On global offensive $k$-alliances in graphs

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Abstract

We investigate the relationship between global offensive $k$-alliances and some characteristic sets of a graph including $r$-dependent sets, $\tau$-dominating sets and standard dominating sets. In addition, we discuss the close relationship that exist among the (global) offensive $k_i$-alliance number of $\Gamma_i$, $i \in \{1, 2\}$ and the (global) offensive $k$-alliance number of $\Gamma_1 \times \Gamma_2$, for some specific values of $k$. As a consequence of the study, we obtain bounds on the global offensive $k$-alliance number in terms of several parameters of the graph.

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1 Introduction

The mathematical properties of alliances in graphs were first studied by P. Kristiansen, S. M. Hedetniemi and S. T. Hedetniemi. They proposed different types of alliances that have been extensively studied during the last four years. These types of alliances are called defensive alliances, offensive alliances, and dual alliances or powerful alliances. A generalization of these alliances called $k$-alliances was presented by K. H. Shafique and R. D. Dutton. We are interested in the study of the mathematical properties of global offensive $k$-alliances.

We begin by stating the terminology used. Throughout this article, $\Gamma = (V, E)$ denotes a simple graph of order $|V| = n$. We denote two adjacent vertices $u$ and $v$ by $u \sim v$. For a nonempty set $S \subseteq V$, $N_S(v)$ denotes the set of neighbors $v$ has in $S$: $N_S(v) := \{u \in S : u \sim v\}$, and the degree of $v$ in $S$ will be denoted by $\delta_S(v) = |N_S(v)|$. We denote the degree of a vertex $v \in V$ by $\delta(v)$, the minimum degree of $\Gamma$ by $\delta$ and the maximum degree by $\Delta$. The complement of the set $S$ in $V$ is denoted by $\overline{S}$ and the boundary of $S$ is defined by $\partial(S) := \bigcup_{v \in S} N_{\overline{S}}(v)$.

A set $S \subseteq V$ is a dominating set in $\Gamma$ if for every vertex $v \in \overline{S}$, $\delta_S(v) > 0$ (every vertex in $\overline{S}$ is adjacent to at least one vertex in $S$). The domination number of $\Gamma$, denoted by $\gamma(\Gamma)$, is the minimum cardinality of a dominating set in $\Gamma$. For $k \in \{2 - \Delta, ..., \Delta\}$, a nonempty set $S \subseteq V$ is an offensive $k$-alliance in $\Gamma$ if

$$\delta_S(v) \geq \delta(\overline{S})(v) + k, \quad \forall v \in \partial(S) \quad (1)$$

or, equivalently,

$$\delta(v) \geq 2\delta(\overline{S})(v) + k, \quad \forall v \in \partial(S). \quad (2)$$

It is clear that if $k > \Delta$, no set $S$ satisfies (1) and, if $k < 2 - \Delta$, all the subsets of $V$ satisfy it. An offensive $k$-alliance $S$ is called global if it is a dominating set. The offensive $k$-alliance number of $\Gamma$, denoted by $a_k^o(\Gamma)$, is defined as the minimum cardinality of an offensive $k$-alliance in $\Gamma$. The global offensive $k$-alliance number of $\Gamma$, denoted by $\gamma_0^o(\Gamma)$, is defined as the minimum cardinality of a global offensive $k$-alliance in $\Gamma$. Notice that $\gamma_k^o(\Gamma) \geq a_k^o(\Gamma)$ and $\gamma_{k+1}^o(\Gamma) \geq \gamma_k^o(\Gamma) \geq \gamma(\Gamma)$.
In addition, if every vertex of $\Gamma$ has even degree and $k$ is odd, $k = 2l - 1$, then every global offensive $(2l - 1)$-alliance in $\Gamma$ is a global offensive $(2l)$-alliance. Hence, in such a case, $\gamma_{2l-1}^o(\Gamma) = \gamma_{2l}^o(\Gamma)$. Analogously, if every vertex of $\Gamma$ has odd degree and $k$ is even, $k = 2l$, then every global offensive $(2l)$-alliance in $\Gamma$ is a global offensive $(2l + 1)$-alliance. Hence, in such a case, $\gamma_{2l}^o(\Gamma) = \gamma_{2l+1}^o(\Gamma)$.

2 The global offensive $k$-alliance number for some families of graphs

The problem of finding the global offensive $k$-alliance number is NP-complete [5]. Even so, for some graphs it is possible to obtain this number. For instance, it is satisfied that for the family of the complete graphs, $K_n$, of order $n$

$$\gamma_k^o(K_n) = \left\lceil \frac{n + k - 1}{2} \right\rceil,$$

for any cycle, $C_n$, of order $n$

$$\gamma_k^o(C_n) = \begin{cases} \left\lceil \frac{n}{k} \right\rceil, & \text{for } k = 0, \\ \left\lceil \frac{n}{2} \right\rceil, & \text{for } k = 1, 2, \end{cases}$$

and for any path, $P_n$, of order $n$

$$\gamma_k^o(P_n) = \begin{cases} \left\lceil \frac{n}{k} \right\rceil, & \text{for } k = 0, \\ \left\lceil \frac{n}{2} \right\rceil + k - 1, & \text{for } k = 1, 2. \end{cases}$$

**Remark 2.1.** Let $\Gamma = K_{r,t}$ be a complete bipartite graph with $t \leq r$. For every $k \in \{2 - r, \ldots, r\}$,

(a) if $k \geq t + 1$, then $\gamma_k^o(\Gamma) = r$.

(b) if $k \leq t$ and $\left\lceil \frac{r+k}{2} \right\rceil + \left\lceil \frac{t+k}{2} \right\rceil \geq t$, then $\gamma_k^o(\Gamma) = t$.

(c) if $-t < k \leq t$ and $\left\lceil \frac{r+k}{2} \right\rceil + \left\lceil \frac{t+k}{2} \right\rceil < t$, then $\gamma_k^o(\Gamma) = \left\lceil \frac{r+k}{2} \right\rceil + \left\lceil \frac{t+k}{2} \right\rceil$.

(d) if $k \leq -t$ and $\left\lceil \frac{r+k}{2} \right\rceil + \left\lceil \frac{t+k}{2} \right\rceil < t$, then $\gamma_k^o(\Gamma) = \min\{t, 1 + \left\lceil \frac{r+k}{2} \right\rceil\}$. 


Proof. (a) Let \( \{V_t, V_r\} \) be the bis-partition of the vertex set of \( \Gamma \). Since \( V_r \) is a global offensive \( k \)-alliance, we only need to show that for every global offensive \( k \)-alliance \( S, V_r \subseteq S \). If \( v \in \overline{S} \) it satisfies \( \delta_S(v) \geq \delta_{\overline{S}}(v) + k > t \), in consequence \( v \in V_t \). Therefore, \( \overline{S} \subseteq V_t \) or, equivalently, \( V_r \subseteq S \). Thus, we conclude that \( \gamma^0_r(\Gamma) = r \).

(b) If \( k \leq t \), it is clear that \( V_t \) is a global offensive \( k \)-alliance, then \( \gamma^0_r(\Gamma) \leq t \). We suppose that \( \lceil \frac{r+k}{2} \rceil + \lceil \frac{t+k}{2} \rceil \geq t \) and there exists a global offensive \( k \)-alliance \( S = A \cup B \) such that \( A \subseteq V_r, B \subseteq V_t \) and \( |S| < t \). In such a case, as \( S \) is a dominating set, \( B \neq \emptyset \). Since \( S \) is a global offensive \( k \)-alliance, \( 2|B| \geq t + k \) and \( 2|A| \geq r + k \). Then we have, \( t > |S| \geq \lceil \frac{r+k}{2} \rceil + \lceil \frac{t+k}{2} \rceil \geq t \), a contradiction. Therefore, \( \gamma^0_r(\Gamma) = t \).

(c) In the proof of (b) we have shown that if there exists a global offensive \( k \)-alliance \( S \) of cardinality \( |S| < t \), then \( |S| \geq \lceil \frac{r+k}{2} \rceil + \lceil \frac{t+k}{2} \rceil \). Taking \( A \subseteq V_r \) of cardinality \( \lceil \frac{r+k}{2} \rceil \) and \( B \subseteq V_t \) of cardinality \( \lceil \frac{t+k}{2} \rceil \) we obtain a global offensive \( k \)-alliance \( S = A \cup B \) of cardinality \( |S| = \lceil \frac{r+k}{2} \rceil + \lceil \frac{t+k}{2} \rceil \).

(d) Finally, if \( S = A \cup B \) where \( A \subseteq V_r, B \subseteq V_t, |A| = \lceil \frac{r+k}{2} \rceil \) and \( |B| = 1 \), then \( S \) is a global offensive \( k \)-alliance. Moreover, \( S \) is a minimum global offensive \( k \)-alliance if and only if \( |S| = 1 + \lceil \frac{r+k}{2} \rceil \leq t \). \( \square \)

3 Global offensive \( k \)-alliances and \( r \)-dependent sets

A set \( S \subseteq V \) is an \( r \)-dependent set in \( \Gamma \) if the maximum degree of a vertex in the subgraph \( \langle S \rangle \) induced by \( S \) is at most \( r \), i.e., \( \delta_S(v) \leq r, \quad \forall v \in S \). We denote by \( \alpha_r(\Gamma) \) the maximum cardinality of an \( r \)-dependent set in \( \Gamma \) [4].

Theorem 3.1. Let \( \Gamma \) be a graph of order \( n \), minimum degree \( \delta \) and maximum degree \( \Delta \).

(a) If \( S \) is an \( r \)-dependent set in \( \Gamma \), \( r \in \{0, \ldots, \lfloor \frac{\delta-1}{2} \rfloor \} \), then \( \overline{S} \) is a global offensive \( (\delta-2r) \)-alliance.

(b) If \( S \) is a global offensive \( k \)-alliance in \( \Gamma \), \( k \in \{2-\Delta, \ldots, \Delta\} \), then \( \overline{S} \) is a \( \lfloor \frac{\Delta-k}{2} \rfloor \)-dependent set.

(c) Let \( \Gamma \) be a \( \delta \)-regular graph (\( \delta > 0 \)). \( S \) is an \( r \)-dependent set in \( \Gamma \), \( r \in \{0, \ldots, \lfloor \frac{\delta-1}{2} \rfloor \} \), if and only if \( \overline{S} \) is a global offensive \( (\delta-2r) \)-alliance.
Proof. (a) Let $S$ be an $r$-dependent set in $\Gamma$, then $\delta_S(v) \leq r$ for every $v \in S$. Therefore, $\delta_S(v) + \delta \leq 2\delta_S(v) + \delta_S(v) \leq 2r + \delta_S(v)$. As a consequence, $\delta_S(v) \geq \delta_S(v) + \delta - 2r$, for every $v \in S$. That is, $S$ is a global offensive $(\delta - 2r)$-alliance in $\Gamma$.

(b) If $S$ is a global offensive $k$-alliance in $\Gamma$, then $\delta(v) \geq 2\delta_S(v) + k$ for every $v \in S$. As a consequence, $\delta_S(v) \leq \frac{\delta(v) - k}{2} \leq \frac{\Delta - k}{2}$ for every $v \in S$, that is, $S$ is a $\lceil \frac{\Delta - k}{2} \rceil$-dependent set in $\Gamma$.

(c) The result follows immediately from (a) and (b).

Corollary 3.2. Let $\Gamma$ be a graph of order $n$, minimum degree $\delta$ and maximum degree $\Delta$.

- For every $k \in \{2 - \Delta, ..., \Delta\}$, $n - \alpha\lceil \frac{\Delta - k}{2} \rceil(\Gamma) \leq \gamma^\tau_k(\Gamma)$.
- For every $k \in \{1, ..., \delta\}$, $\gamma^\tau_k(\Gamma) \leq n - \alpha\lceil \frac{\delta - k}{2} \rceil(\Gamma)$.
- If $\Gamma$ is a $\delta$-regular graph ($\delta > 0$), for every $k \in \{1, ..., \delta\}$, $\gamma^\tau_k(\Gamma) = n - \alpha\lceil \frac{\delta - k}{2} \rceil(\Gamma)$.

4 Global offensive $k$-alliances and $\tau$-dominating sets

Let $\Gamma$ be a graph without isolated vertices. For a given $\tau \in (0, 1]$, a set $S \subseteq V$ is called $\tau$-dominating set in $\Gamma$ if $\delta_S(v) \geq \tau \delta(v)$ for every $v \in S$. We denote by $\gamma_\tau(\Gamma)$ the minimum cardinality of a $\tau$-dominating set in $\Gamma$ [2].

Theorem 4.1. Let $\Gamma$ be a graph of minimum degree $\delta > 0$ and maximum degree $\Delta$.

(a) If $0 < \tau \leq \min\{\frac{k + \delta}{2\delta}, \frac{k + \Delta}{2\Delta}\}$, then every global offensive $k$-alliance in $\Gamma$ is a $\tau$-dominating set.

(b) If $\max\{\frac{k + \delta}{2\delta}, \frac{k + \Delta}{2\Delta}\} \leq \tau \leq 1$, then every $\tau$-dominating set in $\Gamma$ is a global offensive $k$-alliance.

Proof. (a) If $S$ is a global offensive $k$-alliance in $\Gamma$, then $2\delta_S(v) \geq \delta(v) + k$ for every $v \in S$. Therefore, if $0 < \tau \leq \min\{\frac{1}{2}, \frac{k + \delta}{2\delta}\}$, then $\delta_S(v) \geq \frac{\delta(v) + k}{2} \geq \frac{\delta(v)}{2} + \frac{k}{2}$.
\[
\frac{\delta(v)+\delta((2\tau-1)}{2} \geq \tau \delta(v). \text{ Moreover, if } \frac{1}{2} \leq \tau \leq \frac{k+\Delta}{2\Delta}, \text{ then } \delta_S(v) \geq \frac{\delta(v)+k}{2} \geq \tau \delta(v).
\]

(b) Since \( \delta > 0 \), it is clear that every \( \tau \)-dominating set is a dominating set.

If \( \tau \geq \frac{1}{2} \), then \( \delta(2\tau-1) \leq \delta(v)(2\tau-1) \), for every vertex \( v \) in \( \Gamma \). Hence, if \( S \) is a \( \tau \)-dominating set and \( \frac{k+\Delta}{2\Delta} \leq \tau \), we have \( k \leq (2\tau-1)\delta(v) \leq 2\delta_S(v) - \delta(v) \), for every \( v \in S \). Thus, \( S \) is a global offensive \( k \)-alliance in \( \Gamma \).

On the other hand, if \( \tau \leq \frac{1}{2} \), then \( \Delta(2\tau-1) \leq \delta(v)(2\tau-1) \), for every vertex \( v \) in \( \Gamma \). Hence, if \( S \) is a \( \tau \)-dominating set and \( k+\Delta \leq \tau \), we have \( k \leq (2\tau-1)\delta(v) \leq 2\delta_S(v) - \delta(v) \), for every \( v \in S \). Thus, \( S \) is a global offensive \( k \)-alliance in \( \Gamma \).

**Corollary 4.2.** \( S \) is a global offensive \((0)\)-alliance in \( \Gamma \) if, and only if, \( S \) is a \((\frac{1}{2})\)-dominating set.

**Corollary 4.3.** \( S \) is a global offensive \( k \)-alliance in a \( \delta \)-regular graph \( \Gamma \) if, and only if, \( S \) is a \((k+\delta \Delta)\)-dominating set in \( \Gamma \).

**Theorem 4.4.** Let \( \Gamma \) be a graph of order \( n \), minimum degree \( \delta > 0 \) and maximum degree \( \Delta \geq 2 \). For every \( j \in \{2-\Delta, ..., 0\} \) and \( k \leq -\frac{j\delta}{\Delta} \) it is satisfied \( \gamma_o^j(\Gamma) + \gamma_o^j(\Gamma) \leq n \).

**Proof.** If \( j \in \{2-\Delta, ..., 0\} \), then there exists \( \tau \in \left[\frac{1}{\Delta}, \frac{1}{2}\right] \) such that \( j = \Delta(2\tau-1) \). Therefore, if \( S \) is a \( \tau \)-dominating set, then (by Theorem 1 (b)) \( S \) is a global offensive \( j \)-alliance. In consequence, \( \gamma_o^j(\Gamma) \leq \gamma_r(\Gamma) \). Moreover, if \( k \leq -\frac{j\delta}{\Delta} = \delta(1-2\tau) \), then \( 1-\tau \geq \max\{\frac{1}{2}, \frac{k+\Delta}{2\Delta}\} \). Hence, by Theorem 1 (b), we have that every \((1-\tau)\)-dominating set is a global offensive \( k \)-alliance. Thus, \( \gamma_o^k(\Gamma) \leq \gamma_{1-\tau}(\Gamma) \). Using that \( \gamma_r(\Gamma) + \gamma_{1-\tau}(\Gamma) \leq n \) for \( 0 \leq \tau \leq 1 \) (see Theorem 9 \[2\]), we obtain the required result.

Notice that from Theorem 4.4 we have the following result.

**Corollary 4.5.** If \( \Gamma \) is a graph of order \( n \) and minimum degree \( \delta > 0 \), then \( \gamma_o^0(\Gamma) \leq \frac{n}{2} \).

5 Global offensive \( k \)-alliances and standard dominating sets

We say that a global offensive \( k \)-alliance \( S \) is minimal if no proper subset \( S' \subset S \) is a global offensive \( k \)-alliance.
**Theorem 5.1.** Let $\Gamma$ be a graph without isolated vertices and $k \leq 1$. If $S$ is a minimal global offensive $k$-alliance in $\Gamma$, then $S$ is a dominating set in $\Gamma$.

**Proof.** We suppose there exists $u \in S$ such that $\delta_S(u) = 0$ and let $S' = S \setminus \{u\}$. Since $S$ is a minimal global offensive $k$-alliance, and $\Gamma$ has no isolated vertices, there exists $v \in S'$ such that $\delta_S(v) < \delta_S(v) + k$. If $v = u$, we have $\delta_S(u) = \delta_{S'}(u) < \delta_S(v) + k$, a contradiction. If $v \neq u$, we have $\delta_S(v) = \delta_{S'}(v) < \delta_S(v) + k = \delta_S(v) + k$, which is a contradiction too. Thus, $\delta_S(u) > 0$ for every $u \in S$. \hfill $\Box$

In the following result $\bar{\Gamma} = (V, \bar{E})$ denotes the complement of $\Gamma = (V, E)$.

**Lemma 5.2.** Let $\Gamma$ be a graph of order $n$. A dominating set $S$ in $\bar{\Gamma}$ is a global offensive $k$-alliance in $\bar{\Gamma}$ if and only if $\delta_S(v) - \delta_{\bar{S}}(v) + n + k - 1 \leq 2|S|$ for every $v \in \bar{S}$.

**Proof.** We know that a dominating set $S$ in $\bar{\Gamma}$ is a global offensive $k$-alliance in $\bar{\Gamma}$ if and only if $\bar{\delta}_S(v) \geq \bar{\delta}_{\bar{S}}(v) + k$ for every $v \in \bar{S}$, where $\bar{\delta}_S(v)$ and $\bar{\delta}_{\bar{S}}(v)$ denote the number of vertices that $v$ has in $S$ and $\bar{S}$, respectively, in $\bar{\Gamma}$. Now, using that $\bar{\delta}_S(v) = |S| - \delta_s(v)$ and $\bar{\delta}_{\bar{S}}(v) = |\bar{S}| - 1 - \delta_{\bar{S}}(v) = n - |S| - 1 - \delta_{\bar{S}}(v)$, we get that $S$ is a global offensive $k$-alliance in $\bar{\Gamma}$ if and only if $|S| - \delta_s(v) \geq n - |S| - 1 + k - \delta_{\bar{S}}(v)$ or, equivalently, if $\delta_S(v) - \delta_{\bar{S}}(v) + n + k - 1 \leq 2|S|$ for every $v \in \bar{S}$. \hfill $\Box$

**Theorem 5.3.** Let $\Gamma$ be a graph of order $n$, minimum degree $\delta$ and maximum degree $\Delta$.

(a) Every dominating set in $\bar{\Gamma} = (V, \bar{E})$, $S \subseteq V$, of cardinality $|S| \geq \lceil \frac{n+k+\Delta-1}{2} \rceil$ is a global offensive $k$-alliance in $\bar{\Gamma}$.

(b) Every dominating set in $\Gamma = (V, E)$, $S \subseteq V$, of cardinality $|S| \geq \lceil \frac{2n+k-\delta-2}{2} \rceil$ is a global offensive $k$-alliance in $\Gamma$.

**Proof.** If $S$ is a dominating set in $\bar{\Gamma}$ and it satisfies $|S| \geq \lceil \frac{n+k+\Delta-1}{2} \rceil$, then

$$|S| \geq \frac{n+k+\Delta-1}{2} \geq \frac{\delta_S(v) - \delta_{\bar{S}}(v) + n + k - 1}{2}$$

for every vertex $v$. Therefore, by Lemma 5.2 we have that $S$ is a global offensive $k$-alliance in $\bar{\Gamma}$. Thus, the result (a) follows.

Analogously, by replacing $\Gamma$ by $\bar{\Gamma}$ and taking into account that the maximum degree in $\bar{\Gamma}$ is $n - 1 - \delta$, the result (b) follows. \hfill $\Box$
6 The Cartesian product of $k$-alliances

In this section we discuss the close relationship that exist among the (global) offensive $k_i$-alliance number of $\Gamma_i$, $i \in \{1, 2\}$ and the (global) offensive $k$-alliance number of $\Gamma_1 \times \Gamma_2$, for some specific values of $k$.

**Theorem 6.1.** Let $\Gamma_i = (V_i, E_i)$ be a graph of minimum degree $\delta_i$ and maximum degree $\Delta_i$, $i \in \{1, 2\}$.

(a) If $S_i$ is an offensive $k_i$-alliance in $\Gamma_i$, $i \in \{1, 2\}$, then, for $k = \min\{k_2 - \Delta_1, k_1 - \Delta_2\}$, $S_1 \times S_2$ is an offensive $k$-alliance in $\Gamma_1 \times \Gamma_2$.

(b) Let $S_i \subset V_i$, $i \in \{1, 2\}$. If $S_1 \times S_2$ is an offensive $k$-alliance in $\Gamma_1 \times \Gamma_2$, then $S_1$ is an offensive $(k + \delta_2)$-alliance in $\Gamma_1$ and $S_2$ is an offensive $(k + \delta_1)$-alliance in $\Gamma_2$, moreover, $k \leq \min\{\Delta_1 - \delta_2, \Delta_2 - \delta_1\}$.

**Proof.** If $X = S_1 \times S_2$, then $(u, v) \in \partial X$ if and only if, either $u \in \partial S_1$ and $v \in S_2$ or $u \in S_1$ and $v \in \partial S_2$. We differentiate two cases:

Case 1: If $u \in \partial S_1$ and $v \in S_2$, then $\delta_X(u, v) = \delta_{S_1}(u)$ and $\delta_X(u, v) = \delta_{S_1}(u) = \delta(u) + \delta(v)$.

Case 2: If $u \in S_1$ and $v \in \partial S_2$, then $\delta_X(u, v) = \delta_{S_2}(v)$ and $\delta_X(u, v) = \delta(u) + \delta_{S_2}(v)$.

(a) In Case 1 we have $\delta_X(u, v) = \delta_{S_1}(u) \geq \delta_{S_1}(u) + k_1 = \delta_{S_1}(u) - \delta(v) + k_1 \geq \delta_X(u, v) - \Delta_2 + k_1$ and in Case 2 we obtain $\delta_X(u, v) = \delta_{S_2}(v) \geq \delta_{S_2}(v) + k_2 = \delta_{S_2}(v) - \delta(u) + k_2 \geq \delta_X(u, v) - \Delta_1 + k_2$. Hence, for every $(u, v) \in \partial X$, $\delta_X(u, v) \geq \delta_X(u, v) + k$, with $k = \min\{k_2 - \Delta_1, k_1 - \Delta_2\}$. So, the result follows.

(b) In Case 1 we have $\delta_{S_1}(u) = \delta_X(u, v) \geq \delta_{S_1}(u) + k = \delta_{S_1}(u) + \delta(v) + k = \delta_{S_2}(v) + \delta(v) + k$ and in Case 2 we deduce $\delta_{S_2}(v) = \delta_X(u, v) \geq \delta_{S_2}(v) + k = \delta_{S_2}(v) + \delta(u) + \delta_1 + k$. Hence, for every $u \in \partial S_1$, $\delta_{S_1}(u) \geq \delta_{S_1}(u) + \delta_1 + k$ and for every $v \in \partial S_2$, $\delta_{S_2}(v) \geq \delta_{S_2}(v) + \delta_2 + k$. So, the result follows.

\[\square\]

**Corollary 6.2.** Let $\Gamma_i$ be a graph of maximum degree $\Delta_i$, $i \in \{1, 2\}$. Then for every $k \leq \min\{k_1 - \Delta_2, k_2 - \Delta_1\}$, $a_k^o(\Gamma_1 \times \Gamma_2) \leq a_{k_1}^o(\Gamma_1) a_{k_2}^o(\Gamma_2)$. 

8
For the particular case of the graph $C_4 \times K_4$, we have $a_{-3}^2(C_4 \times K_4) = 2 = a_0^2(C_4)a_1^2(K_4)$.

**Theorem 6.3.** Let $\Gamma_2 = (V_2, E_2)$ be a graph of maximum degree $\Delta_2$ and minimum degree $\delta_2$.

(i) If $S$ is a global offensive $k$-alliance in $\Gamma_1$, then $S \times V_2$ is a global offensive $(k - \Delta_2)$-alliance in $\Gamma_1 \times \Gamma_2$.

(ii) If $S \times V_2$ is a global offensive $k$-alliance in $\Gamma_1 \times \Gamma_2$, then $S$ is a global offensive $(k + \delta_2)$-alliance in $\Gamma_1$, moreover, $k \leq \Delta_1 - \delta_2$, where $\Delta_1$ denotes the maximum degree of $\Gamma_1$.

**Proof.**

(i) We first note that, as $S$ is a dominating set in $\Gamma_1$, $X = S \times V_2$ is a dominating set in $\Gamma_1 \times \Gamma_2$. In addition, for every $x_{ij} = (u_i, v_j) \in X$ we have $\delta_X(x_{ij}) = \delta_S(u_i)$ and $\delta_S(u_i) + \Delta_2 \geq \delta_S(u_i) + \delta(v_j) = \delta_X(x_{ij})$, so $\delta_X(x_{ij}) = \delta_S(u_i) \geq \delta_S(u_i) + k \geq \delta_X(x_{ij}) - \Delta_2 + k$. Thus, $X$ is a global offensive $(k - \Delta_2)$-alliance in $\Gamma_1 \times \Gamma_2$.

(ii) From Theorem 6.1 (a) we obtain that $S$ is an offensive $(k + \delta_2)$-alliance in $\Gamma_1$ and $k \leq \Delta_1 - \delta_2$. We only need to show that $S$ is a dominating set. As $S \times V_2$ is a dominating set in $\Gamma_1 \times \Gamma_2$, we have that for every $u \in \overline{S}$ and $v \in V_2$ there exists $(a, b) \in S \times V_2$ such that $(a, b)$ is adjacent to $(u, v)$, hence, $b = v$ and $a$ is adjacent to $u$, so the result follows.

It is easy to see the following result on domination, $\gamma(\Gamma_1 \times \Gamma_2) \leq n_2 \gamma(\Gamma_1)$, where $n_2$ is the order of $\Gamma_2$. An “analogous” result on global offensive $k$-alliances can be deduced from Theorem 6.3 (i).

**Corollary 6.4.** For any graph $\Gamma_1$ and any graph $\Gamma_2$ of order $n_2$ and maximum degree $\Delta_2$, $\gamma_k^{\alpha}(\Gamma_1 \times \Gamma_2) \leq n_2 \gamma_k^{\alpha}(\Gamma_1)$.

We emphasize the following particular cases of Corollary 6.4

**Remark 6.5.** For any graph $\Gamma$,

(a) $\gamma_{k-2}^{\alpha}(\Gamma \times C_t) \leq t \gamma_k^{\alpha}(\Gamma)$,
(b) $\gamma_{k-2}^o(\Gamma \times P_t) \leq t\gamma_k^o(\Gamma)$.

c) $\gamma_{k-t+1}^o(\Gamma \times K_t) \leq t\gamma_k^o(\Gamma)$.

Notice also that if $\Gamma_2$ is a regular graph, Theorem 6.3 (i) can be simplified as follow.

**Corollary 6.6.** Let $\Gamma_2 = (V_2, E_2)$ be a $\delta$-regular graph. A set $S$ is a global offensive $k$-alliance in $\Gamma_1$ if and only if $S \times V_2$ is a global offensive $(k - \delta)$-alliance in $\Gamma_1 \times \Gamma_2$.

### 7 Bounding the global offensive $k$-alliance number

In general, the problem of finding the global offensive $k$-alliance number is $NP$-complete [5]. In the following results we obtain some bounds on this number involving some other parameters of the graphs.

**Remark 7.1.** For every $k \in \{4 - n, \ldots, n - 1\}$,

(a) $\left\lceil \frac{t(n+k-3)}{2} \right\rceil \leq \gamma_k^o(K_n \times C_t) \leq t \left\lceil \frac{n+k+1}{2} \right\rceil$,

(b) $\left\lceil \frac{t(n+k-3)+2}{2} \right\rceil \leq \gamma_k^o(K_n \times P_t) \leq t \left\lceil \frac{n+k+1}{2} \right\rceil$.

**Proof.** (a) Let $S = \bigcup_{i=1}^t S_i \subset V(K_n \times C_t)$, where each $S_i$ ($1 \leq i \leq t$) is a subset of each one of the $t$ copies of $K_n$, respectively. If $S$ is a global offensive $k$-alliance in $K_n \times C_t$, then for every $v \in S$ we have, $|S_i| + 2 \geq \delta_S(v) \geq \delta_{S_i}(v) + k \geq n - 1 - |S_i| + k$, where $S_i$ is the corresponding subset of $S$ included in the same copy of $K_n$ containing $v$. Thus, $|S_i| \geq \frac{n+k-3}{2}$. Hence, for $k > 3 - n$, we obtain that $S_i \neq \emptyset$. Therefore, $|S| = \sum_{i=1}^t |S_i| \geq \frac{t(n+k-3)}{2}$. The upper bound is obtained directly from Remark 6.5. The proof of (b) is completely analogous.

**Theorem 7.2.** Let $\Gamma$ be a graph of order $n$, size $m$, minimum degree $\delta$ and maximum degree $\Delta$. For every $k \in \{2 - \delta, \ldots, \delta\}$, the following inequality holds

$$\gamma_k^o(\Gamma) \geq \left\lceil \frac{(n + 2\Delta + k) - \sqrt{(n + 2\Delta + k)^2 - 4(2m + kn)}}{2} \right\rceil.$$
Proof. To get the bound, we know that
\[ 2m = \sum_{v \in S} \delta(v) + \sum_{v \in S} \delta_S(v) + \sum_{v \in S} \delta_{S'}(v) \]
\[ \leq 2|S|\Delta + \sum_{v \in S} (\delta_S(v) - k) \]
\[ \leq 2|S|\Delta + (n - |S|)(|S| - k). \]
Then, the result follows by solving the inequality
\[ |S|^2 - (n + 2\Delta + k)|S| + 2m + kn \leq 0. \]

Notice the bound is tight, if we consider the cube \( Q_3 \) we obtain \( \gamma^{\circ}_{-1}(Q_3) = \gamma(Q_3) = 2 \) and \( \gamma^{\circ}_{2}(Q_3) = \gamma^{\circ}_{3}(Q_3) = 4. \)

The upper bound in the following theorem have been correctly obtained
in \([5]\) but it appears in the article with a mistake, should be \( \lfloor \frac{\delta - k + 2}{2} \rfloor \) instead
of \( \lceil \frac{\delta - k + 2}{2} \rceil \). So in this article we include the correct result without proof. The
lower bound is an immediate generalization of the previous results obtained
in \([15]\) for \( k = 1 \) and \( k = 2. \)

Theorem 7.3. Let \( \Gamma \) be a graph of order \( n \), size \( m \) and maximum degree \( \Delta \). Then
\[ \left\lfloor \frac{2m + kn}{3\Delta + k} \right\rfloor \leq \gamma^\circ_k(\Gamma) \leq n - \left\lceil \frac{\delta - k + 2}{2} \right\rceil \]

The bounds of Theorem 7.3 are tight. For instance, the lower bound is attained in the case of the 3-cube, \( Q_3 \), for every \( k \). The upper bound is attained, for instance, for the complete graph, \( K_n \), for every \( k \), i.e., \( \gamma^\circ_k(K_n) = \left\lfloor \frac{n + k - 1}{2} \right\rfloor \).

There are graphs in which Theorem 7.2 leads to better results than the
lower bound in Theorem 7.3 and viceversa. For instance, for \( k = 1 \) and
\( \Gamma = K_5 \) the bound in Theorem 7.2 is attained but the lower bound in Theorem 7.3 is not. The opposite one occurs for the case of the 3-cube graph.

Corollary 7.4. Let \( \mathcal{L}(\Gamma) \) be the line graph of a simple graph \( \Gamma \) of size \( m \).
Let \( \delta_1 \geq \delta_2 \geq \cdots \geq \delta_n \) be the degree sequence of \( \Gamma \). Then
\[ \gamma^\circ_k(\mathcal{L}(\Gamma)) \geq \left\lfloor \frac{\sum_{i=1}^{n} \delta_i^2 + m(k - 2)}{3(\delta_1 + \delta_2 - 2) + k} \right\rfloor. \]
Theorem 7.5. Let $\Gamma$ be a graph of order $n$ and maximum degree $\Delta$. For all global offensive $k$-alliance $S$ in $\Gamma$ such that the subgraph $(\overline{S})$ has minimum degree $p$, $|S| \geq \left\lceil \frac{(p+k)n}{\Delta+p+k} \right\rceil$.

Proof. The number of edges in the subgraph $(\overline{S})$ satisfies $m((\overline{S})) \geq \frac{(n-|S|)p}{2}$, hence,

$$\Delta |S| \geq \sum_{v \in S} \delta_S(v) \geq \sum_{v \in S} \delta_{\overline{S}}(v) + k(n-|S|) = 2m((\overline{S}))+k(n-|S|) \geq (p+k)(n-|S|),$$

in consequence, $|S| \geq \frac{(p+k)n}{\Delta+p+k}$.

Notice the bound is attained for the minimal global offensive $k$-alliance in the case of the 3-cube graph $Q_3$ for $k = -1, 2, 3$. For $k = -1$ we have $|S| = 2$ and $p = 2$, and for $k = 2, 3$ we have $|S| = 4$ and $p = 0$.

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