Chaos in PDEs and Lax Pairs of Euler Equations

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Abstract. Recently, the author and collaborators have developed a systematic program for proving the existence of homoclinic orbits in partial differential equations. Two typical forms of homoclinic orbits thus obtained are: (1) transversal homoclinic orbits, (2) Silnikov homoclinic orbits. Around the transversal homoclinic orbits in infinite dimensional autonomous systems, the author was able to prove the existence of chaos through a shadowing lemma. Around the Silnikov homoclinic orbits, the author was able to prove the existence of chaos through a horseshoe construction.

Very recently, there has been a breakthrough by the author in finding Lax pairs for Euler equations of incompressible inviscid fluids. Further results have been obtained by the author and collaborators.

1. Introduction

Unlike chaos in finite dimensional systems, the area of chaos in partial differential equations has been a long standing open field. During the past ten years, there have been some remarkable developments in the area of chaos in partial differential equations by the author and collaborators. Our main results can be briefly described as follows: A systematic program is developed for proving the existence of homoclinic orbits for perturbed soliton equations. The program involves machineries from integrable theory, dynamical systems, and partial differential equations. The types of homoclinic orbits thus obtained can be either transversal or of Silnikov. With regard to transversal homoclinic orbits, shadowing lemmas are utilized or developed to prove the existence of chaos. With regard to Silnikov homoclinic orbits, Smale horseshoes are constructed to prove the existence of chaos. The main machineries from integrable theory are the isospectral theory and Darboux transformations. The main machineries from dynamical systems are persistence of invariant manifolds, Fenichel fibers, Melnikov measurement, and other measurements. The main machineries from partial differential equations are local well-posedness and regularity.

Another exciting development happened to Euler equations of incompressible inviscid fluids. Lax pairs for both 2D and 3D Euler equations was found by the

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author and Steve Childress. A Darboux transformation for the Lax pair of 2D Euler equation was also found by Artym Yurov and the author. The philosophical significance of the existence of Lax pairs for Euler equations is fundamentally important. If one defines integrability of an equation by the existence of a Lax pair, then both 2D and 3D Euler equations are integrable. More importantly, both 2D and 3D Navier-Stokes equations at high Reynolds numbers are near integrable systems. Such a point of view changes our old ideology on Euler and Navier-Stokes equations.

The most important application of the theory on chaos in partial differential equations in theoretical physics will be on the study of turbulence. Our approach of studying turbulence is different from many other studies in which one starts with Stokes equation to prove results on Navier-Stokes equations for small Reynolds number. In our studies, we start with Euler equations and view Navier-Stokes equations at large Reynolds number as singular perturbations of Euler equations. For this goal, we chose the 2D Navier-Stokes equations under periodic boundary conditions to begin a dynamical system study on 2D turbulence. We started with a simple fixed point of 2D Euler equation, and studied the linearized 2D Euler equation at the fixed point. A complete spectral theorem is obtained. In particular, we found unstable eigenvalues. Then, naturally we are interested in the corresponding unstable manifold. Such an unstable manifold forms the prototype of the attractor of Navier-Stokes equations at large Reynolds number. And long term turbulence lives around the attractor. Finding the unstable manifold is not successful yet. Partial success is accomplished in finding the corresponding unstable manifold for some Galerkin truncation. Here the unstable manifold is the 2D surface of an ellipsoid. Stable and unstable manifolds together form a lip-shape configuration. Whether or not the stable and unstable manifolds for 2D Euler equation have the same topology as above is not known. In fact, the existence of stable and unstable manifolds for 2D Euler equation has not been proved.

2. Homoclinic Orbits

In terms of proving the existence of a homoclinic orbit, the most common tool is the so-called Melnikov integral method. This method was subsequently developed by Holmes and Marsden, and most recently by Wiggins. For partial differential equations, this method was mainly developed by Li et al. There are two derivations for the Melnikov integrals. One is the so-called geometric argument. The other is the so-called Liapunov-Schmitt argument. The Liapunov-Schmitt argument is a fixed-point type argument which directly leads to the existence of a homoclinic orbit. The condition for the existence of a fixed point is the Melnikov integral. The geometric argument is a signed distance argument which applies to more general situations than the Liapunov-Schmitt argument. It turns out that the geometric argument is a much more powerful machinery than the Liapunov-Schmitt argument. In particular, the geometric argument can handle geometric singular perturbation problems. I shall also mention an interesting derivation in 4.

In establishing the existence of homoclinic orbits in high dimensions, one often needs other tools besides the Melnikov analysis. For example, when studying orbits homoclinic to fixed points created through resonances in (n \geq 4)-dimensional
near-integrable systems, one often needs tools like Fenichel fibers, as presented in previous chapter, to set up geometric measurements for locating such homoclinic orbits. Such homoclinic orbits often have a geometric singular perturbation nature. In such cases, the Liapunov-Schmitt argument can not be applied. For such works on finite dimensional systems, see for example [32] [33] [50]. For such works on infinite dimensional systems, see for example [48] [40].

2.1. Silnikov Homoclinic Orbits in Nonlinear Schrödinger (NLS) Equation Under Regular Perturbations. Consider the regularly perturbed nonlinear Schrödinger (NLS) equation [48],

\begin{equation}
iq_t = q_{xx} + 2|q|^2q + i\varepsilon[\hat{\partial}_x^2q - \alpha q + \beta],
\end{equation}

where \(q = q(t, x)\) is a complex-valued function of the two real variables \(t\) and \(x\), \(t\) represents time, and \(x\) represents space. \(q(t, x)\) is subject to periodic boundary condition of period \(2\pi\), and even constraint, i.e., \(q(t, x + 2\pi) = q(t, x)\), \(q(t, -x) = q(t, x)\).

\(\omega\) is a positive constant, \(\alpha > 0\) and \(\beta > 0\) are constants, \(\hat{\partial}_x^2\) is a bounded Fourier multiplier,

\begin{equation}
\hat{\partial}_x^2q = -\sum_{k=1}^{N} k^2 \xi_k \hat{q}_k \cos kx,
\end{equation}

\(\xi_k = 1\) when \(k \leq N\), \(\xi_k = 8k^{-2}\) when \(k > N\), for some fixed large \(N\), and \(\varepsilon > 0\) is the perturbation parameter. The following theorem was proved in [48].

**Theorem 2.1.** There exists a \(\varepsilon_0 > 0\), such that for any \(\varepsilon \in (0, \varepsilon_0)\), there exists a codimension 1 surface in the external parameter space \((\alpha, \beta, \omega) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+\) where \(\omega \in (\frac{1}{2}, 1)\), and \(\omega \alpha < \beta\). For any \((\alpha, \beta, \omega)\) on the codimension 1 surface, the regularly perturbed nonlinear Schrödinger equation (2.1) possesses a symmetric pair of Silnikov homoclinic orbits asymptotic to a saddle \(Q_\varepsilon\). The codimension 1 surface has the approximate representation given by \(\alpha = 1/\kappa(\omega)\), where \(\kappa(\omega)\) is plotted in Figure 1.

To prove the theorem, one starts from the invariant plane

\[\Pi = \{ q \mid \partial_x q = 0 \}.\]

On \(\Pi\), there is a saddle \(Q_\varepsilon = \sqrt{T}e^{i\theta}\) to which the symmetric pair of Silnikov homoclinic orbits will be asymptotic to, where

\begin{equation}
I = \omega^2 - \varepsilon \frac{1}{2\omega} \sqrt{\beta^2 - \alpha^2 \omega^2 + \cdots}, \quad \cos \theta = \frac{\alpha \sqrt{T}}{\beta}, \quad \theta \in (0, \frac{\pi}{2}).
\end{equation}

Its eigenvalues are

\begin{equation}
\lambda_n^\pm = -\varepsilon \alpha + \xi_n \omega^2 \pm 2\sqrt{(\frac{n^2}{2} + \omega^2 - I)(3I - \omega^2 - \frac{n^2}{2})},
\end{equation}

where \(n = 0, 1, 2, \ldots\), \(\omega \in (\frac{1}{2}, 1)\), \(\xi_n = 1\) when \(n \leq N\), \(\xi_n = 8n^{-2}\) when \(n > N\), for some fixed large \(N\), and \(I\) is given in (2.2). The crucial points to notice are: (1) only \(\lambda_0^+\) and \(\lambda_1^+\) have positive real parts, \(\text{Re}\{\lambda_0^+\} < \text{Re}\{\lambda_1^+\}\); (2) all the other
eigenvalues have negative real parts among which the absolute value of $\text{Re}\{\lambda^+_1\} = \text{Re}\{\lambda^-_1\}$ is the smallest; (3). $|\text{Re}\{\lambda^+_2\}| < \text{Re}\{\lambda^-_2\}$. Actually, items (2) and (3) are the main characteristics of Silnikov homoclinic orbits.

The unstable manifold $W^u(Q_\varepsilon)$ of $Q_\varepsilon$ has a fiber representation $48$. The Melnikov measurement measures the signed distance between $W^u(Q_\varepsilon)$ and the center-stable manifold $W^c_s$ $48$. By virtue of a Fenichel Fiber Theorem $48$, one can show that, to the leading order in $\varepsilon$, the signed distance is given by the Melnikov integral

$$M = \int_{-\infty}^{+\infty} \int_{0}^{2\pi} \left[ \partial_q F_1(q_0(t))(\partial^2_t q_0(t) - \alpha q_0(t) + \beta) ight. \\
+ \left. \partial_q F_1(q_0(t))(\partial^2_t q_0(t) - \alpha \bar{q}_0(t) + \beta) \right] dx dt,$$

where $q_0(t)$ is a homoclinic orbit for the integrable NLS, which can be generated through Darboux transformations $49$; and $\partial_q F_1$ and $\partial_q F_1$ are Melnikov vectors which can be generated through Floquet discriminant in the isospectral theory of the integrable NLS $49$. The zero of the signed distance implies the existence of an orbit in $W^u(Q_\varepsilon) \cap W^c_s$. The stable manifold $W^s(Q_\varepsilon)$ of $Q_\varepsilon$ is a codimension 1 submanifold in $W^c_s$. To locate a homoclinic orbit, one needs to set up a second measurement measuring the signed distance between the orbit in $W^u(Q_\varepsilon) \cap W^c_s$ and $W^s(Q_\varepsilon)$ inside $W^c_s$. To set up this signed distance, first one can rather easily track the (perturbed) orbit by an unperturbed orbit to an $O(\varepsilon)$ neighborhood of $\Pi$, then one needs to prove the size of $W^s(Q_\varepsilon)$ to be $O(\varepsilon^\nu)$ ($\nu < 1$) with normal form transform. To the leading order in $\varepsilon$, the zero of the second signed distance is given by

$$\beta \cos \gamma = \frac{\alpha \omega (\Delta \gamma)}{2 \sin \frac{\Delta \gamma}{2}},$$

where $\Delta \gamma = -4\theta_0$ and $\theta_0$ is a phase shift of $q_0(t)$. To the leading order in $\varepsilon$, the common zero of the two signed distances satisfies $\alpha = 1/\kappa(\omega)$, where $\kappa(\omega)$ is plotted in Figure $1$. Then the claim of the theorem is proved by virtue of the implicit function theorem. For rigorous details, see $48$.

### 2.2. Silnikov Homoclinic Orbits in NLS Under Singular Perturbations

Consider the singularly perturbed nonlinear Schrödinger equation $40$,

$$(2.4) \quad iq_t = q_{xx} + 2[|q|^2 - \omega^2]q + i\varepsilon[q_{xx} - \alpha q + \beta],$$

where $q = q(t,x)$ is a complex-valued function of the two real variables $t$ and $x$, $t$ represents time, and $x$ represents space. $q(t,x)$ is subject to periodic boundary condition of period $2\pi$, and even constraint, i.e.,

$$q(t, x + 2\pi) = q(t, x), \quad q(t, -x) = q(t, x).$$

$\omega \in (1/2, 1)$ is a positive constant, $\alpha > 0$ and $\beta > 0$ are constants, and $\varepsilon > 0$ is the perturbation parameter. The following theorem was proved in $40$.

**Theorem 2.2.** There exists a $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$, there exists a codimension 1 surface in the external parameter space $(\alpha, \beta, \omega) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ where $\omega \in \left(\frac{1}{2}, 1\right)/S$, $S$ is a finite subset, and $\omega \alpha < \beta$. For any $(\alpha, \beta, \omega)$ on the codimension 1 surface, the singularly perturbed nonlinear Schrödinger equation $(2.4)$ possesses a symmetric pair of Silnikov homoclinic orbits asymptotic to a saddle $Q_\varepsilon$. The codimension 1 surface has the approximate representation given by $\alpha = 1/\kappa(\omega)$, where $\kappa(\omega)$ is plotted in Figure $1$.\"
The proof of the theorem is also completed through two measurements: the Melnikov measurement and the second measurement. But more powerful machineries are needed \[40\].

2.3. Silnikov Homoclinic Orbits in Vector NLS Under Perturbations.

In recent years, novel results have been obtained on the solutions of the vector nonlinear Schrödinger equations \[1, 2, 83\]. Abundant ordinary integrable results have been carried through \[80, 21\], including linear stability calculations \[20\]. Specifically, the vector nonlinear Schrödinger equations can be written as

\[
\begin{align*}
ip_t + p_{xx} + \frac{1}{2}(|p|^2 + \chi|q|^2)p &= 0, \\
iq_t + q_{xx} + \frac{1}{2}(\chi|p|^2 + |q|^2)q &= 0,
\end{align*}
\]

where \(p\) and \(q\) are complex valued functions of the two real variables \(t\) and \(x\), and \(\chi\) is a positive constant. These equations describe the evolution of two orthogonal pulse envelopes in birefringent optical fibers \[62, 63\], with industrial applications in fiber communication systems \[25\] and all-optical switching devices \[28\]. For linearly birefringent fibers \[62\], \(\chi = 2/3\). For elliptically birefringent fibers, \(\chi\) can take other positive values \[63\]. When \(\chi = 1\), these equations are first shown to be integrable by S. Manakov \[59\], and thus called Manakov equations. When \(\chi\) is not 1 or 0, these equations are non-integrable. Propelled by the industrial applications, extensive mathematical studies on the vector nonlinear Schrödinger equations have been conducted. Like the scalar nonlinear Schrödinger equation, the vector nonlinear Schrödinger equations also possess figure eight structures in their phase space. Consider the singularly perturbed vector nonlinear Schrödinger equations,

\[
\begin{align*}
\ip_t + p_{xx} + \frac{1}{2}(|p|^2 + |q|^2) - \omega^2 p &= i\varepsilon[p_{xx} - \alpha p - \beta], \\
iq_t + q_{xx} + \frac{1}{2}(|p|^2 + |q|^2) - \omega^2 q &= i\varepsilon[q_{xx} - \alpha q - \beta],
\end{align*}
\]

where \(p(t, x)\) and \(q(t, x)\) are subject to periodic boundary condition of period \(2\pi\), and are even in \(x\), i.e.

\[
\begin{align*}
p(t, x + 2\pi) &= p(t, x), & p(t, -x) &= p(t, x), \\
q(t, x + 2\pi) &= q(t, x), & q(t, -x) &= q(t, x),
\end{align*}
\]

\(\omega \in (1, 2), \alpha > 0\) and \(\beta\) are real constants, and \(\varepsilon > 0\) is the perturbation parameter. We have

**Theorem 2.3** (\[44\]). There exists a \(\varepsilon_0 > 0\), such that for any \(\varepsilon \in (0, \varepsilon_0)\), there exists a codimension 1 surface in the space of \((\alpha, \beta, \omega) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+\) where \(\omega \in (1, 2)/S\), \(S\) is a finite subset, and \(\alpha \omega < \sqrt{2}\beta\). For any \((\alpha, \beta, \omega)\) on the codimension 1 surface, the singularly perturbed vector nonlinear Schrödinger equations \(\text{[2.5]-[2.6]}\) possesses a homoclinic orbit asymptotic to a saddle \(Q_\varepsilon\). This orbit is also the homoclinic orbit for the singularly perturbed scalar nonlinear Schrödinger equation studied in last section, and is the only one asymptotic to the saddle \(Q_\varepsilon\) for the singularly perturbed vector nonlinear Schrödinger equations \(\text{[2.3]-[2.6]}\). The codimension 1 surface has the approximate representation given by \(\alpha = 1/\kappa(\omega)\), where \(\kappa(\omega)\) is plotted in Figure 7.
2.4. Silnikov Homoclinic Orbits in Discrete NLS Under Perturbations. Consider the following perturbed discrete cubic nonlinear Schrödinger equations \[ iq_n = \frac{1}{h^2} \left[ q_{n+1} - 2q_n + q_{n-1} \right] + |q_n|^2(q_{n+1} + q_{n-1}) - 2\omega^2 q_n + i\varepsilon \left[ -\alpha q_n + \frac{1}{h^2}(q_{n+1} - 2q_n + q_{n-1}) + \beta \right], \] where \( i = \sqrt{-1} \), \( q_n \)'s are complex variables, \( q_{n+N} = q_n \), (periodic condition); and \( q_{-n} = q_n \), (even condition); \( h = \frac{1}{N} \), and
\[
N \tan \frac{\pi}{N} < \omega < N \tan \frac{2\pi}{N}, \quad \text{for } N > 3,
3 \tan \frac{\pi}{3} < \omega < \infty, \quad \text{for } N = 3.
\]
\[ \varepsilon \in (0, \varepsilon_1), \quad \alpha ( > 0), \quad \beta ( > 0) \] are constants.

This is a \( 2(M + 1) \) dimensional system, where
\[
M = N/2, \quad (N \text{ even}); \quad \text{and } M = (N-1)/2, \quad (N \text{ odd}).
\]

This system is a finite-difference discretization of the perturbed NLS \[ 2.4. \] The following theorem was proved in \[ 50. \]

Denote by \( \Sigma_N \) (\( N \geq 7 \)) the external parameter space,
\[
\Sigma_N = \left\{ (\omega, \alpha, \beta) \mid \omega \in \left( \tan \frac{\pi}{N}, \tan \frac{2\pi}{N} \right), \alpha \in (0, \alpha_0), \beta \in (0, \beta_0); \right\}
\]
where \( \alpha_0 \) and \( \beta_0 \) are any fixed positive numbers.

**Theorem 2.4.** For any \( N \) \((7 \leq N < \infty)\), there exists a positive number \( \varepsilon_0 \), such that for any \( \varepsilon \in (0, \varepsilon_0) \), there exists a codimension 1 surface \( E_\varepsilon \) in \( \Sigma_N \); for any external parameters \((\omega, \alpha, \beta)\) on \( E_\varepsilon \), there exists a homoclinic orbit asymptotic to a saddle \( Q_\varepsilon \). The codimension 1 surface \( E_\varepsilon \) has the approximate expression \( \alpha = 1/\kappa \), where \( \kappa = \kappa(\omega; N) \) is shown in Fig.2.

In the cases \((3 \leq N \leq 6)\), \( \kappa \) is always negative. For \( N \geq 7 \), \( \kappa \) can be positive as shown in Fig.2. When \( N \) is even and \( \geq 7 \), there is in fact a symmetric pair of homoclinic orbits asymptotic to a fixed point \( Q_\varepsilon \) at the same values of the external parameters; since for even \( N \), we have the symmetry: if \( q_n = f(n, t) \) solves \[ 2.4 \], then \( q_n = f(n + N/2, t) \) also solves \[ 2.7 \]. When \( N \) is odd and \( \geq 7 \), the study cannot guarantee that two homoclinic orbits exist at the same value of the external parameters.
2.5. Comments on DSII Under Perturbations. Consider the perturbed Davey-Stewartson II equations

\[ iq_t = \Upsilon q + \left[ 2(|q|^2 - \omega^2) + u_y \right] q + i\epsilon f , \]
\[ \Delta u = -4\partial_x q^2 , \]

where \( q \) is a complex-valued function of the three variables \((t, x, y)\), \( u \) is a real-valued function of the three variables \((t, x, y)\), \( \Upsilon = \partial_{xx} - \partial_{yy} \), \( \Delta = \partial_{xx} + \partial_{yy} \), \( \omega > 0 \) is a constant. We also consider periodic boundary conditions. The perturbation \( f \) can be for example a singular perturbation

\[ f = \Delta q - \alpha q + \beta , \]

where \( \alpha > 0, \beta > 0 \) are constants, or a regular perturbation by mollifying \( \Delta \) into a bounded Fourier multiplier

\[ \hat{\Delta} q = -\sum_{k \in \mathbb{Z}^2} \beta_k |k|^2 \hat{q}_k \cos k_1 x \cos k_2 y , \]

in the case of periods \((2\pi, 2\pi)\),

\[ \beta_k = 1, \quad |k| \leq N, \quad \beta_k = |k|^{-2}, \quad |k| > N, \]

for some large \( N \), \( |k|^2 = k_1^2 + k_2^2 \).

Under both regular and singular perturbations, the rigorous Melnikov measurement can be established [42]. It turns out that only local well-posedness is necessary for rigorously setting up the Melnikov measurement, thanks to the fact that the unperturbed homoclinic orbits are classical solutions. The obstacle toward proving the existence of homoclinic orbits comes from a technical difficulty in solving a linear system to get the normal form for proving the size estimate of the stable manifold of a saddle. For details, see [42].

2.6. Transversal Homoclinic Orbits in a Periodically Perturbed Sine-Gordon Equation. Transversal homoclinic orbits in continuous systems often appear in two types of systems: (1) periodic systems where the Poincaré period map has a transversal homoclinic orbit, (2) autonomous systems where the homoclinic orbit is asymptotic to a hyperbolic limit cycle.

Consider the periodically perturbed sine-Gordon (SG) equation,

\[ u_{tt} = c^2 u_{xx} + \sin u + \epsilon [-au_t + u^3 \chi(||u||) \cos t], \]

where

\[ \chi(||u||) = \begin{cases} 1, & ||u|| \leq M, \\ 0, & ||u|| \geq 2M, \end{cases} \]

for \( M < ||u|| < 2M, \chi(||u||) \) is a smooth bump function, under odd periodic boundary condition,

\[ u(x + 2\pi, t) = u(x, t), \quad u(x, t) = -u(x, t), \]

\[ \frac{1}{4} < c^2 < 1, \quad a > 0, \quad \epsilon \] is a small perturbation parameter.

**Theorem 2.5** [48, 69]. There exists an interval \( I \subset \mathbb{R}^+ \) such that for any \( a \in I \), there exists a transversal homoclinic orbit \( u = \xi(x, t) \) asymptotic to \( 0 \) in \( H^1 \).
2.7. Transversal Homoclinic Orbits in a Derivative NLS. Consider the derivative nonlinear Schrödinger equation,

\[ iq_t = q_{xx} + 2|q|^2q + i\varepsilon \left[ \frac{9}{16} - |q|^2 \right] q + \mu |\hat{\partial}_x q|^2\hat{q}, \]

where \(q\) is a complex-valued function of two real variables \(t\) and \(x\), \(\varepsilon > 0\) is the perturbation parameter, \(\mu\) is a real constant, and \(\hat{\partial}_x\) is a bounded Fourier multiplier,

\[
\hat{\partial}_x q = -\sum_{k=1}^{K} k\tilde{q}_k \sin kx, \quad \text{for} \quad q = \sum_{k=0}^{\infty} \tilde{q}_k \cos kx,
\]

and some fixed \(K\). Periodic boundary condition and even constraint are imposed,

\[ q(t, x + 2\pi) = q(t, x), \quad q(t, -x) = q(t, x). \]

**Theorem 2.6 (41).** There exists a \(\varepsilon_0 > 0\), such that for any \(\varepsilon \in (0, \varepsilon_0)\), and \(|\mu| > 5.8\), there exist two transversal homoclinic orbits asymptotic to the limit cycle \(q_c = \frac{3}{4} \exp\{-i(\frac{9}{8}t + \gamma)\}\).

3. Existence of Chaos

The importance of homoclinic orbits with respect to chaotic dynamics was first realized by Poincaré [68]. In 1961, Smale constructed the well-known horseshoe in the neighborhood of a transversal homoclinic orbit [74] [75] [76]. In particular, Smale’s theorem implies Birkhoff’s theorem on the existence of a sequence of structurely stable periodic orbits in the neighborhood of a transversal homoclinic orbit [7]. In 1984 and 1988 [65] [66], Palmer gave a beautiful proof of Smale’s theorem using a shadowing lemma. Later, this proof was generalized to infinite dimensions by Steinlein and Walther [77] [78] and Henry [26]. In 1967, Silnikov proved Smale’s theorem for autonomous systems in finite dimensions using a fixed point argument [72]. In 1996, Palmer proved Smale’s theorem for autonomous systems in finite dimensions using shadowing lemma [15] [67]. In 2002, Li proved Smale’s theorem for autonomous systems in infinite dimensions using shadowing lemma [41].

For nontransversal homoclinic orbits, the most well-known type which leads to the existence of Smale horseshoes is the so-called Silnikov homoclinic orbit [70] [71] [73] [17] [18]. Existence of Silnikov homoclinic orbits and new constructions of Smale horseshoes for concrete nonlinear wave systems have been established in finite dimensions [50] [52] and in infinite dimensions [48] [35] [40].

3.1. Shift Automorphism. Let \(W\) be a set which consists of elements of the doubly infinite sequence form:

\[ a = (\cdots a_{-2}a_{-1}a_0, a_1a_2 \cdots), \]

where \(a_k \in \{l_1, \cdots, l_m\}\), in labels, and \(k \in \mathbb{Z}\). We introduce a topology in \(W\) by taking as neighborhood basis of

\[ a^* = (\cdots a_{-2}^*a_{-1}^*a_0^*, a_1^*a_2^* \cdots), \]

the set

\[ W_\epsilon = \left\{ a \in W \mid a_k = a_k^* (|k| < \epsilon) \right\} \]
for \( j = 1, 2, \ldots \). This makes \( \mathcal{W} \) a topological space. The shift automorphism \( \chi \) is defined on \( \mathcal{W} \) by

\[
\chi : \mathcal{W} \mapsto \mathcal{W},
\]

\[
\forall a \in \mathcal{W}, \quad \chi(a) = b, \quad \text{where} \quad b_k = a_{k+1}.
\]

The shift automorphism \( \chi \) exhibits \textit{sensitive dependence on initial conditions}, which is a hallmark of \textit{chaos}.

\textbf{3.2. NLS Under Regular Perturbations}. We continue from section 2.1 and consider the regularly perturbed nonlinear Schrödinger (NLS) equation \((2.1)\). Starting from the Homoclinic Orbit Theorem 2.1, we have the following theorem on the existence of chaos.

\textbf{Theorem 3.1 (Chaos Theorem, \cite{35}).} Under certain generic assumptions for the perturbed nonlinear Schrödinger system \((2.1)\), there exists a compact Cantor subset \( \Lambda \) of \( H^1 \) (the Sobolev space), \( \Lambda \) consists of points, and is invariant under a Poincaré map \( P \). \( P \) restricted to \( \Lambda \), is topologically conjugate to the shift automorphism \( \chi \) on four symbols \( 1, 2, -1, -2 \). That is, there exists a homeomorphism

\[
\phi : \mathcal{W} \mapsto \Lambda,
\]

such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{W} & \xrightarrow{\phi} & \Lambda \\
\downarrow{\chi} & & \downarrow{P} \\
\mathcal{W} & \xrightarrow{\phi} & \Lambda
\end{array}
\]

Although the symmetric pair of Silnikov homoclinic orbits is not structurally stable, the Smale horseshoes are structurally stable. Thus, the Cantor sets and the conjugacy to the shift automorphism are structurally stable.

\textbf{3.3. NLS Under Singular Perturbations}. We continue from section 2.2 and consider the singularly perturbed nonlinear Schrödinger (NLS) equation \((2.4)\). Starting from the Homoclinic Orbit Theorem 2.2, we have

\textbf{Theorem 3.2 (Chaos Theorem, \cite{43}).} Under certain generic assumptions for the perturbed nonlinear Schrödinger system \((2.4)\), there exists a compact Cantor subset \( \Lambda \) of \( H^1 \), \( \Lambda \) consists of points, and is invariant under a Poincaré map \( P \). \( P \) restricted to \( \Lambda \), is topologically conjugate to the shift automorphism \( \chi \) on four symbols \( 1, 2, -1, -2 \). That is, there exists a homeomorphism

\[
\phi : \mathcal{W} \mapsto \Lambda,
\]

such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{W} & \xrightarrow{\phi} & \Lambda \\
\downarrow{\chi} & & \downarrow{P} \\
\mathcal{W} & \xrightarrow{\phi} & \Lambda
\end{array}
\]
3.4. Discrete NLS Under Perturbations. We continue from section 2.4. Starting from the Homoclinic Orbit Theorem 2.4, we have the theorem on the existence of chaos for the perturbed discrete nonlinear Schrödinger equation 2.7.

**Theorem 3.3 (Chaos Theorem).** Under certain generic assumptions for the perturbed discrete nonlinear Schrödinger system 2.7, there exists a compact Cantor subset \( \Lambda \) of \( \mathbb{R}^{2(M+1)} \), \( \Lambda \) consists of points, and is invariant under a Poincaré map \( P \). \( P \) restricted to \( \Lambda \), is topologically conjugate to the shift automorphism \( \chi \) on two symbols 0, 1. That is, there exists a homeomorphism

\[
\phi : \mathcal{W} \rightarrow \Lambda,
\]

such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{W} & \xrightarrow{\phi} & \Lambda \\
\chi \downarrow & & \downarrow P \\
\mathcal{W} & \xrightarrow{\phi} & \Lambda
\end{array}
\]

3.5. Numerical Simulation of Chaos. The finite-difference discretization of both the regularly and the singularly perturbed nonlinear Schrödinger equations 2.1 and 2.4 leads to the same discrete perturbed nonlinear Schrödinger equation 2.7.

In the chaotic regime, typical numerical output is shown in Figure 3. Notice that there are two typical profiles at a fixed time: (1). a breather type profile with its hump located at the center of the spatial period interval, (2). a breather type profile with its hump located at the boundary (wing) of the spatial period interval. These two types of profiles are half spatial period translate of each other. If we label the profiles, with their humps at the center of the spatial period interval, by “C”; those profiles, with their humps at the wing of the spatial period interval, by “W”, then

\[
\text{“W”} = \sigma \circ \text{“C”},
\]

where \( \sigma \) is the symmetry group element representing half spatial period translate. The time series of the output in Figure 3 is a chaotic jumping between “C” and “W”, which we call “chaotic center-wing jumping”. We interpret the chaotic center-wing jumping as the numerical realization of the shift automorphism \( \chi \) on symbols. We can make this more precise in terms of the phase space geometry. The figure-eight structure of the integrable NLS projected onto the plane of the Fourier component \( \cos x \) is illustrated in Figure 4 and labeled by \( L_C \) and \( L_W \). From the symmetry, we know that

\[
L_W = \sigma \circ L_C.
\]

\( L_C \) has the spatial-temporal profile realization as in Figure 4 with the hump located at the center. \( L_W \) corresponds to the half spatial period translate of the spatial-temporal profile realization as in Figure 4 with the hump located at the boundary (wing). An orbit inside \( L_C \), \( L_{Cin} \) has a spatial-temporal profile realization as in Figure 6. The half period translate of \( L_{Cin} \), \( L_{Win} \) is inside \( L_W \). An orbit outside \( L_C \) and \( L_W \), \( L_{out} \) has the spatial-temporal profile realization as in Figure 7. \( S_l \) and \( S_{l,\sigma} \) are two phase blocks. From these figures, we can see clearly that the chaotic center-wing jumping (Figure 3) is the realization of the shift automorphism on symbols in \( S_l \cup S_{l,\sigma} \).
3.6. Shadowing Lemma and Chaos in Finite-D Periodic Systems

Since its invention [3], shadowing lemma has been a useful tool for solving many dynamical system problems. Here we only focus upon its use in proving the existence of chaos in a neighborhood of a transversal homoclinic orbit. According to the type of the system, the level of difficulty in proving the existence of chaos with a shadowing lemma is different.

A finite-dimensional periodic system can be written in the general form
\[ \dot{x} = F(x, t) , \]
where \( x \in \mathbb{R}^n \), and \( F(x, t) \) is periodic in \( t \). Let \( f \) be the Poincaré period map.

**Definition 3.4.** A doubly infinite sequence \( \{ y_j \} \) in \( \mathbb{R}^n \) is a \( \delta \) pseudo-orbit of a \( C^1 \) diffeomorphism \( f : \mathbb{R}^n \to \mathbb{R}^n \) if for all integers \( j \)
\[ |y_{j+1} - f(y_j)| \leq \delta . \]
An orbit \( \{ f^j(x) \} \) is said to \( \varepsilon \)-shadow the \( \delta \) pseudo-orbit \( \{ y_j \} \) if for all integers \( j \)
\[ |f^j(x) - y_j| \leq \varepsilon . \]

**Definition 3.5.** A compact invariant set \( S \) is hyperbolic if there are positive constants \( K, \alpha \) and a projection matrix valued function \( P(x) \), \( x \in S \), of constant rank such that for all \( x \) in \( S \)
\[ P(f(x))DF(x) = DF(x)P(x) , \]
\[ |DF^j(x)P(x)| \leq Ke^{-\alpha j} , \quad (j \geq 0) , \]
\[ |DF^j(x)(I - P(x))| \leq Ke^{\alpha j} , \quad (j \leq 0) . \]

**Theorem 3.6 (Shadowing Lemma [66]).** Let \( S \) be a compact hyperbolic set for the \( C^1 \) diffeomorphism \( f : \mathbb{R}^n \to \mathbb{R}^n \). Then given \( \varepsilon > 0 \) sufficiently small there exists \( \delta > 0 \) such that every \( \delta \) pseudo-orbit in \( S \) has a unique \( \varepsilon \)-shadowing orbit.

The proof of this theorem by Palmer [66] is overall a fixed point argument with the help of Green functions for linear maps.

Let \( y_0 \) be a transversal homoclinic point asymptotic to a saddle \( x_0 \) of a \( C^1 \) diffeomorphism \( f : \mathbb{R}^n \to \mathbb{R}^n \). Then the set
\[ S = \{ x_0 \} \cup \{ f^j(y_0) : j \in \mathbb{Z} \} \]
is hyperbolic. Denote by $A_0$ and $A_1$ the two orbit segments of length $2m + 1$

$$A_0 = \{x_0, x_0, \ldots, x_0\}, \quad A_1 = \{f^{-m}(y_0), f^{-m+1}(y_0), \ldots, f^{m-1}(y_0), f^m(y_0)\}.$$  

Let

$$a = (\cdots, a_{-1}, a_0, a_1, \cdots),$$

where $a_j \in \{0, 1\}$, be any doubly infinite binary sequence. Let $A$ be the doubly infinite sequence of points in $S$, associated with $a$

$$A = \{\cdots, A_{a_{-1}}, A_{a_0}, A_{a_1}, \cdots\}.$$

When $m$ is sufficiently large, $A$ is a $\delta$ pseudo-orbit in $S$. By the shadowing lemma (Theorem 3.6), there is a unique $\varepsilon$-shadowing orbit that shadows $A$. In this manner, Palmer \[66\] gave a beautiful proof of Smale’s horseshoe theorem.

DEFINITION 3.7. Denote by $\Sigma$ the set of doubly infinite binary sequences

$$a = (\cdots, a_{-1}, a_0, a_1, \cdots),$$

where $a_j \in \{0, 1\}$. We give the set $\{0, 1\}$ the discrete topology and $\Sigma$ the product topology. The Bernoulli shift $\chi$ is defined by

$$[\chi(a)]_j = a_{j+1}.$$

THEOREM 3.8. Let $y_0$ be a transversal homoclinic point asymptotic to a saddle $x_0$ of a $C^1$ diffeomorphism $f : \mathbb{R}^n \mapsto \mathbb{R}^n$. Then there is a homeomorphism $\phi$ of $\Sigma$ onto a compact subset of $\mathbb{R}^n$ which is invariant under $f$ and such that when $m$ is sufficiently large

$$f^{2m+1} \circ \phi = \phi \circ \chi,$$

that is, the action of $f^{2m+1}$ on $\phi(\Sigma)$ is topologically conjugate to the action of $\chi$ on $\Sigma$.

Here one can define $\phi(a)$ to be the point on the shadowing orbit that shadows the midpoint of the orbit segment $A_{a_0}$, which is either $x_0$ or $y_0$. The topological conjugacy can be easily verified. For details, see \[66\]. Other references can be found in \[24, 13, 86, 8\]. There exists also a work on horseshoe construction without shadowing lemma for sinusoidally forced vibrations of buckled beam \[27\].
3.8. Periodically Perturbed Sine-Gordon (SG) Equation. We continue from section 2.6 and use the notations in section 3.6. For the periodically perturbed sine-Gordon equation (2.8), the Poincaré period map is a $C^1$ diffeomorphism in $H^1$. As a corollary of the result in last section, we have the theorem on the existence of chaos.

**Theorem 3.9.** There is an integer $m$ and a homeomorphism $\phi$ of $\Sigma$ onto a compact Cantor subset $\Lambda$ of $H^1$. $\Lambda$ is invariant under the Poincaré period-2$\pi$ map $P$ of the periodically perturbed sine-Gordon equation (2.8). The action of $P^{2m+1}$ on $\Lambda$ is topologically conjugate to the action of $\chi$ on $\Sigma$: $P^{2m+1} \circ \phi = \phi \circ \chi$. That is, the following diagram commutes:

\[
\begin{array}{ccc}
\Sigma & \xrightarrow{\phi} & \Lambda \\
\chi \downarrow & & \downarrow P^{2m+1} \\
\Sigma & \xrightarrow{\phi} & \Lambda
\end{array}
\]

3.9. Shadowing Lemma and Chaos in Finite-D Autonomous Systems. A finite-dimensional autonomous system can be written in the general form

\[\dot{x} = F(x),\]

where $x \in \mathbb{R}^n$. In this case, a transversal homoclinic orbit can be an orbit asymptotic to a normally hyperbolic limit cycle. That is, it is an orbit in the intersection of the stable and unstable manifolds of a normally hyperbolic limit cycle. Instead of the Poincaré period map as for periodic system, one may want to introduce the so-called Poincaré return map which is a map induced by the flow on a codimension 1 section which is transversal to the limit cycle. Unfortunately, such a map is not even well-defined in the neighborhood of the homoclinic orbit. This poses a challenging difficulty in extending the arguments as in the case of a Poincaré period map. In 1996, Palmer [67] completed a proof of a shadowing lemma and existence of chaos using Newton’s method. It will be difficult to extend this method to infinite dimensions, since it used heavily differentiations in time. Other references can be found in [15] [16] [72] [22].

3.10. Shadowing Lemma and Chaos in Infinite-D Autonomous Systems. An infinite-dimensional autonomous system defined in a Banach space $X$ can be written in the general form

\[\dot{x} = F(x),\]

where $x \in X$. In 2002, the author [41] completed a proof of a shadowing lemma and existence of chaos using Fenichel’s persistence of normally hyperbolic invariant manifold idea. The setup is as follows,

- **Assumption (A1):** There exist a hyperbolic limit cycle $S$ and a transversal homoclinic orbit $\xi$ asymptotic to $S$. As curves, $S$ and $\xi$ are $C^3$.
- **Assumption (A2):** The Fenichel fiber theorem is valid at $S$. That is, there exist a family of unstable Fenichel fibers $\{F^u(q) : q \in S\}$ and a family of stable Fenichel fibers $\{F^s(q) : q \in S\}$. For each fixed $q \in S$, $F^u(q)$ and $F^s(q)$ are $C^3$ submanifolds. $F^u(q)$ and $F^s(q)$ are $C^2$ in $q, \forall q \in S$. The unions $\bigcup_{q \in S} F^u(q)$ and $\bigcup_{q \in S} F^s(q)$ are the unstable and stable manifolds of $S$. Both families are invariant, i.e.

\[F^t(F^u(q)) \subset F^u(F^s(q)), \forall t \leq 0, q \in S,\]
where $F^t$ is the evolution operator. There are positive constants $\kappa$ and $\tilde{C}$ such that $\forall q \in S$, $\forall q^- \in F^t(q)$ and $\forall q^+ \in F^t(q)$,
\[
\|F^t(q^-) - F^t(q)\| \leq \tilde{C}e^{\kappa t}\|q^- - q\|, \forall t \leq 0,
\]
\[
\|F^t(q^+) - F^t(q)\| \leq \tilde{C}e^{-\kappa t}\|q^+ - q\|, \forall t \geq 0.
\]
- **Assumption (A3):** $F^t(q)$ is $C^0$ in $t$, for $t \in (-\infty, \infty)$, $q \in X$. For any fixed $t \in (-\infty, \infty)$, $F^t(q)$ is a $C^2$ diffeomorphism on $X$.

**Remark 3.10.** Notice that we do not assume that as functions of time, $S$ and $\xi$ are $C^3$, and we only assume that as curves, $S$ and $\xi$ are $C^3$.

Under the above setup, a shadowing lemma and existence of chaos can be proved \[\text{[41]}\]. Another crucial element in the argument is the establishment of a $\lambda$-lemma (also called inclination lemma) \[\text{[41]}\].

### 3.11. A Derivative Nonlinear Schrödinger Equation

We continue from section \[\text{[24]}\] and consider the derivative nonlinear Schrödinger equation \[\text{[24]}\]. The transversal homoclinic orbit given in Theorem \[\text{[2.6]}\] is a classical solution. Thus, Assumption (A1) is valid. Assumption (A2) follows from the standard arguments in \[\text{[52]}\] \[\text{[48]}\] \[\text{[40]}\]. Since the perturbation in \[\text{[29]}\] is bounded, Assumption (A3) follows from standard arguments. Thus there exists chaos in the derivative nonlinear Schrödinger equation \[\text{[48]}\] \[\text{[31]}\].

### 4. Lax Pairs of Euler Equations of Inviscid Fluids

The governing equations for the incompressible viscous fluid flow are the Navier-Stokes equations. Turbulence occurs in the regime of high Reynolds number. By formally setting the Reynolds number equal to infinity, the Navier-Stokes equations reduce to the Euler equations of incompressible inviscid fluid flow. One may view the Navier-Stokes equations with large Reynolds number as a singular perturbation of the Euler equations.

Results of T. Kato show that 2D Navier-Stokes equations are globally well-posed in $C^0([0, \infty); H^s(R^2))$, $s > 2$, and for any $0 < T < \infty$, the mild solutions of the 2D Navier-Stokes equations approach those of the 2D Euler equations in $C^0([0, T]; H^s(R^2))$ \[\text{[31]}\]. 3D Navier-Stokes equations are locally well-posed in $C^0([0, \tau]; H^s(R^2))$, $s > 5/2$, and the mild solutions of the 3D Navier-Stokes equations approach those of the 3D Euler equations in $C^0([0, \tau]; H^s(R^2))$, where $\tau$ depends on the norms of the initial data and the external force \[\text{[29]}\] \[\text{[30]}\]. Extensive studies on the inviscid limit have been carried by J. Wu et al. \[\text{[81]}\] \[\text{[14]}\] \[\text{[82]}\] \[\text{[9]}\]. There is no doubt that mathematical study on Navier-Stokes (Euler) equations is one of the most important mathematical problems. In fact, Clay Mathematics Institute has posted the global well-posedness of 3D Navier-Stokes equations as one of the one million dollars problems.

V. Arnold \[\text{[5]}\] realized that 2D Euler equations are a Hamiltonian system. Recently, the author found Lax pair structures for Euler equations \[\text{[38]}\] \[\text{[53]}\] \[\text{[45]}\] \[\text{[46]}\] \[\text{[51]}\]. Understanding the structures of solutions to Euler equations is of fundamental interest. Of particular interest is the question on the global well-posedness of 3D Navier-Stokes and Euler equations. Our number one hope is that the Lax pair structures can be useful in investigating the global well-posedness. Our secondary
hope is that the Darboux transformation \[53\] associated with the Lax pair can generate explicit representation of homoclinic structures \[36\].

The philosophical significance of the existence of Lax pairs for Euler equations is even more important. If one defines integrability of an equation by the existence of a Lax pair, then both 2D and 3D Euler equations are integrable. More importantly, both 2D and 3D Navier-Stokes equations at high Reynolds numbers are singularly perturbed integrable systems. Such a point of view changes our old ideology on Euler and Navier-Stokes equations.

4.1. A Lax Pair for 2D Euler Equation. The 2D Euler equation can be written in the vorticity form,

\[\partial_t \Omega + \{\Psi, \Omega\} = 0,\]

where the bracket \(\{,\}\) is defined as

\[\{f, g\} = (\partial_x f)(\partial_y g) - (\partial_y f)(\partial_x g),\]

\(\Omega\) is the vorticity, and \(\Psi\) is the stream function given by,

\[u = -\partial_y \Psi, \quad v = \partial_x \Psi,\]

and the relation between vorticity \(\Omega\) and stream function \(\Psi\) is,

\[\Omega = \partial_x v - \partial_y u = \Delta \Psi.\]

**Theorem 4.1 (Li, \[38\]).** The Lax pair of the 2D Euler equation (4.1) is given as

\[\begin{cases}
L \varphi = \lambda \varphi, \\
\partial_t \varphi + A \varphi = 0,
\end{cases}\]

where

\[L \varphi = \{\Omega, \varphi\}, \quad A \varphi = \{\Psi, \varphi\},\]

and \(\lambda\) is a complex constant, and \(\varphi\) is a complex-valued function.

4.2. A Darboux Transformation for 2D Euler Equation. Consider the Lax pair (4.2) at \(\lambda = 0\), i.e.

\[\begin{cases}
\{\Omega, p\} = 0, \\
\partial_t p + \{\Psi, p\} = 0,
\end{cases}\]

where we replaced the notation \(\varphi\) by \(p\).

**Theorem 4.2 (\[53\]).** Let \(f = f(t, x, y)\) be any fixed solution to the system (4.3, 4.4), we define the Gauge transform \(G_f:\)

\[\tilde{p} = G_f p = \frac{1}{\Omega_x} [p_x - (\partial_x \ln f)p],\]

and the transforms of the potentials \(\Omega\) and \(\Psi:\)

\[\tilde{\Psi} = \Psi + F, \quad \tilde{\Omega} = \Omega + \Delta F,\]

where \(F\) is subject to the constraints

\[\{\Omega, \Delta F\} = 0, \quad \{\Delta F, F\} = 0.\]

Then \(\tilde{p}\) solves the system (4.3, 4.4) at \((\tilde{\Omega}, \tilde{\Psi})\). Thus (4.3) and (4.6) form the Darboux transformation for the 2D Euler equation (4.1) and its Lax pair (4.2, 4.4).
Remark 4.3. For KdV equation and many other soliton equations, the Gauge transform is of the form \[\tilde{p} = p_x - (\partial_x \ln f)p\,.

In general, Gauge transform does not involve potentials. For 2D Euler equation, a potential factor \(1/\Omega_x\) is needed. From (4.3), one has
\[
\frac{p_x}{\Omega_x} = \frac{p_y}{\Omega_y}.
\]
The Gauge transform (4.5) can be rewritten as
\[
\tilde{p} = p_x - \frac{f_x}{\Omega_x} = p_y - \frac{f_y}{\Omega_y} p.
\]
The Lax pair (4.3, 4.4) has a symmetry, i.e. it is invariant under the transform \((t, x, y) \rightarrow (-t, y, x)\). The form of the Gauge transform (4.5) resulted from the inclusion of the potential factor \(1/\Omega_x\), is consistent with this symmetry.

Our hope is to use the Darboux transformation to generate homoclinic structures for 2D Euler equation (36).

4.3. A Lax Pair for Rossby Wave Equation. The Rossby wave equation is
\[
\partial_t \Omega + \{\Psi, \Omega\} + \beta \partial_x \Psi = 0,
\]
where \(\Omega = \Omega(t, x, y)\) is the vorticity, \(\{\Psi, \Omega\} = \Psi_x \Omega_y - \Psi_y \Omega_x\), and \(\Psi = \Delta \Omega\) is the stream function. Its Lax pair can be obtained by formally conducting the transformation, \(\Omega = \tilde{\Omega} + \beta y\), to the 2D Euler equation (38),
\[
\{\Omega, \varphi\} - \beta \partial_x \varphi = \lambda \varphi , \quad \partial_t \varphi + \{\Psi, \varphi\} = 0,
\]
where \(\varphi\) is a complex-valued function, and \(\lambda\) is a complex parameter.

4.4. Lax Pairs for 3D Euler Equation. The 3D Euler equation can be written in vorticity form,
\[
\partial_t \Omega + (u \cdot \nabla)\Omega - (\Omega \cdot \nabla)u = 0,
\]
where \(u = (u_1, u_2, u_3)\) is the velocity, \(\Omega = (\Omega_1, \Omega_2, \Omega_3)\) is the vorticity, \(\nabla = (\partial_x, \partial_y, \partial_z)\), \(\Omega = \nabla \times u\), and \(\nabla \cdot u = 0\). \(u\) can be represented by \(\Omega\) for example through Biot-Savart law.

Theorem 4.4. The Lax pair of the 3D Euler equation (4.8) is given as
\[
\begin{cases}
L \phi = \lambda \phi , \\
\partial_t \phi + A \phi = 0,
\end{cases}
\]
where
\[
L \phi = (\Omega \cdot \nabla)\phi , \quad A \phi = (u \cdot \nabla)\phi ,
\]
\(\lambda\) is a complex constant, and \(\phi\) is a complex scalar-valued function.

Theorem 4.5 (10). Another Lax pair of the 3D Euler equation (4.8) is given as
\[
\begin{cases}
L \varphi = \lambda \varphi , \\
\partial_t \varphi + A \varphi = 0,
\end{cases}
\]
where
\[
L \varphi = (\Omega \cdot \nabla)\varphi - (\varphi \cdot \nabla)\Omega , \quad A \varphi = (u \cdot \nabla)\varphi - (\varphi \cdot \nabla)u ,
\]
\( \lambda \) is a complex constant, and \( \varphi = (\varphi_1, \varphi_2, \varphi_3) \) is a complex 3-vector valued function.

Our hope is that the infinitely many conservation laws generated by \( \lambda \in \mathbb{C} \) can provide a priori estimates for the global well-posedness of 3D Navier-Stokes equations, or better understanding on the global well-posedness \([51]\). For more informations on the topics, see \([53]\) \([51]\).

5. Dynamical System Studies on 2D Euler Equation

5.1. Linearized 2D Euler Equation at a Fixed Point. To begin an infinite dimensional dynamical system study, we investigate the linearized 2D Euler equation at a fixed point \([37]\) \([45]\) \([45]\). We consider the 2D Euler equation (4.1) under periodic boundary condition in both \( x \) and \( y \) directions with period \( 2\pi \).

Expanding \( \Omega \) into Fourier series,

\[ \Omega = \sum_{k \in \mathbb{Z}^2 / \{0\}} \omega_k e^{ik \cdot X}, \]

where \( \omega_{-k} = \overline{\omega_k} \), \( k = (k_1, k_2)^T \), and \( X = (x, y)^T \). The 2D Euler equation can be rewritten as

\[ \dot{\omega}_k = \sum_{k=p+q} A(p, q) \omega_p \omega_q, \]

where \( A(p, q) \) is given by,

\[ A(p, q) = \frac{1}{2} \left( |q|^2 - |p|^2 \right) \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}, \]

where \( |q|^2 = q_1^2 + q_2^2 \) for \( q = (q_1, q_2)^T \), similarly for \( p \).

Denote \( \{\omega_k\}_{k \in \mathbb{Z}^2 / \{0\}} \) by \( \omega \). We consider the simple fixed point \( \omega^* \) \([37]\):

\[ \omega_k^* = \Gamma, \quad \omega_k = 0, \text{ if } k \neq p \text{ or } -p, \]

of the 2D Euler equation \([51]\), where \( \Gamma \) is an arbitrary complex constant. The linearized two-dimensional Euler equation at \( \omega^* \) is given by,

\[ \dot{\omega}_k = A(p, k-p) \Gamma \omega_{k-p} + A(-p, k+p) \overline{\Gamma} \omega_{k+p}. \]

**Definition 5.1 (Classes).** For any \( \hat{k} \in \mathbb{Z}^2 / \{0\} \), we define the class \( \Sigma_{\hat{k}} \) to be the subset of \( \mathbb{Z}^2 / \{0\} \):

\[ \Sigma_{\hat{k}} = \left\{ \hat{k} + np \in \mathbb{Z}^2 / \{0\} \mid n \in \mathbb{Z}, \ p \text{ is specified in } 5.3 \right\}. \]

See Fig 5 for an illustration of the classes. According to the classification defined in Definition 5.1 the linearized two-dimensional Euler equation \([5.4]\) decouples into infinitely many invariant subsystems:

\[ \dot{\omega}_{k+np} = A(p, \hat{k} + (n-1)p) \Gamma \omega_{k+(n-1)p} + A(-p, \hat{k} + (n+1)p) \overline{\Gamma} \omega_{k+(n+1)p}. \]

Let \( L_{\hat{k}} \) be the linear operator defined by the right hand side of \([5.5]\), and \( H^s \) be the Sobolev space where \( s \geq 0 \) is an integer and \( H^0 = \ell_2 \).
The eigenvalues of the linear operator $L_k^{\hat{\Sigma}}$ in $H^s$ are of four types: real pairs $(c, -c)$, purely imaginary pairs $(id, -id)$, quadruples $(\pm c \pm id)$, and zero eigenvalues.

Definition 5.3 (The Disk). The open disk of radius $|p|$ in $\mathbb{Z}^2/\{0\}$, denoted by $D_{|p|}$, is defined as

$$D_{|p|} = \left\{ k \in \mathbb{Z}^2/\{0\} \mid |k| < |p| \right\},$$

and the closure of $D_{|p|}$, denoted by $\bar{D}_{|p|}$, is defined as

$$\bar{D}_{|p|} = \left\{ k \in \mathbb{Z}^2/\{0\} \mid |k| \leq |p| \right\}.$$

Theorem 5.4 (The Spectral Theorem, [37, 47]). We have the following claims on the spectrum of the linear operator $L_k^\Sigma$:

1. If $\Sigma_k \cap \bar{D}_{|p|} = \emptyset$, then the entire $H^s$ spectrum of the linear operator $L_k^{\hat{\Sigma}}$ is its continuous spectrum. See Figure 8 where $b = \left( \frac{4}{\pi} \right)^{\frac{1}{2}} |\Gamma| |p|^{-\frac{3}{2}} \begin{vmatrix} p_1 & \hat{k}_1 \\ p_2 & \hat{k}_2 \end{vmatrix}.$

That is, both the residual and the point spectra of $L_k^{\hat{\Sigma}}$ are empty.

2. If $\Sigma_k \cap \bar{D}_{|p|} \neq \emptyset$, then the entire essential $H^s$ spectrum of the linear operator $L_k^{\hat{\Sigma}}$ is its continuous spectrum. That is, the residual spectrum of $L_k^{\hat{\Sigma}}$ is empty. The point spectrum of $L_k^{\hat{\Sigma}}$ is symmetric with respect to both real and imaginary axes. See Figure 10.

Denote by $L$ the right hand side of (5.4), i.e. the whole linearized 2D Euler operator, the spectral mapping theorem holds.
Theorem 5.5 ([34]).
\[ \sigma(e^{tL}) = e^{t\sigma(L)}, t \neq 0. \]

Let \( \zeta \) denote the number of points \( q \in \mathbb{Z}^2 / \{0\} \) that belong to the open disk of radius \( |p|, D_{|p|} \), and such that \( q \) is not parallel to \( p \).

Theorem 5.6 ([34]). The number of nonimaginary eigenvalues of \( L \) (counting the multiplicities) does not exceed \( 2\zeta \).

Another interesting discussion upon the discrete spectrum can be found in [19].

Since the introduction of continued fractions for calculating the eigenvalues of steady fluid flow, by Meshalkin and Sinai [64], this topic had been extensively explored [84,54,55,56,57,58,6,37]. Rigorous justification on the continued fraction calculation was given in [37,58]. As an example, we take \( p = (1,1)^T \).

When \( \Gamma \neq 0 \), the fixed point has 4 eigenvalues which form a quadruple. These four eigenvalues appear in the invariant linear subsystem labeled by \( \hat{k} = (-3,-2)^T \). One of them is [37]:

\[ \tilde{\lambda} = 2\lambda/|\Gamma| = 0.24822302478255 + i 0.35172076526520. \]

See Figure 11 for an illustration. The essential spectrum (= continuous spectrum) of \( \mathcal{L}_k \) with \( k = (-3,-2)^T \) is the segment on the imaginary axis shown in Figure 11 where \( b = -\frac{1}{4}\Gamma \). The essential spectrum (= continuous spectrum) of the linear 2D Euler operator at this fixed point is the entire imaginary axis. A rather well-known open problem is proving the existence of unstable, stable, and center manifolds. The main difficulty comes from the fact that the nonlinear term is non-Lipschitzian.
5.2. Models. To simplify our study, we study only the case when $\omega_k$ is real, $\forall k \in \mathbb{Z}^2/\{0\}$, i.e. we only study the cosine transform of the vorticity,

$$\Omega = \sum_{k \in \mathbb{Z}^2/\{0\}} \omega_k \cos(k \cdot X).$$

To further simplify our study, we will study a concrete dashed-line model based upon the line of fixed points (5.3) with the mode $p = (1, 1)^T$ parametrized by $\Gamma$, and the only unstable invariant linear subsystem labeled by $\hat{k} = (-3, -2)^T$. The so-called dashed-line model is given by (5.7),

$$\dot{\omega}_n = \epsilon_n - \sum_{m \in Z} A_{n-1} A_{n+1},$$

(5.7)

$$\dot{\omega}_p = -\epsilon_n - \sum_{n \in Z} A_{n-1} A_{n+1},$$

where

$$\omega_n = \omega_{\hat{k} + np}, \quad A_n = A(p, \hat{k} + np), \quad A_{m,n} = A(\hat{k} + mp, \hat{k} + np),$$

$$\epsilon_n = \begin{cases} 1, & \text{if } n \neq 5j, \forall j \in Z, \\ \epsilon, & \text{if } n = 5j, \text{ for some } j \in Z. \end{cases}$$

The model is designed to model the hyperbolic structure of the 2D Euler equation, connected to the line of fixed points (5.3) with $p = (1, 1)^T$. Figure 12 illustrates the collocation of the modes in this model, which has the “dashed-line” nature leading to the name of the model. When $\epsilon = 0$, one can determine the stable and unstable
Figure 11. The spectrum of $\mathcal{L}_k$ with $\hat{k} = (-3, -2)^T$, when $p = (1, 1)^T$.

Figure 12. The collocation of the modes in the dashed-line model.

The manifolds of the fixed point $\omega^*$:

$$\omega_p = \Gamma, \omega_n = 0 \quad (n \in \mathbb{Z}).$$
**Figure 13.** The unstable and stable manifolds of the fixed point \( \omega^* \): (a) and (b) show two “painted eggs” and (c) shows a “lip”.

Using the polar coordinates:

\[
\begin{align*}
\omega_1 &= r \cos \theta, \\
\omega_4 &= r \sin \theta, \\
\omega_2 &= \rho \cos \vartheta, \\
\omega_3 &= \rho \sin \vartheta,
\end{align*}
\]

we have the following explicit expressions for the stable and unstable manifolds of the fixed point \( \omega^* \) and its negative \( -\omega^* \) represented through *perverted heteroclinic orbits*:

\[
\begin{align*}
\omega_p &= \Gamma \tanh \tau, \\
r &= \sqrt{\frac{A_2 - A_1}{A_2}} \Gamma \sech \tau, \\
\theta &= -\frac{A_2}{2\kappa} \ln \cosh \tau + \theta_0, \\
\rho &= \sqrt{-\frac{A_1}{A_2}} r, \\
\theta + \vartheta &= \begin{cases} 
- \arcsin \left[ \frac{1}{2} \sqrt{\frac{A_2}{A_1}} \right], & (\kappa > 0), \\
\pi + \arcsin \left[ \frac{1}{2} \sqrt{\frac{A_2}{A_1}} \right], & (\kappa < 0),
\end{cases}
\end{align*}
\]

where \( \tau = \kappa \Gamma t + \tau_0, (\tau_0, \theta_0) \) are the two parameters parametrizing the two-dimensional stable (unstable) manifold, and

\[
\kappa = \sqrt{-A_1A_2} \cos(\theta + \vartheta) = \pm \sqrt{-A_1A_2} \sqrt{1 + \frac{A_2}{4A_1}}.
\]

The two auxiliary variables \( \omega_0 \) and \( \omega_5 \) have the expressions:

\[
\begin{align*}
\omega_0 &= \frac{\alpha \beta}{1 + \beta^2} \sech \tau \left\{ \sin[\beta \ln \cosh \tau + \theta_0] - \frac{1}{\beta} \cos[\beta \ln \cosh \tau + \theta_0] \right\}, \\
\omega_5 &= \frac{\alpha \beta}{1 + \beta^2} \sech \tau \left\{ \cos[\beta \ln \cosh \tau + \theta_0] + \frac{1}{\beta} \sin[\beta \ln \cosh \tau + \theta_0] \right\},
\end{align*}
\]

where

\[
\alpha = -A_1 \Gamma \kappa^{-1} \sqrt{\frac{A_2}{A_2 - A_1}}, \quad \beta = -\frac{A_2}{2\kappa}.
\]

The explicit expression \((5.10)\) of the perverted heteroclinic orbits shows that the unstable manifold \(W^u(\omega^*)\) of the fixed point \(\omega^*\) is the same as the stable manifold \(W^s(\omega^*)\) of the negative \(-\omega^*\) of \(\omega^*\), and the stable manifold of the fixed point \(\omega^*\) is the same as the unstable manifold of the negative \(-\omega^*\) of \(\omega^*\). Both \(W^u(\omega^*)\) and \(W^s(\omega^*)\) have the shapes of “painted eggs” (Figure 13). \(W^u(\omega^*)\) and \(W^s(\omega^*)\) together form the “lip” (Figure 13) which is a higher dimensional generalization of the heteroclinic connection on plane.

As the first step toward understanding the degeneracy v.s. nondegeneracy of the hyperbolic structure of 2D Euler equation, we are interested in the \(\varepsilon\)-homotopic deformation of such hyperbolic structure for the dashed-line model. We have calculated the Melnikov functions to study the “breaking” or “persistence” of such
hyperbolic structure when $\varepsilon$ is small. It turns out that both the first and the second order Melnikov functions are identically zero [39].

6. Conclusion

Results on chaos in partial differential equations are reported. Results on Lax pairs of Euler equations of incompressible inviscid fluids are also reported. Preliminary results on dynamical system studies of 2D Euler equation are summarized.

References

[1] M. J. Ablowitz, Y. Ohta, and A. D. Trubatch. On Discretizations of the Vector Nonlinear Schrödinger Equation. Phys. Lett. A, 253, no.5-6:287–304, 1999.
[2] M. J. Ablowitz, Y. Ohta, and A. D. Trubatch. On Integrability and Chaos in Discrete Systems. Chaos Solitons Fractals, 11, 1-3:159–169, 2000.
[3] D. V. Anosov. Geodesic Flows on Compact Riemannian Manifolds of Negative Curvature. Proc. Steklov Inst. Math., 90, 1967.
[4] V. I. Arnold. Instability of Dynamical Systems with Many Degrees of Freedom. Sov. Math. Dokl., 5 No. 3:581–585, 1964.
[5] V. I. Arnold. Sur la Geometrie Differentielle des Groupes de Lie de Dimension Infinie et ses Applications a L’hydrodynamique des Fluides Parfaits. Ann. Inst. Fourier, Grenoble, 16,1:319–361, 1966.
[6] L. Belenkaya, S. Friedlander, and V. Yudovich. The Unstable Spectrum of Oscillating Shear Flows. SIAM J. Appl. Math., 59, no.5:1701–1715, 1999.
[7] G. D. Birkhoff. Some Theorems on the Motion of Dynamical Systems. Bull. Soc. Math. France, 40:305–323, 1912.
[8] C. M. Blazquez. Transverse Homoclinic Orbits in Periodically Perturbed Parabolic Equations. Nonlinear Analysis, Theory, Methods and Applications, 10, no.11:1277–1291, 1986.
[9] J. Bona and J. Wu. Zero Dissipation Limit for Nonlinear Waves. Preprint, 1999.
[10] S. Childress. A Lax pair of 3D Euler equation. personal communication, 2000.
[11] S.-N. Chow and J.K. Hale, Methods of Bifurcation Theory, Springer-Verlag, New York-Berlin, 1982.
[12] S.-N. Chow and J. K. Hale and J. Mallet-Paret, An example of bifurcation to homoclinic orbits, J. Differential Equations 37, no.3 (1980), 351.
[13] S. N. Chow, X. B. Lin, and K. J. Palmer. A Shadowing Lemma with Applications to Semilinear Parabolic Equations. SIAM J. Math. Anal., 20, no.3:547–557, 1989.
[14] P. Constantin and J. Wu. The Inviscid Limit for Non-smooth Vorticity. Indiana Univ. Math. J., 45, no.1:67–81, 1996.
[15] B. Coomes, H. Kocak, and K. Palmer. A Shadowing Theorem for Ordinary Differential Equations. Z. Angew. Math. Phys., 46, no.1:85–106, 1995.
[16] B. Coomes, H. Kocak, and K. J. Palmer. Long Periodic Shadowing. Numerical Algorithms, 14:55–78, 1997.
[17] B. Deng. Exponential Expansion with Silnikov's Saddle-Focus. J. Differential Equations, 82, no.1:156–173, 1989.
[18] B. Deng. On Silnikov’s Homoclinic-Saddle-Focus Theorem. J. Differential Equations, 102, no.2:305–329, 1993.
[19] L. D. Faddeev. On the Stability Theory for Stationary Plane-Parallel Flows of Ideal Fluid. Kraevye Zadachi Mat. Fiziki (Zapiski Nauchnykh Seminarov LOMI, v.21), Moscow, Leningrad: Nauka, 5:164–172, 1971.
[20] M. G. Forest, D. W. McLaughlin, D. J. Muraki, and O. C. Wright. Nonfocusing Instabilities in Coupled, Integrable Nonlinear Schrödinger pdes. J. Nonlinear Sci., 10, no.3:291–331, 2000.
[21] M. G. Forest, S. P. Sheu, and O. C. Wright. On the Construction of Orbits Homoclinic to Plane Waves in Integrable Coupled Nonlinear Schrödinger System. Phys. Lett. A, 266, no.1:24–33, 2000.
[22] J. E. Franke and J. F. Selgrade. Hyperbolicity and Chain Recurrence. J. Diff. Eq., 26:27–36, 1977.
[23] J. Guckenheimer and P. J. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Springer-Verlag, Applied Mathematical Sciences, vol.42, 1983.
[24] J. K. Hale and X. B. Lin. Symbolic Dynamics and Nonlinear Semiflows. *Ann. Mat. Pura Appl.*, 144:229–259, 1986.

[25] A. Hasegawa and Y. Kodama. Solitons in Optical Communications. Academic Press, San Diego, 1995.

[26] D. Henry. Exponential Dichotomies, the Shadowing Lemma and Homoclinic Orbits in Banach Spaces. *Resenhas*, 1, no.4:381–401, 1994.

[27] P. J. Holmes and J. E. Marsden. A Partial Differential Equation with Infinitely Many Periodic Orbits: Chaotic Oscillations of a Forced Beam. *Arch. Rat. Mech. Anal.*, 76:135–166, 1981.

[28] M. N. Islam. Ultrafast Fiber Switching Devices and Systems. Cambridge University Press, New York, 1992.

[29] T. Kato. Nonstationary Flows of Viscous and Ideal Fluids in $R^3$. *Journal of Functional Analysis*, 9:296–305, 1972.

[30] T. Kato. Quasi-Linear Equations of Evolution, with Applications to Partial Differential Equations. *Lecture Notes in Mathematics*, 448:25–70, 1975.

[31] T. Kato. Remarks on the Euler and Navier-Stokes Equations in $R^2$. *Proc. Symp. Pure Math.*, Part 2, 45:1–7, 1986.

[32] G. Kovacic, Dissipative dynamics of orbits homoclinic to a resonance band, *Phys Lett A* 167 (1992), 143.

[33] G. Kovacic, Hamiltonian dynamics of orbits homoclinic to a resonance band, *Phys Lett A* 167 (1992), 137.

[34] Y. Latushkin and Y. Li and M. Stanislavova, The spectrum of a linearized 2D Euler operator, *Submitted* (2001).

[35] Y. Li, Smale horseshoes and symbolic dynamics in perturbed nonlinear Schrödinger equations, *Journal of Nonlinear Sciences* 9, no.4 (1999), 363.

[36] Y. Li, Bäcklund-Darboux transformations and Melnikov analysis for Davey-Stewartson II equations, *Journal of Nonlinear Sciences* 10, no.1 (2000), 103.

[37] Y. Li, On 2D Euler equations. I. On the energy-Casimir stabilities and the spectra for a linearized two dimensional Euler equation, *Journal of Mathematical Physics* 41, no.2 (2000), 728.

[38] Y. Li, A Lax pair for the 2D Euler equation, *Journal of Mathematical Physics* 42, no.8 (2001), 3552.

[39] Y. Li, On 2D Euler equations: part II. Lax pairs and homoclinic structures, *Communications on Applied Nonlinear Analysis*, to appear, available at: http://xxx.lanl.gov/abs/math.AP/0010200 or http://www.math.missouri.edu/~cli (2002).

[40] Y. Li, Persistent homoclinic orbits for nonlinear Schrödinger equation under singular perturbation, *Submitted*, available at: http://xxx.lanl.gov/abs/math.AP/0106194 or http://www.math.missouri.edu/~cli (2001).

[41] Y. Li, Chaos and Shadowing Lemma for Autonomous Systems of Infinite Dimensions. *Submitted*, available at: http://xxx.lanl.gov/abs/nlin/0203024 or http://www.math.missouri.edu/~cli 2002.

[42] Y. Li, Melnikov Analysis for Singularly Perturbed DSII Equation. *Submitted*, available at: http://xxx.lanl.gov/abs/math.AP/0206272 or http://www.math.missouri.edu/~cli 2002.

[43] Y. Li, Existence of Chaos for a Singularly Perturbed NLS Equation. *Submitted*, available at: http://xxx.lanl.gov/abs/math.AP/0206270 or http://www.math.missouri.edu/~cli 2002.

[44] Y. Li, Singularly Perturbed Vector and Scalar Nonlinear Schroedinger Equations with Persistent Homoclinic Orbits. *Studies in Applied Mathematics*, 109:19–38, 2002.

[45] Y. Li, Integrable Structures for 2D Euler Equations of Incompressible Inviscid Fluids. *Proc. Institute Of Mathematics of NAS of Ukraine*, 43, 1:332–338, 2002.

[46] Y. Li, Chaos in Partial Differential Equations, *Contemporary Mathematics*, to appear, available at: http://xxx.lanl.gov/abs/math.AP/0205114 or http://www.math.missouri.edu/~cli 2002.

[47] Y. Li, On 2D Euler Equations: III. A Line Model. *Available at*: http://xxx.lanl.gov/abs/math.AP/0206278 or http://www.math.missouri.edu/~cli 2002.

[48] Y. Li et al., Persistent homoclinic orbits for a perturbed nonlinear Schrödinger equation, *Comm. Pure Appl. Math.* XLIX (1996), 1175.

[49] Y. Li and D. W. McLaughlin, Morse and Melnikov functions for NLS pdes, *Comm. Math. Phys.* 162 (1994), 175.
[50] Y. Li and D. W. McLaughlin, Homoclinic orbits and chaos in discretized perturbed NLS system, Journal of Nonlinear Sciences 7 (1997), 211.

[51] Y. Li and R. Shvidkoy, Isospectral Theory of Euler Equations. Submitted, available at: http://xxx.lanl.gov/abs/math.AP/0203125, or http://www.math.missouri.edu/~cli, 2002.

[52] Y. Li and S. Wiggins, Homoclinic Orbits and Chaos in Perturbed Discrete NLS System. Part II Symbolic Dynamics. Journal of Nonlinear Sciences, 7:315–370, 1997.

[53] Y. Li and A. Yurov, Lax pairs and Darboux transformations for Euler equations, Submitted, available at: http://xxx.lanl.gov/abs/math.AP/0101214, or http://www.math.missouri.edu/~cli, 2001.

[54] V. X. Liu. An Example of Instability for the Navier-Stokes Equations on the 2-Dimensional Torus. Comm. Partial Differential Equations, 17, no.11-12:1995–2012, 1992.

[55] V. X. Liu. Instability for the Navier-Stokes Equations on the 2-Dimensional Torus and a Lower Bound for the Hausdorff Dimension of Their Global Attractors. Comm. Math. Phys., 147, no.2:217–230, 1992.

[56] V. X. Liu. A Sharp Lower Bound for the Hausdorff Dimension of the Global Attractors of the 2D Navier-Stokes Equations. Comm. Math. Phys., 158, no.2:327–339, 1993.

[57] V. X. Liu. Remarks on the Navier-Stokes Equations on the Two- and Three-Dimensional Torus. Comm. Partial Differential Equations, 19, no.5-6:873–900, 1994.

[58] V. X. Liu. On Unstable and Neutral Spectra of Incompressible Inviscid and Viscid Fluids on the 2D Torus. Quart. Appl. Math., 53, no.3:465–486, 1995.

[59] V. B. Matveev and M. A. Salle. Darboux Transformations and Solitons, volume 5. Springer Series in Nonlinear Dynamics, 1991.

[60] K. Palmer. Exponential Dichotomies and Transversal Homoclinic Points. J. Differential Equations, 55, no.2:225–256, 1984.

[61] K. Palmer. Exponential Dichotomies and Transversal Homoclinic Points. J. Differential Equations, 55, no.2:225–256, 1984.

[62] K. Palmer. Exponential Dichotomies, the Shadowing Lemma and Transversal Homoclinic Points. Dynamics Reported, 1:265–306, 1988.

[63] K. Palmer. Shadowing and Silnikov Chaos. Nonlinear Anal., 27, no.9:1075–1093, 1996.

[64] H. Poincaré. Les Methodes Nouvelles de la Mecanique Celeste, Vols. 1-3. English Translation: New Methods of Celestial Mechanics, Vols. 1-3, edited by Daniel L. Goroff, American Institute of Physics, New York, 1993. Gauthier-Villars, Paris, 1899.

[65] J. Shatah and C. Zeng. Homoclinic Orbits for the Perturbed Sine-Gordon Equation. Comm. Pure Appl. Math., 53, no.3:283–299, 2000.

[66] L. P. Silnikov. A Case of the Existence of a Countable Number of Periodic Motions. Soviet Math. Doklady, 6:163–166, 1965.

[67] L. P. Silnikov. The Existence of a Denumerable Set of Periodic Motions in Four-dimensional Space in an Extended Neighborhood of a Saddle-Focus. Soviet Math. Doklady, 8:54–58, 1967.

[68] L. P. Silnikov. On a Poincare-Birkhoff Problem. Math. USSR Sb., 3:353–371, 1967.

[69] S. Smale. A Structurally Stable Differentiable Homeomorphism with an Infinite Number of Periodic Points. Qualitative Methods in the Theory of Non-linear Vibrations (Proc. Internat. Sympos. Non-linear Vibrations), II:365–366, 1961.

[70] S. Smale. Differentiable Dynamical Systems. Bull. Amer. Math. Soc., 73:747–817, 1967.
[77] H. Steinlein and H. Walther. Hyperbolic Sets and Shadowing for Noninvertible Maps. *Advanced Topics in the Theory of Dynamical Systems, Academic Press, Boston, MA*, pages 219–234, 1989.

[78] H. Steinlein and H. Walther. Hyperbolic Sets, Transversal Homoclinic Trajectories, and Symbolic Dynamics for \( c^1 \)-Maps in Banach Spaces. *J. Dynamics Differential Equations*, 2, no.3:325–365, 1990.

[79] S. Wiggins, *Global Bifurcations and Chaos: Analytical Methods*, Springer-Verlag, New York, 1988.

[80] O. C. Wright and M. G. Forest. On the Bäcklund-Gauge Transformation and Homoclinic Orbits of a Coupled Nonlinear Schrödinger System. *Phys. D*, 141, no.1-2:104–116, 2000.

[81] J. Wu. The Inviscid Limits for Individual and Statistical Solutions of the Navier-Stokes Equations. *Ph.D. Thesis, Chicago University*, 1996.

[82] J. Wu. The Inviscid Limit of the Complex Ginzburg-Landau Equation. *J. Diff. Equations*, 142, No.2:413–433, 1998.

[83] J. Yang and Y. Tan. Fractal Dependence of Vector-Soliton Collisions in Birefringent Fibers. *Phys. Lett. A*, 280:129–138, 2001.

[84] V. I. Yudovich. Example of the Generation of a Secondary Stationary or Periodic Flow When There is Loss of Stability of the Laminar Flow of a Viscous Incompressible Fluid. *J. Appl. Math. Mech. (PMM)*, 29:527–544, 1965.

[85] W. Zeng. Exponential Dichotomies and Transversal Homoclinic Orbits in Degenerate Cases. *J. Dyn. Diff. Eq., 7*, no.4:521–548, 1995.

[86] W. Zeng. Transversality of Homoclinic Orbits and Exponential Dichotomies for Parabolic Equations. *J. Math. Anal. Appl.*, 216:466–480, 1997.

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