Thermodynamics of resonances and blurred particles

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Abstract

Exact and approximate expressions for thermodynamic characteristics of heated matter, which consists of particles with finite mass-widths, are constructed. They are expressed in terms of Fermi/Bose distributions and spectral functions, rather than in terms of more complicated combinations between real and imaginary parts of the self-energies of different particle species. Therefore thermodynamically consistent approximate treatment of systems of particles with finite mass-widths can be performed, provided spectral functions of particle species are known. Approximation of the free resonance gas at low densities is studied. Simple ansatz for the energy dependence of the spectral function is suggested that allows to fulfill thermodynamical consistency conditions. On examples it is shown that a simple description of dense systems of interacting particle species can be constructed, provided some species can be treated in the quasiparticle approximation and others as particles with widths. The interaction affects quasiparticle contributions, whereas particles with widths can be treated as free. Example is considered of a hot gas of heavy fermions strongly interacting with light bosons, both species with zero chemical potentials. The density of blurred fermions is dramatically increased for high temperatures compared to the standard Boltzmann value. The system consists of boson quasiparticles (with effective masses) interacting with fermion-antifermion blurs. In thermodynamical values interaction terms partially compensate each other. Thereby, in case of a very strong coupling between species thermodynamical quantities of the system, like the energy, pressure and entropy, prove to be such as for the quasi-ideal gas mixture of quasi-free fermion blurs and quasi-free bosons.
1 Introduction

In heavy ion collisions at relativistic energies baryon resonances may play an important role in the reaction dynamics [1]. Therefore the name ”Resonance matter” has been coined for highly excited matter, where a large fraction of nucleons is converted to resonance states. We should specify terms. One deals with a resonance, if the spectral function of the particle has rather sharp peak as function of the energy. In spite of the imaginary part of the self-energy is finite, the pole-like structure of the Green function still remains. In many models resonances are however treated as quasiparticles. The quasiparticle approximation means that one may put imaginary part of the self-energy zero in the expression for the Green function. Then the spectral function becomes the delta-function. Definitely quasiparticle approximation for resonances is only a rough approximation being done for simplification. E.g., in vacuum the $\rho$-meson width at the maximum is as high as about 150 MeV and the $\Delta$-resonance width is about 120 MeV. Resonances with masses up to 2 GeV at RHIC and LHC energies contribute to thermodynamics of the fireball [2]. Some of them have still larger widths. In medium particle widths may get an extra increase due to collision broadening. It may result in a complete blurring of particles. By blurring of the particle we understand here the case, when the Green function is entirely regular. Blurring of fermions may happen, e.g., at sufficiently high temperature and a small baryon concentration [3,4]. Exact solution of the problem is not possible, even if one restricts consideration by a specific choice of the interaction and considers only few particle species. Therefore it is a challenge to construct a simplified but thermodynamically consistent description of heated matter consisting of quasiparticles, resonances and blurred particles.

In this paper we present exact and approximate expressions for thermodynamic characteristics of heated resonance matter that consists of particles with finite mass-widths (resonances and blurs). Thermodynamic quantities are expressed in terms of Fermi/Bose distributions and spectral functions, rather than in terms of more complicated combinations between real and imaginary parts of the self-energies of different particle species, see sect. 2. Solution of the problem is achieved provided spectral functions of particles with widths are known. Different examples, when such a description might be helpful are presented. In sect. 3 we consider gas of free resonances (at low densities). Then in sect. 4 we formulate how one could proceed in description of dense systems of strongly interacting particles with widths in cases, when some species can be treated as quasi-free broad resonances or blurred particles, whereas other species, as interacting quasiparticles. Different examples are considered. In sect. 5 we study example of a strongly interacting hot heavy fermion – light boson system at zero chemical potentials of species. Concluding remarks are presented in sect. 6.
2 Thermodynamic quantities in terms of spectral functions

2.1 General relations

The thermodynamic potential density $\Omega$, pressure $P$, free energy density $F$, energy density $E$ and entropy density $S$ follow thermodynamic relations:

$$E = F + TS, \quad F[f, \omega_0, R_0] = \sum_i \mu_i n_i + \Omega, \quad \Omega = -P,$$  \hspace{1cm} (1)

where $T$ is the temperature and

$$\mu_i = \frac{\partial F}{\partial n_i}$$ \hspace{1cm} (2)

is the chemical potential, and $n_i$ is the density of the $i$-particle species.

The density and the entropy density are the derivatives of the pressure:

$$n_i = \frac{\partial P_i}{\partial \mu_i}|_T, \quad S_i = \frac{\partial P_i}{\partial T}|_{\mu_i}.$$ \hspace{1cm} (3)

These conditions should be fulfilled at any thermodynamically consistent description of properties of the medium.

It is convenient to introduce one-particle spectral and width functions (operators), cf. \cite{6,4},

$$\hat{\Lambda} = -2\text{Im}\hat{G}^R(q) = -2\text{Im}\frac{1}{M + i\hat{\Gamma}/2}, \quad \hat{\Gamma} = -2\text{Im}\hat{\Sigma}^R,$$ \hspace{1cm} (4)

where $\hat{G}^R(q)$ is the full retarded Green function of the resonance, $\hat{\Sigma}^R$ is the fermion retarded self-energy. The quantity

$$\hat{M} = (\hat{G}^{0,R})^{-1} - \text{Re}\hat{\Sigma}^R$$ \hspace{1cm} (5)

demonstrates deviation from the mass shell: $\hat{M} = 0$ on the quasiparticle mass shell in the matter. $\hat{G}^{0,R}$ is the free retarded Green function.
2.2 Non-relativistic particles

For non-relativistic particles (fermions or bosons) spectral function satisfies the sum rule:

\[
\int_{-\infty}^{\infty} A_d\omega \frac{d\omega}{2\pi} = 1.
\]  

(6)

Note that although in (6) the lower integration limit is taken to be \(-\infty\), the spectral function drops to zero at some finite threshold value \(\omega_{th}\). Therefore in reality the lower integration limit is \(\omega_{th}\).

In the quasiparticle limit, when the width \(\Gamma\) is much less than all other relevant physical quantities, the spectral function acquires a \(\delta\) -function shape

\[
A \to A^{q.p.}_{n.r.} = 2\pi\delta(\omega - \omega^{n.r.}_p - \text{Re}\Sigma^R_{n.r.}(\omega, p)),
\]

(7)

\(\omega^{n.r.}_p = p^2/(2m)\), \(m\) is the particle mass. The sum rule renders

\[
\int_{-\infty}^{\infty} A^{q.p.}_{n.r.}\frac{d\omega}{2\pi} = 1 - \frac{\partial\text{Re}\Sigma^R_{n.r.}}{\partial\omega} \mid_{\omega(p)>0},
\]

(8)

where \(\omega(p) = \omega^{n.r.}_p + \text{Re}\Sigma^R_{n.r.}(\omega(p), p)\). To fulfill the exact sum-rule (6) one should take into account a compensating contribution of the whole sea of off-shell modes.

A self-consistent description of non-equilibrium and equilibrium resonance matter beyond the scope of the quasiparticle approximation can be constructed basing on solution of Kadanoff-Baym equations within a \(\Phi\) derivable approach, cf. [5,6,7]. Refs. [5,6] have derived exact Noether expressions for the particle 4-current and the energy-momentum tensor. Results are formulated for non-relativistic particles [6] and for relativistic bosons [5] in case of interactions with non-derivative couplings. For non-relativistic particles:

\[
j^{\mu}_{n.r.} = \text{Tr} \int \frac{d^4p}{(2\pi)^4} v^{\mu} A_{n.r.} f,
\]

(9)

\[
\Theta^{\mu\nu}_{n.r.} = \text{Tr} \int \frac{d^4p}{(2\pi)^4} \epsilon^{\mu\nu} p^\nu A_{n.r.} f + g^{\mu\nu}(\epsilon^{\text{int}}_{n.r.} - \epsilon^{\text{pot}}_{n.r.}),
\]

(10)

\footnote{Generalizations to systems with derivative couplings are done in [7].}
with $\nu^\mu = (1, \vec{v})$, $\vec{v} = \vec{p}/m$. Particle occupations are given by

$$f = \frac{1}{e^{(\omega - \mu_n)/T} \pm 1}, \quad (11)$$

with "+" for fermions, "−" for bosons; for a nucleon resonance such as $\Delta$ from equilibrium condition in respect to the reaction $\Delta \leftrightarrow N + \pi$ it follows that $\mu^{n.r.}_\Delta = \mu^{n.r.}_N$ (since $\mu_\pi = 0$); $\epsilon^{\text{int}}$ and $\epsilon^{\text{pot}}$ are some interaction and potential energies. Explicit expressions for them are presented below. Here and below, considering a hot system, we disregard a contribution of quantum fluctuations.

From (10) we find the energy density

$$\Theta_{00} = E_{n.r.} = N_{n.r.} \int \frac{d^4p}{(2\pi)^4} \omega A_{n.r.} f + \epsilon^{\text{int}}_{n.r.} - \epsilon^{\text{pot}}_{n.r.}, \quad (12)$$

where $N_{n.r.}$ is the degeneracy factor, that appears as the result of taking the trace in (10), and the pressure

$$P_{n.r.} = \frac{1}{3} (\Theta^{11} + \Theta^{22} + \Theta^{33})$$

$$= N_{n.r.} \int \frac{d^4p}{(2\pi)^4} \frac{p^2}{3m} A_{n.r.} f - \epsilon^{\text{int}}_{n.r.} + \epsilon^{\text{pot}}_{n.r.}. \quad (13)$$

Using these expressions and also Eq. (9), from thermodynamical relation (1), Ref. [6] derived exact expression for the entropy density:

$$TS_{n.r.} = N_{n.r.} \int \frac{d^4p}{(2\pi)^4} (\omega + \frac{2}{3} \omega^{n.r.}_p - \mu_{n.r.}) A_{n.r.} f. \quad (14)$$

The interaction terms $\epsilon^{\text{int}}$ and $\epsilon^{\text{pot}}$ have cancelled due to presence of the $g^{\mu\nu}$ factor in (10).

Note that both expressions (9) and (14) for the 4-current and for the entropy are exact and both quantities depend on the specifics of the interaction only through the $A$-spectral function. On the other hand, both the energy (12) and the pressure (13) depend also on a combination $\epsilon^{\text{int}} - \epsilon^{\text{pot}}$ of interaction and potential energies.
2.3 Relativistic bosons

For relativistic spin-less charged bosons\(^2\) the spectral function can be represented in terms of spectral functions of particles and antiparticles, cf. \([7]\),

\[
A_b(\omega, \vec{p}) = A^{(+)}_b(\omega, \vec{p})\theta(\omega) - A^{(-)}_b(-\omega, -\vec{p})\theta(-\omega),
\]

\[(15)\]

\[
A^{(+)}_b(\omega, \vec{p}) = A_b(\omega, \vec{p})\theta(\omega), \quad A^{(-)}_b(\omega, \vec{p}) = -A_b(-\omega, -\vec{p})\theta(\omega),
\]

\[(16)\]

\[
\Sigma^R_b(\omega, \vec{p}) = \Sigma^{R(+)}_b(\omega, \vec{p})\theta(\omega) + \Sigma^{A(-)}_b(-\omega, -\vec{p})\theta(-\omega).
\]

\[(17)\]

Here \(\theta(x)\) is the step-function, \("R"\) denotes retarded and \("A"\), advanced quantity.

The sum rule renders:

\[
\int_0^\infty \frac{d\omega}{2\pi} \left[ A^{(+)}_b(\omega, \vec{p}) + A^{(-)}_b(\omega, -\vec{p}) \right] = 1.
\]

\[(18)\]

In the quasiparticle approximation the spectral function becomes

\[
[A^{q.p.}_b]^{(\pm)} = 2\pi\delta \left[ \omega^2 - (\omega^b_p)^2 - \text{Re}\Sigma^R_b(\omega, p) \right],
\]

\[(19)\]

with \([\omega^b_p]^2 = m^2_b + p^2\), and the sum rule reads

\[
\int_0^\infty [A^{q.p.}_b]^{(\pm)} \omega \frac{d\omega}{2\pi} = \frac{1}{2} \left[ 1 - \frac{\partial \text{Re}\Sigma^R_b(\pm)}{\partial \omega^2} |_{\omega(p)} \right] > 0,
\]

\[(20)\]

where \(\omega^2(p) = m_b^2 + p^2 + \text{Re}\Sigma^R_b(\omega(p), p)\).

Expression for the energy-momentum tensor of the boson sub-system looks similar to Eq. \((10)\), see \([5]\),

\[
\Theta^\mu_\nu^b = \int \frac{d^4p}{(2\pi)^4} 2p^\mu p^\nu A_b f_b + g^\mu\nu (\epsilon^b_{\text{int}} - \epsilon^b_{\text{pot}}),
\]

\[(21)\]

\(^2\) By the charge we mean any conserved quantity like electric charge, strangeness, etc.
where now

$$f_b = \frac{1}{e^{(\omega - \mu_b)/T} - 1}$$  \hspace{1cm} (22)$$

are boson occupations, and $\mu_b$ is the boson chemical potential. Expressions for interaction and potential energies, $\epsilon^{\text{int}}_b$ and $\epsilon^{\text{pot}}_b$, are presented below.

Refs. [5,7] also obtained exact Noether expressions for the charged boson density, the energy density and the pressure:

$$n_b = \int_0^\infty \frac{4\pi p^2 dp}{(2\pi)^3} \int_0^\infty \frac{d\omega}{2\pi} 2\omega A^{(+)}_b f^{(+)}_b - (\mu_b \rightarrow -\mu_b),$$  \hspace{1cm} (23)$$

$$E_b = \int_0^\infty \frac{4\pi p^2 dp}{(2\pi)^3} \int_0^\infty \frac{d\omega}{2\pi} 2\omega^2 A^{(+)}_b f^{(+)}_b + \epsilon^{b(+)}_{\text{int}} - \epsilon^{b(+)}_{\text{pot}} + (\mu_b \rightarrow -\mu_b),$$  \hspace{1cm} (24)$$

$$P_b = \int_0^\infty \frac{4\pi p^2 dp}{(2\pi)^3} \int_0^\infty \frac{d\omega}{2\pi} \frac{2p^2}{3} A^{(+)}_b f^{(+)}_b - \epsilon^{b(+)}_{\text{int}} + \epsilon^{b(+)}_{\text{pot}} + (\mu_b \rightarrow -\mu_b),$$  \hspace{1cm} (25)$$

$$TS_b = \int_0^\infty \frac{4\pi p^2 dp}{(2\pi)^3} \int_0^\infty \frac{d\omega}{2\pi} 2\omega (\omega + \frac{p^2}{3\omega} - \mu_b) A^{(+)}_b f^{(+)}_b + (\mu_b \rightarrow -\mu_b).$$  \hspace{1cm} (26)$$

Antiparticle contributions are included with the help of the replacement $\mu_b \rightarrow -\mu_b$.

### 2.4 Relativistic fermions

For spin 1/2 fermion, spectral function satisfies the sum rule:

$$\frac{1}{4} \text{Tr} \int_0^\infty \gamma_0 \left[ \hat{A}^{(+)}_f (\omega, \vec{p}) + \hat{A}^{(-)}_f (\omega, -\vec{p}) \right] \frac{d\omega}{2\pi} = 2,$$  \hspace{1cm} (27)$$

$\gamma_0$ is the Dirac matrix. The trace is taken over spin degrees of freedom.
In cases when spin degrees of freedom decouple it is sufficient to introduce two functions (for each particle species). E.g.,

\[ \tilde{A}^{(+)}_f = \hat{\rho}\tilde{A}_f + A_{f(1)}, \quad \frac{1}{4} \text{Tr} \gamma_0 \tilde{A}^{(+)}_f = A_f = \tilde{A}_f \omega, \]

(28)

where we explicitly separated \( \omega \)-pre-factor. Then generalizations of non-relativistic expressions for thermodynamic quantities to the case of relativistic fermions are simple. From (9), (10) we recover expressions for the baryon density, the energy density, the pressure and the entropy density:

\[ n_f = N_f \int_0^{\infty} \frac{dp}{(2\pi)^3} \int_0^{\infty} \frac{d\omega}{2\pi} A^{(+)}_f f^{(+)}_f - (\mu_f \rightarrow -\mu_f), \]

(29)

\[ E_f = N_f \int_0^{\infty} \frac{dp}{(2\pi)^3} \int_0^{\infty} \frac{d\omega}{2\pi} \omega A^{(+)}_f f^{(+)}_f + \epsilon^{(+)}_\text{int} - \epsilon^{(+)}_\text{pot} + (\mu_f \rightarrow -\mu_f), \]

(30)

\[ P_f = N_f \text{Tr} \int_0^{\infty} \frac{dp}{(2\pi)^3} \int_0^{\infty} \frac{d\omega}{2\pi} p^2 A^{(+)}_f f^{(+)}_f - \epsilon^{(+)}_\text{int} + \epsilon^{(+)}_\text{pot} + (\mu_f \rightarrow -\mu_f), \]

(31)

\[ TS_f = N_f \int_0^{\infty} \frac{dp}{(2\pi)^3} \int_0^{\infty} \frac{d\omega}{2\pi} (\omega + \frac{p^2}{3\omega} - \mu_f) A^{(+)}_f f^{(+)}_f + (\mu_f \rightarrow -\mu_f), \]

(32)

\[ f^{(+)}_f = \frac{1}{e^{(\omega-\mu_f)/T} + 1}, \]

(33)

\( N_f \) is the degeneracy factor. The replacement \( (\mu_f \rightarrow -\mu_f) \) takes into account antiparticle terms.

2.5 Relations between interaction and potential energies

We will use a relation \[5,6\] between \( \epsilon_{\text{int}} \) and \( \epsilon_{\text{pot}} \):

\[ \epsilon_{\text{int}} = \frac{2}{\alpha} \epsilon_{\text{pot}} \]

(34)
for specific interactions with a certain number $\alpha$ of operators attached to the vertex. This relation is derived using the operator forms of $\hat{\epsilon}_{\text{int}}$ and $\hat{\epsilon}_{\text{pot}}$:

$$\hat{\epsilon}_{\text{int}} = \sum_i \hat{\epsilon}_{\text{int}}^i = -\frac{1}{2} \sum_i (\hat{J}_i^\dagger \hat{\phi}_i + \hat{J}_i \hat{\phi}_i^\dagger),$$

$$G_0^{-1} \hat{\phi}_i = -\hat{J}_i = -\frac{\delta \hat{L}_{\text{int}}}{\delta \phi_i^\dagger}, \quad \hat{\epsilon}_{\text{int}} = -\hat{L}_{\text{int}},$$

(35)

where $\hat{L}_{\text{int}}$ is the interaction term in the Lagrangian density.

For two-body non-relativistic interaction and for relativistic boson $\phi^4$ theory one gets $\alpha = 4$. For a theory with two single-flavor fermions interacting via one-flavor boson (with coupling $\Psi_i^\dagger \Psi_f (\phi_b + \phi_b^\dagger)$) one obtains

$$\epsilon_{\text{int}} = \frac{2}{\alpha_f} (\epsilon_{\text{pot}}^f + \epsilon_{\text{pot}}^b) = \frac{2}{\alpha_f} \epsilon_{\text{pot}}^f = \frac{2}{\alpha_b} \epsilon_{\text{pot}}^b, \quad \alpha = 3, \alpha_f = 2, \alpha_b = 1. \quad (36)$$

For a theory where two fermions with different flavors interact via one-flavor boson, one finds

$$\epsilon_{\text{int}} = 2\epsilon_{\text{pot}}^f = 2\epsilon_{\text{pot}}^2 = 2\epsilon_{\text{pot}}^b. \quad (37)$$

We will also use *exact* expressions of Refs. [6,7] for $\epsilon_{\text{pot}}$, which follow directly from equations of motion:

$$\epsilon_{\text{pot}}^{n.r.} = N_{n.r.} \int_0^\infty \frac{4\pi p^2 dp}{(2\pi)^3} \int_{-\infty}^\infty d\omega \frac{\omega - \omega_p^{n.r.}}{2\pi} A_{n.r. f} f, \quad (38)$$

with $\omega_p^{n.r.} = p^2/2m$ for non-relativistic particles of given species;

$$\epsilon_{\text{pot}}^b = \int_0^\infty \frac{4\pi p^2 dp}{(2\pi)^3} \int_0^\infty d\omega \frac{\omega_b^2 - (\omega_p^b)^2}{2\pi} \left[ A_b^{(+)} f_b^{(+)} + (\mu_b \rightarrow -\mu_b) \right], \quad (39)$$

with $\omega_p^b = \sqrt{p^2 + m_b^2}$ for relativistic spin-less bosons of given species; and

$$\epsilon_{\text{pot}}^f = N_f \int_0^\infty \frac{4\pi p^2 dp}{(2\pi)^3} \int_0^\infty d\omega \frac{\omega_f - \omega_p^f}{2\pi} A_f^{(+)} f_f^{(+)} + (\mu_f \rightarrow -\mu_f), \quad (40)$$

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with \( \omega_p = \sqrt{p^2 + m_f^2} \) for relativistic fermions of the given species (for interactions with a simple spin structure of vertices).

Eqs. (38), (39), (40) demonstrate that in general case even at low densities the potential energy may not cease.

Convenience of expressions for thermodynamic characteristics of the system that we have presented is that all quantities are expressed entirely in terms of spectral functions. Thus the problem of the description of strongly interacting matter would be solved, if spectral functions of all species were known and if there were relations between all partial interaction and potential energy contributions, see (34).

3 Gas of free resonances

3.1 Free relativistic fermion resonances

Specifics of the choice of the interaction between particles enters expression (10) through a difference \( \epsilon_f^{\text{int}} - \epsilon_f^{\text{pot}} \). We introduce the term ”free resonances”, describing the case, when one may neglect the value \( \epsilon_f^{\text{int}} - \epsilon_f^{\text{pot}} \). In this section let us consider thermodynamics of ”free resonances”. Thus we will use Eqs. (29), (30), (31), (32) disregarding in (30), (31) \( \epsilon_f^{\text{int}} - \epsilon_f^{\text{pot}} \) terms.

In the limit of vanishing density, dependence of diagrams determining the resonance width on particle occupations (on \( \mu \) and \( T \)) ceases. Then for relativistic particles the spectral function depends on the energy and momentum through \( s = \omega^2 - p^2 \) variable, cf. [8]. (For non-relativistic particles, instead of \( s \) we would use \( \omega - \omega^{n.r.} \).) Thus, dealing here with relativistic fermions we may assume that

\[
\tilde{A}_f = \tilde{A}_f(s), \quad s = \omega^2 - p^2;
\]

where \( \tilde{A}_f \) is related to the spectral function following Eq. (28).

In the medium \( \tilde{A}_f \) may depend on \( \omega \) and \( \tilde{p} \) separately. Indeed, Green functions enter diagrams, that determine the self-energy of the resonance, together with particle occupations. Thus, also \( \tilde{A}_f \) depends on \( \mu \) and \( T \). At a finite density the full width can be presented as sum of the vacuum and the medium terms. The latter term ceases with decrease of the density. Thereby in the low density limit we can use expression (41).

Now we are able to check that our expressions for thermodynamic quantities
of "free resonances" fulfill thermodynamic consistency conditions (3). Taking derivatives of the pressure (3) we use partial integrations and relations

\[
\frac{\partial A_f}{\partial \omega^2} = -\frac{\partial A_f}{\partial \vec{p}^2}, \quad \frac{\partial f_t}{\partial T} = -\frac{(\omega - \mu)}{T} \frac{\partial f_t}{\partial \omega}, \quad \frac{\partial f_t}{\partial \mu} = -\frac{\partial f_t}{\partial \omega}.
\] (42)

Doing partial integrations we drop "surface terms" that requires a smooth switching of the resonance spectral function at the threshold (due to switching of the width) and at infinity:

\[
A_f(s \to s_{th}) \to 0, \quad A_f(s \to 0) \to 0, \quad A_f(s \to \infty) \to 0.
\] (43)

Thus we arrive at expressions (29) and (32).

Also, a convenient expression for the entropy density can be then recovered with the help of expressions (3), (1), and the relation

\[
\frac{\partial f_t}{\partial T} = -\frac{\partial \sigma_t}{\partial \omega}.
\] (44)

We again use partial integrations and relations (42). Finally we arrive at expression

\[
S_t = N_t \int_0^\infty \frac{4\pi p^2 dp}{(2\pi)^3} \int_0^\infty \frac{d\omega}{2\pi} A_f^{(+)}(+) \sigma_t^{(+)} + (\mu_t \to -\mu_t).
\] (45)

Here

\[
\sigma_t^{(+)} = -(1 - f_t^{(+)}) \ln(1 - f_t^{(+)}) - f_t^{(+)} \ln f_t^{(+)}
\] (46)

is the entropy distribution function. Doing similar manipulations we derive expression (45) from (32), that again proves thermodynamical consistency of our derivations.

Note that we used only assumption \(\epsilon_t^{\text{int}} - \epsilon_t^{\text{pot}} = 0\), an ansatz (28), (11) for the spectral function and conditions (13) to check that thermodynamical quantities that we have derived satisfy thermodynamical consistency conditions. On the other hand, Refs. [5,6] have shown that, even with the full interaction included, the baryon density is presented as the sum of partial Noether contributions. Since we proved thermodynamic consistency of our "free resonance" model and recovered Noether quantities, we could hope that at least in the low density limit our ansatz (28), (11) and assumption, that terms \(\epsilon_t^{\text{int}} - \epsilon_t^{\text{pot}}\) being present in the pressure do not contribute to the particle density, are indeed
justified, i.e. \( \delta n = -\frac{\partial (\epsilon_{\text{int}} - \epsilon_{\text{pot}})}{\partial \mu} = 0 \). However with the help of partial integrations one can show that \( \delta n = -\frac{\partial (\epsilon_{\text{int}} - \epsilon_{\text{pot}})}{\partial \mu} \neq 0 \) and thus there appears an extra contribution to the Noether density. Also the entropy density acquires extra term \( \delta S = -\frac{\partial (\epsilon_{\text{int}} - \epsilon_{\text{pot}})}{\partial T} \). On the other hand, since the entropy density presented in the form (32) should not depend on the interaction terms, we may conclude that \( \delta n \) and \( \delta S \) should be proportional to the density in a higher power than the Noether terms and those interaction terms can be indeed neglected in the virial limit. Below, on explicit examples we show that \( \delta n \to 0 \) and \( \delta S \to 0 \) in the virial limit.

Anyhow, the model of "free resonances" continues to be thermodynamically consistent even at a higher density, where its results, of course, should deviate from exact solutions.

To do the problem tractable, instead of solving a complete set of Dyson equations, we may select a simplified phenomenological expression for \( A_f \) (compare with [8,9]), e.g.,

\[
A_f = \frac{2\xi \omega [2\Gamma_f(s) + \delta]}{(s - m_f^2)^2 + [\Gamma_f(s) + \delta]^2}, \quad \xi = \text{const}, \quad s = \omega^2 - p^2 > s_{\text{th}} > 0, (47)
\]

for \( \delta \to 0 \), with a simple \( s \)-dependence of the width, e.g.,

\[
\Gamma_f(s) = \Gamma_0 m_f^{1-2\alpha} F(s)(s - s_{\text{th}})^\alpha \theta(s - s_{\text{th}}), \quad \Gamma_0 = \text{const}. \quad (48)
\]

Here \( \alpha = l + 1/2 \), i.e., \( \alpha = 1/2 \) for the \( s \)-wave resonance and \( \alpha = 3/2 \) for the \( p \)-wave resonance. The width should vanish at the threshold, i.e. \( \alpha > 0 \). An extra form-factor, \( F(s) \), is introduced to correct the high-energy behavior of the width. One can take \( F^{-1} = 1 + [(s - s_{\text{th}})/\Lambda^2]^\beta \) with \( s_0 \) and \( \beta \) being constants. Appropriate values for the cut-off factor \( \Lambda \) and the power \( \beta \) are: \( \Lambda \sim (0.7 \div 1) \) GeV and \( \beta > \alpha + 1/2 \). The latter inequality provides a decrease of the width at large energies, far off the resonance peak. If a more detailed description is required, one can use a more involved phenomenological expression for the width fitting free parameters from comparison of the resonance shape with experimental data.

The energy dependence of the width may cause a problem. With a simple ansatz (48) for the behavior of \( \Gamma_f(s) \) we get a complicated \( m_f^2(s) \) dependence of the effective resonance mass, as it follows from the Kramers-Kronig relation. However, using that \( m_f^2 \neq 0 \) at the threshold and that \( m_f^2(s) \) is a smooth function of \( s \), we may ignore mentioned complexity and put for simplicity \( m_f^2 \simeq \text{const} \). Factor \( \xi \) is introduced to fulfill the sum-role.
\[
\int_{0}^{\infty} \frac{ds}{4\pi} \tilde{A}_f = 1, \tag{49}
\]
that yields \( \xi \simeq 1 + O(\Gamma_0/m_f^*) \) (for \( m_f^* \gg \Gamma_0, m_f^* > s_{th} \)). Closeness of \( \xi \) to unity (for \( \Gamma_0/m_f^* \ll 1 \)) shows that exact form of the spectral function (4) is corrected by \( \xi \) only slightly.

The baryon number density, the energy density, the pressure and the entropy density are determined by expressions (29), (30), (31), (45) (where for "free resonances" \( \epsilon_f^{\text{int}} - \epsilon_f^{\text{pot}} = 0 \)).

### 3.2 Deficiency of approximation of constant particle width

Note that some thermal models have used the same expressions for \( n \) and \( P \) as we derived for "free resonances". However, these expressions were not derived in those models but introduced using intuitive arguments like a smearing of the \( \delta \)-function spectral density. Then for simplicity one often applies approximation of constant particle width. Here we would like to pay attention to the fact that with assumption of constant width \( (\Gamma_f = \Gamma_0 = \text{const.} \) instead of (48)) conditions (43) are not anymore fulfilled and the model would suffer of thermodynamical inconsistency, if one used mentioned expressions for thermodynamical quantities. E.g., if one found the particle density and the entropy density with the help of thermodynamical consistency conditions (3) using Eq. (31) for the pressure at \( \epsilon_f^{\text{int}} - \epsilon_f^{\text{pot}} = 0 \) and at assumption of the constant width, there would appear extra "surface integral" contributions

\[
\Delta n_f = N_f \int_{0}^{\infty} \frac{4\pi p^4 dp}{(2\pi)^4} \left[ \tilde{A}_f^{(+)} f_f^{(+)} \right] \sqrt{s_{th} + p^2}
\]

\[
- \frac{N_f}{3} \int_{0}^{\infty} \frac{d\omega}{2\pi^2} \tilde{A}_f^{(+)} (s = 0) f_f^{(+)} \omega^4 - (\mu_f \rightarrow -\mu_f), \tag{50}
\]

\[
\Delta S_{th} = - \frac{N_f}{3} \int_{0}^{\infty} \frac{4\pi p^2 dp}{(2\pi)^4} \left[ \tilde{A}_f^{(+)} \sigma_f^{(+)} \right] \omega_{th} p^2
\]

\[
- \frac{N_f}{3} \int_{0}^{\infty} \frac{d\omega}{2\pi^2} \tilde{A}_f^{(+)} (s = 0) \sigma_f^{(+)} \omega^4 + (\mu_f \rightarrow -\mu_f), \tag{51}
\]

compared to Noether values (29) and (45). These extra terms would vanish, if conditions (43) were fulfilled.
On the other hand, instead of derivation of expression for the particle density from expression for the pressure (or $\Omega$), one could start with exact Noether expression for the particle density. Then the pressure and other thermodynamic characteristics would acquire extra ”surface integral” corrections, if one assumed constant widths.

### 3.3 Free relativistic boson resonances

Using ansatz $A = A(s)$ and doing the same replacements of variables (32), (44), as before, we easily check thermodynamical consistency of expressions for thermodynamical quantities: the energy density, the pressure and the entropy density (in a convenient form)

$$E_b = N_b \int_0^\infty \frac{4\pi p^2 dp}{(2\pi)^3} \int_{s_{th}}^\infty \frac{ds}{2\pi} \omega A_b^{(+)} f_b^{(+)} + (\mu_b \rightarrow -\mu_b),$$  \hspace{1cm} (52)

$$P_b = N_b \int_0^\infty \frac{4\pi p^2 dp}{(2\pi)^3} \int_{s_{th}}^\infty \frac{ds}{2\pi} \frac{p^2}{3\omega} A_b^{(+)} f_b^{(+)} + (\mu_b \rightarrow -\mu_b),$$  \hspace{1cm} (53)

$$S_b = N_b \int_0^\infty \frac{4\pi p^2 dp}{(2\pi)^3} \int_{s_{th}}^\infty \frac{ds}{2\pi} A_b^{(+)} \sigma_b^{(+)} + (\mu_b \rightarrow -\mu_b),$$  \hspace{1cm} (54)

now with

$$\sigma_b^{(+)} = (1 + f_b^{(+)}) \ln(1 + f_b^{(+)}) - f_b^{(+)} \ln f_b^{(+)}.$$  \hspace{1cm} (55)

$N_b = 1$ for spin-less bosons.

For practical calculations we may use [9],

$$A_b^{(\pm)} = \frac{\xi[\Gamma_b(s) + \delta]}{(s - m_b^{*2})^2 + [\Gamma_b(s) + \delta]^2/4}, \quad \xi = const,$$  \hspace{1cm} (56)

$$\Gamma_b(s) = 2\Gamma_0 m_1^{-2\alpha} F(s)(s - s_{th})^{\alpha} \theta(s - s_{th}), \quad \Gamma_0 = const.$$  \hspace{1cm} (57)
Thus, taking into account additional pre-factor $v^0 = 2\omega$, that appears in the bosonic case, we see that the bosonic and fermionic spectral functions (56), (47) are similar.

4 Examples of the description of some systems of interacting particles with widths

Above we used ansatze for spectral functions. Also we neglected $\varepsilon_{\text{int}} - \varepsilon_{\text{pot}}$ term. These assumptions are not fulfilled in case of dense systems. Therefore it is worthwhile to obtain exact relations for thermodynamic quantities not doing any assumptions. Below consider the following examples of interacting systems: of one particle species with non-relativistic paired interaction; of single relativistic boson species with $\phi^4$ self-interaction; of two particle species with two single-flavor fermion interacting via one-boson exchange, as $NN\sigma$; and of three species with two-different flavor fermions interacting via one boson exchange, as $\Delta N\pi$.

Note that in most practically interesting situations a broad resonance appears, as a consequence of the interaction between other particle species. Those (other) particle species in many cases acquire much smaller widths than the given broad resonance and thereby they can be treated within the quasiparticle approximation. As example, we may refer to the $\Delta N\pi$ system. In the latter case the $\Delta$ is a broad resonance (the width in vacuum reaches 115 MeV) but nucleons and pions have much smaller widths, at least at not too high temperature. Therefore in many situations nucleons and pions can be considered within the quasiparticle approximation. These observations help us to construct a simplified treatment of the problem.

4.1 Non-relativistic particles interacting via paired potential

Let us start with discussion of example of the system of non-relativistic particles of single species interacting via paired potential ($\alpha = 4$ in (34)). Using (12), (13) and (38) without doing any approximations we find

$$E_{n.r.} = N_{n.r.} \int_0^\infty \frac{4\pi p^2 dp}{(2\pi)^3} \int_{\omega_{\text{th.}}}^\infty \frac{d\omega}{2\pi} \left[ \omega - \frac{(\omega - \omega_{p,n.r.})^2}{2} \right] A_{n.r.} f,$$

(58)
Thermodynamical relations (3) should be fulfilled with exact spectral function, since we did not do yet any approximations. However it is not so easy to satisfy these relations, if one uses a simplified phenomenological ansatze spectral functions instead of the exact ones. E.g., conditions (3) would be violated, if one used the above presented ansatze spectral functions applying Eq. (59) for the case of a dense system.

Within the quasiparticle approximation at low density one has \( A_{n.r.} \rightarrow (2\pi)\delta(\omega - \omega_p^{n.r.}) \) and the interaction term \( \epsilon_{int} - \epsilon_{pot} \) ceases. By this we explicitly demonstrate that the "free resonance term" yields a dominant contribution in the low density limit, at least, if the particle width is sufficiently small. Thereby we arrive at a consistent treatment of the problem for a low density system in the case, when the width is finite but small.

### 4.2 Bosonic \( \phi^4 \) theory

Using \( \lambda\phi^4 \) coupling we may estimate interactions in the pion gas. To be specific consider here spin-less neutral bosons (\( \mu_b = 0 \)). As in case of the paired potential, here \( \alpha = 4 \) in (34). With the help of Eqs. (24), (25), and (39) we find

\[
P_b = \int_0^\infty \frac{4\pi p^2 dp}{(2\pi)^3} \int_0^\infty \frac{d\omega}{2\pi} \left[ \frac{2p^2}{3} + \frac{(\omega^2 - \omega_p^2)}{2} \right] A_b f_b.
\]

As in case with paired potential, if we used a simplified phenomenological ansatze spectral functions instead of the exact ones, we would retain with the same problem with fulfillment of the consistency conditions for a dense system. In the low density limit we again deal with free resonances, at least provided particle width is finite but rather small.

### 4.3 Two single-flavor fermions interacting via one-boson exchange

In this case for relativistic fermions and neutral bosons
\[ TS_f + TS_b = N_f \int_0^\infty \frac{4\pi p_f^2 dp_f}{(2\pi)^3} \int_0^\infty \frac{d\omega_f}{2\pi} \left( \omega_f + \frac{p_f^2}{3\omega_f} \right) A_f^{(+)} f_f^{(+)} - \mu_f^{(+)} n_f^{(+)} \]
\[ + (\mu_f^{(+)} \rightarrow -\mu_f^{(-)}) \]
\[ + N_b \int_0^\infty \frac{4\pi p_b^2 dp_b}{(2\pi)^3} \int_0^\infty \frac{d\omega_b}{2\pi} \left( 2\omega_b^2 + \frac{2p_b^2}{3} \right) A_b^{(+)} f_b^{(+)} . \]

Here to simplify expression we consider the case of \( \mu_b = 0 \). For spin-less bosons \( N_b = 1 \).

Using that following (36) \( \epsilon_{\text{int}} - \epsilon_{\text{pot}} = -\epsilon_{\text{pot}}^f/2 \), and Eq. (40) for relativistic fermions, we find

\[ E_f + E_b = N_f \int_0^\infty \frac{4\pi p_f^2 dp_f}{(2\pi)^3} \int_0^\infty \frac{d\omega_f}{2\pi} \left[ \omega_f - \frac{(\omega_f - \omega_f^f)}{2} \right] A_f^{(+)} f_f^{(+)} \]
\[ + (\mu_f^{(+)} \rightarrow -\mu_f^{(-)}) + N_b \int_0^\infty \frac{4\pi p_b^2 dp_b}{(2\pi)^3} \int_0^\infty \frac{d\omega_b}{2\pi} 2\omega_b^2 A_b^{(+)} f_b^{(+)} , \] (62)

\[ P_f + P_b = N_f \int_0^\infty \frac{4\pi p_f^2 dp_f}{(2\pi)^3} \int_0^\infty \frac{d\omega_f}{2\pi} \left[ \frac{p_f^2}{3\omega_f} + \frac{(\omega_f - \omega_f^f)}{2} \right] A_f^{(+)} f_f^{(+)} \]
\[ + (\mu_f^{(+)} \rightarrow -\mu_f^{(-)}) + N_b \int_0^\infty \frac{4\pi p_b^2 dp_b}{(2\pi)^3} \int_0^\infty \frac{d\omega_b}{2\pi} \frac{2p_b^2}{3} A_b^{(+)} f_b^{(+)} . \] (63)

As we see, the interaction affects the fermion contribution, whereas the boson is described, as the "free resonance" or "free blurred particle". For fermions expressions look similar to the case of the system with paired interaction.

These expressions are very convenient in case when fermions are good quasi-particles but bosons might be broad resonances or blurred particles. In the low density limit, using the quasiparticle spectral function for fermions we see that interaction contributions cease and we retain with a broad boson resonance and the fermion quasiparticle, which interaction with each other is suppressed.

On the other hand, we may correct boson contributions by the interaction. Using that following (36) \( \epsilon_{\text{int}} = \epsilon_{\text{pot}}^f \), i.e. \( \epsilon_{\text{int}} - \epsilon_{\text{pot}} = -\epsilon_{\text{pot}}^b \), and Eq. (39) we find

\[ E_f + E_b = N_f \int_0^\infty \frac{4\pi p_f^2 dp_f}{(2\pi)^3} \int_0^\infty \frac{d\omega_f}{2\pi} \omega_f A_f^{(+)} f_f^{(+)} + (\mu_f^{(+)} \rightarrow -\mu_f^{(-)}) \]
\[ + N_b \int_0^\infty \frac{4\pi p^2_b dp_b}{(2\pi)^3} \int_0^\infty \frac{d\omega_b}{2\pi} \left[ 2\omega_b^2 - (\omega_b^2 - (\omega_p^b)^2) \right] A_b^{(+)} f_b^{(+)}, \] (64)

\[ P_f + P_b = N_f \int_0^\infty \frac{4\pi p^2_f dp_f}{(2\pi)^3} \int_0^\infty \frac{d\omega_f}{\pi} \frac{p_t^f}{3\omega_t} A_f^{(+)} f_f^{(+)} + (\mu_t^{(+)} + -\mu_t^{(-)}) \]
\[ + N_b \int_0^\infty \frac{4\pi p^2_b dp_b}{(2\pi)^3} \int_0^\infty \frac{d\omega_b}{2\pi} \left( \frac{2p_b^2}{3} + \omega_b^2 - (\omega_p^b)^2 \right) A_b^{(+)} f_b^{(+)}. \] (65)

Here interaction affects the boson contribution, whereas the fermion is described, as the "free resonance" or "free blurred particle". For the boson subsystem expressions look the same, as for the $\phi^4$ theory.

Again, in the low density limit the interaction terms cease in (64), (65) provided bosons are good quasiparticles. The problem then is reduced to description of a broad boson resonance and the fermion quasiparticle, not interacting with each other.

4.4 Two different-flavor fermions interacting via one-boson exchange

For a theory with two different-flavor fermion – one boson coupling, using (37) one gets

\[ \epsilon_{\text{int}} - \epsilon_{\text{pot}} = -\epsilon_{\text{pot}}^{f1} = -\epsilon_{\text{pot}}^{f2} = -\epsilon_{\text{pot}}^b. \] (66)

Thus compared to previous example, two particle species can be described as "free resonances" or "free blurred particles", whereas the third particle species contains the interaction term. In case of the $\Delta - N - \pi$ system at not too high baryon density and temperature, only $\Delta$ acquires a large width. Thereby we may treat the $\Delta$, as a broad "free resonance" with the interaction affecting either nucleon or pion sub-systems. Having smaller widths, both nucleon and pion sub-systems can be described in the quasiparticle approximation. Nevertheless, if one wanted to proceed further and incorporate the nucleon (or the pion) width, one could transport the interaction to the sub-system of particles with the smallest width, treating only the latter sub-system within the quasiparticle approximation. This way we may solve a complicated problem, being sure that thermodynamic consistency conditions are approximately fulfilled.

The following remark is in order. Considering $\pi N \Delta$ system in the virial limit and assuming that only $g_{\pi N \Delta}$ coupling retains, Ref. [10] constructed explicit expressions for thermodynamic potential and the net particle density, being
expressed through the so called $B$-spectral function of the $\Delta$ that depends on $\text{Re}\Sigma_\Delta$ and $\text{Im}\Sigma_\Delta$ in a more complicated way than the standard $A$-function. On the other hand, the nucleon and the pion sub-systems are described there, as ideal gases of free particles. In spite of apparent difference with obtained above expressions, both approaches coincide with each other. Indeed, Refs. [5,6] have shown that expression for the total particle density, as the sum of the Noether densities (expressed entirely in terms of the $A$-spectral functions) fulfills thermodynamical consistency condition (first condition (3)). Namely using the latter condition Ref. [10] has derived the net particle density presented there in a different form (using the $B$-function). The $B$-function is expressed in terms of the phase shifts in the $S$-matrix formulation of statistical mechanics [11].

5 Hot matter consisting of strongly interacting light bosons and heavy fermions at zero chemical potentials

In sect. 3 we demonstrated how one can approximately describe dilute heated matter with the help of simple parameterizations of spectral functions. Now consider example, where one can find spectral functions and thermodynamic values of a system of strongly interacting particles avoiding assumption of dilute matter.

Let us consider hot system of two particle species. One species is a light boson and another species is a heavy fermion, both at zero chemical potentials. Let us treat the system in terms of the self-consistent $\Phi$ derivable approximation scheme. Then the $\Phi$ functional of Baym is given by diagrams

\[
i \Phi = \frac{1}{2} \begin{tikzpicture} 
\draw[red,thick,->] (0,0) -- (0.5,0.5) node [above] {\ldots};
\draw[blue,thick,->] (0,0) -- (-0.5,0.5) node [above] {\ldots};
\end{tikzpicture} + \frac{1}{4} \begin{tikzpicture} 
\draw[red,thick,->] (0,0) -- (0.5,0.5) node [above] {\ldots};
\draw[blue,thick,->] (0,0) -- (-0.5,0.5) node [above] {\ldots};
\draw[red,thick,->] (0.5,0.5) to [out=-135,in=180] (0.5,0) node [above] {\ldots};
\draw[blue,thick,->] (-0.5,0.5) to [out=-135,in=180] (-0.5,0) node [above] {\ldots};
\end{tikzpicture} + \ldots
\] (67)

Here both fermion (solid line) and boson (wavy line) Green functions are full Green functions whereas vertices are bare. Let us restrict ourselves by consideration of the simplest $\Phi$ (the first diagram (67)). Then the fermion self-energy is as follows

\[
\begin{tikzpicture} 
\draw[red,thick,->] (0,0) to [out=-135,in=180] (0.5,0) node [above] {\ldots};
\end{tikzpicture}
\] (68)

and the boson self-energy reads
All the multi-particle rescattering processes

\[ \text{---} + \text{---} + \ldots \]  

are then included, whereas processes with crossing of boson lines (correlation effects) like

\[ \text{---} \]

are disregarded.

To simplify derivations consider example of the Yukawa interaction of spin 1/2 non-relativistic heavy fermion with a light relativistic scalar boson. The interaction Lagrangian is given by

\[ L_{\text{int}} = g \bar{\psi} \phi \psi. \]  

Examples of different couplings of relativistic fermions and bosons, including the Yukawa interaction, can be found in [4]. However there we did not calculate thermodynamic quantities, that is our aim now.

5.1 Blurred fermions

Since \( m_f \gg m_b \), one may expect that in a broad temperature range, which we will interested in, \textit{boson occupations are essentially higher than fermion ones}. Then, at such temperatures we may retain in (68) only terms proportional to boson occupations. Using this we find [4]:

\[ \Sigma_f^R(p_f) \simeq \int g^2 \frac{d^3 p_b}{(2\pi)^3} \int_0^\infty \frac{d\omega_b}{2\pi} [G_f^R(p_f + p_b) + G_f^R(p_f - p_b)] A_b(p_b) f_b(\omega_b). \]  

Let us use the \textit{soft thermal loop} (STL) approximation [3] resulting in dropping a \( p_b \)-dependence of fermion Green functions in (73). Then Eq. (73) is simplified as

\[ \Sigma_f^R(p_f) \simeq \int \frac{d^3 p_b}{(2\pi)^3} \int_0^\infty \frac{d\omega_b}{2\pi} [G_f^R(p_f + p_b) + G_f^R(p_f - p_b)] A_b(p_b) f_b(\omega_b). \]
\[
\Sigma_1^R(p_f) \simeq J \cdot G_1^R (p_f), \quad J = 2g^2 \int \frac{d^3p_b}{(2\pi)^3} \int_0^\infty \frac{d\omega_b}{2\pi} A_b(p_b)f_b(\omega_b).
\] (74)

As we will see, at \( T \gtrsim m^*_b(T) \) (further we will consider namely such temperatures) departure of the fermion energy from the mass shell \( \delta \omega_f \sim \sqrt{J} \) is much larger than that for bosons, \( \delta \omega_b \sim \max\{m_b - m^*_b(T), T\} \), and typical fermion momenta \( p_f \sim \sqrt{2m_f T} \) are much higher than typical boson momenta \( p_b \sim \max\{\sqrt{2m^*_b(T)T}, T\} \). Here \( m^*_b(T) \) is an effective boson mass. At these conditions the STL approximation should be valid. Moreover, we assume that \( \sqrt{2m_bT} \ll m_b \) and \( \sqrt{J} \ll m_f \). Then fermions can be treated as non-relativistic particles. The latter approximation allows us to avoid a complicated spin algebra.

As follows from (74), the quantity \( J \) can be expressed through the tadpole diagram

\[
\begin{array}{c}
\text{Diagram}
\end{array}
\] (75)

The latter does not depend on the energy-momentum transfer. It describes fluctuations of virtual (off-mass shell) bosons. On the other hand, \( J \) can be interpreted as the density of quasi-static boson impurities. Due to multiple repetition of this diagram in the Dyson series for the fermion, \( J \) demonstrates the intensity of the multiple quasi-elastic scattering of the fermion on quasi-static boson impurities.

Dyson equation for the retarded fermion Green function is greatly simplified in the STL approximation:

\[
G_1^R = G_0^R f + G_0^R J(G_1^R)^2,
\] (76)

with a simple analytical solution

\[
G_1^R = \frac{\omega_f - \omega_f^I \pm \sqrt{(\omega_f - \omega_f^I)^2 - 4J}}{2J}, \quad \omega_f^I = m_f + \frac{p_f^2}{2m_f}.
\] (77)

In this problem it is convenient to count fermion energies from the mass-shell.

Only negative sign solution satisfies the retarded property and should be retained. For \( (\omega_f - \omega_f^I)^2 \gg 4J \) we recover the quasiparticle (pole-like) solution. Since then typical energies are \( \omega_f - \omega_f^I \sim T \), the quasiparticle approximation is valid for fermions only for \( J \ll T^2 \). Otherwise (for \( J \gtrsim T^2 \)) fermion Green
function is completely regular. Fermions become blurred particles. From (77), using relation \((GR)^{-1}_f = \omega_f - \omega_p^f - \text{Re}\Sigma^R_f + i\Gamma_f/2\), for \(4J > (\omega_f - \omega_p^f)^2\) we find

\[
\text{Re}\Sigma^R_f = \frac{\omega_f - \omega_p^f}{2}, \quad \Gamma_f = \sqrt{4J - (\omega_f - \omega_p^f)^2} \theta \left( 4J - (\omega_f - \omega_p^f)^2 \right),
\]

\[
A_f = \frac{\sqrt{4J - (\omega_f - \omega_p^f)^2} J}{\theta \left( 4J - (\omega_f - \omega_p^f)^2 \right)}. \tag{78}
\]

### 5.2 Intensity of multiple scattering

Now we are able to evaluate the intensity of multiple scattering \(J\). Assume that neutral spin-less bosons under consideration are good quasiparticles in the energy-momentum and temperature region of our interest. Then their spectral function is as follows \(A_b = 2\pi\delta \left( \omega_b^2 - p_b^2 - m_b^2 - \text{Re}\Sigma_b^R(\omega_b, p_b) \right)\), cf. Eq. (19), and from (74) we find

\[
J = \frac{g^2}{2\pi^2} \int_0^\infty \frac{p_b^2 dp_b}{\left[ m_b^2(T) + \beta p_b^2 \right]^{1/2}} \exp \left[ \left( m_b^2(T) + \beta p_b^2 \right)^{1/2} / T \right] - 1,
\]

where in the second line we adopted simple form of the quasiparticle spectrum

\[
\omega_b^2(p_b, T) \simeq m_b^2(T) + \beta(T)p_b^2 + O(p_b^4). \tag{79}
\]

As we will show, in a broad temperature region of our interest \(\beta\) is close to unity and \(m_b^*\) is close to \(m_b\). We present

\[
J = \frac{g^2 T^2}{12\beta^{3/2}} r_s(z), \quad r_s(z) = \frac{6f_0(z)}{\pi^2 z^2}, \quad z = \frac{T}{m_b^*}. \tag{80}
\]

Numerical evaluation of the integral (80) is demonstrated in Fig. 1.

In the limiting case of a high temperature typical values of momenta are \(p_b \sim T \gg m_b^*(T)\) and we obtain

\[
J \simeq \frac{g^2 T^2}{12\beta^{3/2}}, \quad \text{for} \quad T \gg m_b^*(T). \tag{81}
\]

In the limit \(T \ll m_b^*(T)\), the intensity of multiple scattering is exponentially suppressed
Fig. 1. $f_0(z)/z^2$, cf. eq. (80). Dash line demonstrates asymptotic behavior for $z \gg 1$.

$$J \simeq \frac{g^2 T^{3/2} \sqrt{m_b}}{2^{3/2} \pi^{3/2} \beta^{3/2}} \exp \left(-\frac{m_b^*}{T} \right), \quad \text{for } T \ll m_b^*(T). \quad (82)$$

The latter temperature regime is not of our interest here. We are interested to describe the temperature region $T \gtrsim m_b^*(T)$, when $J$ is not exponentially suppressed. In this case the quasiparticle approximation (valid for $T^2 \gg 4J$) would work for fermions only for weak coupling $g \ll \beta^{3/4} \simeq 1$. Since we are interested in description of strongly interacting particles, when $g \gtrsim 1$, we deal with blurred fermions.

The quasiparticle boson density inside the system is given by (we neglect here a weak boson-energy dependence of the self-energy, see below):

$$n_b = \int_0^\infty \frac{4 \pi p^2 dp}{(2 \pi)^3} \int_0^\infty \frac{d\omega}{2 \pi} 2 \omega A_b f_b \simeq \int_0^\infty \frac{p^2 dp}{2 \pi^2} \frac{1}{e^{\sqrt{m_b^2 + p^2}/T} - 1}, \quad (83)$$

cf. Eq. (23), whereas for $T \gg m_b^*$ we find $n_b \simeq T^3 \zeta(3)/(\beta^{3/2} \pi^2)$, $\zeta(3) \simeq 1.202$.

5.3 Density of fermion-antifermion pairs

Consider system with strong interaction, $g \gtrsim 1$, and assume $m_b^*(T) \lesssim T \ll 5m_b \beta^{3/4}/g$. Then we can easily check that, on the one hand, conditions of the STL approximation are fulfilled and, on the other hand, fermions can be treated as non-relativistic particles. As follows from (77), the fermion spectral
function satisfies the full sum-rule (49). Thus, although we used approximations, their consistency is preserved.

Following (29) the 3-momentum fermion distribution is as follows

$$ f_t^{(\pm)}(p_t) = \int_0^\infty \frac{d\omega_t}{2\pi} A_t^{(\pm)} f_t(\omega_t). \quad (84) $$

Replacing (78) into this expression we find the 3-momentum fermion distribution

$$ f_t^{(\pm)}(p_t) = \frac{2}{\sqrt{J}} \int_{-2\sqrt{J}}^{2\sqrt{J}} \frac{d\xi}{\pi} \frac{1}{\sqrt{\omega_t - \omega_p}} \frac{1}{\exp[\xi/\omega_t]} + 1, \quad (85) $$

where we introduced the variable $\xi = \omega_t - \omega_p$ and used that $|\xi| < 2\sqrt{J} \ll m_f$. 

Doing further replacement $\xi = -2\sqrt{J} + Ty$ and using that $m_f \gg T$, we obtain

$$ f_t^{(\pm)}(p_t) \simeq \frac{T^{3/2}}{\pi J^{3/4}} I \left( \frac{4\sqrt{J}}{T} \right) \exp \left[ -\frac{\omega_p^2}{T^2} \right] =: F(T) f_{\text{Bol}}^{(\pm)}(p_t), \quad (86) $$

with

$$ F(T) = \frac{T^{3/2}}{\pi J^{3/4}} I \left( \frac{4\sqrt{J}}{T} \right) \exp \left[ \frac{2\sqrt{J}}{T^2} \right], \quad (87) $$

$$ I(x) = \int_0^x dy \sqrt{y - y^2x^{-1}}, \quad x = 4\sqrt{J}/T, \quad (88) $$

$$ I(x) \simeq \frac{\pi}{8} x^{3/2} \left( 1 - \frac{x}{2} \right), \quad \text{for} \quad x \ll 1, \quad (89) $$

$$ I(x) \simeq \frac{\sqrt{\pi}}{2} \left( 1 - \frac{3}{4x} \right), \quad \text{for} \quad x \gg 1. $$

As we have mentioned, for $g \gtrsim 1$ the condition of the validity of the quasiparticle approximation for fermions, $x \simeq 4\sqrt{J}/T \ll 1$, is not fulfilled at temperatures of our interest.

For $z \gtrsim 1$, and $x \gg 1$ (i.e. for $g \gg 1$), we find
\[ F(T) \simeq \frac{2^{1/2}3^{3/4}\beta^{9/8}}{\pi^{1/2}g^{3/2}r^{3/4}} \exp \left[ \frac{gr^{1/2}}{3^{1/2}\beta^{3/4}} \right], \]  
(90)

and for \( x \gg 1, \ z \gg 1 \) and \( \beta \simeq 1 \):

\[ F(T) \rightarrow \frac{2^{1/2}3^{3/4}}{\pi^{1/2}g^{3/2}} \exp \left[ \frac{g}{3^{1/2}} \right] = \text{const}(T) \gg 1. \]  
(91)

Integrating (86) in momenta we obtain the fermion (antifermion) density:

\[ n_{f}^{(\pm)} \simeq F(T)n_{\text{Bol}}^{(\pm)}, \quad n_{\text{Bol}}^{(\pm)} \simeq N_{f} \left( \frac{m_{f}T}{2\pi} \right)^{3/2} \exp \left[ -\frac{m_{f}}{T} \right], \quad N_{f} = 2. \]  
(92)

We see that with growing parameter \( \sqrt{J}/T \), i.e. with growing temperature, the density of fermion-antifermion pairs increases significantly compared to the standard Boltzmann value.

The result (86) can be interpreted with the help of the relevant quantity

\[ m_{f}^{*}(T) = m_{f} - 2\sqrt{J}, \quad 2\sqrt{J} \ll m_{f}, \]  
(93)

which has the meaning of the effective fermion (antifermion) mass. However, contrary to the usually introduced effective mass, the quantity (93) enters only the exponent in expression (86). We see that \( m_{f}^{*}(T) \) decreases with increase of the intensity of the multiple scattering \( J \), i.e., with increase of the temperature.

Typical temperature, when effective fermion mass decreases significantly, \( \sqrt{J} \sim m_{f} \), is as follows

\[ T \sim T_{\text{bl}} \sim \beta^{3/4}g^{-1}m_{f}r_{s}^{-1/2}(T_{\text{bl}}). \]  
(94)

For \( T \gtrsim T_{\text{bl}} \) the non-relativistic approximation for fermions, that we used, fails. Generalization to the relativistic case can be found in [4]. However exponential increase of the effective fermion mass, that we have demonstrated, starts already for \( T \ll T_{\text{bl}} \), in the region of validity of the non-relativistic approximation. For \( g \gg 1 \) and \( \beta \simeq 1 \) we obtain

\[ T_{\text{bl}} \sim m_{f}/g \ll m_{f}. \]  
(95)

Thus already for comparatively low temperatures the fermion vacuum becomes blurred due to strong interaction between light boson and heavy fermion subsystems.
With the help of expression (77) we are able to calculate the boson self-energy (69), see [4]:

\[
\text{Re}\Sigma^R_b(\omega_b, p_b) \simeq -2g^2 \text{Tr} \int \frac{d^4p_f}{(2\pi)^4} \left[ \text{Re}G^R_f(p_f + p_b) + \text{Re}G^R_f(p_f - p_b) \right] \\
\times \text{Im}G^R_f(p_f) f_f(\omega_f),
\]

(96)

\[
\Gamma^R_b(\omega_b, p_b) \simeq 4g^2 \text{Tr} \int \frac{d^4p_f}{(2\pi)^4} \text{Im}G^R_f(p_f + p_b)\text{Im}G^R_f(p_f) \\
\times \left[ f_f(\omega_f) - f_f(\omega_f + \omega_b) \right].
\]

(97)

Let us present the result in the limit \(x \gg 1\) (i.e., \(g \gg 1\)). Then we find [4]:

\[
\text{Re}\Sigma^R_b(\omega_b, p_b) \simeq -\frac{4g^2n_f^{(+)}(\pm)}{\sqrt{J}} + \alpha(\omega_b^2 - \frac{1}{2}p_b^2), \quad \alpha = \frac{4g^2n_f^{(+)}(\pm)}{Jm_f},
\]

(98)

\[
\frac{\Gamma^R_b}{\text{Re}\Sigma^R_b} = O(T^2/J).
\]

(99)

Using (98) we recover the effective boson mass

\[
m^*_b = m_b^2 - \frac{4g^2n_f^{(+)}(\pm)}{\sqrt{J}}.
\]

(100)

The value \(\alpha\) has extra smallness for \(m_f \gg m_b\) and dependence on it can be neglected.

We see that for temperatures \(T < T_{bd}\) one has \(|\text{Im}\Sigma^R_b/\text{Re}\Sigma^R_b| \ll 1\). Thus, at these temperatures we may treat bosons in the quasiparticle approximation. Although the value \(\text{Re}\Sigma^R_b\) is exponentially small we do not suppress the interaction and do not set \(m^*_b = m_b\) and \(\beta \simeq 1\) because, as we show below, the correction due to the interaction to boson contributions in some thermodynamic quantities, e.g., pressure, can be larger than the fermion contributions.

Following Ref. [4], that treated fermions in relativistic terms, at a somewhat higher temperature, \(T > T_{bd}\), the value \(-\text{Re}\Sigma^R_b\) sharply increases and the effective boson mass substantially decreases.
5.5 Thermodynamic quantities

Since for temperatures of our interest here, $T < T_{bl}$, bosons are good quasi-particles and fermions are blurred particles, it is natural to use expressions (64), (65), with interaction affecting boson terms. Then boson contributions are immediately calculated:

$$E_{b}^{q.p.} = \int_{0}^{\infty} \frac{4\pi p_{b}^{2}dp_{b}}{(2\pi)^{3}} f_{b}(\omega_{b}(p_{b})) \frac{\omega_{b}^{2}(p_{b}) - \frac{1}{2} \Re \Sigma^{R}_{b}(\omega_{b}(p_{b}), p_{b})}{\omega_{b}(p_{b})} \left[ 1 - \frac{\partial \Re \Sigma^{R}_{b}}{\partial \omega^{2}} |_{\omega_{b}(p_{b})} \right]$$

(101)

$$P_{b}^{q.p.} = \int_{0}^{\infty} \frac{4\pi p_{b}^{2}dp_{b}}{(2\pi)^{3}} f_{b}(\omega_{b}(p_{b})) \frac{p_{b}^{2} + \frac{1}{2} \Re \Sigma^{R}_{b}(\omega_{b}(p_{b}), p_{b})}{\omega_{b}(p_{b})} \left[ 1 - \frac{\partial \Re \Sigma^{R}_{b}}{\partial \omega^{2}} |_{\omega_{b}(p_{b})} \right].$$

(102)

Main contributions to the thermodynamic quantities of boson sub-system are given by non-interacting terms:

$$E_{b}^{q.p.} \rightarrow E_{b}^{\text{free}} = \int_{0}^{\infty} \frac{p_{b}^{2}dp_{b}}{2\pi^{2}} \omega_{b}^{3} f_{b}(\omega_{b}), \quad P_{b}^{q.p.} \rightarrow P_{b}^{\text{free}} = \int_{0}^{\infty} \frac{p_{b}^{4}dp_{b}}{6\pi^{2}\omega_{b}^{3}} f_{b}(\omega_{b}).$$

(103)

For $T \gtrsim m_{b}^{*}(T)$ of our interest we estimate $E_{b}^{q.p.} \sim 3P_{b}^{q.p.} \sim \pi^{2}T^{4}/30$.

Let us consider the interaction contribution (by interaction term we mean the term $\propto \epsilon_{b}^{\text{int}} - \epsilon_{b}^{\text{pot}}$):

$$\delta E_{b}^{q.p.} = -\delta P_{b}^{q.p.} \simeq \int_{0}^{\infty} \frac{4\pi p_{b}^{2}dp_{b}}{(2\pi)^{3}} f_{b}(\omega_{b}(p_{b})) \left[ \Re \Sigma^{R}_{b}(\omega_{b}(p_{b}), p_{b}) \right].$$

(104)

We neglected the value $\frac{\partial \Re \Sigma^{R}_{b}}{\partial \omega^{2}} |_{\omega_{b}(p_{b})} = \alpha \ll 1$ for $g \gtrsim 1$ and for temperatures of our interest. Rough estimation yields:

$$\delta E_{b}^{q.p.} \sim -\delta P_{b}^{q.p.} \sim \frac{g^{2}n_{i}^{(\pm)}T^{2}}{\sqrt{J}}.$$

(105)

Simple result can be obtained for $T \gg m_{b}^{*}(T)$:

$$\delta E_{b}^{q.p.} = -\delta P_{b}^{q.p.} \simeq \frac{g^{2}n_{i}^{(\pm)}T^{2}}{6\sqrt{J}}.$$

(106)
Now we need to add to boson quantities (101), (102) the contribution of quasi-free fermion blurs. Integrations in \(\omega_f\) are done similar to that in (85). We use that \(|\xi| < 2\sqrt{J} \ll m_f\) and thus integrating (64), (65) in \(\xi\) we may put \(\omega_f \simeq \omega_p\) in these expressions in the pre-factor of the spectral function. Therefore,

\[
E^{(\pm)}_f \simeq F(T) E^{(\pm)}_{\text{Bol}},
\]

\[
E^{(\pm)}_{\text{Bol}} = N_f \int_0^\infty \frac{4\pi p_f^2 dp_t}{(2\pi)^3} \omega_p f^{(\pm)}_{\text{Bol}}(p_t) \simeq m_f^* n_{\text{Bol}}^{(\pm)} \simeq m_f n_{\text{Bol}}(1 - 2\sqrt{J/m_f}),
\]

\[
P^{(\pm)}_f \simeq F(T) P^{(\pm)}_{\text{Bol}},
\]

\[
P^{(\pm)}_{\text{Bol}} = N_f \int_0^\infty \frac{4\pi p_f^2 dp_t}{(2\pi)^3} \frac{p_t^2}{3\omega_p^2} f^{(\pm)}_{\text{Bol}}(p_t) \simeq T n_{\text{Bol}}^{(\pm)}.
\]

Thus for the fermion sub-system we obtained relations between thermodynamic quantities, similar to those for the Boltzmann gas, however with a common large pre-factor \(F(T)\). This pre-factor demonstrates significantly increased number of produced fermion-antifermion pairs due to strong interaction between fermion-boson species. Thus fermion sub-system represents a gas of blurred fermions. Additionally we kept a higher order term (for \(T \ll T_{\text{bl}}\)) in expression for the energy (which is larger than the main term in the pressure for \(\sqrt{J} \gg T\)) in order further to fulfill thermodynamic consistency conditions.

Note that in spite of extra \(T\) dependence of the factor \(F(T)\) in (107), (108) compared to Boltzmann expressions, thermodynamic consistency conditions remain approximately fulfilled. Indeed, differentiating \(P\) in \(T\) in order to check validity of the consistency condition for the entropy; see second Eq. (3), we may omit terms \(\propto \sqrt{J}\) compared to \(\omega_p \simeq m_f\). It means that with this accuracy we should not differentiate spectral function, but only the Bolzmann distribution. In the limit \(T \gg m_{\text{b}1}^*(T)\), factor \(F(T) \rightarrow \text{const}(T)\) and it does not contribute to the temperature derivative.

On the other hand, we could transport the interaction contribution to the sub-system of the blurred fermions considering bosons, as the gas of free quasiparticles. From (62), (63) we find the interaction term \((\propto \epsilon_{\text{int}}^f - \epsilon_{\text{pot}}^f)\):

\[
\delta E_f = -\delta P_f = 2N_f \int_0^\infty \frac{4\pi p_f^2 dp_t}{(2\pi)^3} \int_0^\infty \frac{d\omega_f}{2\pi} \left[ -\frac{\omega_f}{2} \right] A^{(+)}_{\text{int}} f^{(+)}_f.
\]

Integration is performed, as in (85). Then we obtain

\[
\delta E_f = -\delta P_f \simeq 2\sqrt{J} n_f^{(\pm)},
\]
that for $T \gg m_\star^f$ coincides with (106). Thus we explicitly demonstrated consistency of our derivations.

For $T \gg m_\star^f(T)$ the "free quasiparticle" contributions to the boson pressure and the energy are easily calculated

\[
P_{b}^\text{free q.p.} \simeq \int_0^\infty \frac{4\pi p_b^2 dp_b}{(2\pi)^3} f_b(\omega_b(p_b)) \frac{p_b^2}{3\omega_b(p_b)}
\]

\[
\simeq \frac{\pi^2 T^4}{90} - \frac{m_b^2 T^2}{24} = \frac{\pi^2 T^4}{90} - \frac{m_b^2 T^2}{24} + \frac{g^2 n_t(\pm) T^2}{6\sqrt{J}},
\]

(111)

\[
E_{b}^\text{free q.p.} \simeq \int_0^\infty \frac{4\pi p_b^2 dp_b}{(2\pi)^3} \omega_b(p_b) f_b(\omega_b(p_b))
\]

\[
\simeq \frac{\pi^2 T^4}{30} - \frac{m_b^2 T^2}{24} = \frac{\pi^2 T^4}{30} - \frac{m_b^2 T^2}{24} + \frac{g^2 n_t(\pm) T^2}{6\sqrt{J}}.
\]

(112)

The total pressure, energy density and the entropy calculated following Eq. (11) become

\[
P_{\text{tot}} \simeq \frac{\pi^2 T^4}{90} - \frac{m_b^2 T^2}{24} + 2T n_t(\pm),
\]

(113)

\[
E_{\text{tot}} \simeq \frac{\pi^2 T^4}{30} - \frac{m_b^2 T^2}{24} + 2m_f n_t(\pm),
\]

(114)

\[
T S_{\text{tot}} = E_{\text{tot}} + P_{\text{tot}} \simeq \frac{2\pi^2 T^4}{45} - \frac{m_b^2 T^2}{12} + 2m_f n_t(\pm).
\]

(115)

We see that the interaction term $\delta P_t = \delta P_{b}^\text{q.p.}$ is compensated in the total pressure by the corresponding boson "free quasiparticle" contribution. On the other hand, the interaction term in the energy density doubles the corresponding boson "free quasiparticle" contribution. But this resulting contribution is compensated by the fermion quasiparticle term that distinguishes $m_f$ from $m_\star^f$. Thus the total pressure, the energy and the entropy are approximately given by the sum of contributions of "free" bosons and "free" fermion-antifermion blurs. With these expressions thermodynamic consistency conditions (3) are fulfilled.
Finally we have arrived at the following picture. A hot system of strongly interacting \((g \gg 1)\) light bosons and heavy fermions with zero chemical potentials at temperatures \(T_{\text{bl}} > T \gtrsim m^*_b(T)\) represents a gas mixture of boson quasiparticles and blurred fermions. Blurred heavy fermions undergo rapid \((p_f \sim \sqrt{m_f T} \gg p_b \sim T)\) Brownian motion in the boson quasiparticle gas. The density of blurred fermions is dramatically increased at \(T \sim T_{\text{bl}}\) compared to the standard Boltzmann value. Thermodynamical quantities are such as for the quasi-ideal gas mixture of quasi-free fermion blurs and quasi-free bosons.

5.6 Boson mean field solution

The model with the interaction Lagrangian \((72)\) allows for the mean field solution for the boson field, cf. Walecka model. To get this solution one should replace \(\phi \rightarrow \phi_{\text{cl}} + \phi\). The classical field can be incorporated into expression for the pressure with the help of the replacement \(m_t \rightarrow m_t^\phi = m_t - g\phi_{\text{cl}}\) and by addition of the term \(\delta P_{\text{cl}} = -\frac{1}{2}m^*_b \phi_{\text{cl}}^2\). Variation of the pressure in the classical field produces the equation of motion

\[
m^*_b \phi_{\text{cl}} \simeq 2gn_t^{(\pm)}[m_t^\phi],
\]

where we indicated that the fermion density \(n_t^{(\pm)}\) depends on the classical field through \(m_t^\phi\). The effective fermion mass \((93)\) then becomes

\[
m_t^*(T) = m_t - 2\sqrt{J} - 2g^2n_t^{(\pm)}[m_t^\phi]/m^*_b, \quad 2\sqrt{J} \ll m_t.
\]

The total pressure is

\[
P_{\text{tot}} \simeq \frac{\pi^2 T^4}{90} - \frac{m^2_b T^2}{24} + 2Tn_t^{(\pm)}[m_t^\phi] - \frac{2g^2(n_t^{(\pm)}[m_t^\phi])^2}{m^*_b^2},
\]

compare with Eq. \((113)\). The last term in \((118)\) is the contribution of the classical field \(\delta P_{\text{cl}}\). For all temperatures \(T \lesssim T_{\text{bl}}\) the classical field contribution remains small, as consequence of the smallness of the fermion density, but it drastically increases with decrease of the effective boson mass at higher temperatures.

Note that similar consideration can be performed in the chiral \(\sigma\) model, where in addition to the Yukawa interaction there is the boson self-interaction, and already at zero temperature there exists classical field \(\phi_{\text{cl}} \neq 0\).
5.7 Hot Bose condensation

For $T \gtrsim T_{\text{bl}}$ typical fermion momenta $\sqrt{2m_fT}$ continue to remain much smaller than the mass $m_f$. However non-relativistic approximation for fermions that we have used fails since typical deviation of the fermion energy from the mass-shell becomes comparable with $m_f$. Nevertheless let us extrapolate our results to higher temperatures. From (100) we see that at $\sqrt{\mathcal{J}(T_{CB})} = 4g^2 n_{f}(\pm)/m_b^2$ squared of the effective boson mass reaches zero and it becomes negative for $T > T_{CB}$. Supposing that $T_{CB}$ deviates only a little from the value $T_{\text{bl}}^{n,\text{rel}} = \sqrt{3}T_{\text{bl}}$ (when $m^*_f = 0$) we easily find

$$T_{CB} \approx T_{\text{bl}}^{n,\text{rel}} \left[ 1 - \frac{T_{\text{bl}}^{n,\text{rel}}}{m_f} \ln \frac{\sqrt{2}n_f^{(\pm)}m_f^2}{\pi^2gm_b^2} \right].$$

(119)

In Ref. [4] in realistic relativistic framework we found that $T_{CB}$ is in the vicinity of the value $T_{bl}$, whereas within non-relativistic model we find a larger value for the critical density $T_{CB} \approx T_{bl}^{n,\text{rel}}$ (more precisely $T_{CB}$ is slightly below $T_{bl}^{n,\text{rel}}$ for $1 \ll g \lesssim 10$). This overestimation of the value $T_{CB}$ is a price paid for simplicity of our non-relativistic consideration of heavy fermions here.

Now we may consider a possibility of the hot Bose condensation for $T > T_{CB}$, see [4] where such a possibility has been studied in the framework of relativistic model. In the model with additionally introduced boson self-interaction $L_{\text{int}} = -\lambda \phi^4/4$ (with the coupling $\lambda > 0$) the condensate field is determined by minimization of the pressure in the classical field variable. We find

$$m_b^* \phi + \lambda \phi^3 \approx 2gn_f^{(\pm)}[m_f^\phi].$$

(120)

Let us assume that for $T \gtrsim T_{CB}$ the contribution $gn_f^{(\pm)}[m_f^\phi]$ is small and can be neglected. It is so for $n_f^{(\pm)}[m_f^\phi] \ll |m_b^*|^{3}/(g\sqrt{\lambda})$. Then, we may neglect a small classical field, existing already for $m_b^* > 0$, see (116). A new condensate solution arises then for $T > T_{CB}$, when $m_b^* < 0$. One may say that for $T = T_{CB}$ there occurs second order phase transition leading to the hot Bose condensation. Inclusion into consideration of the term $gn_f^{(\pm)}[m_f^\phi]$ results in a jump in the value of the classical field at $T$ near $T_{CB}$. It means that actually the hot Bose condensation appears as the first order phase transition. However, if the value of the jump is small, one can neglect it and consider the second order phase transition. Note that saturation of the condensate field arises for $T > T_{CB}$ only due to the repulsive boson-boson interaction ($\lambda > 0$).

Boson excitation spectrum is then reconstructed and becomes

$$\omega_b^2 \approx -2m_b^* + p_b^2.$$

(121)
(for $\alpha \ll 1$). The pressure is given by

$$P_{\text{tot}} \simeq \frac{\pi^2 T^4}{90} + \frac{m_b^2 T^2}{12} - \frac{g^2 n_i^{(\pm)} T^2}{2 \sqrt{J}} + \frac{(4g^2 n_i^{(\pm)}/\sqrt{J} - m_b^2)^2}{4\lambda} + 2Tn_i^{(\pm)}.$$  \hspace{1cm} (122)

We see that the pressure acquires a positive contribution $\propto 1/\lambda$, large for small $\lambda$.

One could expect an anomalous enhancement of the boson (e.g. pion and kaon) production at low momenta ($p_b \lesssim T$) and an anomalous behavior of fluctuations, e.g. at LHC conditions, as a signature of the hot Bose condensation for $T > T_{cB}$, if a similar phenomenon occurred in a realistic problem including all relevant particle species.

6 Concluding remarks

We derived exact and simplified expressions for thermodynamic characteristics of the matter of interacting particles with finite widths. First, we disregarded interaction terms and considered system of ”free resonances” using simplified ansatze for spectral functions. Such a consideration might be helpful in description of dilute systems. We have shown that the model is thermodynamically consistent. Our expressions demonstrate deficiency of the use of approximation of constant particle width. Such an approximation is often done to simplify the treatment of the problem. Although at a higher density our ansatz for the spectral function is definitely not valid and interaction and potential energies are not small, the thermodynamical consistency relations continue to be fulfilled.

Then we recovered interaction terms and found simple exact equations for thermodynamic values expressed in terms of spectral functions. In the quasiparticle approximation these expressions transform to the standard quasiparticle expressions. For broad resonances and blurred particles all information on medium effects is hidden in the form of spectral functions. This circumstance allows to model behavior of different systems selecting relevant forms for spectral functions and checking that consistency conditions are approximately fulfilled.

On examples of one-component non-relativistic system interacting by paired potential and the relativistic boson gas with $\phi^4$ interaction, we have shown explicitly that the ”free resonance” term becomes dominating in the low density limit, provided the width is finite but rather small.
We have shown, that in case of a fermion-boson interacting system, the interaction contribution to thermodynamic quantities can be expressed either in terms of the fermion spectral function or in terms of the boson spectral function. Similar conclusion is done for a system of two-flavor fermions interacting with a boson. Here only one species is affected by the interaction, whereas two other species represent free resonances or blurred particles. This observation opens a possibility of approximate description of multi-component systems, provided some species can be described in the quasiparticle approximation and other species represent broad resonances or blurred particles. The interaction energy can be transported to the quasiparticle sub-systems. In the low density limit the interaction part ceases more rapidly compared to a free particle contribution. Then we deal with non-interacting quasiparticles and free broad resonances.

As an illustrative example, we studied behavior of hot strongly interacting \((g \gtrsim 1)\) heavy fermion – light boson sub-systems at zero chemical potentials of species. Here in a broad temperature interval \((T_{bl} > T \gtrsim m^*_b(T))\) bosons can be well described in the quasiparticle approximation, whereas fermions become blurred particles.

The following remark is in order. Although we considered the case \(m_b \ll m_f\), our results are more general, being valid provided \(m_b^* \ll m_f^*\). Thus even if condition \(m_b \ll m_f\) is not fulfilled, with increase of the temperature the system may jump to the regime where blurred fermions undergo the Brownian motion in the bath of effectively much less massive bosons, see [4] for more details.

Fermion particle distributions are significantly increased compared to the standard Boltzmann distributions. The system represents a gas mixture of boson quasiparticles interacting with fermion-antifermion blurs. In thermodynamical values interaction terms partially compensate each other. Thereby, for very strong coupling between species \((g \gg 1)\) thermodynamical quantities of the system, like the energy, pressure and entropy, prove to be such as for the quasi-ideal gas mixture of quasi-free fermion blurs and quasi-free bosons.

The latter observation gives us a hint for construction of equation of state which might be valid at LHC conditions in some temperature interval (below deconfinement temperature). One may conjecture that the pressure is approximately presented as the sum of all possible contributions of quasi-free bosons and quasi-free baryon blurs. The latter contributions differ from the free baryon particle terms only by pre-factors \(F_f(T)\), with subscript ”f” running over the baryon and antibaryon species. The particle density for the given baryon species \(n_f\) is enhanced compared to the corresponding Boltzmann value by the factor \(F_f(T)\). The boson distribution depends on its effective mass which may essentially decrease for the lightest boson species. The ratio of the baryon (antibaryon) to boson densities is expected to be significantly enhanced com-
pared to the ratio of the standard Boltzmann density for the given baryon species to the massless boson density.

For $T > T_{cB}$ there may arise a hot Bose condensation for some boson species. If $T > T_{cB}$ retained at freeze out conditions, Bose condensation could manifest itself in an anomalous enhancement of the corresponding boson production at low momenta (for the $s$-wave condensation discussed here).

Concluding, our treatment of particles with finite widths (resonances and blurred particles) is helpful in cases, when explicit expressions for the spectral functions can be constructed. E.g., description of a system of heavy fermions strongly interacting with light bosons at zero chemical potentials of components represents such an example, cf. [1]. Another example is the low density system. Then a simplified treatment of "free resonances" with simple ansatze spectral functions can be especially helpful provided the broad resonance appears, as the result of the interaction with other particle species, which can be described within the quasiparticle approximation.

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