The puzzle of bulk conformal field theories at central charge $c = 0$

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Non-trivial critical models in 2D with central charge $c = 0$ are described by Logarithmic Conformal Field Theories (LCFTs), and exhibit in particular mixing of the stress-energy tensor with a “logarithmic” partner under a conformal transformation. This mixing is quantified by a parameter (usually denoted $b$), introduced in [V. Gurarie, Nucl. Phys. B 546, 765 (1999)]. The value of $b$ has been determined over the last few years for the boundary versions of these models: $b_{\text{perco}} = -\frac{5}{4}$ for percolation and $b_{\text{poly}} = \frac{5}{6}$ for dilute polymers. Meanwhile, the existence and value of $b$ for the bulk theory has remained an open problem. Using lattice regularization techniques we provide here an “experimental study” of this question. We show that, while the chiral stress tensor has indeed an “experimental study” of this question. We show that, while the chiral stress tensor has indeed an “experimental study” of this question. We show that, while the chiral stress tensor has indeed an “experimental study” of this question. We show that, while the chiral stress tensor has indeed an “experimental study” of this question. We show that, while the chiral stress tensor has indeed an “experimental study” of this question. We show that, while the chiral stress tensor has indeed an “experimental study” of this question. We show that, while the chiral stress tensor has indeed an “experimental study” of this question. We show that, while the chiral stress tensor has indeed an “experimental study” of this question. We show that, while the chiral stress tensor has indeed an “experimental study” of this question. We show that, while the chiral stress tensor has indeed an “experimental study” of this question. We show that, while the chiral stress tensor has indeed an “experimental study” of this question. We show that, while the chiral stress tensor has indeed an “experimental study” of this question. We show that, while the chiral stress tensor has indeed an “experimental study” of this question. We show that, while the chiral stress tensor has indeed an “experimental study” of this question. We show that, while the chiral stress tensor has indeed an “experimental study” of this question. We show that, while the chiral stress tensor has indeed an “experimental study” of this question. We show that, while the chiral stress tensor has indeed an “experimental study” of this question. We show that, while the chiral stress tensor has indeed an “experimental study” of this question. We show that, while the chiral stress tensor has indeed an “experimental study” of this question. We show that, while the chiral stress tensor has indeed an “experimental study” of this question. We show that, while the chiral stress tensor has indeed an “experim...
logarithmic partner \( t(z, \bar{z}) \) such that in the basis \( (T, t) \), the generator of the scale transformation reads
\[
L_0 + \bar{L}_0 = \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}.
\] (1)

We will show later (see also [23]) that the non-diagonalizable term can be decomposed as \( L_0 t = 2 t + T \) and \( L_0 \bar{t} = T \), so that, unlike \( T(z) \) which is purely holomorphic, the field \( t(z, \bar{z}) \) has a non-trivial antiholomorphic part \( \bar{\partial} t \neq 0 \). Using this normalization, we define the number \( b \) as
\[
b = \langle T|t \rangle ,
\] (2)
where we have used the bilinear Virasoro form. Note also that \( L_1 t = 0 \) and \( L_2 t = b I \). Using these relations, global conformal invariance fixes \( b \) the form of the two-point functions to be
\[
\langle T(z)T(0) \rangle = 0
\] (3a)
\[
\langle T(z)t(0,0) \rangle = \frac{b}{z^4}
\] (3b)
\[
\langle t(z,\bar{z})t(0,0) \rangle = \frac{\theta - 2 b \log |z|^2}{z^4},
\] (3c)
where \( \theta \) is an irrelevant constant that can absorbed into a redefinition of the fields, whereas \( b \) is a fundamental number characterizing the logarithmic pair \((T,t)\). We have similar equations for the antiholomorphic counterpart \((\bar{T},\bar{t})\), with also \( \langle \bar{t}\bar{t} \rangle = \langle T\bar{T} \rangle = 0 \).

### Lattice loop models

Our aim is to measure \( b \) in percolation and dilute polymers with PBC. An essential difficulty consists in defining the models; since, for instance, percolation presents many non-local (or apparently non-local) aspects, the consideration of different geometrical features might lead to different, mutually inconsistent, field theoretic descriptions. A safe way out is to use supersymmetry [24], but to keep things simple, we consider—as is quite standard—the observables encoded in a gas of loops, where each loop carries a weight \( n \).

It is well-known how to rewrite the \( Q \)-state Potts model in terms of Fortuin-Kasteleyn (FK) clusters (see e.g. [25]), whose hulls are dense loops with \( n = \sqrt{Q} \). Percolation then arises as the \( n \to 1 \) limit. We choose PBC in both space and imaginary time directions. The corresponding correlations can be obtained using a transfer matrix, or, in the very anisotropic limit, a quantum Hamiltonian in \( 1+1D \), \( H = -\sum_{i=1}^{2N} e_i \), where \( L = 2N \) is the length of the system in the space direction and the \( e_i \) are Temperley-Lieb (TL) generators. These are represented by the contraction \( e_i = \mathcal{X} \) of the sites \( i \) and \( i + 1 \) on the cylinder lattice in Fig. 1. The different sectors of the “Hilbert” space of the Hamiltonian are labeled by the number of through-lines \( 2j \) propagating along the imaginary time direction. Examples of states in the \( j = 1 \) sector on \( L = 6 \) sites are given by \( |\alpha \rangle = \{\bigcup_{j} \{1\}\} \) or \( |\beta \rangle = \{\bigcup_{j} \{\bar{1}\}\} \) imposing PBC. The action of the \( e_i \) can produce winding of through-lines around the cylinder which we replace by a weight \( 1 \); note also that non-contractible and contractible loops have the same weight \( n = 1 \) [24, 25]. Under the action of \( H \), the number of through-lines can only decrease, since \( e_i \) corresponds to a contraction. \( H \) is thus a lower block-triangular matrix which corresponds in the scaling limit to \( L_0 + \bar{L}_0 \) and we expect to see Jordan cells on a finite lattice. In order to study Jordan cells for percolation, we use the same prescription as in the open case (see [14]).

To describe dilute polymers, we use the integrable version of the \( O(n \to 0) \) model on the square lattice [28] which yields a model of dilute loops with fugacity \( n = 0 \). We can write the Hamiltonian as a sum of local densities as before, \( H = -\sum_{i} e_i \), where the operator \( e_i \) is now more complicated (see e.g. [15]). In this case \( j \) denotes the number of through-lines. The states are the same as in the dense case, except that we now allow for empty sites. Examples of configurations are shown in Fig. 1.

### Lattice Jordan cells and scalar products

Both models have trivial partition functions, so their scaling limit is described by conformal theories with \( c = 0 \). The identity operator corresponds to the groundstate of \( H \), henceforth denoted \( |0(N)\rangle \), and can be found in the vacuum \( (j = 0) \) sector. The state corresponding to the stress-energy tensor \( T(z) \) is the only state with conformal weights \( (h, \bar{h}) = (2,0) \) in the vacuum sector. The scaling dimension \( \Delta = h + \bar{h} \) of a given excitation \( \phi \) with energy \( E_\phi(N) \) can be measured numerically from the scaled energy gap, \( E_\phi(N) - E_0(N) = 2\pi v_F^2 \Delta + \ldots \), where \( E_0(N) \) is the energy of the groundstate and \( v_F \) is the Fermi velocity. Bethe ansatz results [23, 30] yield \( v_F = \frac{2\sqrt{3}}{3} \) for percolation and \( v_F = \frac{2}{3} \) for dilute polymers. Both the Hamiltonian and the transfer matrix commute with the operator \( u^2 \) that translates the sites two units to the right. The restriction to a given conformal spin

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**FIG. 1:** Example of dense (left panel) and dilute (right panel) loop configurations. We use periodic boundary conditions in both directions so that the loops actually live on a torus. We also show the corresponding FK clusters for the dense loop model. Each closed loop has a weight \( n \).
\[ s = h - \bar{h} \] can be done using this translational invariance. We can build eigenspaces for \( u^2 \) and introduce a momentum \( k \) to characterize a given sector. It can be argued that the states with conformal spin \( s \) belong to the sector with momentum \( k = \frac{2\pi s}{\nu} \). Using these results, one can readily extract the states with conformal weights \((h, \bar{h}) = (2, 0)\) from the spectrum. We find numerically that the state \(|T^{(N)}\rangle\), which goes to \(|T\rangle = T(z \to 0)|0\rangle\) in the scaling limit \(N \to \infty\) up to normalization issues to be discussed below, is mixed into a rank-2 Jordan cell with its “logarithmic partner” \(|t^{(N)}\rangle\). Note that \(|t^{(N)}\rangle\) belongs to the \( j = 2 \) sector. We normalize the states so that the Hamiltonian in the basis \(|T^{(N)}\rangle, |t^{(N)}\rangle\) reads

\[
H - E_{0}(N) \mathbf{1} = \frac{2\pi v_F}{N} \begin{pmatrix} \Delta^{(N)} & 2 \\ 0 & \Delta^{(N)} \end{pmatrix},
\]

where \(\Delta^{(N)} = \frac{2N}{\pi v_F} (E_{T}(N) - E_{0}(N))\). In order to measure \(b = \langle t|t\rangle\), we also need a “scalar product” which goes to the Virasoro bilinear form in the scaling limit. This issue was already discussed in Refs. [4, 14, 12] and is not modified by the PBC: the natural definition of the scalar product for such loop models corresponds to gluing the mirror image of the first state on top of the second one.

**Virasoro algebra regularization.** It could be tempting to define a lattice version of \(b\) as the scalar product \(\langle t^{(N)}|T^{(N)}\rangle\). However, the Jordan cell in the Hamiltonian is under a simultaneous rescaling of the states \(|T^{(N)}\rangle\) and \(|t^{(N)}\rangle\). Therefore, we need to carefully normalize \(|T^{(N)}\rangle\) so that it goes precisely to the stress-energy tensor in the scaling limit \([11]\); since \(\langle T|T\rangle = 0\), the problem is non-trivial, and has in fact hindered for many years the numerical determination of \(b\). Like in the chiral case \([12]\), we control normalization by using a lattice regularization \(L_{n}^{(N)}, \bar{L}_{n}^{(N)}\) of the Virasoro generators \([31]\). Introducing the notation \(H_{n}^{(N)} = L_{n}^{(N)} + \bar{L}_{n}^{(N)}\), arguments such as those in \([31]\) lead, for both our dense and dilute loop models, to

\[
H_{n}^{(N)} = -\frac{N}{\pi v_F} \sum_{j=1}^{2N} e^{i\pi j/N} (e_i - e_{\infty}) + \frac{c}{12} \delta_{n,0},
\]

where \(e_{\infty} = 1\) (resp. \(e_{\infty} = \sqrt{2}\)) for percolation (resp. dilute polymers) is the groundstate energy density.

**Numerical results.** Gathering all these (admittedly elaborate) pieces, we finally give a lattice expression that goes to \(b\) in the scaling limit \([15]\)

\[
b^{(N)} = \frac{\langle t^{(N)}|H_{-2}^{(N)}|0^{(N)}\rangle^2}{\langle t^{(N)}|T^{(N)}\rangle},
\]

where we have normalized the groundstate so that \(\langle \theta^{(N)}|0^{(N)}\rangle = 1\). Numerical results for percolation and dilute polymers are gathered in Tab. \([1]\) Both theories seem to have \(b = -5\), although the precision is obviously much better for percolation as the Hilbert space is much smaller. Note that we also measured \(b^{(N)}\) in the \(sl(2|1)\)-supersymmetric representation of percolation \([24, 32]\), with the same result \([22]\). Our value of \(b\) thus also applies to the CFT describing the Spin Quantum Hall Transition \([32]\).

The value \(b = -5\) is quite surprising, since the standard argument identifies to a large extent the chiral sector of the bulk theory with the chiral theory \([3]\), and thus one would expect to recover the standard values \(b = -\frac{1}{8}\) or \(\frac{5}{8}\). We now turn to an analytical argument that allows us to derive the value \(b_{\text{bulk}} = -5\) exactly.

**OPE approach.** It was shown \([15]\) that by pushing further the ideas in \([2, 4, 23]\) one can predict the value of indecomposability parameters in the chiral case using operator product expansions (OPEs). The idea is that the identity channel of the OPE of a primary operator \(\Phi_{h, \bar{h}}\) (with conformal weights \((h, \bar{h})\)) with itself is ill-defined as \(c \to 0\)

\[
\Phi_{h, \bar{h}}(z, \bar{z}) \Phi_{h, \bar{h}}(0, 0) \sim \frac{a_{\Phi}}{z^{2h + 2\bar{h}}} \left[ 1 + \frac{2h}{c} z^2 T(0) + \ldots \right],
\]

where \(a_{\Phi}\) is a coefficient and we focused on the holomorphic sector on the right-hand side (a similar reasoning applies to \(\bar{T}\)). The divergence as \(c \to 0\) can be cured if there exists a primary field \(X(z, \bar{z})\) with conformal weights \((h_X, \bar{h}_X) \to (2, 0)\) that “collides” with the stress tensor at \(c = 0\). One can show that a combination of \(X(z, \bar{z})\) and \(T(z)\) becomes the logarithmic partner \(t(z)\) of \(T(z)\) at \(c = 0\), with correlation functions like eqs. \([9]\) consistent with a value of \(b\) given by

\[
b = \lim_{c \to 0} -\frac{\partial^2}{\partial h_{\bar{X}}^2} \log \langle X | X \rangle = \lim_{c \to 0} -c/2h_{\bar{X}}(c).
\]

We assumed that those limits exist and are equal; this turns out to be satisfied in the following.

**Coulomb Gas.** In the remainder of this Letter, we shall denote the bulk fields \(\Phi_{r,s}(z) \otimes \Phi_{r',s'}(\bar{z})\) using Kac labels, meaning that the conformal dimensions fit into

| | Percolation | Dilute Polymers |
|---|---|---|
| \(L = 2N\) | \(b^{(N)}\) | \(L = 2N\) | \(b^{(N)}\) |
| 10 | -4.33296 | 10 | -4.17430 |
| 12 | -4.55078 | 12 | -4.38064 |
| 14 | -4.68234 | 14 | -4.52458 |
| 16 | -4.76634 | 16 | -4.58168 |
| 18 | -4.82256 | 18 | -4.88978 |
| 20 | -4.86168 | 20 | -5.00 ± 0.01 |
| 22 | -4.88978 | 22 | -5.00 ± 0.01 |

Exact | -5 | Exact | -5
the Kac table as \((h = h_{r,s}, \bar{h} = h_{r',s'})\), where we use the parametrization
\[
h_{r,s} = \frac{[x(x+1) - s^2]^{2-1}}{4x(x+1)} \quad \text{and} \quad c = 1 - \frac{6}{x(x+1)}
\]
(our models with \(c = 0\) thus correspond to \(x = 2\)). The crucial step is now to identify the field \(X(z, \bar{z})\), since \(h_X(c)\) then determines \(b\). In the standard approach to this problem \([8, 8]\), one looks for \(X\) among degenerate fields such as \(\Phi_{3,1}\) or \(\Phi_{1,5}\) in the chiral case. Considerations on the underlying lattice model however gives an unambiguous answer \([33]\). First, recall that the dense loop model renormalizes towards a Coulomb Gas (CG) \([34]\) with coupling constant \(g \in [2, 4]\) given by \(Q = 2(1 + \cos \frac{\pi g}{2})\)—recall that \(n = \sqrt{Q}\) is the loop weight.

The associated central charge reads \(c(Q) = 1 - \frac{3(g - 4)^2}{2g}\), and percolation corresponds to \(Q = 1, g = \frac{8}{3}\). Similarly the \(O(n)\) model can be expressed as dilute loops that renormalize towards a Coulomb gas with coupling constant \(g \in [1, 2]\) given by \(n = -2\pi \cos \pi g\), so dilute polymers have \(g = \frac{4}{3}\). The central charge in this case reads \(c(n) = 1 - \frac{6g(g - 2)}{g}\). The operator content of the loop models can then be worked out from their partition function on a torus \([24, 33]\). The candidate for the field \(X(z, \bar{z})\) must be a spin-2 operator, and knowledge from the boundary case, from our numerics, and from replica considerations in the case of the \(O(n)\) model \([2]\) suggests that \(X(z, \bar{z})\) has to be a 2-leg (resp. 4-leg) operator for dilute polymers (resp. percolation). We find that it has conformal weights \((h = \frac{(2 + g)^2}{4g} + \frac{1}{24}, \bar{h} = \frac{(2 - g)^2}{4g} + \frac{1}{24})\) in both cases. Moreover this is the only possible choice in the CG that would yield exponents \((2, 0)\) at \(c = 0\). In terms of Kac labels, we can formally identify this field as \(X(z, \bar{z}) = \Phi_{1, -2}(z) \otimes \Phi_{1, 2}(\bar{z})\), which is non-degenerate. We finally compute \(b\) using the continuation of the critical exponents of \(X\), finding

\[
\begin{align*}
  b_{\text{perco}} &= \lim_{g \to \frac{4}{3}} \frac{c(Q)/2}{(2 - g)^2 + c(Q)} = -5, \\
  b_{\text{poly}} &= \lim_{g \to \frac{4}{3}} \frac{c(n)/2}{(2 - g)^2 + c(n)} = -5,
\end{align*}
\]

in perfect agreement with our numerical results.

Physical discussion. We believe that our result is quite crucial for the general structure of bulk LCFTs at \(c = 0\). In particular, the value \(b = 5\) should have profound consequences on the bulk correlation functions \([4, 33]\). Indeed, the solution of the differential equations for the four-point correlation function of the degenerate bulk field \(\Phi_{2,1}(z, \bar{z}) \equiv \Phi_{2,1}(z) \otimes \Phi_{2,1}(\bar{z})\) (with conformal weights \((\frac{1}{2}, \frac{1}{2})\) or \(\Phi_{1,1}(z, \bar{z}) \equiv \Phi_{1,1}(z) \otimes \Phi_{1,1}(\bar{z})\) (with conformal weights \((\frac{1}{3}, \frac{1}{3})\)), and the resulting conformal blocks lead unambiguously to the values \(b_{\text{poly}} = \frac{5}{2}\) and \(b_{\text{perco}} = -\frac{5}{2}\) respectively (this is one of the arguments used in \([4]\) to predict the value of \(b\) for percolation and polymers), provided that one ignores normalization issues in the bulk. That \(b\) takes none of these values forces one to conclude that, within the \(c = 0\) conformal field theory describing percolation (resp. polymers), the four point correlations of the energy operator \(\Phi_{2,1}\) (resp. \(\Phi_{1,3}\)) must be set equal to zero exactly. This is compatible with the known—but maybe underestimated—fact that one cannot normalize these correlation functions to render them non-trivial and without divergences \([2]\). Note also that we were able to show from both algebraic and numerical arguments that the energy operator \(\Phi_{2,1}\) (resp. \(\Phi_{1,3}\)) is mixed into a rank-2 Jordan cell, and hence has a zero norm-squared. One can deduce that all its correlation functions are actually vanishing \([37]\), and thus irrelevant for the determination of \(b\). Note that the vanishing of the multi-point correlation functions of the energy does not preclude other correlators involving the energy from taking non-zero values. The computation of non-vanishing correlation functions in general nevertheless appears unreachable. Indeed, since \(b = -5\) and \(c = 0\), from our OPE approach, because of mixing with operators having negative Kac labels, progress would presumably require us to control the four-point functions of such operators—a situation for which the differential equation or CG approaches are known to fail.

Note that the crucial point of our calculation is the proper identification of the field \(X(z, \bar{z})\) among primary operators given by the CG. We emphasize that within this natural “minimal” geometrical formulation of the problem, there is no bulk counterpart of the boundary fields \(\Phi_{3,1}\) or \(\Phi_{1,5}\) that played the role of \(X\) in the chiral case. Instead, we have shown that \(X(z, \bar{z})\) must be a non-Kac operator with negative Kac labels, thus yielding a value of \(b\) different from what was expected from, e.g., Refs. \([4, 37]\). The possibility of having a value of \(b\) in the bulk different from the boundary case was suggested in Ref. \([32]\). We emphasize that our geometrical bulk theory admits a unique value of \(b\) \([41]\). Note however that a “non-minimal” formulation—e.g., involving further geometrical observables \([35, 39]\)—might eventually lead to a more complicated Jordan structure for \(T(z)\). Finally, we remark that \(b_{\text{bulk}} = b_{\text{perco}} + b_{\text{poly}}\); this is an intriguing observation indeed, since \(b^{-1}\) is the parameter that appears multiplicatively in correlation functions and OPEs.

In conclusion, we see that the complete picture of the bulk theory of percolation (or polymers) is more complicated than expected, cannot be seen as a simple gluing of two chiral theories, and seems in fact to have very little to do with the nice understanding gained over the last few years in the boundary (chiral) case. Further understanding follows from a detailed algebraic analysis, which will appear elsewhere \([22]\).

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[1] H. Saleur, Nucl. Phys. B 382, 486 (1992).
[2] J. Cardy, *Logarithmic correlations in quenched random magnets and polymers*, *The Stress Tensor in Quenched Random Systems*, cond-mat/9911024.
[3] V. Gurarie and A.W.W. Ludwig, *Conformal field theory at central charge c=0 and two-dimensional critical Systems with quenched disorder*, hep-th/0409105.
[4] V. Gurarie and A.W.W. Ludwig, J. Phys. A 35, L377 (2002).
[5] V. Gurarie, Nucl. Phys. B 410, 535 (1993).
[6] P.A. Pearce, V. Rittenberg, J. de Gier and B. Nienhuis, J. Phys. A 35, L661 (2002).
[7] G. Götz, T. Quella and V. Schomerus, JHEP 03, 003 (2007).
[8] E. Frenkel, A. Losev and N. Nekrasov, arXiv:0803.3302.
[9] L. Rozansky and H. Saleur, Nucl. Phys. B 376, 461 (1992).
[10] V. Gurarie, Nucl. Phys. B 410, 535 (1993).
[11] K. Kytölä and D. Ridout, J. Math. Phys. 50, 123503 (2009).
[12] N. Read and H. Saleur, Nucl. Phys. B 777, 316 (2007).
[13] P. Pearce, J. Rasmussen and J.B. Zuber, J. Stat. Mech. P11017 (2006).
[14] J. Dubail, J.L. Jacobsen and H. Saleur, Nucl. Phys. B 834, 399–422 (2010).
[15] R. Vasseur, J.L. Jacobsen and H. Saleur, Nucl. Phys. B 851, 314 (2011).
[16] P. Mathieu and D. Ridout, Phys. Lett. B 657, 120 (2007).
[17] A.-L. Do and M. Flohr, Nucl. Phys. B 802, 475 (2008).
[18] M.R. Gaberdiel, I. Runkel and S. Wood, J. Phys. A 44, 015204 (2011).
[19] N. Read and H. Saleur, Nucl. Phys. B 777, 263 (2007).
[20] P.P. Martin, *Potts models and related problems in statistical mechanics* (World Scientific, Singapore, 1991); P.P. Martin, Int. J. Mod. Phys. A 7, Supp. 1B, 645 (1992); P.P. Martin and D. McAnally, Int. J. Mod. Phys. A 7, Supp. 1B, 675 (1992).
[21] A.M. Gainutdinov, N. Read and H. Saleur, arXiv:1112.3403 arXiv:1112.3407.
[22] A.M. Gainutdinov, J.L. Jacobsen, N. Read, H. Saleur and R. Vasseur to appear.
[23] I.I. Kogan and A. Nichols, *Stress-energy tensor in c=0 logarithmic conformal field theory*, hep-th/0203207.
[24] N. Read and H. Saleur, Nucl. Phys. B 613, 409 (2001).
[25] R.J. Baxter, *Exactly solved models in statistical mechanics* (Academic Press, London, 1982).
[26] V.F.R. Jones, L’Enseignement Math. 40, 313 (1994).
[27] J.J. Graham and G.I. Lehrer, L’Ens. Math. 44 (1998) 173.
[28] B. Nienhuis, Physica A 163, 152 (1990).
[29] Y.K. Zhou and M.T. Batchelor, Nucl. Phys. B 485, 646 (1997).
[30] C.J. Hamer, G.R.W. Quispel and M.T. Batchelor, J. Phys. A 20, 5677 (1987).
[31] W.M. Koo and H. Saleur, Nucl. Phys. B 426, 459 (1994).
[32] I.A. Gruzberg, A.W.W. Ludwig and N. Read, Phys. Rev. Lett. 82, 4524–4527 (1999).
[33] P. Di Francesco, H. Saleur and J.B. Zuber, Nucl. Phys. B 285, 454 (1987).
[34] B. Nienhuis, in J.L. Jacobsen et al. (eds.), *Exact methods in low-dimensional statistical physics and quantum computing*, Proceedings of the Les Houches Summer School, Session LXXXIX (Oxford University Press, 2009); J.L. Jacobsen, in A.J. Guttmann (ed.), *Polygons, polyminoes and polycubes*, Lecture Notes in Physics 775 (Springer, Dordrecht, 2009).
[35] J.J.H. Simmons and J. Cardy, J. Phys. A 42, 235001 (2009).
[36] Vl.S. Dotsenko and V.A. Fateev, Nucl. Phys. B 240, 312 (1984).
[37] M. Flohr, Int. J. Mod. Phys. A 18, 4497–4592 (2003).
[38] J.L. Jacobsen and P. Zinn-Justin, J. Phys. A: Math. Theor. 35, 2131 (2002); Phys. Rev. E 66, 055102(R) (2002).
[39] J. Dubail, J.L. Jacobsen and H. Saleur, J. Phys. A: Math. Theor. 43, 482002 (2010); R. Vasseur and J.L. Jacobsen, J. Phys. A: Math. Theor. 45, 165001 (2012).
[40] In particular, there is no contradiction with the argument given in Appendix A of Ref. [3].