Hysteresis in flow through porous media

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Abstract. Porous media flow is described by a class of PDEs whose model equation is represented by the Richards equation with soil-moisture hysteresis term. We assume that the hysteresis is represented by the Preisach hysteresis operator. Under suitable assumptions on the hysteresis operator we investigate the existence of the solution of our problem.

1. Introduction
This paper deals with hysteresis in the unsaturated case of the diffusive form of Richards equation. The equation is coupled with initial-boundary data. We investigate the existence of solution of the problem coupled with Preisach operator under the time derivative under appropriate conditions.

The important role of hysteresis has been observed in hydrology and soil physics for a long time. A soil is a collection of grains forming the soil matrix, with interconnected pore spaces forming the channels available for water transport. Water provided by rainfalls or irrigations moves through unsaturated soil, gets into pores and passes through infiltration, drainage, evaporation and absorption by plant roots. The pores in unsaturated soil are filled partly with water and partly with air. The cycle of soil wetting-drying process exhibits hysteresis in porous media, roughly explained by the fact that the volumetric water content has different profiles with respect to the wetting and drying processes. The soil water hysteresis is relation between the volumetric water content and the total potential.

Seventy-nine years ago, Haines [1] observed the presence of hysteresis in the relationship between the water content in the soil and the water potential. For further information about the history and descriptions of soil-moisture hysteresis, see [2, 3, 4, 5]. We refer to Poulavassilis and Childs [6], Mualem [7] and the further references therein for experimental studies of the hysteresis relation.

A model equation with no hysteretic relation between the volumetric water content and the total potential, has been studied by Alt, Luckhaus and Visintin [8], Otto [9]. We refer to the monographs of Brokate and Sprekels [10], Krasnosel’skii and Pokrovskii [11], Krejcí [12], Mayergoyz [13] and Visintin [14] for details about the hysteresis theory. Various studies on the porous media flow with hysteretic behaviour have been published up to now and we mention here the papers [15, 16, 17, 18] and the references given there. In [15] Richards equation is coupled with memory effect in the constitutive law consisting of hysteresis operator of play type and a rate dependent component. The existence result applied also for a Preisach hysteresis operator is presented. Paper [16] describes nonlinear diffusion equation with hysteresis coupled with boundary conditions of Neumann and Signorini type. Existence and asymptotic stability
of a solution for the initial-boundary-value problem are established. In paper [17] the problem of filtration model with hysteresis is formulated in the form of a quasi-variational inequality and a time global solution is proved. In paper [18], well-posedness of the hysteretic model coupled with the potential form of Richards equation with hysteretic hydraulic laws for a wetting-drying cycles is shown. The abstract semigroup theory is used.

The paper is organized as follows. In Section 2 we briefly introduce the concept of model equation. In Section 3, we recall some basic definitions and properties about play and Preisach hysteresis operators. In Section 4 the main result is postulated. The proof of the existence result is led through three steps: approximation, a priori estimates and limit procedure.

2. Model formulation

We assume that the porous medium is rigid, homogeneous and isotropic, that the fluid is inviscid and incompressible.

Richards equation that describes the dynamic flow in an unsaturated zone was deduced by combining generalised Darcy’s law for the flow

$$ q = -k \text{grad} \psi $$

with equation of continuity

$$ \frac{\partial \theta}{\partial t} = -\nabla \cdot q, $$

where $q$ is the flux density, $k$ is the hydraulic conductivity, $\psi$ is the total potential and $\theta$ is the volumetric moisture content. The potential comprises several component potentials. The most important are the matric potential $\psi_m$ and the gravitational potential $\psi_g$. Hence, $\psi = \psi_m + \psi_g$, $\psi \leq 0$. The gravitational potential $\psi_g$ is negative when the position coordinate is taken as positive downwards from the surface, i.e., $\psi_g = -gz$.

Hence, Richards equation representing transient water flow is

$$ \frac{\partial \theta}{\partial t} = \nabla \cdot (k \text{grad} \psi_m - kg). $$

(1)

$\theta_s$ is the volumetric water content at natural saturation and $\theta$ is the volumetric moisture content of the medium. When the soil matrix is perfectly dry then $\theta = 0$, when the matrix is fully saturated with water, then $\theta = \theta_s < 1$ and finally for in-between states the matrix contains both air and water. Hence $\theta$ is bounded between 0 and $\theta_s$, i.e., $0 \leq \theta \leq \theta_s < 1$.

The constitutive relation between the volumetric water content $\theta$ and the matric potential $\psi_m$ is typically represented by a relation of the form

$$ \theta(x,t) \in h(\psi_m)(x,t), $$

where $h : \mathbb{R} \to [0, 1]$ is a maximal monotone graph as in figure 1.

The value $\theta_r$ represents a residual water content, the quantity that remains in a soil after any drainage imposed by the gravitational forced has ceased.

Here we assume hysteretic relation between volumetric water content $\theta$ and the matric potential $\psi_m$ models by the Preisach operator, i.e., $\theta(x,t) = W(\psi_m)(x,t)$, see figure 2.

Heuristically speaking, since $\theta$ is a hysteretic function of $\psi_m$, it follows that the hysteretic behaviour will be involved in all functions depending on $\theta$.

In our case we assume that the hydraulic conductivity depends nonlinearly on matric potential. Moreover, the hydraulic conductivity is single-valued, positive, twice differentiable,
monotonically increasing function of matric potential. Here we can apply the Kirchhoff transformation:

\[ K: \psi_m \mapsto u := \int_0^{\psi_m} k(s)ds. \]

Since \( k(s) \) is positive, this transformation can be inverted and equation (1) can be rewritten in terms of a new variable, \( u := K(\psi_m) \). Defining now

\[ \nabla u = \nabla K(\psi_m) = k(\psi_m) \nabla \psi_m, \]
\[ \tilde{k}(u) = k(K^{-1}(u)) \]

and letting \( e_z \) denote the vertical unit vector, equation (1) becomes

\[ \frac{\partial \theta}{\partial t} = \nabla \cdot (\nabla u - \tilde{k}(u)e_z). \quad (2) \]

By [19], Theorem 4.17, the mapping \( u \mapsto \tilde{W}[\lambda, K^{-1}(u)] \) is again a Preisach operator, hence we obtain (2) with \( \theta = W[\lambda, \cdot] = \tilde{W}[\lambda, K^{-1}(\cdot)] \).

This procedure was used in [8] to prove existence of solution for the problem without hysteresis.

On the other side, if the hydraulic conductivity is a nonlinear function of water content, \( k(\theta) \), then hysteresis in \( \theta \) versus \( \psi_m \) dependence entails occurrence of hysteresis in the \( k \) versus \( \psi_m \) relation. In this case, it is not clear how the Kirchhoff transformation might be extended to this setting, see [15, 16].

**Remark 2.1** We give a general view to the model. In this paper we study model equation only in two dimensions, i.e., equation (2) without gravity term. Three-dimensional Richards equation is studied (paper is in preparation).

### 3. Hysteresis operators

Some remarks concerning hysteresis operators.
3.1. The play operator
Firstly we briefly mention definition and properties of the classical play operator. Definition is

given in this way: for a given input function \( u \in C([0, T]) \) and initial condition \( x_r^0 \in [-r, r] \), we define the output \( \xi := \mathcal{P}_r[x_r^0, u] \in C([0, T]) \cap BV(0, T) \) of the play operator

\[
\mathcal{P}_r : [-r, r] \times C([0, T]) \to C([0, T]) \cap BV(0, T)
\]
as the solution of the variational inequality

\[
\int_0^T [u(t) - \xi(t) - y(t)] \text{d}\xi(t) \geq 0, \quad \forall y \in C([0, T]), \quad \max_{0 \leq \tau \leq T} |y(\tau)| \leq r,
\]

\[
|u(t) - \xi(t)| \leq r, \quad \forall t \in [0, T],
\]

\[
\xi(0) = u(0) - x_r^0. \tag{3}
\]

The concept of memory in connection to hysteresis operators is related to the fact that at any instant time \( t \), the output \( v(t) \) may depend not only on the input \( u(t) \) and the initial condition but also on the previous evolution of input value \( u(t) \).

We notice that we can associate to any \( r \in \mathbb{R} \) the corresponding value \( x_r^0 \); this suggests the idea of making the initial configuration of the play system independent of the initial conditions \( \{x_r^0\}_{r>0} \) for the output function by the introduction of some suitable function of \( r \). More precisely, following [12], Section II.2, let us consider any function \( \lambda \in \Lambda \) where

\[
\Lambda := \left\{ \lambda \in W^{1, \infty}(0, \infty); \left| \frac{d\lambda(r)}{dr} \right| \leq 1 \text{ a.e. in } [-r, r] \right\}.
\]

We also introduce subspaces of \( \Lambda \), i.e.,

\[
\Lambda_K := \{ \lambda \in \Lambda; \lambda(r) = 0 \text{ for } r \geq K \}, \quad \Lambda_0 := \bigcup_{K>0} \Lambda_K. \tag{4}
\]

\( \Lambda \) is called configuration space and the functions \( \lambda \) are called memory configurations.

If \( Q_r : \mathbb{R} \to [-r, r] \) is the projection

\[
Q_r(x) := \text{sign}(x) \min\{r, |x|\} = \min\{r, \max\{-r, x\}\},
\]

then we set

\[
x_r^0 := Q_r(u(0) - \lambda(r)).
\]

This implies that the initial configuration of the play system depends on \( \lambda \) and on \( u(0) \). So we can introduce the following more convenient notation

\[
p_r[\lambda, u] := \mathcal{P}_r[x_r^0, u], \tag{5}
\]

for any \( \lambda \in \Lambda \), \( u \in C([0, T]) \) and \( r > 0 \).

For the sake of completeness \( p_0[\lambda, u] = u \). Moreover, the operator \( p_r : \Lambda \times C([0, T]) \to C([0, T]) \) is Lipschitz continuous in the following sense (see [12], Section II.2, Lemma 2.3).

**Lemma 3.1** For every \( u, w \in C([0, T]) \), \( \lambda, \mu \in \Lambda \) and \( r > 0 \) we have

\[
|p_r[\lambda, u] - p_r[\mu, w]|_\infty \leq \max\{|\lambda(r) - \mu(r)|, \|u - w\|_\infty\}.
\]
The introduction of the function \( \lambda \) plays an important role in the characterization of the memory of the play system, in the sense that, for any given \( \lambda \), we can construct the play operator \( p_r[\lambda, u] \) starting from \( \lambda \) and from a sequence of values \( (t_j, r_j) \) which is so called memory sequence (see [12], Section II.2 or [14], Section III.6) of any input \( u \) at a certain instant \( t \) with respect to the initial configuration \( \lambda \). These values are what one simply has to know in order to evaluate the output of the play operator.

In [20], the play operator is defined in the space \( G_R(0,T) \) of right-continuous regulated functions. This is the space of functions \( u : [0,T] \rightarrow \mathbb{R} \) which admits the left limit \( u(t-) \) at each point \( t > 0 \) and the right limit \( u(t+) \) exists and coincides with \( u(t) \) at each point \( t \geq 0 \). The space \( G_R(0,T) \) is endowed with the norm

\[
\|u\|_{[0,T]} = \sup\{|u(\tau)|; \tau \in [0,T]\} \quad \text{for} \quad u \in G_R(0,T)
\]

so the \( G_R(0,T) \) is a Banach space. By Theorem 2.1 and Proposition 2.4 of [20], this is Lipschitz continuous in the sense that

\[
|p_r[\lambda, u](t) - p_r[\mu, w](t)| \leq \max\{|\lambda(r) - \mu(r)|, \|u - w\|_{[0,T]}\},
\]

for any \( \lambda, \mu \in \Lambda \), \( u, w \in G_R(0,T) \) and \( t \in [0,T] \). For step functions \( u \in G_R(0,T) \) of the form

\[
u(t) = \sum_{k=1}^{m} u_{k-1} \chi_{(t_{k-1}, t_k]}(t) + u_m \chi_{(T]}(t),\]

where \( 0 = t_0 < t_1 < \ldots t_m = T \) is a given division of \([0,T]\) and \( u \) is a function

\[
p_r[\lambda, u](t) = \sum_{k=1}^{m} \xi_{k-1}(r) \chi_{(u_{k-1}, u_k]}(t) + \xi_m(r) \chi_{(T]}(t),
\]

where \( \chi_\omega(t) \) is the characteristic function of a set \( \omega \subset [0,T] \), and

\[
\xi_0(r) = \mathcal{P}[\lambda, u_0](r), \quad \xi_k(r) = \mathcal{P}[\xi_{k-1}, u_k](r),
\]

with \( \mathcal{P} : \Lambda \times \mathbb{R} \rightarrow \Lambda \) defined as

\[
\mathcal{P}[\lambda, w](r) = \max\{w - r, \min\{w + r, \lambda(r)\}\}.
\]

### 3.2. The Preisach operator

Now we briefly recall definition and some basic properties of the Preisach operator. More information about the Preisach operator can be found in [10, 11, 12, 13, 14, 21].

Let us introduce the Preisach half-plane, defines as

\[
\mathbb{R}^{2}_+ := \{(r, w) \in \mathbb{R}^2 : r > 0\}
\]

and assume that a function \( \psi \in L^1_{loc}(\mathbb{R}^{2}_+) \) (the Preisach density) is given with the following property.

**Assumption 3.1** There exist \( \beta_1 \in L^1_{loc}(0, \infty) \), such that

\[
0 \leq \psi(r, w) \leq \beta_1(r) \quad \text{for a.e.} \quad (r, w) \in \mathbb{R}^2_+.
\]

We put

\[
b_1(K) := \int_0^K \beta_1(r) \, dr \quad \text{for} \quad K > 0
\]

and

\[
g(r, w) := \int_0^w \psi(r, z) \, dz \quad \text{for} \quad (r, w) \in \mathbb{R}^2_+.
\]
We define the Preisach operator as follows.

**Definition 3.1** Let \( \psi \in L^1_{\text{loc}}(\mathbb{R}^2_+) \) be given and let \( g \) be as in (14). Then the Preisach operator \( \mathcal{W} : \Lambda_0 \times G_R(0, T) \to G_R(0, T) \) generated by the function \( g \) is defined by the formula

\[
\mathcal{W}[\lambda, u](t) := \int_0^\infty g(r, p_r[\lambda, u](t)) \, dr = \int_0^\infty \int_0^{p_r[\lambda, u](t)} \psi(r, z) \, dz \, dr
\]

for \( \lambda \in \Lambda_0, u \in G_R(0, T) \) and \( t \in [0, T] \).

Then we have the following result (see [12], Section II.3, Proposition 3.11).

**Proposition 3.1** Let Assumption 3.1 be satisfied and let \( K > 0 \) be given. Then for every \( \lambda, \mu \in \Lambda_K \) and \( u, w \in G_R(0, T) \) such that \( \|u\|_{[0,T]} \leq K \), the Preisach operator (15) satisfies

\[
\|\mathcal{W}[\lambda, u] - \mathcal{W}[\mu, w]\| \leq \int_0^K (|\lambda(r) - \mu(r)|/\beta_1(r) + b_1(K)) \|u - w\|_{[0,T]} \forall t \in [0, T].
\]

We introduce the Preisach potential energy \( U \) as

\[
U[\lambda, u] := \int_0^\infty G(r, p_r[\lambda, u]) \, dr,
\]

where

\[
G(r, w) := wg(r, w) - \int_0^w g(r, z) \, dz = \int_0^w \psi(r, z) \, dz,
\]

and the dissipation operator is given by

\[
\mathcal{D}[\lambda, u] := \int_0^\infty rg(r, p_r[\lambda, u]) \, dr.
\]

We recover the following result (see [12], Section II.4, Theorem 4.3).

**Proposition 3.2** Let the Preisach operator \( \mathcal{W} \) satisfy Assumptions 3.1 and let \( K > 0 \) be given. For arbitrary \( \lambda \in \Lambda_K \) and \( u \in W^{1,1}(0, T) \) such that \( \|u\|_{C([0,T])} \leq K \), we put

\[
v := \mathcal{W}[\lambda, u] \quad U := U[\lambda, u] \quad D := \mathcal{D}[\lambda, u].
\]

Then we have

(i) \( U(t) \geq \frac{1}{2b_1(K)} v^2(t) \forall t \in [0, T] \),

(ii) \( \dot{v}(t)u(t) - \dot{U}(t) = |\dot{D}(t)| \, a.e. \)

The following result can be found in [12], Section II.4, Proposition 4.8.

**Proposition 3.3** Let Assumptions 3.1 be satisfied and let \( K > 0 \) be given. Suppose moreover \( b \geq 0, \lambda \in \Lambda_K \), and \( u \in W^{1,1}(0, T) \) be given such that \( \|u\|_{C([0,T])} \leq K \). Put \( v := bu + \mathcal{W}[\lambda, u] \). Then

\[
b \dot{u}^2(t) \leq \dot{v}(t)u(t) \leq (b + b_1(K)) \dot{u}^2(t).
\]
In our equation, both the input function and the initial memory configuration depend on the space variable \( x \). If \( \lambda(x, \cdot) \) belongs to \( \Lambda_0 \) and \( u(x, \cdot) \) belongs to \( C([0, T]) \) for (almost) every \( x \), then we can define

\[
W[\lambda, u](x, t) := \int_0^\infty g(r, p_r[\lambda(x, \cdot), u(x, \cdot)](t)) \, dr.
\]  

In the following we will often write \( W(u) \) instead of \( W[\lambda, u] \) for brevity or when \( \lambda \) is clear from the context.

We conclude this subsection with the convexification of the Preisach operator, i.e., that in a certain region, the convexity of the loops is satisfied (see [12], Section II.4, Proposition 4.22).

Let \( R > 0 \) be fixed, set

\[
D_R := \{(r, w) \in \mathbb{R}_+^2 : |w| + r \leq R\}.
\]

In addition to Assumption 3.1 we prescribe the following conditions.

**Assumption 3.2**

(i) \( \frac{\partial \psi}{\partial w} \in L^\infty_{\text{loc}}(\mathbb{R}_+^2) \);

(ii) \( A_R := \inf \{ \psi(r, w) ; (r, w) \in D_R \} > 0 \).

Furthermore, denote

\[
C_R := \sup \left\{ \left| \frac{\partial}{\partial w} \psi(r, w) \right| ; (r, w) \in D_R \right\}.
\]

Taking possibly a smaller \( R > 0 \), if necessary, we may assume that

\[
K_R := \frac{1}{2} A_R - R C_R > 0.
\]  

We modify the density \( \psi \) outside \( D_R \) and set

\[
\psi_R(r, w) = \begin{cases} 
\psi(r, w) & \text{if } (r, w) \in D_R, \\
\psi(r, -R + r) & \text{if } w < -R + r, r \leq R, \\
\psi(r, R - r) & \text{if } w > R - r, r \leq R, \\
\psi(R, 0) & \text{if } r > R.
\end{cases}
\]  

We define the convexified Preisach operator \( W_R \) by the formula

\[
W_R[\lambda, u](t) = \int_0^\infty \int_0^{p_r[\lambda, u](t)} \psi_R(r, w) \, dw \, dr
\]

for \( \lambda \in \Lambda_0 \) and \( u \in W^{1,1}(0, T) \). The convex character of \( W_R \) will be exploited in Section 4.

**4. Main result**

Consider an open bounded set of Lipschitz class \( \Omega \subset \mathbb{R}^2 \) and set \( Q := \Omega \times (0, T) \). We set \( V := H^1(\Omega) \), \( H := L^2(\Omega) \) and introduce the space of functions with null average in \( \Omega \)

\[
V_* := \{ u \in V : \int_{\Omega} u = 0 \}.
\]

We formally have the equation

\[
v_t - \Delta u = 0,
\]  

which we couple with the boundary condition

\[
\nabla u \cdot \nu = E,
\]

for \( \nu \) the outward normal vector on \( \partial \Omega \).
on $\partial \Omega$, where $\nu$ is the unit outward normal vector, $E \in L^\infty(\partial \Omega \times (0, T))$ is a given outer matric potential. We set $v := W[\nu, u]$. We assume that $u_0(x), v_0(x) \in L^2(\Omega)$ are given initial conditions. The compatibility condition

$$\int_\Omega \nabla u_0(x) \nabla \phi(x) dx - \int_{\partial \Omega} E(x, 0) \phi(x) d\sigma = 0$$

holds for every $\phi \in V$.

We want to solve the following problem.

**Problem 4.1** Let us consider a Preisach operator $W$ and let $u_0 \in L^2(\Omega)$, $\lambda : \Omega \to \Lambda$ be given initial data. We search for a function $u$ such that $u(x, 0) = u_0(x)$ a.e. in $\Omega$ and for any $\phi \in V$, and for a.e. $t \in (0, T)$ we have

$$\int_\Omega \frac{\partial v}{\partial t} \phi dx + \int_\Omega \nabla u \nabla \phi dx = \int_{\partial \Omega} E \phi d\sigma$$

The main result can be stated as follows.

**Theorem 4.1** *(Existence)* Let us assume operator $W$ be the Preisach operator introduced in (20) and satisfying Assumptions 3.1 and 3.2. And let $R > 0$ be fixed as in Subsection 3.2. Let $K \subseteq [0, R]$ and $\lambda : \Omega \to \Lambda_k$ be given. Moreover $E \in L^\infty(\partial \Omega \times (0, T))$, $E_t \in L^2(\partial \Omega \times (0, T))$, $u_0 \in V$, $v_0 \in L^2(\Omega)$ and (26) holds. Set $\alpha := \max\{\|u_0\|_V, \|E\|_{L^\infty(\partial \Omega \times (0, T))}, \|E_t\|_{L^2(\partial \Omega \times (0, T))}\}$. Then there exists a constant $\beta > 0$ such that if $\alpha \leq \beta$, then Problem 4.1 has at least one solution such that

$$u \in C^0(Q),$$

$$u_t \in L^2(0, T; V_\ast).$$

**Proof**

(i) **Approximation.** Let us fix a time step $\tau := \frac{T}{m}$, for some $m \in N$. Furthermore we consider for $k = 1, \ldots, m$ and for any $\phi \in V$ the following recurrant system

$$\frac{1}{\tau} \int_\Omega (v_k - v_{k-1}) \phi dx + \int_\Omega \nabla u_k \nabla \phi dx = \int_{\partial \Omega} E_k \phi dx,$$

where $E_k$ is defined by $E_k(x) = \frac{1}{\tau} \int_{(k-1)\tau}^{k\tau} E(x, t) dt$.

Here we set

$$v_k(x) = \int_0^{r_k} g_R(r, \xi_k(x, r)) dr$$

with

$$g_R(r, w) = \int_0^w \psi_R(r, u') du'.$$

where $\psi_R$ is function introduced in (22), i.e., we are working with the convexified Preisach operator $W_R(u)$. The sequence $\xi_k$ is defined recursively by

$$\xi_0(x, r) := P[\lambda(x, \cdot), u_0(x)](r), \quad \xi_k(x, r) := P[\xi_{k-1}(x, \cdot), u_k(x)](r).$$

Setting

$$\tilde{u}^{(r)}(x, t) = \sum_{k=1}^m u_{k-1}(x) \chi((k-1)\tau, k\tau)(t) + u_m(x) \chi(T)(t),$$
and
\[ \tilde{\xi}_r^{(\tau)}(x, t) = \sum_{k=1}^{m} \xi_{k-1}(x, r)\chi_{[k-1, k)}(t) + \xi_m(x, r)\chi_{[m, \infty)}(t), \]  
(33)

we thus have, in the view of (8) - (11)
\[ \tilde{\xi}_r^{(\tau)}(x, t) = p_r[\lambda, \bar{u}^{(\tau)}](x, t). \]  
(34)

The solution to (28) can be constructed by induction over \( k \), then using the Browder-Minty fixed point theorem, the monotonicity of the mapping \( g(r, \cdot) \) and \( P[\lambda, \cdot] \).

(ii) **A discrete first order energy inequality.** We established here a discrete counterpart of Proposition 3.2. We set \( \xi_k(x) := \xi_k(x, r) \), where \( \xi_k(x, r) \) was introduced in (31). Let \( \psi \) be an arbitrary function satisfying Assumption 3.1. We define a discrete versions of the Preisach potential energy \( U \) and dissipation operator \( D \), introduced in (16) and (18) as
\[ U_k(x) = \int_0^\infty G(r, \xi_k^r(x))dr, \]
\[ D_k(x) = \int_0^\infty rg(r, \xi_k^r(x))dr, \]
with \( G \) given by (17). The discrete version of the first order energy inequality can be stated as follows:
\[ (v_k - v_{k-1})u_k - (U_k - U_{k-1}) \geq \int_0^\infty \int_{\xi_{k-1}}^{\xi_k} r\psi(r, w)dwdr = |D_k - D_{k-1}|. \]  
(35)

(iii) **A discrete second order energy inequality.** The discrete version of the second order energy inequality can be stated as follows: For every \( k = 2, \ldots, n, n \in \{1, \ldots, m\} \) and a.e. \( x \in \Omega \)
\[ (z_k - z_{k-1})q_k \geq \frac{1}{2}(z_kq_k - z_{k-1}q_{k-1}), \]  
(36)
where we set
\[ q_k := \frac{u_k - u_{k-1}}{\tau}, \quad z_k := \frac{v_k - v_{k-1}}{\tau} \quad \text{for } k = 1, \ldots, m; \]
The time continuous case is treated in detail in ([12], Section II.3 and II.4).

(iv) **First a priori estimate.** In the estimates below, let \( C \) denote a constant independent of \( \alpha \) and \( \tau \). Indeed, the value of \( C \) may vary from one formula to another. We choose \( \phi = \tau u_k \) in (28), we obtain
\[ \int_{\Omega} (v_k - v_{k-1})u_k \, dx + \tau \int_{\Omega} |\nabla u_k|^2 \, dx = \tau \int_{\partial\Omega} E_k \, u_k \, d\sigma. \]

Therefore, using (35) and summing up for \( k = 1, \ldots, n \), for every \( n \in \{1, \ldots, m\} \), we obtain
\[ \tau \sum_{k=1}^{n} \int_{\Omega} |\nabla u_k|^2 \, dx \leq \tau \sum_{k=1}^{n} \int_{\partial\Omega} |E_k|^2 \, d\sigma + \frac{1}{4} \tau \sum_{k=1}^{n} \int_{\partial\Omega} |u_k|^2 \, d\sigma + \int_{\Omega} U_0 \, dx. \]
Using the regularity of the initial data and trace operator theorem, it follows that

\[ \tau \sum_{k=1}^{n} \|u_k\|_{V^*}^2 \leq C\alpha^2 \]  \hspace{1cm} (37)

(v) **Second a priori estimate.** Due to the monotonicity and local Lipschitz continuity of the functions \( g(r, \cdot) \) and \( P(\lambda, \cdot)(r) \), we have the pointwise inequality

\[ q_k(x)z_k(x) \geq 0 \quad \text{a.e. in } \Omega, \; k = 1, \ldots, m. \]  \hspace{1cm} (38)

We set for brevity

\[ E_{tk} := E_k - E_{k-1}. \]

We take the incremental ratio in time in (28) and then test by \( \tau q_k \), we get

\[ \int_{\Omega} (z_k - z_{k-1})q_k \, dx + \tau \int_{\Omega} |\nabla q_k|^2 \, dx = \tau \int_{\partial \Omega} E_{tk} q_k \, d\sigma. \]  \hspace{1cm} (39)

Then let us sum for \( k = 1, \ldots, n \), for every \( n \in \{1, \ldots, m\} \), use (36) for \( k \geq 2 \) and Hölder inequality, we obtain

\[ \int_{\Omega} (q_1z_1 - q_1z_0) \, dx + \frac{1}{2} \sum_{k=2}^{n} \int_{\Omega} (q_kz_k - q_{k-1}z_{k-1}) \, dx + \tau \sum_{k=1}^{n} \int_{\Omega} |\nabla q_k|^2 \, dx \]

\[ \leq \tau \sum_{k=1}^{n} \int_{\partial \Omega} |E_{tk}|^2 \, d\sigma + \frac{1}{4} \tau \sum_{k=1}^{n} \int_{\partial \Omega} |q_k|^2 \, d\sigma. \]

Futhermore we use (38), compatibility condition (26), the estimates on the initial data and apply trace operator theorem to get

\[ \tau \sum_{k=1}^{n} \|q_k\|_{V^*}^2 \leq C\alpha^2. \]  \hspace{1cm} (40)

(vi) **Third a priori estimate.** Finally we need \( L^\infty \) bound for our solution. We prove by induction over \( k = 1, \ldots, m \) that there exists \( B > 0 \) independent of \( k \) and \( m \) such that

\[ \|u_k\|_{L^\infty(\Omega)} \leq B\alpha \quad \text{for all } k = 0, \ldots, m. \]  \hspace{1cm} (41)

For \( k = 0, 1, \ldots, m \) we set

\[ B_k^{(m)} = \frac{1}{\alpha} \max\{\|u_j\|_{L^\infty(\Omega)} : j = 0, 1, \ldots, k\}. \]

We have \( B_0^{(m)} \leq C \) independently of \( m \) by hypothesis on initial data. Let now \( 1 \leq k_0 \leq m \) be fixed and assume that \( B_{k_0-1}^{(m)} < \infty \). By direct comparison in equation (28), we derive for \( k = 1, \ldots, k_0 \) the estimate

\[ \|\Delta u_k\|_H \leq \|z_k\|_H. \]  \hspace{1cm} (42)

This yields that \( B_{k_0}^{(m)} < \infty \). Acording to (29) - (31), (22) and by definition of \( \alpha \) and by (40), we obtain from (42) that

\[ \|\Delta u_{k_0}\|_H \leq C\alpha \left(1 + \max\{B_{k_0-1}^{(m)} \alpha, \|u_{k_0}\|_{L^\infty(\Omega)}\}\right). \]  \hspace{1cm} (43)
A combined use of the Poincare inequality, the embedding theorem, Gagliardo-Nirenberg inequality and (37) yield \( \|u_{k0}\|_{L^\infty(\Omega)} \leq C\alpha(1 + \max\{B^{(m)}_{k0-1}, \|u_{k0}\|_{L^\infty(\Omega)}\})^{\frac{2}{3}} \). Assume that \( B^{(m)}_{k0} > B^{(m)}_{k0-1} \). Then

\[
B^{(m)}_{k0} = \frac{1}{\alpha}\max\{\|u_{k0}\|_{L^\infty(\Omega)}\} \leq C(1 + \alpha_0 B^{(m)}_{k0})^{\frac{2}{3}},
\]

hence \( B^{(m)}_{k0} \leq \max\{C, B^{(m)}_{k0-1}\} \) with a constant \( C \) independent of \( k \) and \( m \), and the desired estimate (41) follows. Inequality (43) implies in particular that

\[
\|\Delta u_k\|_{L^2(\Omega)} \leq C\alpha \quad \text{for all } k = 1, \ldots, m.
\] (44)

(v) **Limit procedure.** With the sequence \( \{u_k\} \) constructed above (see (32) - (34)) we define for each fixed time step \( \tau \) their piecewise linear and piecewise constant time interpolates according to the following scheme:

\[
\begin{align*}
\bar{u}^{(\tau)}_+(x,t) &= u_k(x), & \bar{u}^{(\tau)}_-(x,t) &= u_{k-1}(x), \\
\bar{v}^{(\tau)}_+(x,t) &= v_k(x), & \bar{E}^{(\tau)}_+(x,t) &= E_k(x),
\end{align*}
\] (45)

and

\[
\begin{align*}
\hat{u}^{(\tau)}(x,t) &= u_{k-1}(x) + \frac{t-(k-1)\tau}{\tau}(u_k(x) - u_{k-1}(x)) \\
\hat{v}^{(\tau)}(x,t) &= v_{k-1}(x) + \frac{t-(k-1)\tau}{\tau}(v_k(x) - v_{k-1}(x))
\end{align*}
\] (46)

for every \( x \in \Omega \) and \( t \in [(k-1)\tau, k\tau), k = 1, 2, \ldots, m, \) continuously extended to \( t = T \). We have

\[
\bar{v}^{(\tau)}_+ = \mathcal{W}_R[\lambda, \bar{u}^{(\tau)}_+].
\] (47)

As a consequence of the estimates (40), (44) and by comparison of the terms of the equation (28) we get that \( u \in L^\infty(0,T; W^{2,2}(\Omega) \cap V_\alpha), \) \( u_t \in L^2(0,T; V_\alpha) \). The a priori estimates we found allow us to conclude that there exists \( u, u_t \) such that, along a subsequence as \( \tau \to 0 \), we have

\[
\begin{align*}
\hat{u}^{(\tau)} &\to u \quad \text{weakly star in} \quad L^\infty(0,T; W^{2,2}(\Omega)), \\
\hat{u}^{(\tau)}_t &\to u_t \quad \text{weakly in} \quad L^2(0,T; V_\alpha),
\end{align*}
\] (48)

By compact embedding, we have, passing again to a subsequence, if necessary

\[
\begin{align*}
\nabla \hat{u}^{(\tau)} &\to \nabla u \quad \text{strongly in} \quad L^2(Q; \mathbb{R}^2), \\
\hat{u}^{(\tau)} &\to u \quad \text{uniformly in} \quad C^0(Q).
\end{align*}
\] (49)

We further have for every \( \tau \) and every \( (x, t) \in Q \) that

\[
|\hat{u}^{(\tau)}(x,t) - \bar{u}^{(\tau)}_+(x,t)|^2 \leq \max_k |u_k(x) - u_{k-1}(x)|^2 \leq \sum_{k=1}^m |u_k(x) - u_{k-1}(x)|^2,
\]

\[
|\hat{v}^{(\tau)}(x,t) - \bar{v}^{(\tau)}_+(x,t)|^2 \leq \max_k |v_k(x) - v_{k-1}(x)|^2 \leq C \sum_{k=1}^m |v_k(x) - v_{k-1}(x)|^2.
\]
From (40) it follows that
\[ \| \dot{u}^{(\tau)} - \bar{u}^{(\tau)}_\pm \|_{L^2(\Omega; G_R(0,T))} + \| \dot{v}^{(\tau)} - \bar{v}^{(\tau)}_+ \|_{L^2(\Omega; G_R(0,T))} \leq C \sqrt{T}, \]
(50)
\[ \| \nabla \dot{u}^{(\tau)} - \nabla \bar{u}^{(\tau)}_\pm \|_{L^2(\Omega; [0,1]^2)} \leq C \sqrt{T}, \]
(51)
Hence \( \bar{u}^{(\tau)}_\pm \) converge strongly to \( u \) in \( L^2(\Omega; G_R(0,T)) \) as \( T \to 0 \). By Proposition 3.1 we may pass to the limit in (47) and obtain
\[ \bar{v}^{(\tau)}_+ \to v = W_R[\lambda,u] \quad \text{strongly in} \quad L^2(\Omega; G_R(0,T)). \]
(52)
This and (50) yield
\[ \dot{v}^{(\tau)} \to v \quad \text{strongly in} \quad L^2(\Omega; G_R(0,T)). \]
(53)
The convergences (48)-(49), (52)-(53) and inequality (51) enable us to pass to the limit as \( T \to 0 \) and obtain
\[ \int_\Omega (v_t \phi + \nabla u \nabla \phi) \, dx = \int_{\partial \Omega} E \phi \, d\sigma. \]
(54)
At this point we can deduce that there exists a constant \( \beta \) depending on \( R \) such that if \( \alpha \leq \beta \) then \( |u(x,t)| \leq R \) a.e. in \( Q \). This implies that \( W(u) = W_R(u) \) (see Lemma II.2.4 in [12] and the definition of \( \psi_R \) (22)). This completes the proof of Theorem. \( \Box \)

**Remark 4.1** We assume equation (24) with homogenous Dirichlet or Neumann boundary condition and compatibility condition (26) is then without the second term. The system is isolated and starts at the steady state, i.e. compatibility condition holds. Because there are no outer sources on the boundary then the solution stays at the steady state trivially. We can see this from the proof of Theorem 4.1 where the estimate for time derivative is equal to zero.

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**References**

1. Haines W B 1930 J. Agric. Sci. 20 97-116
2. Everett D H 1958 Some Problems in the Investigation of Porosity by Adsorption Methods, The Structure and Properties of Porous Materials (London: Butterworth)
3. Everett D H 1967 The Solid-Gas Interface (New York: Dekker) p 1005
4. Poulovassilis A 1962 Soil Sci. 93 405
5. Flynn D, McNamara H, O’Kane J P and Pokrovskii A 2005 Sci. Hyst. 3 698-744
6. Poulovassilis A and Childs E C 1971 Soil Sci. 112 301-312
7. Mualem Y 1984 Soil Sci. 137 283-291
8. Alt H W, Luckhaus S and Visintin A 1984 Ann. Mat. Pura Appl. 136 303-316
9. Otto F 1997 Adv. Math. Sci. Appl. 7 537-553
10. Brokate M and Sprekels J 1996 Hysteresis and Phase Transitions (Berlin: Springer)
11. Krasnosel’skiıı M and Pokrovskiiıı A 1989 Systems with Hysteresis (New York: Springer)
12. Krejčí P 1996 Hysteresis, Convexity and Dissipation in Hyperbolic Equations (Tokyo: Gakkotosho)
13. Mayergoyz I D 1991 Mathematical Models of Hysteresis (New York: Springer)
14. Visintin A 1994 Differential Models of Hysteresis (Heidelberg: Springer)
15. Bagagiolo F and Visintin A 2000 J. An. Appl. 19 977-997
16. Bagagiolo F and Visintin A 2004 Advan. Math. Sci. Appl. 14 379-403
17. Kubo M 2004 J. Diff. equation 201 75-98
18. Marinoshchi G 2008 Nonlin. An.: Real World Appl. 9 518-535
19. Krejčí P 1991 Appl. Mat. 36 305-326
20. Krejčí P 2006 J. Phys.: Conf. Series 55 144-154
21. Preisach F 1983 Z. Physik 94 277-302