HIDA THEORY FOR SHIMURA VARIETIES OF HODGE TYPE

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Abstract. In this article, we generalize the work of H. Hida and V. Pilloni to construct p-adic families of µ-ordinary modular forms on Shimura varieties of Hodge type $Sh(G, X)$ associated to a Shimura datum $(G, X)$ where $G$ is a connected reductive group over $\mathbb{Q}$ and is unramified at $p$, such that the adjoint quotient $G^{\text{ad}}$ has no simple factors isomorphic to $\text{PGL}_2$.

1. Introduction

The theory of p-adic families of automorphic forms plays an important role in recent developments of algebraic number theory. The first example of such families was considered by Serre in [Ser72], where an Eisenstein series was interpolated continuously in a p-adic family by considering its q-expansion. The work of Hida ([Hi86a, Hid86b]) exploited the p-adic interpolation by a certain family (Hida family) of cuspidal automorphic eigenforms. The corresponding p-adic family of Galois representations led Mazur to develop the theory of Galois deformations ([Maz89]) and these works inspired further breakthroughs in modularity results by Wiles and Taylor-Wiles ([Wil95, TW95]). There are other important applications of the work of Hida (Hida theory), such as the construction of p-adic L-functions in various settings ([EHLS16, Liu15b]) just as the initial motivation for [Ser72]; the proof of certain cases of the Iwasawa Main Conjecture by Skinner-Urban ([SU14]); the proof of the Mazur-Tate-Teitelbaum conjecture by Greenberg-Stevens ([GS93]), etc.

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This article deals with the construction of Hida theory for Shimura varieties of Hodge type, generalizing the works of [Hid02, Pil12], where we work over the $\mu$-ordinary locus of the Shimura variety instead of the ordinary locus (which may be empty). Let $(G, X)$ be a (mixed) Shimura datum where $G$ is a connected reductive group over the rationals $\mathbb{Q}$ and $X$ is one $G(\mathbb{R})$-conjugacy class of homomorphisms from the Deligne torus $\mathbb{T}$ to $G_{\mathbb{R}}$ satisfying certain conditions (see [2] for details). For a compact open subgroup $K \subset G(\mathbb{A}_f)$, suppose that the Shimura variety $Sh_{K}(G, X) = G(\mathbb{Q}) \backslash (X \times G(\mathbb{A}_f))/K$ has an integral model $Sh$ over $\mathcal{O}_p = \mathcal{O}_{E_p}$, where $E$ is the reflex field $E = E(G, X)$ of $(G, X)$ and $p$ is a prime of $E$ over $p$, and assume moreover $Sh$ has a toroidal compactification $Sh^\Sigma$ with respect to a certain cone decomposition $\Sigma$. The general strategy of constructing $p$-adic families of automorphic forms is, according to [Hid02]:

1. define an open dense ordinary locus $Sh^{\Sigma, ord}$ of the compactification $Sh^\Sigma$ and construct a certain Igusa tower $\mathcal{Ig}_{m,n}$ over $Sh^{\Sigma, ord}$.
2. construct the space $\mathcal{V}_{m,n}$ of rational functions on the Igusa tower $\mathcal{Ig}_{m,n}$ and relate these spaces to the space of classical automorphic forms $H^0(Sh, \mathcal{V}_{\lambda})$ of a certain automorphic sheaf $\mathcal{V}_{\lambda}$ associated to a weight $\lambda \in X^*(T)$, via the Hodge-Tate map;
3. construct a certain idempotent operator $e_G$ acting on the spaces $\mathcal{V}_{m,n}$ and $H^0(Sh, \mathcal{V}_{\lambda})$ such that the subspaces $e_G\mathcal{V}_{m,n}$ of $\mathcal{V}_{m,n}$ satisfy certain finite dimensional property.

In general, for a Shimura datum $(G, X)$, the ordinary locus $Sh^{ord}$ of the Shimura variety itself $Sh$ may be empty (in the PEL case, this locus is non-empty if and only if $E_p = \mathbb{Q}_p$, cf. [Wed99 Theorem 1.6.3]). To carry out the first step in the above strategy, one needs a more general notion of ordinarity. The candidate we use in this article is $\mu$-ordinarity. Here $\mu$ refers to a cocharacter $\mu: \mathbb{G}_m \rightarrow G$ associated to the Shimura datum $(G, X)$, which is, in vague terms, the Frobenius-twisted Hodge cocharacter induced from the points in $X$ (see §3.1 for the precise formulation). Now we assume that our Shimura datum is of Hodge type, which means that there is an embedding $(G, X) \rightarrow (\text{GSp}(V), S^\pm)$ into some Siegel Shimura datum. For each geometric point $P$ of $Sh$, one can consider the specialization $A_P$ of the abelian scheme $A$ (the pull-back of the universal abelian scheme from the Siegel Shimura variety associated to $(\text{GSp}(V), S^\pm)$ to $Sh$) over $Sh$ and one has the Frobenius morphism on the Dieudonné module associated to $A_P$. This morphism gives rise to a slope filtration on the Dieudonné module and thus induces a cocharacter $\mu_P \in X_*(T)\mathbb{Z}$, where $T$ is a maximal torus of $G$. Then the $\mu$-ordinary locus $Sh^{\mu}_\mathcal{Ig}$ of $Sh$ is defined to be the reduced subscheme of $Sh$ consisting of those $P$ such that $\mu_P$ is equal to the Galois average $\mu_\mathcal{Ig}$ of $\mu$. One can show that this locus $Sh^{\mu}_\mathcal{Ig}$ is in fact independent of the choice of the embedding $(G, X) \rightarrow (\text{GSp}(V), S^\pm)$ and moreover, it is open and dense in $Sh$ (rigorously speaking, one should work with the base-change $Sh_{\mathbb{T}_\mathbb{Q}}^{\mu}$ of $Sh$ instead of $Sh$ itself). We simplify the presentation in the introduction and refer the readers to [3] for more details and precise formulation). Using Hasse invariants, one can extend the $\mu$-ordinary locus from $Sh$ to the compactification $Sh^\Sigma$.

The construction of the Igusa tower over $Sh^{\Sigma, \mu}_\mathcal{Ig}$ is similar to the PEL case: we consider the group $L_{\mu}$ of isomorphisms of the $p$-divisible group $A_P[p^\infty]$ associated to $A_P$, but considering only the connected component (cf. §4.3). From this one can define the Igusa tower $\mathcal{Ig}_{m,n}$ with $m, n \geq 1$. Then one has the space $\mathcal{V}_{m,n} = H^0(\mathcal{Ig}_{m,n}, \mathcal{O}_{\mathcal{Ig}_{m,n}})$. We define the Hecke operators $T$ by first pulling back the Hecke correspondence from the Siegel Shimura variety and then divide the trace map associated to one projection morphism by a certain explicit power of $p$ (cf. §5). From this one can define the idempotent $\mu$-ordinary operator $e_{\mathbb{T}}$ for the Levi subgroup $\mathbb{T}$ of a parabolic subgroup $\mathbb{P}$ of $G$ with $\mathbb{T} \subset L_{\mu}$. Then one has the space of $\mu$-ordinary $p$-adic automorphic forms $e_{\mathbb{T}}\mathcal{V}_{m,n}$, invariant under the action of $\mathbb{T}_{\mathbb{Q}}^{der}$ (cf. §7). Here $\mathbb{T}_{\mathbb{Q}}^{der}$ is the derived subgroup of $\mathbb{T}_{\mathbb{Q}} = \mathbb{P} \cap L$.

Now we state the main results of this article. First we need some notations. We fix a Borel subgroup $B$ of $G$ and a maximal torus $T$ of $B$. For a parabolic subgroup $P$ containing $B$ as above, write $\tilde{T}_{\mathbb{P}} := \mathbb{P}/\mathbb{P}^{der}$ for the quotient torus of $\mathbb{P}$. Using the natural projection $T \rightarrow \tilde{T}_{\mathbb{P}}$, one can view the group of characters $X^*(\tilde{T}_{\mathbb{P}})$ as a subgroup of $X^*(T)$. For any character $\lambda \in X^*(\tilde{T}_{\mathbb{P}})$, one can construct the space $H^0(Sh^{\Sigma, \mu}_\mathcal{Ig}, \mathcal{V}[\lambda])$ of automorphic forms of weight $\lambda$, of level $K$ (note that $Sh$ depends on the level group $K$). Let $\mathcal{O}_p$ denote the ring of integers of $E_p$ and we write $\mathcal{W}_p = \mathcal{O}_p[[\tilde{T}_{\mathbb{P}}(\mathbb{Z}_p)]]$ and $\mathcal{W}_p^1 := \mathcal{O}_p[[\text{Ker}(\tilde{T}_{\mathbb{P}}(\mathbb{Z}_p) \rightarrow \tilde{T}_{\mathbb{P}}(\mathbb{F}_p))]]$ for the weight space of cuspidal $\mu$-ordinary $p$-adic automorphic forms $\mathbb{M}(\mathbb{P})_{\mathcal{Ig}} = \text{Hom}_{\mathcal{O}_p}(\lim_m e_{\mathbb{T}}\mathcal{V}_{m,n}^{\mathbb{P}^{der}}, E_p/\mathcal{O}_p)$ (cf. (8)).

**Theorem 1.1.** Let $(G, X)$ be as above such that $G$ is unramified at $p$, $G^{\text{ad}}$ has no factors isomorphic to $\text{PGL}_2/\mathbb{Q}$. 

For any character \( \lambda \in X^*(\hat{T}_p) \), the subspace \( e_{\hat{p}}\mathcal{V}_{\Sigma,\mu}^{\text{cusp}}[\lambda^{-1}] \) of \( e_{\hat{p}}\mathcal{V}_{\Sigma,\mu}^{\text{cusp}} \) on which the torus \( \hat{T}_p \) acts by the character \( \lambda \) is a free \( \mathcal{O}_p \)-module of finite rank. Moreover, this rank depends only on the image of \( \lambda \) in the quotient \( X^*(\hat{T}_p)/\mathbb{Z}N_G\lambda_GX^*(\hat{T}_p) \). Here \( \lambda_G \in X^*(\hat{T}_p) \) is a certain character non-trivial associated to some Hodge line bundle on the Shimura variety \( \text{Sh} \) (cf.\([\text{Hid}86b]\)) and \( N_G \geq 0 \) is the Hasse number (cf.\([\text{Hid}67]\)).

For any \( \lambda \in X^*(\hat{T}_p) \) which is dominant as a character of \( T \) and \( (\lambda, \alpha) > 0 \) for at least one positive coroot \( \alpha \) of \( (B, T) \), we have isomorphisms
\[
e_{\hat{p}}H^0(\text{Sh}_{\Sigma,\mu}^\Sigma, \mathcal{V}[\lambda^{-1}]) \simeq e_{\hat{p}}\mathcal{V}_{\Sigma,\mu}^{\text{cusp}}[\lambda^{-1}].
\]

For any \( \lambda \in X^*(\hat{T}_p) \) and any integer \( t \gg 0 \) (depending on \( \lambda \)), we have an isomorphism between the spaces of cuspidal automorphic forms (here \( C \) denotes the cusps):
\[
e_{\hat{p}}H^0(\text{Sh}_{\mu}^\Sigma, \mathcal{V}[(\lambda + t\lambda_G)^{-1}]) \simeq e_{\hat{p}}\mathcal{V}_{\Sigma,\mu}^{\text{cusp}}[(\lambda + t\lambda_G)^{-1}].
\]

The \( \mathcal{W}_{\Sigma}^1 \)-module \( M(\hat{P})_{\text{cusp}} \) is free of finite rank, and moreover for any \( \lambda \in X^*(\hat{T}_p) \), one has the specialization isomorphism
\[
M(\hat{P})_{\text{cusp}} \otimes \mathcal{W}_{\Sigma,\mu}^1, \mathcal{O}_p \simeq \text{Hom}_{\mathcal{O}_p}(e_{\hat{p}}\mathcal{V}_{\Sigma,\mu}^{\text{cusp}}[\lambda^{-1}], \mathcal{O}_p).
\]

The assumption that \( G^{\text{ad}} \) has no factors isomorphic to \( \text{PGL}_2/\mathbb{Q} \) is used when we want to apply Koecher principal (cf.\([\text{Mad}12\] Theorem 5.2.11(5))). This is in fact not a serious restriction since the Hida theory for \( \text{PGL}_2/\mathbb{Q} \) is already well-known ([Hid86d]) and it is easy to combine our result with the case of \( \text{PGL}_2/\mathbb{Q} \) to obtain Hida theory for any \( G \) (whose adjoint quotient may contain \( \text{PGL}_2/\mathbb{Q} \)). We do not give a uniform presentation for these two cases since the article is already very long and the inclusion of \( \text{PGL}_2/\mathbb{Q} \) makes the presentation less readable even though there are no essential technical difficulties involved.

Similar constructions in the case of unitary Shimura varieties of PEL type can be found in\([\text{BR}17, \text{EM}17]\). In\([\text{BR}17]\), the authors use a partial Hasse invariant to treat the problem of lifting automorphic forms from characteristic \( p \) to characteristic 0 and they use a different weight space, which is more in the spirit of rigid geometry. In\([\text{EM}17]\), more results are obtained, including the construction of \( p \)-adic differential operators.

Hida theory has important applications in modularity theorems of automorphic representations and Galois representations (\([\text{Pil}12, \text{Wi}95]\)). Our result for the spin orthogonal groups \( \text{Spin}_{n,2} \) of signature \((n, 2)\) should also be of help to \( p \)-adic modularity results for abelian varieties of higher dimensions, just as the case of elliptic curves and abelian surfaces. Thus this article can be seen as the first step towards the \( p \)-adic modularity problems of abelian varieties, whose following steps we wish to carry out in the near future.

Here is the organization of this article, where we also take the chance to indicate some of the obstacles that we overcome in generalizing\([\text{Hid}02, \text{Pil}12]\). In\(\S 2\) we recall the notion of Shimura varieties of Hodge type and fix various notations that will be used through out this article. In\(\S 3\) we introduce the \( \mu \)-ordinary locus on this Shimura variety \( \text{Sh} \) (of characteristic \( p \)) and then using the Hasse invariant to extend this locus to characteristic 0. The main result is Proposition\([3.3]\) which is crucial to studying the lifting of automorphic forms from char \( p \) to char 0 and also the control theorems. We reduce the problem to the case that \( G \) is simple of adjoint type and for each such \( G \) we give an explicit description of certain element inside \( B(G, \mu) \) in Proposition\([3.10]\). In\(\S 4\) we construct the space of classical automorphic forms on the Shimura variety \( \text{Sh} \) and the spaces of \( (\mu\text{-ordinary}) \) \( p \)-adic automorphic forms by introducing the Igusa tower above the \( \mu \)-ordinary locus \( \text{Sh}_{\Sigma,\mu} \). The first space is an \( L(\mathbb{Z}_p) \)-torsor over the compactified Shimura variety \( \text{Sh}_{\Sigma} \) while the second space is an \( L(\mathbb{Z}_p) \)-torsor over the \( \mu \)-ordinary locus \( \text{Sh}_{\Sigma,\mu} \). It is known that \( L \) and \( L_{\mu} \) are inner twists of each other (this phenomenon seems not appear in preceding works like\([\text{Hid}02, \text{Pil}12, \text{BR}17, \text{EM}17]\) ), and then we can embed the first space into the second using a natural specialization map (the so-called Hodge-Tate map). In\(\S 5\) we carefully define the Hecke operators at \( p \), the main result is Proposition\([3.4]\) which, using the Serre-Tate theory in the \( \mu \)-ordinary case, shows that by dividing out some explicit multiple, the Hecke operators \( \mathbb{T}_e \) preserve the integral structure of the space of \( (p \text{-adic}) \) automorphic forms. Then we show that the space of \( \mu \)-ordinary automorphic forms on the Igusa tower descends to the \( \mu \)-ordinary locus \( \text{Sh}_{\Sigma,\mu} \). In\(\S 6\) we show that one can lift cuspidal automorphic forms of characteristic \( p \) to characteristic 0 for sufficiently regular weights \( \lambda \), which is one of the main ingredients in Hida theory. We first review the notion of toroidal compactification and then show that the mod \( p \) map on the space of automorphic forms is surjective (Proposition\([6.4]\)). Then we proceed to show the finite-dimensionality of the space of \( \mu \)-ordinary automorphic forms. The main ingredient for this is Lemma\([6.8]\) which shows that there is at most one.
subgroup \( H \) of \( D_x[p] \) such that its height and degrees satisfy certain conditions. In the last section §7 we summarize the results obtained in the previous sections and deduce the control theorems on the space of \( p \)-adic automorphic forms and the existence of Hida families. As the reader can see, the presentation and ideas of this article is heavily influenced by [Pil12].

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Notations.

1. We fix an odd prime number \( p \). We fix field embeddings \( \overline{Q} \hookrightarrow C, \overline{Q} \hookrightarrow \overline{Q}_p \) as well as an isomorphism of fields \( C \simeq \overline{Q}_p \) compatible with the previous embeddings. We fix an algebraic closure \( \overline{F}_p \) of \( F_p \). For any field \( k \) of characteristic \( p \), we write \( W(k) \) for its ring of Witt vectors and \( W_n(k) \) for its ring of Witt vectors of length \( n \geq 0 \). In particular, we write \( W = W(\overline{F}_p) \) and \( W_n = W_n(\overline{F}_p) \). Moreover, we identify the total fraction field of \( W \) with the completion of the maximal unramified extension \( \overline{Q}_p \) of \( Q_p \). We write \( A = A_Q \) for the adeles of \( Q \) and \( A_f \) the finite adeles.

2. Let \( G \) be a connected reductive group over \( Q \), \( B \) a Borel subgroup of \( G \) defined over \( Q \) and \( T \) a maximal torus of \( B \) (defined over \( Q \)). We fix a root datum of \( G \)

\[
(X^*(T), \Phi^*(T), X_*(T), \Phi_*(T)),
\]

where \( X^*(T) \) is the group of characters of \( T \), let \( \Phi^*(T) \subset X^*(T) \setminus 0 \) be the subset of roots. We write \( \Delta^*(T) \) for the subset of \( \Phi^*(T) \) consisting of positive roots (with respect to \( B \)) and \( \tilde{\Delta}^*(T) \) for the subset of \( \Delta^*(T) \) consisting of simple roots. Similarly we have the subset of positive co-roots \( \Delta_*(T) \) and \( \tilde{\Delta}_*(T) \) for the subset of \( \Delta_*(T) \) consisting of simple positive co-roots. We denote by \( X^*_{\text{ad}}(T) \) the subset of dominant characters of \( X^*(T), X^*_{\text{ad}}(T) \subset X^*_{\text{ad}}(T) \) the subset consisting of \( \lambda \) such that \( \langle \lambda, \mu \rangle > 0 \) for at least one positive coroot \( \mu \in \Delta_*(T) \). Similarly, \( X_*, \text{ad}(T) \) is the subset of dominant cocharacters of \( X_*(T), X_*, \text{ad}(T) \) consists of dominant coroots \( \mu \) such that \( \langle \lambda, \mu \rangle > 0 \) for at least one positive root \( \lambda \in \Delta^*(T) \). When the context is clear, we will omit \( (T) \) from these notations. Denote \( W_T \) for the Weyl group of \( (G, T) \).

2. Shimura varieties of Hodge type

We recall the notion of Shimura varieties of Hodge type. The main reference is [MH88].

Definition 2.1. Let \( (G, X) \) be a Shimura datum, i.e., \( G \) is a connected reductive group over \( Q \) and \( X \) a \( G(R) \)-conjugacy class of homomorphisms \( S := \text{Res}^C_R G_m \to G_R \) satisfying the following conditions:

1. there exists \( x \in X \) such that the Hodge structure \( h_x : S \to G_R \to GL(g) \) on the Lie algebra \( g \) of \( G \) is of type \( \{(-1, 1), (0, 0), (1, -1)\} \);
2. there exists \( x \in X \) such that \( \text{ad} h_x(i) \) is a Cartan involution on \( G_{\text{ad}} \) where \( G_{\text{ad}} \) is the adjoint quotient of \( G \);
3. \( G_{\text{ad}} \) has no factors defined over \( Q \) whose real points form a compact group;
4. the identity component \( Z(G)^0 \) of the centre \( Z(G) \) of \( G \) splits over a CM-field.

We shall make the following additional assumption, which is not part of the conditions of a Shimura datum but will be used in defining the \( \mu \)-ordinary locus of a toroidal compactification of the Shimura variety associated to \( (G, X) \) (cf. Definition 3.6):

5. \( G_{\text{ad}} \) has no factors isomorphic to \( \text{PGL}_2/Q \).

Let \( K \) be a compact open subgroup of \( G(A_f) \), then define

\[
\text{Sh}_K(G, X) = G(Q) \backslash (X \times G(A_f)/K),
\]

which is a finite disjoint union of locally symmetric spaces. Write \( \mathcal{C} \subset G(A_f) \) for a set of representatives of the quotient \( G(Q) \backslash G(A_f)/K \) and for each \( g \in \mathcal{C} \), let \( \Gamma_g' = gKg^{-1} \cap G(Q)_+ \) and \( \Gamma_g \) its image in \( G_{\text{ad}}(R)^+ \). Let \( X^+ \) be a connected component of \( x \), then \( \text{Sh}_K(G, X) = \bigcup_{g \in \mathcal{C}} \Gamma_g \backslash X^+ \). Write the projective limit

\[
\text{Sh}(G, X) = \lim_K \text{Sh}_K(G, X),
\]

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which is a scheme over $C$.

**Example 2.2.** Consider the following example: let $(V, \psi)$ be a non-degenerated symplectic vector space over $\mathbb{Q}$. The similitude symplectic group $\text{GSp}(V, \psi)$ whose $R$-points are given by

$$\text{GSp}(V, \psi)(R) = \{ g \in \text{GL}(V)(R) \mid \exists \nu \in R^X, \psi(gv, gu) = \nu(v, u), \forall v, u \in V \}.$$  

Let $S^\pm$ be the set of Hodge structures of type $\{-1, 0\}$ on $V$ such that $\pm 2i\pi \psi$ is a polarization. Then $\langle \text{GSp}(V, \psi), S^\pm \rangle$ is a Shimura datum, called a Siegel Shimura datum.

Suppose that there is an embedding $\xi: (G, X) \hookrightarrow (\text{GSp}(V, \psi), S^\pm)$ for some symplectic space $(V, \psi)$, i.e., $\text{Sh}(G, X)$ is a Shimura variety of Hodge type. Let $t = (t_\alpha)_{\alpha \in I}$ be a family of tensors for $V$ (i.e. $t_\alpha \in V^\otimes := \bigoplus_{r,s \in \mathbb{N}} V^{\otimes r \otimes \text{Hom}_Q(V, \mathbb{Q})}$) such that $G$ is the subgroup of $\text{GSp}(V, \psi) \times \mathbb{G}_m$ fixing all these tensors $(t_\alpha)$ (see [Mil88] for the proof of existence of these tensors). Recall the moduli interpretation of $\text{Sh}(G, X)$ (over $\mathbb{C}$): consider the triple $(A, s, \eta)$ consisting of an abelian variety $A$ over $\mathbb{C}$, a family $s = (s_\alpha)_{\alpha \in I}$ of Hodge cycles on $A$ (cf. [Mil88], p.13) and an isomorphism $\eta: \text{V}(\Lambda_f) \rightarrow H_f^2(A) := \otimes_{r,s} H^2(A)$ with $H_f^2(A) := T_f(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ which takes $t_\alpha$ to $s_\alpha$ for all $\alpha \in I$ and there is an isomorphism $i: V \rightarrow H_B(A) = H_1(A(\mathbb{C}), \mathbb{Q})$ sending $t_\alpha$ to $s_\alpha$ and the morphism $\eta \circ i$ is an element in $X$. We say that two triples $(A, s, \eta)$ and $(A', s', \eta')$ are equivalent if there is an isomorphism $\gamma: A \rightarrow A'$ sending $s_\alpha$ to $s'_\alpha$ and $\gamma \circ \eta = \eta'$. Write $\mathcal{M}(G, X, \xi)$ for the set of equivalence classes of these triples $(A, s, \eta)$. Then we have a canonical bijection ([Mil88] Chapter II, Proposition 3.6i):

$$\Phi_\xi: \mathcal{M}(G, X, \xi) \rightarrow \text{Sh}(G, X).$$

Write $E = E(G, X)$ for the reflex field of the Shimura datum $(G, X)$, then for any $K$ as above, there exists a unique canonical model $\text{Sh}_K(G, X)$ over the reflex field $E$ of $\text{Sh}_K(G, X)$ (cf. [Mil88], §II.5). Again we write $\text{Sh}(G, X)$ for the projective limit of $\text{Sh}_K(G, X)$ over all possible $K$ as above, which is then the canonical model over $E$ of $\text{Sh}(G, X)$. From now on, we consider some special compact open subgroups $K$, which can be written as $K = K^p K_p$ with $K^p \subset \text{G}(\Lambda_f^p)$ and $K_p \subset G(\mathbb{Q}_p)$. Write $\text{Ram}(G)$ for the finite set of places of $\mathbb{Q}$ where $G$ is ramified. We assume that

**Hypothesis 2.3.** $G$ is unramified at $p$, i.e. $p \notin \text{Ram}(G)$.

With this hypothesis, we can and will fix a $\mathbb{Z}_p$-model for $G_{\mathbb{Q}_p}$. We denote this model again by $G$. We say that

**Definition 2.4.** $K = K^p K_p$ is $p$-hyperspecial if $K_p$ is hyperspecial in $G(\mathbb{Q}_p)$.

The existence of a $p$-hyperspecial $K$ implies that the reflex field $E$ of the Shimura datum $(G, X)$ is unramified at $p$ ([Mil94], Corollary 4.7i)). In the following, for a place $p$ of $E$ over $p$, we will write $\mathcal{O}_E$, resp., $E_p, \mathcal{O}_p = (\mathcal{O}_E)_p = \mathcal{O}_E/p$ for the ring of integers of $E$, resp., the completion of $E$ at the place corresponding to the prime $p$, the ring of integers of $E_p$. Furthermore, we write $\kappa_p = \mathcal{O}_p/p$ for the residual finite field of $\mathcal{O}_p$.

Write $\text{Sh}_{K^p}(G, X)$ for the projective limit of those $\text{Sh}_{K^p K_p}(G, X)$ with $K_p$ fixed while $K^p$ varying. Let $p$ be a prime of $E(G, X)$ over $p$, then by [Vas99], Theorem 0 and [Kis10], Theorem 1, $\text{Sh}_{K^p}(G, X)$ has an integral model (1)

$$\text{Sh}(G, X, K_p)$$

over $\mathcal{O}_p$. This model inherits an action of $G(\Lambda_f^p)$ and

**Definition 2.5.** We write

$$\text{Sh} := \text{Sh}(G, X, K)$$

for the fixed points of $\text{Sh}(G, X, K_p)$ by $K^p$. In the following, we will fix one such $K^p$ which is sufficiently small (for example neat).

Next we consider the toroidal compactifications of $\text{Sh}$. By [Mad12], for a certain cone decomposition $\Sigma$, one can construct the toroidal compactification $\text{Sh}^\Sigma$ of $\text{Sh}$ over $\mathcal{O}_p$. Moreover, the complement $C^\Sigma = \text{Sh}^\Sigma \setminus \text{Sh}$ is a relative Cartier divisor over $\mathcal{O}_p$. Up to refining $\Sigma$, every complete local ring of $\text{Sh}^\Sigma$ at a geometric point is isomorphic to a complete local ring of $\text{Sh}$. We can apply the Proj construction to a certain graded ring of automorphic forms on $\text{Sh}^\Sigma$ and we get the minimal compactification of $\text{Sh}$:

$$\text{Sh}^{\text{min}}$$
(independent of $\Sigma$) with a unique morphism
\[ \pi^\Sigma: Sh^\Sigma \to Sh^{\text{min}} \]
extending the identity morphism on $Sh$ and compatible with the stratifications on $Sh^\Sigma$ and $Sh^{\text{min}}$. With these integral models, we consider their reductions: for each integer $n$, we write
\[
Sh_n = Sh^\Sigma \otimes_{\mathcal{O}_p} \mathcal{O}_p/\mathfrak{p}^n, \quad ? = \emptyset, \Sigma, \text{min},
\]
\[
\pi^\Sigma_n = \pi^\Sigma \otimes_{\mathcal{O}_p} \mathcal{O}_p/\mathfrak{p}^n: Sh^\Sigma_n \to Sh^{\text{min}}_n.
\]
By construction, we have $(\pi^\Sigma_n)_* (\mathcal{O}_{Sh^\Sigma_n}) = \mathcal{O}_{Sh^{\text{min}}_n}$.

**Remark 2.6.** In the following, we fix an embedding $E \hookrightarrow \overline{\mathbb{Q}}$ and thus we have a corresponding embedding $\mathcal{O}_E \hookrightarrow \mathbb{W}$ since $E$ is unramified at $p$. Moreover, we use the same notations $Sh^\Sigma, C^\Sigma, \pi^\Sigma$ to denote their base changes from $\mathcal{O}_E$ to $\mathbb{W}$ and similarly for the notations with subscript $n$ (base changes from $\mathcal{O}_E/\mathfrak{p}^n$ to $\mathbb{W}_n$).

For $(G, X) = (\text{GSp}(V, \psi), S^\pm)$, we write $K_V = (K_V)_p(K_V)^p$ for a compact open subgroup of $\text{GSp}(V, \psi)(\mathbb{A}_f)$. We write $\mathcal{A}(V, \psi)$ for the universal abelian scheme over $Sh(\text{GSp}(V, \psi), S^\pm, K_V)$. Then write
\[ \pi: \mathcal{A} \to Sh \]
for the universal abelian scheme over $Sh$ via the pullback of $Sh \to Sh(\text{GSp}(V, \psi), S^\pm, K_V)$ (for $K = K_V \cap G(\mathbb{A}_f)$). Write $e: Sh \to \mathcal{A}$ for the unit section and
\[ \omega_{Sh} := \det(e^* \Omega^1_{\mathcal{A}/Sh}) \]
for the Hodge line bundle on $Sh$. By [KW15] Proposition 4.1], we know that $\omega_{Sh}$ is ample over $Sh$.

### 3. $\mu$-ordinary locus and Hasse invariant

#### 3.1. $\mu$-ordinary locus

In this section, we review the theory of $\mu$-ordinary locus for the Shimura varieties $Sh(G, X, K)$ ([Wed99, Wor13]). The integral canonical model $Sh$ in general does not have a moduli interpretation in terms of abelian schemes. Yet one can still define the Newton stratification over $Sh_1$ and Ekedahl-Oort stratification over $Sh \otimes_{\mathcal{O}_p} \mathbb{F}_p$.

We suppose that $G$ is quasi-split over $\mathbb{Z}_p$ and split over a finite étale extension of $\mathbb{Z}_p$. Fix then a Borel subgroup $B$ of $G$ and a maximal torus $T$ of $B$ (both defined over $\mathbb{Z}_p$). We can then identify the quotient $\mathcal{W}_T/\mathcal{X}^*(T)$ with the set of conjugacy classes of co-characters of $G_\mathbb{C}$. Recall that for each $x \in X$, we have a Hodge decomposition $V_\mathbb{C} = V(-1,0) \oplus V(0,-1)$ by the embedding $X \hookrightarrow S^\pm$. Write $\nu_x$ for the co-character of $G_\mathbb{C}$ such that $\nu_x(z)$ acts on $V(-1,0)$ via multiplication by $z$ and on $V(0,-1)$ trivially. Then we write $[\nu]$ for the $G(\mathbb{C})$-conjucacy class of co-characters of $G_\mathbb{C}$ containing all these $\nu_x$ and similarly $[\nu^{-1}]$ containing all these $\nu_x^{-1}$. Then write
\[ \mu \in \mathcal{X}^*(T) \]
for the element such that $Fr^{-1}(\mu) \in [\nu^{-1}]$ where $Fr$ is the Frobenius morphism.

We fix a lattice $\Lambda \subset V$ of $V$ and assume that $t \subset (\Lambda \otimes \mathbb{Z}_p \mathcal{O}(p))^{\mathfrak{g}}$ and $\psi(\Lambda \times \Lambda) \subset \mathbb{Z}$. Write $\Lambda^\vee \subset V^\vee$ for the dual lattice of $\Lambda$ by $\psi$. Recall the moduli interpretation of $Sh(\text{GSp}(V, \psi), S^\pm, K_V)$ for $(K_V)^p$ sufficiently small: let $\mathcal{M}(\text{GSp}(V, \psi), S^\pm, (K_V)^p, \xi)$ be the moduli space over $\mathbb{Z}_{(p)}$ parametrizing abelian schemes $A$ with a polarization of degree $d := [\Lambda^\vee : \Lambda]$ and a $(K_V)^p$-level structure up to isomorphism (which is representable by a quasi-projective scheme over $\mathbb{Z}_{(p)}$, denoted by the same notation). Thus we get an embedding of $\mathbb{Z}_{(p)}$-schemes
\[ Sh(\text{GSp}(V, \psi), S^\pm, K_V) \hookrightarrow \mathcal{M}(\text{GSp}(V, \psi), S^\pm, (K_V)^p, \xi). \]

We make explicit this embedding over $\mathbb{C}$-points: for any
\[ [h, g] \in Sh(\text{GSp}(V, \psi), S^\pm, K_V, \xi)(\mathbb{C}) = \text{GSp}(\mathbb{Q}) \backslash S^\pm \times \text{GSp}(V, \psi)(\mathbb{A}_f)/K_V, \]
we have a Hodge decomposition $V_\mathbb{C} = V(-1,0) \oplus V(0,-1)$ given by $h$. Moreover, there is a unique lattice $\Lambda_g$ of $V$ such that $(\Lambda_g)_\mathbb{Z} = g(\Lambda_\mathbb{Z})$ with $\Lambda_\mathbb{Z} = \Lambda \otimes \mathbb{Z}_p \mathbb{Z}$ and a unique $\mathbb{Q}^\times$-multiple $\psi_{h, g}$ of $\psi$ such that $g(\Lambda^\vee_g)$ is the dual lattice of $g(\Lambda_\mathbb{Z})$ by $\psi_{h, g}$ and such that the bilinear form $(v, w) \mapsto \psi_{h, g}(v, h(i)w)$ is positive definite on $V_\mathbb{Z}$. By Riemann's theorem, we can associate to $[h, g]$ a triple $(A, \lambda, \eta)$ with $A := V(-1,0)/\Lambda$ a complex abelian variety with polarization $\lambda$ induced by $\psi_{h, g}$ and $\eta$ the right $(K_V)^p$-coset of $\Lambda_{2p} \rightarrow g^p(\Lambda_{2p}) = (\Lambda_g)_{2p} \simeq H_1(A, \mathbb{Z})_{2p} \simeq \prod_{\ell \neq p} T_{\ell}(A)$. 


Recall that we have a universal abelian scheme $\pi: A \rightarrow Sh$. We then write

$$V^0 := H^1_{dR}(A/\text{Sh}), \quad V := H^1_{dR}(A \otimes E/\text{Sh}(G,X,K)).$$

Moreover, we have the Hodge filtration on $V^0$:

$$0 = \text{Fil}^{-1}V^0 \subseteq \text{Fil}^0V^0 := \pi_+\Omega^1_{A/\text{Sh}} \subseteq \text{Fil}^1V^0 = V^0.$$

For any field extension $E'/E(G,X)$ embedded in $\mathbb{C}$ and a fixed point $\zeta \in \text{Sh}(E')$, consider the algebraic closure $\overline{E'} \subset \mathbb{C}$ and denote by $\overline{\zeta}$, resp., $\zeta_\mathbb{C}$ the $\overline{E'}$-point, resp., $\mathbb{C}$-point corresponding to $\zeta$. From $\text{Sh} \xrightarrow{\rho} \mathcal{M}(\text{GSp}(V, \psi), S^+, K_V, \xi)$, we have an isomorphism

$$V \cong H_1(A_{\zeta_\mathbb{C}}, \mathbb{Q}).$$

Moreover we have the comparison isomorphisms

$$H^1(A_{\zeta_\mathbb{C}}, \mathbb{Q}) \cong H^1_{dR}(A_{\zeta\mathbb{C}}/\mathbb{C}), \quad H^1(A_{\zeta_\mathbb{C}}, \mathbb{Q})_\mathbb{Q} \cong H^1_{\acute{e}t}(A_{\zeta_\mathbb{C}}, \mathbb{Q}_\mathbb{Q}) \cong H^1_{\acute{e}t}(A_{\zeta}, \mathbb{Q}_\mathbb{Q}),$$

The tensors $t \subset (V^\otimes)^{\otimes} \cong H^1(A_{\zeta_\mathbb{C}}, \mathbb{Q})^{\otimes}$ correspond to $t_{dR,\zeta}$, $t_{\acute{e}t,\zeta}$ which form the Hodge tensors. We can show that there exist sections

$$t_{dR} \subset V^\otimes$$

defined over $E(G,X)$ horizontal with respect to the Gauss-Manin connection $\nabla$ and for each $\zeta \in \text{Sh}(E')$, the pullback of $t_{dR}$ to $\zeta$ is $t_{dR,\zeta} \subset H^1_{dR}(A_{\zeta}/E')^{\otimes}$. One can then extend $t_{dR}$ to integral tensors $t_{dR}^0 \subset (V^\otimes)^{\otimes}$. For any perfect field $k$ of finite transcendental degree over $\overline{E}$ and $L(k) = \text{Frac}(W(k))$, consider the induced points $x \in \text{Sh}(k)$ and $\overline{x} \in \text{Sh}(L(k))$. Write $\mathcal{D}_x$ for the contravariant Dieudonné module of the $p$-divisible group of $A_x$ (equipped with a Frob-linear map $F$ and Frob$^{-1}$-linear map $V$ such that $V = p = VF$). Moreover, we have isomorphisms

$$H^1_{dR}(A_{\overline{x}}/W(k)) \cong H^1_{\text{cris}}(A_x/W(k)) \cong \mathcal{D}_x,$$

$$\Lambda_{\overline{x}p} \cong H_1(A_{\zeta_\mathbb{C}}, Z)p \cong T_p(A_{\zeta_\mathbb{C}}) \cong T_p(A_{\overline{x}}).$$

Dualizing the last isomorphism gives $\Lambda_{\overline{x}p}^\vee \cong T_p(A_{\overline{x}})\nabla(-1) \cong H^1_{\acute{e}t}(A_{\overline{x}}, Z_p)$, which sends $t_{\overline{x}p,\zeta}$ to $t_{\overline{x},\zeta}^\vee \subset H^1_{\acute{e}t}(A_{\overline{x}}, Z_p)$ which are invariant under $\text{Gal}(\mathbb{L}(k)/L(k))$ and whose base change to $H^1_{\acute{e}t}(A_{\overline{x}}, \mathbb{Q}_p)$ are exactly $t_{\overline{x},p,\zeta}$.

From the $p$-adic comparison theorem

$$H^1_{\acute{e}t}(A_{\overline{x}}, Z_p) \otimes Z_p B_{\text{cris}} \cong H^1_{\text{cris}}(A_x/W(k)) \otimes W(k) B_{\text{cris}} \cong \mathcal{D}_x \otimes W(k) B_{\text{cris}},$$

one can get the images $t_{\overline{x},\zeta}^0 \subset (V^\otimes)^{\otimes}$ of these $t_{\overline{x},\zeta}^\vee$. Moreover, we have an isomorphism

$$f: (\Lambda_{\overline{x}p}^\vee, t_{\overline{x}p}(k)) \cong (V^0_{\overline{x}}, t_{dR,\overline{x}})$$

and there is a cocharacter $\lambda$ of $G(W(k))$ such that the filtration $\Lambda_{\overline{x}p}^\vee \supset f^{-1}(\text{Fil}^1V^0_{\overline{x}})$ is induced by $(\cdot)\nabla \circ \lambda$. One can also show that the images $t_{\overline{x},\text{cris}} \subset (\mathcal{D}_x)^{\otimes}$ of $t_{dR,\overline{x}}$ are independent of the choice of $\overline{x}$ over $x$ and we also have an isomorphism

$$\beta: (\Lambda_{\overline{x}p}^\vee, t_{\overline{x}p}(k)) \cong (\mathcal{D}_x, t_{\overline{x},\text{cris}}).$$

To each such $\beta$, one can associate an element $g_{\beta} \in G(L(k))$ as follows: $g_{\beta}$ is the unique element such that the following diagram commutes:

$$\begin{array}{ccc}
\Lambda_{\overline{x}p}^\vee(k) & \xrightarrow{(g_{\beta}^{-1})^t} & \Lambda_{\overline{x}p}^\vee(k) \\
\downarrow{\beta} & & \downarrow{\beta} \\
\mathcal{D}_x^{\sigma} & \xrightarrow{\text{Frob}(1\otimes)^{-1}} & \mathcal{D}_x
\end{array}$$

(2)

Here the transpose inverse $(g_{\beta}^{-1})^t$ is considered in the embedding $g_{\beta} \in G(L(k)) \subset G\text{Sp}(\Lambda(L(k)))$.

We summarize the above discussion in the following lemma

**Lemma 3.1.** For any $x \in \text{Sh}(k)$, let $\beta: (\Lambda_{\overline{x}p}^\vee, t_{\overline{x}p}(k)) \cong (\mathcal{D}_x, t_{\overline{x},\text{cris}})$ be an isomorphism. Then any (other) isomorphism $\beta': (\Lambda_{\overline{x}p}^\vee, t_{\overline{x}p}(k)) \cong (\mathcal{D}_x, t_{\overline{x},\text{cris}})$ is of the form $\beta' = \beta \circ h^\vee$ for some unique $h \in G(W(k))$. Moreover, one has

$$g_{\beta'} = h^{-1}g_{\beta}\sigma(h) \in G(W(k)\mu(p)G(W(k)).$$
Next we define Newton stratifications on $Sh \otimes \kappa_p$. Now suppose that $k$ is algebraically closed. For two elements $g, g' \in G(L(k))$, we say $g$ is $\sigma$-conjugate to $g'$ if $g = h^{-1} g' \sigma(h)$ for some $h \in G(L(k))$. For any $g \in G(L(k))$, we write $[g]$ for the $\sigma$-conjugacy class of $g$ and $B(G)$ for the set of all such $\sigma$-conjugacy classes (this set is independent of the choice of $k$). We have the following Newton map and Kottwitz map of $G$ \cite{Kot85, RR96}:

$$
\nu_G : B(G) \to (W_T \backslash X^*(T)_Q)^{(\sigma)}, \quad \kappa_G : B(G) \to \pi_1(G)_{(\sigma)}.
$$

Here $W_T$ is the Weyl group of the pair $(T, B)$ and $X^*(T)_Q := X^*(T) \otimes \mathbb{Z} \mathbb{Q}$; the superscript $(\sigma)$ means the $\sigma$-invariant and the subscript $(\sigma)$ means the $\sigma$-coinvariant. For later use, we recall some details of the construction of $\nu_G$: for any $b \in G(L(k))$, there is a unique element $\nu_b \in X^*(T)_Q$ such that there exist an integer $s > 0$, an element $c \in G(L(k))$ and a uniformizer $\omega$ of $\mathbb{Q}_p$ (recall that $G$ is defined over $\mathbb{Q}$), such that (among others) $s \nu_b \in X^*(T)$ and $c b \sigma(b) \cdots \sigma^s(b) \sigma^s(c)^{-1} = c(s \nu_b)(\omega)c^{-1}$ \cite{Kot85 §4}. Then we set

$$
\nu_G(b) := \nu_b.
$$

This is the slope homomorphism (cocharacter) associated to $b$. Moreover, the set $(W_T \backslash X^*(T)_Q)^{(\sigma)}$ has a partial order $\preceq$ which is a generalization of the notion of “lying over” for Newton polygons \cite{RR96 §2}. More precisely, recall that $(X^*(T), \Phi^*, X_*(T), \Phi_x)$ is the root datum associated to $(G, T)$. We fix also a basis $\Delta$ of this root datum with respect to $(B, T)$ and denote by $\Delta^\vee$ the set of coroots corresponding to $\Delta$. We then write

$$
\overline{C} := \{x \in X^*(T)_R | \langle x, \alpha \rangle \geq 0, \forall \alpha \in \Delta \},
$$

$$
C^\vee := \{x \in X^*(T)_R | x = \sum_{\alpha^\vee \in \Delta^\vee} n_{\alpha^\vee} \alpha^\vee, 0 \leq n_{\alpha^\vee} \in \mathbb{R} \}
$$

for the Weyl chamber, resp., obtuse Weyl chamber of this root datum. For two elements $x, x' \in X^*(T)_R$, we write $x \preceq x'$ if $x$ lies in the convex hull of the orbit $W_T x'$. One can show that this is equivalent to the fact that for any $\mathbb{Q}$-rational representation $\rho : G \to \text{GL}(W)$ and any maximal torus $T'$ of $\text{GL}(W)$ containing $\rho(T)$, we have $\rho(x) \preceq \rho(x')$ in the usual sense. This relation induces a partial order $\preceq$ on $(W_T \backslash X^*(T))^{(\sigma)}$. Using $\nu_G$, we get a partial order on $B(G)$, still denoted by $\preceq$. Recall that we have a unique dominant element $\mu \in X_*(T)$ such that $\sigma^{-1}(\mu) \in \kappa_G(\nu^{-1})$. Then we set

$$
\overline{\mu} := \frac{1}{r} \sum_{i=0}^{r-1} \sigma^i(\mu) \in (X_*(T)_Q)^{(\sigma)}
$$

where $r$ is a non-zero integer such that $\sigma^r(\mu) = \mu$. We denote the image of $\overline{\mu}$ in $(W_T \backslash X_*(T)_Q)^{(\sigma)}$ again by $\overline{\mu}$. Then we write $\mu^\sharp$ for the image of $\mu$ under the projection

$$
X^*(T) \to \pi_1(G)_Q := (X^*(T)/\langle \alpha^\vee | \alpha^\vee \in \Phi^\vee \rangle)(\sigma), \quad \mu \mapsto \mu^\sharp.
$$

Then we write

$$
B(G, \mu) := \{b \in B(G) | \kappa_G(b) = \mu^\sharp, \nu_G(b) \preceq \overline{\mu} \}.
$$

We know that $B(G, \mu)$ is exactly the image of the double coset $G(W(k)) \mu(p)G(W(k))$ in $B(G)$ (cf. \cite{Wor13} Definition 5.6).

For any point $x \in Sh \otimes \kappa_p$, write $k(x)$ for the residue field of $x$ and $k$ an algebraic closure of $k(x)$ and $\hat{x}$ the associated geometric point over $x$. To each isomorphism $\beta : (\Lambda_{W(k)}^\vee, t_{W(k)}) \simeq (\mathbb{D}_{x}, t_{\text{cris}, x})$, one associates an element $g_\beta \in G(L(k))$. One can show that $[g_\beta]$ is independent of the choices of $\beta$, $k$, and lies in $B(G, \mu)$. Thus we get a well-defined map

$$
(3) \quad \mathcal{N}T : Sh \otimes \kappa_p \to B(G, \mu), \quad x \mapsto [g_\beta].
$$

**Definition 3.2.** For any element $b \in B(G, \mu)$, we write

$$
\mathcal{N}^b := \mathcal{N}T^{-1}(b) \subset Sh \otimes \kappa_p.
$$

It is called the Newton stratum of $b$. The $\mu$-ordinary locus in $Sh \otimes \kappa_p$ is the stratum $\mathcal{N}^{b_{\text{max}}}$ with $b_{\text{max}}$ the maximal element in $B(G, \mu)$ (which can be shown to exist). We write

$$
Sh_{\mu}^{\mu} := \mathcal{N}^{b_{\text{max}}}.
$$
3.2. **Hasse invariant.** In this subsection we recall the construction of the Hasse invariant which cuts out the \( \mu \)-ordinary locus defined above. Moreover, we shall use Hasse invariant to extend the \( \mu \)-ordinary locus from \( Sh \) to a toroidal compactification \( Sh^\Sigma \).

Recall we have a cocharacter \( \mu : \mathbb{G}_m \rightarrow G \) which is defined over \( \mathbb{F}_q \). Write \( P_{\pm} := P_{\pm}(\mu) \) to be the pair of opposite parabolic subgroups of \( G_{\mathbb{F}_q} \), defined by \( \mu \), with common Levi factors \( L \), the centralizer of \( \mu \). Then \((G, P_+, (P_-)^{(\sigma)}, \sigma : G \rightarrow G^{(\sigma)})\) is an algebraic zip datum ([KW15 §2.3]). For any \( x \in P_+ \), write \( \overline{\sigma} \) for its image in the Levi quotient \( P_+ \rightarrow P_+/U_{P_+} \) and similarly for \( y \in (P_-)^{(\sigma)} \), we write \( \overline{\tau} \) for its image in the Levi quotient. We then set the group to be \( E := \{(x, y) \in P_+ \times (P_-)^{(\sigma)} | \sigma(\overline{x}) = \overline{\tau}\} \) which acts on \( G \) by restricting the action of \( P_+ \times (P_-)^{(\sigma)} \) to \( E \). The quotient stack \( [E/G] \) is the stack of \( G \)-zips. More concretely, one has \( E = \{(ux', v\sigma(x')) | u \in U_{P_+}, v \in U_{(P_-)^{(\sigma)}}, x' \in P_+/U_{P_+}\} \). Now we write \( S := \{(x, y) \in E | x = y \} \subset E \), which is the scheme-theoretic stabilizer of \( 1 \in G \) under the action of \( E \). In general \( S \) may not be smooth and we write \( S_{\text{red}} \) to be the underlying reduced group scheme associated to \( S \). Then \( S_{\text{red}} \) is a finite constant group scheme over \( \mathbb{F}_q \). The finite group \( S_{\text{red}}(\mathbb{F}_q) \) can be identified with another group: write \( P_{\pm}^0 = \cap_{n \geq 0} (P_{\pm})^{(n)} \), which are now opposite parabolic subgroups of \( G \) defined over \( \mathbb{F}_q \) with common Levi subgroup \( L_{P_{\pm}} = L_{P_{\pm}^0} \). Then we have ([KW15 Lemma 2.14])

\[
S_{\text{red}}(\mathbb{F}_q) = L_{P_{\pm}}(\mathbb{F}_q).
\]

Consider the character group \( X^*(S_{\text{red}}) \), and we write \( N_G \) to be the exponent of the finite abelian group \( X_*(S_{\text{red}}) \),

\[
N_G = \min \{ 0 < n \in \mathbb{N} | nX_*(S_{\text{red}}) = 0 \},
\]

called the Hasse number of \((G, X)\) ([KW15 Definition 4.11]). Moreover, it is easy to see that the parabolic subgroup \( P_{\overline{\sigma}} \) (defined over \( \mathbb{Z}_p \)) stabilizing the filtration induced by \( \overline{\sigma} \) (which is \( \sigma \)-invariant) gives \( P_{\overline{\sigma}} \otimes_{\mathbb{Z}_p} \mathbb{F}_p = P_{\overline{\sigma}}^0 \). We thus deduce

**Lemma 3.3.** The exponent of the \( \mathbb{F}_p \)-points of the quotient torus \( T_{P_{\overline{\sigma}}} = P_{\overline{\sigma}}/P_{\overline{\sigma}}^0 \) is equal to \( N_G \).

On the other hand, suppose that the splitting field of \( T_{P_{\overline{\sigma}}} \times_{\mathbb{Z}_p} \mathbb{F}_p \) is \( \mathbb{F}_p^{w} \), then

**Corollary 3.4.** We have the following identity:

\[
N_G = p^w - 1.
\]

By ([KW15 4.7]), we know that \( Sh_1^\mu \) is open and dense in \( Sh_1 \). Moreover, one can show that these \( \mathcal{N}^b \) are indeed locally closed (thus form a strata). Moreover, two points \( x_1, x_2 \in Sh_1 \) lie in the same Newton stratum if and only if we have an isomorphism \((\mathbb{D}_{\mathbb{Z}_p}, t_{\text{cris}, \mathbb{Z}_p}) \simeq (\mathbb{D}_{\mathbb{Z}_p}, t_{\text{cris}, \mathbb{Z}_p}) \).

By ([KW15 Definition 4.11]), we know that for the integer \( N := N_G \) (the Hasse number of \((G, X)\)) such that there is a section

\[
H \in H^0(Sh_1, \omega_{Sh}^\Sigma),
\]

whose non-vanishing locus is exactly \( Sh_1^\mu \) ([KW15 Theorem 4.12]). Moreover, by the Koecher principal (or apply [Mad12 Theorem 5.2.11(S)]), \( H \) extends to a unique section

\[
H^\Sigma \in H^0(Sh_1^\Sigma, \omega_{Sh_1^\Sigma}),
\]

whose non-vanishing locus is denoted by \( Sh_1^{\Sigma, \mu} \). If moreover \( G_{\text{ad}} \) has no factor isomorphic to \( \text{PGL}_2/\mathbb{Q} \), then \( H \) extends to a unique section

\[
H^{\min} \in H^0(Sh_1^{\min, \mu}, \omega_{Sh_1^{\min}}),
\]

whose non-vanishing locus is denoted by \( Sh_1^{\min, \mu} \) ([KW15]). If \( Sh_1^? \) is projective, then \( Sh_1^{?} \) is affine (\(? = \emptyset, \Sigma, \text{min}\)).

**Remark 3.5.** We know that a certain power \( H^r \) of the Hasse invariant lifts from the special fibre \( Sh_{1,0} \) to \( Sh \), denoted again by \( H^r \). We will write without further comment the non-vanishing locus of \( H^r \) inside \( Sh \) as \( Sh_1^\mu \). Similarly for \( Sh_1^{\Sigma, \mu} \) and \( Sh_1^{\min, \mu} \).

**Definition 3.6.** We call \( H, H^\Sigma, H^{\min} \) the Hasse invariant of the Shimura variety \( Sh, Sh^\Sigma, Sh^{\min} \).

**Remark 3.7.** We work with the Hasse invariant à la [KW15]. There is another notion of Hasse invariant studied by Hernandez ([Her18]) and others. The latter invariant is a purely local notion and in some sense our \( H \) is a products of local invariants studied in [Her18]. The existence of \( H \) is proved in [KW15], while the local Hasse invariant in [Her18] does not always exist (cf. Remarque 9.23 of loc.cit). The relation between these notions of Hasse invariants remains to be further explored.
The rest of this subsection is devoted to the proof of the following proposition:

**Proposition 3.8.** The Cartier divisor $V(H)$ associated to the Hasse invariant $H$ is reduced on the non $\mu$-ordinary locus $Sh_1^{\mu} := Sh_1 - Sh_1^\mu$.

Note that the non-vanishing locus of $H$ is the $\mu$-ordinary locus $Sh_1^\mu$. The Cartier divisor $V(H)$ lies only inside the complement $Sh_1^{\mu} - Sh_1^{\mu}$ of $Sh_1^\mu$.

In the definition of the Shimura datum $(G, X)$, we require the cocharacters $h_x \in X$ to be minuscule ([Del77, §1.2]). In this case we can describe the set $B(G, \mu)$ explicitly which will enable us to study the properties of the Hasse invariant. Let $T_0 \subset T$ be the maximal split over $\mathbb{Q}_p$. As before, we write $(X(T), \Phi_*, X^*(T), \Phi^*)$ for the absolute root datum with simple roots $\Delta$, simple coroots $\Delta_\vee$, and $(X(T_0), \Phi_{*0}, X^*(T_0), \Phi_{0}^*)$ for the relative root datum with simple roots $\Delta_0$, simple coroots $\Delta_\vee_0$. For a root $\alpha \in \Phi^*$, we denote by $w_\alpha \in X^*(T_0)_{\mathbb{Q}}$ the fundamental weight associated to $\alpha$. Similarly, for a coroot $\beta^\vee \in \Phi_*$, we write $w_{\beta^\vee}$ for the fundamental co-weight associated to $\beta^\vee$. For $\alpha \in \Delta_\vee_0$, put

$$\bar{w}_\alpha := \sum_{\beta} w_\beta \in X^*(T_0)_{\mathbb{Q}},$$

where $\beta$ runs through $\Phi^*$ such that $[\beta]_{T_0} = \alpha$ (which is equal to $w_\alpha$, the fundamental weight associated to $\alpha \in \Phi^*_0$). Then by [CFS] Corollary 4.3,

**Proposition 3.9.** The set $B(G, \mu)$ consists of dominant cocharacters $\nu \in X_{*,\dim}(T_0)_{\mathbb{Q}}$ such that $0 \leq \bar{\mu} - \nu \in \mathbb{Q}_{\geq 0}(\Delta_\vee_0)$ and for any $\alpha \in \Delta_\vee_0$ with $\langle \nu, \alpha \rangle \neq 0$, one has $\langle \bar{\mu} - \nu, w_\alpha \rangle \in \mathbb{N}$.

Note that we have a decomposition

$$X_*(T_0)_{\mathbb{Q}} = (\Phi_{*,0})_{\mathbb{Q}} \bigoplus (\Phi_{0}^*)_{\mathbb{Q}}.$$

According to this decomposition we can write $\bar{\mu} = \bar{\mu}_1 + \bar{\mu}_2$ and similarly $\nu = \nu_1 + \nu_2$ for any $\nu \in B(G, \mu)$. The condition $\bar{\mu} - \nu \in (\Delta_\vee_0)_{\mathbb{Q}}$ shows that $\bar{\mu}_2 = v_2$. Moreover, by definition of $(\Phi_{0}^*)_Q$, we have $\langle \nu_2, \alpha \rangle = 0$ for any $\alpha \in \Delta_\vee_0$. Then we can rewrite $B(G, \mu)$ in the following manners:

$$\{ \nu' \in X_{*,\dim}(T_0)_{\mathbb{Q}} \mid 0 \leq \nu' \in \mathbb{Q}_{\geq 0}(\Delta_\vee_0), \text{ and } \forall \alpha \in \Delta_\vee_0 \text{ with } \langle \bar{\mu} - \nu', \alpha \rangle \neq 0, \langle \nu', w_\alpha \rangle \in \mathbb{N} \}$$

$$= \{ \nu_1 + \nu_2 \in X_{*,\dim}(T_0)_{\mathbb{Q}} \mid 0 \leq \nu_1 - \nu_2 \in \mathbb{Q}_{\geq 0}(\Delta_\vee_0), \text{ and } \forall \alpha \in \Delta_\vee_0 \text{ with } \langle \nu_1, \alpha \rangle \neq 0, \langle \bar{\mu}_1 - \nu_1, w_\alpha \rangle \in \mathbb{N} \}.$$

From this last description we see that the set $B(G, \mu)$ depends only on the root system $(\mathbb{Z}(\Phi_{*,0}), \Phi_{*,0}, \mathbb{Z}(\Phi_{0}^*), \Phi_{0}^*)$ generated by the roots and coroots $\Phi_{*,0}, \Phi_{0}^*$.

For reasons that we will explain later, we will be interested in the difference $\bar{\mu} - \nu = \bar{\mu}_1 - \nu_1$ for $\nu \in B(G, \mu)$. Using the decomposition of root system $(\mathbb{Z}(\Phi_{*,0}), \Phi_{*,0}, \mathbb{Z}(\Phi_{0}^*), \Phi_{0}^*)$, we can assume that the root system is one of the nine classes of indecomposable root systems $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$.

Recall the conditions posed on the cocharacter $\mu$ associated to the Shimura datum $(G, X)$: the cocharacter $\mu: G_m \to G$ is minuscule([Del77, §1.2]). More precisely, let’s suppose that the adjoint group has a decomposition into simple factors $G^{ad} = \prod_{i=1}^{n} G^{(i)}$ and the images of the torus, resp. Borel subgroup, $T^{(i)}, B^{(i)}$ in each factor $G^{(i)}$ are denoted by $T^{(i)}, \text{resp. } B^{(i)}$. Let $\mu^{(i)}$ be the unique $B^{(i)}$-dominant cocharacter of $T^{(i)}$ conjugate to the image of $\mu$ in $G^{(i)}$. Then $\mu$ being minuscule means that there exists at most one simple root $\alpha \in \Delta^{(i)}$ such that $\langle \mu^{(i)}, \alpha \rangle > 0$, in which case $\langle \mu^{(i)}, \alpha \rangle = 1$ and $\alpha$ is special (a simple root which appears with multiplicity one in the highest weight of $(T^{(i)}, B^{(i)})$).

Suppose such a special root $\alpha$ exists, then we have necessarily the fundamental co-weight $\mu^{(i)} = w_{\alpha^\vee}$. Otherwise, we have $\mu^{(i)} = 0$.

**Proposition 3.10.** Suppose that $(\mathbb{Z}(\Phi_{*,0}), \Phi_{*,0}, \mathbb{Z}(\Phi_{0}^*), \Phi_{0}^*)$ is an indecomposable root system such that $\mu = w_{\alpha^\vee}$ for some special root $\alpha \in \Delta^{*}_0$. Then the maximal element in $B(G, \mu) \setminus \{ \bar{\mu} \}$ is $\bar{\mu} - \frac{1}{2} \alpha^\vee$.

**Proof.** We proceed the proof case by case.

(1) $G$ is of type $A_n$. We may identify $G$ with $\text{SL}_{n+1}$ and $B$ is the standard subgroup of upper triangular matrices. Concerning the root datum, for a diagonal element $t = \text{diag}(t_1, \cdots, t_{n+1}) \in \text{SL}_{n+1}$, let $\varepsilon_i \in X^*(T)$ denote the character $\alpha_i(t) = t_i$ $(i = 1, \cdots, n + 1)$. Write $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(i = 1, \cdots, n)$, then the set of simple roots is

$$\Delta^* = \{\alpha_1, \cdots, \alpha_n\}$$
which is a basis for \( X^*(T) \), \( \Phi^* = \{ \varepsilon_i - \varepsilon_j | i \neq j \} \). Similarly, let \( \varepsilon_i^\vee \) denote the cocharacter \( \varepsilon_i^\vee(x) = \text{diag}(1_{i-1}, x, 1_{n+1-i}) \in \text{GL}_{n+1} \) and then \( \alpha_i = \varepsilon_i^\vee - \varepsilon_i^\vee + 1 \).

\[
\Delta^* = \{ \alpha_1^\vee, \ldots, \alpha_n^\vee \},
\]

with \( \Phi^* = \{ \varepsilon_i^\vee - \varepsilon_j^\vee | i \neq j \} \). Moreover, it is easy to see that the fundamental (co-)weights are given by

\[
w_{\alpha_i} = \sum_{j=1}^i \varepsilon_j, \quad w_{\alpha_i^\vee} = \sum_{j=1}^i \varepsilon_j^\vee - \frac{i}{n+1} \sum_{j=1}^{n+1} \varepsilon_j^\vee (i = 1, \ldots, n).
\]

Moreover, the highest weight is \( \varepsilon_1 - \varepsilon_{n+1} = \alpha_1 + \cdots + \alpha_n \). The special roots are therefore \( \alpha_1, \ldots, \alpha_n \). Suppose now that \( \overline{\mu} = w_{\alpha_i} \) for some \( k = 1, 2, \ldots, n \). For any \( \overline{\mu} - \delta \in B(G, \mu) \), we write

\[
\delta = a_1 w_{\alpha_1} + \cdots + a_n w_{\alpha_n} = f_1 \varepsilon_1 + \cdots + f_{n+1} \varepsilon_{n+1} = d_1 \alpha_1^\vee + \cdots + d_n \alpha_n^\vee
\]

\((a_1, f_1, d_1 \in \mathbb{Q})\). We necessarily have \( f_1 + \cdots + f_{n+1} = 0 \). Spelling out the condition in \( B(G, \mu) \), we see that the coefficients \( a_i, f_j \) should satisfy

\[
a_1 = f_1 - f_2 \leq 0, \ldots, a_{k-1} = f_{k-1} - f_k \leq 0, a_k = f_k - f_{k+1} \leq 1,
\]

\[
a_{k+1} = f_{k+1} - f_{k+2} \leq 0, \ldots, a_n = f_n - f_{n+1} \leq 0,
\]

\[
f_1 \geq 0, \ldots, f_1 + \cdots + f_i \geq 0, \ldots, f_1 + \cdots + f_n \geq 0
\]

and moreover (1) if \( a_i \neq 0 \) (with \( i \neq k \)), we have \( f_1 + \cdots + f_i \in \mathbb{N} \) and (2) if \( a_k \neq 1 \), we have \( f_1 + \cdots, f_k \in \mathbb{N} \).

Suppose that the numbers \( a_i \neq 0 \) if and only if \( i = i_1, i_2, \ldots, i_N \) with \( i_1 < i_2 < \cdots < i_N \) elements in \( \{1, 2, \ldots, n\} \).

(a) If \( k = i_j \) for some \( j = 1, \ldots, N \), then we have

\[
0 \leq f_1 = f_2 = \cdots = f_1 < f_{i_j+1} = \cdots = f_{i_2} < \cdots < f_{i_j-1+1} = f_{i_j-1+2} = \cdots = f_{i_j} < 1 + f_{i_j+1} = 1 + f_{i_j+2} = \cdots = 1 + f_{i_j+1} < \cdots < f_{i_N+1} = 1 + f_{i_N+2} = \cdots = 1 + f_{n+1} < 1
\]

such that

\[
d_{i_1} = \ell_1 \in \mathbb{N}, \ldots, d_{i_j} = \ell_j \in \mathbb{N}, d_{i_j+1} = \ell_{j+1} \in \mathbb{N}, \ldots, d_{i_N} = \ell_N \in \mathbb{N}.
\]

One verifies that \( \delta = d_1 \alpha_1^\vee + \cdots + d_n \alpha_n^\vee > \frac{1}{2} \alpha_k^\vee \) unless \( \delta = 0 \).

(b) If \( i_j < k < i_{j+1} \) for some \( j = 0, 1, \ldots, N + 1 \) (with convention \( i_0 = 0, i_{N+1} = n + 1 \)). In this case, \( \overline{\mu} - \delta \in B(G, \mu) \) implies that

\[
0 \leq f_1 = \cdots = f_1 < f_{i_j+1} = \cdots = f_{i_2} < \cdots < f_{i_j-1+1} = f_{i_j-1+2} = \cdots = f_{i_j} < f_{i_j+1} = f_{i_j+2} = \cdots < f_{i_N+1} = 1 + f_{i_N+2} = \cdots = 1 + f_{n+1} < 1
\]

such that

\[
d_{i_1} = \ell_1 \in \mathbb{N}, \ldots, d_{i_j} = \ell_j \in \mathbb{N}, d_{i_j+1} = \ell_{j+1} \in \mathbb{N}, \ldots, d_{i_N} = \ell_N \in \mathbb{N}.
\]

One verifies that \( \delta = d_1 \alpha_1^\vee + \cdots + d_n \alpha_n^\vee \geq \frac{1}{2} \alpha_k^\vee \) and we have \( \overline{\mu} - \frac{1}{2} \alpha_k^\vee \in B(G, \mu) \).

Thus we conclude in the case \( G \) of type \( A_n \) that the maximal element in \( B(G, \mu) \setminus \{ \overline{\mu} \} \) is indeed \( \overline{\mu} - \frac{1}{2} \alpha_k^\vee \).

(2) \( G \) is of type \( B_n \). We identify \( G \) with the special orthogonal group \( SO_{2n+1} \) associated to the symmetric matrix \( \text{antidiag}(1, 1, \cdots, 1) \in \text{GL}_{2n+1} \). Write the diagonal matrix \( t = \text{diag}(t_1, \ldots, t_n, 1, t_n^{-1}, \ldots, t_1^{-1}) \). Denote by \( \varepsilon_i \in X^*(T) \) the character \( \varepsilon_i(t) = t_i \) \( (i = 1, \ldots, n) \) and set \( \alpha_i = \varepsilon_i - \varepsilon_{i+1} \) \( (i = 1, \ldots, n-1) \). Then

\[
\Delta^* = \{ \alpha_1, \cdots, \alpha_{n-1}, \varepsilon_n \}
\]

and \( \Phi^* = \{ \pm(\varepsilon_i \pm \varepsilon_j) | i \neq j \} \cup \{ \pm \varepsilon_i \} \). Similarly, denote by \( \varepsilon_i^\vee \in X_*(T) \) the cocharacter \( \varepsilon_i^\vee(x) = \text{diag}(1_{i-1}, x, 1_{2n+3-2i}, x^{-1}, 1_{i-1}) \in \text{GL}_{2n+1} \) and set \( \alpha_i = \varepsilon_i^\vee - \varepsilon_{i+1}^\vee \). Then

\[
\Delta_* = \{ \alpha_1^\vee, \cdots, \alpha_{n-1}^\vee, 2 \varepsilon_n^\vee \}
\]
and $\Phi_* = \{ \pm (\varepsilon_i^\vee \pm \varepsilon_j^\vee) | i \neq j \} \cup \{ \pm \varepsilon_i^\vee \}$. The fundamental (co-)weights are given by

$$w_{\alpha_i} = \sum_{j=1}^{i} \varepsilon_j, \quad w_{\varepsilon_n} = \frac{1}{2} \sum_{j=1}^{n} \varepsilon_j; \quad w_{\alpha_i^\vee} = \sum_{j=1}^{i} \varepsilon_j^\vee, \quad w_{\varepsilon_n^\vee} = \frac{1}{2} \sum_{j=1}^{n} \varepsilon_j^\vee.$$

In this case, the highest weight is $\varepsilon_1 + \varepsilon_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \cdots + 2\alpha_{n-1} + 2\varepsilon_n$, thus the special root is $\alpha_1$ and therefore $\mathbf{\pi} = w_{\alpha_1^\vee} = \varepsilon_1^\vee$. The computation is exactly the same as above (even simpler): we write

$$\delta = a_1 w_{\alpha_1^\vee} + \cdots + a_n w_{\alpha_n^\vee} = d_1 \alpha_1^\vee + \cdots + d_n \alpha_n^\vee$$

with $a_i, d_j \in \mathbb{Q}$. Thus we have the following conditions on these coefficients

$$a_1 \leq 1, \quad a_2, \ldots, a_n \leq 0, \quad d_1, \ldots, d_n \geq 0$$

and (1) if $a_1 \neq 0$ ($i \neq 1$), then $d_1 \in \mathbb{N}$; (2) if $a_1 \neq 1$, then $d_1 \in \mathbb{N}$. Then one verifies that $\delta = \frac{1}{2} \alpha_1^\vee$ satisfies all these conditions and moreover, $\mathbf{\pi} - \delta$ is the maximal element in $B(G, \mu) \setminus \{ \mathbf{\pi} \}$.

(3) $G$ is of type $C_n$ (the dual case of the preceding one). We identify $\tilde{G}$ with the special symplectic group $Sp_{2n}$ of the symplectic form $\text{antidiag}(J, -J)$ with $J = \text{antidiag}(1, 1, \ldots, 1) \in \text{GL}_n$. Write the diagonal matrix $t = \text{diag}(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1})$. Denote by $\varepsilon_i \in X^*(T)$ the character $\varepsilon_i(t) = t_i$ and $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. Then

$$\tilde{\Delta}^* = \{ \alpha_1, \ldots, \alpha_{n-1}, 2\varepsilon_n \}$$

and $\Phi^* = \{ \pm (\varepsilon_i \pm \varepsilon_j) | i \neq j \} \cup \{ \pm 2\varepsilon_i \}$. Similarly, denote by $\varepsilon_i^\vee \in X_*(T)$ the cocharacter $\varepsilon_i^\vee(x) = \text{diag}(1_{i-1}, x, 1_{2n+2-2i}, x^{-1}, 1_{i-1})$ and $\alpha_i^\vee = \varepsilon_i^\vee - \varepsilon_{i+1}^\vee$. Then

$$\tilde{\Delta}_* = \{ \alpha_1^\vee, \ldots, \alpha_{n-1}^\vee, \varepsilon_n^\vee \}$$

and $\Phi_* = \{ \varepsilon_i^\vee - \varepsilon_j^\vee | i \neq j \} \cup \{ \pm \varepsilon_i \}$. The fundamental (co-)weights are given by

$$w_{\alpha_i} = \sum_{j=1}^{i} \varepsilon_j, \quad w_{2\varepsilon_n} = \sum_{j=1}^{n} \varepsilon_j; \quad w_{\alpha_i^\vee} = \sum_{j=1}^{i} \varepsilon_j^\vee, \quad w_{\varepsilon_n^\vee} = \frac{1}{2} \sum_{j=1}^{n} \varepsilon_j^\vee.$$

Note that the highest weight is $2\varepsilon_1 = 2\alpha_1 + \cdots + 2\alpha_{n-1} + (2\varepsilon_n)$ and thus the special root is $2\varepsilon_n$, therefore we have $\mathbf{\pi} = \frac{1}{2} \sum_{j=1}^{n} \varepsilon_j^\vee = w_{\varepsilon_n^\vee}$. To simplify notations, let’s write $\alpha_n = 2\varepsilon_n$ and $\alpha_n^\vee = \varepsilon_n^\vee$. We write

$$\delta = a_1 w_{\alpha_1^\vee} + \cdots + a_n w_{\alpha_n^\vee} = d_1 \alpha_1^\vee + \cdots + d_n \alpha_n^\vee$$

with $a_i, d_j \in \mathbb{Q}$. The conditions put these coefficients become

$$a_1, \ldots, a_{n-1}, a_n - 1 \leq 0, \quad d_1, \ldots, d_n \geq 0$$

and (1) if $a_i \neq 0$ ($i \neq n$), we have $d_i \in \mathbb{N}$; (2) if $a_n - 1 = 0$, we have $d_n \in \mathbb{N}$. One verifies easily that $\delta = \frac{1}{2} \alpha_n^\vee$ satisfies all these conditions and $\mathbf{\pi} - \delta \in B(G, \mu)$.

(4) $G$ is of type $D_n$. We identify $\tilde{G}$ with the orthogonal group $SO_{2n}$ of the quadratic form $\text{antidiag}(1, \ldots, 1)$. Write the diagonal matrix $t = \text{diag}(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1})$. Denote by $\varepsilon_i \in X^*(T)$ the character $\varepsilon_i(t) = t_i$ and $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ and $\alpha_{n+1}^+ = \varepsilon_{n-1} + \varepsilon_n$. Then

$$\tilde{\Delta}^* = \{ \alpha_1, \ldots, \alpha_{n-1}, \alpha_{n+1}^+ \}$$

and $\Phi^* = \{ \pm (\varepsilon_i \pm \varepsilon_j) | i \neq j \}$. Similarly, denote by $\varepsilon_i^\vee \in X_*(T)$ the cocharacter sending $x$ to the element $\text{diag}(1_{i-1}, x, 1_{2n+2-2i}, x^{-1}, 1_{i-1})$ and $\alpha_i^\vee = \varepsilon_i - \varepsilon_{i+1}$, $(\alpha_{n+1}^+)^\vee = \varepsilon_{n-1} + \varepsilon_n$. Then

$$\tilde{\Delta}_* = \{ \alpha_1^\vee, \ldots, \alpha_{n-1}^\vee, (\alpha_{n+1}^+)^\vee \}$$

and $\Phi_* = \{ \pm (\varepsilon_i^\vee \pm \varepsilon_j^\vee) | i \neq j \}$. The fundamental (co-)weights are given by

$$w_{\alpha_i} = \sum_{j=1}^{i} \varepsilon_j, \quad w_{\alpha_{n+1}^+} = \frac{1}{2} \sum_{j=1}^{n} \varepsilon_j; \quad w_{\alpha_i^\vee} = \sum_{j=1}^{i} \varepsilon_j^\vee, \quad w_{(\alpha_{n+1}^+)^\vee} = \frac{1}{2} \sum_{j=1}^{n} \varepsilon_j^\vee.$$
Note that the highest weight is \( \varepsilon_1 + \varepsilon_2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_{n-1} + \alpha_{n-1}^+ \), thus the special roots are \( \alpha_1, \alpha_{n-1}, \alpha_{n-1}^+ \), and therefore we have three possibilities \( \mu = \varepsilon_1^\vee (w_{\alpha_1^+}) \), \( \sum_{j=1}^n \varepsilon_j^\vee (w_{\alpha_{n-1}^+}) \) or \( \frac{1}{2} \sum_{j=1}^n \varepsilon_j^\vee (w_{\alpha_{n-1}}^+) \). To simplify notations, let’s put \( \alpha_n = \alpha_{n-1}^+ \) and \( \alpha_n^\vee = (\alpha_{n-1}^+)\vee \). We write

\[
\delta = a_1 w_{\alpha_1^+} + \cdots + a_n w_{\alpha_n^+} = d_1 \alpha_1^\vee + \cdots + d_n \alpha_n^\vee
\]

with \( a_i, d_i \in \mathbb{Q} \). We discuss the three cases one by one

(a) For \( \mu = w_{\alpha_1^+} \). The conditions on these coefficients are

\[
a_1 - 1, a_2, \cdots, a_n \leq 0, \quad d_1, \cdots, d_n \geq 0
\]

and (1) if \( a_i \neq 0 \) (i.e. \( i = 1 \)), we have \( d_i \in \mathbb{N} \); (2) if \( a_1 - 1 \neq 0 \), we have \( d_1 \in \mathbb{N} \). Again one verifies that \( \delta = \frac{1}{2} \alpha_1^\vee \) satisfies all these conditions and moreover \( \mu - \delta \in B(G, \mu) \) is the maximal element.

(b) For \( \mu = w_{\alpha_{n-1}^+} \), the conditions on these coefficients are

\[
a_1, \cdots, a_{n-2}, a_{n-1} - 1, a_n \leq 0, \quad d_1, \cdots, d_n \geq 0
\]

and (1) if \( a_i \neq 0 \) (i.e. \( i = n-1 \)), we have \( d_i \in \mathbb{N} \); (2) if \( a_{n-1} - 1 \neq 0 \), we have \( d_{n-1} \in \mathbb{N} \). One verifies as above that \( \delta = \frac{1}{2} \alpha_n \) satisfies all these conditions and \( \mu - \delta \in B(G, \mu) \) is the maximal element.

(c) For \( \mu = w_{\alpha_n^+} \), the conditions on these coefficients are

\[
a_1, \cdots, a_{n-1}, a_n - 1 \leq 0, \quad d_1, \cdots, d_n \geq 0
\]

and (1) if \( a_i \neq 0 \) (i.e. \( i = n \)), we have \( d_i \in \mathbb{N} \); (2) if \( a_n - 1 \neq 0 \), we have \( d_n \in \mathbb{N} \). One verifies that \( \delta = \frac{1}{2} \alpha_n \) satisfies all these conditions and \( \mu - \delta \in B(G, \mu) \) is the maximal element.

(5) \( G \) is of type \( E_6 \). Recall the Dynkin diagram for \( E_6 \):

\[
\begin{array}{cccccc}
\alpha_1 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\
\end{array}
\]

Then one sees that the highest weight is \( \alpha_1 + 2(\alpha_2 + \cdots + \alpha_5) + \alpha_6 \), thus the special roots are \( \alpha_1, \alpha_6 \). If \( \mu = w_{\alpha_1^\vee} \), one can verify as above that \( \mu - \frac{1}{2} \alpha_1^\vee \in B(G, \mu) \setminus \{\mu\} \) is the maximal element. Similarly, if \( \mu = w_{\alpha_6^\vee} \), one verifies that \( \mu - \frac{1}{2} \alpha_6^\vee \in B(G, \mu) \setminus \{\mu\} \) is the maximal element.

(6) \( G \) is of type \( E_7 \). The Dynkin diagram for \( E_7 \) is

\[
\begin{array}{cccccccc}
\alpha_1 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 \\
\end{array}
\]

One verifies that the special root is \( \alpha_1 \) and we have \( \mu = w_{\alpha_1} \). As above, we have \( \mu - \frac{1}{2} \alpha_1^\vee \in B(G, \mu) \setminus \{\mu\} \) is the maximal element.

(7) For other types \( E_8, F_4, G_2 \), there are no special roots.

\[\square\]

Proof. (of Proposition 3.8) We follow the strategy in [PH12, A.3]. Consider a geometric point \( x \in N^b \) for a maximal element \( b \in B(G, \mu \setminus \{\mu\}) \) and \( A_x \) the abelian scheme over \( x \), \( (\mathbb{D}, \mathbb{F}) \) the Dieudonné crystal with \( G \)-structure of \( A_x \) evaluated at the dual numbers \( \mathbb{F}_p[\varepsilon] \), which is of \( \mathbb{F}_p[\varepsilon] \)-rank \( \dim(V) \). By the above proposition, we see that in the decomposition \( G^{\text{ad}} = \prod_{i=1}^7 G^{(i)} \), among the components \( b^{(i)} \) of the dominant co-character \( b \in X_{s, \dim}(T_0) \), exactly one of them (say, the \( i \)-th component) is of the form \( \mu^{(i)} - \frac{1}{2} \alpha_1^\vee \) where \( \alpha_1^\vee \) is the notation in the above proposition. By the embedding \( (G, X) \hookrightarrow (G_0, V, \psi, S^+) \), we see that in the copy \( s_\alpha \) of \( sf_2 \) inside the Lie algebra \( g(\mathbb{F}_p) \) of \( G(\mathbb{F}_p) \), one can choose basis \( B = (e_1, e_2, \cdots, e_n, e_1^\dagger, \cdots, e_6^\dagger) \) of \( \Lambda \) such that the projection of \( \mu \) to this copy is of the form \( \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \) while the projection of \( b \) to this copy is of the form \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \). Suppose that \( s_\alpha \) acts on the subspace \( \mathcal{X}_\alpha = \mathbb{F}_p(e_n, e_n^\dagger) \) of \( \mathcal{X}_\alpha \). We identify \( \mathcal{X}_\alpha \) with \( \mathbb{D} \) thus \( \mathcal{X}_\alpha \) with a subspace of \( \mathbb{D} \). Then the Hodge filtration on \( \mathcal{X}_\alpha \)
is given by \( \text{Fil}^1(\Lambda_n) = \overline{\mathbb{F}}_p(e_n) \). Now the same argument as in \cite[Théorème A.4]{Fili2} shows that the Hasse invariant defines a non-zero linear form on the \( \overline{\mathbb{F}}_p \)-vector space \( X_p(\overline{\mathbb{F}}_p[x]) \) of liftings from \( x \) to \( \overline{\mathbb{F}}_p[x] \). At last note that the union \( \cup_b N^b \) with \( b \) running through all the maximal elements in \( B(G, \mu) \setminus \{ \overline{\mathbb{F}}_p \} \) is open dense in the non-\( \mu \)-ordinary locus \( Sh_1^\mu \) (cf. \cite[Theorem A]{Zha1}.

\[ \square \]

4. Automorphic forms

4.1. Some representations. We first fix some notations. Recall that we have fixed a toroidal compactification \( Sh^\Sigma \) of the integral model \( Sh \) over \( \mathbb{W} \) and we have the Hodge line bundles \( \omega_{Sh^\Sigma} \) and \( \omega_{Sh} \) over these spaces. We fix a point \( x \in X \) and the associated cocharacter \( \nu_x \) of \( G_\Sigma \), which is actually defined over the reflex field \( E \). We write \( P_V \) for the parabolic subgroup of \( GSp(V, \psi) \) stabilizing the Hodge filtration of the Hodge structure \( \mathcal{A}_d \circ (\xi \circ x) \). \( L_V \) the Levi subgroup of \( P_V \) and \( U_V \) the unipotent radical of \( P_V \). Note that \( L_V \) is the centralizer of the cocharacter \( \nu_x \) inside \( GSp(V, \psi) \).

We fix a basis for \( V \) such that the symplectic form \( \psi \) on \( V \) is represented by \( \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix} \) with \( A = \text{anti-diag}(1, 1, \cdots, 1) \).

Then we write \( T_V \) for the maximal torus of \( GSp(V, \psi) \) consisting of diagonal matrices, \( B_V \) the Borel subgroup of \( GSp(V, \psi) \) consisting of upper triangular matrices. We write \( P_V \) for some standard parabolic subgroup of \( GSp(V, \psi) \) containing \( B_V \). Under this basis, \( P_V \) is the maximal parabolic subgroup of \( GSp(V, \psi) \) consisting of matrices of the form \( \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \). Now we consider a standard parabolic subgroup \( \tilde{P}_V \) of \( GSp(V, \varphi) \) such that \( B_V \subset \tilde{P}_V \subset P_V \). Similarly we have a Levi decomposition \( \tilde{P}_V = \tilde{L}_V \tilde{U}_V \). We write \( P^\circ_V \) for the parabolic subgroup of \( GSp(V, \psi) \) opposite to \( P_V \) and similarly \( \tilde{P}^\circ_V \), opposite to \( \tilde{P}_V \). We write \( \tilde{P}^\text{der}_V \) for the derived subgroup of \( \tilde{P}_V \) and we put \( \tilde{T}^\text{der}_V = \tilde{P}_V / \tilde{P}^\text{der}_V \). Then we set

\[ ? = G \cap \tilde{P}_{u.V} \quad \text{with} \quad ? = T, B, P, L, U, \tilde{P}, \tilde{L}, \tilde{U}, \tilde{P}^\circ, \tilde{U}^\circ, \tilde{P}^\text{der}, \tilde{T}^\text{der}_V \quad \text{and} \quad \tilde{T}_V = G \cap \tilde{T}^\text{der}_V = \tilde{P} / \tilde{P}^\text{der} \]

for the corresponding subgroups relative to \( G \).

Remark 4.1. Note in particular that \( L \) is the centralizer of the cocharacter \( \nu_x \) inside \( G_\Sigma \). Recall we have fixed a \( \mathbb{Z}_p \)-model for \( G_{\mathbb{Q}_p} \) (see the paragraph after Hypothesis \( \[ \lambda \rangle \)). We can choose a representative of \( \nu_x \) defined over \( \mathbb{Z}_p \) (which is denoted by the same letter \( \nu_x \)). This gives rise to a \( \mathbb{Z}_p \)-model of the Levi-subgroup \( L \). In the following we will also write this \( \mathbb{Z}_p \)-model by \( L \), when no confusion is possible (see also \cite[2.1.5]{GK1}).

We also write \( \tilde{P}_L = \tilde{P} \cap L \) and thus \( \tilde{T}^\text{der}_V = \tilde{P}_L / \tilde{P}^\text{der}_L \). Moreover, the natural inclusion \( B \subset \tilde{P} \) induces a quotient map \( T \rightarrow \tilde{T}_V \). Thus we can view the character group \( X^*(\tilde{T}_V) \) as a subgroup of \( X^*(T) \): \( X^*(\tilde{T}_V) \subset X^*(T) \). Similarly, we view the cocharacter group \( X_*(\tilde{T}_V) \) as a quotient group of \( X_*(T) \): \( X_*(\tilde{T}_V) \rightarrow X_*(T) \). We then put

\[ X^*_{\text{dm}}(\tilde{T}_V) := X^*(\tilde{T}_V) \cap X^*_\text{dm}(T) \]

\[ X_*_{\text{dm}}(\tilde{T}_V) := \text{Im}(X_*, \text{dm}(T) \rightarrow X_*(\tilde{T}_V)) \]

We write \( \lambda_{GSp(V, \psi)} \) for the character of \( T_V \) sending \((t_1, t_2, \cdots, t_n, \nu/t_n, \nu/t_{n-1}, \cdots, \nu/t_1) \) to \( t_1 t_2 \cdots t_N \) and then we put

\[ \lambda_G = \lambda_{GSp(V, \psi)}|_T \in X^*(T) \]

Definition 4.2. Write \( B_L = B \cap L \). For any \( \mathcal{O}_L \)-algebra \( A \) and any character \( \lambda \in X^*(T) \), we write

\[ R_A[\lambda^{-1}] := \text{Ind}_B^L(\lambda^{-1})/A \]

for the algebraic induction defined as in \cite[§1.3.3]{Lan03}, which is an \( L \)-equivariant line bundle on the flag variety \( L / B_L \). More concretely, for any \( A \)-algebra \( A' \), \( R_A[\lambda^{-1}](A') \) is the set of rational morphisms \( f : L / A' \rightarrow \mathbb{G}_m / A' \) such that \( f(gh) = \lambda^{-1}(h)f(g) \) for any \( g \in L(A'), h \in B_L(A') \).

We have a canonical isomorphism

\[ R_A[\lambda^{-1}] \otimes_A A[1/p] \simeq R_{A[1/p]}[\lambda^{-1}] \]

thus we have a natural inclusion \( R_A[\lambda^{-1}] \hookrightarrow R_{A[1/p]}[\lambda^{-1}] \) and we can view the first space as a lattice in the second space.
Definition 4.3. For a finite flat $\mathcal{O}_p$-algebra $A$, we define an $A$-module

$$R^\text{top}_A[\lambda^{-1}] := \text{Ind}_{B_L(\mathcal{O}_p)}^{L(\mathcal{O}_p)}(\lambda^{-1}, A)$$

to be the set of continuous maps $f : L(\mathcal{O}_p) \to A$ such that $f(gtu) = \lambda^{-1}(t)f(g)$ for any $g \in L(\mathcal{O}_p)$, $t \in T(\mathcal{O}_p)$, $u \in U_L(\mathcal{O}_p)$. Similarly we define an $A[1/p]$-module

$$R^\text{top}_{A[1/p]}[\lambda^{-1}] := \text{Ind}_{B_L(E_p)}^{L(E_p)}(\lambda^{-1}, A[1/p])$$

to be the set of continuous maps $f : L(E_p) \to A[1/p]$ such that $f(gtu) = \lambda^{-1}(t)f(g)$ for any $g \in L(E_p)$, $t \in T(E_p)$ and $u \in U_L(E_p)$.

We let $L$ act on $R_A[\lambda^{-1}]$ by left translation and let $L(\mathcal{O}_p)$ act on $R^\text{top}_A[\lambda^{-1}]$ by left translation.

By the Iwahori decomposition $L(E_p) = L(\mathcal{O}_p)B_L(E_p)$, we see that an element $f \in R^\text{top}_{A[1/p]}[\lambda^{-1}]$ is determined by its restriction to $L(\mathcal{O}_p)$ and thus we have a natural inclusion

$$R^\text{top}_A[\lambda^{-1}] \hookrightarrow R^\text{top}_{A[1/p]}[\lambda^{-1}]$$

therefore we can view the first space as a lattice in the second space by the compactness of $L(\mathcal{O}_p)$.

There are several morphisms among these representations that will be useful later on.

1. We have a natural map

$$\text{ev}_{\mathcal{O}_p} : R_A[\lambda^{-1}] \to R^\text{top}_A[\lambda^{-1}], \quad f \mapsto (f|_{\mathcal{O}_p} : g \in L(\mathcal{O}_p) \mapsto f(g)),$$

which is simply the evaluation of the algebraic representation $R_A[\lambda^{-1}]$ at $\mathcal{O}_p$ (we view $g \in L(\mathcal{O}_p)$ as an element in $L(A)$ via the natural map $\mathcal{O}_p \to A$).

2. We write $A[\lambda^{-1}]$ for the free $A$-module of rank 1 on which $T$ acts by the character $\lambda^{-1}$. For any character $\lambda \in X^* T_\mathfrak{P}(\mathfrak{P})$ (we view it also as a character of $T$), we have a restriction map, which is clearly $T_\mathfrak{P}$-equivariant:

$$\text{Res}_A : R_A[\lambda^{-1}] \to A[\lambda^{-1}], \quad f \mapsto f|_{\mathfrak{P}L}.$$ Denote by $R^0_A[\lambda^{-1}]$ the kernel of this map.

3. Similarly we have another restriction map, which is $T_\mathfrak{P}(\mathcal{O}_p)$-equivariant:

$$\text{Res}_{A^\cdot} : R^\text{top}_A[\lambda^{-1}] \to A[\lambda^{-1}], \quad f \mapsto f|_{\mathfrak{P}L(\mathcal{O}_p)},$$

whose kernel we denote by $R^\text{top,0}_A[\lambda^{-1}]$.

Next we define some operators on these modules, which correspond to the Hecke operators that we will consider later. We write

$$T^\pm_\mathfrak{P}(E_p) \subset T_\mathfrak{P}(E_p)$$

for the sub-monoid generated by the elements $\mu(p) \in T_\mathfrak{P}(E_p)$ for all $\mu \in X_{s,dim}(T_\mathfrak{P})$. We let $T^\pm_\mathfrak{P}(E_p)$ act on $B_L(E_p)$ by inverse conjugation. It is easy to see that $T^\pm_\mathfrak{P}(E_p)$ stabilizes the subgroup $B_L(\mathcal{O}_p)$.

We write the fundamental coroots

$$\{\epsilon'_1, \epsilon'_2, \ldots, \epsilon'_{r'}\}$$

for the monoid $X_{s,dim}(T)$ of dominant cocharacters. Then the element

$$\epsilon_i := \epsilon'_i(p) = \text{diag}(\Lambda_{t_1}, p^{s_i} \lambda^{-1}_{t_1}) \in T(V(\mathcal{O}_p))$$

is a diagonal matrix in $\text{GSp}(V, \psi)(\mathcal{O}_p)$ with $\Lambda_t = \text{diag}(p_{t_1}^{s_1}, \ldots, p_{t_n}^{s_n})$ such that $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n (2n = \text{dim}_\mathbb{Q} V)$ and $s_i > 0$. We denote the images of $\epsilon'_i$ in the projection $X_s(T) \to X_s(T_\mathfrak{P})$ again by $\epsilon'_i$.

Definition 4.4. We fix a finite flat $\mathcal{O}_p$-algebra $A$. For an element $\epsilon \in T^\pm_\mathfrak{P}(E_p)$ and the algebraic representation $R_A[1/p][\lambda^{-1}]$ for some dominant character $\lambda \in X^*_{\text{dim}}(T_\mathfrak{P})$, we define an operator $T_\epsilon$ on $R_{E_p}[\lambda^{-1}]$ as follows: for any element $f \in R_A[1/p][\lambda^{-1}]$, set $T_\epsilon(f) \in R_A[1/p][\lambda^{-1}]$ to be

$$(T_\epsilon f)(g) := f(\epsilon g e^{-1}), \quad \forall g \in L(A[1/p]).$$

We define an operator $T_\epsilon$ on the spaces $R^\text{top}_{A[1/p]}[\lambda^{-1}]$ and $A[1/p][\lambda^{-1}]$ by the same formula.
Proposition 4.5. For a finite flat $O_p$-algebra $A$ with field of fractions, a dominant character $\lambda \in X^*_\text{dm}(\widetilde{T}_P)$ and an element $\epsilon \in \widetilde{T}_P^+(E_p)$, we have $T_{\epsilon'} T_{\epsilon} = T_{\epsilon \epsilon'}$. We have the following observations (cf. [PIII, Proposition 3.1]):

**Proof.** We prove the proposition for $\epsilon = \epsilon_i$. The general situation follows since each $\epsilon$ is non-negative integral combination of these $\epsilon_i$ and $T_{\epsilon \epsilon'} = T_{\epsilon} T_{\epsilon'}$. Recall $\kappa_p$ is the residual field of $O_p$. Let $I_{B_L^p}$ be the set of elements in $L(O_p)$ whose reduction modulo $p$ lies in the Borel subgroup $B_L(\kappa_p)$ (opposite to $B_L(\kappa_p)$ (an Iwahori subgroup of $L(O_p)$)). By the Bruhat decomposition $L(\kappa_p) = \bigsqcup \beta L(\kappa_p) w B_L(\kappa_p)$ where $w$ runs through the double quotient set $W_P \backslash W / W_P$ ([Jan03, §II.1.9]), we have a decomposition

$$L(O_p) = \bigsqcup \ i \ B_w^p w B_L(O_p) = \bigsqcup \ i \ B_w^p w U_L(O_p)$$

where $w$ runs through the same set $W_P \backslash W / W_P$. Now take a function $f \in R_A^\text{top}[\lambda^{-1}]$ and an element $g = i w \in I_{B_L^p} B_L(O_p)$, by definition, we have

$$(T_{\epsilon} f)(g) = f(i w v e^{-1}) = f(i w e^{-1}) = f(e i e^{-1} w \cdot w e^{-1} w e w^{-1}) = \lambda^{-1}(w^{-1} w e w^{-1}) f(e i e^{-1} w).$$

Note that for $\lambda \in X^*_\text{dm}(\widetilde{T}_P)$, we have $\lambda^{-1}(w^{-1} w e w^{-1}) \in O_p$. On the other hand, by our restriction to $\epsilon = \epsilon_i$ and previous assumptions on these $\epsilon_i$, we see that $e_i e^{-1} \in L(O_p)$ and thus $e_i e^{-1} w \in L(O_p)$. We deduce that $(T_{\epsilon} f)(g) \in A$. $\square$

On the other hand, $R_A[\lambda^{-1}]$ may not be preserved by $T_{\epsilon}$ inside $R_{A[1/p]}[\lambda^{-1}]$ (unless $A$ is an unramified extension of $O_p$). This is because we do not have a Bruhat decomposition of $L(A)$ for general $A$. Due to this fact, we need to modify the integral structure on $R_A[\lambda^{-1}]$ which will be stable under the action of $T_{\epsilon}$. The idea is simple: recall that we have a natural map $e\nu_{O_p} : R_A[\lambda^{-1}] \rightarrow R_A^\text{top}[\lambda^{-1}]$ as well as its base change to $A[1/p]$, that is, $e\nu_{E_p} : R_{A[1/p]}[\lambda^{-1}] \rightarrow R_{A[1/p]}^\text{top}[\lambda^{-1}]$.

**Definition 4.6.** Then we set

$$\widetilde{R}_A[\lambda^{-1}] := e\nu_{E_p}^{-1}(R_A^\text{top}[\lambda^{-1}]) \subset R_{E_p}^\text{top}[\lambda^{-1}]$$

which is a lattice in $R_{A[1/p]}[\lambda^{-1}]$ containing $R_A[\lambda^{-1}]$.

Clearly the map $e\nu_{E_p}$ is equivariant for the operator $T_{\epsilon}$. Thus we see that

**Corollary 4.7.** The operator $T_{\epsilon}$ preserves $\widetilde{R}_A[\lambda^{-1}]$.

Next we consider ordinary projectors.

**Definition 4.8.** Consider as above a character $\lambda \in X^*_\text{dm}(\widetilde{T}_P)$ and we put $T_{\epsilon}^P = \prod_{i=1}^{T_{\epsilon}} T_{\epsilon_i}$ and

$$e_P = \lim_{n \rightarrow \infty} (T_{\epsilon}^P)^n!.$$  

It is easy to see that this limit is independent of the choice of the set of generators. Then one has

**Proposition 4.9.** For $\lambda \in X^*_\text{dm}(\widetilde{T}_P)$ and a finite flat $O_p$-algebra $A$, we have an isomorphism of $A[1/p]$-modules:

$$e_P R_{A[1/p]}[\lambda^{-1}] \cong A[\frac{1}{p}] [\lambda^{-1}].$$

Similarly, we have an isomorphism of $A$-modules, which is compatible with the above isomorphism:

$$e_P R_{A}[\lambda^{-1}] \cong A[\lambda^{-1}].$$

**Proof.** Consider the first isomorphism. For the surjectivity, consider the function $f \in R_{A[1/p]}[\lambda^{-1}]$ which is $1$ on $\widetilde{P}_L$ and $0$ otherwise. Since $\widetilde{P}_L \supset B_L$, $f$ is well-defined and in fact $f \in R_{A[1/p]}[\lambda^{-1}] \backslash R_{A[1/p]}^0[\lambda^{-1}]$. Moreover, the action of $T_{\epsilon}$ on $f$ is trivial, and $\text{Res}_{A[1/p]}(f) \neq 0$, thus the surjectivity follows.

For the injectivity, it is enough to show that $e_P R_{A[1/p]}^0[\lambda^{-1}] = 0$. Note that the big cell $B_L^2 P_L$ is dense in $L$ ([Jan03, §II.1.9], over the field $A[1/p]$), so is $\widetilde{P}_L^2 \widetilde{P}_L$. Thus an element $f \in R_{A[1/p]}[\lambda^{-1}]$ is determined by its restriction to $\widetilde{P}_L^2 \widetilde{P}_L$. 

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Moreover the conjugate action of $\prod_{i=1}^r e_i$ contracts $\tilde{P}_L^0$ into $\tilde{P}_L^0 \cap \tilde{P}_L$. Yet by definition, any $f \in R_{A[1/p]}^0[\lambda^{-1}]$ satisfies $f|_{\tilde{P}_L} = 0$ and thus we see that $e_{\tilde{P}}f = 0$.

Now consider the second isomorphism. The surjectivity is the same as above. For the injectivity, any $f \in R_{A[1/p]}^0[\lambda^{-1}]$ is also an element in $R_{A[1/p]}^0[\lambda^{-1}]$ and thus $e_{\tilde{P}}f = 0$. □

Similarly we have

**Proposition 4.10.** For $\lambda \in X_{\text{dm}}^*(\tilde{T}_{\tilde{P}})$ and a finite flat $O_p$-algebra $A$, we have a commutative diagram with isomorphic horizontal arrows

$$
\begin{array}{ccc}
\varepsilon \quad R_{\text{top}}^0[A] & \overset{\sim}{\to} & A[\lambda^{-1}] \\
\downarrow & & \downarrow \\
\varepsilon \quad R_{A[1/p]}^0[\lambda^{-1}] & \overset{\sim}{\to} & A[1/p][\lambda^{-1}] 
\end{array}
$$

**Proof.** The proof is exactly the same as in the above proposition. □

Combining the above propositions, we see

**Corollary 4.11.** For any $\lambda \in X_{\text{dm}}^*(\tilde{T}_{\tilde{P}})$ and a finite flat $O_p$-algebra $A$, we have an isomorphism of $A$-modules

$$
ev_{\tilde{P}}R_{\text{top}}^0[A[\lambda^{-1}] \simeq \varepsilon_{\tilde{P}}R_{A[1/p]}^0[\lambda^{-1}] \simeq A[\lambda^{-1}].$$

#### 4.2. Classical automorphic forms

Now we will globalize the above discussion to the whole Shimura variety $Sh$ (or its the $\mu$-ordinary locus). We define the $G$-torsor

$$
\mathcal{G}^\Sigma := \text{Isom}(\mathcal{O}_{Sh^\Sigma} \otimes_G \Lambda, t), \left( e^*H^1_{\text{dR}}(\mathcal{A}^\Sigma/Sh^\Sigma), t^*_{\text{dR}} \right),
$$

where $e: Sh^\Sigma \to \mathcal{A}^\Sigma$ is the unit section.

Similarly, the cocharacter $\nu_{\Sigma}^{-1}$ induces a filtration $\text{Fil}^iV_{\mathcal{O}_p}$:

$$0 = \text{Fil}^0V_{\mathcal{O}_p} \subset \text{Fil}^1V_{\mathcal{O}_p} \subset \text{Fil}^2V_{\mathcal{O}_p} = V_{\mathcal{O}_p}.$$ 

Then $(P_V)_{\mathcal{O}_p}$ is exactly the stabilizer in $GSp(V, \psi)_{\mathcal{O}_p}$ of this filtration. We define the $P$-torsor as the sub-torsor of $\mathcal{G}^\Sigma$

$$
\mathcal{P}^\Sigma \subset \mathcal{G}^\Sigma
$$

which preserves the filtrations on $\mathcal{O}_{Sh^\Sigma} \otimes_G V(\mathbb{Z})$ (induced from $V$) and on $H^1_{\text{dR}}(\mathcal{A}^\Sigma/Sh^\Sigma)$ (the Hodge filtration). Then we define the $L$-torsor to be the quotient

$$
\mathcal{L}^\Sigma := \mathcal{P}^\Sigma/U.
$$

For any character $\lambda \in X^*(\tilde{T}_{\tilde{P}})$, we write

$$
\mathcal{V}^\Sigma_\lambda := \mathcal{L}^\Sigma \times^L R_{\mathcal{O}_p}[\lambda^{-1}]
$$

for the contracted product, which is a quasi-coherent sheaf on $Sh^\Sigma$. We then define $\mathcal{V}_\lambda$ for the restriction of $\mathcal{V}^\Sigma_\lambda$ from $Sh^\Sigma$ to $Sh$ and $\mathcal{V}_\lambda^\Sigma(-C^\Sigma)$ for the extension of $\mathcal{V}_\lambda$ by zero back to the compactification $Sh^\Sigma$.

**Definition 4.12.** For any $O_p$-algebra $A$, we call $H^0(Sh_1^\Sigma, \mathcal{V}^\Sigma_\lambda)$ the space of modular forms on $Sh^\Sigma$ of weight $\lambda$, of level $K$, of coefficients in $A$, and call $H^0(Sh_1^\Sigma, \mathcal{V}^\Sigma_\lambda(-C^\Sigma))$ the space of cuspidal modular forms on $Sh^\Sigma$ of weight $\lambda$, of level $K$, of coefficients in $A$.

**Remark 4.13.** By the definition of the Hasse invariant $H$, we know that $H \in H^0(Sh_1, \omega_{Sh}^\otimes N_\mathcal{O}_G)$ is a modular form of weight $N_G\lambda_G$. 

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4.3. \textit{p}-adic automorphic forms. In this subsection, we define the space of \(p\)-adic automorphic forms. For any integer \(k \geq 0\) and any \(\mathbb{W}\)-scheme \(Y\), we write \(Y_k := Y \times_{\mathbb{W}} \mathbb{W}_k\) for the base change. We fix a point \(x_0 \in \text{Sh}_1^\mu\) of characteristic \(p\) in the (non-compactified) \(\mu\)-ordinary locus and denote by \(A_{x_0}[p^\infty]\) the \(p\)-divisible group associated to \(A_{x_0}\).

Recall the notion of \(p\)-divisible groups with \(G\)-structure as in [SZ16, Definition 3.1]: for a \(p\)-divisible group \(D\) over a ring \(R\), we write \(\mathbb{D}(D)\) for the contravariant Dieudonné crystal associated to \(D\). Then a \(p\)-divisible group with \(G\)-structure is the data of a \(p\)-divisible group \(D\) over \(\overline{\mathbb{F}}_p\) and tensors \(s_0 = (s_\ell, 0) \subset \mathbb{D}(D)(\mathbb{W})\) such that there exists a finite free \(\mathbb{Z}_p\)-module \(V\) with a \(\mathbb{W}\)-linear isomorphism \(V \otimes_{\mathbb{Z}_p} \mathbb{W} \simeq \mathbb{D}(D)(\mathbb{W})\) and the stabilizer in \(\text{GL}(V)\) of the images of the tensors \(s_0\) in \(V^\infty\) via the above isomorphism is exactly the group \(G\). By [SZ16 Theorem 5.5], for any such point \(x_0\), there is a unique lifting \(\tilde{x}_0 \in \text{Sh}(\mathbb{W})\) such that the action of the subgroup \(I_{x_0} \subset \text{Aut}_{\mathbb{Q}}(A_{x_0})\) on \(A_{x_0}\) lifts to \(A_{\tilde{x}_0}\). Here \(I_{x_0}\) consists of those automorphisms fixing the tensors \(t_{\ell, \ell, x_0}\) for all \(\ell \neq p\) (cf. [SZ16 §5.4]). We call \(A_{\tilde{x}_0}\) the canonical lift of \(A_{x_0}\). For any other point \(y_0 \in \text{Sh}_0^\mu\), we have an isomorphism of the canonical lifts \(A_{\tilde{x}_0}[p^\infty] \simeq A_{y_0}[p^\infty]\). Indeed, we have an isomorphism of \(p\)-divisible groups \(A_{x_0}[p^\infty] \simeq A_{y_0}[p^\infty]\) ([SZ16 Proposition 5.4]). Moreover, by Grothendieck-Messing theory, lifts of \(A_{x_0}[p^\infty]\) to \(\mathbb{W}\) correspond to lifts of the natural filtration \(\mathbb{F}_p\) to \(\mathbb{W}\) on \(\mathbb{D}(A_{x_0}[p^\infty]) \otimes_{\mathbb{Z}_p} \mathbb{F}_p\) induced by the mod \(p\) cocharacter \(\mu\) (mod \(p\)) to \(\mathbb{D}(A_{x_0}[p^\infty])\). Here \(\mu \in X_s(T)\) is the dominant representative of the conjugacy class of \(\nu_{x_0}^{-1}\) in \(X_s(T)\) (independent of \(x\)). In particular, the canonical lift \(A_{\tilde{x}_0}[p^\infty]\) corresponds to the filtration on \(\mathbb{D}(A_{x_0}[p^\infty])\) given by the cocharacter \(\mu\). By comparing the filtrations on both sides \(\mathbb{D}(A_{x_0}[p^\infty]) \simeq \mathbb{D}(A_{y_0}[p^\infty])\), we see that \(A_{\tilde{x}_0}[p^\infty]\) is isomorphic to \(A_{y_0}[p^\infty]\). We define \(p\)-divisible groups

\[\tilde{B}T := A_{\tilde{x}_0}[p^\infty], \quad BT := A_{x_0}[p^\infty].\]

We have a slope decomposition \(BT = \prod_{i=1}^r BT_i\) with each \(BT_i\) isoclinic of slope \(\lambda_i\) and \(1 \geq \lambda_1 > \lambda_2 > \cdots > \lambda_r \geq 0\). This slope filtration lifts to \(\tilde{B}T\):

\[0 = \tilde{B}T_0 \subset \tilde{B}T_1 \subset \tilde{B}T_2 \subset \cdots \subset \tilde{B}T_r = \tilde{B}T\]

such that each graded piece \(\text{gr}_i(\tilde{B}T) = \tilde{B}T_i / \tilde{B}T_{i-1}\) lifts \(BT_i\) from \(\mathbb{F}_p\) to \(\mathbb{W}\).

We consider then the construction of Igusa towers. Set \(r_0 = \lceil \frac{1 + \sqrt{5}}{2} \rceil\). For any \(x \in \text{Sh}_1^\mu\) we consider an isomorphism

\[\varphi : \prod_{i=1}^{r_0} A_x[p^\infty]i \simeq \prod_{i=1}^{r_0} A_{x_0}[p^\infty]i,\]

which, if \(r\) is an odd integer, respects the polarization at the components \(A_x[p^\infty]r_0\) and \(A_{x_0}[p^\infty]r_0\). Then one can extend \(\varphi\) in a unique way to an isomorphism

\[\varphi : \prod_{i=1}^r A_x[p^\infty]i \simeq \prod_{i=1}^r A_{x_0}[p^\infty]i,\]

respecting the polarizations on both sides. Now for a cuspidal label representative \(W\) of rank \(t\) of a component of the boundary of \(\text{Sh}_1^\mu\) of rank \(t\) which intersects non-trivially with the closure of a lifting of \(x\) from \(\text{Sh}_1^\mu\) to \(\text{Sh}_1^\mu\), by [Mad12 §3.2.1], we have an extension of the universal abelian scheme \(\mathcal{A}_W\) to a semi-abelian scheme \(\mathcal{G}\)

\[0 \to \mathbb{G}_m \otimes W \to \mathcal{G} \to \mathcal{A}_W \to 0\]

as well as their \(p\)-divisible groups

\[0 \to \mu^t \to \mathcal{G}[p^\infty] \to \mathcal{A}_W[p^\infty] \to 0.\]

By construction of the toroidal compactification for the integral Siegel Shimura variety \(\text{Sh}(\text{GSp}(V, \psi), S^\pm, K_V)\), we see that the difference between \(\mathcal{A}_W[p^\infty]\) and \(A_{x_0}[p^\infty]\) is \(\mu^t[p^\infty]\) and \((\mathbb{Q}_p / \mathbb{Z}_p)^t\). Thus we see that for any \(y \in W\), any isomorphism \(\varphi\) as above extends in a unique way to an isomorphism

\[(7) \quad \varphi_W : (\mathcal{G}_y[p^\infty])^0 \simeq \prod_{i=1}^{r-1} A_{x_0}[p^\infty]i,\]

where \((\mathcal{G}_y[p^\infty])^0\) denotes the connected component of \(\mathcal{G}_y[p^\infty]\). Translating this into the language of Dieudonné crystals, we have
**Proposition 4.14.** Fix a point $x \in Sh^\mu_1$ and consider an isomorphism $\phi : \prod_{i=1}^{r_0} A_x[p^\infty]_i \simeq \prod_{i=1}^{r_0} A_{x_0}[p^\infty]_i$. We assume moreover that for $r$ odd, $\phi$ respects the polarization at the components $A_x[p^\infty]_r$ and $A_{x_0}[p^\infty]_r$. Then for any $W$ as above and any point $y \in W$, $\phi$ extends in a unique way to an isomorphism of Dieudonné crystals

$$\mathbb{D}(\phi_W)(\mathbb{W}) : \mathbb{D}((G_y[p^\infty])^0) \simeq \mathbb{D}(\prod_{i=1}^{r-1} A_{x_0}[p^\infty]_i)$$

Next we discuss the automorphism group of the $p$-divisible group $A_{x_0}[p^\infty]$. We define an algebraic subgroup $L_\mu \subset G_{Q_p}$ over $\mathbb{Q}_p$ as follows: for any $G_{Q_p}$-algebra $R$,

$$L_\mu(R) := \{g \in G(R \otimes_{\mathbb{Q}_p} \mathbb{W}) \mid g^{-1} \sigma(\mu(p)) \sigma(g) = \sigma(\mu(p))\}.$$ 

It is known that $L_\mu$ has the same rank as $G_{Q_p}$ (cf. [SZ16, §§3.3 and 5.4] and [Kis16, Corollary 2.1.7]). By [SZ16, §3.3], we know also that $L_\mu(Q_p)$ is the set of automorphisms of the $p$-divisible group $A_{x_0}[p^\infty]$ in the isogeny category, fixing the tensors $\mu$. Next we discuss the integral structure of $L_\mu$ over $\mathbb{Z}_p$. We know that $L_\mu$ is an inner form of the centralizer $L_{Q_p} = \text{Cent}_{G}(\tilde{\mathfrak{p}})$ in $G_{Q_p}$ of the cocharacter $\tilde{\mathfrak{p}}$ (by definition of $L_\mu$ or see [Kot97, §4.3]). Since we have fixed a representative of $\mu$ which is defined over $\mathbb{Z}_p$, the $\mathbb{Z}_p$-model $L$ of $L_{Q_p}$ determines a $\mathbb{Z}_p$-model of $L_\mu$ via the inner twist between $L_{Q_p}$ and $L_\mu$. We will fix such a $\mathbb{Z}_p$-model of $L_\mu$ and denote this model by the same letter $L_\mu$, when no confusion is possible. Then one sees immediately (using the Dieudonné crystals associated to $A_{x_0}[p^\infty]$) that $L_\mu(\mathbb{Z}_p)$, resp., $L_\mu(\mathbb{Z}_p/p^m)$, is the set of automorphisms of $A_{x_0}[p^\infty]$, resp., $A_{x_0}[p^m]$ (induced from an automorphism of $A_{x_0}[p^\infty]$) fixing the tensors $\mu$. Lastly it is known that $L_\mu(Q_p)$ respects the slope filtration on $\mathbb{D}(A_{x_0}[p^\infty])$ induced by $\mu$ (cf. the proof of [SZ16, Theorem 3.5]).

From the above discussion we get the following:

**Definition-Proposition 4.15.** We write

$$\text{Ig} := \text{Isom}_{Sh_{\Sigma,\mu}}((A[p^\infty], t), (\overline{BT}, t_{\mathbb{Z}_p}))$$

for the $L_\mu(\mathbb{Z}_p)$-torsor over the $\mu$-ordinary locus $Sh^{\Sigma,\mu}$, consisting of isomorphisms

$$\varphi : A_y[p^\infty] \simeq \overline{BT}$$

as above such that the induced isomorphisms on the $\mathbb{W}$-points of the Dieudonné crystals

$$\mathbb{D}(\varphi)(\mathbb{W}) : \mathbb{D}(A_y[p^\infty])(\mathbb{W}) \to \mathbb{D}(\overline{BT})(\mathbb{W})$$

respect the Hodge tensors $t$ and $t_{x_0}$ on both sides (the associated map on the corresponding Dieudonné crystals, by Proposition 4.14 extends uniquely to an isomorphism on the boundary). Similarly for any $m \geq 0$, we have an $L_\mu(\mathbb{Z}_p)$-torsor

$$\text{Ig}_m \subset \text{Isom}_{Sh_{\Sigma,\mu}}((A[p^\infty], t), (\overline{BT}, t_{\mathbb{Z}_p}))$$

over $Sh^\Sigma_{\mu}$, which is the pull-back of $\text{Ig}$ along the closed embedding $Sh^\Sigma_{\mu} \hookrightarrow Sh^{\Sigma,\mu}$. Moreover for any integer $n \geq 1$, we consider the $L_\mu(\mathbb{Z}_p/p^n)$-torsor

$$\text{Ig}_{m,n} \subset \text{Isom}_{Sh_{\Sigma,\mu}}(A[p^n], \overline{BT}[p^n])$$

consisting of isomorphism $\varphi_n : \prod_{i=1}^{r_0} \text{gr}_i(A_y[p^n]) \simeq \prod_{i=1}^{r_0} \text{gr}_i(\overline{BT})$ such that there is an isomorphism $\varphi : \prod_{i=1}^{r_0} \text{gr}_i(A_y[p^\infty]) \simeq \prod_{i=1}^{r_0} \text{gr}_i(\overline{BT})$ with reduction $\varphi \equiv \varphi_n(\text{mod } p^n)$ and $\mathbb{D}(\varphi)(\mathbb{W})$ respects the Hodge tensors $t$ and $t_{x_0}$ on both sides.

We write $\text{Ig}_\infty$ to be the formal completion of $\text{Ig}$ along the special fibre $\text{Ig}_0$. Similarly we write $Sh^\Sigma_{\mu}$ to be the formal completion of $Sh^{\Sigma,\mu}$ along $Sh^\Sigma_0$. Let $\tilde{P}$ be as above.

**Definition 4.16.** (1) We write

$$\mathbb{V} : = H^0(\text{Ig}, \mathcal{O}_{\text{Ig}})$$

$$\mathbb{V}_m : = H^0(\text{Ig}_m, \mathcal{O}_{\text{Ig}_m})$$

$$\mathbb{V}_m,n : = H^0(\text{Ig}_{m,n}, \mathcal{O}_{\text{Ig}_{m,n}})$$

$$\mathbb{V}_\infty : = \varinjlim_m \mathbb{V}_m.$$
(2) For any $\tilde{P}$ as above, we write $\mathcal{V}^\text{der}_\infty$ for the subspace of $\mathcal{V}_\infty$ consisting of regular functions which are invariant under the action of $\tilde{P}_L^\text{der}$. This is the space of $p$-adic automorphic forms for the parabolic subgroup $\tilde{P} \subset G$ and of level $K$.

(3) For any character $\kappa \in X^*(\tilde{T}_P)$, we write $\mathcal{V}^\text{der}_\infty[\kappa]$ to be the subspace of $\mathcal{V}^\text{der}_\infty$ on which $\tilde{T}_P$ acts by the character $\kappa$. This is the space of $p$-adic automorphic forms for $\tilde{P}$ of level $K$ and of character $\kappa$.

(4) We write $\mathcal{V}^\text{der}_{\text{cusp}, \infty}[\kappa]$ for the subspace of $\mathcal{V}^\text{der}_\infty$ consisting of regular functions which vanish at the cusps $C^\Sigma_c$. This is the space of cuspidal $p$-adic automorphic forms for $\tilde{P}$ of level $K$. Similarly for any $\kappa \in X^*(\tilde{T}_P)$ we have the space of cuspidal $p$-adic automorphic forms $\mathcal{V}^\text{der}_{\text{cusp}, \infty}[\kappa]$ for $\tilde{P}$ of level $K$ and of character $\kappa$.

4.4. Hodge-Tate map. In this subsection, we define the Hodge-Tate map, which relates the Igusa tower $(Ig_{gm})_{m \geq 1}$ to the torsors $\Omega^\Sigma$. For the torsors $\mathcal{V}^\Sigma, \mathcal{L}^\Sigma$, we write $\mathcal{V}^\Sigma_{\mu}, \mathcal{L}^\Sigma_{\mu}$ for the restriction of these torsors from $Sh^\Sigma$ to the $\mu$-ordinary locus $Sh^\Sigma_{\mu}$ ($? = 0, 1, \ldots, 0$). Now for any isomorphism $\varphi: A[p^\infty] \to A_{\infty}[p^\infty]$ over $Sh^\Sigma_{\mu}$ in $Ig_{gm}$, we have an induced isomorphism

$$HT_m(\varphi): e^*H^1_{dR}(A/Sh^\Sigma_{\mu}) \to e^*H^1_{dR}(A_{\infty}/Sh^\Sigma_{\mu})$$

respecting the Hodge tensors $t, t_{dR}$ and the Hodge filtrations on both sides. Then the above process gives us the Hodge-Tate map

**Definition 4.17.** We define a map

$$HT_m: Ig_m \to \Omega^\Sigma_{\mu}, \varphi \mapsto HT_m(\varphi).$$

For a parabolic subgroup $\tilde{P} \subset P$, we write $HT^\tilde{P}_m$ to be the composition

$$HT^\tilde{P}_m: Ig_m \xrightarrow{HT_m} \mathcal{L}^\Sigma_{\mu} \to \mathcal{L}^\tilde{P}_m.$$

Moreover, we write $HT_{\infty}: Ig_{\infty} \to \mathcal{L}^\Sigma_{\mu}$ for the inverse limit of the maps $(HT_m)_m$ and similarly for $HT^\tilde{P}_{\infty}$.

Note that a priori, $Ig_m$ is an $L_{\mu}(\mathbb{Z}_p/p^m)$-torsor and $\Omega^\Sigma_{\mu}$ is an $L(\mathbb{Z}_p/p^m)$-torsor. However by the relation between $L_{\mu}$ and $L$ (cf. preceding Definition-Proposition 4.15), we see that the Hodge-Tate map $HT_m$ is equivariant for the actions of $L_{\mu}(\mathbb{Z}_p/p^m)$ and $L(\mathbb{Z}_p/p^m)$ on both sides.

Using this map, we can associate to a classical modular form $f$ a $p$-adic modular form as follows: for any $f \in H^0(Sh^\Sigma_{\lambda}, V^\Sigma_{\lambda})$ with $A$ an $O_p/p^m$-algebra, by definition $f$ is a global section of $\mathcal{V}^\Sigma_{\mu}$ and thus $B_L$ acts on $f$ by the character $\lambda$. As a result $HT^\Sigma_m(f)$ is a global section of the structural sheaf $\mathcal{O}_{Ig_{gm}}$, on which $B_L$ acts by the character same $\lambda$. In summary we have the following morphism

$$HT^\Sigma_m: H^0(Sh^\Sigma_{\lambda}, V^\Sigma_{\lambda}) \to \mathcal{V}^\text{der}_m[\lambda^{-1}].$$

Moreover, if $f$ is cuspidal, so is $HT^\Sigma_m(f)$.

Consider the quotients $\mathcal{L}^\Sigma_{\mu}/\mathcal{P}^\text{der}_L$ and $\mathcal{L}^\Sigma_{\mu}/\mathcal{P}_L$ of the $\mathcal{P}^\text{der}_L$-torsor $\mathcal{L}^\Sigma_{\mu}$ over $Sh^\Sigma_{\mu}$ and the quotient $Ig_{gm}/\mathcal{P}_L(O_p)$ of $Ig_{gm}$ by $\mathcal{P}_L(O_p)$. Write the fibre product

$$\tilde{Ig}_{gm} := (Ig_{gm}/\mathcal{P}_L(O_p)) \times_{\mathcal{L}^\Sigma_{\mu}/\mathcal{P}_L} \mathcal{L}^\Sigma_{\mu}/\mathcal{P}^\text{der}_L$$

and consider the projection to the first factor $pr: \tilde{Ig}_{gm} \to Ig_{gm}/\mathcal{P}_L(O_p)$. Note that $\tilde{T}_P$ acts on this torsor by its natural action on $\mathcal{L}^\Sigma_{\mu}/\mathcal{P}^\text{der}_L$. We have the following result which relates the Igusa tower and the $L$-torsor (cf. [Pil12 Proposition 4.1]):

**Proposition 4.18.** For any $\lambda \in X^*(\tilde{T}_P)$, the natural morphism

$$\tilde{HT}^\Sigma_P : Ig_{gm}/\mathcal{P}^\text{der}_L(O_p) \to \tilde{Ig}_{gm}$$

induces the following isomorphism:

$$H^0(Ig_{gm}/\mathcal{P}_L(O_p), pr_*(\mathcal{O}_{\tilde{Ig}_{gm}}[\lambda^{-1}])) \simeq \mathcal{V}^\text{der}_m[\lambda^{-1}].$$
Here $O_{\widetilde{\mathfrak{P}}_m}[\lambda^{-1}]$ denotes the subsheaf of $O_{\widetilde{\mathfrak{P}}_m}$ on which $\widetilde{T}_P$ acts by the character $\lambda$.

**Proof.** By definition, we have

$$\forall \gamma \in \widetilde{\mathfrak{P}}_m^{\operatorname{der}}[\lambda^{-1}] = H^0(\widetilde{\mathfrak{P}}_m/\tilde{\mathfrak{P}}_m^{\operatorname{der}}(O_p), \mathcal{O}_{\mathfrak{P}_m/\tilde{\mathfrak{P}}_m^{\operatorname{der}}(O_p)}[\lambda^{-1}]).$$

Note that $\mathfrak{P}_m/\tilde{\mathfrak{P}}_m^{\operatorname{der}}(O_p)$ is a pro-finite étale covering of $\mathfrak{P}_m/\tilde{\mathfrak{P}}_m(O_p)$ of group schemes over $O_p$ (the first being a constant group scheme). Note that $\widetilde{T}_P$ is quasi-split over $O_p$, we see that étale locally over $\mathfrak{P}_m/\tilde{\mathfrak{P}}_m(O_p)$, $\mathfrak{P}^e$ sends $H^0(\widetilde{T}_P, O_{\widetilde{T}_P}[\lambda^{-1}])$ bijectively to $H^0(\widetilde{T}_P(O_p), O_{\widetilde{T}_P}(O_p)[\lambda^{-1}])$ (both are in fact of rank 1 over $O_p$).

5. Hecke operators

5.1. Parahoric Hecke algebra. We introduce first the abstract Hecke algebras associated to the parabolic subgroups $\widetilde{P}$ that we are going to use. We write $\mathbb{Z}[\widetilde{T}^+(P)/(E_p)]$ for the commutative algebra over $\mathbb{Z}$ generated by the elements in $\widetilde{T}^+(P)/(E_p)$.

For any integer $n \geq 1$, denote by $I_{\widetilde{P}}(n)$ for the subset of $P(O_p)$ consisting of elements $g$ such that $g \bmod p^n = \widetilde{P}(O_p/p^n)$. Similarly, denote by $I_{S\widetilde{P}}(n)$ the subset of $P(O_p)$ consisting of $g$ such that $g \bmod p^n \in S\widetilde{P}(O_p/p^n)$. Write

$$C(G(E_p)/I_{\widetilde{P}}(n), \mathbb{Z}), \text{ resp., } C(G(E_p)/I_{S\widetilde{P}}(n), \mathbb{Z})$$

for the set of compact support functions from the double coset $G(E_p)/I_{\widetilde{P}}(n)$, resp., $G(E_p)/I_{S\widetilde{P}}(n)$, to $\mathbb{Z}$. These two sets have natural $\mathbb{Z}$-algebra structures whose product is given by the involution. We then have the canonical projections

$$\pi_{\widetilde{P}}: G(E_p) \rightarrow G(E_p)/I_{\widetilde{P}}(n),$$
$$\pi_{S\widetilde{P}}: G(E_p) \rightarrow G(E_p)/I_{S\widetilde{P}}(n).$$

**Proposition 5.1.** The maps of $\mathbb{Z}$-algebras given by

$$\mathbb{Z}[\widetilde{T}^+(P)/(E_p)] \rightarrow C(G(E_p)/I_{\widetilde{P}}(n), \mathbb{Z}), \quad \epsilon \in \widetilde{T}^+(P)/(E_p) \mapsto 1_{\pi_{\widetilde{P}}(\epsilon)};$$

$$\mathbb{Z}[\widetilde{T}^+(P)/(E_p)] \rightarrow C(G(E_p)/I_{S\widetilde{P}}(n), \mathbb{Z}), \quad \epsilon \in \widetilde{T}^+(P)/(E_p) \mapsto 1_{\pi_{S\widetilde{P}}(\epsilon)}$$

are well-defined and are both injective. Here $1_X$ denotes the characteristic function of a subset $X$.

**Proof.** We treat the first case, the second being similar. We already know that the proposition is true for the case $G = \operatorname{GSp}(V, \psi)$ ([Hid02], §3.6). The proof for the general case is quite similar to this case as in [Hid05] §2. More precisely, write temporarily $H = I_{\widetilde{P}}(n)$, then for each $\epsilon \in \widetilde{T}^+(P)/(E_p)$, we can choose a set of representatives $X_\epsilon \subset H$ such that we have a decomposition

$$H = \bigsqcup_{\alpha \in X_\epsilon} H \epsilon \alpha.$$ 

We can choose $X_\epsilon$ as follows: let $\widetilde{T}^+(P)$ act on the parabolic subgroup $\widetilde{P}(O_p)$ of $G(O_p)$ by inverse conjugation, thus we see that $\epsilon^{-1} \widetilde{P}(O_p) \epsilon \subset \widetilde{P}(O_p)$. Moreover, the following natural inclusion is in fact a bijection:

$$H/(\epsilon^{-1} H \cap H) \hookrightarrow \widetilde{P}(O_p)/\epsilon^{-1} \widetilde{P}(O_p) \epsilon.$$

Then we just take a set $X_\epsilon$ of representatives in $\widetilde{P}(O_p)$ for the quotient $H/(\epsilon^{-1} H \cap H)$. Note that for any two $\epsilon, \epsilon' \in \widetilde{T}^+(P)/(E_p)$,

$$\sharp X_{\epsilon \epsilon'} = [H : (\epsilon \epsilon')^{-1} H (\epsilon \epsilon') \cap H] = [\widetilde{P}(O_p) : (\epsilon \epsilon')^{-1} \widetilde{P}(O_p)(\epsilon \epsilon')]$$
$$= [\widetilde{P}(O_p) : \epsilon^{-1} \widetilde{P}(O_p) \epsilon \cdot (\epsilon \epsilon')^{-1} \widetilde{P}(O_p) \epsilon : (\epsilon \epsilon')^{-1} \widetilde{P}(O_p) \epsilon]$$
$$= \sharp X_{\epsilon} \cdot \sharp X_{\epsilon'}$$.
Combined with the inclusion \( H e l' H \subset H e H e' H \), this shows that \( H e l' H = H e H \cdot H e' H = H e' H \cdot H e H \).

This shows that the map sending \( \epsilon \) to \( 1_{\pi_p(\epsilon)} \) is well-defined and is indeed a morphism of \( \mathbb{Z} \)-algebras. Next we show that it is injective. Note that we have inclusions \( G(E_p) \rightarrow \text{GSp}(V, \psi)(E_p), \tilde{T}_p^+(E_p) \rightarrow \tilde{T}_{LV}^+(E_p) \), the latter of which induces an inclusion \( \mathbb{Z}[\tilde{T}_p^+(E_p)] \rightarrow \mathbb{Z}[\tilde{T}_{LV}^+(E_p)] \). The image \( \text{Im}(\pi_p) \) is a \( \mathbb{Z} \)-algebra generated by the characteristic functions of the double cosets \( \pi_p(\epsilon) \) for all \( \epsilon \in \tilde{T}_p^+(E_p) \). Now consider the following map

\[
\varphi: \text{Im}(\pi_p) \rightarrow \text{Im}(\pi_p), \quad \pi_p(\epsilon) \mapsto \pi_p(\epsilon), \quad \forall \epsilon \in \tilde{T}_p^+(E_p);
\]

It is clear that \( \varphi \) is injective. Moreover, we have a commutative diagram (the isomorphism in the bottom line is [Hid95, Proposition 2.1]):

\[
\begin{array}{c}
\mathbb{Z}[\tilde{T}_p^+(E_p)] \longrightarrow \text{Im}(\pi_p) \\
\downarrow \quad \downarrow \\
\mathbb{Z}[\tilde{T}_{LV}^+(E_p)] \longrightarrow \text{Im}(\pi_{p'}).
\end{array}
\]

From this we see that the top line in the above diagram is injective. \( \square \)

We will identify \( \mathbb{Z}[\tilde{T}_p^+(E_p)] \) with its images inside \( C(G(E_p)/I_p(n), \mathbb{Z}) \) and \( C(G(E_p)/I_{Sp}(n), \mathbb{Z}) \).

5.2. Hecke operators on the Igusa tower. By the discussion after Proposition 4.5, we will also need to modify the integral structure on the space of \( p \)-adic modular forms. We write \( \text{Sh}_m^{\Sigma, \mu} \) for the rigid space associated to the formal scheme \( \text{Sh}_m^{\Sigma, \mu} \). Similarly, write \( \text{Ig}_{m}^{\Sigma} \) for the rigid space associated to \( \text{Ig}_{m}^{\Sigma} \) \( (\Sigma = G, P, L) \). We fix a character \( \lambda \in X_{\text{der}}(\tilde{T}_p) \). Recall we have a sheaf \( \mathcal{V}_{\Sigma}^{\lambda}(-C^2) \) over \( \text{Sh}_m^{\Sigma} \). Then we write \( \mathcal{R}_{m}[\lambda^{-1}] \) for the restriction of \( \mathcal{V}_{\Sigma}^{\lambda}(-C^2) \) to \( \text{Sh}_m^{\Sigma, \mu} \), and \( \mathcal{R}_{\text{rig}}[\lambda^{-1}] \) for the rigid sheaf associated to the formal limit \( \mathcal{R}_{\infty}[\lambda^{-1}] \) of the projective system \( (\mathcal{R}_{m}[\lambda^{-1}])_{m \geq 1} \). Here \( [\lambda^{-1}] \) means the subsheaf on which \( \tilde{T}_p \) acts by the character \( \lambda \). Then we have the relation

\[
H^0(\text{Sh}_m^{\Sigma, \mu}, \mathcal{R}_{\text{rig}}[\lambda^{-1}]) = H^0(\text{Sh}_m^{\Sigma, \mu}, \mathcal{R}_{\infty}[\lambda^{-1}]) \otimes_{\mathcal{O}_p} E_p.
\]

On the other hand, we have a morphism \( \varphi': \text{Ig}_m/\tilde{P}_{L}^{\text{der}}(\mathcal{O}_p) \rightarrow \text{Sh}_m^{\Sigma, \mu} \) and thus a sheaf

\[
\mathcal{R}_{m}^{\text{top}}[\lambda^{-1}] := \varphi'_{*}(\mathcal{O}_{\text{Ig}_m/\tilde{P}_{L}^{\text{der}}(\mathcal{O}_p)[\lambda^{-1}])
\]

over \( \text{Sh}_m^{\Sigma, \mu} \) and the rigid sheaf \( \mathcal{R}_{\text{rig}}^{\text{top}}[\lambda^{-1}] \) associated to the formal sheaf \( \mathcal{R}_{\infty}^{\text{top}}[\lambda^{-1}] \). Again we have a similar identity as above:

\[
H^0(\text{Sh}_m^{\Sigma, \mu}, \mathcal{R}_{\text{rig}}^{\text{top}}[\lambda^{-1}]) = H^0(\text{Sh}_m^{\Sigma, \mu}, \mathcal{R}_{\infty}^{\text{top}}[\lambda^{-1}]) \otimes_{\mathcal{O}_p} E_p.
\]

The natural morphism \( \text{Ig}_m/\tilde{P}_{L}^{\text{der}}(\mathcal{O}_p) \rightarrow \text{Ig}_m/\tilde{P}_L(\mathcal{O}_p) \) gives rise to a morphism \( \mathcal{R}_{\gamma}[\lambda^{-1}] \rightarrow \mathcal{R}_{\gamma}^{\text{top}}[\lambda^{-1}] \) of sheaves over \( \text{Sh}_m^{\Sigma, \mu} \) \( (\Sigma = m, \infty, \text{rig}) \). For an affine formal open subset \( \text{Spf}(A) \) of \( \text{Sh}_m^{\Sigma, \mu} \) and the corresponding affinoid open subset \( \text{Sp}(A[\frac{1}{p}]) \) of the rigid space \( \text{Sh}_m^{\Sigma, \mu} \), we have the following commutative diagram of functions over these open subsets

\[
\begin{array}{c}
\mathcal{R}_{\infty}[\lambda^{-1}](A) \xrightarrow{\text{ev}_{\mathcal{O}_p}} \mathcal{R}_{\text{rig}}[\lambda^{-1}](A[\frac{1}{p}]) \\
\downarrow \quad \downarrow \\
\mathcal{R}_{\infty}^{\text{top}}[\lambda^{-1}](A) \xrightarrow{\text{ev}_{\mathcal{O}_p}} \mathcal{R}_{\text{rig}}^{\text{top}}[\lambda^{-1}](A[\frac{1}{p}]).
\end{array}
\]

As in the previous section, we put

\[
\mathcal{R}_{\infty}[\lambda^{-1}](A) := \text{ev}_{E_p}^{-1}(\text{Im}(t^{\text{top}}))
\]

which is a lattice in \( \mathcal{R}_{\text{rig}}[\lambda^{-1}](A[\frac{1}{p}]) \) containing the lattice \( \mathcal{R}_{\infty}[\lambda^{-1}](A) \). This is analogous to the construction of \( \mathcal{R}_B[\lambda^{-1}] \) where \( B \) is a finite flat \( \mathcal{O}_p \)-algebra. Clearly, this construction is functorial in \( A \) and glues well, thus defines a new formal sheaf \( \mathcal{R}_{\infty}[\lambda^{-1}] \) on the formal scheme \( \text{Sh}_m^{\Sigma, \mu} \).
Next we review the construction of the correspondence on the integral Siegel Shimura variety
\[ \text{Sh}(V) := \text{Sh}(GSp(V, \psi), S^\pm, K_V) \]
(here \( K_V \) is a compact open subgroup of \( GSp(V, \psi)(\mathbb{A}_f) \) such that \( K_V \cap G(\mathbb{A}_f) = K \)) (cf. [PH12] §5.1.3). We fix an element \( \epsilon \in \mathbb{T}_P(E_p) \). Viewed as an element in \( GSp(V, \psi) \), we can write \( \epsilon = \text{diag}(\nu(\epsilon), \nu(\epsilon)^{-1}) \) with \( \epsilon \in L(E_p) \).

Then the correspondence \( \text{Ig}^{GSp(V, \psi)}_{\infty, n} / \mathbb{P}_L(\epsilon) \) over the \( \mu \)-ordinary locus of the (non-compactified) Shimura variety \( \text{Sh}(GSp(V, \psi), S^\pm, K_V)^{\infty}_{\infty} \) parameterizes the following data: for any \( \mathbb{Z}_p \)-algebra \( A \), \( \text{Ig}^{GSp(V, \psi)}_{\infty, n} / \mathbb{P}_L(\epsilon)(A) \) consists of quintuples \((A, \tilde{A}, \pi, \psi_n, \tilde{\psi}_n)\) where

1. \( A \) is a \( \mu \)-ordinary principally polarized abelian scheme over \( \text{Spec}(A) \) and a \( \mathbb{P}_V(\mathbb{O}_p/p^n) \)-coset of isomorphisms \( \psi_n : A_{x_0}[p^n] \cong \tilde{A}[p^n] \) such that there exists an isomorphism \( \psi_\infty : A_{x_0}[p^\infty] \cong \tilde{A}[p^\infty] \) with \( \psi_\infty \equiv \psi_n (\text{mod } p^n) \);
2. \( \tilde{A} \) is a \( \mu \)-ordinary principally polarized abelian scheme over \( \text{Spec}(A) \) and a \( \mathbb{P}_V(\mathbb{O}_p/p^n) \)-coset of isomorphisms \( \tilde{\psi}_n : A_{x_0}[p^n] \cong \tilde{A}[p^n] \) such that there exists an isomorphism \( \tilde{\psi}_\infty : A_{x_0}[p^\infty] \cong \tilde{A}[p^\infty] \) with \( \tilde{\psi}_\infty \equiv \tilde{\psi}_n (\text{mod } p^n) \);
3. \( \pi : A \to \tilde{A} \) is a \( p \)-isogeny of similitude factor \( \nu(\epsilon) \) of abelian schemes such that on the level of Dieudonné crystals, we have \( \mathbb{D}((\psi_\infty)^{-1} \circ \pi \circ \psi_\infty)(W) = \epsilon : \mathbb{D}(A_{x_0}[p^\infty])(W) \to \mathbb{D}(A_{x_0}[p^\infty])(W) \).

We emphasize that we parameterize only the \( \mu \)-ordinary abelian schemes. Now for \( \epsilon \in \mathbb{T}_P^\pm(E_p) \), we define the correspondence
\[ \text{Ig}^{\mu}_{\infty, n} / \mathbb{P}_L(\epsilon) \]
to be the pull-back of \( \text{Ig}^{GSp(V, \psi)}_{\infty, n} / \mathbb{P}_L(\epsilon) \) along the embedding \( \text{Sh}^\mu_{\infty} \hookrightarrow \text{Sh}(V)^{\mu}_{\infty} \). Clearly, as in the Siegel case, we have two natural projections
\[ \text{pr}_1, \text{pr}_2 : \text{Ig}^{\mu}_{\infty, n} / \mathbb{P}_L(\epsilon) \to \text{Ig}^{\mu}_{\infty, n} / \mathbb{P}_L, \]
where \( \text{pr}_1 \) takes the quintuple \((A, \tilde{A}, \pi, \psi_n, \tilde{\psi}_n)\) to \((A, \psi_n)\) and \( \text{pr}_2 \) takes it to \((\tilde{A}, \tilde{\psi}_n)\). Here
\[ \text{Ig}^{\mu}_{\infty, n} / \mathbb{P}_L := \text{Ig}^{\mu}_{\infty, n} / \mathbb{P}_L(\mathbb{O}_p/p^n) \]
is defined in the same manner as the quotient \( \text{Ig}^{\mu}_{\infty} / \mathbb{P}_L(\mathbb{O}_p) \). Similarly we have induced morphisms on the corresponding rigid spaces
\[ \text{pr}_{1, \text{rig}}, \text{pr}_{2, \text{rig}} : \text{Ig}^{\mu}_{\infty, n} / \mathbb{P}_L(\epsilon) \to \text{Ig}^{\mu}_{\infty, n} / \mathbb{P}_L. \]

Now over the correspondence \( \text{Ig}^{GSp(V, \psi)}_{\infty, n} / \mathbb{P}(\epsilon) \), we have two universal abelian schemes \( \mathcal{A}^{GSp(V, \psi)}, \tilde{\mathcal{A}}^{GSp(V, \psi)} \) and a \( p \)-isogeny
\[ \pi^{GSp(V, \psi)} = \pi^{GSp(V, \psi)}, \mathcal{A}^{GSp(V, \psi)} \to \tilde{\mathcal{A}}^{GSp(V, \psi)} \]
whose trivialization (via lifts \( \psi_\infty^{GSp(V, \psi)} \), resp., \( \tilde{\psi}_\infty^{GSp(V, \psi)} \) of \( \psi_\infty^{GSp(V, \psi)} \), resp. \( \tilde{\psi}_\infty^{GSp(V, \psi)} \) is \( \epsilon \). We denote by \( \mathcal{A}^G, \tilde{\mathcal{A}}^G, \psi^G, \tilde{\psi}^G \) the corresponding pull-backs along \( \text{Sh}^\mu \to \text{Sh}(V)^{\mu} \) and \( \pi^G = \pi^{G, \epsilon} : \mathcal{A}^G \to \tilde{\mathcal{A}}^G \). From this, we get a morphism
\[ (\pi^{G, \epsilon})^* : H^1_{\text{dR}}(\tilde{\mathcal{A}}^G/((\text{Ig}^{\mu}_{\infty, n} / \mathbb{P}(\epsilon)))) \to H^1_{\text{dR}}(\mathcal{A}^G/((\text{Ig}^{\mu}_{\infty, n} / \mathbb{P}(\epsilon))). \]

Recall that we have just constructed \( \mathcal{R}_\infty[\lambda^{-1}] \) over \( \text{Sh}^\mu_{\infty} \). Consider the morphism \( \text{Ig}^{\mu}_{m, n} / \mathbb{P}_L \to \text{Sh}^\mu_{m, n} \), and we denote the pull-backs along this morphism of the sheaf \( \mathcal{R}_m[\lambda^{-1}] \) by \( \mathcal{R}_m^{\mu}[\lambda^{-1}] \) and similarly we have \( \mathcal{R}_\infty^{\mu}[\lambda^{-1}], \mathcal{R}_\infty^{\mu}[\lambda^{-1}] \) etc.. Using the above morphism \((\pi^{G, \epsilon})^*\), we can construct a morphism
\[ (\pi^{G, \epsilon})^* : \text{pr}_{2, \text{rig}}^*(\mathcal{R}_m^{\mu}[\lambda^{-1}]) \to \text{pr}_{1, \text{rig}}^*(\mathcal{R}_m^{\mu}[\lambda^{-1}]) \]
as follows: consider an affinoid open subset \( \text{Sp}(A(\frac{1}{p})) \) of \( \text{Ig}^{\mu}_{\infty, n} / \mathbb{P}_L(\epsilon) \) associated to an affine formal open subset \( \text{Spf}(A) \) of \( \text{Ig}^{\mu}_{\infty, n} / \mathbb{P}_L(\epsilon) \). Then \( \text{pr}_{2, \text{rig}}(\mathcal{R}_m^{\mu}[\lambda^{-1}]) \) consists of rational functions on \( G(A(\frac{1}{p})) \) with values in \( A(\frac{1}{p}) \) invariant under the right action of the unipotent subgroup \( \tilde{U}(A(\frac{1}{p})) \) and on which the torus \( \mathbb{T}(A(\frac{1}{p})) \) acts by the character \( \lambda \) (? = 1, 2). Caution that the difference between \( \text{pr}_{1, \text{rig}}(\mathcal{R}_m^{\mu}[\lambda^{-1}]) \) and \( \text{pr}_{2, \text{rig}}(\mathcal{R}_m^{\mu}[\lambda^{-1}]) \) lies in how we identify the unipotent group \( \tilde{U}(A(\frac{1}{p})) \) as explained below. One can identify \( G(A(\frac{1}{p})) \) with the set of isomorphisms \( f : A_{x_0}[p^\infty] \to \tilde{A}[p^\infty] \)
over (or rather, restricted to) \( \text{Sp}(A(\frac{1}{p})) \), preserving the Hodge tensors on both sides. Note that by the definition of the correspondences \( \text{Ig}^{\mu}_{m, n} / \mathbb{P}_L(\epsilon) \), for each such \( f \), there is another \( \tilde{f} : A_{x_0}[p^\infty] \to \tilde{A}[p^\infty] \) over \( \text{Sp}(A(\frac{1}{p})) \), preserving the
Hodge tensors and moreover $\mathbb{D}(f^{-1} \circ \pi \circ f)(\mathcal{W}) = \epsilon^{-1}$ (which is necessarily unique). Then we define the morphism $(\pi^{G, \epsilon})^*$ on the affinoid open subset $\text{Sp}(A[\frac{1}{p}])$ by the formula
\[
(\pi^{G, \epsilon})^* : \text{pr}^*_2(\mathcal{R}_{\text{rig}}^{\text{Ig}}[\lambda^{-1}](A[\frac{1}{p}]) \to \text{pr}^*_1(\mathcal{R}_{\text{rig}}^{\text{Ig}}[\lambda^{-1}](A[\frac{1}{p}]),
\]
\[
F \mapsto (\mathbb{D}(f) \mapsto F(\mathbb{D}(\tilde{f}))).
\]
This process is clearly functorial in the affinoid open subset $\text{Sp}(A[\frac{1}{p}])$ and thus we can globalize the above process and obtain the morphism $(\pi^{G, \epsilon})^*$. In particular, if we fix a geometric point $\mathfrak{p} \in (\text{Ig}_{\infty, n}/\tilde{P}_L(\epsilon))(\mathbb{F}_p)$ and set $A$ to be the strict Henselization of $\text{Ig}_{\infty, n}/\tilde{P}_L(\epsilon)$ at the point $\mathfrak{p}$, then (cf. [Pil12 Lemme 5.1])

**Proposition 5.2.** For $\lambda \in X^*_\text{dr}(\tilde{T}_p)$ and $\mathfrak{p}$ as above, we have a commutative diagram with isomorphic vertical arrows:

\[
\begin{array}{ccc}
R_A[\lambda^{-1}] & \rightarrow & R_A[\lambda^{-1}]
\\
\downarrow & & \downarrow T_x
\\
\text{pr}^*_2(\mathcal{R}_{\text{rig}}^{\text{Ig}}[\lambda^{-1}](A)) & \rightarrow & \text{pr}^*_1(\mathcal{R}_{\text{rig}}^{\text{Ig}}[\lambda^{-1}](A[\frac{1}{p}]), (\pi^{G, \epsilon})^*
\\
F & \mapsto & \mathbb{D}(f) \mapsto F(\mathbb{D}(\tilde{f})).
\end{array}
\]

**Proof.** By the above discussion, we see easily that the vertical arrows are indeed isomorphisms. It remains to show that the second square is commutative. This follows from our definition of $(\pi^{G, \epsilon})^*$. Indeed, for any $f$ as in the definition of $(\pi^{G, \epsilon})^*$ (we identify $f$ and its Dieudonné crystal counterpart), passing to the contravariant Dieudonné crystals, we have an identity $\mathbb{D}(f) \circ \mathbb{D}(\pi) \circ \mathbb{D}(f)^{-1} = \epsilon^{-1}$. Moreover, an element $F \in R_A[\lambda^{-1}]$ is sent to an element $F_1 \in \text{pr}^*_1(\mathcal{R}_{\text{rig}}^{\text{Ig}}[\lambda^{-1}])$, resp., $F_2 \in \text{pr}^*_2(\mathcal{R}_{\text{rig}}^{\text{Ig}}[\lambda^{-1}])$ which is given by

\[
F_1(g \circ \mathbb{D}(\psi_C^{G, \epsilon})) = F(g), \text{ resp., } F_2(g \circ \mathbb{D}(\psi_C^{G})) = F(g), \forall g \in G(\mathbb{F}_p).
\]

Moreover, note that for $\mathbb{D}(\phi) = g \circ \mathbb{D}(\psi_C^{G, \epsilon})$, one has
\[
\mathbb{D}(\phi) = \epsilon \circ g \circ \epsilon^{-1} \circ \mathbb{D}(\psi_C^{G, \epsilon}).
\]

Thus we have
\[
((\pi^{G, \epsilon})^* F_2)(g \circ \mathbb{D}(\psi_C^{G, \epsilon})) = F_2(\epsilon \epsilon^{-1} \mathbb{D}(\psi_C^{G, \epsilon})) = F(\epsilon \epsilon^{-1}) = (T_x F)(g) = (T_x F)_1(g \circ \mathbb{D}(\psi_C^{G, \epsilon})).
\]

From this we conclude that $(\pi^{G, \epsilon})^* F_2 = (T_x F)_1$, thus the commutativity follows. \qed

Moreover, it is also easy to see the above commutative diagram is functorial in $A$ and thus again we can globalize the diagram and obtain the following morphisms
\[
(\pi^{G, \epsilon})^* : \text{pr}^*_2(\mathcal{R}_{\infty, \text{Ig}}^{\text{top}}[\lambda^{-1}]) \rightarrow \text{pr}^*_1(\mathcal{R}_{\infty, \text{Ig}}^{\text{top}}[\lambda^{-1}]),
\]
\[
(\pi^{G, \epsilon})^* : \text{pr}^*_2(\mathcal{R}_{\text{rig}}^{\text{Ig}}[\lambda^{-1}]) \rightarrow \text{pr}^*_1(\mathcal{R}_{\text{rig}}^{\text{Ig}}[\lambda^{-1}]),
\]
\[
(\pi^{G, \epsilon})^* : \text{pr}^*_2(\mathcal{R}_{\infty}^{\text{Ig}}[\lambda^{-1}]) \rightarrow \text{pr}^*_1(\mathcal{R}_{\infty}^{\text{Ig}}[\lambda^{-1}]).
\]

Then we consider the trace map over the formal scheme $\text{Ig}_{\infty, n}/\tilde{P}_L$:
\[
\text{Tr}_{\text{pr}^*_1} : (\text{pr}^*_1)_* (\mathcal{O}_{\text{Ig}_{\infty, n}/\tilde{P}_L(\epsilon)}) \rightarrow \mathcal{O}_{\text{Ig}_{\infty, n}/\tilde{P}_L}.
\]

We want to study the $p$-integrality of this morphism. Recall that we have the unipotent radical $U_{\tilde{P}}$ of the parabolic subgroup $\tilde{P}$ of $G$. Note that $U_{\tilde{P}}(E_p)$ acts by inverse conjugation on $U_{\tilde{P}}(O_p)$.

**Definition 5.3.** For an element $\epsilon \in T_{\tilde{P}}^+(E_p)$, we set
\[
m_\epsilon := \frac{1}{\# U_{\tilde{P}}(O_p)} U_{\tilde{P}}(O_p) \epsilon^{-1} \cap U_{\tilde{P}}(O_p).
\]
Lemma 5.5. We fix an element \( \epsilon \in \tilde{T}_P^+(E_p) \), the morphism

\[
\text{Tr}_\epsilon := \frac{1}{m_e} \text{Tr}_{pr_{1,\infty}} : (pr_{1,\infty})_*(\mathcal{O}_{I_p^\mu_{\infty}/\tilde{P}_L}(\epsilon)) \to \mathcal{O}_{I_p^\mu_{\infty}/\tilde{P}_L}
\]

is well-defined (in other words, the morphism \( \text{Tr}_{pr_{1,\infty}} \) is divisible by \( m_e \)).

Proof. The proof is similar to the ones in [Hid02, Lemma 6.6] and [Pil12, Appendix 1], using the Serre-Tate theory developed in [SZ16]. First we recall some relevant results in [SZ16]. Let \( x_0 \in S_{h_1} \) be a geometric point in the \( \mu \)-ordinary locus (as we have done since the beginning) and \( BT \) be the \( p \)-divisible group associated to the abelian scheme \( A_{x_0} \). Then we write \( \mathcal{D}(BT) \) for the deformation space of \( BT \) over Spec(\( \mathbb{W} \)) and \( \mathcal{D}_G(BT) \) the subspace of \( \mathcal{D}(BT) \) consisting of \( G \)-adapted deformations (cf. [SZ16, §4]). Then by [SZ16, Theorem 4.9], \( \mathcal{D}_G(BT) \) is a shifted subcascade and has a dense subset consisting of CM points (i.e. torsion points). More precisely, recall that we have a slope decomposition \( BT = \prod_{i=1}^r BT_i \) each piece of slope \( 1 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0 \) and its canonical lifting \( \tilde{BT} = \prod_{i=1}^r \tilde{BT}_i \). We write (in the notation of [SZ16, Definition 4.4]) \( E_{i,j} = \text{Ext}(\tilde{BT}_i, \tilde{BT}_j) \) for \( i < j \). Let \( BT_{i,j} \) denote the product \( \prod_{k=1}^j \tilde{BT}_k \) and \( \mathcal{D}_G(BT_{i,j}) \) its corresponding deformation space over Spec(\( \mathbb{W} \)) (\( i \leq j \)). Then [SZ16, Proposition 4.8, Theorem 4.9] show that

\[
(1) \text{ the natural morphisms } \lambda_{i,j} : \mathcal{D}_G(BT_{i,j}) \to \mathcal{D}_G(BT_{i-1,j}) \text{ and } \rho_{i,j} : \mathcal{D}_G(BT_{i,j}) \to \mathcal{D}_G(BT_{i+1,j}) \text{ satisfy the commutative relation } \rho_{i,j-1} \circ \lambda_{i,j} = \lambda_{i+1,j} \circ \rho_{i,j} ;
\]

\[
(2) \mathcal{D}_G(BT_{i,j}) \text{ is a bi-extension of } (\mathcal{D}_G(BT_{i-1,j}) ; \mathcal{D}_G(BT_{i+1,j})) \text{ by } E_{i,j} \times \mathcal{D}_G(BT_{i+1,j-1}).
\]

Recall that over the correspondence \( I_p^\mu_{\infty,1}/\tilde{P}_L(\epsilon) \), we have two abelian schemes \( A^G \) and \( \tilde{A}^G \) and a \( p \)-isogeny \( \pi^{G,\epsilon} : A^G \to \tilde{A}^G \). This \( p \)-isogeny induces an action of \( \epsilon \) on the deformation space \( \mathcal{D}_G(BT) = \mathcal{D}_G(BT_{1,r}) \). More precisely, we view \( \epsilon \) as an element in \( \text{GSp}(V, \psi)(E_p) \) and suppose that the parabolic subgroup \( P_v \) of \( \text{GSp}(V, \psi) \) corresponds to a partition \( \text{dim}(V) = n_1 + n_2 + \cdots + n_r \) (recall \( P_v \) stabilizes the filtration induced by the cocharacter \( \mu \)) and the parabolic subgroup \( \tilde{P}_v \) (contained in \( P_v \)) corresponds to partitions \( n_i = n_{i,1} + n_{i,2} + \cdots + n_{i,s_i} \) for \( i = 1, \cdots, r \). Then \( \epsilon = \text{diag}(\epsilon_1, \cdots, \epsilon_r) \) with each diagonal matrix \( \epsilon_i = \text{diag}(p_{i,1} \cdot 1_{n_{i,1}}, \cdots, p_{i,s_i} \cdot 1_{n_{i,s_i}}) \) such that \( t_{1,1} \leq t_{2,1} \leq \cdots \leq t_{r,1} \leq t_{1,2} \leq \cdots \leq t_{r,s_2} \leq \cdots \leq t_{r,s_r} \). By the equivalence between the category of Honda systems over \( \mathbb{W} \) and the category of \( p \)-divisible groups over \( \mathbb{W} \) (cf. [SZ16, Appendix A]), we can identify \( E_{i,j} \) with \( \tilde{P}(O_p) \cap P_{i,j}(O_p) \), here \( P_{i,j} \) is the subgroup of the unipotent subgroup \( U_{P_v} \) of \( P_v \) where the entries at index \( (i', j') \) vanish for \( i' < j' \) and \( (i', j') \notin \{ \sum_{k=1}^{i-1} n_k + 1, \sum_{k=1}^{i} n_k \} \times \{ \sum_{k=1}^{j-1} n_k + 1, \sum_{k=1}^{j} n_k \} \). We see that \( \epsilon_i \) acts by conjugation on \( E_{i,j} \) via its conjugate action on the Honda system \( \mathbb{D}((\tilde{BT}_j)^{\epsilon}) \) and thus we have (see also [Hid02, p.64])

\[
\frac{\mathcal{D}_G(BT)}{\epsilon \mathcal{D}_G(BT)\epsilon^{-1} \cap \mathcal{D}_G(BT)} = m_e.
\]

From this we deduce that \( \text{Tr}_\epsilon \) is \( p \)-integral.

We have also the following simple observations

Lemma 5.5. We fix an element \( \epsilon \in \tilde{T}_P^+(E_p) \).

(1) There is a subset \( X_\epsilon \subset \tilde{P}(E_p) \) such that for any \( n \geq 1 \):

\[
I_{\tilde{P}}(\epsilon)I_{\tilde{P}}(n) = \bigcup_{x \in X_\epsilon} I_{\tilde{P}}(n)x.*
\]

(2) We have the following identity of double cosets for any \( n \geq 2 \):

\[
I_{\tilde{P}}(\epsilon)I_{\tilde{P}}(n) = I_{\tilde{P}}(n) I_{\tilde{P}}(n-1).
\]

Proof. (1) By definition, \( I_{\tilde{P}}(\epsilon)I_{\tilde{P}}(n) = \bigcup_{x} I_{\tilde{P}}(n)x \) where \( x \) runs through \( (\epsilon^{-1}I_{\tilde{P}}(\epsilon) \cap I_{\tilde{P}}(n)) \). Since \( \epsilon \in \tilde{T}_P^+(E_p) \), we can identify the quotient with \( \epsilon^{-1}U_{\tilde{P}}(O_p) \epsilon \backslash U_{\tilde{P}}(O_p) \), which is independent of \( n \).

(2) The proof is the same as [Pil12, Proposition 5.3]. More precisely, write \( I_{\tilde{P}}(n)^\epsilon := I_{\tilde{P}}(n) \epsilon \cap I_{\tilde{P}}(n) \) and \( I_{\tilde{P}}(n-1)^\epsilon := I_{\tilde{P}}(n) \epsilon \cap I_{\tilde{P}}(n-1) \). Then we see that both \( I_{\tilde{P}}(n) \setminus I_{\tilde{P}}(n-1) \) and \( I_{\tilde{P}}(n-1)^\epsilon \setminus I_{\tilde{P}}(n) \) are in bijection with \( \text{Ker}(U_{\tilde{P}}(O_p)/p^n) \to U_{\tilde{P}}(O_p)/p^{n-1}) \). From this we deduce that the quotients \( I_{\tilde{P}}(n)^\epsilon \setminus I_{\tilde{P}}(n) \) and \( I_{\tilde{P}}(n-1)^\epsilon \setminus I_{\tilde{P}}(n-1) \) are identical using the snake lemma (the former one is \textit{a priori} contained in the latter one).
Definition 5.6. For any sheaf $\mathcal{F}$ over $\text{Ig}^\mu_{\text{rig}, n}/\tilde{P}_L$, we define $\mathbb{T}_\epsilon$ to be the composition of the following morphisms

$$
\begin{align*}
H^0(\text{Ig}^\mu_{\text{rig}, n}/\tilde{P}_L, \mathcal{F}) &\longrightarrow H^0(\text{Ig}^\mu_{\text{rig}, n}/\tilde{P}_L(\epsilon), \text{pr}^*_\text{rig}\mathcal{F}) \\
\downarrow & \downarrow (\epsilon^G)^* \\
H^0(\text{Ig}^\mu_{\text{rig}, n}/\tilde{P}_L, \mathcal{F}) &\longleftarrow H^0(\text{Ig}^\mu_{\text{rig}, n}/\tilde{P}_L(\epsilon), \text{pr}^*_\text{rig}\mathcal{F})
\end{align*}
$$

By considering the affine open formal subsets of $\text{Ig}^\mu_{\text{rig}, n}/\tilde{P}_L$, we define also the operator $\mathbb{T}_\epsilon : H^0(\text{Ig}^\mu_{\text{rig}, n}/\tilde{P}_L, \tilde{R}^\text{Ig}_\infty[\lambda^{-1}]) \rightarrow H^0(\text{Ig}^\mu_{\text{rig}, n}/\tilde{P}_L, \tilde{R}^\text{Ig}_\infty[\lambda^{-1}]).$

By the above arguments, we see that these operators are well-defined.

Proposition 5.7. We have the following morphisms of $\mathbb{Z}$-algebras

$$
\mathbb{Z}[\tilde{T}_P^+(E_p)] \rightarrow \text{End}(H^0(\text{Ig}^\mu_{\text{rig}, n}/\tilde{P}_L, \tilde{R}^\text{Ig}_\infty[\lambda^{-1}])), \epsilon \mapsto \mathbb{T}_\epsilon;
$$

$$
\mathbb{Z}[\tilde{T}_P^+(E_p)] \rightarrow \text{End}(H^0(\text{Ig}^\mu_{\text{rig}, n}/\tilde{P}_L, \tilde{R}^\text{Ig}_\text{rig}[\lambda^{-1}])), \epsilon \mapsto \mathbb{T}_\epsilon.
$$

Corollary 5.8. For any $\lambda \in X^*(\tilde{T}_P)^+\epsilon \in \tilde{T}_P^+(E_p)$ and $n \geq 2$, we have

$$
\mathbb{T}_\epsilon(H^0(\text{Ig}^\mu_{\text{rig}, n}/\tilde{P}_L, \tilde{R}^\text{Ig}_\text{rig}[\lambda^{-1}])) \subset H^0(\text{Ig}^\mu_{\text{rig}, n-1}/\tilde{P}_L, \tilde{R}^\text{Ig}_\text{rig}[\lambda^{-1}]).
$$

Recall we have a projection $\text{pr} : \tilde{\text{Ig}}_{\infty}/\tilde{P}_L \rightarrow \text{Ig}_{\infty}/\tilde{P}_L(\mathcal{O}_p)$. We write

$$
\Omega[\lambda^{-1}] := \text{pr}(\text{Ig}_{\infty}/\tilde{P}_L[\lambda^{-1}]].
$$

Now we give

Definition 5.9. Fix a (finite) set of generators $\epsilon_1, \cdots, \epsilon_r$ of the monoid $\tilde{T}_P^+(E_p)$, we define the $\tilde{P}$-ordinary projector:

$$
\epsilon_{P} := \lim_{n \to \infty} \left( \prod_{i=1}^{r} \mathbb{T}_{\epsilon_i} \right)^{n!} \in \text{End}_{\mathcal{O}_p}(H^0(\text{Ig}^\mu_{\text{rig}, n}/\tilde{P}_L, \Omega[\lambda^{-1}]])
$$

Using the isomorphism in [4.18] and the Koecher principal (by Definition 2.17 Condition 5 and [Lan16 Remark 10.3]), we write

$$
\epsilon_{P} \tilde{V}_\infty^{\text{Pder}}[\lambda^{-1}]
$$

for the space of $\tilde{P}$-ordinary $p$-adic modular forms of weight $\lambda$, of level $K$. Using the Hodge-Tate map $\text{HT}^*_{\infty}$, we write

$$
\epsilon_{P} H^0(Sh^\Sigma, V^\Sigma_\lambda), \text{resp., } \epsilon_{P} H^0(Sh^\Sigma, V^\Sigma_\lambda(-C^\Sigma))
$$

for the images of $H^0(Sh^\Sigma, V^\Sigma_\lambda), \text{resp., } H^0(Sh^\Sigma, V^\Sigma_\lambda(-C^\Sigma))$ in $\epsilon_{P} \tilde{V}_\infty^{\text{Pder}}[\lambda^{-1}]$.

Now we can globalize the result in Proposition 4.9 as follows

Proposition 5.10. For any $\lambda \in X^*_\text{dm}(\tilde{T}_P)$, we have a commutative diagram with horizontal isomorphisms

$$
\begin{align*}
\epsilon_{P} H^0(\text{Ig}^\mu_{\text{rig}, n}/\tilde{P}_L, \tilde{R}^\text{Ig}_\infty[\lambda^{-1}]) &\longrightarrow \epsilon_{P} \tilde{V}_\infty^{\text{Pder}}[\lambda^{-1}] \\
\downarrow & \downarrow \\
\epsilon_{P} H^0(\text{Ig}^\mu_{\text{rig}, n}/\tilde{P}_L, \tilde{R}^\text{Ig}_\text{rig}[\lambda^{-1}]) &\longrightarrow \epsilon_{P} \tilde{V}_\infty^{\text{Pder}}[\lambda^{-1}][1/p]
\end{align*}
$$
Proof. Using the preceding corollary, the bottom isomorphism comes from the isomorphism
\[
e_P H^0(Ig_{\infty}^\mu / \tilde{P}_L, \mathcal{R}_{\tilde{\infty}}^\mu[\lambda^{-1}]) \simeq e_P \mathcal{V}_{\infty}^p[\lambda^{-1}][\frac{1}{p}],
\]
which again comes from the $p$-integral version by considering stalks of the corresponding sheaves on $Ig_{\infty}^\mu / \tilde{P}_L$:
\[
e_P H^0(Ig_{\infty}^\mu / \tilde{P}_L, \mathcal{R}_{\tilde{\infty}}^\mu[\lambda^{-1}]) \simeq e_P \mathcal{V}_{\infty}^p[\lambda^{-1}] .
\]
To show this last isomorphism, again for a point $x \in Ig_{\infty}^\mu / \tilde{P}_L$, consider the stalk $\mathcal{R}_{\tilde{\infty}}^\mu[\lambda^{-1}]_x$ of the sheaf $\mathcal{R}_{\tilde{\infty}}^\mu[\lambda^{-1}]$ at the point $x$. Now for any affine open subset $\text{Spf}(A)$ containing $x$, $\Omega_\infty[\lambda^{-1}](A)$ is given exactly by the module $A[\lambda^{-1}]$. On the other hand, $\mathcal{R}_{\tilde{\infty}}^\mu[\lambda^{-1}](A)$ is just the modified representation $\tilde{R}_A[\lambda^{-1}]$. Thus by Proposition [4.9] we have an isomorphism $e_P \tilde{R}_A[\lambda^{-1}] \simeq A[\lambda^{-1}]$. Write $\mathcal{R}_{\tilde{\infty}}^\mu[\lambda^{-1}]^0$ for the sheaf on $Ig_{\infty}^\mu / \tilde{P}_L$, which is given on affine open subset $\text{Spf}(A)$ by the kernel $\text{Ker}(\tilde{R}_A[\lambda^{-1}] \to A[\lambda^{-1}])$. Then it suffices to show that $e_P((\mathcal{R}_{\tilde{\infty}}^\mu[\lambda^{-1}]^0)_x = 0$ for any point $x$. Clearly the operator $e_P$ commutes with the inductive limit and this last one follows from the fact that $e_P \text{Ker}(\tilde{R}_A[\lambda^{-1}] \to A[\lambda^{-1}]) = 0$. The top isomorphism in the proposition follows also from this isomorphism.  

We want to refine the top isomorphism in the above proposition: more precisely, it is desirable to replace $Ig_{\infty,1}^\mu / \tilde{P}_L$ by the $\mu$-ordinary locus $Sh_{\infty}^\mu$. We set
\[
X_{\text{dd}}^*(\bar{T}_P) := X_{\text{dd}}^*(T) \cap X_{\text{dm}}^*(\bar{T}_P).
\]

Proposition 5.11. For any $\lambda \in X_{\text{dd}}^*(\bar{T}_P)$, the following natural map is an isomorphism:
\[
e_P H^0(Sh_{\infty}^\mu, \mathcal{R}_{\infty}[\lambda^{-1}]) \to e_P H^0(Ig_{\infty,1}^\mu / \tilde{P}_L, \mathcal{R}_{\infty}^\mu[\lambda^{-1}]).
\]

Proof. We follow the strategy in [Pi12] §5.2.3. The idea is to construct a new Hecke operator on a modified correspondence such that this new Hecke operator takes functions on $Ig_{\infty,1}^\mu / \tilde{P}_L$ to functions on $Sh_{\infty}^\mu$, i.e., the image of this Hecke operator are functions invariant under the action $G(O_p / p^n)$. More precisely, we consider the following correspondence $Ig_{\infty,n}^{\text{Sp}(V,\psi),\Delta} / \tilde{P}_L(\epsilon)$ over $Sh(V)_\infty$, whose $A$-points ($A$ is a $\mathbb{Z}_p$-algebra) consists of quintuples $(A, \tilde{A}, \pi, \psi, \tilde{\psi}_n)$ where

1. $\tilde{A}$ is a $\mu$-ordinary principally polarized abelian scheme over $\Spec(A)$ and a $\tilde{P}_V(O_p / p^n)$-coset of isomorphisms
2. $\tilde{A}$ is a $\mu$-ordinary principally polarized abelian scheme over $\Spec(A)$ and a $\tilde{P}_V(O_p / p^n)$-coset of isomorphisms
3. $\pi : A \to \tilde{A}$ is a $p$-isogeny of similitude factor $\nu(\epsilon)$ of abelian schemes such that there are isomorphisms

Then as before, we write $Ig_{\infty,n}^{\mu,\Delta} / \tilde{P}_L(\epsilon)$ to be the pull-back of $Ig_{\infty,n}^{\text{Sp}(V,\psi),\Delta} / \tilde{P}_L(\epsilon)$ along the embedding $Sh_{\infty}^\mu \to Sh(V)_\infty$. We then write
\[
pr_1^\Delta : pr_2^\Delta : Ig_{\infty,n}^{\mu,\Delta} / \tilde{P}_L(\epsilon) \to Ig_{\infty,n}^\mu / \tilde{P}_L
\]
for the two natural projections which takes $(A, \tilde{A}, \pi, \psi_n, \tilde{\psi}_n)$ to $(A, \psi_n)$ resp., $(\tilde{A}, \tilde{\psi}_n)$.

Note that in the definition above, the isomorphism $\psi_n$ may not be congruent to $\psi_n$ modulo $p^n$. Write $W_{\tilde{P}_V}$ to be the Weyl group scheme of $\tilde{P}_V$ (over $\Spec(O_p)$), which is finite étale over $Spec(O_p)$ ([Con11] Proposition 3.2.8). We know that the set of $\tilde{P}_V(O_p / p^n)$-cosets of isomorphisms $\psi_n$ is parameterized by the Weyl group $W_{\tilde{P}_V}(O_p / p^n)$. We claim that there is an isomorphism $\psi_0^\infty : A_{x_0} [p^\infty] \to A[p^\infty]$ such that $\psi_0^\infty \equiv \psi_n \pmod{p^n}$ and $D((\psi_0^\infty)^{-1} \circ \pi \circ \psi_0^\infty) = \epsilon^{-1} \circ w$ for some $w_n \in W_{\tilde{P}_V}(O_p / p^n)$. Indeed, we know that there exists some $w_\infty \in W_{\tilde{P}_V}(O_p / p^n)$ such that $\psi_0^\infty \circ w_\infty \equiv \psi_n \pmod{p}$. Thus we put $\psi_0^\infty = \psi_0^\infty \circ w_\infty$, then one gets the desired claim by setting $w_n$ to be the image of $w_\infty$ in $W_{\tilde{P}_V}(O_p / p^n)$. Moreover, it is clear that $w_\infty \in W_{\tilde{P}_L}$ if and only if the point $x \in Ig_{\infty,n}^{\mu,\Delta} / \tilde{P}_L(\epsilon)$ corresponding to the above quintuple lies in $Ig_{\infty,n}^\mu / \tilde{P}_L(\epsilon)$.
Write \( (\mathcal{A}^{G,\triangle}, \overline{\mathcal{A}}^{G,\triangle}, \pi_G, \triangle, \psi_m, \overline{\psi}_m) \) for the universal quintuple over \( \mathrm{Ig}_{\infty,n}^\mu / \overline{P}_L \). As in the case of \( \mathrm{Ig}_{\infty,n}^\mu / \overline{P}_L(\epsilon) \), one can define a morphism

\[
(\pi^{G,\triangle,\epsilon,*})^* : (\pr_{2,\rig}^\triangle)^* (\mathcal{R}_{\rig}^{\mathrm{Ig}}[\lambda^{-1}]) \to (\pr_{1,\rig}^\triangle)^* (\mathcal{R}_{\rig}^{\mathrm{Ig}}[\lambda^{-1}]), \quad F \mapsto (\mathbb{D}(f) \mapsto F(\mathbb{D}(f))).
\]

For a finite flat \( \mathcal{O}_p \)-algebra \( A \), we define a modified Hecke operator \( T^{\triangle}_\epsilon \) on \( R_{A,\overline{\mathcal{A}}^\mu}[\lambda^{-1}] \) by sending an element \( F \) to \( (T^{\triangle}_\epsilon F)(g) = F(\epsilon w_{\infty} g \epsilon^{-1}) \) for any \( g \in G \). We claim that for \( F \in \epsilon \overline{P} R_{A,\overline{\mathcal{A}}^\mu}[\lambda^{-1}], w_\infty \notin \mathcal{W}_L, \epsilon \overline{P} = \prod_{i=1}^2 \epsilon_i \) and \( \lambda \in X_{\overline{p}}(\mathcal{T}_p) \), we have \( T^{\triangle}_\epsilon F \in p \overline{P} R_{A,\overline{\mathcal{A}}^\mu}[\lambda^{-1}] \). Since \( p \) is supported on \( \overline{P} \), it suffices to show that \( (T^{\triangle}_\epsilon F)(g) \) is divisible by \( p \) for any \( g \in \overline{P} \). In this case, we have

\[
(T^{\triangle}_{\epsilon} F)(g) = F(\epsilon \overline{P} w_{\infty} \epsilon^{-1}) = F(\epsilon \overline{P} w_{\infty} \epsilon^{-1}) = F(\epsilon \overline{P} w_{\infty} \epsilon^{-1}) = \lambda^{-1}(w_{\infty}^{-1} \epsilon \overline{P} w_{\infty} \epsilon^{-1}) F(1).
\]

Note that \( \lambda^{-1}(w_{\infty}^{-1} \epsilon \overline{P} w_{\infty} \epsilon^{-1}) \in pA \), we deduce that \( (T^{\triangle}_\epsilon F)(g) \) is indeed divisible by \( p \) for any \( g \).

Next we can globalization the above argument. The same reasoning as in Proposition 5.2 shows that for any affine open subset \( \text{Spf}(A) \) of \( \mathrm{Ig}_{\infty,n}^\mu / \overline{P}_L \), we have the following commutative diagram

\[
\begin{array}{ccc}
R_{A,\overline{\mathcal{A}}^\mu}^{\lambda^{-1}} & \longrightarrow & (\pr_{2,\rig}^\triangle)^* (\mathcal{R}_{\rig}^{\mathrm{Ig}}[\lambda^{-1}]) (A^{1_p}) \\
\downarrow & & \downarrow \\
R_{A,\overline{\mathcal{A}}^\mu}^{\lambda^{-1}} & \longrightarrow & (\pr_{1,\rig}^\triangle)^* (\mathcal{R}_{\rig}^{\mathrm{Ig}}[\lambda^{-1}]) (A^{1_p}) \\
\downarrow T^{\triangle}_\epsilon & & \downarrow (\pi^{G,\triangle,\epsilon,*})^* \\
R_{A,\overline{\mathcal{A}}^\mu}^{\lambda^{-1}} & \longrightarrow & (\pr_{1,\rig}^\triangle)^* (\mathcal{R}_{\rig}^{\mathrm{Ig}}[\lambda^{-1}]) (A^{1_p})
\end{array}
\]

We write

\[
T^{\triangle}_\epsilon = \frac{1}{m_\epsilon} T^{\omega}_{\epsilon} \pr_{2,\rig}^\triangle.
\]

Then for any \( \mathcal{O}_{\mathrm{Ig}_{\infty,n}^\mu,\overline{P}_L} \)-sheaf \( F \), we write \( T^{\triangle}_\epsilon \) for the composition of morphisms (see the remark below):

\[
H^0(\mathrm{Ig}_{\infty,n}^\mu / \overline{P}_L, F) \longrightarrow H^0(\mathrm{Ig}_{\infty,n}^\mu / \overline{P}_L(\epsilon), (\pr_{2,\rig}^\triangle)^* F) \]

\[
\downarrow T^{\triangle}_\epsilon \]

\[
H^0(\mathrm{Ig}_{\infty,n}^\mu / \overline{P}_L, F) \longrightarrow H^0(\mathrm{Ig}_{\infty,n}^\mu / \overline{P}_L(\epsilon), (\pr_{1,\rig}^\triangle)^* F)
\]

On one hand, for \( \lambda \in X_{\overline{p}}^{\omega}(\mathcal{T}_p) \) and \( F \in \epsilon \overline{P} H^0(\mathrm{Ig}_{\infty,n}^\mu / \overline{P}_L, \mathcal{R}_{\rig}^{\mathrm{Ig}}[\lambda^{-1}]) \), it is easy to see that \( T^{\triangle}_\epsilon F \) is divisible by \( p \): indeed, for any point \( x \in \mathrm{Ig}_{\infty,n}^\mu / \overline{P}_L(\epsilon) \), if this point is in \( \mathrm{Ig}_{\infty,n}^\mu / \overline{P}_L(\epsilon) \), then \( T^{\triangle}_\epsilon \) and \( T^{\triangle}_\epsilon \) coincide and the difference is 0; otherwise, by definition \( (T^{\triangle}_\epsilon F)_x = 0 \) while by the above argument \( (T^{\triangle}_\epsilon F)_x \) is divisible by \( p \). Thus one deduces that

\[
T^{\triangle}_\epsilon F \in p H^0(\mathrm{Ig}_{\infty,n}^\mu / \overline{P}_L, \mathcal{R}_{\rig}^{\mathrm{Ig}}[\lambda^{-1}])
\]

On the other hand, by the definition of the correspondence \( \mathrm{Ig}_{\infty,n}^\mu / \overline{P}_L(\epsilon) \), one sees that \( T^{\triangle}_\epsilon \) sends a section \( F \in H^0(\mathrm{Ig}_{\infty,n}^\mu / \overline{P}_L, \mathcal{R}_{\rig}^{\mathrm{Ig}}[\lambda^{-1}]) \) to a section that is in fact invariant under the whole \( G(\mathcal{O}_p/p^n) \), and thus \( T^{\triangle}_\epsilon F \) is a section in \( H^0(S_{\overline{p}}, \mathcal{R}_{\rig}^{\mathrm{Ig}}[\lambda^{-1}]) \).

Now using the fact that \( T^{\triangle}_\epsilon \) is invertible on \( \epsilon \overline{P} H^0(\mathrm{Ig}_{\infty,n}^\mu / \overline{P}_L, \mathcal{R}_{\rig}^{\mathrm{Ig}}[\lambda^{-1}]) \) and Nakayama’s Lemma, we see that the natural inclusion

\[
e_p H^0(S_{\overline{p}}, \mathcal{R}_{\rig}^{\mathrm{Ig}}[\lambda^{-1}]) \hookrightarrow e_p H^0(\mathrm{Ig}_{\infty,n}^\mu / \overline{P}_L, \mathcal{R}_{\rig}^{\mathrm{Ig}}[\lambda^{-1}])
\]

is indeed an isomorphism. 

\[\square\]

**Remark 5.12.** Here we have implicitly used Koecher’s principle (which is valid by our assumption Definition 2.7 and [Lan16]) in the definition of \( T^{\triangle}_\epsilon \): indeed, write \( \overline{\mathrm{Ig}}_{\infty}^\mu / \overline{P}_L \) for the pull-back of \( \overline{\mathrm{Sh}}_{\infty}^\mu \rightarrow \overline{\mathrm{Sh}}_{\infty}^\mu \) along the projections
Then we apply Koecher's principle to extend to $H^0(\mathcal{I}_G^{\mu}/\widetilde{P}_L, \mathcal{F})$.

Applying Corollary 4.11 we obtain

**Corollary 5.13.** For any $\lambda \in X^*_d(\mathcal{T}_P)$, we have an isomorphism

$$e_F H^0(\text{Sh}_{X^*_d G}[\lambda^{-1}]) \to e_F H^0(\mathcal{I}_G^{\mu}/\widetilde{P}_L, \mathcal{R}_{X^*_d G}[\lambda^{-1}]).$$

6. **Lifting modular forms**

In this section, we will deal with the problem of lifting modular forms with values in $A/pA$ to modular forms with values in $A$ for a finite flat extension $A$ of $\mathcal{O}_F$.

6.1. **Review of toroidal compactification.** We first review some results in [Mac12] and also introduce some notations.

Let $(G, X)$ be our Shimura data of Hodge type and decompose the adjoint group $G^m = \prod_{i=1}^{m} G_i$ into simple factors. Then a parabolic subgroup $P$ of $G$ is said to be admissible if $P_i := \text{Im}(P_i \hookrightarrow G \to G^m \to G_i)$ is a proper maximal parabolic subgroup of $G_i$ for the whole $G_i$ for all $i = 1, \ldots, m$. Such an admissible parabolic subgroup corresponds to a rational boundary component for $(G, X)$. (Mac12 §2.1.3). Recall $U_P$ denotes the unipotent radical of $P$, then we write $W_P = Z(U_P)$ for the centre of $U_P$. Denote by $Q$ the minimal normal subgroup of $P$ such that the morphism $h^*_x : S \to G$ factors through $(Q_P)_R$ (here $x \in X$). Then $Q_P(R)W_P(C)$ acts on the set $\pi_0(X)$ of connected components of $X$ via the natural map $\pi_0(Q_P(R)W_P(C)) \to \pi_0(Q_P(R)) \to \pi_0(G(R))$. For a connected component $X^+$ of $X$, we then write $F_{P, X^+}$ for the $Q_P(R)W_P(C)$-orbit of $(X^+, h^*_x)$ inside $\pi_0(X) \times \text{Hom}(\mathcal{S}_C, (Q_P)_C)$ (Mac12 2.1.5). A cuspidal label representative (clr) for $(G, X)$ is a triple $\Phi = (\Phi, X^+_G, g_\Phi)$ where $\Phi$ is an admissible parabolic subgroup of $G$, $X^+_G$ is a connected component of $X$ and $\Phi$ is an element in $G(\mathbb{A})$ (Mac12 §2.1.7). We put $\Phi = \Phi$, $U_\Phi = U_{\Phi}$, $W_\Phi = W_{\Phi}$, $Q_\Phi = Q_{\Phi}/W_{\Phi}$, $V_\Phi := U_\Phi/W_\Phi$ the unipotent radical of $Q_\Phi$, $G_{\Phi, h} = Q_{\Phi}/U_{\Phi}$ the Levi component of $Q_{\Phi}$, $D_\Phi := F_{P_\Phi, X^+_G}$, $D_\Phi = W_\Phi(C)\backslash D_\Phi, D_{\Phi, h} = V_\Phi(R)\backslash D_\Phi$. Then $(G_{\Phi, h}, D_{\Phi, h})$ is a pure Shimura datum with reflex field $E$ and $(Q_{\Phi}, D_{\Phi})$, $(Q_\Phi, D_\Phi)$ mixed Shimura data with reflex field $E$ (Mac12 2.1.7). Given the Shimura variety $Sh_{K}(G, X)$, we write $Sh_{K}(Q_{\Phi}, D_{\Phi})$ for the (complex) Shimura variety associated to $(Q_{\Phi}, D_{\Phi}, K_{\Phi})$ where $K_{\Phi} = Q_{\Phi}(\mathbb{A}) \cap K_{\Phi, h}$ and $K_{\Phi, p} = P_{\Phi}(\mathbb{A}) \cap g_\Phi Q_{\Phi} g_\Phi^{-1}$. Write $K_{\Phi, U}$ to be the subset of $U_{\Phi}(\mathbb{A})$ which is the image of the set $(z, u) \in Z_{\Phi}(\mathbb{Q}) \times U_{\Phi}(\mathbb{A})|zu \in K_{\Phi})$ under the canonical projection to the second factor $U_{\Phi}(\mathbb{A})$. Then let $K_{\Phi, U} \subset V_{\Phi}(\mathbb{A})$ denote the image of $K_{\Phi, U}$ under the natural quotient map. Thus we obtain a canonical smooth family of abelian varieties (Mac12 §2.1.10)

$$\mathcal{A}_{K}(\Phi)(\mathbb{C}) \to Sh_{K,h}(G_{\Phi, h}, D_{\Phi, h})(\mathbb{C}).$$

Concerning the interaction between our Shimura variety of Hodge type and the Siegel Shimura variety, We change a little bit of notations and write $\lambda : (G, X) \to (\text{GSp}(V, \psi), S^\pm) := (G^1, X^\dagger)$ for the embedding of our Shimura data into the Siegel Shimura data fixed from the beginning. We fix also an element $g_\Phi$ in $G^1(\mathbb{A})$. Then the clr $\Phi$ for $(G, X)$ induces a clr $\Phi^d := (\Phi^d, X^+_\Phi, g_\Phi)$ for $(G^1, X^\dagger)$; $\Phi^d := \iota_\Phi(\Phi)$, $X^+_\Phi$ is the unique connected component of $X^\dagger$ containing $\iota(X_\Phi)$ and $g_\Phi := g_\Phi g_\Phi^{-1}$ (Mac12 2.1.28). Now we continue our discussion. We write $B_K(\Phi) := (W_\Phi(\mathbb{Q}) \cap K_{\Phi, W})(-1)$ and let $E_K(\Phi)$ denote the torus split over $\mathbb{Z}$ with character group $S_K(\Phi) := B_K(\Phi)^\vee = \text{Hom}_{\mathbb{Z}}(B_K(\Phi), \mathbb{Z})$ (Mac12 §2.1.11). From this we get a canonical isomorphism of families of complex groups

$$P_\Phi(0)(\mathbb{Q}) \backslash D_\Phi(0) \times P_\Phi(0)(\mathbb{A})/K_{\Phi}(0) \cong E_K(\Phi)(\mathbb{C}) \times Sh_{\nu_{\Phi}(K_{\Phi, h})}(\mathbb{C}),$$

Here we fix a surjective map of Shimura data $\nu_{\Phi} : (G_{\Phi, h}, D_{\Phi, h}) \to (\mathbb{G}_m, S^\pm(0))$, $P_\Phi(0) := W_\Phi \times \mathbb{G}_m, D_\Phi(0) := \{((\omega, \lambda) \in \text{Hom}(\mathbb{S}_C, P_\Phi(0)C) \times S^\pm(0))| (\pi \circ h)(x, y) = xy\}$, and $K_{\Phi}(0) := K_{\Phi, W} \times \nu_{\Phi}(K_{\Phi, h}) \subset P_\Phi(0)(\mathbb{A})$.

The relation among different choices of clr $\Phi$ for $(G, X)$ is as given (cf. Mac12 §4.1.3). To state the relevant results, we need first introduce some more notions. Recall that for a finite dimensional vector space $V$ over $\mathbb{Q}$, a rational polyhedral cone (rpc) $\sigma \subset V \otimes \mathbb{R}$ is a subset given by $\sigma = \{v \in V \otimes \mathbb{R} | f_i(v) \geq 0, \forall i = 1, 2, \ldots, r\}$ for a finite set $f_1, f_2, \ldots, f_r$ of dual vectors in $V^\vee$. Fix a $\mathbb{Z}$-lattice $X$ of $V$, then $\sigma$ is admissible (with respect to $X$) if the above $f_1, \ldots, f_r$ can be chosen to be part of a basis for $X$. On the other hand, for two admissible parabolic subgroups $P_1, P_2$ of $G$ and an element $\gamma \in G(\mathbb{Q})$, we write $P_1 \gamma P_2$ if $\gamma Q_{P_1} \gamma^{-1} \subset Q_{P_2}$. For two clr $\Phi_1, \Phi_2$ for $(G, X)$ and
elements $\gamma \in G(\mathbb{Q})$, $q_2 \in Q_{\Phi_2}(\mathbb{Q}_l)$, we write $\Phi_1 \xrightarrow{q_2 \otimes \gamma} \Phi_2$ if $P_{\Phi_1} \gamma \subset P_{\Phi_2}$, $\gamma \cdot X_{\Phi_1}^+ \in \pi_0(X)$ lies inside the $Q_{\Phi_2}(\mathbb{Q})$-orbit of $X_{\Phi_2}^+$ and moreover $g_{\Phi_1} \in q_2g_{\Phi_2}K$. Now for a clr $\Phi$, we write $H^*(\Phi) \subset W_{\Phi}(\mathbb{R})(-1)$ to be the union of the images of the cones $\text{Int}(\gamma^{-1}(H^*(\Phi)))$ for all $\Phi \xrightarrow{q_2 \otimes \gamma} \Phi$. Here $H^*(\Phi) := H^*_p, X_{\Phi}^+$ is given in [Mad12 §2.1.6], which is an open non-degenerate self-adjoint convex cone inside $W_{\Phi}(\mathbb{R})(-1)$ corresponding to $X_{\Phi}^+$. A rational polyhedral cone decomposition (rpcd) for $H^*(\Phi)$ is a set $\Sigma(\Phi)$ of rpc $\sigma \subset W_{\Phi}(\mathbb{R})(-1)$ such that (1) $\sigma \subset H^*(\Phi)$ for all $\sigma \in \Sigma(\Phi)$; (2) for any $\sigma \in \Sigma(\Phi)$, any face of $\sigma$ is also in $\Sigma(\Phi)$; (3) for any $\sigma_1, \sigma_2 \in \Sigma(\Phi)$, $\sigma_1 \cap \sigma_2$ is a common face of $\sigma_1$ and $\sigma_2$ ([Mad12 §2.1.22]). An admissible rpcd for $(G, X, K)$ is an assignment to each clr $\Phi$ for $(G, X)$ a rpc $\Sigma(\Phi)$ for $H^*(\Phi)$ and satisfying compatibility conditions: for any $\Phi_1 \xrightarrow{q_2 \otimes \gamma} \Phi_2$, we have $\Sigma(\Phi_2) = \{\sigma \subset W_{\Phi_1}(\mathbb{Q})(-1) | \text{Int}(\gamma^{-1}(\sigma)) \in \Sigma(\Phi_1)\}$.

We fix an admissible rpcd (rpcd) $\Sigma^+$ for $(G^i, X^\alpha, K^\dagger)$ and $\Sigma$ the induced rpcd for $(G, X, K)$, then we have the corresponding morphism of toroidal compactifications of the integral models of the Shimura varieties:

$$Sh^\Sigma \rightarrow Sh(G^i_t, X^\alpha, K^\dagger)^\Sigma,$$

which, by construction, is the normalization of $Sh$ inside $Sh(G^i_t, X^\alpha, K^\dagger)^\Sigma$ ([Mad12 §4.1.4]). We consider pairs $(\Phi, \sigma)$ where $\Phi$ is a clr and $\sigma \in \Sigma^\circ(\Phi)$, i.e. $\sigma^\circ \subset H^*(\Phi)$. We say two such pairs $(\Phi_1, \sigma_1)$ and $(\Phi_2, \sigma_2)$ are equivalent if there is $\Phi_1 \xrightarrow{q_2 \otimes \gamma} \Phi_2$ such that $\gamma P_{\Phi_1} = P_{\Phi_2}$ and $\text{Int}(\gamma)(\sigma_1) = \sigma_2$. Write $\text{Cusp}_K^\Sigma(G, X)$ for the set of such equivalence classes and there is a natural partial order $\preceq$ on this set: two equivalence classes $[(\Phi_1, \sigma_1)] \preceq [(\Phi_2, \sigma_2)]$ if $(\Phi_1, \sigma_1) \xrightarrow{q_2 \otimes \gamma} (\Phi_2, \sigma_2)$ for some $\gamma \in G(\mathbb{Q})$ and $q_2 \in Q_{\Phi_2}(\mathbb{Q}_l)$ ([Mad12 2.1.26]).

Now for any rpc $\sigma^d \subset B_{K^d}(\Phi^d) \otimes \mathbb{R}$, we have the twisted torus embedding ([Mad12 4.1.2], note that our notations are a bit different from loc.cit):

$$Sh(Q_{\Phi^d}, D_{\Phi^d}, K^d_{\Phi^d}) \hookrightarrow Sh(Q_{\Phi^d}, D_{\Phi^d}, K^d_{\Phi^d}, \sigma^d).$$

Let $\sigma = \sigma^d \cap W_{\Phi}(\mathbb{R})(-1)$ and $Sh(Q_{\Phi^d}, D_{\Phi^d}, K^d_{\Phi^d}, \sigma)$ be the normalization of $Sh(Q_{\Phi^d}, D_{\Phi^d}, K^d_{\Phi^d}, \sigma^d)$ in $Sh(Q_{\Phi^d}, D_{\Phi^d}, K^d_{\Phi^d})$. Then we write $\widetilde{Z}(Q_{\Phi^d}, D_{\Phi^d}, K^d_{\Phi^d}, \sigma)$ for the closed stratum in $Sh(Q_{\Phi^d}, D_{\Phi^d}, K^d_{\Phi^d}, \sigma) \otimes_{\mathbb{Q}_l} \mathbb{E}$ and $Z(Q_{\Phi^d}, D_{\Phi^d}, K^d_{\Phi^d}, \sigma)$ for the normalization of $Z(Q_{\Phi^d}, D_{\Phi^d}, K^d_{\Phi^d}, \sigma)$ in $\widetilde{Z}(Q_{\Phi^d}, D_{\Phi^d}, K^d_{\Phi^d}, \sigma)$.

We then have the following commutative diagram ([Mad12 §4.1.4]):

$$
\begin{array}{ccc}
\widetilde{Z}(Q_{\Phi^d}, D_{\Phi^d}, K^d_{\Phi^d}, \sigma) & \xrightarrow{=} & \widetilde{Z}(Q_{\Phi^d}, D_{\Phi^d}, K^d_{\Phi^d}, \sigma^d) \\
\phi & \downarrow \cong & \phi \\
Z_K(\gamma) & \longrightarrow & Z_{K^d}(\gamma) \\
\end{array}
$$

We then write $Z_K(\gamma)$ for the normalization of $Z_K(\gamma)$ inside $\widetilde{Z}(\gamma)$. As such one obtains the structure of the toroidal compactification $Sh^\Sigma$. We refer the reader to [Mad12 Theorem 4.1.5] for the precise statements. By construction, the stratification on $Sh^\Sigma$ is parameterized by $\gamma \in \text{Cusp}_K^\Sigma(G, X)$.

Next we review the construction of minimal compactification $Sh^\min$ of $Sh$. We write $\text{Cusp}_K^\Sigma(G, X)$ for the set of equivalence classes of clr for $(G, X)$ for the relation $\Phi_1 \sim \Phi_2$ given by some $\Phi_1 \xrightarrow{q_2 \otimes \gamma} \Phi_2$ such that $\gamma \cdot P_{\Phi_1} = P_{\Phi_2}$. It also has a natural partial order $\preceq$ given by: $[\Phi_1] \preceq [\Phi_2]$ if there exists some $\Phi_1 \xrightarrow{q_2 \otimes \gamma} \Phi_2$. By [Mad12 §5.2.11], one has

$$Sh^\min = \bigsqcup_{[\Phi] \in \text{Cusp}_K^\Sigma(G, X)} Z_K([\Phi]).$$

Recall that the Hodge line bundle $\omega_{K^d}(\Sigma^d)$ over $Sh(G^i_t, X^\alpha, K^\dagger)^{\Sigma^d}$ is given by ([Mad12 Definition 5.1.1]):

$$\omega_{K^d}(\Sigma^d) = \text{det}(H_{dr}(\Sigma^d)/F^0H_{dr}(\Sigma^d))^{-1}.$$

We write $\omega_K(\Sigma)$ for the restriction of $\omega_{K^d}(\Sigma^d)$ to $Sh^\Sigma$ (independent of the choice of $K^\dagger, \Sigma^d$). We know that $\omega_K(\Sigma)$ is an ample line bundle over $Sh^\Sigma$ and the graded $O^\dagger$-ring $\oplus_{n \geq 0} H^0(Sh^\Sigma, \omega_K(\Sigma)^{\otimes n})$ is finitely generated ([Mad12 §5.1.3]). Let $H_{dr}(\Phi, \sigma)$ be the restriction of $H_{dr}(\Phi^d, \sigma^d)$ to $Sh(Q_{\Phi^d}, D_{\Phi^d}, K^d_{\Phi^d}, \sigma)$, then we have

$$\text{det}(H_{dr}(\Phi, \sigma)/F^0H_{dr}(\Phi, \sigma))^{-1} \simeq \omega_{K}^\dagger(\Phi) \otimes_{\mathbb{Z}} \omega_K^d(\Phi) = \omega_K(\Phi).$$
Here \( \omega^{ab}_K(\Phi) \) is the Hodge line bundle associated with the universal abelian scheme \( A \to Sh(G_{\Phi,h}, D_{\Phi,h}, K_{\Phi,h}) \) (and also its pull-back to \( Sh(Q_{\Phi}, D_{\Phi}, K_{\Phi}, \sigma) \)) and \( \omega^{\triangle}_K(\Phi) := \text{det}(\text{gr}^i \triangledown \mathbb{H}^9(\mathbb{Z}))^{-1} \) ([Mad12, §5.1.4]). Moreover, since \( Sh(Q_{\Phi}, D_{\Phi}, K_{\Phi}) \) is an \( E_K(\Phi) \)-torsor over \( \overline{Sh(Q_{\Phi}, D_{\Phi}, K_{\Phi})} \), we have ([Pil12 p.32])

\[
Sh(Q_{\Phi}, D_{\Phi}, K_{\Phi}) = \text{Spec}(\oplus_{l \in S_K(\Phi)} \Psi^{(l)}_K(\Phi))
\]

where \( \Psi^{(l)}_K(\Phi) \) is a line bundle over \( \overline{Sh(Q_{\Phi}, D_{\Phi}, K_{\Phi})} \). We then write the projective morphism

\[
\pi' : \overline{Sh(Q_{\Phi}, D_{\Phi}, K_{\Phi})} \to Sh(G_{\Phi,h}, D_{\Phi,h}, K_{\Phi,h}).
\]

Now for each \( l \in S_K(\Phi) \), we define a coherent sheaf over \( Sh(G_{\Phi,h}, D_{\Phi,h}, K_{\Phi,h}) \):

\[
FJ^{(l)}_K(\Phi) := \pi'_l(\Psi^{(l)}_K(\Phi))
\]

For a \( \Upsilon = [(\Phi, \sigma)] \) and an integer \( n \geq 0 \), we have an evaluation map \( FJ_{(\Phi, \sigma)} \) which is the composition of the following morphisms ([Mad12, §5.1.5]):

\[
\begin{array}{c}
H^0(Sh^\Sigma, \omega_K(\Sigma)^{\otimes n}) \\
\downarrow \sim \\
H^0(Sh^\Sigma, \omega_K(\Phi)^{\otimes n}) \\
\downarrow \\
\prod_{l \in S_K(\Phi)} H^0(Sh(G_{\Phi,h}, D_{\Phi,h}, K_{\Phi,h}), FJ^{(l)}_K(\Phi) \otimes \omega_K(\Phi)^{\otimes n}).
\end{array}
\]

One can show that these morphisms \( FJ_{(\Phi, \sigma)} \) are compatible with each other ([Mad12 (5.1.6.2)]).

Now let \( P_{\Phi}(\mathbb{Q})_{\gamma} \subset P_{\Phi}(\mathbb{Q}) \) be the stabilizer in \( P_{\Phi}(\mathbb{Q}) \) of the \( Q_{\Phi}(\mathbb{R}) \)-orbit of \( X_{\Phi}^\circ \). We know that there is an element \( q \in Q_{\Phi}(\mathbb{A}) \) such that \( \Phi \xrightarrow{[(\gamma, q)_K]} \Phi \). This induces an isomorphism on \( Sh(Q_{\Phi}, D_{\Phi}, K_{\Phi}) \) which does not depend on the choice of the finite component \( q_l \) of \( q \). Now we define

\[
\triangle_K(\Phi) := P_{\Phi}(\mathbb{Q})_{\gamma} \bigcap (Q_{\Phi}(\mathbb{A})_gKg_{\Phi}^{-1})/Q_{\Phi}(\mathbb{Q}).
\]

Then \( \triangle_K(\Phi) \) acts on the tower of Shimura varieties

\[
Sh(Q_{\Phi}, D_{\Phi}, K_{\Phi}) \to \overline{Sh(Q_{\Phi}, D_{\Phi}, K_{\Phi})} \to Sh(G_{\Phi,h}, D_{\Phi,h}, K_{\Phi,h}).
\]

Moreover, the action of \( \triangle_K(\Phi) \) on this last Shimura variety factors through a finite quotient \( \triangle_{\text{fin}}(\Phi) \). If \( L_{\Phi} := P_{\Phi}/U_{\Phi} \to G_{\Phi,h} := P_{\Phi}/Q_{\Phi} \) has a section (such that \( L_{\Phi} = G_{\Phi,h} \times G_{\Phi,f} \)), then \( \triangle_K(\Phi) \) acts trivially on \( Sh(G_{\Phi,h}, D_{\Phi,h}, K_{\Phi,h}) \) ([Mad12 (2.1.16.2)]). Now write \( P_K(\Phi) \subset S_K(\Phi) \) for the sub-monoid consisting of elements that are non-negative on \( \text{H}(\Phi) \). Then \( FJ_{(\Phi, \sigma)} \) does not depend on the choice of \( \sigma \in \Sigma(\Phi) \) and its image is in the invariant subspace \( \prod_{l \in S_K(\Phi)} H^0(Sh(G_{\Phi,h}, D_{\Phi,h}, K_{\Phi,h}), FJ^{(l)}_K(\Phi) \otimes \omega_K(\Phi)^{\otimes n})/\triangle_K(\Phi) \) ([Mad12, (5.18)]). Now the minimal compactification of \( Sh \) is given by ([Mad12 5.2.1])

\[
Sh^{\text{min}} := \text{Proj}(\oplus_{n \geq 0} H^0(Sh^\Sigma, \omega_K(\Sigma)^{\otimes n}))
\]

together with a canonical map

\[
\phi^\Sigma : Sh^\Sigma \to Sh^{\text{min}}.
\]

Now let \( x \in Sh^{\text{min}}(\overline{\Phi}_{\gamma}) \) be a geometric point lifting to a point \( y \in Z_K([\Phi]) \) and write \( FJ^{(l)}_K(\Phi)^{\gamma}_y \) for the completion of \( FJ^{(l)}_K(\Phi) \) at the point \( y \). Then we have canonical isomorphisms of complete local rings ([Mad12 5.2.8]):

\[
\Omega^{\gamma}_{Sh^{\text{min}},x} \simeq \left( \prod_{l \in S_K(\Phi)} FJ^{(l)}_K(\Phi)^{\gamma}_y \right)^{\triangle_K(\Phi)}.
\]
Moreover, if $\text{bundle } [\text{EGA, II.5.1.12}]$, and thus is affine. From this we get the following isomorphism

$$H^0 \left( \text{Spec} \left( \mathcal{O}_{S_{\text{Sh}^{\text{min}}, \omega}}^\Sigma \right), (\prod_{l \in S_K(\Phi)} H^0(\text{Sh}(G_{\Phi, h}, D_{\Phi, h}, K_{\Phi, h})_{y}, FJ^l_{K}(\Phi) \otimes \omega_K(\Phi)^{\otimes n})) \right) \cong_{K(\Phi)} \left( \prod_{l \in S_K(\Phi)} \text{Hom} \right).$$

6.2. Base change of modular forms. We continue to use the notations in the previous subsection. Now we can show

**Proposition 6.1.** For $y \in Z_K(\Phi)$ as in the last subsection, $\lambda \in X_{\text{dm}}(\overline{T_P})$, $l \in P_K(\Phi)$ positive definite and $n \geq 0$, we have the following isomorphism

$$H^0(\text{Sh}(G_{\Phi, h}, D_{\Phi, h}, K_{\Phi, h})_{y} \otimes \mathcal{O}_y \mathcal{O}_p/p^n, FJ^l_{K}(\Phi) \otimes \omega_K(\Phi)[\lambda^{-1}]) \cong H^0(\text{Sh}(G_{\Phi, h}, D_{\Phi, h}, K_{\Phi, h})_{y}, FJ^l_{K}(\Phi) \otimes \omega_K(\Phi)[\lambda^{-1}]) \otimes \mathcal{O}_p/p^n.$$

Moreover, if $x \in S_{l, n, \mu}(\overline{\mathbb{P}}_p)$ is in the $\mu$-ordinary locus, then we have (write the morphism $\overline{\pi} : \text{Ig}_{\infty}^\mu \otimes \overline{P}_L \rightarrow \text{Sh}_{\infty}^\mu$):

$$\text{Hom}(\text{Sh}_y \otimes \mathcal{O}_y \mathcal{O}_p/p^n, FJ^l_{K}(\Phi) \otimes \overline{\pi}(\mathcal{O}_{\text{Ig}_{\infty}^\mu} \otimes \overline{\mathcal{P}}_L(\lambda^{-1}))) \cong H^0(\text{Sh}_y, FJ^l_{K}(\Phi) \otimes \overline{\pi}(\mathcal{O}_{\text{Ig}_{\infty}^\mu} \otimes \overline{\mathcal{P}}_L(\lambda^{-1}))) \otimes \mathcal{O}_p/p^n.$$

**Proof.** First introduce some notations. Recall that we have fixed a $\mathbb{Z}$-lattice $\Lambda$ of $V$ which is unimodular with respect to the symplectic form $\langle \cdot, \cdot \rangle$. Let $C$ be the set of totally isotropic submodules of $\Lambda$. For any $\Lambda' \in C$, denote by $\Lambda^\perp$ for the submodule of $\Lambda$ orthogonal to $\Lambda'$ and by $(\Lambda/\Lambda^\perp)$ the cone of positive definite symmetric bilinear forms on the vector space $\Lambda/\Lambda^\perp \otimes_{\mathbb{Z}} \mathbb{R}$ whose radical is defined over $\mathbb{Q}$. It is clear that for $\Lambda'' \subset \Lambda'$ in $C$, one has $C(V/V'' \subset C(V/V')$. Consider the equivalence relation on the set of $(C(V/V')$ for all $\Lambda \in C$ generated by this inclusion relation. We write the quotient set of the above set by this equivalence relation by $C$. Note that for any $\Lambda' \in C$, there is a natural filtration $W_\Lambda$ on $\Lambda$

$$W_{-3}\Lambda = 0 \subset W_{-2}\Lambda = \Lambda' \subset W_{-1}\Lambda = \Lambda_1' \subset W_0\Lambda = \Lambda.$$

Then we write $P_{\Lambda'} \subset G$ for the parabolic subgroup stabilizing this filtration $W_\Lambda$ (Mad12 §2.2.1). Conversely, for any admissible parabolic subgroup $P \subset G$, there is a canonical filtration $W_\Lambda$ on $\mathfrak{g}$:

$$W_{-3}\mathfrak{g} = 0 \subset W_{-2}\mathfrak{g} = \text{Lie}(W_P) \subset W_{-1}\mathfrak{g} = \text{Lie}(U_P) \subset W_0\mathfrak{g} = \text{Lie}(P) \subset W_1\mathfrak{g} = \mathfrak{g},$$

which then induces a filtration on $\Lambda$. Now we choose a $\Lambda' \in C$ such that the corresponding parabolic subgroup $P_{\Lambda'}$ is the same as $P_{\Phi}$. Note that $\Psi^l_{K}(\Phi)$ is the pull-back of the Poincaré line bundle on $A \times A'$ (recall $A$ is the universal abelian scheme over $Sh$) along the natural map $c_l : \text{Hom}_\mathbb{Q}(\Lambda/\Lambda^\perp, A') \rightarrow A \times A'$. Moreover, $Sh(\overline{Q}_\Phi, \overline{D}_\Phi, \overline{K}_\Phi)_{l}$ in fact a $\text{Hom}_\mathbb{Q}(\Lambda/\Lambda^\perp, A')$-torsor over the Shimura variety $Sh(G_{\Phi, h}, D_{\Phi, h}, K_{\Phi, h})$. Thus by Mumford’s vanishing theorem ([Mum70] §III.16), we get the first isomorphism.

As for the second isomorphism, we define the following torsor over $Sh(\overline{Q}_\Phi, \overline{D}_\Phi, \overline{K}_\Phi)_{m}$:

$$\pi^\Phi_{m,n} : \text{Ig}_{m,n}^\Phi_{\text{der}} : \text{Isom}_{\text{Sh}(\overline{Q}_\Phi, \overline{D}_\Phi, \overline{K}_\Phi)_{m}}(A[x_0[p^n]^\circ], A[p^n][y]^\circ) \rightarrow \text{Sh}(\overline{Q}_{\Phi}, \overline{D}_{\Phi}, \overline{K}_{\Phi})_{m}.$$ 

Since the morphism $\pi^\Phi_{m,n} / \overline{P}_{\text{der}} : \text{Ig}_{m,n}^\Phi_{\text{der}} / \overline{P}_{\text{der}} \rightarrow \text{Sh}(\overline{Q}_\Phi, \overline{D}_\Phi, \overline{K}_\Phi)$ is finite, thus $(\pi^\Phi_{m,n} / \overline{P}_{\text{der}})^* (FJ^l_{K})$ is an ample line bundle ([EGA II.5.1.12]), and thus is affine. From this we get the following isomorphism

$$H^0((\text{Ig}_{m,n}^\Phi / \overline{P}_{\text{der}})_{x} \otimes \mathcal{O}_x \mathcal{O}_p/p^n, (\pi^\Phi_{m,n} / \overline{P}_{\text{der}})^* FJ^l_{K}) \cong H^0((\text{Ig}_{m,n}^\Phi / \overline{P}_{\text{der}})_{x}, (\pi^\Phi_{m,n} / \overline{P}_{\text{der}})^* FJ^l_{K}) \otimes \mathcal{O}_x \mathcal{O}_p/p^n.$$

This isomorphism corresponds to the second isomorphism in the proposition in the case $\lambda = 0$. In general, since $\mathcal{O}_{\text{Ig}_{m,n}^\Phi / \overline{P}_{\text{der}}}[\lambda^{-1}]$ and $\mathcal{O}_{\text{Ig}_{m,n}^\Phi / \overline{P}_{\text{der}}}[0]$ are equivalent as $\mathcal{O}_{\text{Ig}_{m,n}^\Phi / \overline{P}_{\text{der}}}$-modules, thus we deduce the second isomorphism for any $\lambda \in X_{\text{dm}}(\overline{T_P})$. 

We can deduce the base change of cuspidal modular forms on the minimal compactification:
Proof. As in the proof of \cite[Proposition 6.2]{Pilloni2012}, it suffices to show that the action of the group $\triangle_K(\Phi)$ on these spaces of cuspidal forms is trivial. Indeed, for $l$ positive definite, its stabilizer in $\triangle_K(\Phi)$ is trivial by the assumption that $K$ is neat.

For a scheme $Y$ over $\mathbb{W}$, we write $i_m: Y_m \hookrightarrow Y$ for the closed immersion of the reduction modulo $p^m$ of $Y$ into $Y$. Then we have the following commutative diagram

$$
\begin{array}{ccc}
Sh_m^\Sigma & \xrightarrow{i_m} & Sh^\Sigma \\
\downarrow \pi & & \downarrow \pi \\
Sh_m^{\text{min}} & \xrightarrow{i_m} & Sh_m^{\text{min}}
\end{array}
$$

We have a similar commutative diagram if we take the $\mu$-ordinary locus of each space in the diagram. We write $i_m^\mu$, $\pi^\mu$ for the induced morphisms. From the above proposition, we get

**Corollary 6.3.** The following natural morphisms of automorphic sheaves are isomorphisms

$$
i_m^* \pi_* \int^\Sigma \mathcal{R}_\infty^{\text{Ig}}[\lambda^{-1}][-C^\Sigma] \simeq \pi_* i_m^* \int^\Sigma \mathcal{R}_\infty^{\text{Ig}}[\lambda^{-1}][-C^\Sigma],$$

$$(i_m^\mu)^* (\pi^\mu)_* \int^\Sigma \mathcal{R}_\infty^{\text{top, Ig}}[\lambda^{-1}][-C^\Sigma] \simeq (\pi^\mu)_* (i_m^\mu)^* \int^\Sigma \mathcal{R}_\infty^{\text{top, Ig}}[\lambda^{-1}][-C^\Sigma].$$

From these isomorphisms, one can deduce

**Proposition 6.4.** For any $\lambda \in X^*_{\text{dm}}(\overline{T_P})$ and any integer $m \geq 0$ the following natural morphisms of reduction modulo $p^m$ are surjective

$$
H^0(Ig_m/\mathcal{P}_L, \mathcal{R}_\infty^{\text{Ig}}[\lambda^{-1}][-C^\Sigma]) \rightarrow H^0(Ig_m/\mathcal{P}_L, \mathcal{R}_m^{\text{Ig}}[\lambda^{-1}][-C^\Sigma]),
$$

$$
H^0(Sh_m^{\Sigma, \mu}, \mathcal{V}_\lambda(-C^\Sigma)) \rightarrow H^0(Sh_m^{\Sigma, \mu, \text{cusp, der}}[\lambda^{-1}]) - \mathcal{V}_\lambda(-C^\Sigma).
$$

**Proof.** This follows from the fact that the $\mu$-ordinary locus on the minimal compactification $Sh_m^{\text{min}}$ is affine because the latter is projective by construction.

Recall the Hodge line bundle $\omega$ we used to construct the Hasse invariant $H$ in \cite{section32}(using its power $\omega^{\otimes N_G}$). Similar to the above proposition, one has

**Proposition 6.5.** For each $\lambda \in X^*_{\text{dm}}(\overline{T_P})$, there is a positive integer $N(\lambda)$ such that for any $t \geq N(\lambda)$, the following reduction map is surjective

$$
H^0(Sh_m^{\Sigma, \text{cusp, der}}[\lambda^{-1}]) - \mathcal{V}_\lambda(-C^\Sigma)) \rightarrow H^0(Sh_m^{\Sigma, \mu, \text{cusp, der}}[\lambda^{-1}]) - \mathcal{V}_\lambda(-C^\Sigma)).$$
The proof is the same as above by noting that $\omega$ is an ample line bundle.

Recall that we have a morphism $\tilde{\tilde{HT}} : \tilde{\tilde{P}} : \tilde{\tilde{P}}_{L}(\mathcal{O}_{p}) \tilde{\tilde{P}}_{\mathcal{O}_{p}} \rightarrow S\Sigma^{\mu}$. We deduce the following important corollary

**Corollary 6.6.** The module  
$$ \left( \bigoplus_{\lambda \in X_{\dim}^{\mathcal{T}}(\mathcal{T}_{p})} (\tilde{\tilde{HT}}_{\mathcal{T}})^{*}H^{0}(Sh^{\Sigma^{\mu}}, V^{\Sigma}[-1]) \left[ \frac{1}{p} \right] \right) \cap V_{\mathcal{C}}^{\mathcal{P}_{\mathcal{O}_{p}}} $$

is dense in $V_{\mathcal{C}}^{\mathcal{P}_{\mathcal{O}_{p}}}$.

### 6.3. Finite Dimensionality

In this subsection, we will show that the space of $\mu$-ordinary $p$-adic modular forms is bounded in certain sense. The main strategy is to use Proposition 3.8 (cf. [Pil12, Appendix A.3]).

First note that for a $\mu$-ordinary modular form $f$, the multiplication by Hasse invariant $H(f)$ is again $\mu$-ordinary:

**Lemma 6.7.** We fix a character $\lambda \in X^{\mathcal{T}}_{\dim}(\mathcal{T}_{p})$. For any cuspidal automorphic form $f$ in the space $H^{0}(Sh^{\Sigma^{\mu}}, V^{\Sigma}(-C_\Sigma))$ and any Hecke operator $\mathcal{T}_{\epsilon}$ with $\epsilon \in \mathcal{T}_{p}^{\epsilon} (\mathcal{E}_{p})$, we have  
$$ H(\mathcal{T}_{\epsilon} f) = \mathcal{T}_{\epsilon} H(f). $$

**Proof.** Consider a $\mu$-ordinary point $x \in \text{Sh}^{\mu}$, $\mathcal{A}_{x}$ the abelian scheme over $x$ and $(\mathcal{D}_{x}, \text{Fr})$ the $F$-crystal with $G$-structure associated to the $p$-divisible group $\mathcal{A}_{x}[p^\infty]$ of $\mathcal{A}_{x}$. One can choose a basis  
$$ B = \{ e_{1}, \ldots, e_{n}, e_{n+1}, \ldots, e_{n+2} \} $$

of $\mathbb{D}_{x}$ under which $\text{Fr}^{\mu}$ becomes $\text{diag}(1_{n}, p1_{n})$. Here $2n = \dim(V)$ and $\mathbb{F}_{p}^{\mu}$ is the splitting field of the torus $(\mathcal{T}_{\mathbb{F}_{p}})^{\mathcal{T}_{p}}$. Then $Hf(x)$ is the multiplication of $f(x)$ by the determinant $\det(\text{Fr}^{\mu})$ where $\text{Fr}^{\mu}$ is the induced map on the quotient $\mathbb{D}_{x}/\text{Fil}^{1}\mathbb{D}_{x}$. Applying the Hecke operator $\mathcal{T}_{\epsilon}$ to the form $f$ amounts, in terms of Dieudonné crystals, to consider subcrystals $\mathbb{D}_{x}'$ of $\mathbb{D}_{x}$ such that under the basis $B$ one has $\epsilon \mathbb{D}_{x} = \mathbb{D}_{x}'$. Since the Hodge filtration on $\mathbb{D}_{x}'$ is induced from $\mathbb{D}_{x}$, we see that the determinant $\det(\text{Fr}^{\mu})$ of the Frobenius on the quotient $\mathbb{D}_{x}'/\text{Fil}(\mathbb{D}_{x}')$ is the same as $\det(\text{Fr}^{\mu})$. Thus we conclude that $H(\mathcal{T}_{\epsilon} f) = \mathcal{T}_{\epsilon} H(f)$ for any $f$. \hfill $\square$

Next we turn to canonical subgroups of the $p$-divisible group $\mathcal{A}[p^{\infty}]$ of the abelian scheme $\mathcal{A}$ in a strict neighbourhood of the $\mu$-ordinary locus $\text{Sh}^{\mu}$. We first recall some definitions from [Bij16] (see also [Far10]). Fix a finite extension $L$ of $\mathbb{Q}_{p}$ and also a valuation $v$ on $L$ such that $v(p) = 1$. Let $H$ be a finite flat group scheme of $p$-power order over the ring of integers $\mathcal{O}_{L}$ of $L$. Write $\omega_{H}$ for the co-normal module along the unit section of $H$. The degree of $H$ is defined to be  
$$ \deg(H) := v(\text{Fitt}(\omega_{H})), $$

where Fitt is the Fitting ideal. For two finite flat subgroups $H_{1}, H_{2}$ of $H$, one has (cf. [Bij16 Corollary 1.16]):  
$$ \deg(H_{1}) + \deg(H_{2}) \leq \deg(H_{1} \cap H_{2}) + \deg(H_{1} + H_{2}). $$

Write $\text{ht}(H)$ for the height of $H$. Then the slope of $H$ is related to its height and degree by the formula:  
$$ \lambda(H) = \frac{\deg(H)}{\text{ht}(H)}. $$

Recall that the slopes of the abelian scheme $\mathcal{A}_{x_{0}}$ over a point $x_{0} \in \text{Sh}^{\mu}$ in the $\mu$-ordinary locus are $1 \geq \lambda_{1} > \lambda_{2} \cdots > \lambda_{r} \geq 0$. Write  
$$ \delta = \frac{1}{4} \min_{0 \leq i < r} \lambda_{i} - \lambda_{i+1} > 0. $$

Let $\mathcal{A}$ be the abelian scheme over $\text{Sh}$. For a point $x \in \text{Sh}^{\mu}$ in the $\mu$-ordinary locus, we have a slope filtration of the $p$-divisible group $\mathcal{D}_{x}$ associated to the abelian scheme $\mathcal{A}_{x}$ over the point $x$:  
$$ 0 \subset (\mathcal{D}_{x})_{1} \subset (\mathcal{D}_{x})_{2} \subset \cdots \subset (\mathcal{D}_{x})_{r} \subset \mathcal{D}_{x}. $$

Correspondingly we have a filtration for the $p$-torsion subgroups of the above subgroups $0 \subset (\mathcal{D}_{x})_{i}[p] \subset \cdots \subset (\mathcal{D}_{x})_{r}[p] \subset \mathcal{D}_{x}[p]$. We write $\text{ht}$ for the height of $(\mathcal{D}_{x})_{i}[p]$ ($i = 1, \cdots, r$). Then one has  
$$ \deg((\mathcal{D}_{x})_{i}[p]) = h_{i} \lambda_{1} + (h_{i} - h_{i-1}) \lambda_{i-1} =: d_{i}. $$
Consider a maximal element $b \in B(G, X) \setminus \{ \overline{v} \}$ and a geometric point $x \in \mathcal{N}^b$ in the Shimura scheme. Then the slopes of the abelian scheme $\mathcal{A}_x$ are

$$1 \geq \lambda_1 > \cdots > \lambda_{i-1} > (\lambda_i) > \lambda'_i > (\lambda_{i+1}) > \lambda_{i+2} > \cdots$$

$$> \lambda_{r-i-1} > (\lambda_{r-i}) > \lambda'_{r-i} > (\lambda_{r+1-i}) > \lambda_{r+2-i} > \cdots > \lambda_r \geq 0$$

where $(\lambda_i)$ means that the slope $\lambda_i$ may not exist in the above sequence and $\lambda'_i = \frac{1}{2}(\lambda_i + \lambda_{i+1})$ and similarly $\lambda'_{r-i} = \frac{1}{2}(\lambda_{r-i} + \lambda_{r+1-i})$.

Now we have the following observation (cf. [Bij16, Proposition 1.24]):

**Lemma 6.8.** Let $b$ be $\overline{v}$ or a maximal element in $B(G, X) \setminus \{ \overline{v} \}$ and consider a geometric point $x \in \mathcal{N}^b$. Then for any $i = 1, \ldots, r$, there is at most one subgroup $H$ of the finite flat group scheme $\mathcal{D}_x[p]$ such that $\text{ht}(H) = h_i$ and $\deg(H) > d_i - \delta$.

**Proof.** Suppose that there are two such subgroups $H_1, H_2$ of $H := (\mathcal{D}_x)[p]$. Write $h$ for the height of $H_1 \cap H_2$. Then the height of $H_1 + H_2$ is $2h_i - h$.

First we assume that $b = \overline{v}$. Then the degree $H_1 \cap H_2$ is bounded as follows: if $h_k \leq h \leq h_{k+1}$ (set $h_0 = 0$), then $\deg(H_1 \cap H_2) \leq h_1 \lambda_1 + (h_2 - h_1) \lambda_2 + \cdots + (h_k - h_{k-1}) \lambda_k + (h - h_k) \lambda_{k+1}$. On the other hand, if $h \leq 2h_i - h \leq h_{i+1}$, then similarly $\deg(H_1 + H_2) \leq h_1 \lambda_1 + (h_2 - h_1) \lambda_2 + \cdots + (h_i - h_{i-1}) \lambda_i + (2h_i - h - h_i) \lambda_{i+1}$. Suppose that $h < h_i$, then we have $h_{k+1} \leq h \leq h_i$. Write

$$\deg_{i; k, l} = h_1 \lambda_1 + (h_2 - h_1) \lambda_2 + \cdots + (h_k - h_{k-1}) \lambda_k + (h - h_k) \lambda_{k+1}$$

$$+ h_1 \lambda_1 + (h_2 - h_1) \lambda_2 + \cdots + (h_l - h_{l-1}) \lambda_l + (2h_i - h - h_l) \lambda_{l+1}.$$ From the fact $\deg(H_1) + \deg(H_2) \leq \deg(H_1 \cap H_2) + \deg(H_1 + H_2)$, and the assumption $\deg(H_1), \deg(H_2) > d_i - \delta$, one gets by a simple computation the following inequality:

$$h_{k+1} - h_k - (h - h_k)(\lambda_{k+1} - \lambda_{k+2}) + \cdots + (h_i - h_k - (h - h_k))(\lambda_i - \lambda_{i+1})$$

$$+ (h_i - h_k - (h - h_k + h_{i+1} - h_i)) (\lambda_{i+1} - \lambda_{i+2}) + \cdots + (h_i - h_k - (h - h_k + h_l - h_i))(\lambda_l - \lambda_{l+1})$$

$$+ (h_i - h_l) \lambda_{l+1} < 2 \delta$$

which is impossible by the definition of $\delta$. Thus we get $h = h_i$ and therefore $H_1 = H_2$.

Next consider $b \neq \overline{v}$. Suppose that the slopes of $\mathcal{A}_x$ are

$$\lambda_1 > \cdots > \lambda_{i_0} > \lambda'_{i_0} > \lambda_{i_0+1} > \cdots > \lambda_{r-i_0-1} > \lambda_{r-i_0} > \lambda'_{r-i_0} > \lambda_{r-i_0+1} > \cdots > \lambda_r,$$

and the corresponding heights are

$$h_1 < \cdots < h_{i_0-1} < h_{i_0} < \Delta h < h_{i_0+1} < \cdots$$

$$< h_{r-i_0-1} < h_{r-i_0} - \Delta h < h_{r-i_0+1} < \cdots < h_r.$$ We then put

$$s_1 = d_1, \ldots, s_{i_0-1} = d_{i_0-1},$$

$$s_{i_0} = d_{i_0} - \Delta h \lambda_{i_0}, s'_{i_0} = d_{i_0} + \Delta h \lambda_{i_0+1}, s_{i_0+1} = d_{i_0+1} - \Delta h (\lambda_{i_0} - \lambda_{i_0+1}), \cdots,$$

$$s_{r-i_0-1} = d_{r-i_0-1} - \Delta h (\lambda_{i_0} - \lambda_{i_0+1}), s_{r-i_0} = d_{r-i_0} - \Delta h (\lambda_{i_0} - \lambda_{i_0+1}) - \Delta h \lambda_{r-i_0},$$

$$s'_{r-i_0} = d_{r-i_0+1} - \Delta h (\lambda_{i_0} - \lambda_{i_0+1}) + \Delta h \lambda_{r-i_0+1},$$

$$s_{r-i_0+1} = d_{r-i_0+1} - \Delta h (\lambda_{i_0} - \lambda_{i_0+1}) - \Delta h (\lambda_{r-i_0} - \lambda_{r-i_0+1}), \cdots$$

$$s_r = d_r - \Delta h (\lambda_{i_0} - \lambda_{i_0+1}) - \Delta h (\lambda_{r-i_0} - \lambda_{r-i_0+1}).$$

These quantities are just the heights of the subgroups of $\mathcal{D}_x[p]$ of the corresponding height and slopes as above. Now the same reasoning as above shows that there is at most one subgroup $H$ of $\mathcal{D}_x[p]$ such that $\text{ht}(H) = h_i$, resp., $= h'_i$, $= h'_{r-i_0}$ and $\text{deg}(H) > d_i - \delta$, resp., $> s'_{i_0} - \delta$, resp., $> s'_{r-i_0} - \delta$. If $i < i_0$, then we are done by the definition of $s_i$. If $i = i_0$, then (assume again that $h_k \leq h \leq h_{k+1}$ and $h_l \leq 2h_{i_0} - h \leq h_{l+1}$)

$$2(d_{i_0} - \delta) - 2 \Delta h \lambda_{i_0} = 2(s_{i_0} - \delta) < \deg(H_1) + \deg(H_2)$$

$$\leq \deg(H_1 \cap H_2) + \deg(H_1 + H_2)$$

$$\leq \deg_{s_{i_0}; k, l} - 2 \Delta h \lambda_{i_0}.$$
The last inequality is true regardless of the relation between \( h \) and \( \Delta h \). Thus from the first part, we know that this is impossible unless \( h = h_1 \). Similar reasoning applies to the remaining cases. \( \square \)

Using the arguments from [Bij16, Corollary 1.26], we get the existence and uniqueness of canonical subgroups \( 0 < H_1 < H_2 < \cdots < H_r < D[p] \) with \( x \in Sh \) such that its reduction \( \overline{x} \in N^b \) with \( b \in B(G, X) \) as in the above lemma, the maximal element or next to the maximal element. Here we require that each \( H_i \) is of height \( \text{ht}(H_i) = h_i \) and of degree \( \text{deg}(H_i) > d_i - \delta \). Now one can define Hecke operators \( \mathbb{T}_{H_i} \) for each subgroup \( H_i \) as follows: we write \( Sh_i^1 := \bigcup bN^b \) with \( b \) running through the maximal or next-to-the-maximal elements in \( B(G, X) \). Then set \( Sh_i^0 \subset Sh \), consisting of those \( x \) whose reduction \( \overline{x} \in Sh_i^1 \). For each \( H_i \), there is a \( p \)-isogeny of kernel \( H_i \):

\[
\pi_i : A \rightarrow A_i := A/H_i.
\]

We then define a correspondence \( Sh(V)_{H_i} \) over \( Sh(V) \) as follows: for any \( \mathbb{Z}_p \)-algebra \( A \), \( Sh(V)_{H_i}(A) \) is the set of equivalence classes of triples \((A, H, \psi_{K^\vee})\) where \( A/\phi \) is a principally polarized abelian scheme over \( A, \psi_{K^\vee} \) a level structure on \( A \) of level \( K \) and \( H \subset A[p] \) a finite flat subgroup of height \( \text{ht}(H) = h_i \) and of degree \( \text{deg}(H) > d_i - \delta \). Then this correspondence is representable (again denoted) by \( Sh(V)_{H_i} \). We write \( Sh_{H_i} \) for the pull-back of this correspondence along the embedding \( Sh \hookrightarrow Sh(V) \). We have two natural projections \( pr_1, pr_2 : Sh_{H_i} \rightarrow Sh, pr_1((A, \psi_K, H)) = (A, \psi_K) \) and \( pr_2((A, \psi_K, H)) = (A/H, \psi_K) \). Write \( \epsilon_{H_i} \) for the element in \( \mathcal{T}_p'(E_p) \) which induces the filtration \( 0 \subset H_i \subset A[p] \). Now for any \( \mathcal{O} \)-algebra \( A \), consider the endomorphism \( \mathbb{T}_{H_i} \) of \( H^0(Sh^0, \lambda) \) given by the composition of the following morphisms:

\[
H^0(Sh_{H_i}^0, \lambda) \rightarrow H^0(Sh_{H_1}/A, pr_2^*\lambda) \xrightarrow{\pi^*} H^0(Sh_{H_i}/A, pr_1^*\lambda) \xrightarrow{\mathbb{T}_{H_i}} H^0(Sh_{H_i}^0, \lambda).
\]

On the abelian scheme \( A \) over \( Sh^0 \), there is a filtration of canonical subgroups as described above \( 0 < H_1 < H_2 < \cdots < H_r \subset A[p] \). For each \( x \in Sh^0 \), we write \( (H_i)_x \) for the specialization of \( H_i \) to \( x \). Now recall the argument in the proof of Proposition 5.4 for \( x \) in the \( \mu \)-ordinary locus, \( \pi^* \) corresponds to the diagonal matrix \( \epsilon_{H_i} \); while for \( x \) not in the \( \mu \)-ordinary locus, \( \pi^* \) corresponds to the diagonal matrix \( \epsilon_{H_i} \), which is a product of \( \epsilon_{H_i} \) and another diagonal matrix \( M_i \in \text{GSp}(V, \psi) \). Here on the diagonal of \( M_i \), the \((h_i-1)\)-th entry is given by \( p^{-1 + \text{deg}(H_i)} \) and the \((\dim V - h_i + 1)\)-th entry is given by \( p^{1 - \text{deg}(H_i)} \), the remaining entries are all 1. We put

\[
C = \max_i m_{\epsilon_{H_i}}.
\]

Note that this constant \( C \) depends only on the Shimura datum \( (G, X) \) and the parabolic subgroup \( \tilde{P}_L \).

The next lemma shows that for large weights \( \lambda \in X^1(\tilde{T}_\mathbb{P}) \), the Hecke operators \( \mathbb{T}_{H_i} \) act zero on the non-\( \mu \)-ordinary locus \( Sh_{1}^{\Sigma, n-\mu} = Sh_{1}^{\Sigma} \backslash Sh_{1}^\mu \) (cf. [Pil12, Proposition A.5]):

**Lemma 6.9.** For any \( i = 1, 2, \cdots, r \) and any modular form \( f \in H^0(Sh_{H_i}^1, \lambda) \) of weight \( \lambda \in X^1(\tilde{T}_\mathbb{P}) \) with \( \text{val}_p(\lambda(\epsilon_{H_i})) \geq m_{\epsilon_{H_i}} + 1 \), one has

\[
(\mathbb{T}_{H_i} f)|_{Sh_{1}^{\Sigma, n-\mu}} = 0.
\]

In particular, for any \( f \in H^0(Sh_{H_i}^1, \lambda) \) of weight \( \lambda \in X^1(\tilde{T}_\mathbb{P}) \) with \( \text{min}_i \text{val}_p(\lambda(\epsilon_{H_i})) \geq C + 1 \), then

\[
(e_{\mathbb{P}} f)|_{Sh_{1}^{\Sigma, n-\mu}} = 0.
\]

**Proof.** It suffices to show that \( \mathbb{T}_{H_i} f \) vanishes on \( Sh_{1}^{\Sigma} \backslash Sh_{1}^\mu \). We have the following commutative diagram at each point \( x \) which is not in the \( \mu \)-ordinary locus:

\[
\begin{array}{ccc}
\varepsilon^* H^1_{\text{dR}}(A_x) & \xrightarrow{\pi^*} & \varepsilon^* H^1_{\text{dR}}(A_x/(H_i)_x) \\
\downarrow \mathbb{D} & & \downarrow \mathbb{D}' \\
A \mathcal{W} & \xrightarrow{M_i} & A \mathcal{W}
\end{array}
\]

For \( F \) an element in \( H^0(Sh_{H_i}, \lambda) \), we have the transformation rule: for each \( g \in G(\mathbb{W}[1/\mathfrak{p}]) \)

\[
(\pi^* F)(\mathbb{D} \circ g) = F(\mathbb{D}' \circ M_i^{-1} \circ g).
\]

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By the Iwasawa decomposition of $G$, we can write $M_i^{-1} \circ g = \text{kan}$ with $k \in G(\mathbb{W})$, $a \in \tilde{T}_P(\mathbb{W}[\frac{1}{p}])$ and $n$ a unipotent element. Moreover, we have $\lambda(a) = \lambda(e_{H_i}').$ By assumption $\text{val}_p(\lambda(e_{H_i}')) \geq m'e_{H_i} + 1,$ thus $\pi^*F$ is divisible by $p,$ which gives the first part. The second part follows from the first part. 

It follows from the definition that $\mathbb{T}_{H_i},$ when restricted to the $\mu$-ordinary locus, coincides with the Hecke operator $\mathbb{T}_{e_{H_i}'}$ as in Definition 5.6. Since $Sh^\sim$ is dense in $Sh$ and $Sh^E,$ we can extend $\mathbb{T}_{H_i}$ to act on $H^0(Sh, V_{\lambda}(-C^\Sigma)) = H^0(Sh^E, V_{\lambda}(-C^\Sigma)).$ Moreover, by Proposition 6.5 we know that for each $\lambda \in X^*_\text{dom}(\tilde{T}_P),$ there is a positive integer $N(\lambda)$ such that for any $t \geq N(\lambda),$ the reduction map from $H^0(Sh^E, V_{\lambda+tN_G\lambda_G}(-C^\Sigma))$ to $H^0(Sh^E, V_{\lambda+tN_G\lambda_G}(-C^\Sigma))$ is surjective. Thus for such $\lambda$ and $t,$ the action of $\mathbb{T}_{H_i}$ descends to $H^0(Sh^E, V_{\lambda+tN_G\lambda_G}(-C^\Sigma)).$

By the relation $\lambda_i + \lambda_{i+1} - 1 = 0,$ it is easy to see that $e_{H_i} \neq 1.$ We put

$$n_G := \min \text{val}_p(\lambda_G(e_{H_i}')) > 0,$$

We deduce immediately the following corollary (cf. [Pil12, Proposition A.6]):

**Corollary 6.10.** For a dominant weight $\lambda \in X^*_\text{dom}(\tilde{T}_P)$ and $t \in \mathbb{N}$ with $t \geq \max(N(\lambda), (C+1)/n_G),$ the multiplication by the Hasse invariant $H : H^0(Sh^E, V_{\lambda+tN_G\lambda_G}(-C^\Sigma)) \rightarrow H^0(Sh^E, V_{\lambda+(t+N_G\lambda_G)}(-C^\Sigma))$ induces an isomorphism on the subspaces of $\mu$-ordinary cuspidal modular forms

$$H : e_{\tilde{T}} H^0(Sh^E, V_{\lambda+tN_G\lambda_G}(-C^\Sigma)) \rightarrow e_{\tilde{T}} H^0(Sh^E, V_{\lambda+(t+N_G\lambda_G)}(-C^\Sigma)).$$

**Proof.** By Lemma 6.7 we know that this multiplication map commutes with the Hecke operators $\mathbb{T}_\epsilon$ for $\epsilon \in \tilde{T}_P^+(E_p).$ By the preceding lemma, we know that any $f \in e_{\tilde{T}} H^0(Sh^E, V_{\lambda+tN_G\lambda_G}(-C^\Sigma))$ vanishes on the non-$\mu$-ordinary locus and thus we can divide $f$ by $H$ to get a modular form of weight $\lambda + t\lambda_G.$

Recall the construction of the Hasse invariant from §3.2. It is defined by the $N_G$-th power of the pull-back of the Hodge line bundle $\omega$ from $Sh(V)$ to $Sh.$ Then one has (cf. [Pil12, Corollaire A.3])

**Proposition 6.11.** The space $e_{\tilde{T}} H^0(Sh^E, V_{\lambda}(-C^\Sigma))$ is of finite dimension over $\mathbb{T}_p.$ Moreover, for $t \gg 0$ and any $\lambda \in X^*(\tilde{T}_P),$ we have the following identities ($r = 1, 2, \cdots, \infty$):

$$e_{\tilde{T}} H^0(Sh^E, V_{\lambda+tN_G\lambda_G}(-C^\Sigma)) = e_{\tilde{T}} H^0(Sh^E, V_{\lambda+(t+N_G\lambda_G)}(-C^\Sigma)).$$

**Proof.** We have, by the properties of Hasse invariant $H$,

$$H^0(Sh^E_{\lambda}, V_{\lambda}(-C^\Sigma)) = \bigcup_{t \in \mathbb{N}} H^{-1} H^0(Sh^E_{\lambda}, V_{\lambda+tN_G\lambda_G}(-C^\Sigma)).$$

Note that the multiplication by $H$ embeds $H^0(Sh^E_{\lambda}, V_{\lambda})$ into $H^0(Sh^E_{\lambda}, V_{\lambda+tN_G\lambda_G}),$ thus for the first part in the proposition, it suffices to treat $\lambda' := \lambda + t_0 N_G \lambda_G$ in place of $\lambda$ with some $t_0 \gg 0.$ Therefore we have

$$e_{\tilde{T}} H^0(Sh^E_{\lambda}, V_{\lambda}(-C^\Sigma)) = \bigcup_{t \in \mathbb{N}} e_{\tilde{T}} H^0(Sh^E_{\lambda}, V_{\lambda+tN_G\lambda_G}(-C^\Sigma)).$$

However, the preceding corollary shows that the RHS is equal to $e_{\tilde{T}} H^0(Sh^E_{\lambda}, V_{\lambda}(-C^\Sigma))$, thus we get, for $t \gg 0$,

$$e_{\tilde{T}} H^0(Sh^E_{\lambda}, V_{\lambda+tN_G\lambda_G}(-C^\Sigma)) = e_{\tilde{T}} H^0(Sh^E_{\lambda}, V_{\lambda+tN_G\lambda_G}(-C^\Sigma)).$$

The case for $r = \infty$ follows from Proposition 6.5 with $t \gg 0.$

## 7. Hida theory for Shimura varieties of Hodge type

**7.1. Control theorems.** We retain the notations as in the preceding sections (cf. §4.1): $(G, X)$ is a mixed Shimura datum of Hodge type (with embedding $(G, X) \rightarrow (\text{GSp}(V, \psi), S^\pm), \tilde{P}$ a parabolic subgroup of $G$ (the restriction from a parabolic subgroup $P_V$ of $\text{GSp}(V, \psi)$ to $G$), $T_P = \tilde{P}/\tilde{P}^{\text{der}} = \tilde{P}_L/\tilde{P}^{\text{der}}$ where $\tilde{P}_L = \tilde{P} \cap L$. We need some more notations. Write $(?) = (\emptyset, \text{cusp})$

$$M(\tilde{\mathbb{P}}) := \lim_{\tilde{m}} e_{\tilde{T}} \mathbb{P}^{\tilde{m}} = M(\tilde{\mathbb{P}}) := \text{Hom}_{\mathbb{C}_p}(M(\tilde{\mathbb{P}}), E_p/\mathbb{C}_p)$$

(8)
for the colimit and its Pontryagin dual. $M(\tilde P)$ is the space of $\mu$-ordinary $p$-adic modular forms on $Sh_\Sigma$. We then put

$$\tilde T^1_P := \text{Ker}(\tilde T_P(O_p) \to \tilde T_P(\mathcal{O}/p)), \quad W_p := \mathcal{O}_p[[\tilde T_P(O_p)]], \quad W^1_p := \mathcal{O}_p[[\tilde T^1_P]].$$

$W^1_p$ is the weight space. The decomposition $\tilde T_P(O_p) = \tilde T_P(\mathcal{O}/p) \times \tilde T^1_P$ induced from the Teichmüller lifting shows that we can view the Iwasawa weight algebra $W^1_p$ as a subalgebra of $W_p$, and thus the $W_p$-module structures on $e_pV^\text{der}_{\Sigma,m}$, $M(\tilde P)$, and $M(\tilde P)$? ($? = 0$, cusp) give rise to $W^1_p$-module structures on these spaces. Here we gather some of the results that we have proved or the corollaries of these results.

**Theorem 7.1.**  
(1) For any character $\lambda \in X^*(\tilde T_P)$, the $O_p$-module $e_pV^\text{der}_{\text{cusp,}\infty}[\lambda^{-1}]$ is free of finite rank (denoted by $\text{rk}(\lambda)$). Moreover the rank $\text{rk}(\lambda)$ depends only on the image of $\lambda$ by the natural projection map $X^*(\tilde T_P) \to X^*(\tilde T_P)/\mathbb{Z}N_{G\lambda}G$.

(2) For any $\lambda \in X^*_d(\tilde T_P)$, we have an isomorphism

$$e_pH^0(\mathfrak{I}_G,\tilde R_{\infty} \mathcal{O}_p[\lambda^{-1}]) \simeq e_pV^\text{der}_{\infty}[\lambda^{-1}].$$

If moreover $\lambda \in X^*_d(\tilde T_P)$, we can descent the isomorphism to $Sh_{\Sigma,\mu}$:

$$e_pH^0(Sh_{\Sigma,\mu},\tilde R_{\infty} \mathcal{O}_p[\lambda^{-1}]) \simeq e_pV^\text{der}_{\infty}[\lambda^{-1}].$$

(3) For any $\lambda \in X^*(\tilde T_P)$ and $t \gg 0$ (depending on $\lambda$), we have an isomorphism of spaces of cuspidal modular forms:

$$e_pH^0(Sh_{\Sigma,\mu},\tilde R_{\infty} [\lambda^t \mathcal{O}_p[\lambda^{-1}]] \simeq e_pV^\text{der}_{\text{cusp,}\infty}[(\lambda + t\lambda_{G})^{-1}].$$

(4) For any $\lambda \in X^*(\tilde T_P)$, we have the following specialization isomorphism

$$M(\tilde P)_{\text{cusp}} \otimes W_p,\lambda \mathcal{O}_p \simeq \text{Hom}O_p(e_pV^\text{der}_{\text{cusp,}\infty}[\lambda^{-1}], \mathcal{O}_p);$$

(5) The $W^1_p$-module $e_pV^\text{der}_{\text{cusp,}\infty}$ is free of finite rank.

**Proof.**  
(1) First note that we have the base change isomorphism by Proposition\ref{6.11}

$$e_pV^\text{der}_{\text{cusp,}\infty}[\lambda^{-1}] \otimes \mathcal{O}_p/p \simeq e_pV^\text{der}_{\text{cusp,}\infty}[\lambda^{-1}].$$

Now that the dimension of $e_pV^\text{der}_{\text{cusp,}\infty}[\lambda^{-1}]$ depends on the image of $\lambda$ in $X^*(\tilde T_P)/\mathbb{Z}N_{G\lambda}G$. Thus we can assume that $\lambda \in X^*_d(\tilde T_P)$ and now we can apply Proposition\ref{6.11} to such $\lambda$.

(2) The first point is Proposition\ref{5.10}. The second point follows from Corollary\ref{5.13}.

(3) This is Proposition\ref{6.11}.

(4) By definition, for any $\lambda \in X^*(\tilde T_P)$, we have the following natural isomorphism

$$M(\tilde P)_{\text{cusp}} \otimes W_p,\lambda \mathcal{O}_p \simeq \text{Hom}O_p(M(\tilde P)_{\text{cusp}}[\lambda^{-1}], E_p/\mathcal{O}_p) \simeq \text{Hom}O_p(e_pV^\text{der}_{\text{cusp,}\infty}[\lambda^{-1}] \otimes \mathcal{O}_p, E_p/\mathcal{O}_p) \simeq \text{Hom}O_p(e_pV^\text{der}_{\text{cusp,}\infty}[\lambda^{-1}], \mathcal{O}_p).$$

(5) We follow [Pil12, p.36]. For any character $\chi: \tilde T_P(O_p) \to O_p^\times$ in $X^*(\tilde T_P)$, we write $r(\chi)$ for the $O_p$-rank of $M(\tilde P)_{\text{cusp}} \otimes W_p,\chi \mathcal{O}_p$, which is finite by the points 1 and 4. Then we can define a surjective $W^1_p$-linear map $f_\chi: (W^1_p)^r(\chi) \to e_pV^\text{der}_{\text{cusp,}\infty} \otimes \mathcal{O}_p[\tilde T_P(\mathcal{O}/p)], \mathcal{O}_p$. Now for any other character $\chi': \tilde T_P(O_p) \to O_p^\times$ such that $\chi|_{\tilde T_P(\mathcal{O}/p)} = \chi'|_{\tilde T_P(\mathcal{O}/p)}$, the induced map

$$f_\chi \otimes 1: (W^1_p)^r(\chi) \otimes W_p,\chi' \mathcal{O}_p \to e_pV^\text{der}_{\text{cusp,}\infty} \otimes W_p,\chi \mathcal{O}_p$$
Remark 7.3. By construction, the spherical Hecke algebra at \( \ell \) is a free \( \ell \)-adic algebra of finite rank. Thus we see that \( f_\ell \) is in fact an isomorphism. On the other hand, we have a decomposition

\[
\mathcal{M}(\tilde{P})_{\text{cusp}} = \oplus_{\chi} \mathcal{M}(\tilde{P})_{\text{cusp}} \otimes \mathcal{O}_p[\mathcal{O}_p/(\mathcal{O}/p)]\chi, \quad \mathcal{O}_p,
\]

where \( \chi \) runs through characters indexed by \( \mathcal{O}_p \). Since there are only finitely many such \( \chi \), we get immediately that \( \mathcal{M}(\tilde{P})_{\text{cusp}} \) is free of finite rank over \( \mathcal{O}_p \).

\[\square\]

7.2. Hida families. Recall that \( G \) is a connected reductive group over \( \mathbb{Q} \) which embeds in \( \text{GSp}(V, \psi) \). We have also fixed a compact open subgroup \( K \) of \( G(\mathbb{A}_f) \). Let \( S' \) be the subset of rational primes \( \ell \) of \( \mathbb{Q} \) such that the \( \ell \)-th component of \( K \) is not maximal open compact in \( G(\mathbb{Q}_\ell) \). Write \( S = S' \cup \{p\} \). For each rational prime \( \ell \) of \( \mathbb{Q} \) not in \( S \) such that \( G \) is unramified over \( \ell \), we fix one hyperspecial subgroup \( K_{\ell} \) of \( G(\mathbb{Q}_\ell) \) which is the intersection \( G(\mathbb{Q}_\ell) \cap K_{V,\ell} \) where \( K_{V,\ell} = \text{GSp}(V, \psi)(\mathbb{Z}_\ell) \) is a maximal compact open subgroup of \( \text{GSp}(V, \psi)(\mathbb{Q}_\ell) \). Then we define the local commutative spherical Hecke algebra at \( \ell \) as usual in the following way:

\[
\mathcal{H}_\ell := \mathbb{Z}[G(\mathbb{Q}_\ell)]/[K_\ell].
\]

For each double coset \( K_{V,\ell} \beta K_\ell \) with \( \beta \in G(\mathbb{Q}_\ell) \), one can define a spherical Hecke operator \( \mathbb{T}_\beta \) on the spaces of \( \mu \)-ordinary \( p \)-adic automorphic forms \( \mathcal{M}(\tilde{P})_\ell \) and \( \mathcal{M}(\tilde{P})_{\text{cusp}} \) similar to Definition 5.6 as follows: we first define a correspondence \( \mathcal{X}_{\nu,\ell} \) over \( Sh(V) \) as in \( \text{cf.} \) \( [\text{FC90}, \text{§VII.3}] \). Then we write \( \mathcal{X}_\beta \) for the pull-back of \( \mathcal{X}_{\nu,\ell} \) along the embedding \( Sh \to Sh(V) \). We have two natural projections \( \mathcal{P}_1, \mathcal{P}_2: \mathcal{X}_\beta \to Sh(V) \), a universal isogeny \( \pi^{G,\beta} \) between universal abelian schemes \( \pi^{G,\beta}: A \to A' \). Then for any sheaf \( \mathcal{F} \) over \( Sh \), we put

\[
\mathcal{T}_\beta: H^0(Sh, \mathcal{F}) \to H^0(\mathcal{X}_\beta, \mathcal{P}_1^* \mathcal{F}) \xrightarrow{\pi^{G,\beta}_*} H^0(\mathcal{X}_\beta, \mathcal{P}_2^* \mathcal{F}) \xrightarrow{\mathcal{P}_1} H^0(Sh, \mathcal{F}).
\]

Similarly, one can define \( \mathcal{T}_\beta \) on the Igusa towers. We know moreover that these Hecke operators \( \mathcal{T}_\beta \) commute with each other for different \( \ell \) and also commutes with those operators \( \mathcal{T}_\epsilon \) for \( \epsilon \in \mathcal{T}^+_p(E_p) \) as well as the action of \( \mathcal{T}_{\tilde{P}}(\mathcal{O}_p) \).

Definition 7.2. We define the cuspidal \( \mathcal{P} \)-ordinary Hecke algebra

\[
e_{\mathcal{P}}\mathcal{H}_{\mathcal{P}}^{\text{cusp}} := \mathcal{M}(\mathcal{P})(\mathcal{H}_\ell | \ell \notin S) \subset \text{End}_{\mathcal{W}_p}(\mathcal{M}(\mathcal{P})_{\text{cusp}}).
\]

Each irreducible component of \( \text{Spec}(e_{\mathcal{P}}\mathcal{H}_{\mathcal{P}}^{\text{cusp}}) \) is called a Hida family.

Remark 7.3. By construction, the \( \mathcal{O}_p \)-points of \( \text{Spec}(e_{\mathcal{P}}\mathcal{H}_{\mathcal{P}}^{\text{cusp}}) \) correspond bijectively with the eigen-systems for the action of the \( e_{\mathcal{P}}\mathcal{H}_{\mathcal{P}}^{\text{cusp}} \) on the space \( \mathcal{M}(\mathcal{P})_{\text{cusp}} \). By the results in the preceding subsection, we know that the Hecke algebra \( e_{\mathcal{P}}\mathcal{H}_{\mathcal{P}}^{\text{cusp}} \) is a faithfully flat \( \mathcal{W}_p \)-algebra of finite rank. Thus we see that for any \( \lambda \in X^*(\tilde{P}) \), any eigenform \( f \in e_{\mathcal{P}}\mathcal{V}_{\text{cusp}, \infty}^{\text{der}}[\lambda^{-1}] \otimes_{\mathcal{O}_p} \mathcal{O}_p \) for the Hecke algebra \( e_{\mathcal{P}}\mathcal{H}_{\mathcal{P}}^{\text{cusp}} \), and any other weight \( \lambda' \in X^*(\tilde{P}) \) with the same image as \( \lambda \) in \( X^*(\tilde{P})/\mathbb{Z} \mathcal{N}_G \mathcal{G} \), there is an eigenform \( f' \in e_{\mathcal{P}}\mathcal{V}_{\text{cusp}, \infty}^{\text{der}}[\lambda'^{-1}] \otimes_{\mathcal{O}_p} \mathcal{O}_p \) for this Hecke algebra such that the eigen-systems corresponding to \( f \) and \( f' \) have the same image in \( \text{Spec}(e_{\mathcal{P}}\mathcal{H}_{\mathcal{P}}^{\text{cusp}} \otimes_{\mathcal{O}_p} \mathcal{O}_p) \). Therefore, one can choose a sequence of characters \( \lambda_n \in X^*_d(\tilde{T}_p) \) which converge \( p \)-adically to \( \lambda \) and a sequence of classical modular forms \( f_n \in e_{\mathcal{P}}\mathcal{V}_{\text{cusp}, \infty}^{\text{der}}[\lambda_n^{-1}] \otimes_{\mathcal{O}_p} \mathcal{O}_p \) such that the corresponding eigen-systems of these \( f_n \) converge \( p \)-adically to that of \( f \).

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