A Compact Representation of the Three-Gluon Vertex

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Abstract

The three-gluon vertex is a basic object of interest in nonabelian gauge theory. It contains important structural information, in particular on infrared divergences, and also figures prominently in the Schwinger-Dyson equations. At the one-loop level, it has been calculated and analyzed by a number of authors. Here we use the worldline formalism to unify the calculations of the scalar, spinor and gluon loop contributions to the one-loop vertex, leading to an extremely compact representation. The SUSY - related sum rule found by Binger and Brodsky follows from an off-shell extension of the Bern-Kosower replacement rules. We explain the relation of the structure of our representation to the low-energy effective action.
1 Introduction

The off-shell three-gluon vertex has been under investigation for more than three decades. By an analysis of the nonabelian gauge Ward identities, Ball and Chiu\cite{1} in 1980 found a form factor decomposition of this vertex which is valid at any order in perturbation theory, with the only restriction that a covariant gauge be used. At the one-loop level, they also calculated the vertex explicitly for the case of a gluon loop in Feynman gauge. Later Cornwall and Papavassiliou\cite{2} applied the pinch technique to the non-perturbative study of this vertex. Davydychev, Osland and Sax \cite{3} calculated the massive quark contribution of the one loop three-gluon vertex. Binger and Brodsky\cite{4} calculated the one-loop vertex in the pinch technique and found the following SUSY-related identity between its scalar, spinor and gluon loop contributions,

\[ 3\Gamma_{\text{scalar}} + 2\Gamma_{\text{spinor}} + \Gamma_{\text{gluon}} = 0 \] (1)

In this talk, I present a recalculation of the scalar, spinor and gluon loop contributions to the three-gluon vertex using the worldline formalism \cite{5, 6, 7, 8}. The vertex is shown in fig. 1 (for the fermion loop case).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{three-gluon_vertex.png}
\caption{Three-gluon vertex.}
\end{figure}

Following the notation of \cite{3}, we write

\[ \Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3) = -igf^{a_1a_2a_3}f^{a_3a_2a_1}\Gamma_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3) \] (2)

The gluon momenta are ingoing, such that \( p_1 + p_2 + p_3 = 0 \). There are actually two diagrams differing by the two inequivalent orderings of the three gluons along the loop. Those diagrams add to produce a factor of two.

The Ball-Chiu decomposition of the vertex can be written as
\[\Gamma_{\mu_1\mu_2\mu_3}(p_1, p_2, p_3) = A(p_1^2, p_2^2, p_3^2) g_{\mu_1\mu_2} (p_1 - p_2)_{\mu_3} + B(p_1^2, p_2^2, p_3^2) g_{\mu_1\mu_3} (p_1 - p_2)_{\mu_3}
+ C(p_1^2, p_2^2, p_3^2) \left[ p_{1\mu_2} p_{2\mu_1} - (p_1 \cdot p_2) g_{\mu_1\mu_2} \right] (p_1 - p_2)_{\mu_3}
+ \frac{1}{3} S(p_1^2, p_2^2, p_3^2) \left[ p_{1\mu_3} p_{2\mu_2} p_{3\mu_1} + p_{1\mu_2} p_{2\mu_3} p_{3\mu_1} \right]
+ F(p_1^2, p_2^2, p_3^2) \left[ (p_1 \cdot p_2) g_{\mu_1\mu_2} - p_{1\mu_2} p_{2\mu_1} \right] \left[ p_{1\mu_3} (p_2 \cdot p_3) - p_{2\mu_3} (p_1 \cdot p_3) \right]
+ H(p_1^2, p_2^2, p_3^2) \left\{ - g_{\mu_1\mu_2} \left[ p_{1\mu_3} (p_2 \cdot p_3) - p_{2\mu_3} (p_1 \cdot p_3) \right]
+ \frac{1}{3} \left[ p_{1\mu_3} p_{2\mu_2} p_{3\mu_1} - p_{1\mu_2} p_{2\mu_3} p_{3\mu_1} \right] \right\}
+ \left\{ \text{cyclic permutations of } (p_1, \mu_1), (p_2, \mu_2), (p_3, \mu_3) \right\} \] (3)

Here the A, C and F functions are symmetric in the first two arguments, B anti-symmetric, and H(S) are totally (anti)symmetric with respect to interchange of any pair of arguments. Note that the F and H functions are totally transverse, i.e., they vanish when contracted with any of \(p_{1\mu_1}, p_{2\mu_2} \) or \(p_{3\mu_3}\).

2 The scalar loop case

In the worldline formalism the three-gluon amplitude for the scalar loop case is represented as \([6, 8]\)

\[\Gamma_{\text{scalar}}^{a_1a_2a_3}(p_1, \varepsilon_1; p_2, \varepsilon_2; p_3, \varepsilon_3) = (-ig)^3 \text{tr}(T^{a_1} T^{a_2} T^{a_3}) \int_0^\infty \frac{dT}{T} e^{-m^2 T} \times \int \mathcal{D}x(\tau) \int_0^T d\tau_1 \varepsilon_1 \cdot \dot{x}_1 e^{ip_1 x_1} \int_0^{\tau_1} d\tau_2 \varepsilon_2 \cdot \dot{x}_2 e^{ip_2 x_2} \varepsilon_3 \cdot \dot{x}_3 e^{ip_3 x_3} e^{-\int_0^T \frac{x^2}{2}} \] (4)

Here \(T\) is the total proper time of the loop particle, \(m\) the mass of the loop particle, \(T^a\) a generator of the gauge group in the representation of the scalar, and \(\int \mathcal{D}(x)\) an integral over closed trajectories in Minkowski space-time with periodicity \(T\). Although our calculation will be off-shell, we introduce gluon polarization vectors \(\varepsilon_j\) as a book-keeping device. Each gluon is represented by a vertex operator \(\int d\tau \varepsilon \cdot \dot{x} e^{ip.x}\). Translation invariance in proper-time has been used to set \(\tau_3 = 0\).

The path integral \([4]\) is Gaussian so that its evaluation requires only the standard combinatorics of Wick contractions and the appropriate Green’s function,

\[< x^{\mu_1}(\tau_1) x^{\mu_2}(\tau_2) > = -g_{B12} x^{\mu_1\mu_2}, \quad G_{B12} := G_B(\tau_1, \tau_2) = |\tau_1 - \tau_2| - \frac{(\tau_1 - \tau_2)^2}{T} \] (5)
In this formalism structural simplification can be expected from the removal of all second derivatives $\hat{G}_{\alpha \beta}$'s, appearing after the Wick contractions, by suitable integrations by part (IBP). After doing this we have (see [8] for the combinatorial details of the Wick contraction and IBP procedure)

\[
\Gamma^{\text{scalar}} = \frac{g^3}{(4\pi)^{D/2}} (\Gamma^{3}_{\text{scalar}} + \Gamma^{2}_{\text{scalar}} + \Gamma^{\text{bt}}_{\text{scalar}})
\]

\[
\Gamma^{3}_{\text{scalar}} = -\text{tr}(T^{a_1} [T^{a_2}, T^{a_3}]) \int_0^\infty \frac{dT}{T^D} e^{-m^2 T} \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 Q^3_{3}(\alpha_{1,2} P_{1,2} + \alpha_3 P_3) + 2 \text{ perm}
\]

\[
\Gamma^{2}_{\text{scalar}} = \Gamma^{3}_{\text{scalar}} (Q^3_3 \to Q^3_3)
\]

\[
\Gamma^{\text{bt}}_{\text{scalar}} = -\text{tr}(T^{a_1} [T^{a_2}, T^{a_3}]) \int_0^\infty \frac{dT}{T^D} e^{-m^2 T} \int_0^T d\tau_1 \hat{G}_{B_{12}} \hat{G}_{B_{23}} \left[ \varepsilon_3 \cdot f_1 \cdot \varepsilon_2 e^{G_{B_{12} P_1} \cdot (P_2 + P_3) + 2 \text{ perm}} \right]
\]

\[
Q^3_3 = \hat{G}_{B_{12}} \hat{G}_{B_{23}} \hat{G}_{B_{13}} \text{tr}(f_1 f_2 f_3)
\]

\[
Q^2_3 = \frac{1}{2} \hat{G}_{B_{12}} \hat{G}_{B_{23}} \text{tr}(f_1 f_2) \sum_{k=1,2} \hat{G}_{B_{3k}} \varepsilon_3 \cdot p_k + 2 \text{ perm}
\]

The abelian field strength tensors $f_i^{\mu\nu} := p_i^\mu \varepsilon_i^\nu - \varepsilon_i^\mu p_i^\nu$ appear automatically in the IBP procedure. The $\Gamma^{\text{bt}}_{\text{scalar}}$'s are boundary terms of the IBP.

We rescale to the unit circle, $\tau_i = T u_i$, $i = 1, 2, 3$, and rewrite these integrals in term of the standard Feynman/Schwinger parameters, related to the $u_i$ by

\[
\alpha_1 = \alpha_2 + \alpha_3 \quad , \quad u_2 = \alpha_3 \quad , \quad u_3 = 0 \quad , \quad \alpha_1 + \alpha_2 + \alpha_3 = 1 \quad (7)
\]

For the scalar case, we find

\[
\Gamma^{\text{scalar}} = \frac{g^3}{(4\pi)^{D/2}} \text{tr}(T^{a_1} [T^{a_2}, T^{a_3}]) (\Gamma^{3}_{\text{scalar}} + \Gamma^{2}_{\text{scalar}} + \Gamma^{\text{bt}}_{\text{scalar}})
\]

\[
\gamma^{3}_{\text{scalar}} = \Gamma (3 - \frac{D}{2}) \text{tr}(f_1 f_2 f_3) I^D_{B_{12}} (p_1^2, p_2^2, p_3^2)
\]

\[
\gamma^{2}_{\text{scalar}} = \frac{1}{2} \Gamma (3 - \frac{D}{2}) \left[ \text{tr}(f_1 f_2) \left( \varepsilon_3 \cdot p_1 I^D_{B_{12}} (p_1^2, p_2^2, p_3^2) - \varepsilon_3 \cdot p_2 I^D_{B_{12}} (p_1^2, p_2^2, p_3^2) \right) + 2 \text{ perm} \right]
\]

\[
\gamma^{\text{bt}}_{\text{scalar}} = -\Gamma (2 - \frac{D}{2}) \left[ \varepsilon_3 \cdot f_1 \cdot \varepsilon_2 I^D_{B_{12}} (p_1^2) + 2 \text{ perm} \right]
\]

(8)
where

\[
I_{3,B}^{D}(p_1^2, p_2^2, p_3^2) = \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 \delta(1 - \alpha_1 - \alpha_2 - \alpha_3) \\
\times \frac{(1 - 2\alpha_1)(1 - 2\alpha_2)(1 - 2\alpha_3)}{(m^2 + \alpha_1 \alpha_2 p_1^2 + \alpha_2 \alpha_3 p_2^2 + \alpha_1 \alpha_3 p_3^2)^{3-D/2}}
\]

\[
I_{2,B}^{D}(p_1^2, p_2^2, p_3^2) = \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 \delta(1 - \alpha_1 - \alpha_2 - \alpha_3) \\
\times \frac{(1 - 2\alpha_2)(1 - 2\alpha_1)}{(m^2 + \alpha_1 \alpha_2 p_1^2 + \alpha_2 \alpha_3 p_2^2 + \alpha_1 \alpha_3 p_3^2)^{3-D/2}}
\]

\[
I_{bt,B}^{D}(p^2) = \int_0^1 d\alpha \frac{(1 - 2\alpha)^2}{(m^2 + \alpha(1 - \alpha)p^2)^{2-D/2}}
\]

(9)

3 Fermion and gluon loop calculations

By an off-shell generalization of the Bern-Koswer replacement rules [5], whose correctness for the case at hand we have verified, one can get the results for the spinor and gluon loop from the scalar loop one simply by replacing

\[
\Gamma_{\text{scalar}} \rightarrow \Gamma_{\text{spinor}} : \quad I_{\{3,2, bt\}, B}^{D} \rightarrow I_{\{3,2, bt\}, B}^{D} - I_{\{3,2, bt\}, F}^{D}
\]

\[
\Gamma_{\text{scalar}} \rightarrow \Gamma_{\text{gluon}} : \quad I_{\{3,2, bt\}, B}^{D} \rightarrow I_{\{3,2, bt\}, B}^{D} - 4I_{\{3,2, bt\}, F}^{D}
\]

(10)

where the \(I_{\{\cdot, \cdot\}}^{D}\)'s are three integrals similar to the \(I_{\{\cdot, \cdot\}}^{D}\)'s above (for the spinor loop one must also multiply by a global factor of \(-2\)). From (10) we immediately recover the Binger-Brodsky identity eq.(1).

4 Comparison with the effective action

Finally let us compare our results with the low energy expansion of the QCD effective action induced by a scalar loop,

\[
\Gamma_{\text{scalar}}[F] = \int_0^\infty \frac{dT}{T} \frac{e^{-m^2T}}{(4\pi T)^2} \text{tr} \int dx_0 \sum_{n=2}^{\infty} \frac{(-T)^n}{n!} O_n[F]
\]

(11)

For our comparison we need only \(O_2\) and \(O_3\) which are\[8\]

\[
O_2 = -\frac{1}{6} F_{\kappa \lambda} F^{\kappa \lambda}, \quad O_3 = -\frac{2}{15} i F^\lambda_\kappa F^\kappa_\mu F^\mu_\lambda - \frac{1}{20} D_\kappa F^\kappa_\mu B^{\mu \lambda} F^{\lambda \mu}
\]

(12)
where

\[ F_{\mu\nu} = f_{\mu\nu} + ig[A_{\mu}, A_{\nu}] \, , \quad f_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \, , \quad D_{\mu} = \partial_{\mu} + igA_{\mu} \]  

(13)

We can recognize the correspondences

\[ \gamma^3 (\cdot) \leftrightarrow F^\lambda_\mu F^\mu_\kappa = f^\lambda_\mu f^\mu_\kappa + \text{higher point terms} \]

\[ \gamma^2 (\cdot) \leftrightarrow (\partial + igA)F(\partial + igA)F \]

\[ \gamma^{bt} (\cdot) \leftrightarrow (f + ig[A, A])(f + ig[A, A]) \]

(14)

5 Conclusions and outlook

In our recalculation of the scalar, spinor and gluon contributions to the one-loop three gluon vertex we have achieved a significant improvement over previous calculations both in efficiency and compactness of the result. This improvement is in large part due to the replacement rules (10) whose validity off-shell we have verified. Details and a comparison with the Ball-Chiu decomposition will be presented elsewhere. We believe that along the lines presented here even a first calculation of the four-gluon vertex would be feasible.

References

[1] J. S. Ball and T. W. Chiu, Phys. Rev. D 22, 2550 (1980).
[2] J. M Cornwall and J. Papavassiliou, Phys. Rev. D 40, 3474 (1989).
[3] A. I. Davydychev, P. Osland and L. Saks, JHEP 0108:050 (2001).
[4] M. Binger and S. J. Brodsky, Phys. Rev. D 74, 054016 (2006).
[5] Z. Bern and D. A. Kosower, Phys. Rev. Lett. 66, 1669 (1991); Nucl. Phys. B 379, 451 (1992).
[6] M. J. Strassler, Nucl. Phys. B 385, 145 (1992).
[7] M. Reuter, M. G. Schmidt and C. Schubert, Ann. Phys. (N.Y.) 259, 313 (1997).
[8] C. Schubert, Phys. Rept. 355, 73 (2001).