A priori estimate for a family of semi-linear elliptic equations with critical nonlinearity

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Abstract

We consider positive solutions of $\Delta u - \mu u + Ku^{\frac{n+2}{n-2}} = 0$ on $B_1$ ($n \geq 5$) where $\mu$ and $K > 0$ are smooth functions on $B_1$. If $K$ is very sub-harmonic at each critical point of $K$ in $B_{2/3}$ and the maximum of $u$ in $\bar{B}_{1/3}$ is comparable to its maximum over $\bar{B}_1$, then all positive solutions are uniformly bounded on $\bar{B}_{1/3}$. As an application, a priori estimate for solutions of equations defined on $S^n$ is derived.

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1 Introduction

In this article we study the equation

$$\Delta u - \mu(x)u + K(x)u^\sigma = 0, \quad u > 0, \quad u \in C^2(B_1), \quad n \geq 5, \quad (1.1)$$

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where $B_1$ is the unit ball centered at the origin, $n^* = \frac{n+2}{n-2}$ is the critical power in Sobolev embedding, $\mu$ is a $C^1$ function on $B_1$ and $K \in C^3(B_1)$ is a positive function. We shall derive a priori estimate under natural assumptions on $K$ and $\mu$.

Equation (1.1) has rich connections in physics and geometry. In particular, it is very closely related to the well known Yamabe equation, which has been extensively studied for decades. Many interesting features of the Yamabe equation are also reflected on this equation. When $\mu$ is a non positive constant, many results on the existence of solutions, multiplicity of solutions, a priori estimates, bifurcation phenomena, Harnack type inequalities, etc can be found in the literature. We refer the interested readers to [3][6][9][28][31] and the references therein. On the other hand, much less references can be found for the case $\mu > 0$. A recent paper of Lin-Prapapet [25] discussed the case $\mu = \text{constant} > 0$ and they pointed out that it is also interesting and important to study the following Harnack type inequality:

\[
(\max_{B_1/3} u)(\min_{B_2/3} u) \leq C.
\] (1.2)

In a slightly different setting, they derived this Harnack inequality for $3 \leq n \leq 6$ under some flatness assumptions of $\nabla K$ near its critical points. They also speculated that (1.2) should still hold for higher dimensions under similar assumptions by finer analysis.

The Harnack inequality (1.2) is an important estimate to understand the blowup phenomena of (1.1) and many related equations with the critical Sobolev nonlinearity. The very first discussions of this inequality can be found in [27] and [11]. With this Harnack type inequality, usually the blowup phenomena is greatly simplified and some energy estimates are implied. Moreover, some further results such as a priori estimate, precise description of the blowup bubbles, etc can be obtained.

In this article, we use a very different approach from Lin-Prajapat’s to obtain a priori estimate for general $\mu \in C^1$ and $n \geq 5$. We shall also derive the Harnack type inequality as an intermediate step toward our result. Our idea stems from the author’s joint work with Y. Y. Li [21][23] on the compactness of solutions of the Yamabe equation.

We assume the following on $K$ and $\mu$:

\[
(K, \mu) : \quad C_1^{-1} \leq K(x) \leq C_1, \quad x \in B_1, \quad \|K\|_{C^1(B_1)} \leq C_1, \quad \|\mu\|_{C^1(B_1)} \leq C_0.
\]

In addition, we need the maximum of $u$ in $B_{1/3}$ comparable to its maximum in $B_1$: There exists $C_2 > 0$ such that

\[
\max_{B_{1/3}} u \geq \frac{1}{C_2} \max_{B_1} u.
\] (1.3)
The main result of the paper is

**Theorem 1.1.** Given \((K, \mu)\) and \((1.3)\) there exists \(C_3(C_0, n, C_1, C_2) > 0\) such that for each critical point \(x\) of \(K\) in \(B_{\frac{2}{3}}\), if \(\Delta K(x) > C_3\), any solution \(u\) of \((1.1)\) satisfies

\[
\max_{B_{1/3}} u \leq C_4(C_0, n, C_1, C_2).
\]

If \(\mu\) is a positive constant, Lin-Prajapet proved in [25] without the assumption \((1.3)\) that for \(n = 3\), if \(K\) is Hölder continuous with exponent \(\theta \in \left(\frac{1}{2}, 1\right]\) then \((1.2)\) holds. For \(4 \leq n \leq 6\), if \(K \in C^1\) and in a neighborhood of each critical point \(x_0\) of \(K\),

\[
c|x - x_0|^\theta - 1 \leq |\nabla K(x)| \leq C|x - x_0|^\theta - 1
\]

holds for \(c, C > 0\) and \(\frac{n-2}{2} \leq \theta \leq n - 2\), then \((1.2)\) holds.

Even though \((1.3)\) seems to be a strong restriction for locally defined equations, it can be removed when Theorem 1.1 is applied to globally defined equations. In this case Theorem 1.1 is particularly useful. For example, let \(w\) be a positive solution to

\[
Lw - aw + R w^{\frac{n+2}{n-2}} = 0, \quad \text{on} \quad S^n
\]

where \(L = \Delta_{g_0} - \frac{n(n-2)}{4}\) is the conformal Laplacian operator of \((S^n, g_0)\), \(a\) and \(R\) are positive smooth functions. By using the stereographic projection \(\pi\) from \(S^n\) to \(\mathbb{R}^n\), we set \(K(y) = R(\pi^{-1}(y))\) for \(y \in \mathbb{R}^n\) and \(\mu(y) = a(\pi^{-1}(y))\). Without loss of generality we assume the north pole is not a critical point of \(R\). Similar to \((K, \mu)\) we assume

\[
(K, \mu)_1: \quad C_5^{-1} \leq K \leq C_5, \quad \|K\|_{C^3(\mathbb{R}^n)} \leq C_5, \quad \|\mu\|_{C^1(\mathbb{R}^n)} \leq C_5.
\]

Then we have

**Corollary 1.1.** Given \(n \geq 5\) and \((K, \mu)_1\), then there exists \(C_6(C_5, n) > 0\) such that for each critical point \(y\) of \(K\), if \(\Delta K(y) > C_6\), solution \(w\) of \((1.5)\) satisfies

\[
C_7^{-1} \leq w(x) \leq C_7, \quad x \in S^n
\]

where \(C_7(C_5, n) > 0\) is independent of \(w\).

**Remark 1.1.** If \(a \equiv 0\) in \((1.5)\), then we only need to assume \(\Delta K(y) > 0\) on critical point \(y\) to get the same a priori estimate.
The results in Corollary 1.1 and Remark 1.1 can be compared with closely related results in [10][13][19][20], etc and the references therein. Comparing to these results, Corollary 1.1 gives the a priori estimate under a very short assumption on $K$.

The conclusion of Theorem 1.1 can also be compared with related results on the compactness of solutions of the Yamabe equation. It is proved in [21] and [26] that for the Yamabe equation, blowup point can not appear at places where the Weyl tensor is 0. The role of $\Delta K$ in Theorem 1.1 is similar to that of the Weyl tensor for the Yamabe equation. Please also see [4][5][10][14][18][22][24] for related discussions.

In another work of the author [30], among other things the following result is essentially proved: For $\mu = 0$ and $n \geq 5$, suppose (1.3) holds and $\Delta K(x) > 0$ for all critical point $x$ in $B_{2/3}$, then $\max_{B_{1/3}} u \leq C$ for some $C > 0$.

Based on Theorem 1.1 and the results in [25][30] we propose the following two questions for the case $\mu = \text{constant} > 0$:

1. For $n = 4$, under the assumptions of Theorem 1.1 can one obtain the Harnack type inequality (1.2) or even stronger, the a priori estimate?

2. For $n \geq 5$ under the assumptions $(K, \mu)$ and (1.3), what is the smallest $C > 0$ so that Theorem 1.1 holds for $\Delta K > C$ at each critical point in $B_{2/3}$? We suspect that the best constant only comes from the Pohozaev Identity.

The idea of the proof of Theorem 1.1 is by an iterative use of the well known method of moving spheres (MMS). Some estimates established by Chen-Lin [12] are very crucial to our approach. The reason that we need to apply MMS many times is because we need to construct appropriate test functions for this well known method. The construction of test functions depends on the estimates of some error terms and is closely related to the spectrum of the linearized operator of the equation. At the beginning we only have crude estimates of these error terms. As a consequence we can only construct test functions (and apply MMS) on these small domains. However, MMS and Chen-Lin’s estimate lead to better estimates of error terms, which make it possible to construct test functions on larger domains. After applying this procedure iteratively we obtain the desired estimates on the error terms and MMS can be applied on roughly the whole domain.

More specifically the outline of the proof is as follows. Suppose there is no uniform bound for a sequence of solutions $u_i$. Scale $u_i$ appropriately so that the maximum of the re-scaled function $v_i$ is comparable to 1. Then these functions are defined on very large balls, say, $B(0, \frac{1}{10}M_i^{\frac{2}{n-2}})$, where $M_i$ is the maximum of $u_i$. The first step is to show that $v_i$ is comparable to a standard bubble that takes 1 at 0.
over the range $B(0, M_i^{(n-2)/2})$. The way to prove this step is by an easy application of MMS on this range. Then in step two we show that the difference between $v_i$ and the standard bubble is of the order $O(M_i^{-2})$. The approach for this step is based on Chen-Lin’s argument in [12]. The result in step two helps us to describe some error terms in a better way so that we can use MMS to prove, in step three, that $v_i$ is comparable to the standard bubble over the range $B(0, M_i^{(n-2)/2})$. In this step we need to rewrite some major error terms into a product of spherical harmonics with radial functions. This decomposition allows us to find test functions of the same form. By using the Pohozaev identity in step four we obtain that $|\nabla K|$ at the blowup point must vanish at the order of $O(M_i^{-2})$. Then in step five we apply the Chen-Lin estimate again to show that the closeness between $v_i$ and the standard bubble can be improved to $O(M_i^{-2})$. This is optimal for this closeness. This new estimate helps us again to describe some error terms in a better way so that we can apply MMS again in a bigger domain. In fact in step six we show that in almost the whole domain, $v_i$ is comparable to the standard bubble. In this step, the largeness of $\Delta K$ at the blowup point is used. Finally in step seven we apply the Pohozaev identity over the whole domain, using symmetry and all previous estimates to get a contradiction.

The organization of this paper is as follows. In section two the proof of Theorem 1.1 is presented. The different steps of the proof are contained in different sub-sections. At the end of section two we use Theorem 1.1 to prove Corollary 1.1.

2 The proof of Theorem 1.1

In the proof of Theorem 1.1 we only consider the case $\mu = constant > 0$. In our argument, the difference between $\mu$ being a general $C^1$ function and a constant only produces minor terms in our estimate. In order not to make notations difficult, we leave the general case for the interested readers.

The proof is based on an assumption for contradiction. Suppose a sequence of functions $u_i$ can be found to satisfy

$$\Delta u_i - \mu_i u_i + K_i(x)u_i^{n^*} = 0 \quad B_1$$

with $K_i$ satisfying $(K, \mu)$ and $0 \leq \mu_i \leq C_0$ such that

$$u_i(\bar{x}_i) = \max_{B_1/3} u_i \to \infty.$$  \hspace{1cm} (2.1)
Then by a standard selection process we have $x_i \in B(\bar{x}_i, 1/10)$ such that $x_i$ is a local maximum of $u_i$ and $M_i = u_i(x_i)$ is comparable to the maximum of $u_i$ in $B_1$. Moreover

$$v_i(y) = M_i^{-1}u_i(M_i^{-\frac{2}{n-2}}y + x_i)$$

tends in $C^2_{\text{loc}}(\mathbb{R}^n)$ norm to $U$ which satisfies

$$\Delta U + \lim_{i \to \infty} K_i(x_i)U^{\frac{n+2}{n-2}} = 0, \quad \mathbb{R}^n.$$ 

and $U(0) = 1 = \max_{\mathbb{R}^n} U$. By the well known classification theorem of Caffarelli-Gidas-Spruck we have

$$U(y) = (1 + |y|^2)^{-\frac{n-2}{2}}.$$ 

Here we have assumed without loss of generality that

$$\lim_{i \to \infty} K_i(x_i) = n(n - 2).$$

The standard selection process can be found in quite a few papers, for example [30]. Direct computation shows that the equations for $v_i$ is

$$\Delta v_i - \mu_i M_i^{-\frac{4}{n-2}} v_i + K_i(M_i^{-\frac{2}{n-2}} \cdot + x_i)v_i^{n*} = 0, \quad \Omega_i$$

where $\Omega_i := B(0, \frac{1}{10}M_i^{-\frac{2}{n-2}})$. In the sequel unless we state otherwise a constant always depends on $n, C_0, C_1, C_2$.

### 2.1 Estimate of $v_i$ over $B(0, \delta M_i^{-\frac{2}{(n-2)^2}})$

In this subsection we establish the following estimate:

**Proposition 2.1.** For any $\epsilon > 0$ there exists $\delta_0(\epsilon) > 0$ such that for all large $i$

$$\min_{|y| = r} v_i(y) \leq (1 + \epsilon)r^{2-n}, \quad \forall r \in (0, \delta_0 M_i^{-\frac{2}{(n-2)^2}}).$$

**Proof of Proposition 2.1:** The proof is by a contradiction. Suppose there exists $\epsilon_0 > 0$ and $r_i = o(1)M_i^{-\frac{2}{(n-2)^2}}$ so that

$$\min_{|y| = r_i} v_i(y) \geq (1 + \epsilon_0)r_i^{2-n}. \quad (2.3)$$
Note that by the convergence of $v_i$ to $U$ we certainly have $r_i \to \infty$. We shall use the moving sphere argument. Here we let $\Sigma_\lambda = B_{r_i} \setminus \bar{B}_\lambda$. The boundary condition for $v_i$ on $|y| = r_i$ is (2.3). Let

$$v^\lambda_i(y) = \left( \frac{\lambda}{|y|} \right)^{n-2} v_i(y^\lambda), \quad y^\lambda = \frac{\lambda^2 y}{|y|^2}$$

be the Kelvin transformation of $v_i$ with respect to $\partial B_\lambda$. Note that in this article $\lambda$ is always assumed to stay between two positive constants independent of $i$. The equation for $v^\lambda_i$ is

$$\Delta v^\lambda_i - \mu_i M_i^{-\frac{n}{2}} \left( \frac{\lambda}{|y|} \right)^{4} v^\lambda_i + K_i (M_i^{-\frac{n-2}{2}} y^\lambda + x_i)(v^\lambda_i)^{n^*} = 0 \quad \Sigma_\lambda. \quad (2.4)$$

Set $w_\lambda = v_i - v^\lambda_i$ in $\Sigma_\lambda$. For simplicity we omit $i$ in this notation. We shall apply the moving sphere argument to $w_\lambda$ with a test function. The equation for $w_\lambda$ is

$$T_\lambda w_\lambda = E_\lambda \quad \text{in} \quad \Sigma_\lambda. \quad (2.5)$$

where

$$T_\lambda := \Delta - \mu_i M_i^{-\frac{n}{2}} + n^* K_i (M_i^{-\frac{n-2}{2}} y + x_i) \xi_\lambda^{4}.$$

$\xi_\lambda$ is obtained from the mean value theorem:

$$\xi_\lambda^{\frac{4}{n-2}} = \int_0^1 (tv_i + (1-t)v^\lambda_i) \frac{4}{n-2} dt,$$

$$E_\lambda = \mu_i M_i^{-\frac{n}{2}} v^\lambda_i (1 - \left( \frac{\lambda}{|y|} \right)^{4}) + (K_i (M_i^{-\frac{n-2}{2}} y^\lambda + x_i) - K_i (M_i^{-\frac{n-2}{2}} y + x_i))(v^\lambda_i)^{n^*} \quad (2.6)$$

is the main error term.

For the moving sphere argument we shall find two constants $\lambda_0$ and $\lambda_1$, both independent of $i$ such that $\lambda_0 \in (\frac{1}{2}, 1), \lambda_1 \in (1, 2)$. We shall only consider $\lambda \in [\lambda_0, \lambda_1]$ for the moving sphere method. Test function $h_\lambda$, which depends on $i$ ($\lambda \in [\lambda_0, \lambda_1]$), will be constructed to satisfy

$$h_\lambda|_{\partial B_\lambda} = 0 \quad (2.7)$$

$$h_\lambda = o(1)r^{2-n} \quad \text{in} \quad \Sigma_\lambda, \quad |\nabla h_\lambda| = o(1) \quad \text{in} \quad \Sigma_\lambda \cap B_R, \quad (2.8)$$

$$T_\lambda h_\lambda + E_\lambda \leq 0, \quad \text{in} \quad O_\lambda := \{ y \in \Sigma_\lambda; \quad v_i(y) \leq 2v^\lambda_i(y) \} \quad (2.9)$$

where $R$ in (2.8) is any fixed large constant.
Once such a test function is constructed the moving sphere argument can be applied to get a contradiction to (2.3). In fact first we show that the moving sphere process can get started at \( \lambda_0 \):

\[
w_{\lambda_0} + h_{\lambda_0} > 0 \quad \text{in} \quad \Sigma_{\lambda_0}. \tag{2.10}
\]

To see this, first we state a property of the standard bubble \( U \):

\[
U(y) - U^\lambda(y) \sim (1 - \frac{\lambda}{|y|})(1 - \lambda)|y|^{2-n}, \quad |y| > \lambda,
\]

which implies that for any \( \lambda_0 < 1 \) and \( |y| > \lambda_0 \)

\[
w_{\lambda_0}(y) > C(|y| - \lambda_0)|y|^{1-n}, \quad \lambda_0 < |y| < \bar{R}
\]

for all large \( i \) by the convergence of \( w_{\lambda_0} \) to \( U - U^\lambda \) in \( C^2_{\text{loc}}(\mathbb{R}^n) \). Here \( \bar{R} \) is a large fixed number to be determined. By (2.7) (2.8) one sees easily that \( w_{\lambda_0} + h_{\lambda_0} > 0 \) over \( B_{\bar{R}} \cap \Sigma_{\lambda_0} \).

For \( |y| > \bar{R} \) we observe that

\[
v_i^{\lambda_0}(y) \leq \lambda_0^{n-2}(1 + \epsilon_1)|y|^{2-n}, \quad |y| \geq \bar{R}
\]

where \( \epsilon_1 \) is sufficiently small so that \( \lambda_0^{n-2}(1 + \epsilon_1) < 1 - 5\epsilon_1 \). On the other hand, by the convergence of \( v_i \) to \( U \) we can make \( \bar{R} \) large enough so that

\[
v_i(y) \geq (1 - 2\epsilon_1)|y|^{2-n}, \quad |y| = \bar{R}.
\]

On \( \Sigma_{\lambda_0} \setminus B_{\bar{R}} \) \( v_i \) satisfies

\[
\Delta v_i - \mu_iM_i^{\frac{4}{n-2}} v_i \leq 0, \quad \Sigma_{\lambda_0} \setminus B_{\bar{R}}.
\]

Let

\[
\tilde{O} := \{ y \in \Sigma_{\lambda_0} \setminus B_{\bar{R}} : v_i(y) \leq 2|y|^{2-n} \}.
\]

Clearly \( w_{\lambda_0} + h_{\lambda_0} > 0 \) in \( \Sigma_{\lambda_0} \setminus (B_{\bar{R}} \cup \tilde{O}) \). Let \( G \) be the Green’s function of \( -\Delta \) on \( \Sigma_{\lambda_0} \setminus B_{\bar{R}} \) with respect to the Dirichlet boundary condition, let

\[
\phi(y) = \int_{\tilde{O}} G(y, \eta)\mu_iM_i^{\frac{4}{n-2}} v_i(\eta) d\eta.
\]

Then

\[
-\Delta \phi = \mu_iM_i^{\frac{4}{n-2}} v_i \quad \tilde{O}.
\]
Elementary estimate gives
\[ \phi(y) \leq C(n) \mu_i M_i^{-\frac{4}{n-2}} |y|^{4-n}. \]  
(2.12)
So
\[ \phi(y) \leq \epsilon_1 |y|^{2-n}. \]  
(2.13)
Now we have
\[ \Delta(v_i + \phi) \leq 0 \text{ in } \tilde{O}. \]
By maximum principle
\[ v_i + \phi > (1 - 3\epsilon_1)|y|^{2-n} \text{ in } \tilde{O}. \]
By (2.13)
\[ v_i(y) > (1 - 4\epsilon_1)|y|^{2-n} \text{ in } \tilde{O}. \]
Since \( h_\lambda = o(1)|y|^{2-n} \) for \( \lambda \in (1/2, 2) \), (2.10) is established.

Once the moving sphere process can get started at \( \lambda_0 \), let \( \tilde{\lambda} \) be the critical position where \( w_\lambda + h_\lambda \) ceases to be positive in \( \Sigma_\lambda \). But because of (2.7), (2.8) and (2.9), the moving sphere process can reach \( \lambda_1 \), i.e. \( \tilde{\lambda} \geq \lambda_1 \). Note that \( T_\lambda h_\lambda + E_\lambda \) only needs to be non positive in \( O_\lambda \) because by (2.8) \( w_\lambda + h_\lambda > 0 \) in \( \Sigma_\lambda \setminus O_\lambda \). Since \( \lambda_1 > 1 \), by letting \( i \rightarrow \infty \) we have
\[ U(y) - U^{\lambda_1}(y) \leq 0, \quad |y| \geq \lambda_1, \]
which is contradictory to (2.11).

**Remark 2.1.** What is described above is a general procedure of the application of moving sphere method, which will be used a few times in the sequel. Even though \( \Sigma_\lambda \) and \( h_\lambda \) will be different in different contexts, the important thing is to construct \( h_\lambda \) that satisfies (2.7), (2.8) and (2.9). The way to start the moving sphere process and to apply the maximum principle to get a contradiction from the standard bubbles are just the same and will not be repeated.

To construct \( h_\lambda \) in this subsection we use the following crude estimate of \( E_\lambda \):
\[ |E_\lambda| \leq CM_i^{-\frac{4}{n-2}} r^{2-n} + CM_i^{-\frac{2}{n-2}} r^{-n-1} \leq CM_i^{-\frac{2}{n-2}} r^{2-n}, \quad \Sigma_\lambda. \]  
(2.14)
Note that we use \( r \) to represent \( |y| \). The construction of \( h_\lambda \) in this subsection is not subtle with respect to \( \lambda \), we just set \( \lambda_0 = \frac{1}{2} \) and \( \lambda_1 = 2 \). We need the following non positive function: For \( 2 < \alpha < n \), let
\[ f_{n,\alpha}(r) = -\frac{1}{(n-\alpha)(2-\alpha)}(r^{2-\alpha} - \lambda^{2-\alpha}) - \frac{\lambda^{n-\alpha}}{(n-\alpha)(n-2)}(r^{2-n} - \lambda^{2-n}). \]
By direct computation one verifies that
\[
\begin{align*}
\Delta f_{n,\alpha}(r) &= f''_{n,\alpha}(r) + \frac{n-1}{r} f'_{n,\alpha}(r) = -r^{-\alpha}, \quad r \geq \lambda, \\
f_{n,\alpha}(\lambda) &= f'_{n,\alpha}(\lambda) = 0, \\
0 &\leq -f_{n,\alpha}(r) \leq C(n, \alpha).
\end{align*}
\tag{2.15}
\]

This function is mainly used to control minor terms. We define \( h_\lambda \) as
\[
h_\lambda = Q M_i^{-\frac{2}{n-2}} f_{n,n-2}(\frac{r}{\lambda})
\]
where \( Q \) is a large number to be determined. Here we see that by (2.15), (2.7) and (2.8) hold. Also \( h_\lambda \leq 0 \) in \( \Sigma_\lambda \). Now we verify (2.9). First by choosing \( Q \) large enough we have
\[
\Delta h_\lambda + E_\lambda \leq -\frac{Q}{2} M_i^{-\frac{2}{n-2}} r^{2-n}, \quad \Sigma_\lambda.
\]
Since \( h_\lambda \) is non positive, the term \( n^* K_i(M_i^{-\frac{2}{n-2}} y + x_i) \xi_{\lambda}^{-\frac{2}{n-2}} h_\lambda \) is also non positive. The only thing we need to verify is
\[
-\mu_i M_i^{-\frac{4}{n-2}} h_\lambda \leq \frac{Q}{4} M_i^{-\frac{2}{n-2}} r^{2-n} \quad \text{in} \quad \Sigma_\lambda.
\]
By direct computation this holds. Proposition 2.1 is established. \( \square \)

Next we establish the closeness between \( v_i \) and \( U_i \), which satisfies:
\[
\Delta U_i + K_i(x_i) U_i^{n*} = 0, \quad \mathbb{R}^n. \quad U_i(0) = 1 = \max_\mathbb{R^n} U_i.
\]

**Proposition 2.2.** There exist \( \delta_1 > 0 \) and \( C > 0 \) such that
\[
|v_i(y) - U_i(y)| \leq C M_i^{-\frac{2}{n-2}}, \quad |y| \leq \delta_1 M_i^{\frac{2}{(n-2)^2}}.
\]

**Proof of Proposition 2.2**

The proof of Proposition 2.2 consists of two steps. First we show that there exists \( \delta_2(n, C_0) > 0 \) small, so that
\[
v_i(y) \leq C U_i(y), \quad |y| \leq \delta_2 M_i^{\frac{2}{(n-2)^2}}. \tag{2.16}
\]

The proof of (2.16) is very similar to that of Lemma 3.2 in [12]. The only difference comes from the extra term \(-\mu_i M_i^{\frac{4}{n-2}} v_i\). For this we let \( G_1 \) be the Green’s
function of the operator $-\Delta + \mu M_i^{-\frac{4}{n-2}}$ with respect to the Dirichlet condition on $\Omega_i$ (Recall that $\Omega_i := B(0, \frac{1}{10} M_i^{\frac{2}{n-2}})$). i.e.

$$\begin{cases} (-\Delta y + \mu M_i^{-\frac{4}{n-2}}) G_1(x, y) = \delta_x, & \Omega_i, \\ G_1(x, y) = 0, & y \in \partial \Omega_i. \end{cases}$$

By direct computation

$$G_1(x, y) = \frac{1}{\omega_n(n-2)} |x - y|^{2-n} + \phi(x, y)$$

(2.17)

where $\omega_n$ is the area of $S^{n-1}$, $\phi(x, y)$ satisfies

$$|\phi(x, y)| \leq \mu_i M_i^{-\frac{4}{n-2}} C(n) |x - y|^{4-n}.$$  

(2.18)

Once we have this, the rest of the proof is very similar to that of lemma 3.2 in [12]. For $\epsilon > 0$ small to be determined, there exists some constant $\delta_1 \in (0, 1)$, independent of $i$, such that for large $i$, let $y_1$ be a minimum of $v_i$ on $|y| = \delta_1 M_i^{\frac{2}{n-2}}$, the following estimates hold:

$$v_i(y_1) \geq \int_{\Omega_i} G_1(y_1, \eta) K_i(M_i^{-\frac{2}{n-2}} y + x_i) v_i(\eta)^{\frac{n+2}{n-2}} d\eta$$

$$\geq \int_{B(0, \delta_1 M_i^{\frac{2}{n-2}})} G_1(y_1, \eta) K_i(M_i^{-\frac{2}{n-2}} y + x_i) v_i(\eta)^{\frac{n+2}{n-2}} d\eta,$$

and, using (2.17) and (2.18),

$$G_1(y_1, \eta) \geq \frac{(1 - \epsilon/2)}{(n-2)\omega_n} |y_1 - \eta|^{2-n} \geq \frac{(1 - 3\epsilon/4)}{(n-2)\omega_n} |y_1|^{2-n}, \quad |\eta| \leq \delta_2 |y_1|,$$

if $\delta_2$ is chosen small enough. Now we use $\lim_{i \to -\infty} K_i(x_i) = n(n - 2)$ to get

$$v_i(y_1) \geq \frac{(1 - \epsilon)n}{\omega_n} |y_1|^{2-n} \int_{B(0, \delta_2 |y_1|)} v_i^{\frac{n+2}{n}}(\eta) d\eta$$

On the other hand, by Proposition 2.1

$$v_i(y_1) \leq (1 + \epsilon) |y_1|^{2-n}.$$  

So

$$\int_{B(0, \delta_2 |y_1|)} v_i^{\frac{n+2}{n}}(\eta) d\eta \leq (1 + 4\epsilon) \omega_n/n.$$
A direct computation gives,
\[ \int_{\mathbb{R}^n} U_n^{\frac{n+2}{n-2}} = \frac{\omega_n}{n}. \]
By the convergence of \( v_i \) to \( U \), there exists some \( R_1 \), depending only on \( n \) and \( \epsilon \), such that, for large \( k \),
\[ \int_{R_1 \leq |\eta| \leq \delta_2 |y_1|} v_i^{\frac{n+2}{n-2}} d\eta \leq \frac{5\epsilon}{n} \omega_n. \]
Since \( v_i \leq C_2 \) (by (1.3))
\[ \int_{R_1 \leq |\eta| \leq \delta_2 |y_1|} v_i^{\frac{2n}{n-2}} d\eta \leq C_2 \int_{R_1 \leq |\eta| \leq \delta R_k} v_i^{\frac{n+2}{n-2}} d\eta \leq C \epsilon. \]
For each \( 2R_1 < r < \delta_2 |y_1|/2 \), we consider \( \tilde{v}_i(z) = r^{\frac{n-2}{2}} v_i(rz) \) for \( 1/2 < |z| < 2 \). Then \( \tilde{v}_i \) satisfies
\[ \Delta \tilde{v}_i(z) - \mu_i M_i^{-\frac{4}{n-2}} r^2 \tilde{v}_i(z) + K_i(M_i^{\frac{2}{n-2}} rz + x_i) \tilde{v}_i(z) \tilde{v}_i^{\frac{n+2}{n-2}} = 0, \quad 1/2 < |z| < 2. \]
We know that \( \int_{\frac{1}{2} \leq |z| \leq 2} \tilde{v}_i(z)^{\frac{2n}{n-2}} \leq C \epsilon \). Fix some universally small \( \epsilon > 0 \), we apply the Moser iteration technique to obtain \( \tilde{v}_i(z) \leq C \) for \( \frac{3}{4} \leq |z| \leq \frac{4}{3} \), where \( C \) is independent of \( k \). With this, we apply the Harnack inequality to obtain \( \max_{|z|=1} \tilde{v}_i(z) \leq C \min_{|y|=r} v_i(y) \), i.e., \( \max_{|y|=r} v_i(y) \leq C \min_{|y|=r} v_i(y) \). Then (2.16) is established.

The second part of the proof is essentially the argument of Lemma 3.3 in [12]. We state the outline here. Let \( w_i = v_i - U_i \). Here we recall that \( U_i \) satisfies
\[
\begin{aligned}
\Delta U_i + K_i(x_i)U_i^{n^*} &= 0 \quad \mathbb{R}^n, \\
U_i(0) &= 1 = \max_{\mathbb{R}^n} U_i.
\end{aligned}
\]
The equation for \( w_i \) is
\[
\begin{aligned}
(\Delta - \mu_i M_i^{-\frac{4}{n-2}} + n^* K_i(M_i^{-\frac{2}{n-2}} y + x_i) \xi_i^{\frac{4}{n-2}})w_i &= E_i, \quad r \leq \delta M_i^{\frac{2}{(n-2)^2}} \\
w_i(0) &= 0 = |\nabla w_i(0)|
\end{aligned}
\] (2.19)
where \( \xi_i \) is obtained from the mean value theorem:
\[ \xi_i^{\frac{4}{n-2}} = \int_0^1 (tv_i + (1-t)U_i)^{\frac{4}{n-2}} dt \]
and
\[ E_i = \mu_i M_i^{-\frac{n-4}{2}} U_i + (K_i(x_i) - K_i(M_i^{-\frac{n-2}{2}} y + x_i))U_i^\ast. \]

For \( E_i \) we clearly have
\[ |E_i(y)| \leq CM_i^{-\frac{4}{n-2}}(1 + |y|)^{2-n} + CM_i^{-\frac{2}{n-2}}(1 + |y|)^{-1-n}. \]

Let
\[ \Lambda_i = \max_{B(0,\delta M_i^{(n-2)^2})} \frac{|w_i(y)|}{M_i^{-\frac{n-2}{2}}}. \]

The goal is to prove \( \Lambda_i \leq C \). We shall prove by contradiction. Suppose \( \Lambda_i \to \infty \), let \( y_i \) be the point that \( \lambda_i \) is attained. Let
\[ \bar{w}_i = \frac{w_i}{\Lambda_i M_i^{-\frac{n-2}{2}}}. \]

Then if \( |y_i| \) is bounded, a subsequence of \( \bar{w}_i \) will converge to \( w \) that satisfies
\[
\begin{align*}
\Delta w + n^* U_i^{-\frac{4}{n-2}} w &= 0, \quad \mathbb{R}^n, \\
w(0) &= 0 = |\nabla w(0)|, \quad |w(y)| \leq 1.
\end{align*}
\]

The only function that satisfies (2.20) is 0 (Lemma 2.4 of [12]). This violates \( \bar{w}_i(y_i) = \pm 1 \). This contradiction forces us to assume \( y_i \to \infty \). However by the Green’s representation theorem, the estimate of \( E_i \) makes it impossible to have \( |\bar{w}_i(y_i)| = 1 \). Proposition 2.2 is established. □

### 2.2 Estimate of \( v_i \) over \( B(0, \delta M_i^{\frac{4}{(n-2)^2}}) \)

Proposition 2.2 enables us to improve the estimate of the error term. In fact, from the equation for \( w_i \) we first have
\[ |w_i(y)| \leq CM_i^{-\frac{2}{n-2}}|y|^2, \quad |y| \leq 10. \quad \text{(2.21)} \]

Consequently
\[ v_i^\lambda(y) = U_i^\lambda(y) + O(M_i^{-\frac{2}{n-2}})|y|^{-n}. \]

So we can write \( E_\lambda \) as (see (2.6))
\[
E_\lambda = \mu_i M_i^{-\frac{4}{n-2}} U_i^\lambda (1 - (\frac{\lambda}{r})^4) + (K_i(M_i^{-\frac{2}{n-2}} y^\lambda + x_i) - K_i(M_i^{-\frac{2}{n-2}} y + x_i))(U_i^\lambda)^n_* + O(M_i^{-\frac{6}{n-2}} r^{-n}) + O(M_i^{-\frac{4}{n-2}} r^{-3-n}).
\]

\[ (2.22) \]
Note that the last two terms come from the difference between \( v_i^\lambda \) and \( U_i^\lambda \). These terms will be estimated again later. Now by the Taylor expansion of \( K_i \) we have

\[
E_{\lambda} = E_1 + E_2 + O(M_i^{-\frac{6}{n-2}} r^{1-n}) + O(M_i^{-\frac{4}{n-2}} r^{3-n}). \tag{2.23}
\]

where

\[
E_1 = \mu_i M_i^{-\frac{4}{n-2}} U_i^\lambda(1 - \left(\frac{\lambda}{r}\right)^4) + \frac{1}{2n} M_i^{-\frac{4}{n-2}} \Delta K_i(x_i) \left(\frac{\lambda^4}{r^4} - 1\right) r^2 (U_i^\lambda)^n^* \tag{2.24}
\]

\[
E_2 = M_i^{-\frac{2}{n-2}} \sum_j \partial_j K_i(x_i) \left(\frac{\lambda^2}{r^2} - 1\right) r \partial_j (U_i^\lambda)^n^* + M_i^{-\frac{2}{n-2}} \sum_{j \neq l} \partial_{jl} K_i(x_i) \left(\frac{\lambda^4}{r^4} - 1\right) r^2 \partial_j \theta_l (U_i^\lambda)^n^* + \frac{1}{2} M_i^{-\frac{4}{n-2}} \sum_j \partial_{jj} K_i(x_i) \left(\frac{\lambda^4}{r^4} - 1\right) r^2 \left(\frac{2}{n} - 1\right) (U_i^\lambda)^n^*,
\]

where \( \theta_j = y_j/r \). Note that the term of the order \( M_i^{-\frac{6}{n-2}} \) has been changed due to the expansion of \( K_i \). Each term in \( E_2 \) can be considered as a product of a radial function and an angular function. Each angular function is an eigenfunction of \( -\Delta_\theta \) on \( S^{n-1} \) (\( \theta_j \) corresponds to eigenvalue \( n-1 \), \( \theta_j^2 - \frac{1}{n} \) corresponds to eigenvalue \( 2n \)). By using the ideas in [21, 30] we construct test functions of the same form. The current purpose is to prove Proposition 2.4 in the sequel.

Before we state Proposition 2.4, we include here a proposition whose proof can be found in [21]:

**Proposition 2.3.** For each \( s = 1, 2 \), there exists a unique \( C^2 \) radial function \( g_s \) that satisfies

\[
\begin{align*}
\begin{cases}
g''_s + \frac{n-1}{r} g'_s + (n^* K_i(x_i) \tilde{\xi}_\lambda^{\frac{n}{n-2}} - \frac{\lambda^4}{r^4}) g_s = r^s((1/r)^{2s} - 1)(U_i^\lambda)^n^*, & \lambda < r < M_i^{\frac{2}{n-2}}, \\
g_s(\lambda) = 0, & g_s(M_i^{\frac{2}{n-2}}) = 0.
\end{cases}
\end{align*}
\]

where \( \lambda \in (1 - \epsilon(n), 1 + \epsilon(n)) \), \( \epsilon(n) \) is small, \( \tilde{\lambda}_s = s(n - 2) \), \( \tilde{\xi}_\lambda \) is

\[
\tilde{\xi}_\lambda^{\frac{n}{n-2}} = \int_0^1 (t U_i^\lambda + (1-t) U_i^\lambda) \frac{4}{n-2} dt.
\]

Moreover, there exists a dimensional constant \( C_0(n) \) so that

\[
0 \leq g_s(r) \leq C_0(1 - \frac{\lambda}{r}) r^{2-n}, \quad \lambda < r < M_i^{\frac{2}{n-2}}. \tag{2.25}
\]
By comparing $\tilde{\xi}_\lambda$ and $\xi_\lambda$, we see that $\tilde{\xi}_\lambda$ is radial and is very close to $\xi_\lambda$ for $r \leq \delta_2 M_i^{\frac{2}{n-2}}$. For $r \geq \delta_2 M_i^{\frac{2}{n-2}}$, both terms are comparable to $r^{2-n}$.

Next we show

**Proposition 2.4.** Given $\epsilon > 0$, there exists $\delta_3 > 0$ such that for all large $i$,

$$\min_{|y|=r} v_i(y) \leq (1+\epsilon)r^{2-n}, \quad 10 \leq r \leq \delta_3 M_i^{\frac{4}{n-2}}.$$

**Proof of Proposition 2.4.** We prove this by a contradiction. Suppose there exist $\epsilon_0 > 0$ and $\epsilon_i \to 0$ such that

$$\min_{|y|=r_i} v_i(y) \geq (1 + \epsilon_0)r_i^{2-n}, \quad \text{for some } r_i \in (\delta_2 M_i^{\frac{2}{n-2}}, \epsilon_i M_i^{\frac{4}{n-2}}). \quad (2.26)$$

By the convergence of $v_i$ to $U$, $r_i \to \infty$. Let $\Sigma_\lambda = B(0, \epsilon_i M_i^{\frac{4}{n-2}}) \setminus \bar{B}_\lambda$ and

$$h_1 = -M_i^{-\frac{n-2}{2}} \sum_j \partial_j K_i(x_i) \theta_j g_1(r). \quad (2.27)$$

$$h_2 = -M_i^{-\frac{n-4}{2}} g_2(r) \left( \sum_{j \neq l} \partial_{jl} K_i(x_i) \theta_j \theta_l + \frac{1}{2} \sum_j \partial_{jj} K_i(x_i)(\theta_j^2 - \frac{1}{n}) \right). \quad (2.28)$$

Then we have

$$(\Delta + n^* K_i(x_i) \Delta_{\lambda^{\frac{4}{n-2}}})(h_1 + h_2) + E_2 = 0 \quad (2.29)$$

and, by Proposition 2.3

$$|h_1(y)| \leq C |\nabla K_i(x_i)| M_i^{-\frac{n-2}{2}} r^{2-n},$$

$$|h_2(y)| \leq C M_i^{-\frac{n-4}{2}} r^{2-n}. \quad (2.30)$$

Let

$$h_3(r) = Q M_i^{-\frac{4}{n-2}} f_{n,3}$$

where $Q >> 1$ is to be determined. Then first we notice that $h_3 < 0$ in $\Sigma_\lambda$. Also by the definition of $f_{n,\alpha}$ we have

$$\Delta h_3 = -Q M_i^{-\frac{4}{n-2}} f_{n,3} < 0, \quad \Sigma_\lambda. \quad (2.31)$$
By (2.30) and (2.15), each of $h_j$, $j = 1, 2, 3$ satisfies (2.7) and (2.8). Now by (2.29) and (2.31)
\[ T_\lambda(h_1 + h_2 + h_3) = -E_2 - \mu_i M_i \frac{h_1}{n^\lambda} (h_1 + h_2 + h_3) \]
\[ + n^* (K_i(M_i^{-\frac{2}{n^\lambda}} y + x_i)\xi_{\lambda}^{-\frac{4}{n^\lambda}} - K_i(x_i)\xi_{\lambda}^{-\frac{4}{n^\lambda}})(h_1 + h_2) \]
\[ - Q M_i^{-\frac{4}{n^\lambda}} r^{-3} + n^* K_i(M_i^{-\frac{2}{n^\lambda}} y + x_i)\xi_{\lambda}^{-\frac{4}{n^\lambda}} h_3. \] (2.32)

Note that since $h_3 \leq 0$ the last two terms have a good sign. To estimate other terms in (2.32) we first use (2.25) and (2.15) to get
\[ - \mu_i M_i^{-\frac{4}{n^\lambda}} (h_1 + h_2 + h_3) = O(M_i^{-\frac{6}{n^\lambda}} r^{2-n}) + O(M_i^{-\frac{8}{n^\lambda}}). \] (2.33)

Next by Proposition 2.2 we estimate the following:
\[ K_i(M_i^{-\frac{2}{n^\lambda}} y + x_i)\xi_{\lambda}^{-\frac{4}{n^\lambda}} - K_i(x_i)\xi_{\lambda}^{-\frac{4}{n^\lambda}} \]
\[ = (K_i(M_i^{-\frac{2}{n^\lambda}} y + x_i) - K_i(x_i))\xi_{\lambda}^{-\frac{4}{n^\lambda}} + K_i(x_i)(\xi_{\lambda}^{-\frac{4}{n^\lambda}} - \xi_{\lambda}^{-\frac{4}{n^\lambda}}) \]
\[ = O(M_i^{-\frac{2}{n^\lambda}} r^{-3}) + K_i(x_i)(\xi_{\lambda}^{-\frac{4}{n^\lambda}} - \xi_{\lambda}^{-\frac{4}{n^\lambda}}). \] (2.34)

To estimate the last term of the above, we use
\[ \xi_{\lambda}^{-\frac{4}{n^\lambda}} = \int_0^1 (t v_i + (1-t)v_i^\lambda)\frac{1}{n^\lambda} dt \]
\[ = \int_0^1 t U_i + (1-t)U_i^\lambda + a)^{4/2} n^{-2} dt \]
\[ = \tilde{\xi}_{\lambda}^{-\frac{4}{n^\lambda}} + a r^{n-6} \] (2.35)

where
\[ |a| \leq \begin{cases} 
CM_i^{-\frac{2}{n^\lambda}}, & \lambda < r \leq \delta M_i^{n-2} \\
Cr^{-n}, & r \geq \delta M_i^{n-2}, \quad y \in O_\lambda.
\end{cases} \] (2.36)

Putting (2.30) (2.34) (2.35) and (2.36) together we have
\[ n^* (K_i(M_i^{-\frac{2}{n^\lambda}} y + x_i)\xi_{\lambda}^{-\frac{4}{n^\lambda}} - K_i(x_i)\xi_{\lambda}^{-\frac{4}{n^\lambda}})(h_1 + h_2) = O(M_i^{-\frac{4}{n^\lambda}} r^{-3}), \quad O_\lambda. \] (2.37)
Thus, by \( (2.5) \), \( (2.23) \), \( (2.32) \), \( (2.33) \) and \( (2.37) \) we have
\[
T_\lambda(w_\lambda + h_1 + h_2 + h_3)
\leq E_1 + O(M_i^{-\frac{n-6}{2}r^2 - n}) + O(M_i^{-\frac{n-4}{2}r^{-3}})
+ O(M_i^{-\frac{8}{n-2}} - QM_i^{-\frac{4}{n-2}r^{-3}}), \quad \text{in} \quad O_\lambda.
\]
It is easy to verify that
\[
|E_1(y)| \leq CM_i^{-\frac{4}{n-2}r^{2-n}}, \quad M_i^{-\frac{8}{n-2}} = o(1)M_i^{-\frac{4}{n-2}r^{-3}}.
\]
So by choosing \( Q \) large enough we have
\[
T_\lambda(w_\lambda + h_1 + h_2 + h_3) \leq 0 \quad O_\lambda.
\]
Proposition \ref{prop2.4} is established.

**Remark 2.2.** In the proof of Proposition \ref{prop2.4} we don’t need the sign of \( \Delta K_i(x_i) \).

### 2.3 The vanishing rate of \( |\nabla K_i(x_i)| \)

Next we improve the estimate of \( v_i - U_i \): First we have

**Proposition 2.5.** There exist \( \delta_4 > 0 \) and \( C > 0 \) such that
\[
v_i(y) \leq CU_i(y) \quad |y| \leq \delta_4 M_i^{(n-2)^2}.
\]

The proof is similar to the step one of Proposition \ref{prop2.2}. \( \square \)

To further estimate \( v_i - U_i \) more precisely, we need the following Pohozaev Identity for
\[
\Delta f(x) - tf(x) + H(x)f(x)^n = 0 \quad B_\sigma.
\]

\[
\int_{B_\sigma} \left( \frac{n-2}{2n} (\nabla H \cdot x)f^{\frac{2n}{n-2}} - tf^2 \right)
= \int_{\partial B_\sigma} \left( \frac{n-2}{2n} \sigma H f^{\frac{2n}{n-2}} + \sigma |\frac{\partial f}{\partial \nu}|^2 - \frac{\sigma}{2} |\nabla f|^2 + \frac{n-2}{2} \frac{\partial f}{\partial \nu} f - \frac{t}{2} f^2 \right).
\]

Let \( \bar{v}_i(y) = v_i(y + e) \), where \( e = \frac{\nabla K_i(x_i)}{|
abla K_i(x_i)|} \) is a unit vector. Let
\[
\bar{K}_i(y) = K_i(M_i^{-\frac{n-2}{2}}(y + e) + x_i),
\]
then \( \tilde{v}_i \) satisfies

\[
\Delta \tilde{v}_i(y) - \mu_i M_i^{-\frac{4}{n-2}} \tilde{v}_i(y) + \tilde{K}_i(y) \tilde{v}_i(y)^{n^*} = 0, \quad |y| \leq L_i := \frac{\delta_4}{2} M_i^{\frac{4}{(n-2)^2}}.
\]

So the Pohozaev Identity applied to \( \tilde{v}_i \) over \( B_{L_i} \) gives

\[
\int_{B_{L_i}} \left( \frac{n-2}{2n} \left( \nabla \tilde{K}_i(y) \cdot y \right) \tilde{v}_i^{\frac{2n}{n-2}} - \mu_i M_i^{-\frac{4}{n-2}} \tilde{v}_i^2(y) \right) dy = \int_{\partial B_{L_i}} \left( \frac{n-2}{2n} L_i \tilde{K}_i(y) \tilde{v}_i^{\frac{2n}{n-2}} - \frac{L_i}{2} \left| \nabla \tilde{v}_i \right|^2 + \frac{n-2}{2} \frac{\partial \tilde{v}_i}{\partial \nu} \tilde{v}_i - \frac{\mu_i}{2} M_i^{-\frac{4}{n-2}} L_i \tilde{v}_i^2 \right).
\] (2.38)

By Proposition 2.5 and standard elliptic estimates the right hand side of (2.38) is \( O(M_i^{-\frac{4}{n-2}}) \). Then by using \( e = \nabla K_i(x_i) / |\nabla K_i(x_i)| \) we see that the left hand side of the Pohozaev identity is greater than

\[
C |\nabla K_i(x_i)| M_i^{-\frac{2}{n-2}} + O(M_i^{-\frac{4}{n-2}})
\]

for some \( C > 0 \). Consequently

\[
|\nabla K_i(x_i)| \leq C M_i^{-\frac{2}{n-2}}. \tag{2.39}
\]

Base on (2.39) we can write the equation for \( w_i := v_i - U_i \) as

\[
(\Delta + n^* K_i(M_i^{-\frac{4}{n-2}} y + x_i) \xi_i^{\frac{1}{n^*}}) w_i = O(M_i^{-\frac{4}{n-2}})(1 + r)^{2-n}, \quad r \leq \delta_4 M_i^{\frac{4}{(n-2)^2}}.
\]

Then the same estimate in Proposition 2.2 gives

\[
|v_i(y) - U_i(y)| \leq C M_i^{-\frac{4}{n-2}}, \quad |y| \leq \delta_4 M_i^{\frac{4}{(n-2)^2}}. \tag{2.40}
\]

Using the fact that \( v_i(0) = U_i(0) \) and \( \nabla v_i(0) = \nabla U_i(0) = 0 \), we have

\[
|v_i(y) - U_i(y)| \leq C M_i^{-\frac{4}{n-2}} |y|^2, \quad |y| \leq 10. \tag{2.41}
\]
2.4 Harnack inequality on \( B(0, \delta M_{i}^{\frac{n-2}{2}}) \)

Now we establish

**Proposition 2.6.** For any \( \epsilon > 0 \), there exists \( \delta_{5} > 0 \) depending on \( \epsilon \) and \( n \) such that

\[
\min_{|y|=r} v_{i}(y) \leq (1+\epsilon)r^{2-n}, \quad r \leq \delta_{5}M_{i}^{\frac{n-2}{2}}.
\]

**Proof of Proposition 2.6.** We still prove it by a contradiction by assuming that there exist \( \epsilon_{0} \) and \( \epsilon_{i} \to 0 \) such that

\[
\min_{|y|=\epsilon_{i}M_{i}^{\frac{n-2}{2}}} v_{i}(y) \geq (1+\epsilon_{0})|y|^{2-n}.
\]

As a consequence of (2.41) (see also (2.21))

\[
v_{i}^{\lambda}(y) = U_{i}^{\lambda}(y) + O(M_{i}^{-\frac{n-4}{2}}|y|^{-n}).
\]

Therefore, in stead of (2.23) we now have

\[
E_{\lambda} = E_{1} + E_{2} + O(M_{i}^{-\frac{n-8}{2}}r^{1-n})(1 - \frac{\lambda}{r}) + O(M_{i}^{-\frac{n-6}{2}}r^{-3-n})(1 - \frac{\lambda}{r}).
\]

Note that we include \( 1 - \frac{\lambda}{r} \) deliberately because the dominant term now vanishes on \( \partial B_{\lambda} \). We still construct \( h_{1} \) and \( h_{2} \) as in (2.27) and (2.28). Because of the new rate of \( |\nabla K_{i}(x_{i})| \) we now have

\[
|h_{1}| + |h_{2}| = O(M_{i}^{-\frac{n-4}{2}}r^{2-n}).
\]

We note that (2.29) also holds. Now we have

\[
T_{\lambda}(h_{1} + h_{2}) = -E_{2} - \mu_{i}M_{i}^{-\frac{n-4}{2}}(h_{1} + h_{2})
+ n^{*}(K_{i}(M_{i}^{-\frac{n-2}{2}}y + x_{i})\xi_{\lambda}^{\frac{4}{n-2}} - K_{i}(x_{i})\xi_{\lambda}^{\frac{4}{n-2}})(h_{1} + h_{2}).
\]

By the new rate of \( |\nabla K_{i}(x_{i})| \) we have

\[
|\mu_{i}M_{i}^{-\frac{n-4}{2}}(h_{1} + h_{2})| \leq CM_{i}^{-\frac{n-8}{2}}r^{2-n}(1 - \frac{\lambda}{r}).
\]
Also by using this new rate of $|\nabla K_i(x_i)|$ in (2.34) we have

$$K_i(M_i^{-\frac{n-2}{2}} y + x_i)\xi_{\lambda}^{\frac{4}{n-2}} - K_i(x_i)\xi_{\lambda}^{\frac{4}{n-2}}$$

$$= O(M_i^{-\frac{n-2}{2}} r^{-2}) + K_i(x_i)(\xi_{\lambda}^{\frac{4}{n-2}} - \xi_{\lambda}^{\frac{4}{n-2}}).$$

Corresponding to (2.35) and (2.36) we now have

$$\xi_{\lambda}^{\frac{4}{n-2}} = \tilde{\xi}_{\lambda}^{\frac{4}{n-2}} + ar^{n-6} O_\lambda$$

where

$$|a| \leq \begin{cases} 
O(M_i^{-\frac{4}{n-2}}), & \lambda < r < \delta M_i^{(n-2)/2}, \\
Cr^{2-n}, & \delta M_i^{(n-2)/2} \leq r \leq \epsilon M_i^{\frac{4}{n-2}}.
\end{cases}$$

Consequently

$$n^*(K_i(M_i^{-\frac{n-2}{2}} y + x_i)\xi_{\lambda}^{\frac{4}{n-2}} - K_i(x_i)\xi_{\lambda}^{\frac{4}{n-2}})(h_1 + h_2)$$

$$= O(M_i^{-\frac{8}{n-2}} r^{2-n})(1 - \frac{\lambda}{r}) + \tilde{E}_3.$$

where $\tilde{E}_3$ is 0 when $r \leq \delta_4 M_i^{(n-2)/2}$ and is $O(M_i^{-\frac{4}{n-2}} r^{-2-n})$ when $r \geq \delta_4 M_i^{(n-2)/2}$.

Now we can combine $w_\lambda$ with $h_1, h_2$:

$$T_\lambda(w_\lambda + h_1 + h_2)$$

$$\leq E_1 + O(M_i^{-\frac{8}{n-2}} r^{2-n})(1 - \frac{\lambda}{r}) + O(M_i^{-\frac{6}{n-2}} r^{-3-n})(1 - \frac{\lambda}{r}) + \tilde{E}_3.$$

Now the second term in $E_1$ becomes the dominant term. In fact, since $\Delta K_i(x_i)$ is large we have

$$T_\lambda(w_\lambda + h_1 + h_2)$$

$$\leq \mu_i M_i^{-\frac{4}{n-2}} U^\lambda_i(1 - (\frac{\lambda}{r})^4) - \frac{1}{3n} M_i^{-\frac{4}{n-2}} \Delta K_i(x_i)(1 - (\frac{\lambda}{r})^4)r^2(U^\lambda_i)^n. \quad (2.44)$$

Note that $\Delta K_i(x_i)$ makes the right hand side of (2.44) negative when $r$ is close to, or comparable to $\lambda$. But the right hand side becomes positive when $r$ is large. So we need to construct the following function to deal with this. Let

$$h_3(y) = \int_{\Sigma_\lambda} \mu_i M_i^{-\frac{4}{n-2}} G(y, \eta) U^\lambda_i(\eta)(1 - (\frac{\lambda}{|\eta|})^4)d\eta.$$
For \( h_3 \) we have
\[
0 \leq h_3(y) \leq C(C_0, n)M_i^{-\frac{n}{n+2}}|y|^{4-n},
\]
(so \( h_3 = o(1)|y|^{2-n} \)) and
\[
-\Delta h_3 = \mu_i M_i^{-\frac{n}{n+2}} U_i^\lambda(1 - \frac{\lambda}{|y|})^4, \quad \Sigma_\lambda.
\]
Because of this equation, the bad term now becomes \( n^*K_i(M_i^{-\frac{n}{n+2}}y + x_i)\xi_\lambda^{-\frac{1}{n+2}}h_3. \) To be more precise, we have
\[
T_\lambda(w_\lambda + \sum_{s=1}^{3} h_s)
\leq -\frac{1}{3n}M_i^{-\frac{n+2}{n-2}}\Delta K_i(x_i)(1 - \frac{\lambda}{r})^4r^2(U_i^\lambda)^{n^*} + n^*K_i(M_i^{-\frac{n}{n+2}}y + x_i)\xi_\lambda^{-\frac{1}{n+2}}h_3
\leq -\frac{1}{3n}M_i^{-\frac{n+2}{n-2}}\Delta K_i(x_i)(1 - \frac{\lambda}{r})^4r^2(U_i^\lambda)^{n^*}
+ C(n, C_0)M_i^{-\frac{n}{n-2}}|y|^{4-n}(1 + |y|)^{-4}(1 - \lambda/r)
\leq 0
\]
where in the last step we used the largeness of \( \Delta K_i(x_i) \). With this inequality the moving sphere argument applies as before to get a contradiction. Proposition 2.6 is established. □

2.5 The completion of the proof of Theorem 1.1

By Proposition 2.6 and the step one of the proof of Proposition 2.2 we have, for some \( \delta_5 > 0 \) small and \( C \) large
\[
v_i(y) \leq CU_i(y), \quad |y| \leq \delta_5 M_i^{-\frac{2}{n+2}}. \tag{2.45}
\]

Note that we can not get better estimate on \( v_i - U_i \) as before because terms of order \( O(M_i^{-\frac{n}{n+2}}) \) prevent us from getting estimates better than (2.40) and (2.41). By using the vanishing rate of \( |\nabla K_i(x_i)| \) (2.39) and (2.45), (2.40), (2.41) we shall get a contradiction to (2.1) from the Pohozaev identity as follows.

Let \( L_i = \delta_5 M_i^{-\frac{2}{n+2}} \), we apply the Pohozaev identity to \( v_i \) on \( B_{L_i} \). Then the right hand side is
\[
\int_{\partial B_{L_i}} \left\{ \frac{n-2}{2n}L_iK_i v_i^{2n} - \frac{L_i}{2} |v_i|^2 + \frac{n-2}{2} \frac{\partial v_i}{\partial n} v_i - \frac{\mu_i M_i^{-\frac{n}{n+2}}}{2} L_i v_i^2 \right\}.
\]
By (2.45) and the corresponding gradient estimate the right hand side of the Pohozaev identity is \( O(M_i^{-2}) \).

The left hand side of the Pohozaev identity is

\[
\int_{B_L} \left\{ \frac{n-2}{2n} M_i^{\frac{2}{n-2}} \left( \nabla K_i(M_i^{\frac{2}{n-2}} y + x_i) \cdot y \right) v_i^{2n/(n-2)} - \mu_i M_i^{-\frac{n}{n-2}} v_i^2 \right\}
\]

We call the first term of the above \( L_1 \), the second term \( L_2 \). Clearly by the convergence of \( v_i \) to \( U \), we have

\[
L_2 \geq -\mu_i M_i^{-\frac{n}{n-2}} C(n).
\]

Now we estimate \( L_1 \), for which we first have

\[
\nabla K_i(M_i^{-\frac{2}{n-2}} y + x_i) \cdot y = \sum_j \partial_j K_i(x_i) y_j + M_i^{-\frac{2}{n-2}} \sum_{i,j} \partial_{ji} K_i(x_i) y_j y_i + O(M_i^{-\frac{4}{n-2}} r^3).
\]

Then we can write \( L_1 \) as

\[
L_1 = \frac{n-2}{2n} \int_{B_{L_i}} \sum_j \partial_j K_i(x_i) y_j v_i^{2n/(n-2)} + \frac{n-2}{2n} M_i^{-\frac{2}{n-2}} \int_{B_{L_i}} \sum_{i,j} \partial_{ji} K_i(x_i) y_j y_i v_i^{2n/(n-2)} dy + O(M_i^{-\frac{6}{n-2}})
\]

\[
= L_{11} + L_{12} + O(M_i^{-\frac{6}{n-2}}).
\]

To estimate \( L_{11} \) let \( \bar{L}_i = M_i^{\frac{1}{n-2}} \), then

\[
\int_{B_{L_i}} y_j v_i^{2n/(n-2)} = \int_{B_{L_i}} + \int_{B_{L_i} \setminus B_{L_i}} y_j (U_i + O(M_i^{-\frac{4}{n-2}})) v_i^{2n/(n-2)} dy + O(M_i^{-1})
\]

\[
= O(M_i^{-\frac{4}{n-2}}) + O(M_i^{-1}).
\]

So by (2.39) \( L_{11} = O(M_i^{-\frac{4}{n-2}}) \). By similar estimate we see that the leading term in \( L_{12} \) is \( \frac{n-2}{2n} M_i^{-\frac{2}{n-2}} \sum \Delta K_i(x_i) \int_{\mathbb{R}^n} |y|^2 U_i^{2\frac{n-2}{n}} dy \). Then the largeness of \( \Delta K_i(x_i) \) clearly leads to a contradiction in the Pohozaev identity. Theorem 1.1 is established. \( \square \)
2.6 The Proof of Corollary 1.1

Let
\[
u(y) = \left(\frac{2}{1+|y|^2}\right)^{n+2} w(\pi^{-1}(y)), \quad y \in \mathbb{R}^n
\]
and \(g_0\) denote the standard metric on \(S^n\). In stereographic projection
\[
g_0 = \sum_{i=1}^{n+1} dx_i^2 = \left\{\left(\frac{2}{1+|y|^2}\right)^{n+2}\right\}^{\frac{1}{n+2}} \, dy^2.
\]
Since the north pole is not a critical point of \(R\), we know \(u \sim O(|y|^{2-n})\) at infinity. Then the equation for \(u\) becomes
\[
\begin{align*}
\Delta u(y) - \mu(y)\left(\frac{2}{1+|y|^2}\right)^2 u(y) + K(y) u(y)^{\frac{n+2}{n-2}} &= 0, \quad \mathbb{R}^n, \\
\text{\(u(y) \sim O(|y|^{2-n})\)} &\text{ at } \infty.
\end{align*}
\]
(2.46)

By applying Theorem 1.1 we obtain
\[
u(y) \leq C, \quad y \in \mathbb{R}^n.
\]
(2.47)

If \(\mu \equiv 0\), the result in [30] yields (2.47) only under the assumption \(\Delta K(y) > 0\) for each critical point \(y\). The upper bound on \(u\) gives the upper bound on \(w\), then by Harnack inequality
\[
\frac{1}{C} \leq w(x) \leq C, \quad \text{on } S^n.
\]

Corollary 1.1 is established. □

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