Tetrahedron maps, Yang–Baxter maps, and partial linearisations

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Abstract

We study tetrahedron maps, which are set-theoretical solutions to the Zamolodchikov tetrahedron equation, and Yang–Baxter maps, which are set-theoretical solutions to the quantum Yang–Baxter equation. In particular, we clarify the structure of the nonlinear algebraic relations which define linear (parametric) tetrahedron maps (with nonlinear dependence on parameters), and we present several transformations which allow one to obtain new such maps from known ones. Furthermore, we prove that the differential of a (nonlinear) tetrahedron map on a manifold is a tetrahedron map as well. Similar results on the differentials of Yang–Baxter and entwining Yang–Baxter maps are also presented. Using the obtained general results, we construct new examples of (parametric) Yang–Baxter and tetrahedron maps. The considered examples include maps associated with integrable systems and matrix groups. In particular, we obtain a parametric family of new linear tetrahedron maps, which are linear approximations for the nonlinear tetrahedron map constructed by Dimakis and Müller-Hoissen (2019 Lett. Math. Phys. 109 799–827) in a study of soliton solutions of vector Kadomtsev–Petviashvili equations. Also, we present invariants for this nonlinear tetrahedron map.

Keywords: Zamolodchikov tetrahedron equation, quantum Yang–Baxter equation, parametric tetrahedron maps, parametric Yang–Baxter maps, entwining Yang–Baxter maps, linearisations, differentials of maps

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1. Introduction

The Zamolodchikov tetrahedron equation [51, 52] is a higher-dimensional analogue of the well-celebrated quantum Yang–Baxter equation. They belong to the most fundamental equations in mathematical physics and have applications in many diverse branches of physics and mathematics, including statistical mechanics, quantum field theories, algebraic topology, and the theory of integrable systems. Some applications of the tetrahedron equation can be found in [4, 6, 13, 15, 19, 25, 26, 30, 40, 43, 44, 49, 50] and references therein.

This paper is devoted to tetrahedron maps and Yang–Baxter maps, which are set-theoretical solutions to the tetrahedron equation and the Yang–Baxter equation, respectively. Set-theoretical solutions of the Yang–Baxter equation have been intensively studied by many authors after the work of Drinfeld [16]. Even before that, examples of such solutions were constructed by Sklyanin [48]. A quite general construction for tetrahedron maps first appeared in works of Korepanov (see [38, 39] and references therein) in connection with integrable dynamical systems in discrete time. Presently, the relation of tetrahedron maps and Yang–Baxter maps with integrable systems (including partial differential equations (PDEs) and lattice equations) is a very active area of research (see, e.g. [1, 3, 9, 12, 15, 18, 26, 29, 30, 32, 37, 39, 42, 46, 49, 50]).

This paper is organised as follows.

Section 2 contains the definitions of (parametric) tetrahedron maps, (parametric) Yang–Baxter maps and recalls some basic properties of them.

Sections 3 and 4 are devoted to linear tetrahedron maps and to linear parametric tetrahedron maps with nonlinear dependence on parameters. In particular, we clarify the structure of the nonlinear algebraic relations that define such maps, and we present several transformations which allow one to obtain new such maps from known ones.

The results of sections 3 and 4 on linear (parametric) tetrahedron maps generalise some results of [8] on linear (parametric) Yang–Baxter maps. Examples of linear parametric Yang–Baxter maps related to integrable PDEs of vector Kadomtsev–Petviashvili (KP) and (deformed) nonlinear Schrödinger types were discussed also in [12, 37].

Remark 5.6, corollary 5.5, and examples 5.8, 5.10 show how linear tetrahedron maps appear as linear approximations of nonlinear ones.

Hietarinta [21] considered some special linear tetrahedron maps. A relation of our results with those of [21] is discussed in remark 3.8.

We study also nonlinear Yang–Baxter and tetrahedron maps on manifolds and their differentials defined on the corresponding tangent bundles. For a manifold $M$, its tangent bundle is denoted by $T\!\!M$. When we consider maps of manifolds, we assume that they are either smooth, or complex-analytic, or rational, so that the differential is defined for such a map. Section 5 contains the following results:

- For any Yang–Baxter map $Y : M \times M \to M \times M$, the differential $dY : TM \times TM \to TM \times TM$ is a Yang–Baxter map of the manifold $TM \times TM$. A similar result is valid also for entwining Yang–Baxter maps.
- For any tetrahedron map $T : M \times M \times M \to M \times M \times M$, the differential
  \[ dT : TM \times TM \times TM \to TM \times TM \times TM, \]
  is a tetrahedron map of the manifold $TM \times TM \times TM$.

The above result on the differential of a Yang–Baxter map was used (without proof) in [8].
Examples of the differentials for tetrahedron maps are presented in section 5. The computed differentials are tetrahedron maps (46) and (50). An example of a computation of the differentials for a family of Yang–Baxter maps is given in section 6.

In example 5.10 we consider the nonlinear birational tetrahedron map (51) which was constructed by Dimakis and Müller-Hoissen [12] in a study of soliton solutions of vector KP equations. We present invariants for this map and find for it a linear approximation, which is a family of new linear tetrahedron maps (54) depending nonlinearly on the parameter \( c \in \mathbb{C} \).

The presented method (described in section 5) to derive new linear tetrahedron maps as linear approximations of nonlinear ones is developed further in the preprint [24], where several more parametric families of new linear tetrahedron maps (depending nonlinearly on parameters) are obtained by this method.

Generalising some constructions from [8], in section 6 we present new examples of linear parametric Yang–Baxter maps (with nonlinear dependence on parameters) associated with some matrix groups. Let \( \mathbb{K} \) be either \( \mathbb{C} \) or \( \mathbb{R} \). For \( n \in \mathbb{Z}_{>0} \), consider the matrix group \( \text{GL}_n(\mathbb{K}) \) and an abelian subgroup \( \Omega \subset \text{GL}_n(\mathbb{K}) \). The construction in section 6 involves the computation of the differential of a nonlinear Yang–Baxter map on \( \text{GL}_n(\mathbb{K}) \times \text{GL}_n(\mathbb{K}) \) and restricting the computed differential to the submanifold \((\Omega \times \text{Mat}_n(\mathbb{K})) \times (\Omega \times \text{Mat}_n(\mathbb{K}))\) of the tangent bundle

\[
T(\text{GL}_n(\mathbb{K}) \times \text{GL}_n(\mathbb{K})) \cong T\text{GL}_n(\mathbb{K}) \times T\text{GL}_n(\mathbb{K}) \cong (\text{GL}_n(\mathbb{K}) \times \text{Mat}_n(\mathbb{K})) \\
\times (\text{GL}_n(\mathbb{K}) \times \text{Mat}_n(\mathbb{K})),
\]

of the manifold \( \text{GL}_n(\mathbb{K}) \times \text{GL}_n(\mathbb{K}) \). Furthermore, we extend the obtained Yang–Baxter map to a family of Yang–Baxter maps depending on a nonzero constant \( l \in \mathbb{K} \).

As a result, for any nonzero \( l \in \mathbb{K}, n, p \in \mathbb{Z}_{>0} \), and any abelian subgroup \( \Omega \subset \text{GL}_n(\mathbb{K}) \), we obtain the parametric Yang–Baxter map (59) with parameters \( a, b \in \Omega \). For \( p \geq 2 \) the map (59) is new. For \( p = 1 \) it was presented in [8].

In the construction of (59) we assume that \( \mathbb{K} \) is either \( \mathbb{R} \) or \( \mathbb{C} \), in order to use tangent spaces and differentials. Furthermore, one can verify that (59) is a parametric Yang–Baxter map for any field \( \mathbb{K} \).

Section 7 concludes the paper with comments on how the results of this paper can be extended.

Remark 1.1. According to remarks 2.3 and 5.1 many constructions of this paper involve Yang–Baxter and tetrahedron maps which are ‘partly linear’ in the sense that the maps are linear with respect to some of the variables and nonlinear with respect to the other variables. Thus, informally speaking, one can say that we deal with ‘partial linearisations’ of Yang–Baxter and tetrahedron maps.

2. Preliminaries

2.1. Tetrahedron maps

For any set \( S \) and \( n \in \mathbb{Z}_{>0} \), we use the notation \( S^n = S \times S \times \cdots \times S \).

Let \( W \) be a set. A tetrahedron map is a map

\[
T : W^3 \to W^3, \quad T(x, y, z) = (u(x, y, z), v(x, y, z), w(x, y, z)), \quad x, y, z \in W,
\]
Figure 1. Schematic interpretation of the tetrahedron equation [25, 30, 52].

satisfying the (Zamolodchikov) tetrahedron equation
\[ T^{123} \circ T^{145} \circ T^{246} \circ T^{356} = T^{356} \circ T^{246} \circ T^{145} \circ T^{123}. \] (1)

Here \( T^{ijk} : W^6 \to W^6 \) for \( i, j, k = 1, \ldots, 6, \) \( i < j < k, \) is the map acting as \( T \) on the \( i \)th, \( j \)th, \( k \)th factors of the Cartesian product \( W^6 \) and acting as identity on the remaining factors. For instance,
\[ T^{246}(x, y, z, r, s, t) = (x, u(y, r, t), z, v(y, r, t), s, w(y, r, t)), \] \( x, y, z, r, s, t \in W. \)

The schematic interpretation of the tetrahedron equation is given in figure 1. Every line with a number \( i = 1, \ldots, 6, \) corresponds to one of six copies of the set \( W, \) and every intersection point of lines \( i, j, k \) corresponds to the map \( T^{ijk}. \)

**Proposition 2.1 ([30]).** Consider the permutation map
\[ P^{13} : W^3 \to W^3, \quad P^{13}(a_1, a_2, a_3) = (a_3, a_2, a_1), \quad a_i \in W. \]

If a map \( T : W^3 \to W^3 \) satisfies the tetrahedron equation (1) then \( \tilde{T} = P^{13} \circ T \circ P^{13} \) obeys this equation as well.

**Proposition 2.2 ([30]).** Let \( T : W^3 \to W^3 \) be a tetrahedron map. Suppose that a map \( \sigma : W \to W \) satisfies
\[ (\sigma \times \sigma \times \sigma) \circ T \circ (\sigma \times \sigma \times \sigma) = T, \quad \sigma \circ \sigma = \text{Id}. \]

Then
\[ \tilde{T} = (\sigma \times \text{Id} \times \sigma) \circ T \circ (\text{Id} \times \sigma \times \text{Id}), \]
\[ \tilde{T} = (\text{Id} \times \sigma \times \text{Id}) \circ T \circ (\sigma \times \text{Id} \times \sigma), \]
are tetrahedron maps.
2.2. Parametric tetrahedron maps

Let $\Omega$ and $V$ be sets. Here $\Omega$ is regarded as a set of parameters. Consider a map of the form

$$T : (\Omega \times V) \times (\Omega \times V) \times (\Omega \times V) \to (\Omega \times V) \times (\Omega \times V),$$

$$T(\alpha, \beta, \gamma, x, y, z) = (u(\alpha, x), v(\alpha, y), w(\alpha, z)), \quad \alpha, \beta, \gamma \in \Omega, \quad x, y, z \in V.$$

Equation (5) is called the parametric tetrahedron equation. Thus, the parametric map $T : \Omega \times V \to \Omega \times V \times (\Omega \times V)$ satisfies $T(\alpha, \beta, \gamma, x, y, z) = T(\alpha, \beta, \gamma, \pi(x), \pi(y), \pi(z))$ for all $\alpha, \beta, \gamma \in \Omega$. For instance,

$$T_{\alpha, \beta, \gamma}^{123} \circ T_{\alpha, \beta, \gamma}^{145} \circ T_{\alpha, \beta, \gamma}^{246} \circ T_{\alpha, \beta, \gamma}^{356} = T_{\alpha, \beta, \gamma}^{356} \circ T_{\alpha, \beta, \gamma}^{246} \circ T_{\alpha, \beta, \gamma}^{145} \circ T_{\alpha, \beta, \gamma}^{123} \text{ for all } \alpha, \beta, \gamma \in \Omega. \quad (5)$$

The terms $T_{\alpha, \beta, \gamma}^{123}, T_{\alpha, \beta, \gamma}^{145}, T_{\alpha, \beta, \gamma}^{246}, T_{\alpha, \beta, \gamma}^{356}$ are maps $V^6 \to V^6$ defined similarly to the terms in equation (1), adding the parameters $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$. For instance,

$$T_{\alpha, \beta, \gamma}^{246}(x, y, z, r, s, t) = (x, u_{\alpha, \beta, \gamma}(y, r, t), z, v_{\alpha, \beta, \gamma}(y, r, t), s, w_{\alpha, \beta, \gamma}(y, r, t)), \quad x, y, z, r, s, t \in V.$$

We use the notation

$$T_{\alpha, \beta, \gamma} : V^3 \to V^3,$$

$$T_{\alpha, \beta, \gamma}(x, y, z) = (u_{\alpha, \beta, \gamma}(x, y, z), v_{\alpha, \beta, \gamma}(x, y, z), w_{\alpha, \beta, \gamma}(x, y, z)), \quad \alpha, \beta, \gamma \in \Omega, \quad x, y, z \in V. \quad (6)$$

Thus, $T_{\alpha, \beta, \gamma}$ defined by (6) is a map $V^3 \to V^3$ depending on parameters $\alpha, \beta, \gamma \in \Omega$. Equation (5) is called the parametric tetrahedron equation. The family of maps (6) is called a parametric tetrahedron map if it satisfies equation (5). Then we can say more briefly that $T_{\alpha, \beta, \gamma}$ is a parametric tetrahedron map.

Remark 2.3. Thus, the parametric map $T_{\alpha, \beta, \gamma}$ defined by (6), (3) and (4) obeys the parametric tetrahedron equation (5) if and only if the (nonparametric) map (2) obeys the tetrahedron equation (1).

In section 4 we consider the case when $V$ is a vector space and for any values of $\alpha, \beta, \gamma$ the map $T_{\alpha, \beta, \gamma} : V^3 \to V^3$ is linear. Note that usually $\Omega$ is a subset of another vector space, and the dependence of $T_{\alpha, \beta, \gamma}$ on the parameters $\alpha, \beta, \gamma \in \Omega$ is nonlinear.
Then one can say that in section 4 we study tetrahedron maps of the form (2) which are linear with respect to $V$ and may be nonlinear with respect to $\Omega$. However, it is useful to keep $\alpha, \beta, \gamma$ as parameters and to work with $T_{\alpha,\beta}$ instead of $T$ from (2).

### 2.3. Tetrahedron maps vs Yang–Baxter maps

In this subsection, we recall some simple relations between (parametric) tetrahedron maps and (parametric) Yang–Baxter maps, which are defined below. The results of proposition 2.4 and corollary 2.5 are known and are very simple, but for completeness we present proofs for them. Proposition 2.4 and corollary 2.5 show that (parametric) Yang–Baxter maps can be regarded as a particular class of (parametric) tetrahedron maps.

Let $W$ be a set. A Yang–Baxter map is a map

$$Y: W \times W \to W \times W, \quad Y(x, y) = (u(x, y), v(x, y)), \quad x, y \in W,$$

satisfying the Yang–Baxter equation

$$Y^{12} \circ Y^{13} \circ Y^{23} = Y^{23} \circ Y^{13} \circ Y^{12}. \quad (7)$$

The terms $Y^{12}, Y^{13}, Y^{23}$ in (7) are maps $W^3 \to W^3$ defined as follows

$$Y^{12}(x, y, z) = (u(x, y), v(x, y), z),$$

$$Y^{23}(x, y, z) = (x, u(y, z), v(x, z)),$$

$$Y^{13}(x, y, z) = (u(x, z), y, v(x, z)), \quad x, y, z \in W.$$

**Proposition 2.4.** Let $Y: W^2 \to W^2$ be a Yang–Baxter map. Then the maps

$$Y^{23}: W^3 \to W^3, \quad Y^{12}: W^3 \to W^3,$$

are tetrahedron maps.

**Proof.** Let $T = Y^{23}$. We need to prove (1). Using the identity map $\text{Id}_{W^3}: W^3 \to W^3$, one obtains

$$T^{123} = Y^{23} \times \text{Id}_{W^3}, \quad T^{145} = \text{Id}_{W^3} \times Y^{12},$$

$$T^{246} = \text{Id}_{W^3} \times Y^{13}, \quad T^{356} = \text{Id}_{W^3} \times Y^{23},$$

$$T^{123} \circ T^{145} \circ T^{246} \circ T^{356} = (Y^{23} \times \text{Id}_{W^3}) \circ (\text{Id}_{W^3} \times (Y^{12} \circ Y^{13} \circ Y^{23})), \quad (8)$$

$$T^{356} \circ T^{246} \circ T^{145} \circ T^{123} = (\text{Id}_{W^3} \times (Y^{23} \circ Y^{13} \circ Y^{12})) \circ (Y^{23} \times \text{Id}_{W^3}),$$

$$= (Y^{23} \times \text{Id}_{W^3}) \circ (\text{Id}_{W^3} \times (Y^{23} \circ Y^{13} \circ Y^{12})). \quad (9)$$

Since $Y$ satisfies (7), from (8) and (9) we derive (1) for $T = Y^{23}$. Similarly, one can prove (1) for $T = Y^{12}$. 

Let $\Omega$ and $V$ be sets. A parametric Yang–Baxter map $Y_{\alpha,\beta}$ is a family of maps

$$Y_{\alpha,\beta}: V \times V \to V \times V, \quad Y_{\alpha,\beta}(x, y) = (u_{\alpha,\beta}(x, y), v_{\alpha,\beta}(x, y)), \quad x, y \in V, \quad \alpha, \beta \in \Omega, \quad (10)$$
depending on parameters $\alpha, \beta \in \Omega$ and satisfying the parametric Yang–Baxter equation
\begin{equation}
Y_{\alpha,\beta}^{12} \circ Y_{\alpha,\beta}^{13} \circ Y_{\beta,\gamma}^{23} = Y_{\beta,\gamma}^{23} \circ Y_{\alpha,\beta}^{13} \circ Y_{\alpha,\gamma}^{12} \quad \text{for all } \alpha, \beta, \gamma \in \Omega. 
\end{equation}

The terms $Y_{\alpha,\beta}^{12}$, $Y_{\alpha,\beta}^{13}$, $Y_{\beta,\gamma}^{23}$ in (11) are maps $V^3 \to V^3$ given by
\begin{align*}
Y_{\alpha,\beta}^{12}(x, y, z) &= (u_{\alpha,\beta}(x, y), v_{\alpha,\beta}(x, y), z), \\
Y_{\beta,\gamma}^{23}(x, y, z) &= (x, u_{\beta,\gamma}(y, z), v_{\beta,\gamma}(y, z)), \\
Y_{\alpha,\gamma}^{13}(x, y, z) &= (u_{\alpha,\gamma}(x, z), y, v_{\alpha,\gamma}(x, z)),
\end{align*}
for all $x, y, z \in V$.

A parametric Yang–Baxter map (10) with parameters $\alpha, \beta$ can be interpreted as the following Yang–Baxter map $Y$ without parameters
\begin{align}
Y : (\Omega \times V) \times (\Omega \times V) &\to (\Omega \times V) \times (\Omega \times V), \\
Y((\alpha, x), (\beta, y)) &= ((\alpha, u_{\alpha,\beta}(x, y)), (\beta, v_{\alpha,\beta}(x, y))).
\end{align}

Corollary 2.5. For any parametric Yang–Baxter map (10), the maps
\begin{align}
T_{\alpha,\beta,\gamma} = Y_{\beta,\gamma}^{23} : V^3 &\to V^3, \\
\tilde{T}_{\alpha,\beta,\gamma} = Y_{\alpha,\beta}^{12} : V^3 &\to V^3,
\end{align}
are parametric tetrahedron maps.

Proof. Applying Proposition 2.4 to the (nonparametric) Yang–Baxter map $Y$ given by (12), we see that $Y_{\beta,\gamma}^{23}$, $Y_{\alpha,\beta}^{12}$ are (nonparametric) tetrahedron maps. This is equivalent to the fact that (13) are parametric tetrahedron maps. \hfill \square

3. Linear tetrahedron maps

For any vector space $W$ we denote by End($W$) the set of linear maps $W \to W$.

Let $V$ be a vector space over a field $\mathbb{K}$. Usually $\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{R}$. In this section we consider linear maps $T : V^3 \to V^3$ given by
\begin{equation}
T : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} A & B & C \\ D & E & F \\ K & L & M \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad x, y, z, u, v, w \in V,
\end{equation}
where $A, B, C, D, E, F, K, L, M \in \text{End}(V)$.

Remark 3.1. If $V = \mathbb{K}^n$ for some $n \in \mathbb{Z}_{>0}$ then $A, B, C, D, E, F, K, L, M$ are $n \times n$ matrices.

Proposition 3.2. A map $T \in \text{End}(V^3)$ given by (14) satisfies the tetrahedron equation (1) if and only if the maps $A, B, C, D, E, F, K, L, M \in \text{End}(V)$ in (14) obey the following equations
\begin{align}
DA &= AD + BDA, & AB &= BA + ABD, \tag{15a} \\
ED &= DE + EDB, & BE &= EB + DBE, \\
LE &= EL + FLE, & EF &= FE + FLE, \\
ML &= MLF + LM, & FM &= MF + LFM, \tag{15b}
\end{align}
\[D = DD + EDA, \quad B = BB + ABE,\]
\[L = LL + ML\dot{E}, \quad F = FF + EFM,\]  
\[(15c)\]
\[K = DK + EKA + FKD + FLDA,\]
\[C = CB + ACE + BCL + ABFL,\]  
\[(15d)\]
\[KA = AK + BKA + CKD + CLDA,\]
\[AC = CA + ACD + BCK + ABEK,\]  
\[(15e)\]
\[KK + LKA + MKD + MLD\dot{A} = 0,\]
\[CC + ACF + BCM + ABFM = 0,\]  
\[(15f)\]
\[EA + DBD = AE + BDB, \quad ME + LFL = EM + FLF,\]  
\[(15g)\]
\[K = KL + LKB + MKE + MLDB,\]
\[C = FC + DCF + ECM + DMB,\]  
\[(15h)\]
\[MK = KM + LKC + MKF + MLDC,\]
\[CM = MC + KCF + LCM + KBFM,\]  
\[(15i)\]
\[FD + EFK = DF + EDC, \quad BL + CLE = LB + KBE,\]  
\[(15j)\]
\[LD = DL + EKB + FKE + FLDB,\]
\[BF = FB + DCE + ECL + DBFL,\]  
\[(15k)\]
\[LA + KBD = AL + BKB + CKE + CLDB,\]  
\[(15l)\]
\[AF + BDC = FA + DCD + ECK + DBFK,\]  
\[(15m)\]
\[MD + LFK = DM + EKC + FKE + FLDC,\]  
\[(15n)\]
\[BM + CLF = MB + KCE + LCL + KBFM,\]  
\[(15o)\]
\[MA - AM + KCD + LCK + KBFK = BKC + CKF + CLDC.\]  
\[(15p)\]

**Proof.** This can be proved by substitution of \(T\) in (14) to the tetrahedron equation (1). For any \(x, y, z, r, s, t \in V\), from the left-hand side of (1) we obtain
\[
(T^{123} \circ T^{145} \circ T^{246} \circ T^{356})(x, y, z, r, s, t)
\]
\[
= (AAx + ABDy + ABEr + ABFKz + ABFLs
+ ABFMt + ACEs + ACFt + BAr + BBr + BCKz + BCLS
+ CAz + CBS + CCT + DAx + BDBy + DBEr + DBFKz + DBFLs
+ DBFMt + DCDz + DCEs + DCFt + EAy + EBr + ECKz
+ ECLS + ECMt + FAz + FBS + FCT + KAx + KBDy + KBEr
+ KBFKz + KBFLs + KBFM + KCDz + KCEs + KCFt
+ LAy + LBr + LCKz + LCLS + LCMt + MAz + MBs + MCT, Dx)
\]
while the right-hand side of (1) implies
\[ (T^{356} \circ T^{246} \circ T^{145} \circ T^{123})(x, y, z, r, s, t) \]
\[ = (AAx + ABy + ACz + Br + Cx, ADx + AEy + AFz + BDx + BDAx + BDBy + BDCz + BEr + BFz + BLs + CKDx + CKEy + CKFz + CLDAx + CLDBy + CLDCz + CLEr + CLFs + CMt, DDx + DEy + DFz + EDAx + EDBy + EDCz + EEr) \]
\[ + EFs + Ft, Dkx + DLx + DMz + EKx + EKBy + EKCz + EKDAx + EKDBy + EKDCz + EKLEr + EKLFs + EMLDx + EMLBy + EMLDz + EMLEr + EMLFs + EMLMt, Lka + LKBy + LKCz + LLs + KMDx + KMEy + KMFz + MLDax + MLDBy + MLDCz + MLER + MLFs + MMt). \]

By equating the coefficients of \( x, y, z, r, s, t \) for each component of these vectors, we derive a system of relations equivalent to (1). For instance, consider the coefficients of \( y \) in the first components:
\[ ABD + BA = AB. \]

This is the second equation from (15a). Performing the same actions with all variables and components of the obtained vectors, we get all of relations (15). \( \square \)

**Corollary 3.3.** System (15) implies the matrix equations (16a)–(16d)

\[
\begin{pmatrix}
D & E \\
A & B
\end{pmatrix}
\begin{pmatrix}
D & BE \\
DA & B
\end{pmatrix}
= \begin{pmatrix}
D & BE \\
DA & B
\end{pmatrix}, \quad (16a)
\]

\[
\begin{pmatrix}
L & M \\
E & F
\end{pmatrix}
\begin{pmatrix}
L & FM \\
LE & F
\end{pmatrix}
= \begin{pmatrix}
L & FM \\
LE & F
\end{pmatrix}, \quad (16b)
\]

\[
\begin{pmatrix}
AB & B \\
D & ED
\end{pmatrix}
\begin{pmatrix}
D & E \\
A & B
\end{pmatrix}
= \begin{pmatrix}
AB & B \\
D & ED
\end{pmatrix}, \quad (16c)
\]

\[
\begin{pmatrix}
EF & F \\
L & ML
\end{pmatrix}
\begin{pmatrix}
L & M \\
E & F
\end{pmatrix}
= \begin{pmatrix}
EF & F \\
L & ML
\end{pmatrix}, \quad (16d)
\]
as well as the following
\[
[E-BED, A] = [BD, D + B], \quad [A - DAB, D] = [DB, D + B],
\]
\[
[M - FML, E] = [FM, M + F], \quad [E - LEF, M] = [MF, M + F],
\]
\[
[B + D - DB, E] = 0, \quad [B + D - BD, A] = 0,
\]
\[
[L + F - FL, M] = 0, \quad [L + F - FL, E] = 0,
\]
\[
[E, FK + CL - KB - DC] + [F + L, D + B] + [DB, FL] = 0,
\]
where by \([\cdot, \cdot]\) we denote the commutator \([A, B] = AB - BA\).

Remark 3.4. Equation (16) are equivalent to (15a)–(15c). Thus, equations (15a)–(15c) can be replaced by equation (16), which have more clear structure.

Proposition 3.5. For any vector space \(V\), the set of linear tetrahedron maps (14) is invariant under the following transformations
\[
\begin{pmatrix}
A & B & C \\
D & E & F \\
K & L & M
\end{pmatrix}
\mapsto
\begin{pmatrix}
M & L & K \\
F & E & D \\
C & B & A
\end{pmatrix},
\]
(18)
\[
\begin{pmatrix}
A & B & C \\
D & E & F \\
K & L & M
\end{pmatrix}
\mapsto
\begin{pmatrix}
-A & B & -C \\
D & -E & F \\
-K & L & -M
\end{pmatrix}.
\]
(19)
Let \(V = \mathbb{K}^{n}\) for some \(n \in \mathbb{Z}_{>0}\). Then \(A, B, C, D, E, F, K, L, M\) in (14) are \(n \times n\) matrices. In this case, the set of linear tetrahedron maps (14) is invariant also under the transformation
\[
\begin{pmatrix}
A & B & C \\
D & E & F \\
K & L & M
\end{pmatrix}
\mapsto
\begin{pmatrix}
A & B & C \end{pmatrix}^{T}
\begin{pmatrix}
M & L & K \\
F & E & D \\
C & B & A
\end{pmatrix}
\begin{pmatrix}
A^{T} & D^{T} & K^{T} \\
B^{T} & E^{T} & L^{T} \\
C^{T} & F^{T} & M^{T}
\end{pmatrix}.
\]
(20)
where \(T\) denotes the transpose operation for matrices.

Proof. The statement about the transformation (18) follows from proposition 2.1 with \(W = V\).

The case of the transformation (19) follows from proposition 2.2, if we take \(W = V\) and consider the map \(\sigma : V \rightarrow V, \sigma(v) = -v\).

To prove the statement about the transformation (20), one can apply the transpose operation to both sides of the tetrahedron equation (1) for \(T\) given by (14). \(\square\)

Example 3.6. Let \(V = \mathbb{K}^{2}\). Then \(V^{3} = \mathbb{K}^{6}\). Let \(c \in \mathbb{K}, c \neq 0\). Consider the linear map \(T \in \text{End}(V^{3})\) given by (14) with the matrix
\[
\begin{pmatrix}
A & B & C \\
D & E & F \\
K & L & M
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & \frac{c-1}{c} \\
0 & 1 & \frac{1}{c} \\
0 & 0 & \frac{1}{c}
\end{pmatrix}
\begin{pmatrix}
\frac{c-1}{c} & \frac{c}{1-c} & \frac{c-1}{c} \\
\frac{1}{c} & 0 & \frac{c-1}{c} \\
\frac{1}{c} & \frac{c}{1-c} & \frac{c-1}{c}
\end{pmatrix}.
\]
(21)
Thus, for (21) we have
\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} \frac{c-1}{c} & \frac{(c-1)^2(c+1)}{1-c} \\ 0 & \frac{c}{1-c} \end{pmatrix},
\]
\[
C = \begin{pmatrix} \frac{c-1}{c} & 1-c \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\]
\[
E = \begin{pmatrix} \frac{1}{c} & -(c-1)^2(c+1) \\ 0 & c \end{pmatrix}, \quad F = \begin{pmatrix} \frac{c-1}{c} & c-1 \\ 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\]
\[
L = \begin{pmatrix} 0 & 1-c \\ 0 & 1-c \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

(22)

Using proposition 3.2, one can verify that (21) is a tetrahedron map.

As explained in example 5.10, we have derived this linear tetrahedron map, using the differential of a nonlinear tetrahedron map from [12]. Applying the transformations (18), (19), (20) and their compositions to (21), we obtain several more linear tetrahedron maps.

**Proposition 3.7.** Let \( T_1, T_2 \) be linear tetrahedron maps of the form
\[
T_1 = \begin{pmatrix} A & B \\ D & E \end{pmatrix} 0 \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix}, \quad T_2 = \begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{E} \end{pmatrix} 0 \begin{pmatrix} \tilde{F} & 0 \\ \tilde{L} & \tilde{M} \end{pmatrix}.
\]

Let \( l, m \in \mathbb{K}, l \neq 0 \). Then
\[
T_{1}^{l,m} = \begin{pmatrix} lA & B \\ D & l^{-1}E \end{pmatrix} 0 \begin{pmatrix} 0 & 0 \\ 0 & mM \end{pmatrix}, \quad T_{2}^{l,m} = \begin{pmatrix} mL & 0 \\ 0 & \tilde{L} \end{pmatrix} 0 \begin{pmatrix} \tilde{M} & 0 \\ 0 & \tilde{L}^{-1} \end{pmatrix},
\]

are linear tetrahedron maps as well.

**Proof.** For each \( i = 1, 2 \), the fact that \( T_i \) obeys equation (15) implies that \( T_{i}^{l,m} \) obeys these equations as well. \( \square \)

**Remark 3.8.** Hietarinta [21] studied some special linear tetrahedron maps. In our notation, Hietarinta [21] assumes that \( A, B, C, D, E, F, K, L, M \) in (14) belong to a commutative ring. The assumption that \( A, B, C, D, E, F, K, L, M \) commute simplifies equation (15) very considerably, and this simplified version of (15) appears in [21].

Results of [21] imply that for any \( a, b, c \in \mathbb{K} \) the following matrix determines a linear tetrahedron map \( \mathbb{K}^3 \rightarrow \mathbb{K}^3 \)
\[
\begin{pmatrix} A & B & C \\ D & E & F \end{pmatrix} 0 \begin{pmatrix} 0 & 0 \\ 1-bc & e \end{pmatrix} = \begin{pmatrix} a & 1-ab & 0 \\ 0 & b & 0 \\ 0 & 1-bc & e \end{pmatrix}.
\]

(23)

Hietarinta [21] studied the case when \( a, b, c \) are elements of a commutative ring.
4. Linear parametric tetrahedron maps

Let $V$ be a vector space over a field $\mathbb{K}$. Let $\Omega$ be a set. In this section we study linear maps $T_{\alpha\beta\gamma} \in \text{End}(V)$ depending on parameters $\alpha, \beta, \gamma \in \Omega$. We consider a linear map

$$T_{\alpha\beta\gamma} : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} A_{\alpha\beta\gamma} & B_{\alpha\beta\gamma} & C_{\alpha\beta\gamma} \\ D_{\alpha\beta\gamma} & E_{\alpha\beta\gamma} & F_{\alpha\beta\gamma} \\ K_{\alpha\beta\gamma} & L_{\alpha\beta\gamma} & M_{\alpha\beta\gamma} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad x, y, z, u, v, w \in V,$$

(24)

where $A_{\alpha\beta\gamma}, B_{\alpha\beta\gamma}, C_{\alpha\beta\gamma}, D_{\alpha\beta\gamma}, E_{\alpha\beta\gamma}, F_{\alpha\beta\gamma}, K_{\alpha\beta\gamma}, L_{\alpha\beta\gamma}, M_{\alpha\beta\gamma} \in \text{End}(V)$ for all $\alpha, \beta, \gamma \in \Omega$. Then $T_{\alpha\beta\gamma}$ is called a linear parametric tetrahedron map if it satisfies the parametric tetrahedron equation (5).

Remark 4.1. Note that usually $\Omega$ is a subset of another vector space, and the dependence of $T_{\alpha\beta\gamma}$ on the parameters $\alpha, \beta, \gamma$ is nonlinear. Examples of such maps are presented in section 6.

Proposition 4.2. A parametric map $T_{\alpha\beta\gamma}$ given by (24) satisfies the parametric tetrahedron equation (5) if and only if it obeys the following list of equations for all values of the parameters $\alpha, \beta, \gamma, \delta, \epsilon, \zeta \in \Omega$:

\begin{align*}
A_{\alpha\beta\gamma}A_{\alpha\beta\gamma} &= A_{\alpha\beta\gamma}A_{\alpha\beta\gamma}, \quad E_{\alpha\beta\gamma}E_{\beta\delta\zeta} = E_{\beta\delta\zeta}E_{\alpha\beta\gamma}, \\
M_{\beta\delta\zeta}M_{\gamma\epsilon\zeta} &= M_{\gamma\epsilon\zeta}M_{\beta\delta\zeta}, \tag{25a}
\end{align*}

\begin{align*}
D_{\alpha\beta\gamma}A_{\alpha\beta\gamma} &= A_{\beta\delta\zeta}D_{\alpha\beta\gamma} + B_{\beta\delta\zeta}D_{\alpha\beta\gamma}A_{\alpha\beta\gamma}, \\
A_{\alpha\beta\gamma}B_{\alpha\beta\gamma} &= B_{\alpha\beta\gamma}A_{\alpha\beta\gamma} + A_{\alpha\beta\gamma}B_{\alpha\beta\gamma}D_{\beta\delta\zeta}, \tag{25b}
\end{align*}

\begin{align*}
E_{\alpha\beta\gamma}E_{\beta\delta\zeta} &= D_{\beta\delta\zeta}E_{\alpha\beta\gamma} + E_{\beta\delta\zeta}D_{\alpha\beta\gamma}B_{\beta\delta\zeta}, \\
B_{\beta\delta\zeta}E_{\beta\delta\zeta} &= E_{\alpha\beta\gamma}B_{\beta\delta\zeta} + D_{\alpha\beta\gamma}E_{\beta\delta\zeta}B_{\beta\delta\zeta}, \tag{25c}
\end{align*}

\begin{align*}
L_{\alpha\beta\gamma}E_{\beta\delta\zeta} &= L_{\gamma\epsilon\zeta}E_{\alpha\beta\gamma} + E_{\gamma\epsilon\zeta}L_{\beta\delta\zeta}E_{\alpha\beta\gamma}, \\
E_{\beta\delta\zeta}F_{\beta\delta\zeta} &= F_{\alpha\beta\gamma}E_{\gamma\epsilon\zeta} + E_{\alpha\beta\gamma}F_{\beta\delta\zeta}L_{\gamma\epsilon\zeta}, \tag{25d}
\end{align*}

\begin{align*}
M_{\beta\delta\zeta}M_{\gamma\epsilon\zeta} &= M_{\gamma\epsilon\zeta}M_{\beta\delta\zeta}F_{\alpha\beta\gamma} + L_{\gamma\epsilon\zeta}M_{\alpha\beta\gamma}, \\
F_{\gamma\epsilon\zeta}M_{\beta\delta\zeta} &= M_{\alpha\beta\gamma}F_{\gamma\epsilon\zeta} + L_{\alpha\beta\gamma}M_{\beta\delta\zeta}M_{\gamma\epsilon\zeta}, \tag{25e}
\end{align*}

\begin{align*}
D_{\alpha\beta\gamma} &= D_{\beta\delta\zeta}D_{\alpha\beta\gamma} + E_{\beta\delta\zeta}D_{\alpha\beta\gamma}A_{\alpha\beta\gamma}, \\
B_{\alpha\beta\gamma} &= B_{\alpha\beta\gamma}B_{\beta\delta\zeta} + A_{\alpha\beta\gamma}B_{\alpha\beta\gamma}E_{\beta\delta\zeta}, \tag{25f}
\end{align*}

\begin{align*}
L_{\beta\delta\zeta} &= L_{\gamma\epsilon\zeta}L_{\alpha\beta\gamma} + M_{\gamma\epsilon\zeta}L_{\beta\delta\zeta}E_{\alpha\beta\gamma}, \\
F_{\beta\delta\zeta} &= F_{\alpha\beta\gamma}F_{\gamma\epsilon\zeta} + E_{\alpha\beta\gamma}F_{\beta\delta\zeta}M_{\gamma\epsilon\zeta}, \tag{25g}
\end{align*}

\begin{align*}
K_{\alpha\beta\gamma} &= D_{\gamma\epsilon\zeta}K_{\alpha\beta\gamma} + E_{\gamma\epsilon\zeta}K_{\alpha\beta\gamma}A_{\alpha\beta\gamma} + F_{\gamma\epsilon\zeta}K_{\beta\delta\zeta}D_{\alpha\beta\gamma} + F_{\gamma\epsilon\zeta}L_{\beta\delta\zeta}D_{\alpha\beta\gamma}A_{\alpha\beta\gamma}, \tag{25h}
\end{align*}

\begin{align*}
C_{\alpha\beta\gamma} &= C_{\alpha\beta\gamma}B_{\gamma\epsilon\zeta} + A_{\alpha\beta\gamma}C_{\alpha\beta\gamma}E_{\gamma\epsilon\zeta} + B_{\alpha\beta\gamma}C_{\beta\delta\zeta}L_{\gamma\epsilon\zeta} + A_{\alpha\beta\gamma}B_{\alpha\beta\gamma}F_{\beta\delta\zeta}L_{\gamma\epsilon\zeta}, \tag{25i}
\end{align*}
\[ K_{\alpha \beta \gamma} A_{\alpha \delta \epsilon} = A_{\gamma \zeta} K_{\alpha \beta \gamma} + B_{\alpha \beta} K_{\alpha \delta \epsilon} A_{\alpha \gamma \zeta} + C_{\gamma \zeta} K_{\beta \delta \epsilon} D_{\alpha \beta \gamma} + C_{\gamma \zeta} L_{\beta \delta \epsilon} D_{\alpha \beta \gamma} A_{\alpha \gamma \zeta}, \quad (25j) \]
\[ A_{\alpha \delta \epsilon} C_{\alpha \beta \gamma} = C_{\alpha \beta \gamma} A_{\alpha \delta \epsilon} + A_{\alpha \beta} C_{\alpha \delta \epsilon} D_{\gamma \beta \gamma} + B_{\alpha \beta \gamma} C_{\delta \beta \epsilon} L_{\gamma \beta \gamma} + A_{\alpha \beta \gamma} B_{\alpha \delta \epsilon} F_{\beta \delta \epsilon} K_{\gamma \beta \gamma}, \quad (25k) \]
\[ K_{\gamma \zeta} K_{\alpha \beta \gamma} + L_{\gamma \zeta} K_{\alpha \beta \gamma} A_{\alpha \gamma \zeta} + M_{\gamma \zeta} K_{\beta \delta \epsilon} D_{\alpha \gamma \zeta} + M_{\gamma \zeta} L_{\beta \delta \epsilon} D_{\alpha \beta \gamma} A_{\alpha \gamma \zeta} = 0, \quad (25l) \]
\[ C_{\alpha \beta \gamma} C_{\gamma \zeta} + A_{\alpha \beta} C_{\gamma \zeta} F_{\epsilon \beta \gamma} + B_{\alpha \beta} C_{\beta \delta \epsilon} M_{\gamma \beta \gamma} + A_{\alpha \beta \gamma} B_{\alpha \delta \epsilon} L_{\beta \delta \epsilon} M_{\gamma \beta \gamma} = 0, \quad (25m) \]
\[ E_{\alpha \beta \gamma} A_{\beta \delta \epsilon} + D_{\alpha \beta \gamma} B_{\alpha \beta \gamma} D_{\beta \delta \epsilon} = A_{\beta \delta \epsilon} E_{\alpha \beta \gamma} + B_{\beta \delta \epsilon} D_{\alpha \beta \gamma}, \quad (25n) \]
\[ M_{\alpha \delta \epsilon} E_{\gamma \zeta} + L_{\alpha \delta \epsilon} F_{\beta \delta \epsilon} L_{\gamma \beta \gamma} = E_{\gamma \zeta} M_{\alpha \delta \epsilon} + F_{\gamma \zeta} L_{\beta \delta \epsilon} F_{\gamma \beta \gamma}, \quad (25o) \]
\[ K_{\beta \delta \epsilon} = K_{\gamma \zeta} L_{\beta \delta \epsilon} + L_{\gamma \zeta} K_{\alpha \beta \gamma} B_{\alpha \beta \gamma} + M_{\gamma \zeta} K_{\beta \delta \epsilon} E_{\alpha \beta \gamma} + M_{\gamma \zeta} L_{\beta \delta \epsilon} D_{\alpha \beta \gamma}, \quad (25p) \]
\[ C_{\alpha \beta \gamma} M_{\beta \delta \epsilon} = K_{\alpha \beta \gamma} M_{\beta \delta \epsilon} + L_{\alpha \beta \gamma} K_{\alpha \beta \gamma} C_{\delta \beta \epsilon} + M_{\gamma \zeta} K_{\beta \delta \epsilon} F_{\epsilon \beta \gamma} + M_{\gamma \zeta} L_{\beta \delta \epsilon} D_{\alpha \beta \gamma} C_{\delta \beta \epsilon}, \quad (25q) \]
\[ M_{\beta \delta \epsilon} K_{\gamma \zeta} = M_{\gamma \zeta} M_{\beta \delta \epsilon} + L_{\gamma \zeta} M_{\alpha \beta \gamma} K_{\delta \beta \epsilon} + M_{\gamma \zeta} M_{\beta \delta \epsilon} F_{\beta \delta \epsilon} E_{\alpha \beta \gamma} + M_{\gamma \zeta} L_{\beta \delta \epsilon} D_{\alpha \beta \gamma} K_{\gamma \beta \gamma}, \quad (25r) \]
\[ C_{\gamma \zeta} M_{\beta \delta \epsilon} = M_{\alpha \beta \gamma} C_{\gamma \zeta} + K_{\alpha \beta \gamma} C_{\delta \beta \epsilon} F_{\epsilon \beta \gamma} + L_{\alpha \beta \gamma} C_{\delta \beta \epsilon} M_{\gamma \beta \gamma} + K_{\alpha \beta \gamma} B_{\alpha \delta \epsilon} F_{\beta \delta \epsilon} M_{\gamma \beta \gamma}, \quad (25s) \]
\[ F_{\alpha \beta \gamma} D_{\gamma \zeta} + E_{\alpha \beta \gamma} F_{\beta \delta \epsilon} K_{\gamma \beta \gamma} = D_{\beta \delta \epsilon} F_{\alpha \beta \gamma} + E_{\beta \delta \epsilon} D_{\alpha \beta \gamma} C_{\delta \beta \epsilon}, \quad (25t) \]
\[ B_{\gamma \zeta} C_{\alpha \delta \epsilon} + C_{\gamma \zeta} L_{\beta \delta \epsilon} E_{\alpha \beta \gamma} = L_{\alpha \beta \gamma} B_{\beta \delta \epsilon} + K_{\alpha \beta \gamma} B_{\alpha \delta \epsilon} E_{\beta \delta \epsilon}, \quad (25u) \]
\[ L_{\alpha \beta \gamma} D_{\beta \delta \epsilon} = D_{\gamma \zeta} L_{\alpha \beta \gamma} + E_{\gamma \zeta} K_{\alpha \beta \gamma} B_{\alpha \beta \gamma} + F_{\gamma \zeta} K_{\beta \delta \epsilon} E_{\alpha \beta \gamma} + F_{\gamma \zeta} L_{\beta \delta \epsilon} D_{\alpha \beta \gamma} B_{\alpha \beta \gamma}, \quad (25v) \]
\[ B_{\beta \delta \epsilon} F_{\alpha \beta \gamma} = F_{\alpha \beta \gamma} B_{\gamma \zeta} + D_{\alpha \beta \gamma} C_{\alpha \delta \epsilon} E_{\gamma \zeta} + E_{\alpha \beta \gamma} C_{\beta \delta \epsilon} L_{\gamma \beta \gamma} + D_{\alpha \beta \gamma} B_{\alpha \delta \epsilon} F_{\beta \delta \epsilon} L_{\gamma \beta \gamma}, \quad (25w) \]
\[ L_{\alpha \beta \gamma} A_{\beta \delta \epsilon} + K_{\alpha \beta \gamma} B_{\alpha \delta \epsilon} D_{\beta \delta \epsilon} = A_{\gamma \zeta} L_{\alpha \beta \gamma} + B_{\gamma \zeta} K_{\alpha \beta \gamma} B_{\alpha \beta \gamma} + C_{\gamma \zeta} K_{\beta \delta \epsilon} E_{\alpha \beta \gamma} + C_{\gamma \zeta} L_{\beta \delta \epsilon} D_{\alpha \beta \gamma} B_{\alpha \beta \gamma}, \quad (25x) \]
\[ A_{\beta \delta \epsilon} F_{\alpha \beta \gamma} + B_{\beta \delta \epsilon} D_{\alpha \beta \epsilon} C_{\alpha \beta \gamma} = F_{\alpha \beta \gamma} A_{\gamma \zeta} + D_{\alpha \beta \gamma} C_{\alpha \delta \epsilon} D_{\gamma \beta \gamma} + E_{\alpha \beta \gamma} C_{\delta \beta \epsilon} K_{\gamma \beta \gamma} + D_{\alpha \beta \gamma} B_{\alpha \delta \epsilon} F_{\beta \delta \epsilon} K_{\gamma \beta \gamma}, \quad (25y) \]
\[ M_{\alpha \beta \gamma} D_{\gamma \zeta} + L_{\alpha \beta \epsilon} F_{\beta \delta \epsilon} K_{\gamma \beta \gamma} = D_{\gamma \zeta} M_{\alpha \beta \gamma} + E_{\gamma \zeta} K_{\alpha \beta \gamma} C_{\alpha \beta \gamma} + F_{\gamma \zeta} K_{\beta \delta \epsilon} F_{\alpha \beta \gamma} + F_{\gamma \zeta} L_{\beta \delta \epsilon} D_{\alpha \gamma \zeta} C_{\alpha \beta \gamma}, \quad (25z) \]
\[ B_{\gamma \zeta} M_{\alpha \beta \gamma} + C_{\gamma \zeta} L_{\beta \delta \epsilon} F_{\alpha \beta \gamma} = M_{\alpha \beta \gamma} B_{\gamma \zeta} + K_{\alpha \beta \gamma} C_{\alpha \delta \epsilon} E_{\gamma \zeta} + L_{\alpha \beta \gamma} C_{\beta \delta \epsilon} L_{\gamma \beta \gamma} + K_{\alpha \beta \gamma} B_{\alpha \delta \epsilon} F_{\beta \delta \epsilon} L_{\gamma \beta \gamma}, \quad (25aa) \]
invariantalsounderthetransformation make any permutation of the parameters in these equations. In what follows we deduce some consequences from (25). Note that, since the proof is similar to the proof of proposition 3.2.

**Proof.** As explained in remark 4.3, we are allowed to make any permutation of the parameters. Thus, equations (25b)–(25g) can be replaced by equation (26), which have more clear structure.

**Proposition 4.4.** System (25) implies the following matrix equations

\[
\begin{pmatrix}
D_{\alpha\beta\gamma} & E_{\alpha\beta\gamma} \\
A_{\alpha\beta\gamma} & B_{\alpha\beta\gamma}
\end{pmatrix}
\begin{pmatrix}
D_{\zeta\delta\lambda} & E_{\zeta\delta\lambda} \\
A_{\zeta\delta\lambda} & B_{\zeta\delta\lambda}
\end{pmatrix}
= 
\begin{pmatrix}
D_{\zeta\delta\lambda} & B_{\zeta\delta\lambda} \\
D_{\zeta\delta\lambda} & E_{\zeta\delta\lambda}
\end{pmatrix},
\tag{26a}
\]

\[
\begin{pmatrix}
L_{\alpha\beta\gamma} & M_{\alpha\beta\gamma} \\
E_{\alpha\beta\gamma} & F_{\alpha\beta\gamma}
\end{pmatrix}
\begin{pmatrix}
L_{\gamma\delta\zeta} & M_{\gamma\delta\zeta} \\
E_{\gamma\delta\zeta} & F_{\gamma\delta\zeta}
\end{pmatrix}
= 
\begin{pmatrix}
L_{\gamma\delta\zeta} & F_{\gamma\delta\zeta} \\
L_{\gamma\delta\zeta} & M_{\gamma\delta\zeta}
\end{pmatrix},
\tag{26b}
\]

\[
\begin{pmatrix}
E_{\zeta\delta\lambda} & F_{\zeta\delta\lambda} \\
L_{\gamma\delta\zeta} & M_{\gamma\delta\zeta}
\end{pmatrix}
\begin{pmatrix}
E_{\alpha\beta\gamma} & M_{\alpha\beta\gamma} \\
E_{\alpha\beta\gamma} & F_{\alpha\beta\gamma}
\end{pmatrix}
= 
\begin{pmatrix}
E_{\alpha\beta\gamma} & F_{\alpha\beta\gamma} \\
E_{\alpha\beta\gamma} & M_{\alpha\beta\gamma}
\end{pmatrix}.
\tag{26c}
\]

**Proof.** As explained in remark 4.3, we are allowed to make any permutation of the parameters. Making the permutation \((\alpha, \beta, \gamma, \delta, \epsilon, \zeta) \rightarrow (\zeta, \alpha, \beta, \delta, \gamma)\) in the first of (25b) and the first of (25f) and taking the second from (25c) and the second from (25f), we obtain (26a). Equations (26b)–(26d) can be deduced from (25) similarly.

**Remark 4.5.** One can check that equation (26) is equivalent to (25b)–(25g), up to permutations of the parameters. Thus, equations (25b)–(25g) can be replaced by equation (26), which have more clear structure.

**Proposition 4.6.** For any vector space \(V\), the set of linear parametric tetrahedron maps (24) is invariant under the following transformations

\[
\begin{pmatrix}
A_{\alpha\beta\gamma} & B_{\alpha\beta\gamma} & C_{\alpha\beta\gamma} \\
D_{\alpha\beta\gamma} & E_{\alpha\beta\gamma} & F_{\alpha\beta\gamma} \\
K_{\alpha\beta\gamma} & L_{\alpha\beta\gamma} & M_{\alpha\beta\gamma}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
M_{\gamma\delta\lambda} & L_{\gamma\delta\lambda} & K_{\gamma\delta\lambda} \\
F_{\gamma\delta\lambda} & E_{\gamma\delta\lambda} & D_{\gamma\delta\lambda} \\
C_{\gamma\delta\lambda} & B_{\gamma\delta\lambda} & A_{\gamma\delta\lambda}
\end{pmatrix},
\tag{27}
\]

\[
\begin{pmatrix}
A_{\alpha\beta\gamma} & B_{\alpha\beta\gamma} & C_{\alpha\beta\gamma} \\
D_{\alpha\beta\gamma} & E_{\alpha\beta\gamma} & F_{\alpha\beta\gamma} \\
K_{\alpha\beta\gamma} & L_{\alpha\beta\gamma} & M_{\alpha\beta\gamma}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
-A_{\alpha\beta\gamma} & B_{\alpha\beta\gamma} & -C_{\alpha\beta\gamma} \\
D_{\alpha\beta\gamma} & -E_{\alpha\beta\gamma} & F_{\alpha\beta\gamma} \\
-K_{\alpha\beta\gamma} & L_{\alpha\beta\gamma} & -M_{\alpha\beta\gamma}
\end{pmatrix}.
\tag{28}
\]

Let \(V = \mathbb{K}^n\) for some \(n \in \mathbb{Z}_{>0}\). Then \(A_{\alpha\beta\gamma}, B_{\alpha\beta\gamma}, C_{\alpha\beta\gamma}, D_{\alpha\beta\gamma}, E_{\alpha\beta\gamma}, F_{\alpha\beta\gamma}, K_{\alpha\beta\gamma}, L_{\alpha\beta\gamma}, M_{\alpha\beta\gamma}\) in (24) are \(n \times n\) matrices. In this case, the set of linear parametric tetrahedron maps (24) is invariant also under the transformation

\[
\begin{pmatrix}
A_{\alpha\beta\gamma} & B_{\alpha\beta\gamma} & C_{\alpha\beta\gamma} \\
D_{\alpha\beta\gamma} & E_{\alpha\beta\gamma} & F_{\alpha\beta\gamma} \\
K_{\alpha\beta\gamma} & L_{\alpha\beta\gamma} & M_{\alpha\beta\gamma}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
A_{\alpha\beta\gamma} & B_{\alpha\beta\gamma} & C_{\alpha\beta\gamma} \\
D_{\alpha\beta\gamma} & E_{\alpha\beta\gamma} & F_{\alpha\beta\gamma} \\
K_{\alpha\beta\gamma} & L_{\alpha\beta\gamma} & M_{\alpha\beta\gamma}
\end{pmatrix}^T = 
\begin{pmatrix}
A_{\alpha\beta\gamma}^T & D_{\alpha\beta\gamma}^T & K_{\alpha\beta\gamma}^T \\
B_{\alpha\beta\gamma}^T & E_{\alpha\beta\gamma}^T & L_{\alpha\beta\gamma}^T \\
C_{\alpha\beta\gamma}^T & F_{\alpha\beta\gamma}^T & M_{\alpha\beta\gamma}^T
\end{pmatrix}.
\tag{29}
\]
Proof. The statement about the transformation (27) follows from proposition 2.1 with \( W = \Omega \times V \).

The case of the transformation (28) follows from proposition 2.2, if we take \( W = \Omega \times V \) and consider the map

\[
\sigma : \Omega \times V \to \Omega \times V, \quad \sigma(\xi, v) = (\xi, -v), \quad \xi \in \Omega, \quad v \in V.
\]

To prove the statement about the transformation (29), one can apply the transpose operation to both sides of the parametric tetrahedron equation (5) for \( T_{\alpha\beta\gamma} \) given by (24).

□

Proposition 4.7. Let \( T_{1,\alpha\beta\gamma}, T_{2,\alpha\beta\gamma} \) be linear parametric tetrahedron maps of the form

\[
T_{1,\alpha\beta\gamma} = \begin{pmatrix}
A_{\alpha\beta\gamma} & B_{\alpha\beta\gamma} & 0 \\
D_{\alpha\beta\gamma} & E_{\alpha\beta\gamma} & 0 \\
0 & 0 & M_{\alpha\beta\gamma}
\end{pmatrix}, \quad T_{2,\alpha\beta\gamma} = \begin{pmatrix}
0 & 0 & 0 \\
0 & \tilde{E}_{\alpha\beta\gamma} & \tilde{F}_{\alpha\beta\gamma} \\
0 & \tilde{L}_{\alpha\beta\gamma} & \tilde{M}_{\alpha\beta\gamma}
\end{pmatrix}.
\]

Let \( l, m \in \mathbb{K}, l \neq 0 \). Then

\[
T_{1,\alpha\beta\gamma}^{lm} = \begin{pmatrix}
lA_{\alpha\beta\gamma} & B_{\alpha\beta\gamma} & 0 \\
D_{\alpha\beta\gamma} & E_{\alpha\beta\gamma} & 0 \\
0 & 0 & M_{\alpha\beta\gamma}
\end{pmatrix}, \quad T_{2,\alpha\beta\gamma}^{lm} = \begin{pmatrix}
m\tilde{A}_{\alpha\beta\gamma} & 0 & 0 \\
0 & \tilde{E}_{\alpha\beta\gamma} & \tilde{F}_{\alpha\beta\gamma} \\
0 & \tilde{L}_{\alpha\beta\gamma} & \tilde{M}_{\alpha\beta\gamma}
\end{pmatrix},
\]

are linear parametric tetrahedron maps as well.

Proof. For each \( i = 1, 2 \), the fact that \( T_{i,\alpha\beta\gamma} \) obeys equation (25) implies that \( T_{i,\alpha\beta\gamma}^{lm} \) obeys these equations as well.

□

5. Differentials of Yang–Baxter and tetrahedron maps

In this section, when we consider maps of manifolds, we assume that they are either smooth, or complex-analytic, or rational, so that the differential is defined for such a map.

Let \( \mathcal{M} \) be a manifold. Consider the tangent bundle \( \tau : T\mathcal{M} \to \mathcal{M} \). Then

- The bundle \( \tau \times \tau : T\mathcal{M} \times T\mathcal{M} \to \mathcal{M} \times \mathcal{M} \) can be identified with the tangent bundle of the manifold \( \mathcal{M} \times \mathcal{M} \),
- The bundle \( \tau \times \tau \times \tau : T\mathcal{M} \times T\mathcal{M} \times T\mathcal{M} \to \mathcal{M} \times \mathcal{M} \times \mathcal{M} \) can be identified with the tangent bundle of the manifold \( \mathcal{M} \times \mathcal{M} \times \mathcal{M} \).

Using these identifications and the general procedure to define the differential of a map of manifolds, for any maps

\[
Y : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M}, \quad T : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M} \times \mathcal{M}
\]

we obtain the differentials

\[
dY : T\mathcal{M} \times T\mathcal{M} \to T\mathcal{M} \times T\mathcal{M},
\]
\[
dT : T\mathcal{M} \times T\mathcal{M} \times T\mathcal{M} \to T\mathcal{M} \times T\mathcal{M} \times T\mathcal{M}.
\]
**Remark 5.1.** The maps $dY, dT$ are linear along the fibres of the bundle $TM \rightarrow M$ and, in general, are nonlinear with respect to (local) coordinates on the manifold $M$. The explicit computation of $dY, dT$ in coordinates is presented below.

We need the following well-known property of differentials.

**Lemma 5.2.** Let $M_1, M_2, M_3$ be manifolds. Consider maps $f: M_1 \rightarrow M_2$, $g: M_2 \rightarrow M_3$ and their differentials $df: TM_1 \rightarrow TM_2$, $dg: TM_2 \rightarrow TM_3$.

Then for the differential $(dg \circ f): TM_1 \rightarrow TM_3$ of the composition map $g \circ f: M_1 \rightarrow M_3$ we have $d(g \circ f) = dg \circ df$.

**Proposition 5.3.** Let $M$ be a manifold. For any Yang–Baxter map $Y: M \times M \rightarrow M \times M$, the differential $dY: TM \times TM \rightarrow TM \times TM$ is a Yang–Baxter map of the manifold $TM \times TM$.

**Proof.** Consider the permutation maps

$$
P^{12}: M \times M \times M \rightarrow M \times M \times M, \quad P^{12}(a_1, a_2, a_3) = (a_2, a_1, a_3), \quad a_i \in M,
$$

$$
\tilde{P}^{12}: TM \times TM \times TM \rightarrow TM \times TM \times TM,
$$

$$
\tilde{P}^{12}(b_1, b_2, b_3) = (b_2, b_1, b_3), \quad b_i \in TM,
$$

and the identity maps $Id_M: M \rightarrow M$, $Id_{TM}: TM \rightarrow TM$.

$Y: M \times M \rightarrow M \times M$ obeys the Yang–Baxter equation

$$
Y^{12} \circ Y^{13} \circ Y^{23} = Y^{23} \circ Y^{13} \circ Y^{12}, \quad (30)
$$

where the maps $Y^{12}, Y^{13}, Y^{23}: M \times M \times M \rightarrow M \times M \times M$ can be described as follows

$$
Y^{12} = Y \times Id_M, \quad Y^{23} = Id_M \times Y, \quad (31)
$$

$$
Y^{13} = P^{12} \circ (Id_M \times Y) \circ P^{12}. \quad (32)
$$

We need to prove that the map $dY: TM \times TM \rightarrow TM \times TM$ obeys the Yang–Baxter equation

$$
(dY)^{12} \circ (dY)^{13} \circ (dY)^{23} = (dY)^{23} \circ (dY)^{13} \circ (dY)^{12}, \quad (33)
$$

where

$$
(dY)^{12} = dY \times Id_{TM}, \quad (dY)^{23} = Id_{TM} \times dY,
$$

$$
(dY)^{13} = \tilde{P}^{12} \circ (Id_{TM} \times dY) \circ \tilde{P}^{12}.
$$

By lemma 5.2,

$$
d(Y^{12} \circ Y^{13} \circ Y^{23}) = d(Y^{12}) \circ d(Y^{13}) \circ d(Y^{23}),
$$

$$
d(Y^{23} \circ Y^{13} \circ Y^{12}) = d(Y^{23}) \circ d(Y^{13}) \circ d(Y^{12}). \quad (34)
$$
Taking the differential of (30) and using (34), we obtain

\[ d(Y_{12}) \circ d(Y_{13}) \circ d(Y_{23}) = d(Y_{23}) \circ d(Y_{13}) \circ d(Y_{12}). \] (35)

From (31) one derives

\[ d(Y_{12}) = d(Y \times \text{Id}_M) = dY \times \text{Id}_{TM} = (dY)_{12}, \]
\[ d(Y_{23}) = d(\text{Id}_M \times Y) = \text{Id}_{TM} \times dY = (dY)_{23}. \]

Using lemma 5.2 and the relation \( d(P_{12}) = \tilde{P}_{12} \), from (32) we obtain

\[ d(Y_{13}) = d((P_{12} \circ (\text{Id}_M \times Y) \circ P_{12}) = d(P_{12}) \circ d(\text{Id}_M \times Y) \circ d(P_{12}) \]
\[ = \tilde{P}_{12} \circ (\text{Id}_{TM} \times dY) \circ \tilde{P}_{12} = (dY)_{13}. \]

Thus, we have

\[ d(Y_{12}) = (dY)_{12}, \quad d(Y_{13}) = (dY)_{13}, \quad d(Y_{23}) = (dY)_{23}. \] (36)

Substituting (36) in (35), one obtains (33).

\[ \blacksquare \]

The statement of proposition 5.3 was used (without proof) in [8].

**Proposition 5.4.** Let \( M \) be a manifold. For any tetrahedron map \( T : M \times M \times M \to M \times M \times M \), the differential

\[ dT : TM \times TM \times TM \to TM \times TM \times TM, \]

is a tetrahedron map of the manifold \( TM \times TM \times TM \).

**Proof.** The map \( T : M \times M \times M \to M \times M \times M \) obeys the tetrahedron equation

\[ T^{123} \circ T^{145} \circ T^{246} \circ T^{356} = T^{356} \circ T^{246} \circ T^{145} \circ T^{123}. \] (37)

We need to show that \( dT : TM \times TM \times TM \to TM \times TM \times TM \) satisfies the tetrahedron equation

\[ (dT)^{123} \circ (dT)^{145} \circ (dT)^{246} \circ (dT)^{356} = (dT)^{356} \circ (dT)^{246} \circ (dT)^{145} \circ (dT)^{123}. \] (38)

Here for \( 1 \leq i < j < k \leq 6 \) the map \( (dT)^{ijk} : (TM)^6 \to (TM)^6 \) is constructed from \( dT \) similarly to the construction of \( T^{ijk} \) from \( T \).

Taking the differential of (37) and using lemma 5.2, we derive

\[ d(T^{123}) \circ d(T^{145}) \circ d(T^{246}) \circ d(T^{356}) = d(T^{356}) \circ d(T^{246}) \circ d(T^{145}) \circ d(T^{123}). \] (39)

Similarly to obtaining (36), one can show the following

\[ d(T^{123}) = (dT)^{123}, \quad d(T^{145}) = (dT)^{145}, \]
\[ d(T^{246}) = (dT)^{246}, \quad d(T^{356}) = (dT)^{356}. \] (40)
For example, let us prove $d(T^{246}) = (dT)^{246}$. For any $i, j \in \{1, 2, \ldots, 6\}$, $i < j$, let $P^{ij} : (\mathcal{M})^6 \to (\mathcal{M})^6$ be the permutation map which interchanges the $i$th and $j$th factors of the Cartesian product $(\mathcal{M})^6$. Then

$$P^{ij} = d(P^{ij}) : (T, \mathcal{M})^6 \to (T, \mathcal{M})^6,$$

is the permutation map of the same type for the Cartesian product $(T, \mathcal{M})^6$. We have

$$T^{246} = P^{45} \circ P^{23} \circ P^{34} \circ (\text{id}_{(\mathcal{M})^3} \times T) \circ P^{34} \circ P^{23} \circ P^{45},$$

$$(dT)^{246} = \tilde{P}^{45} \circ \tilde{P}^{23} \circ \tilde{P}^{34} \circ (\text{id}_{(T, \mathcal{M})^3} \times dT) \circ \tilde{P}^{34} \circ \tilde{P}^{23} \circ \tilde{P}^{45}.$$

Using these formulas, lemma 5.2, and the relation $d(P^{ij}) = P^{ij}$, we obtain

$$d(T^{246}) = d \left( P^{45} \circ P^{23} \circ P^{34} \circ (\text{id}_{(\mathcal{M})^3} \times T) \circ P^{34} \circ P^{23} \circ P^{45} \right)$$

$$= d(P^{45}) \circ d(P^{23}) \circ d(P^{34}) \circ d(\text{id}_{(\mathcal{M})^3} \times T) \circ d(P^{34}) \circ d(P^{23}) \circ d(P^{45})$$

$$= \tilde{P}^{45} \circ \tilde{P}^{23} \circ \tilde{P}^{34} \circ (\text{id}_{(T, \mathcal{M})^3} \times dT) \circ \tilde{P}^{34} \circ \tilde{P}^{23} \circ \tilde{P}^{45} = (dT)^{246}.$$

Similarly, one can prove all of (40). Substituting (40) in (39), one obtains (38). \qed

**Corollary 5.5.** Consider a manifold $\mathcal{M}$, a tetrahedron map $T : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M}$, and its differential

$$dT : T, \mathcal{M} \times T, \mathcal{M} \times T, \mathcal{M} \to T, \mathcal{M} \times T, \mathcal{M} \times T, \mathcal{M}.$$

Let $a \in \mathcal{M}$ such that $T((a, a, a)) = (a, a, a)$. Consider the tangent space $T_a \mathcal{M} \subset T, \mathcal{M}$ at the point $a \in \mathcal{M}$. Then we have

$$dT(T_a \mathcal{M} \times T_a \mathcal{M} \times T_a \mathcal{M}) \subset T_a \mathcal{M} \times T_a \mathcal{M} \times T_a \mathcal{M} \subset T, \mathcal{M} \times T, \mathcal{M} \times T, \mathcal{M},$$

and the map

$$dT|_{(a,a,a)} : T_a \mathcal{M} \times T_a \mathcal{M} \times T_a \mathcal{M} \to T_a \mathcal{M} \times T_a \mathcal{M} \times T_a \mathcal{M},$$

is a linear tetrahedron map. Here $dT|_{(a,a,a)}$ is the restriction of the map $dT$ to $T_a \mathcal{M} \times T_a \mathcal{M} \times T_a \mathcal{M}$.

**Proof.** The property $T((a, a, a)) = (a, a, a)$ and the definition of the differential imply (41) and the fact that the map (42) is linear.

By proposition 5.4, the differential $dT$ is a tetrahedron map. Therefore, its restriction $dT|_{(a,a,a)}$ to $T_a \mathcal{M} \times T_a \mathcal{M} \times T_a \mathcal{M}$ is a tetrahedron map as well. \qed

**Remark 5.6.** The definition of the differential implies that the linear tetrahedron map $dT|_{(a,a,a)}$ described in corollary 5.5 can be regarded as a linear approximation of the nonlinear tetrahedron map $T$ at the point $(a, a, a) \in \mathcal{M} \times \mathcal{M} \times \mathcal{M}$. Explicit examples of $dT|_{(a,a,a)}$ are presented in examples 5.8 and 5.10.

Let $W$ be a set. Three maps $G, H, Q : W \times W \to W \times W$ are called entwining Yang–Baxter maps if they satisfy

$$G^{12} \circ H^{13} \circ Q^{23} = Q^{23} \circ H^{13} \circ G^{12},$$

(43)
(see, e.g. [27, 36, 41]). The maps $G^{12}, H^{13}, Q^{23} : W \times W \times W \to W \times W \times W$ are constructed from $G, H, Q$ in the standard way. One has (43), but the maps $G, H, Q$ individually do not necessarily satisfy the Yang–Baxter equation. Similarly to proposition 5.3, one can prove the following.

**Proposition 5.7.** Let $\mathcal{M}$ be a manifold. For any entwining Yang–Baxter maps $G, H, Q : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M}$, the differentials $dG, dH, dQ : T\mathcal{M} \times T\mathcal{M} \to T\mathcal{M} \times T\mathcal{M} \times T\mathcal{M}$ are entwining Yang–Baxter maps of the manifold $T\mathcal{M} \times T\mathcal{M}$.

Let $n \in \mathbb{Z}_{>0}$. Let $\mathcal{M}$ be an $n$-dimensional manifold with (local) coordinates $x_1, \ldots, x_n$. Then $\dim T\mathcal{M} = 2n$, and we have the (local) coordinates $x_1, \ldots, x_n, X_1, \ldots, X_n$ on the manifold $T\mathcal{M}$, where $X_i$ corresponds to the differential $dx_i$, which can be regarded as a function on $T\mathcal{M}$. (Thus, the functions $X_1, \ldots, X_n$ are linear along the fibres of the bundle $T\mathcal{M} \to \mathcal{M}$.)

To study maps of the form

$$\mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M} \times \mathcal{M},$$

$$T\mathcal{M} \times T\mathcal{M} \times T\mathcal{M} \to T\mathcal{M} \times T\mathcal{M} \times T\mathcal{M},$$

we consider

- Six copies of the manifold $\mathcal{M}$ with coordinate systems

  $$(x_1, \ldots, x_n), \quad (y_1, \ldots, y_n), \quad (z_1, \ldots, z_n),$$

  $$(\tilde{x}_1, \ldots, \tilde{x}_n), \quad (\tilde{y}_1, \ldots, \tilde{y}_n), \quad (\tilde{z}_1, \ldots, \tilde{z}_n),$$

- Six copies of the manifold $T\mathcal{M}$ with coordinate systems

  $$(x_1, \ldots, x_n, X_1, \ldots, X_n), \quad (y_1, \ldots, y_n, Y_1, \ldots, Y_n),$$

  $$(z_1, \ldots, z_n, Z_1, \ldots, Z_n), \quad (\tilde{x}_1, \ldots, \tilde{x}_n, \tilde{X}_1, \ldots, \tilde{X}_n),$$

  $$(\tilde{y}_1, \ldots, \tilde{y}_n, \tilde{Y}_1, \ldots, \tilde{Y}_n), \quad (\tilde{z}_1, \ldots, \tilde{z}_n, \tilde{Z}_1, \ldots, \tilde{Z}_n).$$

Here, for each $i = 1, \ldots, n$, the functions $X_i, Y_i, Z_i, \tilde{X}_i, \tilde{Y}_i, \tilde{Z}_i$ correspond to the differentials $dx_i, dy_i, dz_i, d\tilde{x}_i, d\tilde{y}_i, d\tilde{z}_i$. Below we use the following notation

$$x = (x_1, \ldots, x_n), \quad y = (y_1, \ldots, y_n), \quad z = (z_1, \ldots, z_n),$$

$$X = (X_1, \ldots, X_n), \quad Y = (Y_1, \ldots, Y_n), \quad Z = (Z_1, \ldots, Z_n),$$

$$\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n), \quad \tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_n), \quad \tilde{z} = (\tilde{z}_1, \ldots, \tilde{z}_n),$$

$$\tilde{X} = (\tilde{X}_1, \ldots, \tilde{X}_n), \quad \tilde{Y} = (\tilde{Y}_1, \ldots, \tilde{Y}_n), \quad \tilde{Z} = (\tilde{Z}_1, \ldots, \tilde{Z}_n).$$

Consider a tetrahedron map

$$\mathbf{T} : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M} \times \mathcal{M}, \quad (x, y, z) \mapsto (\tilde{x}, \tilde{y}, \tilde{z}),$$

$$\tilde{x}_i = f_i(x, y, z), \quad \tilde{y}_i = g_i(x, y, z), \quad \tilde{z}_i = h_i(x, y, z), \quad i = 1, \ldots, n.$$
Its differential is the following tetrahedron map
\[ d\mathbf{T} : T\mathcal{M} \times T\mathcal{M} \times T\mathcal{M} \to T\mathcal{M} \times T\mathcal{M} \times T\mathcal{M}, \]
\[(x, X, y, Y, z, Z) \mapsto (x, X, y, Y, z, Z), \]
where \( x = f(x, y, z), \ y = g(x, y, z), \ z = h(x, y, z), \ i = 1, \ldots, n, \)
\[ \tilde{x}_i = f_i(x, y, z), \ \tilde{y}_i = g_i(x, y, z), \ \tilde{z}_i = h_i(x, y, z), \]
\[ \tilde{x}_i = \sum_{j=1}^{n} \left( \frac{\partial f(x, y, z)}{\partial x_j} X_j + \frac{\partial f(x, y, z)}{\partial y_j} Y_j + \frac{\partial f(x, y, z)}{\partial z_j} Z_j \right), \]
\[ \tilde{y}_i = \sum_{j=1}^{n} \left( \frac{\partial g(x, y, z)}{\partial x_j} X_j + \frac{\partial g(x, y, z)}{\partial y_j} Y_j + \frac{\partial g(x, y, z)}{\partial z_j} Z_j \right), \]
\[ \tilde{z}_i = \sum_{j=1}^{n} \left( \frac{\partial h(x, y, z)}{\partial x_j} X_j + \frac{\partial h(x, y, z)}{\partial y_j} Y_j + \frac{\partial h(x, y, z)}{\partial z_j} Z_j \right). \]

Note that the map \( d\mathbf{T} \) is linear with respect to \( X, Y, Z \) and, in general, is nonlinear with respect to \( x, y, z \).

**Example 5.8.** Let \( n = \dim \mathcal{M} = 1 \). Consider the well-known electric network transformation
\[ \mathbf{T} : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M} \times \mathcal{M}, \ (x, y, z) \mapsto (\tilde{x}, \tilde{y}, \tilde{z}), \]
\[ \tilde{x} = \frac{xy}{x + z + xyz}, \ \tilde{y} = x + z + xyz, \ \tilde{z} = \frac{yz}{xyz + x + z}, \]
which is a tetrahedron map [26, 47]. Its differential is the following tetrahedron map
\[ d\mathbf{T} : T\mathcal{M} \times T\mathcal{M} \times T\mathcal{M} \to T\mathcal{M} \times T\mathcal{M} \times T\mathcal{M}, \ (x, X, y, Y, z, Z) \mapsto (\tilde{x}, \tilde{y}, \tilde{z}). \]
\[ \tilde{x} = \frac{xy}{x + z + xyz}, \ \tilde{y} = x + z + xyz, \ \tilde{z} = \frac{yz}{xyz + x + z}, \]
\[ \tilde{X} = \frac{-xy(1 + xy)Z + yzX + x(x + z)Y}{(xyz + x + z)^2}, \ \tilde{Y} = X + Z + xyZ + xzY + yzX, \]
\[ \tilde{Z} = \frac{-yz(yz + 1)xZ + x(x + z)Y + xyZ}{(xyz + x + z)^2}. \]

We assume that \( x, y, z, \tilde{x}, \tilde{y}, \tilde{z} \) take values in \( \mathbb{C} \), so \( \mathcal{M} \) is a complex manifold. Consider \( i = \sqrt{-1} \in \mathbb{C} \). Formulas (44) and (45) imply \( \mathbf{T}(i, i, i) = (i, i, i) \).

Let \( a = i \). The coordinate system on \( \mathcal{M} \) gives the isomorphism \( T_a\mathcal{M} \cong \mathbb{C} \). By corollary 5.5, we obtain the linear tetrahedron map \( d\mathbf{T}_{|i,i,i} : \mathbb{C}^3 \to \mathbb{C}^3 \). To compute it, we substitute \( x = y = z = i \) in (47) and (48) and derive the linear map
\[ \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \mapsto \begin{pmatrix} X + 2Y \\ -Y \\ 2Y + Z \end{pmatrix}. \]
with the matrix \[
\begin{pmatrix}
A & B & C \\
D & E & F \\
K & L & M
\end{pmatrix} = \begin{pmatrix}
1 & 2 & 0 \\
0 & -1 & 0 \\
0 & 2 & 1
\end{pmatrix},
\]
which is of the form (23) for \(a = 1, b = -1, c = 1\).

The linear tetrahedron map (49) is a linear approximation of the map (44) and (45) at the point \((i, i, i)\) in the following sense. We have
\[
T((i + \varepsilon X, i + \varepsilon Y, i + \varepsilon Z)) = (i + \varepsilon (X + 2Y) + O(\varepsilon^2), i - \varepsilon Y + O(\varepsilon^2), i + \varepsilon (2Y + Z) + O(\varepsilon^2)).
\]

**Example 5.9.** Let \(n = \dim M = 2\). Consider the Kassotakis–Nieszporski–Papageorgiou–Tongas map (map (33) in [30])
\[
T: M \times M \times M \to M \times M \times M,
\]
\[
(x_1, x_2, y_1, y_2, z_1, z_2) \mapsto (\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2, \tilde{z}_1, \tilde{z}_2),
\]
\[
\tilde{x}_1 = \frac{y_1 + x_1 z_1}{z_1}, \quad \tilde{x}_2 = y_2 z_1, \quad \tilde{y}_1 = x_1 z_1,
\]
\[
\tilde{y}_2 = \frac{(y_1 + x_1 z_1) z_2}{z_1}, \quad \tilde{z}_1 = \frac{y_1 z_1}{y_1 + x_1 z_1}, \quad \tilde{z}_2 = \frac{y_2}{x_1}.
\]
Its differential is the following tetrahedron map
\[
dT: T M \times T M \times T M \to T M \times T M \times T M,
\]
\[
(x_1, x_2, x_1, y_1, y_2, y_1, y_2, z_1, z_2, Z_1, Z_2) \mapsto (\tilde{x}_1, \tilde{x}_2, \tilde{X}_1, \tilde{X}_2, \tilde{Y}_1, \tilde{Y}_2, \tilde{Z}_1, \tilde{Z}_2),
\]
\[
\tilde{X}_1 = X_1 + \frac{1}{z_1} Y_1 - \frac{y_1}{z_1} Z_1, \quad \tilde{X}_2 = z_1 Y_2 + y_2 Z_1,
\]
\[
\tilde{Y}_1 = z_1 X_1 + x_1 Z_1, \quad \tilde{Y}_2 = z_2 X_1 + \frac{z_2}{z_1} Y_1 - \frac{y_1 z_2}{z_1} Z_1 + \frac{y_1 + x_1 z_1}{z_1} Z_2,
\]
\[
\tilde{Z}_1 = -\frac{y_1 z_1^2}{(y_1 + x_1 z_1)^2} X_1 + \frac{x_1 z_1^2}{(y_1 + x_1 z_1)^2} Y_1 + \frac{y_2^2}{(y_1 + x_1 z_1)^2} Z_1,
\]
\[
\tilde{Z}_2 = -\frac{y_2}{x_1} X_1 + \frac{1}{x_1} Y_1.
\]

**Example 5.10.** Let \(n = \dim M = 2\). Dimakis and Müller-Hoissen [12] constructed the tetrahedron map
\[
T: M \times M \times M \to M \times M \times M,
\]
\[
(x_1, x_2, y_1, y_2, z_1, z_2) \mapsto (\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2, \tilde{z}_1, \tilde{z}_2),
\]
Therefore, by corollary 5.5, one obtains the linear tetrahedron map
\[ \tilde{x}_1 = y_1 C, \quad \tilde{x}_2 = \left( y_1 - \frac{A}{x_1} \right) C, \quad \tilde{y}_1 = \frac{x_1}{C}, \quad \tilde{y}_2 = 1 - B, \]
\[ \tilde{z}_1 = \frac{z_1 y_1 (x_1 - x_2)}{A}, \quad \tilde{z}_2 = 1 - \frac{(1 - y_2)(1 - z_2)}{B}. \]
\[ A = y_2 z_1 x_1 - y_2 x_1 - z_1 x_2 + x_1 y_1, \quad B = y_2 z_2 x_1 - y_2 x_1 - z_2 x_2 + 1, \]
\[ C = \frac{A B - A (1 - y_2)(1 - z_2) x_1 - B z_1 (x_1 - x_2)}{A B - A (1 - y_2)(1 - z_2) - B z_1 y_1 (x_1 - x_2)}. \]

We have found the following invariants for this map
\[ I_1(x_1, x_2, y_1, y_2, z_1, z_2) = x_1 y_1, \quad I_2(x_1, x_2, y_1, y_2, z_1, z_2) = (y_2 - 1)(z_2 - 1), \]
(52)
\[ I_3(x_1, x_2, y_1, y_2, z_1, z_2) = (x_1 - x_2)(y_1 - y_2)(z_1 - z_2). \]
(53)
That is, for \( \tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2, \tilde{z}_1, \tilde{z}_2 \) given by the above formulas, one has
\[ I_j(\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2, \tilde{z}_1, \tilde{z}_2) = I_j(x_1, x_2, y_1, y_2, z_1, z_2), \quad j = 1, 2, 3. \]
The invariants \( I_1, I_2, I_3 \) are functionally independent.

We assume that \( x, y, z, \tilde{x}, \tilde{y}, \tilde{z} \) take values in \( \mathbb{C} \), so \( M \) is a complex manifold. One can check that for any nonzero \( c \in \mathbb{C} \) we have \( T((\alpha, a, a)) = (\alpha, a, a) \), where \( \alpha = (c, 0) \in M \).

Therefore, by corollary 5.5, one obtains the linear tetrahedron map
\[ dT_{((\alpha, a, a))}: T_a M \times T_a M \times T_a M \rightarrow T_a M \times T_a M \times T_a M. \]
The coordinate system on \( M \) gives the isomorphism \( T_a M \cong \mathbb{C}^2 \), so we have \( dT_{((\alpha, a, a))}: \mathbb{C}^6 \rightarrow \mathbb{C}^6 \).
Computing \( dT \) and \( dT_{((\alpha, a, a))} \) for \( a = (c, 0) \), one derives that \( dT_{((\alpha, a, a))} \) is given by the matrix
\[
\begin{pmatrix}
1 & 0 & \frac{c-1}{c} & (c-1)^2(c+1) & -\frac{c-1}{c} & 1 - c \\
0 & 1 & \frac{c}{1-c} & 0 & 0 & 0 \\
0 & 0 & 1 & -\frac{(c-1)^2(c+1)}{c} & \frac{c-1}{c} & c-1 \\
0 & 0 & 0 & \frac{c}{1-c} & 0 & 0 \\
0 & 0 & 0 & 1 - c & 1 & 0 \\
0 & 0 & 0 & 1 - c & 0 & 1 \\
\end{pmatrix},
\]
(54)
which coincides with (21). Thus, the linear map \( dT_{((\alpha, a, a))} \) is of the form (21) and (22).

According to remark 5.6, the linear tetrahedron map (54) is a linear approximation of the nonlinear tetrahedron map (51) at the point \( (a, a, a) \in M \times M \times M \) with \( a = (c, 0), c \neq 0. \)

6. Parametric Yang–Baxter maps associated with matrix groups

Let \( G \) be a group and \( p \in \mathbb{Z}_{>0} \). It is known that one has the following Yang–Baxter map
\[ F: G \times G \rightarrow G \times G, \quad F(x, y) = (x, x^p y x^{-p}), \quad x, y \in G, \]
(55)
(see, e.g. [10] and references therein). For \( p = 1 \) this map appeared in [16].

Assume that \( \mathbb{K} \) is either \( \mathbb{R} \) or \( \mathbb{C} \). Let \( n \in \mathbb{Z}_{>0} \) and consider the matrix group \( G = GL_n(\mathbb{K}) \subset Mat_n(\mathbb{K}) \). Then \( G \) is a manifold, and for each \( x \in G = GL_n(\mathbb{K}) \) one has the tangent space
\( T_g G \cong \text{Mat}_n(\mathbb{K}) \). Set \( M = \text{Mat}_n(\mathbb{K}) \). The tangent bundle of the manifold \( G \) can be identified with the trivial bundle \( G \times M \to G \).

For \( G = \text{GL}_n(\mathbb{K}) \), the Yang–Baxter map (55) is an analytic diffeomorphism of the manifold \( G \times G \). The differential \( dF \) of this diffeomorphism \( F \) can be identified with the following map

\[
\begin{align*}
 dF : (G \times M) \times (G \times M) & \to (G \times M) \times (G \times M), \\
 dF((x, M_1), (y, M_2)) & = \left( (x, M_1), \left( x^p y x^{-p}, \frac{\partial}{\partial \varepsilon}(x + \varepsilon M_1 y + \varepsilon M_2(x + \varepsilon M_1^{-p})) \right) \right),
\end{align*}
\]

where \( x, y \in G = \text{GL}_n(\mathbb{K}) \), \( M_1, M_2 \in M = \text{Mat}_n(\mathbb{K}) \).

By proposition 5.3, since \( F \) is a Yang–Baxter map, its differential \( dF \) is a Yang–Baxter map as well.

Let \( \Omega \subset G \) be an abelian subgroup of \( G \). Denote by \( Y : (\Omega \times M) \times (\Omega \times M) \to (\Omega \times M) \times (\Omega \times M) \) the restriction of the map \( dF \) to the subset \( (\Omega \times M) \times (\Omega \times M) \subset (G \times M) \times (G \times M) \). As \( dF \) is a Yang–Baxter map, \( Y \) is a Yang–Baxter map as well.

Let \( a, b \in \Omega \). Since \( ab = ba \), computing (56) for \( x = a \) and \( y = b \), we obtain

\[
Y : (\Omega \times \text{Mat}_n(\mathbb{K})) \times (\Omega \times \text{Mat}_n(\mathbb{K})) \to (\Omega \times \text{Mat}_n(\mathbb{K})) \times (\Omega \times \text{Mat}_n(\mathbb{K})),
\]

\[
Y((a, M_1), (b, M_2)) = \left( (a, M_1), \left( b, a^p b M_2 a^{-p} + \sum_{i=0}^{p-1} (a M_1 a^{-i-1} b - b a M_1 a^{i-1}) \right) \right).
\]

The Yang–Baxter map (57) can be interpreted as the following linear parametric Yang–Baxter map

\[
Y_{a,b} : \text{Mat}_n(\mathbb{K}) \times \text{Mat}_n(\mathbb{K}) \to \text{Mat}_n(\mathbb{K}) \times \text{Mat}_n(\mathbb{K}),
\]

\[
Y_{a,b}(M_1, M_2) = \left( M_1, a^p b M_2 a^{-p} + \sum_{i=0}^{p-1} (a M_1 a^{-i-1} b - b a M_1 a^{i-1}) \right),
\]

with parameters \( a, b \in \Omega \). We need the following result from [8].

**Proposition 6.1** (8). Let \( V \) be a vector space over a field \( \mathbb{K} \). Consider a linear parametric Yang–Baxter map \( Y_{\alpha,\beta} : V \times V \to V \times V \) given by the formula

\[
Y_{\alpha,\beta} : \frac{x}{y} \mapsto \frac{u}{v} = \begin{pmatrix} A_{\alpha,\beta} & B_{\alpha,\beta} \\ C_{\alpha,\beta} & D_{\alpha,\beta} \end{pmatrix} \frac{x}{y}, \quad \alpha, \beta, A_{\alpha,\beta}, B_{\alpha,\beta}, C_{\alpha,\beta}, D_{\alpha,\beta} \in \text{End}(V), x, y \in V.
\]

Then, for any nonzero constant \( l \in \mathbb{K} \), the map

\[
Y^l_{\alpha,\beta} : V \times V \to V \times V,
\]

\[
Y^l_{\alpha,\beta} : \frac{x}{y} \mapsto \frac{u}{v} = \begin{pmatrix} 1 & 0 \\ 0 & l^{-1}D_{\alpha,\beta} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \frac{x}{y}, \quad x, y \in V,
\]

is a parametric Yang–Baxter map as well.
Let \( l \in \mathbb{K}, \ l \neq 0 \). Applying proposition 6.1 to the map (58), we obtain the linear parametric Yang–Baxter map

\[
Y_{a,b}^l : \text{Mat}_n(\mathbb{K}) \times \text{Mat}_n(\mathbb{K}) \to \text{Mat}_n(\mathbb{K}) \times \text{Mat}_n(\mathbb{K}),
\]

\[
Y_{a,b}^l(M_1, M_2) = \left( lM_1, l^{-1} a^p M_2 a^{-p} + \sum_{i=0}^{p-1} (a^i M_1 a^{-i-1} b - b a^i M_1 a^{-i-1}) \right), \tag{59}
\]

\( a, b \in \Omega \), \( \Omega \) is an abelian subgroup of \( \text{GL}_n(\mathbb{K}) \).

In the above construction of (59) we have assumed that \( \mathbb{K} \) is either \( \mathbb{R} \) or \( \mathbb{C} \), in order to use tangent spaces and differentials. Now one can verify that (59) is a parametric Yang–Baxter map for any field \( \mathbb{K} \).

For \( p = 1 \) the maps (57), (58) and (59) were presented in [8]. For \( p \geq 2 \) the maps (57), (58) and (59) are new.

Using corollary 2.5, from the parametric Yang–Baxter map (58) we obtain the following parametric tetrahedron map

\[
T_{a,b,c} : \text{Mat}_n(\mathbb{K}) \times \text{Mat}_n(\mathbb{K}) \times \text{Mat}_n(\mathbb{K}) \to \text{Mat}_n(\mathbb{K}) \times \text{Mat}_n(\mathbb{K}) \times \text{Mat}_n(\mathbb{K}),
\]

\[
T_{a,b,c}(M_1, M_2, M_3) = \left( M_1, a^p M_2 a^{-p} + \sum_{i=0}^{p-1} (a^i M_1 a^{-i-1} b - b a^i M_1 a^{-i-1}) \right), \tag{60}
\]

with parameters \( a, b, c \in \Omega \).

Let \( l, m \in \mathbb{K}, \ l \neq 0 \). Applying proposition 4.7 to the map (60), we derive the parametric tetrahedron map

\[
T_{a,b,c}^{l,m} : \text{Mat}_n(\mathbb{K}) \times \text{Mat}_n(\mathbb{K}) \times \text{Mat}_n(\mathbb{K}) \to \text{Mat}_n(\mathbb{K}) \times \text{Mat}_n(\mathbb{K}) \times \text{Mat}_n(\mathbb{K}), \quad a, b, c \in \Omega,
\]

\[
T_{a,b,c}^{l,m}(M_1, M_2, M_3) = \left( lM_1, l^{-1} a^p M_2 a^{-p} + \sum_{i=0}^{p-1} (a^i M_1 a^{-i-1} b - b a^i M_1 a^{-i-1}) \right), \tag{61}
\]

The tetrahedron map (61) carries almost the same information as the Yang–Baxter map (59), but we present (61) for completeness.

7. Conclusions

In this paper we have presented a number of results on tetrahedron maps and Yang–Baxter maps.

In particular, in sections 3 and 4 we have clarified the structure of the nonlinear algebraic relations which define linear (parametric) tetrahedron maps (with nonlinear dependence on parameters). Using this result, in propositions 3.5, 3.7, 4.6 and 4.7 we have presented several transformations which allow one to obtain new such maps from known ones.

Furthermore, in section 5 we have proved that the differential of a (nonlinear) tetrahedron map on a manifold is a tetrahedron map as well. Similar results on the differentials of Yang–Baxter and entwining Yang–Baxter maps are also presented in section 5.

In remark 5.6, corollary 5.5, and examples 5.8, 5.10 we have shown how linear tetrahedron maps appear as linear approximations of nonlinear ones.
Using the obtained general results, we have constructed a number of new Yang–Baxter and tetrahedron maps.

Example 5.10 is devoted to the nonlinear tetrahedron map (51) from [12]. We have obtained (functionally independent) invariants (52) and (53) for it. Furthermore, we have constructed a family of new linear tetrahedron maps (54), which are linear approximations of the map (51). The family of maps (54) depends on the parameter \( c \in \mathbb{C} \).

In examples 5.8 and 5.9, computing the differentials of some tetrahedron maps from [26, 30, 47], we have obtained new tetrahedron maps (46) and (50).

Let \( \mathbb{K} \) be a field. (For instance, one can take \( \mathbb{K} = \mathbb{C} \) or \( \mathbb{K} = \mathbb{R} \).) In section 6, for any nonzero \( l \in \mathbb{K} \), \( n, p \in \mathbb{Z}_{>0} \), and any abelian subgroup \( \Omega \subset \text{GL}_n(\mathbb{K}) \), we have obtained the parametric Yang–Baxter map (59) with parameters \( a, b \in \Omega \). For \( p \geq 2 \) the map (59) is new. For \( p = 1 \) it was presented in [8].

Motivated by the results of this paper, we suggest the following directions for future research:

- There are several methods in the literature for associating Yang–Baxter and tetrahedron maps to discrete lattice equations [22, 27–30, 46]. It would be interesting to compare discrete integrable systems related to given Yang–Baxter and tetrahedron maps with discrete systems associated with the differentials of these maps. The procedure to compute the differentials is presented in section 5.

Furthermore, in section 5 we have described a method to derive new linear tetrahedron maps as linear approximations of nonlinear ones. Note that linear approximations of nonlinear Yang–Baxter maps were considered in [8, 37]. It would be interesting to understand whether the linear approximations of nonlinear Yang–Baxter and tetrahedron maps correspond to some sort of linear approximations of nonlinear discrete lattice equations.

A result on classification of linear 3D consistent quad-graph equations (quadrilateral lattice equations) in the scalar case is provided in [2]. It would be interesting to generalise this result of [2] to the multicomponent case and to study relations with linear Yang–Baxter and tetrahedron maps.

- The Yang–Baxter and tetrahedron equations are members of the family of \( n \)-simplex equations [5, 13, 21, 40, 43, 44], where they correspond to the cases of two-simplex and three-simplex, respectively. We suggest to try to extend the results of this paper and of [8] to the case of \( n \)-simplex equations for \( n \geq 4 \). Some results on linear set-theoretical solutions to \( n \)-simplex equations (which can be called linear \( n \)-simplex maps) are presented in [11, 21].

- Noncommutative versions and extensions of Yang–Baxter and tetrahedron maps have been recently of particular interest (see, e.g. [14, 15, 20, 33, 34]). In particular, Yang–Baxter maps extended by means of Grassmann algebras were obtained in [20, 35].

One can try to prove that the differentials of Grassmann extended Yang–Baxter and tetrahedron maps are also solutions to the Grassmann extended Yang–Baxter and tetrahedron equations. Moreover, we propose to compare the relation between Yang–Baxter and tetrahedron maps and their differentials versus the relation between the former and the latter in the case of Grassmann algebras.

- It is well known that Bäcklund transformations for integrable partial differential, differential–difference and difference–difference equations can often be constructed by means of chains of Miura-type transformations (also called Miura maps). There is a method to construct Miura-type transformations for differential–difference equations from...
Darboux–Lax representations (DLRs) of such equations [7], using Lie group actions associated with matrices from DLRs. The method in [7] is applicable to a wide class of DLRs and can be extended to difference–difference equations. It is inspired by the results of [17, 23] on Miura-type transformations for (1 + 1)-dimensional evolution PDEs.

On the other hand, there are examples of Yang–Baxter maps [36, 37, 45] and tetrahedron maps [32] arising from matrix refactorisation problems for Darboux matrices corresponding to Darboux–Bäcklund transformations of Lax representations of integrable (1 + 1)-dimensional evolution PDEs. Furthermore, it is well known that such a Darboux–Bäcklund transformation very often gives integrable differential–difference and difference–difference equations (see, e.g. [22, 31, 45]). This suggests to study relations between Miura maps, Yang–Baxter maps, and tetrahedron maps.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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