3D mirror symmetry from S-duality

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We consider type IIB SL(2, Z) symmetry to relate the partition functions of different 5d supersymmetric Abelian linear quiver Yang-Mills theories in the Ω-background and squashed S^5 background. By Higgsing S-dual theories, we extract new and old 3d mirror pairs. Generically, the Higgsing procedure yields 3d defects on intersecting spaces, and we derive new hyperbolic integral identities expressing the equivalence of the squashed S^3 partition functions with additional degrees of freedom on the S^1 intersection.

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I. INTRODUCTION

One of the most beautiful features in the family of 3d gauge theories with $\mathcal{N} = 4$ supersymmetry is the existence of mirror symmetry [1]. When 3d supersymmetric gauge theories admit brane constructions through D3 branes suspended between $(p, q)$ branes [2–6], mirror symmetry can be understood from the SL(2, Z) symmetry of type IIB string theory. From the QFT perspective, mirror symmetry is deeply related to $S$-duality of the boundary conditions in 4d $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) [7], and for Abelian theories it can also be traced back to the existence of a natural SL(2, Z) action on path integrals (functional Fourier transform) [8,9]. For non-Abelian theories, this action can be implemented at the level of localized partition functions [10,11]. Moreover, the class of 3d $\mathcal{N} = 4$ theories can be deformed in many interesting ways to $\mathcal{N} = 2$, such as the inclusion of masses, Fayet-Iliopoulos (FI) parameters for Abelian factors in the gauge group, or superpotential terms. While the reduced supersymmetry implies a weaker control over the dynamics, mirror-like dualities are known to exist for a long time [12–14]. Lately, this has been a very active research field, and significant progress is made possible thanks to the careful analysis of (monopole) superpotentials [15–22]. In many cases, the IR equivalence of proposed dual pairs has been tested using the exact evaluation of supersymmetric observables through localization, such as the (squashed) $S^3$ partition function [23,24]. In fact, over the past few years, the results of supersymmetric localization (see, e.g., [25] for a review) have been systematically exploited to predict and test dual pairs.

In this paper, we continue the study of 3d dualities inherited from the SL(2, Z) symmetry of type IIB string theory. Our strategy is to consider first 5d $\mathcal{N} = 1$ SYM theories with unitary gauge groups engineered by $(p, q)$-webs in type IIB string theory in which the SL(2, Z) action can be manifestly realized, for instance, through the exchange of D5 and NS5 branes (a.k.a. the fiber-base or S-duality [26,27]). Second, we engineer codimension 2 defects of the parent 5d theories by the Higgsing procedure [28,29], and in simple configurations we can identify candidate 3d mirror pairs (this is the perspective also adopted in [30–32]). In order to be able to explicitly test their IR equivalence through the exact evaluation and comparison of the partition functions, we focus on 5d Abelian linear quivers in which the instanton corrections can be easily resummed [33]. In fact, the fiber-base dual picture of such theories provides a very simple duality frame for the resulting 3d theories, which look free. Our reference example is 5d SQED with one fundamental and one anti-fundamental flavors and its fiber-base dual. From this very simple example, we can already extract nontrivial dualities for 3d non-Abelian theories. One of our main results is indeed a non-Abelian version of the basic SQED/XYZ duality. Remarkably, this duality has implicitly appeared in [34] (at the level of the squashed $S^3$ partition function) as an intermediate step to test the mirror dual of $(A_1, A_{2n-1})$ Argyres-Douglas (AD) theories reduced to 3d, which has been shown to follow from an involved cascade of sequential confinement and mirror symmetry [20,21] starting from the 3d reduction of the 4d “Lagrangian” description [35,37]. Here, we provide a first principle derivation of this crucial bridge from the 5d physics viewpoint.

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Another motivation for this paper comes from the recent studies of supersymmetric gauge theories on intersecting spaces [38–44]. In our case, we are interested in pairs of 3d theories supported on two codimension 2 orthogonal spaces in the ambient 5d space (which we take to be either the $\Omega$-background $\mathbb{C}^2_{q,t^{-1}} \times S^1$ or the squashed $S^5$ [45–55]), interacting along a common codimension 4 locus ($S^1$) where additional degrees of freedom live. A natural question is whether 3d mirror symmetry survives in these more complicated configurations. Remarkably, we are able to generalize known dualities to this more refined setup too by studying the relevant compact and noncompact space partition functions using the integral identities descending from the fiber-base duality, up to some subtleties inherent to intersecting theories [36].

The rest of the paper is organized as follows. In Sec. II, we review instanton partition functions of 5d Abelian linear quiver theories on $\mathbb{C}^2_{q,t^{-1}} \times S^1$ through the refined topological vertex, exploiting their $(p,q)$-web realization in type IIB string theory or M-theory on toric Calabi-Yau 3-folds [4–6,56–58] allows us to perform the various computations using the topological vertex formalism [59–62]. In this paper, we mainly follow the conventions of [63], summarized in Appendix D. In a nutshell, in any toric diagram there is a frame in which one associates internal white arrows which point in the same (preferred/instanton) direction and correspond to unitary gauge groups, with the ranks determined by their number in each segment (one in this paper); consecutive gauge groups are coupled through bi-fundamental hypers, while non-compact white arrows correspond to (anti)fundamental hypers.

Our reference examples are the diagrams listed in Fig. 1. By explicit computation, it is easy to verify that the associated topological amplitudes correspond respectively to the instanton partition functions of: (i) the $U(1)$ theory with one fundamental and one antifundamental hypers (SQED); (ii) the theory of four free hypers and "resummed instantons," which will be simply referred to as the "free theory"; (iii) the $U(1) \times U(1)$ theory with one bifundamental hyper. However, we show that generic Higgsings produce 3d/1d coupled theories which live on distinct codimension 2 subspaces mutually intersecting along codimension 4 loci, and we generalize and test the dualities in these cases too. In Sec. IV, we discuss further our results and outline possible applications and extensions for future research. In Appendix A, we collect the definitions of the special functions which we use throughout the paper. In Appendix B and C, we present few technical definitions and derivations. In Appendix D, we collect useful information and notation of the refined topological vertex.

II. 5D INSTANTON PARTITION FUNCTIONS

In this section, we review the instanton partition functions of 5d Abelian linear quiver theories with unitary gauge groups in the $\Omega$-background, usually denoted by $\mathbb{C}^2_{q,t^{-1}} \times S^1$. The geometric engineering of these theories through $(p,q)$-webs in type IIB string theory or M-theory on toric Calabi-Yau 3-folds [4–6,56–58] allows us to perform the various computations using the topological vertex formalism [59–62]. In this paper, we mainly follow the conventions of [63], summarized in Appendix D. In a nutshell, in any toric diagram there is a frame in which one associates internal white arrows which point in the same (preferred/instanton) direction and correspond to unitary gauge groups, with the ranks determined by their number in each segment (one in this paper); consecutive gauge groups are coupled through bi-fundamental hypers, while non-compact white arrows correspond to (anti)fundamental hypers.

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where \( p \equiv q t^{-1} \). The prefactor in front of the instanton sum can be identified with the perturbative or 1-loop contribution from the fundamental hypermultiplets. We refer to Appendix A for the definition of \( q \)-Pochhammer symbols and Nekrasov’s function.

The second diagram, if read naively corresponding to the four free hypermultiplets, has amplitude given by

\[
Z_2 = \frac{Z_{\text{Resummed}}(Q_0, Q_1, Q_2)}{\prod_{i=1}^2 (Q_i p^{1/2}; q, t^{-1})_\infty (Q_0 p^{1/2}; q, t^{-1})_\infty (Q_0 Q_2 p^{1/2}; q, t^{-1})_\infty}.
\]

We note that the factors in the bracket captures the contribution from four free hypermultiplets, however, the resummation of instantons has also produced a factor

\[
Z_{\text{Resummed}}(Q_0, Q_1, Q_2) = (Q_0 Q_1; t^{-1})_\infty (Q_0 Q_2 p; q, t^{-1})_\infty,
\]

in the numerator, which does not have straightforward interpretation in terms of conventional 5d supersymmetric matter content. Here, we simply take the expression (2.2) as a computational result. We refer the readers to [64,65] for more detail on these nonconventional matter, which are termed “non-full spin content.”

Finally, the third diagram, corresponding to the \( U(1) \times U(1) \) theory, has amplitude

\[
Z_3 = \left[ \frac{1}{(Q_0 p^{1/2}; q, t^{-1})_\infty} \right] \sum_{|\alpha|,|\beta|} (p^{-1/2} Q_1)^{|\alpha|} (p^{-1/2} Q_2)^{|\beta|} \frac{N_{\alpha,\beta}(Q_0 p^{1/2}; q, t^{-1})}{N_{\alpha,\beta}(1; q, t^{-1})}.
\]

The prefactor in front of the instanton sum can be identified with the perturbative contribution of the bi-fundamental hyper.

The above computation can be generalized to more complicated toric diagrams. For instance, a strip of 2\( N \) vertices can be associated to three QFT frames, corresponding respectively to: (i) the \( U(1)^{N-1} \) theory coupled to \( N - 2 \) bifundamentals, one fundamental at first node and one anti-fundamental at last node; (ii) the theory of \( 2N \) free hypers and “resummed instantons”; (iii) the \( U(1)^N \) theory coupled to \( N - 1 \) bifundamentals. A similar triality relation among distinct gauge theories has been recently obtained also in 6d [66,67].

A. Duality frames

The three configurations in Fig. 1 share the same toric diagram. In fact, they all give equivalent amplitudes. Let us start by focusing on the first two diagrams in Fig. 1. They can be understood as two different \((p, q)\)-webs related by S-duality in type IIB string theory, under which D5 and NS5 branes are exchanged. Upon a clockwise rotation by 90 degrees, the S-duality is represented by Figure 2.

FIG. 2. S-duality between the two \((p, q)\)-webs.
Since D5s correspond to horizontal (1,0) branes, NS5s correspond to vertical (0,1) branes and diagonal segments correspond to (1,1) branes, the duality map is indeed represented by the $S$ element in $\text{SL}(2,\mathbb{Z})$ acting on the $(p,q)$ charge vectors. In this particularly simple example, we can explicitly check the invariance of the amplitude. We can expand $Z_1$ and $Z_2$ in series of $Q_0$, and both $Z_1$ and $Z_2$ equal

$$Z_1 = \prod_{j=1}^{2} \frac{1}{(Q_jp^{1/2}; q, t^{-1})} \left[ 1 + \frac{Q_0}{(q-1)(t-1)} (qQ_2 + tQ_1 - (qt)^{1/2}(1 + Q_1Q_2)) + \cdots \right]$$

$$= \prod_{j=1}^{2} \frac{1}{(Q_jp^{1/2}; q, t^{-1})} \left[ 1 - \frac{(q^{1/2}Q_2 - t^{1/2})(t^{1/2}Q_1 - q^{1/2})}{(1-q)(1-t)} Q_0 + \cdots \right] = Z_2,$$

(2.5)

confirming one of the predictions. More generally, $(p,q)$ webs constructed by gluing vertically $N$ copies of the left diagram in Fig. 2 or constructed by gluing horizontally $N$ copies of the right diagram are $S$-dual to each other and hence give equivalent amplitudes: the former is nothing but the $U(N)$ SQCD, while the latter is a linear $U(2)^N$ quiver with bifundamental hypers between gauge nodes and one fundamental and one antifundamental hypers at each ends.

The third duality frame is related to the first one by a clockwise rotation by 45 degree of the (1,1) branes, which acts as the $STS^{-1}$ element in $\text{SL}(2,\mathbb{Z})$ on the $(p,q)$ charge vector. One can verify that $Z_1 = Z_2 = Z_3$ using the identities of [63].

The map between the Kähler parameters of the string geometry and the physical masses and coupling constants of the gauge theory depends on the duality frame. First of all, it is convenient to introduce exponential variables

$$q \equiv e^{2\pi i \beta_1}, \quad t \equiv e^{-2\pi i \beta_2}, \quad Q_{1,2} \equiv e^{2\pi i \beta_{1,2}}, \quad Q_0 \equiv e^{2\pi i \beta_0},$$

(2.6)

where $\beta$ measures the $\mathbb{S}^1$ radius. In the frame corresponding to the $U(1)$ theory, we can identify

$$a_1 \equiv -i(\Sigma - \tilde{M}), \quad a_2 \equiv i(\Sigma - M), \quad Q_0(Q_1Q_2)^{1/2} = e^{-2\pi i \beta_0}.$$

(2.7)

where $M, \tilde{M}$ are the 5d fundamental and antifundamental masses, $\Sigma$ is the v.e.v. of the vector multiplet scalar and $g$ is the YM coupling.

Similarly, on the $U(1) \times U(1)$ side we can identify

$$a_0 \equiv i(\Sigma_{12} - M_{\text{bif}}), \quad \mathbf{p}^{-1/2}Q_{1,2} = e^{-2\pi i \beta_{1,2}/\sqrt{g_{1,2}}}.$$

(2.8)

where $M_{\text{bif}}$ is the 5d bi-fundamental mass, $\Sigma_{12} \equiv \Sigma_1 - \Sigma_2$ and $\Sigma_{1,2}$ are the v.e.v.s of the vector multiplet scalars and $g_{1,2}$ are the YM couplings.

B. $S^5$ partition functions

In this section, we use the refined topological string/Nekrasov partition functions in the various duality frames to write $S^5$ partition functions related by type IIB $\text{SL}(2,\mathbb{Z})$ transformations. The study of compact space partition functions is useful because one can get rid of subtleties related to boundary conditions, at the price of introducing an integration over some modulus. The round $S^5 \equiv \{ (z_1, z_2, z_3) \in \mathbb{C}^3 | |z_1|^2 + |z_2|^2 + |z_3|^2 = 1 \}$ admits a toric $U(1)^3$ action given by $z_a \rightarrow e^{i\alpha_a}z_a$. Denoting by $e_a$ the corresponding vector fields, the vector $R = e_1 + e_2 + e_3$ is the so-called Reeb vector, and it describes the Hopf fibration $U(1) \rightarrow S^5 \rightarrow \mathbb{C}P^2$. A useful generalization is obtained by replacing the Reeb vector with $R = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3$ ($\omega_i \in \mathbb{R}, \omega_0$), and the resulting manifold is referred to as the squashed $S^5$ and the $\omega$’s as squashing (or equivariant) parameters. We refer to [68] for further details of this geometry.

The partition functions of 5d $\mathcal{N} = 1$ gauge theories on the (squashed) $S^5$ can be computed via localization. In the Coulomb branch localization scheme [51–54] (as opposed to the Higgs branch scheme [38,44,69]), the result is given in terms of a matrix-like integral over the constant vector multiplet scalar in the Cartan subalgebra of the gauge group. It is known that the integrand can be constructed by gluing three Nekrasov partition functions [51,54,68,70,71], one for each fixed point of the toric action on $\mathbb{C}P^2$, with equivariant parameters $\epsilon_{1,2}$ and radius $\beta$ of the $\Omega$-background related to (complexified) squashing parameters. For each of the fixed points labeled by $\alpha = 1, 2, 3$, where the space looks like a copy of $C^2_{q,t,1} \times S^1_{p}$, we can choose

$$\begin{array}{cccc}
\epsilon_1 & \omega_1 + \omega_2 & \omega_2 + \omega_3 & \omega_1 + \omega_3 \\
\epsilon_2 & \omega_3 & \omega_1 & \omega_2 \\
\beta & 1/\omega_1 & 1/\omega_2 & 1/\omega_3
\end{array}$$

(2.9)

On the $U(1)$ theory side (frame 1), the product of Nekrasov partition functions yields
where $\omega = \omega_1 + \omega_2 + \omega_3$ and $| \cdot |^3$ denotes the product of three objects with parameters related by Table (2.9). Notice that the 1-loop contributions have fused into triple Sine functions (and exponential factors) by using the definition (A13).

On the free theory side (frame 2), the product of $q$-Pochhammer symbols yields

\[
|Z_1|^3 = e^{-\frac{1}{2}(B_{33}(-i(\Sigma - M) + \frac{C}{2}) + B_{33}(i(\Sigma - M) + \frac{C}{2}))} \frac{S_3(-i(\Sigma - \hat{M}) + \frac{g}{2})S_3(i(\Sigma - M) + \frac{g}{2})}{S_3(i(\Sigma - M) + \frac{C}{2})} \times |Z_{\text{inst}}^{C^2}(-1)|^3.
\]

(2.10)

and the “Fourier-like transform” of the squashed $S^5$ partition function of free theory by

\[
Z_{\text{free}}^{S_5} = \int d\Sigma Z_{\text{QED}}^{S_5}(g; \Sigma) \cdot Z_{\text{inst}}^{C^2}(-1) \cdot |Z_{\text{inst}}^{C^2}|^3,
\]

(2.13)

As for the $\Omega$-background case, we simply take this result as a computational fact and we do not attempt to give here a gauge theory interpretation, which is not needed for the purposes of this paper.

On the $\text{U}(1) \times \text{U}(1)$ side (frame 3), we can write

\[
|Z_3|^3 = \frac{e^{-\frac{1}{2}(B_{33}(-i(\Sigma - M) + \frac{C}{2}) + B_{33}(i(\Sigma - M) + \frac{C}{2}))}}{S_3(i(\Sigma_{12} - M_{\text{bit}}) + \frac{g}{2})} \times |Z_{\text{inst}}^{C^2}(-1)|^3
\]

(2.17)

and in order to reproduce the squashed $S^5$ partition function we need to bring the exponential factor on the other side and integrate with the classical action, namely

\[\text{III. MIRROR SYMMETRY}\]

In this section, we will follow type IIB $S$-duality acting on 5d gauge theories, and extract mirror dual partition...
functions of 3d gauge theories defined on the squashed $\mathbb{S}^3$ or on the intersecting space $\mathbb{S}^3(1) \cup \mathbb{S}^3(2) \subset \mathbb{S}^5$. The spheres $\mathbb{S}^3(\alpha)$ are submanifolds associated to the equations $z_\alpha = 0$, $\alpha = 1, 2, 3$. We will focus on $\mathbb{S}^3(1)$ and $\mathbb{S}^3(2)$, which clearly intersect transversally along the circle $|z_3| = 1$. We will denote the squashing parameters of $\mathbb{S}^3(1)$ and $\mathbb{S}^3(2)$ by $b_1 = \sqrt{\omega_2 / \omega_3}$ and $b_2 = \sqrt{\omega_1 / \omega_3}$ respectively, and we will set $Q_\alpha = b_\alpha + b_\alpha^{-1}$ as usual. We will review few aspects of gauge theories on this type of geometries in the following, while for further details we refer to [39,40].

A. Higgsing, residues, and mirror symmetry

Higgsing [28,29] a higher dimensional bulk theory is an effective procedure for accessing lower dimensional supersymmetric theories that preserve half (or fewer) the supercharges that the bulk theory enjoys. More precisely, the resulting lower dimensional supersymmetric field theories are worldvolume theories of codimension 2 Bogomol’nyi-Prasad-Sommerfield defects inserted into the bulk theory. The procedure can be more easily described when there is a (flat space) brane construction. If the 5d theory $T$ admits a construction in terms of an array of D5s suspended between parallel NS5s, for example when $T$ is a unitary linear quiver gauge theory, then one type of Higgsing amounts to aligning the outermost flavor D5 with the adjacent gauge D5, and subsequently pulling the in-between NS5 away from the array while stretching a number of D3s. See Fig. 3 for an example. At the level of the compact space partition function, Higgsing $T$ implies taking the residues at certain poles of the partition function as a meromorphic function of mass parameters. In practice, when the compact space partition function is written as a Coulomb branch integral, this is often equivalent to computing the residues of the integrand at a collection of poles of the perturbative determinant as a function of the v.e.v.’s of the scalars in the vector multiplet(s).

Let us consider the partition function of the SQED on the squashed $\mathbb{S}^5$ expressed as an integral as in (2.13). In the following we will focus on the poles of $Z_{\text{SQED}}(\Sigma)$ of the form

$$i(\Sigma - M) + \omega/2 = -n^{(1)}\omega_1 - n^{(2)}\omega_2 - n^{(3)}\omega_3,$$

where $n^{(\alpha)} \in \mathbb{Z}_{\geq 0}$. It is sufficient to study the cases with $n^{(3)} = 0$ as they already demonstrate many core features of more general cases. The cases $n^{(3)} \neq 0$ are a straightforward generalization. As was extensively discussed in [40], the residue of the integrand can be organized into the partition function of a 5d/3d/1d coupled system. Indeed, upon taking the residue, a few things happen which we now summarize (we refer to [40] for a full account, and to Appendix B for the sketch of a slightly different derivation). The non-perturbative factors and the classical factor are simply evaluated at the pole $\Sigma' = M + i\omega/2 + in^{(1)}\omega_1 + in^{(2)}\omega_2$.

Because of the different $\mathbb{S}^1$ periodicities at the fixed points $|z_\alpha| = 1$ labeled by $\alpha = 1, 2, 3$ as in Table 2.9, the $n^{(1)}$ dependence in the first instanton partition function drops out, and it only depends on $n^{(2)}$, and similarly the second depends only on $n^{(1)}$, while the third depends on both. Therefore, among the three instanton partition functions associated to the three fixed points, two simply reduce to the vortex partition functions of two SQCDAs with gauge groups $U(n^{(2)})$ and $U(n^{(1)})$ supported on $(C_{\tau^1} \times \mathbb{S}^1)_{(1)}$ and $(C_{\tau^2} \times \mathbb{S}^1)_{(2)}$ respectively, while the remaining one encodes the vortex partition functions of the two SQCDAs, now supported on $(C_{\tau^1} \times \mathbb{S}^1)_{(3)}$ and $(C_{\tau^2} \times \mathbb{S}^1)_{(3)}$ respectively, and their intricate interaction along the common $\mathbb{S}^1$ at the origin. Schematically, we have the reduction

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2The intersection is transversal from the perspective of the two C’s in the two individual tubular neighborhoods $C \times \mathbb{S}^1 \subset \mathbb{S}^3(1) \cup \mathbb{S}^3(2)$. Put differently, the two complex planes intersect only at the origin.

3We refer to the $U(n)$ SYM theory coupled to $n_f$ fundamental, antifundamental and 1 adjoint chiral multiplets with SQCD.
where the “extra” factors are remnants that will eventually cancel out in the final result. Also, the residue of the 1-loop factors can be simplified to

$$\text{Res}_{\Sigma-\Sigma'} Z_{\text{1-loop}}^+(\Sigma) = Z_{\text{HM}}^+(\hat{M} - M) \times \ldots,$$

where $Z_{\text{HM}}^+(M) \equiv S_4(iM + a/2)^{-1}$ denotes the 1-loop determinant of a free hyper of mass $M$ on the squashed $S^3$, while the dots denote 1-loop determinant factors similar to those which would arise in a Higgs branch localization computation of SQCDAs on each $S^3$ [72,73], plus interaction terms. Because of the form of the $q$, $t$ parameters at each fixed point and the 3d holomorphic block factorization of $S^3$ partition functions [74–76], one can readily understand that the above reduction describes the partition function of the combined system of two SQCD on $S^3(1)$ and $S^3(2)$, interacting through additional degrees of freedom at the common $S^1$.4

To make our life easier when dealing with the defect theories, it is convenient to recast the above Higgs branch-like representation of the partition function sketched above, into a Coulomb branch-like integral, making the structure of the world volume theories manifest. This is possible thanks to the following nontrivial observation: one can reorganize all the (intricated) factors into an elegant matrix integral, namely

**Proposition 1 (residues).**

$$\text{Res}_{\Sigma-\Sigma'} Z_{\text{1-loop}}^+ (\Sigma) \bigg| Z_{\text{inst}}^{\Sigma} (\Sigma) \bigg| Z_{\text{inst}}^{\Sigma} (\Sigma) \bigg| Z_{\text{inst}}^{\Sigma} (\Sigma) \bigg|^3 = Z_{\text{HM}}^+(\hat{M} - M) 
= \int \prod_{a=1}^{2} \prod_{n=1}^{n_{\text{fin}}(a)} \frac{d\sigma_{a}(n)}{2\pi i n_{\text{fin}}(a)!} Z_{\text{U}(n^{(a)})-\text{SQCD}}^{(a)}(\sigma_{(1)}) 
\times Z_{\text{4d chiral}}^{(1)}(\sigma_{(1)}, \sigma_{(2)}) Z_{\text{U}(n^{(1)})-\text{SQCD}}^{(2)}(\sigma_{(2)}) \equiv Z_{\text{U}(n^{(1)})-\text{SQCD}}^{(1)} U(n^{(1)})-\text{SQCD} \cup U(n^{(2)})-\text{SQCD}. $$

The proof of this equality relies on formal manipulations of Nekrasov’s functions and brute force computational checks, as briefly explained in Appendix B.

To summarize, the result of the residue computation can be naturally interpreted as the partition function of a free hypermultiplet on the squashed $S^3$ in the presence of two Bogomol’nyi-Prasad-Sommerfield codimension 2 defects supported respectively on $S^3(1)$ and $S^3(2)$ which intersect along a common $S^1 = S^3(1) \cap S^3(2)$. Each defect is characterized by its worldvolume theory, being 3d $\mathcal{N} = 2$ U(n$^{(a)})$-SQCD with $a = 1$, 2 respectively. It is crucial to emphasize that the two defect worldvolume theories interact at an $S^1$, which harbors a pair of additional 1d $\mathcal{N} = 2$ chiral multiplets transforming in the bifundamental representation of the two 3d gauge groups. Figure 4 summarizes the quiver structure of the 5d/3d/1d coupled system. Each SQCD on $S^3(a)$ contains one fundamental, one antifundamental and one adjoint chiral multiplet, of masses $m^{(a)}$, $\tilde{m}^{(a)}$, and $m^{(a)}_{\text{adj}}$ respectively. The FI term is turned on with coefficient $\zeta^{(a)}$. These parameters can be identified with the 5d hyper multiplet masses and gauge coupling according to the dictionary

$$m^{(a)} = \lambda_d (M + i/2(\omega + \omega_a)),
\tilde{m}^{(a)} = \lambda_d (\tilde{M} + i/2(\omega - \omega_a)),
m^{(a)}_{\text{adj}} = i\omega_a \lambda_{\text{adj}},
\zeta^{(a)} = \frac{8\pi^2 \lambda_d}{g^2}. $$

FIG. 4. The quiver structure of the 5d/3d/1d theory describing the 5d theory in the presence of intersecting codimension 2 defects. The hyper or chiral multiplets supported on spheres of different dimensions are indicated by their colors.

\[\text{Notice that a generalization to the three-component subspace } S^3(1) \cup S^3(2) \cup S^3(3) \subset S^3 \text{ features in the Higgs branch localization on } S^3 [44]\]
where $\lambda_\alpha \equiv \sqrt{\omega_\alpha/\omega_1 \omega_2 \omega_3}$. The 3d chiral multiplets $q, \tilde{q}$ couple to the bulk hyper multiplet $q_{\text{bulk}}$ via cubic superpotentials $q_{(1)} \tilde{q}_{(1)} q_{\text{bulk}}$ and $q_{(2)} \tilde{q}_{(2)} q_{\text{bulk}}$, leading to the mass relations

\[
\begin{align*}
\begin{align*}
&b_{(1)} m^{(1)} - b_{(2)} m^{(2)} = \frac{i}{2} (b_{[2]}^2 - b_{(1)}^2), \\
&b_{(1)} \tilde{m}^{(1)} - b_{(2)} \tilde{m}^{(2)} = -\frac{i}{2} (b_{[2]}^2 - b_{(1)}^2), \\
&b_{(1)} m_{\text{adj}}^{(1)} - b_{(2)} m_{\text{adj}}^{(2)} = i(b_{[2]}^2 - b_{(1)}^2).
\end{align*}
\end{align*}
\]

(3.6)

In other words, the theories on $S^3_{(1)}$ and $S^3_{(1)}$ share the same $U(1)$ flavor group. The FI parameters in the two theories are also related by $(b_\zeta(1) = (b_\zeta(2)$, indicating that the two theories also share the same $U(1)$ topological symmetry.

Now we are ready to extract candidate 3d mirror pairs. The two sides of the fiber/base duality between frame 1 and 2 share the same poles in the integrand. In fact, the integral equality trivially follows from the equality of the integrand, and therefore by taking the residue at the same pole $\Sigma \rightarrow \Sigma'$ on both sides (and dropping the common factors), we extract a family of nontrivial integral identities labeled by non-negative integers $n^{(1)}$ and $n^{(2)}$, namely

\textbf{Proposition 2 (master identity).}

\[
\begin{align*}
Z^{S^3_{(1)} \cup S^3_{(1)}} & \equiv \text{U}(n^{(1)}) - \text{SQCDA} \cup \text{U}(n^{(2)}) - \text{SQCDA} \\
&= \exp \left[ \frac{8\pi^3}{g^2 \omega_1 \omega_2 \omega_3} (-i(M + \tilde{M}) + \omega)(n^{(1)} \omega_1 + n^{(2)} \omega_2) - \frac{16\pi^3}{g^2 \omega_3} n^{(1)} n^{(2)} \right] \\
&\times \prod_{a=1}^{n^{(1)}} \prod_{k=0}^{n^{(2)}} S_2 (-k + 1) \omega_3 \omega_1 \omega_2 S_2 \left( \pm \frac{8\pi^3}{g^2 \omega_3} + \frac{1}{2} (M - \tilde{M}) + \frac{\omega}{2} + (k - 1) \omega_1 + (\ell - 1) \omega_2 \omega_3 \right),
\end{align*}
\]

(3.7)

where $\gamma = 1, 2$ when $a = 2, 1$. We refer to Appendix A for the definitions of the double Sine and single Sine functions. Notice that this mathematical identity, which we will refer to as the master identity, is new and provides a huge generalization of the hyperbolic identity in Theorem 5.6.8 of [77]. The proof relies on formal manipulations of Nekrasov’s functions and brute force computational checks. We will shortly see that these integral identities, derived from type IIB $S$-duality, capture 3d $\mathcal{N} = 2$ mirror symmetry on intersecting $S^3$’s.

\textbf{B. Warming up: SQED/XYZ duality}

We begin with a warm-up exercise to see that the well-known Abelian mirror symmetry between 3d $\mathcal{N} = 2$ SQED and the XYZ model arises from the integral identities discussed above. For this, we consider $n^{(1)} = 1$ and $n^{(2)} = 0$. Upon substituting in (3.5), the master equality (3.7) implies

\[
Z^{S^3_{(1)}} \text{U}(1) - \text{SQCDA} = \left[ e^{-iZ'(\tau + \bar{m})} s_b \left( \frac{iQ}{2} + m - \bar{m} \right) s_b \left( -\frac{\zeta - \frac{m - \bar{m}}{2}}{2} \right) \right].
\]

(3.8)

We refer to Appendix A for the definition of the double sine function. This integral equality is nothing but the mirror symmetry relation between 3d $\mathcal{N} = 2$ SQED and the XYZ model at the level of $S^3_{(1)}$ partition functions. As expected, the complexified masses of the three free chiral multiplets in the XYZ model, namely (suppressing the label $^{(1)}$)

\[
m_{X,Y} \equiv \pm \zeta - \frac{m - \bar{m}}{2}, \quad m_Z \equiv m - \bar{m},
\]

(3.9)

satisfy

\[
m_X + m_Y + m_Z = -iQ.
\]

(3.10)

signaling the presence of the superpotential $XYZ$. On the SQED side, the additional 1-loop factor signals the presence of a decoupled chiral multiplet $\beta_1$ interacting with the adjoint chiral $\Phi$ through the superpotential $\beta_1 \Phi$. 

\[126002-8\]
C. Generalization: Intersecting SQED/XYZ duality

Now we are ready to generalize the mirror symmetry relation between the SQED and XYZ models to intersecting spheres. Dropping from both sides the common 1-loop factors like $\prod_{i=1}^{2} s_b(iQ/2 + m_{adj})$, the master equality (3.7) with $n(1) = n(2) = 1$ implies a more involved integral identity, namely

$$Z_{\text{SQED}}^S \cup \text{SQED} = \frac{\sin \frac{i\pi}{2} (b(1)m_X(1) + b(2)m_X(2)) \sin \frac{i\pi}{2} (b(1)m_Y(1) + b(2)m_Y(2))}{\sin \frac{i\pi}{2} (b(1)m_{adj}(1) + b(2)m_{adj}(2)) \sin \frac{i\pi}{2} (b(1)(m_X + m_{adj}) + b(2)(m_Y + m_{adj}))} \times \prod_{\alpha=1}^{2} \left[ e^{-i\pi(m+i\tilde{m})} s_b \left( \frac{iQ}{2} + m_X \right) s_b \left( \frac{iQ}{2} + m_Y \right) \right] \tag{3.11}$$

where the masses in the XYZ models are defined as usual by (we suppress the label (\alpha))

$$m_{X,Y} \equiv \pm \tilde{m} - \frac{m - m}{2} - \frac{iQ}{2}, \quad m_Z \equiv m - m. \tag{3.12}$$

The l.h.s. of the above identity is the partition function of two SQED on $S^3(1)$ and $S^3(2)$, coupled through a pair of 1d bi-fundamental chiral multiplets along the common $S^1$ intersection. The r.h.s. can be naturally interpreted as the partition function of two XYZ models on $S^3(1)$ and $S^3(2)$, coupled to a pair of 1d free Fermi multiplets and another pair of 1d chiral multiplets on $S^1$. The fact that the masses of the 1d multiplets are combinations of those of the 3d multiplets indicates the presence of a certain 1d superpotential that involves both the 3d and 1d chiral multiplets. As a result, the 1d multiplets are charged under the 3d global symmetries, in particular, the Fermi multiplets are charged under the 3d topological $U(1)$ symmetry.

Some remarks follow. In the previous subsection, the integral equality reproduces the well-known 3d mirror symmetry between SQED and XYZ model, confirming at the level of partition function that the SQED flows to the free XYZ model as the IR limit. In this subsection, our derived integral identity shall be viewed as a piece of mathematical evidence from which we identify a possible flow from the gauge theory on the intersecting space $S^3(1) \cup S^3(2)$ to a free theory on the same space. However, one should also note that surprises arise in intersection theories [36], which might bring subtleties to such naive RG flow. We leave detailed investigation of these subtleties to future study.

D. Generalization: Non-Abelian SQCDA/XYZ duality

We can now move to discuss more interesting examples, generalizing the previous Abelian examples to non-Abelian gauge groups. Let us start by considering $n(1) > 0$, $n(2) = 0$, in which case the master equality specializes to

$$\prod_{\mu=0}^{n-1} s_b \left( \frac{iQ}{2} + m Y \right) = \frac{1}{\prod_{\mu=1}^{n} s_b \left( iQ/2 - \mu m_{adj} - iQ \right)}, \tag{3.15}$$

and move the denominators to the l.h.s. of (3.13). Defining the leftover mass on the r.h.s.

$$m_Z \equiv m_{Z(n-1)} = m - m + (n - 1)m_{adj}, \tag{3.17}$$

one easily finds the masses satisfy

$$m_X + m_Y + m_Z = -iQ, \quad \mu = 0, \ldots, n - 1. \tag{3.18}$$

which is compatible with the superpotential $\sum_{\mu=1}^{n} X(X \mu Y)$. On the l.h.s., the additional 1-loop factors are compatible with free chiral multiplets $\beta_\mu$ and $\gamma_\mu$ interacting with the
adjoint chiral $\Phi$ and the quarks $q, \bar{q}$ through the superpotential $\sum_{\mu=0}^{n-1} \beta_\mu \Phi^\mu q + \sum_{\mu=1}^{n-1} \beta_\mu \Phi^\mu$.

The mathematical relation (3.13) has implicitly appeared in [34] as an intermediate step to test another duality, involving the SU($n$) theory coupled to one fundamental, one anti-fundamental and one adjoint chiral on the one hand, and the U(1) theory coupled to $n$ hypers on the other hand as shown in Fig. 5, which was motivated by the study of the mirror dual of $(A_1, A_{2n-1})$ AD theories reduced to 3d [20,21]. This duality is simply related to ours by gauging the topological U(1). Hence, we have physically interpreted the FI parameter, viewed as the “mass” for the topological U(1) symmetry.

E. Generalization: Intersecting non-Abelian SQCDA/XYZ duality

It is now straightforward to take the further generalization $n^{(1)}, n^{(2)}>0$. In this case, the master identity yields

$$Z^{S^{(1)}\cup S^{(2)}} U(n^{(1)}) \cup U(n^{(2)}) \cup U(n^{(2)}) \cup U(n^{(2)}) \cup U(n^{(2)}) - \text{SQCDA}$$

where we used the same shorthand notations as before. We can reorganize the factors as we did in the previous subsection, and the difference compared to the previous result (besides the doubling of all factors) is the presence of the additional 1-loop contributions from the 1d matter living on the $S^{(1)}$ intersection, represented by the last line. This picture provides the generalization of the non-Abelian SQCDA/XYZ duality to the more complicated geometry involving 1d degrees of freedom, and we have shown that it also descends from type IIB S-duality.

It is worth noting that one can further integrate over the FI parameters $\zeta^{(a)}$ to obtain the intersecting space version of the SU($n$)-SQCDA/U(1) duality mentioned at the end of the last subsection. However, the fact that the FI parameters on each component space are related by $(b^{(a)}_\zeta)^{(1)} = (b^{(a)}_\zeta)^{(2)}$ implies integration with the constraint $\delta(\sum_{a=1}^{n^{(1)}} (b^{-1} \sigma_a)^{(1)}) + \sum_{a=1}^{n^{(2)}} (b^{-1} \sigma_a)^{(2)}$, whose field theory interpretation remains unclear to us at the moment.

F. Quiver gauge theories

It is possible to generalize the above computations to quiver gauge theories. As shown in Fig. 3, one could start from a 5d linear quiver gauge theory and engineer intersecting codimension 2 defects with quiver world volume theories by multiple Higgsings. For example, it is not hard to convince oneself that by Higgsing twice the 5d linear quiver gauge theory with two U(1) gauge nodes, one will obtain 3d quiver theories of the form depicted in Figure 6. It is possible to apply the Higgsing procedure by taking the residues of the resulting partition functions and their fiber/base dual, and repeat the computations in the previous discussions. However, the technical computations are more involved and we do not consider them here explicitly.

IV. DISCUSSION AND OUTLOOK

In this paper, we have studied a class of 3d $\mathcal{N} = 2$ non-Abelian gauge theories which can be realized as codimension 2 defects in the parent 5d $\mathcal{N} = 1$ Abelian gauge theories, which in turn can be realized in type IIB string theory. Generically, the defect theories are not supported on a single component subspace, instead, they live on mutually orthogonal submanifolds intersecting at codimension 4 loci where additional degrees of freedom live. We have
considered some implications of type IIB SL(2, Z) symmetry for these systems, and we have generalized to this class of more complicated geometries the known fact that type IIB S-duality reduces to 3d mirror symmetry. Using the refined topological vertex, we have been able to test this idea in simple cases where the parent 5d gauge theory is simply the SQED with two flavors, while the dual 3d theories are SQCDA with two chiralos and a generalized XYZ model. Interestingly enough, the QFT/string theory methods have also allowed us to physically explain existing integral identities in the math literature, and moreover, to derive new ones and interpret them as the equivalence of partition functions of mirror dual theories on (intersecting) squashed spheres. One should bare in mind, however, that the field-theoretical interpretation of the intersecting mirror symmetry requires further investigation due to surprises/inconsistences hidden within theories on intersecting spaces.

Along the lines of this paper, one should also be able to study more complicated 5d theories and hence derive new or generalized 3d mirror pairs. As byproduct, one may also obtain new mathematical identities expressing the equivalence of dual partition functions. Moreover, what we have discussed in this paper is expected to have a higher dimensional lift [78] by considering 6d theories engineered by periodic (p, q)-webs [58,79,80] and the resulting 4d/2d defect theories.

Finally, it is worth noting that the type of 3d/1d defects that we have considered in this paper appear in the Higgs branch localization approach to SQCD on S^3 [44], whose partition functions are identified with correlators in the q-Virasoro modular triple [81]. Therefore, another interesting route of investigation would be the study of type IIB SL(2, Z) symmetry from the viewpoint of the Bogomol’nyi-Prasad-Sommerfield/CFT and 5d AGT correspondences [82–96] and the DIM algebra [97,98], whose representation theory is known to govern the topological amplitudes associated to toric CY 3-folds or (p, q)-webs [99–102]. From this perspective, the SL(2, Z) symmetry group is identified with the automorphism group of the DIM algebra, and it would be interesting to systematically study how different q-deformed correlators are related to each other. In turn, this perspective may give powerful tools for handling 3d mirror symmetry very efficiently. This is a topic which deserves further investigations, and in Appendix D we have collected few preliminary comments and background material for the interested readers.

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APPENDIX A: SPECIAL FUNCTIONS

In this Appendix, we recall the definitions of several special functions which we use in the main body. Below, r is a positive integer, and \( \vec{\omega} \equiv (\omega_1, \ldots, \omega_r) \) is a collection of nonzero complex parameters. We frequently take \( r = 1, 2, 3 \) for concreteness. We refer to [103] for further details.

The multiple Bernoulli polynomials \( B_{r n}(X|\vec{\omega}) \) are defined by the generating function

\[
\prod_{i=1}^r (1 - e^{\omega_i t})^{-1} = \sum_{n \geq 0} B_{r n}(X|\vec{\omega}) \frac{t^n}{n!}.
\] (A1)

In particular, we use \( B_{22}(X|\vec{\omega}) \) and \( B_{33}(X|\vec{\omega}) \) in this paper, and they are given explicitly by

\[
B_{22}(X|\vec{\omega}) = \frac{X^2}{\omega_1 \omega_2} - \frac{\omega_1 + \omega_2}{\omega_1 \omega_2} X + \frac{\omega_1^2 + \omega_1 \omega_2 + 3\omega_1 \omega_2}{6\omega_1 \omega_2},
\]

\[
B_{33}(X|\vec{\omega}) = \frac{X^3}{\omega_1 \omega_2 \omega_3} - \frac{3(\omega_1 + \omega_2 + \omega_3)}{2 \omega_1 \omega_2 \omega_3} X^2
+ \frac{\omega_1^2 + \omega_1 \omega_2 + 3\omega_1 \omega_2 + 3\omega_2 \omega_3 + 3\omega_3 \omega_1}{2 \omega_1 \omega_2 \omega_3} X
- \frac{(\omega_1 + \omega_2 + \omega_3)(\omega_1 \omega_2 + \omega_2 \omega_3 + \omega_3 \omega_1)}{4 \omega_1 \omega_2 \omega_3}.
\] (A2)

The \( q \)-Pochhammer symbols are defined as

\[
(x; q_1, \ldots, q_r)_\infty = \prod_{n_1, \ldots, n_r = 0}^{\infty} (1 - x q_1^{n_1} \cdots q_r^{n_r})
\]
when all \( |q_i| < 1 \). (A4)

Other regions in the \( q \)-planes are defined through the replacements

\[
(x; q_1, \ldots, q_r)_\infty \rightarrow \frac{1}{(q_i^{-1} x; q_1, \ldots, q_i^{-1}, \ldots, q_r)_\infty}.
\] (A5)

The multiple Sine functions \( S_r(X|\vec{\omega}) \) can be defined by the \( \zeta \)-regularized product
\begin{align}
S_r(X|\tilde{\omega}) & \approx \prod_{m_1, \ldots, m_r \in \mathbb{N}} (X + \sum_{i=1}^r m_i \omega_i)^{(-1)^{r+1}} \\
& \times \left( -X + \sum_{i=1}^r (m_i + 1) \omega_i \right). \quad (A6)
\end{align}

The reflection property of \( s_b(z) \) is simply
\begin{equation}
S_b(z) s_b(-X) = 1. \quad (A12)
\end{equation}

The triple Sine function \( S_3(X|\tilde{\omega}) \equiv S_3(X) \) also has a useful factorization property. When \( \text{Im}(\omega_i/\omega_j) \neq 0 \) for all \( i \neq j \), then
\begin{equation}
S_3(X) = e^{-\frac{i\pi}{2}B_{31}(X)} \prod_{1 \leq j < k \leq 3} \left( e^{2\pi i j / \omega_k} ; e^{2\pi i \omega_j / \omega_k} ; e^{2\pi i \omega_j / \omega_k} \right)_{\infty}. \quad (A13)
\end{equation}

The Nekrasov function is defined as
\begin{equation}
N_{J\mu}(x; q, t^{-1}) \equiv \prod_{(i,j) \notin \mu} \left( 1 - x q^{-i} q^{j} t^{-i} t^{j} \right) \times \prod_{(i,j) \in \mu} \left( 1 - x q^{-i} q^{j} t^{-i} t^{j} \right), \quad (A14)
\end{equation}

where \( \dagger \) denotes transposition of the Young diagrams.

**APPENDIX B: DERIVATION OF THE \( S^3(1) \cup S^3(2) \) MATRIX MODEL**

Here we sketch how to derive the matrix model (3.4) following the argument given above (3.3). The exact equality between the residue of the \( S^3 \) integrand at the selected poles (3.1) (with \( n^{(3)} = 0 \)) and the \( S^3(1) \cup S^3(2) \) matrix model is established in the next section in the notation used in the main body. See also [40] for another derivation.

We start by rewriting the instanton sum (2.1) using the manipulations considered in [43]. Shown in Fig. 7 is a large Hook Young diagram \( \lambda \) decomposed into an upper-left full rectangle with exactly \( r \) rows and \( c \) columns, an upper-right subdiagram \( Y_R \) with at most \( r \) rows and a lower-left subdiagram \( Y_L \) with at most \( c \) rows. For such a diagram, we can write the corresponding summand in the instanton partition function (2.1) as
where $\eta_{LR}$ are free parameters such that $\eta_L/\eta_R = \sqrt{q}/t$, we defined

$$z^{i\lambda}_y \equiv \eta_L x t^{-r} q^{-i} t^{-j} y^j_t, \quad z^{r\lambda}_y \equiv \eta_R x q^{r} t^{-j} y^j_t,$$

and $N_{\emptyset\emptyset}$ denotes the whole factor beginning in the second line and evaluated for empty diagrams. The nonperturbative instanton partition function is obtained as the weighted sum over $\lambda$ with weight $(p^{-1/2} Q_0)^{|\lambda|}$, where $|\lambda| \equiv \sum_{i \lambda} \lambda_i$ implies the total number of boxes in $\lambda$. The sum can be further decomposed into a form respecting the hook Young diagram decomposition as shown in Eq. 7, namely $\sum_{\lambda} = \sum_{r,c \geq 0} \sum_{r^c,y^r}$, such that $r - c = n$ is a fixed arbitrary integer expressing a linear relation between $r$ and $c$. Note that if we tune $p^{1/2} Q_2 = q^{-n(t^n)}$, the first factor in (2.1) vanishes, and therefore the instanton sum only receives non-vanishing contributions from diagrams $\lambda$ which do not contain the box $(n_2 + 1, n_1 + 1)$, i.e., Hook diagrams with $\lambda_{n_2+1} \leq n_1, \lambda_{n_1+1} \leq n_2$; they include all large hook Young diagrams with an upper-left rectangle of the shape $r = n_2, c = n_1$, and infinitely many diagrams that we call small hook diagrams. Let us focus on the large Hook diagrams. In this case we get the simplification

$$N_{\emptyset\lambda}(Q_1 p^{1/2}; q, t) N_{\emptyset\lambda}(Q_2 p^{1/2}; q, t) = \prod_{i=1}^{r} \prod_{j=1}^{c} \frac{(1 - q^{-i} t^{j})(1 - p^{1/2} Q_1 t^{i} q^{-j})}{(1 - t^{-i} q^{j})(1 - t^{i} q^{-j})} \times \Delta_1(z^{i\lambda}_y, q^{r\lambda}_y) \Delta_2(z^{i\lambda}_y, t^{-1}) \prod_{i=1}^{r} \left( \frac{t^{i\lambda}_y p^{1/2} Q_1 x / z^{i\lambda}_y; q}{t^{i\lambda}_y q^{r\lambda}_y x / z^{r\lambda}_y; q} \right) \frac{(q^{-1} \eta_L p^{1/2} Q_1 x / z^{i\lambda}_y; t^{-1})_{\infty}}{(q^{-1} \eta_L q^{r\lambda}_y x / z^{r\lambda}_y; t^{-1})_{\infty}} \times \prod_{i,j \geq 1} \frac{1 - p^{-1/2} z^{i\lambda}_y / z^{r\lambda}_y (1 - p^{-1/2} z^{r\lambda}_y / z^{i\lambda}_y)}{1 - p^{-1/2} z^{i\lambda}_y / z^{r\lambda}_y (1 - p^{-1/2} z^{r\lambda}_y / z^{i\lambda}_y)}. \tag{B3}$$

Also, the residue of the perturbative factor in (2.1) at a pole $p^{1/2} Q_2 = q^{-n(t^n)}$ reads

$$\prod_{i=1}^{r} \prod_{j=1}^{c} \frac{1}{(p^{1/2} Q_1; q, t^{-1})_{\infty}} \frac{\text{Res}_{z^{i\lambda}_y}(z^{i\lambda}_y, q^{r\lambda}_y, t^{-1})_{\infty}}{(p^{1/2} Q_1; q, t^{-1})_{\infty}} \prod_{i=1}^{r} \prod_{j=1}^{c} \frac{1 - q^{-i} t^{j}}{1 - q^{r\lambda}_y t^{-j}} \prod_{i=1}^{r} \prod_{j=1}^{c} \frac{1}{1 - q^{r\lambda}_y t^{-j}} \prod_{i=1}^{r} \frac{1}{1 - q^{r\lambda}_y t^{-j}}. \tag{B4}$$

Notice that the second factor will cancel against the first factor in the numerator of (B3). We can also set

$$Q_1 = q^{r(t^n)} p^{1/2} w / x, \tag{B5}$$

and redefine

$$z^{i\lambda}_y, q^{-c} t^r = \eta_L q^{r} x q^{-c} t^r / y^j_t \rightarrow z^{i\lambda}_y, \quad z^{r\lambda}_y q^{-c} t^r = \eta_R t^{r} q^{-c} t^r / y^j_t \rightarrow z^{r\lambda}_y, \tag{B6}$$

so that

$$N_{\emptyset\lambda}(Q_1 p^{1/2}; q, t) N_{\emptyset\lambda}(Q_2 p^{1/2}; q, t) = \prod_{i=1}^{r} \prod_{j=1}^{c} \frac{(1 - q^{-i} t^{j})(1 - p t^{i} q^{-j} w / x)}{(1 - t^{-i} q^{j})(1 - t^{i} q^{-j})} \times \Delta_1(z^{i\lambda}_y, q^{r\lambda}_y) \Delta_2(z^{i\lambda}_y, t^{-1}) \prod_{i=1}^{r} \left( \frac{t^{i\lambda}_y p w / z^{i\lambda}_y; q}{t^{i\lambda}_y q^{r\lambda}_y x / z^{r\lambda}_y; q} \right) \frac{(q^{-1} \eta_L p w / z^{i\lambda}_y; t^{-1})_{\infty}}{(q^{-1} \eta_L x / z^{r\lambda}_y; t^{-1})_{\infty}} \times \prod_{i,j \geq 1} \frac{1 - p^{-1/2} z^{i\lambda}_y / z^{r\lambda}_y (1 - p^{-1/2} z^{r\lambda}_y / z^{i\lambda}_y)}{1 - p^{-1/2} z^{i\lambda}_y / z^{r\lambda}_y (1 - p^{-1/2} z^{r\lambda}_y / z^{i\lambda}_y)}. \tag{B7}$$
For convenience, we can also set

\[ t \eta_R p_w \equiv w_R, \quad q^{-1} \eta_L p_w \equiv w_L, \quad t \eta_R x \equiv x_R, \quad q^{-1} \eta_L x \equiv x_L, \quad (B8) \]

so that

\[
\frac{N_{Q_{\bar{Q}}}(Q_1 p^{1/2}; q, t) N_{Q\bar{Q}}(Q_2 p^{1/2}; q, t)}{N_{\bar{Q}Q}(1; q, t)} = \prod_{i=1}^{r} \left( \frac{\prod_{j=1}^{c} (1- q^{-i} t^j) (1- p t^{i-j} q^{-i} w / x)}{(1- t^j q^{-i}) (1- t^{i-j} q^{-j})} \right)
\]

\[
\times \Delta t(z_{\bar{Q}}^i; q) \Delta a_{\bar{Q}}^{-1}(z_{Q}^{i-1}; t^{-1}) \prod_{i \geq 1} \frac{(w_R / z_{\bar{Q}}^i q; q)_{\infty}}{(x_R / z_{Q}^j q; q)_{\infty}} \prod_{i,j \geq 1} \frac{1}{(1- p^{1/2} z_{Q}^j / z_{\bar{Q}}^i)(1- p^{-1/2} z_{\bar{Q}}^i / z_{Q}^j)}, \quad (B9)
\]

Notice that

\[
\frac{\Theta(\xi p^{-1/2} Q_0 / z_{\bar{Q}}^i q; q) \Theta(\xi; q)}{\Theta(\xi; z_{Q}^i q; q) \Theta(\xi p^{-1/2} Q_0; q)} = \frac{\Theta(\xi p^{-1/2} Q_0 / z_{\bar{Q}}^i q; q) \Theta(\xi; q)}{\Theta(\xi; z_{Q}^i q; q) \Theta(\xi p^{-1/2} Q_0; q)} (p^{-1/2} Q_0)^{|y|^i}, \quad (B10)
\]

\[
\frac{\Theta(\xi p^{-1/2} Q_0 / z_{\bar{Q}}^i q; t^{-1}) \Theta(\xi; t^{-1})}{\Theta(\xi; z_{Q}^i q; t^{-1}) \Theta(\xi p^{-1/2} Q_0; t^{-1})} = \frac{\Theta(\xi p^{-1/2} Q_0 / z_{\bar{Q}}^i q; t^{-1}) \Theta(\xi; t^{-1})}{\Theta(\xi; z_{Q}^i q; t^{-1}) \Theta(\xi p^{-1/2} Q_0; t^{-1})} (p^{-1/2} Q_0)^{|y|^i}, \quad (B11)
\]

where \( \xi \) is arbitrary. Since

\[
(p^{-1/2} Q_0)^{|y|} = (p^{-1/2} Q_0)^{|y|} (p^{-1/2} Q_0)^{|y|}, \quad (B12)
\]

we can recognize the weighted sum over the left and right diagrams [second and third line of (B9)] as the vortex part of the partition function

\[
\mathcal{B}_{LR} \equiv \int \prod_{i=1}^{r} \frac{dz_{\bar{Q}R}}{2 \pi i z_{\bar{Q}R}} \prod_{j=1}^{c} \frac{dz_{Q}}{2 \pi i z_{Q}} \gamma_L(z_L) \gamma_R(z_R) = \quad (B13)
\]

\[
= \text{Res}_{z_{\bar{Q}R} = z_{Q}} \gamma_L(z_L) \gamma_R(z_R) \sum_{y^i, y^j} \frac{\gamma_L(z_{\bar{Q}}^i) \gamma_{\text{int}}(z_{\bar{Q}}^i, z_{Q}) \gamma_R(z_{Q})}{\gamma_L(z_{Q}) \gamma_{\text{int}}(z_{Q}, z_{\bar{Q}}^i) \gamma_R(z_{\bar{Q}}^i)}, \quad (B14)
\]

where

\[
\gamma_R(z_{\bar{Q}}) = \prod_{i=1}^{r} \frac{\Theta(\xi p^{-1/2} Q_0 / z_{\bar{Q}R}; q) \Theta(\xi; q)}{\Theta(\xi; z_{Q}^i q; q) \Theta(\xi p^{-1/2} Q_0; q)} \Delta t(z_{\bar{Q}}; q) \prod_{i=1}^{c} \frac{(w_R / z_{\bar{Q}R} q; q)_{\infty}}{(x_R / z_{Q}^j q; q)_{\infty}}, \quad (B15)
\]

\[
\gamma_L(z_{Q}) = \prod_{j=1}^{c} \frac{\Theta(\xi p^{-1/2} Q_0 / z_{Q}; t^{-1}) \Theta(\xi; t^{-1})}{\Theta(\xi; z_{\bar{Q}}^i q; t^{-1}) \Theta(\xi p^{-1/2} Q_0; t^{-1})} \Delta a_{\bar{Q}}^{-1}(z_{Q}; t^{-1}) \prod_{j=1}^{c} \frac{(w_L / z_{Q}^j q; t^{-1})_{\infty}}{(x_L / z_{\bar{Q}}^i q; t^{-1})_{\infty}}, \quad (B16)
\]

\[
\gamma_{\text{int}}(z_{\bar{Q}}^i, z_{Q}) = \prod_{i=1}^{r} \prod_{j=1}^{c} \frac{1}{(1- p^{-1/2} z_{Q}^j / z_{\bar{Q}}^i)(1- p^{-1/2} z_{\bar{Q}}^i / z_{Q}^j)}, \quad (B17)
\]

and the contour is chosen to encircle the poles\(^6\)

\[^6\text{We simply integrate the } \xi \text{'s one after the other, starting from } z_{R,i=w} \text{ around } x_R \text{ and } z_{L,i=\infty} \text{ around } x_L.\]
\[ Z^{(1)}_{\text{L};\text{R}}(\mathcal{Z}) \equiv \prod_{1 \leq i < j \leq r} S_2(Z_{\mathcal{R}_i} - Z_{\mathcal{R}_j} | \omega_1, \omega_3) \times \frac{S_2(Z_{\mathcal{R}_i} - Z_{\mathcal{R}_j}, \omega_1, \omega_3)}{S_2(Z_{\mathcal{L}_i} - Z_{\mathcal{L}_j}, \omega_2, \omega_3)} \]
APPENDIX C: $S^3$ AND $S^3_{(1)}$ U $S^3_{(2)}$ PARTITION FUNCTIONS

In this Appendix, we establish the exact equality between the residue of the $S^3$ integrand at the selected poles (3.1) (with $n^{(3)} = 0$) and the $S^3_{(1)}$ U $S^3_{(2)}$ matrix model (3.4) in the notation used in the main body. We start by recalling useful definitions of partition functions on a squashed spheres or their intersections.

The squashed $S^3$ partition function of a $U(n)$ gauge theory coupled to $n_f = n_{df}$ fundamental and antifundamental chiral multiplets and one adjoint, which we will refer to as $U(n)$-SQCDA, is given by

$$Z_{S^3_{U(n)}} = \int \frac{d^n \sigma}{(2\pi)^n} e^{-\frac{1}{2} \sum_{a=b}^n \sin \pi b(\sigma_a - \sigma_b)\sin \pi b^{-1}(\sigma_a - \sigma_b)} \times \prod_{i=1}^{n} \prod_{a=1}^{n} s_b(i \frac{Q}{2} + \sigma_a - \sigma_b - m_i).$$

As usual, $b$ denotes the squashing parameter, $Q \equiv b + b^{-1}$, while $m_i$, $\tilde{m}_i$, and $m_{adj}$ denote the complexified masses of fundamental, antifundamental, and adjoint chiral multiplets

$$m \equiv m^R - i Q, \quad \tilde{m} \equiv \tilde{m}^R + i Q, \quad m_{adj} \equiv m_{adj}^R - Q_{adj}^R - \frac{i Q}{2},$$

and $\zeta$ is the FI parameter. Let us denote the integrand simply as $Z_{S^3_{U(n)}}(\sigma)$. Then the partition function of a pair of $U(n^{(a)})$-SQCDA on $S^3_{(1)}$ U $S^3_{(2)}$, interacting through a pair of 1d bi-fundamental chiral multiplets at the intersection $S^1 = S^3_{(1)} \cap S^3_{(2)}$, is given by

$$Z_{S^3_{U(n^{(a)}) \times S^3_{U(n^{(b)})}}} = \int \frac{d^n \sigma^{(a)}}{(2\pi)^n} \frac{d^n \sigma^{(b)}}{(2\pi)^n} Z_{S^3_{U(n^{(a)})}, n_i, n_a}(\sigma^{(a)}) Z_{S^3_{U(n^{(b)})}, n_i, n_a}(\sigma^{(b)}) \times Z_{1d\text{chiral}}(\sigma^{(1)}, \sigma^{(2)}) Z_{S^3_{U(n^{(a)})}, n_i, n_a}(\sigma^{(1)}), Z_{S^3_{U(n^{(b)})}, n_i, n_a}(\sigma^{(2)}),$$

where the contribution from the 1d chiral multiplets is captured by

$$Z_{1d\text{chiral}}(\sigma^{(1)}, \sigma^{(2)}) = \prod_{a=1}^{n^{(a)}} \prod_{b=1}^{n^{(b)}} \frac{1}{2 \sinh \frac{\pi b_1}{2} (b_1^{(1)} - b_2^{(2)} + \frac{1}{2} (b_2^{(1)} + b_2^{(2)}))}.$$

In general, the parameters in the two SQCDA are independent, however, when they are the world volume theories of intersecting codimension 2 defects in a bulk 5d $\mathcal{N} = 1$ theory, the masses are likely to be related due to

5d/3d superpotentials, which is indeed the case throughout our paper. For example, we have mass relations

$$b^{(1)} m_i^{(1)} - b^{(2)} m_i^{(2)} = \frac{i}{2} (b_2^{(1)} - b_2^{(2)}).$$

The matrix model (C3) should be understood as a contour integral with a Jeffrey-Kirwan residue prescription. Take $n^{(1)} = 1, n^{(2)} = 1$ as an example. There are two sets of poles, the first of which is given by

$$\sigma^{(1)} = m_i^{(1)} - im^{(1)} b^{(1)} - im^{(1)} b^{(2)}, \quad \sigma^{(2)} = m_i^{(2)} - im^{(2)} b^{(2)} - im^{(2)} b^{(1)},$$

for all $m^{(a)}, n^{(a)} \geq 0$, while the second

$$\sigma^{(1)} = m_i^{(1)} - im^{(1)} b^{(1)} - im^{(1)} b^{(2)}, \quad \sigma^{(2)} = m_i^{(2)} - im^{(2)} b^{(2)} - im^{(2)} b^{(1)},$$

for all $n^{(a)} \geq 0$. Clearly, the second set come from the poles of $Z_{1d\text{chiral}}$, since this set of poles satisfies

$$\sinh \pi \left( b^{(1)} \sigma^{(1)} - b^{(2)} \sigma^{(2)} - \frac{i}{2} (b_2^{(2)} - b_2^{(1)}) \right) = 0,$$

thanks to the mass relations mentioned above. With these definitions, the equality (3.4) and the master identities (3.7) can be explicitly verified (e.g., by using Mathematica).

APPENDIX D: THE REFINED TOPOLOGICAL VERTEX AND DIM ALGEBRA

1. The refined topological vertex

The topological vertex formalism [59] and its refinement [62,63] are powerful tools to study 5d instanton partition functions and their properties. In this paper we will mainly follow the conventions of [63], which we now review.
The relevant vertices\(^7\) are graphically represented in Fig. 8. Note that at each vertex there are two black and one white arrows (the preferred/instanton direction), each labeled by a Young diagram. The three arrows are ordered in a clockwise manner, keeping the white arrow in the middle. For example, in the two diagrams in the Fig.8, the white arrows are labeled with \(\lambda_2\), and is also the second index of the vertex. Lowered/raised indices of the vertex correspond to incoming/outgoing arrows. These graphical vertices represent the following contributions to the full amplitude,

\[
C_{\lambda_1\lambda_2}^{\lambda_3} = P_{\lambda_3}(t^n; q, t) \sum_{\lambda} \frac{\mu[\lambda]}{f_{\lambda_3}(q, t)} \times iP_{\lambda_1/\lambda_2}(t^n q^n; q, t) \quad \text{(D1)}
\]

\[
C^{\lambda_1\lambda_2\lambda_3} = P_{\lambda_1}(t^n q^n; q, t) \sum_{\lambda} \frac{\mu[\lambda]}{f_{\lambda_2}(q, t)} \times iP_{\lambda_1/\lambda_2}(q^n t^n q^n; q, t) \quad \text{(D2)}
\]

The \(P_{\lambda/\mu}(x; q, t)\) is the skew Macdonald function of the sequence of variables \(x = (x_1, x_2, \ldots)\) with Young diagrams \(\lambda = (\lambda_1, \lambda_2, \ldots)\) and \(\mu = (\mu_1, \mu_2, \ldots)\) as parameters, while \(|\lambda| \equiv \sum \lambda_i\) denotes the total number of boxes in the diagram \(\lambda\) and \(t\) is the involution \(i(p_n) = -p_n\) acting on the power sums \(p_n \equiv \sum \lambda_i^n\). The other parameters \(q \equiv e^{i2\pi / n}, t \equiv e^{-i2\pi / n}\) and \(p \equiv qt^{-1}\) are complex numbers.

The vertices can be joined together to form web diagrams corresponding to CY or \((p, q)\)-webs engineering 5d supersymmetric gauge theories. In doing so, each internal line is further associated to a complex parameter \(Q^{[\lambda]}\) and a framing factor \(f_{\lambda}(q, t)^n\) (for us \(n = 0\), and the corresponding Young diagrams are summed over.

### 2. DIM intertwiners

The topological vertex can be interpreted as matrix elements of DIM intertwining operators in the Macdonald basis [104], namely

\[
C^{\mu\nu}_{\lambda}(q, t) = Q_{N,(x)}^{[\lambda]}(r^{-1/2})_{[x]}^{[\frac{1}{4}]} f^{\frac{N}{2}}_{\lambda}(q, t) f_{\mu}^{\frac{N}{2}}(q, t) \\
\times \frac{\|P_{\mu} \|P_{\lambda}}{\langle P_{\mu} \rangle} \frac{(1, N + 1)_{-\nu}}{(0, 1)_{-\nu}} \langle Q_{\lambda} \rangle \quad \text{(D3)}
\]

where we defined \(Q_{N,(x)} \equiv -q(-y)^N/t^{1/2}x\). The state \(|P_{\mu}\rangle\) and its dual \(|\bar{P}_{\mu}\rangle\) give a Fock basis, and the labels \((n, k)_x\) are DIM representations specified by the integer values of the two central charges and the complex spectral parameter. In particular, \((0, 1)_x\) is called vertical, while \((1, N)_x\) is called horizontal. They are isomorphic and related by the so-called spectral duality [101,105,106], a manifestation of the SL(2, \(\mathbb{Z}\)) group of automorphism of the DIM algebra.

In the web diagram, the choice of preferred/white direction correspond to the choice of vertical representation, to which \(\Phi\) or \(\Phi^\dagger\) are attached. See Fig. 9 for an illustration.

As the basic example, let us consider the resolved conifold amplitude with preferred direction or (0,1) representation along the vertical direction

\[
\langle \emptyset | \Phi^\dagger \left( \begin{array}{c} (1, N) \hbar \cr (1, N + 1) \end{array} \right) \Phi \left( \begin{array}{c} (1, N + 1)_{-\nu} \cr (0, 1) \end{array} \right) | \emptyset \rangle = \sum_{\lambda} (v/a)^{[\lambda]} C_{\lambda}^{\emptyset} \Phi^\dagger(q, t) C_{\lambda}^{\emptyset} \Phi(q, t),
\]

where \(uv = ab\). Alternatively, we could have put the preferred direction or (0,1) representation along the horizontal direction

\[
\langle \emptyset | \Phi^\dagger \left( \begin{array}{c} (1, N - 1)_{-\nu} \cr (1, N) \end{array} \right) \Phi \left( \begin{array}{c} (1, N + 1)_{-\nu} \cr (0, 1) \end{array} \right) | \emptyset \rangle \times \Phi \left( \begin{array}{c} (0, 1) \hbar \cr (1, N) \end{array} \right) \left( \begin{array}{c} (1, N - 1)_{-\nu} \cr (1, N) \end{array} \right) | \emptyset \rangle
\]

where \(a'/b = v'/u'\). The two results should agree because of slicing invariance of the topological vertex, and they do provided \(v/a = b'/u' = Q_{0}\), which is the ratio of the outgoing/incoming spectral parameters associated to the (0,1) representations. From the DIM perspective, this should descend from the SL(2, \(\mathbb{Z}\)) automorphism of the algebra, see Fig. 10 for an illustration. A more complicated choice is to assign the preferred direction or (0,1) representation to the diagonal direction. Now the composition of

\[\text{Fig.8. Refined topological vertices.}\]
the intertwiners acts on the tensor product of two Fock spaces, and the corresponding amplitude is

$$
\langle \emptyset | \otimes \emptyset \sum_{\lambda} \frac{1}{\langle P_{\lambda} | P_{\lambda} \rangle} \Phi_{\lambda} \left[ (1, 1 - M)_{b''} (0, 1)_{u''/v''} (1, -M)_{a''} \right] 
\otimes \Phi^*_{\lambda} \left[ (1, -M)_{b''} (0, 1)_{u''/v''} (1, 1 - M)_{a''} \right] | \emptyset \rangle \otimes | \emptyset \rangle
= \sum_{\lambda} (a''/v'')^{|\lambda|} C_{\emptyset}(q, 1) C_{\emptyset}(q, 1),
$$  \tag{D6}

where $b''/a'' = u''/v''$. This corresponds to the Nekrasov partition function of the 5d pure U(1) SYM theory with instanton counting parameter $a''/v''$. This expansion coincides with the previous ones provided we identify $a''/v'' = Q_0$. See Fig. 11 for an illustration.

For the next level of complication, we can consider the geometries considered in the main text. As we discussed, there is a frame corresponding to a U(1) theory with two flavors (Fig. 1 left), a frame corresponding to four free hypers (Fig. 1 center) and a frame corresponding to a U(1) × U(1) theory with one bifundamental hyper (Fig. 1 right). It is now easy to recognize the various topological amplitudes as (vacuum) matrix elements of intertwining operators between various representations, and the fact that they should agree is expected from the SL(2, Z) automorphism of DIM. In particular, we can identify (we neglect the unnecessary labels in order to avoid cluttering)

$$
Z_1 = \langle \emptyset | \otimes \emptyset \sum_{\lambda} \frac{(\Phi_{\emptyset} \otimes \mathbb{1})(\Phi_{\lambda} \otimes \Phi^*)(\mathbb{1} \otimes \Phi_{\emptyset})}{\langle P_{\lambda} | P_{\lambda} \rangle} | \emptyset \rangle \otimes | \emptyset \rangle,
$$  \tag{D7}

Of course, we need suitable identifications between parameters. Anyhow, from the form of the matrix elements it is immediate that $Z_1$ should correspond to a U(1) theory, $Z_2$ to a free theory and $Z_3$ to a U(1) × U(1) theory. Also, since the $W_4$ theory can be represented on the tensor product of two horizontal DIM representations, while $W_4$ can be represented on the tensor product of three horizontal DIM representations, the resulting 5d $\mathcal{N} = 1$ quiver gauge theories match with Kimura-Pestun construction of quiver $W_4$ algebras [94]. In their construction, the basic object is the $Z$ operator, which is an infinite product of the $W_4$ screening charges. From the DIM perspective, we can identify

$$
Z[A_1] = \sum_{\lambda} \frac{\Phi_{\lambda} \otimes \Phi_{\lambda}^*}{\langle P_{\lambda} | P_{\lambda} \rangle},
$$

$$
Z[A_2] = \sum_{\lambda_1, \lambda_2} \frac{(\mathbb{1} \otimes \Phi_{\lambda_1} \otimes \Phi_{\lambda_2}^*)(\Phi_{\lambda_1} \otimes \Phi_{\lambda_2}^* \otimes \mathbb{1})}{\langle P_{\lambda_1} | P_{\lambda_1} \rangle \langle P_{\lambda_2} | P_{\lambda_2} \rangle}.
$$  \tag{D10}

On the other hand, it is known that Kimura-Pestun construction as an analogous for 3d $\mathcal{N} = 2$ quiver gauge theories, which involves a finite number of $W_4$.
screening charges [90,107–109]. An efficient control on the transformation relations between the DIM operators in different duality frames and at specific points in the parameter space (corresponding to complete Higgsing of the 5d theories) would imply an elegant description of some 3d dualities. The peculiar example of the self-mirror $T[\mathbb{U}(N)]$ theory [7] has been recently considered in [30] from the $\mathbb{W}_{q,1}$ perspective.

[1] K. A. Intriligator and N. Seiberg, Mirror symmetry in three-dimensional gauge theories, Phys. Lett. B 387, 513 (1996).
[2] A. Hanany and E. Witten, Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics, Nucl. Phys. B492, 152 (1997).
[3] T. Kitao, K. Ohta, and N. Ohta, Three-dimensional gauge dynamics from brane configurations with $(p, q)$—five-brane, Nucl. Phys. B539, 79 (1999).
[4] O. Aharony, A. Hanany, and B. Kol, Webs of $(p, q)$ five-branes, five-dimensional field theories and grid diagrams, J. High Energy Phys. 01 (1998) 002.
[5] O. Aharony and A. Hanany, Branes, superpotentials and superconformal fixed points, Nucl. Phys. B504, 239 (1997).
[6] B. Kol and J. Rahmfeld, BPS spectrum of five-dimensional field theories, $(p, q)$ webs and curve counting, J. High Energy Phys. 08 (1998) 006.
[7] D. Gaiotto and E. Witten, S-duality of boundary conditions in $N = 4$ super Yang-Mills theory, Adv. Theor. Math. Phys. 13, 721 (2009).
[8] E. Witten, SL(2,Z) action on three-dimensional conformal field theories with Abelian symmetry, arXiv:hep-th/0307041.
[9] A. Kapustin and M. J. Strassler, On mirror symmetry in three-dimensional Abelian gauge theories, J. High Energy Phys. 04 (1999) 021.
[10] D. R. Gulotta, C. P. Herzog, and S. S. Pufu, From necklace quivers to the F-theorem, operator counting, and T(UN), J. High Energy Phys. 12 (2011) 077.
[11] B. Assel, Hanany-Witten effect and SL(2,Z) dualities in matrix models, J. High Energy Phys. 10 (2014) 117.
[12] O. Aharony, IR duality in $d = 3$ $\mathcal{N} = 2$ supersymmetric USp(2N(c)) and U(N(c)) gauge theories, Phys. Lett. B 404, 71 (1997).
[13] O. Aharony, A. Hanany, K. A. Intriligator, N. Seiberg, and M. J. Strassler, Aspects of $N = 2$ supersymmetric gauge theories in three-dimensions, Nucl. Phys. B499, 67 (1997).
[14] J. de Boer, K. Hori, and Y. Oz, Dynamics of $N = 2$ supersymmetric gauge theories in three-dimensions, Nucl. Phys. B500, 163 (1997).
[15] O. Aharony, S. S. Razamat, N. Seiberg, and B. Willett, 3d dualities from 4d dualities, J. High Energy Phys. 07 (2013) 149.
[16] O. Aharony, S. S. Razamat, N. Seiberg, and B. Willett, 3d dualities from 4d dualities for orthogonal groups, J. High Energy Phys. 08 (2013) 099.
[17] A. Amariti and C. Klare, A journey to 3d: Exact relations for adjoint SQCD from dimensional reduction, J. High Energy Phys. 05 (2015) 148.
[18] S. Benvenuti and S. Pasquetti, 3d $\mathcal{N} = 2$ mirror symmetry, pq-webs and monopole superpotentials, J. High Energy Phys. 08 (2016) 136.
[19] F. Benini, S. Benvenuti, and S. Pasquetti, SUSY monopole potentials in $2 + 1$ dimensions, J. High Energy Phys. 08 (2017) 086.
[20] S. Benvenuti and S. Giacomelli, Abelianization and sequential confinement in $2 + 1$ dimensions, J. High Energy Phys. 10 (2017) 173.
[21] S. Benvenuti and S. Giacomelli, Supersymmetric Gauge Theories with Decoupled Operators and Chiral Ring Stability, Phys. Rev. Lett. 119, 251601 (2017).
[22] S. Giacomelli and N. Mekareeya, Mirror theories of 3d $\mathcal{N} = 2$ SQCD, J. High Energy Phys. 03 (2018) 126.
[23] A. Kapustin, B. Willett, and I. Yaakov, Exact results for Wilson loops in superconformal Chern-Simons theories with matter, J. High Energy Phys. 03 (2010) 089.
[24] N. Hama, K. Hosomichi, and S. Lee, SUSY gauge theories on squashed three-spheres, J. High Energy Phys. 05 (2011) 014.
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PHYS. REV. D 98, 126002 (2018)

[25] V. Pestun et al., Localization techniques in quantum field theories, J. Phys. A 50, 440301 (2017).
[26] S. Katz, P. Mayr, and C. Vafa, Mirror symmetry and exact solution of 4D $N = 2$ gauge theories: I., Adv. Theor. Math. Phys. 1, 53 (1997).
[27] L. Bao, E. Pomoni, M. Taki, and F. Yagi, M5-branes, toric diagrams and gauge theory duality, J. High Energy Phys. 04 (2012) 105.
[28] D. Gaiotto, L. Rastelli, and S. S. Razamat, Bootstraping the superconformal index with surface defects, J. High Energy Phys. 01 (2013) 022.
[29] D. Gaiotto and H.-C. Kim, Surface defects and instanton partition functions, J. High Energy Phys. 10 (2016) 012.
[30] A. Nedelin, S. Pasquetti, and Y. Zenkevich, T[U(N)] duality webs: Mirror symmetry, spectral duality and gauge/CFT correspondences, arXiv:1712.08140.
[31] A. Nedelin, S. Pasquetti, and Y. Zenkevich (to be published).
[32] F. Aprile, S. Pasquetti, and Y. Zenkevich (to be published).
[33] N. A. Nekrasov, Instanton partition functions and M-theory, in Proceedings of 15th International Seminar on High Energy Physics (Quarks, 2008).
[34] N. Aghaei, A. Amariti, and Y. Sekiguchi, Notes on integral identities for 3d supersymmetric dualities, J. High Energy Phys. 04 (2018) 022.
[35] K. Maruyoshi and J. Song, Enhancement of Supersymmetry via Renormalization Group Flow and the Superconformal Index, Phys. Rev. Lett. 118, 151602 (2017).
[36] E. Mintun, J. Polchinski, and S. Sun, The field theory of intersecting $D_3$-branes, J. High Energy Phys. 08 (2015) 118.
[37] K. Maruyoshi and J. Song, $\mathcal{N} = 1$ deformations and RG flows of $\mathcal{N} = 2$ SCFTs, J. High Energy Phys. 02 (2017) 075.
[38] Y. Pan and W. Peelaers, Ellipsoid partition function from Seiberg-Witten monopoles, J. High Energy Phys. 10 (2015) 183.
[39] J. Gomis, B. Le Floch, Y. Pan, and W. Peelaers, Intersecting surface defects and two-dimensional CFT, Phys. Rev. D 96, 045003 (2017).
[40] Y. Pan and W. Peelaers, Intersecting surface defects and instanton partition functions, J. High Energy Phys. 07 (2017) 073.
[41] N. Nekrasov, BPS/CFT correspondence II: Instantons at crossroads, moduli and compactness theorem, Adv. Theor. Math. Phys. 21, 503 (2017).
[42] N. Nekrasov, BPS/CFT correspondence III: Gauge origami partition function and qq-characters, Commun. Math. Phys. 358, 863 (2018).
[43] F. Nieri, Y. Pan, and M. Zabzine, 3d Expansions of 5d instanton partition functions, J. High Energy Phys. 04 (2018) 092.
[44] F. Nieri, Y. Pan, and M. Zabzine, Bootstrapping the $S^5$ partition function, EPJ Web Conf. 191, 06005 (2018).
[45] A. Lossev, N. Nekrasov, and S.L. Shatashvili, Testing Seiberg-Witten solution, in Strings, Branes and Dualities: Proceedings of NATO Advanced Study Institute, Cargese, France, 1997 (1997), pp. 359–372.
[46] G. W. Moore, N. Nekrasov, and S. Shatashvili, D particle bound states and generalized instantons, Commun. Math. Phys. 209, 77 (2000).
[47] J. F. Gomis, B. Le Floch, Y. Pan, and W. Peelaers, Intersecting surface defects and two-dimensional CFT, Phys. Rev. D 96, 045003 (2017).
[48] A. Losev, N. Nekrasov, and S. L. Shatashvili, Issues in topological gauge theory, Nucl. Phys. B534, 549 (1998).
[49] N. A. Nekrasov, Seiberg-Witten prepotential from instanton counting, Adv. Theor. Math. Phys. 7, 831 (2003).
[50] N. Nekrasov and A. Okounkov, in Seiberg-Witten theory and random partitions, edited by P. Etingof, V. Retakh, I. M. Singer, The Unity of Mathematics. Progress in Mathematics Vol. 244 (Birkhäuser Boston, 2006).
[51] H.-C. Kim, J. Kim, and S. Kim, Instantons on the 5-sphere and M5-branes, arXiv:1211.0144.
[52] K. Hosomichi, R.-K. Seong, and S. Terashima, Supersymmetric gauge theories on the five-sphere, Nucl. Phys. B865, 376 (2012).
[53] Y. Imamura, Perturbative partition function for squashed $S^3$, Prog. Theor. Exp. Phys. 2013, 073B01 (2013).
[54] G. Lockhart and C. Vafa, Superconformal partition functions and non-perturbative topological strings, J. High Energy Phys. 10 (2018) 051.
[55] J. Källén, J. Qiu, and M. Zabzine, The perturbative partition function of supersymmetric 5D Yang-Mills theory with matter on the five-sphere, J. High Energy Phys. 08 (2012) 157.
[56] K. A. Intriligator, D. R. Morrison, and N. Seiberg, Five-dimensional supersymmetric gauge theories and degenerations of Calabi-Yau spaces, Nucl. Phys. B497, 56 (1997).
[57] N. C. Leung and C. Vafa, Branes and toric geometry, Adv. Theor. Math. Phys. 2, 91 (1998).
[58] T. J. Hollowood, A. Iqbal, and C. Vafa, Matrix models, geometric engineering and elliptic genera, J. High Energy Phys. 03 (2008) 069.
[59] M. Aganagic, A. Klemm, M. Marino, and C. Vafa, The topological vertex, Commun. Math. Phys. 254, 425 (2005).
[60] A. Iqbal and A.-K. Kashani-Poor, The vertex on a strip, Adv. Theor. Math. Phys. 10, 317 (2006).
[61] H. Awata and H. Kanno, Instanton counting, Macdonald functions and the moduli space of D-branes, J. High Energy Phys. 05 (2005) 039.
[62] A. Iqbal, C. Kozcaz, and C. Vafa, The Refined topological vertex, J. High Energy Phys. 10 (2009) 069.
[63] H. Awata and H. Kanno, Refined BPS state counting from Nekrasov’s formula and Macdonald functions, Int. J. Mod. Phys. A 24, 2253 (2009).
[64] L. Bao, V. Mitev, E. Pomoni, M. Taki, and F. Yagi, Non-Lagrangian theories from brane junctions, J. High Energy Phys. 01 (2014) 175.
[65] M. Taki, Seiberg duality, 5d SCFTs and Nekrasov partition functions, arXiv:1401.7200.
[66] B. Bastian, S. Hohenegger, A. Iqbal, and S.-J. Rey, Triality in little string theories, Phys. Rev. D 97, 046004 (2018).
[67] B. Bastian, S. Hohenegger, A. Iqbal, and S.-J. Rey, Beyond triality: Dual quiver gauge theories and little string theories, J. High Energy Phys. 11 (2018) 016.
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PHYS. REV. D 98, 126002 (2018)

[68] J. Qiu, L. Tizzano, J. Winding, and M. Zabzine, Gluing Nekrasov partition functions, Commun. Math. Phys. 337, 785 (2015).

[69] Y. Pan, 5d Higgs branch localization, Seiberg-Witten equations and contact geometry, J. High Energy Phys. 01 (2015) 145.

[70] F. Nieri, S. Pasquetti, F. Passerini, and A. Torrielli, 5D partition functions, $q$-Virasoro systems and integrable spin-chains, J. High Energy Phys. 12 (2014) 040.

[71] J. Qiu and M. Zabzine, Factorization of 5D super Yang-Mills theory on $Y^{p,q}$ spaces, Phys. Rev. D 89, 065040 (2014).

[72] F. Benini and W. Pebelers, Higgs branch localization in three dimensions, J. High Energy Phys. 05 (2014) 030.

[73] J. Gomis and B. Le Floch, M2-brane surface operators and gauge theory dualities in Toda, J. High Energy Phys. 04 (2016) 183.

[74] S. Pasquetti, Factorisation of $N = 2$ theories on the squashed 3-sphere, J. High Energy Phys. 04 (2012) 120.

[75] C. Beem, T. Dimofte, and S. Pasquetti, Holomorphic blocks in three dimensions, J. High Energy Phys. 12 (2014) 177.

[76] M. Taki, Holomorphic blocks for 3d non-Abelian partition functions, arXiv:1012.4620.

[77] J. Ding and K. Iohara, Generalization of drinfeld quantum affine algebras, Lett. Math. Phys. 41, 181 (1997).

[78] K. Miki, A $(q,\gamma)$ analog of the $w_{1+\infty}$ algebra, J. Math. Phys. (N.Y.) 48, 123520 (2007).

[79] A. Mironov, A. Morozov, and Y. Zenkevich, Spectral duality in elliptic systems, six-dimensional gauge theories and topological strings, J. High Energy Phys. 05 (2016) 121.

[80] C. Beem, T. Dimofte, and S. Pasquetti, Holomorphic blocks in three dimensions, J. High Energy Phys. 12 (2014) 177.

[81] A. Mironov, A. Morozov, S. Shakirov, and A. Smirnov, Spectral duality in elliptic systems, six-dimensional gauge theories and topological strings, J. High Energy Phys. 05 (2016) 121.

[82] N. Nekrasov and V. Pestun, Seiberg-Witten geometry of four dimensional $N = 2$ quiver gauge theories, arXiv:1211.2240.

[83] E. Carlsson, N. Nekrasov, and A. Okounkov, Five dimensional gauge theories and vertex operators, Moscow Math. J. 14, 39 (2014).

[84] N. Nekrasov, BPS/CFT correspondence: Non-perturbative Dyson-Schwinger equations and qq-characters, J. High Energy Phys. 03 (2016) 181.

[85] H. Awata and Y. Yamada, Five-dimensional AGT conjecture and the deformed Virasoro algebra, J. High Energy Phys. 01 (2010) 125.

[86] H. Awata and Y. Yamada, Five-dimensional AGT relation and the deformed beta-ensemble, Prog. Theor. Phys. 124, 227 (2010).

[87] A. Mironov, A. Morozov, S. Shakirov, and A. Smirnov, Proving AGT conjecture as HS duality: extension to five dimensions, Nucl. Phys. B855, 128 (2012).

[88] F. Nieri, S. Pasquetti, and F. Passerini, 3d and 5d gauge theory partition functions as $q$-deformed CFT correlators, Lett. Math. Phys. 105, 109 (2015).

[89] A. Nedelin, F. Nieri, and M. Zabzine, $q$-Virasoro modular double and 3d partition functions, Commun. Math. Phys. 353, 1059 (2017).

[90] V. Mitev and E. Pomoni, Toda 3-point functions from topological strings, J. High Energy Phys. 06 (2015) 049.

[91] M. Isachenkov, V. Mitev, and E. Pomoni, Toda 3-point functions from topological strings II, J. High Energy Phys. 08 (2016) 066.

[92] M. Aganagic and N. Haouzi, ADE little string theory on a Riemann surface (and triality), arXiv:1506.04183.

[93] T. Kimura and V. Pestun, Quiver W-algebras, Lett. Math. Phys. 108, 1351 (2018).

[94] S. Benvenuti, G. Bonelli, M. Ronzani, and A. Tanzini, Symmetry enhancements via 5d instantons, $qW$-algebras and (1,0) superconformal index, J. High Energy Phys. 09 (2016) 053.

[95] M. Aganagic, E. Frenkel, and A. Okounkov, Quantum $q$-Langlands correspondence, arXiv:1701.03146.

[96] H. Awata, H. Kanno, T. Matsumoto, A. Mironov, A. Morozov, Y. Okubo, and Y. Zenkevich, Explicit examples of DIM constraints for network matrix models, J. High Energy Phys. 07 (2016) 103.

[97] H. Awata, A. Mironov, A. Morozov, A. Morozov, Y. Okubo, and Y. Zenkevich, Toric Calabi-Yau threefolds as quantum integrable systems. R-matrix and $RTT$ relations, J. High Energy Phys. 10 (2016) 047.

[98] H. Awata, H. Kanno, A. Mironov, A. Morozov, A. Morozov, Y. Okubo, and Y. Zenkevich, Anomaly in RTT relation for DIM algebra and network matrix models, Nucl. Phys. B918, 358 (2017).

[99] A. Narukawa, The modular properties and the integral representations of the multiple elliptic gamma functions, arXiv:math:0306164.

[100] H. Awata, B. Feigin, and J. Shiraishi, Quantum algebraic approach to refined topological vertex, J. High Energy Phys. 03 (2012) 041.

[101] A. Mironov, A. Morozov, Y. Zenkevich, and A. Zotov, Spectral duality in integrable systems from AGT conjecture, Pis'ma Zh. Eksp. Teor. Fiz. 97, 49 (2013) [JETP Lett. 97, 45 (2013)].

[102] A. Mironov, A. Morozov, B. Runov, Y. Zenkevich, and A. Zotov, Spectral dualities in XXZ spin chains and five dimensional gauge theories, J. High Energy Phys. 12 (2013) 034.

[103] M. Aganagic, N. Haouzi, C. Kozcaz, and S. Shakirov, Gauge/Liouville triality, arXiv:1309.1687.

[104] M. Aganagic and S. Shakirov, Gauge/vortex duality and AGT, in New Dualities of Supersymmetric Gauge Theories, edited by J. Teschner (2016), pp. 419–448.

[105] M. Aganagic, N. Haouzi, and S. Shakirov, $A_n$-triality, arXiv:1403.3657.