The Space of Nonpositively Curved Metrics of a Negatively Curved Manifold

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Abstract

We show that the space of nonpositively curved metrics of a closed negatively curved Riemannian $n$-manifold, $n \geq 10$, is highly non-connected.

Section 0. Introduction.

Let $M^n$ be a closed smooth manifold of dimension $\text{dim } M = n$. We denote by $\mathcal{MET}(M)$ the space of all smooth Riemannian metrics on $M$ and we consider $\mathcal{MET}(M)$ with the smooth topology. Also, we denote by $\mathcal{MET}^{\text{sec} < 0}(M)$ the subspace formed by all negatively curved Riemannian metrics on $M$. In [8] we proved that $\mathcal{MET}^{\text{sec} < 0}(M)$ always has infinitely many path-components, provided $n \geq 10$ and it is non-empty. Moreover we showed that all the groups $\pi_{2p-4}(\mathcal{MET}^{\text{sec} < 0}(M))$ are non-trivial for every prime number $p > 2$, and such that $p < \frac{n+5}{6}$ (this is true in every component of $\mathcal{MET}^{\text{sec} < 0}(M)$). In fact, these groups contain the infinite sum $(\mathbb{Z}_p)^\infty$ of $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$'s.

We also showed that $\pi_1(\mathcal{MET}^{\text{sec} < 0}(M))$ contains the infinite sum $(\mathbb{Z}_2)^\infty$ when $n \geq 12$ (see also [9]). All these results follow from the Main Theorem in [8], which states that the orbit map $\Lambda_g : \text{DIFF}(M) \to \mathcal{MET}^{\text{sec} < 0}(M)$ is “very non-trivial” at the $\pi_k$-level. Here $\text{DIFF}(M)$ is the group of self-diffeomorphisms on $M$ and $\Lambda_g(\phi) = \phi_*g$ (see the introduction of [8] for more details).

Let $\mathcal{MET}^{\text{sec} \leq 0}(M)$ be the subspace of $\mathcal{MET}(M)$ formed by all non-positively curved Riemannian metrics on $M$. In this paper we generalize to $\mathcal{MET}^{\text{sec} \leq 0}(M)$ the results mentioned above, provided $\pi_1 M$ is (word) hyperbolic:

Main Theorem. Let $M^n$ be a closed smooth manifold with hyperbolic fundamental group $\pi_1 M$. Assume $\mathcal{MET}^{\text{sec} \leq 0}(M)$ is non-empty. Then

(i) the space $\mathcal{MET}^{\text{sec} \leq 0}(M)$ has infinitely many components, provided $n \geq 10$.

(ii) The group $\pi_1(\mathcal{MET}^{\text{sec} \leq 0}(M^n))$ is not trivial when $n \geq 12$. In fact it contains the infinite sum $(\mathbb{Z}_2)^\infty$ as a subgroup.

(iii) The groups $\pi_{2p-4}(\mathcal{MET}^{\text{sec} \leq 0}(M^n))$ are non-trivial for every prime number $p > 2$, and such that $p < \frac{n+5}{6}$. In fact, these groups contain the infinite sum $(\mathbb{Z}_p)^\infty$ as a subgroup.

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Remarks.
1. The results for $\pi_k \mathcal{MET}^{sec \leq 0}(M), k > 0$, given above are true relative to any base point, that is, for every component of $\mathcal{MET}^{sec \leq 0}(M)$.
2. The decoration “$sec \leq 0$” can be tightened to “$a \leq sec \leq 0$”, for any $a < 0$.
3. The Theorem above follows from a nonpositively curved version of the Main Theorem of [8] (which we do not state to save space). This nonpositively curved version is obtained from the Main Theorem in [8] by replacing $\mathcal{MET}^{sec < 0}(M)$ by $\mathcal{MET}^{sec \leq 0}(M)$ and adding the hypothesis “$\pi_1(M)$ is hyperbolic”. This is the result that we prove in this paper. And, as in [8], we obtain the following corollary.

Corollary. Let $M$ be a closed smooth $n$-manifold with $\pi_1(M)$ hyperbolic. Let $I \subset (-\infty, 0]$ and assume that $\mathcal{MET}^{sec \in I}(M)$ is not empty. Then the inclusion map $\mathcal{MET}^{sec \in I}(M) \hookrightarrow \mathcal{MET}^{sec \leq 0}(M)$ is not null-homotopic, provided $n \geq 10$.

Moreover, the induced maps of this inclusion, at the $k$-homotopy level, are not constant for $k = 0$, and non-zero for $k$ and $n$ as in cases (ii.), (iii.) in the Main Theorem. Furthermore, the image of these maps satisfy a statement analogous to the one in the Addendum to the Main Theorem in [8].

Here $\mathcal{MET}^{sec \in I}$ has the obvious meaning. In particular taking $I = (-\infty, 0)$ we get that the inclusion $\mathcal{MET}^{sec < 0}(M) \hookrightarrow \mathcal{MET}^{sec \leq 0}(M)$ is not nullhomotopic, provided $n \geq 10$ and $M$ admits a negatively curved metric.

In some sense it is quite surprising that we were able to extend the results in [8] to the nonpositively curved case because negative curvature is a “stable” condition (the space $\mathcal{MET}^{sec < 0}(M)$ is open in $\mathcal{MET}(M)$) while $\mathcal{MET}^{sec \leq 0}$ is not stable. Indeed it is not even known whether $\mathcal{MET}^{sec \leq 0}(M)$ is locally contractible or even locally connected. We state these as questions:

Questions.
1. Is the space $\mathcal{MET}^{sec \leq 0}(M)$ of nonpositively curved metrics on $M$ locally contractible?
2. Is the space $\mathcal{MET}^{sec \leq 0}(M)$ of nonpositively curved metrics on $M$ locally connected?

So, we prove here that $\mathcal{MET}^{sec \leq 0}(M)$ is not (globally) connected when it is not empty, $n \geq 10$ and $\pi_1 M$ is hyperbolic. But on the other hand it is not known whether $\mathcal{MET}^{sec \leq 0}(M)$ is locally connected.

In this paper there are two additional obstacles to pass from negative curvature to nonpositive curvature. First, since we can now have parallel geodesic rays emanating perpendicularly from a closed geodesic, the obstructions we defined in [8] (which lie in the pseudoisotopy space of $S^k S^{n-2}$) may not be homeomorphisms at infinity.

The second problem is that we may now have a whole family of closed geodesics freely homotopic to a given one. But in our previous papers we strongly used the fact there is a unique such closed geodesic. Moreover, we strongly used the fact that such unique closed geodesics depend smoothly on the metric. This does not happen in nonpositive curvature. Even worse: there are examples of smooth families $g_t, t \in [0, 1]$, of nonpositively curved metrics such that there is no continuous path of closed $g_t$-geodesics joining a closed $g_1$-geodesic to a closed $g_0$-geodesic (all closed geodesics in the same free homotopy class). See for instance the “swinging neck” in Appendix A. We deal with this by incorporating the closed geodesics into the system, but we pay a price for this: instead of dealing with
discs (to prove that an element is zero in a homotopy group) we have to deal with more complicated spaces which we call “cellular discs”. Because of this the use of shape theory becomes necessary.

In section 1 we define cellular discs and give some preliminary results. In section 2 we prove the Main Theorem (see remark 3 above). We shall refer to [8] for some details.

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Section 1. Preliminaries.

A. Cellular Discs.

We will consider the $k$-disc $D^k = \{ x \in \mathbb{R}^k : |x| \leq 1 \}$ with base point $u_0 = (1, 0, 0, ..., 0)$. A cellular $k$-disc is a metrizable compact pointed topological space $(X, x_0)$ together with a surjective continuous map $\eta : (X, x_0) \to (D^k, u_0)$ such that the pre-image $\eta^{-1}(u)$, $u \in D^k$, is homeomorphic to the $\ell_u$-disc $D^{\ell_u}$, with $0 \leq \ell_u \leq \ell$, for some $\ell < \infty$ and all $u \in D^k$. We write $X_u = \eta^{-1}(u)$, $X_0 = \eta^{-1}(u_0)$ and $\partial X = \eta^{-1}(\partial D^k) = \eta^{-1}(S^{k-1})$.

A pair $((X, x_0), X')$, $x_0 \in X' \subset \partial X$, together with a map $\eta : X \to D^k$ is a cellular $k$-disc pair if $X$ (that is $((X, x_0), \eta)$) is a cellular $k$-disc and $\partial X$ is fibered homeomorphic to $X' \times X_0$, that is, there is a homeomorphism $X' \times X_0 \to \partial X$ that sends $\{x'\} \times X_0$ to $X_u$, $u = \eta(x') \in S^{k-1}$. In particular $\eta|_{X'} : X' \to S^{k-1}$ is a homeomorphism. We identify $X'$ with $S^{k-1}$ and say that $(X, S^{k-1})$ is a cellular $k$-disc pair.

Note that it makes sense to say that a map $h : S^{k-1} \to Y$ extends to a cellular $k$-disc pair $(X, S^{k-1})$.

In the proofs of the following two Propositions we use shape theory (see for instance [3], [12]). Recall that the objects of the shape category are pointed spaces, and for two such objects $A$ and $B$ we denote the set of morphisms by $sh\{A, B\}$. There is a functor, the shape functor, from the pointed homotopy category of topological spaces to the shape category. Hence, for each pair of pointed spaces $A$ and $B$ we get a shape map between $[A, B]$, the set of pointed homotopy classes of maps, and $sh\{A, B\}$. In particular there are shape maps from the homotopy groups of $B$ to the homotopy pro-groups of $B$ (these are the shape versions of the homotopy groups of $B$).

Recall that a metric space $Z$ is $LC^m$ if for every $z \in Z$ and $\epsilon > 0$ there is a $\delta > 0$ such that any continuous map $f : P \to B_\delta(z)$, $P$ a locally finite polyhedron of dimension $\leq m$, is homotopic in $B_\epsilon(z)$ to a constant map. And $Z$ is $LC^\infty$ if it is $LC^m$ for every $m$. We will use the following facts:

**Fact 1.** A cell-like map between finite dimensional spaces is a shape equivalence [13].

**Fact 2.** Let $W$ and $Z$ be pointed spaces. Assume $W$ is finite dimensional and $Z$ is sufficiently nice (for instance $Z$ is $LC^\infty$). Then

$$[W, Z] \xrightarrow{\text{shape}} sh\{W, Z\}$$
is a bijection.

Fact 2 follows from the proof of Lemma 3.1 in [3] (the given proof is for homology but the same proof works for homotopy) and the Whitehead Theorem in pro-homotopy (see [2]).

**Proposition 1.1.** Let \((X, S^{k-1})\) be cellular \(k\)-disc pair with \(X/S^{k-1}\) finite dimensional. Let \(f : (X, S^{k-1}) \rightarrow (Z, z_0)\), \(z_0 \in Z\), where \(Z\) is \(LC^\infty\). If \(\pi_k(Z, z_0) = 0\), \(k \geq 1\), then \(f\) is null-homotopic rel \(S^{k-1}\).

**Proof.** Let \(\eta : X \rightarrow \mathbb{D}^k\) be the map that defines the cellular disc \(X\). Write \(W = X/S^{k-1}\). The map \(\eta\) induces a map \(\eta' : W \rightarrow \mathbb{D}^k/S^{k-1} = S^k\). The map \(\eta'\) is a cell-like map because \(\partial X/S^{k-1}\) is homeomorphic to \(S^{k-1} \times X_0/S^{k-1} \times \{x_0\}\), hence contractible. Moreover, by hypothesis, the space \(W\) is finite dimensional. Therefore, by fact 1 above, the map \(\eta'\) is a shape equivalence, that is, an equivalence in the shape category. Consider the following commutative diagram:

\[
\begin{array}{ccc}
[S^k, Z] & \xrightarrow{\text{shape}} & sh\{S^k, Z\} \\
(\eta')^* \downarrow & & \downarrow (\eta')^* \\
[W, Z] & \xrightarrow{\text{shape}} & sh\{W, Z\}
\end{array}
\]

where \((\eta')^*\) is induced by composition with \(\eta'\). By fact 2 above both horizontal arrows are bijections. Also, since \(\eta'\) is a shape equivalence the right vertical arrow is also a bijection. Hence the left vertical arrow is also a bijection. But \([S^k, Z] = \pi_k(Z, z_0) = 0\), therefore \([W, Z]\) consists of a single element. This proves the Proposition.

**Proposition 1.2.** Let \(Z\) be \(LC^\infty\) and \(f : S^{k-1} \rightarrow Z\). If \(f\) extends to a cellular \(k\)-disc pair \((X, S^{k-1})\), then \(f\) extends to \(\mathbb{D}^k\).

**Proof.** Let \(\eta : X \rightarrow \mathbb{D}^k\) be the map that defines the cellular disc \(X\). The map \(\eta\) is a cell-like map hence it induces a isomorphisms of all \(i\)-th homotopy pro-groups. Therefore all of these pro-groups are trivial for \(i > 0\). It follows that the inclusion \(\iota : S^{k-1} \rightarrow X\) represents zero in the \((k-1)\) homotopy pro-group of \(X\). Consequently \(f \iota\) represents zero in the \((k-1)\) homotopy pro-group of \(Z\). By fact 2 above \(f \iota\) represents zero in \(\pi_{k-1}(Z, z_0)\). This proves the Proposition.

**Proposition 1.3.** Every principal (locally trivial) \(S^1\)-bundle over a finite dimensional cellular disc is trivial.

**Proof.** Such bundles are in one-to-one correspondance with \([X, CP^\infty]\), where \(X\) is the cellular disc base space. Consider the following commutative diagram

\[
\begin{array}{ccc}
[X, CP^\infty] & \rightarrow & sh\{X, CP^\infty\} \\
\uparrow & & \uparrow \\
[\mathbb{D}^k, CP^\infty] & \rightarrow & sh\{\mathbb{D}^k, CP^\infty\}
\end{array}
\]

The two horizontal maps are bijections because of Fact 2, and the right hand vertical map is also a bijection because of Fact 1. Since \([\mathbb{D}^k, CP^\infty]\) consists of a single point, so does \([X, CP^\infty]\). This proves
the Proposition.

B. \( C^k \)-Convergence of \( g \)-Geodesics, with Varying \( g \).

Consider Riemannian metrics \( g \) on a fixed manifold. We need to study how \( g \)-geodesics behave when the Riemannian metric \( g \) changes. We are interested in their \( C^k \)-convergence. In this section \( U \) denotes an open set of \( \mathbb{R}^n \).

**Proposition 1.4.** Let \( S = \{ g^a = (g^a_{ij}) \}_{a \in A} \) be a collection of Riemannian metrics on \( U \). Let \( X = \{ x / x \text{ is a unit speed } g^a \text{-geodesic, } a \in A \} \). Assume that the set \( \{ \det g^a(x) / x \in U, a \in A \} \) is bounded away from zero. Then if \( S \) is \( C^k \)-bounded for some finite \( k \geq 0 \), then the set of all derivatives \( \frac{d^{k+1}}{dt}(t), 1 \leq l \leq k + 1, x = (x_1,...x_n) \in X, t \in \text{Domain of } x, \) is bounded.

**Remarks.**

1. The Riemannian metrics in \( S \) are not assumed to be complete.
2. The geodesics in \( X \) are defined on any interval.
3. Here “\( S \) is \( C^k \)-bounded” means that for \( 0 \leq l \leq k \), all \( l \)-partial derivatives of the \( g^a_{ij} \) are bounded.
4. Note that the conclusion of the Lemma is weaker than “\( X \) is \( C^{k+1} \)-bounded” (which implies \( C^0 \)-boundedness). Indeed, if the open set \( U \) is not bounded, then \( X \) is not \( C^0 \)-bounded, hence not \( C^k \)-bounded either.

**Proof.** We denote by \( (g^a_{ij}) \) the matrix inverse of \( g^a = (g^a_{ij}) \). First note that, since \( \{ \det g^a(x) / x \in U, a \in A \} \) is bounded away from zero and \( S \) is \( C^k \)-bounded, we have that all \( l \)-partial derivatives of the \( g^a_{ij} \), \( 0 \leq l \leq k \), are bounded. Moreover the set \( \{ |v| : g^a(x,v,v) = 1, v \in \mathbb{R}^n, x \in U, a \in A \} \) is bounded. (Here \( |v| \) is the Euclidean length \( (v,v)^{1/2} \)) Hence, the set of Euclidean lengths of the velocity vectors of unit speed geodesics is bounded. Therefore, the set

\[
\left\{ \frac{dx_i(t)}{dt}, x(t) = (x_1(t),...,x_n(t)) \text{ is a unit speed } g^a \text{-geodesic, } a \in A, t \in \text{Domain of } x \right\}
\]

is bounded. This proves the Proposition for \( k = 0 \).

Assume \( S \) is \( C^1 \)-bounded. Let \( x(t) = (x_1(t),...,x_n(t)) \) be a unit speed \( g^a \)-geodesic. Then the \( x_i \)'s satisfy a second order ODE of the form

\[
\frac{d^2 x_i}{dt^2} = \Phi \left( \frac{dx_j}{dt}, \Gamma^a_{st}(x) \right)
\]

where \( \Gamma^a_{st} = (\Gamma^a_{st})^a \) are the Christoffel symbols of the metric \( g^a \) and the function \( \Phi \) is a polynomial function independent of \( x \) and \( a \in A \). But the Christoffel symbols \( (\Gamma^a_{st})^a \) can be written canonically as a polynomial expression on the \( g^a_{ij} \) and the first partial derivatives of the \( g^a_{ij} \). Since all these terms are bounded we conclude that the set of all Christoffell symbols \( (\Gamma^a_{st})^a \) is bounded. Therefore the set

\[
\left\{ \frac{d^2 x_i(t)}{dt^2}, x(t) = (x_1(t),...,x_n(t)) \text{ is a unit speed } g^a \text{-geodesic, } a \in A, t \in \text{Domain of } x \right\}
\]

is bounded. This proves the Proposition for \( k > 0 \).
is also bounded. This proves the Proposition for \( k = 1 \).

Assume \( S \) is \( C^2 \)-bounded. We differentiate the geodesic equation above to obtain the third order ODE

\[
\frac{d^3 x_i}{dt^3} = \Psi \left( \frac{dx_i}{dt}, \frac{dx_i^2}{dt^2}, \Gamma^i_{st}(x), \frac{\partial k}{\partial x^i} \Gamma^r_{st}(x) \right)
\]

which is satisfied by any \( x = (x_1, \ldots, x_n) \in X \). Since \( \Psi \) is a universal polynomial, and \( \Psi \) is applied to a set of bounded variables we conclude that the Proposition holds for \( k = 2 \). Proceeding in this way we prove the Proposition for any \( k \geq 0 \). This proves the Proposition.

In the next two Propositions we use the following notation. For a Riemannian metric \( g_0 \) and sequence of Riemannian metrics \( \{g_n\} \) on \( U \) we write \( g_n \xrightarrow{C^k} g_0 \) to express uniform \( C^k \)-convergence on compact supports. Also, for \( p \in U \), \( v \in \mathbb{R}^n \) we denote by \( \alpha(p,v,g) \) the \( g \)-geodesic with value \( p \) at zero, and velocity \( v \) at zero. Also \( \alpha(p_n,v_n,g_n) \xrightarrow{C^k} \alpha(p,v,g_0) \) means convergence on any closed interval \([a,b]\) where all paths are defined. (Note that in this case there is \( \epsilon > 0 \) such that \( \alpha(p,v,g_0) \) and all \( \alpha(p_n,v_n,g_n) \) are defined on \([-\epsilon, \epsilon]\).

**Lemma 1.5.** If \( g_n \xrightarrow{C^1} g_0 \), \( p_n \rightarrow p \), \( v_n \rightarrow v \), then \( \alpha(p_n,v_n,g_n) \xrightarrow{C^1} \alpha(p,v,g_0) \)

**Proof.** \( C^1 \)-Convergence follows from the general theory of first order ODE with parameters. This proves the Lemma.

**Proposition 1.6.** Let \( g_n \xrightarrow{C^k} g_0 \), \( k \geq 1 \), and \( \alpha_n(t), t \in [a,b], \) be \( g_n \)-geodesics such that \( \alpha_n \xrightarrow{C^0} \alpha_0 \). Then \( \alpha_n \rightarrow \alpha_0 \).

**Proof.** It is enough to prove \( C^1 \)-convergence because then the \( C^k \)-convergence, \( k \geq 2 \), follows using the same argument used in the proof of Proposition 1.4 involving the \( \Phi \), \( \Psi \)\ldots functions. But if \( \alpha_n \) does not \( C^1 \)-converge to \( \alpha_0 \) we arrive, using Lemma 1.5, to a contradiction. This proves the Proposition.

C. Sets of Parallel Lines in a Hadamard Manifold.

Let \( H = H^n \) be a Hadamard manifold and \( \mathcal{L} \) a set of parallel geodesic lines in \( H \). We assume that \( \mathcal{L} \) is ribbon convex, i.e. if \( \ell_0, \ell_1 \in \mathcal{L} \) then \( \ell \in \mathcal{L} \), for every \( \ell \) contained in the flat ribbon bounded by \( \ell_0 \) and \( \ell_1 \) (for the existence of the flat ribbon see [11]). Write \( L = \bigcup \mathcal{L} \). The Flat Ribbon Theorem of A. Wolf [11] implies that \( L \) is a convex set. We choose one of the two points at infinity determined by any \( \ell \in \mathcal{L} \). This choice “orients” all lines \( \ell \in \mathcal{L} \) and we can now make the real line \( \mathbb{R} \) act isometrically on \( L \) by translations: for \( t \in \mathbb{R} \) and \( p \in \ell \in \mathcal{L} \), \( t.p = q \), where \( q \in \ell \), and \( q \) is obtained from \( p \) by a \( t \)-translation.

Now, fix \( p \in \ell_0 \subset L \). Let \( \ell \in \mathcal{L} \). Since \( \ell_0 \) and \( \ell \) bound a flat ribbon there is a unique point \( p_\ell \in \ell \) which is the closest to \( \ell_0 \) and the geodesic segment \([p,p_\ell]\) is perpendicular to both \( \ell_0 \) and \( \ell \). Write \( K = \{ p_\ell \mid \ell \in \mathcal{L} \} \). Note that \( K \cap \ell = p_\ell \).
Proposition 1.7. The set $K$ is convex.

This lemma is proved in [6].

Consider the map $K \times \mathbb{R} \to L$, $(p, t) \mapsto t.p_\ell$. Since $K$ is convex this map is an isometry, where we consider $K \times \mathbb{R}$ with the metric $d((p, t), (p', t')) = \sqrt{d_H(p, p')^2 + d_\mathbb{R}(t, t')^2}$. The inverse of this map is the map $(\pi, T)$, where $\pi(x) = p_\ell$, $x \in \ell$, is the projection onto $K$, and $T(x)$ is the (oriented) distance between $x$ and $p_\ell$.

Corollary 1.8. Assume $L$ is a closed subset of $H$ and that $K$ is compact. Then $K$ is homeomorphic to the closed $k$-disc with smoothly (locally) totally geodesic embedded interior.

Proof. Proposition 1.7 and Theorem 1.6 of [1], p. 418, imply that $K$ is homeomorphic to a compact, contractible $k$-manifold, $0 \leq k \leq n - 1$. Moreover the inclusion $K \hookrightarrow H$ restricted to the (manifold) interior of $K$ is smooth and (locally) totally geodesic. This proves the corollary.

D. Sets of Homotopic Closed Geodesics.

Let $Q = S^1 \times \mathbb{R}^{n-1}$, with a complete nonpositively curved Riemannian metric $g$. Write $\iota : S^1 = S^1 \times \{0\} \hookrightarrow Q$ for the inclusion and

$$\Omega = \{ \alpha \in C^\infty(S^1, Q) / \alpha \simeq \iota \}$$

with the $C^k$ topology, $0 \leq k \leq \infty$. Note that $S^1$ acts freely on $\Omega$ by $z.\alpha(w) = \alpha(zw)$, for $z, w \in S^1 \subset \mathbb{C}$. Write $\Sigma = \Omega/S^1$. It is straightforward to verify that the quotient map $\Omega \to \Sigma$ is a (locally trivial) principal $S^1$-bundle. Note that $\mathbb{R}$ also acts on $\Omega$ by $z = x.\alpha(w) = \alpha(e^{2\pi ix}w)$, for $x \in \mathbb{R}$, $w \in S^1$. Moreover, we also get $\Omega/\mathbb{R} = \Sigma$. Let $\mathcal{C} = \mathcal{C}_g$ be the set of all parametrized closed geodesics homotopic to the inclusion, i.e.

$$\mathcal{C} = \{ \alpha \in \Omega / \alpha \text{ is a } g \text{-geodesic} \}$$

Note that every $\alpha \in \mathcal{C}$ is an embedding. Moreover, by the Flat Ribbon Theorem of J. A. Wolf [11] any two elements in $\mathcal{C}$ either have the same image (hence lie in the same $S^1$-orbit) or have disjoint images.

Let $\mathcal{A} = \mathcal{A}_g$ be the image of $\mathcal{C}$ by the bundle map $\Omega \to \Sigma$. That is, $\mathcal{A}$ is the set of all unparametrized closed geodesics homotopic to the inclusion. Assume that

(a) $\mathcal{C}$ is non-empty.
(b) $\mathcal{C}$ is $C^0$-bounded. Equivalently, the set $C = \bigcup \mathcal{C}$ is contained in a compact set.

Proposition 1.9. Under these assumptions $\mathcal{A}$ is homeomorphic to a closed $k$-disc, $k \leq n - 1$.

Remark. All topologies $C^k$, $0 \leq k \leq \infty$, induce the same topology on $\mathcal{C}$ and $\mathcal{A}$.

Proof. Let $H$ be the universal cover of $Q$. Then $H$ is a Hadamard manifold, the infinite cyclic group $\mathbb{Z}$ acts freely by isometries on $H$ and $Q = H/\mathbb{Z}$. Let $\mathcal{L}$ be the set of all lines in $H$ which cover elements
in $\mathcal{A}$; i.e. all lifts to $H$ of unparametrized closed geodesics homotopic to $\iota$. It is straightforward to check that $\mathcal{L}$ is ribbon-convex and $L = \bigcup \mathcal{L}$ is closed. Let $K$ be constructed from $\mathcal{L}$ as in section C. Using projections we can construct, in the obvious way, a one-to-one, onto, $\mathcal{A}$, continuous map $K \to \mathcal{A}$. This proves the Proposition.

**Proposition 1.10.** The space $\Omega$ deformation retracts to $\mathcal{C}$.

**Proof.** Let $\Omega, Q$ denote the space of all based loops which are based homotopic to $\iota$. Then we have a fibration $\Omega, Q \to \Omega \to Q$, where the last map is the evaluation map (at, say, $1 \in S^1$). Since $\Omega, Q$ is contractible and $Q \sim S^1$ it follows that $\mathcal{C} \to \Omega$ is a homotopy equivalence. This proves the proposition.

Since $\mathcal{A}$ is a disc, the bundle $\Omega \to \Sigma$ restricted to $\mathcal{A} \subset \Sigma$ is trivial. Hence $\mathcal{C}$ is homeomorphic to $\mathcal{A} \times S^1$. Let $s : \mathcal{A} \to \mathcal{C}$ be any section of this bundle (equivalently, a lifting of the identity $1_{\mathcal{A}}$). Write $B = s(\mathcal{A}) \subset \mathcal{C} \subset \Omega$.

**Proposition 1.11.** Let $V$ be an open neighborhood of $B$ in $\Omega$. There the is an open neighborhood $U \subset V$ of $B$ in $\Omega$ such that $U$ deformation retracts, inside $V$, to $B$.

**Proof.** Let $h_t, h_0 = 1_{\Omega}, h_1 : \Omega \to \mathcal{C}$, be a deformation retract. Since $\mathcal{C}$ is homeomorphic to $\mathcal{A} \times S^1$, we can find an open neighborhood $W$ of $B$ in $\mathcal{C}$ that deformation retracts to $B$. And we can assume that this deformation retract happens inside $V$. Denote this deformation retract by $f_t$. Let $U$ be an open neighborhood of $B$ in $\Omega$ small enough so that $h_t(U) \subset V$ and $h_1(U) \subset W$. Then our desired deformation retract is the concatenation of the $h_t$ with the $f_t$. This proves the Proposition.

We will need the following Lemma in the next section.

**Lemma 1.12.** Assume the metric $g$ on $Q$ satisfies assumptions (a) and (b) above. Let $\mathcal{L}$ be the length of a (hence all) closed $g$-geodesic homotopic to $\iota$. Then there is a bounded set $R \subset Q$ such that if the image of an $\alpha \in \Omega$ is not contained in $R$ then the $g$-length of $\alpha$ is larger than $1 + \mathcal{L}$.

**Proof.** Suppose not. Then there is a sequence $\alpha_n$ in $\Omega$, such that: (1) $x_n = \alpha_n(1)$ goes to infinity and (2) all $\alpha_n$ have length $\leq 1 + \mathcal{L}$. Fix $\alpha_0 \in \mathcal{C}$ and write $x = \alpha_0(1)$. Let $s_n$ be a geodesic segment $[x, x_n]$ such that $d(x, x_n)$ is its length and write $s_n(t) = \text{exp}_x(tv_n)$, for some unit length vector $v_n \in T_xQ$. We can assume $v_n \to v$, where $v$ also has unit length. Write $s(t) = \text{exp}_x(tv)$. Let $H$ be the universal cover of $Q$. Fix a lift $\beta_0 : \mathbb{R} \to H$ of $\alpha_0$. Write $y = \beta_0(0)$ and $z = \beta_0(1)$. Let $s'_n, s'$ be liftings of $s_n$ and $s$ beginning at $y$ and $s''_n, s''$ be liftings of $s_n$ and $s$ beginning at $z$, respectively. Note that the endpoints of $s'_n$ and $s''_n$ can be joined by a lifting of $\alpha_n$, hence their distance lies in the interval $[\mathcal{L}, 1 + \mathcal{L}]$. Therefore $d_H(s'_n(t), s''_n(t)) \in [\mathcal{L}, 1 + \mathcal{L}]$. It follows that $d_H(s'(t), s''(t)) \in [\mathcal{L}, 1 + \mathcal{L}]$ for all $t \geq 0$. But the function $t \mapsto d_H(s'(t), s''(t))$ is convex with minimum value at $t = 0$, thus it cannot be a bounded function unless it is constant. But this contradicts assumption (b). This proves the Lemma.

E. Sets of Homotopic Closed $g$-Geodesics, with Varying $g$.

Let $Q, \Omega, \Sigma$ be as in section D. We denote by $\mathcal{MET}^{\sec \leq 0}(Q)$ the space of all complete nonpositively...
curved Riemannian metrics on $Q$, with the weak smooth topology (i.e., the union of the weak $C^s$ topologies, which are the topologies of the $C^s$-convergence on compact sets). Let $\sigma : \mathbb{D}^k \to \mathcal{MET}^{sec \leq 0}(Q)$ be continuous. Write $g_u = \sigma(u)$. Using the methods and the notation of section D, for each $g_u$ we obtain $A_u, B_u, C_u$. Write $C_u = \bigcup C_u$. In what follows we assume that all $g_u$ satisfy assumptions (a) and (b) of section D. In particular, for each $u$ we get a positive number $L(u)$ which is the length of an element in $C_u$, that is, the length of a $g_u$-geodesic homotopic to the inclusion $S^1 \to Q$.

**Lemma 1.13.** The map $L : \mathbb{D}^k \to (0, \infty)$ is upper semi-continuous.

**Proof.** This follows from the following facts: (1) $L(u)$ is the smallest possible length of a curve homotopic to the inclusion $S^1 \to Q$, and (2) for any closed curve $\alpha$, the $g_u$-length of $\alpha$ is close to the $g_u$-length of $\alpha$, provided $u$ is close to $v$. This proves the Lemma.

**Lemma 1.14.** Let $u_n \to u$ in $\mathbb{D}^k$. Then there is a compact set $S$ of $Q$ and a sequence $\alpha_n \in \mathcal{C}_n$ such that $\alpha_n \subset S$, for $n$ sufficiently large.

**Proof.** Let $R$ be as in lemma 1.12 for $g = g_u$ and assume that $R$ is closed. Let $S$ be any compact of $Q$ with $R \subset \text{int} S$. We claim that there is a sequence $\alpha_n \in \mathcal{C}_n$ such that $\alpha_n \subset S$, for $n$ sufficiently large. This would prove the lemma. Suppose not. Then we can assume, by passing to a subsequence, that for every $\alpha_n \in \mathcal{C}_n$ we have $\alpha_n \not\subset S$. Write $L = L(u), g_n = g_{u_n}$ and let $\alpha \in \mathcal{C}_u$. Then the $g$-length $L_g(\alpha)$ of $\alpha$ is $L$. Therefore $L_{g_n}(\alpha_n)$ is close to $L$. For each $n$ let $\alpha_n^t$ be a homotopy with $\alpha_n^0 = \alpha, \alpha_n^1 \in \mathcal{C}_{u_n}$ and $L_{g_n}(\alpha_n^t) \leq L_{g_n}(\alpha_n^s)$, for $t > s$. That is the deformation $t \mapsto \alpha_n^t$ begins in $\alpha$, ends in a $g_u$-geodesic, and is $g_u$-length non-increasing. (Such a deformation can be done in the usual way using evolution equations or using a polygonal deformation.) Note that, by hypothesis, $\alpha_n^1 \not\subset S$ and $\alpha = \alpha_n^0 \subset R \subset S$. This together with the continuity of the deformation imply that there is $s = s_n$ such that $\beta_n = \alpha_n^s \not\subset R$ and $\beta_n \subset S$. But lemma 1.12 together with the convergence $g_n \to g$ imply that $L_{g_n}(\beta_n) > 1/2 + L$ when $n$ is sufficiently large (see remark below). This contradicts the fact that the deformation $t \mapsto \alpha_n^t$ is $g_u$-length non-increasing. This proves the lemma.

**Remark.** In the proof above we are using the following fact: if $|g_n - g|_g \leq \delta$ then for any PD path $\alpha$ we have $|L_{g_n}(\alpha) - L_g(\alpha)| \leq \frac{\delta}{1 - \delta}L_{g_n}(\alpha)$. This fact follows from the definition of length (using integrals) and the triangular inequality.

**Corollary 1.15.** The map $L : \mathbb{D}^k \to (0, \infty)$ is continuous.

**Proof.** Let $u_n \to u$ and $S$ be as in 1.14. Hence there are $\alpha_n \in \mathcal{C}_{u_n}$ with $\alpha_n \subset S$. Since $g_{u_n} \to g_u$ uniformly on $S$ we get from lemma 1.13 (and the remark above) that $L(u_n) = L_{g_{u_n}}(\alpha_n)$ is close to $L_{g_u}(\alpha_n) \geq L(u)$. This shows $L$ is lower semi-continuous. This proves the corollary.

**Proposition 1.16.** The set $\bigcup_{u \in \mathbb{D}^k} \mathcal{C}_u$ is $C^0$-bounded.

That is, the set of all $g_u$-geodesics lie at bounded distance from, say, the inclusion $\iota$, for all $u \in \mathbb{D}^k$.

**Proof.** Suppose not. Then there are $u_n \to u$ and $\alpha_n' \in \mathcal{C}_n = \mathcal{C}_{u_n}$ with $\alpha_n'$ going to infinity, i.e., $\alpha_n' \not\subset K$, for any given compact $K$, provided $n$ is large. Let $S$ be as in 1.14. Hence there are $\alpha_n \in \mathcal{C}_{u_n}$ with $\alpha_n \subset S$. Let $S'$ be a compact such that $S \subset \text{int} S'$. Since $\alpha_n$ and $\alpha_n'$ bound a flat two dimensional cylinder (in the $g_n = g_{u_n}$ metric) we can find $\beta_n \in \mathcal{C}_n$ with $\beta_n \in S'$ and $\beta_n \not\subset S$. By corollary 1.15 we
can assume $\mathcal{L}(u_n) \leq 1/2 + \mathcal{L}(u)$. On the other hand, by 1.12 and the uniform convergence $g_n \to g_u$ on $S'$ we have $\mathcal{L}(u_n) = t_{g_n(\beta_n)}$ is close to $1 + \mathcal{L}(u)$. This is a contradiction. This proves the proposition.

**Proposition 1.17.** The set $\bigcup_{u \in \mathbb{D}^k} \mathcal{C}_u$ is $C^k$-bounded, for any $k$, $0 \leq k < \infty$.

**Proof.** The Proposition follows from Proposition 1.4 by considering $Q$ as an open set of $\mathbb{R}^n$. Note that, by 1.16, we can work on an open set with compact closure, hence all required quantities will be bounded. Note also that in 1.4 the geodesics are assumed to have speed one, but the geodesics in $\mathcal{C}_u$ have speed $\mathcal{L}(u)/2\pi$. This can be fixed by a rescaling of geodesics and using the fact that (by 1.15) the set $\{\mathcal{L}(u)\}_{u \in \mathbb{D}^k}$ is bounded and bounded away from zero. This proves the proposition.

**Proposition 1.18.** Let $V$ be an open neighborhood of $\mathcal{C}_u$ in $\Omega$. Then, for $u' \in \mathbb{D}^k$ close enough to $u$, $\mathcal{C}_{u'} \subset V$.

**Proof.** Suppose not. Then there are $u_n \to u$ and $\alpha_n \in \mathcal{C}_{u_n}$ with $\alpha_n \notin V$. Since $u_n \to u$, Proposition 1.17 (recall we are using the weak smooth topology) implies that the set $\{\alpha_n\}$ is $C^0$-equicontinuous, where we consider $Q$ with metric $g_u$. Moreover we can assume all $\alpha_n$ to be Lipschitz with the same constant. Proposition 1.17 also says that the set $\{\alpha_n\}$ is bounded. By Arzela-Ascoli theorem we can assume the $\alpha_n$ $C^0$-converge to a Lipschitz $\alpha : \mathbb{S}^1 \to Q$. Since $\alpha$ is Lipschitz its length is finite (the length defined as $\sup \sum g_u(\alpha(z_i), \alpha(z_{i+1}))$, the sup taken over all partitions of $\mathbb{S}^1$). Moreover it is straightforward to show that $g_{u_n}$-lengths of the $\alpha_n$ converge to the $g_u$-length of $\alpha$. Proposition 1.15 implies now that $\alpha$ has minimal $g_u$-length, hence it is smooth and $\alpha \in \mathcal{C}_u$. Now using proposition 1.6 we see that $\alpha_n \in V$ for $n$ large enough. This contradiction proves the proposition.

Define $Y = \prod_{u \in \mathbb{D}^k} \{u\} \times A_u \subset \mathbb{D}^k \times \Sigma$, that is $Y = \{ (u, a) \mid u \in \mathbb{D}^k, a \in A_u \}$. Define also $Z = \prod_{u \in \mathbb{D}^k} \{u\} \times \mathcal{C}_u \subset \mathbb{D}^k \times \Omega$. Each $C^k$-topology on $\Omega$, $0 \leq k \leq \infty$, induces a $C^k$-topology on $Z$.

**Proposition 1.19.** All $C^k$-topologies on $Z$ coincide.

**Proof.** This follows from Proposition 1.6. This proves the Proposition.

**Proposition 1.20.** The space $Z$ is compact and metrizable.

**Proof.** The space $Z$ is certainly metrizable. Let $\{(u_n, \alpha_n)\}$ be a sequence in $Z$. We can assume $u_n \to u$. By Proposition 1.17, the sequence $\{\alpha_n\}$ is $C^k$-bounded, for all $k \geq 0$. In particular it is $C^1$-bounded. Therefore the sequence $\{\alpha_n\}$ is equicontinuous. Moreover we can assume all $\alpha_n$ to be Lipschitz with the same constant. Proposition 1.16 says that the set $\{\alpha_n\}$ is $C^0$-bounded. By Arzela-Ascoli Theorem we can assume that $\{\alpha_n\}$ $C^0$-converges to a Lipschitz $\alpha \in \Omega$. Since $\alpha$ is Lipschitz its length is finite, where the length is defined as $\sup \sum g_u(\alpha(z_i), \alpha(z_{i+1}))$, the sup taken over all partitions of $\mathbb{S}^1$. Using this definition of length it is straightforward to show that $g_{u_n}$-lengths of the $\alpha_n$ converge to the $g_u$-length of $\alpha$. Corollary 1.15 implies now that $\alpha$ has minimal $g_u$-length, hence it is smooth and $\alpha \in \mathcal{C}_u$. Therefore $\{(u_n, \alpha_n)\}$ converges to $(u, \alpha) \in Z$. This proves the Proposition.

It follows from Proposition 1.6 that $Y$ is compact. The space $Y$ is also Hausdorff. Therefore the projection $Y \to \mathbb{D}^k$ is a cellular $k$-disc (choose any base point).

**Proposition 1.21.** The space $Y$ is finite dimensional.
**Proof.** Since $Y$ is Hausdorff compact and $Z \to Y$ is locally trivial, the proposition follows from the following claim.

**Claim.** Let $U \subset Y$ be compact and such that $Z \to Y$ is trivial over $U$. Then $U$ is homeomorphic to a compact subset of $\mathbb{R}^{n+k}$. Hence $U$ is finite dimensional.

**Proof of the claim.** Let $U' \subset Z$ be the image of a section of $Z \to Y$ over $U$. Hence the restriction $U' \to U$ is a homeomorphism. As before we are considering $Q$ as an open set of $\mathbb{R}^n$. Now, just define $h: U' \to \mathbb{R}^{k+n}$ as $h(u, \alpha) = (u, \alpha(1))$. (Recall $1 \in S^1 \subset \mathbb{C}$ and $\alpha: S^1 \to Q$.) This is a one-to-one continuous map with compact domain between metric spaces. Hence it is a homeomorphism onto its image. This proves the claim and Proposition 1.21.

By Propositions 1.3 and 1.21 the $S^1$-bundle $Z \to Y$ is trivial (thus $Z$ is homeomorphic to $Y \times S^1$). Take a section $Y \to Z$ of this bundle and let $X$ be the image of this section. Write $\eta: X \to \mathbb{D}^k$ for the projection. Then $\eta: X \to \mathbb{D}^k$ is a cellular $k$-disc. Note that $X$ is formed by honest parametrized $g_u$-geodesics, not unparametrized ones, like the ones in $Y$. And since $X$ and $Y$ are homeomorphic, Proposition 1.21 has the following corollary.

**Corollary 1.22.** The space $X$ is a finite dimensional space.

Recall that the cellular discs $Y$ and $X$ were constructed from a map $\sigma: \mathbb{D}^k \to \mathcal{MET}^{sec \leq 0}(M)$. Now assume that $\sigma|_{S^{k-1}}$ is constant mod $DIFF(M)$; that is $\sigma|_{S^{k-1}}$ factors through the map $\Lambda_{g_0}$ in $\mathbb{S}$. (This map is just the orbit map $\Lambda_{g_0}: DIFF(M) \to \mathcal{MET}^{sec \leq 0}(M)$ given by $\phi \mapsto \phi \ast g_0$.) Hence for $u \in S^{k-1}$ we can write $\sigma_u = (\phi_u)_* g_0$, for some continuous $u \mapsto \phi_u$, and it follows that $A_u = \phi_u (A_{u_0})$, $u \in S^{k-1}$. Therefore we can write $\partial Y = S^{k-1} \times A_{u_0}$ and we can consider $S^{k-1} \subset Y$ by choosing any element in $A_{u_0}$. Analogously for $X$. Hence we obtain cellular $k$-discs pairs $(Y, S^{k-1})$, $(X, S^{k-1})$.

**Corollary 1.23.** Assume $\sigma|_{S^{k-1}}: S^{k-1} \to \mathcal{MET}^{sec \leq 0}(M)$ is constant mod $DIFF(M)$. Then the space $X/S^{k-1}$ is a finite dimensional space.

**Proof.** From the proof of 1.21 we have that the embedding $h$ induces an embedding $X/S^{k-1} \to \mathbb{R}^{k+n}/h(S^{k-1})$ which is clearly finite dimensional.

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**Section 2. Proof of the Main Theorem.**

In this section we shall use the notation and results given in $\mathbb{S}$ to prove the nonpositively curved version of the Main Theorem in $\mathbb{S}$. In turn this version follows from the nonpositively curved version of Theorem 1 of $\mathbb{S}$ together with Theorem 2 of $\mathbb{S}$. Right before Theorem 1 of $\mathbb{S}$ the following diagram is given:
\[ DIFF(\mathbb{S}^1 \times \mathbb{S}^{n-2}) \times I, \partial) \xrightarrow{\Phi} DIFF(M) \xrightarrow{\Lambda_g} \mathcal{MET}^{\sec < 0}(M) \]

\[ \downarrow \]

\[ P(\mathbb{S}^1 \times \mathbb{S}^{n-2}) \]

where \( P(N) \) is the space of topological pseudoisotopies of a manifold \( N \). In [8] it is proved that, under the relevant conditions, \( \text{Ker}(\pi_k(\Lambda_g \Phi)) \subset \text{Ker}(\pi_k(\iota)) \). Before we present the nonpositively curved version of this result we need some notation and definitions.

For a manifold \( L \), \( \text{TOP}(L) \) is the space of all self homeomorphisms of \( L \), \( \text{CELL}(L) \) is the space of cellular maps and \( C(L) \) is the space of all continuous \( L \to L \), with the compact-open topology.

**Remark.** If the manifold \( L \) has boundary then for a map \( f : L \to L \) to be cellular we demand the restriction \( f|_{\partial L} : \partial L \to \partial L \) to be cellular too. See [14].

**Lemma 2.1.** The map \( \pi_k P(N) \to \pi_k \text{TOP}(N \times [0, 1]) \), \( k \geq 0 \), is injective.

**Proof.** Let \( \alpha : \mathbb{S}^k \to P(N) \), \( \beta : \mathbb{D}^{k+1} \to \text{TOP}(N \times [0, 1]) \), with \( \beta|_{\mathbb{S}^k} = \alpha \). For \( f \in \text{TOP}(N \times [0, 1]) \) write \( f_0 : N \to N \) for its bottom, that is, for its restriction to \( N \times \{0\} \). Define \( \gamma : \mathbb{D}^{k+1} \to \text{TOP}(N \times [0, 1]) \) by \( \gamma(u) = (\beta(u)_{0})^{-1} \times 1_{[0,1]} \). Note that \( \gamma(u) = 1_{N \times [0,1]} \) for \( u \in \mathbb{S}^k \). Finally define \( \beta' : \mathbb{D}^{k+1} \to \text{TOP}(N \times [0, 1]) \) by \( \beta'(u) = \gamma(u) \beta(u) \). Then \( \beta'|_{\mathbb{S}^k} = \alpha \). This proves the Lemma because \( \beta' : \mathbb{D}^{k+1} \to P(N) \).

**Lemma 2.2.** Let \( N \) be compact and \( \dim N \neq 3 \). Then the map \( \pi_k \text{TOP}(N \times [0, 1]) \to \pi_k \text{CELL}(N \times [0, 1]) \), \( k \geq 0 \), is an isomorphism.

This is a fiber version of Siebenmann result [14]. The proof follows from proposition 4.1 of B. Haver [10] together with the fact that the closure of \( \text{TOP}(L) \) is \( \text{CELL}(L) \), \( \dim L \neq 4 \), proved by Siebenmann [14]. In the Lemma above (and the Corollary below), for the case \( k = 0 \) “isomorphism” means “bijection”. These two Lemmas imply:

**Corollary 2.3.** Let \( N \) be compact and \( \dim N \neq 3 \). Then the map \( \pi_k P(N) \to \pi_k \text{CELL}(N \times [0, 1]) \), \( k \geq 0 \), is injective.

Now, assume \( M \) is closed and admits a nonpositively curved metric \( g \). Here is the new version of the diagram above:
The map Lemma 2.5. most important change to be made here is a new version of fact 9: definition using quasi-geodesics. And this definition coincides with the definition using geodesics. The $Q$ is positively curved manifold. But, since we are assuming

There are several facts stated for

proposition 4.1 of [10] and the main results in [14] and [5].

We will use the fact that

Remark. We will use the fact that $CELL(L)$ is $LC^\infty$ for a compact $L$, $dim L \neq 3$. This follows from proposition 4.1 of [10] and the main results in [14] and [5].

Changes to Section 2 of [8].

There are several facts stated for $Q$ in section 2 of [8] that are clearly not true for a general non-positively curved manifold. But, since we are assuming $\pi_1 M$ word hyperbolic we can assume that $\tilde{Q}$ is $\delta$-hyperbolic (in the sense of Gromov), hence the definition of $\partial_\infty \tilde{Q}$ remains valid, that is the definition using quasi-geodesics. And this definition coincides with the definition using geodesics. The most important change to be made here is a new version of fact 9:

Lemma 2.5. The map $\tilde{A} : \tilde{N} \to \partial_\infty \tilde{Q} \setminus \partial_\infty \tilde{S}$, given by $\tilde{A}(v) = [c_v]$ is cellular. Furthermore, we can extend $\tilde{A}$ to $\tilde{W} \to (\tilde{Q} \setminus \partial_\infty \tilde{S})$ by defining $\tilde{A}(v) = \tilde{E}(\tilde{\zeta}(|v|)\frac{v}{|v|}) = \exp_\tilde{q}(\tilde{\zeta}(|v|)\frac{v}{|v|})$, for $|v| < 1$, $v \in \tilde{W}_q$. This extension is cellular and a homeomorphism on $\tilde{W} \setminus \tilde{N}$.

Proof. Standard Hadamard manifold techniques show that the map $\tilde{A} : \tilde{N} \to \partial_\infty \tilde{Q} \setminus \partial_\infty \tilde{S}$ is continuous. Let $v \in \tilde{N}_p$, $p \in \tilde{S}$. We now prove that $C = \tilde{A}^{-1}(\tilde{A}(v))$ is homeomorphic to a convex set in $\tilde{S}$. First note that for $v, v' \in \tilde{N}_p$ we have $\tilde{A}(v) \neq \tilde{A}(v')$ because $c_v$ and $c_v'$ are two geodesic rays emanating from the same point. Hence the continuous map $\pi|_C : C \to \pi(C)$ is injective. (Recall $\pi : \tilde{N} \to \tilde{S}$ is the bundle projection).

Let $v' \in \tilde{N}_{p'}$, $p' \neq p$, be such that $\tilde{A}(v') = \tilde{A}(v)$. Let $[p, p']$ be the unique geodesic segment joining $p$ to $p'$. Since the geodesic rays $c_v$, $c_{v'}$ make a right angle with $\tilde{S}$, at $p$ and $p'$ respectively, we have that $c_v$, $c_{v'}$ and $[p, p']$ bound a flat geodesic ribbon. Hence $[p, p'] \subset \pi(C)$ and it follows that $\pi(C)$ is
Lemma 2.6. The map $\tilde{A}$ is onto and proper.

Proof. Let $p \in \tilde{S}$ and $\alpha$ a geodesic ray emanating from $p$ and not contained in $\tilde{S}$. We have to show that there is a $c_v$ (emanating from some $q \in \tilde{S}$ with direction $v$ perpendicular to $\tilde{S}$) such that $\alpha$ and $c_v$ determine the same point in $Q$. Let $c_{v_n}$, at $q_n$, be perpendicular to $\tilde{S}$ and passing through $\alpha(n)$. Using the $\delta$-hyperbolicity of $\tilde{Q}$ we get that $\{q_n\}$ is bounded so we can assume $q_n \to q$. Furthermore we can assume $v_n \to v$. It is straightforward to verify that $c_v$ and $\alpha$ determine the same point at infinity. This proves that $\tilde{A}$ is onto.

We identify $\partial \tilde{Q}$ with the unit sphere $S$ in $T_p \tilde{Q}$, for some $p \in \tilde{S}$. Let $K \subset S - T_p \tilde{S}$ be compact. We now prove $\tilde{A}^{-1}(K)$ bounded. Note that if $\alpha$ is a ray emanating from $p$ with direction $\alpha'(0) \in K$, then the angle between $\alpha'(0)$ and $\tilde{S}$ is bounded away from zero, hence there is a $\kappa > 0$ such that $d(\alpha(t), \tilde{S}) \geq \kappa t$, for every such $\alpha$. Let $v \in \tilde{A}^{-1}(K)$, with $v \in T_p \tilde{S}$. Then $[c_v] = [\alpha]$, for some $\alpha$ as above. Using the $\delta$-hyperbolicity of $\tilde{Q}$ we get that $d(p, q)$ cannot be arbitrarily large. This proves that $\tilde{A}^{-1}(K)$ is bounded. This proves the lemma.

Lemma 2.7. The injectivity radius at $p \in \tilde{Q}$ tends to infinity, as $p$ gets far from $S$.

Proof. Suppose not. Let $\gamma_n$ be non-contractible loops in $S$ with $d(\gamma_n, S) = n$ and the lengths $\ell(\gamma_n)$ bounded (say by $a > 0$). Each $\gamma_n$ is homotopic to a closed geodesic $\beta_n$ in $S$. Lifting to $\tilde{Q}$ we obtain $p_n, p'_n \in \tilde{S}$ and vectors $v_n, v'_n$ such that $d(c_{\gamma_n(n)}, c_{\gamma_n'(n)}) \leq a$ and $b \leq d(p_n, p'_n) \leq a$, where $b > 0$ is the injectivity radius of $S$. Since $\tilde{S}$ has a compact fundamental domain we can assume $p_n \to p$, $p'_n \to p'$, $v_n \to v$ and $v'_n \to v'$. It is straightforward to check that $c_v$ and $c_{v'}$ bound an infinite flat (half) ribbon. We can repeat this process with the set $\{\gamma_n^k\}$, where $\alpha^k = \alpha \ast \ast \ast \alpha$ (concatenation $k$ times). In this way we get that $\tilde{Q}$ contains flat geodesic ribbons isometric to $[0, b] \times [0, \infty)$ with $b \to \infty$ which would contradict the $\delta$-thinness of triangles in the $\delta$-hyperbolic space $\tilde{Q}$. This proves the Lemma.

Lemma 2.2 of [8] remains true. In the proof Lemma 2.6 above is used and there is no need for some of the (now not valid) facts mentioned in section 2 of [8]. We can now descend to $Q$ and define, as in Lemma 2.4 of [8], the map $A$:

Lemma 2.8. The map $A : N \to \partial_{\infty}Q$, given by $A(v) = [c_v]$ is cellular. Furthermore, we can extend $A$ to $W \to \partial_{\infty}Q \cup Q$ by defining $A(v) = E((\varsigma(|v|), \frac{v}{|v|}))$, for $|v| < 1$. This extension is cellular and a homeomorphism on $W \setminus N$.

Proof. Since $\tilde{A}$ covers $A$, it is enough to prove that $\gamma C \cap C = \emptyset$, for $\gamma \in \Gamma$ and $C$ as in the proof of Lemma 2.5. Suppose there is $v \in \tilde{N}_p$ with $A(\tilde{v}) = A(\gamma_{\ast}(v))$, for some nontrivial $\gamma \in \Gamma$. Note that $\gamma_{\ast}(v) \in \tilde{N}_{\gamma_{\ast}(p)}$ and $\gamma(p) \neq p$. But then $A(v) = A(\gamma_{\ast}^n(v))$, $\gamma_{\ast}^n(v) \in \tilde{N}_{\gamma_{\ast}^n(p)}$ and the distance between $\gamma_{\ast}^n(p)$ and $p$ becomes large. Therefore we again obtain large geodesic flat ribbons in $\tilde{Q}$, which cannot happen. This proves the Lemma.
Changes to Section 3 of [8] and Proof of Theorem 2.4.

1. In the first three paragraphs replace $\mathcal{MET}^{sec<0}(M)$ by $\mathcal{MET}^{sec\leq0}(M)$ and delete the second sentence of the fourth paragraph.

2. The remark at the beginning of section 3 will not be needed because the embedding results 1.1-1.4 in section 1 of [8] will not be used directly here. A new argument is given in item 7 below that does not use (directly) these embedding results (even though they are used indirectly: the fiber of the map $Emb(S^1, Q) \hookrightarrow C^\infty(S^1, Q)$ is $(n - 5)$-connected, which follows from 1.4 of [8]).

3. On item (vi) take $c_2 = 0$.

4. In the paragraph after item (vi) choose $\alpha_0$ to be one $\sigma(u_0)$-geodesic homotopic to $\alpha$. Also, for $u \in S^k$, define $\alpha_u = \phi_u(\alpha_0)$. By applying lemma 1.4 of [8] (write $S^k = \mathbb{D}^k/\partial\mathbb{D}^k$) we can assume $\alpha_u = \alpha_0 = \alpha$. Hence, for every $u \in S^k$, $\alpha$ is a $\sigma(u)$-geodesic.

5. The function $h$ defined after item (iv') cannot be defined in our case, but facts 1 and 2 still hold.

6. The proof now continues as follows. Let $X$ be the cellular disc constructed in section 1E. Since $\sigma$ is constant mod $DIFF(M)$ on $\partial\mathbb{D}^k$ we also get that $(X, S^k)$ is a cellular $(k + 1)$-disc pair (see paragraph before 1.23). Recall that the elements of $X$ are pairs $(u, \beta)$, $u \in \mathbb{D}^{k+1}$ and $\beta$ is a $\sigma(u)$-geodesic. We consider $S^k \subset X$ by identifying $u \in S^k = \partial\mathbb{D}^{k+1}$ with $(u, \alpha)$. Now, the function $h$ in [8] is replaced here by $h : X \to Emb(S^1, Q) \subset \Omega = C^\infty(S^1, Q)$, $h(u, \beta) = \beta$. Note that $h(S^k) = \{\alpha\}$.

7. After item (vi) the proof continues as follows. Since $\pi_k(C^\infty(S^1, Q)) = 0$, $k > 1$, and the fiber of $Emb(S^1, Q) \hookrightarrow C^\infty(S^1, Q)$ is $(n - 5)$-connected (this follows from Lemma 1.4 of [8]) we have also that $\pi_k(Emb(S^1, Q)) = 0$. This together with Proposition 1.1 and the fact $Emb(S^1, Q)$ is locally contractible imply that $h : (X, S^k) \to (Emb(S^1, Q), \alpha)$ is relative null-homotopic (i.e the homotopy always sends $S^k$ to $\{\alpha\}$). Hence for each $x = (u, \beta) \in X$ there is an isotopy $h_t(x) : S^1 \to Q$ such that $h_0(x) = \beta$ and $h_1(x) = \alpha$.

Remark. By modifying $X$ a little bit we can also assume the above to be true for $k = 0$: consider the map $X \to Q$, $(u, \beta) \mapsto \beta(1)$. If this map represents $n \in \mathbb{Z} = \tilde{H}_1(S^1) = \tilde{H}_1(Q)$ then just replace $X$ by $\{(x, e^{-n\pi\eta(u)}\beta) \mid (u, \beta) \in X\}$.

Write $\sigma(u, \beta) = \sigma(u)$ (we use the same letter). Hence we are now considering $X$ to be the domain of $\sigma$. Now claim 1 of section 3 of [8] makes sense. The proof is done by extending the isotopies $h_t(x)$ to ambient isotopies as in Lemma 1.4 of [8], and taking the pullback of $\sigma(u)$ by the end of these isotopies.

To complete the proof replace $\mathbb{D}^{k+1}$ by $X$ and $u$ by $x = (u, \beta)$ throughout. Using the same procedure used in [8] to construct, for each $u \in \mathbb{D}^{k+1}$, an $f_u \in P(S^1 \times S^{n-2})$, we can construct a $f_x \in CELL(S^1 \times S^{n-2} \times [0, 1])$, for $x \in X$. And claim 6 of section 3 of [8] still holds with $\iota\theta$ instead of $\iota\theta$. By Proposition 1.2 $\iota\theta$ extends to $\mathbb{D}^{k+1}$. This proves Theorem 2.4.

Remark. We have to be careful here since the original metric is not necessarily negatively curved, hence it may not determine the identity on $S^1 \times S^{n-2} \times [0, \infty]$. To overcome this let $B_t : W \to \partial\infty Q \cup Q$ be a 1-parameter family of cellular maps such that $B_0$ is a homeomorphism and $B_1$ is the map $A$ of
lemma 2.8 corresponding to the base point $u_0 \in S^k$. Then identify $W = S^1 \times S^{n-2} \times [0, \infty]$ with $\partial \infty Q \cup Q$ by $B_0$.

Appendix A. The Swinging Neck.

For a function $h : \mathbb{R} \to (0, \infty)$ denote by $M_h$ the surface of revolution obtained by rotating the graph $\{(x, h(x), 0) : x \in \mathbb{R}\}$ of $h$ around the $x$-axis. We consider $M_h$ with the Riemannian metric induced by $\mathbb{R}^3$.

Let $f : \mathbb{R} \to [1, \infty)$ be a smooth function such that: (1) $f \equiv 1$ on $[-1, 1]$, (2) $f''(x) > 0$, $|x| > 1$, (3) $f''(x) \geq \delta > 0$, for $|x| \geq 2$. Then $M_f$ is nonpositively curved and contains the flat cylinder $[-1, 1] \times S^1$.

Let $\alpha : \mathbb{R} \times [-2, 2] \to [0, 1]$ be a smooth function such that (we write $\alpha_t$ for the function $x \mapsto \alpha(x, t)$): (1) $\alpha \equiv 0$ for $|x| \geq 4$ and all $t$, (2) $\alpha''_t(x) > 0$, for $|x| \leq 3$ and all $t$, (3) $\alpha_t$ has a unique minimum value (equal to 0) on $[-3, 3]$ at $t$, for all $t$.

Define $F : \mathbb{R} \times [0, 1] \to [1, \infty)$ by $F(x, 0) = f(x)$, and for $t \in (0, 1]$ by

$$F(x, t) = f(x) + e^{-1/t} \alpha(x, \sin(1/t))$$

and write $f_t(x) = F(x, t)$. Thus $f_0 = f$. Then $F$ is smooth and for small enough $t > 0$ we have: (1) $f''_t(x) \geq 0$, $\forall x \in \mathbb{R}$ (2) $f_t$ has a unique minimum value at $\sin(1/t)$. (3) $f_t \equiv f$ outside $[-4, 4]$.

Write $M_t = M_{f_t}$ and $M = M_f$. Then $M_t$ is negatively curved and concides with $M$ outside a compact set. Note that all $\{x\} \times S^1, x \in [-1, 1]$, are non-trivial closed geodesics of minimal length in $M$. But $M_t$ has a unique non-trivial closed geodesic $\{\sin(1/t)\} \times S^1$ of minimal length that oscillates between $\{-1\} \times S^1$ and $\{1\} \times S^1$ faster and faster, as $t$ approaches 0.

Note that, with some care, we can fit these “necks” - the relevant parts of $M_t$ and $M$ - on a closed negatively curved surface.

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