A NEW COMPUTABLE VERSION OF HALL’S HAREM THEOREM

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ABSTRACT. We prove a version of Hall’s Harem Theorem where the matching realizes a function with some special properties of its cycles. Using this approach we also prove a new computable version of Hall’s Harem theorem. We apply it to non-amenable computable coarse spaces.

1. Introduction

The Hall Harem theorem describes a condition which is equivalent to the existence of a perfect \((1,k)\)-matching of a bipartite graph, for example see Section III.3 of [Bol79] or Theorem H.4.2. in [CSC10]. Assuming that such a graph is of the form \((\mathbb{N}, \mathbb{N}, E)\), where \(E\) denotes the set of edges between natural numbers, the matching clearly realizes a \(k\) to 1 function, say \(f : \mathbb{N} \rightarrow \mathbb{N}\). In this paper we study properties which can be additionally added to such a function. In particular we are especially interested in fast reaching of cycles of \(f\). This will be formalized below as the property of controlled sizes of cycles.

The original motivation to such investigation comes from computable amenability, a topic which recently appeared in papers of M. Cavaleri [Cav17, Cav18], N. Moryakov [Mor18], and the author (together with A. Ivanov), see [DI22b, DI22a]. In order to obtain a computable version of Schneider’s theorem on existence of \(d\)-regular forests in non-amenable coarse spaces [Sch18] some special form of Hall’s harem theorem is required. In fact we need a matching which is realized by a computable subset of \(\mathbb{N}^2\) with the additional property that there is an algorithm which decides whether two natural numbers \(m\) and \(n\) belong to the same connected component of \(f\). The property of controlled sizes of cycles which was mentioned above guarantees existence of such an algorithm for computable functions.

The version of Hall’s theorem which gives the required collection of properties is proved in Sections 7 - 13. The proof is very complicated. It uses special arguments, which develop the method of Kierstead [Kie83] and the further modification of it given in [DI22a]. This made the author search an intermediate statement which demonstrates the main idea of the construction but avoids computability issues as much as it is possible. The result of this work is the theorem which is called The Main Theorem. This is a classical version of Hall’s harem theorem with additional restrictions on \(f\). The author is convicted that this version is interesting by itself. It does not follow from the Computable Version of the Main Theorem proved in Sections 7 - 13.

The formulation of the classical version of Hall’s harem theorem with controlled sizes of cycles is given in Section 2. See Theorem 2.4. The proof of it is given in Sections 3 - 6.

2. Preliminaries

2.1. General preliminaries. To introduce the reader to the subject we recall Hall’s Harem theorem. We begin with some necessary definitions from graph theory. We mostly follow the notation of [CSC10]. A graph \(\Gamma = (\Gamma, E)\) is called a bipartite graph if the set of vertices \(\Gamma\) is partitioned into sets \(U\) and \(V\) in such a way, that the set of edges \(E\) is a subset of \(U \times V\). We denote such a bipartite graph by \(\Gamma = (U, V, E)\).

From now on we concentrate on bipartite graphs. Note that although the definitions below concern this case, they usually have obvious extensions to all ordinary graphs.

A subgraph of \(\Gamma\) is a triple \(\Gamma' = (U', V', E')\) with \(U' \subseteq U\), \(V' \subseteq V\), \(E' \subseteq E\). When \(\Gamma'\) is such a subgraph but only the sets of its vertices are specified (i.e. for example \(\Gamma' = (U', V')\)), this means that \(E' = E \cap (U' \times V')\). Then we say that the subgraph \(\Gamma'\) is induced in \(\Gamma\) by the set of vertices.
Let $\Gamma = (U, V, E)$. We will say that an edge $(u, v)$ is adjacent to vertices $u$ and $v$. In this case we say that $u$ and $v$ are adjacent too. When two edges $(u, v), (u', v') \in E$ have a common adjacent vertex we say that $(u, v), (u', v')$ are also adjacent. A sequence $x_1, x_2, \ldots, x_n$ is called a path if each pair $x_i, x_{i+1}$ is adjacent, $1 \leq i < n$.

Below we will denote the set of vertices $\Gamma$ by the same letter with the graph as a structure, i.e. $\Gamma$. Given a vertex $x \in \Gamma$ the neighbourhood of $x$ is a set
\[ N_\Gamma(x) = \{ y \in \Gamma : (x, y) \in E \}. \]
For subsets $X \subset U$ and $Y \subset V$, we define the neighbourhood $N_\Gamma(X)$ of $X$ and the neighbourhood $N_\Gamma(Y)$ of $Y$ by
\[ N_\Gamma(X) = \bigcup_{x \in X} N_\Gamma(x) \quad \text{and} \quad N_\Gamma(Y) = \bigcup_{y \in Y} N_\Gamma(y). \]
We drop the subscript $\Gamma$ if it is clear from the context.

Within this paper we always assume that $\Gamma$ is locally finite, i.e. the set $N(x)$ is finite for all $x \in \Gamma$. Moreover we use the following notion of Kierstead, [Kie83]. The subset $X$ of $U$ (resp. of $V$) is called connected if for all $x, x' \in X$ there exist a path $x = p_0, p_1, \ldots, p_k = x'$ in $\Gamma$ such that $p_i \in X \cup N_\Gamma(X)$ for all $i \leq k$. Note here that this definition concerns only bipartite graphs.

For a given vertex $v$ a star of $v$ is a subgraph $S = (\{v\} \cup N_\Gamma(v), E')$ of $\Gamma$, with $E' = (\{v\} \cup N_\Gamma(v)) \times (\{v\} \cup N_\Gamma(v)) \cap E$. A $(1, k)$-fan is a subset of $E$ consisting of $k$ edges adjacent to some vertex $u \in U$. We say that $u$ is a root of the fan, and when $(u, v)$ belongs to the fan we call $v$ a leaf of it.

**Definition 2.1.** An $(1, k)$-matching from $U$ to $V$ is a collection $M$ of pairwise disjoint $(1, k)$-fans.

The $(1, k)$-matching $M$ is called perfect if each vertex from $U$ is a root of a fan from $M$, and each vertex from $V$ belongs to exactly one fan of $M$.

We often view an $(1, k)$-matching as a bipartite graph $M$ where the fan of $u \in U$ is the $M$-star of $u$, i.e. a graph consisting of the set of all vertices and edges adjacent to $u$ in $M$. We emphasize that a perfect $(1, k)$-matching from $U$ to $V$ is a set $M \subset E$ satisfying following conditions:

1. each vertex $u \in U$ there exists exactly $k$ vertices $v_1, \ldots, v_k \in V$ such that $(u, v_1), \ldots, (u, v_k) \in M$;
2. for all $v \in V$ there is an unique vertex $u \in U$ such that $(u, v) \in M$.

**Theorem A** (Hall’s Harem theorem). Let $\Gamma = (U, V, E)$ be a locally finite graph and let $k \in \mathbb{N}, k \geq 1$. The following conditions are equivalent:

(i) For all finite subsets $X \subset U, Y \subset V$ the following inequalities holds: $|N(X)| \geq k|X|$, $|N(Y)| \geq \frac{k}{E}|Y|.$

(ii) $\Gamma$ has a perfect $(1, k)$-matching.

The first condition in this formulation is known as Hall’s $k$-harem condition.

2.2. **Reflections.** Throughout the paper, $d$ is a natural number greater than 1. When $\Gamma = (U, V, E)$ is a bipartite graph, we always assume that $V \subset U \subset \mathbb{N}$, i.e. $V$ is a subset of the right copy of $U$.

The following notation substantially simplifies the presentation. For any vertex $v \in V$ there exist a vertex from $U$ which is a copy of $v$ (i.e. the same natural number), we denote it by $u_v$. If a vertex $u \in U$ has the copy in $V$ then we denote this copy by $v_u$.

**Definition 2.2.** The graph $\Gamma = (U, V, E)$ is called $U$-reflected if $V$ is a subset of the right copy of $U$ and for every edge $(u, v) \in E$ with $v_u \in V$ the edge $(u_v, v_u)$ is in $E$ too. If additionally $V$ is a right copy of $U$ then $\Gamma$ is called a fully reflected bipartite graph.

The main theorem below states that in the case of fully reflected bipartite graphs the Hall’s harem condition allows us to force some additional properties at the expense of obtaining a perfect $(1, (d - 1))$-matching instead of a $(1, d)$-matching. We will now give necessary details.
2.3. **Controlled sizes of cycles. Main theorem.** Let $f$ be a function. If for some $i \neq 0$ we have $f^i(u) = u$ then we will say that $u$ is a periodic point. For such $u$ and the smallest $i \neq 0$ with $f^i(u) = u$, we say that $\{u, f(u), \ldots, f^{i-1}(u)\}$ is a cycle of $f$. Any $(1, (d-1))$ matching can be considered as a $(d-1)$ to 1 function $f : \mathbb{N} \to \mathbb{N}$. Moreover, if this matching is perfect, such a function $f$ is total and surjective. We roughly want to show that given a fully reflected bipartite graph satisfying Hall's $d$-harem condition, there is a perfect $(1, (d-1))$ matching $f : \mathbb{N} \to \mathbb{N}$, such that for each $u$ there exist $i \geq 0$ such that $f^i(u)$ is a periodic point.

**Definition 2.3.** Let $f : \mathbb{N} \to \mathbb{N}$ be a $(d-1)$ to 1 function. We say that $f$ has controlled sizes of its cycles if each of the following conditions holds:

1. $f^2(1) = 1$;
2. if $n \geq 2$ and $f^i(n) = n$ then $i \leq n$;
3. if $n \geq 2$ and for all $i \leq n$ we have $f^i(n) \neq n$ then there exist $k \leq 2n$ and $l \leq n$ such that $f^{k+l}(n) = f^k(n)$.

The following theorem is the main result of the paper.

**Theorem 2.4.** (Main Theorem) Let $\Gamma = (U, V, E)$ be a locally finite bipartite graph such that:

- both $U$ and $V$ are identified with $\mathbb{N} \setminus \{0\}$,
- $E$ does not contain edges of the form $(u, v, u)$,
- $\Gamma$ is fully reflected,
- $\Gamma$ satisfies Hall's $d$-harem condition.

Then there exist a perfect $(1, (d-1))$-matching of $\Gamma$, which realizes a $(d-1)$ to 1 function $f : \mathbb{N} \to \mathbb{N}$ with controlled sizes of its cycles.

The proof of the theorem is based on an inductive construction of the matching.

2.4. **Notation used in the construction.** Since this construction is highly technical, we start with a list of the notation. We do not insist on a thorough inspection of it. In the beginning a hasty view will suffice.

- $M$ is a perfect matching that we construct.
- $M_n$ is a set of $(1, d-1)$-fans added to $M$ at the end of the $n$-th step. Thus $M = \bigcup_{n=1}^{\infty} M_{n-1}$.
- $\Gamma^{(0)} = (U^{(0)}, V^{(0)}, E^{(0)})$ is the original graph $\Gamma$.
- $U^{(n)} := U^{(n-1)} \setminus \{u \in U^{(n-1)} : \exists v \in V^{(n-1)}, (u, v) \in M_{n-1}\}$.
- $V^{(n)} := V^{(n-1)} \setminus \{v \in V^{(n-1)} : \exists u \in U^{(n-1)}, (u, v) \in M_{n-1}\}$.
- $\Gamma^{(n)} = (U^{(n)}, V^{(n)})$. We will see that $\Gamma^{(n)}$ is $U^{(n)}$-reflected.
- After the $n$-th step we obtain decompositions $U^{(n)} = U^{(n)*} \cup U^{(n)\perp}$ and $V^{(n)} = V^{(n)*} \cup V^{(n)\perp}$, where we say that $U^{(n)\perp}$ consists of elements from $U^{(n)}$ which might spoil Hall’s $d$-harem condition for $\Gamma^{(n)}$.
- Put $U^{(0)*} = \emptyset$ and $V^{(0)\perp} = \emptyset$ (since $\Gamma^{(0)}$ is $\Gamma$, i.e. it satisfies Hall’s $d$-harem condition).
- $\Gamma^{(n)*}$ is a graph with the sets of vertices $(U^{(n)*}, V^{(n)*})$ and the set of edges corresponding to $(1, d-1)$-fans with roots denoted by $u^{(n)}_{j, i} \in U^{(n)*}$.
- When $U^{(n)*} \setminus U^{(n-1)*}$ is not empty, $U^{(n)*} \setminus U^{(n-1)*} = \{u^{(n)}_{j, i} : 1 \leq i \leq d-1\}$ and $V^{(n)*} \setminus V^{(n-1)*}$ consists of leaves $\{v^{(n)}_{j, i-1} : 1 \leq i \leq d-1\}$.
- $\Gamma^{(n)*} := (U^{(n)} \setminus U^{(n)*}, V^{(n)} \setminus V^{(n)*})$. We will see that $\Gamma^{(n)*}$ satisfies Hall’s harem condition.
- During the construction of $M_n$ we will define fans $M^j_{n, i}$, $j \leq n + 1$. The graph $M_n$ is the union of them.
- $M^j_{n, i}$ is a fan consisting of edges denoted by $\{(u^j_{n, i}, u^j_{n, i+1}) : i \leq d-1\}$.
- $u^0_n$ is a starting vertex of the $n$-th step, it is also denoted by $u^0_{n, i}$.
- For any subgraph $\Gamma'$ of $\Gamma^{(n)}$ by $\Gamma'(-u^0_{n, i}, \ldots, -u^0_{n, j})$ we denote the graph obtained from $\Gamma'$ by removal of the $(1, d-1)$-fans of $M_n$ with roots $u^0_{n, i}, \ldots, u^0_{n, j}$. 

**Remark:** The condition $f^i(n) \neq n$ is not necessary. It is needed only for the uniqueness of the solution. However, it is still a reasonable condition to consider.
For any subgraph $\Gamma' = (U', V')$ of $\Gamma^{(n)}$ and any $u_j^+ \in U^{(n)} \perp$ by $\Gamma'(+u_j^+)$ we denote the graph induced in $\Gamma^{(n)}$ by the sets of vertices $U' \cup \{u_j^+\}$ and $V' \cup \{v_{j,n}^i : 1 \leq i \leq d - 1\}$.

- For any subgraph $\Gamma' = (U', V')$ of $\Gamma^{(n)}$ and any vertex $v \in V^{(n)}$ by $\Gamma'(v)$ (resp. $\Gamma'(-v)$) we denote the graph induced in $\Gamma^{(n)}$ by the sets of vertices $U'$ and $V' \cup \{v\}$ (resp. $V' \setminus \{v\}$).

$\mathcal{M}_n^1$ denotes a perfect $(1, d)$-matching in $\Gamma^{(n)*}$ which appears in the first part of step $n + 1$.

$\mathcal{M}_n^2$ denotes a perfect $(1, d)$-matching in $\Gamma^{(n)}(-u_{n,1}, \ldots, -u_{n,j}) \cap \Gamma^{(n)*}$ for some $j$, which appears in the second part of step $n + 1$.

Elements adjacent to $u_{n+1}^+$ in the matching $\mathcal{M}_n^2$ are denoted by $\cdot_{n,1}^j, \cdot_{n,d}^j$. The element $\cdot_{n,1}^j$ is a candidate for $u_{n+1}^+$.

The fan $(u_{n+1}^+, \cdot_{n,j}^j | 1 \leq j \leq d - 1)$ usually appears as a part of a fan of the matching $\mathcal{M}_n^2$. We warn the reader that it is possible that $u_{n+1}^+$ does not exist.

- We assume that all of these elements are natural numbers and are ordered according to the standard ordering of the natural numbers.

### 3. The Construction

We assume that $\Gamma = (U, V, E)$ is a bipartite graph satisfying the Hall’s $d$-harem condition, such that:

- both $U$ and $V$ are identified with $\mathbb{N} \setminus \{0\}$;
- $\Gamma$ is fully reflected;
- $E$ does not contain edges of the form $(v, u_0)$.

We now describe an inductive construction which is the heart of the proof of our main theorem. At the end of each step we formulate some claims that certain graphs satisfy Hall’s $d$-harem condition. They support further steps of the construction.

### 3.1. Step 1, part 1.

We take $u_0$, the first element of the set $U$ (it is clear that $u_0 = 1$). Using Hall’s harem theorem we find a perfect $(1, d)$-matching $\mathcal{M}_0^1$. Let $v_{0,1}^0, \ldots, v_{0,d}^0$ be elements of $V$, ordered by its numbers, such that $(u_0, v_{0,i}^0) \in \mathcal{M}_0^1$ for all $i \leq d$. Define the fan $M_0^0$ as the set of edges $(u_0, v_{0,i})$ for $i \leq d - 1$.

Since the graph $\Gamma' = (U \setminus \{u_0\}, V \setminus \{v_{0,1}^0, \ldots, v_{0,d}^0\})$ has an appropriate matching, it satisfies Hall’s $d$-harem condition. Furthermore, the following statement holds.

**Claim 3.1.** $\Gamma^{(0)}(-u_0) := (U \setminus \{u_0\}, V \setminus \{v_{0,1}^0, \ldots, v_{0,d-1}^0\})$ satisfies Hall’s $d$-harem condition.

The graph in the claim is $\Gamma'$ together with $v_{0,d}^0$. See Lemma 5.1 for the proof of this claim.

![Figure 1](image_url). The first part of the first step, $\mathcal{M}_0^1$-fan of $u_0$ in red and green, $M_0^0$ in green.

Before the second part of step 1 let us discuss our local goals. Let the partial function $f_0$ correspond to $M_0$ and let $\Gamma^{(1)}$ be the graph obtained after step 1, i.e. the part of $\Gamma$ after removal $M_0$. We want to force that:
It is clear that (1)-(2) are satisfied if we add the edge \((u_{0,0}, v_0)\) to the matching. This means that \(f_0^2(u_0) = u_0\). We should organize it in a clever way.

3.2. **Step 1, part 2.** We denote \(u_0^1 := u_{0,0}\) and aim to add the edge \((u_0^1, v_0)\), to \(M_0^1\). Note that \((u_0^1, v_0) \in \Gamma(0)(-u_0)\), because \((u_0^0, v_0^0)\) is in \(\Gamma(0)\) and the latter one is \(U^0\)-reflected. Since \(\Gamma(0)(-u_0)\) satisfies Hall’s \(d\)-harem condition, it has a perfect \((1, d)\)-matching \(M_0^2\). We remind the reader that \(\hat{v}_{0,1}, \ldots, \hat{v}_{0,i}^d\) are elements of \(V_d\), ordered by its numbers, such that \((u_0^1, \hat{v}_{0,i}^d) \in M_0^2\) for all \(i \leq d\). Since \(v_0\) has the lowest number in \(V_0\), there are two possible cases:

1. \(v_0 = \hat{v}_{0,1}^d\). We set \(v_{0,i}^d := \hat{v}_{0,i}^d\), \(1 \leq i \leq d - 1\) (i.e. \(\hat{v}_{0,1}^d = v_0\)).

   Then find the fan \((u, v_i) \in M_0^2\), \(1 \leq i \leq d\), such that \(v_{0,i} = v_i\) (here we assume that the ordering of \(v_i\) corresponds to their indexes). We set:
   - \(v_{0,i} := v_{0,1}\) and \(v_{0,i}^d := \hat{v}_{0,i-1}^d\) for \(2 \leq i \leq d - 1\),
   - and define a candidate for \(\Gamma(1)^\perp\):
   - \(\hat{v}_{0,1}^d := u\);
   - \(\hat{v}_{0,i-1}^d := v_i\) for \(2 \leq i \leq d\).

   In either case define the fan \(M_0^1\) as the set of edges \((u_0^1, v_{0,i}^d)\) for \(1 \leq i \leq d - 1\).

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**Figure 2.** Step 1, part 2. \(M_0^2\) is red. We aim to produce a cycle of length 2 in our matching by adding the edge \((u_0^1, v_0)\). Since \(v_0\) is matched with \(u\), we force the situation from Figure 3.

**Figure 3.** \(M_0^1\) is green. It is possible that the purple fan consisting of edges \((\hat{u}_{0,1}^d, \hat{v}_{0,1}^d), (\hat{u}_{0,2}^d, \hat{v}_{0,2}^d)\) will be added to \(\Gamma(1)^\perp\).
Put $M_0 = M_0^0 \cup M_1^1$. We obtain $\Gamma^{(1)}$ by removal of $M_0$ from $\Gamma$. Since $u_0, v_{n_0}, v_{0,1}$ and $u_0 = u_{0,1}$ have been removed, $\Gamma^{(1)}$ is $U^{(1)}$-reflected. It might turn out that it does not satisfy Hall’s $d$-harem condition.

Let $\Gamma'_0 = (U^{(1)} \setminus \{\hat{u}_0\}, V^{(1)} \setminus \{\hat{v}_{0,1}, \ldots, \hat{v}_{0,d-1}\})$. The following claim follows from Lemma 5.5 below.

**Claim 3.2.** At least one of $\Gamma'_0$ or $\Gamma^{(1)}$ satisfies Hall’s $d$-harem condition.

If $\Gamma^{(1)}$ satisfies Hall’s $d$-harem condition, set $\Gamma^{(1)*} := \Gamma^{(1)}$ and $U^{(1)} = \emptyset, V^{(1)} = \emptyset$. If $\Gamma^{(1)}$ does not satisfy Hall’s $d$-harem condition, set $\Gamma^{(1)*} := \Gamma'_0$ and

- $u^\perp := \hat{u}_0$;
- $v_{0,i}^\perp := \hat{v}_{0,i}, 1 \leq i \leq d - 1$;
- $U^{(1)} = \{u_0\}$;
- $V^{(1)} = \{v_{0,i}^\perp : 1 \leq i \leq d - 1\}$.

### 3.3. Step $n+1$, part 1.

At the previous step we constructed graphs $\Gamma^{(n)}$ and $\Gamma^{(n)*}$, where $\Gamma^{(n)}$ is $U^{(n)}$-reflected and $\Gamma^{(n)*}$ satisfies Hall’s $d$-harem condition. Let $\mathcal{W}_n^1$ be an $(1, d)$-matching in $\Gamma^{(n)*}$. Also note that since $|U^{(n)}| - |U^{(n-1)}| \leq 1$, there are at most $n$ vertices $u^\perp$ in $U^{(n)}$ (see Section 2.4 for the corresponding definition).

Take $u_n$, the first element of the set $U^{(n)}$. In order to define $M_n^0$ we have two possible cases:

1. There is $j$ such that $u_n = u^\perp_j \in U^{(n)}$. Then we set $M_n^0$ to consist of all edges of the form $(u^\perp_j, v_{n,j}^\perp)$ and remove the fan with the root $u^\perp_j$ from $\Gamma^{(n)}$. We redefine $U^{(n)}$ and $V^{(n)}$ accordingly (in particular $u^\perp_j$ is removed from $U^{(n)}$).

2. If $u_n \notin U^{(n)}$, then verify whether there is $j$ such that $(u_n, v_{n,j}^0) \in \mathcal{W}_n^1$ and $(u_{n,j}, v_{n,j}^0) \in \Gamma^{(n)}$. By the definition of $\Gamma^{(n)}$ it can happen for at most one $j$. (Note here that it can also happen that $v_{n,j}^0$ is not even in $\Gamma^{(n)}$.) If there is such a $j$, we set $M_n^0$ to consist of $(u_n, v_{n,j}^0)$ and $d - 2$ of the remaining $d - 1$ edges of the form $(u_n, v_{n,i}^0) \in \mathcal{W}_n^1$ (excluding one with the greatest index).

In the case when the corresponding $j$ does not exist, we set $M_n^0$ to consist of edges $(u_n, v_{n,i}^0) \in \mathcal{W}_n^1$ for $i \leq d - 1$.

The following claim follows from Lemma 5.1 below.

**Claim 3.3.** Let $\Gamma^{(n)*}(-u_n)$ be $\Gamma^{(n)}(-u_n) \cap \Gamma^{(n)*}$. One of the following holds:

- $\Gamma^{(n)*}(-u_n)$ satisfies Hall’s $d$-harem condition;
- there exist some vertex $u^\perp_j \in U^{(n)}$ such that the graph $\Gamma^{(n)*}(-u_n, +u^\perp_j)$ satisfies Hall’s $d$-harem condition.

If $\Gamma^{(n)*}(-u_n)$ does not satisfy Hall’s $d$-harem condition, let $u^\perp_j$ be an element from $U^{(n)}$ realizing the second possibility of this claim. We remove the fan of $u^\perp_j$ with its leaves from $(U^{(n)} \setminus V^{(n)})$ and then we put it into $\Gamma^{(n)*}$. Thus the latter graph (and $U^{(n)} \setminus V^{(n)}$) are updated. It is clear that now the redefined $\Gamma^{(n)*}(-u_n)$ satisfies Hall’s $d$-harem condition.

Before the second part of Step $n+1$ we describe the goals which we want to achieve after the step. We want:

1. the partial function $f_n$ corresponding to $\bigcup_{i=0}^n M_i$ has controlled sizes of its cycles, and
2. the graph $\Gamma^{(n+1)}$ obtained at the end of the step is $U^{(n+1)}$-reflected and the corresponding graph $\Gamma^{(n+1)*}$ satisfies Hall’s $d$-harem condition.

In order to achieve the first condition we will organize one of the the following properties:

1. there is a sequence of vertices $u_n^0, u_n^1, u_n^2, \ldots, u_n^j, 1 \leq j \leq n$ such that each edge $(u_n^j, v_{n,j-1})$ belongs to $M_n$, and for some $0 \leq \ell \leq j - 1$ the sequence $(u_n^\ell, u_n^{\ell+1}, \ldots, u_n^j)$ is a cycle;
(ii) there is a sequence of vertices \((u_0^n, u_1^n, u_2^n, \ldots, u_l^n)\), \(j \leq n\) such that the edges of the form 
\((u_i^n, v_{a_{i-1}}^n)\) belong to \(M_n\), and \(v_{a_{i-1}}^n\) is already adjacent to some edge from \(\bigcup_{i=0}^{n-1} M_i\);

3.4. **Step \(n+1\), part 2.** We begin by checking whether \(v_{a_n}\) belongs to \(\Gamma^{(n)\perp}\). If \(v_{a_n} \in \Gamma^{(n)\perp}\) then we denote \(u_n\) by \(u_n^0\) and begin the following process of choosing the consecutive vertices \(u_i^n\).

**First step of iteration.** Assume that for some \(j_1, i\) we have \(v_{a_n} = v_{j_1,i+} \in V^{(n)\perp}\).

Then we set
\[u_1^n = u_{j_1},\ v_{n,k}^i := v_{j_1,k},\ k \leq d - 1,\]
\[M_1^n := \{(u_n^1, v_{n,k}^1) : 1 \leq k \leq d - 1\}\]
and check whether \(v_{a_n}^1 \in \Gamma^{(n)\perp}(-u_n^0, -u_n^1)\).

If it is so we repeat the iteration for \(v_{a_n}^1\). Note that \(v_{a_n}^1 \in \Gamma^{(n)\perp}(-u_n^0, -u_n^1)\) then.

**Single step of iteration.** We verify if \(v_{a_n}^{m+1} \in \Gamma^{(n)\perp}(-u_n^0, -u_n^1, \ldots, -u_n^m)\). If it is so then for some \(j_{m+1}, i\) we have \(v_{a_n}^{m+1} = v_{j_{m+1},i} \in V^{(n)\perp}\). Define
\[u_{n+1}^{m+1} = u_{j_{m+1}},\ v_{n,k}^{m+1} := v_{j_{m+1},k},\ 1 \leq k \leq d - 1,\]
\[M_{n+1}^m := \{(u_{n+1}^{m+1}, v_{n,k}^{m+1}) : 1 \leq k \leq d - 1\}\]
and we repeat the iteration for \(v_{a_n}^{m+1}\). This ends the single iteration step.

Since \(|U^{(n)\perp}| \leq n\), the procedure ends after at most \(n\) iterations. Therefore one of the following cases is realized for some \(l \leq n\):

1. \(v_{a_n}^l \notin \Gamma^{(n)}\);
2. \(v_{a_n}^l \in \Gamma^{(n)}\), but \(v_{a_n}^l \notin \Gamma^{(n)}(-u_n^0, -u_n^1, \ldots, -u_n^l)\) (this case is impossible for \(l = 0\));
3. \(v_{a_n}^l \in \Gamma^{(n)*}\).

In case (1) \(v_{a_n}^l\) was already added to \(M\) at preceding steps and condition (ii) described before this stage is satisfied. We finish step \(n + 1\), so no cycle is constructed at the step.

In case (2) we also finish step \(n + 1\). Note that the last iteration closes some cycle \((u_n^k, \ldots, u_n^l)\) constructed at step \(n + 1\). The length of this cycle is not greater than \(l + 1\).

In case (3) we will construct a cycle \((u_n^k, u_n^{l+1})\) of length 2. This is the most complicated case. It will be considered in the next subsection. Before we start it we give an example of a cycle obtained by construction in case (2).

**Example.** The following pictures show how a cycle of length 3 can be obtained by this procedure in the graph satisfying Hall’s 3-harem condition.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example.png}
\caption{\(\Gamma^{(n)*}\) in black, \(\Gamma^{(n)\perp}\) in purple.}
\end{figure}
3.5. **Case (3) and the end.** In case (3) we will construct a cycle \((u_n^l, u_n^{l+1})\) of length 2 as follows.

We start with a new term \(u_n^{l+1} := u_{n,1}^l\).
Since the edge \((u^l_n, v^l_n)\) is in \(\Gamma(n)\) and is not in \(M^l_n\), applying \(U(n)-\)reflectedness of \(\Gamma(n)\) we see \((u^l_{n+1}, v^l_{n+1})\) is \(G(n)\)-reflected in \(\Gamma(n)\), \(-u_{n+1}^l, \ldots, -u^l_n\). Observe that since \(u^l_j, \ldots, u^l_j\) are in \(U(n)-\)
\[\Gamma(n)\cap (-u_n^l, -u_n^l, \ldots, -u_n^l) = \Gamma(n)\cap (\Gamma(n)\cap (-u_n^l, -u_n^l)) = \Gamma(n)\cap (\Gamma(n)^\ast(-u_n)).\]
We also remind the reader that by the first part of this step the graph \(\Gamma(n)^\ast(-u_n)\) satisfies Hall’s \(d\)-harem condition. Thus we can find an \((1, d)\)-matching \(M^l_n\) in \(\Gamma(n)^\ast(-u_n)\). Let us fix it.

We now check whether there is \(j\) with \(u^l_{n+1} = u^l_j\).
If it is so, we set \(\hat{v}^l_{n+1} := v^l_j\) for \(1 \leq i \leq d - 1\).
If there is no \(j\) such that \(u^l_{n+1} = u^l_j\), then \(\hat{v}^l_{n+1} \ldots \hat{v}^l_{n+1}\) will denote the elements adjacent to \(u^l_{n+1}\) under \(\hat{M}_n^2\).

There are two cases:

A) \((u^l_{n+1}, v^l_n) \in \hat{\mathcal{M}}^l_n\), i.e. \(v^l_{n+1} = \hat{v}^l_{n+1}\) for some \(1 \leq k \leq d\). In this case \(u^l_{n+1}\)
cannot be \(u^l_j\) for any \(j\).

B) \((u^l_{n+1}, v^l_n) \notin \hat{\mathcal{M}}^l_n\), i.e. there exists some \(u \in \Gamma(n)^\ast(-u_n)\), such that \(v^l_{n+1} = v_k\)
for some \(1 \leq k \leq d\), where \(v_1 \ldots v_d\) denote the elements adjacent to \(u\) under \(\hat{\mathcal{M}}^l_n\). In this case it is possible that \(u^l_{n+1} = u^l_j\).

In case A, we produce a cycle of length 2 by including the pair \((u^l_{n+1}, v^l_n)\) into \(\hat{M}^l_n\). If \(\hat{v}^l_{n+1} \ldots \hat{v}^l_{n+1}\) leaves \(U(n)\) with the root \(u^l_{n+1}\) and \((d - 2)\) leaves taken among \(\hat{v}^l_{n+1}\). To be precise we organize it as follows. If \(k = d\), we set \(\hat{v}^l_{n+1} := \hat{v}^l_{n+1}\) for \(2 \leq i \leq d\). If \(k \neq d\) we set \(\hat{v}^l_{n+1} := \hat{v}^l_{n+1}\) for \(1 \leq i \leq d - 1\).
The set \(\hat{M}^l_n\) consists of edges \((u^l_{n+1}, v^l_{n+1})\) for \(1 \leq i \leq d - 1\). The procedure is finished.

In case B, we produce a cycle of length 2 by including the pair \((u^l_{n+1}, v^l_n)\) into \(\hat{M}^l_n\). In fact, as in the previous case we include it into \(\hat{M}^l_n\) together with a fan with the root \(u^l_{n+1}\) and \((d - 2)\) leaves taken among \(\hat{v}^l_{n+1}\), which will be denoted \(v^l_{n+1} = v^l_n\) and \(v^l_{n+1} = \hat{v}^l_{n+1}\), \(1 \leq i \leq d - 2\). We define \(\hat{u}^l_n := u\) and rename the remaining \(d - 1\) vertices \(v_j\) to \(\hat{v}^l_{n+1}\). The procedure is finished. It is worth noting here that if \(u^l_{n+1}\) coincides with some \(u^l_j\), then the vertex \(\hat{v}^l_{n+1}\) does not exist, i.e.
the only vertex \(\hat{v}^l_{n+1}\) from the fan of \(u^l_{n+1}\) would be outside of \(\hat{M}^l_n\).

Let \(M_n = \bigcup_{k=1}^l M^k_n\). We obtain the graph \(\Gamma(n+1)\) from \(\Gamma(n)\) by removal of \(M_n\)-fans. Since for each \(u \in U(n)\) \(\backslash U(n)-\)the element \(v^l_n\) is also removed, then the graph is \(U(n)-\)reflected.

In cases (1), (2) and (3) A) we set \(\mathcal{U} := U(n+1) \backslash U(n)\), \(\mathcal{V} := V(n) \backslash V(n)\). Let \(\mathcal{S} = (\mathcal{U}, \mathcal{V})\) and \(\Gamma(n)^\lambda := \Gamma(n)^\lambda\cap \Gamma(n)\). We note here that in cases (1) and (2) only elements of \(\Gamma(n)^\lambda\) were added to \(M_n\) at the end of the step. Therefore in these cases \(\mathcal{S}\) coincides with \(\Gamma(n)^\lambda\) and satisfies Hall’s \(d\)-harem condition.

In case (3) B) we set \(\mathcal{U} := U(n+1) \backslash (U(n) \cup \{u^l_{n+1}\})\), \(\mathcal{V} := V(n) \backslash (V(n) \cup \{v^l_{n+1}\})\). Let \(\mathcal{S} = (\mathcal{U}, \mathcal{V})\) and \(\Gamma(n)^\lambda := \Gamma(n)^\lambda\cap U(n+1)\). We define \(\Gamma(n)^\lambda\) accordingly:
\(\Gamma(n)^\lambda := (V(n) \cup \{v^l_{n+1}\}) \cap U(n+1)\), and set
\(\hat{\Gamma}(n)^\lambda := (\hat{U}(n) \cup \hat{V}(n)\).\)

The following claim follows from Lemma 5.5 below.

Claim 3.4. At least one of the following holds:

- \(\mathcal{S}\) satisfies Hall’s \(d\)-harem condition;
- there exists some vertex \(u^l_j \in \hat{U}(n)\) such that the graph \(\mathcal{S}(\pm u^l_j)\) satisfies Hall’s \(d\)-harem condition.
- there exist vertices \(u^l_j, u^l_j \in \hat{U}(n)\) such that the graph \(\mathcal{S}(\pm u^l_j, \pm u^l_j)\) satisfies Hall’s \(d\)-harem condition.
Depending on the output of this claim we define the final output of step $n + 1$. In the case (3)B (i.e. $\hat{u}_n^\perp$ exists) if $u_i^\perp \neq \hat{u}_n^\perp \neq u_j^\perp$ (for any output of the claim), we set $u_n^\perp := \hat{u}_n^\perp$ and $v_{n,k}^\perp := \hat{v}_{n,k}^\perp$ for $1 \leq k \leq d - 1$. Otherwise, or in the remaining cases $u_n^\perp, v_{n,k}^\perp$ are not defined.

In the first case of the claim we set $\Gamma^{(n+1)*} := \mathcal{I}$ and

$$U^{(n+1)\perp} := (U^{(n)} \cup \{u_n^\perp\}) \cap U^{(n+1)},$$

$$V^{(n+1)\perp} := (V^{(n)} \cup \{v_{n,k}^\perp : 1 \leq k \leq d - 1\}) \cap V^{(n+1)}.$$ 

In the second case we set $\Gamma^{(n+1)*} := \mathcal{I}(+,u_j^\perp)$ and

$$U^{(n+1)\perp} := ((U^{(n)} \setminus \{u_j^\perp\}) \cup \{u_n^\perp\}) \cap U^{(n+1)},$$

$$V^{(n+1)\perp} := ((V^{(n)} \setminus \{v_{j,k}^\perp : 1 \leq k \leq d - 1\}) \cup \{v_{n,k}^\perp : 1 \leq k \leq d - 1\}) \cap V^{(n+1)}.$$ 

In the third case we set $\Gamma^{(n+1)*} := \mathcal{I}(+,u_r^\perp,+u_j^\perp)$ and

$$U^{(n+1)\perp} := ((U^{(n)} \setminus \{u_r^\perp,u_j^\perp\}) \cup \{u_n^\perp\}) \cap U^{(n+1)},$$

$$V^{(n+1)\perp} := ((V^{(n)} \setminus \{v_{r,k}^\perp,v_{j,k}^\perp : 1 \leq k \leq d - 1\}) \cup \{v_{n,k}^\perp : 1 \leq k \leq d - 1\}) \cap V^{(n+1)}.$$ 

4. Technical Lemmas

The notation used in this section does not correspond the construction above.

Throughout this section $\Gamma = (U,V,E)$ denotes a bipartite graph and $\Gamma^\perp = (U^\perp,V^\perp,E^\perp)$ denotes its subgraph.

The graph $\Gamma^* = (U^*,V^*,E^*)$ is an induced subgraph of $\Gamma$ such that

$$U^* \cap U^\perp = \emptyset = V^* \cap V^\perp.$$ 

Below we always assume that $d$ is a natural number greater than 1. The following situation will arise several times in our arguments in Section 5. Let $X$ be a subset of $V$ such that

$$|N_\Gamma(X)| \geq \frac{1}{d}|X|,$$

but

$$|N_\Gamma(X) \setminus U^\perp| < \frac{1}{d}|X|.$$ 

Thus we can conclude that $N_\Gamma(X) \cap U^\perp \neq \emptyset$.

The following lemma describes typical circumstances which lead to this situation.

**Lemma 4.1.** Let $\Gamma = (U,V,E)$ be $U$-reflected and let $\Gamma^*$ be a subgraph of $\Gamma$ induced by the sets of vertices $U^* = U \setminus U^\perp$, $V^* = V \setminus V^\perp$. Assume that $\Gamma^*$ satisfies Hall’s $d$-harem condition and assume that for each $Y \subseteq U^\perp$ we have $|N_\Gamma(Y)| \geq (d - 1)|Y|$.

Then for any $X \subseteq V$ we have $|N_\Gamma(X)| \geq (d - 1)|X|$. In particular $|N_\Gamma(X)| \geq \frac{1}{d}|X|$

**Proof.** Let $U_X := \{u \in U : v_u \in X\}$. By $U$-reflectedness of $\Gamma$ we have $|U_X| = |X|$. Consider the sets

$$U_X^\perp := U_X \setminus U^\perp$$

and

$$U_X^\perp := U_X \cap U^\perp.$$ 

Using $U$-reflectedness of $\Gamma$ again, we see

$$|N_\Gamma(X)| \geq |N_\Gamma(U_X^\perp)| + |N_\Gamma(U_X^\perp) \cap V^\perp|.$$ 

Since Hall’s $d$-harem condition is satisfied for any subset of $U^*$,

$$|N_\Gamma(U_X^\perp)| \geq d|U_X^\perp|.$$ 

Moreover we have $|N_\Gamma(U_X^\perp) \cap V^\perp| \geq (d - 1)|U_X^\perp|$. Therefore

$$|N_\Gamma(U_X^\perp)| + |N_\Gamma(U_X^\perp) \cap V^\perp| \geq (d - 1)|U_X|.$$
Since \( d \geq 2 \) it follows that
\[
|N_\Gamma(X)| \geq (d - 1)|X| \geq \frac{1}{d}|X|.
\]

This lemma cannot be applied in the situation of Part 1 of the step of the main construction, since some vertices from a \( U \)-reflected graph are removed. In that case, we have a graph \( \Gamma \) which is not \( U \)-reflected, but for some \( v \in V \) its subgraph \( \Gamma(-v) \) is \( U \)-reflected. Because of this change, in the following lemmas both \( \Gamma^\perp \) and \( \Gamma^* \) are the subgraphs of \( \Gamma(-v) \).

**Lemma 4.2.** Let \( v \) be a vertex from \( V \) and \( \Gamma(-v) = (U, V(-v)) \). Assume that \( \Gamma(-v) \) is \( U \)-reflected. Let \( \Gamma^\perp = (U^\perp, V^\perp, E^\perp) \) be a subgraph of \( \Gamma(-v) \) and let \( \Gamma^* \) be defined as the subgraph of \( \Gamma(-v) \) induced by the sets of vertices \( U^* := U \setminus U^\perp, V^* := V(-v) \setminus V^\perp \).

Assume that \( \Gamma^* \) satisfies Hall’s \( d \)-harem condition and assume that for each \( Y \subset U^\perp \) we have
\[
|N_\Gamma(Y) \cap V^\perp| \geq (d - 1)|Y|.
\]

Then for any \( X \subseteq V \setminus \{v\} \) the inequality \( |N_\Gamma(X)| \geq (d - 1)|X| - 1 \) holds.

Moreover, the equality \( |N_\Gamma(X)| = (d - 1)|X| - 1 \) can happen only if \( v \in N_\Gamma(U_X) \) and \( N_{\Gamma^*}(U_X) = \emptyset \), where \( U_X := \{u \in U : v_u \in X\} \).

**Proof.** By \( U \)-reflectedness of \( \Gamma(-v) \) we have \( |U_X| = |X| \) and
\[
|N_\Gamma(X)| + 1 \geq |N_{\Gamma(-v)}(U_X)|.
\]

Furthermore, if \( v \notin N_{\Gamma^*}(U_X) \), then
\[
|N_\Gamma(X)| \geq |N_{\Gamma(-v)}(U_X)|.
\]

Consider the sets
\[
U^\perp_X := U_X \setminus U^\perp \text{ and } U^\perp_X := U_X \cap U^\perp.
\]

Applying the argument of Lemma 4.1 to \( \Gamma(-v) \) we obtain
\[
|N_{\Gamma(-v)}(U_X)| \geq |N_{\Gamma^*}(U_X)| + |N_{\Gamma^*}(U^\perp_X) \cap V^\perp| \geq (d - 1)|U_X|.
\]

Thus
\[
|N_\Gamma(X)| \geq |N_{\Gamma(-v)}(U_X)| - 1 \geq (d - 1)|X| - 1.
\]

Observe that
\[
|N_{\Gamma(-v)}(U_X)| = (d - 1)|U_X|\text{ only if } N_{\Gamma(-v)}(U_X) \subseteq V^\perp, \text{ i.e. } N_{\Gamma^*}(U_X) = \emptyset.
\]

If \( v \notin N_{\Gamma^*}(U_X) \) then
\[
|N_\Gamma(X)| \geq |N_{\Gamma(-v)}(U_X)| \geq (d - 1)|X|.
\]

Therefore
\[
|N_\Gamma(X)| = (d - 1)|X| - 1 \text{ only if } v \in N_{\Gamma^*}(U_X) \text{ and } N_{\Gamma^*}(U_X) = \emptyset.
\]

\( \square \)

The following definition will be useful in the proofs of the lemmas of the next section.

**Definition 4.3.** Let \( \Gamma := (U, V, E) \) be a bipartite graph, and \( \Gamma^* = (U^*, V^*, E^*) \) be a subgraph in \( \Gamma \) satisfying Hall’s \( d \)-harem condition. Assume that \( M \) is a corresponding perfect \((1, d)\)-matching in \( \Gamma^* \).

Let \( X \) be a subset of \( V \) and let \( x \in X \). We say that \( x \) is accessible from \( y \in N_{\Gamma}(X) \) through \( X \) by matching \( M \), (denoted by \( y \xrightarrow{M,X} x \)) if there exist two sequences of vertices \( \{v'_0, \ldots, v'_n\} \subset X \) and \( \{u'_0, \ldots, u'_{n-1}\} \subset N_{\Gamma}(X) \) such that
- \( v'_0 = x; \)
- \( (u'_i, v'_i) \in M \) and \( (u'_i, v'_{i+1}) \in E \setminus M \), where \( i < n; \)
- \( (y, v'_0) \in E. \)

We provide the following technical lemma.

**Lemma 4.4.** Let \( \Gamma := (U, V, E) \) be a bipartite graph. Let \( \Gamma^* = (U^*, V^*, E^*) \) be a subgraph of \( \Gamma \) satisfying Hall’s \( d \)-harem condition and \( M \) be a corresponding perfect \((1, d)\)-matching.

Assume that
- \( \widehat{v} \in N_{\Gamma^*}(U^*) \setminus V^*; \)
- \( X \) is the smallest connected subset in \( V^* \cup \{\widehat{v}\} \) with \( |N_{\Gamma}(X)| < \frac{1}{d}|X|; \)
- \( \widehat{u} \in N_{\Gamma}(X) \setminus U^*; \)
Then $\hat{u} \xrightarrow{M,X} \hat{v}$.

**Proof.** It is clear that $\hat{v} \in X$. Let $X'$ denote the subset of all elements of $X$ that are accessible from $\hat{u}$ through $X$ by the matching $M$. Assume that $\hat{v} \notin X'$. Since $\hat{u} \in N_Γ(X)$, we then see that there exist $u'_0, v'_0$ such that $\hat{u}, u'_0 \in N_Γ(v'_0)$ and $(u'_0, v'_0) \in M$. Therefore $X' \neq \emptyset$. We will show that $|N_Γ(X \setminus X')| < \frac{1}{d}|X \setminus X'|$.

Let $|X'| = l$ and let $U_M(X') = \{u \in U \mid (\exists v \in X')(u, v) \in M\}$. Since elements of $X \setminus X'$ are not accessible from $\hat{u}$ through $X$ by the matching $M$, then $N_Γ(X \setminus X') \cap U_M(X') = \emptyset$.

Using this we see that

$$|N_Γ(X \setminus X')| \leq |N_Γ(X)| - |U_M(X')|.$$ 

Since $M$ is a $(1, d)$-matching, each element of $U_M(X')$ can be matched with at most $d$ elements from $X'$. Therefore $|U_M(X')| \geq \lfloor \frac{l}{d} \rfloor$ and

$$|N_Γ(X \setminus X')| \leq N_Γ(X) - |U_M(X')| \leq |N_Γ(X)| - \lfloor \frac{l}{d} \rfloor \leq \frac{1}{d}|X| - l = \frac{1}{d}|X \setminus X'|.$$ 

So $|N_Γ(X \setminus X')| < \frac{1}{d}|X \setminus X'|$ and $X \setminus X'$ is smaller than $X$, a contradiction with the choice of the latter. As a result $\hat{v} \in X'$, and consequently $\hat{u} \xrightarrow{M,X} \hat{v}$.

\[\square\]

### 5. Graphs constructed in Parts 1 and 2 of Each Step Satisfy Hall's d-Harem Condition

In this section the notation is taken from the construction of Section 3.

#### 5.1. Part 1.

**Lemma 5.1.** For any $n$ one of the following holds:

- $Γ^{(n)}(-u_n)$ satisfies Hall's $d$-harem condition;
- there exist some vertex $u^+_j \in U^{(n)⊥}$ such that the graph $Γ^{(n)}(-u_n, +u^+_j)$ satisfies Hall’s $d$-harem condition.

**Remark 2.** If $U^{(n)⊥} = \emptyset$, then by Lemma 5.1 the graph $Γ^{(n)}(-u_n)$ satisfies Hall’s $d$-harem condition. In particular, this lemma proves Claim 3.1.

**Proof.** We know that the graph $Γ^{(n)}(-u_n)$ satisfies Hall’s $d$-harem condition. Let $v$ denote the only vertex from the set $\{v_{n,1}^0, \ldots, v_{n,d}^0\}$ that belongs to $Γ^{(n)}(-u_n)$. The choice of $v_{n,1}^0, v_{n,2}^0, \ldots, v_{n,d}^0$ ensures that the graph $Γ^{(n)}(-u_n, -v)$ satisfies Hall’s $d$-harem condition as well. Since $U^{(n)}(-u_n) = U^{(n)⊥}(-u_n, -v)$, for any $X \subset U^{(n)}(-u_n)$ we have

$$|N_{Γ^{(n)}(-u_n)}(X)| \geq |N_{Γ^{(n)}(-u_n, -v)}(X)| \geq d|X|.$$ 

The corresponding property holds for all subsets of $V^{(n)⊥}(-u_n)$ that do not contain $v$. Therefore if $Γ^{(n)}(-u_n)$ does not satisfy Hall’s $d$-harem condition, then a witness of this is a finite $X \subset V^{(n)}(-u_n)$ which contains $v$.

If such $X$ is not connected, then the neighbourhood of $X$ is a disjoint union of the neighbourhoods of the connected subsets of $X$. Therefore there exists the smallest connected set $S$ such that $v \in X \subset V^{(n)}(-u_n)$ and $|N_{Γ^{(n)}(-u_n)}(X)| < \frac{1}{d}|X|$. Choose $X$ with these properties. We now want to prove that there is some $u^+_j \in N_{Γ^{(n)}(-u_n)}(X)$. By the inequality of the previous line it suffices to show that

$$|N_{Γ^{(n)}(-u_n)}(X)| \geq \frac{1}{d}|X|. \tag{1}$$

First, we consider the case when $X = \{v\}$. The vertex $u_v$ either belongs to $U^{(n)⊥}(-u_n)$ or to $U^{(n)⊥}$. Since $Γ^{(n)⊥}$ consists of $(1, d - 1)$-fans, in each case we have $|N_{Γ^{(n)}(-u_n)}(u_v)| \geq d - 1$. Moreover, the equality

$$|N_{Γ^{(n)}(-u_n)}(u_v)| = d - 1 \tag{2}$$
implies that \( v_{un} \notin \Gamma^{(n)} \). Indeed, if \( v_{un} \in \Gamma^{(n)} \) then \( v_{un} \in N_{\Gamma^{(n)}(\neg u_n)}(u_\theta) \) by reflectedness. On the other hand equality (2) implies that \((u_\theta, v_{un}) \in \Gamma^{(n)} \). Thus \((u_\theta, v)\) would be added to the matching \( M \) at the first part of \( n + 1\)-st step of the construction, i.e. \( v \notin \Gamma^{(n)\ast}(\neg u_n) \), a contradiction.

Our next observation is

\[
|N_{\Gamma^{(n)}(\neg u_n)}(v)| \geq |N_{\Gamma^{(n)}(\neg u_n)}(u_\theta)| - 1.
\]

This follows by \( U^{(n)}\)-reflectedness of \( \Gamma^{(n)} \): \( v_{un} \) is the only possible element incident to \( u_\theta \) that does not have the left copy in \( U^{(n)}(\neg u_n) \). Moreover, the equality

\[
|N_{\Gamma^{(n)}(\neg u_n)}(v)| = |N_{\Gamma^{(n)}(\neg u_n)}(u_\theta)| - 1
\]

holds only if \( v_{un} \in \Gamma^{(n)}(\neg u_n) \).

Therefore equalities (2), (3) are not consistent, i.e.:

\[
|N_{\Gamma^{(n)}(\neg u_n)}(v)| > (d - 1) - 1 \quad \text{and} \quad |N_{\Gamma^{(n)}(\neg u_n)}(v)| \geq \frac{1}{d}.
\]

Since \( X \) is a singleton,

\[
|N_{\Gamma^{(n)}(\neg u_n)}(X)| \geq \frac{1}{d}|X|.
\]

As we mentioned above this means that there is some \( u_\bot^1 \in N_{\Gamma^{(n)}(\neg u_n)}(X) \).

Assume \( X \neq \{v\} \). Then \( |N_{\Gamma^{(n)\ast}(\neg u_n)}(X)| \geq 1 \), i.e. by the assumption \( |N_{\Gamma^{(n)\ast}(\neg u_n)}(X)| < \frac{1}{d}|X| \), we have \( |X| > d \). Consider this case. Let \( U_X := \{u \in U : v_u \in X\} \) in \( \Gamma^{(n)}(\neg u_n) \). Applying the fact that \( V^{(n)} \) is a subset of the copy of \( U^{(n)} \) we arrive at two possibilities:

(i) \( v_{un} \notin X \) and \(|X| = |U_X|\);

(ii) \( v_{un} \in X \) and \(|X| = |U_X| + 1\).

We will show that in either case the inequality (1) follows from Lemma 4.2.

Let \( S \) denote the fan from the matching \( M^{(n)}_v \) containing \( v_{un} \) and \( S' \) denote \( S \) with \( v_{un} \) removed. The conditions of Lemma 4.2 are satisfied if we consider \( \Gamma^{(n)}(\neg u_n) \) as \( \Gamma \) in that lemma, \( v_{un} \) as \( v \), and \( \Gamma^{(n)}(\neg u_n, \neg v_{un}) \) as \( \Gamma' \). Indeed, \( \Gamma^{(n)}(\neg u_n, \neg v_{un}) \) is \( U^{(n)}(\neg u_n, \neg v_{un})\)-reflected. Moreover, the corresponding graph \( \Gamma' \) from the lemma is obtained by removal of \( \Gamma^{(n)}(\neg u_n, \neg v_{un}) \), therefore it is the same as \( \Gamma^{(n)\ast}(\neg u_n, \neg v) \) with \( S \) removed. Since \( S \) is a fan from the perfect \((1, d)\) matching in \( \Gamma^{(n)\ast}(\neg u_n, \neg v) \), we know that \( \Gamma' \) satisfies Hall’s \( d\)-harem condition.

Therefore in case (i) by Lemma 4.2 we have

\[
|N_{\Gamma^{(n)}(\neg u_n)}(X)| \geq (d - 1)|X| - 1.
\]

This fact combined with inequalities \( d \geq 2 \) and \(|X| > d\) implies

\[
|N_{\Gamma^{(n)}(\neg u_n)}(X)| \geq \frac{1}{d}|X|.
\]

In case (ii) by Lemma 4.2 we have:

\[
|N_{\Gamma^{(n)\ast}(\neg u_n)}(X \setminus \{v_{un}\})| \geq (d - 1)(|X| - 1) - 1.
\]

Therefore

\[
|N_{\Gamma^{(n)}(\neg u_n)}(X)| \geq (d - 1)(|X| - 1) - 1.
\]

Let us show that the inequality is strict. Indeed, by Lemma 4.2 the equality

\[
|N_{\Gamma^{(n)}(\neg u_n)}(U_X \setminus \{v_{un}\})| = (d - 1)(|X| - 1) - 1
\]

implies that \( v_{un} \in N_{\Gamma^{(n)}(\neg u_n)}(U_X \setminus \{v_{un}\}) \) and \( N_{\Gamma^{(n)\ast}(\neg u_n, \neg v)}(U_X \setminus \{v_{un}\}) = \emptyset \). Therefore \( v_{un} \in V^{(n)\perp} \). On the other hand, \( v_{un} \in X \subset \Gamma^{(n)\ast} \), a contradiction with the choice of \( X \).

Again, inequalities \( d \geq 2 \) and \(|X| > d\) imply

\[
|N_{\Gamma^{(n)}(\neg u_n)}(X)| \geq (d - 1)(|X| - 1) \geq \frac{1}{d}|X|.
\]
Therefore, the assumption $|N_{\Gamma(n)^*(-u_n)}(X)| < \frac{1}{d}|X|$ implies

$$N_{\Gamma(n)^*(-u_n)}(X) \cap U(n)_{-u_n} \neq \emptyset,$$

i.e. there exists some $u_j^+ \in N_{\Gamma(n)^*(-u_n)}(X)$.

There are $d-1$ vertices adjacent to $u_j^+$ in $\Gamma(n)_{-u}$. We denote them by $v_{j,1}^+, \ldots, v_{j,d-1}^+$.

Since $\mathcal{M}_n^1$ is a perfect $(1, d)$ matching in the graph $\Gamma(n)^*(-u_n, -v)$, which in turn is a subgraph of the bipartite graph $\Gamma(n)(-u_n)$, we can use Lemma 4.4 for $v$, $X$, $u_j^+$ and arrive at $u_j^+ \rightarrow \mathcal{M}_n^1 \rightarrow v$.

This gives us a sequence of vertices $\{v_j', \ldots, v_n\}, \{u_j', u_{j,d-1}'\}$ as in Definition 4.3.

In order to prove that the graph $\Gamma(n)^*(-u_n, +u_j^+)$ satisfies Hall’s $d$-harem condition, we construct a perfect $(1, d)$-matching in it. We set

$$M' := \mathcal{M}_n^1 \setminus \{(u_n, v_{n,1}^0), \ldots, (u_n, v_{n,d}^0), (u_0, v_0'), (u_0', v_0'), (u_{j,d-1}', v_{j,d-1})\} \cup \{(u_j^+, v_{j,1}^+), \ldots, (u_j^+, v_{j,d-1}^+), (u_j^+, v_0'), (u_0', v_0'), (u_{j,d-1}', v_{j,d-1}^+), (u_{j,d-1}', v_{j,d-1}^+), (u_{j,d-1}', v_{j,d-1}^+), (u_{j,d-1}', v_{j,d-1}^+)\},$$

where $v_n' = v$.

**Figure 8.** We replace the red fans in the matching $\mathcal{M}_n^1$ by the blue fans to obtain the matching $M'$.

We remind the reader that $\mathcal{M}_n^1$ is a perfect $(1, d)$-matching in the graph $\Gamma(n)^*$. We have obtained $M'$ by removing $d$ edges adjacent to $u_n$, adding $d$ edges adjacent to $u_j^+$, and the following replacement: for each of $u_j'$ we replace one edge adjacent to it by another adjacent edge (then $v$ becomes adjacent to one edge in $M'$). It follows that the matching $M'$ is a perfect $(1, d)$-matching in the graph $\Gamma(n)^*(-u_n, +u_j^+)$. Therefore that graph satisfies Hall’s $d$-harem condition.

**5.2. Notation used in proof of Lemma 5.5.** Before stating the second lemma, we remind the notation used in it.

- In Case 3A
  - $\mathcal{U} := U(n+1) \setminus U(n)_{-u}^+$
  - $\mathcal{V} := V(n+1) \setminus V(n)_{-u}^+$
  - $\mathcal{T} := (\mathcal{U}, \mathcal{V}, \mathcal{E})$, where $\mathcal{E}$ is induced in $\Gamma$ by the sets of vertices $\mathcal{U}, \mathcal{V}$.
  - $U(n)_{-u}^+ := U(n)_{-u}^+ \cap U(n+1)$
  - $V(n)_{-u}^+ := V(n)_{-u}^+ \cap V(n+1)$
  - $\Gamma(n)^+ := (U(n)_{-u}^+, V(n)_{-u}^+, \hat{E}(n)^+)$, where $\hat{E}(n)^+$ is defined according to the set of fans of elements from $U(n)_{-u}^+$ putted into $\Gamma(n)^+$.

- In Case 3B
  - $\mathcal{U} := U(n+1) \setminus \{U(n)_{-u}^+ \cup \{\tilde{u}_n^+\}\}$
  - $\mathcal{V} := V(n+1) \setminus \{V(n)_{-u}^+ \cup \{\tilde{v}_{n,i}^+ : 1 \leq i \leq d - 1\}\}$.
  - $\mathcal{T} := (\mathcal{U}, \mathcal{V}, \mathcal{E})$, where $\mathcal{E}$ is induced in $\Gamma$ by the sets of vertices $\mathcal{U}, \mathcal{V}$. 


Lemma 5.3. Depending on the existence of \( \omega_2 \) either the graph \( \Sigma(\omega_2) \) or \( \Sigma(\omega_2) \) satisfies Hall’s \( d \)-harem condition.

Proof. We only consider the case of \( \omega_2 \) existing, the other one is analogous. It follows from the procedure that:

\[
\Omega(-\omega_2) = U^{(n+1)}(-u_n) \setminus \{v^i_{n,1}, \ldots, v^i_{n,d}, \ldots, v^i_{n,l}, \ldots, v^i_{n,d}\},
\]

Since \( \Omega_n \) is a perfect \((1, d)\)-matching in the graph \( \Gamma(n)\omega_n \) and \( \Sigma(-\omega_1, -\omega_2) \) is obtained from \( \Gamma(n)\omega_n \) by removal of two fans from \( \Omega_n \), we know that \( \Sigma(-\omega_1, -\omega_2) \) satisfies Hall’s \( d \)-harem condition. Therefore conditions of Lemma 4.1 are satisfied.

Lemma 5.4. For any \( X \subseteq V^{(n+1)} \), we have

\[
|N_{\Gamma(n+1)}(X)| \geq \frac{1}{d} |X|.
\]

Proof. The inequality follows from Lemma 4.1. Indeed, consider \( \Gamma(n+1) \) to be \( \Gamma \) from that lemma. Note that the construction guarantees that \( \Gamma(n+1) \) is \( U^{(n+1)} \)-reflected. Depending on existence of \( \omega_2 \) consider either \( \Gamma(n+1)(\omega_2) \) or \( \Gamma(n+1)(-\omega_1, +\omega_2) \) to be \( \Gamma \) from that lemma. Then the corresponding graph \( \Gamma \) from the lemma is equal to either \( \Sigma(-\omega_1) \) or \( \Sigma(-\omega_1, -\omega_2) \) and by Lemma 5.3 it satisfies Hall’s \( d \)-harem condition. Therefore conditions of Lemma 4.1 are satisfied.

Lemma 5.5. For any \( n \) one of the following holds:

- \( \Sigma \) satisfies Hall’s \( d \)-harem condition;
- there exist some vertex \( u^i_j \in U^{(n+1)} \) such that the graph \( \Sigma(u^i_{j}) \) satisfies Hall’s \( d \)-harem condition.

Remark 5.6. If \( |U^{(n+1)}| \leq 1 \), then Lemma 5.5 can be restated as follows. One of the following holds:

- \( \Sigma \) satisfies Hall’s \( d \)-harem condition;
- \( \Gamma(n+1) \) satisfies Hall’s \( d \)-harem condition.

Therefore this lemma proves Claim 3.2.

Proof of Lemma 5.5. Assume that \( \Sigma \) does not satisfy Hall’s \( d \)-harem condition. Let \( u^{l+1}_{n,i} \) be the root of the last fan added to the matching \( M_n \) in the second part of the \( n \)-th step; it belongs to the produced cycle of length 2. Recall that \( \omega_1, \omega_2 \) denote the vertices of the form \( v^i_{n,1} \) that were not added to \( M_n \). In particular in Case 3A there is only one such a vertex and in Case 3B there is either one or two such vertices. From now on we consider the case of two additional vertices: \( \omega_1 \) and \( \omega_2 \). The case of only one of them, \( \omega_1 \), is similar.

It is clear that the inequality \( |N_{\Sigma}(X)| \geq |N_{\Sigma}(\omega_1, \omega_2)(X)| \) holds for all \( X \subseteq \Omega \). This inequality also holds for subsets of \( \Omega \) which do not intersect \( \{\omega_1, \omega_2\} \). By Lemma 5.3 \( \Sigma(-\omega_1, -\omega_2) \) satisfies...
Hall’s $d$-harem condition, therefore if $\mathfrak{T}$ does not satisfy Hall’s $d$-harem condition then a witness of this is some finite subset of $\mathfrak{H}$ containing at least one of $\hat{v}_1, \hat{v}_2$.

The rest of the proof is divided into two parts.

**Part 1.** We check whether the graph $\mathfrak{T}(\hat{v}_2)$ satisfies Hall’s $d$-harem condition. If it does, we set

- $M'$ is a perfect $(1, d)$-matching in $\mathfrak{T}(\hat{v}_2)$;
- $\bar{T}^{(n+1)\perp} := \bar{T}^{(n)\perp}$;
- $\Gamma' = \mathfrak{T}$, denoting $\Gamma' = (U', V')$,

and finish Part 1 of the proof.

If it does not, then there exists the smallest connected set $X$ such that $\hat{v}_1 \in X \subset \mathfrak{H}(\hat{v}_2)$ and $|N_{\mathfrak{H}(\hat{v}_2)}(X)| < \frac{1}{d}|X|$. Observe that $|N_{\Gamma^{(n+1)}}(X)| \geq \frac{1}{d}|X|$ by Lemma 5.4

The inequalities

$$|N_{\Gamma^{(n+1)}}(X)| \geq \frac{1}{d}|X| \text{ and } |N_{\mathfrak{T}(\hat{v}_2)}(X)| < \frac{1}{d}|X|$$

imply

$$N_{\Gamma^{(n+1)}}(X) \cap U'^{\perp} \neq \emptyset,$$

i.e. there exists some $u^+ \in N_{\Gamma^{(n+1)}}(X)$. Similarly as in Lemma 5.1 we denote by $v^+_{j,1}, \ldots, v^+_{j,d-1}$ the remaining vertices of the fan from $\bar{T}^{(n)\perp}$ containing $u^+$.

Since $\mathfrak{H}^{(n)}$ is a perfect $(1, d)$-matching in the graph $\mathfrak{T}(\hat{v}_1, \hat{v}_2)$, which in turn is a subgraph of the bipartite graph $\Gamma^{(n+1)}$, we can use Lemma 4.4 for $u^+, X, \hat{v}_1$ and arrive at $u^+ \xrightarrow{\mathfrak{H}^{(n)}X} \hat{v}_1$. This gives us sequences $\{v^+_{0}, \ldots, v^+_{1}\}, \{v^+_{0}, \ldots, v^+_{n-1}\}$ as in Definition 4.3. We now apply an argument similar to one from the proof of Lemma 5.1. We set

$$M' := \mathfrak{H}^{(n)} \setminus \{(u^+_{n+1}, \ldots, u^+_{n+1}), (u^+_{n+1}, v^+_{0}), \ldots, (u^+_{0}, v^+_{0}), (u^+_{n+1}, v^+_{0}), \ldots, (u^+_{0}, v^+_{0})\} \cup$$

$$\{(u^+_{j,1}, \ldots, v^+_{j,d-1}), (u^+_{j,1}, v^+_{j,d-1}), (u^+_{j,1}, v^+_{j,d-1}), (u^+_{j,1}, v^+_{j,d-1})\},$$

where $v^+_{0} = \hat{v}_1$. Observe that

$$(\mathfrak{H} \cup \{u^+_{j}\}, (\mathfrak{H} \cup \{v^+_{j,1}, \ldots, v^+_{j,d-1}\} \setminus \{v^+_{0}\}) = \mathfrak{T}(u^+_{j}) \setminus \{v^+_{0}\} = \mathfrak{T}^+(u^+_{j}, -\hat{v}_2).$$

We remind the reader that $\mathfrak{H}^{(n)}$ is a perfect $(1, d)$-matching in the graph $\Gamma^{(n)}(u^-_{n})$. We have obtained $M'$ from $\mathfrak{H}^{(n)}$ by removing $d$ edges adjacent to $u^+_{n+1}$, adding $d$ edges adjacent to $u^+_{j}$, and the following replacement: for each $u^+_{j}$ we replace one edge adjacent to it by another adjacent edge. It follows that the matching $M'$ is a perfect $(1, d)$-matching in the graph $\mathfrak{T}(u^+_{j}, -\hat{v}_2)$. Therefore that graph satisfies Hall’s $d$-harem condition.

We define $\Gamma' := \mathfrak{T}(u^+_{j}, \hat{v}_2)$, denote $\Gamma' = (U', V')$ and put

$$\bar{T}^{(n+1)\perp} = (\bar{T}^{(n)\perp} \setminus \{u^+_{j}\}, \bar{V}^{(n)\perp} \setminus \{v^+_{j,k} : 1 \leq k \leq d - 1\}).$$

This ends the first part of the proof.

**Part 2.** We check whether the graph $\Gamma'$ satisfies Hall’s $d$-harem condition. If it does, then by Part 1 $\Gamma'$ has to be equal to $\mathfrak{T}(u^+_{j})$ and the proof is finished by the second option of the formulation. If it does not then repeating the reasoning of Part 1 we see that there exists the smallest connected set $X$ such that $\hat{v}_2 \in X \subset V'$ and $|N_{\Gamma'}(X)| < \frac{1}{d}|X|$. Again, using Lemma 5.4 we obtain

$$N_{\Gamma^{(n+1)}}(X) \cap U'^{\perp} \neq \emptyset,$$

i.e. there exists some $v^+_{i} \in N_{\Gamma^{(n+1)}}(X)$. We denote by $v^+_{i,1}, \ldots, v^+_{i,d-1}$ the vertices adjacent to $u^+_{j}$ in $\bar{T}^{(n+1)\perp}$.

The matching $M'$ obtained in the first part of the proof is a perfect $(1, d)$-matching for either the graph $\mathfrak{T}(\hat{v}_2)$, or the graph $\mathfrak{T}(u^+_{j}, -\hat{v}_2)$. Each of them is a subgraph of $\Gamma^{(n+1)}$. Therefore we can use Lemma 4.4 for $u^+_{j}, X, \hat{v}_2$. We have $u^+_{j} \xrightarrow{\mathfrak{H}^{(n)}X} \hat{v}_2$. Again we can apply the argument of Lemma 5.1 to obtain an appropriate matching:

$$M'' := M' \setminus \{(u^+_{0}, v^+_{0}), \ldots, (u^+_{n-1}, v^+_{n-1})\} \cup$$

or the graph $\mathfrak{T}(u^+_{j}, -\hat{v}_2)$. Each of them is a subgraph of $\Gamma^{(n+1)}$. Therefore we can use Lemma 4.4 for $u^+_{j}, X, \hat{v}_2$. We have $u^+_{j} \xrightarrow{\mathfrak{H}^{(n)}X} \hat{v}_2$. Again we can apply the argument of Lemma 5.1 to obtain an appropriate matching:
We have obtained $M''$ from $M'$ by adding $d$ edges adjacent to $u_k$, and replacing one edge in matching for each of $u_k$ in such a way, that $v_2$ is adjacent to one edge in $M''$.

The final argument depends on two possible outputs of Part 1. If $u_j^+$ does exist, then $M''$ is a perfect $(1, d)$-matching in the graph $\mathcal{X}(+u_j^+, u_j^+)$. If it does not, we redefine $u_j^+ := u_j^-$ and then $M''$ becomes a perfect $(1, d)$-matching in the graph $\mathcal{X}(+u_j^-, u_j^-)$. Therefore if $\mathcal{X}$ does not satisfy Hall’s $d$-harem condition, then either $\mathcal{X}(+u_j^+) or $\mathcal{X}(+u_j^-, u_j^-)$ satisfies this condition.

6. Proof of the Main Theorem

Proof of Theorem 2.4. Let us apply the construction of Section 3. This construction works modulo Claims 3.2 and 3.4. Claims 3.1 and 3.3 follow from Lemma 5.1. Claims 3.2 and 3.4 follow from Lemma 5.5. Since for every $n$ the union $\bigcup_{i=1}^{n} M_i$ consist of $(1, d - 1)$-fans, the final set of edges $M$ is an $(1, d - 1)$-matching.

For every $u \in U$ there is a step where an edge adjacent to $u$ is added to $M$. Then if the copy $v_u$ was not added to $M$ earlier, in the second part of this step this copy is added to $M$. It follows that $M$ is a perfect $(1, d - 1)$-matching of the graph $\Gamma$.

It remains to verify that $M$ realizes a function, say $f$, with controlled sizes of its cycles. Since the edges $(u_0, v_{0,1}^0)$ and $(v_{0,1}^0, u_{0,1}^0)$ are added to $M$ at step 1 we see $f^2(u_0) = f(v_{0,1}^0) = u_0$.

Since $u_0 = 1$, condition (i) of Definition 2.3 is satisfied.

Till the end of the proof a natural number $n$ will be used both for vertices and numbers of steps. Note that as a vertex it appears in $M$ at the $n$-th step at latest. This follows from minimality of $u_n$ in $\mathcal{U}^{(n-1)}$. The length of a cycle created at the $n$-th step cannot be greater than $\max\{2, n\}$ so if $n$ is in a cycle then $f^i(n) = n$ for some $i \leq n$ and condition (ii) of Definition 2.3 is satisfied.

It remains to show that condition (iii) is satisfied. Let $f_i$ be the partial function (living in $\mathbb{N}$) realized by $M_i$ and $f_n$ be the partial function realized by $\bigcup_{i=0}^{n} M_i$. Let $F_n = \{(f_n(m), m) : m \in \text{Dom}(f_n)\}$ be the graph of $f_n$ on $\text{Dom}(f_n) \cup \text{Rng}(f_n)$. Then $M_n = \{(f_n(m), m) : m \in \text{Dom}(f_n)\}$ is the graph of $f_n$ and is a subgraph of $F_n$ too.

Observe that $F_n$ has at most $n + 1$ connected components. Indeed, each subgraph $M_i$ has exactly one connected component. Furthermore, one of the following properties holds:

- $M_n$ is a connected component of $F_n$;
- $M_n$ is a fan with a root which already appears in $F_{n-1}$ as a vertex of degree 1 (see the way how fans of $U$-elements are added to $M$).

The construction guarantees that $F_n$ consists only of vertices of degree 1 and $d$. When a vertex has degree 1 its $F_n$-neighbourhood is only its $f_n$-image. When a vertex has degree $d$ its $f_n$-image and $d - 1$ preimages are in $F_n$. In particular each connected component of $F_n$ contains a cycle. The length of the cycle is not greater than $\max\{n, 2\}$.

Since the value $f_{n-1}(n)$ is defined, $n$ belongs to some connected component of $F_{n-1}$. Thus there exist $k$ and $l$ such that $f_{n-1}^{k+l}(n) = f_{n-1}^k(n)$. These $k$ and $l$ work for the equality $f^{k+l}(n) = f^k(n)$. It remains to show that $k$ can be bounded by $2n$ and $l$ by $n$.

The latter estimate is easy: the biggest cycle that can be constructed before the $n$-th step does not have more than $\max\{n - 1, 2\}$ elements. Below we will use the bound $n$ for $l$ for simplicity.

In order to show that $k$ is bounded by $2n$ let us estimate the size of a subset of $U$ without a cycle that can be added to the matching $M$ in the process of the $n$-th step of the construction. It must consist of elements of $U^{(n-1)\perp}$ added to the matching at the iteration of part 2 of the $n$-th step together with $u_n$. Therefore we can bound it by the maximal possible number of elements in $U^{(n-1)\perp}$ increased by 1.

Let us denote the number of elements from $U^{(s)\perp}$ added to $M$ at the $s$-th step by $\ell_s$. Clearly, $f^{\ell_s+1}(n) \in M_j$ for some $s \leq n$ and $j \leq n - 1$. If no cycle is constructed in the $j$-th step then
$f^{\ell+1}(f^{\ell+1}(n)) \in M_i$ for some $i \leq j - 1$. Iterating this argument we arrive at

$$k \leq \sum_{s=1}^{n}(\ell_s + 1).$$

Since at the $n$-th step the size of $\bigcup_{s=1}^{n} U(s)^{\perp}$ does not exceed $n - 1$,

$$\sum_{s=1}^{n} \ell_s \leq n - 1.$$

We see that $k \leq 2n - 1$. Thus condition (iii) of Definition 2.3 is satisfied. \hfill \Box

7. A Computable Version of Hall’s d-Harem Theorem

Before we state a computable version of Theorem 2.4 we describe our approach to computability of classical Hall’s d-harem theorem from [DI22a].

Let us consider the following definition. It is standard in computability theory.

**Definition 7.1.** A graph $\Gamma$ with the set of vertices $\Gamma$ is computable if there exists a bijective function $\nu : \mathbb{N} \to \Gamma$ such that the set

$$R := \{(i, j) : (\nu(i), \nu(j)) \in E\}$$

is computable. A bipartite graph $\Gamma = (U, V, E)$ (with $\Gamma = U \cup V$) is computably bipartite if $\Gamma$ is computable and the set of $\nu$-numbers of $U$ is computable.

To simplify the matter, in the case of bipartite graphs, from now on we will always identify $U$ and $V$ with computable subsets of $\mathbb{N}$ so that $\nu$ is the identity map. Further, admitting that $U \cap V \neq \emptyset$ we distinguish these sets saying that $U$ is taken from the left copy of $\mathbb{N}$ but $V$ is taken from the right one. In particular we often consider graphs where $U = V = \mathbb{N}$.

To formulate a computable version of Hall’s Harem theorem we need the following definition of Kierstead [Kie83].

**Definition 7.2.** A locally finite computable graph $\Gamma$ is called highly computable if the function $g(n) = |N_{\Gamma}(\nu(n))|$, $n \in \mathbb{N}$, is a computable function $g : \mathbb{N} \to \mathbb{N}$.

It is worth noting here that this definition easily implies the existence of an algorithm which for every $m$ and $n$ computes the $n$-ball of $\nu(m)$ in $\Gamma$.

**Definition 7.3.** Let $\Gamma = (U, V, E)$ be a computable bipartite graph. A perfect $(1, k)$-matching $M$ from $U$ to $V$ is called a computable perfect $(1, k)$-matching if $M$ is a computable set of pairs.

Observe that computable perfectness exactly means that there is an algorithm which

- for each $i \in U$, finds the tuple $(i_1, i_2, \ldots, i_k)$ such that $(i, i_j) \in M$, for all $j = 1, 2, \ldots, k$;
- when $i \in V$ it finds $i' \in U$ such that $(i', i) \in M$.

In [DI22a] we introduced the following condition, which implies the existence of computable perfect $(1, k)$-matchings in highly computable bipartite graphs. For $k = 1$ it was formulated earlier in [Kie83].

**Definition 7.4.** A bipartite graph $\Gamma = (U, V, E)$ satisfies the computable expanding Hall’s harem condition with respect to $k$ (denoted c.e.H.h.c.(k)), if and only if there is a total, computable function $h : \mathbb{N} \to \mathbb{N}$ such that:

- $h(0) = 0$
- for all finite sets $X \subseteq U$, the inequality $h(n) \leq |X|$ implies $n \leq |N(X)| - k|X|$
- for all finite sets $Y \subseteq V$, the inequality $h(n) \leq |Y|$ implies $n \leq |N(Y)| - \frac{k}{2}|Y|$.

In [DI22a] we have proven the following.

**Theorem B.** If $\Gamma = (U, V, E)$ is a highly computable bipartite graph satisfying the c.e.H.h.c.(k), then $\Gamma$ has a computable perfect $(1, k)$-matching.
8. A computable version of Hall’s harem Theorem with cycles, preliminaries

We will now present an effective version of the construction of Section 3. As previously, we assume that $\Gamma = (U, V, E)$ is a bipartite graph, such that $V \subseteq U \subseteq \mathbb{N}$, i.e. $V$ is a subset of the right copy of $U$.

8.1. Aim of the construction. We will prove that given a bipartite graph as in Theorem 2.3 which is additionally highly computable and satisfying $c.e.H.h.c.(d)$, there is a computable perfect $(1, (d - 1))$ matching which realizes a function $f : \mathbb{N} \to \mathbb{N}$ with controlled sizes of cycles (Definition 2.3), such that the following conditions are satisfied:

(I) $\{n | n$ is a periodic point $\} = \mathbb{N}$ is computable;

(II) $\{\{n, m \} | n$ is a periodic point , $m$ is in the cycle of $n\} = \mathbb{N}$ is computable;

(III) the set $\{\{n, m \} | f^{m}(n)$ is a periodic point $\} = \mathbb{N}$ is computable and its first coordinates cover $\mathbb{N} \setminus \{0\}$. Moreover, the matching can be constructed by a uniform algorithm.

In fact it is easy to see that a computable $(d - 1)$ to 1 function with controlled sizes of its cycles satisfies conditions (I)–(III).

8.2. Notation used in the construction. The computable version of the construction of Section 3 is very similar. We only list here the differences.

- By $h(x)$ we denote a witness of $c.e.H.h.c.(d)$ for $\Gamma$. At the end of each step, in Lemma 11.5 we produce $\tilde{h}(x)$ (possibly with some index) which is a witness of $c.e.H.h.c.(d)$ for $\Gamma_{n}$, defined with the help of $h$.

- $B(n)(u_{n})$ (resp. $S(n)(u_{n})$) is the ball (resp. sphere) in $\Gamma_{n}$ with the center $u_{n}$ and the radius $\max\{4h(3d(n + 1)) + 3.5\}$.

- $B(n)(u_{n}^{l+1})$ (resp. $S(n)(u_{n}^{l+1})$) is the ball (resp. sphere) in $\Gamma_{n}$ with the center $u_{n}^{l+1}$ and the radius $\max\{4h(3d(n + 1)) + 3.5\}$. $\mathcal{M}_{n}^{1}$ denotes a finite $(1, d)$-matching in the bipartite graph $B(n)^{*}(u_{n}) = B(n)(u_{n}) \cap \Gamma_{n}$.

- It is obtained in such a way, that it satisfies the conditions of the perfect $(1, d)$-matchings for all vertices that are at the distance less than $\max\{4h(3d(n + 1)) + 3.5\}$ from $u_{n}$.

- $\mathcal{M}_{n}^{2}$ denotes a finite $(1, d)$-matching in the bipartite graph $B(n)^{*}(u_{n}^{l+1}) = B(n)(u_{n}^{l+1}) \cap \Gamma_{n}$.

- It is obtained in such a way, that it satisfies the conditions of the perfect $(1, d)$-matchings for all vertices that are at the distance less than $\max\{4h(3d(n + 1)) + 3.5\}$ from $u_{n}^{l+1}$.

The main difference with the classical version of the construction is that instead of $(1, d)$-matchings of infinite graphs we use $(1, d)$-matchings of finite subgraphs (when we produce $(1, d - 1)$-fans to add to the constructed objects). In the Sections 10 and 11 we will show that the choice of the radius allows us to use this finite matchings $\mathcal{M}_{n}^{1}, \mathcal{M}_{n}^{2}$ without spoiling $c.e.H.h.c.(d)$ in the resulting graphs.

9. The computable version of the construction

We assume that $\Gamma = (U, V, E)$ is a highly computable bipartite graph satisfying $c.e.H.h.c.(d)$, such that:

- both $U$ and $V$ are identified with $\mathbb{N} \setminus \{0\}$;

- $\Gamma$ is fully reflected;

- $E$ does not contain edges of the form $(v, u_{n})$.

Before the construction we mention the following easy proposition. It gives an algorithm for finite matchings $\mathcal{M}_{n}^{1}$ and $\mathcal{M}_{n}^{2}$ in $B(n)^{*}(u_{n}^{0})$ and in $B(n)^{*}(u_{n}^{0})$ respectively (see the notation above).

Proposition 9.1. Let $\Gamma = (U, V, E)$ and $\Gamma' = (U', V', E')$ where $\Gamma'$ is a subgraph of $\Gamma$ which is a highly computable bipartite graph satisfying $c.e.H.h.c.(d)$. Let $B(u')$ be a ball of odd radius in $\Gamma$, where $u' \in U'$. Put $B'(u') = B(u') \cap \Gamma'$. 

Then there exists a uniform algorithm finding a $(1, d)$-matching in $\mathcal{B}'(u^*)$, say $M_{u^*}$, which satisfies the conditions of perfect $(1, d)$-matchings for all $u \in U' \cap \mathcal{B}(u^*)$ and $v \in V' \cap \mathcal{B}(u^*) \setminus \mathcal{S}(u^*)$.

From now on, such a matching $M_{u^*}$ will be called perfect $(1, d)$-matching in the ball $\mathcal{B}'(u^*)$.

**Proof.** All elements of $V' \cap (\mathcal{B}(u^*) \setminus \mathcal{S}(u^*))$ have the same neighbourhoods in $\Gamma'$ and $\mathcal{B}'(u^*)$. By Theorem 9.3 there exists a perfect $(1, d)$-matching in $\Gamma'$. Thus there exists a matching in $\mathcal{B}'(u^*)$ which satisfies the conditions of a perfect $(1, d)$-matchings for all $u \in U' \cap \mathcal{B}(u^*)$ and $v \in V' \cap (\mathcal{B}(u^*) \setminus \mathcal{S}(u))$. Since $\Gamma'$ is highly computable and $\mathcal{B}'(u^*)$ is finite we find this matching in a computable way. The algorithm is uniform, i.e. it does not depend on $u^*$.

9.1. **Step 1, part 1.** We take $u_0$, the first element of the set $U$ (it is clear that $u_0 = 1$). Let $\mathcal{B}'(u_0)$ be a ball centered at $u_0$ and of the radius $\max\{4h(3d) + 3, 5\}$.

By Proposition 9.1 we can compute a perfect $(1, d)$-matching $\mathcal{M}_0$ in the ball $\mathcal{B}'(u_0)$. Let $\tilde{v}_0^{0,1}, \ldots, \tilde{v}_0^{0,d}$ be elements of $V$, ordered by its numbers, such that $(u_0, \tilde{v}_0^{0,i}) \in \mathcal{M}_0$ for all $i \leq d$. We include edges $(u_0, v_0,i)$ for $i \leq d - 1$ in $M_0$. The following statement holds.

**Claim 9.2.** $\Gamma(0)(-u_0) := (U \setminus \{u_0\}, V \setminus \{v_0^{0,1}, \ldots, v_0^{0,d-1}\})$ satisfies $c.e.H.h.c.(d)$.

See Lemma 11.1 for the proof of this claim.

9.2. **Step 1, part 2.** As in the classical version of the construction we denote $u_0^1 := u_0 \cup u_0^0$ and aim to add the edge $(u_0^1, v_0^0)$ to $M_0^1$. Note that $(u_0^1, v_0^0) \in \Gamma^0(-u_0)$, because $(u_0, v_0^0) \in \Gamma^0$ and the latter one is $U^0$-reflected.

Let $\mathcal{B}'(u_0^1)$ be the ball in the graph $\Gamma(0)(-u_0)$, with the center $u_0$ and the radius $\max\{4h(3d) + 3, 5\}$. Since $\Gamma(0)(-u_0)$ satisfies $c.e.H.h.c.(d)$, by Proposition 9.1 we can compute a perfect $(1, d)$-matching in the ball $\mathcal{B}'(u_0^1)$. We remind the reader that $\hat{v}_0^{1,0}, \ldots, \hat{v}_0^{1,d}$ are elements of $V$, ordered by its numbers, such that $(u_0^1, \hat{v}_0^{1,i}) \in \mathcal{M}_0^1$ for all $i \leq d$. Since $u_0$ has the least number in $V$, we have two possible cases:

1. $v_{u_0} = \hat{v}_0^{1,0}$. We set $\tilde{v}_0^{1,0} := \hat{v}_0^{1,0}$ (i.e. $v_{u_0}$ is among these elements).
2. $v_{u_0} \neq \hat{v}_0^{1,0}$. Then find the fan $(u, v_i) \in \mathcal{M}_0^2$, $1 \leq i \leq d$, such that $v_{u_0} = v_1$ (here we assume that the ordering of $v_i$ corresponds to their indexes). We set:
   
   \[ \begin{align*}
   & v_0^{1,0} := v_{u_0} \\
   & v_0^{1,i} := \hat{v}_0^{1,i-1} \quad \text{for } i \geq 2,
   \end{align*} \]

   and define a candidate for $\Gamma(1)$:
   
   \[ \begin{align*}
   & \hat{u}_0^{1,i} := u \\
   & \hat{v}_0^{1,i-1} := v_i \quad \text{for } i \geq 2.
   \end{align*} \]

   In either case define the fan $M_0^1$ as the set of edges $(u_0^1, v_0^{1,i})$ for $1 \leq i \leq d - 1$.

   Put $M_0 = M_0^0 \cup M_0^1$. We obtain $\Gamma(1)$ by removal of $M_0$ from $\Gamma$. Since $u_0, v_{u_0}, v_0^{0,1}$ and $u_0^1$ have been removed, it is $U(1)$-reflected. It might turn out that $\Gamma(1)$ does not satisfy Hall’s $h$-arem condition.

   Let $\Gamma' = (U(1) \setminus \{\hat{u}_0^{1,0}\}, V(1) \setminus \{\hat{v}_0^{1,0}, \ldots, \hat{v}_0^{1,d-1}\})$. The following claim follows from Lemma 11.5 below.

**Claim 9.3.** At least one of $\Gamma(1)$ or $\Gamma'$ satisfies $c.e.H.h.c.(d)$.

   If $\Gamma(1)$ satisfies $c.e.H.h.c.(d)$, set $\Gamma'(1) := \Gamma(1)$ and $U(1) := \emptyset, V(1) := \emptyset$. If $\Gamma(1)$ does not satisfy $c.e.H.h.c.(d)$, set $\Gamma'(1) := \Gamma'$ and

   \[ \begin{align*}
   & u_0^{1,0} := u_0^{1,0} \\
   & v_0^{1,i} := \hat{v}_0^{1,i} \\
   & U(1) := \{u_0^{1,0}\} \\
   & V(1) := \{v_0^{1,i} : 1 \leq i \leq d - 1\}.
   \end{align*} \]
9.3. Step $n+1$, part 1. At the previous step we constructed graphs $\Gamma^{(n)}$ and $\Gamma^{(n)*}$ where $\Gamma^{(n)}$ is $U^{(n)}$-reflected and $\Gamma^{(n)*}$ satisfies c.e.H.h.c.$(d)$. Note that since $|U^{(n)} \setminus U^{(n-1)}| \leq 1$, there are at most $n$ vertices $u_i^\perp$ in $U^{(n)}\perp$.

Take $u_n$, the first element of the set $U^{(n)}$. In order to define $M_0^n$ we have two possible cases:

1. There is $j$ such that $u_n = u_j^\perp \in U^{(n)}\perp$. Then we set $M_0^n$ to consist of all edges of the form $(u_j^\perp, v_{j,i}^\perp)$ and remove the fan with the root $u_j^\perp$ from $\Gamma^{(n)}\perp$. We redefine $U^{(n)}\perp$ and $V^{(n)}\perp$ accordingly (in particular $u_j^\perp$ is removed from $U^{(n)}\perp$).

2. If $u_n \notin U^{(n)}\perp$, then take the ball $B^{(n)}(u_n)$ in $\Gamma^{(n)}$ with the center $u_n$ and the radius $\max\{4h(d(3n+1)) + 3, 5\}$. By Proposition 9.1 we can compute $M_1^n$, a perfect $(1,d)$-matching in the ball $B^{(n)*}(u_n) := B^{(n)}(u_n) \cap \Gamma^{(n)*}$. Verify whether there is $j$ with $(u_n, v_{n,j}^\perp) \in \Gamma^{(n)}\perp$. By the definition of $\Gamma^{(n)}\perp$ it can happen for at most one $j$. If it is so, set $M_0^n$ to consist of $(u_n, v_{n,j}^0)$ and $d-2$ of the remaining edges of the form $(u_n, v_{n,i}^0) \in M_1^n$ (excluding the one with the greatest index). If not, set $M_0^n$ to consist of edges $(u_n, v_{n,i}^0) \in M_1^n$ for $i \leq d-1$.

The following claim follows from Lemma 11.1 below.

Claim 9.4. Let $\Gamma^{(n)*}(-u_n) = \Gamma^{(n)}(-u_n) \cap \Gamma^{(n)*}$. Then one of the following statements holds:

- $\Gamma^{(n)*}(-u_n)$ satisfies c.e.H.h.c.$(d)$.
- We can compute $u_j^\perp \in U^{(n)}\perp$ such that the graph $\Gamma^{(n)*}(-u_n + u_j^\perp)$ satisfies c.e.H.h.c.$(d)$.

If $\Gamma^{(n)*}(-u_n)$ does not satisfy c.e.H.h.c.$(d)$, let $u_j^\perp$ be an element from $U^{(n)}\perp$ realizing the second possibility of this claim. We change the place of the fan of $u_j^\perp$ with its leaves from $V^{(n)}\perp$ as follows: we take it from $(U^{(n)}\perp, V^{(n)}\perp)$ and then we put it into $\Gamma^{(n)*}$. Thus the latter (and $U^{(n)}\perp, V^{(n)}\perp$) are updated. It is clear that now the redefined $\Gamma^{(n)*}(-u_n)$ satisfies c.e.H.h.c.$(d)$.

Similarly as in the classical version of this construction, at the second part of Step $n+1$ we want to achieve the following goals after the step:

1. the partial function $f_n$ corresponding to $\bigcup_{i=0}^n M_i$ has controlled sizes of its cycles, and
2. the graph $\Gamma^{(n+1)}$ obtained at the end of the step is $U^{(n+1)}$-reflected and the corresponding graph $\Gamma^{(n+1)*}$ satisfies c.e.H.h.c.$(d)$.

9.4. Step $n+1$, part 2. We begin by checking whether $v_{u_n}$ belongs to $\Gamma^{(n)}\perp$. If $v_{u_n} \in \Gamma^{(n)}\perp$ then we denote $u_n$ by $u_0^n$ and use the same process of choosing the consecutive vertices $u_i^\perp$ as in classical case i.e.

First step of iteration. Assume that for some $j, i$ we have $v_{u_n} = v_{j,i}^\perp \in V^{(n)}\perp$. Then set

$$u_n^1 = u_{j,i}^1, v_{n,k}^1 : = v_{j,k}^\perp, k \leq d - 1,$$

$$M_i^n : = \{ (u_n^1, v_{n,k}^1) : 1 \leq k \leq d - 1 \}$$

and check whether $v_{u_n} \in \Gamma^{(n)}\perp(-u_0^n)$. If it is so we repeat the iteration for $v_{u_n}$. Note that $v_{u_n} \in \Gamma^{(n)}\perp(-u_0^n, -u_1^n)$ then.

Single step of iteration. We verify if $v_{u_n} \in \Gamma^{(n)}\perp(-u_0^n, -u_1^n, \ldots, -u_m^n)$. If it is so then for some $j_{m+1}, i$ we have $v_{u_n} = v_{j_{m+1}, i}^\perp \in V^{(n)}\perp$. Define

$$u_{m+1}^n = u_{j_{m+1}, i}^1, v_{n,k}^m : = v_{j_{m+1},k}^\perp, 1 \leq k \leq d - 1,$$

$$M_m^n : = \{ (u_{m+1}^n, v_{n,k}^m) : 1 \leq k \leq d - 1 \}$$

and we repeat the iteration for $v_{u_n}$. This ends the single iteration step.

Since $|\Gamma^{(n)}\perp| \leq n$, the procedure ends after at most $n$ iterations. Therefore one of the following cases is realized for some $l \leq n$:

1. $v_{u_n} \notin \Gamma^{(n)}$;
2. $v_{u_n} \in \Gamma^{(n)}$, but $v_{u_n} \notin \Gamma^{(n)}(-u_0^n, -u_1^n, \ldots, -u_l^n)$ (this case is impossible for $l = 0$);
(3) \( v_{ud_n} \in \Gamma^{(n)*} \).

In case (1) \( v_{ud_n} \) was already added to \( M \) at preceding steps and (ii) is satisfied. We finish step \( n + 1 \), so no cycle is constructed at the step.

In case (2) we also finish step \( n + 1 \). Note that the last iteration closes some cycle \((u_n^k, \ldots u_n^l)\) constructed at Step \( n + 1 \). The length of this cycle is not greater than \( l + 1 \).

In case (3) we will construct a cycle \((u_n^l, u_n^{l+1})\) of length 2 as follows.

We start with a new term \( u_n^{l+1} := u_{ud_n} \).

Since the edge \((u_n^l, v_n^l)\) is in \( \Gamma^{(n)} \) and is not in \( M_n^l \), applying \( U^{(n)} \)-reflectedness of \( \Gamma^{(n)} \) we see \((u_n^{l+1}, v_n^l) \in \Gamma^{(n)}(-u_n, -u_n^l, \ldots, -u_n^l) \). Observe that since \( u_n^1, \ldots, u_n^{l+1} \) are in \( U^{(n)} \),

\[
\Gamma^{(n)}(-u_n, -u_n^1, \ldots, -u_n^l) \cap \Gamma^{(n)*} = \Gamma^{(n)}(-u_n) \cap \Gamma^{(n)*} = \Gamma^{(n)*}(-u_n).
\]

Recall that by the first part of this step the graph \( \Gamma^{(n)*}(-u_n) \) satisfies c.e.H.h.c.(d). Thus we can take the ball \( B^{(n)}(u_n^{l+1}) \) in \( \Gamma^{(n)}(-u_n, -u_n^1, \ldots, -u_n^l) \) with the center \( u_n^{l+1} \) and the radius \( \max\{4h(3d(n+1)) + 3, 5 \} \). By Proposition \[3.4\] we can compute a \((1, d)\)-perfect matching realizing \( \mathcal{M}_n^2 \) in the ball \( B^{(n)}(u_n^{l+1}) \).

We now check whether for some \( j \) we have \( u_n^{l+1} = u_j^\perp \). If it is so, we set \( v_{j,i}^{l+1} := v_{j,i}^\perp \) for \( 1 \leq i \leq d \). If there is no \( j \) such that \( u_n^{l+1} = u_j^\perp \), then \( v_{j,i}^{l+1} \) will denote the elements adjacent to \( u_n^{l+1} \) under \( \mathcal{M}_n^2 \).

There are two cases:

A) \( \gamma u_n^{l+1}, v_{ud_n} \in \mathcal{M}_n^2 \), i.e. \( v_{ud_n} = u_{l,k}^{l+1} \) for some \( 1 \leq k \leq d \). In this case \( u_n^{l+1} \) cannot be \( u_j^\perp \) for any \( j \).

B) \( \gamma u_n^{l+1}, v_{ud_n} \notin \mathcal{M}_n^2 \), i.e. there exists some \( u \in \Gamma^{(n)*}(-u_n) \), such that \( v_{ud_n} = u_k \) for some \( 1 \leq k \leq d \), where \( v_1 \ldots v_d \) denote the elements adjacent to \( u \) under \( \mathcal{M}_n^2 \). In this case it is possible that \( u_n^{l+1} = u_j^\perp \).

In either case we produce a cycle of length 2 as usual i.e. by including \( (u_n^{l+1}, v_{ud_n}) \) into \( M_n^{l+1} \) together with a fan with the root \( u_n^{l+1} \) and \( (d - 2) \) leaves taken among \( v_{j,i}^{l+1} \). The difference between cases is that in case B) \( u_n^\perp \) exists and is equal to \( u \). We rename by \( \check{v}_{j,i}^\perp \) these \( d - 1 \) vertices \( v_j \) which are not equal to \( v_{ud_n} \). The procedure is finished.

We define all necessary subgraphs of \( \Gamma \) in the same way as in the classical case (see Section \[3.4\]).

The following claim follows from Lemma \[11.5\] below.

Claim 9.5. At least one of the following holds:

- \( \exists \gamma \) satisfies c.e.H.h.c.(d);
- there exist some vertex \( u^\perp \in \overset{\perp}{U}^{(n)} \) such that the graph \( \gamma (u^\perp) \) satisfies c.e.H.h.c.(d).
- there exists vertices \( u^\perp, u^\perp_j \in \overset{\perp}{U}^{(n)} \) such that the graph \( \gamma (u^\perp, u^\perp_j) \) satisfies c.e.H.h.c.(d).

It remains to define the graph \( \Gamma^{(n+1)*} \) and the sets \( \overset{\perp}{U}^{(n+1)} \), \( \overset{\perp}{V}^{(n+1)} \) as in the classical case.

10. TECHNICAL LEMMAS - COMPUTABLE CASE

In the proofs of Lemmas \[11.1\] and \[11.5\] we will regularly use the following fact.

Proposition 10.1 (Removing at most \( k \) fans from a ball). Assume that \( \Gamma = \gamma (U, V, E) \) is a highly computable bipartite graph satisfying c.e.H.h.c.(d) with witness \( h \). Let

- \( u \in U \) and \( B(u) \) be the ball in \( \Gamma \) with the center \( u \) and the radius of at least \( \max\{2h(kd) + 3, 5\} \);
- \( M_u \) be a perfect \((1, d)\)-matching in the ball \( B(u) \);
- \( U_1 \) be a subset of \( U \) from the ball of \( u \) of radius \( \leq 2 \), \( |U_1| \leq k \), and \( V_1 \) be the set of vertices adjacent in \( M_u \) to vertices from \( U_1 \).
Then \( \Gamma' := (U \setminus U_1, V \setminus V_1) \) satisfies Hall’s \( d \)-harem condition.

**Proof.** Let us show that the inequality \(|N_{\Gamma'}(X)| - d|X| \geq 0\) holds for any \( X \subset U \setminus U_1 \). If \( X \) is not connected, then its neighbourhood is a disjoint union of the neighbourhoods of the connected components of \( X \). Thus the inequality for each component implies it for \( X \). Therefore without loss of generality we may also assume that \( X \) is connected. Further, we may assume that \( X \not\subseteq U \cap B(u) \) and \( V_1 \cap N_{\Gamma'}(X) \neq \emptyset \). Indeed, if \( V_1 \cap N_{\Gamma'}(X) = \emptyset \) then \( N_{\Gamma'}(X) = N_{\Gamma}(X) \), i.e. \(|N_{\Gamma'}(X)| \geq d|X|\). On the other hand if \( X \subset U \cap B(u) \) then using existence of an \((1,d)\)-matching \( M_u \) from \( U \cap B(u) \) to \( V \cap B(u) \) which is perfect for subsets of \( U \cap B(u) \), we see that \(|N_{\Gamma'}(X)| - d|X| \geq 0\).

Let us choose \( u_1, u_2 \in X \) such that \( u_1 \in N_{\Gamma'}(V_1) \) and \( u_2 \in X \setminus B(u) \). By the choice of the radius of the ball \( B(u) \) the distance between \( u_1 \) and \( u_2 \) is at least \( 2h(2d) + 1 \). Thus \(|X| \geq h(kd)\). Since \( \Gamma \) satisfies c.e.H.h.c.(\( d \)) we have

\[ |N_{\Gamma}(X)| - d|X| \geq kd. \]

We know that \(|N_{\Gamma'}(X)| \geq |N_{\Gamma}(X)| - kd\), therefore

\[ |N_{\Gamma'}(X)| - d|X| \geq |N_{\Gamma}(X)| - kd - d|X| \geq 0. \]

The proof for the case of subsets of \( V' \) is completely analogous.

\[ \square \]

Lemmas [11.1] and [11.5] below are counterparts of Lemmas [5.1] and [5.5]. As in the latter (classical) case Lemma [4.4] will be essential in the proofs below in order to replace some \((1,d)\)-fans in the matching. Since in the present (computable) case this matching is partial and finite (unlike in the classical case above), we need an additional technical lemma. It will guarantee that in a graph satisfying c.e.H.h.c.(\( d \)) Hall’s \( d \)-harem condition can be preserved after some replacement of \((1,d-1)\)-fans.

**Lemma 10.2** (Replacement of a fan in a finite matching). Assume

- \( \Gamma = (U,V,E) \) is a highly computable bipartite graph satisfying c.e.H.h.c.(\( d \)) with a witness \( h \);
- \( \Gamma^* = (U^*,V^*) \) is a subgraph of \( \Gamma \) such that \( \Gamma \setminus \Gamma^* \) consists of a \((1,d-1)\)-fans;
- \( u \in U \) and \( B(u) \) is a ball in \( \Gamma \) of \( u \) of odd radius of at least \( \max\{4h(d(n+1)) + 3,5\} \);
- \( \mathcal{F}_u \) is an \((1,d-1)\)-fan in \( \Gamma^* \) with the root \( u \);
- \( \mathcal{G} \) is an \((1,d-1)\)-fan in \( \Gamma \setminus \Gamma^* \) with the root \( u^\perp \);
- \( \mathfrak{M} \) is a perfect \((1,d)\)-matching in the ball \(((\Gamma^* \setminus \mathcal{F}_u) \cup \mathcal{G}) \cap B(u) \).

Additionally assume that \( \Gamma^* \) satisfies c.e.H.h.c.(\( d \)) with the witness

\[ \hat{h}(m) = \begin{cases} 
0, & \text{if } m = 0, \\
h(m + nd), & \text{if } m > 0.
\end{cases} \]

but \( \Gamma^* \setminus \mathcal{F}_u \) does not satisfy Hall’s \( d \)-harem condition.

Then \( (\Gamma^* \setminus \mathcal{F}_u) \cup \mathcal{G} \) satisfies Hall’s \( d \)-harem condition.

**Proof.** We start the proof with several claims.

**Claim 10.3.** All subsets of \( U^* \setminus \{u\} \) satisfy Hall’s \( d \)-harem condition in \( \Gamma^* \setminus \mathcal{F}_u \).

**Proof of Claim 10.3.** Assume that \( X \subset U^* \setminus \{u\} \) does not satisfy Hall’s \( d \)-harem condition. If \( X \) is not connected, then its neighbourhood is a disjoint union of the neighbourhoods of the connected components of \( X \). Therefore without loss of generality we may assume that \( X \) is connected. Since \( \Gamma^* \) satisfies Hall’s \( d \)-harem condition, there exists \( u' \in X \) such that distance between \( u' \) and \( u \) is equal to 2. On the other hand the matching \( \mathfrak{M} \) witnesses Hall’s \( d \)-harem condition for all subsets of \( ((U^* \setminus \mathcal{F}_u) \cup \{u^\perp\}) \cap B(u) \). Therefore \( X \not\subseteq ((U^* \setminus \{u\}) \cup \{u^\perp\}) \cap B(u) \). It follows that there exists \( u'' \in X \setminus B(u) \). The distance between \( u' \) and \( u'' \) is at least \( 4h(d(n+1)) + 1 \), i.e. \(|X| \geq h(d(n+1)) + 1\). By the definition of \( h \),

\[ |N_{\Gamma^*}(X)| \geq d|X| + d. \]
Thus
\[ |N_{\Gamma \setminus \mathfrak{S}_u}(X)| \geq |N_{\Gamma^*}(X)| - (d - 1) \geq d|X| + 1, \]
i.e. \( X \) satisfies Hall’s \( d \)-harem condition in \( \Gamma^* \setminus \mathfrak{S}_u \). \( \Box \)

By this claim there is \( X \subset V^* \setminus \mathfrak{S}_u \) that does not satisfy Hall’s \( d \)-harem condition in \( \Gamma^* \setminus \mathfrak{S}_u \). Take the smallest connected one. Observe that \( u \) must be in \( N_{\Gamma^*}(X) \). Indeed, otherwise \( |N_{(\Gamma^* \setminus \mathfrak{S}_u)(X)}| = |N_{\Gamma^*}(X)| \). Since \( \Gamma^* \) satisfies c.e.H.h.c.(\( d \)) we get a contradiction.

**Claim 10.4.** For any vertex \( v \) from \( X \) the distance between \( u \) and \( v \) cannot exceed \( 2h(d(n+1)) - 1 \).

**Proof of Claim 10.4.** Since \( \Gamma \) is bipartite, then distance between \( u \) and any element of the set \( X \) is odd. If for some \( v \in X \) the distance between \( u \) and \( v \) is at least \( 2h(d(n+1)) + 1 \), then \( X \) has at least \( h(d(n+1)) \) elements. Since \( h \) is a witness of c.e.H.h.c.(\( d \)) for \( \Gamma^* \), we have
\[ |N_{\Gamma \setminus \mathfrak{S}_u}(X)| \geq |N_{\Gamma^*}(X)| - 1 \geq \frac{1}{d}|X| + 1 - 1 \geq \frac{1}{d}|X|, \]
which contradicts that \( X \) does not satisfy Hall’s \( d \)-harem condition. \( \Box \)

As a consequence of Claim 10.4 \( X \subset (B(u) \setminus S(u)) \).

**Claim 10.5.** The element \( u^+ \) belongs to \( N_{(\Gamma^* \setminus \mathfrak{S}_u) \cup \mathfrak{S}}(X) \) and the distance between \( u^+ \) and \( u \) is at most \( 2h(d(n+1)) \).

**Proof of Claim 10.5.** Since \( X \subset (B(u) \setminus S(u)) \) and \( \mathfrak{M} \) satisfies the conditions of the perfect \((1, d)\)-matching for all subsets of \( ((\Gamma^* \setminus \mathfrak{S}_u) \cup \mathfrak{S}) \cap (B(u) \setminus S(u)) \), then
\[ |N_{(\Gamma^* \setminus \mathfrak{S}_u) \cup \mathfrak{S}}(X)| \geq \frac{1}{d}|X| > |N_{\Gamma \setminus \mathfrak{S}_u}(X)|. \]
Clearly \( u^+ \) is in \( N_{(\Gamma^* \setminus \mathfrak{S}_u) \cup \mathfrak{S}}(X) \).

If the distance between \( u^+ \) and \( u \) is at least \( 2h(d(n+1)) + 2 \) then there exists an element of \( X \) which is at distance at least \( 2h(d(n+1)) + 1 \) from \( u \). Contradiction with Claim 10.4 \( \Box \)

We now see that \( \{u, u^-\} \subset N_{\Gamma \setminus \mathfrak{S}_u}(X) \). Let us prove that Hall’s \( d \)-harem condition is satisfied in \( (\Gamma^* \setminus \mathfrak{S}_u) \cup \mathfrak{S} \) for subsets of \( (U^* \setminus \{u\}) \cup \{u^+\} \). By Claim 10.3 all subsets of \( U^* \setminus \{u\} \) satisfy Hall’s \( d \)-harem condition in \( \Gamma^* \setminus \mathfrak{S}_u \) and so in \( (\Gamma^* \setminus \mathfrak{S}_u) \cup \mathfrak{S} \) too. Therefore it remains to consider the case of subsets of \( (U^* \setminus \{u\}) \cup \{u^+\} \) that contain \( u^- \). Also note that the matching \( \mathfrak{M} \) witnesses Hall’s \( d \)-harem condition for all subsets of \( (\Gamma^* \setminus \mathfrak{S}_u) \cup \mathfrak{S} \) contained in \( B(u) \setminus S(u) \). In particular trying to verify Hall’s \( d \)-harem condition in \( (\Gamma^* \setminus \mathfrak{S}_u) \cup \mathfrak{S} \) for \( Y \subset (U^* \setminus \{u\}) \cup \{u^+\} \) we may assume that \( Y \not\subseteq B(u) \). As above we also assume that \( Y \) is connected.

Let \( u_1 \in Y \setminus B(u) \). The distance between \( u \) and \( u_1 \) at least \( 4h(d(n+1)) + 4 \). Since \( u^- \in N_{\Gamma \cup \mathfrak{S}}(X) \cap Y \), we see by Claim 10.5 that there is some \( u_2 \in Y \) at the distance \( \leq 2h(d(n+1)) \) from \( u \). Then the distance between \( u_1 \) and \( u_2 \) is at least \( 4h(d(n+1)) + 4 - (2h(d(n+1)) + 2) \), i.e at least \( 2h(d(n+1)) + 2 \). Thus \( |Y| > h(d(n+1)) \), and
\[ |N_{\Gamma}(Y)| - \frac{1}{d}|Y| \geq d(n+1). \]
On the other hand,
\[ |N_{(\Gamma^* \setminus \mathfrak{S}_u) \cup \mathfrak{S}}(Y)| \geq |N_{\Gamma^*}(Y)| - n(d-1), \]
hence:
\[ |N_{(\Gamma^* \setminus \mathfrak{S}_u) \cup \mathfrak{S}}(Y)| - \frac{1}{d}|Y| \geq |N_{\Gamma^*}(Y)| - \frac{1}{d}|Y| - n(d-1) \geq n \geq 0. \]
It follows that Hall’s \( d \)-harem condition holds for any \( Y \subset (U^* \setminus \{u\}) \cup \{u^+\} \) such that \( u^- \in Y \). This finishes the case of subsets of \( (U^* \setminus \{u\}) \cup \{u^+\} \).

It remains to show that Hall’s \( d \)-harem condition is satisfied in \( (\Gamma^* \setminus \mathfrak{S}_u) \cup \mathfrak{S} \) for any \( Y \subset V \cap ((\Gamma^* \setminus \mathfrak{S}_u) \cup \mathfrak{S}) \). We may assume that \( N_{\Gamma \cup \mathfrak{S}}(Y) \cap \{u, u^-\} \neq \emptyset \). Indeed, otherwise apply the assumption that \( \Gamma^* \) satisfies Hall’s \( d \)-harem condition. Also note again, that the matching \( \mathfrak{M} \) witnesses Hall’s \( d \)-harem condition for all subsets of \( (\Gamma^* \setminus \mathfrak{S}_u) \cup \mathfrak{S} \) contained in \( B(u) \setminus S(u) \). In particular we may assume that \( Y \not\subseteq (B(u) \setminus S(u)) \). As above we only consider connected \( Y \subset (V \cap ((\Gamma^* \setminus \mathfrak{S}_u) \cup \mathfrak{S})) \).
Take any element \( v_1 \) of \( Y \setminus (B(u) \setminus S(u)) \). It is clear that the distance from \( u \) to \( v_1 \) is at least \( 4h(d(n+1)) + 3 \). If \( u \in N_{\Gamma^*}(Y) \) then \( Y \) contains a vertex at the distance 1 from \( u \). If \( u \notin N_{\Gamma^*}(Y) \) then by Claim 10.5 \( Y \) contains a vertex at the distance \( 2h(d(n+1)) + 1 \) from \( u \). In any case we can find \( v_2 \in Y \) at the distance at most \( 2h(d(n+1)) + 1 \) from \( u \). We see that the distance between \( v_1 \) and \( v_2 \) is at least \( 2h(d(n+1)) + 2 \). Thus \( Y \) has at least \( h(d(n+1)) \) elements and

\[
|N_{\Gamma(Y)}| - \frac{1}{d}|Y| \geq d(n+1).
\]

On the other hand using

\[
|N_{\Gamma^*\cup S\setminus \delta_u}(Y)| \geq |N_{\Gamma(Y)}| - n
\]

we see that:

\[
|N_{\Gamma^*\cup S\setminus \delta_u}(Y)| - \frac{1}{d}|Y| \geq |N_{\Gamma(Y)}| - n - \frac{1}{d}|Y| \geq d(n+1) - n = n(n-1) + d \geq 0,
\]

i.e. Hall’s \( d \)-harem condition is also satisfied. \( \square \)

The last of technical lemmas concerns finding a witness of \( c.e.H.h.c.(d) \) in a subgraph.

**Lemma 10.6.** Assume that \( \Gamma = (U,V,E) \) is a highly computable bipartite graph satisfying \( c.e.H.h.c.(d) \) with witness \( h \). Let

- \( \mathfrak{M} \) be an \((1,d-1)\)-matching in \( \Gamma \) consisting of \( \ell \) fans,
- \( \Gamma^* = (U^*,V^*) \) be \( \Gamma \) without \( \mathfrak{M} \),
- \( \Gamma^* \) satisfy Hall’s \( d \)-Harem condition.

Then

\[
\hat{h}(m) = \begin{cases} 
0, & \text{if } m = 0, \\
\hat{h}(m + \ell d), & \text{if } m > 0.
\end{cases}
\]

is a witness of \( c.e.H.h.c.(d) \) for \( \Gamma^* \).

**Proof.** Since \( \Gamma^* \) satisfies Hall’s \( d \)-harem condition we may assume that \( m > 0 \). First, we prove that \( \hat{h} \) is a witness of \( c.e.H.h.c.(d) \) for subsets of \( U^* \). The inequality \( |X| \geq \hat{h}(m) \) implies \( |X| \geq h(m + \ell d) \), i.e.

\[
|N_{\Gamma}(X)| - d|X| - \ell d \geq m.
\]

Since \( d \geq 2 \) and

\[
|N_{\Gamma^*}(X)| \geq |N_{\Gamma}(X)| - \ell(d - 1),
\]

we have

\[
|N_{\Gamma^*}(X)| - d|X| \geq |N_{\Gamma}(X)| - \ell(d - 1) - d|X| \geq m.
\]

In order to show that \( \hat{h} \) works for \( V^* \) we start with the observation that for each \( X \subset V^* \),

\[
|N_{\Gamma^*}(X)| \geq |N_{\Gamma}(X)| - \ell.
\]

Therefore using \( d \geq 2 \) we see that when \( |X| \geq \hat{h}(m) \),

\[
|N_{\Gamma}(X)| - \frac{1}{d}|X| - m - \ell d \geq 0.
\]

It follows that:

\[
|N_{\Gamma^*}(X)| - \frac{1}{d}|X| \geq |N_{\Gamma}(X)| - \frac{1}{d}|X| - \ell d \geq m.
\] \( \square \)
11. Graphs constructed in parts 1 and 2 of each step satisfy c.e.H.h.c.(d)

11.1. Claims [9.2, 9.4]. Before stating the lemma concerning claims [9.2, 9.4] we remind the notation related to this part of the n + 1-st step.

- \( \Gamma^{(n)} \) is \( U^{(n)} \)-reflected;
- \( \Gamma^{(n)} \) is a subgraph of \( \Gamma^{(n)} \) obtained by removal of \((1,d - 1)\)-fans with roots belonging to the set \( U^{(n)} \);
- \( \Gamma^{(n)} \) satisfies c.e.H.h.c.(d);
- \( |U^{(n)}| \leq n \);
- \( \mathfrak{M}^1 \) is a finite \((1,d)\)-matching in the bipartite graph \( B^{(n)}(u_n) = B^{(n)}(u_n) \cap \Gamma^{(n)} \). It satisfies the conditions of the perfect \((1,d)\)-matchings for all vertices that are at the distance less than \( \max\{4h(3d(n + 1)) + 3, 5\} \) from \( u_n \).

Lemma 11.1. For any \( n \) the bipartite graph \( \Gamma^{(n)}(u_n) \) is highly computable. Furthermore, one of the following holds:

- \( \Gamma^{(n)}(-u_n) \) satisfies c.e.H.h.c.(d);
- there is \( u^\perp_j \in U^{(n)} \) such that the graph \( \Gamma^{(n)}(-u_n, u^\perp_j) \) satisfies c.e.H.h.c.(d).

In the latter case the element \( u^\perp_j \) can be computed and the bipartite graph \( \Gamma^{(n)}(-u_n, u^\perp_j) \) is highly computable.

Remark 12. If \( U^{(n)} = \emptyset \), then this lemma implies that \( \Gamma^{(n)}(-u_n) \) satisfies c.e.H.h.c.(d). In particular, Claim 9.2 holds.

Proof. Let

\[
\hat{h}_1(m) = \begin{cases} 
0, & \text{if } m = 0, \\
h(m + (3n + 1)d), & \text{if } m > 0.
\end{cases}
\]

and

\[
\hat{h}_2(m) = \begin{cases} 
0, & \text{if } m = 0, \\
h(m + 3nd), & \text{if } m > 0.
\end{cases}
\]

Claim 11.3. Assume that one of the graphs \( \Gamma^{(n)}(-u_n) \) or \( \Gamma^{(n)}(-u_n, u^\perp_j) \) satisfies Hall’s d-harem conditions. Then in the first case the function \( \hat{h}_1 \) witnesses c.e.H.h.c.(d) for \( \Gamma^{(n)}(-u_n) \) and in the second case the function \( \hat{h}_2 \) witnesses c.e.H.h.c.(d) for \( \Gamma^{(n)}(-u_n, u^\perp_j) \).

Proof. Apply Lemma 10.6. In the first case view \( \Gamma^{(0)} \) as \( \Gamma \) and \( \Gamma^{(n)}(-u_n) \) as \( \Gamma^* \) in this lemma. The number \( \ell \) from the lemma is equal to the number of fans in \( \bigcup_{i=1}^n M_{i-1} \) and \( \Gamma^{(n)} \) increased by one, corresponding to the fan of \( u_n \). Thus \( \ell \leq 3n + 1 \). Since \( \Gamma^{(n)}(-u_n) \) satisfies Hall’s d-harem conditions, the lemma works.

In the second case view \( \Gamma^{(0)} \) as \( \Gamma \) and \( \Gamma^{(n)}(-u_n, u^\perp_j) \) as \( \Gamma^* \). To compute the number corresponding to \( \ell \) from the lemma we should take the number of fans in \( \bigcup_{i=1}^n M_{i-1} \) and \( \Gamma^{(n)} \) together, then increase it by one, corresponding to the fan of \( u_n \), and then subtract one, corresponding to the fan of \( u^\perp_j \). Therefore \( \ell \leq 3n \). Since \( \Gamma^{(n)}(-u_n, u^\perp_j) \) satisfies Hall’s d-harem conditions, Lemma 10.6 works again.

It is clear that both \( \Gamma^{(n)}(-u_n) \) and \( \Gamma^{(n)}(-u_n, u^\perp_j) \) are highly computable bipartite graphs (assuming that \( u^\perp_j \) is computed). Thus it is enough to show that one of these graphs satisfies Hall’s d-harem condition (then apply the claim). We know that \( \Gamma^{(n)} \) satisfies Hall’s d-harem condition. Let us denote by \( v \) the only vertex from the set \( \{v_{n,1}^0, \ldots, v_{n,d}^0\} \) that belongs to \( \Gamma^{(n)}(-u_n) \). By Proposition 10.10, the choice of \( u_n, v_{n,1}^0, \ldots, v_{n,d}^0 \) ensures that \( \Gamma^{(n)}(-u_n, -v) \) satisfies Hall’s d-harem condition as well. Let \( \mathfrak{M} \) denotes the perfect \((1,d)\)-matching in \( \Gamma^{(n)}(-u_n, -v) \).

Since \( U^{(n)}(-u_n) = U^{(n)}(-u_n, -v) \), for \( X \subset U^{(n)}(-u_n) \) we have

\[
|N_{\Gamma^{(n)}(-u_n)}(X)| \geq |N_{\Gamma^{(n)}(-u_n, -v)}(X)| \geq d|X|.
\]
The corresponding property holds for all subsets of $V^{(n)}(\uM(\neg u_n))$ that do not contain $u$. Therefore if $\Gamma^{(n)}(\neg u_n)$ does not satisfy Hall’s $d$-harem condition, this is witnessed by a finite $X \subset V^{(n)}(\uM(\neg u_n))$ which contains $u$.

**Claim 11.4.** If $X$ is a connected subset $X$ of $V^{(n)}(\uM(\neg u_n))$ and $|N_{\Gamma^{(n)}(\neg u_n)}(X)| < \frac{1}{d}|X|$, then $X \subset (B(u_n) \setminus S(u_n))$.

**Proof.** Observe that by the choice of the radius of the ball $B(u_n)$, for any connected $X \not\subset (B(u_n) \setminus S(u_n))$ with $u \in X$ we have $|X| \geq 2h(d(3n + 1)) + 1$. Therefore

$$|N_{\Gamma^{(0)}}(X)| - \frac{1}{d}|X| \geq d(3n + 1).$$

On the other hand at every step at most 3 fans from $\Gamma$ are added to the matching, hence $|U^{(0)} \setminus U^{(n)}(\uM(\neg u_n))| \leq 3n + 1$. As a consequence

$$|N_{\Gamma^{(n)}(\neg u_n)}(X)| \geq |N_{\Gamma^{(0)}}(X)| - (3n + 1) \geq |N_{\Gamma^{(0)}}(X)| - d(3n + 1) \geq \frac{1}{d}|X|.$$

□

Since the neighbourhood of any set is decomposed into a disjoint union of the neighbourhoods of connected subsets, applying the claim we see that if $\Gamma^{(n)}(\neg u_n)$ does not satisfy Hall’s $d$-harem condition, this is witnessed by a finite connected $X \subset V^{(n)}(\uM(\neg u_n)) \cap (B(u_n) \setminus S(u_n))$ which contains $u$. Using high computability of $\Gamma^{(n)}$ and finiteness of $B(u_n) \setminus S(u_n)$, we compute $X$ with these properties.

We now know that $|N_{\Gamma^{(n)}(\neg u_n)}(X)| < \frac{1}{d}|X|$ and want to prove that there is some $u_{j}^{\perp} \in N_{\Gamma^{(n)}(\neg u_n)}(X)$.

First, consider the case when $X = \{u\}$. The vertex $u_{0}$ either belongs to $U^{(n)}(\uM(\neg u_n))$ or to $U^{(n)}(\neg u_n)$. By inequality (4) and the fact that $U^{(n)}(\neg u_n)$ consist of $(1,d-1)$-fans it follows that in either case we have $|N_{\Gamma^{(n)}(\neg u_n)}(u_{0})| \geq d - 1$. Moreover the equality

$$|N_{\Gamma^{(n)}(\neg u_n)}(u_{0})| = d - 1$$

implies that $v_{u_{0}} \not\in \Gamma^{(n)}$. To see this just repeat the corresponding argument of Lemma 5.1. Indeed, if $v_{u_{0}} \in \Gamma^{(n)}$ then $v_{u_{0}} \in N_{\Gamma^{(n)}(\neg u_n)}(u_{0})$ (by reflectedness). On the other hand equality (5) implies that $(u_{0}, v_{u_{0}}) \in \Gamma^{(n)}$. Thus $(u_{0}, v)$ would be added to the matching $M$ at the first part of $n + 1$-st step of the construction, i.e. $v \not\in \Gamma^{(n)}(\neg u_n)$, a contradiction.

Similarly as in the proof of Lemma 5.1 we now to observe that

$$|N_{\Gamma^{(n)}(\neg u_n)}(v)| \geq |N_{\Gamma^{(n)}(\neg u_n)}(u_{0})| - 1.$$

This again follows by $U^{(n)}$-reflectedness of $\Gamma^{(n)}$: $v_{u_{0}}$ is the only possible element incident to $u_{0}$ that does not have the left copy in $U^{(n)}(\neg u_n)$. Moreover, the equality

$$|N_{\Gamma^{(n)}(\neg u_n)}(v)| = |N_{\Gamma^{(n)}(\neg u_n)}(u_{0})| - 1$$

holds only if $v_{u_{0}} \in N_{\Gamma^{(n)}(\neg u_n)}(u_{0})$. Therefore equalities (5), (6) are not consistent, i.e.

$$|N_{\Gamma^{(n)}(\neg u_n)}(v)| > (d - 1) - 1 \text{ and } |N_{\Gamma^{(n)}(\neg u_n)}(v)| \geq \frac{1}{d}.$$

We have

$$|N_{\Gamma^{(n)}(\neg u_n)}(X)| \geq \frac{1}{d}.$$

Since $X$ is a singleton,

$$|N_{\Gamma^{(n)}(\neg u_n)}(X)| \geq \frac{1}{d}|X|,$$

and as in Lemma 5.1 this means that there is some $u_{j}^{\perp} \in N_{\Gamma^{(n)}(\neg u_n)}(X)$. Using high computability of $\Gamma^{(n)}(\neg u_n)$ we can compute it.
Assume $X \neq \{v\}$. Then $|N_{\Gamma(n)}(u_{n})(X)| \geq 1$ and by assumptions on $X$, $|X| > d$. Let $U_{X} := \{u \in U : v_{u} \in X\} \in \Gamma(n)^{(-u_{n})}$. Applying the fact that $V(n)$ is a subset of the copy of $U(n)$ we arrive at two possibilities:

1. $v_{u_{n}} \notin X$ and $|X| = |U_{X}|$;
2. $v_{u_{n}} \in X$ and $|X| = |U_{X}| + 1$.

Following the proof of Lemma 5.1, in either case we deduce the inequality $|N_{\Gamma(n)}(u_{n})(X)| \geq (d - 1)|X| - 1$.

This fact combined with inequalities $d \geq 2$ and $|X| > d$ imply

$$|N_{\Gamma(n)}(u_{n})(X)| \geq \frac{1}{d}|X|.$$ 

In case (ii) by Lemma 4.2 we have:

$$|N_{\Gamma(n)}(u_{n})(X)| \geq (d - 1)(|X| - 1) - 1.$$ 

Therefore

$$|N_{\Gamma(n)}(u_{n})(X)| \geq (d - 1)(|X| - 1) - 1.$$ 

Let us show that the inequality is strict. Indeed, by Lemma 4.2 the equality

$$|N_{\Gamma(n)}(u_{n})(U_{X}\{v_{u_{n}}\})| = (d - 1)(|X| - 1) - 1$$

implies that $v_{u_{n}} \in N_{\Gamma(n)^{(-u_{n})}}(U_{X}\{v_{u_{n}}\})$ and $N_{\Gamma(n),(+u_{n})}(U_{X}\{v_{u_{n}}\}) = \emptyset$. Therefore $v_{u_{n}} \in V(n)$. On the other hand, $v_{u_{n}} \in X \subset \Gamma(n)^{(*)}$, a contradiction with the choice of $X$.

Again, inequalities $d \geq 2$ and $|X| > d$ implies

$$|N_{\Gamma(n)}(u_{n})(X)| \geq (d - 1)(|X| - 1) - 1.$$

Therefore, the assumption $|N_{\Gamma(n),(+u_{n})}(X)| \leftarrow \frac{1}{d}|X|$ implies

$$N_{\Gamma(n)}(u_{n})(X) \cap U(n)^{(-u_{n})} \neq \emptyset,$$

i.e. there exists some $u_{j}^{+} \in N_{\Gamma(n)(u_{n})}(X)$. Then the vertex $u_{j}^{+}$ belongs to $N_{\Gamma(n)^{(-u_{n})}}(U_{(u_{n})})$ (recall that $X \subseteq B(u_{n}) \setminus \mathcal{S}(u_{n})$) and we find it effectively. Let use denote by $v_{j,1}, ..., v_{j,d-1}$ the $(d - 1)$-tuple of vertices adjacent to $u_{j}^{+}$ in $\Gamma(n)^{(-u_{n})}$.

Consider the graph $\Gamma^{*}$ induced in the graph $\Gamma(n)^{(*)}$ by the vertices incident to the edges from the matching $M_{x}$. Since $M_{x}$ is a perfect $(1, d)$-matching in the ball $\Gamma(n)^{(*)} \cap B(u_{n})$ it follows that $\Gamma^{*}$ satisfies Hall’s $d$-harem condition. Since $\Gamma^{*}$ is a subgraph of a bipartite graph $\Gamma(n)(u_{n})$, we can apply Lemma 4.4 for $v$, $X$, $u_{j}^{+}$. We see $u_{j}^{+} \rightarrow v$. This gives us sequences of vertices $\{v_{0}, ..., v_{n}\}$, $\{u_{0}, ..., u_{n-1}\}$ as in Definition 4.3.

In order to prove that the graph $\Gamma(n)^{(*)}(u_{n})$ satisfies Hall’s $d$-harem condition, we construct a perfect $(1, d)$-matching in the ball $\Gamma(n)^{(*)}(u_{n}, u_{j}^{+}) \cap B(u_{n})$. Let

$$M' := (M_{x} \setminus \{\{u_{n}, u_{n,1}^{0}, ..., u_{n,1}^{n_{d}}\}, (u_{0}, v_{0})\}) \cup \{(u_{j,1}^{+}, u_{j,1}^{+}), ..., (u_{j,d-1}^{+}, v_{0}), (u_{0}, v_{1}^{1}), ..., (u_{n-1}, v_{n-1})\},$$

where $v_{n} = v$. We have obtained $M'$ by removing $d$ edges adjacent to $u_{n}$, adding $d$ edges adjacent to $u_{j}^{+}$, and the following replacement: for each of $u_{i}$ we replace one edge adjacent to it by another.
adjacent edge (then \(v\) becomes adjacent to one edge in \(M'\)). It follows that the matching \(M'\) satisfies the conditions of the perfect \((1, d)\)-matchings for all subsets of \(\Gamma(n)^\ast(-u_n, +u^+_j) \cap (\mathcal{B}(u_n) \setminus \mathcal{S}(u_n))\).

To finish the proof notice that the conditions of Lemma \ref{10.2} are satisfied if we consider \(\Gamma(0)\) as \(\Gamma\) in this lemma, \(\Gamma(n)^\ast\) as \(\Gamma^\ast\), \(M'\) as \(\mathfrak{M}\) and \(\Gamma(n)^\ast(-u_n, +u^+_j)\) as \((\Gamma^\ast \setminus \mathfrak{G}) \cup \mathfrak{E}\), i.e. \(u_n\) is considered as \(u\) and \(u^+_j\) as \(u^+\). Therefore by Lemma \ref{10.2} \(\Gamma(n)^\ast(-u_n, +u^+_j)\) satisfies Hall’s \(d\)-harem condition.

\[\square\]

### 11.2. Notation of Lemma 11.5 and the proof.

Before stating the second lemma, we remind the notation used in it.

- **In Case 3A**
  - \(\mathfrak{U} := \mathcal{U}(n+1) \setminus \{u^0\}\)
  - \(\mathfrak{V} := \mathcal{V}(n+1) \setminus \{v^0\}\)
  - \(\mathfrak{T} := (\mathfrak{U}, \mathfrak{V}, \mathfrak{E})\), where \(\mathfrak{E}\) is induced in \(\Gamma\) by the sets of vertices \(\mathfrak{U}, \mathfrak{V}\).
  - \(\hat{\mathcal{U}}(n) := \mathcal{U}(n) \cap \mathcal{U}(n+1)\)
  - \(\hat{\mathcal{V}}(n) := \mathcal{V}(n) \cap \mathcal{V}(n+1)\)
  - \(\hat{\Gamma}(n) := (\hat{\mathcal{U}}(n), \hat{\mathcal{V}}(n), \hat{\mathcal{E}}(n))\)

- **In Case 3B**
  - \(\mathfrak{U} := \mathcal{U}(n+1) \setminus \{u^0\}\)
  - \(\mathfrak{V} := \mathcal{V}(n+1) \setminus \{v^0\}\)
  - \(\mathfrak{T} := (\mathfrak{U}, \mathfrak{V}, \mathfrak{E})\), where \(\mathfrak{E}\) is induced in \(\Gamma\) by the sets of vertices \(\mathfrak{U}, \mathfrak{V}\).
  - \(\hat{\mathcal{U}}(n) := \mathcal{U}(n) \cup \{u^\ast_k : 1 \leq k \leq d - 1\}\)
  - \(\hat{\mathcal{V}}(n) := \mathcal{V}(n) \cup \{v^\ast_k : 1 \leq k \leq d - 1\}\)
  - \(\hat{\Gamma}(n) := (\hat{\mathcal{U}}(n), \hat{\mathcal{V}}(n), \hat{\mathcal{E}}(n))\)

- **\(\mathfrak{M}^2\)** is a finite \((1, d)\)-matching in the bipartite graph \(\mathcal{B}(n)^\ast(u^+_{i+1}) = \mathcal{B}(n)(u^+_{i+1}) \cap \Gamma(n)^\ast\). It satisfies the conditions of the perfect \((1, d)\)-matchings for all vertices that are at the distance less than \(\max\{4h(3d(n + 1)) + 3, 5\}\) from \(u^+_{i+1}\).
  - \(\hat{v}_{n,i}^+\), \(1 \leq i \leq d\), are vertices incident to \(u^+_{i+1}\) in \(\mathfrak{M}^2\).
  - \(\hat{v}_1^+, \hat{v}_2^+\) are among \(\hat{v}_{n,i}^+\), \(1 \leq i \leq d\), that were not added to \(\mathcal{M}^2\); there are at most 2 of them.

**Lemma 11.5.** For any \(n\) the bipartite graph \(\mathfrak{T}\) is highly computable. Moreover, one of the following holds:

- \(\mathfrak{T}\) satisfies c.e.H.h.c.(d);
- there is \(u^+_j \in \hat{\mathcal{U}}(n)\) such that the graph \(\mathfrak{T}(+u^+_j)\) satisfies c.e.H.h.c.(d);
- there are \(u^+_i, u^+_j \in \hat{\mathcal{U}}(n)\) such that the graph \(\mathfrak{T}(+u^+_i, +u^+_j)\) satisfies c.e.H.h.c.(d).

In the latter cases, the elements \(u^+_i, u^+_j\) can be computed and the corresponding bipartite graphs are highly computable.

**Remark 11.6.** If \(|\hat{\mathcal{U}}(n)| \leq 1\), then Lemma 11.5 can be restated as follows. One of the following holds:

- \(\mathfrak{T}\) is a highly computable bipartite graph satisfying c.e.H.h.c.(d);
- \(\Gamma(n+1)\) is a highly computable bipartite graph satisfying c.e.H.h.c.(d).

Therefore this Lemma proves Claim 9.3.

**Proof of Lemma 11.5** As in the proof of Lemma 11.1 we begin by showing that each of graphs from the formulation satisfies c.e.H.h.c.(d) as long as it satisfies Hall’s \(d\)-harem condition.

Let

\[\hat{h}_1(m) = \begin{cases} 0, & \text{if } m = 0, \\ h(m + (3n + 3)d), & \text{if } m > 0. \end{cases}\]
\[ \hat{h}_2(m) = \begin{cases} 0, & \text{if } m = 0, \\ h(m + (3n + 2)d), & \text{if } m > 0. \end{cases} \]

and

\[ \hat{h}_3(m) = \begin{cases} 0, & \text{if } m = 0, \\ h(m + (3n + 1)d), & \text{if } m > 0. \end{cases} \]

Claim 11.7. Assume that one of the graphs \( T \), or \( T(+u_j^+) \) or \( T(+u_i^+,+u_j^+) \) satisfies Hall’s \( d \)-harem conditions. Then in the first case the function \( \hat{h}_2 \) witnesses \( c.e.H.h.c.(d) \) for \( T \), in the second case the function \( \hat{h}_2 \) witnesses \( c.e.H.h.c.(d) \) for \( T(+u_j^+) \) and in the third case the function \( \hat{h}_3 \) witnesses \( c.e.H.h.c.(d) \) for \( T(+u_i^+,+u_j^+) \).

Proof. It is a counterpart of Claim 11.3. In the first case view \( \Gamma(0) \) as \( \Gamma \) and \( T \) as \( \Gamma^* \) in Lemma 10.6, i.e. \( \ell \) from this lemma is equal to the number of fans in \( \bigcup_{i=1}^n M_{i-1} \) and \( \Gamma(n) \) increased by three, corresponding to the fans of \( u_n, u_{n+1}^i \) and \( \hat{u}(n) \). Thus \( \ell \leq 3n + 3. \) Since \( T \) satisfies Hall’s \( d \)-harem conditions, the lemma works.

In the second case view \( \Gamma(0) \) as \( \Gamma \) and \( T(+u_j^+) \) as \( \Gamma^* \) i.e. \( \ell \) from Lemma 10.6 is equal to the number of fans in \( \bigcup_{i=1}^n M_{i-1} \) and \( \Gamma(n) \) increased by three, corresponding to the fans of \( u_n, u_{n+1}^i \) and \( \hat{u}(n) \) and decreased by one, corresponding to the fan of \( u_j^+ \). Therefore it is at most \( 3n + 2. \) Since \( T(+u_i^+,+u_j^+) \) satisfies Hall’s \( d \)-harem conditions, Lemma 10.6 works again. In the third case view \( \Gamma(0) \) as \( \Gamma \) and \( T(+u_i^+,+u_j^+) \) as \( \Gamma^* \) i.e. \( \ell \) from this lemma is equal to the number of fans in \( \bigcup_{i=1}^n M_{i-1} \) and \( \Gamma(n) \) increased by three, corresponding to the fans of \( u_n, u_{n+1}^i \) and \( \hat{u}(n) \) and decreased by two, corresponding to the fans of \( u_i^+, u_j^+ \). Therefore it is at most \( 3n + 1. \) Since \( T(+u_i^+,+u_j^+) \) satisfies Hall’s \( d \)-harem conditions, Lemma 10.6 works again.

Since (assuming that \( u_i^+, u_j^+ \) are computed) each of the graphs from the claim is a highly computable bipartite graph, it is enough to show that at least one of them satisfies Hall’s \( d \)-harem condition (then apply the claim). The structure of the rest of the proof corresponds to the proof of Lemma 5.5.

Assume that \( T \) does not satisfy Hall’s \( d \)-harem condition. Let \( u_{n+1}^i \) be the root of the last fan added to the matching \( M_n \) in the second part of the \( n \)-th step; it belongs to the produced cycle of length 2. Recall that \( \hat{v}_1, \hat{v}_2 \) denote the vertices of the form \( \hat{v}_{n+1} \) that were not added to \( M_n \). In particular in Case 3A there is only one such a vertex and in Case 3B there is either one or two such vertices. As in the proof of Lemma 5.5 we only consider the case of two additional vertices: \( \hat{v}_1 \) and \( \hat{v}_2 \).

Note that the statements of Lemmas 5.3 and 5.4 hold in the situation of the present lemma too. Applying Lemma 5.3 we see that the graph \( T(-\hat{v}_1, -\hat{v}_2) \) satisfies Hall’s \( d \)-harem condition. Let \( \mathcal{W} \) denote the perfect \( (1, d) \)-matching in \( T(-\hat{v}_1, -\hat{v}_2) \).

It is clear that the inequality \( |N_T(X)| \geq |N_{\mathcal{W}}(X)| \) holds for all \( X \subseteq \mathcal{W} \). This inequality also holds for subsets of \( \mathcal{W} \) which do not intersect \( \{\hat{v}_1, \hat{v}_2\} \). Therefore if \( T \) does not satisfy Hall’s \( d \)-harem condition then a witness of this is some finite subset of \( \mathcal{W} \) containing at least one of \( \hat{v}_1, \hat{v}_2 \).

As in the proof of the Lemma 5.5, we now divide the proof into two parts.

Part 1. We check whether \( T(-\hat{v}_2) \) satisfies Hall’s \( d \)-harem condition. If it does, we set

- \( M' \) is a matching obtained by Proposition 9.1 in \( T(-\hat{v}_2) \cap \mathcal{B}(u_{n+1}^i) \);
- \( \hat{\Gamma}^{(n+1)}_\perp := \hat{\Gamma}^{(n)}_\perp \);
- \( \Gamma' = \mathcal{W} \), denoting \( \Gamma' = (U', V') \),

and finish Part 1 of the proof.
If it does not, then there exists the smallest connected set \( X \) such that \( \hat{v}_1 \in X \subset \mathfrak{Y}(\hat{v}_2) \) and \( |N_{\hat{T}}(\hat{v}_2)(X)| < \frac{1}{d}|X| \). Analogously to the Claim \[4.4\] it can be showed that such \( X \) is a subset of \((\mathcal{B}(u_{n+1}^l) \setminus \mathcal{S}(u_{n+1}^l))\) and as a consequence we can compute \( X \) with these properties.

The inequality
\[
|N_{\hat{T}(n+1)}(X)| \geq \frac{1}{d}|X|
\]
follows from Lemma \[5.3\].

The inequalities
\[
|N_{\hat{T}(n+1)}(X)| \geq \frac{1}{d}|X| \quad \text{and} \quad |N_{\hat{T}}(\hat{v}_2)(X)| < \frac{1}{d}|X|
\]
imply
\[
N_{\hat{T}(n+1)}(X) \cap \hat{U}^{(n)\bot} \neq \emptyset,
\]
i.e. there exists some \( u_j^l \in N_{\hat{T}(n+1)}(X) \). Since \( \Gamma^{(n+1)} \) is highly computable, the element \( u_j^l \) and its fan from \( \hat{\Gamma}^{(n)\bot} \) can be computed. Let \( v_{j,1}^l, \ldots, v_{j,d-1}^l \) be the remaining vertices of this fan.

Consider the graph \( \Gamma^* \) induced in the graph \( \check{\Sigma}(\hat{v}_2) \) by the vertices incident to the edges of the matching \( \check{\mathfrak{M}}_2^n \). Since this matching is a perfect \((1,d)\) matching in the ball \( \mathcal{B}(u_{n+1}^l) \cap \Gamma^{(n)*} \), it follows that \( \Gamma^* \) satisfies Hall’s \( d \)-harem condition. Since \( \Gamma^* \) is a subgraph of a bipartite graph \( \Gamma^{(n+1)} \), we can use Lemma \[4.4\] for \( \hat{v}_1, X, u_j^l \) and arrive at \( u_j^l \xrightarrow{\check{\mathfrak{M}}_2^n \cap X} \hat{v}_1 \). This gives us a sequences of vertices \( \{v_{0}', \ldots, v_{n}'\}, \{v_{0}', \ldots, v_{n-1}'\} \) as in Definition \[4.3\]

We now apply an argument similar to one from the proof of Lemma \[11.1\]. Set
\[
M' := \check{\mathfrak{M}}_2^n \setminus \{(u_{n+1}^l, v_{j,1}^l+1), \ldots, (u_{n+1}^l, v_{j,d-1}^l), (u_0', v_0'), \ldots, (u_{n-1}'^l, v_{n-1}'^l)\} \cup
\{(u_j^l, v_{j,1}^l), \ldots, (u_j^l, v_{j,d-1}^l), (u_j^l, v_0'), \ldots, (u_j^l, v_{n-1}', v_{n}’)\},
\]
where \( v_n' = \hat{v}_1 \). Observe that
\[
(M \cup \{u_j^l\}, (M \cup \{v_{j,1}^l, \ldots, v_{j,d-1}^l\}) \setminus \{\hat{v}_2\}) = \check{\Sigma}(+u_j^l) \setminus \{\hat{v}_2\} = \check{\Sigma}(+u_j^l, -\hat{v}_2).
\]
We have obtained \( M' \) from \( \check{\mathfrak{M}}_2^n \) by removing the \( d \)-fan of \( u_{n+1}^l \), adding the \( d \)-fan of \( u_j^l \), and the following replacement: for each of \( u_j^l \) we replace one edge adjacent to it by another adjacent edge (then \( \hat{v}_1 \) becomes adjacent to one edge in \( M' \)). Since \( \check{\mathfrak{M}}_2^n \) was a \((1,d)\)-perfect matching in the ball \( \mathcal{B}(u_{n+1}^l) \cap \Gamma^{(n)*} \), it follows that matching \( M' \) satisfies the conditions of the perfect \((1,d)\)-matchings for all subsets of \( \check{\Sigma}(+u_j^l, -\hat{v}_2) \cap (\mathcal{B}(u_{n+1}^l) \setminus \mathcal{S}(u_{n+1}^l)) \). Therefore \( M' \) is a perfect \((1,d)\)-matching in the ball \( \check{\Sigma}(+u_j^l, -\hat{v}_2) \cap \mathcal{B}(u_{n+1}^l) \).

The conditions of Lemma \[10.2\] are satisfied if we consider \( \Gamma^{(0)} = \Gamma \) in this lemma, \( \Gamma^{(n)*}(\check{u}_n) \) as \( \Gamma^* \), \( M' \) as \( \check{\mathfrak{M}}_2^n \) and \( \check{\Sigma}(+u_j^l, -\hat{v}_2) \) as \( (\mathfrak{T}^* \setminus \check{\mathfrak{Y}}_n) \cup \mathfrak{S} \), i.e. \( u_{n+1}^l \) is considered as \( u \) and \( u_j^l \) is considered as \( u_j \) in that lemma. Therefore by Lemma \[10.2\] \( \check{\Sigma}(+u_j^l, -\hat{v}_2) \) satisfies Hall’s \( d \)-harem condition.

We define \( \Gamma' := \check{\Sigma}(+u_j^l) \), denote \( \Gamma' = (U', V') \) and put
\[
\Gamma'^{(n+1)\bot} = (U^{(n)\bot} \setminus \{u_j^l\}, V^{(n)\bot} \setminus \{u_{j,k}^l : 1 \leq k \leq d - 1\}).
\]
This ends Part 1 of the proof.

Part 2. We check whether the graph \( \Gamma' \) satisfies Hall’s \( d \)-harem condition. If it does, then by Part 1 \( \Gamma' \) has to be equal to \( \check{\Sigma}(+u_j^l) \) and the proof is finished by the second option of the formulation.

If it does not then again, by repeating the reasoning of Part 1, we see that we can effectively recognize the smallest connected set \( X \) such that \( \hat{v}_2 \in X \subset V' \cap (\mathcal{B}(u_{n+1}^l) \setminus \mathcal{S}(u_{n+1}^l)) \) and \( |N_{\Gamma'}(X)| < \frac{1}{d}|X| \). Again, the inequality
\[
|N_{\Gamma^{(n+1)}}(X)| \geq \frac{1}{d}|X|
\]
follows from Lemma \[5.4\]. Depending on the possible outputs of Part 1, the corresponding graph \( \Gamma^* \) from the lemma is either equal to \( \check{\Sigma}(\hat{v}_2) \) or to \( \check{\Sigma}(+u_j^l, -\hat{v}_2) \) and in either case satisfies Hall’s \( d \)-harem condition. As a result
\[
N_{\Gamma^{(n+1)}}(X) \cap \hat{U}^{(n)\bot} \neq \emptyset,
\]
i.e. there exists some $u_j^± \in N_{\Gamma(n+1)}(X)$ and by high computability of $\Gamma(n+1)$ it can be computed. As usual, by $v^1_{i,1}, \ldots, v^1_{i,d-1}$ we denote the vertices adjacent to $u^+_{j}$ in $\Gamma(n+1)\) ^{\perp}$. The matching $M'$ obtained in the first part of the proof is a perfect $(1, d)$-matching either in the ball $\bar{\Sigma}(-\hat{v}_2) \cap B(u^+_{n+1})$, or in the ball $\bar{\Sigma}(+u^+_{n+1}, -\hat{v}_2) \cap B(u^+_{n+1})$. Each of these is a subgraph of $\Gamma(n+1)$. Therefore we can use Lemma 4.4 for $u^+_j, X, \hat{v}_2$ to show that $u^+_{j} \xrightarrow{M', X} \hat{v}_2$. This gives us a subgraph of vertices $\{v'_0, \ldots, v'_n\}, \{u^+_{0}, \ldots, u^+_{n-1}\}$ as in Definition 4.3.

As in the Part 1 we apply the argument of Lemma 11.1 to obtain an appropriate matching. Set

$$M'' := M' \setminus \{\langle u'_0, v'_0 \rangle, \ldots, \langle u'_{n-1}, v'_{n-1} \rangle\} \cup \{\langle u^+_{i,1}, v^+_{i,1} \rangle, \ldots, \langle u^+_{i,d-1}, v^+_{i,d-1} \rangle, \langle u^+_0, v'_0 \rangle, \langle u^+_0, v'_1 \rangle, \ldots, \langle u^+_{n-1}, v'_n \rangle\}.$$  

The matching $M''$ is obtained from $M'$ by adding the $d$-fan of $u^+_j$, and the following replacement: for each of $u^+_j$ we replace one edge adjacent to it by another adjacent edge (then $\hat{v}_2$ becomes adjacent to one edge in $M''$). It follows that matching $M''$ is the perfect $(1, d)$-matching in the ball $\bar{\Sigma}(+u^+_{n+1}) \cap B(u^+_{n+1})$.

The conditions of Lemma 10.2 are satisfied if we consider $\Gamma(0)$ as $\Gamma$ in this lemma, $\Gamma(n\ast(-u_n)$ as $\Gamma\ast$, $M''$ as $\mathcal{R}$ and $\Gamma^\ast(+u^+_{n+1})$ as $\Gamma^\ast \setminus \mathcal{B} \cup \mathcal{S}$, i.e. $u^\ast_{n+1}$ is considered as $u$ and $u^\ast_{n+1}$ as $u^\perp$ in that lemma. Therefore by Lemma 10.2, $\Gamma^\ast(+u^+_{n+1})$ satisfies Hall’s $d$-harem condition.

The final argument depends on two possible outputs of Part 1. If $u^+_j$ does exist, then $M''$ is a perfect $(1, d)$-matching in the graph $\bar{\Sigma}(+u^+_j, +u^+_j)$. If it does not, we redefine $u^+_j := u^+_i$ and then $M''$ becomes a perfect $(1, d)$-matching in the graph $\bar{\Sigma}(+u^+_j)$. Therefore if $\bar{\Sigma}$ does not satisfy Hall’s $d$-harem condition, then either $\bar{\Sigma}(+u^+_j)$ or $\bar{\Sigma}(+u^+_j, +u^+_j)$ satisfies this condition. \hfill \Box

12. Computable version of Main Theorem

The following theorem is a kind of amalgamation of Theorem 2.4 and Theorem 13 from Section 7.

**Theorem 12.1.** Let $\Gamma = (U, V, E)$ be a highly computable bipartite graph, such that:

- both $U$ and $V$ are identified with $\mathbb{N} \setminus \{0\}$
- $E$ does not contain edges of the form $(u, v_w)$
- $\Gamma$ is fully reflected
- $\Gamma$ satisfies the c.e.H.h.c.($d$).

Then there exist a computable perfect $(1, d-1)$-matching of $\Gamma$, which realizes a computable $(d-1)$ to 1 function $f : \mathbb{N} \rightarrow \mathbb{N}$ with controlled sizes of its cycles.

**Proof.** Let us apply the construction of Section 9. This construction works modulo Claims 9.2, 9.3, 9.4, 9.5. Claims 9.2 and 9.4 follow from Lemma 11.1. Claims 9.3 and 9.5 follow from Lemma 11.5. Since for every $n$ the union $\bigcup_{i=1}^{n} M_i$ consist of $(1, d-1)$-fans, the final set of edges $M$ is a $(1, d-1)$-matching.

For every $u \in U$ there is a step where an edge adjacent to $u$ is added to $M$. Then if the copy $v_u$ was not added to $M$ earlier, in the second part of this step this copy is added to $M$. It follows that $M$ is a perfect $(1, d-1)$-matching of the graph $\Gamma$. This property also implies computability of this matching. Indeed, the construction guarantees that for any $n$ all edges from $M$ of the form both $(n, j)$ and $(k, n)$ belong to $\bigcup_{i=1}^{n} M_i$. Therefore there is an algorithm which for any $n$ computes all such pairs from $M$. We see that the function $f$ realized by this matching is computable. It remains to show that $f$ has controlled sizes of its cycle. This can be arranged by copying the corresponding place of the proof of Theorem 2.4. \hfill \Box
13. Computable entourages of coarse spaces

We recall some terminology from the coarse geometry. For the relation $E \subseteq X \times X$ on a set $X$ and $x \in X$, let

$$E[x] := \{ y \in x | (x, y) \in E \},$$

and

$$E[A] := \bigcup \{ E[x] | x \in A \}.$$ 

Furthermore, we denote by $\Gamma(E)$ the graph associated with the relation $E$, i.e. $\Gamma(E) = (X, E)$.

**Definition 13.1.** A coarse space is a pair $(X, E)$ consisting of a set $X$, and a collection of subsets of $X \times X$ (called entourages) such that:

- $\Delta_X \in E$;
- if $F \subseteq E \in E$, then also $F \in E$;
- if $E, F \in E$ then $E \cup F, E^{-1}, E \circ F \in E$.

**Definition 13.2.** A coarse space $(X, E)$ is said to have bounded geometry if for each $E \in E$ and every $x \in X$ the set $E[x]$ is finite.

**Definition 13.3.** A coarse space $(X, E)$ of a bounded geometry is called amenable if for every $\theta > 1$ and every $E \in E$ there exists a non-empty finite $F \subseteq X$ such that $|E[F]| \leq \theta |F|$.

F.M. Schneider has proved in [Sch18] (see Theorem 2.2) that a coarse space of bounded geometry is not amenable if and only if there is $E \in E$ such that $\Gamma(E)$ is a $d$-regular forest (i.e. each vertex has valence $d$).

The following proposition explains our original motivation which leaded us to the main results of the paper.

**Proposition 13.4.** Let $(\mathbb{N}, E)$ be a coarse space of bounded geometry and let $E \in E$ be a symmetric entourage. Let $f$ be a computable function realizing a perfect $(1, d-1)$-matching for $d \geq 3$ in the graph $\Gamma = (\mathbb{N}, \mathbb{N}, R)$, where $R = E \setminus \Delta_{\mathbb{N}} \subseteq \mathbb{N} \times \mathbb{N}$.

If $f$ has controlled sizes of its cycles, then there exists a computable $E' \in E$ such that $\Gamma(E')$ is a $d$-regular forest. Moreover there exist an algorithm which for any $m, n \in \mathbb{N}$ recognizes if $m$ and $n$ are in the same connected component of $\Gamma(E')$.

**Proof.** We adapt the proof of Theorem 2.2 of [Sch18].

Since $f$ is a total, computable, surjective, $(d-1)$ to 1 function, the graph of $f$ (denoted by $\Gamma(f)$) is computable and $d$-regular. We remind the reader that $f$ satisfies the following properties:

(i) $f^2(1) = 1$;  
(ii) if $n \geq 2$ and $f^i(n) = n$ then $i \leq n$;  
(iii) if $n \geq 2$ and for all $i \leq n$ we have $f^i(n) \neq n$ then there exist $k \leq 2n$ and $l \leq n$ such that $f^{k+i}(n) = f^k(n)$;  
(iv) for each $n$ the pair $(n, f(n))$ belongs to $R$.

Since $R = E \setminus \Delta_X$ the last property implies that $f$ does not have fixed points. Let

$$P(f) = \{ n \in \mathbb{N} | \exists m \geq 1 (f^m(n) = n) \} \text{ and } P_0(f) = \{ n \in P(f) | \forall m \geq 1 (f^m(n) \geq n) \},$$

i.e. the union of all cycles and the set of minimal representatives of cycles. By property (ii) there is an algorithm which for every $n \in \mathbb{N}$ verifies whether $n \in P(f)$, i.e. $P(f)$ is computable. Observe that $P_0(f)$ is computable too. Indeed, 1 obviously belongs to $P_0(f)$. When $n \geq 2$ and $n \in P(f)$, then verifying if $f^i(n) \geq n$ for all $i \leq n$ we can check whether $n \in P_0(f)$ (apply (ii) again).

Since each component does not have two disjoint cycles we see that whenever $n, m \in P_0(f)$ and $n \neq m$, then $n$ and $m$ do not belong to the same connected component of the graph $\Gamma(f)$.

Thus

$$P(f) = \bigcup_{n \in P_0(f)} \{ f^m(n) \mid m \in \mathbb{N} \}.$$
There is an algorithm which for every \( n \in \mathbb{N} \) finds the \( P_0(f) \)-representative of the connected component of \( n \). Indeed, if for example \( n \notin P(f) \) then applying (iii) we can find \( i \leq 2n \) such that \( f^i(n) \in P(f) \) and later \( j \leq n \) such that \( f^{i+j}(n) \in P_0(f) \).

Based on this we want to construct a new computable function \( f_* \) such that its graph (denoted by \( \Gamma(f_*) \)) is a computable \( d \)-regular forest. Let us start with two auxiliary functions \( g, h : P_0(f) \times \mathbb{N} \to \mathbb{N} \) such that, for all \( n \in P_0(f) \) and \( m \geq 1 \) the following properties hold:

- \( g(n, 0) = n \) and \( h(n, 0) = f(n) \);
- \( \{g(n, m), h(n, m)\} \cap P(f) = \emptyset \);
- \( f(g(n, m)) = g(n, m - 1) \), and \( f(h(n, m)) = h(n, m - 1) \).

We want \( g, h \) to be computable functions. Since \( P_0(f) \) and \( P(f) \) are computable and the graph \( \Gamma(f) \) is computable and \( d \)-regular the following rule gives required algorithm. Given \( n \in P_0(f) \) and \( m \geq 1 \) and having defined \( g(n, m - 1) \) find the minimal \( x \) such that \( x \notin P(f) \) and \( f(x) = g(n, m - 1) \). Then let \( g(n, m) = x \). The definition of \( h(n, m) \) is similar. Clearly \( g \) and \( h \) are injective and have disjoint ranges.

Now we define \( f_* : \mathbb{N} \to \mathbb{N} \) for \( x \in \mathbb{N} \) in the following way:

\[
f_*(x) = \begin{cases} 
  g(y, m+2) & \text{if } x = g(y, m) \text{ for } y \in P_0(f) \text{ and even } m \geq 0, \\
  g(y, m-2) & \text{if } x = g(y, m) \text{ for } y \in P_0(f) \text{ and odd } m \geq 3, \\
  f_0^2(x) & \text{if } x = h(y, m) \text{ for } y \in P_0(f) \text{ and } m \geq 2, \\
  f(x) & \text{otherwise.}
\end{cases}
\]

We remind the reader that for each \( n \) there is \( i \leq 3n \) such that \( f^i(n) \in P_0(f) \). Thus for any \( m' > 3n \) the number \( n \) does not belong to \( \{g(y, m'), h(y, m')\} \). This guarantees that \( f_* \) is computable. In the proof of Theorem 2.2 in [Sch18] it is proved that \( \Gamma(f_*) \in \mathcal{E} \), the function \( f_* \) does not have cycles and for each \( x \in \mathbb{N} \) the size \( |f_*^{-1}(x)| = d - 1 \). Therefore the graph \( \Gamma(f_*) \) is a computable \( d \)-regular forest.

To see the last statement of the proposition note that if \( C \subseteq \mathbb{N} \) is a connected components of \( \Gamma(f) \) then it is also the set of vertices of a connected component of \( \Gamma(f_*) \). Since for each \( n \) one can compute \( m' \in P_0(f) \) such that \( n \) and \( m' \) are in same connected component of \( \Gamma(f) \) we have an algorithm which for any \( n \) and \( m \) verifies whether \( n \) and \( m \) are in the same tree in \( \Gamma(f_*) \).

**Theorem 13.5.** Let \( d \geq 3 \). Let a coarse space \((\mathbb{N}, \mathcal{E})\) of a bounded geometry be non-amenable. Furthermore, assume that there exists a highly computable symmetric \( E \in \mathcal{E} \) such that for any finite \( F \subseteq \mathbb{N} \) we have \( |E[F]| \geq (d+2)|F| \). Then there exists a computable \( E' \in \mathcal{E} \) such that \( \Gamma(E') \) is a \( d \)-regular forest. Moreover, there exist an algorithm which for any \( m, n \in \mathbb{N} \) recognizes if \( m \) and \( n \) are in the same connected component of \( \Gamma(E') \).

To see that this statement is a computable version of Schneider’s result we remind the reader that when \((X, \mathcal{E})\) is not amenable, then for every finite \( d \) there is a symmetric entourage such that \( |E[F]| \geq d|F| \) for every finite \( F \subseteq X \) (see discussion before Proposition 2.1 in [Sch18]).

**Proof of Theorem 13.5.** Theorem 12.1 can be applied under these circumstances. Indeed, let \( E \) be a highly computable symmetric entourage as in Theorem 13.5 and consider the graph \( \Gamma(R) \) defined for the symmetric relation \( R := E \setminus \Delta_\mathbb{N} \subseteq \mathbb{N} \times \mathbb{N} \). Since the coarse space \((\mathbb{N}, \mathcal{E})\) is of bounded geometry, the neighbourhood of each vertex is finite. By high computability of \( E \) it follows that there is an algorithm which computes the sizes of neighbourhoods of vertexes. Clearly \( \Gamma(R) \) is a highly computable graph.

It remains to show that \( \Gamma(R) \) satisfies \( c.e.H.h.c.(d) \). The following inequality holds for \( R \):

\[
|R[F]| \geq |E[F] \setminus F| \geq |E[F]| - |F| \geq (d + 1)|F|.
\]

Thus, for all finite sets \( X \subset \mathbb{N} \), the following holds

\[
|N_{\Gamma(R)}(X)| - d|X| \geq (d + 1)|X| - d|X| = |X|.
\]

Hence \( n \leq |X| \) implies that \( n \leq |N(X)| - d|X| \leq |N(X)| - \frac{1}{d}|X| \). Since the identity map on \( \mathbb{N} \) is a total, computable function, it follows that \( \Gamma(R) \) satisfies \( c.e.H.h.c.(d) \). As a result applying Theorem 12.1 we get a computable function \( f \) realizing a perfect \((1, d-1)\) matching in the graph \( \Gamma(R) \) with controlled sizes of its cycles.
By Proposition 13.4 there exists a computable \( E' \in \mathcal{E} \) such that \( \Gamma(E') \) is a \( d \)-regular forest and there exist an algorithm which for any \( m,n \in \mathbb{N} \) recognizes if \( m \) and \( n \) are in the same connected component of \( \Gamma(E') \). 

We now translate this Theorem into a statement about wobbling groups. Given a coarse space \((X,\mathcal{E})\) we define its wobbling group as 

\[
\mathcal{W}(X,\mathcal{E}) := \{ \alpha \in \text{Sym}(X) | \{(x, \alpha(x)) | x \in X \} \in \mathcal{E} \}.
\]

Let us recall that a subgroup \( G \leq \text{Sym}(X) \) is said to be semi-regular if no non-identity element of \( G \) has a fixed point in \( X \). The following statement is a computable version of Corollary 2.3 [Sch18].

**Corollary 13.6.** Let the coarse space \((\mathbb{N},\mathcal{E})\) of a bounded geometry be non-amenable. Furthermore, assume that there exists a highly computable symmetric \( E \in \mathcal{E} \) such that \( \forall F \subseteq \mathbb{N} \) if \( F \) is finite then \(|E[F]| \geq 6|F|\). Then there are two computable permutations \( \sigma, \pi \in \text{Sym}(\mathbb{N}) \) such that \( \langle \sigma, \pi \rangle \) is a free semi-regular subgroup of \( \mathcal{W}(\mathbb{N},\mathcal{E}) \).

**Proof.** By Theorem 13.5 there is \( E' \in \mathcal{E} \) such that \( \Gamma(E') \) is a computable \( 4 \)-regular forest. Moreover, there is an algorithm which for each \( n, m \in \mathbb{N} \) recognizes if \( n \) and \( m \) are in the same connected component of \( \Gamma(E') \). Further, there is another algorithm which or each \( n \) finds four natural numbers \( n_1 < n_2 < n_3 < n_4 \) such that either \((n, n_i) \in E'\) or \((n_i, n) \in E'\). Since \( \Gamma(E') \) is a computable \( 4 \)-regular forest, for each \( n, m \in \mathbb{N} \), we can compute \( B_m(n) \subset \mathbb{N} \), the \( m \)-ball of \( n \), by a uniform algorithm.

The standard Cayley graph of the free group on two generators is isomorphic to the \( 4 \)-regular tree so it is clear that there exist permutations \( \sigma, \pi \) as in the formulation. We will show that there exist computable ones. The construction is by induction. At every step we construct two finite partial isometries of \( \Gamma(E') \).

We will use the following notion. We say that a partial permutation \( \sigma \) is loxodromic on a finite, connected set \( D \), if:

- it preserves distance i.e. for all \( n, m \in D \) we have \( d(n, m) = d(\sigma(n), \sigma(m)) \);
- there is a maximal line in the set \( D \cup \sigma(D) \) i.e. there exists \( d_0, \ldots, d_k \in D \cup \sigma(D) \) for \( k = \max\{d(a, b) | a, b \in D\} + 1 \), such that \( \sigma(d_0) = d_1, \ldots, \sigma(d_{k-1}) = d_k \) where \( d_0 \in D \setminus \sigma(D) \) and \( d_k \in \sigma(D) \setminus D \).

At the first step of the construction \( \sigma \) and \( \pi \) we find \( 1, 2, 3, 4 \) and set \( \sigma(1) = 1, \sigma(2) = 1, \sigma(1)(1) = 1, \sigma(1)(1_4) = 1 \). Let \( D_1 = B_1(1) = \{1, 1, 1, 2, 3, 4\} \). We extend \( \sigma_1 \) and \( \pi_1 \) to loxodromic maps on \( D_1 \) such that \( \sigma(1)(D_1) = B_1(\sigma(1)) \) and \( \pi(1)(D_1) = B_1(\pi(1)) \).

Assume that before the \( n \)-th step of the construction two finite partial permutations \( \sigma_{n-1} \) and \( \pi_{n-1} \) are defined so that they are loxodromic maps on some set \( D_{n-1} \). For all \( a \in D_{n-1} \) we set \( \sigma_n(a) = \sigma_{n-1}(a) \) and \( \pi_n(a) = \pi_{n-1}(a) \). Let \( a_1 \) be the first number in \( \mathbb{N} \setminus D_{n-1} \). Let \( X \) be the connected component of \( \Gamma(E') \) containing \( a_n \). If it does not meet \( D_{n-1} \) then we find the corresponding \( (a_n)_1, (a_n)_2, (a_n)_3, (a_n)_4 \), and define \( \sigma_n \) and \( \pi_n \) on \( \{a_n, (a_n)_1, (a_n)_2, (a_n)_3, (a_n)_4\} \) as at the first step. Then put \( D_n = D_{n-1} \cup B_1(a_n) \).

It is worth mentioning a this stage that according the assumptions on \( \Gamma(E') \) the question if \( X \cap D_{n-1} \neq \emptyset \) is decidable: just check if \( a_n \) is in the same component with elements of \( D_{n-1} \). Now assume that the answer is positive. Then find the minimal \( i < n \) such that \( a_i \in X \) and the minimal \( m \) such that \( a_m = B_m(a_i) \). Then we extend \( \sigma_n \) and \( \pi_n \) to a loxodromic partial permutations on \( D_n = D_{n-1} \cup B_m(a_i) \) in such way that for any \( a \in D_n \) and any \( i, j \in \{-1, 1\} \) we have \( \sigma_n^i(a) \neq \pi_n^j(a) \).

Let \( \sigma = \lim_{n \to \infty} \sigma_n \) and \( \pi = \lim_{n \to \infty} \pi_n \). These permutations act on each connected component of \( \Gamma(E') \) by translations along some line, with lines corresponding to \( \sigma \) and \( \pi \) having one common vertex which is the first \( n \) that belongs to component. Clearly there are no points which are fixed by non-identity of \( \langle \sigma, \pi \rangle \) and this group is are a free semi-regular subgroup of \( \mathcal{W}(\mathbb{N}, \mathcal{E}) \). Permutations \( \sigma \) and \( \pi \) are computable by the construction. 

\[\square\]
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