The abelian gauge-Yukawa $\beta$-functions at large $N_f$

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ABSTRACT: We study the impact of the Yukawa interaction in the large-$N_f$ limit to the abelian gauge theory. We compute the coupled $\beta$-functions for the system in a closed form at $\mathcal{O}(1/N_f)$. 
1 Introduction

A comprehensive understanding of the UV behaviour of gauge-Yukawa theories has become of key importance with the growing interest in the asymptotic-safety paradigm [1–4]. Prime candidates for these considerations are gauge-Yukawa models with a large number of fermion flavours, $N_f$. Computing the leading large-$N_f$ contribution to the $\beta$-functions was pioneered by evaluating the $O(1/N_f)$ gauge $\beta$-functions [5–7] for $N_f$ fermions charged under the gauge group; see also Refs [8, 9].

We recently computed the $O(1/N_f)$ $\beta$-function for Yukawa-theory [10] inspired by the earlier works [11, 12]. The Yukawa-theory is closely related to the Gross–Neveu model, which has been extensively studied in the past using a different approach; see e.g. Refs [13–16]. For Gross–Neveu–Yukawa model the behaviour near the fixed point in terms of critical exponents is known up to $O(1/N_f^2)$ [17, 18]. However, the strength of our analysis is that we readily achieved a closed form expression of the $\beta$-function, and as shown in the present work, the procedure is straightforwardly generalisable to include gauge interactions.

In this paper, we compute the leading $1/N_f$ contributions to the $\beta$-functions of the gauge-Yukawa system in a closed form. This result is new and sheds light to the impact of the Yukawa interaction to the gauge theory in the large-$N_f$ limit.

The gauge contribution to the Yukawa $\beta$-function was computed in the abelian case in Ref. [11] and later generalised to non-abelian and semi-simple gauge groups in Ref. [12] assuming that only one flavour of fermions couples to the scalar via Yukawa interaction. We relax this assumption and show that it is possible to get a closed form expressions also in the general case. The current result provides a groundwork for several interesting extensions including e.g. non-abelian gauge groups and chiral fermions.

The paper is organized as follows: In Sec. 2 we introduce the framework and notations and in Sec. 3 compute the new contributions to the renormalization constants and $\beta$-functions. In Sec. 4 we collect the results and comment on the structure of the coupled system, and in Sec. 5 we conclude.

2 The framework

We consider the massless U(1) gauge theory with $N_f$ fermion flavours (QED) with a gauge-singlet real scalar field coupling to the fermionic multiplet, $\psi$, via Yukawa interaction:

$$
\mathcal{L} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + i \bar{\psi} \slashed{D} \psi + y \bar{\psi} \psi \phi.
$$

We define the rescaled gauge and Yukawa couplings,

$$
E \equiv \frac{e^2}{4\pi^2} N_f, \quad \text{and} \quad K \equiv \frac{y^2}{4\pi^2} N_f,
$$

We define...
which are kept constant in the limit $N_f \to \infty$. The purpose of this paper is to derive the coupled system of $\beta$-functions for $E$ and $K$ at the $1/N_f$ level:

$$\beta_E \equiv \frac{dE}{d \ln \mu} = E \left( K \frac{\partial}{\partial K} + E \frac{\partial}{\partial E} \right) G_1(K,E),$$

$$\beta_K \equiv \frac{dK}{d \ln \mu} = K \left( K \frac{\partial}{\partial K} + E \frac{\partial}{\partial E} \right) H_1(K,E),$$

where $G_1$ and $H_1$ are defined by

$$\log Z_E \equiv \log Z - \frac{1}{3} = \sum_{n=1}^{\infty} \frac{G_n(K,E)}{e^n},$$

$$\log Z_K \equiv \log(Z_S^{-1}Z_F^{-2}Z_V^2) = \sum_{n=1}^{\infty} \frac{H_n(K,E)}{e^n},$$

and $Z_3, Z_S, Z_F,$ and $Z_V$ are the renormalization constants for the photon, the scalar, and the fermion wave function, and the 1PI vertex, respectively.

The photon wave function renormalization constant, $Z_3$, is given by

$$Z_3 = 1 - \text{div} \left\{ Z_3 \Pi_0(p^2, Z_K K, Z_E E, \epsilon) \right\},$$

where $\Pi_0$ is the self-energy divided by the external momentum squared, $p^2$, and we denote the poles of $X$ in $\epsilon$ by $\text{div} X$. The self-energy can be written as

$$\Pi_0(p^2, K_0, E_0, \epsilon)$$

$$= E_0 \Pi^{(1)}_E(p^2, \epsilon) + \frac{1}{N_f} \sum_{n=2}^{\infty} \left( E_0^n \Pi^{(n)}_E(p^2, \epsilon) + E_0 K_0^{n-1} \Pi^{(n)}_K(p^2, \epsilon) \right) + O(1/N_f^2),$$

where $\Pi^{(1)}_E$ is the one-loop contribution, and $\Pi^{(n)}_E$ and $\Pi^{(n)}_K$ contain the $n$-loop part consisting of $n-2$ fermion bubbles in the gauge and Yukawa chains summing over the topologies given in Fig. 1.

The scalar wave function renormalization constant, $Z_S$, is determined via

$$Z_S = 1 - \text{div} \left\{ Z_S S_0(p^2, Z_K K, Z_E E, \epsilon) \right\},$$

where $S_0$ is the self-energy for the scalar wave function.
Figure 2: Scalar self-energy corrections.

with the scalar self-energy given by

$$S_0(p^2, K_0, E_0, \epsilon) = K_0 S_K^{(1)} (p^2, \epsilon) + \frac{1}{N_f} \sum_{n=2}^{\infty} \left( K_0^n S_K^{(n)} (p^2, \epsilon) + K_0 E_0^{n-1} S_E^{(n)} (p^2, \epsilon) \right) + \mathcal{O}(1/N_f^2), \tag{2.10}$$

where $S_K^{(1)}$ is the one-loop result, and $S_K^{(n)}$ and $S_E^{(n)}$ the $n$-loop terms consisting of $n - 2$ fermion bubbles in the Yukawa and gauge chains summing over the topologies shown in Fig. 2.

For the fermion self-energy and vertex renormalization constants, the lowest non-trivial contributions are already $\mathcal{O}(1/N_f)$, and we have

$$Z_f = 1 - \text{div} \left\{ \Sigma_0 (p^2, Z_K K, Z_E E, \epsilon) \right\}, \tag{2.11}$$

$$\Sigma_0 (p^2, K_0, E_0, \epsilon) = 1 + \frac{1}{N_f} \sum_{n=1}^{\infty} \left( K_0^n \Sigma_K^{(n)} (p^2, \epsilon) + E_0^n \Sigma_E^{(n)} (p^2, \epsilon) \right) + \mathcal{O}(1/N_f^2), \tag{2.12}$$

where $\Sigma_K^{(n)}$ and $\Sigma_E^{(n)}$ are depicted in Fig. 3a with $n - 1$ fermion bubbles. Similarly,

$$Z_V = 1 - \text{div} \left\{ V_0 (p^2, Z_K K, Z_E E, \epsilon) \right\}, \tag{2.13}$$

$$V_0 (p^2, K_0, E_0, \epsilon) = 1 + \frac{1}{N_f} \sum_{n=1}^{\infty} \left( K_0^n V_K^{(n)} (p^2, \epsilon) + E_0^n V_E^{(n)} (p^2, \epsilon) \right) + \mathcal{O}(1/N_f^2), \tag{2.14}$$

where $V_K^{(n)}$ and $V_E^{(n)}$ contain $n - 1$ fermion bubbles and are shown diagrammatically in Fig. 3b.

The term corresponding to pure QED, $\Pi_E^{(n)}$, was computed in Ref. [6], and the pure-Yukawa contributions, $S_K^{(n)}$, $\Sigma_K^{(n)}$ and $V_K^{(n)}$, in Ref. [10]. Their contribution to the $\beta$-functions, Eqs (2.3) and (2.4), is

$$\beta_E (K = 0) = E^2 \left[ \frac{2}{3} + \frac{1}{4N_f} \int_0^{2/3E} \pi_E (t) dt \right] + \mathcal{O}(1/N_f^2), \tag{2.15}$$

$$\beta_K (E = 0) = K^2 \left[ 1 + \frac{1}{N_f} \left( \frac{3}{2} + \int_0^K \xi_K (t) dt \right) \right] + \mathcal{O}(1/N_f^2), \tag{2.16}$$
Figure 3: Gauge and Yukawa contributions to fermion self-energy and the vertex corrections due to a chain of fermion bubbles.

where

\[
\pi_E(t) = \frac{\Gamma(4-t)(1-t)(1-\frac{t}{3})(1+\frac{t}{2})}{\Gamma(2-\frac{t}{2})\Gamma(3-\frac{t}{2})\Gamma(1+\frac{t}{2})},
\]

(2.17)

\[
\xi_K(t) = -\frac{\Gamma(4-t)}{\Gamma(2-\frac{t}{2})\Gamma(3-\frac{t}{2})\pi t} \sin \left(\frac{\pi t}{2}\right)
\]

(2.18)

The impact of the mixed contributions, namely \(\Pi_K^{(n)}\), \(S_E^{(n)}\), \(\Sigma_E^{(n)}\), \(V_E^{(n)}\), is evaluated in the next section.

3 Mixed contributions

In this section we derive the mixed contributions to the renormalization constants for the photon self-energy, the fermion self-energy, the Yukawa vertex, and the scalar self-energy, and eventually compute the coupled \(\beta\)-functions.

3.1 The Yukawa contribution to the QED \(\beta\)-function

The Yukawa contribution to the photon self-energy (depicted in the second row of Fig. 1), is obtained by substituting Eq. (2.8) in Eq. (2.7). We get

\[
Z_3(K) = -\frac{E}{N_f} \text{div} \left\{ \sum_{n=1}^{\infty} (Z_K K)^n \Pi_K^{(n+1)}(p^2, \epsilon) \right\}.
\]

(3.1)

Notice that the diagrams involving a horizontal bubble chain differ from the corresponding ones for the scalar self-energy in Fig. 2 just by an overall factor \((2 - d)\) coming from the
algebra of the $\gamma$-matrices. Altogether, we find

$$\Pi_K^{(n)}(p^2, \epsilon) = (-1)^{n-1} \frac{3}{4(d-1)\epsilon^{n-1}} \pi_K(p^2, \epsilon, n), \tag{3.2}$$

where $\pi_K(p^2, \epsilon, n)$ can be expanded as

$$\pi_K(p^2, \epsilon, n) = \sum_{j=0}^{\infty} \pi_K^{(j)}(p^2, \epsilon)(\epsilon n)^j, \tag{3.3}$$

with $\pi_K^{(j)}(p^2, \epsilon)$ regular for $\epsilon \to 0$. Recalling that $Z_K = (1 - \frac{1}{\epsilon} K)^{-1} + \mathcal{O}(1/N_f)$, we can evaluate $Z_3(K)$ from Eq. (3.1):

$$Z_3(K) = - \frac{E}{N_f} \text{div} \left\{ \sum_{n=1}^{\infty} K^n \sum_{i=0}^{n-1} \frac{n-1}{i} (\Pi_K^{n-i})(p^2, \epsilon) \right\}$$

$$= - \frac{3E}{4N_f} \text{div} \left\{ \sum_{n=1}^{\infty} \frac{(-K)^n}{(d-1)\epsilon^n} \sum_{j=0}^{n-1} \pi_K^{(j)}(p^2, \epsilon) \epsilon^j \right\}$$

$$\times \sum_{i=0}^{n-1} \frac{(n-1)^i}{i} (-1)^i(n-i+1)^{j-1} \right\}$$

$$= - \frac{3E}{4N_f} \text{div} \left\{ \sum_{n=1}^{\infty} \frac{(-K)^n}{(d-1)\epsilon^n} \pi_K^{(0)}(\epsilon) \frac{(-1)^{n+1}}{n(n+1)} \right\}$$

$$= - \frac{3E}{4N_f} \frac{1}{\epsilon} \int_0^K \frac{\pi_K^{(0)}(t)}{t-3} \left(1 - \frac{t}{K}\right) dt,$$

where we used

$$\sum_{i=0}^{n-1} \frac{(n-1)^i}{i} (-1)^i(n-i+1)^{j-1} = \frac{(-1)^{n+1}}{n(n+1)} \delta_{j,0}, \quad j = 0, \ldots, n - 1 \tag{3.5}$$

and restricted ourselves to the $1/\epsilon$ pole. The function $\pi_K^{(0)}$ is independent of $p^2$, as it should, and reads

$$\pi_K^{(0)}(t) = \frac{(t-2)(t-1)\Gamma(5-t)}{6\Gamma(3 - \frac{5}{2})^2\pi t} \sin \left(\frac{\pi t}{2}\right). \tag{3.6}$$

The contribution of $Z_3(K)$ to $\beta_E$, Eq. (2.3), is found to be

$$\beta_E(K \neq 0) = E^2 \frac{3}{4N_f} \int_0^K \pi_K(t) dt. \tag{3.7}$$

where we have defined

$$\pi_K(t) \equiv \frac{\pi_K^{(0)}(t)}{t-3}. \tag{3.8}$$

We show the function $\pi_K(t)$ in Fig. 4. Since $\pi_K(t)$ has a first order pole at $t = 3$, the first singularity of $\beta_E(K \neq 0)$ occurs at $K = 3$ and is a logarithmic one. The next singularity of $\pi_K(t)$ is found at $t = 5$ (first order) and would result in a logarithmic singularity of $\beta_E(K \neq 0)$ at $K = 5$.  

– 6 –
3.2 The QED contribution to the Yukawa $\beta$-function

The QED contribution to the fermion self-energy and to the Yukawa vertex is closely related to the pure-Yukawa case. This is because the gauge chain is equivalent to the Yukawa chain besides an overall factor. In fact, $\Sigma^{(n)}_E$ and $V^{(n)}_E$ are related to $\Sigma^{(n)}_K$ and $V^{(n)}_K$ as

$$\Sigma^{(n)}_E(\not p) = (d-2) \left( \frac{d-2}{d-1} \right)^{n-1} \Sigma^{(n)}_K(\not p),$$

$$V^{(n)}_E(p^2) = -d \left( \frac{d-2}{d-1} \right)^{n-1} V^{(n)}_K(p^2).$$

The factors $(d-2)$ and $-d$ come from the algebra of the $\gamma$-matrices, while $\left( \frac{d-2}{d-1} \right)^{n-1}$ takes into account the difference in replacing $\Pi^{(1)}_E$ with $S^{(1)}_K$. Notice that $g_{\mu\nu}$ is the only relevant Lorentz structure in the photon propagator, since the $k_\mu k_\nu$ term do not contribute to the $\beta$-function.

Making use the relations Eqs (3.9) and (3.10), $\Sigma^{(n)}_E$ and $V^{(n)}_E$ are expanded as

$$\Sigma^{(n)}_E(\not p) = (-1)^{n-1} \left( \frac{2}{3} \right)^n \frac{3}{4\pi^3 n \epsilon} \sigma_E(p^2, \epsilon, n),$$

$$\sigma_E(p^2, \epsilon, n) = \sum_{j=0}^{\infty} \sigma^{(j)}_E(p^2, \epsilon)(n\epsilon)^j,$$

and

$$V^{(n)}_E(p^2) = (-1)^{n-1} \frac{3}{4\pi^3 n \epsilon} \left( \frac{2}{3} \right)^n v_E(p^2, \epsilon, n),$$

$$v_E(p^2, \epsilon, n) = \sum_{j=0}^{\infty} v^{(j)}_E(p^2, \epsilon)(n\epsilon)^j.$$
yield

$$Z_f(E) = -\frac{1}{N_f} \sum_{n=1}^{\infty} \text{div} \left\{ (Z_E E)^n \Sigma^{(n)}_E (p^2, \epsilon) \right\} = -\frac{1}{N_f} \frac{3}{4 \epsilon} \int_0^{2E} \sigma^{(0)}_E (t) dt, \quad (3.15)$$

$$Z_V(E) = -\frac{1}{N_f} \sum_{n=1}^{\infty} \text{div} \left\{ (Z_E E)^n V^{(n)}_E (p^2, \epsilon) \right\} = -\frac{1}{N_f} \frac{3}{\epsilon} \int_0^{2E} v^{(0)}_E (t) dt, \quad (3.16)$$

where we kept only the $1/\epsilon$ pole. The functions $\sigma^{(0)}_E$ and $v^{(0)}_E$ are independent of $p^2$, and are given by

$$\sigma^{(0)}_E (t) = \frac{2 \Gamma(4-t)}{3 \pi \Gamma (1-\frac{t}{2}) \Gamma (3-\frac{t}{2})}, \quad (3.17)$$

$$v^{(0)}_E (t) = \left( \frac{1 - \frac{t}{2}}{1 - \frac{t}{2}} \right)^2 \sigma^{(0)}_E (t). \quad (3.18)$$

The QED contribution to the scalar self-energy is shown in the second row of Fig. 2. The diagrams involving a horizontal gauge chain are related to the ones in the pure-Yukawa case analogously to Eq. (3.9). Altogether, we find

$$S^{(n)}_E (p^2, \epsilon, \sigma) = (-1)^n \left( \frac{27}{4n(n-1) \epsilon} \right)^n s_E (p^2, \epsilon, n), \quad (3.19)$$

$$s_E (p^2, \epsilon, n) = \sum_{j=0}^{\infty} s_E^{(j)} (p^2, \epsilon) n^j. \quad (3.20)$$

The QED contribution in Eq. (2.9) is singled out as follows:

$$Z_S(E) = -K \text{div} \left\{ Z_f(E)^{-2} Z_V(E)^2 S^{(1)}_F (p^2, \epsilon) + \frac{1}{N_f} \sum_{n=1}^{\infty} (Z_E E)^n S^{(n+1)}_E (p^2, \epsilon) \right\}. \quad (3.21)$$

To evaluate the right-hand side of Eq. (3.21), we closely follow the procedure in Ref. [10] for the scalar self-energy:

$$Z_S(E) = -\frac{K}{N_f} \sum_{n=1}^{\infty} E^n \text{div} \left\{ \left( 1 - \frac{2E}{3} \right)^{-n} \left[ 2 S^{(1)}_F \left( \Sigma^{(n)}_E - V^{(n)}_E \right) + S^{(n+1)}_E \right] \right\}$$

$$= -\frac{K}{N_f} \sum_{n=1}^{\infty} E^n \text{div} \left\{ \sum_{i=0}^{n-1} \left( n-1 \right) \left( \frac{2}{3} \right)^i \frac{1}{\epsilon} \right.$$  

$$\times \left[ 2 S^{(1)}_F \left( \Sigma^{(n-i)}_E - V^{(n-i)}_E \right) + S^{(n-i+1)}_E \right] \}$$

$$= -\frac{3K}{N_f} \sum_{n=1}^{\infty} \left( -\frac{2}{3} E \right)^n \text{div} \left\{ \frac{1}{\epsilon^n} \sum_{i=0}^{n-1} \left( n-1 \right) (-1)^i \frac{\xi_E (p^2, \epsilon, n-i)}{(n-i)(n-i+1)\epsilon^{n+1}} \right\},$$

where we defined

$$\xi_E (p^2, \epsilon, n) = \epsilon (n+1) 2 S^{(1)}_F \left( v_E (p^2, \epsilon, n) - \frac{1}{4} \sigma_E (p^2, \epsilon, n) \right) - \frac{3}{2} s_E (p^2, \epsilon, n+1), \quad (3.23)$$
Figure 5: The functions $\tilde{\xi}_E(t)$ (left panel) and $\xi_E(t)$ (right panel) defined in Eqs (3.31) and (3.28), respectively.

and $S_{E}^{(1)}$ is the finite part of the one-loop bubble $S_{K}^{(1)}$. Then, by expanding

$$\xi_E(p^2, \epsilon, n-i) = \sum_{j=0}^{\infty} \epsilon^j (n-i+1)^j \xi_E^{(j)}(p^2, \epsilon),$$  \hspace{1cm} (3.24)

and using

$$\sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^i (n-i+1)^j \frac{1}{n-i} = \begin{cases} \frac{(-1)^n}{n} & j = 0 \\ \frac{(-1)^n}{n+1} & j = 1, \ldots, n \end{cases},$$  \hspace{1cm} (3.25)

we can further simplify the expression to

$$Z_S(E) = \frac{3K}{N_f} \sum_{n=1}^{\infty} \left( \frac{2}{3} \right)^n \text{div} \left\{ \frac{1}{(n+1)\epsilon^{n+1}} \xi_E^{(0)}(p^2, \epsilon) + \frac{1}{n\epsilon^{n+1}} \sum_{j=1}^{\infty} \xi_E^{(j)}(p^2, \epsilon) \epsilon^j \right\}$$

$$= \frac{9K}{2EN_f} \sum_{n=2}^{\infty} \left( \frac{2}{3} \right)^n \text{div} \left\{ \frac{1}{n} \left( \frac{\xi_E^{(0)}(p^2, \epsilon)}{n} + \frac{\xi_E(p^2, \epsilon, 0) - \xi_E^{(0)}(p^2, \epsilon)}{n-1} \right) \right\}$$

$$= \frac{9K}{2EN_f} \int_{0}^{\frac{2}{3}E} \left( \xi_E^{(0)}(t) - \xi_E^{(0)}(0) + \frac{2\xi_E(p^2, t, 0) - \xi_E^{(0)}(t)}{3} E \right) dt,$$  \hspace{1cm} (3.26)

where we kept the $1/\epsilon$ pole only. The function $\xi_E(p^2, t, 0) = \lim_{n \to 0} \xi_E(p^2, t, n)$ has to be independent of $p^2$ for the consistency of the computation. This is indeed the case: we checked that

$$\frac{3}{2} s_E(p^2, t, 1) = 2 \left( 1 + t s_E^{(1)}(p^2, t) \right) \left( v_E^{(0)}(t) - \frac{1}{4} \sigma_E^{(0)}(t) \right),$$  \hspace{1cm} (3.27)

and therefore

$$\xi_E(p^2, t, 0) = -2v_E^{(0)}(t) + \frac{1}{2} \sigma_E^{(0)}(t) \equiv \xi_E(t).$$  \hspace{1cm} (3.28)

Finally, we find:

$$Z_S(E) = \frac{3K}{\epsilon N_f} \left\{ \int_{0}^{\frac{2}{3}E} \left( \xi_E^{(0)}(t) - \xi_E^{(0)}(0) \right) dt + \int_{0}^{\frac{2}{3}E} \frac{\xi_E(t) - \xi_E^{(0)}(t)}{t} dt \right\}.$$  \hspace{1cm} (3.29)
With Eqs (3.15), (3.16) and (3.29) at hand, we can compute the QED contribution to the Yukawa $\beta$-function:

$$\beta_K(E \neq 0) = -\frac{3K^2}{N_f} \left\{ \frac{2}{3} \int_0^{\frac{2}{3}E} \xi_E(t)dt + \frac{1}{2} + \left(1 - \frac{2E}{3K}\right) \xi_E\left(\frac{2}{3}E\right) \right\}. \quad (3.30)$$

where we have defined

$$\tilde{\xi}_E(t) = \frac{\xi_E(t) - \xi_E(0)}{t}. \quad (3.31)$$

The functions $\xi_E(t)$ and $\tilde{\xi}_E(t)$ are explicitly given by

$$\xi_E(t) = -\frac{2(t - 3)^2\Gamma(2 - t)}{3\Gamma(2 - \frac{2}{3})\Gamma(3 - \frac{2}{3})\pi t} \sin\left(\frac{\pi t}{2}\right), \quad (3.32)$$

$$\tilde{\xi}_E(t) = \frac{(15 + t - 5t^2 + t^3)\Gamma(4 - t)}{9(2 - t)\Gamma(2 - \frac{2}{3})\Gamma(3 - \frac{2}{3})\pi t} \sin\left(\frac{\pi t}{2}\right). \quad (3.33)$$

We plot the functions $\tilde{\xi}_E(t)$ and $\xi_E(t)$ in Fig. 5. The first singularity of $\beta_K(E \neq 0)$ is at $E = 15/2$ and consists of a first-order pole coming from $\xi_E(t)$ plus a logarithmic singularity arising from the integration of $\tilde{\xi}_E(t)$, both at $t = 5$.

4 The coupled system

Here we summarize and discuss our results for the coupled system. Combining Eqs (2.15) and (2.16) with the new results in Eqs (3.7) and (3.30), we obtain

$$\frac{\beta_K}{K^2} = 1 - \frac{3}{N_f} \left\{ 1 - \frac{1}{3} \int_0^K \xi_K(t)dt + \int_0^{\frac{2}{3}E} \xi_E(t)dt + \left(1 - \frac{2E}{3K}\right) \xi_E\left(\frac{2}{3}E\right) \right\}, \quad (4.1)$$

$$\frac{\beta_E}{E^2} = -\frac{2}{3} + \frac{1}{4N_f} \left\{ \int_0^{\frac{2}{3}E} \pi_E(t)dt + 3 \int_0^K \pi_K(t)dt \right\}. \quad (4.2)$$

Near the Gaussian fixed point, these can be expanded as

$$\beta_E = \frac{2}{3}E^2 + \frac{1}{2N_f}E^3 - \frac{1}{4N_f}E^2K - \frac{11}{72N_f}E^4 + \frac{7}{32N_f}E^2K^2 \quad (4.3)$$

$$\beta_K = \left(1 + \frac{3}{2N_f}\right)K^2 - \frac{3}{N_f}EK - \frac{3}{2N_f}K^3 + \frac{5}{4N_f}EK^2 + \frac{5}{6N_f}E^2K \quad (4.4)$$

$$+ \frac{7}{16N_f}K^4 - \frac{1}{2N_f}E^2K^2 + \frac{35}{108N_f}E^2K$$

$$+ \frac{11}{96N_f}K^5 + \frac{1}{3888N_f}(-1625 + 1296\zeta_3)E^3K^2 + \frac{1}{648N_f}(83 - 144\zeta_3)E^4K \ldots$$

We have checked that the expansions agree with the known four-loop results [19–23] in the leading order in $N_f$. Furthermore, the $-\frac{2E}{3K}\xi_E\left(\frac{2}{3}E\right)$ part in the last term of Eq. (4.1)
Figure 6: The flow diagram for the coupled system with $N_f = 30$. The arrows point towards UV.

corresponds to the result of Refs [11, 12], and we have checked that our result agrees with those.

The first singularity of the pure-QED $\beta$-function is located at $E = 15/2$, whereas for the pure-Yukawa case it occurs at $K = 5$. These known singularities are now accompanied by the ones from the mixed contributions, Eqs (3.7) and (3.30). As we noticed in Section 3, $\beta_E(K \neq 0)$ has the first singularity at $K = 3$, while $\beta_K(E \neq 0)$ at $E = 15/2$. The former, similarly to the pure gauge and Yukawa cases, is a logarithmic singularity, whereas the latter is a pole of first order.

The $\mathcal{O}(1/N_f)$ coupled system has only the three already known fixed points: the Gaussian fixed point, and the pure-QED (near $E = 15/2$) and pure-Yukawa (near $K = 3$) fixed points.

We show the flow diagram for $N_f = 30$ outside the vicinity of the singularities in Fig. 6. Near $K = 3$, the logarithmic singularity in $\beta_E$ arising from $\pi_K(t)$ dominates making the gauge coupling to increase and approach the value $E = 15/2$. Near $E = 15/2$, however, $\beta_K$ has a pole arising from $\xi_E(t)$ eventually dominating the flow, and driving the Yukawa coupling to zero near $E = 15/2$. The flow may be extended setting $K \equiv 0$ and switching to pure-QED, so that the gauge coupling reaches the fixed point as $E \to 15/2$ in the UV.

5 Conclusions

We have computed the leading $1/N_f$ mixed contributions for the $\beta$-functions for abelian gauge-Yukawa theory with $N_f$ fermion flavours coupling to a gauge-singlet real scalar. Together with the known results for the pure-QED and pure-Yukawa cases, this allows the study of the abelian gauge-Yukawa system.
The flow in the interacting theory leads to the vanishing Yukawa coupling near the gauge coupling value $E = 15/2$ due to the peculiar interplay of the singularities. However, the gauge $\beta$-function is still positive around $(K,E) = (0,15/2)$, and $E$ keeps growing before eventually reaching the fixed point due to the known a logarithmic singularity near $E = 15/2$.

Our work extends the previous results towards a more complete picture of gauge-Yukawa theories in the large-$N_f$ limit.

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