Non-symmetrical separated flow along the parabolic wing

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Abstract. Inviscid incompressible separated flow along the parabolic wing is explored. The partition is placed in the symmetry plane. The problem is considered without an assumption of flow symmetry. Self-similar solution is constructed. Both symmetrical and non-symmetrical solutions are observed. Fields of self-similar trajectories and stability are analysed.

1. Introduction

A flow along a body with a small aspect ratio is known to be non-symmetrical at high angles of attack, even so the shape of the body and its position relative to the flow are symmetrical.

The phenomenon can be explained \cite{1} by the symmetrical flow instability at high angles of attack, so one of the stable non-symmetrical solutions is realized.

In the present study a non-symmetrical separated flow along a parabolic wing is explored. The wing is assumed to be curved on the parabolic law and has the parabolic shape in plan. Also the wing has a partition in the symmetry plane and the partition height grows on the parabolic law. Because of a small aspect ratio the unsteady analogy can be used. It allows us to reduce a steady three-dimensional problem to an unsteady two-dimensional one.

The symmetrical self-similar solution of this problem was obtained \cite{2} and experimentally studied \cite{3}. The solution is two discrete vortex filaments escaped from the wing vertex.

2. Statement of the problem

The parabolic wing geometry (Figure 1) is described by the relationships:

\begin{equation}
\begin{aligned}
\ell(Z) = 2e\alpha Z^{1/2},
Y(Z) = -e\beta Z^{1/2},
h(Z) = e\gamma Z^{1/2}.
\end{aligned}
\end{equation}

Here, \(\ell(Z)\) is the dependence of the wing width on the longitudinal coordinate \(Z\), \(Y(Z)\) is the intersection of the wing and the partition, \(h(Z)\) is the partition height, \(e\) is the dimensional parameter, \(\alpha\), \(\beta\), \(\gamma\) are dimensionless constants, \(0 \leq Z < +\infty\).

Inviscid incompressible steady separated flow along the parabolic wing is explored. Far from the wing the flow is uniform with the velocity \(V_\infty\) along the \(Z\) axis. A solution depends on two dimensionless parameters: the relative partition height \(b\) (the quotient of the partition height to the wing half-width) and the wing flexion \(a_0\) (the quotient of the wing flexure depth to the wing half-width)

\begin{equation}
b = \gamma / \alpha, \quad a_0 = \beta / \alpha.
\end{equation}
A solution should satisfy the Chaplygin–Zhukovsky condition on the sharp edges. The flow is required to be unperturbed far from the wing. The normal component of the velocity to the boundary is to be zero.

\begin{figure}
\centering
\includegraphics[width=0.4\textwidth]{wing_geometry}
\caption{Wing geometry ($a_0 = 5, b = 1.5$).}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=0.4\textwidth]{self_similar_planes}
\caption{Geometry in self-similar and conformal planes.}
\end{figure}

3. Derivation of the equations

The unsteady analogy is valid for $Z \gg \varepsilon Z^{1/2}$ or $Z \gg \varepsilon^2$, when the wing aspect ratio is assumed to be small. Due to the unsteady analogy the problem is reduced to an expanding plate motion. Meanwhile the law, describing the expansion of the plate and its motion along the $Y$ axis, is determined by substitution the expression $Z = V_\infty t$ in equalities (2.1) taking (2.2) into account:

\begin{align*}
l(t) &= 2a_0 t^{1/2}, \quad h(t) = ab t^{1/2}, \quad Y(t) = aa_0 t^{1/2}, \quad a = \varepsilon a V_\infty^{1/2}.
\end{align*}

Choose the coordinate system with the origin in the point of intersection of the partition and the plate. The axis $x_1$ is directed along the plate and the axis $y_1$ is directed along the partition. The free-stream flow velocity along the axis $y_1$ in this coordinate system is

\begin{equation}
\frac{dY}{dt} = \frac{1}{2} a a_0 t^{-1/2}.
\end{equation}

This problem is self-similar because there are not some characteristic scales of space and time. The circulation of vortices escaped from the sharp edges of the plate and the partition does not depend on time. Therefore, a continuous descent of a vortex sheet from the edges is not realized. There are three discrete vortices escaped from the edges at the time origin.

Introduce the complex coordinate in the physical plane $z = x_1 + iy_1$ and in the self-similar plane $z = x + iy$, where $z_1 = at^{1/2} z$. The velocity and circulation of the vortices in the physical and self-similar variables are connected by the relationships

\begin{equation}
u_1 = a t^{-1/2} \nu, \quad \Gamma = a^2 \Gamma_0.
\end{equation}

In the self-similar variables the plate has unity half-width and the free-stream flow velocity is $a_0 / 2$.

Transform the plate with the partition in the plane $z$ to the cylinder of unity radius in the plane $\zeta$ (Figure 2) using the next conformal transformations:
\[ \sigma(z) = \sqrt{z^2 - 1}, \quad \mu(\sigma) = p\sigma + q, \quad \zeta(\mu) = \mu + \sqrt{\mu^2 - 1}, \]  

(3.1)

where

\[ p = \frac{-2i}{1 + \sqrt{1 + b^2}}, \quad q = \frac{1 - \sqrt{1 + b^2}}{1 + \sqrt{1 + b^2}}. \]

The regular branches of the roots are chosen by the conditions: \( \sigma(\pm\infty) = \pm\infty, \quad \zeta(\pm\infty) = \pm\infty. \)

In the conformal plane \( \zeta = \xi + i\eta \) the cylinder is in the free-stream flow with the velocity

\[ u_{\xi} = \frac{1 + \sqrt{1 + b^2}}{2}a_0 \]

directed on the \( \xi \) axis. The edges of the plate with the partition

\[ z_{A,C} = \pm 1, \quad z_B = ib \]
come to the points on the cylinder surface

\[ \zeta_{A,C} = q \pm i\sqrt{1 - q^2}, \quad \zeta_B = 1. \]

A flow complex potential \( w \) in the conformal plane is a superposition of the free-stream flow, the dipole in the center of the cylinder, three vortices escaped from three sharp edges and three adjoint vortices in the inverse points inside the cylinder. The complex adjoint velocity is

\[ \frac{dw}{d\zeta} = u_{\xi}\left(1 - \frac{1}{\zeta^2}\right) + \frac{1}{2\pi i} \frac{\Gamma_j}{\zeta - \zeta_j} - \frac{G_j}{\zeta - 1/\zeta_j}. \]  

(3.2)

Here, \( j = 1, 2, 3 \) is the vortex number, \( G_j \) is the dimensionless intensity (circulation) of the vortex with the number \( j \), \( \zeta_j \) is the complex coordinate of the vortex with the number \( j \).

The vortex moves in the physical plane with the velocity

\[ \frac{d\xi_j}{dr} = \lim_{z_i \to z_{i,j}} \left( \frac{dw}{dz} - \frac{1}{2\pi i z - z_j} \right), \]  

(3.3)

where \( \Gamma_j \) is the dimensional intensity (circulation) of the vortex with the number \( j \), \( w_i \) is the flow complex potential in the physical plane. Taking into account the relationships

\[ z_j = a t^{1/2} z, \quad w_i = a^2 w, \quad \Gamma_j = a^3 G_j, \]

we obtain the equation of the vortex motion in the self-similar plane from the expression (3.3):

\[ \frac{d\xi_j}{dr} + \frac{\xi_j}{2} = \lim_{\zeta \to \zeta_{i,j}} \left( \frac{dw}{dz} - \frac{1}{2\pi i z - z_j} \right). \]  

(3.4)

Hence we obtain the condition of the vortex rest in the self-similar plane

\[ \frac{\xi_j}{2} = \lim_{\zeta \to \zeta_{i,j}} \left( \frac{dw}{dz} - \frac{1}{2\pi i z - z_j} \right). \]

Substitute in this condition the expression (3.2) and take into account the transformations (3.1):

\[ \frac{\xi_j}{2} \frac{dz_j}{d\zeta_j} = u_{\xi}\left(1 - \frac{1}{\zeta_j^2}\right) + \frac{1}{2\pi i} \sum_{k=1}^3 \left( \frac{G_k}{z_j - z_k} - \frac{G_k}{\zeta_j - 1/\zeta_j} \right) - \frac{1}{2\pi i} G_j - \frac{G_j}{\zeta_j - 1/\zeta_j} + \frac{G_j}{2\pi i} A_j \]  

(3.5)

where

\[ \frac{dz_j}{d\zeta_j} = \frac{1 - 1/\zeta_j^2}{2p} \sigma_j, \quad A_j = \lim_{\zeta \to \zeta_{i,j}} \left( \frac{1}{\zeta - \zeta_j} - \frac{1}{z - z_j} \right) \frac{d\zeta_j}{d\zeta} = \frac{1 - 1/\zeta_j^2}{4p\sigma_j z_j - \zeta_j (\zeta_j^2 - 1)}. \]

According to the Chaplygin–Zhukovsky conditions the velocities at the edges \( A, B, C \) are finite
\[
\frac{d\omega_j}{dz_i}\bigg|_{A,B,C} < \infty.
\]

In conformal plane
\[
\frac{d\omega}{d\zeta}\bigg|_{A,B,C} = 0
\]
as
\[
\frac{d\zeta}{dz}\bigg|_{A,B,C} = \infty.
\]

These relationships are reduced to three real equations
\[
4\pi u_\infty \text{Im} \zeta_m + \sum_{j=1}^{3} G_j \left(\frac{\zeta_j^2 - 1}{|\zeta_j - \zeta_m|}\right) = 0
\]
(3.6)

where \( m = A, B, C \) is an edge number.

The system of three complex (3.5) and three real (3.6) equations is reduced by extraction of real and imaginary parts to the system of nine real equations with nine unknown quantities. They are the vortex coordinates in conformal plane and their intensities. We can express the vortex intensities through the coordinates because the equations (3.6) are linear relative to the former. After substitution of these expressions into the equations (3.5) we will have the system of six real equations that can be solved by Newton method.

4. Self-similar trajectories

A self-similar trajectory is a fixed liquid particle trajectory in the self-similar plane \( z \).

Liquid particle velocities in the physical and the self-similar planes are connected by the relation
\[
\frac{dz_i}{dt} = \frac{1}{2} a t^{-1/2} z + a t^{1/2} \frac{dz}{dt}.
\]

Express these velocities through the complex potential
\[
\frac{d\omega_j}{dz_i} = \frac{1}{2} a t^{-1/2} \omega + a t^{1/2} \frac{d\omega}{dz}
\]
or
\[
\frac{dz}{dt} = \frac{d\omega}{dz} - \frac{\overline{z}}{2}.
\]

Multiplied the right and the left parts of the latter equation by an element of self-similar trajectory \( dz \), we obtain the complex equation with real left part. Therefore, the self-similar trajectory equation [2] is
\[
\text{Im} \left[ \frac{d\omega}{dz} - \frac{\overline{z}}{2} \right] dz = 0.
\]

It enables to determine in every point the unity vector \( dz / [dz] \) that is tangential to the trajectory.

In Figure 3 we have reproduced the self-similar trajectories for the wing with the flexion \( a_0 = 4.6 \). It was observed the non-symmetrical solution continuously turns into the symmetrical one if the partition height decreases. Such transition occurs at the critical partition height \( b_c = 1.10666 \). Figure 3 (a) shows a weakly non-symmetrical solution at the partition height slightly more than \( b_c \).

The non-symmetrical solution is proved to tend to a non-symmetrical limit as the partition height tends to infinity. This limit is reached at \( b \approx 4 \). Figure 3 (c) shows the solution at the extremely high partition \( b = 150 \).
Figure 3. Self-similar trajectories, $a_0 = 4.6; b = 1.110$ (a), $b = 1.150$ (b), $b = 150$ (c). 1, 2, 3 – vortices position, 4 – rigid body, 5 – self-similar trajectories.

Figure 4. Self-similar trajectories, $a_0 = 4.7; b = 1.010$ (a), $b = 1.015$ (b), $b = 1.050$ (c), $b = 1.100$ (d), $b = 150$ (e). 1, 2, 3 – vortices position, 4 – rigid body, 5 – self-similar trajectories.

The critical partition height increases as the flexion decreases. It was obtained $b \approx 150$, if $a_0 = 2.816$, and $b \approx 1.5$, if $a_0 = 3.787$. Non-symmetrical solutions at $a_0 \leq 2.8$ are not observed.
A continuous transition of a non-symmetrical solution to a symmetrical one is observed at $2.8 < a_0 \leq 4.6$. If $a_0 \geq 4.7$ a non-symmetrical solution does not tend to a symmetrical one if the partition height decreases. In Figure 4 (a) the vortex escaped from the partition is close to the left edge and its intensity is very small relative to the side vortices intensities. In Figure 4 (e) we have reproduced the limiting non-symmetrical solution at the extremely high partition $b = 150$.

In Table 1 and Table 2 we have presented the intensities of the vortices in Figure 3 and Figure 4.

**Table 1.** Vortices intensities at $a_0 = 4.6$.

| $b$  | $G_1$ | $G_2$ | $G_3$ |
|------|-------|-------|-------|
| 1.110| -20.22| 2.44  | 20.37 |
| 1.150| -19.46| 7.80  | 20.89 |
| 150  | -16.82| 11.12 | 20.34 |

**Table 2.** Vortices intensities at $a_0 = 4.7$.

| $b$  | $G_1$ | $G_2$ | $G_3$ |
|------|-------|-------|-------|
| 1.010| -24.27| 0.18  | 33.89 |
| 1.015| -24.22| 1.40  | 33.85 |
| 1.050| -25.43| 11.22 | 30.48 |
| 1.100| -20.05| 10.08 | 22.47 |
| 150  | -17.17| 11.49 | 20.85 |

**5. Stability analysis**

A vortex sheet generation on the expanding plate is considered to check a stability of the solutions under discussion. As an initial condition at $t_0 = 1$ we take the self-similar solution, i.e. the flow is assumed to be developed without any perturbation till $t_0 = 1$. The vortex sheet is simulated by the discrete vortices method. During the time step $\Delta t$ three vortices are escaped from the edges and move on the position determined by the complex quantity $\hat{z}_m \Delta t$, where $\hat{z}_m$ is an initial vortex velocity that can be calculated from the physical fluid velocity at the edge number $m$ $(m = A, B, C)$. The vortex intensity is determined from the Chaplygin–Zhukovsky conditions.

During the time step the vortices in the calculation area move on the quantity $\hat{z}_j \Delta t$, where $\hat{z}_j$ is calculated using the formula (3.4). Note, the vortex velocity in self-similar plane is inversely proportional to the time.

The vortex sheet is developed by the algorithm till the state shown in Figure 5 (a) (the vortex sheets escaped from the side edges have three quarters of a coil and the vortex sheet escaped from the partition have a quarter of a coil). A further vortex sheet evolution is described by the “vortex-section” approach. If a vortex crosses the horizontal section in Figure 5 (a), it “drops” on a large vortex so the large vortex intensity is incremented on the “dropped” vortex intensity and the position moves in their vorticity center. It is necessary to “drop” vortices because a lot of vortices in a confined area lead to the calculation process instability.
For a symmetrical solution stability analysis a vortex sheet is assumed to escape from the partition due to small perturbations. The solution is understood as stable if the vortex sheet weakly changes intensities and positions of the large vortices. The solution is understood as unstable if third large vortex is generated and a solution will tend to a non-symmetrical one.

The algorithm was used for stability analysis of symmetrical and non-symmetrical solutions shown in Figure 3 (b) and Figure 4 (d), and weakly non-symmetrical solution shown in Figure 3 (a). The vortex sheet was stable at all these cases during long time period and its intensity decreased. While testing the symmetrical solution the vortex sheet escaped from the partition makes many coils (Figure 5 (b)) so this solution is stable in accordance with the criterion mentioned above. The algorithm cannot be used for the stability analysis in cases shown in Figure 4 (a), (c) because of more complicated flow pattern. The stability in these cases is not tested.

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References
[1] Goman M G, Zakharov S B and Hrabrov A N 1985 Symmetrical and non-symmetrical separated flow along a small aspect ratio wing with a fuselage Uchenye zapiski TsAGI 16 (6) 1–8
[2] Nikol’skii A A, Betyaev S K and Malyshev I P 1971 On the limiting separated self-similar flow of an inviscid fluid Problems in Applied Mathematics and Mechanics (Moscow: Nauka) pp 262–268
[3] Bakulin V L and Gayfullin A M 1989 Experimental study of the flow in the cores of a vortex structure Fluid mechanics. Soviet research. 18 (1) 42–46