Kinetic inflation in deformed phase space Brans-Dicke cosmology

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In this paper, by establishing a Brans-Dicke (BD) cosmology by means of a deformed phase space, in the absence of any scalar potential, cosmological constant and ordinary matter, we show that it is feasible to overcome obstacles reported in the corresponding commutative (non-deformed) frameworks. More concretely, by applying the Hamiltonian formalism and introducing a dynamical deformation, between the momenta associated to the FLRW scale factor and the BD scalar field, we obtain the modified equations of motion. In particular, these equations reduce to their standard counterparts when the noncommutative (NC) parameter is switched off. By focusing on a specific branch of solutions, in contrast to standard frameworks (even with a varying BD coupling parameter), we show, that we can obtain an adequate appropriate inflationary epoch, possessing a suitable graceful exit. In other words, in the Jordan frame (JF), such branch of solutions properly satisfy the sufficient condition required for satisfactory inflation, which is equivalent to get an inflationary phase in the conformal Einstein frame (EF) without branch change. Concerning the cosmological dynamics, we further show that our NC framework bears close resemblance to the $R^2$ (Starobinsky) inflationary model.

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I. INTRODUCTION

One of the strongest motivations, in cosmology, for applying various scenarios of scalar tensor theories, including BD theory [1] as the simplest prototype, has been to overcome the problems associated to standard inflationary scenarios [2–6]. In these applications as well as for constructing quintessential scenarios (see, e.g., [7] and related papers) associated to the present accelerating epoch, a scalar potential is usually added by hand into the action of the standard BD theory. Moreover, it should be noted that inflationary scenarios have also been presented by establishing quantum BD cosmology, see, for instance, [12–15].

Nevertheless, in contrast to that standard inflationary strategy, it has also been considered that the accelerating scale factor associated to such epoch can be obtained merely from the dynamics of the scalar field, without employing either the corresponding scalar potential or the cosmological constant [16–20]. However, it has been demonstrated that such kinetic inflationary scenarios, as investigated in the BD theory (even with varying BD coupling parameter), suffer from a pertinent set of problems [18]. More concretely, for the inflating scale factor associated to all the D-branch solutions in the JF, there is no successful transition between this accelerating epoch and the decelerating expansion, which is known as the graceful exit problem. Furthermore, satisfying a sufficient condition for inflation imposes the scale factor of EF to accelerate. However, concerning a kinetic inflationary model in such generalized BD theory, there is no source to yield such an acceleration stage. In this paper we address that obstacle in the context of the BD theory from the perspective of a NC setting and show that it is feasible to overcome such a problem.

For the X-branch, the situation is instead quite different. Let us be more precise. In the absence of any scalar potential and ordinary matter in the standard BD theory (where the BD coupling parameter is constant), the only

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1 Recently, concerning such an ad hoc scalar potential in scalar tensor theories, it has been shown that the geometry of the higher dimensions can yield it, see, for instance, [8–11].
2 The D- and X-branches, will be introduced in section III.
3 Moreover, the accelerating epoch obtained in the context of string theory [21], driven by a scalar inflaton under the influence of a potential, also encounters the same problem. It has been argued that this problem is a consequence of a cosmological singularity being reached in a finite proper time [22]. It has been demonstrated that the singularity can be removed by including quantum effects [23–24].
class of solutions which can yield an accelerating scale factor is the D-branch solutions, with negative values for both the BD coupling parameter and the time derivative of the scalar field. The case where the BD coupling parameter is equal to $-1$, have been investigated in [25, 26]. However, it is impossible to get a gravity-driven acceleration for the X-branch solutions in the strict context of the standard BD theory for the spatially flat FLRW background. One viable alternative to overcome this situation and obtaining gravity-driven acceleration in the X-branch, is applying a generalized BD theory where the BD coupling parameter can vary, see, e.g., [18]. However, these models have, in turn, their shortcomings. For instance, as argued in [18], similarly to the D-branch solutions, we find a lack of accelerating scale factor in the EF, or equivalently, the graceful exit problem remains.

At high energy regimes, to overcome the mentioned obstacles, features as motivated from tentative quantum gravity proposals, have been considered. Among such physical frameworks, a deformation in the phase space structure [27], emerging via a length parameter [which is usually interpreted as the Planck (length) constant] to the background theory, has been considered as a promising approach. It should be emphasized that the standard results are recovered in a proper limit of the length parameter. The implications associated to these frameworks in cosmology [28, 29] have included the removal of the big bang singularity at early times as well as coarse grained effects at late times of the evolution of the universe. Recently, various NC cosmological/gravitational frameworks have been constructed to investigate different shortcomings associated to the corresponding standard models [30–35]. For instance, in [19], two of us have shown that such source can be borrowed from the NC portions accompanying the standard BD setting. Therefore, by focusing on the D-branch solutions, the graceful exit problem has been solved appropriately. NC phase space models have also been proposed within general settings emerging from a generalized uncertainty principle, see, for instance [36–38].

In the context of the above paragraphs, the main objective of the present paper is to show that by applying a BD cosmology and by employing a dynamical deformation in the phase space, it is feasible to overcome the obstacles associated to the X-branch solutions. Moreover, we show that our herein NC model has other salient features which can be related to other interesting inflationary models constructed in either BD theory or general relativity by including quantum corrections. Specifically, (i) the big bang singularity is removed; (ii) an accelerating inflationary phase is obtained for early times, which has a graceful exit; (iii) for the late times, the scale factor is in a zero acceleration phase; (iv) the solutions fully satisfy the nominal and sufficient conditions associated to inflation; (v) the cosmological phase space analysis not only confirms the X-branch to be successful inflation but also provides situations for plausible resemblance to the Starobinsky inflationary model [39]. Furthermore, by analyzing the time behaviours associated to either scale factor in the conformal EF or its equivalent quantity in the JF, we show that our herein NC framework can provide a viable scenario for a real inflation [40]. In addition, deriving the EF equations of motion in a conformably flat background motivates to find definite conditions (at least by means of numerical methods) to obtain a possible correspondence between our NC model and the quantum BD cosmological framework in the EF, in which the scalar field is coupled to Dirac spinors [40].

The paper is briefly outlined as follows. In the next section, the cosmological equations of motion for the generalized BD cosmology, in the presence a general dynamical deformation (noncommutativity), will be obtained. In section III, we first present solutions of the equations of motion for the non-deformed (commutative) case, analytically. Subsequently, by specifying the deformed Poisson bracket, we show that the evolutionary equations depend only on the dynamics of the BD scalar field. Then, we solve analytically the NC equations for particular cases. For the general NC case, we apply numerical methods to depict the time behavior of physical quantities. In section IV, we show that the nominal as well as the sufficient conditions are completely satisfied for the solutions associated to the inflation obtained in III. In section V, we compare the evolutionary equation for the scale factor associated to our herein model with that of the Starobinsky model. Subsequently, by depicting the phase space portrait, we show that the results of the section III for the X-branch solutions are reconfirmed. In section VI, we proceed to derive the results in the conformal EF and present complementary comments. Finally, in section VII, we summarize the main results of the work and present additional discussions.

II. DEFORMED PHASE SPACE BRANS-DICKE SETTING

We consider a background geometry which is described with the spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) line-element as

$$\text{d}s^2 = -N^2(t)\text{d}t^2 + a(t)^2 \left( \text{d}x^2 + \text{d}y^2 + \text{d}z^2 \right),$$

(2.1)

where $a(t)$ is the scale factor, $N(t)$ is a lapse function and $(x, y, z)$ denote the Cartesian coordinates. In the JF, the Lagrangian density associated to the scalar tensor gravity can be written as

$$\mathcal{L}[\gamma, \phi] = \sqrt{-\gamma} \left[ \phi \mathcal{R} - \frac{\omega(\phi)}{\phi} \gamma^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \right] + \sqrt{-\gamma} \mathcal{L}_{\text{matter}},$$

(2.2)

where 

$$\mathcal{R} = \frac{1}{2} \left( \nabla^\nu \nabla_\nu - \nabla^2 \right) \mathcal{R}.$$
where the Greek indices run from zero to 3, $\gamma$ and $R$ stand for the determinant and the Ricci scalar associated to the metric $\gamma_{\mu\nu}$, respectively; $L_{\text{matt}} = 16\pi \rho(a)$ [where $\rho(a)$ is the energy density] is the Lagrangian density associated to ordinary matter; $\phi$ is the (BD) scalar field, which is normalized to $G^{-1}$ (where $G$ is Newton gravitational constant) such that its present value should be equal to unity [5] (throughout this paper, we will use the units where $c = 1 = \hbar$) and $V(\phi)$ is the scalar potential. Moreover, for attractive gravity, we must assume $\phi > 0$.

In a particular case where $\omega(\phi) = \omega_0 = \text{constant}$, the Lagrangian (2.2) reduces to the Lagrangian of the BD theory in the presence of a scalar potential. In the next sections, for the sake of simplicity in analysing cosmological solutions to be studied in the next sections, let us assume that the Lagrangian density of the ordinary matter as well as the scalar potential are absent. Instead, we extend the standard model by investigating a generalized setting in which the dynamical Poisson bracket (2.5) is used.

It is easy to show that the Hamiltonian associated to (2.2) can be written as

$$H = \frac{N}{6\xi^2} \left[ \frac{\omega(\phi)}{6a\phi} P_a^2 a^{-3} \phi P_\phi^2 - a^{-2} P_a P_\phi \right] + Na^3 \left[ V(\phi) - 16\pi \rho(a) \right],$$

(2.3)

where

$$\xi \equiv \left[ 1 + \frac{2\omega(\phi)}{3} \right]^\frac{1}{2},$$

(2.4)

and $P_a$ and $P_\phi$ stand for the conjugate momenta associated to the scale factor and scalar field, respectively.

In order to obtain and investigate an acceleration of the universe, solely driven from the kinematics, which will be studied in the next sections, let us assume that the Lagrangian density of the ordinary matter as well as the scalar potential are absent. Instead, we extend the standard model by investigating a generalized setting in which the Poisson bracket associated to the conjugate momenta of the scale factor and the BD scalar field does not vanish. More precisely, we will generalize the noncommutativity proposed in [30]. It is straightforward to show that such a dynamical NC model can be constructed based upon a dimensional analysis [30,32].

In this work, let us propose the Poisson commutation relation between the momenta as

$$\{P_a, P_\phi\} = \theta \zeta \phi,$$

(2.5)

where $\theta$ is a constant NC parameter and $\zeta = \zeta(a)$ is an arbitrary function of the scale factor; we leave the Poisson brackets associated to the other variables unchanged, namely

$$\{a, \phi\} = 0, \quad \{\phi, P_\phi\} = 1, \quad \{a, P_a\} = 1.$$  

(2.6)

Furthermore, we will work in the comoving gauge; namely, we would like to use the cosmic time and therefore we should set $N(t) = 1$. Moreover, we begin in obtaining the general formula associated to the generalized BD theory in the absence of the ordinary matter and the scalar potential, in which the dynamical Poisson bracket (2.5) is used. Subsequently, in the next section we will employ a constant BD coupling parameter.

By employing the commutation relations (2.5) and (2.6), the modified equations of motion associated to the Hamiltonian (2.3) are given by

$$\dot{a} = -\frac{1}{6\xi^2 a} \left[ \frac{\omega}{3\phi} P_a + a^{-1} P_\phi \right],$$

(2.7)

$$\dot{P}_a = -\frac{1}{36\xi^2 a^4 \phi} \left[ \omega a^2 P_a^2 + 12a\phi P_a P_\phi - 18\phi^2 P_\phi^2 \right] + \frac{\theta \zeta(a) \phi}{6\xi^2 a^3} (2\phi P_\phi - a P_a),$$

(2.8)

$$\dot{\phi} = -\frac{1}{6\xi^2 a} (a^{-1} P_a - 2a^{-2} \phi P_\phi),$$

(2.9)

$$\dot{P}_\phi = \frac{-1}{36\xi^2 a^3 \phi^2} \left[ \omega a^2 P_a^2 + 6\phi^2 P_\phi^2 \right] + \frac{P_a^2}{36\xi^2 a \phi} \frac{d\omega(\phi)}{d\phi} + \frac{\theta \zeta(a)}{18\xi^2 a^2} (3\phi P_\phi + \omega(\phi) a P_a),$$

(2.10)

where an overdot stands for the differentiation with respect to the cosmic time. Moreover, as we have considered the homogeneous and isotropic FLRW universe, we have assumed that the BD scalar field to be merely a function of the

4 Such a choice of dynamical deformation between the conjugate momentum sector might be corroborated by a few physical arguments already presented in [30,32], see also [41].
cosmic time. Additionally, we should notice that, by choosing the noncommutativity, just the equations of motion associated to the conjugate momenta, i.e., (2.8) and (2.10), have been modified. It is important to note that when \( \theta = 0 \), equations (2.7), (2.9), (2.10) correspond to those of the corresponding commutative (non-deformed) counterparts. After some manipulations, it is straightforward to show that the NC equations can be written, as the modified versions of the standard form, as

\[
H^2 = \frac{\omega}{6} \left( \frac{\dot{\phi}}{\phi} \right)^2 - H \left( \frac{\dot{\phi}}{\phi} \right),
\]

\[
\frac{\ddot{a}}{a} = -\frac{\omega}{3} \left( \frac{\dot{\phi}}{\phi} \right)^2 + H \left( \frac{\dot{\phi}}{\phi} \right) + \frac{1}{6\xi^2 \phi} \frac{d\omega(\phi)}{d\phi} \left( \frac{\dot{\phi}}{\phi} \right)^2 + \frac{\theta \zeta(a) \phi}{18\xi^2 a^2} \left( 3H - \frac{\omega \dot{\phi}}{\phi} \right),
\]

\[
\ddot{\phi} + 3H \dot{\phi} = -\frac{1}{3\xi^2} \frac{d\omega(\phi)}{d\phi} \left( \frac{\dot{\phi}}{\phi} \right)^2 - \frac{\theta \zeta(a) \phi}{6\xi^2 a^2} \left( 2H + \frac{\dot{\phi}}{\phi} \right),
\]

where \( H = \dot{a}/a \) is the Hubble parameter. It is important to note that the Hamiltonian constraint was not modified under the dynamical deformation between the momenta, which leads to a power-law relation between the scale factor and the scalar field as in the corresponding case of the standard BD theory \cite{17, 18}. However, the acceleration equation, as well as the wave equation (associated to the BD scalar), have been modified such that in the absence of the NC parameter we recover the corresponding commutative equations \cite{17, 18}. We should point that we have three nonlinear differential equations (which are not independent) with two unknowns, \( a \) and \( \phi \). In this paper, the extra terms, which were produced due to the deformation in the phase space, will mostly require our attention.

III. BRANS-DICKE COSMOLOGICAL VACUUM SOLUTIONS IN DEFORMED PHASE SPACE

Let us focus on the equations (2.11)-(2.13) and work in the context of standard BD theory as the background theory. We should refer to the gravity driven acceleration and kinetic inflation investigated in non-deformed cases of the generalized BD theory [namely, solutions of equations (2.11)-(2.13) by assuming \( \theta = 0 \)] in interesting works by Levin \cite{17, 18}. In our work, we would like to investigate instead how effects imported from deformation in the phase space can overcome the problems of the non-deformed cases.

In analogy with the standard BD theory, we should determine the energy density and the pressure associated to the \( \phi \)-field, e.g., \cite{17}

\[
H^2 = \frac{8\pi}{3\phi} \rho_\phi,
\]

\[
\frac{\ddot{a}}{a} = \frac{4\pi}{3\phi} (\rho_\phi + 3p_\phi).
\]

By comparing equations (2.11) and (2.12) with (3.1) and (3.2), respectively, for a constant \( \omega \), we retrieve

\[
\rho_\phi = \frac{\omega}{16\pi} \frac{(\dot{\phi})^2}{\phi} - \frac{3}{8\pi} H \dot{\phi},
\]

\[
p_\phi = \frac{\omega}{16\pi} \frac{(\dot{\phi})^2}{\phi} - \frac{1}{8\pi} H \dot{\phi} - \frac{\theta \zeta(a) \phi}{24\pi \xi^2} \left( \frac{\dot{\phi}}{\phi} \right)^2 \left( H - \frac{\omega \dot{\phi}}{3 \phi} \right).
\]

Let us proceed our investigation by analysing the solutions associated to field equations (2.11)-(2.13) for both the commutative and NC cases for constant \( \omega \). From (2.11), we obtain

\[
H = \chi \left( \frac{\dot{\phi}}{\phi} \right), \quad a(t) = a_i [\phi(t)]^\chi,
\]

\[
\text{Note that } \zeta = \zeta(a) \text{ is not another independent dynamical variable; indeed, we will proceed to get the solutions by specifying this function.}
\]

\[
\text{Hereafter, we always set } \omega = \omega_0 = \text{constant. For simplicity, let us drop the index 0 from } \omega_0.
\]
where \( a_i \) is an integration constant\(^7\) and

\[
\chi = \chi_{\pm} = -\frac{1}{2} (1 + \lambda \xi), \quad \lambda = \lambda_{\pm} = \pm 1.
\]  
(3.6)

In this paper, \( \lambda_+ \) and \( \lambda_- \) are associated to what we will denominate as the upper (plus sign) and the lower (minus sign) solutions. Here, in analogy with the non-deformed BD theory (either with constant or varying \( \omega \)), we should distinguish the cosmological solutions. It has been shown that in the EF (and standard case), there are two different types of the cosmological solutions correspond to different values of \( \lambda \): in one type, the universe contracts while in the other one, it expands, see, e.g., \([17]\). However, in the JF, for both of the signs, still we can get expanding universe. Nevertheless, these solutions can be distinguished. Let us denominate them following \([17, 19]\): the \( X \) and \( D \) branches correspond to the solutions whose scale factor expands and contracts in the EF, respectively. Throughout this paper, the \( X \) and \( D \) branches correspond to the upper and lower signs, respectively.

According to relations \((3.5)\) and \((3.6)\), we obtain two branches for the Hubble parameter \( H \): (i) By assuming \( \omega < 0 \) (\( \xi < 1 \)), for both of the branches (namely, for both values of \( \lambda \)), we obtain contracting and expanding universe provided that \( \dot{\phi} > 0 \) and \( \dot{\phi} < 0 \), respectively. (ii) Letting \( \omega > 0 \) (\( \xi > 1 \)), we obtain \( H > 0 \), if we choose either the upper sign together with \( \dot{\phi} < 0 \) or the lower sign together with \( \dot{\phi} > 0 \). We should note that due to the presence of the NC parameter in the field equations, it affects on the dynamics of the variables, and, as it will be shown in the next sections, it is not easy to anticipate the time behaviour of the quantities.

In what follows, we will show that it is possible to write both \( \dot{\phi} \) and \( \dot{a}/a \) (in the corresponding differential equations) as functions of the BD scalar field. It is easy to show that equation \((2.13)\) yields

\[
a^{3} \dot{\phi} + c = -\frac{\theta(2\chi + 1)}{12\xi^{2}} \int a\xi(\dot{a})d(\phi^{2}),
\]  
(3.7)

where \( c \) is an integration constant, which can be positive, negative or zero. Using integration by parts and employing \((3.5)\) into the retrieved equation, we obtain

\[
\dot{\phi} = -\frac{c}{\xi a_{i}^{3} \phi^{3}} - \left[ \frac{\theta(2\chi + 1)}{12\xi^{2}} \right] \left[ \phi^{2(1-\chi)} \xi(a) - \frac{1}{a_{i} e^{\phi \xi}} \int \phi^{2} d[a\xi(a)] \right] \equiv f(\phi).
\]  
(3.8)

From relation \((3.5)\), it is seen that the scale factor is a function of the BD scalar field, therefore, the right hand side of \((3.8)\) is just a function of \( \phi(t) \), namely \( \dot{\phi} = f(\phi) \).

Moreover, it is straightforward to show that the quantity \( \dot{a}/a \) can also be written merely as a function of the BD scalar field, as

\[
\frac{\dot{a}}{a} = \left[ 1 + \frac{\theta \xi(a) \phi^{\chi+2}}{6a_{i}^{2} \xi^{2} f(\phi)} \right] \left[ \frac{f(\phi)}{\phi} \right]^{2},
\]  
(3.9)

where we have used \((3.5)\) and \( \dot{\phi} = f(\phi) \) is given by \((3.8)\).

### A. Standard case: \( \theta = 0 \)

In order to compare the solutions associated to the deformed case with those obtained in the standard case, let us review the results associated to the standard BD theory for the spatially flat FLRW metric in vacuum which are known as O’Hanlon-Tupper solutions \([22, 23]\). Substituting \( \theta = 0 \) in \((3.8)\) and using \((3.5)\), we obtain the following solutions

\[
\phi(t) = \frac{[C(t - t_{i})]^{\frac{1}{\chi+1}}}{a_{i}},
\]  
(3.10)

\[
a(t) = a_{i} [C(t - t_{i})]^{\frac{1}{\chi+1}},
\]  
(3.11)

for \( \chi \neq -1/3 \) (\( \omega \neq -\frac{4}{3} \)), where \( C \equiv \frac{-(3\chi+1)}{a_{i}} \) and

\[
\phi(t) = \phi_{0} e^{-\frac{\pi}{\lambda}(t-t_{i})},
\]  
(3.12)

\[
a(t) = a_{i} \phi_{0}^{-\frac{4}{3}} e^{-\frac{\pi}{\lambda}(t-t_{i})},
\]  
(3.13)

\(^7\) Hereafter, situations where \( \chi \) and \( \lambda \) have no explicit index, it means that we discuss both the signs + and −.
for $\chi = -1/3$ ($\omega = -\frac{4}{3}$), where $t_i$ is an integration constant.

Let us analyse the time behaviors of the BD scalar field and the scale factor in solutions (3.10) and (3.11) by considering different conditions for the integration constants and the BD coupling parameter. More concretely, for $C > 0$, $\omega > -4/3$ and $t > t_i$, we obtain two different sets of the solutions. (i) For $\lambda_{-} = -1$, from the progressing cosmic time, the BD field always increases, while $G_{eff} \simeq 1/\phi$ and the scale factor always contract. (ii) For $\lambda_{+} = 1$, as cosmic time increases, the BD field decreases (consequently, $G_{eff}$ increases) whilst the scale factor increases (where always $\ddot{a} < 0$). The solutions associated to $\lambda_{-}$ and $\lambda_{+}$ are known as slow and fast solutions, respectively [40].

Under the well-known duality transformation associated to the spatially flat FLRW metric for the BD cosmology in vacuum [44, 45], it has been shown that the slow and fast solutions are interchanged [40]. Moreover, we should note that for both the cases (i) and (ii), when the BD coupling parameter is restricted to $-3/2 < \omega < -4/3$, we can easily show that $\phi(t)$ always decreases with the cosmic time, while the scale factor always accelerates for the case (i) and decelerates for the case (ii). These non-deformed solutions will be compared with the corresponding NC cases in this paper. Furthermore, it should be emphasized that for all of these cases, there is a big bang singularity as $t \to t_i$.

By assuming $t > t_i$, when the constant $C$ can take negative values, it is worthwhile to look at the above solutions from another perspective. Again, let us assume that $\omega$ is restricted to $-4/3 < \omega < 0$. In this case, the solutions associated to $\lambda_{-}$ correspond to a pre-big bang inflationary scenario where the scale factor $a(t)$ accelerates. In this case, $\phi(t)$ decreases as cosmic time progresses. However, we must obtain an expanding universe after the big bang and also we should respect to the constraints on the evolution of the $G_{eff}$. Consequently, the solutions with such properties are merely provided by considering the solutions case (ii) described above. Namely, to get post-big bang solutions, we need a branch changing, i.e., $\lambda_{-} \to \lambda_{+}$, as well as $C < 0 \to C > 0$. These pre and post-big bang solutions include the solutions obtained in [43] and correspond to low energy limit of some string theories, namely, the particular value of the BD coupling parameter $\omega = -1$.

B. General deformed case with $\zeta(a) = a^n$

Henceforward, let us proceed our investigation with only power-law functions of the scale factor in the deformed Poisson bracket, namely, $\zeta(a) = a^n$ and investigate the corresponding time behaviors for the physical quantities. First, we will obtain general equations of motion and subsequently, we will show how specific values of $n$ produce interesting dynamics.

By substituting $\zeta = a^n$ into (3.8) and using (3.5), it is easy to show that

\[
\dot{\phi} = - \frac{1}{a_i^2 \phi^{\chi}} \left[ c + \frac{a_i^{n+1}(2\chi + 1)\phi^{(n+1)\chi+2}}{6\xi^2[(n+1)\chi+2]} \right] \equiv f_n(\phi),
\]

where we have assumed general values of the BD coupling parameter where $(n+1)\chi \neq -2$. Similarly, by substituting $\zeta(a) = a^n$ and $\dot{\phi}$ from (3.14) into (3.9), we obtain

\[
\frac{\dot{a}}{a} = \frac{3\chi - \omega}{3} \left[ 1 + \frac{\theta a^{n-2}\phi^{[(n+1)\chi+2]} \left[ f_n(\phi) \right]^2}{6\xi^2f_n(\phi)} \right].
\]

Moreover, the energy density and the pressure associated to the BD scalar field are given by

\[
\rho_{\phi} = \frac{\omega - 6\chi}{16\pi\phi} \left[ f_n(\phi) \right]^2,
\]

\[
\rho_{\phi} = \frac{1}{8\pi} \left[ \frac{(\omega - 2\chi)\phi}{2} + \frac{a_i^{n-2}\theta(\omega - 3\chi)\phi^{[(n-2)\chi+3]}}{9\xi^2f_n(\phi)} \right] \left[ f_n(\phi) \right]^2,
\]

where we have used (3.3) + (3.5) and (3.14).

To obtain an accelerating universe the following condition must be satisfied

\[
\frac{3\chi - \omega}{3} \left[ 1 + \frac{\theta a^{n-2}\phi^{[(n+1)\chi+2]} \left[ f_n(\phi) \right]^2}{6\xi^2f_n(\phi)} \right] > 0,
\]

which requires determining the BD scalar field in terms of the cosmic time, which, in turn, is obtained from solving differential equation (3.14). Moreover, we should note that to avoid obtaining ghost models, we must check the values taken by the kinetic energy density according to relation (3.16). By considering the Einstein representation of the standard BD models, for an FLRW line-element, as long as the BD coupling parameter is restricted to $\omega \geq -3/2$, the
kinetic energy density takes positive values \[17\]. We will show, for the X-branch solutions, how the above constraints are satisfied for a short time and then it can exit from the acceleration phase and then turns to a decelerating one.

In this paragraph, let us discuss the simplest case of deformation in the phase space. More concretely, by assuming \( \zeta = 1 \), (2.5) yields

\[
\{P_a, P_\phi\} = \theta \phi. \tag{3.19}
\]

With a quick glance at relations (2.5) and (3.5), it can be easily indicated that BD coupling parameter plays a very important role in our herein deformed model. For instance, for this simplest case \( n = 0 \), when \( |\omega| \) goes to infinity\(^8\), then \( 1/\chi \) vanishes and therefore we get

\[
\{P_a, P_\phi\} = \theta. \tag{3.20}
\]

This very particular case has been investigated in [30]. In that paper, by proposing a special kind of deformation in the Poisson bracket associated to the momenta of the BD scalar field and the FLRW scale factor, in the particular case where \( |\omega| \to \infty \), interesting results have been obtained. For instance, it has been shown that it is possible to overcome the horizon and the graceful exit problems for an accelerating epoch associated to the early times of the universe. However, it has been neither discussed quantitatively nor the general values of the BD coupling parameter were employed to solve the horizon problem. Further investigations on this simplest case will not be of interest in this paper. In this work, not only we will analyse more general solutions in which the BD coupling parameter as well as (integer) \( n \) take arbitrary values but also we will investigate the horizon problem quantitatively. Furthermore, the cosmological phase space analysis as well as the discussions on the EF results will be in our focus.

1. Particular exact solutions of the deformed case with \( \zeta(a) = a^n \)

First, let us discuss concerning the exact analytic solutions which can be obtained from (3.14) for the particular cases and subsequently analyse the solutions for the more general cases.

- **Class I**: \( \chi \neq 1/(2 - n) \)

In this case, when \( c \neq 0 \), solving differential equation (3.14) leads to the hypergeometric functions, which is complicated to deal with as exact analytic solutions. However, we will discuss this case when we will employ a numerical analysis. In the particular case where \( c = 0 \), it is straightforward to show that

\[
t - t_i = \frac{6a_t^{2-n}(n+1)\chi + 2\xi^2\phi^{(2-n)\chi - 1}}{[n-2]^2\chi + 1}[2\chi + 1]^{2}\theta, \tag{3.21}
\]

where \( t_i \) is an integration constant.

- **Class II**: \( \chi = 1/(2 - n) \) **where** \( n \neq 2 \)

In this case, from relations (2.4) and (3.5), we obtain constant values for \( \omega \) and \( \xi \) as

\[
\omega = \frac{6(3 - n)}{(n - 2)^2}, \quad \xi = \frac{(4 - n)\lambda}{n - 2}. \tag{3.22}
\]

Similar to the previous class, we obtain different solutions correspond to whether \( c \neq 0 \) or \( c = 0 \). Namely, when \( c \neq 0 \), we obtain an exact solution as

\[
t - t_i = \left[ \frac{6a_t^{2-n}(n-4)}{(2 - n)\theta} \right] \ln \left[ 1 + \frac{a_t^{n+1}(n-2)^2\theta}{6c(n^2 - 9n + 20)} \phi^{\frac{n-4}{n-2}} \right], \tag{3.23}
\]

and for \( c = 0 \), we get

\[
t - t_i = - \left[ \frac{6a_t^{2-n}(n-4)(n-5)}{(2 - n)^2\theta} \right] \ln \phi, \tag{3.24}
\]

---

\(^8\) We should note that in the particular case of our NC model where \( \omega \to \infty \), then \( \phi \) does not take constant values and therefore relation (3.20) does make sense.
It is easy to rewrite the relations (3.21), (3.23) and (3.24) such that the BD scalar field is stated in terms of the cosmic time. Consequently, for these set of exact solutions, from (3.5), we can obtain exact solutions for the scale factor versus the cosmic time.

2. General numerical results of the deformed case with \( \zeta(a) = a^n \)

Unfortunately, for the general NC case, it is almost impossible to obtain analytic exact solutions. Therefore, let us discuss this case by employing a numerical analysis. For the sake of completeness, we will compare our NC results with those obtained from standard exact solutions already reported. In what follows, let us analyze the time behaviour of the quantities arising from the numerical analysis, associated to the solutions for both the commutative and NC case. Various numerical studies have produced the following interesting results.

Lower solutions \((\lambda = \lambda_- = -1)\): We have already demonstrated that for the non-deformed case, with the values of the BD coupling parameter from interval \(-3/2 < \omega < -4/3\) and assuming \(C > 0\), we do obtain accelerating scale factor. Similarly, by applying our herein NC model, by employing numerical codes, we could find more larger intervals of \(\omega\) (with respect to the standard case) where the scale factor always accelerates\(^9\). However, a mere accelerating scale factor cannot be considered as a sufficient condition for a successful scenario for an inflationary epoch of the early universe.

Let us be more precise. In this case, notwithstanding applying several numerical endeavors, we could not find any condition to solve the graceful exit problem. Indeed, this is the most important problem with the D-branch solution in BD theory (even with varying \(\omega\) \(\text{[18]}\)) as well as in the string theory \(\text{[21]}\). Obviously, as will be shown in the next section, the requirement to either satisfying the sufficient condition or fully overcome the graceful exit problem can be directly related to the time behaviour of either \(a\phi^{1/2}\) (associated to the JF) appearing in the sufficient condition (the most strongest condition which is required to solve the horizon problem in the standard cosmology) or the scale factor associated to the EF. More concretely, in \(\text{[18]}\), it has been emphasized that (i) the D-branch solution suffers from the graceful exit problem in the kinetic inflationary scenario in the BD theory \(\text{[18]}\) as well as in the string theory \(\text{[21]}\), namely, the corresponding inflation cannot exit from the accelerating phase successfully, to enter an expanding universe (with \(\ddot{a} < 0\)); (ii) for the X-branch solution, there may be a possibility to exit it and subsequently to enter into an expanding phase, but, this solution suffers from the flatness problem. Moreover, it has been mentioned that the problems mentioned in (i) and (ii) are due to the lack of ability of those models to obtain an accelerating scale factor in the corresponding EF \(\text{[18]}\). Nevertheless, in \(\text{[19]}\), by employing another noncommutativity for the BD theory, and focusing on the D-branch solutions, we have obtained a successful kinetic inflation without encountering the problem mentioned in (i).

In the present work, for our chosen NC framework, it seems that we cannot overcome the problems associated to the D-branch solutions. Instead, in what follows, let us focus on the X-branch solutions. We will show that the herein NC model can provide conditions to obtain X-branch solutions which can be considered as a successful kinetic inflationary scenario.

Upper solutions \((\lambda = \lambda_+ = +1)\): Before reporting our numerical results for this case, we should mention a few important points. (i) In order to analyse and depict the time behaviour of the quantities, except the NC parameter, we have used the same initial conditions (IC) for both the commutative and NC cases to solve equations \(\text{(2.11)-(2.13)}\) by taking constant values of the BD coupling parameter and \(\zeta = a^n\). Obviously, we just need the initial values for \(\phi, a_i\) and \(\dot{\phi}\) [which is equivalent to the integration constant \(c\) in equation \(\text{(3.14)}\)]. (ii) We have re-scaled the plots of a few quantities associated to the commutative case. (iii) We have assumed that \(C > 0\) in relations \(\text{(3.10) and (3.11)}\) for the standard solutions and \(-3/2 < \omega < 0\) for both the commutative and NC cases. (iv) We have investigated this case by taking very small positive values for the NC parameter \(\theta\).

- For the NC case, we have shown that the BD scalar field has a completely different time behaviour in comparison to the standard BD model. Let us be more precise. We found that in the commutative case, \(\phi\) contracts such that for all times \(\phi > 0\), see figure \(\text{[1]}\). However, with taking any values of the BD coupling parameter such that \(\omega > -3/2\), taking the same initial conditions as for the commutative case, and employing very small positive values of the NC parameter, we have shown that the BD scalar field...
always contracts such that for the early times $\ddot{\phi} < 0$. Whilst, after a short time, it turns to be positive, namely $\ddot{\phi} > 0$. Moreover, for large values of the cosmic time, $\dot{\phi}$ asymptotically vanishes; for instance, see figure 1.

- For the commutative case, the scale factor of the universe always decelerates. Namely, for all the times, the universe expands with $\ddot{a} < 0$. Let us see that how the effects of the noncommutativity change the behaviour of the scale factor. For the same above mentioned initial condition, we see that the scale factor associated to the NC case accelerates for very early times, namely, $\ddot{a} > 0$. Subsequently, after a short time, it turns to decelerate. Finally, for very large values of the cosmic time, our numerical endeavors show that we get zero acceleration for the scale factor of the universe. Moreover, for the deformed case, we see that the big bang singularity is removed; see, for instance, figures 2.

- For the commutative case, the numerical results show that both the energy density and the pressure associated to the BD scalar field decrease with the cosmic time. Moreover, both of them get positive values for all the times of their evolution. For the NC case, like the commutative case, the energy density always gets positive values. However, its time behavior is completely different from the commutative case. Namely, for the early times it increases and after short time when it gets a maximum value, it turns to
Figure 3: The behaviour of the energy density (left panel) and pressure (right panel) associated to the BD scalar field against cosmic time for the commutative case (dashed curves) and NC case (solid curves) for the upper solution ($\lambda = +1$). We have set $\omega = -1.35$, $a_i = 0.07$, $\phi(0) = -0.001$, $\phi(0) = 0.9$, $n = 0$ and $\theta = 0.001$ (for the NC case). For high viability, we re-scaled the plots associated to the commutative case.

Figure 4: The time behaviour of $\dot{a}$ for the deformed case and the upper solution ($\lambda = +1$) for different values of (i) the NC parameter (left panel): $\theta = 0.005$ (black curve), $\theta = 0.001$ (blue curve) and $\theta = 0.0008$ (red curve) where we have set $\omega = -1.42$, $a_i = 0.08$, $\phi(0) = -0.002$, $\phi(0) = 0.8$, $n = 0$. And for (ii) different values of the negative BD coupling parameter (right panel): $\omega = -1.45$ (black curve), $\omega = -1.25$ (blue curve) and $\omega = -1$ (red curve), where $\theta = 0.001$, $a_i = 0.07$, $\phi(0) = -0.001$, $\phi(0) = 0.7$ and $n = 0$.

decrease, such that for large values of the cosmic time it tends to zero. However, the NC pressure always takes negative values such that it starts a decreasing evolution and reaches to its minimum value and then turns to increase. Similar to NC energy density, it asymptotically approaches to zero. For instance, in figure 3, the energy density and pressure associated to the BD scalar field have been plotted for a few initial conditions.

Let us now describe how for different values of $n$, the NC parameter and the BD coupling parameter affect the time behaviour of the physical quantities. In order to show that our general results listed above can also be obtained with extended ranges of the suitable ICs, we would like to plot the following figures with a suitable set of different values of the ICs. Our numerical efforts indicate that the different values of either the positive NC parameter or negative BD coupling parameter or $n$ do not affect on the resulted background features explained above (for instance, getting accelerating scale factor at early times and then decelerating one). However, by taking different values of these parameters, the time interval of acceleration as well as the corresponding amounts of the physical quantities effectively are changed. We have established that the smaller $\theta$ or $|\omega|$ or $|n|$, the larger the time interval of inflation and the smaller the slope of the scale factor associated to the accelerating phase, see, for instance, figures 4 and 5.

Up to now, employing a numerical analysis, we have shown that the X-branch solutions in the JF yield an accelerating scale factor for a short time, which can exit successfully from this phase and subsequently turns to decelerate. However, such results are not sufficient to consider them as a successful inflation. In the next sections, we will investigate whether or not the other required conditions for a successful kinetic inflation are satisfied.

IV. HORIZON PROBLEM

As mentioned, obtaining an accelerating scale factor for the early times of the universe is not solely a sufficient condition for getting a successful inflationary epoch. More concretely, the most important problem with the standard
cosmology, the horizon problem, must be resolved in a successful inflationary scenario. Moreover, in a successful
inflation, the graceful exit problem must be solved. In the previous subsections, we have shown that our herein model
does satisfy the latter. In this section, we would investigate the horizon problem. In order to resolve the horizon
problem, we should show that the extent of the observable universe is encompassed by an inflated causally connected
region. In this respect, we will investigate both the nominal and sufficient conditions associated to the obtained
accelerating universe which is relevant for the casual physics of an inflation.

A. Nominal condition

In analogy with the standard models, the nominal condition can be written as

\[ D_H \equiv \dot{a}_H^\text{tot} - H^{-1} > 0, \tag{4.1} \]

where

\[ \dot{a}_H^\text{tot} \equiv a(t) \int \frac{\dot{a}}{a}. \tag{4.2} \]

We will show that in our model, \( \dot{a}_H^\text{tot} \) contributes two parts, namely, the commutative and NC parts as

\[ \dot{a}_H^\text{tot} = \dot{a}_H^\text{com} + \dot{a}_H^\text{nc}, \tag{4.3} \]

such that when \( \theta = 0 \), it reduces to the its corresponding commutative quantity. In what follows, we will show that
the nominal condition is completely satisfied for the inflationary model obtained for the X-branch.

Following Ref. [46], we employ (2.11) to obtain

\[ \left( H + \frac{\dot{\phi}}{2\phi} \right)^2 = \frac{\xi^2}{4} \left( \frac{\dot{\phi}}{\phi} \right)^2, \tag{4.4} \]

which yields

\[ \frac{d\ln(\phi a^2)}{dt} = \lambda \xi \left( \frac{\dot{\phi}}{\phi} \right). \tag{4.5} \]

Concerning our case where \( \zeta(a) = a^n \), by employing (3.5), (3.14) and integrating (4.5) over \( dt \), it is straightforward
to show that

\[ \dot{a}_H^\text{com} = -\frac{\lambda(1-\delta)a^3\phi}{c_\xi} = -\frac{\lambda (1-\delta)\phi^{3\chi+1}}{c_\xi}, \tag{4.6} \]

\[ \dot{a}_H^\text{nc} = \frac{a_1^{n+1}\theta(2\chi+1)\phi^n}{6c_\xi^2[(n+1)\chi+2]} \int \phi^{n\chi+2} dt, \tag{4.7} \]
where we have included the integration constant $a_i^3$ in $\delta$, which was defined as

$$\delta = \frac{a_i^3}{a^2\phi}. \quad (4.8)$$

Moreover, to calculate $\{4.6\}$ and $\{4.7\}$, we need to obtain $\phi$ in terms of the cosmic time, which should be determined from the differential equation $\{3.14\}$. We should note that by considering an attractive gravity (where $\phi > 0$ forever), then $\delta$ always takes positive values. In this subsection, let us assume $\delta = 0$. However, in the next subsection where the sufficient condition will be probed, we will proceed our calculations with nonzero $\delta$.

Therefore, by calculating the BD scalar field from $\{3.14\}$, then substituting it into relations $\{4.6\}$ and $\{4.7\}$, and finally obtaining the quantity $D_H$ with the assistance of numerical calculations, we can establish whether or not the nominal condition is satisfied in our NC model. Because of the NC part $d_{nc}^H$, we expect that the horizon problem can be solved in our NC model more appropriately than the generalized BD theory in which $\omega = \omega(\phi)$. When $\theta = 0$, then $d_{nc}^H$ vanishes and we retrieve $D_H = d_{com}^H$.

As mentioned, it is impossible to solve the differential equation $\{3.14\}$, analytically, for the general cases. However, using numerical analysis, we have shown that for the NC case, the nominal condition is satisfied. For instance, for the upper solution ($\lambda = +1$), in which we have obtained an inflationary epoch for the early times and the zero acceleration for the scale factor at late times, we have plotted the behaviour of $D_H$ in figure 6.

B. Sufficient condition

In analogy with the non-deformed case of generalized BD theory in JF [18], we can admit the condition associated to the sufficient inflation as

$$\frac{d_{tot}^H}{a_*} > \frac{1}{H_0 a_0}, \quad (4.9)$$

where $d_{tot}^H$ and $a_*$ denote the values of those quantities at an earlier time $t_*$. Moreover, the index 0 denotes the present values of those quantities. Obviously, in the particular case where $\theta = 0$, from relations $\{4.3\}$, $\{4.6\}$ and $\{4.7\}$, it is easy to show that the inequality $\{4.9\}$ reduces to its non-deformed counterpart.

Let us first calculate the left and right hand sides (lhs and rhs) of inequality $\{4.9\}$, separately, and then derive the sufficient condition. The lhs of $\{4.9\}$ is given by

$$\frac{d_{tot}^H}{a_*} = -(a_i^2 \phi_i) \left[ \frac{\lambda (1 - \delta)}{c\xi} + a_i^{n-2}\theta(2\chi + 1)\phi^{-(2\chi+1)} \int \phi^{n\chi+2}dt \right]_{t_*}, \quad (4.10)$$

where the quantity inside brackets is evaluated at the specific time $t_*$ and we have used relations $\{3.5\}$, $\{4.3\}$, $\{4.6\}$ and $\{4.7\}$.

For calculating the rhs of $\{4.9\}$, we will use the following steps employed in [18]: (i) It is feasible to write the relation for the Hubble constant at present time as

$$H_0 = \sqrt{\frac{\alpha_0}{M_{Pl}}} T_0^2, \quad (4.11)$$
where, throughout this paper, $M_{\text{Pl}}$ denotes the Planck mass at present time; $a_0 = \gamma(t_0)\eta_0 = (8\pi/3)(\pi^2/30)g_*\gamma(t_0)\eta_0$; $\rho_0$ denotes the ratio of the energy density in matter to that in radiation at present time. (ii) We will use $t_\text{end}$ to denote a specific time in which the inflation was stopped and entropy was produced. (iii) We will employ the relation

$$a_\text{end}T_\text{end} = a_0T_0$$

which indicates an adiabatically evolution of the universe. (iv) Concerning the heating mechanism, $T_\text{end}$ can be related to the net available kinetic energy density $E_\text{end}$ as $T_\text{end} = \epsilon E_\text{end}$, where $\epsilon$ stands for the efficiency whereby the kinetic energy density is converted to entropy [15]. (v) Finally, let us assume that $E_\text{end}$ is produced merely by the kinetic energy associated to the BD scalar field, namely, $E_\text{end} = (4\pi/3)a_\text{end}^3\rho_\text{end}^{(\phi)}$. Consequently, by using assumptions (i)-(v) and employing relations (3.14) and (3.16), we obtain

$$\frac{1}{H_0\theta_0} = \left[ \frac{12M_{\text{Pl}}a_\text{end}^2\phi_\text{end}}{\epsilon(\omega - 6\chi)\sqrt{\alpha_0T_0}} \right] \left[ \frac{1}{c + \frac{a_\text{end}^{n+1}\theta(2\chi+1)\phi((n+1)\chi+2)}{6\xi^2[(n+1)\chi+2]}} \right]^2. \tag{4.12}$$

Substituting relations (4.10) and (4.12) into (4.9) yields

$$\frac{a^2\phi_\star}{a_\text{end}^2\phi_\text{end}} \gtrsim \left[ \frac{12M_{\text{Pl}}a_0^{-2}}{\epsilon(\omega - 6\chi)T_0} \right] \left[ \frac{1}{c + \frac{a_\text{end}^{n+1}\theta(2\chi+1)\phi((n+1)\chi+2)}{6\xi^2[(n+1)\chi+2]}} \right]^2 \left[ \frac{1}{c + \frac{a_\text{end}^{n+1}\theta(2\chi+1)\phi((n+1)\chi+2)}{6\xi^2[(n+1)\chi+2]}} \right]^2. \tag{4.13}$$

This is the modified (NC) version of the sufficient condition associated to the inflation with respect to that obtained in the generalized BD theory in [18] [where $\omega = \omega(\phi)$].

In [18], in addition to the analytic analysis associated to the sufficient conation in the JF, by choosing two different suitable sets of the constants and the parameters of the model, the following predictions have been demonstrated: (i) For the first choice, it has been argued that just the $D$-branch solutions (in which the quantity $a^2\phi$ always decreases with cosmic time) can satisfy the sufficient condition. However, for the present time, as $\phi$ must take constant value and the scale factor must accelerate, therefore, the $D$-branch does not yield the present expanding universe unless a branch change to be induced. (ii) For the second choice, it has been demonstrated that the X-branch (in which $a^2\phi$ may increase) can satisfy the sufficient condition if one of the integration constants takes very large value.

It has been demonstrated that the sufficient condition associated to inflation can be satisfied if the scale factor in the conformal EF does accelerate [17, 18]. Regarding this point and because of the intricate construction of inequality [4.13], let us skip the intuitive analysis (where we should probe whether or not the sufficient condition, by trying some sets of suitable values for the integration constants and the parameters, is satisfied) and just produce our predictions by analyzing the time behaviour of the quantity $a^2\phi$. Let us also mention a relevant reason: it has been shown that obtaining an accelerating scale factor in the JF is not a sufficient condition for the occurrence of a corresponding inflation in the conformal EF. More concretely, another essential complementary condition must be satisfied: the following time dependent ratio must increase with the cosmic time (in the JF) [40]

$$\frac{a(t)l_0}{l_{\text{Pl}}(t)} = \frac{a(t)l_0}{\sqrt{b/\alpha_{\text{Pl}}}} = \frac{a(t)l_0}{\sqrt{\eta/\rho(\phi)}} \equiv a_{\text{RI}}, \tag{4.14}$$

where $l_{\text{Pl}}(t)$ is the Planck length and $l_0$ is a comoving constant length. There are situations in which, although the scale factor accelerates, it cannot increase faster than the $l_{\text{Pl}}(t)$, see for instance [18, 43]. In [43], the pre-big bang cosmology [47] has been studied in the BD theory for the specific value $\omega = -1$, where the BD theory corresponds to the low energy limit of some string theories.

Let work in Planck units, where we obtain $a_{\text{RI}} = a\phi^{1/2}$. For the resulted inflation associated to the upper solution in section [31] again, by means of numerical analysis, we have shown that the quantity $a_{\text{RI}}(t)$ not only always increases with the cosmic time, but also accelerates at some time prior to the present time. In other words, for the upper solution in which we have taken very small positive values for $\theta$ and the BD coupling parameter is restricted to $-3/2 < \omega < 0$, the sufficient condition is always satisfied. For instance, in figure [7] we have plotted the quantity $a_{\text{RI}}(t)$ versus cosmic time. It is important to note that for the specific case where $\omega = -1$, we have also obtained a real inflation (as denominated in [31]), see figure [7].
Figure 7: The behaviour of $a_{\text{RI}}(t) = a \phi^{1/2}$ against cosmic time; where in the (i) left panel: for different values of the NC parameter: $\theta = 0.0009$ (red curve); $\theta = 0.009$ (blue curve) and $\theta = 0.005$ (black curve), we set $\omega = -1.28$ for all of these three cases; and for (ii) right panel: for different values of the BD coupling parameter: $\omega = -1$ (red curve); $\omega = -0.25$ (blue curve) and $\omega = -0.38$ (black curve) where $\theta = 0.001$ is fixed for all of them. Moreover, we have set the following common parameters for both of the left and right panels: $\lambda = +1$, $a = 0.06$, $\dot{\phi}(0) = -0.002$, $\phi(0) = 0.8$ and $n = 0$.

V. COSMOLOGICAL DYNAMICS IN DEFORMED PHASE SPACE BRANS-DICKE THEORY

In this section, let us first compare the evolutionary equation for the scale factor associated to our herein kinetic inflation, arisen from deformed phase space BD setting, with that obtained in “$R^2$” (“Starobinsky”) inflationary model [39]. Subsequently, we will proceed our discussion to a dynamical analysis and plot the phase plane diagrams.

Employing equations (2.12) and (3.5), it is straightforward to show that the equation for the scale factor associated to $\zeta(a) = a^n$ case can be written as

$$\frac{\ddot{a}}{a} + (4 - n) \frac{\dot{a}^2}{a^2} + \left( \frac{\dot{a}}{a} \right)^2 + A \left( \frac{\dot{a}}{a} \right)^4 = 0 \quad (5.1)$$

where

$$A \equiv (n - 4) - \frac{2}{3(1 + \lambda \xi)} [(n - 4) + (n - 2) \lambda \xi] [6\xi(\xi + \lambda) - \omega]. \quad (5.2)$$

In the commutative case, where $\theta = 0$, equation (5.1) reduces to

$$\frac{\ddot{a}}{a} + \left[ \frac{2\xi(\xi + \lambda)}{(1 + \lambda \xi)^2} \right] \left( \frac{\dot{a}}{a} \right)^2 = 0. \quad (5.3)$$

Moreover, it is worthwhile to employ a few appropriate transformations introduced in [39]. First, let us set $u = a^2 \dot{a}^2$. Then, equation (5.1) transforms into

$$\frac{uu''}{a^2} + (2 - n) \frac{uu'}{a^3} - \frac{u^2}{2a^2} + 2 [A + n - 2] \frac{u^2}{a^4} = 0, \quad (5.4)$$

where a prime stands for a derivative with respect to the scale factor $a$. From equations (5.4), we obtain a partial de Sitter solution:

$$u(a) = H_1^2 a^4, \quad a(t) = a_1 \exp(H_1 t), \quad (5.5)$$

(where $a_1$ and $H_1$ are integration constants) provided that

$$\omega = \begin{cases} -\frac{4}{3} & \text{for } \lambda = -1 \\ \frac{6(3-n)}{(n-2)^2} & \text{(where } n \neq 2) \text{ for } \lambda = \pm 1. \end{cases} \quad (5.6)$$

Second, by letting $g = u^2$ and $z = (12)^{-\frac{1}{2}} a^3$, (5.4) is represented as

$$\frac{d^2 g}{dz^2} - \frac{1}{3g} \left( \frac{dg}{dz} \right)^2 - \left( \frac{n - 4}{3z} \right) \frac{dg}{dz} + [A + n - 2] \frac{g}{6z^2} = 0. \quad (5.7)$$
Finally, by substituting
\[ g = zx, \quad y = z \frac{dx}{dz} \]  
into (5.7), it yields
\[ \frac{dy}{dx} = \frac{y}{3x} - \frac{[2 + n] x}{6y} + \frac{n-5}{3}. \]  
(5.9)

From equations (5.1), (5.4), (5.7) and (5.9), it is seen that, disregarding the absence of the term $H^2$ as well as different multiplications associated to some terms, these equations bear much resemblance to the corresponding ones obtained in [39]. The price we had to pay to obtain such a correspondence between our herein model and that obtained in [39] was to remove the explicit presence of the NC parameter in the equations derived above. Further complicated investigations are required to derive the role of the NC parameter together with BD coupling parameter (as well as other parameters and integration constants) via the employed transformations to construct a physical connection from our model to the elements used in [39]. This investigation is not the scope of the present work.

Let us move to the second stage of this section in which we can see the explicit presence of the NC parameter in our dynamical analysis. Taking a constant BD coupling parameter and employing $\zeta(a) = a^n$ and (3.5), equation (2.12) can be rewritten as
\[ \dot{H} + c_1(\omega) H^2 + \theta c_2(\omega) a^{(n-2+\frac{1}{\chi})} H = 0, \]  
(5.10)

where
\[ c_1(\omega) = \frac{1}{3\chi^2} (3\chi^2 + \omega - 3\chi), \]  
(5.11)
\[ c_2(\omega) = \frac{\omega - 3\chi}{18 a_i^\frac{1}{\chi} \chi \xi^2}. \]  
(5.12)

Let $y = \dot{a}$, then equation (5.10) transforms to
\[ \frac{dy}{da} = \left[ 1 - c_1(\omega) \right] a^{-1} y - \theta c_2(\omega) a^{(n-2+\frac{1}{\chi})}. \]  
(5.13)

This equation enables us to illustrate the difference between the commutative and noncommutative cases. In the left panel of figure 8 we plotted the phase portrait of equation (5.13) when $\theta = 0$ (non-deformed case). We observe that all the solutions $(\dot{a}, a)$ start with large and always decreasing values of $\dot{a}$. When the noncommutative parameter $\theta$ is switched on (see the right panel) the situation changes and all the solutions start with very small values for $\dot{a}$.
which increases until a maximum is reached. Afterwards, \( \dot{a} \) begins its decreasing phase. We should note that the particular solution in figure 2 corresponds to one of the trajectories depicted in figure 8, thus showing that a vast range of solutions allow an acceleration stage. This behavior is in accordance with the inflationary epoch belonging graceful exit, which was already obtained for the NC case associated to the X-branch in section III.

VI. EINSTEIN FRAME

For obtaining the new set of dynamical variables \( (\tilde{\gamma}, \tilde{\phi}) \) associated to the EF,\(^{10}\) we should use the conformal transformation for the metric together with redefinition for the scalar field as \(^{40}\)

\[
\gamma_{\mu\nu} = \Omega^2 \tilde{\gamma}_{\mu\nu}, \quad \Omega = M_{Pl} \phi^{-\frac{1}{2}},
\]

\[
\tilde{\phi}(\phi) = \sqrt{\frac{3}{16\pi}} M_{Pl} \xi \ln \left( \frac{\phi}{\phi_0} \right),
\]

where \( \phi_0 = G^{-1} = M_{Pl}^2 \) (we will use, again, the units where the \( c = \hbar = 1 \)), \( \phi \neq 0 \) and the BD coupling parameter must be restricted as \( \omega \geq -3/2 \) to guarantee to get real values for \( \tilde{\phi} \). Moreover, in order to compare the results of our model with those obtained in the non-deformed case, let us employ the procedure using the same coordinate transformation employed in \(^{18}\)

\[
dt = \Omega \dtilde{t}, \quad \text{and} \quad a = \Omega \tilde{a}.
\]

From transformations \( (6.1)-(6.3) \), the metric in the EF is written as the spatially flat FLRW metric as \(^{18}\)

\[
d\tilde{s}^2 = -d\tilde{t}^2 + \tilde{a}^2 (dx^2 + dy^2 + dz^2),
\]

where the scale factor in the EF is related to that in the JF as \( \tilde{a} \propto a \phi^{1/2} \) and, again, the lapse function has been set equal to unity.

It is straightforward to show that the evolutionary equations associated to the scale factor and the scalar field correspond to the EF can be written as\(^{11}\)

\[
\ddot{\tilde{H}} = -\frac{2\lambda \sqrt{3\pi}}{3M_{Pl}} \frac{d\tilde{\phi}}{dt}, \quad \tilde{H}^2 = \frac{8\pi}{3M_{Pl}^2} \tilde{\rho}_{\tilde{\phi}},
\]

\[
\frac{d^2 \tilde{\phi}}{dt^2} + 3\dot{\tilde{H}} \frac{d\tilde{\phi}}{dt} = -\left( \frac{\theta M_{Pl}^3}{4\sqrt{3\pi} \xi} \right) \tilde{a}^{-n-2} \tilde{H} \exp \left[ \frac{2\sqrt{3\pi}(3-n)\tilde{\phi}}{3\xi M_{Pl}} \right],
\]

where \( \tilde{H} \) is the Hubble parameter in the EF and

\[
\tilde{\rho}_{\tilde{\phi}} = \frac{1}{2} \left( \frac{d\tilde{\phi}}{dt} \right)^2,
\]

is the energy density associated to the scalar field \( \tilde{\phi} \). From \( (6.5) \), we can obtain the relation between the scale factor and the scalar field as

\[
\ddot{a}(t) = \frac{2\lambda \sqrt{3\pi}}{3M_{Pl}} \tilde{\phi}(t),
\]

\(^{10}\) It should be emphasized that the discrepancies/equivalencies of the different conformal frames have been widely discussed, see, for instance, \(^{18}40\) and references therein, and further inquiry as to the details of such notifications will not be in the scope of the present work.

\(^{11}\) In order to obtain the field equations in the EF, we have transformed the corresponding equations associated to the JF, under the conformal transformations. Equivalently, we can derive the herein equations of motion (in EF) by starting from the conformal EF action. Therefore, we must construct the corresponding deformed Poisson bracket in the EF by employing the the required transformation and respecting the dimensional analysis provided that, again, the rhs depends linearly on the NC parameter.
where $\hat{a}_i$ is an integration constant. Therefore, equation (6.6) is easily integrated and yields

$$
\frac{d\hat{\phi}}{dt} = \frac{1}{\hat{a}_i^3} \left[ -\hat{C} + \frac{\theta \hat{A}}{B} \exp \left( \hat{B} \hat{\phi} \right) \right], 
$$

(6.9)

where

$$
\hat{A} \equiv \frac{\lambda M_{Pl}^2 \hat{a}_i^{n+1}}{6\xi}, \quad \hat{B} \equiv -\frac{2\sqrt{3\pi}}{3M_{Pl}\xi} \left[ (n+1)\lambda \xi + (n-3) \right]
$$

and $\hat{C}$ is an arbitrary integration constant. By substituting $\hat{a}$ from (6.8) into equation (6.9), we obtain

$$
\frac{d\hat{\phi}}{dt} = \hat{a}_i^3 \exp \left[ -\frac{2\sqrt{3\pi}}{M_{Pl}} \hat{\phi}(\hat{t}) \right] \left[ -\hat{C} + \frac{\theta \hat{A}}{B} \exp \left( \hat{B} \hat{\phi} \right) \right],
$$

(6.10)

which is a first order differential equation only in terms of scalar field $\hat{\phi}$. However, for the general case, the corresponding solutions lead us to the Hypergeometric functions.

In the particular case where $\theta = 0$, equations (6.5), (6.6), (6.9) and (6.10) reduce to their corresponding standard counterparts in the EF [18, 43]. As the solutions and the pertinent problems associated to the standard case have been widely studied in [18, 43], let us forbear from reviewing them. In what follows, we would like to derive the counterparts in the EF [18, 43]. As the solutions and the pertinent problems associated to the standard case have been widely studied in [18, 43], let us forbear from reviewing them. In what follows, we would like to derive the sufficient condition (4.9) in the EF and see that how it is modified with respect to the non-deformed case. Similar to the method employed in [18], it is straightforward to show that inequality (4.9) associated to our NC model in the EF can also be written as

$$
\frac{\hat{a}_i^{\text{tot}}}{\hat{a}_i} > \frac{1}{H_0 \hat{a}_0},
$$

(6.11)

where using relations (6.5) and (6.6), it is easy to show that

$$
\hat{a}_i^{\text{tot}}(\hat{t}) = \hat{a} \int_{\hat{t}_i}^{\hat{t}} \frac{d\hat{a}}{\hat{a}(\hat{t}')},
$$

(6.12)

where $\delta = (a_i^2 \phi_i)/(a^2 \phi) = \hat{a}_i^2/\hat{a}^2$. Reemploying (6.5), (6.6), using (6.8) and applying the same approximation used in [18], it is easy to show that the sufficient condition associated to the EF in our NC model is given by

$$
\left. \left( \frac{d\hat{a}}{dt} \right) \right|_{\hat{t}_i}^{-1} - \frac{\theta \hat{A}}{BC} \left[ \left( \frac{d\hat{a}}{dt} \right) \right]^{-1} \exp \left( \hat{B} \hat{\phi} \right) \left. \left( \frac{d\hat{a}}{dt} \right) \right|_{0}^{-1},
$$

(6.13)

In order to evaluate this inequality we must find $\hat{\phi}(\hat{t})$ by solving the differential equation (6.10).

The above inequality in the non-deformed case where $\theta = 0$, reduces to the corresponding counterpart obtained in [18], namely,

$$
\left. \frac{d\hat{a}}{dt} \right|_{0} \gtrsim \left. \frac{d\hat{a}}{dt} \right|_{\hat{t}_i}.
$$

(6.14)

It has been demonstrated that for satisfying the sufficient condition, the scale factor associated to EF must accelerate at some time prior to the present time [18, 46]. However, in the standard BD theory it is not possible to get an accelerating scale factor in EF. However, as we have already shown in the previous sections, the time behaviours of scale factors in both the JF and EF (namely, $\dot{a} \propto a \phi^{1/2}$) for the X-branch solutions, have indicated that our NC model yields inflationary epoch possessing graceful exit in both of the frames. Therefore, checking inequality (6.13) in this section will be just a rehash of the same issue.

Before closing our investigations in the EF, let us represent the equations of motion associated to the EF in the conformally flat background where the conformal time $\eta$ is introduced as

$$
d\hat{t} = \hat{a}(\eta) d\eta,
$$

(6.15)
It is straightforward to show that equations (6.5) and (6.6), in terms of the conformal time, can be combined as

\[-\frac{8}{3\xi} \sqrt{\frac{\pi G}{3}} \left[ \frac{12}{16\pi G} \frac{d^2 \tilde{a}}{d\eta^2} + \frac{d\tilde{a}}{d\eta} \left( \frac{d\phi}{d\eta} \right)^2 \right] + \frac{d}{d\eta} \left( \frac{\tilde{a}^2 \frac{d\phi}{d\eta}}{d\eta} \right) = \frac{\theta \tilde{a}^n + 1}{18\xi G B} \left( \frac{3\lambda \xi + 4}{d\eta} \right),\]  

(6.16)

where we have used (6.8). In the non-deformed case where \( \theta = 0 \), equation (6.16) is substantially simplified as

\[\frac{d^2 \tilde{\phi}}{d\eta^2} - \frac{\sqrt{12\pi \lambda}}{M_{Pl}} \left( \frac{d\tilde{\phi}}{d\eta} \right)^2 - \frac{\lambda \xi}{d\eta} \frac{d\tilde{a}}{d\eta} \frac{d\tilde{\phi}}{d\eta} = 0.\]  

(6.17)

With respect to equation (19) of [14], under particular conditions with \( \theta \neq 0 \), which can make the rhs of equation (6.16) equal to zero, we recover that equation. It might be expected, by means of numerical analysis, at the level of equations of motion, we can get a sensible correspondence between our herein NC model with the quantum BD cosmological model proposed in [13, 15], in which the conformally flat background setting for the spinor case in the EF has been established.

VII. CONCLUSIONS

In this paper, we introduced a dynamical deformation\(^{12}\) in the phase space associated to the momenta of the spatially flat FLRW scale factor and the BD scalar field. Using the Hamiltonian formalism, after some computations, we have derived the modified equations, such that when the NC parameter vanishes, we recover the equations correspond to the non-deformed case. In particular, from applying the proposed deformed Poisson bracket, we found that the Hamiltonian constraint is not modified, but other equations contains extra terms explicitly depending on the NC parameter (cf. Section II).

For the sake of comparison with the standard case, we have presented (see Section III) the exact solutions associated to the non-deformed case and described the properties of the solutions. Moreover, we have pointed that the pre and post big-bang solutions obtained in our herein paper does obviously include those obtained for the particular case of the BD theory with \( \omega = -1 \) in Ref. [23].

Subsequently, we proceeded towards our specific NC setting and took a constant BD coupling parameter, set up to work with a general function \( \zeta(a) \) [in the rhs of the deformed Poisson bracket in (2.5)] as well as with the particular case where \( \zeta(a) = a^n \). We have shown how the equations of motion associated to the scale factor, BD scalar field, energy density and pressure (associated to the BD scalar field) only depend on the dynamics of the BD scalar field. We have further obtained general conditions under which the scale factor can describe an accelerating (a decelerating) universe and the constraints determining appropriate energy conditions.

After obtaining particular solutions associated to the NC model, as it was not possible to find exact analytic solutions associated to the complicated coupled differential equations associated to the general NC case, we focused on a numerical analysis. Concerning the D-branch solutions (see Section III), we have shown that it is possible to get a kinetic inflationary epoch for the universe. Despite employing a highly diverse set of initial conditions (IC) in the corresponding numerical endeavors, we could not find any suitable condition which can produce an inflationary epoch possessing satisfactory exit from inflation. More concretely, our herein NC kinetic inflation associated to the D-branch, similar to those obtained in [18], suffers from the graceful exit problem. However, motivated from constructing another distinct NC model in the context of the BD theory [19], where we had previously overcome this problem for the inflationary model associated to D-branch, we have preferred to focus on the X-branch solutions, which in the commutative cases, even with varying \( \omega \), are not immune from the problems [13].

Let us be more precise, so to clarify the above previous sentence. By employing the allowed ranges of the ICs for the X-branch, we have shown that the evolution of the universe for both the early and the late times fits better with the current observational data than those obtained in the non-deformed models. More concretely, concerning the X-branch solutions, in contrary to the models constructed in the context of the BD theory (even with varying \( \omega \)), we have obtained solutions with the following features. In the Jordan frame (JF), there is no big bang singularity for the inflationary phase, occurring at early times of the universe. After a short time, the accelerating phase is replaced by a decelerating epoch which can be assigned to the radiation era. Such a successful graceful exit is emerges from the NC

\(^{12}\) The special case of the deformation (2.5) has been previously proposed in Ref. [20], based upon dimensional analysis; the Jacobi identity being satisfied.
effects, without needing any branch-change process. It should be emphasized that in the standard and/or generalized BD theory, it is not possible to get a kinetic inflation with graceful exit. In contrast to the commutative cases, for late times, the scale factor tends to take a constant value, which might be supposed to signify the effects of the NC effects on the very large scales (coarse grained explanation by the quantum gravity effects). In such a universe, the energy density and pressure associated to the BD scalar field always take positive and negative values, respectively. Moreover, these solutions have been probed by employing a variety of different ICs.

Then, we have focused on the horizon problem (see Section [V]) by considering the nominal as well as sufficient conditions. We have shown that both these conditions are satisfied for our herein X-branch solutions.

Concerning the cosmological dynamics (see Section [V]), we have focused on two different stages. In the first stage, in order to compare our NC model with the well-know Starobinsky model at the level of equations of motion, not only we have constructed the second order differential equation associated to deformed equations for the scale factor, but also we have produced a useful discussion. It has been demonstrated that there might be a close resemblance between our model and the Starobinsky model, at least in the level of the equations of motion. These equations indicate that it might be possible to establish a physical relation between our NC model with the Starobinsky model via deriving a relation which can connect the NC as well as BD coupling parameter to constants associated to quantum field contributions. However, as we had to remove the NC parameter for drawing such a comparison, constructing such relations is a very complicated issue. In the second stage, we have constructed another representation for the equation of the scale factor, in which the NC parameter is present. We have shown that the phase plane diagrams associated to the X-branch solutions confirm the corresponding numerical results.

Furthermore, we have derived the equations of our herein NC model in the Einstein frame (EF). Similar to the standard case, the Hubble parameter is proportional to the corresponding time derivative of the scalar field associated to the LF. However, the wave equation which yields the scalar field has an extra term which comes from the dynamics of the NC portion. From a EF view point, this extra term appropriately assists to obtain a real inflation for the X-branch for our model. For \( \theta = 0 \), we can recover the corresponding exact solutions associated to the non-deformed case. Moreover, we have obtained the sufficient condition for inflation in the EF. It has been further pointed that satisfying the required conditions associated to real inflation in the JF is equivalent to not only getting inflation in the EF with a graceful exit, but also satisfying the sufficient conditions in both the frames. Subsequently, we have established the EF field equations in the conformally flat background, which indicate that we might obtain a sensible correspondence between our herein NC framework with the Einstein representation of the quantum BD setting constructed in the conformally flat background, where the scalar field coupled to the Dirac spinor matter.

In summary, in the herein paper, we have mainly focused on the X-branch solutions in the absence of any scalar potential and the ordinary matter. By extending the present model into more generalized frameworks, in the presence of NC effects, but with more degrees of freedom, we will get further solutions with interesting features. In fact:

(i) Employing the herein model but taking a varying BD coupling parameter, it is expected to get solutions in which it may be feasible to overcome the mentioned problem, associated to the D-branch solutions. Our expectation lies in that due to having NC effects, with varying \( \omega \), it would be possible to generalize the discussions regarding a bouncing universe already presented in [17].

(ii) In the presence of the ordinary matter and considering instead a non-flat FLRW space time (but still assuming vanishing scalar potential), due to the presence of the NC effects, we will have more degrees of freedom with respect to the corresponding standard models (see for instance [18] in the case of varying \( \omega \)). Moreover, by taking constant values for \( \omega \) and considering the equation of state associated to the false vacuum, we can construct a NC model of extended inflation [21, 49]. Furthermore, in the more general cases, by assuming non-vanishing scalar potential and the ordinary matter fields, with varying, and even with constant values of the BD coupling parameter, it will be worthwhile to compare the results of the generalised NC inflationary model with those obtained in the standard cases, see for instance [50].

(iii) Finally, as the FLRW setting is homogeneous and isotropic, it has been claimed that investigating the horizon problem in such framework is an apparent paradox [45]. Respecting this argument, as well as in order to obtain observables for compatibility checking of our model with the detailed observational data from Planck [51, 52], we should construct a cosmological perturbation setup for our herein NC model. However, the lack of a definite Lagrangian corresponding to the modified set of equations of motion (2.11)-(2.13) is an obstruction for applying the linear perturbations theory. Nevertheless, it would be very interesting to see how the Mukhanov-Sasaki equation would be modified with the introduction of a NC parameter. This would enable to get explicitly the power spectrum of the vacuum fluctuations and all the relevant cosmological observational parameters. This is a very intriguing and difficult task that we leave for a forthcoming paper.
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