Mapping Class Group Actions on Quantum Doubles

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Abstract: We study representations of the mapping class group of the punctured torus on the
double of a finite dimensional possibly non-semisimple Hopf algebra that arise in the construction
of universal, extended topological field theories. We discuss how for doubles the degeneracy
problem of TQFT’s is circumvented. We find compact formulae for the $S^\pm 1$-matrices using the
canonical, non degenerate forms of Hopf algebras and the bicrossed structure of doubles rather
than monodromy matrices. A rigorous proof of the modular relations and the computation
of the projective phases is supplied using Radford’s relations between the canonical forms and
the moduli of integrals. We analyze the projective $SL(2,\mathbb{Z})$-action on the center of $U_q(sl_2)$
for $q$ an $l = 2m + 1$-st root of unity. It appears that the $3m + 1$-dimensional representation
decomposes into an $m + 1$-dimensional finite representation and a $2m$-dimensional, irreducible
representation. The latter is the tensor product of the two dimensional, standard representation
of $SL(2,\mathbb{Z})$ and the finite, $m$-dimensional representation, obtained from the truncated TQFT
of the semisimplified representation category of $U_q(sl_2)$.
1. Introduction

Since the seminal paper of Atiyah [A] on the abstract definition of a topological quantum field theory (TQFT) much progress has been made in finding non trivial examples and extended structures. The most interesting developments took place in three dimensions where actual models of quantum field theory, like rational conformal field theories and Chern-Simons theory led to the discovery of new invariants. See [Cr] and [Wi].

In an attempt to counterpart these heuristic theories by mathematically rigorous constructions the field theoretical machinery had been replaced by quasitriangular Hopf algebras, or quantum groups. The resulting invariants are described in [TV] and [RT]. From here it is not hard to understand how to associate a TQFT to a rigid, abelian, monoidal category and an extended TQFT to a braided tensor category (BTC). In order for these theories to be well defined one has to make a few more assumptions. One is that the category shall contain only a finite number of inequivalent, simple objects, i.e., it is rational. The other is a technical non degeneracy condition, called “modularity” in [T], which is to assure that elementary cobordisms are associated to identifications rather than projections. Alternatively, if the modularity condition fails to hold, it is standard in the Atiyah [A] description to define a truncated TQFT by reducing the vectorspaces to the images of the projections.

All of the mentioned TQFT’s are semisimple, i.e., they rely on the decompositions into simple objects. Clearly, semisimplicity cannot be an assumption of fundamental but only of technical nature. In seeking universal constructions of TQFT’s, which do not refer to decompositions, one should thus not only generalize the existing ones to non semisimple theories but also gain a deeper understanding of the structure underlying them.

A partial answer for the genus one case of how a universal TQFT should look like had been given by Lyubachenkov in [Ly]. There representations of the mapping class group $D$ of the punctured torus are constructed as a subgroup of the End set of a coend in a BTC with certain finiteness conditions. For the representation category of a finite dimensional Hopf algebra the coend turns out to be the algebra acting on itself by the adjoint action. A number of explicit formulae for the action of genus one mapping class groups on Hopf algebras had been derived from this by Lyubachenkov and Majid [LyM].

One of the objectives of this paper is to give natural definitions of the modular operators and independent, rigorous proofs of the relations that rely mainly on the theory of integrals on Hopf algebras as developed by Larson, Sweedler and Radford. In doing so we will be able to give the precise relation of the projective phases of the representation to the basic invariant of a Hopf algebra obtained from the moduli.

Starting with nonsemisimple Hopf algebras it is a natural question to ask how the universal TQFT relates to the reduced TQFT defined by the semisimplified representation
category of the same algebra. In the second part we give the precise connection for the mapping class group \( SL(2, \mathbb{Z}) \) of the closed torus and the quantum group \( U_q(sl_2) \). In the universal picture the representation of \( SL(2, \mathbb{Z}) \) is found as the restriction of the action of \( D \) to the center. The usual modular representation will appear in a tensor product with the fundamental, algebraic representation besides an additional, inequivalent finite representation.

In order to give an idea where these results fit into the general framework of a TQFT we give here an outline of the constructions of an extended three dimensional TQFT with BTC’s. The axioms are essentially due to Kazhdan and Reshetikhin, [KR], and differ from other definition in that they make no use of higher algebraic structures like 2-categories. We shall give the objects assigned to compact, oriented surfaces with boundaries both in the case of the TQFT constructed in [RT] for semisimple categories and for the universal TQFT associated to a Hopf algebra \( A \).

Extended Three Dimensional Topological Quantum Field Theories:

As in [A] an extended TQFT is defined as a functor or, more precisely, a collection of functors from cobordism categories to abelian categories over an algebraically closed field \( k \).

To a given one dimensional manifold \( S \) we can associate a cobordism category \( \text{Cob}_S \) as follows: The objects of the category are compact, oriented two-folds \( \Sigma \) with coordinate maps \( S \xrightarrow{\sim} \partial \Sigma \). A morphisms between \( \Sigma_1 \) and \( \Sigma_2 \) is a 3-fold \( M \) whose boundary is parametrized by \( -\Sigma_1 \coprod \Sigma_2 \xrightarrow{\sim} \partial M \). The composition of two morphisms is are given by an identification along a common surface. An extended TQFT assigns to every surface \( S \) a category \( \mathcal{C}_S \) and a functor

\[
\Phi_S : \text{Cob}_S \rightarrow \mathcal{C}_S.
\]

Assuming that \( \mathcal{C}_\emptyset = \text{ Vect}(k) \) this implies the original definition of [A]. We have a natural inclusion of categories \( \text{Cob}_S \times \text{Cob}_{S'} \rightarrow \text{Cob}_{S\cup S'} \). For the respective abelian categories we also assume a functor

\[
\otimes : \mathcal{C}_S \times \mathcal{C}_{S'} \rightarrow \mathcal{C}_{S\cup S'}
\] (1.1)

compatible with \( \Phi \). We require this to be a tensor product of abelian categories in the sense of [D]. Note that this is consistent with \( \mathcal{C}_\emptyset \otimes \mathcal{C} = \text{ Vect}(k) \otimes \mathcal{C} \cong \mathcal{C} \).

A standard consequence of this are representation of mapping class groups. To see this we consider \( M = \Sigma \times I \coprod_\partial \Sigma \) where the relation \( \sim \) is \( (s,t) \sim s \forall s \in \partial \Sigma, \ t \in I \) and \( I \) is the unit interval. For the boundary \( \partial M = \Sigma \coprod_{\partial \Sigma} \Sigma \) we choose different coordinate maps for the two boundary pieces coinciding on \( \partial \Sigma \). If we denote by \( \text{Diff}(\Sigma, \partial \Sigma) \) the group of homeomorphisms of \( \Sigma \) to itself which are identity on the boundary we obtain from these cobordisms a representation:

\[
\pi_o(\text{Diff}(\Sigma, \partial \Sigma)) \rightarrow \text{ End}_{\mathcal{C}_{\partial \Sigma}}(X_\Sigma).
\] (1.2)
Here we denoted by $X_{\Sigma} = \Phi_{\partial \Sigma}((\Sigma, \partial \Sigma))$.

Next we formulate the axiom that leads to lower dimensional cobordism functors. To this end suppose that $S = A \amalg B$ and $S' = B \amalg C$ then for tensor categories the contraction functor $\text{Hom}(1, \otimes) : C_B \times C_B \rightarrow \text{Vect}(k)$ induces a bilinear, covariant functor

$$\mathcal{C}_{AIB} \times \mathcal{C}_{BIC} \rightarrow \mathcal{C}_{AIC}.$$  \hspace{1cm} (1.3)

On the side of the cobordism categories we consider two three manifolds $M$ and $M'$ that belong to $\text{Cob}_{AIB}$ and $\text{Cob}_{BIC}$ respectively. We can consider half tubular neighborhoods of the 1-folds $B$ in the boundaries of $M$ and $M'$. These define oriented ribbon graphs in the boundaries along which we can glue the two manifolds $M$ and $M'$. The result is again a three manifold $M \amalg B M'$. The boundary pieces are the boundary pieces of the individual 3-folds glued along $B$. This way we obtain a cobordism in $\text{Cob}_{AIC}$ from $\Sigma_1 \amalg B \Sigma_1'$ to $\Sigma_2 \amalg B \Sigma_2'$. The assignment

$$\text{Cob}_{AIB} \times \text{Cob}_{BIC} \rightarrow \text{Cob}_{AIC}$$ \hspace{1cm} (1.4)

is easily seen to be a functor. The next axiom of an extended TQFT asserts that the functors $\Phi$ intertwines the two functors in (1.3) and (1.4).

This axiom allows us to define a functor from the category of 2-cobordisms between 1-folds and the category of abelian, tensor categories. The assignment of morphisms is given by the composition:

$$\mathcal{F}_{\Sigma} : C_A \xrightarrow{1 \otimes X_{\Sigma}} C_A \otimes C_A \otimes C_B \xrightarrow{\text{Hom}(1, \otimes) \otimes \text{id}} C_B$$  \hspace{1cm} (1.5)

Here $\Sigma$ denotes a 2-manifold cobording the pieces $A$ and $B$ by some coordinate maps $A \rightarrow \partial \Sigma \leftarrow B$.

In order to check functoriality of $A \rightarrow C_A$ and $\Sigma \rightarrow \mathcal{F}_{\Sigma}$ we consider again the manifold $M = \Sigma \times I \amalg \partial \Sigma$ as in (1.2) now with the same coordinate maps for the boundary pieces but two components for $\partial \Sigma$. Specializing to surfaces of the form $\Sigma = S \times I$ we get as in (1.2) a homomorphism

$$\pi_o(\text{Diff}(S)) \rightarrow \text{End}_{\text{Cat}}(\mathcal{C}_S)$$ \hspace{1cm} (1.6)

For compact $S$ and by (1.1) we easily identify (1.6) as the homomorphism from the permutation group of circles and to the permutations of tensor factors.

The functors associated to the elementary cobordisms, given by spheres with one, two, and three punctures (denoted $P_1$, $P_2$ and $P_3$ respectively) have a specific meaning for the circle category. Since $P_2$, seen as a cobordism from $S^1$ to $S^1$ with the same coordinate maps, is a unit in the cobordism category we want the associated $\mathcal{F}_{P_2}$ to be the identity.
functor in the basic category $\mathcal{C}_1$ of the circle. Regarding $P_3$ as a cobordism from $S^1 \amalg S^1$ to $S^1$ the associated functor defines a tensor product $\mathcal{F}_{P_3} = \otimes : \mathcal{C}_1 \otimes \mathcal{C}_1 \rightarrow \mathcal{C}_1$, which we assume to be same as the one used in (1.5). Finally, the functor of $P_1 : \emptyset \rightarrow S^1$ clearly gives the injection of an identity object with respect to $\otimes$ and $P_1 : S^1 \rightarrow \emptyset$ is assigned to the invariance functor $\text{Hom}(1, -)$.

In an extended TQFT we can also consider 3-cobordisms of 2-cobordisms, which yield natural transformations. More precisely, let $M$ have boundary pieces $\Sigma_i, i = 1, 2$ and $\partial \Sigma_i = A \amalg B$. The functor $\Phi_{\partial \Sigma}$ associates to the surfaces $\Sigma_i$ objects $X_i \in \mathcal{C}_A \otimes \mathcal{C}_B$ and a morphism $f_M \in \text{Hom}(X_1, X_2)$. For an object $Y \in \mathcal{C}_A$ we apply to the morphism $id \otimes f_M : Y \otimes X_1 \rightarrow Y \otimes X_2$ the functor $\text{Hom}(1, - \otimes -) \otimes id$ as in (1.5) to give us a morphism $\tilde{f}_M(Y) : \mathcal{F}_{\Sigma_1}(Y) \rightarrow \mathcal{F}_{\Sigma_2}(Y)$. It is easy to see that this defines a natural transformation $\tilde{f}_M : \mathcal{F}_{\Sigma_1} \rightarrow \mathcal{F}_{\Sigma_2}$ and thereby a functor $\text{Cob}_{A \amalg B} \rightarrow \text{Funct}(\mathcal{C}_A, \mathcal{C}_B)$.

A special type of natural transformations are generated by cobordisms of the form $M = S \amalg (S \times I \times I) \amalg S$ with relations $\alpha : s \sim (s, 0, t)$ and $\beta : (s, 1, t) \sim F(s, t) \forall s \in S t \in I$. Here $F$ is a homotopy in the set of homeomorphisms $\text{Diff}(S)$ of $S$ to itself. Confining ourselves to loops, i.e., $F(s, 1) = F(s, 0) = s$, we obtain a homomorphism

$$\pi_1(\text{Diff}(S)) \rightarrow \text{Nat}(id, id)$$

(1.7)

Reconsidering the elementary cobordism $P_1$, we can discuss some elementary natural transformations that identify the circle category $\mathcal{C}_1$ as a BTC. The $2\pi$ rotation of $S^1$ generating $\pi_1(\text{Diff}(S^1))$ gives us by (1.7) a natural transformation, denoted $\theta \in \text{Nat}(id)$. We can also cobord the surface $P_3$ to $P_3$ with exchanged coordinate maps for the $S^1 \amalg S^1$ piece of the boundary by moving the circles around each other in one of two directions. The TQFT assigns a transformation $\epsilon^{\pm} \in \text{Nat}(\otimes, P \otimes)$. The square of this cobordism is homeomorphic to the one where annuli around the punctures are twisted by $2\pi$ so we obtain the identity of natural transformations:

$$\epsilon(Y, X)\epsilon(X, Y) = \theta(X \otimes Y) \theta(X)^{-1} \otimes \theta(Y)^{-1}$$

(1.8)

This means $\theta$ is a balancing of $\mathcal{C}_1$. The associativity constraint is obtained in a similar way.

Let us discuss for a surface $\Sigma'$ whose boundary is the union on $n$ circles and the corresponding closed surface $\Sigma$ a connection between (1.2) and (1.6). We have fibrations $\text{Diff}(\Sigma', \partial \Sigma') \subseteq \text{Diff}(\Sigma') \rightarrow \text{Diff}(\partial \Sigma')$ and $\text{Diff}(\Sigma') \subseteq \text{Diff}(\Sigma) \rightarrow K_n$. 

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The objects associated to punctured surfaces are found easily by sewing along circles. For example the surface $\Sigma$ is the closed surface. The second homomorphism is given by the invariance functor $\text{Hom}(1, -)^{\otimes n}$ acting on $C_{\Sigma'} \cong C_{\Sigma}^{\otimes n}$ and $V(\Sigma) = \text{Hom}(1, -)^{\otimes n}(X_{\Sigma'})$ is the vector space associated to the closed surface.

Examples and the Degeneracy Problem: The objects associated to punctured surfaces can be identified up to isomorphism for two types of categories. One is a semisimple, rational BTC $C_o$ with simple objects $I$ the other is the representation category $R(A)$ of a finite dimensional Hopf algebra $A$. Quite generally it is possible to produce a semisimple, rational category from $R(A)$ by a generalized GNS construction with respect to a canonical categorical trace $tr$, see for example [K]. Thus in principle there are two ways of constructing TQFT’s from a given Hopf algebra $A$ which will lead to different representations, e.g., of mapping class groups. The precise connection in one example will be discussed in an example in the last chapter.

The assignment of objects for the two punctured sphere is easily inferred from $F_{P_2} = \text{id}$ and formula (1.9). In $C_o$ the answer is $X_{P_2} \cong \sum_{j \in I} j \otimes j^\vee$ and in $R(A)$ the module $X_{P_2}$ is given by $A$ with $A^{\otimes 2}$-action given by $a \otimes b(x) = axS(b)$. Moreover, $F_{P_3} = \otimes$ implies that $X_{P_3} \cong \sum_{ij \in I} i \otimes j \otimes (i \otimes j)^\vee$ or $X_{P_3} = A \otimes A$ with $A^{\otimes 3}$-action $(a \otimes b \otimes c)(x \otimes y) = (ax \otimes by)\Delta(S(c))$. This allows us to identify the object associated to the punctured torus $T'$ with $\partial T' = S^1$ by contracting the objects $X_{P_3}$ and $X_{P_2}$ along the category of the $S^1 \times S^1$-boundary pieces. In $C_o$ the resulting object is $X_{T'} = \sum_{j \in I} j \otimes j^\vee$ and in $R(A)$ by the module $A$ with adjoint action. The objects of all other surfaces are now found easily by sewing along circles. For example the surface $\Sigma_{g,1}$ of genus $g$ with one puncture is assigned to $X_{T'}^{\otimes g}$. The object of the $(n+1)$-punctured sphere $P_{n+1}$ has object $X_{P_{n+1}} = \sum_{i_1 \cdots i_n} \prod_{i_1} (i_1 \otimes \cdots \otimes i_n)\otimes y_i$ and in $R(A)$ the module $X_{P_{n+1}} = A^{\otimes n}$ where the $A^{\otimes (n+1)}$-action is given by the obvious generalization of the cases $n = 2, 3$. The object for a general compact, orientable surface is found by sewing $X_{P_{n+1}}$ and $X_{\Sigma_{g,1}}$. For $R(A)$ this gives for example the module $\text{Hom}_A(A^\otimes g, A^{\otimes n})$ of intertwiners for one of the $A$-actions.

Let us discuss the case $g = 1, n = 1$ in some more detail. The mapping class group $D = \pi_o(\text{Diff}(T', \partial T'))$ maps by (1.2) into $\text{End}(T')$ so that we obtain in $R(A)$ an action
of $D$ on $A$ intertwining the adjoint action. Following (1.9) we obtain a representation of the modular group $\pi_0(\text{Diff}(T))$ on $V(T) = \text{Hom}(1, T)$, which for $R(A)$ is just the restriction of the $D$ action to the center $Z(A) = \text{Hom}(1, A)$. In order to interpret the rest of (1.9) recall that for $R(A)$ the natural transformations of the identity functor are given by the action of central elements of $A$. In particular the generator $\theta$ of $\pi_1(\text{Diff}(S))$ acts on $A$ as $\text{ad}(v)$ where $v = \theta(A)$ is the central “ribbon element”, see [RT]. The Dehn twist along the boundary can also be given by $S^4$ where $S$ is the standard generator of $D$. The restriction of $S^4 = \text{ad}(v)$ to the center is clearly trivial. The second generator of $D$, the Dehn twist at a handle, $T$ is given by the action of $\theta$ on the constituent $X_{P^2}$, i.e., by multiplication of $v$ on $A$.

The definition of a TQFT we presented so far is not quite complete. Clearly, there are many ways of sewing up a surface $\Sigma$ so we have many ways to construct the object $X_{\Sigma}$. For example instead of using the center of $A$ as the vectorspace for the closed torus $V(T)$ we can also choose the space $\text{Hom}(A, 1)$ - which is isomorphic to the space of characters on $A$ - or we could have chosen the endomorphism set $\text{End}(X_{P^2}) = \text{End}_{A^{\otimes 2}}(A)$. These spaces are isomorphic to each other but there is no one canonical isomorphism identifying two of them. Instead the sewing procedure used to find the object defines a surface with a cut diagram or decoration. Thus we should take as objects of the cobordism categories surfaces $\Sigma$ together with a Lagrangian subspace of $H_1(\Sigma, \mathbb{R})$ which must be compatible with the cobording 3-manifolds. The functor of the TQFT is now allowed to have projective phases. This means for two cobordisms $M_1$ and $M_2$ with a common, decorated boundary component that

$$\Phi(M_1M_2) = c^\mu \Phi(M_1)\Phi(M_2)$$

(1.10)

where $\mu$ is the Maslov index of a triple of Lagrangian subspaces defined by the cobordisms. It also measures the non-additivity of the signature of the 4-manifolds cobording the $M_i$ to the corresponding union of handlebodies. If the $M_i$ are invertible morphisms in the cobordism category we obtain projective representations of the modular groups. For details see [T]. The main result of the first chapter is the relation of the phase $c$ to intrinsic invariants of the Hopf algebra $A$.

In order to discuss the modularity condition we recall how the $S$ matrix can be obtained from the [RT]-construction for standard TQFT’s with $S = \emptyset$. The cobordism describing the action of $S$ on $T'$ is a 3-manifold whose boundary is $\partial M = T' \coprod S_1 T'$, the closed surface of genus two, and can thus be considered a cobordism $\Sigma_2 \to \emptyset$. In [RT] the vectorspace associated to $\Sigma_2$ is $\bigoplus_{ij} \text{Hom}(i \otimes i^\vee \otimes j \otimes j^\vee, 1)$. The linear form assigned to $\Sigma_2 \to \emptyset$ is found by computing the the invariant in $S^3$ of the ribbon graph embedded in the outside of $\Sigma_2$. On a vector $f$ its value is

$$1 \xrightarrow{\text{coev}\otimes \text{coev}} i \otimes i^\vee \otimes j \otimes j^\vee \xrightarrow{1\otimes \epsilon \otimes 1} i \otimes i^\vee \otimes j \otimes j^\vee \xrightarrow{f} 1$$
In the description of an extended TQFT we have to consider this as the matrix element of \( S \in \text{End}(T') \cong \bigoplus_{ij} \text{Hom}(j \otimes j^\vee, i \otimes i^\vee) \). Thus on a summand we have

\[
S : j \otimes j^\vee \xrightarrow{\text{coev} \otimes 1} i \otimes i^\vee \otimes j \otimes j^\vee \xrightarrow{1 \otimes \text{ev} \otimes 1} i \otimes i^\vee \otimes j \otimes j^\vee \xrightarrow{1 \otimes c_j} i \otimes i^\vee
\]

where \( c_j \) is proportional to \( j \otimes j^\vee \xrightarrow{\epsilon} j^\vee \otimes j \xrightarrow{\epsilon \text{ev}} 1 \). The generalization of this formula to non semisimple categories is described by [Ly] and will be reviewed in the next chapter.

A priori the operations \( S \) and \( T \) defined for a general semisimple \( C \) do not yield a projective representation of \( SL(2, \mathbb{Z}) \) unless we impose one further condition. This is the rather specialized “modularity condition” introduced in [T] asserting that the \( S \)-matrix is invertible. In case this condition is violated we may still apply Atiyah’s prescription and reduce the space \( k^I \) by the projection \( P = SS^- \) where the matrix \( S^-_{ij} = S_{ij^\vee} \) is assigned by the [RT]-prescription to the inverse cobordism.

For example if \( C \) is a symmetric category the \( S \) matrix is of rank one so the \( SL(2, \mathbb{Z}) \) representation is one dimensional. A degeneracy problem occurs quite generally if \( I \) contains a subset \( I_o \) of irreducible objects, which braid trivially, i.e., \( \epsilon(k, j)\epsilon(j, k) = 1 \) for all \( k \in I_o, j \in I \). In case \( I_o \) is a subgroup of invertibles \( \{\sigma\} \) we have for the natural action of its elements on \( k^I \) that \( S\sigma = S \). Hence \( S \) and \( T \) can be defined on the orbit space \( \text{im}(\sum_{\sigma \in I_o} \sigma) \), where we can hope for the modularity condition to hold.

This situation occurs for the semisimplified representation categories of quantum groups at certain roots of unity. The example we will come back to in the last chapter is \( U_q(sl_2) \) where \( q^{l/2} \) is an \( l = 2m + 1 \) th root of unity. We have \( |I| = 2m \) and the \( 2m \)-th representation braids trivially and is invertible of order two. The truncated theory yields an \( m \)-dimensional representation of \( SL(2, \mathbb{Z}) \).

The problem of degeneracy is resolved in a very natural way in the universal picture for \( R(A) \) by choosing \( A \) to be a double constructed algebra. In this situation we find very simple formulae for \( S \) and its inverse.

**Survey of Contents and Summary of Results:**

In Chapter 2 we define and study the action of operators generating the mapping class group \( D := \pi_0(\text{Diff}(T, D)) \) on the double \( D(A) \) of a finite dimensional Hopf algebra. We start in Section 1 with a review of the bicrossed structure of a double and properties of
an isomorphism $D(A)^* \hookrightarrow D(A)$. These are in particular the relations between traces and characters on $D(A)$ and central and group like elements in $D(A)$. We also recall the definitions of canonical and balancing elements in quasitriangular Hopf algebras. For later application we derive a relation for the monodromy matrix of $D(A)$. The next section is a recollection from [Rd] of relations between several non degenerate bilinear forms and moduli defined by the integrals of a finite dimensional Hopf algebra. This leads for $D(A)$ to the Drinfeld-Radford formula $S^4 = Ad(g)$. In Section 3 we determine the integral and cointegral of a double $D(A)$. In particular we find that the comodulus is trivial and that the modulus is the canonical element $g$. This allows us to show that a pair of non-degenerate, canonical traces on $D(A)$ can be defined very simply from the natural contraction on $D(A)$. The balancing of a double $D(A)$ is related in Section 4 to second order roots of the moduli and a fourth order root $\nu$ of the $\omega$-invariant of $A$. Guided by categorial constructions in [Ly] we define in Section 5 the action of the generators of the mapping class group $D$ on $D(A)$. We obtain an intriguingly simple expression for the actions of $S$ and $S^{-1}$ involving only the non degenerate forms on $A$ and $A^*$ and the bicrossed isomorphism of $D(A)$. Similarly we have a formula for the braided antipode. The results from all previous sections are used in Section 6 to give a rigorous proof of the modular relations and determine the projective phase of the universal TQFT as $\nu^{-3}$.

In Chapter 3 we find the structure of the representation of $SL(2, \mathbb{Z})$ on the center of $U_q(sl_2)$ by restricting the action of $D$. The non-degenerate forms and moduli of the double of $B_q$, the Borel algebra of $U_q(sl_2)$, are given in Section 1. In Section 2 we determine the center of $D(B_q)$, which for the TQFT is the vectorspace of the torus. If $q$ is an $l = 2m + 1$-st primitive root of unity it is given by $\mathbb{C}[\mathbb{Z}/l] \otimes \mathcal{V}$. Here $\mathcal{V}$ is a $3m + 1$-dimensional algebra with a basis of $m + 1$ idempotents and $2m$ nilpotents. The balancing element of $D(B_q)$ is expressed in terms of this basis, see Section 3. In doing so we propose a method to generate new partition identities. In Section 4 we compute the matrix elements of the $SL(2, \mathbb{Z})$-action on the center of $D(B_q)$. This requires us to find transformations from the PBW basis of $U_q(sl_2)$ to the algebra $\mathcal{V} \otimes \mathbb{C}[K]$, where $K$ is a Cartan element. We analyze this representation in Section 6. We find evidence for the decomposition of the representation into two irreducibles. One of which is a finite, $m + 1$ - dimensional representation, the other is the tensor product of the two dimensional standard representation of $SL(2, \mathbb{Z})$ and the finite, $m$-dimensional representation obtained from the semisimplified representation category of $U_q(sl_2)$. The conjecture is verified in the last section. Here we find the decomposition and the explicit finite representations for two non trivial roots of unity.
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2. Mapping Class Group Action on Doubles

1.) Double Algebras and Balancing: In this section we recall some basic facts and notions on Hopf algebras that we will use later. For a finite dimensional Hopf algebra \( A \) over a field \( k \) we denote by \( A^o \) the dual Hopf algebra with opposite comultiplication. We shall always assume that \( A \) has a counit \( \epsilon \) and an invertible antipode \( S \). The antipode of \( A^o \) is thus given by \( S^{-1} \).

For \( \lambda, \mu \in A^o \) and \( a, b \in A \) we have the following relations:

\[
< \lambda\mu, a > =< \lambda \otimes \mu, \Delta(a) > \quad \quad \Delta(\lambda), ab >=< \lambda, b \otimes a > \quad (2.11)
\]

\[
\Delta \otimes id(R) = R^{13} R^{23} \quad \quad id \otimes \Delta(R) = R^{13} R^{12} \quad (2.12)
\]

\[
< S^{+1}(l), a >=< l, S^{-1}(a) > \quad S \otimes id(R) = id \otimes S^{-1}(R) = R^{-1} \quad (2.13)
\]

Here \( <, > : A^o \otimes A \rightarrow k \) is the usual contraction and \( R \) is the canonical element \( R \in A \otimes A^o \).

Thus, if \( \{ e_i \} \) is a basis of \( A \) and \( \{ f_i \} \) the respective dual basis of \( A^o \) we can write

\[
R = \sum_i e_i \otimes f_i
\]

A bicrossed product of the two algebras is a Hopf algebra \( D \) which contains \( A \) and \( A^o \) as sub-Hopf algebras such that \( R \Delta(y) = \Delta'(y) R \) for all \( y \in D(A) \).

Here \( \Delta' = \tau \circ \Delta \) is the opposite comultiplication and \( \tau \) is the flip \( \tau(a \otimes b) = b \otimes a \).

The bicrossed structure is given explicitly by

\[
\therefore: \quad A \otimes A^o \xrightarrow{\tau} A^o \otimes A \xrightarrow{\Delta \otimes \Delta'} A^o \otimes A^o \otimes A \xrightarrow{1 \otimes S \otimes 1 \otimes 1} A^o \otimes A^o \otimes A \otimes A \xrightarrow{1 \otimes <, > \otimes 1} A^o \otimes A \quad (2.15)
\]

If we use the usual abbreviation \( \Delta(a) = \sum_i a'_i \otimes a'_i = a' \otimes a'' \), \( \Delta^2(a) = a' \otimes a'' \otimes a''' \)

\[
\therefore (y \otimes \lambda) = \lambda'' \otimes y'' \langle \lambda', y' \rangle \langle S(\lambda'''), y''' \rangle \quad (2.14)
\]

The inverse is given similarly by

\[
\therefore^{-1}: \quad A^o \otimes A \xrightarrow{\Delta \otimes \Delta'} A^o \otimes A^o \otimes A \otimes A \xrightarrow{1 \otimes S \otimes 1 \otimes 1} A^o \otimes A^o \otimes A \otimes A \xrightarrow{1 \otimes <, > \otimes 1} A^o \otimes A \xrightarrow{\Delta \otimes \Delta'} A^o \otimes A \otimes A \xrightarrow{1 \otimes <, > \otimes 1} A^o \otimes A \quad (2.16)
\]
For a Hopf algebra $D$, with $\dim(D) < \infty$ we denote by $G(D)$ the finite group of group like elements $g$, characterized by $\Delta(g) = g \otimes g$. Also we shall use the notation $Ch(D) \cong G(D^*)$ for the group of one dimensional representations of $D$. For doubles we have the following easy fact:

**Lemma 1** For a Hopf algebra $A$ the multiplication map $\cdot : A \otimes A^* \rightarrow D(A)$ yields a group isomorphism:

$$G(A) \oplus Ch(A) \rightarrow G(D(A))$$

Similarly the sum of restrictions yields:

$$Ch(D(A)) \rightarrow Ch(A) \oplus G(A)$$

**Proof:** For $b \in D(A)$ let $b$ be the corresponding element in $End(A)$. For the coproduct this means $\Delta(b)(x \otimes y) = \Delta(b(y))$ which for $b \in G(A)$ has to equal $b(x) \otimes b(y)$. Inserting $y = 1$ and applying $\epsilon \otimes 1$ we find that $\alpha$ is of rank one and $a = g \cdot \gamma$ where $g = \alpha(1)$ and $\gamma = \epsilon \circ \alpha$. Inserting instead $x = 1$ and $y = 1$ (applying $\epsilon \otimes \epsilon$) shows that $g \in G(A)$ ($\gamma \in Ch(A)$). The adjoint action of $Ch(A)$ on the double $D(A)$ stabilizes $A$ and, there, coincides with the coadjoint action, i.e., we have $\gamma \cdot y \cdot \gamma^{-1} = \gamma \rightarrow y \leftarrow \gamma^{-1}$ for all $y \in A$.

Since the coadjoint action on group likes in $A$ is trivial the images of $G(A)$ and $Ch(A)$ centralize each other and the inclusion factors into the direct sum. Injectivity now follows from linear independence of group likes, see [Ab], and injectivity of $\cdot$.

Here we used the notation $\rightarrow \leftarrow$ as in [Ab] for the left (right) action of $D^*$ on a Hopf algebra $D$. Similarly, we use $(a \cdot \lambda)(y) := \lambda(ya)$ for the left action of $D$ on $D^*$ and $\leftarrow$ for the corresponding right action. We also use the adjoint actions of $D$ on itself given by $ad(a)(y) = a' y S(a'')$ and on $D^*$ given by $ad^*_r(a)(\lambda) = a'' \circ \lambda \circ S(a')$. The invariance in $D$ under the adjoint action is precisely the center $Z(D)$ and the invariance in $D^*$ are the $q$-characters $\overline{C(D)} = \{ \lambda \in D^* : \lambda(xy) = \lambda(S^2(y)x) \}$, which were introduced in [Dr1]. In [Dr1] it is shown that these two spaces are related to each other by the map

$$\mathcal{F} : D^* \rightarrow D : \lambda \rightarrow \lambda \otimes 1(M). \quad (2.17)$$

Here $M \in D^{\otimes 2}$ is the element $M = \tau(R)R = \sum_{ij} f_j e_i \otimes e_j f_i = \sum_k m_k \otimes n_k$. In the case of a double $D = D(A)$, $\{m_k\}$ and $\{n_k\}$ are differents basis of $D(A)$ so that $M$ is nondegenerate. The following is a slightly extended version of a lemma in [Dr1].

**Lemma 2**

1. The map $\mathcal{F} : D(A)^\ast \rightarrow D(A)$ is an isomorphism of $D(A)$-modules with respect to the adjoint actions.

2. $\mathcal{F} : \overline{C(D(A))} \rightarrow Z(D(A))$ is an isomorphism of algebras.

3. $\mathcal{F} : Ch(D(A)) \rightarrow G(D(A))$ is the group isomorphism $(g, \gamma) \mapsto (\gamma, g)$
4. We have $\overline{f}^* \circ S^* = S^{-1} \circ \overline{f}$

Proof: The fact that $\overline{f}$ intertwines the actions of $D(A)$ follows from basic Hopf algebra relations, (2.14) and the identity $(S(y') \otimes 1) M(y'' \otimes 1) = (S(y') \otimes 1) M(y'' (S(y'''))) = (S(y') \otimes 1)(y'' \otimes y''') M(1 \otimes S(y'''')) = (1 \otimes y') M(1 \otimes S(y'''))$.

It is clear that $\overline{f}$ is an isomorphism. In particular we can write it as the composition:

$$D(A)^* \xrightarrow{(\text{iso})^*} (A^* \otimes A)^* = A \otimes A^* \xrightarrow{\ast} D(A)$$

Clearly, the invariances are mapped isomorphically to each other and a computation in [Dr1] shows that this is a homomorphism. In fact we have $f(\chi \lambda) = f(\chi)f(\lambda)$ for any $\chi, \lambda \in D(A)^*$. 3.) follows from Lemma 1 and the form of $M$. Finally, (2.13) implies $S \otimes S(M) = \tau(M)$ and thereby 4.)

It follows from (2.12) that the $R$ matrices satisfy the Yang Baxter equation $R^{1213}R^{231} = R^{2312}$. For later computations of modular relation we derive here an analogous equation for the $M$ matrices.

Lemma 3 For $M$ and bases $\{e_i\}, \{f_i\}, \{n_k\}, \{m_k\}$ as above we have

$$(\tau(M) \otimes 1)(1 \otimes M) = \sum_{k,j} n_kf_j \otimes m_k \otimes S^{-1}(e''_j)n_k' e'_j$$

or equivalently

$$1 \otimes \lambda \otimes 1((\tau(M) \otimes 1)(1 \otimes M)) = \sum_{ij} f_i f_j \otimes S^{-1}(e_j)e_i \overline{f}(\lambda)' e_i.$$

Proof: If we multiply $R$ matrices from the left and right to the Yang Baxter equation and permute the first and third factor we obtain $(R^{-1})^{31} (R^{-1})^{32} R^{21} R^{31} = R^{21} (R^{-1})^{32}$. Applying 1 $\otimes 1 \otimes S^{-1}$ to this equation and using (2.13) we find

$$\sum_{ij} f_i \otimes e_i f_j \otimes e_j = \sum_{i,j,k,l} f_i f_j f_k \otimes f_i e_j \otimes S^{-1}(e_k) e_l e_i$$

Multiplication with $R \otimes 1$ from the left and $1 \otimes R$ from the right yields

$$(\tau(M) \otimes 1)(1 \otimes M) = \sum_{tsijkl} e_{i} f_{s} f_{j} f_{k} \otimes f_{i} f_{e} f_{j} e_{s} \otimes S^{-1}(e_{k}) e_{i} e_{s}$$

by (2.12) = $\sum_{ij} e_{i} f_{j} f''_{k} f_{j} \otimes f_{i} e_{j} \otimes S^{-1}(e_{k}) e_{i} e_{j} f'_{j}$

by (2.14) = $\sum_{ij} e_{i} f_{j} f''_{k} f_{j} \otimes f_{i} e_{j} \otimes S^{-1}(e_{k}) e_{i} f''_{j} e_{i}$.
The formulas follow now from $\Delta(n_{ij}) = e_i^f j^f \otimes e_i^f j^f$ and again (2.12).

Let us also record here the canonical elements from [Dr1] and [Ly1] implementing the square of the antipode. They are defined by

$$u := \sum_i S(f_i) e_i \quad \text{and} \quad \hat{u} := \sum_i S^2(e_i) f_i$$  \hspace{1cm} (2.18)

and satisfy the relations

$$S^2(y) = u y u^{-1} = \hat{u} y \hat{u}^{-1}, \quad \hat{u} = S(u)^{-1}$$ \hspace{1cm} (2.19)

From $u$ and $\hat{u}$ one has two further elements of a quasitriangular Hopf algebra $D$ with special properties:

$$g := u \hat{u}, \quad \text{with} \quad g \in G(D) \quad \text{and} \quad S^4(y) = g y g^{-1}$$ \hspace{1cm} (2.20)

$$z := u \hat{u}^{-1}, \quad \text{with} \quad z \in Z(D) \quad \text{and} \quad M^2 = z \otimes z \Delta(z^{-1})$$ \hspace{1cm} (2.21)

2.) Integrals, Moduli and Radford’s Relations : We start this section with a review of basic facts from Hopf algebra theory and a summary of the formulae in [Rd], we will use in this paper. The analysis of integrals of Hopf algebras in [LSw] is based on the fundamental Theorem of Hopf modules. It asserts that a Hopf module $M$ of a Hopf algebra $D$ is free in the sense that $M^{\text{cov}} \otimes D \rightarrow M$ is an isomorphism of Hopf modules. Here $D$ acts on itself by multiplication and comultiplication, $M^{\text{cov}}$ is the the coinvariance of the coaction and $\rightarrow$ is given by the left action on $M$. It is instructive to apply this to the situation where $M = D^*$ with actions $h. \lambda := \lambda \triangleleft S(h)$ and coaction $\delta(\lambda) = \lambda \otimes 1 \Delta \in \text{End}(D) = D \otimes D^*$. The isomorphism $J \otimes D \rightarrow D^*$ then implies that $J = \{ \lambda : \lambda \otimes 1 \Delta(y) = 1 \lambda(y) \}$ - the space of right integrals - is one dimensional and every nonzero element induces a nondegenerate bilinear form. Analogous statements are found if we use left actions or consider Hopf modules of the dual algebra. Let us fix once and for all a left integral $\mu$ and a left cointegral $x$ with the properties

$$1 \otimes \mu \Delta(h) = 1 \mu(h), \quad hx = \epsilon(h)x, \quad \text{and} \quad \mu(x) = 1.$$  \hspace{1cm} (2.22)

As in [Rd] we use notations for the following isomorphisms:

$$\beta_l, \beta_r : D \rightarrow D^* \quad \text{with} \quad \beta_l(h) = \mu \triangleleft h \quad \text{and} \quad \beta_r(h) = h \triangleright \mu$$  \hspace{1cm} (2.23)

$$\overline{\beta}_l, \overline{\beta}_r : D^* \rightarrow D \quad \text{with} \quad \overline{\beta}_l(\lambda) = x \leftarrow \lambda \quad \text{and} \quad \overline{\beta}_r(\lambda) = \lambda \rightarrow x.$$  \hspace{1cm} (2.24)
They intertwine the right and left actions as in
\[ \beta_l(kh) = \beta_l(k)h \quad \text{and} \quad \beta_r(kh) = k \triangleright \beta_r(h) . \]  
(2.25)

It is obvious that \( xh \) is again a left cointegral for any \( h \in D \). Hence by uniqueness of \( x \), we find \( \alpha \in Ch(D) \) and for the dual situation \( a \in G(D) \) such that
\[ \alpha(h)x = xh \quad \text{and} \quad \mu \otimes 1 \Delta(h) = a \mu(h) . \]  
(2.26)

Since \( D \) is finite dimensional, both the \textit{modulus} \( \alpha \) and the \textit{comodulus} \( a \) are of finite order and
\[ \omega := \alpha(a) \]  
(2.27)
is a root of unity. Note, that in the following we use the opposite comultiplication for \( D^\ast \) so that, e.g., \( S^{-1} = S^\ast \) \((\gamma \text{ in } [Rd]) \). The antipode acts on the integrals as follows:
\[ S^{-1}(\mu) = a \triangleright \mu = \omega S(\mu) = \omega \mu \triangleleft a \]  
(2.28)
\[ S(x) = \alpha \rightarrow x = \omega S^{-1}(x) = \omega x \leftarrow \alpha \]  
(2.29)
The compositions of isomorphisms in (2.23) and (2.24) are given by the following formulae.
Each one can be given on \( D^\ast \) or the adjoint one on \( D \) using \( \beta_l^\ast = \beta_r \):
\[ \beta_r \beta_l(\lambda) = S^{-1}(\lambda) \]  
(2.30)
\[ \bar{\beta}_r \beta_l(\lambda) = S(a \triangleright \lambda) \]  
(2.31)
\[ \beta_l \beta_r(\lambda) = S(\alpha^{-1} \lambda) \]  
(2.32)
\[ \bar{\beta}_l \bar{\beta}_r(\lambda) = \alpha \cdot S^{-1}(\lambda \triangleleft a) \]  
(2.33)
From (2.28)-(2.31) we can derive further useful relations between adjoints:
\[ \beta_l(S(h)) = \omega a^{-1} \triangleright S(\beta_r(h)) \quad \text{and} \quad \beta_l(S^{-1}(h)) = a^{-1} \triangleright S^{-1}(\beta_r(h)) \]  
(2.34)
\[ \beta_l(\alpha \rightarrow h) = \beta_r(S^2(h)) \quad \text{resp} \frac{\mu((\alpha \rightarrow k)h) = \mu(hS^2(k))}{\mu((\alpha \rightarrow k)h) = \mu(hS^2(k))} \]  
(2.35)
\[ \beta_l(S(\lambda)) = \alpha^{-1} \triangleright S(\beta_r(\lambda)) \quad \text{and} \quad \beta_l(S^{-1}(\lambda)) = \omega \alpha^{-1} \rightarrow S^{-1}(\lambda) . \]  
(2.36)
\[ \bar{\beta}_l(S^2(\lambda)) = \bar{\beta}_l(a \triangleright \lambda) \quad \text{resp} \quad S^2 \otimes 1 \Delta'(x) = \Delta(x)(a \otimes 1) \]  
(2.37)
Combining these identities we find Radford’s formula for \( S^4 \):
\[ S^4 = \text{ad}^\ast(\alpha) \circ \text{ad}(a^{-1}) \]  
(2.38)
Since in a double \( D(\mathcal{A}) \) the adjoint action of \( G(\mathcal{A}^o) \) coincides with the coadjoint action of \( Ch(\mathcal{A}) \) on \( \mathcal{A} \) we find from that (2.38) and the corresponding equation on \( \mathcal{A}^o \) that the group like element \( \alpha \otimes a^{-1} \) implements \( S^4 \) on \( D(\mathcal{A}) \). The same is true for the element \( g \) defined in (2.17). In fact we have the following result of Drinfeld for doubles:

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Proposition 4 [Dr1]

\[ g = \alpha \cdot a^{-1} \]

3.) Integrals and Canonical Traces of Doubles: In the construction of representations of the modular group, the integrals of the defining algebra play an important role. The integral \( \mu_D \) and the cointegral \( x_D \) of a double \( D(A) \) clearly have to be related to the integrals and cointegrals of \( A \). In this section, we are also interested in finding the moduli \( \alpha_D \) and \( a_D \). Comparing Proposition 4 to (2.38), we are led to expect that \( \alpha_D = 1 \) and \( a_D \) is the same as \( g \). We shall prove triviality of the comodulus first:

Proposition 5 For left integrals \( \mu \) and \( x \) as in (2.22) define the canonical element in \( D(A) \) by \( p = \mu \cdot S^{-1}(x) \). Then

1. \( S(p) = p \)

2. \( p \) is both a right and left cointegral in \( D(A) \) and \( \alpha_D = 1 \)

3. Let \( P \in \text{End}(A) \) be the image of \( p \) under the map \( \cdot^{-1} : D(A) \longrightarrow A \otimes A^* = \text{End}(A) \) is the projector onto the space of left cointegrals and is given by:

\[
A \xrightarrow{\xrightarrow{R \otimes 1}} A \otimes A^o \otimes A \xrightarrow{1 \otimes \tau} A \otimes A \otimes A^o \xrightarrow{A \otimes A^o \xrightarrow{\Delta \otimes 1}} A \otimes A \otimes A^o \xrightarrow{1 \otimes \tau^{-1} \tau} A
\]

Proof: Since \( S(p) = x \cdot S(\mu) \) and \( S^{-1}(x) \) and \( S(\mu) \) are right integrals it is clear that 1.) implies that \( p \) is an invariant with respect to right and left multiplication of \( A \) and \( A^o \). This shows 2.). Assuming that 1.) is true we show 3.):

\[
\begin{align*}
\otimes^{-1} (\mu \otimes S^{-1}(x)) &= \langle S(\mu'), S^{-1}(x)' \rangle \langle \mu'', S^{-1}(x)'' \rangle S^{-1}(x)''' \otimes \mu''
\text{by (2.28)}
&= \omega^{-1} \langle S(\mu'), S(x'''') \rangle \langle \mu'', S(x') \rangle S(x'') \otimes \mu'' \\
&= \omega^{-1} \sum_{ij} \langle \mu'', e''_j e_i \rangle \langle f_j, S(\mu') \rangle e_j' \otimes f_i \\
&= \omega^{-1} \sum_{ij} \langle f_j, S(\beta_i(\mu')) \rangle e_j' \otimes f_i \\
&= \omega^{-1} \sum_{ij} \langle f_j, S^2(\beta_i e_i) \rangle e_j' \otimes f_i \\
&= x \otimes S(m) \text{ by (1.5) = } x \otimes S(m) \quad (2.40)
\end{align*}
\]
If we apply $1 \otimes S^2$ to both sides of the last equation and use (2.28) we find
\[
P = \sum_{ij} \langle f_j, S^2(e''_i) e'_j \rangle e'_j \otimes f_i = \sum_j \langle f''_j, S^2(e'_j) \rangle e'_j \otimes f'_j
\]
by (2.42) = \sum_{ij} \langle f_j, S^2(e''_j) e'_i \rangle e'_i \otimes f_i ,
\]
which is precisely the equation given in 3.). The same formula has been proven in [Dr1] using the theory of Hopf modules directly. From (2.23) we find $S(\mu)(x) = \mu(ax) = 1$ so that $P$ as in (2.40) is a rank one projection. It remains to show the first part of the proposition. For this purpose we need two identities for the integrals, namely
\[
S^2((S(\mu)'' a^{-1}) \alpha) \otimes S(\mu)' = S^2((S(\mu)'' a^{-1}) \alpha) \otimes (S(\mu)'' a^{-1})' 
\]
by (2.28) = $S^2(\mu''\alpha) \otimes \mu'$
by (2.38) = $\mu' \otimes \mu'' = \Delta(\mu)$ (2.41)
and
\[
(S^2(x'' a^{-1}) \leftarrow \alpha) \otimes x' \quad \text{by (2.37)} = ((x') \leftarrow \alpha) \otimes x''
\]
= $(x \leftarrow \alpha)' \otimes (x \leftarrow \alpha)''$
by (2.28) = $S^{-1}(x)' \otimes S^{-1}(x)'' = \Delta(S(x)^{-1})$ (2.42)
Inserting (2.41) and (2.42) into the expression for the bicrossed product of $p$ in (2.33) we find:
\[
\otimes^{-1} (\mu \otimes S^{-1}(x)) = \langle S(\mu'), S^{-1}(x) \rangle \langle \mu'', S^{-1}(x)'' \rangle S^{-1}(x)'' \otimes \mu''
\]
= $\langle S^3((S(\mu)'' a^{-1}) \alpha), S^2((x'' a^{-1}) \leftarrow \alpha) \rangle \langle S(\mu)'', x'' \rangle x' \otimes S(\mu)'
\]
= $\alpha^{-1}(a \triangleright S(\mu)''''), (x''' a^{-1}) \leftarrow \alpha \rangle \langle S(\mu)''', x''' \rangle x' \otimes S(\mu)'
\]
= $\langle S(\mu)''''S(S(\mu)''), x''' \rangle x' \otimes S(\mu)' = x \otimes S(\mu)$ . (2.43)
Hence $\mu \cdot S^{-1}(x) = S(\mu \cdot S^{-1}(x)) = x \cdot S(\mu)$ and we have shown 1.) of the proposition. □

Thanks to the simple comultiplicative structure of $D(\mathcal{A})$ it is much easier to find the integral. Since $S^{-1}(\mu)$ is a right integral of $\mathcal{A}$ and since we have opposite comultiplication on $\mathcal{A}^\circ$ a right integral of $D(\mathcal{A})$ is given by
\[
\mu_D(\lambda \cdot y) := \lambda(x) \mu(S(y)) \quad \text{for all } \lambda \in \mathcal{A}^\circ, \text{ and } y \in \mathcal{A} .
\]
Its properties are described next:

**Proposition 6** Let $x$ and $\mu$ be the left integrals of $\mathcal{A}$ as in (2.22). Then
1. $\mu_D$ is a right integral of $D(A)$ with $\mu_D(p) = 1$.

2. The modulus (as defined in (2.28)) of $D(A)$ is $a_D = g^{-1}$.

3. $\mu_D \in \overline{C}(D(A))$, in particular $\mu_D(\lambda \cdot y) = \omega \mu_D(y \cdot \lambda) \ \forall \ y \in A, \lambda \in A^o$.

Proof: Part 1.) is clear. Also, it follows directly from the definitions of the moduli of $A$ that $1 \otimes \mu_D \Delta(h) = \alpha \cdot a^{-1} \mu_D(h)$. 2.) follows if we apply the antipode and use that $S(\mu_D)$ is a left integral of $D(A)$. In order to show 3.) we observe that the equation for right integrals analogous to (2.35) is $\mu_D(S^2(k)h) = \mu_D(h(k \leftarrow a^{-1}_D))$. Together with Proposition 5 2.) this shows $\mu_D \in \overline{C}(D(A))$. In the case where $k$ and $h$ are in the special subalgebras we use (2.28) to show $\mu_D(S^2(\lambda)y) = \omega^{-1} \mu_D(\lambda y)$ which yields the last equation in part 3.).

The fact that the right integral of a double algebra is invariant under the coadjoint action allows us to identify as an object in the representation category of $D(A)$, namely with the integral of the “braided algebra” [Ly] of the category. Before we explain this aspect in more detail in the next section let us discuss a few more consequences of Proposition 6 for doubles.

It is easy to see that an element of a Hopf algebra $w \in D$ with $S^2(y) = wyw^{-1}$ provides us with an isomorphism $\overline{C}(D) \rightarrow C_o(D) : \lambda \mapsto \lambda \cdot w$. Here $C_o(D)$ denotes as in [Dr1] the space of traces on $D$. Given the two canonical elements in (2.19) we wish to compute the respective traces for $\mu_D$. To this end define the following linear forms on $D(A)$:

\[ \chi : D(A) \rightarrow A^o \otimes A \xrightarrow{\Delta} k \]
\[ \hat{\chi} : D(A) \rightarrow A^o \otimes A \xrightarrow{1 \otimes S^{-1}} A^o \otimes A \xrightarrow{\Delta} k \]

The forms on $D(A)$ and the canonical elements are now related as follows.

**Proposition 7**

\[ \chi = \mu_D \cdot u , \quad \hat{\chi} = \omega^{-1} \mu_D \cdot \hat{u} , \]

and both $\chi$ and $\hat{\chi}$ are nondegenerate traces on $D(A)$.

Proof: From previous considerations it is clear that $\mu_D \cdot u$ and $\mu_D \cdot \hat{u}$ are traces. The rest of the proof are straightforward computations:

\[
(\mu_D \cdot u)(\lambda \cdot y) = (\mu_D \cdot u)(y \cdot \lambda) = \sum_i \mu_D(S(f_i) \cdot e_i \cdot y \cdot \lambda) \\
= \sum_i \mu_D(S^2(\lambda) \cdot S(f_i) \cdot e_i \cdot y) = \sum_i (S^2(\lambda)S(f_i))(x) \mu(S(y)S(e_i))
\]
\[= \sum_{i} f_i(x \leftarrow S^2(\lambda)) \mu(S(y)e_i) = \mu(S(y)(x \leftarrow S^2(\lambda)))\]

by \([2.30]\),

\[= \mu(S(y)\overline{\beta}_i(S^2(\lambda))) = \beta_r \overline{\beta}_i(S^2(\lambda))(S(y))\]

\[= S(\lambda)(S(y)) = \lambda(y) = \chi(\lambda \cdot y)\].

Similarly,

\[(\mu_D \triangleleft \hat{u})(\lambda \cdot y) = \sum_i \mu_D(f_i \cdot \lambda \cdot y \cdot e_i) = \sum_i f_i \cdot \lambda(x) \mu(S(y)e_i))\]

by \([2.30]\),

\[= \mu(S(y(\lambda \rightarrow x))) = \omega \mu(ay(\lambda \rightarrow x))\]

by \([2.31]\),

\[= \omega \mu(ay \overline{\beta}_r(\lambda)) = \omega \beta_r \overline{\beta}_r(\lambda)(ay)\]

Nondegeneracy of \(\chi\) and \(\hat{\chi}\) follow directly from nondegeneracy of \(\mu_D\).

\[\square\]

4.) Balancing in Doubles: In a rigid BTC any object \(X\) is isomorphic to its double conjugate \(X^{\vee}\). Yet the only isomorphism that is a priori canonical is between \(X\) and \(X^{\vee\vee}\). Thus in addition to the usual axioms defining a BTC one often requires the existence of a \(\otimes\)– natural isomorphism of the functor \(X \rightarrow X^{\vee\vee}\) to the identity, which squares to the canonical one from \(X \rightarrow X^{\vee\vee\vee}\) to the identity. For the representation category of a quasitriangular Hopf algebra \(D\) this is equivalent to the existence of a group-like element \(k\) with:

\[k \in G(D), \quad g = k^2, \quad \text{and} \quad S^2(y) = kyk^{-1}. \quad (2.47)\]

It is clear that a balancing does not have exist since often \(g\) is not a square in \(G(D)\). If it does it is unique up to multiplication with central, group like elements of order two, i.e., elements in \(\Sigma(D) := G(D) \cap Z(D)\).

Equivalently, we can consider the corresponding element \(v := u \cdot k^{-1} = \hat{u}^{-1} \cdot k\). Inspecting \([2.21]\) it is easily verified that \(v\) defines a balancing elements iff

\[v \in Z(D), \quad S(v) = v, \quad \text{and} \quad M = v \otimes v\Delta(v^{-1}) \quad (2.48)\]

From these conditions \(\epsilon(v) = 1\) and \(v^2 = z\) follow. This point of view has been introduced in [RT0] where \(v\) is called a \emph{ribbon element}. In their context the eigenvalue of \(v\) in an irreducible representation yields the framing anomalies of colored link.

For a double \(D(A)\) the existence of a balancing can be phrased as a property of the moduli of \(A\).

**Proposition 8**
1. \( k \) is a balancing of \( D(\mathcal{A}) \) if and only if

\[
k = \sqrt{\alpha} \cdot (\sqrt{\alpha})^{-1}
\]

where \( \sqrt{\alpha} \in G(\mathcal{A}^o) \), \( \sqrt{a} \in G(\mathcal{A}) \) square to \( \alpha \) and \( a \) respectively

and

\[
S^2 = \text{ad}^*(\sqrt{\alpha}) \circ \text{ad}(\sqrt{a}^{-1}) \quad \text{on} \quad \mathcal{A}.
\] (2.49)

2. To a given balancing we associate the number \( \nu \) defined by

\[
\nu^{-1} = \chi(k) = \sqrt{\alpha}(\sqrt{a}).
\] (2.51)

If \( D(\mathcal{A}) \) admits a balancing \( \nu \) is a root of unity, \( \nu^4 = \omega \) and \( \nu^2 \) does not depend on the choice of balancing.

Proof: From Lemma 1 and (2.47) we infer that \( k \) has to be a product of group likes of the special subalgebras. By definition of doubles (2.50) is the same as \( S^2(y) = kyk^{-1} \forall y \in \mathcal{A} \). The inverse adjoint of (2.50) yields the same equation on \( \mathcal{A}^o \) and thereby (2.47). For part 2.) we remark that two balancings \( k \) and \( k' \) are related by \( k' = k \cdot R \) where \( R = \rho \cdot r \in \Sigma(D(\mathcal{A})) \cong \Sigma(\mathcal{A}^o) \oplus \Sigma(\mathcal{A}) \). Then \( \chi(k') \cdot \chi(k)^{-1} = \sqrt{\alpha}(r) \rho(\sqrt{a}) \rho(r) \) which is of order two since \( \rho \) and \( r \) are.

In particular, the last statement implies that once a balancing exists the intrinsic quantity \( \omega \) has a canonical square root.

5.) Representation of Mapping Class Groups on Doubles:

In several papers [Ly] Lyubachenko has developed the notion of a Hopf algebra \( F \) in a braided tensor category \( \mathcal{C} \). It is analogue to the notion of braided group, as defined by Majid [M2]. As an object \( F \) in a category with all limits the algebra is the constant functor of the coend \( \left< \text{Hom} : h : \text{Hom} \rightarrow F \right> \) of the functor \( \text{Hom} : \mathcal{C}^{opp} \times \mathcal{C} \rightarrow \mathcal{C} : (X,Y) \rightarrow X^\vee \otimes Y \). For definitions see [Mc]. The multiplication and comultiplication of \( F \) are induced by certain compositions of dinatural transformations using universality of the coend. As opposed to symmetric categories the definition of the multiplication of \( F \) depends on the choice of a commutativity isomorphism. The same is true for the axiom replacing cohomomorphie of the multiplication. An analogous statement of the fundamental theorem of Hopf modules holds for the braided algebras so that under certain finiteness conditions the algebra has an integral \( \mu \in \text{Hom}(1,F) \). The algebra also possesses a braided antipode \( \Gamma \in \text{End}(F) \). Lyubachenko constructs, in analogy to the definitions for semisimple categories, modular operators \( T, S \in \text{End}(F) \). They are determined by the
coend properties of $F$ and the following commutative diagrams:

$$
\begin{array}{ccc}
X \otimes X & \overset{1 \otimes \epsilon_X}{\longrightarrow} & X \otimes X \\
\downarrow h_X & & \downarrow h_X \\
F & \overset{\tau}{\longrightarrow} & F
\end{array}
\quad
\begin{array}{ccc}
X \otimes X & \overset{\gamma_X}{\longrightarrow} & X \otimes X \\
\downarrow h_X & & \downarrow h_X \\
F & \overset{\Gamma}{\longrightarrow} & F
\end{array}
$$

(2.52)

Here $v_\epsilon \in Nat(id)$ is the balancing and $\gamma_X := q_X \otimes 1 \epsilon(X, X)$, where $\epsilon$ is the commutativity constraint and $q_X : X \to X^\vee \otimes X \overset{1 \otimes \epsilon_X}{\longrightarrow} X^\vee \otimes X \otimes X^\vee \to X$. Furthermore,

$$
\begin{array}{ccc}
Y^\vee \otimes Y & \overset{\mu \otimes 1}{\longrightarrow} & F \otimes Y^\vee \otimes Y \\
\downarrow \quad & & \downarrow \quad \\
F & \overset{S}{\longrightarrow} & F \\
\quad & & \leftarrow h_X \quad \\
& & X \otimes X \\
\end{array}
\quad
\begin{array}{c}
(1 \otimes 1 \otimes ev)(1 \otimes \epsilon^2 \otimes 1)
\end{array}
$$

(2.53)

The coend and integral exist if $\mathcal{C}$ is the representation category of a finite dimensional Hopf algebra $D$. Specifically, we have that $F = D^*$, which is a $D$–module by $ad_\ast$-action. The comultiplication is just the multiplication on $D$. However, the multiplication in $F$ stems from a distorted coproduct $\Delta_{Br}$ on $D$ as the usual one is not $ad_\ast$-covariant. In one convention we have, e.g., $\Delta_{Br}(y) = e''_iy'S^{-1}(e'_i) \otimes f_iy''$. As remarked in [LyM] the right integral for the braided multiplication coincides with the ordinary right integral. This is seen easily, e.g., from the fact that $\mu_D$ for a double is $ad_\ast$-invariant. An antipode $\Gamma \in End(D)$ of the braided multiplication consistent with $\Delta_{Br}$ is

$$
\Gamma(A) := \sum_i S(e_i)S(A)\hat{u}_i.
$$

(2.54)

The triple $(D, \cdot, \Delta_{Br})$ is the prototype of a braided group. For an thorough treatment of this structure which inspired the algebra construction in [Ly] we refer to [M2]. However, the construction of the $S$ and $T$ given in [Ly] can also be translated into the context of ordinary Hopf algebras. The action of $T$ is clearly given by multiplication of a ribbon element $v$. The identity of integrals allows us to derive from (2.53) a formula for $S$ acting on a quasitriangular algebra $D$.

$$
S(A) = S(\overline{\mu}_D\triangleleft A) = \sum_{i,j} \mu_D(Af_je_i)S(f_i)S(e_j)
$$

(2.55)

This formula (with slightly different conventions) has been given in [LyM]. Using the form of the right integral given in (3) and applying the bicross formula (2.15) to order $Af_je_i$ this formula can be worked out further. The formula $S(\lambda \otimes h) = \sum_i f_i \otimes (x'') < \lambda, e''_ix'S^{-1}(e'_i) >$ resulting from this has been given in [M1].
Let us now use the properties of integrals given in the previous section and the identities for the canonical isomorphisms to derive an intriguingly, compact formula for $S$. From this form the invertibility of $S$ for doubles is obvious and the inverse readily computed from the identities (2.31) and following.

**Proposition 9** For a double $D(A)$ over a finite dimensional Hopf algebra $A$ let $\mu_D$ be as in (2.44) and $S \in \text{End}(D(A))$ be defined as in (2.55). Then the following diagrams of isomorphisms commute:

\[
\begin{array}{ccccccc}
A \otimes A & \xrightarrow{\beta_l \circ S^{-1} \otimes S \circ \overline{\beta_l}} & A^o \otimes A & A^o \otimes A & \xrightarrow{\overline{\beta_r} \otimes \beta_l \circ L_a} & A \otimes A \\
D(A) & \xrightarrow{S} & D(A) & D(A) & \xrightarrow{S^{-1}} & D(A)
\end{array}
\]

Here $L_a$ denotes left multiplication with $a$.

**Proof:** The first diagram is verified by direct computation:

\[
S(y \cdot \lambda) = \sum_{ij} \mu_D(y \cdot \lambda f_j e_i) \otimes S(f_i) S(e_j) = \sum_{ij} \mu_D(\lambda f_j e_i S^{-2}(y))
\]

\[
= \sum_{ij} \lambda f_j(x) \mu(S^{-1}(y) e_i) f_i \cdot S(e_j) = \mu \circ S^{-1}(y) \cdot S(x \leftarrow \lambda)
\]

\[
= \beta_l(S^{-1}(y)) \cdot S(\overline{\beta_l}(\lambda))
\]

The second diagram follows immediately from relations (2.30) and (2.31), which allow us to invert $\beta_l$ and $\overline{\beta_l}$.

Let us also give a more convenient form for the braided antipode:

**Lemma 10** Let $\tilde{\Gamma} : A^o \otimes A \rightarrow A \otimes A^o$ be given by

\[
\begin{array}{ccccccc}
A^o \otimes A & \xrightarrow{R \otimes 1^2 \circ R} & A \otimes A^o \otimes A \otimes A^o & A \otimes A^o & \xrightarrow{S \otimes 1 \otimes S^{-1} \otimes 1^3} & A \otimes A^o \otimes A \otimes A \otimes A^o
\end{array}
\]

and $\Gamma$ as in (2.54). Then the following diagram commutes:

\[
\begin{array}{ccccccc}
A^o \otimes A & \xrightarrow{\tilde{\Gamma}} & A \otimes A^o \\
D(A) & \xrightarrow{\Gamma} & D(A)
\end{array}
\]
Proof: Straightforward computation:

\[
\Gamma(\lambda \cdot y) = \sum_i S(e_i) S(y) \cdot S(\lambda) \hat{u}_i = \sum_i S(e_i) S(y) \hat{u} S^{-1}(\lambda)f_i
\]

\[
= \sum_{ij} S(S(f_j) ye_i) \cdot e_j S^{-1}(\lambda)f_i,
\]

which is precisely the above composition. \(\square\)

6.) Proof of Modular Relations and the Projective Phases:

For the square of the braided antipode we easily verify

\[
\Gamma^2 = ad^-(v^{-1}),
\]

where \(v\) is any ribbon element and \(ad^-(y) = S^{-1} \circ ad(y) \circ S\). Proposition 8 and Lemma 10 put us in a position to prove the next lemma. From this we will infer one of the modular relations and the correct projective phase.

Lemma 11 We have the following relation for maps \(A^o \otimes A \rightarrow A^o \otimes A:\)

\[
\beta \circ S^{-1} \otimes S \circ \overline{\beta} \Gamma = \omega \otimes \overline{\beta} \circ (\beta L_a)
\] \(2.61\)

Proof: We shall prove \((2.61)\) by evaluating both sides on \(\lambda \otimes y \in A^o \otimes A\) individually and comparing results. For the right hand side we have

\[
\bigotimes \overline{\beta} \circ (\beta L_a)(\lambda \otimes y) = \bigotimes \left( x' \lambda(x'') \otimes \mu \triangleleft (ay) \right)
\]

\[
= \langle (\mu \triangleleft (ay))', x' \rangle \langle S((\mu \triangleleft (ay))''', x'''') \lambda(x''') (\mu \triangleleft (ay))'''' \otimes x'''
\]

\[
= \langle \mu', x' \rangle \langle \mu'', ay S^{-1}(x''') \lambda(x''') \mu'' \otimes x''
\]

\[
= \sum_i \langle \mu', x' \rangle \langle f_i, x'' \rangle \langle \mu'', ay S^{-1}(e_i) \rangle \lambda(e_{i''}) \mu'' \otimes e_i
\]

\[
= \sum_i \langle \underline{\beta_r}, (f_i)''', ay S^{-1}(e_i) \rangle \lambda(e_{i''}) \beta_r(f_i)', \beta_r (f_i)' \otimes e_i
\]

\[
= \sum_i \langle S(f_i) \triangleleft a^{-1}'', ay S^{-1}(e_i) \rangle \lambda(e_{i''}) \left( S(f_i) \triangleleft a^{-1} \right)' \otimes e_i
\]

\[
= \sum_i \langle S(f_i), y S^{-1}(e_i) \rangle \lambda(e_{i''}) S(f_i)' \otimes e_i
\] \(2.62\)
The evaluation of the left hand side gives:

\[
\beta_l \circ S^{-1} \otimes S \circ \overline{\beta_l} \quad (\lambda \otimes y) = \sum_{ij} \beta_l \circ S^{-1} \otimes S \circ \overline{\beta_l} \left( S(S(e_i)ye_j) \otimes (f_i S^{-1}(\lambda)f_j) \right)
\]

\[
= \sum_{ij} \beta_l(S(e_i)ye_j) \otimes S \circ \overline{\beta_l}(f_i S^{-1}(\lambda)f_j)
\]

\[
= \sum_{ij} \left\langle f_i, x' \right\rangle \left\langle S^{-1}(\lambda)f_j, x'' \right\rangle \mu \triangleleft (S(e_i)ye_j) \otimes S(x''')
\]

by (2.28)

\[
= \omega \sum_{ij} \left\langle f_i, x'' \right\rangle \left\langle S^{-1}(\lambda)f_j, e'_i \right\rangle S \left( (S^{-1}(ye_j)x') \mu \triangleleft a \right) \otimes S(e''_i)
\]

\[
= \omega \sum_{ij} \left\langle S^{-1}(\lambda)f_j, e'_i \right\rangle S \left( (S^{-1}(ye_j)) \mu \triangleleft a \right) \otimes S(e''_i)
\]

by (2.31)

\[
= \omega \sum_{ij} \left\langle S^{-1}(\lambda)f_j, e'_i \right\rangle S \left( (S^{-1}(ye_j)) \mu \triangleleft a \right) \otimes S(e''_i)
\]

\[
= \omega \sum_{ij} \left\langle S^{-1}(\lambda)f_j, e'_i \right\rangle \lambda(e''_i) S(f_i)' \otimes e'_i
\]

Comparison of (2.62) to (2.63) proves the assertion. \(\square\)

The projective phases of the second modular relation arise in the computation of the value of \(S\) on the ribbon element.

**Lemma 12** Suppose \(v\) is a ribbon element of a double \(D(A)\) and \(v\) is the associated fourth root of \(\omega\) (see Prop.8). Then we have for \(S\) as defined in (2.53) :

\[
S(v) = v^{-1} v^{-1} \quad S(v^{-1}) = v^5 v \quad (2.64)
\]

**Proof :** Straightforward computation: Using \(v = uk^{-1}\) we have

\[
S(v) = \sum_{ij} \mu_D(uk^{-1} f_j e_i) S(f_i) S(e_j)
\]

by Prop7. = \(\sum_{ij} \chi(e_i k^{-1} f_j) S(f_i) S(e_j)\)
by multiplication with $v$ in Lemma 3 we find with $\tau$

\[ S(\nu) = \sum_{ij} \mu_D(\hat{a}k e_i) S(f_i) S(e_j) = \omega \sum_{ij} \chi(f_j \sqrt{\alpha^{-1}} \sqrt{\alpha}) S(f_i) S(e_j) \]

\[ = \omega \sum_{ij} \langle f_j \sqrt{\alpha^{-1}}, S(e_i) \sqrt{\alpha^{-1}} \rangle S(f_i) S(e_j) = \omega \sum_{i} f_i S(\sqrt{\alpha^{-1}} \cdot (e_i \sqrt{\alpha^{-1}})) \]

\[ = \omega \sum_{i} f_i \sqrt{\alpha} S(\sqrt{\alpha^{-1}} \cdot e_i) = \omega \sum_{i} f_i \sqrt{\alpha} \cdot \sqrt{\alpha} S(e_i) \]

\[ = \nu^5 k^{-1} u = \nu^5 v \quad . \tag{2.66} \]

Let us now prove the second modular relation:

**Proposition 13** For a double $D(A)$ with balancing, let $S$ be defined as in (2.55) and $T$ by multiplication with $v$. Then

\[ ST^{-1} S = \nu^5 TS T \tag{2.67} \]

**Proof**: If we apply $\eta \circ S^{-1} \otimes 1 \otimes S$ for some $\eta \in \Delta(A)^*$ to both sides of the equation in Lemma 3 we find with $\tau(M) = S \otimes S(M)$ that

\[ S \circ T (\lambda \otimes S \circ T(\eta)) = (\lambda \otimes S)((S(T(\eta)) \otimes 1) M) \]

\[ = \eta \circ S^{-1} \otimes \lambda \otimes S ((\tau(M) \otimes 1) (1 \otimes M)) \]

\[ = \sum_{ij} \eta(S^{-1}(T(\lambda') f_i f_j)) \otimes S(e_i) S(T(\lambda'')) e_j \quad . \tag{2.68} \]

Inserting into (2.68) the forms $\lambda = \mu_D \otimes \rho$ and $\eta = \mu_D \otimes A$ for some $A, \rho \in \Delta(A)$ and by using the definition (2.55) we find:

\[ S \circ L_\rho \circ S(A) = \sum_{ij} \mu_D(A S^{-1}(\langle S^{-1}(S(\rho))', f_i f_j \rangle) S(e_i) S(S^{-1}(S(\rho))') e_j \]

\[ = \sum_{ij} \mu_D(A S^{-1}(f_j) f_i S^{-2}(S(\rho)'')) S^2(e_i) S(\rho)' e_j \quad . \tag{2.69} \]
Here $L_\rho$ is the left multiplication with $\rho$. The left hand side of the assertion \((2.64)\) is now found by specializing $\rho = v^{-1}$ where $L_v = T$. In order to evaluate the right hand side of \((2.69)\) we notice that Lemma 12 implies the following identities:

$$
\Delta(S(v^{-1})) = \nu^5 \Delta(v) = \nu^5 v \otimes v M^{-1} = \nu^5 v \otimes v R^{-1} \tau(R^{-1}) = \nu^5 v \otimes v S^3 \otimes S^2(R) 1 \otimes S(\tau(R)) = \nu^5 \sum_{kl} v S^3(e_k)f_i \otimes v S^2(f_k) S(e_l)
$$

Replacing $S(v^{-1})' \otimes S(v^{-1})''$ in \((2.69)\) by this expression yields the assertion:

$$
ST^{-1} S = \nu^5 \mu_D \left( A S^{-1}(f_j) f_i v f_k S^{-1}(e_l) \right) v S^2(e_i) S^3(e_k) f_i e_j
$$

$$
= \nu^5 \sum_{j l} \sum_{ik} \mu_D \left( v A S^{-1}(f_j)(f_i f_k) S^{-1}(e_l) \right) v S^2(e_i) S(e_k) f_i e_j
$$

by \((2.13)\)

$$
by(2.13) = \nu^5 \sum_{j l} \mu_D \left( v A S^{-1}(f_j) S^{-1}(e_l) \right) v f_i e_j
$$

We readily identify the last equation with the right hand side of \((2.67)\). This completes the proof.

The $S$ matrix was originally defined as an element in the $\text{End}$ -set of the coend of the representation category. As a map on $D(A)$ it therefore intertwines the $ad^-$ - action (see \((2.60)\)) of the algebra on itself. (This property can also be inferred directly from Lemma \(2\).) The same is true for multiplications with central elements as for example for $T$. Hence the center $Z(D(A))$ - which is the invariance of the $ad^-$ - action - is an invariant subspace of both operators. It follows immediately from \((2.59)\) that the restriction of $\Gamma$ to the center is the usual antipode $S$ and thus involutive.

We summarize these observations and the relations found in \((2.60), (2.61)\), and \((2.67)\) in the following theorem:

**Theorem 1** Suppose $D(A)$ is the double of a finite dimensional Hopf algebra. Assume that $D(A)$ admits a balancing and let $\nu$ and $\omega$ be as in Proposition 8. Furthermore, let $T$ be the multiplication with $v$, and $S$ and $\Gamma$ be defined as in \((2.53)\) and \((2.59)\), respectively.

Then

1. The generators define a projective representation of the mapping class group $\mathcal{D} := \pi_o(\text{Diff}(T, D))$ of torus maps fixing a disk with the following relations:

$$
S^2 = \omega \Gamma^{-1} \quad \quad \quad T \Gamma = \Gamma T
$$

$$
(S T)^3 = \nu^3 \Gamma^{-2} = \nu^3 \text{ad}^-(v)
$$

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2. The maps $S$ and $T$ stabilize the center $Z(D(A))$. The restrictions $\overline{S}$ and $\overline{T}$ satisfy

$$(\overline{S}\overline{T})^3 = \nu^3 \quad \overline{S}^2 = \omega S^{\pm 1} \quad \overline{T}S = ST$$

(2.73)

where $S$ is the involutive map given by the restriction of the antipode to the center.

The relations in (2.73) show that $\overline{S}$ and $\overline{T}$ define a projective representation of $SL(2, \mathbb{Z})$. The normalization of the $\overline{S}$-operation was defined by the canonical normalization of $\mu_D$. For the computation of topological invariants it is often more convenient to have a normalization for which the operators are inverted if we invert the braided structure. For a given balancing $k$ let $\overline{S}'$ and $\overline{T}'$ be the analogous operators defined with respect to $R' = \tau(R^{-1})$. Then as $u' = \hat{u}$ we have that $\overline{T}' = \overline{T}^{-1}$ is already correctly normalized. A computation similar to the one in Proposition 9 yields

$$\overline{S}' = \omega \overline{S}^{-1}.$$ 

Thus it is the matrix $S_* := \nu^2 \overline{S}^{-1}$ which inverts under inversion of the braided structure. For these generators we have the relations:

$$S_*^4 = 1 \quad (S_* \overline{T})^3 = \nu^{-3} S_*^2$$

(2.74)

Comparing (2.74) to relations in [T] and [RT] we find that the projective phase $c$ of the functor $\Phi$ in (1.10) for a universal TQFT over a double $D(A)$ is given by:

$$c = \nu^{-3}.$$ 

3. The Relation of Universal and Semisimple TQFT’s: An Example

In this section we shall analyze the proposed representation of the mapping class group $D$ of the punctured torus explicitly in the example of the double of the quantum-$sl_2$-Borel algebra $B_q$.

1.) The Algebra $D(B_q)$: Let $q$ be a primitive $l$-th root of unity where $l = 2m + 1$, $m \in \mathbb{Z}_{\geq 1}$. We denote by $B_q$ the Hopf algebra with generators $e$, $k^{\pm 1}$ and relations:

\[
\begin{align*}
kek^{-1} &= qe, & k^l &= 1 & e^l &= 0 \\
\Delta(k) &= k \otimes k & \Delta(e) &= e \otimes 1 + k^2 \otimes e \\
S(e) &= -k^{-2}e & S(k) &= k^{-1} & \epsilon(e) &= 0 & \epsilon(k) &= 1.
\end{align*}
\]
As PBW-basis for $B_q$ we choose $e^n k^j$ with $n = 0, \ldots, l - 1$ and $j \in \mathbb{Z}/l$. The left cointegral of $B_q$ is given by

$$x = (\sum_{j=0}^{l-1} k^j)e^{l-1} \tag{3.76}$$

and the left integral with normalization $m(x) = 1$ is

$$m(e^n k^j) = q^2 \delta_j \delta_{n,(l-1)}. \tag{3.77}$$

The moduli of these integrals are easily found to be

$$a = k^2 \quad \text{and} \quad \alpha(k) = q, \quad \alpha(e) = 0 \tag{3.78}$$

so that

$$\omega = q^2. \tag{3.79}$$

Since we assumed $l$ to be odd we can choose as generators of the dual algebra $B_q^*$ the modulus $\alpha$ and the linear form $f$ defined by $<f, e^n k^j> = \delta_{n,1}$. The following relations together with those in (3.75) can be used as a definition for the double $D(B_q)$ containing $B_q$ and $B_q^*$ with opposite comultiplication:

$$\begin{align*}
\alpha f \alpha^{-1} &= q^2 f \\
\alpha e \alpha^{-1} &= q^{-2} e \\
f f = f &= k^2 \\
\Delta(\alpha) &= \alpha \otimes \alpha \\
\Delta(f) &= f \otimes \alpha + 1 \otimes f
\end{align*} \tag{3.80}$$

We shall sometimes refer to the $\mathbb{Z}$-gradation of $D(B_q)$ which is defined on the generators by $gr(e) = +1$, $gr(k) = 0$, $gr(f) = -1$, and $gr(\alpha) = 0$. The universal $\mathcal{R}$-matrix of this algebra is

$$\mathcal{R} = (\sum_{n=0}^{l-1} \frac{q^{-n(n-1)}}{n!} e^n \otimes f^n)(\frac{1}{l} \sum_{i,j \in \mathbb{Z}/l} q^{-ij} k^i \otimes \alpha^j) \tag{3.81}$$

Here $[n] = [n][n-1] \ldots [1]$ with $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$. If we compute the expressions in (2.23) and (2.24) for the integrals in (3.76) and (3.77) we obtain the following isomorphisms between $B_q$ and $B_q^*$:

$$\begin{align*}
\beta_l(k^j e^n) &= \frac{q^{(l-1-n)(l-2-n)}}{[l-1-n]!} f^{l-1-n} \frac{1}{l} \sum_{i \in \mathbb{Z}/l} q^{ij} k^i \alpha^{i+1}. \tag{3.82}
\end{align*}$$

and

$$\begin{align*}
\overline{\beta}_l(\alpha^i f^n) &= (-1)^n [n] q^{n(n+3)} 2^{-2i} e^{l-1-n} \sum_{j \in \mathbb{Z}/l} q^{j(i-1)} k^j. \tag{3.83}
\end{align*}$$
As an associative algebra $D(B_q)$ is isomorphic to the product $C[Z/l] \otimes U_q(sl_2)$, where the central group algebra $C[Z/l]$ is generated by
\[ z := \alpha^{-m}k \] (3.84)

The generators of the $U_q(sl_2)$ factor are defined by
\[ E := z^{-1}e, \quad F := -f, \quad K := \alpha^m k \] (3.85)

and obey the relations
\[ KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F, \] (3.86)
\[ EF - FE = K - K^{-1}. \] (3.87)

2.) The Center of $D(B_q)$ : Thanks to the above decomposition the center of $D(B_q)$ is given by $C[Z/l] \otimes V$ where $V$ is the center of $U_q(sl_2)$. In order to give a description of $V$ it is convenient to introduce the projections
\[ \pi_j(K) = \frac{1}{l} \sum_{i \in Z/l} q^{2ij} K^i \quad j \in Z/l \] (3.88)
on the eigenspaces of $K$ with eigenvalue $q^{-2j}$. Furthermore we introduce the projections
\[ T_j = \sum_{s=j+1}^{l-1-j} \pi_s(K) \quad j = 0, \ldots, m - 1. \] (3.89)

The standard quadratic Casimir of $U_q(sl_2)$ is given by:
\[ X = EF + \frac{qK^{-1} + q^{-1}K}{q - q^{-1}}. \] (3.90)

The trivially graded part $U^\circ$ of $U_q(sl_2)$ ($\text{gr}(E)=1, \text{gr}(F)=-1, \text{gr}(K)=0$) is a free module over the ring $C[K]$ with basis $\{X^j\}_{j=0,...,l-1}$ and the minimal equation for $X$ is:
\[ \prod_{j=0}^{l-1} (X - b(j)) = 0, \] (3.91)

where the roots
\[ b(j) = b(l - 1 - j) := \frac{q^{(2j+1)} + q^{-(2j+1)}}{q - q^{-1}} \] (3.92)
are of order two for \( j = 0, \ldots, (m-1) \) and of order one for \( j = m \).

Using the polynomials
\[
\phi_j(X) = \prod_{0 \leq s \leq (l-1); b(s) \neq b(j)} (X - b(s)) \quad j = 0, \ldots, m
\] (3.93)
of order \((l-2)\) for \( j < m \) and of order \((l-1)\) for \( j = m \) we can define the idempotents and nilpotents associated to \( X \):
\[
P_j = \frac{1}{\phi_j(b(j))} \phi_j(X) - \phi'_j(b(j)) (X - b(j)) \phi_j(X) \quad j = 0, \ldots, m
\] (3.94)
\[
N_j^+ = T_j N_j \quad N_j^- = (1 - T_j) N_j
\]

For example a general polynomial \( \Psi(X) \) in \( X \) is expressed in terms of \( P_j \) and \( N_j \) by the formula:
\[
\Psi(X) = \sum_{j=0}^{m-1} \Psi(b(j)) P_j + \sum_{j=0}^{m-1} \Psi'(b(j)) N_j
\] (3.95)

The normalizations in (3.94) can be evaluated explicitly using
\[
\phi_j(b(j)) = \frac{1}{(q-q^{-1})^d [d^\pm_j]^2}
\]
\[
\phi'_j(b(j)) = -\frac{1}{(q-q^{-1})^d [d^\pm_j]^3}
\]
\[
\phi_m(b(m)) = \frac{1}{(q-q^{-1})^{d-1}}
\] (3.96)

The center of the quantum algebra does not only contain the subalgebra generated by \( X \) but also the above combinations of nilpotents with the weight-projectors \( T_j \). More precisely, we have the following lemma:

**Lemma 14** The center, denoted by \( \mathcal{V} \) of \( U_q(sl_2) \) is the \((3m+1)\)-dimensional algebra with basis \( \{ P_i, N_j^\pm : i = 0, \ldots, m; j = 0, \ldots, m-1 \} \) and products:
\[
P_i P_j = \delta_{ij} P_j
\]
\[
P_i N_j^\pm = \delta_{ij} N_j^\pm
\]
\[
N_i^\pm N_j^\pm = N_i^\pm N_j^\mp = 0
\] (3.97)

**Proof:** We use the fact that every element \( y \) in the trivially graded part \( U_o \) has a unique presentation:
\[
y = \sum_{s \in \mathbb{Z}/l} \pi_s(K)p_s(X)
\]

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wher the \( p_s \) are polynomials of order smaller than \( l \). The condition that \( y \) commutes with \( E \) is then:
\[
\sum_{s \in \mathbb{Z}/l} \pi_s(K)(p_s(X) - p_{s-1}(X)) \in \mathcal{I}
\]
Here we denote the ideal \( \mathcal{I} = \{ y \in U^o : Ey = 0 \} \). It is clear that \( \mathcal{I} \) is generated by
\[
E^{l-1}F^{l-1} = \sum_{s \in \mathbb{Z}/l} \pi_{s+1}(K) \prod_{j \in \mathbb{Z}/l, j \neq s} (X - b(j))
\]
The polynomials in \( X \) occurring in this sum are proportional to the nilpotents and idempotents defined in (3.94). The ideal \( \mathcal{I} \) is therefore spanned by the elements
\[
\pi_{s+1}(K)N_s, \ s = 0, \ldots, m - 1 \quad \text{and} \quad \pi_{m+1}F_m
\]
Solving the recursion for the \( p_j \)'s we find that the center is generated by the elements in (3.94). The commutation with \( F \) yields exactly the same conditions. Linear independence of these generators can be shown by choosing special representations of \( X \).

3.) Canonical Elements and Balancing : The canonical, group like element, \( g \), from (2.20) implementing the fourth order of the antipode is obtained from the equations for the moduli:
\[
g = \alpha k^{-2} = K^{-2}
\]
For odd \( l \) this element has precisely one square root in the group like elements,
\[
\sqrt{g} = K^{-1}
\]
so that we have uniqueness of balancing. The fourth root of one associated to this by (2.51) is
\[
\nu = q^{-m}
\]
The canonical element \( u = S(R^{(2)})R^{(1)} \) can be expressed as the following product of commuting elements
\[
u = \sum_{n=0}^{l-1} q^{\frac{n(n+3)}{2}} K^n F^n E^n
\]
30
Here we denote the Gauss sum \( \gamma_q := \frac{1}{\sqrt{l}} \sum_{j=0}^{l-1} q^{mj^2} \), which is a phase for odd \( l \) and can be evaluated explicitly (see e.g. [L]). The unique ribbon element \( v \) can be written as a product of an element in the \( \mathbf{C}[\mathbf{Z}/l] \) factor and an element in the \( U_q(sl_2) \)-factor of the algebra:

\[
v = u_Z v_o
\]

where

\[
v_o = K u_K u_o \quad (3.102)
\]

If we denote by \( \mathcal{T}_Z, \mathcal{T}_o \) and \( \mathcal{T} \) the linear operators on \( \mathbf{C}[\mathbf{Z}/l], U_q(sl_2) \) and \( D(B_q) \) defined by multiplication with \( u_Z, v_o \) and \( v \) respectively, this implies

\[
\mathcal{T} = \mathcal{T}_Z \otimes \mathcal{T}_o \quad (3.103)
\]

We have the following expression for the central element \( v_o \) in terms of the basis given in Lemma 14:

**Lemma 15** The central ribbon element \( v_o \in U_q(sl_2) \) has the Jordan decomposition

\[
v_o = q^m P_m + \sum_{j=0}^{m-1} q^{2(j+1)} \left( P_j + \frac{d^+_j}{[d^-_j]} N^+_j + \frac{d^-_j}{[d^-_j]} N^-_j \right),
\]

(3.104)

Here the basis elements of \( \mathcal{V} \) are the same as in (3.94) and the numbers \( d^+_j = 1, \ldots, l-1 \), are defined for \( j = 0, \ldots, m-1 \) by

\[
d^+_j := 2j + 1 \quad d^-_j := l - (2j + 1).
\]

**Proof**: The computation of these coefficients is most conveniently done by multiplying the expression for \( v_o \) obtained from (3.101) by a weight projector \( \pi_s(K) \). The result can be expressed in terms of a polynomial \( \Psi_s(X) \) of the quadratic Casimir \( X \):

\[
\pi_s(K)v_o = \pi_s(K)\Psi_s(X),
\]

(3.105)

where

\[
\Psi_s(X) = \sum_{n=0}^{l-1} \frac{q^{n(n+1)}}{[n]!} q^{2a(a-n-1)} \prod_{i=l-n}^{l-1} (X - b(i + s)).
\]

(3.106)

From the general expansion (3.99) we see that the coefficient of \( P_j \) is given by \( \Psi_s(b(j)) \) for any \( s \) and the coefficients of \( N^+_j \) and \( N^-_j \) are given by \( \Psi'_s(b(j)) \) where \( s = j + 1, \ldots, l - 1 - j \) and \( s = -j, \ldots, j \) respectively. For a choice of \( s \) with \( b(s-1) = b(j) \) we can avoid one summation in the expressions for \( \Psi_s \) and \( \Psi'_s \). In order to evaluate the remaining sum for \( \Psi' \) we invoke the partition identity for \( t \) with \( t^i \neq 1 \) for \( i = 1, \ldots, d \):

\[
\frac{d}{1 - t^d} = \sum_{n \geq 1} \frac{1}{1 - t^n} \prod_{i=1}^{n-1} (1 - t^{(d-i)})
\]

(3.106)
Remark: From the observation that the coefficients should be independent of the choice of the weight $s$ we are led to new partition identities. For example in the computation of $\Psi_s$ we find the formula:

$$t^{AB} = \sum_{n=0}^{\min(A,B)} \prod_{i=0}^{n-1} \frac{(t^A - t^i)(t^B - t^i)}{t^i(t^{(i+1)} - 1)}.$$  

4.) The $SL(2,\mathbb{Z})$-Action on the Center of $D(B_q)$: We use the formula obtained in (9) to give the explicit action of $S$ on $D(B_q)$. Together with $T$ defined by multiplication with the ribbon element this yields a representation of the mapping class group $D$ on $D(B_q)$. If we insert the expressions for the integrals from (3.82) and (3.83) the action of $S$ can be immediately written if we use both PBW bases $k^j e^n f^p \alpha^s$ and $\alpha^s f^n e^p k^3$ as:

$$S(k^j e^n f^p \alpha^s) = \frac{(-1)^n[p]!}{l-1-n} q^{\frac{(n+1)(n+2)}{2} + (n+1)j + \frac{p(p-1)}{2}} \times$$

$$\left( \frac{1}{l} \sum_{i \in \mathbb{Z}/l} q^{-ij} \alpha^i \right) f^{(l-1-n)} e^{(l-1-p)} \left( \sum_{i \in \mathbb{Z}/l} q^{i(s+p)} k^{-i} \right).$$ \hspace{1cm} (3.107)

A similar formula was obtained in [LyM]. It is immediate from the above form that the $S$-matrix preserves the gradation $n-p \in \mathbb{Z}$ of a basis element. Given that the balancing element is trivially graded and acts by multiplication it follows that the $D$-representation on $D(B_q)$ decomposes into a direct sum of the $2l-1$ spaces corresponding each gradation.

Clearly, the category from which $S$ is obtained is the tensor product of the representation category of $U_q(sl_2)$ and $\mathbb{C}[\mathbb{Z}/l]$ as an abelian category. Also, since the balancing element and hence the monodromy can be factorized into a product of invertible elements from either algebra the $S$-matrix has to factorize too. More precisely we define the following isomorphisms on $C[\mathbb{Z}/l(z)]$,

$$S_Z(z^n) := \frac{1}{\sqrt{l}} \sum_{j \in \mathbb{Z}/l} q^{-jn} z^j.$$ \hspace{1cm} (3.108)

and on $U_q(sl_2)$

$$S_\alpha(K^j E^n F^p) := \frac{(-1)^p[p]!}{l-1-n} q^{\frac{(n-p)(n+1)}{2} + j(2n+1-p)+1} \times$$

$$\left( \frac{1}{\sqrt{l}} \sum_{k \in \mathbb{Z}/l} q^{k(n-k)} K^k \right) E^{(l-1-n)} E^{(l-1-p)}.$$ \hspace{1cm} (3.109)
Using the isomorphism $D(B_q) \cong C[Z/l] \otimes U_q(sl_2)$ defined by the change of basis in (3.84) and (3.85) we can now write the $S$-matrix in the form:

$$S = S_Z \otimes S_o$$  \hspace{1cm} (3.110)

Together with (3.103) this shows that the representation of $D$ on $D(B_q)$ is given by the tensor product of two projective representations of $D$. Since $C[Z/l]$ is central in $D(B_q)$ we expect the representation generated by $T_Z$ and $S_Z$ to factor through a projective representation of $SL(2,Z)$. In fact we easily verify the following relations

$$S_Z T_Z = T_Z S_Z^2, \quad S_Z^2(z^n) = z^{-n}, \quad (S_Z T_Z)^3 = \gamma q I$$  \hspace{1cm} (3.111)

It is clear that the action of $S_o$ on $U_q(sl_2)$ preserves the gradation in the same way as the action of $S$ on $D(B_q)$. For example the restriction on the highest $l-1$ graded subspace defines for each $g_o \in D$ by

$$g_o(aKE^{l-1}) = \hat{g}(a)KE^{l-1}$$

an action $\hat{g}$ on an element $a$ in the group algebra $C[Z/l]$ generated by $K$. It factors into an $SL(2,Z)$ representation and is equivalent to the one defined previously by $T_Z$ and $S_Z$ with $q$ replaced by $q^{-1}$.

In the following we shall focus on the $0$-graded part $U_o$ of $U_q(sl_2)$ from where we wish to compute the restrictions to the center. We determine explicitly the $(3m+1)$-dimensional representation matrices of $SL(2,Z)$ which we obtain by restricting the action of $D$ onto the center $V$ of $U_q(sl_2)$. We choose the basis as in Lemma 14 in the order $P_0, N^+_0, N^-_0, P_1, \ldots, N^-_{(m-1)}, P_m$. On the subspace spanned by $P_j, N^+_j, N^-_j$ we define the Jordan block:

$$\tau_j := q^{2j(j+1)} \begin{bmatrix} 1 & 0 & 0 \\ d^+_j & 1 & 0 \\ d^-_j & 0 & 1 \end{bmatrix} \quad \text{for } j = 0, \ldots, (m-1).$$

Then it is obvious from the formula in Lemma 15 that the $T_o$ matrix defined by multiplication of $v_o$ is given by the direct sum:

$$T_o = \tau_0 \oplus \tau_1 \oplus \ldots \oplus \tau_{(m-1)} \oplus q^m$$  \hspace{1cm} (3.112)

The restriction of $S_o$ to the center is much more complicated and will be dealt with in the rest of this section. It involves finding transformations from the idempotents and nilpotents given in Lemma 14 to polynomials in $X$ and $K$, to the standard PBW basis.
of \( U_q(sl_2) \) and backwards. The transformations between the center and expressions in \( X \) and \( K \) can be obtained from the relations given in (3.94) and (3.95). In order to reexpress polynomials in \( X \) and \( K \) in terms of the basis \( K^l E^m F^n \) and conversely we need the following two technical lemmas. A special case of the first lemma we used already in the computation of the center. The proof is straightforward.

**Lemma 16** Let \( X \) be the quadratic Casimir defined in (3.94) and set
\[
Q_j = \frac{q^{(2j+1)}K^{-1} + q^{-(2j+1)}K}{q - q^{-1}}
\]  
then the following relations hold
\[
E^j F^j = \prod_{s=0}^{j-1} (X - Q_s) \]  
\[
F^j E^j = \prod_{s=l-j}^{l-1} (X - Q_s) .
\]  

Before we give the converse transformations let us state the following identity for general polynomials

**Lemma 17** Suppose \( \lambda_0, \ldots, \lambda_N \) is an ordered set of roots and \( 0 \leq a_1 < a_2 \ldots < a_k \leq N \) are an ordered set of \( k \) indices then we have
\[
\prod_{j=0, j \neq \{a_i\}}^{N} (X - \lambda_j) = \sum_{0 \leq s_1 < s_2 < \ldots < s_k \leq N} \prod_{i=0}^{s_1-1} (X - \lambda_i) \prod_{i=s_1+1}^{s_2-1} (\lambda_{a_{i_1}} - \lambda_i) \ldots \prod_{i=s_k+1}^{N} (\lambda_{a_{i_k}} - \lambda_i)
\]  
reexpressing a polynomial with omitted roots in terms of polynomials with consecutive roots.

Here an empty product is meant to be 1. The proof is a straightforward induction which is most conveniently done by assuming the statement for for all \( (k, N') \) with \( k' < k \) or \( k' = k \) and \( N' \leq N \) and proving it for \( k' = k \) and \( N' = N + 1 \) thus for all pairs with \( k' = k \). This is followed by an induction in \( k \). If we combine Lemma [16] and Lemma [17] we arrive at the following formula for the polynomials defined in (3.93)

**Lemma 18** Let \( \phi_k(X) \) be the polynomials in the quadratic Casimir \( X \) as defined in (3.93), \( \pi_s(K) \) the projector from (3.88) and \( b(j) \) as in (3.92). Then
\[
\pi_t(K)(X - b(k))\phi_k(X) = \sum_{j=0}^{t-1} \prod_{i=j+1}^{t-1} (b(k) - b(i + t)) \pi_t(K)E^j F^j ,
\]  
34
\[ \pi_t(K)\phi_m(X) = \sum_{j=0}^{l-1} \prod_{i=j+1}^{l-1} (b(m) - b(i + t)) \pi_t(K)E^jF^j , \quad (3.118) \]
\[ \pi_t(K)\phi_k(X) = \sum_{j=0}^{l-2} \sum_{s=j+1}^{l-1} \prod_{i=j+1}^{l-1} (b(j) - b(i + t)) \pi_t(K)E^jF^j . \quad (3.119) \]

**Proof**: We apply Lemma 17 to the situation where \( N = l - 1 \), \( X \) is the quadratic Casimir and the roots \( \lambda_j \) are replaced by the elements \( Q_j \) defined in (3.113). The polynomials with consecutive roots on the right hand side of (3.116) are precisely those in (3.114). Thus for \( k = 1, 2 \) we obtain the specializations:

\[ \prod_{j=0, j \neq a}^{l-1} (X - Q_j) = \sum_{j=0}^{l-1} \prod_{s=j+1}^{l-1} (Q_a - Q_s) E^jF^j . \quad (3.120) \]

and

\[ \prod_{j=0, j \neq a, b}^{l-1} (X - Q_j) = \sum_{j=0}^{l-1} \sum_{s=j+1}^{l} \prod_{i=j+1}^{s-1} (Q_p - Q_i) \prod_{i=s+1}^{l} (Q_b - Q_i) E^jF^j . \quad (3.121) \]

Notice that the polynomials \( \phi_k(X) \) and \( (X - b(k))\phi_k(X) \) are obtained from (3.91) by omitting one or two roots. If we multiply equation (3.120) and (3.121) with the projector \( \pi_t(K) \) for suitable choices of \( a \) and \( b \) we obtain these polynomials on the left hand side. The identities (3.117) - (3.119) follow using \( \pi_t(K)Q_j = \pi_t(K)b(j + t) \).

\[ \Box \]

Lemma 18 puts us now in the position to determine the action of \( S_o \) on the polynomials \( \phi_k(X) \) and \( (X - b(k))\phi_k(X) \).

Insertion into (3.109) yields for \( k = 0, \ldots, (m - 1) \):

\[ S_o\left( \pi_t(K)(X - b(k))\phi_k(X) \right) = \sum_{b \in \mathbb{Z}/l} \pi_b(K)\Delta_b^{tk}(X) , \quad (3.122) \]
\[ S_o\left( \pi_t(K)\phi_k(X) \right) = \sum_{b \in \mathbb{Z}/l} \pi_b(K)\Gamma_b^{tk}(X) , \quad (3.123) \]
\[ S_o\left( \pi_t(K)\phi_m(X) \right) = \sum_{b \in \mathbb{Z}/l} \pi_b(K)\Delta_b^{tm}(X) . \quad (3.124) \]

The polynomials \( \Delta_b^{tk}(X) \) and \( \Gamma_b^{tk}(X) \) are defined by

\[ \Delta_b^{tk}(X) := \]  

(3.125)
coefficients of polynomials can be expanded for every weight \( b \) polynomials at \( X \) of projections following quantities: 

\[ S \]

The action of following Lemma 15 this can be used to produce new families of partition identities.

By general construction the \( S \)-matrix has to map \( V \) to itself. As in the remark following Lemma 13 this can be used to produce new families of partition identities.

In order to find the matrix coefficients of the \( SL(2, \mathbb{Z}) \) representation we need the following quantities:

\[ \eta(d_A, d_B) := \]

\[ [d_A]^2 \frac{(q - q^{-1})^l}{l^2} \sum_{j=1}^{\min(d_B, d_A - 1)} \sum_{s=j+1}^{d_A} [l - 1 - j]! \frac{(-1)^j}{[j]} q^{(j - d_B)(j + 1 + d_A - 2s)} \times \]

\[ \prod_{i=1}^{j} (q - q^{-1}) \sum_{i \neq j} [s - i][d_A - s + i] \prod_{i=1}^{j-1} \left( q^{(d_B - i)} - q^{-(d_B - i)} \right) \]

\[ \mu(d_A, d_B) := \]

\[ [d_A]^2 \frac{(q - q^{-1})^l}{l^2} \sum_{s \in \mathbb{Z}/1} \sum_{j=1}^{d_B} \sum_{r=1}^{j} [l - 1 - j]! \frac{(-1)^j}{[j]} q^{(j - d_B)(j + 1 + d_A - 2s)} \times \]

\[ \prod_{i=1}^{j} (q - q^{-1}) \sum_{i \neq j} [s - i][d_A - s + i] \prod_{i=1}^{j-1} \left( q^{(d_B - i)} - q^{-(d_B - i)} \right) \]

and

\[ \rho(d_B) := \frac{(q - q^{-1})^{l-1}}{l^2} \sum_{j=1}^{d_B} \sum_{s=j+1}^{l} [l - 1 - j]! \frac{q^{(j - d_B)(j + 1 - 2s)}}{[j]} \times \]

(3.129)
\[ \prod_{i=1}^{j}(q - q^{-1})[s - i]^2 \prod_{i=1}^{j-1}(q^{(d_B - i)} - q^{-(d_B - i)}) \]

The main result of the previous calculation - the \( SL(2, \mathbb{Z}) \) representation on \( V \) - is described in the next theorem:

**Theorem 2** Let \( P_0, N_0^+, N_0^-, P_1, \ldots, N_{(m-1)}, P_m \) be the ordered basis of \( V \) as defined in Lemma 14. Then the following matrices define a projective \( SL(2, \mathbb{Z}) \) representation.

1. For \( k = 0, \ldots, m - 1 \)

\[ S_o(N_k^\pm) = \frac{q}{\sqrt{l}} \frac{q - q^{-1}}{l} [d_k^\pm]^2 \sum_{p=0}^{m-1} \frac{[d_k^\pm d_p^\mp]}{[d_p^\mp]} P_p \]

\[ + \frac{q}{\sqrt{l}} [d_k^\pm]^2 d_k^\pm P_m \]

2. For \( k = 0, \ldots, m - 1 \)

\[ S_o(P_k) = \frac{q}{\sqrt{l}} \frac{[2d_k^\pm]}{[d_k^\pm]} P_m \]

\[ + \frac{q}{\sqrt{l}} \sum_{p=0}^{m-1} \sum_{\epsilon = \pm} \left( \mu(d_k^\pm, d_p^\mp) + \frac{[2d_k^\pm] \eta(d_k^\pm, d_p^\mp) + \eta(d_k^\pm, d_p^\mp)}{[d_k^\pm]^3} \right) N_p^\epsilon \]

3. \[ S_o(P_m) = \frac{q}{\sqrt{l}} P_m + \frac{q}{\sqrt{l}} \sum_{p=0}^{m-1} \sum_{\epsilon = \pm} \rho(d_p^\pm) N_p^\epsilon \]

These matrices satisfy the relations

\[ (S_o T_o)^3 = \gamma q^{-1} q^{(1-m)} \mathbb{I} \]

\[ (S_o)^2 = q^2 \mathbb{I} \]

Here the superscript * means that either + or - can be inserted yielding the same result.
5.) The Structure of the $SL(2, \mathbb{Z})$-Representation on $\mathcal{V}$: For small values of $l$ the following polynomial identities hold true:

$$\eta(d_k^+ d_p^+ + d_k^- d_p^- + d_k^+ d_p^- = \eta(d_k^- d_p^+ + d_k^+ d_p^-$$

and

$$\rho(\rho_p^+) = \rho(d_p^-) \quad (3.130)$$

In this case it’s easy to see that the representation contains an $m+1$-dimensional subrepresentation spanned by the $N_j$’s and $P_m$.

On this subspace the $T$-matrix is diagonal and has eigenvalues \{q^{2j(j+1)}, j = 0, \ldots, m\}, i.e., one more than the finite $m$-dimensional representation obtained from the semisimplified representation category. For prime $l$ it is not hard to see that the representation is irreducible. Also, for small $l$ we find that it is finite.

It is clear by inspection of the $T$-matrix that a complement to this representation has to contain the linearly independent nilpotents $\tilde{N}_j = \sum_{\epsilon=\pm} \frac{d^\epsilon}{[d_j]} N_j^\epsilon$. Thus it also contains the vectors $S_o(\tilde{N}_j)_k$ where the subindex $k = 0, \ldots, m-1$ means that we take the component in the $k$-th eigenspace of $T$. Since the elements $S_o(\tilde{N}_0)_k$ are linearly independent from the nilpotents, a $2m$-dimensional complement exists only if

$$S_o(\tilde{N}_j)_k = c_{jk} S_o(\tilde{N}_0)_k + b_{jk} \tilde{N}_k$$

(3.131)

for some coefficients $c_{jk}$ and $b_{jk}$. Comparison of the coefficients of the idempotents shows that we need $c_{jk} = \left[\begin{array}{c} d_j & \end{array}\right] \left[\begin{array}{c} d_j & \end{array}\right]$, i.e., the $m \times m$-matrix $c$ defined by these coefficients is equivalent to the $S$-matrix of the semisimple TQFT. We can write polynomial identities similar to (3.130) which are equivalent to (3.131) with $b_{jk} = 0$. Again, for small values of $l$ we know that they hold true. Using that $S^2$ is proportional to the identity they also imply that $S$ decomposes into a tensor product $b \otimes c$, where $b$ is a two by two matrix with vanishing diagonal elements. The $T$-matrix on the second summand has eigenvalues \{q^{2j(j+1)}, j = 0, \ldots, (m-1)\} all of which are doubly degenerate, with non trivial Jordan-bloc. For a suitable normalization we thus expect the second summand to be the tensor product of the two dimensional standard representation and the known $m$-dimensional finite representation.

In a TQFT $\tilde{SL}(2, \mathbb{Z})$ extends to representations of modular groups at higher genus. If these factor through their actions on the homology of the surface the projective $SL(2, \mathbb{Z})$-representation extends to representations of higher symplectic group. It is a fact that for congruence groups as the higher symplectic groups over $\mathbb{Z}$ any irreducible representation is the tensor product of a finite and an algebraic representation, see [Kz]. Thus it is likely that the tensor product presentation described in the previous paragraph can also be inferred from rather general arguments.

We summarize our observations in the following conjecture. In the next section we show that it holds true for the five and seven dimensional representation.
Conjecture 1 The projective, $3m + 1$-dimensional $SL(2, \mathbb{Z})$ representation defined in Theorem (3) decomposes as
\[ \mathcal{V} = \mathcal{V}_N \oplus \mathcal{V}_{\text{stan}} \otimes \mathcal{V}_{\text{semis}} \]

where

1. $\mathcal{V}_N$ is an $(m + 1)$-dimensional, irreducible, finite representation spanned by $N_j = N_j^+ + N_j^-$ and $P_m$, see e.g. (3.134) or (3.138).

2. $\mathcal{V}_S$ is the $2m$-dimensional subrepresentation spanned by

\[ \tilde{N}_j = \sum_{\epsilon = \pm} \frac{d_j^\epsilon}{|d_j^\epsilon|} N_j^\epsilon \]

and the $j$-th $\mathcal{T}$-eigenspace components

\[ (S_0(\tilde{N}_0))_j. \]

This representation is the tensorproduct $\mathcal{V}_S = \mathcal{V}_{\text{stan}} \otimes \mathcal{V}_{\text{semis}}$ of

(a) the two dimensional, algebraic standard representation $\mathcal{V}_{\text{stan}}$ as in (3.135) or (3.140) and

(b) an $m$-dimensional finite representation $\mathcal{V}_{\text{semis}}$ which is isomorphic - up to a projective phase - to the $SL(2, \mathbb{Z})$ representation obtained from the semisimple subquotient category, see for example (3.141).

6.) The Examples $l = 3, 5$ : In this section we verify the conjecture of the previous section for $l = 3$ and $l = 5$. We compute the explicit representation matrices of the various finite representations:

For $l = 3$ the matrices of the $SL(2, \mathbb{Z})$ are given in the basis $P_0, N_0^+, N_0^-, P_1$ by:

\[
\mathcal{T}_o = \begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
-2 & 0 & 1 & 0 \\
0 & 0 & 0 & q
\end{bmatrix}
\]

(3.132)
\[ S_o = \frac{q}{\sqrt{3}} \begin{bmatrix} 0 & -\frac{2}{3}(q - q^{-1}) & -\frac{1}{3}(q - q^{-1}) & 0 \\ \frac{2}{3}(q - q^{-1}) & -1 & 0 & -\frac{2}{3}(q - q^{-1}) \\ -\frac{7}{3}(q - q^{-1}) & -1 & 0 & -\frac{2}{3}(q - q^{-1}) \\ -1 & \frac{2}{3}(q - q^{-1}) & \frac{1}{3}(q - q^{-1}) & 1 \end{bmatrix} \] (3.133)

This representation decomposes into the sum of two irreducible, two-dimensional sub-representations

\[ V = V_N + V_S . \]

Here the subspace \( V_N \) is spanned by \( N_0 = N_0^+ + N_0^- \) and \( P_1 \) with \( S \) and \( T \) acting as:

\[ T_N = \begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix} \quad S_N = \frac{q}{\sqrt{3}} \begin{bmatrix} -1 & -\frac{2}{3}(q - q^{-1}) \\ (q - q^{-1}) & 1 \end{bmatrix} \] (3.134)

This subrepresentation \( V_S \) has basis vectors \( \tilde{P}_o = P_0 + \frac{1}{(q - q^{-1})}N_o \) and \( \tilde{N}_o := N_o^+ - 2N_o^- \) for which the \( S \) and \( T \) matrix have the form of the standard representation:

\[ T_S = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad S_S = q^\gamma q \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \] (3.135)
For \( l = 5 \) the matrices are given for the basis \( P_0, N_0^+, N_0^-, P_1 N_1^+, N_1^-, P_2 \):

\[
\mathcal{T}_o = \begin{bmatrix}
1 & 0 & 0 & : & 0 & 0 & 0 & : & 0 \\
& & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 1 & 0 & : & 0 & 0 & 0 & : & 0 \\
& & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-4 & 0 & 1 & : & 0 & 0 & 0 & : & 0 \\
& \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & : & q^{-1} & 0 & 0 & : & 0 \\
& & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & : & -q^{-1} \frac{3}{[2]} & q^{-1} & 0 & : & 0 \\
& \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & : & q^{-1} \frac{2}{[2]} & 0 & q^{-1} & : & 0 \\
& \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & : & 0 & 0 & 0 & : & q^2 \\
\end{bmatrix}
\tag{3.136}
\]

\[
\mathcal{S}_o = \frac{q}{\sqrt{5}}
\begin{bmatrix}
0 & -\xi & \xi & : & 0 & \xi(-1 + 2[2]) & \xi(1 - 2[2]) & : & 0 \\
\xi(2 + 4[2]) & [2] & 0 & : & \xi(2 - 6[2]) & \frac{1}{5}(-2 + [2]) & \frac{1}{5}(2 - 6[2]) & : & \xi(-4 + 2[2]) \\
\xi(4[2] - 23) & [2] & 0 & : & \xi(-23 + 19[2]) & \frac{1}{5}(-2 + [2]) & \frac{1}{5}(2 - 6[2]) & : & \xi(-4 + 2[2]) \\
& \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & -\xi & \xi & : & 0 & -\xi[2] & \xi[2] & : & 0 \\
-\xi(12 + 14[2]) & -3 - 2[2] & 0 & : & \xi(18 + 16[2]) & \frac{1}{5}(-2 + [2]) & \frac{1}{5}(2 + 4[2]) & : & -\xi(6 + 2[2]) \\
\xi(13 + 11[2]) & -3 - 2[2] & 0 & : & -\xi(7 + 9[2]) & \frac{1}{5}(-2 + [2]) & \frac{1}{5}(2 + 4[2]) & : & -\xi(6 + 2[2]) \\
& \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
[2] & 4\xi & \xi & : & -1 - [2] & \xi(2 - 2[2]) & \xi(3 - 3[2]) & : & 1 \\
\end{bmatrix}
\tag{3.137}
\]
where
\[ \xi = \frac{q - q^{-1}}{5}. \]

This representation decomposes into two irreducible representations \( \mathcal{V}_N \) and \( \mathcal{V}_S \), with \( \dim(\mathcal{V}_N) = 3 \) and \( \dim(\mathcal{V}_S) = 4 \). The three dimensional representation is spanned by the vectors
\[ N_0 = \frac{1}{q - q^{-1}}(N_0^+ + N_0^-) \quad N_1 = \frac{1}{q - q^{-1}}(N_1^+ + N_1^-) \quad \text{and} \quad P_2. \]

The representation matrices are
\[
\mathcal{T}_N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & q^{-1} & 0 \\ 0 & 0 & q^2 \end{bmatrix} \quad \mathcal{S}_N = \frac{q}{\sqrt{5}} \begin{bmatrix} [2] & -[2] & 2 \\ -(3 + 2[2]) & [2] & (4 + 2[2]) \\ 1 & (1 - [2]) & 1 \end{bmatrix} \quad (3.138)
\]

The four dimensional representation is spanned by
\[ \tilde{P}_0 := P_0 - [2]N_0 \quad \tilde{P}_1 := \frac{1}{[2]} (P_1 + (3 + 2[2])N_1) \]
and
\[ \tilde{N}_0 := (N_0^+ - 4N_0^-) \quad \tilde{N}_1 := \frac{1}{[2]^2}(-3N_1^+ + 2N_1^-). \]

With ordering \( \tilde{P}_0, \tilde{N}_0, \tilde{P}_1, \tilde{N}_1 \) we find the matrices:
\[
\mathcal{T}_S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \vdots \\ 1 & 1 & 0 & 0 \\ \vdots \\ 0 & 0 & \vdots & q^{-1} \\ 0 & 0 & q^{-1} & q^{-1} \end{bmatrix} \quad \mathcal{S}_S = \frac{q(q - q^{-1})}{\sqrt{5}} \begin{bmatrix} 0 & -1 & 0 & -[2] \\ \vdots \\ 1 & 0 & [2] & 0 \\ \vdots \\ 0 & -[2] & 0 & 1 \\ [2] & 0 & -1 & 0 \end{bmatrix} \quad (3.139)
\]
Now it is easy to see that this can be written as a tensor product of $SL(2, \mathbb{Z})$ representations:

$$V_S \cong V_{\text{stan}} \otimes V_{\text{semis}}.$$ 

In order to denote the isomorphism

$$\tilde{P}_i \rightarrow v_P \otimes w_i \quad \tilde{N}_i \rightarrow v_N \otimes w_i$$

we introduce bases $\{v_P, v_N\}$ and $\{w_0, w_1\}$ of $V_{\text{stan}}$ and $V_{\text{semis}}$ respectively. For these bases we can write

$$T_N = T_{\text{stan}} \otimes T_{\text{semis}} \quad S_N = qS_{\text{stan}} \otimes S_{\text{semis}}$$

with

$$T_{\text{stan}} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad S_{\text{stan}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and

$$T_{\text{semis}} = \begin{bmatrix} 1 & 0 \\ 0 & q^{-1} \end{bmatrix} \quad S_{\text{semis}} = \frac{(q - q^{-1})}{\sqrt{5}} \begin{bmatrix} 1 & [2] \\ [2] & -1 \end{bmatrix}$$

(3.140)

(3.141)

**Conclusion**

The results of Chapter 2 show that a the construction of a universal TQFT should include two features. One is to avoid degeneracies by considering only doubles. The fact that the projective phases and the proofs of modular relations are most conveniently given in terms the bilinear forms and moduli defined from the integrals is an indication that this is the correct language also for constructions at higher genus. In view of the glueing operations described in the introduction the genus one case can in fact be thought of as a basic building bloc. It should be possible to understand more conceptually the appearance of the finite representation we know from the semisimple theory as a tensor product with the standard representation rather than a sub representation. In particular it should be interesting to see how the representation on general $D(\mathcal{A})$ is modified if we pass to the semisimple quotient of the representation category of $D(\mathcal{A})$ and a possible truncation of the resulting TQFT.

Also, the appearance of algebraic representations is a novel feature of these theories. We expect to find higher dimensional algebraic representations of $SL(2, \mathbb{Z})$ if we start
from higher rank quantum groups for which the orders of nilpotencies of central elements will be higher.

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