Super-Affine Hierarchies
and their Poisson Embeddings

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Abstract

The link between (super)-affine Lie algebras as Poisson brackets structures and integrable hierarchies provides both a classification and a tool for obtaining superintegrable hierarchies. The lack of a fully systematic procedure for constructing matrix-type Lax operators, which makes the supersymmetric case essentially different from the bosonic counterpart, is overcome via the notion of Poisson embeddings (P.E.), i.e. Poisson mappings relating affine structures to conformal structures (in their simplest version P.E. coincide with the Sugawara construction). A full class of hierarchies can be recovered by using uniquely Lie-algebraic notions. The group-algebraic properties implicit in the super-affine picture allow a systematic derivation of reduced hierarchies by imposing either coset conditions or hamiltonian constraints (or possibly both).
1 Introduction.

Affine Lie algebras and conformal algebras have received a great attention in the physicists community in the last several years, mainly due to their relevance to phenomenological models (2d $\sigma$-models in the WZNW description), as well as the more fundamental string approach to the unification of interactions. It is well understood by now that conformal algebras (even the non-linear $W$-type ones) are the output of affine algebras after some construction, hamiltonian reductions or cosets, are performed on them.

While affine-Lie and conformal algebras are universally appreciated, not so much attention has received a truly remarkable property they share, i.e. that they support hierarchies of integrable equations in 1 + 1 dimension in the sense that they provide (one of the) Poisson Brackets (PB for short in the following) for the associated hierarchy.

To my knowledge such a property has not yet found a direct implementation in the string-theory program, however has already found application to physically motivated theories like discretized 2d gravity in the matrix-model approach (see [1] and references therein). There, essentially, $W$-algebras arise as Ward identities known as $W$-constraints and the partition function is a tau-function of an associated integrable hierarchy.

Moreover integrable hierarchies underline such exactly solvable models as 4d $N = 2$ Seiberg-Witten SYM theories (see e.g. [2]).

Due to the above-mentioned results it is clear why a lot of attention continues to be focused on the supersymmetric extensions. It is hoped that their understanding will provide the basis for discretized 2d supergravity (see [3]). However, unlike the purely bosonic theories, the supersymmetric extensions, for reasons we will discuss later, have so far failed being accomodated into a single unifying picture. Due to this basic problem super-hierarchies have been produced by using all sort of tools available, i.e. by direct construction, via Lax operators, bosonic as well as fermionic and both in scalar or matrix form, by coset procedure [4, 5, 6, 7, 8, 9, 10] and so on. We have by now an impressive list of “zoological” data concerning superhierarchies. There is an overwhelming evidence that some order should be made and a single unifying picture should be provided. In this paper we wish to point out a possible tool for both classifying and explicitly constructing a class of super-hierarchies.

Such a tool is based on the super-affine framework and Poisson Embedding (or shortly PE). This means the derivation of super-hierarchies by regarding as fundamental ingredient the Poisson brackets structure furnished by the supersymmetric affinization of a (super)-Lie algebra (more on this notion later). Poisson Embeddings are a special class of Poisson mappings, (i.e. maps between two sets of (super)-fields $f_P : \{\Phi_i\}_I \mapsto \{\tilde{\Phi}_j\}_{II}$ which preserves the PB structures between sets $\{I\}$ and $\{II\}$) having the further property that the PB structure in $\{II\}$ is the one of a given integrable hierarchy.

As a consequence it is possible to define on the super-fields in $\{I\}$ a hierarchy of equations which inherit the integrability property from the one defined on the second set $\{II\}$. This seemingly innocent remark has indeed far-reaching consequences and allows us to produce and identify new hierarchies of equations, even generalizing the set of hierarchies produced in the literature with more dispendious and time consuming methods.

Well-known examples of Poisson maps are the Wakimoto free (super)-fields realization of affine algebras, and the Sugawara-type construction relating affine(super)-fields to the (super)-stress energy tensor. The latter is also a Poisson Embedding, and therefore integrable hierar-
chies are induced both at the level of affine and of Wakimoto free superfields.

A real breakthrough in this context appears to be the realization in [11] that among such mappings there is a (differential) polynomial Poisson map which is an $N = 4$ extension of the Sugawara construction based on super-affine $\mathfrak{sl}(2) \oplus \mathfrak{u}(1)$. Such a mapping, besides being a PB one, is a Poisson Embedding since the Sugawara-produced hierarchy turns out to be the small $N = 4$ SCA carrying the $N = 4$ KdV hierarchy.

The key observation is that superconformal algebras can be more directly identified with the PB structure of a given hierarchy since they are easily accomodated into scalar-type Lax operators (see [9]), which can be constructed with a systematic procedure.

Focus can therefore be put into the construction of generalized Sugawara mappings. Here however we advocate the point of view that we can investigate the properties of already existing Sugawara constructions to identify and classify series of new hierarchies. In particular one of such mappings sends any $N = 2$ (super)-affine algebra into $N = 2$ Virasoro. This is the PB structure for three distinct $N = 2$ KdV hierarchies [3] (associated to a value of parameter $a = 4, -2, 1$). Accordingly, three induced hierarchies are associated to the $N = 2$ affine superfields realizing the PE. In some cases, the three hierarchies collapse into a single one.

The full power of affine algebras gets really appreciated when one realizes that due to stringent group-theoretical reasons one can further reduce such algebras with coset procedures and/or hamiltonian reductions. On conformal algebras themselves these procedures cannot be carried out. For instance the Virasoro algebra itself is already a hamiltonian reduction of affine $\mathfrak{sl}(2)$ [12].

A full bunch of “popping out” hierarchies can therefore find a natural group-theoretical explanation and interpretation.

2 Notations and preliminary remarks.

The class of Poisson brackets structure we will consider is given by the superaffinization of any given semisimple (super)-Lie algebra, defined as follows: Let $\mathcal{G}$ be any finite semisimple Lie algebra, either purely bosonic or supersymmetric, with $n_b$ bosonic and $n_f$ fermionic generators ($n_f = 0$ for standard Lie algebras) collectively denoted as $\tau_\alpha$, for $\alpha = 1, ..., n_b + n_f$, and let $f^\gamma_{\alpha\beta}$ denote the structure constants.

We can introduce the $N = 1$ superfields (in the superspace coordinate $X = x, \theta$, with $\theta$ Grassmann) $\Psi_\alpha(X)$, associated to each generator $\tau_\alpha$ and with opposite statistics w.r.t. $\tau_\alpha$.

The superaffine algebra is defined by assuming the following Poisson brackets

$$\{\Psi(X)_\alpha, \Psi(Y)_\beta\} =_{def} f^\gamma_{\alpha\beta}\Psi(Y)_\gamma\delta(X,Y) + cK_{\alpha\beta}D_Y\delta(X,Y)$$

where we introduced the supersymmetric Dirac’s $\delta$-function

$$\delta(X,Y) = \delta(x - y)(\theta - \eta)$$

and the $N = 1$ superderivative

$$D_Y = \frac{\partial}{\partial \eta} + \eta \frac{\partial}{\partial y}$$

for $Y \equiv y, \eta$. 

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$c$ is the central extension and $K_{\alpha\beta}$ is defined as a supertrace $K_{\alpha\beta} = \text{Str}(\tau_\alpha \tau_\beta)$ in a given representation for $G$, let’s say the adjoint.

In the above relation the Jacobi identities are satisfied and therefore (1) indeed defines a PB structure.

We mention that if the (super)-algebra $G$ of departure admits a complex structure, then (1) is indeed $N = 2$ supersymmetric and can be recasted into a manifestly $N = 2$ formalism (see [13]), here however we do not need such technical improvement.

The bosonic limit of the above (1) affine algebra (realized on a bosonic $G$ and via purely bosonic fields) is the building block for constructing generalized Drinfeld-Sokolov hierarchies, via the association to a matrix type Lax operator $L$ of the kind

$$L = \partial + \sum_\alpha J_\alpha(x) \tau_\alpha + \Lambda$$

where $\Lambda$ is a constant element in the $\tilde{G} = G \otimes \mathbb{C}(\lambda, \lambda^{-1})$ loop algebra of $G$ realized on an auxiliary variable $\lambda$ which plays the role of a spectral parameter. By DS construction, if $\Lambda$ is chosen in such a way to realize the decomposition

$$\tilde{G} = \mathcal{K} \oplus \mathcal{M}$$

with $\mathcal{K} = \text{Ker}_{ad-\Lambda}$ and $\mathcal{M} = \text{Im}_{ad-\Lambda}$ over the adjoint action of $\Lambda$, and if furthermore $\mathcal{K}$ is abelian

$$[\mathcal{K}, \mathcal{K}] = 0$$

then we are guaranteed about the existence of infinite integrals of motion in involution.

The same kind of construction has been generalized in the supersymmetric case by Inami and Kanno in a series of papers (see e.g. [6] and references therein). Now $L$ assumes the form

$$L = D + \sum_\alpha \Psi_\alpha(X) \tau_\alpha + \Lambda$$

the first and second terms in the r.h.s. are fermionic and so $\Lambda$ must be fermionic as well. This constraint puts a very strong restriction on the superhierarchies which can be obtained through DS procedure. In the bosonic case for instance generalized-KdV hierarchies which include among others Boussinesq are defined by taking $\Lambda$ to be the sum over all simple roots of the $\tilde{G}$ algebra. Consequently generalized super-KdV hierarchies can be obtained solely from those superalgebras which admit a Dynkin diagram presentation involving only fermionic simple roots. Admittedly, this is a rather restrict class of superalgebras.

For them however we have a viable and systematic procedure to construct superhierarchies.

A problem arises because very natural hierarchies like the supersymmetric extensions of NLS fail to be accomodated into this scheme due to the fact that their bosonic equivalents are recovered from a $\Lambda$ in (1) belonging to the Cartan generators of $G$. Since the Cartan sector of whatever simple bosonic Lie or super-Lie algebra is in any case bosonic, we have no possibility at all to construct a fermionic $\Lambda$ with the desired properties.

\footnote{this is a slightly imprecise way of saying, in order to talk about simple roots we should introduce the extended affine algebra over the auxiliary loop space parameter $\lambda$, but let’s avoid these technical complications here.}
In effect there exists a class of superalgebras, the so-called strange superalgebras of the $Q(n)$ series, which admit a fermionic Cartan sector (see [14]). But these superalgebras are of no use here for another reason. The fermionic $\Lambda$ taking value in the fermionic Cartan always fail to satisfy either condition (3) or condition (4).

There are some ad hoc procedures to overcome this difficulty (see [15]), but it must be said that even if viable for the practical purpose of computing higher order hamiltonians, they lack a clear and compelling motivation which makes them not attractive for the purpose of classifying hierarchies. Similarly, matrix-type Lax operators have been produced for hierarchies of super-NLS type (or obtainable from super-NLS) reduction. Such Lax operators, unlike the bosonic ones, have entries which are composite superfields. Here again it is hard to justify their appearance in terms of fundamental principles. Rather, they are the signal that a simpler structure should be found behind them. In the next section we will see how an answer (at least a partial one) can be provided.

3 Poisson maps and Poisson embeddings.

In the previous section we have discussed some problems arising in the construction of super-integrable hierarchies. Now we will show how to overcome such problems via the introduction of the notion of Poisson maps and Poisson embeddings, already outlined in the introduction. Evidence will be furnished that superaffine algebras are the right setting to deal and classify superhierarchies.

A point should be clear, the maps we are investigating are polynomial differential maps. In literature non-polynomial maps relating different hierarchies have been considered (in some cases they are not even Poisson maps), but in all known examples they can be recasted into (or derived from) polynomial differential maps. So it seems there is no compelling reason to look beyond the realm of polynomial differential Poisson maps.

Under an $f_P$ Poisson Embedding the infinite series of $H_k$ hamiltonians in involution of the integrable hierarchy ($\{H_k, H_{k'}\}_{(I)} = 0$) can be regarded as hamiltonians in involution w.r.t. the first PB structure (for any couple is indeed $\{H_k, H_{k'}\}_{(I)} = 0$) and it makes sense to define an infinite series of compatible flows for the superfields $\Phi_i$ as:

$$\frac{\partial}{\partial t_k} \Phi_i = \{H_k(f_P(\Phi_j)), \Phi_i\}_{(I)}$$

We will refer to hierarchies of this kind either as the induced or as the embedded hierarchy.

Examples of PE are given by Sugawara-type construction. We recall that for any bosonic and $N = 1$ supersymmetric affine algebra there exists a well-defined procedure which allows us to produce conformal fields satisfying a closed $\mathcal{W}$-algebra structure. They are expressed in terms of the enveloping algebra of the (super)-affine algebra $\mathcal{G}$ and are in 1-to-1 correspondence with each Casimir of $\mathcal{G}$. The most relevant or leading term in the enveloping algebra being given by

$$d^{i_1...i_n}J_{i_1}(x)...J_{i_n}(x)$$

in the bosonic case and

$$d^{i_1...i_n}(D_X\Psi_{i_1}) \cdot ... \cdot (D_X\Psi_{i_{n-1}}) \cdot \Psi_{i_n}(X)$$

(7)
in the $N = 1$ super-case.

Here $d^{i_1...i_n}$ is the symmetric tensor denoting an $n$-th order Casimir and $J_{i_k}$ ($\Psi_{i_k}$) are spin 1 fields (spin $\frac{1}{2}$ superfields respectively).

A full bunch of improvements or covariantization terms must be added to the above “leading order terms” to render the (super)-field a primary or conformal field. For instance, in the case of the order 2 Casimir (which exists for any (super)-Lie algebra) the terms to be added are just the Deift-Fuchs terms which provide a non-vanishing central charge so that the Sugawara-constructed field satisfies the full Virasoro ($N = 1$ superVirasoro) algebra.

As an example, in the case of the $sl(n)$ series the $W$-algebra so produced is the $W_n$ algebra (or its $N = 1$ supersymmetrization). The “miracle” here, which has a group-theoretical explanation, lies in the fact that this is the same algebra arising from Dirac brackets after the Drinfeld-Sokolov hamiltonian reduction sketched in the previous section has been taken into account.

For the moment let me just point out that the (super)-affine algebra, which at the beginning was not associated to any evident hierarchy, has now acquired the status of a PB structure for the induced hierarchy. Any Poisson map onto the super-affine fields is therefore also a Poisson Embedding.

There exists a well-defined prescription on how to realize any super-affine algebra in terms of free (super)-fields. The result is given by the (generalized) (super)-Wakimoto realizations whose origin is traced on the theory of non-linear realizations of algebraic structures. The simplest case of a super-Wakimoto construction is the realization of the $N = 1$ affine $sl(2)$ algebra [8].

The generalized (super)-Wakimoto realizations are all examples of Poisson maps. In full generality the Wakimoto free (super)-fields satisfy induced hierarchies whose origin arises from their “double” embedding into the generalized KdV hierarchies.

4 Induced $N = 2$ and $N = 4$ hierarchies.

In the previous section we have discussed the general theory of Poisson Embeddings and have shown that super-affine Lie algebras are associated with (at least $N = 1$) integrable hierarchies.

Here we further discuss properties of PE and study them in the context of $N = 2$ hierarchies. This is indeed a very interesting case since for the first time appears (as shown by explicit construction in [4]) that one and the same PB structure is associated with different series of integrable hierarchies.

Indeed if we denote as $J(x, \theta, \bar{\theta})$ the real $N = 2$ Virasoro field consisting of two (spin = 1, 2) boson components and a couple of spin $\frac{3}{2}$ fermions, three inequivalent $N = 2$ KdV hierarchies can be recovered (for a parameter $a$ taking values $a = 4, -2, 1$ respectively).

Therefore any Poisson map onto $N = 2$ Virasoro induces three different and in principle inequivalent series of hierarchies.

The case $N = 2$ is interesting also for another reason. While generalized $N = 1$ Sugawara poses no problem in its construction (and in its association with super-KdV hierarchies), much less is known concerning the $N = 2$ case. Apart from the Sugawara construction of the $N = 2$ Virasoro discussed below, to my knowledge no general theorem has been given so far and only explicit examples have been worked out on how to perform Sugawara construction of $N = 2$
\(\mathcal{W}\)-algebras (the main question concerns the feasibility of adding Feigin-Fuchs terms, while maintaining a full \(N = 2\) \(\mathcal{W}\) algebra structure).

Nevertheless what we already have can be exploited to produce and identify interesting classes of induced \(N = 2\) hierarchies.

Let us for the moment discuss just the simplest examples of superaffine algebras producing PE onto \(N = 2\) Virasoro. They are given respectively by the \(N = 2\) supersymmetric and can be reformulated with an \(N = 2\) formalism as stated before of

\(i\) the \(u(1) \oplus u(1)\) algebra;

\(ii\) the \(sl(2) \oplus u(1)\) algebra and

\(iii\) the \(sl(2|1)\) superalgebra.

In the first two examples the original algebra is not a simple one, this is just because we need a complex structure (provided by the extra \(u(1)\)) which enables us to have a second supersymmetry.

\(sl(2|1)\) is the simplest example of an \(N = 2\) Lie superalgebra; it contains \(sl(2)\) as subalgebra and four extra fermionic simple roots.

The first case coincides with the well-known \(N = 2\) version of (the three induced hierarchies of) \(m\)-KdV.

Much more interesting and rich of structure is case \(ii\). The algebra here admits a quaternionic structure which allows a full \(N = 4\) Sugawara construction \((\text{[11]}))\). This has the consequence that, besides the induced \(N = 2\) hierarchies (for \(a = 4, 2, -1\)) recovered from the \(N = 2\) Sugawara through, respectively, \(a\) the \(N = 2\) Virasoro \(J_{u(1) \oplus u(1)}\) associated to the \(u(1) \oplus u(1)\) subalgebra (it coincides with the previous case), \(b\) the full \(N = 2\) Virasoro \(J_{\text{full}}\) given by

\[J_{\text{full}} = H\overline{H} + FF + cD\overline{H} + \sigma\overline{D}H\]

(where the last two are the Feigin Fuchs terms), \(c\) the coset \(N = 2\) Virasoro given by

\[J_{\text{coset}} = J_{\text{full}} - J_{u(1) \oplus u(1)} = FF + cD\overline{H} + \sigma\overline{D}H,
\]

we can defined induced an induced hierarchy based on \(N = 4\) KdV. This hierarchy has a very interesting property, namely that it is consistent with the equations of motion to set

\[H = \overline{H} = 0.\]

The reduced hierarchy on the superfields \(F, F\overline{\cdot}\) coincides with the \(N = 2\) NLS.

In this framework the NLS hierarchy can therefore be directly obtained from its Poisson Embedding properties on \(N = 4\) KdV, in contrast for instance with the original coset construction of \(N = 2\) NLS \((\text{[8]}))\), whose superfields were mapped onto an \(N = 2\) Virasoro without central charge and for that reason integrability had to be proven separately and with different methods.

The \(iii\) case can be treated similarly. This superalgebra contains \(sl(2) \oplus u(1)\) as a subalgebra. All the induced hierarchies defined in the previous case can therefore be consistently extended. They involve extra equations of motions associated to the \(N = 2\) non-linearly (anti)-chiral bosonic superfields of spin \(\frac{1}{2}\) (of “wrong” statistics) associated to the fermionic roots.

It is clear at this point that a full class of \(N = 4\) induced hierarchies can be produced from the superaffinization of any quaternionic super-Lie algebra \(\mathcal{G}_Q\) (i.e. with \(N = 4\) supersymmetric PB), with the following recipe:

\(i\) individuate any \(sl(2) \oplus u(1)\) subalgebra,

\(ii\) construct from the given subalgebra the \(N = 4\) Sugawara leading to \(N = 4\) SCA,
iii) use this concrete realization of the $N = 4$ SCA as PB structure for the $N = 4$ KdV. An $N = 4$ hierarchy is automatically induced on the affine superfields generating the full $G_Q$ algebra. The induced hierarchy is automatically $N = 4$ invariant because by construction both the hamiltonians and the superaffine-$G_Q$ PB structure are $N = 4$ supersymmetric.

Furthermore, on these induced hierarchies it is possible to investigate whether consistent reductions can be imposed both as hamiltonian constraints or coset construction.

5 Conclusions.

In this talk I have investigated the role of Lie-algebraic methods in classifying integrable hierarchies. One of the nice features of the approach based on super-affine Lie algebras is that it allows investigating systematically hamiltonian constraints and coset reductions. The structure of $N = 4$ integrable hierarchies is currently under investigation, with the methods here outlined, in a collaboration with E. Ivanov and S. Krivonos.

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6 Appendix: the classification of cosets.

One of the nice features of the approach based on affine superfields is the fact discussed in the text that it allows to construct reduced hierarchies by exploiting symmetries and group-theoretical properties of the affine algebras. Here we will discuss the class of reductions known as “cosets”, which arises when superfields associated to semisimple (super)-Lie subalgebras are set equal to zero. This class of reductions are classified by all inequivalent (super)-Lie subalgebras which can be embedded into $G$, the (super)-Lie algebra whose superaffinization furnishes the original hierarchy.

The classification scheme to find all inequivalent Lie subalgebra of a given Lie algebra has been given by Dynkin. This scheme provides all possible cosets we can construct out of a given hierarchy. To be specific we specialize our discussion here to the $sl(3)$ case which already contains all the features we are interested in.

The full list of $sl(3)$ subalgebras is given by

$$u(1), \quad u(1) \oplus u(1), \quad sl(2), \quad sl(2) \oplus u(1)$$

(9)

The corresponding coset hierarchies involve $8 - 1 = 7$, $8 - 2 = 6$, $8 - 3 = 5$ and $8 - 4 = 4$ (super)-fields respectively ($8$ is the order of $sl(3)$). This is not the full story however because an abelian $u(1)$ generator ($= v$) can always be chosen to lie in the Cartan sector of $sl(3)$ and be specified by an angle $\phi$:

$$v = \cos \phi H_1 + \sin \phi H_2,$$

(10)
where $H_1, H_2$ are the two Cartan generators of $sl(3)$.

Coset hierarchies w.r.t. an abelian $u(1)$ are therefore labelled by such an angle $\phi$. The angle however is not completely arbitrary and is further required to lie on the interval

$$0 \leq \phi < \frac{\pi}{6}$$

The reason is the presence of the extra discrete symmetries which involve the Cartan decomposition of a Lie algebra. In case of $sl(3)$ there are two such sources of symmetries:

i) the $Out = \frac{Aut}{Int}$ automorphism group, which coincides here with $\mathbb{Z}_2$ and is related to the symmetry of the Dynkin diagram (i.e. the exchange of the two simple roots);

ii) the Weyl group of $sl(3)$, which is the 6-elements $S_3$ permutation group.

The full group of discrete automorphisms coincides here with the 12-elements group, direct product of $S_3$ and $\mathbb{Z}_2$. It can be easily realized that its action on the Cartan subspace spanned by the $H_1, H_2$ generator is that of the finite rotation group generated by 30-degrees rotations, which finally leads to the above restrictions on the angle $\phi$.

For what concerns $sl(2)$ subalgebras there exists only two inequivalent ways of embedding $sl(2)$ on $sl(3)$, up to the $Adj$-action of $sl(3)$ internal automorphism group. They correspond to the two decompositions of the 8 $sl(3)$ generators in terms of the $sl(2)$ representations.

Only the latter $sl(2)$ allows to accommodate a further abelian $u(1)$ subalgebra, and therefore the $sl(3)$ coset over $sl(2) \oplus u(1)$ is unique.

In this way we have listed the full class of coset-hierarchies arising from $sl(3)$. The case of the coset over $sl(2) \oplus u(1)$ is of particular interest because both $sl(3)$ and $sl(2) \oplus u(1)$ are quaternionic algebras. It is likely that its supersymmetric extension (under the appropriate $N = 4$-invariant hamiltonians) would correspond to an $N = 4$ coset hierarchy. This case is currently under investigation in a collaboration with Ivanov and Krivonos.

The above procedure can be carried in full generality and cosets can be classified according to the Dynkin scheme.

It is worth to notice that no restriction involving e.g. symmetric space is required to define consistent coset hierarchies, they are just (a more specialized) example of coset-construction.

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