HOLOGRAPHY, CFT AND BLACK HOLE ENTROPY*

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Abstract

Aspects of holography or dimensional reduction in gravitational physics are discussed with reference to black hole thermodynamics. Degrees of freedom living on Isolated Horizons (as a model for macroscopic, generic, eternal black hole horizons) are argued to be topological in nature and counted, using their relation to two dimensional conformal field theories. This leads to the microcanonical entropy of these black holes having the Bekenstein-Hawking form together with finite, unambiguous quantum spacetime corrections. Another aspect of holography ensues for radiant black holes treated as a standard canonical ensemble with Isolated Horizons as the mean (equilibrium) configuration. This is shown to yield a universal criterion for thermal stability of generic radiant black holes, as a lower bound on the mass of the equilibrium isolated horizon in terms of its microcanonical entropy. Saturation of the bound occurs at a phase boundary separating thermally stable and unstable phases with symptoms of a first order phase transition.

1 Introduction

The laws of black hole mechanics [1] relate changes in the area $A_{hor}$ of a spatial section of the event horizon (EH) of stationary black hole spacetimes to variation in parameters like the change in the ADM mass $M$ and the surface gravity $\kappa_{hor}$ on the EH,

$$\delta A_{hor} \geq 0$$

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\[ \kappa_{\text{hor}} = \text{const} \]
\[ \delta M = \kappa_{\text{hor}} \delta A_{\text{hor}} + \cdots, \]
(1)

These laws appear to have a curious analogy with the zeroth, first and second laws of standard thermodynamics, with the area of the EH. Yet, black hole spacetimes like Schwarzschild and Kerr emerge as exact solutions of the vacuum Einstein equation in complete absence of matter or energy. There is no conceivable source for any kind of microstates usually thought to be as the origin of thermodynamical behaviour. The originators of these laws, understandably, did not venture beyond their formulation and derivation from general relativity.

Bekenstein [2] was the first to argue that these laws must signify thermodynamic behaviour of spacetimes, beyond mere analogy. If an object falls through an EH, there is a net reduction in the entropy of the part of spacetime external to that EH. To be consistent with the second law of standard thermodynamics, these spacetimes themselves must carry entropy, whose increase compensates for the reduction mentioned above, such that the sum of the black hole and external entropies never decreases. This, essentially, is Bekenstein’s statement of the so-called Generalized second law of thermodynamics. In order for this black hole entropy to be consistent with eq. (1), Bekenstein proposed that

\[ S_{bh} = \frac{A_{\text{hor}}}{4l_P^2} (k_B = 1). \]
(2)

where, \( l_P \equiv (G\hbar/c^3)^{1/2} \sim 10^{-33} \text{cm} \) is the Planck length, usually taken to be scale at which quantum gravitational effects dominate physics. The factor 4 in (2) emerges from Hawking’s formulation [3] of black holes in presence of ambient quantum matter as radiating like a thermal black body with a temperature given by \( \kappa_{\text{hor}} \), thus abundantly substantiating Bekenstein’s hypothesis. However, the natural emergence of the Planck length in (2) goes to suggest that thermodynamics associated with classical spacetimes really stems from microstates arising from the quantum ‘atoms’ of such spacetimes, as would be found in a theory of quantum gravitation. What remains a puzzling observation however, is the dependence of black hole entropy upon an area rather than on three dimensional volume as in standard thermodynamics. How and why that happens are the key issues we wish to address in this article.

The next section deals with a discussion of the Holographic hypothesis as formulated by ’t Hooft and our proposal that it may be a consequence of the huge invariance group associated with spacetime in general relativity. This is followed by a description of Weakly Isolated Horizons (WIH) as an inner boundary of spacetime which serves as the protottype of a large class of horizons including but not only, stationary black hole event horizons. How holography emerges in this description is also pointed out, especially in connection with the topological description of the boundary degrees of freedom. A very brief description of Loop Quantum Gravity is next given, with emphasis on the spectrum of certain geometrical observables. How this leads to a quantum theory of WIH and eventually to
the microcanonical entropy of generic, macroscopic, eternal black holes (in 4 dimensions) is in the next section. Another version of holography in spacetime physics is discussed next, within the context of a standard equilibrium statistical mechanics approach to a canonical ensemble of radiant (hence non-isolated) black holes. A rather universal criterion of thermal stability of radiant black holes is discussed, in which the mass of the equilibrium configuration (chosen to be a WIH) is bounded from below by the microcanonical entropy of this configuration. The point of saturation of this bound corresponds to a ‘phase boundary’ between a thermally stable and an unstable phase; the transition has tell-tale signs of a first order phase transition. Brief comparison with the pioneering work of Hawking and Page is given. We end with a list of issues yet unresolved.

2 Holographic Hypothesis

The Holographic hypothesis was postulated by ’t Hooft as one way of understanding how in gravitational physics, information about the full black hole spacetime is encoded on the EH - a three dimensional null hypersurface (which is also an ‘outer trapped surface’and a boundary of the black hole spacetime). The main idea of this hypothesis is best stated in ’t Hooft’s words,

... Given any closed surface, we can represent all that happens inside it by degrees of freedom on this surface itself. This ... suggests that quantum gravity should be described by a topological quantum field theory in which all degrees of freedom are projected onto the boundary.

The questions that immediately arise are

- Is Holography in (quantum) general relativity a consequence of diffeomorphism and local Lorentz invariance of spacetime in presence of boundaries (like the EH) ? If it does arise naturally in spacetime physics, one need not postulate it as an additional hypothesis.

- Is there a link to a topological field theory on a boundary of spacetime ?

- Is there any relation to a two dimensional conformal field theory ?

- How does one compute the entropy of black holes on this basis ?

- Is there any implication for thermal stability of radiant black holes ?

The above list is somewhat biased toward the order of topics in this article, and hence can be construed as a list of contents. We shall show, not always rigorously, that the answer to each of the queries above is most likely in the affirmative, thus obtaining most of what ’t Hooft hypothesized a decade and a half ago.

We begin with the oft-made observation that local gauge invariance is not a statement of symmetry but rather of redundancy of some of the degrees of freedom used to formulate
the theory. E.g., in vacuum Maxwell electrodynamics, the photon field $\mathbf{A}$ admits the decomposition

$$\mathbf{A} = \underbrace{\mathbf{A} - \nabla \int d^4x' G(x - x') \text{div}' \mathbf{A}'(x')}_{\mathbf{A}_T} + \underbrace{\nabla \int d^4x' G(x - x') \text{div}' \mathbf{A}'(x')}_{\mathbf{a}_L}$$

where $\Box G(x - x') = \delta^{(4)}(x - x')$. Under a gauge transformation $\mathbf{A} \to \mathbf{A}^\omega = \mathbf{A} + \nabla \omega$

$$\mathbf{A}_T \to \mathbf{A}_T^\omega = \mathbf{A}_T$$

$$\mathbf{a}_L \to \mathbf{a}_L^\omega = \mathbf{a}_L + \omega ,$$

thus clearly revealing the unphysical nature of the longitudinal degree of freedom $\mathbf{a}_L$. The 4-vector field $\mathbf{A}_T$ is 4-divergence free (transverse in spacetime). Integral curves of this field are closed spacelike curves. It is trivial to show that the Maxwell field strength tensor is uniquely determined by $\mathbf{A}_T$ and is completely independent of $\mathbf{a}_L$. Yet, the standard formulation of the Maxwell theory continues to use this redundancy, perhaps because most feel it is convenient to do so. What is also obvious is the nonexistence of a gauge invariant tensorial conserved Nöther current for vacuum electrodynamics, corresponding to gauge transformations. If gauge invariance were a symmetry, surely such a current would be in existence.

A similar redundancy exists in spacetime physics as portrayed in general relativity. Spacetime is diffeomorphic to itself under coordinate diffeomorphisms, thus underlining the redundancy of coordinate frames. The clinching evidence for this is the nonexistence of a covariantly conserved energy momentum tensor for spacetime. All attempts to construct such a tensor at best yield non-covariant expressions that can hardly be given a true physical meaning. There is thus no such thing as a local ‘gravitational’ energy density in the spacetime of general relativity.

This state of affairs becomes clearer in a canonical formulation of vacuum general relativity. In this formalism, diffeomorphism invariance of the theory leads to a ‘bulk’ Hamiltonian

$$H_{bulk} = \int_M \left[ \nabla \mathcal{H} + \nabla \cdot \mathbf{P} \right],$$

where $M$ is a three dimensional spacelike hypersurface (partial Cauchy surface) on which initial data are specified. This means that the Lapse and Shift functions are Lagrange multipliers, enforcing the First Class constraints (infinitesimal spacetime diffeomorphism generators),

$$\mathcal{H}^{(3g,3\Pi)} \approx 0$$

$$\mathbf{P}^{(3g,3\Pi)} \approx 0 .$$
Here, phase space is spanned by the 3-metric $^3 g$ and 3-momenta $^3 \Pi$. On the constraint surface then one has $H_{\text{bulk}} \approx 0$. There is thus no notion of a ‘bulk’ energy associated with spacetime in general relativity, signifying that diffeomorphism invariance is not a symmetry but a redundancy. In a formulation of general relativity using orthonormal tetrads as local Lorentz frame variables, there is in addition, another first class constraint corresponding to local Lorentz transformations.

Is there any notion at all of ‘gravitational energy’ in general relativity? For spacetimes that are asymptotically flat, i.e., those that in some sense approximate Minkowski spacetime infinitely far away from the EH, both in space and time, it is possible to define globally conserved quantities like energy, momentum or angular momentum at the boundary of spacetime. These definitions of global generators depend crucially on how one describes the asymptotic structure of spacetime. For spacetimes with additional ‘inner’ boundaries like the WIH, one can define an energy associated with the WIH, which is distinct from the energy defined at the asymptotic boundary. It is obvious in any case that the total Hamiltonian for a general relativistic system is given on the constraint surface in phase space by

$$H = H_{\text{bulk}} + H_{\text{bdy}} \approx H_{\text{bdy}},$$

where ‘boundary’ may consist of several disconnected hypersurfaces embedded in spacetime. Our main interest in what follows will be on WIHs as inner boundaries of spacetimes. In any event, the very fact that any notion of gravitational energy or momentum or indeed angular momentum of spacetime refers to the boundary of spacetime rather than the bulk is ample evidence of holography at play.

### 3 Weakly Isolated Horizons

Event horizons of stationary black holes are excessively global. This is implied in the following features of such horizons:

- Event horizons are determined only after the entire spacetime is known.

- Stationarity implies that the black hole metric has global timelike isometry with the corresponding Killing vector field generating time translations at spatial infinity.

- Event horizons are usually treated separately from cosmological horizons like de Sitter horizons. A unifying treatment is desirable.

- The ADM mass featuring in the First law of black hole mechanics in eq. (1) is defined at spatial infinity and is in no way associated with the event horizon.

These features make is necessary to seek generalizations which are characterized locally.
Figure 1: Weakly Isolated Horizon

The particular generalization which we adopt here is known as the weakly Isolated Horizon (IH), developed in [14].

The properties of an IH can be summarized as follows [14]:

- It is a null inner boundary of spacetime with topology $\mathbb{R} \otimes S^2$.
- The area of the IH $A_{\text{hor} \sim S^2} = \text{const.}$ This is what is inherent in the isolation. It also means that the IH is never crossed, even though there may exist matter and radiation arbitrarily close to it.
- It is a marginal outer trapping surface.
- There is no global timelike isometry associated with the IH; this implies that non-stationary generalization of stationary black hole horizons.
- On IH one can define a surface gravity $\kappa_i$, which however is not defined outside of the IH, since there is no global timelike Killing vector field allowing such a definition.
- The ‘Zeroth law of IH mechanics’ $\kappa_i|_{IH} = \text{const}$ can be demonstrated, although the norm of $\kappa_i$ is not fixed since there once again there is no timelike Killing vector.
- On IH, one can define mass $M_{IH} = M_{\text{ADM}} - E_{\text{rad}}$ such that $\delta M_{IH} = \kappa_i \delta A_{\text{hor}} + \ldots$; this may be termed as First law of Isolated Horizon Mechanics.
Such horizons correspond thermodynamically to a microcanonical ensemble with fixed $A_{\text{hor}}$.

Now, because any IH is an inner boundary, the variational principle cannot be applicable without an appropriate boundary term

$$S = S_{\text{EHL}} + S_{\text{IH}}$$

where, the second term is chosen such that its variation cancels the surface term arising from the variation of the Einstein-Hilbert-Lorentz action. On the other hand, since an IH is null, the 3-metric on the IH is degenerate: $\sqrt{g} = 0$. This implies that on the IH one cannot have an action of the usual type $\int f_{\mu} \sqrt{g} L$. Rather, the quantum field theory describing the IH dof must be a 3 dimensional Topological Field Theory (TFT).

The issue that must be addressed now is: which TFT? This question is of considerable importance because the microcanonical black hole entropy $S_{\text{bh}} \equiv \log \dim \mathcal{H}_{\text{TFT}}$, where $\mathcal{H}_{\text{TFT}}$ is the Hilbert space corresponding to the theory describing the dynamics of the IH. For this one needs to briefly touch upon the matter of appropriate canonical variables for general relativity. So far we have written eqns. (5), (6) and (6) symbolically in terms of functions of the 3-metric $g$ and its conjugate 3-momenta. Through canonical transformations and fixing the gauge invariance associated with local Lorentz boosts (time gauge), a more convenient formulation is seen to emerge [15] for a real $SU(2)$ connection $A_{SU(2)}$ on the spacelike Cauchy surface $M$ chosen to supply Cauchy data, with its conjugate momenta being the densitized triads $E$, the pullbacks of tetrads (local frame fields) to $M$.

With IH boundary conditions, the solution space of vacuum Einstein equation admits a closed two-form (symplectic structure)

$$\Omega(\delta_1, \delta_2) = \frac{1}{16\pi l_p^2} \text{tr} \int_M [\delta_1 A \wedge \delta_2 \Sigma - (1 \leftrightarrow 2)]$$

$$+ \frac{A_S}{8\pi\gamma l_p^2} \text{tr} \int_S [\delta_1 A \wedge \delta_2 A - (1 \leftrightarrow 2)]$$

$$\equiv \Omega_{\text{blk}} + \Omega_{\text{bdy}},$$

where, $S$ is spatial foliation of IH by $M$ and $\Sigma \sim E \wedge E$. The solution space corresponding to the $\Omega_{\text{bdy}}$ is the one corresponding to the $SU(2)$ Chern Simons equation with the pullback of $\Sigma$ playing the role of source on the IH. Further pulled back to the foliation $S$ of the IH, this equation is the Chern Simons Gauss law

$$\left(\frac{k}{2\pi} F_{CS}(A) + \Sigma\right)|_S = 0$$

where, $k \equiv [A_S/8\pi l_p^2]$ with $[\ ]$ signifying ‘nearest integer’. Thus, the entire role of bulk spatial degrees of freedom characterized by $\Sigma$ (determined by solving Einstein equation) is
as source for the Chern Simons degrees of freedom (given by $F_{CS}$) characterizing boundary (IH) geometry. It is clear that this is a version of a Holographic picture, albeit in a somewhat subtle way. Certain aspects of the Holographic Hypothesis have already been realized such as primacy of boundary degrees of freedom, a TFT on the inner boundary (IH) and so on. Note that this realization of the holography paradigm has been crucially dependent on the diffeomorphism and local Lorentz invariance of spacetime and the IH dynamics. We shall see another aspect of this picture later.

4 Loop Quantum Gravity: Spin Network Basis in brief

This is a background-independent, non-perturbative approach to quantum general relativity\[16\]. Quantum three dimensional space is pictured as a network with edges carrying spin $j = n/2$, $n \in \mathbb{Z}$ and vertices consisting of invariant $SU(2)$ tensors ('intertwiners'). The operators on this space are nonlocal, being holonomies of the $SU(2)$ connection along edges of the network and smeared densitized triads. The network is not a rigid network but a floating one more like a 3 dimensional fishnet. The length of the edges is not fixed to any scale; nor are the edges required to remain straight. Arbitrary knottings of edges are allowed. The vertices can have any valence consistent with conservation of net spin on the edges meeting at a vertex. Local Lorentz invariance and spatial diffeomorphism invariance require that the network be a closed one with no 'hanging' edges. Each state in this basis is required to be annihilated by the local Lorentz and spatial diffeomorphism generators, and the set of all spinnet states span a kinematical Hilbert space. The physical Hilbert space, consisting of the kernel of the Hamiltonian constraint, is yet to be worked out in detail.

The spinnet basis is the eigenbasis for geometrical operators like length, area and volume in three dimensions. Consider for instance a two dimensional spacelike surface of classical area $A_{cl}$ embedded in a spin network; links of the network will intersect this surface. Assume that the surface is divided into tiny patches such that each patch is pierced by only a single link, whose spin is encoded in the puncture on that patch. The eigenvalues of the area operator are then given, in terms of the spins $j_i$, $i = 1, 2, \ldots, N$ at these punctures (or equivalently, the spin numbers $n_i$) by

$$a(n_1, \ldots, n_N) = \frac{1}{4} \gamma l_P^2 \sum_{p=1}^{N} \sqrt{n_p(n_p + 2)}$$

$$\lim_{N \to \infty} a(n_1, \ldots, n_N) \leq A_{cl} + O(l_P^2)$$

The area operator thus has a bounded, discrete spectrum, even though there is a certain degree of arbitrariness in the scale of discreteness \[16\]. Also, the rate of convergence of the discrete eigenvalues to the continuum value is approached rather rapidly \[16\].
5 Quantum Isolated Horizon

Now, the IH embedded in a spatial geometry represented by spin networks; the spatial section of an IH, assumed spherical here for simplicity, is a two dimensional surface punctured by spin network links which transmit their spins to the punctures. A cartoon depicting this is shown in the following diagram.

With the kind of penetration of spin network links discussed above, the Consistency Condition eq.(9) can be expressed as a condition in terms of the relevant quantum operators operating on the kinematical Hilbert space,

\[
\left( k \hat{F}_{CS} + \hat{E} \times \hat{E} \right)_{S} |\Psi\rangle = 0,
\]

where \(|\Psi\rangle \in \mathcal{H}_{bulk} \otimes \mathcal{H}_{bdy}\). The object now is to compute \(\dim \mathcal{H}_{bdy}\) which is basically the dimensionality of the CS Hilbert space on an IH with \(N\) punctures on the \(S^2\) having spins \(j_i\) acting as pointlike sources.
5.1 Counting of CS states

Having argued that the boundary (IH) degrees of freedom constitute those of an $SU(2)$ Chern Simons theory with pointlike spin-valued sources, we now relate the Hilbert space of the Chern Simons theory to conformal blocks of the $SU(2)_k$ WZW model that lives on the foliation $S$ of the IH, following Witten and others [17]. According to this relationship, 
\[ \dim \mathcal{H}_{CS+\text{sources}} = \Omega(j_1, \ldots, j_N) \]
where $\Omega$ is the number of conformal blocks of the $SU(2)_k$ WZW model 'living' on $S^2$ with point sources carrying spins $j_1, \ldots, j_N$.

Using the Verlinde formula, this number can be computed exactly [6]. Alternatively, one can solve the Chern Simons theory and calculate $\Omega(j_1, \ldots, j_N)$ directly [5]. Our interest is in macroscopically large black holes (IHs), i.e., those IHs whose areas $A_{IH} >> l_P^2$. Thus the deficit angles of punctures made are being required to together represent a smooth $S^2$.

Intuitively, this means that one must maximize the number of punctures for a given fixed classical area, and this requires the spin on each puncture to be as small a possible, i.e., spin $1/2$. With this choice, $\Omega(1/2, 1/2, \ldots, 1/2)$ can be easily computed and its logarithm extracted, giving the microcanonical entropy [7],

\[ S_{\text{micro}} = k_B \left[ \frac{A_{\text{hor}}}{4l_P^2} - \frac{3}{2} \log \left( \frac{A_{\text{hor}}}{4l_P^2} \right) + O \left( \left( \frac{A_{\text{hor}}}{4l_P^2} \right)^{-1} \right) \right]. \]  

(13)

where, the correct normalization for the leading order Bekenstein-Hawking result emerges as a fit for a real parameter (Barbero-Immirzi parameter $\gamma$) invariably multiplying the Planck scale $l_P$. Observe, however, that this is the only ambiguity in the entire infinite series each of whose terms are finite.

Admittedly, the counting presented above is rather crude! One should actually consider a set of $N$ punctures with spins $j_i$, $i = 1, \ldots, N$, count $\Omega(j_1, \ldots, j_N)$ and sum over all possible spins and punctures. Setting all spins equal to $1/2$ is not guaranteed to give the same number, although in view of our argument regarding deficit angles, it may be a reasonable approximation for asymptotically large horizon areas. Various computations of $\Omega$ by this procedure have been reported. The one that appears to resemble a truly statistical mechanical computation is that of [18] which is quite consistent with our final result (13) in so far as leading logarithmic corrections to the area law are concerned.

6 Low-tech way: It from Bit

The idea of It from bit [19, 4] essentially is to think of a floating two dimensional lattice covering a two dimensional sphere $S^2$ which we take to be the horizon. Each plaquette of the lattice is taken to be of Planck area size, so that the area of the horizon in Planckian units is given by an integer $p \equiv A_{\text{hor}}/l_P^2$, which is also the number of plaquettes covering...
the horizon. Place in each plaquette a spin 1/2 object so that two states can be associated it, the $m_s = \pm 1/2$ states. If all states of the horizon with random orientation of spin 1/2 variables are considered, the number of states $\Omega(p) = 2^p$. However, as in the previous subsection, we are interested in states of this system with net spin $j_{tot} = 0$ so that

$$\Omega(p) = \binom{p/2}{p/2} - \binom{p/2 + 1}{p}$$

which yields the same degeneracy and $S_{micro}$ upon using the Stirling approximation for the factorials in eq. (14).

Without having made any use of the Holographic hypothesis, all aspects of it have thus been seen to be realized in the ab initio computation of the microcanonical entropy of IHs which we believe serve as good prototypes of black hole event horizons. The only lacuna in the computation is that this is actually a computation of the dimensionality of the kinematical Hilbert space rather than the physical one. There is thus the crucial assumption that the states we count also belong to the physical Hilbert space. This remains a grey area because of the difficulties associated with discerning the semiclassical behaviour of the states spanning the kernel of the bulk Hamiltonian operator. On the positive side, the calculation we present yields an asymptotic series of terms of decreasing powers of the area of the IH, of which only the first, i.e., the term proportional to $A_{IH}$ had been anticipated by Bekenstein and Hawking. Each of the quantum spacetime correction terms is robust, unambiguous and finite, requiring no ad hoc regularization or indeed renormalization of coupling constants. However, the system under consideration corresponds to a nonradiant
(or nonaccreting) isolated black hole and therefore is unphysical. It is not as yet clear how the foregoing formalism can be extended or generalized to realistic situations. In the following we adopt a pragmatic approach based on equilibrium statistical mechanics of canonical Gibbs ensembles including Gaussian thermal fluctuation corrections, to reach some understanding of radiant black holes, based on our approach to quantum spacetime geometry.

7 Radiant Black Holes

Black holes undergoing Hawking radiation (or indeed thermal accretion) show a runaway behaviour in that as they radiate, their Hawking temperature increases. This is once again in contradistinction with standard thermodynamic systems which radiate and cool down as they approach thermal equilibrium. This instability due to thermal radiation appears to occur for all asymptotically flat spacetimes \[13\] but not necessarily for asymptotically anti-de Sitter spacetimes. In this part of the article, we analyze the situation from our standpoint of equilibrium statistical mechanics within a ‘mean field’ approach incorporating thermal fluctuations. We also choose the mean field or equilibrium configuration to be an IH whose quantum behaviour we believe is on firm footing as explained in the previous sections. Beyond this assumption, our approach is not exactly semiclassical, in contrast to most of the literature on black hole thermodynamics, because we do not use any aspect of classical spacetime geometry, like the form of the metric or even its asymptotic structure. This allows us to derive a general criterion for thermal stability of black holes.

In canonical general relativity, we mentioned that the bulk Hamiltonian is a sum of first class constraints. The total Hamiltonian for any spacetime with boundary is thus

\[ H = H_{bulk} + H_{bdy} \]  \hspace{1cm} (15)

where, \( H_{bdy} \) is the Hamiltonian corresponding to all boundaries including the one at spatial \( \infty \) (i.e., the ADM Hamiltonian) such that

\[ H_{bulk} \approx 0 \]  \hspace{1cm} (16)

In any theory of quantum general relativity, one expects

\[ \hat{H} = \hat{H}_{bulk} + \hat{H}_{bdy} \]  \hspace{1cm} (17)

such that

\[ \hat{H}_{bulk} |\psi_N\rangle_{blk} = 0 \]  \hspace{1cm} (18)

where \( |\psi_N\rangle_{blk} \) are states characterizing bulk space.
Choose as basis eigenstates of the full Hamiltonian

$$|\Psi\rangle = \sum_{N,\alpha} c_{N,\alpha} |\psi_N\rangle_{blk} |\chi_\alpha\rangle_{bdy}$$

(19)

With these properties, the canonical partition function

$$Z = \text{Tr} \exp -\beta (\hat{H}_{blk} + \hat{H}_{bdy})$$

$$= \sum_{N,\alpha} |c_{N,\alpha}|^2 \langle \psi_N | \exp -\beta \hat{H}_{blk} |\psi_N\rangle \langle \chi_\alpha | \exp \beta \hat{H}_{bdy} |\chi_\alpha\rangle$$

$$= \sum_{\alpha} |\tilde{c}_\alpha|^2 \langle \chi_\alpha | \exp \beta \hat{H}_{bdy} |\chi_\alpha\rangle$$

$$= \text{const.} \cdot \text{Tr}_{bdy} \exp -\beta \hat{H}_{bdy}$$

$$\equiv Z_{bdy},$$

(20)

where we have used (18) for the bulk states $|\psi_N\rangle$. This has the rather remarkable ramification that black hole thermodynamics is completely determined by the boundary partition function This is the holographic picture reappearing in another guise, once again originating from the Hamiltonian constraint characterizing temporal diffeomorphism invariance. Consistent with this picture is the preeminence of the area of the horizon rather than the volume in determining the entropy of black hole spacetimes, although this is not directly follow from our arguments.

### 7.1 Saddle Point Approximation

Assume now that

- Boundary in question is a black hole event horizon
- Eigenvalues of $\hat{H}_{bdy}$: $\mathcal{E}_n = \mathcal{E}(a_n)$ where, $a_n = 4\pi \sqrt{3} \gamma l_P^2 \ n$

These imply that

$$Z_{bdy} = \sum_n g(\mathcal{E}(a_n)) \exp \beta \mathcal{E}(a_n)$$

(21)

$$\simeq \int dn \ g(\mathcal{E}(a(n))) \exp -\beta \mathcal{E}(a(n)) \text{ for } n >> 1$$

(22)

$$= \int d\mathcal{E} \ \exp(S_{micro}(\mathcal{E}) - \beta \mathcal{E} - \log |\frac{d\mathcal{E}}{dn}|),$$

(23)

where $g(\mathcal{E}(a_n))$ is the degeneracy associated with the area eigenstate state labelled by $a_n$.

We now make the saddle point approximation, with saddle point chosen to be $\mathcal{E} = M_{TH}$, i.e., equilibrium configuration is chosen to be an isolated horizon with a mass $M(A_{hor})$. 
In saddle point approx using standard formulae of equilibrium statistical mechanics including Gaussian thermal fluctuations,

$$S_{\text{canon}} = S_{\text{micro}}(A_{\text{hor}}) - \frac{1}{2} \log \Delta$$

where,

$$\Delta \propto \frac{M''(A_{\text{hor}})S'_{\text{micro}}(A_{\text{hor}}) - M'(A_{\text{hor}})S''_{\text{micro}}(A_{\text{hor}})}{M'(A_{\text{hor}})S'^2(A_{\text{hor}})}$$

For thermal stability, $S_{\text{canon}} \rightarrow \text{real} \Rightarrow \Delta > 0$; it turns out that this is also a necessary condition which guarantees the positivity of the heat capacity. This

$$\Delta > 0 \Rightarrow M''(A_{\text{hor}})S'_{\text{micro}}(A_{\text{hor}}) > M'(A_{\text{hor}})S''_{\text{micro}}(A_{\text{hor}})$$

Upon integrating this inequality with respect to area and choosing appropriate dimensional constants for simplicity, we get

$$M(A_{\text{hor}}) > S_{\text{micro}}(A_{\text{hor}}),$$

a criterion [8], [9] - [11] described entirely in terms of quantities well-understood within the IH-LQG framework.

Since this criterion has been obtained without any reference to classical spacetime geometry, it is perhaps worthwhile to check that indeed it holds for known semiclassical situations.

### 7.2 Schwarzschild Black Hole

In this case, the area and mass are related by the well-known relation

$$M(A_{\text{hor}}) \sim A_{\text{hor}}^{1/2}$$

It is obvious that, since the $S_{\text{micro}} \sim A_{\text{hor}}$ in this case, for large $A_{\text{hor}}$ the bound is not satisfied, and we expect a thermally unstable situation, as indeed it is. This is consistent with the heat capacity $C < 0$. Unfortunately, this sort of instability is endemic to all asymptotically flat black hole spacetimes, as has been discussed in detail in [13].

### 7.3 AdS Schwarzschild Black Hole

Asymptotically anti-de Sitter spacetimes have a timelike infinity requiring specification of incoming data. The incoming radiation has to precisely cancel the outgoing one in order to completely specify Cauchy data, for a range of black hole parameters (mass $M$ and
cosmological constant $\Lambda$), thereby guaranteeing a stable thermal equilibrium. This range is given by $l \equiv (-\Lambda)^{-1/2} \ll (A_{\text{hor}}/4\pi)^{1/2}$. How do we see that in the criterion derived above? For this we need only use the mass-area relation for AdS Schwarzschild black holes

$$ M(A_{\text{hor}}) = \frac{1}{2} \left( \frac{A_{\text{hor}}}{4\pi} \right)^{1/2} \left( 1 + \frac{A_{\text{hor}}}{4\pi l^2} \right). \quad (30) $$

So long as the area is within the range specified above, it is obvious that $M \sim A_{\text{hor}}^{3/2} > S_{\text{micro}}(A_{\text{hor}})$ which means that the inequality (28) is satisfied. As one approaches the endpoints of the range, i.e., when $l$ approaches $(A_{\text{hor}}/4\pi)^{1/2}$, the system looks more and more like an asymptotically flat Schwarzschild spacetime, and thermal instability begins to set in.

It is obvious however that enroute to instability, the inequality (28) must saturate for certain values of the parameters. This can be shown[12] to correspond to the heat capacity blowing up from the positive side. When the inequality reverses, the heat capacity is definitely negative, exhibiting a sort of behaviour reminiscent of first order phase transitions in statistical mechanics.

As emphasized earlier, no classical metrics have been used anywhere in this analysis, and in this sense the treatment here can be thought of as a generalization of the pioneering semiclassical treatment of Hawking and Page [13]. Observe also that the criterion for thermal stability involves the domination of the black hole mass over the microcanonical entropy which has to do with the disorder associated with the degrees of freedom of the quantum isolated horizon. Thus the transition from a thermally stable to an unstable phase, although not a typical phase transition in statistical mechanics, is still not without similar symptoms.

8 Questions Yet to be Resolved

In this section we list a list of pending issues which await satisfactory resolution:

- The origin of the assumed mass-area relation remains somewhat obscure, although an approach may be to derive it from an analogue of the relation in LQG between the area operator and the bulk Hamiltonian [20].

- An important issue is the possibility of Hawking radiation from an IH. It is to be seen what precisely among the characterizations of isolation can best be discarded to enable this.

- Can one go beyond effective description of black hole as IH (inner boundary of sptm) and consider realistic dynamical collapse?
• Is the thermal nature of the Hawking radiation spectrum an artifact of the semi-classical approximation?

• Does the lowest area quantum $\sim l_p^2$ have implications for the information loss problem?

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