Gradings on Lie algebras with applications to infra-nilmanifolds

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October 16, 2014

Abstract

In this paper, we study positive as well as non-negative and non-trivial gradings on finite dimensional Lie algebras. We give a different proof that the existence of such a grading on a Lie algebra is invariant under taking field extensions, a result very recently obtained by Y. Cornulier. Similarly, we prove that given a grading of one these types and a finite group of automorphisms, there always exist a positive grading which is preserved by this group. From these results we conclude that the existence of an expanding map or a non-trivial self-cover on an infra-nilmanifold depends only on the covering Lie group. Another application is the construction of a nilmanifold admitting an Anosov diffeomorphisms but no non-trivial self-covers and in particular no expanding maps, which is the first known example of this type.

Let $E \subseteq \mathbb{C}$ be a subfield of the complex numbers and $\mathfrak{n}$ a finite dimensional Lie algebra over $E$. A grading of the Lie algebra $\mathfrak{n}$ is a decomposition of $\mathfrak{n}$ as a direct sum $\mathfrak{n} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{n}_i$ of subspaces $\mathfrak{n}_i \subseteq \mathfrak{n}$ such that $[\mathfrak{n}_i, \mathfrak{n}_j] \subseteq \mathfrak{n}_{i+j}$ for all $i, j \in \mathbb{Z}$. We call the grading positive if $\mathfrak{n}_i = 0$ for all $i \leq 0$ and non-negative if $\mathfrak{n}_i = 0$ for all $i < 0$. Every Lie algebra has a non-negative grading given by $\mathfrak{n} = \mathfrak{n}_0$ and we call this the trivial grading of $\mathfrak{n}$. An automorphism $\varphi \in \text{Aut}(\mathfrak{n})$ preserves the grading if $\varphi(\mathfrak{n}_i) = \mathfrak{n}_i$ for all $i \in \mathbb{Z}$.

There is a strong relation between positive gradings and expanding automorphisms of Lie algebras. An automorphism $\varphi \in \text{Aut}(\mathfrak{n})$ is called expanding if $|\lambda| > 1$ for all eigenvalues $\lambda$ of $\varphi$. Let $\mathfrak{n}$ be a Lie algebra over $E$ with a positive grading $\mathfrak{n} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{n}_i$. Given a $\mu \in E$ with $|\mu| > 1$, there exists an expanding automorphism $\varphi$ of $\mathfrak{n}$ which maps an element $x \in \mathfrak{n}_i$ to $\mu^ix$. Moreover, if $\psi$ is an automorphism of $\mathfrak{n}$ which preserves the grading, then $\psi$ will commute with the expanding automorphism $\varphi$.

In [5, Theorem 3.4] it is shown that this construction also works the other way around. Given an expanding automorphism $\varphi$ we find a positive grading such that every automorphism $\psi$ commuting with $\varphi$ preserves the positive grading. The result in [5] was only stated in the case where $E = \mathbb{Q}$ but in fact the proof works for any field $E$.

There is a similar relation between non-trivial and non-negative gradings and partially expanding automorphisms. We call an automorphism partially expanding if for every eigenvalue $\lambda$ of $\varphi$ we have $\lambda = 1$ or $|\lambda| > 1$ and there is at least one eigenvalue $\lambda \neq 1$. Therefore, the study of these two types of gradings on Lie algebras preserved by a given automorphism is equivalent to the study of (partially) expanding automorphisms which commute with this automorphism.

*The author is supported by a Ph.D. fellowship of the Research Foundation – Flanders (FWO). Research supported by the research Fund of the KU Leuven
The automorphism group of a Lie algebra is a linear algebraic group and in the first section of this paper we recall the basis properties of such groups. By using the theory of linear algebraic groups, the main results are proved in the following sections. The first theorem states that the existence of positive gradings is invariant under taking field extensions:

**Theorem 2.2.** Let $E \subseteq F \subseteq \mathbb{C}$ be field extensions and $\mathfrak{n}$ a Lie algebra over the field $E$. Then $\mathfrak{n}$ admits an expanding automorphism if and only if $F \otimes \mathfrak{n}$ admits an expanding automorphism. Equivalently, the Lie algebra $\mathfrak{n}$ admits a positive grading if and only if the Lie algebra $F \otimes \mathfrak{n}$ admits a positive grading.

The next one shows that given a finite group of automorphisms, there always exists a grading preserved by this group:

**Theorem 3.2.** Let $\mathfrak{n}$ be a rational Lie algebra which admits an expanding automorphism and $H \subseteq \text{Aut}(\mathfrak{n})$ a finite subgroup. Then there exists an expanding automorphism of $\mathfrak{n}$ which commutes with every element of $H$. Equivalently, if $\mathfrak{n}$ admits a positive grading, it also admits a positive grading which is preserved by $H$.

The same results for non-trivial and non-negative gradings on Lie algebras are also true.

In the next section, we combine these main results about Lie algebras with [5, Theorem 4.2.] to get the following classification of infra-nilmanifolds admitting an expanding map:

**Theorem 4.1.** Let $\Gamma \backslash G$ be an infra-nilmanifold modeled on a Lie group $G$ with corresponding Lie algebra $\mathfrak{g}$. Then the following are equivalent:

1. $\Gamma \backslash G$ admits an expanding map;
2. $\mathfrak{g}$ has a positive grading;
3. $\mathfrak{g}$ has an expanding automorphism.

Again, there is a similar classification for non-trivial self-covers of $\Gamma \backslash G$, corresponding to non-trivial and non-negative gradings. As a consequence of this theorem we construct a nilmanifold admitting an Anosov diffeomorphism but no expanding map in the last section.

Some of these results were proved independently and by different methods in [2] by Y. Cornulier. There are more detailed references in this paper.

## 1 Linear algebraic groups

We recall some basic properties about linear algebraic groups, a more detailed discussion can be found e.g. in [1, 10, 11].

Let $K$ be any subfield of $\mathbb{C}$, then we define a linear algebraic $K$-group $G$ as a subgroup of $\text{GL}(n, \mathbb{C})$ which is $K$-closed, i.e. $G$ is the zero set of a finite number of polynomials with coefficients in $K$. We denote by $G(K) = G \cap \text{GL}(n, K)$ the subgroup of $K$-rational points in $G$. The connected component of the identity element in $G$ is denoted as $G^0$ and this subspace is a normal subgroup of finite index in $G$. We call an element $x \in G$ expanding if for all the eigenvalues $\lambda$ of $x$, we have $|\lambda| > 1$. We call an automorphism partially expanding if for every eigenvalue $\lambda$ of $x$ we have $\lambda = 1$ or $|\lambda| > 1$ and $x$ has at least one eigenvalue $\neq 1$.

A group morphism between two linear algebraic $K$-groups is said to be defined over $K$ (or is a $K$-morphism) if the coordinate functions are given by polynomials over the field $K$. Denote by $\mathbb{C}^*$ the multiplicative group of the field $\mathbb{C}$, then a character of $G$ is a group morphism $G \to \mathbb{C}^*$. A $K$-torus is a linear algebraic $K$-group which is isomorphic to a closed subgroup of diagonal matrices $D(n, \mathbb{C})$. If the isomorphism is defined over $K$, then we call the torus $K$-split. If a $K$-torus has no non-trivial characters defined over $K$, then we call the torus anisotropic.

The radical $\mathfrak{R}G$ of a linear algebraic $K$-group $G$ is defined as the maximal connected and solvable normal subgroup of $G$. This normal subgroup always exists and is defined over $K$, see [1].
Section 11.21]. The subgroup of $\mathfrak{R}G$ consisting of all its unipotent elements is called the unipotent radical and we denote this normal subgroup as $(\mathfrak{R}G)_u$. If the unipotent radical is trivial, i.e. if $(\mathfrak{R}G)_u = \{e\}$, the group $G$ is called reductive. A Levi subgroup is a (reductive) linear algebraic subgroup $L \subseteq G$ such that $G$ is the semi-direct product of $L$ and $(\mathfrak{R}G)_u$. From [10 Chapter VIII] we know that Levi subgroups always exist in characteristic 0 and that they are unique up to conjugation. Moreover, every reductive linear algebraic subgroup of $G$ is contained in a Levi subgroup. If $G$ is a connected reductive group, it can be decomposed as a product $G = \mathcal{Z} : \mathfrak{D}G$ where $\mathcal{Z} = Z(G)^0$ is the identity component of the center of $G$ and $\mathfrak{D}G = [G,G]$ is the commutator subgroup, see [1] Section 14.2]. The intersection $\mathcal{Z} \cap \mathfrak{D}G$ is a finite subgroup of $G$.

Let $E \subseteq \mathbb{C}$ be any field and $n$ a nilpotent Lie algebra over the field $E$. If $E \subseteq F$ is a field extension of $E$, then we can construct the Lie algebra $n^F = F \otimes_E n$ over the field $F$. The standard example will be the case where $F = \mathbb{C}$ and we call $n^\mathbb{C}$ the complexification of the Lie algebra $n$. The automorphism group $G = \text{Aut}(n^\mathbb{C})$ is a linear algebraic $E$-group and we have that $\text{Aut}(n) = G(E)$.

2 Gradings under field extensions

In this section we show that the existence of a positive grading is invariant under field extensions. First, we give a proof in the more general case of linear algebraic groups:

**Theorem 2.1.** Let $K \subseteq L \subseteq \mathbb{C}$ be field extensions and $G$ a linear algebraic $K$-group. Then $G(K)$ has an expanding element if and only if $G(L)$ has an expanding element.

**Proof.** We have a natural inclusion $G(K) \subseteq G(L)$, so if $G(K)$ has an expanding element, also $G(L)$ has an expanding element. For the other implication, it is sufficient to prove it in the case where $L = \mathbb{C}$. Since every power of an expanding element is again expanding, we can also assume that the group $G$ is connected.

Let $z \in G(\mathbb{C})$ be an expanding element. By the multiplicative Jordan decomposition, we can assume that $x$ is semisimple. Every semisimple element of $G$ lies in a maximal torus, so the existence of an expanding element is equivalent to the existence of an expanding element in a maximal torus. Since all maximal tori are conjugate and $G$ contains also a maximal torus defined over $K$, we can assume that $G$ is a $K$-torus.

From [1] Section 8.15] it follows that every $K$-torus $G$ can be written as $G = G_aG_d$ where $G_a$ is an anisotropic subtorus and $G_d$ is a $K$-split subtorus. Since every $K$-split torus is conjugated over $GL(n, K)$ to a $K$-closed subgroup of $D(n, \mathbb{C})$, we can assume that $G_d$ is a subgroup of $D(n, \mathbb{C})$. Let $x \in G$ be an expanding element and write $x = yz$ with $y \in G_a$ and $z \in G_d$. We show that $z$ is also expanding and hence that the $K$-split torus $G_d$ contains an expanding element. If not, we can write $z$ up to permutation of the eigenvalues and thus up to conjugation by an element of $GL(n, \mathbb{Q}) \subseteq GL(n, K)$ as

$$z = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}$$

where $Z_1$ is a diagonal matrix with only eigenvalues $\leq 1$ in absolute value and $Z_2$ a diagonal matrix with only eigenvalues $> 1$ in absolute value. Since the torus $G_a$ commutes with $z$, every element $t \in G_a$ can be written as

$$t = \begin{pmatrix} A_t & 0 \\ 0 & B_t \end{pmatrix}$$

for some invertible matrices $A_t$ and $B_t$. Consider the character $\chi : G_a \to \mathbb{C}^*$ with $\chi(t) = \det(A_t)$. This morphism is clearly defined over $K$ and hence must be trivial. In particular, for the element $y$ we have $\det(A_y) = 1$. This would imply that $x$ is not expanding, a contradiction and we conclude that $G_d$ contains an expanding element.

So we are left to prove the theorem in the case where $G$ is a $K$-split torus, or equivalently, in the case of a $K$-closed subgroup of $D(n, \mathbb{C})$. Note that the expanding elements form an open subset of $G(\mathbb{R})$ and $G(\mathbb{C})$ for the Euclidean topology. The case $K \subseteq \mathbb{R}$ is then immediate since $G(K)$ forms a dense subset of $G$ for the Euclidean topology. In the case $K \subseteq \mathbb{R}$, we only have
that $G(K)$ is a dense subset of $G(\mathbb{R})$ for the Euclidean topology and therefore it suffices to show that $G(\mathbb{R})$ has an expanding element.

From [1] Section 8.2 it follows that $G$ is defined by character equations, so as the intersection of kernels of characters. Every character $D(n, \mathbb{C}) \to \mathbb{C}^*$ is of the form

$$
\begin{pmatrix}
\lambda_1 & 0 & \ldots & 0 \\
0 & \lambda_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \lambda_n
\end{pmatrix}
\mapsto \lambda_1^{k_1} \lambda_2^{k_2} \ldots \lambda_n^{k_n}
$$

for some $k_i \in \mathbb{Z}$. This implies that if an element

$$
\begin{pmatrix}
\lambda_1 & 0 & \ldots & 0 \\
0 & \lambda_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \lambda_n
\end{pmatrix}
\in G,
$$

then also

$$
\begin{pmatrix}
|\lambda_1| & 0 & \ldots & 0 \\
0 & |\lambda_2| & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & |\lambda_n|
\end{pmatrix}
\in G,
$$

since the latter will also satisfy the same character equations. By applying this to an expanding element of $G$, we get an expanding element of $G(\mathbb{R})$ and this finishes the proof.

By applying Theorem 2.1 to the automorphism group of a Lie algebra, we have the following consequence:

**Theorem 2.2.** Let $E \subseteq F \subseteq \mathbb{C}$ be field extensions and $\mathfrak{n}$ a Lie algebra over the field $E$. Then $\mathfrak{n}$ admits an expanding automorphism if and only if $\mathfrak{n}^F$ admits an expanding automorphism. Equivalently, the Lie algebra $\mathfrak{n}$ admits a positive grading if and only if the Lie algebra $\mathfrak{n}^F$ admits a positive grading.

At this point, we want to remark here that Yves Cornulier presented the first proof of this theorem and this over all fields of characteristic 0 in [1, Theorem 1.4]. The approach we present here was developed independently from [2, Theorem 1.4] and uses different methods.

More or less the same proof also works for partially expanding elements instead of expanding elements. If we write a partially expanding automorphism $x = yz$ with $y \in G_a$ and $z \in G_d$ for an anisotropic torus $G_a$ and a $K$-split torus $G_d$, we can show just as in Theorem 2.1 that $z$ has no eigenvalues of absolute value $< 1$. This implies that the $K$-split torus $G_d$ has a partially expanding element.

One difference in the case of partially expanding elements is that the set of all partially expanding elements does not form an open subset of $G(\mathbb{R})$ or $G(\mathbb{C})$ for the Euclidean topology. If $G$ is a $K$-closed subgroup of $D(n, \mathbb{C})$ containing a partially expanding element with the first $k$ eigenvalues equal to 1, then we restrict to the subtorus

$$
\tilde{G} = \left\{ \begin{pmatrix}
\lambda_1 & 0 & \ldots & 0 \\
0 & \lambda_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \lambda_n
\end{pmatrix} \in G \mid \lambda_1 = \ldots = \lambda_k = 1 \right\} \subseteq G
$$

and the proof works for this subtorus $\tilde{G}$. In particular we get the following result:

**Theorem 2.3.** Let $\mathfrak{n}$ be a Lie algebra over a field $E \subseteq \mathbb{C}$ and $E \subseteq F \subseteq \mathbb{C}$ a field extension. Then $\mathfrak{n}$ admits a partially expanding automorphism if and only if $\mathfrak{n}^F$ admits a partially expanding automorphism. Equivalently, $\mathfrak{n}$ admits a non-trivial and non-negative grading if and only if $\mathfrak{n}^F$ admits a non-trivial and non-negative grading.
3 Gradings preserved by automorphisms

In this section, we study gradings preserved by automorphisms which lie in a reductive subgroup of the automorphism group. Note that all finite subgroups and diagonalizable groups of automorphisms satisfy this property, since they only contain semisimple elements. Again, we first consider the more general case of linear algebraic groups:

**Theorem 3.1.** Let $G$ be a linear algebraic $K$-group with an expanding element and $H \subseteq G(K)$ a subgroup contained in a reductive subgroup of $G$. Then there exists an expanding element of $G(K)$ which commutes with every element of $H$.

**Proof.** The elements of $G$ which commute with every element of $H$ form a $K$-closed subgroup of $G$, so because of Theorem 2.1 it suffices to show that $G$ contains an expanding element which commutes with every element of $H$.

Since $H$ is a subgroup of a reductive subgroup of $G$, there exists a Levi subgroup $L$ of $G$ such that $H \subseteq L$. Every expanding semisimple element of $G$ also lies in a Levi subgroup. Because all Levi subgroups are conjugated, we know that if the group $G$ has an expanding element, the Levi subgroup $L$ (and therefore also $L^0$) has an expanding element.

The group $L^0$ is reductive and thus $L^0$ can be written as

$$L^0 = Z \cdot DL^0$$

with $Z$ the connected center of $L^0$. Take $x$ an expanding element of $L^0$, then we can write it as $x = zy$ with $y \in DL^0$ and $z \in Z$. Just as above in Theorem 2.1 we claim that $z$ is an expanding element. If $z$ is not expanding, we can write $z$ up to conjugation in $GL(n, \mathbb{C})$ as

$$z = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}$$

where $Z_1$ is an invertible matrix with only eigenvalues $\leq 1$ in absolute value and $Z_2$ an invertible matrix with only eigenvalues $> 1$ in absolute value. Since $z$ lies in the center of $L^0$, every element $t \in L^0$ is of the form

$$t = \begin{pmatrix} A_t & 0 \\ 0 & B_t \end{pmatrix}$$

for some invertible matrices $A_t$ and $B_t$. Consider now the character $\chi : L^0 \to \mathbb{C}^*$ given by $\chi(t) = \det(A_t)$. Since $y \in DL^0$, we have $\chi(y) = 1$ and this is a contradiction since $x = zy$ is expanding. We deduce that the connected center of $L^0$ contains an expanding element.

Since $L^0$ has finite index in $L$, there exist elements $l_1, \ldots, l_k \in L$ such that $L = l_1L^0 \sqcup \ldots \sqcup l_kL^0$. The subgroup $Z$ is a characteristic subgroup in $L$, so we have $l_iZl_i^{-1} = Z$ for all $1 \leq i \leq k$. Fix an expanding element $z \in Z$, then the element

$$z_0 = \prod_{1 \leq i \leq k} l_iz_i^{-1} \in Z$$

commutes with every element $l_i$ with $1 \leq i \leq k$ and therefore also with every element of $L$. The group $H$ is a subgroup of $L$ and thus $z_0$ commutes with every element of $H$. Since all the elements $l_iz_i^{-1} \in Z$ are expanding and commute, it follows that $z_0$ is expanding. This finishes the proof.

By applying this theorem to the linear algebraic group $\text{Aut}(n^\mathbb{C})$ of a rational Lie algebra $n$ and taking $H$ a finite subgroup, we get:

**Theorem 3.2.** Let $n$ be a rational Lie algebra which admits an expanding automorphism and $H \subseteq \text{Aut}(n)$ a finite subgroup. Then there exists an expanding automorphism of $n$ which commutes with every element of $H$. Equivalently, if $n$ admits a positive grading, it also admits a positive grading which is preserved by $H$. 


The same result holds for partially expanding maps with the same proof:

**Theorem 3.3.** Let \( \mathfrak{n} \) be a rational Lie algebra which admits a partially expanding automorphism and \( H \subseteq \text{Aut}(\mathfrak{n}) \) a finite subgroup. Then there exists a partially expanding automorphism of \( \mathfrak{n} \) which commutes with every element of \( H \). Equivalently, if \( \mathfrak{n} \) admits a non-negative and non-trivial grading, it also admits a non-negative and non-trivial grading which is preserved by \( H \).

These results were proved independently and by different methods in [2, Corollary 3.26].

### 4 Applications to infra-nilmanifolds

In this section we apply the main results to expanding maps and non-trivial self-covers on infra-nilmanifolds. First we recall the basic definitions about infra-nilmanifolds, more details can be found in [3, 13].

Let \( G \) be a connected and simply connected nilpotent Lie group and \( \text{Aut}(G) \) the group of continuous automorphisms of \( G \). Define the affine group \( \text{Aff}(G) \) as the semi-direct product \( G \rtimes \text{Aut}(G) \). The group \( \text{Aff}(G) \) acts on \( G \) in the following natural way:

\[
\forall \alpha = (g, \delta) \in \text{Aff}(G), \forall h \in G : \quad \alpha h = g \delta(h).
\]

Let \( C \subseteq \text{Aut}(G) \) be a compact subgroup of automorphisms. A subgroup \( \Gamma \subseteq G \rtimes C \) is called an almost Bieberbach group if \( \Gamma \) is a discrete, torsion-free subgroup such that the quotient \( \Gamma \backslash G \) is compact. The quotient space \( \Gamma \backslash G \) is a closed manifold and we call \( \Gamma \backslash G \) an infra-nilmanifold modeled on the Lie group \( G \). Denote by \( p : \Gamma \to \text{Aut}(G) \) the natural projection on the second component, then the group \( H = p(\Gamma) \) is a finite group, which is called the holonomy group of \( \Gamma \).

The subgroup \( N = \Gamma \cap G \) is a uniform lattice in \( G \) and \( \Gamma \) fits in the following exact sequence:

\[
1 \to N \to \Gamma \to H \to 1.
\]

In the case where \( G \) is abelian, i.e. \( G \cong \mathbb{R}^n \) for some \( n \), the manifolds constructed in this way are exactly the closed flat Riemannian manifolds.

Let \( \alpha \in \text{Aff}(G) \) be an affine transformation satisfying \( \alpha \Gamma \alpha^{-1} \subseteq \Gamma \). Then \( \alpha \) induces a differentiable map \( \bar{\alpha} \) on the infra-nilmanifold \( \Gamma \backslash G \), given by

\[
\bar{\alpha}(\Gamma g) = \Gamma^\alpha g.
\]

The induced map \( \bar{\alpha} \) is called an affine infra-nilmanifold endomorphism. The eigenvalues of \( \alpha \) are defined as the eigenvalues of the linear part of \( \alpha \), where eigenvalues of an automorphism of \( G \) are the eigenvalues of the corresponding Lie algebra automorphism on the Lie algebra of \( G \). If \( \alpha \Gamma \alpha^{-1} = \Gamma \), then the map \( \bar{\alpha} \) is a diffeomorphism and we call \( \bar{\alpha} \) an affine infra-nilmanifold automorphism. We call \( \bar{\alpha} \) expanding if \( \alpha \) only has eigenvalues \( > 1 \) in absolute value and hyperbolic if it has no eigenvalue of absolute value \( 1 \).

The exact sequence (1) does not split in general, so there is no natural representation \( H \to \text{Aut}(N) \). By embedding the group \( N \) into its rational Mal’cev completion \( N_\mathbb{Q} \), see [14], we get the following commutative diagram

\[
\begin{array}{cccccc}
1 & \to & N & \to & \Gamma & \to & H & \to & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & N_\mathbb{Q} & \to & \Gamma_\mathbb{Q} & \to & H & \to & 1,
\end{array}
\]

where the bottom exact sequence splits, see [8]. This leads to a natural representation \( H \to \text{Aut}(N_\mathbb{Q}) \) which is called the rational holonomy representation. Sometimes we will identify the group \( H \) with its image under this representation. The rational Mal’cev completion corresponds to a rational nilpotent Lie algebra \( \mathfrak{n} \) and moreover every rational nilpotent Lie algebra occurs in
this way. The Lie algebra $\mathfrak{n}^\mathbb{R}$ is the Lie algebra corresponding to the Lie group $G$. A rational form of the Lie algebra $\mathfrak{n}^\mathbb{R}$ is a rational subalgebra $\mathfrak{m} \subseteq \mathfrak{n}^\mathbb{R}$ such that $\mathfrak{m}^\mathbb{R} = \mathbb{R} \otimes \mathfrak{m} = \mathfrak{n}^\mathbb{R}$.

An expanding map $f : M \to M$ on a closed Riemannian manifold $M$ is a differentiable map such that there exists constants $c > 0$ and $\lambda > 1$ with $\|Df^n(v)\| \geq c\lambda^n\|v\|$ for all $v \in TM$ and all $n \geq 1$. By a result of Gromov, see [8], we know that every expanding map is topological conjugate to an expanding affine infra-nilmanifold endomorphism. So up to homeomorphism, the infra-nilmanifolds are the only manifolds admitting an expanding map. Anosov diffeomorphisms are defined in a similar way, see [14] and every hyperbolic affine infra-nilmanifold automorphism is an Anosov diffeomorphism. It follows from [4, Corollary 3.5.] that the existence of an Anosov diffeomorphism on a nilmanifold $N \setminus G$ depends only on the Lie algebra $\mathfrak{n}$. If $N \setminus G$ admits an Anosov diffeomorphism, we call the Lie algebra $\mathfrak{n}$ Anosov.

In [5] we showed that the existence of an expanding map depends only on the nilpotent Lie algebra $\mathfrak{n}$ and the rational holonomy representation $H \to \text{Aut}(\mathfrak{n})$. By combining Theorem 2.2 and Theorem 3.2, we have a complete algebraic description of the infra-nilmanifolds admitting an expanding map:

**Theorem 4.1.** Let $\Gamma \setminus G$ be an infra-nilmanifold modeled on a Lie group $G$ with corresponding Lie algebra $\mathfrak{g}$. Then the following are equivalent:

1. $\Gamma \setminus G$ admits an expanding map;
2. $\mathfrak{g}$ has a positive grading;
3. $\mathfrak{g}$ has an expanding automorphism.

As a corollary, we see that the existence of an expanding map depends only on the covering Lie group $G$:

**Corollary 4.2.** Let $M_1$ and $M_2$ be two infra-nilmanifolds modeled on the same Lie group. Then $M_1$ admits an expanding map if and only if $M_2$ admits an expanding map.

Similarly, we can combine Theorem 2.3 and Theorem 3.3 with the result [5, Theorem 5.4.] to find a complete algebraic characterization of the infra-nilmanifolds admitting a non-trivial self-cover, i.e. a self-cover which is not a homeomorphism:

**Theorem 4.3.** Let $\Gamma \setminus G$ be an infra-nilmanifold modeled on a Lie group $G$ with corresponding Lie algebra $\mathfrak{g}$. Then the following are equivalent:

1. $\Gamma \setminus G$ admits a non-trivial self-cover;
2. $\mathfrak{g}$ has a non-negative and non-trivial grading;
3. $\mathfrak{g}$ has a partially expanding automorphism.

So also the existence of a non-trivial self-cover depends only on the covering Lie group $G$.

Another application of Theorem 3.1 is the following result:

**Theorem 4.4.** Let $\Gamma \setminus G$ be an infra-nilmanifold admitting an expanding map and an Anosov diffeomorphism. Then there exists an expanding map and an Anosov diffeomorphism on $\Gamma \setminus G$ which commute.

**Proof.** If $\Gamma \setminus G$ admits an Anosov diffeomorphism, then [6, Theorem A] implies that there exists a hyperbolic automorphism $\varphi$ of $\mathfrak{n}$, i.e. with no eigenvalues of absolute value 1, such that $\varphi$ has characteristic polynomial in $\mathbb{Z}[X]$, $|\det(\varphi)| = 1$ and $\varphi$ commutes with every element of $H \subseteq \text{Aut}(\mathfrak{n})$. We can also assume that $\varphi$ is semisimple because of the multiplicative Jordan decomposition. Take $T$ the smallest linear algebraic subgroup of $\text{Aut}(\mathfrak{n}^\mathbb{C})$ which contains $\varphi$. The group $T$ will also commute with every element of $H$ and the group $HT$ forms a reductive subgroup of $\text{Aut}(\mathfrak{n}^\mathbb{C})$. By Theorem 3.1, we conclude that there exists a positive grading $\mathfrak{n} = \bigoplus_{i>0} \mathfrak{n}_i$ on $\text{Aut}(\mathfrak{n})$ which is preserved by $\varphi$ and every element of $H$. 7
From the proof of [6, Theorem A], it follows that some power \( \varphi^k \) of \( \varphi \) satisfies \( \varphi^k \Gamma \varphi^{-k} = \Gamma \) and thus induces a hyperbolic affine infra-nilmanifold automorphism on \( \Gamma \setminus G \). Let \( p \) be any prime and consider the expanding automorphisms \( \psi_p : n \to n \) which are given by \( \psi_p(x) = p^l x \) for all \( x \in n \). By the proof of [3, Theorem 4.2], we know that there exists a prime \( p \) and \( l > 0 \) such that \( \psi_p^l \Gamma \psi_p^{-l} \subseteq \Gamma \). The expanding affine infra-nilmanifold endomorphism induced by \( \psi_p^l \) commutes with the Anosov diffeomorphisms induced by \( \varphi^k \) and this ends the proof.

Theorem [4] is also true for non-trivial self-covers on infra-nilmanifolds by replacing positive grading by non-trivial and non-negative grading in the proof.

5 New type of example

As another application of Theorem [3] we construct a nilmanifold which admits an Anosov diffeomorphism but no non-trivial self-cover and so also no expanding map. This is the first example of a nilmanifold satisfying these properties.

First, we will construct a rational Lie algebra \( n \) as a quotient of a free nilpotent Lie algebra \( g \) by using the Hall basis of such a Lie algebra. Next, we give a general way of constructing automorphisms on this Lie algebra \( n \). In this way, we give a finite group of automorphisms \( H \) such that there exist no partially expanding automorphism of \( n \) commuting with \( H \). By Theorem [2.1] and Theorem [8.1] this shows that there are no partially expanding automorphisms on this Lie algebra nor on any rational form of the Lie algebra \( n^R \). Finally, we use the techniques of [7] to prove that the Lie algebra \( n^R \) has a rational form which is Anosov. So every nilmanifold corresponding to this rational form then has the desired properties.

Construction of the Lie algebra \( n \): Let \( g \) be the free 6-step nilpotent Lie algebra over \( \mathbb{Q} \) on 4 generators. Denote by \( X_1, X_2, X_3, X_4 \) a set of generators for the Lie algebra \( g \). Consider the natural grading \( g_1 \oplus \ldots \oplus g_6 \) for \( g \), where \( g_1 \) is the vector space spanned by \( X_1, \ldots, X_4 \). We say that the vector \( a \) has degree \( i \) if \( a \in g_i \) and we denote this as \( \deg(a) = i \). We will use the shorthand notation \([a, b, c] \) with \( a, b, c \in g \) for the Lie bracket \([a, [b, c]] \) and similarly for longer brackets.

A natural basis for the Lie algebra \( g \) as vector space over \( \mathbb{Q} \) is the Hall basis. The elements of this basis are constructed inductively: given the basis for \( g_1 \), with \( 1 \leq i \leq k \), we build the basis for \( g_{k+1} \). We fix an order relation on the basis vectors of \( g_1, \ldots, g_4 \), assuming that \( a \prec b \) if \( \deg(a) < \deg(b) \). The basis vectors for \( g_{k+1} \) are then given by Lie brackets \([a, b] \in g_k \) with \( \deg(a) + \deg(b) = k + 1 \) and \( a \prec b \) with the extra condition that if \( b = [b_1, b_2] \) then \( a \geq b_1 \). For more details and a proof that these vectors form indeed a basis, we refer to [9]. In our case we will always assume that the order relation satisfies \( X_1 < X_2 < X_3 < X_4 \) on \( g_1 \), so \( X_1 < X_2 \) if and only if \( i_1 < i_2 \).

Let \( \beta \) be the Hall basis for \( g_4 \). Every vector of the form \([X_{i_1}, X_{i_2}], [X_{i_3}, X_{i_4}] \) with \( b \in \beta \), \( i_1 \in \{1, 2, 3, 4\} \) and \( i_1 < i_2, i_3 \geq i_4 \) is by definition an element of the Hall basis of \( g_6 \). Since \([X_{i_1}, X_{i_2}], [X_{i_3}, X_{i_4}] = [X_{i_1}, X_{i_2}] - [X_{i_3}, X_{i_1}] \) we can replace the vector \([X_{i_1}, X_{i_2}, [X_{i_3}, X_{i_4}] \) of the Hall basis by \([X_{i_1}, X_{i_2}, b] \) for \( i_1 < i_2 \). This shows that the vectors given by \([X_{i_1}, X_{i_2}, b] \) with \( b \in \beta \) and \( i_1, i_2 \in \{1, 2, 3, 4\} \) are linearly independent in \( g_6 \). This also implies that if \( \beta \) is another basis for \( g_4 \), then the vectors \([X_{i_1}, X_{i_2}, b] \) with \( b \in \beta \) and \( i_1, i_2 \in \{1, 2, 3, 4\} \) are linearly independent.

Every permutation \( s \in S_4 \) determines an automorphism \( \varphi_s \) of \( g \) which is given by the relations \( \varphi_s(X_i) = X_{s(i)} \) on the generators. Consider the automorphism \( \varphi \in \text{Aut}(g) \) of order 4 which is induced by the permutation (1234) \( s_4 \). Let \( I \) be the smallest ideal of \( g \) such that

\[ [X_1, X_1, X_3, X_2, X_2, X_4] \] and \([X_1, X_1, X_2, X_3, X_2, X_4] \in I \]

for all \( i, j \in \{1, \ldots, 4\} \) with the indices \( i, j \) distinct. From the Jacobi identity it follows that also

\[ [X_{i_1}, X_{i_2}, X_{i_3}, X_{i_4}] = -[X_{i_1}, X_{i_2}, X_{i_1}, X_{i_1}, X_{i_1}, X_{i_2}] \] and \([X_{i_1}, X_{i_2}, X_{i_2}, X_{i_2}, X_{i_2}, X_{i_2}, X_{i_2}] \in I \]

for all distinct \( i, j \in \{1, \ldots, 4\} \).
Assume that \( \phi \) is a generator of \( I \) under \( \alpha \) is again a generator of \( I \), it follows that \( I \) is invariant under \( \alpha \), i.e. \( \alpha(I) = I \). Let \( \mathfrak{n} \) be the quotient Lie algebra \( \mathfrak{g}/I \) and denote by \( \tilde{\alpha} : \mathfrak{g} \to \mathfrak{n} \) the natural projection map. Since \( I \) is invariant under \( \alpha \), the automorphism \( \alpha \) induces an automorphism \( \alpha \) \( \mathfrak{n} \) and denote the induced map as \( \tilde{\alpha} : \mathfrak{n} \to \mathfrak{n} \).

Consider the vector \( v = [X_1, X_2, X_3, X_4, X_2, X_3] \in \mathfrak{g} \). We claim that the vector \( v \) satisfies \( \tilde{v}(v) \neq 0 \) or equivalently that \( v \notin \mathfrak{g}_0 \). By using the Hall basis of \( \mathfrak{g}_4 \), it’s an exercise to check that \( [X_4, X_3, X_3, X_3] \notin \mathfrak{g}_4 \). If \( \beta \) is a basis for the subspace \( \mathfrak{g}_4 \), this is equivalent to saying that the set \( \{[X_4, X_4, X_2, X_3] \cup \beta \} \) is linearly independent. The existence of a Hall basis then implies that also the set \( \{[X_1, X_2, X_3, X_4, X_2, X_3] \cup \{X_{i_1, i_2}, b \mid i_1, i_2 \in \{1, 2, 3, 4\}, b \in \beta \} \} \) is linearly independent. Since \( \mathfrak{g}_0 = [\mathfrak{g}_1, \mathfrak{g}_1, \mathfrak{g}_4] \), this implies that \( v = [X_1, X_2, X_4, X_2, X_3] \notin \mathfrak{g}_0 \). In a similar way one can check more generally that the vectors \( \tilde{v}(v), \tilde{v}(\alpha(v)), \tilde{v}(\alpha^2(v)) \) and \( \tilde{v}(\alpha^3(v)) \) are linearly independent in \( \mathfrak{n} \).

The vector space spanned by \( \tilde{v}(v) \) \( \in \mathfrak{n} \) is an ideal by the definition of the Lie algebra \( \mathfrak{n} \). Let \( J \) be the smallest ideal of \( \mathfrak{n} \) containing this vector and which is invariant under \( \tilde{\alpha} \). Consider the Lie algebra \( \mathfrak{n} = \mathfrak{g}/J \) with projection map \( \pi : \mathfrak{g} \to \mathfrak{n} \). It follows from the previous paragraph that \( \pi(v) \neq 0 \). The automorphism \( \tilde{\alpha} \) induces an automorphism \( \tilde{\alpha} \in \text{Aut}(\mathfrak{n}) \) of order 4.

**Automorphisms on \( \mathfrak{n} \):** By the explicit construction of the Lie algebra \( \mathfrak{n} \) as a quotient of the free Lie algebra \( \mathfrak{g} \) we can give a general way of constructing automorphisms on \( \mathfrak{n} \) for any field extension \( E \supseteq \mathbb{Q} \). Consider the linear subspace \( \mathfrak{g}_4 \) of \( \mathfrak{g}_4 \) spanned by \( X_1, \ldots, X_4 \). Take \( \lambda_1, \ldots, \lambda_4 \in E \) such that \( \lambda_1 \lambda_2 \lambda_3 \lambda_4 = 1 \) and consider the linear map \( \mathfrak{g}_4 \to \mathfrak{g}_4 \) given by

\[
X_i \mapsto \lambda_i X_i.
\]

This map uniquely extends to an automorphism \( \varphi : \mathfrak{g}_4 \to \mathfrak{g}_4 \). Moreover, since \( \mathfrak{g} \) is generated by eigenvectors of \( \varphi \), it also induces a linear automorphism \( \tilde{\varphi} : \mathfrak{n} \to \mathfrak{n} \). The vector \( \tilde{v}(v) = [X_2, X_4] \in \mathfrak{n} \) is also an eigenvector of the map \( \tilde{\varphi} \) since \( \lambda_1 \lambda_2 \lambda_3 \lambda_4 = 1 \) and thus \( \tilde{\varphi} \) induces a map \( \tilde{\varphi} \) on the Lie algebra \( \mathfrak{n} \).

Take the basis \( X_1, \ldots, X_4 \) for the vector space \( \mathfrak{n} = \mathfrak{g}/[\mathfrak{n}, \mathfrak{n}] \). Under the natural projection map \( \pi : \text{Aut}(\mathfrak{n}) \to \text{Aut}(\mathfrak{n}) \cong \text{GL}(4, E) \), the automorphism \( \tilde{\varphi} \) is mapped to the diagonal matrix with eigenvalues \( \lambda_1, \lambda_2, \lambda_3 \) and \( \lambda_4 \). This will be an important construction of automorphisms on the Lie algebra \( \mathfrak{n} \).

As a consequence of this construction for automorphisms, we have the following proposition:

**Proposition 5.1.** The Lie algebra \( \mathfrak{n} \) has no partially expanding automorphisms.

**Proof.** Let \( H \) be the subgroup of \( \text{GL}(4, \mathbb{Q}) \) generated by the matrices

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}.
\]

This subgroup is isomorphic to \( \mathbb{Z}_2 \otimes \mathbb{Z}_2 \) and the centralizer of \( H \) in \( \text{GL}(4, \mathbb{Q}) \) is given by the diagonal matrices \( D(n, \mathbb{Q}) \). As described just above this theorem, each of the generators of \( H \) above induces an automorphism of the Lie algebra \( \mathfrak{n} \) and thus we get a faithful representation \( i : H \to \text{Aut}(\mathfrak{n}) \).

Assume that \( \mathfrak{n} \) does have a partially expanding automorphism \( \varphi \). By Theorem 5.2, we can assume that \( \varphi \) commutes with every element of the finite group \( i(H) \). Consider the vector space \( \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}] \) with basis \( X_1, \ldots, X_4 \) and the natural projection map \( \pi : \text{Aut}(\mathfrak{n}) \to \text{Aut}(\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]) \cong \text{GL}(4, E) \)....
GL(4, Q). Since $\pi(\varphi)$ lies in the centralizer of $H$, we know that

$$\pi(\varphi) = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}.$$  

The vector $p(v)$, with $v$ as in the definition of $n$ above, is then an eigenvector of $\varphi$ with eigenvalue $\lambda_1 \lambda_2^2 \lambda_3^2 \lambda_4^2$ and $p([X_2, X_4])$ is an eigenvector of eigenvalue $\lambda_2 \lambda_4$. Since $p(v) = p([X_2, X_4]) \neq 0$, it must hold that $\lambda_1 \lambda_2 \lambda_3 \lambda_4 = 1$. This is a contradiction since $\varphi$ is a partially expanding automorphism. 

**Anosov Lie algebra:** In [7], a general way of constructing Anosov Lie algebras is given. We recall the following result [7, Corollary 2.7] which allows us to show that a real Lie algebra has a rational form which is Anosov:

**Theorem 5.2.** Let $n$ be a rational Lie algebra and assume there exists a decomposition of $n$ into subspaces

$$n = \bigoplus_{\lambda \in E} V_{\lambda}$$

such that $[V_{\lambda_1}, V_{\lambda_2}] \subseteq V_{\lambda_1 \lambda_2}$. Let $\rho : Gal(E, \mathbb{Q}) \to \text{Aut}(n)$ be a representation such that $\rho(\sigma)(V_{\lambda}) = V_{\sigma(\lambda)}$ for all $\sigma \in Gal(E, \mathbb{Q})$. Then the linear map $f : n^E \to n^E$ given by $f(X) = \lambda X$ for all $X \in V_{\lambda}$ induces an automorphism on some rational form of $n^E$. If every $\lambda$ is an algebraic unit of absolute value different of 1, then this rational form is Anosov.

Let $E$ be a real Galois extension of $\mathbb{Q}$ of degree $n$, then we call an algebraic integer a Pisot number if $\mu > 1$ and for all $1 \neq \sigma \in Gal(E, \mathbb{Q})$, it holds that $|\sigma(\mu)| < 1$. A Pisot number which is also an algebraic unit of $E$ is called a unit Pisot number. An algebraic unit $\mu$ with Galois conjugates $\mu_1 = \mu, \ldots, \mu_n$ satisfies the full rank condition if for all $d_1, \ldots, d_n$ integers with

$$\prod_{j=1}^{n} \mu_j^{d_j} = \pm 1,$$

it holds that $d_1 = d_2 = \ldots = d_n$. Every unit Pisot number satisfies the full rank condition, see [12] Proposition 3.6. From the appendix of [7] it follows that every real Galois extension $E$ has unit Pisot numbers.

By combining the existence of unit Pisot numbers with Theorem 5.2, we have the following consequence:

**Proposition 5.3.** The Lie algebra $n^R$ has a rational form which is Anosov.

**Proof.** Take $Q \subseteq E \subseteq R$ a field extension with Galois group $Gal(E, Q) \cong \mathbb{Z}_4$ and denote by $\sigma$ a generator of $Gal(E, Q)$. Let $\mu$ be a unit Pisot number in $E$ and write $\mu_i = \sigma_i^{-1}(\mu)$. By squaring $\mu$ if necessary, we can assume that $\mu_1 \mu_2 \mu_3 \mu_4 = 1$. Take $\varphi : n \to n$ the automorphism induced by the linear map that maps

$$X_i \to \mu_i X_i,$$

as explained above Proposition 5.1.

All eigenvalues of the automorphism $\varphi$ are products of the algebraic units $\mu_i$ of length at most 6. The only possibility to get an eigenvalue of absolute value 1 is $\mu_1 \mu_2 \mu_3 \mu_4$, since $\mu$ satisfies the full rank condition, see above. By construction of the Lie algebra $n$, all the eigenvectors with eigenvalue $\mu_1 \mu_2 \mu_3 \mu_4$ lie in $I$, so this eigenvalues does not occur. Hence, $\varphi$ has no eigenvalues of absolute value 1.

Consider the representation $\rho : Gal(E, Q) \to \text{Aut}(n)$ given by $\rho(\sigma) = \sigma$. The maps $\rho$ and $\varphi$ satisfy the conditions of Theorem 5.2, where the subspaces $V_{\lambda}$ are given the eigenspaces of the map $\rho$ and the map $f$ of the theorem is equal to $\varphi$. This implies that $n^E$ (and therefore also $n^R$) has a rational form which is Anosov. \qed
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