j_m\} \subseteq \{l_1, l_2, \ldots, l_m, l_{m+1}, \ldots, l_{m+t_2-t_1}\} \).

The first inequality says the fact that adding \( (t_2 - t_1) \) rows can at most increase \( (t_2 - t_1) \) for the rank. The second inequality is from the definition of \( R_{t_1}(U^{AB}) \). This proves (4.2).

When \( t = d-1 \), from Definition 3, \( m \) can only take \( m = 1 \). Since \( U^{AB} \) is unitary, then every row vector and every column vector of \( U^{AB} \) are all nonzero. Hence, \( R_{d-1}(U^{AB}) = 0 \). This proves (4.3).

(4.4) is a direct result of (4.2) and (4.3). We then completed this proof.

With Proposition 4, we propose the definition of the index of rank deficiency of the transition matrix \( U^{AB} \).

**Definition 5.** We define the index of rank deficiency of the transition matrix \( U^{AB} \) as

\[
\tau_{AB} := \min_{t \in [0, d-1]} \{ t \mid R_t(U^{AB}) = 0 \} - 1.
\]

Clearly, \( \tau_{AB} \in [-1, d-2] \). When \( \tau_{AB} \in [0, d-2] \), \( \tau_{AB} \) is the maximal \( t \) for which \( R_t(U^{AB}) > 0 \). When \( R_0(U^{AB}) = 0 \), we have that \( \tau_{AB} = -1 \). For \( \tau_{AB} = -1 \), every \( m \times m \) submatrix \( \left( j_1, j_2, \ldots, j_m \right) \) of rank \( m \), particularly, every element \( U^{AB} = a_{jk} \neq 0 \). In Ref. [14], it is shown that when \( A \) and \( B \) are completely incompatible, i.e., \( \chi_{AB} = d+1 \), then it holds that \( \tau_{AB} = -1 \). Theorem 6 below shows a more general result.

**Theorem 6.** Suppose \( A = \{a_{ij}\}_{j=1}^d \) and \( B = \{b_{j}\}_{j=1}^d \) are two orthonormal bases of \( d \)-dimensional complex Hilbert space \( H \). Then the incompatibility order \( \chi_{AB} \) and the index of rank deficiency \( \tau_{AB} \) have the relation

\[
\chi_{AB} + \tau_{AB} = d.
\]

**Proof.** The case of \( \chi_{AB} = d+1 \) (or equivalently \( \tau_{AB} = -1 \)) has been proved in Ref. [14], then we only consider the case of \( 2 \leq \chi_{AB} \leq d \) (or equivalently \( 0 \leq \tau_{AB} \leq d-2 \)). Suppose the incompatibility order is \( \chi_{AB} \), then (1.2) in Definition 1 holds, that is, there exist \( \emptyset \neq S_A \subseteq A \) and \( \emptyset \neq S_B \subseteq B \) such that \( |S_A| + |S_B| = \chi_{AB} \) and \( \text{span}(S_A) \cap \text{span}(S_B) \neq \{0\} \). Then there exists a pure state \( \psi \in \text{span}(S_A) \cap \text{span}(S_B) \). Without loss of generality, we assume \( S_A = \{a_{ij}\}_{j=1}^d, S_B = \{b_{j}\}_{k=1}^d \). We explicitly write \( U^{AB} \) as
Expanding $|\psi\rangle$ in $S_A$ and $S_B$ we get that

$$|\psi\rangle = \sum_{j=1}^{[S_A]} x_j |a_j\rangle = \sum_{k=1}^{[S_B]} y_k |b_k\rangle,$$

where \{x_j\}_{j=1}^{[S_A]} are complex numbers not all vanishing, \{y_k\}_{k=1}^{[S_B]} are complex numbers not all vanishing. Consequently,

$$\langle \psi | b_k \rangle = 0 \text{ for all } |S_B| + 1 \leq k \leq d,$$

$$\langle a_j | \psi \rangle = 0 \text{ for all } |S_A| + 1 \leq j \leq d.$$

Further

$$\sum_{j=1}^{[S_A]} x_j^* \langle a_j | b_k \rangle = 0 \text{ for all } |S_A| + 1 \leq k \leq d,$$

$$\sum_{k=1}^{[S_B]} y_k \langle a_j | b_k \rangle = 0 \text{ for all } |S_A| + 1 \leq j \leq d,$$

where $x_j^*$ is the complex conjugate of $x_j$. These say that the $|S_A| \times (d - |S_B|)$ submatrix \((1,2,\ldots,[S_A])\) has linearly dependent row vectors, and the $(d - |S_A|) \times |S_B|$ submatrix \((|S_B| + 1,|S_B| + 2,\ldots,d)\) has linearly dependent column vectors. Since $2 \leq \tau_{AB} \leq d$, then $|S_A| + |S_B| \leq d$, $|S_A| \leq d - |S_B|$, and $|S_B| \leq d - |S_A|$. These further imply that $R_{d - |S_A| - |S_B|}(U^{AB}) > 0$ and $\tau_{AB} \geq d - \chi_{AB}$.

Conversely, suppose the index of rank deficiency is $\tau_{AB}$, then there exist $1 \leq m \leq d - \tau_{AB}$ and $(k_1,k_2,\ldots,k_m)$ such that $m > \text{rank}(1,2,\ldots,[S_A])$. Then there exist $1 \leq n \leq d - \tau_{AB}$ and $(j_1,j_2,\ldots,j_n)$ such that $n > \text{rank}((k_1,k_2,\ldots,k_{m + \tau_{AB}}))$. We consider the former case, the latter can be discussed similarly. For the former, case, without loss of generality, we assume $(1,2,\ldots,m) = (j_1,j_2,\ldots,j_n)$. Since $m > \text{rank}(1,2,\ldots,m + \tau_{AB})$, then there exist $(z_j)_{j=1}^m$ being complex numbers and not all vanishing such that

$$\sum_{j=1}^m z_j \langle a_j | b_k \rangle = 0 \text{ for all } 1 \leq k \leq m + \tau_{AB}.$$

Let $|\varphi\rangle = \sum_{j=1}^m z_j^* |a_j\rangle$, then $|\varphi\rangle \neq 0$ and

$$\langle \varphi | b_k \rangle = 0 \text{ for all } 1 \leq k \leq m + \tau_{AB}.$$}

It follows that $n_A(\varphi) \leq m$, $n_B(\varphi) \leq d - m - \tau_{AB}$, and

$$\chi_{AB} = n^{\text{min}}_{AB} \leq n_A(\varphi) + n_B(\varphi) \leq d - \tau_{AB}.$$}

We then finished this proof. □

Theorem 6 and Theorem 2 provide a way to determine $\chi_{AB}$ and $n^{\text{min}}_{AB}$ via $\tau_{AB}$. We will use these relations to prove the uncertainty principle in Eq. (4).

IV. QUANTIFICATION OF INCOMPATIBILITY

The incompatibility order $\chi_{AB}$, or the minimal support uncertainty $n^{\text{min}}_{AB}$, provides only a classification for incompatibility of two orthonormal bases $A$ and $B$, not a quantification. For example, when $\chi_{AB} = n^{\text{min}}_{AB} = 2$, it must hold that there is at least one element in the intersection of $A = \{|a_j\rangle\}_{j=1}^d$ and $B = \{|b_k\rangle\}_{k=1}^d$. We assume that $|a_1\rangle \langle a_1 | = |b_1\rangle\langle b_1 | \in \overline{A} \cap \overline{B}$. We write $A = \{|a_j\rangle\}_{j=1}^d \cup A'$ with $A' = \{|a_j\rangle\}_{j=d+1}^d$, $B = \{|b_k\rangle\}_{k=1}^d \cup B'$ with $B' = \{|b_k\rangle\}_{k=d+1}^d$. We see that $\chi_{AB} = 2$ only captures the fact $|a_1\rangle \langle a_1 | = |b_1\rangle\langle b_1 | \in \overline{A} \cap \overline{B}$, without any further information about $A'$ or $B'$, respectively. To make this more precise parameterize incompatibility, we ask the question that how to quantify incompatibility, that is, how to find a real-valued functional which can serve as an incompatibility measure. Two orthonormal bases $A$ and $B$ are compatible iff $\overline{A} = \overline{B}$, then the incompatibility measure should describe how far the distance between $\overline{A}$ and $\overline{B}$.

We propose three necessary conditions as follows which any incompatibility measure $C(A,B)$ should satisfy.

(i). $C(A,B) \geq 0$ and $C(A,B) = 0$ iff $\overline{A} = \overline{B}$.

(ii). $C(A,B) = C(A_1,B_1) + C(A_2,B_2)$ if $A = A_1 \cup A_2$, $B = B_1 \cup B_2$, $|A_1| = |B_1|$ and $U^{AB} = U^{A_1B_1} \oplus U^{A_2B_2}$.

(iii). For given dimension $d$, $C(A,B)$ reaches the maximum if $\{|a_j\rangle\} = 1/\sqrt{d}$.

Condition (i) is a natural requirement for $C(A,B)$ being a faithful incompatibility measure. Condition (ii) is an additivity for direct sum of

$$U^{AB} = U^{A_1B_1} \oplus U^{A_2B_2}, \quad (11)$$

where $U^{AB} = U^{A_1B_1} \oplus U^{A_2B_2}$ means that $\text{span}(A_1) \subset \text{span}(B_2)$, $\text{span}(A_2) \subset \text{span}(B_1)$, or $\text{span}(A_1) = \text{span}(B_1)$ and $\text{span}(A_2) = \text{span}(B_2)$. For this case, it is reasonable to think that incompatibility exists only between $A_1$ and $B_1$, between $A_2$ and $B_2$, but does not exist between $A_1$ and $B_2$, between $A_2$ and $B_1$. In condition (iii), two orthonormal bases $A$ and $B$ satisfying $\{|a_j\rangle\} = 1/\sqrt{d}$ are called mutually unbiased bases (MUBs). See reviews on MUBs in Refs. [29, 30] and some applications of MUBs recently reported such as in Refs. [31–33]. We view two MUBs as maximally incompatible bases.

Below we propose some incompatibility measures satisfying (i)-(iii).

**Definition 7.** For $0 < \alpha \neq 2$, we define the incompatibility measure based on $l_\alpha$ norm as

$$C_{\alpha}(A,B) = \frac{1}{2 - \alpha} \sum_{k=1}^{d} \left( \sum_{j=1}^{d} |\langle a_j | b_k \rangle|^\alpha \right)^{1/\alpha} - d. \quad (12)$$

In particular, when $\alpha = 1$, we have the incompatibility
measure based on $l_1$ norm as

$$C_1(A, B) = \sum_{j,k=1}^{d} |\langle a_j | b_k \rangle| - d. \quad (13)$$

We show that $C_\alpha(A, B)$ satisfies (i)-(iii). For complex vector $\mathbf{x} = (x_1, x_2, \ldots, x_d)$, the $l_1$ norm of $\mathbf{x}$ is $||\mathbf{x}||_1 = (\sum_{j=1}^{d} |x_j|)^{1/\alpha}$. We know that when $0 < \alpha < \beta$, it holds that $||\mathbf{x}||_1 \leq ||\mathbf{x}||_\alpha$ and $||\mathbf{x}||_\beta$ iff there exists at most one nonzero element in $\{x_j\}_{j=1}^{d}$. Since $U^{AB}$ is unitary, then $\sum_{j=1}^{d} |\langle a_j | b_k \rangle|^2 = 1$, $\sum_{j=1}^{d} |\langle a_j | b_k \rangle|^{\alpha} \geq 1$ for $0 < \alpha < 2$ and $\sum_{j=1}^{d} |\langle a_j | b_k \rangle|^{\alpha} \leq 1$ for $\alpha > 2$, and also $\sum_{j=1}^{d} |\langle a_j | b_k \rangle|^{\alpha} = 1$ iff there is only one nonzero element in $\{\langle a_j | b_k \rangle\}_{j=1}^{d}$, this nonzero element must have modulus 1. Consequently, $C_\alpha(A, B) = 0$ implies that every column of $U^{AB}$ has just one nonzero element. $U^{AB}$ thus has totally $d$ nonzero elements, and every row also must have just one nonzero element since $U^{AB}$ is of full rank. As a result, $\mathbf{A} = \mathbf{B}$. We then showed $C_\alpha(A, B)$ satisfies (i). $C_\alpha(A, B)$ satisfying (ii) is obvious by the definition of $C_\alpha(A, B)$, $C_\alpha(A, B)$ satisfying (iii) can be proved by the method of Lagrange multipliers with the constraints $\{\sum_{j=1}^{d} |\langle a_j | b_k \rangle|^2 = 1\}_{k=1}^{d}$.

Definition 8. We define the incompatibility measure based on noncommutability as

$$C_{NC}(A, B) = \sum_{j,k=1}^{d} |\langle a_j | b_k \rangle| \sqrt{1 - |\langle a_j | b_k \rangle|^2}. \quad (14)$$

In Ref. [34], the authors defined a quantity $\Upsilon_{p}(E, F)$ based on the noncommutability of two measurements $(E, F)$. In the case of two rank-1 projective measurements $(\mathbf{A}, \mathbf{B})$, this quantity $\Upsilon_{p}(E, F)$ yields $C_{NC}(A, B)$ in Eq. (14). We see that $C_{NC}(A, B)$ evidently satisfies (i) and (ii), $C_{NC}(A, B)$ satisfying (iii) is a result of $\Upsilon_{p}(E, F)$ being maximized by MUBs.

We give two simple examples to illustrate the calculation of incompatibility order and incompatibility measure.

Example 1. For qubit system, $d = 2$,

$$U^{AB} = \begin{pmatrix} \langle a_1 | b_1 \rangle & \langle a_1 | b_2 \rangle \\ \langle a_2 | b_1 \rangle & \langle a_2 | b_2 \rangle \end{pmatrix}. \quad (15)$$

The incompatibility measure based on $l_1$ norm is

$$C_1(A, B) = 2b\cos \theta.$$

The incompatibility measure based on noncommutability is

$$C_{NC}(A, B) = 2\sin 2\theta,$$

with $\sin \theta = |\langle a_1 | b_1 \rangle|$, $\cos \theta = |\langle a_1 | b_2 \rangle| = |\langle a_2 | b_1 \rangle|$, $\theta \in [0, \frac{\pi}{2}]$. When $\theta = 0$ or $\frac{\pi}{2}$, $C_1(A, B) = 0$, $\mathbf{A} = \mathbf{B}$. When $\theta = \frac{\pi}{4}$, $C_1(A, B)$ reaches the maximum.

$U^{AB}$ is unitary, then is of full rank. When $\theta = 0$ or $\frac{\pi}{2}$, $\sin \theta = 0$ or $\cos \theta = 0$, we have $\tau_{AB} = 0$, $\chi_{AB} = 2$, $\mathbf{A} = \mathbf{B}$. When $0 \neq \theta \neq \frac{\pi}{2}$, we have $\tau_{AB} = -1$, $\chi_{AB} = 3 = d + 1$, $A$ and $B$ are completely incompatible.

Example 2. We turn to the DFT matrix $F$. $F = U^{AB}$ is defined as $F_{jk} = \langle a_j | b_k \rangle = \frac{1}{\sqrt{d}} e^{i\frac{2\pi}{d} jk}$, with $i = \sqrt{-1}$, $j \in [0, d - 1]$, $k \in [0, d - 1]$. With direct algebra, we get that $C_1(A, B) = d(\sqrt{d} - 1)$ and $C_{NC}(A, B) = d\sqrt{d} - 1$. Since $|\langle a_j | b_k \rangle| = \frac{1}{\sqrt{d}}$ for any $j, k$, then $A$ and $B$ are maximally incompatible.

It is shown that $A, B$ are completely incompatible $(\chi_{AB} = d + 1)$ iff $d$ is a prime [13, 14]. We now consider the general case that $d$ is not necessarily a prime. We have Theorem 9 below.

Theorem 9. For $d$-dimensional DFT, it holds that

$$\chi_{AB} = d' + d/d'. \quad (15)$$

with $d'$ defined in Eq. (5).

We provide a proof for Theorem 9 in Appendix. Where we will see that Theorem 9 immediately implies the uncertainty principle in Eq. (4).

V. SUMMARY AND OUTLOOK

For two orthonormal bases $A, B$ of a quantum system, we proposed the notion of incompatibility order $\chi_{AB}$, which resulted in a classification for incompatibility. We proposed the notion of the index of rank deficiency of the transition matrix $U^{AB}$, denoted by $\tau_{AB}$. We established a link between $\chi_{AB}$ and minimal support uncertainty $n_{AB}^{\text{min}}$, and established a link between $\chi_{AB}$ and $\tau_{AB}$. With these relations, we derived an uncertainty principle of DFT which generalized the well-known uncertainty principle of Tao. We also proposed a framework to quantify the amount of incompatibility.

Donoho-Stark uncertainty principle has been generalized to finite abelian groups [35] and compact groups [36]. Based on the results of the uncertainty principle in Eq. (4), future potential research may include its applications in different fields, and finding corresponding results on more general sets (abelian group, non-abelian group, compact group, etc.) instead of the set $\{j\}_{j=1}^{d}$.

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Appendix: Proof of Theorem 9

Let the pure state $|\psi\rangle = \sum_{j=0}^{d-1} f(j)|a_j\rangle$, then we have that $\text{supp} f = n_A(|\psi\rangle)$. 

Rewrite
\[
|\psi\rangle = \sum_{j=0}^{d-1} f(j)|a_j\rangle
= \sum_{j,k=0}^{d-1} f(j)|b_k\rangle|a_j\rangle
= \sum_{j,k=0}^{d-1} F_{jk} f(j)|b_k\rangle
= \sum_{k=0}^{d-1} \tilde{f}(k)|b_k\rangle,
\]
(A1)
then \(|\text{supp} \tilde{f}| = n_B|\langle \psi |\rangle|.

According to Theorem 6, to prove Theorem 9, it suffices to prove
\[
\tau_{AB} = d - (d' + d/d').
\]
(A2)
When \(d\) is a prime, Theorem 9 returns to Eq. (3) in main text, which has been proved in [13]. Then we only consider the case that \(d\) is not prime. Note that \(F = F^1\), thus \(R_{t,r}(F) = R_{t,c}(F)\).

Suppose \(d = d_1d_2\),

(A3)
with \(d_1|d, d_2|d\) and \(1 < d_1 \leq d_2 < d\). We rewrite the index sets \(\{j\}_{j=0}^{d-1}\) and \(\{k\}_{k=0}^{d-1}\) as
\[
j = j_0 + j'd_2, j_0 \in [0,d_2-1], j' \in [0,d_1-1],\]
\[
k = k_0 + k'd_1, k_0 \in [0,d_1-1], k' \in [0,d_2-1],\]
then
\[
F_{jk} = \frac{1}{\sqrt{d}} e^{2\pi i jk_0d_2^{d_1}} e^{2\pi i jk'd_1d_1} e^{2\pi i j_0k_0d_2},
\]
(A6)
where we have used the fact \(e^{2\pi i jk'd_1d_2} = 1\). As pointed out in Ref. [28], Eq. (A6) implies that
\[
\text{rank} \left( \begin{array}{c}
\{j_0 + j'd_2\}_{j' \in [0,d_1-1]}; \\
\{k_0 + k'd_1\}_{k' \in [0,d_2-1]}
\end{array} \right) = 1
\]
(A7)
since
\[
\left( \begin{array}{c}
\{j_0 + j'd_2\}_{j' \in [0,d_1-1]}; \\
\{k_0 + k'd_1\}_{k' \in [0,d_2-1]}
\end{array} \right) = \frac{1}{\sqrt{d}} e^{2\pi i j_0k_0d_2} F_{j_0}^t F_{k_0},
\]
(A8)
where we have denoted the row vector
\[
F_{k_0} = (1, e^{2\pi i k_0d_2}, e^{2\pi i k_0d_2}, \ldots, e^{2\pi i (d_1-1)d_2}),
\]
and denoted the transpose of \(F_{k_0}\) by \(F_{k_0}^t\).

The submatrix
\[
\left( \begin{array}{c}
\{j_0 + j'd_2\}_{j' \in [0,d_1-1]}; \\
\{k_0 + k'd_1\}_{k_0 \in [1,d_1-1], k' \in [0,d_2-1]}
\end{array} \right)
\]
can be viewed as the column union of the submatrices
\[
\left\{ \begin{array}{c}
\{j_0 + j'd_2\}_{j' \in [0,d_1-1]}; \\
\{k_0 + k'd_1\}_{k_0 \in [1,d_1-1], k' \in [0,d_2-1]}
\end{array} \right\}_{k_0 \in [1,d_1-1]},
\]
thus the column rank (and then the rank)
\[
\text{rank} \left( \begin{array}{c}
\{j_0 + j'd_2\}_{j' \in [0,d_1-1]}; \\
\{k_0 + k'd_1\}_{k_0 \in [1,d_1-1], k' \in [0,d_2-1]}
\end{array} \right) \leq d_1 - 1.
\]
(A10)
Since
\[
\left( \begin{array}{c}
\{j_0 + j'd_2\}_{j' \in [0,d_1-1]}; \\
\{k_0 + k'd_1\}_{k_0 \in [1,d_1-1], k' \in [0,d_2-1]}
\end{array} \right)
\]
has \(d_1\) rows, thus
\[
\left( \begin{array}{c}
\{j_0 + j'd_2\}_{j' \in [0,d_1-1]}; \\
\{k_0 + k'd_1\}_{k_0 \in [1,d_1-1], k' \in [0,d_2-1]}
\end{array} \right)
\]
is rank deficient for rows. By the definition of \(\tau_{AB}\), it follows that \(\tau_{AB} \geq (d_1 - 1)d_2 - d_1\), that is
\[
\tau_{AB} \geq d - (d_1 + d_2).
\]
(A11)
Minimizing \(d_1 + d_2\) over all \(d_1\) under Eq. (A3) will yield
\[
\tau_{AB} \geq d - (d' + d/d'),
\]
(A12)
\[
d' : = \max\{d_1|1 < d_1 \leq \sqrt{d}, d_1|d\}.
\]
(A13)
Next, we prove that the equality in Eq. (A12) holds, then Theorem 9 is true. For simplicity of notation, we let \(d'/d'' = d''\). By Definition 5, we need to show that
\[
R_{t}(F) = 0 \text{ when } t = (d' + d'' + 1).
\]
(A14)
That is, for any submatrix \((j_1,j_2,...,j_m)\) of \(F, 0 \leq j_1 < j_2 < ... < j_m \leq d, 0 \leq k_1 < k_2 < ... < k_{m+1} \leq d, t = d - (d' + d'') + 1, \) we need to prove that \(\text{rank} (k_1,k_2,...,k_{m+1}) = m\). To prove \(\text{rank} (k_1,k_2,...,k_{m+1}) = m\), it suffices to find a nonsingular \(m \times m\) submatrix of \((k_1,k_2,...,k_{m+1})\).

Since \(t = d - (d' + d'') + 1, \) thus \(m + t = d - (d' + d'') + 1 + m\). When \(m = 1\), \((k_1,k_2,...,k_{m+1})\) has only one row, then \(\text{rank} (k_1,k_2,...,k_{m+1}) = 1\) evidently holds. When \(m + t = d, \) i.e., \(m = d' + d'' - 1, \) since \(F\) is unitary, then \(\text{rank} (k_1,k_2,...,k_{m+1}) = 1\) evidently holds. Therefore, below we only need to consider \(1 < m < d' + d'' - 1\).

We set two situations for \(m\) to complete this proof.
\((9.1), \) \(m \in [1,d'] \cup [d''', d'' + d'' - 1].\)

Lemma 1 (Proposition 3.6 in [37].) For the \(m \times m\) submatrix \((j_1,j_2,...,j_m)\) of \((k_1,k_2,...,k_{m+1})\) of \(d\)-dimensional DFT matrix \(F,\)
\(0 \leq j_1 < j_2 < ... < j_m \leq d, 0 \leq k_1 < k_2 < ... < k_m \leq d,\)
if \((j_1,j_2,...,j_m)\) or \((k_1,k_2,...,k_m)\) are consecutive (or say, have common difference 1) then \((k_1,k_2,...,k_{m+1})\) is nonsingular.

For the submatrix \((j_1,j_2,...,j_m)\), there are \(-m + t \right) = (d' + d'' - m - 1) columns of \(F\) which do not appear in \((k_1,k_2,...,k_{m+1})\). We partition the columns \((0,1,...,d-1)\) of \(F\) into \([d/m]\) parts as \((0,1,...,m-1), (m+1,...,2m-1), ..., (d-1)\) with the first \([d/m]\) parts (or all parts if \([d/m] = d/m\) each having \(m\) consecutive columns. Here for real number \(x, \lfloor x \rfloor\) denotes the floor function of \(x\) which is the greatest integer less than or equal to \(x\), and \([x]\) denotes the ceiling function of \(x\) which
is the least integer greater than or equal to \( x \). We assert that there must exist one part in the first \( \lfloor d/m \rfloor \) parts (or in all parts if \( \lfloor d/m \rfloor = d/m \)) its columns all appear in \((k_1, k_2, \ldots, k_{m+t})\). Otherwise, each part in the first \( \lfloor d/m \rfloor \) parts (or in all parts if \( \lfloor d/m \rfloor = d/m \)) has at least one column which does not appear in \((k_1, k_2, \ldots, k_{m+t})\), thus there are totally at least \( \lfloor d/m \rfloor \) columns in \((0, 1, \ldots, d - 1)\) which do not appear in \((k_1, k_2, \ldots, k_{m+t})\). Consequently \( \lfloor d/m \rfloor \leq d' + d'' - m - 1 \). However, below we will show that

$$\lfloor d/m \rfloor > d' + d'' - m - 1$$  \hspace{1cm} (A15)

where \( m \in [1, d'] \cup [d'', d' + d'' - 1] \).

For the function \( f(x) = \frac{x}{d} + x, x > 0 \), we know that \( f(x) \) strictly decreases when \( x \) increases in \([1, \sqrt{d}]\) and \( f(x) \) strictly increases when \( x \) increases in \([\sqrt{d}, d]\). Using this fact, we have that \( d/m + m \geq d' + d'' \) whenever \( m \in [1, d'] \cup [d'', d' + d'' - 1] \). Eq. (A15) then follows.

Since there must exist one part in the first \( \lfloor d/m \rfloor \) parts (or in all parts if \( \lfloor d/m \rfloor = d/m \)) its columns all appear in \((k_1, k_2, \ldots, k_{m+t})\), employing Lemma 1, we get that \( \text{rank}(j_1, j_2, \ldots, j_m; k_1, k_2, \ldots, k_{m+t}) = m \).

(9.2). \( d' < m < d'' \).

**Lemma 2**: Let \( d_1 < d_2 \) be two consecutive divisors of \( d \). If \( d_1 \leq |\supp f| \leq d_2 \) then

$$|\supp f| \geq \frac{d}{d_1 d_2} (d_1 + d_2 - |\supp f|).$$ \hspace{1cm} (A16)

For the submatrix \((j_1, j_2, \ldots, j_m; k_1, k_2, \ldots, k_{m+t})\), if \( \text{rank}(j_1, j_2, \ldots, j_m; k_1, k_2, \ldots, k_{m+t}) < m \), then there must exist a row vector \((x_1, x_2, \ldots, x_m)\) with \( \{x_j\}_{j=1}^m \) complex numbers and not all zero, and \((x_1, x_2, \ldots, x_m)(j_1, j_2, \ldots, j_m; k_1, k_2, \ldots, k_{m+t}) = 0 \).

We denote by \(|\{x_j\}_{j=1}^m|\) the number of nonzero elements in \(\{x_j\}_{j=1}^m\). If \( |\{x_j\}_{j=1}^m| < m \), without loss of generality, we assume \(|\{x_j\}_{j=1}^m| = m_1 < m \) and \( \{x_j \neq 0\}_{j=1}^{m_1} \). \( \{x_j = 0\}_{j=m_1+1}^m \). Then we see that \((x_1, x_2, \ldots, x_m)(j_1, j_2, \ldots, j_m; k_1, k_2, \ldots, k_{m+t}) = 0 \), this implies \(\text{rank}(j_1, j_2, \ldots, j_m; k_1, k_2, \ldots, k_{m+t}) < m_1 \). If \( m_1 < d' \), \(\text{rank}(j_1, j_2, \ldots, j_m; k_1, k_2, \ldots, k_{m+t}) < m_1 \) contradicts the result of (9.1). If \( d' < m_1 \), then repeat this process of \( m \) to \( m_2 \). As a result, there must exist \( d' < m_1 < m \) such that there exist a row vector \((y_1, y_2, \ldots, y_m)\) with \( \{y_j\}_{j=1}^m \) complex numbers and all nonzero, and \((y_1, y_2, \ldots, y_m)(j_1, j_2, \ldots, j_m; k_1, k_2, \ldots, k_{m+t}) = 0 \). Since \( \{|y_j|\}_{j=1}^m = m_1 \), \( d' < m_1 < d' \), and \( d'' \) are consecutive divisors of \( d \) (if \( d_0 |d \) and \( d' < d'' < d_0 \) then \( d_0 + d/d_0 \geq d' + d'' \)), this contradicts the definition of \( d' \), employing Lemma 2, we have that \(|(y_1, y_2, \ldots, y_m)(j_1, j_2, \ldots, j_m; k_1, k_2, \ldots, k_{m+t})| \geq d' + d'' - m_1 \), here \( |\cdot| \) denotes the number of nonzero elements in the row vectors. Together with \((y_1, y_2, \ldots, y_m)(j_1, j_2, \ldots, j_m; k_1, k_2, \ldots, k_{m+t}) = 0 \), then we have

$$|(y_1, y_2, \ldots, y_m)(j_1, j_2, \ldots, j_m; k_1, k_2, \ldots, k_{m+t})| \geq d' + d'' - m_1,$$

here \([0, d-1]\)(\(k_1, k_2, \ldots, k_{m+t}\)) is the complement set of \((k_1, k_2, \ldots, k_{m+t})\) in \([0, d-1]\). However, Eq. (A17) is impossible since \(|(0, d-1)\)(\(k_1, k_2, \ldots, k_{m+t})\) = \( d - (m + t) = d' + d'' - m - 1 \).

Combining (9.1) and (9.2), we then proved Theorem 9, and the uncertainty principle in Eq. (4) in main text immediately follows.

[1] P. J. Coles, M. Berta, M. Tomamichel, and S. Wehner, Entropic uncertainty relations and their applications, Rev. Mod. Phys. 89, 015002 (2017).
[2] Y. Huang, Variance-based uncertainty relations, Phys. Rev. A 86, 024101 (2012).
[3] P. Busch, P. Lahti, and R. F. Werner, Proof of heisenberg’s error-disturbance relation, Phys. Rev. Lett. 111, 160405 (2013).
[4] S. Friedland, V. Gheorghiu, and G. Gour, Universal uncertainty relations, Phys. Rev. Lett. 111, 230401 (2013).
[5] L. Rudnicki, Z. Puchala, and K. Życzkowski, Strong majorization entropic uncertainty relations, Phys. Rev. A 89, 052115 (2014).
[6] M. Ringbauer, D. N. Biggerstaff, M. A. Broome, A. Fedrizzi, C. Branciard, and A. G. White, Experimental joint quantum measurements with minimum uncertainty, Phys. Rev. Lett. 112, 020401 (2014).
[7] L. Maccone and A. K. Pati, Stronger uncertainty relations for all incompatible observables, Phys. Rev. Lett. 113, 260401 (2014).
[8] J.-L. Li and C.-F. Qiao, The generalized uncertainty principle, Annalen der Physik 533, 2000335 (2021).
[9] Y. Xiao, K. Sengupta, S. Yang, and G. Gour, Uncertainty principle of quantum processes, Phys. Rev. Research 3, 023077 (2021).
[10] G. Tóth and F. Fröwis, Uncertainty relations with the variance and the quantum fisher information based on convex decompositions of density matrices, Phys. Rev. Research 4, 013075 (2022).
[11] S.-H. Chiew and M. Gessner, Improving sum uncertainty relations with the quantum fisher information, Phys. Rev. Research 4, 013076 (2022).
[12] D. L. Donoho and P. B. Stark, Uncertainty principles and signal recovery, SIAM Journal on Applied Mathematics 49, 906 (1989).
[13] T. Tao, An uncertainty principle for cyclic groups of prime order, Mathematical research letters 12, 121 (2005).
[14] S. De Bièvre, Complete incompatibility, support uncertainty, and kirkwood-dirac nonclassicality, Phys. Rev. Lett. 127, 190404 (2021).
[15] T. Heinosaari and M. M. Wolf, Nondisturbing quan-
tum measurements, Journal of Mathematical Physics 51, 092201 (2010).
[16] T. Heinosaari, T. Miyadera, and M. Ziman, An invitation to quantum incompatibility, Journal of Physics A: Mathematical and Theoretical 49, 123001 (2016).
[17] S. Designolle, M. Farkas, and J. Kaniewski, Incompatibility robustness of quantum measurements: a unified framework, New Journal of Physics 21, 113053 (2019).
[18] C. Carmeli, T. Heinosaari, and A. Toigo, Quantum random access codes and incompatibility of measurements, EPL (Europhysics Letters) 130, 50001 (2020).
[19] F. Buscemi, E. Chitambar, and W. Zhou, Complete resource theory of quantum incompatibility as quantum programmability, Phys. Rev. Lett. 124, 120401 (2020).
[20] R. Uola, T. Kraft, S. Designolle, N. Miklin, A. Tavakoli, J.-P. Pellenpää, O. Gühne, and N. Brunner, Quantum measurement incompatibility in subspaces, Phys. Rev. A 103, 022203 (2021).
[21] S.-L. Chen, N. Miklin, C. Budroni, and Y.-N. Chen, Device-independent quantification of measurement incompatibility, Phys. Rev. Research 3, 023143 (2021).
[22] L. M. Johansen, Quantum theory of successive projective measurements, Phys. Rev. A 76, 012119 (2007).
[23] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information: 10th Anniversary Edition (Cambridge University Press, 2010).
[24] C.-T. Claude, D. Bernard, and L. Franck, Quantum Mechanics (Wiley, 2015).
[25] E. Matusiak, M. Özaydin, and T. Przebinda, The donoho-stark uncertainty principle for a finite abelian group, Acta Mathematica Universitatis Comenianae. New Series 73, 155 (2004).
[26] S. Ghobber and P. Jaming, On uncertainty principles in the finite dimensional setting, Linear Algebra and its Applications 435, 751 (2011).
[27] A. Wigderson and Y. Wigderson, The uncertainty principle: variations on a theme, Bulletin of the American Mathematical Society 58, 225 (2021).
[28] S. Delvaux and M. Van Barel, Rank-deficient submatrices of fourier matrices, Linear algebra and its applications 429, 1587 (2008).
[29] M. Planat, H. C. Rosu, and S. Perrine, A survey of finite algebraic geometrical structures underlying mutually unbiased quantum measurements, Foundations of Physics 36, 1662 (2006).
[30] T. Durt, B.-G. Englert, I. Bengtsson, and K. Zyczkowski, On mutually unbiased bases, International Journal of Quantum Information 08, 535 (2010).
[31] J. S. Lundeen, B. Sutherland, A. Patel, C. Stewart, and C. Bamber, Direct measurement of the quantum wavefunction, Nature 474, 188 (2011).
[32] E. A. Aguilar, J. J. Borka, P. Mironowicz, and M. Pawlowski, Connections between mutually unbiased bases and quantum random access codes, Phys. Rev. Lett. 121, 050501 (2018).
[33] M. Farkas and J. Kaniewski, Self-testing mutually unbiased bases in the prepare-and-measure scenario, Phys. Rev. A 99, 032316 (2019).
[34] K. Mordasewicz and J. Kaniewski, Quantifying incompatibility of quantum measurements through non-commutativity, arXiv:2110.10646 (2021).
[35] K. T. Smith, The uncertainty principle on groups, SIAM Journal on Applied Mathematics 50, 876 (1990).
[36] G. Alagic and A. Russell, Uncertainty principles for compact groups, Illinois Journal of Mathematics 52, 1315 (2008).
[37] F. Krahmer, G. E. Pfander, and P. Rashkov, Support size conditions for time-frequency representations on finite abelian groups, Jacobs Univ. Bremen, Tech. Rep 13 (2007).
[38] R. Meshulam, An uncertainty inequality for finite abelian groups, European Journal of Combinatorics 27, 63 (2006).