On a Theory of Multi-gap Superfluidity
Based on Fermi–liquid Approach

A.I. Akhiezer, A.A. Isayev, S.V. Peletminsky and
A.A. Yatsenko

Kharkov Institute of Physics and Technology, Kharkov, 310108, Ukraine

Superfluid-to-superfluid phase transitions in a Fermi-liquid leading to the emergence of two–gap superfluid states have been studied. There are considered the systems of fermions of one and two (nuclear matter) sorts. The self-consistent equations describing new two-gap superfluid states have been obtained. These equations differ essentially from the equations of the BCS theory. The solutions of the self-consistent equations corresponding to one–gap and two–gap superfluid states have been found.

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1. INTRODUCTION

As a rule, superfluid states in a Fermi-liquid (FL) are considered as arising as a result of a phase transition from the normal state. However, phase transitions that might also take place in a superfluid FL can lead to the emergence of new superfluid states. Thus, we are speaking of phase transitions from one superfluid state to another. A new superfluid state can be characterized by not one but a few order parameters. In this report we consider the systems of fermions of one and two sorts. In the first case the possibility of a phase transition to the state for which the spin of a Cooper pair has no definite value but is equal with certain probability to zero or unity is studied (singlet-triplet pairing of fermions). In the second case a phase transition in superfluid nuclear matter to the state corresponding to the superposition of states with singlet-triplet and triplet-singlet pairing of nucleons (in spin and isotopic spaces) is considered.
Our analysis is based on the theory of a superfluid FL. A Fermi superfluid is described by the normal distribution functions \( f_{\kappa_1 \kappa_2} = \text{Tr} \rho a_{\kappa_2}^+ a_{\kappa_1} \) and anomalous distribution functions \( g_{\kappa_1 \kappa_2} = \text{Tr} \rho a_{\kappa_2} a_{\kappa_1} \) (\( \rho \) is the density matrix of the system, \( a^+_{\kappa} \) and \( a_{\kappa} \) are the creation and annihilation operators of a fermion in the state with momentum \( \mathbf{p} \), spin projection \( \sigma \); for a nucleon in nuclear matter the state is characterized also by the isotopic spin projection \( \tau \)).

The energy of the system, \( E = E(f, g) \), determines the fermion one-particle energy and the matrix order parameter

\[
\varepsilon_{\kappa_1 \kappa_2} = \frac{\partial E}{\partial f_{\kappa_2 \kappa_1}}, \quad \Delta_{\kappa_1 \kappa_2} = 2 \frac{\partial E}{\partial g_{\kappa_2 \kappa_1}},
\]

(1)

The distribution functions \( f \) and \( g \) are related with the quantities \( \varepsilon \) and \( \Delta \) by the formulas:

\[
f = \frac{1}{2} \left( 1 - \frac{\xi}{E} \tanh \frac{Y_0 E}{2} \right), \quad g = -\frac{1}{2E} \tanh \frac{Y_0 E}{2} \cdot \Delta;
\]

(2)

\[
E = \sqrt{\xi^2 + \Delta \Delta^*}, \quad \xi = \varepsilon + \frac{Y_4}{Y_0}
\]

Here \( Y_0 = 1/T, \quad -Y_4/Y_0 = \mu \), \( T \) is temperature, \( \mu \) is chemical potential.

For a superfluid FL with the singlet-triplet (ST) pairing of fermions the matrix order parameter has the form

\[
\Delta(p) = \Delta_0(p) \sigma_2 + \vec{\Delta}(p) \vec{\sigma} \sigma_2,
\]

(3)

\( \sigma_i \) being the Pauli matrices in spin space. The quantities \( \Delta_0 \) and \( \Delta_i \) in Eq. (3) determine the singlet and triplet components of the order parameter \( \Delta \), respectively. In what follows, we shall assume that the structure of \( \Delta_0 \) and \( \Delta_i \) is such that \( \Delta_0(p) = \Delta_0(p), \Delta_i(p) = R_{ik} \Delta(p) \), where \( R_{ik} \) is a real rotation matrix. The wave function of a Cooper pair for singlet-triplet pairing reads as

\[
g(p) = g_0(p) \sigma_2 + \vec{g}(p) \vec{\sigma} \sigma_2
\]

(4)

To derive the self-consistent equations it is necessary to specify the energy functional which we set in the form

\[
E(f, g) = E_0(f) + E_{int}(g), \quad E_0(f) = 2 \sum_p \varepsilon_0(p) f_0(p), \quad \varepsilon_0(p) = \frac{p^2}{2m}
\]

(5)

\[
E_{int}(g) = \frac{1}{V} \sum_{pp'} g_0^*(p) L_s(p, p') g_0(p') + \frac{1}{V} \sum_{pp'} \vec{g}^*(p) L_t(p, p') \vec{g}(p')
\]
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Here $f_0$ is the coefficient of the unit matrix $\sigma_0$ in expansion of the distribution function $f$ in the Pauli matrices; $L_s$, $L_t$ are the singlet and triplet anomalous FL interaction amplitudes with the structure $L_s(p, p') = L_s(p, p') = L_t(p, p') = L_t(p, p') = \sqrt{p}p^0$. The use of Eqs. (1), (2), (3) allows to represent the self-consistent equations in the form

$$\Delta_0(p) = -\frac{1}{4V} \sum_q L_s(p, q) \left\{ \frac{\Delta_+(q)}{E_+(q)} \tanh \frac{Y_0E_+(q)}{2} + \frac{\Delta_-(q)}{E_-(q)} \tanh \frac{Y_0E_-(q)}{2} \right\},$$

$$\Delta(p) = -\frac{1}{4V} \sum_q L_t(p, q) \left\{ \frac{\Delta_+(q)}{E_+(q)} \tanh \frac{Y_0E_+(q)}{2} - \frac{\Delta_-(q)}{E_-(q)} \tanh \frac{Y_0E_-(q)}{2} \right\}.$$  \hfill (6)

Here $\Delta_\pm = \Delta_0 \pm \Delta$, $E_\pm = \sqrt{\xi^2 + \Delta^2}$. Eqs. (6) represent themselves the system of the integral equations for determining the singlet and triplet order parameters in the case of ST pairing of quasiparticles in a superfluid FL.

Let us give an analysis of Eqs. (6), using the model representations of the BCS theory (the amplitudes $L_s$ and $L_t$ are not equal to zero only in a narrow layer near the Fermi surface: $|\xi| \leq \theta$, $\theta \ll \varepsilon_F$). Then the quantities $\Delta_0 \equiv \Delta_0(p = p_F)$, $\Delta \equiv \Delta(p = p_F)$ can be found from the relations

$$\Delta_0 = \frac{1}{2} x(1 + d(x, T)), \quad \Delta = \frac{1}{2} x(1 - d(x, T)) \hfill (7)$$

where $x$ is the solution of the equation

$$d(x, T) \cdot d(x \cdot d(x, T), T) \equiv D(x, T) = 1, \hfill (8)$$

$$d(x, T) = \frac{4g_s g_t^l \lambda(x, T) - g_s - g_t}{g_t - g_s}, \quad \lambda(x, T) = \int_{-\theta}^\theta \frac{d\xi}{E} \tanh \frac{E}{2T}, \quad E = \sqrt{\xi^2 + x^2}, \quad g_t^l = \frac{g_t}{3}, \quad g_s, t = -\frac{\nu_F L_{s, t}(p = p_F, q = p_F)}{4}$$

One–gap solutions are obtained as solutions of the equations $d(x, T) = 1$ (singlet pairing) and $d(x, T) = -1$ (triplet pairing), while two-gap solutions correspond to the case $d(x, T) \neq \pm 1$ (singlet-triplet pairing). The critical temperatures of the singlet and triplet superfluid transitions are found from the equations $d(0, T_s) = 1$, $d(0, T_t) = -1$ respectively. Assume for definiteness, that $g_s > g_t^l$. An analysis of the behaviour of the function $D(x, T)$ shows (see Fig. 1) that equation $D(x, T) = 1$ has no one-gap solutions at temperatures $T > T_s$, only one singlet solution exists for $T_t < T < T_s$, while for $T_{st} < T < T_t$ the system is characterized by two one-gap (singlet and triplet) solutions. Finally, at $T < T_{st}$ we have two new ST solutions in addition to the previous solutions. For determining the critical temperature $T_{st}$ of transition to the state with ST fermion pairing we have the equations

$$d(x, T) = -1, \quad xd'_x(x, T) = 2 \hfill (9)$$
Fig. 1. Behaviour of the function $D(x, T)$ at different temperatures. The critical temperatures of various phase transitions in the model case with $g_s = 0.25$, $g'_t = 0.2$ and $\theta = 0.01\varepsilon_F$ are as follows: $T_s = 0.0045\varepsilon_F$, $T_t = 0.0033\varepsilon_F$ and $T_{st} = 0.0028\varepsilon_F$.

The first of these equations indicates that ST solutions are continuously branched off the one-gap triplet solution, while the second equation is the condition that the derivative $D'_x(x, T)$ vanishes at the branching point. Calculation of the second derivative $D''_{xx}(x, T)$ in the critical point gives $D''_{xx}(x_{st}, T_{st}) = 0$, i.e., the mechanism of branching of ST solutions is the formation of inflection on the curve $z = D(x, T_{st})$ for $x = x_{st}$.

The numerical analysis of the equations for determining $\Delta_0, \Delta$ in the model case considered above is given in Fig. 2. Note that if the coupling constants satisfy the inequality $g'_t > g_s$ then ST solutions branch off the singlet one-gap solution.

3. SYSTEMS OF FERMIONS OF TWO Sorts

The problem of studying of phase transitions in nuclear matter has encouraged a great interest (see Refs. 3,4 and references therein). However, earlier the normal-to-superfluid phase transition in nuclear matter has been studied only. In principle, at further lowering of temperature a superfluid
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Fig. 2. Order parameters $\Delta_0$ and $\Delta$ vs temperature. $st(s_+), st(s_-)$ and $st(t_{\pm})$ are notations for the dependences of the order parameters $\Delta_0$ and $\Delta$ in two pair ($\pm \Delta_0, \Delta$) of ST solutions of self-consistent equations.

Further we shall study the unitary states of superfluid nuclear matter, for which the product $\Delta \Delta^+$ is proportional to the product of the unit matrices in the spin and isospin spaces, $\Delta \Delta^+ \propto \sigma_0 \tau_0$. We consider as a particular
case two-gap unitary states, for which the order parameter reads as

$$\Delta(p) = \Delta_{30}(p)\sigma_3\tau_2 + \Delta_{03}(p)\sigma_2\tau_3$$  \hspace{1cm} (10)$$

In this case the wave function of a Cooper pair describes the superposition of states with the triplet-singlet and singlet-triplet pairing of nucleons (TS-ST states) with the projections of total spin and isospin $S_z = T_z = 0$.

To derive the self–consistent equations of a nucleon superfluid FL it is necessary to specify the energy functional, which we set in the form

$$E(f, g) = E_0(f) + E_{\text{int}}(g), \quad E_0(f) = 4\sum_p \varepsilon_0(p)f_{00}(p), \quad \varepsilon_0(p) = \frac{p^2}{2m}. \hspace{1cm} (11)$$

$$E_{\text{int}}(g) = \frac{2}{V}\sum_{p,q}\{g^*_{30}(p)V_1(p,q)g_{30}(q) + g^*_{03}(p)V_2(p,q)g_{03}(q)\}$$

Here $m$ is the effective nucleon mass, $f_{00}$ is the coefficient of the product $\tau_0\sigma_0$ in expansion of the distribution function $f$ in the Pauli matrices, $V_1, V_2$ are the anomalous FL interaction amplitudes, which have the symmetry properties $V_i(-p,q) = V_i(p,-q) = V_i(p,q), i = 1, 2$. The quantity $m$ contains account of the normal FL effects and represents itself the mass of a free nucleon, renormalized by the normal FL interaction. The use of Eqs. (1), (2), (11) allows to obtain the self–consistent equations in the form of Eqs. (6), where $\Delta_{\pm} = \Delta_{30} \pm \Delta_{03}$, $E_{\pm} = \sqrt{\xi^2 + \Delta^2_{\pm}}$ and it is necessary to do the substitution $L_s \rightarrow V_1, L_t/3 \rightarrow V_2$. In what follows, we shall choose the Skyrme forces as the amplitudes of NN interaction

$$V_{1,2}(p,q) = t_0(1 \pm x_0) + \frac{1}{6}t_3n^{\alpha}(1 \pm x_3) + \frac{1}{2\hbar^2}t_1(1 \pm x_1)(p^2 + q^2), \hspace{1cm} (12)$$

where $n$ is the density of symmetrical nuclear matter, $t_i, x_i, \alpha$ are some phenomenological parameters. Note that the amplitudes $V_1, V_2$ contain no dependence from the parameters $t_2, x_2$, because the amplitudes $V_1, V_2$ are the even functions of the arguments $p, q$ (see details in Ref. 2). There are sets of parameters $t_i, x_i, \alpha$, which differ for various versions of the Skyrme forces (we shall use the Ska, SkM, SkM* and RATP potentials as well as the SkP potential).

The temperature dependence of the order parameters $\Delta_{30} \equiv \Delta_{30}(p = p_F), \Delta_{03} \equiv \Delta_{03}(p = p_F)$ is determined from Eq. (3), (8) in which it is necessary to do the above mentioned substitution of the amplitudes as well as the substitution of the coupling constants $g_s \rightarrow g_1, g'_s \rightarrow g_2$. The results of numerical integration for the potential SkM* at $n = 0.4fm^{-3}$ are given in Fig. 3. It is seen that two-gap solutions branch off from one-gap singlet-triplet solution. The results of calculations for the potentials SkM, Ska have
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Fig. 3. Order parameters $\Delta_{30}, \Delta_{03}$ vs temperature. $tsst(ts_+), tsst(ts_-)$ and $tsst(st_{\pm})$ are notations for the dependences of the order parameters in two pair $(\pm \Delta_{30}, \Delta_{03})$ of TS-ST solutions of self-consistency equations.

the form being analogous to the form in Fig. for the potentials RATP and SkP the self-consistent equations have no two-gap solutions.

Note that since the coupling constants depend on density of nuclear matter, there exists the possibility of phase transitions in density to multi-gap states of superfluid nuclear matter as well. It is demonstrated on Fig. 4 where the results of numerical determination of the order parameters $\Delta_{30}, \Delta_{03}$ as functions of density at $T = 0$ are presented. The results of calculations for the potentials SkM, Ska have the same form whereas for the potentials RATP, SkP the self-consistent equations have no two-gap solutions.

4. CONCLUSION

Thus, we have pointed out the possibility of phase transitions in superfluid FL, consisting of fermions of one and two sorts, to two-gap superfluid states, corresponding to the superposition of fermion pairings with different values of spin (and isotopic spin). The self-consistent equations for these states differ essentially from the equations of the BCS theory and contain the one-gap solutions as some particular cases. Phase transitions to multi-
gap superfluid states can take place either in temperature or density if the coupling constants in various pairing channels depend on $n$. Note, that mixed pairing of quasiparticles (but not in spin space) was considered in Ref. 7 (superposition of $d_{x^2-y^2}$ and $d_{xy}$ states in orbital space for HTSC) and for superconductors with overlapping bands, first, in Ref. 8 and in some recent works (see, for example, Ref. 9 and references therein).

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