Tails of probability density for sums of random independent variables

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The exact expression for the probability density \( p_N(x) \) for sums of a finite number \( N \) of random independent terms is obtained. It is shown that the very tail of \( p_N(x) \) has a Gaussian form if and only if all the random terms are distributed according to the Gauss Law. In all other cases the tail for \( p_N(x) \) differs from the Gaussian. If the variances of random terms diverge the non-Gaussian tail is related to a Lévy distribution for \( p_N(x) \). However, the tail is not Gaussian even if the variances are finite. In the latter case \( p_N(x) \) has two different asymptotics. At small and moderate values of \( x \) the distribution is Gaussian. At large \( x \) the non-Gaussian tail arises. The crossover between the two asymptotics occurs at \( x \) proportional to \( N \). For this reason the non-Gaussian tail exists at finite \( N \) only. In the limit \( N \) tends to infinity the origin of the tail is shifted to infinity, i.e., the tail vanishes. Depending on the particular type of the distribution of the random terms the non-Gaussian tail may decay either slower than the Gaussian, or faster than it. A number of particular examples is discussed in detail.

For the sake of simplicity in what follows sums of independently and identically distributed random variables with zero mean values are considered. Extension of the approach to more general cases is straightforward.

First of all notice that, of course, nothing is wrong with CLT. The point is that CLT says that \( p(x) \) converges to GD asymptotically at \( N \to \infty \). On the other hand in any practical case \( N \) is a finite quantity. At finite \( N \) the profile of probability density for \( x \) may have nontrivial dependence on \( N \). To stress this dependence \( p(x) \) at finite \( N \) will be denoted below as \( p_N(x) \). Then, if \( p_N(x) \) has a non-Gaussian tail, which begins at \( |x| \gg x_c(N) \) and if \( x_c(N) \to \infty \) at \( N \to \infty \), it does not contradict CLT. What I am going to show is that the occurrence of such a tail is a generic feature of the problem.

To begin with let us consider two following examples

\[
\begin{align*}
  f(\xi) &= \begin{cases} 
    0 & \text{at } \xi < -1/2 \\
    1 & \text{at } -1/2 \leq \xi \leq 1/2 \\
    0 & \text{at } \xi > 1/2 
  \end{cases} \\
  f(\xi) \Xi A/|\xi|^m \text{ at } |\xi| \gg x_c,
\end{align*}
\]

where \( f(\xi) \) denotes the probability density for random \( \xi \). In Eq. (3) \( A, m \) and \( x_c \) stand for certain positive constants.

In case of Eq. (2) the conditions of CLT hold. However, it is evident that at any finite \( N \) \( p_N(x) \) vanishes identically at \( |x| \geq N/2 \) instead of exhibiting the exponential Gaussian decay at \( x \to \infty \). In this case one has a superlight tail (actually, no tail at all).

To see a non-Gaussian tail in case of Eq. (3) let us first calculate the probability \( p_1(x)dx \) that \( x \) has a certain large value from the interval \( x_1 \leq x \leq x_1 + dx \) because of contribution of a single, anomalously large term, while all other terms in sum Eq. (3) are much smaller than this single term. The probability that a term in Eq. (3) has the value \( \xi_n = x \), where \( x \) belongs to the specified region is \( f(x)dx \). The probability, that \( |\xi_n| \) is much smaller then \( |x| \) is \( \int_{-x}^{x} f(\xi')d\xi' \), \( \xi \ll |x| \).

Due to independence of \( \xi_n \) the probability that, say, \( \xi_1 = x \)

It is well known that if a random quantity \( x \) is a sum of a number of independent random variables \( \xi \)

\[
x = \sum_{n=1}^{N} \xi_n, \quad (1)
\]

the corresponding probability density \( p(x) \) is described by a Gaussian distribution (GD), provided the conditions of the Central Limit Theorem (CLT) of Probability Theory [1] are fulfilled. However, in reality while GD fits well the profile of \( p(x) \) at small and moderate values of \( x \) the very tail of \( p(x) \) often deviates from the Gauss Law. Most of the observed deviations are related to the so-called heavy tails [2, 5], which exhibit a decay slower than the Gaussian (usually the heavy tails decay as a certain power of \( x \)). It should be stressed that despite the heavy tails do not influence much the average characteristics of \( x \), such as, e.g., different moments of it, large fluctuations of \( x \) are described by the tails entirely. Therefore the tails of \( p(x) \) play an important role in those problems, where rare but large fluctuations are of special interest, for example for temperature fluctuations in a nuclear reactor, where too big fluctuations may result in damage of the reactor.

The heavy tails have been observed in a wide diversity of phenomena ranging from the statistics of solar flares [2, 3] to the one of casualties of wars [4] and from market price fluctuations [5] to fluid dynamics [6]. It provides grounds to suppose that the phenomenon should have a general cause, which is not related to a particular realization of a random sum in a given problem. However, to the best of my knowledge such a cause has not been revealed yet.

The present contribution is an attempt to fill the gap. It is shown that occurrence of non-Gaussian tails is quite a common case, which does not require any special conditions to be realized.\( \star \)\( \star \)\( \star \)

Visa versa, special (and very strict) conditions are required for realization of Gaussian tails. It is shown also that depending on the distribution of random \( \xi \) the tails of \( p(x) \) could be either heavy, or light. In the latter case the tails decay faster than the Gaussian. Examples of tails of both types are considered in detail.

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Due to independence of \( \xi_n \) the probability that, say, \( \xi_1 = x \) would have been...
and \( |\xi_n| \ll |x| \) at \( 2 \leq n \leq N \) is \( \int_\xi^\infty f(\xi')d\xi' \rangle^{N-1} f(x)dx \). Finally, bearing in mind that there are \( N \) terms in sum Eq. (3) and every one may be large, we obtain the following expression for the total probability of the desired event

\[
p_1(x)dx = \left[ \int_\xi^\infty f(\xi')d\xi' \right]^{N-1} N f(x)dx = \left[ 1 - \int_{-\infty}^{-\xi} f(\xi')d\xi' - \int_\xi^\infty f(\xi')d\xi' \right]^{N-1} N f(x)dx.
\]

Let us suppose that \( x \) is so large that the following conditions hold

\[
N \left( \int_{-\infty}^\infty f(\xi')d\xi' + \int_{-\infty}^{-\xi} f(\xi')d\xi' \right) \ll 1; \quad \xi_c \ll \xi \ll x.
\]

In this case the probability density \( p_1(x) \) is reduced to the expression

\[
p_1(x) \equiv N f(x), \quad (4)
\]

i.e., a heavy tail arises, see Eq. (3).

Note, that according to CLT the probability density \( p(z) \) for the normalized sum

\[
z \equiv x/(\sigma \sqrt{N}),
\]

where \( \sigma^2 \equiv \langle \xi^2 \rangle \), should converge to GD at \( m > 3 \), viz.

\[
p(z) \equiv p_0(z) \equiv \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{z^2}{2} \right) \text{ at } N \to \infty. \quad (5)
\]

Here and in what follows \( \langle \ldots \rangle \) denotes average over \( \xi \).

GD Eq. (5) describes the probability density for the sum to have a value \( x \) due to contribution of a big number of individual terms. Naturally, it cannot be smaller than the probability density related to contribution of a single anomalously big term. In other words GD Eq. (5) certainly fails to describe the actual situation, when the condition \( p_0(z) \geq p_1(x(z))dx/dz \) is violated [8]. It provides us with the applicability condition for the GD \( |z| \ll z_0 \), where \( z_0 \) stands for the solution of the equation \( p_0(z) = p_1(x(z))dx/dz \). At large \( N \) the approximate solution of this equation obtained by iterations is in the following

\[
z_0 \approx [(m-3) \ln N + m \ln ((m-3) \ln N + \ldots)]^{1/2} \gg 1.
\]

Thus, for the problem in question the GD is invalid when the value of \( x \) becomes much greater than the standard deviation for the given GD.

In what follows I am going to prove that the above result is a generic property of the problem, which is not connected with the particular type of \( f(\xi) \). Naturally, it is supposed that \( f(\xi) \) decays at big \( |\xi| \) sharp enough, so that the variance \( \sigma^2 \) remains finite. Otherwise instead of GD one will obtain a Lévy distribution (LD). The case of LD will be considered later separately.

Let us calculate the characteristic function \( \varphi_N(\omega) \equiv \langle \exp(i\omega z) \rangle \), where \( \omega \) is a real quantity. By definition

\[
\varphi_N(\omega) \equiv \int_{-\infty}^{\infty} \exp(i\omega z) p_N(z)dz \quad (6)
\]

On the other hand, employing Eq. (3) and independence of \( \xi_n \) one can obtain

\[
\varphi_N(\omega) \equiv \left\langle \prod_{n=1}^{N} \exp \left[ \frac{i\omega \xi_n}{\sigma \sqrt{N}} \right] \right\rangle = \prod_{n=1}^{N} \left\langle \exp \left[ \frac{i\omega \xi_n}{\sigma \sqrt{N}} \right] \right\rangle \equiv [g_N(\omega)]^N,
\]

where

\[
g_N(\omega) \equiv \int_{-\infty}^{\infty} \exp \left[ \frac{i\omega \xi}{\sigma \sqrt{N}} \right] f(\xi) d\xi. \quad (7)
\]

Eqs. (5) – (7) supplemented with the inverse Fourier transform give rise to the following expression for \( p_N(z) \)

\[
p_N(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[-i\omega z + N \ln g_N(\omega)]d\omega. \quad (8)
\]

It should be stressed that Eqs. (8) – (8) are just a sequence of identities. Thus, expression Eq. (8) is an exact result, valid at any value of \( N \) [8].

\[\text{FIG. 1. Contour of integration for Eq. (11)}\]

To obtain GD from Eqs. (7) – (8) let us consider the limit of large \( N \) and small \( \omega \). Expansion of the integrand in Eq. (7) in powers of \( \omega/\sqrt{N} \) and integration (I remind the reader that \( \langle \xi \rangle = 0 \)) yield

\[
g_N(\omega) = 1 - \frac{\omega^2}{2N} + O \left( \frac{\omega^2}{2N} \right) \quad (9)
\]

\[
N \ln g_N(\omega) = -\frac{\omega^2}{2} + o(\omega^2) \quad (10)
\]

Then, inserting Eq. (10) in Eq. (8) one obtains

\[
p_N(z) \equiv \int_{-\infty}^{\infty} \exp \left[ -\frac{1}{2}(\omega + iz)^2 - \frac{z^2}{2} \right] d\omega. \quad (11)
\]
The integral is taken by integration in the plane of complex \( \omega \) along the contour shown in Fig. 1. The integrand is an analytic function inside the contour, hence the integral over the entire contour is zero. The integrals along the vertical segments vanish exponentially at \( \omega' \to \pm \infty \). Therefore the integral Eq. (11) equals the one along the line \( -\infty - i \zeta < \omega < \infty - i \zeta \). It eventually results in GD Eq. (5).

However, here we are focusing on the applicability conditions of these transformations rather than on the well-known result of the integration. The transformation of the integral Eq. (11) into the one in the complex plane implies that expansion Eq. (10) transforms into an identity. Equalizing Eq. (7) and Eq. (13) and employing the inverse Fourier transform, one can recover the only applicable condition for the expansion say that \( |\omega|^2/N \ll 1 \), where \( \omega \equiv \omega' + i \omega'' \). This condition must hold at least for those \( \omega ' s \), which make the main contribution to the integral. For the integral along the line \( -\infty - i \zeta < \omega < \infty - i \zeta \) the main contribution is made by \( \omega' = O(1) \). However, the imaginary part of these \( \omega ' s \) is \( -\zeta \). We are interested in the tail of \( p_N(z) \), where \( |\zeta| \gg 1 \), so for \( \omega \) with \( \omega' = O(1) \ll |\zeta| \) we have \( |\omega| \cong |\zeta| \), which finally yields the following applicability condition for GD Eq. (5)

\[
z^2 \ll N \tag{12}
\]

Generally speaking, it means that the tail of actual \( p_N(z) \) given by Eq. (3) differs from that for Eq. (5) at \( z^2 \) of order of (or grater than) \( N \). The only exception from the rule is the case when

\[
g_N(\omega) = \exp \left( -\frac{\omega^2}{2N} \right), \tag{13}
\]

and expansion Eq. (10) transforms into an identity. Equalizing Eq. (7) and Eq. (13) and employing the inverse Fourier transform, one can recover the only \( f(\xi) \), which brings about a Gaussian tail for \( p_N(x) \). This \( f(\xi) \) is as follows

\[
f(\xi) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{\xi^2}{2\sigma^2} \right).
\]

It is worth mentioning that contrary to all other cases, when GD is an asymptotic law valid at large \( N \) only, now Eq. (5) is the exact result, which is good at any \( N \) and any \( z \). This property reflects the well-known fact of self-similarity of a Gaussian distribution.

Thus, we have arrived at a number of important conclusions

- A sum of any finite number of independently and identically distributed random variables has a Gaussian tail if and only if the random variables themselves have a Gaussian distribution.
- In any other situation the probability density \( p_N(z) \) for the normalized sum \( z \) has a tail, which differs from the Gaussian.
- Contrary to the GD for the normalized sum \( z \) the non-Gaussian tail is not universal. Its profile depends on the distribution of the random terms contributing to the sum and may be either heavier than the Gaussian, or lighter than it.
- The non-Gaussian tail begins at \( |z| \) of order of \( \sqrt{N} \) and lasts up to infinity.
- The non-Gaussian tails exist at finite \( N \) only. It disappears in the limit \( N \to \infty \).

To illustrate these general conclusions let us consider in detail several particular examples of \( f(\xi) \) and obtain the corresponding profiles of \( p_N(z) \) based upon Eqs. (7) – (8). In case of Eq. (4) simple calculations yield

\[
g_N(\omega) = \frac{2\sigma \sqrt{N}}{\omega} \sin \frac{\omega}{2\sigma \sqrt{N}}, \quad \sigma^2 = \frac{1}{12},
\]

so that

\[
p_N(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\sigma \sqrt{N}}{\omega} \sin \frac{\omega}{2\sigma \sqrt{N}} \exp \left( -i\omega z \right) d\omega. \tag{14}
\]

First I show that the above expression for \( p_N(z) \) does fulfill the condition \( p_N(z) = 0 \) at \( |z| \geq \sqrt{N}/(2\sigma) \). To this end let us extend the integration in Eq. (14) to the plane of complex \( \omega \) and note that the integrand is an analytic function of \( \omega \) at any finite \( |\omega| \). For this reason the corresponding integral over any closed contour in the complex plane is identical zero. For definitiveness in what follows I suppose that \( z > 0 \) (the case of negative \( z \) is analyzed analogously). Then, I consider a closed contour consisting of two elements — a segment of the real axis \( -R \leq \omega' \leq R \) and a circular arc with the radius \( R \) lying in the lower half-plane, which connects the edges of the segment. Next, note that the greatest term of the integrand in the lower half-plane has the form

\[
\frac{\text{const}}{\omega^{N}} \exp \left[ i\omega \left( \frac{\sqrt{N}}{2\sigma} - z \right) \right]
\]

It is straightforwardly seen from the above expression that the integral over the arc tends to zero at \( z > \sqrt{N}/(2\sigma) \) and \( R \to \infty \). Finally, bearing in mind that the integral over the entire closed contour is zero, I conclude that the integral Eq. (14) does turn into identical zero at \( z > \sqrt{N}/(2\sigma) \). As an example the plot \( p_N(z) \), obtained by numerical integration of Eq. (14) at \( N = 10 \) is presented in Fig. (3).

Next, we discuss \( p_N(z) \) at \( f(\xi) \) resulting in heavy tails. Specifically we consider the normalized probability density \( f(\xi) \) in the form

\[
f(\xi) = \frac{A}{1+\xi^2}, \quad A \equiv \frac{l}{\pi} \sin \frac{\pi}{2l}, \tag{15}
\]

where \( l \) is any positive integer number. Thus, the case in question is a particular realization of \( f(\xi) \) given by Eq. (3). Straightforward calculation yields the following expression for the variance at \( l \geq 2 \)
\[ \sigma^2 = \left( \sin \frac{\pi}{2l} / \sin \frac{3\pi}{2l} \right)^2 \]

At \( l = 1 \) the variance diverges. It is trivial to obtain \( p(z) \) at \( l = 1 \). The only modification to the described approach is that the normalized sum \( z \) in this case should be defined as \( z/N \). For this \( z \) Eq. (7) yields \( g_N(\omega) = \exp(-|\omega|/N) \). Next, Eq. (8) brings about the final formula (a Lévy distribution)

\[
p(z) = \frac{1}{\pi} \frac{1}{1 + z^2}.
\] (16)

It should be stressed (i) the expression Eq. (16) is an exact result, valid at any \( N \), and (ii) the profile of \( p(z) \) in Eq. (16) coincides with that for \( f(\xi) \) in Eq. (13) at \( l = 1 \) — a Lévy distribution possesses the same self-similar properties as a Gaussian does [3]. However, here we are focused on the tail behavior rather than on the self-similarity of distributions. Following this line, I would like to emphasize that the tail of \( p_N(x) \), which Eq. (16) leads to, has the form of Eq. (4) and obviously may be explained in the same terms [4].

![FIG. 2. Plots of Gaussian distribution Eq. (5) for the normalized sum \( z \) (thin line) and probability density \( p_N(z) \) calculated numerically according to Eq. (14) at \( N = 10 \) (thick line) exhibiting a tail lighter than the Gaussian. Note, that in agreement with Eq. (13) deviation of \( p_N(z) \) from the GD begins at \( z \approx \sqrt{N} \).](image)

To calculate \( p_N(z) \) in case of arbitrary \( l \leq 2 \) let us employ the fact that \( g_N(\omega) \) is a real even function of \( \omega \) at any real \( f(\xi) \), see Eq. (3), and hence for such \( f(\xi) \) Eq. (8) may be transformed as follows

\[
p_N(z) = \frac{1}{\pi} \text{Re} \int_0^\infty (g_N(\omega))^N e^{-i\omega z} dz.
\] (17)

Thus, it suffices to calculate \( g_N(\omega) \) for positive \( \omega \) only. Next, extending integral Eq. (8) into the plane of complex \( \xi \), note that the for \( f(\xi) \) given by Eq. (13) integrand has \( l \) simple poles in the upper half-plane. The poles are located in the points

\[ \xi_j = \exp \left[ \frac{\pi}{2l} (2j + 1) \right], \quad 0 \leq j \leq l - 1. \]

Then, the integral is equal the following expression

\[
g_N(\omega) = i \left( \sin \frac{\pi}{2l} \right) \sum_{j=0}^{l-1} \exp \left[ \frac{i\omega}{\pi \sqrt{N}} \exp \left[ \frac{i\pi}{2l} (2j + 1) \right] \right] \] (18)

Calculation of \( p(z) \) according to Eqs. (17), (18) is rather cumbersome, see Appendix. It yields the following result

\[ p_N(z) \cong \frac{lN}{\pi(\sigma \sqrt{N})^{l-1}} \frac{\sin \frac{\pi}{2l} z}{z^l} \quad \text{at} \ z \gg \sqrt{N}. \] (19)

Plots of \( p_N(z) \) obtained by numerical integration of Eqs. (17), (18) at \( l = 2 \) and two values of \( N (N = 10^2, 10^4 \text{, respectively}) \) are presented in Fig. 3. The crossover from GD to heavy tails Eq. (19) is seen clearly. In agreement with the above discussion the larger is \( N \) the longer the corresponding curve follows GD.

![FIG. 3. Plots of Gaussian distribution Eq. (5) shown as a thin full line and probability densities \( p_N(z) \) calculated numerically according to Eqs. (17), (18) at \( l = 2 \) for \( N = 10^2 \) (thick dashed) and \( N = 10^4 \) (thick dot-dashed). Both \( p_N(z) \) exhibit tails heavier than the Gaussian. The corresponding tail asymptotics, Eq. (15) are drawn as thin dashed and dot-dashed lines respectively.](image)

Comparing Eq. (19) with Eq. (4) one can see that the two expressions yield exactly the same asymptotic behavior for the probability densities [4], including the values of the normalizing multiplier [see the definition of \( A \) in Eq. (15)]. It should be stressed, however, that this is not the case for any \( f(\xi) \) with a tail lighter the the Gaussian. Eq. (4) describes the probability density for a sum to have a great value because of contribution of a single extremely great term. Meanwhile if \( f(\xi) \) decays sharp enough the probability of such an event is very small. In this case it could be more probable to obtain the great value of the sum due to contribution of several great but not extremely great terms. It means, that the actual tail of \( p_N(x) \) just cannot be lighter than that given by Eq. (4) but, in principle, it could be heavier. To elucidate this issue let us consider the following profile of \( f(\xi) \)

\[ f(\xi) = \frac{1}{\pi \cosh \xi}. \]

The corresponding variance \( \sigma^2 \) equals \( \pi^2/4 \). Function \( g_N(\omega) \) is calculated according to Eq. (8) with standard methods of integration in a complex plane. The integration yields

\[ g_N(\omega) = \frac{1}{\cosh \frac{\omega}{2\sigma \sqrt{N}}} = \frac{1}{\cosh \frac{\sqrt{2\sigma^2}}{N}}. \]
Then, \( p_\sigma(z) \) is given by the following integral
\[
p_\sigma(z) = \frac{\sigma \sqrt{N}}{\pi^2} \int_{-\infty}^{\infty} \exp\left(-2i\zeta \frac{\sigma \sqrt{N}}{\pi} \right) \cosh^N \zeta \, d\zeta,
\] (20)
where \( \zeta \equiv \omega \pi / (2\sigma \sqrt{N}) \). To find the asymptotics of integral Eq. (20) at \( z \to \infty \) let us extend the integration to the plane of complex \( \zeta \) and consider a standard contour, where the edges of the segment \( -R \leq \zeta \leq R \) are connected through the lower half-plane by a circular arc with the radius \( R \). The integrand has infinite number of poles of order \( N \) lying in points \( \zeta = (2n+1)\pi/2 \) of the imaginary axis. Here \( n \) is any integer. Bearing in mind that the integral over the arc tends to zero as \( R \to \infty \) and the direction of circulation about the contour is negative one obtains
\[
p_\sigma(z) = -2\pi i \frac{\sigma \sqrt{N}}{\pi^2} \sum_{n=0}^{\infty} \text{res} \left[ \frac{\exp(-2iz\zeta \frac{\sigma \sqrt{N}}{\pi})}{\cosh^N \zeta}; -i\pi \frac{2n+1}{2} \right].
\] (21)
In its turn
\[
\text{res} \left[ \frac{\exp(-2iz\zeta \frac{\sigma \sqrt{N}}{\pi})}{\cosh^N \zeta}; -i\pi \frac{2n+1}{2} \right] = \lim_{\zeta \to -i\pi(2n+1)/2} \left( \zeta + i\pi \frac{(n+\frac{1}{2})}{\cosh \zeta} \right)^N \exp\left(-2iz\zeta \frac{\sigma \sqrt{N}}{\pi} \right)
\] (22)
We are interested in the leading (in \( z \gg \sqrt{N} \)) term of Eq. (22). Note, that each differentiation of (\ldots)^N contributes a multiplier of order \( N \), while each differentiation applied to \( \exp(\ldots) \) yields the one of order of \( z\sqrt{N} \gg N \). Thus, to obtain the leading term of Eq. (22) one must apply all \( N-1 \) derivatives to the exponent. It gives rise to the following expression for the residue
\[
\text{res} \left[ \frac{\exp(-2iz\zeta \frac{\sigma \sqrt{N}}{\pi})}{\cosh^N \zeta}; -i\pi \frac{2n+1}{2} \right] \approx \left( -i \right)^n \frac{N!}{(N-1)!} \frac{2z \sigma \sqrt{N}}{\pi}^{N-1} \exp\left[ -(2n+1)z\sigma \sqrt{N} \right] + \ldots,
\] (23)
where dots denote dropped higher order (in \( \sqrt{N}/z \)) terms.

Substitution of Eq. (23) in Eq. (21) transforms the sum there into a geometric progression, which may be summarized. However, the summarizing is not required — due to the exponential decay of the sum’s terms the sum equals to its first term with the exponential accuracy. Finally the desired tail of the probability density at \( z \gg \sqrt{N} \) is given by the following formula
\[
p_\sigma(z) \approx \frac{(2\sigma \sqrt{N}/\pi)^N}{(N-1)!} z^{N-1} \exp\left(-z\sigma \sqrt{N} \right) = \frac{N^{N/2}}{(N-1)!} z^{N-1} \exp\left(-\frac{\pi z \sqrt{N}}{2} \right)
\] (24)
(I remind the reader that \( \sigma = \pi/2 \)). It should be emphasized that the above expression gives rise to \( p_\sigma(z) \), which is much greater than that following from Eq. (20). A plot of \( p_\sigma(z) \) obtained by numerical integration of Eq. (20) as well as its comparison with GD and asymptotic Eq. (24) are shown in Fig. 4.

Thus, the above consideration has shown that the very tail of the probability density \( p_\sigma(x) \) for a sum \( x \to N \) random terms \( \xi_n \) at any finite \( N \) is defined by the tail of the probability density \( f(\xi) \). The tail is lighter than a Gaussian, if \( f(\xi) \) decays at large \( \xi \) faster than GD, and heavier than it in the opposite case. The only situation, when the tail of \( p_\sigma(x) \) is the Gaussian, corresponds to a Gaussian \( f(\xi) \). For \( f(\xi) \) decaying as a certain power of \( \xi \) large fluctuations of \( x \) occur because of contribution to the sum of a single anomalously great term, rather than due to the ones of a number of terms of moderate values \( f(\xi) \), while for exponentially decaying \( f(\xi) \) this is not the case.

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APPENDIX:

Here I discuss the derivation of Eq. (13). Note, that if \( \omega \) is regarded as a complex variable, then \( g_\omega(\omega) \) given by Eq. (18) is an analytical function of it at any finite \( |\omega| \). Since \( N \) is a positive integer number the same is true for \( (g_\omega(\omega))^N \). For this reason \( \int_{C} (g_\omega)^N \exp(-i\omega z) d\omega = 0 \), where \( C \) stands for any closed contour in the plane of complex \( \omega \).

To obtain the asymptotic behavior of \( p(z) \) at large \( z \) let us consider contour \( C \) consisting of segments of real \( (0 \leq \omega' \leq R) \) and imaginary \( (0 \geq \omega'' \geq -R) \) axes and a circular arc.
with the radius \( R \), connecting the edges of the segments. According to Eq. (13) the integrand decays along the real axis as \( \exp \left[ - (\omega' \sqrt{N}/\sigma) \sin(\pi/2l) \right] \). In the lower half-plane it decays not slower than \( \exp \left[ |\omega| \sqrt{N}/\sigma - \omega'' z \right] \). Then, it is easy to show that at \( z \gg \sqrt{N}/\sigma \) the integral over the circular arc tends to zero at \( R \to \infty \). Thus, in the specified limit \( z \gg \sqrt{N}/\sigma \) one obtains

\[
p_N(z) \approx \left[ - \frac{1}{\pi} \text{Re} \int_{-\infty}^{0} (g_N(\omega))^N e^{-i\omega z} d\omega \right]^{\prime}
\]

\[
= \frac{1}{\pi} \text{Im} \int_{0}^{\infty} (g_N(-i\omega''))^N e^{-\omega'' z} d\omega''
\]

Next, note that \( \exp(-\omega'' z) \) is a sharp function relative to \( (g_N(-i\omega''))^N \). In this case to find the leading asymptotic of \( p_N(z) \) at \( z \to \infty \) one may expand \( g_N(-i\omega'') \) in powers of \( \omega'/(\sigma \sqrt{N}) \), truncating the expansion when the first term with nontrivial contribution to \( p_N(z) \) is obtained.

The replacement in Eq. (13) with \( -i\omega'' \) and the expansion of exponents in the numerators results in

\[
g_N(-i\omega'') = -i \left( \sin \frac{\pi}{2l} \right) \sum_{j=0}^{l-1} \sum_{k=0}^{\infty} \left( \frac{\omega''}{\sigma \sqrt{N}} \right)^k \exp \left[ \frac{i\pi}{2l} (2j + 1)(k - 2l + 1) \right]
\]

\[
= -i \left( \sin \frac{\pi}{2l} \right) \left[ \sum_{k=0}^{\infty} \frac{(\omega''/\sigma \sqrt{N})^k}{k!} \exp \left[ \frac{i\pi}{2l} (k + 1) \right] \right] \times \sum_{j=0}^{l-1} \exp \left[ \frac{i\pi(k + 1)}{l} j \right]
\]

The latter sum over \( j \) is a geometric progression. Summarizing it up one obtains

\[
\sum_{j=0}^{l-1} \exp \left[ \frac{i\pi(k + 1)}{l} j \right]
\]

\[
= \begin{cases} 
2 / [1 - \exp \left( i\pi \frac{k+1}{l} \right)] & \text{at } k = 2p \\
0 & \text{at } k = 2p + 1 \neq 2ql - 1 \\
1 & \text{at } k = 2ql - 1
\end{cases}
\]

Here \( p = 0, 1, 2, \ldots \); \( q = 1, 2, 3, \ldots \). Then, Eq. (A2) may be transformed as follows

\[
g_N(-i\omega'') = \sum_{p=0}^{\infty} r_p + \sum_{q=1}^{\infty} s_q,
\]

where

\[
r_p = \frac{1}{(2p)!} \left( \frac{\omega''}{\sigma \sqrt{N}} \right)^{2p} \sin \frac{\pi}{2l} \left( \frac{\omega''}{\sigma \sqrt{N}} \right)^p,
\]

\[
s_q = \left( \frac{1}{2l} \right) q^{-1} \left( \frac{\omega''}{\sigma \sqrt{N}} \right)^{2q-1}.
\]

At small \( \omega \), retaining just the first terms in both the sums in Eq. (A3) one arrives at the following result \( g_N(-i\omega'') \approx 1 + \ldots + i\epsilon_1 + \ldots \), and next, \( \text{Im} \left( g_N(-i\omega'') \right)^N \approx \epsilon_1 + \ldots \), where dots denote dropped higher order in \( \omega''/(\sigma \sqrt{N}) \) terms. Substitution it into Eq. (A1) and integration yield Eq. (15).