Galactic warps induced by cosmic infall

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ABSTRACT
Recent ideas for the origin and persistence of the warps commonly observed in disc galaxies have focused on cosmic infall. We present N-body simulations of an idealized form of cosmic infall onto a disc galaxy and obtain a warp that closely resembles those observed. The inner disc tilts remarkably rigidly, indicating strong cohesion due to self-gravity. The line of nodes of the warp inside $R_{26.5} \sim 4.5R_d$ is straight, while that beyond $R_{26.5}$ generally forms a loosely-wound, leading spiral in agreement with Briggs’s rules. We focus on the mechanism of the warp and show that the leading spiral arises from the torques from the misaligned inner disc and its associated inner oblate halo. The fact that the line of nodes of most warps forms a leading spiral might imply that the disc mass is significant in the centre. If the line of nodes can be traced to very large radii in future observations, it may reveal information on the mass distribution of the outer halo. The warp is not strongly damped by the halo because the precession rate of the inner disc is slow and the inner halo generally remains aligned with the inner disc. Thus even after the imposed quadrupolar perturbation is removed, the warp persists for a few Gyrs, by which time another infall event can be expected.

Key words: stellar dynamics — galaxies: evolution — galaxies: kinematics and dynamics — galaxies: structure

1 INTRODUCTION
The optically visible parts of galactic discs are usually remarkably thin and flat, whereas the more extended HI discs of many edge-on galaxies appear noticeably warped with an integral sign shape. Stellar warps do exist in some galaxies (e.g. Reshetnikov et al. 2002), but are much less pronounced than the warps in the extended gaseous discs. We infer that warps are a gravitational phenomenon because weak stellar warps, if present, follow the same tended gaseous discs. We infer that warps are a gravitational phenomenon because weak stellar warps, if present, follow the same tended gaseous discs. We infer that warps are a gravitational phenomenon because weak stellar warps, if present, follow the same tended gaseous discs. We infer that warps are a gravitational phenomenon because weak stellar warps, if present, follow the same tended gaseous discs. We infer that warps are a gravitational phenomenon because weak stellar warps, if present, follow the same tended gaseous discs. We infer that warps are a gravitational phenomenon because weak stellar warps, if present, follow the same tended gaseous discs. We infer that warps are a gravitational phenomenon because weak stellar warps, if present, follow the same tended gaseous discs. We infer that warps are a gravitational phenomenon because weak stellar warps, if present, follow the same tended gaseous discs. We infer that warps are a gravitational phenomenon because weak stellar warps, if present, follow the same tended gaseous discs. We infer that warps are a gravitational phenomenon because weak stellar warps, if present, follow the same tended gaseous discs. We infer that warps are a gravitational phenomenon because weak stellar warps, if present, follow the same tended gaseous discs. We infer that warps are a gravitational phenomenon because weak stellar warps, if present, follow the same tended gaseous discs. We infer that warps are a gravitational phenomenon because weak stellar warps, if present, follow the same tended gaseous discs. We infer that warps are a gravitational phenomenon because weak stellar warps, if present, follow the same tended gaseous discs. We infer that warps are a gravitational phenomenon because weak stellar warps, if present, follow the same tended gaseous discs. We infer that warps are a gravitational phenomenon because weak stellar warps, if present, follow the same tended gaseous discs. We infer that warps are a gravitational phenomenon because weak stellar warps, if present, follow the same tended gaseous discs. We infer that warps are a gravitational phenomenon because weak stellar warps, if present, follow the same tended gaseous discs. We infer that warps are a gravitational phenomenon because weak stellar warps, if present, follow the same tended gaseous discs. We infer that warps are a gravitational phenomenon because weak stellar warps, if present, follow the same tended gaseous discs.

Warp modes can be detected kinematically even when the system is not edge-on. Briggs (1990) studied a sample of 12 warped galaxies with high-quality 21-cm data, and found that galactic warps obey some fairly simple rules:

(i) the HI layer typically is coplanar inside radius $R_{25}$, the radius where the B-band surface brightness is $25$ magnitudes arcsec$^{-2} = 25\mu$B, and the warp develops between $R_{25}$ and $R_{26.5}$ (aka the Holmberg radius);
(ii) the line of nodes (LON) is roughly straight inside $R_{26.5}$;
(iii) the LON takes the form of a loosely-wound leading spiral outside $R_{26.5}$.

The origin and persistence of warps still presents a puzzle. Hunter & Toomre (1969), before the discovery of dark matter halos, studied the bending dynamics of isolated thin discs. They found that discrete warp modes in a cold, razor-thin disc do not exist unless the edge is unrealistically sharp. Such modes (standing waves) can be realized by the superposition of outgoing and ingoing waves, provided that waves incident on the edge can reflect. However, a disc with a smooth edge does not reflect bending waves (Toomre 1983).

Subsequent ideas of warp formation rely in some way on the interaction between the disc and its dark matter halo. Dekel & Shlosman (1983) and Toomre (1983) suggested that a flattened halo misaligned with the disc could form a long-lasting warp.
Sparke & Casertano (1988) and Kuijken (1991) invoked a rigid flattened halo and obtained persisting warps (dubbed “modified tilt modes”), which were insensitive to the details of the disc edge. Lovelace (1998) studied the tilting dynamics of a set of rings also in a rigid halo, but generally assumed that the inner disc lay in the symmetry plane of the spheroidal halo.

Dark matter halos are not rigid, however, and a responsive halo alters the dynamics in several ways. Nelson & Tremaine (1995) showed that the precession of a misaligned disc inside a flattened halo would sap energy from the warp mode through dynamical friction, damping it on timescales much shorter than a Hubble time. Numerical studies using N-body halos also found the discrete warp mode does not survive: Dabinski & Kuijken (1995) found that the inner halo and disc quickly settle into alignment. Binney et al. (1998) confirm that a discrete warp mode in a rigid halo does not survive in a simulation with a responsive halo, and conclude that the inner halo could never be significantly misaligned from the inner disc.

Ostriker & Binney (1989) drew attention to the likely misalignment with the disc axes of late infalling material in hierarchical galaxy formation models, and proposed that warps arise due to the slewing of the galactic potential as material with misaligned angular momentum is accreted. Quinn & Binney (1992) showed, for a realistic cosmic scenario, that the mean spin axis of a galaxy must slew as late arriving material rains down on the early disc. The less-than-critical matter density in modern ΛCDM universe models implies that infall is less pervasive at later times, but it manifestly continues to the present day in gravitationally bound environments such as that of the Local Group.

Jiang & Binney (1999, hereafter JB99) present results of an experiment in which a disc is subjected to the torque from a misaligned, massive torus at a large radius. This well-defined perturbation is a very crude model of an outer halo that is rotationally flattened, with a spin axis misaligned with that of the disc. It is misaligned and farther out because, in hierarchical scenarios, the mean angular momentum of the later arriving outer halo is probably misaligned from that of the original inner halo and disc. They concede that the accretion axis is in reality likely to slew continuously over time, so a model with a constant inclination is somewhat unrealistic.

Here we use high quality N-body simulations of a model of this type. We improve on the work by JB99 in the following main aspects:

- We use a disc of particles with random motion, whereas JB99 employed a disc composed of rigid rings coupled only by gravitation. Thus JB99 did not include random motion of the disc stars, which has been shown to add to the stiffness of the disc (Debattista & Sellwood 1999).
- We use a much more extensive disc so that we can study the warp – JB99 truncated the disc at 4Rd. The behaviour of the LON beyond this radius is important for comparing our results with observations of warps (e.g. Briggs’s rules);
- In §2 we remove the forcing perturbation at later times and follow the evolution of the distorted disc fully self-consistently.

We obtain flat inner discs and long-lasting warps in the outer disc that match all of Briggs’s rules quite well. We present a detailed analysis of the dynamics and show, for example, that the persistence of warps is not nearly as perplexing as is currently believed.

2 MODEL SETUP AND SIMULATION DETAILS

Following JB99, our simulations include three distinct mass components: a disc, a halo, and an accreted torus. The plane of the torus is inclined to that of the disc, as sketched in Figure 1

Our simulations are set up as follows: The initial disc has the exponential surface density profile:

$$\Sigma(r) = \frac{M_d}{2\pi R_d^2} \exp \left( -\frac{r}{R_d} \right),$$

where $R$ is the cylindrical radius and $R_d$ and $M_d$ are the scale radius and the total mass of the disc, respectively. We truncate the disc at $R = 8R_d$, spread the particles vertically as the isothermal sheet with a locally-defined equilibrium vertical velocity dispersion, and embed it in the halo; the rms vertical thickness of the initial disc is 0.1Rd at any radius. We set the initial velocities of the disc particles such that the Toomre stability parameter $Q \simeq 1.5$ and the disc is in rotational balance.

The halo is set up so as to be in equilibrium with the disc using the procedure described in Appendix A of Debattista & Sellwood (2000). The distribution function has a King-model form with $\Psi(0)/\sigma^2 = 2.0$, where $\sigma$ is the central velocity dispersion and $\Psi(0)$ is the relative potential at the centre (Binney & Tremaine 1987), which in our case includes a contribution from the disc. We set the tidal radius $r_t = 18R_d$, and total mass $M_h$. The King radius is $r_h \sim 5.6R_d$ and the half-mass radius of the halo is $r_h \sim 5R_d$. The inner halo is mildly oblate initially due to the gravity of the disc. The halo we adopted here is massive, as we attempt to use a similar halo profile used by JB99 in which $M_h/M_d = 5.0$ for $r < 3.2R_d$. Other experiments with halos having different mass profiles, to be reported elsewhere, have revealed that the conclusions presented here are not strongly affected by the choice of halo; warping behaviour is less sensitive to the density profile or total mass of the halo than to the masses and sizes of the disc and torus.

Following JB99, we gradually inject particles into an inclined torus, causing its mass to increase linearly from 0 at $t = 0$, to $M_t = 2.5M_d$ at $t = 200$. The torus, which has radius $R_t$, is uniform in azimuth and has Gaussian density profile in cross-section, with a standard deviation $b$; in our case, $R_t = 15.0R_d$ and $b = 0.2R_d$. The torus plane is inclined at 15° relative to the initial disc plane and we adopt the line of nodes as the $y$-axis of our coordinate system.

The initial velocities of injected torus particles are equal to the local circular speed in magnitude and directed tangentially around the torus, as in JB99. Thus the torus has very large angular momentum, and therefore does not precess significantly from its original plane in response to the torque from the disc. However, a cold torus is dynamically unstable and breaks into radially extensive spiral filaments after a few Gyr when allowed to evolve self-consistently. The breakup of the cold torus depends on many factors that we do not fully understand — e.g., tori with the same parameters dissolve
differently and at different times in runs with rigid and live halos. In order to avoid the inconvenient complication of time-varying, non-axisymmetric perturbing forces acting on the disc, we generally keep the torus particles fixed in their initial positions relative to the galactic centre. This stratagem enables us to compare the dynamical evolution of warps in different experiments with the same torus parameters.

All 1.25 million particles belonging to massive components in our simulations have equal masses.

We adopt $R_d$ and $M_d$ as our units of length and mass, respectively, and our time units are therefore dynamical times $(R_d^3/GM_d)^{1/2}$. Generally all quantities are expressed in units such that $G = M_d = R_d = 1$ unless otherwise noted. These units can be scaled to physical values as desired; we adopt one possible scaling, choosing $R_d = 2.5$ kpc and a unit of time of 10 Myr implies $M_d = 3.47 \times 10^{10} M_{\odot}$; our unit of velocity scales to 244 km s$^{-1}$.

We use the hybrid particle-mesh scheme described in detail in Sellwood (2003, Appendix B). The self-gravity of the disc is computed on a high-resolution cylindrical polar grid (see Sellwood & Valluri 1997; Shen & Sellwood 2004), while that of the halo is computed using a surface harmonic expansion on a spherical grid. We have carefully checked that the preferred plane of the cylindrical polar grid does not introduce artificial forces between a tilted disc and the grid, which requires a fine polar grid.

Our adopted numerical parameters are summarized in Table 1.

Over a period of 400 dynamical times, energy is conserved to about 0.02%, angular momentum components ($L_x$, $L_y$, $L_z$) change no more than 1%, and the absolute change of linear momenta ($p_x$, $p_y$, $p_z$) are less than 0.06 in units we adopted. Without the misaligned torus, these global integrals are conserved with a precision almost one order of magnitude better.

Our main warp diagnostic is the tip-LON diagram introduced by Briggs (1990). We construct it from an analysis of the disc particles, which are binned into annuli. The inner disc is very stiff and remains very closely coplanar within the innermost $3R_d$; we therefore use a single bin for all particles inside this radius and adopt cylindrical bins (spherical bins are preferred if run for a very long time) of equal radial widths $0.7R_d$ outside this radius to $R = 7.9R_d$. (The bins remain aligned with the original disc axes.) We then compute the eigenvectors of the inertia tensor to determine the orientation of the disc element, $\theta$ and $\phi_{\text{LON}}$ in each bin. We use the orientation of the inner bin to define the plane of the inner disc. We have experimented with various bin widths and found this binning scheme gives us the best compromise between spatial resolution and noise in estimation of the disc orientation (especially for thicker “hotter” inner discs). We describe the diagram and its meaning in §3.

**Table 1. Numerical parameters used in the canonical simulation**

| Parameter               | Cylindrical grid | Spherical grid |
|-------------------------|------------------|----------------|
| Grid size               | $(N_R, N_\phi, N_z)$ | $N_r = 401$ |
|                         | $= (102, 128, 125)$ |                |
| Angular compnts . . .   | $0 \leq \theta \leq \pi$ | $0 \leq \theta \leq \pi$ |
| Outer radius ($R_d$) . . | 8.0              | 18.0           |
| z-spacing ($R_d$) . . .  | 0.02             |                |
| Softening length ($R_\phi$) | 0.02       |                |
| $N$ . . . . . . . . . . . | 0.1M             | 0.9M           |
| Time step . . . . . . . . | 0.04             | 0.04           |

**Figure 2.** (a) The warp at $t = 400$ in our simulation; its morphology closely resembles the observed HI warp of NGC 4013 (Bottinelli 1996 reproduced with permission), shown in (b). The length unit shown is the scale length $R_d$ of the exponential disc. Note that we have oriented the model so that the inner ($R < 3R_d$) disc lies in the $x-y$ plane, which is perpendicular to the paper.

**Figure 3.** (a) Disc tip angle $\Theta$ as a function of radius at intervals of 40 time units ($t = 0$ to $t = 320$ from bottom up). (b) as for (a), but the warp angle $\theta$ relative to the inner disc. The inner disc is defined by the radial range $0 < R < 3R_d$. 
3 A CANONICAL SIMULATION

Without the torus, the disc in our model heats mildly; $Q$ rises from $\sim 1.5$ to $\sim 1.8$ over 300 dynamical times, the disc thickens by $\sim 20\%$ and, of course, shows no signs of warping. The virial ratio stays at 0.5, confirming the equilibrium of our initial model. The disc remains unbarred until very late times.

We grow the inclined torus in our canonical experiment, in which the initially thin and flat disc experiences a growing torque. Since the disc is spinning rapidly, it tends to precess about the symmetry axis of the distant torus; differential precession will make the disc warp. We present a quantitative discussion of the precession in §3.3. The much larger moment of inertia of the accreted torus implies that the precession rate of the simulated warp is low – collisional dissipation in a gas layer becomes important only when the differential precession rate is high (Gurney 1979; Tohline et al. 1983). Furthermore gas often condenses into very compact clouds, which behave quasi-ballistically in a similar manner to a stellar system (Binney 1992).

3.2 Warp angle

Figure 3(a) shows $\Theta(R)$, the tip angle relative to the $z$-axis of the initial disc, at various times. In Figure 3(b) we transform to warp angles $\theta$, which are measured relative to the orientation of inner disc ($R < 3R_d$) at each time. The warp amplitude at later times in Figure 3(b) seems to compare well with that in NGC 4013, both visually and quantitatively (cf. Bottema 1996, Figure 6).

3.3 LON of the warp is a leading spiral

Consistent with our convention for warp angles, we use $\phi_{LON}$ for the azimuth of the line of nodes between the plane of the inner disc and that of the annular bin, and $\Phi_{LON}$ for the corresponding line of nodes with the initial unperturbed disc plane.

Figure 4 shows the evolution of LON of the warp, in the form of tip-LON plots (Briegel 1990). The points (small open circles) indicate the angle $(\theta, \phi_{LON})$ of the symmetry axis of an annulus of disc material, relative to the inner disc plane at any given time. The radial coordinate is the warp angle (inclination), $\theta$, and the azimuthal coordinate indicates the azimuth, $\phi_{LON}$, of the best-fit plane to a disc annulus. Points at successively larger ring radii are joined sequentially by straight line segments. Note that since the warp angle, $\theta$, increases with radius (Figure 4), the plotted points refer to radially ordered annuli from the centre out.

Figure 5 shows that the LON of the warp precesses in the retrograde direction, while curving gradually into a leading spiral (i.e., the LON advances in the direction of galaxy rotation for successively larger radii), consistent with the third rule of Briggs (1990). We have run many other experiments with various parameters and found that the LON to $8R_d$ always forms a leading spiral when the disc is perturbed by an outer inclined torus. The reason for the leading spiral is explained in detail in the next section. Also the LON is fairly loosely wound throughout our simulation. Bottema (1996) concluded that the LON of NGC 4013 winds modestly, by about $20^\circ$, which is consistent with that in our simulated warp, in addition to the nice visual resemblance in Figure 2.

4 IN-DEPTH ANALYSIS ON THE SHAPE OF LON

Since there are many factors that affect the warp morphology, we break the dynamics into parts. We first study the simplest aspect analytically, and then gradually add in separate pieces of the physics to illustrate their effects. These simplified experiments are very helpful for developing understanding of warp dynamics and the reason for the curved LON. We stress that our adopted massive
torus at a very large radius has so much angular momentum that its precession due to the torques from the disc and flattened inner halo can be neglected.

4.1 A test particle disc in a rigid halo

We first study a very simple model, in which the halo is both rigid and spherical. We also replace the self-gravitating disc with massless particles, and is grown from 8 particles at each radius; the heavy solid lines are predictions based on the fourth-order approximation of \( \omega_p \) (Equation A3); the dashed lines are predictions based on the second-order approximation (Equation A9). The fourth-order approximations are almost invisible because they overlap the curves measured from the 8 particles, indicating excellent agreement, while the second-order approximations are clearly inadequate at large radii. Note that \( \Phi_{\text{LON}} \) has been set to \(-90^\circ\) for \( \Theta < 1^\circ \), to avoid large scatter at early times.

\[
\Phi_{\text{LON}} \approx -\pi/2 - \Delta \Phi_p / 2 \quad \text{for small } i
\]

(5)

For test particles at a given radius, the tip angle \( \Theta \) and \( \Phi_{\text{LON}} \) can be measured from the orientation of their angular momentum vectors. The angles, \( \Phi_{\text{LON}} \) and \( \Theta \), measured from particles at two distinct radii are shown in Figure 6 each particle in an annulus has its own pair of angles at each instant, which vary systematically with phase giving rise to the spread. The mean is in qualitative agreement with these second-order predictions, shown by the dashed curves while the fourth order (Equation A3) predictions (solid curves) agree very well. Since \( \Phi_{\text{LON}} \) can have any value when \( \Theta \approx 0 \), we set \( \Phi_{\text{LON}} \) to \(-90^\circ\) for \( \Theta < 1^\circ \), to suppress noise in these figures.

The increasing rate of retrograde precession with radius causes the warp angle \( \theta \) and \( \Phi_{\text{LON}} \) to increase outwards also. We observe this behaviour in our test-particle simulation as shown in Figure 7. In this simplified model, the warp appears as the result of the differential precession and the LON forms a trailing spiral.

This simulation also serves as a nice check of parts of our numerical code: forces from the torus particles and the time integration of the test particle disc are calculated accurately.

4.2 Including disc self-gravity

We separate a self-gravitating disc into an inner disc and an outer disc. The dense inner disc, which contains most of the disc mass, tilts rigidly as a whole; it is strongly coupled together due both to self-gravity and to radial velocity spread of stars, which communicate stress across the disc via epicyclic excursions (Debattista & Sellwood 1999). The gradual locking of the disc due to self-gravity was also found in Lovelace (1998). The low-density outer disc, on the other hand, is much less cohesive because self-gravity is weak and epicycles are small; the outer disc behaves more as a collection of test particles therefore.

As the torus mass rises, the whole disc starts to precess as de-
Figure 7. Tip-LON plots of the test particle disc model in different reference frames. (a) The top two rows show Θ and Φ_LON with respect to the fixed coordinate system of Figure 6. (b) The bottom two rows show the angles θ and φ_LON relative to the inner disc frame at each time, which is more appropriate for comparisons with observations. The radial binning scheme is the same as that used in Figure 4, but the concentric dotted circles are drawn at intervals of 10°.

The torque from the torus causes the inner disc, which is strongly cohesive, to precess slowly as a whole in a retrograde manner about the symmetry axis of the torus, while the outer disc precesses more rapidly. The developing misalignment between the inner and outer disc causes the particles in the outer disc to feel an additional torque from the massive inner disc. The rate of precession of the outer disc due to the torque from the inner disc follows a different rule from that due to the torus.

The disc potential at larger radii can be approximated as the sum of a monopole and a quadrupole term (Binney & Tremaine 1987, Equation 6-84),

\[
\Phi_d \approx -\frac{GM_d}{r} + \frac{G}{2r^3} P_2(\cos \theta) \cdot \int_0^\infty 2\pi R^3 \Sigma(R')dR'.
\]  

Using equation (1) for an exponential disc, we have

\[
\Phi_d \approx -\frac{GM_d}{r} + \frac{GM_d}{r} \frac{3R_0^2}{r^2} P_2(\cos \theta).
\]  

In the same manner as for the torus (Appendix A, see also Kahn & Woltjer 1959; Gunn 1979), we find the precession rate of test rings due to the disc is

\[
\omega_{p, \text{disc}} \sim \left(\frac{3}{2} \cdot \frac{3R_0^2}{r^2} \cdot \frac{GM_d}{r}\right) (rV_c)^{-1} \propto r^{-4} \quad \text{for} \quad V_c \sim \text{const.}
\]  

While still producing retrograde precession, the rate from this additional forcing term varies with radius in the opposite sense of that from the torus. In the case of forcing by the inner disc, the precession rate Equation (8) decreases with increasing radius, which in isolation would tend to make the LON develop a leading spiral.

The outer disc is now subject to two torques, from the torus and from the inner disc, both of which cause retrograde precession, but at rates that vary radially in opposite senses. Whether the resulting spiral in the warp LON should lead or trail over the range of interest depends on relative magnitudes of the torques, and is best determined from simulations.

We have therefore performed a series of simulations with increasingly massive discs to show that the torque from the inner disc dominates in our canonical simulation, which develops a leading spiral in the LON (see Figure 4). Figure 8 presents the tip-LON diagrams from four experiments with increasingly larger disc masses (from \(M_d = 0.1\) to \(2.0\)) to demonstrate the effect of the disc mass. These experiments continue to employ the same spherical rigid halo and torus parameters as above. The general trend confirms that the leading spiral gradually dominates the inner LON as \(M_d\) is increased; the LON has become completely leading in Figure 8(d) where \(M_d = 2.0\). (Note that the result for \(M_d = 1\) in a rigid halo differs from that in Figure 3 where the halo is responsive.)

4.3 A test particle disc in a live halo

The rigid spherical halo in the above experiments, cannot respond to the precessing disc, and therefore does not affect how the LON curves. A halo composed of live particles should acquire a distorted shape in response to the fields of the torus and the disc. These effects are both included in a fully self-consistent simulation. Here
we study these effects separately and demonstrate that the live halo does not alter the conclusions we reached in §4.1 and §4.2.

In a further experiment, we replace the rigid halo in the model of §3.1 with a live one with nearly the same potential. Figure 9 shows that in this experiment the LON becomes trailing, as we found in §3.1 where the halo is spherically rigid. This again supports our argument that the massive disc, instead of other factors like a live halo, is the primary reason why the LON forms a leading spiral for warps induced by cosmic infall. The small differences in the angle of precession between Figures 8 and 9(b) indicate that the oblateness of the halo (induced by the torus) contributes only slightly to the precession of the disc.

We discuss the response of a live halo with a massive disc in §5.

4.4 Spirality twist of LON at large radius

In Figure 8(c), we can clearly see the spirality of LON twists from leading to trailing in the tip-LON plot at \( t = 500 \). Here we show that such spirality twist is a generic feature for warps caused by an external quadrupole field giving outward-increasing rate of retrograde precession.

As shown above, the sense of spirality of the LON is determined by the radial variation of the precession rate \( \omega_p \), and is measured by \( \Phi_{LON} \) in the tip-LON plots. From Equation (8) and (9), we have

\[
\Phi_{LON} \approx \text{const} - \frac{1}{2} \int \omega_p(R, t) \, dt
\]

the negative sign indicates that the precession is retrograde. Whether the spiral leads or trails depends on whether \( \omega_p \) falls or rises as \( R \) increases, as illustrated in Figure 10.

We have argued that the torque due to the massive inner disc, which causes \( \omega_p \sim \omega_{p, \text{disc}} \propto R^{-4} \) to dominate, causing the leading spiral found in our simulations. Beyond some critical radius, \( R_{tw} \), precession is dictated by the accreting torus, \( \omega_p \sim \omega_{p, \text{torus}} \propto R \), and a trailing LON should be expected. The critical radius \( R_{tw} \) can be estimated as follows: The magnitude ratio of the torques exerted on a test ring at radius \( R \), due to the torus (the first term of Equation A7 as the second-order approximation) and to the massive inner disc (Equation 8), is approximately

\[
\frac{\tau_{\text{torus}}}{\tau_{\text{disc}}} \approx \left( \frac{3 GM_d R_t^2}{4 R_i^4} \right) \left( \frac{9 GM_d R_t^3}{2 R R^2} \right)^{-1} = \frac{1}{6} \frac{M_d}{M_i} R^5 \frac{R_t}{R_i^3}
\]

The radius of the spirality twist is approximately where

\[
\frac{\tau_{\text{torus}}}{\tau_{\text{disc}}} \mid_{R=R_{tw}} \sim 1.
\]

So we have

\[
R_{tw} \sim \left[ 6 \left( \frac{M_d}{M_i} \right) \right]^{1/5} \left[ \frac{R_t}{R_i^3} \right]^{1/5}.
\]

For our torus parameters \((M_i = 2.5, R_c = 15)\), we find \( R_{tw} \sim 6.1 R_i \). In Figure 8(c) where \( M_d = 1.0 \), the spirality twist occurs near the sixth point from the centre \((i.e., \text{near the annulus centered on } R = 6.15 R_i)\), so it agrees well with the expected \( R_{tw} \).

\( R_{tw} \) can also be understood in the context of a Laplacian surface, where the net torque vanishes \cite{Binney1987, chap. 6.6}. It is easy to show that \( R_{tw} \) is also the transition radius of the Laplacian surface; for \( R \ll R_{tw} \), the Laplacian surface almost coincides with the inner disc, which dominates the precession for small radii, while for \( R \gg R_{tw} \), the Laplacian surface follows the equatorial plane of the torus, which dominates the precession at large radii.

The inner part of a responsive halo becomes oblate due to the massive inner disc. The co-aligned oblate inner halo adds significantly to the torque from the inner disc, causing \( R_{tw} \) to be larger than estimated above, to perhaps \( R_{tw} \lesssim 7 R_i \). So the LON spirality twist could occur at radii \( \gtrsim 2 R_{10} \).

Thus the spirality twist can be seen only if there is a tracer at very large radii. We do not see the spirality twist in Figure 11 because
very few particles lie beyond $\sim 8R_d$. To reveal the expected spi-
rality twist, we add into our canonical simulation 20,000 massless
test particles in the radial range of $3R_d < R < 12R_d$, to mimic
the outer gas layer. We used 14 radial bins to cover all particles; the
binning scheme is similar to that described in §3. The spirality
twist shows up in Figure 11 near the seventh from outermost point
(i.e., near the annulus centered on $R = 7.55R_d$). As expected, the
spirality twist grows more pronounced with time.

Note that a twist can occur only if the forcing causes an
outward-increasing rate of retrograde precession, as for our sim-
plified torus. More realistic perturbations may not have this effect.

5 THE HALO RESPONSE

In this section, we return to our canonical simulation (§4) with the
live halo and the massive disc ($M_d = 1.0$).

5.1 Disc–inner halo alignment

Figure 12 shows the orientation of the halo in three spherical bins
$(0 < r < 4R_d$, $4R_d < r < 8R_d$, and $8R_d < r < 12R_d)$
measured from the canonical simulation. The inner halo (heavy
line) follows the inner disc (dashed) closely, while the outer halo
(light line) quickly aligns with the outer torus, which has a tip an-
gle $\Theta \sim 15^\circ$ and azimuth $\Phi_{LON} \sim 180^\circ$. The halo orientation
in the middle bin is intermediate. It is worth stressing that the halo
mass interior to $5R_d$ is almost five times that of the disc, yet its
orientation does not follow that of the outer halo and massive torus,
but instead follows that of the lighter disc, which precesses around
the axis of the torus because of its angular momentum. Because
the massive inner halo is oblate and aligned with the inner disc,
it adds significantly to the precession and curving of the LON in
the outer disc. Binney et al. [1998] also found that the inner halo
quickly realigns with the disc, even when a large initial misalign-
ment is imposed.

5.2 Halo damping

Figure 13 shows that halo damping of the warp by the live halo is
weak. Now for the convenience of description, we name the canonical
run as “Run CL” (i.e. canonical run live halo). We introduce
“Run CR” (i.e. canonical run rigid halo) with the almost same halo
potential as in the canonical run except that the halo is rigid and stays
exactly spherical. Panel (a) shows that nested rings of test
particles on circular orbits in a rigid halo precess at constant incli-
nation, $\theta_i$, to the plane of the torus. Since rings precess at different
rates, the apparent deviations from $15^\circ$ at later times are caused by
averaging rings with the same inclinations, yet different azimuths,
over a finite radial range. (We have verified that the inclinations of
individual particles with respect to the torus plane indeed remain
close to $15^\circ$, as expected.) Panel (b) shows that self-gravity in a
massive disc causes differential precession about the axis of the
inner disc (Run CR). The quasi-periodic modulation of the inclina-
tions of the outer rings in (b) is a manifestation of the extra torque
from the inner disc. The initial decrease of the inclinations of most
annuli to the torus plane (between $t = 0$ and 600) has nothing to
do with dynamical friction or halo damping since the halo is rigid
in this case. The final panel, (c), shows the effect of using a live
halo (the canonical Run CL). The precession due to the disc is aug-
mented by the aligned flattening of the inner halo, which is partly
responsible for the differences from panel (b), but a full explana-
tion of all the details in this Figure is complicated; e.g., we would
need to take into account the inclination of the halo at intermediate
radii. The important result is that even after 2000 dynamical times,
the mean inclination of the inner disc to the torus has not decreased
by more than $3^\circ$; i.e., damping from the live halo is very weak.

Our result differs from the conclusions of Dubinski & Kuijken
[1995] and Nelson & Tremaine [1995] because the scenarios studied
are different. Their models have a very large initial misalign-
ment between the halo and the inner disc and a significant pre-
cession rate calculated according to the “modified tilt mode” in
Sparks & Casertano [1988]. Another difference is that the halos in
their models were given no time to adjust to the strong and rapidly-
precessing warp initially inserted in.

In our simulations, the precession rate of the inner disc is
very low, and the inner halo follows its motion closely, as shown in
Figure 12 which is why dynamical friction between the two
components is negligible. Only the outer halo is misaligned and
damping comes mainly from coupling between the inner and outer
halo. Such coupling is weak for the isotropic halo models we used
[Nelson & Tremaine 1992]. This is also consistent with the recent

\footnote{Note that the two potentials cannot be exactly the same, because the live
halo is slightly oblate due to the gravity of the disc initially, and its oblate-
ness evolves with time.}
cosmological hydrodynamical simulations by Bailin et al. (2005), which found that the relative orientations of inner ($r < 0.1 r_{vir}$) and outer ($r > 0.1 r_{vir}$) halo are uncorrelated.

6 DISCUSSION

6.1 The source of the external torque

The torque in our model arises from a uniform, massive torus centered on the galaxy and inclined to the disc, as originally adopted by Jiang & Binney (1999). The warp is driven by the quadrupole field of the torus, which varies as $M_t/R_t^2$, and therefore a less massive torus at a smaller radius has a similar effect, as we have verified in other experiments. However, a weaker perturbation would give rise to a milder warp that would take longer to form.

A natural direct interpretation of the torus is an idealized representation of the stream of stars and dark matter from a disrupted orbiting satellite, if the mass is sufficiently large. There are numerous examples of accreted satellites, or companion galaxies orbiting in planes that are misaligned with that of the disc of the host galaxy. The Sgr Dwarf and the Magellanic Clouds are clear examples in the Milky Way, but their masses are generally on the low side unless their halos are substantial.

On the other hand, the torque from the torus is similar to that of a flattened outer halo, which may also have misaligned angular momentum (Quinn & Binney 1992). Conceptually, one could separate the halo into two mass components: an outer halo flattened in a plane misaligned with the disc and having a uniform, quasispherical core, and the remainder – an inner halo with an arbitrary radial mass profile, that may be strongly peaked to the centre of the galaxy. Regardless of how flattened the inner halo may be, it exerts no torque on the inner disc, since it is generally aligned with the disc plane (see § 6.2). In this picture, the gravitational potential of the outer oblate halo alone might approximate the form: $\Phi_{OH}(R, z) = v_0^2/2 \cdot \ln (R^2 + z^2 + q_0^2)$, which arises from a flattened isothermal envelope with a quasi-spherical, uniform core of radius $\sim R_c$, with a density scale set by $v_0$. The ellipticity of the equipotential surfaces $\epsilon_q = 1 - q_0$ is approximately $\epsilon_p/3$, where $\epsilon_p$ is the ellipticity of the halo density. It is easy to show the torque due to this outer halo exerted on a ring in the disc at radius $r < R_c$ is $(\tau(r) \sim r^2 v_0^2(q_0^2 - 1)/(2R_c^{2q}))$, which has the same radial dependence as the torque of a uniform torus. The effects discussed in § 6.1 and 6.2 are dependent on the outward-increasing rate of retrograde precession of our model. The torque from our adopted torus is of the same order of magnitude as such an outer halo with a typical flattening in the range $0.6 < q_0 < 0.9$ depending on $R_c$.

Cosmological simulations that include baryonic infall have not yet settled on a set of robust predictions for the shape and alignment of the outer halo. A misaligned potential of the form just described is highly idealized, but has served as a useful theoretical exercise to understand warps. It has yielded some insight that may capture features of the torques that drive real warps.

6.2 Persistence of warps

The windup rate of a warp depends on the radial variation of the net precession rate. (The amount of the windup can be measured by $\cot i = \Delta t \cdot R \cdot d\omega_t/dR$, see Binney & Tremaine 1987.) Forcing by our adopted torus causes differential precession of the opposite sign from that of the disc, making the total retrograde precession less differential, especially in the warp region ($4R_d < R < 10R_d$). Thus the stiff inner disc and the two precessions work together to slow the windup of the warp.

However, it is more interesting to study the evolution of a prewarped massive disc without any external forcing. We therefore conducted two further experiments in which we removed the forc-
The LON of Run runs, with a rigid halo, had the exactly same LON at \( t = 400 \).

In these two runs, a warp was already well developed by \( t = 400 \). Ramping down the torque from the torus allows us to study the unforced behaviour of the warped disc. In run CR–torus, the disc is affected by its self-gravity only (because the halo is rigidly spherical), but in run CL–torus, the effect of self-gravity of the live halo is also included.

In Figure 14 and Figure 15, we compare the LON between runs in which the torus was left in place and the cases in which it was removed. Figure 14 shows the two cases at \( t = 700 \) for the rigid halo, and Figure 15 shows the corresponding two cases with the live halo. Note that both experiments in each pair are identical at \( t = 0 \) and 600 after which, no torus particles remain. (Note that in all runs the torus mass was grown linearly to its maximum between \( t = 0 \) and 200.)

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7 CONCLUSIONS

Now that the role of a live halo has been properly taken into account, the idea that galaxy warps are manifestations of long-lived warp modes appears to have reached a dead end. But warps as responses to external forcing remain viable, provided that suitable perturbations are frequent enough. Hierarchical galaxy formation scenarios guarantee late infall within gravitationally bound subgroups, both diffuse and in lumps, and the infalling matter is likely to share the angular momentum vector of the early disc and halo (Ostriker & Binney 1989; Quinn & Binney 1992).

The simulations presented here are motivated by this idea, but are limited to an in-depth study of one highly idealized case. We examine how an initially flat particle-disc with random motion, embedded in a live halo, gradually acquires a warp as a result of a steady, applied external torque. Rather than striving for realism, we have endeavoured to understand how the warp develops and why the LON generally forms a loosely-wound, leading spiral.

Despite having focused mainly on one case, we find a long-lived, large-amplitude warp that resembles those observed in many respects. The disc is flat in the inner part, and starts to warp at \( r \sim 5R_d \) into an open leading spiral, which is consistent with the warping rules found by Briggs (1990).

The external torque causes the disc to precess in a retrograde sense. The warp develops because the precession rate of the inner disc, which is strongly cohesive due to its self-gravity and random motion, is slower than that of the outer disc. The growing misalignment between the inner and outer disc gives rise to a new torque acting on the outer disc from the massive inner disc, and the inner halo that is coupled to it. The torque from the interior mass is responsible for the leading spiral of the line of nodes – our adopted external field alone would produce a trailing spiral. The fact that the LON of most warps forms a leading spiral over an extended radial range seems to imply a massive disc (e.g. Figure 8). On the other hand, the leading spiral tells us little about the cause of the warp, but such information might be revealed by the behaviour of the warp at large radii. We show that the LON twists to trailing at very large radii from our experiments with a perturbation that alone would cause the warp to form a trailing spiral. Thus better observational data on the shape of the LON at very large radii may reveal the radial dependence of the torque that created the warp, and thereby provide information on the shape of the halo at large radii.

Because the two separate components of the torque in our model cause differential precessions in opposite senses, the net retrograde precession of the warp is less differential than when only one is present. The moment we removed the external torque, the warp began to wind, although its amplitude did not decrease. Thus the quasi-steady warp we observe that lasts for more than a Hubble time is a consequence of steady external forcing.

Even though the disc precesses due to the external torque, its motion is hardly damped over many Gyr, in contrast to the expectations from Nelson & Tremaine (1995). Damping is weak because the slow precession rate allows the inner halo to remain closely aligned with the disc, which therefore causes little drag. The very
weak damping seems to be caused more by the relative precession of the inner and outer halo.

A fixed outer torus is clearly unrealistic and external forcing is likely to be strongly time-dependent in reality. We cannot say much about the rate at which misalignments in real halos will settle, but we have shown that the warp survives for a few gigayears after the torus is removed. Time dependence may anyway be a side issue if the halo axis is constantly slewing, as argued by Quinn & Binney (1992). Warps formed this way can be repeatedly regenerated when a new infall event happens. Since cosmic infall and mergers are more likely to happen in a denser environment, warps can be induced more frequently in such an environment, which is consistent with the warp statistics in García-Ruiz et al. (2002).

Since warps are ubiquitous and a gravitational phenomenon closely related to hierarchical galaxy formation, their properties may possibly be able to constrain small-scale cosmic structure. Clearly further experimentation with various external perturbations, halo profiles and masses, disc properties (random motions, barredness), etc., is required to discover what might be inferred from the existence of warps in galaxies.

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APPENDIX A: THE PRECESSION RATE DUE TO THE TORUS

Suppose the disc is composed of many rigid rings (or annuli). Here we calculate the retrograde precession rate of one such ring, at radius \( r \) and inclined relative to the torus symmetry plane with the angle \( \theta \), due to the potential of the accreting torus (see Figure A1).

The potential of the accreting torus (its thickness ignored) can be obtained as the following, by solving Laplace’s equation.

\[
\Phi(r, \theta) = -\frac{GM}{R_t} \left[ 1 - \frac12 \left( \frac{r}{R_t} \right)^2 P_2(\cos \theta) + \frac38 \left( \frac{r}{R_t} \right)^4 P_4(\cos \theta) + O \left( \frac{r^6}{R_t^6} \right) \right],
\]

where \( \theta \) is the conventional polar angle in the polar coordinate system based on the torus; \( R_t \) and \( M \) are the radius and mass of the torus, respectively.

We define

\[
T(r) = \frac12 \frac{GM}{R_t} \left( \frac{r}{R_t} \right)^2 \quad \text{and} \quad Q(r) = \frac{3GM}{8 R_t} \left( \frac{r}{R_t} \right)^4
\]

for convenience, so to the order of \( r^4/R_t^4 \) we have

\[
\Phi(r, \theta) = -\frac{GM}{R_t} + T(r)P_2(\cos \theta) - Q(r)P_4(\cos \theta).
\]

The only force component relevant to the calculation of torque on the annulus is \( \Phi_{\theta} \):

\[
F_\theta = -\frac{1}{r} \frac{\partial \Phi}{\partial \theta} = -\frac{T(r) dP_2(\cos \theta)}{r \cos \theta} \cdot (-\sin \theta) + \frac{Q(r) dP_4(\cos \theta)}{r \cos \theta} \cdot (-\sin \theta).
\]

Since

\[
P_2(x) = \frac12 (3x^2 - 1); \quad \frac{dP_2(x)}{dx} = 3x;
\]

\[
P_4(x) = \frac18 (35x^4 - 30x^2 + 3); \quad \text{and} \quad \frac{dP_4(x)}{dx} = \frac12 (35x^3 - 15x),
\]

we find

\[
F_\theta = 3\frac{T(r)}{r} \cos \theta \sin \theta + \frac12 \frac{Q(r)}{r} (15 \cos \theta - 35 \cos^3 \theta) \sin \theta.
\]

The torque exerted on a point mass in the rigid ring due to the torus depends on the azimuth of the point mass; the maximum value of the torque occurs at azimuth \( \phi = 0 \) or \( \pi \) where \( \theta = \pi/2 - i \).

\[
\tau_{\max} = |\mathbf{r} \times \mathbf{F}| = r F_\theta = 3T(r) \cos \theta \sin \theta + \frac12 Q(r)(15 - 35 \cos^2 \theta) \cos \theta \sin \theta
\]

\[
= 3T(r) \sin i \cos i + \frac12 Q(r)(15 - 35 \sin^2 i) \sin i \cos i.
\]

The azimuthally average torque on the rigid ring is therefore

\[
\langle \tau \rangle = \int_0^{2\pi} \tau_{\max} \cos^2 \phi d\phi \frac{2\pi}{2\pi} = \frac{3}{2} T(r) \sin i \cos i + \frac14 Q(r)(15 - 35 \sin^2 i) \sin i \cos i.
\]

We can obtain the total retrograde precession rate due to the torus (up to the fourth order):

\[
\omega_p = \frac{\langle \tau \rangle}{L \sin i} = \frac{3}{2} \frac{T(r)}{r \cdot V_c} \cos i + \frac14 \frac{Q(r)(15 - 35 \sin^2 i)}{r \cdot V_c} \cos i
\]

\[
= \frac{3GM}{4} \frac{r}{R_t^3} \cos i \left[ 1 + \frac18 \left( \frac{r}{R_t} \right)^2 (15 - 35 \sin^2 i) \right]
\]
\[ = \omega_p^{(2)} \left[ 1 + \frac{1}{8} \left( \frac{r}{R_t} \right)^2 \left( 15 - 35 \sin^2 i \right) \right]. \] (A8)

For qualitative purposes, we generally use the following simpler second-order approximation \( \omega_p^{(2)} \), unless the radius of the annulus is large.

\[ \omega_p \approx \omega_p^{(2)} = \frac{3}{4} \frac{GM_t r}{R_t V_c^2} \cos i. \] (A9)

Note that same results can be obtained for the nodal precession rate of a test particle’s circular orbit (as opposed to a rigid ring), the only difference here is that the average is taken as the time average over one orbit (the orbit-average procedure can be found in Steiman-Cameron & Durisen 1984, Appendix).

**APPENDIX B: THE DERIVATION OF TIP ANGLE OF RINGS AND AZIMUTHAL ANGLE OF THEIR LON**

We first calculate the tip angle \( \Theta \) relative to the original z-axis. We know \( 2\overline{AB} \sin \frac{\Delta \Phi_p}{2} = 2\overline{OB} \sin i \sin \frac{\Delta \Phi_p}{2} \) (B1)

here \( \Delta \Phi_p = \int \omega_p(t) \, dt \) is angle that the annulus has precessed. So obviously the equation to determine \( \Theta \) is

\[ \sin \frac{\Theta}{2} = \sin i \sin \frac{\Delta \Phi_p}{2}. \] (B2)

If we define the azimuthal angle of the LON of a ring as the counter-clockwise angle from +x axis, \( \Phi_{LON} = \angle ADC - \pi \). It can be shown, with some trigonometry on triangles in Figure B1 that

\[ \angle ADC = \cos^{-1} \left[ \frac{\cos i \sin \frac{\Delta \Phi_p}{2}}{\sqrt{1 - \sin^2 i \sin^2 \frac{\Delta \Phi_p}{2}}} \right]. \] (B3)
So

\[ \Phi_{LON} = \angle ADC - \pi = \cos^{-1} \left[ \frac{\cos i \sin \frac{\Delta \phi_p}{2}}{\sqrt{1 - \sin^2 i \sin^2 \frac{\Delta \phi_p}{2}}} \right] - \pi \]  

(B4)

For small \( i \), we have

\[ \Phi_{LON} \approx -\frac{\pi}{2} - \frac{\Delta \phi_p}{2} \]  

(B5)

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