REVIEW ARTICLE

Dynamics of Higher-order Bright and Dark Rogue Waves in a New (2+1)-Dimensional Integrable Boussinesq Model

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Abstract
This work deals with the dynamics of higher-order rogue waves in a new integrable (2+1)-dimensional Boussinesq equation governing the evolution of high and steep gravity water waves. To achieve this objective, we construct rogue wave solutions by employing Bell polynomial and Hirota’s bilinearization method, along with the generalized polynomial function. Through the obtained rogue wave solutions, we explore the impact of various system and solution parameters in their dynamics. Primarily, these parameters determine the characteristics of rogue waves, including the identification of their type, bright or dark type localized structures, and manipulation of their amplitude, depth, and width. Reported results will be encouraging to the studies on the rogue wave in higher dimensional systems as well as to experimental investigations on the controlling mechanism of rogue waves in optical systems, atomic condensates, and deep water oceanic waves.

KEYWORDS:
Higher dimensional nonlinear model; Boussinesq Equation; Rogue Waves; Rational Solitons

1 | INTRODUCTION

Oceanic rogue waves are one of the significant nonlinear wave structures arising in the deep sea, and they played an essential role in the naval transport, which has been recorded in various instances\textsuperscript{1}. These rogue waves are nothing but a new kind of nonlinear type coherent structures localized in all dimensions/direction and short-lived in time. Due to this non-trivial behavior, it has been defined as ‘wave coming from nowhere and disappears with no trace’\textsuperscript{2}. Such doubly localized (both in spatial and temporal dimensions) excited wave structures can also be referred to as “freak waves, extreme waves, monster waves, killer waves, and giant waves” in the literature. An important reason for these definitions is their exceptionally high amplitude with multi-fold magnitudes on a steady sea state, which provides the ability to destroy even bigger boats and ships during their transport in the deep sea as well as the oil platforms\textsuperscript{3-7}. In contrary to other nonlinear waves that have stable long-living characteristics, for example, solitary waves or solitons, the significance/effect of these rogue waves are much less. Also, due to their occurrence, they have attracted limited interest until a few decades ago. However, their presence in other physical systems, especially optical communication systems and atomic condensates, ignited interest among the researchers working along with those fields. Since then, numerous results on the dynamics and importance of those rogue waves are being reported regularly, particularly in the past two decades. Further, it has been recreated in an artificial water tank experiment recently by executing the familiar nonlinear
Schrödinger model. Also, wind-perturbed rogue waves in the hydrodynamics system and an annular water flume that was also experimented in laboratory. Additionally, the initially recorded Draupner wave was recently recreated in the laboratory to understand the role of breaking in crossing seas.

Although various observations confirm the existence of rogue waves, the origin of rogue waves remains a debatable subject. Several theoretical studies and certain experimental investigations showed the modulation instability as a pioneering phenomenon in their generation. Further, the synchronization of several coherent structures has also been understood as another aspect of these rogue wave generation. Their behavior is mysterious and it can also be explained with the chaotic phenomenon. Also, it has been shown that rogue waves can appear in a wider range of systems apart from the hydrodynamic model including nonlinear optics and lasers, plasma physics, and matter waves (Bose-Einstein condensate). These investigations have given further impulse, and the interest in rogue waves is now a well-motivated multidisciplinary research area. Additionally, a steady increase in the occurrence of these monster waves in some ocean regions due to extreme weather because of global warming has led to a focus on their intensive exploration. Still, there exist several open questions on their formation and dynamics, that enable several researchers towards a continuous study on these rogue waves in recent years.

Mathematically, the rogue waves can be well defined by a set of nonlinear differential equations of scalar (single) as well as multi-component systems. Such rogue waves also arise in higher-dimensional systems. It is important to note that some of the (2+1)-dimensional systems have fundamental rogue waves which further contain a line profile known as line rogue waves studied with analytical and numerical methods. There is a considerable difference between the profile of fundamental rogue waves in a (1+1)-dimensional system and that of (2+1)-dimensional systems. The central peak is surrounded by several gradually decreasing peaks in (1+1)-dimensional system which are quite distinct in non-fundamental rogue waves. There are various reports on the rogue wave solutions for different evolution equations constructed using different analytical methods, including the famous Inverse spectral transform, Darboux transformation, Hirota method, dressing method, and several ansatz approaches. The results on the rogue waves reveal that their dynamics in (2+1)-dimensional systems is a fascinating topic and deserves further investigations in different soliton equations. Due to the crucial role of nonlinearity in ocean wave dynamics, the formation of rogue waves and the difficulty in solving the corresponding nonlinear models by analytical/semi-analytic methods is a tough task. However, the involvement of extensive computation makes it little convenient to take the challenge in obtaining multiple rogue wave solutions. Thus, continuous analysis of the rogue waves will help in the enrichment of a complete understanding of the mysterious phenomenon. The interest of researchers has now been shifted to explore multiple rogue wave solutions in addition to multi-soliton solutions. Interaction between these rogue waves and soliton solutions gives rise to a new kind of solution which is of further interest in recent years.

One of the fundamental nonlinear wave equations which describes the flow in shallow inviscid layer is the Boussinesq model:

\[
-u_{tt} - u_{xx} - \beta u_{xxxx} - \gamma u_{xxxxx} = 0
\]

where \(\beta\) is the vertical extent of fluid, and \(\gamma\) represents the velocity of wave profiles. Several integrable/nonintegrable Boussinesq-type equations are proposed with dispersion, temporality, and nonlinearity to model the various phenomena in coastal areas, oceanic rogue waves and tsunami waves, and analytical solutions are obtained. As our objective in this work is to investigate the dynamics of multiple rogue waves in a higher-dimensional nonlinear model, we consider the following (2+1)-dimensional integrable Boussinesq equations proposed by Wazwaz and Lakhveer governing the gravity waves and collisions of surface water waves:

\[
u_{ttt} - u_{xx} - \beta(u^2)_{xx} - \gamma u_{xxxx} + \alpha^2 u_{yy} + \alpha u_{yy} = 0,
\]

where \(\alpha, \beta, \) and \(\gamma\) are nonzero constants. The above generalized equation have two extra terms than classical Boussinesq equation (1), one spatial term \(u_{yy}\) and another spatio-temporal \(u_{yy}\), which is found to be integrable and passes Painlevé integrability test under the influence of the constant \(\alpha^2/4\) and \(\alpha\). It is important to note that the above equation shall reduces to different versions of Boussinesq and even Benjamin-Ono type model for different choices of \(\alpha, \beta, \) and \(\gamma\). Simply, for an example, Eq. (2) takes the classical Boussinesq equation for \(\alpha = 0, \beta = 3, \gamma = 1\) studied in, and shall be listed a few other versions too which are discussed in the literature. However, to the best of our knowledge, only the soliton (solitary wave) solutions are reported for the above integrable model. Still, solutions and dynamics of various other types of nonlinear coherent structures are not available/reported. So, we devote our analysis to construct rogue wave solutions and a detailed study on their dynamics in this manuscript and leave other nonlinear wave solutions for a future investigation. On the other hand, there are certain results on solitary waves, rational solutions, periodic and lump solutions of a different type of (2+1)-dimensional Boussinesq equation and its simple classical model in recent times, to name a few. Our motive in this
research is to bring light on the rogue wave solutions and their dynamical characteristics for the considered equation (2), since, nowadays, theoretical analysis of such waves has become an integral segment of the field nonlinear sciences. The Hirota bilinear method is found successful in investigating various nonlinear evolution equations to obtain rogue wave solutions and especially it was adopted for low order rogue wave solutions in these studies. Despite the high difficulty level in exploring multiple rogue waves, still, there are few works of literature on it. The symbolic computation approach and polynomials reported in recent years help to study the multiple rogue wave solutions of such equations.

This paper is divided into various sections consisting of the development of the bilinear equation by using the Hirota bilinear transformation method in Section 2. Next in section 3, we construct the one, two and three-rogue wave solutions of Eq. (2) by implementing a direct and effective way with an establishment of the generalized polynomial function for the bilinear equation (2). Further, discussion on the mechanism of rogue waves by controlling the system and solution parameters are also presented. After giving a few important remarks regarding the present work, we provide a brief conclusion in the final section 4.

2 | BILINEARIZATION OF THE (2+1)D INTEGRABLE BOUSSINESQ EQUATION

This section is devoted for fabricating bilinear form of equation (2) by making use of the Bell polynomials. A crisp description about the Bell polynomials can be studied with the help of reference for better understanding. Using the properties of Bell polynomials and considering the transformation

\[ u(x, t) = \frac{3y}{\beta} q_{xx} + u_0, \]  

with \( r = y + \alpha t, \) \( x = x, \) and an auxiliary function \( q = q(x, t) = 2 \ln(R(x, r)), \) the bilinear equation of Eq. (2) can be obtained as

\[ \left( a_1^2 + \frac{\alpha^2}{4} + \alpha a_1 \right) D_x^2 - (2\beta u_0 + 1) D_x^2 - \gamma D_t^4 \right) R \cdot \mathcal{R} = 0. \]  

Here \( D \) represents the Hirota bilinear operators defined as

\[ D^a D^b(\Lambda \cdot \Theta) = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^a \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^b \Lambda(x, t)\Theta(x', t')|_{x'=x, t'=t}. \]  

For appropriate functions of \( \mathcal{R} \) different types of nonlinear wave solutions can be derived. Here we focus only the construction of rogue wave solution by using the above bilinear form and leave other nonlinear wave solutions for a future investigation.

3 | ROGUE WAVES: SOLUTION AND DYNAMICS

In this section, we construct rogue wave solution of Boussinesq equation (2). For this purpose, we wish to realize a generalized solution structure from the following rogue wave solution of the simplified Boussinesq equation (1) for \( \alpha = 0, \beta = 3, \) and \( \gamma = 1 \) reported in Ref.

\[ u(x, t) = 2 \frac{\partial}{\partial x} \ln(x^2 - t^2 - 3) \Rightarrow \frac{-4(3 + x^2 + t^2)}{(3 - x^2 + t^2)^2}. \]  

Further, the following form of rational solution for the classical Boussinesq model is proposed in:

\[ u_r(x, t) = 2 \frac{\partial}{\partial x^2} \ln R_r(x, t), \quad r \geq 1, \]  

where \( R_r(x, t) \) is a polynomial in \( x^2 \) and \( t^2 \) of degree \( r(r + 1)/2 \) and it is expressed as:

\[ R_r(x, t) = \sum_{n=0}^{r(r+1)/2} \sum_{i=0}^{n} c_{i,n} x^{2i} t^{2(n-i)}, \]  

where \( c_{i,n} \) are the constant to be determined by solving equations arising as the coefficients of different powers of \( x \) and \( t \). Further in Ref. the generalization of rational solution for the model is proved to be of the form

\[ \ddot{u}_r(x, t; \lambda, \mu) = 2 \frac{\partial^2}{\partial x^2} \ln \dot{R}_r(x, t; \lambda, \mu), \quad \text{for} \quad r \geq 1 \]
where

\[ \bar{R}_{r+1}(x, t; \lambda, \mu) = R_{r+1}(x, t) + 2itF_r(x, t) + 2\mu x G_r(x, t) + (\lambda^2 + \mu^2)R_{r-1}(x, t). \]

Here \( R_r(x, t) \) take the form of as given in [8], while \( F_r(x, t) \) and \( G_r(x, t) \) is obtained as

\begin{align*}
F_r(x, t) &= \sum_{n=0}^{r(r+1)/2} \sum_{i=0}^{n} e_{r,n} x^{2n} y^{2(n-i)}, \\
G_r(x, t) &= \sum_{n=0}^{r(r+1)/2} \sum_{i=0}^{n} h_{r,n} x^{2n} y^{2(n-i)}.
\end{align*}

(9c)

Based on the above solution, a generalized ansatz with more number of control parameters was proposed recently in 2018 for constructing the multiple rogue waves under the similar setting as given below, but for a different model Kadomtsev-Petviashvili (KP) type equations [5].

\[ R = R_{r+1}(x, \tau; \lambda, \mu) = R_{r+1}(x, \tau) + 2\lambda \tau F_r(x, \tau) + 2\mu x G_r(x, \tau) + (\lambda^2 + \mu^2)R_{r-1}(x, \tau), \]

(10a)

with

\begin{align*}
R_{r}(x, \tau) &= \sum_{n=0}^{r(r+1)/2} \sum_{i=0}^{n} c_{r(n-1)-2n,2i} x^{r(r+1)-2n} y^{2i}, \\
F_{r}(x, \tau) &= \sum_{n=0}^{r(r+1)/2} \sum_{i=0}^{n} e_{r(n-1)-2n,2i} x^{r(r+1)-2n} y^{2i}, \\
G_{r}(x, \tau) &= \sum_{n=0}^{r(r+1)/2} \sum_{i=0}^{n} h_{r(n-1)-2n,2i} x^{r(r+1)-2n} y^{2i},
\end{align*}

(10b, 10c, 10d)

where \( \lambda, \mu, e_{p,q}, c_{p,q}, \) and \( h_{p,q} (p, q = 0, 2, 4, \ldots, r(r + 1)) \) are arbitrary real parameters to be obtained.

Motivated from the above solution structure, we are interested to construct a generalized multi-rogue wave solution of the (2+1)D Boussinesq equation [2] by following the polynomial functions given above in Eq. (10) and discuss their dynamics in detail in the rest of the article. This method has not been utilized much for obtaining rogue wave solutions, in particular for the present (2+1)D Boussinesq equation.

### 3.1 First Order Rogue wave

In this subsection, we obtain rogue wave solution of first order to equation (2) via the bilinear form (4) and the above polynomial function for \( R \). For this purpose, we put \( r = 0 \) in (10a), which results into the form of \( R \) as

\[ R = R_1(x, \tau) = c_{0,0} + c_{0,2} x^2 + c_{2,0} x^2. \]

(11)

Without loss of generality, we take \( c_{0,0} = 1 \). Substituting Eq. (11) into the bilinear form and equating the coefficients at different powers of \( x \) and \( \tau \) to zero, yield the following system of equations:

\begin{align*}
\frac{1}{2} c_{0,0}^2 (2a_1 + a)^2 + 2c_{0,2}(1 + 2u_0\beta) &= 0, \\
\frac{1}{2} c_{0,0} c_{0,2} (2a_1 + a)^2 - 2c_{0,0}(1 + 2u_0\beta) - 12\gamma &= 0.
\end{align*}

(12a, 12b)

Solving the above Eq. (12), we get the rogue wave parameter as

\[ c_{0,2} = \frac{-4(1 + 2u_0\beta)}{(2a_1 + a)^2}, \quad c_{0,0} = \frac{-3\gamma}{(1 + 2u_0\beta)}. \]

(13)

Therefore, we obtain the explicit solution of bilinear equation (4) as given below.

\[ R_1 = (x - \lambda)^2 - \frac{4(1 + 2u_0\beta)}{(2a_1 + a)^2}(y - \mu)(x - \lambda) - \frac{3\gamma}{(1 + 2u_0\beta)}. \]

(14)

Thus by using the above form of \( R_1 \) and bilinear transformation (3), the first order rogue wave solution of Boussinesq equation is obtained as

\[ u = u_0 + \frac{12\gamma}{\beta} \left( \frac{\frac{-3\lambda}{(1 + 2u_0\beta)} - (x - \lambda)^2 - \frac{4(1 + 2u_0\beta)}{(2a_1 + a)^2}(y + a_1t - \mu)^2}{\left( \frac{\frac{-3\lambda}{(1 + 2u_0\beta)} + (x - \lambda)^2 - \frac{4(1 + 2u_0\beta)}{(2a_1 + a)^2}(y + a_1t - \mu)^2 \right)^2} \right). \]

(15)
The above rogue wave solution describes the dynamics of a localized excitations appearing in the considered (2+1)D Boussinesq equation, which is characterized by seven arbitrary parameters \( u_0, \beta, \gamma, \lambda, \alpha, \mu, \text{ and } a_1 \). A categorical analysis of this solution reveals that these arbitrary parameters contribute to the dynamics and manipulation of obtained rogue waves under constraint conditions \( 2a_1 + \alpha \neq 0 \) and \( 1 + 2u_0\beta < 0 \) that results into singularity of the solution. The evolution of the constructed solution takes a variety of coherent structures in different dimensional planes ranging from a doubly-localized rogue wave to spatially localized rational solitons (solitary waves). Particularly, the present solution exhibits a doubly localized rogue wave structure along \( x - t \) as well as \( x - y \) planes while it admits a rational soliton form in the \( y - t \) plane. Such rogue and rational wave structures are depicted in Figs. 1–3 and the explicit roles of the arbitrary parameters are discussed below.

**FIGURE 1** Bright and dark type first order rogue waves through solution (15). The parameter choice for bright rogue wave is \( u_0 = -0.95, \gamma = 0.5, \beta = 1.5, \lambda = 0.2, a_1 = 1.5, \mu = 0.05, \text{ and } \alpha = 0.5 \), while that of the dark rogue wave is \( u_0 = 1.04, \gamma = 0.15, \beta = -0.55, \lambda = 0.2, a_1 = -0.35, \mu = 0.5, \text{ and } \alpha = 0.5 \) at \( y = 0.05 \).

**FIGURE 2** Propagation of bright rogue wave observed at different time \( t = -7.5, t = 0.0, \text{ and } t = 7.5 \) for \( u_0 = -0.95, \gamma = 0.5, \beta = 1.5, \lambda = 0.2, a_1 = 1.5, \mu = 0.05, \text{ and } \alpha = 0.5 \).
Substituting the above arguments can be visualized/confirmed through Fig. 4 for clear understanding.

3.2 Second Order Rogue wave

In continuation to the first-order rogue wave solution, here we consider \( r = 1 \) in (10a) to construct the second-order rogue wave of (2+1)D Boussinesq equation (2). Thus, we get the approximate form of \( R \) as

\[
R = R_2(x, \tau; \alpha, \beta, \gamma) \Rightarrow R_2(x, \tau) + 2\alpha \tau F_1(x, \tau) + 2\mu x G_1(x, \tau) + (\lambda^2 + \mu^2)R_0,
\]

\[
R_2 = \left( c_{0,0} + c_{0,2}x^2 + c_{0,4}x^4 + c_{0,6}x^6 \right) + (c_{2,0} + c_{2,2}x^2 + c_{2,4}x^4)x^2 + \left( c_{4,0} + c_{4,2}x^2 \right)x^4
\]

\[
+ x^5 + 2\alpha \tau (e_{0,0} + e_{0,2}x^2 + e_{2,0}x^2) + 2\mu x (h_{0,0} + h_{0,2}x^2 + h_{2,0}x^2 + (\lambda^2 + \mu^2)).
\]

Substituting the above \( R \) form into the bilinear equation (4) and solving all the resultant equations arising at different powers of \( x \) and \( \tau \), we obtained the following relations among the parameters:

\[
c_{0,0} = \frac{-1}{144(1 + 2u_0\beta)^3} \left( 270000 \gamma^3 + (1 + 2u_0\beta)(4(1 + 2u_0\beta)(36 + 4\alpha^2)e_{2,0}^2 + 4a_1e_{2,0}^2a + e_{2,0}^2a^2 + 72u_0\beta) \lambda^2 \right.
\]

\[
- (16a_1^2h_{0,2}^2 + 32a_1^2h_{0,2}^2a + 24a_1^2h_{0,2}^2a^2 + 8a_1h_{0,2}^2a^3 + h_{0,2}^2a^4 - 144(1 + 2u_0\beta)^2 \mu^2)) \right),
\]

\[
c_{0,2} = \frac{-1900\gamma^2}{(2a_1 + \alpha)^2(1 + 2u_0\beta)}, \quad c_{0,4} = \frac{-272(1 + 2u_0\beta)^3}{(2a_1 + \alpha)^4}, \quad c_{0,6} = \frac{-64(1 + 2u_0\beta)^3}{(2a_1 + \alpha)^6}, \quad c_{2,0} = \frac{-125\gamma^2}{(1 + 2u_0\beta)^2},
\]

\[
c_{2,2} = \frac{36\gamma}{(2a_1 + \alpha)^2}, \quad c_{2,4} = \frac{48(1 + 2u_0\beta)^2}{(2a_1 + \alpha)^4}, \quad c_{4,0} = \frac{-25\gamma}{1 + 2u_0\beta}, \quad c_{4,2} = \frac{-12(1 + 2u_0\beta)}{(2a_1 + \alpha)^2},
\]

\[
e_{0,0} = \frac{-5e_{2,0}\gamma}{3(1 + 2u_0\beta)}, \quad e_{0,2} = \frac{4e_{2,0}^2}{3(2a_1 + \alpha)}, \quad h_{0,0} = \frac{h_{0,2}(2a_1 + \alpha)^2\gamma}{12(1 + 2u_0\beta)^2}, \quad h_{2,0} = \frac{h_{0,2}(2a_1 + \alpha)^2}{12(1 + 2u_0\beta)}.
\]
FIGURE 4 Impact of various arbitrary parameters $a_1$, $\beta$, $\gamma$, $\mu$, $\alpha$, and $\lambda$ in the first order rogue wave at $y = 0.05$ and $t = 0.05$ with other parameters $u_0 = -0.95$, $\gamma = 0.5$, $\beta = 1.5$, $\lambda = 0.20$, $a_1 = 1.5$, $\alpha = 1.0$, and $\mu = 0.5$.

From the above parameters explicit form of $R_2$ required in the bilinear equation (4) can be obtained. Thus the resultant second-order rogue wave solution of (2+1)D Boussinesq equation (2) can be constructed straightforwardly from the bilinear transformation

$$u = u_0 + \frac{\partial}{\partial y} (\ln R_2)_{xx}$$

with the help of explicit $R_2$ given above in Eq. (16).

The second-order rogue wave (16-17) consists of nine arbitrary parameters $u_0$, $\beta$, $\gamma$, $\lambda$, $\alpha$, $\mu$, $a_1$, $h_{0,2}$, and $e_{2,0}$. Here also, one can obtain both bright and dark type rogue waves for appropriate choice of parameters, which consists of doubly localized dual-peak/dip and triple-dip/peak bright/dark profiles, as shown in Figs. 5-6. Note that their appearance in $y-t$ and $x-y$ planes takes different structures, wherein the former it admits M-shaped (W-shaped) rational soliton. At the same time, it exhibits a doubly-localized rogue wave profile but with a longer tail in the later, respectively, for the bright (dark) solution. Additionally, the explicit impact of each arbitrary parameters can be unveiled, and our analysis shows that they play similar roles as in the case of first-order rogue waves. It starts from the manipulation and control of rogue wave amplitude, width, tail-depth, and localization. Further, the above solution also reveals the characteristics of rational solitons (solitary waves) of either propagating one as well as stationary structures. For completeness and an easy understanding, such stationary rational solitons are also demonstrated in Fig. 7.
FIGURE 5 Second order bright rogue wave in x-t plane at $y = 0.05$ and M-shaped rational soliton profile in y-t plane at $x = 0.05$ obtained through solution for $u_0 = -0.59; \gamma = 0.5; \beta = 1.50; \lambda = 0.20; a1 = 1.5; \mu = 0.05; \alpha = 0.50; h_{0,2} = 0.2$; and $e_{2,0} = 0.4$.

FIGURE 6 Dark type second order rogue waves in x-t plane at $y = 0.05$ and W-shaped rational soliton profile in y-t plane at $x = 0.05$ obtained through solution for $u_0 = 1.04; \gamma = 0.15; \beta = 0.55; \lambda = 0.2; a1 = -0.45; \mu = 0.5; \alpha = 0.5; h_{0,2} = 0.2$; and $e_{2,0} = 0.4$. 
3.3 | Third Order Rogue wave

Similar to the first- and second-order rogue waves, here let us take $r = 2$ in \( 10a \) to extract third-order rogue wave solution of Boussinesq equation (7), which leads to the form of \( R \) as

\[
R = R_3(x, \tau; \mu) = R_1(x, \tau) + 2 \mu \frac{\tau}{2} F(x, \tau) + 2 \mu x G_2(x, \tau) + (\lambda^2 + \mu^2) R_1.
\]

\[
R_3 = (c_{0,0} + c_{0,2} \tau^2 + c_{0,4} \tau^4 + c_{0,6} \tau^6 + c_{0,8} \tau^8 + c_{0,10} \tau^{10} + c_{0,12} \tau^{12})
\]

\[
+ (c_{4,0} + c_{4,2} \tau^2 + c_{4,4} \tau^4 + c_{4,6} \tau^6 + c_{4,8} \tau^8) x^2
\]

\[
+ (c_{8,0} + c_{8,2} \tau^2 + c_{8,4} \tau^4 + c_{8,6} \tau^6) x^6 + (c_{10,0} + c_{10,2} \tau^2) x^{10} + x^{12}
\]

\[
+ 2 \mu x (h_{0,0} + h_{0,2} \tau^2 + h_{0,4} \tau^4 + h_{0,6} \tau^6 + (h_{2,0} + h_{2,2} \tau^2 + h_{2,4} \tau^4) x^2
\]

\[
+ (h_{4,0} + h_{4,2} \tau^2) x^4 + x^6) + R_1(x, \tau)(\lambda^2 + \mu^2).
\]

In a similar manner, by substituting the above \( R \) in the bilinear equation and solving the resulting set of equations at different powers of \( x \) and \( \tau \), we get the following system of relations among the parameters:

\[
c_{0,0} = \frac{1}{147456(2 a_1 + a)^2 (2 + 2u_0)^8} \left( 1769472 a_1^6 \gamma \lambda^2 + 14155776 a_1^5 \gamma \lambda^2 + 53084160 a_1^4 \gamma \lambda^2
\]

\[
+ 123863040 a_1^3 \gamma \lambda^2 + 201277440 a_1^2 \gamma \lambda^2 + 241532928 a_1^1 \gamma \lambda^2 + 221405184 a_1^0 \gamma \lambda^2
\]

\[
+ 158146560 a_1^9 \gamma \lambda^2 + 88957440 a_1^8 \gamma \lambda^2 + 39536640 a_1^7 \gamma \lambda^2 + 13837824 a_1^6 \gamma \lambda^2
\]

\[
+ 3773952 a_1^5 \gamma \lambda^2 + 786240 a_1^4 \gamma \lambda^2 + 120960 a_1^3 \gamma \lambda^2 + 27 a_1^2 \gamma \lambda^2
\]

\[
+ 589824 (1 + 2 u_0) \gamma \mu (\lambda^2 + \mu^2) - 16384 (\alpha + 2 u_0) \beta^2 (-878826025 \gamma \lambda^2
\]

\[
+ 9 (1 + 2 u_0) \beta^2 (\lambda^2 + \mu^2)) + 32 a_1 (1799835699200 (1 + 2 u_0) \gamma^2 \lambda^6 + 27 a_1^4
\]

\[
+ 2048 (1 + 2 u_0) \gamma \lambda^2 - 18432 (1 + 2 u_0) \beta^2 (\lambda^2 + \mu^2)) + 32 a_1^2 (1799835699200 (1 + 2 u_0) \gamma^2 \lambda^6
\]

\[
+ 27 (15 a_1^4 + 2048 (1 + 2 u_0) \gamma^2 \lambda^2 - 18432 (1 + 2 u_0) \beta^2 (\lambda^2 + \mu^2)))
\]

\[
(19a)
\]

\[
c_{0,2} = \frac{1}{12288 (2 a_1 + a)^2 (2 + 2 u_0)^8} \left( 492989235200(1 + 2 u_0) \beta^2 \gamma^3 + 3 (16384 a_1^{14} + 114688 a_1^{12} a
\]

\[
+ 372736 a_1^{10} a^2 + 745472 a_1^8 a^3 + 1025024 a_1^6 a^4 + 1025024 a_1^4 a^5 + 768768 a_1^2 a^6 + 439296 a_1^4 a^7
\]

\[
+ 192192 a_1^5 a^8 + 64064 a^9 + 16016 a^{10} + 2912 a^{11} + 364 a^{12} + 28 a_1 a^{13} + a^{14}
\]

\[
+ 16384 (1 + 2 u_0) \gamma \lambda^2)
\]

\[
(19b)
\]
\[
c_{2,0} = \frac{1}{49152(1 + 2u_0\beta)^7} \left( -2617942835200\gamma^3 - 10477171340800u_0\beta\gamma^3 - 10477171340800u_0^2\beta^2\gamma^3 
- 49152\lambda^2 - 49152a_1^{14}\lambda^2 - 344064a_1^{13}\alpha\lambda^2 - 1118208a_1^{12}\alpha^2\lambda^2 - 2236416a_1^{11}\alpha^3\lambda^2 
- 3075072a_1^{10}\alpha^4\lambda^2 - 3075072a_1^9\alpha^5\lambda^2 - 2306304a_1^8\alpha^6\lambda^2 - 1317888a_1^7\alpha^7\lambda^2 - 576576a_1^6\alpha^8\lambda^2 
- 192192a_1^5\alpha^9\lambda^2 - 48048a_1^4\alpha^{10}\lambda^2 - 8736a^3\alpha^{11}\lambda^2 - 1092a_1^2\alpha^{12}\lambda^2 - 84a_1\alpha^{13}\lambda^2 - 3\alpha^{14}\lambda^2 
- 688128u_0\alpha^6\lambda^2 - 4128768u_0^2\alpha^7\lambda^2 - 13762560u_0^3\alpha^8\lambda^2 - 2752520u_0^4\alpha^9\lambda^2 - 3303014u_0^5\alpha^{10}\lambda^2 
- 2202096u_0^6\alpha^{11}\lambda^2 - 6291456u_0^7\alpha^{12}\lambda^2 \right), 
\]

\[
c_{0,4} = \frac{262267600\gamma^4}{3(2a_1 + \alpha)^4(1 + 2u_0\beta)^4}, \quad c_{0,6} = \frac{51134720\gamma^3}{3(2a_1 + \alpha)^6}, \quad c_{0,8} = \frac{1109760(1 + 2u_0\beta)^2\gamma^2}{(2a_1 + \alpha)^{10}}, 
\]

\[
c_{0,10} = \frac{59392(1 + 2u_0\beta)^4\gamma}{(2a_1 + \alpha)^{10}}, \quad c_{0,12} = \frac{4096(1 + 2u_0\beta)^6}{(2a_1 + \alpha)^{12}}, \quad c_{2,2} = -\frac{2263800\gamma^4}{(2a_1 + \alpha)^{14}(1 + 2u_0\beta)^4}, 
\]

\[
c_{2,4} = \frac{235200\gamma^3}{(2a_1 + \alpha)^3(1 + 2u_0\beta)^5}, \quad c_{2,6} = -\frac{2266880(\gamma^2 + 2u_0\beta\gamma^2)}{(2a_1 + \alpha)^6}, \quad c_{2,8} = -\frac{145920(1 + 2u_0\beta)^3\gamma}{(2a_1 + \alpha)^{10}}, 
\]

\[
c_{2,10} = -\frac{6144(1 + 2u_0\beta)^5\gamma}{(2a_1 + \alpha)^{10}}, \quad c_{4,0} = \frac{5187875\gamma^4}{3(1 + 2u_0\beta)^2}, \quad c_{4,2} = \frac{882000\gamma^3}{(2a_1 + \alpha)^4(1 + 2u_0\beta)^2}, 
\]

\[
c_{4,4} = \frac{599200\gamma^2}{(2a_1 + \alpha)^4}, \quad c_{4,6} = \frac{93440(1 + 2u_0\beta)^2\gamma}{(2a_1 + \alpha)^6}, \quad c_{4,8} = \frac{3840(1 + 2u_0\beta)^4}{(2a_1 + \alpha)^8}, 
\]

\[
c_{6,0} = \frac{-75460\gamma^3}{3(1 + 2u_0\beta)^3}, \quad c_{6,2} = \frac{-74480\gamma^2}{(2a_1 + \alpha)^2(1 + 2u_0\beta)^2}, \quad c_{6,4} = \frac{-24640(\gamma + 2u_0\beta\gamma)}{(2a_1 + \alpha)^4}, 
\]

\[
c_{6,6} = -\frac{1280(1 + 2u_0\beta)^3\gamma}{(2a_1 + \alpha)^6}, \quad c_{8,0} = \frac{735\gamma^2}{(1 + 2u_0\beta)^2}, \quad c_{8,2} = \frac{2760\gamma}{(2a_1 + \alpha)^4}, \quad c_{8,4} = -\frac{24(1 + 2u_0\beta)}{(2a_1 + \alpha)^8}, 
\]

\[
c_{8,8} = \frac{240(1 + 2u_0\beta)^2\gamma}{(2a_1 + \alpha)^8}, \quad c_{10,0} = -\frac{98\gamma}{1 + 2u_0\beta}, \quad c_{10,2} = -\frac{24(1 + 2u_0\beta)}{(2a_1 + \alpha)^2}, 
\]

\[
e_{0,0} = \frac{1}{192(2a_1 + \alpha)^2(1 + 2u_0\beta)^6\lambda} \left( 4829440\alpha^8\gamma^3\lambda + 19317760a_1^7a_1\gamma^3\lambda + 3380680a_1^6a_1^2\gamma^3\lambda 
+ 3380680a_1^5a_1^3\gamma^3\lambda + 21128800a_1^4a_1^4\gamma^3\lambda + 8451520a_1^3a_1^5\gamma^3\lambda + 2112880a_1^2a_1^6\gamma^3\lambda 
+ 301840a_1a_1^7\gamma^3\lambda + 18865a_1^2\gamma^3\lambda - 768(1 + 2u_0\beta)^2\mu(\lambda^2 + \mu^2) \right), 
\]

\[
e_{0,2} = \frac{665(2a_1 + \alpha)^6\gamma^2}{64(1 + 2u_0\beta)^5}, \quad e_{0,4} = \frac{105(2a_1 + \alpha)^6\gamma}{64(1 + 2u_0\beta)^4}, \quad e_{0,6} = -\frac{5(2a_1 + \alpha)^6}{64(1 + 2u_0\beta)^3}, 
\]

\[
e_{2,0} = -\frac{245(2a_1 + \alpha)^4\gamma^2}{16(1 + 2u_0\beta)^4}, \quad e_{2,2} = \frac{95(2a_1 + \alpha)^4\gamma}{8(1 + 2u_0\beta)^3}, \quad e_{2,4} = -\frac{5(2a_1 + \alpha)^4}{16(1 + 2u_0\beta)^2}, 
\]

\[
e_{4,0} = -\frac{7(2a_1 + \alpha)^2\gamma^2}{4(1 + 2u_0\beta)^2}, \quad e_{4,2} = \frac{9(2a_1 + \alpha)^2}{4(1 + 2u_0\beta)}, 
\]

\[
h_{0,0} = \frac{3(\lambda + 2u_0\beta)^3\mu + 3(1 + 2u_0\beta)^3\lambda^2\mu^2}{3(1 + 2u_0\beta)^3\mu}, \quad h_{0,2} = -\frac{2140\gamma^2}{(2a_1 + \alpha)^2(1 + 2u_0\beta)^2}, 
\]

\[
h_{0,4} = -\frac{720(\gamma + 2u_0\beta\gamma)}{(2a_1 + \alpha)^4}, \quad h_{0,6} = -\frac{320(1 + 2u_0\beta)^3\lambda}{(2a_1 + \alpha)^6}, \quad h_{2,0} = -\frac{245\gamma^2}{(1 + 2u_0\beta)^2}, 
\]

\[
h_{2,2} = -\frac{920\gamma}{(2a_1 + \alpha)^2}, \quad h_{2,4} = -\frac{80(1 + 2u_0\beta)^2\gamma}{(2a_1 + \alpha)^2}, \quad h_{4,0} = -\frac{13\gamma}{1 + 2u_0\beta}, \quad h_{4,2} = \frac{36(1 + 2u_0\beta)}{(2a_1 + \alpha)^2}.
\]

Thus the third-order rogue wave solution of Boussinesq equation (19) is obtained by deducing the explicit form of \( R_s \) from the above set of relations (19) and substituting it in the bilinear transformation (5) \( u = u_0 + \frac{\gamma}{\mu} (\ln R_s)_{xx} \).

Our analysis on the above third-order rogue wave solution of the considered (2+1)D Boussinesq equation (2) shows that it contains a number of arbitrary parameters \( u_0, \beta, \gamma, \lambda, \alpha, \mu, \) and \( a_1 \). The role of these parameters in defining the characteristics of the rogue waves is usual, as in the first- and second-order of rogue wave dynamics, which includes the alteration of multi-peak-amplitude and multi-hole-depth of the bright and dark rogue waves respectively. In the present third order rogue wave, we have
obtained two symmetric peaks on either side of a central maximum peak in the bright type localized wave profile. In contrast, the dark wave structure consists of the merely same number of symmetrically spaced amplitude-holes/dips (lowest amplitude) instead of peaks. This behavior is quite similar in $x-t$ and $x-y$ planes except for the localization in different directions. But, the solution supports a multi-peak (with a central maximum) rational solitons of both bright and dark types along the $y-t$ plane. For illustrative purposes, we have given such bright and dark rogue waves as well as rational solitons in Figs. 8-9. Additionally, one can control $a_1$ parameter to obtain stationary rational solitons as well apart from the moving/traveling rational solitons.

**FIGURE 8** Third order bright rogue waves along $x-t$ and $x-y$ planes at $y = 0.05$ and $t = 0.05$ with multi-peak doubly localized structures. A multi-peak rational soliton in $y-t$ plane at $x = 0.05$. Other parameters are chosen as $u_0 = -0.9, \gamma = 1.5, \beta = 1.5, \lambda = 0.2, a_1 = 1.5, \mu = 0.05$ and $\alpha = 0.5$.

**FIGURE 9** Third order dark rogue waves along $x-t$ and $x-y$ planes at $y = 0.05$ and $t = 0.05$ with multi-hole doubly localized structures. A multi-hole with central deep hole rational soliton in $y-t$ plane at $x = 0.05$. Other parameters are chosen as $u_0 = 1.04, \gamma = 0.15, \beta = -0.55, \lambda = 0.2, a_1 = 1.2, \mu = 0.5$, and $\alpha = -0.5$. 
It is a straightforward exercise to construct an arbitrary $N$-rogue wave solutions ($N \geq 4$) by following the similar procedure given above. Still, as it involves a tedious task and with lengthy mathematical relations, which restricted us to present/discuss them here in this manuscript.

**REMARKS**

- It should be mentioned that the occurrence of $N$-soliton solution ($N \geq 3$) is an alternative approach to confirm the integrability nature of a model. But here, the occurrence of $N$-rogue wave solutions ($N \geq 3$) does not guarantee the integrability. Precisely, the $N$-rogue wave solutions ($N \geq 3$) for any model (integrable/non-integrable) can be obtained if it admits the following type of bilinear form:

\[
(A_1 D_x^4 + A_2 D_x^2 + A_3 D_x^2) R \cdot R = 0,
\]

where $A_1$, $A_2$, and $A_3$ are parameters associated with the considered system, and if the bilinear from is free from any mixed partial derivatives, as mentioned in Ref. 48. In this direction, it would be an interesting study to find out, which kind of bilinear form can show multiple rogue wave solution.

- The present generalized computation approach is better than the method proposed in Ref. 49, because of the fact that the current solutions utilizes mathematical expression to generate the polynomial test function for constructing any arbitrary $N$-th order rogue wave solutions. But, in the case of Ref. 49, two different test functions are necessary to construct rogue wave solutions of order one and two only.

- The rogue wave parameters are calculated by the obtained determined/overdetermined systems. In the case of first order rogue wave, we have obtained the determined system while in construction of second and third order rogue wave we have obtained the overdetermined systems. On the other hand, the number of arbitrary parameters affecting the evolution mechanism of rogue waves are seven, nine and seven for first, second and third order rogue wave solutions respectively.

- It is interesting to point out that all the obtained rogue wave solutions are having the characteristics

\[
\lim_{|x| \to \infty} u(x, y, t) = u_0 \text{ and } \lim_{|y| \to \infty} u(x, y, t) = u_0.
\]

**4 | CONCLUSIONS**

We have considered an integrable (2+1)-dimensional Boussinesq equation and constructed higher-order rogue wave solutions by utilizing generalized polynomial test functions implemented through Bell polynomial and Hirota’s bilinearization. The dynamics of rogue waves are investigated by carrying out a categorical analysis of the obtained solutions. We have explored the evolution dynamics of these rogue waves along with W-shaped, M-shaped, and multi-peak rational solitons. We found that the arbitrary parameters ($u_0$, $\beta$, $\gamma$, $\lambda$, $\alpha$, $\mu$, and $a_1$) available in the obtained solutions help to manipulate the dynamics of rogue waves which enable one to control the amplitude/depth, width, tail-depth, and localization of the bright and dark rogue waves. Also, we found that $\alpha$ and $a_1$ parameters enact the transformation of the rogue wave into stationary rational solitons in first as well as all the higher-order solutions. Further, the position shifting of the localized structures along a particular dimension/direction is also possible through these parameters. The results presented in this work will be encouraging to the studies on the rogue waves on other higher dimensional systems. Also, it will be helpful to various experimental investigations on the controlling mechanism of rogue waves in optical systems, atomic condensates, deep water oceanic waves., and other related coherent wave systems.

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