LIMITING BEHAVIOR OF TRAJECTORIES OF COMPLEX POLYNOMIAL VECTOR FIELDS

S. IVASHKOVICH

Abstract. We prove that every trajectory of a polynomial vector field on the complex projective plane accumulates to the singular locus of the vector field. This statement represents a holomorphic version of the Poincaré-Bendixson theorem and solves the complex analytic counterpart of Hilbert’s 16th problem. The main result can be also reformulated as the nonexistence of “exceptional minimals” of holomorphic foliations on \( \mathbb{P}^2 \) and, in particular, implies the nonexistence of real analytic Levi flat hypersurfaces in the complex projective plane. Finally, we describe (in the first approximation) the way a minimal complex trajectory approaches the singular locus of the vector field.

Section 1. Introduction. ................................................. pp. 2–10

1.1. Statement of the main result. 1.2. BLM-trichotomy. 1.3. The role of the holonomy group. 1.4. Reduction to nef models. 1.5. Minimal sets in projective spaces. 1.6. Levi flat hypersurfaces. 1.7. Exceptional minimals of Briot-Bouquet foliations. 1.8. Exceptional minimals and Levi flats in Hirzebruch surfaces. 1.9. Approach of a leaf to the singular locus. 1.10. Notes. 1.11. Acknowledgement.

Section 2. Pseudoconvexity of Poincaré domains. ......................... pp. 10–21

2.1. Minimal sets of holomorphic foliations. 2.2. Poincaré domains. 2.3. Docquier-Grauert criterion and Fujita theorems. 2.4. Local pseudoconvexity of Poincaré domains. 2.5. Nef models of holomorphic foliations. 2.6. Obstructions to the existence of universal covering Poincaré domains. 2.7. A Rothstein type extension theorem.

Section 3. Holomorphic representations of the fundamental group. ...... pp. 21–30

3.1. A germ of the holomorphic representation. 3.2. Expansion of the holomorphic representation. 3.3. Universal covering Poincaré domains of hyperbolic foliations. 3.4. Imbedding of the expanded Poincaré domain into \( \mathbb{C}^2 \). 3.5. Universal covering Poincaré domains of parabolic foliations.

Section 4. Proofs of the main results. ........................................ pp. 30–40

4.1. BLM-trichotomy. 4.2. Proof of Theorem B 4.3. Limiting behavior of leaves with hyperbolic holonomy. 4.4. Minimal leaves in the product of projective lines. 4.5. Briot-Bouquet foliations. 4.6. Levi problem in Hirzebruch surfaces. 4.7. Exceptional minimals and Levi flats in Hirzebruch surfaces. 4.8. Minimal sets in projective spaces.
Section 5. Pseudoconvexity vs. rational curves. .............................. pp. 40–46

5.1. Analytic objects. 5.2. Extension of analytic objects. 5.3. Pseudocon-\v{c}vexity of the universal covering Poincaré domains. 5.4. Proof of Corollary

References. ......................................................... pp. 46–49

1. INTRODUCTION

1.1. Statement of the main result. In this paper we study the limiting behavior of trajectories of polynomial vector fields. Let \( v = P(x,y)\partial/\partial x + Q(x,y)\partial/\partial y \) be a complex vector field on \( \mathbb{C}^2 \) with polynomial coefficients. Trajectories of \( v \) define a holomorphic foliation \( \mathcal{L} \), which naturally extends onto the complex projective plane \( \mathbb{P}^2 \). Vice versa, every holomorphic foliation on \( \mathbb{P}^2 \) is defined as the set of trajectories of a polynomial vector field starting from an appropriately chosen affine chart. In what follows we shall not distinguish between trajectories of polynomial vector fields and leaves of holomorphic foliations. Denote by \( \text{Sing} \mathcal{L} \) the singular locus of a holomorphic foliation \( \mathcal{L} \) on a compact complex surface \( X \), i.e., the set where the corresponding vector field vanishes. This is a non empty finite subset if \( X = \mathbb{P}^2 \). For a point \( m \not\in \text{Sing} \mathcal{L} \) the leaf \( \mathcal{L}_m \) through \( m \) is, by definition, the leaf of the smooth foliation \( \mathcal{L}^\text{reg} := \mathcal{L}|_{X^\text{reg}} \), where \( X^\text{reg} := X \setminus \text{Sing} \mathcal{L} \). If \( m \in \text{Sing} \mathcal{L} \) then leaves through \( z \) are not defined, i.e., a stationary point is not considered as being a trajectory.

The main goal of this paper is to prove the following:

**Theorem 1.** Let \( \mathcal{L} \) be a holomorphic foliation on \( \mathbb{P}^2 \) and let \( \mathcal{L}_m \) be any of its leaves. Then

\[
\overline{\mathcal{L}_m \cap \text{Sing} \mathcal{L}} \neq \emptyset.
\]

(1.1)

The **limiting set** of the leaf \( \mathcal{L}_m \) is defined as

\[
\lim \mathcal{L}_m := \bigcap_{K \in \mathcal{L}_m} \mathcal{L}_m \setminus K,
\]

(1.2)

where \( K \) runs over all compact subsets of \( \mathcal{L}_m \). Theorem 1 can be restated also in a following way: the limiting set of a leaf of a holomorphic foliation on \( \mathbb{P}^2 \) intersects the singularity set of the foliation. Recall that the Poincaré - Bendixson theorem, in the form it was originally proved by Poincaré in [P], states the following: Let \( v \) be a polynomial vector field on \( \mathbb{R}^2 \) (or, on \( S^2 \)) and let \( \gamma \) be its trajectory. Then, either \( \gamma \) is a periodic trajectory (an orbit), or for each of the limiting sets \( \lim^\pm \gamma \) the following holds: either \( \lim^\pm \gamma \) is an orbit, or \( \lim^\pm \gamma \cap \text{Sing} v \neq \emptyset \). In its turn the second part of the 16-th Hilbert’s problem asks: *If the number of limiting orbits of a polynomial vector field on \( \mathbb{R}^2 \) of degree \( n \) is bounded by some number, depending only on \( n \)?* See [H].

The answer is still unknown. Theorem 1 should be viewed as an answer to a complex analytic counterpart of Hilbert’s problem: *the limiting set of a complex trajectory always accumulates to the singular locus of the vector field.* It was known already for a long time that in the complex case there exist no ”orbits”, i.e., algebraic invariant curves in \( X^\text{reg} \), this readily follows from the Camacho-Sad formula, see Appendix in [CS]. The problem was: if there could exist some leaves with **massive** limiting sets away from singularities? This question is mentioned in [CLS1] and was explicitly posed in [Ca] on ICM-1990. Theorem 1 tells that such sets **do not exist.**
1.2. BLM trichotomy. Our point of departure for the proof of Theorem 1.2. will be the following trichotomy due to Bonatti-Langevin-Moussu, see \[B\text{LM}\]. A non-empty subset \(I \subset X\) is called \(L\)-\textit{invariant}, if it is not contained in \(\text{Sing} L\) and if for every point \(m \in I \setminus \text{Sing} L\) the leaf \(L_m\) of \(L\) through \(m\) is entirely contained in \(I\). For example, the closure of each leaf is a closed invariant set. Limiting set of every leaf is also a closed invariant set, unless it is entirely contained in \(\text{Sing} L\). But singular points \(m \in \text{Sing} L\) are not considered as invariant sets. A closed invariant set \(M\) is called \(\text{minimal}\) if it doesn’t contain any proper closed invariant subset. Every closed invariant set contains a (non unique in general) minimal subset. Finally, every minimal set is the closure of any of its leaves. If the leaf \(L_m\) is such that \(L_m\) is minimal we call it a \(\text{minimal leaf}\).

Remark 1.1. "Exceptional minimals" is a common name for minimal sets of holomorphic foliations, which do not intersect the singularity set.

Let \((X, L)\) be a foliated pair, where \(X\) is a projective surface, and let \(M = \overline{L_m}\) be an exceptional minimal for \(L\) in \(X\). Recall that every foliation on a complex projective surface is defined by a \textit{global} meromorphic form \(\Omega\). For such a form the divisors of poles \(\text{Pole}(\Omega)\) and zeroes \(\text{Zero}(\Omega)\) are correctly defined and both of them contain \(\text{Sing} L\).

According to the result from \([B\text{LM}]\) if \(M = \overline{L_m}\) is a leaf of a holomorphic foliation \(L\) such that \(\overline{L_m}\) doesn’t intersects at least one of \(\text{Pole}(\Omega)\) or \(\text{Zero}(\Omega)\) then one has only the following possibilities:

- the defining meromorphic from \(\Omega\) is (algebraically) closed;
- \(L_m\) is a compact leaf;
- there exists a point \(n \in \overline{L_m}\) with \(\overline{L_n} = \overline{L_m}\) such that the leaf \(L_n\) contains a loop with hyperbolic holonomy.

First two possibilities do produce exceptional minimals, see examples in Section 4. But in \(\mathbb{P}^2\) they are obviously not possible, see Subsection 4.1. Therefore we must exclude the third case. Recall that a loop \(\gamma \in \pi_1(L_m, m)\) is said to have a hyperbolic holonomy if the derivative of its holonomic representative is by modulus less then one.

1.3. The role of the holonomy group. For the sake of simplicity we suppose at most instances along this discussion that \(X = \mathbb{P}^2\). Let \(L_m\) be a minimal leaf and let \(D\) be a Poincaré disc through \(m\), i.e., an image of \(\Delta \subset \mathbb{C}\) under a holomorphic imbedding into \(X\), such that it is transversal to the leaves of \(L\). Let \(P_D\) be the Poincaré domain of \(L\) over \(D\), i.e., \(P_D = \bigcup_{z \in D} L_z\) - the union of leaves cutting \(D\). This is an open subset of \(X\), if \(\text{Sing} L \neq \emptyset\) (ex. \(X = \mathbb{P}^2\)) then this set is also a proper subset of \(X\). Would it so happen that \(P_D\) is pseudoconvex (in \(\mathbb{P}^2\) that means Stein) and \(\overline{L_m}\) doesn’t intersects \(\text{Sing} L\) we would get a contradiction. Indeed, from one side \(\overline{L_m}\) should be contained in \(P_D\) by minimality, from another side - this contradicts to the maximum principle. Therefore we are left with the option when \(P_D\) is not pseudoconvex for every \(D\). Corollary 2.2 now states the following:

\text{If} \(X = \mathbb{P}^2\) and \(L\) is a holomorphic foliation, which contains an exceptional minimal \(M = \overline{L_m}\),\n\text{then for a given Poincaré disc} \(D \ni m\) the Poincaré domain \(P_D\) can be non pseudoconvex only when all points of \(\text{Sing} L\) are the isolated boundary points for \(P_D\). Moreover, in that case \(P_D = \mathbb{P}^2 \setminus \text{Sing} L\).

The only use that we shall make from this first observation is that \(P_D\) is simply connected in this case and contains a lot of rational curves. Examples with \(P_D = \mathbb{P}^2 \setminus \text{Sing} L\)
do exist, see [LR]. But up to this moment we didn’t use the restrictive properties of the holonomy group. Now, up to taking another point \( n \in \mathcal{L}_m \) and a leaf \( \mathcal{L}_n \) (with the same closure) we can suppose that \( \text{Hln}(\mathcal{L}_m, m) \) contains a hyperbolic (i.e., contracting) element \( \alpha \). Therefore one can choose a complex coordinate \( t \) in \( \mathcal{D} \) such that this element will become to be a multiplication by some \( \alpha \in \mathbb{C}^* \) with \( |\alpha| < 1 \) (but the whole group \( \text{Hln}(\mathcal{L}_m, m) \) might be even non abelian).

**Remark 1.2.** It is an appropriate moment to mention the remarkable dichotomy of Cerveau, who proved that either the whole holonomy group \( \text{Hln}(\mathcal{L}_m, m) \) is abelian or, our exceptional minimal \( \mathcal{M} = \mathcal{L}_m \) is an \( \mathcal{L} \)-invariant Levi flat hypersurface! See [C]. Unfortunately we are not able to make any use from \( \mathcal{M} \) being a Levi flat hypersurface and our proof uses only the existence of some loop in \( \pi_1(\mathcal{L}_m, m) \) with contractible holonomy. Levi flats will be ruled out together will exceptional minimals.

Following Ilyashenko [Iy1], we consider the universal covering Poincaré domain (a ”skew cylinder“ in the terminology of [Iy1]): \( \mathcal{P}_D := \bigcup_{z \in D} \mathcal{L}_z \) - the union of universal coverings of leaves cutting \( \mathcal{D} \). The natural topology on \( \mathcal{P}_D \) might be non Hausdorff. The obvious reason is the possible presence of vanishing cycles. But the more deeper reason is the presence of some special \( \mathcal{L} \)-invariant rational curves in \( X \). After bringing \( (X, \mathcal{L}) \) to a nef model \( (Y, \mathcal{F}) \) we get reed from this problem. In the case when \( \mathcal{P}_D \) is Hausdorff the natural universal covering maps \( \mathcal{P}_z : \mathcal{L}_z \to \mathcal{L}_z \) glue together to a locally biholomorphic foliated projection \( \tilde{\mathcal{P}} : \mathcal{P}_D \to \mathcal{P}_D \subset X \). This aloud to consider the pair \( (\mathcal{P}_D, \tilde{\mathcal{P}}) \) as a Riemann domain over \( X \).

One should take care at this point, because the nef model can have singularities other than \( \text{Sing} \mathcal{L} \), i.e., the surface \( X \) itself can become singular. Also one should check that nothing changes in our initial data. All this is not really difficult and therefore we continue the explanation of our proof assuming that \( (X, \mathcal{L}) \) is already nef.

The key observation of this paper can be described as follows: using \( \alpha \) we expand the universal covering Poincaré domain \( \mathcal{P}_D \) to a foliated domain \( \mathcal{P}_C \) over the whole complex plane \( \mathbb{C} \). Let us formulate the precise statement. By a foliated domain we mean a triple \( (W, \pi, S) \), where \( W \) is a connected complex surface, \( S \) is a complex curve and \( \pi : W \to S \) is a holomorphic submersion with connected fibers. For example the universal covering \( \mathcal{P}_D \) of a Poincaré domain \( \mathcal{P}_D \) is foliated over \( D \), the corresponding map is denoted as \( \tilde{\pi} : \mathcal{P}_D \to D \), see Section 2 for more details. In Section 3 we prove the following:

**Theorem 2.** Let \( \mathcal{L}_m \) be a leaf of a holomorphic foliation on a compact complex surface such that \( \text{Hln}(\mathcal{L}_m, m) \) contains a hyperbolic element \( \alpha \). Suppose furthermore, that \( \mathcal{L}_D \) is Hausdorff and Rothstein. Then there exists a foliated domain \( \mathcal{P}_C \) over \( \mathbb{C} \) such that:

i) \( \mathcal{P}_D \) is a foliated subdomain of \( \mathcal{P}_C \), i.e., \( \tilde{\pi} \) extends to \( \mathcal{P}_C \), and, moreover, \( \tilde{\mathcal{P}} : \mathcal{P}_D \to X \) also extends to \( \mathcal{P}_C \);

ii) \( \mathcal{P}_C \) is periodic, i.e., \( \alpha \) lifts to a foliated, \( \tilde{\mathcal{P}} \)-invariant biholomorphism \( \tilde{\alpha} \) of \( \mathcal{P}_C \).

For the notion of rothsteiness we refer to Subsection 2.1. For the moment let us say that every Stein manifold (or, normal space) is Rothstein.

### 1.4. Reduction to nef models.

Having in our disposal the expanded Poincaré domain we are in the position to deploy the powerful "Noncommutative Mori theory" of McQuillan, see [McQ1], [McQ2]. One of the main results of this theory tells that, after reducing \( (X, \mathcal{L}) \) to the nef model, we find ourselves under the following alternative:
• either all leaves of $\mathcal{L}$ are parabolic, i.e., covered by $\mathbb{C}$;
• or, the set \{parabolic leaves\} $\cup \text{Sing } \mathcal{L}$ is a proper algebraic subset $\mathcal{A}$ of $X$, and, moreover, the hyperbolic distance along the (hyperbolic!) leaves in $X \setminus \mathcal{A}$ is continuous.

This is not a very precise statement. One should exclude the case of rational quasi-fibrations and one should also take care about the items already mentioned above (i.e., of the fact that the nef model may have cyclic singularities). But all this doesn’t really matter and therefore we continue the explanation of our proof assuming that $(X, \mathcal{L})$ is already a nef model, i.e., that dichotomy of McQuillan actually takes place already for our $\mathcal{L}$.

4. Parabolic case. In parabolic case we have the following possibilities for a leaf with hyperbolic holonomy, see Theorem 3.2:

- $\mathcal{L}$ is a rational quasi-fibration (not excluded by McQuillan’s alternative).
- $\mathcal{L}_m$ is compact, i.e., is a torus or, a projective line.
- $\mathcal{L}_m$ is biholomorphic to $\mathbb{C}^*$ and is a locally closed analytic subset of $X \setminus \text{Sing } \mathcal{L}$.

Remark 1.3. When stating these possibilities we do not suppose that $\mathcal{L}_m$ is neither exceptional nor even that it is minimal.

In all these cases it is easy to determine what happens with our $\mathcal{M} = \overline{\mathcal{L}_m}$ when $X = \mathbb{P}^2$. First case is trivial. In the case of $\mathbb{P}^2$ the second cases doesn’t happens. Third case obviously leads to a contradiction in the case when minimality and exceptionality are additionally assumed.

5. Hyperbolic case. In hyperbolic case we prove, see Theorem 3.1, that:

- $\tilde{\mathcal{P}} : \tilde{\mathcal{P}}_\mathcal{C} \to \mathbb{P}_D$ is a regular covering.

If, for example, $X$ is simply connected (ex. $\mathbb{P}^2$ is such) this last fact quickly leads to a contradiction. Indeed, if $X = \mathbb{P}^2$ we have $\mathcal{P}_D = \mathbb{P}^2 \setminus \text{Sing } \mathcal{L}$, as it was noticed, i.e., $\mathcal{P}_D$ is also simply connected. Therefore $\mathcal{P}_\mathcal{C} = \mathcal{P}_D$ by monodromy. But $\mathcal{P}_\mathcal{C}$ is foliated over $\mathbb{C}$ by a submersive holomorphic map $\tilde{\pi} : \tilde{\mathcal{P}}_\mathcal{C} \to \mathbb{C}$. This map then extends through $\text{Sing } \mathcal{L}$ onto $\mathbb{P}^2$ and, therefore, should be constant. Contradiction.

In reality the last argument is a bit more technical and makes more use of rational curves in $\mathbb{P}^2 \setminus \text{Sing } \mathcal{L}$ then of simple connectivity, see Subsection 4.2 for more details.

1.5. Minimal sets in projective spaces. Starting from a polynomial vector field on $\mathbb{C}^2$ one can compactify it to a holomorphic foliation as well on $\mathbb{P}^2$ as on $\mathbb{P}^1 \times \mathbb{P}^1$, for example. Therefore let us state the following:

Corollary 1. Let $\mathcal{L}$ be a holomorphic foliation on $\mathbb{P}^1 \times \mathbb{P}^1$ and let $\mathcal{L}_m$ be its minimal leaf. Then:

i) either $\overline{\mathcal{L}_m} \cap \text{Sing } \mathcal{L} \neq \emptyset$;
ii) or, $\mathcal{L}$ is a rational fibration and $\mathcal{L}_m$ is its fiber.

The fact that the closure of a leaf of a codimension one holomorphic foliation on $\mathbb{P}^n$, for $n \geq 3$, must intersect $\text{Sing } \mathcal{L}$ was proved in [L]. In fact, the principal result of [L] reads as follows: the singular set $\text{Sing } \mathcal{L}$ of a holomorphic codimension one foliation on $\mathbb{P}^n$, $n \geq 2$, has at least one irreducible component of codimension two. Since the complement to $\overline{\mathcal{L}_m}$ in $\mathbb{P}^n$ is Stein by Fujita’s theorem, we see readily that $\text{Sing } \mathcal{L} \cap \overline{\mathcal{L}_m} \neq \emptyset$ provided $n \geq 3$. 
In dimension two, however, the Stein domain $\mathbb{P}^2 \setminus \Sigma_m$ may well contain codimension two analytic sets = finite sets of points. Therefore the Theorem 1 could be true only because the closures of leaves of holomorphic foliations on $\mathbb{P}^n$ posed more restrictive properties with respect to the singular locus of the foliation then just to “intersect it”. Indeed, if one looks on the statement of Theorem 1 from the point of view of higher dimensions then it states that: $\Sigma_m$ contains at least one irreducible component of $\text{Sing} L$ of codimension two. This should be true in general. In this paper we provide a partial step in this direction:

**Corollary 2.** Let $\mathcal{L}$ be a codimension one holomorphic foliation on the complex projective space $\mathbb{P}^n$, $n \geq 3$, and let $L_m$ be any of its leaves. Then $L_m$ intersects at least one irreducible component of $\text{Sing} \mathcal{L}$ of codimension two by a closed set of positive $(2n - 4)$-dimensional Hausdorff measure.

The proof follows from Theorem 1 by taking generic sections and is given in Subsection 4.8.

1.6. Levi flat hypersurfaces. A special case (of a special interest) of the ”exceptional minimals problem“ is the question of (non)existence of Levi flat hypersurfaces in certain complex manifolds (like $\mathbb{P}^2$). Recall that a real hypersurface $M$ in a complex manifold $X$ is called Levi flat if $M$ is locally foliated by complex hypersurfaces. Equivalently, if $M$ locally divides $X$ onto pseudoconvex parts. In this paper we consider only real analytic $M$-s, if the opposite is not explicitly stated, and, one more point: our hypersurfaces will be always compact.

To start with let us remark that given a real analytic Levi flat hypersurface $M$ in a complex manifold $X$ there exists a neighborhood $U$ of $M$ and a holomorphic foliation $\mathcal{L}$ on $U$ by complex hypersurfaces which extends the Levi foliation of $M$.

**Remark 1.4.** If $M$ appeared as a minimal set of a holomorphic foliation, the existence of $\mathcal{L}$ doesn’t comes into a question. But it is also true (and easy) that a Levi flat real analytic $M$ itself induces a holomorphic foliation, but only on a neighborhood of it, in general.

1. **A Levi-type problem.** Recall that the Levi problem on a (not necessarily compact) complex manifold $X$ consists in finding the necessary and sufficient conditions on a relatively compact subdomain $D \Subset X$ to be Stein. The first step in our approach can be described as follows:

- Either all components of $X \setminus M$ are ”convex enough“ (ex. Stein), or the foliated pair $(U, L)$ is degenerate in a neighborhood of $M$.

We are not going to make this statement more precise: a substantial amount of classical results on the Levi problem show that the ”convex“ case is quite typical in this setting. Therefore the failure of $X \setminus M$ to be ”convex enough“ is the first raison d’être for a Levi flat hypersurface in a compact complex manifold. Remark that the example of Grauert, see [Na2], was viewed as an example of pseudoconvex manifold which doesn’t carries non-constant holomorphic functions. In this example one has a Levi flat hypersurface $M$ in a complex torus $T^2$ of dimension two such that $D = T^2 \setminus M$ is the said pseudoconvex manifold (and the corresponding foliation is clearly ”degenerate“).

2. **BLM-trichotomy.** On the other side, if the components of $X \setminus M$ are ”convex enough“ (like in the case of $\mathbb{P}^2$), then the Levi foliation extends to a holomorphic foliation $\mathcal{L}$ on the whole of $X$. 

• In that case $M$ should contain an "exceptional minimal" $\mathcal{M}$ of $\mathcal{L}$.

This is the second raison d'être for a Levi flat hypersurface in a compact complex manifold. Remark that examples of Levi scrolls of Ohsawa, see [Oh3], are of that kind. Then one should check BLM-trichotomy for $\mathcal{M}$. In our setting this simply means that either $\mathcal{M} = M$ and then we apply our machine directly to $M$, or $\overline{L_m}$ is a proper closed invariant subset of $M$ and we derive the global behavior of $M$ from that of $\mathcal{M} = \overline{L_m}$. In this way one easily obtains the following corollaries.

**Corollary 3.** Complex projective plane $\mathbb{C}P^2$ doesn't contains any real analytic Levi flat hypersurface.

**Corollary 4.** Let $M$ be a real analytic Levi flat hypersurface in $\mathbb{P}^1 \times \mathbb{P}^1$. Then $M = \gamma \times \mathbb{P}^1$ (or $\mathbb{P}^1 \times \gamma$), where $\gamma$ is a closed, real analytic curve in $\mathbb{P}^1$.

Proofs are given in Remarks 4.3 and 4.4.

**1.7. Exceptional minimals of Briot-Bouquet foliations.** Let us emphasize that neither $\mathbb{P}^2$ nor $\mathbb{P}^1 \times \mathbb{P}^1$ are always natural manifolds to carry a holomorphic foliation, which comes from algebraic differential equation. Consider, for example, the polynomial equation of Briot and Bouquet, studied for the first time in [BB]. It is an equation of the form

$$F(z, z') = 0,$$

(1.3)

where $F$ is an irreducible polynomial of two complex variables, non constant both in $z$ and $z'$. This class of equations includes the Riccati equation, equation for elliptic curves and so on. In [Dn], after the geometrization of the problem, it was showed that (1.3) naturally raises to a holomorphic foliation on the product $\Sigma_g \times \mathbb{P}^1$, where $\Sigma_g$ is the compact Riemann surface of genus $g \geq 1$. The affine part $\Sigma_g^0$ of $\Sigma_g$, in an appropriate affine coordinates $p, q$, is given by the equation $F(p, q) = 0$. It is natural to call holomorphic foliations on $\Sigma_g \times \mathbb{P}^1$ - the Briot-Bouquet foliations. We prove the following:

**Corollary 5.** The only exceptional minimals of Briot-Bouquet foliations are fibers $E_1 := \{pt\} \times \mathbb{P}^1$ or $E_2 := \Sigma_g \times \{pt\}$.

The proof is given in Subsection 4.5.

**1.8. Exceptional minimals and Levi flats in Hirzebruch surfaces.** In Subsections 4.6 and 4.7 we classify the exceptional minimals and Levi flats in Hirzebruch surfaces:

**Corollary 6.** Let $\mathcal{M}$ be a minimal set of a holomorphic foliation on the Hirzebruch surface $H_k$, $k \geq 1$. Then:

i) either $\mathcal{M} \cap \text{Sing} \mathcal{L} \neq \emptyset$;

ii) or, $\mathcal{L}$ is the rational fibration $\mathcal{L}_s = \pi^{-1}(s)$ and $\mathcal{M}$ is one of its fibers.

**Corollary 7.** Let $M$ be a real analytic Levi flat hypersurface in Hirzebruch surface $H_k$. Then $M = \pi^{-1}(\gamma)$, where $\gamma$ is a real analytic imbedded loop in $\mathbb{P}^1$.

**1.9. Approach of a leaf to the singular locus.** The classical Poincaré - Bendixson theory apart of the description of the limiting behavior of a trajectory of a vector field on the real plane, which stays away from the singular locus of the vector field (it accumulates to an orbit), describes also the way a trajectory behaves when accumulating to the singular locus of a vector field in question. Since we proved that in the complex case the first option is impossible, i.e., a complex trajectory always accumulates to the singular locus of the
vector field, it is natural to provide a step towards the description of a way a complex trajectory approaches the singular locus.

In the first approximation we describe the limiting behavior of a minimal leaf of a holomorphic foliation in terms of its invariant neighborhoods - Poincaré domains $\mathcal{P}_D$ associated to transversal discs cutting $\mathcal{L}_m$ at $m$ (or at any other point) and their leafwise universal coverings. Our goal is to specify the behavior of the Poincaré domains themselves and/or the universal covering Poincaré domains (at least), and, even more, of the leaf $\pi$ through some point $m$ that approaches the singular locus.

Let us see that the case of hyperbolic holonomy is practically done. Let $\text{Hln} : \pi_1(\mathcal{L}_m, m) \to \text{Diff}(D, m)$ be the holonomy representation. Its image is the holonomy group $\text{Hln}(\mathcal{L}_m, m)$ of the leaf $\mathcal{L}_m$. For $\gamma \in \pi_1(\mathcal{L}_m, m)$ one can take the derivative $\text{Hln}(\gamma)'(m)$ of its holonomy representative at $m$ and thus obtain a homomorphism $\text{Hln}' : \pi_1(\mathcal{L}_m, m) \to \mathbb{C}^*$.

**Definition 1.1.** We say that the holonomy of $\mathcal{L}_m$ is parabolic if $\text{Hln}'(\pi_1(\mathcal{L}_m, m)) \subset \mathbb{S}^1$. If it is not the case we say that the holonomy of $\mathcal{L}_m$ is hyperbolic.

The latter case means that there exists a loop $\gamma \subset \mathcal{L}_m$, starting and ending at $m$, such that $|\text{Hln}'(\gamma)(m)| < 1$. One says also that such $\gamma$ has the contracting or hyperbolic holonomy.

**Corollary 8.** Let $\mathcal{L}_m$ be a minimal leaf of a holomorphic foliation on $\mathbb{P}^2$ with hyperbolic holonomy. Then:

i) either $\overline{\mathcal{L}_m} \setminus \text{Sing} \mathcal{L}$ posed a Stein invariant neighborhood;

ii) or, $\overline{\mathcal{L}_m}$ is a rational $\mathcal{L}$, which cuts $\text{Sing} \mathcal{L}$ by at least two points;

iii) or, $\overline{\mathcal{L}_m}$ is an elliptic curve, which cuts $\text{Sing} \mathcal{L}$ by at least one point.

Options (ii) and (iii) mean, in another words, that $\mathcal{L}_m = C \setminus \{p_1, \ldots, p_k\}$, where $p_j \in \text{Sing} \mathcal{L}$ for every $j$ and $C$ is a rational or elliptic curve in $\mathbb{P}^2$ such that $C \setminus \{p_1, \ldots, p_k\}$ is imbedded as a closed analytic subset in $\mathbb{P}^2 \setminus \{p_1, \ldots, p_k\}$. Moreover, for every $p_j$ there exists a closed subset $E_j$ in $\mathcal{L}_m$ such that $E_j$ is biholomorphic to the punctured disc $\Delta := \{z \in \mathbb{C} : 0 < |z| < 1\}$ (the so called vanishing end). Finally, $\overline{E_j} \cap \text{Sing} \mathcal{L} = \{p_j\}$. Note that the possibility that $p_i = p_j$ for some $i \neq j$ is not excluded (but $E_i \cap E_j = \emptyset$ for all $i \neq j$).

The proof of this corollary repeats step to step the proof of Theorem 1 and is given in Subsection 4.3. In general, i.e., without the assumption of hyperbolicity we can prove a much weaker result.

**Theorem 3.** Let $\mathcal{L}$ be a holomorphic foliation on $\mathbb{P}^2$ and let $\mathcal{L}_m$ be a minimal leaf of $\mathcal{L}$ through some point $m \in X^{\text{ne}} := \mathbb{P}^2 \setminus \text{Sing} \mathcal{L}$. Then:

i) either for a sufficiently small transversal disc $D$ through $m$ the universal covering Poincaré domain $\hat{\mathcal{P}}_D$ is a Stein;

ii) or, the closure $\overline{\mathcal{L}_m}$ of $\mathcal{L}_m$ is a rational curve, cutting $\text{Sing} \mathcal{L}$ by exactly one point.

The second case represents a satisfactory description of a limiting behavior of a minimal leaf. At the same time it gives a precise obstruction for $\hat{\mathcal{P}}_D$ to be Stein. The proof of this theorem requires much more then that of the preceding corollary (modulo Theorem 1 of course) and in given in the last Section 5 of this paper.

Theorem 3 aloud to make more precise the case (i) of Corollary 8. The point is that the Stein invariant neighborhood, which occurs there is equal to the Poincaré domain $\mathcal{P}_D$ for a sufficiently small disc through $m$ plus a finite number $\{s_1, \ldots, s_d\}$ of isolated boundary points of $\mathcal{P}_D$. The nature of this points is given by the following:
Corollary 9. In the case (i) of Corollary 8 all \( \{s_1, \ldots, s_d\} \) are dicritical for \( \mathcal{L} \).

The proof is given in Subsection 5.4.

1.10. Notes. We are certainly even not trying here to give any sort of review on an unobservable amount of literature on general, i.e., not only holomorphic Poincaré-Bendixson theory. Issues concerning Hilbert’s 16th problem the interested reader may consult in \([Iy3]\). We only want to track a bit the **incomplete in all senses** history of the holomorphic case.

1. The nature of “periodic” solutions of complex polynomial equations is studied since the time when these equation are studied themselves. For example, \([BB]\) stays continuously to be a paper of reference, see ex. \([ELN]\) and references there.

2. In his celebrated paper \([P]\) Poincaré, influenced by \([BB]\) and preceding works of Cauchy (see his recours on p. 385 of \([P]\)) proved his famous theorem for polynomial vector fields in real dimension two.

3. The paper \([Be]\) of Bendixson, where he proved a \( C^1 \)-generalization of the Poincaré’s theorem is, in fact, mostly devoted to the equations with holomorphic coefficients: Ch. I - to, what is now known as Poincaré-Bendixson theorem, Ch-s II - VII - to holomorphic differential equations. At the end of the Introduction Bendixson shows that he is well aware that the situation with the Poincaré’s famous theorem is unclear in the complex case, saying that: “Nous nous bornons ici à cette remarque, voulant dans ce mémoire traiter seulement les courbes intégrales réelles des équations différentielles”. Non-integral, i.e., local trajectories, which he calls characteristics, are studied in \([Be]\) for the case of holomorphic vector fields.

4. In \([AGH]\) the authors discovered that flows on nilmanifolds provide a rich source of examples of minimal sets. In \([Sa]\) it was proved that if a connected solvable Lie group acts holomorphically on a compact Kähler manifold \( X \) with \( H^1(X, \mathbb{C}) = 0 \) then this action has a fixed point in every invariant complex subspace. In \([Bd]\) further study of minimal sets of holomorphic actions of Lie groups on compact Kähler manifolds was undertaken.

5. After the appearance of \([CLS1]\) the problem of the existence of minimal invariant sets, which do not intersect the singular locus of the foliation, became very popular. Let us mention only \([BLM]\), \([C]\) and, finally, for \( n \geq 3 \) the statement of Theorem 1 was proved in \([L]\).

6. After the dichotomy of Cerveau was discovered, see \([C]\), the attention was switched mainly to the question of (non)existence of Levi flats in \( \mathbb{P}^2 \). But it should be said that the Levi flat hypersurfaces first appeared in complex analysis long before, to my best knowledge in the example of Grauert of pseudoconvex manifold without non constant holomorphic functions, see \([Na1]\) and \([Na2]\).

7. Levi flats as common boundaries of two pseudoconvex domains were studied in \([Shc]\) and other papers. As natural boundaries of envelopes of holomorphy of bounded tube domains they appeared \([Vf]\). Based on these papers and several others (see \([Va]\) for more details on this activity) the question of existence of Levi flats in \( \mathbb{P}^2 \) was actively discussed on the seminar on Complex Analysis in Moscow University in mid 80-s. \([Ml]\) is a track of this activity. This list is very far from being complete, see ex. \([Bd, R]\).

8. The case of smooth, i.e., non real analytic hypersurfaces in dimension \( \geq 3 \) was excluded by Siu in \([SB]\). The case of dimension 2 remained open despite of several very
clever attempts made in [Oh1, Si4] and several other papers dependant on these ones, see historical sketch in [IM].

9. Reduction to nef models was developed in the series of prominent works, among which I bound myself with mentioning Seidenberg [Sei], Miyaoka [Miy], McQuillan [McQ1], [McQ2] and Brunella [Br5]. Poincaré domains (under the name “skew cylinders”) where introduced in the case of foliations on Stein manifolds by Ilyashenko in [Iy1] and studied by him and his students. In the non Stein setting Poincaré domains (under the name of “covering tubes”) were studied by Brunella. The study of a way how a nearby trajectory cuts a transversal interval on the real plane - is the central idea of the paper of Poincaré [P]. Therefore I think it is right to associate the analogous objects, also in complex setting, with his name.

1.11. Acknowledgement. I am grateful to Dmitri Markushevich for numerous helpful discussions around complex algebraic aspects of this paper, as well as to Stefan Nemirovski for sending me his unpublished preprint [Ne2] and, especially, for pointing out to me the gap in the first version of this paper. From Frank Loray and Bertrand Deroin I learned interesting examples of foliations on $\mathbb{P}^2$. I am especially grateful to Michael McQuillan for giving to me explanations on his Noncommutative Mori theory. Also to Vsevolod Shevchishin and Alexandre Sukhov, who were the first to listen my exposés on the subject of this paper, for their patience and criticism.

2. Pseudoconvexity of Poincaré domains

2.1. Minimal sets of holomorphic foliations. We consider foliated manifolds, i.e., pairs $(X, \mathcal{L})$, where $X$ is a complex manifold and $\mathcal{L}$ is a (singular) codimension one holomorphic foliation on $X$. Later we shall aloud $X$ to have the so called cyclic quotient singularities, but this will not introduce any complications into the our exposition. There are several equivalent ways to define a holomorphic foliation on a complex manifold. We shall use two of them. First reads as follows. A codimension one holomorphic foliation $\mathcal{L}$ on a complex manifold $X$ is given by:

i) an open covering $\{U_j\}$ of $X$ and non identically zero holomorphic 1-forms $\omega_j$ on $U_j$;

ii) forms $w_j$ satisfy integrability condition $w_j \wedge dw_j = 0$ (no condition if $\dim X = 2$);

iii) on intersections $U_j \cap U_k$ the defining forms $w_j$ are proportional, i.e., $\omega_j = f_{jk} \omega_k$ for some $f_{jk} \in \mathcal{O}^*(U_j \cap U_k)$.

The set $\{\omega_j = 0\}$ is the singularity set $\text{Sing} \mathcal{L}$ of $\mathcal{L}$. Up to reducing common factors of coefficients of $\omega_j$ the singularity set can be supposed to be of complex codimension at least two. In regular part $X^* := X \setminus \text{Sing} \mathcal{L}$ of $X$ the leaves $\mathcal{L}_z$ of $\mathcal{L}$ are defined by the equations $\omega_j|_{\mathcal{L}_z} = 0$. The cocycle $\{f_{jk}\} \in H^1(X, \mathcal{O}^*)$ defines the conormal bundle $\mathcal{N}_\mathcal{L}$ of $\mathcal{L}$. Second definition (in dimension two) reads as follows:

i) given an open covering $\{U_j\}$ of $X$ and non identically zero holomorphic vector fields $v_j$ on $U_j$;

ii) on intersections $U_j \cap U_k$ the vector fields $v_j$ are proportional, i.e., $v_j = g_{jk} v_k$ for some $g_{jk} \in \mathcal{O}^*(U_j \cap U_k)$.

Leaves of $\mathcal{L}$ are locally the complex curves tangent to $v_j$, $\text{Sing} \mathcal{L} = \{v_j = 0\}$. The cocycle $\{g_{jk}\} \in H^1(X, \mathcal{O}^*)$ defines a holomorphic line bundle, which is called the canonical bundle of $\mathcal{L}$. It will be denoted as $\mathcal{K}_\mathcal{L}$ or, simply as $\mathcal{K}$ if no misunderstanding could occur.
Definition 2.1. A subset $I \subset X$ of a foliated manifold is called $\mathcal{L}$-invariant or, simply invariant, if $\mathcal{L}$ is clear from the context, if:

1) $I \setminus \text{Sing}\mathcal{L}$ is non empty;
2) for every point $z \in I \setminus \text{Sing}\mathcal{L}$ the leaf $\mathcal{L}_z$ is entirely contained in $I$.

Would we ask only the condition (ii) in this definition then, since no leaves are defined through singular points, any subset $S \subset \text{Sing}\mathcal{L}$ would formally satisfy such definition. But sets in $\text{Sing}\mathcal{L}$ are not the objects of our study and they are not considered to be the invariant sets.

Definition 2.2. A closed invariant set, that doesn’t contains any proper closed invariant subset is called a minimal set of $\mathcal{L}$.

A minimal set is, obviously, the closure of any of its leaves. A leaf $\mathcal{L}_m$ such that $\overline{\mathcal{L}_m}$ is minimal we shall call a minimal leaf. Let us remark the following:

Proposition 2.1. Let $I$ be a compact invariant set of a codimension one holomorphic foliation $\mathcal{L}$ on a complex manifold $X$. Then there exists a minimal $\mathcal{L}$-invariant set $M$, which is contained in $I$.

Proof. Consider the partially ordered by inclusion set $\mathcal{I}$ of closed invariant subsets of $I$. Let $A$ be its linearly ordered subset. Let us see that there exists a smallest element in $A$. Indeed, take $I_0 := \bigcap_{i \in A} I_i$. It is a non empty compact in $X$ and is obviously invariant unless $I_0 \subset \text{Sing}\mathcal{L}$. We need to prove that $I_0 \not\subset \text{Sing}\mathcal{L}$.

Let $U$ be an $(n-1)$-complete neighborhood of $\text{Sing}\mathcal{L}$ in $X$, see [Ba]. Any one of $I_i$ cannot be contained entirely in $U$ by the maximum principle. Therefore $I_0 \cap (X \setminus U)$ is non empty. By Zorn’s Lemma $\mathcal{I}$ contains a minimal element and this element is our minimal set.

2.2. Poincaré domains. Let $X$ be a complex surface and let $\mathcal{L}$ be a (singular) holomorphic foliation by curves on $X$. We closely follow constructions from [Y1] [Y2] [Sz] [Br5]. For a point $m \in X^{\text{reg}} := X \setminus \text{Sing}\mathcal{L}$ denote by $\mathcal{L}_m$ the leaf of $\mathcal{L}^{\text{reg}} := \mathcal{L}|_{X^{\text{reg}}}$ through $m$. Again, let us underline that throughout this paper we are not considering any sort of leaves through the singular points of $\mathcal{L}$. Take a smooth, locally closed disc $D$ in $X^{\text{reg}}$, transversal to $\mathcal{L}^{\text{reg}}$ and cutting the leaf $\mathcal{L}_m$ at $m$. Therefore we require that $D$ is closed in some subdomain $Y$ of $X^{\text{reg}}$ and that for every point $z \in D$ the intersection $D \cap \mathcal{L}_z$ is transversal. We shall call such discs - the Poincaré discs.

Denote by $\hat{\mathcal{L}}_m$ the holonomy covering of $\mathcal{L}_m$, i.e., the covering with respect to the kernel of the holonomy representation $\text{Hln} : \pi_1(\mathcal{L}_m, m) \to \text{Diff}(D, m)$. The image $\text{Hln}(\mathcal{L}_m, m) := \text{Hln}(\pi_1(\mathcal{L}_m, m))$ of the fundamental group of $\mathcal{L}_m$ under this representation is called the holonomy group of $\mathcal{L}$ along the leaf $\mathcal{L}_m$. It is a subgroup of the group $\text{Diff}(D, m)$ of germs of biholomorphisms of the transversal $D$ fixing the point $m$. Up to a conjugation $\text{Hln}(\mathcal{L}_m, m)$ doesn’t depends on the choice of the transversal through $m$. We refer to [Gd] for generalities on holonomy groups of foliations. But in a moment we shall recall some features of the holonomy representation, which will be crucial for us along this paper. By $\hat{\mathcal{L}}_m$ we denote the universal covering of $\mathcal{L}_m$. Consider the following sets:

$$
\mathcal{P}_D := \bigcup_{z \in D} \mathcal{L}_z, \quad \hat{\mathcal{P}}_D := \bigcup_{z \in D} \hat{\mathcal{L}}_z, \quad \check{\mathcal{P}}_D := \bigcup_{z \in D} \check{\mathcal{L}}_z.
$$

(2.1)
The first set we shall name the Poincaré domain of $\mathcal{L}$ over the transversal $D$. The second is a holonomy covering Poincaré domain ("tube normaux" in the terminology of \[SZ\]) and the third the universal covering Poincaré domain ("skew cylinder" in the terminology of \[LY1, LY2\]). We shall often call them simply holonomy or covering domains for short. If no misunderstanding can occur, we shall also call $\mathcal{P}_D, \hat{\mathcal{P}}_D, \hat{\mathcal{P}}_D$ - the Poincaré domains of the Poincaré disc $D$.

Poincaré domain $\mathcal{P}_D$ is an open connected subset of $X^{**}$ and therefore of $X$. $D$ admits a tautological imbedding $i : D \to \mathcal{P}_D$, namely $i : z \to z$. Note that $i(D)$ is not closed in $\mathcal{P}_D$ in most interesting cases. Poincaré domains $\hat{\mathcal{P}}_D$ and $\hat{\mathcal{P}}_D$ come together with the natural topologies and foliations on them. Let us briefly recall this. Consider the topological space $\mathcal{C}(D, \mathcal{L})$ of paths $\gamma_{z,w}$ starting from points $z \in D$, ending at $w \in \mathcal{L}_z$ and contained in $\mathcal{L}_z$, i.e., we consider paths inside leaves of $\mathcal{L}$ only. The topology in $\mathcal{C}(D, \mathcal{L})$ is the topology of uniform convergence on the space of continuous maps from the unit interval $[0,1]$ to $X$. Paths $\gamma_{z,w}$ and $\beta_{z,w}$ (with the same ends) are equivalent if the holonomy along $\beta_{z,w}^{-1} \circ \gamma_{z,w}$ is trivial. The holonomy covering Poincaré domain is the quotient of $\mathcal{C}(D, \mathcal{L})$ under this equivalence relation.

To check that this quotient is Hausdorff let us recall what is the holonomy representation. Let $\gamma$ be a closed path in $\mathcal{L}_m$, which starts and ends at $m$. If one “displaces” $\gamma$ to a nearby leaf $\mathcal{L}_z$, i.e., if one takes a point $z \in D$ close to $m$ and draws a path $\beta$ starting from $z$ in $\mathcal{L}_z$ close to $\gamma$, then $\beta$ certainly hits $D$, but in general by a point $z'$ different from $z$. This way one obtains a mapping $z \to z'$, which is called the holonomy representation of $\gamma$. It depends only on the homotopy class of $\gamma$ and is holomorphic. To see this one covers $\gamma$ by foliated charts and realizes that the holonomy representative map of $\gamma$ is a composition of obvious holomorphic maps in these local foliated charts, see [Gd] for more details. The representation obtained we denote as $\text{Hln} : \pi_1(\mathcal{L}_m, m) \to \text{Diff}(D, m)$. This is a formalization of d’application de premier retour of Poincaré.

Now it is easy to see that $\hat{\mathcal{P}}_D$ is Hausdorff. Suppose that there exists a sequence $\gamma_{z_n,w_n}$, $z_n \to z_0$ in $D, \mathcal{L}_{z_n} \ni w_n \to w_0 \in \mathcal{L}_{z_0}$ in $\mathcal{C}(D, \mathcal{L})$ which, after factorization, converges to the two limit points $\gamma_{z_0,w_0}$ and $\beta_{z_0,w_0}$. That means that $\gamma_{z_n,w_n}$ uniformly converge to $\gamma_{z_0,w_0}$ and there is another sequence $\beta_{z_n,w_n}$ which uniformly converge to $\beta_{z_0,w_0}$, such that:

\begin{enumerate}
  \item $\gamma_{z_n,w_n}$ and $\beta_{z_n,w_n}$ are equivalent for all $n$,
  \item while $\gamma_{z_0,w_0}$ and $\beta_{z_0,w_0}$ are not.
\end{enumerate}

In another words the holonomy along the closed path $\beta_{z_0,w_0}^{-1} \circ \gamma_{z_0,w_0}$ is non trivial, but at the same time $\text{Hln}(\beta_{z_0,w_0}^{-1} \circ \gamma_{z_0,w_0})(z_n) = z_n$. We got a contradiction with the uniqueness theorem for holomorphic functions.

The natural map $\hat{p}(\gamma_{z,w}) = w$ is locally homeomorphic and therefore the pair $(\hat{\mathcal{P}}_D, \hat{p})$ is a Riemann domain over $X$. The map $i : D \to \hat{\mathcal{P}}_D$, defined as $i : z \to \gamma_{z,z}$, is a holomorphic imbedding and its image is a closed disc in $\hat{\mathcal{P}}_D$ - the base of the holonomy Poincaré domain. $\hat{\mathcal{P}}_D$ admits also a natural projection $\hat{\pi}$ onto $D$ defined as $\hat{\pi}(\hat{\mathcal{L}}_z) = z$. Holonomy Poincaré domain $\hat{\mathcal{P}}_D$ inherits a natural foliation $\hat{\mathcal{L}}$ with leaves $\hat{\mathcal{L}}_z$ (the holonomy foliation) and the locally biholomorphic map $\hat{p} : (\hat{\mathcal{P}}_D, \hat{\mathcal{L}}) \to (X, \mathcal{L})$ is foliated, i.e., sends leaves to leaves. Foliation $\hat{\mathcal{L}}$ on $\hat{\mathcal{P}}_D$ has no holonomy by construction.

The same construction can be repeated with the following equivalence relation: paths $\gamma_{z,w}$ and $\beta_{z,w}$ are equivalent if $\beta_{z,w}^{-1} \circ \gamma_{z,w}$ is homotopic to the constant path $\gamma_{z,z}$ inside of the leaf $\mathcal{L}_z$. The quotient (if it is Hausdorff!) is the universal covering Poincaré domain.
Pseudoconvexity of Poincaré domains

\( \tilde{\mathcal{P}}_D \) in question. The corresponding objects are marked as \( \tilde{p}, \tilde{\pi}, i \) and \( \tilde{\mathcal{C}} \) - the last we shall call the **universal foliation** on \( \tilde{\mathcal{P}}_D \). We shall need to see this construction starting from pathes in \( \mathcal{P}_D \). Consider the space of pathes \( \mathcal{C}(\{0,1\}, \mathcal{L}) \) starting from points in \( D \) inside of leaves of the holonomy foliation on the holonomy covering Poincaré domain \( \tilde{\mathcal{L}}_D \). Two pathes \( \gamma_{z,w} \) and \( \beta_{z,w} \) are equivalent if they are homotopic inside \( \mathcal{L}_z \) to a constant path \( \gamma_{z,z} \). The quotient is the universal covering Poincaré domain \( \tilde{\mathcal{P}}_D \). It possesses a natural projection \( p: \tilde{\mathcal{P}}_D \to \mathcal{P}_D \) sending \( \gamma_{z,w} \) to \( w \). Composition \( \tilde{p} := \tilde{p} \circ p : \tilde{\mathcal{P}}_D \to \mathcal{P}_D \) is the mapping, which one naturally obtains when constructing \( \tilde{\mathcal{P}}_D \) starting from pathes in \( \mathcal{P}_D \).

In general, it is useful to point out the following items:

- The universal covering Poincaré domain might be non Hausdorff.
- The Poincaré domain \( \mathcal{P}_D \) in most cases cannot be projected to \( D \), simply because the same leaf \( \mathcal{L}_z \) may intersect \( D \) in several (even, in infinite number of) points.
- Both \( \tilde{p}: \tilde{\mathcal{P}}_D \to \mathcal{P}_D \) and \( \tilde{\mathcal{P}}_D \to \mathcal{P}_D \) are not regular coverings in most interesting cases (even if \( \tilde{\mathcal{P}}_D \) exists). They are regular coverings only along the leaves.
- Both \( (\tilde{\mathcal{P}}_D, \tilde{p}) \) and \( (\mathcal{P}_D, \tilde{p}) \) are usually not locally pseudoconvex over \( X \).

### 2.3. Docquier-Grauert criterion and Fujita’s Theorems.

Recall that a domain \( R \) in a complex manifold \( X \) is called pseudoconvex at each of its boundary points.

Through this paper we shall repeatedly use the following remark able Docquier-Grauert criterion:

**Theorem 2.1. (Docquier-Grauert)** Let \( (R,p) \) be a Riemann domain over a Stein manifold \( X \). If every holomorphic imbedding \( h : H^n_\varepsilon \to R \) extends to a locally biholomorphic mapping \( h : \Delta^n_\varepsilon \to R \) then \( R \) is a Stein manifold.

In [DG] this type of convexity of a domain over a Stein manifold was called \( pt \)-convexity. As an obvious corollary from this criterion one gets one theorem of K. Stein: a regular cover of a Stein manifold is Stein itself. Remark that the inverse is not true: think about \( \mathbb{C}^2 \) covering a torus \( \mathbb{T}^2 \).
We shall crucially use in our proofs the following results of R. Fujita:

**Theorem 2.2. (Fujita)** i) Let \((R, p)\) be a locally pseudoconvex Riemann domain over \(\mathbb{P}^n\). If \(p : R \to \mathbb{P}^2\) is not a homeomorphism then \(R\) is a Stein manifold.

ii) Let \((R, p)\) be a locally pseudoconvex Riemann domain over \(\mathbb{P} = \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_k}\). If \(p\) is not a homeomorphism of \(R\) onto a domain, which up to a permutation, contains \(\mathbb{P}^{n_1} \times \{pt\}\), then \(R\) is a Stein manifold.

For the proof see [F1] and [F2]. We shall repeatedly use the following statement:

**Lemma 2.1.** Let \(\mathcal{L}\) be a codimension one holomorphic foliation in \(\Delta^n\). Then every leaf of \(\mathcal{L}\) intersects the Hartogs figure \(H^n_\varepsilon\) (for any \(\varepsilon > 0\)).

**Proof.** Suppose that the leaf \(\mathcal{L}_m\) is such that \(\overline{\mathcal{L}_m} \cap H^n_\varepsilon = \emptyset\). Take any curve \(\gamma : [0, 1] \in \Delta^{n-1}\), coming from 0 to the boundary of \(\Delta^{n-1}\), and such that \(\gamma([0, 1])\) doesn’t intersects the projection \(S\) of the singularity set of \(\mathcal{L}\) onto \(\Delta^{n-1}\) (\(S\) is at most countable union of locally closed hypersurfaces in \(\Delta^{n-1}\)). Consider the family of analytic discs

\[
\Delta_{\gamma(t)} := \{\gamma(t)\} \times \Delta, \quad t \in [0, 1].
\]

Remark that boundaries of these discs never intersect \(\overline{\mathcal{L}_m}\). Suppose that this family intersects \(\overline{\mathcal{L}_m}\). Let \(t_0\) be the first value of \(t\) such that \(\Delta_{\gamma(t)} \cap \overline{\mathcal{L}_m} \neq \emptyset\). \(t_0\) exists because \(\overline{\mathcal{L}_m}\) is closed and \(t_0 \neq 0\) because \(\Delta_{\gamma(0)} \cap \overline{\mathcal{L}_m} = \emptyset\) by assumption. We get a contradiction with the positivity of intersections of complex varieties. Therefore \(\overline{\mathcal{L}_m}\) should be contained in \(S \times \Delta\). But in that case it should be an irreducible component of this set. And as such necessarily intersects \(H^n_\varepsilon\). Contradiction.

\[\square\]

2.4. **Local pseudoconvexity of Poincaré domains.** In what follows \(\dim X = 2\) if the opposite is not explicitly stated. \(\mathcal{L}\) is a singular holomorphic foliation by curves on \(X\). We start from the following:

**Lemma 2.2.** Let \(z_0\) be a non-isolated boundary point of \(\mathcal{P}_D\). Then \(\mathcal{P}_D\) is pseudoconvex at \(z_0\).

**Proof.** For a holomorphic foliation \(\mathcal{L}\) on \(X\) and an open, connected subset \(U \subset X\) we denote by \(\mathcal{L}|_U\) the *restriction* of \(\mathcal{L}\) to \(U\). The meaning is obvious, but let us point out that the leaves of \(\mathcal{L}|_U\) are not the intersections of the leaves of \(\mathcal{L}\) with \(U\) in general. They are the connected components of these intersections. To make a clear distinction between leaves of \(\mathcal{L}|_U\) and intersections \(\mathcal{L}_z \cap U\) we will set \(\mathcal{F} := \mathcal{L}|_U\) and denote the leaf of \(\mathcal{F}\) passing through \(z \in U\) as \(\mathcal{F}_z\). Again, let us repeat that all our leaves are defined outside of singular points. Now consider two cases.

**Case 1.** \(z_0 \notin \text{Sing} \mathcal{L}\). Take an \(\mathcal{L}\)-foliated neighborhood \(U\) of \(z_0\), i.e., \(U\) is biholomorphic to \(\Delta \times \Delta\) with \(\mathcal{F}_{z_1} := \{z_1\} \times \Delta\) being the leaves of \(\mathcal{F} = \mathcal{L}|_U\). Since \(\mathcal{P}_D\) is an \(\mathcal{L}\)-invariant domain the intersection \(\mathcal{P}_D \cap U\) is a union of leaves of \(\mathcal{F}\), i.e., has the form \(U_1 \times \Delta\) for some open subset \(U_1 \subset \Delta\). Therefore \(\mathcal{P}_D \cap U\) is pseudoconvex.

**Remark 2.1.** Remark that from the considerations, made above, it follows that \(z_0 \in \partial \mathcal{P}_D \setminus \text{Sing} \mathcal{L}\) cannot be an isolated point of \(\partial \mathcal{P}_D\). Indeed, if \(z_0 \in \partial \mathcal{P}_D\) then \(\mathcal{L}_{z_0} \subset \partial \mathcal{P}_D\).
Case 2. $z_0 \in \text{Sing } L$. Let $U$ be a neighborhood of $z_0$ biholomorphic to a ball, which doesn’t contain any other then $z_0$ singular point of $L$. Take some point $z \in \partial P_D \cap U$ ($z$ may be well $z_0$), but in our case, it is not an isolated point of $\partial P_D$. Therefore we can find a sequence $z_n \in \partial P_D \cap U$, $z_n \notin \text{Sing } L$, converging to $z$. After going to a subsequence, $\mathcal{F}_{z_n}$ converge in the Hausdorff metric to a closed in $U$ set $L$. $L$ clearly contains $z$, is connected and $L \subset \partial P_D \cap U$ because every point in $L$ is a limit of boundary points, namely points of $\mathcal{F}_{z_n}$. Moreover, $L$ is $\mathcal{F}$-invariant. Indeed, a Hausdorff limit of closed invariant sets is, obviously, a closed invariant set.

We proved that $(\partial P_D \cap U) \setminus \{z_0\}$ is the union of leaves of $\mathcal{F} = L|_U$. Let $h : H^2_\varepsilon \rightarrow P_D \cap U$ be a holomorphic imbedding. Then $h$ induces a smooth holomorphic foliation $\mathcal{E} := h^*\mathcal{F}$ on $H^2_\varepsilon$.

Remark 2.2. By pulling back a foliation $\mathcal{F}$ from a manifold $U$ by a (locally) biholomorphic map $h : \Omega \rightarrow U$ from a complex manifold $\Omega$ into $U$ we mean the following. $\mathcal{F}$ is defined locally, on open sets $U_j$ by holomorphic forms $\omega_j$ satisfying the usual compatibility conditions. Take pull backs $h^* \omega_j$ of $\omega_j$. Then these holomorphic 1-forms will define a foliation in $\Omega$. It is this foliation we mean when writing $h^*(\mathcal{F})$. The leaf of $\mathcal{E} := h^*(\mathcal{F})$ through a point $z \in \Omega \setminus \text{Sing } \mathcal{E}$ will be denoted by $\mathcal{E}_z$.

$\mathcal{E}$ extends to a (singular) holomorphic foliation $\check{\mathcal{E}}$ on $\Delta^2$. $h$ extends onto $\Delta^2$ and $h(\Delta^2) \subset U$ (by the usual Hartogs theorem, because it is a mapping into $\mathbb{C}^2$). Suppose that $h(\Delta^2) \cap \partial P_D \neq \emptyset$. Then $h(\Delta^2)$ contains some $z \in \partial P_D \setminus \{z_0\}$, again because $z_0$ is supposed to be a non isolated boundary point. Let $w \in \Delta^2$ be some $h$-preimage of $z$. Then $h(\check{\mathcal{E}}_w) \subset \mathcal{F}_z \subset \partial P_D$. But $\check{\mathcal{E}}_w$ intersects $H^2_{\varepsilon}$, see Lemma 2.1. Therefore $h(H^2_{\varepsilon})$ should also intersect $\partial P_D$. Contradiction. By the Docquier-Grauert criterion we conclude that $P_D \cap U$ is Stein.

Since at its isolated boundary point a domain cannot be locally pseudoconvex, we obtain the following:

Corollary 2.1. The Poincaré domain $P_D$ is not locally pseudoconvex at $z_0 \in \partial P_D$ if and only if $z_0$ is an isolated point of $\partial P_D$ and $z_0 \in \text{Sing } L$.

This implies immediately the following:

Corollary 2.2. Suppose that $\mathcal{M} = \overline{L_m}$ is an exceptional minimal for a holomorphic foliation $L$ in $\mathbb{P}^2$. Then for every Poincaré disc $D \ni m$ one has $P_D = \mathbb{P}^2 \setminus \text{Sing } L$.

Proof. If not then there exists at least one boundary point $z_1$ of $P_D$, which is not an isolated point of $\partial P_D$. Adding to $P_D$ all its isolated boundary points we get a new domain $\overline{P_D}$, which is pseudoconvex by Corollary 2.1 and different from $\mathbb{P}^2$ - it doesn’t contains $z_1$. Therefore $\overline{P_D}$ is Stein by Fujita’s theorem and at the same time it contains an invariant compact $\mathcal{M}$. Contradiction with the maximum principle.

To analyze the situation with the failure of pseudoconvexity of Poincaré domains we shall employ the universal covering Poincaré domains. However the main problem with $\overline{P_D}$ is that the natural topology on it might not be separable in general. In [Ly1, Ly2] and [IL] the following statement was proved:
Proposition 2.2. Let $\mathcal{L}$ be a holomorphic foliation by curves on a Stein manifold $X$, let $\mathcal{L}_m$ be a leaf of $\mathcal{L}$ and $D$ a transversal hypersurface through $m$. Then $\hat{\mathcal{P}}_D$ is Hausdorff. Moreover, if $D$ is Stein then $\hat{\mathcal{P}}_D$ is also Stein.

In the forthcoming Subsections we shall describe the obstructions to the existence (i.e., separability of the topology) of universal coverings of Poincaré domains in compact complex surfaces. We shall also replace the steiness of universal covering Poincaré domains by another property appropriate for our needs - we call it rothsteiness. But before doing that we shall need to aloud our complex surfaces to have some mild singularities.

2.5. Nef models of holomorphic foliations. Recall that a two-dimensional complex space $X$ has a cyclic quotient singularity at point $a \in X$ if there exists a neighborhood $U \ni a$ which is biholomorphic to the quotient $\chi^{l,d} := \Delta^2 / \Gamma_{l,d}$. Here, for the relatively prime $1 \leq l < d$ the group $\Gamma_{l,d}$ is defined by acting on $\Delta^2$ as follows: $(z_1, z_2) \to (e^{\frac{2\pi l}{d}} z_1, e^{\frac{2\pi d}{l}} z_2)$. Such neighborhood $\chi^{l,d}$ carries a natural foliation $\mathcal{L}^\nu$ - we call it the vertical foliation - it is defined as such that lifts to the standard vertical foliation $\mathcal{L}^v := \{ z_1 \} \times \Delta$ on $\Delta^2$ under the natural cyclic covering map $\pi_{l,d} : \Delta^2 \to \chi^{l,d}$.

Let $\mathcal{L}$ be a holomorphic foliation on a projective surface $X$ with at most cyclic singularities. Our standing assumption on $\mathcal{L}$ is that at every singular point of $X$ our foliation $\mathcal{L}$ is biholomorphic to the vertical one. A leaf of such $\mathcal{L}$ is still the leaf of $\mathcal{L}^\ast := \mathcal{L}|_{X \setminus \operatorname{Sing} \mathcal{L}}$. In more colloquial terms that means that we do not consider the cyclic points of $X$ as singular points of $\mathcal{L}$ because $\mathcal{L}$, by our standing assumption, is as good as a smooth foliation in a neighborhood of a cyclic point. That means in its turn that a cyclic point $a$ does belongs to a certain leaf of $\mathcal{L}$, one may note this leaf as $\mathcal{L}_a$ as we always do. But we shall never consider a Poincaré disc through a cyclic point. Finally, let us underline once more that singular points of $\mathcal{L}$ do not belong to any of leaves of $\mathcal{L}$.

Let us briefly discuss some specific features of leaves passing through the cyclic points. Take the loop $\gamma = \pi_{l,d}(\{0\} \times \{\frac{1}{d} e^{i\theta} : \theta \in [0, 2\pi)\}) \subset \chi^{l,d}$ and take the Poincaré disc $D = \pi_{l,d}(\Delta \times \{1\})$ at $m = \pi_{l,d}(0, -\frac{1}{l})$. Then the holonomy of $\mathcal{L}^\nu$ along $\gamma$ is periodic with period $d$. Remark also that $\Delta^2 = \hat{\mathcal{P}}_D$ is the holonomy covering Poincaré domain for $\mathcal{P}_D^\nu = \chi^{l,d}$ and $\hat{\mathcal{P}}^\nu : \mathcal{P}_D^\nu \to \mathcal{P}_D^\nu$ is nothing but the natural cyclic covering map $\pi_{l,d}$. Therefore we construct the holonomy covering $\hat{\mathcal{P}}_D$ for an arbitrary Poincaré disc $D$ situated away from both $\operatorname{Sing} \mathcal{L}$ and $\operatorname{Sing} X$ literally in the same way as we deed in Subsection 2.2. Every cyclic point $a$ will contribute in a way that $\hat{p} : \hat{\mathcal{L}}_a \to \mathcal{L}_a$ will be a ramified covering of order $d$ over $a$. More precisely, if the leaf $\mathcal{L}_z$ is such that it passes through a cyclic point $a$ then $\hat{p}|_{\hat{\mathcal{L}}_z} : \hat{\mathcal{L}}_z \to \mathcal{L}_z$ is ramified with the same order $d$ at every point over $a$. The mapping $\hat{p} : \mathcal{P}_D \to \mathcal{P}_D$ as a whole will be locally biholomorphic over non-cyclic points and will behave as the standard cyclic covering $\pi_{l,d}$ over the cyclic points.

The universal covering Poincaré domain $\hat{\mathcal{P}}_D$ is now constructed from the holonomy covering domain $\hat{\mathcal{P}}_D$ again as in Subsection 2.2. The natural projection $\hat{p}$ will behave in the same way as $\hat{p}$ does and its restriction $\hat{p}|_z : \hat{\mathcal{L}}_z \to \mathcal{L}_z$ to a leaf over a cyclic point $z$ will have ramifications of the same (corresponding to $z$) order $d$ at all points over $z$. I.e., $\hat{p}|_{\hat{\mathcal{L}}_z} : \hat{\mathcal{L}}_z \to \mathcal{L}_z$ will be an orbifold covering.

Definition 2.3. A leaf $\mathcal{L}_z$ is called hyperbolic if $\hat{\mathcal{L}}_z$ is a disc. It is called parabolic if $\hat{\mathcal{L}}_z$ is $\mathbb{C}$ or $\mathbb{P}^1$.
From now on the hyperbolicity/parabolicity of leaves will be understood in the sense defined above. If at least one leaf of \( L \) has \( \mathbb{P}^1 \) as its (orbifold) universal covering then \( L \) is a rational quasi-fibration. We are usually excluding this exceptional (and trivial) case from our statements.

**Remark 2.3.** A holomorphic foliation \( L \) on a projective surface \( X \) with at most cyclic singularities is called a rational quasi-fibration if the closure of its generic fiber is a rational curve. For such foliations our theorem is obvious, therefore in the sequel we shall suppose that \( L \) is not a rational quasi-fibration whenever it will be convenient for us.

**Remark 2.4.** Results on pseudoconvexity of Poincaré domains were proved in this Section for foliations on smooth surfaces only, and only in that special case they will be used along this paper.

Recall furthermore the following:

**Definition 2.4.** Let \( L \) be a holomorphic foliation on a projective surface \( X \) with at most cyclic singularities such that \( \text{Sing} \ L \cap \text{Sing} \ X = \emptyset \), and let \( K_L \) denotes its canonical bundle. \( L \) is called nef (numerically effective) if \( K_L \) is nef, i.e., if for any irreducible algebraic curve \( C \subset X \) one has \( K_L \cdot C \geq 0 \).

Our guideline in the proof of the main theorem of this paper will be the following statement, which is one of the main results of McQuillan's "Noncommutative Mori Theory" of holomorphic foliations:

**Theorem 2.3. (McQuillan)** Let \( L \) be a holomorphic foliation by curves on an algebraic surface \( X \) with at most cyclic singularities, which is not a rational quasi-fibration. Then there exists a bimeromorphic transformation of \( X \) onto a projective surface \( Y \) with at most cyclic singularities, such that the transformed foliation \( \mathcal{F} \) is nef and therefore enjoys the following alternative:

i) either all leaves of \( \mathcal{F} \) are parabolic;

ii) or, the set \( \text{Sing} \mathcal{F} \cup \{ \text{parabolic leaves} \} \) is a proper algebraic subset \( A \) of \( X \) and, moreover, the Lobatchevski distance is continuous on \( X \setminus A \).

Of course, the condition \( \text{Sing} \mathcal{F} \cap \text{Sing} \ X = \emptyset \) is also preserved. The bimeromorphic transformation from \( X \) to \( Y \) consists of two steps: first - blowing up singular points of \( L \) to make all singularities reduced; second - contracting invariant rational curves violating the nef condition. In the process of the proof of Theorem 1 we shall go through these two steps paying a specific attention to what is happening with exceptional minimal leaves of \( L \) and with the Poincaré domains containing them.

**Remark 2.5.**

1. It should be remarked that a rational quasi-fibrations cannot be brought to the nef model.
2. The reference for Theorem 2.3 is [McQ1] and, more specifically [McQ2]. A more analytically motivated reader might find helpfull consulting the nice expositions in [Br1] [Br2].

**2.6. Obstructions to the existence of universal covering Poincaré domains.** We shall need the existence of universal covering Poincaré domains in more general cases then foliations on Stein manifolds.

A foliated holomorphic immersion between foliated pairs \((X, L)\) and \((Y, F)\) is a holomorphic immersion \( f : X \to Y \) which sends leaves to leaves. A foliated meromorphic
immersion is a meromorphic map which is a holomorphic foliated immersion outside of its indeterminacy set. All immersions, considered in this paper will be mappings between manifolds of the same dimension, i.e., locally biholomorphic over the smooth points, and they will be aloud to behave as standard cyclic covering over the cyclic points.

Let $\Delta^2 = \Delta \times \Delta$ be a bidisc in $\mathbb{C}^2$. Recall that the foliation $\mathcal{L}^v$ in $\Delta^2$ with leaves $L^v_\lambda := \{ \lambda \} \times \Delta$, $\lambda \in \Delta$ we called vertical.

**Definition 2.5.** A foliated pair $(X, \mathcal{L})$, with $\dim X = 2$, will be called straight if any foliated meromorphic immersion $h : (\Delta^2, \mathcal{L}^v) \to (X, \mathcal{L})$ is, in fact, holomorphic.

The class of straight foliated pairs we shall denote as $S_T$. An important for us observation is due to Brunella, see Lemma 1 in [Br5].

**Lemma 2.3.** Let $(X, \mathcal{L})$ be a nef foliated pair and let $p : (\Delta^2, \mathcal{L}^v) \to (X, \mathcal{L})$ be a foliated meromorphic immersion. Then $p$ is holomorphic.

In another words a nef foliated pair is straight. Let us emphasis that in both Definition 2.5 and Lemma 2.3 the foliated pair $(X, \mathcal{L})$ is aloud to have cyclic singularities.

Another class of straight foliated pairs on smooth manifolds appears as follows (and will be needed later in Section 5). Let us denote by $B_H$ the class of connected, Hausdorff, countable at infinity complex manifolds $X$ such that every locally biholomorphic map $h : \Omega \to X$ from a Stein domain $\Omega$ of dimension $n = \dim X$ extends to a locally biholomorphic map of the envelope of holomorphy $\hat{\Omega}$ into $X$. Note that vacuously all 1-dimensional complex manifolds are in $B_H$ simply because in dimension one all maps are holomorphic (meromorphic functions are also holomorphic as mappings with values in $\mathbb{P}^1$). All Stein manifolds belong to $B_H$. Also, if $\hat{X}$ is a cover of $X$ then $X$ and $\hat{X}$ belong, or not to $B_H$ simultaneously. In particular, all tori are in $B_H$. By a covering between smooth manifolds we mean a local homeomorphism/biholomorphism $\pi : \tilde{X} \to X$ such that $\pi^{-1}$ is extendable along all pathes in $X$ for all initial values. More generally to $B_H$ do belong all complex manifolds without rational curves etc.

In the future we shall need the following, less obvious, observation:

**Proposition 2.3.** i) Complex projective space $\mathbb{P}^n$ belongs to $B_H$ for every $n \geq 1$;

ii) If $X_1, X_2 \in B_H$ then $X_1 \times X_2 \in B_H$;

iii) in particular, if $S_1, ..., S_n$ are Rieamann surfaces then $S_1 \times ... \times S_n \in B_H$.

For (i) see [Iv2], for (ii) and (iii) - [Iv3]. Of course, only the holomorphic extendability is a problem here. If a map extends holomorphically (and not meromorphically (l), as one should expect in general) then this extension will be necessarily also locally biholomorphic.

We have in our disposal the following three subclasses in $S_T$.

- If $(X, \mathcal{L})$ is nef then $(X, \mathcal{L})$ is straight.
- $X \in B_H$ then $(X, \mathcal{L}) \in S_T$ for any $\mathcal{L}$.
- If a foliated pair $(X, \mathcal{L})$ doesn’t contains invariant rational curves then $(X, \mathcal{L}) \in S_T$.

About the second item it should be said that there exist compact complex manifolds $X \not\in B_H$ carrying holomorphic foliations by curves $\mathcal{L}$ such that $(X, \mathcal{L}) \in S_T$, see Example 5.1 in [Iv6].

**Corollary 2.3.** Let $(X, \mathcal{L})$ be a straight foliated pair and $\mathcal{P}_D$ a Poincaré domain. Then the universal covering Poincaré domain is Hausdorff. In particular, this is always true:
ii) for a nef foliated pair;

ii) for any \((X, \mathcal{L})\) with \(X = \mathbb{P}^2\), \(\mathbb{P}^1 \times \mathbb{P}^1\) and all from Proposition 2.3.

**Proof.** We follow [Br3]. Obstructions to the separability of the topology on \(\hat{\mathcal{P}}_D\) are vanishing cycles, see Section 3 in [Iv6]. A vanishing cycle, which appears this way, can be supposed to be imbedded, see Lemma 3.4 in [Iv6]. It means that there exists an imbedded loop \(\gamma\) in \(\hat{\mathcal{L}}_m\) (starting and ending at \(m\)) such that:

- \(\gamma\) is not bounding a disc in \(\hat{\mathcal{L}}_m\);
- But there exist a sequence \(m_k \in D, m_k \to m\) and imbedded loops \(\gamma_k \subset \hat{\mathcal{L}}_{m_k}\) which uniformly converge to \(\gamma\) and such that every \(\gamma_k\) bounds a disk \(L_k\) in \(\hat{\mathcal{L}}_{m_k}\).

Perturbing slightly, if necessary, we can include this sequence of loops into a continuous family \(\Gamma = \{\gamma_z : z \in D\}\) (one might need to take a smaller \(D\)). Then one constructs a “generalized Hartogs figure” \(W\) around \(\Gamma\), see Section 2 of [Iv6] for more details. \(W\) is a special foliated subdomain in \(\hat{\mathcal{P}}_D\). Then one replaces \(W\) by another foliated domain \((V, \pi)\) over the same disc \(D\) and transfers the map \(\hat{p}|_W : W \to X\) to a foliated map \(\hat{q} : V \to X\). The domain \(V\), in contrast to \(W\), is foliated over the disc \(D\) with all fibers being discs.

One more feature of \(V\) is that the fiber \(V_z\) of \(V\) over \(z \in D\) is mapped by \(\hat{q}\) into the leaf \(\mathcal{L}_z\) (with the same \(z\)).

The map \(\hat{q}\) extends to \(V\), but the extended map \(\hat{q}\) might become meromorphic, i.e., it might have a discrete set \(R\) of points of indeterminacy in \(V\). This clearly doesn’t happen in the straight case, because \(\hat{q}\) is foliated and locally biholomorphic outside of its (eventual) points of indeterminacy. Since \(V_m\) is mapped into \(\mathcal{L}_m\) by \(\hat{q}\) we see that \(\gamma\) also bounds a disc, namely \(\hat{p}^{-1}(\hat{q}(V_m))\), i.e., \(\gamma\) is not a vanishing cycle.

\(\square\)

### 2.7. A Rothstein type extension theorem.

We shall need a non standard version of the Rothstein’s extension theorem. Recall that the classical Rothstein’s theorem states the following, see [Si2]:

**Theorem 2.4. (Rothstein)** Let \(f\) be a holomorphic/meromorphic function in the unit bidisc \(\Delta^2 = \Delta \times \Delta\). Suppose that for every \(z_1 \in \Delta\) the restriction \(f_{z_1} := f(z_1, \cdot)\) extends as a holomorphic/meromorphic function of the variable \(z_2\) onto the disc \(\Delta_R\) with \(R > 1\). Then \(f\) holomorphically/meromorphically extends onto the bidisc \(\Delta \times \Delta_R\) as a function of both variables \((z_1, z_2)\).

Let us call a complex manifold (or, a normal complex space) \(X\) a Rothstein manifold (space) if the statement of Rothstein’s theorem is valid for holomorphic mappings with values in \(X\). Stein manifolds are obviously Rothstein. If \(\tilde{X} \to X\) is a covering then \(\tilde{X}\) and \(X\) are Rothstein, or not simultaneously. We send the interested reader to Subsection 2.5 in [Iv5] for more information when the Rothstein-type theorem is valid. Then compare with Lemma 6 from [Iv4] to derive that this property is invariant under the regular coverings (the property of being Stein is not invariant).

In this paper we are motivated by a more general statement, which basically says that all complex manifolds are “almost Rothstein”.

**Proposition 2.4.** Let \(X\) be a complex manifold (or, a normal complex space). Then \(X\) is ”almost Rothstein” in the sense that every holomorphic/meromorphic mapping \(f : \Delta^2 \to X\), such that for all \(z_1 \in \Delta\) the restriction \(f_{z_1} := f(z_1, \cdot)\) holomorphically/meromorphically
extends onto $\Delta_R$, $R > 1$, is holomorphically/meromorphically extendable onto $(\Delta \times \Delta_R) \setminus (E \times \Delta_R)$, where $E$ is a closed polar subset of $\Delta$.

For the proof see Corollary 2.5.1 in [Iv5].

**Definition 2.6.** A foliated pair $(\mathcal{P}, \mathcal{L})$, where $X$ is a (not necessarily compact) smooth complex surface, we shall call Rothstein if for every foliated holomorphic immersion $f : (\Delta^2, \mathcal{L}^0) \to (\mathcal{P}, \mathcal{L})$ such that for every $z_1 \in \Delta$ the restriction $f_{z_1}$ extends to a holomorphic immersion onto $\Delta_R, R > 1$, the map $f$ extends to a holomorphic immersion onto $\Delta \times \Delta_R$ as a mapping of both variables $(z_1, z_2)$.

By a Riemann domain over a complex surface $Y$ with at most cyclic singularities we understand a smooth complex surface $R$ together with a holomorphic mapping $p : R \to Y$ such that:

i) $p$ is locally biholomorphic over non cyclic points;

ii) for every cyclic point $b \in Y$ and every $a \in p^{-1}(b)$ there exist neighborhoods $W \ni a$ and $V \ni b$ such that the restriction $p_{|W} : W \to V$ is the standard cyclic covering.

Let us state now a variation of a Rothstein theorem, which will be needed in this paper. Let $\mathcal{P}_D$ be a universal covering Poincaré domain of a holomorphic foliation on a projective surface with at most cyclic singularities and let $\tilde{p} : (\mathcal{P}_D, \tilde{\mathcal{L}}) \to (X, \mathcal{L})$ be the canonical foliated projection, i.e., $(\mathcal{P}_D, \tilde{p})$ is a Riemann domain over $X$ (provided that $\mathcal{P}_D$ is Hausdorff).

**Proposition 2.5.** Let $(X, \mathcal{L})$ be a straight foliated pair, where $X$ is a projective surface with at most cyclic singularities and let $\mathcal{P}_D$ be a Poincaré domain in $(X, \mathcal{L})$. Then the universal covering foliated pair $(\mathcal{P}_D, \tilde{\mathcal{L}})$ is Rothstein.

**Proof.** First of all $(\mathcal{P}_D, \tilde{\mathcal{L}})$ exists as a Hausdorff topological space by Corollary 2.3. Let $f : (\Delta^2, \mathcal{L}^0) \to (\mathcal{P}_D, \tilde{\mathcal{L}})$ be a foliated immersion such that for every $z_1 \in \Delta$ the restriction $f_{z_1}$ extends as a holomorphic immersion onto $\Delta_R$. The composition $g := \tilde{p} \circ f$ is meromorphic on $\Delta \times \Delta_R$ by the classical theorem of Rothstein 2.4, because we assumed $X$ to be projective. Let us check that $g$ is, moreover, a meromorphic immersion. If not then let $C$ be the critical set of $g$ in $\Delta \times \Delta_R$. $C$ cannot contain a component of the form \{ $z_1$ \} $\times \Delta_R$ because $g$ is a holomorphic immersion on $\Delta \times \Delta$. Therefore the intersection of $C$ with $(\Delta \setminus E) \times \Delta_R$ it non empty. Here $E$ stands for the polar set of Proposition 2.4 for $f$.

Let $c_1 \notin E$ be such that \{ $c_1$ \} $\times \Delta_R \cap C$ contains some $c_2$. Since $f_{z_1}$ for all $z_1$ is supposed to be an immersion then $f$ could fail to be an immersion in a neighborhood of $c = (c_1, c_2)$ only if $f$ contracts to a point some local component $C_1$ of $C$, which passes through $c$. But this is impossible because the universal foliation $\tilde{\mathcal{L}}$ on $\mathcal{P}_D$ has no singularities. Therefore $f$ is an immersion in a neighborhood of $c$ and so must be also $g = \tilde{p} \circ f$. Contradiction.

Therefore $g$ is a meromorphic immersion. By assumed straightness of $(X, \mathcal{L})$ our $g$ is holomorphic everywhere. Take a point $a = (a_1, a_2) \in \Delta \times \partial \Delta$. Let $V$ be a cyclic neighborhood of $b := g(a)$. Taking a sufficiently small neighborhood $U$ of $a$ of the form $\Delta_r(a_1) \times \Delta_p(a_2)$ and an appropriate coordinates $(v_1, v_2) \in V \ni b$, we can suppose that the mapping $g_{|U} : U \to V$ has the standard cyclic form. Let \{ $W_j$ \} be the at most countable set of all connected components of $\tilde{p}^{-1}(V)$. By connectivity and the fact that $f$ is a foliated holomorphic (on both variables) immersion on $\Delta_r(a_1) \times (\Delta_p(a_2) \cap \Delta)$ we see readily that there exists such $j_0$ that for all $z_1 \in \Delta_r(a_1)$ we have that $f_{z_1}(\Delta_p(a_2)) \subset W_{j_0}$, i.e., that the
restriction of $f$ to $\Delta_r(a_1) \times \Delta_r(a_2)$ takes its values in $W_{j_0}$. Moreover, this restriction is jointly holomorphic on $\Delta_r(a_1) \times (\Delta_r(a_2) \cap \Delta)$.

The disc $f_a(\Delta_r(a_2))$ standardly covers the disc $g_{a_1}(\Delta_r(a_2))$: like $z \mapsto z^d$ (everywhere in this text $d = 1$ is not excluded). Therefore shrinking both $W_{j_0}$ and $V$ we obtain a cyclic covering $\tilde{\rho}|_W : W \to V$ such that for all $z_1 \in \Delta_r(a_1)$ (with some smaller $r > 0$) $f_{z_1}(\Delta_r(a_2)) \subset W$. This is because $f_{z_1}(\Delta_r(a_2) \cap \Delta) \subset W$. But $W$ is a bidisc. Therefore our $f$ extends as a holomorphic map of two variables onto $\Delta_r(a_1) \times \Delta_r(a_2)$ by Rothstein’s theorem. The rest is obvious. Remark only that the “vertical size” $\rho$ in our construction depends only on “vertical size” of the cyclic neighborhood $V$ of $b$ (and on $g$), but not on $f$.

\[
\square
\]

3. Holomorphic representation of the fundamental group

We have in mind a certain “unification” of two representations of the fundamental group $\pi_1(\mathcal{L}_m, m)$ of a leaf of a holomorphic foliation $\mathcal{L}$. The first is the standard one - the holonomy representation, it was briefly recalled in Subsection 2.2, the second is the representation by the deck transformations of the universal covering $\mathcal{L}_m \to \mathcal{L}_m$. Both are one dimensional representations in the sense that the space of these representations is either the one dimensional complex disc $D$ - i.e., the Poincaré disc, or the Riemann surface $\mathcal{L}_m$ itself.

In this Section we shall construct one more holomorphic representation of $\pi_1(\mathcal{L}_m, m)$. It will be full dimensional and will act by foliated biholomorphisms on the universal (and holonomy) covering Poincaré domains. This ”unified” representation will be our principal tool in proving the main results of this paper.

Moreover, using the hyperbolic feature of the holonomy group in or setting we shall prove that under these circumstances we can expand the universal (and holonomy also) covering Poincaré domain to some “Poincaré domain” $\tilde{\mathcal{P}}_\mathbb{C}$ foliated over $\mathbb{C}$ and it will (in the case of hyperbolic $\mathcal{L}$) regularly cover $\mathcal{P}_D$. In the case of $X = \mathbb{P}^2$ it will lead to a contradiction and this will finish the proof. The case of parabolic $\mathcal{L}$ will be excluded with a different argument (but also using the expanded domain $\tilde{\mathcal{P}}_\mathbb{C}$).

3.1. A germ of the holomorphic representation. Let $\mathcal{L}$ be a singular holomorphic foliation by curves on a compact complex surface $X$ with at most cyclic singularities. Fix some Riemannian metric $r$ on $X$. Fix some point $m \in X^0 := X \setminus (\text{Sing}\mathcal{L} \cup \text{Sing}\, X)$ and let $\mathcal{L}_m$ be the leaf of $\mathcal{L}^{\text{reg}} := \mathcal{L}|_{X^{\text{reg}}}$ through $m$. Here $X^{\text{reg}} := X \setminus \text{Sing}\, \mathcal{L}$ and may contain cyclic points. Take a small disc $m \in D \subset X^0$, transversal to $\mathcal{L}^{\text{reg}}$. By saying “a small disc” we mean that $D$ is a disc of a small geodesic radius with center in $m$. Of course, the transversality to $\mathcal{L}$ will be always supposed. By saying “a smaller” subdisc of $D$ we mean a subdisc of smaller geodesic radius and with the same center $m$. In this context writing $D_k \subset D$ we mean that $D_k$ has radius $1/k$. Our discs will be always situated in $X^0$.

Take a point $w \in \tilde{\mathcal{L}}_m$ such that $\tilde{\rho}(w) = m$. Denote by $[\gamma]$ the element of $\pi_1(\mathcal{L}_m, m)$ which realizes $w$. Take a foliated neighborhood $U \ni m$ in $\mathcal{P}_D$ and let $U_0 \ni m$ be its biholomorphic image in $X$ under the canonical foliated projection $\tilde{\rho}$. Take a foliated neighborhood $V \ni w$ in $\mathcal{P}_D$ such that $\tilde{\rho}|_V : V \to U_0$ is a foliated biholomorphism (to achieve this one might need to shrink $U_0$ and therefore $U$).
First we shall extend it to a "more global" germ. Namely, denote by Bihol(\hat{\mathcal{P}}_D, \hat{\mathcal{L}}_m) the group of foliated biholomorphic imbeddings of foliated neighborhoods of \hat{\mathcal{L}}_m into \hat{\mathcal{P}}_D, which send \hat{\mathcal{L}}_m onto itself. Therefore an element \varphi of Bihol(\hat{\mathcal{P}}_D, \hat{\mathcal{L}}_m) is a biholomorphic foliated mapping \varphi : \hat{\mathcal{P}}_V \rightarrow \hat{\mathcal{P}}_W, where V and W are some neighborhoods of m in D, such that \varphi(\hat{\mathcal{L}}_m) = \hat{\mathcal{L}}_m. Here for V \subset D we set \hat{\mathcal{P}}_V = \tilde{\pi}^{-1}(V) and call it a foliated neighborhood of \hat{\mathcal{L}}_m when V is a neighborhood of m.

**Lemma 3.1.** Suppose that \hat{\mathcal{P}}_D is Rothstein. Then the germ \varphi_\gamma extends to a foliated imbedding of a foliated neighborhood of \hat{\mathcal{L}}_m into \hat{\mathcal{P}}_D such that \varphi_\gamma(\hat{\mathcal{L}}_m) = \hat{\mathcal{L}}_m. It depends only on the homotopy class \([\gamma]\) of \gamma. The map

\[ \tilde{\Phi} : \pi_1(\mathcal{L}_m, m) \rightarrow \text{Bihol}(\hat{\mathcal{P}}_D, \hat{\mathcal{L}}_m) \]

\[ \gamma \mapsto \varphi_\gamma \]

is a monomorphism of groups.

**Proof.** Take a path \gamma in \hat{\mathcal{L}}_m from m to the point w, which defines our germ \varphi_\gamma as in (3.1). Let U, U_0 and V be as above. Denote by D_0 a sufficiently small subdisc in D such that U is foliated over D_0 by \tilde{\pi}|_U : U \rightarrow D_0. Restrictions \tilde{p}|^{-1}_V \circ \hat{\varphi}_z of our germ onto the leaves, initially defined on \hat{\mathcal{L}}_m \cap U with values in \hat{\mathcal{L}}_m \cap U, extend along any path in \hat{\mathcal{L}}_m which starts at z. This extensions clearly gave us a single-valued biholomorphic maps of \hat{\mathcal{L}}_m into \hat{\mathcal{L}}_m - deck transformations of the universal coverings of leaves in question. Since \hat{\mathcal{P}}_D is Rothstein these extensions glue together for \gamma = \gamma_{hln}(z) of \gamma is a germ of a foliated biholomorphism in a foliated neighborhood of \hat{\mathcal{L}}_m.

The fact that \tilde{\Phi} is a homomorphism of groups is obvious, because \varphi_\gamma restricted to \hat{\mathcal{L}}_m is nothing but the deck transformation of the universal covering \hat{\varphi}_z : \hat{\mathcal{L}}_m \rightarrow \hat{\mathcal{L}}_m. From this point of view our representation is an extension of the deck transformation group to a neighborhood of \hat{\mathcal{L}}_m in \hat{\mathcal{P}}_D. The extension \varphi_\gamma of every deck transformation is subduced to the condition that it is \hat{p} - equivariant by the construction. From this remark it becomes obvious that \varphi_\gamma is uniquely determined by its restriction \varphi_\gamma|_{\hat{\mathcal{L}}_m}. This proves that our representation is a monomorphism of groups.

**Definition 3.1.** Monomorphism \tilde{\Phi} : \pi_1(\mathcal{L}_m, m) \rightarrow \text{Bihol}(\hat{\mathcal{P}}_D, \hat{\mathcal{L}}_m) we shall call the holomorphic representation of the fundamental group of the leaf \mathcal{L}_m.

It is also can be called the holomorphic extension of the deck transformation group and is more precise then the holonomy representation (which is not a monomorphism in general).

**Remark 3.1.** Perhaps the most comprehensive view on \tilde{\Phi} is that it is a "unification" of the two "orthogonal" representations of the fundamental group of the leaf \mathcal{L}_m: one is the holonomy representation, the second - is the representation by the deck transformations of the universal covering \hat{\mathcal{L}}_m \rightarrow \mathcal{L}_m.

Up to now our exposition was quite general. We newer used any specific features of the holonomy group that will appear in the following Subsection.
3.2. Expansion of the holomorphic representation. Now we are going to explore the fact that in our case $\text{Hln}(L_m,m)$ contains a hyperbolic element. Denote this hyperbolic element as $\alpha$ and its holomorphic representative $\tilde{\alpha}$, i.e., $\tilde{\alpha} := \Phi(\alpha)$. Now let us fix a coordinate $t \in \Delta$ such that $\alpha$ becomes to be a multiplication by the complex number $\alpha$, $0 < |\alpha| < 1$ in this coordinate. Rescaling $t$, if necessary, we can suppose that $D = D_{|\alpha|^{-2}}$, where by $D_r$ we denote the disc $\{ t : |t| < r \}$ in $\mathbb{C}$. Set $A_{r_1, r_2} := D_{r_1} \setminus D_{r_2}$ - the annulus of radii $0 < r_1 < r_2$. Fix some $0 < \varepsilon < 1 - |\alpha|$. For every integer $n \geq 0$ consider the foliated domain

$$\left( \tilde{P}_{A_{|\alpha|^{-n+1}(-1,1)}, |\alpha|^{-2n}} \right) \quad \text{over the annulus} \quad A_{|\alpha|^{-2n+1}(-1,1), |\alpha|^{-2n+2}}. \quad (3.3)$$

Mappings $\alpha^k \tilde{\pi} : \tilde{P}_{A_{|\alpha|^{-n+1}(-1,1), |\alpha|^{-2n}}} \rightarrow A_{|\alpha|^{-k}(-1,1), |\alpha|^{-2n+2}}$ for $k \in \mathbb{Z}$ and $w \in \tilde{P}_{A_{|\alpha|^{-n+1}(-1,1), |\alpha|^{-2n}}}$. The domain $(\tilde{P}_{A_{|\alpha|^{-n+1}(-1,1), |\alpha|^{-2n}}}, \alpha^0 \tilde{\pi})$ we consider as identical to $\tilde{P}_{A_{|\alpha|^{-n+1}(-1,1), |\alpha|^{-2n}}} \subset \tilde{P}_D$. Glue domain $(\tilde{P}_{A_{|\alpha|^{-n+1}(-1,1), |\alpha|^{-2n}}}, \alpha^{-2n} \tilde{\pi})$ to $(\tilde{P}_{A_{|\alpha|^{-n+1}(-1,1), |\alpha|^{-2n}}}, \alpha^{-2n-2} \tilde{\pi})$ by the biholomorphism $\tilde{\alpha}^2 : \tilde{P}_{A_{|\alpha|^{-n+1}(-1,1), |\alpha|^{-2n}}} \rightarrow \tilde{P}_{A_{|\alpha|^{-n+1}(-1,1), |\alpha|^{-2n}}}$, see diagram (3.3) below.

\[ \begin{array}{ccc}
\tilde{P}_{A_{|\alpha|^{-n+1}(-1,1), |\alpha|^{-2n}}} & \supset & \tilde{P}_{A_{|\alpha|^{-n-1}(-1,1), |\alpha|^{-2n}}} \\
\alpha^{-2n} \tilde{\pi} & \downarrow & \alpha^{-2n-2} \tilde{\pi} \\
A_{|\alpha|^{-2n+1}(-1,1), |\alpha|^{-2n+2}} & \subset & A_{|\alpha|^{-2n-1}(-1,1), |\alpha|^{-2n-2}} \\
\equiv & \tilde{P}_{A_{|\alpha|^{-n+1}(-1,1), |\alpha|^{-2n}}} & \subset \tilde{P}_{A_{|\alpha|^{-n+1}(-1,1), |\alpha|^{-2n}}} \\
\equiv & A_{|\alpha|^{-n-1}(-1,1), |\alpha|^{-2n-2}} & \subset A_{|\alpha|^{-n-1}(-1,1), |\alpha|^{-2n-4}}. \\
\end{array} \quad (3.4) \]

The obtained Poincaré domain over $\mathbb{C}$ denote as $\tilde{P}_C$ and call it the expanded universal covering Poincaré domain. The projection $\tilde{\pi}$ obviously extends to a holomorphic projection $\tilde{\pi} : \tilde{P}_C \rightarrow \mathbb{C}$ by construction: the extended $\tilde{\pi}$ on each $(\tilde{P}_{A_{|\alpha|^{-n+1}(-1,1), |\alpha|^{-2n}}}, \alpha^{-2n} \tilde{\pi})$ is simply equal to $\alpha^{-2n} \tilde{\pi}$.

**Lemma 3.2.** The germ $\tilde{\alpha}$ extends to a global foliated biholomorphism of the expanded Poincaré domain, which commutes with $\tilde{\pi}$. The canonical foliated projection $\tilde{p} : \tilde{P}_D \rightarrow X$ also extends to a foliated immersion $\tilde{p} : \tilde{P}_C \rightarrow \tilde{P}_D$ and this extension stays to be $\tilde{\alpha}$-equivariant.

**Proof.** The proof of extendability of $\tilde{\alpha}$ consists in checking of the correctness of its natural definitions on the overlapping subsets. Let us do it for $\tilde{\alpha}^{-1}$ instead of $\tilde{\alpha}$. Subdomain $(\tilde{P}_{A_{|\alpha|^{-n-1}(-1,1), |\alpha|^{-2n-2}}})$ is identified with the domain $(\tilde{P}_{A_{|\alpha|^{-n+1}(-1,1), |\alpha|^{-2n+2}}}, \alpha^{-2n-2} \tilde{\pi})$ by $\tilde{\alpha}^2$, see (3.1). Therefore $w \in (\tilde{P}_{A_{|\alpha|^{-n-1}(-1,1), |\alpha|^{-2n-2}}}, \alpha^{-2n-2} \tilde{\pi})$ is identified with $\tilde{\alpha}^2 w \in \left( \tilde{P}_{A_{|\alpha|^{-n+1}(-1,1), |\alpha|^{-2n+2}}}, \alpha^{-2n-2} \tilde{\pi} \right)$, see the upper horizontal line in the diagram (3.5). One can act by $\tilde{\alpha}^{-1}$ both on $w$ and on its twin $\tilde{\alpha}^2 w$, see down arrows in the diagram below.

\[ \begin{array}{ccc}
(\tilde{P}_{A_{|\alpha|^{-n-1}(-1,1), |\alpha|^{-2n-2}}}) & \ni & w \\
\alpha^{-1} & \downarrow & \alpha^{-1} \\
(\tilde{P}_{A_{|\alpha|^{-n+1}(-1,1), |\alpha|^{-2n+2}}}) & \ni & \alpha^{-2n} \tilde{\pi} \ni \alpha^{-1} w \\
\alpha^{-1} & \downarrow & \alpha^{-1} \\
(\tilde{P}_{A_{|\alpha|^{-n+1}(-1,1), |\alpha|^{-2n+2}}}) & \ni & \alpha^{-2n-2} \tilde{\pi}
\end{array} \quad (3.5) \]

On the left one gets $\tilde{\alpha}^{-1} w$, on the right $\tilde{\alpha} w$. But they are identified by $\tilde{\alpha}$. Therefore $\tilde{\alpha}^{-1}$ is clearly defined globally. It is commuting with $\tilde{\pi}$ by construction.
Since the gluing maps involved in the expansion of the universal covering Poincaré domain are \( \tilde{p} \)-equivariant the map \( \tilde{p} \) extends obviously to a locally biholomorphic foliated map \( \tilde{p} : \mathcal{P}_C \to \mathcal{P}_D \) and in its turn stays to be \( \tilde{\alpha} \)-equivariant. Moreover, we have that for every \( t \in \mathbb{C} \) the restriction \( \tilde{p}|_t \) is the universal covering of \( \mathcal{L}_{\alpha^N t} \) for \( N \) big enough, namely \( \alpha^N t \) should be in \( D \). As we already remarked the image of \( \mathcal{P}_C \) under \( \tilde{p} \) is nothing but \( \mathcal{P}_D \) due to the periodicity of the expanded domain.

\[ \square \]

3.3. Universal covering Poincaré domains of hyperbolic foliations. As we explained in the Introduction the proof of the Theorem \([\text{II}]\) will be done separately for two different cases: when \( \mathcal{L} \) is parabolic and when it is hyperbolic. In this Subsection we consider the hyperbolic case.

Therefore let \( \mathcal{L} \) be a hyperbolic holomorphic foliation on a projective surface \( X \) with at most cyclic singularities, \textit{i.e.}, at least one leaf of \( \mathcal{L} \) is hyperbolic. In that case one can locally define the following hyperbolic norm on vectors tangent to \( \mathcal{L} \):

\[
\| v \|_1 = \inf \{ 1/r : \exists \text{ holomorphic } u : \Delta(r) \to \mathcal{L}_z, u(0) = z, u'(0) = v \}. \tag{3.6}
\]

If the leaf \( \mathcal{L}_z \) passes through a cyclic point then one defines the hyperbolic norm on the (orbifold) universal covering \( \tilde{\mathcal{L}}_z \) of \( \mathcal{L}_z \) and pushes it down to a singular (at cyclic point) metric on \( \mathcal{L}_z \). In fact we shall nod need to push it down and will work on \( \tilde{\mathcal{L}}_z \) (more precisely on \( \mathcal{P}_D \)). One calls (depending on traditions) \( \|v\|_1 \) the Lobatchevski/ Poincaré/Kobayashi/hyperbolic norm of \( v \in T\mathcal{L}_z \). Function \([\text{3.70}]\) is well defined on tangent vectors to leaves, \( L(v) := \ln \|v\|_1 \) is finite if \( v \) is tangent to a hyperbolic leaf, and is equal to \(-\infty\) if it is tangent to a parabolic leaf. We say that Lobatchevski metric is continuous if for any local holomorphic vector filed \( v \) tangent to \( \mathcal{L} \) the local function \( \|v\|_1 \) is continuous.

If we suppose that a sufficiently small transversal disc through \( m \) doesn’t cuts any parabolic leaf of \( \mathcal{L} \), \textit{i.e.}, that all leaves \( \mathcal{P}_z \) for \( z \in D \) are hyperbolic, then, by construction, the same holds for all leaves of \( \mathcal{P}_C \). As one see from the McQuillan’s alternative we can suppose that the Lobatchevski norm is \textit{continuous} on \( \mathcal{P}_C \) (provided \( (X, \mathcal{L}) \) is \textit{nef}).

Our aim in this Subsection is to prove that \( \tilde{p} : \mathcal{P}_C \to \mathcal{P}_D \) is a regular covering in the hyperbolic case.

**Lemma 3.3.** Suppose that \( \mathcal{L}_m \) has a hyperbolic holonomy and that all leaves \( \mathcal{L}_z \) for \( z \in D \) are hyperbolic. Moreover, suppose that that \( \mathcal{P}_C \) is Hausdorff and Rothstein and that the Lobatchevski metric is continuous on \( \mathcal{P}_C \). If for some \( a_1 \in \tilde{\mathcal{L}}_{z_1} \) and \( b_1 \in \tilde{\mathcal{L}}_{w_1} \) one has that \( \tilde{p}(a_1) = \tilde{p}(b_1) \) then there exists a global \( \tilde{p} \)-equivariant automorphism \( \varphi \) of \( \mathcal{P}_C \) such that \( \varphi(a_1) = b_1 \).

**Proof.** Applying \( \tilde{\alpha}^N \) to \( a_1 \) and \( b_1 \) with \( N \) big enough we can suppose that corresponding \( z_1, w_1 \) belong to \( D \). Take foliated neighborhoods \( U \ni a_1, V \ni b_1 \) and \( U_0 \ni \tilde{p}(a_1) \) such that \( \tilde{p}|_U : U \to U_0 \) and \( \tilde{p}|_V : V \to U_0 \) are foliated biholomorphisms. As in the proof of Lemma \([\text{3.3}]\) the composition \( \tilde{p}|_U^{-1} \circ \tilde{p}|_V \) extends along the leaves to a foliated biholomorphism \( \varphi \) of some \( \mathcal{P}_{D_0} \) and \( \mathcal{P}_{B_0} \), where \( D_0 \ni z_1 \) and \( B_0 \ni w_1 \). Move \( a_1 \) to \( z_1 \) inside \( \tilde{\mathcal{L}}_{z_1} \) and follow this move by the move of \( b_1 \) inside \( \tilde{\mathcal{L}}_{w_1} \) in order to still have \( \tilde{p}(a_1) = \tilde{p}(b_1) \) with \( a_1 = z_1 \) this time.

Let us prove that \( \varphi \) extends along any path in \( \mathbb{C} \) in the sense that for any path \( \gamma : [0,1] \to \mathbb{C}, \gamma(0) = z_1 \), there exists a continuous family of discs \( D_t \) with centers at \( \gamma(t) \) of radii
r(t) (continuously depending on t), and there exist foliated $\tilde{p}$-invariant biholomorphisms $\varphi_t : \tilde{P}_{D_t} \to \tilde{P}_{B_t}$ (for appropriate domains $B_t$) such that $\varphi_{t_1}$ coincide with $\varphi_{t_2}$ on $\tilde{P}_{D_{t_1} \cap D_{t_2}}$ for $t_1 - t_2$ small enough. Of course we mean that $\varphi_0 = \varphi$.

Let $t_0$ be the supremum of $t$-s such that $\varphi$ extends up to $t$. All we need to prove is that $\varphi$ extends also to a neighborhood of $t_0$. Set $\beta(t) := \tilde{\pi}(\varphi(\gamma(t)))$. Let us prove first that $\beta(t)$ stays in a compact part of $\mathbb{C}$ as $t \nearrow t_0$. If not then there exists a sequence $0 < t_1 < t_2 < \cdots < t_{2n-1} < t_{2n} < \cdots \to t_0$ such that $|\beta(t_{2n-1})| = |\alpha|^{-2k_n+1}(1 - \varepsilon)$, $|\beta(t_{2n})| = |\alpha|^{-2k_n-2}$ and $\beta([t_{2n-1}, t_{2n}]) \subset A_{|\alpha|^{-2k_n+1}(1 - \varepsilon)}, |\alpha|^{-2k_n-2}}$. Remark that applying $\tilde{\alpha}_N^N$ with $N$ big enough (from the very beginning) we can suppose that $\gamma([t_1, t_0]) \subset A_{|\alpha|(1 - \varepsilon)}, |\alpha|}$.

Applying $\tilde{\alpha}_N^N$ to every piece $\varphi(\gamma([t_{2n-1}, t_{2n}]))$ and taking a subsequence we obtain that $\tilde{\pi}(\tilde{\alpha}_N^N(\varphi(\gamma([t_{2n-1}, t_{2n}])))$ converge in Hausdorff sense to a continuum $C \subset A_{|\alpha|(1 - \varepsilon)}, |\alpha|}$. Let us prove now the following:

Claim. $\mathcal{L}_{\gamma(t_0)}$ coincides with $\mathcal{L}_c$ for every $c \in C$. Indeed, take a sequence $\tau_n \nearrow t_0$ such that $c_n := \tilde{\pi}(\tilde{\alpha}^{k_n}(\varphi(\gamma(\tau_n)))) \to c$. Denote by $r_n$ the geodesic distance from $\tilde{p}(c_n, c_n)$ to $\tilde{p}(\gamma(\tau_n), \gamma(\tau_n))$ in the leaf $\mathcal{L}_{\gamma(\tau_n)}$. The sequence $\{r_n\}$ is obviously bounded. This results from the fact that $\tilde{p}$ is defined and holomorphic in a neighborhood of both of $t_0$ and $c$ and the fact that the hyperbolic distance along the leaves is continuous in $\mathcal{P}_D$. But that means that we can find a point $b_n \in \mathcal{L}_{c_n}$ on a distance not more then $r_n$ such that $\tilde{p}(c_n, b_n) = \tilde{p}(\gamma(\tau_n), \gamma(\tau_n))$. After taking a subsequence $b_n \to b_0 \in \mathcal{L}_c$. But then $\tilde{p}(c) = \tilde{p}(\gamma(t_0), \gamma(t_0))$ by continuity of $\tilde{p}$ and the Claim is proved.

But this (what we get in the Claim) is impossible because one leaf can cut $D$ only by at most countable set. Therefore $\beta(t)$ stays bounded when $t \to t_0$.

Applying $\tilde{\alpha}_N$ once more we can suppose that $\beta(t)$ stays in $D_{1/2}$. Limiting set of $\beta(t)$ when $t \to t_0$ can be either a point or a continuum. But the latter is impossible by the reason already explained above. Therefore $\beta(t) \to w_0$ for some $w_0 \in D$ when $t \to t_0$. Mapping $\varphi$ along $\gamma$ writes as $\varphi = \tilde{p}^{-1} \circ \tilde{p}$ for some choice of $\tilde{p}^{-1}$.

Both $\tilde{p}\mid_{\mathcal{L}_{w_0}}$ and $\tilde{p}\mid_{\mathcal{L}_{\gamma(t_0)}}$ cover that same leaf $\mathcal{L}_{\gamma(t_0)}$ (see the Claim). Let $d_l(\cdot)$ be the Lobatchevski distance along the leaves of $\tilde{\mathcal{L}}$. Remark that $d_l(\tilde{p}^{-1} \circ \tilde{p}\mid_{\gamma(t)}, \beta(t))$ is continuous up to $t_0$. Indeed, it is nothing else but the distance from $\tilde{p}(\gamma(t))$ to $\tilde{p}(\beta(t))$ and the latter is continuous up to $t_0$. Therefore the limiting set of $\varphi(\gamma(t))$ when $t \nearrow t_0$ is a compact $K$ in $\mathcal{L}_{w_0}$. This implies that biholomorphisms $\varphi_{\mathcal{L}_{\gamma(t)}} : \tilde{\mathcal{L}}_{\gamma(t)} \to \tilde{\mathcal{L}}_{\beta(t)}$ converge to a biholomorphism $\tilde{\mathcal{L}}_{\gamma(t_0)} \to \tilde{\mathcal{L}}_{w_0}$ as $t \nearrow t_0$. Now we can easily extend $\tilde{p}^{-1} \mid_{\tilde{\mathcal{L}}_{w_0}} \circ \tilde{p}\mid_{\mathcal{L}_{\gamma(t_0)}}$ to foliated neighborhoods as it was done at the beginning of the proof of the Theorem. 

Denote by $G$ the group of all $\tilde{p}$-equivariant foliated biholomorphisms of $\tilde{\mathcal{P}}_{\mathbb{C}}$. We have the following:

**Theorem 3.1.** If all leaves of $\mathcal{P}_D$ are hyperbolic and the hyperbolic distance is continuous on $\mathcal{P}_D$ then $\tilde{p} : \tilde{\mathcal{P}}_{\mathbb{C}} \to \mathcal{P}_D$ is a regular covering.

**Proof.** Let us underline that by saying that $\tilde{p} : \tilde{\mathcal{P}}_{\mathbb{C}} \to \mathcal{P}_D$ is a regular covering we mean that $\mathcal{P}_D = \tilde{\mathcal{P}}_{\mathbb{C}}/G$, in particular, it is the standard cyclic covering over the cyclic points. Recall, that an action of a discrete group $G$ on a complex manifold $\tilde{\mathcal{P}}_{\mathbb{C}}$ is called proper discontinuous if for every compacts $K_1, K_2 \subset \tilde{\mathcal{P}}_{\mathbb{C}}$ the set

$$\{g \in G : gK_1 \cap K_2 \neq \emptyset\}$$

is finite.
is finite. But in our case we have a $G$-equivariant local biholomorphism $\tilde{p} : \tilde{P}_\mathbb{C} \to X$. Suppose there exist $w_n \to w_0$ in $K_1$ such that $g_n(w_n) = v_n \to v_0$ in $K_2$, and $g_n \in G$ are distinct. Take neighborhoods $W \ni w_0$ and $V \ni v_0$ such that $(\tilde{p}|_V)^{-1} \circ \tilde{p}|_W : W \to V$ is a biholomorphism. Since $\tilde{p}$ is $G$-invariant we get that for $n >> 1$ $g_n|_W = (\tilde{p}|_V)^{-1} \circ \tilde{p}|_W$, i.e., all $g_n$ for $n > 1$, are equal to each other. Contradiction.

Action of $G$ is cyclic, i.e., every point $a \in \tilde{P}_\mathbb{C}$ admits a neighborhood $U$ such that $G_a := \{g \in G : gU \cap U \neq \emptyset\}$ is isomorphic to $\Gamma_{l,d}$ for the appropriate $1 \leq l < d$. Indeed, take $N >> 1$ in order to have $b := \alpha^N a \in D_x^C$. It is obviously sufficient to find a needed neighborhood $U$ of $b$. Take $U$ such that $\tilde{p}|_U : U \to X$ is cyclic. Since $\tilde{p}$ is $G$-equivariant and the last acts by global biholomorphisms one cannot have any other behavior of $G$ at $a$, because then $\tilde{p}$ would not be cyclic on $U$.

$\square$

### 3.4. Imbedding of the expanded Poincaré domain into $\mathbb{C}^2$.

Before considering the case of parabolic foliations we need few preparatory lemmas.

**Definition 3.2.** By a foliated domain we shall mean a triple $(W, \pi, D)$ where $D$ is a domain in $\mathbb{C}$, $W$ is a connected complex surface, and $\pi : W \to \mathbb{C}$ is a holomorphic submersion with connected fibers.

A holomorphic section of a foliated domain $(W, \pi, D)$ is a holomorphic map $\sigma : D \to W$ such that $\pi \circ \sigma = \text{Id}$. Remark that our covering Poincaré domains $\tilde{P}_D$ do admit holomorphic sections, namely the natural map $i : D \to \tilde{P}_D$ defined as $i : z \to \gamma_{z,\gamma}$ is such a section. The complex linear space $\mathbb{C}^2$ we shall also see as a foliated domain $(\mathbb{C}^2, \pi_1)$, where $\pi_1 : (z_1, z_2) \to z_1$ is the natural "vertical" projection. By a foliated holomorphic imbedding of $(W, \pi, D)$ into $(\mathbb{C}^2, \pi_1)$ we mean an imbedding $H : W \to \mathbb{C}^2$ such that every leaf $W_z$ is mapped into the leaf $\mathbb{C}_z := \{z\} \times \mathbb{C}$, i.e., $H$ has the form

$$ (z, \cdot) \to (z, h(z, \cdot)), \quad (3.7) $$

where for every fixed $z$ the function $h(z, \cdot)$ realizes a conformal imbedding of the domain $W_z$ into $\mathbb{C}_z$.

The following result is due to Brunella, see [Br4]:

**Lemma 3.4.** Let $\tilde{P}_D$ be a universal covering Poincaré domain over a simply connected transversal $D$ such that:

1. all fibers $\mathcal{L}_z$ for $z \in D$ are biholomorphic to $\mathbb{C}$;
2. the foliated pair $(X, \mathcal{L})$ is straight.

Then there exists a foliated holomorphic imbedding $H : \tilde{P}_D \to (\mathbb{C}^2, \pi_1)$ sending a leaf $\mathcal{L}_z$ to $\{z\} \times \mathbb{C}$ for all $z \in D$.

**Remark 3.2.** The statement of [Br4] is more general, but $\tilde{P}_D$ is understood there also in a more general sense. Namely, one should add to some leaves of $\mathcal{L}$ certain "vanishing ends" in order for $\tilde{P}_D$ to become Hausdorff. But in the straight case one doesn’t need to do that according to Lemma 2.3.

Remark that if at least one leaf of $\tilde{P}_D$ is $\mathbb{P}^1$ then all are such. Therefore we exclude this case in what follows.

**Corollary 3.1.** If all leaves of $\tilde{P}_\mathbb{C}$ are parabolic and different form $\mathbb{P}^1$ then $(\tilde{P}_\mathbb{C}, \tilde{\pi})$ is leafwise biholomorphic to $(\mathbb{C}^2, \pi_1)$. 
Proof. First let us check that \((\bar{P}_C, \pi, C)\) satisfies the Gromov’s spray condition, see [Gro] or [Fo] for definitions. For that take a point \(z \in C\) and such a small disc \(B \ni z\) that \(\pi : \bar{P}_B \to B\) admits a section over \(B\). Then one can apply Lemma 3.4 and imbed the restriction \((\bar{P}|_B, \pi, B)\) leafwise into \(C^2\). After that the spray condition becomes obvious. This condition via the theorem of Gromov, see the same sources, provides us a global section of our fibration.

After that one can employ the result of Siu that every Stein submanifold (a section in our case) admits a Stein neighborhood, which is, in addition biholomorphic to a neighborhood of the zero section in the normal bundle to this submanifold, see Corollary 1 in [Si5].

Therefore the proof of Brunella, given in [Br4], applies again and finishes the proof of our Corollary.

\[\square\]

3.5. Universal covering Poincaré domains of parabolic foliations. We now ready to consider the case when \(L\) is parabolic, i.e., every leaf \(L_z\) of \(L\) has as its (orbifold) universal covering either \(\mathbb{P}^1\) or \(\mathbb{C}\). The former case means that \(L\) is a rational quasi-fibration and therefore we don’t need to consider it. The cases \(\bar{L}_m = \mathbb{P}^1, T^1\) are also obvious, in the setting of Theorem 1, but we shall include them for the sake of completeness.

Since our coverings \(\bar{ho}_z : \bar{L}_z \to L_z\) are, in fact, orbifold coverings it is useful to recall the formula relating their Euler characteristics:

\[
\chi(\tilde{L}_z, \nu) = \chi(L_z) + \sum_j \left( \frac{1}{\nu(r_j)} - 1 \right),
\]

(3.8)

where \(\chi(L_z)\) is the Euler characteristic of the underlying Riemann surface \(L_z\) and \(\nu(r_j)\) is the value of ramification function \(\nu\) at ramification point \(r_j\), see [Mil] and references there. Parabolicity of \(\tilde{L}_z\) means that \(\chi(\tilde{L}_z, \nu) \geq 0\). This leaves very few possibilities for \(L_z\) and the ramification function \(\nu\). When \(L_z\) is noncompact we have only the following ones:

r) \(L_z = \mathbb{C}\) with one ramification point, or with two ramification points of order two.

n) \(L_z = \mathbb{C}^*\) without ramifications.

Theorem 3.2. Let \(L_m\) be a parabolic leaf with the hyperbolic holonomy of a holomorphic foliation \(L\) on a projective surface \(X\) with at most cyclic singularities. Suppose that for a sufficiently small Poincaré disc \(D\) all leaves cutting \(D\) are also parabolic and that \(\bar{P}_D\) is Hausdorff and Rothstein. Then:

i) either \(\bar{L}_m\) is a rational or elliptic curve;

ii) or, \(L_m = \mathbb{C}^*\) and is an imbedded analytic subset in some open subset of \(X \setminus \text{Sing}\). \(L\).

Proof. Let \(\bar{L}_m\) be not algebraic and therefore such are all \(\bar{L}_z\) for \(z\) in a sufficiently small Poincaré disc \(D\). Note that \(L_m\) cannot be \(\mathbb{C}\) because \(\pi_1(L_m)\) contains an element \(\alpha\) with hyperbolic holonomy. Therefore \(L_m = \mathbb{C}^*\) and the universal covering \(\tilde{L}_m \to L_m\) is unramified.

According to Lemma 3.4 in this case \((\bar{P}_C, \tilde{\pi})\) is leafwise biholomorphic to \((\mathbb{C}^2, \pi_1)\). Up to an affine change of the coordinate \(z_2\) we can suppose that the holomorphic representative
Lemma 3.5. There exist \( \varepsilon > 0 \) and \( 0 < C < \infty \) such that the automorphism \( \tilde{\alpha} \) satisfies for all \( n \) and all \( (z_1, z_2) \) with \( |z_1| < \varepsilon \) the relation
\[
\tilde{\alpha}^n(z_1, z_2) = (\alpha^n z_1, z_2 + n + a_n(z_1) + b_n(z_1) z_2),
\] (3.10)
where
\[
|a_n(z_1)| \leq \frac{C|z_1|}{(1 - |\alpha|)^2} \quad \text{and} \quad |b_n(z_1)| \leq \frac{C|z_1|}{1 - |\alpha|}.
\] (3.11)
Moreover, \( a_n \to a_0 \), \( b_n \to b_0 \) uniformly on \( z_1 \) in a neighborhood of zero. Here \( a_0 \) and \( b_0 \) are holomorphic in a neighborhood of zero.

Proof. Since for every fixed \( z_1 \) the function \( h(z_1, z_2) \) should be an automorphism of \( \mathbb{C} \) and for \( z_1 = 0 \) it has the form as in (3.9) we obtain that \( h(z_1, z_2) = z_2 + 1 + a_1(z_1) + b_1(z_1) z_2 \) with \( a_1(0) = 0 \) and \( b_1(0) = 0 \). Set \( A = a_1 \), \( B = b_1 \) and write
\[
\begin{aligned}
\tilde{\alpha}(z_1, z_2) &= (\alpha z_1, z_2 + 1 + A(z_1) + B(z_1) z_2), \\
\text{where } |A(z_1)|, |B(z_1)| &\leq C|z_1| \text{ for } |z_1| < \varepsilon.
\end{aligned}
\] (3.12)
Here \( 0 < C < \infty \) and \( \varepsilon > 0 \) are some constants. Let us prove by induction that for every \( n \in \mathbb{N} \) one has
\[
\begin{aligned}
\tilde{\alpha}^n(z_1, z_2) &= (\alpha^n z_1, z_2 + n + \sum_{k=1}^{n} [k A_{n,k}(z_1) + z_2 B_{n,k}(z_1)]), \\
\text{with } |A_{n,k}(z_1)|, |B_{n,k}(z_1)| &\leq C|\alpha|^{k-1} \prod_{j=1}^{n-1} [1 + C|\alpha|^j |z_1|] |z_1| \text{ for } 1 \leq k \leq n.
\end{aligned}
\] (3.13)
For \( n = 1 \) (3.13) is nothing but (3.12). Next write, using (3.12) and (3.13) the second component \( \tilde{\alpha}^{n+1} \) of \( \tilde{\alpha}^{n+1} \) as follows:
\[
\begin{aligned}
\tilde{\alpha}^{n+1}(z_1, z_2) &= z_2 + n + 1 + \sum_{k=1}^{n} [k A_{n,k}(z_1) + z_2 B_{n,k}(z_1)] + A(\alpha^n z_1) + (z_2 + n + \sum_{k=1}^{n} [k A_{n,k}(z_1) + z_2 B_{n,k}(z_1)]) B(\alpha^n z_1) \\
&\quad + z_2 \sum_{k=1}^{n} B_{n,k}(z_1) [1 + B(\alpha^n z_1)] + z_2 B(\alpha^n z_1).
\end{aligned}
\]
Set \( A_{n+1,n+1}(z_1) := \frac{1}{n+1} A(\alpha^n z_1) + \frac{n}{n+1} B(\alpha^n z_1) \). The estimate \( |A_{n+1,n+1}(z_1)| \leq C|\alpha|^n |z_1| \) follows. Analogously set \( B_{n+1,n+1}(z_1) := B(\alpha^n z_1) \) and get \( |B_{n+1,n+1}(z_1)| \leq C|\alpha|^n |z_1| \).

For \( 1 \leq k \leq n \) set \( A_{n+1,k}(z_1) := A_{n,k}(z_1) [1 + B(\alpha^n z_1)] \). Then by induction we have
\[
|A_{n+1,k}(z_1)| = |A_{n,k}(z_1) [1 + B(\alpha^n z_1)]| \leq C|\alpha|^k \prod_{j=1}^{n-1} [1 + C|\alpha|^j |z_1|] |z_1| (1 + C|\alpha|^n |z_1|),
\]
which gives us (3.13). Further set \( B_{n+1,k}(z_1) := B_{n,k}(z_1) [1 + B(\alpha^n z_1)] \) and get the same estimate. (3.13) is proved.
Since $|\prod_{j=1}^{n-1} |1 + C|\alpha|^j|z_1|| \leq K$ we get the estimates:

$$|A_{n,k}(z_1)|, |B_{n,k}(z_1)| \leq C|\alpha|^{k-1}|z_1|. \quad (3.14)$$

These estimates plus the usual summation in $(3.13)$ gave the proof of the lemma.

Now we can get more information about the global behavior of $\tilde{p}$. Suppose that for some $z_1 \neq 0$ the restriction $\tilde{p}_{z_1} : \mathbb{C}_{z_1} \to X$ has a nontrivial period, i.e., that there exists a non-zero complex number $a(z_1)$ such that $\tilde{p}|_{z_1}$ is invariant under the translation $z_2 \to z_2 + a(z_1)$ on the complex line $\mathbb{C}_{z_1}$.

**Lemma 3.6.** If for some $z_1 \neq 0$ the restriction $\tilde{p}_{z_1}$ has a nontrivial period, then there exists $\varepsilon > 0$ and a non-vanishing holomorphic function $a$ in $\Delta(0, \varepsilon)$ such that $a(0) = 1$ and

$$\tilde{p}(z_1, z_2) \equiv \tilde{p}(z_1, z_2 + a(z_1)) \quad (3.15)$$

for all $(z_1, z_2) \in \Delta(0, \varepsilon) \times \mathbb{C}$.

**Proof.** Denote by $L_{z_1}$ the leaf which is covered by $\tilde{p}_{z_1}$. Since $L_{z_1}$ is also covered by every $\tilde{p}_{a^n z_1}$ we can suppose that $|z_1| < \varepsilon$ where $\varepsilon > 0$ is from Lemma 3.5. From $\tilde{a}$-invariance of $\tilde{p}$ we see that for every $z_2$ the point

$$\tilde{a}(z_1, z_2 + a(z_1)) = (\alpha z_1, z_2 + a(z_1) + 1 + a_1(z_1) + b_1(z_1)a(z_1) + b_1(z_1)z_2)$$

should be a translation by some $d_1 \cdot a(\alpha z_1)$ of the point

$$\tilde{a}(z_1, z_2) = (\alpha z_1, z_2 + 1 + a_1(z_1) + b_1(z_1)z_2)$$

on the line $\mathbb{C}_{\alpha z_1}$. Here $a(\alpha z_1)$ is a notation for the period of this translation. Therefore

$$a(z_1) + a_1(z_1) + b_1(z_1)a(z_1) + b_1(z_1)z_2 = a_1(z_1) + b_1(z_1)z_2 + d_1 \cdot a(\alpha z_1).$$

From here we get that

$$a(\alpha z_1) = \frac{1}{d_1} [1 + b_1(z_1)] a(z_1). \quad (3.16)$$

Likewise, using the formula $(3.10)$, we get that

$$a(\alpha^n z_1) = \frac{1}{d_n} [1 + b_n(z_1)] a(z_1). \quad (3.17)$$

Here, again, $a(\alpha^n z_1)$ is a notation for the period of the corresponding translation in $\mathbb{C}_{\alpha^n z_1}$.

**Remark 3.3.** Periods of $\tilde{p}_{a^n z_1}$ (we always mean *minimal* periods) are uniquely defined, because $p_{a^n z_1}$ can be supposed to be noncompact.

Recall that $b_n \to b_0$. Would $d_n$ be non bounded, some subsequence $a(\alpha^{n_k} z_1)$ would converge to zero. This contradicts to the local biholomorphicity of $\tilde{p}$ at the origin. Therefore $[1 + b_0(z_1)] a(z_1) = \lim_{k \to \infty} [1 + b_{n_k}(z_1)] a(z_1)$ is a (may be, non minimal) period of $\tilde{p}_0$, and it is a limit of (may be not minimal) periods of $\tilde{p}_{a^n z_1}$. Would be this period different from 1 (i.e., equal to some $d \geq 2$) this would contradict to the fact that the holonomy along the loop $\tilde{p}_0([0,1])$ is contractible. Therefore we get that

$$[1 + b_0(z_1)] a(z_1) = 1. \quad (3.18)$$

In the same way one gets

$$[1 + b_0(\alpha^n z_1)] a(\alpha^n z_1) = 1. \quad (3.19)$$
for all \( n \in \mathbb{N} \). The relation (3.18) means that \( a(z_1) \) extends to a holomorphic function to a neighborhood of zero, which can defined by

\[
a(z_1) = 1/[1 + b_0(z_1)].
\]  

(3.20)

Relation (3.19) means that this extension in all points \( \{\alpha^n z_1\} \) is a period of \( \tilde{p}_{\alpha^n z_1} \). Therefore for every \( z_2 \) the holomorphic equation

\[
\tilde{p}(z_1, z_2) = \tilde{p}(z_1, z_2 + a(z_1))
\]

has a converging to zero sequence of solutions (3.19). I.e., (3.15) is proved.

Lemma 3.7. \( \mathcal{L}_m \) is a locally closed analytic subset of \( X \setminus \text{Sing} \mathcal{L} \).

Proof. Let us prove that \( \mathcal{L}_m \) is such in a neighborhood of \( m \). If not, i.e., if \( \mathcal{L}_m \) cuts \( D \) by a sequence of points \( m_k \to m \) then all \( \tilde{p}|_{m_k} : \mathcal{L}_{m_k} \to \mathcal{L}_{m_k} \) are the periodic coverings of the same leaf \( \mathcal{L}_m \) and Lemma 3.6 applies. But this means that holonomy along the path \( \gamma = \tilde{p}_0(0,1]) \) should be trivial. Indeed, for every \( z_1 \) close to zero \( \tilde{p}_{z_1}(0) = \tilde{p}_{z_1}(a(z_1)) \). Therefore, if we denote by \( U \) a foliated neighborhood of zero in \( \mathbb{C}^2 = \mathcal{P}_\mathbb{C} \), by \( U_0 \) its image under \( \tilde{p} \), and by \( V \) the corresponding foliated neighborhood of the point \( (0,1) \in \mathbb{C}^2 \) then \( \tilde{p} \circ \tilde{p}_{|U}^{-1} \circ \tilde{p}_{|U/D} = \text{id} \). But the latter is a holonomy map corresponding to \( \gamma \). At the same time the holonomy along \( \gamma \) is the multiplication by \( \alpha \). Contradiction.

Therefore \( \mathcal{L}_m \) cuts our initial disc \( D \) in a neighborhood of \( m \) only finitely many times \( m \). Therefore \( \mathcal{L}_m \) is an imbedded analytic set in a neighborhood of \( m \) in \( X \). But \( D \) can be taken through any point of \( \mathcal{L}_m \setminus \text{Sing} X \). The case of a cyclic point obviously follows. This proves that \( \mathcal{L}_m \) is an imbedded curve in some open subset of \( X \setminus \text{Sing} \mathcal{L} \).

Theorem 3.2 is proved.

4. Proofs of the main results

4.1. BLM-trichotomy. Observe, first of all, that on a projective surface \( X \) every holomorphic foliation can be defined by a global meromorphic 1-form. Indeed, from the exact sequence

\[
0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{M}^*/\mathcal{O}^* \to 0,
\]

were \( \mathcal{M}^* \) is a sheaf of non zero meromorphic functions on \( X \), we get

\[
0 \to \mathcal{C}^* \to H^0(X, \mathcal{M}^*) \to H^0(X, \mathcal{M}^*/\mathcal{O}^*) \to H^1(X, \mathcal{O}^*) \to \ldots
\]

(4.1)

(4.2)

Here \( H^0(X, \mathcal{M}^*/\mathcal{O}^*) \) is the group of divisors on \( X \) and \( H^1(X, \mathcal{O}^*) \) is the group of holomorphic line bundles on \( X \). From the other hand on projective \( X \) every holomorphic line bundle admits a meromorphic section and as such is a bundle canonically associated with a divisor of zeroes and poles of this section. Therefore the map \( H^0(X, \mathcal{M}^*/\mathcal{O}^*) \to H^1(X, \mathcal{O}^*) \) in (4.2) is onto.

Let now \( \omega_j \) be defining holomorphic 1-forms of \( \mathcal{L} \) on an open subsets \( U_j \) with \( \omega_k = f_{kj} \omega_j \). The meromorphic section of the normal bundle \( \mathcal{N}_X^* \), i.e., of the bundle defined by the cocycle \( \{f_{kj}\} \in H^1(X, \mathcal{O}^*) \), is a couple \( \{f_j\} \) of meromorphic functions on open sets \( U_j \) such that \( f_k = f_{kj}f_j \) - i.e., \( \{f_j\} \) is a section of \( H^0(X, \mathcal{M}^*/\mathcal{O}^*) \). But then \( \Omega := \{\Omega_j := f_j^{-1}\omega_j\} \) is a globally defined meromorphic form on \( X \) and it still defines our \( \mathcal{L} \). If \( \{v_j\} \) is another set of holomorphic forms which define the same \( \mathcal{L} \) then it is straightforward to see that the corresponding global meromorphic form it equal to \( \Omega \) modulo a non-zero complex number.
Proofs of the main results

Therefore the notion of $\Omega$ being (algebraically) closed is correctly defined.

Following [BLM] we set

$$\text{Pole}_\Omega = \bigcup_j \text{Pole} (f_j^{-1}) \cup \text{Sing } \mathcal{L},$$

$$\text{Zero}_\Omega = \bigcup_j \text{Zero} (f_j^{-1}) \cup \text{Sing } \mathcal{L}.$$ 

The precise formulation of BLM-trichotomy is as follows, see Section IV in [BLM]:

**Theorem 4.1. (Bonatti-Langevin-Moussu)** Let $\mathcal{L}$ be a holomorphic foliation on a compact complex surface $X$, which can be defined by a global meromorphic form $\Omega$. Let $L_m$ be a minimal leaf of $\mathcal{L}$ such that $L_m$ doesn’t intersects either $\text{Zero}_\Omega$ or $\text{Pole}_\Omega$. Then the following three cases are possible:

i) $\Omega$ is algebraically closed.

ii) $L_m$ is a compact leaf of $\mathcal{L}$.

iii) there exists another leaf with $L_n$ with the same closure, i.e., $L_n = L_m$, such that the holonomy group $H_{ln}(L_m,m)$ contains a hyperbolic element.

Let now $x,y$ coordinates in an affine chart $U_0$ of $X$ (i.e., $X$ is quite special there-fore, like $\mathbb{P}^2$) in which $\mathcal{L}$ is defined by the vector field $v = P(x,y) \frac{\partial}{\partial x} + Q(x,y) \frac{\partial}{\partial y}$. The corresponding polynomial defining form is then $\omega_0 = P(x,y)dy - Q(x,y)dx$. Write $\omega_0 = P(dx - Q/Pdy) =: P \cdot \omega_0$. Perform the standard coordinate change in $\mathbb{P}^2$: $x = 1/u, y = v/u$, and get that

$$\omega_0 = 1/u^{2+\max\{\deg P,\deg Q\}} \omega_1,$$

where

$$\omega_1 = u^{1+\max\{\deg P,\deg Q\} - \deg P} P_1(u,v) \left( dv + \frac{Q_1(u,v)}{u^{1+\max\{\deg P,\deg Q\} - \deg P} P_1(u,v)} du \right)$$

is the polynomial form in the chart $(U_1,u,v)$ which defines $\mathcal{L}$ there. Here the exact form of $Q_1$ plays no role, whereas $P_1(u,v) = u^{\deg P} \cdot P(1/u, v/u)$. Set

$$\Omega_1 = \frac{1}{u} \left( dv + \frac{Q_1(u,v)}{u^{1+\max\{\deg P,\deg Q\} - \deg P} P_1(u,v)} du \right)$$

(4.3)

and check, finally, that $\Omega_0 = \Omega_1$ on $U_1 \cap U_2$. I.e., $\Omega = \{\Omega_j\}_{j=0}^2$ is the global meromorphic form which defines $\mathcal{L}$. The term $\Omega_2$ (in the chart $U_2$) doesn’t needs to be computed, because $\mathbb{P}^2 \setminus (U_0 \cup U_1)$ is a point and a meromorphic form, which is globally defined on $\mathbb{P}^2 \setminus \{ \text{point} \}$ extends to $\mathbb{P}^2$. All this was needed for just to say that the defining meromorphic form for $\mathcal{L}$ in the chart $(U_0,x,y)$ is given by

$$\Omega_0 = dx - \frac{Q}{P} dy = P^{-1} \omega_0,$$

(4.4)

and in $(U_1,u,v)$ by (4.3). As a result we see that

$$\text{Pole}_\Omega = \text{Zero}(P) \cup \{u = 0\} \quad \text{and} \quad \text{Zero}_\Omega = \emptyset$$

(4.5)

for every holomorphic foliation on $\mathbb{P}^2$. Even in the case $P \equiv 0$, i.e., when $\mathcal{L}$ is a rational quasi-fibration, the corresponding meromorphic form $dx$ has no zeroes. Therefore the BLM-trichotomy applies to foliations on $\mathbb{P}^2$. 


Let us consider the case (i) i.e., that the defining \( \mathcal{L} \) meromorphic form \( \Omega \) is algebraically closed. Supposing that \( \mathcal{L} \) is not a rational fibration take the standard affine chart with coordinates \((x, y)\) and write our form as \( \Omega_0 = dy + f(x, y)dx \), where \( f \) is rational. The closeness of \( \Omega \) means that \( f \) is a function of \( x \) only: \( f(x) = p(x)/q(x) \) with \( p \) and \( q \) relatively prime. If \( q \) is not constant, i.e., has a zero, say \( x_0 \), then the projective line \( \{x = x_0\} \) is tangent to \( \mathcal{L} \). Now either either it cuts \( \text{Sing} \mathcal{L} \), or it is a leaf. In the latter case \( \mathcal{L} \) must be a rational fibration. But such in \( \mathbb{P}^2 \) doesn’t exists. If \( q \) is constant then \( \Omega \) is exact: \( \Omega = d(y + \int p) = dF \) with \( F \) being rational. Then \( \mathcal{L} \) is a quasi-fibration by level sets of a rational function. A minimal set in this case is a lever set of \( F \) and these level sets do intersect, i.e., \( \mathcal{M} \) cannot be away from the \( \text{Sing} \mathcal{L} \).

The case (ii) of the trichotomy is ruled out by the Camacho-Sad formula: \( \mathcal{L}_m : \mathcal{L}_m = 0 \).

Therefore we are left with the case (iii) of the trichotomy. Now we are in the position to apply the results of Sections 2 and 3 via the reduction to the \textit{nef} models.

### 4.2. Proof of Theorem

Let \( X \) be now the complex projective plane \( \mathbb{P}^2 \). Suppose that \( \mathcal{M} = \overline{\mathcal{L}_m} \) is a minimal leaf, which doesn’t intersects \( \text{Sing} \mathcal{L} \). Our aim is to arrive to a contradiction. It will be convenient for the future references to present this proof as a sequence of steps. Since for the case when \( \mathcal{L} \) is a rational quasi-fibration our theorem is obvious, we may suppose along the forthcoming considerations that this case doesn’t happens. One more remark, if \( \overline{\mathcal{L}_m} \) is an algebraic curve in \( \mathbb{P}^2 \) then our theorem is again trivial. The same for any other leaf in \( \mathcal{M} \). Therefore we will suppose that this is also not the case.

**Step 1.** \( \mathcal{P}_D = \mathbb{P}^2 \setminus \text{Sing} \mathcal{L} \) for every Poincaré disc through \( m \). Consequently \( \mathbb{P}^2 \) doesn’t contains \( \mathcal{L} \)-invariant algebraic curves. The first assertion was proved in Corollary \( \ref{corollary2.2} \). Suppose \( C \) is an \( \mathcal{L} \)-invariant algebraic curve. Then it is the closure of some leaf of \( \mathcal{L} \). Since \( \mathcal{P}_D \) intersects this leaf it should contain it. Therefore \( C = \overline{\mathcal{L}_z} \) for some \( z \in D \). But the set \( C \cap D \) cannot accumulate to \( m \). Therefore taking a smaller disc \( D_k \) we can arrange that \( C \cap D_k = \varnothing \) and, consequently, \( \mathcal{L}_z \cap \mathcal{P}_{D_k} = \varnothing \). But \( \mathcal{P}_{D_k} \) is still \( \mathbb{P}^2 \setminus \{\text{finite set}\} \). Contradiction.

**Step 2.** Seidenberg’s reduction. Let us perform the first step in transforming of the foliated pair \((X, \mathcal{L})\) to its nef model: \textit{reduction} of singularities. This reduction consists in blowing up singular points of \( \mathcal{L} \). I.e., one performs the first blow up \( \pi_1 : X^1 \to X \) with center at some non-reduced singular point of \( \mathcal{L} \) to get a new foliation \( \mathcal{L}_1 \) on \( X^1 \), and one does this with \((X^1, \mathcal{L}_1)\) and so on, until one gets a foliated pair \((X^N, \mathcal{L}^N) =: (Z, \mathcal{E})\) with only reduced singularities. The finiteness of this procedure is the content of the theorem of Seidenberg. Denote by \( \pi_Z := \pi_1 \circ \ldots \circ \pi_N \) the resulting modification. Now we want to remark few things.

i) First: the proper preimage \( \mathcal{M}^\mathcal{E} \) of \( \mathcal{M} \) under \( \pi_Z : Z \to X \) doesn’t intersects neither \( \text{Sing} \mathcal{E} \) nor \( E^Z \), where \( E^Z \) is the exceptional divisor of \( \pi_Z \).

ii) Second: if \( D \ni m \) is a Poincaré disc for \( \mathcal{L} \) through \( \mathcal{L}_m \) with \( \overline{\mathcal{L}_m} = \mathcal{M} \), then it lifts under \( \pi_Z \) to a Poincaré disc for \( \mathcal{E} \).

These observations are obvious and result from the fact that neither \( \mathcal{M} \) nor \( D \) do not intersect points in \( \text{Sing} \mathcal{L} \). And the blowing up process goes only over these points. Denote by \( \mathcal{P}_D^\mathcal{E} \) the Poincaré domain of \( \mathcal{E} \) corresponding to \( \pi_Z^{-1}(D) \) (the latter can be identified with \( D \) due to the remark (ii) ). Decompose \( E^Z = E^Z_1 \cup E^Z_2 \), where \( E^Z_1 \) is the union of
$\mathcal{E}$-invariant components of $E^Z$ and $E^Z_\delta$ - of dicritical ones, i.e., such that $\mathcal{E}$ is generically transverse to $E^Z_\delta$.

iii) Third: $\mathcal{P}^Z_D = Z \setminus (E^Z_\delta \cup \text{Sing } \mathcal{E})$ and $\mathcal{P} = \mathcal{P}^Z_D \setminus E^Z_\delta$.

More precisely we mean that $\pi_Z|_{\mathcal{P}^Z_D \setminus E^Z_\delta} : \mathcal{P}^Z_D \setminus E^Z_\delta \to \mathcal{P}$ is a foliated biholomorphism. Indeed, an invariant component of $E^Z_\delta$ is a closure of a leaf of $\mathcal{E}$, which cannot cut $D$. What concerns the dicritical components, take a leaf $L$ of $\mathcal{E}$ (not contained in $E^Z_\delta$), which cuts one of them. Then, since $L \setminus E^Z$ is a leaf of $\mathcal{L}$ it should intersect $D$. Therefore $E^Z_\delta \setminus E^Z_i \subset \mathcal{P}^Z_D$. We conclude this remark by noticing that $\mathcal{P}$ is equal to $\mathcal{P}^Z_D$ minus a divisor.

**Step 3. Contracting invariant rational curves.** Let $(Z, \mathcal{E})$ be a Seidenberg’s reduction of our foliated pair $(X, \mathcal{L})$. Suppose that there exists an irreducible algebraic curve $C \subset Z$ such that $K_{\mathcal{E}} \cdot C < 0$, i.e., that $C$ violates the nefness of $K_{\mathcal{E}}$. Then $C$ can be only a smooth $\mathcal{E}$-invariant rational curve with negative self-intersection, which contains exactly one point from $\text{Sing } \mathcal{E}$. But moreover, in our special case $C$ should be a component of $E^Z_i$ (in general this is not the case). Otherwise it should be an $\mathcal{L}$-invariant algebraic curve, and this was already prohibited in Step 1.

Contracting $C$ to a point we get a new surface with at most cyclic singularity $a$ and a foliation downstairs, which is non-singular at $a$. Indeed, by the Step 1 there simply happens, because all modifications took place away from $D$.

Proof. $T_i$ doesn’t intersects any other $\mathcal{E}$-invariant curve. Indeed, would $C$ be such, then its image downstairs in $Y$ would be an $\mathcal{L}$-invariant curve passing through $c_i$, and therefore it would be a closure of a leaf $\mathcal{F}$ from $\mathcal{P}^\mathcal{F}_D$ and as such would intersect $D$. But this cannot happen, because all modifications took place away from $D$.

Taking $V_i$ small enough we can insure that $\text{Sing } \mathcal{E} \cap V_i \subset T_i$ and that no component of $E^Z_i$ other then some from $T_i$ intersects $V_i$. Now the both (a) and (b) become clear.

Item (c) readily follows from (a), because $T_i$-s are disjoint from $\mathcal{P}^\mathcal{F}_D$. 

$\square$
Remark 4.1. For a cyclic point \( c \notin \mathcal{P}_D \) the contracting to it tree \( T \) may well intersect some other \( \mathcal{E} \)-invariant curves, which are not subject of contraction. But these points are out from our process.

The universal covering Poincaré domain \( \tilde{\mathcal{P}}_D^F \) is Hausdorff and Rothstein, because \((Y, \mathcal{F})\) is nef, see Corollary 2.3 and Proposition 2.5. Therefore from this point we can split our proof onto parabolic and hyperbolic cases.

Step 4. End of the proof: parabolic case. We suppose that \((Y, \mathcal{F})\) is parabolic, i.e., all its leaves are parabolic. Therefore all our constructions from Sections 2 and 3 do apply. We obtain that our minimal leaf \( \mathcal{F}_m = \mathcal{L}_m \) closes to an algebraic curve, or it is \( \mathbb{C}^* \) and in the latter case it is a locally closed analytic set in \( Y \setminus \text{Sing} \mathcal{F} \). The case of a compact curve was prohibited in the Step 1. What concerns the second possibility let us recall that at the same time \( \overline{\mathcal{F}_m} = \mathcal{M}^F \) is a compact set in \( Y \setminus (\text{Sing} \mathcal{F} \cup \text{Sing} Y) \) by Step 3. The limiting set \( \lim \mathcal{F}_m \) cannot contain \( \mathcal{F}_m \) because \( \mathcal{F}_m \) is closed in some open subset of \( Y \setminus (\text{Sing} \mathcal{F} \cup \text{Sing} Y) \). Therefore it should be empty by the minimality of \( \mathcal{F}_m \). Therefore \( \mathcal{F}_m \) is a closed leaf of \( \mathcal{F} \) and then such is also \( \mathcal{L}_m \). But this was again prohibited by Step 1.

Step 5. End of the proof: hyperbolic case. Now consider the case when \( \mathcal{F} \) is hyperbolic. If our leaf \( \mathcal{F}_m \) happen to belong to the exceptional set \( \mathcal{A} \) in McQuillan’s alternative then the closure of \( \mathcal{L}_m \) is also an algebraic curve in \( \mathbb{P}^2 \) and we are done.

Therefore we can suppose that \( \mathcal{F}_m \) is hyperbolic and \( \mathcal{P}_D^F \subset Y \setminus \mathcal{A} \). Indeed, if \( \mathcal{P}_D^F \) still contains a leaf from \( \mathcal{A} \) then we take a smaller \( D \). Therefore the Lobatchevski-Poincaré norm is continuous on \( \mathcal{P}_{\mathcal{C}}^F \) and the Theorem 3.1 applies. We have that \( \tilde{p}^F : \mathcal{P}_\mathcal{C}^F \to \mathcal{P}_D^F \) is a regular (cyclic) covering. But \( \mathcal{P}_D^F \supset \mathcal{P}_E^F \supset \mathcal{P}_D \) and therefore contains a lot of rational curves, also such that do not pass through the cyclic points. Therefore \( \mathcal{P}_\mathcal{C}^F \) also contains them. But \( \mathcal{P}_\mathcal{C}^F \) posed a holomorphic surjective projection onto \( \mathbb{C} \) and therefore these curves should be the fibers. This is impossible, because the fibers of \( \mathcal{P}_\mathcal{C}^F \) are Kobayashi hyperbolic. Contradiction.

Theorem 4.1 is proved.

Remark 4.2. If \( \{c_i\} \) is non empty then \( \tilde{\mathcal{P}}_D^\mathcal{C} \) is not Hausdorff and we cannot work with it.

Remark 4.3. What concerns the proof of Corollary 3 let us just remark the following items. Convexity of \( \mathbb{P}^2 \setminus M \) is guaranteed by Fujita’s theorem. Therefore the Levi foliation \( \mathcal{L} \) of \( M \) extends onto \( \mathbb{P}^2 \). Now \( M \) should contain a minimal leaf \( \mathcal{M} \) of \( \mathcal{L} \). But already \( \mathcal{M} \) intersects \( \text{Sing} \mathcal{L} \) due to the Theorem 1. Likewise \( M \) does and a fortiori it cannot be smooth. Corollary 2 is proved.

4.3. Limiting behavior of leaves with hyperbolic holonomy. We shall prove now the Corollary 3 from the Introduction, i.e., we shall detect the reasons for the failure of steiness of \( \mathcal{P}_D \) of a minimal leaf with hyperbolic holonomy. This will be done as in the proof of Theorem 1 along the reduction to the nef model. First of all let us remark that if \( \mathcal{L} \) is a rational quasi-fibration then any of its leaves cannot have a hyperbolic holonomy.

Furthermore, according to Corollary 2.1 \( \mathcal{P}_D \) is not pseudoconvex at some boundary point \( z_0 \) if and only if \( z_0 \) is an isolated boundary point of \( \mathcal{P}_D \) and \( z_0 \in \text{Sing} \mathcal{L} \). In the sequel we denote as \( \{z_i\}_{i=1}^b \) the set of all such points. By \( \{w_i\}_{i=1}^l \) we denote the set of points of \( \text{Sing} \mathcal{L} \) which belong to \( \mathcal{M} \), i.e., \( \mathcal{L}_m \) approaches to them, but which are not isolated points of \( \partial \mathcal{P}_D \). I.e., \( \partial \mathcal{P}_D \) is a sort of a Levi flat ”cone” with vertices at these \( \{w_i\} \).
Step 1. If the set \( \{ w_i \} \) is not empty then \( \mathcal{M} \) admits a Stein invariant neighborhood. If it is empty then \( \mathcal{P}_D = \mathbb{P}^2 \setminus \text{Sing} \mathcal{L} \). If \( \{ w_i \} \) is non-empty we can add to \( \mathcal{P}_D \) all \( \{ z_i \}_{i=1}^k \) and obtain a domain \( \mathcal{P}_D \) which is still different from \( \mathbb{P}^2 \), because it doesn’t contain any of \( \{ w_i \} \). At the same time \( \mathcal{P}_D \) is pseudoconvex by Lemma 4.2 and is obviously invariant. Therefore it is a Stein invariant neighborhood as claimed. If \( \{ w_i \} \) is empty we obtain in the same manner a pseudoconvex domain \( \mathcal{P}_D \) which contains an invariant compact \( \mathcal{M} \). Would \( \mathcal{P}_D \) be different from \( \mathbb{P}^2 \) it would be Stein. Contradiction: a Stein domain cannot contain a compact invariant set. Therefore we proved that \( \mathcal{P}_D = \mathbb{P}^2 \setminus \text{Sing} \mathcal{L} \) in this case.

Suppose that there exist an invariant rational curve \( C \) in \( \mathbb{P}^2 \). If \( C \) is not the closure of \( \mathcal{L}_m \) then we get a contradiction exactly as in the Step 1 of the proof of Theorem 1. If \( C \) is the closure of \( \mathcal{L}_m \) then the case (ii) of our Corollary occurs.

From now on we can suppose that \( \mathcal{P}_D = \mathbb{P}^2 \setminus \text{Sing} \mathcal{L} \) for all Poincaré discs and therefore \( \mathbb{P}^2 \) doesn’t contains invariant rational curves.

Step 2. Seidenberg’s reduction. Let \((Z, \mathcal{E})\) be a Seidenberg’s reduction of \((X, \mathcal{L})\) and \( \pi_Z: Z \to X \) the corresponding modification. Then as in the Step 2 before we have that \( \mathcal{P}_D = \mathcal{P}_D^Z \setminus E_Z^Z \). If \( \mathcal{E}_m \) is the leaf of \( \mathcal{E} \) through \( m \) then \( \mathcal{L}_m = \mathcal{E}_m \setminus \{ e_j \} \), where \( \{ e_j \} \) is a discrete (in the topology of \( \mathcal{E}_m \)) set of vanishing ends of \( \mathcal{L}_m \).

Step 3. Contraction of invariant rational curves. When contracting invariant rational curves, which violate the nefness of the canonical bundle of \( \mathcal{E} \) let us observe that any of \( \{ e_j \} \) cannot arrive to a cyclic point of \((Y, \mathcal{F})\) - the nef model of \((X, \mathcal{L})\).

Moreover, \( \mathcal{P}_D^F = \mathcal{P}_D^F \setminus \{ c_i \} \), where \( \{ c_i \} \) are some cyclic points of \( Y \). We have the following two possibilities for \( \mathcal{F}_m \).

1) \( \mathcal{F}_m \) cuts some of \( \{ c_i \} \), then we get some more vanishing ends (finite number).

2) \( \mathcal{F}_m \) accumulates to some of \( \{ c_i \} \) - this case is irrelevant for us.

Step 4. Parabolic case. If the nef model \((Y, \mathcal{F})\) is parabolic we have two possibilities for \( \mathcal{F}_m \).

1) \( \mathcal{F}_m \) is a torus or, a sphere. Then so is also \( \mathcal{L}_m \) and the number of ends occurred along the previous steps is finite, i.e., cases (ii) and (iii) of our Corollary occur.

2) \( \mathcal{F}_m = \mathbb{C}^* \) and is a locally closed analytic subset in \( Y \setminus \text{Sing} \mathcal{F} \). The limiting set again cannot contain \( \mathcal{F}_m \) and therefore should be contained in \( \text{Sing} \mathcal{F} \) - a finite set. By the theorem of Remmert-Thullen \( \mathcal{F}_m \) closes to a rational curve with two vanishing ends. \( \mathcal{L}_m \) might get more of them. I.e., the case (ii) occurs again.

Step 5. Hyperbolic case. If \( \mathcal{F}_m \) is contained in an exceptional set \( \mathcal{A} \) of McQuillan’e theorem then we have again one of the cases (ii) , or (iii) . If not, then the same proof as in the corresponding Step 5 before shows that this option is impossible.

Corollary 4.1 is proved.

4.4. Minimal leaves in the product of projective lines. We shall prove now Corollary 4.1. Let \( \mathcal{L}_m \) be a minimal leaf of a holomorphic foliation \( \mathcal{L} \) on \( \mathbb{P}^1 \times \mathbb{P}^1 \). Denote by \( \mathcal{M} \) its closure \( \overline{\mathcal{L}_m} \). Suppose that \( \mathcal{M} \) doesn’t intersects \( \text{Sing} \mathcal{L} \). Computations (simpler) as that from Subsection 4.1 show that a defining meromorphic form of a holomorphic foliation on \( \mathbb{P}^1 \times \mathbb{P}^1 \) has no zeroes. Therefore the BLM-trichotomy applies.

Step 1. The use of BLM-trichotomy. According to this trichotomy we must study three cases. Let \( \Omega \) be a global meromorphic defining form of \( \mathcal{L} \).
Case 1. \( \Omega \) is closed. Supposing that \( \mathcal{L} \) is not a rational fibration take the standard affine chart with coordinates \((x, y)\) (see the Subsection 4.1) and write our form as \( \Omega_0 = dy + f(x, y)dx \), where \( f \) is rational. It means that \( f \) is a function of \( x \) only: \( f(x) = p(x)/q(x) \) with \( p \) and \( q \) relatively prime. If \( q \) is not constant, i.e., has a zero, say \( x_0 \), then \( \mathbb{P}_{x_0} := \{x_0\} \times \mathbb{P}^1 \) is tangent to \( \mathcal{L} \). Now either it cuts \( \text{Sing} \mathcal{L} \), or it is a leaf. In the latter case \( \mathcal{L} \) is an obvious rational fibration. I.e., the case (ii) of Corollary occurs. If \( q \) is constant then \( \Omega \) is exact: \( \Omega = d(y + \int p) = dF \) with \( F \) being rational. Then \( \mathcal{L} \) is a quasi-fibration by level sets of a rational function. A minimal set in this case is a leaf and this leaf can not to intersect the singularity locus = indeterminacy set, if and only if \( \mathcal{L} \) is one of two rational fibrations.

We still have two cases.

Case 2. \( \mathcal{M} \) is a leaf. Write \( \mathcal{M} = aE_1 + bE_2 \), where \( E_1 = \{pt\} \times \mathbb{P}^1 \) and \( E_2 = \mathbb{P}^1 \times \{pt\} \) - generators of \( H^2(X, \mathbb{Z}) \). Then \( \mathcal{M}^2 = 2ab \) and this should be zero. Therefore, again \( \mathcal{M} = \mathcal{L}_m \) is \( \{pt\} \times \mathbb{P}^1 \) or vice versa and we find ourselves in the case (ii) of our Corollary.

Case 3. \( \text{Hln}(\mathcal{L}_m, m) \) contains a hyperbolic element. More precisely, this is so for \( a \), may be, some other leaf \( \mathcal{L}_n \) with the same closure. Take the expanded Poincaré domain \( \tilde{\mathcal{P}}_C \). If it is hyperbolic the proof is identical to that of case \( \mathbb{P}^2 \), because \( \mathbb{P}^1 \times \mathbb{P}^1 \) also has sufficiently many rational curves, with only one difference - eventual appearance of domains of the form \( D \times \mathbb{P}^1 \) when studying pseudoconvexity of \( \mathcal{P}_D \) or of \( \tilde{\mathcal{P}}_D \). But then \( \mathcal{L} \) is the obvious rational fibration and the case (ii) of Corollary occurs. Therefore hyperbolic case cannot happen. If \( \tilde{\mathcal{P}}_C \) is parabolic then everything is the same as in \( \mathbb{P}^2 \).

Corollary is proved.

Remark 4.4. What concerns Corollary let us remark that if \( \mathbb{P}^1 \times \mathbb{P}^1 \setminus \mathcal{M} \) is not Stein then by Fujita’s theorem \( \mathcal{M} \) is already of the form \( \gamma \times \mathbb{P}^1 \) (or \( \mathbb{P}^1 \times \gamma \)). The rest is obvious.

4.5. Briot-Bouquet foliations. For the study of Briot-Bouquet foliations we shall need an analog of Fujita’s theorems for pseudoconvex domains in \( \Sigma_g \times \mathbb{P}^1 \), where \( \Sigma_g \) is a compact complex curve of genus \( g \geq 1 \).

Proposition 4.1. Let \( D \) be a locally pseudoconvex domain over \( \Sigma_g \times \mathbb{P}^1 \). Then:

i) either \( D = D_1 \times \mathbb{P}^1 \);

ii) or \( D = \Sigma_g \times D_1 \);

ii) or \( D \) is Stein.

Proof. Denote by \( \hat{\mathbb{C}}^2 \) the punctured \( \mathbb{C}^2 \), i.e., \( \hat{\mathbb{C}}^2 := \mathbb{C}^2 \setminus \{0\} \). Take the canonical projection \( p : \Sigma_g \times \hat{\mathbb{C}}^2 \rightarrow \Sigma_g \times \mathbb{P}^1 \). Let \( \hat{D} \) be the preimage of \( D \) under \( p \). Consider \( \hat{D} \) as a domain over \( \Sigma_g \times \hat{\mathbb{C}}^2 \). It is locally pseudoconvex there over all points apart, may be, of the points in \( \Sigma_g \times \{0\} \). It is also \( \mathbb{C}^* \)-invariant under the action

\[
\lambda \cdot [s; w_1, w_2] = [s; \lambda w_1, \lambda w_2].
\]

Denote by \( \hat{D} \) the pseudoconvex envelope of \( \hat{D} \). By the Theorem of Grauert-Remmert, see [GR1], \( \hat{D} \) might be different from \( \hat{D} \) only over some points over \( \Sigma_g \times \{0\} \). It is also invariant under the action in question. We have the following three cases.

Case 1. \( \hat{D} \) contains a point over the \( \Sigma_g \times \{0\} \). Let \( (s, 0) \) be this point. But then we see, acting by \( \mathbb{C}^* \), that \( \hat{D} \) contains the fiber \( \{s\} \times \hat{\mathbb{C}}^2 \). That means that \( D \) is of the form \( V \times \mathbb{P}^1 \) for some domain \( V \subset \Sigma \).
Case 2. \( \hat{D} = \hat{D} \) but is not Stein. By theorem of Brun, see [Brn], \( \hat{D} = \Sigma \times B \) for some \( B \subset \mathbb{C}^2 \). But then \( D = \Sigma_g \times V \) for the projection \( V \) of \( B \).

Case 3. \( \hat{D} = \hat{D} \) and is Stein. Applying the theorem from [MM] we get that \( D \) is Stein itself.

Now let us accomplish the classification of exceptional mininals in Briot-Bouquet foliations. Let \( \mathcal{M} \) be an exceptional minimal of a holomorphic foliation \( \mathcal{L} \) on \( \Sigma_g \times \mathbb{P}^1 \), \( g \geq 1 \).

Let \( \hat{\Sigma}_g \) be the universal covering of \( \Sigma_g \) (i.e., \( \hat{\Sigma}_g \) is \( \mathbb{C} \) or \( \Delta \)). Denote by \( y \) the natural coordinate on \( \hat{\Sigma}_g \) and by \( x \) an affine coordinate on an appropriate chart of \( \mathbb{P}^1 \). Let \( \Omega = dy + f(x,y)dx \) be a meromorphic form defining \( \mathcal{L} \). Here \( f(x,y) \) is rational with respect to \( x \) and automorphic with respect to \( y \). The same analysis as in Subsection 4.4 shows that \( \Omega \) has no zeroes and BLM-trichotomy applies.

Identically to the Step 1 of Subsection 4.4 we get that if \( \Omega \) is closed then \( \mathcal{M} \) can only a fiber \( E_1 := \{ pt \} \times \mathbb{P}^1 \) or \( E_2 := \Sigma_g \times \{ pt \} \). The same holds also for the second option in BLM-trichotomy.

Therefore we are left with the case when \( \mathcal{M} = \mathcal{L}_m \) and the leaf \( \mathcal{L}_m \) contains a loop with contractible hyperbolic holonomy. All results of this paper are applicable because we have a replacement of the Fujita theorem - Proposition 4.1. Let \( \mathcal{P}_D \) be the Poincaré domain of a Poincaré disc \( D \) through \( m \). If it is locally pseudoconvex but non Stein then the situation is clear via Proposition 4.1 - our \( \mathcal{L} \) is a fibration. If \( \mathcal{P}_D \) is Stein then we have three following cases to consider. Remark that the case when \( \mathcal{L} \) is a rational fibration is obvious, as usual, and will be ignored (but it produces the possibility: \( \mathcal{L}_s \) is Stein).

Case 1. The nef model \( (Y,F) \) of \( (X,L) \) is parabolic and \( g \geq 2 \). Projections of leaves onto \( \Sigma_g \) can be only constants. Therefore \( \mathcal{L} \) is again a rational fibration, the already excluded case.

Case 2. The nef model of \( \mathcal{L} \) is parabolic and \( \Sigma_1 = \mathbb{T} \). If \( \mathcal{L}_m \) is a torus or, a sphere then this case was already considered. Otherwise \( \mathcal{L}_m \) is \( \mathbb{C}^* \) imbedded as a locally closed analytic subset to \( Y \setminus \text{Sing} \mathcal{L} \) and its closure doesn’t intersect \( \text{Sing} \mathcal{L} \). This is again a contradiction unless \( \mathcal{L}_m \) is not a fiber.

Case 3. The nef model of \( \mathcal{L} \) is hyperbolic. As in the case of \( \mathbb{P}^2 \), or \( \mathbb{P}^1 \times \mathbb{P}^1 \) we have that \( \mathcal{P}_D = X \setminus \text{Sing} \mathcal{L} \), otherwise we would either get a contradiction with maximum principle, or conclude that \( \mathcal{L} \) is a trivial fibration \( \mathcal{L}_p = \Sigma_g \times \{ pt \} \) or \( \{ pt \} \times \mathbb{P}^1 \). But \( \hat{p} : \mathcal{P}_C \rightarrow \mathcal{P}_D = X \setminus \{ \text{points} \} \) is a regular covering by Theorem 3.1 But then \( \mathcal{P}_D \) contains a lot of rational curves, and so the \( \mathcal{P}_C \) does. Contradiction.

Therefore we proved the Corollary 5 from the Introduction.

4.6. Levi problem in Hirzebruch surfaces. As in the case with Briot-Bouquet foliations we need to discuss the Levi problem first. Recall that for \( k \geq 1 \) the Hirzebruch surface \( H_k \) is the projectivization of the bundle \( E = \mathcal{O} \oplus \mathcal{O}(-k) \rightarrow \mathbb{P}^1 \). We need to understand the description of locally pseudoconvex domains in \( H_k \), which are not Stein.

Set \( \hat{\mathbb{C}}^2 := \mathbb{C}^2 \setminus \{ 0 \} \) and consider the domain \( U := \hat{\mathbb{C}}^2 \times \hat{\mathbb{C}}^2 \subset \mathbb{C}^4 \). Coordinates in \( \mathbb{C}^4 \) we denote as \( (z_1, z_2; w_1, w_2) \), or as \( (z, w) \). Set furthermore \( E_1 := \hat{\mathbb{C}}^2 \times \{ 0 \} \) and \( E_2 := \{ 0 \} \times \hat{\mathbb{C}}^2 \).
Denote by $G$ the compact Lie group $S^1 \times S^1$ and by $G_C = \mathbb{C}^* \times \mathbb{C}^*$ its complexification. Elements of $G_C$ we denote as $(\lambda, \mu)$, where $\lambda$ and $\mu$ are non-zero complex numbers. Consider the following action of $G_C$ on $U$:

\[
\begin{align*}
\lambda \cdot [z_1, z_2; w_1, w_2] &= [\lambda z_1, \lambda z_2; w_1, \lambda w_2], \\
\mu \cdot [z_1, z_2; w_1, w_2] &= [z_1, z_2; \mu w_1, \mu w_2].
\end{align*}
\]

(4.6)

It is not difficult to check that $U/G_C \equiv H_k$. Denote by $p : U \rightarrow H_k$ the natural projection. The image of $\{w_1 = 0\}$ is the exceptional curve in $H_k$, it will be denoted as $E$. By $\pi : H_k \rightarrow \mathbb{P}^1$ we denote the natural projection coming from the projection $(z, w) \rightarrow z$ after a factorization. $\pi$ realizes $H_k$ as a ruled surface over $\mathbb{P}^1$.

Let us gather the information we need about the Levi problem in Hirzebruch surfaces:

**Proposition 4.2.** Let $D$ be a locally pseudoconvex domain in a Hirzerbruch surface $H_k$, $k \geq 1$. Then the following cases are possible:

i) $D = \pi^{-1}(V)$ for some domain $V \subset \mathbb{P}^1$ (this includes the case $D = H_k$);

ii) $D$ is a 1-complete neighborhood of the exceptional curve $E$;

iii) $D = B \setminus C$, where $B$ is a 1-complete neighborhood of the exceptional curve $E$;

iv) $D = H_k \setminus E$;

v) $D$ is Stein.

**Proof.** Here by saying that $D$ (or $B$) is 1-complete we mean that after the contraction the exceptional curve $E$ it becomes Stein. Let $D$ be a locally pseudoconvex domain in $H_k$. Denote by $\hat{D}$ its preimage $p^{-1}(D)$. $\hat{D}$ is a domain in $\mathbb{C}^4$, which is:

- invariant under the action of $G_C$;

- locally pseudoconvex at every boundary point except, may be, points in $E_1 \cup E_2$.

Denote by $\tilde{D}$ the envelope of holomorphy of $\hat{D}$. This is a locally pseudoconvex domain over $\mathbb{C}^4$, which is invariant under the action of $G_C$. Again, due to [GR1] $\tilde{D}$ might be different form $\hat{D}$ only by some points over over $E_1 \cup E_2$. We have several cases to consider.

**Case 1.** $\tilde{D}$ contains a point over $E_1$. Let $(z^0, 0)$ be this point. Since a schlicht neighborhood over $(z^0, 0)$ is contained then in $\tilde{D}$ we get, by acting with $\mu$ on this neighborhood, that $\tilde{D}$ contains $\{z^0\} \times \mathbb{C}^2$. In fact $\tilde{D}$ contains all fibers $\{z\} \times \mathbb{C}^2$ for $z$ in a neighborhood of $z^0$. That means that $D$ has the form $\pi^{-1}(v)$ for some $V \subset \mathbb{P}^1$, i.e., the case (i) of our Proposition occurs.

**Case 2.** $\tilde{D}$ contains a point over $E_2$. Let $(0, w^0)$ be this point. Then $\tilde{D}$ contains a schlicht neighborhood over the polydisc $\Delta^2_\varepsilon(0) \times \Delta^2_\varepsilon(w^0)$ for some $\varepsilon > 0$. Acting on this set by $\lambda$ we easily get that $\tilde{D}$ contains a schlicht domain over a neighborhood of $\Delta^2_\varepsilon(0) \times \Delta_\varepsilon(w^0) \times \{|w_2| \geq \varepsilon |w_1|\}$.

**Subcase 2a.** $w^0_1 = 0$. Then, acting by $\mu$ we obtain that $\tilde{D}$ contains a schlicht domain over the cone

\[
\Delta^2_\varepsilon \times \{|w_2| \geq \varepsilon |w_1|\}.
\]

(4.7)

In this case $D$ is a 1-complete neighborhood of the exceptional curve $E$, i.e., the case (ii) occurs.
Proofs of the main results

Subcase 2b. $w_0^0 \neq 0$. Acting again by $\mu$ we obtain that $\hat{D}$ contains a schlicht domain over a neighborhood of the cone

$$\Delta^2 \times \{|w_2| \geq \frac{|w_0^0|}{|w_1^0|}|w_1| \setminus \{w_1 = 0\}\}.$$  

In this case $D = B \setminus E$ for some 1-complete domain $B$ containing $E$, i.e., the case (iii) occurs if $w_0^0 \neq 0$ and the case (iv) if $w_0^0 = 0$.

Case 3. $\tilde{D} = \check{D}$, i.e., $\tilde{D}$ is Stein. In this case $D$ is a factor of a Stein domain by a free holomorphic action of a complexification of a compact Lie group. Theorem of Matsushima and Morimoto, see [MM], assures then that $D$ is Stein itself.

4.7. Exceptional minimals and Levi flats in Hirzebruch surfaces. In the very same spirit one can clarify the situation with exceptional minimals in Hirzebruch surfaces $H_k$ for $k \geq 1$, we keep notations of Subsection 4.6.

Let us prove the Corollary 6 from the Introduction. BLM-trichotomy is applicable in this case to and gives the following possibilities. If the defining meromorphic form is closed then $L$ can be only the canonical rational fibration. To understand the second case let $C$ be a smooth, irreducible algebraic curve in $H_k$, which is a leaf of some holomorphic foliation $L$ on $H_k$. If $E$ denotes the exceptional curve and $F$ - the fiber, then write $C = nE + lF$. Suppose that $C$ is neither $E$ nor $F$. Then it should intersect $E$ non-negatively:

$$(nE + lF) \cdot E = -kn + l \geq 0$$  

and this implies $l \geq nk$. (4.9)

At the same time by Camacho-Sad formula we have that

$$0 = C^2 = (nE + lF)^2 = -n^2k + 2ln \geq -n^2k + 2n^2k = n^2k > 0.$$  

Therefore an imbedded curve cannot be a leaf of a holomorphic foliation unless it is a fiber of the canonical rational fibration.

The case with the leaf with hyperbolic holonomy is identical to the already considered cases of $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and $\Sigma_g \times \mathbb{P}^1$. We shall not repeat it.

4.8. Minimal sets in projective spaces. Now we shall prove Corollary 2 from Introduction. Let $L$ be a codimension one holomorphic foliation in $\mathbb{P}^n$. First we shall make a preparatory step.

Step 1. Generic sections by hyperplanes. We follow [CLS2] and [BLM]. An open subset of $\mathbb{P}^n$ is called generic if its complement is thin, i.e., is contained in at most countable
union of locally closed proper analytic subsets. Recall that if \( w = [w_0 : ... : w_n] \) is a point in the dual \( \mathbb{P}^n^* \), then the corresponding hyperplane \( E_w \) in \( \mathbb{P}^n \) is given by the equation \( w_0z_0 + ... + w_nz_n = 0 \). There is a generic subset \( G_1^* \) in the dual \( \mathbb{P}^n^* \) such that for every \( w \in G_1^* \) the plane \( E_w \) from this subset the following holds:

i) \( E_w \) is not contained in any leaf of \( \mathcal{L} \).

ii) All components of \( \text{Sing} \mathcal{L}^w \) have codimension at least two. Here \( \mathcal{L}^w := \mathcal{L}|_{E_w} \) - the restriction of \( \mathcal{L} \) to \( E_w \).

For the proof see Lemma 10 in [CLS2]. In [BLM], Proposition in Section III, it is proved that in addition to (i) , (ii) one has the following assertion. There exists a generic subset \( G_2^* \subset \mathbb{P}^n^* \) and an integer number \( d \geq 0 \) such that for every \( w_0 \in G_2^* \) the hyperplane \( E_{w_0} \) satisfies also the following:

iii) there exist exactly \( d \) points \( \{z_1, ..., z_d\} \) in \( E_{w_0} \setminus \text{Sing} \mathcal{L} \) where \( E_{w_0} \) is tangent to \( \mathcal{L} \).

Moreover, for every \( w_0 \in G_2^* \) there exists a neighborhood \( w_0 \in W \subset G_2^* \) and a neighborhood \( Z \) of \( \{z_1, ..., z_d\} \), \( Z \setminus \text{Sing} \mathcal{L} = \emptyset \), such that for any \( w \in W \) all points in the plane \( E_w \) in which \( E_w \) is tangent to \( \mathcal{L} \), are contained in \( Z \). In particular all these points are contained in a compact away from \( \text{Sing} \mathcal{L} \).

**Step 2. Points of tangency.** Let \( \mathcal{M} \) be a closed invariant set of our foliation \( \mathcal{L} \). For \( w \in G_1^* \cap G_2^* \) denote by \( T_w \) the finite set of points where \( \mathcal{L} \) is tangent to \( E_w \). Let us see that:

iv) there exists open and dense subset \( G_3^* \subset \mathbb{P}^n^* \) such that for every \( w \in G^* \) one has \( T_w \cap \mathcal{M} = \emptyset \).

Indeed, take \( w \) as above and let \( T_w = \{z_1, ..., z_d\} \). Suppose that \( z_d \in \mathcal{M} \). Since \( \mathcal{M} \) has empty interior (by minimality) we can take \( z'_d \) close to \( z_d \), which doesn’t belongs to \( \mathcal{M} \). Take \( w' \) close to \( w \) such that \( \mathcal{L} \) is tangent to \( T_w' \) at \( z'_d \). We obtained that \( T_w' \cap \mathcal{M} \) has at most \( d-1 \) points and the same for hyperplanes in a neighborhood of \( T_w' \). After \( d \) steps we obtain an open dense set as in (iv).

**Step 3. Measure positivity of boundaries of invariant sets.** We shall prove the statement of Corollary 2 by induction. For \( n = 2 \) Corollary 2 is proved by Theorem 1. Therefore, from now on \( n \geq 3 \).

For \( w \in G^* = G_3^* \) from the Step 2 denote \( \mathcal{L}^w \) the restriction of \( \mathcal{L} \) to the hyperplane \( E_w \). Then:

i) \( \text{Sing} \mathcal{L}^w \subset (\text{Sing} \mathcal{L} \cap E_w) \cup T_w \), where \( T_w \) is disjoint from \( \text{Sing} \mathcal{L} \) and from \( \mathcal{M} \);

ii) \( \mathcal{M}^w := \mathcal{M} \cap E_w \) is a closed invariant set of \( \mathcal{L}^w \) (may be not minimal).

\( \mathcal{M}^w \) cuts \( \text{Sing} \mathcal{L}^w \) by a set of positive \( 2n-6 \)-measure for every \( w \) from the set of full measure. If \( n \geq 4 \) we are done, because this set can be contained only in \( \text{Sing} \mathcal{L} \cap E_w \) (and not in \( T_w \), which is just finite). If \( n = 3 \) we have that \( \mathcal{M}^w \) contains at least one point from \( \text{Sing} \mathcal{L}^w \) and this point is not from \( T_w \), i.e., it can be only from \( \text{Sing} \mathcal{L} \cap E_w \), and we are done again.

Corollary 2 is proved.

5. Pseudoconvexity vs. rational curves

5.1. Analytic objects. Throughout this paper we used extension properties of some "analytic objects" like holomorphic/meromorphic mappings, foliations etc. These analytic objects have the following two decisive properties:
A1) The Hartogs type extension theorem is valid for them. I.e., if any of these objects is given on the Hartogs figure $H^2$ then it extends to the same type of object onto the bidisc $\Delta^2$.

A2) They obey the uniqueness theorem. I.e., if two of them $\sigma_1$, $\sigma_2$ are defined on a connected manifold (or space) $U$ and for some open $V \subset U$ one has $\sigma_1|_V = \sigma_2|_V$, then $\sigma_1 = \sigma_2$.

Remark 5.1. 1. A holomorphic object can became to be a meromorphic after extending in a non-Stein case. In the Stein case such thing cannot happen for functions, holomorphic forms, sections of holomorphic bundles. But it can happen, even in the Stein case, for mappings with values in Kähler manifolds, for example.

2. A holomorphic foliation on $X$ (resp. singular holomorphic foliation) is defined by a holomorphic section (resp. meromorphic section) of the projectivized tangent bundle $\mathbb{P}(TX)$. The products $U \times \mathbb{P}(T_x X) \equiv \mathbb{P}(TX)|_U$, where $U$ is a local chart, are Kähler. Therefore the extension works. Involutibility, being a holomorphic condition, is preserved by extension. The extended foliation might become singular.

3. Obvious generalizations to $n > 2$ will be not needed for us in this paper.

Definition 5.1. A sheaf on analytic objects on a complex manifold (or, a normal space) $X$ is a sheaf of sets which sections obey properties (A1) and (A2).

5.2. Extension of analytic objects. First, we recall the following:

Definition 5.2. A real hypersurface $\Sigma$ in 2-dimensional complex manifold $X$ is called strictly 1-convex if it locally admits a smooth defining function $\rho : X \cap U \to \mathbb{R}$, $\Sigma \cap U = \{\rho = 0\}$, such that the eigenvalues of the Levi form of $\rho$ are strictly positive at each point. Such function is called strictly 1-convex, or strictly plurisubharmonic. More precisely, $\Sigma$ is called to be strictly 1-convex from the side $U^-$ = $\{z \in U : \rho(z) < 0\}$.

One has the following:

Theorem 5.1. Let $D$ be a domain in a complex surface $\hat{D}$ and let $\Sigma_t$ be a continuous family of 1-convex hypersurfaces in $\hat{D}$, $t \in (t_1, t_0)$. Suppose that for every $t \in (t_1, t_0)$ the intersection $\Sigma_t \cap (\hat{D} \setminus D)$ is contained in a relatively compact part of $\hat{D} \setminus D$ and suppose that $\Sigma_{t_0} \subset D$. Let $E$ be a sheaf of analytic objects on $D$ and let $E$ be an element of $E|_D$. Then $E$ extends along the family $\{\Sigma_t\}_{t \in (t_0, t_1)}$.

The proof is standard for the standard analytic objects and will be not given. Now let describe a somewhat non standard situation in which it will be applied. The idea of the following construction is inspired by §3 from [IV3]. For $\epsilon > 0$ and $\alpha \in (0, \infty)$ consider the following smooth functions

$$\rho_{\epsilon, \alpha}(z) = |z_1|^2 - \frac{\epsilon^2}{4} - \left(1 - \frac{\epsilon^2}{4}\right)|z_2|^{2\alpha},$$

domains

$$D_{+, \alpha}^\epsilon = \{(z_1, z_2) \in \Delta^2 : \rho_{\epsilon, \alpha}(z) < 0\} \quad \text{and} \quad D_{-, \alpha}^\epsilon := \Delta^2 \setminus D_{+, \alpha}^\epsilon,$$

and hypersurfaces

$$\Gamma_{+, \alpha}^\epsilon = \{(z_1, z_2) \in \Delta^2 : \rho_{\epsilon, \alpha}(z) = 0\},$$

separating $D_{+, \alpha}^\epsilon$ from $D_{-, \alpha}^\epsilon$, see Figure 1.
Lemma 5.1. i) For all $\varepsilon > 0$ and $\alpha > 0$ hypersurfaces $\Gamma_{\varepsilon, \alpha}$ are strictly pseudoconvex in $\Delta^2 \setminus \{|z_1| = \frac{\varepsilon}{2}, z_2 = 0\}$ from the side of $D_{\varepsilon, \alpha}$.

ii) For a fixed $\varepsilon > 0$ the domain $D_{\varepsilon, \alpha}$ is contained in $H^2_\varepsilon$ for $\alpha$ big enough and

$$
\bigcup_{\alpha > 0} D^+_{\varepsilon, \alpha} = \Delta^2 \setminus (A_{\frac{\varepsilon}{2}} \times \{0\}).
$$

Proof. The Levi form of $\rho_{\varepsilon, \alpha}$ at $z$ is

$$
H_z(\rho_{\varepsilon, \alpha}) = \begin{pmatrix}
1 & 0 \\
0 & -\alpha^2(1-\varepsilon^2/4)|z_2|^{2\alpha-2}
\end{pmatrix}
$$

Since $\partial \rho_{\varepsilon, \alpha} = (\bar{z}_1, -\alpha(1-\varepsilon^2/4)\bar{z}_2|z_2|^{2\alpha-2})$, a complex tangent vector to $\Gamma_{\varepsilon, \alpha}$ at $z$ is $v_z = (\alpha(1-\varepsilon^2/4)z_2|z_2|^{2\alpha-2}, z_1)$. And therefore

$$
H_z(\rho_{\varepsilon, \alpha})(v_z, v_z) = (1-\varepsilon^2/4)\alpha^2|z_2|^{2\alpha-2}[(1-\varepsilon^2/4)|z_2|^{2\alpha} - |z_1|^2] = -\alpha^2\varepsilon^2/4(1-\varepsilon^2/4)|z_2|^{2\alpha-2}.
$$

The term on the right hand side is negative and therefore the assertion (i) of the Lemma is proved. Assertion (ii) is left to the reader. □

Condition $\rho_{\varepsilon, \alpha} > 0$ one rewrites as

$$
\rho_1(z) := \frac{\ln|z_1|^2 - \varepsilon^2}{\ln|z_2|^2} < \alpha.
$$

The preceding calculations mean that the complex Hessian of $\rho_1$ is strictly positive along the complex directions tangent to the level sets of $\rho_1$. Taking a sufficiently convex function $\psi : (0, \infty) \rightarrow (0, \infty)$ one get a strictly plurisubharmonic function

$$
\rho := \psi(\rho_1)
$$

in $\Delta \times \Delta^*$ with the same level sets as $\rho_1$.

Let us formulate the statement we need.

Lemma 5.2. Let $(X, \mathcal{L})$ be a straight foliated pair on a projective surface and let $(\hat{P}_D, \hat{\mathcal{L}})$ be a universal covering Poincaré domain. Let $(\Delta^2, \mathcal{E})$ be a foliation given by the level sets of a holomorphic function in $\Delta^2$ with a discrete critical set $\text{Crit}(\pi) \subset \Delta^2 \setminus H^2_\varepsilon$. Then every foliated immersion $\hat{h} : (H^2_\varepsilon, \mathcal{E}) \rightarrow (\hat{P}_D, \hat{\mathcal{L}})$, extends to $\Delta^2 \setminus \text{Crit}(\pi)$. 
Poincaré domain over $X$ and $\tilde{z}$.

This will be done in several steps.

**Proof. Step 1. Suppose first that Crit($\pi$) is empty.** Take the function $\rho$ as in \((5.7)\) and let $D^\pm_\alpha$ be its upper/lower level sets. Suppose that $\hat{h}$ is extended up to a point $y_0 \in \partial D^+_\alpha$. Take the leaf $\mathcal{E}_{y_0}$ through $y_0$. It cannot be contained in $D^-_\alpha$ in any neighborhood of $y_0$, because $\partial D^-_\alpha$ is strictly pseudoconvex at $y_0$. Take a point $y_1 \in D^-_\alpha \cap \mathcal{E}_{y_0}$ close to $y_0$. Trace a small transversal $L$ to $\mathcal{E}$ at $y_1$ such that $L \subset D^+_\alpha$. Let $W$ be a foliated bidisc for $\mathcal{E}$ based on $L$, i.e., for every $y \in L$ one has $\mathcal{E}_y \cap W$ is the vertical disc $W_y$. Set $h := \tilde{p} \circ \hat{h}$. It extends to a neighborhood of $y_0$ by the usual Hartogs. Using the fact that $\tilde{p}$ is the universal covering we can extend $\hat{h}|_{W_y \cap D^+_\alpha} = \tilde{p}^{-1} \circ h|_{W_y \cap D^+_\alpha}$ with the given initial value $h(y)$ from $W_y \cap D^+_\alpha$ onto $W_y$ for every $y$. Using straightness of $(X, \mathcal{L})$ and therefore rothsteiness of $(\mathcal{P}_D, \hat{\mathcal{L}})$, see Proposition \(2.3\) we get the extension of $\hat{h}$ to a punctured neighborhood of $y_0$. This way we extend $\hat{h}$ to $\Delta^2 \setminus A_{1,1} \times \{0\}$. Removal of $A_{1,1} \times \{0\}$ (i.e., the Thullen type extension theorem) doesn’t represent any difficulties in this context. I.e., $\hat{h}$ is extended onto $\Delta^2$.

**Step 2. Now suppose that $\pi$ has just one critical point $c_1$, situated away from the Hartogs figure.** I.e., such that $c \in \Delta^2 \setminus H^2_{\varepsilon}$. Without loss of generality we may suppose that $c = (1/2, 0)$ (with $\varepsilon << 1/2$, see the Figure \[1]). Let $\mathcal{E}$ be the foliation on $\Delta^2 \setminus \{c\}$ by level sets of $\pi$. Take $(\Delta^2 \setminus \{c\}, \mathcal{E})$ as $(X, \mathcal{E})$ and repeat the Step 1 of the proof. Then $\hat{h}$ is extended again to $\Delta^2 \setminus A_{1,1} \times \{0\}$. Removal of $(A_{1,1} \times \{0\}) \setminus \{c\}$ is again obvious.

**Step 3. The Crit($\pi$) is discrete.** Then do the same up to arriving to the first critical point $c_1$ at value $\alpha_1$. Then place appropriately a Hartogs figure into $D^+_\alpha$ in order to obtain the situation of Step 2 and therefore extend $\hat{h}$ to a punctured neighborhood of $c_1$. The rest is obvious.

\[\square\]

### 5.3. Pseudoconvexity of the universal covering Poincaré domains.

We shall prove first a general statement about appearance of invariant rational curves as obstructions to local pseudoconvexity of covering Poincaré domains.

**Theorem 5.2.** Let $(X, \mathcal{L})$ be a straight projective pair and let $m$ be a point in $X^{\text{reg}}$. Then the following statements are equivalent:

1. For every Poincaré disc $D$ through $m$ $\hat{\mathcal{P}}_D$ is not locally pseudoconvex over $X$:
2. $\overline{\mathcal{L}_m}$ is a rational curve, cutting $\text{Sing} \mathcal{L}$ exactly at one point (it is the very same point over which all $\mathcal{P}_D$ are not pseudoconvex).

**Proof.** This will be done in several steps.

**Step 1. Pseudoconvexity of $\hat{\mathcal{P}}_D$ over non-singular points.** $\hat{\mathcal{P}}_D$ was defined in Subsection \(2.2\) and $\tilde{p} : \mathcal{P}_D \to \mathcal{P}_D \subset X^{\text{reg}}$ denotes the canonical map. The pair $(\mathcal{P}_D, \tilde{p})$ is a Riemann domain over $X$. First of all let us remark that, as in the case with $\mathcal{P}_D$ the covering Poincaré domain $\hat{\mathcal{P}}_D$ is always pseudoconvex over non singular points of $\mathcal{L}$.

**Lemma 5.3.** If $z_0 \notin \text{Sing} \mathcal{L}$ then $\hat{\mathcal{P}}_D$ is locally pseudoconvex over $z_0$.

**Proof.** Pseudoconvexity of $\mathcal{P}_D$ at such $z_0$ was proved in Lemma \(2.2\). Let us see that an analogous proof goes through also for $\hat{\mathcal{P}}_D$. Indeed, let $z_0 \notin \text{Sing} \mathcal{L}$. Take then, as in the proof of Lemma \(2.2\) a foliated bidisc $U \ni z_0$. Set, as above, $\mathcal{F} := \mathcal{L}|_U$. Let $U_1$ be a connected component of $\tilde{p}^{-1}(U)$ and let $V$ be the image of $U_1$ under $\tilde{p}|_{U_1} : U_1 \to U$. If $z = (z_1, z_2) \in V$ is the image of some $w \in U_1$ then $\tilde{p}^{-1}(\mathcal{F}_{z_1})$ (with initial value $w$) extends along $\mathcal{F}_{z_1} := \{z_1\} \times \Delta$ because of simple connectivity of $\Delta$, and $\tilde{p}^{-1}(\mathcal{F}_{z_1}) \subset U_1$ because
of the connectivity of $U_1$. Therefore for every $z = (z_1, z_2) \in V$ we have that $\mathcal{F}_{z_1} \subset V$ and $\tilde{p}^{-1}[\mathcal{F}_{z_1}]$ is a disjoint union of discs, each of them is mapped by $\tilde{p}$ biholomorphically onto $\mathcal{F}_{z_1}$. Since $V$ is connected, we have that $V = V_1 \times \Delta$ for some open, connected $V_1 \subset \Delta$. A connected component $U_1$ is also a connected component of $\tilde{p}^{-1}(V)$ and it is foliated by discs - preimages of leaves of $\mathcal{F}$. The restriction $\tilde{p}|_{U_1} : U_1 \to V$ is a foliated local biholomorphism, i.e., $(U_1, \tilde{p}|_{U_1})$ is a Riemann domain over a Stein manifold $V = V_1 \times \Delta$. Foliation on $U_1$ we denote as $\bar{\mathcal{F}}$ - it this the restriction of the universal foliation $\mathcal{L}$ to $U_1$.

To prove that $U_1$ is $p_\tau$-convex consider a holomorphic imbedding $\bar{h} : H^2_\epsilon \to U_1$. Let $\mathcal{E}$ be the pull back of $\bar{\mathcal{F}}$ to $H^2_\epsilon$ by $\bar{h}$. Since it is the same as pull back of $\mathcal{L}$ by the extended map $h : \Delta^2 \to V$ (here $h$ stands for the extension of $\tilde{p} \circ \bar{h}$) we conclude that $\mathcal{E}$ extends to a smooth foliation domain on $\mathcal{E}$. Let $w$ be a point in $\Delta^2$. Take a leaf $\mathcal{E}_w$. It intersects $H^2_\epsilon$ by Lemma 5.1. Let $u$ be some point of this intersection and let $v = h(u)$. Take the lift $\tilde{p}|_{\mathcal{F}}$ with the initial value $h(u)$. The image of this lift is a disc - a leaf of $\bar{\mathcal{F}}$ and it is entirely contained in $U_1$ by preceding considerations. Since $\tilde{h}|_{\mathcal{E}_w \cap H^2_\epsilon} = \tilde{p}^{-1}|_{\mathcal{F}} \circ h|_{\mathcal{E}_w}$ we see that $\tilde{h}|_{\mathcal{E}_w \cap H^2_\epsilon}$ extends onto $\mathcal{E}_w$ as a mapping with values in $U_1$ (because $\tilde{p}^{-1}|_{\mathcal{F}}$ takes its values in $U_1$). This proves, via Rothstein-type Proposition 2.4, the extendability of $\tilde{h}$ onto $\Delta^2$ as a mapping with values in $U_1$. Indeed, $(U_1, \mathcal{L})$ is Rothstein, because it is a foliated domain over the Stein pair $(U, \mathcal{L})$.

We proved that $U_1$ is $p_\tau$-convex and therefore it is Stein.

\textbf{Step 2. Nearby rational curves.} Now we shall prove the following:

\textbf{Lemma 5.4.} Let $(X, \mathcal{L})$ be a straight foliated pair on a projective surface $X$. Let $m$ be a point in $X^{reg}$ and $D \ni m$ a transversal to $\mathcal{L}$, locally closed disc. Suppose that the universal covering Poincaré domain $(\mathcal{D}_D, \tilde{p})$ is not locally pseudoconvex over a point $z_0 \in X$. Then:

\begin{enumerate}
  \item $z_0 \in \text{Sing} \mathcal{L}$ and $z_0$ is an isolated point of $\partial \mathcal{D}_D$;
  \item for some $m_1 \in D$ the closure $\overline{\mathcal{L}_{m_1}}$ is a rational curve passing through $z_0$.
\end{enumerate}

\textbf{Proof.} We already know that such $z_0$ should belong to $\text{Sing} \mathcal{L}$ and, in particular, $z_0$ should be a boundary point of $\mathcal{D}_D$. Take some small neighborhood $U$ of $z_0$ biholomorphic to a ball and not containing any other then $z_0$ points of $\text{Sing} \mathcal{L}$.

Let $U_1$ be some connected component of $\tilde{p}^{-1}(U)$ and let $\bar{h} : H^2_\epsilon \to U_1$ be a holomorphic imbedding. Take $\tilde{p} \circ \bar{h}$ and extend it by Hartogs theorem to a locally biholomorphic mapping $h : \Delta^2 \to U \subset X$. Let $\mathcal{E} := h^*\mathcal{F}$ be the induced foliation on $\Delta^2$, where $\mathcal{F} = \mathcal{L}|_{U}$. Set $S = \text{Sing} \mathcal{E}$, it is clear that $S = h^{-1}(z_0)$ if it is nonempty, i.e., if $z_0 \in h(\Delta^2)$. Moreover, it is clear that $\mathcal{E}|_{H^2_\epsilon} = h^*\bar{\mathcal{F}}$, where $\bar{\mathcal{F}} := \bar{\mathcal{L}}|_{U_1}$. Remark furthermore that the universal foliation $\bar{\mathcal{L}}$ of $\bar{\mathcal{D}}$ possesses a first integral, namely $\bar{\pi}$. Therefore $\mathcal{E}$ possesses it to, it is nothing but $\bar{\pi} \circ h$ extended from $H^2_\epsilon$ to $\Delta^2$. Denote this integral as $\pi$.

Shrinking $\Delta^2$ arbitrarily slightly we can suppose that $S$ is finite. It is clear that $S = \text{Crit}(\pi)$. Applying Lemma 5.2 to our $\pi$ we extend $\bar{h}$ as a locally biholomorphic map $\bar{h} : \Delta^2 \setminus S \to U_1$.

Now we have two cases.

\textbf{Case 1. The point $z_0$ is not an isolated boundary point of $\mathcal{D}_D$.} In that case $U \cap \mathcal{D}_D$ is Stein, as we already know from Lemma 2.2 and consequently $h(\Delta^2) \subset U \cap \mathcal{D}_D$. Therefore $S = \emptyset$.
and \( \tilde{h} \) extends to a locally biholomorphic mapping \( \tilde{h} : \Delta^2 \to U_1 \). I.e., \( U_1 \) is \( p_1 \)-convex and the Docquier-Granert criterion provides the steiness of \( U_1 \).

**Case 2.** \( z_0 \in \text{Sing} \mathcal{L} \) and is an isolated boundary point of \( \mathcal{P}_D \). Let \( U \) be as above. Suppose that there is a connected component \( U_1 \) of \( \tilde{p}^{-1}(\mathcal{P}_D \cap U) \) which is not Stein. Let \( \tilde{h} : H^2_\Delta \to U_1 \) and \( h := \tilde{p} \circ \tilde{h} \) be as above and let \( \hat{h} : \Delta^2 \setminus S \to U_1 \subset \mathcal{P}_D \) be the extension of \( \tilde{h} \) constructed at the beginning of the proof.

This extension is proper near \( S \). Remark that \( S \) cannot be empty due to the assumed non Steininess of \( U_1 \). What concerns properness, indeed, let \( y_n \in \Delta^2 \) be a sequence such that \( y_n \to s_0 \in S \). We need to prove that \( \hat{h}(y_n) \) leave every compact in \( \mathcal{P}_D \). But \( h = \tilde{p} \circ \tilde{h} \) is a locally biholomorphic map to \( X \). Therefore it is biholomorphic in a neighborhood of \( s_0 \) and \( h(s_0) = z_0 \). Take neighborhoods \( W \ni s_0 \) and \( V \ni z_0 \), biholomorphic to a ball, such that \( h|_W : W \to V \) is a biholomorphism. Set \( z_n := h(y_n) \) and \( x_n := \hat{h}(y_n) \in U_1 \). Then \( \hat{p}(x_n) = z_n \to z_0 \). Would \( x_n \) stay in a compact part of \( \mathcal{P}_D \) their images \( z_n = \hat{p}(z_n) \) could not approach a singular point \( z_0 \) of our foliation. Therefore \( \hat{h}(y_n) \) leaves every compact in \( \mathcal{P}_D \).

Now we can attach the set \( S \) to \( U_1 \) and get a completed domain \( \bar{U}_1 \) over \( X \), and consequently we can complete \( \mathcal{P}_D \). Indeed, for \( s_0 \) and \( z_0 = h(s_0) \) take \( W \) and \( V \) as above. Then \( \tilde{p}^{-1} \circ h \) lifts over \( W \setminus \{ s_0 \} \) to an injective map \( \tilde{p}^{-1} \circ h|_{W \setminus \{ s_0 \}} : W \setminus \{ s_0 \} \to U_1 \), which is proper at \( s_0 \). Now we can attach \( s_0 \) to \( U_1 \) by this map.

The projection \( \tilde{p} \) will extend as a locally biholomorphic map to \( \bar{U}_1 \). We can repeat this procedure with all components of \( \tilde{p}^{-1}(U) \) which are not Stein. The obtained Riemann domain over \( X \) we denote by \( (\mathcal{P}_D, \tilde{p}) \) (\( \tilde{p} \) stands for the extension of \( p \)).

\( \mathcal{P}_D \setminus \mathcal{P}_D \) is a discrete, non empty subset of \( \mathcal{P}_D \). We have the holomorphic projection \( \pi : \mathcal{P}_D \to D \) onto the base \( D \) of the Poincaré domain \( \mathcal{P}_D \) and it holomorphically extends to \( \mathcal{P}_D \). Denote by \( \tilde{\pi} \) the extended map. Take some point \( s_0 \in \mathcal{P}_D \setminus \mathcal{P}_D \). Set \( t_0 = \tilde{\pi}(s_0) \) and consider the complex curve \( C_{t_0} := \tilde{\pi}^{-1}(t_0) \). It can be nothing else but a 1-point compactification of a simply connected leaf \( \mathcal{L}_{t_0} \) by \( s_0 \). I.e., \( C_{t_0} \) can be only a rational curve. Note that \( C_{t_0} \) passes through \( s_0 \). Therefore \( \overline{\mathcal{L}_{t_0}} = \tilde{p}(C_{t_0}) \) is an invariant rational curve in \( X \) passing through \( z_0 \). All what is left is to set \( m = t_0 \).

**Step 3. Rationality of \( \overline{\mathcal{L}_m} \).** Write \( D_1 := D \) and \( C_1 := C_{m_1} \). Let \( D_k \) denotes the subdisc of \( D_1 \) of geodesics radius \( 1/k \) with the same center \( m \). \( \mathcal{P}_{D_k} \) is then a subdomain of \( \mathcal{P}_{D_k} \). Suppose that for all \( k \) Poincaré domains \( \mathcal{P}_{D_k} \) are not pseudoconvex over \( z_0 \), in particular, that \( z_0 \) is an isolated boundary point for all \( \mathcal{P}_{D_k} \).
Repeating the previous considerations for every \( k \) we get \( m_k \in D_k \), a rational curve \( C_k \) in \( \mathcal{P}_{D_k} \subset \mathcal{P}_D \), which is a one point compactification of \( L_{m_k} \). \( C_k \) passes through \( s_k \) such that \( \tilde{p}(s_k) = z_0 \) for all \( k \). More accurately: \( \tilde{p}_k(s_k) = z_0 \). But \( \tilde{p}_k \) is the restriction of \( \tilde{p} \) to \( \mathcal{P}_{D_k} \). The images \( \tilde{p}(C_k) = \overline{L_{m_k}} \) are invariant rational curves in \( X \) passing through \( z_0 \). And \( z_0 \) is the only point of \( \overline{L_{m_k}} \cap \text{Sing} L \).

**Remark 5.2.** All \( s_k \) may be well distinct. It is true that they all project to \( z_0 \) under \( \tilde{p} \), but \( \mathcal{P}_D \) might be infinite sheeted over \( X \).

If some subsequence of \( \tilde{p}(C_k) \) stabilizes then it is equal to \( \overline{L_m} \) in fact, and we are done. If not, then we got a sequence of distinct invariant rational curves through \( z_0 \). But then \( L \) is a rational quasi-fibration and again we are done. We proved that in this case \( \overline{L_m} \) is an invariant rational curve passing through \( z_0 \) as predicts Part (ii) of Theorem 5.4.

\[ \square \]

**5.4. Proof of Corollary 9.** Suppose that in the conditions of Corollary 8 the case (i) occurs and, moreover, that \( \mathcal{P}_D \setminus \mathcal{P}_D \neq \emptyset \). Denote by \( \{s_1, \ldots, s_n\} \) the set of all nondicritical points of this set. We shall prove that this set is empty. If not let \( s \in \text{Sing} L \) appeared to be an isolated boundary point of the Poincaré domain \( \mathcal{P}_D \) such that \( s \) is not dicritical.

Consider first the case when the nef model \((X, F)\) of \((X, L)\) is parabolic. From the minimality of \( \overline{L_m} \) we obtain readily that \( \lim \mathcal{L}_m \subset \text{Sing} L \) and therefore \( \overline{L} \) is a rational curve cutting \( \text{Sing} L \) by at least two points, i.e., the case (ii) of Corollary 8 occurs.

Let us turn to the hyperbolic case. Using contractibility of the holonomy we can, as before, forbid any invariant rational curves entering to the Poincaré domains appearing in the process of the nef reduction (by taking smaller \( D \)). Let \( T \) be the tree of rational curves appeared in the process of Seidenberg’s reduction of singularities over \( s \). If \( T \) is entirely contracted by the subsequent modification then all what happens to \( s \) is that it is replaced by a cyclic point. But then it can be only a smooth point. In this case the universal Poincaré domain \( \mathcal{P}_D \) cannot be Hausdorff. Contradiction.

Therefore \( T \) is divided to subtrees \( T_i \) of invariant rational curves, subsequently contracted to cyclic points \( c_i \) and connected by some other rational chains, which are not contracted. Through each of \( c_i \) passes then an \( F \)-invariant rational curve, which cannot belong to \( \mathcal{P}^F_D \) and therefore no one of \( c_i \) doesn’t belong to \( \mathcal{P}_D \), as well as all these ”connecting” invariant curves. Therefore \( \mathcal{P}^F_D \) contains a punctured ball. In the hyperbolic case that means that \( \mathcal{P}^F_C \) is not Stein. \( \mathcal{P}_C \) is the same as \( \mathcal{P}^F_D \) over this ball, i.e., is also not Stein. But then \( \mathcal{L}_m \) is a rational curve cutting \( \text{Sing} L \) by exactly one point (this point is \( s \)). This is impossible because \( \pi_1(L_m, m) \) cannot be trivial. Contradiction.

Corollary 9 is proved.

**References**

[AGH] Auslander L., Green L., Hahn F.: *Flows on homogeneous spaces. With the assistance of L. Markus and W. Massey, and an appendix by L. Greenberg.* Annals of Mathematics Studies, No. 53 Princeton University Press, Princeton, N.J. (1963).

[Bar] Barlet D. *Base de voisinages n-complets pour un sous-ensemble analytique compact de dimension n.* Applications. C. R. Acad. Sci. Paris Ser. A-B 286, no. 17, A751A753 (1978).

[Bd] Bedford E.: *Holomorphic continuation of smooth functions over Levi-flat hypersurfaces.* Trans. Amer. Math. Soc. 232 323-341 (1977).
[Be] Bendixson I.: Sur les courbes définies par des équation différentielles. Acta Math. 24 1-88 (1901).

[Bo1] Bochner S.: A theorem on analytic continuation of functions in several variables. Ann. of Math. (2) 39, no. 1, 14–19 (1938).

[Bo2] Bochner, S.: Analytic and meromorphic continuation by means of Green’s formula. Ann. of Math. (2) 44, 652–673 (1943).

[BLM] Bonatti C., Langevin R., Moussu R.: Feuilletages de $\mathbb{C}P^n$: de l’holonomie hyperbolique pour les minimaux exceptionnels. Publ. I.H.E.S., no 75, 123 - 134 (1992).

[BB] Briot C., Bouquet J.-C.: Mémoire sur l’intégration des équations différentielles au moyen des fonctions elliptiques. Journ. Ecole Polytechnique 21, 199-253 (1851).

[Br1] Brunella M.: Birational geometry of foliations. IMPA Mathematical Publications (2004).

[Br2] Brunella M.: Foliations on complex projective surfaces. Dynamical systems, Part II, 49-77, Publ. Cent. Ric. Mat. Emnio Giorgi, Scuola Norm. Sup., Pisa (2003).

[Br3] Brunella M.: On entire curves tangent to a foliation. J. Math. Kyoto Univ. 47, no 4, 717-754 (2007).

[Br4] Brunella M.: Some Remarks on Parabolic Foliations. Contemp. Math., 389, 91-102 (2005).

[Br5] Brunella M.: Subharmonic variation of the leafwise Poincaré metric. Invent. math. 152, 119-148 (2003).

[Ca] Camacho C.: Problems on limit sets of foliations on complex projective spaces. Proc. ICM, Kyoto 1990, 1235-1239, Math. Soc. Japan, Tokyo (1991).

[CLS1] Camacho C., Lins Neto A., Sad P.: Minimal sets of foliations on complex projective spaces. Publ. Math. I.H.E.S., 68, 187-203 (1988).

[CLS2] Camacho C., Lins Neto A., Sad P.: Foliations with algebraic limit sets. Annals of Math. 136 429-446 (1992).

[CS] Camacho C., Sad P.: Invariant varieties through singularities of holomorphic vector fields. Annals of Math. 115 3-105 (1982).

[CR1] Cerqueiro D.: Minimaux des feuilletages algébriques de $\mathbb{C}P^n$. Ann. Inst. Fourier 43 1535-1543 (1993).

[Di] Diener F.: Feuilletage de Briot et Bouquet. Collect. Math. 28, no. 2, 101-144 (1977).

[DG] Docquier F., Grauert H.: Levi's Problem and Runge's Satz for Teiltgebiete Steinischer Mannigfaltigkeiten. Math. Ann. 140, 94-123, (1960).

[Ez] Eremenko E., Liao L., Ng T.: Meromorphic solutions of higher order Briot-Bouquet differential equations. Math. Proc. Cambridge Philos. Soc. 146, no. 1, 197-206 (2009).

[Ha] Hartshorne R.: Algebraic geometry. Springer (1977).

[Fo] Forstneric F. The Oka principle for sections of subelliptic submersions. Math. Z. 241, no. 3, 527–551 (2002).

[F] Fujita R.: Domaines sans point critique intérieur sur l’espace projectif complexe. J. Math. Soc. Japan 15 443-473 (1963).

[H] Fujita R.: Domaines sans point critique intérieur sur l’espace produit. J. Math. Kyoto Univ. 3-4 493-514 (1965).

[GD] Godbillon C.: Feuilletages. Études géométriques. Birkhäuser (1991).

[G1] Grauert H.: On Levi’s problem and imbedding of real analytic manifolds. Ann. of Math. 68 460-472 (1958).

[G2] Grauert H.: Über Modifikationen und exceptionelle analytische Mengen. Math. Ann. 462 331-368 (1962).

[GR1] Grauert H., Remmert R.: Konvextät in der komplexen Analysis. Nicht-holomorph- konvexe Holomorphiegebiete und Anwendungen auf die Abbildungstheorie. Comment. Math. Helv. 31, 152-183 (1956).
[GR2] Gunning R., Rossi H.: Analytic Functions of Several Complex Variables. Prentice-Hall, N.J. (1965).

[Gro] Gromov M.: Oka's principle for holomorphic sections of elliptic bundles. J. Amer. Math. Soc. 2, 851-897 (1989).

[H] Hilbert D.: Mathematische probleme. Arch. Math. Phys., 3, 213-237 (1901) English transl., http://www.ams.org/bull/2000-37-04/S0273-0979-00-00881-8.

[Iy1] Ilyashenko J.: Fiberings into analytic curves. Math. USSR Sbornik 17 no 4 551-569 (1972).

[Iy2] Ilyashenko J.: Covering manifolds for analytic families of leaves of foliations by analytic curves. Topol. Methods Nonlinear Analysis 11, no. 2, 361–373 (1998).

[Iy3] Ilyashenko Yu.: Centennial history of Hilbert’s 16th problem. BAMS 39, 301-354 (2002).

[Iy4] Ilyashenko Yu.: Some open problems in real and complex dynamical systems. Nonlinearity 21, no. 7, 101-107 (2008).

[IS] Ilyashenko Yu., Shcherbakov A.: Remarks on the paper: “Covering manifolds for analytic families of leaves of foliations by analytic curves” [Topol. Methods Nonlinear Anal. 11 (1998), no. 2, 361–373; MR1659438] by Ilyashenko. Topol. Methods Nonlinear Anal. 23, no. 2, 377–381 (2004).

[IM] Jordan A., Matthey F.: Régularité de l’opérateur $\overline{\partial}$ et théorème de Siu sur la nonexistence d’hypersurfaces Levi-plates dans d’espace projectif complexe $\mathbb{CP}^n$, $n \geq 3$. C.R. Acad. Sci. Paris, Ser. I 346 395-400 (2008).

[It] Ito T.: A Poincaré-Bendixson Type Theorem for Holomorphic Vector Fields RIMS publication, Kyoto, n 878 (1994).

[Iv1] Ivashkovich S.: Envelopes of holomorphy of some tube sets in $\mathbb{C}^2$ and the monodromy theorem. Math. USSR Izvestija 19, N 1, 189-196 (1982).

[Iv2] Ivashkovich S.: Extension of locally biholomorphic mappings of domains into complex projective space. Math. USSR Izvestija, V.22, N 1, 181-189 (1984).

[Iv3] Ivashkovich S.: Extension of locally biholomorphic mappings into a product of complex manifolds. Math. USSR Izvestija 27, N 1, 193-199 (1986).

[Iv4] Ivashkovich S.: The Hartogs phenomenon for holomorphically convex Kähler manifolds. Math. USSR Izvestija, V.29, N 1, 225-232 (1987).

[Iv5] Ivashkovich S.: An example concerning extension and separate analyticity properties of meromorphic mappings. Amer. J. Math. 121 97-130 (1999).

[Iv6] Ivashkovich S.: Vanishing Cycles in Holomorphic Foliations by Curves and Foliated Shells. mathCV/0812.2114 , 11 Dec (2008).

[Lb] Lieberman D.: A Poincaré-Bendixson theorem for holomorphic vector fields. Lect. Notes Math. 712 60-65 (1979).

[L] Lins Neto A.: A note on projective Levi flats and minimal sets of algebraic foliations. Ann. Inst. Fourier 49 1369-1385 (1999).

[LR] Loray F., Rebelo J.: Minimal, rigid foliations by curves on $\mathbb{CP}^n$. J. Eur. Math. Soc. 5, no. 2, 147–201 (2003).

[McQ1] MacQuillan M.: Noncommutative Mori theory. I.H.E.S. Preprint M01’42 (2001).

[McQ2] MacQuillan M.: Canonical models of foliations. Pure Appl. Math. Q. 4, no. 3, part 2, 877-1012 (2008).

[MM] Matsushima Y., Morimoto A.: Sur certains espaces fibrés holomorphes sur une variété de Stein. Bull. Soc. Math. France, 88 137-155 (1960).

[Mi] Milnor J.: Dynamics in One Complex Variable. Annals of Math. Studies 60 (2006).

[Miy] Miyaoka Y.: Deformation of a morphism along a foliation and applications. Alg. Geom., Bowdoin, Proc. Symp. Pure Math. 47, 245-268 (1987).

[Mi] Mishchenko, M.: Three-dimensional compact analytic manifolds embedded in $\mathbb{CP}^2$ with identically vanishing Levi form. Math. Notes 54, no. 3-4, 1035–1044 (1994).

[Na1] Narasimhan R.: The Levi problem for complex spaces. Math. Ann. 142 355-365 (1961).
\[\text{Na2}\] Narasimhan R.: *The Levi problem in the theory of functions of several complex variables.* Proc. Internat. Congr. Mathematicians, Stockholm 1962, pp. 385–388 (1963).

\[\text{Ne1}\] Nemirovski S.: *Stein Domains with Levi-Flat Boundaries on Compact Complex Surfaces.* Mathematical Notes 66, N 4, 522-524 (1999).

\[\text{Ne2}\] Nemirovski S.: *Levi problem and semistable quotients.* Preprint.

\[\text{Oh1}\] Ohsawa T.: *Nonexistence of real analytic Levi flat hypersurfaces in } \mathbb{P}^2. \text{ Nagoya Math. J. 158, 95-98 (2000).}

\[\text{Oh2}\] Ohsawa T.: *Remark on pseudoconvex domains with analytic complements in compact Kähler manifolds.* J. Math. Kyoto Univ. 47, n 1, 115-119 (2007).

\[\text{Oh3}\] Ohsawa T.: *On the Levi-flats in Complex Tori of Dimension Two.* Publ. RIMS, Kyoto Univ. 42 361-377 (2006).

\[\text{Ra}\] Ransford T.: *Potential Theory in the Complex Plane.* London Math. Soc. Student Texts 28 (1995).

\[\text{R}\] Rea C.: *Levi-flat submanifolds and holomorphic extension of foliations.* Ann. Scuola Norm. Sup. Pisa 26, N 3 665681 (1972).

\[\text{P}\] Poincaré H.: *Sur les courbes définies par une équation différentielle.* Journal de Mathématiques Pures et Appliquées 7 375-422 (1881), 8 251-296 (1882) see also Oeuvres, t. 1, Paris (1892).

\[\text{Sei}\] Seidenberg A.: *Reduction of singularities of the differential equation \( Ady = Bdx. \) Amer. J. Math. 89, 248-269 (1967).

\[\text{Shc}\] Shcherbina N.: *An example of two pseudoconvex domains that are not separable by a Levi-flat hypersurface.* Math. Notes 45 no. 1-2, 174-177 (1989).

\[\text{Si1}\] Siu Y.-T.: *Extending coherent analytic sheaves.* Ann. of Math. 90 108-143 (1969).

\[\text{Si2}\] Siu Y.-T.: *Techniques of extension of analytic objects.* Marcel Dekker, NY (1974).

\[\text{Si3}\] Siu Y.-T.: *Nonexistence of smooth Levi-flat hypersurfaces in complex projective spaces of dimension \( \geq 3. \) Ann. of Math. 151 1217-1243 (2000).

\[\text{Si4}\] Siu Y.-T.: *Regularity for weakly pseudoconvex domains in compact Hermitian symmetric spaces with respect to invariant metrics.* Ann. of Math. 156 595-621 (2002).

\[\text{Si5}\] Siu Y.-T.: *Every Stein Subvariety Admits a Stein Neighborhood.* Invent. math. 38, 89-100 (1976).

\[\text{ST}\] Siu Y.-T., Trautmann G.: *Gap-Sheaves and Extension of Coherent Analytic Subsheaves.* Lect. Notes. Math 172 Springer (1971).

\[\text{So}\] Sommese A.: *Holomorphic vector-fields on compact Kaehler manifolds.* Math. Ann. 210, 75-82 (1974).

\[\text{Sz}\] Suzuki M.: *Sur les intégrales premières de certains feuilletages analytiques.* Séminaire F. Norguet 1975/77, 31-79, LNM 670 (1978).

\[\text{Ue}\] Ueda T.: *Pseudoconvex domains over Grassmann manifolds.* J. Math. Kyoto Univ. 20, no. 2, 391-394 (1980).

\[\text{Vi}\] Vitushkin A.: *Real-analytic hypersurfaces of complex manifolds.* Uspekhi Mat. Nauk 40, no. 2(242), 3-31 (1985).