Multi-graded Galilean conformal algebras

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Abstract

Galilean conformal algebras can be constructed by contracting a finite number of conformal algebras, and enjoy truncated Z-graded structures. Here, we present a generalisation of the Galilean contraction procedure, giving rise to Galilean conformal algebras with truncated $Z^{\sigma}$-gradings, $\sigma \in \mathbb{N}$. Detailed examples of these multi-graded Galilean algebras are provided, including extensions of the Galilean Virasoro and affine Kac-Moody algebras. We also derive the associated Sugawara constructions and discuss how these examples relate to multivariable extensions of Takiff algebras. We likewise apply our generalised contraction prescription to tensor products of $W_3$ algebras and obtain new families of higher-order Galilean $W_3$ algebras.
1 Introduction

The Galilean Virasoro algebra appears in studies of asymptotically flat three-dimensional spacetimes, see [1] and references therein, and can be constructed [2–6] as a contraction of a pair of Virasoro algebras. Similarly, the Galilean \( W_3 \) algebra [7–11] follows by contracting a pair of \( W_3 \) algebras, while more general Galilean conformal algebras with extended symmetries have been constructed in [9,11,12] and are known as Galilean \( W \)-algebras. Non-relativistic systems with (typically non-affine) conformal symmetry were previously studied in [13–19].

Following ideas put forward in [11], higher-order Galilean contractions were developed in [20], generalising the contraction procedure from pairs of symmetry algebras (or equivalently vertex algebras) to any finite number of symmetry algebras (or vertex algebras). In the case of Virasoro or affine Kac-Moody, the usual second-order Galilean algebras have been found [20] to be isomorphic to the Takiff algebras [21] considered in [22,23], while the higher-order counterparts provide \( N \)-th-order generalisations (where \( N \) is the number of inputted symmetry algebras \( A \)). These higher-order Galilean algebras thus enjoy a truncated \( \mathbb{Z} \)-grading whose truncation is determined by the order \( N \) of the contraction.

Here, we modify the higher-order contraction procedure to let it depend on a factorisation of \( N \), where \( N = N_1, \ldots, N_\sigma \) is a finite sequence of positive integers such that \( N = N_1 \cdots N_\sigma \). We thus organise the \( N \)-fold tensor product \( A \otimes N \) in terms of \( N_\ell \)-fold tensor-product factors,

\[
A \otimes N = (A \otimes N_1) \otimes \cdots \otimes (A \otimes N_\sigma),
\]

(1.1)

and apply the higher-order contraction prescription of [20] to the factors ‘simultaneously’. We find that the ensuing Galilean algebra, \( A^N \), is \( \mathbb{Z}^{\sigma} \)-graded, truncated according to the sequence \( N_1, \ldots, N_\sigma \). Because of this graded structure, we refer to the generalised contraction as multi-graded contraction. We also observe that the contractions are independent of the ordering of the factors in the factorisation of \( N \). In case the mode algebra underlying \( A \) is a Lie algebra, we find that \( A^N \) is isomorphic to a multivariable generalisation of the Takiff algebras discussed in [20], with the number of variables given by the length \( \sigma \) of the contraction sequence \( N \).

In Section 2, we outline the multi-graded contraction procedure and illustrate it by working out the corresponding Galilean Virasoro and affine Kac-Moody algebras. We also discuss the ensuing grading structures and relate the corresponding Galilean algebras to a multivariable generalisation of the Takiff algebras. In Section 3, we construct a Sugawara operator for each of the multi-graded Galilean Kac-Moody algebras; its central charge is given by the product of the contraction order \( N \) and the dimension of the underlying Lie algebra. We also show that the Sugawara construction commutes with the contraction procedure. In Section 4, we apply multi-graded contractions to the \( W_3 \) algebra and thereby obtain a new class of Galilean \( W_3 \) algebras. Section 5 contains some concluding remarks.

2 Contraction procedure

We find it advantageous to describe the multi-graded contractions and ensuing algebras in the language of operator-product algebras (OPAs), and refer to [11,24] for details on the structure of an OPA. We say that an OPA is of Lie type if the corresponding mode algebra is a Lie algebra, as is the case for the Virasoro and Kac-Moody algebras. Throughout, \( I \) denotes the identity field and \( \Delta_A \) the conformal weight of the scaling field \( A \).

2.1 Star relations in OPAs

For the space of quasi-primary fields in the OPA \( A \), we let \( B_A \) denote a basis consisting of quasi-primary fields only. Only keeping the non-singular terms, the operator-product expansion of \( A, B \in B_A \) can
then be expressed as

\[ A(z)B(w) \sim \sum_{Q \in \mathcal{B}_A} C_{A,B}^Q \left( \sum_{n=0}^{\Delta_A+\Delta_B-\Delta_Q} \frac{\beta_{\Delta_A,\Delta_B}^{\Delta_Q,n} \partial^n Q(w)}{(z-w)^{\Delta_A+\Delta_B-\Delta_Q-n}} \right), \]

(2.1)

with structure constants \( C_{A,B}^Q \in \mathbb{C} \) and

\[ \beta_{\Delta_A,\Delta_B}^{\Delta_Q,n} = \frac{(\Delta_A - \Delta_B + \Delta_Q)_n}{n!(2\Delta_Q)_n}, \quad (x)_n = \prod_{j=0}^{n-1} (x+j). \]

(2.2)

Convenient for our purposes, the essential part of the operator-product expansion \[2.1\] is synthesised in the so-called star relation

\[ A \ast B \simeq \sum_{Q \in \mathcal{B}_A} C_{A,B}^Q \{Q\}, \]

(2.3)

where \( \{Q\} \) represents the sum over \( n \) displayed in \[2.1\].

For example, the Virasoro algebra \( \mathfrak{Vir} \) of central charge \( c \) is of Lie type and generated by \( T \), with star relation

\[ T \ast T \simeq \frac{c}{2} \{1\} + 2\{T\}. \]

(2.4)

Likewise, the nontrivial star relations in an affine Kac-Moody algebra \( \widehat{\mathfrak{g}} \) are given by

\[ J^a \ast J^b \simeq \kappa^{ab} k \{1\} + f^{abc} \{J^c\}, \]

(2.5)

where \( f^{abc} \in \mathbb{C} \) are structure constants, \( k \in \mathbb{C} \) the level and \( \kappa \) the Killing form of the underlying finite-dimensional complex Lie algebra \( \mathfrak{g} \). As is customary, we do not display summations over repeated group indices, here the summation over \( c \in \{1, \ldots, \dim \mathfrak{g}\} \). We note that \( \widehat{\mathfrak{g}} \) is of Lie type.

### 2.2 Higher-order contractions

Higher-order Galilean contractions were developed in \[11\]. Here, we recast them in a notation suitable for their multi-graded generalisation introduced in Section \[2.3\]. Thus, for \( N \in \mathbb{N} \), let

\[ \mathcal{A}^{\otimes N} = \bigotimes_{i=0}^{N-1} \mathcal{A}_{(i)}, \]

(2.6)

where \( \mathcal{A}_{(0)}, \ldots, \mathcal{A}_{(N-1)} \) are copies of the same OPA \( \mathcal{A} \), up to the values of their central parameters (such as central charges or levels). In effect, we are viewing the central parameters of \( \mathcal{A} \) as independent indeterminants. We then write

\[ \mathcal{A}_* = \left( \begin{array}{c} A_{(0)} \\ \vdots \\ A_{(N-1)} \end{array} \right), \quad \mathcal{C}_* = \left( \begin{array}{c} c_{(0)} \\ \vdots \\ c_{(N-1)} \end{array} \right), \]

(2.7)

where \( A_{(i)} \) (respectively \( c_{(i)} \)) denotes the field \( A \in \mathcal{A}_{(i)} \) (respectively a central parameter of \( \mathcal{A}_{(i)} \)). For \( \epsilon \in \mathbb{C} \), we also let

\[ U_N(\epsilon, \omega) = D_N(\epsilon)U_N(\omega), \quad D_N(\epsilon) = \text{diag}(\epsilon^0, \epsilon^1, \ldots, \epsilon^{N-1}) \]

(2.8)

and

\[ U_N(\omega) = (\omega^{ij})_{0 \leq i,j \leq N-1} = \begin{pmatrix} \omega^0 & \omega^0 & \cdots & \omega^0 \\ \omega^0 & \omega^1 & \cdots & \omega^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ \omega^0 & \omega^{N-1} & \cdots & \omega^{(N-1)^2} \end{pmatrix}, \]

(2.9)
where $\omega$ is the principal $N$th root of unity,

$$\omega = e^{2\pi i/N}. \quad (2.10)$$

It follows that (for $\epsilon \neq 0$)

$$D_N^{-1}(\epsilon) = D_N(\epsilon^{-1}), \quad U_N^{-1}(\omega) = \frac{1}{N} U_N(\omega^{-1}). \quad (2.11)$$

Thus, with

$$A_\epsilon = \begin{pmatrix} A_{0,\epsilon} \\ \vdots \\ A_{N-1,\epsilon} \end{pmatrix} = U_N(\epsilon, \omega) A_\epsilon, \quad c_\epsilon = \begin{pmatrix} c_{0,\epsilon} \\ \vdots \\ c_{N-1,\epsilon} \end{pmatrix} = U_N(\epsilon, \omega) c_\epsilon, \quad (2.12)$$

the map

$$\mathcal{A}^\otimes N \to \mathcal{A}^\otimes N, \quad A_\epsilon \mapsto A_\epsilon, \quad c_\epsilon \mapsto c_\epsilon, \quad (2.13)$$

is invertible for $\epsilon \neq 0$. For $\epsilon = 0$, on the other hand, the map is singular (unless $N = 1$). If a well-defined OPA arises in the limit $\epsilon \to 0$, where

$$A_\epsilon \to A = \begin{pmatrix} A_0 \\ \vdots \\ A_{N-1} \end{pmatrix}, \quad c_\epsilon \to c = \begin{pmatrix} c_0 \\ \vdots \\ c_{N-1} \end{pmatrix}, \quad (2.14)$$

the ensuing algebra is known \[ \mathcal{A} \] as the $N$th-order Galilean OPA $\mathcal{A}_G^N$. Note that $\mathcal{A}_G^1 \cong \mathcal{A}$. For small $N$, the Galilean Virasoro algebras $\mathcal{W}_G^N$ also appeared in \[ \text{[25].} \]

### 2.3 Generalised higher-order contractions

Fix $\sigma \in \mathbb{N}$. As in Section \[ \text{[11]} \] we denote complex-number sequences of length $\sigma$ by $S = S_1, \ldots, S_\sigma$ etc, with $0 = 0, \ldots, 0$ the zero sequence. Linear combinations are readily formed

$$\alpha i + \beta j = \alpha i_1 + \beta j_1, \ldots, \alpha i_\sigma + \beta j_\sigma, \quad \alpha, \beta \in \mathbb{C}, \quad (2.15)$$

and two sequences can be compared as

$$i \leq j \quad \text{if} \quad i_1 \leq j_1, \ldots, i_\sigma \leq j_\sigma; \quad i < j \quad \text{if} \quad i_1 < j_1, \ldots, i_\sigma < j_\sigma. \quad (2.16)$$

If every element of $S$ is nonzero, we let $S^{-1}$ denote the sequence $S_1^{-1}, \ldots, S_\sigma^{-1}$.

The set

$$I_N = \{ i \in \mathbb{Z}^{\otimes \sigma} \mid 0 \leq i < N \} \quad (2.17)$$

of integer sequences bounded strictly by $N$ admits the canonical order where $i$ appears before $j$ if and only if for each $m$ such that $i_m > j_m$ there exists $\ell < m$ such that $i_\ell < j_\ell$. This corresponds to the usual ordering of basis vectors for the tensor product space $V = V_1 \otimes \ldots \otimes V_\sigma$, where, for each $\ell \in \{1, \ldots, \sigma\}$, $V_\ell$ is an $N_\ell$-dimensional vector space with ordered basis $\{ e^\ell_1, \ldots, e^\ell_{N_\ell} \}$. That is, in the $N$-vector formed by the components $\{ v_i \mid 0 \leq i < N \}$ in the decomposition

$$v = \sum_{0 \leq i < N} v_i e_i \in V, \quad (2.18)$$

the components are ordered according to the canonical ordering of the multi-indices $i$. For example, for $N = 2, 3$, the components are ordered as

$$v_{0,0}, v_{0,1}, v_{0,2}, v_{1,0}, v_{1,1}, v_{1,2}. \quad (2.19)$$
Using the same ordering prescription, we now form the $N$-dimensional vectors

$$A_\epsilon = (A_{i\epsilon})_{0 \leq i < N}, \quad c_\epsilon = (c_{i\epsilon})_{0 \leq i < N},$$

where $A_{i\epsilon}$ (respectively $c_{i\epsilon}$) denotes the field $A \in A_{i\epsilon}$ (respectively a central parameter of $A_{i\epsilon}$). With

$$\omega = \omega_1, \ldots, \omega_\sigma, \quad \omega_\ell = e^{2\pi i/N_\ell}, \quad \ell = 1, \ldots, \sigma,$$

and for

$$\epsilon = \epsilon_1, \ldots, \epsilon_\sigma, \quad \epsilon_\ell \in \mathbb{C}, \quad \ell = 1, \ldots, \sigma,$$

we also introduce

$$U_N(\epsilon, \omega) = U_{N_1}(\epsilon_1, \omega_1) \otimes \cdots \otimes U_{N_\sigma}(\epsilon_\sigma, \omega_\sigma) = D_N(\epsilon)U_N(\omega),$$

where

$$D_N(\epsilon) = D_{N_1}(\epsilon_1) \otimes \cdots \otimes D_{N_\sigma}(\epsilon_\sigma), \quad U_N(\omega) = U_{N_1}(\omega_1) \otimes \cdots \otimes U_{N_\sigma}(\omega_\sigma).$$

The map

$$\mathcal{A}^{\otimes N} \to \mathcal{A}^{\otimes N}, \quad A_i \mapsto A_\epsilon = (A_{i\epsilon})_{0 \leq i < N} = U_N(\epsilon, \omega)A_i, \quad c_i \mapsto c_\epsilon = (c_{i\epsilon})_{0 \leq i < N} = U_N(\epsilon, \omega)c_i,$$

is invertible if and only if $\epsilon_1, \ldots, \epsilon_\sigma \neq 0$, in which case

$$U_N^{-1}(\epsilon, \omega) = \frac{1}{N}U_N(\omega^{-1})D_N(\epsilon^{-1}).$$

If a well-defined ($N$th-order Galilean) OPA arises in the limit $\epsilon \to 0$, where

$$A_\epsilon \to A, \quad c_\epsilon \to c,$$

we denote it by $A_G^N$.

For $U_N(\epsilon, \omega)$ invertible, using

$$\sum_{n=0}^{N_\ell-1} \omega_\ell^n = N_\ell \delta_{k,0 \mod N_\ell}, \quad \ell = 1, \ldots, \sigma,$$

we see that, for $0 \leq i, j, m < N$,

$$\sum_{0 \leq k < N} U_N(\epsilon, \omega)_{ijk} U_N(\epsilon, \omega)_{jk} U_N(\epsilon^{-1}, \omega^{-1})_{km} = N \delta_{m,i+j}.$$

In Section [2.4] we use this result to determine the structure of $A_G^N$ for $A$ of Lie type.

### 2.4 Multi-grading

Still treating central parameters as indeterminants, we assign the following grades to the generators and parameters of the Galilean algebras:

$$\text{gr} : A_G^N \to I_N, \quad A_i \mapsto i, \quad c_i \mapsto i.$$  

The action of gr is then extended linearly and to rational functions of the central parameters and normal-ordered products and derivatives of the fields, with $\text{gr}(\partial) = 0$, so that, for instance,

$$\text{gr} \left( c_{0,0} \partial B_{3,1} - 35 \frac{c_{1,1}c_{2,1} + 4c_{3,2}}{c_{1,4}} (A_{0,1}B_{1,2}) \right) = 3, 1.$$  

5
We say the algebra is *multi-graded* if the grading is compatible with the product structure of operator-product expansions, in the sense that

\[ \text{gr}(A_i \ast B_j) = i + j. \]  

(2.32)

A priori, it is not guaranteed that all terms appearing in the decomposition of \( A_i \ast B_j \) have a well-defined grade, let alone the same grade. However, as we will argue, all Galilean algebras of the type \( \mathcal{A}_G^N \) are, in fact, multi-graded. Moreover, the grading is finitely truncated by \( N \) in the sense that \( A_i \ast B_j = 0 \) unless \( i + j < N \).

Let \( A, B \in B_A \) and consider the star relation (2.3). If \( \mathcal{A} \) is of Lie type, then the only structure constants \( C^Q_{AB} \) that can depend on central parameters have \( Q = I \), as in (2.4) and (2.5). To indicate this, we write

\[ A \ast B \simeq C^I_{A,B}(c)\{I\} + \sum_{Q \in B_A \setminus \{I\}} C^Q_{A,B}\{Q\}, \]  

(2.33)

where \( C^I_{A,B}(c) \) is linear in \( c \),

\[ C^I_{A,B}(c) = f_{A,B} c, \quad f_{A,B} \in \mathbb{C}, \]  

(2.34)

while \( C^Q_{A,B} \) is independent of \( c \) for all \( Q \in B_A \setminus \{I\} \). From (2.29), it follows that

\[ A_{i,\epsilon} \ast B_{j,\epsilon} = f_{A,B} c_{i+j,\epsilon} \{I\} + \sum_{Q \in B_A \setminus \{I\}} C^Q_{AB}\{Q_{i+j,\epsilon}\}, \]  

(2.35)

so, in the Galilean algebra \( \mathcal{A}_G^N \),

\[ A_i \ast B_j = f_{A,B} c_{i+j} \{I\} + \sum_{Q \in B_A \setminus \{I\}} C^Q_{AB}\{Q_{i+j}\} = \sum_{Q \in B_A} C^Q_{AB}\{Q_{i+j}\}. \]  

(2.36)

Thus, \( \mathcal{A}_G^N \) is multi-graded if \( \mathcal{A} \) is of Lie type.

The nonlinearity of an OPA that is not of Lie type obscures the question of its grading structure, as witnessed in sections 3 and 4. However, as already indicated, all the Galilean algebras we have analysed are nevertheless multi-graded in the sense outlined above.

**Virasoro algebras:** The multi-graded Galilean Virasoro algebra \( \mathfrak{Vir}_G^N \) is generated by the fields \( \{T_i \mid 0 \leq i < N\} \) and has central parameters \( \{c_i \mid 0 \leq i < N\} \), with star relations given by

\[ T_i \ast T_j \simeq \left( \frac{\epsilon i j}{2} \{I\} + 2 \{T_{i+j}\}, \quad i + j < N, \right. \]  

\[ 0, \quad \text{otherwise}. \]  

(2.37)

Note that \( T_0 \) generates a subalgebra isomorphic to \( \mathfrak{vir} \) with central charge \( c_0 \), and that, for every \( i, T_i \) is quasi-primary with respect to \( T_0 \).

**Affine Kac-Moody algebras:** The multi-graded Galilean Kac-Moody algebra \( \hat{\mathfrak{g}}_G^N \) is generated by \( \{J^a_i \mid a = 1, \ldots, \dim \mathfrak{g}; 0 \leq i < N\} \), with nontrivial star relations

\[ J^a_i \ast J^b_j \simeq \kappa^{ab} k_{i+j} \{I\} + f^{ab}_{\epsilon} c^{J^c}_{i+j} \{J^c_{i+j}\}, \quad i + j < N. \]  

(2.38)

Note that \( \{J^a_0 \mid a = 1, \ldots, \dim \mathfrak{g}\} \) generates a subalgebra isomorphic to \( \hat{\mathfrak{g}} \) at level \( k_0 \).
2.5 Permutation invariance

In all the Galilean algebras we have analysed, we observe that

\[ A_i \ast B_j \simeq A_{i'} \ast B_{j'} \quad \text{if} \quad i + j = i' + j'. \]  

(2.39)

Together with the grading property, this implies that all inequivalent decompositions of star relations arise as \( A_0 \ast B_j \) for some \( A, B \in \mathcal{A} \) and \( 0 \leq j < N \). It also implies that the multi-graded contraction procedure is independent of the \textit{ordering} of the elements in the contraction sequence. That is,

\[ \mathcal{A}_G^N \cong \mathcal{A}_G^{\pi(N)}, \quad \pi(N) = N_{\pi_1}, \ldots, N_{\pi_\sigma}, \]  

(2.40)

where \( \pi = \pi_1, \ldots, \pi_\sigma \) is a permutation of the integers \( 1, \ldots, \sigma \). Moreover, as the tensorial structure of the contraction process ensures that

\[ \mathcal{A}_G^{N_1, N_2} \cong (\mathcal{A}_G^{N_1})_{N_2}^G, \]  

(2.41)

we see that

\[ (\ldots ((\mathcal{A}_G^{N_1})_{N_2}^G) \ldots )_{N_\sigma}^G \cong \mathcal{A}_G^{N_{\pi_1}, \ldots, N_{\pi_\sigma}} \cong (\ldots ((\mathcal{A}_G^{N_1})_{N_2}^G) \ldots )_{N_\sigma}^G. \]  

(2.42)

2.6 Multivariable Takiff algebras

For some \( R_{\ell}^l \in \mathbb{R} \), \( \ell, \ell' = 1, \ldots, \sigma \), we let \( N \to \infty \) denote the limit where

\[ N_\ell \to \infty, \quad \frac{N_\ell}{N_{\ell'}} \to R_{\ell'}, \quad \ell, \ell' = 1, \ldots, \sigma. \]  

(2.43)

In this limit, the algebra \( \mathcal{g}_G^N \) becomes \( \mathcal{g}_G^\infty \) generated by \( \{ J_1^a | a = 1, \ldots, \dim \mathfrak{g}; i \geq 0 \} \), with nontrivial star relations

\[ J_1^a \ast J_j^b \simeq \kappa^{ab} k_{i+1} \{1\} + f^{abc}_i J_j^c + \kappa^{ab} k_{i+1} \{1\} \]  

(2.44)

This \( \mathbb{Z}^{\infty} \)-graded algebra is seen to be isomorphic to a multivariable polynomial ring,

\[ \mathcal{g}_G^\infty \cong \mathcal{g} \otimes \mathbb{C}[t_1, \ldots, t_\sigma], \]  

(2.45)

and we likewise recognise the isomorphism

\[ \mathcal{g}_G^N \cong \mathcal{g} \otimes \mathbb{C}[t_1, \ldots, t_\sigma]/(t_1^{N_1}, \ldots, t_\sigma^{N_\sigma}). \]  

(2.46)

This extends to multiple variables the Takiff algebras considered in [20], themselves extensions to general order \( N \) of the second-order (one-variable) Takiff algebras considered in [22, 23]. We similarly have

\[ \text{Vir}_G^\infty \cong \text{Vir} \otimes \mathbb{C}[t_1, \ldots, t_\sigma], \quad \text{Vir}_G^N \cong \text{Vir} \otimes \mathbb{C}[t_1, \ldots, t_\sigma]/(t_1^{N_1}, \ldots, t_\sigma^{N_\sigma}). \]  

(2.47)

Further generalisations of the Galilean and Takiff algebras are obtained as follows. Let \( s = \{s_1, \ldots, s_\rho\} \) denote a subset of \( \{1, \ldots, \sigma\} \) and \( N \to \infty \) the limit where

\[ N_{s_1}, \ldots, N_{s_\rho} \to \infty, \quad \frac{N_{s_i}}{N_{s_j}} \to R_{s_j}^{s_i}, \quad s_i, s_j \in s. \]  

(2.48)

In this limit, the Galilean algebra \( \mathcal{A}_G^N \) becomes \( \mathcal{A}_G^{\infty \circ} \), where, for example,

\[ \mathcal{g}_G^{\infty \circ} \cong \mathcal{g} \otimes \mathbb{C}[t_1, \ldots, t_\sigma]/(t_1^{N_{s_1}}, \ldots, t_\sigma^{N_{s_\rho}}), \]  

(2.49)

\[ \text{Vir}_G^{\infty \circ} \cong \text{Vir} \otimes \mathbb{C}[t_1, \ldots, t_\sigma]/(t_1^{N_{s_1}}, \ldots, t_\sigma^{N_{s_\rho}}). \]  

(2.50)
3 Generalised Sugawara construction

The objective here is to construct a Sugawara operator for each Galilean affine Kac-Moody algebra \( \hat{g}^N_G \) and to show that this process commutes with the Galilean contraction procedure, thereby establishing the commutativity of the diagram

\[
\begin{array}{c}
\hat{g}^N_G \xrightarrow{\text{Gal Sug}} \mathcal{Q}_{\text{Vir}}^N_G \\
\downarrow \text{Gal} \downarrow \text{Gal} \\
\hat{g}^N_G \xrightarrow{\text{Sugawara}} \mathcal{Q}_{\text{Vir}}^N_G
\end{array}
\]

The lower branch is analysed in Section 3.1, the upper one in Section 3.2.

3.1 Galilean Sugawara construction

For the Sugawara generators of \( \mathcal{Q}_{\text{Vir}}^N_G \), we make the ansatz

\[
T_i = \sum_{0 \leq r, s < N} \lambda^{rs}_i \kappa_{ab}(J^a_r J^b_s), \quad 0 \leq i < N, \tag{3.1}
\]

where \( \kappa_{ab} \) are elements of the inverse Killing form on \( g \), and our goal is to determine the coefficients \( \lambda^{rs}_i \) such that

\[
T_i \star J^a_j \simeq \begin{cases} 
\{J^a_{i+j}\}, & 0 \leq i + j < N, \\
0, & \text{otherwise}. 
\end{cases} \tag{3.2}
\]

To this end, we compute the operator-product expansion

\[
J^a_j(z)T_i(w) \sim \frac{1}{(z - w)^2} \sum_{0 \leq r, s < N} \lambda^{rs}_i \kappa_{bc}[k_{j+r}a_j^a(w) + k_{j+s}b_j^a(w) + 2h^\vee J^a_{j+r+s}(w)]
+ \frac{1}{z - w} \sum_{0 \leq r, s < N} \lambda^{rs}_i \kappa_{bc}[f^{ab}_d(J^d_j c_j^c(w) + f^{ac}_d(J^b_j J^d_j c_j^c)(w)), \tag{3.3}
\]

where the dual Coxeter number \( h^\vee \) of \( g \) has arisen through

\[
\kappa_{bc}f^{ab}_d f^{dc}_e = 2h^\vee \delta^a_e. \tag{3.4}
\]

To satisfy (3.2), the sum multiplying the single pole in (3.3) must be zero while the sum multiplying the double pole must equal \( J^a_{i+j}(w) \). The single-pole constraint implies that

\[
\lambda^{rs}_i = \begin{cases} 
\lambda^{n N-1}_i, & r + s = N - 1 + n \quad (0 \leq n < N), \\
0, & \text{otherwise}, 
\end{cases} \tag{3.5}
\]

where \( 1 = 1, \ldots, 1 \). For each \( 0 \leq i < N \), this fixes all but the \( N \) coefficients \( \lambda^{n N-1}_i \) labelled by \( 0 \leq n < N \). The double-pole constraint then requires that

\[
2 \sum_{j \leq m < N} \sum_{0 \leq n \leq m-j} \lambda^{n N-1}_i k_{N-1-m+j+n}^a m + 2N h^\vee \lambda^{0 N-1}_i \delta_{i,0} J^a_{N-1} = \begin{cases} 
J^a_{i+j}, & i + j < N, \\
0, & \text{otherwise}. 
\end{cases} \tag{3.6}
\]

For each \( i \), the conditions (3.6) for \( j \neq 0 \) are all repetitions of conditions appearing for \( j = 0 \), so it suffices to consider (3.6) for \( j = 0 \):

\[
2 \sum_{0 \leq m < N} \sum_{0 \leq n \leq m} \lambda^{n N-1}_i (k_{N-1-m+n} + Nh^\vee \delta_{n,0} \delta_{m,N-1}) J^a_m = J^a_i, \tag{3.7}
\]
As the generators \( \{ J_\mathbf{m}^a \mid 0 \leq \mathbf{m} < \mathbf{N} \} \) are linearly independent, the constraint (3.7) translates into a lower-triangular system of linear equations in the variables \( \{ \lambda_\mathbf{n}^{\mathbf{m}+1} \mid 0 \leq \mathbf{n} < \mathbf{N} \} \). Indeed, considering \( \lambda_\mathbf{n}^{\mathbf{m}+1} \) as the \( \mathbf{N} \)-vector with components \( \lambda_\mathbf{n}^{\mathbf{m}+1} \) ordered canonically according to \( \mathbf{n} \in \mathcal{I}_\mathbf{N} \), such that

\[
M \lambda_\mathbf{n}^{\mathbf{m}+1} = (\delta_{\mathbf{j},\mathbf{i}})_{0 \leq \mathbf{j} < \mathbf{N}}, \quad 0 \leq \mathbf{i} < \mathbf{N},
\]

the coefficient matrix \( M \) is given by

\[
M_{\mathbf{m},\mathbf{n}} = \begin{cases} 2k_{\mathbf{N}-1-m+n}, & 0 \leq \mathbf{m} - \mathbf{n} < \mathbf{N}, \\ 0, & \text{otherwise}, \end{cases}
\]

for all \( \mathbf{i} \), where

\[
k_{\mathbf{m}}' = k_{\mathbf{m}} + N \mathbf{h}^{\mathbf{u}} \delta_{\mathbf{m},0}, \quad 0 \leq \mathbf{m} < \mathbf{N}.
\]

All the diagonal entries are thus given by \( 2k_{\mathbf{N}-1} \). The only nonzero component on the righthand side of (3.8) is a 1 in the position corresponding to \( \mathbf{i} \in \mathcal{I}_\mathbf{N} \).

The structure of \( M \) resembles a lower-triangular Toeplitz matrix, but with some entries set to 0. Indeed,

\[
M = \begin{pmatrix} M_1 \\ \vdots \\ M_{i_1} \\ \vdots \\ M_{N_1} \end{pmatrix} = \begin{pmatrix} M_{i_1,1} \\ \vdots \\ M_{i_1,i_2} \\ \vdots \\ M_{i_1,N_2} \end{pmatrix},
\]

where each \( M_{i_1} \in \{ M_1, \ldots, M_{N_1} \} \) is an \( \frac{\mathbf{N}}{N_1} \times \frac{\mathbf{N}}{N_1} \) lower-triangular matrix (recall that \( \mathbf{N} = N_1 \ldots N_\sigma \)) of the form

\[
M_{i_1} = \begin{pmatrix} M_{i_1,1} \\ \vdots \\ M_{i_1,i_2} \\ \vdots \\ M_{i_1,N_2} \end{pmatrix},
\]

where each \( M_{i_1,i_2} \in \{ M_{i_1,1}, \ldots, M_{i_1,N_2} \} \) is an \( \frac{\mathbf{N}}{N_1 N_2} \times \frac{\mathbf{N}}{N_1 N_2} \) lower-triangular matrix of similar form, and so on. The innermost lower-triangular matrices appearing in this nested description of \( M \) are \( N_\sigma \times N_\sigma \) Toeplitz matrices of the form

\[
M_{i_1,\ldots,i_\sigma-1} = \begin{pmatrix} 2k_{i_1,\ldots,i_\sigma-1,N_\sigma-1} \\ \vdots \\ 2k_{i_1,\ldots,i_\sigma-1,N_\sigma-\delta} \\ \vdots \\ 2k_{i_1,\ldots,i_\sigma-1,0} \end{pmatrix},
\]
The Galilean Sugawara construction (3.1) is thus given by
\[ M = 2 \]
written using the simplified notation \( k_{i_1,i_2,i_3} = k_{i_1,i_2,i_3} \).

The inverse of \( M \),
\[ M^{-1} = (b_{m,n})_{0 \leq m,n < N}, \]
has the same nested Toeplitz-like structure, with all diagonal entries given by \( 1/(2k_{N-1}) \). With this, we solve (3.8) and find
\[ \lambda_i^{n,N-1} = b_{n,i}. \]

The Galilean Sugawara construction (3.1) is thus given by
\[ T_i = \sum_{i \leq n < N} b_{n,i} \sum_{0 \leq t < N-n} \kappa_{ab}(J_{n+t}^a J_{N-1-t}^b). \]

For each \( i \), the value of the central parameter \( c_i \) follows from the leading pole in the OPE
\[ T_0(z)T_i(w) \sim \sum_{i \leq n < N} b_{n,i} \sum_{0 \leq t < N-n} \kappa_{ab} \kappa_n k_{N-1+n}^a \kappa_n k_{N-1}^b \frac{2T_i(w)}{(z-w)^4} + \frac{2T_i(w)}{(z-w)^2} + \frac{\partial T_i(w)}{z-w}. \]

Since \( k_h = 0 \) unless \( h < N \), the only contribution to the leading-pole term appears for \( n = 0 \), hence for \( i = 0 \). The term thus reduces to
\[ \kappa_{0,h} \kappa_{N-1} \frac{2N \dim g \delta_{1,0}}{(z-w)^4}, \]
from which it follows that
\[ c_0 = N \dim g, \quad c_i = 0, \quad i \neq 0. \]

This result for the central charge \( c_0 \) resembles similar results for Sugawara constructions associated with so-called double extensions.

\( \mathfrak{g}_G^{2,3} \) algebra: To illustrate, we consider the contraction sequence \( N = 2, 3 \). In this case, \( \sigma = 2 \) and \( N = 6 \), while the canonical ordering is
\[ I_{2,3} : 0, 0; 0, 1; 0, 2; 1, 0; 1, 1; 1, 2. \]
The corresponding Galilean Sugawara construction is given by
\[
T_{0,0} = \frac{\kappa_{ab}}{2k_{1,2}} [(J^0_{0,0}J^b_{1,2}) + (J^0_{0,1}J^b_{1,1}) + (J^0_{0,2}J^b_{1,0}) + (J^0_{1,0}J^b_{0,2}) + (J^0_{1,1}J^b_{0,1}) + (J^0_{1,2}J^b_{0,0})]
\]
\[
- \frac{k_{1,2}}{k_{1,2}} [(J^0_{0,1}J^b_{1,2}) + (J^0_{0,2}J^b_{1,1}) + (J^0_{0,1}J^b_{0,2}) + (J^0_{1,2}J^b_{0,1}) + (J^0_{1,2}J^b_{1,0})]
\]
\[
- \frac{k_{0,2}}{k_{1,2}} [(J^0_{1,0}J^b_{1,2}) + (J^0_{1,1}J^b_{1,1}) + (J^0_{1,2}J^b_{1,0}) + 2k_{0,2k_{1,1}k_{1,2}} \frac{k_{0,1}k_{1,2}}{(k_{1,2})^2} (J^0_{0,2}J^b_{1,2})]
\]
\[
- \frac{3k_{0,2}(k_{1,2})^2 - 2k_{0,2k_{1,1}k_{1,2}}k_{1,2}^2k_{0,1}k_{1,2}^2}{(k_{1,2})^2} (J^0_{1,2}J^b_{1,2})]
\]
\[
T_{0,1} = \frac{\kappa_{ab}}{2k_{1,2}} [(J^0_{0,1}J^b_{1,2}) + (J^0_{0,2}J^b_{1,1}) + (J^0_{1,0}J^b_{1,2}) + (J^0_{0,0}J^b_{0,2}) - \frac{k_{1,2}}{k_{1,2}} (J^0_{0,2}J^b_{1,2}) + (J^0_{1,2}J^b_{0,2})]
\]
\[
T_{0,2} = \frac{\kappa_{ab}}{2k_{1,2}} [(J^0_{0,2}J^b_{1,2}) + (J^0_{1,2}J^b_{0,2}) - \frac{k_{0,2}}{k_{1,2}} (J^0_{1,2}J^b_{1,2})]
\]
\[
T_{1,0} = \frac{\kappa_{ab}}{2k_{1,2}} [(J^0_{1,0}J^b_{1,2}) + (J^0_{1,1}J^b_{1,1}) + (J^0_{1,2}J^b_{1,0}) - \frac{k_{1,2}}{k_{1,2}} (J^0_{1,1}J^b_{1,2}) + (J^0_{1,2}J^b_{1,1})] + \frac{k_{1,2}}{k_{1,2}} (J^0_{1,2}J^b_{1,2})]
\]
\[
T_{1,1} = \frac{\kappa_{ab}}{2k_{1,2}} [(J^0_{1,1}J^b_{1,2}) + (J^0_{1,2}J^b_{1,1}) - \frac{k_{1,2}}{k_{1,2}} (J^0_{1,1}J^b_{1,2})]
\]
\[
T_{1,2} = \frac{\kappa_{ab}}{2k_{1,2}} (J^0_{1,2}J^b_{1,2})
\]
and has central parameters
\[
c_{0,0} = 6 \dim \mathfrak{g}, \quad c_{0,1} = c_{0,2} = c_{1,0} = c_{1,1} = c_{1,2} = 0.
\]

### 3.2 Sugawara before Galilean contraction

As above, let \( N = N_1, \ldots, N_\sigma \) and \( N = N_1 \cdots N_\sigma \). Accordingly, on the individual factors of \( \mathfrak{g} \otimes N \), we denote the Sugawara construction by
\[
T^{(i)} = \frac{\kappa_{ab}}{2(k^{(i)} + h^{\nu})} (J_{ab}^{(i)})^b (J_{ab}^{(i)})^b, \quad c^{(i)} = \frac{k^{(i)} \dim \mathfrak{g}}{k^{(i)} + h^{\nu}}, \quad 0 \leq i < N,
\]
let \( T_\epsilon \) and \( c_\epsilon \) denote the corresponding \( N \)-vectors formed as in (2.20), and change basis as in (2.25):
\[
T_\epsilon = (T_{i,\epsilon})_{0 \leq i < N} = U_N(\epsilon, \omega)T_\epsilon, \quad c_\epsilon = (c_{i,\epsilon})_{0 \leq i < N} = U_N(\epsilon, \omega)c_\epsilon.
\]
It follows that
\[
T_{i,\epsilon} = \frac{\sum_{0 \leq m, n' < N} (\prod_{\ell=1}^{\sigma}(\epsilon_{\ell} \omega_{\ell}^{j_{\ell}})^N \cdot \eta_{N-1-\eta_{\ell}-n_{\ell} - n_{\ell}'}) \kappa_{ab}(J_{ab}^{(i)})^b (J_{ab}^{(i)})^b)}{2Nk_{N-1,\epsilon} \sum_{0 \leq m < N} a_m \prod_{\ell=1}^{\sigma} (\epsilon_{\ell} \omega_{\ell}^{j_{\ell}})^m},
\]
where
\[
a_m = \frac{k_{N-1-m, m + Nh^{\nu}}}{k_{N-1,\epsilon}}, \quad 0 \leq m < N.
\]
Since \( a_0 = 1 \), there exist \( \delta_m, 0 \leq m < N \), where \( \delta_0 = 1 \), such that
\[
\left( \sum_{0 \leq m < N} a_m \prod_{\ell=1}^{\sigma} (\epsilon_{\ell} \omega_{\ell}^{j_{\ell}})^m \right)^{-1} = \sum_{0 \leq m < N} \delta_m \prod_{\ell=1}^{\sigma} (\epsilon_{\ell} \omega_{\ell}^{j_{\ell}})^m + O(\epsilon_{N_1}^{N_1}, \ldots, \epsilon_{N_\sigma}^{N_\sigma}).
\]
We can thus write
\[ T_{1,\epsilon} = \frac{1}{2NkN^{-1,\epsilon}} \sum_{0 \leq j, n, n', m < N} \hat{a}_m \left( \prod_{\ell=1}^{\sigma} (\epsilon_{\ell} \omega_{\ell}^{j_{\ell}})^{N_{\ell}-1+i_{\ell}-n_{\ell}-n'_{\ell}+m_{\ell}} \right) \kappa_{ab}(J_{n,\epsilon}^{a} J_{n'}^{b}) + \mathcal{O}(\epsilon_{1}^{N_1}, \ldots, \epsilon_{\sigma}^{N_{\sigma}}), \tag{3.34} \]

For each \( \ell \in \{1, \ldots, \sigma\} \), the summation over \( j_{\ell} \) yields a factor of the form
\[ \sum_{j_{\ell}=0}^{N_{\ell}-1} \omega_{\ell}^{j_{\ell}(N_{\ell}-1+i_{\ell}-n_{\ell}-n'_{\ell}+m_{\ell})} = \begin{cases} N_{\ell}, & N_{\ell} - 1 + i_{\ell} - n_{\ell} - n'_{\ell} + m_{\ell} \equiv 0 \pmod{N_{\ell}}, \\ 0, & N_{\ell} - 1 + i_{\ell} - n_{\ell} - n'_{\ell} + m_{\ell} \not\equiv 0 \pmod{N_{\ell}}, \end{cases} \tag{3.35} \]
so
\[ \sum_{0 \leq j < N_{\ell}} \prod_{\ell=1}^{\sigma} \omega_{\ell}^{j(N_{\ell}-1+i_{\ell}-n_{\ell}-n'_{\ell}+m_{\ell})} = \begin{cases} N_{\ell}, & N - 1 + i - n - n' + m \equiv 0 \pmod{N}, \\ 0, & N - 1 + i - n - n' + m \not\equiv 0 \pmod{N}. \end{cases} \tag{3.36} \]

Since \( N - 1 + i - n - n' + m > -N \), it follows that the \( T_{1,\epsilon} \)-coefficients to negative powers of any of the \( \epsilon_i \)'s are all zero. The limit \( \epsilon \to 0 \) is therefore well-defined, and we find
\[ T_{1,\epsilon} \to T_1 = \frac{1}{2NkN^{-1}} \sum_{0 \leq n, n', m < N} \hat{a}_m \kappa_{ab}(J_{n}^{a} J_{n'}^{b}) \delta_{N-1+i-n-n'+m,0} = \sum_{i \leq n < N} \frac{\hat{a}_{n-i}}{2NkN^{-1}} \sum_{0 \leq t < N-n} \kappa_{ab}(J_{n+t}^{a} J_{N-1-t}^{b}). \tag{3.37} \]

This is seen to agree with (3.17) if
\[ b_{n,i} = \frac{\hat{a}_{n-i}}{2NkN^{-1}} \tag{3.38} \]
for all \( i \leq n < N \), that is, if
\[ \lambda_i^{n,N^{-1}} = \frac{\hat{a}_{n-i}}{2NkN^{-1}}. \tag{3.39} \]

In the affirmative, the relations (3.39) follow from the similarity in structures of \( M_{m,n} \) in (3.8) and \( a_m \) in (3.32). We have thus established the commutativity of Galilean contractions and Sugawara constructions, without explicit knowledge of the coefficients \( b_{n,i} \) and \( a_n \) appearing in the inversion of the coefficient matrix \( M \) and the series expansion of the denominator of \( T_{1,\epsilon} \), respectively. The coefficients are readily obtained case by case, but cumbersome to express for general parameters. For \( \sigma = 1 \), the coefficients are given in [20].

For the central parameters, we have
\[ c_{1,\epsilon} = \sum_{0 \leq j, n < N} \left( \prod_{\ell=1}^{\sigma} (\epsilon_{\ell} \omega_{\ell}^{j_{\ell}})^{N_{\ell}-1+i_{\ell}-n_{\ell}} \right) k_{n,\epsilon} \dim g \]
\[ = \frac{\dim g}{kN^{-1,\epsilon}} \sum_{0 \leq m < N} a_m \prod_{\ell=1}^{\sigma} (\epsilon_{\ell} \omega_{\ell}^{j_{\ell}})^{m_{\ell}} \]
\[ = \frac{\dim g}{kN^{-1,\epsilon}} \sum_{0 \leq n, m < N} \hat{a}_m k_n \prod_{\ell=1}^{\sigma} (\epsilon_{\ell} \omega_{\ell}^{j_{\ell}})^{N_{\ell}-1+i_{\ell}-n_{\ell}+m_{\ell}} + \mathcal{O}(\epsilon_{1}^{N_1}, \ldots, \epsilon_{\sigma}^{N_{\sigma}}), \tag{3.40} \]
so in the limit \( \epsilon \to 0 \),
\[ c_{1,\epsilon} \to c_1 = \frac{N \dim g}{kN^{-1}} \sum_{0 \leq n, m < N} \hat{a}_m k_n \delta_{N-1+i-n+m,0} = N \dim g \delta_{1,0}, \tag{3.41} \]
in accordance with (3.20).
4 Galilean $W_3$ algebras

Our generalised Galilean contractions can also be applied to W-algebras. Below, we present the results for the factorisation sequence $N = 2, 3$ applied to the $W_3$ algebra, giving rise to the sixth-order Galilean algebra $(W_3)_G^3$. In preparation, we first recall the structure of the $W_3$ algebra and its second- and third-order Galilean counterparts.

4.1 $W_3$ algebra

The $W_3$ algebra [28] of central charge $c$ is generated by a Virasoro field $T$ and a primary field $W$ of conformal weight 3, with star relations

$$T \ast T \simeq \frac{4}{7} \{\|\} + 2\{T\}, \quad T \ast W \simeq 3\{W\}, \quad W \ast W \simeq \frac{2}{3} \{\|\} + 2\{T\} + \frac{32}{22 + 5c} \{\Lambda\}, \quad (4.1)$$

where

$$\Lambda = (TT) - \frac{3}{10} \partial^2 T \quad (4.2)$$

is quasi-primary.

4.2 Galilean $W_3$ algebras of type $(W_3)_G^N$

Following [20] and Section 2.2 the $N$th-order Galilean algebra $(W_3)_G^N$ is generated by the fields $\{T_i, W_i \mid i = 0, \ldots, N - 1\}$ and has central parameters $\{c_i \mid i = 0, \ldots, N - 1\}$. The star relations involving the $T_i$ fields are

$$T_i \ast T_j \simeq \frac{c_{i+j}}{2} \{\|\} + 2\{T_{i+j}\}, \quad T_i \ast W_j \simeq 3\{W_{i+j}\}, \quad i + j \in \{0, \ldots, N - 1\}, \quad (4.3)$$

and

$$T_i \ast T_j \simeq T_i \ast W_j \simeq 0, \quad i + j \geq N. \quad (4.4)$$

As will become clear below, it is convenient to introduce

$$c_0' = c_0 + \frac{22N}{5}. \quad (4.5)$$

$(W_3)_G^2$ algebra: For $N = 2$, the star relations between the $W$-fields are given by

$$W_0 \ast W_0 \simeq \frac{c_0}{3} \{\|\} + 2\{T_0\} + \frac{64}{5c_1} \{\Lambda_{0,1}\} - \frac{32c_0'}{5c_1} \{\Lambda_{1,1}\}, \quad (4.6)$$

$$W_0 \ast W_1 \simeq \frac{c_1}{3} \{\|\} + 2\{T_1\} + \frac{32}{5c_1} \{\Lambda_{1,1}\}, \quad (4.7)$$

$$W_1 \ast W_1 \simeq 0, \quad (4.8)$$

where

$$\Lambda_{0,1} = (T_0 T_1) - \frac{3}{10} \partial^2 T_1, \quad \Lambda_{1,1} = (T_1 T_1) \quad (4.9)$$

are quasi-primary with respect to $T_0$.

$(W_3)_G^3$ algebra: For $N = 3$, the star relations between the $W$-fields are given by

$$W_0 \ast W_0 \simeq \frac{c_0}{3} \{\|\} + 2\{T_0\} + \frac{64}{5c_2} \{\Lambda_{0,2}\} + \frac{32}{5c_2} \{\Lambda_{1,1}\} - \frac{64c_1}{5(c_2)^2} \{\Lambda_{1,2}\} - \frac{32(c_1)^2 - c_0'c_2}{5(c_2)^3} \{\Lambda_{2,2}\}, \quad (4.10)$$

$$W_0 \ast W_1 \simeq \frac{c_1}{3} \{\|\} + 2\{T_1\} + \frac{64}{5c_2} \{\Lambda_{1,2}\} - \frac{32c_1}{5(c_2)^2} \{\Lambda_{2,2}\}, \quad (4.11)$$

$$W_0 \ast W_2 \simeq \frac{c_2}{3} \{\|\} + 2\{T_2\} + \frac{32}{5c_2} \{\Lambda_{2,2}\}, \quad (4.12)$$
and

\[ W_1 \ast W_1 \simeq W_0 \ast W_2, \quad W_1 \ast W_2 \simeq W_2 \ast W_2 \simeq 0, \quad (4.13) \]

where

\[ \Lambda_{0,2} = (T_0 T_2) - \frac{3}{10} \partial^2 T_2, \quad \Lambda_{1,1} = (T_1 T_1) - \frac{3}{10} \partial^2 T_2, \quad \Lambda_{1,2} = (T_1 T_2), \quad \Lambda_{2,2} = (T_2 T_2) \quad (4.14) \]

are quasi-primary with respect to \( T_0 \). Both \( (W_3)_G^3 \) and \( (W_3)_G^2 \) are readily seen to be (multi-)graded, truncated according to their order (2 and 3, respectively).

### 4.3 Galilean algebra \( (W_3)_G^{2,3} \)

The \( N \)th-order Galilean algebra \( (W_3)_G^N \) is generated by the fields \( \{T_i, W_i \mid 0 \leq i < N\} \) and has central parameters \( \{c_i \mid 0 \leq i < N\} \), with the star relations involving the \( T \) fields given by (4.3) and

\[ T_i \ast W_j \simeq \begin{cases} 3\{W_{i+j}\}, & i+j < N, \\ 0, & \text{otherwise}. \end{cases} \quad (4.15) \]

Extending (4.3), for all \( N \), it is convenient to introduce

\[ c'_0 = c_0 + \frac{22N}{5}. \quad (4.16) \]

Comparing this with (3.10), one may view \( h^\vee = \frac{22}{5} \) as the associated “dual Coxeter number”.

\( (W_3)_G^{2,3} \) algebra: For the contraction sequence \( N = 2, 3 \), up to the equivalences (2.39), the inequivalent nontrivial star relations involving the \( W \) fields are given by

\[
\begin{align*}
W_{0,0} \ast W_{0,0} & \simeq \frac{c_0}{3} \{I\} + 2\{T_{0,0}\} + \frac{64}{5c_{1,2}} \{\Lambda_{0,0;1,2} + \Lambda_{0,1;1,1} + \Lambda_{0,2;1,0}\} - \frac{64c_{1,1}}{5(c_{1,2})^2} \{\Lambda_{0,1;1,2} + \Lambda_{0,2;1,1}\} \\
& \quad + \frac{32(c_{1,1}^2 - c_{1,0} c_{1,2})}{5(c_{1,2})^3} \{\Lambda_{1,0;2,1}\} - \frac{32c_{0,2}}{5(c_{1,2})^3} \{2\Lambda_{1,0;2,1} + \Lambda_{1,1;1,1}\} + \frac{64[2c_{0,2} c_{1,1} - c_{0,1} c_{1,2}]}{5(c_{1,2})^3} \{\Lambda_{1,1;1,2}\} \\
& \quad - \frac{32[2c_{1,2} - 2c_{1,1}] c_{1,2} c_{0,1}}{5(c_{1,2})^3} \{\Lambda_{1,2;1,2}\}, \quad (4.17) \\
W_{0,0} \ast W_{0,1} & \simeq \frac{c_0}{3} \{I\} + 2\{T_{0,1}\} + \frac{64}{5c_{1,2}} \{\Lambda_{0,1;1,2} + \Lambda_{0,2;2,1}\} - \frac{64c_{1,1}}{5(c_{1,2})^2} \{\Lambda_{0,2;1,2}\} - \frac{64c_{0,2}}{5(c_{1,2})^3} \{\Lambda_{1,1;1,2}\} \\
& \quad + \frac{32[2c_{0,2} c_{1,1} - c_{0,1} c_{1,2}]}{5(c_{1,2})^3} \{\Lambda_{1,2;1,2}\}, \quad (4.18) \\
W_{0,0} \ast W_{0,2} & \simeq \frac{c_0}{3} \{I\} + 2\{T_{0,2}\} + \frac{64}{5c_{1,2}} \{\Lambda_{0,2;1,2}\} - \frac{32c_{0,2}}{5(c_{1,2})^3} \{\Lambda_{1,2;1,2}\}, \quad (4.19) \\
W_{0,0} \ast W_{1,0} & \simeq \frac{c_0}{3} \{I\} + 2\{T_{1,0}\} + \frac{32}{5c_{1,2}} \{2\Lambda_{1,0;2,1} + \Lambda_{1,1;1,1}\} - \frac{64c_{1,1}}{5(c_{1,2})^2} \{\Lambda_{1,1;1,2}\} \\
& \quad + \frac{32[(c_{1,1}^2 - c_{1,0} c_{1,2})]}{5(c_{1,2})^3} \{\Lambda_{1,2;1,2}\}, \quad (4.20) \\
W_{0,0} \ast W_{1,1} & \simeq \frac{c_0}{3} \{I\} + 2\{T_{1,1}\} + \frac{64}{5c_{1,2}} \{\Lambda_{1,1;1,2}\} - \frac{32c_{1,1}}{5(c_{1,2})^3} \{\Lambda_{1,2;1,2}\}, \quad (4.21) \\
W_{0,0} \ast W_{1,2} & \simeq \frac{c_0}{3} \{I\} + 2\{T_{1,2}\} + \frac{32}{5c_{1,2}} \{\Lambda_{1,2;1,2}\}, \quad (4.22) \\
\end{align*}
\]

where, for \( i+j \geq N - 1 \),

\[ \Lambda_{i,j} = (T_i \ast T_j) - \frac{3}{10} \partial^2 T_{N-1} \delta_{i+j,N-1} \quad (4.23) \]

is quasi-primary with respect to the Virasoro generator \( T_0 \). It follows that the sixth-order Galilean algebra \( (W_3)_G^{2,3} \) is multi-graded, with truncation dictated by the sequence 2, 3.
5 Discussion

In our continued exploration [9, 11, 20] of Galilean contractions, we have presented a generalisation of the contraction procedure to multi-graded Galilean algebras. Our construction uses factorisations of the order parameter $N$, and has resulted in whole new families of higher-order Galilean conformal algebras, including Virasoro, affine Kac-Moody and $W_3$ algebras. We have also discussed how some of these algebras are related to a multivariable extension of Takiff algebras.

$W$-algebras related to Takiff algebras were introduced in [29, 30] and constructed as principal $W$-algebras built on the centralizer of a nilpotent element in $\mathfrak{gl}(n)$. The construction is carried out in the context of (Poisson) vertex algebras, and it appears natural that it is linked to the one presented here. In particular, the nilpotent element being characterised by a partition $\lambda = \lambda_1, \lambda_2, \ldots$, the algebras have an indexation comparable to $N = N_1, N_2, \ldots$ used in our multi-graded contraction procedure.

Other avenues for future work include asymmetric contractions and free-field realisations. Asymmetric Galilean $N = 1$ superconformal algebras were constructed in [10, 31–33] from a Galilean contraction of the tensor product $\mathfrak{SVir} \otimes \mathfrak{Vir}$, where one contracts the Virasoro subalgebra of an $N = 1$ superconformal algebra, $\mathfrak{SVir}$, with a separate Virasoro algebra. The ensuing Galilean superconformal algebra can be viewed as encoding a $(1, 0)$ supersymmetry. This was extended in [20] to a contraction of the asymmetric tensor product $W_3 \otimes \mathfrak{Vir}$, giving rise to a Galilean $W_3$ algebra generated by fields $T_0, T_1, W$. There is significant freedom in such contractions, and we hope to return elsewhere with a partial classification of the inequivalent Galilean algebras that can arise this way.

Free-field realisations [34–43] are ubiquitous in conformal field theory, and we find it natural to expect that they will continue to play a central role when Galilean conformal symmetries are present. Some work on this has been done [33, 44, 45], but a systematic approach and general results remain outstanding. We hope to report such advances in the near future.

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