Selective decay in fluids with advected quantities: MHD and Hall MHD

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Abstract

Modifications of the equations of ideal fluid dynamics with advected quantities are introduced for either the energy $h$ or the Casimir quantities $C$ in the Lie-Poisson formulation. The dissipated quantity (energy or Casimir, respectively) is shown to decrease in time until the modified system reaches a state that holds for ideal energy-Casimir equilibria, namely $\delta(h + C) = 0$. The result holds for Lie-Poisson equations in general, independently of the Lie algebra and the choice of Casimir. This process is illustrated with a number of selected decay examples that pass to stable energy-Casimir equilibria for magnetohydrodynamics (MHD) and Hall MHD in 2D and 3D by decay of either the energy or the Casimirs.

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1 Introduction

Historically, the hypothesis of selective decay in MHD turbulence assumed that the total energy was to be minimized, subject to the conservation of certain ideal invariants. This hypothesis was consistent with the observed long term evolution of freely decaying MHD turbulence at high magnetic Reynolds number, $R_m = U L / \eta$, where $U$ is a typical velocity scale of the flow, $L$ is a typical length scale of the flow and $\eta$ is the magnetic resistivity. This situation was called an inverse cascade Frisch, Pouquet, Leorat and Mazure [1975], because the energy flux is predominantly toward small scales while the flux of the ideal invariant known as the magnetic helicity passes toward the larger scales and ostensibly creates the spiral structures observed in the decay of MHD turbulence. See Matthaeus and Montgomery [1980]; Montgomery and Bates [1999] for further historical discussion of the selective decay hypothesis for MHD turbulence.

The aim of this paper is to impose selective decay of either energy or the invariants of the Lie-Poisson bracket of the ideal theory by using the MHD Hamiltonian structure, following the work of Gay-Balmaz and Holm [2013] for geophysical fluid dynamics. We interpret the resulting modifications of the equations as a means of dynamically and nonlinearly parameterizing the interactions between disparate scales, by introducing new nonlinear pathways to dissipation, based on selective decay. Remarkably, the theory developed here for selective decay of either the energy $h$ or the Casimir $C$ does not always take the modified system toward the standard energy-Casimir equilibria of the ideal equations, obtained from a critical point of the sum $\delta(h+C) = 0$. Our selective decay theory is always consistent with the energy-Casimir equilibrium conditions, but its decay process sometimes tends towards only some of the energy-Casimir conditions, not all of them. This is explained in the proof of the main result, Theorem 2.3, which shows that, although the asymptotic tendency to zero of the decay terms in the modified system always recovers energy-Casimir equilibrium conditions, the number of decay terms is sometimes less than the number of energy-Casimir equilibrium conditions. Consequently, the selective decay mechanism introduced here does not necessarily approach fully time-independent solutions of the unmodified ideal equations. Several examples of this feature are given for compressible and incompressible MHD and Hall MHD in sections 4 and 5.

Casimirs

A Poisson manifold is a manifold $P$ with a Poisson bracket $\{ \cdot, \cdot \}$ defined on the space of smooth functions on $P$; see, e.g., Marsden and Ratiu [1994]. A Poisson system with Hamiltonian $h : P \to \mathbb{R}$ yields the time-evolution of any smooth function $f : P \to \mathbb{R}$ by computing the solution curves of the dynamical equation $df/dt = \{f, h\}$. Poisson systems often arise by Lie-group reduction of Hamiltonian systems with symmetry on Lie groups, in which case the Poisson bracket is called a Lie-Poisson bracket. Examples include the Euler equations of ideal incompressible fluid dynamics, for which the symmetry group is the particle-relabeling group Arnold and Khesin [1998], and the equations for a heavy top, for which the symmetry group is the Euclidean group Holm [2011].

Definition 1.1 (Casimirs). Casimirs on a Poisson manifold $(P, \{ \cdot, \cdot \})$ are functions $\mathcal{C}$ that satisfy $\{ \mathcal{C}, h \} = 0$, for all $h$, that is they are constant under the flow generated by the Poisson bracket for any choice of the Hamiltonian. The existence of Casimirs is thus due to the degeneracy of the Poisson bracket. In the case of reduction by symmetry on Lie groups, this degeneracy arises
when passing from the canonical Hamiltonian formulation in terms of Lagrangian variables to the Lie–Poisson formulation in terms of symmetry-reduced variables.

An example is the reduction of the Hamiltonian description of rigid body dynamics from the six-dimensional phase space $T^*SO(3)$ of the Euler angles for $SO(3)$ rotations, to the three-dimensional space of angular momenta in $\mathfrak{so}(3)^* \simeq \mathbb{R}^3$. For ideal fluids, the reduction is from the Lagrangian variables to the Eulerian variables, which are invariant under relabeling of Lagrangian particles. Thus, Casimir conservation is a property of the Lie–Poisson bracket that results from the reduction by symmetry, not the choice of Hamiltonian. Indeed, the Casimirs commute under the Lie–Poisson bracket with any Hamiltonian that is expressed in terms of the symmetry-reduced variables. This means the motion generated by the Lie–Poisson bracket in the symmetry-reduced variables takes place along intersections of level sets of the Hamiltonian $h$ and a Casimir $C$.

Casimirs have been used in stability analyses of fluid and plasma equilibria which extend traditional energy methods to the energy-Casimir method Arnold [1969]; HMRW [1985]. This method applies in determining the stability of a certain class of equilibrium solutions $p_e \in P$ under the Poisson flow $df/dt = \{f, h\}$ defined on the Poisson manifold $P$ and generated by a Hamiltonian $h$. Namely, the energy-Casimir method supposes that there is a function $C$, the Casimir, which is constant under the flow generated by the Poisson bracket (since $\{C, h\} = 0$ for all $h$) and that the equilibrium solution $p_e$ is a critical point of the sum $h_C := h + C$, so that $\delta h_C = \langle D h_C(p_e), \delta p \rangle = 0$ for a nondegenerate pairing $\langle \cdot, \cdot \rangle$. Linear Lyapunov stability follows if the critical point $p_e$ is a local extremum of $h_C$, that is, if $\delta^2 h_C(p_e)$ is positive definite or negative definite. This condition implies formal stability since $\delta^2 h_C(p_e)$ is conserved by the linearized equations around the equilibrium solution $p_e$ and if it is either positive definite or negative definite, then it defines a norm for Lyapunov stability, see HMRW [1984]. Nonlinear Lyapunov stability follows by a further argument, which is available when the functional $h_C$ is convex in the neighborhood of $p_e$. One may use the energy-Casimir method to seek stable equilibrium states. Such equilibrium states are called stable energy-Casimir equilibria. Because of the exchange symmetry of $h_C := h + C$ under $C \leftrightarrow h$ these equilibrium states may be regarded as either extrema of the energy $h$ on a level set of a Casimir $C$, or vice versa, as extrema of a Casimir $C$ on a level set of the energy $h$.

**Aim of the paper**

The aim of the present paper is to introduce a dissipative modification of the Lie–Poisson flow whose dynamics will tend toward conditions verified by energy-Casimir equilibria of the ideal unmodified equations, starting from any initial state on $P$. In particular, the present paper investigates the effects of a particular approach to imposing selective decay of a particular Casimir while preserving the energy, and vice versa, imposing selective decay of energy while preserving the chosen Casimir. This is accomplished by using the Lie–Poisson structure of the ideal theory, and interpreting the resulting modifications of the equations as nonlinear pathways for selective dissipation that parameterize the observed effects of the interactions among disparate scales of motion. This type of modification is computable at a single time scale, so it may be useful in situations where it would be computationally prohibitive to rely on the slower, indirect effects of viscosity and other types of diffusivity which typically affect both the energy and the Casimir. In particular, the present paper takes the Casimir dissipation approach of Gay-Balmaz and Holm [2013] further by applying it in several fluid models, in both 2D and 3D flows, including compressible fluids, magnetohydrodynamics (MHD) and Hall MHD.
The new feature of the present paper is its introduction of modified Lie–Poisson equations that describe the selective decay of ideal fluids with *advected quantities*. Mathematically, ideal flows that advect fluid properties such as mass, heat and magnetic field may be described by the *combined* actions of Lie groups on their dual Lie algebras and also on the vector spaces in which the advected quantities are defined HMR [1998]. The Casimirs for such flows differ from the Casimirs of simple ideal fluid motion. These differences introduced by the advective flow of fluid properties yield new fluid equilibria and new nonlinear mechanisms for selective decay of either energy or Casimirs. In particular, the proof of Theorem 2.3 of the present paper shows that selective nonlinear dissipation in an advective flow enforces a condition consistent with stable energy-Casimir equilibria of the *unmodified* fluid equations satisfying \( \delta(h + C) = 0 \), for extrema of the energy \( h \) on a level set of a Casimir \( C \), or vice versa, for extrema of a Casimir \( C \) on a level set of the energy \( h \). These extrema may be either maxima or minima, depending on the choice of sign of a parameter \( \theta \) and the sign of the Casimir appearing in the modified equations.

Various types of modifications of the Poisson bracket for Hamiltonian systems have previously been proposed that produce energy dissipation. These proposed modifications include: (i) adding a symmetric bilinear form, Kaufman [1984]; Morrison [1984]; Grmela [1984]; Öttinger [2005]), and (ii) application of a double bracket, Brockett [1991]; Bloch et al. [1996]; Kandrup [1991]; Holm, Putkaradze and Tronci [2008]; Brody, Ellis and Holm [2008]. See also Vallis, Carnevale, and Young [1989]; Shepherd [1990], in which a modification of the transport velocity was used to impose energy dissipation with fixed Casimirs in an incompressible fluid.

The theory we develop here uses the Lie-Poisson Hamiltonian framework for ideal fluids to treat either Casimir decay at fixed energy, or energy decay at fixed Casimirs. That is, the same theory is used here to treat selective decay of either quantity, so that the choice of which mechanism to investigate can be made, motivated for example by the effects seen in large-scale numerical simulations of the fully dissipative equations. The switch from decay of the chosen Casimir \( C \) at fixed Hamiltonian \( h \) to decay of the Hamiltonian \( h \) at fixed Casimir \( C \) is accomplished by a simple exchange \( C \leftrightarrow h \) in one of the key formulas, e.g., in equation (1.5).

**Selective decay of Casimirs.** Selective decay of Casimirs is an effect that was first observed in numerical simulations of 2D incompressible turbulence cascades by Matthaeus and Montgomery [1980] as the rapid decay of the enstrophy (turbulence intensity) while the energy stayed essentially constant. This observed disparity in the time scales for decay of the energy and the enstrophy in 2D turbulence has a profound effect on its energy spectrum. In Gay-Balmaz and Holm [2013], selective decay by Casimir dissipation was introduced by modifying the vorticity equation, based on the well-known Lie–Poisson structure of the Hamiltonian formulation for vorticity dynamics in the case of 2D incompressible flows of ideal fluids Arnold [1966, 1969, 1978]; HMR [1998]. In this framework, the earlier work of Vallis, Carnevale, and Young [1989]; Shepherd [1990] on selective decay of energy at fixed values of the Casimirs was recovered by the exchange of the Casimirs and the Hamiltonian in formula (1.5) for the modified vorticity dynamics. This earlier work studied selective decay for the purpose of finding stable equilibrium states. As we show here, imposing selective decay in fluid flows with advected quantities may also lead to stable energy-Casimir equilibrium states; see Theorem 2.3.

In 3D incompressible fluid turbulence, the energy tends to decay more rapidly than the Casimirs do. This is contrary to selective decay of turbulence in 2D, so a different modeling approach is required in 3D. Thus, a comprehensive theory must be capable of passing within the same
framework from selective decay of the Casimir in 2D to selective decay of the energy in 3D. This is accomplished in the present theory by taking advantage of its exchange symmetry under $C \leftrightarrow h$.

1.1 Parameterizing subgridscale effects on macroscales

In a previous paper Gay-Balmaz and Holm [2013] the problem of parameterizing the interactions of disparate scales in fluid flows was addressed by considering a property of two-dimensional incompressible turbulence. The property considered was a type of selective decay, in which a Casimir of the ideal formulation (enstrophy, in the case of 2D incompressible flows) was observed to decay rapidly in time, compared to the much slower decay of energy. That is, the Casimir was observed to decay, while the energy stayed essentially constant. The previous paper introduced a nonlinear fluid mechanism that produced the selective decay by enforcing Casimir dissipation at constant energy. This mechanism introduced an additional geometric feature into the description of the flow; namely, it introduced a Riemannian inner product on the space of Eulerian fluid variables. The resulting dissipation mechanism based on decay of enstrophy in 2D flows turned out to be related to the numerical method of anticipated vorticity discussed in Sadourny and Basdevant [1981, 1985]. Several examples were given and a general theory of selective decay was developed that used the Lie–Poisson structure of the ideal theory. A scale-selection operator allowed the resulting modifications of the fluid motion equations to be interpreted in these examples as parameterizing the nonlinear dynamical interactions between disparate scales. The type of modified fluid equations that was derived in the previous paper was also proposed for turbulent geophysical flows, where it is computationally prohibitive to rely on the slower, indirect effects of a realistic viscosity, such as in interactions between large-scale, coherent, oceanic flows and the much smaller eddies.

The selective decay mechanism discussed in the previous paper was based on Casimir dissipation in the example of 2D incompressible flows, treated as a dynamical parameterization of the interactions between disparate scales. Following that example, the paper discussed the general theory of selective decay by Casimir dissipation in the Lie algebraic context that underlies the Lie–Poisson Hamiltonian formulation of ideal fluid dynamics, as explained in, e.g., HMR [1998]. In particular, it developed the Kelvin circulation theorem and Lagrange-d’Alembert variational principle for Casimir dissipation. In the Lagrange-d’Alembert formulation, the modification of the motion equation to impose selective decay was seen as an energy-conserving constraint force. Finally, the previous paper extended the Casimir dissipation theory to include fluids that possess advected quantities such as heat, mass, buoyancy, magnetic field, etc., by using the standard method of Lie–Poisson brackets for semidirect-product actions of Lie groups on vector spaces reviewed in HMRW [1985]. The main subsequent examples were the rotating shallow water equations and the 3D Boussinesq equations for rotating stratified incompressible fluid flows.

Plan of the paper. The present paper pursues further the selective decay approach based on Gay-Balmaz and Holm [2013], whose main results are reviewed in the remainder of this Introduction. The formulations of selective decay of either Casimirs or energy on semidirect products is summarized in section 2, then applied in other fluid models that possess advected quantities in the subsequent sections. The applications in this paper include 3D compressible fluids, discussed in section 3, 2D and 3D magnetohydrodynamics (MHD), in section 4 and 2D and 3D Hall MHD in section 5. In each case, we derive the equations that enforce either selective decay of Casimirs at fixed energy, or vice versa.
1.2 Summary of key equations in Gay-Balmaz and Holm [2013]

Let us recall that ideal incompressible 2D fluid flows admit a Hamiltonian formulation in terms of a Lie–Poisson bracket \{·, ·\}_+, given by Arnold [1966, 1969, 1978]

\[
\frac{df(\omega)}{dt} = \{f, h\}_+(\omega) = \left\langle \omega, \left[ \frac{\delta f}{\delta \omega}, \frac{\delta h}{\delta \omega} \right] \right\rangle := \int_D \omega \left\{ \frac{\delta f}{\delta \omega}, \frac{\delta h}{\delta \omega} \right\} dxdy. \tag{1.1}
\]

Here \(\omega\) is the vorticity of the flow, the bracket \{·, ·\} is the 2D Jacobian, written as \(\{f, h\} = J(f, h) = f_x h_y - h_x f_y\), and the angle bracket \(\langle ·, · \rangle\) in (1.1) is the \(L^2\) pairing in the domain \(D\) of the \((x, y)\) plane. For convenience, we shall take the domain \(D\) to be periodic, so we need not worry about boundary terms arising from integrations by parts. Two types of conservation laws are associated with the Hamiltonian formulation. The first one is the conservation of energy, i.e., the Hamiltonian \(h(\omega)\). Conservation law of energy arises from the antisymmetry of the Lie–Poisson bracket as

\[
\frac{dh(\omega)}{dt} = \{h, h\}(\omega) = 0,
\]

for any given choice of \(h\). The second type of conservation law arises because the Lie–Poisson bracket has a kernel (i.e., is degenerate), which means there exist functions \(C(\omega)\) for which

\[
\frac{dC(\omega)}{dt} = \{C, h\}(\omega) = 0, \tag{1.2}
\]

for any Hamiltonian \(h(\omega)\). Functions that satisfy this relation for any Hamiltonian are called Casimir functions. (Lie called them distinguished functions, according to Olver [2000].) For example, the Casimirs for the Lie–Poisson bracket (1.1) in the Hamiltonian formulation of 2D incompressible ideal fluid motion are

\[
C_\Phi(\omega) = \int_D \Phi(\omega) dxdy,
\]

for any smooth function \(\Phi\), Arnold [1966, 1969, 1978].

Ideal 3D fluids also admit this type of Lie–Poisson bracket, given by

\[
\{f, g\}_+(\mathbf{u}) = \int_D \mathbf{u} \cdot \left[ \frac{\delta f}{\delta \mathbf{u}}, \frac{\delta g}{\delta \mathbf{u}} \right] d^3x,
\]

where \(\mathbf{u}\), with \(\text{div } \mathbf{u} = 0\), is the velocity and \([·, ·]\) denotes the Lie bracket of vector fields, i.e., \([\mathbf{u}, \mathbf{v}] = \mathbf{v} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{v}\).

Lie–Poisson brackets and Casimirs. In this paper, we shall denote by \(\mathfrak{g}\) a Lie algebra, with Lie brackets \([·, ·]\), and by \(\mathfrak{g}^\ast\) a space in weak nondegenerate duality with \(\mathfrak{g}\). That is, there exists a bilinear map (called a pairing) \(\langle ·, · \rangle : \mathfrak{g}^\ast \times \mathfrak{g} \to \mathbb{R}\), such that for any \(\xi \in \mathfrak{g}\), the condition \(\langle \mu, \xi \rangle = 0\), for all \(\mu \in \mathfrak{g}^\ast\) implies \(\xi = 0\) and, similarly, for any \(\mu \in \mathfrak{g}^\ast\), the condition \(\langle \mu, \xi \rangle = 0\) for all \(\xi \in \mathfrak{g}\) implies \(\mu = 0\). Recall that \(\mathfrak{g}^\ast\) carries a natural Poisson structure, called the Lie–Poisson structure, and given in terms of the pairing by

\[
\{f, h\}_+(\mu) = \left\langle \mu, \left[ \frac{\delta f}{\delta \mu}, \frac{\delta h}{\delta \mu} \right] \right\rangle, \tag{1.3}
\]
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(see, e.g., Marsden and Ratiu [1994]). Here \(f, g \in \mathcal{F}(\mathfrak{g}^*)\) are real valued functions defined on \(\mathfrak{g}^*\), and \(\delta f/\delta \mu \in \mathfrak{g}\) denotes the functional derivative of \(f\), defined through the duality pairing \(\langle \cdot, \cdot \rangle\), by

\[
\left\langle \frac{\delta f}{\delta \mu}, \delta \mu \right\rangle = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} f(\mu + \varepsilon \delta \mu).
\]

The Lie-Poisson bracket in (1.3) is obtained by symmetry reduction of the canonical Poisson structure on the phase space \(T^*G\) of the Lie group \(G\) with Lie algebra \(\mathfrak{g}\). The symmetry underlying this reduction is given by right translation by \(G\) on \(T^*G\). In the case of ideal fluid motion, this symmetry corresponds to relabeling symmetry of the Lagrangian in Hamilton’s principle.

**Lie–Poisson (LP) equations.** The Lie–Poisson (LP) equations with Hamiltonian \(h : \mathfrak{g}^* \to \mathbb{R}\) are, by definition, the Hamilton equations associated to the Poisson structure (1.3), i.e.,

\[
\frac{df}{dt} = \{f, h\}_+ \quad \text{for all} \quad f \in \mathcal{F}(\mathfrak{g}^*).
\]

They are explicitly written as

\[
\partial_t \mu + \text{ad}^*_{\frac{\delta h}{\delta \mu}} \mu = 0,
\]

where \(\text{ad}^*_{\xi} : \mathfrak{g}^* \to \mathfrak{g}^*\) is the coadjoint operator defined by \(\langle \text{ad}^*_{\xi} \mu, \eta \rangle = \langle \mu, [\xi, \eta] \rangle\). One recalls that the coadjoint operator is equivalent to the Lie derivative, i.e., \(\text{ad}^*_{\xi} \mu = L_{\xi} \mu\), when \(\mu \in \mathfrak{g}^* \simeq \Omega^1 \otimes \text{dVol}\) is a 1-form density, as occurs in the case when \(\mu\) is the momentum density in ideal fluid dynamics.

**Casimir functions.** A function \(C : \mathfrak{g}^* \to \mathbb{R}\) is called a Casimir function for the Lie–Poisson structure (1.3) if it verifies \(\{C, f\}_+ = 0\) for all functions \(f \in \mathcal{F}(\mathfrak{g}^*)\) or, equivalently

\[
\text{ad}^*_{\frac{\delta C}{\delta \mu}} \mu = 0,
\]

for all \(\mu \in \mathfrak{g}^*\). A Casimir function \(C\) is therefore a conserved quantity for Lie–Poisson equations associated to any choice of the Hamiltonian \(h\).

**Symmetric bilinear form.** Below, we will denote by \(\gamma_{\mu}\) a (possibly \(\mu\)-dependent, \(\mu \in \mathfrak{g}^*\)) symmetric bilinear form \(\gamma_{\mu} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}\). This form is said to be positive if

\[
\gamma_{\mu}(\xi, \xi) \geq 0, \quad \text{for all} \quad \xi \in \mathfrak{g}.
\]

**Definition 1.2** (Casimir-dissipative LP equation). Given a Casimir function \(C(\mu)\), for \(\mu \in \mathfrak{g}^*\), a positive symmetric bilinear form \(\gamma_{\mu}\), and a real number \(\theta > 0\), we consider the following modification of the Lie–Poisson (LP) dynamical equation (1.4) to produce the Casimir dissipative LP equation:

\[
\frac{df(\mu)}{dt} = \{f, h\}_+ - \theta \gamma_{\mu} \left( \left[ \frac{\delta f}{\delta \mu}, \frac{\delta h}{\delta \mu} \right], \left[ \frac{\delta C}{\delta \mu}, \frac{\delta h}{\delta \mu} \right] \right),
\]

for arbitrary functions \(f, h : \mathfrak{g}^* \to \mathbb{R}\).

Equation (1.5) yields the following equation for \(\mu\),

\[
\partial_t \mu + \text{ad}^*_{\frac{\delta h}{\delta \mu}} \mu = \theta \text{ad}^*_{\frac{\delta h}{\delta \mu}} \left[ \frac{\delta C}{\delta \mu}, \frac{\delta h}{\delta \mu} \right],
\]

(1.6)
where $b : g \to g^*$ is the flat operator associated to $\gamma_\mu$, that is, for $\xi \in g$, the linear form $\xi^\flat \in g^*$ is defined by $\langle \xi^\flat, \eta \rangle = \gamma_\mu(\xi, \eta)$, for all $\eta \in g$.

Note that the flat operator $b$ need not be either injective or surjective. Note also that in equation (1.6) above, the flat operator is evaluated at $\mu$. It is important to observe that the modification term depends on both the given Hamiltonian function $h$ and the chosen Casimir $C$. It is convenient to write (1.6) as

$$\partial_t \mu + \text{ad}^*_b \delta h \delta \mu = 0,$$

for the modified momentum

$$\tilde{\mu} := \mu + \theta \left[ \frac{\delta h}{\delta \mu}, \frac{\delta C}{\delta \mu} \right]^b.$$

The energy is preserved by the dynamics of equations (1.5) and (1.6), since we have

$$\frac{dh(\mu)}{dt} = \{h, h\} - \theta \gamma_\mu \left( \left[ \frac{\delta h}{\delta \mu}, \frac{\delta h}{\delta \mu} \right], \left[ \frac{\delta C}{\delta \mu}, \frac{\delta h}{\delta \mu} \right] \right) = 0.$$ 

However, when $\theta > 0$ the Casimir function $C$ is dissipated since

$$\frac{dC(\mu)}{dt} = \{C, h\} - \theta \gamma_\mu \left( \left[ \frac{\delta C}{\delta \mu}, \frac{\delta h}{\delta \mu} \right], \left[ \frac{\delta C}{\delta \mu}, \frac{\delta C}{\delta \mu} \right] \right) = - \theta \left\| \left[ \frac{\delta C}{\delta \mu}, \frac{\delta h}{\delta \mu} \right] \right\|_\gamma,$$

where $\|\xi\|_\gamma^2 := \gamma_\mu(\xi, \xi)$ is the quadratic form (possibly degenerate) associated to the positive bilinear form $\gamma_\mu$.

**Remark 1.3** (Left-invariant case). Recall that the Lie–Poisson structure (1.3) is associated to right $G$-invariance on $T^*G$. We have made this choice because ideal fluids are naturally right-invariant systems in the Eulerian representation. Other systems, such as rigid bodies, are left $G$-invariant. In this case, one obtains the Lie–Poisson brackets $\{f, g\}_-(\mu) = - \left\langle \mu, \left[ \frac{\delta f}{\delta \mu}, \frac{\delta h}{\delta \mu} \right] \right\rangle$ and this leads to the following change of sign in the Casimir-dissipative LP equation (1.6):

$$\partial_t \mu - \text{ad}^*_b \delta h \delta \mu = \theta \text{ad}^*_b \left[ \frac{\delta C}{\delta \mu}, \frac{\delta h}{\delta \mu} \right]^b.$$ 

The modified momentum is now

$$\tilde{\mu} := \mu - \theta \left[ \frac{\delta h}{\delta \mu}, \frac{\delta C}{\delta \mu} \right]^b,$$

and we have, in comparison with (1.5),

$$\frac{df(\mu)}{dt} = \{f, h\}_+ - \theta \gamma_\mu \left( \left[ \frac{\delta f}{\delta \mu}, \frac{\delta h}{\delta \mu} \right], \left[ \frac{\delta C}{\delta \mu}, \frac{\delta h}{\delta \mu} \right] \right).$$

**Definition 1.4** (Energy-dissipative LP equation). As discussed in Gay-Balmaz and Holm [2013], by simply exchanging $h$ and $C$ in the $\theta$-term of equation (1.5) one obtains an energy-dissipative LP equation that preserves the chosen Casimir $C$:

$$\frac{df(\mu)}{dt} = \{f, h\}_+ - \theta \gamma_\mu \left( \left[ \frac{\delta f}{\delta \mu}, \frac{\delta C}{\delta \mu} \right], \left[ \frac{\delta h}{\delta \mu}, \frac{\delta C}{\delta \mu} \right] \right),$$

for arbitrary functions $f, h : g^* \to \mathbb{R}$. 

Remark 1.5 (Energy-dissipative formulation). In the energy-dissipative formulation (1.12), we have $dC/dt = 0$ and energy decay given by

$$\frac{dh(\mu)}{dt} = \{h, h\} + \theta \gamma (\left[ \frac{\delta h}{\delta \mu}, \frac{\delta C}{\delta \mu} \right], \left[ \frac{\delta h}{\delta \mu}, \frac{\delta C}{\delta \mu} \right]) = \theta \left\| \left[ \frac{\delta h}{\delta \mu}, \frac{\delta C}{\delta \mu} \right] \right\|_\gamma^2.$$  

(1.13)

By symmetry under the exchange $C \leftrightarrow h$, the two rates of decay are the same in (1.9) and (1.13).

Equation (1.12) leads to the following equation for $\mu$,

$$\partial_t \mu + \text{ad}^*_{\delta h/\delta \mu} \mu = -\theta \text{ad}^*_{\delta C/\delta \mu} \left[ \frac{\delta C}{\delta \mu}, \frac{\delta h}{\delta \mu} \right],$$  

(1.14)

where $\gamma : \mathfrak{g} \to \mathfrak{g}^*$ is again the flat operator associated to $\gamma$. Of course, these equations follow from (1.6) by exchanging $C$ and $h$ in the $\theta$-term.

Remark 1.6 (Double-bracket formulations). There are apparent similarities in the Lie algebraic formulations of the present energy dissipation and the double-bracket formulations mentioned earlier and reviewed, for example, in Bloch et al. [2012]. Indeed, for general Lie algebras the double bracket dissipation equations can be written as

$$\frac{df(\mu)}{dt} = \{f, h\} + \theta \gamma^* \left( \text{ad}^*_{\delta h/\delta \mu} \mu, \text{ad}^*_{\delta f/\delta \mu} \mu \right),$$  

(1.15)

(compare with equation (1.12) to see the differences) where $\gamma$ is a inner product on $\mathfrak{g}$, $\gamma^*$ is the inner product induced on $\mathfrak{g}^*$, and $k : \mathfrak{g}^* \to \mathbb{R}$ is a given function. One readily checks that Casimirs are preserved while, in the special case $k = h$, the energy dissipates. In that case, the equation of motion arising from the double bracket in (1.15) is given by

$$\partial_t \mu + \text{ad}^*_{\delta h/\delta \mu} \mu = \theta \text{ad}^* \left( \text{ad}^*_{\delta h/\delta \mu} \mu \right),$$  

(1.16)

where $\sharp : \mathfrak{g}^* \to \mathfrak{g}$ is the sharp operator associated to $\gamma$. See Bloch et al. [1996]; Holm, Putkaradze and Tronci [2008] for discussions of double bracket dissipation of energy.

Remark 1.7 (Comparison with double-bracket dissipation). Formula (1.16) for double-bracket dissipation coincides in some particular cases with equations (1.14), obtained from our approach after exchanging the functions $C$ and $h$ in (1.5). More precisely, this coincidence occurs when the Casimir is quadratic in $\mu$ and the inner product $\gamma_\mu$ in (1.12) is ad-invariant on the Lie algebra, as in the case of the Killing form for semi-simple Lie algebras. In that case, $(\text{ad}^*_{\xi} \mu)^\sharp = -\text{ad}_{\xi} \mu^\sharp$ and equation (1.16) takes the double-bracket form,

$$\partial_t \mu^\sharp - \text{ad}^*_{\delta h/\delta \mu} \mu^\sharp = \theta \text{ad} \left( \text{ad}^*_{\delta h/\delta \mu} \mu \right),$$  

(1.17)

Examples. Examples of double-bracket formulations in fluid dynamics include Vallis, Carnevale, and Young [1989]; Shepherd [1990], in which a modification of the transport velocity was used to impose energy dissipation with fixed Casimirs. The Lie algebraic nature of this modification of the transport velocity becomes clear by rewriting the special case (1.16) of the double bracket motion equation (1.15) as

$$\partial_t \mu + \text{ad}^*_{\xi} \mu = 0 \quad \text{with transport velocity} \quad v = \frac{\delta h}{\delta \mu} - \theta \left( \text{ad}^*_{\delta h/\delta \mu} \mu \right)^\sharp.$$  

(1.18)
Outlook. In the remainder of this paper, we will first concentrate on Casimir dissipation at constant energy, and then treat the opposite case of energy dissipation at a constant Casimir by simply switching $h$ and $C$ as in passing from equation (1.5) to equation (1.12). After this switch, we can reduce further to the double bracket form seen previously in the literature for the case of a quadratic Casimir and an Ad-invariant inner product on the Lie algebra. In the cases of 3D MHD and 3D Hall MHD, the explicit formulas for selective energy decay at fixed values of the Casimir will be discussed in detail and compared with historical treatments of the selective decay hypothesis for MHD, such as Brown, Canfield and Pertsoy [1999].

1.2.1 Lagrange-d’Alembert variational principle

Equations (1.6) and (1.14) provide the constraint forces that will guide the ideal MHD system into a particular class of equilibria, by decreasing, respectively, either a particular choice of Casimir at constant energy, or vice versa. The balance between the Casimir and energy that occurs at a critical point of their sum determines the class of equilibria that is achievable by a given choice of constraint force. The existence of a constraint force that will dynamically guide an MHD system into a certain class of equilibria (or preserve it once it has been obtained) may be useful in the design and control of magnetic confinement devices.

The Lagrange-d’Alembert variational principle extends Hamilton’s principle to the case of forced systems, including nonholonomically constrained systems (Bloch [2004]). We now explain following Gay-Balmaz and Holm [2013] how the Casimir-dissipative LP equations (1.6) and energy-dissipative LP equations (1.14) can be obtained from the Lagrange-d’Alembert principle. Consider the Lagrangian $\ell : g \to \mathbb{R}$ related to $h$ via the Legendre transform, that is, we have

$$h(\mu) = \langle \mu, \xi \rangle - \ell(\xi), \quad \mu := \frac{\delta \ell}{\delta \xi},$$

where we have assumed that the second relation yields a bijective correspondence between $\xi$ and $\mu$. In terms of $\ell$, equation (1.6) for Casimir dissipation reads

$$\partial_t \mu + \text{ad}^*_\xi \mu = \theta \text{ad}^*_\xi \left[ \frac{\delta C}{\delta \mu}, \xi \right]^b, \quad \mu := \frac{\delta \ell}{\delta \xi}.$$ (1.19)

These equations can be obtained by applying the Lagrange-d’Alembert variational principle

$$\delta \left[ \int_0^T \ell(\xi) dt \right] + \theta \int_0^T \gamma \left( \left[ \frac{\delta C}{\delta \mu}, \xi \right], [\xi, \zeta] \right) dt = 0, \quad \text{for variations } \delta \xi = \partial_t \zeta - [\xi, \zeta],$$

where $\zeta \in g$ is an arbitrary curve vanishing at $t = 0, T$. Thus, in the Lagrange-d’Alembert formulation, the modification of the motion equation to impose selective decay of the Casimir is seen as an energy-conserving constraint force.

Remark 1.8. Similarly, the energy-dissipative LP equation (1.14) admits the variational formulation

$$\delta \left[ \int_0^T \ell(\xi) dt \right] + \theta \int_0^T \gamma \left( \left[ \xi, \frac{\delta C}{\delta \mu} \right], \left[ \frac{\delta C}{\delta \mu}, \zeta \right] \right) dt = 0, \quad \text{for variations } \delta \xi = \partial_t \zeta - [\xi, \zeta],$$

where $\zeta \in g$ is an arbitrary curve vanishing at $t = 0, T$. 
1.2.2 Kelvin-Noether theorem

The well-known Kelvin circulation theorems for fluids can be seen as reformulations of Noether’s theorem and, therefore, they have an abstract Lie algebraic formulation (the Kelvin-Noether theorems), see HMR [1998]. We now discuss the abstract Kelvin circulation theorem for Casimir-dissipative LP equation (1.6).

In order to formulate the Kelvin-Noether theorem, one has to choose a manifold \( C \) on which the group \( G \) acts on the left and consider a \( G \)-equivariant map \( K : C \to \mathfrak{g}^{**} \), i.e. \( \langle K(gc), \text{Ad}^*_g \nu \rangle = \langle K(c), \nu \rangle, \forall g \in G \). Here \( gc \) denotes the action of \( g \in G \) on \( c \in C \) and \( \text{Ad}^*_g \) denotes the coadjoint action defined by \( \langle \text{Ad}^*_g \mu, \xi \rangle = \langle \mu \text{Ad}_g \xi \rangle \), where \( \mu \in \mathfrak{g}^* \), \( \xi \in \mathfrak{g} \), and \( \text{Ad}_g \) is the adjoint action of \( G \) on \( \mathfrak{g} \). Given \( c \in C \) and \( \mu \in \mathfrak{g}^* \), we will refer to \( \langle K(c), \mu \rangle \) as the Kelvin-Noether quantity (HMR [1998]). In application to fluids, \( C \) is the space of loops in the fluid domain and \( K \) is the circulation around this loop, namely

\[
\langle K(c), u \cdot dx \rangle := \oint_c u \cdot dx.
\]

The Kelvin-Noether theorem for Casimir-dissipative LP equations is formulated as follows.

**Proposition 1.9.** Fix \( c_0 \in C \) and consider a solution \( \mu(t) \) of the Casimir-dissipative LP equation (1.6). Let \( g(t) \in G \) be the curve determined by the equation \( \frac{\delta h}{\delta \mu} = \dot{g} g^{-1} \), \( g(0) = e \). Then the time derivative of the Kelvin-Noether quantity \( \langle K(g(t)c_0), \mu(t) \rangle \) associated to this solution is

\[
\frac{d}{dt} \langle K(g(t)c_0), \mu(t) \rangle = \theta \left\langle K(g(t)c_0), \text{ad}^*_{\delta h} \left[ \frac{\delta C}{\delta \mu}, \frac{\delta h}{\delta \mu} \right] \right\rangle.
\]

Note that \( g(t) \in G \) is the motion in Lagrangian coordinates associated to the evolution of the momentum \( \mu(t) \in \mathfrak{g}^* \) in Eulerian coordinates. The \( \theta \) term is an extra source of circulation with a double commutator. This term is absent in the ordinary Lie–Poisson case (i.e., for \( \theta = 0 \)) and therefore in this case the Kelvin-Noether quantity \( \langle K(g(t)c_0), \mu(t) \rangle \) is conserved along solutions.

**Corollary 1.10.** In the case of the energy-dissipative LP equation (1.14), the Kelvin-Noether theorem is found from the exchange \( C \leftrightarrow h \) to be

\[
\frac{d}{dt} \langle K(g(t)c_0), \mu(t) \rangle = \theta \left\langle K(g(t)c_0), \text{ad}^*_{\delta C} \left[ \frac{\delta h}{\delta \mu}, \frac{\delta C}{\delta \mu} \right] \right\rangle.
\]

1.3 Example: the rigid body

The Lie–Poisson bracket on the dual Lie algebra of \( \mathfrak{so}(3) \) may be written on \( \mathbb{R}^3 \) as

\[
\{ F, h \}_{-} (\Pi) = - \Pi \cdot \frac{\delta F}{\delta \Pi} \times \frac{\delta h}{\delta \Pi}, \tag{1.20}
\]

with \( \Pi \in \mathbb{R}^3 \). The corresponding Lie–Poisson motion equation is, for the left-invariant case,

\[
\frac{d}{dt} \Pi - \Pi \times \frac{\delta h}{\delta \Pi} = 0.
\]
This equation describes the motion of a rigid body with Hamiltonian \( h(\Pi) = \frac{1}{2} \Pi \cdot I^{-1} \Pi \) and symmetric positive-definite moment of inertia tensor \( I \) whose principle moments are assumed to be ordered as \( I_1 > I_2 > I_3 \). The Casimir for this Lie–Poisson bracket is

\[
C(\Pi) = \frac{1}{2} |\Pi|^2 \quad \text{with} \quad \frac{\delta C}{\delta \Pi} = \Pi,
\]

and one may check that the Lie-Poisson bracket (1.20) yields \( \{ C, h \} = 0 \) for any Hamiltonian \( h \).

**Selective Casimir decay for the rigid body.** The modified momentum induced by the principle of selective decay of Casimirs is found from equation (1.11) in this case to be

\[
\tilde{\Pi} = \Pi + \theta \left( \frac{\delta C}{\delta \Pi} \times \frac{\delta h}{\delta \Pi} \right)^b = \Pi + \theta \left( \Pi \times \frac{\delta h}{\delta \Pi} \right)^b.
\]

The angular velocity of the rigid body is given by \( \frac{\delta h}{\delta \Pi} = \Omega = I^{-1} \Pi \). Choosing the usual inner product on \( \mathbb{R}^3 \) for the bilinear form \( \gamma_\Pi \) yields \( b = I_d \), which implies from equation (1.10) that

\[
\frac{d}{dt} \Pi - \Pi \times \Omega = \theta (\Pi \times \Omega) \times \Omega, \quad \text{so that} \quad \frac{d}{dt} \frac{1}{2} \Pi^2 = -\theta |\Omega \times \Pi|^2 \leq 0. \tag{1.21}
\]

One might also have chosen the inner product \( \gamma_I \) associated with the inertia tensor \( I \), in which case

\[
\frac{d}{dt} \Pi - \Pi \times \Omega = \theta I (\Pi \times \Omega) \times \Omega, \quad \frac{d}{dt} \frac{1}{2} \Pi^2 = -\theta I (\Omega \times \Pi) \cdot (\Omega \times \Pi) \leq 0.
\]

**Selective energy decay for the rigid body.** Upon choosing instead to dissipate the energy at a fixed value of the Casimir and taking the usual inner product on \( \mathbb{R}^3 \) for the bilinear form \( \gamma_\Pi \) so that \( b = I_d \), the modified Lie–Poisson motion equation for selective decay of energy is found from equation (1.14), written in the left-invariant case as

\[
\partial_t \mu - \text{ad}^*_h \mu = -\theta \text{ad}^*_C \left[ \frac{\delta C}{\delta \mu} : \frac{\delta h}{\delta \mu} \right]^b.
\]

Thus, for the rigid body Hamiltonian \( h(\Pi) = \frac{1}{2} \Pi \cdot I^{-1} \Pi \) and Casimir \( C(\Pi) = \frac{1}{2} |\Pi|^2 \), this becomes

\[
\frac{d}{dt} \Pi + \Omega \times \Pi = \theta \Pi \times (\Pi \times \Omega), \tag{1.23}
\]

which is the Landau-Lifshitz equation for spatially homogeneous dynamics of magnetization (\( \Pi \)) at the microscopic scale Landau and Lifshitz [1935]. This equation may also be rewritten in the form

\[
\frac{d}{dt} \Pi + \Xi \times \Pi = 0 \quad \text{with transport velocity} \quad \Xi = \Omega - \theta \Omega \times \Pi. \tag{1.24}
\]

Consequently, for this choice of the inner product given by the bilinear form \( \gamma_\Pi \) the rigid body energy decays as

\[
\frac{d}{dt} \left( \frac{1}{2} \Omega \cdot \Pi \right) = \Omega \cdot \frac{d}{dt} \Pi = -\theta |\Omega \times \Pi|^2 \leq 0.
\]

**Remark 1.11.** By the exchange symmetry of the dynamics of (1.5) and (1.12) under \( h \leftrightarrow C \), the energy of the rigid body decays at constant Casimir at the same rate as its Casimir decays at constant energy. The decay of either the energy or the Casimir ends at an equilibrium of the rigid body flow, at which the angular frequency \( \Omega \) and the \( \Pi \) angular momentum are aligned, so that \( \Omega \times \Pi = 0 \), as expected from applying the energy-Casimir stability method in the example of the rigid body flow, HMRW [1984].
Figures for selective decay of rigid-body energy at constant Casimir. Equation (1.23) governs energy decay of the rigid body flow while preserving the Casimir whose level set defines angular momentum spheres in $\mathbb{R}^3$. Figures 1.1 to 1.4 show solution curves of the energy-dissipative equation (1.23) on a particular angular momentum sphere. These figures show how the solution behavior depends on various initial positions and values of the parameter $\theta$. All of the figures show the $\theta = 0$ curves in black. The solution curves for a small value of $\theta$ are plotted in red and the solution curves for a larger value of $\theta$ are plotted in magenta. These first four figures show that the solution curves of the $\theta \neq 0$ flow of (1.23) all eventually spiral into one of the two stable equilibria of the $\theta = 0$ rigid body flow that have the lowest energy (longest principle axis) lying at opposite points on the angular momentum sphere. The basins of attractions for the two North (green) and South (blue) least energy states are shown in Figures 1.5 and 1.6 for two different values of $\theta$. Along the basin boundaries in these figures, a slight change in the initial conditions may result in approaches to diametrically opposite equilibrium states asymptotically in time. Figure 1.7 shows Casimir decay at fixed energy for rigid body flow.
Figure 1.1: This Figure shows the solution curves of (1.23) which governs energy decay of the rigid body flow while preserving the angular momentum sphere. Color code: black is for $\theta = 0$ curves; red for a small value of $\theta$; and magenta for a larger value of $\theta$. The small black square shows the initial angular momenta of both cases with $\theta \neq 0$. Here the solution curves for $\theta \neq 0$ start close to the stable equilibrium for the $\theta = 0$ rigid body flow on the short axis of the energy ellipsoid, wind near its unstable heteroclinic point on the intermediate axis, then approach its stable equilibrium on the long axis. This shows that the $\theta \neq 0$ rigid body flow of (1.23) spirals into the stable equilibrium of the $\theta = 0$ rigid body flow that has the lowest energy.
Figure 1.2: This Figure shows that upon making a slight change in the larger value of $\theta$ in the previous case the solution curves can reach equilibria on opposite points on the sphere, starting from the same initial point. Color code: black is for $\theta = 0$ curves; red for a small value of $\theta$; and magenta for a larger value of $\theta$. The small black square shows the initial angular momenta of both cases with $\theta \neq 0$. 
Figure 1.3: In this Figure the solution curves of the $\theta \neq 0$ rigid body flow of (1.23) on the angular momentum sphere start close to the intermediate axis and wind around the stable equilibrium of the $\theta = 0$ rigid body flow that has the lowest energy as they approach it. Color code: black is for $\theta = 0$ curves; red for a small value of $\theta$; and magenta for a larger value of $\theta$. The small black circle shows the initial angular momenta of both cases with $\theta \neq 0$. 
Figure 1.4: In this Figure the solution curves start from a point chosen at random on the angular momentum sphere. The Figure shows that the solution curves of (1.23) with different values of $\theta$ can spiral into equilibria of the rigid body flow at opposite points on the angular momentum sphere on the long axis of the energy ellipsoid, even though they started from the same initial point. Color code: black is for $\theta = 0$ curves; red for a small value of $\theta$; and magenta for a larger value of $\theta$. The small black square shows the initial angular momenta of both cases with $\theta \neq 0$. 
Figure 1.5: For the solution curves of (1.23) with $\theta = 0.1$, this Figure shows the basins of attraction of the North (green) and South (blue) least energy states (of longest principle axis) lying at opposite points on the angular momentum sphere. Initial conditions starting in the blue (resp. green) region stay in the blue (resp. green) region. Along the basin boundaries, a slight change in the initial conditions may result in asymptotic approaches to diametrically opposite equilibrium states.
Figure 1.6: For the solution curves of (1.23) with $\theta = 0.3$, this Figure shows the basins of attraction of the North (green) and South (blue) least energy states (of longest principle axis) lying at opposite points on the angular momentum sphere. Initial conditions starting in the blue (resp. green) region stay in the blue (resp. green) region. Along the basin boundaries, a slight change in the initial conditions may result in asymptotic approaches to diametrically opposite equilibrium states.
Figure 1.7: This figure shows Casimir decay at fixed energy. The solution curves (red and magenta) both start from a point near the long axis of the energy ellipsoid on the initial angular momentum sphere. We see them winding around the long axis followed by winding around the short axis, after making a transition near the middle axis. These solution curves of (1.23) with different values of $\theta$ spiral into the stable rigid-body equilibria on the short axis of the energy ellipsoid at opposite points on the angular momentum sphere, even though they both started from the same initial point. Color code: black is for $\theta = 0$ curves; red for a small value of $\theta$; and magenta for a larger value of $\theta$. The small black square shows the initial angular momenta of both cases with $\theta \neq 0$. 

2 Selective decay of Casimirs or energy on semidirect products

2.1 Semidirect products

The Hamiltonian structure of fluids that possess advected quantities such as heat, mass, buoyancy, magnetic field, etc., can be understood by using Lie–Poisson brackets for semidirect-product actions of Lie groups on vector spaces Marsden, Ratiu and Weinstein [1984].

In this setting, besides the Lie group configuration space \( G \), one needs to include a vector space \( V \) on which \( G \) acts linearly. Its dual vector space \( V^* \) contains the advected quantities. One then considers the semidirect product \( G \circledast V \) with Lie algebra \( \mathfrak{g} \circledast V \), and the Hamiltonian structure is given by the Lie–Poisson bracket (1.3), written on \( (\mathfrak{g} \circledast V)^* \) instead of \( \mathfrak{g}^* \). We refer to Marsden, Ratiu and Weinstein [1984], HMR [1998] for a detailed treatment. Given a Hamiltonian function \( h = h(\mu, a) \) with \( h : (\mathfrak{g} \circledast V)^* \to \mathbb{R} \) one thus obtains the Lie–Poisson equations

\[
\frac{\partial}{\partial t} \mu + \text{ad}^*_{\frac{\delta h}{\delta \mu}} \mu + \frac{\delta h}{\delta a} \diamond a = 0, \quad \frac{\partial}{\partial t} a + a \frac{\delta h}{\delta \mu} = 0, \tag{2.1}
\]

for \( \mu(t) \in \mathfrak{g}^* \) and \( a(t) \in V^* \). More explicitly, making use of the expression for the \( \text{ad}^* \)-operator in the semidirect product case, these equations read

\[
\frac{\partial}{\partial t} \mu + \text{ad}^*_{\frac{\delta h}{\delta \mu}} \mu + \frac{\delta h}{\delta a} \diamond a = 0, \quad \frac{\partial}{\partial t} a + a \frac{\delta h}{\delta \mu} = 0, \tag{2.2}
\]

where the operator \( \diamond : V \times V^* \to \mathfrak{g}^* \) is defined by

\[
\langle v \diamond a, \xi \rangle := -\langle a \xi, v \rangle, \tag{2.3}
\]

and \( a \xi \in V^* \) denotes the (right) Lie algebra action of \( \xi \in \mathfrak{g} \) on \( a \in V^* \).

Casimir-dissipative LP equation. We can now extend the Casimir dissipation approach to the semidirect product case. Fixing a Casimir function \( C = C(\mu, a) \) for the Lie–Poisson bracket on the semidirect product and a (possibly (\mu, a)-dependent) positive symmetric bilinear form \( \gamma(\mu, a) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R} \), we extend the semidirect-product Lie–Poisson system (2.2) naturally to allow for Casimir dissipation by setting

\[
\frac{\partial}{\partial t} \mu + \text{ad}^*_{\frac{\delta h}{\delta \mu}} \mu + \frac{\delta h}{\delta a} \diamond a = \theta \text{ad}^*_{\frac{\delta C}{\delta \mu}} \frac{\delta h}{\delta \mu}, \quad \frac{\partial}{\partial t} a + a \frac{\delta h}{\delta \mu} = 0, \tag{2.4}
\]

where \( \flat : \mathfrak{g} \to \mathfrak{g}^* \) is the flat operator associated to \( \gamma \). In Lie–Poisson form, this equation reads

\[
\frac{df(\mu, a)}{dt} = \{f, h\}_+ (\mu, a) - \theta \gamma \left( \frac{\delta f}{\delta \mu}, \frac{\delta h}{\delta \mu} \right), \tag{2.5}
\]

which for \( f = h \) shows that the energy is conserved while the Casimir dissipates as in (1.9) when one sets \( f = C \) in (2.5).
Energy-dissipative LP equation. One can exchange $h$ and $C$ as done in (1.14) to obtain the energy-dissipative LP equation for semidirect product Lie groups,

$$\partial_t \mu + \text{ad}^*_{\frac{\delta h}{\delta a}} \mu + \frac{\delta h}{\delta a} \diamond a = -\theta \text{ad}^*_{\frac{\delta C}{\delta \mu}} \left[ \frac{\delta C}{\delta \mu}, \frac{\delta h}{\delta \mu} \right]^b, \quad \partial_t a + \frac{\delta h}{\delta \mu} = 0.$$  \hfill (2.6)

In Lie–Poisson form, equation (2.6) reads

$$\frac{df(\mu, a)}{dt} = \{f, h\} + (\mu, a) - \theta \gamma \left( \frac{\delta f}{\delta \mu}, \frac{\delta C}{\delta \mu} \right),$$  \hfill (2.7)

which for $f = h$ shows that the energy decays

$$\frac{dh(\mu, a)}{dt} = -\theta \left\| \frac{\delta h}{\delta \mu}, \frac{\delta C}{\delta \mu} \right\|_\gamma^2,$$  \hfill (2.8)

while the Casimir is seen to be conserved when one chooses $f = C$ in (2.6).

Lagrange-d’Alembert variational principle for semidirect products. In the case of equations (2.4) the Lagrange-d’Alembert principle reads

$$\delta \left[ \int_0^T \ell(\xi, a) \, dt \right] + \theta \int_0^T \gamma \left( \frac{\delta C}{\delta \mu}, \xi, \left[ \xi, \zeta \right] \right) \, dt = 0,$$  \hfill (2.9)

for variations $\delta \xi = \partial_t \zeta - \left[ \xi, \zeta \right]$, $\delta a = -a_\zeta$, where $\zeta$ is an arbitrary curve in $g$ vanishing at $t = 0, T$. Recall that $\mu = \delta \ell/\delta \xi$ in (2.9). This principle only yields the first equation in (2.4). The second equation follows from the assumption that $a$ is an advected quantity, i.e., $a(t) = a_0 g(t)^{-1}$, where $a_0$ is the initial condition and $\dot{g}(t) g(t)^{-1} = \xi(t)$ with $g(0) = e$.

The energy-dissipative LP equation (2.6) admits a similar Lagrange-d’Alembert formulation.

Kelvin-Noether theorem for semidirect products. To formulate the Kelvin-Noether theorem for semidirect products, we consider a $G$-equivariant map $K : \mathcal{C} \times V^* \to g^*$, i.e., a map that satisfies

$$\langle K(gc, ag^{-1}), \text{Ad}_{g}^* \nu \rangle = \langle K(c, a), \nu \rangle, \forall \ g \in G.$$

Given $c_0 \in \mathcal{C}$ and a solution $\mu(t)$, $a(t)$ of (2.4), the associated Kelvin-Noether quantity reads $\langle K(g(t)c_0, a(t)), \mu(t) \rangle$ and similarly with Proposition 1.9 we get the Kelvin-Noether theorem

$$\frac{d}{dt} \langle K(g(t)c_0, a(t)), \mu(t) \rangle = \left\langle K(g(t)c_0, a(t)), \theta \text{ad}^*_{\frac{\delta h}{\delta a}} \left[ \frac{\delta C}{\delta \mu}, \frac{\delta h}{\delta \mu} \right]^b - \frac{\delta h}{\delta a} \diamond a \right\rangle.$$  \hfill (2.10)

In the energy-dissipative case (2.6), the Kevin-Noether theorem for semidirect products reads

$$\frac{d}{dt} \langle K(g(t)c_0, a(t)), \mu(t) \rangle = \left\langle K(g(t)c_0, a(t)), \theta \text{ad}^*_{\frac{\delta h}{\delta a}} \left[ \frac{\delta h}{\delta \mu}, \frac{\delta C}{\delta \mu} \right]^b - \frac{\delta h}{\delta a} \diamond a \right\rangle.$$  \hfill (2.11)
Natural generalization - Casimir dissipation for semidirect products. Note that the Casimir-dissipative LP equation (2.4) was obtained from the Lie–Poisson equations on the semidirect product by modifying the momentum $\mu \in g^*$ only, while keeping the quantity $a \in V^*$ unchanged. From the Lie algebraic point of view, however, the direct generalization of (1.6) to semidirect product Lie groups would be

$$
\partial_t (\mu, a) + \text{ad}_{(\frac{\delta h}{\delta \mu}, \frac{\delta h}{\delta a})}^\ast (\delta C, \delta C) \left[ \left( \frac{\delta h}{\delta \mu}, \frac{\delta h}{\delta a} \right), \left( \frac{\delta h}{\delta \mu}, \frac{\delta h}{\delta a} \right) \right]^b ,
$$

(2.12)

where the flat operator $b : g \times V \rightarrow g \times V^*$ is associated to a positive symmetric bilinear map $\gamma(\mu, a) : (g \times V) \times (g \times V) \rightarrow \mathbb{R}$. Using the expression $[(\xi, v), (\eta, w)] = [(\xi, \eta), v\eta - w\xi]$ for the Lie bracket on $g \circledast V$, we can write (2.12) as

$$
\partial_t (\mu, a) + \text{ad}_{(\frac{\delta h}{\delta \mu}, \frac{\delta h}{\delta a})}^\ast (\tilde{\mu}, \tilde{a}) = 0,
$$

(2.13)

in which both $\mu$ and $a$ are modified as

$$(\tilde{\mu}, \tilde{a}) = (\mu, a) + \theta \left[ \left( \frac{\delta h}{\delta \mu}, \frac{\delta h}{\delta a} \right), \left( \frac{\delta h}{\delta \mu}, \frac{\delta h}{\delta a} \right) \right]^b .
$$

(2.14)

By using the formula $\text{ad}_{(\xi, v)}^\ast (\mu, a) = (\text{ad}_{\xi}^\ast \mu + v \circ a, a\xi)$ in equation (2.13), one finds the explicit Casimir-dissipative system

$$
\partial_t \mu + \text{ad}_{\frac{\delta h}{\delta \mu}}^\ast \mu + \frac{\delta h}{\delta a} \circ \tilde{a} = 0, \quad \partial_t a + \frac{\delta h}{\delta \mu} \tilde{a} = 0.
$$

(2.15)

When $\gamma$ is diagonal on the Cartesian product $g \times V$, we can write $(\xi, v)^\flat = (\xi^\flat, v^\flat)$ and (2.15) can be written explicitly as

$$
\begin{cases}
\partial_t \mu + \text{ad}_{\frac{\delta h}{\delta \mu}}^\ast \mu + \frac{\delta h}{\delta a} \circ a + \theta \text{ad}_{\frac{\delta h}{\delta \mu}}^\ast \left[ \frac{\delta h}{\delta \mu}, \frac{\delta h}{\delta a} \right]^b + \theta \frac{\delta h}{\delta a} \circ \left( \frac{\delta h}{\delta a} \frac{\delta C}{\delta \mu} - \frac{\delta C}{\delta a} \frac{\delta h}{\delta \mu} \right)^b = 0 \\
\partial_t a + a \frac{\delta h}{\delta \mu} + \theta \left( \frac{\delta h}{\delta a} \frac{\delta C}{\delta \mu} - \frac{\delta C}{\delta a} \frac{\delta h}{\delta \mu} \right)^b \frac{\delta h}{\delta \mu} = 0 .
\end{cases}
$$

(2.16)

The system (2.15) recovers (2.4) in the case when the bilinear form $\gamma$ vanishes on $V$, since in this case we have $(\xi, v)^\flat = (\xi^\flat, 0)$.

As above for (2.5), one may verify that the modified semidirect-product Lie–Poisson system (2.12) dissipates the Casimir $C$ while keeping energy conserved, under the modification of both $\mu$ and $a$. Namely, one computes the Lie–Poisson form

$$
\frac{df(\mu, a)}{dt} = \{f, h\} + (\mu, a) - \theta \gamma \left[ \left( \frac{\delta f}{\delta \mu}, \frac{\delta f}{\delta a} \right), \left( \frac{\delta h}{\delta \mu}, \frac{\delta h}{\delta a} \right) \right] ,
$$

(2.17)

which for $f = h$ shows that the energy is conserved while for $f = C$ shows that the Casimir dissipates.

Remark 2.1 (A simplification for $\frac{\delta C}{\delta \mu} = 0$). Note that the modification of $\mu$ in the system (2.15) implies a modification of the $\mu$-equation only. However, a modification of $a$ alone will yield a
modification of both the $\mu$- and $a$-equations. For example, if $\frac{\delta C}{\delta \mu} = 0$ then equation (2.14) reduces to

$$\tilde{\mu} = \mu, \quad \tilde{a} = a - \theta \left( \frac{\delta C}{\delta a} \frac{\delta h}{\delta \mu} \right),$$

(2.18)

and equation (2.16) simplifies to

$$\partial_t \mu + \text{ad}^*_{\frac{\delta h}{\delta a}} \mu + \frac{\delta h}{\delta a} \circ a = \theta \frac{\delta h}{\delta a} \circ \left( \frac{\delta C}{\delta a} \frac{\delta h}{\delta \mu} \right), \quad \partial_t a + \frac{\delta h}{\delta a} \circ \left( \frac{\delta C}{\delta a} \frac{\delta h}{\delta \mu} \right) = \theta \frac{\delta h}{\delta a} \circ \left( \frac{\delta C}{\delta a} \frac{\delta h}{\delta \mu} \right).$$

(2.19)

Natural generalization - energy dissipation for semidirect products. Exchanging the role of $h$ and $C$ in the $\theta$-term of (2.12), we get the energy-dissipative LP equation which preserves the Casimir $C$ for semidirect product Lie groups,

$$\partial_t (\mu, a) + \text{ad}^*_{\frac{\delta h}{\delta a}} (\mu, a) = \theta \text{ad}^*_{\frac{\delta C}{\delta a}} \left( \left( \frac{\delta h}{\delta a} \frac{\delta h}{\delta \mu} \right), \left( \frac{\delta C}{\delta a} \frac{\delta C}{\delta \mu} \right) \right),$$

(2.20)

In Lie–Poisson form this becomes

$$\frac{df(\mu, a)}{dt} = \{f, h\} + (\mu, a) - \theta \gamma \left( \left[ \left( \frac{\delta f}{\delta \mu} \frac{\delta f}{\delta a} \right), \left( \frac{\delta C}{\delta a} \frac{\delta C}{\delta \mu} \right) \right], \left[ \left( \frac{\delta h}{\delta \mu} \frac{\delta h}{\delta a} \right), \left( \frac{\delta C}{\delta a} \frac{\delta C}{\delta \mu} \right) \right] \right),$$

(2.21)

which for $f = h$ shows that the energy dissipates as,

$$\frac{dh(\mu, a)}{dt} = -\theta \left\| \left[ \left( \frac{\delta h}{\delta \mu} \frac{\delta h}{\delta a} \right), \left( \frac{\delta C}{\delta a} \frac{\delta C}{\delta \mu} \right) \right] \right\| \gamma ^2$$

$$\quad \quad \quad = -\theta \left\| \left[ \frac{\delta h}{\delta \mu} \frac{\delta C}{\delta \mu} \right] \right\| \gamma ^2 - \theta \left\| \frac{\delta h}{\delta a} \frac{\delta C}{\delta \mu} - \frac{\delta C}{\delta a} \frac{\delta h}{\delta \mu} \right\| \gamma ^2,$$

(2.22)

while for $f = C$ equation (2.21) shows that the Casimir is conserved under the dynamics of (2.21).

After using the formula $\text{ad}^*_{(\xi, v)} (\mu, a) = (\text{ad}^*_{\xi} \mu + v \circ a, a \xi)$ for the coadjoint operator of the semidirect product $g \circ V$, and assuming that $\gamma$ is diagonal on the Cartesian product $g \times V$, the system (2.20) is explicitly given by

$$\begin{align*}
\left\{ \begin{array}{l}
\partial_t \mu + \text{ad}^*_{\frac{\delta h}{\delta a}} \mu + \frac{\delta h}{\delta a} \circ a + \theta \text{ad}^*_{\frac{\delta C}{\delta a}} \left[ \frac{\delta h}{\delta a} \frac{\delta C}{\delta \mu} \right] + \theta \frac{\delta C}{\delta a} \circ \left( \frac{\delta h}{\delta a} \frac{\delta C}{\delta \mu} - \frac{\delta C}{\delta a} \frac{\delta h}{\delta \mu} \right) = 0 \\
\partial_t a + \frac{\delta h}{\delta a} \circ \left( \frac{\delta h}{\delta a} \frac{\delta C}{\delta a} \frac{\delta h}{\delta \mu} - \frac{\delta C}{\delta a} \frac{\delta h}{\delta \mu} \right) = 0
\end{array} \right. 
\end{align*}$$

(2.23)

In comparing equations (2.16) and (2.23) one observes that the structure and interpretation of the modified equations change in passing from Casimir-dissipative flows to energy-dissipative flows. Instead of modifying the transported momentum and advected quantities as in the Casimir-dissipative case, the modified equation in the energy-dissipative case when $\frac{\delta C}{\delta \mu} \neq 0$ may be interpreted as modifying the transport velocity and the potential energy terms. When $\frac{\delta C}{\delta \mu} = 0$, the latter may be interpreted as modifying only the potential energy terms. Consult Gay-Balmaz and Holm [2013] for more discussion of these matters.
Remark 2.2 (Simplifications for $\frac{\delta C}{\delta \mu} = 0$). When $\frac{\delta C}{\delta \mu} = 0$, the energy-dissipative system (2.23) simplifies to

$$
\partial_t \mu + \text{ad}^*_{\frac{\delta h}{\delta a}} \mu + \frac{\delta h}{\delta a} \circ a = \theta \frac{\delta C}{\delta a} \circ \left( \frac{\delta C \delta h}{\delta a \delta \mu} \right)^b, \quad \partial_t a + a \frac{\delta h}{\delta \mu} = 0 ,
$$

(2.24)
cf. equation (2.19) for the corresponding simplification in the Casimir-dissipative case. Note that contrary to the Casimir dissipative case (Remark 2.1), the advection equation is left unchanged. The LP form of (2.24) may be obtained by substitution, to find

$$
\frac{df}{dt} = \left\langle \frac{\delta f}{\delta \mu}, \partial_t \mu \right\rangle + \left\langle \frac{\delta f}{\delta a}, \partial_t a \right\rangle = \{f, h\}_+ (\mu, a) - \theta \left( \frac{\delta C \delta h}{\delta a \delta \mu}, \frac{\delta C \delta f}{\delta a \delta \mu} \right) = \{f, h\}_+ (\mu, a) - \theta \gamma \left( \frac{\delta C \delta h}{\delta a \delta \mu}, \frac{\delta C \delta f}{\delta a \delta \mu} \right).
$$

(2.25)

Setting $f = h$ in the final equation of (2.25) gives the energy dissipation equation

$$
\frac{dh(\mu, a)}{dt} = -\theta \gamma \left( \frac{\delta C \delta h}{\delta a \delta \mu}, \frac{\delta C \delta h}{\delta a \delta \mu} \right) = -\theta \left\| \frac{\delta C \delta h}{\delta a \delta \mu} \right\|_{\gamma}^2
$$

(2.26)

which may also be obtained by setting $f = h$ and $\frac{\delta C}{\delta \mu} = 0$ in the modified energy-dissipative LP equation (2.21), to find

$$
\frac{dh(\mu, a)}{dt} = -\theta \left\| \left( 0, \frac{\delta C \delta h}{\delta a \delta \mu} \right) \right\|_{\gamma}^2.
$$

(2.27)

Finally, setting $f = C$ and using $\frac{\delta C}{\delta \mu} = 0$ in the final equation of (2.25) shows that the energy-dissipative system (2.24) preserves the Casimir $C$.

2.2 Convergence to steady states of the unmodified LP equations

In the discussions below, we shall assume that the solutions of the modified (dissipative) equations possess long-time existence. That is, we shall work formally from the viewpoint of mathematical analysis, and ignore the possibility of blow up in finite time.

**Theorem 2.3** (Steady states). For either Casimir-dissipative or energy-dissipative LP equations for semidirect product Lie groups, under the modified dynamics (2.12) or (2.20), the dissipated quantity (Casimir or energy, respectively), assumed to be positive,\(^1\) decreases in time until the modified system reaches a state verified by energy-Casimir equilibria, namely $\delta (h + C) = 0$, independently of the Lie algebra and the choice of Casimir.

\(^1\)As discussed in Gay-Balmaz and Holm [2013], one may assume $C \geq 0$, knowing that if $C \neq 0$ is indefinite, one may replace it in these formulas by its square, $C \rightarrow C^2$, since the squares of Casimirs are still Casimirs.
Proof. Although a shorter proof of this theorem can be given, we choose to present it in three different cases, depending on how the advected variables are treated. This allows us to make several relevant comments in the proof. These cases are the following:

(I) The advected variables $a$ are absent; 

(II) all of the variables $\mu, a$ are modified; and

(III) the advected variables $a$ are present but are left unmodified.

(I) The first class is the case in which the advected variables $a$ are absent, so that $h = h(\mu)$. In this case, for (1.6), resp. (1.14), we have

$$
\frac{d}{dt}C(\mu) = -\theta \left\| \frac{\delta h}{\delta \mu} \frac{\delta C}{\delta \mu} \right\|_\gamma^2 \quad \text{resp.} \quad \frac{d}{dt} h(\mu) = -\theta \left\| \frac{\delta h}{\delta \mu} \frac{\delta C}{\delta \mu} \right\|_\gamma^2.
$$

Thus, if $h, C \geq 0$ and $\gamma$ is nondegenerate, both solutions converge to an asymptotic state with

$$
\left[ \frac{\delta h}{\delta \mu}, \frac{\delta C}{\delta \mu} \right] = 0.
$$

This condition holds for steady states $\mu_e$ that satisfy the energy-Casimir equilibrium condition $\delta (h + C)/\delta \mu = 0$ at $\mu = \mu_e$, independently of the Lie algebra and the choice of Casimir. Note also that (2.28) means that the $\theta$-term in the modified equation (2.25) tends to zero.

(A) In the special case when an ad-invariant pairing $\kappa$ exists (e.g. if $g$ is semisimple) then $\text{ad}_\xi^* \mu = [\mu, \xi]$ and if we choose the Casimir $C(\mu) = \frac{1}{2} \kappa(\mu, \mu)$, then

$$
\left[ \frac{\delta h}{\delta \mu}, \frac{\delta C}{\delta \mu} \right] = -\text{ad}_{\frac{\delta h}{\delta \mu}}^*_\mu,
$$

and in this case the solutions of both the Casimir dissipative and energy dissipative LP equations converge to a steady state, for any choice of the Hamiltonian $h$ and for all equilibria, not just for energy-Casimir equilibria. This is the case for the rigid body and for the 2D ideal fluid.

(B) The above setting is not the only one in which this occurs. For example, for ideal incompressible 3D fluids with the helicity Casimir $C = \int u \cdot \text{curl} u \, d^3 x$, the condition (2.28) becomes

$$
[u, \text{curl} u] = 0 \quad \text{i.e.,} \quad \text{curl}(u \times \text{curl} u) = 0, \quad \text{i.e.,} \quad u \times \omega = \nabla p.
$$

These equilibria are the steady Lamb flows, in which the level sets of pressure $p$ form symplectic manifolds Arnold and Khesin [1998]. In this situation, both the energy-dissipative and Casimir-dissipative LP equations converge to a steady state of the unmodified equations. The latter holds for the case that the Casimir is taken to be helicity-squared, cf. the footnote above.

(II) For a semidirect product LP system in which all variables are modified, as in (2.12) and (2.20), and if $\gamma$ is nondegenerate on $g \times V$, the equations converge as in (2.22) to a state with both

$$
\left[ \frac{\delta h}{\delta \mu}, \frac{\delta C}{\delta \mu} \right] = 0 \quad \text{and} \quad \frac{\delta h \, \delta C}{\delta a \, \delta \mu} - \frac{\delta C \, \delta h}{\delta a \, \delta \mu} = 0.
$$

These conditions mean that the $\theta$-term in the modified equation (2.25) tends to zero. Again this pair of conditions is satisfied for steady states $(\mu_e, a_e)$ that satisfy the energy-Casimir equilibrium.
condition $\delta(h+C)/\delta(\mu, a) = 0$, at $(\mu, a) = (\mu_e, a_e)$, independently of the Lie algebra and the choice of Casimir, provided $\frac{\delta C}{\delta \mu} \neq 0$.

When $\frac{\delta C}{\delta \mu} = 0$, condition (2.29) reduces to

$$0 = \frac{\delta C \delta h}{\delta a \delta \mu}, \quad (2.30)$$

which is an equilibrium state of the selective decay equation of energy (resp. Casimir) for $\frac{\delta C}{\delta \mu} = 0$, as given in (2.26).

The requirement $\frac{\delta C}{\delta \mu} = 0$ restricts the choice of Casimirs for either the modified LP equations or the energy-Casimir equilibrium conditions of the unmodified equations. However, this case still retains some physically important cases, such as magnetic helicity for MHD, discussed among the examples in the later sections of the paper, particularly for Example 2 in §4.3.1 and in §4.3.2.

(III) For semidirect product LP equations with variables $(\mu, a)$ in which only the momentum equation is modified, as in (2.4) or (2.6), and for which $\gamma$ is nondegenerate on $g$, the solution converges to a solution with

$$\left[ \frac{\delta h}{\delta \mu}, \frac{\delta C}{\delta \mu} \right] = 0. \quad (2.31)$$

Once more, this condition holds for the class of equilibria for which the energy-Casimir method applies; namely, the criticality condition $\delta(h+C) = 0$.

**Remark 2.4.** Here is a summary sketch diagraming the various lines of reasoning used in the proof of Theorem 2.3.

In all cases, we have the following diagram

$$\delta(h+C) = 0 \iff \frac{\delta h}{\delta \mu} + \frac{\delta C}{\delta \mu} = 0 \& \quad \frac{\delta h}{\delta a} + \frac{\delta C}{\delta a} = 0 \implies (2.29)$$

$$\text{steady state} \iff \text{ad}^\gamma_{\left(\frac{\delta h}{\delta \mu}, \frac{\delta h}{\delta a} \right)}(\mu, a) = 0 ,$$

where implications are denoted by $A \implies B$, and equivalences are denoted by $A \iff B$.

In the special case $\frac{\delta C}{\delta \mu} = 0$, the diagram becomes

$$\delta(h+C) = 0 \iff \frac{\delta h}{\delta \mu} = 0 \& \quad \frac{\delta h}{\delta a} + \frac{\delta C}{\delta a} = 0 \implies (2.30)$$

$$\text{steady state} \iff \text{ad}^\gamma_{\left(\frac{\delta h}{\delta \mu}, \frac{\delta h}{\delta a} \right)}(\mu, a) = 0 .$$

In the case $h = h(\mu)$ the diagram becomes

$$\delta(h+C) = 0 \iff \frac{\delta h}{\delta \mu} + \frac{\delta C}{\delta \mu} = 0 \implies (2.28)$$

$$\text{steady state} \iff \text{ad}^\gamma_{\frac{\delta h}{\delta \mu}} \mu = 0 .$$
Note the directions of the implications. Namely, the critical point conditions \( \delta(h + C) = 0 \) imply that the energy, or Casimir, decay rate vanishes, but not necessarily vice versa. This will become clear, later, when we discover that the number of critical point conditions obtained from \( \delta(h + C) = 0 \) may in some cases exceed the number of asymptotically vanishing decay rate terms obtained from the modified equations, as in Remark 4.5, for example.

2.3 Examples

2.3.1 Heavy top

The Hamiltonian of the heavy top is the sum of its kinetic and potential energies,

\[
h(\Pi, \Gamma) = \frac{1}{2} \Pi \cdot I^{-1}\Pi + mg\chi \cdot \Gamma,
\]

(2.32)

in which \( \Pi \) is the body angular momentum, \( \chi \) is the constant vector in the body from its point of support to its centre of mass, \( mg \) is its weight and \( \Gamma(t) = O^{-1}(t)\hat{z} \) is the vertical direction, as seen from the body. For this case, the condition (2.29) reads

\[
\frac{\delta h}{\delta \Pi} \times \frac{\delta C}{\delta \Pi} = 0 \quad \text{and} \quad \frac{\delta h}{\delta \Gamma} \times \frac{\delta C}{\delta \Pi} - \frac{\delta C}{\delta \Gamma} \times \frac{\delta h}{\delta \Pi} = 0.
\]

(2.33)

If we choose the Casimir \( C(\Pi, \Gamma) = (\Pi \cdot \Gamma)^2 \), then the above condition is the steady state condition for the heavy top flow. This fact is further explained in §2.3.3 below.

However, if we choose the Casimir \( C(\Pi, \Gamma) = \frac{1}{2}|\Gamma|^2 \) then the energy-Casimir critical point condition yields a trivial equilibrium, since

\[
\delta(h + C) = I^{-1}\Pi \cdot \delta \Pi + (mg\chi + \Gamma) \cdot \delta \Gamma = 0.
\]

(2.34)

In this case \( \frac{\delta C}{\delta \Pi} = 0 \) and the critical point condition \( \delta(h + C) = 0 \) requires \( \frac{\delta h}{\delta \Pi} = 0 \), so the equilibrium conditions in (2.33) are satisfied trivially. Had we invoked (2.29) instead of using the critical point condition (2.34), we would have found for the Casimir \( C(\Pi, \Gamma) = \frac{1}{2}|\Gamma|^2 \) that \( \Gamma \times I^{-1}\Pi = 0 \), which by (2.33) is verified for equilibria of the heavy top equations.

2.3.2 2D incompressible MHD

For 2D incompressible MHD with \( B \) in the \((x,y)\) plane, that will be discussed in Section 4, exactly the same situation arises and the conclusion again depends on which Casimir is used. (In the MHD case, the heavy top Casimir \( \Pi \cdot \Gamma \) corresponds to the planar MHD Casimir \( \int \omega A \, dx \, dy \), and \( \frac{1}{2}|\Gamma|^2 \) corresponds to \( \int A^2 \, dx \, dy \).) This conclusion also applies for 2D incompressible MHD with \( B \) perpendicular to the plane, also discussed in Section 4.

2.3.3 A general class

Both of the previous examples belong to the same general class. These are semidirect products of the type \( g \circledast V \), where \( V = g \) is acted on by the adjoint action and \( g \) admits an Ad-invariant
pairing $\kappa$. In this case, there are the two Casimirs

$$C(\mu, a) = \kappa(\mu, a) \quad \text{and} \quad C(\mu, a) = \frac{1}{2} \kappa(a, a).$$

When the first Casimir is used, the condition (2.29) is equivalent to a steady state condition, so both the energy dissipative and Casimir dissipative (with squared Casimir) converge to a steady state. If the second Casimir is used, then (2.29) reads $[a, \frac{\delta h}{\delta \mu}] = 0$ and the critical point condition $\delta (h + C) = 0$ requires $\frac{\delta h}{\delta \mu} = 0$, so the corresponding energy-Casimir equilibrium is trivial. If only the momentum $\mu$ is modified, then only the first Casimir should be used since the second one yields no changes in the equation. When the first Casimir is used, the asymptotic solution verifies $[a, \frac{\delta h}{\delta \mu}] = 0$, which implies that $a$ reaches a time independent state.

3 Selective decay for 3D compressible fluids

In the Hamiltonian formulation of 3D compressible fluids, the variables are the fluid momentum density $m \in \mathcal{X}(\mathcal{D})^*$, the mass density $\rho \in \mathcal{F}(\mathcal{D})^*$, and the specific entropy $\eta \in \mathcal{F}(\mathcal{D})$. The Lie–Poisson bracket reads

$$\{f, g\} + (m, \rho, \eta) = \int_\mathcal{D} m \cdot \left[\frac{\delta f}{\delta m} - \frac{\delta g}{\delta m}\right] d^3x + \int_\mathcal{D} \rho \left(\frac{\delta g}{\delta \rho} - \frac{\delta f}{\delta \rho}\right) \cdot \nabla \left(\frac{\delta f}{\delta \rho} - \frac{\delta g}{\delta \rho}\right) d^3x$$

$$+ \int_\mathcal{D} \eta \left(\text{div} \left(\frac{\delta f}{\delta \eta} \frac{\delta g}{\delta \rho} - \frac{\delta f}{\delta \rho} \frac{\delta g}{\delta \eta}\right) - \text{div} \left(\frac{\delta \rho}{\delta \eta} \frac{\delta g}{\delta \rho}\right)\right) d^3x \quad (3.1)$$

and this bracket yields the Lie-Poisson equations

$$\begin{cases}
\partial_t \left(\frac{m}{\rho}\right) + \left(\nabla \frac{\rho}{m}\right) \times \frac{\delta h}{\delta m} + \nabla \left(\frac{\delta h}{\delta m} \cdot \frac{m}{\rho}\right) + \nabla \delta h = 0, \\
\partial_t \rho + \text{div}(\rho \mathbf{u}) = 0, \quad \partial_t \eta + \mathbf{u} \cdot \nabla \eta = 0.
\end{cases} \quad (3.2)$$

These equations follow from the Lie–Poisson equation (2.1) with $\mathfrak{g}^* = \mathcal{X}(\mathcal{D})$ and $\mathcal{V}^* = \mathcal{F}(\mathcal{D})^* \times \mathcal{F}(\mathcal{D})$. By using the Hamiltonian defined by the total fluid energy, given by

$$h(m, \rho, \eta) = \int_\mathcal{D} \left(\frac{1}{2\rho} |m|^2 + \rho e(\rho, \eta)\right) d^3x, \quad (3.3)$$

with variational derivatives

$$\frac{\delta h}{\delta m} = \frac{m}{\rho} = \mathbf{u}, \quad \frac{\delta h}{\delta \rho} = -\frac{1}{2} |\mathbf{u}|^2 + e + \rho \frac{\partial e}{\partial \rho}, \quad \frac{\delta h}{\delta \eta} = \frac{\partial e}{\partial \eta}, \quad (3.4)$$

Consequently, (3.2) yields the compressible ideal fluid equations

$$\rho(\partial_t u + u \cdot \nabla u) = -\nabla p, \quad \partial_t \rho + \text{div}(\rho \mathbf{u}) = 0, \quad \partial_t \eta + \mathbf{u} \cdot \nabla \eta = 0,$$

where $p = \rho^2 \partial e/\partial \rho$ is the fluid pressure and $\mathbf{u}$ is the fluid velocity.
3.1 Selective Casimir decay for 3D compressible fluids

In this case, the Casimirs for the Lie–Poisson bracket (3.1) are

\[ C(m, \rho, \eta) = \int_D \rho \Phi(\eta, q) \, d^3x, \quad \text{where} \quad q := \rho^{-1} \text{curl}(m/\rho) \cdot \nabla \eta, \]  

(3.5)

with variational derivatives

\[ \frac{\delta C}{\delta m} = \rho^{-1} \text{curl}(\Phi_q \nabla \eta), \quad \frac{\delta C}{\delta \rho} = \Phi - q \Phi_q - \rho^{-2} m \cdot \text{curl}(\Phi_q \nabla \eta), \quad \frac{\delta C}{\delta \eta} = \rho \Phi_\eta - \text{div}(\Phi_q \text{curl}(m/\rho)), \]

where \( \Phi_q = \partial \Phi / \partial q \) and \( \Phi_\eta = \partial \Phi / \partial \eta \). Consequently, the modified momentum (2.14) is

\[ \tilde{m} = m + \theta \left[ \frac{\delta h}{\delta m} \cdot \frac{\delta C}{\delta m} \right]^\flat = m + \theta \left[ \rho^{-1} m, \rho^{-1} \text{curl}(\Phi_q \nabla \eta) \right]^\flat. \]  

(3.6)

One can also modify \( \rho \) and \( \eta \) according the right hand side in formula in (2.14). In any case, Theorem 2.3 guarantees that the modifications of the advected variables vanish under the critical point conditions \( \delta (h + C) = 0 \) corresponding to energy-Casimir equilibria of the unmodified equations in the compressible fluid case. In the present example, the critical point condition \( \delta (h + C)/\delta m = 0 \) is sufficient for the modification of the momentum in (3.6) to vanish.

As an explicit case, if we choose the Casimir

\[ C(m, \rho, \eta) = \frac{1}{2} \int_D \rho q^2 \, d^3x \]  

(3.7)

and the positive bilinear form \( \gamma_{\rho,\eta}(u, v) = \int_D \rho (u \cdot v) \, d^3x \) which is weighted by the mass density, then the modified momentum in (3.6) becomes

\[ \tilde{m} = m + \theta X, \quad X := \rho [u, \rho^{-1} \text{curl}(q \nabla \eta)] \]  

(3.8)

and the modified Lie–Poisson system in (2.4) becomes

\[ \begin{cases} \rho (\partial_t u + u \cdot \nabla u) + \nabla p = -\theta (u \cdot \nabla X + \nabla u^T \cdot X + X \text{div} u), \\ \partial_t \rho + \text{div}(\rho u) = 0, \quad \partial_t \eta + u \cdot \nabla \eta = 0. \end{cases} \]  

(3.9)

In a slight abuse of notation for the Lie derivative of a one-form density, we set

\[ {\mathcal{L}}_u X := u \cdot \nabla X + \nabla u^T \cdot X + X \text{div} u. \]

The first equation in (3.9) can then be written in this notation simply as

\[ \rho (\partial_t u + u \cdot \nabla u) + \nabla p = -\theta {\mathcal{L}}_u X. \]

This system preserves energy \( h \) in (3.3), mass \( \int_D \rho \, d^3x \) and total entropy \( \int_D \rho \eta \, d^3x \), while it dissipates the Casimir (3.7) given by the integrated squared PV as follows,

\[ \frac{d}{dt} \frac{1}{2} \int_D \rho q^2 \, d^3x = -\theta \int_D \rho |\text{ad}_u W|^2 \, d^3x = -\theta \int_D \frac{1}{2} |X|^2 \, d^3x = -\theta \gamma_{\rho,\eta}(X/\rho, X/\rho). \]  

(3.10)
In the energy-Casimir stability method for compressible fluid flow HMRW [1985], the velocity equilibrium condition is given by

\[ \frac{\delta (h + C)}{\delta m} = u + \rho^{-1} \text{curl} (q \nabla \eta) = 0, \quad \text{at equilibrium.} \]  

(3.11)

Under this condition, the commutator vanishes in the definition of the 1-form density \( X \) in (3.8) when the flow velocity achieves its equilibrium form. Thus, \(|X|^2 = 0\) in (3.10) vanishes for energy-Casimir equilibria of ideal compressible fluid flow. This is an example of Case III of Theorem 2.3.

3.1.1 Kelvin-Noether theorem and Casimir-dissipative constraint force

**Kelvin-Noether theorem.** The Kelvin-Noether theorem for the modified Lie–Poisson system (3.9) shows that the right hand side of the motion equation influences the circulation on loops moving with the fluid, according to

\[ \frac{d}{dt} \oint_{c(u)} u \cdot dx = \oint_{c(u)} (Td\eta - \theta \rho^{-1} L_u X), \]  

(3.12)

where \( T = \partial e / \partial \eta \rho \) is temperature, while \( L_u \) represents Lie derivative with respect to the velocity vector field \( u \), and \( X \) is the modified momentum 1-form density defined in (3.8), which vanishes at the equilibria (3.11) of the unmodified equations. This can be obtained via equation (2.10) by choosing \( \langle K(c, \rho), m \rangle := \oint_c \rho^{-1} m \cdot dx \) in that equation.

**Casimir-dissipation constraint force.** The selection of Casimir dissipation exerts a force on the system (3.9) obtained from its momentum equation

\[ \partial_t (\rho u_i) = - \partial_j (\rho u_i + \theta X_i u^j + \delta^j_i p) - \theta X_j \partial_i u^j, \]  

(3.13)

so that the total force is obtained from

\[ \frac{d}{dt} \int_D \rho u d^3 x = - \theta \int_D \nabla u^T \cdot X d^3 x. \]  

(3.14)

Thus, the Casimir-dissipative force does no work, because it conserves energy, but as a constraint force it does change the total momentum. From the definition of \( X \) in (3.8), the constraint force density \(- \theta \nabla u^T \cdot X \) vanishes at the equilibria (3.11) of the unmodified equations, for any finite shear gradient \( \nabla u \).

**Remark 3.1** (Rotating 3D compressible fluid flows). Rotation may also be included into the example of 3D compressible fluids, as was done for the shallow water case in Gay-Balmaz and Holm [2013].

3.2 Selective energy decay for 3D compressible fluids

We recall that the energy (3.3) for 3D compressible fluids is

\[ h(m, \rho, \eta) = \int_D \left( \frac{1}{2\rho} |m|^2 + \rho e(\rho, \eta) \right) d^3 x, \]

(3.15)
and the Casimir is given in (3.5). To introduce energy dissipation into the motion equation only, we will use equations (2.6), which we also recall here

$$\partial_t \mu + \text{ad}^*_h \mu + \frac{\delta h}{\delta a} \circ a = \theta \text{ ad}^*_C \left[ \frac{\delta h}{\delta \mu}, \frac{\delta C}{\delta \mu} \right], \quad \partial_t a + \frac{\delta h}{\delta \mu} = 0. \quad (3.16)$$

As in the previous subsection, the bracket term is given by

$$\left[ \frac{\delta h}{\delta m}, \frac{\delta C}{\delta m} \right] = \left[ \rho^{-1} m, \rho^{-1} \text{ curl} (\partial_\Phi \nabla \eta) \right]. \quad (3.17)$$

In particular, if we choose the enstrophy

$$C(m, \rho, \eta) = \frac{1}{2} \int_D \rho q^2 \, d^3x \quad (3.18)$$

as the Casimir, and the positive bilinear form $\gamma_{\rho, \eta}(u, v) = \int_D \rho (u \cdot v) \, d^3x$, then the bracket term in (3.17) is given by

$$X := \rho [u, w] \quad \text{where} \quad w = \rho^{-1} \text{ curl}(q \nabla \eta) = \frac{\delta C}{\delta m}. \quad (3.19)$$

Consequently, the modified Lie–Poisson system in (3.16) becomes

$$\begin{cases} 
\rho(\partial_t u + u \cdot \nabla u) + \nabla p = \theta L_w X, \\
\partial_t \rho + \text{ div}(\rho u) = 0, \quad \partial_t \eta + u \cdot \nabla \eta = 0.
\end{cases} \quad (3.20)$$

This system preserves the enstrophy Casimir in (3.18), as well as mass $\int \rho \, d^3x$ and total entropy $\int \rho \eta \, d^3x$, while it dissipates the energy in (3.15) as follows,

$$\frac{d}{dt} \int_D \left( \frac{1}{2} \rho |u|^2 + \rho e(\rho, \eta) \right) \, d^3x = -\theta \int_D \frac{1}{\rho} |X|^2 \, d^3x. \quad (3.21)$$

As expected from Theorem 2.3, equations (3.10) and (3.21) have the same right hand side, which vanishes for energy-Casimir equilibria of ideal compressible fluid flow.

## 4 Selective decay for MHD

In the barotropic (resp. incompressible) magnetohydrodynamics (MHD) approximation, plasma motion in three dimensions is governed by the following system of equations, see HMRW [1985] and references therein:

$$\rho(\partial_t u + u \cdot \nabla u) = -\nabla p + J \times B, \quad \partial_t B = -\text{ curl} E, \quad \partial_t \rho + \text{ div}(\rho u) = 0, \quad \text{ div} B = 0, \quad (4.1)$$

where $p = p(\rho)$, (resp., $\text{ div} u = 0$). Here $B$ denotes the magnetic field, $J := \text{ curl} B$ is the electric current density and $E := -u \times B$ expresses the electric field in a frame moving with the fluid.

The pressure $p$ in the barotropic case is a given function of the mass density $\rho$: $p = p(\rho)$. In contrast, $p$ is determined for the incompressible case by requiring that the condition $\text{ div} u = 0$ be preserved in time.
The barotropic MHD equations (4.1) can be augmented by including the specific entropy \( \eta \) verifying the advection equation \( \partial_t \eta + \mathbf{u} \cdot \nabla \eta = 0 \), and by considering pressure \( p \) as resulting from the First Law of Thermodynamics,

\[
de = \rho^{-2} p \, d\rho + T \, d\eta
\]

for a given equation of state \( e = e(\rho, \eta) \) for the internal energy per unit mass. The resulting isentropic MHD equations are given in (4.48), and their properties under selective decay will be treated in section 4.3.2.

### 4.1 Planar MHD with \( B \) in the plane

In this section we consider two-dimensional MHD taking place in a domain \( \mathcal{D} \) in the \((x,y)\) plane with \( B \) parallel to the plane, whose unit normal vector is denoted \( \hat{\mathbf{z}} = \nabla z \).

#### 4.1.1 Homogeneous incompressible case

We write \( \mathbf{u} = \text{curl}(\psi \hat{\mathbf{z}}) \) and \( \mathbf{B} = \text{curl}(A \hat{\mathbf{z}}) \). The electric field and electric current are given in this geometry by the following vectors normal to the plane of motion,

\[
\mathbf{E} = E \hat{\mathbf{z}} = \{A, \psi\} \hat{\mathbf{z}}, \quad \mathbf{J} = J \hat{\mathbf{z}} = -\Delta A \hat{\mathbf{z}},
\]

(4.2)

where \( \Delta A \) denotes the Laplacian of the scalar function \( A \). For the Lie–Poisson bracket

\[
\{f, g\}_+(\omega, A) = \int_{\mathcal{D}} \left[ \omega \left( \frac{\delta f}{\delta \omega} \frac{\delta g}{\delta \omega} \right) + A \left( \left\{ \frac{\delta f}{\delta \omega}, \frac{\delta g}{\delta A} \right\} - \left\{ \frac{\delta g}{\delta \omega}, \frac{\delta f}{\delta A} \right\} \right) \right] \, dx \, dy,
\]

(4.3)

the Hamiltonian dynamical equations are Holm and Kupershmidt [1983a,b]; HMRW [1985]

\[
\partial_t \omega + \left\{ \omega, \frac{\delta h}{\delta \omega} \right\} + \left\{ A, \frac{\delta h}{\delta A} \right\} = 0, \quad \partial_t A + \left\{ A, \frac{\delta h}{\delta \omega} \right\} = 0.
\]

(4.4)

Thus, \( A \) is an advected scalar function. For the Hamiltonian

\[
h(\omega, A) = \frac{1}{2} \int_{\mathcal{D}} (\omega (-\Delta^{-1} \omega) + |\nabla A|^2) \, dx \, dy,
\]

(4.5)

one finds

\[
\frac{\delta h}{\delta \omega} = -\Delta^{-1} \omega = \psi \quad \text{and} \quad \frac{\delta h}{\delta A} = -\Delta A = J,
\]

(4.6)

so the Lie–Poisson equations (4.4) become

\[
\partial_t \omega + \left\{ \omega, \psi \right\} + \left\{ A, J \right\} = 0, \quad \partial_t A + \left\{ A, \psi \right\} = 0.
\]

(4.7)

Using the formula \( \{A, J\} \hat{\mathbf{z}} = \text{curl}(\mathbf{B} \times \mathbf{J}) \) and \( \text{curl}(\{A, \psi\} \hat{\mathbf{z}}) = \text{curl} \mathbf{E} \), we readily see that these equations agree with (4.1) in the 2D incompressible case, when \( \mathbf{B} = \text{curl}(A \hat{\mathbf{z}}) \) and \( \mathbf{u} = \text{curl}(\psi \hat{\mathbf{z}}) \).

The Casimirs for the Lie–Poisson bracket (4.3) are

\[
C(\omega, A) = \int_{\mathcal{D}} (\omega \Phi(A) + \Psi(A)) \, dx \, dy
\]

(4.8)
whose variational derivatives are
\[ \frac{\delta C}{\delta \omega} = \Phi(A), \quad \frac{\delta C}{\delta A} = \omega \Phi'(A) + \Psi'(A). \]  
(4.9)

As a consequence of (2.14) the modified momenta are
\[ \tilde{\omega} = \omega + \theta L \left\{ \frac{\delta h}{\delta \omega}, \frac{\delta C}{\delta \omega} \right\} = \omega + \theta L \{ \psi, \Phi(A) \} \]
\[ \tilde{A} = A + \theta K \left( \left\{ \frac{\delta h}{\delta \omega}, \frac{\delta C}{\delta \omega} \right\} - \{ \delta h, \delta \omega \} \right) \]
\[ = A + \theta K \left( \{ \psi, \omega \Phi'(A) + \Psi'(A) \} - \{ \Phi(A), J \} \right), \]
where \( L \) in \( \tilde{\omega} \) and \( K \) in \( \tilde{A} \) are the flat operators, taken here to be positive self-adjoint linear differential operators. Using (2.15), we obtain the dissipative Casimir LP equations as, cf. (4.7),
\[ \partial_t \omega + \{ \tilde{\omega}, \psi \} + \{ \tilde{A}, J \} = 0, \quad \partial_t A + \{ \tilde{A}, \psi \} = 0. \]  
(4.10)

These equations preserve the energy \( h \) and dissipate the Casimir \( C(\omega, A) \) in (4.8) as
\[ \frac{d}{dt} C(\omega, A) = -\theta \left\| \left( \frac{\delta C}{\delta \omega}, \frac{\delta C}{\delta A} \right) \right\|^2_{\gamma} \]
\[ = -\theta \left\| \left( \frac{\delta C}{\delta \omega}, \frac{\delta h}{\delta \omega} \right) \right\|^2_L - \theta \left\| \left( \frac{\delta C}{\delta \omega}, \frac{\delta h}{\delta A} \right) \right\|^2_K \]
\[ = -\theta \left\| \{ \Phi(A), \psi \} \right\|^2_L - \theta \left\| \{ \Phi(A), J \} - \{ \psi, \omega \Phi'(A) + \Psi'(A) \} \right\|^2_K. \]  
(4.11)

Thus, if all of the variables are modified and if \( C(\omega, A) \geq 0 \), then the solution of (4.10) converges to a solution with
\[ \{ \Phi(A), \psi \} = 0 \quad \text{and} \quad \{ \Phi(A), J \} - \{ \psi, \omega \Phi'(A) + \Psi'(A) \} = 0. \]  
(4.12)

Thus, the Casimir dissipation enforces a condition consistent with stable energy-Casimir equilibria for ideal planar 2D MHD with \( B \) in the plane, governed by equations (4.7). Namely, upon assuming \( \Phi'(A) \neq 0 \), vanishing of the first term in (4.12) yields the equilibrium condition \( \{ A, \psi \} = 0 \) for (4.7). Substituting this into the second term in (4.12) yields
\[ \Phi'(A) \left( \{ A, J \} + \{ \omega, \psi \} \right) = 0, \]
which is the other equilibrium condition for (4.7).

We will treat the case \( \Phi'(A) = 0 \) in Example 1, below.

The modified bracket is obtaining by adding to the Lie–Poisson bracket \( \{ f, h \}_+ \) in (4.3), the expression
\[ -\theta \gamma \left[ \left( \frac{\delta f}{\delta \omega}, \frac{\delta f}{\delta A} \right), \left( \frac{\delta h}{\delta \omega}, \frac{\delta h}{\delta A} \right) \right] \cdot \left[ \left( \frac{\delta C}{\delta \omega}, \frac{\delta C}{\delta A} \right), \left( \frac{\delta h}{\delta \omega}, \frac{\delta h}{\delta A} \right) \right] \]
\[ = -\theta \int_D \left\{ \frac{\delta f}{\delta \omega}, \frac{\delta h}{\delta \omega} \right\} L \left\{ \frac{\delta C}{\delta \omega}, \frac{\delta h}{\delta A} \right\} dx dy \]
\[ - \theta \int_D \left\{ \frac{\delta f}{\delta A} - \frac{\delta h}{\delta A} \right\} K \left\{ \frac{\delta C}{\delta \omega}, \frac{\delta h}{\delta A} \right\} dx dy \]  
(4.13)
One can choose to modify \( \omega \) only and keep \( A \) unchanged as in (2.4), in which case instead of (4.10) the equations are simply
\[
\partial_t \omega + \{ \tilde{\omega}, \psi \} + \{ A, J \} = 0, \quad \partial_t A + \{ A, \psi \} = 0,
\]
where \( \tilde{\omega} = \omega + \theta L \{ \psi, \Phi(A) \} \), and we have the Casimir dissipation rate,
\[
\frac{d}{dt} C(\omega, A) = -\theta \left\| \frac{\delta C}{\delta \omega}, \frac{\delta h}{\delta \omega} \right\|_L^2 = -\theta \left\| \{ \Phi(A), \psi \} \right\|_L^2.
\]
Hence, if we modify only the momentum variables, the Casimir dissipates, and the solution approaches an equilibrium solution for planar incompressible 2D MHD with \( B \) in the plane. This condition is verified by all energy-Casimir equilibria since \( \delta(h + C)/\delta \omega = 0 \) implies \( \psi + \Phi(A) = 0 \), and if \( \{ A, \psi \} = 0 \) holds, then the second of the unmodified 2D incompressible MHD equations in (4.14) implies that \( A \) is time independent.

**Example 1: Alfvén and Grad–Shafranov equilibria.** Using the Casimir
\[
C(\omega, A) = \int_D \Psi(A) \, dx \, dy
\]
we have \( \delta C/\delta \omega = 0 \), so that only \( A \) is modified:
\[
\tilde{\omega} = \omega \quad \text{and} \quad \tilde{A} = A + \theta K \left\{ \frac{\delta h}{\delta \omega}, \frac{\delta C}{\delta A} \right\} = A + \theta K \{ \psi, \Psi'(A) \}.
\]
From (4.10) the modified equations become
\[
\partial_t \omega + \{ \omega, \psi \} + \{ \tilde{A}, J \} = 0, \quad \partial_t A + \{ \tilde{A}, \psi \} = 0,
\]
and the Casimir dissipation rate is given by
\[
\frac{d}{dt} C(\omega, A) = -\theta \left\| \{ \psi, \Psi'(A) \} \right\|_K^2.
\]
For example, with \( \Psi(A) = \frac{1}{2} A^2 \) and \( K = Id \), equations (4.16) specialize to
\[
\partial_t \omega + \{ \omega, \psi \} + \{ A + \theta \{ \psi, A \}, J \} = 0, \quad \partial_t A + \{ A + \theta \{ \psi, A \}, \psi \} = 0,
\]
and
\[
\frac{d}{dt} \frac{1}{2} \int_D A^2 \, dx \, dy = -\theta \int_D \{ \psi, A \}^2 \, dx \, dy.
\]
Therefore, the solution of the modified equations again converges to a steady state consistent with the conditions for an energy-Casimir equilibrium of the unmodified equations (4.7), but it does not enforce all of the equilibrium conditions. In particular, it does not enforce the equilibrium condition \( \{ \omega, \psi \} = 0 \). Note that the energy-Casimir equilibria associated to the Casimir (4.15) are the Grad–Shafranov equilibria \( \psi = 0 \) and \( J + \Psi'(A) = 0 \) discussed in Chandrasekhar [1961].

**Remark 4.1.** Rewriting the equation set (4.17) in the previous example suggests that it provides a type of anticipated vector potential formulation for planar incompressible 2D MHD with \( B \) in the plane. The anticipated vector potential is \( \tilde{A} = A - \theta K(u \cdot \nabla A) \), as in Sadourny and Basdevant [1981] for anticipated vorticity in 2D incompressible flow. Namely, (4.17) becomes
\[
\partial_t \omega + \{ \omega, \psi \} + \{ A - \theta K(u \cdot \nabla A), J \} = 0, \quad \partial_t A = -u \cdot \nabla(A - \theta K(u \cdot \nabla A)).
\]
Example 2: Equilibria associated with cross helicity Chandrasekhar [1961]. Upon choosing the Casimir (cross helicity) \[ C(\omega, A) = \int_D \omega A \, dx \, dy = \int_D \mathbf{u} \cdot \mathbf{B} \, dx \, dy, \] (4.20)
we have \( \delta C/\delta \omega = A \) and \( \delta C/\delta A = \omega \), so that
\[ \tilde{\omega} = \omega + \theta L \{ \psi, A \} \quad \text{and} \quad \tilde{A} = A + \theta K (\{ \psi, \omega \} - \{ A, J \}). \]
From (4.10), we find
\begin{align*}
\partial_t \omega + \{ \omega + \theta L \{ \psi, A \}, \psi \} + \{ A + \theta K (\{ \psi, \omega \} - \{ A, J \}), J \} &= 0, \\
\partial_t A + \{ A + \theta K (\{ \psi, \omega \} - \{ A, J \}), \psi \} &= 0. \tag{4.21}
\end{align*}
The Casimir dissipation rate for these modified equations is obtained from (4.11) as
\[ \frac{d}{dt} C(\omega, A) = -\theta \| \{ A, \psi \} \|^2_L - \theta \| A, J \| - \{ \psi, \omega \} \|^2_K. \]
Therefore, by considering the square of this Casimir, it follows that the solution always converges to a steady state given by \( \{ A, \psi \} = 0 \) and \( \{ A, J \} + \{ \omega, \psi \} = 0 \) of the unmodified equation. This is a special instance of the situation considered in §2.3.3.

Of course, one can choose to modify \( \omega \) only, in which case, we have
\[ \frac{d}{dt} \int_D \omega A \, dx \, dy = -\theta \| \{ \psi, A \} \|_L^2. \]
This is a special case of (4.14).

4.1.2 2D Compressible MHD with \( \mathbf{B} \) in the plane
As before, since \( \text{div} \mathbf{B} = 0 \), we may write \( \mathbf{B} = \text{curl}(A \hat{z}) \), and we have \( \mathbf{J} = J \hat{z} = -\Delta A \hat{z} \).

The Lie–Poisson bracket for compressible MHD in HMRW [1985]
\[ \{ f, g \}_+(m, \rho, A) = \int_D m \cdot \left[ \frac{\delta f}{\delta m} \frac{\delta g}{\delta m} \right] \, dx \, dy + \int_D \rho \left( \frac{\delta g}{\delta m} \cdot \nabla \frac{\delta f}{\delta \rho} - \frac{\delta f}{\delta m} \cdot \nabla \frac{\delta g}{\delta \rho} \right) \, dx \, dy \]
\[ + \int_D A \left( \text{div} \left( \frac{\delta f}{\delta A} \frac{\delta g}{\delta m} \right) - \text{div} \left( \frac{\delta g}{\delta A} \frac{\delta f}{\delta m} \right) \right) \, dx \, dy \] (4.22)
yields the Lie–Poisson equations,
\begin{align*}
\partial_t \left( \frac{m}{\rho} \right) + \text{curl} \left( \frac{m}{\rho} \right) \times \frac{\partial h}{\partial m} + \nabla \left( \frac{m}{\rho} \cdot \frac{\partial h}{\partial m} \right) &= -\nabla \frac{\partial h}{\partial \rho} + \frac{1}{\rho} \frac{\partial h}{\partial A} \nabla A, \\
\partial_t \rho + \text{div} \left( \rho \frac{\partial h}{\partial m} \right) &= 0, \quad \partial_t A + \frac{\partial h}{\partial m} \cdot \nabla A = 0. \tag{4.23}
\end{align*}
where, for a two dimensional vector field \( \mathbf{u} \), the curl operator is \( \text{curl} \mathbf{u} = \omega \hat{z} \), with \( \omega \) the scalar vorticity associated to \( \mathbf{u} \).
Note that the Lie–Poisson bracket (4.22) consistently recovers the Lie–Poisson bracket (4.3) in the incompressible case when the variable $\rho$ is absent.

Consider the Hamiltonian for compressible MHD with $\mathbf{B}$ in the plane. This is the sum of energies, kinetic, plus potential, plus magnetic,

$$h(\mathbf{m}, \rho, A) = \int_D \left( \frac{1}{2\rho} |\mathbf{m}|^2 + \rho e(\rho) + \frac{1}{2} |\nabla A|^2 \right) \, dx \, dy,$$

(4.24)

where $e(\rho)$ is the specific internal energy. Since

$$\frac{\delta h}{\delta \mathbf{m}} = \frac{\mathbf{m}}{\rho} = \mathbf{u}, \quad \frac{\delta h}{\delta \rho} = -\frac{1}{2} |\mathbf{u}|^2 + e + \rho \frac{\partial e}{\partial \rho} \quad \text{and} \quad \frac{\delta h}{\delta A} = -\Delta A = J,$$

(4.25)

the Lie–Poisson equations in this situation read

$$\rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + J \mathbf{A}, \quad \partial_t \rho + \text{div}(\rho \mathbf{u}) = 0, \quad \partial_t A + \mathbf{u} \cdot \nabla A = 0,$$

and they agree with (4.1) in the two-dimensional case, since in that case

$$\mathbf{J} \times \mathbf{B} = J \hat{z} \times \text{curl}(\hat{A} \hat{z}) = J \nabla A,$$

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B} = -\mathbf{u} \times \text{curl}(\hat{A} \hat{z}) = (\mathbf{u} \cdot \nabla A) \hat{z}.$$

(4.26)

The Casimirs for this case are

$$C(\mathbf{m}, \rho, A) = \int_D (\omega \Phi(A) + \rho \Psi(A)) \, dx \, dy, \quad \omega := \hat{z} \cdot \text{curl}(\mathbf{m}/\rho),$$

(4.27)

and they have the variational derivatives

$$\frac{\delta C}{\delta \mathbf{m}} = \text{curl}(\Phi(A) \hat{z}) = \Phi'(A) \nabla A \times \hat{z}, \quad \frac{\delta C}{\delta \rho} = \Psi(A), \quad \frac{\delta C}{\delta A} = \omega \Phi'(A) + \rho \Psi'(A).$$

Consequently, the modified variables in (2.14) here become

$$\mathbf{m} = \mathbf{m} + \theta \left[ \frac{\delta h}{\delta \mathbf{m}}, \frac{\delta C}{\delta \mathbf{m}} \right]^b = \mathbf{m} + \theta \left[ \mathbf{u}, \Phi'(A) \nabla A \times \hat{z} \right]^b,$$

$$\rho = \rho + \theta \left( \frac{\delta C}{\delta \mathbf{m}} \cdot \nabla \frac{\delta h}{\delta \rho} - \frac{\delta h}{\delta \mathbf{m}} \cdot \nabla \frac{\delta C}{\delta \rho} \right)^b$$

$$= \rho + \theta \left( (\Phi'(A) \nabla A \times \hat{z}) \cdot \left( -\frac{1}{2} |\mathbf{u}|^2 + e + \rho \frac{\partial e}{\partial \rho} \right) - \mathbf{u} \cdot \Phi'(A) \nabla A \right)^b,$$

(4.28)

$$A = A + \theta \left( \text{div} \left( \frac{\delta h}{\delta A} \frac{\delta C}{\delta \mathbf{m}} \right) - \text{div} \left( \frac{\delta C}{\delta A} \frac{\delta h}{\delta \mathbf{m}} \right) \right)^b$$

$$= A + \theta \left( \text{div} \left( J \Phi'(A) \nabla A \times \hat{z} \right) - \text{div} \left( (\omega \Phi'(A) + \rho \Psi'(A)) \mathbf{u} \right) \right)^b.$$

The equations resulting from (2.12) in this case preserve the energy $h$ in (4.24) and dissipate the Casimir $C$ in (4.27) as

$$\frac{d}{dt} C(\mathbf{m}, \rho, A) = -\theta \left\| \begin{bmatrix} \frac{\delta C}{\delta \mathbf{m}} & \frac{\delta C}{\delta \rho} & \frac{\delta C}{\delta A} \end{bmatrix} \cdot \begin{bmatrix} \frac{\delta h}{\delta \mathbf{m}} & \frac{\delta h}{\delta \rho} & \frac{\delta h}{\delta A} \end{bmatrix} \right\|_\gamma^2$$

$$= -\theta \left\| \begin{bmatrix} \frac{\delta h}{\delta \mathbf{m}} & \frac{\delta h}{\delta \rho} \end{bmatrix} \right\|_1^2 - \theta \left\| \begin{bmatrix} \frac{\delta C}{\delta \mathbf{m}} & \frac{\delta C}{\delta \rho} \end{bmatrix} \cdot \nabla \frac{\delta \mathbf{m}}{\delta \rho} - \frac{\delta C}{\delta \mathbf{m}} \cdot \nabla \frac{\delta \mathbf{m}}{\delta \rho} \right\|_2^2$$

$$- \theta \left\| \text{div} \left( \frac{\delta C}{\delta A} \frac{\delta h}{\delta \mathbf{m}} \right) - \text{div} \left( \frac{\delta C}{\delta A} \frac{\delta h}{\delta \mathbf{m}} \right) \right\|_3^2.$$

(4.29)
Each of the three summands in (4.29) vanishes for equilibrium solutions of equations (4.23). Since the square of the Casimir also reaches a minimum for these conditions, the Casimir-dissipative approach to equilibrium tends toward conditions consistent with energy-Casimir equilibria for planar compressible MHD flows with $\mathbf{B}$ in the plane.

**Remark 4.2.** The modified variables in (4.28) provide a wide variety of possible approaches for applying Casimir dissipation in 2D compressible MHD with $\mathbf{B}$ in the plane. For example, we can always choose to modify only the momentum and keep any number of the other variables in (4.28) unchanged. Moreover, we can choose any functions $\Psi$ and $\Phi$, including zero in (4.27). We are also free to choose the $b$ operation differently in each of the three modified variables, including choosing $b = id$. The natural choice for $b$ in the case of $\tilde{m}$ is $(\cdot)^b = \rho(\cdot)$, i.e. (multiplication by the advected density), as chosen for the rotating shallow water equations in Gay-Balmaz and Holm [2013].

### 4.2 Planar MHD with $\mathbf{B}$ normal to the plane

In this section we consider two-dimensional MHD taking place in a domain $D$ in the $xy$ plane with $\mathbf{B}$ normal to the plane, that is,

$$\mathbf{B} = B\hat{z}.$$  

#### 4.2.1 Homogeneous incompressible case

As before, we write $\mathbf{u} = \text{curl}(\psi \hat{z})$. The electric field and electric current are now given by, cf. equation (4.2),

$$\mathbf{E} = -\text{curl}(\psi \hat{z}) \times B\hat{z} = B\nabla \psi, \quad \mathbf{J} = \text{curl}(B\hat{z}).$$  

(4.30)

The Lie–Poisson bracket is given by Holm and Kupershmidt [1983a,b]; HMRW [1985] as

$$\{ f, g \} + (\omega, B) = \int_D \left[ \omega \left( \frac{\delta f}{\delta \omega}, \frac{\delta g}{\delta \omega} \right) + B \left( \frac{\delta f}{\delta \omega}, \frac{\delta g}{\delta B} \right) - \left( \frac{\delta g}{\delta \omega}, \frac{\delta f}{\delta B} \right) \right] \, dx \, dy,$$

(4.31)

and the associated Lie–Poisson equations are

$$\partial_t \omega + \left\{ \omega, \frac{\delta h}{\delta \omega} \right\} + \left\{ B, \frac{\delta h}{\delta B} \right\} = 0, \quad \partial_t B + \left\{ B, \frac{\delta h}{\delta B} \right\} = 0.$$

For the Hamiltonian given by the sum of kinetic and magnetic energies,

$$h(\omega, B) = \frac{1}{2} \int_D (\omega(-\Delta^{-1}\omega) + B^2) \, dx \, dy,$$

since $\delta h/\delta \omega = \psi$ and $\delta h/\delta B = B$ the Lie–Poisson equations are

$$\partial_t \omega + \{ \omega, \psi \} = 0, \quad \partial_t B + \{ B, \psi \} = 0.$$

(4.32)

Using the formula $\{ B, \psi \} \hat{z} = (\mathbf{u} \cdot \nabla B)\hat{z} = \text{curl}(\mathbf{B} \times \mathbf{u})$, we readily see that these equations agree with (4.1) in the 2D incompressible case, when $\mathbf{B} = B\hat{z}$ and $\mathbf{u} = \text{curl}(\psi \hat{z})$. Note that $\mathbf{J} \times \mathbf{B} = \text{curl}(B\hat{z}) \times B\hat{z} = -B\nabla B = -\frac{1}{2} \nabla B^2$ in (4.1) is a gradient and therefore does not appear in the vorticity formulation.
The Casimirs in this case are
\[
C(\omega, B) = \int_D (\omega \Phi(B) + \Psi(B)) \, dx \, dy,
\]
and we have the variational derivatives
\[
\frac{\delta C}{\delta \omega} = \Phi(B), \quad \frac{\delta C}{\delta B} = \omega \Phi'(B) + \Psi'(B).
\]
Consequently, from (2.14) the modified momenta are expressible as
\[
\tilde{\omega} = \omega + \theta L \left( \frac{\delta h}{\delta \omega}, \frac{\delta C}{\delta \omega} \right) = \omega + \theta L \left\{ \psi, \Phi(B) \right\} = \omega - \theta L \left( u \cdot \nabla \Phi(B) \right),
\]
\[
\tilde{\omega} = B + \theta K \left( \frac{\delta h}{\delta \omega}, \frac{\delta C}{\delta B} \right) = B - \theta K \left( u \cdot \nabla (\omega \Phi'(B) + \Psi'(B)) \right),
\]
where \( L \) and \( K \) are positive self-adjoint linear differential operators and we have used the fact that \( \left\{ \frac{\delta C}{\delta \omega}, \frac{\delta h}{\delta B} \right\} = \{ \Phi(B), B \} = 0 \). The Casimir-dissipative Lie–Poisson equations are
\[
\partial_t \omega + \left\{ \tilde{\omega}, \frac{\delta h}{\delta \omega} \right\} + \left\{ \tilde{B}, \frac{\delta h}{\delta B} \right\} = 0, \quad \partial_t B + \left\{ \tilde{B}, \frac{\delta h}{\delta \omega} \right\} = 0.
\]
These equations preserve the energy \( h \) and dissipate the Casimir \( C \) as
\[
\frac{d}{dt} C(\omega, B) = -\theta \left\| \left[ \frac{\delta C}{\delta \omega}, \frac{\delta h}{\delta B} \right] \right\|^2_L - \theta \left\| \left[ \frac{\delta C}{\delta \omega}, \frac{\delta h}{\delta B} \right] - \left[ \frac{\delta C}{\delta \omega}, \frac{\delta C}{\delta B} \right] \right\|^2_K.
\]
Thus, if all of the variables are modified and if \( C(\omega, B) \geq 0 \), then the solution converges to a solution with
\[
\{ \Phi(B), \psi \} = 0 \quad \text{and} \quad \{ \Phi(B), B \} - \{ \psi, \omega \Phi'(B) + \Psi'(B) \} = 0.
\]
By Theorem 2.3, this condition holds for energy-Casimir equilibria \( \delta (h + C) / \delta (\omega, B) = 0 \), associated to the chosen Casimir \( C \). By an appropriate choice of the Casimir, we can achieve the asymptotic case to be a steady solution, see Example 1 below.

One can choose to modify \( \omega \) only and keep the \( B \) advection equation unchanged, as in (2.4), in which case the equations are
\[
\partial_t \omega + \left\{ \omega, \frac{\delta h}{\delta \omega} \right\} + \left\{ \theta L \left( \frac{\delta h}{\delta \omega}, \Phi(B) \right), \psi \right\} + \left\{ B, \frac{\delta h}{\delta B} \right\} = 0, \quad \partial_t B + \left\{ B, \frac{\delta h}{\delta \omega} \right\} = 0
\]
and
\[
\frac{d}{dt} C(\omega, B) = -\theta \left\| \left[ \frac{\delta C}{\delta \omega}, \frac{\delta h}{\delta \omega} \right] \right\|^2_L.
\]
Thus, the Casimir-dissipative approach enforces a condition consistent with stable energy-Casimir equilibria for planar MHD flows with \( B \) normal to the plane. One need not worry that these magnetic Casimirs are sign indefinite, since their squares also decay, as shown in Gay-Balmaz and Holm [2013].
**Example 1.** For the choices $\Phi(B) = B$ and $\Psi(B) = \text{const}$, these equations take a form that extends the anticipated vorticity approach of Sadourny and Basdevant [1981] to the case of incompressible 2D MHD with $B = B\hat{z}$ normal to the $(x, y)$ plane, namely,
\[
\tilde{\omega} = \omega - \theta L(u \cdot \nabla B) = \omega + \theta L\{\psi, B\}, \\
\tilde{B} = B - \theta K(u \cdot \nabla \omega) = B + \theta K\{\psi, \omega\}.
\]
This could be called a *cross-anticipated method*. The equations are
\[
\partial_t \omega + \{\tilde{\omega}, \psi\} + \{\tilde{B}, B\} = 0, \quad \partial_t B + \{\tilde{B}, \psi\} = 0.
\]
For this cross-anticipated method we get the decay rate
\[
\frac{d}{dt} \int_D \omega B \, dx \, dy = -\theta \|\{B, \psi\}\|_L^2 - \theta \|\{\psi, \omega\}\|_K^2.
\]
In accordance with the general theory in §2.3.3, the solution converges to a steady state consistent with the energy-Casimir equilibria of the unmodified equations (4.32). ♦

**Example 2.** One can also consider $\Phi(B) = 0$ and $\Psi(B) = \frac{1}{2}B^2$ in (4.33), so that
\[
\tilde{\omega} = \omega \quad \text{and} \quad \tilde{B} = B - \theta K(u \cdot \nabla B) = B + \theta K\{\psi, B\},
\]
and
\[
\partial_t \omega + \{\omega, \psi\} + \{B + \theta K\{\psi, B\}, B\} = 0, \quad \partial_t B + \{B + \theta K\{\psi, B\}, \psi\} = 0.
\]
In this case, we find that the magnetic energy decays as
\[
\frac{d}{dt} \frac{1}{2} \int_D B^2 \, dx \, dy = -\theta \|\{\psi, B\}\|_K^2.
\]
Therefore, the solution of the modified equations again converges to a steady state consistent with the conditions for an energy-Casimir equilibrium of the unmodified equations (4.32), but it does not enforce all of the equilibrium conditions. In particular, it does not enforce the equilibrium condition $\{\omega, \psi\} = 0$. ♦

### 4.2.2 Planar compressible barotropic MHD with B normal to the plane

The Lie–Poisson bracket for this case is
\[
\{f, g\}_+(m, \rho, A) = \int_D m \cdot \left[ \frac{\delta f}{\delta m}, \frac{\delta g}{\delta m} \right] \, dx \, dy + \int_D \rho \left( \frac{\delta g}{\delta \rho} \cdot \nabla \frac{\delta f}{\delta \rho} - \frac{\delta f}{\delta \rho} \cdot \nabla \frac{\delta g}{\delta \rho} \right) \, dx \, dy \\
+ \int_D B \left( \frac{\delta g}{\delta B} \cdot \nabla \frac{\delta f}{\delta B} - \frac{\delta f}{\delta B} \cdot \nabla \frac{\delta g}{\delta B} \right) \, dx \, dy,
\]
Holm and Kupershmidt [1983a,b]; HMRW [1985], and the associated Lie–Poisson equations are
\[
\partial_t \left( \frac{m}{\rho} \right) + \text{curl} \left( \frac{m}{\rho} \right) \times \frac{\delta h}{\delta m} + \nabla \left( \frac{m}{\rho} \cdot \frac{\delta h}{\delta m} \right) = -\nabla \frac{\delta h}{\delta \rho} - \frac{1}{\rho} B \nabla \frac{\delta h}{\delta B},
\]
\[
\partial_t \rho + \text{div} \left( \rho \frac{\delta h}{\delta m} \right) = 0, \quad \partial_t B + \text{div} \left( B \frac{\delta h}{\delta m} \right) = 0.
\]
Consider the Hamiltonian

\[ h(m, \rho, A) = \int_D \left( \frac{1}{2\rho} |m|^2 + \rho e(\rho) + \frac{1}{2} B^2 \right) \, dx \, dy, \]

where \( e(\rho) \) is the specific internal energy. Since \( \delta h/\delta m = m/\rho = u, \delta h/\delta \rho = -\frac{1}{2} |u|^2 + e + \rho \frac{\partial e}{\partial \rho}, \) and \( \delta h/\delta B = B, \) the Lie–Poisson equations read

\[ \rho(\partial_t u + u \cdot \nabla u) = -\nabla p - \frac{1}{2} \nabla |B|^2, \quad \partial_t \rho + \text{div}(\rho u) = 0, \quad \partial_t B + \text{div}(Bu) = 0. \]

These equations agree with (4.1) in the two-dimensional case and when \( B = B \hat{z}, \) since \( J \times B = -\frac{1}{2} \nabla |B|^2 \) and \( \text{curl} E = \text{div}(Bu) \) in that case. The pressure is again \( p = \rho^2 \partial e / \partial \rho. \)

The Casimirs are

\[ C(\rho, B) = \int_D \rho \Phi(B/\rho) \, dx \, dy, \tag{4.39} \]

and its variational derivatives are

\[ \frac{\delta C}{\delta m} = 0, \quad \frac{\delta C}{\delta \rho} = \Phi(B/\rho) - \frac{B}{\rho} \Phi'(B/\rho), \quad \frac{\delta C}{\delta B} = \Phi'(B/\rho). \]

Consequently, from (2.14) we have \( \tilde{m} = m \) and

\[ \tilde{\rho} = \rho + \theta \left( \frac{\delta C}{\delta m} \cdot \nabla \frac{\delta h}{\delta \rho} - \frac{\delta h}{\delta m} \cdot \nabla \frac{\delta C}{\delta \rho} \right)^b = \rho - \theta \left( u \cdot \nabla \left( \frac{\Phi(B/\rho) - B}{\rho} \Phi'(B/\rho) \right) \right)^b, \]

\[ \tilde{B} = B + \theta \left( \frac{\delta C}{\delta m} \cdot \nabla \frac{\delta h}{\delta B} - \frac{\delta h}{\delta m} \cdot \nabla \frac{\delta C}{\delta B} \right)^b = B - \theta (u \cdot \nabla (\Phi'(B/\rho)))^b. \]

For example, the choice \( \Phi(B/\rho) = \frac{1}{2} (B/\rho)^2 \) leads to

\[ \tilde{\rho} = \rho + \theta u \cdot \nabla \frac{1}{2} (B/\rho)^2, \quad \tilde{B} = B - \theta u \cdot \nabla (B/\rho). \tag{4.40} \]

**Remark 4.3.** Note that the Casimir functions in (4.39) do not depend on the fluid momentum \( m, \) therefore \( \tilde{m} = m \) in all cases. That is, the momentum \( m \) is not modified in imposing Casimir dissipation in this case. This example relies on (2.14) in which the advected quantities \( a \in V^*, \) given here by \( \rho \) and \( B, \) are also modified.

The modified equations for planar compressible barotropic MHD with \( B \) normal to the plane for the Casimir (4.39) with \( \Phi(B/\rho) = \frac{1}{2} (B/\rho)^2 \) are given by

\[ \partial_t m + \underbar{L}_{\tilde{m}} m = -\tilde{\rho} \nabla \frac{\delta h}{\delta \rho} - \tilde{B} \nabla \frac{\delta h}{\delta B}, \]

\[ \partial_t \rho + \text{div} \left( \tilde{\rho} \frac{\delta h}{\delta m} \right) = 0, \quad \partial_t B + \text{div} \left( \tilde{B} \frac{\delta h}{\delta m} \right) = 0, \]

where \( \tilde{\rho} \) and \( \tilde{B} \) are given in (4.40).
4.3 Selective decay for three-dimensional MHD

4.3.1 3D homogeneous incompressible MHD

For the Lie–Poisson bracket Holm and Kupershmidt [1983a,b]; HMRW [1985]

\[ \{ f, g \} + (m, B) = \int_D m \cdot \left[ \frac{\delta f}{\delta m}, \frac{\delta g}{\delta m} \right] d^3x + \int_D \left( \text{curl} \left( B \times \frac{\delta f}{\delta m} \right) \cdot \frac{\delta g}{\delta B} - \text{curl} \left( B \times \frac{\delta g}{\delta m} \right) \cdot \frac{\delta f}{\delta B} \right) d^3x, \]

with \( \text{div} B = 0 \), the Lie–Poisson equations are

\[ \partial_t m + \text{curl} m \times \frac{\delta h}{\delta m} = \text{curl} \frac{\delta h}{\delta B} \times B - \nabla p, \quad \partial_t B + \text{curl} \left( B \times \frac{\delta h}{\delta m} \right) = 0. \]

In vorticity form, they read

\[ \partial_t \omega + \text{curl} \left( \omega \times \text{curl} \frac{\delta h}{\delta \omega} \right) = \text{curl} \left( \frac{\delta h}{\delta B} \times B \right), \quad \partial_t B + \text{curl} \left( B \times \frac{\delta h}{\delta \omega} \right) = 0, \]

where we have written \( \frac{\delta h}{\delta m} = \text{curl} \frac{\delta h}{\delta \omega} \), with \( \frac{\delta h}{\delta \omega} \) divergence-free.

For the Hamiltonian

\[ h(m, B) = \int_D \left( \frac{1}{2} \left| m \right|^2 + \frac{1}{2} \left| B \right|^2 \right) d^3x, \]

we have \( u = \delta h/\delta m = m \) and \( \delta h/\delta B = B \), and the Lie–Poisson equations recover (4.1) in the incompressible case, upon redefining the pressure.

**Casimir dissipative modified equations.** The modified momenta are

\[ \tilde{m} = m + \theta \left[ \frac{\delta h}{\delta m}, \frac{\delta C}{\delta m} \right]^b = m + \theta \text{curl} \left( \frac{\delta h}{\delta m} \times \frac{\delta C}{\delta m} \right)^b, \]

\[ \tilde{B} = B + \theta \left( \text{curl} \frac{\delta h}{\delta B} \times \frac{\delta C}{\delta m} - \text{curl} \frac{\delta C}{\delta B} \times \frac{\delta h}{\delta m} - \nabla \phi \right)^b, \] (4.41)

where \( \nabla \phi \) is such that the term inside the parenthesis is divergence free, and the Casimir dissipates as

\[ \frac{d}{dt} C(m, B) = -\theta \int_D \left| \text{curl} \left( \frac{\delta h}{\delta m} \times \frac{\delta C}{\delta m} \right) \right|^2 d^3x - \theta \int_D \left| \text{curl} \frac{\delta h}{\delta B} \times B - \text{curl} \frac{\delta C}{\delta B} \times \frac{\delta h}{\delta m} - \nabla \phi \right|^2 d^3x, \] (4.42)

when the standard inner product is used.

**Example 1: Cross helicity.** One Casimir for incompressible 3D MHD is given by the cross helicity

\[ C(m, B) = \int_D m \cdot B d^3x, \]
for which we have $\frac{\delta C}{\delta \mathbf{m}} = \mathbf{B}$ and $\frac{\delta C}{\delta \mathbf{B}} = \mathbf{m}$, so that from (4.41) the modified momenta are

$$
\tilde{\mathbf{m}} = \mathbf{m} + \theta \left[ \frac{\delta \mathbf{h}}{\delta \mathbf{m}}, \frac{\delta C}{\delta \mathbf{m}} \right]^b = \mathbf{m} + \theta \left[ \mathbf{m}, \mathbf{B} \right]^b = \mathbf{m} + \theta \text{curl}(\mathbf{m} \times \mathbf{B}),
$$

$$
\tilde{\mathbf{B}} = \mathbf{B} + \theta (\text{curl} \mathbf{B} \times \mathbf{B} - \text{curl} \mathbf{m} \times \mathbf{m} - \nabla \phi).
$$

In this case, equation (4.42) becomes

$$
\frac{d}{dt} C(\mathbf{m}, \mathbf{B}) = -\theta \int_D |\text{curl} (\mathbf{m} \times \mathbf{B})|^2 d^3x - \theta \int_D |\text{curl} \mathbf{B} \times \mathbf{B} - \text{curl} \mathbf{m} \times \mathbf{m} - \nabla \phi|^2 d^3x. \tag{4.43}
$$

When the squared Casimir is considered, the solutions converge to an equilibrium state of the unmodified equations.

**Example 2: Magnetic helicity.** Another Casimir for incompressible 3D MHD is the magnetic helicity, given by

$$
C(\mathbf{B}) = \frac{1}{2} \int_D \mathbf{B} \cdot \text{curl}^{-1} \mathbf{B} \ d^3x,
$$

which is well-defined for $\text{div} \mathbf{B} = 0$ and $H^1(\mathcal{D}) = H^2(\mathcal{D}) = 0$. We have $\frac{\delta C}{\delta \mathbf{m}} = 0$ and $\frac{\delta C}{\delta \mathbf{B}} = \text{curl}^{-1} \mathbf{B}$. In this case, the modified momenta are

$$
\tilde{\mathbf{m}} = \mathbf{m} \quad \text{and} \quad \tilde{\mathbf{B}} = \mathbf{B} - \theta (\mathbf{E} + \nabla \phi), \quad \text{with} \quad \mathbf{E} := -\mathbf{u} \times \mathbf{B}.
$$

The dissipative LP equations (2.12) associated to the magnetic helicity read

$$
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mathbf{J} \times (\mathbf{B} - \theta (\mathbf{E} + \nabla \phi)), \quad \partial_t \mathbf{B} = -\text{curl}(\mathbf{E} + \theta \mathbf{u} \times (\mathbf{E} + \nabla \phi)),$$

where $\text{div} \mathbf{u} = 0$, $\text{div} \mathbf{B} = 0$, and we recall that $\mathbf{J} = \text{curl} \mathbf{B}$, $\mathbf{E} = -\mathbf{u} \times \mathbf{B}$, and $\nabla \phi$ is such that $\mathbf{E} + \nabla \phi$ is divergence free. The magnetic helicity dissipates as

$$
\frac{d}{dt} \frac{1}{2} \int_D \mathbf{B} \cdot \text{curl}^{-1} \mathbf{B} \ d^3x = -\theta \int_D |\mathbf{u} \times \mathbf{B} - \nabla \phi|^2 d^3x = -\theta \int_D |\mathbf{E} + \nabla \phi|^2 d^3x.
$$

**Remark 4.4.** Although the fluid momentum $\mathbf{m}$ is not modified, the modification of $\mathbf{B}$ produces changes in both the $\mathbf{u}$- and $\mathbf{B}$-equations, as explained in §2.1. Note also that applying (2.4) here would not yield any changes in the original equations, since $\frac{\delta C}{\delta \mathbf{m}} = 0$. However, selective decay by Casimir dissipation in this case does produce loss of magnetic helicity. This means the introduction of Casimir dissipation causes a loss of linkages in the $\mathbf{B}$ field lines, due to reconnection.

**Energy-dissipative case – magnetic Lamb surfaces.** From (2.23), we get the equations

$$
\partial_t \mathbf{m} + \text{curl} \mathbf{m} \times \frac{\delta \mathbf{h}}{\delta \mathbf{m}} + \mathbf{B} \times \text{curl} \frac{\delta \mathbf{h}}{\delta \mathbf{B}} = -\theta \text{curl} \left( \text{curl} \left( \frac{\delta \mathbf{h}}{\delta \mathbf{m}} \times \frac{\delta C}{\delta \mathbf{m}} \right)^b \times \frac{\delta C}{\delta \mathbf{m}} \right)
$$

$$
- \theta \left( \text{curl} \frac{\delta \mathbf{h}}{\delta \mathbf{B}} \times \frac{\delta C}{\delta \mathbf{m}} - \text{curl} \frac{\delta C}{\delta \mathbf{B}} \times \frac{\delta \mathbf{h}}{\delta \mathbf{m}} - \nabla \phi \right)^b \times \frac{\delta C}{\delta \mathbf{B}} - \nabla p,
$$

$$
\partial_t \mathbf{B} + \text{curl} \left( \mathbf{B} \times \frac{\delta \mathbf{h}}{\delta \mathbf{m}} \right) + \theta \text{curl} \left( \left( \text{curl} \frac{\delta \mathbf{h}}{\delta \mathbf{B}} \times \frac{\delta C}{\delta \mathbf{m}} - \text{curl} \frac{\delta C}{\delta \mathbf{B}} \times \frac{\delta \mathbf{h}}{\delta \mathbf{m}} - \nabla \phi \right)^b \times \frac{\delta C}{\delta \mathbf{m}} \right) = 0,
$$
where $\nabla \phi$ is such that the term inside the parenthesis is divergence free. For the MHD Hamiltonian, when the cross helicity is chosen (as in Example 1 above), relative to the usual inner product, these equations read

$$
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{B} \times \mathbf{J} = \theta (\text{curl} \, \text{curl} \, \mathbf{E}) \times \mathbf{B} - \theta (\text{curl} \, \mathbf{B} \times \mathbf{B} - \text{curl} \, \mathbf{u} \times \mathbf{u} - \nabla \phi) \times \text{curl} \, \mathbf{u} - \nabla p,
$$

and

$$
\partial_t \mathbf{B} + \text{curl} (\mathbf{B} \times \mathbf{u}) + \theta \, \text{curl} \left( (\text{curl} \, \mathbf{B} \times \mathbf{B} - \text{curl} \, \mathbf{u} \times \mathbf{u} - \nabla \phi) \times \mathbf{B} \right) = 0.
$$

The energy dissipates as

$$
\frac{d}{dt} \int_D \left( \frac{1}{2} |\mathbf{u}|^2 + \frac{1}{2} |\mathbf{B}|^2 \right) \, d^3 x = -\theta \int_D |\mathbf{u} \times \mathbf{B}|^2 \, d^3 x - \theta \int_D |\mathbf{B} \times \mathbf{B} - \text{curl} \, \mathbf{u} \times \mathbf{u} - \nabla \phi|^2 \, d^3 x.
$$

Thus, the solution once again converges to a steady state of the unmodified 3D incompressible MHD equations.

When the magnetic helicity is used (Example 2 above) for the energy-dissipative case, the modified equations become

$$
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{B} \times \mathbf{J} = \theta (\mathbf{B} \times \mathbf{u} + \nabla \phi) \times \mathbf{B} - \nabla p,
$$

and

$$
\partial_t \mathbf{B} + \text{curl} (\mathbf{B} \times \mathbf{u}) = 0.
$$

We note that the advection equation for $\mathbf{B}$ again remains unmodified. The energy dissipates as

$$
\frac{d}{dt} \int_D \left( \frac{1}{2} |\mathbf{u}|^2 + \frac{1}{2} |\mathbf{B}|^2 \right) \, d^3 x = -\theta \int_D |\mathbf{u} \times \mathbf{B} - \nabla \phi|^2 \, d^3 x = -\theta \int_D |\mathbf{E} + \nabla \phi|^2 \, d^3 x.
$$

This dynamics approaches an equilibria in which $\mathbf{E} = -\mathbf{u} \times \mathbf{B} = \nabla \phi$. In this equilibrium the velocity vector $\mathbf{u}$ and the magnetic field vector $\mathbf{B}$ are both tangent to each level set of the electrical potential $\phi$. These level sets may be called magnetic Lamb surfaces, in analogy to the well-known Lamb surfaces for incompressible Euler fluid equilibria. Thus, when the magnetic helicity Casimir is chosen, the energy-dissipative dynamics approaches a condition consistent with energy-Casimir equilibria, but it does not enforce all of the energy-Casimir equilibrium conditions.

**The effects of rotation.** Rotation can be easily included in the Lie–Poisson formulation, by considering the Hamiltonian

$$
h(\mathbf{m}, \mathbf{B}) = \int_D \left( \frac{1}{2} |\mathbf{m} - \mathbf{R}|^2 + \frac{1}{2} |\mathbf{B}|^2 \right) \, d^3 x,
$$

where $\text{curl} \, \mathbf{R} = 2\Omega$ is the Coriolis parameter (i.e., twice the angular rotation frequency). The equations for Casimir dissipation can be derived by using $\frac{\delta h}{\delta \mathbf{m}} = \mathbf{m} - \mathbf{R} = \mathbf{u}$ and $\frac{\delta h}{\delta \mathbf{B}} = \mathbf{B}$. For example, for the magnetic helicity, we get, exactly as before,

$$
\dot{\mathbf{m}} = \mathbf{m} \quad \text{and} \quad \dot{\mathbf{B}} = \mathbf{B} - \theta (\mathbf{E} + \nabla \phi).
$$

The modified equations now read

$$
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + 2\Omega \times \mathbf{u} = -\nabla p + \mathbf{J} \times (\mathbf{B} - \theta (\mathbf{E} + \nabla \phi))
$$

and

$$
\partial_t \mathbf{B} = -\text{curl} (\mathbf{E} + \theta \mathbf{u} \times (\mathbf{E} + \nabla \phi)).
$$
4.3.2 3D compressible isentropic MHD

Here we include the specific entropy $\eta$ in the equations. For the Lie–Poisson bracket

$$\{f, g\}_+(m, \rho, \eta, B) = \int_D m \cdot \left[ \frac{\delta f}{\delta m}, \frac{\delta g}{\delta m} \right] d^3x + \int_D \rho \left( \frac{\delta g}{\delta m} \cdot \nabla \frac{\delta f}{\delta \rho} - \frac{\delta f}{\delta m} \cdot \nabla \frac{\delta g}{\delta \rho} \right) d^3x$$

$$+ \int_D \eta \left( \nabla \left( \frac{\delta f}{\delta \rho} \frac{\delta g}{\delta \eta} \frac{\delta m}{\delta \rho} \right) - \nabla \left( \frac{\delta g}{\delta \rho} \frac{\delta f}{\delta \eta} \frac{\delta m}{\delta \rho} \right) \right) d^3x$$

$$+ \int_D \left( \text{curl} \left( B \times \frac{\delta f}{\delta m} \right) \cdot \frac{\delta g}{\delta B} - \text{curl} \left( B \times \frac{\delta g}{\delta m} \right) \cdot \frac{\delta f}{\delta B} \right) d^3x,$$

with $\text{div} B = 0$, the Lie–Poisson equations are

$$\partial_t \left( \frac{m}{\rho} \right) + \text{curl} \left( \frac{m}{\rho} \right) \times \frac{\delta h}{\delta m} + \nabla \left( \frac{m}{\rho} \cdot \frac{\delta h}{\delta m} \right) = -\nabla \frac{\delta h}{\delta \rho} + \frac{1}{\rho} \frac{\delta h}{\delta \eta} \nabla \eta + \frac{1}{\rho} \text{curl} \frac{\delta h}{\delta B} \times B,$$

$$\partial_t \rho + \text{div} \left( \rho \frac{\delta h}{\delta m} \right) = 0, \quad \partial_t \eta + \frac{\delta h}{\delta m} \cdot \nabla \eta = 0, \quad \partial_t B + \text{curl} \left( B \times \frac{\delta h}{\delta m} \right) = 0.$$

Consider the Hamiltonian

$$h(m, \rho, \eta, B) = \int_D \left( \frac{1}{2\rho}|m|^2 + \rho e(\rho, \eta) + \frac{1}{2}|B|^2 \right) d^3x,$$

where $e(\rho, \eta)$ is the specific internal energy. The variational derivatives of this Hamiltonian are given by

$$\frac{\delta h}{\delta m} = \frac{m}{\rho} = u, \quad \frac{\delta h}{\delta \rho} = -\frac{1}{2} |u|^2 + e + \rho \frac{\partial e}{\partial \rho}, \quad \frac{\delta h}{\delta \eta} = \rho \frac{\partial e}{\partial \eta}, \quad \frac{\delta h}{\delta B} = B.$$

Hence, the Lie–Poisson equations (4.45) recover (4.1) with an additional advection equation for the specific entropy variable $\eta$. Namely,

$$\rho(\partial_t u + u \cdot \nabla u) = -\nabla p + J \times B, \quad \partial_t B = -\text{curl} E,$$

$$\partial_t \rho + \text{div}(\rho u) = 0, \quad \partial_t \eta + u \cdot \nabla \eta = 0, \quad \text{div} B = 0.$$  

Here again $B$ denotes the magnetic field, $J := \text{curl} B$ is the electric current density, $E := -u \times B$ expresses the electric field in a frame moving with the fluid, and now the pressure $p(\rho, \eta)$ is determined from the equation of state $e(\rho, \eta)$ by the First Law, as $p(\rho, \eta) = \rho^2 \partial e/\partial \rho$.

**Example 1: Potential B-field.** Three-dimensional compressible isentropic MHD possesses a scalar material invariant called potential B-field, defined as

$$q_B := B/\rho \cdot \nabla \eta,$$

in analogy to the potential vorticity for isentropic compressible fluid flow. Material invariance of the scalar quantity $q_B$ may be derived by combining the three MHD advection laws for magnetic flux, $B \cdot \delta S$, specific entropy, $\eta$, and mass, $\rho \delta^3x$. Combining the first two of these three advected quantities into an advected volume-form yields a material-invariant 3-form in addition to the mass 3-form, so that

$$(\partial_t + F_u)(B \cdot \delta S \wedge d\eta) = 0 = (\partial_t + F_u)(\rho \delta^3x).$$
Hence, the scalar quantity $q_B$ is advected, according to

$$(\partial_t + \mathcal{L}_u)q_B = \partial_t q_B + u \cdot \nabla q_B = 0$$

That is, the quantity $q_B$ is a scalar material invariant (i.e., the scalar $q_B$ is conserved on particles). The material invariance of $q_B$ implies an additional class of Casimir functions for 3D compressible MHD, given by functionals of the potential $B$-field

$$C(m, \rho, \eta, B) = \int_D \rho \Phi(q_B) \, d^3x = \int_D \rho \Phi(B/\rho \cdot \nabla \eta) \, d^3x,$$

for an arbitrary smooth function, $\Phi$. This Casimir has the following variational derivatives

$$\frac{\delta C}{\delta m} = 0, \quad \frac{\delta C}{\delta \rho} = \Phi(q_B) - \Phi'(q_B)q_B,$$

$$\frac{\delta C}{\delta \eta} = -\text{div}(\Phi'(q_B)B), \quad \frac{\delta C}{\delta B} = \mathbb{P} (\Phi'(q_B)\nabla \eta),$$

where $\mathbb{P}$ is the Hodge projection onto divergence-free vector fields. Note that, necessarily, $\tilde{m} = m$ with these Casimirs. So we may again use (2.12) to introduce Casimir dissipation and (2.20) to introduce energy dissipation.

We note that since $\delta C/\delta \tilde{m} = 0$, the energy-dissipative approach keeps all the advection equations unchanged, see (2.23). Although the explicit equations of motion may have complicated expressions, from (2.26), we easily obtain the energy dissipation in this case as

$$\frac{d}{dt} h(m, \rho, \eta, B) = -\theta \int_D \left| u \cdot \nabla \frac{\delta C}{\delta \rho} \right|^2 + \left| \text{div}\left( u \frac{\delta C}{\delta \eta} \right) \right|^2 + \left| \mathbb{P} \left( \text{curl} \frac{\delta C}{\delta B} \times u \right) \right|^2 \, d^3x.$$

**Example 2: Magnetic helicity.** Another Casimir for compressible MHD (either with or without the specific entropy variable $\eta$) is given as in the case of incompressible MHD by the magnetic helicity

$$C(m, \rho, \eta, B) = \frac{1}{2} \int_D B \cdot \text{curl}^{-1} B \, d^3x. \quad (4.51)$$

1. In the **Casimir-dissipative case**, the variational derivatives of the magnetic helicity imply the modified variables $\tilde{m} = m$, $\tilde{\rho} = \rho$, $\tilde{\eta} = \eta$, and

$$\tilde{B} = B + \theta \mathbb{P} \left( \frac{\delta h}{\delta \tilde{m}} \times \text{curl} \frac{\delta C}{\delta B} - \frac{\delta C}{\delta B} \times \text{curl} \frac{\delta h}{\delta B} \right) = B - \theta \mathbb{PE},$$

when $\gamma$ is chosen as the standard dot product, as in (4.51).

As with the incompressible case, we find the Casimir dissipation equations

$$\rho(\partial_t u + u \cdot \nabla u) = -\nabla p + J \times (B - \theta \mathbb{PE}), \quad \partial_t B = -\text{curl}(E + \theta u \times \mathbb{PE}),$$

which results in the dissipation of magnetic helicity,

$$\frac{d}{dt} \frac{1}{2} \int_D B \cdot \text{curl}^{-1} B \, d^3x = -\theta \int_D |\mathbb{PE}|^2 \, d^3x. \quad (4.52)$$
2. In the energy-dissipative case, since $\delta C/\delta m = 0$, only the momentum equation is modified. It reads

$$\frac{\partial t}{\rho} \frac{m}{\rho} + \text{curl} \left( \frac{m}{\rho} \times \frac{\delta h}{\delta m} \right) + \nabla \left( \frac{m}{\rho} \cdot \frac{\delta h}{\delta m} \right) = -\nabla \frac{\delta h}{\delta \rho} + \frac{1}{\rho} \frac{\delta h}{\delta \eta} \nabla \eta + \frac{1}{\rho} \text{curl} \frac{\delta h}{\delta B} \times B + \frac{\theta}{\rho} (\mathbb{P}(B \times u)) \times B.$$ 

Correspondingly, the energy dissipates as

$$\frac{d}{dt} h(m, \rho, \eta, B) = -\theta \int_D |\mathbb{P}(B \times u)|^2 d^3x = -\theta \int_D |\mathbb{P}E|^2 d^3x. \quad (4.53)$$

Thus, the asymptotic solution verifies $\mathbb{P}E = E + \nabla \phi = 0$ with $\text{div}(\mathbb{P}E) = 0$.

**Remark 4.5** (Force-free compressible MHD equilibria). When the Casimir is chosen to be the magnetic helicity in (4.51), the associated energy-Casimir equilibrium conditions arising from $\delta (h + C) = 0$ read

$$\frac{\delta (h + C)}{\delta m} = 0 \quad \text{and} \quad \frac{\delta (h + C)}{\delta B} = 0.$$ 

These equilibrium conditions yield both $u = 0$ and $B + \text{curl}^{-1} B = 0$ with $\text{div} B = 0$. Thus, in this case, $\delta (h + C) = 0$ implies both

$$J \times B = 0 \quad \text{(upon using} \ J := \text{curl} B \ \text{)} \quad \text{and} \quad \mathbb{P}E = 0. \quad (4.54)$$

These are the “force-free” equilibria, introduced in Woltjer [1958, 1959, 1960] and discussed in the context of toroidal $z$ pinch operation in Taylor [1974, 1986]. See Greene and Karlson [1969] for a review of the earliest work in this field and Brown, Canfield and Pertsoy [1999] for discussions of later work.

In contrast to the two energy-Casimir conditions in (4.54), the energy-dissipative approach for constant magnetic helicity implies in (4.53) only the single condition $\mathbb{P}E = 0$ at equilibrium. Thus, as stated in Theorem 2.3, the energy-dissipative equilibrium conditions are consistent with the associated energy-Casimir equilibrium conditions. However, the latter may contain more conditions and thus be more restrictive.

For energy dissipation at constant magnetic helicity, the diagram with the various implications discussed in Remark 2.4 after Theorem 2.3 reads

$$\delta (h + C) = 0 \iff u = 0 \ & \ B + \text{curl}^{-1} B = 0 \implies (2.30) : \mathbb{P}E = 0,$$

steady state $\iff \text{ad}^\eta (\frac{\delta h}{\delta m}, \frac{\delta h}{\delta B})(m, B) = 0$.

That is, the implications need not always be equivalences.

**Case 3: Cross helicity.** When the specific entropy variable $\eta$ is absent, the cross-helicity is a Casimir, given by

$$C(m, \rho, B) = \int_D \frac{1}{\rho} m \cdot B \ d^3x, \quad \frac{\delta C}{\delta m} = \frac{1}{\rho} B, \quad \frac{\delta C}{\delta \rho} = -\frac{1}{\rho^2} m \cdot B, \quad \frac{\delta C}{\delta B} = \frac{1}{\rho} m.$$
In this case, the modified variables are found from (2.14) to be

\[ \tilde{m} = m + \theta \left[ \frac{\delta h}{\delta m}, \frac{\delta C}{\delta m} \right]^b = m + \theta \left[ u, \rho^{-1} B \right]^b, \]

\[ \rho = \rho + \theta \left( \frac{\delta C}{\delta \rho} \cdot \nabla \frac{\delta h}{\delta \rho} - \frac{\delta h}{\delta \rho} \cdot \nabla \frac{\delta C}{\delta \rho} \right)^b = \rho + \theta \left( \rho^{-1} B \cdot \nabla \left( e + \rho \frac{\partial e}{\partial \rho} - \frac{1}{2} |u|^2 \right) - u \cdot \nabla (\rho^{-1} B \cdot u) \right)^b, \]

\[ \tilde{B} = B + \theta \left( \frac{\delta h}{\delta m} \times \text{curl} \frac{\delta C}{\delta m} - \frac{\delta C}{\delta m} \times \text{curl} \frac{\delta h}{\delta m} - \nabla \phi \right)^b = B + \theta \left( u \times \text{curl} u - \rho^{-1} B \times \text{curl} B - \nabla \phi \right)^b. \]

For vectors in \( \mathbb{R}^3 \), the \( ^b \) operation (flat) is the identity, so we may drop it. Our formalism allows us to select which among these variables will be modified. For example, modifying \( m \) only, while keeping \( \rho, \eta, B \) unchanged, and using the positive bilinear form \( \gamma_{\rho}(u, v) = \int_D \rho \cdot (u \cdot v) \, d^3x \) produces the following modification of the \( u \)-equation

\[ \rho (\partial_t u + u \cdot \nabla u) = -\nabla p + J \times B - \theta \mathcal{L}_u X, \quad (4.55) \]

where \( X = \rho[u, \rho^{-1} B]^b \), as compared with (3.9) for Casimir dissipation in 3D compressible fluids.

1. In the **Casimir-dissipative case**, we find the decay rate for cross helicity to be

\[ \frac{d}{dt} \int_D u \cdot B \, d^3x = -\theta \int_D \frac{1}{\rho} |X|^2 \, d^3x. \quad (4.56) \]

In the energy-Casimir stability method for compressible MHD HMRW [1985], the velocity equilibrium condition is given by

\[ \frac{\delta (h + C)}{\delta m} = u + \rho^{-1} B = 0 \quad \text{at equilibrium.} \quad (4.57) \]

Under this condition, and consistently with Theorem 2.3, the commutator vanishes in the definition of the 1-form density \( X = \rho[u, \rho^{-1} B] \) when the flow velocity achieves its equilibrium form. That is, \( |X|^2 = 0 \) in (4.56) for energy-Casimir equilibria of ideal compressible fluid flow.

2. We now consider the **energy-dissipative case**. As before, we consider only the modification in the momentum equations, see (2.6). Using the same positive bilinear form \( \gamma_{\rho}(u, v) = \int_D \rho \cdot (u \cdot v) \, d^3x \) as before, we get, cf. equation (4.55),

\[ \rho (\partial_t u + u \cdot \nabla u) = -\nabla p + J \times B + \theta \mathcal{L}_u X, \quad (4.58) \]

where \( w = \delta C/\delta m = \rho^{-1} B \) and \( X = \rho[u, \rho^{-1} B]^b \). Finally, one finds the energy decay rate, cf. (4.56),

\[ \frac{d}{dt} h(m, \rho, \eta, B) = -\theta \int_D \frac{1}{\rho} |X|^2 \, d^3x. \quad (4.59) \]

As one expects from Theorem 2.3, the energy and Casimir decay rates for the cross-helicity case are again the same.
5 Selective decay for Hall MHD

5.1 Three-dimensional compressible Hall MHD

The Lie–Poisson bracket for Hall MHD in 3D is given in Holm [1987] by

\[
\{ f, g \}_{+}(\mathbf{m}, \rho; \eta; \mathbf{p}, \mathbf{n}) = \int_{\mathcal{D}} \mathbf{m} \cdot \left[ \frac{\delta f}{\delta \mathbf{m}} \frac{\delta g}{\delta \mathbf{m}} \right] d^{3}x + \int_{\mathcal{D}} \rho \left( \frac{\delta g}{\delta \mathbf{m}} \cdot \nabla \frac{\delta f}{\delta \rho} - \frac{\delta f}{\delta \mathbf{m}} \cdot \nabla \frac{\delta g}{\delta \rho} \right) d^{3}x \\
+ \int_{\mathcal{D}} \eta \left( \text{div} \left( \frac{\delta f}{\delta \eta} \frac{\delta g}{\delta \mathbf{m}} \right) - \text{div} \left( \frac{\delta g}{\delta \eta} \frac{\delta f}{\delta \mathbf{m}} \right) \right) d^{3}x \\
+ \int_{\mathcal{D}} \mathbf{p} \cdot \left[ \frac{\delta f}{\delta \mathbf{p}} \frac{\delta g}{\delta \mathbf{p}} \right] d^{3}x + \int_{\mathcal{D}} \mathbf{n} \left( \frac{\delta g}{\delta \mathbf{n}} \cdot \nabla \frac{\delta f}{\delta \mathbf{p}} - \frac{\delta f}{\delta \mathbf{n}} \cdot \nabla \frac{\delta g}{\delta \mathbf{p}} \right) d^{3}x,
\]

(5.1)

with momentum density \( \mathbf{p} := \mathbf{n} \mathbf{A} / \mathbf{R} \), where physically \( \mathbf{n} \) is the electron number density, \( \mathbf{A} \) is the magnetic vector potential and \( \mathbf{R} \) is the Hall scaling parameter. The Lie–Poisson bracket is the sum of the Lie–Poisson bracket on the dual Lie algebra \((\mathbf{X}(\mathcal{D}) \otimes (\mathbf{F}(\mathcal{D}) \times \mathbf{F}(\mathcal{D})))^{*} \ni (\mathbf{m}, \rho, \eta)\) and the Lie–Poisson bracket on the dual Lie algebra \((\mathbf{X}(\mathcal{D}) \otimes \mathbf{F}(\mathcal{D}))^{*} \ni (\mathbf{p}, \mathbf{n})\). The last two expressions may also be written directly in terms of \( \mathbf{n} \) and \( \mathbf{A} \) as Holm [1987]; Kawazura and Hameiri [2012]

\[
R \int_{\mathcal{D}} \frac{1}{\mathbf{n}} \left( \left( \frac{\delta f}{\delta \mathbf{A}} \cdot \nabla \mathbf{A} \right) \cdot \frac{\delta g}{\delta \mathbf{A}} - \left( \frac{\delta g}{\delta \mathbf{A}} \cdot \nabla \mathbf{A} \right) \cdot \frac{\delta f}{\delta \mathbf{A}} \right) d^{3}x + R \int_{\mathcal{D}} \left( \frac{\delta g}{\delta \mathbf{n}} \text{div} \left( \frac{\delta f}{\delta \mathbf{A}} - \frac{\delta f}{\delta \mathbf{p}} \text{div} \frac{\delta g}{\delta \mathbf{A}} \right) \right) d^{3}x.
\]

(5.2)

We refer to [Gay-Balmaz and Ratiu, 2009, §8.3] for the associated Poisson reduction approach. The associated Lie–Poisson equations are

\[
\begin{align*}
\partial_{t} \left( \frac{\mathbf{m}}{\rho} \right) + \text{curl} \left( \frac{\mathbf{m}}{\rho} \right) \times \frac{\delta h}{\delta \mathbf{m}} + \nabla \left( \frac{\mathbf{m}}{\rho} \cdot \frac{\delta h}{\delta \mathbf{m}} \right) &= -\nabla \frac{\delta h}{\delta \rho} + \frac{1}{\rho} \frac{\delta h}{\delta \eta} \nabla \eta, \\
\partial_{t} \rho + \text{div} \left( \rho \frac{\delta h}{\delta \mathbf{m}} \right) &= 0, \\
\partial_{t} \eta + \frac{\delta h}{\delta \mathbf{m}} \cdot \nabla \eta &= 0, \\
\partial_{t} \left( \frac{\mathbf{p}}{\mathbf{n}} \right) + \text{curl} \left( \frac{\mathbf{p}}{\mathbf{n}} \right) \times \frac{\delta h}{\delta \mathbf{p}} + \nabla \left( \frac{\mathbf{p}}{\mathbf{n}} \cdot \frac{\delta h}{\delta \mathbf{p}} \right) &= -\nabla \frac{\delta h}{\delta \mathbf{n}}, \\
\partial_{t} \mathbf{n} + \text{div} \left( \mathbf{n} \frac{\delta h}{\delta \mathbf{p}} \right) &= 0.
\end{align*}
\]

(5.3)

The Hamiltonian for Hall MHD is

\[
\begin{align*}
\mathcal{H}(\mathbf{m}, \rho, \eta; \mathbf{p}, \mathbf{n}) &= \int_{\mathcal{D}} \frac{1}{2\rho} \left| \mathbf{m} - \frac{a\rho}{\mathbf{n}} \mathbf{A} \right|^{2} d^{3}x + \int_{\mathcal{D}} \rho e(\rho, \eta) d^{3}x + \frac{R^{2}}{2} \int_{\mathcal{D}} \left| \text{curl} \frac{\mathbf{p}}{\mathbf{n}} \right|^{2} d^{3}x \\
&:= \int_{\mathcal{D}} \frac{1}{2\rho} \left| \mathbf{m} - \frac{a\rho}{\mathbf{R}} \mathbf{A} \right|^{2} d^{3}x + \int_{\mathcal{D}} \rho e(\rho, \eta) d^{3}x + \frac{1}{2} \int_{\mathcal{D}} |\mathbf{B}|^{2} d^{3}x,
\end{align*}
\]

(5.4)

where we have used the relations

\[
\mathbf{A} := \frac{\mathbf{p}}{\mathbf{n}} \quad \text{and} \quad \mathbf{B} := \text{curl} \mathbf{A},
\]

(5.5)

for, respectively, the magnetic vector potential and the magnetic field, the constants \( a, \mathbf{R} \) are respectively the ion charge-to-mass ratio and the Hall scaling parameter. The functional derivatives
of the Hall MHD Hamiltonian (5.21) are computed to be
\[
\mathbf{u} := \frac{\delta h}{\delta \mathbf{m}} = \frac{1}{\rho} \mathbf{m} - \frac{a}{R} \mathbf{A}, \quad \frac{\delta h}{\delta \rho} = -\frac{1}{2} |\mathbf{u}|^2 - \frac{a}{R} \mathbf{A} \cdot \mathbf{u} + e + \rho \frac{\partial e}{\partial \rho}, \quad \frac{\delta h}{\delta \eta} = \rho \frac{\partial e}{\partial \eta} =: \rho T,
\]
and
\[
\mathbf{v} := \frac{\delta h}{\delta \mathbf{p}} = -\frac{a \rho}{\tilde{n}} \mathbf{u} + \frac{R}{\tilde{n}} \text{curl} \mathbf{B}, \quad \frac{\delta h}{\delta \tilde{n}} = \frac{a \rho}{\tilde{n}^2} \mathbf{p} \cdot \mathbf{u} - \frac{R}{\tilde{n}^2} \mathbf{p} \cdot \text{curl} \mathbf{B} = -\frac{1}{R} \mathbf{A} \cdot \mathbf{v}.
\]
These equations involve two fluid velocities: \(\mathbf{u}\) (ion fluid velocity) and \(\mathbf{v}\) (electron fluid velocity).

From the third equation of (5.3) we deduce the Ohm’s Law
\[
\partial_t \mathbf{A} + \text{curl} \mathbf{A} \times \mathbf{v} = 0 .
\]
(5.6)
Taking its curl yields
\[
\partial_t \mathbf{B} + \text{curl} (\mathbf{B} \times \mathbf{v}) = 0
\]
which shows that the magnetic field lines are tied to the flow of the electron fluid. Substituting this Ohm’s Law into the first equation in (5.3) gives
\[
\partial_t \mathbf{u} + \text{curl} \mathbf{u} \times \mathbf{u} + \frac{1}{2} \nabla |\mathbf{u}|^2 = -\frac{a}{R} \mathbf{B} \times (\mathbf{u} - \mathbf{v}) - \frac{1}{\rho} \nabla p,
\]
where \(p = \rho^2 \partial e / \partial \rho\). The \(\rho\) and \(\tilde{n}\) equations read
\[
\partial_t \rho + \text{div}(\rho \mathbf{u}) = 0 \quad \text{and} \quad \partial_t \tilde{n} - \text{div}(a \rho \mathbf{u}) = 0 .
\]
Together they imply
\[
\partial_t (\tilde{n} + a \rho) = 0 .
\]
This means that these equations preserve local charge neutrality at every point, if it holds initially. Subject to the spatially preserved charge neutrality assumption \(\tilde{n} + a \rho = 0\), the equations for \(\mathbf{u}\) and \(\mathbf{A}\) are found to simplify to
\[
\partial_t \mathbf{u} = -\mathbf{u} \cdot \nabla \mathbf{u} - \frac{1}{\rho} \nabla p - \frac{1}{\rho} \mathbf{B} \times \text{curl} \mathbf{B} ,
\]
\[
\partial_t \mathbf{A} = \mathbf{u} \times \mathbf{B} + \frac{R}{a \rho} \mathbf{B} \times \text{curl} \mathbf{B} .
\]
(5.7)
In terms of these variables, the Kelvin-Noether theorems for compressible 3D Hall MHD is given by
\[
\frac{d}{dt} \oint_{c(\mathbf{u})} (\mathbf{u} + \frac{a}{R} \mathbf{A}) \cdot \mathbf{d}x = \frac{d}{dt} \oint_{c(\mathbf{u})} \rho^{-1} \mathbf{m} \cdot \mathbf{d}x = \oint_{c(\mathbf{u})} T \mathbf{d}\eta .
\]
(5.8)
and from equation (5.6),
\[
\frac{d}{dt} \oint_{c(\mathbf{v})} \mathbf{A} \cdot \mathbf{d}x = 0 .
\]
(5.9)
As we shall see below, this implies that magnetic helicity in the form
\[
\frac{d}{dt} \int \mathbf{A} \cdot \text{curl} \mathbf{A} \, d^3 x = 0 ,
\]
is also preserved in 3D compressible Hall MHD.
Remark 5.1. The Lie-Poisson bracket in (5.1) written as a direct sum of two fluid Lie-Poisson
brackets may be rewritten equivalently in terms of the familiar fluid momentum $\rho \mathbf{u}$ and magnetic
field contribution $\frac{\tilde{n}}{\tilde{R}} \mathbf{A}$, by changing momentum variables as

$$
\mathbf{m} \to \mathbf{n} := \mathbf{m} + p = \rho \mathbf{u} \quad \text{and} \quad p \to \mathbf{p},
$$

where $\rho \mathbf{u} = m - \frac{\partial \mathbf{A}}{\partial \tilde{R}} = \mathbf{m} + \frac{\tilde{n}}{\tilde{R}} \mathbf{A} = \mathbf{m} + \mathbf{p}$. This linear transformation yields the following
equivalent entangled form of the Lie-Poisson bracket (5.1)

$$
\{f, g\} + (\mathbf{n}, \rho, \eta; \mathbf{p}, \tilde{n}) = \int_D \mathbf{n} \cdot \left[ \frac{\delta f}{\delta \mathbf{n}} \frac{\delta g}{\delta \mathbf{n}} \right] d^3x + \int_D \rho \left( \frac{\delta g}{\delta \rho} \frac{\delta f}{\delta \mathbf{n}} \cdot \nabla \frac{\delta f}{\delta \rho} - \frac{\delta f}{\delta \mathbf{n}} \cdot \nabla \frac{\delta g}{\delta \rho} \right) d^3x \\
+ \int_D \eta \left( \text{div} \left( \frac{\delta f}{\delta \eta} \frac{\delta g}{\delta \mathbf{n}} \right) - \text{div} \left( \frac{\delta g}{\delta \eta} \frac{\delta f}{\delta \mathbf{n}} \right) \right) d^3x \\
+ \int_D \mathbf{p} \cdot \left[ \frac{\delta f}{\delta \mathbf{p}} \frac{\delta g}{\delta \mathbf{p}} \right] d^3x + \int_D \mathbf{p} \cdot \left( \left[ \frac{\delta f}{\delta \mathbf{n}} \frac{\delta g}{\delta \mathbf{p}} \right] + \left[ \frac{\delta f}{\delta \mathbf{p}} \frac{\delta g}{\delta \mathbf{n}} \right] \right) d^3x \\
+ \int_D \tilde{n} \left( \left( \frac{\delta g}{\delta \mathbf{p}} + \frac{\delta g}{\delta \mathbf{n}} \right) \cdot \nabla \frac{\delta f}{\delta \mathbf{n}} - \left( \frac{\delta f}{\delta \mathbf{p}} + \frac{\delta f}{\delta \mathbf{n}} \right) \cdot \nabla \frac{\delta g}{\delta \mathbf{n}} \right) d^3x.
$$

Case 1. A class of Casimir functions for the first Lie–Poisson bracket in (5.1) (i.e. the Lie–Poisson
bracket given by the first three terms, with variables $(\mathbf{m}, \rho, \eta)$) is given by

$$
C(\mathbf{m}, \rho, \eta) = \int_D \rho \Phi(\eta, q) d^3x, \quad \text{where} \quad q := \rho^{-1} \text{curl} (\mathbf{m}/\rho) \cdot \nabla \eta.
$$

Since (5.1) is a direct sum of two Poisson brackets, $C$ is a Casimir for the Poisson bracket (5.1).

Remark 5.2. In order to preserve charge neutrality $\tilde{n} + \rho a = 0$, we shall always use the Casimir
dissipation expression (2.4) for Hall MHD, instead of the more general form in (2.12).

As in §3, if we choose $C(\mathbf{m}, \rho, \eta) = \frac{1}{2} \int_D \rho q^2 d^3x$, and the bilinear form $\gamma_{\rho,\eta}(\mathbf{u}, \mathbf{v}) = \int_D \rho (\mathbf{u} \cdot \mathbf{v}) d^3x$, then the modified momentum is

$$
\tilde{\mathbf{m}} = \mathbf{m} + \theta \mathbf{X}, \quad \mathbf{X} := \rho \left[ \mathbf{u}, \rho^{-1} \text{curl}(q \nabla \eta) \right]^{\flat}
$$

and the $\mathbf{u}$-equation is modified as

$$
\rho (\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\mathbf{B} \times \text{curl} \mathbf{B} - \nabla p - \theta \mathcal{L}_u \mathbf{X},
$$

whereas the other equations are unchanged. We have

$$
\frac{d}{dt} \frac{1}{2} \int_D \rho q^2 d^3x = - \theta \int_D \frac{1}{2} |\mathbf{X}|^2 d^3x.
$$

As for ideal compressible fluid flow, the quantity $|\mathbf{X}|^2$ also vanishes for compressible Hall MHD
energy-Casimir equilibria.
Using (2.10) applied to the \((m, \rho, \eta)\)-Lie–Poisson bracket, we get the Kelvin-Noether circulation result
\[
\frac{d}{dt} \oint_{c(u)} (u + \frac{a}{R} A) \cdot dx = \frac{d}{dt} \oint_{c(u)} \frac{1}{\rho} m \cdot dx = \oint_{c(u)} \rho^{-1} \left( -\frac{\delta h}{\delta \rho} \circ \rho - \frac{\delta h}{\delta \eta} \circ \eta - \theta \mathcal{L}_u X \right) = \oint_{c(u)} \left( Td\eta - \theta \rho^{-1} \mathcal{L}_u X \right).
\]
(5.13)

When applied to the \((p, \tilde{n})\) Lie–Poisson bracket, we get
\[
\frac{d}{dt} \oint_{c(v)} A \cdot dx = R \frac{d}{dt} \oint_{c(v)} \frac{1}{\tilde{n}} p \cdot dx = R \oint_{c(v)} \frac{1}{\tilde{n}} \left( -\frac{\delta h}{\delta \tilde{n}} \circ \tilde{n} \right) = R \oint_{c(v)} \nabla \frac{\delta h}{\delta \tilde{n}} \cdot dx = 0.
\]
(5.14)

Note that only the circulation theorem (5.8) of ordinary Hall MHD is affected. The circulation theorem (5.9) remains unchanged. This follows from the fact that the Casimir \(C\) for 3D compressible Hall MHD does not depend on \((p, \tilde{n})\).

**Energy dissipation.** Exchanging the role of \(h\) and \(C\), we get the following modification of the \(u\)-equation
\[
\rho (\partial_t u + u \cdot \nabla u) = -B \times \text{curl} B - \nabla p + \theta \mathcal{L}_w X,
\]
where \(X\) has been defined above and \(w := \delta C/\delta m = \rho^{-1} \text{curl}( (\partial \Phi/\partial q) \nabla \eta)\). The other equations are unchanged.

**Case 2.** Another Casimir may be deduced from the Ohm’s Law (5.7) and its curl. This is the magnetic helicity,
\[
C(p, \tilde{n}) = \frac{1}{2} \oint A \cdot \text{curl} A \ d^3 x = \frac{R^2}{2} \int_D (p/\tilde{n}) \cdot \text{curl}(p/\tilde{n}) \ d^3 x.
\]

This is also easily seen by found from (5.2). Using \(\frac{\delta C}{\delta p} = R \tilde{n} \text{curl} A\) and \(\frac{\delta C}{\delta \tilde{n}} = -\frac{1}{\tilde{n}} A \cdot \text{curl} A\) and choosing the bilinear form \(\gamma(u, v) = -\int_D \tilde{n} (u \cdot v) \ d^3 x\) (the minus sign is chosen since the variable \(\tilde{n}\) turns out to be negative because of the neutrality assumption), we get
\[
\tilde{p} = p + \theta \left[ \frac{\delta h}{\delta p}, \frac{\delta C}{\delta p} \right] = p - \theta \tilde{n} R [v, \tilde{n}^{-1} B],
\]
where \(v = \delta h/\delta p\) is the electron fluid velocity. This yields a modification of the \(p\)-equation, which implies
\[
\partial_t A + \text{curl} A \times v = -\frac{R}{\tilde{n}} \theta \mathcal{L}_v Y, \quad \text{where} \quad Y := -\tilde{n} R [v, \tilde{n}^{-1} B].
\]
(5.15)

When used in the \(u\)-equation, together with the charge neutrality assumption \(\tilde{n} + a \rho = 0\), this yields
\[
\rho (\partial_t u + u \cdot \nabla u) = -\nabla p - B \times \text{curl} B - \theta \mathcal{L}_v Y,
\]
where
\[
v = u - \frac{R}{a \rho} \text{curl} B \quad \text{and} \quad Y = -\tilde{n} R [v, \tilde{n}^{-1} B] = -\rho R [u - (a \rho)^{-1} \text{curl} B, \rho^{-1} B].
\]
Substituting the neutrality assumption into the \( p \)-equation yields

\[
\partial_t A = u \times B + \frac{R}{a \rho} B \times \text{curl} B + \frac{R \theta}{a \rho} \mathcal{L}_v Y.
\]

Consequently, one finds the Casimir dissipation formula

\[
\frac{d}{dt} \frac{1}{2} \int_D A \cdot \text{curl} A \, d^3x = -\theta \int_D \frac{1}{a \rho} |Y|^2 \, d^3x.
\]

By the evolution equations for \( \tilde{n} \) and \( A \) one finds that the sum \( (v + \tilde{n}^{-1} B) \) vanishes for Hall MHD equilibria. Therefore, \(|Y|^2 = 0\) for energy-Casimir equilibria of compressible 3D Hall MHD flow.

Using (2.10) applied to the \((m, \rho, \eta)\)-Lie–Poisson bracket, we get the Kelvin-Noether circulation result

\[
\frac{d}{dt} \oint_{c(u)} (u + \frac{a}{R} A) \cdot d\mathbf{x} = R \frac{d}{dt} \oint_{c(v)} \frac{1}{\tilde{n}} m \cdot d\mathbf{x} = \oint_{c(u)} \left( -\frac{\delta h}{\delta \rho} \otimes \rho - \frac{\delta h}{\delta \eta} \otimes \eta \right) \, d\mathbf{x} = \oint_{c(u)} T d\eta. \quad (5.16)
\]

When applied to the \((p, \tilde{n})\)-Lie–Poisson bracket, we get

\[
\frac{d}{dt} \oint_{c(v)} A \cdot d\mathbf{x} = R \frac{d}{dt} \oint_{c(v)} \frac{1}{n} p \cdot d\mathbf{x} = R \oint_{c(v)} \frac{1}{\tilde{n}} \left( -\frac{\delta h}{\delta n} \otimes \tilde{n} - \theta \mathcal{L}_v Y \right) \, d\mathbf{x} = \frac{R \theta}{a} \oint_{c(v)} \rho^{-1} \mathcal{L}_v Y.
\]

This time, only the circulation theorem (5.9) is affected, whereas (5.8) remains unchanged. This follows from the fact that the Casimir \( C \) does not depend on \((m, \rho, \eta)\).

### 5.2 Compressible Hall MHD with electron entropy

Following Kawazura and Hameiri [2012], we extend the electron part of the Lie–Poisson bracket (5.1) to account for electron entropy \( \tilde{\eta} \), and write the \((m, \rho, \eta; p, \tilde{n}, \tilde{\eta})\) Lie–Poisson bracket as

\[
\{f, g\}_+(m, \rho, \eta; p, \tilde{n}, \tilde{\eta}) = \int_D m \cdot \left[ \frac{\delta f}{\delta m} - \frac{\delta g}{\delta m} \right] \, d^3x + \int_D \rho \left( \frac{\delta g}{\delta \rho} \cdot \nabla \frac{\delta f}{\delta \rho} - \frac{\delta f}{\delta \rho} \cdot \nabla \frac{\delta g}{\delta \rho} \right) \, d^3x
\]

\[
+ \int_D \eta \left( \text{div} \left( \frac{\delta f}{\delta \eta} \frac{\delta g}{\delta \eta} m \right) - \text{div} \left( \frac{\delta g}{\delta \eta} \frac{\delta f}{\delta \eta} m \right) \right) \, d^3x
\]

\[
+ \int_D \tilde{n} \left( \frac{\delta f}{\delta \tilde{n}} \frac{\delta g}{\delta \tilde{n}} \right) \, d^3x + \int_D \tilde{n} \left( \frac{\delta g}{\delta \tilde{n}} \cdot \nabla \frac{\delta f}{\delta \tilde{n}} - \frac{\delta f}{\delta \tilde{n}} \cdot \nabla \frac{\delta g}{\delta \tilde{n}} \right) \, d^3x
\]

\[
+ \int_D \tilde{\eta} \left( \text{div} \left( \frac{\delta f}{\delta \tilde{\eta}} \frac{\delta g}{\delta \tilde{\eta}} \right) - \text{div} \left( \frac{\delta g}{\delta \tilde{\eta}} \frac{\delta f}{\delta \tilde{\eta}} \right) \right) \, d^3x.
\]

The Casimirs for this Lie–Poisson bracket are

\[
C_\Phi = \int \rho \Phi(q, \eta) \, d^3x \quad \text{and} \quad C_{\tilde{\Phi}} = \int \tilde{n} \tilde{\Phi}(\tilde{q}, \tilde{\eta}) \, d^3x,
\]

where the quantities

\[
q = \rho^{-1} \nabla \eta \cdot \text{curl}(m/\rho) \quad \text{and} \quad \tilde{q} = \tilde{n}^{-1} \nabla \tilde{\eta} \cdot \text{curl}(p/\tilde{n})
\]

(5.19)
are advected respectively along the flows of the vector fields \( \mathbf{u} := \delta h/\delta \mathbf{m} \) and \( \mathbf{v} := \delta h/\delta \mathbf{p} \). The associated Lie–Poisson equations are

\[
\partial_t \left( \frac{\mathbf{m}}{\rho} \right) + \text{curl} \left( \frac{\mathbf{m}}{\rho} \right) \times \frac{\delta h}{\delta \mathbf{m}} + \nabla \left( \frac{\mathbf{m}}{\rho} \cdot \frac{\delta h}{\delta \mathbf{m}} \right) = -\nabla \frac{\delta h}{\delta \rho} + \frac{1}{\rho} \frac{\delta h}{\delta \eta} \nabla \eta, \\
\partial_t \rho + \text{div} \left( \rho \frac{\delta h}{\delta \mathbf{m}} \right) = 0, \\
\partial_t \left( \frac{\mathbf{p}}{\tilde{\eta}} \right) + \text{curl} \left( \frac{\mathbf{p}}{\tilde{\eta}} \right) \times \frac{\delta h}{\delta \mathbf{p}} + \nabla \left( \frac{\mathbf{p}}{\tilde{\eta}} \cdot \frac{\delta h}{\delta \mathbf{p}} \right) = -\nabla \frac{\delta h}{\delta \tilde{\eta}} + \frac{1}{\tilde{\eta}} \frac{\delta h}{\delta \tilde{\eta}} \nabla \tilde{\eta}, \\
\partial_t \tilde{\eta} + \text{div} \left( \tilde{\eta} \frac{\delta h}{\delta \mathbf{p}} \right) = 0,
\]

The functional derivatives of the Hamiltonian (5.21) may then be computed as

\[
\delta h / \delta \rho = -1/2 |\mathbf{u}|^2 - a R \mathbf{A} \cdot \mathbf{u} + k_{tot}, \\
\delta h / \delta \tilde{\eta} = a \rho \tilde{\eta}^2 \mathbf{u} - R \tilde{\eta}^2 \mathbf{p} \cdot \text{curl} \mathbf{B} = -1/R \mathbf{A} \cdot \mathbf{v}
\]

The Hamiltonian for Hall MHD with electron entropy \( \tilde{\eta} \) is, cf. equation (5.21),

\[
h(\mathbf{m}, \rho, \eta; \mathbf{p}, \tilde{\eta}, \tilde{\eta}) := \frac{1}{2} \int_{\mathcal{D}} \frac{1}{\rho} |\mathbf{m} - \frac{a \rho}{R} \mathbf{A}|^2 d^3x + \int_{\mathcal{D}} \rho (e(\rho, \eta) + \tilde{e}(\rho, \tilde{\eta})) d^3x + \frac{1}{2} \int_{\mathcal{D}} |\mathbf{B}|^2 d^3x,
\]

where the quantities

\[
\mathbf{A} := R \frac{\mathbf{p}}{\tilde{\eta}} \quad \text{and} \quad \mathbf{B} = \text{curl} \mathbf{A},
\]

are, respectively, the magnetic vector potential and the magnetic field, while the constants \( a, R \) are respectively the ion charge-to-mass ratio and the Hall scaling parameter.

To compute variational derivatives of the Hamiltonian, we will need to specify the equation of state and the thermodynamic First Law for this system. We consider an equation of state in which the total internal energy \( e_{tot} \) is the sum of the internal energies for ions \( e \) and electrons \( \tilde{e} \), each with its own thermodynamic dependence, so that

\[
e_{tot}(\rho, \eta, \tilde{\eta}) := e(\rho, \eta) + \tilde{e}(\rho, \tilde{\eta}).
\]

We then write the sum of their two First Laws as

\[
d(e(\rho, \eta) + \tilde{e}(\rho, \tilde{\eta})) = -(p + \tilde{p}) d\rho^{-1} + T d\eta + \tilde{T} d\tilde{\eta}.
\]

In terms of \( e_{tot} := e + \tilde{e} \) and \( p_{tot} := p + \tilde{p} \) this sum is written as

\[
d e_{tot}(\rho, \eta, \tilde{\eta}) := -p_{tot} d\rho^{-1} + T d\eta + \tilde{T} d\tilde{\eta}.
\]

The functional derivatives of the Hamiltonian (5.21) may then be computed as

\[
u := \frac{\delta h}{\delta \mathbf{p}} = \frac{a \rho}{\tilde{\eta}} \mathbf{u} + \frac{R}{\tilde{\eta}} \text{curl} \mathbf{B}, \\
\frac{\delta h}{\delta \rho} = \frac{a}{R} \mathbf{A} \cdot \mathbf{u} + k_{tot}, \\
\frac{\delta h}{\delta \tilde{\eta}} = a \rho \tilde{\eta}^2 \mathbf{u} - R \tilde{\eta}^2 \mathbf{p} \cdot \text{curl} \mathbf{B} = -1/R \mathbf{A} \cdot \mathbf{v}
\]

where

\[
k_{tot} := e_{tot} + \rho \frac{\partial e_{tot}}{\partial \rho} = e_{tot} + \frac{p_{tot}}{\rho}, \quad dk_{tot} = \rho^{-1} dp_{tot} + T d\eta + \tilde{T} d\tilde{\eta}.
\]
These equations involve two fluid velocities: \( \mathbf{u} \) (ion fluid velocity) and \( \mathbf{v} \) (electron fluid velocity).

The \( \rho \) and \( \tilde{n} \) equations read
\[
\partial_t \rho + \text{div}(\rho \mathbf{u}) = 0 \quad \text{and} \quad \partial_t \tilde{n} - \text{div}(a\rho \mathbf{u}) = 0. \tag{5.27}
\]
Together they imply
\[
\partial_t (\tilde{n} + a\rho) = 0. \tag{5.28}
\]
This means that these equations preserve local charge neutrality at every point, if it holds initially.

From the third equation of (5.20) and the charge neutrality assumption \( \tilde{n} + a\rho = 0 \), we deduce the \( \mathbf{A} \)-equation
\[
\partial_t \mathbf{A} + \text{curl} \, \mathbf{A} \times \mathbf{v} = 0 + \frac{R\rho}{\tilde{n}} \tilde{T} \nabla \tilde{n} = \left( -\frac{R}{a} \right) \mathbf{T} \nabla \tilde{n}. \tag{5.29}
\]
Taking its curl yields the \( \mathbf{B} \)-equation
\[
\partial_t \mathbf{B} + \text{curl}(\mathbf{B} \times \mathbf{v}) = 0 + \frac{R}{a} \text{curl} \left( \rho^{-1} \nabla \tilde{p} \right), \tag{5.30}
\]
where we have used the First Law for the electrons
\[
d \tilde{e} = -\tilde{p} \rho^{-1} + \tilde{T} d \tilde{n} = -d(\tilde{p} \rho^{-1}) + \rho^{-1} d\tilde{p} + \tilde{T} d \tilde{n},
\]
to simplify the right-hand side. Substituting the \( \mathbf{A} \)-equation (5.29) into the first equation of (5.3) gives the motion equation
\[
\partial_t \mathbf{u} + \text{curl} \, \mathbf{u} \times \mathbf{u} + \frac{1}{2} \nabla |\mathbf{u}|^2 = -\frac{a}{R} \mathbf{B} \times (\mathbf{u} - \mathbf{v}) - \frac{1}{\rho} \nabla p_{tot}, \tag{5.31}
\]
where \( p_{tot} = \rho^2 \partial e_{tot}/\partial \rho = p + \tilde{p} \). From the charge neutrality assumption \( \tilde{n} + a\rho = 0 \), we have \( \mathbf{u} - \mathbf{v} = \frac{R}{a \rho} \text{curl} \, \mathbf{B} \), the equations for \( \mathbf{u} \) and \( \mathbf{A} \) are found to simplify to
\[
\begin{align*}
\partial_t \mathbf{u} &= -\mathbf{u} \cdot \nabla \mathbf{u} - \frac{1}{\rho} \nabla p_{tot} - \frac{1}{\rho} \mathbf{B} \times \text{curl} \, \mathbf{B}, \\
\partial_t \mathbf{A} &= \mathbf{u} \times \mathbf{B} + \frac{R}{a \rho} \mathbf{B} \times \text{curl} \, \mathbf{B} - \frac{R}{a} \tilde{T} \nabla \tilde{n}. \tag{5.32}
\end{align*}
\]
Equations (5.32) imply the Kelvin-Noether theorem for compressible 3D Hall MHD, given by
\[
\frac{d}{dt} \int_{c(u)} \left( \mathbf{u} + \frac{a}{R} \mathbf{A} \right) \cdot d\mathbf{x} = \frac{d}{dt} \int_{c(u)} \rho^{-1} \mathbf{m} \cdot d\mathbf{x}
\]
\[
= -\int_{c(u)} \rho^{-1} dp_{tot} + \tilde{T} d \tilde{n} = \int_{c(u)} T d \eta, \tag{5.33}
\]
where we have used the First Law expression for \( dk_{tot} \) in the last step. The last step yields what we had already known from the semidirect-product form of the Lie-Poisson bracket (5.17). Namely, that the ion potential vorticity \( q \) in (5.19) is advected by the \( \mathbf{u} \)-flow. Since the Kelvin theorem (5.8) and advection of \( \eta \) by the \( \mathbf{u} \)-flow imply
\[
(\partial_t + \mathcal{L}_u) (\rho^{-1} \mathbf{m} \cdot d\mathbf{x} \wedge d\eta) = T \eta d \eta \wedge d\eta = 0, \tag{5.34}
\]
we find
\[
(\partial_t + \mathcal{L}_u) d (\rho^{-1} \mathbf{m} \cdot d\mathbf{x} \wedge d\eta) = (\partial_t + \mathcal{L}_u)(\nabla \eta \cdot \text{curl}(\rho^{-1} \mathbf{m}) d^3 x) = 0, \tag{5.35}
\]
thereby recovering advection of $q$ in (5.19) from the continuity equation for $\rho$ in (5.3). The advection of $q$ is a property of the Lie-Poisson bracket. It is independent of the choice of both Hamiltonian and equation of state. The same statement applies to the advection of the electron potential vorticity $\tilde{q}$ in (5.19) along the $v$-flow.

Applying the Kelvin-Noether for the $(p, \tilde{n}, \tilde{\eta})$- Lie–Poisson bracket, we get

$$\frac{d}{dt} \oint_{c(v)} A \cdot d\mathbf{x} = R \oint_{c(v)} \tilde{T} d\tilde{\eta}.$$  

**Remark 5.3.** One also notes that with the new entropy variable in the Poisson bracket (5.17), the magnetic helicity

$$C(p, \tilde{n}) = \frac{1}{2} \int A \cdot \text{curl} A \, d^3x$$

is no longer a Casimir, because the electrons are no longer tied to the magnetic field lines. The resulting slippage between them is the classical Hall effect. Vice versa, the magnetic field is no longer frozen into the $v$-flow of the electrons, as shown by the new Ohm’s law in (5.29).

**Selective Casimir decay.** We now briefly describe the Casimir-dissipative equations associated to the Casimir functions in (5.18).

1. If we choose the Casimir $C(m, \rho, \eta) = \int \rho \Phi(q, \eta) \, d^3x$ and the bilinear form $\gamma_{\rho, \eta}(u, v) = \int_D \rho (u \cdot v) \, d^3x$, then the modified momentum reads

$$\tilde{m} = m + \theta \mathbf{X}, \quad \mathbf{X} = \left[ \frac{\delta h}{\delta m}, \frac{\delta C}{\delta m} \right] \flat = \rho \left[ u, \rho^{-1} \text{curl} \nabla \eta \partial \Phi / \partial q \right].$$

We get the following modification of (5.32):

$$\begin{align*}
\partial_t u &= -u \cdot \nabla u - \frac{1}{\rho} \nabla p_{\text{tot}} - \frac{1}{\rho} \mathbf{B} \times \text{curl} \mathbf{B} - \frac{\theta}{\rho} \mathbf{L}_u \mathbf{X}, \\
\partial_t A &= u \times \mathbf{B} + \frac{R}{\alpha \rho} \mathbf{B} \times \text{curl} \mathbf{B} - \frac{R}{\alpha} \tilde{T} \nabla \tilde{\eta}.  
\end{align*}$$

Consequently, one finds the Casimir dissipation formula

$$\frac{d}{dt} \frac{1}{2} \int_D \int \rho \Phi(q, \eta) \, d^3x = -\theta \int_D \frac{1}{\rho} |\mathbf{X}|^2 \, d^3x.$$  

If the Casimir is positive, then the solution tends to a state with $\mathbf{X} = 0$.

The corresponding Kelvin-Noether circulation theorems (see (2.10)) are

$$\frac{d}{dt} \oint_{c(u)} \left( u + \frac{a}{R} A \right) \cdot d\mathbf{x} = \oint_{c(u)} (TD\eta - \theta \rho^{-1} \mathbf{L}_u \mathbf{X})$$

and

$$\frac{d}{dt} \oint_{c(v)} \mathbf{A} \cdot d\mathbf{x} = R \oint_{c(v)} \tilde{T} d\tilde{\eta}.$$
2. If we choose the Casimir \( C(p, \tilde{n}, \tilde{\eta}) = \int \tilde{n} \Phi(\tilde{q}, \tilde{\eta}) \, d^3x \) and the bilinear form \( \gamma_{\tilde{n}, \tilde{\eta}}(u, v) = -\int_D \tilde{n} (u \cdot v) \, d^3x \), then the modified momentum reads

\[
\tilde{p} = p + \theta Y, \quad Y = \left[ \frac{\delta h}{\delta p} + \frac{\delta C}{\delta p} \right] = -\tilde{n} \left[ v, \tilde{n}^{-1} \nabla \tilde{\eta} \partial \Phi / \partial \tilde{\eta} \right]. \tag{5.37}
\]

We get the following modifications of (5.32):

\[
\begin{align*}
\partial_t u &= -u \cdot \nabla u - \frac{1}{\rho} \nabla p_{tot} - \frac{1}{\rho} B \times \text{curl} B - \frac{\theta}{\rho} \mathcal{L}_v Y, \\
\partial_t A &= u \times B + \frac{R}{a \rho} B \times \text{curl} B - \frac{R}{a} \tilde{T} \nabla \tilde{\eta} + \frac{R \theta}{a \rho} \mathcal{L}_v Y. \tag{5.38}
\end{align*}
\]

The Casimir dissipation formula reads

\[
\frac{d}{dt} \frac{1}{2} \int \tilde{n} \Phi(\tilde{q}, \tilde{\eta}) \, d^3x = -\theta \int_D \frac{1}{\rho} |Y|^2 \, d^3x.
\]

If the Casimir is positive, then the solution tends to a state with \( Y = 0 \).

The corresponding Kelvin-Noether circulation theorems (see (2.10)) are

\[
\frac{d}{dt} \int_{c(u)} (u + \frac{a}{R} A) \cdot d\mathbf{x} = \int_{c(u)} T d\eta
\]

and

\[
\frac{d}{dt} \int_{c(v)} A \cdot d\mathbf{x} = R \int_{c(v)} \left( \tilde{T} d\tilde{\eta} + \frac{\theta}{a \rho} \mathcal{L}_v Y \right).
\]

**Remark 5.4.** Note that the quantities \( X \) and \( Y \) both vanish for energy-Casimir equilibria of compressible Hall MHD with electron entropy.

**Energy-dissipative case.**

1. If we choose the Casimir \( C(m, \rho, \eta) = \int \rho \Phi(q, \eta) \, d^3x \) and the bilinear form \( \gamma_{\rho, \eta}(u, v) = \int_D \rho (u \cdot v) \, d^3x \), then we obtain the following modification of (5.32):

\[
\begin{align*}
\partial_t u &= -u \cdot \nabla u - \frac{1}{\rho} \nabla p_{tot} - \frac{1}{\rho} B \times \text{curl} B + \frac{\theta}{\rho} \mathcal{L}_w X, \\
\partial_t A &= u \times B + \frac{R}{a \rho} B \times \text{curl} B - \frac{R}{a} \tilde{T} \nabla \tilde{\eta},
\end{align*}
\]

where \( w = \frac{\delta C}{\delta m} = \rho^{-1} \nabla \eta \partial \Phi / \partial q \) and \( X = \left[ \frac{\delta h}{\delta m} + \frac{\delta C}{\delta m} \right] = \rho |u, w| \). Energy dissipates as

\[
\frac{d}{dt} h(m, \rho, \eta; p, \tilde{n}, \tilde{\eta}) = -\theta \int_D \frac{1}{\rho} |X|^2 \, d^3x.
\]

If we choose the Casimir \( C(p, \tilde{n}, \tilde{\eta}) = \int \tilde{n} \Phi(\tilde{q}, \tilde{\eta}) \, d^3x \) and the bilinear form \( \gamma_{\tilde{n}, \tilde{\eta}}(u, v) = -\int_D \tilde{n} (u \cdot v) \, d^3x \), then we get the following modifications of (5.32):

\[
\begin{align*}
\partial_t u &= -u \cdot \nabla u - \frac{1}{\rho} \nabla p_{tot} - \frac{1}{\rho} B \times \text{curl} B + \frac{\theta}{\rho} \mathcal{L}_w Y, \\
\partial_t A &= u \times B + \frac{R}{a \rho} B \times \text{curl} B - \frac{R}{a} \tilde{T} \nabla \tilde{\eta} - \frac{R \theta}{a \rho} \mathcal{L}_w Y,
\end{align*}
\]
where \( w = \frac{\delta C}{\delta p} = \tilde{n}^{-1} \operatorname{curl}(\nabla \tilde{q} \partial \Phi / \partial q) \) and \( Y = \left[ \frac{\delta h}{\delta p} \frac{\delta C}{\delta p} \right]^b = -\tilde{n} [v, w]^b \). Energy dissipates as

\[
\frac{d}{dt} h(m, \rho, \eta; p, \tilde{n}, \tilde{q}) = -\theta \int_D \frac{1}{a \rho} |Y|^2 d^3 x.
\]

### 5.3 Homogeneous 3D incompressible Hall MHD

The Lie–Poisson equations are

\[
\begin{align*}
\partial_t m + \operatorname{curl} m \times \frac{\delta h}{\delta m} &= -\nabla p, \quad \text{div } m = 0, \quad (5.39) \\
\partial_t p + \operatorname{curl} p \times \frac{\delta h}{\delta p} &= -\nabla q, \quad \text{div } p = 0,
\end{align*}
\]

where the pressures \( p, q \) are determined by the incompressibility constraints \( \text{div } m = \text{div } p = 0 \). The Lie–Poisson bracket is the sum of two Lie–Poisson brackets on the dual Lie algebra \( (\mathfrak{g}_{\text{vol}}(D))^* \). The Hamiltonian for incompressible homogeneous Hall MHD is

\[
\begin{align*}
h(m; p) := \frac{1}{2} & \int_D \left| m - a \frac{A}{R} \right|^2 d^3 x + \frac{1}{2} \int_D |B|^2 d^3 x, \\
& (5.40)
\end{align*}
\]

where, as above \( B = \operatorname{curl} A \) and \( A := -R p/a \).

The functional derivatives are computed to be

\[
\begin{align*}
u &:= \frac{\delta h}{\delta m} = m - a \frac{A}{R}, \quad \quad v := \frac{\delta h}{\delta p} = u - \frac{a}{R} \operatorname{curl} B.
\end{align*}
\]

From the second equation of (5.39) we deduce \( \partial_t A + \operatorname{curl} A \times v = -\nabla q \). Substituting this into the first equation yields

\[
\partial_t u + u \cdot \nabla u = -\frac{a}{R} B \times (u - v) - \nabla p.
\]

Then, since \( u - v = a^{-1} R \operatorname{curl} B \), we find that equations (5.39) are equivalent to

\[
\begin{align*}
\partial_t u + u \cdot \nabla u &= -B \times \operatorname{curl} B - \nabla p, \quad \text{div } u = 0, \\
\partial_t A &= u \times B + \frac{R}{a} B \times \operatorname{curl} B - \nabla q, \quad \text{div } A = 0.
\end{align*}
\]

(5.41)

To facilitate the solution for the second pressure \( q \), we choose \( \text{div } A = 0 \). That is, we choose the standard Coulomb gauge for the magnetic vector potential.

**Case 1.** One Casimir function for the incompressible homogeneous Hall MHD equations (5.39) or (5.41) is

\[
C(m, p) = \frac{1}{2} \int_D (m \cdot \operatorname{curl} m) d^3 x.
\]

The modified momentum reads

\[
\tilde{m} = m + \theta \left[ \frac{\delta h}{\delta \mathbf{m}} \frac{\delta C}{\delta \mathbf{m}} \right]^b = m + \theta [u, \operatorname{curl} u + \frac{a}{R} B]^b = m + \theta \operatorname{curl}(u \times (\operatorname{curl} u + \frac{a}{R} B)).
\]
The $m$-equation is modified for Casimir dissipation into
\[ \partial_t m + \text{curl} \ m \times u = -\nabla p - \theta \text{curl} X \times u, \]
where $X = \text{curl}(u \times (\text{curl} \ u + \frac{a}{R} \mathbf{B}))$ and $m = u + \frac{a}{R} \mathbf{A}$. The $p$-equation is unchanged and yields
\[ \partial_t A + \text{curl} \ A \times v = -\nabla q. \]

When used in the $m$-equation, this equation yields
\[ \partial_t u + \text{curl} \ u \times u = -B \times \text{curl} \ B - \nabla p - \theta \text{curl} X \times u, \quad \partial_t A = u \times B + \frac{R}{a} \mathbf{B} \times \text{curl} B - \nabla q \quad (5.42) \]
and
\[ \frac{d}{dt} \frac{1}{2} \int_D m \cdot \text{curl} m \ d^3x = -\theta \int_D |X|^2 \ d^3x. \]

**Energy dissipation.** Exchanging the role of $h$ and $C$, we get the following modification of the $u$-equation
\[ \partial_t u + \text{curl} \ u \times u = -B \times \text{curl} \ B - \nabla p + \theta \text{curl} X \times w, \quad w = \frac{\delta h}{\delta m} = \text{curl} \ u + \frac{a}{R} \mathbf{B}. \]

**Case 2.** Another Casimir function for incompressible homogeneous Hall MHD is given by
\[ C(m, p) = \frac{1}{2} \int_D (p \cdot \text{curl} p) \ d^3x. \]

The modified momentum in this case reads
\[ \tilde{p} = p + \theta \left[ \frac{\delta h}{\delta p}, \frac{\delta C}{\delta p} \right]^b = p + \theta \left[ v, -\frac{a}{R} \mathbf{B} \right] = p - \theta \text{curl}(v \times \frac{a}{R} \mathbf{B}). \]

Thus, $\text{div} \ p = 0$ implies that $\text{div} \ \tilde{p} = 0$, as well. The $p$-equation is modified as
\[ \partial_t p + \text{curl} \ p \times v = -\nabla q - \theta \text{curl} Y \times v, \]
where $Y = -\theta \frac{a}{R} \text{curl}(v \times \mathbf{B})$ and $p = -\frac{a}{R} \mathbf{A}$. Using this equation together with $m = u + \frac{a}{R} \mathbf{A}$, we can write the $m$-equation as
\[ \partial_t u + \text{curl} \ u \times u + \frac{a}{R} (B \times (u - v)) + \frac{R \theta}{a} \text{curl} Y \times v = -\nabla p. \]

Finally, we obtain the equations
\[ \partial_t u + \text{curl} \ u \times u = -B \times \text{curl} B - \theta \text{curl} Y \times v - \nabla p, \quad \partial_t A + \text{curl} A \times v = \frac{R \theta}{a} \text{curl} Y \times v - \nabla q \quad (5.43) \]
with dissipation
\[ \frac{a^2}{2R^2} \frac{d}{dt} \int_D A \cdot \text{curl} A \ d^3x = -\theta \int_D |Y|^2 \ d^3x. \]
Energy dissipation. Exchanging the role of \( h \) and \( C \), we get the following modification of the equations

\[
\partial_t \mathbf{u} + \text{curl} \mathbf{u} \times \mathbf{u} = -\mathbf{B} \times \text{curl} \mathbf{B} + \theta \text{curl} \mathbf{Y} \times \mathbf{w} - \nabla p,
\]

\[
\partial_t \mathbf{A} + \text{curl} \mathbf{A} \times \mathbf{v} = -\frac{R \theta}{a} \text{curl} \mathbf{Y} \times \mathbf{w} - \nabla q,
\]

for the same vector field \( \mathbf{Y} \) as above and where \( \mathbf{w} = \frac{\delta C}{\delta p} = -\frac{a R}{B} \).

5.4 Planar Hall MHD with \( B \) normal to the plane

5.4.1 Incompressible case of 2D Hall MHD with \( B \) normal to the plane

Planar 2D Hall MHD requires \( B \) to be normal to the plane of flow. This is because the electron momentum density \( \mathbf{p} \) must lie in the plane of the 2D flow. However, \( \mathbf{p} \) is proportional to the magnetic vector potential \( \mathbf{A} \) for 2D Hall MHD, which means that the magnetic field vector \( \mathbf{B} = \text{curl} \mathbf{A} \) must be normal to the plane of flow. Therefore, in this section we consider two-dimensional Hall MHD taking place in a domain \( D \) in the \( xy \) plane with \( B \) normal to the plane, that is, \( \mathbf{B} = B \hat{z} \).

The Lie-Poisson bracket for 2D Hall MHD is given by

\[
\{f, g\}_+ (\omega, \lambda) = \int_D \left[ \omega \left\{ \frac{\delta f}{\delta \omega}, \frac{\delta g}{\delta \omega} \right\} + \lambda \left\{ \frac{\delta f}{\delta \lambda}, \frac{\delta g}{\delta \lambda} \right\} \right] dx \, dy,
\]

(5.44)

where \( \omega = \hat{z} \cdot \text{curl} \mathbf{m} \) and \( \lambda = \hat{z} \cdot \text{curl} \mathbf{p} \), with \( \mathbf{m}, \mathbf{p} \in X_{\text{vol}}(D) \), and the bracket \( \{\cdot, \cdot\} \) in the integrand is the canonical Poisson bracket in \((x, y)\). The unadorned canonical bracket operation \( \{\cdot, \cdot\} \) should not be confused with the subscripted Lie-Poisson bracket, denoted \( \{\cdot, \cdot\}_+ \) because the first acts on pairs of functions, while the second acts on pairs of functionals.

The magnetic vector potential is defined as in the three dimensional case by \( \mathbf{A} := -\frac{1}{R} \mathbf{p} \), where \( a \) is the charge to mass ratio of the ions and \( R \) is the Hall scaling parameter. The magnetic vector field is \( \mathbf{B} = B \hat{z} = \text{curl} \mathbf{A} \), so we have \( B = -\frac{1}{R} \mathbf{A} \lambda \). The Lie–Poisson equations that follow from the bracket in (5.44) are then expressed in terms of \( \omega \) and \( \lambda \) as

\[
\partial_t \omega + \left\{ \omega, \frac{\delta h}{\delta \omega} \right\} = 0,
\]

\[
\partial_t \lambda + \left\{ \lambda, \frac{\delta h}{\delta \lambda} \right\} = 0.
\]

(5.45)

In the 2D case, the Hamiltonian (5.40) can be written as

\[
h(\omega, \lambda) = \frac{1}{2} \int_D \left( -\Delta^{-1} (\omega - a B/R) \right) (\omega - a B/R) \, dx \, dy + \frac{1}{2} \int_D |B|^2 \, dx \, dy.
\]

Using \( \psi = \delta h/\delta \omega = -\Delta^{-1} \omega + R^{-1} a \Delta^{-1} B \) and \( \delta h/\delta \lambda = \psi - a^{-1} R B \), we obtain the equations

\[
\partial_t ( -\Delta \psi ) + \{ -\Delta \psi, \psi \} = 0 \quad \partial_t B + \{ B, \psi \} = 0.
\]

(5.46)

Note that \( \omega = -\Delta \psi + a B/R \) is the vorticity associated to the vector field \( \mathbf{u} + a \mathbf{A}/R \), where \( \psi \) is the stream function of the velocity \( \mathbf{u} \), i.e., \( \mathbf{u} = \text{curl}(\psi \hat{z}) \). For the electron fluid velocity we have \( \mathbf{v} = \text{curl}((\psi + RB/a)\hat{z}) \) with vorticity \(-\Delta \psi + R \Delta B/a\).
Remark 5.5. In 2D the $B = B\hat{z}$ term in the momentum equation may be absorbed into the gradient pressure, since in this case $B \times \text{curl} B = B\nabla B = \frac{1}{2}\nabla |B|^2$. This simplification also occurred for 2D incompressible MHD in §4.2 because the magnetic field was frozen into the incompressible fluid flow. In the 2D Hall MHD case, it occurs because the magnetic field is frozen into the ion flow.

Case 1. Using the Casimir functions

$$C(\omega, \lambda) = \int_D \Phi(\omega) \, dx \, dy,$$

we obtain the modified momentum

$$\tilde{\omega} = \omega + \theta \left\{ \delta h \frac{\delta C}{\delta \omega} \right\}^b = \omega + \theta L \{ \psi, \Phi'(\omega) \}$$

and the modified equations

$$\partial_t \omega + \left\{ \omega, \frac{\delta h}{\delta \omega} \right\} = -\theta \left\{ L \left\{ \frac{\delta h}{\delta \omega}, \frac{\delta C}{\delta \omega} \right\}, \frac{\delta h}{\delta \omega} \right\}, \quad \partial_t \lambda + \left\{ \lambda, \frac{\delta h}{\delta \lambda} \right\} = 0.$$

For example, if $\Phi(\omega) = \frac{1}{2} \omega^2$ (enstrophy), we have

$$\partial_t (-\Delta \psi) + \{-\Delta \psi, \psi\} = -\theta \left\{ L \{ \psi, -\Delta \psi + aB/R \}, \psi \right\}, \quad \partial_t B + \{B, \psi\} = 0$$

which implies the following formula for the Casimir dissipation

$$\frac{d}{dt} \int_D (-\Delta \psi + aB/R)^2 \, dx \, dy = -\theta \{ \psi, -\Delta \psi + aB/R \} \|_L^2.$$

Energy dissipation. Exchanging the role of $h$ and $C$, we get the following modification of the equations

$$\partial_t \omega + \left\{ \omega, \frac{\delta h}{\delta \omega} \right\} = -\theta \left\{ L \left\{ \frac{\delta C}{\delta \omega}, \frac{\delta h}{\delta \omega} \right\}, \frac{\delta C}{\delta \omega} \right\}, \quad \partial_t \lambda + \left\{ \lambda, \frac{\delta h}{\delta \lambda} \right\} = 0,$$

which, in the case of the enstrophy, become

$$\partial_t (-\Delta \psi) + \{-\Delta \psi, \psi\} = -\theta \left\{ L \{-\Delta \psi + aB/R, \psi\}, -\Delta \psi + aB/R \right\}, \quad \partial_t B + \{B, \psi\} = 0$$

Case 2. Using the Casimir functions

$$C(\omega, \lambda) = \int_D \Phi(\lambda) \, dx \, dy,$$

we get the modified momentum

$$\tilde{\lambda} = \lambda + \theta \left\{ \frac{\delta h}{\delta \lambda}, \frac{\delta C}{\delta \lambda} \right\}^b = \lambda + \theta L \left\{ \psi - \frac{R}{a} B, \Phi'(\lambda) \right\} = \lambda + \theta L \{ \psi, \Phi'(\lambda) \}$$
and the modified equations
\[ \frac{\partial \omega}{\partial t} + \left\{ \omega, \frac{\delta h}{\delta \omega} \right\} = 0, \quad \frac{\partial \lambda}{\partial t} + \left\{ \lambda, \frac{\delta h}{\delta \lambda} \right\} = -\theta \left\{ L \left\{ \frac{\delta C}{\delta \lambda}, \frac{\delta h}{\delta \lambda} \right\}, \frac{\delta h}{\delta \lambda} \right\}. \]

For example, if \( \Phi(\lambda) = \frac{1}{2} \lambda^2 \) (enstrophy), we obtain the equations
\[ \frac{\partial (-\Delta \psi)}{\partial t} + \left\{ -\Delta \psi, \psi \right\} = -\frac{\theta}{R} \left\{ L \left\{ \psi, B \right\}, \psi - RB/a \right\}, \quad \frac{\partial \psi}{\partial t} + \left\{ B, \psi \right\} = -\theta \left\{ L \left\{ \psi, B \right\}, \psi - RB/a \right\}. \]

**Energy dissipation.** Exchanging the role of \( h \) and \( C \), we get the following modification of the equations
\[ \frac{\partial (-\Delta \psi)}{\partial t} + \left\{ -\Delta \psi, \psi \right\} = -\theta \left\{ L \left\{ B, \psi \right\}, B \right\}, \quad \frac{\partial B}{\partial t} + \left\{ B, \psi \right\} = \frac{R \theta}{a} \left\{ L \left\{ B, \psi \right\}, B \right\}. \]

**Remark 5.6.** Note that although the equations for incompressible Hall MHD in 2D (5.46) happen to coincide with the equations for 2D MHD (4.32), their modifications to include Casimir dissipation do not coincide, because they have different Casimirs.

### 5.4.2 Compressible case of 2D Hall MHD with \( B \) normal to the plane

The Lie–Poisson bracket for this case has the same form as (5.3) and the Hamiltonian is (5.21), except that now all variables are defined on the two-dimensional manifold \( D \). The equations are identical to (5.7). In terms of \( B \), they read
\[ \rho \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) = -\nabla \left( p + \frac{1}{2} B^2 \right), \]
\[ \frac{\partial A}{\partial t} - u \times B = \frac{R}{2a \rho} \nabla B^2, \]
\[ \frac{\partial \rho}{\partial t} + \text{div}(\rho u) = 0, \quad \frac{\partial \eta}{\partial t} + u \cdot \nabla \eta = 0, \]

where \( p = \rho^2 \partial e/\partial \rho \).

One class of Casimir functions is given by
\[ C(\rho, \tilde{n}) = \int_\mathcal{D} \tilde{n} \Psi(\zeta/\tilde{n}) \, dx \, dy, \quad \zeta := \hat{z} \cdot \text{curl}(\rho \tilde{n}) = B/R, \]

since by (5.5) \( \rho \tilde{n} = A/R \) and \( B = B \hat{z} \). If we choose the bilinear form
\[ \gamma(\rho, v) = -\int_\mathcal{D} \tilde{n} (u \cdot v) \, d^3x, \]

then the modified momentum reads
\[ \tilde{p} = p + \theta Y, \quad Y = \left[ \frac{\delta h}{\delta \tilde{p}} \frac{\delta C}{\delta \tilde{p}} \right]^b = -\tilde{n} \left[ v, \tilde{n}^{-1} \text{curl}(\Psi'(\zeta/\tilde{n})\hat{z}) \right]. \]
In a similar way with §5.1 (case 2), under the neutrality assumption \( \tilde{n} + a\rho = 0 \), we get the equations

\[
\rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p - \mathbf{B} \times \text{curl} \mathbf{B} - \theta \mathbf{L}_v \mathbf{Y}, \quad \partial_t \mathbf{A} = \mathbf{u} \times \mathbf{B} + \frac{R}{a\rho} \mathbf{B} \times \text{curl} \mathbf{B} + \frac{R\theta}{a\rho} \mathbf{L}_v \mathbf{Y},
\]

where \( \mathbf{v} = \mathbf{u} - \frac{R}{a\rho} \text{curl} \mathbf{B} \) and \( \mathbf{Y} \) is defined in equation (5.50).

For example, if \( \Psi(\zeta/\tilde{n}) = -\frac{1}{2}(\zeta/\tilde{n})^2 \), we have \( \mathbf{X} = -\tilde{n}[\mathbf{v}, \tilde{n}^{-1} \text{curl}(\zeta/\tilde{n})\hat{z}] \) and

\[
\frac{d}{dt} \frac{1}{2} \int_D \rho \zeta^2 \, dx \, dy = -\theta \int_D \rho |\mathbf{Y}|^2 \, dx \, dy.
\]

**Energy dissipation.** Exchanging the role of \( h \) and \( C \), we get the following modification of the equations

\[
\rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p - \mathbf{B} \times \text{curl} \mathbf{B} + \theta \mathbf{L}_w \mathbf{Y}, \quad \partial_t \mathbf{A} = \mathbf{u} \times \mathbf{B} + \frac{R}{a\rho} \mathbf{B} \times \text{curl} \mathbf{B} - \frac{R\theta}{a\rho} \mathbf{L}_w \mathbf{Y},
\]

where \( \mathbf{w} = \delta C/\delta p = \tilde{n}^{-1} \text{curl}(\Psi'(\zeta/\tilde{n})\hat{z}) \).

If the entropy variable \( \eta \) is absent, then the same family of Casimir functions

\[
C(\mathbf{m}, \rho) = \int \rho \Psi(\omega/\rho) \, dx \, dy, \quad \omega := \hat{z} \cdot \text{curl}(\mathbf{m}/\rho),
\]

can be used for the variables \((\mathbf{m}, \rho)\).

### 6 Conclusions

This paper has introduced modifications of the equations of ideal fluid dynamics with advected quantities to allow selective decay of either the energy \( h \) or the Casimir quantities \( C \) in its Lie-Poisson formulation. The dissipated quantity (energy or Casimir, respectively) is shown to decrease in time until the modified system reaches a state that holds for ideal energy-Casimir equilibria, namely \( \delta(h + C) = 0 \). The result holds for Lie-Poisson equations in general, independently of the Lie algebra and the choice of Casimir. The selective decay results are always consistent with the ideal energy-Casimir equilibrium conditions. However, the selective decay process sometimes tends towards only some of the energy-Casimir conditions, not all of them. This is explained in the proof of the main result, Theorem 2.3. Thus, the selective decay mechanism introduced here does not necessarily approach fully time-independent solutions of the unmodified ideal equations.

We interpret the selective decay modifications of the equations as dynamical and nonlinear parameterizations of the interactions among different scales, obtained by introducing new nonlinear pathways to dissipation. The modification process is illustrated with a number of selected decay examples that pass to stable energy-Casimir equilibria for magnetohydrodynamics (MHD) and Hall MHD in 2D and 3D by decay of either the energy or the Casimirs.

The balance between the Casimir and energy that occurs at a critical point of their sum determines the class of equilibria that is achievable by a given choice of constraint force. Equations
(1.6) and (1.14) provide the constraint forces that will guide the ideal MHD system into a particular class of equilibria, by decreasing, respectively, either the Casimir at constant energy, or vice versa. The existence of a constraint force that will dynamically guide the ideal MHD system into a certain class of equilibria (or preserve it once it has been obtained) may provide useful ideas in the design and control of magnetic confinement devices.

The Lagrange-d’Alembert variational principle extends Hamilton’s principle to the case of forced systems, including nonholonomically constrained systems (Bloch [2004]). We have explained following Gay-Balmaz and Holm [2013] in section 1.2.1 how the constraint forces for Casimir-dissipative LP equations (1.6) and energy-dissipative LP equations (1.14) can be obtained from the Lagrange-d’Alembert principle.

The selective decay approach for fluid flows with advected quantities could also be useful in generating higher-order stable dissipative numerical schemes that generalize the 2D anticipated vorticity method to 3D and include the advected quantities. Thus, the numerical implementation of the present approach may lead to improved numerical schemes that dissipate energy, but conserve Casimirs. This could lead to generalizations for MHD of the anticipated vorticity method for parametrizing subgrid scale barotropic and baroclinic eddies in quasi-geostrophic models, Sadourny and Basdevant [1985].

In all of the present discussions, we have assumed that the solutions of the modified (dissipative) equations possess long-time existence. However, from the viewpoint of mathematical analysis, the possibility of blow up in finite time for these equations remains an open problem.

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