The Geometric Construction of WZW Effective Action in Non-commutative Manifold

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Abstract. By constructing close one cochain density $\Omega^{1}{}_{2\alpha}$ in the gauge group space we get WZW effective Lagrangian on high dimensional non-commutative space. Especially consistent anomalies derived from this WZW effective action in non-commutative four-dimensional space coincides with those by L.Bonora etc[1].

Keywords: Non-commutative Space, WZW Effective Action, Close Cochain, Consistent Anomaly.

1 Introduction

The growing interest in non-commutative geometry is calling our attention upon some old problems of YM theory in non-commutative manifold. We would like to know whether or to what extent old problems, solutions or algorithms in commutative YM theories fit in the new non-commutative framework. One of these is what form the WZW effective action takes in the new non-commutative setting. This has partially been answered via a direct computation in some special conditions in non-commutative low dimension space [2]. But the generalization to high dimensions is very difficult, even impractical. So we try to construct the WZW effective action in non-commutative high dimensions via its geometric property. In fact, in this letter WZW effective action appears as the close-one-cochain of the gauge group in geometry [3, 4, 5, 6, 7, 8], which is invariant and only dependent on the topology of gauge group instead of the local geometry, for example, commutator. In sequence, this makes the calculation of the WZW effective action to be possible through the close-one-cochain of gauge group in the non-commutative high dimensions space.
2 Non-commutative Gauge Theory and WZW Effective Action

In the non-commutative geometry formalism, geometric spaces are described by a $C^*$-algebra, which is in general not commutative and realized by Moyal product:

$$f(x) \star g(x) \equiv e^{i\theta_{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu}} f(x + \xi) g(x + \zeta) \bigg|_{\xi=\zeta=0},$$

(1)

where $\theta_{\mu\nu}$ is a real antisymmetric constant background, and reflects the non-commutativity of the coordinates of $\mathbb{R}^D$:

$$[x_\mu, x_\nu]_* = i\theta_{\mu\nu}. $$

(2)

Then we have the following identities:

$$\int_{-\infty}^{+\infty} d^D x \ f(x) \star g(x) = \int_{-\infty}^{+\infty} d^D x \ g(x) \star f(x) = \int_{-\infty}^{+\infty} d^D x \ f(x) g(x),$$

(3)

$$\int_{-\infty}^{+\infty} d^D x \ (f_1 \star f_2 \star f_3) (x) = \int_{-\infty}^{+\infty} d^D x \ (f_3 \star f_1 \star f_2) (x) = \int_{-\infty}^{+\infty} d^D x \ (f_2 \star f_3 \star f_1) (x).$$

(4)

Especially, in integral with one compact space there is the following circular relation, i.e. the following relation can work up to a space-time derivative [1]:

$$\text{Tr}(E_1 \star E_2 \star \ldots \star E_n) = \text{Tr}(E_n \star E_1 \star \ldots \star E_{n-1}) (-1)^{k_n(k_1+\ldots+k_{n-1})},$$

(5)

where $E_i$ is the $k_i$ form in the base compact space. Then we will take such equivalence in this sense below, since the final result is required to integrate with one compact space. Additionally, we assume that the physics fields disappear in infinite distance point, so $\mathbb{R}^D$ the base space-time can be treated as $S^D$ space. Throughout this papers all calculation will be taken in $S^D$ space.

In path-integral formalism, the partition function of a system involving fermion fields and gauge fields is [13, 14, 15]:

$$Z = \int DA_\mu^a D\bar{\psi} D\psi e^{iS_G - \int \bar{\psi} \star (\bar{\psi} + A)^\star \psi},$$

(6)

where

$$S_G = -\frac{1}{4} \int d^D x \ F_{\mu\nu}(x) \star F^{\mu\nu}(x),$$

(7)

$$F_{\mu\nu}(x) \equiv \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + ig[A_\mu(x), A_\nu(x)]_*, $$

(8)

Set

$$D = \gamma^\mu \partial_\mu + \gamma^\mu [A_\mu, \ ]_*, $$

(9)
As usual, we first integrate the fermionic fields and treat the gauge fields as the background fields, then finish out all quantization.

\[ Z = \int \mathcal{D}A \det D e^{i S_G}, \quad \det D = \int \mathcal{D}\bar{\psi} D\psi e^{-i \int \bar{\psi} \left( i \partial + A \right) \psi}. \]  

(10)

From the \( \det D \) one can define the WZW effective action generated functional [7]:

\[ W[A] = -i \ln(\det D). \]  

(11)

Under the gauge transformation \( \det D \) changes into:

\[ \operatorname{Det} \left( \partial + A + g \right) = e^{-i \alpha^1(A; g)} \operatorname{Det} \left( \partial + A \right), \]  

(12)

where \( \alpha^1(A; g) \) is assumed to exist, which is thought reasonable from the existence of \( \alpha^1(A; g) \) in non-commutative two dimensions [3, 4, 5, 6, 7, 8] and the true consistent anomalies deduced from it in non-commutative four dimensions, and then one can prove that it is the close-one-cochain of the gauge group, which reveals the topology properties of the gauge group and does not dependent on the local property of the gauge group [3, 7, 8]. Then we can generalize it in non-commutative space, i.e. \( S^6 \).

3 The WZW effective action in non-commutative four dimensions space

The close-one-cochain of the gauge group in non-commutative four dimensions space, \( \mathcal{G}^4 \), behaves as the first order topology obstacle, which can be derived from the topology properties of the gauge group in higher by two dimensions, i.e. six-dimensions. So we now start with the non-commutative \( S^6 \). A-S index formula in commutative geometry may not work in non-commutative space as usual:

\[ Z = c_3 \int_{S^6} \operatorname{Tr}(F^3), \quad c_3 = \frac{i^3}{3!(2\pi)^3}. \]  

(13)

We still call \( Z \) Chen-Character though \( Z \) maybe is not an integer here [11]. When \( Z \neq 0 \), \( \Omega^{-1}_6 \equiv c_3 \operatorname{Tr}(F \star F \star F) \) is not an exact form in \( S^6 \), moreover the gauge potential \( A \) cannot be well-defined in \( S^6 \). Insteadly, one can well-define \( \Omega^{-1}_6(F) \) which is the close-zero-cochain in \( S^6 \).

We cover the manifold \( M = S^6 \) with two open sets: \( U_0 \cup U_1 = M \). Each open set \( U_j (j = 0, 1) \) is homomorph to the six-dimensional plate \( D_6 \) which is a trivial in topology. Then we can write \( \Omega^{-1}_6(F) \) as an exact form in each trivial area \( U_j \) due to Poincare lemma:

\[ \Omega^{-1}_6(F = dA_j + A_j \ast A_j) = d\Omega^0_5(A_j), \]  

(14)

and we get easily

\[ \Omega^0_5(A) = c_3 \operatorname{Tr} \left( A \ast dA \ast dA + \frac{3}{2} dA \ast A \ast A + \frac{3}{5} A \ast A \ast A \ast A \ast A \right) \]

\[ = c_3 \operatorname{Tr} \left( A \ast F \ast F - \frac{1}{2} A \ast A \ast A \ast F + \frac{1}{10} A \ast A \ast A \ast A \ast A \right). \]  

(15)
\Omega^0_5(A_j) can only be defined in a trivial area \(U_j\) but in whole manifold \(S^6\). The \(\Omega^0_5(A_j)\) is a \(\check{\text{Cech}}\) zero-cochain instead of \(\check{\text{Cech}}\) close-zero-cochain. In overlap area \(U_{01} = U_0 \cap U_1\), the difference of \(\Omega^0_5(A_j)\) is non-vanishing:

\[
\triangle \Omega^0_5(A_0, A_1) \equiv \Omega^0_5(A_1) - \Omega^0_5(A_0).
\] (16)

Notice that \(U_{01}\) and \(S_5\) are homomorphic, so \(U_{01}\) could be condensed into \(S_5\). Then

\[
Z = \int_{S_5} \triangle \Omega^0_5(A_0, A_1).
\] (17)

Since the gauge potentials \(\{A_j\}\) in two areas \(\{U_j\}\) are equivalent up to a gauge transformation, we can set:

\[
A_0 = A, A_1 = g^{-1} \ast A \ast g + g^{-1} \ast dg.
\] (18)

Thus we have

\[
\begin{align*}
\triangle \Omega^0_5(A, A^g) &= \frac{c_3}{2} d\text{Tr}(A \ast dA \ast (dg \ast g^{-1}) + dA \ast A \ast (dg \ast g^{-1}) \\
&\quad - A \ast (dg \ast g^{-1}) \ast (dg \ast g^{-1}) \ast (dg \ast g^{-1}) \\
&\quad - \frac{1}{2} A \ast (dg \ast g^{-1}) \ast A \ast (dg \ast g^{-1}) + A \ast A \ast A \ast (dg \ast g^{-1}) \\
&\quad + \frac{c_3}{10} \text{Tr}[(dg \ast g^{-1}) \ast (dg \ast g^{-1}) \ast (dg \ast g^{-1}) \ast (dg \ast g^{-1})].
\end{align*}
\] (19)

\[
Z = \frac{c_3}{10} \int_{S_5} \text{Tr}[(dg \ast g^{-1}) \ast (dg \ast g^{-1}) \ast (dg \ast g^{-1}) \ast (dg \ast g^{-1})],
\] (20)

In fact, what we have done above is to map strictly the differential form \(\wedge^*(S_6)\) in the manifold \(S_6\) into each area \(U_j\), and then take the \(\check{\text{Cech}}\) difference of mapping, i.e. \(\triangle\), which leads to Mayer-Vietoris series when it works constantly:

\[
0 \longrightarrow \wedge^*(S_6) \rightarrow \wedge^*(D_6^{(0)}) \oplus \wedge^*(D_6^{(1)}) \rightarrow \wedge^*(S_5) \rightarrow 0,
\] (21)

where we should notice that \(\Omega^{-1}_6(A_j) = d\Omega^0_5(A_j)\) is the \(\check{\text{Cech}}\) close-zero-cochain. Additionally, \(\check{\text{Cech}}\) difference \(\triangle\) and differential operator \(d\) commute, which makes \(\triangle \Omega^0_5(A_0, A_1)\) close form. Then we can have \(\triangle - d\) bi-complex.

In the same way we can cover \(S_6\) with three, four, \ldots, areas homomorphic with \(D_6\) and operate them with \(d - \triangle\) orderly. Thus we have the analogous series \(\Omega^1(A, A^{g_1}), \Omega^2(A, A^{g_1}, A^{g_2}) \ldots\) which satisfy:

\[
\begin{align*}
\triangle \Omega^{k-1}_{2n-k}(A^{g_0}, A^{g_1}, A^{g_2}, \ldots, A^{g_k}) &\equiv \sum_{i=0}^{k} (-1)^i \Omega^{k-1}_{2n-k}(A^{g_0}, A^{g_1}, \ldots, \hat{A}^{g_i}, \ldots, A^{g_k}), \\
\triangle \Omega^{k-1}_{2n-k}(A^{g_0}, A^{g_1}, A^{g_2}, \ldots, A^{g_k}) &= d\Omega^k_{2n-k-1}(A^{g_0}, A^{g_1}, A^{g_2}, \ldots, A^{g_k}), \\
&\quad (1 \leq k \leq 2n, 2n = 6),
\end{align*}
\] (22)

where \(\Omega^k_{2n-k-1}\) denotes the close-cochain density of gauge group space. Integrating with appropriate base space will give the close-cochains of the gauge group in the base space, that is

\[
\alpha^1(A; g) = 2\pi \int_{S_4} \Omega^1(A, A^g)
\] (24)
This is the close-one-cochain of the gauge group, $G_4^4$ in the $S_4$ (i.e. physics $R^4$). So we get our WZW effective action:

$$W[A, g] = 2\pi \int_{S_4} \Omega^1_4(A, A^g) = \alpha^1(\Gamma_1, S_4),$$  \hspace{1cm} (25)$$

$$\Omega^1_4(A, A^g) = \frac{c_3}{2} \text{Tr}[A \star dA \star (dg \star g^{-1}) + dA \star A \star (dg \star g^{-1}) - A \star (dg \star g^{-1}) \star (dg \star g^{-1}) \star (dg \star g^{-1}) - \frac{1}{2} A \star (dg \star g^{-1}) \star A \star (dg \star g^{-1}) + A \star A \star A \star (dg \star g^{-1})] + \frac{c_3}{10} d^{-1} \text{Tr}[(dg \star g^{-1}) \star (dg \star g^{-1}) \star (dg \star g^{-1}) \star (dg \star g^{-1}) \star (dg \star g^{-1})],$$  \hspace{1cm} (26)$$

where $d^{-1}$ is a homotopic operator and only well-defined in a trivial area in topology. Furthermore, we can argue the anomalies as expected if we expand $g$ around the unit $g = I + \Theta$:

$$W[A, I + \Theta] = -\frac{2\pi c_3}{2} \int_{S_4} \text{Tr}(d\Theta \star A \star dA + d\Theta \star dA \star A + d\Theta \star A \star A \star A).$$  \hspace{1cm} (27)$$

This is consistent with the results of L. Bonora and others [1, 16].

4 WZW Effective Action in Non-commutative high dimensions space

In principle, the similar analysis can be generalized into high dimensions. However, it will be very difficult to calculate the WZW effective action. In this section, we will take another way to get the close-one-cochain, even any order close-cochain. We can calculate the close-cochains via the relations between the close-cochains of connection space $U = \{A(x)\}$ and gauge group, then the $\Omega^1_2n(A, A^g)$ and the WZW effective action.

At first, we introduce the close-cochain of the connection space $U = \{A(x)\}$ which is a poly manifold in an affine space of infinite dimensions. So we have a linear insertion among $k$ connections which have no linear relation:

$$A_t = A_0 + \sum_{i=1}^{k} t_i \eta_i = A_\mu(x, t) dx^\mu, \quad \eta_i = A_i - A_0, \quad 0 \leq \sum_{i=1}^{k} t_i \leq 1. \hspace{1cm} (28)$$

Then a connection is set in $M \otimes R(t)^k$:

$$\mathcal{A} = (A_\mu(x, t) dx^\mu; 0) = (A_t; 0),$$  \hspace{1cm} (29)$$

where $R(t)^k$ is a commutative linear space which commutates with base space $M$ too. The field strength is

$$\mathcal{F} = (d + \delta)\mathcal{A} + \mathcal{A} \star \mathcal{A} = dA_t + \sum_{i=1}^{k} \delta t_i \eta_i + A_t \star A_t = F_t + H,$$  \hspace{1cm} (30)$$

where $d$ is the derivative operator in $M$ and $\delta$ the derivative operator with the arguments $\{t_i\}$ in arguments space $R(t)^k$, which satisfy $d^2 = \delta^2 = 0, \quad (d + \delta)^2 = d\delta + \delta d = 0$ and

$$F_t = dA_t + A_t \star A_t, \quad H = \sum_{i=1}^{k} \delta t_i \eta_i \equiv \delta t \cdot \eta,$$  \hspace{1cm} (31)$$
Expanding the trace of $F^n$ in the power of $H$, one can get

$$\text{Tr}(F^n) = \sum_{k=1}^{n} q_k^{2n-k}, \quad (32)$$

with

$$q_k^{2n-k} = \text{Tr}((H^k \ast F_t^{n-k}) + (H^{-1} \ast F_t \ast H \ast F_t^{n-k-1}) + \cdots + (F_t^{n-k} \ast H^k)], \quad (33)$$

where there are $\frac{n!}{k!(n-k)!}$ terms on the right hand side of Equ.$(33)$. On the other hand, since

$$(\delta + d)\text{Tr}(F^n) = 0. \quad (34)$$

multiplying $(d + \delta)$ on the both sides of Equ.$(32)$ gives the descent equation:

$$dq_k^{2n-k} = -\delta q_k^{-1}^{2n-k+1}, \quad k = 1, 2, \cdots, n \quad (35)$$

$$dq_0^{2n} = d\text{Tr}(F^n) = 0, \quad \delta q_n^{n} (A_0, A_1, \cdots, A_n) = \delta \text{Tr}(H^n) = 0. \quad (36)$$

Integrate $q_k^{2n-k}$ with the k-order chain $\Gamma_k$ in the connection space one can obtain the functional $Q_2n-k$ of $\Gamma_k$:

$$\Gamma_k = A_0 + \sum_{i=1}^{k} t_i \eta_i \equiv \Gamma_k(A_0, \cdots, A_k), \quad (0 \leq \sum_{i=1}^{k} t_i \leq 1), \quad (37)$$

$$Q_2n-k(A_0, \cdots, A_k) = Q_2n-k(\Gamma_k) = \int_{\Gamma_k} q_k^{2n-k}$$

$$\Gamma_{(i)}^{k-1} \equiv \Gamma^{k-1}(A_0, \cdots, \tilde{A}_i, \cdots, A_k), \quad \left(\Delta q_0^{2n} = 0, \quad (\partial \Gamma_k) \right)$$

where $Q_2n-k$ is the k-cochain in the connection space, called the generalized Chen-Simons form ($Q$ series). We can justify easily:

$$dQ_2n-k(\Gamma_k) = (-1)^{(k-1)} Q_2n-k+1(\partial \Gamma_k). \quad (39)$$

Define

$$Q_2n-k+1(\partial \Gamma_k) = \sum_{i=1}^{k} (-1)^i Q_2n-k+1(\Gamma_{(i)}^{k-1}) \equiv \Delta Q_2n-k+1(\Gamma_k), \quad (40)$$

we have

$$\Delta^2 = 0, \quad (41)$$

where $\Delta$ is called the coedge operator in the functional space of connections. So we have our homology in the connection space. In the following, we pay our attention on the expressions of $Q$ series.

According to the following exact series between the homotopy groups via injective and inclusion map:

$$\pi_{n+1}(U) \longrightarrow \pi_{n+1}(U/G) \longrightarrow \pi_n(G) \longrightarrow \pi_{n+1}(U), \quad (42)$$
we can get the cohomology group of gauge group if we calculate the cohomology group of connection space, i.e. Q series. In the expression of \( \Omega \) we can depart two parts similar to Equ.(29): the Q series including the gauge potential \( A \) and R series including only the pure gauge potential \( v \),

\[
\Omega^{k}_{2n-k-1}(A, A^{g_{1}}, \ldots, A^{g_{1}\cdots g_{k}}) = c_{n}(-1)^{k}Q_{2n-k-1}(0, A, v_{1}, \cdots, v_{k}) + R^{k}_{2n-k-1}(0, v_{1}, \cdots, v_{k}),
\]

where \( c_{n} \) is defined similar to \( c_{3} \) before and

\[
v_{1} = g_{1} * dg_{1}^{-1}, \quad v_{2} = (g_{1} * g_{2})d(g_{1} * g_{2})^{-1}, \cdots,
\]

where Q series have explicit expressions in (38). In order to calculate the R series we introduce the generalized coedge operator \( \bar{\Omega} \):

\[
(-1)^{k-1}\bar{\Omega}Q_{2n-k}(0, A, v_{1}, \cdots, v_{k}) = \Delta Q_{2n-k}(0, A, v_{1}, \cdots, v_{k}) - Q_{2n-k}(0, v_{1}, \cdots, v_{k}).
\]

Then we apply the coedge operator on the left hand side of the Equ.(43) but on the right hand side Q series must be applied with the generalized coedge operator \( \bar{\Omega} \):

\[
\Delta \Omega^{k-1}_{2n-k}(A, A^{g_{1}}, \cdots, A^{g_{1}\cdots g_{k}}) = c_{n}(-1)^{k-1}\bar{\Omega}Q_{2n-k}(0, A, v_{1}, \cdots, v_{k})
+ \Delta R^{k-1}_{2n-k}(0, v_{1}, \cdots, v_{k}),
\]

Additionally, we act on both sides of Equ.(43) with the derivative operator \( d \):

\[
\Delta \Omega^{k-1}_{2n-k}(A, A^{g_{1}}, \cdots, A^{g_{1}\cdots g_{k}}) = c_{n}\Delta Q_{2n-k}(0, A, v_{1}, \cdots, v_{k}) + dR^{k}_{2n-k-1}(0, v_{1}, \cdots, v_{k}).
\]

Comparing Equ.(46) and Equ.(47) we have the formula satisfied by R series:

\[
dR^{k}_{2n-k-1}(0, v_{1}, \cdots, v_{k}) = c_{n}Q_{2n-k}(0, v_{1}, \cdots, v_{k}) + \Delta R^{k-1}_{2n-k}(0, v_{1}, \cdots, v_{k}),
\]

i.e. the second term of Equ.(46) is a concrete example of above formula. With this formula we can calculate all order expressions of R series in the trivial area by the homotopy operator \( d^{-1} \), then we can have all explicit expressions of \( \Omega \) if the explicit expressions of R series are substituted in Equ.(26). Thus, we obtain the \( \Omega^{2n}_{2n} \), i.e. WZW effective Lagrangian and the WZW effective action when it is integrated.

5 Summary

In this article we study the geometric property of the WZW effective action and then construct the WZW effective action in non-commutative space via calculating the close-one-cochain of the gauge group in non-commutative space. We adopt two methods to construct the close-one-cochain in low dimensions and high dimensions. In four dimensions we cover the six-dimensional \( S^{6} \) with several trivial areas in topology to derive the close-one-cochain in the non-commutative four dimensions space. This method will become very difficult when it is generalized into high dimensions space for it is too difficult to express a complex expression with an exact form in a trivial area in topology of high dimensions so we chose another way to construct the close-cochains of gauge group. By calculating the cohomology group of the connection space we obtained the close-cochains of gauge group in arbitrary even dimensions and further calculated the WZW effective action. Though our work stopped here but we could work more since we have all close-cochains of the gauge group in the non-commutative high dimensions space. The close cochains may be used to discuss some physics problems related to the topology properties of non-commutative high dimensions space, such as the consistent anomalies [1, 17, 18, 9], Hamiltonian anomalies and Jacobi anomalies.
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