A DG ALGEBRA RESOLUTION OF TRIMMINGS OF GORENSTEIN IDEALS

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Abstract. Let \((R, m, k)\) be a regular local ring of dimension 3. Let \(I\) be a Gorenstein ideal of \(R\) of grade 3. Buchsbaum and Eisenbud proved that there is a skew-symmetric matrix of odd size such that \(I\) is generated by the sub-maximal pfaffians of this matrix. Let \(J\) be the ideal obtained by multiplying some of the pfaffian generators of \(I\) by \(m\); we say that \(J\) is a trimming of \(I\). In this paper we construct an explicit free resolution of \(R/J\) with a DG algebra structure. Our work builds upon a recent paper of Vandebogert. We use our DG algebra resolution to prove that recent conjectures of Christensen, Veliche and Weyman on ideals of class \(G\) hold true in our context and to address the realizability question for ideals of class \(G\).

1. Introduction

Let \((R, m, k)\) be a regular local ring and \(I\) a perfect ideal of grade 3. Buchsbaum and Eisenbud showed in [5] that a minimal free resolution \(F_\bullet\) of \(R/I\) has a differential graded (DG) algebra structure. This DG algebra induces a graded algebra structure on \(\text{Tor}^R_\bullet(R/I, k) := H_\bullet(F_\bullet \otimes_R k)\). By results of Weyman [12] and of Avramov, Kustin, and Miller [4], this graded algebra structure does not depend on the DG algebra structure on \(F_\bullet\) and may be classified into one of five distinct classes.

Of particular interest to us are ideals whose Tor algebra is of class \(G(r)\), where \(r\) is a parameter defined as the rank of the map

\[
\text{Tor}^R_2(R/I, k) \to \text{Hom}_k(\text{Tor}^R_1(R/I, k), \text{Tor}^R_3(R/I, k)).
\]

Every Gorenstein ideal of grade 3 that is not a complete intersection ideal is of class \(G\). Moreover, by a result of Buchsbaum and Eisenbud [5], every Gorenstein ideal of grade 3 is generated by the sub-maximal pfaffians of a skew-symmetric matrix of odd size. It was conjectured by Avramov in [3] that every ideal of class \(G\) is Gorenstein. A counterexample to this conjecture was found by Christensen, Veliche and Weyman in [6]. This counterexample was constructed by a process referred to as “trimming”, which consists of replacing a generator \(g\) of the ideal by \(mg\).

In [7], Christensen, Veliche and Weyman give conjectures on the parameter \(r\) for an ideal \(J\) of class \(G(r)\) that is not Gorenstein. Using results from [6], these conjectures can be refined for ideals \(J\) of class \(G(r)\) arising by trimming a Gorenstein ideal in a regular local ring of dimension 3. The aim of our paper is to prove these conjectures hold for ideals obtained by trimming the pfaffian generators of a Gorenstein ideal in a regular local ring of dimension 3.

The main tool used in this paper to prove the conjectures mentioned above is a DG algebra resolution of the trimmed ideal \(J\). This DG algebra resolution was originally constructed by Vandebogert in [11] in a more general setting. The product formulae given in [11, Theorem 3.2] are defined implicitly by lifting certain cycles through the differential of the resolution. In this paper we make use of Vandebogert’s construction and of pfaffian identities proved by Knuth [9] to give a more explicit product on the resolution constructed by Vandebogert.

The paper is organized as follows. In Section 2 we recall background information on perfect ideals of grade 3 and their classification, on DG algebra resolutions of Gorenstein ideals, and on pfaffians and their identities. In Section 3 we apply Vandebogert’s construction to give a more explicit free resolution of an ideal obtained by trimming pfaffian generators of a Gorenstein ideal of grade 3. In Section 4 we use the product table provided by Vandebogert and the pfaffian identities provided by Knuth to give more explicit products on the free resolution constructed in Section 3. In Section 5 we use the DG algebra resolution constructed in Section 4 to study the Betti numbers and the parameter \(r\) for an ideal of class \(G(r)\) arising by trimming the pfaffian generators of a Gorenstein ideal of grade 3. We use these results to prove that the conjectures of Christensen, Veliche and Weyman hold true under these hypotheses. Finally, in Section 6 we address the realizability question for ideals of class \(G\).
2. Background and Notation

Throughout the paper $(R, \mathfrak{m}, k)$ will denote a regular local ring of dimension 3.

2.1. Let $I$ be a perfect ideal of $R$ of grade 3. We say that $I$ has format $(1, m, m + n - 1, n)$ if the minimal free resolution $F_\bullet$ of $R/I$ is of the form

$$F_\bullet: 0 \rightarrow R^n \rightarrow R^{m+n-1} \rightarrow R^m \rightarrow R \rightarrow 0.$$ 

We fix bases

$$\{e_i\}_{i=1,...,m}, \quad \{f_i\}_{i=1,...,m+n-1}, \quad \{g_i\}_{i=1,...,n}$$

of $F_1, F_2$ and $F_3$ respectively.

It was proved in [5] that there is a skew-symmetric matrix $T$ on $\text{Tor}_2 R$.

We also set a notation for the pfaffian of the submatrix of $T$ sublists of size 2. By convention we set $k$ only considering the rows and columns in positions

$$\text{sgn}\text{ denotes the sign of a permutation and the sum is taken over all the partitions of } n \text{ in } k \text{ subsets of size 2.}$$

Throughout the paper ($A$ DG algebra structure on this resolution has been found in [2]. Let

$$\{0 \rightarrow R \rightarrow R^m \rightarrow R^m \rightarrow R \rightarrow 0, \text{ where } D_1 = (p_1(T)) \rightarrow (p_2(T)) \rightarrow (p_3(T)) \rightarrow (p_4(T)) \rightarrow (p_5(T)).$$

Let $\{e_i\}_{i=1,...,m}, \{f_i\}_{i=1,...,m}$ and $\{g\}$ be bases in degrees 1, 2 and 3 respectively of the resolution $F_\bullet$. We denote by $(m)$ the sequence $(1 2 \cdots m),$
and by \((m)\setminus\{i_1,\ldots,i_n\}\), the sequence \((m)\) with the elements \(i_1,\ldots,i_n\) removed. A product on \(F_*\) is given by the following formulae

\[
e_i e_j = \sum_{r=1}^{m} (-1)^{i+1} \text{sgn}(m)\setminus\{i\}) \text{pf}_{i,j,r}(T) f_r, \quad \text{for } i < j,
\]

\[
e_{i} f_{j} = \delta_{i,j} g,
\]

where \(\delta_{i,j}\) is the Kronecker delta.

2.4. For the convenience of the reader we record properties of pfaffians. We first recall the formula in [8, (A.1.5)], if \(\beta\) is a word in \(1,\ldots,m\) and \(b\) an element of \(\{1,\ldots,m\}\), then

\[
\mathcal{P}[\beta] = \sum_{r \in \beta} \text{sgn}(\beta r \setminus\{b, r\}) \mathcal{P}[\beta \setminus\{b, r\}] \mathcal{P}[br].
\]

Let \(1 \leq i, j \leq m\) with \(i \neq j\), by taking \(\beta = (m)\setminus\{i\}\) and \(b = j\) in (2.3) one gets

\[
\text{pf}_{i,r}^f(T) = \sum_{r=1}^{m} \text{sgn}(m)\setminus\{i\} \text{pf}_{i,j,r}^f(T).
\]

Let \(1 \leq i, j, k \leq m\) be distinct indices. Set \(\beta = k(m)\setminus\{i,j\}\). We point out that \(k\) appears twice in \(\beta\). In terms of permutations signs, we treat the two appearances of \(k\) as being distinct and set \(b\) to be the first \(k\) in \(\beta\). Applying (2.3) one gets

\[
\sum_{r=1}^{m} \text{sgn}(k(m)\setminus\{i,j\}) T_{k, r} \text{pf}_{i,j,r}^f(T) = 0.
\]

Let \(1 \leq i, j, r, k \leq m\) be distinct indices. By taking \(\beta = (m)\setminus\{i,j,r\}\) and \(b = k\) in (2.3) one gets

\[
\text{pf}_{i,j,r}^k(T) = \sum_{h=1}^{m} \text{sgn}(m)\setminus\{i,j,r\} T_{k, h} \text{pf}_{i,j,r,k}^h(T).
\]

2.5. Let \(\sigma\) be a permutation in the symmetric group \(S_n\). We write \(\sigma\) in matrix form as

\[
\begin{pmatrix}
1 & 2 & \cdots & n \\
\sigma(1) & \sigma(2) & \cdots & \sigma(n)
\end{pmatrix},
\]

and we draw a line connecting \(i\) and \(\sigma(i)\) for each \(i\). The number of pairs of lines intersecting is called the crossing number of \(\sigma\), and we denote it by \(c(\sigma)\). It is well known, see for example [11, Pages 27-29], that

\[
\text{sgn}(\sigma) = (-1)^{c(\sigma)}.
\]

2.6. We recall that the unit step function, also known as the Heaviside step function, is a function \(\theta : \mathbb{R}\setminus\{0\} \to \{0,1\}\) defined as

\[
\theta(x) = \begin{cases} 
0 & \text{if } x < 0 \\
1 & \text{if } x > 0.
\end{cases}
\]

Let \(m\) be a positive integer and let \(i,j,r\) be distinct elements of \(\{1,\ldots,m\}\). We introduce the following notation

\[
\sigma_{i,j,r} := (-1)^{i+1} \text{sgn}(m)\setminus\{i\} \text{pf}_{i,j,r}(T).
\]

Using the crossing number of the permutation above, it is easy to check that \(\sigma_{i,j,r}\) can be written as

\[
\sigma_{i,j,r} = (-1)^{i+j+r+1} \theta(r-i) + \theta(r-j) + \theta(j-i).
\]

The following relations satisfied by \(\sigma_{i,j,r}\) follow from the previous expression; we record them here for later reference

\[
\sigma_{i,j,r} = -\sigma_{j,i,r}, \quad \sigma_{i,j,r} = \sigma_{r,i,j}, \quad \sigma_{j,i,r} = \sigma_{i,r,j} = \sigma_{r,j,i}.
\]

We point out that using crossing numbers, the sign in (2.7) is, up to a power of \(-1\) that does not depend on \(r\), equal to \(\sigma_{i,j,r}\), therefore when \(i,j \neq k\) we get the following formula
We introduce the following notation for the sign in (2.6)
\[
\sigma_{i,j,r,h,k} := \text{sgn} \left( \begin{pmatrix} \langle m \rangle \setminus \{i, j, r \} \\ k h(m) \setminus \{i, j, r, k, h\} \end{pmatrix} \right).
\]
Using the crossing number of the permutation above one can check that the following equality holds
\[
\sigma_{i,j,r,h,k} = (-1)^{h+k+1+\theta(k-\i)+\theta(k-\j)+\theta(\k-\r)+\theta(h-\i)+\theta(h-\j)+\theta(h-\r)}.
\]
We point out that \(\sigma_{i,j,r,h,k}\) is independent of the order of \(i,j,r\).

2.7. Let \(i,h,s,k,j \in \{1,\ldots,m\}\) with \(i,h,s,k \neq j\). Then, similar to (2.9), one has the following equation
\[
\sum_{r=1}^{m} \sigma_{i,r,h,s,k,r} T_{j,r} \text{pf}_{1,r,h,s,k,r}(T) = 0.
\]
Indeed let \(\beta = j(m) \setminus \{i,h,s,k\}\). Applying (2.3) with \(b\) equals to the first appearance of \(j\) in \(\beta\) yields
\[
0 = \mathcal{P}[\beta] \quad \text{since \(\beta\) has a repeated \(j\)}
\]
\[
= \sum_{r=1}^{m} \text{sgn} \left( \begin{pmatrix} j & r \end{pmatrix} \{i,h,s,k\} \right) T_{j,r} \mathcal{P}[\beta \setminus \{r\}] \quad \text{by (2.3)}
\]
\[
= \sum_{r=1}^{m} \text{sgn} \left( \begin{pmatrix} j(m) \setminus \{i,h,s,k\} \\ r(m) \setminus \{i,h,s,k,r\} \end{pmatrix} \right) T_{j,r} \text{pf}_{1,r,h,s,k,r}(T)
\]
\[
= \sum_{r=1}^{m} \text{sgn} \left( \begin{pmatrix} \langle m \rangle \setminus \{i,h,s,k\} \\ r(m) \setminus \{i,h,s,k,r\} \end{pmatrix} \right) T_{j,r} \text{pf}_{1,r,h,s,k,r}(T)
\]
\[
= \sum_{r=1}^{m} (-1)^{r+1+\theta(\r-\i)+\theta(\r-\h)+\theta(\r-\s)+\theta(\r-\k)} T_{j,r} \text{pf}_{1,r,h,s,k,r}(T) \quad \text{using the crossing number,}
\]
and up to a power of \(-1\) that does not depend on \(r\) this coincides with the left side of (2.11).

3. A FREE RESOLUTION OF A TRIMMED PFAFFIAN IDEAL

We fix a generating set for the maximal ideal \(m = (z_1, z_2, z_3)\). Let \(I\) be a Gorenstein ideal of \(R\) of grade 3 that is not a complete intersection. It has been proved in [5] Theorem 2.1] that there is a odd sized skew-symmetric \(m \times m\) matrix \(T\) such that \(I = (y_1, \ldots, y_m)\) where \(y_i := (-1)^i+1 R \text{pf}_i(T)\) for \(i = 1, \ldots, m\). Let \(t\) be an integer such that \(1 \leq t \leq m\), and let \(J := y_t m + \cdots + y_1 m + (y_{t+1}, \ldots, y_m)\); following the terminology set in [6] we say that \(J\) is obtained from \(I\) by trimming the first \(t\) generators of \(I\). The aim for this section is to obtain a free resolution of \(R/J\).

In [11] Theorem 2.6], Vandebogert constructs a free resolution of a trimmed ideal \(J\) starting from the free resolution of a (general) ideal \(R/I\). We apply his construction to the resolution of a Gorenstein ideal of grade 3 given in [2,3].

Let \((G_*,δ_*)\) be the Koszul resolution of \(k\) over \(R\), with
\[
G_1 = Ru_1 \oplus Ru_2 \oplus Ru_3, \quad G_2 = Rv_{1,2} \oplus Rv_{1,3} \oplus Rv_{2,3}, \quad G_3 = Rw,
\]
\[
δ_1 = (z_1 z_2 z_3), \quad δ_2 = \begin{pmatrix} -z_2 & -z_3 & 0 \\ z_1 & 0 & -z_3 \\ 0 & z_2 & z_3 \end{pmatrix}, \quad δ_3 = \begin{pmatrix} z_3 \\ -z_2 \\ z_1 \end{pmatrix}.
\]
We will need \(t\) copies of \((G_*, δ_*)\) which we will denote by \((G^{k}_*, δ^{k}_*)\) for \(k = 1, \ldots, t\). We denote the generators of \(G^{k}_1\) by \(u_1^k, u_2^k, u_3^k\), and similarly for \(G^{k}_2, G^{k}_3\). We set \(v_{\beta, \alpha}^k = -v_{\alpha, \beta}^k\) for \(\alpha < \beta\) and \(\alpha, \beta \in \{1,2,3\}\).
Let $c_{i,j,l}$, for $i, j = 1, \ldots, m$ and $l = 1, 2, 3$, be elements of $R$ satisfying the following equality

\begin{equation}
T_{j,l} = \sum_{i=1}^{3} c_{i,j,l} z_{l}.
\end{equation}

We define constants that will be used in the statement of the main theorem of this section. Let $\alpha, \beta \in \{1, 2, 3\}$ with $\alpha < \beta$, and let $k = 1, \ldots, t$, then we define

\begin{equation}
d_{k,\alpha,\beta} := \sum_{i=1}^{m} \sum_{r=1}^{m} \alpha_{i,k,r} c_{i,k,\beta} c_{r,k,\alpha} \sigma_{r,k,r}(T).
\end{equation}

We use the notation set in \cite{2,3} for the minimal free resolution $(F_{\bullet}, D_{\bullet})$ of $R/I$.

We define $F_{1}'$ by the decomposition $F_{1} = (\oplus_{i=1}^{t} Re_{i}) \oplus F_{1}'$. Similarly, we define $D_{2}'$ by the decomposition

$$D_{2} \in \text{Hom}_{R}(F_{2}, F_{1}) = (\oplus_{i=1}^{t} \text{Hom}_{R}(F_{2}, Re_{i})) \oplus \text{Hom}_{R}(F_{2}, F_{1}')$$

by setting $D_{2} = D_{1}^{3} + D_{0}^{2} + \cdots + D_{0}^{0} + D_{2}'$.

We denote the composition

$$F_{2} \xrightarrow{D_{0}^{i}} Re_{i} \longrightarrow R$$

as $D_{0}'^{i}$ for $1 \leq i \leq t$, where the second map sends $e_{i} \mapsto 1$.

**Theorem 3.1.** Using the notation set so far, a free resolution of $R/I$ is given by

\begin{equation}
0 \longrightarrow F_{3} \oplus (\oplus_{k=1}^{3} G_{k}^{1}) \xrightarrow{\partial_{3}} F_{2} \oplus (\oplus_{k=1}^{t} G_{k}^{2}) \xrightarrow{\partial_{2}} F_{1}' \oplus (\oplus_{k=1}^{t} G_{k}^{1}) \xrightarrow{\partial_{1}} R
\end{equation}

where the maps $q_{1}^{k} : F_{2} \to G_{k}^{1}, q_{2}^{k} : F_{3} \to G_{2}^{k}$ for $k = 1, \ldots, t$ are defined by

$$q_{1}^{k}(f_{i}) = \sum_{i=1}^{3} c_{i,k,l} u_{l}^{k},$$

$$q_{2}^{k}(g) = d_{1,2}^{k} v_{1,2}^{k} + d_{1,3}^{k} v_{1,3}^{k} + d_{2,3}^{k} v_{2,3}^{k}.$$

**Proof.** After checking that the choices of $q_{1}^{k}$ and $q_{2}^{k}$ above give commutative diagrams

the result follows from \cite{11} Theorem 2.6. First we check the commutativity left-hand diagram:

$$D_{0}'^{k}(f_{i}) = T_{k,i},$$

$$\delta_{k}^{1}(q_{1}^{k}(f_{i})) = \delta_{k}^{1} \left( \sum_{i=1}^{3} c_{i,k,l} u_{l}^{k} \right) = \sum_{i=1}^{3} c_{i,k,l} z_{l} = T_{k,i}.$$

Now we check the commutativity of the right-hand diagram:

$$q_{1}^{k}(D_{3}(g)) = q_{1}^{k} \left( \sum_{i=1}^{m} y_{i} f_{i} \right) = \sum_{i=1}^{m} \sum_{l=1}^{3} y_{i} c_{i,k,l} u_{l}^{k} = \left( \sum_{i=1}^{m} y_{i} c_{i,k,1} \right) u_{1}^{k} + \left( \sum_{i=1}^{m} y_{i} c_{i,k,2} \right) u_{2}^{k} + \left( \sum_{i=1}^{m} y_{i} c_{i,k,3} \right) u_{3}^{k},$$

and

$$q_{2}^{k}(g) = d_{1,2}^{k} v_{1,2}^{k} + d_{1,3}^{k} v_{1,3}^{k} + d_{2,3}^{k} v_{2,3}^{k}.$$
\[ \delta_k(\sigma_2(g)) = \delta_k(d_{1,2}^k u_1^k + d_{1,3}^k u_1^k + d_{2,3}^k u_2^k) = \delta_k(z_1 u_1^k - z_2 u_1^k) + d_{1,3}^k(z_1 u_1^k - z_3 u_1^k) + d_{2,3}^k(z_2 u_1^k - z_3 u_1^k) = -z_2 d_{1,2}^k - z_3 d_{1,3}^k + (z_1 d_{1,2}^k - z_3 d_{2,3}^k)u_2^k + (z_1 d_{1,3}^k + z_2 d_{2,3}^k)u_3^k. \]

We check that \( \sum_{i=1}^{m} y_i c_{i,k,1} = -z_2 d_{1,2}^k - z_3 d_{1,3}^k \); the remaining two equalities are similarly checked.

It follows from (2.4) and the definition of \( \sigma_{i,k,r} \) that for every \( k \in \{1, \ldots, m\} \setminus \{i\} \) the following equality holds

\[ y_i = \sum_{r=1}^{m} \sigma_{i,k,r} T_{k,r} \overline{p_{r,k,i}(T)}. \]

Expanding \( \sum_{i=1}^{m} y_i c_{i,k,1} \) using (3.3) and (3.1) gives

\[ \sum_{i=1}^{m} \sum_{r=1}^{m} \sigma_{i,k,r} c_{r,k,1} c_{i,k,1} \overline{p_{r,k,i}(T)} z_1 + \sum_{i=1}^{m} \sum_{r=1}^{m} \sigma_{i,k,r} c_{r,k,2} c_{i,k,1} \overline{p_{r,k,i}(T)} z_2 + \sum_{i=1}^{m} \sum_{r=1}^{m} \sigma_{i,k,r} c_{r,k,3} c_{i,k,1} \overline{p_{r,k,i}(T)} z_3. \]

We notice that the coefficient of \( z_1 \) is 0 since, by (2.8),

\[ \sigma_{i,k,r} c_{r,k,1} c_{i,k,1} \overline{p_{r,k,i}(T)} = -\sigma_{i,k,r} c_{r,k,1} c_{i,k,1} \overline{p_{r,k,i}(T)}, \]

and that the coefficients of \( z_2 \) and \( z_3 \) are \(-d_{1,2}^k\) and \(-d_{1,3}^k\) respectively.

\[ \square \]

4. A DG Algebra Structure

We start by defining several constants that will be used in the statement of the main theorem of this section, together with the constant defined in (3.2). Let \( \alpha, \beta \in \{1, 2, 3\} \) with \( \alpha < \beta \), let \( k \) be an integer between 1 and \( t \), and let \( i, j \) be integers between 1 and \( m \).

We first define the constants \( d_{\alpha,\beta}^{k,i,j} \) which will be used in products \( b \) and \( e \) of the theorem

\[ d_{\alpha,\beta}^{k,i,j} := \sum_{r=1}^{m} \sigma_{i,j,r} \sum_{h=1}^{m} \sigma_{i,j,r,h,k} \overline{p_{r,j,h,k}(T)} c_{r,k,\beta} c_{h,k,\alpha}. \]

Let \( l \in \{1, 2, 3\} \). The next set of constants will be used in product \( b \).

\[ d_{\alpha,\beta}^{k,i,j} = z_i d_{\alpha,\beta}^{k,i,j}, \quad \text{for } k \neq i, \]
\[ d_{\alpha,\beta}^{k,i,j} = 0 \quad \text{if } \{\alpha, \beta, l\} = \{1, 2, 3\}, \]
\[ d_{\alpha,\beta}^{k,i,j} = \sum_{r=1}^{m} \sigma_{i,j,r} \overline{p_{r,j,h,k}(T)} c_{r,k,\beta}, \]
\[ d_{\alpha,\beta}^{k,i,j} = -\sum_{r=1}^{m} \sigma_{i,j,r} \overline{p_{r,j,h,k}(T)} c_{r,k,\alpha}. \]

Let \( l, s \in \{1, 2, 3\} \). The next set of constants will be used in product \( d \).

\[ d_{\alpha,\beta}^{k,i,j,l,s} = z_l z_s d_{\alpha,\beta}^{k,i,j}, \quad \text{for } k \neq i, j, \]
\[ d_{\alpha,\beta}^{k,i,j,l,s} = z_s d_{\alpha,\beta}^{k,i,j}, \]
\[ d_{\alpha,\beta}^{k,i,j,l,s} = 0 \quad \text{if } \{\alpha, \beta, s\} = \{1, 2, 3\}, \]
\[ d_{\alpha,\beta}^{k,i,j,l,s} = z_l \sum_{r=1}^{m} \sigma_{i,j,r} \overline{p_{r,j,h,k}(T)} c_{r,i,\alpha}, \]
\[ d_{\alpha,\beta}^{k,i,j,l,s} = -z_l \sum_{r=1}^{m} \sigma_{i,j,r} \overline{p_{r,j,h,k}(T)} c_{r,i,\alpha}. \]

We will use the constants defined in (3.2) in product \( i \). We set \( d_{\beta,\alpha} = -d_{\alpha,\beta} \) for any number of superscripts.
The DG algebra structure on $G^k_*$ (with $1 \leq k \leq t$) is the usual exterior product of the Koszul complex, namely
\[ u^k_1 u^k_2 = v^k_{1,2}, \quad u^k_1 u^k_3 = v^k_{1,3}, \quad u^k_2 u^k_3 = v^k_{2,3}, \quad u^k_1 u^k_2 u^k_3 = w^k, \]
and the usual skew-commutativity rules. A product in the DG algebra $G^k_*$ will be denoted as $- \cdot G^k -$. In the statement (and proof) of the theorem we will denote a product in the DG algebra $F_*$ as $- \cdot F -$. We will denote products on the resolution given in Theorem 3.1 by an unadorned $\cdot -$.

**Theorem 4.1.** A DG algebra structure on the resolution constructed in Theorem 3.1 is given by the following product rules

a. $F'_1 \otimes F'_1 \to F_2 \oplus (\oplus_{k=1}^t G^k_2)$

\[ e_i \cdot e_j := e_i \cdot F e_j + \sum_{k=1}^t d^{k,i,j}_{1,2} v^k_{1,2} + d^{k,i,j,k}_{1,3} v^k_{1,3} + d^{k,i,j,k}_{2,3} v^k_{2,3}, \]
for $t + 1 \leq i, j \leq m$.

b. $F'_1 \otimes G^j_1 \to F_2 \oplus (\oplus_{k=1}^t G^k_2)$

\[ e_j \cdot u^i_1 := z_i e_i \cdot F e_j + \sum_{k=1}^t d^{k,i,j,l}_{1,2} v^k_{1,2} + d^{k,i,j,l,k}_{1,3} v^k_{1,3} + d^{k,i,j,l,k}_{2,3} v^k_{2,3}, \]
for $1 \leq i \leq t$ and $t + 1 \leq j \leq m$.

c. $G^j_1 \otimes G^j_1 \to F_2 \oplus (\oplus_{k=1}^t G^k_2)$

\[ u^i_1 \cdot u^i_s := -y_{i,s} u^i_s, \]
for $1 \leq i \leq t$ and $l < s$.

d. $G^j_1 \otimes G^j_1 \to F_2 \oplus (\oplus_{k=1}^t G^k_2)$ with $i < j$

\[ u^i_1 \cdot u^j_4 := z_i z_s e_i \cdot F e_j + \sum_{k=1}^t d^{k,i,j,l,s}_{1,2} v^k_{1,2} + d^{k,i,j,l,s,k}_{1,3} v^k_{1,3} + d^{k,i,j,l,s,k}_{2,3} v^k_{2,3}, \]
for $1 \leq i, j \leq t$.

e. $F'_1 \otimes F_2 \to F_3 \oplus (\oplus_{k=1}^t G^k_3)$

\[ e_i \cdot f_j := \begin{cases} e_i \cdot F f_j, & t + 1 \leq j \leq m \\ -\sum_{r=1}^m c^{r,i,p}_{i,j} u^j_1 w^r, & 1 \leq j \leq t \end{cases}, \]
for $t + 1 \leq i \leq m$.

f. $F'_1 \otimes G^j_2 \to F_3 \oplus (\oplus_{k=1}^t G^k_3)$

\[ e_j \cdot v^i_{\alpha,\beta} := (-1)^p \sum_{r=1}^m \sigma^{r,i,p}_{i,j} \rho^{j,r}_{1,3}(T) c^{r,i,p} w^i, \]
with $p \in \{1, 2, 3\} \setminus \{\alpha, \beta\}$, $1 \leq i \leq t$ and $t + 1 \leq j \leq m$.

g. $G^j_1 \otimes G^j_2 \to F_3 \oplus (\oplus_{k=1}^t G^k_3)$

\[ u^i_1 \cdot v^i_{\alpha,\beta} := -y_{i} u^i_{\alpha,\beta}, \]
for $1 \leq i \leq t$. 
h. $G^i_1 \otimes G^j_2 \rightarrow F_3 \oplus (\oplus_{k=1}^t G^k_3)$ with $i \neq j$

$$u^i_1 \cdot v_{\alpha, \beta} := (-1)^{p+1} z_i \sum_{r=1}^m \sigma_{i,j,r} p_{i,j,r}(T)c_{r,j,p} w^j$$

with $p \in \{1, 2, 3\} \setminus \{\alpha, \beta\}, 1 \leq i, j \leq t$.

i. $G^i_1 \otimes F_2 \rightarrow F_3 \oplus (\oplus_{k=1}^t G^k_3)$

$$u^i_1 \cdot f_j := -z_i e_i \cdot_F f_j + (-1)^{j+1} \delta_{i,j} d_{i,F}^p w^j,$$

with $\{\phi, \psi, l\} = \{1, 2, 3\}$ and $\phi < \psi$, $1 \leq i \leq t$ and $1 \leq j \leq m$.

Proof.

a. $F^i_1 \otimes F^i_1 \rightarrow F_2 \oplus (\oplus_{k=1}^t G^k_2)$

By [11] Theorem 3.2, it suffices to show that

$$q^k_1(e_i \cdot_F e_j).$$

Indeed, the left side is equal to

$$(-d_{1,2}^{k,i,j} z_2 - d_{1,3}^{k,i,j} z_3)u^k_1 + (d_{1,2}^{k,i,j} z_1 - d_{2,3}^{k,i,j} z_3)u^k_2 + (d_{1,3}^{k,i,j} z_1 + d_{2,3}^{k,i,j} z_2)u^k_3,$$

while the right side is equal to

$$q^k_1(e_i \cdot_F e_j) = q^k_1 \left( \sum_{r=1}^m \sigma_{i,j,r} p_{i,j,r}(T)f_r \right)$$

$$= \sum_{r=1}^m \sigma_{i,j,r} p_{i,j,r}(T)(c_{r,k,1} u^k_1 + c_{r,k,2} u^k_2 + c_{r,k,3} u^k_3)$$

$$= \sum_{r=1}^m \sigma_{i,j,r} \sum_{h=1}^m \sigma_{i,j,r,h,k} T_{k,h} p_{i,j,r,h,k}(T) (c_{r,k,1} u^k_1 + c_{r,k,2} u^k_2 + c_{r,k,3} u^k_3)$$

$$= \sum_{r=1}^m \sigma_{i,j,r} \sum_{h=1}^m \sigma_{i,j,r,h,k} (c_{h,k,1} z_1 + c_{h,k,2} z_3 + c_{h,k,3} z_3) p_{i,j,r,h,k}(T) (c_{r,k,1} u^k_1 + c_{r,k,2} u^k_2 + c_{r,k,3} u^k_3),$$

where the first equality comes from the product in $F_\bullet$, the second equality comes from applying $q^k_1$ as defined in Theorem 3.1, the third equality comes from (2.9), and the fourth equality comes from (3.1). We show that the coefficients of $u^k_1$ are the same; a similar argument can be used for the coefficients of $u^k_2$ and $u^k_3$. We need to show that the following equality holds

$$-d_{1,2}^{k,i,j} z_2 - d_{1,3}^{k,i,j} z_3 = \sum_{r=1}^m \sigma_{i,j,r} \sum_{h=1}^m \sigma_{i,j,r,h,k} p_{i,j,r,h,k}(T)c_{r,k,1} c_{h,k,1} z_1$$

$$+ \sum_{r=1}^m \sigma_{i,j,r} \sum_{h=1}^m \sigma_{i,j,r,h,k} p_{i,j,r,h,k}(T)c_{r,k,1} c_{h,k,2} z_2$$

$$+ \sum_{r=1}^m \sigma_{i,j,r} \sum_{h=1}^m \sigma_{i,j,r,h,k} p_{i,j,r,h,k}(T)c_{r,k,1} c_{h,k,3} z_3.$$

The coefficient of $z_1$ on the right hand side of the display above is zero since

$$\sigma_{i,j,r} \sigma_{i,j,r,h,k} = -\sigma_{i,j,r} \sigma_{i,j,r,h,k},$$

which can be verified using (2.7) and (2.10). The coefficients of $z_2$ and $z_3$ match for the same reason.

b. $F^i_1 \otimes G^i_1 \rightarrow F_2 \oplus (\oplus_{k=1}^t G^k_3)$

By [11] Theorem 3.2, it suffices to show that

$$x_2^{1,i,j,l} v_{1,i} + y_2^{1,i,j,l} v_{1,3} + y_2^{2,i,j,l} v_{2,3} = y_j u^i_1 + z_i q^j_1 (e_i \cdot_F e_j),$$

(4.1)
We first prove (4.1). The left side is equal to

\[-d_{1,2}^{i,j,l} z_2 - d_{1,3}^{i,j,l} z_3 u_1 + (d_{1,2}^{i,j,l} z_1 - d_{2,3}^{i,j,l} z_3) u_2 + (d_{1,3}^{i,j,l} z_1 + d_{2,3}^{i,j,l} z_2) u_3,
\]

while the right side is equal to

\[y_j u_1^i + z_i q_1^i (e_i \cdot \mathbf{e}_j) = y_j u_1^i + z_i q_1^i \left( \sum_{r=1}^m \sigma_{i,j,r} T_{r,j}(T) f_r \right) = y_j u_1^i + z_i \sum_{r=1}^m \sigma_{i,j,r} T_{r,j}(T) (c_{r,i,1} u_1^i + c_{r,i,2} u_2^i + c_{r,i,3} u_3^i) = y_j u_1^i + z_i \left( \sum_{r=1}^m \sigma_{i,j,r} T_{r,j}(T) (c_{r,i,1} u_1^i + \sum_{r=1}^m \sigma_{i,j,r} T_{r,j}(T) c_{r,i,2} u_2^i + \sum_{r=1}^m \sigma_{i,j,r} T_{r,j}(T) c_{r,i,3} u_3^i) \right),\]

where the first equality comes from the product in \( F \), the second equality comes from applying \( q_1^i \) as defined in Theorem [X.X] and the third equality comes from rewriting the previous line. To prove that the coefficients of \( u_1^1 \), \( u_2^1 \) and \( u_3^1 \) match, we assume that \( l = 1 \); the remaining cases are proved similarly. The coefficients of \( u_3^2 \) and \( u_3^3 \) match by the definition of the constants \( d_{1,2}^{i,j,l} \), \( d_{1,3}^{i,j,l} \) and \( d_{2,3}^{i,j,l} \). Expanding \( y_j \) using (2.3) and (2.8), it follows that the coefficient of \( u_1^1 \) in \( y_j u_1^i + z_i q_1^i (e_i \cdot \mathbf{e}_j) \) is

\[- z_1 \sum_{r=1}^m \sigma_{i,j,r} T_{r,j}(T) c_{r,i,1} - z_2 \sum_{r=1}^m \sigma_{i,j,r} T_{r,j}(T) c_{r,i,2} - z_3 \sum_{r=1}^m \sigma_{i,j,r} T_{r,j}(T) c_{r,i,3} = -z_1 \sum_{r=1}^m \sigma_{i,j,r} T_{r,j}(T) c_{r,i,1}.\]

Now we prove (4.2). By the definitions of \( d_{1,2}^{i,j,l} \), \( d_{1,3}^{i,j,l} \) and \( d_{2,3}^{i,j,l} \) when \( k \neq i \), the left hand side is equal to

\[\delta_2^k (z_i d_{1,2}^{k,i,j,l} v_{1,2}^k + z_i d_{1,3}^{k,i,j,l} v_{1,3}^k + z_i d_{2,3}^{k,i,j,l} v_{2,3}^k) = z_i \delta_2^k (d_{1,2}^{k,i,j,l} v_{1,2}^k + d_{1,3}^{k,i,j,l} v_{1,3}^k + d_{2,3}^{k,i,j,l} v_{2,3}^k) = z_i q_1^k (e_i \cdot \mathbf{e}_j),\]

where the last equality was shown in the proof of product \( a \).

c. \( G_1^1 \otimes G_1^1 \to F_2 \oplus (\oplus_{k=1}^3 G_2^k) \)

This product follows immediately from [11, Theorem 3.2].

d. \( G_1^i \otimes G_1^j \to F_2 \oplus (\oplus_{k=1}^3 G_2^k) \) with \( i \neq j \)

By [11, Theorem 3.2], it suffices to show that

\[(4.3) \quad \delta_2^i (d_{1,2}^{i,j,l,s,i} v_{1,2}^i + d_{1,3}^{i,j,l,s,i} v_{1,3}^i + d_{2,3}^{i,j,l,s,i} v_{2,3}^i) = z_i s_1 q_1^i (e_i \cdot \mathbf{e}_j) + z_i y_j u_1^i,
\]

\[(4.4) \quad \delta_2^j (d_{1,2}^{i,j,l,s,j} v_{1,2}^j + d_{1,3}^{i,j,l,s,j} v_{1,3}^j + d_{2,3}^{i,j,l,s,j} v_{2,3}^j) = z_i s_1 q_1^j (e_i \cdot \mathbf{e}_j) - z_i y_j u_2^j,
\]

\[(4.5) \quad \delta_2^k (d_{1,2}^{i,j,l,s,k} v_{1,2}^k + d_{1,3}^{i,j,l,s,k} v_{1,3}^k + d_{2,3}^{i,j,l,s,k} v_{2,3}^k) = z_i s_1 q_1^k (e_i \cdot \mathbf{e}_j), \quad \text{for } k \neq i, j.
\]

We first prove (4.3). By the definitions of \( d_{1,2}^{i,j,l,s} \), \( d_{1,3}^{i,j,l,s} \) and \( d_{2,3}^{i,j,l,s} \), the left hand side is equal to

\[z_i \delta_2^i (d_{1,2}^{i,j,l,i} v_{1,2}^i + d_{1,3}^{i,j,l,i} v_{1,3}^i + d_{2,3}^{i,j,l,i} v_{2,3}^i)
\]

\[= z_i \delta_2^i (d_{1,2}^{i,j,l,i} v_{1,2}^i + d_{1,3}^{i,j,l,i} v_{1,3}^i + d_{2,3}^{i,j,l,i} v_{2,3}^i)
\]

\[= z_i (y_j u_1^i + z_i q_1^i (e_i \cdot \mathbf{e}_j)).\]
where the last equality was shown in the proof of (4.11) in product b. Now we prove (4.3). The left hand side is equal to

\[-d_{1,2}^{i,j,l,s} z_2 + d_{2,3}^{i,j,l,s} z_3 u_1^j + (d_{1,2}^{i,j,l,s} z_1 - d_{2,3}^{i,j,l,s} z_3) u_2^j + (d_{1,3}^{i,j,l,s} z_1 + d_{2,3}^{i,j,l,s} z_2) u_3^j,
\]

while the right side is equal to

\[z_i(z_s q_i^j (e_i \cdot F e_j) - y_i u_s^j) = z_i \left( z_s q_i^j \left( \sum_{r=1}^{m} \sigma_{i,j,r} p_{r,j}^k (T) f_r \right) - y_i u_s^j \right) = z_i \left( z_s \sum_{r=1}^{m} \sigma_{i,j,r} p_{r,j}^k (T) (c_{r,j,1} u_1^j + c_{r,j,2} u_2^j + c_{r,j,3} u_3^j) - y_i u_s^j \right).\]

To prove that the coefficients of \( u_1^j, u_2^j \) and \( u_3^j \) match, we assume that \( s = 1 \); the remaining cases are proved similarly. The coefficients of \( u_1^j \) and \( u_2^j \) match by the definition of the constants \( d_{1,2}^{i,j,l,s}, d_{1,3}^{i,j,l,s} \) and \( d_{2,3}^{i,j,l,s} \). Expanding \( y_i \) using (2.4), it follows that the coefficient of \( u_1^j \) in \( z_i z_s q_i^j (e_i \cdot F e_j) - y_i z_s u_s^j \) is

\[z_i z_s \sum_{r=1}^{m} \sigma_{i,j,r} p_{r,j}^k (T) c_{r,j,1} = z_i \sum_{r=1}^{m} \sigma_{i,j,r} p_{r,j}^k (T) c_{r,j,1} - z_i z_s \sum_{r=1}^{m} \sigma_{i,j,r} p_{r,j}^k (T) c_{r,j,1} - z_i z_s \sum_{r=1}^{m} \sigma_{i,j,r} p_{r,j}^k (T) c_{r,j,3} = -d_{1,2}^{i,j,l,1} z_2 - d_{1,3}^{i,j,l,1} z_3.
\]

Finally we prove (4.3). By the definitions of \( d_{1,2}^{k,i,j,l,s}, d_{1,3}^{k,i,j,l,s} \) and \( d_{2,3}^{k,i,j,l,s} \), the left hand side is equal to

\[d_{1,2}^k (z_s q_{1,2}^{k,i,j,l,s} v_1^k + z_s q_{1,3}^{k,i,j,l,s} v_1^k + z_s q_{2,3}^{k,i,j,l,s} v_2^k) = z_i z_s d_{1,2}^k (v_1^k + v_1^k + v_2^k) = z_i z_s q_{1,2}^k (e_i \cdot F e_j),
\]

where last equality was shown in the proof of product a.

\[e_i \cdot (F^t \otimes F_2) \rightarrow F_3 \oplus (\otimes_{k=1}^t G_k^t)
\]

Set

\[g_2^k := \left( \sum_{r=t+1}^{m} T_{r,j} d_{1,2}^{k,i,r} \right) v_1^k + \left( \sum_{r=t+1}^{m} T_{r,j} d_{1,3}^{k,i,r} \right) v_1^k + \left( \sum_{r=t+1}^{m} T_{r,j} d_{2,3}^{k,i,r} \right) v_2^k, \quad \text{with } 1 \leq k \leq t,
\]

\[g_{2,r}^k := \left( \sum_{l=1}^{3} c_{j,k,l} d_{1,2}^{r,k,i,l} \right) v_1^r + \left( \sum_{l=1}^{3} c_{j,k,l} d_{1,3}^{r,k,i,l} \right) v_1^r + \left( \sum_{l=1}^{3} c_{j,k,l} d_{2,3}^{r,k,i,l} \right) v_2^r, \quad \text{with } 1 \leq k, r \leq t.
\]

By (11) Theorem 3.2, it suffices to show that

\[(4.6) \quad e_i \cdot \left( d_{2}^k (f_j) - \sum_{k=1}^{t} q_1 (f_{j,k}) \right) = e_i \cdot F d_{2}^k (f_j) - \sum_{k=1}^{t} g_2^k - \sum_{k=1}^{t} \left( d_{1,3}^{k,i,j,l,s} e_i \cdot F e_i + \sum_{r=1}^{t} g_{2,r}^k \right),
\]

\[(4.7) \quad q_{2}^k (e_i \cdot F f_j) + g_2^k - \sum_{r=1}^{t} g_{2,r}^k = 0, \quad \text{with } k \neq j,
\]

\[(4.8) \quad q_{2}^k (e_i \cdot F f_j) + g_2^k - \sum_{r=1}^{t} g_{2,r}^j = d_{3}^j \left( \sum_{r=1}^{m} c_{r,j,3} d_{1,2}^{j,i,r} w^j \right).
\]
We first prove (4.6). The left hand side is equal to
\[
e_i \cdot \sum_{r=t+1}^m T_{r,j} e_r - e_i \cdot \sum_{k=1}^t \sum_{l=1}^3 c_{j,k,l} t_i^k
\]
\[
= \sum_{r=t+1}^m T_{r,j} e_i \cdot e_r + \sum_{r=t+1}^m T_{r,j} \left( d_{1,1}^{r,k,i} v_{1,1}^{r,k} + d_{1,1}^{r,k,i} v_{1,3} + d_{2,3}^{r,k,i} v_{2,3} \right)
\]
\[
- \sum_{k=1}^t \sum_{l=1}^3 c_{j,k,l} t_i^{k,i} v_{1,2} + \sum_{k=1}^t \sum_{l=1}^3 c_{j,k,l} d_{1,1}^{r,k,i,l} v_{1,2} + d_{1,3}^{r,k,i,l} v_{1,3} + d_{2,3}^{r,k,i,l} v_{2,3},
\]
while the right hand side is equal to
\[
e_i \cdot F \sum_{r=t+1}^m T_{r,j} e_r + \sum_{k=1}^t \sum_{l=1}^3 \left( c_{j,k,l} t_i^{k,i} v_{1,2} + d_{1,3}^{r,k,i,l} v_{1,3} + d_{2,3}^{r,k,i,l} v_{2,3} \right),
\]
which coincides with the left hand side. Now we prove (4.7). We show that the coefficient of \(v_{1,2}^k\) on the left hand side of (4.7) is zero; the remaining cases are similarly checked. The coefficient of \(v_{1,2}^k\) is
\[
\delta_{i,j} \sum_{r=1}^m \sum_{h=1}^m \sigma_{r,k,h} e_{r,k,h} c_{h,k,1} p_{r,k,h}(T) + \sum_{r=t+1}^m T_{r,j} d_{1,1}^{r,k,i} - \sum_{r=1}^t \sum_{l=1}^3 c_{j,r,l} d_{1,1}^{r,k,i,l}
\]
\[
= \delta_{i,j} \sum_{r=1}^m \sum_{h=1}^m \sigma_{r,k,h} e_{r,k,h} c_{h,k,1} p_{r,k,h}(T) + \sum_{r=t+1}^m T_{r,j} d_{1,1}^{r,k,i} - \sum_{r=1}^t \sum_{l=1}^3 c_{j,r,l} d_{1,1}^{r,k,i,l} - \sum_{l=1}^3 c_{j,k,l} d_{1,1}^{r,k,i,l}
\]
\[
= \delta_{i,j} \sum_{r=1}^m \sum_{h=1}^m \sigma_{r,k,h} e_{r,k,h} c_{h,k,1} p_{r,k,h}(T) + \sum_{r=t+1}^m T_{r,j} d_{1,1}^{r,k,i} - \sum_{r=1}^t \sum_{l=1}^3 c_{j,r,l} d_{1,1}^{r,k,i,l} - \sum_{l=1}^3 c_{j,k,l} d_{1,1}^{r,k,i,l}
\]
\[
= \delta_{i,j} \sum_{r=1}^m \sum_{h=1}^m \sigma_{r,k,h} e_{r,k,h} c_{h,k,1} p_{r,k,h}(T) + \sum_{r=t+1}^m T_{r,j} d_{1,1}^{r,k,i} - \sum_{r=1}^t \sum_{l=1}^3 c_{j,k,l} d_{1,1}^{r,k,i,l}
\]
\[
= \delta_{i,j} \sum_{r=1}^m \sum_{h=1}^m \sigma_{r,k,h} e_{r,k,h} c_{h,k,1} p_{r,k,h}(T) + \sum_{r=1}^m T_{r,j} d_{1,1}^{r,k,i} - \sum_{r=1}^t \sum_{l=1}^3 c_{j,k,l} d_{1,1}^{r,k,i,l}
\]
\[
- \sum_{r=1}^m \sigma_{k,i,r} p_{k,i,r}(T)c_{j,k,1} c_{r,k,2} + \sum_{r=1}^m \sigma_{k,i,r} p_{k,i,r}(T)c_{r,k,1} c_{j,k,2}
\]
\[
= \delta_{i,j} \sum_{r=1}^m \sum_{h=1}^m \sigma_{r,k,h} e_{r,k,h} c_{h,k,1} p_{r,k,h}(T) + \sum_{r=1}^m T_{r,j} d_{1,1}^{r,k,i} + \sum_{r=1}^m \sigma_{k,i,r} p_{k,i,r}(T)(c_{r,k,1} c_{j,k,2} - c_{j,k,1} c_{r,k,2})
\]
where the first equality follows from separating the case \( r = k \), the second equality follows from the definition of \( d^{k,i,r}_{1,2} \), the third equality follows from (3.1), the fourth equality follows from definition of \( k^{i,k,i,l}_{1,2} \), the fifth equality follows from combining summations, the sixth and seventh equalities follow from definition of \( d^{1,2}_{k} \), and the eighth equality follows from combining summations. First, we consider the case \( i \neq j \), in which the coefficient of \( u^{k}_{1,2} \) becomes

\[
\sum_{r=1}^{m} T_{r,j} d^{k,i,r}_{1,2} + \sum_{r=1}^{m} \sigma_{k,i,r} p f_{k,i,r}(T)(c_{r,k,1} c_{j,k,2} - c_{j,k,1} c_{r,k,2})
\]

\[
= \sum_{r=1}^{m} T_{r,j} \sum_{r=1}^{m} \sigma_{i,r,h} \sum_{s=1}^{m} \sigma_{i,r,h,s,k} p f_{i,r,h,s,k}(T) c_{s,k,1} c_{h,k,2} + \sum_{r=1}^{m} \sigma_{k,i,h} p f_{k,i,h}(T)(c_{h,k,1} c_{j,k,2} - c_{j,k,1} c_{h,k,2})
\]

\[
= \sum_{r=1}^{m} T_{r,j} \sum_{r=1}^{m} \sigma_{i,r,h} \sum_{s=1}^{m} \sigma_{i,r,h,s,k} p f_{i,r,h,s,k}(T) c_{s,k,1} c_{h,k,2} + \sum_{r=1}^{m} T_{r,j} \sigma_{i,j,r} \sum_{r=1}^{m} \sigma_{i,r,h,j,k} p f_{i,r,h,j,k}(T) c_{h,k,1} c_{j,k,2}
\]

\[
+ \sum_{h=1}^{m} \sigma_{k,i,h} \sum_{r=1}^{m} T_{r,j} \sigma_{i,k,h,r,j} p f_{k,i,h,r,j}(T)(c_{h,k,1} c_{j,k,2} - c_{j,k,1} c_{h,k,2})
\]

\[
= \sum_{r=1}^{m} T_{r,j} \sum_{r=1}^{m} \sigma_{i,r,h} \sum_{s=1}^{m} \sigma_{i,r,h,s,k} p f_{i,r,h,s,k}(T) c_{s,k,1} c_{h,k,2} + \sum_{r=1}^{m} T_{r,j} \sigma_{i,j,r} \sum_{r=1}^{m} \sigma_{i,r,h,j,k} p f_{i,r,h,j,k}(T) c_{h,k,1} c_{j,k,2}
\]

\[
- \sum_{h=1}^{m} \sigma_{k,i,h} \sum_{r=1}^{m} T_{r,j} \sigma_{i,k,h,r,j} p f_{k,i,h,r,j}(T)(c_{h,k,1} c_{j,k,2} - c_{j,k,1} c_{h,k,2})
\]

\[
= \sum_{r=1}^{m} T_{r,j} \sum_{r=1}^{m} \sigma_{i,r,h} \sum_{s=1}^{m} \sigma_{i,r,h,s,k} p f_{i,r,h,s,k}(T) c_{s,k,1} c_{h,k,2} + \sum_{r=1}^{m} T_{r,j} \sigma_{i,j,r} \sum_{r=1}^{m} \sigma_{i,r,h,j,k} p f_{i,r,h,j,k}(T) c_{h,k,1} c_{j,k,2}
\]

\[
+ \sum_{r=1}^{m} T_{r,j} \sum_{r=1}^{m} \sigma_{k,i,h} \sigma_{k,i,h,r,j} p f_{k,i,h,r,j}(T) c_{j,k,1} c_{h,k,2}
\]

\[
= \sum_{r=1}^{m} T_{r,j} \sum_{r=1}^{m} \sigma_{i,r,h} \sum_{s=1}^{m} \sigma_{i,r,h,s,k} p f_{i,r,h,s,k}(T) c_{s,k,1} c_{h,k,2} + \sum_{r=1}^{m} T_{r,j} \sigma_{i,j,r} \sum_{r=1}^{m} \sigma_{i,r,j,h,k} p f_{i,r,j,h,k}(T) c_{j,k,1} c_{h,k,2}
\]

\[
= \sum_{r=1}^{m} T_{r,j} \sum_{r=1}^{m} \sigma_{i,r,h} \sum_{s=1}^{m} \sigma_{i,r,h,s,k} p f_{i,r,h,s,k}(T) c_{s,k,1} c_{h,k,2} + \sum_{r=1}^{m} T_{r,j} \sigma_{i,j,r} \sum_{r=1}^{m} \sigma_{i,r,j,h,k} p f_{i,r,j,h,k}(T) c_{j,k,1} c_{h,k,2}
\]

\[
= \sum_{r=1}^{m} T_{r,j} \sum_{r=1}^{m} \sigma_{i,r,h} \sum_{s=1}^{m} \sigma_{i,r,h,s,k} p f_{i,r,h,s,k}(T) c_{s,k,1} c_{h,k,2} + \sum_{r=1}^{m} T_{r,j} \sigma_{i,j,r} \sum_{r=1}^{m} \sigma_{i,r,j,h,k} p f_{i,r,j,h,k}(T) c_{j,k,1} c_{h,k,2}
\]

\[
= \sum_{r=1}^{m} T_{r,j} \sum_{r=1}^{m} \sigma_{i,r,h} \sum_{s=1}^{m} \sigma_{i,r,h,s,k} p f_{i,r,h,s,k}(T) c_{s,k,1} c_{h,k,2} + \sum_{r=1}^{m} T_{r,j} \sigma_{i,j,r} \sum_{r=1}^{m} \sigma_{i,r,j,h,k} p f_{i,r,j,h,k}(T) c_{j,k,1} c_{h,k,2}
\]

\[
= \sum_{r=1}^{m} \sum_{r=1}^{m} \sigma_{r,h} c_{r,k,2} c_{h,k,1} p f_{r,k,s}(T)
\]

\[
= 0,
\]

where the first equality follows from the definition of \( d^{k,i,r}_{1,2} \) and reindexing, the second equality follows from separating the case \( h = j \), reindexing and using (2.6), the third equality changes \( T_{r,j} \) in the third term with \( -T_{r,j} \), the fourth equality follows from cancellation via (2.7) and (2.10), the fifth equality follows from separating the case \( s = j \), the sixth equality follows from cancellation via (2.7) and (2.10), the seventh equality follows from rearranging the summation, and the eighth equality follows from (2.11).

We move to consider the case \( i = j \), which forces \( k \neq j \), and the coefficient of \( u^{k}_{1,2} \) is

\[
\sum_{r=1}^{m} \sum_{r=1}^{m} \sigma_{r,h} c_{r,k,2} c_{h,k,1} p f_{r,k,s}(T) + \sum_{r=1}^{m} T_{r,j} d^{k,i,r}_{1,2} + \sum_{r=1}^{m} \sigma_{k,i,r} p f_{k,i,r}(T)(c_{r,k,1} c_{i,k,2} - c_{i,k,1} c_{r,k,2})
\]
\[
\sum_{r=1}^{m} \sum_{h=1}^{m} \sigma_{r,k,h} c_{r,k,2} c_{h,k,1} p_{r,k,h}(T) + \sum_{r=1}^{m} T_{r,i} \sum_{s=1}^{m} \sigma_{r,i,s} \sum_{h=1}^{m} \sigma_{i,r,s,h,k} p_{r,i,s,h,k}(T) c_{s,k,2} c_{h,k,1}
\]
\[+ \sum_{r=1}^{m} \sigma_{s,i,r} p_{s,i,r}(T) (c_{r,k,1} c_{i,k,2} - c_{i,k,1} c_{r,k,2}) \]
\[= \sum_{r=1}^{m} \sum_{h=1}^{m} \sigma_{r,k,h} c_{r,k,2} c_{h,k,1} p_{r,k,h}(T) + \sum_{h=1}^{m} \sigma_{i,r,s,h,k} p_{i,r,s,h,k}(T) c_{s,k,2} c_{h,k,1}
\]
\[+ \sum_{r=1}^{m} T_{r,i} \sum_{s=1}^{m} \sigma_{i,r,s} \sum_{h=1}^{m} \sigma_{i,r,s,h,k} p_{i,r,s,h,k}(T) c_{s,k,2} c_{h,k,1} + \sum_{h=1}^{m} \sigma_{i,r,s} p_{i,r,s}(T) (c_{r,k,1} c_{i,k,2} - c_{i,k,1} c_{r,k,2}) \]
\[= \sum_{r=1}^{m} \sum_{h=1}^{m} \sigma_{r,k,h} c_{r,k,2} c_{h,k,1} p_{r,k,h}(T) + \sum_{h=1}^{m} \sigma_{i,r,s,h,k} p_{i,r,s,h,k}(T) c_{s,k,2} c_{h,k,1}
\]
\[+ \sum_{r=1}^{m} T_{r,i} \sum_{s=1}^{m} \sigma_{i,r,s} \sum_{h=1}^{m} \sigma_{i,r,s,h,k} p_{i,r,s,h,k}(T) c_{s,k,2} c_{h,k,1} - \sum_{h=1}^{m} \sigma_{i,r,s} p_{i,r,s}(T) c_{r,k,2} c_{i,k,1} \]
\[= \sum_{r=1}^{m} \sum_{h=1}^{m} \sigma_{r,k,h} c_{r,k,2} c_{h,k,1} p_{r,k,h}(T) + \sum_{h=1}^{m} T_{r,i} \sum_{s=1}^{m} \sigma_{i,r,s} \sum_{h=1}^{m} \sigma_{i,r,s,h,k} p_{i,r,s,h,k}(T) c_{s,k,2} c_{h,k,1}
\]
\[+ \sum_{r=1}^{m} \sum_{h=1}^{m} \sigma_{r,k,h} c_{r,k,2} c_{h,k,1} p_{r,k,h}(T) + \sum_{h=1}^{m} \sigma_{i,r,s,h,k} p_{i,r,s,h,k}(T) c_{s,k,2} c_{h,k,1}
\]
\[= \sum_{r=1}^{m} \sum_{h=1}^{m} \sigma_{r,k,h} c_{r,k,2} c_{h,k,1} p_{r,k,h}(T) + \sum_{h=1}^{m} T_{r,i} \sum_{s=1}^{m} \sigma_{i,r,s} \sum_{h=1}^{m} \sigma_{i,r,s,h,k} p_{i,r,s,h,k}(T) c_{s,k,2} c_{h,k,1}
\]
\[\sum_{r=1}^{m} \sum_{h=1}^{m} \sigma_{r,k,h} c_{r,k,2} c_{h,k,1} p_{r,k,h}(T) + \sum_{h=1}^{m} \sigma_{i,r,s,h,k} p_{i,r,s,h,k}(T) c_{s,k,2} c_{h,k,1}
\]
\[= 0 \]

where the first equality follows from the definition of \(a_{k,i,r}^{j} \), the second equality follows from separating the case \(r = i \), the third equality follows from cancellation via (2.8), the fourth equality follows from separating the case \(h = i \), the fifth equality follows from cancellation via (2.6), the sixth equality follows from (2.6), the seventh equality follows from reindexing, the eighth equality changes \(T_{r,i} \) in the second term with \(-T_{i,r} \), and the ninth equality follows from cancellation via (2.7) and (2.10). Notice that we can take \(r, s, h \neq i \) in the second term of the eighth equality as the pfaffian will be zero otherwise.

Now we show (1.8). We show that the coefficient of \(v_{i,2}^{j} \) is the same on both sides of (1.8), the remaining cases are similarly checked. The coefficient of \(v_{i,2}^{j} \) on the left hand side of (1.8) is computed as in (1.7), and is equal to

\[
\delta_{i,j} \sum_{r=1}^{m} \sum_{h=1}^{m} \sigma_{r,j,h} c_{r,j,2} c_{h,j,1} p_{r,j,h}(T) + \sum_{r=1}^{m} T_{r,j} \delta_{i,2} + \sum_{r=1}^{m} \sigma_{j,i,r} p_{j,i,r}(T) (c_{r,j,1} c_{j,j,2} - c_{j,j,1} c_{r,j,2}) \]
\[= \delta_{i,j} \sum_{r=1}^{m} \sum_{h=1}^{m} \sigma_{r,j,h} c_{r,j,2} c_{h,j,1} p_{r,j,h}(T) + \sum_{r=1}^{m} T_{r,j} \delta_{i,2} \]

W
Notice that \(i \neq j\), since in (4.8), the element \(g_j^i\) is only defined for \(1 \leq j \leq t\) and we have \(t + 1 \leq i \leq m\). Hence the coefficient above is computed as in (4.7) and is equal to

\[
\sum_{s=1}^{m} c_{s,j,1} \sum_{h=1}^{m} c_{h,j,2} \sum_{r=1}^{m} \sigma_{t,r,h} \sigma_{t,r,h,s,j} T_{r,j} \text{pf}_{t,r,h,s,j}(T)
\]

\[= \sum_{s=1}^{m} c_{s,j,1} \sum_{h=1}^{m} c_{h,j,2} \sum_{r=1}^{m} \sigma_{t,r,h} \sigma_{t,r,h,s,j} \text{pf}_{t,r,h,s,j}(T) \sum_{l=1}^{3} c_{j,r,l} z_l
\]

\[= z_1 \sum_{s=1}^{m} c_{s,j,1} \sum_{h=1}^{m} c_{h,j,2} \sum_{r=1}^{m} c_{s,j,1} c_{j,r,1} \sigma_{t,r,s} \sigma_{t,r,s,h,j} \text{pf}_{t,r,s,h,j}(T)
\]

\[+ z_2 \sum_{s=1}^{m} c_{s,j,1} \sum_{h=1}^{m} c_{h,j,2} \sum_{r=1}^{m} c_{h,j,2} c_{j,r,2} \sigma_{t,r,s} \sigma_{t,r,s,h,j} \text{pf}_{t,r,s,h,j}(T)
\]

\[+ z_3 \sum_{s=1}^{m} c_{s,j,1} \sum_{h=1}^{m} c_{h,j,2} \sum_{r=1}^{m} c_{j,r,3} \sigma_{t,r,s} \sigma_{t,r,s,h,j} \text{pf}_{t,r,s,h,j}(T)
\]

\[= z_1 \sum_{s=1}^{m} c_{s,j,1} \sum_{h=1}^{m} c_{h,j,2} \sum_{r=1}^{m} \sigma_{t,r,h} \sigma_{t,r,s,h,j} \text{pf}_{t,r,h,s,j}(T) c_{s,j,1} c_{j,r,1} z_1
\]

\[= \sum_{r=1}^{m} c_{r,j,3} \sum_{h=1}^{m} c_{i,r,h} \sum_{s=1}^{m} \sigma_{t,r,h} \sigma_{t,r,h,s,j} \text{pf}_{t,r,h,s,j}(T) c_{s,j,1} c_{j,r,1} z_1
\]

where the first equality follows from (4.1), the second equality follows from rearranging the summations, the third equality follows from (2.7) and (2.10), the fourth equality follows from cancellation, the fifth equality follows from rearranging the summation, and the sixth equality follows from definition of \(d_{1,2}^{i,j}\). Hence the nonzero coefficients of \(v_{1,2}^1\), \(v_{1,3}^1\), and \(v_{2,3}^1\) in (4.8) are, respectively,

\[
\left( \sum_{r=1}^{m} c_{r,j,3} d_{1,2}^{i,j,r} \right) z_3, \quad \left( \sum_{h=1}^{m} c_{h,j,2} d_{1,3}^{i,j,h} \right) z_2, \quad \text{and} \quad \left( \sum_{s=1}^{m} c_{s,j,1} d_{2,3}^{i,j,s} \right) z_1.
\]

One can check that the quantities in the parentheses in the display above are the same.

\[f : F_1^t \otimes G_2^t \rightarrow F_3^t \oplus (\oplus_{k=1}^{t} G_3^k)\]

Set

\[g_k^2 = \left( z_0 d_{1,2}^{k,i,j,\beta} - z_0 d_{1,2}^{k,i,j,\alpha} \right) v_{1,2}^k + \left( z_0 d_{1,3}^{k,i,j,\beta} - z_0 d_{1,3}^{k,i,j,\alpha} \right) v_{1,3}^k + \left( z_0 d_{2,3}^{k,i,j,\beta} - z_0 d_{2,3}^{k,i,j,\alpha} \right) v_{2,3}^k, \quad 1 \leq k \leq t.\]
By [11] Theorem 3.2, it suffices to show that

\begin{equation}
\delta_i(e_j)v_{i,\alpha} - e_j \cdot \delta_2^t(v_{i,\alpha}) = y_jv_{i,\alpha} - \sum_{k=1}^{t} g_{2k}^k
\end{equation}

\begin{equation}
g_{2k}^k = 0, \text{ if } k \neq i.
\end{equation}

\begin{equation}
\delta_3 \left( (-1)^p \sum_{r=1}^{m} \sigma_{i,j,r} \text{pf}_{\tau_{i,j,r}}(T)_{cr_i} \cdot \sigma_{i,j,r}' \right) = y_jv_{i,\alpha} - g_{2i}.
\end{equation}

We first prove (4.9).

\[
y_jv_{i,\alpha} - e_j \cdot (z_\alpha v_\beta - z_\beta v_\alpha)
\]

\[
= y_jv_{i,\alpha} - z_\alpha z_\beta c_i \cdot \sigma \cdot e_j - z_\alpha \sum_{k=1}^{t} d_{1,2}^{k,i,j,\beta} v_{1,2} + d_{1,3}^{k,i,j,\beta} v_{1,3} + d_{2,3}^{k,i,j,\beta} v_{2,3}
\]

\[
+ z_\beta z_\alpha c_i \cdot \sigma \cdot e_j + z_\beta \sum_{k=1}^{t} d_{1,2}^{k,i,j,\alpha} v_{1,2} + d_{1,3}^{k,i,j,\alpha} v_{1,3} + d_{2,3}^{k,i,j,\alpha} v_{2,3}
\]

\[
= y_jv_{i,\alpha} - z_\alpha \sum_{k=1}^{t} d_{1,2}^{k,i,j,\beta} v_{1,2} + d_{1,3}^{k,i,j,\beta} v_{1,3} + d_{2,3}^{k,i,j,\beta} v_{2,3} + z_\beta \sum_{k=1}^{t} d_{1,2}^{k,i,j,\alpha} v_{1,2} + d_{1,3}^{k,i,j,\alpha} v_{1,3} + d_{2,3}^{k,i,j,\alpha} v_{2,3}
\]

\[
= y_jv_{i,\alpha} - \sum_{k=1}^{t} g_{2k}^k,
\]

where the first equality follows from applying product b, the second equality follows from cancellation, and the third equality follows from definition of $g_{2k}^k$.

For (4.10) we notice that if $k \neq i$, then by the definition of $d_{1,2}^{k,i,j,\beta}$, we have that $g_{2k}^k$ is equal to

\[
z_\alpha z_\beta \left( d_{1,2}^{k,i,j,\beta} v_{1,2} + d_{1,3}^{k,i,j,\beta} v_{1,3} + d_{2,3}^{k,i,j,\beta} v_{2,3} \right) - z_\alpha z_\beta \left( d_{1,2}^{k,i,j,\beta} v_{1,2} + d_{1,3}^{k,i,j,\beta} v_{1,3} + d_{2,3}^{k,i,j,\beta} v_{2,3} \right) = 0.
\]

Now we prove (4.11). We assume that $\alpha = 1$ and $\beta = 2$; the remaining two cases are proved similarly.

The left side is equal to

\[
- \sum_{r=1}^{m} \sigma_{i,j,r} \text{pf}_{\tau_{i,j,r}}(T)_{cr_i} \sum_{r=1}^{t} g_{2r}^r(u^r)
\]

\[
= - \sum_{r=1}^{m} \sigma_{i,j,r} \text{pf}_{\tau_{i,j,r}}(T)_{cr_i} (z_\alpha v_\beta - z_\beta v_\alpha)
\]

\[
= - z_3 \sum_{r=1}^{m} \sigma_{i,j,r} \text{pf}_{\tau_{i,j,r}}(T)_{cr_i} v_{1,2} + z_2 \sum_{r=1}^{m} \sigma_{i,j,r} \text{pf}_{\tau_{i,j,r}}(T)_{cr_i} v_{1,3} - z_1 \sum_{r=1}^{m} \sigma_{i,j,r} \text{pf}_{\tau_{i,j,r}}(T)_{cr_i} v_{2,3},
\]

while the right hand side is equal to

\[
(y_j + z_2 d_{1,2}^{i,j,1} v_{1,2} + z_2 d_{1,3}^{i,j,2} v_{1,3}) v_{1,2} + (z_2 d_{2,3}^{i,j,1} v_{1,3} + z_2 d_{2,3}^{i,j,2} v_{2,3}) v_{1,3} + (z_2 d_{1,2}^{i,j,1} v_{1,2} + z_2 d_{1,3}^{i,j,2} v_{1,3} + z_2 d_{2,3}^{i,j,2} v_{2,3}) v_{2,3}
\]

\[
= - z_1 \sum_{r=1}^{m} \sigma_{i,j,r} \text{pf}_{\tau_{i,j,r}}(T)_{cr_i} v_{1,2} - z_2 \sum_{r=1}^{m} \sigma_{i,j,r} \text{pf}_{\tau_{i,j,r}}(T)_{cr_i} v_{1,3} - z_3 \sum_{r=1}^{m} \sigma_{i,j,r} \text{pf}_{\tau_{i,j,r}}(T)_{cr_i} v_{2,3}
\]

\[
+ z_2 \sum_{r=1}^{m} \sigma_{i,j,r} \text{pf}_{\tau_{i,j,r}}(T)_{cr_i} v_{1,3} + z_3 \sum_{r=1}^{m} \sigma_{i,j,r} \text{pf}_{\tau_{i,j,r}}(T)_{cr_i} v_{1,2}
\]

\[
+ z_2 \sum_{r=1}^{m} \sigma_{i,j,r} \text{pf}_{\tau_{i,j,r}}(T)_{cr_i} v_{2,3} - z_1 \sum_{r=1}^{m} \sigma_{i,j,r} \text{pf}_{\tau_{i,j,r}}(T)_{cr_i} v_{1,3} - z_2 \sum_{r=1}^{m} \sigma_{i,j,r} \text{pf}_{\tau_{i,j,r}}(T)_{cr_i} v_{1,2}
\]

\[
- z_3 \sum_{r=1}^{m} \sigma_{i,j,r} \text{pf}_{\tau_{i,j,r}}(T)_{cr_i} v_{2,3}.
\]
where the first equality follows from the definition of $d_{1,3}^{i,j,2}$ and $d_{2,3}^{i,j,1}$, the second equality follows from (4.4) and from the definition of the constants $d_{1,3}^{i,j,1}$, and the third equality follows from cancellation.

\[ g_{i} \otimes G_{2} \rightarrow F_{3} \oplus (\oplus_{k=1}^{t} G_{k}^{i}) \]
This product follows immediately from [11, Theorem 3.2].

\[ h. G_{i} \otimes G_{2} \rightarrow F_{3} \oplus (\oplus_{k=1}^{t} G_{k}^{i}) \text{ with } i \neq j \]
For $1 \leq k \leq t$ set
\[
g_{2}^{k} = \left( z_{\alpha}d_{1,2}^{k,i,j,l,\alpha} - z_{\beta}d_{1,2}^{k,i,j,l,\beta} \right) v_{1,2}^{k} + \left( z_{\alpha}d_{1,3}^{k,i,j,l,\alpha} - z_{\beta}d_{1,3}^{k,i,j,l,\beta} \right) v_{1,3}^{k} + \left( z_{\alpha}d_{2,3}^{k,i,j,l,\alpha} - z_{\beta}d_{2,3}^{k,i,j,l,\beta} \right) v_{2,3}^{k}. \]
By [11, Theorem 3.2], it suffices to show that
\[
\delta_{j} \left( (1)^{p+1} z_{i} \sum_{r=1}^{m} \sigma_{i,j,r} p_{r} c_{r,i,p} w^{j} \right) = g_{2}^{j} + y_{i}z_{i}v_{\alpha,\beta}^{j}. \]
We first prove (4.12)
\[
\partial_{1}(u_{i}^{j})v_{\alpha,\beta}^{j} - u_{i}^{j} \cdot \partial_{2}(v_{\alpha,\beta}^{j}) = -y_{i}z_{i}v_{\alpha,\beta}^{j} - \sum_{k=1}^{t} g_{2}^{k}. \]

\[
(4.13)\quad g_{2}^{k} = 0 \text{ if } k \neq j; \]

\[
(4.14)\quad \delta_{j} \left( (1)^{p+1} z_{i} \sum_{r=1}^{m} \sigma_{i,j,r} p_{r} c_{r,i,p} w^{j} \right) = g_{2}^{j} + y_{i}z_{i}v_{\alpha,\beta}^{j}. \]

where the first equality follows from distribution, the second equality follows from applying product $d$, the third equality follows from cancellation, and the fourth equality follows from the definition of $g_{2}^{k}$. For (4.13) we notice that if $k \neq j$, then $g_{2}^{k}$ is equal to
\[
\left( z_{\alpha}z_{\beta}d_{1,2}^{k,i,j,l} - z_{\alpha}z_{\beta}d_{1,2}^{k,i,j,l} \right) v_{1,2}^{k} + \left( z_{\alpha}z_{\beta}d_{1,3}^{k,i,j,l} - z_{\alpha}z_{\beta}d_{1,3}^{k,i,j,l} \right) v_{1,3}^{k} + \left( z_{\alpha}z_{\beta}d_{2,3}^{k,i,j,l} - z_{\alpha}z_{\beta}d_{2,3}^{k,i,j,l} \right) v_{2,3}^{k} = 0. \]

The proof of (4.14) is similar to the proof of (4.11) and is therefore omitted.

\[ i. G_{1}^{i} \otimes F_{2} \rightarrow F_{3} \oplus (\oplus_{k=1}^{m} G_{3}^{i}) \]
Set
\[
g_{k}^{2} = \sum_{r=t+1}^{m} T_{r,j} \left( d_{1,2}^{k,i,r,l} v_{1,2}^{k} + d_{1,3}^{k,i,r,l} v_{1,3}^{k} + d_{2,3}^{k,i,r,l} v_{2,3}^{k} \right), \quad 1 \leq k \leq t, \]
\[
g_{k}^{h} = \sum_{s=1}^{3} c_{j,k,s} \left( d_{1,2}^{h,i,k,l,s} v_{1,2}^{h} + d_{1,3}^{h,i,k,l,s} v_{1,3}^{h} + d_{2,3}^{h,i,k,l,s} v_{2,3}^{h} \right), \quad 1 \leq k, h \leq t. \]
By [11] Theorem 3.2, it suffices to show that

\begin{equation}
(4.15) \quad \partial_1(u^i_l)f_j - u^i_l \cdot \partial_2(f_j) = -ziD_3(e_i \cdot f_j) - y_iu^i_l \cdot \mathcal{G}^i \ q^i_1(f_j) + g^k_2 + \sum_{k=1 \atop k \neq j}^{t} \left( g^k_2 + \sum_{h=1}^{t} g^{h,k}_2 \right).
\end{equation}

\begin{equation}
(4.16) \quad g^k_2 + \sum_{h=1 \atop h \neq 1}^{t} g^{h,k}_2 + ziq^i_2(e_i \cdot f_j) = 0.
\end{equation}

\begin{equation}
(4.17) \quad g^k_2 + \sum_{h=1 \atop h \neq 1}^{t} g^{h,k}_2 + ziq^i_2(e_i \cdot f_j) - y_iu^i_l \cdot \mathcal{G}^i \ q^i_1(f_j) = \delta_3^3 \left( (-1)^{t+1} \partial \Omega^3 d^{(i)} \Omega^j \right).
\end{equation}

We first prove (4.15).

\[
\partial_1(u^i_l)f_j - u^i_l \cdot \partial_2(f_j) \\
= -y_i z_i f_j - u^i_l \left( D^i_2(f_j) - \sum_{k=1}^{t} q^i_k(f_j) \right) \\
= -y_i z_i f_j + \sum_{r=t+1}^{m} T_{r,j} e_i \cdot e_r + u^i_l \sum_{k=1}^{t} \sum_{s=1}^{3} c_{j,k,s} u^k_s \\
= -y_i z_i f_j + \sum_{r=t+1}^{m} T_{r,j} z_2 e_i \cdot e_r + \sum_{r=t+1}^{m} T_{r,j} \sum_{k=1}^{t} \left( d_{1,2}^{k,i,l} v^k_{1,2} + d_{1,3}^{k,i,l} v^k_{1,3} + d_{2,3}^{k,i,l} v^k_{2,3} \right) \\
+ \sum_{k=1 \atop k \neq j}^{t} \sum_{s=1}^{3} c_{j,k,s} u^k_s + \sum_{s=1}^{3} c_{j,i,s} u^i_l \cdot u^i_s \\
= -y_i z_i f_j + \sum_{r=t+1}^{m} T_{r,j} z_2 e_i \cdot e_r + \sum_{r=t+1}^{m} T_{r,j} \sum_{k=1}^{t} \left( d_{1,2}^{k,i,l} v^k_{1,2} + d_{1,3}^{k,i,l} v^k_{1,3} + d_{2,3}^{k,i,l} v^k_{2,3} \right) \\
+ \sum_{k=1 \atop k \neq j}^{t} \sum_{s=1}^{3} c_{j,k,s} z_2 e_i \cdot e_k + \sum_{k=1 \atop k \neq j}^{t} \sum_{s=1}^{3} c_{j,k,s} \sum_{h=1}^{3} \left( d_{1,2}^{h,i,k,l,s} v^h_{1,2} + d_{1,3}^{h,i,k,l,s} v^h_{1,3} + d_{2,3}^{h,i,k,l,s} v^h_{2,3} \right) - \sum_{s=1}^{3} c_{j,i,s} y_i v^i_s \\
= -y_i z_i f_j + \sum_{r=t+1}^{m} T_{r,j} z_2 e_i \cdot e_r + \sum_{r=t+1}^{m} T_{r,j} \sum_{k=1}^{t} \left( d_{1,2}^{k,i,l} v^k_{1,2} + d_{1,3}^{k,i,l} v^k_{1,3} + d_{2,3}^{k,i,l} v^k_{2,3} \right) \\
+ \sum_{k=1 \atop k \neq j}^{t} \sum_{s=1}^{3} c_{j,k,s} \sum_{h=1}^{3} \left( d_{1,2}^{h,i,k,l,s} v^h_{1,2} + d_{1,3}^{h,i,k,l,s} v^h_{1,3} + d_{2,3}^{h,i,k,l,s} v^h_{2,3} \right) - \sum_{s=1}^{3} c_{j,i,s} y_i v^i_s \\
= z_i \sum_{r=1}^{m} T_{r,j} e_i \cdot e_r - y_i f_j \\
+ \sum_{r=t+1}^{m} T_{r,j} \sum_{k=1}^{t} \left( d_{1,2}^{k,i,l} v^k_{1,2} + d_{1,3}^{k,i,l} v^k_{1,3} + d_{2,3}^{k,i,l} v^k_{2,3} \right) \\
+ \sum_{k=1 \atop k \neq j}^{t} \sum_{s=1}^{3} c_{j,k,s} \sum_{h=1}^{3} \left( d_{1,2}^{h,i,k,l,s} v^h_{1,2} + d_{1,3}^{h,i,k,l,s} v^h_{1,3} + d_{2,3}^{h,i,k,l,s} v^h_{2,3} \right) - \sum_{s=1}^{3} c_{j,i,s} y_i v^i_s \\
= -z_i D_3(e_i \cdot f_j) + \sum_{k=1}^{t} g^k_2 + \sum_{k=1 \atop k \neq j}^{t} \sum_{h=1}^{t} g^{k,h}_2 - y_i u^i_l \cdot \mathcal{G}^i \ q^i_1(f_j) \\
= -z_i D_3(e_i \cdot f_j) - y_i u^i_l \cdot \mathcal{G}^i \ q^i_1(f_j) + g^k_2 + \sum_{k=1 \atop k \neq j}^{t} g^{k,h}_2 - g^k_2 - \sum_{h=1}^{t} g^{h,k}_2.
\]
where the first and second equalities follow from applying the differential maps, the third equality follows from applying product \( b \) and separating the case \( k = i \), the fourth equality follows from applying products \( c \) and \( d \), the fifth equality follows from reindexing and \((3.1)\), the sixth equality follows from factoring out a \( z_i \), the seventh equality follows from the definitions of \( D_{\alpha} \), \( y_{j,k} \), \( d_{k,h} \) and the product in the Koszul complex, and the eighth equality is obvious. Now we show \((4.16)\). Notice that if we fix \( \alpha < \beta \), then the coefficient of \( v_{\alpha,\beta}^k \) on the left hand side of \((4.16)\) is equal to

\[
\sum_{r=t+1}^{m} T_{r,j}d_{\alpha,\beta}^{k,i,r,l} + \sum_{h=1}^{3} l \sum_{s=1}^{t} c_{j,h,s}d_{\alpha,\beta}^{k,i,h,l,s} + \delta_{i,j}z_{i}d_{\alpha,\beta}^{k}
\]

\[
= z_{i} \sum_{r=t+1}^{m} T_{r,j}d_{\alpha,\beta}^{k,i,r} + \sum_{h=1}^{3} l \sum_{s=1}^{t} c_{j,h,s}d_{\alpha,\beta}^{k,i,h,l,s} + \delta_{i,j}z_{i}d_{\alpha,\beta}^{k}
\]

\[
= z_{i} \sum_{r=t+1}^{m} T_{r,j}d_{\alpha,\beta}^{k,i,r} + \sum_{h=1}^{3} l \sum_{s=1}^{t} c_{j,h,s}d_{\alpha,\beta}^{k,i,h,l,s} + \delta_{i,j}z_{i}d_{\alpha,\beta}^{k}
\]

\[
= z_{i} \sum_{r=t+1}^{m} T_{r,j}d_{\alpha,\beta}^{k,i,r} + \sum_{h=1}^{3} l \sum_{s=1}^{t} c_{j,h,s}d_{\alpha,\beta}^{k,i,h,l,s} + \delta_{i,j}z_{i}d_{\alpha,\beta}^{k}
\]

\[
= z_{i} \sum_{r=t+1}^{m} T_{r,j}d_{\alpha,\beta}^{k,i,r} + \sum_{h=1}^{3} l \sum_{s=1}^{t} c_{j,h,s}d_{\alpha,\beta}^{k,i,h,l,s} + \delta_{i,j}z_{i}d_{\alpha,\beta}^{k}
\]

\[
= z_{i} \cdot 0
\]

\[
= 0
\]

where the first equality follows from the definition of \( d_{\alpha,\beta}^{k,i,r,l} \) and separating the case \( h = k \), the second equality follows from definition of \( d_{\alpha,\beta}^{k,i,h,l,s} \) and expanding the third term, the third equality follows from \((3.1)\) and the definition of \( d_{\alpha,\beta}^{k,i,k,l,-} \), the fourth equality follows from combining summations, the fifth equality follows from factoring out \( z_i \), and the sixth equality follows from the computations in the proof of \((4.17)\). Finally we show \((4.17)\). We fix \( \alpha < \beta \) and we calculate the coefficient of \( v_{\alpha,\beta}^i \) on both sides of \((4.17)\). There are three cases to consider. First, if \( \{l, \alpha, \beta\} = \{1, 2, 3\} \), then the coefficient of \( v_{\alpha,\beta}^i \) on the left hand side of \((4.17)\) is

\[
\sum_{r=t+1}^{m} T_{r,j}d_{\alpha,\beta}^{i,i,r,l} + \sum_{h=1}^{3} l \sum_{s=1}^{t} c_{j,h,s}d_{\alpha,\beta}^{i,i,h,l,s} + \delta_{i,j}z_{i}d_{\alpha,\beta}^{i}
\]

as \( u_{i}^{i} \cdot G_{i} \) \( u_{i}^{i} \cdot G_{i} = u_{i}^{i} \cdot \alpha = v_{i,\alpha}^{i} \) and \( u_{i}^{i} \cdot G_{i} = u_{i}^{i} \cdot \beta = v_{i,\beta}^{i} \). Notice that since \( d_{\alpha,\beta}^{i,i,r,l} = 0 \) and \( d_{\alpha,\beta}^{i,i,h,l,s} = 0 \), the above display is equal to \( \delta_{i,j}z_{i}d_{\alpha,\beta}^{i} \), which coincides with the coefficient of \( v_{\alpha,\beta}^i \) on the right hand side of \((4.17)\). Now if \( l = \alpha \) and \( \{\alpha, \beta, \epsilon\} = \{1, 2, 3\} \), then the coefficient of \( v_{\alpha,\beta}^i \) in the left hand side of \((4.17)\) is

\[
= \sum_{r=t+1}^{m} T_{r,j}d_{\alpha,\beta}^{i,i,r,l} + \sum_{h=1}^{3} l \sum_{s=1}^{t} c_{j,h,s}d_{\alpha,\beta}^{i,i,h,l,s} + \delta_{i,j}z_{i}d_{\alpha,\beta}^{i} - y_{i}c_{j,i,\beta}
\]

\[
= \sum_{r=t+1}^{m} T_{r,j}d_{\alpha,\beta}^{i,i,r,l} + \sum_{h=1}^{3} l \sum_{s=1}^{t} c_{j,h,s}d_{\alpha,\beta}^{i,i,h,l,s} + \delta_{i,j}z_{i}d_{\alpha,\beta}^{i} - y_{i}c_{j,i,\beta}
\]

\[
= \sum_{r=t+1}^{m} T_{r,j}d_{\alpha,\beta}^{i,i,r,l} + \sum_{h=1}^{3} l \sum_{s=1}^{t} c_{j,h,s}d_{\alpha,\beta}^{i,i,h,l,s} + \delta_{i,j}z_{i}d_{\alpha,\beta}^{i} - y_{i}c_{j,i,\beta}
\]
where the first equality follows from definition of $d_{\alpha,\beta}^{i,i,h,l}$, the second equality follows from (3.1), the third equality follows from reindexing, the fourth equality follows from combining summations, the fifth equality follows from definition of $d_{\alpha,\beta}^{i,i,r,l}$, the sixth equality follows from separating the case $h = j$, the seventh equality follows since $T$ is skew-symmetric and from (2.8), the eighth equality follows from (3.3), and the ninth equality follows from cancellation. If $i \neq j$, then by (2.4),

$$
\sum_{r=1}^{m} T_{r,1} \sum_{h=1}^{m} \sigma_{i,r,h} \text{pf}_{t,r,h}(T)c_{h,i,\beta} = 0,
$$

showing (4.17) when $l = \alpha$ and $i \neq j$. If $i = j$, then we have

$$
\sum_{r=1}^{m} T_{r,i} \sum_{h=1}^{m} \sigma_{i,r,h} \text{pf}_{t,r,h}(T)c_{h,i,\beta} + z_\alpha d_{\alpha,\beta}^{i,i,i,i} = 0
$$

$$
= \sum_{r=1}^{m} T_{r,i} \sum_{h=1}^{m} \sigma_{i,r,h} \text{pf}_{t,r,h}(T)c_{h,i,\beta} + z_\alpha \sum_{h=1}^{m} \sum_{r=1}^{m} \sigma_{h,i,r} \text{pf}_{h,r,h}(T)c_{r,i,\alpha}c_{h,i,\beta}
$$

$$
= - \sum_{r=1}^{m} \sum_{s=1}^{m} c_{r,i,s} \sum_{h=1}^{m} \sigma_{i,r,h} \text{pf}_{t,r,h}(T)c_{h,i,\beta} + z_\alpha \sum_{h=1}^{m} \sum_{r=1}^{m} \sigma_{h,i,r} \text{pf}_{h,r,h}(T)c_{r,i,\alpha}c_{h,i,\beta}
$$

$$
= - z_\alpha \sum_{r=1}^{m} \sum_{h=1}^{m} \sigma_{i,r,h} \text{pf}_{t,r,h}(T)c_{r,i,\alpha}c_{h,i,\beta} - z_\beta \sum_{r=1}^{m} \sum_{h=1}^{m} \sigma_{i,r,h} \text{pf}_{t,r,h}(T)c_{r,i,\beta}c_{h,i,\beta}
$$

$$
- z_\gamma \sum_{r=1}^{m} \sum_{h=1}^{m} \sigma_{i,r,h} \text{pf}_{t,r,h}(T)c_{r,i,\gamma}c_{h,i,\beta} + z_\alpha \sum_{h=1}^{m} \sum_{r=1}^{m} \sigma_{h,i,r} \text{pf}_{h,r,h}(T)c_{r,i,\alpha}c_{h,i,\beta}
$$
\[
\begin{align*}
&= -z_\beta \sum_{r=1}^m \sum_{h=1}^m \sigma_{i,r,h} \text{pf}_{t,r,h} (T) c_{r,i,\beta} c_{h,i,\beta} - z_\epsilon \sum_{r=1}^m \sum_{h=1}^m \sigma_{i,r,h} \text{pf}_{t,r,h} (T) c_{r,i,\epsilon} c_{h,i,\beta} \\
&= -z_\epsilon \sum_{r=1}^m \sum_{h=1}^m \sigma_{i,r,h} \text{pf}_{t,r,h} (T) c_{r,i,\epsilon} c_{h,i,\beta} \\
&= \epsilon z_\epsilon \sum_{r=1}^m \sum_{h=1}^m \sigma_{r,i,h} \text{pf}_{t,r,h} (T) c_{r,i,\epsilon} c_{h,i,\beta} \\
&= \epsilon z_\epsilon d_{i,\epsilon}^h \\
\end{align*}
\]
where the first equality follows from definition of \( d_{i,\alpha}^h \), the second equality follows from \( \text{eq.} \), the third equality follows from separating the cases \( s = \alpha \), \( s = \beta \) and \( s = \epsilon \), the fourth equality follows from cancellation, the fifth equality follows from \( \text{eq.} \) and cancellation, the sixth equality follows from \( \text{eq.} \), and the seventh equality follows from definition of \( d_{i,\beta}^h \). The case \( l = \beta \) follows similarly and is therefore omitted.

Hence the coefficient of \( v_{i,\alpha}^h \) is \( (-1)^{i+1} \delta_{i,\beta} d_{i,\psi}^h \) when \( \{ \phi, \psi, l \} = \{ 1, 2, 3 \} \) and \( \phi < \psi \).

5. Applications

In this section we show that the conjectures in [7.4 Conjectures] on ideals of class \( G(r) \) hold true for ideals obtained by trimming the pfaffian generators of a Gorenstein ideal of grade 3, which is not a complete intersection, in a regular local ring of dimension 3. Moreover, it will also follow that \( r \) is always given by the lower bound provided in [6] (2.4 Theorem).

**Notation 5.1.** Let \( T \) be a skew-symmetric matrix of odd size \( m \) with entries in \( R \), and let \( t \) be an integer between 1 and \( m \). We denote by \( Q_t \) the transpose of the matrix \((q_1^t, \cdots, q_m^t)\) and by \( p(T, t) \) the number of pivot columns of \( Q_1 \otimes q_2 \& \cdots \& q_m \) among the last \( m - t \) columns. Moreover, we denote by \((C, \partial_e)\) the resolution given in Theorem 3.1 and by a bar the residue class modulo \( m \).

**Convention 5.2.** To make the results of this section easier to read, we will denote the classes \( H(0, 0) \) and \( H(0, 1) \) by \( G(0) \) and \( G(1) \), respectively. In particular, we say that ideals of class \( H(0, 0) \) and \( H(0, 1) \) are of class \( G \).

**Proposition 5.3.** Let \( I \) be a Gorenstein ideal of grade 3 generated by \(((-1)^{t+1} \text{pf}(T))_{t=1, \ldots, 5} \) for some skew-symmetric matrix \( T \) of size 5 with entries in \( R \). Let \( t \) be an integer between 1 and 5. \( J \) be the ideal obtained by trimming the first \( t \) generators of \( I \). Then \( J \) is of class \( G \) if and only if the \( 2 \times 2 \) minors of the matrix

\[
\begin{pmatrix}
\frac{\partial h}{k} & \frac{\partial h}{k_2} \\
\frac{\partial r}{k} & \frac{\partial r}{k_2}
\end{pmatrix}
\]

are zero for every \( i, j, k \) distinct with \( t + 1 \leq i, j \leq 5 \) and \( 1 \leq k \leq t \), \( \{ h, r \} = [5] \setminus \{ k, i, j \} \).

**Proof.** The only products of the ones listed in Theorem 3.1 that are possibly nonzero modulo \( m \) are products \( a \) and \( e \). Using the convention and the product tables in [2] the ideal \( J \) is of class \( G \) if and only if product \( a \) is zero. In product \( a \) the coefficient of \( e_{i,\beta}^h \) is

\[
\sigma_{i,j,r} \sigma_{i,j,r,h,k} \frac{\partial h}{k} \frac{\partial h}{k_2} c_{r,k,\beta} + \sigma_{i,j,h} \sigma_{i,j,h,r,k} \frac{\partial r}{k} \frac{\partial r}{k_2} c_{h,k,\beta},
\]

where \( h \) and \( r \) are defined as in the statement of the proposition. Using [2.7] and [2.10], the previous display is equal to

\[
\sigma_{i,j,r} \sigma_{i,j,r,h,k} \left( c_{h,k,\alpha} c_{r,k,\beta} - c_{r,k,\alpha} c_{h,k,\beta} \right).
\]

Hence the coefficient of \( v_{i,\alpha}^h \) is zero exactly when \( c_{h,k,\alpha} c_{r,k,\beta} - c_{r,k,\alpha} c_{h,k,\beta} = 0 \), which holds exactly when the condition in the statement of the proposition is satisfied.

**Lemma 5.4.** Let \( T \) be a skew-symmetric matrix of size 5 with entries in \( R \) and let \( t \) be an integer between 1 and 3. Consider the following two conditions:
(1) For every $i, j, k$ distinct with $t + 1 \leq i, j \leq 5$ and $1 \leq k \leq t$, set $\{r, h\} = \{5\} \setminus \{i, j\}$. The $2 \times 2$ minors of the matrix 
\[
\begin{pmatrix}
\sigma_{r,k,1} & \sigma_{r,k,2} & \sigma_{r,k,3} \\
\sigma_{i,k,1} & \sigma_{i,k,2} & \sigma_{i,k,3}
\end{pmatrix}
\]
are zero.

(2) Let $P$ be the subset of $\{5\}$ corresponding to the pivot columns of $T$. For every $i \in \{5\}$ and $k \in \{5\} \setminus P$ with $1 \leq k \leq t$ and $t + 1 \leq i \leq 5$, set $\{h, s, r\} = \{5\} \setminus \{i, k\}$. The determinant of the matrix 
\[
\begin{pmatrix}
\sigma_{r,k,1} & \sigma_{r,k,2} & \sigma_{r,k,3} \\
\sigma_{i,k,1} & \sigma_{i,k,2} & \sigma_{i,k,3}
\end{pmatrix}
\]
is zero.

Then condition (1) implies condition (2).

Proof. Since $t \leq 3$, we may choose $s \in \{5\} \setminus \{i, k\}$ such that $t + 1 \leq s \leq 5$. Expanding the determinant of the matrix in (2) along the second column immediately shows that (1) implies (2).

Lemma 5.5. Let $I$ be a Gorenstein ideal of grade 3 generated by $((-1)^{i+1}p_f(T))_{i=1,\ldots,m}$ for some skew-symmetric matrix $T$ of size 5 with entries in $R$. Let $t$ be an integer between 1 and 5. Let $J$ be the ideal obtained by trimming the first $t$ generators of $I$. Then $C_T C_2 \subseteq F_3$ if and only if condition (2) in Lemma 5.4 is satisfied.

Proof. Under the hypothesis $m = 5$, the only product $C_T C_2$ of the ones listed in Theorem 4.1 that is possibly nonzero modulo $m$ is product $c$. In product $c$, for $C_T C_2 \subseteq F_3$, we must have $\sum_{r=1}^{m} \sigma_{s,k,3} c_{r,k,r}^{p,v} = 0$ for all $t + 1 \leq i \leq m$ and $1 \leq k \leq t$ with $k \in \{5\} \setminus P$. Fixing $\{r, h, s\} = \{5\} \setminus \{i, k\}$, we can expand this summation as 
\[
\sigma_{i,k} = \sigma_{r,h} \sigma_{r,h,s,k} (\sigma_{s,k,1} c_{r,k,1} - \sigma_{r,k,1} c_{s,k,2}) \\
+ \sigma_{i,k} \sigma_{r,h} \sigma_{r,h,s,k} c_{h,k} \sigma_{s,k,1} c_{r,k,2} - \sigma_{r,k,1} c_{s,k,2} \\
+ \sigma_{s,k,1} (\sigma_{r,h} \sigma_{r,h,s,k} c_{h,k} \sigma_{s,k,1} c_{r,k,2}) \\
= - \sigma_{r,h} \sigma_{r,h,s,k} (\sigma_{s,k,1} c_{h,k,1} c_{r,k,2} - \sigma_{h,k,1} c_{s,k,2}) \\
+ \sigma_{s,k,1} (\sigma_{r,h} \sigma_{r,h,s,k} c_{h,k} \sigma_{s,k,1} c_{r,k,2}) \\
= - \sigma_{r,h} \sigma_{r,h,s,k} \det \begin{pmatrix}
\sigma_{h,k,1} & \sigma_{h,k,2} & \sigma_{h,k,3} \\
\sigma_{r,k,1} & \sigma_{r,k,2} & \sigma_{r,k,3}
\end{pmatrix},
\]
which shows that condition (2) in Lemma 5.4 is equivalent to $C_T C_2 \subseteq F_3$.

Theorem 5.6. Let $I$ be a Gorenstein ideal of grade 3 generated by $((-1)^{i+1}p_f(T))_{i=1,\ldots,m}$ for some skew-symmetric matrix $T$ of odd size $m \geq 5$ with entries in $R$. Let $t$ be an integer between 1 and $m$. Let $J$ be the ideal obtained by trimming the first $t$ generators of $I$. Then $J$ is an ideal of format 
\[
(1, m + 2t - \text{rank}(Q_1 \otimes_R k), m + 3t - \text{rank}(Q_1 \otimes_R k), 1 + t).
\]
Moreover,

(1) If $m = 5$, then $J$ is of class $G$ if and only if condition (1) in Lemma 5.4 is satisfied.

(2) If $m \geq 7$, then $J$ is of class $G$.

Furthermore, if $J$ is of class $G(r)$, then $r = m - t - p(T, t)$.
*Proof.* We first show that the format of $J$ is the one given by the formula above. A resolution of $R/J$ is given in Theorem 3.1. The ranks of the free modules in this resolution are $1, m + 2t, m + 3t,$ and $1 + t$ for degree $0, 1, 2,$ and $3,$ respectively. The only differential that may contain units is $\partial_2.$ The only submatrix of $\partial_2$ that may contain units is $-Q_1,$ therefore when passing to a minimal resolution, one needs to remove exactly rank($Q_1 \otimes_R k$) copies of $R$ from the domain and codomain of $\partial_2.$

Now we consider when $J$ is of class $G.$ If $m = 5,$ then Proposition 5.3 shows that $J$ is of class $G$ if and only if condition (1) in Lemma 5.4 is satisfied. If $m \geq 7,$ then the only product of the ones listed in Theorem 4.1 that is possibly nonzero modulo $m$ is product $e.$ Using Convention 5.2 and the product tables in 2.1, this forces $J$ to be of class $G.$

We now assume that $J$ is of class $G(r)$ and we calculate $r.$ Notice that the only possible nonzero product in homology is given by product $e.$

First we consider the case where $m = 5$ and $t = 5.$ In this case, $m - t - p(T, t) = 0$ and there are no products of type $e,$ so $J$ is of class $G(0).$

Now we show that in all other cases, $C_1 C_2 \subseteq F_3.$ In case where $m = 5$ and $t = 4,$ we assume by contradiction that condition (2) of Lemma 5.4 is not satisfied. Then there exists some $k \in [5] \setminus P$ with $1 \leq k \leq 4$ such that the determinant of the matrix

$$
\begin{pmatrix}
\ell_{h,k,1} & \ell_{s,k,1} & \ell_{r,k,1} \\
\ell_{h,k,2} & \ell_{s,k,2} & \ell_{r,k,2} \\
\ell_{h,k,3} & \ell_{s,k,3} & \ell_{r,k,3}
\end{pmatrix}
$$

is nonzero, where $P$ is the subset of $[5]$ corresponding to the pivot columns of $T$ and $\{h, s, r\} = [4] \setminus \{k\}.$ This shows that $Q_1 \otimes_R k$ has a pivot in the columns $h, s$ and $r$ coming from the block corresponding to $q_t^k.$ By the skew-symmetry of the matrix $T$ it follows that $Q_1 \otimes_R k$ also has a pivot in column $k,$ which contradicts the choice of $k.$ Hence condition (2) of Lemma 5.4 is satisfied and by Lemma 5.4 it follows that $C_1 C_2 \subseteq F_3.$

If $m = 5$ and $t \leq 3,$ then by Lemma 5.3 and Lemma 5.3 we also have $C_1 C_2 \subseteq F_3.$ Finally, if $m \geq 7,$ then $C_1 C_2 \subseteq F_3$ as well.

Hence we assume that $J$ is of class $G(r)$ and that $C_1 C_2 \subseteq F_3$ and show that $r = m - t - p(T, t).$ In order to split off the nonminimal part of $C_* \otimes_R k,$ we need to apply row and column operations to $\partial_2 \otimes_R k.$ Applying row operations changes the basis elements $u^k_i,$ which do not give contribution to the product, where $u^k_i$ is the basis element in homology corresponding to $u^k_i.$ Therefore we can assume that $\partial_2 \otimes_R k$ is in its reduced row echelon form. Now one needs to apply column operations so that every pivot is the only nonzero element in its row. We denote by $f_1, \ldots, f_m$ the basis elements in homology corresponding to $f_1, \ldots, f_m,$ and we set up a similar notation for the remaining basis elements of $\text{Tor}_R^R(R/J, k).$ The new basis obtained after applying these column operations is $f'_1, \ldots, f'_m,$ where $f'_i = f_i - \sum_{j \in X} f_{ji},$ where $X$ is a subset of the set of indices of pivot columns. After splitting off the nonminimal part of $C_* \otimes_R k,$ a basis for $\text{Tor}_R^R(R/J, k)$ in degree 2 is given by $\{e_{ij} \mid i \in [m] \setminus P \cup \{V^1_{1,2}, V^1_{2,3}, V^3_{2,3}\} \}_{i=1}^{m}$, where $P$ is the subset of $[m]$ corresponding to the indices of the pivot columns of $Q_1 \otimes_R k.$ Now we can compute the rank of the map

$$\eta : \text{Tor}_R^R(R/J, k) \to \text{Hom}_R(\text{Tor}_R^R(R/J, k), \text{Tor}_R^R(R/J, k)).$$

The only basis elements of $\text{Tor}_R^R(R/J, k)$ that may give a contribution to the rank are $\{e_{ij} \mid i \in [m] \setminus P, j > t\}.$ The only basis elements of $\text{Tor}_R^R(R/J, k)$ that have a nonzero product with the basis elements of $\text{Tor}_R^R(R/J, k)$ are $\{e_{ij} \mid i = t+1, \ldots, m\}.$ If $f'_j = f_j - \sum_{i \in X} f_{ji}$ with $j \in [m] \setminus P$ and $j > t,$ then $\eta(f'_j)$ is a matrix with only one nonzero row (the one corresponding to $g$). This nonzero row has 1 in position $j - t$ (coming from its product with $e_j$) and $-1$ in positions $i - t$ with $i \in X$ (coming from its product with $e_i$), moreover it has zeros on the right of the 1.

We can represent $\eta(f'_j)$ with a row vector given by its nonzero row and then as many zeros as are needed. To calculate the rank of $\eta$ it suffices to calculate the rank of the matrix whose rows are $\eta(f'_j).$ This matrix has zeroes on the right of the 1’s and it never has two 1’s in the same column, therefore it has as many pivots as 1’s. The number of 1’s is the cardinality of the basis $\{f'_j \mid i \in [m] \setminus P, j > t\},$ which is $m - t - p(T, t).$ \hfill \qed

Remark 5.7. We point out that, adopting the notation and hypotheses of Theorem 5.6 if the entries of $T$ are in $m^2,$ then the ideal $J$ is always of class $G(m - t).$ This also follows from [11 Corollary 4.12].

Remark 5.8. With Convention 5.2 in place, the conjectures given by Christensen, Veliche and Weyman in [7 7.4 Conjectures], together with [9 (2.4) Theorem] can be restated as follows for trimmed ideals:
If $I$ is a Gorenstein ideal of grade 3 that is not a complete intersection and $J$ an ideal of class $G(r)$ obtained from $I$ by trimming $t$ times, then

1. if $t = 1$, then $r = \mu(J) - 3$ where $\mu(J)$ denotes the minimal number of generators of $J$.
2. if $t \geq 2$, then $r \leq \mu(J) - 4$.

In the following corollary we prove that these conjectures hold when trimming the pfaffian generators of $I$.

**Corollary 5.9.** Let $I$ be a Gorenstein ideal of grade 3 generated by $((-1)^{i+1}\text{pf}_{i}(T))_{i=1,\ldots,m}$ for some skew-symmetric matrix $T$ of odd size $m \geq 5$ with entries in $R$. Let $J$ be the ideal obtained by trimming the first $t$ generators of $I$. If $J$ is of class $G(r)$, then

1. if $t = 1$, then $r = \mu(J) - 3$.
2. if $t \geq 2$, then $r \leq \mu(J) - 4$.

**Proof.** By Theorem 5.6, we have $\mu(J) = m + 2t - \text{rank}(Q_{1} \otimes_{R} k)$ and $r = m - t - p(T,t)$.

1. If $t = 1$, since $T_{1,1} = 0$, it follows by definition that $\text{rank}(Q_{1} \otimes_{R} k) = p(T,t)$. Therefore, $r = m - 1 - p(T,1) = (m + 2 - p(T,1)) - 3 = \mu(J) - 3$.

2. By definition it follows that $\text{rank}(Q_{1} \otimes_{R} k) - p(T,t) \leq t$.

If $t \geq 2$, then $t \leq 3t - 4$, therefore $\text{rank}(Q_{1} \otimes_{R} k) - p(T,t) \leq 3t - 4$,

which is equivalent to $m - t - p(T,t) \leq m + 2t - \text{rank}(Q_{1} \otimes_{R} k) - 4$;

hence $r \leq \mu(J) - 4$.

\[\square\]

**Remark 5.10.** We point out that in Corollary 5.9 one does not need to necessarily trim the first $t$ generators of $I$, since one can always reduce to this case by conjugating the matrix $T$ by a permutation matrix. In particular, if one wishes to trim generators $i_{1}, i_{2}, \ldots, i_{t}$, one conjugates $T$ by the matrix corresponding to the permutation

\[
\begin{pmatrix}
i_{1} & i_{2} & \cdots & i_{t} \\
1 & 2 & \cdots & t
\end{pmatrix}.
\]

Indeed, it is an easy linear algebra exercise to show that if $E$ is an elementary matrix corresponding to a swap operation, then the sequence of principal minors of the matrix $EAE$ is obtained by applying the same swap operation to the sequence of principal minors of $A$. Therefore, if $A$ is skew-symmetric then the sequence of pfaffians of $EAE$ is obtained by applying the same swap operation to the sequence of sub-maximal pfaffians of $A$, up to a sign. Now one uses that a permutation matrix is a product of elementary matrices corresponding to swap operations.

If one wishes to understand the format and class of an ideal obtained by trimming any $t$ pfaffian generators of a Gorenstein ideal of grade 3, then by the observation above one can reduce to the case of trimming the first $t$ generators and apply Theorem 5.6.

### 6. Realizability

In this section we address the realizability question, i.e. for what integers $\ell, n, r$ can one find an ideal of format $(1, \ell, \ell + n - 1, n)$ and of class $G(r)$? If the ideal is obtained from a Gorenstein ideal of grade 3 that is not a complete intersection by trimming $t$ pfaffian generators, then $n = t - 1$. Moreover, it follows from Theorem 5.6 and from the observation that

\[0 \leq \text{rank}(Q_{1} \otimes_{R} k) - p(T,t) \leq t,\]

that $2t \leq \ell - r \leq 3t$. In Corollary 5.9, the realizability of ideals with $\ell - r = 3t$, $\ell - r = 3t - 2$, and $\ell - r = 3t - 3$ was already shown. In this section we show that $\ell - r = 3t - 1$ is not possible and construct a family of examples realizing $\ell - r = 2t$. 

Proposition 6.1. Let \( J \) be an ideal obtained by trimming \( t \) pfaffian generators of a Gorenstein ideal of grade 3 that is not a complete intersection. If \( J \) is minimally generated by \( \ell \) elements and is of class \( \textbf{G}(r) \), then \( \ell - r \neq 3t - 1 \).

Proof. Without loss of generality, we can assume that \( J \) is obtained by trimming the first \( t \) pfaffian generators of a Gorenstein ideal of grade 3, see Remark 5.10. Seeking a contradiction, we assume that \( \ell - r = 3t - 1 \). By Theorem 5.6, it follows that

\[
\text{rank}(Q_1 \otimes_R k) - p(T, t) = 1,
\]

i.e. the matrix \( Q_1 \otimes_R k \) has exactly one pivot column among the first \( t \) columns. We argue that this is not possible. Recall that \( Q_1 \) is a block matrix with \( t \) blocks, see Notation 5.1. Let \( i \) be the pivot column among the first \( t \) columns. Let \( j \) be the the block containing the first nonzero entry in column \( i \). By the skew-symmetry of the matrix presenting the Gorenstein ideal, it follows that \( i \neq j \) and that there is a nonzero entry in column \( j \) and block \( i \) with zeroes to the left of it. This forces column \( j \) to be a pivot column, which is a contradiction.

\[ \square \]

6.2. Set \( R = k[x, y, z] \) and let \( s \) be a positive integer. We define the \( s \times s \) matrix \( U_s \) as the matrix

\[
U_s := \begin{pmatrix}
x & z & \cdots & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
x & z & \cdots & \cdots & \cdots & \cdots \\
z & y & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix},
\]

where the empty entries are 0. Let \( V_s \) be the \((2s + 1) \times (2s + 1)\) matrix given by

\[
V_s := \begin{pmatrix}
O & O_z & U_s \\
-(O_z)^T & 0 & y^2 O \\
-U_s & -(y^2 O)^T & O \\
\end{pmatrix}
\]

where \( O_z \) is an appropriately sized column matrix with an \( x \) at the bottom and zeroes elsewhere and \( y^2 O \) is an appropriately sized row matrix with \( y^2 \) in the leftmost entry and zeroes elsewhere. Let \( T_s \) be the \((4s + 3) \times (4s + 3)\) matrix given by

\[
T_s := \begin{pmatrix}
V_s & O_x & U_{2s+1} \\
-(O_x)^T & 0 & y^2 O \\
-U_{2s+1} & -(y^2 O)^T & O \\
\end{pmatrix}
\]

Let \( I_s \) be the ideal \(((−1)^{t+1} \text{pf}_s(T_s)))_{i=1,\ldots,4s+3} \), and let \( J_s \) be the ideal obtained by trimming the first \( 2s + 1 \) generators of \( I_s \).

Lemma 6.3. The ideals \( I_s \) are Gorenstein ideals of grade 3.

Proof. By [5] Theorem 2.1 it suffices to show that these ideals contain a regular sequence of length 3. We claim that \( \text{pf}(T_s), \text{pf}_{2s+1}(T_s) \) and \( \text{pf}_{4s+3}(T_s) \) form a regular sequence.

The matrix obtained by removing the 1st row and 1st column from \( T_s \) has determinant \( y^{8s+4} \), it follows that \( \text{pf}(T_s) = \pm y^{4s+2} \).

By [10] Theorem 2], the determinant of the matrix obtained by removing the \( 2s + 1 \)st row and \( 2s + 1 \)st column from \( T_s \) is \((\det U_{2s+1})^2\), and therefore \( \text{pf}_{2s+1}(T_s) = \pm \det U_{2s+1} \), which contains \( z^{2s+1} \) as a summand by a proof similar to [6] (3.2 Lemma).

By taking the cofactor expansion of the determinant of the matrix obtained from \( T_s \) by removing the \( 4s + 3 \)rd row and column, it is clear that this determinant has \( \pm x^{4s+2} \) as a summand. It follows that \( \text{pf}_{4s+3}(T_s) \) has \( \pm x^{2s+1} \) as a summand.

\[ \square \]

Proposition 6.4. The ideal \( J_s \) has format \((1, 6s + 2, 8s + 3, 2s + 2)\) and is of class \( \textbf{G}(2s) \).
Proof. An elementary analysis of the matrix $Q_1 \otimes_R k$ shows that the $x$’s and $-x$’s in the block $V_s$ correspond to $2s$ pivot columns among the first $2s + 1$ columns of $Q_1 \otimes_R k$. The only other pivot among the first $2s + 1$ columns corresponds to the $z$ in the first row of $V_s$. The remaining pivots of $Q_1 \otimes_R k$ correspond to the $z$ in the $s + 1$st row of $T_s$ and to the $x$ in the $2s + 1$st row of $T_s$. Therefore $\text{rank}(Q_1 \otimes_R k) = 2s + 3$ and $p(T_s, 2s + 1) = 2$. The result now follows from Theorem 5.6. \hfill \square

6.5. Set $R = k[[x, y, z]]$ and let $s$ be an integer larger than or equal to 2. The $s \times s$ matrix $U_s$ is defined as in [6.2]. Let $V'_s$ be the $(2s) \times (2s)$ matrix given by

$$V'_s := \begin{pmatrix} O & U_s \\ -U_s & O \end{pmatrix}.$$ 

Let $T'_s$ be the $(4s + 1) \times (4s + 1)$ matrix given by

$$T'_s := \begin{pmatrix} V'_s & O_x & U_{2s} \\ -(O_x)^T & 0 & y^2 O \\ -U_{2s} & -(y^2 O)^T & 0 \end{pmatrix},$$

where $O_x$ is an appropriately sized column matrix with an $x$ at the bottom and zeroes elsewhere and $y^2 O$ is an appropriately sized row matrix with $y^2$ in the leftmost entry and zeroes elsewhere. Let $I'_s$ be the ideal $((-1)^{s+1}p_{Q}(T'_s)))_{s=1, \ldots, 4s+1}$, and let $J'_s$ be the ideal obtained by trimming the first $2s$ generators of $I'_s$.

The proofs of the next two results are similar to the proofs of Lemma 6.6 and Proposition 6.7 and are therefore omitted.

**Lemma 6.6.** The ideals $I'_s$ are Gorenstein ideals of grade 3.

**Proposition 6.7.** The ideal $J'_s$ has format $(1, 6s - 1, 8s - 1, 2s + 1)$ and is of class $G(2s - 1)$.

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