Lobachevsky geometry of (super)conformal mechanics

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Abstract

We give a simple geometric explanation for the similarity transformation mapping one-dimensional conformal mechanics to free-particle system. Namely, we show that this transformation corresponds to the inversion of the Klein model of Lobachevsky space (non-compact complex projective plane) \( \mathbb{CP}^1 \). We also extend this picture to the \( \mathcal{N} = 2k \) superconformal mechanics described in terms of Lobachevsky superspace \( \mathbb{CP}^{1|k} \).

Introduction

Conformal symmetry plays an important role in modern field theory. So, the study of various aspects of simple (super)conformal invariant models could be useful for more complicated systems. Since the middle of seventies, after [1], it was realized that even the one-dimensional one-particle mechanics given by the Hamiltonian

\[
H = \frac{p^2}{2} + \frac{g^2}{2x^2}
\]

is a good polygon for the study of the possible consequences of the conformal symmetry. This Hamiltonian together with the generators

\[
D = px, \quad K = \frac{x^2}{2}
\]

forms the conformal algebra \( \text{so}(1, 2) \) with respect to canonical Poisson brackets \( \{p, x\} = 1 \):

\[
\{H, D\} = H, \quad \{H, K\} = 2D, \quad \{K, D\} = K.
\]

Here \( D \) is the dilatation, and \( K \) is the conformal boost. By that reason, this system is called in literature a one-dimensional ”conformal mechanics”. Clearly, it is a conformal symmetric system in the field-theoretical context (i.e. its action functional possesses a conformal symmetry provided that the time reparametrizations are admitted) but not in the sense of integrable mechanical systems.

Although this model is quite simple, it inherits some properties of more complicated conformally symmetric mechanical systems. The study of its various supersymmetric and superconformal extensions is of special importance. It was initiated in [2], and has been continuing in the various directions up to now (see, e.g., [3] and references therein). Let us mention the article [5] (and the related ones [6]), where it was observed that the motion of the (super)particle near horizon of the extremal Raissner-Nordström black hole is described by the (super)conformal mechanics.

The conformal mechanics can be considered as a two-particle Calogero model, which is a one-dimensional multiparticle integrable system with inverse-square interaction [7]. This model has attracted much attention due to numerous applications in the wide area of physics, as well as due to rich internal structure (see, e.g., the recent review [8] and references therein). Already in the pioneering paper [7] it was observed that the spectrum of the Calogero model with additional oscillator potential is similar to the spectrum of free \( \mathcal{N} \)-dimensional oscillator. It was claimed there that a similarity transformation to free oscillator system may exist, at least, in the part of Hilbert space. However, this transformation has been written explicitly only decades after [9]. Its elegant group-theoretical explanation has been given in [10], where the similarity transformation is related to the conformal group \( SU(1, 1) \). Exploiting this similarity, the authors built not only \( \mathcal{N} = 2 \) supersymmetric Calogero model [11] but also suggested an algorithm for the construction of \( \mathcal{N} = 4 \) superconformal Calogero model, which was unknown. An explicit expressions for the \( \mathcal{N} = 4 \) superconformal Calogero system were presented in [12].

In this note we present a simple geometric view on this ”decoupling” transformation for the conformal mechanics (i.e. for two-particle Calogero system) and for its \( \mathcal{N} = 2k \) superconformal extension. First, we parameterize the phase space of conformal mechanics by the Klein model of Lobachevsky space (which is a Kähler space) in such a way that the generators [1, 2] become the isometries of the Kähler structure of the Klein model. Then we show that the decoupling transformation corresponds to the inversion transformation of the Klein model. The quantum counterpart of this picture can be get by the standard procedure of the geometric quantization.
Then, using the above picture, we construct the \( N = 2k \) superconformal mechanics and give a similar description for its decoupling transformation as well. For this purpose, we consider a linear action of the \( u(1, 1|k) \) algebra on the Euclidean superspace \( \mathbb{C}^{1,1|k} \). Then, performing the Hamiltonian reduction of this (phase) space by the \( U(1) \) group, we arrive at the Lobachevsky superspace \( \mathbb{C}P^{(1|k)} \). The superconformal algebra \( su(1,1|k) \) defines the isometries of its Kähler structure. The Hamiltonian of the superconformal mechanics and the generators of superconformal algebra play the role of the Killing potentials. The construction of the superconformal mechanics by the reduction from the Euclidean superspace allows us immediately extend the decoupling procedure to the supersymmetric system as well.

### Conformal mechanics

Let us start from the description of the conformal mechanics in a form, which is suitable for our purposes. It is convenient to describe the algebra (1) in terms of the generators

\[
J_0 = H + K, \quad J_1 = D, \quad J_3 = H - K: \quad \{J_a, J_b\} = -2\varepsilon_{abc}J^c, \quad a, b, c = 0, 1, 3,
\]

where the indices are upped by the use of \( (1 + 2) \)-dimensional Euclidean metrics \( \gamma_{ab} = \text{diag}(1, -1, -1) \). It is easy to see that these generators define the upper sheet of two-sheet hyperboloid (Lobachevsky plane) with the radius \( g_1 \):

\[
J_a J^a = g_2.
\]

It can be parameterized by the complex coordinate

\[
w = \frac{p}{x} + \frac{ig}{x^2}, \quad \text{Im } w > 0: \quad \{w, \bar{w}\} = \frac{-1}{g} (w - \bar{w})^2.
\]

So, the phase space of the system is the Klein model of the Lobachevsky plane.

In the above parametrization, the \( \text{so}(1,2) \) generators (4) take the form

\[
J_0 = ig \frac{w\bar{w} + 1}{w - \bar{w}}, \quad J_1 = ig \frac{w + \bar{w}}{w - \bar{w}}, \quad J_3 = ig \frac{w\bar{w} - 1}{w - \bar{w}}.
\]

They are precisely the Hamiltonian generators defining the isometries of the Kähler structure of the Lobachevsky plane (Killing potentials), which is given by the following metric and potential

\[
ds^2 = -\frac{gdwd\bar{w}}{(w - \bar{w})^2}, \quad K = g \log i(\bar{w} - w).
\]

This Kähler structure is invariant under the discrete transformation (which is, obviously, a canonical one)

\[
w \rightarrow -\frac{1}{w},
\]

whereas the Killing potentials (7) are not invariant with respect to this transformation:

\[
J_0 \rightarrow J_0, \quad J_1 \rightarrow -J_1, \quad J_3 \rightarrow -J_3.
\]

Taking into account (4), one can rewrite this transformation in terms of initial generators:

\[
H = \frac{g w\bar{w}}{i(w - \bar{w})} \rightarrow K = \frac{g}{i(w - \bar{w})}, \quad K \rightarrow -H, \quad D \rightarrow -D.
\]

So, the presented canonical transformation maps the conformal mechanics to the free particle system in the momentum space. Let us write down its explicit form in \( p, x \) coordinates:

\[
\frac{\dot{x}}{\dot{p}} + \frac{g}{p^2 + \frac{g^2}{x^2}} = \frac{-px + ig}{p^2 + \frac{g^2}{x^2}}.
\]

Hence, we get the following canonical transformation of the initial phase space

\[
\tilde{p} = \pm \sqrt{p^2 + \frac{g^2}{x^2}}, \quad \tilde{x} = \mp \frac{px}{\sqrt{p^2 + \frac{g^2}{x^2}}}.
\]

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1 We were informed by E. Ivanov, that this had been observed many years ago in his old paper with Krivonos and Leviant [13].
Remark. One can consider the quantum counterpart of our picture performing the geometric quantization of \( su(1, 1) \) on Lobachevsky space. The relevant formulae can be found in Ref. [14], where the quantization was done on the framework of Poincaré model of Lobachevsky space with the following metric and Killing potentials

\[
ds^2 = \frac{g d\bar{w} d\bar{\bar{w}}}{(1 - \bar{w}\bar{\bar{w})^2}}, \quad J_0 = g \frac{1 + \bar{w}\bar{\bar{w}}}{1 - \bar{w}\bar{\bar{w}}}, \quad J_3 + i J_1 = \frac{2g\bar{w}\bar{\bar{w}}}{1 - \bar{w}\bar{\bar{w}}}, \quad |w| < 1.
\]

Performing the conformal transformation

\[w \rightarrow \frac{w - i}{w + i},\]

we will arrive at the Klein model of Lobachevsky space.

Superconformal mechanics

Our construction can be extended straightforwardly to the \( \mathcal{N} = 2k \) superconformal algebra \( su(1, 1|k) \) with some real central charge. Apart from the \( so(1, 2) \) generators, this superconformal algebra contains \( 2k \) pairs of Grassmanian odd generators \( \Theta_\alpha^A = (Q^A, S^A), \Theta_\alpha^A = (\bar{Q}^A, \bar{S}^A), \alpha = 1, 2, A = 1, 2, \ldots, k \), which are Hermitian conjugates of each other, and the \( u(k) \) generators \( R^{AB} \). Notice, that \( Q^A \) and \( \bar{Q}^A \) define the supersymmetry generators, and \( S^A \) and \( \bar{S}^A \) the superconformal ones.

In order to define this algebra on the superextension of the Lobachevsky space, let us first consider the \( \mathbb{C}^{1,1|k} \) Euclidean superspace equipped with the canonical Kähler (and symplectic) two-form

\[i(dz_1 \wedge d\bar{z}_0 - dz_0 \wedge d\bar{z}_1) + d\eta^A \wedge d\bar{\eta}^A.\]

The Poisson brackets are given by the following non-vanishing relations and their complex conjugates:

\[
\{z_0, \bar{z}_1\} = 1, \quad \{z_1, \bar{z}_0\} = -1, \quad \{\eta^A, \bar{\eta}^B\} = \delta^{AB}.
\]

The rotational symmetries of this Kähler structure defined by the following Killing potentials

\[
\mathcal{J} = i(z_1\bar{z}_0 - z_0\bar{z}_1) + i\eta\bar{\eta},
\]

\[
\mathcal{J}_a = z\sigma^a \bar{\bar{z}}, \quad R^{AB} = i\eta^A \bar{\eta}^B, \quad \Theta_\alpha^A = \bar{z} \eta^A,
\]

which form the \( u(1, 1|k) \) superalgebra:

\[
\{\mathcal{J}_a, \mathcal{J}_b\} = -2\varepsilon_{abc}\mathcal{J}^c, \quad \{\Theta_\alpha^A, \bar{\Theta}_\beta^B\} = \frac{1}{2}\delta^{AB}(\sigma^a_{\alpha\beta}\mathcal{J}_a + i\epsilon_{\alpha\beta}\mathcal{J}) + i\epsilon_{\alpha\beta}\left(R^{AB} - \frac{1}{2}\delta^{AB}R_0\right),
\]

\[
\{\mathcal{J}_a, \Theta_\alpha^A\} = -\epsilon_{\alpha\beta}\sigma^a_{\beta\gamma}\Theta_\gamma^A, \quad \{R^{AB}, \Theta_\alpha^A\} = i\delta^{CB}R_0^{AB}, \quad \{R^{AB}, R^{CD}\} = i\delta^{CB}R_0^{AD} - i\delta^{AD}R_0^{CB},
\]

\[
\{\mathcal{J}_a, \mathcal{J}_b\} = \{\mathcal{J}, \Theta_\alpha^A\} = \{\mathcal{J}, R^{AB}\} = 0, \quad \{\mathcal{J}_a, R^{AB}\} = 0,
\]

where \( \sigma^a = \sigma^0, \sigma^1, \sigma^3 \) are "Minkowskian" Pauli matrices (\( \sigma^0 \) is the identity matrix) and \( R_0 = \sum_A R^{AA} \). The generator \( \mathcal{J} \) defines the center of the superalgebra \( u(1, 1|k) \), while the other generators form the superconformal algebra \( su(1, 1|k) \). Hence, we can reduce the \( \mathbb{C}^{1,1|k} \) superspace by the action of the generator \( \mathcal{J} \) to the Lobachevsky superspace, whose isometry superalgebra is \( su(1, 1|k) \). We refer to the [15] for the details, where the complex projective superspace \( \mathbb{C}P^{n|k} \) has been constructed by the Hamiltonian reduction from the Euclidean space \( \mathbb{C}^{n+1|k} \). The procedure presented below is just its noncompact counterpart. The details of reduction of \( \mathbb{C}^{1,1|k} \) to the Poincare and Klein models can be found in [16]. Clearly, the reduced superspace is \( (2|2k)_{\mathbb{R}} \)-dimensional. One can parameterize it by

\[
\bar{w} = \frac{z_0}{z_1}, \quad \theta^A = \frac{\eta^A}{z_1}; \quad \{\bar{w}, \mathcal{J}\} = \{\theta^A, \mathcal{J}\} = 0.
\]
On the level surface $J = g$ the following relation yields:

$$|z_1|^2 = \frac{g}{i(\bar{w} - \tilde{w}) + i\theta \bar{\theta}} \quad (23)$$

Calculating the Poisson brackets between coordinates (22) and restricting them to the level surface $J = g$, we obtain the Poisson brackets on the reduced phase space:

$$\{\tilde{w}, \bar{\tilde{w}}\} = \frac{i(\tilde{w} - \bar{w}) + i\theta \bar{\theta}}{g} (\tilde{w} - \bar{w}), \quad \{\tilde{w}, \tilde{\theta}^A\} = -i \frac{i(\tilde{w} - \bar{w}) + i\theta \bar{\theta}}{g} \tilde{\theta}^A, \quad \{\theta^A, \bar{\theta}^B\} = \frac{i(\tilde{w} - \bar{w}) + i\theta \bar{\theta}}{g} \delta^{AB} \quad (24)$$

These Poisson brackets define the Kähler structure given by the following Kähler potential:

$$\tilde{K} = g \log(i(\tilde{w} - \bar{w}) + i\theta \bar{\theta}). \quad (25)$$

This supermanifold is a superextension of the Klein model of the Lobachevsky space, $\mathbb{CP}^{1|k}$.

Now, let us write down the Killing potentials of the $su(1,1|k)$ superalgebra obtained by the restriction of (19) to the level surface $J = g$

$$\mathcal{J}_0 = \mathcal{H} + \mathcal{K} = g \frac{\bar{w} \tilde{w} + 1}{i(\tilde{w} - \bar{w}) + i\theta \bar{\theta}}, \quad \mathcal{J}_1 = \mathcal{D} = g \frac{\bar{w} \tilde{w} - 1}{i(\tilde{w} - \bar{w}) + i\theta \bar{\theta}}, \quad \mathcal{J}_3 = \mathcal{H} - \mathcal{K} = g \frac{\bar{w} + \tilde{w}}{i(\tilde{w} - \bar{w}) + i\theta \bar{\theta}} \quad (26)$$

$$\Theta_1^A = Q^A = \frac{\tilde{\theta}^A}{i(\tilde{w} - \bar{w}) + i\theta \bar{\theta}}, \quad \Theta_2^A = S^A = g \frac{\bar{w} \tilde{\theta}^A}{i(\tilde{w} - \bar{w}) + i\theta \bar{\theta}}, \quad R^{AB} = g \frac{\theta^A \bar{\theta}^B}{i(\tilde{w} - \bar{w}) + i\theta \bar{\theta}} \quad (27)$$

These generators form the algebra (20) with respect to the reduced Poisson brackets, where the generator $J$ is replaced by the central charge $g$. It is clear from our consideration that the supersymmetric extension of the similarity transformation looks as follows

$$\tilde{w} \rightarrow -\frac{1}{w}, \quad \theta^A \rightarrow -\frac{\theta^A}{w}. \quad (28)$$

It yields the following transformation of the generators of $\mathcal{N} = 2k$ superconformal algebra

$$\mathcal{H} \rightarrow \mathcal{K}, \quad \mathcal{K} \rightarrow \mathcal{H}, \quad \mathcal{D} \rightarrow -\mathcal{D}, \quad R^{AB} \rightarrow R^{AB}, \quad Q^A \rightarrow S^A, \quad S^A \rightarrow Q^A. \quad (29)$$

Finally, we pass to the coordinates, which split the fermionic and bosonic sectors of the Poisson brackets:

$$\chi^A = \frac{\sqrt{g} \tilde{\theta}^A}{\sqrt{i(\tilde{w} - \bar{w}) + i\theta \bar{\theta}}}, \quad w = \tilde{w} - \frac{1}{2} \theta \bar{\theta} \quad (30)$$

$$\{w, \tilde{w}\} = -i \frac{(w - \bar{w})^2}{g}, \quad \{\chi^A, \chi^B\} = \delta^{AB}, \quad \{w, \chi^A\} = \{w, \chi^B\} = 0. \quad (31)$$

Substituting these expressions in (26), (27) and expressing $\tilde{w}$ via canonical coordinates $(p, x)$ as in (6), we get the $\mathcal{N} = 2k$ superconformal mechanics. In the same manner as in the pure bosonic case, we obtain the expression of the similarity transformations (28).

**Summary and discussion**

In conclusion, let us emphasize the main statements of the current article.

- We identified the Killing potentials of the Klein model of the Lobachevsky space with the Hamiltonian of classical one-dimensional conformal mechanics and with the generators of conformal boost and dilatation. The inversion transformation (with minus sign) corresponds to the canonical transformation of the Hamiltonian to the generator of conformal boost, which describes the one-dimensional free particle. In other words, the inversion transformation of the Lobachevsky space defines the "decoupling transformation" of conformal mechanics.

- Using the method of Hamiltonian reduction, we constructed the one-dimensional $\mathcal{N} = 2k$ superconformal mechanics. The generators of its dynamical symmetry superalgebra $su(1,1|k)$ define the Killing potentials of the superextension of the Klein model of the Lobachevsky space corresponding to the noncompact complex projective superplane $\mathbb{CP}^{1|k}$. We found the decoupling transformation for this superconformal system as well.
It seems that the presented picture can be extended without much efforts to higher-dimensional conformal mechanics and Calogero model. It would be interesting to find a similar description for systems with less trivial potentials, and for the higher-dimensional superconformal mechanics with the Dirac monopole (the example of such system was suggested in \cite{4}). Particularly, we expect that noncompact complex projective space $\mathbb{C}P^N$ can be related with integrable multi-particle systems.

Note that the dilatation operator, providing the phase space of the superconformal mechanics with a constant curvature, does not play any role in our consideration. Hence, one can suppose that a similar construction may be realized for the specific case of two-dimensional surfaces like the Special Kähler surfaces of the local type with two isometries.

Finally, let us draw attention to the pretty simple form of the superconformal mechanics given by the Poisson brackets \cite{24} and the generators \cite{26}, \cite{27}. The simplicity is due to non-canonical Poisson bracket with obvious geometrical meaning, which we used instead of canonical ones. In fact, the quite transparent geometrical nature of these expressions, as well as the way, via which they were found (Hamiltonian reduction), allow us to believe that this could be the shortest and most natural way for the construction of the $\mathcal{N} \geq 4$ superconformal Calogero model.

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