Abstract

In this work we study orthogonal polynomials via polynomial mappings in the framework of the $H_q$-semiclassical class. We consider two monic orthogonal polynomial sequences $\{p_n(x)\}_{n\geq 0}$ and $\{q_n(x)\}_{n\geq 0}$ such that

$$p_{kn}(x) = q_n(x^k), \quad n = 0, 1, 2, \ldots,$$

where $k \geq 2$ is a fixed integer number, and we prove that if one of the sequences, $\{p_n(x)\}_{n\geq 0}$ or $\{q_n(x)\}_{n\geq 0}$, is $H_q$-semiclassical, then so is the other one. In particular, we show that if $\{p_n(x)\}_{n\geq 0}$ is $H_q$-semiclassical of class $s \leq k - 1$, then $\{q_n(x)\}_{n\geq 0}$ is $H_q^k$-classical. This fact allows us to recover and extend recent results in the framework of cubic transformations ($k = 3$). We also provide illustrative examples of $H_q$-semiclassical sequences of classes 1 and 2 involving little $q$-Laguerre and little $q$-Jacobi polynomials, including discrete measure representations for some of the considered examples.

Keywords Orthogonal polynomials · $q$-Polynomials · $H_q$-Semiclassical orthogonal polynomials · Polynomial mappings · $q$-Difference equations

Mathematics Subject Classification 42C05 · 33C45

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1 Introduction

Polynomial mappings constitute an interesting topic in the theory of orthogonal polynomials since their relations with Julia sets and almost periodic Jacobi matrices. Given a sequence of orthogonal polynomials with respect to a probability measure \( \mu \) supported on a set \( I \subseteq [-1, 1] \), polynomial mappings provide a general approach to the analysis of polynomials orthogonal with respect to a measure defined by a polynomial transformation such that the inverse of \( I \) is a real set that, in general, will be the union of a finite number of intervals such that any two of these intervals have at most one common point. These polynomials appear in the study of sieved orthogonal polynomials by using blocks of recurrence relations, see [5,6]. Applications of polynomial mappings in quantum chemistry, solid state physics, and generalized Fibonacci sequences can be found in [34], [30], and [29], respectively. A general framework is provided in [12] and the updated related works [10,28].

The case of quadratic mappings has been studied by many authors. In particular, in [7] the following problem is solved: Given a sequence of orthogonal polynomials \( \{p_n(x)\}_{n \geq 0} \) to find a symmetric sequence of orthogonal polynomials \( \{q_n(x)\}_{n \geq 0} \) such that \( q_{2n}(x) = p_n(x^2) \). In this case, \( q_{2n+1}(x) = K_n(x^2) \), where \( K_n(x) \) is the so-called kernel polynomial of degree \( n \) (see [8]). Later one, in [9] a quite general problem concerning the orthogonality of sequences of polynomials \( \{R_n(x)\}_{n \geq 0} \) defined by \( R_{2n}(x) = p_n(x^2) + \theta_{2n} x K_{n-1}(x^2) \), \( R_{2n+1}(x) = x K_n(x^2) + \theta_{2n+1} p_n(x^2) \), \( n \geq 0 \), is analyzed. Indeed, necessary and sufficient conditions for such an orthogonality are deduced. A more general situation is described in [23], where the study of general quadratic decompositions of sequences of monic orthogonal polynomials \( \{B_n(x)\}_{n \geq 0} \) such that \( B_{2n}(x) = p_n(x^2) + x a_{n-1}(x^2) \), \( B_{2n+1}(x) = x R_n(x^2) + b_n(x^2) \), \( n \geq 0 \), with \( p_n(x), R_n(x) \) polynomials of degree \( n \), and \( a_n(x), b_n(x) \) polynomials of degree at most \( n \), is analyzed. Necessary and sufficient conditions for the orthogonality of the sequences of polynomials \( \{p_n(x)\}_{n \geq 0} \) and \( \{R_n(x)\}_{n \geq 0} \) are given. This idea of quadratic decomposition in a more general framework is the topic presented in [25]. In [18,19], given a quadratic polynomial \( \pi_2(x) \) the orthogonality of sequences of monic polynomials \( \{B_n(x)\}_{n \geq 0} \) such that either \( B_{2n}(x) = p_n(\pi_2(x)) \) or \( B_{2n+1}(x) = (x-c)p_n(\pi_2(x)) \) is studied and the relation between the corresponding linear functionals is obtained. Finally, in [17] the general quadratic decomposition was studied, where instead of a decomposition expressed by \( x^2 \) the authors consider a general polynomial of degree two.

The study of the cubic case comes back to the pioneer work [2] where assuming that \( \{p_n(x)\}_{n \geq 0} \) is a symmetric sequence of monic orthogonal polynomials, then necessary and sufficient conditions for the orthogonality of a symmetric sequence of orthogonal polynomials \( \{B_n(x)\}_{n \geq 0} \) such that \( B_{3n}(x) = p_n(x^3 + bx) \) are given. Such constrains (symmetry and the particular choice of the cubic polynomials) have been removed in [20,21], where the authors consider the problem of orthogonality of sequences \( \{B_n(x)\}_{n \geq 0} \) such that \( B_{3n+m}(x) = \theta_m(x) p_n(\pi_3(x)) \), \( m \in \{0, 1, 2\} \), where \( \pi_3(x) \) is a fixed cubic polynomial and \( \theta_m(x) \) is a fixed polynomial of degree \( m \). The problem of general cubic decompositions has been studied in [26] following the hints of the quadratic case.
Nevertheless, questions related to cubic decompositions of orthogonal polynomial sequences satisfying some extra conditions as their semiclassical character have not been considered in the literature up to the recent contributions [31], [32], and [33] for particular cases of semiclassical and \( H_q \)-semiclassical orthogonal polynomials of class one and two. A more general framework concerning semiclassical orthogonal polynomials, including some particular polynomial mappings, is presented in [4].

The aim of the present contribution is to analyze sequences of monic orthogonal polynomials \( \{ p_n(x) \}_{n \geq 0} \) and \( \{ q_n(x) \}_{n \geq 0} \) such that \( p_{nk}(x) = q_n(x^k) \), \( k \geq 2 \), and to study how the \( H_q \)-semiclassical character of the sequences is preserved. The novelty of our results is related to the analysis of \( H_q \)-semiclassical orthogonal polynomials generated by such polynomial mappings by using the connection between the corresponding Stieltjes functions as a method to generate new examples of sequences of \( H_q \)-semiclassical orthogonal polynomials of class greater than or equal to 1, taking into account that the classification of such sequences, even for class 1, remains an open problem.

The structure of the manuscript is the following. In Sect. 2 we present the basic background concerning \( H_q \)-semiclassical linear functionals as well as some properties about sequences of orthogonal polynomials defined by polynomial mappings. In Sect. 3 we deal with the stability, i.e., the preservation of the semiclassical character, of \( H_q \)-semiclassical linear functionals when polynomial mappings are introduced. The key idea is the consideration of the formal Stieltjes series associated with both linear functionals. In particular, the case of the polynomial mapping \( \pi_k(x) = x^k \), \( k \geq 2 \) is studied and the class of the associated linear functional is discussed (Theorem 3.5). Finally, in Sect. 4 some illustrative examples of \( H_q \)-semiclassical sequences of orthogonal polynomials of classes 1 and 2 involving little \( q \)-Laguerre and little \( q \)-Jacobi polynomials are deeply studied, including discrete measure representations for some of the considered examples.

2 Background

In this section we recall some basic facts concerning the general theory of orthogonal polynomials (OP) that will be needed in the sequel. In particular, we review some basic definitions and operations in the space of all polynomials and its dual, including the definition of Hahn’s operator and the definition and some properties concerning semiclassical OP and functionals with respect to Hahn’s operator. Our main references concerning these topics are [8,15,24]. We also recall some results on OP and polynomial mappings recovered mainly from reference [10].

2.1 Basic definitions

We denote by \( P \) the vector space of polynomials with coefficients in \( \mathbb{C} \) and by \( P^* \) its dual space. The action of a functional \( \mathbf{u} \in P^* \) over a polynomial \( f \in P \) will be represented by \( \langle \mathbf{u}, f \rangle \). In particular, \( u_n := \langle \mathbf{u}, x^n \rangle \) is the moment of order \( n \) of \( \mathbf{u} \). In
we define the \( q \)-derivative of a functional \( u \) by
\[
\langle H_q u, f \rangle := -\langle u, H_q f \rangle,
\]
where \( H_q \) is the Hahn’s operator defined as
\[
(H_q f)(x) := \frac{f(qx) - f(x)}{(q - 1)x}, \quad f \in \mathcal{P}, \quad x \neq 0,
\]
being \( q \in \tilde{\mathbb{C}} := \{ z \in \mathbb{C} \mid z^n \neq 1, \quad n = 1, 2, \ldots \} \) \([13,15,16]\). In particular, this yields
\[
(H_q u)_n := \langle H_q u, x^n \rangle = -[n]_q u_{n-1}, \quad n \in \mathbb{N}_0,
\]
where \([n]_q\) denotes the nonsymmetric \( q \)-number, defined by
\[
[0]_q := 0, \quad [n]_q := 1 + q + \cdots + q^{n-1} = \frac{q^n - 1}{q - 1}, \quad n \in \mathbb{N}.
\]

Given \( u \in \mathcal{P}^* \) and \( \phi \in \mathcal{P} \), the dilation of \( u \) and the left multiplication of a polynomial \( \phi \) by \( u \), are the functionals \( h_d u, \phi u \in \mathcal{P}^* \) \((d \in \mathbb{C} \setminus \{0\})\) defined by
\[
\langle h_d u, f \rangle := \langle u, h_d(f) \rangle = \langle u, f(dx) \rangle, \quad \langle \phi u, f \rangle := \langle u, \phi f \rangle, \quad f \in \mathcal{P}.
\]

Let \( u \in \mathcal{P}^* \). A sequence \( \{p_n(x)\}_{n \geq 0} \) in \( \mathcal{P} \) is said to be an orthogonal polynomial sequence (OPS) with respect to \( u \) if the following two conditions hold:

(i) \( \deg p_n = n \) for each \( n \in \mathbb{N}_0 \);

(ii) \( \langle u, p_n p_m \rangle = k_n \delta_{n,m} \) for every \( m, n \in \mathbb{N}_0 \),

where \( \{k_n\}_{n \geq 0} \) is a sequence of nonzero complex numbers and \( \delta_{n,m} \) is the Kronecker symbol. We say that \( u \) is regular (or quasi-definite) if there exists an OPS with respect to \( u \) \((\text{cf. [8,24]}\)).

### 2.2 \( H_q \)-Semiclassical OPS

A linear functional \( u \in \mathcal{P}^* \) is called \( H_q \)-semiclassical if it is regular and there exist two nonzero polynomials \( \Phi \) and \( \Psi \) such that
\[
\deg \Psi \geq 1, \quad (2.1)
\]
and \( u \) satisfies the functional equation
\[
H_q(\Phi u) = \Psi u. \quad (2.2)
\]

If \( \{p_n(x)\}_{n \geq 0} \) is an OPS with respect to a \( H_q \)-semiclassical functional, then \( \{p_n(x)\}_{n \geq 0} \) is called a \( H_q \)-semiclassical OPS. We point out the following useful criterion \((\text{see, e.g., [11, Lemma 3.1, p.40]}\)).

\[\text{Springer}\]
Table 1 The pairs \( (\Phi, \Psi) \) for the canonical forms corresponding to the monic little \( q \)-Laguerre OPS and to the monic little \( q \)-Jacobi OPS, including the regularity conditions (RC) fulfilled by the associated linear functionals

| \( p_n(x) \) | \( p_n(x; a|q) \) | \( p_n(x; a, b|q) \) |
|----------------|-----------------|-----------------|
| \( \Phi \)    | \( x \)          | \( x(x - b^{-1}q^{-1}) \) |
| \( \Psi \)    | \( a^{-1}q^{-1}(q - 1)^{-1}(x - 1 + aq) \) | \( a^{-1}b^{-1}q^{-2}(q - 1)^{-1}((abq^2 - 1)x + 1 - aq) \) |
| RC            | \( a \neq 0; a \neq q^{-n-1}, n \geq 0 \) | \( ab \neq 0, q^{-n}; a, b \neq q^{-n-1}, n \geq 0 \) |

Proposition 1 Let \( u \in \mathcal{P}^* \) be regular. Then \( u \) is \( H_q \)-semiclassical if and only if there exist two polynomials \( \Phi \) and \( \Psi \), with at least one of them nonzero, such that (2.2) holds. Moreover, under these conditions, necessarily both \( \Phi \) and \( \Psi \) are nonzero and \( \Psi \) satisfies (2.1).

Let \( u \) be a \( H_q \)-semiclassical functional. We say that a pair \( (\Phi, \Psi) \) of nonzero polynomials is admissible (for \( u \)) if (2.1)–(2.2) holds. We denote by \( \mathcal{A}_u \) the set of all admissible pairs \( (\Phi, \Psi) \). The nonnegative integer number

\[
s := \min_{(\Phi, \Psi) \in \mathcal{A}_u} \max\{\deg \Phi - 2, \deg \Psi - 1\}
\]

is called the class of \( u \). The pair \( (\Phi, \Psi) \in \mathcal{A}_u \) where the class of \( u \) is attained is unique (see [15, Proposition 2.3]). If \( \{p_n\}_{n \geq 0} \) is an OPS with respect to a \( H_q \)-semiclassical functional of class \( s \), then \( \{p_n(x)\}_{n \geq 0} \) is called a \( H_q \)-semiclassical OPS of class \( s \). In particular, when \( s = 0 \) (so that \( \deg \Phi \leq 2 \) and \( \deg \Psi = 1 \) \( \{p_n(x)\}_{n \geq 0} \) is called a \( H_q \)-classical OPS.

Table 1 summarizes the two canonical forms for the pairs \( (\Phi, \Psi) \) corresponding to the \( H_q \)-classical OPS \( \{p_n(x; a|q)\}_{n \geq 0} \) and \( \{p_n(x; a, b|q)\}_{n \geq 0} \), known as the little \( q \)-Laguerre polynomials and the little \( q \)-Jacobi polynomials, respectively. We only need to include these two sequences of \( H_q \)-classical OPS because they are the only ones that will appear in the examples given at the last Section. For more details, see [1,14,16,27].

Among several very well-known characterizations of \( H_q \)-semiclassical OPS, we recall the following one [15, Proposition 3.1]: \( u \in \mathcal{P}^* \) is \( H_q \)-semiclassical if and only if it is regular and the associated Stieltjes formal series,

\[
S_u(z) := -\sum_{n=0}^{\infty} \frac{u_n}{z^{n+1}},
\]

satisfies (formally) the equation

\[
A(z) \left( H_{q^{-1}}S_u \right)(z) = C(z)S_u(z) + D(z),
\]
Table 2 The polynomials $A$, $C$, and $D$ appearing in Eq. (2.4) corresponding to the families in Table 1

| $p_n(x)$ | $p_n(x; a|q)$ | $p_n(x; a, b|q)$ |
|----------|----------------|------------------|
| $A$      | $x$            | $x(x - b^{-1})$  |
| $C$      | $qa^{-1}(q - 1)^{-1}(x - 1 + aq) - q$ | $qa^{-1}b^{-1}(q - 1)^{-1}((abq^2 - 1)x + 1 - aq)$ |
| $D$      | $u_0qa^{-1}(q - 1)^{-1}$ | $u_0(qa^{-1}b^{-1}(q - 1)^{-1}(abq^2 - 1) - q^2)$ |

where $A$, $C$ and $D$ are polynomials, $A$ being nonzero. Moreover, if $u$ satisfies (2.2), then the polynomials $A$, $C$ and $D$ in (2.4) are given in terms of the polynomials $\Phi$ and $\Psi$ as follows:

$$
A(z) = q^{\deg \Phi} \left( h_{q^{-1}} \Phi \right)(z), \quad C(z) = q^{\deg \Phi} \left( q \Psi(z) - (H_{q^{-1}} \Phi)(z) \right), \\
D(z) = q^{\deg \Phi} \left( q \left( u \Theta_0 \Psi \right)(z) - (H_{q^{-1}}(u \Theta_0 \Phi))(z) \right).
$$

where, for each $f \in \mathcal{P}$ and $u \in \mathcal{P}^*$, $\Theta_0 f$ and $uf$ are polynomials, defined by

$$
\Theta_0 f(x) := \frac{f(x) - f(0)}{x}, \quad uf(x) := \left\{ u_y, \frac{xf(x) - yf(y)}{x - y} \right\}.
$$

(The notation $u_y$ means that the functional $u$ acts on polynomials in the variable $y$.) Furthermore, if the polynomials $A$, $C$, and $D$ appearing in (2.4) are co-prime (i.e., there is no common zero to these three polynomials), then the class of $u$ is given by (see [15, Proposition 3.1])

$$
s = \max \{ \deg C - 1, \deg D \} \quad (2.5)
$$

and the polynomials $\Phi$ and $\Psi$ that appear in (2.2) are

$$
\Phi(z) = q^{-\deg A} (h_q A)(z) \quad \text{and} \quad \Psi(z) = q^{-\deg A} \left\{ (H_q A)(z) + q^{-1} C(z) \right\}. \quad (2.6)
$$

Table 2 gives the polynomials $A$, $C$ and $D$ appearing in the $q$-difference equation fulfilled by the formal Stieltjes series for the $H_q$-classical functionals ($s = 0$) corresponding to the little $q$-Laguerre OPS and to the little $q$-Jacobi OPS, given in Table 1.

2.3 OP via polynomial mappings

Concerning the study of polynomial mappings in the framework of the theory of OP, several works deal with the analysis of quadratic and cubic transformations (see, e.g., [2,7,18–22,31,32]). For a general polynomial mapping, the corresponding sequences of OP have been studied by Geronimo and Van Assche [12], Charris et al. [5,6], Peherstorfer [28], and de Jesus and Petronilho [10]. In order to describe this mapping,
let \( \{ p_n(x) \} \) be a monic OPS, characterized by its three-term recurrence relation, expressed in terms of blocks as

\[
(x - b_n^{(j)}) p_{nk+j}(x) = p_{nk+j+1}(x) + a_n^{(j)} p_{nk+j-1}(x)
\]

\( (j = 0, 1, \ldots, k - 1; \ n = 0, 1, 2, \ldots), \tag{2.7} \]

satisfying initial conditions \( p_{-1}(x) := 0 \) and \( p_0(x) := 1 \). Without loss of generality, we take \( a_0^{(0)} := 1 \). In general, the coefficients \( a_n^{(j)} \)'s and \( b_n^{(j)} \)'s are complex numbers with \( a_n^{(j)} \neq 0 \) for every \( n \) and \( j \). As a consequence, we can construct determinants \( \Delta_n(i, j; x) \), as introduced by Charris and Ismail [5], and by Charris et al. [6], so that

\[
\Delta_n(i, j; x) := \begin{cases} 
0 & \text{if } j < i - 2, \\
1 & \text{if } j = i - 2, \\
x - b_n^{(j-1)} & \text{if } j = i - 1,
\end{cases} \tag{2.8}
\]

and, if \( j \geq i \geq 1 \), \( \Delta_n(i, j; x) \) is a polynomials of degree \( j - i + 2 \) given by

\[
\Delta_n(i, j; x) := \begin{vmatrix} 
x - b_n^{(j-1)} & 1 & 0 & \ldots & 0 & 0 \\
a_n^{(i)} & x - b_n^{(i)} & 1 & \ldots & 0 & 0 \\
0 & a_n^{(i+1)} & x - b_n^{(i+1)} & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & x - b_n^{(j-1)} & 1 \\
0 & 0 & 0 & \ldots & a_n^{(j)} & x - b_n^{(j)}
\end{vmatrix} \tag{2.9}
\]

for every \( n \in \mathbb{N}_0 \). These determinants play a key role in the theory of OP via polynomial mappings. Taking into account that \( \Delta_n(i, j; x) \) is a polynomial whose degree may exceed \( k \), and since in (2.7) the coefficients \( a_n^{(j)} \)'s and \( b_n^{(j)} \)'s were defined only for \( 0 \leq j \leq k - 1 \), we adopt the convention

\[
b_n^{(k+j)} := b_n^{(j)}, \quad a_n^{(k+j)} := a_{n+1}^{(j)} \quad (i, j, n \in \mathbb{N}_0), \tag{2.10}
\]

and so the following useful equality holds:

\[
\Delta_n(k + i, k + j; x) = \Delta_{n+1}(i, j; x). \tag{2.11}
\]

**Theorem 1** [10, Theorem 2.1] Let \( \{ p_n(x) \} \) be a monic OPS characterized by the general blocks of recurrence relations (2.7). Fix \( r_0 \in \mathbb{C}, m \in \mathbb{N}_0, k \in \mathbb{N}, \) and \( k \geq 2, \) with \( 0 \leq m \leq k - 1 \). Then, there exist polynomials \( \pi_k(x) \) and \( \theta_m(x) \) of degrees \( k \) and \( m \), respectively, and a monic OPS \( \{ q_n(x) \} \) such that \( q_1(0) = -r_0 \) and

\[
p_{kn+m}(x) = \theta_m(x) q_n(\pi_k(x)), \quad n = 0, 1, 2, \ldots, \tag{2.12}
\]

if and only if the following conditions hold:
Lemma 1 \cite{4, Lemma 3.3} Under the conditions of Theorem 1, the formal Stieltjes series \( S_\mathbf{u}(z) := -\sum_{n=0}^{\infty} u_n z^{-n+1} \) and \( S_\mathbf{v}(z) := -\sum_{n=0}^{\infty} v_n z^{-n+1} \) associated with the regular moment linear functionals \( \mathbf{u} \) and \( \mathbf{v} \) with respect to which the sequences \( \{p_n(x)\}_{n \geq 0} \) and \( \{q_n(x)\}_{n \geq 0} \) are orthogonal (resp.) are related by

\[
S_\mathbf{u}(z) = \frac{\mu_0}{v_0} - v_0 \Delta_0(2, m - 1; z) + \left( \prod_{j=1}^{m} a_0^{(j)} \right) \eta_{k-1-m}(z) S_\mathbf{v}(\pi_k(z)) \theta_m(z) \quad (2.17)
\]
3 Polynomial mappings and $H_q$-semiclassical OP

For fixed $\pi \in \mathcal{P}$, let $\sigma_\pi : \mathcal{P} \to \mathcal{P}$ be the linear operator such that $\sigma_\pi[f] := f \circ \pi$ for every $f \in \mathcal{P}$, and define $\sigma_\pi^* : \mathcal{P}^* \to \mathcal{P}^*$ by duality. Henceforth,

$$\sigma_\pi[f](x) := f(\pi(x)), \quad \langle \sigma_\pi^*(u), f \rangle := \langle u, \sigma_\pi[f] \rangle, \quad f \in \mathcal{P}, \ u \in \mathcal{P}^*.$$

Lemma 2 For fixed $f \in \mathcal{P}$ and $u \in \mathcal{P}^*$, the following relations hold:

$$f \sigma_\pi^*(u) = \sigma_\pi^*[f]u, \quad (3.1)$$

$$H_q(\sigma_\pi^*[f])(x) = [k]_q x^{k-1} \sigma_\pi^*[H_qf](x), \quad (3.2)$$

$$\sigma_\pi^*(H_qu) = [k]_q H_q\sigma_\pi^*[x^{k-1}u]. \quad (3.3)$$

Proof Relation (3.1) was stated in [32, Lemma 2.1] for $k = 3$, being the proof similar for any positive integer number $k$. Relation (3.2) holds, since

$$H_q(\sigma_\pi^*[f])(x) = \frac{f(q^kx^k) - f(x^k)}{(q-1)x} = (H_qf)(x^k)[k]_q x^{k-1}$$

$$= [k]_q x^{k-1} \sigma_\pi^*[H_qf](x).$$

Finally, (3.3) follows from (3.2) taking into account the equality

$$\langle \sigma_\pi^*(H_qu), f \rangle = -\langle u, (H_qf) \rangle.$$

Lemma 3 [4, Lemma 3.2] Let $\{p_n(x)\}_{n \geq 0}$ and $\{q_n(x)\}_{n \geq 0}$ be two monic OPS satisfying

$$p_{nk}(x) = q_n(x^k), \quad n = 0, 1, 2, \ldots.$$

Let $u$ and $v$ be the regular functionals in $\mathcal{P}^*$ with respect to which $\{p_n(x)\}_{n \geq 0}$ and $\{q_n(x)\}_{n \geq 0}$ are orthogonal (resp.), and let $\{a_n\}_{n \geq 0}$ and $\{b_n\}_{n \geq 0}$ be the associated dual basis. Then the following relations hold

$$\sigma_\pi^*(a_{nk+j}) = \delta_{j,0}b_n \quad (j = 0, 1, \ldots, k - 1, \ n = 0, 1, 2, \ldots), \quad (3.4)$$

$$\sigma_\pi^*(p_{jk}u) = \delta_{j,0} v_0^{-1} u_0 v \quad (j = 0, 1, \ldots, k - 1). \quad (3.5)$$

Lemma 4 Let $\{p_n(x)\}_{n \geq 0}$ and $\{q_n(x)\}_{n \geq 0}$ be two monic OPS satisfying

$$p_{nk}(x) = q_n(x^k), \quad n = 0, 1, 2, \ldots.$$

(Hence one has $\pi_k(x) := x^k, \theta_n = 1, \text{ and } m = 0$ in Theorem 1, with conditions (i)–(iv) therein.) Let $u$ and $v$ be the regular functionals in $\mathcal{P}^*$ with respect to which $\{p_n(x)\}_{n \geq 0}$
and \( \{q_n(x)\}_{n \geq 0} \) are orthogonal (resp.). Then, the associated formal Stieltjes series \( S_u(z) \) and \( S_v(z) \) satisfy

\[
[k]^{-1}z^{k-1}\eta_{k-1}(q^{-1}z)(H_{q^{-k}}S_v)(z^k) = \frac{v_0}{u_0} (H_{q^{-1}}S_u)(z) - (H_{q^{-1}}\eta_{k-1})(z) S_v(z^k).
\] (3.6)

**Proof** By Lemma 1 we have

\[
S_u(z) = \frac{u_0}{v_0} \eta_{k-1}(z) S_v(z^k),
\] (3.7)

and hence

\[
\frac{v_0}{u_0} (H_{q^{-1}}S_u)(z) = \frac{v_0}{u_0} S_u(q^{-1}z) - S_u(z)
\]
\[
= \frac{\eta_{k-1}(q^{-1}z) S_v(q^{-k}z^k) - \eta_{k-1}(z) S_v(z^k)}{(q^{-1} - 1)z}
\]
\[
= \eta_{k-1}(q^{-1}z) \frac{S_v(q^{-k}z^k) - S_v(z^k)}{(q^{-1} - 1)z} [k]^{-1}z^{k-1} + S_v(z^k) \frac{\eta_{k-1}(q^{-1}z) - \eta_{k-1}(z)}{(q^{-1} - 1)z}
\]
\[
= \eta_{k-1}(q^{-1}z) (H_{q^{-1}}S_v)(z^k) [k]^{-1}z^{k-1} + S_v(z^k) (H_{q^{-1}}\eta_{k-1})(z).
\]

\( \Box \)

**Lemma 5** [4, Lemma 3.4] Let \( \phi \) be a polynomial and \( B_k := \{p_0, p_1, \ldots, p_{k-1}\} \) a finite set of monic OPs (i.e., \( \{p_0, p_1, \ldots, p_{k-1}\} \) is a linearly independent subset of \( \mathcal{P} \) such that \( \deg p_j = j \) for each \( j = 0, 1, \ldots, k-1 \)). Then, to the pair \( (\phi, B_k) \) we may associate \( k \) polynomials \( \phi_0, \phi_1, \ldots, \phi_{k-1} \), with \( \deg \phi_j \leq \lfloor (\deg \phi)/k \rfloor \) for each \( j = 0, 1, \ldots, k-1 \), such that

\[
\phi(x) = \sum_{j=0}^{k-1} p_j(x) \sigma_{x^k} \phi_j.
\] (3.8)

**Theorem 2** Let \( \{p_n(x)\}_{n \geq 0} \) and \( \{q_n(x)\}_{n \geq 0} \) be monic OPs satisfying

\[
p_{nk}(x) = q_n \left( x^k \right), \quad n = 0, 1, 2, \ldots.
\] (3.9)

Then the following hold:

(i) If \( \{p_n(x)\}_{n \geq 0} \) is \( H_q \)-semiclassical of class \( s \), then \( \{q_n(x)\}_{n \geq 0} \) is \( H_{q^k} \)-semiclassical of class \( \bar{s} \), with \( \bar{s} \leq \lfloor s/k \rfloor \).

(ii) If \( \{q_n(x)\}_{n \geq 0} \) is \( H_{q^k} \)-semiclassical of class \( \bar{s} \), then \( \{p_n(x)\}_{n \geq 0} \) is \( H_q \)-semiclassical of class \( s \), with \( s \leq (\bar{s} + 3)k - 3 \).

**Proof** Denote by \( u \) and \( v \) the regular linear functionals with respect to which \( \{p_n(x)\}_{n \geq 0} \) and \( \{q_n(x)\}_{n \geq 0} \) are OPs, respectively.
(i) Assume that \( \{ p_n(x) \}_{n \geq 0} \) is \( H_q \)-semiclassical of class \( s \). Then there exist two nonzero polynomials \( \Phi(x) \) and \( \Psi(x) \), with \( \deg \Psi(x) \geq 1 \), such that

\[
H_q(\Phi u) = \Psi u,
\]

being \( s = \max \{ \deg \Phi - 2, \deg \Psi - 1 \} \). Set \( \ell := 1 + \lfloor s/k \rfloor \) and \( p := \ell k - 1 - s \). Then \( p \in \mathbb{N}_0 \) and we deduce

\[
\sigma^{*}_{x_{k}}(x^p H_q(\Phi u)) = \begin{cases}
[k]_q H_q^k \left( \sigma^{*}_{x_{k}}(x^{k-1} \Phi u) \right), & \text{if } p = 0, \\
q^p[k]_q H_q^k \left( \sigma^{*}_{x_{k}}(x^{k+p-1} \Phi u) \right) - [p]_q \sigma^{*}_{x_{k}}(x^{p-1} \Phi u), & \text{if } p \geq 1.
\end{cases}
\]

In fact, assume that \( p \geq 1 \). Then for each \( f \in \mathcal{P} \) we have

\[
\left\{ \sigma^{*}_{x_{k}}(x^p H_q(\Phi u)), f \right\} = \left\{ H_q(\Phi u), x^p f(x^k) \right\}
\]

\[
= -\left\{ \Phi u, H_q \left( x^p f(x^k) \right) \right\} = -\left\{ \Phi u, \frac{q^p x^p f(q^k x^k) - x^p f(x^k)}{(q-1)x^k} \right\}
\]

\[
= -q^p[k]_q \left\{ \sigma^{*}_{x_{k}}(x^{k+p-1} \Phi u), H_q^k f(x^k) \right\} - [p]_q \left\{ x^{p-1} \Phi u, f \right\}
\]

\[
= q^p[k]_q \left\{ \sigma^{*}_{x_{k}}(x^{k+p-1} \Phi u), H_q^k f(x^k) \right\} - [p]_q \sigma^{*}_{x_{k}}(x^{p-1} \Phi u), f \right\}.
\]

This proves (3.11) for \( p \geq 1 \). The proof is similar for \( p = 0 \). It follows from (3.10) and (3.11) that

\[
H_q^k \left( \sigma^{*}_{x_{k}}(x^{k+p-1} \Phi u) \right) = \begin{cases}
[k]_q^{-1} \sigma^{*}_{x_{k}}(\Psi u), & \text{if } p = 0, \\
q^{-p}[k]_q^{-1} \sigma^{*}_{x_{k}}((x^p \Psi + [p]_q x^{p-1} \Phi) u), & \text{if } p \geq 1.
\end{cases}
\]

Next, Lemma 5 ensures the existence of polynomials \( f_j(x) (j = 0, 1, \ldots, k-1) \), with each \( f_j(x) \) not necessarily of degree \( j \), fulfilling

\[
x^{k+p-1} \Phi(x) = \sum_{j=0}^{k-1} p_j(x) \sigma_{x_k}[ f_j ].
\]

Applying the operator \( \sigma^{*}_{x_{k}} \) and using Lemma 3, we obtain

\[
\sigma^{*}_{x_{k}}(x^{k+p-1} \Phi u) = \sum_{j=0}^{k-1} \sigma^{*}_{x_{k}}(p_j \sigma_{x_k}[ f_j ] u) = v_0^{-1} u_0 f_0 v.
\]
Similarly, consider polynomials \( g_j(x) \) \((j = 0, 1, \ldots, k-1)\), with each \( g_j(x) \) not necessarily of degree \( j \), such that

\[
q^{-p}[k]^{-1}_q (x^p \Psi(x) + [p]_q x^{p-1} \Phi(x)) = \sum_{j=0}^{k-1} p_j(x) \sigma_{x^k}[g_j],
\]

(3.15)

and proceed as above to deduce

\[
\sigma_{x^k}^\ast \left( q^{-p}[k]^{-1}_q (x^p \Psi + [p]_q x^{p-1} \Phi)u \right) = v_0^{-1} u_0 g_0 v.
\]

(3.16)

From (3.12), (3.14), and (3.16), we obtain

\[
H_{q^k} (f_0 v) = g_0 v.
\]

(3.17)

Since \( s = \max \{ \deg \Phi - 2, \deg \Psi - 1 \} \), then either \( \deg \Phi = s + 2 \) and \( \deg \Psi \leq s + 1 \) or else \( \deg \Phi < s + 2 \) and \( \deg \Psi = s + 1 \). In the first case, the polynomial appearing in the left-hand side of (3.13) has degree \((\ell + 1)k\), and hence from the right-hand side of (3.13) we deduce \( \deg f_0 = \ell + 1 \geq 2 \). In the second case, the polynomial appearing in the left-hand side of (3.15) has degree \( \ell k \), and hence \( \deg g_0 = \ell \geq 1 \). We conclude that, in any situation, at least one of the polynomials \( f_0 \) or \( g_0 \) is different from zero. Thus, since \( v \) is regular and fulfills (3.17), it follows from Proposition 1 that \( v \) is \( H_{q^k} \)-semiclassical (being both \( f_0 \) and \( g_0 \) different from zero, and \( \deg g_0 \geq 1 \)). It remains to prove that the class \( \tilde{s} \) of \( v \) satisfies \( \tilde{s} \leq \lfloor s/k \rfloor \). Notice first that

\[
k \deg f_0 \leq \max_{0 \leq j \leq k-1} \{ j + k \deg f_j \} = \deg \left\{ x^{k+p-1} \Phi \right\} \leq (\ell + 1)k,
\]

where the equality follows from (3.13) and the last inequality holds since \( p = \ell k - 1 - s \) and \( \deg \Phi \leq s + 2 \), and hence \( \deg f_0 \leq \ell + 1 \). In the same way, using (3.15), we deduce

\[
k \deg g_0 \leq \deg \left\{ x^p \Psi + [p]_q x^{p-1} \Phi \right\} \leq \ell k,
\]

so \( \deg g_0 \leq \ell \). But, taking into account the conclusions of the discussion above involving the two possible cases (concerning the degrees of \( \Phi \) and \( \Psi \)), at least one of the equalities \( \deg f_0 = \ell + 1 \) or \( \deg g_0 = \ell \) holds. Therefore,

\[
\max \{ \deg f_0 - 2, \deg g_0 - 1 \} = \ell - 1 = \lfloor s/k \rfloor,
\]

(3.18)

and so from (3.17) we obtain \( \tilde{s} \leq \max \{ \deg f_0 - 2, \deg g_0 - 1 \} = \lfloor s/k \rfloor \).
(ii) Assume now that \( \{q_n(x)\}_{n \geq 0} \) is \( H_{q^k} \)-semiclassical of class \( \tilde{s} \). Then the associated formal Stieltjes series, \( S_v(z) := -\sum_{n=0}^{\infty} v_n z^{-n-1} \), satisfies the (formal) first-order linear \( q \)-difference equation

\[
\tilde{A}(z) (H_{q^{-k}} S_v)(z) = \tilde{C}(z) S_v(z) + \tilde{D}(z),
\]

(3.19)

with \( \tilde{A}, \tilde{C}, \) and \( \tilde{D} \) co-prime polynomials, \( \tilde{A} \) nonzero with \( \deg \tilde{A} \leq \tilde{s} + 2 \), and \( \tilde{s} = \max \{ \deg \tilde{C} - 1, \deg \tilde{D} \} \). Replacing in (3.19) by \( z^k \) and taking into account (3.6) and (3.7), we deduce that \( S_u(z) \) satisfies

\[
A(z) (H_{q^{-1}} S_u)(z) = C(z) S_u(z) + D(z),
\]

where

\[
A(z) := v_0 \eta_{k-1}(z) \tilde{A}(z^k),
\]

\[
C(z) := v_0 \left( [k]_{q^{-1}} z^{k-1} \eta_{k-1}(q^{-1}z) \tilde{C}(z^k) + (H_{q^{-1}} \eta_{k-1})(z) \tilde{A}(z^k) \right),
\]

\[
D(z) := u_0 [k]_{q^{-1}} z^{k-1} \eta_{k-1}(q^{-1}z) \eta_{k-1}(z) \tilde{D}(z^k).
\]

(3.20)

Thus \( u \) is a \( H_{q^k} \)-semiclassical functional. Let us prove that the class \( s \) of \( u \) satisfies \( s \leq (\tilde{s} + 3)k - 3 \). Indeed, we have

\[
\deg C \leq \max \{ k - 2 + k \deg \tilde{A}, 2(k - 1) + k \deg \tilde{C} \} \leq k(\tilde{s} + 3) - 2,
\]

\[
\deg D = 3(k - 1) + k \deg \tilde{D} \leq k(\tilde{s} + 3) - 3.
\]

Therefore, \( s \leq \max \{ \deg C - 1, \deg D \} \leq (\tilde{s} + 3)k - 3 \).

\( \square \)

**Remark 1** Theorem 2 has been stated in [3] for the particular case \( k = 2 \).

**Corollary 1** Let \( \{p_n(x)\}_{n \geq 0} \) and \( \{q_n(x)\}_{n \geq 0} \) be two monic OPS satisfying (3.9). If \( \{p_n(x)\}_{n \geq 0} \) is \( H_{q^k} \)-semiclassical of class \( s \leq k - 1 \), then \( \{q_n(x)\}_{n \geq 0} \) is \( H_{q^k} \)-classical.

**Proof** It is a straightforward consequence of part (i) in Theorem 2. \( \square \)

**Corollary 2** Let \( \{p_n\}_{n \geq 0} \) and \( \{q_n\}_{n \geq 0} \) be monic OPS satisfying (3.9), and let \( u \) and \( v \) be the corresponding regular functionals. If there exist nonzero polynomials \( \Phi \) and \( \Psi \) such that \( H_q(\Phi v) = \Psi u \), with \( \deg \Phi \geq 1 \), then

\[
H_q (\Phi u) = \Psi u,
\]

where \( \Phi \) and \( \Psi \) are polynomials given by

\[
\Phi(x) := q^{1-k} \eta_{k-1}(q x) \tilde{\Phi}(x^k),
\]

\[
\Psi(x) := q^{1-k} \left( [k]_{q^x} x^{k-1} \eta_{k-1}(q^{-1}x) \tilde{\Psi}(x^k) + \left( (H_q \eta_{k-1})(x) + q^{-1}(H_{q^{-1}} \eta_{k-1})(x) \right) \tilde{\Phi}(x^k) \right).
\]
Proof The statement follows immediately from (2.6) and (3.20) in the proof of Theorem 2, and taking into account the following relations:

\[
[k]_{q^{-1}} = [k]_{q} q^{1-k},
\]

\[
v_0^{-1}(H_q A)(x) = \eta_{k-1} (q x) (H_q x^{k} \tilde{A})(x^{k}) [k]_{q} x^{k-1} + \tilde{A}(x^{k}) (H_q \eta_{k-1})(x),
\]

\[
(qv_0)^{-1} C(x) = \eta_{k-1} (q^{-1} x) q^{-k} C(x^{k}) [k]_{q} x^{k-1} + q^{-1} \tilde{A}(x^{k}) (H_{q^{-1}} \eta_{k-1})(x).
\]

\[\square\]

4 Examples

In this section we give examples of \(H_q\)-semiclassical OPS of classes 1 and 2 obtained via cubic transformations. In particular, we confirm the results contained in the recent work [32] where the authors considered the problem of determining all the \(H_q\)-semiclassical monic OPS of class 1, \(\{p_n(x)\}_{n \geq 0}\), such that the cubic decomposition

\[p_{3n}(x) = q_n(x^3), \quad n = 0, 1, 2, \ldots, \quad (4.1)\]

holds, being \(\{q_n(x)\}_{n \geq 0}\) a monic OPS. In [32, Theorem 4.2] it was stated that property (4.1) is fulfilled only if \(\{q_n(x)\}_{n \geq 0}\) coincides with some specific family of \(H_q\)-3-classical OPS (which have been determined explicitly, all the possible families being special cases of little \(q^3\)-Laguerre and little \(q^3\)-Jacobi polynomials, up to affine changes of the variable). It is clear from Corollary 1 that, indeed, only \(H_q\)-3-classical OPS \(\{q_n(x)\}_{n \geq 0}\) may appear as solutions of such a problem. Moreover, we see immediately that considering the analog problem demanding \(\{p_n(x)\}_{n \geq 0}\) to be \(H_q\)-semiclassical of class 2, then again only \(H_q\)-3-classical OPS \(\{q_n(x)\}_{n \geq 0}\) may appear fulfilling such cubic transformation. Thus, in the next we present several examples involving as \(\{q_n(x)\}_{n \geq 0}\) the little \(q^3\)-Laguerre or the little \(q^3\)-Jacobi polynomials and, in each case, we give the corresponding families \(\{p_n(x)\}_{n \geq 0}\) which are semiclassical of class 1. Thus we recover the families presented in [32]) or of class 2 by giving new examples.

4.1 Description of the semiclassical families \(\{p_n\}_{n \geq 0}\) of class \(s \leq 2\)

We start by making the assumption that \(\{p_n(x)\}_{n \geq 0}\) is \(H_q\)-semiclassical of class \(s \leq 2\), and noticing that (4.1) corresponds to a polynomial mapping such that \(k = 3\) and \(m = 0\), being \(\pi_3(x) = x^3\) and \(\theta_0 \equiv 1\). Thus, by Corollary 1, \(\{q_n(x)\}_{n \geq 0}\) is a \(H_q\)-3-classical monic OPS. We assume that \(\{q_n(x)\}_{n \geq 0}\) is (up to an affine change of variables) one of the families of the little \(q^3\)-Laguerre or little \(q^3\)-Jacobi polynomials, described in Table 1. We analyze these two cases separately. Before performing this analysis, notice that according to the expression of \(\eta_{k-m-1} \equiv \eta_2\) given in Theorem 1, one has

\[\eta_2(x) = \Delta_0(2, 2) = x^2 - (b_0^{(1)} + b_0^{(2)}) x + b_0^{(1)} b_0^{(2)} - a_0^{(2)}. \quad (4.2)\]
On the other hand, by (2.13),
\[ x^3 = \pi_3(x) = (x - b_0^{(1)}) \eta_2(x) - a_0^{(1)}(x - b_0^{(2)}) + r_0. \] (4.3)
Therefore, setting
\[ \tau := b_0^{(0)}, \quad k_\tau := a_0^{(1)} + \tau^2, \] (4.4)
using (4.2) and (4.3), we may write
\[ \eta_2(x) = x^2 + \tau x + k_\tau = p_2^{(1)}(x). \] (4.5)

From now on we assume that \( \{q_n(x)\}_{n \geq 0} \) is \( H_q^3 \)-classical and coincides with one of the families of the little \( q^3 \)-Laguerre or little \( q^3 \)-Jacobi polynomials. Since \( \{p_n(x)\}_{n \geq 0} \) fulfills the cubic decomposition (4.1), it follows from the proof of Theorem 2 that the formal Stieltjes series \( S_\mathbf{u}(z) \) satisfies
\[ A(z) (H_{q^{-1}} S_\mathbf{u})(z) = C(z) S_\mathbf{u}(z) + D(z), \]
with
\[
\begin{align*}
A(z) & := v_0 \eta_2(z) \tilde{A}(z^3), \\
C(z) & := v_0 \left( [3]_{q^{-1}} \eta_2(q^{-1}z) \tilde{C}(z^3) + \tilde{A}(z^3) \right) (H_{q^{-1}} \eta_2)(z), \\
D(z) & := u_0 [3]_{q^{-1}} z^2 \eta_2(q^{-1}z) \eta_2(z) \tilde{D}(z^3),
\end{align*}
\] (4.6)
where the polynomials \( \tilde{A}(x), \tilde{C}(x), \) and \( \tilde{D}(x) \) are the polynomials \( A, C, \) and \( D \) appearing in Table 2 with \( q \) replaced by \( q^3 \).

In Tables 3 and 4 we describe all the possible monic OPS \( \{p_n(x)\}_{n \geq 0} \) \( H_q \)-semiclassical of classes 1 or 2 such that (4.1) holds, by giving the polynomials \( \Phi \) and \( \Psi \) appearing in the \( q \)-difference equation for each associated functional \( \mathbf{u} \), assuming that \( \{q_n(x)\}_{n \geq 0} \) is \( H_q^3 \)-classical and coincides with one of the families of the little \( q^3 \)-Laguerre or little \( q^3 \)-Jacobi polynomials. On the construction of such tables we have made use of MAPLE. As we see on these tables, there are 13 possible cases. The three cases corresponding to class \( s = 1 \) are the ones that have been obtained in [32, Theorem 4.2], up to affine change of the variables. For class \( s = 2 \) the examples given in cases (4)–(13) are new. Next we only give the details for obtaining the results in cases (1) and (4). The procedure for the remaining cases is similar.

Let \( \{q_n(x)\}_{n \geq 0} \) be the little \( q^3 \)-Laguerre monic OPS, i.e., let \( q_n(x) = p_n(x; a|q^3) \), for an arbitrary parameter \( a \). Write \( \eta_2(z) = (z - z_1)(z - z_2) \). Assume that \( z_1 \neq z_2 \). By (4.6), \( v_0 z^2 \) is a common factor of \( A, C, \) and \( D \). The division of these three polynomials by this factor gives (for simplicity, we still use \( A, C, \) and \( D \), although in fact these are the polynomials obtained by dividing the above ones by the common factor)
\[
\begin{align*}
A(z) & := z(z - z_1)(z - z_2), \\
C(z) & := [3]_{q^{-1}} (q^{-1}z - z_1)(q^{-1}z - z_2)\left( \ell(z^3 - 1 + aq^3) - q^3 \right) \\
& \quad + z((q^{-1} + 1)z - z_1 - z_2), \\
D(z) & := u_0 [3]_{q^{-1}} \ell(q^{-1}z - z_1)(q^{-1}z - z_2)(z - z_1)(z - z_2),
\end{align*}
\] (4.7)
Table 3  Description of all possible $H_q$-semiclassical OPS \( \{p_n(x)\}_{n \geq 0} \) of class \( s \leq 2 \) obtained via a cubic transformation such that \( p_{3n}(x) = q_n(x^3) \) for all \( n \geq 0 \), being \( \{q_n(x)\}_{n \geq 0} \) either the sequence of little $q^3$-Laguerre polynomials \( \{p_n(x; a|q^3)\}_{n \geq 0} \) or the sequence of little $q^3$-Jacobi polynomials \( \{p_n(x; a, b|q^3)\}_{n \geq 0} \).  

| Case | \( s \) | \( q_n(x) \) | \( b_0^{(0)} \) | \( a_0^{(1)} \) | Constraints |
|------|--------|--------------|--------------|--------------|-------------|
| (1)  | 1      | \( p_n(x; a|q^3) \) | \( \tau \)    | \(-\tau^2\)  | \( a = q^{-1}, \tau^3 = -1 \) |
| (2)  | \( p_n(x; a, b|q^3) \) | \( \tau \)    | \(-\tau^2\)  | \( a = q^{-1}, \tau^3 = -1 \) |
| (3)  | \( p_n(x; a, b|q^3) \) | \( \tau \)    | \(-\tau^2\)  | \( a = q^{-1}, \tau^3 \neq -1, b = -\tau^{-3}q^{-3}, \tau \neq 0 \) |
| (4)  | 2      | \( p_n(x; a|q^3) \) | \( \tau \)    | \(-\tau^2\)  | \( a = q^{-1}, \tau \neq 0, \tau^3 \neq -1 \) |
| (5)  | \( p_n(x; a|q^3) \) | \( \tau \)    | \(-\tau^2\)  | \( a \neq q^{-1}, \tau^3 = -1 \) |
| (6)  | \( p_n(x; a, b|q^3) \) | \( \tau \)    | \(-\tau^2\)  | \( \tau^3 = -(1 + q)^3 \) |
| (7)  | \( p_n(x; a, b|q^3) \) | \( \tau \)    | \(-\tau^2\)  | \( a \neq q^{-1}, \tau^3 = -1 \) |
| (8)  | \( p_n(x; a, b|q^3) \) | \( \tau \)    | \(-\tau^2\)  | \( a = q^{-1}, \tau^3 \neq -1, b \neq -\tau^{-3}q^{-3}, \tau \neq 0 \) |
| (9)  | \( p_n(x; a, b|q^3) \) | \( \tau \)    | \(-\tau^2\)  | \( a \neq q^{-1}, \tau^3 \neq -1, b = -\tau^{-3}q^{-3}, \tau \neq 0 \) |
| (10) | \( p_n(x; a, b|q^3) \) | \( \tau \)    | \(-\tau^2\)  | \( \tau^3 = -(1 + q)^3 \) |
| (11) | \( p_n(x; a, b|q^3) \) | \( \tau \)    | \(-\tau^2\)  | \( \tau^3 = -q^{-3}(1 + q)^3, b = 1 \) |
| (12) | \( p_n(x; a, b|q^3) \) | \( \tau \)    | \(-\tau^2\)  | \( \tau \neq 0, \tau^3 \neq -(1 + q)^3, b = -(1 + q)^3 \tau^{-3}q^{-6} \) |
| (13) | \( p_n(x; a, b|q^3) \) | \( \tau \)    | \(-\tau^2\)  | \( c \neq 0, c \neq -q^{-1}/(1 + q), c^2 + \tau c + \tau^2 \neq 0, (\tau + c)^3 = -1, b = c^{-3}q^{-3} \) |

This polynomial mapping depends on the choice of the parameters \( a, b, c \) and \( \tau \), which may be chosen arbitrarily in \( \mathbb{C} \) subject to the given constraints.
The polynomials $\Phi$ and $\Psi$ appearing in the canonical distributional $q$-difference equation $H_q(\Phi u) = \Psi u$ satisfied by the functional $u$ with respect to which $(p_n(x))_{n \geq 0}$ is an OPS, in accordance with each case described in Table 3.

| Case  | $\Phi$                                                                 | $\Psi$                                                                 |
|-------|------------------------------------------------------------------------|------------------------------------------------------------------------|
| (1)   | 1                                                                      | $q^{-1}(q-1)^{-1}x^2 + \tau x - \tau^2$                               |
| (2)   | $x^3 - q^{-3}b^{-1}$                                                   | $q^{-4}b^{-1}((q-1)^{-1}(q^4b-1))x^2 - \tau x + \tau^2$               |
| (3)   | $x^3 + q^{-1}(1-q)\tau x^2 - q^{-1}(1-q)\tau^2x + q^{-1}\tau^3$       | $(q-1)^{-1}(1-q^{-2}b^{-1})x^2 - q^{-1}\tau x + q^{-1}\tau^2$         |
| (4)   | $x + \tau q^{-1}$                                                     | $q^{-2}(q-1)^{-1}(x^3 + \tau qx^2 + q^2 - 1)$                         |
| (5)   | $x                                                                      | $q^{-3}a^{-1}(q-1)^{-1}x^3 + \tau x^2 - \tau^2x + q(q-1)^{-1}(aq^2 - 1)$ |
| (6)   | $x                                                                      | $q^{-3}a^{-1}(q-1)^{-1}x^3 + \tau x^2 - \tau^2(x + q(q-1)^{-1}(aq - 1q^{-1}))$ |
| (7)   | $x^4 - q^{-3}b^{-1}x$                                                  | $q^{-6}(q-1)^{-1}b^{-1}(aqb^6 - 1)x^3 + \tau(q^{-1})x^2 + \tau^2(q-1)x + q - aq^3$ |
| (8)   | $x^4 + q^{-1}\tau x^3 - q^{-3}b^{-1}x - q^{-4}b^{-1}\tau$             | $q^{-6}(q-1)^{-1}b^{-1}((aqb^6 - 1)x^3 + \tau(q^{-1})x^2 + \tau^2(q-1)x + q - aq^3)$ |
| (9)   | $x^4 + q^{-1}\tau(1-q)x^3 + \tau^2q^{-1}(q-1)x^2 + \tau^3q^{-1}x$    | $q^{-6}(q-1)^{-1}b^{-1}a^{-1}((aqb^6 - 1)x^3 + abq^{-1}(1-q)x^2 + abq^{-1}(q^{-1})x + abq^{-1}\tau^2(q-1)x + abq^{-1}\tau^3(q-1) + 1 - aq)$ |
| (10)  | $x^4 - q^{-3}b^{-1}x$                                                  | $a^{-1}b^{-1}q^{-6}((q-1)^{-1}(aqb^6 - 1)x^3 - \tau x^2 + (q + 1)^{-1}\tau^2x + q^2(q-1)^{-1}(1 - aq))$ |
| (11)  | $x^4 + \tau q^{-1}(q + 1)^{-1}(1-q)x^3$                               | $a^{-1}q^{-4}(q^{-2}(q-1)^{-1}(aqb^6 - 1)x^3 - \tau q^{-1}(q + 1)^{-1}(aq^4 + 1)x^2 + \tau^2(q + 1)^{-1}(aq^3 + 1)x + (q + 1)^{-3}(q-1)^{-1}a^{-1}(aq^3q^2(1) + (q + 1)^3(1 - aq)))$ |
| (12)  | $x^4 + \tau q^{-1}(1-q)x^3 + \tau^2(q + 1)^{-1}(q-1)x^2 + q^3(q + 1)^{-3}x$ | $a^{-1}(q-1)^{-1}(q + 1)^{-3}(\tau x^3 - a(q^3(1) + 3aq^2(1))x^2 - \tau q^{-1}x^2 + \tau^2(q + 1)^{-1}x + (q - 1)^{-3}\tau^3(a^2 - 1)$ |
| (13)  | $x^4 + c q^{-1}(q-1)x^3 + c^2 q^{-1}(q-1)x^2 - c^3 q^{-1}x$           | $a^{-1} b^{-1} q^{-6} ((q - 1)^{-1}(aqb^5 - 1)x^3 + (c(abq^5 - 1 - \tau)x^2 + (c^2(abq^5 + 1 + c)(\tau + 2c) - (q - 1)^{-1}(c^2 q^{-1}(q - 1) + q(a - 1)))$ |
where $\ell := a^{-1}(q^3 - 1)^{-1}q^3$. Now, we see that $C(0) = D(0) = 0$ if and only if $z_1z_2 = 0$. If $z_1 = 0$ (the reasoning hereafter is similar if $z_2 = 0$), then from (4.5) we obtain $z_2 = -\tau \neq 0$ (since we are assuming $z_1 \neq z_2$). Under such conditions, $z$ is a common factor of the polynomials $A, C$, and $D$ given by (4.7), and hence the division of these polynomials by $z$ yields

$$
A(z) := z(z + \tau), \\
C(z) := q^{-1}(q - 1)^{-1}a^{-1}(z^4 + \tau qz^3 + (aq^2 - 1)z + \tau q(aq - 1)), \\
D(z) := u_0(q - 1)^{-1}(z + \tau q)(z + \tau).
$$

Now, for these polynomials (4.8), we have $C(0) = D(0) = 0$ if and only if $a = q^{-1}$. Under such conditions, $z$ is a common factor of the polynomials $A, C$, and $D$ given by (4.8), and so dividing these polynomials by $z$ we obtain

$$
A(z) := z + \tau, \\
C(z) := (q - 1)^{-1}(z^3 + \tau qz^2 + q - 1), \\
D(z) := u_0(q - 1)^{-1}(z + \tau q)(z + \tau).
$$

Then $C(-\tau) = D(-\tau) = 0$ if and only if $\tau^3 = -1$. Under such conditions, $z + \tau$ is a common factor of $A, C$, and $D$ given by (4.9). Hence, dividing by $z + \tau$, we obtain

$$
A(z) := 1, \\
C(z) := (q - 1)^{-1}(z^2 - \tau(1-q)z + \tau^2(1-q)), \\
D(z) := u_0(q - 1)^{-1}(z + \tau q).
$$

We may conclude that if $z_1 = 0$, $z_2 = -\tau \neq 0$, $a = q^{-1}$, and $\tau^3 = -1$, then $u$ is semiclassical of class $s = 1$. Moreover, under these conditions, from (2.6) we obtain

$$
\Phi(x) = 1, \quad \Psi(x) = q^{-1}(q - 1)^{-1}x^2 + \tau x - \tau^2.
$$

This gives case (1) appearing in Table 3 and recovers the first solution presented in [32, Theorem 4.2]. If $z_1 = 0$, $z_2 = -\tau \neq 0$, $a = q^{-1}$, and $\tau^3 \neq -1$, then it is clear from (4.9) that $u$ is semiclassical of class $s = 2$, and from (2.6) we deduce

$$
\Phi(x) = x + \tau q^{-1}, \quad \Psi(x) = q^{-2}(q - 1)^{-1}(x^3 + \tau qx^2 + q^2 - 1).
$$

This gives case (4) appearing in Table 3. We also note that if $z_1 = z_2$, then using a similar reasoning we can show that the class of $u$ is greater than two.

### 4.2 Discrete measure representation

Next, we provide a discrete measure representation for the functional $u$ with respect to which $(p_n(x))_{n \geq 0}$ is a monic OPS (given by Tables 3 and 4, when $0 < q < 1$). These representations may be obtained using the following proposition, which generalizes [32, Lemma 4.3].
Lemma 6 Let \( \{p_n(x)\}_{n \geq 0} \) and \( \{q_n(x)\}_{n \geq 0} \) be monic OPS satisfying (4.1), and let \( u \) and \( v \) be the corresponding regular linear functionals in \( P^* \) (respectively). Assume further that \( v \) has a discrete measure representation

\[
v = \sum_{\ell=0}^{+\infty} a_\ell \delta_{\mu_\ell^3},
\]

being \( \{a_\ell\}_{\ell \geq 0} \) and \( \{\mu_\ell\}_{\ell \geq 0} \) sequences of complex numbers, with \( \mu_\ell \neq 0 \) for each \( \ell = 0, 1, 2, \ldots \), such that

\[
\left| \sum_{\ell=0}^{+\infty} a_\ell \mu_\ell^{n-2} \right| < +\infty, \quad \forall n \in N_\tau,
\]

where \( N_\tau := \mathbb{N} \) if \( k_\tau = 0 \), and \( N_\tau := \mathbb{N}_0 \) if \( k_\tau \neq 0 \), being \( k_\tau \) a constant defined by (4.4). Then \( u \) has the discrete measure representation

\[
uu = u_0 \sum_{\ell=0}^{+\infty} a_\ell \delta_{\mu_\ell^3},
\]

where \( j = e^{2\pi i/3} \) and \( \eta_2 \) is the polynomial given by (4.5).

Proof By Lemma 5, for each polynomial \( f(x) \), there are polynomials \( f_0(x) \), \( f_1(x) \), and \( f_2(x) \) such that the decomposition

\[
f(x) = f_0(x^3) + p_1(x)f_1(x^3) + p_2(x)f_2(x^3)
\]

holds. Therefore, by Lemma 3 we have

\[
\langle u, f \rangle = v_0^{-1}u_0 \langle v, f_0 \rangle.
\]

Since \( \eta_2(x) = x^2 + \tau x + k_\tau \) and \( 1 + j + j^2 = 0 \), we compute \( \eta_2(\mu_\ell) + j\eta_2(j\mu_\ell) + j^2\eta_2(j^2\mu_\ell) = 3\mu_\ell^2 \), and hence

\[
\sum_{p=0}^{2} \langle j^p \eta_2(j^p \mu_\ell) \delta_{j^p \mu_\ell}, f_0(x^3) \rangle = 3\mu_\ell^2 f_0(\mu_\ell^3).
\]

Similarly, using \( p_1(x) = x - \tau \) and \( p_2(x) = x^2 - (b_0^{(0)} + b_0^{(1)})x + b_0^{(0)}b_0^{(1)} - a_0^{(1)} \), together with (4.4), we show that

\[
\sum_{p=0}^{2} \langle j^p \eta_2(j^p \mu_\ell) \delta_{j^p \mu_\ell}, p_1(x)f_1(x^3) \rangle = 0
\]
and
\[
\sum_{p=0}^{2} \left\{ j^p \eta_2 \left( j^p \mu_\ell \right) \delta_{j^p \mu_\ell}, p_2(x) f_2(x^3) \right\} = 0. \tag{4.18}
\]
Thus (4.13) follows from (4.14)–(4.18), taking into account (4.11) and (4.12).

Remark 2 If \( k_\tau := a_0^{(1)} + \tau^2 = 0 \), then we recover [32, Lemma 4.3].

Lemma 6 may be applied to give a discrete measure representation of the functional \( u \) in each case quoted in Tables 3 and 4. We will present two illustrative examples. The first one recovers the first discrete measure representation for the functional \( u \) given in [32, Theorem 4.4], which corresponds to the case described in line (1) appearing in Tables 3 and 4. Using the data in line (1) of Table 3, we have \( \eta_2(x) = x(x + \tau) \) and \( k_\tau = 0 \). From the discrete representation of the functional \( \mathcal{L}(a, q) \) given in [32, Appendix 1], it is clear that conditions (4.11) and (4.12) are fulfilled for \( 0 < q < 1 \), where

\[
a_\ell := (q^2, q^3)^{\infty} \frac{q^{2\ell}}{(q^3, q^3)_\ell}, \quad \mu_\ell := q^\ell, \quad \ell = 0, 1, 2, \ldots.
\]

Here \((a, q)_n\) denotes the \( q \)-analogue of the Pochhammer symbol, defined by

\[
(a, q)_0 := 1, \quad (a, q)_n := \prod_{j=0}^{n-1} (1 - aq^j), \quad n = 1, 2, \ldots,
\]

and

\[
(a, q)^{\infty} := \prod_{n=0}^{\infty} (1 - aq^n).
\]

Therefore, by (4.13), we deduce

\[
u = \frac{\mu_0}{3} (q^2, q^3)^{\infty} \sum_{\ell=0}^{\infty} \frac{q^{\ell}}{(q^3, q^3)_\ell} \left( (q^\ell + \tau) \delta_{q^\ell} + (q^\ell + j^2 \tau) \delta_{j^2 q^{\ell}} \right),
\]

recovering (up to an affine change of variables) the first solution presented in [32, Theorem 4.4]. Discrete measure representation for functionals \( u \) corresponding to other lines in Tables 3 and 4 may be obtained by a similar process. For instance, for the functional \( u \) of class \( s = 2 \) described in line (13), we have \( \eta_2(x) = x^2 + \tau x - c(c + \tau) \neq 0 \). Thus, for the discrete representation of the functional \( \mathcal{U}(a, b, q) \) given in [32, Appendix 1], it is clear that conditions (4.11) and (4.12) are fulfilled for \( 0 < q < 1 \) and \( 0 < a < q^{-1} \), where

\[
a_\ell := \frac{(aq^3, q^3)^{\infty}}{(ac^{-3}q^3, q^3)^{\infty}} \frac{(c^{-3}, q^3)^{\ell}}{(q^3, q^3)_\ell} (aq^3)^{\ell}, \quad \mu_\ell := q^\ell, \quad \ell = 0, 1, 2, \ldots.
\]
Hence, using Lemma 6, we deduce
\[
    u = \frac{u_0}{3} \frac{(aq^3, q^3)_{\infty}}{(ac^{-3}q^3, q^3)_{\infty}} \sum_{\ell=0}^{+\infty} \left( \frac{c^{-3}, q^3}{q^3, q^3} \right)_\ell (aq^3)^{\ell} \left( (q^{2\ell} + \tau q^\ell - c(c + \tau)) \delta_{q^\ell} + (q^{2\ell} + j^2 \tau q^\ell - j^2 c(c + \tau)) \delta_{j^2 q^\ell} \right)
\]

4.3 Further remarks

We conclude this work with some remarks concerning the analysis presented here. In the previous section, we started from the knowledge of the monic OPS \( \{q_n(x)\}_{n \geq 0} \), and we found the corresponding monic OPS \( \{p_n(x)\}_{n \geq 0} \) satisfying the cubic transformation (4.1), requiring \( \{p_n(x)\}_{n \geq 0} \) to be \( H_q \)-semiclassical of class at most 2. We point out that, conversely, starting from a given \( H_q \)-semiclassical monic OPS \( \{p_n(x)\}_{n \geq 0} \) of class at most 2—e.g., as described by Tables 3 and 4, we can find the corresponding monic OPS \( \{q_n(x)\}_{n \geq 0} \) fulfilling (4.1), provided we know \textit{a priori} that such a cubic transformation exists. We will illustrate this procedure considering the monic OPS \( \{p_n(x)\}_{n \geq 0} \) described in case (13) appearing in Tables 3 and 4 (and so, we already know that a cubic transformation exists). In this case, from Table 3, we have \( b_0(0) = r \) and \( a_0(1) = (c^2 + \tau c + \tau^2) \). Since the polynomials \( \Phi \) and \( \Psi \) satisfy the q-difference equation \( H_q(\Phi u) = \Psi u \), then \( \langle u, \Psi \rangle = 0 \). On the other hand, we also have \( p_1(x) = x - \tau, p_2(x) = \Delta_0(1, 1; x) = x^2 - (b_0(1) + \tau)x + \tau b_0(1) + c^2 + \tau c + \tau^2 \), and \( p_3(x) = \pi_3(x) - r_0 = c^3 - r_0. \) Therefore, from the system of four equations \( \langle u, \Psi \rangle = \langle u, p_j \rangle = 0 \) (for \( j = 1, 2, 3 \)), we deduce, after some computations, that
\[
    r_0 = c^3(c^3 - aq^3)^{-1}(1 - aq^3).
\]

Furthermore, using (4.2), (4.3), and (4.5), we find
\[
    b_0(1) = -\tau - b_0(2), \quad b_0(2) = \tau + \frac{\tau^3 - r_0}{a_0(1)}, \quad a_0(2) = b_0(1)a_0(2) - a_0(1) - \tau^2. \tag{4.19}
\]

Replacing in (4.19) the above expression for \( r_0 \) and taking into account the constraints appearing in case (13) of Table 3, we obtain
\[
    b_0(1) = c + \frac{aq^3(c^3 - 1)}{(c^3 - aq^3)(c^2 + \tau c + \tau^2)}, \quad b_0(2) = c + \frac{c^3(1 - c^3)}{(c^3 - aq^3)(c^2 + \tau c + \tau^2)}, \quad a_0(2) = \frac{c^3 q^3 a(1 - c^3)}{(c^3 - aq^3)^2(c^2 + \tau c + \tau^2)^2}.
\]

On the other hand, using again the data in case (13) of Table 4, we may write
\[
    x^2 \Phi(x) = x^6 + A_1 x^5 + A_2 x^4 + A_3 x^3, \quad \Psi(x) = B_1 x^3 + B_2 x^2 + B_3 x + B_4,
\]
where the coefficients $A_i$ and $B_i$ are

$$
A_1 := c(1 - q^{-1}), \quad A_2 := c^2(1 - q^{-1}), \quad A_3 := -c^3 q^{-1},
$$
$$
B_1 := a^{-1} q^{-3}(q - 1)(aq^3 - c^3), \quad B_2 := a^{-1} q^{-3}c(aq^2 - \tau c^2 - c^3),
$$
$$
B_3 := a^{-1} q^{-3}c^2(aq^2 + \tau c^2 + 2\tau c^2 + c^3), \quad B_4 := a^{-1} q^{-2}(q - 1)^{-1}c^3(aq - 1).
$$

The above computations yield

$$
x^2 \Phi(x) = p_0(x)(x^6 + (A_1(\tau^2 + a_0^{(1)}) + A_2 \tau + A_3)x^3)
+ p_1(x)(A_2 + (\tau + b_0^{(1)})A_1)x^3 + p_2(x)A_1x^3,
$$
$$
\Psi(x) = p_0(x)(B_1 x^3 + B_2(\tau^2 + a_0^{(1)}) + B_3 \tau + B_4)
+ p_1(x)(B_3 + (\tau + b_0^{(1)})B_2) + p_2(x)B_2.
$$

Therefore, using (3.13), (3.15) and (3.17) with $k = 3$ and $m = p = 0$ (we notice that while proving (3.13) and (3.15) we considered $p \geq 1$, but by direct inspection we see that the formulas also hold for $p = 0$), we conclude that

$$
f_0(x^3) = x^6 + (A_1(\tau^2 + a_0^{(1)}) + A_2 \tau + A_3)x^3 = x^3(x^3 - c^3),
$$
$$
g_0(x^3) = [3]_q^{-1}(B_1 x^3 + B_2(\tau^2 + a_0^{(1)}) + B_3 \tau + B_4)
= q^{-3}a^{-1}(q^3 - 1)^{-1}(aq^3 - c^3)x^3 + c^3(1 - aq^3)).
$$

Hence the functional $\nu$ fulfills

$$
H_q^3(x(x - c^3) \nu) = \left[q^{-3}a^{-1}(q^3 - 1)^{-1}(aq^3 - c^3)x + c^3(1 - aq^3)\right] \nu.
$$

As a consequence, by Table 1, $q_n(x) = p_n(x; a, c^{-3}q^{-3}|q^3)$. This confirms the result given on line (13) in Table 3.

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