D-branes in
Singular Calabi-Yau $n$-fold
and $\mathcal{N} = 2$ Liouville Theory

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Abstract

Making use of the $\mathcal{N} = 2$ Liouville theory and world-sheet techniques, we study the properties of D-branes wrapped around vanishing SUSY cycles of singular Calabi-Yau $n$-folds ($n = 2, 3, 4$). After constructing boundary states describing the wrapped branes, we evaluate the disc amplitudes corresponding to the periods of SUSY cycles. We use the old technique of KPZ scaling in Liouville theory and derive holomorphicity and scaling behavior of vanishing cycles which are in agreement with geometrical considerations.

We also discuss the open string Witten index using the $\mathcal{N} = 2$ Liouville theory and obtain the intersection numbers among SUSY cycles which also agree with geometrical expectation.
1 Introduction

Recently the study of string compactification on singular Calabi-Yau manifolds has been receiving a lot of attentions [1, 2, 3, 4, 5, 6]. When a CY manifold approaches a singular limit, some of its cycles become degenerate and various non-perturbative phenomena take place. In particular when CY manifold $X_n$ ($n$ denotes the complex dimension) has an isolated A-D-E type singularity, we expect to obtain a scale invariant theory in $d = 10 - 2n$ dimensions which is decoupled from gravity [4]. It has been proposed that such a space-time theory may be studied by looking at its dual string theory [7, 5, 8, 9]: this dual theory possesses a linear dilaton background and involves the $\mathcal{N} = 2$ Liouville theory [10] (or $SL(2, \mathbb{R})/U(1)$ theory) and the $\mathcal{N} = 2$ minimal model. Liouville field $\phi$ describes the radial direction of the throat of singular CY manifold and another scalar field $Y$ in $\mathcal{N} = 2$ Livouville sector parameterizes the circular direction of the throat.

Such a dual string theory is reminiscent of the Gepner model of exactly soluble string compactification [11]. In the present case the minimal model is coupled to Liouville sector which is non-compact and the non-compactness makes it non-trivial to see if such a string background is in fact consistent. Recently, modular invariance for these backgrounds has been studied and it has been shown that there exist a series of modular invariant partition functions of the A-D-E type which are in one to one correspondence with the CY manifolds with A-D-E singularities [12, 13, 14, 15].

It is quite interesting that the Liouville theory plays a novel role in the description of singular CY compactification. In this paper we will borrow the technique from the old Liouville theory [16, 17, 18] and apply it to the analysis of the scaling behavior of singular CY manifold. In particular we will study the scaling laws of the periods of vanishing cycles as the deformation parameter $\mu$ of the singularity is turned off.

In order to carry out such an analysis we use the method of boundary states which describes the SUSY cycles from the world-sheet point of view. Technologies of describing D-branes using world-sheet theory have recently been developed considerably [19, 20]. In particular the boundary state approach to D-branes in Gepner models has been discussed by various authors [21, 22, 23]. In this paper we will make a slight generalization of this construction to cope with the case of non-compact CFT consisting of Liouville field.

We will consider two-point functions of boundary states with chiral primary fields which
correspond to periods of vanishing cycles. These correlation functions vanish identically due to Liouville momentum conservation unless the system is perturbed by some Liouville potential terms. We shall show that when the theory is perturbed by the cosmological constant operator, correlation functions become non-zero and behave exactly as the periods of vanishing cycles. We shall show that we can in fact identify the deformation parameter $\mu$ as the coefficient of the cosmological constant term in Liouville theory. We also compute the intersection numbers among vanishing cycles using the Liouville theory. We obtain results which agree completely with geometrical considerations and also agree with the results of [24, 25] based on the $SL(2, \mathbb{R})/U(1)$ approach.

This paper is organized as follows: In section 2, starting from a brief review of the setup of world-sheet description of [3], we construct the supersymmetric boundary states. In section 3 we will evaluate the periods of the D-branes wrapped around the vanishing cycles by means of the world-sheet techniques. In section 4 we compute the intersection matrix (open string Witten index [26]) of vanishing cycles. We will also discuss the relation of Liouville theory to the $SL(2, \mathbb{R})/U(1)$ approach in section 5. In section 6 we summarize our results and present some discussions.

2 World-sheet Description of D-branes in Singular CY Compactification

2.1 World-sheet Approach to Singular CY Compactification

Throughout this paper we consider type II string theory on the background $\mathbb{R}^{d-1,1} \times X_n$, where $X_n$ is a CY $n$-fold $(2n + d = 10)$ with an isolated singularity of A-D-E type defined by a polynomial equation $F(z_1, z_2, \ldots, z_{n+1}) = 0$. (More precisely, we shall consider IIA string for the cases $d = 2, 4$, and IIB string for $d = 4$). For simplicity we focus on the $A_{N-1}$-type singularity in this paper;

\[
F(z_1, \ldots, z_n) = z_1^N + z_2^2 + \cdots + z_{n+1}^2.
\] (2.1)

According to [3] the decoupled limit of string theory on $\mathbb{R}^{d-1,1} \times X_n$ is described as a string theory on

\[
\mathbb{R}^{d-1,1} \times (\mathbb{R}_\phi \times S^1) \times LG(W = F) \cong \mathbb{R}^{d-1,1} \times (\mathbb{R}_\phi \times S^1) \times M_N
\]
where $LG(W = F)$ stands for the $\mathcal{N} = 2$ Landau-Ginzburg model with a superpotential $W = F$, and $M_N$ denotes the $\mathcal{N} = 2$ minimal model with central charge $c = 3(N - 2)/N$. The sector $R_\phi \times S^1$ is described by the $\mathcal{N} = 2$ Liouville theory \cite{10} whose field contents consist of bosonic variables $\phi$ (parameterizing $R_\phi$), $Y$ (parameterizing $S^1$) and their fermionic partners $\Psi^+, \Psi^-$. "$R_\phi$" indicates a linear dilaton background with the background charge $Q(>0)$. Adjustment of the central charge gives the following values of the background charge in dimensions $d = 6, 4, 2$

\[
d = 6; \quad Q = \sqrt{\frac{2}{N}},
\]

\[
d = 4; \quad Q = \sqrt{\frac{N + 2}{N}}, \quad (2.2)
\]

\[
d = 2; \quad Q = \sqrt{\frac{2(N + 1)}{N}}.
\]

The superconformal currents read as

\[
\begin{align*}
T &= -\frac{1}{2} (\partial Y)^2 - \frac{1}{2} (\partial \phi)^2 - \frac{Q}{2} \partial^2 \phi - \frac{1}{2} (\Psi^+ \partial \Psi^- - \partial \Psi^+ \Psi^-), \\
G^\pm &= -\frac{1}{\sqrt{2}} \Psi^\pm (i \partial Y \pm \partial \phi) \mp \frac{Q}{\sqrt{2}} \partial \Psi^\pm, \\
J &= \Psi^+ \Psi^- - Q i \partial Y,
\end{align*}
\]

which generate the $\mathcal{N} = 2$ superconformal algebra (SCA) with $\hat{c}(\equiv \frac{c}{3}) = 1 + Q^2$.

Since we have a linear dilaton background, $\Phi(\phi) = -\frac{Q}{2} \phi$, the theory is weakly coupled in the "near boundary region" $\phi \sim +\infty$. On the other hand, in the opposite end $\phi \sim -\infty$ (near the singularity) the string coupling blows up. In order to avoid the region $\phi \sim -\infty$ one should add a "Liouville potential" term $\sim e^{-\alpha \phi}$ ($\alpha > 0$) to the action. Compatibility with the $\mathcal{N} = 2$ superconformal symmetry leads us to the following three types of Liouville potentials, two of them are (anti-)chiral and one is non-chiral;

\[
S^\pm = \int d^2z \, \Psi^+ \tilde{\Psi}^+ e^{-\frac{Q}{2} (\phi \pm i Y)}, \quad (2.4)
\]

\[
S_{nc} = \int d^2z \, (\partial \phi - i \partial Y - Q \Psi^+ \Psi^-) (\bar{\partial} \phi + i \bar{\partial} Y + Q \bar{\Psi}^+ \Psi^-) e^{-Q \phi}. \quad (2.5)
\]

They commute with all the generators of $\mathcal{N} = 2$ superconformal algebra and thus are the screening charges of SCA. It is easy to recognize these as the super- and Kähler potential

\textsuperscript{1}The present normalization of the background charge is defined so that the Liouville central charge equals $c_\phi = 1 + 3Q^2$. 

3
terms in the $\mathcal{N} = 2$ theory

\begin{align*}
S^\pm &= \int d^2z d^2\theta^\pm e^{-\frac{1}{2}\Phi^\pm}, \\
S_{\text{nc}} &= \int d^2z d^2\theta^+ d^2\theta^- e^{-\frac{1}{2}(\Phi^+ + \Phi^-)},
\end{align*}

(2.6) (2.7)

where $\Phi^\pm$ denotes the (anti-)chiral superfield whose scalar component is $\phi \pm iY$. Thus the operators $S^\pm$ describe the deformation of the superpotential or the complex structure of the CY manifold $X_n$ while $S_{\text{nc}}$ provides the deformation of the Kähler structure of the theory. We can identify the operators $S^\pm$ as the analogues of the cosmological constant term in the bosonic Liouville theory. On the other hand, as we discuss later, the operator $S_{\text{nc}}$ may be identified as the screening operator of $SL(2, \mathbb{R})$ current algebra or the black hole mass operator in the $SL(2, \mathbb{R})/U(1)$ WZW model of 2D black hole [27].

In the following we shall concentrate on the perturbation by the cosmological constant operators $S^\pm$ ($S_L$ denotes the Liouville action)

\[ S_L \Rightarrow S_L + \mu S^+ + \bar{\mu} S^- . \]

(2.8)

Note that $\mu$ here is a complex parameter as opposed to the bosonic Liouville theory where $\mu$ is a real parameter.

It has been suggested by various authors [2, 5] that the perturbation (2.8) corresponds to the most relevant deformation of the singularity

\[ F(z_1, \ldots, z_{n+1}) + \mu = 0. \]

(2.9)

If we consider the case of $n = 3$ for instance, supersymmetric 3-cycles in CY threefold are degenerate in the singular limit $\mu = 0$. These cycles become inflated and acquire a finite size when the perturbation $\mu$ is turned on. In the following sections we compute correlation functions in Liouville theory which correspond to the sizes and intersection numbers of the vanishing cycles. These correlation functions in fact vanish by the Liouville momentum conservation at $\mu = 0$, however, they acquire finite values when the system is perturbed by the cosmological constant operator. We use the technique of the Liouville theory developed in 2D gravity theory [16, 18] and compute the correlation functions by inserting a suitable number of cosmological constant operators. It turns out that we obtain scaling relations for the vanishing cycles which are in complete agreement with the geometrical considerations.
2.2 Supersymmetric Boundary States in $N = 2$ Liouville Theory

It is known that in the $\mathcal{N} = 2$ theory there are two types of boundary states preserving the superconformal symmetry $[19]$. In the context of superstring theory they correspond to the D-branes wrapped around special Lagrangian submanifolds or holomorphic cycles in the CY manifolds. We describe them using the mode expansion in the closed string channel.

**A-type** (middle-dimensional cycles, special Lagrangian submanifolds)

\[
(J_n - \tilde{J}_{-n}) |B; \eta\rangle = 0 \quad (2.10)
\]

\[
(G^\pm_r - i\eta \tilde{G}_\mp^r) |B; \eta\rangle = 0 \quad (2.11)
\]

**B-type** (holomorphic even-dimensional cycles)

\[
(J_n + \tilde{J}_{-n}) |B; \eta\rangle = 0 \quad (2.12)
\]

\[
(G^\pm_r - i\eta \tilde{G}_\mp^r) |B; \eta\rangle = 0 \quad (2.13)
\]

where $\eta = +1$ or $-1$. Signature $\eta$ is related to the choice of NS or R boundary conditions in the open string channel.

In both cases we impose the superconformal invariance at the boundary;

\[
(L_n - \tilde{L}_{-n}) |B\rangle = 0 \quad (2.14)
\]

\[
(G_r - i\eta \tilde{G}_{-r}) |B\rangle = 0 \quad (2.15)
\]

where $G = G^+ + G^-$. 

Now let us consider the supersymmetric boundary states in the $\mathcal{N} = 2$ Liouville theory. By inspecting the mode expansion of operators of SCA $[23]$ it is easy to see that the A-type and B-type boundary conditions require

- **A-type**: $\phi$-direction must obey the Neumann boundary condition (b.c.), and $Y$-direction must be Dirichlet b.c.

- **B-type**: $\phi$-direction must obey the Neumann b.c., and $Y$-direction must be Neumann b.c.
Note that the Liouville direction $\phi$ must always obey the Neumann b.c., which means that

\[
(\alpha_n^\phi + \tilde{\alpha}_{-n}^\phi)|B\rangle = 0, \quad (n \neq 0) \quad (2.16)
\]

\[
\alpha_0^\phi|B\rangle = \tilde{\alpha}_0^\phi|B\rangle = \frac{iQ}{2}|B\rangle, \quad (2.17)
\]

\[
(\Psi_r^\phi + \imath \eta \tilde{\Psi}_{-r}^\phi)|B\rangle = 0, \quad \Psi^\phi = \frac{\imath}{\sqrt{2}}(\Psi^+ - \Psi^-) \quad (2.18)
\]

where $\imath \partial_\phi(z) = \sum_n \frac{\alpha_n^\phi}{z^{n+1}}$. As is discussed in [28], shift $iQ/2$ of Liouville momentum in (2.17) follows from the requirement of conformal invariance at the boundary; $(L_n - \bar{L}_{-n})|B\rangle = 0$.

In fact the commutation relation

\[
[L_m, \alpha_n^\phi] = -n\alpha_{m+n}^\phi - i\frac{Q}{2}m(m + 1)\delta_{m+n,0}, \quad (2.19)
\]

leads to

\[
[L_m - \bar{L}_{-m}, \alpha_n^\phi + \tilde{\alpha}_{-n}^\phi] = -n(\alpha_{m+n}^\phi + \tilde{\alpha}_{-(m+n)}^\phi) - iQ\delta_{m+n,0}. \quad (2.20)
\]

Thus $(\alpha_0^\phi + \tilde{\alpha}_0^\phi)|B\rangle = iQ|B\rangle$. Since $\phi$, being a non-compact boson, has no winding mode, we have $\alpha_0^\phi = \tilde{\alpha}_0^\phi$. The fact that the supersymmetric boundary states have the fixed Liouville momentum $p_\phi = \frac{iQ}{2}$ plays an important role in deriving the scaling behavior of supersymmetric cycles.

It is interesting to consider the Dirichlet condition for the Liouville direction and this case has been discussed by several authors [28, 29]. However, we only consider the Neumann condition in this paper.

### 2.3 Ishibashi States in Singular $CY_n$ Theory

Now we explicitly construct the supersymmetric boundary states in the singular $CY_n$ theory. In the following we take the light-cone point of view and consider only the transverse directions $\mathbb{R}^{d-2} \times (\mathbb{R}_\phi \times S^1_Y) \times M_N$ of the theory. We focus on the A-type boundary condition which corresponds to the $Dn$-brane wrapped around the middle-dimensional SUSY cycles in $CY_n$ (special Lagrangian submanifold);

- All the $\mathbb{R}^{d-2}$-directions $\longrightarrow$ Dirichlet b.c., characterized by the zero-mode momentum

\[
\alpha_0^\mu|B\rangle = \tilde{\alpha}_0^\mu|B\rangle = p_0^\mu|B\rangle \quad (2.21)
\]
• $R_\phi$-direction $\rightarrow$ Neumann b.c., and as we have already observed, with the zero-mode momentum fixed as
\[ \alpha_0^\phi |B\rangle = \tilde{\alpha}_0^\phi |B\rangle = \frac{iQ}{2} |B\rangle \] (2.22)

• $S^1_Y$-direction $\rightarrow$ Dirichlet b.c., characterized by the Kaluza-Klein momentum
\[ \alpha_0^Y |B\rangle = \tilde{\alpha}_0^Y |B\rangle = p |B\rangle, \quad p = \frac{n}{NQ} \quad (n \in \mathbb{Z}) \] (2.23)

(The radius of $S^1_Y$ is equal to $NQ$. After imposing GSO projection, effective radius becomes $Q/4$).

• $M_N$-sector $\rightarrow$ A-type boundary condition. We denote the Ishibashi state $|30\rangle$ in this sector as $|l,m,s\rangle_I$ ($l = 0, 1, \ldots, N - 2$, $m \in \mathbb{Z}_{2N}$, $s \in \mathbb{Z}_4$, $l + m + s \in 2\mathbb{Z}$) which is associated to the character $\chi_l^m_s(q)$ of the representation of the $\mathcal{N} = 2$ minimal model with central charge $3(N - 2)/N$. $s$ represents different spin structures of the $\mathcal{N} = 2$ theory.

Tensoring all the sectors we can obtain the Ishibashi state $|p_0, s_0; l, m, s; p\rangle_I$ which diagonalizes the closed string Hamiltonian $H^{(c)}$
\[ I\langle p', s_0'; l', m', s'; p'|\tilde{q}H^{(c)}|p_0, s_0; l, m, s; p\rangle_I = \frac{\tilde{q}e^{2\pi i\tau s_0}}{\eta(\tilde{q})^d} \chi_{S_0}^{SO(d)}(\tilde{q})^d \chi_l^m_s(q) \delta(p_0 - p') \delta_{s_0, s'_0} \delta_{l, l'} \cdots, \] (2.24)

where $\tilde{q} = e^{-2\pi i\tau}$ and $\chi_{S_0}^{SO(d)}(\tilde{q})$ denotes the character of $\tilde{S_O}(d)_1$ ($s_0 = 0, 1, 2, -1$ correspond respectively to the basic, spinor, vector, co-spinor representations). Note that $d - 2$ free fermions of the transverse directions $\mathbb{R}^{d-2}$ are combined with the two fermions coming from the $\mathcal{N} = 2$ Liouville sector and generate the $SO(d)$ current algebra at level 1.

We now impose the GSO conditions which enforces the integrality of the $U(1)_R$ charge and determine the spectrum of $Y$-momentum $p$ in the Ishibashi state $|p_0, s_0; l, m, s; p\rangle_I$.

• In all of the cases $d = 2, 4, 6$, the GSO conditions for the NS sectors is given by
\[ -\frac{s_0}{2} - \frac{s}{2} + \frac{m}{N} - Qp \in 2\mathbb{Z} + 1, \quad (s_0, s = 0, 2). \] (2.25)

• For the R sector, the conditions are
\[ d = 2 \quad -\frac{s_0}{2} - \frac{s}{2} + \frac{m}{N} - Qp \in 2\mathbb{Z} + 1 \] (2.26)
\[ d = 4 \quad - \frac{s_0}{2} - \frac{s}{2} + \frac{m}{N} - Q_p \in 2\mathbb{Z} + \frac{1}{2} \quad (2.27) \]
\[ d = 6 \quad - \frac{s_0}{2} - \frac{s}{2} + \frac{m}{N} - Q_p \in 2\mathbb{Z} \quad (2.28) \]

Shift in the right-hand-side of GSO conditions in the R sector is due to the presence of \( U(1)_R \) charge \((d - 2)/4\) in Ramond ground state of the \( \mathbb{R}^{d-2} \) sector.

### 2.4 Cardy States in Singular \( CY_n \) Theory

It is well-known that open string amplitudes factorize when Cardy states are used for the boundary states [31]. Following the standard prescription we construct the Cardy state as the "Fourier transform" of an Ishibashi state,

\[ |L, M, S; R, X_0, S_0\rangle_C = \int d^{d-2}p_0 e^{-ip_0 X_0} \sum_{l,m,s,s_0,p} \frac{S_{Lt}}{\sqrt{S_{0l}(2N)^{1/4}}} \frac{e^{-is_0 S_0/2}}{\sqrt{2}} \frac{e^{-is_0 S_0/2}}{\sqrt{2}} e^{-ip_R} |p_0, s_0; l, m, s; p\rangle_I, \]

where the sum over \( p \) runs over the values dictated by the GSO conditions. \( S_{Lt} \) denotes the matrix element of the \( S \)-transformation of \( SU(2) \) characters

\[ S_{Lt} = \sqrt{\frac{2}{N}} \sin \left( \frac{\pi (L + 1)(\ell + 1)}{N} \right). \quad (2.30) \]

\( R \) denotes the location around \( S^1 \) described by the field \( Y \). From now on we only treat the boundary states with the center of mass coordinate \( X_0 = 0 \) and suppress the \( \mathbb{R}^{d-2} \) sector. We then write the Cardy state as \( |L, M, S; R, S_0\rangle_C \).

By combining spin structures \( S, S_0 \) we construct the boundary states \( |L, M, R; \eta\rangle^{(NS)}_C \), \( |L, M, R; \eta\rangle^{(R)}_C \) as

\[ |L, M, R; +1\rangle^{(NS)}_C = \frac{1}{4} \sum_{\alpha,\beta=0,1} |L, M, 2\alpha; R, 2\beta\rangle_C, \]
\[ |L, M, R; -1\rangle^{(NS)}_C = \frac{1}{4} \sum_{\alpha,\beta=0,1} |L, M, 2\alpha + 1; R, 2\beta + 1\rangle_C, \quad (2.31) \]
\[ |L, M, R; +1\rangle^{(R)}_C = \frac{1}{4} \sum_{\alpha,\beta=0,1} (-1)^{\alpha+\beta} |L, M, 2\alpha; R, 2\beta\rangle_C, \]
\[ |L, M, R; -1\rangle^{(R)}_C = \frac{1}{4} \sum_{\alpha,\beta=0,1} (-1)^{\alpha+\beta} |L, M, 2\alpha + 1; R, 2\beta + 1\rangle_C. \quad (2.32) \]
One can check that \(|L, M, R; \eta\rangle^{(NS)}_C\) in fact contains only NSNS components of Ishibashi states \((s, s_0 = 0, 2)\) and similarly \(|L, M, R; \eta\rangle^{(R)}_C\) contains the RR components \((s, s_0 = \pm 1)\). We also notice the following boundary conditions in the fermionic sector

\[
\left( G^\pm - i \eta \tilde{G}^\mp_r \right) |L, M, R; \eta\rangle^{(s)}_C = 0, \tag{2.33}
\]

\[
\left( \Psi^\pm - i \eta \tilde{\Psi}^\mp_r \right) |L, M, R; \eta\rangle^{(s)}_C = 0. \tag{2.34}
\]

We will drop the superscript \((R)\) from \(|L, M, R; \eta\rangle^{(R)}_C\) in the following sections where we will work exclusively in the Ramond sector.

## 3 Boundary State Approach to the Periods

In this section let us study the periods or the central charges of the D-branes wrapped around the vanishing cycles. We start with simple geometrical computations of the periods and scaling behaviors of vanishing cycles when the CY manifold approaches the A-D-E singularities.

### 3.1 Geometrical Analysis of Periods

Let us consider the perturbed Landau-Ginzburg potential

\[
F = X^N + \sum_{m=1}^{N-2} g_m X^m + \mu + z_1^2 + \cdots + z_n^2 \equiv P(X; g_m, \mu) + z_1^2 + \cdots + z_n^2. \tag{3.1}
\]

\(F(X, z_i) = 0\) describes the CY\(_n\) with an isolated \(A_{N-1}\)-type singularity. In the following discussions the coupling constants \(g_m (\forall m)\) are sometimes set to zero for the sake of simplicity.

The holomorphic \(n\)-form of \(X_n\) is defined by

\[
\Omega = \frac{dX \wedge dz_1 \wedge \cdots \wedge dz_{n-1}}{\partial F / \partial z_n}. \tag{3.2}
\]

When we assign the \(U(1)\)-charge 1 to the superpotential \(F\), \(n\)-form \(\Omega\) has the \(U(1)\)-charge \(r_\Omega = \frac{Q^2}{2}\). We also introduce the derivative of the \(n\)-form in the coupling constant \(g_m\)

\[
\Omega_m = \frac{\partial \Omega}{\partial g_m} |_{g_i = 0}, \tag{3.3}
\]

which has the \(U(1)\)-charge; \(r_\Omega + \frac{m}{N} - 1\). Since the variables \(z_i\), \(i = 1, \cdots n\) enter quadratically in \(P(X; g_m, \mu)\) they may be integrated out in the evaluation of periods. Up to a proportionality
constant we have
\[
\int \Omega = \int P(X; g_m, \mu)^{\frac{2-n}{2}} dX = \int P(X; g_m, \mu)^{\frac{n-d}{2}} dX.
\] (3.4)

Let us now consider the roots of the equation \( P(X; g_m, \mu) = 0 \) and denote them as \( X_a, a = 1, \ldots, N \). We introduce an arbitrary path \( C_{ab} \) in the complex \( X \)-plane connecting a given pair of roots \( X_a \) and \( X_b \). Let us then consider the integral of the \( n \)-form \( \Omega \) and compare it with the integral of \( \lvert \Omega \rvert \) along the path \( C_{ab} \). Periods are identified as the central charges carried by the particles obtained by wrapping the D-brane on vanishing cycles. On the other hand the integral of the absolute value of the \( n \)-form gives the masses of these particles. We have an obvious inequality
\[
\int_{C_{a,b}} P(X)^{\frac{n-d}{2}} dX \geq \int_{C_{a,b}} P(X)^{\frac{n-d}{2}} dX \iff \text{mass} \geq \lvert \text{central charge} \rvert.
\] (3.5)

This inequality is saturated and we obtain BPS states iff the phase of \( P(X)^{(6-d)/2}dX \) is constant along the path \( C_{ab} \). If we suppose, for instance, that the contour \( C_{ab} \) is on the real axis and \( P(X) \) is real and negative along \( C_{ab} \). In this case by taking the real part of the equation \( F(X, z_i) = 0 \) we find a fibration of a sphere \( S^{n-1} \) over the interval \( C_{ab} \) whose radius vanishes at the end points \( X_a, X_b \). Altogether it describes an \( n \)-dimensional cycle with a minimum volume and thus a special Lagrangian submanifold of \( X_n \) [32, 4].

In general it is a non-trivial problem to find out if there exists a Lagrangian submanifold in a given (relative) homology class \( [C_{ab}] \). One has to solve the differential equation
\[
\frac{dX}{dt} = \frac{\alpha}{P(X)^{\frac{n-d}{2}}},
\] (3.6)
and construct an integral curve which passes through both \( X_a \) and \( X_b \). \( \alpha \) in (3.6) specifies the phase of the period integral in question. We cannot always expect the existence of such an integral curve for an arbitrary deformed polynomial \( P(X; g_m, \mu) \) and a pair of roots \( X_a, X_b \). To establish the existence of such a curve is an important problem in the study of stable BPS states, and some of its features have already been discussed in [3]. In our simplest case with the polynomial \( P(X; g_m, \mu) = X^N + \mu \), there exists a unique solution of (3.6) for any choice of \( X_a, X_b \). Since each pair \( X_a, X_b \) may be identified as a root of \( SU(N) \), vanishing cycles corresponding to the roots of \( A_{N-1} \) provide the stable BPS states in our coupling region.

Let us now parameterize pairs of roots of \( P(X) = X^N + \mu = 0 \) as
\[
X_a = (-\mu)^{1/N} e^{i\pi(M+L+1)/N}, \quad X_b = (-\mu)^{1/N} e^{i\pi(M-L-1)/N},
\] (3.7)
where $L = 0, 1, \ldots, \left[\frac{N}{2} - 1\right]$, $M \in \mathbb{Z}_{2N}$, $L + M \in 2\mathbb{Z} + 1$. It turns out that parameters $L, M$ correspond to the quantum numbers $L, M$ of the $\mathcal{N} = 2$ minimal model. Let us denote the SUSY cycle corresponding to the pair $(X_a, X_b)$ as $\gamma_{LM}$.

We next evaluate the integral of holomorphic $n$-form $\Omega$ over a path $C_{LM}$ connecting $X_a, X_b$

$$
\int_{C_{LM}} (X^N + \mu)^{\frac{6-d}{4}} dX
$$

Unlike the computation of the mass of a particle, period integral depends only on the homology class $[C_{LM}]$ and is independent of the choice of the contour. We may introduce the ”Morse function” $W(X) \equiv W(X; \mu)$ by

$$
dW(X) = P(X)^{\frac{6-d}{4}} dX,
$$

and then the period integral is simply expressed as

$$
\int_{C_{LM}} \Omega = \int_{C_{LM}} P(X)^{\frac{6-d}{4}} dX = W(X_b) - W(X_a).
$$

We note that $W(X)$ is no other than the superpotential for the space-time superconformal theory discussed in [4]. The definition (3.9) leads to the following scaling properties of $W(X)$;

$$
W(\alpha^{1/N}X; \alpha \mu) = \alpha^{\frac{1}{N} + \frac{6-d}{4}} W(X; \mu),
$$

$$
W(e^{2\pi i/n}X; \mu) = e^{2\pi i/N} W(X; \mu).
$$

We thus obtain the following behavior of the periods (up to an overall constant independent of $L, M$);

$$
\int_{C_{LM}} \Omega \approx \mu^\Omega e^{i\pi \frac{M}{N}} \sin \left(\frac{\pi (L + 1)}{N}\right).
$$

Moreover, since the functions $W_m(X) \equiv \left. \frac{\partial W}{\partial g_m}\right|_{g_i=0}$ have similar scaling properties,

$$
W_m(\alpha^{1/N}X; \alpha \mu) = \alpha^{\frac{m+1}{N} + \frac{2d}{4}} W_m(X; \mu),
$$

$$
W_m(e^{2\pi i/n}X; \mu) = e^{2\pi i/(m+1)n} W_m(X; \mu),
$$

we also obtain

$$
\int_{C_{LM}} \Omega_m \approx \mu^\Omega e^{i\pi \frac{M(m+1)}{N}} \sin \left(\frac{\pi (L + 1)(m + 1)}{N}\right),
$$

$$
= \mu^{\Omega(1-\Delta(g_m))} e^{i\pi \frac{M(m+1)}{N}} \sin \left(\frac{\pi (L + 1)(m + 1)}{N}\right).
$$
Here $\Delta(g_m) = \frac{N - m}{N r_\Omega}$ represents the scaling dimension of the coupling constant $g_m$. In ref. [1, 3] the coupling constant $g_m$ is identified as the VEV of a chiral operator in the space-time superconformal theory. For example, in the $d = 4$ case $\Delta(g_m) = 2^N - m + 2$, which agrees with the spectrum of scaling dimensions [33] at the Argyres-Douglas points [34]. Since the unitarity bound imposes $\Delta(g_m) > 1$, $m$ is bounded from above; $\bar{m} \equiv \frac{N - 2}{2} > m$.

3.2 Boundary State Approach to the Periods of Vanishing Cycles

Let us now turn to the stringy computation of the periods of vanishing cycles by means of the $\mathcal{N} = 2$ Liouville theory. We make use of the boundary states constructed in the previous section and compute their correlation functions by inserting the cosmological constant operators $S^\pm$. We shall show that the two point function of the boundary state and the chiral field depends only on $\mu$ (not on $\bar{\mu}$) due to chirality conditions and behave exactly as in (3.16).

Following [19] we first introduce suitable disk amplitudes which correspond to period integrals in CY manifolds. Fix a SUSY cycle $\gamma$, and let $|\gamma\rangle$ be the corresponding boundary state. We also introduce a $(c, c)$ chiral primary field $\phi_\alpha$. We then define the periods of $\gamma$ with respect to $\phi_\alpha$ by the following two-point function

$$
\Pi_\alpha^{\gamma} = \lim_{T \to +\infty} \langle 0; + | \phi_\alpha e^{-TH^{(c)}} | \gamma \rangle_{RR}.
$$

(3.17)

$|0; \pm\rangle$, $\langle 0; \pm|$ denote the $RR$ vacua obtained from the identity operator by the spectral flow. They possess the hermiticity property

$$
|0; +\rangle^\dagger = \langle 0; -|, \quad |0; -\rangle^\dagger = \langle 0; +|.
$$

(3.18)

and are expressed explicitly as

$$
|0; \pm\rangle = |l = 0, m = \pm 1, s = \pm 1\rangle_{MN} \otimes e^{\pm \frac{1}{2} H_0 \pm i \frac{Q}{2} Y_0} |0\rangle_L,
$$

$$
\otimes |l = 0, m = \pm 1, s = \pm 1\rangle_{MN} \otimes e^{\pm \frac{1}{2} H_0 \pm i \frac{Q}{2} Y_0} |0\rangle_L,
$$

$$
\langle 0; \pm| = \langle l = N - 2, m = \pm(N - 1), s = \pm 1|_{MN} \otimes \langle 0|_L e^{\pm \frac{1}{2} H_0 \pm i \frac{Q}{2} Y_0}
$$

$$
\otimes \langle l = N - 2, m = \pm(N - 1), s = \pm 1|_{MN} \otimes \langle 0|_L e^{\pm \frac{1}{2} H_0 \pm i \frac{Q}{2} Y_0}.
$$

(3.19)

Here the bosonic field $H$ is related to the fermions $\Psi^\pm$ in the Liouville sector as $\Psi^\pm = e^{\pm iH}$ and $Y_0, H_0$ denote the zero modes of $Y, H$. These RR ground states are annihilated by $G^\pm_0, \tilde{G}^\pm_0$. 

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Let us explicitly construct chiral fields $\phi_\alpha$ in the combined $\mathcal{N} = 2$ Liouville and minimal theories. A general chiral field is obtained by taking a chiral field of the minimal model $\phi^M_N$ (corresponding to the monomial $X^m$ in the Landau-Ginzburg description) and by "gravitationally" dressing it by the Liouville field as

$$\phi_{m,r} = \phi^M_N e^{p_{m,r}(\phi + iY)}, \quad m = 0, 1, \ldots, N - 2, \quad r \in \mathbb{Z}. \quad (3.20)$$

Momentum $p$ of the Liouville and $Y$ field takes values dictated by the GSO condition

$$p = p_{m,r} = \frac{m}{QN} - \frac{r}{Q} \quad (3.21)$$

where $r$ runs over odd integers (in NS sector). Total $U(1)_R$ charge of $\phi_{m,r}$ is readily computed as

$$U(1)-\text{charge}(\phi_{m,r}) = r. \quad (3.22)$$

Fractional charge $\frac{m}{N}$ from the minimal sector has been canceled by the $Y$-momentum of the Liouville sector. Then $r$ can freely run over (odd) integers.

We can likewise introduce the $(a,a)$ type chiral primary fields $\phi^*_m$ and then the states $\phi_{m,r}|0; +\rangle, \phi^*_m|0; -\rangle$ span the complete set of RR vacua.

For each fixed value of $m$ it is natural to consider the value $r = r^*$ which gives the minimum $U(1)_R$ charge and identify $\phi_{m,r^*}$ as the "primary operator". Then the other operators $\phi_{m,r} (r \neq r^*)$ become its "descendant" fields. Namely, we adopt $\phi_{m,r^*} = 1$ as the primary field for each fixed value of $m$. Quantum number $r$ originates from the momentum around $S^1$ and thus it represents the KK mode. It may also be identified as the number of $D0$ branes attached to the SUSY cycles. These descendant fields appear quite similar to the gravitational descendants in 2D gravity theory described by Landau-Ginzburg formulation [35]. It is curious to see how far we can push the analogy to 2D gravity. The appearance of descendants has also been noted in $SL(2, \mathbb{R})/U(1)$ formulation in a somewhat different context [25].

We now want to propose that the periods $\Pi^\gamma_\alpha \quad (3.17)$ with $\phi_\alpha$ replaced by the primary field $\phi_{m,r^*}$ correspond to the periods of $\Omega_m$ defined geometrically in (3.16). We will now see how this correspondence works.

Let us first examine the general properties of the correlation function

$$\Pi^\gamma_\alpha = \lim_{T \to +\infty} \langle 0; +|\phi_{m,r} e^{-T H(c)}|L, M, R; \eta\rangle_C \quad (3.23)$$
(α stands for (m, r) and γ for (L, M, R)). First we should note that the period Π_α vanishes in general due to Liouville momentum conservation unless the system is perturbed by Liouville potential terms. In fact the boundary state has a fixed (imaginary) Liouville momentum \( p_\phi = -Q/2 \) while the primary field (3.20) carries \( p_\phi = p_{m,r} \). and thus the net Liouville momentum becomes \( p_{m,r} - Q/2 \) in \( \Pi_\alpha \). On the other hand due to the presence of the background charge Liouville momentum has to add up to \(-Q\) on the sphere in order to give a non-vanishing correlation function. We find that the momentum conservation law \( p_{m,r} - Q/2 = -Q \) has no solution with \( m = 0, \ldots, N - 2, \ r \in 2\mathbb{Z} + 1 \) and thus the period \( \Pi_\alpha \) has to vanish.

As we have discussed in section 2, there are three different types of Liouville potentials \( S_{nc}, S^\pm \) which may be used for perturbation. It is shown in the appendix that

1. Correlation function \( \Pi_\alpha \) remains zero when the non-chiral operator \( S_{nc} \) is inserted. In general \( \Pi_\alpha \) does not depend on the perturbation of the Kähler potential. This follows from the chirality properties of the RR vacua and also from the boundary condition satisfied by the boundary state.

2. Correlation function also remains zero under the perturbation by the anti-chiral operator \( S^- \).

On the other hand, as we shall see in the following, the insertion of chiral operator \( S^+ \) into the correlation function gives a non-zero result. Therefore when the system is perturbed by the cosmological constant operators \( S^\pm \), \( \Pi_\alpha \) depends only on \( \mu \) and not on \( \bar{\mu} \) and thus has the holomorphicity in \( \mu \). In the following we derive the scaling laws of the periods by inserting the cosmological constant operator \( S^+ \).

Let us now recall some technology of the computation of correlation functions in Liouville theory. In the standard treatment [18] one first integrates over the zero-mode of the Liouville field and obtains the insertion of the Liouville potential term \( \sim \Gamma(-n)\mu^n(S^+)^n \) into the correlation function. Here \( n \) is the number of insertions of the cosmological constant and is given by

\[
n = Qp_{m,r} + \frac{Q^2}{2}. \tag{3.24}
\]

Note that this agrees with the value given by the Liouville momentum conservation. We compute correlation functions first by assuming as if \( n \) was an integer and then continue the results to the (generically) fractional value (3.24). This method of analytic continuation is somewhat heuristic, however, it is known to reproduce the correct results at least in bosonic Liouville theory. Recently the procedure has been made more rigorous by approaches based
on the conformal bootstrap [36]. We assume that the same prescription works also in the case of fermionic Liouville theory.

Now, the evaluation of the disc amplitude goes as follows: As we see in the appendix, insertion of the operator $S^−$ gives a vanishing result at $T = ∞$ and we need to consider only the insertion of the $S^+$ operators. We use the bosonic representation of fermions $\Psi^\pm = e^{±iH}$ and the algebra of vertex operators to obtain

\[
\langle 0; + | φ_{m,r} (μS^+)^n e^{−TH^{(c)}} | L, M, R; η \rangle_C = μ^n \langle 0; + | φ^{MN}_{m,r} e^{(p_{m,r} − 2)}(φ_0 + iy_0 + ˜φ_0 + 3y_0) \\
× e^{−in(H_0 + B_0)} \int \prod_{i=1}^{n} d^2 z_i \prod_{i<j} | z_i − z_j |^2 \prod_{j=1}^{n} \bar{V}(+) (ζ_j) \cdot V(+) (z_j) e^{−TH^{(c)}} | L, M, R; η \rangle_C
\]

(3.25)

where $V^±(z) = exp \left( −(φ^±(z) + iY^±(z))/Q − iH^±(z) \right)$ and $φ^+(Y^+), H^+(z)$ are positive frequency parts of the fields $φ, Y, H$. Negative frequency parts of $φ, Y, H$ have already been annihilated by the vacuum $⟨ 0; + |$. We recall that the vacuum $⟨ 0; + |$ carries a $Y$-momentum $Q/2$ and the boudary state $| L, M; R, η \rangle$ has an (imaginary) $φ$-momentum $−Q/2$. Since the net $φ$ momentum must equal $−Q$ to have a non-zero correlation function, boundary state $| L, M; R, η \rangle$ carries an effective $φ$ momentum $Q/2$. We then find the zero mode factors $φ_0, Y_0$ cancell exactly in (3.25) when the insertion number $n$ takes the value (3.24).

When $V(+) (z_j) (j = 1, \ldots, n)$ hit the boudary state in (3.25), they are converted into negative frequency parts $V(−)$ due to the boundary conditions. For the Liouville field we have

\[
φ^+(z) e^{−TH^{(c)}} | L, M, R; η \rangle = ˜φ^−(z^{-1} e^{−T}) e^{−TH^{(c)}} | L, M, R; η \rangle,
\]

(3.26)

and a similar equation holds for $Y, H$ with a $−$ sign in front of RHS. We also note that $H$ zero mode obeys a relation

\[
e^{−iH_0} | L, M, R; η \rangle = iη e^{iH_0} | L, M, R; η \rangle
\]

(3.27)

in order to reproduce (2.34).

When we next move these negative frequency operators to the left so that they get annihilated by the vacuum $⟨ 0; + |$, we pick up commutator terms with $\bar{V}(+) (ζ_j) (j = 1, \ldots, n)$. Collecting these factors we find

\[
Γ(−n) ⟨ 0; + | φ_{m,r} (μS^+)^n e^{−TH^{(c)}} | L, M, R; η \rangle_C
\]

\[
= \frac{c(n)}{2 \sqrt{N} \sin \left( \frac{π(m+1)}{N} \right)} μ^n (iη)^n \sin \left( \frac{π (L + 1)(m + 1)}{N} \right) e^{π M_{m+1}}
\]

(3.28)

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where we have introduced a notation
\[ c(n) \equiv \Gamma(-n)e^{-\frac{n(n+1)T}{2}} \prod_{i=1}^{n} \int d^2z_i \prod_{i<j} |z_i - z_j|^2 \prod_{i,j} \frac{1}{1 - z_i z_j e^{-T} |z^2_{i,j}|^{\frac{1}{2}}} \]  
(3.29)

Note that the RHS of (3.28) does not depend on \( R \) and we will henceforth suppress the label \( R \) in the Cardy state.

In the limit of \( T \rightarrow +\infty \) we have
\[ c(n) = \Gamma(-n)e^{-\frac{n(n+1)T}{2}} \prod_{i=1}^{n} \int d^2z_i \prod_{i<j} |z_i - z_j|^2 \approx e^{-\frac{n(n+1)T}{2} A n^{\frac{n(n+1)}{2}}} \]  
(3.30)

where \( A \) is the area of disk (world-sheet). The important point here is that the numerical factors \( c(n), \sqrt{N \sin \left( \frac{\pi (m+1)}{N} \right)} \) etc. are independent of the parameters \( L, M \) characterizing the boundary states. Thus we may absorb them into the normalization of the chiral field \( \phi_{m,r} \).

After the analytic continuation in \( n \) using the selection rule (3.24), we obtain
\[ \lim_{T \rightarrow +\infty} \langle 0; + | \phi_{m,r} e^{-TH^{(c)}} | L, M; \eta \rangle_C \approx \mu^{\frac{m^2}{2} - r^2} \sin \left( \pi \frac{(L+1)(m+1)}{N} \right) e^{i\pi M_{m+1}} \]  
(3.31)

Recall that we regard \( \phi_{m,r=1} \) as the primary fields. For these operators we find
\[ \lim_{T \rightarrow +\infty} \langle 0; + | \phi_{m,1} e^{-TH^{(c)}} | L, M; \eta \rangle_C \approx \mu^{r_{1\Omega}} \sin \left( \pi \frac{(L+1)(m+1)}{N} \right) e^{i\pi M_{m+1}}, \]  
(3.32)

where \( r_{\Omega} \equiv \frac{Q^2}{2}, \Delta(g_m) = (N-m)/Nr_{\Omega} \). (3.32) reproduces exactly the geometrical calculation (3.16) when we make the identification
\[ |L, M\rangle_C \iff \gamma_{L,M}, \phi_{m,1} \iff \Omega_m. \]  
(3.33)

We would like to make a few remarks:

1. With our convention Ramond ground states are mutually orthogonal in the sense that
\[ \langle 0; + | \phi_{m,r} \phi_{m',r'} | 0; + \rangle = 0, \langle 0; - | \phi^*_{m,r} \phi^*_{m',r'} | 0; - \rangle = 0, \]  
\[ \langle 0; + | \phi_{m,r} \phi^*_{m',r'} | 0; - \rangle \propto \delta_{m,m'} \delta_{r,r'}, \langle 0; - | \phi^*_{m,r} \phi_{m',r'} | 0; + \rangle \propto \delta_{m,m'} \delta_{r,r'}. \]  
(3.34)
(3.35)
As one may easily check, these properties follow from the conservation laws of Liouville and \( Y \)-momentum and also of the fermion number when we evaluate the inner products by inserting Liouville cosmological terms.
2. In view of the result (3.33), it is quite natural to suppose that the marginal perturbation by the \((c,c)\) operator

\[
\int d^2 z \int d\bar{w} G^{-}(w) \oint z \cdot d\bar{w} \bar{G}^{-}(\bar{w}) \phi_{m,r^*}(z,\bar{z}),
\]  

(3.36)
corresponds to the deformation of the singularity by a monomial \(X^m\)

\[
X^N + \mu \rightarrow X^N + g_m X^m + \mu.
\]  

(3.37)

In order to test this identification we may compare the norm of the differential form \(\Omega_m\),

\[
\int_{CY_n} |\Omega_m|^2
\]

with the Zamolodchikov metric \(\langle 0; + | \phi_{m,1} \phi_{m,1}^* | 0; - \rangle\) in the moduli space of \(N = 2\) theory. Using a similar method of calculation as in the case of periods, Zamolodchikov metric is evaluated as

\[
\langle 0; + | \phi_{m,1} \phi_{m,1}^* | 0; - \rangle \approx \langle 0; + | \mu S^+ e^{-2T H^{(e)}} (\bar{\mu} S^-)^n \phi_{m,1}^* | 0; - \rangle \approx e^{-(n+1)T} |c(n)|^2 |\mu|^{2r \alpha (1-\Delta(g_m))}.
\]  

(3.38)

(3.38) agrees with the geometrical relation \(\int_{CY_n} |\Omega_m|^2 \approx |\mu|^{2r \alpha (1-\Delta(g_m))}\).

It is worthwhile to note that the normalizability condition of the operator \(\phi_{m,r^*}\) at \(\phi \rightarrow +\infty\) in the context of Liouville theory \([37]\) corresponds to the inequality \(\Delta(g_m) > 1\). This in turn corresponds to the divergence of the metric \(\int_{CY_n} |\Omega_m|^2\) in the \(\mu \rightarrow 0\) limit. As discussed in \([4, 6]\), this behavior is equivalent to the requirement that \(\Omega_m\) is supported by a normalizable cohomology class localized at the singularity, so that the coupling \(g_m\) can be realized as the VEV of some dynamical fields in space-time conformal theory. We summarize the normalizability condition of primary fields in various dimensions;

\[\begin{align*}
d = 6; & \text{ All the primary fields } \phi_{m,1} \text{ are normalizable.} \\
d = 4; & \phi_{m,1} \text{ is normalizable, iff } 0 \leq m < \tilde{m} \equiv \frac{N - 2}{2}. \\
d = 2; & \text{ All the primary fields } \phi_{m,1} \text{ are non-normalizable.}
\end{align*}\]

3. It turns out that in the conifold case, \(d = 4, N = 2\), one has \(n = 0 (m = 0, r^* = 1)\) and the factor \(\Gamma(-n)\) coming from the \(\phi_0\) integral becomes divergent. We then obtain a scaling violation

\[
\mu^{1+\epsilon} \Gamma(-1 - \epsilon) \sim \frac{\text{const}}{\epsilon} + \mu \ln \mu + O(\epsilon).
\]  

(3.39)

Analogy of the conifold theory to 2D gravity coupled to \(c = 1\) matter has been stressed in \([2]\).
4 Open String Witten Index

4.1 Generalities of Open String Witten Index

Let us next look at intersection numbers among SUSY cycles in singular CY manifold and try to reproduce them using the perturbed Liouville theory. Intersection numbers among boundary states are computed by the open string Witten index in world-sheet theory. Open string Witten index is in general defined as the cylinder amplitude \[ I_{\gamma; \gamma'} = \langle \gamma'; \eta | \gamma; -\eta \rangle_{RR}, \] (4.1)

where \( \gamma, \gamma' \) corresponds to the SUSY cycles described by the Cardy states. Opposite signs for the signatures between in and out states correspond to the insertion of \((-1)^F\). This amplitude can be evaluated by inserting the complete set of the string states \(|n\rangle\) in the RR sector

\[ I_{\gamma; \gamma'} = \sum_n \langle RR | \gamma'; \eta | n \rangle \langle n | \gamma; -\eta \rangle_{RR}. \] (4.2)

According to the standard argument on supersymmetry index, one may keep only the RR vacua in the intermediate states of this amplitude. In our formulation, RR-vacua are mutually orthogonal but are not normalized. Thus we consider

\[ I_{\gamma; \gamma'} = \sum_{n \in RR \text{ vacua}} \frac{\langle RR | \gamma'; \eta | n \rangle \langle n | \gamma; -\eta \rangle_{RR}}{\langle n | n \rangle}. \] (4.3)

This gives the starting point of our evaluation of the index.

4.2 Open String Witten Index for the Singular CY Theory

Let us now compute the Witten index

\[ I(L', M', R'; L, M, R) = \sum_{n \in RR \text{ vacua}} \lim_{T \to \infty} \frac{c \langle L', M', R'; +1 | e^{-TH^{(c)}} | n \rangle \times \langle n | e^{-TH^{(c)}} | L, M, R; -1 \rangle_C}{\langle n | e^{-2TH^{(c)}} | n \rangle}. \] (4.4)

Ordinarily we should be able to compute (4.4) for any \( T \) and obtain a result which is \( T \)-independent. Unfortunately, in Liouville theory we can compute it only in the limit of \( T = +\infty \). In the following we evaluate the index at \( T = +\infty \) and assume that the result gives the correct topological invariant.
If we use the field identification in the minimal sector

\[ |\ell, m, s\rangle = |N - 2 - \ell, m + N, s + 2\rangle, \]  

we can set \( s = s_0 \) and the RR vacua constructed in the previous section span the complete set of states to be inserted in (4.4). We introduce the following notation from now on.

Here we use the abbreviated notations

\[ |m, r; s' + 1\rangle = \phi_{m, r}|0; +\rangle = e^{pm_r(\phi + iY)}\phi_m^M|0; +\rangle, \]  

\[ |m, r; s' - 1\rangle = \phi^*_{m, r}|0; -\rangle = e^{pm_r(\phi - iY)}\phi_M^M|0; -\rangle, \]  

\[ \langle -m, -r; s' = +1 | = \langle 0; +|\phi_{m, r} = \langle 0; +|e^{pm_r(\phi + iY)}\phi_m^N, \]  

\[ \langle -m, -r; s' = -1 | = \langle 0; -|\phi_{m, r} = \langle 0; -|e^{pm_r(\phi - iY)}\phi_M^N. \]  

Here \( s' = s = s_0 \) and \( m \) runs over the values \( m = 0, \ldots, N - 2 \). The numerator of (4.4) is then evaluated as

\[ C\langle L', M', R'; +1|e^{-TH^{(c)}}(\mu_{s'}S^{s'})^n|m, r; s'\rangle \approx (\mu_{s'})^n e^{-\frac{n(n+1)T}{2(2N)^{1/4}}} \frac{c(n)}{S_{L_0, m}} \frac{S_{L', m} e^{-i\pi \epsilon'_{m+1} M'}}{S_{0, m}} \]  

\[ \langle -m, -r; -s'|(\mu_{-s}S^{-s'})^n e^{-TH^{(c)}}|L, M, R; -1\rangle_C \approx (\mu_{-s'})^n e^{-\frac{n(n+1)T}{2(2N)^{1/4}}} \frac{c(n)}{S_{L_0, m}} \frac{S_{L, m} e^{i\pi \epsilon'_{m+1} M} e^{-i\pi n}}{S_{0, m}}. \]  

Here we use the abbreviated notations

\[ \mu_{s'} = \begin{cases} \mu & (s' = +1) \\ \bar{\mu} & (s' = -1) \end{cases}, \quad S^{s'} = \begin{cases} S^+ & (s' = +1) \\ S^- & (s' = -1) \end{cases}. \]  

The number \( n \) of insertions of Liouville potentials is given by

\[ n = Qp_{m, r} + \frac{Q^2}{2} \]  

as before. There is again no \( R, R' \) dependences in these amplitudes and we omit these labels from now on.

An additional factor \( e^{-i\pi n} \) in (4.11) follows from the fact that the signature \( \eta \) has opposite signs in the in-coming and out-going boundary states. Namely, when we use the fermionic boundary condition (2.34), we pick up an additional \( - \) sign for each mode \( j = 1, \ldots, n \) in (4.11) as compared with (4.10) and we obtain \( (-1)^n \).

We also evaluate the denominator of the formula (4.4) as

\[ \langle -m, -r; -s'|(\mu_{-s}S^{-s'})^n e^{-2TH^{(c)}}(\mu_{s'}S^{s'})^n|m, r; s'\rangle \approx e^{-n(n+1)T} c(n)^2 |\mu|^{2n}. \]
By combining these calculations the formula (4.4) now reads (up to the overall normalization)

\[ I(L', M'; L, M) = \sum_r \sum_{m=0}^{N-2} \sum_{s' = \pm 1} S_{L, m} S_{L', m} e^{i\pi \frac{(m+1)}{N} (M-M')} e^{-i\pi n}, \]

\[ = \sum_r \sum_{l=0}^{N-2} \sum_{m=0}^{N-2} N_{L', L}^{l} S_{l, m} e^{i\pi \frac{(m+1)}{N} (M-M')} e^{-i\pi n}. \]

(4.15)

\( S_{L, m} \) is the matrix of the \( S \)-transformation of \( \hat{SU}(2) \) and we have used the Verlinde formula for \( \hat{SU}(2)_{N-2} \). \( N_{L', L}^{l} \) denotes the fusion coefficients of \( \hat{SU}(2)_{N-2} \).

We find that the factors \( c(n), e^{-n(n+1)T} \) etc. and the powers of \( \mu, \bar{\mu} \) exactly cancel between the numerator and denominator in the formula. This is consistent with the fact that we are calculating a topological index.

The GSO conditions (2.26), (2.27), (2.28) for \(|m, r; s'\rangle, \langle -m, -r; -s'|\) are now written as

\[ d = 2, 6; \quad r \in 2\mathbb{Z} + 1, \]

(4.16)

\[ d = 4; \quad r \in \begin{cases} 2\mathbb{Z} & (s' = +1), \\ 2\mathbb{Z} + 1 & (s' = -1). \end{cases} \]

(4.17)

We evaluate the crucial phase factor \( e^{-i\pi n} \) in (4.15) by using the GSO conditions. We easily find

\[ d = 2; \quad e^{-i\pi n} = e^{-i\pi \frac{(m+1)}{N}}, \]

(4.18)

\[ d = 4; \quad e^{-i\pi n} = e^{\frac{\pi i}{2} s'} e^{-i\pi \frac{(m+1)}{N}}, \]

(4.19)

\[ d = 6; \quad e^{-i\pi n} = -e^{-i\pi \frac{(m+1)}{N}}. \]

(4.20)

We hence obtain the index for \( d = 6, \)

\[ I(L', M'; L, M) = \sum_r \sum_{l=0}^{N-2} \sum_{m=0}^{N-2} N_{L', L}^{l} S_{l, m} \left( -e^{i\pi \frac{(m+1)}{N}} \right) \left( e^{i\pi \frac{(m+1)(M-M')}{N}} + e^{-i\pi \frac{(m+1)(M-M')}{N}} \right), \]

\[ = \sum_r \sum_{l=0}^{N-2} \sum_{m=0}^{N-2} N_{L', L}^{l} S_{l, m} e^{-i\pi \frac{(m+1)(M-M')}{N}} \left( e^{i\pi \frac{(m+1)}{N}} - e^{-i\pi \frac{(m+1)}{N}} \right), \]

\[ \approx \sum_r \sum_{\ell=0}^{N-2} \sum_{\alpha, \beta = \pm 1} N_{L', L}^{\ell} (-1)^{\frac{\alpha+\beta}{2}} \delta^{(2N)}(M - M' + \alpha(\ell + 1) + \beta), \]

(4.21)

where we have used the relation \( \ell + M - M' \equiv L + L' + M - M' \equiv 0 \) (mod 2). \( \delta^{(2N)} \) denotes the Kronecker-delta modulo \( 2N \). In the case of \( d = 2 \), calculation of the index is completely parallel to the \( d = 6 \) case.
In the case of \( d = 4 \), we instead obtain

\[
I(L', M'; L, M) = \sum_r \sum_{l=0}^{N-2} \sum_{m=0}^{N-2} N^l_{L'} S_{lm} e^{-i\pi \frac{m+1}{N}} (ie^{i\pi \frac{(m+1)(M-M')}{N}} - ie^{-i\pi \frac{(m+1)(M-M')}{N}}),
\]

\[
\approx \sum_r \sum_{l=0}^{N-2} \sum_{\alpha,\beta=\pm 1} N^l_{L'} (-1)^{\frac{\alpha}{2}} \delta^{(2N)}(M - M' + \alpha(l + 1) + \beta). \quad (4.22)
\]

We have several comments to make:

1. The above formulas (4.21), (4.22) involve a decoupled sum over \( r \), i.e. descendants, which generates an overall infinite factor. It is not clear to us if this divergence really exists in the theory: it is conceivable that these descendants are singular vectors in Liouville theory and may not appear as intermediate states in correlation functions. This is in fact the case in 2D topological gravity. We would like to clarify this issue in a future publication.

2. If we ignore the divergent summation, our result reproduces the prediction based on the geometrical calculation under the correspondence (3.33). It also coincides with the formula recently proposed by [24, 25] using the \( SL(2; R)/U(1) \) Kazama-Suzuki model in the \( d = 6 \) case. In these papers, however, the validity of a formal analytic continuation in the level \( k \) of \( SU(2)_k \) WZW theory to a negative value has been assumed. It will be interesting to see if an operator insertion calculation like ours may be carried out in \( SL(2; R)/U(1) \) theory and confirm their result.

3. As a consistency check, we notice that the matrix \( I(L', M'; L, M) \) for \( d = 2, 6 \) is symmetric while the one for \( d = 4 \) is anti-symmetric with respect to \( L, M \) and \( L', M' \). This is consistent with the fact that \( I(L', M'; L, M) \) is identified as the intersection matrix among the even (odd) dimensional cycles for \( d = 2, 6 \) \( (d = 4) \).

For example, in the simplest case \( L = L' = 0 \), we obtain

\[
I_{M,M'} \approx \delta^{(2N)}(M - M' + 2) + \delta^{(2N)}(M - M' - 2) - 2\delta^{(2N)}(M - M'), \quad d = 2, 6 \quad (4.23)
\]

\[
I_{M,M'} \approx \delta^{(2N)}(M - M' + 2) - \delta^{(2N)}(M - M' - 2), \quad d = 4. \quad (4.24)
\]

Especially, in the \( d = 2, 6 \) cases we have the extended Cartan matrix of the \( A_{N-1} \)-type [24].
4. Although we have restricted ourselves to the $A$-type singularity, it is straightforward to extend our analysis to the more general $A$-$D$-$E$ cases since it is known how to construct the Cardy states based on $A$-$D$-$E$ type modular invariants \cite{39}. In the general $A$-$D$-$E$ case our computation of Witten indices gives rise to characteristic factors of the form $e^{\frac{i\pi m}{h}} - e^{-\frac{i\pi m}{h}} \ (d = 2, 6)$, $e^{\frac{i\pi m}{h}} + e^{-\frac{i\pi m}{h}} \ (d = 4)$ ($h$ denotes the Coxeter number of $A$-$D$-$E$ algebras) and we obtain the same results as in \cite{25}.

5 Relation to the $SL(2; \mathbb{R})/U(1)$ Approach

In this section we examine the perturbation of the Liouville theory by the non-chiral operator $S_{nc}$, which corresponds to the screening charge in the Wakimoto representation of $SL(2, \mathbb{R})$ current algebra \cite{10}. As we have pointed out, this operator does not affect the complex structure of the theory, but it modifies its Kähler structure, i.e. the metric of the target space. Bosonic part of the action of the perturbed theory is given by

$$S_L = \frac{1}{8\pi} \int d^2z \left\{ (\partial_a \phi)^2 + (\partial_a Y)^2 \right\} - \frac{Q}{8\pi} \int \phi R$$

$$+ \lambda \int d^2z (\partial \phi - i\partial Y)(\bar{\partial} \phi + i\bar{\partial} Y)e^{-Q\phi}. \quad (5.1)$$

Here $\lambda$ is the coupling constant. By performing the T-duality transformation along the $Y$-direction, we obtain an action

$$S_L^{\text{dual}} = \frac{1}{8\pi} \int d^2z \left\{ (\partial_a \phi)^2 + (\partial_a X)^2 \right\} - \frac{Q}{8\pi} \int \phi R$$

$$+ \lambda \int d^2z (\partial \phi - i\partial X)(\bar{\partial} \phi - i\bar{\partial} X)e^{-Q\phi}, \quad (5.2)$$

where the dual coordinate is denoted as $X$. This deformed action (5.2) is exactly the one considered in \cite{11} in connection with the $SL(2, \mathbb{R})/U(1)$ model of 2D black hole \cite{27}. There the coupling constant $\lambda$ is interpreted as the black hole mass parameter. In fact, by changing the coordinates as

$$\begin{cases}
\phi = \frac{2}{Q} \log \cosh r + \phi_0 \\
X = \frac{2}{Q} (\theta - i \log \tanh r)
\end{cases} \quad (5.3)$$

and tuning the coupling as $\lambda = e^{Q\phi_0}$, the above action is transformed into

$$S_{2DBH} = \frac{k}{4\pi} \int d^2z \left\{ (\partial_a r)^2 + \tanh^2 r (\partial_a \theta)^2 \right\} - \frac{k}{4\pi} \int R \log \cosh r, \quad (5.4)$$

22
where we set $k = \frac{2}{Q^2}$. This is the exactly the (Euclidean) 2D black-hole which describes the non-compact cigar geometry.

Our observation above seems to support the conjecture of [8] that the $SL(2; \mathbb{R})/U(1)$ model and the $\mathcal{N} = 2$ Liouville theory should be related by T-duality. Here, however, is a problem: the counter part of the cosmological term $S^\pm$ is missing in the $SL(2, \mathbb{R})$ side. We do not have an operator in $SL(2, \mathbb{R})/U(1)$ theory which deforms the complex structure of singular CY manifold. It seems likely that the deformed CY manifolds no longer possess $SL(2, \mathbb{R})$ symmetry.

6 Discussions and Concluding Remarks

In this paper we have studied the scaling behaviors and intersection numbers of vanishing cycles in singular CY manifold making use of the supersymmetric Liouville theory. We have obtained results which are in good agreement with geometrical considerations.

It is well-known that in the case of $d = 6$ dimensions, the T-dual of the ALE space is given by a collection of parallel NS five-branes with its throat region being described by the $SU(2)$ WZW model [3, 42]. On the other hand in the case of $d = 4$ dimensions we expect the T-dual of a singular CY threefold to be given by a collection of NS five-branes wrapped around a Riemann surface. Explicit description of such a configuration has not been worked out. Instead a configuration of intersecting NS five-branes has been proposed to represent the geometry of the conifold [43]. However, the position of the singularity is delocalized in this description and intersecting-brane picture does not seem fit to our framework.

It has recently been pointed out [44] that the geometry of the conifold may be approximately described by an $SU(2) \times SU(2)_{U(1)}$ WZW model of the type proposed in [45]. The sigma model metric of this WZW model is given by

$$ds^2 = k_1(d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2) + k_2(d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2) + k_1(d\psi + \cos \theta_1 d\phi_1 + \sqrt{\frac{k_2}{k_1}} \cos \theta_2 d\phi_2)^2$$ (6.1)

where $k_1$ and $k_2$ are the levels of two $SU(2)$ current algebras. $\phi_i, \theta_i (i = 1, 2)$ and $\psi$ are the angular variables of two and three sphere, $S^2, S^3$. On the other hand the angular part of conifold metric is given by [46]

$$ds^2 = \frac{1}{6}(d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2) + \frac{1}{6}(d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2) + \frac{1}{9}(d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)^2.$$ (6.2)
Since the target space and the sigma model metric of the WZW model should agree only at large $N$ or levels of current algebra, we can not unambiguously determine the levels $k_i, i = 1, 2$ by comparing the above two expressions. However, in our construction of modular invariant partition functions \cite{12}, we have encountered an extra parafermionic degrees of freedom in $d = 4$ theory. This seems quite consistent with the presence of an extra $SU(2)/U(1)$ factor in the above construction as compared with the $d = 6$ case. It will be very interesting if it is possible to find a metric for singular CY threefolds at large $N$.

Very recently a paper [hep-th/0011091] by K.Sugiyama and S.Yamaguchi has appeared which discusses a subject related to this article.

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Appendix

A Holomorphy of Disc Amplitudes

Let us recall the definition of our disc amplitude

$$\Pi^\alpha_{\gamma} = \lim_{T \to +\infty} \langle 0; +| \phi^\alpha e^{-TH^{(c)}} | \gamma \rangle_{RR},$$  \hspace{1cm} (A.1)$$

where the $RR$ vacuum $\langle 0; +|$ is obtained by the spectral flow from the identity operator in NS sector $\langle 0; +| = NS \langle 0|U$. $\phi^\alpha$ is a chiral field of $(c, c)$ type. $|\gamma \rangle_{RR}$ denotes the $RR$-component of the boundary state describing the SUSY cycle $\gamma$. Note that acting on the vacuum $\langle 0; +|$, modes of fermionic operators are shifted as in the topological formulation of the theory $U^{-1}G^\pm_n U = G^\pm_{n+\frac{1}{2}}$.

1. Deformation of the Kähler moduli

The Kähler deformations are generated by operators of the form

$$t_a \int d^2 z \int d\hat{w} G^+(w) \int d\tilde{z} \tilde{G}^{-}(\tilde{w}) \hat{\phi}_a(z, \tilde{z}) + (h.c.).$$ \hspace{1cm} (A.2)$$

Here $\hat{\phi}_a$ is a $(a, c)$ type chiral field. h.c. means the complex conjugation and corresponds to the $(c, a)$ field. In the following we discuss the case of $(a, c)$ type operator. Treatment of $(c, a)$ case is exactly similar.

Acting on the vacuum $\langle 0; +|$ the 1st term of (A.2) is rewritten as

$$t_a \int d^2 z \left\{ G^+_0, [\tilde{G}^-_1, \hat{\phi}_a(z, \tilde{z})] \right\}.$$ \hspace{1cm} (A.3)$$

We then obtain

$$\langle 0; +|\left\{ G^+_0, [\tilde{G}^-_1, \hat{\phi}_a]\right\} e^{-TH^{(c)}} |\gamma; \eta \rangle = -\langle 0; +|\hat{\phi}_a \tilde{G}^-_1 G^+_0 e^{-TH^{(c)}} |\gamma; \eta \rangle$$

$$= -i \eta \langle 0; +| \hat{\phi}_a \tilde{G}^-_1 G^+_0 e^{-TH^{(c)}} |\gamma; \eta \rangle = i \eta \langle 0; +| \tilde{G}^-_1 \hat{\phi}_a G^+_0 e^{-TH^{(c)}} |\gamma; \eta \rangle = 0.$$ \hspace{1cm} (A.4)$$

Thus the period $\Pi^\alpha_{\gamma}$ is independent of the Kähler modulus,

$$\frac{\partial}{\partial t_a} \Pi^\alpha_{\gamma} = 0.$$ \hspace{1cm} (A.5)$$
In the above we have used the relation
\[
\langle 0; + | G_n^+ = \langle 0; + | \tilde{G}_n^+ = 0 \ (n \leq 0), \ \langle 0; + | G_n^- = \langle 0; + | \tilde{G}_n^- = 0 \ (n \leq 1) \quad (A.6)
\]
and the A-type boundary condition
\[
(G_0^\pm - i\eta \tilde{G}_0^\mp | \gamma; \eta) = 0. \quad (A.7)
\]

2. Deformation of the complex structure moduli

Here we would like to derive the holomorphicity of \( \Pi_\alpha \) on the complex structure moduli. Let us consider an \((a,a)\) type deformation,

\[
\bar{g}_\alpha \int d^2z \oint z \oint \bar{z} \phi^*_\alpha(z, \bar{z}), \quad (A.8)
\]
where \( \phi^*_\alpha \) is an \((a,a)\) type anti-chiral field. Its insertion into the disc amplitude is evaluated as,

\[
\langle 0; + | \{ G_0^+ , \tilde{G}_0^+ , \phi^*_\alpha \}) e^{-TH(c)} | \gamma; \eta \rangle = \langle 0; + | \phi^*_\alpha \tilde{G}_0^+ G_0^+ e^{-TH(c)} | \gamma; \eta \rangle
\]
\[
= i\eta \langle 0; + | \phi^*_\alpha \tilde{G}_0^+ G_0^+ e^{-TH(c)} | \gamma; \eta \rangle = i\eta \langle 0; + | \phi^*_\alpha \{ \tilde{G}_0^+ , G_0^- \} e^{-TH(c)} | \gamma; \eta \rangle
\]
\[
= i\eta \langle 0; + | \phi^*_\alpha \left( L_0 - \frac{\hat{c}}{8} \right) e^{-TH(c)} | \gamma; \eta \rangle = 0 \quad \text{in the limit } T \to +\infty. \quad (A.9)
\]

Hence
\[
\frac{\partial}{\partial \bar{\mu}} \Pi_\alpha = 0. \quad (A.10)
\]
We can also prove in the same way that the period \( \bar{\Pi}_\alpha \) defined with respect to the vacuum \( \langle 0; - | \) is independent of the perturbation of the \((c,c)\) type. Hence we have established that the holomorphicity of the period \( \Pi_\alpha \) with respect to the complex structure moduli.
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