New CP Phase and Exact Oscillation Probabilities of Dirac Neutrino derived from Relativistic Equation

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ABSTRACT: We present a new formulation deriving the neutrino oscillation probabilities relativistically based on not the Schrödinger equation but the Dirac equation. In two generations, we calculate the oscillation probabilities exactly in the case that there exists only the Dirac mass term. We find that two kinds of new terms appear in the oscillation probabilities derived from the Dirac equation. One is the term dependent on the absolute value of neutrino mass. Although it has been considered that the oscillation probabilities depend only on the mass squared differences until now, we could observe the absolute value of mass through neutrino oscillations in principle. The other is the term including a new CP phase. If there are some interactions to distinguish the flavors of right-handed neutrinos beyond the Standard Model, we could also observe this new CP phase in principle even in the framework of two generations. We discuss the possibility to observe the contribution of these terms by the neutrino oscillations of atomic size. On the other hand, it is negligible in the usual short and long-baseline experiments, and there is no contradiction with previous experiments.
1 Introduction

In 1957, Pontecorvo proposed the idea of neutrino-antineutrino oscillations inspired by the idea of $K^0-\bar{K}^0$ oscillations if lepton number violation exists [1]. In 1962, after the discovery of muon neutrinos, Maki, Nakagawa, and Sakata proposed the oscillations between neutrinos with different flavors [2]. Afterward, the various theories of neutrino oscillations have been developed, for example in the cases of three generations [3], with matter effect [4, 5], with magnetic fields [6] and with non-standard interactions [7–11].

In 1998, the neutrino oscillation was discovered by the atmospheric neutrino experiment in Super-Kamiokande. After that, the evidence of neutrino oscillations has been accumulated by the solar neutrino experiments [12–14], the long-baseline neutrino experiments [15–17] and the reactor experiments [18–21]. The analysis of these experiments clarified the values of two mass squared differences and three mixing angles. Furthermore, the latest result in T2K experiment showed a preference for values of the Dirac CP phase which is near maximal CP violation [22]. Next-generation experiments [23, 24] are planned in order to investigate the CP phase and the neutrino mass ordering. In these experiments, we need to estimate the matter effects as accurately as possible for the precise measurement of neutrino parameters. For this purpose, the theory of neutrino oscillation including matter effects was formulated exactly by using the Schrödinger equation [25–30].

The oscillations confirmed in the previous experiments are all those without chirality-flip. On the other hand, the oscillations between neutrinos with different chiralities have
been also suggested. Actually, there is a possibility for changing the chirality because ν_L and ν_R are included in one multiplet when we consider the Dirac equation. In one generation Dirac neutrino, Fukugita and Yanagida calculated the probability of ν_L → ν_R oscillation [31]. Also in the case of Majorana neutrino [32], the oscillation probabilities between neutrino and anti-neutrino were derived in both two and three-generation schemes [33–38]. These also belong to oscillations between neutrinos with different chiralities. In the oscillations with chirality-flip, the oscillation probabilities are proportional to m^2/E^2, where E and m are neutrino energy and mass respectively. Therefore, it is considered to be difficult to measure this tiny effect. However, if there exist the oscillations with chirality-flip in addition to the established oscillations between the neutrinos with different flavors and same chirality, the sum of the probabilities for ν_eL → ν_eL, ν_eL → ν_μL and ν_eL → ν_τL is not equal to one, in other words, the unitarity must be violated. This is due to the use of the Schrödinger equation including only left-handed neutrinos.

In this paper, we formulate the theory of neutrino oscillation by using the relativistic Dirac equation, in order to unify the oscillations with and without chirality-flip. We consider the case of two-generation neutrinos with the Dirac mass term in vacuum. The obtained results can be extended to the case of Majorana neutrinos, n generations, and the case with matter effects or magnetic fields. We show that the unitarity exactly holds by considering the oscillations with and without chirality-flip. We find that two new terms appear in derived probabilities, one is the term dependent on the absolute mass of neutrino and the other is the term including a new CP phase. This new CP phase appears in the oscillations with chirality-flip and we can interpret that the Majorana CP phase is a kind of this CP phase.

As a result, we can measure not only the mass squared differences but also the absolute mass of neutrinos by neutrino oscillation experiments in principle. Furthermore, we find that the new CP phase can be measured if there are some interactions to distinguish the flavors of right-handed neutrinos even in two generations. The contribution of these terms can be non-negligible effects in experiments of atomic size. On the other hand, the contribution of the new terms can be negligible in short, medium, and long-baseline experiments, so it is consistent with the previous results.

The paper is organized as follows. In section II, we define our notations used in this paper. In section III, we review the derivation of neutrino oscillation probabilities developed in the previous papers by using the Schrödinger equation. In section IV, we present the derivation of neutrino oscillation probabilities from the Dirac equation relativistically and check the unitarity. In section V, we summarize our results obtained in this paper.

2 Notation

In this section, we write down the notation used in this paper. We mainly use the chiral representation because neutrinos are measured through weak interactions. In chiral representation, the gamma matrices with 4 × 4 form are given by

\[ \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]

(2.1)
where $2 \times 2$ $\sigma$ matrices are defined by
\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\] (2.2)
We also define 4-component spinors $\psi$, $\psi_L$ and $\psi_R$ as
\[
\psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad \psi_L = \frac{1 - \gamma_5}{2} \psi = \begin{pmatrix} 0 \\ \eta \end{pmatrix}, \quad \psi_R = \frac{1 + \gamma_5}{2} \psi = \begin{pmatrix} \xi \\ 0 \end{pmatrix},
\] (2.3)
and 2-component spinors $\xi$ and $\eta$ as
\[
\xi = \begin{pmatrix} \nu_R' \\ \nu_R \end{pmatrix}, \quad \eta = \begin{pmatrix} \nu_L' \\ \nu_L \end{pmatrix}.
\] (2.4)
Furthermore, we use the subscript $\alpha$ and $\beta$ for flavor, $L$ and $R$ for chirality, the number $1$ and $2$ for generation and superscript $\pm$ for energy. Because of negligible neutrino mass, mass eigenstate has been often identified with energy eigenstate in many papers. But in the future, we should differentiate these two kinds of eigenstates because of the finite neutrino mass. More concretely, we use the following eigenstates;
\[
\text{chirality-flavor eigenstates} : \quad \nu_{\alpha L}, \nu_{\alpha R}, \nu_{\beta L}, \nu_{\beta R},
\] (2.5)
\[
\text{chirality-mass eigenstates} : \quad \nu_{1 L}, \nu_{1 R}, \nu_{2 L}, \nu_{2 R},
\] (2.6)
\[
\text{energy-helicity eigenstates} : \quad \nu^+_1, \nu^-_1, \nu^+_2, \nu^-_2.
\] (2.7)
It is noted that chirality-mass eigenstates are not exactly the eigenstates of the Hamiltonian. We use the term, eigenstates, in the sense that the mass submatrix in the Hamiltonian is diagonalized. Judging from common sense, one may think it strange that the chirality and the mass live in the same eigenstate. Details will be explained in the subsequent section. We also define the spinors for anti-neutrino as charge conjugation of neutrino $\psi^c = i\gamma^2 \psi^*$. The charge conjugations for left-handed and right-handed neutrinos are defined by
\[
\psi^c_L \equiv (\psi_L)^c = \begin{pmatrix} \nu_L' \\ \nu_L \end{pmatrix} \equiv i\gamma^2 \psi^*_L = i\gamma^2 \frac{1 - \gamma_5}{2} \psi^*
\]
\[
= \frac{1 + \gamma_5}{2} (i\gamma^2 \psi^*) = (\psi^c)_R = \begin{pmatrix} i\sigma_2 \eta^* \\ 0 \end{pmatrix} = \begin{pmatrix} \nu^*_L \\ -\nu^*_R \\ 0 \\ 0 \end{pmatrix},
\] (2.8)
\[
\psi^c_R \equiv (\psi_R)^c = \begin{pmatrix} 0 \\ \nu_R' \\ \nu_R \end{pmatrix} \equiv i\gamma^2 \psi^*_R = i\gamma^2 \frac{1 + \gamma_5}{2} \psi^*
\]
\[
= \frac{1 - \gamma_5}{2} (i\gamma^2 \psi^*) = (\psi^c)_L = \begin{pmatrix} 0 \\ -i\sigma_2 \xi^* \\ 0 \\ -\nu^*_R \end{pmatrix}.
\] (2.9)
It is noted that the chirality is flipped by taking the charge conjugation.

3 Conventional Derivation of Oscillation Probabilities

In this section, we consider the case of Dirac neutrinos and review the derivation of the oscillation probabilities performed in previous papers.

3.1 Conventional Derivation of Chirality non-flipped Probabilities

At first, we review how the oscillation probabilities without chirality-flip are derived by the Schrödinger equation. In vacuum, time evolution of chirality-flavor eigenstates (weak eigenstates), $\nu_{\alpha L}$ and $\nu_{\beta L}$ after the time $t$, is given by

$ \begin{pmatrix} \nu_{\alpha L}(t) \\ \nu_{\beta L}(t) \end{pmatrix} = \begin{pmatrix} U_{\alpha 1} & U_{\alpha 2} \\ U_{\beta 1} & U_{\beta 2} \end{pmatrix} \begin{pmatrix} \nu^+_1(t) \\ \nu^+_2(t) \end{pmatrix} = \begin{pmatrix} e^{-iE_1t} & 0 \\ 0 & e^{-iE_2t} \end{pmatrix} \begin{pmatrix} \nu^+_1 \\ \nu^+_2 \end{pmatrix}, \quad (3.1) $ where $U_{\alpha i}$ and $U_{\beta i}$ are the elements of unitary matrix, $E_i$ is the energy of $\nu^+_i$ and $\nu^+_i$ means $\nu^+_i(0)$. It is noted that the energy-helicity eigenstates and the chirality-mass eigenstates are identified with so-called mass eigenstate in conventional derivation. Rewriting the relations of the fields to those of the one particle states, we obtain

$ |\nu_{\alpha L}(t)| = U^*_{\alpha 1} e^{-iE_1t} |\nu^+_1| + U^*_{\alpha 2} e^{-iE_2t} |\nu^+_2|, \quad (3.2) $ $ |\nu_{\beta L}(t)| = U^*_{\beta 1} e^{-iE_1t} |\nu^+_1| + U^*_{\beta 2} e^{-iE_2t} |\nu^+_2|, \quad (3.3) $ and the conjugate states

$ \langle \nu_{\alpha L} | = U_{\alpha 1} |\nu^+_1| + U_{\alpha 2} |\nu^+_2|, \quad (3.4) $ $ \langle \nu_{\beta L} | = U_{\beta 1} |\nu^+_1| + U_{\beta 2} |\nu^+_2|, \quad (3.5) $ Therefore, we have the transition amplitudes of $\nu_{\alpha L}$ to $\nu_{\alpha L}$ and $\nu_{\beta L}$,

$ A(\nu_{\alpha L} \rightarrow \nu_{\alpha L}) = \langle \nu_{\alpha L} | \nu_{\alpha L}(t) \rangle = |U_{\alpha 1}|^2 e^{-iE_1t} + |U_{\alpha 2}|^2 e^{-iE_2t}, \quad (3.6) $ $ A(\nu_{\alpha L} \rightarrow \nu_{\beta L}) = \langle \nu_{\beta L} | \nu_{\alpha L}(t) \rangle = U^*_{\alpha 1} U_{\beta 1} e^{-iE_1t} + U^*_{\alpha 2} U_{\beta 2} e^{-iE_2t}. \quad (3.7) $ As the mixing matrix, $U$ in two generations is $2 \times 2$ form and has 4 components, it can be parametrized as

$ U = \begin{pmatrix} e^{i\rho_1} & 0 \\ 0 & e^{i\rho_2} \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix} = \begin{pmatrix} e^{i\rho_1} \cos \theta & e^{i\rho_1+\phi} \sin \theta \\ -e^{i\rho_2} \sin \theta & e^{i\rho_2+\phi} \cos \theta \end{pmatrix}. \quad (3.8) $ By squaring the amplitudes of (3.6) and (3.7), we obtain the well-known form of the oscillation probabilities,

$ P(\nu_{\alpha L} \rightarrow \nu_{\alpha L}) = |U_{\alpha 1}|^4 + |U_{\alpha 2}|^4 + 2|U_{\alpha 1} U_{\alpha 2}|^2 \text{Re}[e^{i(E_2-E_1)t}] $ $ = c^4 + s^4 + 2s^2 c^2 \cos(E_2 - E_1)t $ $ = 1 - 2s^2 c^2 [1 - \cos(E_2 - E_1)t] = 1 - \sin^2 2\theta \sin^2 \frac{(E_2 - E_1)t}{2}, \quad (3.9) $ $ P(\nu_{\alpha L} \rightarrow \nu_{\beta L}) = |U_{\alpha 1} U_{\beta 1}|^2 + |U_{\alpha 2} U_{\beta 2}|^2 + 2 \text{Re}[U^*_{\alpha 1} U_{\beta 1} U_{\alpha 2} U_{\beta 2} e^{i(E_2-E_1)t}] $ $ = 2s^2 c^2 [1 - \cos(E_2 - E_1)t] = \sin^2 2\theta \sin^2 \frac{(E_2 - E_1)t}{2}. \quad (3.10) $
where we use the abbreviation $c = \cos \theta$ and $s = \sin \theta$. One can see that the phase included in the mixing matrix $U$ does not appear in the oscillation probabilities in conventional derivation. The oscillation probabilities without chirality-flip obtained here are interpreted as the transition occurred by the deviation between the chirality-flavor eigenstates and chirality-mass (energy-helicity) eigenstates.

### 3.2 Conventional Derivation of Chirality flipped Probabilities

Next, we review the oscillation probability with chirality-flip calculated by Fukugita-Yanagida in one generation [31]. Fukugita and Yanagida suggested the possibility for the chirality-flip $\nu_L$-$\nu_R$ oscillation in the case that the neutrino has finite mass and non-diagonal matrix elements exist in the Hamiltonian. The Dirac equation in one generation is given by

$$i\gamma^\mu \partial_\mu \psi - m \psi = 0,$$

where $m$ is the Dirac mass. Multiplying $\gamma^0$ from left, we rewrite the above equation as

$$i\partial_0 \psi + i\gamma^0 \gamma^i \partial_i \psi - m \gamma^0 \psi = 0.$$  \hspace{1cm} (3.12)

Here, if we take the equal momentum assumption, $\psi(x,t) = e^{i\vec{p} \cdot \vec{x}}(\nu'_R, \nu_R, \nu'_L, \nu_L)^T$ and choose as $\vec{p} = (0,0,p)$, we obtain the matrix form,

$$i \frac{d}{dt} \begin{pmatrix} \nu'_R \\ \nu_R \\ \nu'_L \\ \nu_L \end{pmatrix} = \begin{pmatrix} p & 0 & m & 0 \\ 0 & -p & 0 & m \\ m & 0 & -p & 0 \\ 0 & m & 0 & p \end{pmatrix} \begin{pmatrix} \nu'_R \\ \nu_R \\ \nu'_L \\ \nu_L \end{pmatrix},$$  \hspace{1cm} (3.13)

One can see that the Dirac mass term exist in the non-diagonal components, which mixes $\nu_L$ and $\nu_R$ ($\nu'_L$ and $\nu'_R$). We expect the transition between $\nu_L$ and $\nu_R$ through the Dirac mass term. Extracting the part related to $\nu_L$ and $\nu_R$, we obtain the equation,

$$i \frac{d}{dt} \begin{pmatrix} \nu_R \\ \nu_L \end{pmatrix} = \begin{pmatrix} -p & m \\ m & p \end{pmatrix} \begin{pmatrix} \nu_R \\ \nu_L \end{pmatrix},$$  \hspace{1cm} (3.14)

and the chirality-flavor eigenstates (weak eigenstates) are related to the energy-helicity eigenstates by the matrix diagonalizing the Hamiltonian in (3.14),

$$\begin{pmatrix} \nu_R \\ \nu_L \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{E+p}{2E}} & \sqrt{\frac{E-p}{2E}} \\ -\sqrt{\frac{E-p}{2E}} & \sqrt{\frac{E+p}{2E}} \end{pmatrix} \begin{pmatrix} \nu^- \\ \nu^+ \end{pmatrix},$$  \hspace{1cm} (3.15)

where

$$E = \sqrt{p^2 + m^2}. \hspace{1cm} (3.16)$$
eigenstates are different in the case of massive neutrino. Therefore, we need to distinguish these eigenstates in future experiments. The time development of energy-helicity eigenstates are given by

\[
\frac{d}{dt} \left( \nu^- \right) = \begin{pmatrix} -E & 0 \\ 0 & E \end{pmatrix} \left( \nu^- \right).
\]

(3.17)

Rewriting these relations for fields to those for one particle states, the chirality-mass eigenstates after the time \( t \) are given by

\[
|\nu_R(t)\rangle = \sqrt{E + p^2} E e^{iEt} |\nu^-\rangle + \sqrt{E - p^2} E e^{-iEt} |\nu^+\rangle,
\]

(3.18)

\[
|\nu_L(t)\rangle = -\sqrt{E - p^2} E e^{iEt} |\nu^-\rangle + \sqrt{E + p^2} E e^{-iEt} |\nu^+\rangle,
\]

(3.19)

and the conjugate states are also given by,

\[
\langle \nu_R | = \sqrt{E + p^2} E |\nu^-\rangle + \sqrt{E - p^2} E |\nu^+\rangle,
\]

(3.20)

\[
\langle \nu_L | = -\sqrt{E - p^2} E |\nu^-\rangle + \sqrt{E + p^2} E |\nu^+\rangle.
\]

(3.21)

So, the amplitudes after the time \( t \) become

\[
A(\nu_L \rightarrow \nu_L) = \langle \nu_L | \nu_L(t) \rangle = \frac{E - p}{2E} e^{iEt} + \frac{E + p}{2E} e^{-iEt}, \]

(3.22)

\[
A(\nu_L \rightarrow \nu_R) = \langle \nu_R | \nu_L(t) \rangle = -\frac{m}{2E} (e^{iEt} - e^{-iEt}) = -i \frac{m}{E} \sin(Et).
\]

(3.23)

Then, the oscillation probabilities are calculated by squaring these amplitudes as

\[
P(\nu_L \rightarrow \nu_L) = \left( \frac{E - p}{2E} \right)^2 + \left( \frac{E + p}{2E} \right)^2 + \frac{E^2 - p^2}{4E^2} (e^{2iEt} + e^{-2iEt})
\]

\[
= \frac{E^2 + p^2}{2E^2} + \frac{m^2}{2E^2} \cos(2Et) = 1 - \frac{m^2}{2E^2} + \frac{m^2}{2E^2} \cos(2Et)
\]

\[
= 1 - \frac{m^2}{2E^2} \{1 - \cos(2Et)\} = 1 - \left( \frac{m}{E} \right)^2 \sin^2(Et), \]

(3.24)

\[
P(\nu_L \rightarrow \nu_R) = \left( \frac{m}{E} \right)^2 \sin^2(Et).
\]

(3.25)

Fukugita and Yanagida derived these results by using the approximation [31]. But the results are found to be exact in the above calculation. About the oscillation between \( \nu'_L \) and \( \nu'_R \), we obtain the probabilities by the replacement, \( p \rightarrow -p \) in (3.14). The probabilities do not depend on \( p \) but \( p^2 \), therefore the same probabilities are obtained also in the case \( \nu'_L \) and \( \nu'_R \). Note that the oscillation probabilities with chirality-flip obtained here are interpreted as the transition occurred by the deviation between the chirality-mass (chirality-flavor) eigenstates and energy-helicity eigenstates. It is considered that \( \nu_L \rightarrow \nu_R \) oscillations hard to happen in the atmospheric neutrino, solar neutrino, reactor neutrino, and the accelerator neutrino because the neutrino mass \( m \) is tiny compared with the energy \( E \). The smallness of
the probability is due to the small non-diagonal components in the Hamiltonian compared to the difference between diagonal components. On the other hand, it has been pointed out that the oscillations with chirality-flip have significant consequences on the detection of the cosmological relic neutrinos [39, 40].

4 New Derivation of Oscillation Probabilities

As reviewed in the previous section, the oscillations with and without chirality-flip were calculated separately in previous papers. If both types of oscillations exist, the unitarity does not hold by considering only one of them. We can interpret the oscillation without chirality-flip occurred by the deviation between the chirality-flavor eigenstates and the chirality-mass (energy-helicity) eigenstates, and the oscillation with chirality-flip occurred by the deviation between the chirality-mass (chirality-flavor) eigenstates and the energy-helicity eigenstates. We would like to consider these oscillations in a unified way in this section. This can be realized by using the Dirac equation. We show that the unitarity holds only when both types of oscillations are considered simultaneously.

4.1 Oscillation Probabilities of Neutrino

The lagrangian for two-generation neutrino is represented as

\[
L = i\bar{\psi}_a \gamma^\mu \partial_\mu \psi_a + i\bar{\psi}_a R \gamma^\mu \partial_\mu \psi_a + i\bar{\psi}_b \gamma^\mu \partial_\mu \psi_b + i\bar{\psi}_b R \gamma^\mu \partial_\mu \psi_b \\
- [\bar{\psi}_a m_{\alpha\alpha} \psi_a + \bar{\psi}_b m_{\beta\beta} \psi_b + \bar{\psi}_a m_{\beta\alpha} \psi_b + \bar{\psi}_b m_{\alpha\beta} \psi_a] \\
- [\bar{\psi}_a R m_{\alpha\alpha} \psi_a + \bar{\psi}_b R m_{\beta\beta} \psi_b + \bar{\psi}_a R m_{\beta\alpha} \psi_b + \bar{\psi}_b R m_{\alpha\beta} \psi_a], \tag{4.1}
\]

by using 4-component spinors. The Euler-Lagrange equation for \( \bar{\psi}_a \),

\[
\frac{\partial L}{\partial \bar{\psi}_a} - \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \bar{\psi}_a)} \right) = 0 \tag{4.2}
\]
gives us a Dirac equation,

\[
i\gamma^\mu \partial_\mu \psi_a - m^*_{\alpha\alpha} \psi_a - m^*_{\beta\alpha} \psi_b = 0. \tag{4.3}
\]

Multiplying \( \gamma^0 \) from left, we rewrite the above equation as

\[
i\partial_0 \bar{\psi}_a + i\gamma^0 \gamma^i \partial_i \bar{\psi}_a - m^*_{\alpha\alpha} \gamma^0 \psi_a - m^*_{\beta\alpha} \gamma^0 \psi_b = 0. \tag{4.4}
\]

If we represent this equation by two-components spinors \( \xi \) and \( \eta \), we obtain the following matrix form,

\[
i\partial_0 \begin{pmatrix} 0 \\ \eta_\alpha \end{pmatrix} - i \sigma_i \partial_i \begin{pmatrix} 0 \\ \eta_\alpha \end{pmatrix} - m^*_{\alpha\alpha} \begin{pmatrix} 0 \\ \xi_\alpha \end{pmatrix} - m^*_{\beta\alpha} \begin{pmatrix} 0 \\ \xi_\beta \end{pmatrix} = 0. \tag{4.5}
\]

We extract the lower part of this equation,

\[
i\partial_0 \eta_\alpha - i\sigma_i \partial_i \eta_\alpha - m^*_{\alpha\alpha} \xi_\alpha - m^*_{\beta\alpha} \xi_\beta = 0. \tag{4.6}
\]
In the same way, we also obtain other three equations,

\[ i\partial_t \eta_\beta - i\sigma_i \partial_i \eta_\beta - m_\beta^* \xi_\beta - m_\alpha^* \xi_\alpha = 0, \]  
\[ i\partial_t \xi_\alpha + i\sigma_i \partial_i \xi_\alpha - m_\alpha \eta_\alpha - m_\beta \eta_\beta = 0, \]  
\[ i\partial_t \xi_\beta + i\sigma_i \partial_i \xi_\beta - m_\beta \eta_\beta - m_\alpha \eta_\alpha = 0. \]  

Here, if we choose \( \vec{p} = (0, 0, p) \), the equations (4.6)-(4.9) can be combined into the following matrix form,

\[
\begin{pmatrix}
\nu'_{\alpha R} \\
\nu_{\alpha R} \\
\nu'_{\alpha L} \\
\nu_{\alpha L} \\
\nu'_{\beta R} \\
\nu_{\beta R} \\
\nu'_{\beta L} \\
\nu_{\beta L}
\end{pmatrix} = \begin{pmatrix}
p & 0 & m_\alpha & 0 & 0 & 0 & m_\alpha^* & 0 \\
0 & -p & 0 & m_\alpha & 0 & 0 & 0 & m_\beta \\
m_\alpha^* & 0 & -p & 0 & m_\beta^* & 0 & 0 & 0 \\
0 & m_\alpha^* & 0 & p & 0 & m_\beta^* & 0 & 0 \\
0 & 0 & m_\beta & 0 & p & 0 & m_\beta & 0 \\
0 & 0 & 0 & m_\beta & 0 & -p & 0 & m_\beta \\
m_\beta^* & 0 & 0 & 0 & m_\beta & 0 & -p & 0 \\
0 & m_\beta^* & 0 & 0 & 0 & m_\beta^* & 0 & p
\end{pmatrix} \begin{pmatrix}
\nu'_{\alpha R} \\
\nu_{\alpha R} \\
\nu'_{\alpha L} \\
\nu_{\alpha L} \\
\nu'_{\beta R} \\
\nu_{\beta R} \\
\nu'_{\beta L} \\
\nu_{\beta L}
\end{pmatrix}, \tag{4.12}
\]

It is expected that the oscillation probabilities for \( \nu_L \rightarrow \nu_R \) take a non-zero value because of the finite non-diagonal components of the Hamiltonian in (4.12) which mix \( \nu_L \) and \( \nu_R \). We will show it in the following calculation. On the other hand, the non-diagonal components of the Hamiltonian which mix \( \nu_L \) and \( \nu_L \) takes a value of zero. Therefore, the oscillations without chirality-flip seem to vanish or small. However, it is shown that the oscillation probabilities become large contrary to this expectation. Exchanging some rows and some columns, we rewrite (4.12) as

\[
\begin{pmatrix}
\nu'_{\alpha R} \\
\nu_{\beta R} \\
\nu'_{\alpha L} \\
\nu_{\beta L} \\
\nu'_{\alpha R} \\
\nu_{\beta R} \\
\nu'_{\alpha L} \\
\nu_{\beta L}
\end{pmatrix} = \begin{pmatrix}
p & 0 & m_\alpha & m_\beta & 0 & 0 & 0 & 0 \\
0 & p & m_\beta & m_\beta & 0 & 0 & 0 & 0 \\
m_\alpha^* & m_\beta^* & -p & 0 & 0 & 0 & 0 & 0 \\
m_\alpha^* & m_\beta^* & 0 & -p & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -p & 0 & m_\alpha & m_\alpha^* \\
0 & 0 & 0 & 0 & 0 & -p & m_\beta & m_\beta \\
m_\alpha^* & m_\beta^* & 0 & 0 & p & 0 & m_\beta & m_\beta \\
0 & 0 & 0 & 0 & m_\alpha^* & m_\beta^* & 0 & p
\end{pmatrix} \begin{pmatrix}
\nu'_{\alpha R} \\
\nu_{\beta R} \\
\nu'_{\alpha L} \\
\nu_{\beta L} \\
\nu'_{\alpha R} \\
\nu_{\beta R} \\
\nu'_{\alpha L} \\
\nu_{\beta L}
\end{pmatrix}, \tag{4.13}
\]

where \( \nu_R \) represents the state with right-handed chirality and the flavor of \( \nu_R \) cannot be distinguished in the Standard Model. However, we assign the flavor for also the right-handed
neutrinos in preparation for some distinguishable interactions included in the physics beyond the Standard Model. In the above $8 \times 8$ matrix, the upper-left $4 \times 4$ part is completely separated from the lower-right $4 \times 4$ part and never mix in this setting even if time passed. In the following calculations, we consider the lower-right $4 \times 4$ part,

$$\frac{d}{dt} \begin{pmatrix} \nu_{\alpha R} \\ \nu_{\beta R} \\ \nu_{\alpha L} \\ \nu_{\beta L} \end{pmatrix} = \begin{pmatrix} -p & 0 & m_{\alpha \alpha} & m_{\alpha \beta} \\ 0 & -p & m_{\beta \alpha} & m_{\beta \beta} \\ m_{\alpha \alpha}^* & m_{\beta \alpha}^* & p & 0 \\ m_{\alpha \beta}^* & m_{\beta \beta}^* & 0 & p \end{pmatrix} \begin{pmatrix} \nu_{\alpha R} \\ \nu_{\beta R} \\ \nu_{\alpha L} \\ \nu_{\beta L} \end{pmatrix}, \quad (4.14)$$

where $m_{\alpha \alpha}$, $m_{\beta \beta}$ and $m_{\alpha \beta}$ are complex in general. The chirality-flavor eigenstates are related to the chirality-mass eigenstates as

$$\begin{pmatrix} \nu_{\alpha R} \\ \nu_{\beta R} \\ \nu_{\alpha L} \\ \nu_{\beta L} \end{pmatrix} = \begin{pmatrix} V_{\alpha 1} & 0 & 0 & 0 \\ 0 & V_{\beta 1} & 0 & 0 \\ 0 & 0 & U_{\alpha 1} & U_{\alpha 2} \\ 0 & 0 & U_{\beta 1} & U_{\beta 2} \end{pmatrix} \begin{pmatrix} \nu_{1 R} \\ \nu_{2 R} \\ \nu_{1 L} \\ \nu_{2 L} \end{pmatrix}. \quad (4.15)$$

(If we cannot distinguish $\nu_{\alpha R}$ with $\nu_{\beta R}$ by any means, we can define $\nu_{1 R}$ and $\nu_{2 R}$ as $\nu_{\alpha R}$ and $\nu_{\beta R}$, and then we can regard V as the identity matrix. In this case, the CP phase cannot be observed, but we consider the general case here.) The Dirac mass term of the Hamiltonian is diagonalized by the above mixing matrix as

$$\begin{pmatrix} V_{\alpha 1}^* & V_{\beta 1}^* \\ V_{\alpha 2}^* & V_{\beta 2}^* \end{pmatrix} \begin{pmatrix} -p & 0 & m_{\alpha \alpha} & m_{\alpha \beta} \\ 0 & -p & m_{\beta \alpha} & m_{\beta \beta} \\ m_{\alpha \alpha}^* & m_{\beta \alpha}^* & p & 0 \\ m_{\alpha \beta}^* & m_{\beta \beta}^* & 0 & p \end{pmatrix} \begin{pmatrix} V_{\alpha 1} & 0 & 0 & 0 \\ 0 & V_{\beta 1} & 0 & 0 \\ 0 & 0 & U_{\alpha 1} & U_{\alpha 2} \\ 0 & 0 & U_{\beta 1} & U_{\beta 2} \end{pmatrix} = \begin{pmatrix} -p & 0 & m_1 & 0 \\ 0 & -p & 0 & m_2 \\ m_1 & 0 & p & 0 \\ m_2 & 0 & p & 0 \end{pmatrix}. \quad (4.16)$$

It is emphasized here that the chirality does not change in the transformation (4.15). In other words, $\nu_{\alpha L}$ and $\nu_{\beta L}$ with left-handed chirality can be represented by the linear combination of $\nu_{1 L}$ and $\nu_{2 L}$ with also left-handed chirality. This is the reason why we name these two kinds of states “chirality-flavor eigenstates” and “chirality-mass eigenstates”. The time evolution equation for the chirality-mass eigenstates is given by

$$i \frac{d}{dt} \begin{pmatrix} \nu_{1 R} \\ \nu_{2 R} \\ \nu_{1 L} \\ \nu_{2 L} \end{pmatrix} = \begin{pmatrix} -p & 0 & m_1 & 0 \\ 0 & -p & 0 & m_2 \\ m_1 & 0 & p & 0 \\ 0 & m_2 & 0 & p \end{pmatrix} \begin{pmatrix} \nu_{1 R} \\ \nu_{2 R} \\ \nu_{1 L} \\ \nu_{2 L} \end{pmatrix}. \quad (4.17)$$

Next, let us rewrite the above equation to that for the energy-helicity eigenstates to diagonalize the Hamiltonian completely. Exchanging some rows and some columns in (4.17), we obtain the equation,

$$i \frac{d}{dt} \begin{pmatrix} \nu_{1 R} \\ \nu_{1 L} \\ \nu_{2 R} \\ \nu_{2 L} \end{pmatrix} = \begin{pmatrix} -p m_1 & 0 & 0 \\ m_1 p & 0 & 0 \\ 0 & -p m_2 & 0 \\ 0 & 0 & m_2 p \end{pmatrix} \begin{pmatrix} \nu_{1 R} \\ \nu_{1 L} \\ \nu_{2 R} \\ \nu_{2 L} \end{pmatrix}. \quad (4.18)$$
We further diagonalize the Hamiltonian in the above equation by replacing the relation between the chirality-mass eigenstates and the energy-helicity eigenstates,

\[
\begin{pmatrix}
\nu_{1R} \\
\nu_{1L} \\
\nu_{2R} \\
\nu_{2L}
\end{pmatrix} =
\begin{pmatrix}
\sqrt{\frac{E_1 + p}{2E_1}} & \sqrt{\frac{E_1 - p}{2E_1}} & 0 & 0 \\
-\sqrt{\frac{E_1 - p}{2E_1}} & \sqrt{\frac{E_1 + p}{2E_1}} & 0 & 0 \\
0 & 0 & \sqrt{\frac{E_2 + p}{2E_2}} & \sqrt{\frac{E_2 - p}{2E_2}} \\
0 & 0 & -\sqrt{\frac{E_2 - p}{2E_2}} & \sqrt{\frac{E_2 + p}{2E_2}}
\end{pmatrix}
\begin{pmatrix}
\nu_1^- \\
\nu_1^+ \\
\nu_2^- \\
\nu_2^+
\end{pmatrix},
\]  

(4.19)

and the time evolution equation for the energy-helicity eigenstates is given by

\[
\frac{d}{dt} \begin{pmatrix}
\nu_1^- \\
\nu_1^+ \\
\nu_2^- \\
\nu_2^+
\end{pmatrix} =
\begin{pmatrix}
-E_1 & 0 & 0 & 0 \\
0 & E_1 & 0 & 0 \\
0 & 0 & -E_2 & 0 \\
0 & 0 & 0 & E_2
\end{pmatrix}
\begin{pmatrix}
\nu_1^- \\
\nu_1^+ \\
\nu_2^- \\
\nu_2^+
\end{pmatrix},
\]  

(4.20)

where

\[
E_1 = \sqrt{p^2 + m_1^2}, \quad E_2 = \sqrt{p^2 + m_2^2}.
\]  

(4.21)

From the equations (4.15) and (4.19), the chirality-flavor eigenstates are related by the energy-helicity eigenstates as

\[
\begin{pmatrix}
\nu_{\alpha R} \\
\nu_{\beta R} \\
\nu_{\alpha L} \\
\nu_{\beta L}
\end{pmatrix} =
\begin{pmatrix}
V_{\alpha 1} & V_{\alpha 2} & 0 & 0 \\
V_{\beta 1} & V_{\beta 2} & 0 & 0 \\
0 & 0 & U_{\alpha 1} & U_{\alpha 2} \\
0 & 0 & U_{\beta 1} & U_{\beta 2}
\end{pmatrix}
\begin{pmatrix}
\nu_1^- \\
\nu_1^+ \\
\nu_2^- \\
\nu_2^+
\end{pmatrix},
\]  

(4.22)

Rewriting these relations for fields to those for one particle states, the chirality-flavor eigenstates after the time \(t\) are given by

\[
|\nu_{\alpha R}(t)\rangle = V_{\alpha 1}^* \sqrt{\frac{E_1 + p}{2E_1}} e^{iE_1 t} |\nu_1^-\rangle + V_{\alpha 2}^* \sqrt{\frac{E_1 - p}{2E_1}} e^{-iE_1 t} |\nu_1^+\rangle + V_{\alpha 1} \sqrt{\frac{E_2 + p}{2E_2}} e^{iE_2 t} |\nu_2^-\rangle + V_{\alpha 2} \sqrt{\frac{E_2 - p}{2E_2}} e^{-iE_2 t} |\nu_2^+\rangle,
\]  

(4.23)

\[
|\nu_{\beta R}(t)\rangle = V_{\beta 1}^* \sqrt{\frac{E_1 + p}{2E_1}} e^{iE_1 t} |\nu_1^-\rangle + V_{\beta 2}^* \sqrt{\frac{E_1 - p}{2E_1}} e^{-iE_1 t} |\nu_1^+\rangle + V_{\beta 1} \sqrt{\frac{E_2 + p}{2E_2}} e^{iE_2 t} |\nu_2^-\rangle + V_{\beta 2} \sqrt{\frac{E_2 - p}{2E_2}} e^{-iE_2 t} |\nu_2^+\rangle,
\]  

(4.24)
\[ |\nu_{\alpha L}(t)\rangle = -U^*_{\alpha 1}\sqrt{\frac{E_1 - p}{2E_1}}e^{iE_1 t}\langle\nu_1^- | + U^*_{\alpha 1}\sqrt{\frac{E_1 + p}{2E_1}}e^{-iE_1 t}\langle\nu_1^+ | \]
\[ -U^*_{\alpha 2}\sqrt{\frac{E_2 - p}{2E_2}}e^{iE_2 t}\langle\nu_2^- | + U^*_{\alpha 2}\sqrt{\frac{E_2 + p}{2E_2}}e^{-iE_2 t}\langle\nu_2^+ | \]
\[ |\nu_{\beta L}(t)\rangle = -U^*_{\beta 1}\sqrt{\frac{E_1 - p}{2E_1}}e^{iE_1 t}\langle\nu_1^- | + U^*_{\beta 1}\sqrt{\frac{E_1 + p}{2E_1}}e^{-iE_1 t}\langle\nu_1^+ | \]
\[ -U^*_{\beta 2}\sqrt{\frac{E_2 - p}{2E_2}}e^{iE_2 t}\langle\nu_2^- | + U^*_{\beta 2}\sqrt{\frac{E_2 + p}{2E_2}}e^{-iE_2 t}\langle\nu_2^+ | , \] (4.25)

and we also have the conjugate states,

\[ \langle\nu_{\alpha R} | = V_{\alpha 1}\sqrt{\frac{E_1 - p}{2E_1}}\langle\nu_1^- | + V_{\alpha 1}\sqrt{\frac{E_1 + p}{2E_1}}\langle\nu_1^+ | + V_{\alpha 2}\sqrt{\frac{E_2 - p}{2E_2}}\langle\nu_2^- | + V_{\alpha 2}\sqrt{\frac{E_2 + p}{2E_2}}\langle\nu_2^+ | , \] (4.27)
\[ \langle\nu_{\beta R} | = V_{\beta 1}\sqrt{\frac{E_1 - p}{2E_1}}\langle\nu_1^- | + V_{\beta 1}\sqrt{\frac{E_1 + p}{2E_1}}\langle\nu_1^+ | + V_{\beta 2}\sqrt{\frac{E_2 - p}{2E_2}}\langle\nu_2^- | + V_{\beta 2}\sqrt{\frac{E_2 + p}{2E_2}}\langle\nu_2^+ | , \] (4.28)
\[ \langle\nu_{\alpha L} | = -U_{\alpha 1}\sqrt{\frac{E_1 - p}{2E_1}}\langle\nu_1^- | + U_{\alpha 1}\sqrt{\frac{E_1 + p}{2E_1}}\langle\nu_1^+ | - U_{\alpha 2}\sqrt{\frac{E_2 - p}{2E_2}}\langle\nu_2^- | + U_{\alpha 2}\sqrt{\frac{E_2 + p}{2E_2}}\langle\nu_2^+ | , \] (4.29)
\[ \langle\nu_{\beta L} | = -U_{\beta 1}\sqrt{\frac{E_1 - p}{2E_1}}\langle\nu_1^- | + U_{\beta 1}\sqrt{\frac{E_1 + p}{2E_1}}\langle\nu_1^+ | - U_{\beta 2}\sqrt{\frac{E_2 - p}{2E_2}}\langle\nu_2^- | + U_{\beta 2}\sqrt{\frac{E_2 + p}{2E_2}}\langle\nu_2^+ | . \] (4.30)

It is crucial that the chirality-flavor eigenstates are constituted by not only positive energy states but also negative energy states. This is because the positive energy states and the negative energy states are included in the same multiplet. Then, the amplitudes are calculated by

\[ A(\nu_{\alpha L} \rightarrow \nu_{\alpha L}) = \langle\nu_{\alpha L} | \nu_{\alpha L}(t)\rangle \]
\[ = |U_{\alpha 1}|^2 \sqrt{\frac{E_1 - p}{2E_1}}e^{iE_1 t} + |U_{\alpha 1}|^2 \sqrt{\frac{E_1 + p}{2E_1}}e^{-iE_1 t} + |U_{\alpha 2}|^2 \sqrt{\frac{E_2 - p}{2E_2}}e^{iE_2 t} + |U_{\alpha 2}|^2 \sqrt{\frac{E_2 + p}{2E_2}}e^{-iE_2 t} \]
\[ = |U_{\alpha 1}|^2 \cos(E_1 t) - i|U_{\alpha 1}|^2 \cdot \frac{p}{E_1} \sin(E_1 t) + |U_{\alpha 2}|^2 \cos(E_2 t) - i|U_{\alpha 2}|^2 \cdot \frac{p}{E_2} \sin(E_2 t) , \] (4.31)
\[ A(\nu_{\alpha L} \rightarrow \nu_{\beta L}) = \langle\nu_{\beta L} | \nu_{\alpha L}(t)\rangle \]
\[ = U^*_{\alpha 1}U_{\beta 1} \left( \sqrt{\frac{E_1 - p}{2E_1}}e^{iE_1 t} + \sqrt{\frac{E_1 + p}{2E_1}}e^{-iE_1 t} \right) + U^*_{\alpha 2}U_{\beta 2} \left( \sqrt{\frac{E_2 - p}{2E_2}}e^{iE_2 t} + \sqrt{\frac{E_2 + p}{2E_2}}e^{-iE_2 t} \right) \]
\[ = U^*_{\alpha 1}U_{\beta 1} \left\{ \cos(E_1 t) - i \cdot \frac{p}{E_1} \sin(E_1 t) \right\} + U^*_{\alpha 2}U_{\beta 2} \left\{ \cos(E_2 t) - i \cdot \frac{p}{E_2} \sin(E_2 t) \right\} , \] (4.32)
\[ A(\nu_{\beta L} \rightarrow \nu_{\alpha L}) = \langle\nu_{\alpha L} | \nu_{\beta L}(t)\rangle \]
\[ = -U^*_{\beta 1}V_{\alpha 1} \frac{m_1}{E_1} (e^{iE_1 t} - e^{-iE_1 t}) - U^*_{\beta 2}V_{\alpha 2} \frac{m_2}{E_2} (e^{iE_2 t} - e^{-iE_2 t}) \]
\[ = -iU^*_{\beta 1}V_{\alpha 1} \frac{m_1}{E_1} \sin(E_1 t) - iU^*_{\beta 2}V_{\alpha 2} \frac{m_2}{E_2} \sin(E_2 t) , \] (4.33)
\[ A(\nu_{\beta L} \rightarrow \nu_{\beta L}) = \langle\nu_{\beta L} | \nu_{\beta L}(t)\rangle \]
\[ = U^*_{\beta 1}V_{\beta 1} \frac{m_1}{E_1} (e^{iE_1 t} - e^{-iE_1 t}) - U^*_{\beta 2}V_{\beta 2} \frac{m_2}{E_2} (e^{iE_2 t} - e^{-iE_2 t}) \]
\[ = -iU^*_{\beta 1}V_{\beta 1} \frac{m_1}{E_1} \sin(E_1 t) - iU^*_{\beta 2}V_{\beta 2} \frac{m_2}{E_2} \sin(E_2 t) , \] (4.34)
Furthermore, we obtain the oscillation probabilities by squaring these amplitudes,

\[
P(\nu_{aL} \rightarrow \nu_{aL}) = \{ |U_{a1}|^2 \cos(E_1 t) + |U_{a2}|^2 \cos(E_2 t) \}^2 \\
+ \left\{ |U_{a1}|^2 \cdot \frac{p}{E_1} \sin(E_1 t) + |U_{a2}|^2 \cdot \frac{p}{E_2} \sin(E_2 t) \right\}^2, \quad (4.35)
\]

\[
P(\nu_{aL} \rightarrow \nu_{3\beta L}) = |U_{a1} U_{\beta 1}|^2 \left\{ \cos^2(E_1 t) + \frac{p^2}{E_1^2} \sin^2(E_1 t) \right\} \\
+ |U_{a2} U_{\beta 2}|^2 \left\{ \cos^2(E_2 t) + \frac{p^2}{E_2^2} \sin^2(E_2 t) \right\} \\
+ 2 \text{Re} \left[ U_{a1}^* U_{\beta 1} \left\{ \cos(E_1 t) - i \frac{p}{E_1} \sin(E_1 t) \right\} U_{a2}^* U_{\beta 2} \left\{ \cos(E_2 t) + i \frac{p}{E_2} \sin(E_2 t) \right\} \right], \quad (4.36)
\]

\[
P(\nu_{aL} \rightarrow \nu_{aR}) = |U_{a1} V_{a1}|^2 \frac{m_2^2}{E_1^2} \sin^2(E_1 t) + |U_{a2} V_{a2}|^2 \frac{m_2^2}{E_2^2} \sin^2(E_2 t) \\
+ 2 \text{Re} [U_{a1}^* U_{a2}^* V_{a1} V_{a2} \left( \frac{m_1 m_2}{E_1 E_2} \sin(E_1 t) \sin(E_2 t) \right)], \quad (4.37)
\]

\[
P(\nu_{aL} \rightarrow \nu_{\beta R}) = |U_{a1} V_{\beta 1}|^2 \frac{m_2^2}{E_1^2} \sin^2(E_1 t) + |U_{a2} V_{\beta 2}|^2 \frac{m_2^2}{E_2^2} \sin^2(E_2 t) \\
+ 2 \text{Re} [U_{a1}^* U_{a2}^* V_{\beta 1} V_{\beta 2} \left( \frac{m_1 m_2}{E_1 E_2} \sin(E_1 t) \sin(E_2 t) \right)], \quad (4.38)
\]

In general, 2 × 2 unitary matrices \(U\) and \(V\) are parametrized by

\[
U = \begin{pmatrix}
    e^{i\rho_{1L}} & 0 \\
    0 & e^{i\rho_{2L}}
\end{pmatrix}
\begin{pmatrix}
    \cos \theta_L & \sin \theta_L \\
    -\sin \theta_L & \cos \theta_L
\end{pmatrix}
\begin{pmatrix}
    1 & 0 \\
    0 & e^{i\phi_L}
\end{pmatrix} = \begin{pmatrix}
    e^{i\rho_{1L} \cos \theta_L} e^{i(\rho_{1L} + \phi_L) \sin \theta_L} \\
    -e^{i\rho_{2L} \sin \theta_L} e^{i(\rho_{2L} + \phi_L) \cos \theta_L}
\end{pmatrix}, \quad (4.39)
\]

\[
V = \begin{pmatrix}
    e^{i\rho_{1R}} & 0 \\
    0 & e^{i\rho_{2R}}
\end{pmatrix}
\begin{pmatrix}
    \cos \theta_R & \sin \theta_R \\
    -\sin \theta_R & \cos \theta_R
\end{pmatrix}
\begin{pmatrix}
    1 & 0 \\
    0 & e^{i\phi_R}
\end{pmatrix} = \begin{pmatrix}
    e^{i\rho_{1R} \cos \theta_R} e^{i(\rho_{1R} + \phi_R) \sin \theta_R} \\
    -e^{i\rho_{2R} \sin \theta_R} e^{i(\rho_{2R} + \phi_R) \cos \theta_R}
\end{pmatrix}, \quad (4.40)
\]

respectively. Substituting these representations into (4.35)-(4.38), the oscillation probabilities for \(\nu_{aL}\) to \(\nu_{aL}, \nu_{\beta L}, \nu_{aR}\) and \(\nu_{\beta R}\) are given by

\[
P(\nu_{aL} \rightarrow \nu_{aL}) = 1 - 4s_{L}^2 c_{L}^2 \sin^2 \left( \frac{E_2 - E_1}{2} t \right) \\
- \left[ c_{L}^4 \cdot \frac{m_2^2}{E_1^2} \sin^2(E_1 t) + s_{L}^4 \cdot \frac{m_2^2}{E_2^2} \sin^2(E_2 t) + 2s_{L}^2 c_{L}^2 \left( 1 - \frac{p^2}{E_1 E_2} \right) \sin(E_1 t) \sin(E_2 t) \right], \quad (4.41)
\]

\[
P(\nu_{aL} \rightarrow \nu_{\beta L}) = 4s_{L}^2 c_{L}^2 \sin^2 \left( \frac{E_2 - E_1}{2} t \right) \\
- s_{L}^2 c_{L}^2 \left[ \frac{m_2^2}{E_1^2} \sin^2(E_1 t) + \frac{m_2^2}{E_2^2} \sin^2(E_2 t) - 2 \left( 1 - \frac{p^2}{E_1 E_2} \right) \sin(E_1 t) \sin(E_2 t) \right], \quad (4.42)
\]

\[
P(\nu_{aL} \rightarrow \nu_{aR}) = c_{L}^2 s_{L}^2 \frac{m_2^2}{E_1} \sin^2(E_1 t) + s_{L}^2 s_{R}^2 \frac{m_2^2}{E_2} \sin^2(E_2 t) \\
+ 2s_{L} s_{R} c_{L} c_{R} \cos(\phi_L - \phi_R) \frac{m_1 m_2}{E_1 E_2} \sin(E_1 t) \sin(E_2 t), \quad (4.43)
\]

\[
P(\nu_{aL} \rightarrow \nu_{\beta R}) = c_{L}^2 s_{R}^2 \frac{m_2^2}{E_1} \sin^2(E_1 t) + s_{L}^2 c_{R}^2 \frac{m_2^2}{E_2} \sin^2(E_2 t) \\
- 2s_{L} s_{R} c_{L} c_{R} \cos(\phi_L - \phi_R) \frac{m_1 m_2}{E_1 E_2} \sin(E_1 t) \sin(E_2 t), \quad (4.44)
\]
where we use the abbreviation \( \sin \theta_L = s_L \) and \( \cos \theta_L = c_L \) and so on. As shown above, we derived the oscillation probabilities with and without chirality-flip in a unified way by using the Dirac equation. The oscillation probabilities obtained above are exact ones based on the equal momentum assumption. Taking the limit \( m/E \to 0 \), one can see that these oscillation probabilities coincide with the well-known probabilities derived by the Schrödinger equation.

Here, we list some crucial points in order. First, in the probabilities without chirality-flip, the new terms \( (4.42) \) and \( (4.44) \) appear in addition to the well-known terms \( (4.41) \) and \( (4.43) \). At the same time, the probabilities with chirality-flip are in order \( O(m^2/E^2) \). As shown later, these terms in order \( O(m^2/E^2) \) cancel out each other and the sum of all probabilities is kept in one. Second, we found that the oscillation probabilities depend on not only the mass squared differences but also the absolute value of mass in exact expressions. Third, the new CP phase appears in the probabilities even for Dirac neutrino in two generations if there exist some new interactions which distinguish the flavor of right-handed neutrinos. This new phase appears in the probabilities with chirality-flip and can be interpreted to be a comprehensive phase including the Majorana phase.

The transition probability between \( \nu_\alpha \) and \( \nu_\beta \) was calculated relativistically in [42] but approximetely up to order \( O(m^2/E^2) \). In refs. [43–45], the transition probability was also derived by the wave-packet formalism without any approximations and after some calculations, their result is in agreement with our expressions \( (4.43) \) and \( (4.44) \). However, they didn’t mention the new CP phase effect and the mixing of right-handed neutrinos. We derive the exact and complete oscillation probabilities relativistically and present the dependence on the new CP phase and the mixing of right-handed neutrinos for the first time.

Let us comment about \( \nu'_L \) and \( \nu'_R \) in the end of this subsection. In order to obtain the probabilities for \( \nu' \), we only have to replace \( p \) to \( -p \) in \( (4.41)-(4.46) \). As the oscillation probabilities only depend on \( p^2 \), we obtain the same probabilities as those for \( \nu_L \) and \( \nu_R \) in this framework.

### 4.2 Energy and Baseline Where New Terms give Non-trivial Contribution

From the results obtained above, it is very interesting to explore the case that the new terms have a non-trivial contribution. We discuss under what situations the contribution of the new terms is comparative to that of usual terms. Replacing the time \( t \) to the distance \( L \) that neutrinos travel in the time \( t \), the leading terms contributing to usual neutrino oscillations have the magnitude of order \( \sin^2(m^2L/E) \). On the other hand, the new terms have the contribution with the magnitude of order \( m^2/E^2 \sin^2(EL) \). Below, let us divide into three cases by the value of \( EL \) and \( m^2L/E \), and compare the magnitude of the two kinds of terms.

(i) Case for \( EL < 1 \), i.e. case for \( L < 10^{-6} \text{m} \cdot \frac{\text{eV}}{E} \)

If the contribution of the new terms and the usual terms are nearly equal, we have the relation, \( \sin^2 \left( \frac{m^2L}{E} \right) \simeq \frac{m^2}{E^2} \sin^2(EL) \) and that is approximated by
\[
\left( \frac{m^2 L}{E} \right)^2 \simeq \frac{m^2}{E^2} (EL)^2 \iff \frac{m^2}{E^2} \simeq 1. 
\]
This means that new terms become important for the case that the mass of neutrino is comparative with the energy.

(ii) Case for \(|EL| > 1\) and \(\frac{m^2 L}{E} < 1\), i.e. case for \(10^{-6} \text{m} \cdot \frac{\text{eV}}{E} < L < 10^{-6} \text{m} \cdot \frac{E}{\text{eV}} \cdot \frac{\text{eV}^2}{m^2}\).

The relation, \(\sin^2 \left( \frac{m^2 L}{E} \right) < \frac{m^2}{E^2} \sin^2 (EL) \) is approximated by \(\left( \frac{m^2 L}{E} \right)^2 < \frac{m^2}{E^2} \iff mL < 1 \iff m(eV^2) \cdot L(\text{m}) < 10^{-6}\). If we set the typical neutrino mass about \(m^2 \simeq 10^{-3}(\text{eV}^2)\), the contribution of the new terms is important in the range \(L(\text{m}) < 10^{-3}\) almost without depending on energy.

(iii) Case for \(|EL| > 1\) and \(\frac{m^2 L}{E} > 1\), i.e. case for \(L > 10^{-6} \text{m} \cdot \frac{\text{eV}}{E}\) and \(L > 10^{-6} \text{m} \cdot \frac{E}{\text{eV}} \cdot \frac{\text{eV}^2}{m^2}\).

The relation, \(\sin^2 \left( \frac{m^2 L}{E} \right) \simeq \frac{m^2}{E^2} \sin^2 (EL) \) is approximated by \(\frac{m^2}{E^2} \simeq 1\).

We found that the new terms are important for the case that the neutrino mass is comparative with the energy as in (i). However, if the distance \(L\) becomes too long the wave packets of light neutrinos and heavy neutrinos separate, and the interference, i.e. neutrino oscillations do not occur.

In the previous neutrino experiments, the baseline length is comparatively large and the neutrino mass is small enough to ignore compared to neutrino energy. Accordingly, the contribution of the new terms can be neglected. However, we may observe the contribution of the new terms and obtain the information of the absolute mass of neutrino if we measure the neutrino oscillation probabilities in atomic size like \(0\nu\beta\beta\) decay experiments. Besides, the contribution of the new CP phase and the mixing angle may also appear if some interactions which distinguish the flavor of right-handed neutrinos exist. An example, in which \(\nu_R\) has weak interaction as \(\nu_L\), is the case of Majorana neutrinos. In this case, \(\nu_R\) is identified with \(\nu^c_L\) and therefore, the mixing and the CP phase of \(\nu_R\) relate to those of \(\nu_L\) like \(\phi_R = -\phi_L\). See our next paper for detail. At the end of this subsection, let us comment that the left-right symmetric models were considered in the framework of the Dirac neutrinos [41] and there is a possibility for the mixing angle and the CP phase of \(\nu_R\) to be independent of \(\nu_L\).

4.3 Unitarity Check of Oscillation Probabilities

In the Standard Model, \(\nu_R\) can be identified with the mass eigenstate as it does not have weak interactions. In this situation, \(V\) becomes the identity matrix and we can choose \(s_R = 0\). Therefore, the mixing angle for \(\nu_R\) and the CP phase do not appear and then only the sum of two oscillation probabilities,

\[
P(\nu_{\alpha L} \to \nu_R) = P(\nu_{\alpha L} \to \nu_{\alpha R}) + P(\nu_{\alpha L} \to \nu_{\beta R}) = c_{\alpha 1}^2 \frac{m_1^2}{E_1^2} \sin^2 (E_1 t) + s_{\alpha 1}^2 \frac{m_2^2}{E_2^2} \sin^2 (E_2 t) \tag{4.47}
\]
can be observable. Adding also the two oscillation probabilities to $\nu_{\alpha L}$ and $\nu_{\beta L}$, we obtain

$$P(\nu_{\alpha L} \rightarrow \nu_{\beta L}) = P(\nu_{\alpha L} \rightarrow \nu_{\beta L}) + P(\nu_{\alpha L} \rightarrow \nu_{\beta L})$$

$$= \left\{ c_{\alpha L}^2 \cos^2(E_1 t) + s_{\alpha L}^2 \cos^2(E_2 t) \right\} + \left\{ c_{\alpha L}^2 \cdot \frac{p^2}{E_1^2} \sin^2(E_1 t) + s_{\alpha L}^2 \cdot \frac{p^2}{E_2^2} \sin^2(E_2 t) \right\}.$$ (4.48)

Furthermore, we calculate the total of the oscillation probabilities from $\nu_{\alpha L}$.

$$P(\nu_{\alpha L} \rightarrow \nu_R) + P(\nu_{\alpha L} \rightarrow \nu_L) = c_{\alpha L}^2 \frac{m_2^2}{E_1^2} \sin^2(E_1 t) + s_{\alpha L}^2 \frac{m_3^2}{E_2^2} \sin^2(E_2 t)$$

$$+ \left\{ c_{\alpha L}^2 \cos^2(E_1 t) + s_{\alpha L}^2 \cos^2(E_2 t) \right\} + \left\{ c_{\alpha L}^2 \cdot \frac{E_2^2 - m_3^2}{E_1^2} \sin^2(E_1 t) + s_{\alpha L}^2 \cdot \frac{E_2^2 - m_3^2}{E_2^2} \sin^2(E_2 t) \right\}$$

$$= \left\{ c_{\alpha L}^2 \cos^2(E_1 t) + s_{\alpha L}^2 \cos^2(E_2 t) \right\} + \left\{ c_{\alpha L}^2 \sin^2(E_1 t) + s_{\alpha L}^2 \sin^2(E_2 t) \right\} = c_{\alpha L}^2 + s_{\alpha L}^2 = 1.$$ (4.49)

We have confirmed that the total sum of the probabilities with and without chirality-flip becomes one exactly as expected.

### 4.4 Oscillation Probabilities of Right-Handed Neutrinos

If the flavor of right-handed neutrinos can be measured by some methods beyond the Standard Model, $\nu_R$ oscillations will be observed. The probabilities of these oscillations can be calculated in the same way. For completeness, we write down these probabilities.

Exchanging $L$ to $R$ in the probabilities of $\nu_L$, we have the following expressions,

$$P(\nu_{\alpha R} \rightarrow \nu_{\alpha R}) = 1 - 4s_{\alpha R}^2c_{\alpha R}^2 \sin^2\left(\frac{E_2 - E_1}{2} t\right)$$

$$- \left[ c_{\alpha R}^2 \cdot \frac{m_2^2}{E_1^2} \sin^2(E_1 t) + s_{\alpha R}^2 \frac{m_3^2}{E_2^2} \sin^2(E_2 t) + 2s_{\alpha R}^2c_{\alpha R}^2 \left(1 - \frac{p^2}{E_1 E_2}\right) \sin(E_1 t) \sin(E_2 t) \right], (4.50)$$

$$P(\nu_{\alpha R} \rightarrow \nu_{\beta R}) = 4s_{\alpha R}^2c_{\alpha R}^2 \sin^2\left(\frac{E_2 - E_1}{2} t\right)$$

$$- 2s_{\alpha R}^2c_{\alpha R}^2 \left[ \frac{m_2^2}{E_1^2} \sin^2(E_1 t) + \frac{m_3^2}{E_2^2} \sin^2(E_2 t) - 2 \left(1 - \frac{p^2}{E_1 E_2}\right) \sin(E_1 t) \sin(E_2 t) \right], \quad (4.51)$$

$$P(\nu_{\alpha R} \rightarrow \nu_{\alpha L}) = c_{\alpha L}^2 \frac{m_1^2}{E_1^2} \sin^2(E_1 t) + s_{\alpha L}^2 \frac{m_2^2}{E_2^2} \sin^2(E_2 t)$$

$$+ 2s_{\alpha L}c_{\alpha L} \frac{m_3^2}{E_1} \cos(\phi_R - \phi_L) \frac{m_1 m_2}{E_1 E_2} \sin(E_1 t) \sin(E_2 t), \quad (4.52)$$

$$P(\nu_{\alpha R} \rightarrow \nu_{\beta L}) = c_{\alpha R}^2 \frac{m_1^2}{E_1^2} \sin^2(E_1 t) + c_{\alpha R}^2 \frac{m_2^2}{E_2^2} \sin^2(E_2 t)$$

$$- 2s_{\alpha L}c_{\alpha L} \frac{m_3^2}{E_1} \cos(\phi_R - \phi_L) \frac{m_1 m_2}{E_1 E_2} \sin(E_1 t) \sin(E_2 t). \quad (4.53)$$

The new mixing angle of the right-handed neutrinos can be observed through $\nu_R \rightarrow \nu_R$ oscillations if $W_R$ exists in the left-right symmetric model. In this case, one may produce $\nu_R$ beam by decaying $W_R$ generated from the collision of the high energy pi-mesons to the target. If $W_R$ reacts the matter in the detector and reproduces the charged lepton with right-handed through $W_R$, we have a chance to measure this new mixing angle without suppression due to small neutrino masses.
4.5 Oscillation Probabilities of Anti-Neutrinos

Next, we summarize the oscillation probabilities of anti-neutrinos. Charge conjugation of $\psi$,

$$\psi^c = (\psi_L)^c + (\psi_R)^c \equiv \begin{pmatrix} \nu^c_L \\ \nu^c_R \end{pmatrix} = \begin{pmatrix} i\sigma_2\eta^* \\ -i\sigma_2\xi^* \end{pmatrix} = \begin{pmatrix} \nu^*_L \\ -\nu^*_R \end{pmatrix}, \quad (4.54)$$

also satisfies a Dirac equation,

$$i\gamma^\mu \partial_\mu \psi^c_{cL} - m_\alpha \psi^c_{cR} - m_\beta \psi^c_{\beta R} = 0. \quad (4.55)$$

It is noted that the Dirac equation satisfied by the charge conjugated field $\psi^c$ is a little bit different from that for the original field $\psi$ because the mass terms are complex in general. Namely, the Dirac equation for $\psi^c$ has the complex conjugate mass terms against the original equation. Multiplying $\gamma^0$ from the left, we obtain

$$i\partial_0 \psi^c_{cL} + i\gamma^0 \gamma^i \partial_i \psi^c_{cL} - m_\alpha \gamma^0 \psi^c_{cR} - m_\beta \gamma^0 \psi^c_{\beta R} = 0. \quad (4.56)$$

If we represent this equation by two-components spinors $\xi$ and $\eta$, we obtain the following matrix form,

$$i\partial_0 \left( \begin{array}{c} i\sigma_2\eta^* \\ 0 \end{array} \right) + \left( \begin{array}{c} \sigma_1 \partial_i (i\sigma_2\eta^*_\alpha) \\ 0 \end{array} \right) - m_\alpha \left( \begin{array}{c} -i\sigma_2\xi^*_\alpha \\ 0 \end{array} \right) - m_\beta \left( \begin{array}{c} -i\sigma_2\xi^*_\beta \\ 0 \end{array} \right) = 0, \quad (4.57)$$

and extracting the upper part, we obtain

$$i\partial_0 (i\sigma_2\eta^*_\alpha) + i\sigma_1 \partial_i (i\sigma_2\eta^*_\alpha) - m_\alpha (-i\sigma_2\xi^*_\alpha) - m_\beta (-i\sigma_2\xi^*_\beta) = 0. \quad (4.58)$$

In the same way, we also obtain three equations,

$$i\partial_0 (i\sigma_2\eta^*_\beta) + i\sigma_1 \partial_i (i\sigma_2\eta^*_\beta) - m_\beta (-i\sigma_2\xi^*_\beta) - m_\alpha (-i\sigma_2\xi^*_\alpha) = 0, \quad (4.59)$$

$$i\partial_0 (-i\sigma_2\xi^*_\alpha) - i\sigma_1 \partial_i (-i\sigma_2\xi^*_\alpha) - m_\alpha (i\sigma_2\eta^*_\alpha) - m_\beta (i\sigma_2\eta^*_\beta) = 0, \quad (4.60)$$

$$i\partial_0 (-i\sigma_2\xi^*_\beta) - i\sigma_1 \partial_i (-i\sigma_2\xi^*_\beta) - m_\beta (i\sigma_2\eta^*_\beta) - m_\alpha (i\sigma_2\eta^*_\alpha) = 0. \quad (4.61)$$

Here, we take the complex conjugate of (4.10) and (4.11),

$$\eta^*_\alpha (x, t) = e^{-i\vec{p}.\vec{x}} \eta^*_\alpha (t) = e^{-i\vec{p}.\vec{x}} \begin{pmatrix} \nu^*_L \\ \nu^*_R \end{pmatrix}, \quad \xi^*_\alpha (x, t) = e^{-i\vec{p}.\vec{x}} \xi^*_\alpha (t) = e^{-i\vec{p}.\vec{x}} \begin{pmatrix} \nu^*_L \\ \nu^*_R \end{pmatrix}, \quad (4.62)$$

$$\eta^*_\beta (x, t) = e^{-i\vec{p}.\vec{x}} \eta^*_\beta (t) = e^{-i\vec{p}.\vec{x}} \begin{pmatrix} \nu^*_{\beta L} \\ \nu^*_{\beta R} \end{pmatrix}, \quad \xi^*_\beta (x, t) = e^{-i\vec{p}.\vec{x}} \xi^*_\beta (t) = e^{-i\vec{p}.\vec{x}} \begin{pmatrix} \nu^*_{\beta L} \\ \nu^*_{\beta R} \end{pmatrix}. \quad (4.63)$$
and choose \( \vec{p} = (0, 0, p) \), the equations (4.58)-(4.61) are rewritten by the matrix form,

\[
\begin{align*}
    i\hbar \dot{\nu}^c_{\alpha L} + p \begin{pmatrix}
        \nu^c_{\alpha L} \\
        \nu^c_{\alpha R}
    \end{pmatrix} & - m_{\alpha\alpha} \begin{pmatrix}
        \nu^c_{\alpha L} \\
        \nu^c_{\alpha R}
    \end{pmatrix} - m_{\beta\alpha} \begin{pmatrix}
        \nu^c_{\beta R} \\
        \nu^c_{\beta L}
    \end{pmatrix} = 0, \\
    i\hbar \dot{\nu}^c_{\beta L} + p \begin{pmatrix}
        \nu^c_{\beta L} \\
        \nu^c_{\beta R}
    \end{pmatrix} & - m_{\beta\beta} \begin{pmatrix}
        \nu^c_{\beta L} \\
        \nu^c_{\beta R}
    \end{pmatrix} - m_{\alpha\beta} \begin{pmatrix}
        \nu^c_{\alpha R} \\
        \nu^c_{\alpha L}
    \end{pmatrix} = 0, \\
    i\hbar \dot{\nu}^c_{\alpha R} - p \begin{pmatrix}
        \nu^c_{\alpha R} \\
        \nu^c_{\alpha L}
    \end{pmatrix} & - m_{\alpha\alpha}^* \begin{pmatrix}
        \nu^c_{\alpha R} \\
        \nu^c_{\alpha L}
    \end{pmatrix} - m_{\alpha\beta}^* \begin{pmatrix}
        \nu^c_{\beta R} \\
        \nu^c_{\beta L}
    \end{pmatrix} = 0, \\
    i\hbar \dot{\nu}^c_{\beta R} - p \begin{pmatrix}
        \nu^c_{\beta R} \\
        \nu^c_{\beta L}
    \end{pmatrix} & - m_{\alpha\alpha}^* \begin{pmatrix}
        \nu^c_{\beta R} \\
        \nu^c_{\beta L}
    \end{pmatrix} - m_{\beta\beta}^* \begin{pmatrix}
        \nu^c_{\alpha R} \\
        \nu^c_{\alpha L}
    \end{pmatrix} = 0.
\end{align*}
\]  

Equations obtained above can be combined to one matrix form,

\[
\frac{d}{dt} \begin{pmatrix}
    \nu^c_{\alpha R} \\
    \nu^c_{\beta R} \\
    \nu^c_{\alpha L} \\
    \nu^c_{\beta L}
\end{pmatrix} = \begin{pmatrix}
    -p & 0 & m_{\alpha\alpha}^* & m_{\alpha\beta}^* \\
    0 & -p & m_{\beta\alpha}^* & m_{\beta\beta}^* \\
    m_{\alpha\alpha} m_{\beta\alpha} & p & 0 & 0 \\
    m_{\beta\alpha} m_{\beta\beta} & 0 & -p & 0
\end{pmatrix} \begin{pmatrix}
    \nu^c_{\alpha R} \\
    \nu^c_{\beta R} \\
    \nu^c_{\alpha L} \\
    \nu^c_{\beta L}
\end{pmatrix}. \tag{4.68}
\]

Note that this equation is for the anti-neutrinos with negative momentum, \(-p\). Changing the sign of momentum in order to calculate the oscillation probabilities with \(p\), the above equation is rewritten as

\[
\frac{d}{dt} \begin{pmatrix}
    \nu^c_{\alpha R} \\
    \nu^c_{\beta R} \\
    \nu^c_{\alpha L} \\
    \nu^c_{\beta L}
\end{pmatrix} = \begin{pmatrix}
    p & 0 & m_{\alpha\alpha} & m_{\alpha\beta} \\
    0 & p & m_{\beta\alpha} & m_{\beta\beta} \\
    m_{\alpha\alpha} m_{\beta\alpha} & -p & 0 & 0 \\
    m_{\beta\alpha} m_{\beta\beta} & 0 & p & 0
\end{pmatrix} \begin{pmatrix}
    \nu^c_{\alpha R} \\
    \nu^c_{\beta R} \\
    \nu^c_{\alpha L} \\
    \nu^c_{\beta L}
\end{pmatrix}. \tag{4.69}
\]

Comparing this to (4.13), one can see that the matrix for anti-neutrinos can be obtained by the replacement as follows;

\[
\begin{pmatrix}
    \nu_{\alpha R} \\
    \nu_{\beta R} \\
    \nu_{\alpha L} \\
    \nu_{\beta L}
\end{pmatrix} \rightarrow \begin{pmatrix}
    \nu'_{\alpha R} \\
    \nu'_{\beta R} \\
    \nu'_{\alpha L} \\
    \nu'_{\beta L}
\end{pmatrix}, \quad \begin{pmatrix}
    \nu'_{\alpha R} \\
    \nu'_{\beta R} \\
    \nu'_{\alpha L} \\
    \nu'_{\beta L}
\end{pmatrix} \rightarrow \begin{pmatrix}
    \nu_{\alpha R} \\
    \nu_{\beta R} \\
    \nu_{\alpha L} \\
    \nu_{\beta L}
\end{pmatrix}, \quad m \rightarrow m^*.
\]

As seen from the replacement, the CP conjugate state of \( \nu_L \) is considered to be \( \nu_L' \). In the anti-neutrino case, the unitary matrices \( U \) and \( V \) diagonalizing the mass part are changed
into $U^*$ and $V^*$ according to the complexation of the mass terms. Therefore, the sign of the phase $\phi$ is converted. So, the oscillation probabilities for anti-neutrinos are given by

$$P(\nu^c_{\alpha L} \to \nu^c_{\alpha L}) = P(\nu_{\alpha L} \to \nu_{\alpha L})$$

$$= 1 - 4s^2_{L}c^2_{L} \sin^2 \frac{(E_2 - E_1)t}{2} - \left[ c^4_L \cdot \frac{m^2}{E^2_1} \sin^2(E_1t) + s^4_L \cdot \frac{m^2}{E^2_2} \sin^2(E_2t) + 2s^2_L c^2_L \left( 1 - \frac{p^2}{E_1 E_2} \right) \sin(E_1t) \sin(E_2t) \right], \quad (4.71)$$

$$P(\nu^c_{\alpha L} \to \nu^c_{\beta L}) = P(\nu_{\alpha L} \to \nu_{\beta L})$$

$$= 4s^2_{L}c^2_{L} \sin^2 \frac{(E_2 - E_1)t}{2} - s^2_{L}c^2_{L} \left[ \frac{m^2}{E^2_1} \sin^2(E_1t) + \frac{m^2}{E^2_2} \sin^2(E_2t) - 2 \left( 1 - \frac{p^2}{E_1 E_2} \right) \sin(E_1t) \sin(E_2t) \right], \quad (4.72)$$

$$P(\nu^c_{\alpha L} \to \nu^c_{\alpha R}) = P(\nu_{\alpha L} \to \nu_{\alpha R})$$

$$= c^2_{L}c^2_{R} \cdot \frac{m^2}{E^2_1} \sin^2(E_1t) + s^2_{L}c^2_{R} \cdot \frac{m^2}{E^2_2} \sin^2(E_2t)$$

$$+ 2s_{L}s_{R}c_{L}c_{R} \cos(\phi_R - \phi_L) \frac{m_1 m_2}{E_1 E_2} \sin(E_1t) \sin(E_2t), \quad (4.73)$$

$$P(\nu^c_{\alpha L} \to \nu^c_{\beta R}) = P(\nu_{\alpha L} \to \nu_{\beta R})$$

$$= c^2_{L}s^2_{R} \cdot \frac{m^2}{E^2_1} \sin^2(E_1t) + s^2_{L}c^2_{R} \cdot \frac{m^2}{E^2_2} \sin^2(E_2t)$$

$$- 2s_{L}s_{R}c_{L}c_{R} \cos(\phi_R - \phi_L) \frac{m_1 m_2}{E_1 E_2} \sin(E_1t) \sin(E_2t). \quad (4.74)$$

One can see that the probabilities of anti-neutrinos are the same as those for neutrinos in two generation scheme. The point is that $U$ and $V$ become complex conjugates in the case of anti-neutrinos. This is the origin of the difference between neutrinos and anti-neutrinos in more than three generations.

5 Summary

To summarize the paper, we formalized the theory of neutrino oscillations exactly by using the Dirac equation. In this paper, we considered the case of two-generation Dirac neutrino in vacuum. We understood the neutrino oscillations with and without chirality-flip in a unified way, which was discussed separately in previous papers. We have also shown that the unitarity holds only after these two kinds of oscillations are taken into account.

The relativistic method developed in this paper extended to the case for the oscillations with $n$ generations, matter effects, and magnetic fields, etc. We found that the two kinds of new terms appear in the oscillation probabilities. One is the term dependent on the absolute mass of neutrino and the other is the term including a new CP phase although we consider only two generations. This phase is comprehensive in the sense of including the Majorana phase [36, 46–48]. The first new term means that we can measure not only the mass squared differences but also the absolute mass by neutrino oscillation experiments in principle. The new CP phase effect may be detectable if there are some interactions to
distinguish the flavor of the right-handed neutrinos beyond the Standard Model. These new terms can be comparatively large in neutrino oscillations of atomic size like $\nu_{e}/\beta\beta$ decay. On the other hand, the contribution of these new terms is negligible and it is consistent with previous neutrino experiments. We succeeded to extend the theory of neutrino oscillation relativistically in a natural way.

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