Gauged Supergravity and Holographic Field Theory

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Abstract

This is a slightly expanded version of my talk at Future Perspectives in Theoretical Physics and Cosmology, Stephen Hawking's 60th Birthday Worshop. I describe some of the issues that were important in gauged supergravity in the 1980's and how these, and related issues have once again become important in the study of holographic field theories.

* This work was supported in part by funds provided by the DOE under grant number DE-FG03-84ER-40168.
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1 Gauged supergravity and a thesis project

I became one of Stephen’s students a little over 20 years ago. At the time, Stephen was formulating and developing many of his ideas of Euclidean quantum gravity, and was also greatly interested in, and supportive of, other approaches to quantum gravity. Most particularly, he was an enthusiastic advocate of supergravity theories. Supergravity theories grew out of particle physics and were not the standard fare of Relativists. It is a testament to Stephen’s broad interests and the atmosphere in the Cambridge Relativity Group that the lines between these disciplines were completely blurred and that it was possible – indeed encouraged – for me as a student to move from my first work in relativity and Euclidean quantum gravity to the study of symmetry breaking in gauged supergravity. I remain very grateful to Stephen and to the Relativity Group for this opportunity and, most particularly, for the excitement, interest and enthusiasm with which they pursued the “Grand Enterprise” of looking for a viable quantum theory of gravity.

The primary problem of quantum gravity is the infinite number of different kinds of divergence in the perturbation theory of pure quantum gravity. It was well known that divergences caused by fermions generically have the opposite sign to similar divergences caused by bosons. In a supersymmetric theory, bosons are paired with fermions in just such a manner that these divergences have the same structure and tend to cancel one another in the quantum theory. In a supergravity theory, the graviton is paired with one, or more, fermionic partners, called gravitini. The number of gravitini, $N$, denotes the amount of (extended) supersym-

\footnote{I would like to apologize in advance to the large number of people who made huge contributions to the development of supergravity and string theory, but my allotted space makes it impossible to adequately and correctly reference them all.}
metry. The more supersymmetry the larger the spectrum of the theory: 
$\mathcal{N} = 1$ supergravity has simply a graviton and one gravitino, and the
maximal supergravity in four dimensions has $\mathcal{N} = 8$, with a spectrum
of one graviton, eight gravitini, 28 vector bosons, 56 spin-$\frac{3}{2}$ particles and
70 scalar fields. One cannot go beyond $\mathcal{N} = 8$ since the spectrum would
then have to include particles whose spin is higher than that of the gravi-
ton and, as far as we know, non-stringy higher spin theories appear to
be inconsistent. The important point is that, in general, the more super-
symmetry there is, the more the divergences are cancelled. The crucial
question of the early 1980’s was whether $\mathcal{N} = 8$ supergravity was finite.
It was ultimately shown that even maximal supergravity very probably
had divergences at higher orders [1, 2], but despite this difficulty, it was
clear that supergravity was a very important step in the right direction
and that it had to be part of the right answer. Subsequent history has
born this prejudice out in that supergravity is the low energy limit of
string theory, which is (as far as we can tell) a finite quantum theory of
gravity.

In the early 1980’s, when we were still trying to turn supergravity into
the Theory of Everything, there was another important issue: Maximal
supergravities contained vector bosons, but in the original formulations
all these vector bosons in these theories were Abelian. To describe the real
world one needs non-abelian gauge symmetry, and many people painstak-
ingly constructed the so-called gauged supergravity theories, that is, the-
ories with non-abelian gauge symmetry mediated by the vector bosons.
This work culminated in the eventual construction of gauged, maximal
($\mathcal{N} = 8$) supergravity theory [3]. This theory also had the maximal possi-
ble gauge group of $SO(8)$.

It is a remarkable and rather attractive feature of these gauged su-
pergravity theories that the joint requirements of gauge symmetry and
supersymmetry also requires a non-linear potential in the scalar sector.
Moreover, the more the supersymmetry, the more rigid the potential, and
indeed the potential is completely fixed in the maximal theory. Thus, the
maximal gauged theory determines its own symmetry breaking structure
completely; there are no choices and no arbitrary parameters to be fixed.

In the early days of supergravity the focus, for obvious reasons, was
primarily upon supergravity in four space-time dimensions. However, it
became very important to study supergravity in every possible dimension.
There are several reasons for this (some of which will be described later),
but one of the primary reasons was that the higher dimensional maxi-
mal theories are generically simpler, and thus easier to construct. Indeed
the lower dimensional maximal theories were often first constructed by
dimensionally reducing the higher dimensional theories (see, for exam-
ple [4, 5]). Here I will be concerned mainly with the gauged maximal
supergravities in four dimensions and in five dimensions; both of these theories have $\mathcal{N} = 8$ supersymmetry. The latter theory has a spectrum of one graviton, eight gravitini, 15 vector bosons, 12 tensor gauge fields, 48 “spin-$\frac{1}{2}$” particles and 42 scalar fields; the gauge group is $SO(6)$. The five-dimensional theory was also the last gauged maximal supergravity to be constructed [6, 7, 8]. This was, to some extent, because of technical issues, but also because it seemed of the least phenomenological interest. It is thus a wonderful irony that this situation is now reversed.

As I walked home with him one wet winter’s day, Stephen suggested that I should study the four-dimensional, maximal gauged $\mathcal{N} = 8$ theory and try to understand its symmetry breaking structure. This was to be the last project of my PhD. At the time, neither I nor Stephen remotely suspected that, rather than finding its most important application directly within quantum gravity, this work would prove important 20 years later in determining part of the phase diagram of strongly coupled, large $\mathcal{N}$, $\mathcal{N} = 4$ Yang-Mills theory: A theory that is a distant relative of QCD, which describes the force that underpins the strong nuclear interaction. It is the purposes of this talk to outline how all of this came about, and describe how the ideas of gravity and supergravity can be used to give beautiful and remarkable insights into field theory via the idea holography on branes.

2 The up’s and down’s of maximal gauged supergravity

In any supergravity theory, or string theory, one wants to find interesting, and hopefully viable, ground states for the theory. In gauged supergravity one is thus naturally led to study the scalar fields and their potential. Both the four and five dimensional maximal gauged supergravity theories have a completely determined scalar potential, and these potentials have somewhat similar structural features. Thus, while my remarks in this section will be directed toward the four-dimensional theory, many of the results have five-dimensional parallels that will be important later.

2.1 The universe is not anti-de Sitter

It is a fundamental property of the superalgebra underlying gauged supergravity that if a ground state preserves supersymmetry, and that supersymmetry transforms under a non-abelian gauge symmetry, then the ground state must necessarily have a negative cosmological constant. This cosmological constant, $\Lambda$, is of order $-g^2$ in Planck units, where $g$ is the gauge coupling constant in the supergravity.

This fact means that the easiest (i.e. most symmetric) ground states to find will generically be associated with anti-de Sitter (AdS) space with
Fig. 1.1. A generic contour map of a two-dimensional section through the maximally symmetric critical point of a supergravity potential. The hump in the middle is the maximally supersymmetric local maximum.

Planck scale $\Lambda$. In particular, this includes the maximally supersymmetric ground state, in which all the scalars vanish. An optimist would note that $\mathcal{N} = 0, 1$ or $\mathcal{N} = 2$ supersymmetric vacua with zero cosmological constant are still allowed since such theories have either no gauge symmetry action, or an (abelian) $U(1)$ action on the residual supersymmetry generators. Thus a phenomenologically interesting ground state is not excluded by the AdS superalgebra. But there is still the problem of explaining the transition from the maximally supersymmetric AdS vacuum to the (hopefully) flat vacuum.

2.2 Stability of ground states

There was also the more immediate problem that all the ground states appeared to be pathologically unstable, even the maximally supersymmetric ground state. Figure 1.1 shows a contour map of a typical two-parameter section through the maximally supersymmetric critical point. This point is a local maximum of the potential, and is thus naively unstable.

It turns out that any supersymmetric critical point is completely classically (and semi-classically) stable [4]. One can prove this by establishing an AdS positive mass theorem. The key physical insight is that because the potential is Planck scale, the gravitational back-reaction is
very strong. Indeed Breitenlohner and Freedman showed very generally that in AdS space, a scalar field was perturbatively stable so long as its mass-squared is not “too negative” \cite{10}. This bound in an anti-de Sitter space, \(AdS_d\), of dimension \(d\) and radius \(R\) is \cite{10, 11}:

\[
m^2 \geq -\frac{(d-1)^2}{4R^2}.
\]

\(2.3\) The standard model and chiral fermions

The maximal possible gauge symmetry of maximal supergravity is \(SO(8)\), and this group is not large enough to contain the gauge groups of the standard model: \(SU(3) \times SU(2) \times U(1)\). There were suggestions that the physical gauge groups might emerge through some composite mechanism, or via a large \(SU(8)\) local composite symmetry of the supergravity theory. These ideas have remained just that, and were not given substance in four dimensions.

The other problem was that the fermions come in real representations of the \(SO(8)\) gauge symmetry, and even if one could break to \(SU(3) \times U(1)\) (hoping that \(SU(2)\) would emerge somehow at low energy) then the fermions would still be in non-chiral (real) representations of these groups. Thus \(SO(8)\) seemed to be fairly hopeless in terms of phenomenology (in spite of some remarkable numerology that originated with Gell-Mann: See \cite{12} for details).

Thus the “real-world” possibilities were very remote for this most divergence-free theory of all supergravities. Moreover, by this time, a consensus emerged that this theory was also most probably not finite. Thus the gauged maximal supergravity slowly faded from notice and interest.

\(3\) Exploring higher dimensions

Parallel to the development of maximal supergravity in four-dimensions was the development and construction of supergravity in higher dimensions. As I remarked earlier, the ungauged maximal theories were first obtained by “trivial” (i.e. on tori) dimensional reduction of the eleven-dimensional supergravity. It was one of the major industries of the early 1980’s to see what low-dimensional theories might be obtained by compactifying higher-dimensional maximal theories on other manifolds.

Initially the focus was upon Kaluza-Klein methods, in which gauge symmetries in lower dimensions emerged via isometries of the compactifying manifold. As with gauged supergravity, there were huge phenomenological problems with this. First, isometries on the compactifying manifold tended to lead to AdS space-times. Furthermore, it was finally shown
that one could never get chiral fermions though Kaluza-Klein without having chiral fermions \textit{ab initio}.

In the mid 1980’s string theory emerged from a long hibernation, and with the invention of the heterotic string it appeared that one could finally get a finite theory of quantum gravity with large enough gauge groups, chiral fermions and no anomalies. The focus thus turned to the compactification of string theory, some of which entailed finding compactifications of the corresponding low-energy supergravity theories (often coupled to supersymmetric matter).

3.1 Sphere compactifications

One particular focus of supergravity was the compactification of eleven-dimensional supergravity on $S^7$ down to $AdS_4$. It was believed, and indeed subsequently proven, that the low energy sector (i.e. essentially the lowest Fourier modes on $S^7$) of this theory is, in fact, maximal gauged $\mathcal{N} = 8$ supergravity in four dimensions. The complete set of Fourier modes of the $S^7$ compactification would thus extend maximal gauged $\mathcal{N} = 8$ supergravity by infinite towers of massive states.

A variant of this idea is of particular relevance today. There is a maximal supergravity theory, called IIB supergravity, in ten dimensions that has two chiral fermions [16] (and so is not a trivial dimensional reduction of eleven-dimensional supergravity). This theory is a low-energy limit of the ten-dimensional, IIB superstring. This supergravity, and the corresponding string theory theory have a compactification on $S^5$ down to five dimensional $AdS_5$, and this almost certainly (though it was never fully proven) yields, in the massless sector, the maximal gauged $\mathcal{N} = 8$ theory in five dimensions.

This string back-ground also arises in another very interesting manner.

3.2 Brane backgrounds

A very important class of stringy backgrounds are $p$-branes, which are the higher dimensional analogues of the extreme Reissner-Nordström black holes. That is, they are ($p$, 1)-dimensional objects that minimally couple to a ($p+1$)-form gauge potential. If the gauge field is a Ramond-Ramond (RR) field of the string then they are called $Dp$-branes, and most significantly, in a closed string theory there can also be open strings that end on the $D$-branes [17]. The lowest modes of the closed string generically describe the graviton supermultiplet, and the lowest modes of the open string generically describe a vector supermultiplet. The directions of the oscillations of the string give rise to the polarization tensors (see figure 1.2).
Exploring higher dimensions

The consequence of this is that a closed string in the presence of $D_p$-branes will induce a Yang-Mills theory on the $D$-branes. If there are $N$ coincident $D$-branes then the Yang-Mills theory has an $SU(N)$ gauge group, and the amount of supersymmetry on the brane is half of that of the bulk theory. Thus IIB superstrings induce $\mathcal{N} = 4$ supersymmetric, $SU(N)$ Yang-Mills theory on a stack of $N$ coincident $D3$-branes (whose world-volume is four dimensional). This particular Yang-Mills theory is also a conformal field theory (CFT), even as a strongly coupled quantum theory.

The extreme Reissner-Nordström black hole has a “near-horizon” limit that is essentially an infinitely long throat described by $AdS_2 \times S^2$. In the same manner, the near-brane limit of the $D3$-brane background is $AdS_5 \times S^5$. The metric in this near-brane limit may be written:

$$ds_{10}^2 = e^{2A(r)} \left( \eta_{\mu\nu} \, dx^\mu \, dx^\nu \right) + dr^2 + ds_5^2,$$

where $A(r) = r/L$ and $ds_5^2$ is the metric of an $S^5$ of radius $L$. The factor $\eta_{\mu\nu} \, dx^\mu \, dx^\nu$ is the flat metric on the $\mathbb{R}^{3,1}$ slices parallel to the $D3$-branes. The first two terms in (1.2) combine to make the $AdS_5$ metric in which $L = (4\pi g_s \alpha' N)^{\frac{1}{4}}$ is the AdS radius. Here $N$ is the number of $D3$ branes, $g_s$ is the string coupling and $\alpha'$ is the string tension parameter. Thus, as depicted in Figure 1.3, this maximally supersymmetric background of
the IIB superstring emerges as an infinitely long throat as one tries to approach the horizon of the D3-brane background.

The next leap forward was made by Maldacena [18] in the late 1990’s, and it was to conjecture a deep relationship between the physics of the closed IIB superstring theory (and hence IIB supergravity) in the “near-brane” background and the physics of the Yang-Mills theory on the D3-branes.

4 Holographic field theory and the AdS/CFT correspondence

As discussed by other speakers at this meeting, the basic idea in holographic field theory is that, in the presence of gravity, physics in a region of space-time can be encoded on a lower-dimensional surface surrounding that region. In particular, the quantum properties of matter that has formed a black hole can be (and have been) holographically encoded on its horizon [19, 20]. The Maldacena Conjecture makes precise computational proposals for several such holographic encodings. One of these states that there is a precise quantum duality between IIB superstring theory in an $AdS_5 \times S^5$ background and the CFT on D3-branes, that is, $\mathcal{N} = 4$ Yang-Mills theory. The holographic principle was originally intended as a way of studying gravity by looking at a holographic field

\footnote{By now it is sufficiently widely believed and tested that we should probably call it a theory or principle.}
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theory on a surface, but here I will be using the holographic principle in reverse to derive field theory results on the brane using IIB string theory, or more precisely supergravity.

The heart of this AdS/CFT correspondence is to realize that a gauge invariant operator, $O(x)$, in the Yang-Mills theory on the brane will act as a source for closed strings in the bulk. In particular, conserved currents on the brane must couple to local gauge fields in the string background. For example, the energy-momentum tensor of the Yang-Mills theory couples to (or is dual to) the graviton in the string theory, and the currents of the global $SO(6)$ $R$-symmetry of Yang-Mills are dual to the vector bosons of the gauged supergravity. This idea extends, via supersymmetry, to a complete correspondence of fields in the Yang-Mills energy-momentum supermultiplet with the fields in the graviton supermultiplet. The latter are precisely the massless fields that constitute the spectrum of maximal gauged $\mathcal{N} = 8$ supergravity in five dimensions. In particular, the 42 scalar fields of the gauged supergravity are dual to bilinears of fundamental Yang-Mills fields, or roughly the Yang-Mills gauge coupling, $\theta$-angle, and all the mass terms for the Yang-Mills fermions and scalar fields.

More generally, if one introduces a generating function for correlation functions of gauge invariant Yang-Mills operators, then the AdS/CFT correspondence states that [18, 21, 22]:

\[
\left\langle \exp \left( - \int O_j(x_j) \, d^4x \right) \right\rangle_{\text{brane}} = Z_{\text{string}}[\varphi_k] \\
\rightarrow Z_{\text{supergravity}}[\varphi_k] \\
\rightarrow \exp \left( - S[\varphi_k] \right).
\] (1.3)

In this equation the left-hand side is the generating function with operators integrated against arbitrary density functions, $\varphi_j^{(0)}$, on the branes. The function $Z_{\text{string}}[\varphi_k]$ is the string path integral evaluated in the AdS background fields, $\varphi_k$, that satisfy boundary conditions

\[
\varphi_k(x^\mu, r) \rightarrow \varphi_j^{(0)}(x^\mu) \quad \text{as} \quad r \rightarrow \infty,
\] (1.4)

where $r$ is $AdS_5$ radial coordinate transverse to the branes, and is defined in (1.2). The first limit in (1.3) reduces the string theory to its supergravity limit, and this may be done by a combination of taking the string tension to infinity ($\alpha' \rightarrow 0$) and sending $g_s N \rightarrow \infty$. The second limit in (1.3) is obtained by making the saddle-point approximation ($g_s \rightarrow 0$) of the supergravity path integral, $Z_{\text{supergravity}}[\varphi_k]$, and thus one simply evaluates the supergravity action on the classical supergravity solution that satisfies the boundary conditions (1.4). The strongest form of the correspondence is, of course, the string theory expression, but, in practice, it is much easier to work with supergravity actions, and it is in this context that most (but not all) of the testing has been done.
The “bottom line” is that the classical action of IIB supergravity in ten dimensions should holographically capture the large-\(N\) limit of strongly coupled quantum Yang-Mills theory. Moreover, if one restricts to the operators of the energy-momentum tensor supermultiplet (mass insertions, gauge coupling and \(\theta\) angle) then the behaviour of this large-\(N\) strongly quantum theory on the brane, under such perturbing operators, should be captured entirely by gauged, \(\mathcal{N} = 8\) supergravity in five dimensions.

5 Bulk gravity and brane renormalization: Where are the branes?

From the gravitational perspective the branes must be at the core of the solution, that is, at the bottom of the infinitely long throat. Therefore, the boundary condition (1.4) as \(r \to \infty\) seems very counterintuitive: The branes are at \(r = -\infty\) and not at \(r = +\infty\). However, locating the branes is not so simple in the holographic perspective: In holography one considers the field theory on any of the (3 + 1)-dimensional Poincaré invariant slice of the space-time. The idea is that one is then “sampling” the field theory on a brane located at some radius, \(r\), in the metric (1.3), and the choice of the radius represents a choice of the renormalization scale in the field theory. I want to illustrate this in several ways since this idea is central to some of the more remarkable tests and applications of the AdS/CFT correspondence.

First, one should recall that \(AdS_5\) is invariant under \(SO(4, 2)\), and that this is the conformal group in (3 + 1) dimensions. If one takes a general \(D3\)-brane slice of (1.2) then fixing a finite value of \(r\) breaks \(SO(4, 2)\) down to the Poincaré invariance on the slice, and in particular, scale invariance has been broken. However, the slice at \(r = \infty\) is special: It is fixed under the action of \(SO(4, 2)\), and so the field theory induced on this slice is conformally invariant. Thus we associate the brane at infinity with a conformal, ultra-violet fixed point of the brane field theory, and a brane at finite \(r\) with the theory at some cut-off scale set by \(r\). Indeed, to implement many computations using (1.3) one has to regulate the Green functions, and this can be done explicitly by setting \(r = 1/\epsilon\) and sending \(\epsilon \to 0\). (See, for example [21, 22, 23]). As one might expect, the true physical scale on the brane is set by the cosmological scale factor, \(e^{A(r)}\), in (1.2), and not so much by \(r\) itself.

5.1 Coarse-graining and red-shifts

The foregoing is a rather formal justification, and other more physical arguments can be given (see, for example, Lenny Susskind’s contribution to these proceedings). One way that one can see this rather clearly is to
consider two parallel slices at different values of \( r \), as depicted in Figure 1.4. If one considers the correlator of two operators on a brane then the correlator is only sensitive to the region of the space-time that is relatively near both operators. If the operators are widely separated, as are \( O_1 \) and \( O_2 \) in Figure 1.4, then they sample deeply into the interior of the space-time (down to lower values of \( r \)). If the operators are close together like \( O_3 \) and \( O_4 \) then they are only sensitive to details of the space-time at large \( r \). Thus we see the UV-IR duality of holography: Short distance behaviour on the brane depends upon large \( r \), while large distance behaviour on the brane depends upon small values of \( r \).

One can take a more active view of the same process. Consider a fixed volume, say \( 1m^3 \), on the \( D3 \)-brane measured in the Poincaré metric, but consider this volume at two different values of \( r \). This is depicted in Figure 1.4 by the large squares on the left and right. If one uses supergravity to evolve data from large \( r \) to smaller \( r \) then it will be blue-shifted. Thus
if one uses supergravity equations to determine evolution in $r$, studying physics on a Poincaré scale of $1m^3$ at smaller $r$ will involve averaging over data from several regions of size $1m^3$ at the larger value of $r$. This is precisely Wilsonian coarse-graining and it comes out of holography via gravitational red (blue)-shifts. The evolution to smaller values of $r$ is thus renormalization group flow to the infra-red.

5.2 Central charge and cosmological entropy

There are several ways in which one can put computational flesh on these ideas, but one of the simplest and most beautiful is the the holographic $c$-theorem.

In a conformal field theory the central charge essentially counts the number of massless degrees of freedom in the theory. It may be defined in terms of the leading singularity in the correlator, or operator product, of two energy-momentum tensors. In $d$ dimensions the energy-momentum tensor, $T(z)$, has dimension $d$, and so to leading order one has

$$T(x)\ T(y) \sim \frac{C}{|x - y|^{2d}}.$$  \hfill (1.5)

(I am suppressing all the details of tensor indices.) If $T(x)$ is canonically normalized then the constant $C$ is essentially the central charge, and it is proportional to the number of degrees of freedom within the theory: It simply counts the ways in which energy can be transmitted.

In a holographic theory one can argue that the central charge is dual to a power of the “effective cosmological constant,” that is,

$$C(r) \sim \frac{1}{A'(r)^{d-1}}.$$  \hfill (1.6)

where $A(r)$ is the function in (1.3). The central charge thus depends upon the scale $r$. Moreover, it can shown [24] that if the matter in the supergravity theory obeys a weak energy condition, then $C(r)$ monotonically decreases as $r$ decreases, and is only stationary in an AdS vacuum, and hence at a conformal fixed point on the brane. In other words, the number of dynamical degrees of freedom monotonically decreases as one flows to the infra-red, unless the flow reaches a conformal fixed point and then the number of degrees of freedom remains constant. This is precisely what one should expect as a result of coarse-graining: a gradual loss of non-scale invariant degrees of freedom.

In cosmology, where the evolution is over time, the quantity analogous to $A'(r)^{d-1}$ is the entropy density of the universe, and this is monotonically increasing. Thus there is at least a formal link between the entropy in
cosmology and the central charge in holographic renormalization group flows. Moreover, both the flow of entropy and the flow of the central charge reflect a loss of information about the detailed finer structure of the matter within the space-time.

5.3 **Universality and black branes**

The foregoing shows that there are remarkable links between classical results of relativity and quantum properties of field theory on the brane. Much has been done to develop this, but given that this meeting is to celebrate Stephen’s contributions to science, there is one speculation that seems very appropriate.

In quantum field theory the idea of universality loosely states that the (infra-red) end-point of a renormalization group flow does not depend upon ultra-violet details. In low dimensions one can go further and argue that infra-red renormalization group fixed points do not depend upon details of interactions, but merely depend upon the symmetries of, and number of degrees of freedom in the physical system. If one thinks holographically, then this says that the supergravity solution for small values of $r$ does not depend upon the details of the matter at large values of $r$. This begins to sound like a “no-hair” theorem, particularly when one thinks of the stronger low-dimensional idea of universality. If the flow solution evolves (as many of them do) to a black brane in the core, then indeed universality of the field theory on brane, and a “no hair” theorem for the black brane are trying to capture exactly the same physical ideas.

It should, of course, be remembered that in holography of four-dimensional field theories we are interested in radial evolution in five, or even ten, dimensions. As Gary Horowitz’s talk illustrates, the collapse of black-branes in higher dimensions can lead to rather exotic end-states. There is, however, a very interesting convergence: One would like to know all the infra-red fixed points of Yang-Mills theories, and it is intriguing to think that (at least for large $N$) it might rest upon understanding the possible end-states of collapse of black branes.

6 **Holgraphic renormalization group flows: An example**

One of the simplest ways to exhibit a holographic renormalization group flow is to take a conformally invariant holographic theory and perturb it by a relevant operator and then use supergravity to study the flow. One may also seek out fixed points of such a flow by looking for new, non-trivial AdS vacua that might be approached in the infra-red ($r \to -\infty$).

Gauged $\mathcal{N} = 8$ supergravity in five dimensions provides a powerful tool for analyzing a sub-class of flows in $\mathcal{N} = 4$ Yang-Mills theory \cite{25, 26, 27}.
The 42 scalars of the supergravity are dual to bilinear operators in the Yang-Mills theory, and thus by choosing the appropriate supergravity scalar one can introduce a gauge invariant mass term for any field in the Yang-Mills theory. The flow to the infra-red will then correspond to integrating the massive field out of the Yang-Mills action, leaving a reduced number of degrees of freedom, and perhaps some new, non-trivial interactions. In the supergravity theory the scalars corresponding to the mass terms must be Poincaré invariant, but will depend upon $r$, and their evolution will be determined by the supergravity equations of motion. The potential in the supergravity will thus determine the flow, and critical points of the potential will, in principle, determine non-trivial conformal fixed points of a flow. Therefore, the (rigidly-determined) supergravity potential represents and characterizes the phase diagram and flows of $\mathcal{N} = 4$ Yang-Mills theory under mass perturbations.

6.1 Stability and unitarity

There are several subtleties involved in holographic duality, and the identification of flows, and one of these involves the stability issue. There are several known critical points of the gauged $\mathcal{N} = 8$ supergravity potential [27], and several of them fail the Breitenlohner-Freedman stability condition (1.1). They are thus not even perturbatively stable supergravity vacua.

It turns out that the holographic correspondence yields a direct relationship between the mass of small oscillations in the supergravity and the dimension of conformal operators on the brane [21, 22]. If a supergravity scalar fails the Breitenlohner-Freedman condition then the corresponding operator has imaginary conformal dimension, and the field theory cannot be unitary. Moreover, if one carefully examines a flow from the maximally symmetric critical point to an unstable critical point one can see that it appears to correspond to an infinite energy deformation involving a mass and a vacuum expectation value for the same operator.

This raises the question as to what critical points and flows are physically sensible. The general belief is that if the critical point is stable then it is a physical vacuum for both the supergravity and the theory on the brane, but there is a further issue. It should always be remembered that the supergravity description is really only valid at large $N$, and so it may be that a flow, or a critical point might represent a “large $N$ pathology” that may not be physical for finite $N$. Fortunately string theory gives us an answer: At finite $N$ one must use the full string path integral, and so a flow solution in supergravity will be valid at finite $N$ if we can demonstrate that it is a good string vacuum. The latter is not so easy to do, but based upon experience in string compactification in the late 1980’s we
know that supergravity vacua are usually good approximations to string vacua if they are supersymmetric. Such vacua also have the virtue of being completely semi-classically stable.

We are thus led to the following proposal: Any supersymmetric supergravity flow solution will reflect a real physical flow at finite $N$ for the theory on the brane. This proposal has passed several very non-trivial tests, and I will now outline one of them.

6.2 A supersymmetric flow

In $\mathcal{N} = 4$ Yang-Mills theory there is one gauge field, $A_\mu$, four fermions, $\lambda^a$, and six scalars, $X^I$, all in the adjoint of $SU(N)$. Consider the following mass perturbation on the brane:

$$\Delta \mathcal{L} = m_1 \text{Tr} (\lambda^1 \lambda^1) + m_2^2 \text{Tr} ((X^1)^2 + (X^2)^2).$$

(1.7)

If $m_1 = m_2$ then this perturbation preserves $\mathcal{N} = 1$ supersymmetry on the brane. Moreover, in field theory at finite $N$, this particular flow is known to lead to a non-trivial, $\mathcal{N} = 1$ supersymmetric fixed point [28].

![Fig. 1.5. The contour diagrams of part of the supergravity potential and superpotential. The mass parameters $\phi_1 = m_2^2$ and $\phi_2 = m_1$ are plotted along the horizontal and vertical axes respectively. Critical points are numbered on the plot of the potential, and the line on the second contour plot shows the steepest descent between critical points of the superpotential.](image)

In supergravity the perturbation (1.7) is represented by two scalars, $\phi_1 = m_2^2$ and $\phi_2 = m_1$. On this sub-sector the supergravity potential is easily computed, and it is in fact shown in Figure 1.1. The contours of this potential are shown in Figure 1.5.

The supergravity equations of motion are, of course, second order and involve the potential, $\mathcal{V}$. However, if one seeks the supersymmetric flow
then this is given by solving a first-order system involving a superpotential. That is one solves:

\[
\frac{d\varphi_j}{dr} = \frac{1}{L} \frac{\partial W}{\partial \varphi_j}, \quad \frac{dA(r)}{dr} = -\frac{2}{3L} W, \tag{1.8}
\]

where \(L\) is the AdS radius as \(r \to \infty\), and

\[
W \equiv \frac{1}{4\rho^2} \left[ \cosh(2\varphi_2) \left( \rho^6 - 2 \right) - (3\rho^6 + 2) \right], \quad \rho \equiv e^{\sqrt{6} \varphi_1}. \tag{1.9}
\]

The supergravity potential is then given by:

\[
V = \frac{1}{2L^2} \sum_{j=1}^{2} \left| \frac{\partial W}{\partial \varphi_j} \right|^2 - \frac{4}{3L^2} |W|^2, \tag{1.10}
\]

The contour diagrams in Figure 1.3 show several critical points. The central one is the maximally supersymmetric vacuum in which all the supergravity scalars vanish. The other vacua come in pairs related by a trivial reflection. The vacua labelled 2 and 3 are unstable, while the vacua labelled by 4 and 5 are \(N = 2\) supersymmetric in the bulk (\(N = 1\) supersymmetric on the brane) and preserve \(SU(2) \times U(1) \subset SO(6)\). Only the supersymmetric vacua show up on the contour plot of the superpotential, \(W\). Thus there is a good candidate supergravity vacuum state for the field theory fixed point of [28]. Indeed it is easy to verify that the unbroken supersymmetry and \(R\)-symmetry at the non-trivial critical point exactly matches that found in [28].

Equations (1.8) show that the flow is given by steepest descent on \(W\), and that the cosmological function, \(A(r)\), is completely determined by the steepest descent. This steepest descent is shown in Figure 1.3. One can see that near the central critical point one has \(\varphi_1 \sim \varphi_2^2\), which is consistent with \(m_1 = m_2\) as required by supersymmetry. At the non-trivial critical point one can examine the linearized supergravity spectrum and one finds that it matches perfectly with holographically dual operators that one expects to find in the field theory [24]. Most significantly, the supergravity predicts the following ratios of central charges:

\[
\frac{C_{IR}}{C_{IR}} = \left( \frac{A'(-\infty)}{A'(+\infty)} \right)^3 = \frac{27}{32}, \tag{1.11}
\]

This can be checked against a direct, and rather non-trivial, anomaly computation within the field theory, and there is perfect agreement.
6 Holographic renormalization group flows: An example

6.3 An important open problem

One does not have to find restrict one’s attention to flows to non-trivial fixed points: There are many physically interesting families of flows that approach singular metrics at finite values of $r$. Among the most interesting of these are the half-maximal supersymmetric flows ($\mathcal{N} = 2$ on the brane $\mathcal{N} = 4$ in the bulk). These are particularly important because part of the quantum effective action is exactly known [29, 30], and thus one can perform some extremely non-trivial tests of such holographic flows.

![Figure 1.6](image.png)

Fig. 1.6. Half-maximal supersymmetric flows appear as steepest descents on a supergravity superpotential. The physically most interesting flow follows the ridge line. This flow is singular at finite $r$, and corresponds to a “disk-like” smearing out of the $D3$-brane distribution.

Rather surprisingly, only a little is known about this class of solutions in $IIB$ supergravity. Indeed only one such flow solution is known, and this was constructed again using the techniques of gauged $\mathcal{N} = 8$ supergravity, and the result was then lifted to the full ten-dimensional theory [31]. Physically one again looks at steepest descents on another supergravity superpotential, as shown in Figure 1.6. In the corresponding ten-dimensional solution the branes are now located at finite $r$ and are spread apart over a uniform “disk-like” distribution. The asymptotic behavior of this solution can be studied in detail [32] and one does indeed find results that are beautifully consistent with those of the quantum field theory.

The solution of [31] represents only one fixed (in fact, the most uniform) distribution of smeared out $D3$-branes. The quantum field theory results of [29, 30] allow, and give exact results, for an arbitrary two-dimensional distribution of the $D3$-branes. Thus the solution of [31] is but a sin-
gle point in an infinite moduli space of half-maximally supersymmetric solutions. No other solutions in this class are known.

For this reason, and for several others it would be very nice to find general results for half-maximal supersymmetric backgrounds. There are theorems about hyper-Kähler manifolds leading to such backgrounds in the absence of branes, and so the open question is to find the analogue of the hyper-Kähler condition for half-maximal supersymmetry in the presence of branes and the Ramond-Ramond fluxes that they generate.

7 Final Comments

In the late 1980’s an eminent particle physicist publicly compared string theory to President Reagan’s rather misguided Strategic Defence Initiative, commonly called “Star Wars.” The suggestion was that both programs were making extravagant claims based upon little, or no experimental evidence. While this commentary probably grew out of some close-mindedness on one side and a certain amount of hubris on the other, it serves to underscore some much broader issues that concern string theory and string theorists.

First, the problem of quantum gravity is very unusual in science: the usual methodology is that one develops theories and chooses between them based on experimental data. In quantum gravity we have the problem that there are fundamental inconsistencies between two classes of well tested theory, and the problem has been to find any theory that solves the most basic of theoretical constraints. Stephen’s work on black hole radiance and upon information loss has been extremely important in making us aware of some central aspects of what happens when quantum mechanics meets gravity. In particle physics, supergravity was a very important step in addressing the problem of divergences, and string theory is almost certainly finite. String theory remains the only viable theory of quantum gravity that we have found after 50 years of trying. This is a remarkable achievement, but it is has remained rather esoteric, and perhaps not as widely appreciated as it might, and should be.

This is further complicated by the fact that research in string theory does not fit the classical “scientific method:” Predict and then experimentally test. Instead we test string theory in as many ways as we can computationally, and we study it in many, many limits. What is remarkable is that when we do this we often get new and deeper insights not just into quantum gravity but into particle physics and mathematics: String theory created the field of mirror symmetry in algebraic geometry; it has given new methods of computing multi-gluon amplitudes in QCD; it has spun-off a whole industry on “brane-worlds” and, as other speakers at this meeting have described, string theory and $D$-branes have given
new insight into the apparent information loss in black holes. In this talk I have tried to describe how string theory has given us a new way of looking at and analyzing strongly coupled quantum field theories via holography. There are beautiful dualities between renormalization group flows, coarse graining and cosmological redshifts and entropy. There are simple classical calculations from which one obtains insight into the phase structure of strongly coupled quantum field theory. This, and all the other developments, are not experimental confirmation, but they do represent major progress in theoretical understanding. I believe that over the last few years the physics community has begun to appreciate the remarkable list of string theory spin-offs because there currently seems to be a more broadly based enthusiasm for string theory than there was in the 1980’s.

This still leaves open the issue of experimental confirmation: It should be remembered that a quantum theory of gravity only becomes important at energy scales vastly above the reach of any conceivable particle accelerator. So, to a purist, the experimental confirmation of string theory may take a very, very long time. On the other hand, supersymmetry is one of the foundational components of string theory, and current experimental data slightly favours the minimal supersymmetric standard model. At this meeting Edward Witten predicted that within the next decade we will see supersymmetry at the LHC. In terms of a solution to quantum gravity it is by no means essential that this happens, but I think it is very important to further strong support of string theory within the research community.

Finally, it is worth remembering that for about a decade after they were first invented, supersymmetric theories were considered by many to be mathematical freaks with particle spectra that are obviously ridiculous. Today, the study of supersymmetric field theories is not only respectable, but it lies at the core of particle physics phenomenology. In 1980 it was not very clear what the future of supersymmetry would be, but it was obviously a very important, new theoretical idea. Stephen has always had a very good sense for the important issues and the right questions to ask. I am therefore very grateful to Stephen for his interest, support and encouragement in the pursuit of a body of ideas that has grown into one of the most exciting theoretical fields of the last, and hopefully the next, twenty years.
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