Higher order generalized Euler characteristics and generating series *

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Abstract

For a complex quasi-projective manifold with a finite group action, we define higher order generalized Euler characteristics with values in the Grothendieck ring of complex quasi-projective varieties extended by the rational powers of the class of the affine line. We compute the generating series of generalized Euler characteristics of a fixed order of the Cartesian products of the manifold with the wreath product actions on them.

Let $X$ be a topological space (good enough, say, a quasi-projective variety) with an action of a finite group $G$. For a subgroup $H$ of $G$, let $X^H = \{x \in X : Hx = x\}$ be the fixed point set of $H$. The orbifold Euler characteristic $\chi_{orb}(X, G)$ of the $G$-space $X$ is defined, e.g., in [1], [10]:

$$
\chi_{orb}(X, G) = \frac{1}{|G|} \sum_{(g_0, g_1) \in G \times G} \chi(X^{(g_0, g_1)}) = \sum_{[g] \in G^*} \chi(X^{(g)}/C_G(g)),
$$

(1)

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where $G_*$ is the set of conjugacy classes of elements of $G$, $C_G(g) = \{ h \in G : h^{-1}gh = g \}$ is the centralizer of $g$, and $\langle g \rangle$ and $\langle g_0, g_1 \rangle$ are the subgroups generated by the corresponding elements.

The higher order Euler characteristics of $(X, G)$ (alongside with some other generalizations) were defined in [3], [13].

**Definition:** The Euler characteristic $\chi^{(k)}(X, G)$ of order $k$ of the $G$-space $X$ is

$$\chi^{(k)}(X, G) = \frac{1}{|G|} \sum_{g \in G^{k+1}, \ g_ig_j = g_jg_i} \chi(X^{\langle g \rangle}) = \sum_{[g] \in G_*} \chi^{(k-1)}(X^{\langle g \rangle}, C_G(g)),$$

where $g = (g_0, g_1, \ldots, g_k)$, $\langle g \rangle$ is the subgroup generated by $g_0, g_1, \ldots, g_k$, and $\chi^{(0)}(X, G)$ is defined as $\chi(X/G)$.

The usual orbifold Euler characteristic $\chi^{\text{orb}}(X, G)$ is the Euler characteristic of order 1, $\chi^{(1)}(X, G)$.

The higher order generalized Euler characteristics takes values in the Grothendieck ring of complex quasi-projective varieties extended by the rational powers of the class of the affine line. Let $K_0(\text{Var}_C)$ be the Grothendieck ring of complex quasi-projective varieties. This is the abelian group generated by the isomorphism classes $[X]$ of quasi-projective varieties modulo the relation:

— if $Y$ is a Zariski closed subvariety of $X$, then $[X] = [Y] + [X \setminus Y]$.

The multiplication in $K_0(\text{Var}_C)$ is defined by the Cartesian product. The class $[X]$ of a variety $X$ is the universal additive invariant of quasi-projective varieties and can be regarded as a generalized Euler characteristic of $X$. Let $\mathbb{L}$ be the class $[\mathbb{A}^1_C]$ of the affine line and let $K_0(\text{Var}_C)[\mathbb{L}^{1/m}]$ be the extension of the Grothendieck ring $K_0(\text{Var}_C)$ by all the rational powers of $\mathbb{L}$.

The formula for the generating series of the generalized orbifold Euler characteristics of the pairs $(X^n, G_n)$ in [9] uses the (natural) power structure over the Grothendieck ring $K_0(\text{Var}_C)$ (and over $K_0(\text{Var}_C)[\mathbb{L}^{1/m}]$) defined in [7]. (See also [8] and [9] for some generalizations of this concept.) This means that for a power series $A(T) \in 1 + t \cdot R[[t]]$ ($R = K_0(\text{Var}_C)$ or $K_0(\text{Var}_C)[\mathbb{L}^{1/m}]$) and for an element $m \in R$ there is defined a series $(A(T))^m \in 1 + t \cdot R[[t]]$ so that all the properties of the exponential function hold. For a quasi-projective variety $M$, the series $(1 - t)^{-[M]}$ is the Kapranov zeta-function of $M$: $\zeta_{[M]}(t) := (1 - t)^{-[M]} = 1 + [M] \cdot t + [\text{Sym}^2 M] \cdot t^2 + [\text{Sym}^3 M] \cdot t^3 + \ldots$, where $\text{Sym}^k M = M^k/S_k$ is the $k$-th symmetric power of the variety $M$. A
geometric description of the power structure over the over the Grothendieck ring $K_0(\text{Var}_C)$ is given in [7] or [9]. The (natural) power structures over $K_0(\text{Var}_C)$ and over $K_0(\text{Var}_C)[\mathbb{L}^{1/m}]$ possess the following properties:

1) $(A(t^s))^m = (A(t))^m |_{t \rightarrow t^s}$;

2) $(A(\mathbb{L}^s t))^m = (A(t))^{L^s m}$.

One can define a power structure over the ring $\mathbb{Z}[u_1, \ldots, u_r]$ of polynomials in $r$ variables with integer coefficients in the following way. Let $P(u_1, \ldots, u_r) = \sum_{k \in \mathbb{Z}_{\geq 0}} p_k u^k \in \mathbb{Z}[u_1, \ldots, u_r]$, where $k = (k_1, \ldots, k_r)$, $u = (u_1, \ldots, u_r)$, $u^k = u_1^{k_1} \cdots u_r^{k_r}$, $p_k \in \mathbb{Z}$. Define

$$(1 - t)^{-P(u_1, \ldots, u_r)} := \prod_{k \in \mathbb{Z}_{\geq 0}^r} (1 - u^k t)^{-p_k},$$

where the power (with an integer exponent $-p_k$) means the usual one. This gives a $\lambda$-structure on the ring $\mathbb{Z}[u_1, \ldots, u_r]$ and therefore a power structure over it (see, e.g., [9, Proposition 1])

i.e., for polynomials $A_i(u)$, $i \geq 1$, and $M(u)$, there is defined a series $(1 + A_1(u) t + A_2(u) t^2 + \ldots)^M(u)$ with the coefficients from $\mathbb{Z}[u_1, \ldots, u_r]$. Let $r = 2$, $u_1 = u$, $u_2 = v$. Let $e : K_0(\text{Var}_C) \rightarrow \mathbb{Z}[u, v]$ be the ring homomorphism which sends the class $[X]$ of a quasi-projective variety $X$ to its Hodge–Deligne polynomial $e(X; u, v) = \sum h^{ij}_X (-u)^i (-v)^j$.

**Remark.** Let $R_1$ and $R_2$ be rings with power structures over them. A ring homomorphism $\varphi : R_1 \rightarrow R_2$ induces the natural homomorphism $R_1[[t]] \rightarrow R_2[[t]]$ (also denoted $\varphi$) by $\varphi(\sum a_i t^i) = \sum \varphi(a_i) t^i$. In [9] Proposition 2], it was shown that if a ring homomorphism $\varphi : R_1 \rightarrow R_2$ is such that $(1 - t)^{-\varphi(m)} = \varphi((1 - t)^{-m})$ for any $m \in R$, then $\varphi((A(t))^{-m}) = (\varphi(A(t)))^{\varphi(m)}$ for $A(t) \in 1 + tR[[t]]$, $m \in R$.

There are two natural homomorphism from the Grothendieck ring $K_0(\text{Var}_C)$ to the ring $\mathbb{Z}$ of integers and to the ring $\mathbb{Z}[u, v]$ of polynomials in two variables: the Euler characteristic (with compact support) $\chi : K_0(\text{Var}_C) \rightarrow \mathbb{Z}$ and the Hodge–Deligne polynomial. Both possesses the following well known identities:

(1) the formula of I.G. Macdonald [12]:

$$\chi(1 + [X] t + [\text{Sym}^2 X] t^2 + [\text{Sym}^3 X] t^3 + \ldots) = (1 - t)^{-\chi(X)},$$
and the corresponding formula for the Hodge–Deligne polynomial (see [4, Proposition 1.2]):

\[
e(1 + [X]t + [\text{Sym}^2 X]t^2 + \ldots) = (1 - T)^{-e(X;u,v)} = \prod_{p,q} \left( \frac{1}{1 - u^p v^q t} \right)^{e^{p,q}(X)}.
\]

These properties and the previous remark imply that the corresponding homomorphisms respect the power structures over the corresponding rings: \(K_0(\text{Var}_\mathbb{C})\) and \(\mathbb{Z}[u,v]\) respectively, see [8].

A generalization of the orbifold Euler characteristic to the orbifold (or stringly) Hodge numbers and the orbifold Hodge–Deligne polynomial (for an action of a finite group \(G\) on a non-singular quasi-projective variety \(X\)) was defined in [5], [15], [2].

Let \(X\) be a smooth quasi-projective variety of dimension \(d\) with an (algebraic) action of the group \(G\). For \(g \in G\), the centralizer \(C_G(g)\) of \(g\) acts on the manifold \(X^{(g)}\) of fixed points of the element \(g\). Suppose that its action on the set of connected components of \(X^{(g)}\) has \(N_g\) orbits, and let \(X_{1}^{(g)}, X_{2}^{(g)}, \ldots, X_{N_g}^{(g)}\) be the unions of the components of each of the orbits. At a point \(x \in X_{\alpha_g}^{(g)}, 1 \leq \alpha_g \leq N_g\), the differential \(dg\) of the map \(g\) is an automorphism of finite order of the tangent space \(T_x X\). Its action on \(T_x X\) can be represented by a diagonal matrix \(\text{diag}(\exp(2\pi i \theta_1), \ldots, \exp(2\pi i \theta_d))\) with \(0 \leq \theta_j < 1\) for \(j = 1, 2, \ldots, d\) (\(\theta_j\) are rational numbers). The shift number \(F_{\alpha_g}^{(g)}\) associated with \(X_{\alpha_g}^{(g)}\) is \(F_{\alpha_g}^{(g)} = \sum_{j=1}^{d} \theta_j \in \mathbb{Q}\). (It was introduced in [15].)

**Definition:** The generalized orbifold Euler characteristic of the pair \((X, G)\) (see [9]) is

\[
[X, G] = \sum_{g \in G} \sum_{\alpha_g = 1}^{N_g} [X_{\alpha_g}^{(g)}/C_G(g)] \cdot \mathbb{L}^{F_{\alpha_g}^{(g)}} \in K_0(\text{Var}_\mathbb{C})[\mathbb{L}^{1/m}]. \tag{3}
\]

Since the Euler characteristic and the Hodge–Deligne polynomial are additive invariants they factor through \(K_0(\text{Var}_\mathbb{C})[\mathbb{L}^{1/m}]\) and the Euler characteristic morphisms sends \([X, G]\) to the orbifold Euler characteristic \(\chi^{\text{orb}}(X, G)\). The Hodge–Deligne polynomial morphism sends it to the orbifold Hodge–Deligne polynomial from [2], [14].

Let \(G^n = G \times \ldots \times G\) be the Cartesian power of the group \(G\). The symmetric group \(S_n\) acts on \(G^n\) by permutation of the factors: \(s(g_1, \ldots, g_n) = \ldots\)
The wreath product \( G_n = G \wr S_n \) is the semidirect product of the groups \( G^n \) and \( S_n \) defined by the described action. Namely the multiplication in the group \( G_n \) is given by the formula \((g, s)(h, t) = (g \cdot s(h), st)\), where \( g, h \in G^n \), \( s, t \in S_n \). The group \( G^n \) is a normal subgroup of the group \( G_n \) via the identification of \( g \in G^n \) with \( (g, 1) \in G^n \). For a variety \( X \) with a \( G \)-action, there is the corresponding action of the group \( G_n \) on the Cartesian power \( X^n \) given by the formula
\[
((g_1, \ldots, g_n), s)(x_1, \ldots, x_n) = (g_1 x_{s^{-1}(1)}, \ldots, g_n x_{s^{-1}(n)}),
\]
where \( x_1, \ldots, x_n \in X \), \( g_1, \ldots, g_n \in G \), \( s \in S_n \). One can see that the quotient \( X^n / G_n \) is naturally isomorphic to the space \( \text{Sym}^n(X/G) = (X/G)^n / S_n \). In particular, in the Grothendieck ring of complex quasi-projective varieties one has \([X^n / G_n] = [(X/G)^n / S_n] = [\text{Sym}^n(X/G)]\).

A formula for the generating series of the \( k \)-th order Euler characteristics of the pairs \((X^n, G_n)\) in terms of the \( k \)-th order Euler characteristics of the \( G \)-space \( X \) was given in [13] (see also [3]).

The generating series of the orbifold Hodge–Deligne polynomials \( e(X^n, G_n; u, v) \) of the pairs \((X^n, G_n)\) was computed in [14].

A reformulation of the result of [14] in terms of the generalized orbifold Euler characteristic with values in \( K_0(\text{Var}_C)[L^{1/m}] \) was given in [9]. Using properties of the power structure one has ([9, Theorem 4]):
\[
\sum_{n \geq 0} [X^n, G_n] t^n = \left( \prod_{r=1}^{\infty} \left(1 - L^{(r-1)d/2} t^r \right) \right)^{-[X,G]}.
\]

Here we define higher order generalized Euler characteristics of a pair \((X, G)\) (with \( X \) non-singular) and give a formula for the generating series of the \( k \)-th order generalized Euler characteristic of the pairs \((X^n, G_n)\).

Before giving the definition of the higher order generalized Euler characteristic of a pair \((X, G)\) we discuss some versions of the definition (3) and of the equation (4).

For a \( G \)-variety \( X \) (not necessarily non-singular) its inertia stack (or rather class) \( I(X, G) \) is defined by
\[
I(X, G) := \sum_{[g] \in G} [X^g / C_G(g)]
\]
\( (g_{s^{-1}(1)}, \ldots, g_{s^{-1}(n)}) \).
(see e.g. [11], [6]). One can see that it is an analogue of the generalized orbifold Euler characteristic (3) without the shift factor $L^{F(g)}$. This inspires the following version of the definition (3).

**Definition:** For a rational number $\varphi_1$, let

$$ [X, G]_{\varphi_1} := \sum_{[g] \in G, \alpha_g = 1}^{N_g} \sum_{\alpha_g = 1}^{[X^{(g)}(g)/C_G(g)]} \prod_{r = 1}^\infty (1 - L^{F(g)} \varphi_1 (r-1)d/2 r^r) \in K_0(Var_C)[L^{1/m}], \quad (6) $$

That is the Zaslow shift $F(g)$ is multiplied by $\varphi_1$. For $\varphi_1 = 1$ one gets the generalized Euler characteristic $[X, G]$ from (3), for $\varphi_1 = 0$ one gets the inertia class $I(X, G)$. The arguments from [9] easily give the following version of Equation (4).

**Proposition 1**

$$ \sum_{n \geq 0} [X^n, G_n]_{\varphi_1} t^n = \left( \prod_{r = 1}^\infty (1 - L^{F(g)} \varphi_1 (r-1)d/2 r^r) \right)^{-[X, G]}.$$

Thus multiplication of Zaslow’s shift by a number (at least by 1 or 0) makes sense. For the corresponding definition of the higher order generalized Euler characteristic one can use factors $\varphi_k$ depending on the order of the Euler characteristic.

Let $X$ be a non-singular $d$-dimensional quasi-projective variety with a $G$ action and let $\varphi = (\varphi_1, \varphi_2, \ldots)$ be a fixed sequence of rational numbers. We use the notations introduced before (3).

**Definition:** The **generalized orbifold Euler characteristic of order $k$** of the pair $(X, G)$ is

$$ [X, G]^k_{\varphi} := \sum_{[g] \in G, \alpha_g = 1}^{N_g} \sum_{\alpha_g = 1}^{[X^{(g)}(g)/C_G(g)]} \prod_{r = 1}^\infty (1 - L^{F(g)} \varphi_1 (r-1)d/2 r^r) \in K_0(Var_C)[L^{1/m}], \quad (8) $$

where $[X, G]^1_{\varphi} := [X, G]_{\varphi_1}$ is the (modified) generalized orbifold Euler characteristic given by (6).

**Remark.** The definition (2) (as well as (1)) contains two equivalent versions. One can say that here we formulate an analogues of the second one. A formula
analogous to the first one (with the factor $\frac{1}{|G|}$ in front) cannot work directly, at least without tensoring the ring $K_0(\text{Var}_C)[L^{1/m}]$ by the field $\mathbb{Q}$ of rational numbers. Moreover, it seems that there is no analogue of Theorem 1 in terms of the power structure. This gives the hint that a definition of this sort makes small geometric sense (if any).

Taking the Euler characteristic, one gets $\chi([X, G]_\varphi^k) = \chi^{(k)}(X, G)$.

To prove the formula for the generating series of $[X^n, G_n]_{\varphi^k}$ we will use some technical statements.

Lemma 1

$$[X' \times X'', G' \times G'']_{\varphi^k} = [X', G']_{\varphi^k} \times [X'', G'']_{\varphi^k}. \tag{9}$$

The proof is obvious.

Let $X_1$ and $X_2$ be two $G$-manifolds and let $X_1^m \times X_2^{n-m}$ be embedded into $(X_1 \coprod X_2)^n$ in the natural way: a pair of elements $(x_1, \ldots, x_m) \in X_1^m$ and $(x_2, \ldots, x_{2,n-m}) \in X_2^{n-m}$ is identified with $(x_1, \ldots, x_m, x_2, \ldots, x_{2,n-m}) \in (X_1 \coprod X_2)^n$. Let $\text{Sym}^n(X_1^m \times X_2^{n-m})$ be the orbit of $X_1^m \times X_2^{n-m}$ under the $S_n$-action on $(X_1 \coprod X_2)^n$. The wreath product $G_n$ acts on $\text{Sym}^n(X_1^m \times X_2^{n-m})$.

Lemma 2

$$[\text{Sym}^n(X_1^m \times X_2^{n-m}), G_n]_{\varphi^k} = [X_1^m, G_m]_{\varphi^k} \times [X_2^{n-m}, G_{n-m}]_{\varphi^k}. \tag{10}$$

Proof. An element $(g, s) \in G_n$ has fixed points on $\text{Sym}^n(X_1^m \times X_2^{n-m})$ if and only if it is conjugate to an element $(g', s') \in G_n$ such that $s' = (s_1, s_2) \in S_m \times S_{n-m} \subset S_n$ and the element $(g', s') = ((g_1, g_2), (s_1, s_2))$ has fixed points on $X_1^m \times X_2^{n-m}$ (and only on it). The centralizer of the element $(g', s')$ is $C_{G_n}((g_1, s_1)) \times C_{G_{n-m}}((g_2, s_2))$. The components of $(X_1^m \times X_2^{n-m})_{((g', s'))}$ are the products $(X_1^m)_{((g_1, s_1))} \times (X_2^{n-m})_{((g_2, s_2))}$ of the components of $(X_1^m)_{((g_1, s_1))}$
and \( (X_{2}^{n-m})^{(g_{s},s_{2})} \). The shift \( F_{\alpha\beta}^{(g',s')} \) is equal to \( F_{\alpha}^{(g_{1},s_{1})} + F_{\beta}^{(g_{2},s_{2})} \). Therefore

\[
[\text{Sym}^{n} (X_{1}^{m} \times X_{2}^{n-m}), G_{n}]_{\varphi}^{k} = \sum_{\alpha\beta} \sum_{[g',s'] \in \alpha\beta} [(X_{1}^{m} \times X_{2}^{n-m})^{(g',s')}, C_{G_{m}} ((g_{1}, s_{1})) \times C_{G_{n-m}} ((g_{2}, s_{2}))]_{\varphi}^{k-1} \cdot \mathbb{L} (F_{\alpha}^{(g_{1}, s_{1})} + F_{\beta}^{(g_{2}, s_{2})})
\]

\[
= \sum_{\alpha} \sum_{[g_{s}, s_{1}] \in \alpha} [(X_{1}^{m})^{(g_{s}, s_{1})}, C_{G_{m}} ((g_{1}, s_{1}))]_{\varphi}^{k-1} \cdot \mathbb{L} F_{\alpha}^{(g_{1}, s_{1})} 
\]

\[
= \sum_{\beta} \sum_{[g_{s_{2}}, s_{2}] \in \beta} [(X_{1}^{m})_{g_{s_{2}}, s_{2}}, C_{G_{m}} ((g_{2}, s_{2}))]_{\varphi}^{k-1} \cdot \mathbb{L} F_{\beta}^{(g_{2}, s_{2})}
\]

\[
= [X_{1}^{m}, G_{m}]_{\varphi}^{k} \times [X_{2}^{n-m}, G_{n-m}]_{\varphi}^{k}.
\]

\[\square\]

Let \( X \) be a \( G \)-manifold and let \( c \) be an element of \( G \) acting trivially on \( X \). Let \( r \) be a fixed positive integer. Denote by \( G \cdot \langle a \rangle \) the group generated by \( G \) and the additional element \( a \) commuting with all the elements of \( G \) and such that \( \langle a \rangle \cap G = \langle c \rangle, c = a^{r} \). Define the action of the group \( G \cdot \langle a \rangle \) on \( X \) (an extension of the \( G \)-action) so that \( a \) acts trivially.

**Lemma 3** (cf. [13, Lemma 4-1]) In the described situation one has

\[ [X, G \cdot \langle a \rangle]_{\varphi}^{k} = r^{k} [X, G]_{\varphi}^{k}. \]

**Proof.** We shall use the induction on \( k \). For \( k = 0 \) this is obvious (since \( [X, G]_{\varphi}^{0} = [X/G] \)). Each conjugacy class of elements from \( G \cdot \langle a \rangle \) is of the form \( [g]a^{s} \), where \( [g] \in G_{s}, 0 \leq s < r \). The fixed point set of \( ga^{s} \) coincides with \( X^{0} \), the Zaslow shift \( F_{\alpha}^{\langle a \rangle} \) at each component of \( X^{0} \) coincides with \( F_{\alpha}^{g} \) (since \( a \) acts trivially). The centralizer \( C_{G \cdot \langle a \rangle} (ga^{s}) \) is \( C_{G} (g) \cdot \langle a \rangle \). Therefore

\[ [X, G \cdot \langle a \rangle]_{\varphi}^{k} = \sum_{[g] \in G_{s}} r^{N_{g}} \sum_{\alpha = 1}^{N_{g}} [X_{\alpha}, C_{G} (g) \cdot \langle a \rangle]_{\varphi}^{k-1} \cdot \mathbb{L} F_{\alpha}^{g} = r^{k} [X, G]_{\varphi}^{k}. \]

\[\square\]

**Theorem 1** Let \( X \) be a smooth quasi-projective variety of dimension \( d \) with a \( G \)-action. Then

\[ \sum_{n \geq 0} [X^{n}, G_{n}]_{\varphi}^{k} \cdot t^{n} = \left( \prod_{r_{1}, \ldots, r_{k} \geq 1} \left( 1 - \mathbb{L} \Phi_{\alpha} (\mathbb{Z}) / 2 \cdot t^{r_{1} r_{2} \cdots r_{k}} \right) r_{1}^{2} \cdots r_{k}^{2} \right) \right) - [X, G]_{\varphi}^{k}, \]

(11)
where
\[ \Phi_k(r_1, \ldots, r_k) = \varphi_1(r_1 - 1) + \varphi_2 r_1(r_2 - 1) + \ldots + \varphi_k r_1 r_2 \cdots r_{k-1} (r_k - 1). \]

**Proof.** To a big extent we shall follow the lines of the proof of Theorem A in [13]. We shall use the induction on the order \( k \). For \( k = 1 \) the equation coincides with the one from Proposition [1]. Assume that the statement is proved for the generalized Euler characteristic of order \( k - 1 \). One has
\[
\sum_{n \geq 0} [X^n, G_n]_k^k t^n = \sum_{n \geq 0} t^n \left( \sum_{[(g, s)] \in G_n^{\ast}} \sum_{\text{comp}} [(X^n)_{\text{comp}}^\langle (g, s) \rangle, C_G_n((g, s))]_k^{k-1} \cdot \mathbb{F}_{\text{comp}}^{\langle (g, s) \rangle} \right),
\]
where the sums are over all the conjugacy classes \([ (g, s) ] \) of elements of \( G_n \) and over all the components of \( (X^n)^\langle (g, s) \rangle \) (or rather unions of components from an orbit of the \( C_G_n((g, s)) \)-action on the components of it).

The conjugacy classes \([ (g, s) ] \) of elements of \( G_n \) are characterized by their types. Let \( a = (g, s) \in G_n, g = (g_1, \ldots, g_n) \). Let \( z = (i_1, \ldots, i_r) \) be one of the cycles in the permutation \( s \). The cycle-product of the element \( a \) corresponding to the cycle \( z \) is the product \( g_{i_r} g_{i_{r-1}} \cdots g_{i_1} \in G \). The conjugacy class of the cycle-product is well-defined by the element \( g \) and the cycle \( z \) of the permutation \( s \). For \( [c] \in G_\ast \) and \( r \geq 0 \), let \( m_r(c) \) be the number of \( r \)-cycles in the permutation \( s \) whose cycle-products lie in \([c]\). One has
\[
\sum_{[c] \in G_\ast} rm_r(c) = n.
\]
The collection \( \{m_r(c)\}_{r,c} \) is called the type of the element \( a = (g, s) \in G_n \). Two elements of the group \( G_n \) are conjugate to each other if and only if they are of the same type.

In [13] (see also [14]) it is shown that, for an element \( (g, s) \in G_n \) of type \( \{m_r(c)\} \), the subspace \( (X^n)^\langle (g, s) \rangle \) can be identified with
\[
\prod_{[c] \in G_\ast} \prod_{r \geq 1} (X^c)^{m_r(c)}. \tag{12}
\]
By [13] Theorem 3.5] the centralizer of the element \( (g, s) \in G_n \) is isomorphic to
\[
\prod_{[c] \in G_\ast} \prod_{r \geq 1} \{(C_G(c) \cdot \langle a_{r,c} \rangle) \cap S_{m_r(c)} \}
\]
(acting on the product component-wise) where \( C_G(c) \cdot \langle a_{r,c} \rangle \) is the group generated by \( C_G(c) \) and an element \( a_{r,c} \) commuting with all the elements of \( C_G(c) \) and such that \( a_{r,c} = c \), \( \langle a_{r,c} \rangle \cap C_G(c) = \langle c \rangle \), and \( a_{r,c} \) acts on \( (X^c)^{m_r(c)} \) trivially.

The components of \( (X^c)^{m_r(c)} \) (with respect to the \( C_G(c) \cdot \langle a_{r,c} \rangle \)-action) are \( \text{Sym}^{m_r(c)} \left( \prod_{\alpha=1}^{N_a} (X^{c}\alpha)^{m_{r,c}(\alpha)} \right) \), where \( \sum_{\alpha=1}^{N_a} m_{r,c}(\alpha) = m_r(c) \). Here and below the sum over \( \text{comp} \) means the summation over all the components indicated in the summands. Therefore

\[
\sum_{n \geq 0} [X^n, G_n]_G t^n = \sum_{n \geq 0} t^n \left( \sum_{\{m_{r,c}(\alpha)\}} \prod_{[c], r} \left\{ (X^c)^{m_{r,c}(\alpha)} \right\}_{\text{comp}, [c], r} \prod_{[c], r} \left\{ (C_G(c) \cdot \langle a_{r,c} \rangle) \cdot S_{m_r(c)} \right\}_{\varphi}^{k-1} \cdot \mathbb{L}^{F^{(g_s)}_{\text{comp}}} \right)
\]

Iterating Lemma 2 one gets

\[
\sum_{\{m_{r,c}(\alpha)\}} \prod_{[c], r} \left\{ (X^c)^{m_{r,c}(\alpha)} \right\}_{\text{comp}, [c], r} \prod_{[c], r} \left\{ (C_G(c) \cdot \langle a_{r,c} \rangle) \cdot S_{m_r(c)} \right\}_{\varphi}^{k-1} \times \\
\mathbb{L} \phi_k \left( \sum_{[c], r} m_{r,c}(\alpha)(F_{\alpha}^{c} + \frac{(r-1)d}{2}) \right)
\]

\[
= \prod_{[c], r} \left( \sum_{\{m_{r,c}(\alpha)\}} \prod_{\alpha=1}^{N_a} \left\{ (X^{c}\alpha)^{m_{r,c}(\alpha)} \right\}_{\text{comp}, [c], r} \prod_{[c], r} \left\{ (C_G(c) \cdot \langle a_{r,c} \rangle) \cdot S_{m_r(c)} \right\}_{\varphi}^{k-1} \times \\
\mathbb{L} \phi_k \left( \sum_{[c], r} m_{r,c}(\alpha)(F_{\alpha}^{c} + \frac{(r-1)d}{2}) \right) \right)
\]
By the induction one gets

\[ \prod_{r_1, \ldots, r_{k-1} \geq 1} \prod_{[c], r \alpha=1} \left( 1 - \Phi_{k-1}(r) \frac{d}{2} \phi_k \left( L \phi_k \left( F_{\alpha} + \frac{(r-1)d}{2} \right) \right)_{r_1 \cdots r_{k-1}} r_2 r_3^2 \cdots r_{k-1}^{k-2} \right)^{-1} \]

\[ = \prod_{r_1, \ldots, r_{k-1} \geq 1} \left( 1 - \Phi_{k-1}(r) \frac{d}{2} \phi_k \left( r_1 \cdots r_{k-1} \right) \right)^{r_2 r_3^2 \cdots r_{k-1}^{k-2}} \]

(Here we use the properties of the power structure.)

\[ = \prod_{r_1, \ldots, r_{k-1} \geq 1} \left( 1 - \Phi_k \left( \frac{d}{2} \phi_k \right) \right)^{r_2 r_3^2 \cdots r_{k-1}^{k-2}} \]

\[ = \prod_{r_1, \ldots, r_{k-1} \geq 1} \left( 1 - \Phi_k \left( \frac{d}{2} \phi_k \right) \right)^{r_2 r_3^2 \cdots r_{k-1}^{k-2}} \]

In the last two equations \( r \) is substituted by \( r_k \). □

**Remark.** For \( \varphi = 0 \), i.e. if \( \varphi_i = 0 \) for all \( i \), the definition of the higher order generalized Euler characteristics does not demand \( X \) to be smooth. This way one gets the definition of a sort of higher order inertia classes and the statement of Theorem 11 holds for an arbitrary \( G \)-variety \( X \).

Since \( \chi([X^n, G_n]^k) = \chi(X, G) \), \( \chi(\mathbb{L}) = 1 \), taking the Euler characteristic of the both sides of the equation (11) one gets Theorem A of [13]:

\[ \sum_{n \geq 0} \chi^{(k)}(X^n, G_n) \cdot t^n = \left( \prod_{r_1, \ldots, r_{k-1} \geq 1} \left( 1 - t^{r_1 r_2 \cdots r_{k-1}} r_2 r_3^2 \cdots r_{k-1}^{k-2} \right) \right)^{-1} \]

Let \( e^{(k)}(X, G; u, v) := e([X, G]^k; u, v) \) be the higher order Hodge–Deligne polynomial of \( (X, G) \) (of order \( k \)). Applying the Hodge–Deligne polynomial homomorphism, one gets a generalization of the main result in [14]:

\[ \sum_{n \geq 0} e^{(k)}(X^n, G_n; u, v) \cdot t^n = \left( \prod_{r_1, \ldots, r_{k-1} \geq 1} \left( 1 - (uv)^{d/2} \cdot t^{r_1 r_2 \cdots r_{k-1}} r_2 r_3^2 \cdots r_{k-1}^{k-2} \right) \right)^{-e^{(k)}(X, G; u, v)} \]
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