New constructions of Hadamard matrices

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Abstract. In this paper, we obtain a number of new infinite families of Hadamard matrices. Our constructions are based on four new constructions of difference families with four or eight blocks. By applying the Wallis-Whiteman array or the Kharaghani array to the difference families constructed, we obtain new Hadamard matrices of order $4(uv + 1)$ for $u = 2, 3$ and $5$. For $u = 2$, $v \in (\Phi_1 \cup \Phi_2 \cup \Phi_3 \cup \Phi_4)$; for $u = 3$ and $v \in (\Phi_1 \cup \Phi_2 \cup \Phi_3 \cup \Phi_4)$. Here, $\Phi_1 = \{q^2 : q \equiv 1 \pmod{4} \text{ is a prime power}\}$, $\Phi_2 = \{n^4 : n \in \mathbb{N}\} \cup \{9n^4 : n \in \mathbb{N}\}$, $\Phi_3 = \{5\}$ and $\Phi_4 = \{13, 37\}$. Moreover, our construction also yields new Hadamard matrices of order $8(uv + 1)$ for any $u \in (\Phi_1 \cup \Phi_2)$ and $v \in (\Phi_1 \cup \Phi_2 \cup \Phi_3)$. 

1. Introduction

Throughout this paper, we fix the following notations. Let $(G, +)$ be an additively written abelian group of order $v$ and let $G^* := G \setminus \{0_G\}$. For any subset $D$ in $G$, we denote $D^{(-1)} := \{-x : x \in D\}$, $\overline{D} := G^* \setminus D$, and $D^c := G \setminus D$. Furthermore, we also denote $\sum_{x \in D} x \in \mathbb{Z}[G]$ by $D$ when there is no confusion.

Let $B_i$, $i = 1, 2, \ldots, \ell$, be $k_i$-subsets of $G$ and $B = \{B_i : i = 1, 2, \ldots, \ell\}$. Any subset $A$ of $G$ is said to be symmetric if $A = A^{(-1)}$. A family $B$ is said to be symmetric if each $B_i$ is symmetric. $B$ is said to be a difference family with parameters $(v; k_1, k_2, \ldots, k_\ell; \lambda)$ in $G$ if the list of differences “$x - y, x, y \in B_i, x \neq y, i = 1, 2, \ldots, \ell$” represents every element of $G \setminus \{0_G\}$ exactly $\lambda$ times; or equivalently

$$\sum_{i=1}^{\ell} B_i B_i^{(-1)} = \lambda G + \left(\sum_{i=1}^{\ell} k_i - \lambda\right) \cdot 0_G.$$

Remark 1.1. To avoid possible confusion, we write $\lambda \cdot 0_G$ as the element in the group ring such that the coefficient of $0_G$ is $\lambda$. This notation will be used throughout this paper.

Each subset $B_i$ is called a block of $B$. If there is only one block in a difference family, the block is a difference set.

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In this paper, our main idea is to construct a new difference family in \( G \times G' \) from difference families in \( G \). Thus, we need the difference families in \( G \) to satisfy certain properties. We first define the following:

**Definition 1.2.** Suppose \( B = \{ B_i : i = 1, 2, \ldots, \ell \} \) is a difference family in \( G \).

(i) For \( \ell = 2, 4, 8 \), \( B \) is said to be of type \( H^\ell \) if \( \sum_{i=1}^{\ell} k_i - \ell(\lvert G \rvert + 1)/4 = \lambda \).

(ii) For \( \ell = 4 \), \( B \) is said to be of type \( H \) if \( \sum_{i=1}^{4} k_i - \lvert G \rvert = \lambda \).

Let us first summarize how a difference family can be used to construct a Hadamard matrix. It is well known that if there is a difference family of type \( H \) in \( G \), then we have a Hadamard matrix of order \( 4\lvert G \rvert \) by plugging the circulant \((-1, 1)\) matrices obtained from its blocks into the Goethals-Seidel array \([3]\). In literature, a number of difference families of type \( H \) have been extensively studied \([5, 14, 18, 20, 22]\).

For any difference family of type \( H^\ell \) in \( G \) for \( \ell = 2 \) or 4, we construct a Hadamard matrix of order \( \ell(\lvert G \rvert + 1) \) by plugging the circulant \((-1, 1)\) matrices obtained from its blocks into the Szekeres array \([12, \text{Theorem 2.4}]\) or the Wallis-Whiteman array \([12, \text{Theorem 4.17}]\), respectively. In the case where \( \ell = 8 \), we may use the Kharaghani array under the assumption that the circulant \((-1, 1)\) matrices \( M_i, i = 0, 1, \ldots, 7 \), obtained from its blocks are amicable, i.e., \( \sum_{i=0}^{3}(M_{2i}M_{2i+1}^T - M_{2i+1}M_{2i}^T) = O \). (See, e.g., \([4,7, \text{Lemma 4.20}], [8, \text{Page 12}]\).) Note that the amicability condition becomes trivial if the difference family is symmetric. That explains why we are interested in difference families of type \( H^4 \) and symmetric difference families of type \( H^8 \) in this paper.

As mentioned before, if a difference family consists of only one block, the block is actually a difference set. However, the parameters of the single block must be of the form \((4n^2; 2n^2 - n; n^2 - n)\). In literature, a difference set with parameters \((4n^2; 2n^2 - n; n^2 - n)\) is called a Menon difference set. For any Menon difference set of order \( 4n^2 \), we consider an incidence matrix of \( 2\lambda \) matrices obtained from its blocks into \( G \). It is known that for any \( s \geq 0 \) and \( k \geq 1 \), a Menon difference set of order \( 4 \cdot 2^s k^4 \) or \( 4 \cdot 9 \cdot 2^s k^4 \) exists. Moreover, Turyn \([11]\) has shown how to construct a Menon difference set of order \( 4n^2 \) from Menon difference sets of order \( 4u^2 \) and \( 4n^2 \). It is then natural to ask if we can extend those results to difference families. It turns out that we need some extra conditions imposed on the differences families concerned. We first list the conditions required.

**Definition 1.3.** Let \( D = \{ D_i : i = 0, 1, 2, 3 \} \) be a difference family of type \( H \) in an abelian group \( G \). We define conditions (c1)-(c5) as follows:

(c1) \( D_0D_2^{(-1)} + D_2D_0^{(-1)} + D_1D_3^{(-1)} + D_3D_1^{(-1)} = (\sum_{i=0}^{3} \lvert D_i \rvert - \lvert G \rvert )G \).
(c2) For each $i = 0, 1, 2, 3$, $D_i$ is symmetric; and

$$D_0D_2^{(-1)} + D_2D_0^{(-1)} + D_1D_3^{(-1)} + D_3D_1^{(-1)} = \left(\sum_{i=0}^{3} |D_i| - |G|\right) G.$$ 

(c3) For each $i = 0, 1, 2, 3$, $D_i$ is symmetric; and for $(h, i, j, k) = (0, 1, 2, 3), (0, 2, 1, 3), (0, 3, 1, 2),$

$$D_hD_i^{(-1)} + D_iD_h^{(-1)} + D_jD_k^{(-1)} + D_kD_j^{(-1)} = \left(\sum_{i=0}^{3} |D_i| - |G|\right) G.$$ 

(c4) $2|\left(\sum_{i=0}^{3} D_i\right)|$ in $\mathbb{Z}[G]$, i.e. all coefficients in $\sum_{i=0}^{3} D_i$ is even.

(c5) $D_0D_2^{(-1)} = D_2D_0^{(-1)}$ and $D_1D_3^{(-1)} = D_3D_1^{(-1)}$.

It has been shown in [17] that a difference family of type $H$ satisfying (c4) yields a “T-matrix”, which has played an important role in constructions of Hadamard matrices [10, 17]. It is also known that a difference family of type $H$ satisfying (c1) yields a difference family with parameters $(2v; k_1, k_2; k_1 + k_2 - v)$ [15]. We may also interpret that such a difference family with two blocks belongs to the class is in between difference families of type $H$ and Menon difference sets. The following ‘product constructions’ using difference families of type $H$ were first studied in [15, 16, 17].

**Theorem 1.4.** Let $D$ and $D'$ be difference families of type $H$ in an abelian group of order $u$ and $n$ respectively. If any of the following conditions is satisfied, then there exists a difference family of type $H$ in an abelian group of order $un$.

1. [15] $D$ satisfies (c1) and $D'$ satisfies (c3).
2. [15] $D$ satisfies (c1) and $D'$ satisfies (c2).
3. [17] $D$ satisfies (c4) and $D'$ satisfies (c5).
4. [17] $D$ satisfies (c4) and $D'$ satisfies (c1).

Moreover, if (1) is satisfied, then there exists a difference family of type $H$ that satisfies (c1).

**Remark 1.5.** Note that Theorem 1.4 (2) is not stated explicitly in [15]. But it can be proved easily by using a similar argument as in the proof of Theorem 2 in [15].

Unfortunately, Theorem 1.4 is not always applicable. A somewhat different ‘product construction’ for “Paley type partial difference sets” has been considered by J. Polhill [6]. A *Paley type partial difference set* is a symmetric subset $D$ of size $(|G| - 1)/2$ defined on a finite group $G$ satisfying $D^2 = \left(\frac{|G|-1}{2}\right) \cdot 0_G - D + \frac{|G|-1}{4} G^*$. To construct Paley type partial difference sets in $G \times G'$, it is not sufficient just to make use of one Paley type partial difference sets in $G$ and one in $G'$. Indeed, in Theorem 4.1 [6], one needs a difference family of type $H$ and a Paley partial difference set in each group $G$ and $G'$. On the other hand, if $P$ is a Paley type partial difference set, then $\{P, P\}$ is symmetric difference family of type $H_2^*$. Therefore, his construction can be
viewed as a ‘product construction’ involves two difference families, one is of type $H^*$ and the other is of type $H^*_4$. It is therefore natural to ask the following:

**Problem 1.** Instead of using only one difference family of type $H$ in $G$ and one in $G'$, is it possible to use two difference families satisfying certain conditions in $G$ and an appropriate difference family in $G'$ to construct a difference family of type $H^*_4$ or $H^*_8$ in the group $G \times G'$?

To address the problem above, we need to work with difference families that satisfy certain conditions.

**Definition 1.** Let $\mathcal{D} = \{D_i : i = 0, 1, 2, 3\}$ be a difference family of type $H$ in a group $G$. We define conditions (d1)-(d5) as follows:

- (d1) $0_G \notin \bigcup_{i=0}^{3} D_i$.
- (d2) There are subsets $E_0, E_1$ of $G^*$ such that $D_0 + D_1 - D_2 - D_3 = E_0 - D_0$, $D_0 + D_3 - D_1 - D_2 = E_1 - D_1$, and the set of $E_0, E_1, E_0, E_1$ forms a difference family of type $H^*_4$.
- (d3) $D_0 D_2^{(-1)} + D_2 D_0^{(-1)} + D_1 D_3^{(-1)} + D_3 D_1^{(-1)} = (\sum_{i=0}^{3} |D_i| - |G| + 1)G$.
- (d4) The coefficient of each element $x \in G^*$ in $\sum_{i=0}^{3} D_i$ is one or three, and the coefficient of $0_G$ in $\sum_{i=0}^{3} D_i$ is even but not two in exactly one of $\sum_{i=0,1} D_i$, $\sum_{i=2,3} D_i$, $\sum_{i=0,3} D_i$ and $\sum_{i=1,2} D_i$.
- (d5) $|D_0| + |D_3| = |D_1| + |D_2|$.

Our first goal is to show the following:

**Theorem 1.** Let $\mathcal{D} = \{D_i : i = 0, 1, 2, 3\}$ be a difference family of type $H$ in a group $G$.

1. If $\mathcal{D}$ satisfies (d3); and there exists a difference family of type $H^*_4$ satisfies (c1) in $G$, then there exists a difference family of type $H^*_4$ in $G \times Z_2$.
2. If $\mathcal{D}$ is symmetric and it satisfies (d3) and there exists a difference family of type $H^*_2$ in $G$, then there exists a difference family of type $H^*_2$ in $G \times Z_3$.
3. If $\mathcal{D}$ is symmetric and it satisfies (d2) and (d5), then there exists a difference family of type $H^*_4$ in $G \times Z_5$.
4. If $\mathcal{D}$ is symmetric and it satisfies (d2); and there exists a symmetric difference family $\mathcal{S}$ of type $H^*$ in $G \times Z_3$ satisfying (d1) and (d2) in an abelian group $N$, then there exists a symmetric difference family of type $H^*_8$ in an abelian group of order $G \times N$.

Before we will prove Theorem 1.8 in Section 3, we will first study some general properties on difference families of type $H$ in Section 2. But in order to apply Theorem 1.8, we construct explicitly difference families of type $H^*$ in Section 4 that are applicable. In the final section, we prove our main results concerning the existence of Hadamard matrices.

**Theorem 1.** Let $\Phi_1 = \{q^2 : q \equiv 1 \mod 4\}$ is a prime power, $\Phi_2 = \{n^4 : n \in N\} \cup \{9n^4 : n \in N\}$, $\Phi_3 = \{5\}$ and $\Phi_4 = \{13, 37\}$. Then, the following hold:
Let us first give an example.

**Example 2.1.** Let \( v = 5 \). Define

\[
D_0 = \{0\}, D_1 = \{1, 4\}, D_2 = \{0\}, D_3 = \{2, 3\}.
\]

Then, the family \( \mathcal{D} = \{D_i : i = 0, 1, 2, 3\} \) is a symmetric difference family of type \( H \) in \( \mathbb{Z}_5 \) that satisfies (d2). Note that \( \mathcal{D} \) also satisfies (d5) but not (d1).

In view of the above example, it is then interesting to explore how condition (d2) and other conditions are related. It is straightforward to check the following:

**Proposition 2.2.** Let \( \mathcal{D} = \{D_i : i = 0, 1, 2, 3\} \) be a symmetric difference family of type \( H \) satisfying (d2) in \( G \) and that \( 0_G \in \bigcap_{i=0}^3 D_i \). Then \( \mathcal{D}' = \{G \setminus D_i : i = 0, 1, 2, 3\} \) is a symmetric difference family of type \( H \) that satisfies (d1) and (d2).

From now on, we always assume \( \mathcal{D} = \{D_i : i = 0, 1, 2, 3\} \) is a symmetric difference family of type \( H \) satisfying (d2) in \( G \); and we define \( E_i \)'s as defined in Definition 1.7. To simplify our notation, we set

\[
W := D_0D_2 + D_1D_3;
\]
\[
X_0 := D_0 + D_1 - D_2 - D_3;
\]
\[
X_1 := D_0 + D_3 - D_1 - D_2;
\]
\[
T_0 := (G + D_2 - D_0)E_0 + (G - D_2 + D_0)\overline{E}_0 + (G + D_1 - D_3)E_1 + (G - D_1 + D_3)\overline{E}_1;
\]
\[
T_1 := (G + D_3 - D_1)E_0 + (G - D_3 + D_1)\overline{E}_0 + (G + D_2 - D_0)E_1 + (G - D_2 + D_0)\overline{E}_1;
\]
\[
T_2 := (G + D_0 - D_2)E_0 + (G - D_0 + D_2)\overline{E}_0 + (G + D_3 - D_1)E_1 + (G - D_3 + D_1)\overline{E}_1;
\]
\[
T_3 := (G + D_1 - D_3)E_0 + (G - D_1 + D_3)\overline{E}_0 + (G + D_0 - D_2)E_1 + (G - D_0 + D_2)\overline{E}_1
\]

throughout this section.

**Proposition 2.3.** Suppose \( \mathcal{D} \) is symmetric and of type \( H \). Then \( \mathcal{D} \) satisfies (d2) if and only if \( \mathcal{D} \) satisfies (d3) and (d4).
PROOF. Suppose \( D \) satisfies (d2). Then \( X_0 = E_0 - \overline{E}_0, \) \( X_1 = E_1 - \overline{E}_1 \), and the set of \( E_0, E_1, \overline{E}_0, \overline{E}_1 \) forms a difference family of type \( H_4^1 \). It is clear that \( D \) satisfies (d4). Observe that all \( D_i \)'s are symmetric. Hence,
\[
(2.1) \quad 4W = 2 \sum_{i=0}^{3} D_i^2 - X_0^2 - X_1^2 = 2 \sum_{i=0}^{3} D_i^2 - (E_0 - \overline{E}_0)^2 - (E_1 - \overline{E}_1)^2.
\]
By assumption, \( D \) is of type \( H \), therefore,
\[
(2.2) \quad \sum_{i=0}^{3} D_i^2 = \left( \sum_{i=0}^{3} |D_i| \right) \cdot 0_G + \left( \sum_{i=0}^{3} |D_i| - |G| \right) G^*.
\]
On the other hand,
\[
(E_0 - \overline{E}_0)^2 + (E_1 - \overline{E}_1)^2 = \sum_{i=0,1} (E_i^2 + \overline{E}_i^2) - 2 \sum_{i=0,1} E_i \overline{E}_i
\]
\[
= \sum_{i=0,1} (E_i^2 + \overline{E}_i^2) + \left( \sum_{i=0,1} (E_i + \overline{E}_i)^2 - \sum_{i=0,1} (E_i^2 + \overline{E}_i^2) \right)
\]
\[
= 2(|G| - 1) \cdot 0_G - 2G^*.
\]
Combining (2.1), (2.2) and (2.3), it is easy to check that (d3) is satisfied.

Conversely, we assume \( D \) satisfies (d3) and (d4). By (d3), we have
\[
(2.4) \quad \sum_{i=0,1} X_i^2 = 2 \sum_{i=0}^{3} D_i^2 - 4W = (2|G| - 2) \cdot 0_G - 2G^*.
\]
We consider the coefficients \( a_x, b_x \) of any \( x \in G \) in \( D_0 + D_1 - D_2 - D_3 \) and \( D_0 + D_3 - D_1 - D_2 \) respectively. By condition (d4), if \( x = 0_G \), \( a_x = b_x = 0 \). If \( x \neq 0_G \), then \( a_x = \pm 1 \) and \( b_x = \pm 1 \). Thus, there are symmetric subsets \( E_0, E_1 \) of \( G^* \) such that \( X = E_0 - \overline{E}_0 \) and \( X_1 = E_1 - \overline{E}_1 \). Furthermore, we have
\[
(2.5) \quad \sum_{i=0,1} X_i^2 = 2 \sum_{i=0,1} (E_i^2 + \overline{E}_i^2) - 2 \cdot 0_G - 2(|G| - 2)G.
\]
Then, combining (2.4) and (2.5), we have
\[
\sum_{i=0,1} (E_i^2 + \overline{E}_i^2) = (2|G| - 2) \cdot 0_G + (|G| - 3)G^*.
\]
Therefore, the set of \( E_0, E_1, \overline{E}_0, \overline{E}_1 \) forms a difference family of type \( H_4^1 \) in \( G \). \( \square \)

**Lemma 2.4.** Let \( D \) be a symmetric difference family of type \( H \) that satisfies (d2) in \( G \). Then
\[ T_0 = T_1 = (|G| - 1) \cdot 0_G + (2|G| - 1)G^* \] and \( T_2 = T_3 = 3(|G| - 1) \cdot 0_G + (2|G| - 3)G^* \).

**Proof.** The proof for obtaining all \( T_i \)'s are similar. Here, we just work with \( T_0 \). By (d2), we see that
\[ D_0 + D_1 - D_2 - D_3 = E_0 - \overline{E}_0 \] and \( D_0 + D_3 - D_1 - D_2 = E_1 - \overline{E}_1 \).
Therefore,

\[
T_0 = (|E_0| + |E_1| + |E_0|)G + (D_2 - D_0)(E_0 - E_0) + (D_1 - D_3)(E_1 - E_1)
\]

\[
= 2(|G| - 1)G - (D_0 - D_2)(D_0 + D_1 - D_2 - D_3) - (D_3 - D_1)(D_0 + D_3 - D_1 - D_2)
\]

(2.6) \[= 2(|G| - 1)G + 2W - \sum_{i=0}^3 D_i^2 = (|G| - 1) \cdot 0_G + (2|G| - 1)G^*.
\]

Note that the last equality in (2.6) follows from Proposition 2.3 and the assumption that \( D \) is of type \( H \).

It turns out that a symmetric difference family of type \( H \) that satisfies (d2) can be constructed using a family of symmetric subsets that satisfies certain conditions.

**Definition 2.5.** We say that a family \( \mathcal{A} = \{A_i : i = 0, 1, \ldots, 7\} \) of symmetric subsets defined on \( G \) is a building family if it satisfies the following conditions:

(a1) \( A_i \cap A_j = \emptyset \) for \( i \neq j \);

(a2) \( \bigcup_{i=0}^7 A_i = G \setminus \{0_G\} \);

(a3) the set of \( \bigcup_{i=0}^7, \bigcup_{i=2,3,4,5} A_i, \bigcup_{i=0,3,5,6} A_i, \bigcup_{i=1,2,4,7} A_i \) forms a difference family of type \( H_4^* \) in \( G \);

(a4) \( \sum_{i=0}^7 A_i^2 - \sum_{i=0}^7 A_i A_{i+4} = (|G| - 1) \cdot 0_G + \sum_{i=0}^3 A_i - \sum_{i=0}^7 A_i, \) where the subscript of \( A_{i+4} \) is reduced modulo 8.

**Example 2.6.** Let \( G = \mathbb{Z}_4 \times \mathbb{Z}_3 \). Define

\( A_0 = \{(0,1), (0,2)\}, A_1 = \{(1,0), (2,0)\}, A_2 = \{(2,1), (1,2)\}, A_3 = \{(1,1), (2,2)\} \)

and \( A_4 = A_5 = A_6 = A_7 = \emptyset \). Then, the family \( \mathcal{A} = \{A_i : i = 0, 1, \ldots, 7\} \) is a building family.

**Proposition 2.7.** Let \( \mathcal{A} = \{A_i : i = 0, 1, \ldots, 7\} \) and \( \mathcal{D} = \{D_0, D_1, D_2, D_3\} \) be families of symmetric subsets in \( G \) such that

\( D_0 = A_1 \cup A_2 \cup A_3, \) \( D_1 = A_5 \cup A_6 \cup A_7 \cup A_3, \) \( D_2 = A_6 \cup A_7 \cup A_1 \cup A_3, \) \( D_3 = A_7 \cup A_0 \cup A_1 \cup A_2. \)

Then \( \mathcal{A} \) is a building family if and only if \( \mathcal{D} \) is a symmetric difference family of type \( H \) that satisfies (d1) and (d2).

**Proof.** Let \( \mathcal{A} \) and \( \mathcal{D} \) be as defined above. First, we prove two claims.

Claim 1. Suppose \( \mathcal{A} \) satisfies (a2). Then \( \mathcal{D} = \{D_i : i = 0, 1, 2, 3\} \) forms a symmetric difference family of type \( H \) if and only if (a4) is satisfied.

Observe that

\[
\sum_{i=0}^3 D_i^2 = \sum_{i=4}^7 A_i^2 + 3 \sum_{i=0}^3 A_i^2 + 2 \sum_{i,j=0: i \neq j} A_i A_j + 2 \sum_{i,j=0: i \neq j} A_i A_{i+4}
\]
(2.7) \[ = \sum_{i=0}^{7} A_i^2 + 2\left(\sum_{i=0}^{3} A_i\right)^2 + 2 \sum_{i,j=0; i \neq j}^{3} A_i A_{j+4}. \]

On the other hand,

(2.8) \[ \frac{1}{2}\sum_{i=0}^{7} A_i A_{i+4} + (|G| - 1) \cdot 0_G + \sum_{i=0}^{3} A_i - \sum_{i=4}^{7} A_i + 2\left(\sum_{i=0}^{3} A_i\right)^2 + 2 \sum_{i,j=0; i \neq j}^{3} A_i A_{j+4} \]

\[ = (|G| - 1) \cdot 0_G + \sum_{i=0}^{3} A_i - \sum_{i=4}^{7} A_i + 2\left(\sum_{i=0}^{3} A_i\right)G^* \]

\[ = (|G| - 1) \cdot 0_G + 2\left(\sum_{i=0}^{3} |A_i|\right)G - \sum_{i=0}^{7} A_i \]

\[ = \left(\sum_{i=0}^{3} |D_i|\right) \cdot 0_G + \left(\sum_{i=0}^{3} |D_i| - |G|\right)G^* \]

since \( \mathcal{A} \) satisfies (a1), (a2) and \( \sum_{i=0}^{3} |D_i| - |G| + 1 = 2 \sum_{i=0}^{3} |A_i| \).

Now, by comparing the two sums (2.7) and (2.8), it is then easy to see that \( \mathcal{D} \) is a symmetric difference family of type \( H \) if and only if (a4) is satisfied.

Claim 2. Suppose \( \mathcal{A} \) satisfies (a1) and (a2). Then (a3) holds for \( \mathcal{A} \) if and only if \( \mathcal{D} \) satisfies (d2).

By assumption, we have

\[ D_0 + D_1 - D_2 - D_3 = \sum_{i=2,3,4,5} A_i - \sum_{i=0,1,6,7} A_i, \]

\[ D_0 + D_3 - D_1 - D_2 = \sum_{i=1,2,4,7} A_i - \sum_{i=0,3,5,6} A_i. \]

As \( \mathcal{A} \) satisfies (a1) and (a2), we may write

\[ E_0 = \bigcup_{i=2,3,4,5} A_i \text{ and } E_1 = \bigcup_{i=1,2,4,7} A_i. \]

Our claim is now clear.

Now, assume \( \mathcal{A} \) is a building family. By Claims 1 and 2, \( \mathcal{D} \) is a symmetric difference family of type \( H \) that satisfies (d2). Finally, (d1) follows easily from (a2).

Conversely, we assume \( \mathcal{D} = \{D_i : i = 0, 1, 2, 3\} \) is a symmetric difference family of type \( H \) in \( G \) satisfying (d1) and (d2). By Claims 1 and 2, it suffices to show that (a1) and (a2) hold. Note that

\[ A_i = \bigcap_{j=0; j \neq i}^{3} D_j, \ i = 0, 1, 2, 3, \]
and

\[ A_i = \bigcap_{j=0,j\neq i-4}^3 (G^* \setminus D_j), \quad i = 4, 5, 6, 7. \]

It is clear that

\[ A_i \cap A_j = \bigcap_{h=0}^3 D_h = \emptyset, \quad i, j = 0, 1, 2, 3, i \neq j, \]

\[ A_i \cap A_j = \bigcap_{h=0}^3 (G^* \setminus D_h) = G^* \setminus \bigcup_{h=0}^3 D_h = \emptyset, \quad i, j = 4, 5, 6, 7, i \neq j, \]

and

\[ A_i \cap A_j = \left( G \setminus \bigcup_{h=0}^3 D_h \right) \cap \left( \bigcap_{h=0}^3 D_h \right) = \emptyset, \quad i = 0, 1, 2, 3, j = 4, 5, 6, 7. \]

Hence, (a1) holds. Next, for any \( x \in G^* \), the coefficient of \( x \) in \( \sum_{i=0}^3 D_i \) is either 1 or 3 as (d2) is satisfied. If the coefficient of \( x \) in \( \sum_{i=0}^3 D_i \) is 1, then \( x \in \bigcup_{i=4}^7 A_i \). On the other hand, if the coefficient of \( x \) in \( \sum_{i=0}^3 D_i \) is 3, then \( x \in \bigcup_{i=0}^3 A_i \). Hence, \( \bigcup_{i=0}^7 A_i = G^* \) and (a2) is satisfied. \( \square \)

**Lemma 2.8.** Let \( A = \{ A_i : i = 0, \ldots, 7 \} \) be a building family of in an abelian group \( G \). Then,

\[ U := (A_0 - A_4)(A_6 - A_2) + (A_1 - A_5)(A_7 - A_3) = -\sum_{i=0}^3 A_i. \]

**Proof.** We continue with our notations used in Proposition 2.7. Then, we have

\[ X_0 = E_0 - \overline{E_0} = \sum_{i=0,1,6,7} A_i - \sum_{i=2,3,4,5} A_i \text{ and } X_1 = E_1 - \overline{E_1} = \sum_{i=0,3,5,6} A_i - \sum_{i=1,2,4,7} A_i. \]

By Proposition 2.7, \( D \) satisfies (d2). Hence, by (2.2), we have

\[(2.9) \quad \sum_{i=0,1} X_i^2 = 2(|G| - 1) \cdot 0_G - 2G^*. \]

On the other hand, by expanding \( X_0^2 \) and \( X_1^2 \), we obtain

\[ \sum_{i=0,1} X_i^2 = 2 \sum_{i=0}^7 A_i^2 - 2 \sum_{i=0}^7 A_i A_{i+4} + 4U. \]

Then, by (a4) and (a1), we get

\[(2.10) \quad \sum_{i=0,1} X_i^2 = 2(|G| - 1) \cdot 0_G + 2 \sum_{i=0}^3 A_i - 2 \sum_{i=4}^7 A_i + 4U = 2(|G| - 1) \cdot 0_G + 4 \sum_{i=0}^3 A_i - 2G^* + 4U. \]

Hence, by (2.9) and (2.10), we obtain \( U = -\sum_{i=0}^3 A_i. \) \( \square \)

To recap, we have shown that if \( D \) is a symmetric difference family of type \( H \) in \( G \), then we have

(a) \( D \) satisfies (d2) if and only if it satisfies both (d3) and (d4).
(b) $\mathcal{D}$ satisfies (d2) if and only if $\mathcal{A}$ defined in Proposition 2.7 is a building family.

As we will show in the next section, condition (d3) and the notion of building family play a very important role in constructing difference families from a group $G$ to a group $G \times G'$.

3. Constructing difference families in $G \times G'$ from difference families in $G, G'$

In this section, we exploit a similar idea used in Polhill’s paper to construct difference families of type $H_i^*$ in $G \times G'$ for some suitable group $G'$. Our constructions are based on the existence of difference families in $G$ that satisfy certain conditions. First, we consider difference families of type $H$ in $G$ that satisfy (d3).

**Example 3.1.**

(a) Let $v = 13$. Define

$$D_0 = \{1, 2, 3, 5, 6, 9\}, \quad D_1 = \{1, 12, 3, 4, 9, 10\}, \quad D_2 = \{0, 2, 5, 6\}, \quad D_3 = \{0, 1, 3, 9\}.$$

Then, $\mathcal{D} = \{D_i : i = 0, 1, 2, 3\}$ is a difference family of type $H$ in $\mathbb{Z}_{13}$ that satisfies (d3).

(b) Let $v = 37$ and $\omega$ be a suitable primitive element in $\mathbb{F}_{37}$, define

$$D_0 = \{0\} \cup \{\omega^{i+12j} : i \in \{0, 3, 7, 8, 10\}, j \in \{0, 1, 2\}\},$$

$$D_1 = \{\omega^{i+12j} : i \in \{0, 1, 7, 8, 10\}, j \in \{0, 1, 2\}\},$$

$$D_2 = \{0\} \cup \{\omega^{i+12j} : i \in \{0, 1, 5, 7, 9\}, j \in \{0, 1, 2\}\},$$

$$D_3 = \{\omega^{i+12j} : i \in \{1, 3, 4, 5, 7\}, j \in \{0, 1, 2\}\}.$$

Then, $\mathcal{D} = \{D_i : i = 0, 1, 2, 3\}$ is a difference family of type $H$ in $\mathbb{Z}_{37}$ that satisfies (d3).

Note that in both examples above, $\mathcal{D}$ constructed is not symmetric.

For convenience, we will abuse the definition of $\times$. For any $X = \sum a_g g \in \mathbb{Z}[G]$ and $Y = \sum b_h h \in \mathbb{Z}[G']$, we define $X \times Y = \sum_{g \in G} \sum_{h \in G'} a_g b_h (g, h) \in \mathbb{Z}[G \times G']$. It is also understood that for any subset $Z$ in $G'$, $X \times Z = X \times \sum_{h \in Z} h$.

**Theorem 3.2.** Suppose $\mathcal{D} = \{D_i : i = 0, 1, 2, 3\}$ is a difference family of type $H$ in an abelian group $G$ that satisfies (d3). If there exists a difference family $S = \{S_i : i = 0, 1, 2, 3\}$ of type $H_i^*$ in an abelian group $G$ satisfying (c1), then there exists a difference family of type $H_i^*$ in $G \times \mathbb{Z}_2$.

**Proof.** Define

$$B_0 = (D_0 \times \{0\}) \cup (D_5^x \times \{1\}),$$

$$B_1 = (D_1 \times \{0\}) \cup (D_5^x \times \{1\}),$$

$$B_2 = (S_0 \times \{0\}) \cup (S_2 \times \{1\}),$$
It is clear that \( \sum_{i=0}^{3} B_{i} B_{i}^{(-1)} = X \times 0 + Y \times 1 \) where

\[
X = \left( \sum_{i=0}^{1} D_{i} D_{i}^{(-1)} + \sum_{i=2}^{3} D_{i}^{c} D_{i}^{c(-1)} + \sum_{i=0}^{3} S_{i} S_{i}^{(-1)} \right)
\]

and

\[
Y = 2(|D_{0}| + |D_{1}|)G - D_{2} D_{0}^{(-1)} - D_{0} D_{2}^{(-1)} - D_{3} D_{1}^{(-1)} - D_{1} D_{3}^{(-1)}
+ S_{0} S_{2}^{(-1)} + S_{2} S_{0}^{(-1)} + S_{1} S_{3}^{(-1)} + S_{3} S_{1}^{(-1)}.
\]

As \( \mathcal{D} \) is a type \( H \) difference family,

\[
\sum_{i=0}^{3} D_{i} D_{i}^{(-1)} + \sum_{i=2}^{3} D_{i}^{c} D_{i}^{c(-1)} = \sum_{i=0}^{3} D_{i} D_{i}^{(-1)} + 2(|G| - |D_{2}| - |D_{3}|)G
= \left( \sum_{i=0}^{3} |D_{i}| + |G| - 2|D_{2}| - 2|D_{3}| \right) G + |G| \cdot 0_{G}.
\]

Since \( \mathcal{S} \) is of type \( H_{4}^{*} \),

\[
\sum_{i=0}^{3} S_{i} S_{i}^{(-1)} = \left( \sum_{i=0}^{3} |S_{i}| - |G| - 1 \right) G + (|G| + 1) \cdot 0_{G}.
\]

Hence,

\[
X = \left( \sum_{i=0}^{3} (|D_{i}| + |S_{i}|) - 2|D_{2}| - 2|D_{3}| - 1 \right) G + (2|G| + 1) \cdot 0_{G}.
\]

Furthermore, as \( \mathcal{D} \) satisfies (d3) and \( \mathcal{S} \) satisfies (c1), we have

\[
Y = 2(|D_{0}| + |D_{1}|)G - \left( \sum_{i=0}^{3} |D_{i}| - |G| + 1 \right) G + \left( \sum_{i=0}^{3} |S_{i}| - |G| \right) G
= \left( \sum_{i=0}^{3} (|D_{i}| + |S_{i}|) - 1 - 2|D_{2}| - 2|D_{3}| \right) G.
\]

Note that

\[
\sum_{i=0}^{3} |B_{i}| - |G \times \mathbb{Z}_{2}| - 1 = \sum_{i=0}^{3} (|D_{i}| + |S_{i}|) - 2|D_{2}| - 2|D_{3}| - 1.
\]

Hence, \( B = \{ B_{i} : i = 0, 1, 2, 3 \} \) forms a difference family of type \( H_{4}^{*} \) in \( G \times \mathbb{Z}_{2} \).

In order to apply the above result, we need a difference family of type \( H_{4}^{*} \) satisfying (c1). One easy way to construct such families is by using Paley type partial difference sets.

**Lemma 3.3.** Let \( P \) be a Paley type partial difference set in \( G \). Then the family

\[
\mathcal{S} = \{ P, P \cup \{ 0_{G} \}, P \cup \{ 0_{G} \}, G \setminus P \}
\]

is a difference family of type \( H_{4}^{*} \) that satisfies (c1).
In particular, there is a difference family of type $H^*_4$ satisfying (c1) for $|G| = 13, 37$. At the same time, we have shown in Example 3.1 that there exist difference families of type $H$ that satisfy (d3). We have thus shown the following:

**Proposition 3.4.** There exists a difference family of type $H^*_4$ in $\mathbb{Z}_v \times \mathbb{Z}_2$ for $v = 13, 37$.

Next, we consider the case when $G' = \mathbb{Z}_3$.

**Theorem 3.5.** Suppose $D = \{D_i : i = 0, 1, 2, 3\}$ is a symmetric difference family of type $H$ in an abelian group $G$ satisfying (d3). If there exists a difference family $S = \{S_i : i = 0, 1\}$ of type $H^*_2$ in an abelian group $G$. Then, there exists a difference family of type $H^*_4$ in $G \times \mathbb{Z}_3$.

**Proof.** Define

$$B_0 = (D_0 \times \{1\}) \cup (D^c_2 \times \{2\}) \cup (S_0 \times \{0\}),$$
$$B_1 = (D_3 \times \{1\}) \cup (D^c_3 \times \{2\}) \cup (S_1 \times \{0\}),$$
$$B_2 = (D^c_0 \times \{1\}) \cup (D_2 \times \{2\}) \cup (S_0 \times \{0\}),$$
$$B_3 = (D^c_0 \times \{1\}) \cup (D_1 \times \{2\}) \cup (S_1 \times \{0\}).$$

Since $D_i$'s are symmetric, it is straightforward to check that

$$\sum_{i=0}^{3} B_i B_i^{(-1)} = \left( \sum_{i=0}^{3} (D_i^2 + D_i^{c2}) + 2 \sum_{i=0,1} S_i S_i^{(-1)} \right) \times \{0\}$$
$$+ (D_2 D_0 + D_0 D_2 + D_3 D_3 + D_3 D_1) \times \{1, 2\}$$
$$+ (S_0^{(-1)}(D_0 + D_0^c) + S_3^{(-1)}(D_3 + D_3^c) + S_0(D_2 + D_2^c) + S_1(D_1 + D_1^c)) \times \{1\}$$
$$+ (S_0(D_0 + D_0^c) + S_1(D_3 + D_3^c) + S_0^{(-1)}(D_2 + D_2^c) + S_1^{(-1)}(D_1 + D_1^c)) \times \{2\}.$$

By using the assumptions for $D$ and $E$, we have

$$\sum_{i=0}^{3} (D_i^2 + D_i^{c2}) + 2 \sum_{i=0,1} S_i S_i^{(-1)} = \left( 2 \sum_{i=0,1} |S_i| + |G| - 1 \right) G^* + \left( 4|G| + 2 \sum_{i=0,1} |S_i| \right) \cdot 0_G.$$

Furthermore, as $D$ satisfies (d3), we have

$$D_2 D_0 + D_0^c D_2 + D_3 D_3 + D_3 D_1 = \left( \sum_{i=0}^{3} |D_i| \right) G - \left( \sum_{i=0}^{3} |D_i| - |G| + 1 \right) G = (|G| - 1) G.$$

Finally, it is clear that
in an abelian group $G$ need a stronger condition on $\mathcal{D}$. Summing up, we obtain
\[
S_0^{(-1)}(D_0 + D_0^c) + S_1^{(-1)}(D_3 + D_3^c) + S_0(D_2 + D_2^c) + S_1(D_1 + D_1^c)
= S_0(D_0 + D_0^c) + S_1(D_3 + D_3^c) + S_0^{(-1)}(D_2 + D_2^c) + S_1^{(-1)}(D_1 + D_1^c)
= 2\left(\sum_{i=0,1} |S_i|\right)G.
\]

This completes the proof of the theorem. \(\Box\)

Unfortunately, condition (d3) is not sufficient to deal with other $G'$. When $G' = \mathbb{Z}_5$, we need a stronger condition on $\mathcal{D}$.

**Theorem 3.6.** Suppose $\mathcal{D} = \{D_i : i = 0, 1, 2, 3\}$ is a symmetric difference family of type $H$ in an abelian group $G$ satisfying (d2) and (d5). Then, there exists a difference family of type $H_4^*$ in $G \times \mathbb{Z}_5$.

**Proof.** Let $I_0 = \{1, 2\}$, $I_1 = \{3, 4\}$, $I_2 = \{2, 4\}$, and $I_3 = \{1, 3\}$. Furthermore, we let $E_0, E_1$ be as defined in condition (d2). We define
\[
B_0 = (D_0 \times I_0) \cup (D_0^c \times I_1) \cup (\overline{E_0} \times \{0\}),
B_1 = (D_3 \times I_0) \cup (D_3^c \times I_1) \cup (\overline{E_1} \times \{0\}),
B_2 = (D_0^c \times I_2) \cup (D_2 \times I_3) \cup (E_0 \times \{0\}),
B_3 = (D_3^c \times I_2) \cup (D_1 \times I_3) \cup (E_1 \times \{0\}).
\]

Note that $D_i$'s and $E_i$'s are symmetric. For convenience, we write
\[
\sum_{i=0}^{3} B_i B_i^{(-1)} = \sum_{i=1}^{6} X_i
\]
where
\[
X_1 = (D_0^2 + D_3^2) \times (I_0 I_0^{(-1)} + I_2 I_2^{(-1)}) + (D_1^2 + D_2^2) \times (I_1 I_1^{(-1)} + I_3 I_3^{(-1)})
X_2 = (\|D_0\| + |D_3|)G \times D_0 D_2 - D_1 D_3 \times (I_0 I_0^{(-1)} + I_1 I_1^{(-1)})
X_3 = (\|D_1\| + |D_2|)G \times D_0 D_2 - D_1 D_3 \times (I_2 I_2^{(-1)} + I_3 I_3^{(-1)})
X_4 = \left(2|G| - 2 \sum_{i=1,2} |D_i|\right)G \times I_1 I_1^{(-1)} + \left(2|G| - 2 \sum_{i=0,3} |D_i|\right)G \times I_2 I_2^{(-1)}
X_5 = [\overline{E_0}(D_0 + D_0^c) + \overline{E}_1(D_3 + D_3^c) + E_0(D_2 + D_0^c) + E_1(D_1 + D_0^c)] \times \mathbb{Z}_5
X_6 = \sum_{i=0,1} (E_i^2 + \overline{E}_i^2) \times 0.
\]
Since \( I_0I_0^{(-1)} + I_2I_2^{(-1)} = I_1I_1^{(-1)} + I_3I_3^{(-1)} = 4 \cdot 0 + \mathbb{Z}_5^* \) and \( \mathcal{D} \) forms a difference family of type \( H \),

\[
X_1 = \left( \sum_{i=0}^{3} D_i^2 \right) \times (4 \cdot 0 + \mathbb{Z}_5^*) = [(\sum_{i=0}^{3} |D_i|) \cdot 0_G + (\sum_{i=0}^{3} |D_i| - |G|)G^*] \times (4 \cdot 0 + \mathbb{Z}_5^*).
\]

Note that \( |D_0| + |D_3| = |D_1| + |D_2| \), and \( I_0I_0^{(-1)} + I_1I_0^{(-1)} + I_2I_3^{(-1)} + I_3I_2^{(-1)} = 4\mathbb{Z}_5^* \). Moreover, \( \mathcal{D} \) satisfies (d3) by Proposition 2.3. Hence,

\[
2(D_0D_2 + D_1D_3) = (\sum_{i=0}^{3} |D_i| - |G| + 1)G.
\]

Therefore,

\[
X_2 + X_3 = \frac{1}{2}(|G| - 1)G \times (I_2I_3^{(-1)} + I_3I_2^{(-1)}) = 2(|G| - 1)(G \times \mathbb{Z}_5^*).
\]

Again, as \( |D_0| + |D_3| = |D_1| + |D_2| \),

\[
2|G| - 2 \sum_{i=1, 2} |D_i| = 2|G| - 2 \sum_{i=0, 3} |D_i| = 2|G| - \sum_{i=0}^{3} |D_i|.
\]

Therefore,

\[
X_4 = (2|G| - \sum_{i=0}^{3} |D_i|)G \times (I_1I_1^{(-1)} + I_2I_2^{(-1)}) = (2|G| - \sum_{i=0}^{3} |D_i|)G \times (4 \cdot 0 + \mathbb{Z}_5^*).
\]

For \( X_5 \), since \( \mathcal{D} \) satisfies (d2), it follows from Lemma 2.4 that

\[
\overline{E}_0(D_0 + D_2^c) + \overline{E}_1(D_3 + D_1^c) + E_0(D_2 + D_0^c) + E_1(D_4 + D_3^c) = T_0 = (|G| - 1) \cdot 0_G + (2|G| - 1)G^*.
\]

Therefore,

\[
X_5 = [(|G| - 1) \cdot 0_G + (2|G| - 1)G^*] \times \mathbb{Z}_5^*.
\]

Finally, as \( E_0, E_1, \overline{E}_0, \overline{E}_2 \) forms a difference family of type \( H_4^* \), we get

\[
X_6 = 2(|G| - 1) \cdot 0_G + (|G| - 3)(G^* \times \mathbb{Z}_5^*).
\]

Summing up, we obtain \( \sum_{i=0}^{3} B_iB_i^{(-1)} = (10|G| - 2) \cdot 0_{G \times \mathbb{Z}_5} + (5|G| - 3)(G \times \mathbb{Z}_5)^* \). This completes the proof of the theorem.

It seems difficult to generalize the above constructions for other \( G' \). However, instead of constructing symmetric difference families of type \( H_4^* \), it is possible to construct symmetric difference families of type \( H_5^* \) under stronger assumptions. In the previous examples, we simply make use of the sets \( D_i \) or \( D_i^c \). But in the next construction, we need to make use of a building family in \( G' \).

**Theorem 3.7.** Suppose \( \mathcal{D} = \{ D_i : i = 0, 1, 2, 3 \} \) is a symmetric difference family in an abelian group \( G \) satisfying the (d2). If there exists a building family \( \mathcal{B} = \{ B_i : i = 0, 1, \ldots, 7 \} \) in an abelian group \( G' \), then, there exists a symmetric difference family of type \( H_5^* \) in \( G \times G' \).
Proof. As $D$ satisfies (d2), we let $E_0, E_1$ be as defined in (d2). We define

$C_0 = (D_0 \times B_0) \cup (D_1 \times B_1) \cup (D_2^* \times B_2) \cup (D_3^c \times B_3)$
$\cup (D_0^c \times B_4) \cup (D_1^c \times B_5) \cup (D_2 \times B_6) \cup (D_3 \times B_7) \cup (\overline{E_0} \times \{0_G\})$,

$C_1 = (D_0 \times B_1) \cup (D_1 \times B_4) \cup (D_2^* \times B_3) \cup (D_3^c \times B_6)$
$\cup (D_0^c \times B_5) \cup (D_1^c \times B_0) \cup (D_2 \times B_7) \cup (D_3 \times B_2) \cup (\overline{E_1} \times \{0_G\})$,

$C_2 = (D_0 \times B_2) \cup (D_1 \times B_3) \cup (D_2^* \times B_0) \cup (D_3^c \times B_1)$
$\cup (D_0^c \times B_6) \cup (D_1^c \times B_7) \cup (D_2 \times B_4) \cup (D_3 \times B_5) \cup (\overline{E_0} \times \{0_G\})$,

$C_3 = (D_0 \times B_3) \cup (D_1 \times B_6) \cup (D_2^* \times B_1) \cup (D_3^c \times B_4)$
$\cup (D_0^c \times B_7) \cup (D_1^c \times B_2) \cup (D_2 \times B_5) \cup (D_3 \times B_0) \cup (\overline{E_1} \times \{0_G\})$,

$C_4 = (D_0 \times B_4) \cup (D_1 \times B_5) \cup (D_2^* \times B_6) \cup (D_3^c \times B_7)$
$\cup (D_0^c \times B_0) \cup (D_1^c \times B_1) \cup (D_2 \times B_2) \cup (D_3 \times B_3) \cup (E_0 \times \{0_G\})$,

$C_5 = (D_0 \times B_5) \cup (D_1 \times B_0) \cup (D_2^* \times B_7) \cup (D_3^c \times B_2)$
$\cup (D_0^c \times B_1) \cup (D_1^c \times B_4) \cup (D_2 \times B_3) \cup (D_3 \times B_6) \cup (E_1 \times \{0_G\})$,

$C_6 = (D_0 \times B_6) \cup (D_1 \times B_7) \cup (D_2^* \times B_4) \cup (D_3^c \times B_5)$
$\cup (D_0^c \times B_2) \cup (D_1^c \times B_3) \cup (D_2 \times B_0) \cup (D_3 \times B_1) \cup (E_0 \times \{0_G\})$,

$C_7 = (D_0 \times B_7) \cup (D_1 \times B_2) \cup (D_2^* \times B_5) \cup (D_3^c \times B_0)$
$\cup (D_0^c \times B_3) \cup (D_1^c \times B_6) \cup (D_2 \times B_1) \cup (D_3 \times B_4) \cup (E_1 \times \{0_G\})$.

It is clear that each $C_i$ is symmetric. We now prove that

\[(3.1) \quad \sum_{i=0}^{7} C_i^2 = 4(|G| \cdot |G'| - 1) \cdot 0_{G \times G'} + 2(|G| \cdot |G'| - 3)(G \times G')^* .\]

To simplify notations, we set

\[P = \sum_{i=0}^{7} B_i^2 - \sum_{i=0}^{7} B_i B_{i+4},\]
\[Q = \left( \sum_{i=2,3,4,5} B_i \right) \left( \sum_{i=0,1,6,7} B_i \right) + \left( \sum_{i=0,3,5,6} B_i \right) \left( \sum_{i=1,2,4,7} B_i \right),\]
\[R = \left( \sum_{i=2,3,4,5} B_i \right)^2 + \left( \sum_{i=0,1,6,7} B_i \right)^2 + \left( \sum_{i=0,3,5,6} B_i \right)^2 + \left( \sum_{i=1,2,4,7} B_i \right)^2 .\]

As before, we let $W, T_0, T_1, T_2, T_3$ be as defined in Section 2 and

\[U := (B_0 - B_4)(B_6 - B_2) + (B_1 - B_5)(B_7 - B_3).\]
Then, we have
\[
\sum_{i=0}^{7} B_i^2 = 2\left(3 \sum_{i=0}^{3} D_i^2\right) \times P + 8W \times U + 2\left(3 \sum_{i=0}^{3} |D_i|\right) G \times Q
\]
\[
+ (2|G| - \sum_{i=0}^{3} |D_i|) G \times R + 2\left(\sum_{i=0,1} E_i^2 + \sum_{i=0,1} E_i^2\right) \times 0_{G'}
\]
\[
+ 2[T_0 \times (B_0 + B_2) + T_1 \times (B_1 + B_3) + T_2 \times (B_4 + B_6) + T_3 \times (B_5 + B_7)].
\]

By (a4), we have
\[
2\left(3 \sum_{i=0}^{3} D_i^2\right) \times P = [2\left(3 \sum_{i=0}^{3} |D_i|\right) \cdot 0_G + 2\left(3 \sum_{i=0}^{3} |D_i| - |G|\right) G^*] \times [(|G'| - 1) \cdot 0_{G'} + \sum_{i=0}^{3} B_i - \sum_{i=4}^{7} B_i].
\]

Next, by Proposition 2.3 and Lemma 2.8, we have
\[
8W \times U = -4\left(3 \sum_{i=0}^{3} |D_i| - |G| + 1\right) G \times \left(3 \sum_{i=0}^{3} B_i\right).
\]

Furthermore, by (a3), we have
\[
2\left(3 \sum_{i=0}^{3} |D_i|\right) G \times Q + \left(2v - \sum_{i=0}^{3} |D_i|\right) G \times R
\]
\[
= 2\left(3 \sum_{i=0}^{3} |D_i| + |G|(|G' - 3)\right) G \times (G^* + 2(|G' - 1) \cdot (2|G| - \sum_{i=0}^{3} |D_i|)(G \times 0_{G'}).
\]

Moreover, by (d2),
\[
2\left[\sum_{i=0,1} E_i^2 + \sum_{i=0,1} E_i^2\right] \times 0_{G'} = [4(|G| - 1) \cdot 0_G + 2(|G| - 3) G^*] \times 0_{G'}.
\]

Finally, by Lemma 2.4, we have
\[
2[T_0 \times (B_0 + B_2) + T_1 \times (B_1 + B_3) + T_2 \times (B_4 + B_6) + T_3 \times (B_5 + B_7)]
\]
\[
= [2(|G| - 1) \cdot 0_G + (2|G| - 1) G^*] \times \left(3 \sum_{i=0}^{3} B_i\right) + 2[3(|G| - 1) \cdot 0_G + (2|G| - 3) G^*] \times \left(7 \sum_{i=4}^{7} B_i\right).
\]

Summing up, we obtain (3.1). This completes the proof of the theorem. \(\square\)

In view of Theorems 3.2, 3.5, 3.6 and 3.7, we have then completed the proof of Theorem 1.8.

4. Symmetric difference families of type H that satisfy (d1) and (d2)

To apply the constructions we obtain in the previous section, it remains to find groups where symmetric difference families of type H satisfying (d1) and (d2) exist in. In [14], the authors constructed difference families of type H in \((\mathbb{F}_{q^2}, +)\) for \(q \equiv 1 (\text{mod } 4)\). Our first result is to show that those difference families actually satisfy (d1) and (d2). Let us recall the construction.
Let \( q = 4m + 1 \) be a prime power and let \( \omega \) be a primitive element of \( \mathbb{F}_q^2 \). Let \( N \) be a divisor of \( q^2 - 1 \). We define

\[
C_i^{(N,q^2)} = \omega^i \langle \omega^N \rangle, \quad i = 0, 1, \ldots, N - 1,
\]

and

\[
D_0 := \left( \bigcup_{\ell=0}^{m-1} \bigcup_{u=1}^3 C_{4\ell+2}^{(8m+4,q^2)} \right) \cup \left( \bigcup_{j=0}^m C_{4j-2}^{(8,4,q^2)} \right), \quad D_i = \omega^{(2m+1)i}D_0, \quad i = 1, 2, 3.
\]

It has been shown in [14] that \( D = \{ D_i : i = 0, 1, 2, 3 \} \) forms a difference family of type \( H \) in \( (\mathbb{F}_q^2,+) \). It is clear that each \( D_i \) is symmetric and \( 0 \notin \bigcup_{i=0} D_i \). Therefore, (d1) is satisfied. Furthermore, we have

\[
D_0 + D_1 - D_2 - D_3 = \left( \sum_{\ell=0}^{m-1} \sum_{u=2,3} C_{4\ell+2}^{(8m+4,q^2)} \right) + \sum_{j=0}^m C_{4j-2}^{(8,4,q^2)}
\]

\[
- \left( \sum_{\ell=0}^{m-1} \sum_{u=0,1} C_{4\ell+2}^{(8m+4,q^2)} \right) + \sum_{j=0}^m \sum_{u=2,3} C_{4j-2}^{(8m+4,q^2)}
\]

\[
= (C_2^{(4,q^2)} + C_{2m+3}^{(4,q^2)}) - (C_0^{(4,q^2)} + C_{2m+1}^{(4,q^2)}),
\]

\[
D_0 + D_3 - D_1 - D_2 = \left( \sum_{\ell=0}^{m-1} \sum_{u=1,2} C_{4\ell+2}^{(8m+4,q^2)} \right) + \sum_{j=0}^m \sum_{u=0,3} C_{4j-2}^{(8m+4,q^2)}
\]

\[
- \left( \sum_{\ell=0}^{m-1} \sum_{u=0,3} C_{4\ell+2}^{(8m+4,q^2)} \right) + \sum_{j=0}^m \sum_{u=1,2} C_{4j-2}^{(8m+4,q^2)}
\]

\[
= (C_2^{(4,q^2)} + C_{2m+1}^{(4,q^2)}) - (C_0^{(4,q^2)} + C_{2m+3}^{(4,q^2)}).
\]

It is known (cf. [22]) that the set of \( E_0 = C_2^{(4,q^2)} \cup C_{2m+3}^{(4,q^2)} \), \( E_0 = C_0^{(4,q^2)} \cup C_{2m+1}^{(4,q^2)} \), \( E_1 = C_2^{(4,q^2)} \cup C_{2m+1}^{(4,q^2)} \), \( E_1 = C_0^{(4,q^2)} \cup C_{2m+3}^{(4,q^2)} \) forms a difference family of type \( H^* \) in \( (\mathbb{F}_q^2,+) \). Hence, \( D \) satisfies (d2). Note that all \(|D_i|\)'s are equal. Therefore, (d5) is satisfied. As \( D \) is also symmetric, it follows from Proposition 2.3 that (d3) and (d4) are also satisfied. We have thus proved the following:

**Theorem 4.1.** Let \( q \equiv 1 (\mod 4) \) be a prime power. Then, there exists a symmetric difference family of type \( H \) in \( (\mathbb{F}_q^2,+) \) that satisfies (d1)-(d5).

Another construction of symmetric difference family \( D \) of type \( H \) in \( G = (\mathbb{F}_q^4,+) \) are found in [2, 13, 19, 21].

**Proposition 4.2.** Let \( G = (\mathbb{F}_0,+) \) or \( G = (\mathbb{F}_q^4,+) \) where \( q \) is an odd prime power. There exists a symmetric difference family \( D \) of type \( H \) in \( G \) such that the following conditions are satisfied.

(i) For any \( i = 0, 1, 2, 3 \), \(|D_i| = \frac{|G|-\sqrt{|G|}}{2} \).
(ii) For any nontrivial character ψ of G, exactly one of the values ψ(D_i), i = 0, 1, 2, 3, is nonzero; and is equal to ±√|G|.

(iii) \( H_0 + H_1 + H_4 + H_5 = G + 1 \cdot 0_G \) and \( 0_G \notin \bigcup_{i=0}^{3} D_i \) or \( 0_G \in \bigcap_{i=0}^{3} D_i \) if we set

\[
H_0 = D_0 \cap D_1, \quad H_1 = D_0^c \cap D_1^c, \quad H_2 = D_0 \cap D_1^c, \quad H_3 = D_0^c \cap D_1, \\
H_4 = D_2 \cap D_3, \quad H_5 = D_2^c \cap D_3^c, \quad H_6 = D_2 \cap D_3^c, \quad H_7 = D_2^c \cap D_3.
\]

Furthermore, we may assume \( 0_G \notin \bigcup_{i=0}^{3} D_i \) if \( G = (\mathbb{F}_q^4, +) \) and \( 0_G \in \bigcap_{i=0}^{3} D_i \) if \( G = (\mathbb{F}_9, +) \).

**Remark 4.3.**

(a) Note that by using conditions (i) and (ii) above and Fourier transform, it is clear that \( \mathcal{D} \) satisfies (d3) as we have

\[
2(D_0 D_2 + D_1 D_3) = (\sqrt{|G|} - 1)^2 G = (\sum_{i=0}^{3} |D_i| - |G| + 1) G.
\]

(b) \( \mathcal{D} \) satisfies (d4) if condition (iii) is satisfied since \( \sum_{i=0}^{3} D_i = 2(H_0 + H_4) + H_2 + H_3 + \)
\[H_6 + H_7 = 2(H_0 + H_4) + G^*.
\]

(c) Therefore, by Proposition 2.3, \( \mathcal{D} \) satisfies (d2) if \( \mathcal{D} \) satisfies conditions (i), (ii) and (iii) above.

In view of the above remark, we obtain another symmetric difference families of type \( H \) that satisfy (d2). However, we do not obtain new parameters from these families. To construct one with new parameters, we employ a standard technique.

Suppose \( \mathcal{D} = \{ D_i : i = 0, 1, 2, 3 \} \) and \( \mathcal{F} = \{ F_i := i = 0, 1, 2, 3 \} \) are symmetric difference families of type \( H \) in abelian group \( G \) and \( N \) respectively. If both \( \mathcal{D} \) and \( \mathcal{F} \) satisfy conditions (i), (ii) and (iii) in Proposition 4.2, we then define the following:

\[
B_0 = H_1 \times F_0 + H_0 \times F_0^c + H_3 \times F_2 + H_2 \times F_2^c \\
B_1 = H_1 \times F_1 + H_0 \times F_1^c + H_2 \times F_3 + H_3 \times F_3^c \\
B_2 = H_6 \times F_0 + H_7 \times F_0^c + H_5 \times F_2 + H_4 \times F_2^c \\
B_3 = H_7 \times F_1 + H_6 \times F_1^c + H_5 \times F_3 + H_4 \times F_3^c.
\]

Clearly, \( \mathcal{B} \) satisfies condition (i). As shown in (cf. [6]), \( \mathcal{B} = \{ B_i : i = 0, 1, 2, 3 \} \) is indeed a symmetric difference family of type \( H \) in \( G \times N \) satisfying condition (ii). We now check that \( \mathcal{B} \) satisfies condition (iii) under the assumption that \( 0_N \notin \bigcup_{i=0}^{3} F_i \):

\[
(B_0 \cap B_1) + (B_0^c \cap B_1^c) + (B_2 \cap B_3) + (B_2^c \cap B_3^c) \\
= H_1 \times (F_0 \cap F_1) + H_0 \times (F_0^c \cap F_1^c) + H_3 \times (F_2 \cap F_3) + H_2 \times (F_3 \cap F_2^c) \\
+ H_1 \times (F_0^c \cap F_1^c) + H_0 \times (F_0 \cap F_1) + H_3 \times (F_2 \cap F_3^c) + H_2 \times (F_3^c \cap F_2) \\
+ H_6 \times (F_0 \cap F_1^c) + H_7 \times (F_0^c \cap F_1) + H_5 \times (F_2^c \cap F_3) + H_4 \times (F_3^c \cap F_2^c) \\
+ H_6 \times (F_0^c \cap F_1^c) + H_7 \times (F_0 \cap F_1) + H_5 \times (F_2^c \cap F_3^c) + H_4 \times (F_3^c \cap F_2) \\
+ H_6 \times (F_0^c \cap F_1^c) + H_7 \times (F_0 \cap F_1) + H_5 \times (F_2^c \cap F_3^c) + H_4 \times (F_3^c \cap F_2)
\]
(4.1)

\[ (H_0 + H_1) \times ( (F_0 \cap F_1) \cup (F_0^c \cap F_1^c)) + (H_2 + H_3) \times ( (F_2 \cap F_3) \cup (F_2^c \cap F_3^c)) \]

\[ + (H_4 + H_5) \times ( (F_4 \cap F_5) \cup (F_4^c \cap F_5^c)) \]

Since \( H_0 + H_1 + H_4 + H_5 = G + 1 \cdot 0_G \) and \( \sum_{i=0}^{3} H_i = \sum_{i=4}^{7} H_i = G \), we have

\[ H_6 + H_7 = H_0 + H_1 - 1 \cdot 0_G \] and \( H_2 + H_3 = H_4 + H_5 - 1 \cdot 0_G \).

It then follows from (4.1) that we have

\[ (B_0 \cap B_1) \cup (B_0^c \cap B_1^c) \cup (B_2 \cap B_3) \cup (B_2^c \cap B_3^c) \]

\[ = (H_0 + H_1 + H_4 + H_5) \times N - 0_G \times ((F_2 \cap F_3) \cup (F_2^c \cap F_3^c) \cup (F_0 \cap F_1) \cup (F_0^c \cap F_1^c)) \]

\[ = (G + 1 \cdot 0_G) \times N - 0_G \times N^* = G \times N + 1 \cdot 0_{G \times N} \]

as \( (F_2 \cap F_3) \cup (F_2^c \cap F_3^c) \cup (F_0 \cap F_1) \cup (F_0^c \cap F_1^c) = N^* \). Finally, we need to compute \( \sum_{i=0}^{3} B_i \).

Observe that the coefficient of \( 0_{G \times N} \) in \( \sum_{i=0}^{3} B_i \) is zero if \( 0_G \not\in \bigcup_{i=0}^{3} D_i \) and four if \( 0_G \in \bigcap_{i=0}^{3} D_i \). That means \( 0_G \not\in \bigcup_{i=0}^{3} B_i \) or \( 0_G \in \bigcap_{i=0}^{3} B_i \). Therefore, \( \mathcal{B} \) satisfies (iii).

By Remark 4.3, the symmetric difference family \( \mathcal{B} \) satisfies (d2). However, it is not clear if it satisfies (d1). But in view of Proposition 2.2, we conclude that either \( \mathcal{B} \) or \( \mathcal{B}' = \{ G \times N - B_i : i = 0, 1, 2, 3 \} \) is a symmetric difference family of type \( H \) that satisfies (d1) and (d2). Finally, as \( \mathcal{B} \) and \( \mathcal{B}' \) satisfy (i), (d5) holds. To summarize, we have shown the following:

**Proposition 4.4.** Suppose there exist symmetric difference families \( \mathcal{D} = \{ D_i : i = 0, 1, 2, 3 \} \) and \( \mathcal{F} = \{ F_i : i = 0, 1, 2, 3 \} \) of type \( H \) satisfying conditions (i), (ii) and (iii) in abelian groups \( G \) and \( N \), respectively. Furthermore, we assume \( 0_N \not\in \bigcup_{i=0}^{3} F_i \). Then, there exists a symmetric difference family \( \mathcal{B} = \{ B_i : i = 0, 1, 2, 3 \} \) of type \( H \) satisfying conditions (i), (ii) and (iii) in \( G \times N \). Moreover, either \( \mathcal{B} \) or \( \mathcal{B}' = \{ G \setminus B_i : i = 0, 1, 2, 3 \} \) is a symmetric difference family of type \( H \) that satisfies (d1)-(d5).

By applying Propositions 4.2 and 4.4 repeatedly, we obtain the following theorem.

**Theorem 4.5.** Let \( p_1, p_2, \ldots, p_s \) be distinct primes and \( G = N \times \mathbb{Z}_{p_1}^{4t_1} \times \mathbb{Z}_{p_1}^{4t_1} \times \cdots \times \mathbb{Z}_{p_s}^{4t_s} \) where \( N = \{ 0 \} \) or \( \mathbb{Z}_2^2 \). Then, there exists a symmetric difference family of type \( H \) in \( G \) satisfying (d1)-(d5).

**5. Conclusion**

In this section, we prove Theorem 1.9. Recall that

\[ \Phi_1 = \{ q^2 : q \equiv 1 (\text{mod} \ 4) \text{ is a prime power} \}, \]

\[ \Phi_2 = \{ n^4 : n \in \mathbb{N} \} \cup \{ 9n^4 : n \in \mathbb{N} \}, \Phi_3 = \{ 5 \} \text{ and } \Phi_4 = \{ 13, 37 \}. \]

Note that for \( v \in \Phi_1 \cup \Phi_2 \), it follows from Theorems 4.1 and 4.5 that there exists a symmetric difference family of type \( H \) that satisfies (d1)-(d5). For \( v \in \Phi_3 \cup \Phi_4 \), difference families constructed explicitly in Examples 2.1 and 3.1 satisfy (d3).
On the other hand, it is known that a Paley type partial difference set $P$ exists in an abelian group of order $v$ for any $v \in (\Phi_1 \cup \Phi_2 \cup \Phi_3 \cup \Phi_4)$ (see, e.g., [6]). By Lemma 3.3, there exists a difference family of type $H_4^*$ satisfying (c1) in an abelian group of order $v$. Also, $\{P, \bar{P}\}$ is a difference family of type $H_2^*$. Therefore, by applying Theorems 3.2, 3.5 and 3.6, we obtain the following:

**Theorem 5.1.** There exists a difference family of type $H_4^*$ in an abelian group of order $2v$ for any $v \in (\Phi_1 \cup \Phi_2 \cup \Phi_3 \cup \Phi_4)$.

**Theorem 5.2.** There exists a difference family of type $H_4^*$ in an abelian group of order $3v$ for any $v \in (\Phi_1 \cup \Phi_2 \cup \Phi_3)$.

**Theorem 5.3.** There exists a difference family of type $H_4^*$ in an abelian group of order $5v$ for any $v \in (\Phi_1 \cup \Phi_2 \cup \Phi_3)$.

Note that in Theorems 5.2 and 5.3, $v \notin \Phi_4$ as the family constructed in Example 3.1 is not symmetric. The next result is now obvious in view of Theorem 3.7.

**Theorem 5.4.** There exists a symmetric difference family of type $H_8^*$ in an abelian group of order $uv$ for any $u \in (\Phi_1 \cup \Phi_2)$ and $v \in (\Phi_1 \cup \Phi_2 \cup \Phi_3)$.

Finally, by using the Wallis-Whiteman array together with the difference families in Theorems 5.1, 5.2 and 5.3, we obtain (1), (2) and (3) in Theorem 1.9. By using the Kharaghani array together with the difference families in Theorem 5.4, we obtain (4) in Theorem 1.9.

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