Multiple Equilibria for an SIRS Epidemiological System

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Abstract

An SIRS type model of disease transmission in an open environment is discussed. We use the Poincaré index together with a perturbation method to show that the endemic proportions need not be unique.

Keywords: Epidemiological model, disease transmission, endemic proportions, perturbation, Poincaré index, structural stability.

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1 Introduction

The social mixing structure of a population or a group of interacting populations play a crucial role in the dynamics of a disease transmission [8]. The most well-known examples of epidemics which are spread by means of the interaction between different populations are those related to venereal diseases. These diseases are transmitted by sexual contacts between two or more different populations. An epidemiological model which considers these interactions is called multigroup model. A large number of multigroup models have been described in [7]. These models usually lead to some high-dimensional systems of differential equations with a probably complicated dynamics [14]. In order to avoid these complications, we consider a single group and we assume that there is a disease transmission from the outside. This assumption is natural, since we are living in an open environment with a limited control on the outside. The effect of this assumption has been studied by using numerical and statistical methods. For example in [2], it has been claimed that most of Swedish HIV-patients with heterosexual transmission had been in contact with a partner of foreign extraction. The effect of the immigrant subpopulation has been considered by many others too [1, 4, 11]. A statistical approach to this concept in [11] showed
that the subpopulation with a high prevalence of HIV-infected individuals is the immigrant one.

In this paper, we consider a simple SIRS type model of disease transmission in an open environment. In an SIRS model, we divide the population into three classes consisting of Susceptible, Infective and Removed individuals. By an open environment we mean that our population has contacts with other populations and there is a disease transmission from the outside [15]. We assume that a proportion of susceptible individuals are infected in this way. We also assume that there is a special control on the people who are frequently in contact with foreigners and a proportion of those who are infected in this way are known and removed. In our model, the incidence function is of proportionate mixing type introduced by Nold [18]. The demographic assumptions are also very simple. Natural births and deaths are assumed to be proportional to the class numbers with all newborns susceptibles and the excess deaths due to the disease among infectives and removeds are proportional too. We could have considered more complicated demographic or vertical transmission assumptions [3, 4], but these parameters have no mathematical significance and one can easily conclude that our main results would still be valid. We want to avoid these complications for two reasons. The first one is that even in this simple case, multiple endemic equilibria may occur. Indeed we shall show that for some suitable values of parameters, the proportions system admits two sinks and one saddle point in the feasibility region. The existence of multiple equilibria or limit cycles has been shown for more complicated systems such as multigroup models [14] or a single group with nonlinear incidence functions [12, 16, 17]. The second reason is related to the technique used here to determine the number of endemic equilibria. We state a simple model in order that we can exhibit our technique more clearly. The technique used here is based on a careful choice of Jordan curves and counting the number of rest points inside them. This technique has no hard analysis and can be easily applied to other similar systems [21, 22]. The reader can verify that our results hold for similar SIR and SIRI systems as well.

We first in the next section, state the model and some results concerning the non-existence of certain types of solutions. In Section 3, we use Poincaré index to obtain some partial results in a special case. Then in Section 4, we use these results together with a perturbation method to examine the general case. Our analysis relies on an index lemma concerning the Poincaré index of a class of Jordan curves. We provide the proof of this result at the end of this paper.
2 The Model

We consider a model of disease transmission in a nonconstant population of size \( N \) divided into three classes: susceptibles, infectives and removeds, the number of each class is given by \( S, I, R \), respectively. We set \( N = S + I + R \) and use the following parameters which are assumed to be positive unless otherwise specified:

- \( b = \) per capita birth rate,
- \( d = \) per capita disease free death rate,
- \( \varepsilon_1 = \) excess per capita death rate of infectives,
- \( \varepsilon_2 = \) excess per capita death rate of removeds,
- \( \lambda = \) effective per capita contact rate of infectives,
- \( \alpha = \) per capita removal rate of infectives,
- \( \gamma = \) per capita recovery rate of removeds.

Here as mentioned before, we assume that there is a disease transmission from the outside to our population. Let \( \beta > 0 \) be the per capita transfer rate of the disease into this population. Furthermore the susceptible individuals which are infected in this way, enter to the classes \( I \) or \( R \) of proportions \( \beta_1 \) and \( \beta_2 \) where \( \beta_1 + \beta_2 = \beta \). The above hypotheses lead to the following system of differential equations in \( \mathbb{R}^3_+ \), where “′” denotes the derivative with respect to \( t \), the time.

\[
\begin{align*}
S' &= bN - (d + \beta)S + \gamma R - \frac{\lambda IS}{N}, \\
I' &= \beta_1 S - (d + \varepsilon_1 + \alpha)I + \frac{\lambda IS}{N}, \\
R' &= \beta_2 S - (d + \varepsilon_2 + \gamma)R + \alpha I,
\end{align*}
\]

where \( \frac{\lambda I}{N} \) is of the proportionate mixing type introduced by Nold [18]. The total population equation is obtained by adding the above three equations:

\[ N' = (b - d)N - \varepsilon_1 I - \varepsilon_2 R. \] (2 - 4)

Now if we set \( s = \frac{S}{N}, i = \frac{I}{N} \) and \( r = \frac{R}{N} \), the equations (2-1)-(2-3) yield

\[
\begin{align*}
s' &= b - (b + \beta)s + \gamma r + (\varepsilon_1 - \lambda)is + \varepsilon_2 rs, \\
i' &= \beta_1 s - (b + \varepsilon_1 + \alpha)i + \lambda is + \varepsilon_1 i^2 + \varepsilon_2 ir, \\
r' &= \beta_2 s - (b + \varepsilon_2 + \gamma)r + \alpha i + \varepsilon_1 ir + \varepsilon_2 r^2.
\end{align*}
\] (2 - 1)'
(2 - 2)'
(2 - 3)'}
We will determine the asymptotic behaviour of the solutions of this system. Prior to this we have the following concepts of ODE’s related to our system.

Given an autonomous system of ordinary differential equations in $\mathbb{R}^n$

$$\frac{dx}{dt} = f(x)$$

We will denote by $x.t$ the value of the solution of this system at time $t$ that is $x$ initially. For $V \subset \mathbb{R}^n, J \subset \mathbb{R}$, we let $V.J = \{x.t : x \in V, t \in J\}$. The set $V$ is called invariant if $V.\mathbb{R} = V$ and it is called positively invariant if $V.\mathbb{R}_+ = V$. For $Y \subset \mathbb{R}^n$ the $\omega$-limit set of $Y$ is defined to be the maximal invariant set in the closure of $Y.\mathbb{R}_+$. A closed curve connecting several rest points whose segments between successive rest points are heteroclinic orbits is called phase polygon. By a sink we mean a rest point at which all the eigenvalues of the linearized system have negative real parts. A rest point is called a source if these eigenvalues have positive real parts and it is called a saddle point if some of these eigenvalues have positive real parts and the others have negative real parts. A rest point is called nondegenerate if all of these eigenvalues are nonzero and it is called hyperbolic if all of its eigenvalues have nonzero real parts.

Now we continue the analysis of the system $(2-1)' - (2-3)'$. If we set $\sum = s + i + r$ then $\sum' = (1 - \sum)(b - \varepsilon_1 i - \varepsilon_2 r)$. Therefore the plane $\sum = 1$ is invariant. We consider the feasibility region

$$D = \{(s, i, r)|s + i + r = 1, s \geq 0, i \geq 0, r \geq 0\},$$

which is a triangle and on its sides we have

$$s = 0 \Rightarrow s' = b + \gamma r \quad , \quad i = 0 \Rightarrow i' = \beta_1 s \quad , \quad r = 0 \Rightarrow r' = \beta_2 s + \gamma i.$$  

Since all parameters are positive, $D$ is positively invariant and any solutions of the system $(2-1)' - (2-3)'$ with initial point in $\partial D$ immediately enters $\overset{\circ}{D}$, where $\partial D$ and $D$ are the boundary and the interior of $D$, respectively. From now on, we examine the dynamics of this system in the feasibility region $D$.

Using the relation $s + i + r = 1$, we see that the system is essentially two dimensional. Thus we can eliminate one of the variables to arrive at the following quadratic planner system:

$$\begin{cases}
  s' = b + \gamma + (\varepsilon_2 - b - \beta - \gamma)s - \gamma i - \varepsilon_2 s^2 + (\varepsilon_1 - \varepsilon_2 - \lambda)i, \\
  i' = \beta_1 s + (\varepsilon_2 - \varepsilon_1 - \alpha - b)i + (\lambda - \varepsilon_2)i, \\
  r' = \beta_2 s + \gamma i.
\end{cases}$$

(2-6)

(2-7)
Notice that this planner system has at most four rest points. Moreover the dynamics of the system \((2 - 1)' - (2 - 3)'\) on \(D\) is equivalent to the dynamics of this system in the positively invariant region \(D_1 = \{(s, i) | s \geq 0, i \geq 0, s + i \leq 1\}\).

The following theorem is a special case of the results of [6] concerning the non-existence of certain types of solutions.

**Theorem 2.1.** Let in (2-5), \(f\) be a smooth vector field in \(\mathbb{R}^3\). Let \(\Gamma(t)\) be a closed piece-wise smooth curve, which is the boundary of an orientable smooth surface \(S \subset \mathbb{R}^3\). Suppose \(g: U \to \mathbb{R}^3\) is defined and smooth in a neighborhood \(U\) of \(S\) with \(g(\Gamma(t)).f(\Gamma(t)) \geq 0\) and \(\text{curl}(g).n < 0\), where \(n\) is the unit normal to \(S\). Then \(\Gamma\) is not a finite union of the orbits of the system (2.5).

**Proposition 2.2.** The system \((2 - 1)' - (2 - 3)'\) has no periodic orbits, homoclinic orbits, or phase polygons in \(\partial D\).

**Proof.** In order to apply the above theorem, we define \(g = g_1 + g_2 + g_3\) where

\[
\begin{align*}
g_1(i, r) &= \left[0, -\frac{f_3(i, r)}{ir}, \frac{f_2(i, r)}{ir}\right], \\
g_2(s, r) &= \left[\frac{f_3(s, r)}{sr}, 0, -\frac{f_1(s, r)}{sr}\right], \\
g_3(s, i) &= \left[-\frac{f_2(s, i)}{si}, \frac{f_1(s, i)}{si}, 0\right],
\end{align*}
\]

and \(f_1\), \(f_2\) and \(f_3\) are the right hand side of \((2 - 1)'\), \((2 - 2)'\) and \((2 - 3)'\) reduced to functions of two variables by using \(\sum = 1\) respectively. Now after some computations [12], we get

\[
\text{curl}g(s, i, r).(1, 1, 1) = -\left(\frac{b + \gamma}{is^2} + \frac{b}{rs^2} + \frac{\beta_1}{r^2} + \frac{\beta_2}{ir^2} + \frac{\alpha}{sr^2}\right) < 0.
\]

Since here \(g.f = 0\), the proof is complete by Theorem 2.1. \(\Box\)

**Corollary 2.3.** The \(\omega\)-limit set of any orbit of the system \((2 - 1)' - (2 - 3)'\) with initial point in \(D\) is a rest point.

**Proof.** Since the vector field related to the system \((2 - 1)' - (2 - 3)'\) is inward on \(\partial D\) and \(D\) is compact, the \(\omega\)-limit set of each orbit with initial point in \(D\), is a nonempty subset of \(\bar{D}\). By generalized Poincaré-Bendixon theorem [13], [20] and Proposition 2.2., this set must be a rest point. \(\Box\)
Corollary 2.4. The system $(2-1)' - (2-3)'$ has no source in $\tilde{D}$.

Proof. Suppose there is a source in $\tilde{D}$ for this system. Since $D$ is positively invariant and there are finitely many rest points in $D$ (at most four, since our system is quadratic), there must be infinitely many heteroclinic orbit running from this source to another rest point. So there is a 2-gons in $\tilde{D}$ which is impossible by Proposition 2.2. □

Corollary 2.5. Every nondegenerate rest point of the system (2-6), (2-7) in $\tilde{D}_1$ is hyperbolic.

Proof. Let $L$ be the linearization of this system at a rest point $(s^*, i^*)$ in $\tilde{D}_1$. We have to show that if $\det L > 0$, then $\text{trace } L \neq 0$. We compute $\text{trace } L$ at a rest point.

\[
\begin{cases}
\frac{\partial s'}{\partial s} = (\varepsilon_2 - b - \beta - \gamma) - 2\varepsilon_2 s + (\varepsilon_1 - \varepsilon_2 - \lambda)i, \\
\frac{\partial i'}{\partial i} = (\varepsilon_2 + \varepsilon_1 - \alpha - b) + (\lambda - \varepsilon_2)s + 2(\varepsilon_1 - \varepsilon_2)i.
\end{cases}
\]

From $i' = 0$ and $s' = 0$ at $(s^*, i^*)$, we have

\[
\text{trace } L = -\frac{b + \gamma(1 - i^*)}{s^*} - \varepsilon_2 s^* - \frac{\beta_1 s^*}{i^*} + (\varepsilon_1 - \varepsilon_2)i^*.
\]

If $\text{trace } L = 0$, then we can slightly increase $\varepsilon_1$ to get $\text{trace } L > 0$ and determine $\lambda$ and $\alpha$ so that $(s^*, i^*)$ remains a rest point for the new values of parameters. We may also assume that $\det L > 0$ at this rest point. Thus we obtain a source in $\tilde{D}_1$ which contradicts Corollary 2.4. □

Remark 2.6. A nondegenerate rest point of the system $(2-6), (2-7)$ is obtained by a transversal intersection of two conic sections $s' = 0$ and $i' = 0$. In Section 4, we shall prove that this intersection is almost always transversal.

3 A Special Case

In this section we consider the planner system (2-6), (2-7) in the case $b = \beta_1 = \gamma = 0$. These assumptions yields the following system of equations

\[
\begin{cases}
s' = s(\varepsilon_2 - \beta - \varepsilon_2 s + (\varepsilon_1 - \varepsilon_2 - \lambda)i), \\
i' = i(\varepsilon_2 - \varepsilon_1 - \alpha + (\lambda - \varepsilon_2)s + (\varepsilon_1 - \varepsilon_2)i).
\end{cases}
\]

(3-1)  \hspace{1cm} (3-2)
First of all notice that there are two invariant lines $s = 0$ and $i = 0$ with three rest points $(0, 0)$, $(0, 1 - \frac{\alpha}{\varepsilon_2 - \varepsilon_1})$ and $(1 - \frac{\beta}{\varepsilon_2}, 0)$. The matrix of the linearized system at $(0, 0)$ is

$$
\begin{bmatrix}
\varepsilon_2 - \beta & 0 \\
0 & \varepsilon_2 - \varepsilon_1 - \alpha
\end{bmatrix}
$$

with the eigenvalues $T_0 := \varepsilon_2 - \beta$ and $T_1 := \varepsilon_2 - \varepsilon_1 - \alpha$.

The matrix of the linearized system at $(0, 1 - \frac{\alpha}{\varepsilon_2 - \varepsilon_1})$ is

$$
\begin{bmatrix}
\varepsilon_2 - \beta + (\varepsilon_1 - \varepsilon_2 - \lambda)(1 - \frac{\alpha}{\varepsilon_2 - \varepsilon_1}) & 0 \\
(\lambda - b - \varepsilon_2)(1 - \frac{\alpha}{\varepsilon_2 - \varepsilon_1}) & \varepsilon_1 - \varepsilon_2 + \alpha
\end{bmatrix}
$$

with the eigenvalues $-T_1$ and

$$
T_2 := \varepsilon_2 - \beta + \frac{(\varepsilon_1 - \varepsilon_2 - \lambda)(\varepsilon_2 - \varepsilon_1 - \alpha)}{\varepsilon_2 - \varepsilon_1}.
$$

Notice that the rest point $(0, 1 - \frac{\alpha}{\varepsilon_2 - \varepsilon_1})$ belongs to $D_1$ if and only if $T_1 \geq 0$ and coincides with $(0, 0)$ in the case of equality. It is easy to see that when $T_1 \leq 0$, the origin attracts the segment $D_1 \cap \{s = 0\}$ and if $T_1 > 0$, then $(0, 1 - \frac{\alpha}{\varepsilon_2 - \varepsilon_1})$ attracts $(D_1 \cap \{s = 0\}) - \{(0, 0)\}$.

The matrix of linearization of $(3 - 1), (3 - 2)$ at $(1 - \frac{\beta}{\varepsilon_2}, 0)$ is

$$
\begin{bmatrix}
\beta - \varepsilon_2 & (\varepsilon_1 - \varepsilon_2 - \lambda)(1 - \frac{\beta}{\varepsilon_2}) \\
0 & (\varepsilon_2 - \varepsilon_1 - \alpha) - (\varepsilon_2 - \lambda)(1 - \frac{\beta}{\varepsilon_2})
\end{bmatrix},
$$

with the eigenvalues $-T_0$ and

$$
T_3 := \varepsilon_2 - \varepsilon_1 - \alpha - \frac{(\varepsilon_2 - \lambda)(\varepsilon_2 - \beta)}{\varepsilon_2}.
$$

Notice that the rest point $(1 - \frac{\beta}{\varepsilon_2}, 0)$ belongs to $D_1$ if and only if $T_0 \geq 0$ and coincides with $(0, 0)$ in the case of equality. It is easy to see that when $T_0 \leq 0$, the origin attracts the segment $D_1 \cap \{i = 0\}$ and if $T_1 > 0$, then $(1 - \frac{\beta}{\varepsilon_2}, 0)$ attracts $(D_1 \cap \{i = 0\}) - \{(0, 0)\}$.
Proposition 3.1. If $T_0 < 0$ and $T_1 < 0$, then the origin is the only rest point of the system (3-1), (3-2).

Proof. In this case $(0, 0)$ is a sink and the rest points $(0, 1 - \frac{\alpha}{\varepsilon_2 - \varepsilon_1})$ and $(1 - \frac{\beta}{\varepsilon_2}, 0)$ are outside of $D_1$. Since our planar system is quadratic, there are at most one rest point in $\bar{D}_1$. Moreover if such a rest point exists, it must be nondegenerate, hence its Poincaré index is $\pm 1$. For a Jordan curve $C$, let $\mu_+(C)$ and $\mu_-(C)$ denote the number of rest points inside of $C$ with the Poincaré index +1 and -1 respectively. Here let $C$ be the curve as shown in Figure (3-1) which is the boundary of the union of $D_1$ and a small disk centered at the origin. Then on this Jordan curve, our vector field is always tangent or inward. It follows from Lemma 5.1. (cf. Section 5) that $I_C(X) = 1$ where $X$ is the vector field related to the system (3-1), (3-2) and $I_C(X)$ is the Poincaré index of $X$ with respect to $C$. Now we can use the Poincaré theorem to obtain $\mu_+(C) - \mu_-(C) = 1.$ Moreover we have shown that $\mu_+(C) + \mu_-(C) \leq 2.$ Therefore $\mu_+(C) = 1$ and $\mu_-(C) = 0$ which finishes the proof. □

![Figure (3-1)](image)

The Jordan curve related to Proposition 3.1.

Proposition 3.2. If $T_0 > 0$, $T_1 > 0$, $T_2 < 0$ and $T_3 < 0$, then the system (3-1), (3-2) has a saddle point in $\bar{D}_1$.

Proof. These assumptions mean that the origin is a source and $(0, 1 - \frac{\alpha}{\varepsilon_2 - \varepsilon_1})$ and $(1 - \frac{\beta}{\varepsilon_2}, 0)$ are sinks. It is also concluded that the system (3-1),(3-2) has only nondegenerate rest points. Let $B_1, B_2$ and $B_3$ be small disks centered at the above three rest points respectively. Let $C = \partial \Delta$ and $\Delta = (D_1 \cup B_2 \cup B_3) - B_1$. (See Figure (3-2).) The Poincaré index of the Jordan curve $C$ is 1 by Lemma 5.1. and we can use Poincaré theorem to obtain $\mu_+(C) - \mu_-(C) = 1.$ Moreover we have
\[ \mu_+(C) + \mu_-(C) \leq 3 \text{ and } \mu_+(C) \geq 2. \] Therefore \( \mu_+(C) = 2 \) and \( \mu_-(C) = 1 \) which means that there is a saddle point in \( \bar{D}_1 \). \( \square \)

**Remark 3.3.** The assumptions of the above proposition do not contradict. To see this suppose that \( T_0 > 0 \) and \( T_1 > 0 \). Then it is easy to check that

\[
T_2 < 0 \Leftrightarrow \frac{T_0}{T_1} < 1 + \frac{\lambda}{\varepsilon_2 - \varepsilon_1}, \quad T_3 < 0 \Leftrightarrow \frac{T_0}{T_1} > (1 - \frac{\lambda}{\varepsilon_2})^{-1}
\]

\[
(1 - \frac{\lambda}{\varepsilon_2})^{-1} < (1 + \frac{\lambda}{\varepsilon_2 - \varepsilon_1}) \Leftrightarrow \varepsilon_1 > \lambda.
\]

Now if \( \varepsilon_2 > \varepsilon_1 > \lambda \), one can choose \( \alpha > 0 \) and \( \beta > 0 \) so that \( T_0 > 0 \), \( T_1 > 0 \) and \( \frac{T_0}{T_1} \in ((1 - \frac{\lambda}{\varepsilon_2})^{-1}, 1 + \frac{\lambda}{\varepsilon_2 - \varepsilon_1}) \) to satisfy the assumptions of Proposition 3.2. This would be helpful for the reader who is more interested in numerical simulations.

### 4 The General Case

In this section we investigate the dynamics of the proportions system \( (2-1)'-(2-3)' \) in the general case. In order to do this, we discuss the existence and stability of the rest points of the planner system (2.6), (2.7). Recall that the feasibility region \( D_1 \) is positively invariant and this system has no rest point on \( \partial D_1 \). Indeed the vector field corresponding to the system (2.6), (2.7) is strictly inward on \( \partial D_1 \). Thus by Lemma 5.1., the Poincaré index of this vector field with respect to \( \partial D_1 \) equals 1. This is the first step to prove the main result of this paper.
**Theorem 4.1.** If the system (2-6), (2-7) has only nondegenerate rest points in $\tilde{D}_1$, then one of the following statements holds:

(A) There exists a unique rest point $\tilde{D}_1$ which is a sink and attracts $D_1 (GAS)$.

(B) There are two sinks and a saddle point in $\tilde{D}_1$.

Moreover both of them occur for suitable values of the involved parameters.

**Proof.** All rest points in $\tilde{D}_1$ are hyperbolic by Corollary 2.5. Let $\mu_0$, $\mu_1$ and $\mu_2$ be the number of sinks, saddles and sources in $\tilde{D}_1$. Since the index of $\partial D_1$ is 1, we have $\mu_0 - \mu_1 + \mu_2 = 1$ in $D_1$. Furthermore $\mu_0 + \mu_1 + \mu_2 \leq 4$ and $\mu_2 = 0$ by Corollary 2.4. Thus we have either $\mu_0 = 1, \mu_1 = \mu_2 = 0$ or $\mu_0 = 2, \mu_1 = 1, \mu_2 = 0$. Now the first conclusion gives (A) and the second one gives (B). In order to see that each of the two above statements occurs, recall that in the case of Theorem 3.1., there is neither a saddle nor a nonhyperbolic rest point in $D_1$. Thus for small values of $b_1$ and $\gamma$, there cannot be any saddle point or nonhyperbolic rest point in $\tilde{D}_1$. This means that (A) occurs. Similarly, under small perturbation, the saddle point obtained in Theorem 3.2 remains in $\tilde{D}_1$ and yields the case (B) of this theorem. □

**Remark 4.2.** It is well-known that the basin of attraction of a sink is a connected open set. In the case (B) of the above result, $D_1$ is the distinct union of the basins of attraction of these two sinks and the stable manifold of the saddle point in $D_1$. The basins of attraction are open and $D_1$ is connected. Thus the stable manifold of the saddle point separates them. In order to specify these sets, we can numerically find those two points at which $\omega$-limit set changes from one sink to another when someone moves on $\partial D_1$. (See Figure 4.1.)

![Figure (4-1)](image_url)

The phase portrait of the case (B) of Theorem 4.1.
Remark 4.3. The non-uniqueness of endemic equilibrium proportions yields some interesting conclusions. The most significant one is that the initial condition of the population may also be important besides the involved parameters. The effect of the initial condition is more crucial when the population equations (2-1)-(2-4) is considered. From (2-4) we get

$$\frac{N'}{N} = b - d - \varepsilon_1 i - \varepsilon_2 r.$$ 

Now suppose that a solution \((s(t), i(t), r(t))\) of the system \((2-1)' - (2-3)'\) tends to an equilibrium \((s^*, i^*, r^*)\). If we set \(T = \frac{b}{d + \varepsilon_1 i^* + \varepsilon_2 r^*}\), then \(N(t) \to \infty\) if \(T > 1\) and \(N(t) \to 0\) if \(T < 1\). (See [5] for more details.) Now each endemic equilibrium gives a \(T\) and when there are two endemic equilibria, we may get different values for \(T\) at these two points.

It remains to consider the case in which there is a degenerate rest point in \(D_1\) for our planar system. Let \(\Omega\) be the parameters space of the system (2-6), (2-7) as an open subset of \(\mathbb{R}^8_+\) and \(\Omega_1\) be the set of all possible values of parameters for which the system (2-6), (2-7) has a nonhyperbolic (or equivalently degenerate by Corollary 2.5.) rest point in \(D_1\). The following fact about \(\Omega_1\) shows that our problem has fairly been solved.

Proposition 4.4. With the above notations, \(\Omega_1\) is a closed nonempty subset of \(\Omega\) with zero measure.

Proof. We first show that \(\Omega_1\) is closed and nonempty. Since \(D_1\) is compact and all rest points in the statements (A) and (B) of Theorem 4.1 are hyperbolic, both (A) and (B) occur in open subsets of \(\Omega\). (In other words, our system is structurally stable in the nondegenerate case.) Since \(\Omega\) is connected, it cannot be the union of these two distinct open subsets. Thus \(\Omega_1\) is closed, nonempty and indeed large enough to separate two open subsets. We use Sard’s theorem [3] to show that \(\Omega_1\) has zero measure. Notice that from (2-6), we can write \(i\) in terms of \(s\) if \(\gamma + (\varepsilon_2 - \varepsilon_1 + \lambda)s \neq 0\) and from (2-7) we can write \(s\) in terms of \(i\) if \(\beta_1 + (\lambda - \varepsilon_2)i \neq 0\). If \(\beta_1 + (\lambda - \varepsilon_2)i = 0\) then from \(i' = 0\) in (2-7), we have \((\varepsilon_2 - \varepsilon_1 - b - \alpha) + (\varepsilon_1 - \varepsilon_2)i = 0\). Thus \(\varepsilon_2 - \varepsilon_1 - b - \alpha > 0\) and hence \(\gamma + (\varepsilon_2 - \varepsilon_1 + \lambda)s > 0\). It follows that either \(i\) can be written in terms of \(s\) from (2-6) or \(s\) in terms of \(i\) from (2-7). In the first case we have a root for the equation \(h_1(s) = \beta_1\) and in the latter case a root for \(h_2(i) = \gamma\).
where:

\[
\begin{align*}
  h_1(s) &= (b + \varepsilon_1 - \varepsilon_2 + \alpha)i + (\varepsilon_2 - \lambda)i + (\varepsilon_2 - \varepsilon_1)i^2, \\
  h_2(i) &= \frac{b}{r} + (b + \beta)s + (\lambda - \varepsilon_1)i - \varepsilon_2s,
\end{align*}
\]

and \( r = 1 - s - i \) as before. Notice that in the first equation \( i \) has been written in terms of \( s \) and in the second equation \( s \) has been written in terms of \( i \). Since at a degenerate rest point of the system (2-6), (2-7), the curves \( i' = 0 \) and \( s' = 0 \) are not transverse, it makes \( \beta_1 \) a critical value of \( h_1 \) or \( \gamma \) a critical value of \( h_2 \). In other words, whenever the system (2-6),(2-7) has a degenerate rest point in \( D_1 \), either \( \beta_1 \) or \( \lambda \) belongs to the set of critical values which has zero measure by Sard’s theorem.

Therefore \( \Omega_1 \) is contained in the union of two sets with zero measure. □

5 The Index Lemma

A basic fact which has been used during the proof of our results is that the Poincaré index of a piece-wise smooth Jordan curve on which the vector field is either tangent or inward is always 1. Here we provide the proof of this fact. The reader is referred to [19] for more details about the Poincaré index.

Let \( \gamma : [0, 1] \to \mathbb{R}^2 \) be a piecewise smooth Jordan curve i.e. there exists a sequence \( 0 = t_0 < t_1 < \ldots < t_n = 1 \) such that \( \gamma \) is smooth on \( (t_i, t_{i+1}) \) for \( 0 \leq i \leq n - 1 \). Suppose \( \gamma' \) is always nonzero on \( (t_i, t_{i+1}) \), moreover the left and right derivatives of \( \gamma \) at \( t_i \) exist and both are nonzero. With these assumptions we can define the external angle \( \theta_i \in (-\pi, \pi) \) at \( t_i \). Also the inward normal vector \( N(t) \) is defined for \( t \neq t_i \) and its right and left limit \( N(t_{i-}^\pm) \) exist at each \( t_i \). For such a curve \( \gamma \) we prove the following lemma.

**Lemma 5.1.** Let \( U \) be a neighborhood of the image of \( \gamma \) and \( X : U \to \mathbb{R}^2 \) be a smooth vector field which does not vanish on the image of \( \gamma \) and satisfies \( N(t) \cdot X(\gamma(t)) \geq 0 \) for \( t \neq t_i \). Then \( I_\gamma(X) = 1 \).

**Proof.** We define a new curve \( \gamma_1 : [0, n + 1] \to \mathbb{R}^2 \) by

\[
\gamma_1(t) = \begin{cases} 
  \gamma(t - i) & i + t_i \leq t \leq i + t_{i+1}, \\
  \gamma(t_{i+1}) & i + t_{i+1} < t < i + 1 + t_{i+1}.
\end{cases} \quad (0 \leq i \leq n - 1)
\]

Notice that \( \gamma_1 \) moves like \( \gamma \) but stops at \( \gamma(t_i) \) for a unit of time, hence \( I_\gamma(X) = I_{\gamma_1}(X) \). Now we use this unit of time to rotate \( N(t_{i-}^\pm) \) to arrive at \( N(t_{i+}^\pm) \). To do
this, we define a continuous function $N_1 : [0, n + 1] \to \mathbb{R}^2$ by

$$
N_1(t) = \begin{cases} 
N(t - i) & i + t < t < i + t_{i+1}, \\
R_{\theta_i}(N(t^-_i)) & i + t_{i+1} \leq t \leq i + 1 + t_{i+1}, 
\end{cases}
$$

where $R_\alpha$ is the rotation function with the angle $\alpha$. Now both $X(\gamma_1(t))$ and $N_1(t)$ are continuous and do not vanish on $[0, n + 1]$.

**Claim:** $N_1(t) \cdot X(\gamma_1(t)) \geq 0$.

Since $N(t) \cdot X(\gamma(t)) \geq 0$ for $t \neq t_i$, by continuity we have $N(t_i^+) \cdot X(\gamma(t_i)) \geq 0$. Moreover $\theta_i \in (-\pi, \pi)$, thus for any $t \in [0, 1]$, there exists $t' \in [0, 1]$ such that

$$
R_{t\theta_i}(N(t^-_i)) = \frac{t'N(t^-_i) + (1 - t')N(t^+_i)}{||t'N(t^-_i) + (1 - t')N(t^+_i)||}.
$$

So $R_{t\theta_i}(N(t^-_i)).X(\gamma(t_i)) \geq 0$. As a result of the above claim

$$
|\Delta \Theta(X(\gamma_1(t))) - \Delta \Theta N_1(t)| \leq \pi \tag{5 - 1}
$$

where $\Delta \Theta$ is the total change of the angle on $[0, n + 1]$. Then we can write

$$
\Delta \Theta(X(\gamma_1(t))) = \Delta \Theta(X(\gamma(t))) = 2\pi I_\gamma(X).
$$

$$
\Delta \Theta(N_1(t)) = \sum_{i=0}^{n-1} \Delta \Theta(N_1|_{(i+t_i,i+t_{i+1})}) + \sum_{i=0}^{n-1} \Delta \Theta(N_1|_{(i+t_{i+1},i+1+t_{i+1})})
$$

$$
= \sum_{i=0}^{n-1} \Delta \Theta(\gamma'|_{(i+t_i,i+t_{i+1})}) + \sum_{i=1}^{n} \theta_i
$$

$$
= 2\pi.
$$

by Gauss-Bonnet theorem. Now from (5-1) we get $|2\pi I_\gamma(X) - 2\pi| < \pi$, hence $|I_\gamma(X) - 1| < \frac{1}{2}$. Since $I_\gamma(X)$ is an integer, we get $I_\gamma(X) = 1$. □

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