$e^+e^-$ annihilation on the stochastic background of squeezed primordial gravitational waves

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Abstract

The evolution of primordial gravitational waves is studied. The $e^+e^-$ annihilation leads to modification in the spectrum of primordial gravitational waves. This effect is studied for primordial gravitational waves in the squeezed vacuum state. It is found that while $e^+e^-$ leads to a reduction of $\sim 10\%$ in the amplitude and of $\sim 25\%$ in the energy of gravitational waves without squeezing effect for frequency $\geq 10^{-12}\text{ Hz}$, it leads to enhancement by the same amount in the amplitude and energy respectively for frequency $\leq 10^{-10}\text{ Hz}$ for the gravitational waves in the squeezed state. The combined effect of $e^+e^-$ annihilation and squeezing effect produces $\sim 70\%$ enhancement in the amplitude and $\sim 140\%$ enhancement in the energy of the primordial gravitational waves compared to the unmodified cases.

1 Introduction

Primordial gravitational waves generated during inflation have traversed the universe throughout its expansion and evolution. As such, primordial gravitational waves carry information about the very early universe and its subsequent evolution that are otherwise difficult or even impossible to be probed through other means. They are believed to form a stochastic background with standing wave pattern over all frequency with spectrum and energy depending on the evolutionary stages of the universe [1–3]. Hence they are present everywhere at anytime at all frequency. However, as their wavelengths have also stretched along with the expansion of universe, their energy have diminished significantly and are the most difficult form of gravitational waves to detect, whether directly or indirectly.

If detected, the stochastic gravitational waves would provide an evidence of and information on the physical processes taking place during the very early stages of the universe. For primordial gravitational waves generated during inflation, the amplitude is directly related to the energy scale during inflation. The lowest frequency mode corresponds to the present-day horizon size while the highest frequency mode of the spectral energy density corresponds to the energy scale of reheating, a process which follows the decay of the inflaton, the scalar field that drives the inflation, after the end of the inflationary process. Thus the spectrum of primordial gravitational waves can provide information on the very early processes of the universe.

The spectrum of the primordial gravitational waves is also believed to be affected by the changes in the physical conditions in the universe. For instance, the $e^+e^-$ annihilation that occurs at frequency $\sim 10^{-10}\text{ Hz}$ causes a change in the number of relativistic degrees of freedom [4]. This further affects the expansion behavior of the universe and subsequently leads to a step in the spectrum of primordial gravitational waves. This is because the growth rate of the Hubble radius gets reduced during the process of $e^+e^-$ annihilation which leads to the change in the rate of horizon re-entry of the modes at that instant, which is within the radiation dominated era and a step in the gravitational wave spectrum appears at the frequency of the order of Hubble rate at that instant. This step is about $\sim 10\%$ in the amplitude and about $\sim 20\%$ in the energy spectrum [4–6].

The primordial gravitational waves produced during the inflationary period are believed to exist in a specific quantum state, known as squeezed vacuum state [7–9]. This is due to the fact that the inflationary quantum vacuum field fluctuations generated non-zero variance for quantum fluctuations which, due to parametric amplification, transformed the initial vacuum state into a quantum state with multiple particles, a state known as the squeezed vacuum state [10–11]. Due to this process, the phase variance of the wave mode is being squeezed while there is increase in the variance of its amplitude at the same time so that the uncertainty product is being held. If this is the case, then the squeezing effect is also expected to be reflected on the spectrum of primordial gravitational waves.

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In this paper, we placed the primordial gravitational waves in the squeezed vacuum state and study the effect of $e^+ e^-$ annihilation on the resulting spectrum. The paper is organized as follows. In section 2, we reviewed the epochs of expansion of the universe starting from inflation up to the present accelerating stage including the $e^+ e^-$ annihilation. In section 3, we introduced the squeezed state of primordial gravitational waves and calculated the parameters which characterize the squeezing effect which themselves are affected by each evolutionary stage of the universe. In section 4, we showed the resultant amplitude and energy spectrum and estimated the amount of change in the spectrum. We calculated the fractional energy density for various $\beta$ models and compared them with the big bang nucleosynthesis (BBN) bound [12–15]. We also checked the detectability of the resultant amplitude with several ongoing and proposed gravitational wave detectors like Advanced LIGO, Advanced Virgo, LIGO A+, KAGRA, Einstein Telescope and Cosmic Explorer [16–27]. We presented our conclusions in section 5. In appendix A, we gave a brief account of the physics of particles before and after the $e^+ e^-$ annihilation, concentrating on the effective relativistic number of species [3][28].

2 Epochs of expansion of the universe

The universe has undergone several stages of evolution throughout its expansion. The successive epochs of the evolution can be characterized by the scale factor $a$ in power-law form as follows [6]:

The inflationary stage:

$$a(\varsigma) = l_0 |\varsigma|^{1+\beta}, \quad -\infty \leq \varsigma \leq \varsigma_1, \quad \text{(1)}$$

where $l_0$ and $\beta$ are arbitrary constants, $\varsigma_1 < 0$ and $\beta < -1.77$. $\varsigma = \frac{2}{H} t$ is the conformal time, $t$ being the cosmic time.

The reheating stage:

$$a(\varsigma) = a_r |\varsigma - \varsigma_0|^{1+\beta}, \quad \varsigma_1 \leq \varsigma \leq \varsigma_s, \quad \text{(2)}$$

where $\beta_s$ is the reheating parameter. The subscript $z$ is used to denote the reheating stage.

The radiation-dominated stage:

$$a(\varsigma) = a_e (\varsigma - \varsigma_e), \quad \varsigma_s \leq \varsigma \leq \varsigma_y, \quad \text{(3)}$$

where the subscript $e$ denotes this stage.

The $e^+ e^-$ annihilation:

$$a(\varsigma) = a_v (\varsigma - \varsigma_y)^{1+v}, \quad \varsigma_y \leq \varsigma \leq \varsigma_z, \quad \text{(4)}$$

where the subscript $v$ denotes the period for $e^+ e^-$ annihilation which starts from $\varsigma_y$ and ends at $\varsigma_z$. This phenomenon occurs during radiation-dominated period. Throughout this paper, the power index $v = 0.063$ will be used for calculation purposes.

After $e^+ e^-$ annihilation and before matter domination:

$$a(\varsigma) = a_g (\varsigma - \varsigma_g), \quad \varsigma_z \leq \varsigma \leq \varsigma_2, \quad \text{(5)}$$

where the subscript $g$ is used to denote this period which is still in the radiation dominated era.

The matter-dominated stage:

$$a(\varsigma) = a_m (\varsigma - \varsigma_m) \gamma, \quad \varsigma_2 \leq \varsigma \leq \varsigma_E, \quad \text{(6)}$$

where the subscript $m$ denotes this stage.

The accelerating stage up to the present time:

$$a(\varsigma) = l_H (\varsigma - \varsigma_a)^{-1}, \quad \varsigma_E \leq \varsigma \leq \varsigma_H, \quad \text{(7)}$$

where $l_H$ is the present day Hubble radius and is taken as,

$$l_H = \left(\frac{a^2}{a'}\right) \approx \frac{1}{H}. \quad \text{(8)}$$

The power index in eq. (7) is often given by $\gamma$ which depends on the dark energy [29]. Even though the matter component also exists, since the dark matter is dominant, we shall take the approximation $\gamma = 1$ to the current acceleration behavior throughout this paper for calculation purposes.

The conformal time instances $\varsigma_1, \varsigma_s, \varsigma_0, \varsigma_z, \varsigma_2, \varsigma_E, \varsigma_H$ denote the various successive evolutionary stages of the universe. By choosing the overall normalization $|\varsigma_H - \varsigma_a| = 1$, the continuous joining of the functions $a(\varsigma)$
and $a'(\varsigma)$, where $'$ is derivative with respect to conformal time, at these points of transition provide the link between the conformal time instances in eqs. 11-14 as:

$\varsigma_0 - \varsigma_E = \xi_E$,  
$\varsigma_E - \varsigma_m = 2\xi_E$,  
$\varsigma_2 - \varsigma_m = 2\xi_E\xi_2^{-1/2}$,  
$\varsigma_2 - \varsigma_0 = \xi_E\xi_2^{-1/2}$,  
$\varsigma_2 - \varsigma_g = \xi_E\xi_2^{-1/2}$,  
$\varsigma_2 - \varsigma - \xi_E\xi_2^{-1/2}\zeta_3^{-1}$,  
$\varsigma_2 - \varsigma_c = (1 + v)\xi_E\xi_2^{-1/2}\zeta_3^{-1}$,  
$\varsigma_0 - \varsigma_c = (1 + v)\xi_E\xi_2^{-1/2}\zeta_3^{-1}\epsilon_{(1 + v)}$,  
$\varsigma_2 - \varsigma_y = \xi_E\xi_2^{-1/2}\zeta_3^{-1}\epsilon_{(1 + v)}$,  
$\varsigma_2 - \varsigma_s = \xi_E\xi_2^{-1/2}\zeta_3^{-1}\epsilon_{(1 + v)}$,  
$\varsigma_2 - \varsigma_p = (1 + \beta_s)\xi_E\xi_2^{-1/2}\zeta_3^{-1}\epsilon_{(1 + v)}$,  
$\varsigma_2 - \varsigma_s = (1 + \beta_s)\xi_E\xi_2^{-1/2}\zeta_3^{-1}\epsilon_{(1 + v)}\epsilon_{(1 + v)}$,  
$\varsigma_2 = (1 + \beta)\xi_E\xi_2^{-1/2}\zeta_3^{-1}\epsilon_{(1 + v)}\epsilon_{(1 + v)}$,  

(9)

and the arbitrary constants as:

$\alpha_n = \frac{1}{4} \xi_E^{-3}$,  
$\alpha_g = \frac{1}{4} \xi_E^{-2}\xi_2^{-1/2}$,  
$\alpha_v = \frac{1}{4} \xi_E^{-2}\xi_2^{-1/2}$,  
$\alpha_s = \frac{1}{4} \xi_E^{-2}\xi_2^{-1/2}$,  
$\alpha_y = \frac{1}{4} \xi_E^{-2}\xi_2^{-1/2}$,  
$\alpha_p = \frac{1}{4} \xi_E^{-2}\xi_2^{-1/2}$,  
$\alpha_{p+0} = \frac{1}{4} \xi_E^{-2}\xi_2^{-1/2}$,  
$\alpha_{p+1} = \frac{1}{4} \xi_E^{-2}\xi_2^{-1/2}$,  

(10)

where,

$\xi_E = \frac{a(\varsigma_H)}{a(\varsigma_E)}$,  
$\xi_2 = \frac{a(\varsigma_2)}{a(\varsigma_2)}$,  
$\xi_3 = \frac{a(\varsigma_3)}{a(\varsigma_3)}$,  
$\xi_4 = \frac{a(\varsigma_4)}{a(\varsigma_4)}$,  
$\xi_5 = \frac{a(\varsigma_5)}{a(\varsigma_5)}$,  

(11)

3 Primordial gravitational waves in the squeezed vacuum state

The perturbed metric of a flat FLRW universe in the presence of gravitational waves is,

$dS^2 = a^2(\varsigma)[-d\varsigma^2 + (\delta_{ij} + h_{ij})dx^i dx^j]$,  

(12)

where $|h_{ij}| \ll \delta_{ij}$ is a transverse-traceless perturbation of space-time, $\delta_{ij} h_{ij} = 0$, $\delta_{ij} h_{ij} = 0$, $\delta_{ij}$ being the flat space metric.

The gravitational wave field $h_{ij}(x, \varsigma)$ can be expanded over spatial Fourier harmonics $e^{\pm i k \cdot x}$ as

$h_{ij}(x, \varsigma) = \frac{D}{(2\pi)^2} \int_{-\infty}^{+\infty} \frac{d^3k}{\sqrt{2k}} \sum_{p=1}^{2} [h^{(p)}_k (\varsigma) e^{(p)}_i (k) e^{(p)+}(k) e^{ik \cdot x} + h^{(p)+}_k (\varsigma) e^{(p)+} (k) e^{(p)+}(k)e^{-ik \cdot x}]]$,  

(13)

where $D = \sqrt{16\pi l_p}$ is the normalization constant, $l_p = \sqrt{G}$ is Planck’s length. The wave number $k$ is related to the wave vector $k$ as $k = (\delta_{ij} k^i k^j)^{1/2}$ and is related to wavelength $\lambda$ by $\lambda = 2\pi k$.

The two linear polarization states $e^{(p)}_i$, $p = 1, 2$, are symmetric and transverse-traceless and satisfy the conditions $e^{(p)}_{ij} \delta^{ij} = 0$, $e^{(p)}_{ij} k^i = 0$, $e^{(p)}_{ij} e^{(p')}_{ij} = 2\delta_{pp'}$, $e^{(p)}_{ij} (k) = e^{(p)}_{ij} (k)$. Since the contributions from both these polarizations are same, the superscript $p$ is dropped from here onward for convenience.

The creation and annihilation operators $e^\dagger_k$ and $e_k$ satisfy the relationships $[e_k, e_{k'}] = \delta^3(k - k')$, $[e_k, e^\dagger_{k'}] = [e_k, e_{k'}] = 0$ and $c_0^0 = 0$, where $|0\rangle$ is the initial vacuum state.
The wave equation of the primordial gravitational waves in the flat FLRW universe in terms of the mode $h_k$ can be written as,

$$h''_k(\varsigma) + \frac{a''}{a}h'_k(\varsigma) + k^2h_k(\varsigma) = 0. \quad (14)$$

The function $h_k(\varsigma)$ is a complex time-dependent function which represents the time-evolution of all $k$ belonging to a given $k$ and can be rescaled in terms of mode function for $k$ as

$$h_k(\varsigma)a(\varsigma) = \mu_k(\varsigma), \quad (15)$$

where the mode functions can have the following form

$$\mu_k(\varsigma) = u_k(\varsigma) + v_k^*(\varsigma), \quad (16)$$

where $u(k)$ and $v(k)$ are complex functions and can be represented in terms of three real functions - the squeezing parameter $r_k$, squeezing angle $\phi_k$ and the rotation angle $\theta_k$ as $[30],$

$$u_k = e^{i\theta_k} \cosh r_k, \quad v_k = e^{-i(\theta_k-2\phi_k)} \sinh r_k. \quad (17)$$

The power spectrum of the gravitational waves is defined by two-point correlation function of the field $h_{ij}$,

$$\langle 0 | h_{ij}(\mathbf{x}, \varsigma)h_{ij}^*(\mathbf{x}, \varsigma) | 0 \rangle = \frac{D^2}{2\pi^2} \int_0^\infty k^2|h_k(\varsigma)|^2 \frac{dk}{k} = \int_0^\infty h^2(k, \varsigma) \frac{dk}{k}, \quad (18)$$

where $h^2(k, \varsigma) = \frac{D^2}{2\pi^2}k^2|h_k(\varsigma)|^2$ gives the mean-square value of the gravitational waves with interval $k$,

$$h^2(k, \varsigma) = \frac{1}{2}|h(k, \varsigma)|^2, \quad \text{and,} \quad |h(k, \varsigma)| = \frac{D}{\pi}k|h_k(\varsigma)|. \quad (19)$$

Using eqs. (16) and (17) in eq. (15), we get

$$h^2_0(\varsigma) = \frac{1}{a^2(\varsigma)}(\cosh 2r_k + \cos 2\phi_k \sinh 2r_k). \quad (20)$$

Then, using eq. (19), we get

$$h(k, \varsigma) = \frac{4l_{pl}}{\sqrt{\pi}a(\varsigma)}(1 + 2 \sinh^2 r_k + \sinh 2r_k \cos 2\phi_k)^{1/2}. \quad (21)$$

The wave number corresponding to the current Hubble radius is,

$$k_H = \frac{2\pi a(\varsigma_H)}{l_{H}}, \quad (22)$$

Then the amplitude of the squeezed primordial gravitational waves for the wave interval $k_{E} \leq k \leq k_1$ becomes

$$h(k, \varsigma_H) = 8\sqrt{\pi} \left(\frac{l_{pl}}{l_H}\right) \left(\frac{k}{k_H}\right)^{1/2}(1 + 2 \sinh^2 r_k + \sinh 2r_k \cos 2\phi_k)^{1/2}. \quad (23)$$

where the expression $(1 + 2 \sinh^2 r_k + \sinh 2r_k \cos 2\phi_k)$ represents the squeezing effect. The amplitude is dimensionless.

### 3.1 Determining the parameters

The squeezing parameter and squeezing angle are growing functions of time. They evolve along with the evolution of each stage of expansion. The squeezing parameter $r_k$ grows as $[2],$

$$r_k \approx \ln \frac{a_{\pi}(k)}{a_{\pi}(k)} \quad (24)$$

where $a_{\pi}$ is the value of $a(\varsigma)$ at $\varsigma_{\pi}$, the start of conformal time of each stage, i.e., the higher frequency end of the range since $\varsigma \propto 1/k$ and $a_{\pi}$ denotes $a(\varsigma)$ at $\varsigma_{\pi}$, the end of conformal time of the stage, i.e., the lower frequency end of range.
For the high frequency mode $k \geq k_1$, $a_s = a_{ss} = a(\zeta_1)$ which gives $r_k = 0$. Thus the high frequency modes $k > k_1$ are not in the amplifying regime. The amplifying regime starts from $k = k_1$. The squeezing parameter for the rest of the frequency range are calculated as descending order as

$$r_k = \ln \left( \frac{k}{k_1} \right)^{\beta - \beta_s}, \quad k_s \leq k \leq k_1,$$  \hspace{1cm} (25)

$$r_k = \ln \left( \frac{k^y}{k_s} \right)^{\beta - \beta_s} \left( \frac{k_s}{k_1} \right)^{\beta - \beta_s}, \quad k_y \leq k \leq k_s,$$  \hspace{1cm} (26)

$$r_k = \ln \left( \frac{k^y}{k} \right)^{\beta} \left( \frac{k_s}{k_1} \right)^{\beta - \beta_s}, \quad k_s \leq k \leq k_y,$$  \hspace{1cm} (27)

$$r_k = \ln \left( \frac{k^y}{k} \right)^{\beta - 1} \left( \frac{k_2}{k_1} \right)^{\beta} \left( \frac{k_s}{k_1} \right)^{-\beta_s}, \quad k_2 \leq k \leq k_z,$$  \hspace{1cm} (28)

$$r_k = \ln \left( \frac{k^y}{k} \right)^{\beta} \left( \frac{k_2}{k_1} \right)^{\beta - 1} \left( \frac{k_s}{k_1} \right)^{-\beta_s}, \quad k_2 \leq k \leq k_2,$$  \hspace{1cm} (29)

$$r_k = \ln \left( \frac{k^y}{k} \right)^{\beta} \left( \frac{k_2}{k_s} \right)^{\beta} \left( \frac{k_2}{k_1} \right)^{\beta - 1} \left( \frac{k_s}{k_1} \right)^{-\beta_s}, \quad k_{E} \leq k \leq k_{H},$$  \hspace{1cm} (30)

$$r_k = \ln \left( \frac{k^y}{k} \right)^{\beta} \left( \frac{k_E}{k_s} \right)^{\beta} \left( \frac{k_2}{k_s} \right)^{\beta} \left( \frac{k_s}{k_1} \right)^{-\beta_s}, \quad k \leq k_{E}.$$  \hspace{1cm} (31)

The reheating parameter $\beta_s$ is present in the expressions (24)-(31) of the squeezing parameter for each stage, hence reheating also affects the amplitude of the entire range of frequency through the effect of squeezing. Also, the index $v$ that denotes the $e^+ e^-$ annihilation also appears in each expression for $k \leq k_y$, where $k_y$ is the start of the $e^+ e^-$ annihilation. Hence this phenomenon affects the spectrum for $k \leq k_y$.

In the adiabatic regime for a given mode, the wavelength is shorter than the Hubble radius, therefore $k$ is dominant. Thus the squeezing angle can be given by,

$$\phi_k = -k(\zeta + \delta_k) = -\frac{k}{a(\zeta)} \left( 1 + a_s(\zeta) \right) \left( \frac{a_s(\zeta)}{a(\zeta_H)} \right), \quad k > k_H.$$  \hspace{1cm} (32)

In the long wavelength regime, $k$ can be neglected, and the squeezing angle becomes

$$\phi_k \propto \tan^{-1} \left( \frac{1}{a^2(\zeta)} \right), \quad k \leq k_H.$$  \hspace{1cm} (33)

Using eqs.(32) and (33), the squeezing angle for each frequency intervals are calculated as,

$$\phi_k = -k \left[ 1 + \left( \frac{k_s}{k} \right)^{\beta_s} \left( \frac{k_2}{k_s} \right)^{\beta} \left( \frac{k_H}{k_2} \right)^{k_s} \left( \frac{k_E}{k_H} \right)^{3} \right], \quad k_s \leq k \leq k_1,$$  \hspace{1cm} (34)

$$\phi_k = -k \left[ 1 + \left( \frac{k_s}{k_y} \right)^{\beta_s} \left( \frac{k_H}{k_y} \right)^{\beta} \left( \frac{k_2}{k_H} \right)^{k_y} \left( \frac{k_E}{k_2} \right)^{3} \right], \quad k_y \leq k \leq k_s,$$  \hspace{1cm} (35)

$$\phi_k = -k \left[ 1 + \left( \frac{k_s}{k} \right)^{\beta_s} \left( \frac{k_H}{k} \right)^{\beta} \left( \frac{k_2}{k_H} \right)^{k} \left( \frac{k_E}{k_2} \right)^{3} \right], \quad k_s \leq k \leq k_y,$$  \hspace{1cm} (36)

$$\phi_k = -k \left[ 1 + \left( \frac{k_H}{k} \right)^{\beta_s} \left( \frac{k_2}{k_H} \right)^{\beta} \left( \frac{k_E}{k} \right)^{k} \right], \quad k_2 \leq k \leq k_z,$$  \hspace{1cm} (37)

$$\phi_k = -k \left[ 1 + \left( \frac{k_H}{k} \right)^{2} \left( \frac{k_E}{k_H} \right)^{3} \right], \quad k_H \leq k \leq k_2,$$  \hspace{1cm} (38)

$$\phi_k = \tan^{-1}(k_{H}^2), \quad k_E \leq k \leq k_{H},$$  \hspace{1cm} (39)

$$\phi_k = \tan^{-1}(k_{E}^2), \quad k \leq k_{E}.$$  \hspace{1cm} (40)

Hence through the squeezing parameter and squeezing angle for successive frequency intervals, it can be seen that the reheating and $e^+ e^-$ annihilation affect the amplitude of the primordial gravitational waves in the squeezed state.

The upper bound on the inflationary index $\beta$ is -1.77. Observational constraints give a restriction $b l^2_{pl} \approx 10^{-6}$, $b$ being $|1 + \beta|^{-(1+\beta)}$ which, using eq.(30) and $l_{pl}/l_{H} = 9.276 \times 10^{59}$, leads to the values of $\beta_s =$
The wave number $k$ is proportional to the frequency $\nu$, so the ratios of the wave numbers can be replaced by the ratios of the frequencies. The Hubble frequency is $H_\nu \equiv \frac{1}{\nu H} \simeq 2 \times 10^{-18}$ Hz. For other values of frequency, we choose $\nu_E = 1.5 \times 10^{-18}$ Hz which is in the long wavelength regime, $\nu_2 = 117 \times 10^{-18}$ Hz, $\nu_3 = 10^8$ Hz, and $\nu_1 = 10^{10}$ Hz. The $e^+ e^-$ annihilation takes place around the frequency $10^{-12} - 10^{-10}$ Hz, therefore, we used $\nu_y = 10^{-10}$ Hz and $\nu_z = 10^{-12}$ Hz.

Figures (a) show the amplitude $h(\nu, \varsigma_H)$ of stochastic background of primordial gravitational waves for the model $\beta = -2.02$ with $\beta_s = -1.903$ and $\nu = 0.063$. Figure (b) is just an enlargement of figure (a) over some frequency range to see the effect of $e^+ e^-$ annihilation. The dashed purple line is the spectrum with no modification, the gray line shows the spectrum without squeezing effect but with $e^+ e^-$ annihilation. The dashed black line shows the spectrum for squeezed gravitational waves while the red line shows the squeezed spectrum with $e^+ e^-$ annihilation. The orange part in the gray line shows the frequency range where $e^+ e^-$ annihilation takes place for the unmodified spectrum while the blue part in the red line shows the same for the squeezed spectrum. It can be seen that while there is a reduction of $\sim 10\%$ of the amplitude at $\nu > 10^{-12}$ Hz due to $e^+ e^-$ annihilation, there is a step of enhancement due to squeezing effect and a further enhancement on the squeezed amplitude due to $e^+ e^-$ annihilation. The step starts at $\nu = 10^{-10}$ Hz with the overall enhancement $\sim 70\%$ as compared to the unmodified amplitude. This continues up to $\nu \sim 10^{-16}$ Hz and there is divergence for $\nu \lesssim 10^{-16}$ Hz where the amplitude without squeezing is flat while the amplitude with squeezing keeps increasing. Repeating the same analysis with other models like $\beta = -1.8, -1.9, -2.0$ give the same result.

Figures (c) show the energy $\Omega_{gw}(\nu)$ of primordial gravitational waves for the model $\beta = -2.02$ with $\beta_s = -1.903$ and $\nu = 0.063$. Figure (d) shows the energy for the entire frequency range while figure (e) shows its enlargement for $10^{-14} \lesssim \nu \lesssim 10^{-8}$ Hz. Color assignments are same as those for the amplitude. The $e^+ e^-$
annihilation causes a step of reduction in energy by $\sim 25\%$ at $\nu > 10^{-12}$ Hz while it causes a step of enhancement by the same amount in the energy of the squeezed gravitational waves. The steps starts at $\nu = 10^{-10}$ Hz with the overall enhancement $\sim 140\%$ as compared to the energy of the unmodified spectrum upto $\nu \sim 10^{-16}$ Hz. Repeating the same analysis with other models like $\beta = -1.8, -1.9, -2.0$ yield the same result with the same amount of enhancement.

The fractional energy density of gravitational waves relative to the critical density of the universe can be defined in terms of spectral energy density as [3]:

$$\Omega_{GW} = \frac{\rho_{gw}}{\rho_c} = \int \frac{\Omega_{gw}(\nu)}{\nu} \, d\nu,$$  \hspace{1cm} (42)

where $\rho_{gw}$ is the energy density of the gravitational waves and $\rho_c$ is the critical energy density of the universe. Since we are assuming spatially flat spacetime, then we must have, $\rho_{gw}/\rho_c < 1$. However, the big bang nucleosynthesis (BBN) bound further gives

$$\Omega_{GW} h_0^2 < 8.1 \times 10^{-6}$$  \hspace{1cm} (43)

where $h_0 \sim 0.719$ is the Hubble parameter.

| Model | $\Omega_{GW} h_0^2$ for each $\beta$ model. |
|-------|------------------------------------------|
| $\beta = -1.8$ | $4.8 \times 10^{-4}$, $e^+e^-$ annihilation, $3.59 \times 10^{-4}$, squeezing effect, $3.53 \times 10^{-5}$ |
| $\beta = -1.9$ | $1.03 \times 10^{-6}$, $5.75 \times 10^{-7}$, $9.4 \times 10^{-6}$, squeezing effect, $9.41 \times 10^{-6}$ |
| $\beta = -2.0$ | $3.92 \times 10^{-7}$, $2.19 \times 10^{-7}$, $5.16 \times 10^{-6}$, squeezing effect, $5.17 \times 10^{-6}$ |
| $\beta = -2.02$ | $3.47 \times 10^{-7}$, $1.94 \times 10^{-7}$, $5.05 \times 10^{-6}$, squeezing effect, $5.05 \times 10^{-6}$ |

Table 1 shows the calculated values of $\Omega_{GW} h_0^2$ over the entire frequency range for each inflationary index for the spectrum with no modification and with modifications from $e^+e^-$ annihilation, squeezing effect and the combined $e^+e^-$ and squeezing effects. If only the case $\rho_{gw}/\rho_c < 1$ is considered, all the models are consistent. However, if the BBN bound is imposed, the model $\beta = -1.8$ is ruled out in every case while $\beta = -1.9$ is ruled out for the squeezed state and the combined $e^+e^-$ and squeezing effect while the rest remain consistent.

Figures 3 show the root mean square amplitude per root Hertz, i.e., $h(\nu, \varsigma_H)\nu^{-1/2}$ for models $\beta = -1.8, -1.9, -2.0, -2.02$ with $\beta = 0.598, -0.538, -1.676, -1.903$ respectively and $v = 0.063$ with and without modifications compared with the sensitivity curves of Einstein Telescope (green), Advanced LIGO (blue), Advanced Virgo (magenta), LIGO A+ (black), KAGRA (orange) and Cosmic Explorer (dark green). Color assignments for the spectrum are same as before. It can be seen that for $\beta = -1.8$, amplitudes in all cases are within the sensitivity curves of the given detectors. For $\beta = -1.9$, the amplitudes for the squeezed gravitational waves, both with and without $e^+e^-$ annihilation are barely within the sensitivity curves of Einstein Telescope and Cosmic Explorer around 10 Hz. The rest are below the sensitivity curves. However, considering the BBN bound, the amplitudes which are within the sensitivity curves of the detectors are the ones disfavored. Thus, considering the current and presently proposed detectors, the models which are safely within the BBN bound will be difficult to be tested as of present.
Figure 3: Amplitudes for different models with modifications tested with the sensitivity curves of Einstein Telescope, Advanced LIGO, Advanced Virgo, LIGO A+, KAGRA and Cosmic Explorer.

5 Discussions and conclusions

From the calculations of the parameters of squeezing, it is found that while reheating affects the entire spectrum for squeezed primordial gravitational waves, the $e^+e^-$ annihilation also affects the spectrum for frequency $\lesssim 10^{-10}$ Hz. Due to this effect, there is enhancement in the amplitude and energy for the squeezed primordial gravitational waves. The amount of enhancement in each case is observed graphically in figures 1b and 2b.

This amount of enhancement continues until the end of the frequency corresponding to the radiation dominated stage after which there is continued increase in the amplitude for the squeezed spectrum. This is due to the fact that for squeezed modes, the variance of the mode’s phase is strongly squeezed while its amplitude is strongly increased and this feature continues up to the very end of the amplifying regime. Thus, the squeezing parameter grows with time and this growth continues up to the very present accelerating stage of the universe. Also, there is strong oscillations towards the higher frequency. This is also due to the squeezing effect - the oscillatory factor $\cos \phi_k$, to be precise, which reflects the squeezing acquired by the modes.

Figures 3 show the root mean square amplitude per root Hertz, i.e., $h(\nu, \varsigma_H)\nu^{-1/2}$ for models $\beta = -1.8, -1.9, -2.0, -2.02$ with and without modifications with the sensitivity curves of Einstein Telescope, Advanced LIGO, Advanced Virgo, LIGO A+, KAGRA and Cosmic Explorer (stage 2) which operate in the frequency range $\mathcal{O}(10^{-10})$ Hz.

| Table 2: $h(\nu, \varsigma_H)\nu^{-1/2}$ for each $\beta$ model in the frequency range $\mathcal{O}(10^{-10})$ Hz. |
|-----------------|--------------------------|--------------------------|
| Model          | no modification          | $e^+e^-$ and squeezing    |
| $\beta = -1.8$ | $2.15 \times 10^{-25} - 2.71 \times 10^{-27}$ | $8.38 \times 10^{-24} - 1.05 \times 10^{-26}$ |
| $\beta = -1.9$ | $2.91 \times 10^{-25} - 1.83 \times 10^{-29}$ | $1.14 \times 10^{-24} - 7.19 \times 10^{-29}$ |
| $\beta = -2.0$ | $3.92 \times 10^{-27} - 1.24 \times 10^{-31}$ | $1.53 \times 10^{-26} - 4.85 \times 10^{-31}$ |
| $\beta = -2.02$| $1.66 \times 10^{-27} - 4.57 \times 10^{-31}$ | $6.50 \times 10^{-27} - 1.79 \times 10^{-31}$ |

The detectors Advanced LIGO, Advanced Virgo, LIGO A+ and KAGRA can probe up to amplitudes of the order of $10^{-24}$ (and above) while Einstein Telescope and Cosmic Explorer (Stage 2) can probe up to $\mathcal{O}(10^{-25})$. 


(and above) within their operating range. Table 2 shows the r.m.s amplitude per root Hertz for these models without modification and with the combined $e^+ e^-$ annihilation and squeezing effects within the frequency range of operation of these detectors.

The amplitudes for $\beta = -1.8$ with and without modifications and $\beta = -1.9$ with squeezing effect (with and without $e^+ e^-$ annihilation) which are within the sensitivity ranges of some of these detectors are those disfavored by the BBN bound. On the other hand, the models which are below the BBN bound have amplitudes much lower than the sensitivity curves of the given detectors. Thus, considering the sensitivities of the currently operating and proposed detectors, the models which are safely within the BBN bound will be difficult to be tested as of present.

\section{Particle physics before and after $e^+ e^-$ annihilation}

During the radiation dominated era, the particles which interact with the photon are in thermal equilibrium. Entropy per unit comoving volume is conserved in the adiabatic system,

$$ S(T) = s(T) a^3(T) = \text{constant}, \quad (44) $$

for

$$ s(T) = \rho_r + p_r = g_{ss}(T) \frac{2\pi^2}{45} T^3, \quad (45) $$

where $s(T)$ is the entropy density and the energy density $\rho_r(T)$ and pressure $p_r(T)$ are,

$$ \rho_r(T) = g_e(T) \frac{\pi^2}{30} T^4, \quad (46) $$

$$ p_r(T) = \frac{1}{3} \rho_r(T), \quad (47) $$

respectively, $g_{ss}$ and $g_e$ being the effective number of relativistic species contributing to the entropy and energy density respectively given by

$$ g_{ss} = \sum_{i=\text{boson}} g_i \left( \frac{T_i}{T} \right)^3 + \frac{7}{8} \sum_{i=\text{fermion}} g_i \left( \frac{T_i}{T} \right)^3, \quad (48) $$

$$ g_e = \sum_{i=\text{boson}} g_i \left( \frac{T_i}{T} \right)^4 + \frac{7}{8} \sum_{i=\text{fermion}} g_i \left( \frac{T_i}{T} \right)^4, \quad (49) $$

where $g_i$ is the number of spin or helicity states of $i$-th species, $T_i$ its temperature and $T$ for photon temperature $T_\gamma$. $g_e(T)$ and $g_{ss}(T)$ are equal for temperature $T \gtrsim 0.1$ MeV. Then, using eqs. (11) and (15) in eq. (49), we get

$$ \rho_r \propto g_e g_{ss}^{-4/3} a^{-4}. \quad (50) $$

Just before $e^+ e^-$ annihilation, all particles were in equilibrium at the same temperature such that $g_e(T) = g_{ss} T$ and the plasma consists of photons (2-helicity states), electrons and positrons (2 spin states each) and three species each of neutrino and antineutrino (1-helicity state each) all at the same temperature $T_1$, say, then

$$ s = \frac{2\pi^2}{45} T_1^3 \left[ 2 + \frac{7}{8} (2 + 2 + 6) \right]. \quad (51) $$

Then, from eqs. (45) and (51), we get

$$ g_e = g_{ss} = 10.75. \quad (52) $$

$e^+ e^-$ annihilation starts around $T \sim 0.5$ MeV and ends around $T \sim 0.1$ MeV \cite{56}. Just after $e^+ e^-$ annihilation and while the three neutrino species remain relativistic, we have only photons and three families each of neutrinos and antineutrinos with two different temperatures $T_\nu$ and $T_\bar{\nu}$ respectively since at temperatures below the $e^+ e^-$ annihilation temperature, neutrinos are cooler than photons which gain heat from the $e^+ e^-$ annihilation, and

$$ s = \frac{2\pi^2}{45} \left[ 2 T_\gamma^3 + \frac{7}{8} \times 6 T_\nu^3 \right], \quad (53) $$

where $T_\nu = (4/11)^{1/3} T_\gamma$. Then, using eqs. (48), (49) and (52), we get the values of $g_{ss}$ and $g_e$ after $e^+ e^-$ annihilation but before the heaviest neutrino becomes non-relativistic as,

$$ g_{ss} = 3.9091, \quad (54) $$

$$ g_e = 3.363. \quad (55) $$
Now, from Friedmann equation, during radiation domination, we have,
\[
\left(\frac{a'}{a^2}\right)^2 = \frac{8\pi G}{3}\rho_r. \tag{56}
\]

Then, using eq.(50), we get
\[
a' = \frac{g}{2} - \frac{2}{3}s. \tag{57}
\]

Taking the conformal time rate of expansion $a'$ before and after the $e^+e^-$ annihilation from eqs.(3) and (5) and using them in eq.(7), we get the ratio
\[
a \frac{g}{a} = \frac{g_{\eta}^{1/2}(\varsigma_y)g_{\varsigma}^{2/3}(\varsigma_y)}{g_{\eta}^{1/2}(\varsigma_z)g_{\varsigma}^{2/3}(\varsigma_z)} \simeq 1.1. \tag{58}
\]

Also, from eqs.(10), we get the ratio
\[
a \frac{g}{a} = \xi_{e^0}^{1/4}. \tag{59}
\]

The $e^+e^-$ annihilation takes place during the temperatures $T \sim (0.5 \sim 0.1)$ MeV, and since $T \propto 1/a(\varsigma)$, we have
\[
\xi_{e^0} = \frac{a(\varsigma_z)}{a(\varsigma_y)} = 5. \tag{60}
\]

Using this with eqs.(58) and (59) yields $v \approx 0.063$, the index that characterizes the $e^+e^-$ annihilation stage.

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