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Fundamental classes in motivic homotopy theory. (English) Zbl 1483.14040

J. Eur. Math. Soc. (JEMS) 23, No. 12, 3935-3993 (2021).

Following ideas of Fulton and MacPherson, the authors define the bivariant theory with coefficients in a motivic (ring) spectrum $E$, graded by integers $n \in \mathbb{Z}$ and virtual vector bundles $v$ on a scheme $X$, as the following group:

$$E_n(X/S, v) := \text{Hom}_{\mathcal{SH}(S)}(\text{Thom}_X(v)[n], p^!(E))$$

for any finite type separated morphism $p : X \to S$. These groups satisfy the usual properties: functoriality, covariance for proper maps, contravariance for étale maps, intersection product. The twist by a virtual vector bundle $v$ is essential because the motivic homotopy category $\mathcal{SH}(S)$ contains theories which are not oriented, such as Chow-Witt groups, Milnor-Witt motivic cohomology or hermitian $K$-theory.

The main result of the paper is the construction of a canonical fundamental class $\eta_f \in E_n(X/S, v)$ for any smoothable lci morphism $f : X \to Y$, where $(L_f)$ is the virtual tangent bundle of $f$, and satisfying an associativity condition and an excess intersection formula. The case of a smooth morphism comes from Morel and Voevodsky’s homotopy purity theorem, which asserts that, for smooth closed pairs $(X, Z)$, the homotopy type of $X$ with support in $Z$ is isomorphic to $\text{Th}Z(N_ZX)$, the Thom space of the normal bundle of $Z$ in $X$. More subtle, the case of a regular closed immersion is achieved thanks the technique of deformation to the normal cone of Fulton. Both cases are then glued together in a straightforward manner.

The last part of the article shows how the fundamental class gives rise to Gysin morphisms (i.e. wrong-way variance). The associativity (resp. excess intersection) property above corresponds to the compatibility with composition (resp. excess intersection formula) satisfied by these Gysin morphisms, as in Chow theory. This is related to the absolute purity property.

The absolute purity conjecture, stated for étale torsion sheaves and $l$-adic sheaves, has been a difficult problem since its formulation by Grothendieck in the mid-sixties (published in 1977 in SGA5). For some time, only the case of one-dimensional regular schemes was known thanks to Deligne. A complete proof was found decades later by Gabber, using a refinement of De Jong resolution of singularities.

For triangulated mixed motives, modeled on the previous étale setting by Beilinson, this conjecture was implicit in the expected property. It was first formulated and proved in the rational case by Cisinski-Déglise. Later the absolute purity property was explicitly highlighted in [D.-C. Cisinski and F. Déglise, “Integral mixed motives in equal characteristic”, Preprint, arXiv:1410.6359], and proven for integral étale motives. It became apparent that this important property should hold in greater generality, and philosophically be an addition to the six functors formalism.

This property has been obtained in several contexts (rational motives, étale motives, KGL-modules) and is studied in greater generality in the present article. New examples can be found in subsequent work by the authors in e.g. [F. Déglise et al., J. Éc. Polytech., Math. 8, 533–583 (2021; Zbl 1471.14052)], and also in [M. Frankland and M. Spitzweck; “Towards the dual motivic Steenrod algebra in positive characteristic”, Preprint, arXiv:1711.05230].

An important consequence of this work is a motivic Gauss-Bonnet formula, computing Euler characteristics in the motivic homotopy category. This result is a generalization of a theorem of Levine which explores the idea of refining classical formulas to the quadratic setting (see also the work of Fasel, Hoyois, Kass, Wickelgren and many others).

Reviewer: Niels Feld (Toulouse)
MSC:

14F42 Motivic cohomology; motivic homotopy theory
14C17 Intersection theory, characteristic classes, intersection multiplicities in algebraic geometry
19E15 Algebraic cycles and motivic cohomology (K-theoretic aspects)

Cited in 7 Documents

Keywords:

fundamental class; motivic homotopy; Gysin morphism; Euler class; motivic Gauss-Bonnet formula

Full Text: DOI arXiv

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