The Eigenvalue Distribution of Discrete Periodic Time-Frequency Limiting Operators

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Abstract—Bandlimiting and timelimiting operators play a fundamental role in analyzing bandlimited signals that are approximately timelimited (or vice versa). In this paper, we consider a time-frequency (in the discrete Fourier transform (DFT) domain) limiting operator whose eigenvectors are known as the periodic discrete prolate spheroidal sequences (PDPSSs). We establish new nonasymptotic results on the eigenvalue distribution of this operator. As a byproduct, we also characterize the eigenvalue distribution of a set of submatrices of the DFT matrix, which is of independent interest.

Keywords—periodic discrete prolate spheroidal sequences, partial discrete Fourier transform matrix, eigenvalue distribution, time-frequency analysis.

I. INTRODUCTION

A series of seminal papers by Landau, Pollak, and Slepian explore the degree to which a bandlimited signal can be approximately timelimited [1]–[3]. The key analysis involves a very special class of functions—the prolate spheroidal wave functions (PSWFs) in the continuous case and the discrete prolate spheroidal sequences (DPSSs) in the discrete case. These functions are the eigenvectors of the corresponding composition of bandlimiting and timelimiting operators and provide a natural basis to use in a wide variety of applications involving bandlimiting and timelimiting [1]–[7]; see also [8] and the references therein for applications using PSWFs and see [9] and the references therein for applications using DPSSs.

The periodic discrete prolate spheroidal sequences (PDPSSs), introduced by Jain and Ranganath [10] and Grünbaum [11], are the counterparts of the PSWFs in the finite dimensional case. The PDPSSs are the finite-length vectors whose discrete Fourier transform (DFT) is most concentrated in a given bandwidth. Being simultaneously concentrated in the time and frequency domains makes these vectors useful in a number of signal processing applications. For example, Jain and Ranganath used PDPSSs for extrapolation and spectral estimation of periodic discrete-time signals [10]. PDPSSs were also used for limited-angle reconstruction in tomography [11], for Fourier extension [12], and in [13], the bandpass PDPSSs were used as a numerical approximation to the bandpass PSWFs for studying synchrony in sampled EEG signals.

The distribution of the eigenvalues of a time-frequency limiting operator dictate the (approximate) dimension of the space of signals which are bandlimited and approximately timelimited [1], [4]. Such distributions are known for the case of PSWFs and DPSSs. Specifically, an asymptotic expression for the PSWF eigenvalues was given in [14], and more recently, Israel [15] provided a non-asymptotic bound. Slepian [4] first provided an asymptotic expression for the DPSS eigenvalues. In [16], we recently provided a non-asymptotic result for the distribution of the DPSS eigenvalues (which improves upon a previous result in [9]).

There exist comparatively few results concerning the PDPSS eigenvalues. In [17], it was shown that unlike the PSWF and DPSS eigenvalues, the PDPSS eigenvalues can exactly achieve 0 and 1 and are degenerate in some cases. A non-asymptotic result on the distribution of the PDPSS eigenvalues was given in [18]. The special distribution of the PDPSS eigenvalues (See Figure 1) has been exploited for fast computing Fourier extensions of arbitrary extension length in [12]. In this paper, we provide a finer non-asymptotic result that improves upon the expression in [18]. We also characterize the spectrum of submatrices of the DFT matrix (see Corollary 1), which is of independent interest in signal processing. For example, the low rank of DFT submatrices can be exploited for efficiently computing DFT [18].

II. MAIN RESULTS

A. Time and Band-limiting Operators

To begin, let \( T_N : \mathbb{C}^M \to \mathbb{C}^M \) denote a time-limiting operator that only keeps the first \( N \leq M \) entries of a vector, i.e., for any \( x \in \mathbb{C}^M \),

\[
(T_N(x))[n] := \begin{cases} x[n], & 0 \leq n \leq N - 1, \\ 0, & N \leq n \leq M - 1. \end{cases}
\]

The DFT of any \( x \in \mathbb{C}^M \), denoted by \( \hat{x} \in \mathbb{C}^M \), is defined as

\[
\hat{x}[n] := \frac{1}{\sqrt{M}} \sum_{m=0}^{M-1} x[m] e^{-j \frac{2\pi nm}{M}}, \quad n \in [M],
\]

where \( [M] = \{0, \ldots, M - 1\} \). Given \( \hat{x} \), \( x \) can be recovered by taking the inverse DFT (IDFT), i.e.,

\[
x[m] = \frac{1}{\sqrt{M}} \sum_{n=0}^{M-1} \hat{x}[n] e^{j \frac{2\pi nm}{M}}, \quad m \in [M].
\]

Suppose \( K \in \mathbb{N} \) such that \( 2K + 1 < M \). Let \( B_K : \mathbb{C}^M \to \mathbb{C}^M \) denote a band-limiting operator that first zeros out the DFT of a vector outside the index range \( \hat{I}_K := \{0, \ldots, K\} \cup \{M - K, \ldots, M - 1\} \).
\{M - K, \ldots, M - 1\}, then returns the corresponding signal in the time domain by taking the IDFT. That is

\[
(B_K(x))[m] := \frac{1}{\sqrt{M}} \sum_{k \in \mathbb{Z}_K} \hat{x}[k] e^{j2\pi km/M} = \frac{1}{M} \sum_{k \in \mathbb{Z}_K} \sum_{n=0}^{M-1} x[n] e^{-j2\pi nk/M} e^{j2\pi km/M} = \sum_{n=0}^{M-1} \frac{\sin((2K+1)(m-n)\pi/M)}{M \sin((m-n)\pi/M)} x[n].
\]

Denote \(W = \frac{2K+1}{2M} < \frac{1}{2}\). Let \(B_{M,W} \in \mathbb{C}^{M \times M}\) denote a prolate matrix with entries

\[
B_{M,W}[m,n] = \frac{\sin(2\pi W(m-n))}{M \sin((m-n)\pi/M)}, \quad m,n \in [M].
\]

Note that \(B_K\) is equivalent to \(B_{M,W}\), whose eigenvectors are given by the PDPSSs [10], [11].

Let \([B_{M,W}]_N \in \mathbb{C}^{N \times N}\) be the leading principal submatrix of \(B_{M,W}\) constructed by removing the last \(M - N\) rows and columns from \(B_{M,W}\). Composing the time- and band-limited operators, we obtain the linear operator \(T_N B_K T_N: \mathbb{C}^M \to \mathbb{C}^M\), which has the same non-zero eigenvalues as \([B_{M,W}]_N\). Similar to the case for the DPPSSs which can be obtained efficiently and numerically stably by computing the eigenvectors of a tridiagonal matrix [4], Grünbaum [11] showed that the prolate matrix \([B_{M,W}]_N\) commutes with a tridiagonal matrix, providing a stable and reliable method for computing the PDPSSs.

B. Eigenvalue Concentration

In the rest of the paper, we assume \(2K + 1 < M\), which is of practical interest for applications (e.g., [10], [11], [13]). Let 

\[1 \geq \lambda_N^{(0)} \geq \lambda_N^{(1)} \geq \cdots \lambda_N^{(N-1)} \geq 0\]

denote the eigenvalues of \([B_{M,W}]_N\), where the upper and lower bounds follow because

\[\|x\|^2 \geq x^* B_{M,W} x = \sum_{k \in \mathbb{Z}_K} |\hat{x}[k]|^2 \geq 0\]

for all \(x \in \mathbb{C}^M\), indicating that the eigenvalues of \(B_{M,W}\) are between 0 and 1 (and thus so are the eigenvalues of \([B_{M,W}]_N\) by the Sturmian separation theorem [10]). We note that when \(2K + 1 > M\), it is possible that some eigenvalues \(\lambda_N^{(j)} > 1\); see [17] for more discussion on this.

We establish the following results concerning the eigenvalue distribution for \([B_{M,W}]_N\).

**Theorem 1.** (Spectrum concentration for \([B_{M,W}]_N\)) For any \(M, N, K \in \mathbb{N}\), suppose \(N < M\) and \(W = \frac{2K+1}{2M} < \frac{1}{2}\). Then for any \(\epsilon \in (0, \frac{1}{2})\), we have

\[
\lambda_N^{(2NW)-R(N,M,\epsilon)} \geq 1 - \epsilon, \quad \lambda_N^{(2NW)+R(N,M,\epsilon)+1} \leq \epsilon,
\]

and

\[\#\{\ell : \epsilon < \lambda_N^{(\ell)} < 1 - \epsilon\} \leq 2R(N,M,\epsilon),\]

where

\[
R(N,M,\epsilon) = \left(\frac{4}{\pi^2} \log(8N) + 6\right) \log \left(\frac{16}{\epsilon}\right) + 2 \max \left(\frac{-\log \left(8\pi \left(\left(\frac{M}{N}\right)^2 - 1\right) \epsilon\right)}{\log \left(\frac{M}{N}\right)}, 0\right).
\]

In words, Theorem 1 implies that the first \(\approx \frac{(2K+1)N}{M}\) eigenvalues tend to cluster very close to one, while the remaining eigenvalues tend to concentrate about zero, after a narrow transition band of width \(O(\log \frac{1}{\epsilon} \log N)\) [14]. Figure 4 presents an example to illustrate this phenomenon. We note that this phenomenon has been utilized in [12] for efficiently computing Fourier extensions. Figure 2 shows the size of the eigenvalue gap \#\{\ell : \epsilon < \lambda_N^{(\ell)} < 1 - \epsilon\} for different \(N\) and \(\epsilon\), illustrating that the size is proportional to \(\log \frac{1}{\epsilon} \log N\).

![Figure 1](image1.png)

**Figure 1.** Eigenvalues of the prolate matrix \([B_{M,W}]_N\) with \(M = 1024, N = 256, K = 128\) so that \(\frac{(2K+1)N}{M} \approx 64\) (dashed line).

![Figure 2](image2.png)

**Figure 2.** Width of the transition band \#\{\ell : \epsilon < \lambda_N^{(\ell)} < 1 - \epsilon\} for \(N = \frac{1}{2} M, K = \frac{1}{8} M, \) and \(\epsilon = 10^{-3}, 10^{-6}, 10^{-9}, 10^{-12}\).

Theorem 1 also has implications regarding the distribution of singular values of submatrices of the DFT matrix. Specifically, let \(F_M\) be the normalized DFT matrix with entries given by

\[
F_M[m,n] = \frac{1}{\sqrt{M}} e^{-j2\pi mn/M}, \quad m,n \in [M].
\]

\(^1O(\cdot)\) denotes the standard “big-O” notation.
Let $L = \frac{M}{p}$ be an integer and let $F_{M,p}$ denote an $L \times L$ submatrix of $F_M$ obtained by deleting any consecutive $M - L$ columns and any consecutive $M - L$ rows of $F_M$. Edelman et al. [18] proposed an approximate algorithm for DFT computations with lower communication cost based on the compressibility (low rank) of the blocks of $F_M$, i.e., $F_{M,p}$. Let $1 \geq \sigma(0) \geq \sigma(1) \geq \cdots \geq \sigma(L-1) \geq 0$ denote the singular values of $F_{M,p}$. For any $\epsilon \in (0, \frac{1}{2})$, a bound on the number of singular values such that $\sqrt{\epsilon} \leq \sigma(\ell) \leq \sqrt{1-\epsilon}$ is given in [18], which shows $\{ \ell : \sqrt{\epsilon} \leq \sigma(\ell) \leq \sqrt{1-\epsilon} \} \sim O(\log L)$. This bound highlights the logarithmic dependence on $L$, but ignores the dependence on $\epsilon$. A finer non-asymptotic bound on the width of this transition region is given as follows.

**Corollary 1.** For any $M, p \in \mathbb{N}$ such that $L = \frac{M}{p}$ is an integer, let $F_{M,p}$ denote an $L \times L$ submatrix of the normalized DFT matrix $F_M$ obtained by deleting any consecutive $M - L$ columns and any consecutive $M - L$ rows of $F_M$. Let $\sigma(0) \geq \sigma(1) \geq \cdots \geq \sigma(L-1)$ denote the singular values of $F_{M,p}$. Then for any $\epsilon \in (0, \frac{1}{2})$,

$$\sigma(2^{\ell+1} R(L,M,\epsilon)) \geq \sqrt{1-\epsilon}, \quad \sigma(2^{\ell+1} R(L,M,\epsilon)+1) \leq \sqrt{\epsilon},$$

and

$$\# \{ \ell : \sqrt{\epsilon} \leq \sigma(\ell) \leq \sqrt{1-\epsilon} \} \leq 2 R(L,M,\epsilon),$$

where $R(\cdot,\cdot,\cdot)$ is specified in Theorem 1.

**III. PROOF OF THE MAIN RESULTS**

**A. Supporting Results on DPSSs**

For any $N \in \mathbb{N}$ and $W \in (0, \frac{1}{2}]$, define $B_{N,W} \in \mathbb{C}^{N \times N}$ as a prolate matrix with elements

$$B_{N,W}[m,n] = \sin \left( \frac{2\pi W (m-n)}{\pi (m-n)} \right)$$

for all $m, n \in \{0, \ldots, N-1\}$. Both $B_{N,W}$ and $[B_{M,W}]_N$ are $N \times N$ Toeplitz matrices and they have very similar eigenvalues—the former has sinc elements while the latter has Dirichlet elements. Our idea for proving Theorem 1 is to utilize the spectrum concentration of $B_{N,W}$ and exploit the similarity between $B_{N,W}$ and $[B_{M,W}]_N$.

Let $\lambda_{N,W}^{(\ell)}$ denote the eigenvalues of $B_{N,W}$ placed in decreasing order. The DPSS vectors are the eigenvectors of $B_{N,W}$ [4]. Let $F_{N,W}$ denote the partial normalized DFT matrix with the lowest $2\lfloor NW \rfloor + 1$ frequency DFT vectors of length $N$, i.e.,

$$F_{N,W} = \left[ \frac{1}{\sqrt{N}} e^{-i \frac{|W|}{N}} \cdots \frac{1}{\sqrt{N}} e^{-i \frac{NW}{N}} \right],$$

where, for $f \in [-\frac{1}{2}, \frac{1}{2}]$,

$$e_f := [e^{i 2\pi f / 0} e^{i 2\pi f / 1} \cdots e^{i 2\pi f (N-1)}]^T \in \mathbb{C}^N.$$

are the sampled exponentials. The following result establishes that the difference between $B_{N,W}$ and $F_{N,W}^* F_{N,W}$ has an effective rank of $O(\log N \log \frac{1}{\epsilon})$.

**Lemma 1.** [16, Theorem 1] Let $N \in \mathbb{N}$ and $W \in (0, \frac{1}{2})$ be given. Then for any $\epsilon \in (0, \frac{1}{2})$, there exist $N \times N$ matrices $L_1$ and $E_1$ such that

$$B_{N,W} = F_{N,W} F_{N,W}^* + L_1 + E_1,$$

where

$$\text{rank}(L_1) \leq \left( \frac{4}{\pi^2} \log(8N) + 6 \right) \log \left( \frac{15}{\epsilon} \right) \quad \text{and} \quad \|E_1\| \leq \epsilon.$$

**B. Proof of Theorem 1**

Lemma 1 implies that the difference between $B_{N,W}$ and $F_{N,W}^* F_{N,W}$ is effectively low rank. The main idea is to first show that the difference between the two prolates matrices $B_{N,W}$ and $[B_{M,W}]_N$ is also effectively low rank. By using the Taylor series

$$\frac{1}{\sin x} - \frac{1}{x} = \sum_{r=1}^{\infty} \frac{2(1-2^{-(2r-1)})\zeta(2r)}{\pi^{2r}} x^{2r-1},$$

where $\zeta$ denotes the Riemann-Zeta function, the $(m,n)$-th entry of the difference $[B_{M,W}]_N - B_{N,W}$ is given by

$$([B_{M,W}]_N - B_{N,W})[m,n] = \sin \left( \frac{2\pi W (m-n)}{\pi (m-n)} \right) - \frac{\sin \left( \frac{2\pi W (m-n)}{\pi (m-n)} \right)}{M} \sin \left( \frac{(m-n)\pi}{M} \right) \frac{\sin \left( \frac{2\pi W (m-n)}{\pi (m-n)} \right)}{M}$$

$$= \sum_{r=1}^{\infty} t(r; m-n) = L_2[m,n] + E_2[m,n]$$

for all $m, n = 0, 1, \ldots, N-1$. Here $t(r;k) := 2^{\frac{k}{2}} (1-2^{1-2r}) \zeta(2r) \left( \frac{k}{M} \right)^{2r-1} \sin(2\pi W k)$, and $L_2$ and $E_2$ are $N \times N$ matrices with entries

$$L_2[m,n] = \sum_{r=1}^{R} t(r; m-n), \quad E_2[m,n] = \sum_{r=R+1}^{\infty} t(r; m-n).$$

Define $D \in \mathbb{R}^{2R \times 2R}$ to have entries

$$D[2r - 1 - p, p] = \frac{2}{M^2} \left( 1 - 2^{-(2r-1)} \right) \zeta(2r)(-1)^p (2r - 1)$$

for $1 \leq r \leq R$ and $0 \leq p \leq 2r - 1$. Also define $U, V \in \mathbb{R}^{N \times 2R}$ such that

$$U[n,r] = \left( \frac{n}{N} \right)^r \sin(2\pi W n), \quad V[n,r] = \left( \frac{n}{N} \right)^r \cos(2\pi W n)$$
for all \( 0 \leq r \leq 2R - 1 \) and \( 0 \leq n \leq N - 1 \). The rank of \( L_2 \) then can be identified by noting that
\[
L_2[m, n] = \sum_{r=1}^{R} \sum_{p=0}^{2r-1} D[2r - 1 - p, p] \left( \frac{m}{M} \right)^{2r-1-p} \left( \frac{n}{M} \right)^p .
\]
\[
(\sin(2\pi W m) \cos(2\pi W n) - \cos(2\pi W m) \sin(2\pi W n))
\]
\[=\sum_{r=1}^{R} \sum_{p=0}^{2r-1} D[2r - 1 - p, p] (U[m, 2r - 1 - p]V[n, p] - V[m, 2r - 1 - p]U[p, n]).
\]
\[
= (UDV^*)[m, n] - (VDU^*)[m, n].
\]
This implies \( L_2 = UDV^* - VDU^* \) and \( \text{rank}(L_2) \leq 4R \).

Also note that \( 1 - 2^{1-s} \zeta(s) = \eta(s) \) is the Dirichlet eta function satisfying \( 0 < \eta(s) < \infty \) for all \( s \geq 1 \). We now turn to bound the entries in \( E_2 \) as
\[
|E_2(m, n)| \leq \sum_{r=R+1}^{\infty} t(r; m - n) \leq \frac{2}{\pi M} \left( \frac{N}{M} \right)^{2r-1}
\]
\[
= \frac{2 \left( \frac{N}{M} \right)^{2R+2}}{\pi N} - \frac{2 \left( \frac{N}{M} \right)^{2R}}{\pi N} \leq \frac{2 \left( \frac{N}{M} \right)^{2R}}{\pi N} - 1.
\]
Choosing \( R = \max \left( \frac{-\log 8(\log(\frac{2}{\pi})^2 - 1)}{2 \log \frac{2}{\pi}}, 0 \right) \), we obtain that \( |E_2(m, n)| \leq \frac{\epsilon}{16} \). It follows from the Gershgorin circle theorem that
\[
\|E_2\| \leq \max_{m} \max_{n} |E_2(m, n)| \leq \frac{\epsilon}{16}.
\]

By Lemma [1] there exist \( N \times N \) matrices \( L_1 \) and \( E_1 \) with
\[
\text{rank}(L_1) \leq \left( \frac{4}{\pi^2} \log(8N) + 6 \right) \log \left( \frac{16N}{\epsilon} \right), \quad \|E_1\| \leq \frac{4}{16} \epsilon
\]
such that \( B_{N,W} = F_{N,W} F_{N,W}^* + L_1 + E_1 \).

Denoting \( L = L_1 + L_2 \) and \( E = E_1 + E_2 \), we obtain
\[
B_{M,W} = B_{N,W} + L_2 + E_2 = F_{N,W} F_{N,W}^* + L + E,
\]
where
\[
\text{rank}(L) \leq \text{rank}(L_1) + \text{rank}(L_2)
\]
\[\leq \left( \frac{4}{\pi^2} \log(8N) + 6 \right) \log \left( \frac{16N}{\epsilon} \right) + 2 \max \left( \frac{-\log 8(\log(\frac{2}{\pi})^2 - 1)}{2 \log \frac{2}{\pi}}, 0 \right)
\]
and
\[
\|E\| \leq \|E_1\| + \|E_2\| \leq \frac{15}{16} \epsilon + \frac{1}{16} \epsilon = \epsilon.
\]

Now we utilize the fact that \( F_{N,W} F_{N,W}^* \) has only eigenvalues 1 and 0 to obtain the eigenvalue distribution for \( B_{M,W} \).

For all integers \( \ell \in [N] \), the Weyl-Courant minimax representation of the eigenvalues can be stated as
\[
\lambda_N^{(\ell)} = \left\{ \begin{array}{ll}
\min_{S_{\ell}} \max_{y_1, y_2 \in S_{\ell}} |y_1 \cdot y_2| \left( \frac{B_{M,W}, N}{y_1, y_2} \right), & \text{if } \ell \leq \frac{N}{2}, \\
\max_{y_{\ell+1}} \min_{y_{\ell+1} \in S_{\ell+1}} |y_1 \cdot y_2| \left( \frac{B_{M,W}, N}{y_1, y_2} \right), & \text{if } \ell > \frac{N}{2},
\end{array} \right.
\]
where \( S_{\ell} \) is an \( \ell \)-dimensional subspace of \( \mathbb{C}^N \).

Recall that both \( [B_{M,W}]_{N} \) and \( L \) are symmetric, which implies that \( E \) is also symmetric. Let \( \tilde{S} \) be the column space of \( [F_{N,W}, L] \) and \( d = \dim(\tilde{S}) \).

\[
\lambda_N^{(d)} = \min_{S_{d}} \max_{y_1, y_2 \in S_{d}} \langle \frac{B_{M,W}, N}{y_1, y_2} \rangle
\]
\[\leq \max_{\|y\| = 1} \langle \frac{B_{M,W}, N}{y} \rangle
\]
\[= \max_{\|y\| = 1} \langle \left( \frac{B_{M,W}, N - F_{N,W} F_{N,W}^* - L}{y} \right) y, y \rangle
\]
\[\leq \|E\| \leq \epsilon.
\]
We have \( d = \dim(\tilde{S}) \leq 2|NW| + R(N, M, \epsilon) + 1 \), giving
\[
\lambda_N^{(d)} \leq \epsilon.
\]
On the other hand, let \( \tilde{S} \) be the intersection of the column space of \( F_{N,W} \) and the null space of \( L \). Denote \( \tilde{d} = \dim(\tilde{S}) \). We have
\[
\lambda_N^{(\tilde{d})} = \max_{S_{\tilde{d}}} \min_{y_1, y_2 \in S_{\tilde{d}}} \langle \frac{B_{M,W}, N}{y_1, y_2} \rangle
\]
\[\geq \max_{\|y\| = 1} \langle \frac{B_{M,W}, N}{y} \rangle
\]
\[= \max_{\|y\| = 1} \langle \left( \frac{B_{M,W}, N - F_{N,W} F_{N,W}^* - L}{y} \right) y, y \rangle
\]
\[+ \langle \frac{E}{y} \rangle
\]
\[\geq 1 - \|E\| \geq 1 - \epsilon.
\]
Noting that \( \tilde{d} \geq 2|NW| - R(N, M, \epsilon) + 1 \), we obtain
\[
\lambda_N^{(\tilde{d})} \geq 1 - \epsilon.
\]

C. Proof of Corollary [7]

We first consider the spectrum of \( [F_{M,L}] \), the top-left principal submatrix of \( F_{M} \). It is clear that its singular values are between 0 and 1 since it is a submatrix of \( F_{M} \). We first observe that the gram matrix of \( [F_{M,L}] \) is identical to \( [B_{M,1/2p}] \) up to unitary phase factors, i.e.,
\[
\langle \frac{F_{M,L}, [F_{M,L}]}{m, n} \rangle = e^{-\pi \frac{m-n}{2r_{M,1/2p}}} \frac{\sin((\pi m - n)/p)}{M \sin((\pi m - n)/M)}.
\]
This implies \( [F_{M,L}]_{[F_{M,L}]} \) has the same eigenvalue distribution to \( [B_{M,1/2p}] \). Thus, Corollary [7] holds for \( [F_{M,L}] \) trivially by following Theorem [1].

Now note that any submatrix \( F_{M,L} \) obtained by deleting any consecutive \( M - L \) columns and any consecutive \( M - L \) rows of \( F_{M} \) is identical to \( [F_{M,L}] \) up to unitary phase factors
\[
F_{M,L} = \text{diag}(a_{\xi}) [F_{M,L}] \text{diag}(a_{\eta}),
\]
where \( \xi, \eta \) depend on the locations of the submatrix \( F_{M,L} \) and
\[
a_{\xi} = \begin{bmatrix} 1 & e^{-2\pi i \frac{1}{M}} & \cdots & e^{-2\pi i (L-1)/M} \end{bmatrix}^T.
\]
Thus, any submatrix \( F_{M,L} \) has the same spectrum as \( [F_{M,L}] \).
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