ANALYSIS OF A MIXED FINITE ELEMENT METHOD FOR THE QUAD-CURL PROBLEM
GANG CHEN∗, WEIFENG QIU †, AND LIWEI XU‡

Abstract. Quad-curl term plays an essential role in the numerical analysis of the resistive magnetohydrodynamics (MHD) and the forth order inverse electromagnetic scattering problem. It is desirable to develop simple and efficient numerical methods for the quad-curl problem. In this paper, we firstly give a regularity result for the quad-curl problem on Lipschitz polyhedron domains, which is new in literatures. Then, we propose a mixed finite element method for the quad-curl problem. With novel discrete Sobolev embedding inequalities for the piecewise polynomials, we obtain stability results and derive optimal error estimates relying on a low regularity assumption of the exact solution. To the best of our knowledge, this low regularity assumption is lower than the regularity requirements in existing works.

Key words. Mixed finite element method, quad-curl problem, Lipschitz domain, low regularity, discrete Sobolev embedding equality

AMS subject classifications. 65N15, 65N30

1. Introduction. Let Ω be a bounded simply-connected Lipschitz polyhedron in \(\mathbb{R}^3\) with connected boundary \(\partial \Omega\). We consider the following quad-curl (fourth order) problem:

Find the vector \(u\) and the Lagrange mutliplier \(p\) such that

\[
\begin{align*}
\nabla \times (\nabla \times (\nabla \times (\nabla \times u))) + \nabla p &= f \quad \text{in } \Omega, \\
\nabla \cdot u &= 0 \quad \text{in } \Omega, \\
n \times u &= 0 \quad \text{on } \partial \Omega, \\
n \times (\nabla \times u) &= 0 \quad \text{on } \partial \Omega, \\
p &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where \(f \in [L^2(\Omega)]^3\). This model problem arises in many different applications, such as in the resistive magnetohydrodynamics (MHD) and in the inverse electromagnetic scattering theory.

The resistive MHD system reads [28, 13]:

Find the velocity \(u\), the pressure \(p\) and the magnetic induction field \(B\) such that

\[
\begin{align*}
(1.2a) \quad \rho (u_t + (u \cdot \nabla) u) + \nabla p &= \frac{1}{\mu_0} (\nabla \times B) + \mu \Delta u \quad \text{in } \Omega, \\
(1.2b) \quad B_t - \nabla \times (u \times B) &= -\frac{\eta}{\mu_0} (\nabla \times (\nabla \times B))
\end{align*}
\]

∗School of Mathematics Sciences, University of Electronic Science and Technology of China, Chengdu, China. The first author’s work is supported by National Natural Science Foundation of China (NSFC) grant no. 11801063, China Postdoctoral Science Foundation project no. 2018M633339, and Key Laboratory of Numerical Simulation of Sichuan Province (Neijiang, Sichuan Province) grant no. 2017KF003. (cglwdm@uestc.edu.cn)
†Department of Mathematics, City University of Hong Kong, 83 Tat Chee Avenue, Hong Kong, China. The second author’s research is supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China no. CityU 11304017. (weifeqiu@cityu.edu.hk).
‡School of Mathematics Sciences, University of Electronic Science and Technology of China, Chengdu, China. The third author’s research is supported in part by a Key Project of the Major Research Plan of NSFC grant no. 91630202 and NSFC grant no. 11771068. Corresponding author: Liwei Xu (xul@uestc.edu.cn).
\[-\frac{d_i}{\mu_0} \nabla \times ((\nabla \times B) \times B)\]
\[-\frac{\eta_2}{\mu_0} \nabla \times (\nabla \times (\nabla \times (\nabla \times B)))\]
in \(\Omega\),
(1.2c) \[\nabla \cdot u = 0\]
in \(\Omega\),
(1.2d) \[\nabla \cdot B = 0\]
in \(\Omega\),

with some proper boundary conditions. Here \(\rho\) is the mass density, \(\eta\) is the resistivity, \(\eta_2\) is the hyper-resistivity, \(\mu_0\) is the magnetic permeability of free space, and \(\mu\) is the viscosity.

In the inverse electromagnetic scattering theory, the transmission eigenvalue problem for the anisotropic Maxwell's equations can be formulated in the following fourth order problem [21]:

Find the vector \(u\) and the number \(k\) such that

\[(\nabla \times (\nabla \times (\nabla \times (\nabla \times u))) - k^2 N (N - I)^{-1} (\nabla \times (\nabla \times (\nabla \times (\nabla \times u))) - k^2 u)) = 0\]
in \(\Omega\),
(1.3a) \[n \times u = 0\]
on \(\partial \Omega\),
(1.3b) \[n \times (\nabla \times u) = 0\]
on \(\partial \Omega\),
(1.3c)

where \(N\) is a given real matrix field and \(I\) is the identity matrix. The leading term in both (1.2) and (1.3) is \(\nabla \times (\nabla \times (\nabla \times (\nabla \times u)))\). Therefore, it is important to investigate numerical methods solving the quad-curl term, such as (1.1).

There are vast literatures on numerical methods on the MHD model without the quad-curl term \(\nabla \times (\nabla \times (\nabla \times (\nabla \times u)))\), see [3, 10, 15, 11, 24, 16, 9, 19] and references therein for detailed information. However, when the quad-curl term \(\nabla \times (\nabla \times (\nabla \times (\nabla \times u)))\) presents, it becomes more difficult to design and analyze numerical methods, since the regularity estimates for (1.1) are an open question, and the curl-curl conforming elements in three dimensions are still unknown (see [26] for curl-curl conforming elements in two dimensions). If the curl-curl conforming elements are considered, it would be complicated and far from obvious (since the conforming elements for the biharmonic problem are quite complicated even in two dimensions, see [8] for example). Therefore, it is worth devising simple and efficient numerical methods for the quad-curl problem, and it thus makes it possible to apply these methods to the resistive MHD system and the forth order inverse electromagnetic scattering problem.

There are already many works devoted to the study on the quad-curl problem in the past decades. In [28], a nonconforming finite element method for the quad-curl model with low order terms was studied under the regularity assumption

\[u \in [H^4(\Omega)]^3.\]

A discontinuous Galerkin (DG) method using \(H(\text{curl})\) conforming elements for the quad-curl model problem was investigated in [13], where the following regularity requirement was assumed:

\[u \in [H^2(\Omega)]^3, \quad \nabla \times u \in [H^2(\Omega)]^3.\]

A mixed finite element method for the quad-curl eigenvalue problem was introduced and analyzed in [25] given that the following regularity

\[u \in [H^3(\Omega)]^3, \quad \nabla \times u \in [H^3(\Omega)]^3,\]
MIXED METHOD FOR QUAD-CURL

holds. Besides, a finite element method for the quad-curl problem in two dimensions was studied in [4] based on the Hodge decomposition. In [23], the author proved that: when the domain has no point and edge singularities, it holds that \( u \in [H^4(\Omega)]^3 \); when the domain has point and edge singularities, \( u \) does not belong to \([H^3(\Omega)]^3\) in general. In [27], the author proved that on convex polyhedral domains, if \( \nabla \cdot f = 0 \), there holds

\[
\begin{align*}
    u &\in [H^2(\Omega)]^3, \\
    \nabla \times u &\in [H^2(\Omega)]^3, \\
    p &\in H^p(\Omega),
\end{align*}
\]

However, there are no regularity results available for non-convex Lipschitz polyhedral domains in three dimensions. Therefore, there are no numerical analysis available on general Lipschitz domains (which can be non-convex), or at least available under regularity lower than (1.4).

In this paper, concerning numerical investigation on the model problem (1.1). We firstly give the regularity of quad-curl problem on general Lipschitz domains, which is new in literatures. Then, we propose a mixed finite element method for the quad-curl model problem (1.1). Finally, we prove the convergence results under a relative low regularity, i.e.:

\[
\begin{align*}
    u &\in [H^{r_{u_0}}(\Omega)]^3, \\
    \nabla \times u &\in [H^{r_{u_1}}(\Omega)]^3, \\
    \nabla \times (\nabla \times u) &\in [H^{r_{u_2}}(\Omega)]^3, \\
    p &\in H^{r_{p}}(\Omega),
\end{align*}
\]

where \( r_{u_0} > 1/2, r_{u_1} \geq 1, r_{u_2} > 1/2, \) and \( r_{p} > 3/2 \). This regularity requirement is lower than those in any other existing works, especially we only need \( u \in [H^{r_{u_0}}(\Omega)]^3 \) with \( r_{u_0} > 1/2 \), and all the other works need at least \( r_{u_0} \geq 2 \). We point out that, even though our proposed numerical scheme is similar to the one in [13], the research in [13] dealt with the quad-curl problem with a reaction term, which benefits the theoretical analysis, and the regularity requirement in [13] is also much higher than ours.

We also establish a novel discrete Sobolev embedding inequality in piecewise \( H^1 \) norm:

\[
\sum_{K \in \mathcal{T}_h} \| v_h \|_{1,K}^2 \leq C \left[ \sum_{K \in \mathcal{T}_h} (\| \nabla \times v_h \|_{0,K}^2 + \| \nabla \cdot v_h \|_{0,K}^2) + \sum_{F \in \mathcal{E}_h} h_{F}^{-1} \| [v_h] \|_{0,F}^2 \right],
\]

where \( v_h \) is a piecewise polynomial of a fixed order. This inequality could be a useful tool in the numerical analysis of nonlinear problems, and it would be reported in our future works.

The rest of this paper is organized as follows. In section 2, we present the stability and regularity for the quad-curl partial differential equations (PDEs). In section 3, we propose a mixed method for quad-curl problem and interpolations will be used in analysis. In section 4, we obtain a novel discrete Sobolev inequality and stability results for underlying mixed methods. In section 5, we prove the convergence result. In section 6, we give estimates in \( H(\text{curl}) \) norm.

Throughout this paper, we use \( C \) to denote a positive constant independent of mesh size, not necessarily the same at its each occurrence.

2. Stability and regularity for PDEs. For any bounded domain \( \Lambda \subset \mathbb{R}^s \) \((s = 2, 3)\), and any two function \( u, v \in L^2(\Lambda) \), we denote the \( L^2(\Lambda) \) inner product and norm by

\[
(u, v)_\Lambda := \int_{\Lambda} uv \, dx, \quad \| u \|_{0,\Lambda} := (u, u)_\Lambda^{1/2},
\]
and

\[(u, v) := (u, v)_{\Omega}, \quad \|u\|_0 := \|u\|_{0, \Omega},\]

for simplicity. We denote \(W^{m,p}(\Lambda)\) and \(W^{m,p}_0(\Lambda)\) by the Sobolev spaces defined on \(\Lambda\), and denote \(|v|_{m,p,\Lambda}\) and \(||v||_{m,p,\Lambda}\) by its semi-norm and norm, respectively. When \(p = 2\) we omit \(p\) in \(|v|_{m,p,\Lambda}\) and \(||v||_{m,p,\Lambda}\); when \(\Lambda = \Omega\) we omit \(\Lambda\) in \(|v|_{m,p,\Lambda}\) and \(||v||_{m,p,\Lambda}\).

For conventional notation, we denote \(H^m(\Lambda) := W^{m,2}(\Lambda)\) and \(H^m_0(\Lambda) := W^{m,2}_0(\Lambda)\).

In particular, when \(\Lambda \in \mathbb{R}^2\), we use \(\langle \cdot, \cdot \rangle_\Lambda\) to replace \((\cdot, \cdot)_\Lambda\) for distinguish. The bold face fonts will be used for vector (or tensor) analogues of the Sobolev spaces along with vector-valued (or tensor-valued) functions. Define the spaces

\[H^m(\text{curl}; \Omega) := \{v \in [L^2(\Omega)]^3 : \nabla \times v \in [L^2(\Omega)]^3\},\]

\[H^m(\text{div}; \Omega) := \{v \in [L^2(\Omega)]^3 : \nabla \cdot v \in L^2(\Omega)\},\]

\[H^m_0(\text{curl}; \Omega) := \{v \in H^m(\text{curl}; \Omega) : n \times v|_{\partial \Omega} = 0\},\]

\[H^m_0(\text{div}; \Omega) := \{v \in H^m(\text{div}; \Omega) : n \cdot v|_{\partial \Omega} = 0\}.\]

The following embedding theory is quite standard but useful in the analysis of \(H(\text{curl})\) space.

**Lemma 2.1.** [2] If the domain \(\Omega\) is a Lipschitz polyhedron, then \(X_T(\Omega)\) and \(X_N(\Omega)\) are continuously embedded in \([H^s(\Omega)]^3\) for a real number \(s \in (1/2, 1]\), where the spaces \(X_N(\Omega)\) and \(X_T(\Omega)\) are defined as

\[X_N := H^0(\text{curl}; \Omega) \cap H(\text{div}; \Omega), \quad X_T := H(\text{curl}; \Omega) \cap H^0(\text{div}; \Omega).\]

With the above statement, we are ready to prove the following regularity on Lipschitz polyhedron domains.

**Theorem 2.2.** Let \(\Omega\) be a simply-connected Lipschitz polyhedron domain in \(\mathbb{R}^3\) and \((u, p)\) be the solution of (1.1), then we have the stability

\[\|\nabla \times (\nabla \times (\nabla \times u))\|_0 + \|\nabla \times (\nabla \times u)\|_0 + \|\nabla \times u\|_1 + \|\nabla p\|_0 \leq C\|f\|_0,\]

and the regularity

\[\|u\|_{r_{u_0}} \leq C\|f\|_0,\]

with the regularity index \(r_{u_0} \in (1/2, 1]\). In addition, if \(\nabla \times (\nabla \times u) \in [H^{r_{u_1}}(\Omega)]^3\), we have the following regularity

\[\|\nabla \times u\|_{r_{u_1}} \leq C\|\nabla \times (\nabla \times u)\|_{r_{u_1}-1},\]

holds true with \(r_{u_1} \in [1, 3/2)\) but close to \(3/2\). If \(f \in H(\text{div}; \Omega)\), there exists a constant \(r_p \in (3/2, 2]\) such that

\[\|p\|_{r_p} \leq C\|\nabla \cdot f\|_0.\]
Proof. We present our proof in following several steps.

• Proof of $\|\nabla p\|_0 \leq \|f\|_0$: it holds
  \[
  \|\nabla p\|_0^2 = (f - \nabla \times (\nabla \times (\nabla \times u)), \nabla p) \tag{1.1a}
  \]
  by integration by parts.

  Then $\|\nabla p\|_0 \leq \|f\|_0$ follows immediately.

• Proof of $\|\nabla \times (\nabla \times (\nabla \times (\nabla \times u)))\|_0 \leq \|f\|_0$: it follows from (1.1a) that
  \[
  \|\nabla \times (\nabla \times (\nabla \times (\nabla \times u)))\|_0 = \|f - \nabla p\|_0 \leq \|f\|_0 + \|\nabla p\|_0 \leq 2\|f\|_0.
  \]

• Proof of $\|\nabla \times (\nabla \times (\nabla \times u))\|_0 \leq \|f\|_0$: using (1.1a), (1.1c), (1.1d) and integration by parts, one has
  \[
  \|\nabla \times (\nabla \times (\nabla \times u))\|^2_0 + (\nabla \times (\nabla \times u), u) = (f, u).
  \]
  We use Lemma 2.1 twice to get
  \[
  \|\nabla \times (\nabla \times u)\|_0^2 = (f, u) - (\nabla p, u)
  \leq C(\|f\|_0 + \|\nabla p\|_0)\|u\|_0
  \leq C\|f\|_0\|\nabla \times u\|_0
  \leq C\|f\|_0\|\nabla \times (\nabla \times u)\|_0,
  \]
  which leads to
  \[
  \|\nabla \times (\nabla \times u)\|_0 \leq C\|f\|_0.
  \]

• Proof of $\|\nabla \times u\|_1 \leq C\|f\|_0$: We define $\sigma := \nabla \times u$. From (1.1c) we have,
  \[
  n \cdot \sigma|_{\partial \Omega} = n \cdot (\nabla \times u)|_{\partial \Omega} = \nabla|_{\partial \Omega} \cdot (n \times u) = 0,
  \]
  which cooperates with $n \times \sigma|_{\partial \Omega} = 0$ from (1.1d), leads to
  \[
  \sigma = 0 \text{ on } \partial \Omega. \tag{2.1}
  \]

  Using the fact $\nabla \cdot \sigma = 0$, we can get
  \[
  \Delta \sigma = \nabla(\nabla \cdot \sigma) - \nabla \times (\nabla \times \sigma) = -\nabla \times (\nabla \times \sigma) \text{ in } \Omega. \tag{2.2}
  \]

  Utilizing (2.1), (2.2), and the regularity for the elliptic problem in [6], yields
  \[
  \|\nabla \times u\|_1 = \|\sigma\|_1 \leq C\|\nabla \times (\nabla \times \sigma)\|_{-1} = C\|\nabla \times (\nabla \times (\nabla \times u))\|_{-1}
  \leq C\|\nabla \times (\nabla \times u))\|_0
  \leq C\|f\|_0.
  \]

• Proof of $\|\nabla \times u\|_{r_{u_1}} \leq C\|\nabla \times (\nabla \times (\nabla \times u))\|_{r_{u_1-2}}$: utilizing (2.1), (2.2) and the regularity for the elliptic problem in [17, Theorem 1.1], there exists a constant $r_{u_1} \in [1, 3/2)$ but close to 3/2 such that
  \[
  \|\nabla \times u\|_{r_{u_1}} = \|\sigma\|_{r_{u_1}} \leq C\|\nabla \times (\nabla \times \sigma)\|_{r_{u_1}-2}
  = C\|\nabla \times (\nabla \times (\nabla \times u))\|_{r_{u_1}-2}
  \leq C\|\nabla \times (\nabla \times u)\|_{r_{u_1}-1}.
  \]
• Proof of \( \|p\|_{r_p} \leq C\|\nabla \cdot f\|_0 \): we apply \( \nabla \cdot \) on (1.1a) and combine (1.1e) to get

\[
\begin{align}
(2.3a) \quad & \Delta p = \nabla \cdot f \quad \text{in } \Omega, \\
(2.3b) \quad & p = 0 \quad \text{on } \partial \Omega.
\end{align}
\]

Utilizing the stability result of the three-dimensional elliptic problem in [6], there exists a constant \( r_p \in (3/2, 2] \) such that

\[
\|p\|_{r_p} \leq C\|\nabla \cdot f\|_0.
\]

3. A Mixed finite element method. Let \( \mathcal{T}_h = \bigcup \{ K \} \) be a shape-regular and quasi-uniform partition of the domain \( \Omega \) consists of simplex. For any \( K \in \mathcal{T}_h \), let \( h_K \) be the infimum of the diameters of spheres containing \( K \) and denote the mesh size \( h := \max_{K \in \mathcal{T}_h} h_K \). Let \( \mathcal{E}_h = \bigcup \{ F \} \) be the set of all faces of the mesh \( \mathcal{T}_h \). For all \( K \in \mathcal{T}_h \) and \( F \in \mathcal{F}_h \), we denote by \( n_T \) and \( n_F \) the unit outward normal vector to \( \partial K \) and face \( F \), respectively. Let \( F = \partial K \cap \partial K' \) be an interior face shared by element \( K \) and element \( K' \) and \( n_F \) is from \( K \) point to \( K' \). Let \( \phi \) be a piecewise smooth function.

We define the average and jump of \( \phi \) on \( F \) as

\[
\{ \{ \phi \} \} := \frac{1}{2} (\phi + \phi'), \quad \lbrack \phi \rbrack := \phi - \phi',
\]

On a boundary face \( F = \partial K \cap \partial \Omega \), we set \( \{ \{ \phi \} \} := \phi \) and \( \lbrack \phi \rbrack := \phi \). For any bounded domain \( \Lambda \) and any integer \( \ell \geq 0 \), \( P_{\ell}(\Lambda) \) denotes the set of all polynomials defined on \( \Lambda \) with degree no greater than \( \ell \). We use \( P_{\ell}(\mathcal{T}_h) \) denote the piecewise polynomial of order \( \ell \) with respect to the decomposition \( \mathcal{T}_h \).

3.1. Mixed methods. For any integer \( k \geq 1 \), we define the finite element spaces

\[
E_h := H_0^1(\text{curl}; \Omega) \cap [P_{k+1}(\mathcal{T}_h)]^3, \quad Q_h := H_0^1(\Omega) \cap P_{k+2}(\mathcal{T}_h).
\]

Then our mixed method is to find \( u_h \in E_h \) and \( p_h \in Q_h \) such that

\[
\begin{align}
\sum_{K \in \mathcal{T}_h} (\nabla \times (\nabla \times u_h), \nabla \times (\nabla \times v_h))_K + (\nabla p_h, v_h) \\
- \sum_{F \in \mathcal{E}_h} \langle \{ \{ \nabla \times (\nabla \times u_h) \} \}, n \times \lbrack \nabla \times v_h \rbrack \rangle_F \\
\pm \sum_{F \in \mathcal{E}_h} \langle \{ \{ \nabla \times (\nabla \times v_h) \} \}, n \times \lbrack \nabla \times u_h \rbrack \rangle_F \\
+ \sum_{F \in \mathcal{E}_h} \frac{\tau}{h_F} (n \times \lbrack \nabla \times u_h \rbrack, n \times \lbrack \nabla \times v_h \rbrack)_F = (f, v_h), \\
(3.1a) \quad (u_h, \nabla q_h) = 0,
\end{align}
\]

holds for all \( (v_h, q_h) \in E_h \times Q_h \). The stabilization parameter \( \tau > 0 \) is independent of the mesh size. In the following text, we consider the analysis only for the symmetry case (i.e., we replace ‘\( \mp \)’ by ‘\( - \)’ in (3.1a)), since the proof of the non-symmetry case is similar to the symmetry case.
3.2. Interpolations. For integer \( \ell \geq 1 \), we denote \( \Pi_{h,\ell}^{\text{curl}} \) the standard \( H \)-conforming interpolation from \( H^s(\text{curl}; \Omega) \) to \( H(\text{curl}; \Omega) \cap [\mathcal{P}_\ell(T_h)]^3 \) with \( s > 1/2 \), and thus also from \( H^s(\text{curl}; \Omega) \cap H_0(\text{curl}; \Omega) \) to \( H_0(\text{curl}; \Omega) \cap [\mathcal{P}_\ell(T_h)]^3 \) with \( s > 1/2 \). From the results in [22, 1, 20], it holds

\[
\|u - \Pi_{h,\ell}^{\text{curl}} u\|_{0,K} \leq C h_{K}^t (\|u\|_{t,K} + \|
abla \times u\|_{t,K}),
\]

\[
\|
abla \times (u - \Pi_{h,\ell}^{\text{curl}} u)\|_{0,K} \leq C h_{K}^t \|
abla \times u\|_{t,K},
\]

with \( u \in H^t(\text{curl}; \Omega), t \in (1/2, \ell] \). We define the \( L^2 \)-projection from \( L^2(\Omega) \) onto \( Q_h \) as: for any \( p \in L^2(\Omega) \), find \( \Pi_h^Q p \in Q_h \) such that

\[
(\Pi_h^Q p, q) = (p, q) \quad \forall q \in Q_h.
\]

The following approximation result holds true

\[
\|p - \Pi_h^Q p\|_j \leq C h_j^{-1} \|p\|_j,
\]

for \( j \in [1, k+3] \) and \( p \in H^j(\Omega) \). Next, we introduce an interpolation from [5]: for any \( v \in H^s(\text{curl}; \Omega) \) with \( s > 1/2 \), we define \( \Pi_h^E v \in E_h \) such that:

\[
\Pi_h^E v = \Pi_{h,k+1}^{\text{curl}} v + \nabla \sigma_h,
\]

where \( \sigma_h \in Q_h \) satisfies the elliptic problem:

\[
(\nabla \sigma_h, \nabla q_h) = (v - \Pi_{h,k+1}^{\text{curl}} v, \nabla q_h) \quad \forall q_h \in Q_h.
\]

Utilizing (3.3) and (3.4), we get the following result immediately.

**Lemma 3.1.** We have the following orthogonality

\[
(\Pi_h^E v, \nabla q_h) = 0,
\]

holds for all \( v \in H^s(\text{curl}; \Omega) \) (\( s > 1/2 \)) and \( q_h \in Q_h \). In addition, it holds the approximation property

\[
\|v - \Pi_h^E v\|_0 \leq 2\|v - \Pi_{h,k+1}^{\text{curl}} v\|_0
\]

In analysis, we need the following \( H_0(\text{curl}) \)-conforming and \( H_0^1 \)-conforming interpolations.

**Lemma 3.2.** [14, Proposition 4.5] For any integer \( \ell \geq 1 \), let \( v_h \in [\mathcal{P}_\ell(T_h)]^3 \), there exists a function \( \Pi_{h,\ell}^{\text{curl},c} v_h \in [\mathcal{P}_\ell(T_h)]^3 \cap H_0(\text{curl}; \Omega) \) such that

\[
\|\Pi_{h,\ell}^{\text{curl},c} v_h - v_h\|_0 \leq C \left( \sum_{F \in E_h} h_F \|\mathbf{n} \times [v_h]\|_{0,F}^2 \right)^{1/2},
\]

\[
\left( \sum_{K \in T_h} \|
abla \times (\Pi_{h,\ell}^{\text{curl},c} v_h - v_h)\|_{0,K}^2 \right)^{1/2} \leq C \left( \sum_{F \in E_h} h_F^{-1} \|\mathbf{n} \times [v_h]\|_{0,F}^2 \right)^{1/2},
\]

with a constant \( C > 0 \) independent of mesh size.
LEMMA 3.3. [18, Theorem 2.2] For any integer \( \ell \geq 1 \), let \( \mathbf{v}_h \in [P_{\ell}(T_h)]^3 \), there exists a function \( \mathbf{I}_{h,\ell} \mathbf{v}_h \in [P_{\ell}(T_h)]^3 \cap \left[ H_0^1(\Omega) \right]^3 \) such that

\[
\| \mathbf{I}_{h,\ell} \mathbf{v}_h - \mathbf{v}_h \|_0 \leq C \left( \sum_{F \in E_h} h_F \| \mathbf{v}_h \|_{0,F}^2 \right)^{1/2},
\]

\[
\left( \sum_{K \in T_h} \| \nabla (\mathbf{I}_{h,\ell} \mathbf{v}_h - \mathbf{v}_h) \|_{0,K}^2 \right)^{1/2} \leq C \left( \sum_{F \in E_h} h_F^{-1} \| \mathbf{v}_h \|_{0,F}^2 \right)^{1/2},
\]

with a constant \( C > 0 \) independent of mesh size.

4. Stability for the mixed method. In this section, we will derive stability results for our mixed method. To this purpose, we develop novel discrete Sobolev embedding inequalities. These inequalities will become an efficient tool for nonlinear problems, and we will report it in our future works.

4.1. Novel discrete Sobolev embedding inequalities. We present a novel discrete Sobolev in the following subsection, which is the first to be reported in literatures.

THEOREM 4.1. There is a constant \( C > 0 \), such that

\[
\sum_{K \in T_h} \| \mathbf{v}_h \|_{1,K}^2 \leq C \left[ \sum_{K \in T_h} \left( \| \nabla \times \mathbf{v}_h \|_{0,K}^2 + \| \nabla \cdot \mathbf{v}_h \|_{0,K}^2 \right) + \sum_{F \in E_h} h_F^{-1} \| \mathbf{v}_h \|_{0,F}^2 \right],
\]

\[
\| \mathbf{v}_h \|_{0,6} \leq C \left[ \sum_{K \in T_h} \left( \| \nabla \times \mathbf{v}_h \|_{0,K}^2 + \| \nabla \cdot \mathbf{v}_h \|_{0,K}^2 \right) + \sum_{F \in E_h} h_F^{-1} \| \mathbf{v}_h \|_{0,F}^2 \right],
\]

holds for all \( \mathbf{v}_h \in [P_{\ell}(T_h)]^3 \) with \( \ell \geq 1 \) being an integer.

Before proving this theorem, we begin with a well-known discrete Sobolev embedding inequality in [7, Theorem 2.1] and a continuous Sobolev embedding inequality.

LEMMA 4.2. [7, Theorem 2.1] There is a constant \( C > 0 \), such that

\[
\| \mathbf{v}_h \|_{0,6} \leq C \left[ \sum_{K \in T_h} \| \nabla \mathbf{v}_h \|_{0,K}^2 + \sum_{F \in E_h} h_F^{-1} \| \mathbf{v}_h \|_{0,F}^2 \right]^{1/2},
\]

holds for all \( \mathbf{v}_h \in [P_{\ell}(T_h)]^3 \) with \( \ell \geq 1 \) being an integer.

LEMMA 4.3. There is a constant \( C > 0 \), such that

\[
\| \mathbf{v} \|_1 \leq C \left( \| \nabla \times \mathbf{v} \|_0 + \| \nabla \cdot \mathbf{v} \|_0 \right),
\]

holds for any \( \mathbf{v} \in [H_0^1(\Omega)]^3 \).

Proof. Since \( \mathbf{v} \in [H_0^1(\Omega)]^3 \), it holds \( \Delta \mathbf{v} \in [H^{-1}(\Omega)]^3 \). Using the regularity result from [17], we have

\[
(1) \quad \| \mathbf{v} \|_1 \leq C \| \Delta \mathbf{v} \|_{-1}.
\]

So, we only need to estimate \( \| \Delta \mathbf{v} \|_{-1} \). Utilizing integration by parts and the Cauchy-Schwarz inequality, for any \( \mathbf{w} \in [H_0^1(\Omega)]^3 \), we get

\[
-(\Delta \mathbf{v}, \mathbf{w}) = (\nabla \times (\nabla \times \mathbf{v}), \mathbf{w}) - (\nabla (\nabla \cdot \mathbf{v}), \mathbf{w})
\]
Then it arrives
\begin{equation}
\|\Delta v\|_{-1} = \sup_{0 \neq w \in [H^1_0(\Omega)]^3} \frac{(\Delta v, w)}{\|w\|_1} \leq C (\|\nabla \times v\|_0 + \|\nabla \cdot v\|_0) \|w\|_1.
\end{equation}

The proof can be obtained immediately from the combination of inequalities (4.1) and (4.2).

Now, we are in the position to prove Theorem 4.1:

Proof. Let \( I_{h,\ell}^v \) be as in Lemma 3.3, then for any \( v_h \in [P_\ell(T_h)]^3 \), we have \( I_{h,\ell}^v \in [H^1_0(\Omega)]^3 \). Using Lemma 4.3 and the estimates in Lemma 3.3, we arrive at
\begin{align}
\|I_{h,\ell}^v v_h\|_1^2 &\leq C (\|\nabla \times I_{h,\ell}^v v_h\|_0^2 + \|\nabla \cdot I_{h,\ell}^v v_h\|_0^2) \\
&\leq C \left( \sum_{K \in T_h} (\|\nabla \times v_h\|_{0,K}^2 + \|\nabla \cdot v_h\|_{0,K}^2 + \|\nabla (I_{h,\ell}^v v_h - v_h)\|_{0,K}^2) \right) \\
&\leq C \left( \sum_{K \in T_h} (\|\nabla \times v_h\|_{0,K}^2 + \|\nabla \cdot v_h\|_{0,K}^2) + \sum_{F \in \mathcal{E}_h} h_F^{-1} \|v_h\|_{0,F}^2 \right).
\end{align}

Due to the above estimate, the triangle inequality and the estimates in Lemma 3.3, we have the first part of our theorem. The second part of this lemma is just followed by Lemma 4.2 and the first part of this theorem.

4.2. \( L^3 \) stability of \( u_h \) and \( \nabla \times u_h \). We recall the novel discrete Sobolev embedding result developed in [24].

Lemma 4.4. There exists a positive constant \( C \) such that for any \( v_h \in [P_{k+1}(T_h)]^3 \)
\begin{equation}
(v_h, \nabla q_h) = 0 \quad \forall q_h \in Q_h = H^1_0(\Omega) \cap P_{k+2}(T_h),
\end{equation}
then we have
\begin{equation}
\|v_h\|_{0,3} \leq C \left( \sum_{K \in T_h} \|\nabla \times v_h\|_{0,K}^2 + \sum_{F \in \mathcal{E}_h} \frac{1}{h_F^3} \|\nabla \times [v_h]\|_{0,F}^2 \right)^{1/2}.
\end{equation}

With the above lemma, we can derive the discrete Sobolev inequalities for \( H(\text{curl}) \) conforming functions.

Lemma 4.5. For all \( v_h \in E_h = H_0(\text{curl}; \Omega) \cap [P_{k+1}(T_h)]^3 \), if there holds
\begin{equation}
(v_h, \nabla q_h) = 0 \quad \forall q_h \in Q_h = H^1_0(\Omega) \cap P_{k+2}(T_h),
\end{equation}
then we have
\begin{equation}
\|v_h\|_{0,3} \leq C \|\nabla \times v_h\|_0,
\end{equation}
and
\begin{equation}
\|\nabla \times v_h\|_{0,3} \leq C \left( \sum_{K \in T_h} \|\nabla \times (\nabla \times v_h)\|_{0,K}^2 + \sum_{F \in \mathcal{E}_h} \frac{1}{h_F^3} \|\nabla \times [v_h]\|_{0,F}^2 \right)^{1/2}.
\end{equation}
Lemma 4.4

Using a standard energy argument, one can get the following result.

Lemma 4.6. Let \((u_h, p_h)\) be the solution of (3.1), then when \(\tau > 0\) is a sufficient large constant, we have

\[
\sum_{K \in T_h} \| \nabla \times (\nabla \times u_h) \|^2_{0,K} + \sum_{F \in E_h} \frac{\tau}{h_F} \| n \times [\nabla \times u_h] \|^2_{0,F} \leq C(f, u_h).
\]

Proof. We take \(v_h = u_h \in E_h\) in (3.1a), \(q_h = p_h \in Q_h\) in (3.1b), then the combination of the two equalities leads to

\[
\sum_{K \in T_h} \| \nabla \times (\nabla \times u_h) \|^2_{0,K} - 2 \sum_{F \in E_h} \langle \langle \nabla \times (\nabla \times u_h) \rangle, n \times [\nabla \times u_h] \rangle_F + \sum_{F \in E_h} \frac{\tau}{h_F} \| n \times [\nabla \times u_h] \|^2_{0,F} = (f, u_h).
\] (4.6)

Since \(\tau > 0\) is a sufficient large constant, then we have

\[
\left| 2 \sum_{F \in E_h} \langle \langle \nabla \times (\nabla \times u_h) \rangle, n \times [\nabla \times u_h] \rangle_F \right| \leq 2 \sum_{F \in E_h} \| \langle \langle \nabla \times (\nabla \times u_h) \rangle \|_{0,F} \| n \times [\nabla \times u_h] \|_{0,F} \] \leq \frac{1}{2} \sum_{K \in T_h} \| \nabla \times (\nabla \times u_h) \|^2_{0,K} + \frac{1}{2} \sum_{F \in E_h} \| n \times [\nabla \times u_h] \|^2_{0,F}.
\] (4.7)

Combining (4.6) and (4.7) yields

\[
\sum_{K \in T_h} \| \nabla \times (\nabla \times u_h) \|^2_{0,K} + \sum_{F \in E_h} \frac{\tau}{h_F} \| n \times [\nabla \times u_h] \|^2_{0,F} \leq C(f, u_h).
\]

With the above lemma, we are ready to prove the following stability result.

Theorem 4.7. Let \((u_h, p_h)\) be the solution of (3.1), then when \(\tau > 0\) is a sufficient large constant, we have the following stability result

\[
\| u_h \|_{0,3} + \| \nabla \times u_h \|_{0,3} + \left( \sum_{K \in T_h} \| \nabla \times (\nabla \times u_h) \|^2_{0,K} + \sum_{F \in E_h} \frac{\tau}{h_F} \| n \times [\nabla \times u_h] \|^2_{0,F} \right)^{1/2} \leq C\| f \|_{0,4}.
\]

Proof. Using Lemma 4.5, one obtains

\[
\| u_h \|_{0,3} \leq C\| \nabla \times u_h \|_{0} \leq C\| \nabla \times u_h \|_{0,3}.
\] (4.8)
and

\begin{equation}
\|\nabla \times u_h\|_{0,3} \leq C \left( \sum_{K \in \mathcal{T}_h} \|\nabla \times (\nabla \times u_h)\|_{0,K}^2 + \sum_{F \in \mathcal{E}_h} \frac{\tau}{h_F^2} \|n \times [\nabla \times u_h]\|_{0,F}^2 \right)^{1/2}.
\end{equation}

Thus, using (4.8), (4.9), Lemma 4.6 and the Cauchy-Schwarz inequality, we get the stability result

\[
\|u_h\|_{0,3} + \|\nabla \times u_h\|_{0,3} + \left( \sum_{K \in \mathcal{T}_h} \|\nabla \times (\nabla \times u_h)\|_{0,K}^2 + \sum_{F \in \mathcal{E}_h} \frac{\tau}{h_F^2} \|n \times [\nabla \times u_h]\|_{0,F}^2 \right)^{1/2} \leq C \|f\|_{0,\frac{1}{2}}. \quad \Box
\]

### 4.3. $L^6$ and discrete $H^1$ stability of $\nabla \times u_h$

Next, we are ready to prove the discrete $H^1$ stability of $\nabla \times u_h$.

**Theorem 4.8.** Let $(u_h, p_h)$ be the solution of (3.1), then when $\tau > 0$ is a sufficient large constant, we have the following stability

\[
\|\nabla \times u_h\|_{0,6}^2 + \sum_{K \in \mathcal{T}_h} \|\nabla \times (\nabla \times u_h)\|_{0,K}^2 + \sum_{F \in \mathcal{E}_h} h_F^{-1} \|\nabla \times u_h\|_{0,F}^2 \\
+ \|\nabla \times (\nabla \times u_h)\|_{0,K}^2 + \sum_{F \in \mathcal{E}_h} h_F^{-1} \|n \times [\nabla \times u_h]\|_{0,F}^2 \leq C \|f\|_{0,\frac{1}{2}}^2.
\]

**Proof.** Since $u_h \in H_0(\text{curl}; \Omega)$, then there holds $\nabla \times u_h \in H(\text{div}; \Omega)$ and $\nabla \cdot (\nabla \times u_h) = 0$. Using the triangle inequality, one arrives at

\[
\|(\nabla \times u_h)\|_{0,F} = \|(n \times [\nabla \times u_h]) \times n + (n \cdot [\nabla \times u_h]) \cdot n\|_{0,F} \\
\leq \|(n \times [\nabla \times u_h]) \times n\|_{0,F} + \|(n \cdot [\nabla \times u_h]) \cdot n\|_{0,F} \\
\leq C \|n \times [\nabla \times u_h]\|_{0,F}.
\]

Using the above estimate, Theorem 4.1 with $v_h = \nabla \times u_h$, and Lemma 4.6, we have

\[
\|\nabla \times u_h\|_{0,6}^2 + \sum_{K \in \mathcal{T}_h} \|\nabla \times (\nabla \times u_h)\|_{0,K}^2 + \sum_{F \in \mathcal{E}_h} h_F^{-1} \|\nabla \times u_h\|_{0,F}^2 \\
\leq \sum_{K \in \mathcal{T}_h} \|\nabla \times (\nabla \times u_h)\|_{0,K}^2 + \sum_{F \in \mathcal{E}_h} h_F^{-1} \|\nabla \times u_h\|_{0,F}^2 \\
\leq C \sum_{K \in \mathcal{T}_h} \left( \|\nabla \times (\nabla \times u_h)\|_{0,K}^2 + \|\nabla \cdot (\nabla \times u_h)\|_{0,K}^2 \right) + \sum_{F \in \mathcal{E}_h} h_F^{-1} \|\nabla \times u_h\|_{0,F}^2 \\
= C \left[ \sum_{K \in \mathcal{T}_h} \|\nabla \times (\nabla \times u_h)\|_{0,K}^2 + \sum_{F \in \mathcal{E}_h} h_F^{-1} \|n \times [\nabla \times u_h]\|_{0,F}^2 \right] \\
\leq C(f, u_h) \\
\leq C\|f\|_{0,\frac{1}{2}} \|u_h\|_{0,3}.
\]

From Lemma 4.5, one has

\[
\|u_h\|_{0,3} \leq C\|\nabla \times u_h\|_0 \leq C\|\nabla \times u_h\|_{0,3}
\]
\[ \leq C \left( \sum_{K \in T_h} \| \nabla (\nabla \times \mathbf{u}_h) \|_{0,K}^2 + \sum_{F \in \mathcal{E}_h} h_F^{-1} \| \nabla \times \mathbf{u}_h \|_{0,F}^2 \right)^{1/2}. \]

The desired result is followed by combining the above two inequities.

4.4. Discrete inf-sup condition. The result to be presented below can be derived based on the fact \( \nabla Q_h \subset E_h \) and the fact \( \nabla \times (\nabla q_h) = 0 \) for all \( q_h \in Q_h \).

**Lemma 4.9.** The following discrete inf-sup condition

\[ \sup_{0 \neq \mathbf{v}_h \in E_h} \frac{\langle \mathbf{v}_h, \nabla q_h \rangle}{\| \mathbf{v}_h \|} \geq \| \nabla q_h \|_0, \]

holds for all \( q_h \in Q_h \), where

\[ \| \mathbf{v}_h \|^2 := \| \mathbf{v}_h \|_0^2 + \| \nabla \times \mathbf{v}_h \|_0^2 + \sum_{K \in T_h} \| \nabla \times (\nabla \times \mathbf{v}_h) \|_{0,K}^2 \]

\[ + \sum_{F \in \mathcal{E}_h} \frac{\tau}{h_F} \| n \times [\nabla \times \mathbf{v}_h] \|_{0,F}^2. \]

5. Main error estimates. In the rest parts of this paper, we will assume that the following regularity holds true:

\[ \mathbf{u} \in [H^{r_0}(\Omega)]^3, \quad \nabla \times \mathbf{u} \in [H^{r_1}(\Omega)]^3, \quad \nabla \times (\nabla \times \mathbf{u}) \in [H^{r_2}(\Omega)]^3, \quad p \in H^r(\Omega), \]

where \( r_0 \in (1/2, \infty), \ r_1 \in [1, \infty), \ r_2 \in (1/2, \infty), \) and \( r_p \in (3/2, \infty) \). We may as well assume that \( r_0 \leq r_1 \leq r_0 + 1 \) to simplify the notation in error analysis.

**Lemma 5.1.** Let \((\mathbf{u}, p)\) be the solution of (1.1), then

\[ \sum_{K \in T_h} (\nabla \times (\nabla \times \mathbf{u}), \nabla \times (\nabla \times \mathbf{v}_h))_K + (\nabla p, \mathbf{v}) \]

\[ - \sum_{F \in \mathcal{E}_h} \langle [\nabla \times (\nabla \times \mathbf{u})], n \times [\nabla \times \mathbf{v}_h] \rangle_F \]

\[ - \sum_{F \in \mathcal{E}_h} \langle [\nabla \times (\nabla \times \mathbf{v}_h)], n \times [\nabla \times \mathbf{u}] \rangle_F \]

\[ + \sum_{F \in \mathcal{E}_h} \frac{\tau}{h_F} (n \times [\nabla \times \mathbf{u}], n \times [\nabla \times \mathbf{v}_h])_F = (f, \mathbf{v}_h), \quad (u, \nabla q_h) = 0, \]

holds for all \((\mathbf{v}_h, q_h) \in E_h \times Q_h\).

**Proof.** For any \( \mathbf{v}_h \in E_h \), it follows from (1.1a) that

\[ (\nabla \times (\nabla \times (\nabla \times \mathbf{u})), \mathbf{v}_h) + (\nabla p, \mathbf{v}_h) = (f, \mathbf{v}_h). \]

By doing integration by part, we have

\[ (\nabla \times (\nabla \times (\nabla \times \mathbf{u})), \nabla \times \mathbf{v}_h) + (\nabla p, \mathbf{v}_h) = (f, \mathbf{v}_h). \]

By doing integration by part again, we have

\[ \sum_{K \in T_h} (\nabla \times (\nabla \times \mathbf{u}), \nabla \times (\nabla \times \mathbf{v}_h))_K \]
\[ \nabla \times (\nabla \times u) = \nabla \times (\nabla \times v_h) \]

where we have used the fact \( \nabla \times (\nabla \times u) = \nabla \times (\nabla \times v) \) because \( \nabla \times (\nabla \times u) \in [H^{s_2} \cap \mathcal{K}]^3 \) with \( s_{u_2} > 1/2 \). Since \( [\nabla \times u] = 0 \) because \( \nabla \times u \in [H^{s_1} \cap \mathcal{K}]^3 \) with \( s_{u_1} \geq 1 \), then (5.1a) follows immediately. For any \( q_h \in Q_h \), by (1.1b), it holds

\[ (\nabla \cdot u, q_h) = 0. \]

We use integration by parts on the above equality to get (5.1b). Thus we finish our proof. \( \square \)

**Theorem 5.2.** Let \((u, p)\) and \((u_h, p_h)\) be the solution of (1.1) and (3.1), respectively, then we have the following error estimates

\[
\|u - u_h\|_0 + \|\nabla \times (u - u_h)\|_0 + \|\nabla (p - p_h)\|_0 \\
+ \left( \sum_{K \in T_h} \|\nabla \times (\nabla \times (u - u_h))\|_{0, K}^2 + \sum_{F \in E_h} \frac{r}{h_F} \|n \times [\nabla \times (u - u_h)]\|_{0, F}^2 \right)^{1/2} \\
\leq C(h^{s_0} \|u\|_{s_0} + h^{s_1 - 1}\|\nabla \times u\|_{s_1} + h^{s_2}\|\nabla \times (\nabla \times u)\|_{s_2} + h^{s_p - 1}\|p\|_{s_p}),
\]

and the discrete \(H^1\) norm error estimate for \(\nabla \times (u - u_h)\):

\[
\left( \sum_{K \in T_h} \|\nabla \times (u - u_h)\|_{0, K}^2 + \sum_{F \in E_h} h_F^{-1}\|\nabla \times (u - u_h)\|_{0, F}^2 \right)^{1/2} \\
\leq C(h^{s_0} \|u\|_{s_0} + h^{s_1 - 1}\|\nabla \times u\|_{s_1} + h^{s_2}\|\nabla \times (\nabla \times u)\|_{s_2} + h^{s_p - 1}\|p\|_{s_p}),
\]

where \( s_{u_0} \in (1/2, \min(r_{u_0}, k+1)] \), \( s_{u_1} \in [1, \min(r_{u_1}, k+1)] \), \( s_{u_2} \in (1/2, \min(r_{u_2}, k+1)] \), \( s_p \in (3/2, \min(r_p, k+3)] \).

**Proof.** We subtract (3.1) from (5.1) to get: for all \((v_h, q_h) \in E_h \times Q_h\) it holds

\[
\sum_{K \in T_h} (\nabla \times (\nabla \times (u - u_h)), \nabla \times (\nabla \times v_h))_K + (\nabla (p - p_h), v_h) \\
- \sum_{F \in E_h} \langle \nabla \times (\nabla \times (u - u_h)) \rangle_F, n \times [\nabla \times v_h]\rangle_F \\
- \sum_{F \in E_h} \langle \nabla \times (\nabla \times v_h) \rangle_F, n \times [\nabla \times (u - u_h)]\rangle_F \\
+ \sum_{F \in E_h} \frac{r}{h_F} (n \times [\nabla \times (u - u_h)], n \times [\nabla \times v_h])_F = 0,
\]

(5.2a)

(5.2b)

\[ (u - u_h, \nabla q_h) = 0. \]

To simplify the notation, we define

\[ e^u_h := \Pi_h^E u - u_h, \quad e^p_h := \Pi_h^Q p - p_h. \]

By taking \(v_h = e^u_h \in E_h\) in (5.2a) and \(q_h = e^p_h \in Q_h\) in (5.2b), we can get

\[
\sum_{K \in T_h} (\nabla \times (\nabla \times (u - u_h)), \nabla \times (\nabla \times e^u_h))_K + (\nabla (p - p_h), e^u_h)
\]
Lemma 4.6, we arrive at

\[- \sum_{F \in \mathcal{E}_h} \langle \langle \nabla \times (\nabla \times (u - u_h)) \rangle, n \times [\nabla \times e_h^u] \rangle_F \]

\[- \sum_{F \in \mathcal{E}_h} \langle \langle \nabla \times (\nabla \times e_h^u) \rangle, n \times [\nabla \times (u - u_h)] \rangle_F \]

\[+ \sum_{F \in \mathcal{E}_h} \frac{\tau}{h_F} \langle n \times [\nabla \times (u - u_h)], n \times [\nabla \times e_h^u] \rangle_F = 0, \tag{5.3a} \]

\[(u - u_h, \nabla e_h^p) = 0, \tag{5.3b} \]

Rearranging (5.3) and by noticing that \((\Pi_h^E u - u, \nabla e_h^p) = 0\) from (3.6), it leads to

\[\sum_{K \in \mathcal{T}_h} \langle \nabla \times (\nabla \times e_h^u), \nabla \times (\nabla \times e_h^u) \rangle_K + \langle \nabla e_h^p, e_h^p \rangle \]

\[- 2 \sum_{F \in \mathcal{E}_h} \langle \langle \nabla \times (\nabla \times e_h^u) \rangle, n \times [\nabla \times e_h^u] \rangle_F \]

\[+ \sum_{F \in \mathcal{E}_h} \frac{\tau}{h_F} \langle n \times [\nabla \times e_h^u], n \times [\nabla \times e_h^u] \rangle_F \]

\[= \sum_{K \in \mathcal{T}_h} \langle \nabla \times (\nabla \times (\Pi_h^E u - u)), \nabla \times (\nabla \times e_h^u) \rangle_K + \langle \nabla (\Pi_h^Q p - p), e_h^u \rangle \]

\[- \sum_{F \in \mathcal{E}_h} \langle \langle \nabla \times (\nabla \times (\Pi_h^E u - u)) \rangle, n \times [\nabla \times e_h^u] \rangle_F \]

\[- \sum_{F \in \mathcal{E}_h} \langle \langle \nabla \times (\nabla \times e_h^u) \rangle, n \times [\nabla \times (\Pi_h^E u - u)] \rangle_F \]

\[+ \sum_{F \in \mathcal{E}_h} \frac{\tau}{h_F} \langle n \times [\nabla \times (\Pi_h^E u - u)], n \times [\nabla \times e_h^u] \rangle_F, \tag{5.4a} \]

\[(e_h^u, \nabla e_h^p) = 0, \tag{5.4b} \]

Utilizing (5.4) and the similar argument in the proof of Lemma 4.6, we arrive at

\[\frac{1}{2} \sum_{K \in \mathcal{T}_h} \| \nabla \times (\nabla \times e_h^u) \|^2_{0,K} + \frac{1}{2} \sum_{F \in \mathcal{E}_h} \frac{\tau}{h_F} \| n \times [\nabla \times e_h^u] \|^2_{0,F} \]

\[\leq \sum_{K \in \mathcal{T}_h} \langle \nabla \times (\nabla \times (\Pi_h^E u - u)), \nabla \times (\nabla \times e_h^u) \rangle_K + \langle \nabla (\Pi_h^Q p - p), e_h^u \rangle \]

\[- \sum_{F \in \mathcal{E}_h} \langle \langle \nabla \times (\nabla \times (\Pi_h^E u - u)) \rangle, n \times [\nabla \times e_h^u] \rangle_F \]

\[- \sum_{F \in \mathcal{E}_h} \langle \langle \nabla \times (\nabla \times e_h^u) \rangle, n \times [\nabla \times (\Pi_h^E u - u)] \rangle_F \]

\[+ \sum_{F \in \mathcal{E}_h} \frac{\tau}{h_F} \langle n \times [\nabla \times (\Pi_h^E u - u)], n \times [\nabla \times e_h^u] \rangle_F \]

\[= : R_1 + R_2 + R_3 + R_4 + R_5. \tag{5.5} \]

Now we estimate \(\{R_i\}_{i=1}^5\) in details. Let \(\Pi_{h,k}\) be the \(L^2\)-projection from \(L^2(\Omega)\) to space \(P_k(T_h)^3\). Then by the triangle inequality, the inverse inequality and the estimate (3.2b), there holds

\[\| \nabla \times (\nabla \times \Pi_{h,k+1}^\text{pair} u) - \nabla \times (\nabla \times u) \|_{0,K} \]
\[ \leq \| \nabla \times (\nabla \times \Pi_{h,k+1}^{\text{curl}} u - \Pi_{h,k} (\nabla \times u)) \|_{0,K} + \| \nabla \times (\Pi_{h,k} (\nabla \times u) - \nabla \times u) \|_{0,K} \]
\[ \leq C h_{K}^{-1} \| \nabla \times \Pi_{h,k+1}^{\text{curl}} u - \Pi_{h,k} (\nabla \times u) \|_{0,K} + \| \Pi_{h,k} (\nabla \times u) - \nabla \times u \|_{1,K} \]
\[ \leq C h_{K}^{s_{u} - 1} \| \nabla \times u \|_{s_{u},K}. \]
(5.6)

From the above estimate, we get

\[ |R_1| \leq \sum_{K \in T_h} (\nabla \times (\nabla \times \Pi_{h,k+1}^{\text{curl}} u) - \nabla \times (\nabla \times u), \nabla \times (\nabla \times e_h^u))_K \quad \text{by (3.4)} \]
\[ \leq C h_{K}^{s_{u} - 1} \| \nabla \times e_h^u \|_{0,K} \quad \text{by (5.6)}. \]
(5.7)

We use the approximation property for \( \Pi_{h}^{Q} \) in (3.3) and Lemma 4.5 to get

\[ R_2 \leq C h_{K}^{s_{u} - 1} \| p \|_{s_p} \| e_h^u \|_0 \]
\[ \leq C h_{K}^{s_{u} - 1} \| p \|_{s_p} \| e_h^u \|_{0,3} \]
\[ \leq C h_{K}^{s_{u} - 1} \| p \|_{s_p} \| \nabla \times e_h^u \|_0 \]
\[ \leq C h_{K}^{s_{u} - 1} \| p \|_{s_p} \| \nabla \times e_h^u \|_{0,3} \]
\[ \leq C h_{K}^{s_{u} - 1} \| p \|_{s_p} \left( \sum_{K \in T_h} \| \nabla \times (\nabla \times e_h^u) \|_{0,K}^2 + \sum_{F \in E_h} \frac{1}{h_F} \| n \times [\nabla \times e_h^u] \|_{0,F}^2 \right)^{1/2}. \]
(5.8)

Using the triangle inequality, the inverse inequality, and the approximation property of \( \Pi_{h,k+1}^{\text{curl}} \) in (3.2b), one has

\[ \sum_{F \in E_h} h_F \| \nabla \times (\nabla \times (\Pi_{h,k+1}^{\text{curl}} u - u)) \|_{0,F}^2 \]
\[ \leq 2 \sum_{F \in E_h} h_F \| \nabla \times (\nabla \times \Pi_{h,k+1}^{\text{curl}} u) - \Pi_{h,k} \nabla \times (\nabla \times u) \|_{0,F}^2 \]
\[ + 2 \sum_{F \in E_h} h_F \| \Pi_{h,k} \nabla \times (\nabla \times u) - \nabla \times (\nabla \times u) \|_{0,F}^2 \]
\[ \leq C \sum_{K \in T_h} \| \nabla \times (\nabla \times \Pi_{h,k+1}^{\text{curl}} u) - \Pi_{h,k} \nabla \times (\nabla \times u) \|_{0,K}^2 \]
\[ + Ch_{K}^{2s_{u} - 2} \| \nabla \times (\nabla \times u) \|_{s_{u},2}^2 \]
\[ \leq C \sum_{K \in T_h} \| \nabla \times (\nabla \times \Pi_{h,k+1}^{\text{curl}} u) - \nabla \times (\nabla \times u) \|_{0,K}^2 \]
\[ + C \sum_{K \in T_h} \| \nabla \times (\nabla \times u) - \Pi_{h,k} \nabla \times (\nabla \times u) \|_{0,K}^2 \]
\[ + Ch_{K}^{2s_{u} - 2} \| \nabla \times (\nabla \times u) \|_{s_{u},2}^2 \]
\[ \leq C (h_{K}^{s_{u} - 1} \| \nabla \times u \|_{s_{u},1} + h_{K}^{s_{u}} \| \nabla \times (\nabla \times u) \|_{s_{u},2}^2 \). \]
We use the above estimate to get
\[
|R_3| = \left| \sum_{F \in \mathcal{E}_h} \left( \parallel \nabla \times (\Pi_{h,k+1}^\text{curl} u - u) \parallel \cdot \parallel \nabla \times e_h^p \parallel \right)_F \right|
\leq C \left( h^{s_{u_1} - 1} \parallel \nabla \times u \parallel_{s_{u_1}} + h^{s_{u_2}} \parallel \nabla \times \nabla \times u \parallel_{s_{u_2}} \right)
\times \left( \sum_{F \in \mathcal{E}_h} \frac{\tau}{h_F} \parallel \nabla \times e_h^p \parallel_{0,F}^2 \right)^{1/2}.
\]
(5.9)

Again, by the triangle inequality, the approximation property of \( \Pi_{h,k+1}^\text{curl} \) in (3.2b), it holds
\[
\sum_{F \in \mathcal{E}_h} h_F^{-1} \parallel \nabla \times (\Pi_{h,k+1}^\text{curl} u - u) \parallel_{0,F}^2
\leq 2 \sum_{F \in \mathcal{E}_h} h_F^{-1} \parallel \nabla \times (\Pi_{h,k+1}^\text{curl} u - \Pi_{h,k} \nabla \times u) \parallel_{0,F}^2
+ 2 \sum_{F \in \mathcal{E}_h} h_F^{-1} \parallel \Pi_{h,k} \nabla \times u - \nabla \times u \parallel_{0,F}^2
\leq C \sum_{K \in \mathcal{T}_h} h_K^{-2} \parallel \nabla \times (\Pi_{h,k+1}^\text{curl} u - \Pi_{h,k} \nabla \times u) \parallel_{0,K}^2
+ Ch^{2(s_{u_1} - 1)} \parallel \nabla \times u \parallel_{s_{u_1}}^2
\leq C \sum_{K \in \mathcal{T}_h} h_K^{-2} \parallel \nabla \times (\Pi_{h,k+1}^\text{curl} u - \nabla \times u) \parallel_{0,K}^2
+ C \sum_{K \in \mathcal{T}_h} h_K^{-2} \parallel \nabla \times u - \Pi_{h,k} \nabla \times u \parallel_{0,K}^2
+ Ch^{2(s_{u_1} - 1)} \parallel \nabla \times u \parallel_{s_{u_1}}^2
\leq Ch^{2(s_{u_1} - 1)} \parallel \nabla \times u \parallel_{s_{u_1}}^2.
\]

Using the above estimate and the inverse inequality, it arrives
\[
|R_4| = \left| \sum_{F \in \mathcal{E}_h} \left( \parallel \nabla \times (\nabla \times e_h^p) \parallel \cdot \parallel \nabla \times (\Pi_{h,k+1}^\text{curl} u - u) \parallel \right)_F \right|
\leq Ch^{s_{u_1} - 1} \parallel \nabla \times u \parallel_{s_{u_1}} \left( \sum_{K \in \mathcal{T}_h} \parallel \nabla \times (\nabla \times e_h^p) \parallel_0 \right)^{1/2},
\]
(5.10)

and
\[
|R_5| = \left| \sum_{F \in \mathcal{E}_h} \frac{\tau}{h_F} \left( \parallel \nabla \times (\Pi_{h,k+1}^\text{curl} u - u) \parallel \cdot \parallel \nabla \times e_h^p \parallel \right)_F \right|
\leq Ch^{s_{u_1} - 1} \parallel \nabla \times u \parallel_{s_{u_1}} \left( \sum_{F \in \mathcal{E}_h} \frac{\tau}{h_F} \parallel \nabla \times e_h^p \parallel_{0,F}^2 \right)^{1/2}.
\]
(5.11)

Thus, from (5.5), (5.7), (5.8), (5.9) and (5.10), one can get
\[
\sum_{K \in \mathcal{T}_h} \parallel \nabla \times (\nabla \times e_h^p) \parallel_{0,K}^2
+ \sum_{F \in \mathcal{E}_h} \frac{\tau}{h_F} \parallel \nabla \times e_h^p \parallel_{0,F}^2
\leq C(h^{s_{u_1} - 1} \parallel \nabla \times u \parallel_{s_{u_1}} + h^{s_{u_2}} \parallel \nabla \times (\nabla \times u) \parallel_{s_{u_2}} + h^{s_p - 1} \parallel p \parallel_{s_p}).
\]
We conclude the result for the estimates of \( u - u_h \) by the fact

\[
\| e_h^u \|_0 + \| \nabla \times e_h^u \|_0 \\
\leq C \left( \sum_{K \in T_h} \| \nabla \times (\nabla \times e_h^u) \|_{0,K}^2 + \sum_{F \in E_h} \frac{T}{h_F} \| n \times [\nabla \times e_h^u] \|_{0,F}^2 \right)^{1/2},
\]

from Lemma 4.5 and the triangle inequality. Using (5.4a) and the estimates for \( u - u_h \) above, one can get

\[
-(\nabla e_h^p, v_h) = \sum_{K \in T_h} (\nabla \times (\nabla \times (u - u_h)), \nabla \times (\nabla \times v_h))_K + \langle \nabla(p - \Pi_h^Q p), v_h \rangle \\
- \sum_{F \in E_h} \langle \nabla \times (\nabla \times (u - u_h)), n \times [\nabla \times v_h] \rangle_F \\
- \sum_{F \in E_h} \langle \nabla \times (\nabla \times v_h), n \times [\nabla \times (u - u_h)] \rangle_F \\
+ \sum_{F \in E_h} \frac{T}{h_F} (n \times [\nabla \times (u - u_h)], n \times [\nabla \times v_h])_F \\
\leq C(h^{s_u_0} \| u \|_{s_u_0} + h^{s_u_1-1} \| \nabla \times u \|_{s_u_1}) \| v_h \| \\
+ C(h^{s_p_2} \| \nabla \times (\nabla \times u) \|_{s_p_2} + h^{s_p-1} \| p \|_{s_p}) \| v_h \|.
\]

By the discrete inf-sup condition (4.10) and (5.12), one can get

\[
\| \nabla e_h^p \|_0 \leq \sup_{0 \neq v_h \in E_h} \frac{\langle v_h, \nabla e_h^p \rangle}{\| v_h \|} \\
\leq C(h^{s_u_0} \| u \|_{s_u_0} + h^{s_u_1-1} \| \nabla \times u \|_{s_u_1} + h^{s_p_2} \| \nabla \times (\nabla \times u) \|_{s_p_2} + h^{s_p-1} \| p \|_{s_p}) \quad \square
\]

A triangle inequality will finish the proof of estimates for \( \nabla(p - p_h) \).

6. \( H(\text{curl}) \) error estimate. To derive the \( H(\text{curl}) \) error estimate, we need the following dual problem: find \((\Phi, \Psi)\) such that

\[
(6.1a) \quad \nabla \times (\nabla \times (\nabla \times \Phi))) + \nabla \Psi = 0 \quad \text{in } \Omega, \\
(6.1b) \quad \nabla \cdot \Phi = 0 \quad \text{in } \Omega, \\
(6.1c) \quad n \times \Phi = 0 \quad \text{on } \partial \Omega, \\
(6.1d) \quad n \times (\nabla \times \Phi) = 0 \quad \text{on } \partial \Omega, \\
(6.1e) \quad \Psi = 0 \quad \text{on } \partial \Omega.
\]

From Theorem 2.2, when \( \nabla \cdot \Phi = 0 \), we have \( \Psi = 0 \). We assume the following regularity

\[
(6.2) \quad \| \Phi \|_{\beta} + \| \nabla \times (\nabla \times \Phi) \|_{\beta} + \| \nabla \times \Phi \|_{1+\gamma} \leq C_{\text{reg}} \| \Theta \|_0.
\]

in which \( \beta \in (1/2, 1] \), \( \gamma \in [0, 1] \), \( \gamma \leq \beta \), and \( C_{\text{reg}} \) is a constant independent of mesh size. We notice that when \( \Omega \) is convex, there holds \( \gamma = \beta = 1 \) from the regularity result in [27].

**Lemma 6.1.** Let \((u, p)\) and \((u_h, p_h)\) be the solution of (1.1) and (3.1), respectively, and let \( \nabla \cdot \Theta = 0 \). Then we have the following error estimates

\[
(\Theta, u - u_h) \leq Ch^\sigma (h^{s_u_0} \| u \|_{s_u_0} + h^{s_u_1-1} \| \nabla \times u \|_{s_u_1} + h^{s_p-1} \| p \|_{s_p}) \| \Theta \|_0,
\]

where \( \sigma = \min(\beta, \gamma) \), and \( \beta, \gamma \) are defined in (6.2).
Lemma 5.1, we have the following equations:

\[
\sum_{K \in \mathcal{T}_h} (\nabla \times (\nabla \times \Phi), \nabla \times (\nabla \times (u - u_h)))_K + (\nabla \Psi, (u - u_h)) \\
- \sum_{F \in \mathcal{E}_h} (\nabla \times (\nabla \times \Phi), n \times [\nabla \times (u - u_h)])_F \\
- \sum_{F \in \mathcal{E}_h} (\nabla \times (\nabla \times (u - u_h)), n \times [\nabla \times \Phi])_F \\
(6.3a) \\
+ \sum_{F \in \mathcal{E}_h} \frac{\tau}{h_F} (n \times [\nabla \times \Phi], n \times [\nabla \times (u - u_h)])_F = (\Theta, (u - u_h)), \\
(6.3b) \\
(\Phi, \nabla q) = 0.
\]

We use the fact \(\Psi = 0\), we can get

\[
(\Theta, u - u_h) = \sum_{K \in \mathcal{T}_h} (\nabla \times (\nabla \times (u - u_h)), \nabla \times (\nabla \times \Phi))_K \\
- \sum_{F \in \mathcal{E}_h} (\nabla \times (\nabla \times \Phi), n \times [\nabla \times (u - u_h)])_F \\
- \sum_{F \in \mathcal{E}_h} (\nabla \times (\nabla \times (u - u_h)), n \times [\nabla \times \Phi])_F \\
(6.4) \\
+ \sum_{F \in \mathcal{E}_h} \frac{\tau}{h_F} (n \times [\nabla \times \Phi], n \times [\nabla \times (u - u_h)])_F.
\]

Let \((\Phi_h, \Psi_h) \in E_h \times Q_h\) be the solution of the following system:

\[
\sum_{K \in \mathcal{T}_h} (\nabla \times (\nabla \times \Phi_h), \nabla \times (\nabla \times v_h))_K + (\nabla \Psi_h, v_h) \\
- \sum_{F \in \mathcal{E}_h} (\nabla \times (\nabla \times \Phi_h), n \times [\nabla \times v_h])_F \\
- \sum_{F \in \mathcal{E}_h} (\nabla \times (\nabla \times v_h), n \times [\nabla \times \Phi_h])_F \\
(6.5a) \\
+ \sum_{F \in \mathcal{E}_h} \frac{\tau}{h_F} (n \times [\nabla \times \Phi_h], n \times [\nabla \times v_h])_F = (f, v_h), \\
(6.5b) \\
(\Phi_h, \nabla q_h) = 0,
\]

holds for all \((v_h, q_h) \in E_h \times Q_h\). Using Theorem 5.2, and the fact \(\Psi = 0\), we can get the error estimate for \((\Phi_h, \Psi_h)\) in (3.1):

\[
\|\Phi - \Phi_h\|_0 + \|\nabla \times (\Phi - \Phi_h)\|_0 + \|\nabla (\Psi - \Psi_h)\|_0 \\
+ \sum_{K \in \mathcal{T}_h} \|\nabla \times (\nabla \times (\Phi_h))\|_{0,K}^2 + \sum_{F \in \mathcal{E}_h} \frac{\tau}{h_F} \|n \times [\nabla \times (\Phi - \Phi_h)]\|_{0,F}^2 \\
\leq C [h^\beta (\|\Phi\|_\beta + \|\nabla \times (\nabla \times \Phi)\|_\beta) + h^\gamma \|\nabla \times \Phi\|_{1+\gamma}].
\]

From (5.2), we can get

\[
\sum_{K \in \mathcal{T}_h} (\nabla \times (\nabla \times (u - u_h)), \nabla \times (\nabla \times \Phi_h))_K + (\nabla (p - p_h), \Phi_h)
\]
Using the Cauchy-Schwarz inequality, the triangle inequality, the inverse inequality,
\begin{equation}
(6.7a)
\end{equation}
and the error estimate in \((6.8)\), one can get
\begin{equation}
(6.7b)
\end{equation}
By comparing \((6.4)\) and \((6.7)\), and the fact \((\nabla (p - p_h), \Phi_h) = (\nabla (p - p_h), \Phi_h - \Phi)\),
we get
\begin{equation}
(\Theta, u - u_h) = \sum_{K \in T_h} (\nabla \times (\nabla \times (u - u_h)), \nabla \times (\nabla \times (\Phi - \Phi_h)))_K
- \sum_{F \in E_h} (\nabla \times (\nabla \times (\Phi - \Phi_h))), n \times [\nabla \times (u - u_h)]_F
- \sum_{F \in E_h} (\nabla \times (\nabla \times (u - u_h))), n \times [\nabla \times (\Phi - \Phi_h)]_F
+ \sum_{F \in E_h} \frac{\tau}{h_F} (n \times [\nabla \times (\Phi - \Phi_h)], n \times [\nabla \times (u - u_h)]_F
+ (\nabla (p - p_h), \Phi - \Phi_h)
\end{equation}
Now we estimate \(\{T_i\}_{i=1}^5\) in detail. To simply the notation, we define
\[
M := (h^{s_u} \| u \|_{s_u} + h^{s_u-1} \| \nabla \times u \|_{s_u} + h^{s_u+2} \| \nabla \times (\nabla \times u) \|_{s_u} + h^{s_p-1} \| p \|_{s_p}).
\]
We use the Cauchy-Schwarz inequality, the estimate in \textbf{Theorem 5.2}, the error estimate \((6.6)\) and the regularity \((6.2)\) to get
\begin{align}
|T_1| &\leq \left( \sum_{K \in T_h} \| \nabla \times (\nabla \times (u - u_h)) \|_{0,K} \right)^{1/2} \left( \sum_{K \in T_h} \| \nabla \times (\nabla \times (\Phi - \Phi_h)) \|_{0,K} \right)^{1/2}
\leq CM (h^\beta (\| \Phi \|_\beta + \| \nabla \times (\nabla \times \Phi) \|_\beta) + h^\gamma \| \nabla \times \Phi \|_{1+\gamma})
\leq Ch^\gamma M \| \Theta \|_0.
\end{align}
Using the Cauchy-Schwarz inequality, the triangle inequality, the inverse inequality, the error estimate in \((6.6)\), \((5.2)\), and the regularity \((6.2)\), we get
\begin{align}
|T_2| &\leq \sum_{F \in E_h} h_F^{1/2} \| \nabla \times (\nabla \times (\Phi - \Phi_h)) \|_{0,F} h_F^{-1/2} \| n \times [\nabla \times (u - u_h)] \|_{0,F}
\leq \sum_{F \in E_h} \| \nabla \times (\nabla \times \Phi - \Pi_{h,k+1} \nabla \times \Phi) \|_{0,F} \| n \times [\nabla \times (u - u_h)] \|_{0,F}
+ \sum_{F \in E_h} \| \nabla \times (\Pi_{h,k+1} \nabla \times \Phi) - \nabla \times \Phi_h \|_{0,F} \| n \times [\nabla \times (u - u_h)] \|_{0,F}
\leq Ch^\beta M \| \nabla \times \Phi \|_\beta + Ch^\gamma M \| \nabla \times \Phi \|_{1+\gamma}.
\end{align}
\[ + CM \left( \sum_{K \in T_h} \| \nabla \times (\Pi_{\text{curl}}^{h+1}(\nabla \times \Phi) - \nabla \times \Phi_h) \|^0_{0,K} \right)^{1/2} \leq CM \left( h^\beta (\| \Phi \|_\beta + \| \nabla \times (\nabla \times \Phi) \|_\beta) + h^\gamma \| \nabla \times \Phi \|_{1+\gamma} \right) \leq Ch^\sigma M \| \Theta \|^0_0. \]

(6.10)

Similar to (6.10), one can get
\[ |T_3| \leq \sum_{F \in E_h} \| \nabla \times (\nabla \times (u - u_h)) \|_{0,F} \| n \times \nabla \times (\Phi - \Phi_h) \|^0_{0,F} \leq \sum_{F \in E_h} \| \nabla \times (\nabla \times u - \Pi_{\text{curl}}^{h+1}(\nabla \times u)) \|_{0,F} \| n \times \nabla \times (\Phi - \Phi_h) \|^0_{0,F} + \sum_{F \in E_h} \| \nabla \times (\Pi_{\text{curl}}^{h+1}((\nabla \times u) - \nabla \times u_h)) \|_{0,F} \| n \times \nabla \times (\Phi - \Phi_h) \|^0_{0,F} \leq CM \left( h^\beta (\| \Phi \|_\beta + \| \nabla \times (\nabla \times \Phi) \|_\beta) + h^\gamma \| \nabla \times \Phi \|_{1+\gamma} \right) \leq Ch^\sigma M \| \Theta \|^0_0, \]

(6.11)

and
\[ |T_4| \leq CM \left( h^\beta (\| \Phi \|_\beta + \| \nabla \times (\nabla \times \Phi) \|_\beta) + h^\gamma \| \nabla \times \Phi \|_{1+\gamma} \right) \leq Ch^\sigma M \| \Theta \|^0_0. \]

(6.12) We also use the Cauchy-Schwarz inequality, the estimate in Theorem 5.2, the error estimate in (6.6) and the regularity (6.2) to get
\[ |T_5| \leq \| \nabla p - \nabla p_h \|_0 \| \Phi - \Phi_h \|_0 \leq CM \left( h^\beta (\| \Phi \|_\beta + \| \nabla \times (\nabla \times \Phi) \|_\beta) + h^\gamma \| \nabla \times \Phi \|_{1+\gamma} \right) \leq Ch^\sigma M \| \Theta \|^0_0. \]

(6.13)

The desired result follows by (6.8), (6.9), (6.10), (6.11), (6.12) and (6.13).

In the following two subsection we are going to estimate \( \| u - u_h \|_0 \) and \( \| \nabla \times (u - u_h) \|_0 \) based on dual argument, thus, we will obtain our proof of \( \mathcal{H}(\text{curl}; \Omega) \) error estimates.

6.1. \( L^2 \) error estimate.

THEOREM 6.2. Let \( (u, p) \) and \( (u_h, p_h) \) be the solution of (1.1) and (3.1), respectively, then we have the following error estimates
\[ \| u - u_h \|_0 \leq Ch^\sigma (h^{s_1-1} \| \nabla \times u \|_{s_{u_1}} + h^{s_2} \| \nabla \times (\nabla \times u) \|_{s_{u_2}} + h^{s-p-1} \| p \|_{s_p}) + Ch^{s_0} (\| u \|_{s_{u_0}} + \| \nabla \times u \|_{s_{u_0}}), \]

where \( \sigma = \min(\alpha, \beta, \gamma) \), \( \alpha \) is defined in Lemma 2.1, \( \beta \) and \( \gamma \) are defined in (6.2).
Proof. We take $\Theta$ as: find $\Theta \in H(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$ satisfying
\[
\nabla \times \Theta = \nabla \times (\Pi_h^E u - u_h) \quad \text{in } \Omega, \\
\nabla \cdot \Theta = 0 \quad \text{in } \Omega, \\
n \times \Theta = 0 \quad \text{on } \partial \Omega.
\]
It follows (3.1a) and (6.1) that
\[
(\Pi_h^E u - u_h, \nabla q_h) = (u, \nabla q_h) = - (\nabla \cdot u, q_h) = 0,
\]
for all $q_h \in Q_h$. Due to the result in [12, Lemma 4.5] one has
\[
(6.15) \quad \| (\Pi_h^E u - u_h) - \Theta \|_0 \leq C h^\alpha \| \nabla \times (\Pi_h^E u - u_h) \|_0.
\]
in which $\alpha$ is defined in Lemma 2.1. Thus, from the triangle inequality, the estimate (6.15), and $\sigma \leq \alpha$, one can get
\[
\| \Theta \|_0 \leq \| (\Pi_h^E u - u_h) - \Theta \|_0 + \| \Pi_h^E u - u_h \|_0 \\
\leq C h^\alpha \| \nabla \times (\Pi_h^E u - u_h) \|_0 + \| u - u_h \|_0 \\
+ C s_{s_0} (\| u \|_{s_{s_0}} + \| \nabla \times u \|_{s_{s_0}}) \\
\leq C h^\sigma M + C h^{s_{s_0}} (\| u \|_{s_{s_0}} + \| \nabla \times u \|_{s_{s_0}}) + \| u - u_h \|_0,
\]
From the triangle inequality, (6.16), Lemma 6.1 and (6.15), it arrives
\[
\| u - u_h \|_0^2 = (u - u_h, u - u_h) \\
= (\Theta, u - u_h) + ((\Pi_h^E u - u_h) - \Theta, u - u_h) + (u - \Pi_h^E u, u - u_h) \\
\leq C h^\sigma M \| \Theta \|_0 + C h^\sigma \| \nabla \times (\Pi_h^E u - u_h) \|_0 \| u - u_h \|_0 \\
+ C s_{s_0} (\| u \|_{s_{s_0}} + \| \nabla \times u \|_{s_{s_0}}) \| u - u_h \|_0 \\
\leq C h^{2\sigma} M^2 + C h^{2s_{s_0}} (\| u \|_{s_{s_0}}^2 + \| \nabla \times u \|_{s_{s_0}}^2) + \frac{1}{2} \| u - u_h \|_0^2,
\]
which leads to
\[
\| u - u_h \|_0 \leq C (h^\sigma (h^{s_{s_1}} - 1) \| \nabla \times u \|_{s_{s_1}} + h^s_{s_2} \| p \|_{s_2}) + h^s_{s_0} (\| u \|_{s_{s_0}} + \| \nabla \times u \|_{s_{s_0}}).
\]
Therefore, we finish our proof. \ \Box

6.2. Curl operator error estimate.

Theorem 6.3. Let $(u, p)$ and $(u_h, p_h)$ be the solution of (1.1) and (3.1), respectively, then we have the following error estimates
\[
\| \nabla \times (u - u_h) \|_0 \\
\leq C h^\sigma (h^{s_{s_0}} \| u \|_{s_{s_0}} + h^{s_{s_1}} \| \nabla \times u \|_{s_{s_1}} + h^{s_{s_2}} \| \nabla \times (\nabla \times u) \|_{s_{s_2}} + h^{s_{s_2}} \| p \|_{s_2}),
\]
where $\sigma = \min(\alpha, \beta, \gamma)$, $\alpha$ is defined in Lemma 2.1, $\beta$ and $\gamma$ are defined in (6.2).

Proof. We take $\Theta$ as: find $\Theta \in H(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$ satisfying
\[
\nabla \times \Theta = \nabla \times (\Pi_{h,k+1}^\text{curl} (\nabla \times u) - \Pi_{h,k+1}^\text{curl} (\nabla \times u_h)) \quad \text{in } \Omega, \\
\nabla \cdot \Theta = 0 \quad \text{in } \Omega,
\]
\( n \times \Theta = 0 \) on \( \partial \Omega \).

Integration by parts yields the following equality

\[
(\nabla \times (\Pi_{h,k+1}^{\text{curl}}(\nabla \times u) - \Pi_{h,k+1}^{\text{curl},c}(\nabla \times u_h)), \nabla q_h) = 0,
\]

for all \( q_h \in Q_h \). Again, due to the result in [12, Lemma 4.5] one has

\[
\|(\Pi_{h,k+1}^{\text{curl}}(\nabla \times u) - \Pi_{h,k+1}^{\text{curl},c}(\nabla \times u_h)) - \Theta\|_0
\]

\((6.17)\)

\[
\leq Ch^\alpha \|\nabla \times (\Pi_{h,k+1}^{\text{curl}}(\nabla \times u) - \Pi_{h,k+1}^{\text{curl},c}(\nabla \times u_h))\|_0.
\]

where \( \alpha \) is defined in Lemma 2.1. Utilizing the triangle inequality and (6.17), one can get

\[
\|(\Pi_{h,k+1}^{\text{curl}}(\nabla \times u) - \Pi_{h,k+1}^{\text{curl},c}(\nabla \times u_h)) - \Theta\|_0
\]

\[
\leq Ch^\sigma \|\nabla \times (\Pi_{h,k+1}^{\text{curl}}(\nabla \times u) - \Pi_{h,k+1}^{\text{curl},c}(\nabla \times u_h))\|_0
\]

\[
+ \|\Pi_{h,k+1}^{\text{curl},c}(\nabla \times u) - \Pi_{h,k+1}^{\text{curl},c}(\nabla \times u_h)\|_0
\]

\((6.18)\) \( \leq Ch^\sigma \mathcal{M} + \|\nabla \times u - \nabla \times u_h\|_0. \)

where we have used the fact \( \sigma \leq 1 \). Integration by parts gives

\[
\|\nabla \times (u - u_h)\|_0^2 = (\nabla \times (u - u_h), \nabla \times (u - u_h))
\]

\[
= (\nabla \times u - \Pi_{h,k+1}^{\text{curl}}(\nabla \times u), \nabla \times (u - u_h))
\]

\[
+ (\Pi_{h,k+1}^{\text{curl}}(\nabla \times u) - \Pi_{h,k+1}^{\text{curl},c}(\nabla \times u_h), \nabla \times (u - u_h))
\]

\[
+ (\Pi_{h,k+1}^{\text{curl},c}(\nabla \times u) - \nabla \times u_h, \nabla \times (u - u_h))
\]

\[
= (\nabla \times u - \Pi_{h,k+1}^{\text{curl}}(\nabla \times u), \nabla \times (u - u_h))
\]

\[
+ (\nabla \times (\Pi_{h,k+1}^{\text{curl}}(\nabla \times u) - \Pi_{h,k+1}^{\text{curl},c}(\nabla \times u_h)), u - u_h)
\]

\[
+ (\Pi_{h,k+1}^{\text{curl},c}(\nabla \times u) - \nabla \times u_h, \nabla \times (u - u_h))
\]

\((6.19)\)

\[
= S_1 + S_2 + S_3.
\]

It follows from the Cauchy-Schwarz inequality that

\[
|S_1| \leq Ch^{s_1} \|\nabla \times u\|_{s_1} \|\nabla \times (u - u_h)\|_0 \leq Ch^{2\sigma} \mathcal{M}^2 + \frac{1}{4} \|\nabla \times (u - u_h)\|_0^2.
\]

We use the result in Lemma 6.1 and the estimate (6.18) to get

\[
|S_2| = \langle \Theta, u - u_h \rangle
\]

\[
\leq Ch^\sigma \mathcal{M} \|\Theta\|_0
\]

\((6.20)\)

\[
\leq Ch^{2\sigma} \mathcal{M}^2 + \frac{1}{4} \|\nabla \times (u - u_h)\|_0^2.
\]
Using the approximation property $\Pi_{h,k+1}^{\text{curl,e}}$, one has
\[
|S_3| \leq Ch \left( \sum_{F \in \mathcal{E}_h} \frac{1}{h_F} \|n \times [u_h]\|_F^2 \right)^{1/2} \| \nabla \times (u - u_h) \|_0
\]
\[
= Ch \left( \sum_{F \in \mathcal{E}_h} \frac{1}{h_F} \|n \times [u - u_h]\|_F^2 \right)^{1/2} \| \nabla \times (u - u_h) \|_0
\]
\[
\leq ChM \| \nabla \times (u - u_h) \|_0
\]
\[
\leq Ch^2 M^2 + \frac{1}{4} \| \nabla \times (u - u_h) \|_0^2.
\]
Thus, the desired result can be obtained from (6.19), (6.20), (6.21) and (6.22).

When $\Omega$ is convex and then solution of (1.1) is sufficient smooth, from the previous results, we can get the following optimal error estimates in $H(\text{curl})$ norm.

**Proposition 6.4.** Let $(u, p)$ and $(u_h, p_h)$ be the solution of (1.1) and (3.1), respectively, when $\Omega$ is convex and $(u, p)$ are sufficient smooth, then we have the following error estimates
\[
\|u - u_h\|_0 + \| \nabla \times (u - u_h) \|_0 \leq Ck^{k+1}(\|u\|_{k+1} + \|\nabla \times u\|_{k+1} + \|p\|_{k+1}).
\]

**REFERENCES**

[1] A. Alonso and A. Valli, *An optimal domain decomposition preconditioner for low-frequency time-harmonic Maxwell equations*, Math. Comp., 68 (1999), pp. 607–631.

[2] C. Amrouche, C. Bernardi, M. Dauge, and V. Girault, *Vector potentials in three-dimensional non-smooth domains*, Math. Methods Appl. Sci., 21 (1998), pp. 823–864.

[3] N. Ben Salah, A. Soulaimani, and W. G. Habashi, *A finite element method for magneto-hydrodynamics*, Comput. Methods Appl. Mech. Engrg., 190 (2001), pp. 5867–5892.

[4] S. C. Brenner, J. Sun, and L.-Y. Sung, *Hodge decomposition methods for a quad-curl problem on planar domains*, J. Sci. Comput., 73 (2017), pp. 495–513.

[5] H. Chen, W. Qiu, and K. Shi, *A priori and computable a posteriori error estimates for an HDG method for the coercive Maxwell equations*, Comput. Methods Appl. Mech. Engrg., 333 (2018), pp. 287–310.

[6] M. Dauge, *Elliptic boundary value problems on corner domains*, vol. 1341 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1988. Smoothness and asymptotics of solutions.

[7] D. A. Di Pietro and A. Ern, *Discrete functional analysis tools for discontinuous Galerkin methods with application to the incompressible Navier-Stokes equations*, Math. Comp., 79 (2010), pp. 1303–1330.

[8] J. Douglas, Jr., T. Dupont, P. Percell, and R. Scott, *A family of $C^1$ finite elements with optimal approximation properties for various Galerkin methods for 2nd and 4th order problems*, RAIRO Anal. Numér., 13 (1979), pp. 227–255.

[9] H. Gao and W. Qiu, *A semi-implicit energy conserving finite element method for the dynamical incompressible magnetohydrodynamics equations*, Computer Methods in Applied Mechanics and Engineering, (2018).

[10] C. Greif, D. Li, D. Schötzau, and X. Wei, *A mixed finite element method with exactly divergence-free velocities for incompressible magnetohydrodynamics*, Comput. Methods Appl. Mech. Engrg., 199 (2010), pp. 2840–2855.

[11] J.-L. Guermond, R. Laguerre, J. Léorat, and C. Nore, *An interior penalty Galerkin method for the MHD equations in heterogeneous domains*, J. Comput. Phys., 221 (2007), pp. 349–369.

[12] R. Hiptmair, *Finite elements in computational electromagnetism*, Acta Numer., 11 (2002), pp. 237–339.

[13] Q. Hong, J. Hu, S. Shu, and J. Xu, *A discontinuous Galerkin method for the fourth-order curl problem*, J. Comput. Math., 30 (2012), pp. 565–578.

[14] P. Houston, I. Perugia, A. Schneebeli, and D. Schötzau, *Interior penalty method for the indefinite time-harmonic Maxwell equations*, Numer. Math., 100 (2005), pp. 485–518.
[15] K. Hu, Y. Ma, and J. Xu, Stable finite element methods preserving \( \nabla \cdot B = 0 \) exactly for MHD models, Numer. Math., 135 (2017), pp. 371–396.

[16] K. Hu, W. Qiu, K. Shi, and J. Xu, Magnetic-Electric Formulations for Stationary Magneto-hydrodynamics Models, ArXiv e-prints, (2017).

[17] D. Jerison and C. E. Kenig, The inhomogeneous Dirichlet problem in Lipschitz domains, J. Funct. Anal., 130 (1995), pp. 161–219.

[18] O. A. Karakashian and F. Pascal, A posteriori error estimates for a discontinuous Galerkin approximation of second-order elliptic problems, SIAM J. Numer. Anal., 41 (2003), pp. 2374–2399.

[19] B. Li, J. Wang, and L. Xu, A convergent linearized Lagrange finite element method for the magnetohydrodynamic equations in 2D nonsmooth and nonconvex domains, submitted.

[20] P. Monk, Finite element methods for Maxwell’s equations, Numerical Mathematics and Scientific Computation, Oxford University Press, New York, 2003.

[21] P. Monk and J. Sun, Finite element methods for Maxwell’s transmission eigenvalues, SIAM J. Sci. Comput., 34 (2012), pp. B247–B264.

[22] J.-C. Nédélec, A new family of mixed finite elements in \( \mathbb{R}^3 \), Numer. Math., 50 (1986), pp. 57–81.

[23] S. Nicaise, Singularities of the quad curl problem, J. Differential Equations, 264 (2018), pp. 5025–5069.

[24] W. Qiu and K. Shi, A Mixed DG method and an HDG method for incompressible magnetohydrodynamics, ArXiv e-prints, (2017).

[25] J. Sun, A mixed FEM for the quad-curl eigenvalue problem, Numer. Math., 132 (2016), pp. 185–200.

[26] Q. Zhang, L. Wang, and Z. Zhang, An \( H^2(\text{curl}) \)-conforming finite element in 2D and its applications to the quad-curl problem, ArXiv e-prints, (2018).

[27] S. Zhang, Mixed schemes for quad-curl equations, ESAIM Math. Model. Numer. Anal., 52 (2018), pp. 147–161.

[28] B. Zheng, Q. Hu, and J. Xu, A nonconforming finite element method for fourth order curl equations in \( \mathbb{R}^3 \), Math. Comp., 80 (2011), pp. 1871–1886.