HYPERBOLIC MODEL REDUCTION FOR KINETIC EQUATIONS

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ABSTRACT. We make a brief historical review to the moment model reduction to the kinetic equations, particularly Grad’s moment method for the Boltzmann equation. The focus is on the hyperbolicity of the reduced model, which is essential to the existence of its classical solution as a Cauchy problem. The theory of the framework we developed in the past years is then introduced, which may preserve the hyperbolic nature of the kinetic equations with high universality. Some latest progress on the comparison between models with/without hyperbolicity is presented to validate the hyperbolic moment models for rarefied gases.

1. HISTORICAL OVERVIEW

Let us emphasize that a thorough overview of the model reduction of kinetic equations is impossible for us due to the huge amount of literature. Therefore we strictly limit ourselves only to the topics related to hyperbolicity. Even so, only a tiny part of the contributions in the history will be mentioned below.

According to Sir J.H.Jeans [29], the kinetic picture of a gas is “a crowd of molecules, each moving on its own independent path, entirely uncontrolled by forces from the other molecules, although its path may be abruptly altered as regards both speed and direction, whenever it collides with another molecule or strikes the boundary of the containing vessel.” In order to describe the evolution of non-equilibrium gases using the phase-space distribution function, the Boltzmann equation was proposed [1] as a non-linear seven-dimensional partial differential equation. The independent variables of the distribution function include the time, the spatial coordinates, and the velocity.

In most cases, the full Boltzmann equation cannot be solved even numerically. One has to characterize the motion of the gas by resorting to various approximation methods to describe the evolution of macroscopic quantities. One successful way to find approximate solutions is the Chapman-Enskog method [15, 18], which uses a power series expansion around the Maxwellian to describe slightly non-equilibrium gases. The method assumes that the distribution function can be approximated up to any precision only using equilibrium variables and their derivatives. Alternatively, Grad’s moment method [24] was developed in the late 1940s. In this method, by taking velocity moments of the Boltzmann equation, transport equations for macroscopic averages are obtained. The difficulty of this method is that the governing equations for the components of the $n$-th velocity moment also depend on components of the $(n+1)$-th moment. Therefore, one has to use a certain closing relation to get a closed system after the truncation.

Among the models given by Grad’s method [24], Grad’s 13-moment system is the most basic one beyond the Navier-Stokes equations. In [23], it was commented
that Grad’s moment method could be regarded as mathematically equivalent to the Chapman-Enskog method in certain cases. Thus the deduction of Grad’s 13-moment system can be regarded as an application of perturbation theory to the Boltzmann equation around the equilibrium. Therefore, it is natural to hope the 13-moment system is valid in the vicinity of equilibrium, although it was not expected to be valid far away from the equilibrium distribution [25]. However, due to its complex mathematical expression, it is even not easy to check if the system is hyperbolic, as pointed out in [2]. As late as in 1993, it was eventually verified in [35, 36] that the 1D reduction of Grad’s 13-moment equations is hyperbolic around the equilibrium.

In 1958, Grad wrote an article “Principles of the kinetic theory of gases” in Encyclopedia of Physics [26], where he collected his own method in the class of “more practical expansion techniques”. However, successful applications of the 13-moment system had been hardly seen within two decades after Grad’s classical paper in 1949, as mentioned in the comments by Cercignani [14]. One possible reason was found by Grad himself in [25], where it was pointed out that there may be unphysical sub-shocks in a shock profile for Mach number greater than a critical value. However, the appearance of sub-shocks cannot give any hints on the underlying reason why Grad’s moment method does not work for slow flows. Nevertheless, Grad’s moment method was still pronounced to “open a new era in gas kinetic theory” [27].

In our paper [5], it was found astonishingly that in the 3D case, the equilibrium is NOT an interior point of the hyperbolicity region of Grad’s 13-moment model. Consequently, even if the distribution function is arbitrarily close to the local equilibrium, the local existence of the solution of the 13-moment system cannot be guaranteed as a Cauchy problem of a first-order quasi-linear partial differential system without analytical data. The defects of the 13-moment model due to the lack of hyperbolicity had never been recognized as so severe a problem. The absence of hyperbolicity around local equilibrium is a candidate reason to explain the overall failure of Grad’s moment method.

After being discovered, the lack of hyperbolicity is well accepted as a deficiency of Grad’s moment method, which makes the application of the moment method severely restricted. “There has been persistent efforts to impose hyperbolicity on Grad’s moment closure by various regularizations” [39], and lots of progress has been made in the past decades. For example, Levermore investigated the maximum entropy method and showed in [33] that the moment system obtained with such a method possesses global hyperbolicity. Unfortunately, it is difficult to put it into practice due to the lack of a finite analytical expression, and the equilibrium lies on the boundary of the realizability domain for any moment system containing heat flux [30]. Based on Levermore’s 14-moment closure, an affordable 14-moment closure is proposed in [34] as an approximation, which extends the hyperbolicity region to a great extent. Let us mention that actually in [5], we also derived a 13-moment system with hyperbolicity around the equilibrium.

It looks highly non-trivial to gain hyperbolicity even around the equilibrium, while things changed not long ago. Besides the achievement of local hyperbolicity around the equilibrium, the study on the globally hyperbolic moment systems with large numbers of moments was also very successful in the past years. In the 1D case with both spatial and velocity variables being scalar, a globally hyperbolic moment
system was derived in [3] by regularization. Motivated by this work, another type of globally hyperbolic moment systems was then derived in [31] using a different strategy. The model in [3] is obtained by modifying only the last equation and the model in [31] revises only the last two equations in Grad’s original system. The characteristic fields of these models can be fully clarified and the wave speeds are formally a natural extension of Euler equations.

In [4], the regularization method in [3] is extended to multi-dimensional cases. Here the word “multi-dimension” means that the dimensions of spatial coordinates and velocity are any positive integers and can be different. The complicated multi-dimensional models with global hyperbolicity based on a Hermite expansion of the distribution function up to any degree were systematically proposed in [4]. The wave speeds and the characteristic fields can be clarified, too. Later on, the multi-dimensional model for an anisotropic weight function with global hyperbolicity was derived in [20].

Achieving global hyperbolicity was definitely encouraging, while it sounded like a huge mystery for us how the regularization worked in the aforementioned cases. Particularly, the method cannot be applied to moment systems based on a spherical harmonic expansion of distribution function such as Grad’s 13-moment system. As we pointed out, the hyperbolicity is essential for a moment model, while it is hard to obtain by a direct moment expansion of kinetic equations. To overcome such a problem, we in [6] fortunately developed a systematic framework to perform moment model reduction that preserves global hyperbolicity. The framework works not only for the models based on Hermite expansions of the distribution function in the Boltzmann equation, but also works for any ansatz of the distribution function in the Boltzmann equation. Actually, the framework even works for kinetic equations in a fairly general form.

The framework developed in [6] was further presented in the language of projection operators in [19], where the underlying mechanism of how the hyperbolicity is preserved during the model reduction procedure was further clarified. This is the basic idea of our discussion in the next section.

2. Theoretical Framework

In this section, we briefly review the framework in [19] to construct globally hyperbolic moment systems from kinetic equations, as well as its variants and further development. To clarify the statement, we present the definition of the hyperbolicity used in this report.

**Definition 1.** The first-order equations

$$\frac{\partial w}{\partial t} + \sum_{d=1}^{D} A_d(w) \frac{\partial w}{\partial x_d} = 0, \quad w \in \mathbb{G}$$

is hyperbolic at $w_0$, if for any unit vector $n \in \mathbb{R}^D$, the matrix $\sum_{d=1}^{D} n_d A_d(w_0)$ is real diagonalizable; is globally hyperbolic if the system is hyperbolic for any $w \in \mathbb{G}$.

Based on the definition, the hyperbolicity problem for moment systems is converted to the real diagonalizability in linear algorithm. However, the coefficient matrices in the moment systems are symbol matrices, so that few tools could be candidates to study such the problem. The most direct tool is that for the matrix $A \in \mathbb{R}^{n \times n}$,
all its eigenvalues are real and it has \( n \) linearly independent eigenvectors.

As its sufficient conditions, one can also consider following two tools:

**Tool 2.** all the eigenvalues of the matrix are real and distinct;

**Tool 3.** the matrix is symmetric.

Grad [24] investigates the characteristic structure of the 1D reduction of Grad’s 13-moment system, whose hyperbolicity is further studied in [36] based on the tool 2. Afterward, this tool is adopted on the proof of the hyperbolicity of the regularized moment system for the 1D case in [3]. It is worth noting that the tool 2 usually requires the characteristic polynomial of the coefficient matrix of the moment system, and thus the analysis on the hyperbolicity is a bit complex. The property that all the eigenvalues are distinct limits the capability of this tool in the 1D case.

To study the hyperbolicity in multi-dimensional cases, we apply the tool 1 in [5] to show that Grad’s 13-moment system is not hyperbolic even in any small neighbor of the equilibrium, and in [4] to prove the global hyperbolicity of the regularized moment system for the multi-dimensional case. Since the requirement on the eigenvectors, the proof based on the tool 1 is rather long and complex. By contrast, the tool 3 is easy to check, and based on it, Levermore provided a concise and clear proof of the hyperbolicity of the maximum entropy moment system in [33]. In the following, we re-study the hyperbolicity of the regularized moment system in [3, 4] based on the tool 3 and then generate it to the framework in [19].

### 2.1. Review of globally hyperbolic moment system

For the Boltzmann equation

\[
\frac{\partial f}{\partial t} + \sum_{d=1}^{D} v_d \frac{\partial f}{\partial x_d} = Q(f),
\]

denote its equilibrium by \( f_{eq} \). The key idea of Grad’s moment method is to expand the distribution as

\[
f(t, x, v) = \sum_{|\alpha| \leq M} f_{eq}(t, x, v) f_\alpha(t, x) H_{\alpha}(t, x, v) = \sum_{|\alpha| \leq M} f_\alpha(t, x) H_{\alpha}(t, x, v)
\]

for a given integer \( M \geq 2 \), where for the multi-index \( \alpha \in \mathbb{N}^D, |\alpha| = \sum_{d=1}^{D} \alpha_d \), \( H_{\alpha} \) is the Hermite polynomials corresponding to the equilibrium \( f_{eq} \), and the basis function \( H_{\alpha} = f_{eq} H_{\alpha} \). With this expansion, Grad’s moment system can be obtained by substituting the expansion into the Boltzmann equation and matching the coefficient of \( H_{\alpha} \) with \( |\alpha| \leq M \). To clearly describe this procedure, we assume that the distribution \( f \) is defined on a space \( \mathbb{H} \) containing all the basis \( H_{\alpha} \) and let \( \mathbb{H}_M := \text{span}\{H_{\alpha} : |\alpha| \leq M\} \). Then one can introduce the projection from \( \mathbb{H} \) to \( \mathbb{H}_M \) as

\[
\mathcal{P} f = \sum_{|\alpha| \leq M} f_\alpha H_{\alpha} \quad \text{with} \quad f_\alpha = \langle f, H_{\alpha} \rangle,
\]

where the inner product is defined as \( \langle f, g \rangle = \int_{\mathbb{R}^D} f g / f_{eq} \; dv \). The projection accurately describes Grad’s expansion [22] and provides a tool to study the operators in the space \( \mathbb{H}_M \), for example, the matching coefficients of the basis \( H_{\alpha} \) with \( |\alpha| \leq M \).
can be understood as projecting the system into the space $\mathbb{H}_M$. Hence, Grad’s moment system is written as

$$P\frac{\partial Pf}{\partial t} + \sum_{d=1}^{D} P_{vd} \frac{\partial Pf}{\partial x_d} = PQ(Pf). \tag{4}$$

We arrange the basis $H_\alpha$ with $|\alpha| \leq M$ by a given order as a vector, denoted by $H$. Since $Pf$ is a function in $\mathbb{H}_M$, one can collect all the independent variables in $Pf$ and denote it by $w$ with its length equal to the dimension of $\mathbb{H}_M$. Thanks to the definition of the projection operator $P$, there exist the square matrices $D$ and $B_d$, $d = 1, \ldots, D$ such that

$$P\frac{\partial Pf}{\partial t} = H^T D \frac{\partial w}{\partial t}, \quad P_{vd} \frac{\partial Pf}{\partial x_d} = H^T B_d \frac{\partial w}{\partial x_d}. \tag{5}$$

Accordingly, letting $Q$ by the vectorization of $Q_\alpha$ such that $PQ(Pf) = H^T Q$, one can rewrite Grad’s moment system as

$$D\frac{\partial w}{\partial t} + \sum_{d=1}^{D} B_d \frac{\partial w}{\partial x_d} = Q. \tag{6}$$

Actually, the system (6) is the vector form of (4) in $\mathbb{H}_M$ with the basis $H_\alpha$. Precisely, we have the following correspondence

$$w \leftrightarrow Pf, \quad D\frac{\partial}{\partial t} \leftrightarrow P\frac{\partial}{\partial t}, \quad B\frac{\partial}{\partial x_d} \leftrightarrow P_{vd} \frac{\partial}{\partial x_d}, \quad \text{in } \mathbb{H}_M \text{ with basis } H_\alpha. \tag{7}$$

Further, we diagram the procedure of Grad’s moment system in Fig. 1a. It’s argued in [19] that the time derivative part and the space derivative part are different, since there is an extra projection after the time derivative. The key observation

**Figure 1.** Diagram of the procedure of Grad’s and regularized moment system.
in [19] is that one should add a projection between the multiplying velocity operator and the space derivative operator, as illustrated in [Fig. 1b]. The corresponding moment system is

\[(8)\]
\[
P \frac{\partial P f}{\partial t} + \sum_{d=1}^{D} P v_d P \frac{\partial P f}{\partial x_d} = P Q(P f).
\]

Using (5), one can claim that there exist the square matrices \(M_d\), \(d = 1, \ldots, D\) such that

\[(9)\]
\[
P v_d P \frac{\partial P f}{\partial x_d} = H^T M_d \frac{\partial w}{\partial x_d},
\]

and obtain the vector form of the regularized moment system as

\[(10)\]
\[
D \frac{\partial w}{\partial t} + \sum_{d=1}^{D} M_d D \frac{\partial w}{\partial x_d} = Q.
\]

Similar as the correspondence in (7), we have one more correspondence as

\[(11)\]
\[
M_d \leftrightarrow P v_d \cdot , \text{ in } \mathbb{H}_M \text{ with basis } \mathcal{H}_\alpha,
\]

that is to say, the matrices \(M_d\) is the restriction of the operator \(v_d \cdot\) on \(\mathbb{H}_M\). Easy to check that if the basis \(\mathcal{H}_\alpha\) is orthonormal, the matrices \(M_d\) are symmetric; otherwise, \(M_d\) is similar as a symmetric matrix. For each case, the linear combination of the matrices \(M_d\) is real diagonalizable. Easy to check the matrix \(D\) is invertible. Hence \(D^{-1} M_d D\) is similar as \(M_d\) so that the system \(\text{(10)}\) is globally hyperbolic. Moreover, without loss of generalization, we assume the basis is orthonormal. Then if one multiply \(D^T\) on both sides of \(\text{(10)}\), the resulting system

\[(12)\]
\[
D^T D \frac{\partial w}{\partial t} + \sum_{d=1}^{D} D^T M_d D \frac{\partial w}{\partial x_d} = D^T Q
\]

is a symmetric hyperbolic system in the sense of balance laws.

2.2. Hyperbolic regularization framework. Till now, the hyperbolicity of \(\text{(10)}\) has been proved using the tool 3. If looking back on the whole procedure, one can easily find that the key point of the hyperbolic regularization is the extra projection between the multiplying velocity operator and the space derivative operator in [8]. Meanwhile, many elements in [8] can be relaxed and generalized to much more general cases. For example, the Boltzmann equation could be extended to the kinetic equation with the form

\[(13)\]
\[
\frac{\partial f(t, x, v)}{\partial t} + \sum_{d=1}^{D} \xi_d(v) \frac{\partial f(t, x, v)}{\partial x_d} = Q(f)(t, x, v),
\]

where as an example, the velocity \(\xi_d(v) = \frac{v}{|v|}\) for the radiative transfer equation. The local equilibrium \(f_{eq}\) could be replaced by a nonnegative weight function \(\omega\), \(\mathcal{H}_{\omega}\) is replaced by the orthogonal polynomial \(\phi_\alpha\) with respect to \(\omega\), and \(\mathcal{H}_\alpha\) is replaced by \(\Phi_\alpha := \omega \phi_\alpha\). By letting \(\mathbb{H}_M := \text{span}\{\Phi_\alpha : |\alpha| \leq M\}\), one can similar define the projection operator \(P\) as in [3]. As an extension of the globally hyperbolic moment system, we obtain

\[(14)\]
\[
P \frac{\partial P f}{\partial t} + \sum_{d=1}^{D} P \xi_d(v) P \frac{\partial P f}{\partial x_d} = P Q(P f).
\]
We note here that if the corresponding matrix \( D \) as in (6) is invertible, the resulting moment system is globally hyperbolic. We refer readers to [6, 19, 21] for more details of the above formulas and the framework.

This framework provides a concise and clear procedure to derive the hyperbolic moment system from kinetic equations. One can directly apply it to different fields. This framework has been applied to many fields, including anisotropic hyperbolic moment system for Boltzmann equation [20], semiconductor device simulation [7], plasma simulation [11], density functional theory [8], quantum gas kinetic theory [16], and rarefied relativistic Boltzmann equation [32].

2.3. Further progress. The upper framework provides an approach to handle the hyperbolicity of the moment system. However, the hyperbolicity is not the only concerned property. How to maintain the hyperbolicity and more other properties at the same time is a critical issue. In the following, we list some recent trials in this direction.

Navier-Stokes laws is a fundamental property for the high-order moment system in the hydrodynamic limit. Thus how to maintain the hyperbolicity and Navier-Stokes laws simultaneously is an important issue. For most applications in the past, the Navier-Stokes laws are always automatically satisfied by the regularized moment system. However, for the quantum Boltzmann equation, the equilibrium has a very special form. The moment system directly derived from the framework by taking the equilibrium as the weight function disobey the Navier-Stokes laws [16]. In this case, the authors of [16] proposed a called locally linearization method to regularize the moment system. In precise, we assume the Grad-type system has the form as (6) and let \( \hat{M}(w) = B(w)D(w)^{-1} \). In the regularization, the matrix \( M(w) \) is replaced by \( M := \hat{M}(w_{eq}) \) with \( w_{eq} \) to be \( w \) corresponding to the equilibrium. This specific method allows us to acquire both the hyperbolicity and Navier-Stokes laws at the same time, while the symmetry of \( M \) is lost such that we have to use the tool 1 to prove the hyperbolicity.

Another exception is the nonlinear moment equation for radiative transfer equation in [22, 21]. In order to retain the diffusion limit (corresponding to Navier-Stokes laws in hydrodynamics), the authors point out that the projection operators in (14) at different places do not have to be same and revise (14) to be

\[
\hat{P} \frac{\partial Pf}{\partial t} + \sum_{d=1}^{D} \hat{P} \xi_d(v) \hat{P} \frac{\partial Pf}{\partial x_d} = \hat{P} Q(\mathcal{P} f).
\]

Both the operators \( \mathcal{P} \) and \( \hat{\mathcal{P}} \) are orthogonal projections and the space \( \tilde{H}_M \) (for \( \hat{\mathcal{P}} \)) is constructed based on \( \frac{\partial \phi}{\partial \xi_d} \) for each \( \phi \in \mathbb{H} \). Due to the specific selection of \( \tilde{H}_M \), the diffusion limit is achieved, and the symmetry of \( M \) corresponding to that in (10) is kept, and thus the hyperbolicity is retained. This generalization extends the application range of the hyperbolic regularization framework and also permits us to take properties of the kinetic equation into account of the regularization.

Except for the hyperbolicity for the convection term, there are more conditions for the whole moment system with the collision term, for example, Yong’s first stability condition [38], which includes the constraints on the convection term, collision term, and the coupling of both terms. This stability condition is showed to be critical for the existence of the solutions in [37]. In [17], the authors study
multiple Grad-type moment systems and confirm that all of these systems satisfy Yong’s first stability condition.

Under this concise and flexible framework, one may wonder what is sacrificed for the hyperbolicity. By writing out the equations, one can immediately observe that the form of balance law is ruined by the hyperbolic regularization. A natural question is: how to define the discontinuity in the solution? More generally, one may ask: what is the effect of such a regularization on the accuracy of the models? In the following section, we will provide some clues for these questions using numerical experiments.

3. Numerical Validation

The application of the framework in the gas kinetic theory has been investigated in a number of works \cite{3, 9, 10, 12}, where many one- and two-dimensional examples have been numerically studied to show the validity of hyperbolic moment equations. However, these globally hyperbolic models, as an improvement of Grad’s original models, have never been compared with Grad’s models in terms of the modeling accuracy. The only direct comparison seen in the literature is in \cite{10}, wherein for a shock tube problem with a density ratio of 7.0, the simulation of Grad’s moment equations breaks down and the corresponding hyperbolic moment equations appear to be stable. Without a straightforward comparison using the same example for which both models work, it could be questioned whether we lose accuracy when fixing the hyperbolicity. Such doubt may arise since the globally hyperbolic models can be considered as a partial linearization of Grad’s models about local Maxwellians.

In this section, we will make the comparison by testing both models on the same numerical examples. For simplicity, we only consider the one-dimensional physics, for which both $x$ and $v$ are scalars. In this case, the characteristic polynomial for the Jacobian of the flux function has an explicit formula \cite{3}, so that the hyperbolicity of Grad’s equation can be easily checked. The underlying kinetic equation used in our test is the Boltzmann-BGK equation with a constant relaxation time

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = \frac{1}{Kn} (f_{eq} - f).$$

The ansatz of the distribution function is given by \cite{4}, so that \cite{4} stands for Grad’s moment system, and \cite{8} stands for the hyperbolic moment system. Below we are going to use two benchmark tests to show the performance of both types of models. In the numerical results, we will mainly focus on the equilibrium variables including density $\rho$, velocity $u$, and temperature $\theta$, which are defined by

$$\rho(t, x) = \int_{\mathbb{R}} f(t, x, v) \, dv,$$

$$u(t, x) = \frac{1}{\rho(t, x)} \int_{\mathbb{R}} v f(t, x, v) \, dv,$$

$$\theta(t, x) = \frac{1}{\rho(t, x)} \int_{\mathbb{R}} [v - u(t, x)]^2 f(t, x, v) \, dv.$$

3.1. Shock structure. The structure of plane shock waves is frequently used as a benchmark test in the gas kinetic theory. The computational domain is $(-\infty, +\infty)$.
so that no boundary condition is involved, and the initial data are
\[ f(0, x, v) = \begin{cases} 
\frac{\rho_l}{\sqrt{2\pi\theta_l}} \exp\left(-\frac{(v - u_l)^2}{2\theta_l}\right), & \text{if } x < 0, \\
\frac{\rho_r}{\sqrt{2\pi\theta_r}} \exp\left(-\frac{(v - u_r)^2}{2\theta_r}\right), & \text{if } x > 0,
\end{cases} \]
where all the equilibrium variables are determined by the Mach number \( Ma \):
\[ \rho_l = 1, \quad u_l = \sqrt{3} Ma, \quad \theta_l = 1, \]
\[ \rho_r = \frac{2Ma^2}{Ma^2 + 1}, \quad u_r = \frac{\sqrt{3} Ma}{\rho_r}, \quad \theta_r = \frac{3Ma^2 - 1}{2\rho_r}. \]

We are interested in the steady-state of this problem. Since the parameter \( Kn \) only introduces a uniform spatial scaling, it does not affect the shock structure. Therefore we simply set it to be 1. The problem is solved numerically by the finite volume method. We refer the readers to \cite{12} for the details on the numerical method.

3.1.1. Case 1: \( Ma = 1.4 \) and \( M = 4 \). In this case, both Grad’s system and the hyperbolic moment system work due to the relatively small Mach number. The numerical results are shown in Figure 2. By convention, we plot the normalized density, velocity, and temperature defined by
\[ \bar{\rho}(x) = \frac{\rho(x) - \rho_l}{\rho_r - \rho_l}, \quad \bar{u}(x) = \frac{u(x) - u_r}{u_l - u_r}, \quad \bar{\theta}(x) = \frac{\theta(x) - \theta_l}{\theta_r - \theta_l}, \]
so that the value of all variables are generally within the range \([0, 1] \), unless the temperature overshoot is observed.

Figure 2. Left: The comparison of shock structures of two solutions with Mach number 1.4 and \( M = 4 \). Right: The green area is the hyperbolicity region (horizontal axis: \( \hat{f}_{M-1} \), vertical axis: \( \hat{f}_M \)), and the red loop is the parametric curve \((\hat{f}_{M-1}, \hat{f}_M)\) with parameter \( x \).

Figure 2a shows the hyperbolicity region of Grad’s moment equations. It has been proven in \cite{3} that for the one-dimensional physics, the hyperbolicity region...
can be characterized by the following two dimensionless quantities:

\[
\hat{f}_{M-1} = \frac{f_{M-1}}{\rho \theta (M-1)^{1/2}}, \quad \hat{f}_M = \frac{f_M}{\rho \theta M^{1/2}},
\]

where \( f_M \) and \( f_{M-1} \) are the last two coefficients in the expansion (3). The red curve in Figure 2b provides the trajectory of Grad’s solution in this diagram. It can be seen that for such a small Mach number, the whole solution is well inside the hyperbolicity region, so that the simulation of Grad’s moment equations is stable. Figure 2a shows that both methods provide smooth shock structures, and the predictions for all the equilibrium variables are similar. This example confirms the applicability of both systems in weakly non-equilibrium regimes. Note that for one-dimensional physics, Grad’s equations do not suffer from the loss of hyperbolicity near equilibrium.

3.1.2. Case 2: \( Ma = 2.0 \) and \( M = 4 \). Now we increase the Mach number to introduce stronger non-equilibrium. The same plots are provided in Figure 3. In this example, despite the numerical diffusion, discontinuities can be identified without difficulty from the numerical solutions. These discontinuities, also known as subshocks, appear due to the insufficient characteristic speed in front of the shock wave, meaning that both systems are insufficient to describe the physics. This example shows significantly different shock structures predicted by both methods. For Grad’s moment equations, the subshock locates near \( x = -7 \), while for hyperbolic moment equations, the subshock appears near \( x = -5 \). The wave structures also differ a lot. By focusing on the high-density region, we find that the solution of hyperbolic moment equations is smoother, showing the possibly better description of the physics.

![Figure 3](image.png)

**Figure 3.** Left: The comparison of shock structures of two solutions with Mach number 2.0 and \( M = 4 \). Right: The green area is the hyperbolicity region (horizontal axis: \( \hat{f}_{M-1} \), vertical axis: \( \hat{f}_M \)), and the red loop is the parametric curve \((\hat{f}_{M-1}, \hat{f}_M)\) with parameter \( x \).

Here we remind the readers that the wave structure of hyperbolic moment equations may depend on the numerical method, due to its non-conservative nature.
However, we would like to argue that it is meaningless to justify any discontinuous solution for the hyperbolic moment equations. In practice, the appearance of discontinuous solutions is an indication of the inadequate truncation of series, which inspires us to increase $M$ to get more reliable solutions.

Figure 3b shows that Grad’s solution still locates within the hyperbolicity region, although the curve is already quite close to the boundary of the region. This example shows that even in its hyperbolicity region, Grad’s moment method may lose its validity.

3.1.3. Case 3: $Ma = 2.0$ and $M = 6$. Now we try to increase $M$ and carry out the simulation again for Mach number 2.0. The results are given in Figure 4. With the hope that a larger $M$ can provide a better solution, we actually see that Grad’s moment equations lead to computational failure. The numerical solution before the computation breaks down is plotted in Figure 4a. Figure 4b clearly shows that this is caused by the loss of hyperbolicity. We believe that this implies the non-existence of the solution.

3.1.4. Case 4: $Ma = 1.7$ and $M = 6$. In this example, we decrease the Mach number so that the shock structure of Grad’s equations can be found. Figure 5a shows that the results of both systems generally agree with each other, but it can be observed that hyperbolic moment equations provide smoother solutions than Grad’s system, so that it is likely to be more accurate. Therefore, despite the higher nonlinearity of Grad’s system, it does not necessarily help provide better solutions.

**Figure 4.** Left: The shock structure of hyperbolic moment equations for Mach number 2.0 and $M = 6$, and Grad’s solution before computational failure ($t = 1.0$). Right: The green area is the hyperbolicity region (horizontal axis: $\hat{f}_{M-1}$, vertical axis: $\hat{f}_M$), and the red loop is the parametric curve ($\hat{f}_{M-1}, \hat{f}_M$) with parameter $x$.
Interestingly, when looking at the phase diagram plotted in Figure 5b, we see that Grad’s solution has run out of the hyperbolicity region. It is to be further studied why the solution is still stable. Here we would like to conjecture that the collision term and the numerical diffusion help stabilize the numerical solution in the evolutionary process, and for the steady-state equations, solutions for non-hyperbolic equations may still exist. Nevertheless, all the above numerical tests show the superiority of hyperbolic moment equations for both accuracy and stability.

3.1.5. Case 5: \( Ma = 2.0 \) and \( M = 10 \). In this example, we would like to show the failure of both systems for a larger \( M \). In Figure 6, we plot the results at \( t = 0.8 \), where both numerical solutions contain negative temperatures. In [25], the reason for such a phenomenon has been explained, which lies in the divergence of the approximation (3) as \( M \) tends to infinity. It is rigorously shown in [13] that when \( \theta_r > 2\theta_l \), for the solution of the steady-state BGK equation, the limit of \( Pf \) (see (3)) as \( M \to \infty \) does not exist. Here for \( Ma = 2.0 \), the temperature behind the shock wave is \( \theta_r = 55/16 > 2 = 2\theta_l \). Thus for a large \( M \), the divergence leads to a poor approximation of the distribution function, and it is reflected as a negative temperature in the numerical results. Such a divergence issue is independent of the subshock and the hyperbolicity, and should be regarded as a defect for both systems. The work on fixing the issue is ongoing.

3.2. Fourier flow. In this test, we are interested in the performance of both methods with wall boundary conditions. The fluid we are concerned about is between
two fully diffusive walls locating at \( x = -1/2 \) and \( x = 1/2 \). For the Boltzmann-BGK equation \((16)\), the boundary condition is

\[
\begin{align*}
    f(t, -1/2, v) &= \frac{\rho_l}{\sqrt{2\pi\theta_l}} \exp \left( -\frac{v^2}{2\theta_l} \right), & v > 0, \\
    f(t, 1/2, v) &= \frac{\rho_r}{\sqrt{2\pi\theta_r}} \exp \left( -\frac{v^2}{2\theta_r} \right), & v < 0,
\end{align*}
\]

where \( \theta_{l,r} \) stands for the temperature of the walls, and \( \rho_{l,r} \) is chosen such that

\[
\int_{\mathbb{R}} vf(t, \pm 1/2, v) \, dv = 0.
\]

Following \([24]\), the boundary conditions of moment equations can be derived by taking odd moments of the diffusive boundary condition. We choose the initial condition as

\[
(17) 
\begin{align*}
    f(0, x, v) &= \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{v^2}{2} \right)
\end{align*}
\]

for all \( x \). Again we are concerned only about the steady-state of the solution.

In our numerical experiments, we choose \( Kn = 0.3, \theta_l = 1 \) and \( M = 11 \). Two test cases with \( \theta_r = 1.9 \) and \( \theta_r = 2.7 \) are considered. For the smaller temperature ratio \( \theta_r = 1.9 \), the numerical results are given in Figure 7, where two solutions mostly agree with each other. The reference solution, computed using the discrete velocity model, is also provided in Figure 7a. It can be seen that both models provide reasonable approximations to the reference solution. The good behavior of Grad’s solutions can also be predicted by the phase diagram in Figure 7b, from which one can observe that the whole solution locates in the central area of the hyperbolicity region.

For \( \theta_r = 2.7 \), the results are plotted in Figure 8. In this case, if we start the simulation of Grad’s equations from the initial data \((17)\), the computation will break down due to the loss of hyperbolicity in the evolitional process. Therefore, we first run the simulation for hyperbolic moment equations from the initial data \((17)\) and evolve the solution to the steady-state. Afterward, this steady-state solution serves as the initial data of Grad’s equations. Although the steady-state solution of Grad’s equations can be found using this technique, the approximation looks...
poorer than hyperbolic moment equations. The phase diagram (Figure 8b) shows that the solution near the left wall is outside the hyperbolicity region, so that the validity of boundary conditions on the left wall becomes unclear. In contrast, the hyperbolic moment equations still provide reliable approximation despite the high temperature ratio.

**Figure 7.** Left: Steady Fourier flow for $\theta_r = 1.9$ (left vertical axis: $\rho$, right vertical axis: $\theta$). Right: The green area is the hyperbolicity region (horizontal axis: $\hat{f}_{M-1}$, vertical axis: $\hat{f}_M$), and the red line is the parametric curve $(\hat{f}_{M-1}, \hat{f}_M)$ with parameter $x$.

**Figure 8.** Left: Steady Fourier flow for $\theta_r = 2.7$ (left vertical axis: $\rho$, right vertical axis: $\theta$). Right: The green area is the hyperbolicity region (horizontal axis: $\hat{f}_{M-1}$, vertical axis: $\hat{f}_M$), and the red line is the parametric curve $(\hat{f}_{M-1}, \hat{f}_M)$ with parameter $x$. 
3.3. A summary of numerical experiments. In all the above numerical experiments, we see that despite the loss of some nonlinearity, the hyperbolicity fix does not appear to lose accuracy in any of the numerical tests. In regimes with moderate non-equilibrium effects, Grad’s equations may provide solutions outside the hyperbolicity region without numerical instability. In this situation, our experiments show that the hyperbolicity fix is likely to improve the accuracy of the model. It has also been demonstrated that other issues, such as subshocks and divergence, are not related to the hyperbolicity, and these issues have to be addressed independently.

4. Conclusion

The loss of hyperbolicity, as one of the major obstacles for the model reduction in gas kinetic theory, is almost cleared through the research works in recent years. With a handy framework introduced in Section 2 we can safely move our focus of model reduction to other properties such as the asymptotic limit, the stability, and the convergence issues. Our numerical experiments show that the hyperbolic regularization does not harm the accuracy of the model. It is our hope that such a framework can inspire more thoughts in the development of dimensionality reduction even beyond the kinetic theory.

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