A NEW LUMINOSITY FUNCTION FOR GALAXIES AS GIVEN BY THE MASS–LUMINOSITY RELATIONSHIP

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ABSTRACT

The search for a luminosity function for galaxies both alternative or companion to a Schechter function is a key problem in the reduction of data from catalogs of galaxies. Two luminosity functions for galaxies can be built starting from two distributions of mass as given by the fragmentation. A first overall distribution function is the Kiang function, which represents a useful description of the area and volume distribution of the Poisson Voronoi diagrams. The second distribution, which covers the case of low-mass galaxies, is the truncated Pareto distribution: in this model we have a natural bound due to the minimum mass/luminosity observed and an upper bound (function of the considered environment) represented by the boundary with the observed mass/luminosity overall behavior. The mass distribution is then converted into a luminosity distribution through a standard mass–luminosity relationship. The mathematical rules to convert the probability density function are used and the two new functions are normalized to the total number of galaxies per unit volume. The test of the two new luminosity functions for galaxies that cover different ranges in magnitude was made on the Sloan Digital Sky Survey (SDSS) in five different bands; the results are comparable to those of the Schechter function. A new parameter, which indicates the stellar content, is derived. The joint distribution in redshift and flux, the mean redshift and the number density connected with the first luminosity function for galaxies are obtained by analogy with the Schechter function. A new formula, which allows us to express the mass as a function of the absolute magnitude, is derived.

Key words: galaxies: fundamental parameters – galaxies: luminosity function, mass function – galaxies: statistics

1. INTRODUCTION

Over the years the search for a luminosity function for galaxies has played a relevant role in the analysis of data from catalogs. A model for the luminosity of galaxies is the Schechter function

$$\Phi(L) dL = \left( \frac{\Phi^*}{L^*} \right) \left( \frac{L}{L^*} \right)^{\alpha} \exp \left( -\frac{L}{L^*} \right) dL,$$

where $\alpha$ sets the slope for low values of $L$, $L^*$ is the characteristic luminosity and $\Phi^*$ is the normalization. This function was suggested by Schechter (1976) in order to substitute other analytical expressions, see for example Formula (3) in Kiang (1961). Over the years this function has also been applied to describe physical quantities related to the optical luminosity, such as the CO luminosity for galaxies (Keres et al. 2003) and the baryonic mass function of galaxies (Bell et al. 2003a).

An astronomical form of Equation (1) can be deduced by introducing the distribution in absolute magnitude

$$\Phi(M) dM = 0.4(10) \Phi^* \left( 10^{0.4(a+1)(M^*-M)} \right) \times \exp(-10^{0.4(a+1)(M^*-M)}) dM,$$

where $M^*$ is the characteristic magnitude as derived from the data. This distribution has a maximum at

$$M_{p,\text{max}} = M^* - 1.085 \ln(\alpha + 1.0),$$

where $p,\text{max}$ means position of the maximum. In approaching this value the function will progressively flatten.

At present this function is widely used and Table 1 reports, as an example, the parameters from three catalogs.

- The 2dF Galaxy Redshift Survey (2dFGRS) based on a sample of 75,589 galaxies, see the first line of Table 3 in Madgwick et al. (2002).

- The $r^*$-band luminosity function for a sample of 147,986 galaxies at $z = 0.1$ from the Sloan Digital Sky Survey (SDSS), see Blanton et al. (2003).

- The galaxy luminosity function for a sample of 10095 galaxies from the Millennium Galaxy Catalogue (MGC), see Driver et al. (2005).

Over the years many modifications have been made to the standard Schechter function in order to improve its fit: we report three of them. When the fit of the rich clusters luminosity function is not satisfactory a two-component Schechter-like function is introduced, see Driver & Phillips (1996). This two-component function is the Schechter function when $L_{\text{Dwarf}} < L < L_{\text{max}}$ and has $(L/L_{\text{Dwarf}})^{\alpha_{\text{Dwarf}}}$ dependence when $L_{\text{min}} < L < L_{\text{Dwarf}}$. $L_{\text{Dwarf}}$ represents the magnitude where dwarfs dominate over giants, $\Phi_{\text{Dwarf}}$ the faint slope parameter for the dwarf population, and the indices $\alpha_{\text{min}}$ and $\alpha_{\text{max}}$ denote the minimum and the maximum.

Another example is the hybrid Schechter + power-law fit to fit the faint-end of the K-band, see Bell et al. (2003b).

Another function introduced in order to fit the case of extremely low luminosity galaxies is the double Schechter function, see Blanton et al. (2005), where the parameters $\Phi^*$ and $\alpha$ that characterize the Schechter function have been doubled in $\Phi_{\text{Dwarf}}$. The previous efforts focus attention on two ranges in luminosity for galaxies: an overall zone from high luminosity to low luminosity and the low-luminosity zone. This situation recalls the case of stars in which three zones are considered, see Scalo (1986), Kroupa et al. (1993), and Binney & Merrifield (1998); in this case the range of existence of the zones as well as the exponent that characterizes the power-law behavior are functions of the investigated environment.

These three zones in the mass distribution of the stars have been investigated in the light of the physical processes in Elmegreen (2004); they correspond to brown dwarf masses $\approx 0.02M_\odot$, to intermediate-mass stars and high-mass stars.
The starting point of this work is a statistical distribution in the mass of the galaxies, \( M \) as given by a standard gamma variate with range \( 0 < M < \infty \). This distribution describes the area of the irregular Voronoi polygons. This distribution in mass can be converted to a new statistical distribution for the luminosity of galaxies through an analogy with the physics of the stars. This new distribution in luminosity, \( L \), is characterized by the range \( 0 < L < \infty \) and a local maximum, called a mode. A second distribution in the masses starts from a truncated Pareto distribution, after Pareto (1896), with range \( L_{\min} < L < L_{\max} \).

In Section 2 the redshift dependence of the Schechter function and the first new function are explored in detail. Section 3 reports a first test based on the SDSS photometric catalog. In Section 4 the redshift dependence of the Schechter function as well as a new formula for the limiting mass.

Table 1: The Parameters of the Schechter Function from the 2dFGRS, SDSS, and MGC

| Parameter | 2dFGRS | SDSS \((r^\prime)\) band | MGC |
|-----------|--------|--------------------------|-----|
| \( M^* \) (mag) | \(-19.79 \pm 0.04\) | \(-20.44 \pm 0.01\) | \(-19.60 \pm 0.04\) |
| \( \alpha \) | \(-1.19 \pm 0.01\) | \(-1.05 \pm 0.01\) | \(-1.13 \pm 0.02\) |
| \( \Phi^* (h \text{Mpc}^{-3}) \) | \((1.59 \pm 0.1) \times 10^{-2}\) | \((1.49 \pm 0.04) \times 10^{-2}\) | \((1.77 \pm 0.15) \times 10^{-2}\) |

The averaged observed diameter of the galaxies is

\[
\overline{D}^{\text{obs}} \approx 0.6 D^{\text{obs}}_{\text{max}} = 2700 \text{ km s}^{-1} = 27 \text{ Mpc},
\]

where \( D^{\text{obs}}_{\text{max}} = 4500 \text{ km s}^{-1} \) corresponds to the extension of the maximum void visible on the CFA2 slices. In the framework of the theory of primordial explosions, see Charlton & Schramm (1986) and Zaninetti & Ferraro (1990), this means that the mean observed area of a bubble, \( A^{\text{obs}} \), is

\[
\overline{A}^{\text{obs}} \approx 4\pi \left( \frac{D^{\text{obs}}_{\text{max}}}{2} \right)^2 = 2290 \text{ Mpc}^2.
\]

The averaged area of a face of Voronoi polyhedra, \( \overline{A}_V \), is

\[
\overline{A}_V = \frac{A^{\text{obs}}}{N_F},
\]

where \( N_F \) is the averaged number of irregular faces of the Voronoi polyhedra, i.e. \( N_F = 16 \), see Okabe et al. (1992) and Zaninetti (2006). The averaged side of a face of an irregular polyhedron, \( L_V \), is

\[
\overline{L}_V \approx \sqrt{\pi A^{\text{obs}}} \approx 12 \text{ Mpc}.
\]

The thickness of the layer, \( \delta \), can be derived from shock theory, see Bowers & Deeming (1984), and is 1/12 of the radius of the advancing shock,

\[
\delta = \frac{D^{\text{obs}}_{\text{max}}}{2 \times 12} \approx 1.12 \text{ Mpc}.
\]

The number of galaxies in this typical layer, \( N_G \), can be found by multiplying \( n_\pi \approx 0.1 \), the density of galaxies, for the volume of the cube of side 12 Mpc: i.e. \( N_G \approx 172 \).

The more common way to insert the seeds of the Voronoi polygons is a random sequence in the X and Y directions, see Figure 1.

The distribution of the area of the irregular Voronoi polygons is fitted with a Kiang function, see formula (A5) in Appendix A,

\[
H(x; c) = \frac{c}{\Gamma(c)} (cx)^{-c} \exp(-cx),
\]
The mass–luminosity relationship in the case of the stars is well established both from a theoretical point of view, \( L \propto M^3 \) or \( L \propto M^4 \), see Lang (1999), and from an observational point of view, \( L \propto M^{1.43} \) in the case of MAIN_V, see Zaninetti (2005) for further details. A power law which is introduced by analogy regulates the relationship between mass and luminosity of galaxies, but in this case the regulating parameter \( a \) does not have a theoretical counterpart. The second transformation is

\[
\frac{M}{M^*} = \left( \frac{L}{L^*} \right)^{\frac{1}{a}},
\]

where \( 1/a \) is an exponent that connects the mass to the luminosity. The pdf (13) is therefore transformed into the following:

\[
\Psi(L) dL = \left( \frac{1}{a \Gamma(c)} \right) \Psi^* \left( \frac{L}{L^*} \right)^{\frac{c}{a} - 1} e^{-\frac{L}{L^*}} \left( \frac{L}{L^*} \right)^{\frac{c}{a} - 1} dL,
\]

where \( \Psi^* \) is a normalization factor which defines the overall density of galaxies, a number per cubic Mpc. The mathematical range is \( 0 \leq L < \infty \); conversely the astronomical range is \( L_{\text{min}} < L < L_{\text{max}} \).

The relationship connecting the absolute magnitude, \( M \), of a galaxy with its luminosity is

\[
\frac{L}{L^*} = 10^{0.4(M_{\text{bol},\odot} - M)},
\]

where \( M_{\text{bol},\odot} \) is the bolometric luminosity of the Sun, which according to Cox (2000) is \( M_{\text{bol},\odot} = 4.74 \).

The third and last transformation connects the luminosity with the absolute magnitude

\[
\Psi(M) dM = \left( 0.4 \ln 10 \frac{1}{a \Gamma(c)} \right) \Psi^* 10^{0.4(M - M^*)} \left( \frac{M}{M^*} \right)^{\frac{c - 1}{a} + 1} e^{-\frac{M}{M^*}} \left( \frac{M}{M^*} \right)^{\frac{c - 1}{a} + 1} dM.
\]

This data-oriented function contains the parameters \( M^* \), \( a \), \( c \), and \( \Psi^* \) which can be derived from the operation of fitting the observational data. Other interesting quantities are the mean luminosity per unit volume, \( j \),

\[
j = \int_0^\infty L \Psi(L) dL = L^* \Psi^* \frac{\Gamma(c + a)}{\Gamma(c)},
\]

and the averaged luminosity, \( \langle L \rangle \),

\[
\langle L \rangle = \frac{j}{\Psi^*} = L^* \frac{\Gamma(c + a)}{\Gamma(c)}.
\]

The density of galaxies is

\[
n_* = \frac{j}{L^*},
\]

and the mean separation between galaxies,

\[
d_* = n_*^{-1/3}.
\]
The symbols $j, n_*$, and $d_*$ are introduced as in Padmanabhan (1996).

Another way to compute the density of galaxies, now $n_*$, of the $M - L$ function is

$$n_* = \Psi^*.$$  

(23)

The position of the maximum in magnitudes is at

$$M_{p,\text{max}} = M^* - 1.085 \ln(c) a.$$  

(24)

2.3. The Luminosity Distribution for Low-Luminosity Galaxies

The Pareto distribution can model non-negative data with a power-law probability tail. In many practical applications, it is natural to consider an upper bound that truncates the tail (Cohen & Whitten 1988; Devoto & Martínez 1998; Aban et al. 2006). The truncated Pareto distribution has a wide range of applications: data analysis Aban et al. (2006) and Rehfeldt et al. (1992); forest fire area in the Australian Capital Territory, fault offsets in the Vernejoul coal field, hydrocarbon volumes in the Frio Strand Plain, exploration play and fault lengths on Venus, see Burroughs & Tabbens (2001).

In the case of stars, the low mass distribution of masses, see Salpeter (1955), can be represented by a law of the type $p(M_S) \propto M_S^{-2.35}$, where $p(M_S)$ represents the probability of having a mass between $M_S$ and $M_S + dM_S$. By analogy we introduce a truncated Pareto distribution, see Appendix B, for the mass of galaxies

$$\Psi_{LL}(M)dM = \frac{C}{(\frac{M}{M^*})^{\alpha+1}}d\left(\frac{M}{M^*}\right),$$  

(25)

where the index $LL$ stands for low luminosity and the range is $M_{\text{min}} \leq M \leq M_{\text{max}}$ where min and max denote the minimum and maximum mass. Once the constant $C$ is computed as in Appendix B we obtain

$$\Psi_{LL}(M)dM = \frac{d}{\left(\frac{M_{\text{min}}}{M^*}\right)^{-d} - \left(\frac{M_{\text{max}}}{M^*}\right)^{-d}}\left(\frac{M}{M^*}\right)^{d}d\left(\frac{M}{M^*}\right).$$  

(26)

Exactly as in the previous case we introduce the transformation represented by Equation (15) that connects the mass with the luminosity, and the distribution in luminosity is

$$\Psi_{LL}(L)dL = \Psi_{LL}^* \frac{d\left(\frac{L}{L^*}\right)^{-d}}{\left(\frac{L_{\text{min}}}{L^*}\right)^{-d} - \left(\frac{L_{\text{max}}}{L^*}\right)^{-d}}d\left(\frac{L}{L^*}\right),$$  

(27)

with the range $L_{\text{min}} \leq L \leq L_{\text{max}}$ and $\Psi_{LL}^*$ representing the normalization. The mean luminosity per unit volume, $j$, is

$$j = \int_{L_{\text{min}}}^{L_{\text{max}}} L \Psi_{LL}(L) dL = \Psi_{LL}^* d \left(\frac{L_{\text{min}}}{L^*}\right)^{-d} + L_{\text{min}}^2 \frac{d \left(\frac{L_{\text{min}}}{L^*}\right)^{-d}}{(d-a)\left(\frac{L_{\text{min}}}{L^*}\right)^{-2} - \left(\frac{L_{\text{max}}}{L^*}\right)^{-2}}.$$  

(28)

The distribution in magnitude is

$$\Psi_{LL}(M)dM = \Psi_{LL}^* \frac{0.4 d 10^{-0.4 (\frac{M}{M^*} - \alpha)} \ln(10)}{10^{-0.4 (\frac{M_{\text{min}}}{M^*} - \alpha)} - 10^{-0.4 (\frac{M_{\text{max}}}{M^*} - \alpha)}} dM,$$  

(29)

with the range $M_{\text{min}} \leq M \leq M_{\text{max}}$. This distribution in magnitude contains the parameters $M_{\text{min}}$ and $M_{\text{max}}$ which are the minimum and maximum mass of the considered catalog and the parameters $a$, $d$, and $\Psi_{LL}^*$ which are derived from the fitting of the data.

3. APPLICATION TO A REAL SAMPLE OF GALAXIES

The data of the luminosity function for galaxies in five bands of the SDSS are available at http://cosmo.nyu.edu/blanton/lf.html and are discussed from an astronomical point view in Blanton et al. (2001).

The analysis of the new luminosity function was split into two parts. The data from high luminosities to low luminosities was fitted by $\Psi(M)$, Equation (18). The data were processed via the Levenberg–Marquardt method (subroutine MRQMIN in Press et al. 1992) in order to find the three parameters $a$, $M^*$, $\Psi^*$ conversely is introduced by hand. In order to associate a statistical probability to each fit we have chosen a range in magnitude such as $M < M_{\text{max}}$ where $M_{\text{max}}$ represents the selected maximum magnitude of the sample.

The results are reported in Table 2 together with the derived quantities $j, n_*, d_*$, and their uncertainties. Table 2 also reports $M_{\text{max}}$, the number of elements $N$ belonging to the sample, the merit function $\chi^2$ and the associated $p$-value that has to be understood as the maximum probability to obtain a better fitting, see Formula (15.2.12) in Press et al. (1992):

$$p = 1 - \text{GAMMQ}\left(\frac{N - 3}{2}, \frac{\chi^2}{2}\right),$$  

(30)

where GAMMQ is a subroutine for the incomplete gamma function.

The uncertainties are found by implementing the error propagation Equation (often called the law of Gaussian errors). The low-luminosity range conversely was fitted through $\Psi_{LL}(M)$, Equation (29) and the results are reported in Table 3.

The Schechter function, conversely, fits all the range in luminosities and Table 4 reports the data that come out from the fitting procedure. Table 4 also reports $M_{p,\text{max}}$, the value in magnitude where the Schechter function peaks; this value is defined when $\alpha = -1$; otherwise we leave the entry blank.

Table 5 reports the $\chi^2$ of the two zones of the new physical function, their sum and $\chi^2$ of the Schechter luminosity function.
Table 2
Parameters of Fits to Luminosity Function in SDSS Galaxies via the $M - L$ Function

| Band | $a^*$ | $g^*$ | $r^*$ | $i^*$ | $z^*$ |
|------|-------|-------|-------|-------|-------|
| $c^*$ | 1.1 ± 0.2 | 1.0 ± 0.2 | 1.1 ± 0.2 | 2 ± 0.2 | 1.7 ± 0.2 |
| $M^*$ (mag) | −16.58 ± 0.018 | −18.29 ± 0.008 | −18.77 ± 0.007 | −18.26 ± 0.01 | −18.79 ± 0.004 |
| $\Psi^*$ (h Mpc$^{-3}$) | 0.069 ± 0.001 | 0.043 ± 0.0003 | 0.043 ± 0.0002 | 0.037 ± 0.0002 | 0.034 ± 0.003 |
| $a$ | 1.40 ± 0.007 | 1.32 ± 0.003 | 1.5 ± 0.002 | 1.74 ± 0.003 | 1.70 ± 0.014 |
| $j$ (mag) | 1.40 $L^*\Psi^*$ | 1.18 $L^*\Psi^*$ | 1.50 $L^*\Psi^*$ | 4.39 $L^*\Psi^*$ | 3.2 $L^*\Psi^*$ |
| $n_*$ (Mpc$^{-3}$) | 0.097 ± 0.022 | 0.051 ± 0.012 | 0.066 ± 0.016 | 0.14 ± 0.02 | 0.11 ± 0.04 |
| $d$ (Mpc) | 2.17 ± 0.16 | 2.67 ± 0.21 | 2.47 ± 0.19 | 1.91 ± 0.09 | 2.06 ± 0.14 |
| $M^*_{\text{max}}$ (mag) | −15.78 | −18.2 | −19 | −19.3 | −20 |
| $N$ | 483 | 404 | 400 | 471 | 442 |
| $\chi^2$ | 321 | 386 | 233 | 325 | 649 |
| $p = 1 - \text{GAMMQ} \left( \frac{\chi^2}{2}, \frac{N - 3}{2} \right)$ | 0 | 0.31 | 0 | 1.19 $10^{-7}$ | 1.0 |

Table 3
Parameters of Fits to Luminosity Function in SDSS Low-Luminosity Galaxies via the $\Psi_{LL}$ Function

| Band | $g^*$ | $r^*$ | $i^*$ | $z^*$ |
|------|-------|-------|-------|-------|
| $d^*$ | 0.3 ± 0.1 | 0.4 ± 0.1 | 0.5 ± 0.1 | 0.9 ± 0.1 |
| $M^*_{\text{LL}}$ (mag) | −17.2 ± 0.0 | −18.8 ± 0.1 | −17.39 ± 0.1 | −19.3 ± 0.1 |
| $\Psi^*_{\text{LL}}$ (h Mpc$^{-3}$) | 0.043 ± 0.0043 | 0.040 ± 0.0040 | 0.026 ± 0.0032 | 0.035 ± 0.003 |
| $a^*$ | 2.2 ± 0.1 | 1.3 ± 0.1 | 2.7 ± 0.1 | 2.3 ± 0.1 |
| $\chi^2$ | 194 | 273 | 237 | 297 |
| $p = 1 - \text{GAMMQ} \left( \frac{\chi^2}{2}, \frac{N - 3}{2} \right)$ | 204 | 379 | 476 | 313 |

Table 4
Parameters of Fits to Luminosity Function in the SDSS via the Schechter Function

| Band | $a^*$ | $g^*$ | $r^*$ | $i^*$ | $z^*$ |
|------|-------|-------|-------|-------|-------|
| $\alpha$ | −0.90 ± 0.01 | −0.88 ± 0.007 | −1.04 ± 0.004 | −0.99 ± 0.005 | −1.07 ± 0.02 |
| $M^*$ (mag) | −17.92 ± 0.006 | −19.38 ± 0.004 | −20.43 ± 0.003 | −20.81 ± 0.004 | −21.18 ± 0.017 |
| $M_{\text{g,max}}$ (mag) | −17.92 | −19.38 | −20.81 |
| $\Phi^*$ (h Mpc$^{-3}$) | 0.030 ± 0.0003 | 0.021 ± 0.0001 | 0.015 ± 0.00008 | 0.0147 ± 0.00008 | 0.0135 ± 0.00006 |
| $j$ (mag) | 0.95 $L^*\Phi^*$ | 0.92 $L^*\Phi^*$ | 1.02 $L^*\Phi^*$ | 0.99 $L^*\Phi^*$ | 1.04 $L^*\Phi^*$ |
| $n_*$ (Mpc$^{-3}$) | 0.029 ± 0.0003 | 0.020 ± 0.0001 | 0.015 ± 0.00007 | 0.014 ± 0.00008 | 0.014 ± 0.00007 |
| $d$ (Mpc) | 3.24 ± 0.013 | 3.64 ± 0.006 | 4.02 ± 0.006 | 4.08 ± 0.008 | 4.12 ± 0.007 |
| $\chi^2$ | 483 | 599 | 674 | 709 | 740 |
| $p = 1 - \text{GAMMQ} \left( \frac{\chi^2}{2}, \frac{N - 3}{2} \right)$ | 330 | 753 | 2260 | 2282 | 3245 |

Table 5
Synoptic $\chi^2$

| Band | $g^*$ | $r^*$ | $i^*$ | $z^*$ |
|------|-------|-------|-------|-------|
| $\chi^2$ physical luminosity function | 321 | 386 | 233 | 325 | 649 |
| $\chi^2$ luminosity function low luminosities | 0 | 204 | 379 | 476 | 313 |
| $\chi^2$ sum of two zones | 321 | 590 | 612 | 801 | 962 |
| $\chi^2$ Schechter luminosity function | 330 | 753 | 2260 | 2282 | 3245 |

The Schechter function, the new function, and the data are reported in Figures 3–7 when the $u^*$, $g^*$, $r^*$, $i^*$, and $z^*$ bands are considered; Figures 8–12 report the residuals of the $u^*$, $g^*$, $r^*$, $i^*$, and $z^*$ bands. We have used $H_0 = 100h$ km s$^{-1}$ Mpc$^{-1}$, with $h = 1$ in all the numerical evaluations. Due to the testing phase of the new $M - L$ function, we have omitted the propagation of other values of $h$ on the derived quantities; see the discussion in Blanton et al. (2001).

The value obtained for the parameter $a$ should be compared with that of the normal stars which is 3 or 4 as suggested by the theory, see for example Lang (1999), or $\approx 3.8$ as suggested by

![Figure 4](image-url)
the observations for $M > 0.2 M_\odot$, see for example Cox (2000). When the three classes of stars are considered we have $a = 3.43$.
The parameters of the $M - L$ luminosity function based on 2dFGRS data (Table 1), reviewed; by analogy new formulae for the first part of the $M - L$ function are derived. 

4.1. The Behavior of the Schechter Function

The parameters of the $M - L$ function based on 2dFGRS data (Table 1), reviewed; by analogy new formulae for the first part of the $M - L$ function are derived.

Table 6

| Parameter | 2dFGRS |
|-----------|--------|
| $c$       | 0.1    |
| $M^*$ (mag) | $-19 \pm 0.1$ |
| $\Phi^* (h\,\text{Mpc}^{-3})$ | $0.4 \pm 0.01$ |
| $a$       | $1.3 \pm 0.1$ |

The residuals of the fits to the SDSS($z^*$) data. The open stars represent our two luminosity functions (18) and (29) and the filled points represent the Schechter function.

Figure 11. The residuals of the fits to the SDSS($z^*$) data. The open stars represent our two luminosity functions (18) and (29) and the filled points represent the Schechter function.

Figure 12. The residuals of the fits to the SDSS($z^*$) data. The open stars represent our two luminosity functions (18) and (29) and the filled points represent the Schechter function.

The variation of $j$ when the range in magnitude is finite rather than infinite can be evaluated by coupling together Formulae (17) and (18)

$$j = \int_{-15.78}^{-21} 10^{0.4(M_{\text{bol},\odot} - M_{\text{bol}})} \Phi(M) dM. \quad (31)$$

On inserting the parameters of the SDSS band $u^*$ (which is the case in which the $M - L$ function covers the whole magnitude range of the data, see Table 2) and $M_{\text{bol},\odot} = M_{u^*\odot} = 6.39$, $j = 1.4 \times 10^8 L_\odot$ is obtained. This value increases by 5.63% when the range is infinite; see a similar discussion concerning Schechter’s function around Formula (33) in Lin et al. (1996).

In the absence of observational data representing the luminosity function, we can generate them through Schechter’s parameters, see Table 1. This is done, for example for the 2dFGRS, see Cross et al. (2001), and the data of the Schechter function in Table 1. The parameters of the $M - L$ function are reported in Table 6 where the required errors on the values of luminosity are the same as the considered value.

4. TESTS INVOLVING $z$

Some useful formulae connected with the Schechter function in a Euclidean, non-relativistic and homogeneous universe are

$$f = \frac{L}{4\pi r^2}, \quad (32)$$

where $r$ represents the distance to the galaxy. The joint distribution in $z$ and $f$ for galaxies, see Formula (1.104) in Padmanabhan (1996), is

$$\frac{dN}{d\Omega dz df} = 4\pi \left(\frac{c}{H_0}\right)^4 z^4 \Phi \left(\frac{z}{z_{\text{crit}}}\right). \quad (33)$$

where $d\Omega$, $dz$, and $df$ represent the differential of the solid angle, the redshift, and the flux, respectively. The difference in $L^*$ between the previous formula and Formula (1.104) in Padmanabhan (1996) is due to the small difference in the definition of $\Phi$.

The formula for $z_{\text{crit}}$ is

$$z_{\text{crit}}^2 = \frac{H_0^2 L^*}{4\pi f c_L}\, \quad (34)$$

where $c_L$ represents the light velocity; the CODATA recommends $c_L = 299792.458 \text{ km s}^{-1}$. The mean redshift of galaxies with a flux $f$ (see Formula (1.105) in Padmanabhan 1996) is

$$\langle z \rangle = z_{\text{crit}} \frac{\Gamma(3/2 + \alpha)}{\Gamma(5/2 + \alpha)} \quad (35)$$

The number density of galaxies per unit flux interval (see Formula (1.106) in Padmanabhan 1996) is

$$\frac{dN}{df} = \Phi^* \left(\frac{L^*}{4\pi f}\right)^{3/2} \Gamma \left(\frac{5}{2} + \alpha\right). \quad (36)$$

The number of galaxies in $z$ and $f$ as given by Formula (33) has a maximum at $z = z_{\text{max}}$, where

$$z_{\text{max}} = z_{\text{crit}} \sqrt{\alpha + 2}. \quad (37)$$

The value of $z_{\text{max}}$ can be derived from the histogram of the observed number of galaxies expressed as a function of $z$. For practical purposes we analyzed the 2dFGRS data release available at: http://msowww.anu.edu.au/2dFGRS/. In particular we added together the file parent.ngp.txt that contains 145,652 entries for north Galactic pole (NGP) strip sources and the file parent.sgp.txt that contains 204,490 entries for south Galactic pole (SGP) strip sources. Once the heliocentric redshift was selected we processed 219,107 galaxies with $0.001 \leq z \leq 0.25$. 

![Figure 11](image1.png)

![Figure 12](image2.png)
A comparison between the observed and theoretical number of galaxies as a function of $z$ is reported in Figure 13. Another interesting catalog is the 6dF Galaxy Survey that has measured around 150,000 redshifts and 15,000 peculiar velocities from galaxies over the southern sky, see Jones et al. (2006). It is available at: http://vizier.u-strasbg.fr/viz-bin/VizieR?-source=VII/249. We selected the re-calibrated $bJ$ magnitude and the recession velocity $cz$. Figure 14 reports the observed and theoretical number of galaxies as a function of $z$ for the 6dF Galaxy Survey.

4.2. The Behavior of the $M - L$ Function

The joint distribution in $z$ and $f$, in the presence of the $M - L$ luminosity (Equation (16)) is

$$\frac{dN}{d\Omega dz df} = 4\pi \left( \frac{c}{H_0} \right)^5 z^4 \Psi \left( \frac{z^2}{z_{\text{crit}}} \right).$$

(38)

The mean redshift is

$$\langle z \rangle = z_{\text{crit}} \frac{2 \Gamma(2a + c) 2^{2a} a}{\Gamma(c + 3/2a)}.$$  

(39)

The number density of galaxies per unit flux interval is

$$\frac{dN}{d\ln f} = \frac{L^2 a^2}{16 \pi^2 f^2 \Gamma(3/2) \Gamma(3/2)}.$$  

(40)

The number of galaxies as given by Formula (38) has a maximum at $z_{\text{max}}$ where

$$z_{\text{max}} = z_{\text{crit}}(c + a)^{3/2}.$$  

(41)

A comparison between the observed and theoretical number of galaxies as given by the $M - L$ function is reported in Figure 13 where the 2dFGRS is considered and in Figure 14 where the 6dF Galaxy Survey is considered.

5. MASS EVALUATION

One method to deduce the mass of a star by its absolute visual magnitude is presented; the mass of a galaxy is deduced by analogy. In the case of galaxies, the bolometric correction of the stars will be replaced by the Sun’s absolute magnitude and mass–luminosity ratio different in each selected band.

5.1. The Case of Stars

In the case of stars it is possible to parameterize the mass of the star, $M_\odot$, as a function of the observable color $(B - V)$; see Zaninetti (2005). The first equation connects the $(B - V)$ color with the temperature

$$(B - V) = K_{\text{BV}} + \frac{T_{\text{BV}}}{T},$$

(42)

where $T$ is the temperature, and $K_{\text{BV}}$ and $T_{\text{BV}}$ are two parameters that can be derived by implementing the least-squares method on a series of calibrated data. The second equation describes the bolometric correction, BC,

$$BC = M_\odot - M_V = -\frac{T_{\text{BC}}}{T} - 10 \log_{10} T + K_{\text{BC}}.$$  

(43)

where $M_\odot$ is the absolute bolometric magnitude, $M_V$ is the absolute visual magnitude, and $T_{\text{BC}}$ and $K_{\text{BC}}$ are two parameters that can be derived through the general linear least-squares method applied to a series of calibrated data. The third equation is the usual formula for the luminosity

$$\log_{10} \left( \frac{L}{L_\odot} \right) = 0.4(7.47 - M_\odot).$$  

(44)
where $L$ is the luminosity of the star and $L_\odot$ the luminosity of the Sun. The fourth equation is the usual mass–luminosity relationship for stars

$$\log_{10} \left( \frac{L}{L_\odot} \right) = a_{LM} + b_{LM} \log_{10} \left( \frac{M_S}{M_\odot} \right) \quad \text{for} \ M > 0.2 M_\odot,$$

where $M_S$ is the mass of the star and $M_\odot$ is the mass of the Sun.

With these four equations the mass of the star is

$$\log_{10} \frac{M_S}{M_\odot} = \frac{-0.4 M_V - 0.4 K_{BC} + 4.0 \ln \left( \frac{r_{57}}{(B-V) + K_{BC}} \right) \ln(10)^{-1}}{b_{LM}} - \frac{0.4 \ln(-10(B-V) + K_{BC}) + 1.896 - a_{LM}}{b_{LM}},$$

with the various coefficients as given by Table 1 in Zaninetti (2005). As an example, the mass of a star belonging to MAIN SEQUENCE V is

$$\log_{10} \frac{M_S}{M_\odot} = -7.769 + 0.8972 \ln \left( \frac{7361}{(B-V) + 0.6411} \right).$$

We can now express the color $(B - V)$ as a function of the absolute visual magnitude $M_V$ and the following formula for $M_V$ when $-5.8 < M_V < 11.8$,

$$\log_{10} \frac{M_S}{M_\odot} = \frac{-7.769 + 0.8972 \ln \left( \frac{9378}{W(9378 - 8.496 + 0.2972 M_V)} \right)}{W},$$

MAIN SEQUENCE V when $-5.8 < M_V < 11.8$, (48)

where $W$ is the Lambert $W$-function, after Lambert (1758). A test of the previous formula can be done at the two boundaries: when $M_V = -0.58$, $\log_{10} \frac{M_S}{M_\odot} = 1.63$ against the calibrated value $\log_{10} \frac{M_S}{M_\odot} = 1.6$ and when $M_V = 11.8$, $\log_{10} \frac{M_S}{M_\odot} = -0.56$ against the calibrated value $\log_{10} \frac{M_S}{M_\odot} = -0.66$, see Table 3.1 in Bowers & Deeming (1984).

5.2. The Case of Galaxies

The mass of a galaxy can be evaluated once the mass–luminosity ratio, $R$, is given as

$$R = \frac{M}{L}.$$

(49)

Some values of $R$ are now reported: $R \lesssim 20$ by Kiang (1961) and Persic & Salucci (1992), $R = 20$ by Padmanabhan (1996), $R = 5.93$ by van der Marel (1991). Later, Bell & de Jong (2001) (amongst others) demonstrated that $M/L$ varies as a function of galaxy color, and therefore type. If the bright end of the luminosity function is dominated by massive, evolved, red galaxies, and the faint end by low-mass, blue galaxies, then $M/L \propto L^{-0.64}$ (GIANTS III) at the bright end and $M/L \propto L^{-0.7}$ (MAIN SEQUENCE V) at the faint end; see Table 1 in Zaninetti (2005). Then $M/L$ will almost certainly not be constant due to different prevailing populations of stars at the boundaries of the luminosity function. The scatter in the models by Bell & de Jong (2001) is a starting point when evaluating the validity of assuming a constant $M/L$. Generally, near-infrared $M/L$ ratios are more constant than the optical passband, but still vary with luminosity. In our framework, we made $R$ a function of the passband, in order to have the same results for the masses of the galaxies once the absolute magnitude is given; see Figure 15. In our framework $R$ can be expressed as

$$R = \frac{\langle M \rangle}{L_\odot}. \quad (50)$$

On inserting Formula (14) and Formula (20) in the previous ratio the following formula for $M^*$ is obtained:

$$M^* = RL^* \frac{\Gamma(c + a) M_\odot}{\Gamma(c) L_\odot}.$$  

(51)

From Equations (15) and (51) a formula for the mass of the galaxy is

$$M = R 10^{0.4 M_{bol,\odot} - 0.4 M^*} \left( \frac{\Gamma(c + a) (10^{-0.4 M + 0.4 M^*}) c^{-1}}{\Gamma(c)} \right) M_\odot.$$

(52)

An application of the previous formula is reported in Figure 15, where the mass of galaxies as a function of the absolute magnitude in the five bands of the SDSS is presented. In this figure $M_{bol,\odot}$ is different for each selected band and equal to the value suggested in Equation (16) of Blanton et al. (2001).

The new Formula (52) allows us to deduce the mass of the galaxy from its absolute magnitude and can be easily particularized in different pass-bands. As an example, with the data of the SDSS in the five bands reported in Table 2, $M_{bol,\odot}$ as in Equation (16) of Blanton et al. (2001) and $R$ as in Figure 15, we have

$$M = 215.234 e^{-0.6579 M_{bol,\odot}} M_\odot$$

for the $u^*$ band when $-20.6 \leq M \leq -15.7$

$$M = 97.216 e^{-0.6978 M_{bol,\odot}} M_\odot$$

for the $g^*$ band when $-22.0 \leq M \leq -18.2$

$$M = 490,000 e^{-0.6141 M_{bol,\odot}} M_\odot$$

for the $r^*$ band when $-23.0 \leq M \leq -19$. 

Figure 15. Logarithm of the mass of the galaxy as a function of the absolute magnitude. The SDSS bands are $u^*$ with $M_{bol,\odot} = M_{bol,\odot} = 6.39$ and $R = 6$ (full line), $g^*$ with $M_{bol,\odot} = M_{bol,\odot} = 5.07$ and $R = 13$ (dashed), $r^*$ with $M_{bol,\odot} = M_{bol,\odot} = 4.62$ and $R = 16$ (dot–dash), $i^*$ with $M_{bol,\odot} = M_{bol,\odot} = 4.52$ and $R = 15$ (dotted), $z^*$ with $M_{bol,\odot} = M_{bol,\odot} = 4.48$ and $R = 14$ (dash–triple-dot).
et al. (1998). In the Tully–Fisher framework the mass of a galaxy belonging to a given catalog can be evaluated in the following way. The limiting apparent magnitude is computed and inserted into Equation (52). The limiting mass for galaxies, $M_L$, is

$$M_L = \frac{R}{10^{0.4M_{bol} - 0.4M^*}} \frac{10^{-0.4M_{bol} + 2.5\log_{10}(c^2 \Gamma(c)) + 10.0 + 0.4M^*}}{c(1 + c)} \times M_*,$$

(55)

where $c_L z$ is the radial distance expressed in km s$^{-1}$. In order to see how the parameter $z$ influences the limiting mass, Table 7 reports the range of observable masses as a function of $z$.

### 6. CONCLUSIONS

We have split the analysis of the luminosity function in two parts. The analysis of the main new luminosity function, Formula (18), from low luminosities up to maximum magnitude shows the following (see Table 2).

1. The parameter $c$ varies between 0.1 and 2. It must be remembered that the theory predicts 2, 4, and 6 for the 1D, 2D, and 3D fragmentation, respectively.

2. Parameter $a$ varies between 1.32 and 1.74. The numerical mass–luminosity relationship for the stars gives values of the parameter $a$ between 2.43 and 3.43.

3. The $M - L$ function represents a better fit of the observational data in comparison with the Schechter function once the concept of maximum magnitude of the sample is introduced. Without this limiting magnitude the situation is inverted.

The case of low-luminosity galaxies was described by a truncated Pareto-type luminosity function, see Formula (29). This new luminosity function is described by two physical parameters, $d$ and $a$, denoting respectively the distribution in mass and the mass–luminosity connection. The analysis of the data for low luminosity galaxies as reported in Table 3 shows the following.

1. The parameter $d$ varies between 0.3 and 0.9. This value should be compared with $d$ of the stars which is 2.3 - $\Gamma_3$; see Kroupa (2001).

2. The parameter $a$ varies between 1.3 and 2.7.

The theoretical number of galaxies as a function of the redshift presents a maximum that is a function of $a$ and $f$ for the Schechter function and $c$, $a$, and $f$ for the first $M - L$ function; the agreement with the maximum in the observed number of galaxies is acceptable.

The observable range in masses can be parameterized as a function of $z$ and the ratio between maximum and minimum luminosity is 232 at $z = 0.001$ but drops to 1.1 at $z = 0.15$, see Table 7.

Perhaps a more comprehensive way of comparing the mass estimates of Equation (54) (Tully–Fisher relation) with those given here (Equations (52) and (53)) would be as follows. We take the same sample of galaxies for which the luminosity function was computed in Section 3 and compute their mass function according to a given value of $R$. This mass function can be compared with those of other galaxies in a common passband and, also in this case, the distribution is expressed through a Schechter function. Three cases are now analyzed.

1. The Bell case (see Bell et al. 2003a), where $\Phi^* = 0.01$ Mpc$^{-3}/\log_{10}(M)$, $M^* = 5.310^{10} M_\odot$ and $\alpha = -1.21$.

### Table 7

The Limiting Mass for the SDSS Catalog, $u^*$ Band, when $R$ and $M_{bol,0}$ are those of Figure 15

| Mass range | $z$ |
|------------|-----|
| $1.4110^{9} M_\odot < M < 1.710^{11} M_\odot$ | 0.001 |
| $3.7910^{9} M_\odot < M < 1.710^{11} M_\odot$ | 0.01 |
| $1.010^{11} M_\odot < M < 1.710^{11} M_\odot$ | 1 |
| $1.410^{11} M_\odot < M < 1.710^{11} M_\odot$ | 0.13 |

Note. The limiting apparent magnitude is $m = 17.6$.

The method suggested here to deduce the mass of the galaxy can be compared with the formula that comes out from the Tully–Fisher relation, see Tully & Fisher (1977) and Tully et al. (1998). In the Tully–Fisher framework the mass of a rotating galaxy can be parameterized as

$$M = 50V_f^4 M_\odot s^{-4},$$

where $V_f$ is the rotational velocity expressed in km s$^{-1}$; see McGaugh (2005). The mass-to-light ratio in our framework scales $\propto L^{(1/a)-1}$ with $a$ depending on the selected catalog and band. This ratio oscillates, referring to the SDSS data, between a minimum dependence in the $g^*$ band, $M/L \propto L^{-0.24}$, and a maximum dependence in the $i^*$ band, $M/L \propto L^{-0.42}$. A comparison should be made with $M/L \propto L^{0.35}$ in van der Marel (1991) for a sample of 37 bright elliptical galaxies; this result was obtained by implementing axisymmetric dynamical models. The completeness of the mass sample of the galaxies belonging to a given catalog can be evaluated in the following way. The limiting apparent magnitude is known, and is different for each catalog. In the case of the SDSS $(r^*)$ band it is $m = 17.6$; see Blanton et al. (2001).

The observable range in masses can be parameterized as

$$L_{bol} = \frac{50}{10^{0.4M_{bol} - 0.4M^*}} \frac{(c + a)(10^{-0.4M_{bol} + 2.5\log_{10}(c^2 \Gamma(c)) + 10.0 + 0.4M^*)}}{c\Gamma(c)} \times M_*,$$

(55)

where $c_L z$ is the radial distance expressed in km s$^{-1}$. In order to see how the parameter $z$ influences the limiting mass, Table 7 reports the range of observable masses as a function of $z$.

### Figure 16

Baryonic mass function of galaxies: our $\Psi(M)/\log_{10}(M)$ (full line), Bell case (dashed), Bottema case (dot–dash), and Kennicutt–Kroupa case (dotted). In our case we considered the SDSS band $u^*$ with $M_{bol,0} = M_{bol,0} = 6.39$ and $R = 6.0$.
2. The Bottema case (see Bottema 1997), where $\Phi^* = 0.014 \, \text{Mpc}^{-3}/\log_{10}(M)$, $M^* = 2.24\times10^{10}\,M_{\odot}$ and $\alpha = -1.20$.

3. The Kennicutt–Kroupa case (see Kennicutt 1983; Kroupa et al. 1993), where $\Phi^* = 0.011 \, \text{Mpc}^{-3}/\log_{10}(M)$, $M^* = 3.78\times10^{10}\,M_{\odot}$ and $\alpha = -1.22$.

Figure 16 reports the already cited standard distributions as well as our $\Psi(M)/\log_{10}(M)$ when the range in masses is that given by the conversion from luminosity to mass. The analysis of the two new functions for the luminosity for galaxies derived here gives a marginally better fit but certainly the Schechter function for its simplicity and fewer parameters still represents a good model for the luminosity function for galaxies. At present the study of the Schechter function is not yet concluded and two new equations have been derived: Equation (3) that represents the maximum in magnitude distribution and Equation (33) that gives the value of $z$ at which the observed number of galaxies is maximum.

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APPENDIX A

THE KIANG FUNCTION

The starting point is the distribution in length, $s$, of a segment in a random fragmentation

$$p(s) = \lambda \exp(-\lambda s)ds,$$  \hspace{1cm} (A1)

where $\lambda$ is the hazard rate of the exponential distribution. Given the fact that the sum, $u$, of two exponential distributions is

$$p(u) = \lambda^2 u \exp(-\lambda u)du$$ \hspace{1cm} (A2)

the distribution of 1D Voronoi segments, $l$ (the midpoint of the sum of the two segments), can be found from the previous formula by inserting $u = 2l$

$$p(l) = 2\lambda l \exp(-2\lambda l)(2\lambda l).$$ \hspace{1cm} (A3)

On transforming into normalized units $x = l/\lambda$ we obtain

$$p(x) = 2x \exp(-2x)dx/(2\lambda).$$ \hspace{1cm} (A4)

After a century of studies on the Voronoi diagrams, see the two memoirs Voronoi (1907) and Voronoi (1908), the law of the segments in 1D is the unique analytical result on the field. When this result is expressed as a gamma variate we obtain Formula (5) of Kiang (1966)

$$H(x; c) = \frac{c}{\Gamma(c)}(cx)^{c-1}\exp(-cx),$$ \hspace{1cm} (A5)

where $0 \leq x < \infty$, $c > 0$ and $\Gamma(c)$ is the gamma function with argument $c$; in the case of 1D Voronoi diagrams $c = 2$. It was conjectured that the area in 2D and the volumes in 3D of the Voronoi diagrams may be approximated as the sum of two and three gamma variates of argument 2. Due to the fact that the sum of $n$ independent gamma variates with shape parameter $c_i$ is a gamma variate with a shape parameter $c = \sum_{i=1}^{n} c_i$, the area and the volumes are supposed to follow a gamma variate of argument 4 and 6. This hypothesis was later named “Kiang’s conjecture,” and Equation (A5) used as a fitting function; see Kumar et al. (1992) and Zaninetti (2006), or as a hypothesis to accept or reject using the standard procedures of the data analysis, see Tanemura (1988, 2003). A new way to parametrize the 1D, 2D, and 3D cells on the base of the considered dimensionality has been introduced; see Formula (12) in Ferenc & Néda (2007).

APPENDIX B

THE TRUNCATED PARETO DISTRIBUTION

The starting pdf is the Pareto distribution (Pareto 1896; Evans et al. 2000), $P$,

$$P(x; a, c) = \frac{ca^c}{x^{c+1}},$$ \hspace{1cm} (B1)

where $a \leq x < \infty$, $a > 0$, $c > 0$. The average value is

$$\bar{x} = \frac{ca}{c - 1},$$ \hspace{1cm} (B2)

which is defined for $c > 1$, and the variance is

$$\sigma^2 = \frac{a^c}{(c-2)(c-1)^2},$$ \hspace{1cm} (B3)

which is defined for $c > 2$. The presence of an upper bound, $b$, allows us to introduce the following pdf, named the truncated Pareto, $P_T$,

$$P_T(x; a, b, c) = \frac{1}{1 - \left(\frac{b}{a}\right)^c} \frac{ca^c}{x^{c+1}},$$ \hspace{1cm} (B4)

where $a \leq x \leq b$, $a > 0$, $b > 0$, $b > a$ and $c > 0$. The distribution function of the truncated Pareto is

$$F(x; a, b, c) = \frac{1 - \left(\frac{x}{a}\right)^c}{1 - \left(\frac{b}{a}\right)^c}.$$ \hspace{1cm} (B5)

The average value of the truncated Pareto pdf is

$$\bar{x} = \frac{ca}{c - 1} \frac{1 - \left(\frac{b}{a}\right)^{c-1}}{1 - \left(\frac{b}{a}\right)^c},$$ \hspace{1cm} (B6)

and its variance is

$$\sigma^2 = \frac{\text{numerator}}{\text{denominator}},$$ \hspace{1cm} (B7)

with

$$\text{denominator} = (c-2)(c-1)^2(a^2c - 2b^ca^c + b^2c),$$ \hspace{1cm} (B8)

and

$$\text{numerator} = cb^2a^2c^2 + 2a^2bc^2b^2 - 4c^2d^{c+1}b^{c+1} + 2c^2b^{2c}a^c - a^{2c}c^2b^c + 2c^2d^{c+1}b^{c+1} - a^{2c}c^2b^c - c^3b^{2c}a^c - cb^{2c}a^c + ca^2b^c.$$
we would have obtained a given set of observations if given a particular set of values of the distribution parameters, \(c_i\),

\[
L(c) = f(x_1 \ldots x_n | c_1 \ldots c_n).
\]  

(B9)

If we assume that the \(n\) random variables are independently and identically distributed, then we may write the likelihood function as

\[
L(c) = f(x_1 | c_1 \ldots c_p) \ldots f(x_n | c_1 \ldots c_p) = \prod_{i=1}^{n} f(x_i | c_1 \ldots c_p).
\]  

(B10)

The maximum likelihood estimates for the \(c_i\) are obtained by maximizing the likelihood function, \(L(c)\). Equivalently, we may find it easier to maximize \(\ln L(x_i)\), termed the log-likelihood. So, for a random sample \(x_1 \ldots x_n\) from a truncated Pareto distribution, the likelihood function is given by

\[
L(c) = \prod_{i=1}^{n} c (a x_i^{-1} - b x_i^{-1})^{-1} (c x_i^{-1}).
\]  

(B11)

In this model we have assumed that \(a = \min(x_1 \ldots x_n)\) and \(b = \max(x_1 \ldots x_n)\).

Using logarithms, we obtain the log-likelihood

\[
\ln L(c) = nc \ln(a) + n \ln \frac{c}{1 - \left(\frac{x}{b}\right)^{c-1}} - \sum_{i=1}^{n} \ln x_i.
\]  

(B12)

Taking the first derivative, we get

\[
\frac{\partial}{\partial c} \ln L(c) = 0
\]

\[n \ln a + \frac{n}{c} \left(\frac{x}{b}\right)^{c-1} \ln \left(\frac{x}{b}\right) - \sum_{i=1}^{n} \ln x_i = 0.
\]  

(B13)

The parameter \(c\) can be found by solving numerically the previous nonlinear equation.

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