On stability of a new model of wormhole

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(Dated: December 21, 2013)

Abstract

We investigate the stability of the wormhole of the Mors-Thorne-Ellis-Bronnikov type. In our model the matter inside it consists of a radial magnetic field and an ideal phantom-fluid. Properties of the matter are described in section II and III of this paper. We consider spherical perturbations only and find examples of the stable wormholes against these perturbations.

I. INTRODUCTION

The problem of the stability of the wormholes has been discussed in many papers (see [1–10]).

The simplest model of a wormhole [11–13] (phantom scalar field with the negative kinetic term) proved to be unstable (in spite of erroneous papers [1, 2]) and in accordance with more recent (independent) correct research [3–7, 14].

A more complex model of a wormhole (with the same metric [6, 7] — radial magnetic field and the phantom dust with the negative energy density) is almost stable under all types of the spherical perturbation (except for the longitudinal radial motion of the dust on the inertia). The growth of unstable type is sufficiently slow: it increases proportionally to the time [8, 9]. Therefore, the authors of this paper it has been suggested that the unstable type can be easily crushed by the introduction into the model additional parameters. In the paper [16] it was found a stable thin-shell traversable wormholes model, with a thin shell of

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phantom matter on its throat (by using the cut-and-paste procedure).

All of these studies provided hope for the existence and finding fully stable solution (under all spherical perturbation types).

In this paper we investigate the stability (under the spherical perturbations) the similar model of the Mors-Thorne-Ellis-Bronnikov type wormhole (MTEB), see also [15].

The matter of our model of the wormhole (WH) consists of the radial monopole magnetic field (with topological charge $q$) and the ideal phantom fluid. In the stationary case (respect to which will be a study on stability) the energy density $\varepsilon$ of this fluid is negative and equal to twice of the absolute value of the energy density of the magnetic field. The pressure $p$ of the fluid in a stationary case is equal to zero (phantom dust). In the case of deviation of the energy density $\varepsilon$ from its stationary value — the pressure $p$ is proportional to this deviation $f \propto \varepsilon - \varepsilon_0$. As already mentioned, without pressure, this model is linearly unstable over time [8, 9].

II. EQUATIONS OF THE MODEL

It is convenient to choose the metric tensor in the spherically symmetric case like the following$^1$:

$$ds^2 = e^{\nu} dt^2 - e^{\lambda} dx^2 - e^{\alpha}(d\theta^2 + \sin^2 \theta d\varphi^2).$$  \hspace{1cm} (1)

Here $e^{\alpha} = r^2$, where $4\pi r^2$ - area of a sphere around the center of the system, $r$, $\nu$ and $\lambda$ are functions $x$ and $t$.

The Einstein equations corresponding to the metric (1), in the comoving reference sys-

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$^1$ Units are chosen as: $c = 1$ and $G = 1$ (the speed of light and gravitational constant).
The stationary wormhole MTEB metric tensor is defined by

$$ds^2 = dt^2 - dx^2 - (q^2 + x^2) \cdot (d\theta^2 + \sin^2 \theta \, d\varphi^2). \tag{6}$$

This WH has the energy-momentum tensor which corresponds to a mixture of monopole electric (or magnetic) field and the phantom dust:

$$8\pi T_m^m = \text{diag} \left\{ \frac{+q^2}{(q^2 + x^2)^2}, \frac{+q^2}{(q^2 + x^2)^2}, \frac{-q^2}{(q^2 + x^2)^2}, \frac{-q^2}{(q^2 + x^2)^2} \right\} + \text{diag} \left\{ \frac{-2q^2}{(q^2 + x^2)^2}, 0, 0, 0 \right\} \tag{7}$$

The first term on the right side corresponds to the energy-momentum tensor of the electric (or magnetic) field with a charge $q$; second - dust matter with negative energy density.

## III. LINEARIZATION OF THE EQUATIONS

Let us consider small spherical perturbations of matter and metrics of the MTEB wormhole.

We introduce the notations:

$$8\pi \varepsilon = -\frac{2q^2}{(q^2 + x^2)^2} + f(x,t), \quad e^\alpha = (q^2 + x^2)e^{\eta(x,t)}, \quad \xi^2 = (q^2 + x^2), \quad 8\pi p = hf. \tag{8}$$

Here $h(x)$ — arbitrary function, its physical meaning — a square of sound speed $v_s^2$ (in this fluid).

We introduce the dimensionless coordinates: namely we put $q = 1$.

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2 The derivation of these equations can be found, for example in [17] (task 5 in §100), prime denotes the derivative with respect to $x$, and dot - with respect to $t$. 
We write the equations (2-5) in the linear approximation for small perturbations: \( \nu, \lambda, \eta, f \):

\[
f = \frac{\lambda - \eta - 3x \eta' + x \lambda'}{\xi^2} - \eta'' + \frac{2\eta + \lambda}{\xi^4},
\]

(9)

\[
hf + \ddot{\eta} + \frac{\eta(1 - x^2)}{\xi^4} + \frac{\lambda x^2}{\xi^4} - \frac{x(\nu' + \eta')}{\xi^2} = 0,
\]

(10)

\[
hf = \frac{\nu'' + \eta'' - \ddot{\lambda} - \ddot{\eta}}{2} + \frac{2\eta - \lambda}{\xi^4} + \frac{x(\eta' - \lambda'/2 + \nu'/2)}{\xi^2},
\]

(11)

\[
\dot{\eta}' + x(\dot{\eta} - \dot{\lambda})/\xi^2 = 0.
\]

(12)

From (12) we have:

\[
\eta - \lambda = F_1(x) - \frac{\xi^2}{x} \eta'
\]

(13)

If the pressure \( p \) is isotropic, then from equations (2-5) we can obtain two useful relations which derived directly from the formula \( T^{i}_{k, i} = 0 \) (consequence of Bianchi identities):

\[
\dot{\lambda} + 2\dot{\alpha} = -2\dot{\varepsilon}/(p + \varepsilon), \quad \nu' = -2p'/(p + \varepsilon).
\]

(14)

Or in the linear approximation:

\[
\dot{\lambda} + 2\dot{\eta} = \xi^4 \dot{f}, \quad \nu' = \xi^4(hf)'.
\]

(15)

From (15) we have:

\[
\xi^4f = \lambda + 2\eta + F_2(x)
\]

(16)

Expressing \( \lambda \) from (13), and taking into account (16) from (9) we have:

\[
F_2(x) = -\xi^2(x F_1)'
\]

(17)

The arbitrary functions \( F_1(x) \) and \( F_2(x) \) are determined by the choice of initial conditions (at \( t = 0 \)) to small perturbations \( \lambda, f \) and \( \nu \) (for given initial condition at \( \eta(x, t) \)). Therefore, renaming at \( t = 0 \) in (13) and (16) for small perturbations: \( \lambda \rightarrow \bar{\lambda} - F_1 \) and \( f \rightarrow \bar{f} + F_2/\xi^4 \) then we can put the function \( F_1 \) and \( F_2 \) equal to zero.

From (9), (10) and (13) we have:

\[
f = 3\eta/\xi^4 + \eta'/(x \xi^2) = [\xi^3 \eta]'/(x \xi^5),
\]

(18)

\[
\ddot{\eta} + hf - x\xi^2(hf)' + \eta/\xi^4 = 0,
\]

(19)

\[
\lambda = \frac{\xi}{x} [\xi \eta']'.
\]

(20)
From the equations (18-19) we obtain:

\[ \ddot{\eta} - h\eta'' + \eta' \left[ \frac{2h}{x\xi^2} - h' \right] + \eta U(x) = 0 \tag{21} \]

where

\[ U(x) \equiv \frac{12hx^2 + 3h - 3h'x\xi^2 + 1}{\xi^4} \tag{22} \]

Equation (21) at \( h > 0 \) is a differential equation of hyperbolic type. At infinity \( (x \to \pm\infty) \) the potential \( U(x) \) tends to zero and equation (21) becomes an equation for sound waves with the speed of sound \( v_s = \sqrt{h} \).

**IV. STABILITY INVESTIGATION**

We transform the equation (21) to the canonical form. To do this we change variables \( x \to z \). Then \( \partial_x \to \beta \partial_z \), where \( \beta \equiv \frac{\partial z}{\partial x} \). Equation (21) can be rewritten as:

\[ \ddot{\eta} - \left[ h\beta^2 \eta_{zz} + h\beta' \eta_z - \beta \eta_z \left( \frac{2h}{x\xi^2} - h' \right) \right] + \eta U = 0 \tag{23} \]

Now choose a function \( h(x) \) and \( \beta(x) \), so that in the square brackets expressions (23) was only value of \( \eta_{zz} \), i.e. \( h\beta^2 = 1 \) and \( h\beta' - \beta[2h/(x\xi^2) - h'] = 0 \).

From these conditions we obtain \( h = h_0 x^4/\xi^4 \), where \( h_0 = \text{const} > 0 \).

Then the hyperbolic equation (21) can be rewritten in the canonical form:

\[ \ddot{\eta} - \eta_{zz} + \eta U(z) = 0 \tag{24} \]

where

\[ U(z) = \frac{12h_0x^6 - 9h_0x^4 + (1 + x^2)^2}{(1 + x^2)^4} \tag{25} \]

Here we took into account that now the variable \( x \) depends from the variable \( z \) by the following equation:

\[ x^2 - \sqrt{h_0}zx - 1 = 0 \tag{26} \]

and

\[ x^\pm(z) = \sqrt{h_0}z \pm \frac{\sqrt{h_0}z^2 + 4}{2}, \tag{27} \]
Here the sign "+" corresponds to the \( x > 0 \), and the sign "−" — respectively \( x < 0 \). We first investigate the region \( x > 0 \).

Asymptotics of \( z \to -\infty \) corresponds to \( x \to (\sqrt{h_0}|z|)^{-1} \to 0 \), and asymptotics \( z \to +\infty \) corresponds to \( x \to (\sqrt{h_0}z) \to +\infty \).

The solution of equation (24) can be obtained by separation of variables:

\[
\eta(z, t) = \sum_{n=0}^{\infty} T_n(t)\Psi_n(z),
\]

\[
\frac{\ddot{T}_n}{T_n} = \frac{\Psi_{n,zz}}{\Psi_n} - U(z) = -w_n^2.
\]

From (29) we obtain: \( T_n = \exp(iw_n t) \). Here, the quantity \( w_n = \text{const} \) has a physical sense to the harmonic oscillation frequency with the number \( n \), for the small perturbation \( \eta \).

From (29) we obtain for each harmonic:

\[
\Psi_{n,zz} + w_n^2\Psi_n - U(z)\Psi_n = 0
\]

The expression (30) is a stationary Schrödinger equation for the whole numerical axis \((-\infty, +\infty)\) with the potential \( U(z) \).

As is well known (see. [18], §18) energy levels \( E_n = (w_n h)^2/2 \) in the Schrödinger operator spectrum are always positive, if the effective potential \( U(z) \) is regular and non-negative (for all \( z \)) and \( U(z) \to 0 \) at \( z \to \infty \). From (24) follows that \( U(z) \) satisfies these conditions at \( 0 < h_0 < h_{00} \approx 2.8 \) — see figure. However, the physical constraint on the parameter \( h_0 \) is the maximum speed of sound, which can not exceed the speed of light. Since the square of the speed of sound is \( \frac{dp}{d\epsilon} = h \leq c^2 \), then there must be \( h_{\max} = h_0 \leq 1 \).

The result is \( w_n^2 \geq 0 \), ie oscillation frequency must be real value.

The proof for region \( x < 0 \) is analogous.

Thus we demonstrated the existence of the matter model in which the function \( \eta(x, t) \) is a nonincreasing.

Since each term of (28) nonincreasing for time evolution, there must be a non-increasing, and the derivative by \( x \) from each member of this series (as derivative by \( x \) does not affect on the time component \( T_n \), which responsible for the time dependence). Therefore, we can make a statement that the quantity \( \eta'(x, t) \) well as nonincreasing (at \( 0 < h_0 \leq 1 \)). Therefore, the functions \( \nu(x, t) \), \( f(x, t) \) and \( \lambda(x, t) \) are nonincreasing (according to expressions (15, 18, 20)). Thus, the complete solution is stable.
FIG. 1: View of the surface for the potential $U(x, h_0)$ for $h(x) = h_0 x^4/\xi^4$. This shows that region $U > 0$ corresponds to the range $0 < h_0 < h_{00} \approx 2.8$ for any values of $x$.

The case $h = 0$ should be considered individually: Eq. (23) at $h = 0$ becomes an equation of parabolic type and the variables $(x, t)$ are not separated, as decomposition in the form (28) becomes inapplicable. The solution of equation (21) with $h = 0$ is easily found in the form: $\eta_{h=0}(x, t) = \eta_0 \exp \left( \frac{it}{\xi^2} \right)$. This solution is also stable. However, the derivative of this solution $\eta'_{h=0} = -2itx_0 \eta_{h=0}/\xi^4$ at $x \neq 0$ is not asymptotically stable function, and has a linear instability with respect to time. Consequently, the functions $\lambda_{h=0}(x, t)$ and $f_{h=0}(x, t)$ at $x \neq 0$ are also linearly unstable (see. [8, 9]).

V. CONCLUSIONS

We have demonstrated that in general relativity, it is possible to construct a model of static and traversable wormhole, which will be stable with respect to small spherical perturbations.

In this paper we do not perform investigation on the stability by non-spherical perturbation types, but it is known (see. [19]), that the non-spherical perturbation types, seems to be more stable than the spherical, because they have a centrifugal (and other higher-multipole)
barriers in the effective potential for perturbations.

Acknowledgements

The authors are grateful to Kirill Bronnikov, Kip Thorne and Aaron Zimmerman for discussions.

This work was supported in part by the Federal Program “Scientific-Pedagogical Innovational Russia 2009-2011” and the program by the presidium of RAS “The origin, structure and evolution of the universe 2011”.

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