Additional geometric investigation of carved Monge surfaces

M Gil-oulbê*, A I Ndomilep
Department of Civil Engineering, RUDN University, Moscow, Russia

gil-oulbem@hotmail.com

Abstract. The large number of scientific papers on carved Monge surfaces give an idea of the dynamism of the research activity on them. Based on a definition of the concerned surfaces, several investigations have been carried out and many scientific works produced. The methods used by scientists to investigate the geometry of these surfaces are those of the differential geometry. Additional investigations, by mean of the kinematic method gave as a result, an alternative definition: a carved surface is generated by the motion of some plane curve (generatrix) along another arbitrary curve (directrix) so that the generatrix curve lies in the normal plane of the directrix line and is rigidly connected with it. However, it should be noted that the latter condition is necessary, but not sufficient for the formation of a carved surface. To achieve this goal, methods of differential geometry are used. In this article, the equation of the generatrix curve in the polar coordinate system is used and the parametric equation of carved Monge surfaces is specified that allows to more study their inner and outer geometries. New equations are obtained for more kinds of carved Monge surfaces that are classified and plotted by mean of the software Mathcad. These new forms of carved Monge surfaces can be used as middle surfaces for thin elastic shells that will be expressive and cover large spans.

Keywords: carved Monge surfaces, generatrix curves, plane curves, directrix curve, binormal, coefficients of the quadratic form.

1. Introduction

In the theory of surfaces one examines the shape of a surface, its curvature, the properties of various types of curves on a surface, aspects of deformation, the existence of a surface with given internal or external features, etc.

Studies in the theory of surfaces involve examining various classes of surfaces, such as surfaces of the second order, surfaces of screw motion, helicoids, Catalan surfaces, conoids, regression surfaces, canal surfaces, Dupin cyclides, Enneper surfaces, Weingarten surfaces, carved Monge surface etc.

Gaspar Monge [1] defined carved surfaces as surfaces generated by the motion of a flat curve lying in a plane that rolls without sliding along a certain unfolding surface. It is easy to show that every point lying in a plane rolling without slipping on a developing surface makes at each moment a motion orthogonal to this plane. Consequently, an alternative definition of a carved surface can be given: a carved surface is generated by the motion of some plane curve (generatrix) along another arbitrary curve (directrix) so that the generatrix curve lies in the normal plane of the directrix line and is rigidly connected with it. This definition is schematized by figure 1. A similar approach to the definition of carved Monge surfaces was adopted in [2-5]. It should be noted, however, that the latter condition is necessary, but not sufficient for the formation of a carved surface.

Many scientists [6-8] have investigated the geometry of carved surfaces. Investigations have shown that a system of plane curves that form surfaces when moving along a directrix is a system of lines of curvature of the carved Monge surface. The generatrix curves are the geodesic lines of the surface, i.e. the normals to the surface lie in the plane of the generatrix curves and are their normals. In a number of
works, carved surfaces are defined as surfaces for which the planes of the same family of lines of curvature are orthogonal to the surface.

It is necessary to find out the rotation conditions of the generatrix curve so that its successive positions and the trajectories of its points generate a network of lines of curvature and give rise to additional geometric investigation of carved Monge surfaces.

2. Materials and methods

The figure 1 materialize the alternative definition of carved Monge surfaces. It is the representation of the kinematic of generating these surfaces. A circle of variable radius (the generatrix) slides along a line (the directrix) that always passes through its centre. The circle is rigidly always perpendicular to that directrix. The different positions of the circle during its slide motion confer to each of its points a vector position $\rho(u, v)$, function of the Cartesian coordinates and curvilinear coordinates.

The kinematic method for constructing carved surfaces makes it possible to divide them into two additional groups depending on the generatrices: parabolic sinusoidal carved surfaces and parabolic-cosine carved surfaces.

3. Geometry

The surface generated by the motion of a plane curve in the normal plane of an arbitrary curve is shown in figure 1.

Let us write the surface equation in vector form

$$\rho(u, v) = r(u) + R(v)e(u, v),$$

where $\rho(u, v)$ is the radius vector of the surface; $r(u)$ is the radius of the vector of the guide curve; $R(v)$ - equation of the generatrix curve in the polar coordinate system; $e(u, v) = e_0(u)\cos v + g_0(u)\sin v$ - the equation of the circle of the unit radius (Figure 1. b) in the normal plane of the guide curve; $e_0(u), g_0(u)$ are unit initial vectors in the normal plane of the guiding curve.

In equation (1), the equation of the generatrix curve in the polar coordinate system is used. The parametric equation of carved Monge surfaces is

$$\rho(u, v) = r(u) + X(v)e_0(u) + Y(v)g_0(u)$$

(2)

Obviously, it is not difficult to move from one form of the equation to another, however, for the ongoing research it is more convenient to use the surface equation in the form (1).

Given that the generatrix curve lies in the normal plane of the guide curve, i.e. in the plane containing the normal $v$ and binormal $\beta$ directrix curve, the unit circle equation can be written as

$$e(u, v) = v\cos \omega + \beta\sin \omega = e(w, \omega),$$

(3)

where $\omega = v + \theta(u)$; $\theta(u)$ - the angle between the normal of the directrix curve and the initial vector of $e_0(u)$ the moving coordinate system to which the generatrix curve is attached. As the generatrix curve moves along the directrix, the system of unit vectors $e_0(u), g_0(u)$ can rotate around a tangent to the directrix curve with respect to the normal and binormal $w = w[u, \omega(u, v)]$.

Given that vector functions $e(u, v), g(u, v)$ are complex functions when they are differentiated with respect to $u, v$, we use the relations:
We obtain a formula for calculating the coefficient of the second quadratic form, which characterizes the conjugacy of the surface coordinate system

\[ D^2 = EG - F^2 = \left( R^2 + \dot{R}^2 \right) \left[ s' - k_x R \left( e + g \right) \right] + \dot{R}^2 \left( \theta' + \chi_s \right)^2. \]

We obtain a formula for calculating the coefficient of the second quadratic form, which characterizes the conjugacy of the surface coordinate system

\[ M = (m \rho_{uv}) \frac{1}{D} = \left( \begin{array}{ccc} s' - k_x R (e + g) & 0 & R \cdot (\theta' + \chi_s) \\ 0 & R & R \\ k_x [R (e + g) - \dot{R}] & R(\theta' + \chi_s) & \dot{R} (\theta' + \chi_s) \end{array} \right) = \left( \begin{array}{ccc} R^2 & \dot{R} & \dot{R} \left( \theta' + \chi_s \right) \\ \dot{R} & R & R \left( \theta' + \chi_s \right) \end{array} \right). \]
It is known from the theory of surfaces that if the coefficient of the first quadratic form $F$ is equal to zero, the surface coordinate system is orthogonal, and if the coefficient of the second quadratic form $M$ is equal to zero, the surface coordinate system will be conjugate. If both coefficients are equal to zero, the surface coordinate system will be a system of lines of the main curvatures of the surface. Since the coefficients of the quadratic forms $F$ and $M$ are not equal to zero, in General, the surface described by the formula (2) is not a carved Monge surface. However, the formulas (8), (9) for the coefficients $F$ and $M$ show that they have a common multiplier, equating it to zero; we get an orthogonal, conjugate coordinate system. Therefore, in order for the surface generated by the motion of the generatrix flat curve along the directrix line to be a carved Monge surface, the generatrix curve must lie in the normal plane of the directrix curve and the condition must be satisfied:

$$\theta'(u) + \chi_s(u) = 0 \quad \text{or} \quad \theta(u) = \int \chi_s(u) \, du + \theta_0,$$

that means the coordinate system $e_\theta(u), g_\theta(u)$, which describes the equation of a plane generatrix curve in the normal plane of the directrix curve, forms the angle $\theta(u)$ with the normal of the directrix curve $v(u)$, which varies during the motion of the generatrix curve along the directrix according to the law (10), $\theta_0$ is the integration constant, the initial angle between the vectors $e_\theta(u)$ and $v$.

As follows from formula (10), the angle $\theta(u)$ depends on the curvature of the torsion of the directrix line. For a flat directrix line, $\chi = 0$, $\theta(u) = \theta_0$, i.e. the generatrix curve moves in the normal plane of the directrix without rotation.

4. Results and discussion
Taking into account the formula (10), we obtain the coefficient formulas of the first quadratic form of the carved Monge surface:

$$A = \sqrt{E} = s' - k_1 R(ev) = s' - k_1 R \cos \omega; \quad B = \sqrt{G} = \sqrt{R^2 + R'^2} = s'_0; \quad F = 0,$$

$$D = AB = \sqrt{(R^2 + R'^2)}[s' - k_1 R(ev)];$$

(11)

$s'_0$ - parameter of the length of the generatrix curve,

$$m = [s' - Rk_1(ev)] \left( -Re + \tilde{R}g \right) \frac{1}{D} = -\frac{Re + \tilde{R}g}{B}$$

(12)

- normal to the surface, which lies in the plane of the generatrix curve defined by the vectors $e$, $g$ and, therefore, the generatrices of the carved surface are geodesic lines of the surface. We obtain the coefficients of the second quadratic form of the carved Monge surface and its main curvatures:

$$\rho_{uu} = s'' - R[k_1'(ev) + k_4 \theta'(ev)]; \quad \tilde{\rho} + k_1[s' - k_1 R(ev)][(ev) - (e\tilde{v})];$$

$$L = -[R(ev) + \tilde{R}(e\tilde{v})][s' - k_1 R(ev)]^2 - \frac{k_4}{D} = \frac{(R \cos \omega + \tilde{R} \sin \omega)}{k_3} A, k_3 B,$$

$$N = \frac{R(\tilde{R} - R) - 2\tilde{R}^2}{B}; \quad M = 0,$$

$$k_4 = \frac{R \cos \omega + \tilde{R} \sin \omega}{AB} A, \quad k_5 = -\frac{R(\tilde{R} - R) - 2\tilde{R}^2}{B^3} = \frac{R(\tilde{R} - R) + 2\tilde{R}^2}{(R^2 + \tilde{R}^2)^{3/2}} = k_0,$$

(13)

where $k_0$ is the curvature of the generator curve.

The formulas for the coefficients of quadratic forms and radii of curvature (13) are obtained for generatrices lines expressed in the polar coordinate system. When using the equation of the generatrix curve in the parametric form, $x = X(v), y = Y(v)$, we obtain

$$B = s'_0 = \sqrt{X^2 + Y^2}; \quad k_2 = k_0 = \frac{\bar{X} \cdot \bar{Y} - \bar{X} \cdot \bar{X}}{(X^2 + Y^2)^{3/2}}.$$

(14)
To write the formulas of the coefficient $A$ of the first quadratic form and curvature $k_1$, we use the transition from the polar coordinate system to the Cartesian rectangular coordinate system:

$$
\dot{X} = R \cos \theta - R \sin \theta; \quad \dot{Y} = R \sin \theta + R \cos \theta; \\
R \cos \omega = R \cos (\theta + \theta_0) = R \cos \theta \cos \theta_0 - R \sin \theta \sin \theta = X \cos \theta - Y \sin \theta , \\
\begin{align*}
R \cos \omega + \dot{R} \sin \omega &= R \left( \cos \omega \cos \theta - \sin \omega \sin \theta \right) + \dot{R} \left( \sin \omega \cos \theta + \cos \omega \sin \theta \right) \\
&= \left( R \cos \omega - R \sin \omega \right) \sin \theta + \left( \dot{R} \sin \omega + R \cos \omega \right) \cos \theta = \dot{X} \sin \theta + \dot{Y} \cos \theta 
\end{align*}
$$

and therefore,

$$
A = s' - k_s \left[ X(\theta) \cos \theta - Y(\theta) \sin \theta \right]; \quad k_1 = \frac{\dot{X} \sin \theta + \dot{Y} \cos \theta}{AB} . \quad (15)
$$

Let us consider several special cases of carved surfaces.

- If the directrix curve is a circle, then

$$
r(u) = a (\cos u + j \sin u); \quad s' = a; \quad k = 1/a; \quad k_s = 1. \quad (16)
$$

A carved surface is a surface of revolution.

- If the generating curve is a circle, then

$$
R(v) = a = \text{const}; \quad \dot{R} = 0; \quad B = R = a; \quad k_1 = \frac{k_s}{A} \cos \omega; \quad k_2 = \frac{1}{a} . \quad (17)
$$

The carved surface is a tubular surface of circular cross section.

- If the directrix is also a circle, then the carved surface will be a torus.

Take a straight line for the generatrix line. In this case:

$$
R(v) = a \frac{\sin \nu}{\cos \nu}; \quad \dot{R} = a \frac{\cos \nu}{\cos^2 \nu} = R \nu \sin \nu; \quad B = \frac{R}{\cos \nu}; \\
\ddot{R} - \dot{R} = R \nu = R \left( \frac{\nu^2}{\cos^2 \nu} + \frac{1}{\cos \nu} \right) - 1 = 2 R \nu^2; \quad k_2 = 0 . \quad (18)
$$

A carved surface is a deployable (torso) surface. In [9-15] are shown carved surfaces with different directrices and generatrices curves.

Shell shaping based on parabolic sinusoidal carved surfaces.

Let us consider carved surfaces with a directrix parabola and a generatrix sinusoid $X(\nu) = v$. Here $b$ is the amplitude; $c$ is the half-wavelength of the sine wave. Since the parabola is a plane curve, the position of the generatrix of the parabola guide in the normal plane is determined by a constant angle $\theta_0$. Depending on the magnitude of the angle $\theta_0$, which determines the relative position of the parabola and the sinusoid, various shells can be constructed, including half-wave and multi-wave. A parabola in a vertical plane can, if necessary, be replaced by a parabola in a horizontal or inclined plane, which does not affect the geometric characteristics of the surface, $s' = \left( 1 + 4 a^2 u^2 \right)^{1/2}; \quad k = \frac{2 a^2}{s'^2};$

$$
k_s = \frac{2 a^2}{s'^2} = \frac{2 a^2}{1 + 4 a^2 u^2} \text{ - geometric characteristics of the directrix parabola;}
$$

$$
\begin{align*}
s_0' &= \left( 1 + \frac{b^2}{c^2} \pi^2 \cos^2 \left( \frac{\nu}{c} \right) \right)^{1/2}; \quad k_0 = \frac{b \pi^2}{c^2 s_0'}; \quad k_{s0} = k_0 = \frac{b \pi^2}{c^2 s_0'}
\end{align*}
$$

- geometric characteristics of the generatrix sinusoid. The geometric characteristics for the surface are determined by the formulas (14), (15):
\[ A = s' - k \varphi(v, \theta_0); \quad B = s'_0; \quad k_1 = \frac{\psi(v, \theta_0)}{AB}; \quad k_2 = k_0, \]  

Moreover
\[ \varphi(v, \theta_0) = v \cos \theta - b \sin \left( \frac{\pi v}{c} \right) \sin \theta; \quad \psi(v, \theta_0) = v \sin \theta + b \sin \left( \frac{\pi v}{c} \right) \cos \theta \]

- parametric equations of a sinusoid rotated by an angle \( \theta_0 \) counterclockwise relative to the normal of the parabola.

Let us consider a half-wave surface with a directrix parabola in a vertical plane. Figure 2 presents the types of parabolic-sinusoidal shells with different parameters of the generatrix of a sinusoid with a length of one half-wave at \( \theta_0 = 90^\circ \). The parameters of the parabola in all figures are assumed to be the same: \( a = -0.05 \text{ m}^2, \quad -15 \leq u \leq 15 \ (\text{m}) \) (reference span of the parabola in the plan of 30 m), elevation height \( f = 11.25 \text{m} \). The parameters of the sinusoid are shown in Figure 2.

For \( b < 0 \) (\( a < 0 \)) - the vertex of the parabola is directed upwards. We obtain shells of positive Gaussian curvature (Figure 2, a, c, e, g). For \( b > 0 \) (\( a < 0 \)) we obtain shells of negative Gaussian curvature (Figure 2, b, d, e, h). Depending on the ratio of the amplitude \( b \) and the half-wavelength of the sine wave, we obtain gently sloping or uplifting in the transverse direction of the shell.

If the origin of the sinusoid is shifted by \( \pi / 2 \), we obtain
\[ Y(v) = b \sin \left( \frac{\pi v}{c} + \frac{\pi}{2} \right) = b \cos \left( \frac{\pi v}{c} \right), \]

**Figure 2.** Parabolic sinusoidal carved surfaces, Directrix - Parabola \( z = -0.05 y^2, \quad -15 \leq y \leq 15 \);
Generatrix - sine wave \( z = b \sin \left( \frac{\pi x}{c} \right), \quad 0 \leq x \leq c \).

Shell shaping based on parabolic cosine carved surfaces.
That means, such a shift is similar to replacing the generatrix of a sinusoid with a cosine. In this case, we obtain a shell with zones of positive and negative Gaussian curvature in the section \( \pi / 2 \). The surfaces are presented in figure 3.
Figure 3. Parabolic-cosine carved surfaces
Directrix - Parabola $z = -0.05y^2$, $-15 \leq y \leq 15$;
Generatrix - sine wave $a-b = \bar{z} = b \cdot \sin(2\pi x/c)$;
$d-f = \bar{z} = b \cdot \sin(2\pi x/c)$, $0 \leq x \leq c$. 
The shape of the parabolic-cosine shells is not affected by the sign of the amplitude of the cosine (for \(0 \leq x \leq \pi\)), since the cosine is inversely symmetric with respect to the cross section \(x = \pi / 2\). A change in sign leads to a 180° turn of the shell.

In figure 3, c-e are shown the segments of parabolic-cosine shells with the generatrix of the cosine wave on the full wave of the cosine wave.

The sign of the Gaussian curvature of the shell changes in sections \(v = c / 2\) and \(v = 3c / 2\). Cosine shells, in contrast to sinusoidal ones, have a horizontal tangent in the reference sections. The reference angle of the parabolic-sinusoidal curves depends on the ratio of the amplitude of the sine wave to the half-wave length, \(\omega\) is the angle formed by the tangent generatrix of the sinusoid with the axis of the sinusoid at the point of contact of the sinusoid of the reference parabola.

The additional geometric investigation of carved Monge surfaces is also the one of the surfaces of complex geometry. The new obtained surfaces, interesting and expressive can be the median ones of thin elastic shells. These surfaces can be used to design more types of thin elastic shells. Because these elastic shells are thin, their geometry is also the one of their median carved Monge surfaces. The thin elastic shells design in the shapes of these surfaces can be expressive, durable, buckling resistant and cover large spans. They can find their application in the architecture of civil and industrial building and in mechanical engineering. The multitude of the additional expressive forms of carved Monge surfaces
should call for their application in construction industry by architects. In [9-17], more can be found on carved Monge surfaces.

5. Conclusion
The additional investigation on carved Monge surfaces, in which their definition has been specified, gives rise to more surfaces of several forms. They can be the median ones of expressive and large span strong thin elastic shells with needed buckling resistance. Their interesting forms can allow the application of classic and newly appearing building materials.

Many investigations on the geometry of carved surfaces have been carried out by several scientists. As a result, an alternative definition has been given: a carved surface is generated by the motion of some plane curve (generatrix) along another arbitrary curve (directrix) so that the generatrix curve lies in the normal plane of the guide line and is rigidly connected with it. However, it should be noted that the latter condition is necessary, but not sufficient for the formation of a carved surface.

6. References
[1] Monge G 1936 Application of analysis to geometry (Moscow: Publishing house ONTI)
[2] Ivanov V N, Rizvan M 2002 Geometry of the carved surfaces of Monge and the construction of shells (Moscow: Publishing house DIA)
[3] Ivanov V N, Rizvan M 2003 Analysis of the coating of a sports facility with a shell in the form of a carved Monge surface (Moscow: Publishing house DIA)
[4] Rizvan M 2003 Structure of shells in the form of carved Monge surfaces (Moscow: Publishing house DIA)
[5] Rizvan M 2004 Geometry, design and investigation of the stress-strain state of shells in the form of carved Monge surfaces general appearance (PhD thesis) RUDN University, Russia.
[6] Ramsey A J 2018 Automatic Modulation Classification Using Cyclic Features via Compressed Sensing (PhD thesis) Rochester Institute of Technology, USA
[7] Wolfram S 2002 A New Kind of Science (Illinois, USA: Wolfram Media Inc.)
[8] Hestenes D 2011 The Shape of Differential Geometry in Geometric Calculus (Springer) 393–410
[9] Krivoshapko S N, Ivanov V N 2015 Encyclopedia of Analytical Surfaces (Springer)
[10] Brander D and Gravesen J 2017 Surfaces foliated by planar geodesics: A model for curved wood design Proceedings of Bridges 2017 (Arizona, USA: Tessellations Publishing) 487–490
[11] Chicone C and Kalton N 2002 Flat embeddings of the Möbius strip in R^3 Comm. Appl. Nonlinear Anal. 9 31–50
[12] Marriott P, Salmon M 2000 Applications of Differential Geometry to Econometrics (Cambridge University Press)
[13] Manton J H 2005 On the role of differential geometry in signal processing IEEE International Conference on Acoustics, Speech, and Signal Processing 2005 pp. 1021–24
[14] Micheli M 2008 The Differential Geometry of Landmark Shape Manifolds: Metrics, Geodesics, and Curvature (PhD Thesis) University of Padova, Italy.
[15] Love D J, Heath R W 2003 Grassmannian Beamforming for Multiple-Input Multiple-Output Wireless Systems IEEE Transactions on Information Theory 49(10) 2735–47
[16] Marriott P, Salmon M 2000 Applications of Differential Geometry to Econometrics (Cambridge University Press)
[17] Bullo F, Lewis A 2010 Geometric Control of Mechanical Systems: Modeling, Analysis, and Design for Simple Mechanical Control Systems (Springer)