Regularity of Solution to Axis-symmetric Navier-Stokes Equations with a Slightly Supercritical Condition

Xinghong Pan *

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(Department of Mathematics and IMS, Nanjing University, Nanjing 210093, China.)

Abstract

Consider an axis-symmetric suitable weak solution of 3D incompressible Navier-Stokes equation with nontrivial swirl, \( v = v_r e_r + v_\theta e_\theta + v_z e_z \). Let \( z \) denote the axis of symmetry and \( r \) be the distance to the \( z \)-axis. If the solution satisfies a slightly supercritical assumption (that is, \( |v| \leq C (\ln \ln r)^\alpha \) for \( \alpha \in [0, 0.028] \) when \( r \) is small), then we prove that \( v \) is regular. This extends the results in [5],[16],[18] where regularities under critical assumptions, such as \( |v| \leq C \), were proven.

Keywords: axis-symmetric Navier-Stokes equation, slightly supercritical, De Giorgi-Nash-Moser iteration, regularity.

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1 Introduction

The incompressible Navier-Stokes equations in cartesian coordinates are given by

\[
\partial_t v + (v \cdot \nabla) v + \nabla p = \Delta v, \quad \nabla \cdot v = 0,
\]

where \( v \) is the velocity field and \( p \) is the pressure. We consider the axis-symmetric solution of the equations. That means, in cylindrical coordinates \( r, \theta, z \) with \( x = (x_1, x_2, x_3) = (r \cos \theta, r \sin \theta, z) \), the solution is of this form

\[
v = v_r e_r + v_\theta e_\theta + v_z e_z,
\]

*E-mail:math.scrat@gmail.com.
where the basis vectors $e_r, e_\theta, e_z$ are
\[ e_r = \left( \frac{x_1}{r}, \frac{x_2}{r}, 0 \right), \quad e_\theta = \left( -\frac{x_2}{r}, \frac{x_1}{r}, 0 \right), \quad e_z = (0, 0, 1), \]
and the components $v_r, v_\theta, v_z$ do not depend on $\theta$.

Recall $v_r, v_\theta, v_z$ satisfy
\[
\begin{aligned}
\partial_t v_r + (b \cdot \nabla) v_r - \frac{(v_\theta)^2}{r} + \partial_r p &= (\Delta - \frac{1}{r^2}) v_r; \\
\partial_t v_\theta + (b \cdot \nabla) v_\theta + \frac{v_\theta v_r}{r} &= (\Delta - \frac{1}{r^2}) v_\theta; \\
\partial_t v_z + (v \cdot \nabla) v_z + \partial_z p &= \Delta v_z; \\
b = v_r e_r + v_z e_z, \quad \nabla \cdot b &= \partial_r v_r + \frac{v_r}{r} + \partial_z v_z = 0.
\end{aligned}
\] (1.1)

In this paper we study the axis-symmetric Navier-Stokes equations under a slightly super-critical assumption on the drift term $b$. To be precise, we consider $b$ such that
\[
|b| = \sqrt{v_r^2 + v_z^2} \leq \begin{cases} 
\frac{(\ln |\ln \frac{r}{3}|)^\alpha}{r} & \text{if } r \leq 1; \\
\frac{C}{r} & \text{if } r > 1.
\end{cases}
\] (1.2)

Here $\alpha \in [0, 0.028]$ is any fixed constant. Later we will see how 0.028 is obtained.

Recall that the quantity $\Gamma = r v_\theta$ satisfies
\[
\partial_t \Gamma + (b \cdot \nabla) \Gamma - \Delta \Gamma + \frac{2}{r} \partial_r \Gamma = 0. \] (1.3)

Our assumption on $b$ is closely related to a counterexample in [31]. In [31], the authors consider elliptic equation of this form
\[- \Delta u + (b \cdot \nabla) u = 0. \] (1.4)

They construct a counterexample to state that (1.4) does not have Liouville theorem when the divergence-free vector field $b$ satisfies $|b| \leq \frac{\ln \ln |x|}{|x|}$ for large $|x|$. Moreover, Hölder continuity, as well as Harnack inequality, to solutions of (1.4) are also not to be expected. So under the assumption of (1.2), we do not expect a Hölder continuity to solutions of (1.3) even if the exponent $\alpha$ is small.

Therefore, under the current techniques, the "lnln" supercritical assumption on $b$ seems to be the best that one can expect for some continuity results(weaker than the hölder continuity) to solutions of (1.3) which can be used to prove the regularity of solutions to (1.1).

Our main result is the following.

**Theorem 1.1** Let $(v, p)$ be a suitable weak solution of the axis-symmetric Navier-Stokes equation (1.1) in $\mathbb{R}^3 \times [-1, 0)$. Assume that $b$ satisfies (1.2) and $\sup_{x \in \mathbb{R}^3} |\Gamma(\cdot, -1)| < +\infty$. Then we have
\[
\sup_{(x,t) \in \mathbb{R}^3 \times [-1, 0)} |v| < +\infty.
\]
Here and throughout the paper, we will use $c$ and $C$ to denote a generic constant. It may be different from line to line. Also we use $A \lesssim B$ to denote $A \leq CB$.

**Remark 1.1** We note that $\Gamma$ satisfies the equation (1.3) which enjoys the maximal principle. So the assumption $\sup_{x \in \mathbb{R}^3} |\Gamma(\cdot, -1)| < +\infty$ can assure that $|\Gamma| \leq C$ for all $t \in (-1, 0)$ for some positive constant $C$.

Readers can refer to [4] for the definition of suitable weak solutions.

Recall the natural scaling of Navier-Stokes equations: If $(v, p)$ is a solution of equations (1.1), then for any $\lambda > 0$, the following rescaled pair is also a solution:

$$v^\lambda(x, t) = \lambda v(\lambda x, \lambda^2 t), \quad p^\lambda(x, t) = \lambda^2 p(\lambda x, \lambda^2 t).$$

Denote $b^\lambda = v_\theta^\lambda e_r + v_z^\lambda e_z$, we say our assumption (1.2) is supercritical which means, for a fixed point $x_0 = (r_0 \cos \theta_0, r_0 \sin \theta_0, z_0)$, $b^\lambda(x_0)$ satisfies a bound

$$|b^\lambda(x_0)| \leq \frac{(\ln |\ln \frac{\rho x_0}{r_0}|)^\alpha}{r_0}.$$ 

When $\lambda \to 0$, the bound goes to infinity. That is, when one zooms in at a point, the bound on the drift term becomes worse, so the regularity of our solution must be handled more carefully.

Global in-time regularity of the solution to the axis-symmetric Navier-Stokes equations is still open. Under the no swirl assumption, $v^\theta = 0$, Ladyzhenskaya [17] and Ukhovskii-Yudovich [33] independently proved that weak solutions are regular for all time. When the swirl $v^\theta$ is non-trivial, recently, some efforts and progress have been made on the regularity of the axis-symmetric solutions. In [5], Chen-Strain-Yau-Tsai proved that the suitable weak solutions are regular if the solution satisfies $|v| \leq C/r < \infty$. Their method is based on Nash [27], Moser [26] and De Giorgi [8]. Also, Koch-Nadirashvili-Seregin-Sverak in [16] proved the same result using a Liouville theorem and scaling-invariant property. Lei and Zhang in [18] proved regularity of the solution under a more general assumption on the drift term $b$ where $b \in L^\infty([-1, 0), BMO^{-1})$.

It seems that their assumptions on $b$ are critical (for a fixed point, after scaling, the bound on $b$ is invariant). So using a standard linear estimate, they can prove the Hölder continuity of $\Gamma$ from equation (1.3) which breaks the scaling-invariant bound of the angular component $v_\theta$, making the bound on $b$ to a subcritical one. This is very important in proving regularity of the solution $v$. But under our supercritical assumption (1.2), only a logarithmic modulus of continuity, rather than the Hölder continuity, can be obtained which indicates, near $r = 0$, the $L^\infty$ norm of $\Gamma$ has a logarithmic decay with respect to $r$. Note that this also breaks the scaling-invariant bound of $v_\theta$ and is enough to prove the regularity of $v$, but requiring more efforts and more complicated computation.

Our proof of Theorem 1.1 is initially inspired by [5] and [18]. In the appendix of [5], the authors give a time-independent bound to the axis-symmetric weak solution under the assumption $|v| \leq C/r$. Stimulated by their idea, we will give a similar proof to get the regularity of $v$ under the assumption (1.2). In the process, more detailed computation and careful handling will be
needed, especially when we deal with the estimate to the fundamental solution of (1.6) due to the critical term $2\partial_r$. Our procedures of proof are as follows.

First, we will follow [13], using Nash-Moser type method to prove continuity of $\Gamma$ at $r = 0$. It satisfies a log decay near $r = 0$, that is

$$|\Gamma| \leq C \ln \frac{r}{3}^{-c_0} \quad \text{when } r \leq 1,$$

for some small positive $c_0$. See Theorem 1.2. This estimate breaks the scaling-invariant bound of $v_\theta$.

Next we explore the relationship between $v_\theta$ and $w_\theta$, the angular component of the vorticity $w = \nabla \times v$.

Here

$$w(x, t) = w_r e_r + w_\theta e_\theta + w_z e_z$$

and

$$w_r = -\partial_z v_\theta, \quad w_\theta = \partial_z v_r - \partial_r v_z, \quad w_z = (\partial_r + \frac{1}{r})v_\theta,$$

where $w_\theta$ satisfies

$$\left[ \partial_t + b \cdot \nabla - \Delta - \frac{v_r}{r} \right] w_\theta - \partial_r \left( \frac{v_\theta^2}{r} \right) + \frac{w_\theta}{r^2} = 0.$$

Let $\Omega = \frac{w_\theta}{r}$, then $\Omega$ satisfies

$$(\partial_t - L)\Omega = r^{-2} \partial_z (v_\theta)^2, \quad L = \Delta + \frac{2}{r} \partial_r - b \cdot \nabla.$$ (1.6)

Combining (1.5) and (1.6), we can get an estimate of $w_\theta$. Handling of (1.6) involves an estimate to the fundamental solution of $(\partial_t - L)u = 0$ which will be described in Theorem 1.3.

Recall that $b$ satisfies the vector identity

$$-\Delta b = \text{curlcurl} b - \nabla \text{div} b,$$

and note that

$$\text{div} b = 0, \quad \text{curl} b = w_\theta e_\theta,$$

then we have

$$-\Delta b = \nabla \times (w_\theta e_\theta).$$

At last, using the regularity theory of elliptic equations, we can get the boundedness of $b$. This will prove the regularity of our solution $v$.

**Theorem 1.2** For the divergence-free drift term $b$, define a zero-dimensional integral norm

$$E_R(b) \triangleq \sup_{-R^2 \leq t \leq 0} \left\{ \frac{1}{R^{3-p}} \int_{B_R} |b|^p \, dx \right\}^{\frac{1}{p}},$$ (1.7)
where $\frac{5}{3} < p \leq 2$ and $R \leq 1$, $B_R$ is the ball of radius $R$ centered at $x = 0$. If

$$E_R(b) \leq C \left( \ln \left| \ln \frac{R}{3} \right| \right)^{\frac{3p-5}{7p-12p}(1-\beta)} \forall R \in (0, 1],$$

(1.8)

for any $\beta > 0$, then the weak solution of (1.3) is continuous at $(0, 0)$ and it has a log decay near $r = 0$. That is, there exists some positive $c_0$, such that

$$|\Gamma| \leq C |\ln \frac{r}{3}|^{-c_0} \quad \text{for } r \leq 1.$$

**Remark 1.2** We note that when $|b| \leq (\ln |\ln \frac{r}{3}|)\alpha$, $\alpha \in [0, 0.028]$, there exists a $p_0 \in (\frac{5}{3}, 2]$ and $\beta_0$ small such that the assumption (1.8) is satisfied. So Theorem 1.2 infers the log continuity of $\Gamma$.

We compute it as follows,

$$E^p_{R}(b) \leq \frac{1}{R^{3-p}} \int_{B_R} \left[ \left( \ln \frac{\ln \frac{R}{3}}{r} \right)^\alpha \right]^p dx$$

$$\leq \frac{2\pi}{R^{3-p}} \int_0^R \int_0^R \left[ \left( \ln \frac{\ln \frac{R}{3}}{r} \right)^\alpha \right]^p r dr dz$$

$$\leq \frac{C}{R^{2-p}} \int_0^R (\ln |\ln \frac{r}{3}|)^{\alpha p} r^{1-p} dr$$

$$\leq \frac{C}{R^{2-p}} \int_1^\infty (\ln \ln 3s)^{\alpha p} s^{p-3} ds \quad \text{(let } s = 1/r)$$

$$\leq \frac{C}{R^{2-p}} \left\{ \int_\frac{1}{\pi}^{\frac{1}{\pi} e^{(\ln 3/\pi)^{1+\frac{\alpha p}{p}}}} (\ln \ln 3s)^{\alpha p} s^{p-3} ds \right. \right.$$  

$$\left. \left. + \int_{\frac{1}{\pi} e^{(\ln 3/\pi)^{1+\frac{\alpha p}{p}}}}^\infty (\ln \ln 3s)^{\alpha p} s^{p-3} ds \right\} \right.$$  

$$\leq \frac{C}{R^{2-p}} \left\{ \int_\frac{1}{\pi}^{\frac{1}{\pi} e^{(\ln 3/\pi)^{1+\frac{\alpha p}{p}}}} (\ln \ln 3s)^{\alpha p} s^{p-3} ds \right.$$  

$$\left. \left. + \int_{\frac{1}{\pi} e^{(\ln 3/\pi)^{1+\frac{\alpha p}{p}}}}^\infty \frac{\ln 3s}{3s} \ln s^{\alpha p} (3s)^{p-3+\alpha p} ds \right\}.$$

Here $\ln \ln 3s$ is a monotone-increasing function while $\frac{\ln \ln 3s}{3s}$ a monotone-decreasing function, so

$$E^p_{R}(b) \leq \frac{C}{R^{2-p}} \left\{ \left( \ln \frac{3}{R} \right)^{\alpha p+\epsilon p} \int_\frac{1}{\pi}^{\frac{1}{\pi} e^{(\ln 3/\pi)^{1+\frac{\alpha p}{p}}}} s^{p-3} ds \right.$$
\[
+ \frac{(\ln \ln \frac{3}{R})^{\alpha_p + \varepsilon}}{\left(e^{\left(\ln \ln \frac{3}{R}\right)^{1 + \frac{\varepsilon}{R}}}(e^{\left(\ln \ln \frac{3}{R}\right)^{1 + \frac{\varepsilon}{R}}}(3s)^{p-3+\alpha_p} ds\right)}
\]

we need choose a \( p \in \left(\frac{5}{7}, 2\right] \) such that \( p - 3 + \alpha_p < -1 \), that is \( \alpha < \frac{2}{p} - 1 \).

For such a \( p \)

\[
E_R^p(b) \leq \frac{C}{R^{2-p}} \left\{ \left(\ln \ln \frac{3}{R}\right)^{\alpha_p + \varepsilon} \left(\frac{1}{R}\right)^{p-2} + \frac{1}{2 - \alpha_p - p} \left(e^{\left(\ln \ln \frac{3}{R}\right)^{1 + \frac{\varepsilon}{R}}} \right)^{\alpha_p} \right\}
\]

here \( e^{\left(\ln \ln \frac{3}{R}\right)^{1 + \frac{\varepsilon}{R}}} \geq \frac{3}{R} \). So

\[
E_R^p(b) \leq \frac{C}{R^{2-p}} \left(\ln \ln \frac{3}{R}\right)^{\alpha_p + \varepsilon} \left(\frac{1}{R}\right)^{p-2} \leq C \left(\ln \ln \frac{3}{R}\right)^{\alpha_p + \varepsilon} .
\]

Let \( \varepsilon \) and \( \beta_0 \) are sufficiently small to make \( \alpha + \varepsilon \leq \frac{3p-5}{77p-120}(1 - \beta_0) \). This means, \( \alpha < \frac{3p-5}{77p-120} \)

So

\[
\alpha < \min \left\{ \frac{3p-5}{77p-120}, \frac{2}{p} - 1 \right\} .
\]

Let \( f(p) = \min \left\{ \frac{3p-5}{77p-120}, \frac{2}{p} - 1 \right\} \), \( f(p_0) = \max_{\frac{5}{7} < p \leq 2} f(p) \). We compute that

\[
p_0 = \frac{279 + \sqrt{1041}}{160} \approx 1.945 .
\]

We choose such \( p_0 \) to ensure \( \alpha < \min \left\{ \frac{3p_0-5}{77p_0-120}, \frac{2}{p_0} - 1 \right\} \approx 0.028 \). This is nearly the maximum value we can choose for \( \alpha \).

The next theorem gives an upper bound estimate to the fundamental solution of equation

\[
\partial_t u = (\Delta + \frac{2}{r} \partial_r - b \cdot \nabla)u, \quad \nabla \cdot b = 0 \quad (1.9)
\]

under certain bound for \( b \), which will be useful in the proof of Theorem 1.1. Due to the term \( \frac{2}{r} \partial_r \), the result is not covered by the standard theory.

Before stating the theorem, we give the definition of fundamental solutions to (1.9).
Definition 1.1 Let \( Q = \{(x,t) | x \in \mathbb{R}^3, t > s \} \), we say \( 0 \leq p(x,t; y,s) \in L_{\text{loc}}(Q) \cap C^2(Q \setminus (y,s)) \) is a fundamental solution of (1.9) in \( Q \) if it satisfies

1. for any \( \psi(x,t) \in C^\infty_0(Q) \) and \( \psi|_{r=0} = 0 \),
\[
\int_s^{+\infty} \int_{\mathbb{R}^3} p(x,t; y,s)(\psi_t + \Delta \psi + b \cdot \nabla \psi - \frac{2}{r} \partial_r \psi) dx dt = 0.
\]

2. for any \( \phi(x) \in C^\infty_0(\mathbb{R}^3) \),
\[
\lim_{t \to s} \int_{\mathbb{R}^3} p(x,t; y,s)\phi(y) dy = \phi(x).
\]

3. Let \( y = (y_1, y_2, y_3) \) and denote \( y' = (y_1, y_2, 0) \), we require
\[
p(x,t; y,s)|_{y'=0} = 0.
\]

This third condition marks an important difference with the standard theory where fundamental solutions are positive everywhere. Our choice of this fundamental solution coincides with some quantities in the axis-symmetric Navier-Stokes equations, such as \( \Gamma, w_\theta \).

Remark 1.3 Due to our assumption \( p(x,t; y,s) \in C^2(\mathbb{R}^3) \), \( p(x,t; y,s) \) satisfies (1.9) in classical sense except at point \((y,s)\).

Theorem 1.3 Let \( p(x,t; y,s) \) be a fundamental solution of (1.9) and the divergence-free smooth vector function \( b(x,t) \) satisfies \(|b| \leq C_0 + \frac{1}{r} \). Then we have
\[
p(x,t; y,s) \leq C(t-s)^{-3/2} \exp \left\{ -C_1 \frac{|x-y|^2}{t-s} \left( 1 - C_0 \frac{t-s}{|x-y|} \right)^2 \right\}
\]
for some positive constants \( C, C_1 \). Moreover,
\[
\int_{\mathbb{R}^3} p(x,t; y,s) dx \leq 1, \quad \int_{\mathbb{R}^3} p(x,t; y,s) dy = 1.
\]

Remark 1.4 The idea of proving Theorem 1.3 is based on Theorem 5 of [6], but due to the term \( \frac{2}{r} \partial_r \), the proof will be more complicated. In [6], the authors consider the equation
\[
\partial_t u = \Delta u - b \cdot \nabla u.
\]
In their proof, the Davies-type exponent \( \tau(t) \) can map from \([0, T]\) to \([1, \infty)\) and with the help of the logarithmic Sobolev inequality, a \( L^1 \to L^\infty \) estimate to the solution of (1.11) can be obtained. But for (1.9), we must deal with a singular term \( \frac{2}{r} \partial_r \) which will create some difficulties when using their method to estimate the solution. However, stimulated by Fabes-Stroock[10] and Davies [7], we use a dual technique from harmonic analysis to overcome this difficulty. We proceed as follows.
First, we choose \( r(t) : [0, T] \to [2, \infty) \) to get a \( L^2 \to L^\infty \) estimate of the fundamental solution \( p(x, t; y, s) \), then the same estimate can be applied to the adjoint \( p^*(x, t; y, s) \) of \( p(x, t; y, s) \). By duality, we get \( L^1 \to L^\infty \) estimate of \( p(x, t; y, s) \). This will prove our Theorem 1.3.

Estimates to the kernels of parabolic equations have had a long history especially when the drift \( b \) is a divergence-free singular term. Under different assumptions on \( b \), Osada H. [29], Liskevich-Zhang [23], Zhang Qi S. [36] give bounds for the fundamental solution of (1.11). Readers can refer to their papers and their References for more information. Here we add a singular term \( 2r \frac{\partial}{\partial r} \) in the equation and give an upper bound to the fundamental solution. We hope our estimate can not only be applied to the axis-symmetric Navier-Stokes equations, but also to other related incompressible fluid fields.

We now recall some regularity results on the axis-symmetric Navier-Stokes equations. In the presence of swirl, from the partial regularity theory of [2], any singular points of the axis-symmetric suitable weak solution of (1.11) can only lie on the \( z \) axis. In [1], Burke-Zhang give a priori bounds for the vorticity of axially symmetric solutions which indicates that the result of [2] can be applied to a large class of weak solutions. Chan-Vasseur in [34] give a logarithmically improved Serrin criterion for global regularity to solutions of Navier-Stokes equations. See also an extension in Zhou-Lei [35]. Neustupa and Pokorny[6] proved certain regularity of one component(either \( v^0 \) or \( v^r \)) imply regularity of the other components of the solutions. Chae-Lee[11] proved regularity assuming a zero-dimensional integral norm on \( w^\theta \): \( w^\theta \in L^1_t L^q_x \) with \( 3/q + 2/s = 2 \). Also regularity results come from the work of Jiu-Xin [12] under the assumption that another zero-dimensional scaled norms \( \int_Q R^{-1}|w^\theta|^2 + R^{-3}|v^\theta|^2)dz \) is sufficiently small for \( R > 0 \) is small enough. On the other hand, Lei-Zhang[19] give a structure of singularity of 3D axis-symmetric equation near maximum point. Tian-Xin[32] constructed a family of singular axi-symmetric solutions with singular initial datas. Recently, Hou-Li[14] construct a special class of global smooth solutions. See also a recent extension: Hou-Lei-Li[13].

The paper is organized as follows: In section 2, we establish a local maximum estimate using a Moser’s iteration. Based on the local maximal estimate, In section 3, we obtain the continuity of \( \Gamma \) and prove Theorem1.2 by Nash’s method. In section 4, we prove Theorem1.1. The argument is based on [5]. In section 5, we give the proof of Theorem 1.3.

## 2 Local Maximum Estimate

In this section, Using Moser’s iteration, we prove a local maximum estimate of \( \Gamma \) which will be used to obtain continuity of \( \Gamma \) in the next section. The main idea is to exploit the divergence-free property of \( b(x, t) \) and a special cut-off function. We learn from Lei-Zhang[18] and [4] where the authors treated the term \( 2\frac{\partial}{\partial r} \Gamma \) and \( b \cdot \nabla \Gamma \).

We first derive a parabolic De Giorgi type energy estimates of (1.3). Set \( \frac{1}{2} \leq \sigma_2 < \sigma_1 \leq 1 \).
and consider a test function \( \psi(y, s) = \phi(|y|)\eta(s) \) satisfying
\[
\begin{aligned}
\text{supp } \phi &\subset B(\sigma_1), \phi = 1 \text{ in } B(\sigma_2), 0 \leq \phi \leq 1; \\
\text{supp } \eta &\subset (-\sigma_1^2, 0], \eta = 1 \text{ in } (-\sigma_2^2, 0], 0 \leq \eta \leq 1; \\
|\eta'| &\lesssim \frac{1}{(\sigma_1 - \sigma_2)^2}, \left| \frac{\nabla \phi}{\sqrt{\phi}} \right| \leq \frac{1}{\sigma_1 - \sigma_2}.
\end{aligned}
\tag{2.1}
\]

We will also use the following notations to denote our domains. Let \( R > 0 \) and \( R \in (0, 1) \). We write \( B_R = B(0, R) \) and
\[
P(R) = B_R \times (-R^2, 0], \quad P(R_1, R_2) = B_{R_1}/B_{R_2} \times (-R_2^2, 0] \text{ for } R_1 > R_2.
\]

Consider the function \( u = |\Gamma|^q, q > 1 \) and the test function \( \psi_R(y, s) = \phi_R(|y|)\eta_R(s) = \phi(\frac{y}{R})\eta(\frac{s}{\sqrt{R}}) \). Testing (1.3) by \( q|\Gamma|^{2q-2}\Gamma \psi^2_R \) gives
\[
\frac{1}{2} \int \int (\partial_s u^2 + b \cdot \nabla u^2 + \frac{2}{r} \partial_r u^2) \psi^2_R = q \int \int \Delta \Gamma |\Gamma|^{2q-2} \Gamma \psi^2_R. \tag{2.2}
\]

Using Cauchy-Schwartz’s inequality and integration by parts, we compute
\[
q \int \int \Delta \Gamma |\Gamma|^{2q-2} \Gamma \psi^2_R = q \int \int |\nabla \Gamma|^q \Gamma^{2q-2} \psi_R^2 + |\nabla \Gamma||\Gamma|^{2q-1} \nabla \psi_R^2
\]
\[
= -q \int \int \frac{2q-1}{q} |\nabla \Gamma|^q \psi_R^2 + 2 \psi_R |\nabla \psi_R|^q |\nabla \Gamma|^q
\]
\[
= -q \int \int \frac{2q-1}{q} |\nabla u|^q \psi_R^2 + 2 \psi_R u \nabla \psi_R \cdot \nabla u
\]
\[
= -q \int \int \frac{2q-1}{q} |\nabla (u \psi_R)|^2 + (2 - \frac{1}{q}) u \nabla \psi_R \cdot \nabla (u \psi_R) - \frac{1}{q} u^2 |\nabla \psi_R|^2
\]
\[
\lesssim \int \int |\nabla (u \psi_R)|^2 + \int \int u^2 |\nabla \psi_R|^2, \tag{2.3}
\]

and
\[
\frac{1}{2} \int \int \partial_s u^2 \psi_R^2 = \frac{1}{2} \int_{B(\sigma_1 R)} u^2 (\cdot, t) \psi_R^2 - \frac{1}{2} \int \int u^2 \partial_s \psi_R^2. \tag{2.4}
\]

Moreover, by the fact that \( \Gamma = 0 \) on the axis \( r = 0 \), we have
\[
- \frac{1}{r} \int \int \partial_r u^2 \psi_R^2 = -2\pi \int_{-\infty}^{+\infty} \int_{0}^{+\infty} \partial_r u^2 \psi_R^2 drdz.
\]
Consequently, using (2.2) and combining (2.3), (2.4) and (2.5), we get
\[
\int_{B(\sigma, R)} u^2(\cdot, t) \psi_R^2 + \int \int |\nabla(u \psi_R)|^2 \\
\lesssim \int \int u^2(|\nabla \psi_R|^2 + |\nabla^2 \psi_R| + u \partial_r \psi_R (\partial_r (u \psi_R) - u \partial_r \psi_R)) \\
\lesssim \int \int u^2(|\nabla \psi_R|^2 + |\nabla^2 \psi_R| + \frac{1}{4} \int \int |\nabla(u \psi_R)|^2. \tag{2.5}
\]

Consequently, using (2.2) and combining (2.3), (2.4) and (2.5), we get
\[
\int \int_{B(\sigma, R)} u^2(\cdot, t) \psi_R^2 + \int \int |\nabla(u \psi_R)|^2 \\
\lesssim \int \int u^2(|\nabla \psi_R|^2 + |\nabla^2 \psi_R| + |\partial_x \psi_R^2|) - \frac{1}{2} \int \int b \cdot \nabla u^2 \psi_R^2 \\
\lesssim \frac{1}{(\sigma_1 - \sigma_2)^2 R^2} \int \int_{P(\sigma, R)} u^2 - \frac{1}{2} \int \int b \cdot \nabla u^2 \psi_R^2. \tag{2.6}
\]

By the divergence-free property of the drift term \(b\) and using (2.1), we have
\[
\int \int (\psi_R u)^{\frac{3}{2} - \frac{2}{p}} b \cdot \nabla \psi_R \\
= \int \int \psi_R b \cdot \nabla \psi_R \\
= \int \int (\psi_R u)^{\frac{3}{2} - \frac{2}{p}} b \cdot \nabla \psi_R \\
\lesssim \frac{1}{(\sigma_1 - \sigma_2)^2 R} \int \int_{P(\sigma, R)} |b| (\psi_R u)^{\frac{3}{2}} u^{\frac{1}{2}} \\
\leq \frac{1}{(\sigma_1 - \sigma_2)^2 R} \int \left[ \left( \int |b|^p dy \right)^{\frac{1}{p}} \left( \int (\psi_R u)^6 dy \right)^{\frac{1}{3}} \left( \int u^{\frac{2p}{3p - 4}} dy \right)^{\frac{3p - 4}{4p}} \right] ds \\
\leq \frac{\|b\|_{L^p(B\sigma,\sigma_2, R)}^p}{(\sigma_1 - \sigma_2)^2 R} \left( \int \int |\nabla(\psi_R u)|^2 dy ds \right)^{\frac{3p - 4}{4p}} \left( \int \int u^{\frac{2p}{3p - 4}} dy ds \right)^{\frac{3p - 4}{4p}} \left( \int |\nabla(\psi_R u)|^2 ds \right)^{\frac{2 - p}{2p}} \\
\leq \frac{\left( (\sigma_1 - \sigma_2) R \right)^{2 - \frac{2p}{3p - 4}} \|b\|_{L^p(B\sigma,\sigma_2, R)}^p}{(\sigma_1 - \sigma_2)^2 R} \left( \int \int |\nabla(\psi_R u)|^2 dy ds \right)^{\frac{3p - 4}{4p}} \left( \int \int u^{\frac{2p}{3p - 4}} dy ds \right)^{\frac{3p - 4}{4p}} \frac{1}{2} \int \int |\nabla(\psi_R u)|^2 ds \\
\leq \frac{1}{[(\sigma_1 - \sigma_2) R]^{2 - \frac{2p}{p}}} \left[ \left( \int \int |\nabla(\psi_R u)|^2 dy ds \right)^{\frac{3p - 4}{p}} + \frac{1}{2} \int \int |\nabla(\psi_R u)|^2 dy ds. \tag{2.7}
\]
Lemma 2.1 Suppose we get the estimate Combining (2.6) and (2.7), using the Cauchy-Schwartz inequality, we get

\begin{equation}
\int_{B(\sigma_1 R)} u^2(\cdot, t) \psi_R^2 + \int_{\sigma_1 R} |\nabla (u \psi_R)|^2 \\
\lesssim \frac{1}{(\sigma_1 - \sigma_2)^2 R^2} \int_{\sigma_1 R} u^2 dyds + \frac{E_R^4(b)}{[(\sigma_1 - \sigma_2) R]^{12 - \frac{20}{p}}} \left( \int_{\sigma_1 R} u^{2p} dyds \right)^{\frac{3p-4}{p}} \\
\lesssim \frac{1}{(\sigma_1 - \sigma_2)^2 R^2} \left( \int_{\sigma_1 R} u^{\frac{2p}{3p-4}} dyds \right)^{\frac{3p-4}{p}} \left( \int_{\sigma_1 R, \sigma_2 R} u^{\frac{2p}{3p-4}} dyds \right)^{\frac{4-2p}{p}} \\
\lesssim \frac{1 + E_R^4(b)}{[(\sigma_1 - \sigma_2) R]^{12 - \frac{20}{p}}} \left( \int_{\sigma_1 R} u^{\frac{2p}{3p-4}} dyds \right)^{\frac{3p-4}{p}}.
\end{equation}

At last, we get the estimate

\begin{equation}
\sup_{-\sigma_1^2 R^2 \leq t \leq 0} \int_{B(\sigma_1 R)} u^2(\cdot, t) \psi_R^2 + \int_{\sigma_1 R} |\nabla (u \psi_R)|^2 \\
\lesssim (1 + E_R(b))^4 \left( \int_{\sigma_1 R} u^{\frac{2p}{3p-4}} dyds \right)^{\frac{3p-4}{p}}.
\end{equation}

Our next step is to derive a mean value inequality based on (2.8) using Moser’s iteration.

Lemma 2.1 Suppose \( u \) satisfies (2.8) for \( p \in \left( \frac{5}{3}, 2 \right) \), then for \( 0 < R \leq 1 \), there is the estimate

\begin{equation}
\sup_{P(\frac{R}{4})} \Gamma \lesssim (1 + E_R(b))^\frac{3p}{2(3p-5)} \left( \int_{P(R)} \frac{1}{R^2} \Gamma^2 \right)^{\frac{1}{2}}.
\end{equation}

Proof. By Hölder inequality and Sobolev imbedding theorem, we have

\begin{align*}
\int_{\sigma_1 R} u^{\frac{10}{3}} \\
\lesssim \int \left( \|u \psi_R(\cdot, s)\|_{L^2(B(\sigma_1 R))}^{\frac{4}{7}} \|\nabla (u \psi_R)\|_{L^2(B(\sigma_1 R))}^{\frac{2}{7}} \right) ds \\
\lesssim \sup_{-\sigma_1 R^2 \leq t \leq 0} \|u \psi_R(\cdot, t)\|_{L^2(B(\sigma_1 R))}^{\frac{4}{7}} \|\nabla (u \psi_R)\|_{L^2(B(\sigma_1 R))}^{\frac{2}{7}}.
\end{align*}

Using (2.1) and (2.8), we get

\begin{equation}
\int_{\sigma_1 R} u^{\frac{10}{3}} \lesssim \left( \frac{(1 + E_R(b))^4}{[(\sigma_1 - \sigma_2) R]^{12 - \frac{20}{p}}} \right)^{\frac{3}{10}} \left( \int_{P(\sigma_1 R)} u^{\frac{2p}{3p-4}} \right)^{\frac{5(3p-4)}{3p}}.
\end{equation}
Remember \( u = \Gamma^q \), then we obtain
\[
\int \int_{P(\sigma_2 R)} \left( \frac{2p}{3p-4} \right)^{\frac{3p-4}{3p}} \Gamma^{q} \frac{\partial}{\partial t} \frac{1}{\Gamma^{2(\frac{1}{3p-4})}} \left( \frac{(1 + E_R(b))}{[\frac{(\sigma_1 - \sigma_2) R]}^{12-\frac{20}{p}}} \right)^{\frac{p}{3}} \left( \int \int_{P(\sigma_1 R)} \Gamma^{2(\frac{1}{3p-4})} \frac{\partial}{\partial t} \right)^{\frac{1}{3p}} \cdot (2.10)
\]

For convenience of computation, we denote \( \kappa = \frac{p}{3p-4} \), then (2.10) is
\[
\int \int_{P(\sigma_2 R)} \left( \Gamma^{2q} \right)^{\frac{\kappa-1}{\kappa}} dyds
\]
\[
\leq \left\{ \frac{(1 + E_R(b))}{[\frac{(\sigma_1 - \sigma_2) R]}^{12-\frac{20}{p}}} \right\}^{\frac{p}{3}} \left( \int \int_{P(\sigma_1 R)} \Gamma^{2q} dyds \right)^{\frac{\kappa-1}{\kappa}} \cdot (2.11)
\]

For integer \( j \geq 0 \) and a constant \( \sigma = \frac{1}{2} \), set \( \sigma_2 = \frac{1}{2}(1 + \sigma^{j+1}) \) and \( \sigma_1 = \frac{1}{2}(1 + \sigma^j) \). Let \( q = (\frac{2}{3}\kappa)^j \), then we get
\[
\left( \int \int_{P(\frac{R}{2}(1+\sigma^{j+1}))} \Gamma^{2q} \left( \frac{\kappa-1}{\kappa} \right)^{j+1} dyds \right)^{\frac{1}{2\kappa}} \leq (1 + E_R(b))^{\frac{20}{12p-20}} \frac{1}{2\kappa} \left( \frac{\sigma^{j+1}}{2\kappa} \right)^{j+1} \times \left( \int \int_{P(\frac{R}{2}(1+\sigma^j))} \Gamma^{2q} \left( \frac{\kappa-1}{\kappa} \right)^{j} dyds \right)^{\frac{1}{2\kappa}} \cdot (2.10)
\]

By iterating \( j \), the above inequality gives
\[
\left( \int \int_{P(\frac{R}{2}(1+\sigma^{j+1}))} \Gamma^{2q} \left( \frac{\kappa-1}{\kappa} \right)^{j+1} dyds \right)^{\frac{1}{2\kappa}} \leq (1 + E_R(b))^{\frac{15p-20}{6p-10}} \frac{1}{2\kappa} \sum_{j=0}^{(1+\sigma)^{j+1}} \frac{1}{2\kappa} \left( \frac{\sigma^{j+1}}{2\kappa} \right)^{j+1} \times \left( \int \int_{P(\frac{R}{2}(1+\sigma^j))} \Gamma^{2q} dyds \right)^{\frac{1}{2\kappa}} \cdot (2.11)
\]

Note that \( \frac{2}{3}\kappa \in \left[ \frac{5}{6}, 1 \right) \) when we assume \( \kappa = \left( \frac{5}{6}, 2 \right) \). So all the sums on the above are convergent, let \( j \to \infty \) yield that
\[
\sup_{(x,t) \in P(\frac{R}{2})} |\Gamma| \leq (1 + E_R(b))^{\frac{10p-20}{6p-10}} \left( \int \int_{P(R)} \Gamma^{2q} dyds \right)^{\frac{1}{2\kappa}}
\]
\[ (1 + E_R(b)) \frac{15p - 20}{6p - 10} \left( \int \int_{P(R)} \frac{1}{R^5} \Gamma^{2\nu} dy ds \right)^{\frac{1}{2\nu}} \]
\[ \lesssim (1 + E_R(b)) \frac{15p - 20}{6p - 10} \left( \int \int_{P(R)} \frac{1}{R^5} \Gamma^{\frac{2p}{3p - 4}} dy ds \right)^{\frac{3p - 4}{2p}}. \quad (2.12) \]

Next we use (2.12) and an algebraic trick to improve our estimate (2.12). This is from Li-Schoen [21]. From the process of proving (2.12), we have for \( \gamma \in (0, \frac{1}{2}] \), \( \theta \in [\frac{1}{2}, 1 - \gamma] \)

\[
\sup_{P(\theta R)} \Gamma^{\frac{2p}{3p - 4}} \lesssim (1 + E_R(b)) \frac{5p}{3p - q} K \left( \sup_{P((\theta + \gamma) R)} \Gamma^{\frac{2p}{3p - q}} \right)^{1 - \frac{3p - 4}{p}}.
\]

Define \( M(\theta) = \sup_{P(\theta R)} \Gamma^{\frac{2p}{3p - 4}} \), then we yield that

\[
M(\theta) \lesssim (1 + E_R(b)) \frac{5p}{3p - q} KM(\theta + \gamma) K^\lambda,
\]
where \( \lambda = 1 - \frac{3p - 4}{p} \). Choosing \( \theta_0 = \frac{1}{2}, \theta_i = \theta_{i-1} + \frac{1}{2^i + 1} \) and \( \gamma = \frac{1}{2^i + 1} \), then we get

\[
M(\theta_0) \lesssim \left[ K(1 + E_R(b)) \frac{5p}{3p - q} \right] \sum_{i=1}^{j} \lambda^{i-1} M(\theta_j) \lambda^i.
\]

For \( \lambda < 1 \), letting \( j \to \infty \), then we have

\[
M(\theta_0) \lesssim \left[ K(1 + E_R(b)) \frac{5p}{3p - q} \right]^{\frac{p}{3p - 4}}.
\]

That is

\[
\sup_{P(\frac{1}{2} R)} \Gamma^{\frac{2p}{3p - 4}} \lesssim \left[ K(1 + E_R(b)) \frac{5p}{3p - q} \right]^{\frac{p}{3p - 4}}.
\]

So

\[
\sup_{P(\frac{1}{2} R)} |\Gamma| \lesssim (1 + E_R(b))^{\frac{5p}{2(3p - 4)}} K^{\frac{1}{2}}
\]
\[
\lesssim (1 + E_R(b))^{\frac{5p}{2(3p - 4)}} \left( \int \int_{P(R)} \frac{1}{R^5} \Gamma^2 dy ds \right)^{\frac{1}{2}}.
\]

This proves our Lemma.
3 Proof of Theorem 1.2

In this section we study the continuity of \( \Gamma \) using the local maximum estimates (2.9) and Nash type method for parabolic equations. First let us introduce some notations.

For \( 0 < R \leq 1 \), we define

\[
m_R = \inf_{P(R)} \Gamma, \quad M_R = \sup_{P(R)} \Gamma, \quad J_R = M_R - m_R.
\]

Define

\[
u = \begin{cases}
\frac{2(M_R - \Gamma)}{J_R} & \text{if } M_R > -m_R, \\
\frac{2(\Gamma - m_R)}{J_R} & \text{else},
\end{cases}
\] (3.1)

hence

\[
0 \leq \nu \leq 2, \quad a \triangleq \nu|_{r=0} \geq 1.
\] (3.2)

Lower bound on \( \|u\|_{L^q} \).

We give a lemma to state that there is a lower bound on \( \|u\|_{L^q} \) where \( q \in (0, 1) \). This bound depends on our \( E_R(b) \) norm and will serve as an input for Nash’s argument as we will describe it later on.

**Lemma 3.1** If \( \nu \) is a solution of (1.3) and satisfies (3.2). Then for \( \forall q \in (0, 1) \), we have

\[
\frac{1}{R^2} \frac{\nu}{\nu(P(R))} \gtrsim a(1 + E_R(b))^{-\frac{q}{q'}}.
\] (3.3)

**Proof.** Since the lemma is scaling invariant, we just take \( R = 1 \) in the proof. Let \( \psi(x, t) = \phi(x)\eta(t) \), where \( \phi \in C_0^\infty \) s.t. \( \phi = 1 \) in \( B_{\frac{1}{2}} \) and \( \phi = 0 \) in \( \bar{B}_1^c \). \( \frac{\nabla \phi}{\sqrt{\phi}} \) and \( \nabla(\frac{\phi}{\sqrt{\phi}}) \) are bounded. \( \eta \in C_0^\infty \) s.t. \( \eta = 1 \) in \( [-\frac{7}{8}, -\frac{1}{8}] \) and \( \eta \) is supported in \( (-1, 0) \). Let us test (1.3) by \( qu^{q-1}\psi^2 \), where \( q \in (0, \frac{1}{2}) \). Then we have

\[
\int \int (\partial_t u^q + b \cdot \nabla u^q + \frac{2}{r} \partial_r u^q) \psi^2 dyds = q \int \int \Delta u u^{q-1} \psi^2 dyds.
\] (3.4)

Similarly as in [18], we have

\[- \int \int \frac{2}{r} \partial_r u^q \psi^2 dyds = - \int \int 2 \partial_r u^q \psi^2 drdzds
\]

\[= \int_{-1}^{0} \int 2u^q \psi^2 |_{r=0} dzds + \int \int \frac{4}{r} u^q \psi \partial_r \psi dyds
\]

\[\geq -C \int \int u^q dyds + \frac{3}{2} a^q.
\] (3.5)
Here we note that $\frac{\partial_r \psi}{r} = \frac{\partial_\rho \psi}{\rho}$ for $\phi$ is a radial function. Because $\phi = 1$ near $\rho = 0$, $\frac{\partial_r \psi}{r}$ has no singularity.

Moreover

$$
\int \int (-\partial_s u^q + q\Delta u u^{q-1})\psi^2 dyds
= \int \int 2u^q[\psi_\rho \psi + |\nabla \psi|^2 - \frac{q-2}{q}\psi \Delta \psi]dyds - \frac{4(q-1)}{q} \int \int |\nabla (u^q \psi)|^2 dyds
\geq -C \int \int u^q dyds - \frac{4(q-1)}{q} \int \int |\nabla (u^q \psi)|^2 dyds.
$$

(3.6)

For the term involving $b$, we compute the same as (2.7)

$$
-\int \int b \cdot \nabla u^q \psi^2 dyds
\geq -CE_R^4(b) \left( \int \int u^{\frac{q}{2q-4}} dyds \right)^{\frac{3p-4}{p}} - \frac{4(q-1)}{q} \int \int |\nabla (u^q \psi)|^2.
$$

(3.7)

Combining (3.4), (3.5), (3.6) and (3.7), we derive

$$
\int \int u^q dyds + E_R^4(b) \left( \int \int u^{\frac{q}{2q-4}} dyds \right)^{\frac{3p-4}{p}} \gtrsim a^q.
$$

Using Hölder inequality, we have

$$
\left( \int \int u^{2q} dyds \right)^{\frac{1}{2}} + E_R^4(b) \left( \int \int u^{2q} dyds \right)^{\frac{1}{2}} \gtrsim a^q,
$$

then we have

$$
(1 + E_R^4(b))^{\frac{1}{2}} \left( \int \int u^{2q} \right)^{\frac{1}{2q}} \gtrsim a.
$$

So

$$
\left( \int \int u^{2q} dyds \right)^{\frac{1}{2q}} \gtrsim a(1 + E_R(b))^{-\frac{1}{q}}.
$$

This proves our lemma, since $q \in (0, \frac{1}{2})$ is arbitrary.

**Nash’s lower bound**

Before proving the Nash’s lower bound estimates, we recall a Nash inequality, whose proof can be found in [5].

**Lemma 3.2** Let $M \geq 1$ be a constant and $\mu$ be a probability measure. Then for all $0 \leq f \leq M$, there holds

$$
|\ln \int f d\mu - \int \ln f d\mu| \leq \frac{M \|g\|_{L^2}}{\int f d\mu}
$$

where $g = \ln f - \int \ln f d\mu$. 
Now we come to prove Nash’s lower bound estimate. We define a Lipschitz continuous cut-off function such that
\[ \zeta = 1 \text{ in } B(\frac{1}{2}), \zeta = 0 \text{ in } B(1)^c, \int_{\mathbb{R}^3} \zeta^2 dx = 1. \]

In fact we take
\[ \zeta = \begin{cases} 
1 & \text{in } B(\frac{1}{2}); \\
2(1 - |x|) & \text{in } B(1)/B(\frac{1}{2}). 
\end{cases} \tag{3.8} \]

where \( c \) is a constant to ensure \( \int_{\mathbb{R}^3} \zeta^2 dx = 1 \).

Lemma 3.3 Let \( 0 \leq u \leq 2 \) be a solution of (1.3) in \( P(R) \) which is assumed to satisfy
\[ \|u\|_{L^1(P(\frac{1}{2}))} \geq c_1(1 + E_R(b))^{-8} R^5. \tag{3.9} \]

Moreover, we assume that \( u|_{r=0} \) is a constant bigger than 1, then there exists a \( \tau > 0 \) such that
\[ -\int_{\mathbb{R}^3} \ln u \zeta_R^2 dx \lesssim (1 + E_R(b))^{24}, \quad \text{for } -\tau R^2 \leq t < 0. \]

Proof. First, let us define \( u_R(x, t) = u(Rx, R^2t), \ b_R(x, t) = Rb(Rx, R^2t). \) It is clear that \( u_R(x, t) \) solves the equation
\[ \partial_t u_R + b_R \cdot \nabla u_R + \frac{2}{r} \partial_r u_R = \Delta u_R \quad \text{in } P(1) \]

and \( 0 \leq u_R \leq 2, \|u_R\|_{L^1(P(\frac{1}{2}))} \geq c_1(1 + E_1(b_R))^{-8} \). The estimate we are going to get is
\[ -\int_{\mathbb{R}^3} \ln u_R \zeta_R^2 dx \lesssim (1 + E_1(b_R))^{24}, \quad \text{for } -\tau \leq t < 0. \]

For convenience, we shall drop all \( R \) and the subscript from now on and set \( R = 1 \).

Also denote
\[ E = E_1(b_R). \]

Let \( v = -\ln u \). It is easy to see that \( v \) solves the equation
\[ \partial_s v + b \cdot v + \frac{2}{r} \partial_r v - \Delta v + |\nabla v|^2 = 0. \tag{3.10} \]

Testing (3.10) by \( \zeta^2 \), we have
\[ \int \partial_s v \zeta^2 dx + \int |\nabla v|^2 \zeta^2 dx = \int \Delta v - \frac{2}{r} \partial_r v - b \cdot v \zeta^2 dx. \tag{3.11} \]

Using the Cauchy-Schwartz inequality and integration by parts, we have
\[ \int \Delta v \zeta^2 dx = -2 \int \nabla v \cdot \nabla \zeta \zeta dx \]
\[
\begin{align*}
\leq \frac{1}{4} \int |\nabla v|^2 \zeta^2 + 4 \int |\nabla \zeta|^2 dx \\
\leq \frac{1}{4} \int |\nabla v|^2 \zeta^2 + C.
\end{align*}
\] (3.12)

Let \( \overline{v}(t) = \int v(\cdot, t) \zeta^2 dx \), by recalling the assumption that \( u|_{r=0} \) is a non-zero constant and the weighted poincaré inequality

\[
\int |v - \overline{v}|^2 \zeta^2 dx \leq C \int |\nabla v|^2 \zeta^2 dx,
\] (3.13)

one can estimate

\[
-\int \frac{2}{r} \partial_r v \zeta^2 = -\int \frac{2}{r} \partial_r (v - \overline{v}) \zeta^2 dx \\
= -4\pi \int_{-\infty}^{+\infty} \int_{0}^{+\infty} \partial_r (v - \overline{v}) \zeta^2 r dr dz \\
= -4\pi \int_{-\infty}^{+\infty} \zeta^2 (v - \overline{v})|_{r=0} dz + 4\pi \int_{-\infty}^{+\infty} (v - \overline{v}) 2\zeta \partial_r \zeta r dr dz \\
= 4\pi \int_{-\infty}^{+\infty} v \zeta^2 |_{r=0} dz - 4\pi \overline{v} \int_{-\infty}^{+\infty} \zeta^2 dz + 4 \int (v - \overline{v}) \partial_r \zeta \frac{r \partial_r \zeta}{r} r dr d\theta dz \\
\leq C - C\overline{v}(s) + \frac{1}{4} \int |\nabla v|^2 \zeta^2 dx.
\] (3.14)

Before estimating the term involving \( b \), we need a more general weighted poincaré inequality.

Let \( B_R \) be a ball centered at 0 in \( \mathbb{R}^n \). Let \( 1 \leq r \leq q < \infty \) satisfy \( \frac{1}{r} \geq \frac{1}{q} - \frac{1}{n} \) and \( 1 - \frac{n+2}{r} + \frac{n+2}{q} \leq 0 \), then we have

\[
\left( \frac{1}{|B_R|} \int_{B_R} (v - \overline{v}) q \zeta^2_R \zeta_R^2 dx \right)^{\frac{1}{q}} \leq CR^{1 - \frac{2}{q} + \frac{2}{n}} \left( \frac{1}{|B_R|} \int_{B_R} |\nabla v|^r \zeta_R^2 dx \right)^{\frac{1}{r}},
\] (3.15)

here \( |B_R| \) means the Lebesgue measure of the ball \( B_R \) and \( C \) depends only on \( q, r, n \). One can see [9] for its proof. Hence, due to the divergence-free of \( b \) and Hölder inequality, we have

\[
-\int (b \cdot \nabla) v \zeta^2 dx = \int 2\zeta (v - \overline{v}) b \cdot \nabla \zeta dx \\
\leq \left( \int |b|^p |\nabla \zeta|^p dx \right)^{\frac{1}{p}} \left( \int (v - \overline{v}) q \zeta^q dx \right)^{\frac{1}{q}} \\
\leq \left( \int |b|^p |\nabla \zeta|^p dx \right)^{\frac{1}{p}} \left( \int (v - \overline{v}) q \zeta^2 dx \right)^{\frac{1}{q}},
\]

here \( \frac{1}{p} + \frac{1}{q} = 1 \).
In (3.15), let $R = 1$, $r = 2$, $n = 3$. When $p \in (\frac{5}{3}, 2]$, $q = \frac{p}{p-1}$ can satisfy $\frac{1}{q} \geq \frac{1}{r} - \frac{1}{n}$ and $1 - \frac{n+2}{r} + \frac{n+2}{q} \leq 0$. So

$$\left(\int (v - \overline{v})^q \zeta^2 dx\right)^{\frac{1}{q}} \leq C \left(\int |\nabla v|^2 \zeta^2 dx\right)^{\frac{1}{2}}.$$ 

Hence

$$-\int (b \cdot \nabla) v \zeta^2 dx \leq C \left(\int |b|^p |\nabla \zeta|^p dx\right)^{\frac{1}{p}} \left(\int |\nabla v|^2 \zeta^2 dx\right)^{\frac{1}{2}} \leq C \left(\int_{B(1)} |b|^p dx\right)^{\frac{1}{p}} + \frac{1}{4} \int |\nabla v|^2 \zeta^2 dx \leq C E^2 + \frac{1}{4} \int |\nabla v|^2 \zeta^2 dx. \quad (3.16)$$

Combining (3.11), (3.12), (3.14) and (3.16), we have

$$\partial_s \int v \zeta^2 dx + C \varpi(s) \leq -\frac{1}{4} \int |\nabla v|^2 \zeta^2 dx + (1 + E)^2.$$ 

Now we apply the Nash inequality, taking $f = u$, $d\mu = \zeta^2 dx$ in Lemma 3.2, one has

$$|\ln \int u \zeta^2 dx + \int v \zeta^2 dx|^2 \left(\int u \zeta^2 dx\right)^2 \leq M^2 \int |v + \int v \zeta^2 dy|^2 \zeta^2 dx,$$

here $M = 2$ is the upper bound of $u$. Using the weighted Poincaré inequality (3.13) once again, we have

$$|\ln \int u \zeta^2 dx + \int v \zeta^2 dx|^2 \left(\int u \zeta^2 dx\right)^2 \leq C \int |\nabla v|^2 \zeta^2 dx.$$

Then finally we obtain

$$\partial_s \varpi(s) + C_0 \varpi(s) \leq (1 + E)^2 - \frac{1}{4C} |\ln \int u \zeta^2 dx + \varpi(s)|^2 \left(\int u \zeta^2 dx\right)^2.$$ 

Recalling (3.9)

$$\|u\|_{L^1(P(\frac{1}{2}))} \geq c_1 (1 + E)^{-8}.$$ 

Let $\chi$ be the characteristic function of the non-empty set

$$W = \left\{ s \in [-\frac{1}{4}, 0] : \|u\|_{L^1(B(\frac{1}{2}))} \geq \frac{c_1 (1 + E)^{-8}}{10} \right\}.$$ 

We assert that $|W| \geq \frac{c_1(1+E)^{-8}}{20}$. In fact, if $|W| \leq \frac{c_1(1+E)^{-8}}{20}$, then

$$\|u\|_{L^1(P(\frac{1}{2}))} < \int_W 2 |B(\frac{1}{2})| ds + \int_{W^c} \frac{c_1(1+E)^{-8}}{10} ds.$$
\[
\frac{\pi}{3} |W| + \frac{c_1(1 + E)^{-8}}{40} < \left(\frac{\pi}{60} + \frac{1}{40}\right)c_1(1 + E)^{-8} < c_1(1 + E)^{-8},
\]
this is a contradiction with (3.9). Thus, one has
\[
\partial_s \overline{v}(s) + C_0 \overline{v}(t) \leq (1 + E)^2 - \frac{1}{4C} \chi(s) |\ln \int u\zeta^2 dx + \overline{v}(s)|^2 \left(\int u\zeta^2 dx\right)^2
\leq (1 + E)^2 - \frac{1}{4C} \chi(s) |\ln \int u\zeta^2 dx + \overline{v}(s)|^2 \frac{c_1^2(1 + E)^{-16}}{100}. \quad (3.17)
\]
The last inequality is due to
\[
\left(\int u\zeta^2 dx\right)^2 \geq \left(\int_{B(\frac{1}{2})} u dx\right)^2 \geq \frac{c_1^2(1 + E)^{-16}}{100},
\]
when \(s \in W\).

From (3.17), we first have \(\partial_s \overline{v}(s) + C_0 \overline{v}(s) \leq (1 + E)^2\). This gives, for \(-\frac{1}{4} \leq s_1 \leq s_2 \leq 0\),
\[
\overline{v}(s_2) \leq e^{C_0(s_1-s_2)}\overline{v}(s_1) + e^{-s_2} \int_{s_1}^{s_2} (1 + E)^2 ds
\leq e^{C_0|s_1-s_2|}\overline{v}(s_1) + C(1 + E)^2. \quad (3.18)
\]
Now we consider two cases.

Case one: if there exists some \(s_0 \in [-\frac{1}{4}, -\frac{c_1}{40}]\), such that
\[
\overline{v}(s_0) \leq \frac{2}{C_0} (1 + E)^2 + 4|\ln \frac{10}{c_1(1 + E)^{-8}}|.
\]
Then for \(s \in (s_0, 0)\), from (3.18), we have
\[
\overline{v}(s) \lesssim \overline{v}(s_0) + (1 + E)^2
\lesssim (1 + E)^2 + \ln \frac{10}{c_1} + \ln(1 + E)
\lesssim (1 + E)^2.
\]
Choosing \(\tau = s_0\), this completes the proof of the Lemma.

Case two: if for any \(s \in [-\frac{1}{4}, -\frac{c_1}{40}]\),
\[
\overline{v}(s) \geq \frac{2}{C_0} (1 + E)^2 + 4|\ln \frac{10}{c_1(1 + E)^{-8}}|.
\]
Then when \(s \in W \cap [-\frac{1}{4}, -\frac{c_1}{40}]\),
\[
\ln \int u\zeta^2 dx \geq \ln \int_{B(\frac{1}{2})} u\zeta^2 \geq \ln \frac{c_1(1 + E)^{-8}}{10}.
\]
So we have
\[ \varpi(s) + \ln \int u \zeta^2 dx \geq C \varpi(s). \]

From (3.17), we have
\[ \partial_s \varpi(s) + C_0 \varpi(s) \lesssim -C \chi(s) \frac{c_1^2 (1 + E)^{-16}}{100} \varpi(s). \]

Then integrating the above inequality from \([-\frac{1}{4}, -\frac{c_1}{40}]\), one gets
\[ \varpi(-\frac{1}{4})^{-1} - \varpi(-\frac{c_1}{40})^{-1} \lesssim -c_1^2 (1 + E)^{-16} \int \chi ds \]
\[ \lesssim -c_1^2 (1 + E)^{-16} |W| \]
\[ \lesssim -c_1^3 (1 + E)^{-24}. \]

Since \( \varpi(-\frac{1}{4}) \geq 0 \) in this case, we have
\[ \varpi(-\frac{c_1}{40}) \lesssim (1 + E)^{24} c_1^3. \]

Then we use (3.18), for \( s \in [-\frac{c_1}{40}, 0] \),
\[ \varpi(s) \lesssim (1 + E)^2 + (1 + E)^{24} \lesssim (1 + E)^{24}. \]

So we can take \( \tau = \frac{c_1}{40} \), this proves the lemma.

As a corollary of Lemma 3.3, we derive a lower bound of positive solution of (1.3).

**Corollary 3.1** Let \( u, \tau \) be given in Lemma 3.3 and \( E_R(b) \) satisfies the assumption (1.8). Then there exists a \( \delta \in (0, 1) \), depending only on \( R \), such that
\[ \inf_{P(\frac{\tau^2}{3} R)} u \geq \frac{1}{2} \delta(R). \] (3.19)

In fact, we take \( \delta = | \ln \frac{R}{3} |^{-1} \).

**Proof.** Using Lemma (3.3), one has
\[ (1 + E_R(b))^{24} \gtrsim - \int \zeta_R^2(x) \ln u(x,t) dx \]
\[ = - \int_{\delta < u \leq 1} \zeta_R^2(x) \ln u dx - \int_{u \leq \delta} \zeta_R^2(x) \ln u dx - \int_{1 < u \leq 2} \zeta_R^2(x) \ln u dx \]
\[ \geq - \int_{u \leq \delta} \zeta_R^2(x) \ln u dx - \ln 2 \int_{1 < u \leq 2} \zeta_R^2(x) dx \]
\[ \geq - \int_{u \leq \delta} \zeta_R(x) \ln u \, dx - \ln 2. \]

This implies that
\[ - \int_{u \leq \delta} \zeta_R(x) \ln u \, dx \lesssim (1 + E_R(b))^{24}. \]

For \( t \in [-\tau R^2, 0] \), consequently, one has
\[ \left| \left\{ x \in B(\frac{R}{2}) | u \leq \delta \right\} \right| \lesssim \frac{R^3}{-\ln \delta}(1 + E_R(b))^{24}. \]

Using the mean value inequality (2.9), one has
\[ \sup_{P(\sqrt{\tau} R)} (\delta - u)_+ \lesssim (1 + E_R(b))^{\frac{5p}{2(5p-5)}} \left( \int \int_{P(\sqrt{\tau} R)} \frac{1}{(\sqrt{\tau} R)^5} (\delta - u)^2 \, dy \, ds \right)^{\frac{1}{2}} \]
\[ \lesssim (1 + E_R(b))^{\frac{5p}{2(5p-5)}} \delta \left[ \frac{(\sqrt{\tau} R)^2}{(\sqrt{\tau} R)^5 - \ln \delta} (1 + E_R(b))^{24} \right]^{\frac{1}{2}} \]
\[ \lesssim (1 + E_R(b))^{\frac{5p}{2(5p-5)}} + 12 \frac{\delta}{\sqrt{-\ln \delta}}. \]

This gives
\[ \inf_{P(\sqrt{\tau} R)} u \geq \delta \left[ 1 - C \frac{(1 + E_R(b))^{\frac{5p}{2(5p-5)}} + 12}{\sqrt{-\ln \delta}} \right]. \]

Under the assumption (1.8), one has
\[ \inf_{P(\sqrt{\tau} R)} u \geq \delta \left[ 1 - C \left( \frac{\ln(\ln \frac{3}{R})}{\tau^{\frac{3p-5}{12p-120}}(1-\beta)} \right)^{\frac{12p-120}{12p-10}} \right], \]
when \( R \in (0, 1) \). We can take \( \delta(R) = (\ln \frac{3}{R})^{-1} \) to ensure \( \inf_{P(\sqrt{\tau} R)} u \geq \frac{1}{2} \delta(R) \). This proves the corollary.

**Proof of theorem 1.2**

We define
\[ m_\tau = \inf_{P(\sqrt{\tau} R)} \Gamma, \quad M_\tau = \sup_{P(\sqrt{\tau} R)} \Gamma. \]

Then from (3.1) and (3.19), one has
\[ \frac{1}{2} \delta(R) \leq \inf_{P(\sqrt{\tau} R)} u = \begin{cases} 2(M_R - M_\tau)/J_R & \text{if } M_R > -m_R; \\ 2(m_\tau - m_R)/J_R & \text{else}. \end{cases} \]
We add the two cases together to get that
\[
\delta(R) \leq \frac{4}{J_R} \left\{ J_R - \text{osc}(\Gamma, \frac{\sqrt{\tau}}{2} R) \right\},
\]
here \( \text{osc}(\Gamma, \frac{\sqrt{\tau}}{2} R) = M_r - m_r \) and \( \text{osc}(\Gamma, R) = M_R - m_R = J_R \). So
\[
\text{osc}(\Gamma, \frac{\sqrt{\tau}}{2} R) \leq \left( 1 - \frac{\delta(R)}{4} \right) \text{osc}(\Gamma, R).
\]
We write it also as
\[
J_{\frac{\sqrt{\tau}}{2} R} \leq \left( 1 - \frac{\delta(R)}{4} \right) J_R.
\]
For any small \( R \), there exists some integer \( j \geq 0 \) such that \((\frac{\sqrt{\tau}}{2})^{j+1} < R/3 \leq (\frac{\sqrt{\tau}}{2})^j \). Using the above inequality, an iteration argument gives
\[
J_{R/3} \leq J_{(\frac{\sqrt{\tau}}{2})^j} \leq \prod_{k=0}^{j-1} \left[ 1 - \frac{\delta((\frac{\sqrt{\tau}}{2})^k)}{4} \right] J_1.
\]
Noting that \( \ln(1 - x) \leq -x \) for sufficiently small positive \( x \), one has
\[
J_{R/3} \leq \exp \ln J_{R/3} \\
\leq J_1 \exp \sum_{0}^{j-1} \ln \left[ 1 - \frac{\delta((\frac{\sqrt{\tau}}{2})^k)}{4} \right] \\
\leq J_1 \exp \left\{ -\frac{1}{4} \sum_{0}^{j-1} \delta((\frac{\sqrt{\tau}}{2})^k) \right\} \\
\leq J_1 \exp \left\{ -\frac{1}{4} \sum_{0}^{j-1} \left[ \ln \frac{3}{(\frac{\sqrt{\tau}}{2})^k} \right]^{-1} \right\} \\
\leq J_1 \exp \left\{ -\frac{1}{4} \sum_{0}^{j-1} \left[ \ln 3 + k \ln(\frac{\sqrt{\tau}}{2}) \right]^{-1} \right\} \\
\leq J_1 \exp \left\{ -\frac{1}{4} \sum_{0}^{j-1} \left[ C(k+1) \right]^{-1} \right\} \\
\leq J_1 \exp \left\{ -c_0 \ln j \right\} \\
\leq J_1 j^{-c_0} \\
\leq J_1 |\ln \frac{R}{3}|^{-c_0}.
\]
Since, \( \Gamma|_{r=0} = 0 \), the above estimate proves our Theorem 1.2.
4 Proof of Theorem 1.1

In this section we will prove Theorem 1.1 and get the regularity of the solution under the assumption (1.2). The idea comes from [Chen-Strain-Tsai-Yau]'s proof where they assume $|v| \leq Cr^{-1}$.

We divide the proof into 3 steps.

**Step one: scaling of the solution and set up of a equation**

Let $M$ be the maximum of $|v|$ up to a fixed time $t_0$ and we may assume $M > 1$ is large. Define the scaled solution

$$v^M(X, T) = M^{-1}v\left(\frac{X}{M}, \frac{T}{M^2}\right), \quad X = (X_1, X_2, Z).$$

Denote $x = (x_1, x_2, z)$ and $X = (X_1, X_2, Z)$, $r = \sqrt{x_1^2 + x_2^2}$ and $R = \sqrt{X_1^2 + X_2^1}$. We have the following estimate for $r$ and $R$ for time $t < t_0$ and $T < M^2t_0$:

$$|\nabla^k v^M| \leq C_k.$$  \hspace{1cm} (4.1)

This inequality follows from $\|v^M\|_{L^{\infty}} \leq 1$ for $t < t_0$ and the standard regularity theorem of Navier-Stokes equations. Its angular component (we omit the time dependence below) $v^M_\theta(R, Z)$ satisfies $v^M_\theta(0, Z) = 0 = \partial_Z v^M_\theta(0, Z)$ for all $Z$. By mean value theorem and (4.1),

$$|v^M_\theta(R, Z)| \lesssim R, \quad |\partial_Z v^M_\theta(R, Z)| \lesssim R \quad \text{for } R \leq 1.$$  \hspace{1cm} (4.2)

Together with (4.1) for $R \geq 1$, we get

$$|v^M_\theta| \lesssim \min\{R, 1\}, \quad |\partial_Z v^M_\theta| \lesssim \min\{R, 1\}. \hspace{1cm} (4.2)$$

Due to Theorem 1.2,

$$|\Gamma| = |rv_\theta(r, z)| \lesssim \begin{cases} \left(\ln \frac{3}{r}\right)^{-\alpha_0} & \text{for } r \leq 1; \\ 1 & \text{for } r > 1. \end{cases}$$

That is

$$|v_\theta(r, z)| \lesssim \begin{cases} \left(\ln \frac{3}{r}\right)^{-\alpha_0} & \text{for } r \leq 1; \\ \frac{r}{1} & \text{for } r > 1. \end{cases} \hspace{1cm} (4.3)$$

Then $v^M_\theta(R, Z)$ satisfies the estimate

$$|v^M_\theta(R, Z)| = M^{-1}|v_\theta\left(\frac{X}{M}, \frac{T}{M^2}\right)| \lesssim \begin{cases} \left(\ln \frac{3M}{R}\right)^{-\alpha_0} & \text{for } R \leq M; \\ \frac{1}{R} & \text{for } R > M. \end{cases}$$
Combining this with (4.2), one has

\[
|v^M_{\theta}(R, Z)| \lesssim \begin{cases} 
\min \left\{ R, \frac{(\ln \frac{3M}{R})^{-c_0}}{R} \right\} & \text{for } R \leq 1; \\
\min \left\{ 1, \frac{(\ln \frac{3M}{R})^{-c_0}}{R} \right\} & \text{for } 1 < R < M; \\
\frac{1}{R} & \text{for } R \geq M.
\end{cases}
\] (4.4)

Now consider the angular component of the rescaled vorticity. Recall \( \Omega = \frac{w_{\theta}}{r} \). Let

\[
f = \Omega^M(X, T) = M^{-3} \Omega \left( \frac{X}{M}, \frac{T}{M^2} \right) = M^{-3} w_{\theta}(\frac{X}{M}, \frac{T}{M^2}) \frac{M}{R}
\]

where

\[
w^M_{\theta}(X, T) = w_{\theta}(\frac{X}{M}, \frac{T}{M^2}) M^{-2}.
\]

Note that \( w^M_{\theta} \) and \( \nabla w^M_{\theta} \) are bounded by (4.1) and also \( w^M_{\theta}|_{R=0} = 0 \), so one has

\[
|f| \lesssim \frac{1}{1 + R}.
\]

From the equation (1.6), \( f \) satisfies

\[(\partial_T - L)f = g, \quad L = \Delta + \frac{2}{R} \partial_R - b^M \cdot \nabla,\]

where \( g = R^{-2} \partial_Z(v^M_{\theta})^2 \) and \( b^M = v^M_{R} e_R + v^M_{Z} e_Z, |b^M| \leq 1 \).

Combining the estimates (4.2) and (4.4), one has

\[
g = \frac{2}{R^2} v^M_{\theta} \partial_Z(v^M_{\theta}) \lesssim \begin{cases} 
\min \left\{ 1, \frac{(\ln \frac{3M}{R})^{-c_0}}{R^2} \right\} & \text{for } R \leq 1; \\
\min \left\{ \frac{1}{R^2}, \frac{(\ln \frac{3M}{R})^{-c_0}}{R^3} \right\} & \text{for } 1 < R < M; \\
\frac{1}{R^3} & \text{for } R \geq M.
\end{cases}
\] (4.5)

Let \( P(X, T; Y, S) \) be the kernel of \( \partial_T - L \). By Duhamel’s formula

\[
f(X, T) = \int P(X, T; Y, S)f(Y, S)dY + \int_S^T \int P(X, T; Y, \tau)g(Y, \tau)dYd\tau \]

\[
:= I + II.
\] (4.6)
Step two: bounding of $f$

In the following, we will estimate (4.6) and give a bound for $f(X, T)$.

The kernel $P(X, T; Y, S)$ satisfies $P \geq 0$, $\int P(X, T; Y, S)dY \leq 1$ and

$$P(X, T; Y, S) \leq C(T - S)^{-3/2} \exp \left\{ -C \frac{|X - Y|^2}{T - S} \left( 1 - \frac{T - S}{|X - Y|} \right) \right\}. \quad (4.7)$$

The proof of estimate (4.7) is based on [6], but due to the singularity of the term $\frac{2}{r} \partial_r$, the proof is more involved. For completeness of our paper, we will prove it in Section 5 as Theorem 1.3.

Now we give estimates of $P$ in two cases.

Case one: when $1 - \frac{T - S}{|X - Y|} > \frac{1}{2}$, that is $|X - Y| > 2(T - S)$,

$$\exp \left\{ -\frac{|X - Y|^2}{T - S} \left( 1 - \frac{T - S}{|X - Y|} \right) \right\} \leq \exp \left\{ -\frac{1}{4} \frac{|X - Y|^2}{T - S} \right\}$$

$$\leq \begin{cases} 
\exp \left\{ -\frac{1}{4} \frac{|X - Y|}{T - S} \right\} & \text{for } |X - Y| \geq 1; \\
\exp \left\{ -\frac{1}{4} \frac{|X - Y|^2}{T - S} \right\} & \text{for } |X - Y| < 1.
\end{cases}$$

Case two: when $1 - \frac{T - S}{|X - Y|} \leq \frac{1}{2}$, that is $|X - Y| \leq 2(T - S)$,

$$\exp \left\{ -\frac{|X - Y|^2}{T - S} \left( 1 - \frac{T - S}{|X - Y|} \right) \right\} \leq 1 \leq e^2 \exp \left\{ -\frac{|X - Y|}{T - S} \right\}.$$

With these estimates and Hölder inequality, one gets, for $I$ in (4.6),

$$|I| \leq \left[ \int P(X, T; Y, S)|f(Y, S)|^3dY \right]^{\frac{1}{3}} \left[ \int P(X, T; Y, S)dY \right]^{\frac{2}{3}}$$

$$\leq \left\{ \int_{-\infty}^{+\infty} \int_{0}^{+\infty} (T - S)^{-\frac{1}{2}} \left[ e^{-\frac{|X - Y|}{T - S}} + e^{-\frac{|X - Y|^2}{T - S}} \right] dY \right\}^{1/3}$$

$$\leq \left\{ \int_{-\infty}^{+\infty} \left[ e^{-\frac{|X - Y|}{T - S}} + e^{-\frac{|X - Y|^2}{T - S}} \right] dY \right\}^{1/3}$$

$$\leq \left\{ (T - S)^{-\frac{1}{2}} \left[ (T - S) + (T - S)^{\frac{1}{2}} \right] \right\}^{1/3}$$

$$\leq \left\{ (T - S)^{-\frac{1}{2}} \right\} \quad (4.8)$$

for $T - S \geq 1$, next

$$|II| \leq \int_{T}^{s} (T - \tau)^{-\frac{3}{2}} \left\{ \int_{|X - Y| \leq 2(T - \tau)} e^{-\frac{|X - Y|}{T - \tau}} |g|dY \right\}.$$
\[ + \int_{|X-Y| \geq 2(T-\tau), |X-Y| > 1} e^{-\frac{1}{4} \frac{|X-Y|^1}{T-\tau}} |g| dY + \int_{|X-Y| \geq 2(T-\tau), |X-Y| < 1} e^{-\frac{1}{4} \frac{|X-Y|^2}{T-\tau}} |g| dY \} d\tau \\
\quad := \int_S (T-\tau)^{-\frac{3}{2}} \{ II_1 + II_2 + II_3 \} d\tau. \] (4.9)

We deal with \( II_1, II_2, II_3 \) in (4.9) as follows,

\[ II_1 + II_2 \lesssim \int_{-\infty}^{+\infty} \int_{0}^{+\infty} \left( e^{-\frac{1}{4} \frac{|X-Y|^1}{T-\tau}} + e^{-\frac{1}{4} \frac{|X-Y|^2}{T-\tau}} \right) |g| RdRdY_3 \]
\[ \lesssim (T-\tau) \int_{0}^{+\infty} |g| RdR \]
\[ \lesssim (T-\tau) \left\{ \int_{0}^{1} \min \left\{ R, \frac{(\ln 3M)^{1-c_0}}{R} \right\} dR + \int_{1}^{M} \min \left\{ \frac{1}{R}, \frac{(\ln 3M)^{1-c_0}}{R^2} \right\} dR + \int_{M}^{+\infty} \frac{1}{R^2} dR \right\} \]
\[ := (T-\tau)(III_1 + III_2 + III_3) \] (4.10)

For \( III_1 \), when \( R \in (0, 1] \), the function \( R \) is increasing while \( \frac{(\ln 3M)^{1-c_0}}{R} \) is decreasing. Let \( R_0 \) be such that

\[ R_0 = \frac{(\ln 3M)^{1-c_0}}{R_0}. \]

This makes

\[ \left( \frac{1}{R_0} \right)^2 (1-c_0) = \ln 3M + \ln \frac{1}{R_0} \leq \ln 3M + \frac{c_0}{2} \left( \frac{1}{R_0} \right)^2. \]

That is

\[ (1 - \frac{c_0}{2}) \left( \frac{1}{R_0} \right)^2 \leq \ln 3M \leq \left( \frac{1}{R_0} \right)^2. \]

So, there exists a \( C > 1 \), such that

\[ C^{-1}(\ln 3M)^{-\frac{c_0}{2}} \leq R_0 \leq C(\ln 3M)^{-\frac{c_0}{2}}. \] (4.11)

Then

\[ \min \left\{ R, \frac{(\ln 3M)^{1-c_0}}{R} \right\} = \begin{cases} R & \text{for } 0 \leq R \leq R_0; \\ \frac{(\ln 3M)^{1-c_0}}{R} & \text{for } R_0 < R < 1. \end{cases} \]

By (4.11), the control of \( III_1 \) in (4.10) is

\[ III_1 \leq \int_0^{R_0} RdR + \int_{R_0}^{1} \frac{(\ln 3M)^{1-c_0}}{R} dR \]
\[ \leq \frac{1}{2} R_0^2 + (\ln 3M)^{1-c_0} \int_{R_0}^{1} \frac{1}{R} dR \]
\[ \lesssim (\ln 3M)^{-c_0} (1 + \frac{1}{R_0}) \]
\[ \lesssim (\ln 3M)^{-c_0} (1 + \frac{1}{R_0}) \]
\[ \lesssim (\ln 3M)^{-c_0} (1 + (\ln 3M)^{\frac{c_0}{2}}) \]
\[ \lesssim (\ln 3M)^{-\frac{c_0}{2}}. \]  

(4.12)

For \( III_2 \) in (4.10), one has

\[
III_2 \leq \int_1^M \frac{(\ln \frac{3M}{R})^{-c_0}}{R^2} dR
\]
\[
= M^{-1} \int_{M^{-1}}^1 \frac{(\ln \frac{3}{R})^{-c_0}}{R^2} dR
\]
\[
= M^{-1} \left( \int_{M^{-1}}^{M^{-\frac{1}{2}}} + \int_{M^{-\frac{1}{2}}}^1 \right) \frac{(\ln \frac{3}{R})^{-c_0}}{R^2} dR
\]
\[
\leq M^{-1} \left\{ \left( \frac{3}{M^{-\frac{1}{2}}} \right)^{-c_0} \int_{M^{-1}}^{M^{-\frac{1}{2}}} \frac{1}{R^2} dR + \int_{M^{-\frac{1}{2}}}^1 \frac{(R \ln \frac{3}{R})^{-c_0}}{R^2-c_0} dR \right\}.
\]

Since \( R \ln \frac{3}{R} \) is increasing when \( R \in (0, 1) \), one has,

\[
III_2 \lesssim M^{-1} \left\{ (\ln 3M)^{-c_0} M + (M^{-\frac{1}{2}} \ln 3M)^{-c_0} M^{-\frac{c_0}{2}} \right\}
\]
\[
\lesssim (\ln 3M)^{-c_0} + M^{-\frac{1}{2}} (\ln 3M)^{-c_0}
\]
\[
\lesssim (\ln 3M)^{-c_0}. \]  

(4.13)

For \( III_3 \) in (4.10), obviously

\[
III_3 \lesssim M^{-1}. \]  

(4.14)

Hence, Combining (4.12), (4.13) and (4.14), from (4.10), one has

\[
II_1 + II_2 \lesssim (T - \tau) (\ln 3M)^{-\frac{c_0}{2}}. \]  

(4.15)

For \( II_3 \) in (4.9), using the Cauchy-Schwarz inequality, one has

\[
II_3 = \int_{|X - Y| \geq 2(T - \tau), |X - Y| < 1} e^{-\frac{1}{4} \frac{|X - Y|^2}{T - \tau}} |g| dY
\]
\[
= \int e^{-\frac{1}{4} \frac{|X - Y|^2}{T - \tau} - \frac{1}{4} \frac{|X_3 - Y_3|^2}{T - \tau} - \frac{1}{4} \frac{|X_2 - Y_2|^2}{T - \tau}} |g| dY
\]
\[
\leq \left( \int e^{-\frac{1}{2} \frac{|X - Y|^2}{T - \tau} - \frac{1}{2} \frac{|X_3 - Y_3|^2}{T - \tau} - \frac{1}{2} \frac{|X_2 - Y_2|^2}{T - \tau}} dY \right)^\frac{1}{2} \left( \int e^{-\frac{1}{4} \frac{|X_3 - Y_3|^2}{T - \tau}} g^2 dY \right)^\frac{1}{2}
\]
\[
\leq \left( \int e^{-\frac{1}{2} \frac{|X - Y|^2}{T - \tau} - \frac{1}{2} \frac{|X_3 - Y_3|^2}{T - \tau} - \frac{1}{2} \frac{|X_2 - Y_2|^2}{T - \tau}} dY \right)^\frac{1}{2} \left( \int_0^{+\infty} \int_0^{+\infty} e^{-\frac{1}{4} \frac{|X_3 - Y_3|^2}{T - \tau}} g^2 R dR dY_3 \right)^\frac{1}{2}
\]
\[
\leq (T - \tau)^\frac{7}{4} \cdot (T - \tau)^\frac{1}{4} \left( \int_0^{+\infty} \sup_{Y_3} g^2 R dR \right)^{\frac{1}{2}}
\]
\[
\leq (T - \tau) \left( \int_0^{+\infty} \sup_{Y_3} g^2 R dR \right)^{\frac{1}{2}}.
\]

Following (4.5) one has
\[
g^2 \leq \left\{ \begin{array}{ll}
\min \left\{ 1, \frac{(\ln \frac{3M}{R})^{-c_0}}{R^2}, \frac{(\ln \frac{3M}{R})^{-2c_0}}{R^4} \right\} & \text{for } R \leq 1; \\
\min \left\{ \frac{1}{R^4}, \frac{(\ln \frac{3M}{R})^{-c_0}}{R^5}, \frac{(\ln \frac{3M}{R})^{-c_0}}{R^6} \right\} & \text{for } 1 < R < M; \\
\frac{1}{R^6} & \text{for } R \geq M.
\end{array} \right.
\]

As the previous proof for (4.15), one can get
\[
II_3 \lesssim (T - \tau)(\ln 3M)^{-\frac{c_0}{2}}.
\] (4.16)

Inserting (4.15) and (4.16) into (4.9), one has
\[
|II| \lesssim \int_S^T (T - \tau)^{-\frac{1}{2}} (\ln 3M)^{-\frac{c_0}{2}} d\tau \lesssim (T - S)^{\frac{1}{2}} (\ln 3M)^{-\frac{c_0}{2}}.
\] (4.17)

So, combining (4.8) and (4.17), from (4.6), one has
\[
|f(X, T)| \lesssim (T - S)^{-\frac{1}{2}} + (T - S)^{\frac{1}{2}} (\ln 3M)^{-\frac{c_0}{2}}.
\]

Let \( S = T - (\ln 3M)^{-\frac{2}{3} c_0} > -M^2 \) (hence \( f \) is defined), so
\[
|f(X, T)| \lesssim (\ln 3M)^{-\frac{1}{3} c_0}.
\]

**Step three: bounding the solution \( v \) from \( f \)**

First
\[
|w_\theta(x, t)| \leq M^2 |w_\theta^M(rM, zM, tM^2)| \leq |\Omega^M(rM, zM, tM^2)| M^2 rM \leq CM^3 r(\ln 3M)^{-\frac{1}{2} c_0}.
\]

Therefore
\[
|w_\theta(x, t)| \leq CM^2 (\ln 3M)^{-\frac{1}{2} c_0}, \quad \text{for } r \leq M^{-1}(\ln 3M)^{\frac{1}{3} c_0}.
\] (4.18)

In the following, we bound \( b = v_r e_r + v_z e_z \).

Denote \( B_\rho(x_0) = \{ x : |x - x_0| < \rho \} \), where \( \rho > 0 \) to be determined later. By Biot-Savart law, \( b \) satisfies
\[
- \Delta b = \text{curl}(w_\theta e_\theta).
\]
From the estimates of elliptic equation\(^{11}\), for \(q > 1\),
\[
\sup_{B_{\rho}(x_0)} |b| \leq C \left( \rho^{-\frac{3}{q}} \|b\|_{L^q(B_{2\rho}(x_0))} + \rho \sup_{B_{2\rho}(x_0)} |w_\theta| \right). \tag{4.19}
\]

Take \(\rho = M^{-1}(\ln 3M)^{\frac{1}{12}}\), \(x_0 \in \{(r, \theta, z) : r < \rho\}\) and \(1 < q < 2\).

By the assumption (1.2) on \(b\),
\[
\rho^{-3q} \|b\|_{L^q(B_{2\rho}(x_0))} \leq C \rho^{-\frac{3}{q}} \left[ \int_{2\rho}^{3\rho} (\ln \frac{\rho}{r})^{\alpha q} \frac{r}{\rho^q} dr \right]^{\frac{1}{q}} \tag{4.20}
\]

We compute \(\int_0^{3\rho} \frac{(\ln |\ln \frac{r}{3}|)^{\alpha q}}{r^{q-1}} dr\) as follow,
\[
\int_0^{3\rho} \frac{(\ln |\ln \frac{r}{3}|)^{\alpha q}}{r^{q-1}} dr = \int_{\frac{1}{3\rho}}^{+\infty} \frac{(\ln \ln 3r)^{\alpha q} r^{-3}}{r} dr \quad r \text{ replaced by } \frac{1}{r}
\]
\[
= \left( \int_{\frac{1}{3\rho}}^{+\infty} \frac{3}{r^{q-1}} + \int_{\frac{1}{3\rho}}^{+\infty} \frac{3}{r^{q-1}} \left( \ln \ln 3r \right)^{\alpha q} r^{-3} dr \right) \left( \ln \ln \frac{3}{\rho} \right)^{2\alpha q} + \int_{\frac{1}{3\rho}}^{+\infty} \frac{3}{r^{q-1}} \left( \ln \ln 3r \right)^{\alpha q} r^{-3+\alpha q} dr.
\]

Here \(\ln \ln \frac{3}{r}\) is a decreasing function in the integral domain. Also we can pick a \(q \in (1, 2)\) such that \(q - 3 + \alpha q < -1\). So
\[
\int_0^{3\rho} \frac{(\ln |\ln \frac{r}{3}|)^{\alpha q}}{r^{q-1}} dr \leq C \left( \ln \ln \frac{3}{\rho} \right)^{2\alpha q} \left( \frac{1}{\rho} \right)^{q-2} + \frac{(\ln \ln \frac{3}{\rho})^{2\alpha q}}{\frac{3}{\rho} e^{\left( \ln \ln \frac{3}{\rho} \right)^2}} \alpha q \left( \frac{1}{3} \right)^{2\alpha q} \left( \frac{1}{\rho} \right)^{q-2+\alpha q} \tag{4.21}
\]

The last inequality holds because \(e^{(\ln \ln \frac{3}{\rho})^2} > \frac{3}{\rho}\).

Since \(\rho = M^{-1}(\ln 3M)^{\frac{1}{12}}\) and \(\rho^{-1} \leq M\), so, from (4.20) and (4.21),
\[
\rho^{-\frac{3}{q}} \|b\|_{L^q(B_{2\rho}(x_0))} \leq C \rho^{-\frac{2}{q}} \left( \ln \ln \frac{3}{\rho} \right)^{2\alpha q} \left( \frac{1}{\rho} \right)^{1-\frac{q}{q}}
\]
While, due to (4.18), when \(x_0 \in B_\rho(x_0)\),

\[
\sup_{B_\rho(x_0)} |w_\theta| \leq M^{-1}(\ln 3M)^{-\frac{a_0}{\alpha}} M^2(\ln 3M)^{-\frac{a}{\alpha}} \leq C M (\ln 3M)^{-\frac{a}{\alpha}}. \tag{4.23}
\]

Combining (4.19), (4.22), and (4.23), we have

\[
|b| \leq C M (\ln 3M)^{-\frac{a_0}{\alpha}} \quad \text{for } r < M^{-1}(\ln 3M)^{\frac{1}{\alpha} a_0}; \tag{4.24}
\]

next, when \(M^{-1}(\ln 3M)^{\frac{a_0}{\alpha}} \leq r < 1\),

\[
|b| \leq C \left( \frac{\ln 3}{r} \right)^{\alpha} \leq C M (\ln 3M)^{-\frac{a_0}{\alpha}} (\ln 3)^{\alpha} \leq C M (\ln 3M)^{-\frac{a_0}{\alpha}}; \tag{4.25}
\]

thirdly, when \(1 \leq r\),

\[
|b| \leq C \frac{r}{r} \leq C. \tag{4.26}
\]

Combining (4.24), (4.25), and (4.26), we get, for any \(r > 0\),

\[
|b| \leq C M (\ln 3M)^{-\frac{a_0}{\alpha}}. \tag{4.27}
\]

In the following, we bound \(v_\theta\).

Recall that \(v_\theta\) satisfies (4.3), then

\[
v_\theta(r, z) = M|v_\theta(rM, zM)| \leq M \left\{ \min \left\{ \frac{(\ln \frac{3}{rM})^{-a_0}}{rM}, rM \right\} \right. \quad \text{for } r < \frac{1}{M};
\]

\[
\left. \min \left\{ \frac{(\ln \frac{3}{rM})^{-a_0}}{rM}, 1 \right\} \right. \quad \text{for } \frac{1}{M} < r < 1.
\]

Firstly, when \(r < \frac{1}{M}\), \(\frac{(\ln \frac{3}{rM})^{-a_0}}{rM}\) is a decreasing function while \(rM\) is an increasing function with respect to \(r\). Let \(r_0\) be such that

\[
\left( \frac{\ln \frac{3}{r_0M}}{r_0M} \right)^{-a_0} = r_0 M.
\]

This gives

\[
\left( \frac{1}{r_0M} \right)^{\frac{a_0}{\alpha}} = \ln \frac{3}{r_0} > \ln 3M.
\]

So

\[
r_0 < (\ln 3M)^{-\frac{a_0}{\alpha}} M^{-1}.
\]
Then

\[ |v_\theta| \lesssim r_0 M^2 \lesssim (\ln 3M)^{-\frac{20}{27}} M. \]  \hspace{1cm} (4.28)

Next, when \( \frac{1}{M} \leq r < 1 \),

\[ |v_\theta| \lesssim \frac{(\ln \frac{3}{r})^{-c_0}}{r} \lesssim (\ln 3M)^{-c_0} M. \]  \hspace{1cm} (4.29)

Thirdly, when \( 1 \leq r \),

\[ |v_\theta| \leq C. \]  \hspace{1cm} (4.30)

Combining (4.28), (4.29), and (4.30), we have, for any \( r > 0 \),

\[ |v_\theta| \leq C M (\ln 3M)^{-c_0}. \]  \hspace{1cm} (4.31)

Since \( M \) is the maximum of \( |v| \), \( M = \max\{\sup |b|, \sup |v_\theta|\} \). Due to the estimates (4.27) and (4.31), we get

\[ M \leq C M (\ln 3M)^{-c_0}. \]

This gives an upper bound for \( M \) which completes the proof of Theorem 1.1. 

\[ \blacksquare \]

5 Proof of Theorem 1.3

In this section, we prove Theorem 1.3 and give the estimate (4.7) of the fundamental solution.

Following Davies [7] and Carlen-Loss [6], for a fixed constant vector \( \alpha \in \mathbb{R}^3 \), let \( \psi(x) = \alpha \cdot x \). For any \( f \in C_0^\infty(\mathbb{R}^3; (0, +\infty)) \), define

\[
P_{t,s}^\psi f(x) = e^{-\psi(x)} \int f(y)p(x, t; y, s)e^{\psi(y)}dy = e^{-\alpha x} \int f(y)p(x, t; y, s)e^{\alpha y}dy \equiv f_t(x).
\]

In fact, let \( Q_R = B_R(0) \times (s, +\infty) \) and define the Dirichlet fundamental solution \( p^R(x, t; y, s) \) in \( Q_R \) the same as Definition 1.1 which satisfies the boundary condition

\[ p^R(x, t; y, s)|_{(x, t) \in \partial B_R \times (s, +\infty)} = 0. \]

Due to the maximum principle, we have

\[ p^{R_1}(x, t; y, s) \leq p^{R_2}(x, t; y, s) \leq p(x, t; y, s), \quad \text{when } R_1 < R_2. \]

Also

\[ \lim_{R \to +\infty} p^R(x, t; y, s) = p(x, t; y, s). \quad \text{a.e.} \]

In a rigorous computation, all the integrals in the following should be done in \( B_R(0) \) with the function \( f_t(x) \) replaced by \( f_t^R(x) \equiv e^{-\alpha x} \int_{\mathbb{R}^3} f(y)p^R(x, t; y, s)e^{\alpha y}dy \) which satisfies
for simplicity, we just carry out this process on $f_t(x)$ and assume that $f_t(x)$ vanishes on the boundary which means, $f_t(x)|_{|x|=\infty} = 0$.

We divide the proof into 3 parts.

**Part one: $L^2 \to L^\infty$ estimate of $P\psi_{t,s}$**

Let $k(t) : [s, T] \to [2, \infty]$ be a continuously differentiable increasing function to be determined later. By direct computation, we have

\[
k(t)^2\|f_t\|_{k(t)}^{k(t)-1} \frac{d}{dt}\|f_t\|_{k(t)} = k'(t) \int f_t^{k(t)} \ln \left( \frac{f_t^{k(t)}}{\|f_t\|_{k(t)}} \right) dx + k(t)^2 \int f_t^{k(t)-1} \frac{d}{dt} f_t dx
\]

\[
= k'(t) \int f_t^{k(t)} \ln \left( \frac{f_t^{k(t)}}{\|f_t\|_{k(t)}} \right) dx + k(t)^2 \int (f_t^{k(t)-1} e^{-\alpha x})(\Delta + \frac{2}{r} \partial_r - b \cdot \nabla)(e^{\alpha x} f_t) dx
\]

\[
= k'(t) \int f_t^{k(t)} \ln \left( \frac{f_t^{k(t)}}{\|f_t\|_{k(t)}} \right) dx + k(t)^2 \left\{ - \int \nabla (f_t^{k(t)-1} e^{-\alpha x}) \cdot \nabla (e^{\alpha x} f_t) + \int \frac{2}{r} f_t^{k(t)-1} (\partial_r f_t + \partial_r(\alpha x) f_t) - \int f_t^{k(t)-1} (b \cdot \alpha f_t + b \cdot \nabla f_t) \right\}
\]

\[
\triangleq k'(t) \int f_t^{k(t)} \ln \left( \frac{f_t^{k(t)}}{\|f_t\|_{k(t)}} \right) dx + k(t)^2 \{ I + II + III \}. \tag{5.1}
\]

Using Cauchy-Schwartz inequality, we have

\[
I = - \int \left[ (k(t) - 1) f_t^{k(t)-1} \nabla f_t - f_t^{k(t)-1} \alpha \right] \cdot [\nabla f_t + \alpha f_t]
\]

\[
= - \int \frac{4(k(t) - 1)}{k(t)^2} \|\nabla f_t^{k(t)}\|^2 - \int (k(t) - 2) f_t^{k(t)-1} \alpha \cdot \nabla f_t + \int \alpha^2 f_t^{k(t)}
\]

\[
= - \int \frac{4(k(t) - 1)}{k(t)^2} \|\nabla f_t^{k(t)}\|^2 - \int \frac{2(k(t) - 2)}{k(t)} f_t^{k(t)} \alpha \cdot \nabla f_t^{k(t)} + \int \alpha^2 f_t^{k(t)}
\]

\[
\leq - \int \frac{4(k(t) - 1)}{k(t)^2} \|\nabla f_t^{k(t)}\|^2 + \int \frac{(k(t) - 2)^2 \varepsilon}{k(t)^2} \|\nabla f_t^{k(t)}\|^2 + \int \left( 1 + \frac{1}{\varepsilon} \right) \alpha^2 f_t^{k(t)}, \tag{5.2}
\]

where $\varepsilon > 0$ is to be determined later on.

Also,

\[
II = \int \frac{2}{r} f_t^{k(t)-1} (\partial_r f_t + \partial_r(\alpha x) f_t)
\]

\[
\leq \int \frac{2}{k(t)} \partial_r f_t^{k(t)} dr d\theta dz + \int |\alpha| \frac{2}{r} f_t^{k(t)}
\]
\[
\frac{2}{k(t)} \int dz d\theta f_t^{k(t)}|_{r=0} + \int |\alpha| \frac{2}{r} f_t^{k(t)} 
\leq \int |\alpha| \frac{2}{r} f_t^{k(t)}. \tag{5.3}
\]

The estimate (5.3) is due to our choice of \( f \geq 0 \), so \( f_t|_{r=0} \geq 0 \).
Moreover, due to the divergence-free property of \( b \),
\[
III = -\int f_t^{k(t)-1}(b \cdot \alpha f_t + b \cdot f_t) 
= -\int f_t^{k(t)} b \cdot \alpha 
\leq (C_0 + \frac{1}{r}) |\alpha| f_t^{k(t)}, \tag{5.4}
\]
while using integration by parts,
\[
\int \frac{|\alpha|}{r} f_t^{k(t)} = \int \int |\alpha| f_t^{k(t)} d\theta d\phi 
= -\int \int 2|\alpha| r f_t^{k(t)} \partial_r f_t^{k(t)} d\theta d\phi 
\leq 2|\alpha| f_t^{k(t)} \partial_r f_t^{k(t)} dx 
\leq \varepsilon \int \left| \nabla f_t^{k(t)} \right|^2 + \frac{\alpha^2}{\varepsilon} \int f_t^{k(t)}. \tag{5.5}
\]

Thus combining (5.3), (5.4) and (5.5), we have
\[
II + III \leq 3\varepsilon \int \left| \nabla f_t^{k(t)} \right|^2 + 3\frac{\alpha^2}{\varepsilon} \int f_t^{k(t)} + C_0|\alpha| \int f_t^{k(t)}. \tag{5.6}
\]

Now we recall the 3-d log–Sobolev inequality.
For all functions \( u \) on \( \mathbb{R}^3 \), together with their distributional gradients \( \nabla u \) are square integrable, then
\[
\int u^2 \ln \left( \frac{u^2}{\|u\|^2_2} \right) + \left( 3 + \frac{3}{2} \ln a \right) \int u^2 \leq \frac{a}{\pi} \int |\nabla u|^2 \tag{5.7}
\]
for all \( a > 0 \).
Inserting (5.2), (5.6) and (5.7) into (5.1), one has
\[
k(t)^2 \|f_t\|_{k(t)}^{k(t)-1} \frac{d}{dt} \|f_t\|_{k(t)} \leq k'(t) \left[ \frac{a}{\pi} \int \left| \nabla f_t^{k(t)} \right|^2 - \left( 3 + \frac{3}{2} \ln a \right) \int f_t^{k(t)} \right] 
+ k(t)^2 \left\{ \left( \frac{-4(k(t) - 1) + \varepsilon(k(t) - 2)^2}{k(t)^2} + 3\varepsilon \right) \int \left| \nabla f_t^{k(t)} \right|^2 
+ \left( 1 + \frac{4}{\varepsilon} \right) \alpha^2 + C_0|\alpha| \right\} \int f_t^{k(t)},
\]
then

\[
\|f_t\|_{k(t)}^{k(t)-1} \frac{d}{dt}\|f_t\|_{k(t)} \leq \left[\frac{k'(t) a}{k(t)^2 \pi} - \frac{4(k(t) - 1)}{k(t)^2} + \frac{(k(t) - 2)^2}{k(t)^2} \varepsilon + 3\varepsilon\right] \int |\nabla f_t^{k(t)}|^2 \\
+ \left[(1 + \frac{4}{\varepsilon})\alpha^2 + C_0|\alpha| - \frac{k'(t)}{k(t)^2} \left(3 + \frac{3}{2} \ln a\right)\right] \int f_t^{k(t)}.
\]

Here we can not choose \(k(t) : [s, T] \to [1, +\infty]\) as Carlen-Loss in [6] do. Because when \(k(s) = 1\), the coefficient of \(\int |\nabla f_t^{k(t)}|^2\) is \(k'(s)\frac{a}{\pi} + 4\varepsilon\) which is obviously positive when \(k(t)\) is a continuously differentiable increasing function and \(a > 0\). It can not reach zero as Carlen-Loss in [6] do. So we choose \(k(t) : [s, T] \to [2, +\infty]\) to ensure the coefficient of \(\int |\nabla f_t^{k(t)}|^2\) is zero. When \(k(t) \in [2, \infty)\),

\[
\frac{k(t) - 1}{k(t)} \geq \frac{1}{2}, \quad \frac{(k(t) - 2)^2}{k(t)^2} < 1.
\]

So

\[
\|f_t\|_{k(t)}^{k(t)-1} \frac{d}{dt}\|f_t\|_{k(t)} \leq \left[\frac{k'(t) a}{k(t)^2 \pi} - \frac{2}{k(t)} + 4\varepsilon\right] \int |\nabla f_t^{k(t)}|^2 \\
+ \left[(1 + \frac{4}{\varepsilon})\alpha^2 + C_0|\alpha| - \frac{k'(t)}{k(t)^2} \left(3 + \frac{3}{2} \ln a\right)\right] \int f_t^{k(t)}.
\]

Let \(k(t) = 2\sqrt{\frac{T}{T+s-t}}\), \(4\varepsilon = \frac{k'(t)}{k(t)^2} = \frac{1}{k(t)}\), then

\[
a = \frac{\pi k(t)}{k'(t)} = 2\pi(T+s-t),
\]

\[
- \frac{k'(t)}{k(t)^2} = -\frac{1}{4\sqrt{T(T+s-t)}}.
\]

Then

\[
\|f_t\|_{k(t)}^{k(t)-1} \frac{d}{dt}\|f_t\|_{k(t)} \leq \left[\left(1 + 32\sqrt{\frac{T}{T+s-t}}\right)\alpha^2 + C_0|\alpha| \\
- \frac{3}{4\sqrt{T(T+s-t)}} - \frac{3}{8\sqrt{T(T+s-t)}}\right].
\]

Integrating the above inequality in \([s, T]\), we get

\[
\ln \|f_T\|_\infty - \ln \|f_s\|_2 \leq \int_s^T \left[\left(1 + 32\sqrt{\frac{T}{T+s-t}}\right)\alpha^2 + C_0|\alpha| \\
- \frac{3}{4\sqrt{T(T+s-t)}} - \frac{3}{8\sqrt{T(T+s-t)}}\right] dt.
\]
So

\[ \| f_I \|_\infty \leq (2\pi(T - s))^{\frac{3}{4}} \exp\{(65\alpha^2 + C_0\alpha)(T - s)\} \| f \|_{L^2}. \]

That is

\[ \| P_{t,s}^{\psi} f \|_\infty \leq (2\pi(t - s))^{\frac{3}{4}} \exp\{(65\alpha^2 + C_0\alpha)(t - s)\} \| f \|_{L^2}. \] (5.8)

**Part two:** \( L^2 \rightarrow L^\infty \) estimate of the adjoint \( (P_{t,s}^{\psi})^* \) of \( P_{t,s}^{\psi} \).

Now we come to investigate the adjoint \( (P_{t,s}^{\psi})^* \) of \( P_{t,s}^{\psi} \), for any \( f, g \in C_0^\infty(\mathbb{R}^3) \),

\[
(P_{t,s}^{\psi} f(x), g(x)) = \int g(x) e^{-\psi(x)} \int f(y) e^{\psi(y)} p(x, t ; y, s) \, dy \, dx
\]

\[= \int f(y) e^{\psi(y)} \, dy \int g(x) e^{-\psi(x)} p(x, t ; y, s) \, dx \]

\[\triangleq ((P_{t,s}^{\psi})^* g(y), f(y)) \]

So

\[
(P_{t,s}^{\psi})^* g(y) = e^{\psi(y)} \int g(x) e^{-\psi(x)} p(x, t ; y, s) \, dx.
\]

Here, note that we do not require \( t \geq s \). We denote \( y = (y_1, y_2, y_3) \) and \( y' = (y_1, y_2, 0) \).

Let \( p(x, t ; y, s) \) be the fundamental solution of (1.9), that is

\[
\partial_t p(x, t ; y, s) = \Delta_x p(x, t ; y, s) - b \cdot \nabla_x p(x, t ; y, s) + \frac{2}{r_x} \partial_{r_x} p(x, t ; y, s),
\]

when \( t > s \). Here \( r_x = \sqrt{x_1^2 + x_2^2} \). Then \( p(x, t ; y, s) \), with respect to \( (y, s) \), satisfies

\[- \partial_s p(x, t ; y, s) = \Delta_y p(x, t ; y, s) + b \cdot \nabla_y p(x, t ; y, s) + \frac{2}{r_y} \partial_{r_y} p(x, t ; y, s).
\]

Let \( \rho = -t, \tau = -s \). \( p(x, \rho ; y, \tau) \), with respect to \( (y, \tau) \), satisfies

\[
\partial_{\tau} p(x, \rho ; y, \tau) = \Delta_y p(x, \rho ; y, \tau) + b \cdot \nabla_y p(x, \rho ; y, \tau) - \frac{2}{r_y} \partial_{r_y} p(x, \rho ; y, \tau).
\]

Let \( p^*(y, \tau ; x, \rho) = p(x, \rho ; y, \tau) \), then

\[
(P_{\rho,\tau}^{\psi})^* g(y) = e^{\psi(y)} \int g(x) e^{-\psi(x)} p^*(y, \tau ; x, \rho) \, dx.
\]

When \( \tau > \rho \), \( p^*(y, \tau ; x, \rho) \) satisfies

\[
\partial_{\tau} p^*(y, \tau ; x, \rho) = \Delta_y p^*(y, \tau ; x, \rho) + b \cdot \nabla_y p^*(y, \tau ; x, \rho) - \frac{2}{r_y} \partial_{r_y} p^*(y, \tau ; x, \rho).
\]
Then \( p^*(y, \tau; x, \rho) \) is a fundamental solution of

\[
\partial_r v = \Delta v + b \cdot \nabla v - \frac{2}{r} \partial_r v,
\]

(5.9)

with respect to variables \((y, \tau)\) and \( e^{-\psi(y)(P_{b,\tau}^\psi)^* g(y)} \) is a solution of (5.9).

We now restrict the solution \( v \) of (5.9) such that \( v(y, \tau) ||y'| = 0 \). The reason is the following: let \( v = rh \), then by direct computation, \( h \) satisfies

\[
\partial_s h = \Delta h - \frac{1}{r^2} h + b \cdot \nabla h + \frac{b_r}{r} h,
\]

(5.10)

where \( b = b_1 e_r + b_\theta e_\theta + b_2 e_z \).

If \( |b| \leq C_0 + \frac{1}{r^2} \), using Nash-Moser iteration argument as in the section 2 and noting that the integral of \( \frac{b_r}{r} h \) can be absorbed by that of \(- \frac{1}{r^2} h\) which is a good term in the energy estimate due to its minus sign. We can derive that the weak solution of (5.10) is bounded. So we can assume

\[
v||y'| = 0 = rh||y'| = 0.
\]

Then we have \((P_{t,s}^\psi)^* g(y)||y'| = 0 = 0 \) when \( s \geq t \).

Now we can follow the proof of \( L^2 \rightarrow L^\infty \) estimate for \( P_{t,s}^\psi \) to derive the \( L^2 \rightarrow L^\infty \) estimate for \((P_{t,s}^\psi)^* \). \((P_{t,s}^\psi)^* \) has nearly the same form as \( P_{t,s}^\psi \), but the signs on the terms \( \frac{2}{r} \partial_r \) and \( b \cdot \nabla \) are reversed. This makes the estimate a little different when we deal with the term II. If we denote

\[
(P_{t,s}^\psi)^* g(y) = g_s(y)
\]

then

\[
II = -\int \frac{2}{r} g_s^{k(t)-1}[\partial_r g_s + \partial_r(\alpha x)]
\]

\[
\leq -\int \frac{2}{k(t)} \partial_r g_t^{k(t)} drd\theta dz + \int |\alpha| \frac{2}{r^2} g_s^{k(t)}
\]

\[
= \frac{2}{k(t)} \int dzd\theta g_s^{k(t)}|_{r=0} + \int |\alpha| \frac{2}{r^2} g_s^{k(t)}
\]

\[
= \int |\alpha| \frac{2}{r^2} g_s^{k(t)}.
\]

Due to the vanishing property of \((P_{t,s}^\psi)^* g(y)\) at \( |y'| = 0 \), we can also get the estimate (5.3) for \((P_{t,s}^\psi)^* g(y)\), so the \( L^2 \rightarrow L^\infty \) estimate (5.8) is also right to \((P_{t,s}^\psi)^* g(y)\).

\[
\| (P_{t,s}^\psi)^* g \|_{L^\infty} \leq (2\pi|t - s|)^{\frac{3}{4}} \exp\{(65\alpha^2 + C_0|\alpha|)|t - s|\} \| g \|_{L^2}.
\]

Part three: \( L^1 \rightarrow L^\infty \) estimate of \( P_{t,s}^\psi \).

Using the duality, we have the \( L^1 \rightarrow L^2 \) estimate of \( P_{t,s}^\psi \).

\[
\| (P_{t,s}^\psi f \|_{L^2} \leq (2\pi(t - s))^{\frac{3}{4}} \exp\{(65\alpha^2 + C_0|\alpha|)(t - s)\} \| f \|_{L^1}.
\]
So

\[ \| (P_{t,s}^\psi) f \|_{L^\infty} = \| (P_{t,\frac{t+s}{2},s}^\psi P_{t+s,s}^\psi f) \| \leq (2\pi(t - \frac{t+s}{2}))^{-\frac{3}{2}} \exp \{(65\alpha^2 + C_0|\alpha|)(t - \frac{t+s}{2})\} \| P_{t+s,s}^\psi f \|_{L^2} \]

\[ \leq (\pi(t - s))^{-\frac{3}{2}} \exp \{(65\alpha^2 + C_0|\alpha|)(t - s)\} \| P_{t+s,s}^\psi f \|_{L^2}. \]

\[ \leq (\pi(t - s))^{-\frac{3}{2}} \exp \{(65\alpha^2 + C_0|\alpha|)(t - s)\} \| f \|_{L^1}. \]

This is equivalent to

\[ p(x, t; y, s) \leq (\pi(t - s))^{-\frac{3}{2}} \exp \{(65\alpha^2 + C_0|\alpha|)(t - s)\} \exp \{\alpha(x - y)\} \]

\[ \leq (\pi(t - s))^{-\frac{3}{2}} \exp \{65\alpha^2(t - s) + C_0|\alpha|(t - s) + \alpha(x - y)\}. \]

Let \( \alpha = -\frac{1}{65(t-s)} \frac{x-y}{|x-y|} \left[ |x-y| - C_0(t-s) \right]_+ \). With this choice of \( \alpha \), we have

\[ \alpha \cdot (x - y) + C_0|\alpha|(t - s) + 65\alpha^2(t - s) = -\frac{1}{65t} \left[ |x-y| - C_0(t-s) \right]_+^2, \]

then

\[ p(x, t; y, s) \leq (\pi(t - s))^{-\frac{3}{2}} \exp \left\{ -\frac{1}{65(t-s)} \left[ |x-y| - C_0(t-s) \right]_+^2 \right\}. \]

This gives the estimate (1.10) of \( p(x, t; y, s) \).

Moreover, when \( t \geq s \),

\[ \frac{\partial_t}{\partial t} \int_{\mathbb{R}^3} p(x, t; y, s) \, dx = \int_{\mathbb{R}^3} \partial_t p(x, t; y, s) \, dx \]

\[ = \int_{\mathbb{R}^3} \left( \Delta_x + \frac{2}{r_x} \partial_{r_x} - b \cdot \nabla_x \right) p(x, t; y, s) \, dx \]

\[ = \int_{-\infty}^{+\infty} \int_0^{+\infty} \left\{ \int_{-\infty}^{+\infty} \frac{2}{r_x} \partial_{r_x} p(x, t; y, s) r_x \, dr_x \, d\theta \, dz \right\} \]

\[ \leq 0, \]

so

\[ \int_{\mathbb{R}^3} p(x, t; y, s) \, dx \leq \int_{\mathbb{R}^3} p(x, s; y, s) \, dx = 1. \]

Also, when \( t \leq s \),

\[ \frac{\partial_s}{\partial s} \int_{\mathbb{R}^3} p(x, t; y, s) \, dy = \int_{\mathbb{R}^3} \partial_s p(x, t; y, s) \, dy \]

\[ = \int_{\mathbb{R}^3} \left( \Delta_y - \frac{2}{r_y} \partial_{r_y} + b \cdot \nabla_y \right) p(x, t; y, s) \, dy \]

\[ = \int_{-\infty}^{+\infty} \int_0^{+\infty} \left\{ \int_0^{+\infty} -\frac{2}{r_y} \partial_{r_y} p(x, t; y, s) r_y \, dr_y \, d\theta \, dz \right\}. \]
\[ = \int_{-\infty}^{+\infty} \int_{0}^{2\pi} 2p(x, t; y, s)|_{r_y=0} d\theta dz = 0, \]

so
\[ \int_{\mathbb{R}^3} p(x, t; y, s) dy = \int_{\mathbb{R}^3} p(x, t; y, t) dy = 1. \]

Thus we complete the proof of Theorem 1.3. ■

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