HIERARCHIES OF SUBSYSTEMS OF WEAK ARITHMETIC

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Abstract. We completely characterize the logical hierarchy of various subsystems of weak arithmetic, namely: ZR, ZR + N, ZR + GCD, ZR + Bez, OI + N, OI + GCD, OI + Bez.

1. Introduction

In 1964 Shepherdson [6] introduced a weak system of arithmetic, Open Induction (OI), in which the Tennenbaum phenomenon does not hold. More precisely, if we restrict induction just to open formulas (with parameters), then we have a recursive nonstandard model. Since then several authors have studied Open Induction and its related fragments of arithmetic. For instance, since Open Induction is too weak to prove many true statements of number theory (it cannot even prove the irrationality of \( \sqrt{2} \)), a number of algebraic first order properties have been suggested to be added to OI in order to obtain closer systems to number theory. These properties include: Normality [9] (abbreviated by N), having the GCD property [8], being a Bezout domain [3, 8] (abbreviated by Bez), and so on. We mention that GCD is stronger than N, Bez is stronger than GCD and Bez is weaker than \( IE_1 \) (\( IE_1 \) is the fragment of arithmetic based on the induction scheme for bounded existential formulas and by a result of Wilmers [11], does not have a recursive nonstandard model). Boughattas in [1, 2] studied the non-finite axiomatizability problem and established several new results, including: (1) OI is not finitely axiomatizable, (2) OI + N is not finitely axiomatizable. To show that, he defined and considered the subsystems \((OI)_p\) of OI and \((N)_n\) of N \((1 \leq p, n < \omega)\) (See the next section for the definitions) and proved:

Theorem 1.1 (Boughattas [1]). (1) \((OI)_p\) is finitely axiomatizable,
(2) Suppose \((p!, p') = 1\), then \((OI)_p \not\models (OI)_{p'}\).

Theorem 1.2 (Boughattas [2], Theorem 2). Suppose \((p!, p') = 1\) and \((n!, n') = 1\),
(1) \(N + (OI)_p \not\models (OI)_{p'}\),
(2) \((N)_n + (OI) \not\models (N)_{n'}\),
(3) \((OI)_p + \neg(OI)_{p'} + (N)_n + \neg(N)_{n'}\) is consistent.

In [4] we strengthened Theorem 1.1 (2) to completely characterize the logical hierarchy of OI, by showing that \((OI)_p \not\models (OI)_{p+1}\) iff \(p \neq 3\). In this paper by modifying Boughattas’ original proofs, we also strengthen Theorem 1.2 in two directions.

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and completely characterize the logical hierarchy of $OI + N$, $OI + GCD$, $OI + Bez$:

**Theorem C.** $Bez + (OI)_p \not\vdash (OI)_{p+1}$, when $p \neq 3$.

**Theorem D.** $(OI)_p + \neg (OI)_{p+1} + (N)_n + \neg (N)_{n+1}$ is consistent, when $p \neq 3$.

So we will have the following immediate consequences:

**Corollary E.**
1. $N + (OI)_p \not\vdash (OI)_{p+1}$, when $p \neq 3$.
2. $GCD + (OI)_p \not\vdash (OI)_{p+1}$, when $p \neq 3$.
3. All of the following subsystems of arithmetic are non-finite axiomatizable: $OI$, $OI + N$, $OI + GCD$, $OI + Bez$, $(OI)_p + N$, $OI + (N)_n$.

In Theorems A and B of this paper, we consider the $ZR$ versions of the above theorems. $ZR$ is a subsystem of arithmetic that allows Euclidean division over each non-zero natural number $n \in \mathbb{N}$. $ZR$ is introduced by Wilkie [10] in which he proved that $ZR$ and $OI$ have the same $\forall_1$-consequences. Later developments showed that $ZR$ had very important role in constructing models of $OI$ (See Macintyre-Marker [3], Smith [8]). $ZR + N$ has also been studied in [5]. In Theorem A, we study natural subsystems $(ZR)_S$ of $(ZR)$, for a nonempty subset $S$ of the set of prime numbers $\mathbb{P}$ (see the next section for definition) and show that:

**Theorem A.** Suppose $S$ is a nonempty subset of $\mathbb{P}$ and $q$ is a prime number such that $q \notin S$, then $(ZR)_S + Bez \not\vdash (ZR)_q$.

Boughattas in ([2], Lemma 5) proved that DOR + N and ZR + N are not finitely axiomatizable. More precisely he showed that:

**Theorem 1.3** (Boughattas [2], Lemma 5). Suppose $(n!, n') = 1$. Then ZR + $(N)_n \not\vdash (N)_{n'}$.

We modify Boughattas’ proof and strengthen the above theorem in Theorem B:

**Theorem B.** Suppose $S$ is a nonempty subset of $\mathbb{P}$ and $q$ is a prime number such that $q \notin S$, then $(ZR)_S + (N)_n + \neg (ZR)_q + \neg (N)_{n+1}$ is consistent.

Therefore we will have the following immediate implications:

**Corollary F.** Suppose $S$ is a nonempty subset of $\mathbb{P}$ and $q$ is prime number such that $q \notin S$, then
1. $(ZR)_S + N \not\vdash (ZR)_q$.
2. $(ZR)_S + GCD \not\vdash (ZR)_q$.
3. All of the following subsystems of arithmetic are non-finite axiomatizable: $ZR$, $ZR + N$, $ZR + GCD$, $ZR + Bez$, $(ZR)_S + N$, $ZR + (N)_n$, when $S$ is an infinite subset of the set of prime numbers.
2. Preliminaries

Let \( L \) be the language of ordered rings based on the symbols \(+, -, \cdot, 0, 1, \leq\). We write \( \mathbb{N}^* \) for \( \mathbb{N} \setminus \{0\} \). We will work with the following set of axioms in \( L \):

**DOR:** discretely ordered rings, i.e., axioms for ordered rings and
\[
\forall x(0 < x < 1).
\]

**ZR:** discretely ordered \( \mathbb{Z} \)-rings, i.e., DOR and for every \( n \in \mathbb{N}^* \)
\[
\forall x \exists q, r(x = nq + r) \land 0 \leq r < n).
\]

We denote the sentence “DOR + \( \forall x \exists q, r(x = nq + r) \land 0 \leq r < n) \)” by \((ZR)_n\).

Suppose \( \mathbb{P} \) denote the set of prime numbers of \( \mathbb{N} \). Let \( S \) be a nonempty subset of \( \mathbb{P} \). We define the subsystem \((ZR)_S\) of \( ZR \) as the below:

**ZR\(_S\):** DOR + for every \( p \in S \)
\[
\forall x \exists q, r(x = qp + r) \land 0 \leq r < p).
\]

If \( S = \{p_1, \ldots, p_n\} \) is a finite subset of \( \mathbb{P} \), we write \((ZR)_{p_1, \ldots, p_n}\) instead of \((ZR)_{\{p_1, \ldots, p_n\}}\). This is consistent with the above notation \((ZR)_n\).

**OI:** open induction, i.e., DOR and for every open \( L \)-formula \( \psi(\vec{x}, y) \)
\[
\forall \vec{x}(\psi(\vec{x}, 0) \land \forall y \geq 0(\psi(\vec{x}, y) \rightarrow (\psi(\vec{x}, y + 1) \rightarrow \forall y \geq 0(\psi(\vec{x}, y)))).
\]

By considering the fact that in discretely ordered rings an open \( L \)-formula \( \varphi(\vec{x}, y) \) can be written as a Boolean combination of polynomial equalities and inequalities with the variable \( y \) and the parameters \( \vec{x} \), there exist natural numbers \( m, n \) such that:
\[
\varphi(\vec{x}, y) = \bigwedge_{i \leq m} \bigvee_{j \leq n} p_{ij}(\vec{x}, y) \leq q_{ij}(\vec{x}, y),
\]
we can define the degree of \( \varphi(\vec{x}, y) \) relative to \( y \) by
\[
\deg \varphi(\vec{x}, y) = \max \{\deg_y p_{ij}(\vec{x}, y), \deg_y q_{ij}(\vec{x}, y)\} \leq m, j \leq n\}.
\]

**\((OI)_p\):** open induction up to degree \( p \) (i.e., DOR and for every open \( L \)-formula \( \psi(\vec{x}, y) \) with \( \deg \psi(\vec{x}, y) \leq p \)
\[
\forall \vec{x}(\psi(\vec{x}, 0) \land \forall y \geq 0(\psi(\vec{x}, y) \rightarrow (\psi(\vec{x}, y + 1) \rightarrow \forall y \geq 0(\psi(\vec{x}, y)))).
\]

**N:** normality (i.e., being domain and integrally closed in its fraction field, namely for every \( n \in \mathbb{N}^*, \forall x, y, z_1, \ldots, z_n \)
\[
(y \neq 0 \land x^n + z_1 x^{n-1} y + \ldots + z_{n-1} xy^{n-1} + z_n y^n = 0 \rightarrow \exists z(yz = x))).
\]

**\((N)_{n}\):** normality up to degree \( n \in \mathbb{N}^* \) (i.e., being domain and for every \( m \in \mathbb{N}^*, m \leq n, \forall x, y, z_1, \ldots, z_m \)
\[
(y \neq 0 \land x^m + z_1 x^{m-1} y + \ldots + z_{m-1} xy^{m-1} + z_m y^m = 0 \rightarrow \exists z(yz = x))).
\]

It is clear that any domain satisfies \((N)_1\).

**GCD:** having greatest common divisor (i.e., the usual axioms for being a domain plus
\[
\forall x, y(x = y = 0 \lor \exists z(z \land z \lor (\forall t(\forall t(x \land t(y) \rightarrow (t(x \land t(y)))))),
\]
where \( x|y \) is an abbreviation for \( \exists t(t \cdot x = y)) \).

**Bez:** the usual axioms for being a domain plus the **Bezout property**.
\[ \forall x, y \exists z, t((xz + yt) | x \land (xz + yt) | y), \]

namely, every finitely generated ideal is principal.

It is known that Bez ⊥ GCD ⊥ N, and OI ⊄ OI + N ⊄ OI + GCD ⊄ OI + Bez (Smith [7], Lemmas 1.9 and 1.10).

Also we will need another algebraic property, though it is not first-order expressible:

**DCC**: let \( M \) be a domain. \( M \) has the divisor chain condition (DCC) if \( M \) contains no infinite sequence of elements \( a_0, a_1, a_2, \ldots \) such that each \( a_{i+1} \) is a proper divisor of \( a_i \) (i.e., \( a_i/a_{i+1} \) is a nonunit).

Let \( M \) be an ordered domain (resp. a domain), then \( RC(M) \) (resp. \( AC(M) \)) will denote the real closure (resp. the algebraic closure) of its fraction field. It is well known that \( AC(M) = RC(M)[\sqrt{-1}] \). Let \( p \in \mathbb{N}^* \) and \( F \) be an ordered field (resp. a field), we define the \( p \)-real closure (resp. the \( p \)-algebraic closure) of \( F \), denoted by \( RC_p(F) \) (resp. \( AC_p(F) \)), to be the smallest subfield of \( RC(F) \) (resp. \( AC(F) \)) containing \( F \) such that every polynomial of degree \( \leq p \) with coefficients in \( RC_p(M) \) (resp. \( AC_p(F) \)) which has a root in \( RC(F) \) (resp. \( AC(F) \)) also has a root in \( RC_p(F) \) (resp. \( AC_p(F) \)). Similarly if \( M \) be an ordered domain (resp. a domain), then \( RC_p(M) \) (resp. \( AC_p(M) \)) will denote the \( p \)-real closure (resp. the \( p \)-algebraic closure) of its fraction field. It can be shown that \( AC_p(M) = RC_p(M)[\sqrt{-1}] \). Similar to real closed fields and algebraic closed fields, it is also easily seen that:

1. If \( P(x) \) is a polynomial of degree \( \leq p \) with the coefficients in \( RC_p(F) \) and \( P(a) < 0 < P(b) \), for some \( a < b \) in \( RC_p(F) \), then there exists \( a < c < b \) such that \( P(c) = 0 \).

2. If \( P(x) \) is a polynomial of degree \( \leq p \) with the coefficients in \( AC_p(F) \), then \( P(x) \) can be represented as a product of linear factors with coefficients in \( AC_p(F) \).

Properties (1) and (2) can define and axiomatize the notions of \( p \)-real closed field and \( p \)-algebraic closed field, denoted by \( (RCF)_p \) and \( (ACF)_p \), respectively.

Given two ordered domains \( I \subset K \) we say that \( I \) is an integer part of \( K \) if \( I \) is discrete and for every element \( \alpha \in K \), there exists an element \( a \in I \) such that \( 0 \leq \alpha - a < 1 \). We call \( a \), the integer part of \( \alpha \), and sometimes denote it by \( [\alpha]_I \). Shepherdson and Boughattas characterized models of \( (OI)_p \), in terms of \( p \)-real closed fields \( (1 \leq p \leq \omega) \):

**Theorem 2.1** (Shepherdson [9]). Let \( M \) be an ordered domain. \( M \) is a model of \( OI \) iff \( M \) is an integer part of \( RC(M) \).

**Theorem 2.2** (Boughattas [1, 2]). Let \( M \) be an ordered domain. \( M \) is a model of \( (OI)_p \) iff \( M \) is an integer part of \( RC_p(M) \).

We also need a fact from Puiseux series:

**Definition 2.3**. Let \( K \) be a field. The following is the field of Puiseux series in descending powers of \( x \) with coefficients in \( K \):

\[ \forall x, y \exists z, t((xz + yt) | x \land (xz + yt) | y), \]
$K((x^{1/N})) = \{ \sum_{k \leq m} a_k x^{k/r} : m \in \mathbb{Z}, r \in \mathbb{N}^*, a_k \in K \}$.

**Theorem 2.4** (Boughattas [1]). (1 ≤ p ≤ ω) $K$ is a p-real (resp. p-algebraically) closed field iff $K((x^{1/N}))$ is a p-real (resp. p-algebraically) closed field.

3. **The main results**

3.1. **Proof of Theorem A.** Suppose $S$ is a subset of the set of prime numbers $\mathbb{P}$. We present here a relative to $S$ version of some theorems of (Smith [8]) that is needed for proving theorem A. Interestingly, all proofs of (Smith [8]) remain valid, if we make routine changes which will be explained. We mention that when $S = \mathbb{P}$, we get the original definitions and theorems. We first define $\mathbb{Z}_S$ we get the original definitions and theorems. We first define $\mathbb{Z}_S$ for $n \in \mathbb{N}$ for $n \leq m$.

**Lemma 3.1.** $M_{\varphi,S} \models (ZR)_S$.

Parsimony of homomorphisms plays a very important role in Smith’s constructions. Therefore we have the following definition:

**Definition 3.2.** Let $M$ be a discretely ordered ring with $\varphi : M \to \mathbb{Z}_S$ a homomorphism and assume that all standard primes remain prime in $M$. We form a new ring $M_{\varphi,S} = \{ a/n : a \in M, n \in \langle S \rangle \}$ and $n | \varphi(a)$ in $\mathbb{Z}_S$. We extend $\varphi$ to $M_{\varphi,S}$ in the obvious way. We say that $M_{\varphi,S}$ is obtained from $M$ by the $\mathbb{Z}_S$-construction. By relativizing the proof of (Macintyre-Marker [3], Lemma 3.1) we get:

**Lemma 3.3.** If $\varphi : M \to \mathbb{Z}_S$ is $S$-parsimonious, then the extension of $\varphi$ to $M_{\varphi,S}$ is $S$-parsimonious.

**Proof.** The proof is the $S$-relativization of Smith’s proof of Lemma 5.1. in [8]. Let $0 \neq a/n \in M_{\varphi,S}$, where $a \in M$, $n \in \langle S \rangle$. Suppose $\psi_k(a/n) = 0$, for a $k \in \langle S \rangle$. Since $M_{\varphi,S}$ is a model of $(ZR)_S$, we have $k | a/n$ in $M_{\varphi,S}$, so in particular $k | a$ in $M_{\varphi,S}$. Thus $\psi_k(a) = 0$. Since $\varphi : M \to \mathbb{Z}_S$ is $S$-parsimonious, there are only finitely many possibilities for $k \in \langle S \rangle$.

The following theorem says that in the presence of having a $S$-parsimonious map the $\mathbb{Z}_S$-construction preserves GCD and DCC.
**Theorem 3.4.** Let $M$ be a discretely ordered ring with the GCD (DCC). Let $\varphi : M \rightarrow \hat{\mathbb{Z}}_S$ be $S$-parsimonious and in the DCC case the standard primes remain prime in $M$. Then $M_{\varphi,S}$ has the GCD (DCC).

**Proof.** We leave the proof to the reader as an easy and instructive exercise to adopt Smith’s proofs of Theorems 5.3. and 5.5. in [8]. Just replace everywhere in the proof, $\mathbb{Z}$-ring by a model of $(\mathbb{Z}R)_S$, $\varphi : M \rightarrow \hat{\mathbb{Z}}$ by $\varphi : M \rightarrow \hat{\mathbb{Z}}_S$, parsimonious by $S$-parsimonious, $M_{\varphi}$ by $M_{\varphi,S}$, and check that the arguments remain valid! \(\square\)

Transcendental extensions preserve GCD and DCC.

**Theorem 3.5** (Smith [8], Theorems 6.8. and 6.10.). Let $M$ be a GCD (DCC) domain and suppose $x$ is transcendental over $M$. Then $M[x]$ is a GCD (DCC) domain.

By the same adaptation of Theorem 6.12. of (Smith [8]), we see that $S$-parsimonious maps can be extended to transcendental extensions. More precisely:

**Theorem 3.6.** Let $M$ be a countable model of $(\mathbb{Z}R)_S$ and suppose the remainder homomorphism $\varphi : M \rightarrow \hat{\mathbb{Z}}_S$ is $S$-parsimonious. Let $x$ be transcendental over $M$ and suppose $M[x]$ is discretely ordered (and this ordering restricts to the original ordering on $M$). Then $\varphi$ can be extended to a $S$-parsimonious $\varphi : M[x] \rightarrow \hat{\mathbb{Z}}_S$, such that $\varphi(x)$ is a unit of $\hat{\mathbb{Z}}_S$.

We will need in this paper to consider the property of factoriality (a factorial domain has the property that any nonunit has a factorization into irreducible elements, and this factorization is unique up to units). We will use the following theorem:

**Theorem 3.7** (Smith [8], Theorem 1.5.). $M$ is factorial iff $M$ has both of the GCD property and DCC.

In order to gain a Bezout domain the F-construction in Macintyre-Marker paper [3] has a crucial role. By combining Theorems 8.5 and 8.7 from (Smith [8]), Lemma 3.26 of (Macintyre-Marker [3]) and its proof, we have:

**Theorem 3.8.** Let $M$ be a discretely ordered domain with DCC (GCD) and suppose $v, w \in M$ are primes and $x$ is larger than any element of $M$. Let $M^* = M[x, \frac{1-x}{w}]$. Then $M^*$ is a discretely ordered domain with DCC (GCD).

In the following theorem we see that $S$-parsimony can be extended in F-constructions:

**Theorem 3.9.** Let $M$ be a countable model of $(\mathbb{Z}R)_S$ and the remainder homomorphism $\varphi : M \rightarrow \hat{\mathbb{Z}}_S$ is $S$-parsimonious. Let $v, w \in M$ be primes of $M$ and $w$ is nonstandard. Suppose $x$ is transcendental over $M$, and the discrete ordering of $M$ extends to discrete ordering on $M^* = M[x, \frac{1-x}{w}]$. Then $\varphi$ can be extended to $S$-parsimonious $\varphi : M^* \rightarrow \hat{\mathbb{Z}}_S$, such that $\varphi(x)$ is a unit of $\hat{\mathbb{Z}}_S$.

**Proof.** See the proof of Theorem 8.9. of (Smith [8]). \(\square\)

The next theorem guarantees the preservation of the GCD property and DCC in chains constructed by alternative applications of the F-construction and the $\hat{\mathbb{Z}}_S$-construction via parsimonious maps. We express the theorems in a more restricted and more suitable form which is adequate for us:
Theorem 3.10. Suppose $M_0$ is a (GCD) DCC countable model of $(ZR)_S$ and there is $S$-parsimonious $\varphi : M_0 \rightarrow \hat{Z}_S$. Let $\{M_i : i \in \mathbb{N}\}$ be a chain of discretely ordered domains such that $M_{2i+1}$ is constructed from $M_{2i}$ by the $\hat{Z}_S$-construction, and $M_{2i+2}$ is constructed from $M_{2i+1}$ by the $F$-construction. In addition we suppose that in the DCC case, in the whole process of extending rings at most finitely many irreducibles have been killed (this means that only finitely many irreducibles will become reducible in later stages). Then $M = \bigcup_{i \in \mathbb{N}} M_i$ is a model of (GCD) DCC.

Proof. See Theorems 9.4. and 9.8. in (Smith [8]).

By the following series of easy lemmas, we will not worry about DCC in our chain of models in the proof of Theorem A:

Lemma 3.11 (Smith [8], Lemma 3.8.). Let $M$ be a GCD domain. Then $p \in M$ is irreducible iff it is prime.

Of course the following lemma needs an easy $S$-adaptation of Lemma 3.2 in (Macintyre-Marker [3]):

Lemma 3.12. Let $M \models DOR$ and $\varphi : M \rightarrow \hat{Z}_S$ be a ring homomorphism and assume that all standard primes remain prime in $M$. If $q \in M$ is irreducible and $\varphi(q)$ is unit in $\hat{Z}_S$, then $q$ is irreducible in $M_{\varphi,S}$.

Lemma 3.13 (Macintyre-Marker [3], Lemma 3.27). If $q$ is irreducible in $M$, then $q$ is irreducible in $M^*$, constructed in Theorem 3.8 (by the $F$-construction).

Now we have gathered all preliminaries to prove Theorem A:

Theorem A. Suppose $S$ is a nonempty subset of $\mathbb{P}$ and $q$ is prime number such that $q \notin S$, then $(ZR)_S + Bez \varphi (ZR)_q$.

Proof. We do a suitable and modified version of Smith’s process to construct a Bezout model of open induction (Smith [8] Theorem 10.7.). We shall inductively construct an $\omega$-chain of models $M_i$ such that $\bigcup M_i = M_\omega$ will be a model of $(ZR)_S + Bez + \neg(ZR)_q$. We work inside the ordered field $\mathbb{Q}(x_1,\ldots,x_i,\ldots)$ so that for each $i \in \omega$, $x_{i+1}$ is larger than any element of $\mathbb{Q}(x_1,\ldots,x_i)$ and $x_1$ is infinitely large. We will do the $F$-construction at odd stages and the $\hat{Z}_S$-construction at even stages.

Take $M_0 = \mathbb{Z}$ together the natural remainder $S$-parsimonious homomorphism $\varphi : M_0 \rightarrow \hat{Z}_S$. Let us show what we do at stages $2k+1$. Suppose $M_{2k}$ and a $S$-parsimonious map $\varphi : M_{2k} \rightarrow \hat{Z}_S$, have been constructed. At this stage we consider a pair of distinct primes $v$ and $w$ belonging to $M_{2k}$ such that $w$ is nonstandard. (Of course we do this in such a way that every such pair of primes in $M_\omega$ will have been considered at some stage $2k+1$). Thus $(v,w) = 1$ in $M_{2k}$. We define $M_{2k+1} = M_{2k}[x_k,\frac{1-\omega x}{w}]$ according to Theorem 3.8. Suppose $y_k = \frac{1-\omega x}{w}$, then we have $x_k v + y_k w = 1$ in $M_{2k+1}$. So $(v,w)_B = 1$ in $M_{2k+1}$. $(v,w)_B$ is the Bezout greatest common divisor of $v$ and $w$, it means that $(v,w)_B|v$ and $(v,w)_B|w$ and there exist $r$ and $s$ in $M_{2k+1}$ such that $rv + su = 1$. We refer to (Smith [8], Section 3) for the basic related definitions and theorems. By Theorem 3.9 $\varphi$ is extended to a $S$-parsimonious map $\varphi : M_{2k+1} \rightarrow \hat{Z}_S$. At stage $2k+2$, we employ Lemma 3.1 and define $M_{2k+2} = (M_{2k+1})_{\varphi,S}$ which is a model of $(ZR)_S$. Lemma
3.3 gives us the desired parsimonious extensions \( \varphi : M_{2k+2} \rightarrow \hat{\mathbb{Z}}_S. \) Since \((ZR)_S\) is a \(\forall\exists\)-theory, then it is preserved in chains, therefore \(M_\omega \models (ZR)_S.\)

Now we show that \(M_\omega\) is a Bezout domain. The proof is similar to (Smith [8], Theorem 10.7) with a minor change. By Theorems 3.4 and 3.8, each \(M_\omega\) has the GCD and DCC, so by Theorem 3.10 \(M_\omega\) has both the GCD and DCC (by Lemmas 3.11, 3.12 and 3.13 we know that no irreducible is killed) and from Theorem 3.7 we conclude that \(M_\omega\) is a factorial domain. In order to show that \(M_\omega\) is a Bezout domain, by considering the fact that \(M_\omega\) has the GCD property, it suffices to prove that any two elements of \(M_\omega\) has the Bezout greatest common divisor. Let \(a, b \in M_\omega\) and let \(c = (a, b)\) in \(M_\omega\). We can assume \(a, b > 1\). Let \(a = a_1 c, b = b_1 c\) in \(M_\omega\). So \((a_1, b_1) = 1\) in \(M_\omega\). Since \(M_\omega\) is factorial, we can write \(a' = mP_1^{e_1} \cdots P_k^{e_k}\) and \(b' = nQ_1^{f_1} \cdots Q_l^{f_l}\), where \(m, n \in \mathbb{N}\) are nonzero, \(k, l \geq 0\) and the \(P_i, Q_j\) are nonstandard primes such that \(P_i \neq Q_j\) for all \(i, j\). We will show that \((a', b')_B = 1\). Clearly \((m, n)_B = 1\).

Now we show that \((ZR)_q\) fails in \(M_\omega\). We first observe that in the first step of our construction, namely, when passing from \(M_0 = \mathbb{Z}\) to \(M_1\), there is no nonstandard prime in \(M_0\). So \(M_1\) is just \(\mathbb{Z}\{x_1\}\) and we have no \(y_1\). On the other hand from the construction it is evident that elements of \(M_\omega\) are of the form \(f(x_1, x_2, y_2, \ldots, x_k, y_k)\), for some \(k\), where \(f\) is a polynomial with the coefficients in the set \(\mathbb{Z}(S) = \{a/k; a \in \mathbb{Z} \text{ and } k \in \langle S \rangle \}\). Now for a contradiction, suppose \(M_\omega\) is a model of \((ZR)_q\). Then there is a \(b \in M_\omega\) such that \(x_1 = bq + r\) with \(0 \leq r < q\). Take \(b = f(x_1, x_2, y_2, \ldots, x_k, y_k)\), so we have \(x_1 = f(x_1, x_2, y_2, \ldots, x_k, y_k)q + r\). Observe that \(x_2, y_2, \ldots, x_k, y_k\) are transcendental over \(\mathbb{Q}(x_1)\), then \(f\) does not depend on them, so we can assume \(x_1 = f(x_1)q + r\). Since \(x_1\) is also transcendental over \(\mathbb{Q}\), it follow that the degree of \(f\) must be one. Thus \(f(x_1) = ax_1\) and \(a \in \mathbb{Z}(S)\). So \(x_1 = ax_1q + r\) and then \(x_1(1 - aq) = r\), which implies that \(a = 1/q\) and this is in contradiction with \(a \in \mathbb{Z}(S)\), since \(q \notin \langle S \rangle\). 

\[\square\]

3.2. Proof of Theorem B. Now we prove:

**Theorem B.** Suppose \(S\) is a nonempty subset of \(\mathbb{P}\) and \(q\) is a prime number such that \(q \notin S\), then \((ZR)_S + (N)_n + \neg(ZR)_q + \neg(N)_{n+1}\) is consistent.

**Proof.** In [4], we proved that if \(n \neq 3\), there is a \(\lambda\) which is real algebraic of degree \(n + 1\) over \(\mathbb{Q}\) and doesn’t belong to \(RC_n(\mathbb{Q})\). Now suppose \(x\) is an infinitely large element. For \(n \neq 3\), fix \(\lambda\) as above. For \(n = 3\) we choose \(\lambda\) as a root of an irreducible
polynomial of degree 4 such that \( \lambda \notin RC_d(\mathbb{Q}) \). Let \( A \) be the ring of integers of the algebraic number field \( \mathbb{Q}(\lambda) \). Form \( A_{(S)} = \{ a/k; a \in A \text{ and } k \in (S) \} \). It is an elementary fact from algebraic number theory that \( A \) is a normal ring. Since \( A_{(S)} \) is a localization of \( A \) relative to a multiplicative set, then it is also normal. Let \( M = \mathbb{Z}[x]; r \in A_{(S)} \). We claim that \( M \) witnesses Theorem B. It is obvious that \( M \models (ZR)_{\mathbb{S}} \). By an argument similar to the last paragraph of the proof of theorem \( A \), it is easily shown that \( M \models -\mathbb{Z}(QR)_{\mathbb{S}} \).

Now we prove \( M \models -\mathbb{N}^{n+1} \). Let \( v \in \mathbb{N} \) be such that \( v\lambda \) is an algebraic integer. Suppose \( P(t) \in \mathbb{Z} \) is its minimal polynomial of degree \( n+1 \) which is monic. Obviously \( v\lambda x \in M \). But we have \( P(v\lambda x / x) = 0 \), while \( v\lambda \notin M \). So \( M \) is not a model of \( \mathbb{N}^{n+1} \).

It remains to show that \( M \models \mathbb{N}^n \). Let \( u,v \) be nonzero elements of \( M \) such that

\[
(u/v)^s + z_1(u/v)^{s-1} + \cdots + z_s = 0 \quad (z_1, \ldots, z_s \in M, s \leq n).
\]

We will show that \( u/v \in M \). Notice that elements of \( M \) are those elements of \( A_{(S)}[x] \) with integer constant coefficient. \( A_{(S)} \) is normal, so is \( A_{(S)}[x] \). Thus \( u/v \in A_{(S)}[x] \).

On the other hand, since \( \mathbb{Q}(\lambda)[x] \) is a factorial ring, \( u/v \) can be written as:

\[
u/v = \rho \prod_{i \in I} P_i \prod_{j \in J} Q_j,
\]

in which \( \rho \in \mathbb{Q}(\lambda) \), the \( P_i \)'s are irreducible in \( \mathbb{Q}(\lambda)[x] \), without constant coefficient and \( Q_j \)'s are irreducible in \( \mathbb{Q}(\lambda)[x] \) with the constant coefficient one. If \( I \) is nonempty, then \( \rho \prod_{i \in I} P_i \prod_{j \in J} Q_j \) has no constant coefficient and thus \( u/v \in M \). Now suppose \( I = \emptyset \). Put \( x = 0 \) in \( u, v, z_1, \ldots, z_s \). Therefore \( \rho \) is an algebraic integer with the degree, equal or less than \( n \) over \( \mathbb{Z} \). We show it is one. If \( n = 1 \) there is nothing to prove. If not, we have \( |\mathbb{Q}(\lambda):\mathbb{Q}(\rho)| < n + 1 \). But

\[
|\mathbb{Q}(\lambda):\mathbb{Q}(\rho)||\mathbb{Q}(\rho):\mathbb{Q}| = |\mathbb{Q}(\lambda):\mathbb{Q}| = n + 1.
\]

Then \( |\mathbb{Q}(\lambda):\mathbb{Q}(\rho)| \) divides \( |\mathbb{Q}(\lambda):\mathbb{Q}| = n+1 \). So we have a chain of field extensions, \( \mathbb{Q} \subset \mathbb{Q}(\rho) \subset \mathbb{Q}(\lambda) \) such that \( |\mathbb{Q}(\lambda):\mathbb{Q}(\rho)| \leq n - 1 \) and \( |\mathbb{Q}(\rho):\mathbb{Q}| \leq n - 1 \). This implies that \( \lambda \in RC_{n-1}(\mathbb{Q}) \) which is in contradiction with the choice of \( \lambda \). Hence \( \rho \) is an algebraic integer of degree one. So \( \rho \in \mathbb{Z} \) and this implies that \( u/v \in M \), which means that \( M \) is model of \( \mathbb{N}^n \). This completes the proof of Theorem B. \( \Box \)

3.3. Proofs of Theorems C and D.. In order to demonstrate Theorem C, we need a generalization of a theorem of Bougattas. In ([2], Theorem V.1.) Bougattas proved that every saturated ordered field admits a normal integer part. But we show that:

**Lemma 3.14.** Every \( \omega_1 \)-saturated ordered field admits a Bezout integer part.

**Proof.** (Sketch) Suppose \( K \) is an \( \omega_1 \)-saturated ordered field. Bougattas [2] in a series of three Lemmas: Pricipal, Integer Part and Construction, showed that we can build an \( \omega_1 \)-chain of countable discretly ordered rings \( M_i, i < \omega \) such that \( M = \bigcup_{i < \omega_1} M_i \) is an integer part of \( K \). Furthermore he considers an arbitrary subset \( \Lambda \subset K \) of real algebraic elements which plays a role in the construction of the \( M_i \)'s. Varying \( \Lambda \) gives us various kinds of integer parts. When \( \Lambda = \emptyset \), we obtain a normal integer part and it is implicit in the paper that in this case, the \( M_i \)'s are obtained by alternative applications of the Wilkie-construction and the \( \hat{\mathbb{Z}} \)-construction. But it must be noticed that even in this case the procedure of doing the \( \hat{\mathbb{Z}} \)-construction is different from the original one, because it is no longer
assumed that the ground field is dense in its real closure. To gain a Bezout integer part, we observe that we can do the procedure of the Theorems 10.7 and 10.8 of Smith [8] inside K. In this process we need the extra F-construction. Since any \( M_i, i < \omega_1 \) is countable and \( K \) is \( \omega_1 \)-saturated, then there is always an element \( b_i \) in \( K \) which is larger than any element of \( M_i \). By Lemma 3.26 of (Macintyre-Marker [3]) we are sure that we can do the F-construction. To obtain an integer part of \( K \), suppose \( (b_\alpha, \alpha < \omega_1) \) be an enumeration of elements of \( K \). Let \( M_i \) has been constructed and at step \( i + 1 \) we want to do the Wilkie-construction. We seek the least ordinal \( \alpha_i \), such that \( b_{\alpha_i} \) has not an integer part in \( M_i \). Then by combining the Integer Part Lemma of Boughattas [2] with the \( \hat{\alpha} \)-construction, we obtain \( M_{i+1} \) with its parsimonious homomorphism extension to \( \hat{\mathbb{Z}} \), such that \( b_{\alpha_i} \) has an integer part in \( M_{i+1} \). Also suppose \( M_j \) has been constructed and at stage \( j + 1 \) we want to do the F-construction. We seek the least ordinal \( \alpha_j \), such that \( b_{\alpha_j} \) is larger than any element of \( M_j \). Then the F-construction can be done at this step. At limit stages we take union. Moreover, Lemma 9.1, Theorem 9.4 and Theorem 9.8 of (Smith [8]) will guarantee preserving parsimony of homomorphisms and factoriality at limit stages of length \( \leq \omega_1 \). Now there is no obstacle for \( M = \bigcup_{i<\omega_1} M_i \) to be a Bezout integer part of \( K \).

\[ \square \]

**Theorem C.** Be\( \varepsilon \) + (\( OI \))\( _p \) \( \not\models \) (\( OI \))\( _{p+1} \), when \( p \neq 3 \).

**Proof of Theorem C.** In [4], we showed that if \( p \neq 3 \), there is an irreducible polynomial \( P(t) \) of degree \( p + 1 \) over \( \mathbb{Q} \) such that \( P(t) \) has no root in \( RCp(\mathbb{Q}) \). For \( p \geq 4 \), \( P(t) \) was a polynomial with Galois group \( A_{p+1} \). It is well known that we can take \( P(t) \) as a monic polynomial with integer coefficients such that \( P(0) < 0 \). Let \( T \) be the following theory in the language of ordered field with the additional constant symbol \( a \):

\[ T \equiv (RCF)_p + \{ a > k; k \in \mathbb{N} \} + \forall y -(Q(y) \leq 0 < Q(y + 1)) \]

where \( Q(y) = a^{p+1}P(y/a) \). We show that the field of Puiseux power series \( RCp(\mathbb{Q})((x^{1/\mathbb{N}})) \) is a model of \( T \), when interpreting \( a \) by \( x \). Clearly by Theorem 2.4, \( RCp(\mathbb{Q})((x^{1/\mathbb{N}})) \) is a \( p \)-real closed field. Also on the contrary suppose that there exists \( y \in RCp(\mathbb{Q})((x^{1/\mathbb{N}})) \) such that \((Q(y) \leq 0 < Q(y + 1))\). Therefore \( P(y/x) \leq 0 < P((y + 1)/x) \). It is easily seen that degree \( y/x \) must be zero. So let \( y/x = \lambda + \sum_{-\infty < i < 0} c_i x^{i/q} \) in \( RCp(\mathbb{Q})((x^{1/\mathbb{N}})) \). This leads to \( P(\lambda) = 0 \), but \( \lambda \in RCp(\mathbb{Q}) \), which is in contradiction to the choice of \( P(t) \). So \( RCp(\mathbb{Q})((x^{1/\mathbb{N}})) \not \models T \).

Now that \( T \) is consistent, let \( K \) be an \( \omega_1 \)-saturated model of \( T \). By Lemma 3.14 \( K \) has a Bezout integer part. Call it \( M \). Since \( K \models (RCF)_p \), then \( M \models (OI)_p \). On the other hand there is \( n \in \mathbb{N} \) such that \( M \models Q(0) < 0 < Q(n[a]_M) \), where \( [a]_M \) is the integer part of \( a \) in \( M \). But \( K \models \forall y -(Q(y) \leq 0 < Q(y + 1)) \), then \( M \models \forall y -(Q(y) \leq 0 < Q(y + 1)) \), so by (Boughattas [1], Proposition A.1), \( M \models \neg (OI)_{p+1} \). This ends the proof of Be\( \varepsilon \) + (\( OI \))\( _p \) \( \not\models \) (\( OI \))\( _{p+1} \).}
Theorem D. \((OI)_p + \neg(OI)_{p+1} + (N)_n + \neg(N)_{n+1}\) is consistent, when \(p \neq 3\).

Proof of Theorem D. We work with the same theory \(T\) and its saturated model \(K\) as in the proof of Theorem C. Choose \(\Lambda = \{\lambda\}\) and fix \(\lambda\) as in the proof of Theorem B, namely, if \(n \neq 3\), \(\lambda \in RC_{n+1}(\mathbb{Q})\), \(\lambda \notin RC_n(\mathbb{Q})\) and if \(n = 3\) choose \(\lambda\) as a root of an irreducible polynomial of degree 4 such that \(\lambda \notin RC_2(\mathbb{Q})\). Then by Theorem 3.15, there exists \(X \subset K\) such that \(K\) has the integer part \(M = \mathbb{Z}[\{rx; r \in \mathbb{Q}(\lambda)\text{ and } x \in X\}]\). To show \(M \models (N)_n + \neg(N)_{n+1}\), we can repeat the proof of Theorem B, just replace \(x\) by \(X\) and replace \(A(S)\) by \(\mathbb{Q}(\lambda)\). \(\mathbb{Q}(\lambda)[X]\) remains factorial and normal, so the proof works. By the last paragraph of the proof of the Theorem C, it is obvious that \(M \models (OI)_p + \neg(OI)_{p+1}\) ■

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References

1. Sedki Boughattas, L’arithmtique ouverte et ses modles non-standards, J. Symbolic Logic 56 (1991), 700–714.
2. ________, L’induction ouverte dans les anneaux discrets ordonnes et normaux n’est pas finiment axiomatisable, J. London Math. Soc. (2) 53 (1996), no. 3, 455–463.
3. Angus Macintyre and David Marker, Primes and their residue rings in models of open induction, Ann. Pure Appl. Logic 43 (1989), no. 1, 57–77.
4. Shahram Mohsenipour, A note on subsystems of open induction, J. Symbolic Logic 72 (2007), no. 4, 1318–1322.
5. Margarita Otero, Generic models of the theory of normal \(\mathbb{Z}\)-rings, Notre Dame J. Formal Logic 33 (1992), no. 3, 322–331.
6. J. C. Shepherdson, A non-standard model for a free variable fragment of number theory, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 12 (1964), 79–86.
7. Stuart T. Smith, Prime numbers and factorization in \(\mathbb{IE}_1\) and weaker systems, J. Symbolic Logic 57 (1992), no. 3, 1057–1085.
8. ________, Building discretely ordered Bezout domains and GCD domains, J. Algebra 159 (1993), no. 1, 191–239.
9. Lou van den Dries, Some model theory and number theory for models of weak systems of arithmetic, Model theory of algebra and arithmetic (Proc. Conf., Karpacz, 1979), Lecture Notes in Math., vol. 834, Springer, Berlin, 1980, pp. 346–362.
10. A. J. Wilkie, Some results and problems on weak systems of arithmetic, Logic Colloquium ’77 (Proc. Conf., Wroclaw, 1977), Stud. Logic Foundations Math., vol. 96, North-Holland, Amsterdam, 1978, pp. 285–296.
11. George Wilmers, Bounded existential induction, J. Symbolic Logic 50 (1985), no. 1, 72–90.

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