Full-sky lensing reconstruction of gradient and curl modes from CMB maps

Toshiya Namikawa,\textsuperscript{a} Daisuke Yamauchi\textsuperscript{b} and Atsushi Taruya\textsuperscript{c,d}

\textsuperscript{a}Department of Physics, Graduate School of Science, The University of Tokyo
Tokyo 113-0033, Japan
\textsuperscript{b}Institute for Cosmic Ray Research, The University of Tokyo
5-1-5 Kashiwa-no-ha, Kashiwa City, Chiba 277-8582, Japan
\textsuperscript{c}Research Center for the Early Universe, School of Science, The University of Tokyo
Bunkyo-ku, Tokyo 113-0033, Japan
\textsuperscript{d}Institute for the Physics and Mathematics of the Universe, The University of Tokyo
Kashiwa, Chiba 277-8568, Japan

E-mail: namikawa@utap.phys.s.u-tokyo.ac.jp, yamauchi@icrr.u-tokyo.ac.jp, ataruya@utap.phys.s.u-tokyo.ac.jp

Abstract. We present a method of lensing reconstruction on the full sky, by extending the optimal quadratic estimator proposed by Okamoto & Hu (2003) to the case including the curl mode of deflection angle. The curl mode is induced by the vector and tensor metric perturbations, and the reconstruction of the curl mode would be a powerful tool to not only check systematics in the estimated gradient mode but also probe any vector and tensor sources. We find that the gradient and curl modes can be reconstructed separately, thanks to the distinctive feature in the parity symmetry between the gradient and curl modes. We compare our estimator with the flat-sky estimator proposed by Cooray \textit{et al} (2005). Based on the new formalism, the expected signal-to-noise ratio of the curl mode produced by the primordial gravitational-waves and a specific model of cosmic strings are estimated, and prospects for future observations are discussed.
1 Introduction

Ongoing, upcoming and next-generation experiments of CMB, such as PLANCK [1], POLARBEAR [2], ACTPol [3], SPTPol [4], CMBPol [5], and CoRe [6] would resolve not only the fine structure of the primary CMB anisotropies but also tiny effects that are important at small scales. One of the most important signals in those effects is the weak lensing: the deflection of CMB photons coming from the last-scattering surface by metric perturbations along our line-of-sight. Recent studies show that the lensing fields involved in the CMB anisotropies are reconstructed from the cross-correlations between CMB and large-scale structure [7, 8], and CMB maps alone [9-11], and the lensing information from upcoming experiments would provide us an opportunity to probe the late-time evolution of the structure in the Universe (e.g., [12-19]).

Several studies have investigated a method to reconstruct the deflection angle, which characterizes the effect of weak lensing on CMB maps (e.g., [20-28]). In general, the deflection
angle in a direction $\hat{n}$, $d(\hat{n})$, where $\hat{n}$ is the unit vector defined on the unit sphere, is decomposed into gradient and curl part as (e.g., [26])

$$d(\hat{n}) = \nabla \phi(\hat{n}) + (\ast \nabla) \varpi(\hat{n}),$$

where the first term, $\nabla \phi(\hat{n})$, and second term, $(\ast \nabla) \varpi(\hat{n})$, represent gradient and curl mode of deflection angle, respectively, and, in the polar coordinate, the covariant derivative on the unit sphere is given by $\nabla = e_{\varphi}(\partial/\partial \varphi) + (e_{\vartheta}/\sin \theta) (\partial/\partial \vartheta)$ with $e_{\varphi}$ and $e_{\vartheta}$ describing the basis vectors in the polar coordinate. The symbol, $\ast$, denotes a operation which rotates the angle of two-dimensional vector counterclockwise by 90-degree; for a vector on the unit sphere expressed in terms of the basis vectors, $a = a_{\vartheta}e_{\vartheta} + a_{\varphi}e_{\varphi}$, the operator, $\ast$, act on $a$ as $(\ast a) = a_{\vartheta}e_{\varphi} - a_{\varphi}e_{\vartheta}$ [26]. Hereafter, we call the potentials, $\phi$ and $\varpi$, “scalar lensing potential” and “pseudo-scalar lensing potential”, respectively.

The scalar metric perturbations such as the matter density fluctuations at linear order produce only the gradient mode, and the curl mode is usually neglected in the algorithm of lensing reconstruction. However, the curl mode can be induced by vector and/or tensor metric perturbations. In the conformal Newton gauge, the line element in the polar coordinate system is given as

$$ds^2 = a^2(\eta)\{- (1 + 2A)d\eta^2 - 2B_i d\eta dx^i + [(1 + 2C)\gamma_{ij} + 2D_{ij}] dx^i dx^j\},$$

where $a$ is the scale factor, $\eta$ is the conformal time, $A$ and $C$ are the scalar components, $B_i$ is the vector component ($B_{ij} = 0$), and $D_{ij}$ is the tensor component ($D_{iij} = 0$, and $D_{i}i = 0$). The unperturbed spatial metric, $\gamma_{ij}$, is given by

$$\gamma_{ij} dx^i dx^j = d\chi^2 + \chi^2(d\theta^2 + \sin^2 \theta d\varphi^2).$$

Then, the pseudo-scalar lensing potential is described by [29]

$$\varpi(\hat{n}) = (\ast \nabla)^{-2} \int_0^{\chi_s} \frac{d\chi}{\chi^2 \sin \theta} \left[ \frac{\partial \Omega_\varphi(\eta_0 - \chi, \chi \hat{n})}{\partial \varphi} - \frac{\partial \Omega_\vartheta(\eta_0 - \chi, \chi \hat{n})}{\partial \theta} \right],$$

where the quantities, $\chi$ and $\chi_s$, are the comoving distance and comoving distance at the last scattering surface, respectively, $\eta_0$ is the conformal time today, and the quantities, $\Omega_\vartheta$ and $\Omega_\varphi$, are defined as

$$\Omega_a(\eta, \chi \hat{n}) = B_a(\eta, \chi \hat{n}) + 2D_{\chi a}(\eta, \chi \hat{n}) \quad (a = \theta, \varphi).$$

According to the above equations, the primordial gravitational-waves produce the curl mode (e.g.,[30]). The cosmic strings can also produce the curl mode through the vector and tensor perturbations (e.g.,[29, 31]). This implies that the curl mode is a smoking gun of cosmic strings and other vector or tensor sources. Even if these sources are absent, the higher-order density perturbations and foreground contaminations generate not only the gradient mode but also the curl mode. These sources would cause systematics in the estimation of the gradient mode. Thus, the evaluation of the contaminations of these sources in the curl mode would be helpful to estimate the contributions of these sources in the gradient mode.

In this paper, we present a method to reconstruct both the gradient and curl modes of deflection angle. The reconstruction of the curl mode has been previously discussed in Refs.[26, 32]. Ref.[26] proposed an algorithm based on the likelihood analysis. Including
polarizations, this estimator, in principle, suppresses the noise contribution for the reconstructed potentials, but numerically cost, compared to the estimator in Ref.[24]. On the other hand, in Ref.[32], they consider the flat-sky limit and empirically define a quadratic estimator of the curl mode. Since current and future CMB missions will cover nearly entire sky, a full-sky algorithm including the curl mode are highly desirable. In this paper, we derive a full-sky estimator of the gradient and curl modes on the full sky, extending the full-sky formalism for the gradient mode in Ref.[25]. Then, we compare the flat-sky estimator with the one on the full sky, and show that the empirically defined estimator in Ref.[32] can be derived from the full-sky estimator. In addition, based on our full-sky estimator, possible implications to detection of curl mode from primordial gravitational-waves and cosmic strings are discussed.

This paper is organized as follows. In section 2, we briefly summarize the lensing effect on the CMB anisotropies. In section 3, we extend the quadratic estimator of Ref.[25] to the case including both the gradient and curl modes of deflection angle. In section 4, we compute the noise spectrum of the estimator in both the full- and flat-sky cases. In section 5, we discuss implications for primordial gravitational-waves and cosmic strings by reconstructing the curl mode. Section 6 is devoted to summary and conclusion.

In this paper we adopt the cosmological parameters assuming a flat Lambda-CDM model consistent with the results obtained from Ref.[33]; the density parameter of baryon $\Omega_b h^2 = 0.022$, of matter $\Omega_m h^2 = 0.13$, dark energy density $\Omega_\Lambda = 0.72$, scalar spectral index $n_s = 0.96$, scalar amplitude $A_s = 2.4 \times 10^{-9}$ and the optical depth, $\tau = 0.086$. In Table.1, we summarize the meaning and definition of the quantities used to reconstruct the lensing potentials from observed CMB maps.

## 2 Weak lensing of the CMB

Here we briefly review the lensing effect on the CMB anisotropies in the full-sky case, including both the gradient and curl modes of deflection angle. We first discuss the lensing effect on CMB temperature in section 2.1, and the similar discussion for polarizations is given in section 2.2. The detailed calculation of lensing effect is presented in, e.g., Ref.[34] in the absence of the curl mode, and in Ref.[30] including the curl mode.

### 2.1 Lensing effect on CMB anisotropies: temperature

Let us first discuss the lensing effect on the temperature anisotropies. The lensed temperature fluctuations in a direction $\hat{n}$, $\Theta(\hat{n})$, are transformed into the harmonic space according to

$$\tilde{\Theta}_{\ell,m} = \int d\hat{n} \, Y_{\ell,m}^* (\hat{n}) \tilde{\Theta}(\hat{n}),$$

with the quantities, $\tilde{\Theta}_{\ell,m}$ and $\, Y_{\ell,m}(\hat{n})$, describing the harmonic coefficients, and the spin-0 spherical harmonics, respectively. The lensed temperature fluctuations are related to the unlensed temperature fluctuations, $\Theta(\hat{n})$, through $\tilde{\Theta}(\hat{n}) = \Theta(\hat{n} + \mathbf{d})$, where $\mathbf{d}$ is the deflection angle. Usually, the deflection angle is a small perturbed quantity, $|\mathbf{d}| \ll 1$, and the lensed temperature fluctuations may be expressed as

$$\tilde{\Theta}(\hat{n}) = \Theta(\hat{n}) + \mathbf{d} \cdot \nabla \Theta(\hat{n}) + \mathcal{O}(|\mathbf{d}|^2),$$

(2.2)
Table 1. Notations for quantities used to reconstruct the lensing potentials from observed CMB maps. The symbols are divided into three categories. The quantities in the middle eight rows are needed to compute the optimal quadratic estimator. In the bottom three rows, we describe the quantities needed to compute the optimal combination.

| Symbol | Definition | Meaning |
|--------|------------|---------|
| Full sky / Flat sky | Full sky / Flat sky | Scalar or pseudo-scalar lensing potential |
| $\alpha$ (or $\beta$) | A pair of two CMB maps, $X$ and $Y$ |
| $\tilde{x}_{\ell,m}^{(a)} / \tilde{x}_{\ell}^{(a)}$ | Eq.(3.10) / (3.42) | Estimator |
| $F_{\ell,L,L'}^{x,(a)} / F_{\ell,L,L'}^{x,(c)}$ | Eq.(3.27) / (3.43) | Weight function |
| $N_{\ell}^{x,(a)} / N_{\ell}^{x,(c)}$ | Eq.(3.26) / (3.41) | Noise spectrum |
| $g_{\ell,L,L'}^{x,(a)} / g_{\ell,L,L'}^{x,(c)}$ | Eq.(3.21) / (3.40) | - |
| $f_{\ell,L,L'}^{x,(a)} / f_{\ell,L,L'}^{x,(c)}$ | Table.2 / 3 | - |
| $\pm 2S^x$ | Eq.(2.13) | - |
| $S^x$ and $\mp S^x$ | Eq.(2.16) | - |
| $\tilde{x}_{\ell,m}^{(c)} / \tilde{x}_{\ell}^{(c)}$ | Eq.(3.29) / (3.45) | Optimal combination |
| $N_{\ell}^{x,(c)} / N_{\ell}^{x,(c)}$ | Eq.(3.30) / (3.46) | Noise spectrum for optimal combination |
| $N_{\ell}^{x,(a,\beta)} / N_{\ell}^{x,(a,\beta)}$ | Eq.(3.31) / (3.47) | Noise cross-spectrum |

where we expand $\Theta(\hat{n} + \mathbf{d})$ in terms of the deflection angle $\mathbf{d}$. Hereafter we neglect the contributions of $O(|\mathbf{d}|^2)$ in Eq.(2.2). The harmonic coefficients of the lensed quantities are obtained by transforming Eq.(2.2) into the harmonic space, according to Eq.(2.1). Using the expression of deflection angle (1.1), the lensed temperature anisotropies in the harmonic space are given by [30]

$$
\tilde{\Theta}_{L,M} = \Theta_{L,M} + \int d\hat{n} \, 0\mathcal{Y}^*_{L,M}(\hat{n}) \left[ \nabla \phi(\hat{n}) + (\star \nabla) \varpi(\hat{n}) \right] \cdot \nabla \Theta(\hat{n})
$$

$$
= \Theta_{L,M} + \sum_{\ell,m} \sum_{\ell',m'} \Theta_{\ell',m'} (-1)^M \left( \begin{array}{c} L \\ -M \\ m \\ m' \end{array} \right) \sum_{x = \phi, \varpi} 0S^x_{L,\ell,\ell'} x_{\ell,m}, \quad (2.3)
$$

where the quantities, $0S^\phi_{L,\ell,\ell'}$ and $0S^\varpi_{L,\ell,\ell'}$, are defined as

$$
(-1)^M \left( \begin{array}{c} L \\ -M \\ m \\ m' \end{array} \right) 0S^\phi_{L,\ell,\ell'} = \int d\hat{n} \, 0\mathcal{Y}^*_{L,M}(\hat{n}) \left[ \nabla_0 \mathcal{Y}_{\ell,m}(\hat{n}) \right] \cdot \left[ \nabla_0 \mathcal{Y}_{\ell',m'}(\hat{n}) \right], \quad (2.4)
$$

$$
(-1)^M \left( \begin{array}{c} L \\ -M \\ m \\ m' \end{array} \right) 0S^\varpi_{L,\ell,\ell'} = \int d\hat{n} \, 0\mathcal{Y}^*_{L,M}(\hat{n}) \left[ (\star \nabla)_0 \mathcal{Y}_{\ell,m}(\hat{n}) \right] \cdot \left[ \nabla_0 \mathcal{Y}_{\ell',m'}(\hat{n}) \right]. \quad (2.5)
$$
As shown in appendix A (Eqs. (A.7) and (A.8)), the quantities, $0S^\phi_{L,\ell,\ell'}$ and $0S^\varpi_{L,\ell,\ell'}$, are expressed in terms of the Wigner-3$j$ symbols, and the results are

$$0S^\phi_{L,\ell,\ell'} = \sqrt{(2\ell + 1)(2\ell' + 1)(2L + 1)} [16\pi]^{-1} [\ell(L + 1) + \ell(\ell + 1)] \left( \begin{array}{ccc} L & \ell & \ell' \\ 0 & 0 & 0 \end{array} \right),$$  

(2.6)

$$0S^\varpi_{L,\ell,\ell'} = -i \sqrt{(2\ell + 1)(2\ell' + 1)(2L + 1)} \sqrt{\ell(\ell + 1)} \sqrt{\ell'(\ell' + 1)} \times \left[ \begin{array}{cc} L & \ell & \ell'' \\ 0 & -1 & 1 \end{array} \right] - \left( L & \ell & \ell'' \\ 0 & 1 & -1 \end{array} \right].$$  

(2.7)

We note that the quantities $0S^\phi_{L,\ell,\ell'}$ and $0S^\varpi_{L,\ell,\ell'}$ satisfy

$$0S^\phi_{L,\ell,\ell'} = (-1)^{L+\ell+\ell'} 0S^\phi_{L,\ell,\ell'}, \quad 0S^\varpi_{L,\ell,\ell'} = -(-1)^{L+\ell+\ell'} 0S^\varpi_{L,\ell,\ell'}. \quad (2.8)$$

The above equations come from the parity symmetry of $\Theta$, $\phi$ and $\varpi$; the temperature anisotropies and the scalar lensing potential are even parity, while the pseudo-scalar lensing potential is odd parity. In fact, Eq.(2.8) is checked by changing the variable, $\hat{\mathbf{n}} \rightarrow -\hat{\mathbf{n}}$ in the right-hand side of Eqs.(2.4) and (2.5). Under this transformation, the spin-0 spherical harmonics are multiplied by a factor $(-1)^\ell$, and the derivatives become $\nabla \rightarrow -\nabla$ and $(\Theta \nabla) \rightarrow -(\Theta \nabla)$, respectively. As a result, the right-hand sides of Eq.(2.4) and Eq.(2.5) are multiplied by a factor of $(-1)^{L+\ell+\ell'}$ and $(-1)^{L+\ell+\ell'}$, respectively. Eq.(2.8) is also checked with the formulas of the Wigner-3$j$ symbols (see Eq.(A.2) in appendix A).

From Eq.(2.8), $0S^\phi_{L,\ell,\ell'}$ becomes zero if $L + \ell + \ell'$ is an odd integer, and the coefficient $0S^\varpi_{L,\ell,\ell'}$ vanishes when $L + \ell + \ell'$ is an even integer. These properties are essential for a separate reconstruction of gradient and curl modes in subsequent analysis.

### 2.2 Lensing effect on CMB anisotropies: polarizations

Next we consider the lensing effect on the CMB polarizations, $Q(\hat{\mathbf{n}}) \pm iU(\hat{\mathbf{n}})$. We are especially concerned with the rotationally invariant combinations, i.e., $E$- and $B$-mode polarizations [30];

$$[\hat{E} \pm i\hat{B}]_{L,M} = [E \pm iB]_{L,M} + \int d\hat{\mathbf{n}} \pm 2Y^2_{L,M}(\hat{\mathbf{n}}) \left[ \nabla \phi(\hat{\mathbf{n}}) + (\Theta \nabla) \varpi(\hat{\mathbf{n}}) \right] \nabla (Q \pm iU)(\hat{\mathbf{n}})$$

$$= [E \pm iB]_{L,M} + \sum_{\ell,m} \sum_{\ell',m'} [E \pm iB]_{\ell,\ell'} (-1)^{\ell} \left( \begin{array}{ccc} L & \ell & \ell' \\ -M & m & m' \end{array} \right)$$

$$\times \sum_{x=\phi,\varpi} \pm 2S^x_{L,\ell,\ell'} \psi_{\ell,m} \psi_{\ell,m'}, \quad (2.9)$$

with the quantities, $\pm 2S^\phi_{L,\ell,\ell'}$ and $\pm 2S^\varpi_{L,\ell,\ell'}$, defined by

$$(-1)^{\ell} \left( \begin{array}{ccc} L & \ell & \ell' \\ -M & m & m' \end{array} \right) \pm 2S^\phi_{L,\ell,\ell'} = \int d\hat{\mathbf{n}} \pm 2Y^2_{\ell,M}(\hat{\mathbf{n}}) \left[ \nabla \phi_{\ell,m}(\hat{\mathbf{n}}) \right] \cdot \left[ \nabla \pm 2\varpi_{\ell,m'}(\hat{\mathbf{n}}) \right], \quad (2.10)$$

$$(-1)^{\ell} \left( \begin{array}{ccc} L & \ell & \ell' \\ -M & m & m' \end{array} \right) \pm 2S^\varpi_{L,\ell,\ell'} = \int d\hat{\mathbf{n}} \pm 2Y^2_{\ell,M}(\hat{\mathbf{n}}) \left[ (\Theta \nabla) \phi_{\ell,m}(\hat{\mathbf{n}}) \right] \cdot \left[ \nabla \pm 2\varpi_{\ell,m'}(\hat{\mathbf{n}}) \right]. \quad (2.11)$$
The quantity \( \pm 2Y_{\ell',m}(\hat{n}) \) denotes the spin-\( \pm 2 \) spherical harmonics. Similar to the case of temperature anisotropies, the quantities, \( \pm 2S_{L,\ell,\ell'}^\phi \) and \( \pm 2S_{L,\ell,\ell'}^\psi \), are written as \[2.12\]

\[
\pm 2S_{L,\ell,\ell'}^\phi = \sqrt{\frac{(2\ell + 1)(2\ell' + 1)(2L + 1)}{16\pi}} \left[ (\ell + 1) + \ell'(\ell' + 1) - L(L + 1) \right] \left( L \ell \ell' \pm 2 0 \mp 2 \right),
\]

\[
\pm 2S_{L,\ell,\ell'}^\psi = -i \sqrt{\frac{(2\ell + 1)(2\ell' + 1)(2L + 1)}{16\pi}} \left[ (\ell + 1) \sqrt{(\ell' + 2)(\ell' + 1 + 2)} \right] \times \left[ \sqrt{\ell' + 1 \mp 2} \left( L \ell \ell' \pm 2 1 \mp 1 \right) - \sqrt{\ell' + 1 \mp 2} \left( L \ell \ell' \pm 2 1 \mp 2 \right) \right].
\]

Eq.(2.9) is rewritten in the separable form for \( E \)- and \( B \)-mode polarizations:

\[
\tilde{E}_{L,M} = E_{L,M} + \sum_{\ell,m,\ell',m'} (-1)^M \left( L \ell \ell' \right) \sum_{x=\phi,\psi} x_{\ell,m} \{ \oplus S_{L,\ell,\ell'}^x E_{\ell',m'} - \ominus S_{L,\ell,\ell'}^x B_{\ell',m'} \},
\]

\[
\tilde{B}_{L,M} = B_{L,M} + \sum_{\ell,m,\ell',m'} (-1)^M \left( L \ell \ell' \right) \sum_{x=\phi,\psi} x_{\ell,m} \{ \ominus S_{L,\ell,\ell'}^x E_{\ell',m'} + \oplus S_{L,\ell,\ell'}^x B_{\ell',m'} \},
\]

where we define

\[
\oplus S_{L,\ell,\ell'}^x = \frac{2S_{L,\ell,\ell'}^x + -2S_{L,\ell,\ell'}^x}{2}, \quad \ominus S_{L,\ell,\ell'}^x = \frac{2S_{L,\ell,\ell'}^x - 2S_{L,\ell,\ell'}^x}{2i}.
\]

Note again that, for an even integer of \( L + \ell + \ell' \), the coefficients \( \oplus S^\psi \) and \( \ominus S^\phi \) vanish. On the other hand, the quantities \( \oplus S^\phi \) and \( \ominus S^\psi \) vanish when \( L + \ell + \ell' \) is an odd integer. Similar to the case of temperature, these properties come from the fact that \( E \)-mode polarization and scalar lensing potential are even parity, while \( B \)-mode polarization and pseudo-scalar lensing potential are odd parity.

3 Reconstruction of deflection angle in the presence of curl mode

In this section, we present a reconstruction method for \( \phi_{\ell,m} \) and \( \omega_{\ell,m} \), based on the quadratic statistics (e.g., Refs. [21, 23–25]). We frequently use the formulas for Wigner-3j symbols summarized in appendix A. In what follows, the lensed temperature or polarizations, i.e., \( \hat{\Theta}, \hat{E} \) or \( \hat{B} \), are symbolically denoted by \( \hat{X} \) (and \( \hat{Y} \)).

3.1 Full-sky formalism

3.1.1 Lensing field as quadratic statistics
come from the parity symmetry in the lensing potentials and primary CMB anisotropies. With Eqs. (3.1), we can reconstruct the lensing potentials by extracting the off-diagonal terms of the primary CMB anisotropies. Thus, it is possible to distinguish the lensing potentials (see Eqs. (3.1)) from the usual meaning of the ensemble average, \( \langle \cdots \rangle_{\text{CMB}} \). Under the situation, in the correlation of lensed CMB anisotropies, \( \langle X_{L,M} Y_{L',M'} \rangle_{\text{CMB}} \), the lensing potentials are included in the non-zero off-diagonal terms \((L \neq L', M \neq -M')\). This is because the lensing effect causes a non-trivial mode-coupling between the primary CMB anisotropies and lensing potentials (see Eqs. (2.3), (2.14) and (2.15)), and the lensed CMB anisotropies are not statistically isotropic for a given realization of the lensing potentials. Thus, it is possible to reconstruct the lensing potentials by extracting the off-diagonal terms of \( \langle X_{L,M} Y_{L',M'} \rangle_{\text{CMB}} \).

To get insight into the reconstruction of the lensing potentials, we consider an idealistic situation; we can take the ensemble average over primary CMB anisotropies alone, under a given realization of the lensing potentials. Hereafter, we denote this average by \( \langle \cdots \rangle_{\text{XMB}} \). To extract the off-diagonal terms and find the solutions for \( \langle \cdots \rangle_{\text{XMB}} \), summarized in Table 2. The conditions, “even” and “odd”, in Table 2 come from the parity symmetry in the lensing potentials and primary CMB anisotropies. To extract the off-diagonal terms and find the solutions for \( \phi_{\ell,m} \) and \( \omega_{\ell,m} \), we obtain

\[
\langle X_{L,M} Y_{L',M'} \rangle_{\text{CMB}} = C_{LL'}^{XY} \delta_{L',M'} \delta_{M,-M'} (-1)^M + \sum_{\ell,m} (-1)^m \left[ f_{\ell,L,L'}^{\phi,(XY)} \phi_{\ell,m} + f_{\ell,L,L'}^{\omega,(XY)} \omega_{\ell,m} \right],
\]

with the coefficients, \( f_{\ell,L,L'}^{\phi,(XY)} \) and \( f_{\ell,L,L'}^{\omega,(XY)} \), summarized in Table 2.
\( \omega_{\ell,m} \), we multiply

\[
(-1)^{m'} \begin{pmatrix} \ell' & L & L' \\ -m' & M & M' \end{pmatrix} f^{\phi, (XY)}_{\ell', L, L'},
\]

in both sides of Eq.(3.1). Note that the multipoles, \( L \) and \( L' \), are chosen so that \( f^{\phi, (XY)}_{\ell', L, L'} \neq 0 \). Then, summing up the equation over \( M \) and \( M' \), and using the formulas, Eqs.(A.3) and (A.4), we find \(^1\)

\[
\phi_{\ell,m} = \frac{2\ell + 1}{f^{\phi, (XY)}_{\ell, L, L'}} \sum_{M, M'} (-1)^m \begin{pmatrix} \ell & L & L' \\ -m & M & M' \end{pmatrix} \langle \tilde{X}_{L,M} \tilde{Y}_{L', M'} \rangle_{CMB}.
\]

Notice that the term involving \( \varpi \) in Eq.(3.1) vanishes. This is because, for all \( \ell, L \) and \( L' \), the parity symmetry of \( f^{\phi, (XY)}_{\ell, L, L'} \) and \( f^{\varpi, (XY)}_{\ell, L, L'} \) (Table 2) leads to

\[
f^{\phi, (XY)}_{\ell, L, L'} f^{\varpi, (XY)}_{\ell, L, L'} = 0.
\]

Similarly, following the procedure described in Eq.(3.2) below, but replacing \( f^{\phi, (XY)}_{\ell, L, L'} \) with \( f^{\varpi, (XY)}_{\ell, L, L'} \), the solution for \( \varpi_{\ell,m} \) is obtained, and the result is

\[
\varpi_{\ell,m} = \frac{2\ell + 1}{f^{\varpi, (XY)}_{\ell, L, L'}} \sum_{M, M'} (-1)^m \begin{pmatrix} \ell & L & L' \\ -m & M & M' \end{pmatrix} \langle \tilde{X}_{L,M} \tilde{Y}_{L', M'} \rangle_{CMB}.
\]

The above equations, (3.3) and (3.5), can not be used for a definition of the estimator of lensing potentials, because these equations include the ensemble average over the primary CMB anisotropies alone, \( \langle \cdots \rangle_{CMB} \). But, the above equations imply that, by summing the quadratic combination of lensed fields over multipoles appropriately, it is possible to separately construct the estimators for the scalar and pseudo-scalar lensing potentials, \( \phi \) and \( \varpi \).

### 3.1.2 Estimator

Based on Eqs.(3.3) and (3.5), we first naively define the estimator for the lensing potential \( x = \phi \) or \( \varpi \) as follows:

\[
\frac{2\ell + 1}{f^{x, (\alpha)}_{\ell, L, L'}} \sum_{M, M'} (-1)^m \begin{pmatrix} \ell & L & L' \\ -m & M & M' \end{pmatrix} \tilde{X}_{L,M} \tilde{Y}_{L', M'},
\]

where the subscript, \( \alpha \), means a pair of two CMB maps, e.g., \( \alpha = \Theta \Theta \) or \( EB \). The multipoles, \( L \) and \( L' \), are chosen so that \( f^{x, (\alpha)}_{\ell, L, L'} \neq 0 \). With Eqs.(3.3) and (3.5), the estimator is rewritten as

\[
\text{Eq.(3.6)} = x_{\ell,m} + n^{x, (\alpha)}_{\ell,m, L, L'},
\]

\(^1\) In deriving Eq.(3.3), we have ignored the zero-mode \( (C_0^{XY}) \), arising from the first term in Eq.(3.1) .
where the quantity, \( n^{x,(a)}_{L,L'} \), is given by

\[
n^{x,(a)}_{L,L'} = (-1)^m \frac{2\ell + 1}{f^{x,(a)}_{L,L',M,M'}} \sum_{M,M'} \left( \frac{\ell}{-m} \frac{L}{M} \frac{L'}{M'} \right) \left( \tilde{X}_{L,M} \tilde{Y}_{L',M'} - \langle \tilde{X}_{L,M} \tilde{Y}_{L',M'} \rangle_{\text{CMB}} \right) \tag{3.8}
\]

Note that the above equation can be expressed without the quantity, \( \langle \cdots \rangle_{\text{CMB}} \). For instance, in the case using the temperature anisotropies alone (i.e., \( \alpha = \Theta^{\Theta} \)), the above equation is rewritten as

\[
n^{x,(\Theta)}_{L,L,L'} = (-1)^m \frac{2\ell + 1}{f^{x,(\Theta)}_{L,L',M,M'}} \sum_{M,M'} \left\{ \left( \frac{\ell}{-m} \frac{L}{M} \frac{L'}{M'} \right) (1 - \delta_{M,-M'}) \Theta_{L,M} \Theta_{L',M'} + \sum_{x = \phi, \varpi} \sum_{\ell',m'} \sum_{M''} \left\{ (-1)^{M'} \left( \frac{L'}{M'} \frac{\ell'}{m'} \frac{L''}{M''} \right) 0 S_{L',\ell',L''}^{x'} \Theta_{L'',M''} \Theta_{L,M} \right. \\
\left. + (-1)^{M} \left( \frac{L}{-M} \frac{\ell}{m} \frac{L''}{M''} \right) 0 S_{L,\ell,M}^{x} \Theta_{L'',M''} \Theta_{L',M'} \right\} \right\}. \tag{3.9}
\]

The above estimator (3.6) suffers from several drawbacks in practical application to observation. At first, we should know about the primary CMB angular power spectra included in \( F^{x,(a)}_{L,L',L'} \), a priori. Another problem is that the estimator includes the contribution from the term \( n^{x,(a)}_{L,L,L'} \), which leads to a noisy reconstruction of the lensing potentials. Nevertheless, for the former point, the primary CMB angular power spectrum can be theoretically inferred if we know a set of fiducial cosmological parameters from other observations. On the other hand, for the latter point, we redefine the estimator for \( x, \varpi \) by introducing a weight function, \( F^{x,(a)}_{L,L',L'} \), in order to reduce the contribution from \( n^{x,(a)}_{L,L,L'} \). Summing up all combination of \( L \) and \( L' \), we write the estimator of the lensing potentials as

\[
\hat{x}^{(a)}_{L,m} = \sum_{L,L'} \sum_{M,M'} \left( \frac{\ell}{-m} \frac{L}{M} \frac{L'}{M'} \right) \tilde{X}_{L,M} \tilde{Y}_{L',M'}.
\]

The functional form of the weight function is determined so that the noise contribution is minimized.

In what follows, we determine the functional form of the weight function so that the estimator is unbiased and the noise term is minimized. Eq. (3.10) can be recast as

\[
\hat{x}^{(a)}_{L,m} = \sum_{L,L'} F^{x,(a)}_{L,L',L'} \sum_{x = \phi, \varpi} \frac{f^{x,(a)}_{L,L',L'}}{2\ell + 1} x^{(a)}_{L,m} + n^{x,(a)}_{L,m},
\]

where the inner product \([a^x, b^{x'}]_{\ell}^{(a)}\) for arbitrary two quantities, \( a^{x,(a)}_{L,L'} \) and \( b^{x,(a)}_{L,L'} \), is defined by

\[
[a^x, b^{x'}]_{\ell}^{(a)} = \frac{1}{2\ell + 1} \sum_{L,L'} a^{x,(a)}_{L,L'} b^{x',(a)}_{L,L'}.
\]

\[
a = \frac{1}{2\ell + 1} \sum_{L,L'} a^{x,(a)}_{L,L'} b^{x',(a)}_{L,L'}.
\]

\[
[a^x, b^{x'}]_{\ell}^{(a)} = \frac{1}{2\ell + 1} \sum_{L,L'} a^{x,(a)}_{L,L'} b^{x',(a)}_{L,L'}.
\]

\[
[a^x, b^{x'}]_{\ell}^{(a)} = \frac{1}{2\ell + 1} \sum_{L,L'} a^{x,(a)}_{L,L'} b^{x',(a)}_{L,L'}.
\]

\[
[a^x, b^{x'}]_{\ell}^{(a)} = \frac{1}{2\ell + 1} \sum_{L,L'} a^{x,(a)}_{L,L'} b^{x',(a)}_{L,L'}.
\]
The quantity, $n_{\ell ,m}^{\alpha }$, is defined as

$$n_{\ell ,m}^{\alpha } \equiv \sum_{L,L'} F^{\alpha }_{L,L',L'} f^{\alpha }_{\ell ,L,L'} n_{\ell ,m,L,L'}^{\alpha } .$$  \hspace{1cm} (3.13)

Eq.(3.11) implies that the estimator would be an unbiased estimator if we impose the following condition:

$$[F^x , f^{x'} ]_{\ell }^{(\alpha )} = \delta _{\ell ,x} .$$  \hspace{1cm} (3.14)

Mathematically, this is equivalent to $\langle x^{(\alpha )}_{\ell ,m} \rangle _{\text{CMB}} = x_{\ell ,m}$. Also, we wish to suppress the noise contributions, $n_{\ell ,m}^{\alpha }$, imposing the following condition:

$$\frac{\delta}{\delta F^{x}_{\ell ,L,L'}} \left\langle |n_{\ell ,m}^{(\alpha )}|^2 \right\rangle = 0 .$$  \hspace{1cm} (3.15)

Let us determine the functional form of the weight function under the conditions, (3.14) and (3.15), with the Lagrange-multiplier method. The variance of $n_{\ell ,m}^{(\alpha )}$ is given by

$$\left\langle |n_{\ell ,m}^{(\alpha )}|^2 \right\rangle = \frac{1}{2\ell + 1} \sum_{L,L'} (F^{x}_{L,L'})^* \times \left( F^{x}_{L,L'} C^{(\alpha )}_{L,L'} + (\ell + L') F^{x}_{L',L'} C^{(\alpha )}_{L,L'} \right) \left( F^{x}_{L,L'} C^{(\alpha )}_{L',L'} + (\ell + L) F^{x}_{L,L'} C^{(\alpha )}_{L',L'} \right) ,$$  \hspace{1cm} (3.16)

where the quantity, $C^{(\alpha )}_{L,L'}$, is the lensed angular power spectrum including the contributions from instrumental noise. The detailed calculation for the noise variance, $\left\langle |n_{\ell ,m}^{(\alpha )}|^2 \right\rangle$, is presented in appendix B. Then, Eq.(3.15) under the constraint (3.14) is equivalent to

$$\frac{\delta}{\delta F^{x}_{\ell ,L,L'}} \left\{ \frac{1}{2\ell + 1} \sum_{L,L'} (F^{x}_{L,L'})^* \times \left( F^{x}_{L,L'} C_{L,L'}^{(\alpha )} + (\ell + L') F^{x}_{L',L'} C_{L,L'}^{(\alpha )} \right) \right\} = 0 .$$  \hspace{1cm} (3.17)

The quantities, $\lambda ^x_{\phi ,\omega }$ and $\lambda ^x_{\omega ,\omega }$, are the Lagrange multiplier whose functional form is specified below. Eq.(3.17) leads to

$$(F^{x}_{\ell ,L',L'} C_{L,L'}^{(\alpha )})^* C_{L,L'}^{(\alpha )} + (F^{x}_{\ell ,L',L'} C_{L,L'}^{(\alpha )})^* (\ell + L') C_{L,L'}^{(\alpha )} + \sum_{x' = \phi ,\omega }^{x = \ell ,L',L'} \lambda ^x_{x',\ell ,L',L'} = 0 .$$  \hspace{1cm} (3.18)

In the above, interchanging $L$ and $L'$, we also obtain

$$(F^{x}_{\ell ,L,L'} C_{L,L'}^{(\alpha )})^* C_{L,L'}^{(\alpha )} + (F^{x}_{\ell ,L,L'} C_{L,L'}^{(\alpha )})^* (\ell + L') C_{L,L'}^{(\alpha )} + \sum_{x' = \phi ,\omega }^{x = \ell ,L',L'} \lambda ^x_{x',\ell ,L',L'} = 0 .$$  \hspace{1cm} (3.19)

Multiplying the factors $C_{L,L'}^{X Y}$ and $-(\ell + L') C_{L,L'}^{X Y}$ with Eq.(3.18) and Eq.(3.19), respectively, the sum of Eqs.(3.18) and (3.19) gives

$$F^{x}_{\ell ,L,L'} + \sum_{x' = \phi ,\omega }^{x = \ell ,L',L'} (\lambda ^x_{x',\ell ,L',L'}) g^{x'}_{\ell ,L,L'} = 0 ,$$  \hspace{1cm} (3.20)
where we define
\[ g_{\ell, L, L'}^{(a)} = \frac{(f_{\ell, L, L'}^{(a)})^* \tilde{C}_L^{XY} \tilde{C}_L^{XY'} - (-1)^{\ell + L + L'} \tilde{C}_L^{XX'} \tilde{C}_L^{YY'} (f_{\ell, L, L'}^{(a)})^*}{\tilde{C}_L^{XX} \tilde{C}_L^{YY'} \tilde{C}_L^{XX'} \tilde{C}_L^{YY} - (\tilde{C}_L^{XX} \tilde{C}_L^{YY'})^2}. \] (3.21)

Substituting Eq. (3.20) into Eq. (3.14), we obtain
\[ -\sum_{x''}(\lambda_{x'}^{x''})^*[g_{x''}, f_{x'}^{(a)}]_\ell = \delta_{xx'}. \] (3.22)

From Eq. (3.4), we find
\[ [g_{x''}, f_{x'}^{(a)}]_\ell = \delta_{x''x'}[g_{x'}, f_{x'}^{(a)}]. \] (3.23)

Combining the above equation with Eq. (3.22), we obtain the explicit form of the Lagrange multiplier
\[ (\lambda_{x'}^{x''})^* = -\frac{\delta_{xx'}}{[f_{x''}, f_{x'}^{(a)}]_\ell}. \] (3.24)

Then, from Eq. (3.20), we finally obtain the expression for the weight function:
\[ F_{\ell, L, L'}^{x^{(a)}} = \frac{g_{\ell, L, L'}^{x^{(a)}}}{[f_{x''}, f_{x'}^{(a)}]_\ell}. \] (3.25)

Note that, with the explicit expression (3.25), the noise variance, \( N_{\ell}^{x^{(a)}} \), given in Eq. (3.16) becomes
\[ N_{\ell}^{x^{(a)}} \equiv \left( |n_{\ell, m}^{x^{(a)}}|^2 \right) = \frac{1}{2\ell + 1} \frac{1}{[f_{x''}, f_{x'}^{(a)}]_\ell} \sum_{L, L'} (g_{\ell, L, L'}^{x^{(a)}})^* \times \left( F_{\ell, L, L'}^{x^{(a)}} \tilde{C}_L^{XY} \tilde{C}_L^{XY'} + F_{\ell, L', L}^{x^{(a)}} (-1)^{\ell + L + L'} \tilde{C}_L^{XX'} \tilde{C}_L^{YY'} \right) = \frac{1}{[f_{x''}, f_{x'}^{(a)}]_\ell}, \] (3.26)

where we use the relations given in Eqs. (3.18) and (3.24). Thus, the weight function can be recast as
\[ F_{\ell, L, L'}^{x^{(a)}} = N_{\ell}^{x^{(a)}} g_{\ell, L, L'}^{x^{(a)}}. \] (3.27)

With the weight function given above, the estimators defined by Eq. (3.10) become optimal, i.e., the noise contribution is minimized. Eq. (3.25) or Eq. (3.27) is one of the main results in this paper. Note that, if the curl mode is absent, \( \varpi = 0 \), the resultant form of the weight function for \( \phi \) exactly coincides with the one obtained in Ref. [25]. The difference appears when the angular power spectrum of the pseudo-scalar lensing potential, \( C_\ell^{\varpi \varpi} \), included in the lensed angular power spectrum becomes non-vanishing. Note again that, in practical case, to use the estimator, the angular power spectrum of primary CMB anisotropies should be a priori known (i.e., \( f_{\ell, L, L'}^{\varphi^{(a)}} \) and \( f_{\ell, L, L'}^{\varpi^{(a)}} \) are given).
3.1.3 Optimal combination

As discussed in Ref.[25], the noise contribution can be further suppressed by combining multiple observables. Summing up the whole possible combination of temperature and polarization anisotropies, the optimal combination of the minimum variance estimators are given by

\[ \hat{x}_{\ell,m}^{(c)} = \sum_{\alpha} W^{x,(\alpha)}_{\ell,m} \hat{x}_{\ell,m}^{(\alpha)} \quad (x = \phi, \varpi). \]  

(3.28)

The weight functions, \( W^{x,(\alpha)} \), are determined so that the estimator satisfies the unbiased condition \( (\langle \hat{x}_{\ell,m}^{(c)} \rangle_{\text{CMB}} = x_{\ell,m}) \), and the variance of the noise contribution is minimum. The optimal combination of the minimum variance estimator is then determined by the same analogy as in Ref.[25], and the result is

\[ \hat{x}_{\ell,m}^{(c)} = N_{\ell}^{x,(c)} \sum_{\alpha,\beta} \{ (N_{\ell}^{x})^{-1} \}^{\alpha,\beta} \hat{x}_{\ell,m}^{(\alpha)}, \]  

(3.29)

where the variance, \( N_{\ell}^{x,(c)} \), is defined by

\[ \frac{1}{N_{\ell}^{x,(c)}} = (N_{\ell}^{x})^{-1} \sum_{\beta,\beta'} \{ (N_{\ell}^{x})^{-1} \}^{\beta,\beta'}. \]  

(3.30)

The component of the matrix, \( \{ N_{\ell}^{x}\}^{\alpha,\beta} \), is the covariance of \( n_{\ell,m}^{x,(\alpha)} \) and \( n_{\ell,m}^{x,(\beta)} \) which is given by

\[ N_{\ell}^{x,(\alpha,\beta)} = \langle (n_{\ell,m}^{x,(\alpha)})^* n_{\ell,m}^{x,(\beta)} \rangle = \frac{1}{2\ell + 1} \sum_{L,L'} (F_{\ell,L,L'}^{x,(\alpha)})^* \times \left( F_{\ell,L,L'}^{x,(\beta)} \tilde{C}_{L}^{XX'} \tilde{C}_{L'}^{YY'} + F_{\ell,L',L}^{x,(\beta)} (-1)^{\ell+L+L'} \tilde{C}_{L}^{XY'} \tilde{C}_{L'}^{XY} \right). \]  

(3.31)

The derivation of the above equation is given in appendix B.

3.2 Flat-sky limit

The quadratic estimator for the curl mode has been empirically derived in previous work [32], based on the flat-sky approximation. Here, we show that our full-sky estimator can reproduce the flat-sky estimator of Ref.[32] (Eqs.(10)-(12) of Ref.[32]), in the flat-sky limit, \( \ell, L, L' \ll 1 \).

Let us first rewrite the full-sky estimator (3.10) in Fourier space. In the flat-sky limit, we usually adopt the plane wave as a harmonic basis, and spin-0 quantity, \( A(\hat{n}) \), such as, \( \Theta, \phi \) and \( \varpi \), is described in Fourier space as [34]

\[ A_{\ell} = \int d^2 \hat{n} e^{-i\ell \cdot \hat{n}} A(\hat{n}), \]  

(3.32)

where the two-dimensional vector, \( \ell \), is given by \( (\ell \cos \varphi, \ell \sin \varphi) \). Similarly, in the flat-sky limit, \( E \) and \( B \) modes are related to the Stokes parameters \( Q(\hat{n}) \) and \( U(\hat{n}) \) as [34]

\[ E_{\ell} \pm iB_{\ell} = -\int d^2 \hat{n} e^{\pm 2i(\varphi_\ell \phi)} e^{-i\ell \cdot \hat{n}} [Q \pm iU](\hat{n}). \]  

(3.33)
Table 3. Functional forms of $\bar{f}_{\ell,L,L'}^{\phi,(X,Y)}$ and $\bar{f}_{\ell,L,L'}^{\psi,(X,Y)}$ in the flat-sky case.

| $XY$ | $\bar{f}_{\ell,L,L'}^{\phi,(X,Y)}$ | $\bar{f}_{\ell,L,L'}^{\psi,(X,Y)}$ |
|------|----------------------------------|----------------------------------|
| $\Theta$ | $C_{L}^{\Theta} \ell \cdot L + C_{L'}^{\Theta} \ell \cdot L'$ | $C_{L}^{\Theta} (\ast \ell) \cdot L + C_{L'}^{\Theta} (\ast \ell) \cdot L'$ |
| $\Theta E$ | $C_{L}^{\Theta E} \ell \cdot L \cos 2\varphi_{L,L'} + C_{L'}^{\Theta E} \ell \cdot L'$ | $C_{L}^{\Theta E} (\ast \ell) \cdot L \cos 2\varphi_{L,L'} + C_{L'}^{\Theta E} (\ast \ell) \cdot L'$ |
| $\Theta B$ | $C_{L}^{\Theta E} \ell \cdot L \sin 2\varphi_{L,L'}$ | $C_{L}^{\Theta E} (\ast \ell) \cdot L \sin 2\varphi_{L,L'}$ |
| $EE$ | $[\ell \cdot LC_{L}^{EE} + \ell \cdot L' C_{L'}^{EE}] \cos 2\varphi_{L,L'}$ | $[(\ast \ell) \cdot LC_{L}^{EE} + (\ast \ell) \cdot L' C_{L'}^{EE}] \cos 2\varphi_{L,L'}$ |
| $EB$ | $[\ell \cdot LC_{L}^{EE} - \ell \cdot L' C_{L'}^{BB}] \sin 2\varphi_{L,L'}$ | $[(\ast \ell) \cdot LC_{L}^{EE} - (\ast \ell) \cdot L' C_{L'}^{BB}] \sin 2\varphi_{L,L'}$ |
| $BB$ | $[\ell \cdot LC_{L}^{BB} + \ell \cdot L' C_{L'}^{BB}] \cos 2\varphi_{L,L'}$ | $[(\ast \ell) \cdot LC_{L}^{BB} + (\ast \ell) \cdot L' C_{L'}^{BB}] \cos 2\varphi_{L,L'}$ |

Eqs. (3.32) and (3.33) imply that the Fourier coefficients $Z_{\ell}$ ($= A_{\ell}$, $E_{\ell}$ and $B_{\ell}$) are related to the harmonic coefficients, $Z_{\ell,m}$ ($= A_{\ell,m}$, $E_{\ell,m}$ and $B_{\ell,m}$), through [34]:

$$Z_{\ell} = \sqrt{\frac{4\pi}{2\ell + 1}} \sum_{m} i^{-m} Z_{\ell,m} e^{im\varphi_{\ell}}, \quad (3.34)$$

$$Z_{\ell,m} = \sqrt{\frac{2\ell + 1}{4\pi}} i^{m} \frac{1}{2\pi} \int \frac{d\varphi_{\ell}}{2\pi} e^{-im\varphi_{\ell}} Z_{\ell}. \quad (3.35)$$

Using the above equations (3.34) and (3.35), and combining Eq. (3.27), the full-sky estimator (3.10) in Fourier space is re-expressed as

$$\hat{x}_{\ell}^{(a)} = \sum_{L,L'} LL' \int \frac{d\varphi_{L}}{2\pi} \int \frac{d\varphi_{L'}}{2\pi} \bar{T}_{\ell,L,L'} N_{\ell}^{x,(a)} g_{\ell,L,L'} \tilde{X}_{L} \tilde{Y}_{L'}, \quad (3.36)$$

where we define

$$\bar{T}_{\ell,L,L'} \equiv \left( \frac{(2\ell + 1)(2\ell' + 1)}{4\pi (2\ell + 1)(LL')} \right)^{1/2} \sum_{M, M'} (-1)^{m} \left( \begin{array}{c} \ell \\ -m \\ L \\ \ell' \\ M \\ M' \end{array} \right) \times e^{im\varphi_{\ell}} e^{-iM\varphi_{L}} e^{-iM'\varphi_{L'}} e^{-iM'\varphi_{L'}}. \quad (3.37)$$

To go further, we approximate the quantities, $\bar{T}_{\ell,L,L'} g_{\ell,L,L'}^{x,(a)}$ and $N_{\ell}^{x,(a)}$, taking the flat-sky limit. To do this, we use the following relation valid under the flat-sky approximation,
\[ \ell \gg 1 \quad [34]: \]
\[ e^{\pm i(\phi \ell - \varphi)} e^{i\ell \hat{n}} \simeq (\pm i)^s \sqrt{\frac{2\pi}{\ell}} \sum_{m} i^m \mathcal{Y}_{\ell,m}(\hat{n}) e^{-im\varphi} \quad (s = 0, 2). \quad (3.38) \]

We also note that the delta function is given by [34]:
\[ \delta_{\ell} = \int \frac{d^2 \hat{n}}{(2\pi)^2} e^{i\ell \hat{n}}. \quad (3.39) \]

Using Eqs.(3.39), (3.38), (2.4), (2.5), (2.10), and (2.11), we find that, under the flat-sky approximation,
\[ \mathcal{T}_{\ell,L,L'} \hat{g}_{\ell,L,L'}^{x,(a)} \simeq \delta_{L+L'-\ell} \left\{ \frac{\tilde{C}_{XY} \tilde{C}_{YY} (\tilde{F}_{\ell,L,L'})^{x,(a)} s - \tilde{C}_{LY} \tilde{C}_{XY} (\tilde{F}_{\ell,L,L'})^{x,(a)}}{\tilde{C}_{XX} \tilde{C}_{YY} \tilde{C}_{XY} \tilde{C}_{XYY} - (\tilde{C}_{XY} \tilde{C}_{YY})^2} \right\} \]
\[ \equiv \delta_{L+L'-\ell} \tilde{g}_{\ell,L,L'}^{x,(a)}, \quad (3.40) \]
\[ N_{\ell}^{x,(a)} \simeq \left\{ \int \frac{d^2 L}{(2\pi)^2} \int d^2 L' \delta_{L+L'-\ell} \mathcal{T}_{\ell,L,L'} \hat{g}_{\ell,L,L'}^{x,(a)} \right\}^{-1} \]
\[ \equiv \tilde{N}_{\ell}^{x,(a)}, \quad (3.41) \]

where the function, \( \mathcal{T}_{\ell,L,L'} \hat{g}_{\ell,L,L'}^{x,(a)} \), is given in Table.3 for each \( x \) and \( \alpha \). Note that the quantity, \( \tilde{N}_{\ell}^{x,(a)} \), is the flat-sky counterpart of the minimum variance, \( N_{\ell}^{x,(a)} \). The detailed derivation of Eqs.(3.40) and (3.41) is given in appendix C. Then, Eq.(3.36) is rewritten as
\[ \hat{x}_{\ell}^{(a)} = \int \frac{d^2 L}{(2\pi)^2} \int d^2 L' \delta_{L+L'-\ell} \mathcal{T}_{\ell,L,L'} \hat{g}_{\ell,L,L'}^{x,(a)} \tilde{X}_L \tilde{Y}_{L'} \quad (x = \phi, \varphi), \quad (3.42) \]

where we define the function, \( \mathcal{T}_{\ell,L,L'}^{x,(a)} \), as
\[ \mathcal{T}_{\ell,L,L'}^{x,(a)} = \tilde{N}_{\ell}^{x,(a)} \mathcal{T}_{\ell,L,L'}^{x,(a)}. \quad (3.43) \]

Eq.(3.42) is the flat-sky counterpart of Eq.(3.10), which exactly coincides with that empirically defined in Ref.[32]. Ref.[32] mentioned that their estimator does not satisfy the unbiased condition, and may detect some non-zero signals of \( \varphi \) even in the absence of the curl mode. But our result show that their estimator satisfies the condition, \( \langle \hat{x}_{\ell}^{(a)} \rangle_{\text{CMB}} = x_{\ell} \), and becomes zero when the pseudo-scalar lensing potential vanishes.\(^2\)

Before closing this section, we also give the expression for the optimal combination of the flat-sky estimator used in the next section. With Eq.(3.35), the optimal combination (3.29) is rewritten as
\[ \hat{x}_{\ell}^{(c)} = N_{\ell}^{x,(c)} \sum_{\alpha,\beta} \left\{ (N_{\ell}^{x})^{-1} \right\}^{\alpha,\beta} \hat{x}_{\ell}^{(a)}, \quad (3.44) \]

\(^2\)In fact, the integrand in Eq.(14) of Ref.[32] is an odd function in terms of the angle of \( \ell_1 \), and the right-hand side of Eq.(14) vanishes.
Table 4. Experimental specifications for the PLANCK and ACTPol used in this paper. The quantity $\theta_\nu$ is the beam size, and $\sigma_\nu$ represents the sensitivity of each channel to the temperature $\sigma_{\nu,T}$ or polarizations $\sigma_{\nu,P}$, depending on the power spectrum of temperature ($X = \Theta$) or polarizations ($X = E$ or $B$). The quantity $\nu$ means a channel frequency.

| Experiment | $f_{\text{sky}}$ [GHz] | $\nu$ [arcmin] | $\theta_\nu$ [arcmin] | $\sigma_{\nu,T}$ [$\mu$K/pixel] | $\sigma_{\nu,P}$ [$\mu$K/pixel] |
|------------|------------------------|----------------|-----------------------|-----------------------------|-----------------------------|
| PLANCK     | 1                      | 0.65           | 30                    | 33                          | 4.4                         |
|            |                        |                | 44                    | 23                          | 6.5                         |
|            |                        |                | 70                    | 14                          | 9.8                         |
|            |                        |                | 100                   | 9.5                         | 6.8                         |
|            |                        |                | 143                   | 7.1                         | 6.0                         |
|            |                        |                | 217                   | 5.0                         | 13.1                        |
|            |                        |                | 353                   | 5.0                         | 40.1                        |
| ACTPol     | 3                      | 0.1            | 148                   | 1.4                         | 3.6                         |

and the minimum variance is obtained from Eq. (3.30). Then, if we denote the flat-sky counterpart of $N_{\ell}^{x,(c)}$ and $N_{\ell}^{x}$ as $N_{\ell}^{x,(c)}$ and $N_{\ell}^{x}$, respectively, the optimal combination in the flat-sky limit is described by

$$
\hat{x}_\ell^{(c)} = N_{\ell}^{x,(c)} \sum_{\alpha,\beta} \left\{ (N_{\ell}^{x})^{-1} \right\}_{\alpha,\beta} \hat{x}_\ell^{(\alpha)},
$$

(3.45)

with the variance, $N_{\ell}^{x,(c)}$, given by

$$
\frac{1}{N_{\ell}^{x,(c)}} = \sum_{\beta,\beta'} \left\{ (N_{\ell}^{x})^{-1} \right\}_{\beta,\beta'}.
$$

(3.46)

The component of the matrix, $\{N_{\ell}^{x}\}_{\alpha,\beta}$, is obtained by computing the flat-sky counterpart of Eq. (3.31) and the result is

$$
N_{\ell}^{x,(\alpha,\beta)} = \int \frac{d^2L}{(2\pi)^2} \int d^2L' \delta_{L+L'} - \ell (F_{\ell,L,L'}^{x,(\alpha)})^* \\
\times \left( \mathcal{T}_{\ell,L,L'}^{(\beta)} C_{L}^{XX'} + \mathcal{T}_{\ell,L,L'}^{(\beta)} C_{L'}^{X'Y'} \right).
$$

(3.47)

The detailed calculation of Eq. (3.47) is given in appendix C.

4 Noise spectrum

In this section, as a first step to estimate the feasibility to detect the curl mode based on the quadratic estimator, we compute the noise spectrum of the full-sky estimator, in the following cases; ACTPol combined with PLANCK (ACTPol+PLANCK), and cosmic-variance limit (CV-limit). We also numerically evaluate the difference between the full- and flat-sky noise spectra.
Figure 1. The noise spectra of the scalar (left) and pseudo-scalar (right) lensing potentials for the lensing reconstruction from the temperature map alone, $\alpha = \Theta \Theta (T; \text{red lines})$, and from the temperature and polarization maps ($T+P$; green lines). We assume two cases; ACTPol combined with PLANCK (ACTPol+PLANCK; solid lines), and cosmic-variance limit (CV-limit; dashed lines). We compute the noise spectra according to Eqs. (3.26) for $T$ and Eq. (3.30) for $T+P$, with $\ell_{\text{max}} = 3000$, and take into account the effect of finite sky coverage.

Figure 2. The fractional difference of the noise spectrum between full- and flat-sky estimators (4.3), in the case with ACTPol+PLANCK. The left and right panels show the fractional difference for scalar and pseudo-scalar lensing potentials, respectively.
In the full-sky case, the noise spectrum is given by Eqs. (3.26) and (3.30). On the other hand, the noise in the flat-sky limit is obtained from Eqs. (3.41) and (3.46). To compute the noise spectrum, we assume that the lensing effect on the CMB comes only from the large-scale structure, and no source to produce the pseudo-scalar lensing potential is present. We use the modified version of FuturCMB code [35]. In computing noise spectra, we further need the lensed, unlensed and instrumental noise angular power spectra. The lensed and unlensed power spectra are computed by CAMB [36] with the fiducial value of cosmological parameters described in section 1. Owing to the assumption, the angular power spectrum for the pseudo-scalar lensing potential, $C_\ell^{\sigma\sigma}$, is set to zero. On the other hand, the angular power spectrum of the scalar lensing potential is obtained by integrating the matter power spectrum along the line-of-site, for which we adopt the fitting formula of the non-linear matter power spectrum given in Ref. [37]. The instrumental noise power spectra are given by

$$N_{\ell}^{XX} = \left[ \sum_\nu (N_{\ell,\nu}^{XX})^{-1} \right]^{-1}; \quad N_{\ell,\nu}^{XX} = \left( \frac{\sigma_\nu \theta_\nu}{T_{\text{CMB}}} \right)^2 \exp \left[ \frac{\ell (\ell + 1) \theta_\nu}{8 \ln 2} \right],$$  \hspace{1cm} (4.1)$$

with $T_{\text{CMB}} = 2.7K$ being mean temperature of CMB. Here, the quantity $\theta_\nu$ is the beam size, and $\sigma_\nu$ represents the sensitivity of each channel to the temperature $\sigma_\nu T$ or polarizations $\sigma_\nu P$. The specific values for PLANCK and ACTPol are summarized in Table 4. Note that, for ACTPol+PLANCK, we assume that the survey region of ACTPol is entirely overlapped with that of PLANCK, and plot the following noise spectrum

$$N_{\ell}^{x,(a)} = \left[ f_{\text{ACTPol}}^{\text{sky}} N_{\ell,\text{ACTPol}}^{x,(a)} f_{\text{PLANCK}}^{\text{sky}} + f_{\text{sky}}^{\text{ACTPol}} - f_{\text{sky}}^{\text{PLANCK}} \right]^{-1/2},$$  \hspace{1cm} (4.2)$$

where $f_{\text{ACTPol}}^{\text{sky}} = 0.1$ and $f_{\text{PLANCK}}^{\text{sky}} = 0.65$ are the fractional sky-coverage of ACTPol and PLANCK, respectively, and the label, $a$, represents a pair of temperature maps, “TT” or the optimal combination, “c”. In computing the noise of reconstruction within the survey region of ACTPol, $N_{\ell,\text{ACTPol}}^{x,(a)}$, we use the lensed angular power spectra from PLANCK instead of ACTPol, at $\ell < 700$, in order to remedy a large uncertainty at large angular scales arising from the atmospheric temperature fluctuations. On the other hand, the noise, $N_{\ell,\text{PLANCK}}^{x,(a)}$, is calculated with PLANCK experimental specification. For cosmic-variance limit, the reconstruction noise is computed with the instrumental noise being $N_{\ell}^{XX} = 0$.

In Fig. 1, we plot the expected noise spectrum in the full-sky case. The left and right panels show the noise spectra, $N_{\ell}^{\phi\phi,(a)}$ and $N_{\ell}^{\sigma\sigma,(a)}$, respectively. The resultant noise spectra for the pseudo-scalar lensing potential have amplitude comparable to those for the scalar lensing potential. In the cosmic-variance limit, the reconstruction noise is improved by more than an order of magnitude compared to the case with ACTPol+PLANCK. Notice that the noise spectra for the pseudo-scalar lensing potential is sensitive to the inclusion of polarizations compared to the estimator of the scalar lensing potential. In our calculation, the angular power spectrum of the pseudo-scalar lensing potential is set to zero. Although the primordial gravitational-waves or cosmic strings induces the pseudo-scalar lensing potential, as shown in the next section, the amplitude of $C_\ell^{\sigma\sigma}$ is at least two orders of magnitude smaller than that of $C_\ell^{\phi\phi}$, as long as we consider the model parameters consistent with observations. Thus, the inclusion of the pseudo-scalar lensing potential would hardly change the result in Fig. 1.

[3]http://lpsc.in2p3.fr/perotto/
Figure 3. The angular power spectrum of the pseudo-scalar lensing potential from primordial gravitational-waves with the tensor-to-scalar ratio, \( r = 0.1 \) (top left), and cosmic strings with \( G\mu = 10^{-8} \), \( P = 0.001 \) (top right). The error boxes in each figure show the expected variance of angular power spectrum from ACTPol combined with PLANCK (red) and cosmic-variance limit (green). The bottom two panels show the signal-to-noise ratio as a function of maximum multipole, for primordial gravitational-waves (bottom left) and cosmic strings with \( G\mu = 10^{-8} \), \( P = 0.001 \) (green).

In Fig. 2, to show the difference of the noise spectra between the flat- and full-sky estimators, we consider ACTPol+PLANCK and plot the following quantity as a function of \( \ell \):

\[
\Delta x_{\ell}^{(a)} \equiv \frac{N_{x_{\ell}}^{(a)} - N_{x_{\ell}}^{(a)}}{N_{x_{\ell}}^{(a)}}.
\] (4.3)

The fractional difference becomes significant at large scales, and the difference becomes \( \geq 10\% \) at \( \ell \lesssim 10 \) for both \( \phi \) and \( \varphi \). The results do not sensitively depend on whether we include the polarization data for the reconstruction or not. Note that the fractional difference for \( \phi \) is roughly consistent with Ref.[25]. These results show that the flat-sky estimator becomes invalid on large scales (\( \ell \lesssim 10 \)) and the full-sky lensing reconstruction is essential to extract information on primordial gravitational-waves and cosmic strings as discussed in the next section.

5 Implications for primordial gravitational-waves and cosmic strings

In this section, we illustrate the usefulness of the pseudo-scalar lensing potential as a diagnosis of the vector/tensor perturbations. Here we specifically focus on the pseudo-scalar lensing potential induced by two cases; primordial gravitational-waves produced during inflation, and cosmic strings (e.g., Refs.[39, 40]).

The gravitational waves can produce the metric perturbations which have odd parity symmetry. This means that the lensing effect induced by the gravitational waves cause the
curl mode of deflection angle. There are several studies on the curl mode of deflections (or the pseudo-scalar lensing potential) induced by the primordial gravitational-waves ([30, 32, 41, 42]), and the angular power spectrum of the pseudo-scalar lensing potential is given by [42]

\[ C_\ell^{\varpi \varpi} = \frac{8}{\pi \ell^2 (\ell + 1)^2} \left( \frac{\ell + 2}{\ell + 2} \right)^2 \int \frac{dk}{k} \left| \Delta_\ell^{GW}(k, \eta) \right|^2 , \]  

with

\[ \Delta_\ell^{GW}(k, \eta) = \int_{\eta_0}^{\eta} k d\eta \left( \frac{3j_1(k\eta)}{k\eta} \right) \left( \frac{j_\ell(x)}{x} \right) \bigg|_{x=k(\eta_0-\eta)} . \]  

The quantity, \( \eta_* \), denotes the conformal time at the last scattering surface, and the quantity \( r \) is the tensor-to-scalar ratio and we assume \( r = 0.1 \) which is close to the current upper bound [43].

On the other hand, extended objects such as cosmic strings produce not only the scalar perturbations but also the vector/tensor perturbations, and induce the pseudo-scalar lensing potential. For simplicity, we focus on the pseudo-scalar lensing potential only from vector perturbations. To compute the pseudo-scalar lensing potential, we consider a string network described by the velocity-dependent one-scale model [39, 40, 44–47], characterized by the intercommuting probability, \( P \). The loop formation process is required for the energy loss of cosmic strings and the scaling solution. Also, this process is related to the number density of strings. Note that the constraint on \( P \) has an important role to distinguish between the cosmic strings as the topological defect (\( P = 1 \)) and cosmic super-strings generated from the stringy inflation (\( P \ll 1 \)) [48, 49]. We assume that the string segments are distributed randomly between the last scattering surface and observer, consistently with the scaling model, or, in other word, we neglect the correlation between two different segments [40].

If the string tension, \( G\mu \), and intercommuting probability, \( P \), are given, the pseudo-scalar lensing potential from cosmic strings are computed as follows [29]:

\[ C_\ell^{\varpi \varpi} = \frac{4}{\ell(\ell + 1)} (16\pi G\mu)^2 \sqrt{\frac{2}{3\pi}} \frac{v^2}{1 - v^2} \int \frac{\Delta_\ell^{CS}(k, \eta)}{k} d\eta \Delta_\ell^{CS}(k, \eta)^2 , \]  

with [29]

\[ \Delta_\ell^{CS}(k, \eta) = \int_{\eta_*}^{\eta_0} k d\eta \left[ \frac{4\pi a^3 k}{H} \left( \frac{a}{k\xi} \right)^5 \right] \left[ \text{erf} \left( \frac{k\xi}{2\sqrt{6}a} \right) \right]^{1/2} \left( \frac{j_\ell(x)}{x} \right) \bigg|_{x=k(\eta_0-\eta)} . \]  

The quantity, \( H \), is the Hubble expansion rate, and the quantities, \( \xi \) and \( v \), are the correlation length and root-mean-square velocity, respectively, and determined from [39]

\[ 2v^2 \left( 1 + \frac{\pi}{2\sqrt{2}} \frac{2.33}{1 - 8v^6} \right) = 1 , \]  

\[ \xi = \frac{2\sqrt{2}}{\pi} \frac{1 - 8v^6}{2Hv(1 + 8v^6)} . \]  

The detailed description of model for the pseudo-scalar lensing potential from cosmic strings is described in the forthcoming paper [29].

In the left two panels of Fig.3, the angular power spectrum from primordial gravitational-waves and the signal-to-noise ratio are shown. The plotted errors are estimated from

\[ \Delta C_\ell = \frac{C_\ell^{\varpi \varpi} + N_\ell^{\varpi \varpi}^{(c)}}{\sqrt{(\ell + 1/2) f_{\text{sky}} \Delta \ell}} , \]  

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where $\Delta \ell$ is the size of multipole bin, and we set $\Delta \ell = (i + 1)^3 - i^3$ for $i$-th bin, just for illustration. For ACTPol combined with PLANCK, we evaluate the errors as

$$
\Delta C_\ell = \{(\Delta C_{\ell,\text{ACTPol}})^2 + (\Delta C_{\ell,\text{PLANCK}})^2\}^{1/2}, \tag{5.8}
$$

where the errors arising from ACTPol survey region, $\Delta C_{XY}^{\ell,\text{ACTPol}}$, are computed according to Eq. (5.7) with $f_{\text{sky}} = 0.1$ and $N_{\ell} = N_{\ell}^{\text{ACTPol}}$. Similarly, the errors from PLANCK survey area, $\Delta C_{XY}^{\ell,\text{PLANCK}}$, are obtained from Eq. (5.7) with $f_{\text{sky}} = 0.55$ and $N_{\ell} = N_{\ell}^{\text{PLANCK}}$. On the other hand, for the cosmic-variance limit, we compute the errors with $f_{\text{sky}} = 1.0$ and the instrumental noise power spectra being zero. The signal-to-noise ratio for angular power spectrum of the pseudo-scalar lensing potential is defined by

$$
\left( \frac{S}{N} \right)_{<\ell} = \left\{ \sum_{\ell'=2}^{\ell} \left( \frac{C_{\ell'}}{\Delta C_{\ell'}} \right)^2 \right\}^{1/2}. \tag{5.9}
$$

As is expected, it is hard to detect the signature of primordial gravitational-waves from lensing reconstruction. For ACTPol+PLANCK, the signal-to-noise ratio is less than 0.1. Even with CV-limit, the signal-to-noise ratio is $\sim 2$. For the tensor-to-scalar ratio below the current upper limit [43], the signal-to-noise ratio would be further worsened. This is true as long as we adopt the quadratic estimator. These conclusions are consistent with the one obtained in Ref. [32], where they discussed the detectability in the flat-sky limit. Note that, recently, Ref. [50] discussed the detectability of the primordial gravitational-waves using the odd-parity bipolar spherical harmonics. Although they consider the temperature alone, their estimator (Eqs. (12) and (13) of Ref. [50]) and the variance (Eqs. (14) and (15) of Ref. [50]) coincide with our estimator (3.10) and the variance (3.26), respectively. However, the estimated amplitude of the noise spectrum for curl mode, assuming the PLANCK experiment alone, is several orders of magnitude smaller than our estimate based on ACTPol+PLANCK (red lines plotted in Fig. 1). Then, they claimed that the PLANCK would detect the primordial gravitational-waves even from the curl mode if the signal is comparable to a currently detectable level, $r \sim 0.24$ [43]. This apparently contradicts Ref. [32] and our result. The discrepancy of the results may come from the experimental specifications of PLANCK adopted in Ref. [50] which may differ from ours, but the reason is unclear.

On the other hand, in the right panels of Fig. 3, we show the case of the cosmic strings, with $G\mu = 10^{-8}$ and $P = 0.001$ [51]. Note that these values are consistent with the constraint from the temperature angular power spectrum using the Gott-Kaiser-Stebbins (GKS) effect [52, 53] induced by the gravitational potential of a moving string [39]. Although the result depends on the parameters of comic strings, the signal-to-noise ratio is $\sim 3$ even for ACTPol+PLANCK case. In the cosmic variance limit, the signal-to-noise ratio becomes $\sim 30$. Note that, we have ignored the tensor metric perturbations from cosmic strings which also induce the pseudo-scalar lensing potential, and the inclusion of the contributions from the tensor perturbations would further increase the signal-to-noise ratio. The GKS effect observed via temperature map would have larger signal-to-noise ratio than the pseudo-scalar lensing potential [39]. However, the temperature anisotropies at small angular scales are usually dominated by the contributions from point sources and the Sunyaev-Zel’dovich (SZ) effect [54]. In this respect, the reconstruction of pseudo-scalar lensing potential is useful to check systematics and biases in the derived constraints on cosmic strings from GKS effect. Note finally that the information on large scales would be important for detecting the
pseudo-scalar lensing potential from cosmic strings, and the full-sky formalism for lensing reconstruction is indispensable.

6 Summary and discussion

In this paper, we presented a full-sky algorithm for reconstructing the lensing potential of scalar (gradient mode) and pseudo-scalar (curl mode) components. We defined the estimator as a quadratic combination of observed anisotropies, and introduced the weight function to reduce the noise contribution (see Eq.3.10). The resultant form of the weight function which minimizes the noise contribution is given by Eq.3.27 with Eqs.3.26, and 3.21. Thanks to the different parity symmetry between scalar and pseudo-scalar lensing potentials, the gradient and curl modes can be separately reconstructed. Note that the quantities used to reconstruct the lensing potentials are summarized in Table.1. In the flat-sky limit, we showed that the estimator reduces to the one empirically defined in Ref.32. We explicitly evaluated the noise spectra, and showed that the noise contribution for the pseudo-scalar lensing potential is comparable to that for the gradient mode. Further, prospects for reconstructing the curl mode is discussed, and signal-to-noise ratio for the pseudo-scalar lensing potential is computed, especially focusing on primordial gravitational-waves and cosmic strings.

In this paper, we specifically focused on the lensing reconstruction based on the quadratic estimator proposed in Ref.25. On the other hand, in Ref.26, for experiments sensitive to B-mode polarization, it would be possible to improve the precision of the lensing potential with the estimator based on the likelihood analysis. This may be also true for the pseudo-scalar lensing potential. Extending the estimator of Ref.26 to the full-sky case, the signal-to-noise ratio estimated in section 5 would be improved, and upcoming or next generation experiments may detect the pseudo-scalar lensing potential even from primordial gravitational-waves. This will be investigated in our future work.

Throughout this paper, we have assumed several idealizations, i.e., the higher-order terms of the deflection angle are negligible, and observed CMB maps are given on the full sky without foregrounds and the inhomogeneous noise. However, at small scales, the lowest-order approximation (e.g., ignoring the higher-order terms in Eq.2.2 for the temperature anisotropies) would not be valid for high resolution experiments 55. The higher-order terms of deflection angle produce additional contributions in the lensed anisotropies, and the estimated scalar and pseudo-scalar lensing potential include the contributions from the higher-order terms of deflection angle 56. Also, the masking effect 27, 57, 58, foreground contaminations from point sources and thermal/kinematic SZ effect 10, 55, and the inhomogeneous noise 59 induce the additional non-zero off-diagonal terms in Eq.3.1 and the estimated lensing potentials would be biased. Thus, in practical cases, to reduce the systematic bias in the estimated lensing potentials, an accurate treatment of these practical problems for the lensing reconstruction is required and worth investigating.

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A Useful formula

In this appendix, following Refs. [60] and [61], we summarize the formulas used in this paper.

A.1 Symmetry of Wigner-3j symbols

Symmetric properties of the Wigner-3j symbols are described by

\[
\begin{align*}
\binom{\ell_1 \ell_2 \ell_3}{m_1 m_2 m_3} & = \binom{\ell_2 \ell_3 \ell_1}{m_2 m_3 m_1} = \binom{\ell_3 \ell_1 \ell_2}{m_3 m_1 m_2} = (-1)^{\ell_1+\ell_2+\ell_3} \binom{\ell_3 \ell_2 \ell_1}{m_3 m_2 m_1}, \\
\binom{\ell_1 \ell_2 \ell_3}{m_1 m_2 m_3} & = (-1)^{\ell_1+\ell_2+\ell_3} \binom{\ell_1 \ell_2 \ell_3}{-m_1 -m_2 -m_3}.
\end{align*}
\]  

(A.1)

(A.2)

A.2 Summation of Wigner-3j symbols

Throughout the paper, we frequently use the following property of the Wigner-3j symbols:

\[
\sum_{M'} (-1)^{L+M} \binom{\ell}{-m M -M} = \delta_{\ell,0} \delta_{m,0} \sqrt{\frac{2L+1}{2\ell+1}},
\]  

(A.3)

\[
\sum_{M,M'} \binom{\ell}{-m M} \binom{\ell'}{m M'} = \frac{1}{2\ell+1} \delta_{\ell,\ell'} \delta_{m,m'}.
\]  

(A.4)

A.3 Relations between Wigner-3j symbols and spherical harmonics

Let us define

\[
\begin{align*}
\mathcal{X}^{m,m',M}_{\ell,\ell',L} & = \int d\hat{n} \, Y_{\ell,M}^* (\hat{n}) [\nabla_0 Y_{\ell,m}(\hat{n})] \cdot [\nabla_0 Y_{\ell',m'}(\hat{n})], \\
\mathcal{J}^{m,m',M}_{\ell,\ell',L} & = \int d\hat{n} \, Y_{\ell,M}^* (\hat{n}) [(\times \nabla)_0 Y_{\ell,m}(\hat{n})] \cdot [\nabla_0 Y_{\ell',m'}(\hat{n})].
\end{align*}
\]

(A.5)

(A.6)

These two integrals are related to the Wigner-3j symbols as

\[
\begin{align*}
\mathcal{X}^{m,m',M}_{\ell,\ell',L} & = \sqrt{\frac{(2L_1+1)(2L_2+1)(2L_3+1)}{16\pi}} \binom{\ell_1 \ell_2 \ell_3}{m_1 m_2 m_3} \binom{\ell_1 \ell_2 \ell_3}{s 0 -s} \\
& \quad \times \left[ (-\ell_1 + s)(\ell_1 - s + 1) + \ell_2(\ell_2 + 1) + (\ell_3 - s)(\ell_3 + s + 1) \right], \\
\mathcal{J}^{m,m',M}_{\ell,\ell',L} & = -i \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{16\pi}} \binom{\ell_1 \ell_2 \ell_3}{m_1 m_2 m_3} \\
& \quad \times \left[ \sqrt{\ell_2(\ell_2 + 1)(\ell_3 - s)(\ell_3 - s - 1)} \binom{\ell_1 \ell_2 \ell_3}{s -1 -s -1} \\
& \quad - \sqrt{\ell_2(\ell_2 + 1)(\ell_3 - s)(\ell_3 + s + 1)} \binom{\ell_1 \ell_2 \ell_3}{s 1 -s 1} \right].
\end{align*}
\]  

(A.7)

(A.8)

B Noise covariance

Here we derive Eqs. (3.16) and (3.31). Since Eq. (3.16) is obtained by setting \( \beta = \alpha \), in what follows, we derive the expression for the noise covariance given in Eq. (3.31).
B.1 Relation between noise and estimator covariance

First, we rewrite the noise variance using the variance of the estimator. Since $n_{\ell,m}^{x,(\alpha)} = \hat{x}_{\ell,m}^{(\alpha)} - x_{\ell,m}$, the noise variance is rewritten as

$$
\langle (n_{\ell,m}^{x,(\alpha)})^2 \rangle = \langle (\hat{x}_{\ell,m}^{(\alpha)})^2 \rangle - \langle (x_{\ell,m})^2 \rangle = \langle (\hat{x}_{\ell,m}^{(\alpha)})^2 \rangle - \langle (\hat{x}_{\ell,m}^{(\alpha)} x_{\ell,m}) \rangle + C_{\ell}^{xx}. \tag{B.1}
$$

Note here that

$$
\langle (x_{\ell,m})^2 \hat{X}_{L,M} \hat{Y}_{L',M'} \rangle = \sum_{x'} \sum_{\ell',m'} \sum_{L,M,M'} \sum_{\Theta,E,B} \left\{ \left( -1 \right)^M \left( \begin{array}{ccc} L' & \ell' & L'' \\ -M' & m' & M'' \end{array} \right) s_{L,\ell',L''}^{x,(Z)} \left( x_{\ell,m} \right)^x \hat{Z}_{L'',M''}^{x} \hat{X}_{L,M} \right. \\
\left. \left. + \left( -1 \right)^M \left( \begin{array}{ccc} L & \ell & L'' \\ -M & m & M'' \end{array} \right) s_{L,\ell,L''}^{x,(Z)} \left( x_{\ell,m} \right)^x \hat{Z}_{L'',M''}^{x} \hat{Y}_{L,M} \right\} , \tag{B.2}
$$

where we use Eqs. (2.3), (2.14) and (2.15), and introduce the coefficients, $s_{L,\ell,L'}^{x,(XY)}$, defined by

$$
s_{L,\ell,L'}^{x,(\Theta)} = \delta s_{L,\ell,L'}^x, \quad s_{L,\ell,L'}^{x,(EE)} = s_{L,\ell,L'}^{x,(BB)} = \delta s_{L,\ell,L'}^x, \quad s_{L,\ell,L'}^{x,(EB)} = -s_{L,\ell,L'}^{x,(BE)} = \delta s_{L,\ell,L'}^x, \tag{B.3}
$$

and $s_{L,\ell,L'}^{x,(XY)} = 0$ for the other combinations of $XY$. Assuming that the lensing potentials and primary CMB anisotropies are a random Gaussian field, and the correlations between the lensing potentials and primary CMB anisotropies are negligible, the above equation (B.2) reduces to

$$
\langle (x_{\ell,m})^x \hat{X}_{L,M} \hat{Y}_{L',M'} \rangle = \left( -1 \right)^m \left( \begin{array}{ccc} \ell & L & L' \\ -m & M & M' \end{array} \right) f_{\ell,\ell',L,L'}^{x,(XY)} C_{\ell}^{xx}. \tag{B.4}
$$

With Eq.(A.4), this leads to the following equation:

$$
\langle (x_{\ell,m})^x \hat{x}_{\ell,m}^{(\alpha)} \rangle = \sum_{L,L'} F_{\ell,\ell',L,L'}^{x,(\alpha)} \sum_{M,M'} \left( \begin{array}{ccc} \ell & L & L' \\ -m & M & M' \end{array} \right) f_{\ell,\ell',L,L'}^{x,(XY)} C_{\ell}^{xx} = C_{\ell}^{xx}. \tag{B.5}
$$

As a result, Eq.(B.1) becomes

$$
\langle (n_{\ell,m}^{x,(\alpha)})^2 \rangle = \langle (\hat{x}_{\ell,m}^{(\alpha)})^2 \rangle - \langle (\hat{x}_{\ell,m}^{(\alpha)} x_{\ell,m}) \rangle = C_{\ell}^{xx}. \tag{B.6}
$$

B.2 Estimator covariance

Next we compute the estimator covariance, $\langle (\hat{x}_{\ell,m}^{(\alpha)})^2 \hat{x}_{\ell,m}^{(\beta)} \rangle$. From Eq.(3.10), the covariance is given by

$$
\langle (\hat{x}_{\ell,m}^{(\alpha)})^2 \hat{x}_{\ell,m}^{(\beta)} \rangle = \sum_{L_1,L'_1} \sum_{L_2,L'_2} \sum_{M_1,M'_1} \sum_{M_2,M'_2} \left( \begin{array}{ccc} \ell & L_1 & L'_1 \\ -m & M_1 & M'_1 \end{array} \right) \left( \begin{array}{ccc} \ell & L_2 & L'_2 \\ -m & M_2 & M'_2 \end{array} \right) \times \left( F_{\ell,L_1,L'_1}^{x,(\alpha)} F_{\ell,L_2,L'_2}^{x,(\beta)} \hat{X}_{L_1}^{x} \hat{Y}_{L'_1}^{x} \hat{Z}_{L_2} \hat{W}_{L'_2} \right). \tag{B.7}
$$
To compute the right-hand side of Eq. (B.7), we need to evaluate the four-point correlation of the observed CMB anisotropies:
\[ \left\langle \tilde{X}_{L_1,M_1} \tilde{Y}^*_{L_1,M_1} \tilde{Z}_{L_2,M_2} \tilde{W}^*_{L_2,M_2} \right\rangle. \] (B.8)

In general, the four-point correlation (B.8) is described by
\[ \left\langle \tilde{X}_{L_1,M_1} \tilde{Y}^*_{L_1,M_1} \tilde{Z}_{L_2,M_2} \tilde{W}^*_{L_2,M_2} \right\rangle = \left\langle \tilde{X}_{L_1,M_1} \tilde{Y}^*_{L_1,M_1} \tilde{Z}_{L_2,M_2} \tilde{W}^*_{L_2,M_2} \right\rangle_G + \left\langle \tilde{X}_{L_1,M_1} \tilde{Y}^*_{L_1,M_1} \tilde{Z}_{L_2,M_2} \tilde{W}^*_{L_2,M_2} \right\rangle_C. \] (B.9)

The first term is defined by
\[ \left\langle \tilde{X}_{L_1,M_1} \tilde{Y}^*_{L_1,M_1} \tilde{Z}_{L_2,M_2} \tilde{W}^*_{L_2,M_2} \right\rangle_G = \tilde{c}_{L_1} \tilde{c}_{L_1}^{ZY} \Delta_{(1,1'),(2,2')} + \tilde{c}_{L_1} \tilde{c}_{L_1}^{XY} \Delta_{(1,2),(1',2')} + \tilde{c}_{L_1} \tilde{c}_{L_1}^{YW} \Delta_{(1,2'),(1',2')}, \] (B.10)

where the quantity \( \tilde{c} \) is the observed angular power spectrum, and \( \Delta_{(1,1'),(2,2')} \), \( \Delta_{(1,2),(1',2')} \) and \( \Delta_{(1,2'),(1',2')} \) denote
\[ \Delta_{(1,1'),(2,2')} \equiv \delta_{L_1,L_1'} \delta_{L_2,L_2'} \delta_{M_1,-M_1'} \delta_{M_2,-M_2'}, \] (B.11)
\[ \Delta_{(1,2),(1',2')} \equiv \delta_{L_1,L_1'} \delta_{L_2,L_2'} \delta_{M_1,M_1'} \delta_{M_2,M_2'}, \] (B.12)
\[ \Delta_{(1,2'),(1',2')} \equiv \delta_{L_1,L_1'} \delta_{L_2,L_2'} \delta_{M_1,M_1'} \delta_{M_2,M_2'}. \] (B.13)

The other terms included in the four-point correlation are represented by the second term in Eq. (B.9). If the quantities, \( \tilde{X}_{L_1,M_1}, \tilde{Y}^*_{L_1,M_1}, \tilde{Z}_{L_2,M_2} \) and \( \tilde{W}^*_{L_2,M_2} \), are random Gaussian fields, the second term vanishes. From Eq. (B.9), the covariance is decomposed into two parts:
\[ \left\langle \tilde{X}_{L_1,M_1} \tilde{Y}^*_{L_1,M_1} \tilde{Z}_{L_2,M_2} \tilde{W}^*_{L_2,M_2} \right\rangle = G_{L_1,M_1} + \Delta_{L_1,M_1}, \] (B.14)

where we define the Gaussian and connected parts, \( G_{L_1,M_1} \) and \( \Delta_{L_1,M_1} \), as
\[ G_{L_1,M_1} = \sum_{L_1',M_1'} \left\langle \tilde{X}_{L_1,M_1} \tilde{Y}^*_{L_1,M_1} \tilde{Z}_{L_2,M_2} \tilde{W}^*_{L_2,M_2} \right\rangle_G, \] (B.15)
\[ \Delta_{L_1,M_1} = \sum_{L_1',M_1'} \left\langle \tilde{X}_{L_1,M_1} \tilde{Y}^*_{L_1,M_1} \tilde{Z}_{L_2,M_2} \tilde{W}^*_{L_2,M_2} \right\rangle_C. \] (B.16)

Let us first compute the Gaussian part, \( G_{L_1,M_1} \). Substituting Eq. (B.10) into Eq. (B.15), the Gaussian part of the covariance is given by
\[ G_{L_1,M_1} = \sum_{L_1',M_1'} \left\langle \tilde{X}_{L_1,M_1} \tilde{Y}^*_{L_1,M_1} \tilde{Z}_{L_2,M_2} \tilde{W}^*_{L_2,M_2} \right\rangle_G \times \left( F_{L_1,L_1'}^{XY} \right)^* F_{L_2,L_2'}^{YW}, \] (B.17)
Using Eq. (A.3), the term proportional to $\Delta_{(1,1'),(2,2')}$ gives $\delta_{\ell,0}$, and we neglect this term to consider $\ell > 0$. Then, Eq. (B.17) becomes

$$
G_{\ell, m}^{x, (\alpha, \beta)} = \sum_{L_1, L_1', L_2, L_2', M_1, M_1', M_2, M_2'} \left( \frac{\ell}{-m} \right) \left( \begin{array}{cc} L_1 & L_1' \\ M_1 & M_1' \end{array} \right) \left( \begin{array}{cc} \ell & L_2 \\ -m & M_2 M_2' \end{array} \right) \\
\times (F_{\ell, L_1, L_1'})^{(\alpha)} (F_{\ell, L_2, L_2'})^{(\beta)} \left\{ \tilde{C}_{L_1}^{XY} \tilde{C}_{L_1'}^{YZ} \Delta_{(1,2),(1',2')} + \tilde{C}_{L_1}^{XY} \tilde{C}_{L_1'}^{YZ} \Delta_{(1,2'),(1',2)} \right\} \\
= \sum_{L_1, L_1'} \sum_{M_1, M_1'} \left( F_{\ell, L_1, L_1'}^{x, (\alpha)} \right)^{*} \\
\times \left\{ F_{\ell, L_1, L_1'}^{x, (\beta)} \tilde{C}_{L_1}^{XY} \tilde{C}_{L_1'}^{YZ} \left( \frac{\ell}{-m} \right) \left( \begin{array}{cc} L_1 & L_1' \\ -m & M_1 M_1' \end{array} \right) \left( \begin{array}{cc} \ell & L_1 \\ -m & M_1 M_1' \end{array} \right) \\
+ F_{\ell, L_1, L_1'}^{x, (\beta)} \tilde{C}_{L_1}^{XY} \tilde{C}_{L_1'}^{YZ} \left( \frac{\ell}{-m} \right) \left( \begin{array}{cc} L_1 & L_1' \\ -m & M_1 M_1' \end{array} \right) \left( \begin{array}{cc} \ell & L_1 \\ -m & M_1 M_1' \end{array} \right) \right\}. 
$$

(B.18)

Using Eq. (A.1), the above equation reduces to

$$
G_{\ell, m}^{x, (\alpha, \beta)} = \sum_{L_1, L_1'} \left( F_{\ell, L_1, L_1'}^{x, (\alpha)} \right)^{*} (F_{\ell, L_1, L_1'}^{x, (\beta)} \tilde{C}_{L_1}^{XY} \tilde{C}_{L_1'}^{YZ} + (-1)^{\ell+L_1+L_1'} F_{\ell, L_1, L_1'}^{x, (\beta)} \tilde{C}_{L_1}^{XY} \tilde{C}_{L_1'}^{YZ}) \\
\times \sum_{M_1, M_1'} \left( \frac{\ell}{-m} \right) \left( \begin{array}{cc} L_1 & L_1' \\ -m & M_1 M_1' \end{array} \right) \left( \begin{array}{cc} \ell & L_1 \\ -m & M_1 M_1' \end{array} \right) \\
= \frac{1}{2\ell + 1} \sum_{L_1, L_1'} \left( F_{\ell, L_1, L_1'}^{x, (\alpha)} \right)^{*} \\
\times \left( F_{\ell, L_1, L_1'}^{x, (\beta)} \tilde{C}_{L_1}^{XY} \tilde{C}_{L_1'}^{YZ} + F_{\ell, L_1, L_1'}^{x, (\beta)} (-1)^{\ell+L_1+L_1'} \tilde{C}_{L_1}^{XY} \tilde{C}_{L_1'}^{YZ} \right). 
$$

(B.19)

Note that we use (A.4) from the first to the last equation.

Next we compute the connected part of the covariance (B.16). Even if the primordial CMB temperature and polarization anisotropies are random Gaussian fields, the connected part of the four-point correlations is arising from the secondary effects such as the weak lensing. Assuming that the connected part is arising from the lensing effect, the connected part of the four-point correlations is given as

$$
\left\langle \tilde{X}_{L_1, M_1}^{*} \tilde{Y}_{L_1', M_1'}^{*} \tilde{Z}_{L_2, M_2} \tilde{W}_{L_2', M_2'}^{*} \right\rangle_C \\
\approx \sum_{\ell''} \sum_{m''} \left( \frac{\ell''}{-m''} \right) \left( \begin{array}{cc} L_1 & L_1' \\ -m'' & M_1 M_1' \end{array} \right) \left( \begin{array}{cc} \ell'' & L_2 \\ -m'' & M_2 M_2' \end{array} \right) \sum_{x} \left( f_{\ell, L_1, L_1'}^{x, (\alpha)} \right)^{*} f_{\ell, L_2, L_2'}^{x, (\beta)} \tilde{C}_{x,x'}. 
$$

(B.20)

Other terms included in the connected part of four-point correlation, such as the non-Gaussian terms introduced in Ref. [62], induce additional noise term in Eq. (3.31), but the terms may be an order of magnitude smaller than $G_{\ell}^{x, (\alpha, \beta)}$ [56, 62, 63]. Substituting Eq. (B.20)
Using Eq. (A.4), the above equation is rewritten as

\[ C_{\ell,m}^{x,(\alpha,\beta)} = \sum_{L_1, L_2} \sum_{L_1', L_2'} \sum_{m,M} \sum_{m',M'} \frac{\delta_{\ell,L_1} \delta_{\ell,L_2} \delta_{m,M} \delta_{m',M'}}{2\ell + 1} \left( \begin{array}{c} L_1 \\ m \\ M \\ \end{array} \right) \left( \begin{array}{c} L_2 \\ m' \\ M' \end{array} \right) \]

\[ \times \left( F_{\ell,L_1,L_1'}^{x,(\alpha)} \right)^* F_{\ell,L_2,L_2'}^{x,(\beta)} \sum_{\ell''} \left( \begin{array}{c} \ell'' \\ -m'' \\ M'' \end{array} \right) \left( \begin{array}{c} \ell'' \\ -m'' \\ M'' \end{array} \right) \]

\[ \times \sum_{x'} \left( f_{\ell',L_1,L_1'}^{x,(\alpha)} \right)^* f_{\ell',L_2,L_2'}^{x,(\beta)} C_{\ell',x'} \]  \hspace{1cm} (B.21)

With Eq. (A.4), the above equation is rewritten as

\[ C_{\ell,m}^{x,(\alpha,\beta)} = \sum_{L_1, L_2} \sum_{L_1', L_2'} \sum_{m,M} \sum_{m',M'} \frac{\delta_{\ell,L_1} \delta_{\ell,L_2} \delta_{m,M} \delta_{m',M'}}{2\ell + 1} \left( \begin{array}{c} L_1 \\ m \\ M \\ \end{array} \right) \left( \begin{array}{c} L_2 \\ m' \\ M' \end{array} \right) \]

\[ \times \left( F_{\ell,L_1,L_1'}^{x,(\alpha)} \right)^* F_{\ell,L_2,L_2'}^{x,(\beta)} \left( f_{\ell',L_1,L_1'}^{x,(\alpha)} \right)^* f_{\ell',L_2,L_2'}^{x,(\beta)} C_{\ell',x'} \]

\[ = \sum_{x'} \left( [F_{\ell}^{x}, f_{\ell}^{x}]^{(\alpha)} \right)^* [F_{\ell}^{x}, f_{\ell}^{x}]^{(\beta)} C_{\ell,x'} \]  \hspace{1cm} (B.22)

Using Eq. (3.14), we obtain

\[ C_{\ell,m}^{x,(\alpha,\beta)} = C^{xx}_{\ell,\ell'} \]  \hspace{1cm} (B.23)

Finally, substituting the resultant form of the Gaussian (B.19) and connected parts (B.23) into Eq. (B.6), we obtain Eq. (3.31).

C Flat-sky limit

In this appendix, using Eqs. (3.34)-(3.38), we derive Eqs. (3.40), (3.41) and the noise cross-spectrum in the flat-sky limit (3.47).

We first consider the expression for \( T_{\ell,L,L'} g_{\ell,L,L'}^{x,(\alpha)} \) (3.40) in the flat-sky limit. From Eq. (3.21), we obtain

\[ T_{\ell,L,L'} g_{\ell,L,L'}^{x,(\alpha)} = \frac{\tilde{C}_{L}^{XX} \tilde{C}_{L}^{YY} T_{\ell,L,L'} \left( f_{\ell,L,L'}^{x,(\alpha)} \right)^* - \tilde{C}_{L}^{XY} \tilde{C}_{L}^{XY} T_{\ell,L,L'} \left( f_{\ell,L,L'}^{x,(\alpha)} \right)^*}{\tilde{C}_{L}^{XX} \tilde{C}_{L}^{YY} \tilde{C}_{L}^{XY} \tilde{C}_{L}^{XY} - \left( \tilde{C}_{L}^{XY} \tilde{C}_{L}^{XY} \right)^2}. \]  \hspace{1cm} (C.1)

Note that, we use

\[ T_{\ell,L,L'} = (-1)^{\ell+L+L'} T_{\ell,L,L'}. \]  \hspace{1cm} (C.2)

Then, we need to compute the quantity, \( (T_{\ell,L,L'})^{*} f_{\ell,L,L'}^{x,(\alpha)} \). As shown in Table 2, this quantity includes \( (T_{\ell,L,L'})^{*} S_{\ell,L}^{x} \) and \( (T_{\ell,L,L'})^{*} S_{L,L'}^{x} \) where \( s = 0 \) or \( \pm 2 \). From Eqs. (2.4), (2.5), (2.10), and (2.11), we obtain

\[ (T_{\ell,L,L'})^{*} S_{\ell,L}^{x} = \left( \frac{2L+1}{2\ell+1} \right)^{1/2} \sum_{m,M,M'} e^{-i\phi_{\ell}\phi_{L}\phi_{M}\phi_{L'}(-1)^{m+M'}} \]

\[ \times e^{im-M-M'} \int d^{2}n_{x} Y_{L,-M}(n) Y_{L',M'}(n) \nabla_{0} Y_{L,-m}(n) \nabla_{x} Y_{L,M}(n). \]  \hspace{1cm} (C.3)
where, for arbitrary two vectors, \( \mathbf{a} \) and \( \mathbf{b} \), we define the products, \( \odot \) and \( \oplus \), as

\[
\mathbf{a} \odot \mathbf{b} \equiv \mathbf{a} \cdot \mathbf{b}, \quad \mathbf{a} \oplus \mathbf{b} \equiv (\ast \mathbf{a}) \cdot \mathbf{b} = -(\ast \mathbf{b}) \cdot \mathbf{a}.
\]  
(C.4)

Note here that, the right-hand side of Eqs.(2.4) and (2.10) is a real number, since the left-hand side of these equations is a real number. Similarly, the right-hand sides of Eqs.(2.5) and (2.11) is a purely imaginary number. Thus, Eq.(C.3) is rewritten as

\[
(T_{\ell, L, L'})^x s_{L', \ell, L} = \left( \frac{(2L + 1)(2L' + 1)}{4\pi(2\ell + 1)(LL')^2} \right)^{1/2} \sum_{m, M, M'} e^{-im\varphi_L} e^{iM\varphi_L'} (-1)^m + M' \times \left( \frac{\ell(2L + 1)(2L' + 1)}{2(2L + 1)(LL')} \right)^{1/2} \times \sum_{m, M, M'} \sqrt{\frac{2\pi}{\ell}} e^{-im\varphi_L} \sqrt{\frac{2\pi}{L}} e^{iM\varphi_L} \sqrt{\frac{2\pi}{L'}} e^{iM'\varphi_L'} \times \int \frac{d^2 \mathbf{n}}{(2\pi)^2} s_{L', \ell, L} \nabla s_{L', M}(\mathbf{n}) \odot [\nabla \mathbf{v}_{L', m}(\mathbf{n})].
\]  
(C.5)

Using Eq.(3.38) and assuming \( \ell, L, L' \gg 1 \), the above equation reduces to

\[
(T_{\ell, L, L'})^x s_{L', \ell, L} \simeq e^{i(\varphi_{L' - \varphi_L}) L} \odot \ell \int \frac{d^2 \mathbf{n}}{(2\pi)^2} e^{i(\ell - L - L') \mathbf{n}}.
\]  
(C.6)

From Eq.(3.39), we obtain

\[
(T_{\ell, L, L'})^x s_{L', \ell, L} \simeq \delta_{\ell + L' - \ell} L \odot x \ell \times \begin{cases} 
1 & (s = 0) \\
\cos 2\varphi_{L, L'} & (s = \oplus) \\
-\sin 2\varphi_{L, L'} & (s = \ominus)
\end{cases}.
\]  
(C.7)

where we denote \( \varphi_{L, L'} = \varphi_L - \varphi_{L'} \). Similarly, the flat-sky counterpart of \( (T_{\ell, L, L'})^x s_{L', \ell, L} \) is obtained by interchanging \( L \) and \( L' \) in Eq.(C.7) if \( \ell + L + L' \) is an even integer. If \( \ell + L + L' \) is an odd integer, we further multiply it by minus sign. From Eq.(C.7), we can define the following quantity:

\[
\overline{T}_{\ell, L, L'}^x(\alpha) = (T_{\ell, L, L'})^x s_{L', \ell, L'}.
\]  
(C.8)

The functional form of \( \overline{T}_{\ell, L, L'}^x(\alpha) \) is summarized in Table.3 for each \( x \) and \( \alpha \). Substituting Eq.(C.8) into Eq.(C.1), we obtain the following expression:

\[
\overline{T}_{\ell, L, L'}^x(\alpha) = \delta_{\ell + L' - \ell} \frac{\tilde{C}_{L'}^{XY}(\overline{T}_{\ell, L, L'})^x - \tilde{C}_{L}^{XY}(\overline{T}_{\ell, L, L'})^x}{\tilde{C}_{L}^{XY} \tilde{C}_{L'}^{XY} \tilde{C}_{L}^{XX} \tilde{C}_{L'}^{XX} - (\tilde{C}_{L}^{XY} \tilde{C}_{L'}^{XY})^2}.
\]  
(C.9)
with the quantity, \( \tilde{g}_{\ell,LL}^{x,(a)} \), being
\[
\tilde{g}_{\ell,LL}^{x,(a)} = \frac{\partial XX C_{L}^{YY} (\mathcal{F}_{\ell,LL}^{x,(a)})^{*} - \partial XX C_{L}^{YY} (\mathcal{F}_{\ell,LL}^{x,(a)})}{C_{L}^{XX} C_{L}^{YY} C_{L}^{YY} - (C_{L}^{XX} C_{L}^{YY})^2}.
\] (C.10)

Next, we consider the flat-sky counterpart of Eq.(3.31), and show Eqs.(3.41) and (3.47). Using Eq.(A.4), the covariance (3.31) is rewritten as
\[
N_{\ell}^{x,(a,\beta)} = \sum_{m,M,m',M',M''} \sum_{\ell,L,L'} \frac{(-1)^{m+m'}}{2\ell+1} \left( \frac{L}{M} \frac{L'}{M'} \right) \left( \frac{-m}{M} \frac{L}{M'} \right) \times \sum_{L,L'} \left( \int \frac{d\varphi_{\ell}}{2\pi} e^{-i(m-m')\varphi} \int \frac{d\varphi_{L,L'}}{2\pi} e^{-i(M-M')\varphi_{L,L'}} \times \left( F_{\ell,L,L'}^{x,(a)} \right)^* \left( F_{\ell,L,L'}^{x,(\beta)} \right) \right) \times \sum_{L,L'} \left( \int \frac{d\varphi_{\ell}}{2\pi} e^{-i(M-M')\varphi_{L,L'}} \times \left( F_{\ell,L,L'}^{x,(a)} \right)^* \left( F_{\ell,L,L'}^{x,(\beta)} \right) \right),
\] (C.11)
where, for arbitrary integers, \( M_1 \) and \( M_2 \), we use the following equation:
\[
\delta_{M_1,M_2} = \int \frac{d\varphi}{2\pi} e^{-i(M_1-M_2)\varphi}.
\] (C.12)

With the quantity defined in Eq.(3.37), the expression for the covariance (C.11) is rewritten as
\[
N_{\ell}^{x,(a,\beta)} = \int \frac{d\varphi_{\ell}}{2\pi} \sum_{L,L'} \left( \frac{(2L+1)(2L'+1)}{4\pi(L'L')^2} \right) \int \frac{d\varphi_{L,L'}}{2\pi} \left( \mathcal{T}_{L,L,L'}^{(a)} \right)^* \mathcal{T}_{L,L,L'}^{(a,\beta)} \times \left( F_{\ell,L,L'}^{x,(a)} \right)^* \left( F_{\ell,L,L'}^{x,(\beta)} \right) \left( \mathcal{T}_{L,L,L'}^{(a)} \right)^* \mathcal{T}_{L,L,L'}^{(a,\beta)} \left( F_{\ell,L,L'}^{x,(a)} \right)^* \left( F_{\ell,L,L'}^{x,(\beta)} \right),
\] (C.13)
Note here that, in the flat-sky limit, we can define the following quantity:
\[
\delta_{L+L'-\ell}^{x,(\beta)} \approx \mathcal{T}_{L,L,L'}^{(a,\beta)}.
\] (C.14)
This is because, from Eqs.(C.9), the right-hand side of the above equation is proportional to the delta function, \( \delta_{L+L'-\ell} \). Using Eq.(C.14), and \( \delta_0 = 1/\pi \), and assuming \( \ell, L, L' \gg 1 \), Eq.(C.13) becomes
\[
N_{\ell}^{x,(a,\beta)} \approx \int \frac{d\varphi_{\ell}}{2\pi} \sum_{L,L'} \left( \frac{(2L+1)(2L'+1)}{4\pi(L'L')^2} \right) \int \frac{d\varphi_{L,L'}}{2\pi} \delta_{L+L'-\ell} \times \left( F_{\ell,L,L'}^{x,(a)} \right)^* \left( \mathcal{T}_{L,L,L'}^{(a,\beta)} \right)^* \mathcal{T}_{L,L,L'}^{(a,\beta)} \left( F_{\ell,L,L'}^{x,(a)} \right)^* \left( F_{\ell,L,L'}^{x,(\beta)} \right).
\] (C.15)
Note that, in the right-hand side, we can choose two-dimensional coordinate system for the variables of integration, \( L \) and \( L' \), so that the integrand of \( \varphi_\ell \) does not depend on \( \varphi_\ell \). Then, the above equation reduces to

\[
\mathcal{N}^{(\alpha,\beta)}_{\ell} = \int \frac{d^2 L}{(2\pi)^2} \int d^2 L' \delta_{L+L'-\ell} \left( F^{(\alpha)}_{\ell,L,L'} \right)^* \left( F^{(\beta)}_{\ell,L,L'} \hat{C}_L^{XY} \hat{C}_{L'}^{XY} + F^{(\alpha)}_{\ell,L,L'} \hat{C}_L^{XX} \hat{C}_{L'}^{YY} \right). \tag{C.16}
\]

The noise in the flat-sky limit, \( \mathcal{N}^{(\alpha)}_{\ell} \), is obtained by \( \alpha = \beta \) in the above equation:

\[
\mathcal{N}^{(\alpha)}_{\ell} = \int \frac{d^2 L}{(2\pi)^2} \int d^2 L' \delta_{L+L'-\ell} \left( F^{(\alpha)}_{\ell,L,L'} \right)^* \left( \hat{C}_L^{XX} \hat{C}_{L'}^{YY} + \hat{C}_L^{XY} \hat{C}_{L'}^{XY} \right). \tag{C.17}
\]

Note that the quantity \( F^{(\alpha)}_{\ell,L,L'} \) is described by

\[
F^{(\alpha)}_{\ell,L,L'} = \mathcal{N}^{(\alpha)}_{\ell} \mathcal{F}^{(\alpha)}_{\ell,L,L'}. \tag{C.18}
\]

Substituting Eq. (C.18) into Eq. (C.17), we obtain the expression for the noise spectrum in the flat-sky limit:

\[
\mathcal{N}^{(\alpha)}_{\ell} = \left\{ \int \frac{d^2 L}{(2\pi)^2} \int d^2 L' \delta_{L+L'-\ell} \mathcal{F}^{(\alpha)}_{\ell,L,L'} \mathcal{F}^{(\alpha)}_{\ell,L,L'} \right\}^{-1}. \tag{C.19}
\]

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