We prove that there are only finitely many compatibly split closed subschemes of a Frobenius split scheme.

1 Introduction

Let $k$ be an algebraically closed field of characteristic $p > 0$ and let $X$ be a scheme over $k$ (always assumed to be separated of finite type over $k$). The following is the main theorem of this note and we give here its complete and self-contained proof.

Theorem 1.1. Assume that $X$ is Frobenius split by a splitting $\sigma \in \text{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$, where $F$ is the absolute Frobenius morphism (cf. [1, Section 1.1]). Then, there are only finitely many closed subschemes of $X$ that are compatibly split (under $\sigma$).

2 Proof of Theorem 1.1

We first prove the following proposition that is of independent interest.

Proposition 2.1. Let $X$ be a nonsingular irreducible scheme that is Frobenius split by $\sigma \in \text{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) \simeq H^0(X, F_*(\omega_X^{1-p}))$, where $\omega_X$ is the dualizing sheaf of $X$ (cf. [1,
Proposition 1.3.7], and let $Y \subset X$ be a compatibly split closed subscheme of $X$. Then,

$$Y \subset Z(\tilde{\sigma}),$$

where $Z(\tilde{\sigma})$ denotes the set of zeroes of $\tilde{\sigma}$ and $\tilde{\sigma}$ is the section of $F_*(\omega^{1-p}_X)$ obtained from $\sigma$ via the above identification.

It may be remarked that $Z(\tilde{\sigma})$ is not compatibly split in general unless $\tilde{\sigma}$ is a $(p-1)$th power (cf. [1, Proposition 1.3.11]).

Proof. Since any irreducible component of a compatibly split closed subscheme is compatibly split (cf. [1, Proposition 1.2.1]), we can assume that $Y$ is irreducible. Assume, for contradiction, that $Y \cap (X \setminus Z(\tilde{\sigma})) \neq \emptyset$. Then, $Y^{\text{reg}} \cap (X \setminus Z(\tilde{\sigma})) \neq \emptyset$, where $Y^{\text{reg}}$ is the nonsingular locus of $Y$.

Take $y \in Y^{\text{reg}} \cap (X \setminus Z(\tilde{\sigma}))$. Choose a system of local parameters $\{t_1, \ldots, t_m, t_{m+1}, \ldots, t_n\}$ at $y \in X$ such that $\{t_1, \ldots, t_m\}$ restrict to a system of local parameters at $y \in Y$ and $\langle t_{m+1}, \ldots, t_n \rangle$ is the completion of the defining ideal of $Y$ in $X$ at $y$. (This is possible since both $X$ and $Y$ are nonsingular at $y$.) By assumption, $\tilde{\sigma}$ is a unit in the local ring $O_{X,y}$. Moreover, $\sigma$ induces a splitting $\hat{\sigma}$ of the power series ring $k[[t_1, \ldots, t_n]]$ compatibly splitting the ideal $\langle t_{m+1}, \ldots, t_n \rangle$. Now, since $\hat{\sigma}$ does not vanish at $y$, $\hat{\sigma}(t_1 \cdots t_n)^{p-1}$ is a unit in the ring $k[[t_1, \ldots, t_n]]$ (cf. [1, Proposition 1.3.7]). In particular, $\hat{\sigma}$ does not keep the ideal $\langle t_{m+1}, \ldots, t_n \rangle$ stable. This is a contradiction to the assumption. Hence, $Y \subset Z(\tilde{\sigma})$, proving the proposition.

Proof of Theorem 1.1. We prove Theorem 1.1 by induction on the dimension of $X$. If $\dim X = 0$, then the theorem is clear. So assume that $\dim X = n$ and the theorem is true for schemes of dimension $< n$. By [1, Proposition 1.2.1], we can assume without loss of generality that $X$ is irreducible. Let $Y \subset X$ be a compatibly split irreducible closed subscheme. Then, either $Y \subset X^{\text{sing}}$ (where $X^{\text{sing}}$ is the singular locus of $X$) or $Y \cap X^{\text{reg}} \neq \emptyset$. In the latter case, by Proposition 2.1,

$$Y \cap X^{\text{reg}} \subset Z(\tilde{\sigma}^o),$$

where $Z(\tilde{\sigma}^o)$ denotes the set of zeroes of the splitting $\tilde{\sigma}^o$ of the open subset $X^{\text{reg}}$ of $X$ viewed as a section of $F_*(\omega^{1-p}_{X^{\text{reg}}})$. Thus, in this case,

$$Y \subset \overline{Z(\tilde{\sigma}^o)},$$

$\overline{Z(\tilde{\sigma}^o)}$ being the closure of $Z(\tilde{\sigma}^o)$ in $X$. Hence, in either case,

$$Y \subset \overline{Z(\tilde{\sigma}^o)} \cup X^{\text{sing}}.$$  

(1)
Considering the irreducible components, the same inclusion (1) holds for any compatibly split closed subscheme \( Y \subset X \) such that \( Y \neq X \).

Let \( \{ Y_i \}_{i \in I} \) be the collection of all the distinct compatibly split closed subschemes \( Y_i \subset X \) and let \( Y := \bigcup_{i \in I} Y_i \). Observe that \( X \) being a scheme of finite type over \( k \) and \( Y \) a subscheme of \( X \), \( Y \) has only finitely many irreducible components. Since the ideal sheaf \( \mathcal{I}_Y = \bigcap_{i \in I} \mathcal{I}_{Y_i} \) and each \( \mathcal{I}_{Y_i} \) is stable under the splitting \( \sigma \) of \( X \), the closed subscheme \( Y \) is compatibly split. In particular, by (1), for each \( i \in I \),

\[
Y_i \subset \overline{Z(\sigma o)} \cup X^{\text{sing}},
\]

and hence \( Y \subset \overline{Z(\sigma o)} \cup X^{\text{sing}} \). Since \( \dim(\overline{Z(\sigma o)} \cup X^{\text{sing}}) < \dim X \); in particular, one has \( \dim Y < \dim X \). Thus, by the induction hypothesis (applying the theorem with \( X \) replaced by \( Y \)), \( I \) is a finite set. This completes the proof of the theorem. ■

**Remark 2.2.** Karl Schwede has also obtained the above theorem in a recent preprint [4]. His argument (obtained independently of our work) is similar to ours but he uses the theory of tight closure and test ideals to obtain a replacement for our Proposition 2.1. As pointed out by Schwede, when \( X \) is projective, the theorem also follows from [2, Corollary 3.2]. (See also [5] for another proof via the study of Frobenius actions on Artinian modules.)

**Acknowledgments**

We thank A. Knutson for the correspondences, which led to this work. We also thank the two referees for the suggestions to improve the exposition. The first author was partially supported by a Focused Research Groups grant from the National Science Foundation.

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