LIPSCHITZ COHOMOLOGY, NOVIKOV CONJECTURE, AND EXPANDERS

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Abstract. We present sufficient conditions for the cohomology of a closed aspherical manifold to be proper Lipschitz in sense of Connes-Gromov-Moscovici [CGM]. The conditions are stated in terms of the Stone-Čech compactification of the universal cover of a manifold. We show that these conditions are formally weaker than the sufficient conditions for the Novikov conjecture given in [CP]. Also we show that the Cayley graph of the fundamental group of a closed aspherical manifold with proper Lipschitz cohomology cannot contain an expander in the coarse sense. In particular, this rules out a Lipschitz cohomology approach to the Novikov Conjecture for recent Gromov's examples of exotic groups.

§1 Introduction

Lipschitz cohomology classes of a group were defined by Connes, Gromov and Moscovici [CGM]. They proved the Novikov Higher signature conjecture for such classes. Lipschitz cohomology are defined as the images under certain slant product homomorphisms denoted in [CGM] as $\alpha_\Gamma$. Since the analytic assembly map is a slant product in K-theory (with the Mischenko line bundle)[FRR], it allows to make a connection between the Lipschitz cohomology and the Novikov Conjecture (see [CGM]). As application Connes, Gromov and Moscovici proved the Novikov conjecture for the hyperbolic groups. In the same spirit using Lipschitz cohomology T. Kato proved the Novikov conjecture for the combable groups [K1],[K2]. In these cases all cohomology classes of a group $\Gamma$ are Lipschitz and moreover, they belong to the image of one homomorphism $\alpha_\Gamma$. In such situation we say that a group $\Gamma$ has canonically Lipschitz cohomology. We consider the case when $\Gamma$ admits a compact classifying space $B\Gamma$. In this case the cohomologies of a

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group $\Gamma$ coincide with the cohomologies of the space $B\Gamma$ and hence we can speak about Lipschitz cohomology of a space. A typical example of a space with the canonically Lipschitz cohomology is a closed manifold $M$ with a nonpositive sectional curvature [CGM]. In this case the construction of corresponding homomorphism $\alpha_\cap$ relies on the following fact: Every such manifold $M$ is aspherical, i.e. its universal cover $X$ is contractible, and the inverse of the exponential map $\log : X \to T_x$ is $1$-Lipschitz for every $x \in X$. In this paper we consider closed aspherical manifolds with canonically Lipschitz cohomologies.

The main result of the paper (Theorem 1) is a reformulation of the property of a manifold $M$ to have Lipschitz cohomology in terms of compactifications of its universal cover $X$. Clearly that an action of $\Gamma$ on $X$ extends to an action on the Stone-Čech compactification. We show that for an $n$-manifold $M = B\Gamma$ to have canonically Lipschitz rational cohomology is equivalent to possess an essential $\Gamma$-invariant map of the corona of the Stone-Čech compactification of $X$ onto the $n - 1$-dimensional sphere. We also reformulate this condition in terms of a compactification of $X$ introduced by Hurder [Hu] which can be defined in simple terms (Proposition 4) as the maximal equivariant compactification of $X$ for the fundamental group action such that the corona is the fixed point set. Theorem 1 implies in particular that the Novikov Conjecture holds for the group $\pi_1(M)$ if the Hurder corona of $X$ carries a cohomology class which hits the fundamental class of $X$ under the coboundary homomorphism $\delta : \hat{H}^{n-1}(\bar{X} \setminus X) \to H^n_c(X)$.

There are several approaches to the Novikov Conjecture formulated in terms of coarse geometry and coarse topology due to Carlsson-Pedersen, Ferry-Weinberger, Gromov, Higson-Roe, Hurder and others (see [FRR]). It turns out that many of them can be reduced to the Lipschitz cohomology approach though I failed to recover Gromov’s approach via hypereuclidean manifolds ([G2], page 152) on this way. One of the corollaries of Theorem 1 (Theorem 2) deals with a reduction which is due to Carlsson-Pedersen. We show that the Carlsson-Pedersen conditions on $M$ imply that $M$ has Lipschitz cohomologies.

Perhaps the most advanced results on the Novikov Conjecture is a theorem of Guoliang Yu which states that the Novikov Conjecture holds true for a group that admits as a metric space with the word metric a coarse imbedding in a Hilbert space [Yu],[H]. Clearly, after this theorem the question by Gromov [G1] whether every metric separable space is coarsely imbeddable in a Hilbert space became very important. Unfortunately the answer to Gromov’s question turned out to be negative. First example of a countable discrete metric spaces which is not coarsely embeddable in the Hilbert space was constructed in [DGLY]. Then Gromov noticed that expanders form a natural counterexample to his question. An expander is a growing sequence of graphs $X_n$ with uniformly bounded valence and with uniform spectral gap for the combinatorial Laplacians for the first
nonzero eigenvalues: $\lambda_1(X_n) \geq \delta > 0$.

Gromov constructed finitely presented groups $\Gamma$ that contain expander in some coarse sense. Moreover, Gromov constructed a closed aspherical 4-manifold $M$ with the fundamental group containing an expander $[G3],[G4]$. We should note that Gromov’s imbedding conditions are weaker than those of a quasi-isometric imbedding or a coarse imbedding. The latter means an imbedding in the coarse category $[R01],[R02]$ which is often called in the literature by a misleading name uniform imbedding $[G1]$ used for a different thing in the classical analysis. We will call here Gromov’s imbedding of expander by quasi-coarse imbedding.

Thus, expanders form an obstacle for proving the Novikov conjecture using Yu’s theorem. Also they were used to disprove some versions of the Baum-Connes conjecture $[HLS]$. In this paper we show that they are obstructions for proving the Novikov conjecture via Lipschitz cohomology as well (Theorem 3). We show that a group with the canonically Lipschitz cohomology cannot contain an expander. Probably I should mention here that this fact was independently discovered by Gromov (oral communication). Certainly, it was known in the main partial case. Namely, Gromov $[G4]$ and Higson (unpublished) proved independently that the fundamental group of a manifold with non-positive sectional curvature cannot contain an expander in the coarse (and Gromov’s) sense.

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§2 Proper Lipschitz cohomology

In this paper all homology and cohomology will be taken with rational coefficients $\mathbb{Q}$. Suppose that a discrete group $\Gamma$ acts freely and properly on a space $X$. By $H^*_\Gamma(X)$ we denote the equivariant homology of $X$, i.e. the homology of the chain complex $C_{lf}^*(X)^\Gamma$ of locally finite equivariant (singular) chains. Let $q : X \to M$ be the projection onto the orbit space. Let $c$ be a (singular) simplex of $M$ and let $\bar{c}$ be its lift. Then the formula $\mu(c) = \sum_{g \in \Gamma} g\bar{c}$ defines an isomorphism of chain complexes $\mu = \mu_q : C_*(M) \to C_{lf}^*(X)^\Gamma$ provided $M$ is compact. Then $\mu$ defines an isomorphism $\mu_* : H_*(M) \to H_\Gamma^*(X)$. If $M$ is not compact $\mu$ defines an isomorphism $\mu_* : H^\Gamma_*(M) \to H^\Gamma_*(X)$ of the homologies defined by locally finite chains. The inclusion $C_{lf}^\Gamma(X) \subset C_{lf}^*(X)$ defines a forgetful homomorphism $\nu_* : H^\Gamma_*(X) \to H^\Gamma_*(X)$. The composition $\tau_* = \nu_* \mu_* : H^\Gamma_*(X/\Gamma) \to H^\Gamma_*(X)$ is called the transfer. The equivariant cohomology $H^\Gamma_*(X)$ of $X$ are defined by the chain complex $\text{Hom}(C_{lf}^*(X)^\Gamma, \mathbb{Q})$. There are the dual homomorphisms $\mu^*, \nu^*$ and $\tau^*$. 
The above definition can be extended to nonregular covering maps. If \( p : X \to Y \) is a covering map then the homomorphism \( \mu : C^i_\mu(Y) \to C^i_\mu(X) \) can be defined by the formula \( \mu(c) = \sum_{\tilde{c} \in L(c)} \tilde{c} \) where \( L(c) \) is the set of all lifts of the singular simplex \( c \). Then the chain complex \( \mu(C^i_\mu(Y)) \) defines the \( p \)-invariant homology of \( X \) and the dual complex defines the \( p \)-invariant cohomology. We will use the following notations for them: \( H^p_\mu(X) \) and \( H^*_\mu(X) \). Note that they are isomorphic to \( H^i_\mu(Y) \) and to \( H_c^*(Y) \) respectively by means of isomorphisms \( \mu_* \) and \( \mu^* \) induced by \( \mu \).

Assuming the diagonal action on \( X \times Y \) one can define an equivariant slant product \( H^p_\mu(X \times Y) \otimes H^k_\mu(Y) \to H^{n-k}_\mu(X) \) in the same manner as the standard slant product: 
\[
[\omega]/[a] = [\omega/a] \quad \text{where} \quad (\omega/a)(b) = \omega(a \times b), \quad \omega \in \text{Hom}(C^i_\mu(X \times Y)^\Gamma; \mathbb{Q}), \quad a \in C^k_\mu(Y)^\Gamma, \\
\text{and} \quad b \in C^{k-i}_\mu(X)^\Gamma.
\]
One of the features of the equivariant slant product is the equality 
\[
[\omega]/[a] = [\omega']/[a] \quad \text{where} \quad [\omega'] \text{ is the image of } [\omega] \text{ under the homomorphism } H^*_\mu(X \times Y) \to H^*_\mu(\Gamma; \Gamma). 
\]
We also recall the main formula for the slant product: \( a \otimes b/c = a(b,c) \).

We consider the case of geometrically finite groups \( \Gamma \), i.e. groups with \( B \Gamma \) finite complex. Let \( X = E \Gamma \) denote the universal cover of \( B \Gamma \). Proper Lipschitz cohomology classes of \( \Gamma \) are detected by means of the following data \([CGM]\): a finite dimensional complex \( P \) with a proper \( \Gamma \)-action and a map \( \alpha : X \times P \to \mathbb{R}^N \) satisfying the conditions

1. \( \alpha \) is invariant with respect to the diagonal action of \( \Gamma \) on \( X \times P \);
2. the restriction \( \alpha|_{x \times P} \) is proper for all \( x \in X \).

Then the slant product for equivariant (co)homology \( H^N_\Gamma(X \times P) \otimes H^i_\Gamma(P) \to H^{N-i}_{\Gamma}(X) \) defines a homomorphism
\[
\alpha \cap : H^*_\Gamma(P) = H^i_\mu(P/\Gamma) \to H^*_\Gamma(X) = H^*(B \Gamma)
\]
by means of the class \( \alpha^*(\omega) \in H^N_\Gamma(X \times P) \) where \( \omega \in H^N_c(\Gamma; \mathbb{Q}) \) is a generator. Thus,
\[
\alpha \cap (b) = \alpha^*(\omega)/b.
\]

By the definition all classes in \( \alpha \cap(H^i_\Gamma(P/\Gamma; \mathbb{Q})) \subset H^*(\Gamma; \mathbb{Q}) \) are called proper Lipschitz cohomology classes of \( \Gamma \) provided the following condition holds

3. the restriction \( \alpha|_{X \times P} \) is 1-Lipschitz for all \( p \in P \).

Now we consider the case when \( B \Gamma = M \) is a closed orientable \( n \)-manifold, \( X \) is its universal cover, \( \Gamma = \pi_1(M) \).

We note that if \( \Gamma \) acts on \( P = X \times \mathbb{R}^k \) in such a way that the projection \( pr : P \to X \) is equivariant then \( H^i_\mu(P) = H^i_\mu(X \times \Gamma \mathbb{R}^k) = H_{*+k}(M) \) by means of the Thom isomorphism for the \( \mathbb{R}^k \)-bundle \( X \times \Gamma \mathbb{R}^k \to M \).
Proposition 1. Suppose that \( \Gamma \) acts on \( P = X \times \mathbb{R}^k \) such that the projection \( pr : X \times \mathbb{R}^k \to X \) is equivariant and let \( \alpha : X \times P \to \mathbb{R}^{n+k} \) be a map satisfying the conditions (1)-(3). Then the following are equivalent:

1. \( \alpha \mid_{x \times P} \) is isomorphism in all dimensions;
2. \( \alpha \mid_{x \times P} \neq 0 \);
3. \( \alpha \mid_{x \times P} \) is essential for all \( x \in X \).

Proof. (1) \( \Rightarrow \) (2). Obvious.

(2) \( \Rightarrow \) (3). Since \( \alpha \) is \( \Gamma \)-invariant, it induces a map \( \hat{\alpha} : X \times \Gamma P \to \mathbb{R}^{n+k} \), \( \alpha = \hat{\alpha} \circ \bar{q} \) where \( \bar{q} : X \times P \to X \times \Gamma P \) is the projection to the orbit space. The condition (2) implies that \( \alpha^*(\omega) \neq 0 \) and hence \( \hat{\alpha}^*(\omega) \neq 0 \) where \( \omega \in H_c^n(R^{n+k}) \) is the fundamental class. Being the product of a contractible manifold with a Euclidean space, \( P \) is homeomorphic to \( R^{n+k} \).

Since the \( P \)-bundle \( \pi : X \times \Gamma P \to M \) is orientable, the inclusion \( \pi^{-1}(y) \subset X \times \Gamma P \) induces a map \( \pi^{-1}(y) \subset X \times \Gamma P \) induces an isomorphism of \((n+k)\)-dimensional cohomology groups with compact supports. Hence the restriction \( \hat{\alpha} \) to \( \pi^{-1}(y) \) is essential for all \( y \in M \). Therefore the restriction of \( \alpha \) to \( x \times P \) is essential for all \( x \in X \).

(3) \( \Rightarrow \) (1). The essentiality in (3) implies that \( \hat{\alpha}^*(\omega) \) is a generator of the group

\[ H^{n+k}_c(X \times \Gamma P; \mathbb{Q}) = \mathbb{Q} \]

By the proposition conditions \( P/\Gamma = E \) is the total space of an \( \mathbb{R}^k \)-bundle \( \pi_2 : E \to M \). Moreover, \( \pi \) is the pull-back of \( \pi_2 \) and \( \pi_1 : X \times \Gamma X \to M \). Let \( u \in H^{n+k}_c(X \times \Gamma P) \) be the Thom class of \( \pi \). Thus, \( \hat{\alpha}^*(\omega) = \lambda u \) for some \( \lambda \in \mathbb{Q} \setminus \{0\} \).

First we consider the case when \( P = X \). The projection \( \bar{\pi} : X \times X \to M \times M \) onto the orbit space of \( \Gamma \times \Gamma \)-action is factored as \( \bar{\pi} = \pi' \circ \bar{q} \) where \( \bar{q} \) is projection to the orbit space of the diagonal action and \( \pi' : X \times \Gamma X \to M \times M \) is the induced covering map. Let \( N(M) \) denote an open \( r \)-neighborhood of \( M \) imbedded in \( X \times \Gamma X \) as the image \( \bar{q}(X) \) of the diagonal in \( X \times X \). We will call this imbedding a zero section of \( \pi \).

Let \( j : N(M) \to X \times \Gamma X \) denote the imbedding. If \( r \) is small enough, the composition \( j_1 = \pi' \circ j \) is an imbedding as well. We may assume that the Thom class \( u \) of \( \pi \) is supported in \( N(M) \). It is known [MS] that \( w = j_1^*(u) \) is the Poincare dual in \( M \times M \) to the diagonal homology class \( M \subset M \times M \). Also it is known [MS] that the slant product with \( w \) in \( H^n(M \times M) \otimes H_i(M) \to H^{n-i}(M) \) defines the Poincare duality for \( M \). The equivariant slant product with \( \bar{w} = (\mu^*)^{-1}(w) \) in

\[ H^n_{\Gamma \times \Gamma}(X \times X) \otimes H^n_{\Gamma}(X) \to H^{n-i}_{\Gamma}(X) \]

also defines an isomorphism. Namely, \( \mu^* \circ (\bar{w}/-) \circ \mu_* = w/- \). The following diagram is
commutative

\[
\begin{array}{ccc}
\text{H}^*_c(N(M)) & \xrightarrow{j^*} & H^*_c(X \times \Gamma X) & \xleftarrow{\mu^*} & H^*_\Gamma(X \times X) \\
\downarrow j^*_i & & \downarrow \nu^* & & \downarrow \nu^*_i \\
H^*(M \times M) & \xleftarrow{\mu^*} & H^*_\pi(X \times \Gamma X) & \xleftarrow{\mu^*} & H^*_{\Gamma \times \Gamma}(X \times X)
\end{array}
\]

We check that \( j^*_1 = \mu^* \nu^* j^* \). Indeed, \( j^*_1(a) = a(j_1^{-1}(c)_{j_1(N(M))}) = q(\mu^*(c)_{|N(M)}) = a(\mu^*(c)) = \mu^*a(c) = \mu^* \nu^* j^*(a)(c) \). Here \( a \in Hom(C^f_{\ast}(X \times_{\Gamma} X), \mathbb{Q}) \) with the support in \( N(M) \), \( c \in C_{\ast}(M) \) with the mesh \( \epsilon \) such that the \( \epsilon \)-neighborhood of the image under \( j_1 \) of the support of \( a \) is contained in \( j_1(N(M)) \). The restriction \( c|_W \) of a chain to a neighborhood \( W \) is a chain defined by simplices that are contained in \( W \). Let \( \bar{u} = (\mu^*)^{-1}(u) \). Then \( \nu^*_1(\bar{u}) = \bar{w} \). Therefore \( \bar{u}/_{-1} : H^i_{\Gamma}(X) \rightarrow H^{n-i}_{\Gamma}(X) \) is an isomorphism. Since \( \alpha^*(\omega) = \lambda \bar{u} \), the homomorphism \( \alpha_{\Gamma} \) is an isomorphism.

In general case \( P = X \times \mathbb{R}^k \) we still can define the diagonal imbedding \( M \rightarrow M \times E \) using a zero section \( s : M \rightarrow E \) of the bundle \( \pi_2 \). Then similarly one can define the maps \( j, j_1 \) and \( \pi' \). We note that the class \( w \in H^*(M \times M) \) can be presented as \( \bigoplus_l (b_l \otimes c_l) \) where \( \dim b_l + \dim c_l = n \). Then the image under \( j^*_1 \) of the Thom calss \( u \) in the general case equals \( w' = \bigoplus_l (b_l \otimes (c_l \cup v)) \) where \( v \) is the Thom class of \( \pi_2 \). The slant product with \( w' \) in

\[
H^*_{\Gamma}(M \times E) \otimes H^{lj}_{\Gamma}(E) \rightarrow H^{n+k-i}(M)
\]

defines the composition of the homology Thom isomorphism

\[
v \cap - : H^{lj}_{\Gamma}(E) \rightarrow H_{n-k}(M)
\]

with the Poincare duality. Indeed, for \( a \in H^{lj}_{\Gamma}(E) \) we have

\[
w'/a = \bigoplus_{\dim c_l = i-k} b_l(c_l \cup v, a) = \bigoplus_{\dim c_l = i-k} b_l(c_l, a \cap v) = w/(a \cap v) = PD(a \cap v).
\]

Let \( \bar{w}' = (\mu^*)^{-1}(w) \). Then the slant product with \( \bar{w}' \) in

\[
H^*_{\Gamma}(X \times P) \otimes H^*_{\Gamma}(P) \rightarrow H^*_{\Gamma}(X)
\]
is an isomorphism. The commutative diagram

\[
\begin{array}{ccc}
\text{H}^*_c(N(M)) & \xrightarrow{j^*} & H^*_c(X \times \Gamma P) & \xleftarrow{\mu^*} & H^*_\Gamma(X \times P) \\
\downarrow j^*_i & & \downarrow \nu^* & & \downarrow \nu^*_i \\
H^*_c(M \times E) & \xleftarrow{\mu^*} & H^*_\pi(X \times \Gamma P) & \xleftarrow{\mu^*} & H^*_{\Gamma \times \Gamma}(X \times P)
\end{array}
\]
implies that the slant product with $\alpha^*(\omega) = \lambda \bar{q}^*(u)$ equals the slant product with $\lambda \bar{w}'$. Hence $\alpha \cap$ is an isomorphism. \hfill \Box

**Definition.** We say that a manifold $M$ has *canonically Lipschitz cohomology* if there exists a map $\alpha$ as in Proposition 1. The map $\alpha$ is called *Lipschitz representation* of $H^*(M)$.

Some version of the following was stated in [CGM] (9.A).

**Assertion 1.** For every closed orientable aspherical manifold $M$ with the fundamental group $\Gamma$ and the universal cover $X$ there exists a free proper action of $\Gamma$ on $P = X \times \mathbb{R}^k$ for some $k$ with an equivariant projection to the first factor and a map $\alpha : X \times P \to \mathbb{R}^{n+k}$ satisfying (1)-(2) and such that $\alpha \cap$ is an isomorphism.

**Proof.** The diagonal $\Gamma$-action defines a bundle $\pi_1 : X \times_{\Gamma} X \to M$ with the fiber $X$ and the structure group $\Gamma$. We multiply $X \times_{\Gamma} X$ by $\mathbb{R}$ to obtain a topological $\mathbb{R}^{n+1}$-bundle over $M$. There is a vector bundle such that its "Whitney sum" with our topological vector bundle gives a trivial topological vector bundle over $M$. It means that there exists a vector bundle $\nu : E \to M$ such that the pull-back diagram

$$
\begin{array}{ccc}
E' & \xrightarrow{\pi'} & E \\
\nu' \downarrow & & \nu \downarrow \\
X \times_{\Gamma} X & \xrightarrow{\pi_1} & M.
\end{array}
$$

defines a topologically trivial bundle $\pi : E' \to M$, $\pi = \pi_1 \circ \nu'$ with the fiber $\mathbb{R}^{n+k}$.

Let $k = \dim E$. The following pull-back diagram defines $P$ together with an action of $\Gamma$ on it

$$
\begin{array}{ccc}
P & \xrightarrow{\tilde{\nu}} & X \\
q' \downarrow & & q \downarrow \\
E & \xrightarrow{\nu} & M.
\end{array}
$$

Since $X$ is contractible, $\tilde{\nu}$ is a trivial $\mathbb{R}^k$-bundle, i.e. $P$ is homeomorphic to $X \times \mathbb{R}^k$ and $\tilde{\nu}$ is an equivariant projection onto the first factor. It is easy to see that the orbit space of the diagonal action of $\Gamma$ on $X \times P$ is $E'$. Let $p : E' \to \mathbb{R}^{n+k}$ be a trivialization. We define $\alpha = p \circ \tilde{q}$ where $\tilde{q} : X \times P \to E'$ is the orbit map. The argument (3) $\Rightarrow$ (1) of Proposition 1 completes the proof. \hfill \Box

To insure condition (3) for the above $\alpha$ one needs the following.

**Displacement Bound Condition** [CGM]: $\|p(x) - p(\gamma x)\| \leq \|\gamma\|_S$ for all $\gamma \in \Gamma$.  

Here $\|\gamma\|_S$ is the minimal length of $x \in \Gamma$ with respect to a given finite symmetric set of generators $S$. We assume that $S$ is fixed and we let $d_S(x,y) = \|x^{-1}y\|_S$ to denote the word metric on $\Gamma$.

**Proposition 2 [CGM].** Let $q : \Gamma \times P \to P$ be projection to the orbit space of the diagonal action. Then the following conditions for a map $p : P \to \mathbb{R}^N$ are equivalent:

1. $p$ satisfies the Displacement Bound Condition;
2. The restriction $p \circ q|_{\Gamma \times x}$ is 1-Lipschitz for all $x \in P$ with respect to the word metric on $\Gamma$.

**Proof.** We identify $P$ with $1 \times P \subset \Gamma \times P$. Suppose that $p$ satisfies the Displacement Bound Condition. Then $\|p \circ q(\gamma_1 \times x) - p \circ q(\gamma_2 \times x)\| = \|p \circ q(1 \times \gamma_1^{-1}x) - p \circ q(1 \times \gamma_2^{-1}x)\| = \|p(\gamma_1^{-1}x) - p(\gamma_2^{-1}x)\| = \|p(\gamma_1^{-1}x) - p((\gamma_2^{-1}\gamma_1)(\gamma_1^{-1}x))\| \leq \|\gamma_2^{-1}\gamma_1\| = d_S(\gamma_1, \gamma_2)$. 

Now if $p \circ q|_{\Gamma \times x}$ is 1-Lipschitz, we have $\|p(\gamma x) - p(\gamma x)\| = \|p(q(1 \times x) - p \circ q(1 \times x))\| = \|p \circ q(1 \times x) - p \circ q(\gamma^{-1} \times x)\| \leq d_S(\gamma^{-1}) = \|\gamma\|$. $\square$

**Proposition 3.** If the above map $p$ is proper and satisfies the Displacement Bound Condition then the map $p \circ q$ generates a map $\alpha : X \times P \to \mathbb{R}^N$ that satisfies the conditions (1)-(3).

**Proof.** We may assume that $\Gamma$ is isometrically imbedded in $X$ as an orbit $\Gamma x_0$ and $\Gamma$ is $K$-dense in $X$. Let $P(\Gamma)$ be the space of measures on $\Gamma$ with finite supports. Thus $P(\Gamma)$ consists of formal finite sums $\Sigma \lambda_i x_i$, $\lambda_i \geq 0$, $x_i \in \Gamma$. Let $1 - Lip(\Gamma)$ denote the set of real-valued 1-Lipschitz functions on $\Gamma$. We endow $P(\Gamma)$ with the Kantorovich-Rubinstein metric:

$$d_P(\Sigma \lambda_i x_i, \Sigma \mu_i x_i) = \sup\{|\Sigma(\lambda_i - \mu_i)f(x_i)| \mid f \in 1 - Lip(\Gamma)\}.$$

Then $\Gamma$ is isometrically imbedded in $P(\Gamma)$ by means of Dirac measures. Every map $g : \Gamma \to \mathbb{R}^N$ has a natural extension to $\bar{g} : P(\Gamma) \to \mathbb{R}^N$ by the formula $\bar{g}(\Sigma \lambda_i x_i) = \Sigma \lambda_i g(x_i)$. We note that for every 1-Lipschitz map $g$ the extension $\bar{g}$ is $N$-Lipschitz. Indeed, $\|\bar{g}(\Sigma \lambda_i x_i) - \bar{g}(\Sigma \mu_i x_i)\| = \|\Sigma \lambda_i g(x_i) - \Sigma \mu_i g(x_i)\| \leq$ $\Sigma_{k=1}^N |\Sigma_i(\lambda_i - \mu_i)g_k(x_i)| \leq N d_P(\Sigma \lambda_i x_i, \Sigma \mu_i x_i)$

where $g = (g_1, \ldots, g_N)$. Thus, $p \circ q$ defines a map $\bar{p} \circ \bar{q} : P(\Gamma) \times P \to \mathbb{R}^N$ such that the restriction to $P(\Gamma) \times x$ is $N$-Lipschitz for all $x$.

A cover of $X$ by $K$-balls $B_K(x)$ centered at points $x \in \Gamma$ defines a map $\psi : X \to P(\Gamma)$ by the formula $\psi(x) = \Sigma \lambda_i x_i$ where $\lambda_i = d_X(x, X \setminus B_K(x_i))$. If $d_X(x, y) \leq K$, then

$$d_P(\psi(x), \psi(y)) = |\Sigma(d_X(x, X \setminus B_K(x_i)) - d_X(y, X \setminus B_K(x_i)))f(x_i)| \leq \Sigma d_X(x, y)|f(x_i)|$$
for some 1-Lipschitz real-valued function \( f \) with \( f(x) = 0 \). Hence \( |f(x_i)| \leq d_X(x_i, x) \leq 3K \). Then \( d_P(\psi(x), \psi(y)) = 2C(B_K)d_X(x, y)(3K) = Ld_X(x, y) \) where \( L = 6kC(B_K) \) and \( C(B_K) \) is the capacity of a \( K \)-ball in \( \Gamma \). Thus, the map \( \psi \) is locally \( L \)-Lipschitz. Since \( X \) is a geodesic metric space, \( \psi \) is \( L \)-Lipschitz.

We define \( \alpha(x, z) = r \circ p \circ q \circ (\psi \times 1_P) \) where \( r : \mathbb{R}^N \to \mathbb{R}^N \) is a rescaling that makes \( \alpha \) 1-Lipschitz on \( X \times X \), \( x, x' \in P \).

A variation of a continuous function \( f : X \to \mathbb{R} \) on a metric space \( X \) at point \( x \in X \) in an \( R \)-ball is defined as follows

\[
\text{var}_R(f)(x) = \max\{|f(x) - f(x')| \mid x' \in B_R(x)\}.
\]

We recall that the Higson corona \( \nu X \) of a proper metric space \( X \) is the corona \( \bar{X} \setminus X \) of the compactification of \( X \) defined by the ring of bounded functions with the property \( \lim_{\text{dist}(x, x_0) \to \infty} \text{var}_R(f)(x) = 0 \) for every \( R \) and fixed (every) \( x_0 \in X \).

Suppose that a finitely generated group \( \Gamma \) acts properly on a locally compact space \( X \). We define a \( \Gamma \)-variation of a continuous function \( f : X \to \mathbb{R} \) at point \( x \in X \) in an \( R \)-ball as follows

\[
\Gamma - \text{var}_R(f)(x) = \max\{|f(x) - f(x')| \mid x' = \gamma x, \|\gamma\| \leq R\}.
\]

The algebra of bounded continuous functions \( f \) on \( X \) with the property that for every \( R > 0 \) the \( \Gamma \)-variation \( \Gamma - \text{var}_R(f)(x) \) tends to zero as \( x \) approaches infinity defines a compactification \( \bar{X} \) of \( X \) which we will call the Hurder compactification (cf. [Hu]). The corona \( \partial \Gamma X = \bar{X} \setminus X \) will be called the Hurder corona.

Let \( \Gamma \times \mathbb{R} \) be the compactification of \( \Gamma \times \mathbb{R} \) by the suspension \( \Sigma(\nu \Gamma) \) of the Higson corona. When \( X/\Gamma \) is compact the space \( \Gamma \times \mathbb{R} \times \Gamma X = X \times \mathbb{R} \) with the corona \( \Sigma(\nu \Gamma) \times \Gamma \).

**Proposition 4.** Suppose that the action of a finitely generated group \( \Gamma \) on a space \( X \) is free. Then

1. The Hurder compactification of \( X \) is the maximal \( \Gamma \)-equivariant compactification for which the corona is the fixed point set;
2. \( \partial \Gamma X = \nu \Gamma \times \Gamma X \), provided the action is cocompact;
3. There is a map of coronas \( \partial \Gamma (X \times \mathbb{R}) \to \Sigma(\nu \Gamma) \times \Gamma X \) which extends continuously to \( X \times \mathbb{R} \) by the identity map.

**Proof.**

(1) Clearly the action of \( \Gamma \) on the Hurder corona is trivial. It remains to show that every bounded function \( f : X \to \mathbb{R} \) satisfying the property \( f - f \circ \gamma \in C_0(X) \) for all
\(\gamma \in \Gamma\) belongs to the Hurder algebra, i.e. \(\lim_{x \to \infty} \Gamma - \text{var}_R(f)(x) = 0\). The later follows from the fact that \(\lim_{x \to \infty}(f(x) - f(\gamma x)) = 0\) and the faintness of the \(R\)-ball \(B_R(e)\) in \(\Gamma\).

(2) The proof is contained in [Hu], Lemma 3.1. We add one comment here. This corona is obtained by taking the Higson corona of every orbit \(\Gamma x, x \in X\), supplied by the right invariant word metric \(\rho\). Then it fibered over \(M\) with the fiber \(\nu \Gamma\) as it stated in (2). We note the right invariant metric on \(\Gamma\) is defined as \(d\), and for every orbit \(\Gamma x\) the formula \(\rho(\gamma' x, \gamma x) = d^{R}(\gamma', \gamma)\) defines a metric on \(\Gamma x\) together with an isometry to \((\Gamma, d^{R})\). As a metric space \((\Gamma, d^{R})\) is isometric to \(\Gamma\) with the left invariant metric by taking the inverse of the elements. Hence the Higson corona of every orbit \(\Gamma x\) is just \(\nu \Gamma = \nu X\). We note that the left action of every \(\gamma \in \Gamma\) defines a translation on \(\Gamma x\) bounded by \(\|\gamma\|_S\) for every \(x \in X\). This makes the action of \(\Gamma\) on the corona \(\nu(\Gamma) \times_{\Gamma} X\) trivial.

(3) We consider a compactification of \(X \times \mathbb{R}\) by taking the Higson corona of \((\Gamma x, d^{R}) \times \mathbb{R}\) for every orbit \(\Gamma x\). It is easy to check that the corona of this compactification is \(\nu(\Gamma \times \mathbb{R}) \times_{\Gamma} X\). The action of \(\Gamma\) on this corona is trivial by the reason given in (2). By (1) there is a domination \(\beta: \partial_\Gamma(X \times \mathbb{R}) \to \nu(\Gamma \times \mathbb{R}) \times_{\Gamma} X\). The obvious domination \(\nu(\Gamma \times \mathbb{Z}) \to \Sigma(\nu \Gamma)\) defines a map \(\nu(\Gamma \times \mathbb{Z}) \times_{\Gamma} X \to \Sigma(\nu \Gamma) \times_{\Gamma} X\) which together with \(\beta\) gives us a required map. \(\Box\)

**Proposition 5.** Let \(\phi: \partial_\Gamma X \to S^{n-1}\) be an essential map of the Hurder corona of a contractible \(n\)-manifold \(X\) supplied with a \(\Gamma\)-action such that \(\delta(\{\phi\}) \neq 0\), \(\delta: \mathcal{H}^{n-1}(\partial_\Gamma X) \to H^n_\mathcal{R}(X)\). Then there exists an essential map \(p: X \to \mathbb{R}^n\) satisfying the Displacement Bound Condition.

**Proof.** There is an extension \(\tilde{\phi}: W \to S^{n-1}\) of \(\phi\) to an open neighborhood of \(\partial_\Gamma X \subset W \subset \bar{X}\). Let \(r_0 > 0\) be such that \(X \setminus W \subset B_{r_0}(x_0)\) for some \(x_0 \in X\). Let \(S = S^{-1}\) be a finite generating set of \(\Gamma\). We define

\[\nu(t) = \max\{\rho(\tilde{\phi}(x), \tilde{\phi}(sx)) \mid s \in S, \|x\| \geq t\}\]

where \(\rho\) is the standard metric on the unit sphere and \(\|x\| = d_X(x, x_0)\). Since \(\lim_{\|x\| \to \infty} \Gamma - \text{var}_1(\tilde{\phi})(x) = 0\), we have \(\lim_{t \to \infty} \nu(t) = 0\). There exists a smooth function \(f: \mathbb{R}_+ \to \mathbb{R}_+\) such that \(f(t) = 0\) for \(t \leq r_0\), \(f(t) \leq 1/\nu(t)\) and \(f'(t) < 1\) for all \(t\). We define \(p: X \to \mathbb{R}^n\) in polar coordinates as \(p(x) = (1/2f(\|x\|), \tilde{\phi}(x))\). Note that \(\|x\| + 1 \geq \|sx\| \geq \|x\| - 1\) for every \(s \in S\). Then \(\|p(x) - p(sx)\| \leq 1/2f(\|sx\|) - f(\|x\|) + 1/2f(\|x\|)\sin(\rho(\tilde{\phi}(x), \tilde{\phi}(sx)) \leq 1/2(1 + f(\|x\|)\nu(\|x\|)) \leq 1\).
If \( \| \gamma \|_S = k \) and \( \gamma = s_1 \ldots s_k \), we obtain

\[
\| p(x) - p(\gamma x) \| \leq \| p(x) - p(s_k x) \| + \cdots + \| p(s_2 \ldots p_k x) - p(s_1 \ldots p_k x) \| \leq k = \| \gamma \|_S.
\]

Thus the Displacement Bound Condition is satisfied.

To show that \( p \) is essential we consider a map \( g : X \to B^n \) defined as \( g(x) = p(x)/(1 + p(x)) \). The map \( g \) admits a continuous extension \( \bar{g} \) to the Hurder corona by means of \( \phi \). Then the diagram

\[
\begin{array}{ccc}
\check{H}^{n-1}(\partial \Gamma X) & \xrightarrow{\delta} & \check{H}^{n}(\check{X}, \partial \Gamma X) \\
\phi^* \uparrow & & \bar{g}^* \uparrow \\
H^{n-1}(S^n) & \xrightarrow{\delta} & H^{n}(B^n, S^{n-1})
\end{array}
\]

implies that \( g \) is essential. Therefore \( p \) is essential. \( \square \)

We recall that \( \beta Y \) is the standard notation for the Stone-Čech compactification of \( Y \). We denote by \( \check{\beta}Y = \beta Y \setminus Y \) the Stone-Čech corona.

**Theorem 1.** The following conditions for an \( n \)-dimensional closed orientable aspherical manifold \( M \) with the fundamental group \( \Gamma = \pi_1(M) \) and the universal cover \( X \) are equivalent:

1. There exist \( k \) and a \( \Gamma \)-action on \( X \times \mathbb{R}^k \) with an equivariant projection onto \( X \) such that the corona of the Stone-Čech compactification of \( X \times \mathbb{R}^k \) admits a \( \Gamma \)-invariant essential map onto \( n + k - 1 \)-dimensional sphere \( f : \check{\beta}(X \times \mathbb{R}^k) \to S^{n+k-1} \).
2. There exist \( k \) and a \( \Gamma \)-action on \( X \times \mathbb{R}^k \) with an equivariant projection onto \( X \) such that the boundary homomorphism \( \delta : \check{H}^{n+k-1}(\partial \Gamma (X \times \mathbb{R}^k); \mathbb{Q}) \to H^{n+k}_c(X \times \mathbb{R}^k; \mathbb{Q}) \) for the Hurder corona is an epimorphism;
3. There exist a \( \Gamma \)-action on \( X \times \mathbb{R}^k \) for some \( k \) with an equivariant projection onto \( X \) and a proper essential map \( p : X \times \mathbb{R}^k \to \mathbb{R}^{n+k} \) satisfying the Displacement Bound Condition;
4. The manifold \( M \) has the rational cohomology canonically Lipschitz.

**Proof.** (1) \( \Leftrightarrow \) (2). Let \( f \) be as in (1), then it defines a compactification of \( X \times \mathbb{R}^k \) with the corona \( S^{n+k-1} \) and the trivial \( \Gamma \)-action on it. In view of Proposition 4 (1) this compactification is dominated by the Hurder compactification. Hence we have the
following commutative diagram:

\[
\begin{array}{ccc}
\tilde{H}^{n+k-1}(\tilde{\beta}(X \times \mathbb{R}^k)) & \xrightarrow{\delta_1} & H^{n+k}_c(X \times \mathbb{R}^k) \\
\uparrow f_1^* & & \uparrow id \\
\tilde{H}^{n+k-1}(\partial \Gamma(X \times \mathbb{R}^k)) & \xrightarrow{\delta} & H^{n+k}_c(X \times \mathbb{R}^k) \\
\uparrow f_1^* & & \uparrow id \\
\tilde{H}^{n+k-1}(S^{n+k-1}) & \xrightarrow{\delta_2} & H^{n+k}_c(X \times \mathbb{R}^k)
\end{array}
\]

By the Calder-Siegel theorem [CS] $\delta_1$ is an isomorphism for $n + k > 2$. Since the map $f = f_2 \circ f_1$ is essential, $\delta_1 \circ f^*$ is a rational isomorphism. Hence $\delta$ is an epimorphism.

Assume that the condition (2) is satisfied. We may assume that $n + k - 1$ is odd. Then an Eilenberg-Maclane complex $K(\mathbb{Q}, n + k - 1)$ may be regarded as an infinite telescope of $(n + k - 1)$-spheres. Then there exists a map $f_2 : \partial \Gamma(X \times \mathbb{R}^k) \to S^{n+k-1}$ such that $\delta \circ f_2^* \neq 0$. Let $f_1 : \tilde{\beta}(X \times \mathbb{R}^k) \to \partial \Gamma(X \times \mathbb{R}^k)$ be a map that comes from the universality of the Stone-Čech compactification. Then we have the above diagram which together with Calder-Siegel theorem imply that $f = f_2 \circ f_1$ is essential.

(2) $\iff$ (3). If (2) holds for $k$ then it holds for all $k' \geq k$. We take $k$ such that $n + k - 1$ is odd. Then like in the argument for (2) $\iff$ (3) there exists a map $\phi : \partial \Gamma(X \times \mathbb{R}^l) \to S^{n+k-1}$ such that $\delta([\phi]) \neq 0$. Then (3) holds by Proposition 5.

Let $p : X \times \mathbb{R}^k \to \mathbb{R}^{n+k}$ be as in (3). We define a map $f : X \times \mathbb{R}^k \to B^{n+k}$ as $f(x) = p(x)/(1 + ||p(x)||)$. It is easy to show that $f$ satisfies the condition $\lim_{x \to \infty} \Gamma - var_R(f)(x) = 0$. Then $f$ is extendible to the Hurder corona. Then the essentiality of the map $f : (X \times \mathbb{R}^k, \partial \Gamma(X \times \mathbb{R}^k)) \to (B^{n+k}, \partial B^{n+k})$ and the cohomology exact sequence of pair imply (2).

(3) $\iff$ (4). By Proposition 3 there exists $\alpha : X \times P \to \mathbb{R}^{n+k}$ with $P = X \times \mathbb{R}^k$ satisfying (1)-(3). Since $p$ is essential, in view of Proposition 1 $\alpha$ is a Lipschitz representation of $H^*(M)$.

If $\alpha : X \times P \to \mathbb{R}^{n+k}$ gives a Lipschitz representation of $H^*(M)$, the restriction $\alpha' |_{\Gamma \times \alpha \times P}$ is factored through a map $p : \Gamma \times P = P \to \mathbb{R}^{n+k}$, $\alpha' = q \circ p$, which satisfies the Displacement Bound Condition by Proposition 2. □

**DEFINITION.** *We say that the universal cover $X$ of a closed aspherical manifold $M$ with the metric induced from $M$ satisfies the condition CP if it admits a rationally acyclic $\pi_1(M)$-equivariant Higson dominated compactification* [CP], [D].
**Theorem 2.** Suppose that the universal cover $X$ of a closed orientable aspherical $n$-manifold $M$ satisfies the conditions CP. Then the cohomology groups $H^*(M; \mathbb{Q})$ are canonically Lipschitz.

*Proof.* Let $Y$ be the above compactification of $X$ and let $Z = Y \setminus X$. Since $H^*(Y; \mathbb{Q}) = 0$, we have $H^*(Z; \mathbb{Q}) = H^*(S^{n-1}; \mathbb{Q})$ for the reduced Čech cohomology. Moreover, since the action of $\Gamma = \pi_1(M)$ is trivial on $H^*(X; \mathbb{Q})$ and $\delta : H^*(Z; \mathbb{Q}) \to H^{*+1}(X; \mathbb{Q})$ is an equivariant isomorphism, the action of $\Gamma$ on $H^*(Z; \mathbb{Q})$ is trivial.

Let $\Sigma Z$ be the suspension. We consider a locally trivial fibration $\pi : \Sigma Z \times_{\Gamma} X = S \to M$ with the fiber $\Sigma Z$ induced by the projection $pr : \Sigma Z \times X \to X$. The term $E_2^{0,n}$ in the spectral sequence of this fibration (with local coefficients) is isomorphic to $H^n(\Sigma Z; \mathbb{Q}) = \mathbb{Q}$ since the action of $\Gamma$ is trivial on $H^n(\Sigma Z; \mathbb{Q})$. By dimensional reason the term $E_2^{0,n}$ is not transgressive. Therefore the inclusion of the fiber $\Sigma Z \subset S$ induces an epimorphism of $n$-dimensional rational cohomology groups.

The space $S$ is contained in the space $\tilde{E} = \tilde{X} \times \mathbb{R} \times_{\Gamma} X$ where $\tilde{X} \times \mathbb{R}$ is the natural compactification of $X \times \mathbb{R}$ by $Z \ast S^0 = \Sigma Z$. The exact sequence of pair and the inclusion $(\tilde{X} \times \mathbb{R}, \Sigma Z) \subset (\tilde{X} \times \mathbb{R} \times_{\Gamma} X, S)$ defines the commutative diagram for rational cohomology:

$$
\begin{array}{ccc}
\tilde{H}^n(S) & \xrightarrow{\delta} & H_c^{n+1}((X \times \mathbb{R}) \times_{\Gamma} X) \\
\downarrow{i^*} & & \downarrow{j^*} \\
\tilde{H}^n(\Sigma Z) & \xrightarrow{\delta_1} & H_c^{n+1}(X \times \mathbb{R}).
\end{array}
$$

We proved that there is $a \in \tilde{H}^n(S)$ such that $i^*(a)$ is a generator of $\tilde{H}^n(\Sigma Z) = \mathbb{Q}$. Then $\delta(a)$ is a generator of $H_c^{n+1}((X \times \mathbb{R}) \times_{\Gamma} X) = \mathbb{Q}$. The later equality follows from the fact that $(X \times \mathbb{R}) \times_{\Gamma} X$ is the total space of a topological orientable $n+1$-dimensional vector bundle over $M$. Then $\delta(a)$ is the rational Thom class for two $X \times \mathbb{R}$-bundles over $M$ generated by the projections of $X \times \mathbb{R} \times X$ onto both factors $X$. Therefore the restriction of $\delta(a)$ onto any fiber of the first bundle is nontrivial. Let $x_0 \in M$, then the fiber for the first bundle at $x_0$ can be presented as $X \times \mathbb{R} = \Gamma x_0 \times_{\Gamma} (\mathbb{R} \times X) \subset \tilde{X} \times \mathbb{R} \times_{\Gamma} X$.

We compactify $X \times \mathbb{R} \times_{\Gamma} X$ by $\Sigma \nu X \times_{\Gamma} X$. This also gives a compactification with the above corona of a fiber from the first bundle $X \times \mathbb{R} = \Gamma x_0 \times_{\Gamma} \mathbb{R} \times X$. It is easy to check that the action of $\Gamma$ on $X \times \mathbb{R}$ give rise to a trivial action on the corona. In view of proposition 4 the above compactification is Higson dominated.

Since $Z$ is Higson dominated, there is a map of the Higson corona $\xi : \nu X = \nu \Gamma \to Z$ that extends to $X$ continuously by the identity map. This map generates a map $\xi : \Sigma \nu \Gamma \times_{\Gamma} X \to \Sigma Z \times_{\Gamma} X = S$ fixed on $X \times \mathbb{R}$. By Proposition 4 there is a map
\[ \xi_1 : \partial_t (X \times \mathbb{R}) \to \Sigma \nu \Gamma \times \Gamma X \] that extends continuously to \( X \times \mathbb{R} \) by the identity map.

The following diagram

\[
\begin{array}{ccc}
\check{H}^n(S) & \xrightarrow{\delta'} & H^{n+1}_c(X \times \mathbb{R}) \\
(\xi_1)^* \downarrow & & \downarrow \text{id} \\
\check{H}^n(\partial_t (X \times \mathbb{R})) & \xrightarrow{\delta} & H^{n+1}_c(X \times \mathbb{R}).
\end{array}
\]

and the fact that \( \delta' \) is an epimorphism implies that \( \delta \) is an epimorphism and hence the condition (2) of Theorem 1 is satisfied. □

**Lemma 1.** Let \( X \) be the universal cover \( X \) of a closed aspherical Riemannian \( n \)-dimensional manifold \( M \) whose rational cohomologies are canonically Lipschitz and let \( f : X \times P \to \mathbb{R}^N, \) \( P = X \times \mathbb{R}^k, \) be their Lipschitz presentation. Then there exists a function \( c(r) \) such that \( \text{diam}(f^{-1}(B_r(0)) \cap X \times v) \leq c(r) \) for all \( v \in P \) where \( B_r(0) \subset \mathbb{R}^N \) is the \( r \)-ball centered at the origin.

**Proof.** Let \( \bar{q} : X \times P \to X \times \Gamma P \) be the orbit map, then \( f = \hat{f} \circ \bar{q} \) where \( \hat{f} \) is a proper map. Hence \( Y = \hat{f}^{-1}(B_r(0)) \) is compact. Let \( \mathcal{U} \) be a finite cover of \( M \) by evenly covered open maps with respect to the universal cover \( q : X \to M. \) We can subdivide \( Y = \bigcup_{U \in \mathcal{U}} Y_U \) of compact sets such that \( Y_U \subset \pi^{-1}(U) \) where \( \pi : X \times \Gamma P \to M \) is a fibration induced by the projection of \( X \times P \) onto the first factor. Let \( \pi_2 : X \times \Gamma X \to M \) be a fibration induced by the projection \( pr_2 : X \times X \to X \) onto the second factor and let \( h_U : \pi_2^{-1}(U) \to X \) be a trivialization defined by means of a lift \( s_U \) of \( U \) to \( X. \) Let \( pr' : X \times \Gamma P \to X \times \Gamma X \) be a map induced by the projection \( pr : P \to X. \) Let \( d' \) be a metric on \( M, \) We consider the product metric \( \bar{d} \) on \( U \times X \) and define

\[ c(r) = \sum_{U \in \mathcal{U}} \text{diam}_{\bar{d}}(h_U \times 1_U)(pr'Y_U). \]

Then

\[ \text{diam}_{d_X} f^{-1}(B_r(0)) \cap (X \times v) = \text{diam}_{d_X} Y \cap \pi^{-1}(q(pr(v))) \leq \text{diam}_{d_X} pr'(Y) \cap \pi_2^{-1}(q(pr(v))) \leq \sum_{U \in \mathcal{U}} \text{diam}_{d_X} pr'(Y_U) \cap \pi_2^{-1}(q(pr(v))) \leq \sum_{U \in \mathcal{U}} \text{diam}_{\bar{d}}(h_U \times 1_U)(pr'(Y_U)) = c(r). \]

□
§ 3 Expanders

Let $X$ be a finite graph, we denote by $V$ the set of vertices and by $E$ the set of edges in $X$. We will identify the graph $X$ with its set of vertices $V$. Every graph is a metric space with respect to the natural metric where every edge has the length one. For a subset $A \subset X$ we define the boundary $\partial A = \{ x \in X \mid \text{dist}(x, A) = 1 \}$. Let $|A|$ denote the cardinality of $A$.

**Definition [Lu].** An expander with a conductance number $c$ and the degree $d$ is an infinite sequence of finite graphs $\{ X_n \}$ with the degree $d$ such that $|X_n|$ tends to infinity and for every $A \subset X_n$ with $|A| \leq |X_n|/2$ there is the inequality $|\partial A| \geq c|A|$.

Let $X$ be a finite graph, we denote by $P$ all nonordered pairs of distinct points in $X$. For every nonconstant map $f : X \to l_2$ to the Hilbert space we introduce the number

$$D_f = \frac{1}{|P|} \sum_{\{x,y\} \in P} \| f(x) - f(y) \|^2$$

$$- \frac{1}{|E|} \sum_{\{x,y\} \in E} \| f(x) - f(y) \|^2.$$ 

If $X$ is a graph with the degree $d$ and with $|X| = n$, then $|P| = n(n-1)/2$ and $|E| = dn/2$. The following lemma can be derived from [Ma, Proposition 3]. It also can be obtain from the equality

$$\lambda_1(X) = \inf \{ \frac{\| df \|^2}{\| f \|^2} \mid \sum f(x) = 0 \}$$

for the first positive eigenvalue of the Laplacian on $X$ and the Cheeger’s inequality (see Proposition 4.2.3 in [Lu]).

**Lemma 2.** Let $\{ X_n \}$ be an expander. Then there is a constant $c_0$ such that $D_{f_n} \leq c_0$ for all $n$ for all possible maps $f_n : X_n \to l_2$ to the Hilbert space $l_2$.

**Corollary 1.** For every sequence of 1-Lipschitz maps $f_n : X_n \to l_2$ there is the inequality

$$\frac{1}{|P_n|} \sum_{P_n} \| f_n(x) - f_n(y) \|^2 \leq c_0$$

for every $n$.

**Proof.** In the case of 1-Lipschitz map we have

$$\frac{1}{|E|} \sum_{\{x,y\} \in E} \| f(x) - f(y) \|^2 \leq 1.$$ 

Then the required inequality follows. □
**Corollary 2.** Assume that for a sequence of 1-Lipschitz maps $f_n : X_n \to l_2$ we have $\sum_{x \in X_n} f_n(x) = 0$ for every $n$, then

$$\frac{1}{|X_n|} \sum_{x \in X_n} \|f_n(x)\|^2 \leq c_0/2$$

for all $n$.

**Proof.** We assume that the sequence $\{X_n\}$ is enumerated by a subsequence on $\mathbb{N}$ such that $|X_n| = n$.

According to Corollary 1 we have

$$n(n - 1)c_0/2 = \sum_{P_n} \|f_n(x) - f_n(y)\|^2 = \sum_{P_n} \|f_n(x) - f_n(y)\|^2 + \sum_{X_n} (\sum f(x_n))^2 =$$

$$\sum_{P_n} (f_n(x) - f_n(y))^2 + \sum_{X_n} f_n(x)^2 + \sum_{P_n} 2(f_n(x), f_n(y)) = |X_n| \sum_{X_n} f_n(x)^2 = n \sum_{X_n} \|f_n(x)\|^2.$$

Then the required inequality follows. □

**Corollary 3.** Assume that for a sequence of 1-Lipschitz maps $f_n : X_n \to l_2$ we have $\sum_{x \in X_n} f_n(x) = 0$ for every $n$, then there is $R$ such that $|f_n^{-1}(B_R(0))| > |X_n|/2$ for all $n$.

**Proof.** Take $R > \sqrt{c_0}$. Assume that $|f_n^{-1}(B_R(0))| \leq |X_n|/2$.

By Corollary 2 we have the contradiction:

$$\frac{c_0}{2} |X_n| \geq \sum_{x \in X_n} \|f(x)\|^2 \geq R^2 |X_n|/2 > \frac{c_0}{2} |X_n|.$$

□

Let $\{X_n\}$ be an expander and let $X$ be a metric space. We call a sequence of 1-Lipschitz maps $\xi_n : X_n \to X$ a coarse quasi-embedding of an expander $\{X_n\}$ into $X$ if there is a positive function $\rho : \mathbb{R}_+^2 \to \mathbb{R}_+$ such that

$$\lim_{t \to \infty} \frac{\rho(r, t)}{t} = 0$$
for every $r$, and
\[ |\xi^{-1}_n(B_r(x))| < \rho(r, |X_n|) \]
for all $n \in \mathbb{N}$ and for all $x \in X$.

When $X$ is a space of bounded geometry, for instance $X$ is a finitely generated group with the word metric, then this condition is equivalent to the following condition defined in [HLS]
\[ \lim_{n \to \infty} \frac{|\xi^{-1}_n(x)|}{|X_n|} = 0. \]

**Lemma 3.** Suppose a metric space $(X,d)$ has a bounded geometry. Let $f : X \times \mathbb{R}^N \to \mathbb{R}^N$ be a map satisfying the conditions:

1. the restriction $f |_{X \times w} : X \times w \to \mathbb{R}^N$ is $1$-Lipschitz for all $w \in \mathbb{R}^N$;
2. the restriction $f |_{x \times \mathbb{R}^N} : x \times \mathbb{R}^N \to \mathbb{R}^N$ is essential for all $x$;
3. there is a function $c(r)$ such that $\text{diam}_d(f^{-1}(B_r(0)) \cap (X \times w)) \leq c(r)$ for all $w \in \mathbb{R}^N$.

Then no expander is coarsely quasi-embeddable in $X$.

**Proof.**

Assume that $\{\xi_n : X_n \to X\}$ is a coarse quasi-embedding of an expander $\{X_n\}$ into $X$ and let $\rho$ be a corresponding function.

Given $n$ we show that there is $w_n \in \mathbb{R}^N$ such that
\[ \sum_{v \in X_n} f(\xi_n(v), w_n) = 0. \]

We consider a map $F : \mathbb{R}^N \to \mathbb{R}^N$ defined as $F(y) = \frac{1}{|X_n|} \sum_{v \in X_n} f(\xi_n(v), y)$. The condition (1) implies $\|f(\xi_n(v), y) - f(x_0, y)\| \leq d(\xi_n(v), x_0)$. Hence
\[ \|F(y) - f(x_0, y)\| = \frac{1}{|X_n|} \| \sum_{v \in X_n} (f(\xi_n(v), y) - f(x_0, y)) \| \leq \frac{1}{|X_n|} \sum_{v \in X_n} \| (f(\xi_n(v), y) - f(x_0, y)) \| \leq d(\xi_n(x_0), x_0), \]
where $d(\xi_n(x_0), x_0) = \max\{d(\xi(v), x_0) \mid v \in X_n\}$. Thus the maps $F$ and $f |_{x_0 \times \mathbb{R}^N}$ are in a finite distance.
Therefore they are properly homotopic. Then by the condition (2) the map $F$ is essential. Hence there is $w_n$ with $F(w_n) = 0$. Let $\epsilon_n : X \to X \times w_n$ be a natural identification.

We apply Corollary 3 to the sequence of maps

$$(f \mid_{X \times w_n}) \circ \epsilon_n \circ \xi_n : X_n \to \mathbb{R}^N$$

to obtain a number $R > 0$ with the property

$$|((f \mid_{X \times w_n}) \circ \epsilon_n \circ \xi_n)^{-1}(B_R(0))| \geq |X_n| / 2$$

for all $n$. By the condition (3) the preimage $(f \mid_{X \times w_n})^{-1}(B_R(0))$ is contained in a ball $B_{c(R)}(x_n)$ in $X \times w_n$ of radius $c(R)$ centered at some point $x_n$.

Then

$$|\xi_n^{-1} \epsilon_n^{-1}(B_{c(R)}(x_n))| \geq |X_n| / 2.$$  

Thus,

$$|X_n| / 2 \leq |\xi_n^{-1}(B_{c(R)}(x_n))| < \rho(c(R), |X_n|)$$

for all $n$. This contradicts with the condition $\lim_{t \to \infty} \frac{\rho(c(R), t)}{t} = 0$. □

Proposition 1, Lemma 1 and Lemma 3 imply the following.

**Theorem 3.** Suppose that a closed orientable aspherical manifold $M$ has canonically Lipschitz cohomology. Then no expander admits a coarse quasi-embedding into its fundamental group.

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