MATRIX PAINLEVÉ II EQUATIONS

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We use the Painlevé–Kovalevskaya test to find three matrix versions of the Painlevé II equation. We interpret all these equations as group-invariant reductions of integrable matrix evolution equations, which allows constructing isomonodromic Lax pairs for them.

Keywords: Painlevé equation, Lax representation, symmetric reduction

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1. Introduction

The second of the six famous Painlevé equations is

\[ y'' = 2y^3 + zy + a, \]

where where a prime denotes the derivative with respect to \( z \) and \( a \) is an arbitrary complex constant. A matrix version of the Painlevé–Kovalevskaya test was proposed in [1], where it was proved that it holds for (1) in the case where \( y(z) \) is an arbitrary \( n \times n \) matrix and \( a \) is a scalar matrix.

One possible generalization of matrix equation (1) is as follows. It is clear that in the noncommutative case, we can change the principal differential-homogeneous part \( y'' = 2y^3 \) of this equation by adding a term of the same weight \( \kappa [y, y'] \), \( \kappa \in \mathbb{C} \). In particular, it follows from the results in Sec. 2 that the equation

\[ y'' = \kappa [y, y'] + 2y^3 + zy + a, \quad a \in \mathbb{C}, \]

satisfies the matrix Painlevé–Kovalevskaya test if and only if \( \kappa = 0, \pm 1, \pm 2 \). We note that equations with opposite signs of \( \kappa \) are related by the changes \( y \rightarrow -y, a \rightarrow -a \), or \( y \rightarrow y^\top \).

Another direction for generalizations was suggested by the results in [2], where the Painlevé II (PII) equation with matrix coefficients appeared. More precisely, the term linear in \( y \) was written in that paper as \((zy + yz)/2\), where \( z \) denoted a noncommutative dependent variable such that \( z' = 1 \). For our purposes, it is more convenient to replace this variable with \( z + 2b \), where \( z \) is a commutative independent variable and \( b \) is a matrix constant.

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Here, we study matrix generalizations of the PII equation of the general form

$$y'' = \kappa[y, y'] + 2y^3 + zy + b_1y + yb_2 + a,$$

where $a$, $b_1$, and $b_2$ are matrix constants and $\kappa$ is a scalar constant.

**Remark 1.** Equation (2) is invariant under the change

$$b_1 \to b_1 + \beta_1 I, \quad b_2 \to b_2 + \beta_2 I, \quad z \to z - \beta_1 - \beta_2,$$

where $\beta_i \in \mathbb{C}$ and $I$ is the identity matrix.

In Sec. 2, we use the matrix version of the Painlevé–Kovalevskaya test to find the sets of $\kappa$, $a$, $b_1$, and $b_2$ for which Eq. (2) can be integrated, and we prove the following statement.

**Theorem 1.** Equation (2) satisfies the Painlevé–Kovalevskaya test if and only if it is reduced by transformation (3) to one of the cases

$$y'' = 2y^3 + zy + by + yb + \alpha I, \quad \alpha \in \mathbb{C}, \quad b \in \text{Mat}_n,$$

$$y'' = \pm[y, y'] + 2y^3 + zy + a, \quad a \in \text{Mat}_n,$$

$$y'' = \pm 2[y, y'] + 2y^3 + zy + by + yb + a, \quad a, b \in \text{Mat}_n, \quad [b, a] = \pm 2b.$$

We note that (P$^0_2$) is exactly the equation in [2]. Equations (P$^1_2$) and (P$^2_2$) seem new.

In Sec. 3, we justify the integrability of the found cases by constructing isomonodromic Lax pairs

$$A' = B_\zeta + [B, A],$$

where $\zeta$ is a spectral parameter and $A(z, \zeta)$ and $B(z, \zeta)$ are 2×2 matrices with noncommutative elements. Such a representation for Eq. (P$^2_2$) was found in [3].

One method for obtaining representations (4) is based on the known observation that invariant solutions of evolution equations admitting zero-curvature representations satisfy ordinary differential equations of the Painlevé type. In this case, Lax pair (4) is obtained from the zero-curvature representation by a standard procedure. In particular, it is well known that the PII equation corresponds to Galilean-invariant solutions of the nonlinear Schrödinger (NLS) equation and also to scale-invariant solutions of the modified Korteweg–de Vries (mKdV) equation. Here, we generalize these reductions to the matrix case.

One source of noncommutative coefficients in (2) is the arbitrary matrices contained in the symmetry groups of non-Abelian [4], [5] evolutionary equations. For example, the matrix NLS equation

$$u_t = u_{xx} + 2uvu, \quad v_t = -v_{xx} - 2vuv$$

admits transformations of the form $u \to b_1ub_2$, $v \to b_2^{-1}vb_1^{-1}$, where $b_i \in \text{Mat}_n$. Two well-known matrix generalizations of the mKdV equation [4], [6]

$$u_t = u_{xxx} - 3u^2u_x - 3uu_x^2,$$

$$u_t = u_{xxx} + 3[u, u_{xx}] - 6uu_xu,$$

obviously admit the transformation group $u \to bub^{-1}$. 
One more source of matrix coefficients in (2) is related to matrix coefficients in integrable evolution equations themselves. Here, we use the integrable version of the matrix mKdV equation

\[
    u_t = u_{xxx} + 3[u, u_{xx}] - 6uu_x u + (u_x + u^2)b + b(u_x - u^2), \quad b \in \text{Mat}_n,
\]

whose zero-curvature representation is given in Sec. 3. We do not know whether this generalization has appeared in the literature.

In Sec. 3, we show that Eqs. (P\textsuperscript{2})\textsubscript{i} in Theorem 1 can be obtained from the above matrix evolution equations by some self-similar reductions. This allows finding isomonodromic Lax pairs (4) for the matrix PII equations.

2. Proof of Theorem 1

2.1. The Painlevé–Kovalevskaya matrix test. The Painlevé–Kovalevskaya matrix test [1] for Eq. (2) is based on counting the arbitrary scalar constants in a formal solution of the form

\[
    y = \frac{p}{z - z_0} + c_0 + c_1(z - z_0) + \cdots, \quad p, c_j \in \text{Mat}_n, \quad z_0 \in \mathbb{C}.
\]  

(6)

One such arbitrary constant is \( z_0 \). For the series \( y \) to represent a generic solution, the matrices \( p \) and \( c_j \) must contain \( 2n^2 - 1 \) more arbitrary constants. We assume that the Painlevé–Kovalevskaya test is satisfied if such matrices exist. We note that there might also exist other formal solutions of form (6) containing fewer arbitrary constants.

Remark 2. For any nondegenerate matrix \( T \), the series \( TyT^{-1} \) satisfies Eq. (2), where \( b_i \to \tilde{b}_i \overset{\text{def}}{=} Tb_iT^{-1} \) and \( a \to \tilde{a} \overset{\text{def}}{=} TaT^{-1} \). Therefore, together with Eq. (2), the equation corresponding to the coefficients \( \tilde{b}_i \) and \( \tilde{a} \) also satisfies the Painlevé–Kovalevskaya test.

Substituting the series in the equation and collecting the coefficients of powers of \( z - z_0 \), we obtain relations of the form

\[
    p^3 = p,
\]  

(7)

\[
    L_{(j+1)/2}(c) = \frac{j(j - 1)}{2} c_j = f_j(z_0, p, c_0, \ldots, c_{j-1}), \quad j \geq 0,
\]  

(8)

where

\[
    L_\sigma(c) \overset{\text{def}}{=} p^2 + pc + cp^2 + \sigma(p - cp).
\]

we can easily calculate the first few functions \( f_i \) explicitly:

\[
    f_0 = 0, \quad f_1 = -pc_0^2 - capc_0 - c_0^2p - \frac{1}{2}(z_0p + b_1p + pb_2).
\]

Because \( f_j \) in the right-hand sides do not contain \( c_j \), we can calculate the matrices \( c_j \) in Eq. (8) with the number \( j \). If the linear operator in the left-hand side is invertible, then \( c_j \) is uniquely determined. If the corresponding operator is degenerate for some \( j \), then, first, the answer contains as many arbitrary constants as the dimension of the kernel, and, second, the solvability conditions lead to some constraints on \( f_j \) from which we can extract the conditions on the parameters \( b_1, b_2, \) and \( a \).

We start with calculating the possible total number of arbitrary constants. The matrix \( p \) has as many arbitrary constants as the dimension of the orbit of its Jordan form. It is easy to see that the Jordan form of any matrix \( p \) satisfying (7) is

\[
    p = \text{diag}(E_k, -E_m, 0_{n-k-m}).
\]  

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Under the action of a group on a manifold, the dimension of the orbit of a point is equal to the difference of the manifold dimension and the dimension of the stabilizer of this point, i.e., the subgroup that leaves the point fixed. In our case, the dimension of the manifold is $n^2$, and the stabilizer comprises nondegenerate matrices commuting with $p$. It is easy to see that its dimension is $k^2 + m^2 + (n - k - m)^2$, whence it follows that the dimension of the orbit of $p$ is equal to $2m(n - m) + 2k(n - k) - 2km$.

We next calculate the dimensions of the eigensubspaces of the linear operator $L_\sigma$: $\text{Mat}_n \rightarrow \text{Mat}_n$.

**Lemma 1.** The eigenvalues of the operator $L_\sigma$ belong to the set

$$
\lambda_0 = 0, \quad \lambda_{\pm 1} = 1 \pm \sigma, \quad \lambda_{\pm 2} = 1 \pm 2\sigma, \quad \lambda_3 = 3.
$$

The space $\text{Mat}_n$ decomposes into the direct sum of the eigensubspaces

$$
\text{Mat}_n = V_0 \oplus V_{-1} \oplus V_1 \oplus V_{-2} \oplus V_2 \oplus V_3, \quad L_\sigma V_i = \lambda_i V_i,
$$

with the dimensions

$$
dim V_0 = (n - k - m)^2, \quad dim V_{\pm 1} = k(n - k) + m(n - m) - 2km, \quad dim V_{\pm 2} = km, \quad dim V_3 = k^2 + m^2.
$$

In the case where the eigenvalues coincide, the dimensions of the corresponding eigensubspaces are added.

**Proof.** We represent $c$ as a $3 \times 3$ block matrix with the block sizes determined by the Jordan form of $p$:

$$
c = \begin{pmatrix}
  k & m & l \\
  k & c_{11} & c_{12} & c_{13} \\
  l & c_{21} & c_{22} & c_{23} \\
  l & c_{31} & c_{32} & c_{33}
\end{pmatrix}, \quad l = n - k - m.
$$

It is then easy to verify that

$$
L_\sigma(c) = \begin{pmatrix}
  3c_{11} & (1 + 2\sigma)c_{12} & (1 + \sigma)c_{13} \\
  (1 - 2\sigma)c_{21} & 3c_{22} & (1 - \sigma)c_{23} \\
  (1 - \sigma)c_{31} & (1 + \sigma)c_{32} & 0
\end{pmatrix},
$$

i.e., each eigensubspace comprises one or two blocks of the matrix $c$. The dimensions of the eigenspaces are calculated by summing the block sizes. The lemma is proved.

The operator in the left-hand side of (8) degenerates if one of the eigenvalues of the operator $L_{\kappa(j+1)/2}$ coincides with $j(j-1)/2$ (we call this the resonance condition). In particular, the eigenvalue $\lambda_0 = 0$ appears twice (for $j = 0$ and for $j = 1$), and the eigenvalue $\lambda_3 = 3$ appears once for $j = 2$. The conditions that the eigenvalues $\lambda_{\pm 1}$ and $\lambda_{\pm 2}$ are resonant for some index $j$ are respectively given by the equations

$$
1 \pm \frac{\kappa(j + 1)}{2} = \frac{j(j - 1)}{2} \quad \text{and} \quad 1 \pm \frac{\kappa(j + 1)}{2} = \frac{j(j - 1)}{2}.
$$

Shortening this by $j + 1$ (we recall that $j \geq 0$), we obtain the resonance conditions

$$
\lambda_{\pm 1}: \ j = 2 \pm \kappa, \quad \lambda_{\pm 2}: \ j = 2 \pm 2\kappa.
$$
We see that for a fixed $\kappa$, any of the eigenspaces $V_{\pm 1}$ and $V_{\pm 2}$ can occur at its resonant $j$ no more than once. This allows estimating their total dimension, i.e., the number of possible arbitrary constants from above. Namely, if both $\lambda_{\pm 1}$ and $\lambda_{\pm 2}$ appear in (8), then the total sum of dimensions is

$$2 \dim V_0 + \dim V_{-1} + \dim V_1 + \dim V_{-2} + \dim V_2 + \dim V_3 = \dim V_0 + n^2 = (n - k - m)^2 + n^2.$$ 

By adding the dimension of the orbit of $p$, we obtain the total number of arbitrary constants $2n^2 - k^2 - m^2$. This is equal to $2n^2 - 1$ only for $k = 1$ and $m = 0$ or for $k = 0$ and $m = 1$. The second case reduces to the first by the change $y \to -y$. In the first case, the Jordan form of $p$ is $\text{diag}(1, 0, \ldots, 0)$, and its orbit $O$ comprises matrices of the form $uv^T$, where $u$ and $v$ are column vectors such that $u^Tv = 1$.

Therefore, we assume that $p \in O$. Then the eigenvalues $\lambda_{\pm 2}$ in Lemma 1 disappear. To provide the required number of arbitrary constants, series (6) must exist for a general element of the orbit. More precisely, we require that: for any column vectors $u$ and $v$ such that $u^Tv = 1$, there exists a formal solution of the form

$$y = \frac{uv^T}{z - z_0} + c_0 + c_1(z - z_0) + \ldots, \quad p, c_j \in \text{Mat}_n, \quad z_0 \in \mathbb{C},$$

such that its coefficients $c_i$ contain in total $2n^2 - 2n + 1$ arbitrary constants.

**Proof of Theorem 1.** The above counting of arbitrary constants shows that a necessary condition for passing the Painlevé–Kovalevskaya test is that both eigenvalues $\lambda_{\pm 1}$ must be resonant. The resonance conditions for $j = 2 \pm \kappa$ imply that the parameter $\kappa$ must be such that both numbers $2 \pm \kappa$ are nonnegative integers. This leaves the admissible values $\kappa = 0, \pm 1, \pm 2$. The case $\kappa < 0$ reduces to $\kappa > 0$ by the change $y \to y^T$. Therefore, it suffices to analyze Eqs. (8) for each of the three cases $\kappa = 0, 1, 2$.

**2.2. Case $\kappa = 0$.** Let $p = \text{diag}(1, 0, \ldots, 0)$. In the case $\kappa = 0$, the operator $L_\sigma$ has the form

$$L_\sigma: \begin{pmatrix} 1 & n - 1 \\ n - 1 & 0 \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \mapsto \begin{pmatrix} 3c_{11} & (1 + \sigma)c_{12} \\ (1 - \sigma)c_{21} & 0 \end{pmatrix}.$$ 

To analyze Eqs. (8), we write them blockwise by dividing the involved matrices into blocks of appropriate sizes:

$$c_j = \begin{pmatrix} c_{j,11} & c_{j,12} \\ c_{j,21} & c_{j,22} \end{pmatrix}, \quad b_1 = \begin{pmatrix} b_{1,11} & b_{1,12} \\ b_{1,21} & b_{1,22} \end{pmatrix}, \quad b_2 = \begin{pmatrix} b_{2,11} & b_{2,12} \\ b_{2,21} & b_{2,22} \end{pmatrix}, \quad a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$ 

Conditions on the matrices $b_1$, $b_2$, and $a$ arise from the requirement that solutions of inhomogeneous linear systems (8) exist for the resonance values of $j$. In this case, the homogeneous system has a nontrivial solution, and the conditions of the Kronecker–Capelli theorem must be satisfied.

We consider Eqs. (8) for $j = 0, 1, 2, 3$. For $j = 0$, we find that $c_{0,11} = c_{0,12} = c_{0,21} = 0$ and the block $c_{0,22}$ is arbitrary.

Next, for $j = 1$, we obtain

$$c_{1,11} = -\frac{1}{6}(2z_0 + b_{1,11} + b_{2,11}), \quad c_{1,12} = -\frac{1}{2}b_{2,12}, \quad c_{1,21} = -\frac{1}{2}b_{1,21},$$ 

and $c_{1,22}$ is arbitrary. So far, no conditions for the coefficients $b_1$, $b_2$, and $a$ appear.

For $j = 2$, by substituting the obtained values, we find that

$$c_{2,11} = -\frac{1}{4}(1 + a_{11}), \quad c_{2,22} = \frac{1}{2}(a_{22} + b_{1,22}c_{0,22} + c_{0,22}b_{2,22} + z_0c_{0,22}) + c_{0,22}^3.$$
and the blocks $c_{2,12}$ and $c_{2,21}$ are arbitrary. In addition, we obtain conditions on the coefficients in the form of the relations

$$(b_{2,12} - b_{1,12})c_{0,22} = a_{12}, \quad c_{0,22}(b_{1,21} - b_{2,21}) = a_{21}.$$ 

Because the block $c_{0,22}$ is arbitrary, this implies that

$$b_{1,12} = b_{2,12}, \quad b_{1,21} = b_{2,21}, \quad a_{12} = a_{21} = 0. \quad (9)$$

Finally, for $j = 3$, it suffices to write the condition for the 1 block

$$b_{1,12}b_{1,21} - 2b_{2,12}b_{1,21} + b_{2,12}b_{2,21} = 0 \quad (10)$$

because all other equations are solved uniquely with respect to $c_{3,12}$, $c_{3,21}$, and $c_{3,22}$. Condition (10) follows from (9). For $j > 3$, there are no resonances, and all equations for $c_j$ are therefore solved uniquely.

We have thus proved that Eqs. (8) with $\kappa = 0$ are solved with arbitrary blocks $c_{0,22}$, $c_{1,22}$, $c_{2,12}$, $c_{2,21}$, and $c_{3,11}$, which gives $2(n - 1)^2 + 2(n - 1) + 1$ arbitrary constants, as can be easily seen. Moreover, we obtained the solvability conditions in the form of relations (9), which mean that the matrices $a$ and $b_1 - b_2$ commute with $p = \text{diag}(1, 0, \ldots, 0)$.

By choosing matrices of the form diag$(0, 0, \ldots, 1, \ldots, 0)$, which belong to $\mathcal{O}$, as $p$, we find that $a$ and $b_1 - b_2$ must commute with any such matrix, which means that they are diagonal. According to Remark 1, the matrices $TaT^{-1}$ and $T(b_1 - b_2)T^{-1}$ must be diagonal for any nondegenerate $T$. This means that $a$ and $b_1 - b_2$ are scalar matrices. Obviously, the condition $b_1 = b_2 + 2\beta I$ reduces to $b_1 = b_2$ by transformation (3). As a result, we obtain Eq. (P_0).

2.3. Case $\kappa = 1$. As in the case $\kappa = 0$, it suffices to analyze Eqs. (8) for $j = 0, 1, 2, 3$.

Let $p = \text{diag}(1, 0, \ldots, 0)$. Then we find that for $j = 0$, $c_{0,11} = c_{0,12} = c_{0,21} = 0$ and the block $c_{0,22}$ remains arbitrary.

For $j = 1$, we have

$$c_{1,11} = -\frac{1}{6}(b_{1,11} + b_{2,11} + z_0), \quad c_{1,12} = -\frac{1}{4}b_{2,12}.$$ 

Both blocks $c_{1,21}$ and $c_{1,22}$ are arbitrary, and $b_1$ must satisfy the condition

$$b_{1,21} = 0.$$ 

We now repeat the reasoning in Sec. 2.2. Choosing the matrices diag$(0, \ldots, 1, \ldots, 0)$ as $p$, we find that $b_1$ must be diagonal. We then prove that it is scalar. Finally, we use transformation (3) and set $b_1 = 0$.

Next, for $j = 2$, all blocks $c_2$ are uniquely determined:

$$c_{2,11} = -\frac{1}{4}(1 + a_{11}), \quad c_{2,12} = \frac{1}{12}(b_{2,12}c_{0,22} - 4a_{12}),$$

$$c_{2,21} = \frac{1}{3}(a_{21} + c_{0,22}b_{2,21} + 3c_{0,22}c_{1,21}),$$

$$c_{2,22} = \frac{1}{2}(a_{22} + z_0c_{0,22} + c_{0,22}b_{2,22} + [c_{0,22}, c_{1,22}] + c_{0,22}^3).$$

For $j = 3$, the blocks $c_{3,11}$ and $c_{3,12}$ remain arbitrary, and the blocks $c_{3,21}$ and $c_{3,22}$ are uniquely determined. Moreover, as the solvability conditions for the linear system, we obtain the relations

$$b_{2,12}(b_{2,21} + 4c_{1,21}) = 0, \quad b_{2,12}(z_0 + b_{2,22} + 2c_{1,22} + 2c_{0,22}^2) = 0.$$
Taking the arbitrariness of $c_{1,22}$ and $c_{0,22}$ into account, we see that this amounts just to the condition $b_{2,12} = 0$, which, as before, implies that the matrix $b_2$ is also scalar and can also be set to zero. There are no resonance values for $j > 3$, and all other coefficients of series (6) are uniquely found. Therefore, Eqs. (8) with $\kappa = 1$ are solvable for vanishing $b_1$ and $b_2$ and with arbitrariness in the blocks $c_{0,22}, c_{1,21}, c_{1,22}, c_{3,11}$, and $c_{3,12}$. As in the preceding case, this gives the required number of arbitrary constants. There are no conditions for the matrix $a$ because it is not involved in the solvability conditions at all, and we obtain Eq. (P$^1_2$).

2.4. Case $\kappa = 2$. In the case $\kappa = 2$, we must analyze the equations for $j = 0, 1, 2, 3, 4$.

For $j = 0$, we find that $c_{0,11} = c_{0,12} = 0$. In contrast to the preceding cases, two blocks $c_{0,21}$ and $c_{0,22}$ remain arbitrary.

For $j = 1$, we have

$$c_{1,11} = -\frac{1}{6} (b_{1,11} + b_{2,11} + z_0), \quad c_{1,12} = -\frac{1}{6} b_{2,12}, \quad c_{1,21} = \frac{1}{2} b_{1,21} + c_{0,22} c_{0,21}.$$ 

The block $c_{1,22}$ is arbitrary, and no solvability condition appears.

For $j = 2$, all blocks of the matrix $c_2$ are found uniquely:

$$c_{2,11} = -\frac{1}{4} (a_{11} + 1) + \frac{1}{12} (b_{2,12} - 3 b_{1,12}) c_{0,21}, \quad c_{2,12} = -\frac{1}{6} (a_{12} + b_{1,12} c_{0,22}), \quad c_{2,21} = \frac{1}{6} (a_{21} + b_{1,22} c_{0,21} - c_{0,21} b_{1,11} + c_{0,22} (2 b_{1,21} + b_{2,21})) + c_{0,22}^2 c_{0,21}, \quad c_{2,22} = \frac{1}{2} (a_{22} + z_0 c_{0,22} + b_{1,22} c_{0,22} + c_{0,22} b_{2,22}) + [c_{0,22}, c_{1,22}] + \frac{1}{6} c_{0,21} b_{2,12} + c_{0,22}^3.$$ 

For $j = 3$, the block $c_{3,11}$ is arbitrary, the other blocks are uniquely determined, and the first solvability condition appears,

$$3 b_{1,12} b_{1,21} - 2 b_{2,12} b_{1,21} - b_{2,12} b_{2,21} + 6 (b_{1,12} - b_{2,12}) c_{0,22} c_{0,21} = 0. \quad (11)$$ 

Because $c_{0,22}$ and $c_{0,21}$ are arbitrary, we hence conclude that $b_{1,12} - b_{2,12} = 0$. By repeating the same reasoning as in the preceding cases, we prove that the matrices $b_1$ and $b_2$ coincide up to a matrix const.·$I$.

We can then set $b_1 = b_2 = b$ by transformation (3). It is easy to see that the remaining terms in relation (11) then cancel.

Finally, for $j = 4$, we can choose the block $c_{4,12}$ arbitrarily if the condition

$$b_{11} a_{12} + b_{12} a_{22} = a_{11} b_{12} + a_{12} b_{22} + 2 b_{12}$$

is satisfied. This is just $(|b, a| - 2 b)_{12} = 0$, and by the reasoning based on the arbitrariness of $p$, we again conclude that the matrices $a$ and $b$ must satisfy the relation$^1$

$$[b, a] = 2 b.$$ 

The remaining blocks of $c_4$ and all subsequent $c_j$ are found uniquely. Therefore, under the above condition, the coefficients of the series are determined with the arbitrariness in the blocks $c_{0,21}, c_{0,22}, c_{1,22}, c_{3,11}$, and $c_{4,12}$, which have the required total dimension. As the result, we obtain Eq. (P$^2_2$).

The sufficiency of the obtained conditions on the coefficients $\kappa$, $a$, and $b$ for the existence of series (6) with an arbitrary matrix $p \in \mathcal{O}$ and the required number of arbitrary constants is proved by the same reasoning as in each of the three preceding cases. Because series (6) with $p = \text{diag}(1, 0, \ldots, 0)$ exists for any $\kappa$, $a$, and $b$ satisfying the conditions of the theorem, we can replace them with $\kappa, \bar{a} = TaT^{-1}$ and $\bar{b} = T b T^{-1}$, where $T$ is any nondegenerate matrix. The series $T^{-1} y T$ then satisfies the equation with the parameters $\kappa$, $a$, and $b$ and has an arbitrary element of the orbit $\mathcal{O}$ as $p$. The theorem is proved.

$^1$Under transposition, $\kappa = 2$ is replaced with $\kappa = -2$, and the sign of the commutator also changes.
3. Reductions of partial differential equations

3.1. Matrix NLS equation. Matrix NLS equation (5) admits the reduction (which defines solutions that are invariant with respect to some one-parametric subgroup in the group generated by the Galilean transformation, the translation of \( t \), and conjugation by the matrix exponent of a constant matrix)

\[
\begin{align*}
  u &= e^r p(z), & v &= q(z)e^{-r}, & r &= \frac{1}{6}(t^3 - 3xt) - tb, \\
  z &= x - \frac{1}{2}t^2, & b &\in \text{Mat}_n.
\end{align*}
\]

(12)

It is easy to verify that this substitution reduces Eqs. (5) to the system of ordinary differential equations (cf. Eq. (3.1) in [2])

\[
\begin{align*}
  p'' &= -\frac{1}{2}zp - bp - 2qp, & q'' &= -\frac{1}{2}zq - qb - 2pq, & b &\in \text{Mat}_n.
\end{align*}
\]

(13)

The order of this system can be reduced by one because of the first integral

\[
qp' - q'p = c, \quad c &\in \text{Mat}_n.
\]

(14)

Another way to reduce the order is by passing to the logarithmic derivatives of \( p \) and \( q \). It turns out that to use both methods, we can retain only one of the matrix constants \( b \) or \( c \); the other should be chosen to be scalar. We consider these two possibilities.

1. Let \( b &\in \text{Mat}_n \) and \( 2c = \gamma &\in \mathbb{C} \). Then we introduce the variables

\[
\begin{align*}
  f &= p'p^{-1}, & g &= q^{-1}q', & h &= 2pq,
\end{align*}
\]

(15)

and system (13) becomes

\[
\begin{align*}
  f' &= -f^2 - \frac{1}{2}z - b - h, & g' &= -g^2 - \frac{1}{2}z - b - h, & h' &= fh + hg.
\end{align*}
\]

Because \( \gamma \) is scalar, first integral (14) can also be expressed in terms of \( f \), \( g \), and \( h \):

\[
f - g = \gamma h^{-1}.
\]

Using it, we eliminate \( g \) and obtain the equations

\[
\begin{align*}
  f' &= -f^2 - \frac{1}{2}z - b - h, & h' &= fh + hf - \gamma.
\end{align*}
\]

(16)

Finally, we find \( h \) from the first equation and substitute it in the second. This yields Eq. (P\(_0\)) for the variable \( y = f \) with \( a = \gamma - 1/2 \).

Remark 3. The symmetry of the equations with respect to \( f \) and \( g \) immediately implies that \( g \) also satisfies the same equation with the free term \(-\gamma - 1/2\). Combining this with the change of sign \( g \rightarrow -g \), we obtain a Bäcklund transformation for (P\(_0\)). We can prove that iterations of this transformation in terms of \( p \) or \( q \) are governed by a non-Abelian Toda lattice. Conversely, it is possible to derive (P\(_0\)) starting from the Toda lattice and imposing the corresponding reduction (see [2], [7]).
2. Now let \( c \in \text{Mat}_n \) and \( b \) be a scalar. In this case, according to (3), we can set \( b = 0 \). We now use another substitution,
\[
f = p^{-1}p', \quad g = q'q^{-1}, \quad h = 2qp
\]
(alternatively, we can use (14) as before, but replace (14) with another first integral \( p'q - pq' = \tilde{c} \), which appears for \( b = 0 \)). As a result, system (13) with \( b = 0 \) becomes
\[
f' = -f^2 - \frac{1}{2}z - h, \quad g' = -g^2 - \frac{1}{2}z - h, \quad h' = hf + gh,
\]
and first integral (14) becomes
\[
hf - gh = 2c.
\]
Eliminating \( g \), we obtain the system
\[
f' = -f^2 - \frac{1}{2}z - h, \quad h' = 2hf - 2c.
\]
In this case, \( f \) satisfies Eq. (P\(_1\)) with \( \kappa = -1 \) and \( a = 2c - 1/2 \).

We can easily obtain an isomonodromic Lax pair for system (13) by extending substitutions (12) with the change \( \zeta = \lambda - t/4 \) and applying this to the well-known representation \( U_t = V_x + [V, U] \) of system (5) with the matrices
\[
U = \begin{pmatrix} \lambda & -v \\ u & -\lambda \end{pmatrix}, \quad V = -2\lambda U + \begin{pmatrix} -vu & vx \\ u_x & uv \end{pmatrix}.
\]
After some simple algebraic manipulations, this leads to representation (4) for Eq. (13) with the matrices
\[
B = \begin{pmatrix} \zeta & -q \\ p & -\zeta \end{pmatrix}, \quad A = \begin{pmatrix} 8\zeta^2 + 4qp + z & -8\zeta q - 4q' \\ 8\zeta p - 4p' & -8\zeta^2 - 4pq - z - 4b \end{pmatrix}.
\]
Hence, we can also obtain the Lax pairs for the above systems in the variables \( f \) and \( h \) by applying gauge transformations. For system (16), we have
\[
B = \begin{pmatrix} \zeta + f & -\frac{1}{2}h \\ 1 & -\zeta \end{pmatrix}, \quad A = \begin{pmatrix} 8\zeta^2 + 2h + z & -4\zeta h - 2hf + 2\gamma \\ 8\zeta - 4f & -8\zeta^2 - 2h - z - 4b \end{pmatrix}.
\]
We note that Eq. (P\(_2\)) also admits another representation (4) with
\[
B = \begin{pmatrix} \zeta & y \\ y & -\zeta \end{pmatrix}, \quad A = -4\zeta B + \begin{pmatrix} 2y^2 + z + 2b & -2y' - \frac{a}{\zeta} \\ 2y' - \frac{a}{\zeta} & -2y^2 - z - 2b \end{pmatrix},
\]
which is equivalent to the representation in [3]. We were unable to establish a gauge equivalence between these two Lax pairs.

In the case of system (18), we obtain (4) with
\[
B = \begin{pmatrix} \zeta & -\frac{1}{2}h \\ 1 & -\zeta - f \end{pmatrix}, \quad A = \begin{pmatrix} 8\zeta^2 + 2h + z & -4\zeta h - 2hf + 2c \\ 8\zeta - 4f & -8\zeta^2 - 2h - z \end{pmatrix}.
\]
This representation for Eq. (P\(_3\)) is equivalent to the Lax pair obtained in the next section starting from the mKdV equation.
3.2. Matrix mKdV equations. The scalar PII equation can also be obtained from the mKdV equation
\[ u_t = u_{xxx} - 6u^2u_x \]
by a self-similar reduction related to the scaling group. This yields the equation \( y''' = 6y^2y' + y + zy' \), which reduces to the PII equation by integration. As usual, the zero-curvature representation for mKdV equation becomes an isomonodromic Lax pair. We describe this procedure in the matrix setting (cf. [8]).

The first matrix analogue of the mKdV equation [4] is
\[ u_t = u_{xxx} - 3u^2u_x - 3u_xu^2 \]  
(19)
and admits the representation \( U_t = V_x + [V, U] \) with
\[
U = \begin{pmatrix} \lambda u & u \\ u & -\lambda \end{pmatrix}, \quad V = 4\lambda^2 U + \begin{pmatrix} -2\lambda u^2 + [u, u_x] & 2\lambda u_x + u_{xx} - 2u^3 \\ -2\lambda u_x + u_{xx} - 2u^3 & 2\lambda u^2 + [u, u_x] \end{pmatrix}.
\]

In the matrix case, we combine the scaling group with the group of conjugations and apply the self-similar ansatz
\[ u = \varepsilon^\tau e^{\log(\tau)d}y(z)e^{-\log(\tau)d}, \quad \tau = t^{-1/3}, \quad z = \varepsilon\tau x, \quad 3\varepsilon^3 = -1, \quad d \in \text{Mat}_n, \]  
(20)
which converts Eq. (19) to
\[ y''' = 3y^2y' + 3y'y^2 + y + zy' + [d, y]. \]  
(21)
In contrast to the scalar case, this equation does not have a first integral even for \( d = 0 \). Nevertheless, an order reduction is still possible because of a partial first integral. It turns out that (20) admits special solutions described by a second-order equation. This can be easily proved by direct elimination of the third and second derivatives by virtue of an equation of general form (2). The coefficients are determined by equating the remaining terms, and we find that (21) is satisfied by Eq. (P_{12}) with \( a = -\kappa d \). To obtain the corresponding Lax pair, we supplement substitution (19) with the change \( \lambda = \varepsilon^\tau \zeta \). The dependence of \( U \) and \( V \) on \( \tau \) is then separated:
\[
U = \varepsilon^\tau e^{\log(\tau)d}B e^{-\log(\tau)d}, \quad V = (\varepsilon^\tau)^3 e^{\log(\tau)d}K e^{-\log(\tau)d},
\]
where \( B \) and \( K \) depend on \( \zeta, z, y, \) and \( y' \) (the second derivative in \( K \) is replaced according to (P_{12})). For the derivations, this change yields
\[
\partial_t = (\varepsilon^\tau)^3 (\tau \partial_{\tau} + z \partial_z - \zeta \partial_{\zeta}), \quad \partial_z = \varepsilon^\tau \partial_{\tau}.
\]
As a result, the equation for \( U \) and \( V \) becomes
\[
-\zeta B\zeta + zB' + B + [dI, B] = K' + [K, B],
\]
and the further change \( A = -\zeta^{-1}(K - zB - dI) \) brings it to the standard form of Lax pair (4). For Eq. (P_{12}), the calculations by this scheme give
\[
B = \begin{pmatrix} \zeta & y \\ y & -\zeta \end{pmatrix}, \quad A = -4\zeta B + \begin{pmatrix} 2y^2 + z & -2y' \\ 2y' & -2y^2 - z \end{pmatrix} - \zeta^{-1}(\kappa[y, y'] + a) \begin{pmatrix} \kappa & 1 \\ 1 & \kappa \end{pmatrix}.
\]
The second matrix analogue of the mKdV equation was introduced in [6]. It turns out that in contrast to (19), an arbitrary matrix constant can be introduced directly into this equation:

\[ u_t = u_{xxx} + 3[u, u_{xx}] - 6uu_x u - 3(u_x + u^2)c - 3c(u_x - u^2), \quad c \in \text{Mat}_n. \]  

(22)

The origin of this constant is related to the Miura map for the matrix KdV equation, which is constructed according to the function \( \psi \) satisfying the linear Schrödinger equation

\[ \psi'' + v\psi + \psi c = 0, \quad c \in \text{Mat}_n. \]

The zero-curvature representation for (22) is given by the matrices

\[
U = \begin{pmatrix}
0 & 1 \\
c - \lambda & -2u
\end{pmatrix}, \\
V = 2 \begin{pmatrix}
2u(c - \lambda) \\
(u_x - u^2 - c - 2\lambda)(c - \lambda) & -u_x - u^2 - c - 2\lambda
\end{pmatrix}. 
\]

At first glance, introducing the terms with \( c \) in (22) makes self-similar substitution (20) impossible because the homogeneity of the equation is violated. But we can complement this substitution as

\[ c = (\varepsilon \tau)^2 e^{\log(\tau)d} c_0 e^{-\log(\tau)d}, \]

where \( c_0 \) is an arbitrary constant matrix such that \( 2c_0 + [d, c_0] = 0 \). Indeed, differentiation with respect to \( \tau \) shows that this relation implies that the matrix \( c \) is also constant (we note that for scalars, this gives \( c = c_0 = 0 \), i.e., this trick is only possible in the non-Abelian case). As a result, Eq. (22) reduces to

\[ y''' = 3[y'', y] + 6yy'y + y + zy' + 3(y' + y^2)c_0 + 3c_0(y' - y^2) + [d, y]. \]  

(23)

As in the case of Eq. (21), its order can be reduced by a partial first integral. Eliminating \( y''' \) and \( y'' \) by virtue of an equation of form (2), we find that if \( \kappa = -2 \), \( 3c_0 = b \), and \( d = -a \), then (23) is a consequence of Eq. (P2). Further manipulations with the matrices \( U \) and \( V \) differ from the preceding example only in insignificant details (we must set \( \lambda = (\varepsilon \tau)^2 \zeta \) and apply the conjugation by a suitable constant matrix), and this leads to Lax representation (4) with

\[
B = \begin{pmatrix}
0 & 1 \\
\frac{1}{3}b - \zeta & -2y
\end{pmatrix}, \\
A = \frac{1}{\zeta} \begin{pmatrix}
2\zeta y - \frac{2}{3}yb - \frac{1}{2}(a + 1) \\
(2\zeta - y' + y^2 + \frac{1}{3}b + \frac{z}{2})\left(\frac{1}{3}b - \zeta\right) & 2\zeta + y' + y^2 + \frac{1}{3}b + \zeta + \frac{1}{2}
\end{pmatrix}. 
\]

for Eq. (P2) with \( \kappa = -2 \).
4. Conclusion

We have demonstrated that there are at least three matrix generalizations of the second Painlevé equation that satisfy the Painlevé–Kovalevskaya test and admit isomonodromic Lax pairs. Of course, a similar diversity should also be expected for other Painlevé equations. Although the literature on their non-Abelian generalizations is quite rich, the question of the number of different versions remains open, in particular, the question of how the non-Abelian constants can enter the equations.

As one example, we present a matrix version of the PIV equation

$$y'' = \frac{y'^2}{2y} + \frac{3}{2}y^2 + 4zy^2 + 2(z^2 - \alpha)y + \frac{\beta}{y}$$

containing the matrix constant $c \in \text{Mat}_n$ and the scalar constant $\alpha \in \mathbb{C}$:

$$y'' = \frac{1}{2}(y' + 2c)y^{-1}(y' - 2c) + \frac{1}{2}[y, y'] + \frac{3}{2}y^2 + 4zy^2 + 2(z^2 - \alpha)y + cy + yc.$$  

This equation can be obtained by eliminating $h$ from the equivalent system

$$-y' = y'^2 + 2hy + 2zy + 2c, \quad h' = h^2 + yh + hy + 2zh + \alpha - 1,$$

which admits representation (4) with the matrices

$$B = \begin{pmatrix} -2\zeta^2 & 2\zeta \\ \zeta h & y - h \end{pmatrix}, \quad A = \begin{pmatrix} 4\zeta^3 - 2\zeta(h + 2z) + 2\zeta^{-1}c & -4\zeta^2 + 2y + 2h + 2z \\ -2\zeta^2h - hy - \alpha + 1 & 2\zeta h + \alpha\zeta^{-1} \end{pmatrix}.$$  

This example and also a set of other versions of the PIV equation can be obtained by self-similar reduction from equations of NLS type (in this case, we used the equation $S_5^0(0, 1)$ in [9]). For such equations, the reduction procedure leads to systems of two second-order equations, and a further order reduction is usually more complicated than in the scalar setting.

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