Quantum simulation of the spinor-4 Dirac equation with an artificial gauge field

Jean Claude Garreau and Véronique Zehnêl
Université de Lille, CNRS, UMR 8523 -- PhLAM -- Laboratoire de Physique des Lasers Atomes et Molécules, F-59000 Lille, France
(Dated: March 13, 2019)

A two-dimensional spatially and temporarily modulated Wannier-Stark system of ultracold atoms in optical lattices is shown to mimic the behavior of a Dirac particle. Suitable additional modulations generate an artificial gauge field which simulates a magnetic field and imposes the use of the full spinor-4 Dirac equation.

I. INTRODUCTION

The Dirac equation, unifying quantum mechanics and special relativity, is a major achievement in physics. It has large implications in several fields, e.g. particle physics where it describes spin-1/2 leptons and their interactions with radiation, condensed matter where it is used as a model for several types of quasiparticles, and also in the fast progressing field of topological insulators.

Although the Dirac equation represents a paradigm for modern field theory, this physics remained relatively elusive, restricted to high-energy situations or to exotic materials. Recent developments both in condensed matter and ultracold-atom systems have generated a burst of interest, in particular, the concept of “quantum simulator” opened a new window in the study of Dirac physics [1–13]. Inspired by an original insight of Feynman [14], a quantum simulator is a “simple” and controllable system that can mimic (some aspects of) the behavior of a more complex or less accessible ones. The flexibility of ultracold-atom systems also prompted for innovative ideas like the generation of the so-called “artificial” gauge fields acting on (neutral) atoms that mimic the effects of an electromagnetic field [15], allowing for the study of quantum magnetism [16], engineered spin-orbit coupling, and topological systems [17]. The mixing of these ideas with the physics of ultracold atoms in optical lattices has proven extremely fruitful [15, 18–23].

The Dirac equation is the most complete formulation describing a relativistic charged fermion of spin 1/2. It leads to a spinor-4 description which automatically includes the spin and the electron antiparticle, the positron. In the rest frame of the free particle, which always exists for a massive particle, the spinor-4 components can be separated in two spinor-2 obeying equivalent sets of equations. This separation is not possible in the presence of electromagnetic fields, which leads to a richer physics. The aim of the present work is the elaboration of a minimal model simulating a 2D Dirac equation in the presence of an artificial gauge field (related to a vector potential \( \mathbf{A} \) with non-zero rotational). In this case the spinor-4 description is mandatory, and we show that the main characteristics of a Dirac particle are obtained.

We combine time- and space-modulated optical potentials acting on independent atoms in order to obtain a 2D Dirac Hamiltonian of the form

\[
H_D = c \mathbf{\alpha} \cdot (\mathbf{p} - q \mathbf{A}) + \beta m_D c^2
\]

where \( c \) is the velocity of light, \( m_D \) and \( q \) are the Dirac particle’s mass and charge, \( \mathbf{p} \) is the momentum and \( \mathbf{A} \) the vector potential. The \( 4 \times 4 \) Dirac matrices constructed from Pauli matrices \( \sigma_i \) \( (i = x, y, z) \) and the \( 2 \times 2 \) unit matrix 1:

\[
\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

In a previous work [2] we introduced a general model allowing to quantum-simulate a spinor-4 Dirac Hamiltonian in 1D without magnetic field. In the present work we shall i) generalize this approach to the 2D case, ii) introduce an artificial gauge field, and iii) demonstrate its ability to simulate known behaviors of the Dirac equation.

II. SPINOR-4 DIRAC QUANTUM SIMULATOR IN 2D

We first construct a modulated 2D tilted optical lattice model that mimics a 2D spinor-4 Dirac equation with no field. The model builds on the general approach introduced for the 1D case in Ref. [2]. A 2D “tilted” (or Wannier-Stark) lattice in the \( x, z \) plane [24] is described by the 2D (dimensionless) Hamiltonian

\[
H_0 = \frac{p_x^2 + p_z^2}{2m^*} + V_L(x, z) + F_x x + F_z z
\]

with \( \mathbf{p} = p_x \mathbf{x} + p_z \mathbf{z} \) the momentum in 2D \( (\mathbf{x}, \mathbf{z}) \) are unit vectors in the directions \( x, z \), \( \mathbf{F} = F_x \mathbf{x} + F_z \mathbf{z} \) a constant force and \( V_L \) a square lattice formed by orthogonal standing waves

\[
V_L(x, z) = -V_1 (\cos(2\pi x) + \cos(2\pi z))
\]

where space is measured in units of the step \( a [25] \) of the square lattice \( V_L \), \( a = \lambda_L/2 \) if the lattice is formed with propagating beams of wavelength \( \lambda_L = 2\pi/k_L \). Time is measured in units of \( \hbar/E_R \) where \( E_R = \hbar^2 k_L^2/2M \) is the
atom’s recoil energy (M is the atom’s mass). With these choices, \( m^* = \frac{\hbar^2}{2} \), \( \hbar = 1 \) and \( p_j = -i\partial_j \) \((j = x, z)\).

Since the Hamiltonian Eq. (2) is separable, its eigenstates can be factorized in terms of localized Wannier-Stark (WS) functions [26, 27]:

\[
\Phi_{n,m}(x,z) = \psi_n^{(x)}(x)\psi_m^{(z)}(z),
\]

where the index \( n \) indicates the lattice site along the \( x \) direction where the eigenfunction \( \psi_n^{(x)}(x) \) is centered (resp. \( m \) in the \( z \) direction). The energies of the system are then

\[
E_{n,m} = E_0 + n\omega_B^{(x)} + m\omega_B^{(z)}
\]

(3)

where \( E_0 \) is an energy offset with respect to the bottom of the central well \( n = m = 0 \). The energy spacing in directions \( x \) and \( z \) define the so-called Bloch frequencies, \( \omega_B^{(x)} = F_x \) (\( = F_x a / h \) in dimensioned units) and \( \omega_B^{(z)} = F_z \), where we intentionally choose \( F_z \neq F_x \). This defines a “Wannier-Stark ladder” of energy levels separated by integer multiples of \( \omega_B^{(x,z)} \), Eq. (3). As in Ref. [2], we assume here that the excited levels are never populated and that the dynamics depends only on the lowest-ladder WS states sited at each site.

The Hamiltonian Eq. (2) is invariant over a translation of an integer multiple \( n \) of the lattice step in the \( x \) direction provided that the energy is also shifted by \( nF_x \) (resp. \( mF_z \) in the \( z \) direction), hence the eigenstates are such that

\[
\psi_n^{(x)}(x) = \psi_0^{(x)}(x-n),
\]

(4)

(resp. \( \psi_m^{(z)}(z) = \psi_0^{(z)}(z-m) \) in the \( z \) direction).

Our aim is to obtain an effective evolution equation for the system equivalent to a 2D Dirac equation Eq. (1) for a particle of mass \( m \)

\[
i\partial_t \Psi = \left\{ co_x (p_x - A_x) + co_z (p_z - A_z) + \beta mc^2 \right\} \Psi
\]

(5)

where \( \Psi \) denotes a spinor-4, and \( A = A_x \hat{x} + A_z \hat{z} \). The effective speed of light is \( c = hc/\hbar eB \), the gauge field \( A = qaA/h \), the mass \( m = \hbar e^2 a^2 m_B / \hbar^2 \), and effective charge is set to unity. We first consider the case without gauge field (\( A = 0 \)).

We induce controlled dynamics [2] in the system by adding to \( H_0 \), Eq. (2), a resonant time-dependent perturbation of the form

\[
V(x, z, t) = V_x \cos (2\pi x) \sin (\omega_B^{(x)} t) + V_z \cos (\pi x) \cos (2\pi z) \sin (\omega_B^{(z)} t).
\]

(6)

The utility of the term proportional to \( \cos (\pi x) \) with spatial periodicity 2 will appear below. The time modulation at the Bloch frequencies \( \omega_B^{(x)} \) (resp. \( \omega_B^{(z)} \)) resonantly couples first-neighbor WS states proportionally in the \( x \) direction (resp. \( z \) direction). This coupling is proportional to the overlap amplitudes

\[
\left\langle \psi_n^{(x)}(x) \cos (2\pi x) \bigg| \psi_{n+1}^{(x)} \right\rangle = \left\langle \psi_0^{(x)}(x) \cos (2\pi x) \bigg| \psi_1^{(x)} \right\rangle \equiv \Omega_x
\]

[cf. Eq. (4)] in the \( x \) direction, and

\[
\left\langle \varphi_n^{(z)}(x) \cos (\pi x) \bigg| \varphi_{n+1}^{(z)} \right\rangle = \left\langle \varphi_0^{(z)}(x) \cos (2\pi z) \bigg| \varphi_1^{(z)} \right\rangle = (-1)^n \Omega_z
\]

in the \( z \) direction. The \(-1)^n\) contribution comes from the spatial modulation in \( \cos (\pi x) \) in Eq. (6) and introduces a distinction between even and odd sites in the \( x \) direction. Note that as \( F_x \neq F_z \), the dynamics in each direction can be controlled independently.

The general solution of the Schrödinger equation corresponding to the Hamiltonian \( H_0 + V \) can be written as

\[
\psi(x, z, t) = \sum_{n,m} c_{n,m}(t) \varphi_n^{(x)} \varphi_m^{(z)}
\]

from which one obtains a set of differential equations with nearest neighbor couplings

\[
id_t c_{n,m} = T_x [c_{n+1,m} - c_{n-1,m}]
\]

\[
+ (-1)^n T_z [c_{n,m+1} - c_{n,m-1}]
\]

(7)

where \( T_x = -iV_x \Omega_z / 2 \) and \( T_z = -iV_z \Omega_x / 2 \).

Assuming that the wave packet is large and smooth on the scale of the lattice step one can take the continuous limit of Eq. (7), and transform the discrete amplitudes \( c_{n,m}(t) \) in two continuous functions. Note that the site parity dependence in Eq. (7) must be taken into account in defining such smooth amplitudes. We thus introduce the functions \( s_{ee}(x, z, t) \) which is the envelope of \( c_{n,m} \) for \( n, m \) even, \( s_{oo}(x, z, t) \) which is the envelope of \( c_{n,m} \) for \( n \) even and \( m \) odd, and analogously for \( s_{eo} \) and \( s_{go} \). These functions can be arranged as components of a spinor-4 \( \Psi = (s_{ee}, s_{oo}, s_{eo}, s_{go})^T \) describing 4 coupled sub-lattices, which, from Eq. (7), obey a Hamiltonian equation

\[
id_t [\Psi(x, z, t)] = H_S [\Psi(x, z, t)]
\]

(8)

where \( H_S \) can be easily shown to be a 2D “simulated” Dirac Hamiltonian of the form \( H_S = c_S \alpha \cdot p \) with an effective velocity of light \( c_S = V_x \Omega_x = V_z \Omega_z \) (couplings can be made equal by tuning the modulation amplitudes \( V_{x,z} \) adequately, but it is worth noting that one can also create an anisotropic model, with the possibility of an effective violation of Lorentz invariance [1]). The above system can be broken into two equivalent set of equations [2] which correspond to the well-known massless Weyl spinor-2 fermion.

The twofold-degenerate dispersion relation deduced from Eq. (8) is \( \omega(k) = \pm (\Omega_x^2 k_x^2 + \Omega_z^2 k_z^2)^{1/2} \) and corresponds, as expected, to a Dirac cone.

A massive particle can be simulated by adding a static perturbation \( V_0 \cos (\pi x) \cos (\pi z) \) to Eq. (6). Then, neglecting terms in \( \partial^2_{x,z} c_{n,m} \) or higher, Eq. (7) takes the
where and leads to This term can be treated in the same way as in Sec. II by adding to Eq. (6) an additional perturbation sized. The Dirac Hamiltonian of Eq. (1) can be realized in a lattice.

In this section, we show how a vector potential \( \mathbf{A} = A_z \mathbf{x} + A_x \mathbf{z} \) with \( A_z \propto z \) and \( A_x \propto x \), can be synthesized. The Dirac Hamiltonian of Eq. (1) can be realized in a lattice by adding to Eq. (6) an additional perturbation

\[
\hat{V}_A(x, z, t) = V_A^z \cos(2\pi x) \cos(\omega_B^z t) + V_A^x \cos(\pi x) \cos(2\pi z) \cos(\omega_B^x t). \tag{10}
\]

This term can be treated in the same way as in Sec. II and leads to

\[
id_t c_{n,m} = (-1)^{n+m} T_0 + (-1)^n T_z \left[ c_{n+1,m} - c_{n,m-1} \right] + T_x \left[ c_{n+1,m} - c_{n-1,m} \right] + T_A^z m \left[ c_{n+1,m} + c_{n-1,m} \right] + (-1)^n T_A^x \left[ c_{n+1,m} + c_{n,m-1} \right] \tag{11}
\]

where \( T_A^A = \frac{1}{2} \Omega_x V_A^A, T_x^A = \frac{1}{2} \Omega_x V_A^A \). The additional terms on the last two lines in Eq. (11) are due to the potential Eq. (10) which generates the slopes proportional to \( m \) (resp. \( n \)) in direction \( z \) (resp. \( x \)).

In the continuous limit, and neglecting second order (and higher) derivatives of the spinor components, we obtain the Hamiltonian of Eq. (1), with an artificial gauge potential Eq. (10) which generates the slopes proportional to \( m \) (resp. \( n \)) in direction \( z \) (resp. \( x \)). The symmetric gauge potential considered in the next section can be realized by tuning the modulation amplitudes in \( V_A \) so that \( T_A^z = -T_A^x \), thus \( \mathbf{A} = 2T_A^z (\mathbf{x} \times \mathbf{x})/c_S \), corresponding to a uniform magnetic field in the \( y \) direction \( \mathbf{B} = 4T_A^y \mathbf{y}/c_S \).

In the following, we write the spinor-4 as \( |\Psi\rangle = |\phi, \chi\rangle \) where \( \phi, \chi \) are spinor-2s. With this convention, the Dirac equation can be decomposed in 2 coupled equations

\[
(E - mc^2) \phi = c \left[ \sigma \cdot (\mathbf{p} - \mathbf{A}) \right] \chi \tag{12}
\]

\[
(E + mc^2) \chi = c \left[ \sigma \cdot (\mathbf{p} - \mathbf{A}) \right] \phi. \tag{13}
\]

These equations are symmetric under the transposition \( \phi, \chi \leftrightarrow -\phi, -\chi \), \( \phi \) so that the negative energy states are easily deduced from their positive energy counterparts.
Eliminating $\chi$ from Eqs. (13) gives $(E^2 - m^2c^4)\phi = e [\sigma \cdot (p - A)]^2 \phi$, and, after some straightforward algebra,

$$(E^2 - m^2c^4)\phi = e^2 \left[ \left( \frac{p^2}{2\mu} + \frac{\mu}{2}(x^2 + z^2) + L_yB \right) + \sigma_yB \right] \phi$$

where $L_y = (zp - xp_z)$ is the angular momentum component in the $y$ direction, with a "diamagnetic" term proportional to $x^2 + z^2$ and "paramagnetic" terms of the type $L \cdot B$ and $\sigma \cdot B$. This equation has "spin up" $\phi_+$ and "spin down" $\phi_-$ spinor-2 solutions

$$\phi_{\pm} = (1, \pm i)T \psi_{\pm}(x, z)$$

where $\psi_{\pm}(x, z)$ obey the following equation

$$\left[ \frac{p^2}{2\mu} + \frac{\mu}{2}(x^2 + z^2) + L_y \right] \psi_{\pm} = (\tilde{E} \pm 1)\psi_{\pm}$$

with $\tilde{E} = (E^2 - m^2c^4)/(Bc^2)$ and $\mu = B/2$, which strongly evokes a 2D harmonic oscillator of mass $\mu$ and natural frequency $\omega = 1$ in a magnetic field.

Solutions of Eq. (16) are the well-known “Landau levels”, with the spectrum

$$E = \sqrt{m^2c^4 + 2Bc^2n}$$

where $n \in \mathbb{N}^*$ (not to be confused with the site index $n$, which does not appear in the present section) for "spin up" solution $\phi_+$ and $n \in \mathbb{N}$ for "spin down" $\phi_-$. For each $n$ the corresponding energy is infinitely degenerated.

The solutions for $\psi_{\pm}$ Eq. (16) are well known, and together with Eq. (15) and Eq. (13) lead to solutions for the full eigenspinors $[\Psi] = [\phi, \chi]$. In polar coordinates $x = r \cos \varphi$, $z = r \sin \varphi$ and $\rho = r / \ell$, where $\ell = \sqrt{2/B}$ is an effective dimensionless “magnetic length”, an example of solution for the fundamental level $E = mc^2$ is

$$[\phi, \chi] = [(1, -i) \exp(-\rho^2/2); (0, 0)]$$

with a negative energy $E = -mc^2$ counterpart $[-\chi; \phi]$. An example of an eigenstate belonging to the $n = 1$ excited state family $E = (m^2c^4 + 2Bc^2)^{1/2}$ is

$$[\phi, \chi] = [(1, i); (1, -i)\beta \rho \exp(-i\varphi)] \exp(-\rho^2/2)$$

where $\beta = -2c/ [\ell(mc^2 + E)]$ [30].

Numerical simulations of the discrete model reproduce the characteristics of the above solutions of the Dirac equation. According to Eq. (17), plateaus are obtained at integer values $n = (E^2 - m^2c^4)/2Bc^2$. As shown in Fig. 1, the numerical solution of the eigenproblem Eq. (11) displays such a behavior, with plateaus (red disks) appearing as expected at integer values. The simulation also shows eigenenergies which fall at non-integer values (blue circles); we checked that these are edge states due to finite size effects, as the numerical simulation is performed in a square box containing finite number of lattice sites (80×80).

Figure 2 shows a numerical example of the the four components of the eigenspinor $[\Psi] = [\psi_1, \psi_2, \psi_3, \psi_4]$, with $[\psi_1]$ and $[\psi_3]$ being dominant.
\((s_{ex}(x, z, t), s_{eo}(x, z, t), s_{ox}(x, z, t), s_{eo}(x, z, t))\), corresponding to the analytical solution of the Dirac excited mode of Eq. (18); injecting the magnetic length \(\ell = 10\) in Eq. (18), we get a quantitative agreement with the simulation result.

\section{Conclusion}

The present work demonstrates the ability of ultracold-atom systems to mimic a Dirac particle in the presence of a magnetic field, putting together quantum simulations of “exotic” dynamics (from the point of view of low-energy physics) and artificial gauge fields. This is intrinsically expected to lead to topological systems, as the Dirac cone is one of the main (and simplest) examples of topology in physics. This opens new ways to explore exciting new possibilities in this very active field.

\section*{Acknowledgments}

This work was supported by Agence Nationale de la Recherche through Research Grants K-BEC No. ANR-13-B504-0001-01 and MANYLOK No. ANR-18-CE30-0017, the Labex CEMPI (Grant No. ANR-11-LABX-0007-01), the Ministry of Higher Education and Research, Hauts-de-France Council and European Regional Development Fund (ERDF) through the Contrat de Projets Etat-Region (CPER Photonics for Society, P4S).

\begin{thebibliography}{99}
\bibitem{1} C.-H. Park, L. Yang, Y.-W. Son, M. L. Cohen, and S. G. Louie, “Anisotropic behaviours of massless Dirac fermions in graphene under periodic potentials,” Nat. Phys. \textbf{4}, 213–217 (2009).
\bibitem{2} J. C. Garreau and V. Zehnlé, “Simulating Dirac models with ultracold atoms in optical lattices,” Phys. Rev. A \textbf{96}, 043627 (2017).
\bibitem{3} X. Lopez-Gonzalez, J. Sisti, G. Pettini, and M. Modugno, “Effective Dirac equation for ultracold atoms in optical lattices: Role of the localization properties of the Wannier functions,” Phys. Rev. A \textbf{89}, 033608 (2014).
\bibitem{4} C. Qu, C. Hamner, M. Gong, C. Zhang, and P. Engels, “Observation of Zitterbewegung in a spin-orbit-coupled Bose-Einstein condensate,” Phys. Rev. A \textbf{88}, 021604 (2013).
\bibitem{5} L. Tarruell, D. Greif, T. Uehlinger, G. Jotzu, and T. Esslinger, “Creating, moving and merging Dirac points with a Fermi gas in a tunable honeycomb lattice,” Nature (London) \textbf{483}, 302–305 (2012).
\bibitem{6} L. Mazza, A. Bermudez, N. Goldman, R. Mizzi, M. A. Martin-Delgado, and M. Lewenstein, “An optical-lattice-based quantum simulator for relativistic field theories and topological insulators,” New J. Phys. \textbf{14}, 015007 (2012).
\bibitem{7} D.-W. Zhang, Z.-D. Wang, and S.-L. Zhu, “Relativistic quantum effects of Dirac particles simulated by ultracold atoms,” Front. Phys. \textbf{7}, 31–53 (2012).
\bibitem{8} R. Gerritsma, B. P. Lanyon, G. Kirchmair, F. Zähringer, C. Hempel, J. Casanova, J. J. Garcia-Ripoll, E. Solano, R. Blatt, and C. F. Roos, “Quantum Simulation of the Klein Paradox with Trapped Ions,” Phys. Rev. Lett. \textbf{106}, 060503 (2011).
\bibitem{9} L. Lamata, J. Casanova, R. Gerritsma, C. F. Roos, J. J. Garcia-Ripoll, and E. Solano, “Relativistic quantum mechanics with trapped ions,” New J. Phys. \textbf{13}, 095003 (2011).
\bibitem{10} R. Gerritsma, G. Kirchmair, F. Zähringer, E. Solano, R. Blatt, and C. F. Roos, “Quantum simulation of the Dirac equation,” Nature (London) \textbf{463}, 68–71 (2010).
\bibitem{11} S. Longhi, “Photonic analog of Zitterbewegung in binary waveguide arrays,” Opt. Lett. \textbf{35}, 235–237 (2010).
\bibitem{12} F. Dreisow, M. Heinrich, R. Keil, A. Tünnemann, S. Nolte, S. Longhi, and A. Szameit, “Classical Simulation of Relativistic Zitterbewegung in Photonic Lattices,” Phys. Rev. Lett. \textbf{105}, 143902 (2010).
\bibitem{13} D. Witthaut, T. Salger, S. Kling, C. Grossert, and M. Weitz, “Effective Dirac dynamics of ultracold atoms in bichromatic optical lattices,” Phys. Rev. A \textbf{84}, 033601 (2011).
\bibitem{14} R. P. Feynman, “Simulating Physics with Computers,” Int. J. Theor. Phys. \textbf{21}, 467–488 (1982).
\bibitem{15} J. Dalibard, F. Gerbier, G. Juzeliūnas, and P. Öhberg, “Artificial gauge potentials for neutral atoms,” Rev. Mod. Phys. \textbf{83}, 1523–1543 (2011).
\bibitem{16} M. Aidelsburger, M. Atala, M. Lohse, J. T. Barreiro, B. Paredes, and I. Bloch, “Realization of the Hofstadter Hamiltonian with Ultracold Atoms in Optical Lattices,” Phys. Rev. Lett. \textbf{111}, 185301 (2013).
\bibitem{17} V. Galitski and I. B. Spielman, “Spin-orbit coupling in quantum gases,” Nature (London) \textbf{494}, 49–54 (2013).
\bibitem{18} I. Bloch, J. Dalibard, and W. Zwerger, “Many-body physics with ultracold gases,” Rev. Mod. Phys. \textbf{80}, 885–964 (2008).
\bibitem{19} I. Bloch, J. Dalibard, and S. Nascimbene, “Quantum simulations with ultracold quantum gases,” Nat. Phys. \textbf{8}, 267–276 (2014).
\bibitem{20} I. M. Georgescu, S. Asshab, and F. Nori, “Quantum simulation,” Rev. Mod. Phys. \textbf{86}, 153–185 (2014).
\bibitem{21} N. Goldman, J. C. Budich, and P. Zoller, “Topological quantum matter with ultracold gases in optical lattices,” Nat. Phys. \textbf{12}, 639–645 (2016).
\bibitem{22} C. Gross and I. Bloch, “Quantum simulations with ultracold atoms in optical lattices,” Science \textbf{357}, 995–1001 (2017).
\bibitem{23} J. C. Garreau, “Quantum simulation of disordered systems with cold atoms,” Compt. Rendus Phys. \textbf{18}, 31 – 46 (2017).
\bibitem{24} As will be shown above, this choice makes the writing in terms of the conventional Pauli matrices possible. We will later introduce an artificial magnetic field in the \(y\) direction.
\bibitem{25} We use sans serif characters to represent dimensioned quantities, except when no ambiguity is possible, e.g., \(\hbar\).
\bibitem{26} G. H. Wannier, “Wave Functions and Effective Hamiltonian for Bloch Electrons in an Electric Field,” Phys. Rev. \textbf{117}, 432–439 (1960).
\end{thebibliography}
[27] C. Zener, “A Theory of Electrical Breakdown of Solid Dielectrics,” Proc. R. Soc. (London) A 145, 523—1 (1934).

[28] Experimentally, such linear terms can be obtained by superposing two laser fields of frequencies separated by $\delta \omega$, which generate a beat note varying in space as $\sin (\delta \omega x/c) \sim \delta \omega x/c$, if $\delta \omega$ is small enough, the linear approximation can be valid over a large number of neighbor sites.

[29] The Landau gauge can be obtained by setting $T_z^A = 0$ (or $T_x^A = 0$).

[30] Note that these states are also eigenstates of the angular momentum operator $L_y + \Sigma_y$, where $\Sigma_y = \frac{1}{2} \begin{pmatrix} \sigma_y & 0 \\ 0 & \sigma_y \end{pmatrix}$ is the spin operator.