A Consumer-Theoretic Characterization of Fisher Market Equilibria

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Abstract. In this paper, we bring consumer theory to bear in the analysis of Fisher markets whose buyers have arbitrary continuous, concave, homogeneous (CCH) utility functions representing locally non-satiated preferences. The main tools we use are the dual concepts of expenditure minimization and indirect utility maximization. First, we use expenditure functions to construct a new convex program whose dual, like the dual of the Eisenberg-Gale program, characterizes the equilibrium prices of CCH Fisher markets. We then prove that the subdifferential of the dual of our convex program is equal to the negative excess demand in the associated market, which makes generalized gradient descent equivalent to computing equilibrium prices via tâtonnement. Finally, we run a series of experiments which suggest that tâtonnement may converge at a rate of \(O\left(\frac{1+\epsilon}{t^2}\right)\) in CCH Fisher markets that comprise buyers with elasticity of demand bounded by \(E\). Our novel characterization of equilibrium prices may provide a path to proving the convergence of tâtonnement in Fisher markets beyond those in which buyers utilities exhibit constant elasticity of substitution.

Keywords: Market Equilibrium · Market Dynamics · Fisher Market.

1 Introduction

One of the seminal achievements in mathematical economics is the proof of existence of equilibrium prices in Arrow-Debreu competitive economies\(^1\). This result, while celebrated, is non-constructive, and thus provides little insight into the computation of equilibrium prices. The computational question dates back to Léon Walras, a French economist, who in 1874 conjectured that a decentralized price-adjustment process he called tâtonnement, which reflects market behavior, would converge to equilibrium prices\(^2\). An early positive result in this vein was provided by Arrow, Block and Hurwicz, who showed that a continuous version of tâtonnement converges in markets with an aggregate demand function that satisfies the weak gross substitutes (WGS) property\(^2\). Unfortunately, following this initial positive result, Herbert Scarf provided his eponymous example of an economy for which the tâtonnement process does not converge, dashing all hopes of the tâtonnement process justifying the concept of market equilibria in general\(^3\). Nonetheless, further study of tâtonnement in simpler models than a full-blown Arrow-Debreu competitive economy remains important, as some real-world markets are indeed simpler\(^2\).

For market equilibria to be justified, not only should they be backed by a natural price-adjustment process such as tâtonnement, as economists have long argued, they should also be computationally efficient. As Kamal Jain put it, “If your laptop cannot find it, neither can the market”\(^2\). A detailed inquiry into the computational properties of market equilibria was initiated by Devanur et al.\(^17\)\(^18\), who studied a special case of the Arrow-Debreu competitive economy known as the Fisher market\(^6\). This model, for which Irving Fisher computed equilibrium prices using a hydraulic machine in the 1890s, is essentially the Arrow-Debreu model of a competitive economy in which there are no firms, and buyers are endowed with an artificial currency\(^30\). Devanur et al.\(^17\) discovered a connection between the Eisenberg-Gale convex program and Fisher markets in which buyers have linear utility functions, thereby providing a (centralized) polynomial time algorithm for equilibrium computation in these markets\(^17\)\(^18\).

Their work was built upon by Jain, Vazirani, and Ye\(^24\), who extended the Eisenberg-Gale program to all Fisher markets whose buyers have \textit{continuous, concave, and homogeneous (CCH)} utility functions. Further, they proved that the equilibrium of Fisher markets for buyers with CCH utility functions can be computed in polynomial time by interior point methods\(^14\). Even more recently, Gao and Kroer\(^22\) go beyond

\(^1\) We refer to Fisher markets that comprise buyers with a certain utility function by the name of the utility function, e.g., we call a Fisher market that comprise buyers with CCH utility functions a CCH Fisher market.
interior point methods to develop algorithms that converge in linear, quasilinear, and Leontief Fisher markets. However, unlike tâtonnement, these methods provide little insight into how markets reach equilibria.

More recently, Cole and Fleischer [12, 13], and Cheung, Cole, and Devanur [10] showed the fast convergence of tâtonnement in Fisher markets where the buyers’ utility functions satisfy weak gross substitutes with bounded elasticity of demand, and the constant elasticity of substitution (CES) properties respectively, the latter of which is a subset of the class of CCH utility functions [10, 12, 13]. Aside from tâtonnement being a plausible model of real-world price movements due to its decentralized nature, Cole and Fleischer argue for the plausibility of tâtonnement by proving that it is an abstraction for in-market processes in a real-world-like model called the ongoing market model [12, 13]. The plausibility of tâtonnement as a natural price-adjustment process has been further supported by Gillen et al. [23], who demonstrated the predictive accuracy of tâtonnement in off-equilibrium trade settings [23]. This theoretical and empirical evidence for tâtonnement makes it even more important to understand its convergence properties, so that we can better characterize those markets for which we can predict price movements and, in turn, equilibria.

Another price-adjustment process that has been shown to converge to market equilibria in Fisher markets is proportional response dynamics, first introduced by Wu and Zhang for linear utilities [35]; then expanded upon and shown to converge by Zhang for all CES utilities [36]; and very recently shown to converge in Arrow-Debreu exchange markets with linear utilities by Brânzei, Devanur, and Rabani [7]. The study of the proportional response process was proven fundamental when Birnbaum, Devanur, and Xiao noticed its relationship to gradient descent. This discovery opened up a new realm of possibilities in analyzing the convergence of market equilibrium processes. For example, it allowed Cheung, Cole, and Tao [11] to generalize the convergence results of proportional response dynamics to Fisher markets for buyers with mixed CES utilities. This same idea was applied by Cheung, Cole, and Devanur [10] to prove the convergence of tâtonnement in Leontief Fisher markets, using the equivalence between generalized gradient descent on the dual of the Eisenberg-Gale program and tâtonnement, first observed by Devanur et al. [18].

**Our Approach and Findings** In consumer theory [28], consumers/buyers are assumed to solve the utility maximization problem (UMP), in which each buyer maximizes its utility constrained by its budget, thereby discovering its optimal demand. Dual to this problem is the expenditure minimization problem (EMP), in which each buyer minimizes its expenditure constrained by its desired utility level, an alternative means of discovering its optimal demand. These two problems are intimately connected by a deep mathematical structure, yet most existing approaches to computing market equilibria focus on UMP only.

In this paper, we exploit the relationship between EMP and equilibrium prices to provide a new convex program, which like the seminal Eisenberg-Gale program characterizes the equilibrium prices of Fisher markets assuming buyers with arbitrary CCH utility functions. Additionally, by exploiting the duality structure between UMP and EMP, we provide a straightforward interpretation of the dual of our program, which also sheds light on the dual of the Eisenberg-Gale program. In particular, while it is known that an equilibrium allocation that solves the Eisenberg-Gale program is one that maximizes the buyers’ utilities given their budgets at equilibrium prices (UMP; the primal), we show that equilibrium prices are those that minimize the buyers’ expenditures at the utility levels associated with their equilibrium allocations (EMP; the dual).

Our characterization of CCH Fisher market equilibria via UMP and EMP also allows us to prove that the subdifferential of the dual of our convex program is equal to the negative excess demand in the corresponding market [10]. Consequently, solving the dual of our convex program via generalized gradient descent is equivalent to tâtonnement (just as generalized gradient descent on the dual of the Eisenberg-Gale program is equivalent to tâtonnement [18]).

Finally, we run a series of experiments which suggest that tâtonnement may converge at a rate of $O((1+E)/t^2)$ in CCH markets where buyers have bounded elasticity of demand (BED) with elasticity parameter $E$, a class of markets that includes CES Fisher markets. Assuming bounded elasticity of demand, bounded changes in prices result in bounded changes in demand. A summary of all known tâtonnement convergence rate results, as well as this conjecture, appears in Figure 1.

$^2$ Similarly, it is known that the subdifferential of the dual of the Eisenberg-Gale program is equal to the negative excess demand in the corresponding market [10]. Our result also implies this known result, since the two programs’ objective functions differ only by a constant.
Roadmap In Section 2 we introduce essential notation and definitions, and summarize our results. In Section 3 we derive the dual of the Eisenberg-Gale program and propose a new convex program whose dual characterizes equilibrium prices in CCH Fisher markets via expenditure functions. In Section 4 we show that the subdifferential of the dual of our new convex program is equivalent to the negative excess demand in the market, which implies an equivalence between generalized gradient descent and tâtonnement. In Section 5 we include an empirical analysis of tâtonnement in CCH Fisher markets.

2 Preliminaries and an Overview of Results

We use Roman uppercase letters to denote sets (e.g., $X$), bold uppercase letters to denote matrices (e.g., $X$), bold lowercase letters to denote vectors (e.g., $p$), and Roman lowercase letters to denote scalar quantities (e.g., $c$). We denote the $i^{th}$ row vector of any matrix (e.g., $X$) by the equivalent bold lowercase letter with subscript $i$ (e.g., $x_i$). Similarly, we denote the $j^{th}$ entry of a vector (e.g., $p$ or $x_i$) by the corresponding Roman lowercase letter with subscript $j$ (e.g., $p_j$ or $x_{ij}$). We denote the set of numbers $\{1, \ldots, n\}$ by $[n]$, the set of natural numbers by $\mathbb{N}$, the set of non-negative real numbers by $\mathbb{R}_+$ and the set of strictly positive real numbers by $\mathbb{R}_{++}$. We denote by $\Pi_X$ the Euclidean projection operator onto the set $X \subseteq \mathbb{R}^n$: i.e., $\Pi_X(x) = \arg\min_{z \in X} \|x - z\|_2$. We also define some set operations. Unless otherwise stated, the sum of a scalar by a set and of two sets is defined as the Minkowski sum, i.e., $c + A = \{c + a \mid a \in A\}$ and $A + B = \{a + b \mid a \in A, b \in B\}$, and the product of a scalar by a set and two sets is defined as the Minkowski product, e.g., $cA = \{ca \mid a \in A\}$ and $AB = \{ab \mid a \in A, b \in B\}$.

2.1 Consumer Theory

In this paper, we consider the general class of utility functions $u_i : \mathbb{R}^m \rightarrow \mathbb{R}$ that are continuous, concave and homogeneous. Recall that a set $U$ is open if for all $x \in U$ there exists an $\epsilon > 0$ such that the open ball $B_\epsilon(x)$ centered at $x$ with radius $\epsilon$ is a subset of $U$, i.e., $B_\epsilon(x) \subseteq U$. A function $f : \mathbb{R}^m \rightarrow \mathbb{R}$ is said to be continuous if $f^{-1}(U)$ is open for every $U \subseteq \mathbb{R}$. A function $f : \mathbb{R}^m \rightarrow \mathbb{R}$ is said to be concave if $\forall \lambda \in (0, 1), x, y \in \mathbb{R}^m, f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$ and strictly concave if strict inequality holds. A function $f : \mathbb{R}^m \rightarrow \mathbb{R}$ is said to be homogeneous of degree $k \in \mathbb{N}_+$ if for all $x \in \mathbb{R}^m$, $\lambda > 0$, $f(\lambda x) = \lambda^k f(x)$. Unless otherwise indicated, without loss of generality a homogeneous function is assumed to be homogeneous of degree $1$.

A utility function $u : \mathbb{R}^m \rightarrow \mathbb{R}$ assigns a real value to elements of $\mathbb{R}^m$, i.e., to every possible allocation of goods. Every utility function then represents some preference relation $\succeq$ over goods such that if for two bundle of goods $x, y \in \mathbb{R}^m$, $u(x) \geq u(y)$ then $x \succeq y$.
A preference relation \( \geq \) is said to be **locally non-satiated** iff for all \( \mathbf{x} \in \mathbb{R}^m \) and \( \epsilon > 0 \), there exists \( \mathbf{y} \in B_{\epsilon}(\mathbf{x}) \) such that \( \mathbf{y} \succ \mathbf{x} \). Throughout this paper, we assume utility functions represent locally non-satiated preferences. If a buyer’s utility function represents locally non-satiated preferences, there always exists a better bundle for that buyer if their budget increases. We define some important concepts that pertain to utility functions. The **indirect utility function** \( v_i : \mathbb{R}^m_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \) takes as input prices \( \mathbf{p} \) and a budget \( b_i \) and outputs the maximum utility the buyer can achieve at those prices given that budget, i.e., \( v_i(p, b_i) = \max_{\mathbf{x} \in \mathbb{R}^m_+: \mathbf{p} \cdot \mathbf{x} \leq b_i} u_i(\mathbf{x}) \). If the utility function is continuous, then the indirect utility function is continuous and homogeneous of degree 0 in \( \mathbf{p} \) and \( b_i \) jointly, i.e., \( \forall \lambda > 0, v_i(\lambda \mathbf{p}, \lambda b_i) = v_i(\mathbf{p}, b_i) \) is non-increasing in \( \mathbf{p} \), strictly increasing in \( b_i \), and convex in \( \mathbf{p} \) and \( b_i \).

The **Marshallian demand** is a correspondence \( d_i : \mathbb{R}^m_+ \times \mathbb{R}_+ \to \mathbb{R}^m_+ \) that takes as input prices \( \mathbf{p} \) and a budget \( b_i \) and outputs the utility maximizing allocation of goods at budget \( b_i \), i.e., \( d_i(\mathbf{p}, b_i) = \arg \max_{\mathbf{x} \in \mathbb{R}^m_+: \mathbf{p} \cdot \mathbf{x} \leq b_i} u_i(\mathbf{x}) \). The Marshallian demand is convex-valued if the utility function is continuous and concave, and unique if the utility function is continuous and strictly concave.

The **expenditure function** \( e_i : \mathbb{R}^m_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \) takes as input prices \( \mathbf{p} \) and a utility level \( \nu_i \) and outputs the minimum amount the buyer must spend to achieve that utility level at those prices, i.e., \( e_i(\mathbf{p}, \nu_i) = \min_{\mathbf{x} \in \mathbb{R}^m_+: u_i(\mathbf{x}) \geq \nu_i} \mathbf{p} \cdot \mathbf{x} \). If the utility function \( u_i \) is continuous, then the expenditure function is continuous and homogeneous of degree 1 in \( \mathbf{p} \) and \( \nu_i \) jointly, \( \forall \lambda > 0, e_i(\lambda \mathbf{p}, \lambda \nu_i) = \lambda e_i(\mathbf{p}, \nu_i) \) is non-decreasing in \( \mathbf{p} \), strictly increasing in \( \nu_i \), and concave in \( \mathbf{p} \) and \( \nu_i \).

The **Hicksian demand** is a correspondence \( h_i : \mathbb{R}^m_+ \times \mathbb{R}_+ \to \mathbb{R}^m_+ \) that takes as input prices \( \mathbf{p} \) and a utility level \( \nu_i \) and outputs the cost-minimizing allocation of goods at utility level \( \nu_i \), i.e., \( h_i(\mathbf{p}, \nu_i) = \arg \min_{\mathbf{x} \in \mathbb{R}^m_+: u_i(\mathbf{x}) \geq \nu_i} \mathbf{p} \cdot \mathbf{x} \). The Hicksian demand is convex-valued if the utility function is continuous and concave, and unique if the utility function is continuous and strictly concave.

In consumer theory, the demand of buyers can be determined by studying two dual problems, the **utility maximization problem (UMP)** and the **expenditure minimization problem (EMP)**. The UMP refers to the buyer’s problem of maximizing its utility constrained by its budgets (i.e., optimizing its indirect utility function) in order to obtain its optimal demand (i.e., Marshallian demand), while the EMP refers to the buyer’s problem of minimizing its expenditure constrained by its desired utility level (i.e., optimizing its expenditure function) in order to obtain its optimal demand (i.e., Hicksian demand). When the utilities are continuous, concave and represent locally non-satiated preferences the UMP and EMP are related through the following identities, which we use throughout the paper:

\[
\forall b_i \in \mathbb{R}_+ \quad e_i(\mathbf{p}, v_i(\mathbf{p}, b_i)) = b_i \tag{1}
\]

\[
\forall \nu_i \in \mathbb{R}_+ \quad v_i(\mathbf{p}, e_i(\mathbf{p}, \nu_i)) = \nu_i \tag{2}
\]

\[
\forall b_i \in \mathbb{R}_+ \quad h_i(\mathbf{p}, v_i(\mathbf{p}, b_i)) = d_i(\mathbf{p}, b_i) \tag{3}
\]

\[
\forall \nu_i \in \mathbb{R}_+ \quad d_i(\mathbf{p}, e_i(\mathbf{p}, \nu_i)) = h_i(\mathbf{p}, \nu_i) \tag{4}
\]

A good \( j \in [m] \) is said to be a **gross substitute (complement)** for a good \( k \in [m] \setminus \{j\} \) if \( \sum_{i \in [n]} d_{ij}(\mathbf{p}, b_i) \) is increasing (decreasing) in \( p_k \). If the aggregate demand, \( \sum_{i \in [n]} d_{ij}(\mathbf{p}, b_i) \), for good \( k \) is instead weakly increasing (decreasing), good \( j \) is said to be a **weak gross substitute (complement)** for good \( k \).

The class of homogeneous utility functions includes the well-known **linear**, **Cobb-Douglas**, and **Leontief** utility functions, each of which is a special case of the **Constant Elasticity of Substitution (CES)** utility function family, parameterized by \(-\infty \leq \rho \leq 1 \), and given by \( u_i(x_i) = \left( \sum_{j \in [m]} w_{ij} x_j^\rho \right)^{1/\rho} \). Linear utility functions are obtained when \( \rho = 1 \), while Cobb-Douglas and Leontief utility functions are obtained when \( \rho \) approaches 0 and \(-\infty \), respectively. For \( 0 < \rho \leq 1 \), goods are gross substitutes, e.g., Sprite and Coca-Cola, for \( \rho = 1 \); goods are perfect substitutes, e.g., Pepsi and Coca-Cola; and for \( \rho < 0 \), goods are complementary, e.g., left and right shoes.

The **(price) elasticity of demand** reflect how demand varies in response to a change in price. More specifically, buyer \( i \)'s elasticity of demand for good \( j \in [m] \) with respect to the price of good \( k \in [m] \) is defined as \( \frac{\partial d_{ij}(\mathbf{p}, b_i)}{\partial p_k} \frac{p_k}{d_{ij}(\mathbf{p}, b_i)} \). A buyer is said to have **bounded elasticity of demand** with elasticity parameter \( E \) if \( \min_{\mathbf{p} \in \mathbb{R}^m_+, j \in [m]} \left\{ \left| \frac{\partial d_{ij}(\mathbf{p}, b_i)}{\partial p_k} \frac{p_k}{d_{ij}(\mathbf{p}, b_i)} \right| \right\} = E < \infty \).
2.2 Fisher Markets

A Fisher market comprises \( n \) buyers and \( m \) divisible goods. As is usual in the literature, we assume that there is one unit of each good available. Each buyer \( i \in [n] \) has a budget \( b_i \in \mathbb{R}_+ \) and a utility function \( u_i : \mathbb{R}_+^m \to \mathbb{R} \). An instance of a Fisher market is thus given by a tuple \((n, m, U, b)\) where \( U = \{u_1, \ldots, u_n\} \) is a set of utility functions, one per buyer, and \( b \in \mathbb{R}_+^n \) is the vector of buyer budgets. We abbreviate as \((U, b)\) when \( n \) and \( m \) are clear from context.

When the buyers’ utility functions in a Fisher market are all of the same type, we qualify the market by the name of the utility function, e.g., a Leontief Fisher market. Considering properties of goods, rather than buyers, a (Fisher) market satisfies gross substitutes (resp. gross complements) if all pairs of goods in the market are gross substitutes (resp. gross complements). A Fisher market is mixed if all pairs of goods are either gross complements or gross substitutes. A Fisher market exhibits bounded elasticity of demand with parameter \( E \), if the elasticity of demand of the buyer with highest elasticity of demand is \( E < \infty \).

An allocation \( X \) is a map from goods to buyers, represented as a matrix s.t. \( x_{ij} \geq 0 \) denotes the quantity of good \( j \in [m] \) allocated to buyer \( i \in [n] \). Goods are assigned prices \( p \in \mathbb{R}_+^m \). A tuple \((X^*, p^*)\) is said to be a competitive (or Walrasian) equilibrium of Fisher market \((U, b)\) if 1. buyers are utility maximizing constrained by their budget, i.e., for all \( i \in [n] \), \( x_i^* \in d_i(p^*, b_i) \); and 2. the market clears, i.e., for all \( j \in [m] \), \( p_j^* > 0 \) implies \( \sum_{i \in [n]} x_{ij}^* = 1 \); and \( p_j^* = 0 \) implies \( \sum_{i \in [n]} x_{ij}^* \leq 1 \).

If \((U, b)\) is a CCH Fisher market, then the optimal solution \( X^* \) to the Eisenberg-Gale program constitutes an equilibrium allocation, and the optimal solution to the Lagrangian that corresponds to the allocation constraints (Equation (6)) are the corresponding equilibrium prices \[17, 14, 21\]:

\[
\text{Primal} \quad \begin{align*}
\max & \quad \sum_{i \in [n]} b_i \log (u_i(x_i)) \\
\text{subject to} & \quad \sum_{i \in [n]} x_{ij} \leq 1 \quad \forall j \in [m] \quad (6)
\end{align*}
\]

We define the excess demand correspondence \( z : \mathbb{R}^m \to \mathbb{R}^m \), of a Fisher market \((U, b)\), which takes as input prices and outputs a set of excess demands at those prices, as the difference between the demand for each good and the supply of each good: \( z(p) = \sum_{i \in [n]} d_i(p, b_i) - 1_m \).

where \( 1_m \) is the vector of ones of size \( m \).

The discrete tâtonnement process for Fisher markets is a decentralized, natural price adjustment, defined as:

\[
\begin{align*}
p(t + 1) &= p(t) + G(g(t)) \quad \text{for } t = 0, 1, 2, \ldots \quad (7) \\
g(t) &= z(p(t)) \quad \text{(8)} \\
p(0) &\in \mathbb{R}_+^m \quad \text{(9)}
\end{align*}
\]

where \( G : \mathbb{R}^m \to \mathbb{R}^m \) is a coordinate-wise monotonic function s.t. for all \( j \in [m], x, y \in \mathbb{R}^m \), if \( x_j \geq y_j \), then \( G_j(x) \geq G_j(y) \). Intuitively, tâtonnement is an auction-like process in which the seller of \( j \in [m] \) increases (resp. decreases) the price of a good if the demand (resp. supply) is greater than the supply (resp. demand).

2.3 Subdifferential Calculus and Generalized Gradient Descent

We say that a vector \( g \in \mathbb{R}^n \) is a subgradient of a continuous function \( f : U \to \mathbb{R} \) at \( a \in U \) if for all \( x \in U \), \( f(x) \geq f(a) + g^T(x - a) \). The set of all subgradients \( g \) at a point \( a \in U \) for a function \( f \) is called the subdifferential and is denoted by \( \partial_x f(a) = \{ g \mid f(x) \geq f(a) + g^T(x - a) \} \). If \( f \) is convex, then its subdifferential exists everywhere. If additionally, \( f \) is differentiable at \( a \), so that its subdifferential is a singleton at \( a \), then the subdifferential at \( a \) is equal to the gradient. In this case, for notational simplicity, we write \( \partial_x f(a) = \nabla f \); in other words, we take the subdifferential to be vector-valued rather than set-valued. When both \( f \) and \( \hat{f} : U \to \mathbb{R} \) are continuous and convex, subdifferentials satisfy the additivity property: \( \partial_x (f + \hat{f})(a) = \partial_x f(a) + \partial_x \hat{f}(a) \). If \( f \) is again continuous and convex, and if \( \tilde{f} : \mathbb{R} \to \mathbb{R} \) is
continuous, convex, and differentiable with derivative \( \hat{f}' \), subdifferentials satisfy the composition property:

\[ \partial_x (f \circ f(a)) = \hat{f}'(f(a)) \cdot \partial_x f(a). \]

Consider the optimization problem \( \min_{x \in V} f(x) \), where \( f : \mathbb{R}^n \to \mathbb{R} \) is a convex function that is not necessarily differentiable and \( V \) is the feasible set of solutions. Let \( \ell_f(x, y) \) be the linear approximation of \( f \) at \( y \), that is \( \ell_f(x, y) = f(y) + g^T(x - y) \), where \( g \in \partial_x f(y) \). A standard method for solving this problem is the mirror descent \([29]\) update rule as follows:

\[
x(t + 1) = \arg\min_{x \in V} \{ \ell_f(x, x(t)) + \gamma_t \delta_h(x, x(t)) \} \quad \text{for } t = 0, 1, 2, \ldots \tag{10}
\]

\[
x(0) \in \mathbb{R}^n \tag{11}
\]

Here, as above, \( \gamma_t > 0 \) is the step size at time \( t \) and, \( \delta_h(x, x(t)) \) is the Bregman divergence of a convex differentiable kernel function \( h(x) \) defined as \( \delta_h(x, y) = h(x) - h(y) - \ell_h(x, y) \) \([8]\). When the kernel is the scaled weighted entropy \( h(x) = c \sum_{i \in [n]} (x_i \log(x_i) - x_i) \), given \( c > 0 \), then the Bregman divergence reduces to the scaled generalized Kullback-Leibler divergence: \( \delta_{\text{KL}}(x, y) = c \sum_{i \in [n]} \left( x_i \log \left( \frac{x_i}{y_i} \right) - x_i + y_i \right) \), which, when \( V = \mathbb{R}^n_+ \), yields the following simplified update rule, where as usual \( g(t) \in \partial_x f(x(t)) \):

\[
\forall j \in [m] \quad x_j(t + 1) = x_j(t) \exp \left\{ \frac{-g_j(t)}{\gamma_t} \right\} \quad \text{for } t = 0, 1, 2, \ldots \tag{12}
\]

\[
x_j(0) \in \mathbb{R}_+^n \tag{13}
\]

Equations (12) to (13) do not include a projection step, because when the initial iterate is within \( \mathbb{R}^n_+ \), the update rule guarantees that subsequent iterates remain within this set.

### 2.4 A High-Level Overview of Our Contributions

In this paper, we bring consumer theory to bear in the analysis of CCH Fisher markets. In so doing, we first derive the dual of the Eisenberg-Gale program for arbitrary CCH Fisher markets, generalizing the special cases of linear and Leontief markets, which are already understood \([18]\). We then provide a new convex program whose dual also characterizes equilibrium prices in CCH Fisher markets via expenditure functions. This program is of interest because the subdifferential of the objective function of its dual is equal to the negative excess demand in the market, which implies that mirror descent on this objective is equivalent to solving for equilibrium prices in the associated market via tâtonnement. Finally, we conjecture a convergence rate of \( O((1 + E)/n^2) \) for CCH Fisher markets in which the elasticity of buyer demands is bounded by \( E \).

Although the Eisenberg-Gale convex program dates back to 1959, its dual for arbitrary CCH Fisher markets is still not yet well understood. Our first result is to derive the Eisenberg-Gale program’s dual, generalizing the two special cases identified by Cole et al. \([15]\) for linear and Leontief utilities.

**Theorem 1.** The dual of the Eisenberg-Gale program for any CCH Fisher market \((U, b)\) is given by:

\[
\min_{p \in \mathbb{R}^m_+} \sum_{j \in [m]} p_j + \sum_{i \in [n]} \left[ b_i \log \left( \nu_i(p, b_i) \right) - b_i \right] \tag{14}
\]

We then propose a new convex program whose dual characterizes the equilibrium prices of CCH Fisher markets via expenditure functions. We note that the optimal value of this convex program differs from the optimal value of the Eisenberg-Gale program by a constant factor.

**Theorem 2.** The optimal solution \((X^*, p^*)\) to the primal and dual of the following convex programs corresponds to equilibrium allocations and prices, respectively, of the CCH Fisher market \((U, b)\):

**Primal**

\[
\max_{x \in \mathbb{R}^{n \times m}_+} \sum_{i \in [n]} b_i \log \left( \frac{u_i(x_i)}{b_i} \right) \quad \text{subject to } \forall j \in [m], \sum_{i \in [n]} x_{ij} \leq 1
\]

**Dual**

\[
\min_{p \in \mathbb{R}^m_+} \sum_{j \in [m]} p_j - \sum_{i \in [n]} b_i \log \left( \partial_{\nu_i} e_i(p, \nu_i) \right)
\]
This convex program formulation for CCH Fisher markets is of particular interest because its subdifferential equals the negative excess demand in the market. As a result, solving this program via (sub)gradient descent is equivalent to solving the market via tâtonnement.

**Theorem 3.** The subdifferential of the objective function of the dual of the program given in Theorem 2 for a CCH Fisher market $(U, b)$ at any price $p$ is equal to the negative excess demand in $(U, b)$ at price $p$:

$$\partial_p \left( \sum_{j \in [m]} p_j - \sum_{i \in [n]} b_i \log \nu_i e_i(p, \nu) \right) = -z(p)$$

(15)

To prove Theorem 3 we make use of standard consumer theory, specifically the duality structure between UMP and EMP, as well as a generalized version of Shepherd’s lemma [31, 32]. We also provide a new, simpler proof of this generalization of Shepherd’s lemma via Danskin’s theorem [16].

Finally, we conduct an experimental investigation of the convergence of the tâtonnement process defined by the mirror descent rule with KL-divergence and fixed step sizes in CCH Fisher markets. This particular process was previously studied by Cheung, Cole, and Devanur [10] in Leontief Fisher markets. They showed a worst-case lower bound of $\Omega(1/\epsilon^2)$ to complement an $O(1/\epsilon)$ worst-case upper bound. These results suggest a possible convergence rate of $O(1/\epsilon^2)$ or $O(1/\epsilon)$ for entropic tâtonnement in a class of Fisher markets that includes Leontief Fisher markets. Our experimental results support the conjecture that a worst-case convergence rate of $O(1/\epsilon^2)$ might hold, not only in Leontief and CES Fisher markets, but in CCH Fisher markets where buyers’ elasticity of demand is bounded by $E$.

## 3 A New Convex Program for CCH Fisher Markets

In this section, we provide an alternative convex program to the Eisenberg-Gale program, which also characterizes the equilibria of CCH Fisher markets. Of note, our program characterizes equilibrium prices via expenditure functions. For CCH Fisher markets, the Eisenberg-Gale program’s primal allows us to calculate the equilibrium allocations, while its dual yields the corresponding equilibrium prices [9]. Cole et al. [15] provide dual formulations of the Eisenberg-Gale program for linear and Leontief utilities [15], and in unpublished work, Cole and Tao [14] present a generalization of the Eisenberg-Gale dual for arbitrary CCH utility functions. However, as we show in Example 2 (Appendix A), the optimal value of the objective of the Eisenberg-Gale program’s primal differs from the optimal value of the dual provided by Cole and Tao [14] by a constant factor, despite their dual characterizing equilibrium prices accurately. Hence, their dual is technically not the dual of the Eisenberg-Gale program for which strong duality holds. The proof of the following theorem stating the Eisenberg-Gale program’s dual can be found in Appendix A.

**Theorem 1.** The dual of the Eisenberg-Gale program for any CCH Fisher market $(U, b)$ is given by:

$$\min_{p \in \mathbb{R}_+^m} \sum_{j \in [m]} p_j + \sum_{i \in [n]} [b_i \log (v_i(p, b_i)) - b_i]$$

(14)

Before presenting our program, we present several preliminary lemmas. All omitted proofs can be found in Appendix A.

The next lemma establishes an important property of the indirect utility and expenditure functions in CCH Fisher markets that we heavily exploit in this work, namely that the derivative of the indirect utility function with respect to $b_i$—the bang-per-buck—is constant across all budget levels. Likewise, the derivative of the expenditure function with respect to $v_i$—the buck-per-bang—is constant across all utility levels. In other words, both functions effectively depend only on prices. Not only are the bang-per-buck and the buck-per-bang constant, they equal $v_i(p, 1)$ and $e_i(p, 1)$, respectively, namely their values at exactly one unit of budget and one unit of (indirect) utility.

An important consequence of this lemma is that, by picking prices that maximize a buyer’s bang-per-buck, we not only maximize their bang-per-buck at all budget levels, but we further maximize their total indirect utility, given their known budget. In particular, given prices $p^*$ that maximize a buyer’s bang-per-buck at...
budget level 1, we can easily calculate the buyer’s total (indirect) utility at budget \( b_i \) by simply multiplying their bang-per-buck by \( b_i \): i.e., \( v_i(p^*, b_i) = b_i v_i(p^*, 1) \). Here, we see quite explicitly the homogeneity assumption at work.

Analogously, by picking prices that maximize a buyer’s buck-per-bang, we not only maximize their buck-per-bang at all utility levels, but we further maximize the buyer’s total expenditure, given their unknown optimal utility level. In particular, given prices \( p^* \) that minimize a buyer’s buck-per-bang at utility level 1, we can easily calculate the buyer’s total expenditure at utility level \( \nu_i \) by simply multiplying their buck-per-bang by \( \nu_i \): i.e., \( e_i(p^*, \nu_i) = \nu_i e_i(p^*, 1) \). Thus, solving for optimal prices at any budget level, or analogously at any utility level, requires only a single optimization, in which we solve for optimal prices at budget level, or utility level, 1.

**Lemma 1.** If \( u_i \) is continuous and homogeneous of degree 1, then \( v_i(p, b_i) \) and \( e_i(p, \nu_i) \) are differentiable in \( b_i \) and \( \nu_i \), resp. Further, \( \partial_{b_i} v_i(p, b_i) = \{v_i(p, 1)\} \) and \( \partial_{\nu_i} e_i(p, \nu_i) = \{e_i(p, 1)\} \).

The next lemma provides further insight into why CCH Fisher markets are easier to solve than non-CCH Fisher markets. The lemma states that the bang-per-buck, i.e., the marginal utility of an additional unit of budget, is equal to the inverse of its buck-per-bang, i.e., the marginal cost of an additional unit of utility. Consequently, by setting prices so as to minimize the buck-per-bang of buyers, we can also maximize their bang-per-buck. Since the buck-per-bang is a function of prices only, and not of prices and allocations together, this lemma effectively decouples the calculation of equilibrium prices from the calculation of equilibrium allocations, which greatly simplifies the problem of computing equilibria in CCH Fisher markets.

**Corollary 1.** If buyer \( i \)’s utility function \( u_i \) is CCH, then

\[
\frac{1}{e_i(p, 1)} = \frac{1}{\partial_{\nu_i} e_i(p, \nu_i)} = \partial_{b_i} v_i(p, b_i) = v_i(p, 1) \, .
\]

We can now present our characterization of the dual of the Eisenberg-Gale program via expenditure functions. While Devanur et al. [19] provided a method to construct a similar program to that given in Theorem 2 for specific utility functions, their method does not apply to arbitrary CCH utility functions. The proof of this theorem can be found in Appendix A.

**Theorem 2.** The optimal solution \((X^*, p^*)\) to the primal and dual of the following convex programs corresponds to equilibrium allocations and prices, respectively, of the CCH Fisher market \((U, b)\):

**Primal**

\[
\max_{X \in \mathbb{R}^{n \times m}} \sum_{i \in [n]} b_i \log \left( \frac{u_i(x_i)}{b_i} \right)
\]

subject to \( \forall j \in [m], \sum_{i \in [n]} x_{ij} \leq 1 \)

**Dual**

\[
\min_{p \in \mathbb{R}^n} \sum_{j \in [m]} p_j - \sum_{i \in [n]} b_i \log (\partial_{\nu_i} e_i(p, \nu_i))
\]

Our new convex program for CCH Fisher markets, which characterizes equilibrium expenditure functions, makes plain the duality structure between utility functions and expenditure functions that is used to compute “shadow” prices for allocations. In particular, \( e_i(p, \nu_i) \) is the Fenchel conjugate of the indicator function \( \chi_{\{x: u_i(x_i) \geq \nu_i\}} \), meaning the utility levels and expenditures are dual (in a colloquial sense) to one another. Therefore, equilibrium utility levels can be determined from equilibrium expenditures, and vice-versa, which implies that allocations and prices can likewise be derived from one another through this duality structure.

4 Equivalence of Mirror Descent and Tâtonnement

Cheung, Cole, and Devanur [10] have shown via the Lagrangian of the Eisenberg-Gale program, i.e., without constructing the precise dual, that the subdifferential of the dual of the Eisenberg-Gale program is equal to the negative excess demand in the associated market, which implies that mirror descent equivalent

\footnote{A more in-depth analysis of this duality structure can be found in Blume [5].}
to a subset of tâtonnement rules. In this section, we use a generalization of Shephard’s lemma to prove that the subdifferential of the dual of our new convex program is equal to the negative excess demand in the associated market. Our proof also applies to the dual of the Eisenberg-Gale program, since the two duals differ only by a constant factor.

Shephard’s lemma tells us that the rate of change in expenditure with respect to prices, evaluated at prices $p$ and utility level $\nu_i$, is equal to the Hicksian demand at prices $p$ and utility level $\nu_i$. Alternatively, the partial derivative of the expenditure function with respect to the price $p_j$ of good $j$ at utility level $\nu_i$ is simply the share of the total expenditure allocated to $j$ divided by the price of $j$, which is exactly the Hicksian demand for $j$ at utility level $\nu_i$.

While Shephard’s lemma is applicable to utility functions with singleton-valued Hicksian demand (i.e., strictly concave utility functions), we require a generalization of Shephard’s lemma that applies to utility functions that are not strictly concave and that could have set-valued Hicksian demand. An early proof of this generalized lemma was given by Tunaka in a discussion paper [32]; a more modern perspective can be found in a recent survey by Blume [3]. For completeness, we also provide a new, simple proof of this result via Danskin’s theorem (for subdifferentials) [16] in Appendix B.

**Lemma 2. Shephard’s lemma, generalized for set-valued Hicksian demand** [5, 31, 32] Let $e_i(p, \nu_i)$ be the expenditure function of buyer $i$ and $h_i(p, \nu_i)$ be the Hicksian demand set of buyer $i$. The subdifferential $\partial_p e_i(p, \nu_i)$ is the Hicksian demand at prices $p$ and utility level $\nu_i$, i.e., $\partial_p e_i(p, \nu_i) = h_i(p, \nu_i)$.

The next lemma plays an essential role in the proof that the subdifferential of the dual of our convex program is equal to the negative excess demand. Just as Shephard’s Lemma related the expenditure function to Hicksian demand via (sub)gradients, this lemma relates the expenditure function to Marshallian demand via (sub)gradients. One way to understand this relationship is in terms of the Marshallian consumer surplus, the area under the Marshallian demand curve, i.e., the integral of Marshallian demand with respect to prices.

Specifically, by applying the fundamental theorem of calculus to the left-hand side of Lemma 3, we see that the Marshallian consumer surplus equals $b_i \log (\partial_\nu e_i(p, \nu_i))$. The key takeaway is thus that any objective function we might seek to optimize that includes a buyer’s Marshallian consumer surplus is thus optimizing their Marshallian demand, so that optimizing this objective yields a utility-maximizing allocation for the buyer, constrained by their budget.

**Lemma 3.** If buyer $i$’s utility function $u_i$ is CCH, then $\partial_p (b_i \log (\partial_\nu e_i(p, \nu_i))) = d_i(p, b_i)$.

**Remark 1.** Lemma 3 makes the dual of our convex program easy to interpret, and thus sheds light on the dual of the Eisenberg-Gale program. Specifically, we can interpret the dual as specifying prices that minimize the distance between the sellers’ surplus and the buyers’ surplus. The left hand term is simply the sellers’ surplus, and by Lemma 3 the right hand term can be seen as the buyers’ total Marshallian surplus.

**Remark 2.** The lemmas we have proven in this section and the last provide a possible explanation as to why no primal-dual type convex program is known that solves Fisher markets when buyers have non-homogeneous utility functions, in which the primal describes optimal allocations while the dual describes equilibrium prices. By the homogeneity assumption, a CCH buyer can increase their utility level (resp. decrease their spending) by $c\%$ by increasing their budget (resp. decreasing their desired utility level) by $c\%$ [Lemma 5]; see Appendix A. This observation implies that the marginal expense of additional utility, i.e., “bang-per-buck”, and the marginal utility of additional budget, i.e., “buck-per-bang”, are constant [Lemma 1]. Additionally, optimizing prices to maximize buyers’ “bang-per-buck” is equivalent to optimizing prices to minimize their “buck-per-bang” [Corollary 1]. Further, optimizing prices to minimize their “buck-per-bang” is equivalent to maximizing their utilities constrained by their budgets [Lemma 3]. Thus, the equilibrium prices computed by the dual of our program, which optimize the buyers’ buck-per-bang, simultaneously optimize their utilities constrained by their budgets. In particular, equilibrium prices can be computed without reference to equilibrium allocations [Corollary 1 + Lemma 3]. In other words, assuming homogeneity, the computation of the equilibrium allocations and prices can be isolated into separate primal and dual problems.

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5 We note that the definition of Marshallian consumer surplus for multiple goods requires great care and falls outside the scope of this paper. More information on consumer surplus can be found in Levin [26], and Vives [33].
Next, we show that the subdifferential of the dual of our convex program is equal to the negative excess demand in the associated market.

**Theorem 3.** The subdifferential of the objective function of the dual of the program given in Theorem 2 for a CCH Fisher market \((U, b)\) at any price \(p\) is equal to the negative excess demand in \((U, b)\) at price \(p\):

\[
\partial_p \left( \sum_{j \in [m]} p_j - \sum_{i \in [n]} b_i \log \partial_{\nu_i} e_i(p, \nu_i) \right) = -z(p)
\]

Cheung, Cole, and Devanur \cite{Cheung:2020} define a class of markets called **convex potential function (CPF)** markets. A market is a CPF market, if there exists a convex potential function \(\phi\) such that \(\partial_p \phi(p) = -z(p)\). They then prove that Fisher markets are CPF markets by showing, through the Lagrangian of the Eisenberg-Gale program, that its dual is a convex potential function \cite{Cheung:2020}. Likewise, Theorem 3 implies the following:

**Corollary 2.** All CCH Fisher markets are CPF markets.

**Proof.** A convex potential function \(\phi : \mathbb{R}^m \to \mathbb{R}\) for any CCH Fisher market \((U, b)\) is given by:

\[
\phi(p) = \sum_{j \in [m]} p_j - \sum_{i \in [n]} b_i \log (\partial_{\nu_i} e_i(p, \nu_i))
\]

Fix a kernel function \(h\) for the Bregman divergence \(\delta_h\). If the mirror descent procedure given in Equations (10) to (11) is run on Equation (17) (i.e., choose \(f = \phi\)), it is then equivalent to the tâtonnement process for some monotonic function of the excess demand \cite{Cheung:2020}:

\[
p(t + 1) = \min_{p \in \mathbb{R}^m} \{p(t) + g(t) (p - p(t)) + \gamma_t \delta_h(p, p(t))\}
\]

for \(t = 0, 1, 2, \ldots\)

\[
g(t) \in \partial_p \phi(p(t))
\]

\[
p(0) \in \mathbb{R}^m
\]

Thus, by varying the kernel function \(h\) of the Bregman divergence we can obtain different tâtonnement rules. For instance, if \(h = \frac{1}{2}||x||_2^2\), the mirror descent process reduces to the classic tâtonnement rule given by \(G(x) = \gamma_t x\), for \(\gamma_t > 0\) and for all \(t \in \mathbb{N}\), in Equations (7) to (9).

### 5 Convergence of Discrete Tâtonnement

In this section, we conduct an experimental investigation of the rate of convergence of entropic tâtonnement, which corresponds to the tâtonnement process given by mirror descent with the scaled generalized Kullback-Leibler (KL) divergence, specifically \(6_{KL}(p, q)\), as the Bregman divergence, and a fixed step size \(\gamma\). This particular update rule, which reduces to Equations (12) to (13) has been the focus of previous work \cite{Goktas:2021}. Interest in this update rule stems from the fact that prices can never reach 0, which ensures that demands, and as a consequence, excess demands, are bounded throughout the tâtonnement process. This is because the demand for any good \(j\) is always upper bounded by \(\sum_{i \in [n]} b_{ij}\). Before presenting experimental results for entropic tâtonnement, we note that the process is not guaranteed to converge in all CCH Fisher markets. It does not converge, for example, in linear Fisher markets, where there is a slightly modified version of an example provided by Cole and Tao \cite{Cole:2021}.

**Example 1.** Consider a linear Fisher market with two goods and one buyer with utility function \(u(x_1, x_2) = x_1 + x_2\) and budget \(b = 1\). Assume initial prices of \(p_1(0) = 1\) and \(p_2(0) = e^{1/\gamma}\), for any \(\gamma > 0\). In the first iteration, the demand for the first good is 1, while demand for the second is 0. Therefore, the prices during the second iteration are \(p_1((1)) = (1)e^{1/\gamma} = e^{1/\gamma}\) and \(p_2((1)) = e^{1/\gamma}e^{-1/\gamma} = 1\). As the prices cycle, so too does tâtonnement.

\[\text{Our code can be found on } \text{https://github.com/denizalp/fisher-tatonnement.git}\]

\[\text{We thank an anonymous reviewer of an earlier version of this paper for providing this counterexample.}\]
Cheung, Cole, and Devanur \cite{10} proved a worst-case lower bound of $\Omega(1/t^2)$ to complement their $O(1/t)$ worst-case upper bound for the convergence rate of entropic tâtonnement in Leontief markets. These results suggest a possible convergence rate of $O(1/t^2)$ or $O(1/t)$ for entropic tâtonnement for a class of Fisher markets that includes Leontief markets. The goal of our experiments is to better understand the class of Fisher markets for which entropic tâtonnement converges, and to see if a worst-case convergence rate of $O(1/t^2)$ or $O(1/t)$ might hold, not only for Leontief, but for a larger class of CCH Fisher, markets.

In all our experiments, we randomly generated mixed CES Fisher markets, each with 70 buyers and 30 goods. The buyers’ values for goods, and their budgets, were drawn uniformly between 2 and 3. We drew initial prices uniformly in the range [2, 3]. In our first two experiments, we initialized 10,000 mixed CES markets, and we chose the $\rho$ parameter uniformly at random with 1/2 probability in the range [1/4, 3/4] and with 1/2 probability in the range $[-1, -101]$ \footnote{We ruled out values of $\rho$ close to 0 and 1 to ensure numerical stability.} Note that this range for $\rho$ ensures that the elasticity of demand $E$ of the market is bounded above by 4. Under these conditions, we ran the entropic tâtonnement process with a step size of $2$ in each market.

In our first set of experiments, we assigned each buyer, uniformly at random, either CES, Cobb-Douglas, or Leontief utilities, with $E \leq 4$. We observed convergence in all experiments, at the rate depicted in Figure 2a. These results suggest that the sublinear convergence rate of $O(1/t)$ could be improved to $O(1/t^2)$ for entropic tâtonnement in Leontief markets, and could perhaps even be extended to a larger class of Fisher markets, beyond Leontief. (The inner frame in Figure 2a is a closeup of iterations 0 to 10, intended to highlight that the average trajectory of the objective value throughout entropic tâtonnement decreases at a rate faster than $O(1/t^2)$.)

Second, to try to better understand the behavior of tâtonnement in mixed CCH Fisher markets where the elasticity of demand of buyers might be unbounded, we ran almost the same experiment again, but this time, each buyer was assigned, uniformly at random, either CES, Cobb-Douglas, Leontief, or linear utilities. We then checked, for each market, if the process converged. We show a sample entropic tâtonnement trajectory for one mixed CCH Fisher market in Figure 2a. We see that the objective value decreases initially (at a rate slower than $1/t^2$), but then, after about 10 iterations, it begins to oscillate. While at times the market may be tending toward an equilibrium, it is unable to settle at one.

We then ran the same experiment with buyers with linear utilities included (i.e., unbounded elasticity of demand), and found that out of 10,000 experiments, 9889 of them did not converge. This result is unsurprising in light of Example 1, since, in expectation, buyers with linear utilities make up a quarter of this market.

Finally, we ran experiments in which we varied the elasticity of demand. To do so, we ran tâtonnement in markets with elasticities of demand $E \in \{0.1, 0.2, \ldots, 0.9\}$, and we varied the step size $\gamma \in \{1, 2, \ldots, 9\}$. The results are presented in Figure 2b. In this heat map, purple signifies that all experiments converged, while yellow signifies that no experiments converged. Interestingly, as the elasticity of demand of the market increased, prices still converged, albeit only with a sufficiently large step size, thus at a slower rate.

In the light of the results of our experiments, we conjecture that tâtonnement converges at a rate of $O((1+E)/t^2)$ in CCH Fisher markets. We recall that for Leontief utilities $E = 0$, for weak gross complements markets $E \leq 1$, for weak gross substitutes markets $E \geq 1$, and for linear utilities $E = \infty$. Our conjecture thus implies that a convergence rate of $O(1/t^2)$ applies for Leontief Fisher markets, i.e., perfect complements, and that this rate deteriorates as the market’s elasticity of demand increases, ultimately leading to non-convergence in markets of perfect substitutes, i.e., linear Fisher markets. That is, the convergence rate of tâtonnement in CCH Fisher markets can be seen as a combination of the convergence rates of two types of extreme markets: perfect complements, i.e., Leontief, and perfect substitutes, i.e., linear, Fisher markets.

6 Conclusion

In this paper, we introduced a new convex program whose dual characterizes the equilibrium prices of CCH Fisher markets via expenditure functions. We also related this dual to the dual of the Eisenberg-Gale program. The dual of our program is easily interpretable, and thus allows us to likewise interpret the Eisenberg-Gale dual. In particular, while it is known that an equilibrium allocation that solves the Eisenberg-Gale program (the primal) is one that maximizes the Nash social welfare, we show that equilibrium prices—the solution to the dual—minimize the distance between the sellers’ surplus and the buyers’ Marshallian surplus. Building
on the results of Cheung, Cole, and Devanur [10], who showed that the subdifferential of the dual of the Eisenberg-Gale program is equal to the negative excess demand, we show the same for the dual of our convex program, which implies that solving our convex program via generalized gradient descent is equivalent to solving a Fisher market by means of tâtonnement.

The main technical innovation in this work is to express equilibrium prices via expenditure functions. This insight could allow us to prove the convergence of tâtonnement for more general classes of CCH utility functions, beyond CES. To this end, we ran experiments that supported the conjecture that tâtonnement converges at a rate of \( O\left(\frac{(1+E)}{t^2}\right) \) in CCH Fisher markets with elasticity of demand bounded by \( E \). If this result holds in general, it would improve upon and generalize prior results for Leontief markets to a larger class of CCH markets, which includes nested and mixed CES utilities. In future work, we plan to continue to investigate this conjecture, using the insights gained from our consumer-theoretic characterization of the equilibrium prices of Fisher markets.

We believe that our analysis offers important insights about the Eisenberg-Gale program. We observe that in CCH markets, maximizing the bang-per-buck is equivalent to minimizing the buck-per-bang, and moreover, the buck-per-bang and bang-per-buck are constant across utility levels and budgets. Additionally, optimizing prices to minimize buyers’ buck-per-bang is equivalent to maximizing their utilities constrained by their budgets. As a result, equilibrium prices can be determined by minimizing the buck-per-bang of buyers, which depends only on prices. In other words, the computation of equilibrium prices can be decoupled from the computation of equilibrium allocations. Indeed, there exists a primal-dual convex program for these markets. The challenge in solving Fisher markets where buyers’ utility functions can be non-homogeneous seems to stem from the fact that the buck-per-bang and bang-per-buck vary across utility levels and budget, which in turn means that the computation of prices and allocations cannot be decoupled. As a result, we suspect that a primal-dual convex program formulation that solves Fisher markets for buyers with non-homogeneous utility functions may not exist.

An interesting direction for future work would be to devise market dynamics that adjust allocations and prices together in search of equilibria. We believe that such dynamics may be necessary to find equilibria in Fisher markets beyond CCH (e.g., continuous and concave but not necessarily homogeneous utilities), the next frontier in this line of research. Related, to the best of our knowledge, Marshallian consumer surplus in Fisher markets is not well understood. On the contrary, Marshallian consumer surplus is mostly studied in markets with a unique good, and other than Vives [33], not much effort has been put into obtaining explicit expressions for Marshallian consumer surplus. In our work, we have shown that in CCH Fisher markets, the...
Marshallian consumer surplus can be expressed as a function of the expenditure of the buyers. It remains to be seen if a similar explicit expression can be obtained for utility functions beyond CCH. We believe that future work aimed at understanding the Marshallian consumer surplus in Fisher markets could further our understanding of these markets, perhaps beyond CCH Fisher markets.

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A Section 3 Omitted Proofs

Recall that the dual proposed by Cole and Tao [14] is given by:

$$\min_{\mathbf{p} \in \mathbb{R}^m} \sum_{j \in [m]} p_j + \sum_{i \in [n]} b_i \log \left( \max_{\mathbf{x}_i \in \mathbb{R}^m : \mathbf{x}_i \cdot \mathbf{p} \leq b_i} u_i(\mathbf{x}_i) \right)$$

This dual’s optimal differs from the optimal value of the Eisenberg-Gale program by a constant factor (of \( \sum_{i \in [n]} b_i \)) as shown by the following example:

**Example 2.** Consider a linear Fisher market with only one good and one buyer with a utility of 1 for the good and a budget of 1 as well. The equilibrium of this market is given by \( x_{11}^* = 1, p_1^* = 1 \). The primal of the Eisenberg-Gale program thus evaluates to \( b_1 \log(x_{11}^*) = (1) \log(1) = 0 \), while the dual given by Cole and Tao evaluates to \( 1 \log(1) + 1 = 1 \). Hence, the optimal primal value is not equal to the optimal dual value of the dual given by Cole and Tao, so this dual is not exactly the dual of the Eisenberg-Gale program.

We now derive the dual of the Eisenberg-Gale program. We begin with an essential lemma, which states that UMP for CCH utility functions can be expressed as an unconstrained optimization problem.

**Lemma 4.** The optimization problem

$$\max_{\mathbf{x}_i \in \mathbb{R}^m : \mathbf{x}_i \cdot \mathbf{p} \leq b_i} b_i \log(u_i(\mathbf{x}_i))$$

is equivalent to the optimization problem

$$\max_{\mathbf{x}_i \in \mathbb{R}^m} \{ b_i \log(u_i(\mathbf{x}_i)) + b_i - \mathbf{x}_i \cdot \mathbf{p} \}.$$  \hfill (22)

**Proof [Lemma 4]**. The Lagrangian associated with \( \max_{\mathbf{x}_i : \mathbf{x}_i \cdot \mathbf{p} \leq b_i} b_i \log(u_i(\mathbf{x}_i)) \) is given by:

$$L(\mathbf{x}_i, \lambda, \mu) = b_i \log(u_i(\mathbf{x}_i)) + \lambda (b_i - \mathbf{x}_i \cdot \mathbf{p}) + \mu^T \mathbf{x}_i,$$

where \( \lambda \in \mathbb{R}_+ \) and \( \mu \in \mathbb{R}^m \) are slack variables.

Let \( (\mathbf{x}_i^*, \lambda^*, \mu^*) \) be an optimal solution to the Lagrangian. From the KKT stationarity condition for this Lagrangian [25], it holds that, for all \( j \in [m] \),

$$\frac{b_i}{u_i(\mathbf{x}_i^*)} \left[ \frac{\partial u_i}{\partial x_{ij}} \right]_{\mathbf{x}_i = \mathbf{x}_i^*} x_{ij} - \lambda^* p_j + \mu_j^* = 0,$$

$$\frac{b_i}{u_i(\mathbf{x}_i^*)} \left[ \frac{\partial u_i}{\partial x_{ij}} \right]_{\mathbf{x}_i = \mathbf{x}_i^*} x_{ij} - \lambda^* p_j x_{ij} + \mu_j^* x_{ij} = 0,$$

$$\frac{b_i}{u_i(\mathbf{x}_i^*)} \left[ \frac{\partial u_i}{\partial x_{ij}} \right]_{\mathbf{x}_i = \mathbf{x}_i^*} x_{ij} - \lambda^* x_{ij} = 0.$$

The penultimate line is obtained by multiplying both sides by \( x_{ij}^* \), and the last line, by the KKT complementarity condition, namely \( \mu_j^* x_{ij}^* = 0 \).

Summing up across all \( j \in [m] \) on both sides yields:

$$\frac{b_i}{u_i(\mathbf{x}_i^*)} \sum_{j \in [m]} \left[ \frac{\partial u_i}{\partial x_{ij}} \right]_{\mathbf{x}_i = \mathbf{x}_i^*} x_{ij}^* - \lambda^* \sum_{j \in [m]} p_j x_{ij} = 0,$$

$$\frac{b_i}{u_i(\mathbf{x}_i^*)} u_i(\mathbf{x}_i^*) - \lambda^* \sum_{j \in [m]} p_j x_{ij}^* = 0,$$

$$b_i - \lambda^* b_i = 0,$$

$$\lambda^* = 1.$$
where the second line is obtained from Euler’s theorem for homogeneous functions \[27\], and the last line, from the KKT complementarity condition again, namely \( \lambda^* \left( \sum_{j \in [m]} b_i - p_j x^*_j \right) = 0 \).

Hence, plugging \( \lambda^* = 1 \) back into the Lagrangian restricted to \( \mathbb{R}^m_+ \), we get:

\[
\max_{x \in \mathbb{R}^n_+ : x \cdot p \leq b} b_i \log(u_i(x_i)) = \max_{x \in \mathbb{R}^m_+} b_i \log(u_i(x_i)) + \lambda^* (b_i - x_i \cdot p) = \max_{x \in \mathbb{R}^m_+} b_i \log(u_i(x_i)) + b_i - x_i \cdot p.
\]

With this lemma in hand, we can now derive the dual of the Eisenberg-Gale program.

**Theorem 1.** The dual of the Eisenberg-Gale program for any CCH Fisher market \((U, b)\) is given by:

\[
\min_{p \in \mathbb{R}^m_+} \sum_{j \in [m]} p_j + \sum_{i \in [n]} \left[ b_i \log(v_i(p, b_i)) - b_i \right]
\]  

(14)

**Proof (Theorem 1).** The Lagrangian dual function \( g : \mathbb{R}^m \to \mathbb{R} \) of the Eisenberg-Gale primal is given by:

\[
g(p) = \max_{X \in \mathbb{R}^{n \times m}_+} L(X, p)
\]

\[
= \max_{X \in \mathbb{R}^{n \times m}_+} \left\{ \sum_{i \in [n]} b_i \log(u_i(x_i)) + \sum_{j \in [m]} p_j \left( 1 - \sum_{i \in [n]} x_{ij} \right) \right\}
\]

\[
= \sum_{j \in [m]} p_j + \max_{X \in \mathbb{R}^{n \times m}_+} \left\{ \sum_{i \in [n]} b_i \log(u_i(x_i)) - \sum_{j \in [m]} p_j x_{ij} \right\}
\]

\[
= \sum_{j \in [m]} p_j + \sum_{i \in [n]} \max_{x \in \mathbb{R}^m_+} \left\{ b_i \log(u_i(x_i)) - p_j x_{ij} \right\}
\]

\[
= \sum_{j \in [m]} p_j + \sum_{i \in [n]} \max_{x \in \mathbb{R}^m_+: x \cdot p \leq b_i} \left\{ b_i \log(u_i(x_i)) + b_i - p \cdot x_i - b_i \right\}
\]

\[
= \sum_{j \in [m]} p_j + \sum_{i \in [n]} \left[ b_i \log\left( \max_{x \in \mathbb{R}^m_+: x \cdot p \leq b_i} u_i(x_i) \right) - b_i \right]
\]

(14)

\[
= \sum_{j \in [m]} p_j + \sum_{i \in [n]} \left( b_i \log(v_i(p, b_i)) - b_i \right)
\]

The order of the max and the sum over all buyers can be interchanged in this proof because prices are given, which renders the maximization problem for buyer \( i \) independent of that of buyer \( i' \). Therefore, the Eisenberg-Gale dual is \( \min_{p \in \mathbb{R}^m_+} g(p) = \min_{p \in \mathbb{R}^m_+} \sum_{j \in [m]} p_j + \sum_{i \in [n]} \left( b_i \log(v_i(p, b_i)) - b_i \right) \).

**Lemma 5.** Suppose that \( u_i \) is homogeneous, i.e., \( \forall \lambda > 0, u_i(\lambda x_i) = \lambda u_i(x_i) \). Then, the expenditure function and the Hicksian demand are homogeneous in \( \nu_i \), i.e., for all \( \forall \lambda > 0, c_i(p, \lambda \nu_i) = \lambda c_i(p, \nu_i) \) and \( h_i(p, \lambda \nu_i) = \lambda h_i(p, \nu_i) \). Likewise, the indirect utility function and the Marshallian demand are homogeneous in \( b_i \), i.e., for all \( \forall \lambda > 0, v_i(p, \lambda b_i) = \lambda v_i(p, b_i) \) and \( d_i(p, \lambda b_i) = \lambda d_i(p, b_i) \).

**Proof (Lemma 5).** Without loss of generality, assume \( u_i \) is homogeneous of degree 1.\(^9\)

For Hicksian demand, we have that:

\[
h_i(p, \lambda \nu_i)
\]

(23)

\(^{9}\) If the utility function is homogeneous of degree \( k \), we can use a monotonic transformation, namely take the \( k^{th} \) root, to transform the utility function into one of degree 1, while still preserving the preferences that it represents.
\[
\begin{align*}
&= \arg\min_{x_i: u_i(x_i) \geq \nu_i} p \cdot \left( \frac{x_i}{\nu_i} \right) \\
&= \lambda \arg\min_{x_i: u_i(x_i) \geq \nu_i} p \cdot \left( \frac{x_i}{\nu_i} \right) \\
&= \arg\min_{x_i: u_i(x_i) \geq \nu_i} p \cdot x_i \\
&= \lambda h_i(p, \nu_i) \quad \text{(27)}
\end{align*}
\]

The first equality follows from the definition of Hicksian demand; the second, by the homogeneity of \( u_i \); the third, by the nature of constrained optimization; and the last, from the definition of Hicksian demand again. This result implies homogeneity of the expenditure function in \( \nu_i \):

\[
e_i(p, \lambda \nu_i) = h_i(p, \lambda \nu_i) \cdot p = \lambda h_i(p, \nu_i) \cdot p = \lambda e_i(p, \nu_i)
\]

The first and last equalities follow from the definition of the expenditure function, while the second equality follows from the homogeneity of Hicksian demand (Equation (27)).

The proof in the case of Marshallian demand and the indirect utility function is analogous.

**Lemma 1.** If \( u_i \) is continuous and homogeneous of degree 1, then \( v_i(p, b_i) \) and \( e_i(p, \nu_i) \) are differentiable in \( b_i \) and \( \nu_i \), resp. Further, \( \partial_{b_i} v_i(p, b_i) = \{ v_i(p, 1) \} \) and \( \partial_{\nu_i} e_i(p, \nu_i) = \{ e_i(p, 1) \} \).

**Proof (Lemma 1).** We prove differentiability from first principles:

\[
\lim_{h \to 0} \frac{e_i(p, \nu_i + h) - e_i(p, \nu_i)}{h} = \lim_{h \to 0} \frac{e_i(p, (1)(\nu_i + h)) - e_i(p, (1)\nu_i)}{h} = \lim_{h \to 0} \frac{e_i(p, (1)(\nu_i + h) - e_i(p, (1)\nu_i)}{h} = \lim_{h \to 0} \frac{e_i(p, (1)(\nu_i + h - \nu_i))}{h} = \lim_{h \to 0} \frac{e_i(p, (1)(h))}{h} = e_i(p, 1)
\]

The first line follows from the definition of the derivative; the second line, by homogeneity of the expenditure function (Lemma 5), since \( u_i \) is homogeneous; and the final line follows from the properties of limits. The other two lines follow by simple algebra.

Hence, as \( e_i(p, \nu_i) \) is differentiable in \( \nu_i \), its subdifferential is a singleton with \( \partial_{\nu_i} e_i(p, \nu_i) = \{ e_i(p, 1) \} \). The proof of the analogous result for the indirect utility function’s derivative with respect to \( b_i \) is similar.

**Corollary 1.** If buyer i’s utility function \( u_i \) is CCH, then

\[
\frac{1}{e_i(p, 1)} = \frac{1}{\partial_{\nu_i} e_i(p, \nu_i)} = \partial_{b_i} v_i(p, b_i) = v_i(p, 1).
\]

**Proof (Corollary 1).** By Lemma 1 we know that \( e_i(p, \nu_i) \) is differentiable in \( \nu_i \) and that \( \partial_{\nu_i} e_i(p, \nu_i) = \{ e_i(p, 1) \} \). Similarly, by Lemma 1 we know that \( \partial_{b_i} v_i(p, b_i) \) is differentiable in \( b_i \) and that \( \partial_{b_i} v_i(p, b_i) = \{ v_i(p, 1) \} \). Combining these facts yields:

\[
\partial_{b_i} e_i(p, \nu_i) \cdot \partial_{b_i} v_i(p, b_i) = e_i(p, 1) \cdot v_i(p, 1) = e_i(p, v_i(p, 1)) = 1
\]

Therefore, \( \frac{1}{\partial_{\nu_i} e_i(p, \nu_i)} = \partial_{b_i} v_i(p, b_i) \). Combining this conclusion with Lemma 1 we obtain the result.
Lemma 6. Given a CCH Fisher market \((U, b)\), the dual of our convex program (Theorem 2) and that of Eisenberg-Gale differ by a constant, namely \(\sum_{i \in [n]} (b_i \log b_i - b_i)\). In particular,

\[
\min_{p \in \mathbb{R}_+^m} \left\{ \sum_{j \in [m]} p_j - \sum_{i \in [n]} b_i \log (\partial_{\nu_i} c_i(p, \nu_i)) \right\} = \min_{p \in \mathbb{R}_+^m} \sum_{j \in [m]} p_j + \sum_{i \in [n]} (b_i \log (v_i(p, b_i)) - b_i) - \sum_{i \in [n]} (b_i \log b_i - b_i)
\]

Proof [Lemma 6].

\[
\min_{p \in \mathbb{R}_+^m} \sum_{j \in [m]} p_j + \sum_{i \in [n]} (b_i \log (v_i(p, b_i)) - b_i) = \min_{p \in \mathbb{R}_+^m} \sum_{j \in [m]} p_j + \sum_{i \in [n]} b_i \log (v_i(p, 1)) - \sum_{i \in [n]} b_i
\]

(\text{Lemma 5})

\[
= \min_{p \in \mathbb{R}_+^m} \left\{ \sum_{j \in [m]} p_j + \sum_{i \in [n]} b_i \log (v_i(p, 1)) \right\} - \sum_{i \in [n]} b_i \log b_i - \sum_{i \in [n]} b_i
\]

\[
= \min_{p \in \mathbb{R}_+^m} \left\{ \sum_{j \in [m]} p_j - \sum_{i \in [n]} b_i \log \left( \frac{1}{v_i(p, 1)} \right) \right\} + \sum_{i \in [n]} b_i \log b_i - \sum_{i \in [n]} b_i
\]

(\text{Corollary 1})

\[
= \min_{p \in \mathbb{R}_+^m} \left\{ \sum_{j \in [m]} p_j - \sum_{i \in [n]} b_i \log (\partial_{\nu_i} c_i(p, \nu_i)) \right\} + \sum_{i \in [n]} b_i \log b_i - \sum_{i \in [n]} b_i
\]

(\text{Lemma 1})

Theorem 2. The optimal solution \((X^*, p^*)\) to the primal and dual of the following convex programs corresponds to equilibrium allocations and prices, respectively, of the CCH Fisher market \((U, b)\):

\[
\text{Primal} \qquad \max_{x \in [X^\ast]_{\ast}} \sum_{i \in [n]} b_i \log \left( \frac{u_i(x_i)}{b_i} \right) \quad \text{subject to} \quad \forall j \in [m], \sum_{i \in [n]} x_{ij} \leq 1
\]

\[
\text{Dual} \qquad \min_{p \in \mathbb{R}_+^m} \sum_{j \in [m]} p_j - \sum_{i \in [n]} b_i \log (\partial_{\nu_i} c_i(p, \nu_i))
\]

Proof [Theorem 2]. By Lemma 6 our dual and the Eisenberg-Gale dual differ by a constant, which is independent of the decision variables \(p \in \mathbb{R}_+^m\). Hence, the optimal prices \(p^*\) of our dual are the same as those of the Eisenberg-Gale dual, and thus correspond to equilibrium prices in the CCH Fisher market \((U, b)\). Finally, the objective function of our convex program’s primal is:

\[
\sum_{i \in [n]} b_i \log (u_i(x_i)) - \sum_{i \in [n]} (b_i \log b_i - b_i) = \sum_{i \in [n]} b_i \log u_i \left( \frac{x_i}{b_i} \right) + \sum_{i \in [n]} b_i
\]

B Danskin’s Theorem Section 4 Omitted Proofs

Danskin’s theorem [16] offers insights into optimization problems of the form: \(\min_{x \in X} f(x, p)\), where \(X \subset \mathbb{R}^m\) is compact and non-empty. Among other things, Danskin’s theorem allows us to compute the subdifferential of value of this optimization problem with respect to \(p\).
Theorem 4 (Danskin’s Theorem [16]). Consider an optimization problem of the form: \( \min_{x \in X} f(x, p) \), where \( X \subset \mathbb{R}^m \) is compact and non-empty. Suppose that \( X \) is convex and that \( f \) is concave in \( x \). Let \( V(p) = \min_{x \in X} f(x, p) \) and \( X^*(p) = \arg\min_{x \in X} f(x, p) \). Then the subdifferential of \( V \) at \( \mathbf{\hat{p}} \) is given by \( \partial_p V(\mathbf{\hat{p}}) = \{ \nabla_p f(x^*(\mathbf{\hat{p}}), \mathbf{\hat{p}}) \mid x^*(\mathbf{\hat{p}}) \in X^*(\mathbf{\hat{p}}) \} \).

**Lemma 2. Shephard’s lemma, generalized for set-valued Hicksian demand** [5, 31, 32]. Let \( e_i(p, \nu_i) \) be the expenditure function of buyer \( i \) and \( h_i(p, \nu_i) \) be the Hicksian demand set of buyer \( i \). The subdifferential \( \partial_p e_i(p, \nu_i) \) is the Hicksian demand at prices \( p \) and utility level \( \nu_i \), i.e., \( \partial_p e_i(p, \nu_i) = h_i(p, \nu_i) \).

**Proof.** Recall that \( e_i(p, \nu_i) = \min_{x \in X : u_i(x) \geq \nu_i} p \cdot x \). Without loss of generality, we can assume that consumption set is bounded from above, since utilities are assumed to represent locally non-satiated preferences, i.e., \( \min_{x \in X : u_i(x) \geq \nu_i} p \cdot x \) where \( X \subset \mathbb{R}^m_+ \) is compact. Using Danskin’s theorem:

\[
\partial_p e_i(p, \nu_i) = \left\{ \nabla_p (p \cdot x) (x^*(p, \nu_i)) \mid x^*(p, \nu_i) \in h_i(p, \nu_i) \right\} \quad \text{(Danskin’s Thm)}
\]

The first equality follows from Danskin’s theorem, using the facts that the objective of the expenditure minimization problem is affine and the constraint set is compact. The second equality follows by calculus, and the third, by the definition of Hicksian demand.

**Theorem 3.** The subdifferential of the objective function of the dual of the program given in Theorem 2 for a CCH Fisher market \( (U, b) \) at any price \( p \) is equal to the negative excess demand in \( (U, b) \) at price \( p \):

\[
\partial_p \left( \sum_{j \in [m]} p_j - \sum_{i \in [n]} b_i \log \partial_{\nu_i} e_i(p, \nu_i) \right) = -z(p)
\]

**Proof.** For all goods \( j \in [m] \), we have:

\[
\partial_{p_j} \left( \sum_{j \in [m]} p_j - \sum_{i \in [n]} b_i \log \partial_{\nu_i} e_i(p, \nu_i) \right) = \{1\} - \partial_{p_j} \left( \sum_{i \in [n]} b_i \log \partial_{\nu_i} e_i(p, \nu_i) \right) = \{1\} - \sum_{i \in [n]} \partial_{p_j}(b_i \log \partial_{\nu_i} e_i(p, \nu_i)) = \{1\} - \sum_{i \in [n]} \partial_{p_j}(z_i(p, b_i)) = -z_j(p)
\]

**Lemma 3.** If buyer \( i \)'s utility function \( u_i \) is CCH, then \( \partial_p (b_i \log (\partial_{\nu_i} e_i(p, \nu_i))) = d_i(p, b_i) \).

**Proof.** Without loss of generality, we can assume \( u_i \) is homogeneous of degree 1. Then:

\[
\partial_p (b_i \log (\partial_{\nu_i} e_i(p, \nu_i))) = \left( \frac{b_i}{\partial_{\nu_i} e_i(p, \nu_i)} \right) \partial_p (\partial_{\nu_i} e_i(p, \nu_i)) = b_i \left( \partial_{b_i} v_i(p, b_i) \right) \partial_p v_i(p, b_i) \partial_p \left( \partial_{\nu_i} e_i(p, \nu_i) \right) \quad \text{(Corollary 1)}
\]

\[
= b_i \left( \partial_{b_i} v_i(p, b_i) \right) \partial_p v_i(p, b_i) \partial_p (\partial_{\nu_i} e_i(p, \nu_i)) \quad \text{(Lemma 1)}
\]

\[
= b_i \left( \partial_{b_i} v_i(p, b_i) \right) h_i(p, 1) \quad \text{(Shephard’s Lemma)}
\]

\[
= b_i v_i(p, b_i) h_i(p, 1) \quad \text{(Lemma 1)}
\]

\[
= v_i(p, b_i) h_i(p, 1) \quad \text{(Lemma 5)}
\]

\[
= h_i(p, v_i(p, b_i)) \quad \text{(Lemma 5)}
\]

\[
= d_i(p, b_i) \quad \text{(Equation (3))}
\]