GREEN FUNCTIONS FOR PRESSURE OF STOKES SYSTEMS

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Abstract. We study Green functions for the pressure of stationary Stokes systems in a (possibly unbounded) domain \( \Omega \subset \mathbb{R}^d \), where \( d \geq 2 \). We construct the Green function when coefficients are merely measurable in one direction and have Dini mean oscillation in the other directions, and \( \Omega \) is such that the divergence equation is solvable there. We also establish global pointwise bounds for the Green function and its derivatives when coefficients have Dini mean oscillation and \( \Omega \) has a \( C^{1,\text{Dini}} \) boundary. Green functions for the flow velocity of Stokes systems are also considered.

1. Introduction

We study Green functions and fundamental solutions for stationary Stokes systems with variable coefficients. Let \( \mathcal{L} \) be a second order elliptic operator in divergence form

\[
\mathcal{L} u = D_\alpha (A^{\alpha \beta} D_\beta u)
\]

acting on column vector valued functions \( u = (u_1, \ldots, u_d)^T \), defined on a domain \( \Omega \subset \mathbb{R}^d \), where \( d \geq 2 \). Unlike elliptic systems, Stokes systems have two types of Green functions. One is a pair \((G, \Pi) = (G(x, y), \Pi(x, y))\), which we call Green function for the flow velocity, satisfying

\[
\begin{cases}
\mathcal{L} G(\cdot, y) + \nabla \Pi(\cdot, y) = \delta_y I & \text{in } \Omega, \\
\operatorname{div} G(\cdot, y) = 0 & \text{in } \Omega, \\
G(\cdot, y) = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Here, \( G \) is a \( d \times d \) matrix-valued function, \( \Pi \) is a \( d \times 1 \) vector-valued function, \( I \) is the \( d \times d \) identity matrix, and \( \delta_y \) is the Dirac delta function concentrated at \( y \). For a more precise definition of the Green function for the flow velocity, see Section 2.

The other one is a pair \((G, P) = (G(x, y), P(x, y))\), which we call Green function for the pressure, satisfying (for instance, when \( |\Omega| < \infty \))

\[
\begin{cases}
\mathcal{L} G(\cdot, y) + \nabla P(\cdot, y) = 0 & \text{in } \Omega \setminus \{y\}, \\
\operatorname{div} G(\cdot, y) = \delta_y - \frac{1}{|\Omega|} & \text{in } \Omega, \\
G(\cdot, y) = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Here, \( G \) is a \( d \times 1 \) vector-valued function and \( P \) is a real-valued function. For a more precise definition of the Green function for the pressure, see Section 2.

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An observation is that if there exist Green functions for the flow velocity and the pressure, then the pair \((u, p)\) given by

\[
\begin{align*}
u(y) &= \int_{\Omega} G(x, y)^{\top} f(x) \, dx, \\
p(y) &= -\int_{\Omega} G(x, y) \cdot f(x) \, dx
\end{align*}
\]

is a weak solution of the problem

\[
\begin{align*}
\mathcal{L}^* u + \nabla p &= f \quad \text{in } \Omega, \\
\text{div } u &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where \(\mathcal{L}^*\) is the adjoint operator of \(\mathcal{L}\).

There is a large body of literature concerning Green function for the flow velocity satisfying (1.1). Regarding the classical Stokes system with the Laplace operator \(\mathcal{L} = \Delta\), i.e.,

\[
A_{\alpha\beta} = [\delta_{\alpha\beta}\delta_{ij}]_{i,j=1}^{d} \quad (\delta_{ij} \text{ is the usual Kronecker delta symbol}),
\]

we refer the reader to Ladyzhenskaya [21], Maz’ya-Plamenevskiǐ [24, 25], and D. Mitrea-I. Mitrea [27]. In [21], the author provided explicit formulas of fundamental solutions in \(\mathbb{R}^2\) and \(\mathbb{R}^3\). In [24, 25], the authors established pointwise estimates for the Green function and its derivatives in a piecewise smooth domain in \(\mathbb{R}^3\). In [27], the authors proved the existence of the Green function in a Lipschitz domain in \(\mathbb{R}^d\), where \(d \geq 2\). For further related results on fundamental solutions and Green functions, one can refer to the book [14] and references therein. See also [28] for Green functions satisfying mixed boundary conditions in domains in \(\mathbb{R}^3\) and \(\mathbb{R}^2\).

Regarding Stokes systems with variable coefficients, i.e.,

\[
A_{\alpha\beta} = [A_{\alpha\beta}(x)]_{i,j=1}^{d},
\]

we refer the reader to [8, 9, 7]. In [8], the authors established the existence and pointwise bound of the Green function on a bounded \(C^1\) domain when \(d \geq 3\) and \(A_{\alpha\beta}\) have vanishing mean oscillations (VMO). The corresponding results were obtained in [9] in the whole space and a half space when \(A_{\alpha\beta}\) are merely measurable in one direction and have small mean oscillations in the other directions (partially BMO). In [7], the authors constructed the Green function in a two dimensional domain when \(A_{\alpha\beta}\) are measurable and bounded. They also considered pointwise bounds of the Green function and its derivatives. For further related results, one can refer to [17] for the Green function of the Stokes system with oscillating periodic coefficients and [5] for the Green function satisfying the conormal derivative boundary condition.

In contrast to Green functions for the flow velocity, there are relatively few results on Green functions for the pressure satisfying (1.2). In particular, we are not able to find any literature dealing with Green functions for the pressure of the Stokes system with variable coefficients. Restricted to the Stokes system with the Laplace operator, we refer the reader to [29], where the authors proved the pointwise estimate for the Green function (and its derivatives) of the Dirichlet problem in a three dimensional domain. The corresponding results for the mixed problem were obtained in [26].

In this paper, we are concerned with both Green functions and fundamental solutions for the pressure of Stokes systems with variable coefficients. The class of coefficients \(A_{\alpha\beta}\) we consider is called of \emph{partially Dini mean oscillation}, which means
that $A^{\alpha\beta}$ are merely measurable in one direction and have Dini mean oscillation in the other directions; see Definition 2.1. Stokes systems with irregular coefficients of this type may be used to describe the motion of inhomogeneous fluids with density dependent viscosity and two or multiple fluids with interfacial boundaries; see [20, 22, 1, 12].

Let $\Omega$ be a (possibly unbounded) domain in $\mathbb{R}^d$ satisfying an exterior measure condition when $d = 2$, and assume that the divergence equation is solvable in $\Omega$. We prove that if coefficients are of partially Dini mean oscillation, then there exists a unique Green function $(\mathcal{G}, \mathcal{P})$ for the pressure in $\Omega$. The Green function satisfies global pointwise bounds

$$|\mathcal{G}(x, y)| \leq C|x - y|^{1-d}, \quad x \neq y,$$

$$|D_x \mathcal{G}(x, y)| + |P(x, y)| \leq C|x - y|^{-d}, \quad x \neq y$$

if we assume further that coefficients are of Dini mean oscillation in all directions and $\Omega$ has a $C^1,\text{Dini}$ boundary. Especially, the fundamental solution ($d \geq 3$) and the Green function ($d \geq 2$) in a half space have the global pointwise bounds (1.3) under a weaker condition that coefficients are of partially Dini mean oscillation. For further details, see Section 3.

We also deal with the Green function $(G, \Pi)$ (and the fundamental solution) for the flow velocity of Stokes systems. As mentioned above, its existence and pointwise bound, i.e.,

$$|G(x, y)| \leq C|x - y|^{2-d}, \quad x \neq y,$$

were obtained in [8, 9] when $d \geq 3$. In this paper, we extend the results in [8, 9] by showing that

$$|D_x G(x, y)| + |\Pi(x, y)| \leq C|x - y|^{1-d}, \quad x \neq y,$$

under the stronger assumption that the coefficients are of Dini mean oscillation in a domain having a $C^1,\text{Dini}$ boundary. Moreover, we verify a symmetric property of Green functions for the flow velocity and the pressure. For further details, see Section 7.

The theory regarding the existence and estimates of Green functions for Stokes systems is closely related to regularity theory of solutions to

$$\begin{cases}
  L u + \nabla p = f & \text{in } \Omega, \\
  \text{div } u = g & \text{in } \Omega.
\end{cases}$$

(1.4)

When dealing with Green functions for the flow velocity in [8, 9, 5], the authors used $L^\infty$ or $C^\alpha$-estimates of solutions $u$, which can be obtained from $W^{1,q}$-estimates for the system (1.4). See [8, 13, 12, 6] for $W^{1,q}$-regularity results with $q \in (1, \infty)$. This approach was introduced by Hofmann-Kim [18] to deal with Green functions and fundamental solutions for elliptic systems with VMO coefficients. In this paper, to construct the Green function for the pressure, we utilize $L^\infty$-estimates of not only $u$ but also $(Du, p)$. Thus, we are not able to apply the aforementioned $W^{1,q}$-estimates. Instead, we employ the recent results in [4, 4], where we proved $W^{1,\infty}$ and $C^1$-estimates for Stokes systems with coefficients having (partially) Dini mean oscillation. This argument allows us to get pointwise bounds of the Green function as well as its derivatives.

The remainder of this paper is organized as follows. We introduce some notation and definitions in the next section. In Section 3, we state the main theorems regarding Green functions for the pressure. In Section 4, we present some preliminary
results, and in Sections 5 and 6 we provide the proofs of the main theorems. We devote Section 7 to Green functions for the flow velocity. In Appendix, we provide the proofs of $L^\infty$-estimates, which are crucial for proving our main theorems.

2. Preliminaries

In this section, we introduce some notation and definitions used throughout the paper.

2.1. Notation. We use $x = (x_1, x')$ to denote a point in the Euclidean space $\mathbb{R}^d$, where $d \geq 2$ and $x' = (x_2, \ldots, x_d) \in \mathbb{R}^{d-1}$. We also write $y = (y_1, y')$ and $x_0 = (x_{01}, x'_0)$, etc. Balls in $\mathbb{R}^d$ and $\mathbb{R}^{d-1}$ are defined by

$$B_r(x) = \{ y \in \mathbb{R}^d : |x - y| < r \}, \quad B_r'(x') = \{ y' \in \mathbb{R}^{d-1} : |x' - y'| < r \}.$$  

Let $\Omega$ be a domain in $\mathbb{R}^d$. We write $\Omega_r(x) = \Omega \cap B_r(x)$ for all $x \in \mathbb{R}^d$ and $r > 0$.

For $q \in (0, \infty]$, we define

$$\tilde{L}^q(\Omega) = \{ u \in L^q(\Omega) : (u)_{\Omega} = 0 \},$$

where $L^q(\Omega)$ is the set of all measurable functions on $\Omega$ that are $q$-th integrable, and $(u)_{\Omega}$ is the average of $u$ over $\Omega$, i.e.,

$$(u)_{\Omega} = \left\{ \begin{array}{ll} \int_{\Omega} u \, dx = \frac{1}{|\Omega|} \int_{\Omega} u \, dx & \text{if } |\Omega| < \infty, \\ 0 & \text{if } |\Omega| = \infty. \end{array} \right.$$  

Note that $\tilde{L}^q(\Omega) = L^q(\Omega)$ if $|\Omega| = \infty$.

For $q \in [1, \infty]$, we denote by $W^{1,q}(\Omega)$ the usual Sobolev space and by $W_0^{1,q}(\Omega)$ the completion of $C^\infty_0(\Omega)$ in $W^{1,q}(\Omega)$, where $C^\infty_0(\Omega)$ is the set of all infinitely differentiable functions with compact supports in $\Omega$.

For $q \in [1, d]$, the space $Y^{1,q}(\Omega)$ is defined as the set of all measurable functions $u$ on $\Omega$ having a finite norm

$$\|u\|_{Y^{1,q}(\Omega)} = \|u\|_{L^q/(d-q)}(\Omega) + \|Du\|_{L^q(\Omega)},$$

and the space $Y_0^{1,q}(\Omega)$ is defined as the completion of $C^\infty_0(\Omega)$ in $Y^{1,q}(\Omega)$. Note that

$$Y_0^{1,q}(\mathbb{R}^d) = Y^{1,q}(\mathbb{R}^d), \quad W_0^{1,q}(\Omega) \subset Y_0^{1,q}(\Omega),$$

and $Y_0^{1,2}(\Omega)$ ($d \geq 3$) is a Hilbert space with inner product

$$\langle u, v \rangle = \int_{\Omega} Du \cdot Dv \, dx. \tag{2.1}$$

When $d = 2$, we denote by $Y_0^{1,2}(\Omega)$ the set of all weakly differentiable functions $u$ on $\Omega$ such that $\nabla u \in L^2(\Omega)$ and $\eta u \in W_0^{1,2}(\Omega)$ for any $\eta \in C^\infty_0(\mathbb{R}^2)$. In this case, if $\Omega$ is a Green domain in $\mathbb{R}^2$, i.e.,

$$\{ u \chi_\Omega : u \in C^\infty_0(\mathbb{R}^2) \} \not\subset W_0^{1,2}(\Omega),$$

then $Y_0^{1,2}(\Omega)$ is also a Hilbert space with inner product (2.1) and $C^\infty_0(\Omega)$ is a dense subset of $Y_0^{1,2}(\Omega)$. We note that $\mathbb{R}^2$ is not a Green domain, but $\mathbb{R}^2_+$ is. If $|\Omega| < \infty$, then $\Omega$ is a Green domain and $W_0^{1,2}(\Omega) = Y_0^{1,2}(\Omega)$. For more discussions of the space $Y_0^{1,q}(\Omega)$, see [23 §1.3.4].
We say that a measurable function \( \omega : [0, a] \to [0, \infty) \) is a Dini function provided there are constants \( c_1, c_2 > 0 \) such that
\[
c_1 \omega(t) \leq \omega(s) \leq c_2 \omega(t) \quad \text{whenever} \quad 0 < \frac{t}{2} \leq s \leq t < a
\]
and that \( \omega \) satisfies the Dini condition
\[
\int_0^a \frac{\omega(t)}{t} \, dt < \infty.
\]

**Definition 2.1.** Let \( f \in L^1_{loc}(\Omega) \).

(a) We say that \( f \) is of partially Dini mean oscillation with respect to \( x' \) in the interior of \( \Omega \) if there exists a Dini function \( \omega : [0, 1] \to [0, \infty) \) such that for any \( x \in \Omega \) and \( r \in (0, 1] \) satisfying \( B_{2r}(x) \subset \Omega \), we have
\[
\left| \int_{B_r(x)} f(y) - \int_{B_r(x')} f(y_1, z') \, dz' \right| \, dy \leq \omega(r).
\]
(b) We say that \( f \) is of Dini mean oscillation in \( \Omega \) if there exists a Dini function \( \omega : [0, 1] \to [0, \infty) \) such that for any \( x \in \Omega \) and \( r \in (0, 1] \), we have
\[
\left| \int_{\Omega_r(x)} f(y) - \int_{\Omega_r(x')} f(z) \, dz \right| \, dy \leq \omega(r).
\]

We define a domain with a \( C^{1,\text{Dini}} \) boundary by locally the graph of a \( C^1 \) function whose derivatives are uniformly Dini continuous.

**Definition 2.2.** Let \( \Omega \) be a domain in \( \mathbb{R}^d \). We say that \( \Omega \) has a \( C^{1,\text{Dini}} \) boundary if there exist a constant \( R_0 \in (0, 1] \) and a Dini function \( \varrho_0 : [0, 1] \to [0, \infty) \) such that the following holds: For any \( x_0 = (x_{01}, x'_0) \in \partial \Omega \), there exist a \( C^1 \) function \( \chi : \mathbb{R}^{d-1} \to \mathbb{R} \) and a coordinate system depending on \( x_0 \) such that in the new coordinate system, we have
\[
|\nabla x' \chi(x'_0)| = 0, \quad \Omega_{R_0}(x_0) = \{ x \in B_{R_0}(x_0) : x_1 > \chi(x') \},
\]
and
\[
\varrho_\chi(r) \leq \varrho_0(r) \quad \text{for all} \quad r \in [0, R_0],
\]
where \( \varrho_\chi \) is the modulus of continuity of \( \nabla x' \chi \), i.e.,
\[
\varrho_\chi(r) = \sup \{ |\nabla x' \chi(y') - \nabla x' \chi(z')| : y', z' \in \mathbb{R}^{d-1}, |y' - z'| \leq r \}.
\]

### 2.2 Stokes system

Let \( \mathcal{L} \) be a strongly elliptic operator of the form
\[
\mathcal{L} u = D_\alpha (A^{\alpha \beta} D_\beta u),
\]
where the coefficients \( A^{\alpha \beta} = A^{\alpha \beta}(x) \) are \( d \times d \) matrix-valued functions on \( \mathbb{R}^d \) satisfying the strong ellipticity condition, i.e., there is a constant \( \lambda \in (0, 1] \) such that for any \( x \in \mathbb{R}^d \) and \( \xi_\alpha \in \mathbb{R}^d, \alpha \in \{1, \ldots, d\} \), we have
\[
|A^{\alpha \beta}(x)| \leq \lambda^{-1}, \quad \sum_{\alpha, \beta=1}^d A^{\alpha \beta}(x) \xi_\beta \cdot \xi_\alpha \geq \lambda \sum_{\alpha=1}^d |\xi_\alpha|^2. \tag{2.2}
\]
The adjoint operator \( \mathcal{L}^* \) of \( \mathcal{L} \) is defined by
\[
\mathcal{L}^* u = D_\alpha ((A^{\beta \alpha})^T D_\beta u),
\]
where \( (A^{\beta \alpha})^T \) is the transpose of \( A^{\beta \alpha} \). Note that the coefficients of \( \mathcal{L}^* \) also satisfy the ellipticity condition (2.2) with the same constant \( \lambda \).
Let $\Omega$ be a domain in $\mathbb{R}^d$. We say that $(u, p) \in W^{1,1}_{\text{loc}}(\Omega)^d \times L^1_{\text{loc}}(\Omega)$ is a weak solution of

$$L u + \nabla p = f + D\alpha f\alpha$$

in $\Omega$ if

$$\int_{\Omega} A^{\alpha\beta} D_{\beta}u \cdot D_\alpha \phi \, dx + \int_{\Omega} p \, \text{div} \, \phi \, dx = - \int_{\Omega} f \cdot \phi \, dx + \int_{\Omega} f_\alpha \cdot D_\alpha \phi \, dx$$

holds for any $\phi \in C^\infty_0(\Omega)$.

Similarly, we say that $(u, p) \in W^{1,1}_{\text{loc}}(\Omega)^d \times L^1_{\text{loc}}(\Omega)$ is a weak solution of

$$L^* u + \nabla p = f + D\alpha f\alpha$$

in $\Omega$ if

$$\int_{\Omega} A^{\alpha\beta} D_{\beta}\phi \cdot D_\alpha u \, dx + \int_{\Omega} p \, \text{div} \, \phi \, dx = - \int_{\Omega} f \cdot \phi \, dx + \int_{\Omega} f_\alpha \cdot D_\alpha \phi \, dx$$

holds for any $\phi \in C^\infty_0(\Omega)^d$.

### 2.3. Green function for the pressure

In this subsection, we state the definition of a Green function for the pressure. In the definition below, $G = G(x, y)$ is a $d \times 1$ vector-valued function and $P = P(x, y)$ is a real-valued function on $\Omega \times \Omega$.

**Definition 2.3.** Let $d \geq 2$ and $\Omega$ be a domain in $\mathbb{R}^d$. We say that $(G, P)$ is a Green function for the pressure of $L$ in $\Omega$ if it satisfies the following properties.

1. For any $y \in \Omega$ and $R > 0$,
   
   $G(\cdot, y) \in L^1_{\text{loc}}(\Omega)^d$, 
   
   $(1 - \eta)G(\cdot, y) \in Y^{1,2}_0(\Omega)^d$, 

   where $\eta$ is a smooth function satisfying $\eta \equiv 1$ on $B_R(y)$. Moreover,

   $$P(\cdot, y) \in L^1_{\text{loc}}(\Omega) \cap L^2(\Omega \setminus B_R(y))$$

   for any $q \in (0, 1)$.

2. For any $y \in \Omega$ and $R > 0$, $(G(\cdot, y), P(\cdot, y))$ satisfies

   $$\begin{cases}
   L G(\cdot, y) + \nabla P(\cdot, y) = 0 & \text{in } \Omega \setminus B_R(y), \\
   \text{div} G(\cdot, y) = \delta_y - \frac{1}{|\Omega|} & \text{in } \Omega, \\
   G(\cdot, y) = 0 & \text{on } \partial \Omega,
   \end{cases}$$

   (2.3)

   where $\frac{1}{|\Omega|} = 0$ if $|\Omega| = \infty$.

3. If $(u, p) \in Y^{1,2}_0(\Omega)^d \times L^2(\Omega)$ is a weak solution of the problem

   $$\begin{cases}
   L^* u + \nabla p = f & \text{in } \Omega, \\
   \text{div} u = g - (g)_{\Omega} & \text{in } \Omega, \\
   u = 0 & \text{on } \partial \Omega,
   \end{cases}$$

   (2.4)

   where $f \in C^\infty_0(\Omega)^d$ and $g \in C^\infty_0(\Omega)$, then for a.e. $y \in \Omega \setminus \text{supp}\, g$, we have

   $$p(y) = - \int_{\Omega} G(x, y) \cdot f(x) \, dx + \int_{\Omega} P(x, y) g(x) \, dx.$$
We note that in (2.3), the divergence equation is understood as
\[ \int_{\Omega} G(\cdot, y) \cdot \nabla \varphi \, dx = -\varphi(y) + (\varphi)_{\Omega}, \quad \forall \varphi \in C_0^\infty(\Omega). \]

Since \( G(\cdot, y) \in Y^{1,2}(\Omega \setminus B_r(y))^d \) for any \( r > 0 \), the above identity implies that
\[ \text{div} G(\cdot, y) = -\frac{1}{|\Omega|} \quad \text{for a.e. } x \in \Omega \setminus \{y\}. \]

We also note that the property (c) in Definition 2.3 together with the unique solvability of the problem (2.4) gives the uniqueness of a Green function. More precisely, under Assumption 3.1 below, by the solvability result in Lemma 4.1, we get the uniqueness of a Green function in the sense that if \((\tilde{G}, \tilde{P})\) is another Green function satisfying the properties in Definition 2.3 then for any \( \phi \in C_0^\infty(\Omega)^d \) and \( \varphi \in C_0^\infty(\Omega) \), we have
\[ \int_{\Omega} (G(x, y) - \tilde{G}(x, y)) \cdot \phi(x) \, dx = \int_{\Omega} (P(x, y) - \tilde{P}(x, y)) \varphi(x) \, dx = 0 \]
for a.e. \( y \in \Omega \setminus \text{supp } \varphi \).

3. Main results

In this section, we state our main results concerning Green function for the pressure of Stokes system. For this, we impose the following solvability assumption of the divergence equation.

**Assumption 3.1.** There exists a constant \( K_0 > 0 \) such that the following holds: For any \( g \in \tilde{L}^2(\Omega) \), there exists \( u \in Y^{1,2}_0(\Omega)^d \) satisfying
\[ \text{div } u = g \quad \text{in} \quad \Omega, \quad \|Du\|_{L^2(\Omega)} \leq K_0\|g\|_{L^2(\Omega)}. \]

**Remark 3.2.** It is known that Assumption 3.1 holds in a bounded John domain; see [2, Theorem 4.1]. Here and throughout the paper, a domain is said to be bounded if it has finite diameter. Note that a bounded domain \( \Omega \) having a \( C^{1,\text{Dini}} \) boundary as in Definition 2.2 is a John domain as in [2, Definition 2.1] with respect to \((x_0, L) = (x_0, L)(d, R_0, \theta_0) \). Thus by [2, Theorem 4.1], \( \Omega \) satisfies Assumption 3.1 with \( K_0 = K_0(d, R_0, \theta_0, \text{diam}(\Omega)) \).

A simple example of unbounded domain satisfying Assumption 3.1 is the whole space. Indeed, based on a scaling argument and the existence of solutions to the divergence equation in a ball, one can verify that Assumption 3.1 holds with \( K_0 = K_0(d) \) when \( \Omega = \mathbb{R}^d \) and \( d \geq 3 \). By the same reasoning, Assumption 3.1 holds when
\[ \Omega = \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d : x_1 > 0, x_2 > 0, \text{ or } x_d > 0\}, \quad d \geq 2. \]

Exterior domains with Lipschitz boundary also satisfy the assumption; see [14, Theorem III.3.6, p. 189].

In the theorem below and throughout the paper, we denote
\[ d_y = \text{dist}(y, \partial \Omega), \quad d_y^* = \min\{1, d_y\}. \]
Note that \( d_y = \infty \) if \( \Omega = \mathbb{R}^d \).
**Theorem 3.3.** Let \( d \geq 2 \) and \( \Omega \) be a domain in \( \mathbb{R}^d \). When \( d = 2 \), \( \Omega \) is assumed to be a Green domain satisfying
\[
|B_R(x) \setminus \Omega| \geq \theta R^2, \quad \forall x \in \partial \Omega, \quad \forall R \in (0, \infty).
\] (3.1)
Suppose that the coefficients \( A^{\alpha \beta} \) of \( \mathcal{L} \) are of partially Dini mean oscillation with respect to \( x' \) in the interior of \( \Omega \) satisfying Definition 2.1(a) with a Dini function \( \omega = \omega_A \). Then, under Assumption 3.1, there exists a unique Green function \((G, \mathcal{P})\) for the pressure of \( \mathcal{L} \) in \( \Omega \) such that for any \( y \in \Omega \),
\[
G(\cdot, y) \text{ is continuous in } \Omega \setminus \{y\}
\]
and
\[
(G(\cdot, y), \mathcal{P}(\cdot, y)) \in W^{1, \infty}(\Omega \setminus \{y\})^d \times L^{\infty}(\Omega \setminus \{y\}).
\]
Moreover, for any \( x, y \in \Omega \) with \( 0 < |x - y| \leq d_y^* \), we have
\[
|G(x, y)| \leq C|x - y|^{1-d},
\] (3.2)
\[
\text{ess sup}_{B_{|x-y|/2}(x)} (|DG(\cdot, y)| + |\mathcal{P}(\cdot, y)|) \leq C'|x - y|^{-d},
\] (3.3)
where \((C, C') = (C, C')(d, \lambda, \omega_A, K_0)\) and \( C \) depends also on \( \theta \) when \( d = 2 \). The same results hold if \( \mathcal{L} \) is replaced with its adjoint operator \( \mathcal{L}^* \).

**Remark 3.4.** From the proof of Theorem 3.3, we get the following estimates for all \( y \in \Omega \).

(a) For any \( R \in (0, d_y^*] \), we have that
\[
\|G(\cdot, y)\|_{L^\infty(\Omega \setminus B_R(y))} \leq CR^{-d/2},
\]
\[
\|DG(\cdot, y)\|_{L^2(\Omega \setminus B_R(y))} + \|\mathcal{P}(\cdot, y)\|_{L^2(\Omega \setminus B_R(y))} \leq C'R^{-d/2},
\]
where \( q^* = 2d/(d-2) \) if \( d \geq 3 \) and \( q^* = \infty \) if \( d = 2 \).

(b) We have that
\[
|\{x \in \Omega : |G(x, y)| > c_0t\}| \leq C t^{-d/(d-1)}, \quad t \geq (d_y^*)^{-d},
\]
\[
|\{x \in \Omega : |D_x G(x, y)| > t\}| + |\{x \in \Omega : |\mathcal{P}(x, y)| > t\}| \leq C' t^{-1}, \quad t \geq (d_y^*)^{-d},
\]
where \( c_0 = 1 \) if \( d \geq 3 \) and \( c_0 = c_0(\lambda, \omega_A, K_0, \theta) > 0 \) if \( d = 2 \).

(c) For any \( R \in (0, d_y^*] \), we have that
\[
\|G(\cdot, y)\|_{L^q(B_R(y))} \leq C_q R^{1-d+d/q}, \quad \forall q \in (0, d/(d-1)),
\]
\[
\|DG(\cdot, y)\|_{L^q(B_R(y))} + \|\mathcal{P}(\cdot, y)\|_{L^q(B_R(y))} \leq C'_q R^{1-d+d/q}, \quad \forall q \in (0, 1).
\]
In the above, \((C, C') = (C, C')(d, \lambda, \omega_A, K_0), (C_q, C'_q) = (C_q, C'_q)(d, \lambda, \omega_A, K_0, q),\) and \((C, C_q)\) depends also on \( \theta \) when \( d = 2 \).

**Remark 3.5.** In Theorem 3.3, if the coefficients \( A^{\alpha \beta} \) of \( \mathcal{L} \) are of Dini mean oscillation with respect to all the directions in \( \Omega \) satisfying Definition 2.1(b), then by Definition 2.3(b) and Assumption 3.4, we see that \( DG(\cdot, y) \) and \( \mathcal{P}(\cdot, y) \) are continuous in \( \Omega \setminus \{y\} \). Hence, “ess sup” in (3.3) can be replaced with “sup”. Therefore, for any \( x, y \in \Omega \) with \( 0 < |x - y| \leq d_y^*/2 \), we have
\[
|D_x G(x, y)| + |\mathcal{P}(x, y)| \leq C|x - y|^{-d},
\]
where \( C = C(d, \lambda, \omega_A, K_0) \).
Remark 3.6. In the case when $|\Omega| < \infty$, the condition (3.1) can be replaced with the condition
\[ |B_R(x) \setminus \Omega| \geq \theta_R R^d, \quad \forall x \in \partial \Omega, \quad \forall R \in (0, 1). \] (3.4)
Indeed, (3.6) and (3.4) are equivalent because if (3.4) holds, then (3.1) also holds with $\theta = \theta(\theta_0, |\Omega|)$.

In the next theorem, we prove the global pointwise estimates for the Green function and its derivatives in a domain having a $C^{1, \text{Dini}}$ boundary.

**Theorem 3.7.** Let $d \geq 2$ and $\Omega$ be a domain in $\mathbb{R}^d$ having a $C^{1, \text{Dini}}$ boundary as in Definition 2.2. When $d = 2$, $\Omega$ is assumed to be a Green domain satisfying (2.3). Suppose that the coefficients $A^{\alpha \beta}$ of $\mathcal{L}$ are of Dini mean oscillation in $\Omega$ satisfying Definition 2.1(b) with a Dini function $\omega = \omega_A$. Let $(G, P)$ be the Green function for the pressure of $\mathcal{L}$ constructed in Theorem 3.3 under Assumption 3.1. Then for any $y \in \Omega$,
\[ G(\cdot, y) \] is continuously differentiable in $\overline{\Omega} \setminus \{y\}$
and
\[ P(\cdot, y) \] is continuous in $\overline{\Omega} \setminus \{y\}$.
Moreover, for any $x, y \in \Omega$ with $0 < |x - y| \leq 1$, we have
\[ |G(x, y)| \leq C|x - y|^{1-d}, \] (3.5)
\[ |D_x G(x, y)| + |P(x, y)| \leq C|x - y|^{-d}, \] (3.6)
where $C = C(d, \lambda, \omega_A, K_0, R_0, \theta_0)$ and $C$ depends also on $\theta$ when $d = 2$. Furthermore, if $(G^*, P^*)$ is the Green function for the pressure of $\mathcal{L}^*$ in $\Omega^*$, then for any $y \in \Omega$, there exists a measure zero set $N_y \subset \Omega$ containing $y$ such that
\[ P(x, y) = P^*(y, x) \] for all $x \in \Omega \setminus N_y$. (3.7)

**Remark 3.8.** Note that any domain $\Omega \subset \mathbb{R}^d$ having a $C^{1, \text{Dini}}$ boundary as in Definition 2.2 satisfies
\[ |B_R(x) \setminus \Omega| \geq \theta R^d, \quad \forall x \in \partial \Omega, \quad \forall R \in (0, 1), \] (3.8)
where $\theta = \theta(d, R_0, \theta_0)$. Therefore, by Remark 3.6, the condition (3.1) can be removed in Theorem 3.7 when $d = 2$ and $|\Omega| < \infty$. In this case, the constant $C$ in (3.8) and (3.9) also depends on $|\Omega|$ instead of $\theta$.

**Remark 3.9.** Because the Green function satisfies the zero Dirichlet boundary condition, we have a better estimate for $G$ than (3.5) near the boundary of $\Omega$. Indeed, for any $x, y \in \Omega$ with $0 < |x - y| \leq 1$, we have
\[ |G(x, y)| \leq C \min\{d_x, |x - y|\} \cdot |x - y|^{1-d}, \] where $C = C(d, \lambda, \omega_A, K_0, R_0, \theta_0)$ and $C$ depends also on $\theta$ when $d = 2$. For further details, see the proof of Theorem 3.7 in Section 4.

**Remark 3.10.** From the proof of Theorem 3.7 we get the following estimates for any $y \in \Omega$.
(a) For any $R \in (0, 1]$, we have that
\[ \|G(\cdot, y)\|_{L^\infty(\Omega \setminus B_R(y))} \leq CR^{-d/2}, \]
\[ \|DG(\cdot, y)\|_{L^2(\Omega \setminus B_R(y))} + \|P(\cdot, y)\|_{L^2(\Omega \setminus B_R(y))} \leq C'R^{-d/2}, \]
where $q^* = 2d/(d - 2)$ if $d \geq 3$ and $q^* = \infty$ if $d = 2$. 

(b) For any \( t > 0 \), we have that
\[
\left| \{ x \in \Omega : |G(x, y)| > t \} \right| \leq \frac{C}{t^{d/(d-1)}},
\]
\[
\left| \{ x \in \Omega : |D_x G(x, y)| > t \} \right| + \left| \{ x \in \Omega : |P(x, y)| > t \} \right| \leq C' t^{-1}.
\]

(c) For any \( R \in (0, 1] \), we have that
\[
\|G(\cdot, y)\|_{L^q(B_R(\Omega))} \leq C R^{1-d+\delta/q}, \quad \forall q \in (0, d/(d-1)),
\]
\[
\|DG(\cdot, y)\|_{L^q(B_R(\Omega))} + \|P(\cdot, y)\|_{L^q(B_R(\Omega))} \leq C' R^{-\delta/q}, \quad \forall q \in (0, 1).
\]

In the above,
\[
(C, C') = (C, C')(d, \lambda, \omega_A, K_0, R_0, g_0), \quad (C_q, C'_q) = (C_q, C'_q)(d, \lambda, \omega_A, K_0, R_0, g_0, q),
\]
and \((C, C_q)\) depends also on \( \theta \) when \( d = 2 \).

We finish this section with the following theorem, where we extend the results in Theorem 3.12, \( R^d \) up to the boundary in a half space \( R^d \), defined by
\[
R^d = \{ x : (x_1, x') \in R^d : x_1 > 0, x' \in R^{d-1} \}, \quad d \geq 2,
\]
when the measurable direction of the coefficients is perpendicular to the boundary \( \partial R^d \). For the reader’s convenience and future reference, we present the results together with the case when \( \Omega = R^d \) with \( d \geq 3 \).

**Definition 3.11.** Let \( f \in L^1_{loc}(\overline{\Omega}) \), where \( \Omega = R^d \) or \( R^d \). We say that \( f \) is of \textit{partially Dini mean oscillation with respect to} \( x' \) in \( \Omega \) if there exists a Dini function \( \omega : (0, 1] \rightarrow (0, \infty) \) such that for any \( x \in \overline{\Omega} \) and \( r \in (0, 1] \), we have
\[
\int_{\Omega_\epsilon(x)} |f(y) - \int_{B_r(x')} f(y_1, z') \, dz'| \, dy \leq \omega(r).
\]

**Theorem 3.12.** Let \( \Omega = R^d \) with \( d \geq 3 \) or \( \Omega = R^d \) with \( d \geq 2 \). Suppose that the coefficients \( A^{\alpha\beta} \) of \( L \) are of partially Dini mean oscillation with respect to \( x' \) in \( \Omega \) satisfying Definition 3.11, \( \omega = \omega_A \). Then there exists a unique Green function \( (G, P) \) for the pressure of \( L \) in \( \Omega \) such that for any \( y \in \Omega \) and \( r > 0 \),
\[
G(\cdot, y) \text{ is continuous in } \overline{\Omega} \setminus \{ y \}
\]
and
\[
(G(\cdot, y), P(\cdot, y)) \in W^1,\infty(\Omega \setminus B_r(\overline{\Omega})) \times L^\infty(\Omega \setminus B_r(\overline{\Omega})). \tag{3.9}
\]
Moreover, for any \( x, y \in \Omega \) satisfying \( 0 < |x - y| \leq 1 \), we have
\[
|G(x, y)| \leq C \min\{ d_x, |x - y| \} \cdot |x - y|^{-d}, \tag{3.10}
\]
\[
\text{ess sup}_{\Omega_{x-y/4}(x)} (|DG(\cdot, y)| + |P(\cdot, y)|) \leq C |x - y|^{-d}, \tag{3.11}
\]
where \( C = C(d, \lambda, \omega_A) \). Furthermore, the same estimates in Remark 3.10 hold with
\[
C = C(d, \lambda, \omega_A) \text{ and } C_q = C_q(d, \lambda, \omega_A, q). \text{ The same results hold if } L \text{ is replaced with its adjoint operator } L^*.
\]
4. Preliminary results

In this section, we prove some preliminary results. We do not impose any regularity assumptions on the coefficients $A^{\alpha\beta}$ of the operator $\mathcal{L}$. The following lemma concerns the solvability of Stokes system in $Y_0^{1,2}(\Omega) \times \tilde{L}^2(\Omega)$.

**Lemma 4.1.** Let $d \geq 2$ and $\Omega$ be a domain in $\mathbb{R}^d$. When $d = 2$, $\Omega$ is assumed to be a Green domain. Suppose that $f, g \in L^2(\Omega)^d$, $\alpha, \beta \in \mathbb{R}^d$ and

$$f \in \begin{cases} L^{2d/(d+2)}(\Omega)^d & \text{if } d \geq 3, \\ L^q(\Omega)^2 & \text{for some } q > 1 \text{ with a compact support} & \text{if } d = 2. \end{cases}$$

Then, under Assumption 3.1, there exists a unique $(u, p) \in Y_0^{1,2}(\Omega)^d \times \tilde{L}^2(\Omega)$ satisfying

$$\begin{cases} \mathcal{L}u + \nabla p = f + \partial_{\alpha} f, & \text{in } \Omega, \\ \text{div } u = g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, we have that for $d \geq 3$,

$$\|Du\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)} \leq C(\|f\|_{L^q(\Omega)} + \|f, \alpha\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}),$$

and for $d = 2$,

$$\|Du\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)} \leq C_q\|f\|_{L^q(\Omega)} + C(\|f, \alpha\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}),$$

where $C = C(d, \lambda, K_0)$ and $C_q = C_q(\lambda, K_0, \Omega, \text{supp } f, q)$.

**Proof.** We only present here the detailed proof of the case when $d = 2$ and $|\Omega| = \infty$ because the others are the same as the proof of [3] Lemma 3.2, where the authors proved the $W^{1,2}$-solvability in a bounded domain.

Let $f, f, \alpha$, and $g$ satisfy the hypothesis of the lemma. Suppose that supp $f \subset B_{r_0} = B_{r_0}(0)$ for some $r_0 > 0$. Since $f$ has a compact support, we may assume that $q \in (1, 2)$. We define a Hilbert space $H(\Omega)$ as the set of all functions $u \in Y_0^{1,2}(\Omega)^2$ satisfying div $u = 0$ in $\Omega$. Let $H^+(\Omega)$ be the orthogonal complement of $H(\Omega)$ in $Y_0^{1,2}(\Omega)^2$ and $P$ be the orthogonal projection from $Y_0^{1,2}(\Omega)^2$ onto $H^+(\Omega)$. By Assumption 3.1, there exists $w \in Y_0^{1,2}(\Omega)^2$ such that

$$\text{div}(Pw) = \text{div } w = g \text{ in } \Omega,$$

and

$$\|D(Pw)\|_{L^2(\Omega)} \leq \|Du\|_{L^2(\Omega)} \leq K_1\|g\|_{L^2(\Omega)}.$$  \hfill (4.2)

Now we set

$$\mathcal{F}(\phi) = -\int_{\Omega} f \cdot \phi dx + \int_{\Omega} f, \alpha \cdot D_{\alpha} \phi dx - \int_{\Omega} A^{\alpha\beta} D_{\beta}(Pw) \cdot D_{\alpha} \phi dx, \quad \phi \in H(\Omega).$$

By Hölder’s inequality, the Poincaré inequality, and [23] Lemma 1.84, we have

$$\left| \int_{\Omega} f \cdot \phi dx \right| \leq \|f\|_{L^q(\Omega)} \|\phi_{\chi_{B_{r_0}} - \phi(\chi_{B_{r_0}})}\|_{L^{2q/(3q-2)}(B_{r_0})}$$

$$+ C\|f\|_{L^q(\Omega)} \|\phi\|_{L^2(\Omega)}$$

$$\leq C\|f\|_{L^q(\Omega)} \|D(\phi_{\chi_{B_{r_0}}})\|_{L^{2q/(3q-2)}(B_{r_0})} + C\|f\|_{L^q(\Omega)} \|\phi\|_{L^2(\Omega)}$$

$$\leq C_0\|f\|_{L^q(\Omega)} \|D\phi\|_{L^2(\Omega)}.$$
where \( C_0 = C_0(\Omega, B_{r_1}, q) \). From this together with (1.2), we see that \( \mathcal{F} \) is a bounded linear functional on \( H(\Omega) \) satisfying
\[
|\mathcal{F}(\phi)| \leq CM\|D\phi\|_{L^2(\Omega)}, \quad M := C_0\|f\|_{L^q(\Omega)} + \|f_a\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)},
\]
where \( C = C(\lambda, K_1) \). Thus by the Lax-Milgram theorem, there exists \( v \in H(\Omega) \) satisfying
\[
\|Dv\|_{L^2(\Omega)} \leq CM, \quad \int_\Omega A^{\alpha\beta}D_\beta v \cdot D_\alpha \phi dx = \mathcal{F}(\phi), \quad \forall \phi \in H(\Omega).
\]

Therefore, the function \( u = v + Pw \) satisfies
\[
\int_\Omega A^{\alpha\beta}D_\beta u \cdot D_\alpha \phi dx = -\int_\Omega f \cdot \phi dx + \int_\Omega f_a \cdot D_\alpha \phi dx, \quad \forall \phi \in H(\Omega). \tag{4.4}
\]
To find \( p \), we set
\[
\mathcal{K}(\varphi) = -\int_\Omega A^{\alpha\beta}D_\beta u \cdot D_\alpha (P\Phi) dx - \int_\Omega f \cdot (P\Phi) dx + \int_\Omega f_a \cdot D_\alpha (P\Phi) dx,
\]
where \( \varphi \in L^2(\Omega) \) and \( \Phi \in Y^{1,2}_0(\Omega)^2 \) such that
\[
div(P\Phi) = \div \Phi = \varphi \quad \text{in} \quad \Omega
\]
and
\[
\|D(P\Phi)\|_{L^2(\Omega)} \leq \|D\Phi\|_{L^2(\Omega)} \leq K_1\|\varphi\|_{L^2(\Omega)}.
\]
Then it can be easily seen that \( \mathcal{K} \) is a bounded linear functional in \( L^2(\Omega) \) with the estimate
\[
|\mathcal{K}(\varphi)| \leq C(\|Du\|_{L^2(\Omega)} + M)\|\varphi\|_{L^2(\Omega)} \leq CM\|\varphi\|_{L^2(\Omega)}.
\]

Therefore, by the Riesz representation theorem, there exists \( p \in L^2(\Omega) \) such that
\[
\|p\|_{L^2(\Omega)} \leq CM, \quad \int_\Omega p\varphi dx = \mathcal{K}(\varphi), \quad \forall \varphi \in L^2(\Omega).
\]
By the definition of \( \mathcal{K} \) and \( P \), we have
\[
\int_\Omega A^{\alpha\beta}D_\beta u \cdot D_\alpha \phi dx + \int_\Omega p \div \phi dx = -\int_\Omega f \cdot \phi dx + \int_\Omega f_a \cdot D_\alpha \phi dx, \quad \forall \phi \in H^+,(\Omega).
\]
This together with (4.3) and (4.4) proves the lemma when \( d = 2 \). \( \square \)

In the two dimensional case, the \( L^2 \)-estimate in Lemma 4.1 is not well suited to proving optimal estimates of Green functions. Hence, instead of the \( L^2 \)-estimate, we shall use a \( L^q \)-estimate for some \( q > 2 \) (see Lemma 4.2 below), which is an easy consequence of the following reverse Hölder’s inequality.

**Lemma 4.2.** Let \( \Omega \) be a Green domain in \( \mathbb{R}^2 \) satisfying (5.1). Then, under Assumption (5.1), there exists \( q_0 = q_0(\lambda, K_0, \theta) > 2 \) such that if \( (u, p) \in Y^{1,2}_0(\Omega)^d \times \tilde{L}^2(\Omega) \) satisfies
\[
\begin{cases}
Lu + \nabla p = D_\alpha f_a & \text{in} \quad \Omega, \\
\div u = g & \text{in} \quad \Omega, \\
u = 0 & \text{on} \quad \partial \Omega,
\end{cases}
\]
where \( f_a \in L^{q_0}(\Omega)^2 \) and \( g \in \tilde{L}^{q_0}(\Omega) \), then for any \( x \in \overline{\Omega} \) and \( R \in (0, \infty) \), we have
\[
(\|Du\|_{\Omega_R/2(x)}^{q_0} + |p|_{\Omega_R/2(x)}^{q_0})^{1/q_0} \leq C((\|Du\|_{\Omega_R}^{2} + |p|^{2})_{\Omega_R(x)}^{1/2} + C((|f_a|_{\Omega_R}^{q_0} + |g|_{\Omega_R}^{q_0})^{1/q_0}_{\Omega_R(x)}), \tag{4.5}
\]
where $C = C(\lambda, K_0, \theta)$.

Proof. For the proof of the lemma, we refer the reader to that of [12, Lemma 3.8], where the authors proved the reverse Hölder’s inequality in a bounded Reifenberg flat domain. The argument in the proof of [12, Lemma 3.8] is general enough to allow domains to satisfy (3.1) and Assumption 3.1.

We note that our statement is slightly different from that of [12, Lemma 3.8]. In [12, Lemma 3.8], the exponent $q_0 \in (2, q_1)$ depends also on $q_1$ under the assumption that the data are $q_1$-th integrable. Indeed, if $q_1$ is sufficiently close to 2, then $q_0$ can be chosen as $q_1$. This follows by using Proposition 4.3 below instead of [12, Proposition 3.7].

**Proposition 4.3.** Let $1 < q < s$, $\Phi \in L^q(Q)$, and $\Psi \in L^s(Q)$ be nonnegative functions, where $Q$ is a d-dimensional cube. Suppose that

$$
\int_{B_R(x_0)} \Phi^q \, dx \leq C_0 \left( \int_{B_{4R}(x_0)} \Phi \, dx \right)^q + \int_{B_{4R}(x_0)} \Psi^q \, dx + \delta \int_{B_{4R}(x_0)} \Phi^q \, dx
$$

for any $B_{4R}(x_0) \subset Q$ with $R \in (0, R_1)$, where $R_1 > 0$, $C_0 > 1$, and $\delta \in [0, 1)$. Then there exist positive constants $\varepsilon$ and $C$, depending only on $d$, $q$, $C_0$, and $\delta$, such that

$$
\Phi \in L^{q_0}_{loc}(Q) \quad \forall q_0 \in [q, \min\{s, q + \varepsilon\}]
$$

and

$$
\left( \int_{B_R(x_0)} \Phi^{q_0} \, dx \right)^{1/q_0} \leq C \left( \int_{B_{4R}(x_0)} \Phi^q \, dx \right)^{1/q} + C \left( \int_{B_{4R}(x_0)} \Psi^{q_0} \, dx \right)^{1/q_0}
$$

for any $B_{4R}(x_0) \subset Q$ with $R \in (0, R_1]$.

Proof. See, for instance, [15, Ch.V] for the proof of the proposition.

**Lemma 4.4.** Let $\Omega$ be a Green domain in $\mathbb{R}^2$ satisfying (3.1). Let $q_0 = q_0(\lambda, K_0, \theta) > 2$ be the constant from Lemma 4.2 under Assumption 3.1. If $(u, p) \in Y^{1.2}_0(\Omega)^d \times \tilde{L}^2(\Omega)$ satisfies

$$
\begin{align*}
\mathcal{L} u + \nabla p &= f + D_\alpha f_\alpha \quad \text{in } \Omega, \\
\text{div } u &= g \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega,
\end{align*}
$$

where $f \in L^{2q_0/(2+q_0)}(\Omega)^2$, $f_\alpha \in L^{q_0}(\Omega)^2$, and $g \in \tilde{L}^{q_0}(\Omega)$, then we have

$$
\|Du\|_{L^{q_0}(\Omega)} + \|p\|_{L^{q_0}(\Omega)} \leq C \left( \|f\|_{L^{2q_0/(2+q_0)}(\Omega)} + \|f_\alpha\|_{L^{q_0}(\Omega)} + \|g\|_{\tilde{L}^{q_0}(\Omega)} \right),
$$

where $C = C(\lambda, K_0, \theta)$.

Proof. If $f \equiv 0$, then the lemma follows by letting $R \to \infty$ in (4.5). On the other hand, if $f \not\equiv 0$, then by the existence of solutions to the divergence equation in the whole space (see, for instance, [9, Lemma 3.2]), there exist functions $F_\alpha \in Y^{2q_0/(2+q_0)}(\Omega)^2$, $\alpha \in \{1, 2\}$, satisfying

$$
D_\alpha F_\alpha = f \chi_{\Omega} \quad \text{in } \mathbb{R}^2
$$

and

$$
\|F_\alpha\|_{L^{q_0}(\mathbb{R}^2)} + \|DF_\alpha\|_{L^{2q_0/(2+q_0)}(\mathbb{R}^2)} \leq C \|f\|_{L^{2q_0/(2+q_0)}(\Omega)},
$$

where $C$ is an universal constant. Since $(u, p)$ satisfies (4.6) with $D_\alpha(F_\alpha + f_\alpha)$ in place of $f + D_\alpha f_\alpha$, we get the desired estimate from (4.7). The lemma is proved.
5. Approximated Green function

Hereafter in the paper, we shall use the following notation.

Notation 5.1. For a given function \( f \), if there is a continuous version of \( f \), that is, there is a continuous function \( \tilde{f} \) such that \( \tilde{f} = f \) in the almost everywhere sense, then we replace \( f \) with \( \tilde{f} \) and denote the version again by \( f \).

In this section, we assume that the hypotheses in Theorem 3.3 hold. Under these hypotheses, we shall construct approximated Green functions for the pressure of the Stokes system. We recall the notation that if \( |\Omega| = \infty \), then \( 1/|\Omega| = 0 \) and \( (u)_{\Omega} = 0 \) for any \( u \in L^q(\Omega) \).

Let us fix a smooth function \( \Phi \) defined on \( \mathbb{R}^d \) such that

\[
0 \leq \Phi \leq 2, \quad \text{supp} \Phi \subset B_1(0), \quad \int_{\mathbb{R}^d} \Phi \, dx = 1.
\]

For \( y \in \Omega \) and \( \varepsilon \in (0, 1] \), we define

\[
\Phi_{\varepsilon,y}(x) = \varepsilon^{-d} \Phi((x - y)/\varepsilon).
\]

By Lemma 5.1, there exists a unique \( (\mathcal{G}_\varepsilon(\cdot, y), \mathcal{P}_\varepsilon(\cdot, y)) \in Y_0^{1,2}(\Omega)^d \times \tilde{L}^2(\Omega) \) satisfying

\[
\begin{cases}
\mathcal{L}\mathcal{G}_\varepsilon(\cdot, y) + \nabla \mathcal{P}_\varepsilon(\cdot, y) = 0 & \text{in } \Omega, \\
\text{div } \mathcal{G}_\varepsilon(\cdot, y) = \Phi_{\varepsilon,y} - (\Phi_{\varepsilon,y})_{\Omega} & \text{in } \Omega, \\
\mathcal{G}_\varepsilon(\cdot, y) = 0 & \text{on } \partial \Omega.
\end{cases}
\]

(5.1)

Moreover, using the fact that

\[
\|\Phi_{\varepsilon,y} - (\Phi_{\varepsilon,y})_{\Omega}\|_{L^2(\Omega)} \leq 2 \|\Phi_{\varepsilon,y}\|_{L^2(\Omega)} \leq C \varepsilon^{-d/2},
\]

we have

\[
\|D\mathcal{G}_\varepsilon(\cdot, y)\|_{L^2(\Omega)} + \|\mathcal{P}_\varepsilon(\cdot, y)\|_{L^2(\Omega)} \leq C \varepsilon^{-d/2},
\]

(5.2)

where \( C = C(d, \lambda, K_0) \). Throughout the paper, we call \( (\mathcal{G}_\varepsilon(\cdot, y), \mathcal{P}_\varepsilon(\cdot, y)) \) the approximated Green function for the pressure of \( \mathcal{L} \) in \( \Omega \).

In the lemma below, we prove a \( L^1 \)-estimate for \( \mathcal{G}_\varepsilon(\cdot, y) \).

Lemma 5.1. Let \( y \in \Omega \) and \( \varepsilon \in (0, 1] \). Then for any \( x \in \Omega \) and \( R \in (0, d_\varepsilon^*] \), we have

\[
\|\mathcal{G}_\varepsilon(\cdot, y)\|_{L^1(B_\varepsilon(x))} \leq CR,
\]

where \( C = C(d, \lambda, \omega_A, K_0) \) and \( C \) depends also on \( \Theta \) when \( d = 2 \).

Proof. We consider the following two cases:

i. \( 2\varepsilon \leq R \) and \( 2\varepsilon > R \).

ii. \( 2\varepsilon \leq R \): Set

\[
f = \chi_{B_\varepsilon(x)} \left( \text{sgn } \mathcal{G}_\varepsilon^1(\cdot, y), \ldots, \text{sgn } \mathcal{G}_\varepsilon^d(\cdot, y) \right)^	op.
\]

Then by Lemma 4.1, there exists a unique \( (u, p) \in Y_0^{1,2}(\Omega)^d \times \tilde{L}^2(\Omega) \) satisfying

\[
\begin{cases}
\mathcal{L}^* u + \nabla p = -f & \text{in } \Omega, \\
\text{div } u = 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

(5.3)

We apply \( u \) and \( \mathcal{G}_\varepsilon(\cdot, y) \) as test functions to (5.1) and (5.3), respectively, to get

\[
\int_{\Omega} f \cdot \mathcal{G}_\varepsilon(\cdot, y) \, dx = \int_{\Omega} p \Phi_{\varepsilon,y} \, dx,
\]
which implies that (using \( \varepsilon \leq R/2 \))
\[
\| \mathcal{G}_\varepsilon(\cdot, y) \|_{L^1(B_R(x))} \leq \| p \|_{L^\infty(B_{R/2}(y))}.
\]
Hence, we get from (A.2) that
\[
\| \mathcal{G}_\varepsilon(\cdot, y) \|_{L^1(B_R(x))} \leq CR^{-d}(\| Du \|_{L^1(B_R(y))} + \| p \|_{L^1(B_R(y))}) + CR,
\]
where \( C = C(d, \lambda, \omega_A) \). If \( d \geq 3 \), then by (A.1), we have
\[
\| Du \|_{L^2(\Omega)} + \| p \|_{L^2(\Omega)} \leq CR^{(d+2)/2},
\]
where \( C = C(d, \lambda, K_0) \). From this together with (5.3) and Hölder’s inequality, we get the desired estimate. On the other hand, if \( d = 2 \), then by Lemma 4.4 applied to (5.3), we have
\[
\| Du \|_{L^2(\Omega)} + \| p \|_{L^2(\Omega)} \leq C\| f \|_{L^{2q_0/(2+q_0)}(\Omega)} \leq CR^{(2+q_0)/q_0},
\]
where \( C = C(\lambda, K_0, \theta) \) and \( q_0 = q_0(\lambda, K_0, \theta) > 2 \). Thus, from (5.3) and Hölder’s inequality, we get the desired estimate.

ii. \( 2 \varepsilon > R \): If \( d \geq 3 \), then by Hölder’s inequality, the Sobolev inequality, and (5.2), we have
\[
\| \mathcal{G}_\varepsilon(\cdot, y) \|_{L^1(B_R(x))} \leq CR^{(d+2)/2}\| \mathcal{G}_\varepsilon(\cdot, y) \|_{L^{2q_0/(d+2)}(B_R(x))} \\
\leq CR^{(d+2)/2}\| D\mathcal{G}_\varepsilon(\cdot, y) \|_{L^2(\Omega)} \leq CR^{(d+2)/2}\varepsilon^{-d/2} \leq CR,
\]
which gives the desired estimate.

Now we assume \( d = 2 \). Let \( f_\alpha \in C_0^\infty(\Omega)^2 \) and \( g \in C_0^\infty(\Omega) \). Then by Lemma 4.1, there exists a unique \((v, \pi) \in Y_0^{1/2}(\Omega)^2 \times \tilde{L}^2(\Omega)\) satisfying
\[
\begin{align*}
L^\star v + \nabla \pi &= D_\alpha f_\alpha \quad \text{in } \Omega, \\
\text{div } v &= g - (g)_{\Omega} \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]
We also get from Lemma 4.4 that
\[
\| Du \|_{L^{q_0}(\Omega)} + \| \pi \|_{L^{q_0}(\Omega)} \leq C\| f_\alpha \|_{L^{q_0}(\Omega)} + \| g \|_{L^{q_0}(\Omega)},
\]
where \( C = C(\lambda, K_0, \theta) \) and \( q_0 = q_0(\lambda, K_0, \theta) > 2 \). We test (5.5) and (5.5) with \( v \) and \( \mathcal{G}_\varepsilon(\cdot, y) \), respectively, to obtain
\[
\int_\Omega (f_\alpha \cdot D_\alpha \mathcal{G}_\varepsilon(\cdot, y) + g \mathcal{P}_\varepsilon(\cdot, y)) \, dz = \int_{B_\varepsilon(y)} \pi \Phi_{\varepsilon, y} \, dz.
\]
Then by Hölder’s inequality, (5.6), and \( R < 2\varepsilon \leq 2 \), we have
\[
\left| \int_{\Omega} (f_\alpha \cdot D_\alpha \mathcal{G}_\varepsilon(\cdot, y) + g \mathcal{P}_\varepsilon(\cdot, y)) \, dz \right| \leq \| \Phi_{\varepsilon, y} \|_{L^{q_0/(q_0-1)}(B_\varepsilon(y))} \| \pi \|_{L^{q_0}(\Omega)} \\
\leq CR^{-2/q_0} \left( \| f_\alpha \|_{L^{q_0}(\Omega)} + \| g \|_{L^{q_0}(\Omega)} \right).
\]
Since the above inequality holds for all \( f_\alpha \in C_0^\infty(\Omega)^2 \) and \( g \in C_0^\infty(\Omega) \), we get
\[
\| D\mathcal{G}_\varepsilon(\cdot, y) \|_{L^{q_0/(q_0-1)}(\Omega)} + \| \mathcal{P}_\varepsilon(\cdot, y) \|_{L^{q_0/(q_0-1)}(\Omega)} \leq CR^{-2/q_0},
\]
and thus, we obtain by the Sobolev inequality that
\[
\| \mathcal{G}_\varepsilon(\cdot, y) \|_{L^{2q_0/(q_0-2)}(\Omega)} \leq CR^{-2/q_0}.
\]
This together with Hölder’s inequality yields the desired estimate.

The lemma is proved. \( \square \)
We establish the following estimates uniformly in $\varepsilon \in (0, 1]$.

**Lemma 5.2.** Let $y \in \Omega$, $\varepsilon \in (0, 1]$, $R \in (0, d^*_y]$, and

$$q^* = \frac{2d}{d-2} \quad \text{if} \ d \geq 3, \quad q^* = \infty \quad \text{if} \ d = 2.$$  

Then we have

$$\| G_{\varepsilon}(\cdot, y) \|_{L^{q^*}(\Omega \setminus B_R(y))} \leq CR^{-d/2}$$  \hspace{1cm} (5.7)

and

$$\| DG_{\varepsilon}(\cdot, y) \|_{L^2(\Omega \setminus B_R(y))} + \| P_{\varepsilon}(\cdot, y) \|_{L^2(\Omega \setminus B_R(y))} \leq C'R^{-d/2},$$  \hspace{1cm} (5.8)

where $(C, C') = (C, C')(d, \lambda, \omega_A, K_0)$ and $C$ depends also on $\theta$ when $d = 2$.

**Proof.** We first prove the estimate (5.8). If $\varepsilon > R/2$, then (5.8) follows immediately from (5.2). Now we assume $\varepsilon \leq R/2$. Set

$$f_\alpha = \chi_{\Omega \setminus B_R(y)} D_\alpha G_{\varepsilon}(\cdot, y), \quad g = \chi_{\Omega \setminus B_R(y)} P_{\varepsilon}(\cdot, y).$$

Then by Lemma 4.1 there exists a unique $(u, p) \in \mathcal{Y}_{*}^{1,2}(\Omega)^d \times L^2(\Omega)$ satisfying

$$\begin{aligned}
\mathcal{L}^* u + \nabla p &= D_\alpha f_\alpha \quad \text{in} \ \Omega, \\
\text{div} u &= g - (g)_\Omega \quad \text{in} \ \Omega, \\
u &= 0 \quad \text{on} \ \partial \Omega.
\end{aligned}$$  \hspace{1cm} (5.9)

Moreover, we have

$$\begin{aligned}
\| Du \|_{L^2(\Omega)} + \| p \|_{L^2(\Omega)} &\leq C \left( \| f_\alpha \|_{L^2(\Omega)} + \| g \|_{L^2(\Omega)} \right) \\
&\leq C \left( \| DG_{\varepsilon}(\cdot, y) \|_{L^2(\Omega \setminus B_R(y))} + \| P_{\varepsilon}(\cdot, y) \|_{L^2(\Omega \setminus B_R(y))} \right),
\end{aligned}$$  \hspace{1cm} (5.10)

where $C = C(d, \lambda, K_0)$. Since $(u, p)$ satisfies

$$\begin{aligned}
\mathcal{L}^* u + \nabla p &= 0 \quad \text{in} \ B_R(y), \\
\text{div} u &= -(g)_\Omega \quad \text{in} \ B_R(y),
\end{aligned}$$

by (5.2) and Hölder’s inequality, we obtain

$$\| p \|_{L^\infty(B_{R/2}(y))} \leq CR^{-d/2} \left( \| Du \|_{L^2(\Omega)} + \| p \|_{L^2(\Omega)} \right) + C|\Omega|^{-1/2} \| g \|_{L^2(\Omega)},$$  \hspace{1cm} (5.11)

where $C = C(d, \lambda, \omega_A)$. Combining (5.10) and (5.11), and using $R^d \leq C(d)|\Omega|$, we see that

$$\| p \|_{L^\infty(B_{R/2}(y))} \leq CR^{-d/2} \left( \| DG_{\varepsilon}(\cdot, y) \|_{L^2(\Omega \setminus B_R(y))} + \| P_{\varepsilon}(\cdot, y) \|_{L^2(\Omega \setminus B_R(y))} \right),$$  \hspace{1cm} (5.12)

where $C = C(d, \lambda, \omega_A, K_0)$. We apply $u$ and $G_{\varepsilon}(\cdot, y)$ as test functions to (5.1) and (5.2), respectively, to get

$$\| DG_{\varepsilon}(\cdot, y) \|_{L^2(\Omega \setminus B_R(y))} + \| P_{\varepsilon}(\cdot, y) \|_{L^2(\Omega \setminus B_R(y))} \| = \int_{B_{R/2}(y)} p \Phi_{e,y} \, dx.$$  \hspace{1cm} (5.13)

This together with (5.12) gives (5.8).

We now turn to the proof of (5.7) when $d \geq 3$. Let $\eta$ be a smooth function on $\mathbb{R}^d$ satisfying

$$0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ on } B_{R/2}(y), \quad \text{supp} \eta \subset B_R(y), \quad |\nabla \eta| \leq CR^{-1}.$$
Then by the Sobolev inequality, we have
\[
\| (1 - \eta) G_\varepsilon (\cdot, y) \|_{L^{2(d/(d-2))}(\Omega)} \\
\leq C \| D((1 - \eta) G_\varepsilon (\cdot, y)) \|_{L^2(\Omega)} \\
\leq C \left( \| D G_\varepsilon (\cdot, y) \|_{L^2(\Omega \setminus B_{R/2}(y))} + R^{-1} \| G_\varepsilon (\cdot, y) \|_{L^2(B_R(y) \setminus B_{R/2}(y))} \right).
\]

Notice from the Poincaré inequality and Lemma 5.1 that
\[
\| \check{G}_\varepsilon (\cdot, y) \|_{L^2(B_R(y) \setminus B_{R/2}(y))} \\
\leq \| \check{G}_\varepsilon (\cdot, y) - (\check{G}_\varepsilon (\cdot, y))_{B_R(y) \setminus B_{R/2}(y)} \|_{L^2(B_R(y) \setminus B_{R/2}(y))} + CR^{-d/2} \| \check{G}_\varepsilon (\cdot, y) \|_{L^1(B_R(y))} \\
\leq CR \| D \check{G}_\varepsilon (\cdot, y) \|_{L^2(B_R(y) \setminus B_{R/2}(y))} + CR^{-d/2+1},
\]
where \( C = C(d, \lambda, \omega_A, K_0) \). Combining these together and using (5.8), we obtain
\[
\| (1 - \eta) G_\varepsilon (\cdot, y) \|_{L^{2(d/(d-2))}(\Omega)} \leq CR^{-d/2},
\]
which implies (5.7) when \( d \geq 3 \).

Next, we prove that (5.7) holds when \( d = 2 \). In this case, it suffices to show that
\[
\| G_\varepsilon (\cdot, y) \|_{L^\infty(\Omega \setminus \overline{B_R(y)})} \leq CR^{-1}, \quad \forall x \in \Omega \setminus \overline{B_R(y)},
\]
where \( C = C(\lambda, \omega_A, K_0, \theta) \). We consider the following two cases:

1. \( B_{R/8}(x) \cap \partial \Omega = \emptyset \) and \( B_{R/8}(x) \cap \partial \Omega \neq \emptyset \). Hereafter in this proof, we let \( q_0 = q_0(\lambda, K_0, \theta) > 2 \) be the constant from Lemma 4.2.

i. \( B_{R/8}(x) \cap \partial \Omega = \emptyset \): Let \( \zeta \) be a smooth function on \( \mathbb{R}^2 \) satisfying

\[
0 \leq \zeta \leq 1, \quad \zeta \equiv 1 \quad \text{on} \quad B_{R/16}(x), \quad \text{supp} \ \zeta \subset B_{R/8}(x), \quad |\nabla \zeta| \leq CR^{-1}.
\]

Then the pair \((v, \pi)\) given by
\[
(v, \pi) = \zeta (G_\varepsilon (\cdot, y) - (G_\varepsilon (\cdot, y))_{B_{R/8}(x)}, P_\varepsilon (\cdot, y))
\]
satisfies
\[
\begin{cases}
L v + \nabla \pi = f + D_\alpha f_\alpha & \text{in} \ B_{R/8}(x), \\
\text{div} v = g & \text{in} \ B_{R/8}(x), \\
v = 0 & \text{on} \ \partial B_{R/8}(x),
\end{cases}
\]
(5.14)

where
\[
f = A^{\alpha\beta} D_\beta G_\varepsilon (\cdot, y) D_\alpha \zeta + \nabla \zeta P_\varepsilon (\cdot, y),
\]
\[
f_\alpha = A^{\alpha\beta} D_\beta G_\varepsilon (\cdot, y) - (G_\varepsilon (\cdot, y))_{B_{R/8}(x)},
\]
\[
g = \nabla \zeta \cdot (G_\varepsilon (\cdot, y) - (G_\varepsilon (\cdot, y))_{B_{R/8}(x)}) + \zeta (\Phi_{\varepsilon,y} - (\Phi_{\varepsilon,y}))_\Omega.
\]

Since it holds that
\[
|(\Phi_{\varepsilon,y})_\Omega| = |\Omega|^{-1} \leq CR^{-2}
\]
and
\[
|\zeta (\Phi_{\varepsilon,y})| = 0 \quad \text{if} \ \varepsilon < \frac{7R}{8}, \quad |\zeta (\Phi_{\varepsilon,y})| \leq CR^{-2} \quad \text{if} \ \varepsilon \geq \frac{7R}{8},
\]
we obtain
\[
|g| \leq CR^{-1} |G_\varepsilon (\cdot, y) - (G_\varepsilon (\cdot, y))_{B_{R/8}(x)}| + CR^{-2}.
\]

Using this together with (5.8), Hölder’s inequality, the Poincaré inequality, and the fact that
\[
\text{supp} \ \zeta \subset B_{R/8}(x) \subset \Omega \setminus \overline{B_{R/2}(y)},
\]


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one can easily check
\[
\|f\|_{L^{2q_0/(2+q_0)}(B_{R/8}(x))} + \|f_\alpha\|_{L^{q_0}(B_{R/8}(x))} + \|g\|_{L^{q_0}(B_{R/8}(x))} \leq CR^{2/q_0-2}, \quad (5.15)
\]
where \( C = C(\lambda, \omega_A, K_0, q_0) = C(\lambda, \omega_A, K_0, \theta) \). Thus by Lemma 4.4 applied to (5.14), we have
\[
\|Dv\|_{L^{q_0}(B_{R/8}(x))} \leq CR^{2/q_0-2}.
\]
Therefore, from the Morrey inequality, Lemma 5.1 and the fact that \( v \in W^{1,q}_{0}(B_{R/8}(x))^2 \), we get
\[
\|\mathcal{G}_e(\cdot, y)\|_{L^\infty(B_{R/16}(x))} \leq \|v\|_{L^\infty(B_{R/8}(x))} + \|\mathcal{G}_e(\cdot, y)\|_{B_{R/16}(x)} \leq CR^{-1}.
\]

ii. \( B_{R/8}(x) \cap \partial \Omega \neq \emptyset \): We take a point \( x_0 \in \partial \Omega \) such that \( d_x = |x - x_0| < R/8 \). Observe that
\[
B_{R/8}(x) \subset B_{R/4}(x_0) \subset B_{R/2}(x_0) \subset \Omega \setminus \overline{B_{3R/8}(y)}.
\]
Let \( \tilde{\zeta} \) be a smooth function on \( \mathbb{R}^2 \) satisfying
\[
0 \leq \tilde{\zeta} \leq 1, \quad \tilde{\zeta} \equiv 1 \text{ on } B_{R/4}(x_0), \quad \text{supp} \tilde{\zeta} \subset B_{R/2}(x_0), \quad |\nabla \tilde{\zeta}| \leq CR^{-1}.
\]
Then the pair \( (\tilde{v}, \tilde{\pi}) \) given by
\[
(\tilde{v}, \tilde{\pi}) := \tilde{\zeta} \left( \mathcal{G}_e(\cdot, y), \mathcal{P}_e(\cdot, y) \right)
\]
satisfies
\[
\begin{cases}
\mathcal{L}\tilde{v} + \nabla \tilde{\pi} = \tilde{f} + D_\alpha \tilde{f}_\alpha \quad \text{in } \Omega, \\
\text{div } \tilde{v} = \tilde{g} \quad \text{in } \Omega, \\
\tilde{v} = 0 \quad \text{on } \partial \Omega,
\end{cases}
\]
where
\[
\tilde{f} = A^\alpha\beta D_\beta \mathcal{G}_e(\cdot, y) D_\alpha \tilde{\zeta} + \nabla \tilde{\zeta} \cdot \mathcal{P}_e(\cdot, y), \quad \tilde{f}_\alpha = A^\alpha\beta D_\beta \tilde{\zeta} \mathcal{G}_e(\cdot, y),
\]
\[
\tilde{g} = \nabla \tilde{\zeta} \cdot \mathcal{G}_e(\cdot, y) + \tilde{\zeta} \left( \Phi_{e,y} - (\Phi_{e,y})_\Omega \right).
\]
Similar to (5.15), by using (5.8), (5.10), H"older’s inequality, and the following boundary Poincaré inequality
\[
\|\mathcal{G}_e(\cdot, y)\|_{L^{q_0}(\Omega_{R/2}(x_0))} \leq C\|D\mathcal{G}_e(\cdot, y)\|_{L^{2q_0/(2+q_0)}(\Omega_{R/2}(x_0))},
\]
we have
\[
\|\tilde{f}\|_{L^{2q_0/(2+q_0)}(\Omega)} + \|\tilde{f}_\alpha\|_{L^{q_0}(\Omega)} + \|\tilde{g}\|_{L^{q_0}(\Omega)} \leq CR^{2/q_0-2}, \quad (5.18)
\]
where \( C = C(\lambda, \omega_A, K_0, \theta) \). Thus by Lemma 4.4 applied to (5.17), we have
\[
\|D\tilde{v}\|_{L^{q_0}(\Omega_{R/2}(x_0))} \leq CR^{2/q_0-2}.
\]
Therefore, we get from the Morrey inequality and \( \tilde{v}(x_0) = 0 \) that
\[
\|\mathcal{G}_e(\cdot, y)\|_{L^\infty(\Omega_{R/8}(x))} \leq \|\tilde{v}\|_{L^\infty(\Omega_{R/2}(x_0))} \leq CR^{-1}.
\]
The lemma is proved. \( \square \)

From Lemma 5.2, we obtain the following uniform weak type estimates.
Lemma 5.3. Let $y \in \Omega$ and $\varepsilon \in (0, 1]$. Then we have
\[ |\{x \in \Omega : |G_{\varepsilon}(x, y)| > c_0t\}| \leq Ct^{-d/(d-1)}, \quad t \geq (d_0^*)^{-1-d} \]
and
\[ |\{x \in \Omega : |D_{x}G_{\varepsilon}(x, y)| > t\} + |\{x \in \Omega : |P_{\varepsilon}(x, y)| > t\}| \leq C't^{-1}, \quad t \geq (d_0^*)^{-d}, \]
where $c_0 = 1$ if $d \geq 3$ and $c_0 = c_0(\lambda, \omega_A, K_0, \theta) > 0$ if $d = 2$. In the above, $(C, C') = (C, C')(d, \lambda, \omega_A, K_0)$ and $C$ depends also on $\theta$ when $d = 2$.

**Proof.** We only prove the first inequality because the other is the same with obvious modifications. Let us set
\[ \mathcal{A}_t = \{x \in \Omega : |G_{\varepsilon}(x, y)| > c_0t\}, \quad (5.19) \]
where $c_0$ is a constant to be chosen below. We consider the following two cases:

i. $d \geq 3$: Let $c_0 = 1$ and $t \geq (d_0^*)^{-1-d}$. Then by (5.7) with $R = t^{-1/(d-1)}$, we have
\[ |\mathcal{A}_t| = |\mathcal{A}_t \cap B_R(y)| + |\mathcal{A}_t \setminus B_R(y)| \leq CR^d + \frac{1}{\epsilon'} \int_{A \setminus B_R(y)} |G_{\varepsilon}(\cdot, y)|^2d/(d-1) \leq Ct^{-d/(d-1)}. \]

ii. $d = 2$: Let $t \geq (d_0^*)^{-1}$. Then by (5.7) with $R = t^{-1}$, there exists a constant $c_0 = c_0(\lambda, \omega_A, K_0, \theta) > 0$ such that
\[ \|G_{\varepsilon}(\cdot, y)\|_{L^\infty(\Omega \setminus B_R(y))} \leq c_0R^{-1}. \]
Therefore, we have
\[ |\mathcal{A}_t| = |\mathcal{A}_t \cap B_R(y)| \leq CR^2 = Ct^{-2}. \]
The lemma is proved.\hfill \square

The following lemma is a simple consequence of Lemma 5.3.

**Lemma 5.4.** Let $y \in \Omega$, $\varepsilon \in (0, 1]$, and $R \in (0, d_0^*)$. Then we have
\[ \|G_{\varepsilon}(\cdot, y)\|_{L^q(B_R(y))} \leq CR^{1-d+d/q}, \quad 0 < q < \frac{d}{d-1}, \]
and
\[ \|D\mathcal{G}_{\varepsilon}(\cdot, y)\|_{L^q(B_R(y))} + \|\mathcal{P}_{\varepsilon}(\cdot, y)\|_{L^q(B_R(y))} \leq C'R'^{-d+d/q}, \quad 0 < q < 1, \]
where $(C, C') = (C, C')(d, \lambda, \omega_A, K_0)$ and $C$ depends also on $\theta$ when $d = 2$.

**Proof.** We only prove the first inequality because the other is the same with obvious modifications. Let $q \in (0, d/(d-1))$ and recall the notation (1.19). Then by the first inequality with $t = R^{-d/(d-1)}$ in Lemma 5.3, we have
\[
\int_{B_R(y)} |G_{\varepsilon}(\cdot, y)|^q \, dx = q \int_0^\infty s^{q-1} \int_{B_R(y)} \{x \in B_R(y) : |G_{\varepsilon}(x, y)| > s\} \, ds
= c_0q \int_0^\infty (c_0s)^{q-1} \{x \in B_R(y) : |G_{\varepsilon}(x, y)| > c_0s\} \, ds
\leq C \int_0^t s^{q-1} |B_R(y)| \, ds + C \int_t^\infty s^{q-1} |\mathcal{A}_s| \, ds
\leq CR^{1-d+d/q},
\]
which gives the desired estimate. □

6. Proofs of main theorems

This section is devoted to the proofs of the theorems in Section 3. Throughout this section, we denote by \((G_\varepsilon(\cdot, y), P_\varepsilon(\cdot, y))\) the approximated Green function constructed in Section 5.

6.1. Proof of Theorem 3.3 By Lemmas 5.2 and 5.4, the weak compactness theorem, and a diagonalization process, we easily see that there exist a sequence \(\{\varepsilon_\rho\}_{\rho=1}^\infty\) tending to zero and a pair \((G(\cdot, y), P(\cdot, y))\) such that for any \(R \in (0, d_y^*)\),

\[
(1 - \eta_R)G_{\varepsilon_\rho}(\cdot, y) \rightharpoonup (1 - \eta_R)G(\cdot, y) \quad \text{weakly in } Y_0^{1,2}(\Omega)^d, \\
P_{\varepsilon_\rho}(\cdot, y) \rightharpoonup P(\cdot, y) \quad \text{weakly in } L^2(\Omega \setminus B_R(y)),
\]

where \(\eta_R\) is any smooth function in \(\mathbb{R}^d\) satisfying \(\eta_R \equiv 1\) on \(B_R(y)\), and that for fixed \(q \in \left(1, \frac{d}{d-1}\right)\),

\[
G_{\varepsilon_\rho}(\cdot, y) \rightharpoonup G(\cdot, y) \quad \text{weakly in } L^q(B_{d^*/2}(y))^d. \tag{6.2}
\]

Moreover, we obtain the following uniform convergence.

Lemma 6.1. For given compact set \(K \subset \Omega \setminus \{y\}\), there is a subsequence of \(\{G_{\varepsilon_\rho}(\cdot, y)\}\) that converges to \(G(\cdot, y)\) uniformly on \(K\).

Proof. Let \(B_R(x) \subset \Omega \setminus \{y\}\) and \(0 < \varepsilon < \min\{d_y^*, \text{dist}(y, B_R(x))\}\). Note that

\[
\begin{cases}
\mathcal{L}G_{\varepsilon}(\cdot, y) + \nabla P_{\varepsilon}(\cdot, y) = 0 & \text{in } B_R(x), \\
\text{div } G_{\varepsilon}(\cdot, y) = -|\Omega|^{-1} & \text{in } B_R(x).
\end{cases}
\]

By Lemmas 5.2 and 5.4, we see that \(|G_{\varepsilon}(\cdot, y)||_{W^{1,\infty}(B_{R/2}(x))} \leq C\), where \(C\) does not depend on \(\varepsilon\). This implies that \(\{G_{\varepsilon}(\cdot, y)\}\) is equicontinuous and uniformly bounded on \(B_{R/2}(x)\). Therefore, we get the desired conclusion from the Arzelà-Ascoli theorem. □

The pair \((G, P)\) satisfies the estimates in Remark 5.4. Indeed, the estimates in the assertion (a) are simple consequences of Lemma 5.2 and the weak lower semicontinuity. Then by following the same steps used in Lemmas 5.3 and 5.4, we see that \((G, P)\) satisfies the estimates in the assertions (b) and (c).

Now we shall show that \((G, P)\) satisfies the properties (a)–(c) in Definition 2.3 so that it is a Green function for the pressure of \(\mathcal{L}\) in \(\Omega\).

Obviously, \((G, P)\) satisfies the property (a). To verify the property (b), we apply \(\phi \in C_0^\infty(\Omega \setminus B_r(y))^d\) as a test function to (6.1) and use (6.4) to get

\[
\int_{\Omega \setminus B_r(y)} A^{\alpha\beta} D_{\beta} G(\cdot, y) \cdot D_{\alpha} \phi \, dx + \int_{\Omega \setminus B_r(y)} P(\cdot, y) \text{div } \phi \, dx
\]

\[
= \lim_{\rho \to \infty} \left( \int_{\Omega \setminus B_r(y)} A^{\alpha\beta} D_{\beta} G_{\varepsilon_\rho}(\cdot, y) \cdot D_{\alpha} \phi \, dx + \int_{\Omega \setminus B_r(y)} P_{\varepsilon_\rho}(\cdot, y) \text{div } \phi \, dx \right) = 0.
\]

This implies that

\[
\mathcal{L}G(\cdot, y) + \nabla P(\cdot, y) = 0 \quad \text{in } \Omega \setminus B_r(y).
\]
Similarly, by applying \( \varphi \in C_0^\infty(\Omega) \) as a test function to the divergence equation in (5.1), and using (6.1) and (6.2), we have

\[
\int_\Omega \mathcal{G}(\cdot, y) \cdot \nabla \varphi \, dx = \int_{B_{d^*_y}(y)} \mathcal{G}(\cdot, y) \cdot \nabla(\eta \varphi) \, dx + \int_{\Omega \setminus B_{d^*_y}(y)} \mathcal{G}(\cdot, y) \cdot \nabla((1 - \eta) \varphi) \, dx \\
= -\lim_{\rho \to \infty} \left( \int_\Omega \text{div} \, \mathcal{G}_{\rho}(\cdot, y) \varphi \, dx + \int_\Omega \text{div} \, \mathcal{G}_{\rho}(\cdot, y) (1 - \eta) \varphi \, dx \right) \\
= -\lim_{\rho \to \infty} \left( \int_\Omega \Phi_{\rho, \eta} \varphi \, dx - \left( \Phi_{\rho, \eta} \right)_\Omega \int_\Omega \varphi \, dx \right) \\
= -\varphi(y) + \int_\Omega \varphi \, dx,
\]

where \( \eta \) is a smooth function on \( \mathbb{R}^d \) satisfying \( \eta \equiv 1 \) on \( B_{d^*_y}(y) \) and \( \text{supp} \, \eta \subset \Omega \). This implies that

\[
\text{div} \, \mathcal{G}(\cdot, y) = \delta_y - \frac{1}{|\Omega|} \quad \text{in} \quad \Omega,
\]

and thus \( (\mathcal{G}, \mathcal{P}) \) satisfies the property (b). Therefore, by applying Lemma \[A.1\] to (2.3), \( \mathcal{G}(\cdot, y) \) is continuous in \( \Omega \setminus \{ y \} \) and

\[
(\mathcal{G}(\cdot, y), \mathcal{P}(\cdot, y)) \in W^{1, \infty}_\text{loc}(\Omega \setminus \{ y \})^d \times L^\infty_\text{loc}(\Omega \setminus \{ y \}).
\]

To verify the property (c) in Definition 2.3, suppose that \( (u, p) \in Y_0^{1, 2}(\Omega)^d \times \tilde{L}^2(\Omega) \) is a unique weak solution of (2.4). By testing (2.4) and (5.1) with \( \mathcal{G}(\cdot, y) \) and \( u \), respectively, we have

\[
\int_\Omega p \Phi_{\rho, \eta} \, dx = -\int_\Omega \mathcal{G}_{\rho}(x, y) \cdot f(x) \, dx + \int_\Omega \mathcal{P}_{\rho}(x, y) g(x) \, dx. \tag{6.3}
\]

Notice from (6.1) and (6.2) that for any \( y \in \Omega \setminus \text{supp} \, \mathcal{G} \), the right-hand side of (6.3) converges to

\[
-\int_\Omega \mathcal{G}(x, y) \cdot f(x) \, dx + \int_\Omega \mathcal{P}(x, y) g(x) \, dx \quad \text{as} \quad \rho \to \infty.
\]

On the other hand, the left-hand side of (6.3) converges to \( p(y) \) for any \( y \) in the Lebesgue set of \( p \). This implies that \( (\mathcal{G}, \mathcal{P}) \) satisfies the property (c) in Definition 2.3.

To complete the proof of the theorem, it remains to show the estimates (5.2) and (5.3). Let \( x, y \in \Omega \) with \( 0 < |x - y| \leq d^*_y/2 \), and set \( R = |x - y|/2 \). Since \( (\mathcal{G}(\cdot, y), \mathcal{P}(\cdot, y)) \) satisfies

\[
\begin{cases}
\mathcal{L} \mathcal{G}(\cdot, y) + \nabla \mathcal{P}(\cdot, y) = 0 & \text{in} \quad \Omega \setminus B(x), \\
\text{div} \, \mathcal{G}(\cdot, y) = -\frac{1}{|\Omega|} & \text{in} \quad \Omega \setminus B(x),
\end{cases}
\]

we obtain by \[A.2\] and \( \mathcal{L} \) that

\[
\|D \mathcal{G}(\cdot, y)\|_{L^\infty(B_{d^*_y}(x))} + \|\mathcal{P}(\cdot, y)\|_{L^\infty(B_{d^*_y}(x))} \\
\leq CR^{-d} (\|D \mathcal{G}(\cdot, y)\|_{L^1(B(x))} + \|\mathcal{P}(\cdot, y)\|_{L^1(B(x))}) + CR^{-d},
\]

where \( C \) is a constant depending only on \( \Omega \).
where $C = C(d, \lambda, \omega_A)$. Thus from Hölder’s inequality, $B_R(x) \subset \Omega \setminus B_R(y)$, and Remark 3.3 (a), we get

\[
\begin{align*}
\|DG(\cdot, y)\|_{L^\infty(B_{R/2}(x))} + \|P(\cdot, y)\|_{L^\infty(B_{R/2}(x))} & \leq CR^{-d/2}\left(|DG(\cdot, y)|_{L^2(\Omega \setminus B_R(y))} + \|P(\cdot, y)\|_{L^2(\Omega \setminus B_R(y))}\right) + CR^{-d} \\
& \leq CR^{-d},
\end{align*}
\]

where $C = C(d, \lambda, \omega_A, K_0)$. This implies \ref{3.2}. Similarly, by \ref{A.3} and Remark 3.3 (a), we have that

\[
\begin{align*}
\|G(\cdot, y)\|_{L^\infty(B_{R/2}(x))} & \leq CR^{-d}\|G(\cdot, y)\|_{L^1(B_{R/2}(x))} \\
& + CR^{1-d}\left(|DG(\cdot, y)|_{L^1(\partial B_R(x))} + \|P(\cdot, y)\|_{L^1(\partial B_R(x))}\right) + CR^{1-d} \\
& \leq CR^{1-d/2}\|G(\cdot, y)\|_{L^2(\Omega \setminus B_R(y))} \\
& + CR^{1-d/2}\left(|DG(\cdot, y)|_{L^2(\Omega \setminus B_R(y))} + \|P(\cdot, y)\|_{L^2(\Omega \setminus B_R(y))}\right) + CR^{1-d} \\
& \leq CR^{1-d},
\end{align*}
\]

where $C = C(d, \lambda, \omega_A, K_0)$ and $C$ depends also on $\theta$ when $d = 2$. This together with the continuity of $G(\cdot, y)$ in $B_{R/2}(x)$ gives \ref{3.2}. The theorem is proved. \hfill \Box

### 6.2. Proof of Theorem 3.7

Let $(G, P)$ be the Green function for the pressure of $\mathcal{L}$ constructed in Theorem \ref{5.3}. Obviously, by Lemma \ref{A.2} and Definition 2.3 (b), we see that

\[
(G(\cdot, y), P(\cdot, y)) \in C^1(\Omega \setminus B_R(y)) \times C(\Omega \setminus B_R(y))
\]

for all $y \in \Omega$ and $R > 0$.

We divide the proof into several steps.

**Step 1.** In this step, we establish various boundary estimates for the approximated Green function $(G_\varepsilon, P_\varepsilon)$. The following lemma is about the $L^1$-estimate for $G_\varepsilon$, which is the counterpart of Lemma \ref{5.1}.

**Lemma 6.2.** Let $y \in \Omega$ and $\varepsilon \in (0, 1]$. Then for any $x \in \Omega$ and $R \in (0, 1]$, we have

\[
\|G_\varepsilon(\cdot, y)\|_{L^1(\Omega \setminus B_R(x))} \leq CR,
\]

where $C = C(d, \lambda, \omega_A, K_0, R_0, \theta_0)$ and $C$ depends also on $\theta$ when $d = 2$.

**Proof.** Due to the interior estimate in Lemma \ref{5.1} it suffices to consider the case when $d_\varepsilon < R \leq 1$. Moreover, because the proof of Lemma \ref{5.1} still works for $2\varepsilon > R$, we only need to prove the lemma with $2\varepsilon \leq R$.

Now, we assume that $d_\varepsilon < R \leq 1$ and $2\varepsilon \leq R$. Let $(u, p) \in W^{1,2}_0(\Omega)^d \times \tilde{L}^2(\Omega)$ be a unique weak solution of the problem

\[
\begin{cases}
\mathcal{L}^*u + \nabla p = -f & \text{in } \Omega, \\
\text{div } u = 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where $f = \chi_{\Omega_R(x)}(\text{sgn} G_\varepsilon^1(\cdot, y), \ldots, \text{sgn} G_\varepsilon^d(\cdot, y))^\top$. Then by using \ref{A.13} and following the same argument used in deriving \ref{5.4}, we have

\[
\|G_\varepsilon(\cdot, y)\|_{L^1(\Omega \setminus B_R(x))} \leq CR^{-d-1}\|u\|_{L^1(\Omega_R(y))} + CR^{-d}\left(Du\|_{L^1(\Omega_R(y))} + \|p\|_{L^1(\Omega_R(y))}\right) + CR,
\]

(6.4)
where $C = C(d, \lambda, \omega_A, R_0, g_0)$. If $d \geq 3$, then by Lemma 4.4 and the Sobolev inequality, we have
\[ \|u\|_{L^{2d/(d-2)}(\Omega)} + \|Du\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)} \leq CR^{(d+2)/2}. \]
From this together with (6.4) and Hölder’s inequality, we get the desired estimate.

On the other hand, if $d = 2$, then by using the Morrey inequality and the fact that $G_\varepsilon(\cdot, y) = 0$ on $\partial \Omega \cap B_R(x)$, we have
\[ \|u\|_{L^\infty(\Omega_R(x))} \leq CR_\varepsilon^{1-2/q_0}\|Du\|_{L^{q_0}(\Omega_R(x))}, \]
where $q_0 = q_0(\lambda, K_0, \theta) > 2$ is the constant from Lemma 4.2. From this together with Lemma 6.2 and (6.6), and following the same steps as in the proof of Lemma 5.2, it suffices to consider the case of $d = 2$.

The following lemma is an analog of Lemma 5.2.

**Lemma 6.3.** Let $y \in \Omega$, $\varepsilon \in (0, 1]$, $R \in (0, 1]$, and
\[ q^* = \frac{2d}{d-2} \quad \text{if} \quad d \geq 3, \quad q^* = \infty \quad \text{if} \quad d = 2. \]
Then we have
\[ \|G_\varepsilon(\cdot, y)\|_{L^q(\Omega, \overline{B_R(y)})} \leq CR^{-d/2} \quad \text{(6.5)} \]
and
\[ \|DG_\varepsilon(\cdot, y)\|_{L^2(\Omega, \overline{B_R(y)})} + \|P_\varepsilon(\cdot, y)\|_{L^2(\Omega, \overline{B_R(y)})} \leq C'R^{-d/2}, \quad \text{(6.6)} \]
where $(C, C') = (C, C')(d, \lambda, \omega_A, K_0, R_0, g_0)$ and $C$ depends also on $\theta$ when $d = 2$.

**Proof.** By utilizing Lemma 5.2 and (6.6), and following the same steps as in the proof of (5.7), one can show that (6.5) holds. Thus we only prove (6.6). Due to (5.8) in Lemma 5.2, it suffices to consider the case of $d_y < R \leq 1$.

Let $y \in \Omega$, $\varepsilon \in (0, 1]$, and $d_y < R \leq 1$. If $\varepsilon > R/2$, then (6.6) follows immediately from (5.2). Now we assume $\varepsilon \leq R/2$. Set
\[ f_\alpha = \chi_{\Omega_y \setminus B_R(y)} D_\alpha G_\varepsilon(\cdot, y), \quad g = \chi_{\Omega_y \setminus B_R(y)} P_\varepsilon(\cdot, y). \]
Then by Lemma 4.1 there exists a unique solution $(u, p) \in Y_0^{1,2}(\Omega)^d \times \tilde{L}^2(\Omega)$ of the problem
\[
\begin{cases}
L^*u + \nabla p = D_\alpha f_\alpha & \text{in} \ \Omega, \\
\text{div} \ u = g - (g)_{\Omega_y} & \text{in} \ \Omega, \\
u = 0 & \text{on} \ \partial \Omega,
\end{cases}
\]
satisfying
\[ \|Du\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)} \leq C\left(\|DG_\varepsilon(\cdot, y)\|_{L^2(\Omega, \overline{B_R(y)})} + \|P_\varepsilon(\cdot, y)\|_{L^2(\Omega, \overline{B_R(y)})}\right), \quad \text{(6.7)} \]
where $C = C(d, \lambda, K_0)$. Moreover, it follows from (5.11) that (see (5.11))
\[ \|p\|_{L^\infty(\Omega_R(y))} \leq CR^{d/2-1}\|u\|_{L^2(\Omega_R(y))} \]
\[ + C R^{-d/2}\left(\|Du\|_{L^2(\Omega_R(y))} + \|p\|_{L^2(\Omega_R(y))}\right) + C|\Omega|^{-1/2}\|g\|_{L^2(\Omega)}, \]
where $C = C(d, \lambda, \omega_A, R_0, g_0)$. Hence by using the fact that
\[ R^d \leq C(d, R_0, g_0)|\Omega|, \quad \text{(6.8)} \]
we have
\[
\|p\|_{L^\infty(\Omega_R^2(y))} \leq CR^{-d/2 - 1}\|u\|_{L^2(\Omega_R(y))} + CR^{-d/2}\left(\|Du\|_{L^2(\Omega_R(y))} + \|p\|_{L^2(\Omega_R(y))} + \|g\|_{L^2(\Omega)}\right).
\] 
(6.9)

Observe that
\[
\|u\|_{L^2(\Omega_R^2(y))} \leq CR\|Du\|_{L^2(\Omega_{2R}(y))},
\] 
(6.10)

where \(C = C(d, R, \rho_0)\). Indeed, if we take a point \(y_0 \in \partial \Omega\) satisfying \(d_y = |y - y_0| < R\), then by (5.8) and \((B_R(y_0) \setminus \Omega) \subset (B_2R(y) \setminus \Omega)\), we have
\[
|B_2R(y) \setminus \Omega| \geq |B_R(y_0) \setminus \Omega| \geq \theta R^d,
\]
where \(\theta = \theta(d, R, \rho_0)\). This together with the boundary Poincaré inequality (see, for instance, [10, Eq. (7.45)]) gives (6.10). Combining (6.7), (6.9), and (6.10), we have
\[
\theta \leq CR^{-d/2}\left(\|DG_\ell(\cdot, y)\|_{L^2(\Omega, B_R(y))} + \|P_\ell(\cdot, y)\|_{L^2(\Omega, B_R(y))}\right),
\]
where \(\theta = C(d, \lambda, \omega_A, K_0, R_0, \rho_0)\). Therefore, by using the identity (see (5.13))
\[
\|DG_\ell(\cdot, y)\|_{L^2(\Omega, B_R(y))} + \|P_\ell(\cdot, y)\|_{L^2(\Omega, B_R(y))} = \int_{\Omega_\ell(y)} p\Phi_{\ell, y} \, dx,
\]
we conclude (6.6). The lemma is proved. \(\square\)

**Step 2.** In this step, we prove the estimates in the theorem and Remarks 3.9 and 3.10. The estimates in the assertion (a) in Remark 3.10 are simple consequences of Lemma 6.3 and the weak semi-continuity. Then by following the same steps used in Lemmas 5.3 and 5.4, one can easily obtain the estimates in the assertions (b) and (c) in Remark 3.10.

To prove (3.9) and (3.10), let \(x, y \in \Omega\) with \(0 < |x - y| \leq 1\), and set \(R = |x - y|/2\). Since \((G(\cdot, y), P(\cdot, y))\) satisfies
\[
\begin{align*}
\mathcal{L}G(\cdot, y) + \nabla P(\cdot, y) &= 0 \quad \text{in} \quad \Omega_R(x), \\
\text{div} \, G(\cdot, y) &= -\frac{1}{|\Omega|} \quad \text{in} \quad \Omega_R(x), \\
G(\cdot, y) &= 0 \quad \text{on} \quad \partial \Omega \cap B_R(x),
\end{align*}
\]
by (5.5), (A.13), Hölder’s inequality, and the fact that \(B_R(x) \subset \Omega \setminus B_R(y)\), we have
\[
R^{-1}\|G(\cdot, y)\|_{L^\infty(\Omega_R(x))} + \|DG(\cdot, y)\|_{L^\infty(\Omega_R^2(x))} + \|P(\cdot, y)\|_{L^\infty(\Omega_R^2(x))}
\]
\[
\leq CR^{-d/2}\|G(\cdot, y)\|_{L^2(\Omega, B_R(y))} + \|DG(\cdot, y)\|_{L^2(\Omega, B_R(y))} + \|P(\cdot, y)\|_{L^2(\Omega, B_R(x))} + CR^{-d}
\]
\[
= CR^{-d/2}\left(\|DG(\cdot, y)\|_{L^2(\Omega, B_R(y))} + \|P(\cdot, y)\|_{L^2(\Omega, B_R(x))}\right) + CR^{-d},
\]
where \(q^* = 2d/(d - 2)\) if \(d \geq 3\) and \(q^* = \infty\) if \(d = 2\). Here, the constant \(C\) depends only on \(d, \lambda, \omega_A, K_0, R_0, \rho_0\), and \(\theta\). From this inequality and Remark 3.10 (a), we get
\[
R^{-1}\|G(\cdot, y)\|_{L^\infty(\Omega_R^2(x))} + \|DG(\cdot, y)\|_{L^\infty(\Omega_R^2(x))}
\]
\[
+ \|P(\cdot, y)\|_{L^\infty(\Omega_R^2(x))} \leq CR^{-d},
\] 
(6.11)

where \(C = C(d, \lambda, \omega_A, K_0, R_0, \rho_0)\) and \(C\) depends also on \(|\Omega|\) when \(d = 2\). Therefore, by the continuity of \(G(\cdot, y), DG(\cdot, y),\) and \(P(\cdot, y)\), we conclude (3.5) and (3.6).

We now turn to the proof of the estimate in Remark 3.9. Let \(x, y \in \Omega\) with \(0 < |x - y| \leq 1\), and set \(R = |x - y|/2\). We assume \(d_x < R/2\) and extend \(G(\cdot, y)\) by
zero on \( \mathbb{R}^d \setminus \Omega \). Then by taking \( x_0 \in \partial \Omega \) such that \( d_x = |x - x_0| \), and using (6.11) and \( G(x_0, y) = 0 \), we have

\[
|G(x, y)| = |G(x, y) - G(x_0, y)| \leq C d_x \|DG(\cdot, y)\|_{L^\infty(B_{R/2}(x))} \leq C d_x R^{-d}.
\]

From this together with (3.5), we get

\[
|G(x, y)| \leq C \min\{d_x, |x - y|/4\} \cdot |x - y|^{-d},
\]

which yields the estimate in Remark 3.9.

**Step 3.** In this step, we prove that (3.7) holds. Let \((G^*, P^*)\) be the Green function for the pressure of the adjoint operator \(L^*\) and \((G^*_{\sigma}, P^*_{\sigma})\) be its approximated Green function. More precisely, for given \(x \in \Omega\) and \((\sigma, x)\) on \(\partial \Omega\) converging to \( (0, 1) \), \((G^*_{\sigma}(\cdot, x), P^*_{\sigma}(\cdot, x))\) is a unique weak solution of the problem

\[
\begin{cases}
L^* G^*_{\sigma}(\cdot, x) + \nabla P^*_{\sigma}(\cdot, x) = 0 & \text{in } \Omega, \\
\text{div} G^*_{\sigma}(\cdot, x) = \Phi_{\sigma,x} - (\Phi_{\sigma,x})_{\Omega} & \text{in } \Omega,
\end{cases}
(6.12)
\]

where \(\Phi_{\sigma,x}\) is as in Section 5. By proceeding similarly as in the proof of Theorem 6.3 there exists a sequence \(\{\sigma_\tau\}_{\tau=1}^\infty\) satisfying the natural counterparts of (6.1) and (6.2).

We first claim that

\[
\lim_{\rho \to \infty} P_{\varepsilon_\rho}(x, y) = P^*(y, x) \quad \text{for all } x, y \in \Omega \text{ with } x \neq y.
(6.13)
\]

We apply \(G^*_{\sigma}(\cdot, x)\) and \(G_{\varepsilon_\rho}(\cdot, y)\) as test functions to (6.1) and (6.12), respectively, to get

\[
\int_\Omega P_{\varepsilon_\rho}(z, y) \Phi_{\varepsilon_\rho,z}(z) \, dz = \int_\Omega P^*(z, y) \Phi_{\varepsilon_\rho,y}(z) \, dz \quad \text{for any } x, y \in \Omega.
\]

By the continuity of \(P_{\varepsilon_\rho}(\cdot, y)\), the left-hand side of the above inequality converges to \(P_{\varepsilon_\rho}(x, y)\) as \(\tau \to \infty\). On the other hand, by the counterpart of (6.1), the right-hand side converges to

\[
\int_\Omega P^*(z, x) \Phi_{\varepsilon_\rho,y}(z) \, dz \quad \text{as } \tau \to \infty
\]

if \(x \neq y\) and \(0 < \varepsilon_\rho \leq \min\{1, |x - y|/2\}\). Combining these together, we have

\[
P_{\varepsilon_\rho}(x, y) = \int_\Omega P^*(z, x) \Phi_{\varepsilon_\rho,y}(z) \, dz \quad \text{if } 0 < \varepsilon_\rho \leq \min\{1, |x - y|/2\},
\]

and thus, from the continuity of \(P^*(\cdot, x)\) on \(\Omega \setminus \{x\}\), we get the claim (6.13).

Next, we claim that

\[
P^*(y, \cdot) \in L^1_{\text{loc}}(\overline{\Omega} \setminus \{y\}) \quad \text{for any } y \in \Omega.
(6.14)
\]

Let \(x, y \in \Omega\) with \(x \neq y\), \(0 < R \leq \min\{1, |x - y|/2\}\), and \(\varepsilon_\rho \in (0, R]\). Since it holds that

\[
\begin{cases}
L G_{\varepsilon_\rho}(\cdot, y) + \nabla P_{\varepsilon_\rho}(\cdot, y) = 0 & \text{in } \Omega_R(x), \\
\text{div} G_{\varepsilon_\rho}(\cdot, y) = -1 & \text{in } \Omega_R(x),
\end{cases}
(6.11)
\]

\[
G_{\varepsilon_\rho}(\cdot, y) = 0 \quad \text{on } \partial \Omega \cap B_R(x),
\]

and
by using (A.13) and Lemma 6.3, we have
\[
\|P_{x_0}(\cdot, y)\|_{L^\infty(\Omega_{R/2}(x))} \\
\leq C\left(\|G_{x_0}(\cdot, y)\|_{L^1(\Omega_R(x))} + \|P_{x_0}(\cdot, y)\|_{L^1(\Omega_R(x))} + 1\right) \\
\leq C\left(\|G_{x_0}(\cdot, y)\|_{L^1(\Omega_R(y))} + \|P_{x_0}(\cdot, y)\|_{L^1(\Omega_R(y))} + 1\right) \\
\leq C.
\]

From this inequality together with (6.13) and the dominated convergence theorem, we see that \(P^*(y, \cdot) \in L^1(\Omega_{R/2}(x))\) and
\[
\lim_{\rho \to \infty} \int_{\Omega_{R/2}(x)} P_{x_0}(z, y) \, dz = \int_{\Omega_{R/2}(x)} P^*(y, z) \, dz < \infty. \tag{6.15}
\]
Since \(P^*(y, \cdot) \in L^1(\Omega_{R/2}(x))\) for all \(x \in \Omega \setminus \{y\}\) and \(0 < R \leq \min\{1, |x - y|/2\}\), we get (6.14).

We are ready to complete the proof of (3.7). Fix \(y \in \Omega\), and let \(x \in \Omega \setminus \{y\}\) be a Lebesgue point of \(P^*(y, \cdot)\). For \(0 < R \leq \min\{1, |x - y|/2\}\), we see that (using (6.11))
\[
\lim_{\rho \to \infty} \int_{\Omega_{R/2}(x)} P_{x_0}(z, y) \, dz = \int_{\Omega_{R/2}(x)} P(z, y) \, dz.
\]
Combining this identity and (6.15), we get
\[
\int_{\Omega_{R/2}(x)} P(z, y) \, dz = \int_{\Omega_{R/2}(x)} P^*(y, z) \, dz.
\]
Therefore, by taking \(R \to 0\), and using the continuity of \(P(\cdot, y)\) on \(\Omega \setminus \{y\}\), we concluded that
\[
P(x, y) = P^*(y, x).
\]
This implies (3.7). The theorem is proved. \(\square\)

6.3. Proof of Theorem 3.12 We only consider the case when \(\Omega = \mathbb{R}^d_+\) with \(d \geq 2\). Let \((G, P)\) be the Green function for the pressure of \(L\) in \(\Omega\) constructed in Theorem 3.3. Then by the same reasoning as in the proof of the estimates in Remark 3.10 (using Lemma A.4 instead of Lemma A.2), we have the following estimates for any \(y \in \Omega\):

(a) For any \(R \in (0, 1]\), we have that
\[
\|G(\cdot, y)\|_{L^q(\Omega_{B_R(y)})} \leq CR^{-d/2}, \\
\|DG(\cdot, y)\|_{L^q(\Omega_{B_R(y)})} + \|P(\cdot, y)\|_{L^q(\Omega_{B_R(y)})} \leq CR^{-d/2},
\]
where \(q^* = 2d/(d - 2)\) if \(d \geq 3\) and \(q^* = \infty\) if \(d = 2\).

(b) For any \(t > 0\), we have that
\[
|\{x \in \Omega : G(x, y) > t\}| \leq Ct^{-d/(d - 1)}, \\
|\{x \in \Omega : |D_x G(x, y)| > t\}| + |\{x \in \Omega : |P(x, y)| > t\}| \leq Ct^{-1}.
\]

(c) For any \(R \in (0, 1]\), we have that
\[
\|G(\cdot, y)\|_{L^q(\Omega_R(y))} \leq C_q R^{1-d/q}, \quad 0 < q < \frac{d}{d - 1}, \\
\|DG(\cdot, y)\|_{L^q(\Omega_R(y))} + \|P(\cdot, y)\|_{L^q(\Omega_R(y))} \leq C_q R^{-d/q}, \quad 0 < q < 1.
\]
In the above, $C = C(d, \lambda, \omega_A)$ and $C_q$ depends also on $q$. Observe from Definition 2.3 (b) that for any $x \in \Omega$ and $R \in (0, 1]$ satisfying $|x - y| \geq R$, $(\mathcal{G}(\cdot, y), \mathcal{P}(\cdot, y))$ satisfies
\[
\begin{cases}
  \mathcal{L}\mathcal{G}(\cdot, y) + \nabla\mathcal{P}(\cdot, y) = 0 & \text{in } \Omega_R(x), \\
  \text{div } \mathcal{G}(\cdot, y) = 0 & \text{in } \Omega_R(x), \\
  \mathcal{G}(\cdot, y) = 0 & \text{on } \partial\Omega \cap B_R(x).
\end{cases}
\]

By Lemma A.4 Hölder’s inequality, and the estimates in (a), we obtain that
\[
\begin{aligned}
  R^{-1}\|\mathcal{G}(\cdot, y)\|_{L^\infty(\Omega_{R/4}(x))} + \|D\mathcal{G}(\cdot, y)\|_{L^\infty(\Omega_{R/4}(x))} + \|\mathcal{P}(\cdot, y)\|_{L^\infty(\Omega_{R/4}(x))} \\
  \leq CR^{-d/2}\|\mathcal{G}(\cdot, y)\|_{L^2(\Omega_{R/4}(x))} + CR^{-d/2}\|D\mathcal{G}(\cdot, y)\|_{L^2(\Omega_{R/4}(x))} + C\|\mathcal{P}(\cdot, y)\|_{L^2(\Omega_{R/4}(x))}
\end{aligned}
\]

where $C = C(d, \lambda, \omega_A)$. Since the above inequality holds for any $x \in \Omega$ and $R \in (0, 1]$ satisfying $|x - y| \geq R$, we see that
\[
(\mathcal{G}(\cdot, y), \mathcal{P}(\cdot, y)) \in W^{1,\infty}(\Omega \setminus B_r(y))^d \times L^\infty(\Omega \setminus B_r(y)), \quad \forall r > 0,
\]
which gives 3.9.

To verify (3.10) and (3.11), let $x, y \in \Omega$ with $0 < |x - y| = R \leq 1$. Then we get (3.11) from (6.16) immediately. We also have that
\[
|\mathcal{G}(x, y)| \leq CR^{1-d} = C|x - y|^{1-d}.
\]

If $d_x < R/4$, then we take $x_0 \in \partial\Omega$ such that $\text{dist}(x, x_0) = d_x$. Since $\mathcal{G}(x_0, y) = 0$, we obtain by (6.16) that
\[
|\mathcal{G}(x, y)| = |\mathcal{G}(x, y) - \mathcal{G}(x_0, y)| \leq C d_x \|D\mathcal{G}(\cdot, y)\|_{L^\infty(\Omega_{R/4}(x))} \leq C d_x |x - y|^{-d}.
\]

This together with (6.17) yields that
\[
|\mathcal{G}(x, y)| \leq C \min\{d_x, |x - y|/4\} \cdot |x - y|^{-d},
\]
which gives 3.10. The theorem is proved. \hfill \Box

7. Green function for the flow velocity

In this section, we deal with Green function and fundamental solution for the flow velocity of Stokes system. In the definitions below, $G = G(x, y)$ is a $d \times d$ matrix-valued function and $\Pi = \Pi(x, y)$ is a $d \times 1$ vector-valued function on $\Omega \times \Omega$.

Definition 7.1 (Green function for the flow velocity). Let $\Omega$ be a domain in $\mathbb{R}^d$. We say that $(G, \Pi)$ is a Green function for the flow velocity of $\mathcal{L}$ in $\Omega$ if it satisfies the following properties.

(a) For any $y \in \Omega$ and $R > 0$,
\[
G(\cdot, y) \in W^{1,1}_{\text{loc}}(\Omega)^{d \times d}, \quad (1 - \eta)G(\cdot, y) \in W^{1,2}_0(\Omega)^{d \times d},
\]
where $\eta$ is a smooth function satisfying $\eta \equiv 1$ on $B_R(y)$. Moreover,
\[
\Pi(\cdot, y) \in L^1_{\text{loc}}(\Omega)^d \cap L^2(\Omega \setminus B_R(y))^d.
\]
(b) For any \( y \in \Omega \), \((G(\cdot, y), \Pi(\cdot, y))\) satisfies
\[
\begin{aligned}
\mathcal{L}G(\cdot, y) + \nabla \Pi(\cdot, y) &= \delta_y I \quad \text{in } \Omega, \\
\text{div } G(\cdot, y) &= 0 \quad \text{in } \Omega, \\
G(\cdot, y) &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

(c) If \((u, p) \in Y_0^{1, 2}(\Omega)^d \times \tilde{L}^2(\Omega)\) is a weak solution of the problem
\[
\begin{aligned}
\mathcal{L}^* u + \nabla p &= f + D_\alpha f_\alpha \quad \text{in } \Omega, \\
\text{div } u &= g - (g)_{\Omega} \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
where \(f, f_\alpha \in C_0^\infty(\Omega)^d\), and \(g \in C_0^\infty(\Omega)\), then for a.e. \( y \in \Omega \), we have
\[
u(y) = \int_{\Omega} G(x, y)^\top f(x) \, dx - \int_{\Omega} D_\alpha G(x, y)^\top f_\alpha(x) \, dx - \int_{\Omega} \Pi(x, y) g(x) \, dx. \quad (7.1)
\]
The Green function for the adjoint operator \(\mathcal{L}^*\) is defined similarly. The Green function in \(\Omega = \mathbb{R}^d\) is called the fundamental solution.

Remark 7.2. In the definitions above, \(\mathcal{L}G(\cdot, y) + \nabla \Pi(\cdot, y) = \delta_y I\) is understood as
\[
\int_{\Omega} A_{i\beta}^\alpha D_\beta G^{ik}(x, y) D_\alpha \phi^i \, dx + \int_{\Omega} \Pi^{k}(x, y) \text{div } \phi \, dx = -\phi^k(y)
\]
for any \(\phi \in C_0^\infty(\Omega)^d\) and \(k \in \{1, \ldots, d\}\).

7.1. Main results. In this subsection, we state the main results concerning Green function \((G, \Pi)\) for the flow velocity. Note that in \([8]\), the authors proved the global pointwise bound
\[
|G(x, y)| \leq C|x - y|^{2-d}, \quad x \neq y,
\]
when the coefficients are VMO in a bounded \(C^1\) domain. See also \([9]\) for the corresponding results in unbounded domains. In the theorem below, we extend the results in \([8]\) and \([9]\) by showing the pointwise bounds \((7.4)\) and \((7.5)\) under the stronger assumption that the coefficients are of Dini mean oscillation in a domain having a \(C^{1, \text{Dini}}\) boundary.

Theorem 7.3. Let \(d \geq 3\) and \(\Omega\) be a domain in \(\mathbb{R}^d\) having a \(C^{1, \text{Dini}}\) boundary as in Definition \([22]\). Suppose that the coefficients \(A^{\alpha\beta}\) of \(\mathcal{L}\) are of Dini mean oscillation in \(\Omega\) satisfying Definition \([22]\)/(b) with a Dini function \(\omega = \omega_A\). Then under Assumption \([7.7]\) there exists a unique Green function \((G, \Pi)\) for the flow velocity of \(\mathcal{L}\) in \(\Omega\) such that for any \(y \in \Omega\),
\[
G(\cdot, y) \text{ is continuously differentiable in } \overline{\Omega} \setminus \{y\} \quad (7.2)
\]
and
\[
\Pi(\cdot, y) \text{ is continuous in } \overline{\Omega} \setminus \{y\}. \quad (7.3)
\]
Moreover, for any \(x, y \in \Omega\) with \(0 < |x - y| \leq 1\), we have that
\[
|G(x, y)| \leq C \min\{d_x, d_y\} \cdot |x - y|^{-d}, \quad (7.4)
\]
\[
|D_x G(x, y)| + |\Pi(x, y)| \leq C|x - y|^{-d}, \quad (7.5)
\]
where \(C = C(d, \lambda, \omega_A, K_0, R_0, \varrho_0)\). Furthermore, if \((G^*, \Pi^*)\) is the Green function for the flow velocity of \(\mathcal{L}^*\) in \(\Omega\), then we have
\[
G(x, y) = G^*(y, x)^\top \quad \text{for all } x, y \in \Omega, \quad x \neq y. \quad (7.6)
\]
The corresponding results for the case with \( d = 2 \) was proved in [7].

**Theorem 7.4.** [7 Theorems 3.2 and 3.7] Let \( \Omega \) be a bounded domain in \( \mathbb{R}^2 \) having a \( C^{1,\text{Dini}} \) boundary as in Definition 2.2. Suppose that the coefficients \( A^{\alpha\beta} \) of \( \mathcal{L} \) are of Dini mean oscillation in \( \Omega \) satisfying Definition 2.7(b) with a Dini function \( \omega = \omega_A \). Then there exists a unique Green function \( (G, \Pi) \) for the flow velocity of \( \mathcal{L} \) in \( \Omega \) such that for any \( y \in \Omega \),

\[
G(\cdot, y) \text{ is continuously differentiable in } \Omega \setminus \{y\}
\]

and

\[
\Pi(\cdot, y) \text{ is continuous in } \Omega \setminus \{y\}.
\]

Moreover, for any \( x, y \in \Omega \) with \( x \neq y \), we have

\[
|G(x, y)| \leq C \left( 1 + \log \left( \frac{\text{diam}(\Omega)}{|x - y|} \right) \right),
\]

\[
|D_x G(x, y)| + |\Pi(x, y)| \leq C|x - y|^{-1},
\]

where \( C = C(\lambda, \omega_A, R_0, \rho_0, \text{diam}(\Omega)) \). Furthermore, if \( (G^*, \Pi^*) \) is the Green function for the flow velocity of \( \mathcal{L}^* \) in \( \Omega \), then we have

\[
G(x, y) = G^*(y, x)^\top \text{ for all } x, y \in \Omega, \quad x \neq y.
\] (7.7)

Based on Theorems 3.7, 7.3, and 7.4 we have the following corollary, the proof of which is given in Section 7.3.

**Corollary 7.5.** Suppose that the same hypothesis of Theorem 7.3 (resp. Theorem 7.4) hold. Let \( (G, \Pi) \) and \( (\mathcal{G}, \mathcal{P}) \) be the Green functions for the flow velocity and the pressure of \( \mathcal{L} \) in \( \Omega \) derived from Theorems 7.3 (resp. Theorem 7.4) and 7.5 respectively. Then for \( f \in C_0^\infty(\Omega)^d \), the pair \( (u, p) \) given by

\[
u(y) = \int_\Omega G(x, y)^\top f(x) \, dx, \quad p(y) = -\int_\Omega \mathcal{G}(x, y) \cdot f(x) \, dx \quad (7.8)
\]

is a unique weak solution in \( Y_0^{1,2}(\Omega)^d \times L^2(\Omega) \) of the problem

\[
\begin{aligned}
\mathcal{L}^* u + \nabla p & = f \quad \text{in } \Omega, \\
\text{div } u & = 0 \quad \text{in } \Omega, \\
u & = 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

Moreover, if we define \( (d + 1) \times (d + 1) \) matrix-valued functions by

\[
G = \begin{pmatrix}
G^{11} & G^{12} & \ldots & G^{1d} & -G^1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
G^{d1} & G^{d2} & \ldots & G^{dd} & -G^d \\
\Pi^1 & \Pi^2 & \ldots & \Pi^d & \mathcal{P}
\end{pmatrix}
\]

and

\[
G^* = \begin{pmatrix}
(G^*)^{11} & (G^*)^{12} & \ldots & (G^*)^{1d} & -(G^*)^1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
(G^*)^{d1} & (G^*)^{d2} & \ldots & (G^*)^{dd} & -(G^*)^d \\
(\Pi^*)^1 & (\Pi^*)^2 & \ldots & (\Pi^*)^d & \mathcal{P}^*
\end{pmatrix},
\]
where \((G^*, \Pi^*)\) and \((G^*, \mathcal{P}^*)\) are the Green functions for the flow velocity and the pressure of \(\mathcal{L}^*\) in \(\Omega\), respectively, then for any \(y \in \Omega\), there exists a measure zero set \(N_y \subset \Omega\) containing \(y\) such that we have

\[
G(x, y) = G^*(y, x)^\top \quad \text{for all} \quad x \in \Omega \setminus N_y.
\]

(7.9)

7.2. Proof of Theorem 7.3. We first prove the existence of the Green function for the flow velocity. In the case when \(|\Omega| < \infty\), we shall follow the arguments in [8], where the authors proved the existence of the Green function in a bounded Lipschitz domain under the following assumption.

Assumption 7.6 ((A1 in [8] Section 2.1)). There exists constants \(\mu \in (0, 1]\) and \(A_1 > 0\) such that the following holds: If \((u, p) \in W^{1,2}(B_R(x_0))^d \times L^2(B_R(x_0))\) satisfies either

\[
\begin{cases}
\mathcal{L} u + \nabla p = 0 & \text{in } B_R(x_0), \\
\text{div } u = 0 & \text{in } B_R(x_0),
\end{cases}
\]

or

\[
\begin{cases}
\mathcal{L}^* u + \nabla p = 0 & \text{in } B_R(x_0), \\
\text{div } u = 0 & \text{in } B_R(x_0),
\end{cases}
\]

where \(x_0 \in \Omega\) and \(R \in (0, d_{x_0}]\), then we have

\[
\sup_{x, y \in B_{R/2}(x_0)} \frac{|u(x) - u(y)|}{|x - y|^\mu} \leq A_1 R^{-\mu} \left( \int_{B_R(x_0)} |u|^2 \, dx \right)^{1/2}.
\]

(7.10)

Note that because of \(|\Omega| < \infty\), we have

\[
\sup_{x \in \Omega} |dx| \leq M(|\Omega|) < \infty.
\]

Hence, under the hypothesis of Theorem 7.3 we can show that Assumption 7.6 holds with \(\mu = 1\) and \(A_1 = A_1(d, \lambda, \omega_A, M)\). Indeed, since \((u, p - (p)_{B_{R/4}(x_0)})\) satisfies the same system, by [A2] with a covering argument and Hölder’s inequality, we have

\[
\sup_{x, y \in B_{R/2}(x_0)} \frac{|u(x) - u(y)|}{|x - y|} \leq \|Du\|_{L^\infty(B_{R/2}(x_0))} \leq CR^{-d/2} \left( \|Du\|_{L^2(B_{R/4}(x_0))} + \|p - (p)_{B_{R/4}(x_0)}\|_{L^2(B_{R/4}(x_0))} \right),
\]

where \(C = C(d, \lambda, \omega_A, M)\). Thus we get (7.10) from the above inequality and Caccioppoli’s inequality (see, for instance, [8 Lemma 3.3]). Moreover, it is easy to check that, under Assumptions 3.1 and 7.6, the proof of [8 Theorem 2.3] still works for the domain \(\Omega\). Therefore, by the existence result in [8 Theorem 2.3] of a Green function, there exist Green functions \((G, \Pi)\) and \((G^*, \Pi^*)\) for the flow velocity of \(\mathcal{L}\) and \(\mathcal{L}^*\), respectively, satisfying the properties in Definition 7.1 and (7.6). Notice from Definition 7.1 (b) that

\[
\begin{cases}
\mathcal{L} G(\cdot, y) + \nabla \Pi(\cdot, y) = 0 & \text{in } \Omega_R(x), \\
\text{div } G(\cdot, y) = 0 & \text{in } \Omega_R(x), \\
G(\cdot, y) = 0 & \text{on } \partial \Omega \cap B_R(x)
\end{cases}
\]

for any \(x \in \Omega\) and \(R > 0\) satisfying \(y \notin B_R(x)\). Thus by (A.12), we get (7.2) and (7.3). Similarly, the existence of the Green function in a domain \(\Omega\) with \(|\Omega| = \infty\) follows from [8 Theorem 10.4].
We now turn to the proof of (7.4). Let \( x, y \in \Omega \) with \( 0 < |x - y| \leq 1 \), and set \( R = |x - y|/2 \). Suppose that \((u, p) \in Y^{1,2}_0(\Omega) \times L^2(\Omega)\) is the weak solution of
\[
\begin{align*}
L^* u + \nabla p &= f & \text{in } \Omega, \\
\text{div} u &= 0 & \text{in } \Omega,
\end{align*}
\]
where \( f \in C^0(\Omega \setminus \overline{B_R(y)})^d \). Since \( f \equiv 0 \) in \( \Omega_R(y) \), by (A.13) and Lemma 4.1, we have
\[
\|u\|_{L^\infty(\Omega_R/2(y))} \leq CR^{-d/2}\|u\|_{L^1(\Omega_R(y))} + CR^{1-d/2}\|Du\|_{L^1(\Omega_R(y))} + \|p\|_{L^1(\Omega_R(y))}
\]
\[
\leq CR^{-d/2}\|u\|_{L^{2d/(d-2)}(\Omega)} + \|Du\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)}
\]
\[
\leq CR^{1-d/2}\|f\|_{L^{2d/(d+2)}(\Omega \setminus \overline{B_R(y)})},
\]
where \( C = C(d, \lambda, \omega_A, K_0, R_0, \theta_0) \). Since \( u \) is continuous at \( y \), we get from (7.14) and the above inequality that
\[
\left|\int_{\Omega \setminus \overline{B_R(y)}} G(x, y) \top f(x) \, dx\right| \leq CR^{1-d/2}\|f\|_{L^{2d/(d+2)}(\Omega \setminus \overline{B_R(y)})}.
\]
Thus by the duality, we have
\[
\|G(\cdot, y)\|_{L^{2d/(d-2)}(\Omega \setminus \overline{B_R(y)})} \leq CR^{1-d/2}.
\] (7.12)
Similarly, we obtain
\[
\|DG(\cdot, y)\|_{L^2(\Omega \setminus \overline{B_R(y)})} + \|\Pi(\cdot, y)\|_{L^2(\Omega \setminus \overline{B_R(y)})} \leq CR^{1-d/2}.
\] (7.13)
From (7.12), (7.13), and (A.13) applied to (7.11), we get
\[
R^{-1}\|G(\cdot, y)\|_{L^\infty(\Omega_R/2(x))} + \|DG(\cdot, y)\|_{L^\infty(\Omega_R/2(x))}
\]
\[+ \|\Pi(\cdot, y)\|_{L^\infty(\Omega_R/2(x))} \leq CR^{1-d}.
\] (7.14)
This gives (7.5) and
\[
|G(x, y)| \leq C|x - y|^{2-d}.
\] (7.15)
To prove (7.4), we use the idea in the proof of [19, Theorem 3.13], where the authors obtained pointwise bounds for Green functions of elliptic systems near the boundary. We claim that for any \( x, y \in \Omega \) with \( 0 < |x - y| \leq 1 \), we have
\[
|G(x, y)| \leq C \min\{d_x, |x - y|\} \cdot |x - y|^{1-d},
\] (7.16)
\[
|G(x, y)| \leq C \min\{d_y, |x - y|\} \cdot |x - y|^{1-d}.
\] (7.17)
We denote \( R = |x - y|/2 \) and extend \( G(\cdot, y) \) by zero on \( \mathbb{R}^d \setminus \Omega \). Assume \( d_x < R/2 \), and take \( x_0 \in \partial \Omega \) such that \( \text{dist}(x, x_0) = d_x \). Since \( G(x_0, y) = 0 \), we obtain by (7.14) that
\[
|G(x, y)| = |G(x, y) - G(x_0, y)| \leq d_x \|DG(\cdot, y)\|_{L^\infty(\Omega_R/2(x))} \leq C d_x |x - y|^{1-d}.
\] (7.18)
From (7.15) and (7.18), we get
\[
|G(x, y)| \leq C d_x R^{1-d} \leq C \min\{d_x, |x - y|/4\} \cdot |x - y|^{1-d},
\]
which gives (7.10). By the same reasoning, we have
\[
|G^*(y, x)| \leq C \min\{d_y, |x - y|\} \cdot |x - y|^{1-d}.
\]
This together with (7.6) yields (7.17).
We are ready to prove (7.4). We again let \( x, y \in \Omega \) with \( x \neq y \), and set \( R = |x - y|/2 \). Assume \( d_x < R/16 \), and we take \( x_0 \in \partial \Omega \) such that \( \text{dist}(x, x_0) = d_x \).

Note that

\[
\Omega_{R/8}(x) \subset \Omega_{R/4}(x_0) \subset \Omega_{R/2}(x_0) \subset \Omega_R(x).
\]

Since \((G(\cdot, y), \Pi(\cdot, y))\) satisfies (7.11), we have the following boundary Caccioppoli inequality

\[
\|DG(\cdot, y)\|_{L^2(\Omega_{R/4}(x_0))} + \|\Pi(\cdot, y)\|_{L^2(\Omega_{R/4}(x_0))} \leq CR^{-1}\|G(\cdot, y)\|_{L^2(\Omega_{R/2}(x_0))}.
\]

Using this together with (7.2) that

\[
\text{when } x \in \Omega \text{ and } y \in \Omega, \text{ we have }
\]

\[
\|DG(\cdot, y)\|_{L^2(\Omega_{R/4}(x_0))} + \|\Pi(\cdot, y)\|_{L^2(\Omega_{R/4}(x_0))} \leq CR^{-1}\|G(\cdot, y)\|_{L^2(\Omega_{R/2}(x_0))}.
\]

Note that for any \( z \in \Omega_R(x) \), we have \( R < |z - y| < 3R \). Thus, it follows from (7.14) that

\[
|G(z, y)| \leq C \min\{d_y, |z - y|\} \cdot |z - y|^{1-d} \\
\leq C \min\{d_y, |x - y|\} \cdot |x - y|^{1-d}.
\]

This together with (7.16) yields

\[
|G(x, y)| \leq Cd_x \min\{d_y, |x - y|\} \cdot |x - y|^{-d}.
\]

Finally, combining (7.17) and the above inequality, we get the desired estimate (7.3). The theorem is proved. \(\square\)

7.3. Proof of Corollary 7.5

The representation formula (7.3) follows immediately from Definition 7.1(c) and Definition 7.1(c). To verify (7.4), let \( x, y \in \Omega \) with \( x \neq y \) and \((G_2(\cdot, x), \Pi_2(\cdot, x)) \in Y_0^{1,2}(\Omega)^d \times L^2(\Omega)\) be the approximated Green function for the pressure of \( L^* \) satisfying (6.12). Then by Definition 7.1(c) and the continuity of \( G_2^*(\cdot, x) \), we have

\[
G_2^*(y, x) = -\int_\Omega \Pi(z, y) \Phi_{\sigma, x}(z) dz.
\]

Due to the continuity of \( \Pi(\cdot, y) \), the right-hand side of (7.20) converges to \(-\Pi(x, y)\) as \( \sigma \to 0 \). On the other hand, by the counterpart of Lemma 6.4 there exists a subsequence of \( \{G_2^*(y, x)\} \) that converges to \( G^*(y, x) \). Therefore, we conclude that

\[
G^*(y, x) = -\Pi(x, y).
\]

From this together with (7.7) and (7.6) (resp. (7.7)), we conclude that (7.9) holds when \( d \geq 3 \) (resp. \( d = 2 \)). \(\square\)

Appendix A. \( L^\infty \)-estimates

In this section, we prove \( L^\infty \)-estimates of solutions and its derivatives, which are crucial for proving our main results. Denote \( B_r = B_r(0) \) for any \( r > 0 \). The following lemma concerns interior estimates, the proof of which is based on the \( W^{1,\infty} \)-regularity result in [3].

Lemma A.1. Let $R \in (0,1]$. Suppose that the coefficients $A^\alpha\beta$ of $\mathcal{L}$ are of partially Dini mean oscillation with respect to $x'$ in the interior of $B_R$ satisfying Definition 2.4(a) with a Dini function $\omega = \omega_A$. If $(u,p) \in W^{1,2}(B_R)^d \times L^2(B_R)$ satisfies
\[
\begin{cases}
\mathcal{L}u + \nabla p = f & \text{in } B_R, \\
\text{div} u = \ell & \text{in } B_R,
\end{cases}
\]
where $f \in L^\infty(B_R)^d$ and $\ell \in \mathbb{R}$, then we have
\[
(u,p) \in W^{1,\infty}(B_{R/2})^d \times L^\infty(B_{R/2}) \tag{A.1}
\]
with the estimates
\[
\|Du\|_{L^\infty(B_{R/2})} + \|p\|_{L^\infty(B_{R/2})} \leq C \left( R^{-d} \left( \|Du\|_{L^{1}(B_R)} + \|p\|_{L^{1}(B_R)} \right) + R \|f\|_{L^\infty(B_R)} + |\ell| \right), \tag{A.2}
\]
and
\[
\|u\|_{L^\infty(B_{R/2})} \leq CR^{-d} \|u\|_{L^1(B_{R/2})} + C \left( R^{-d} \left( \|Du\|_{L^{1}(B_R)} + \|p\|_{L^{1}(B_R)} \right) + R^2 \|f\|_{L^\infty(B_R)} + R|\ell| \right), \tag{A.3}
\]
where $C = C(d,\lambda,\omega_A)$. If we further assume that $A^\alpha\beta$ are of Dini mean oscillation with respect to all direction in $B_R$ satisfying Definition 2.4(b), then we have
\[
(u,p) \in C^\infty(B_{R/2})^d \times C(B_{R/2}). \tag{A.4}
\]

Proof. Note that (A.1) and (A.4) are easy consequences of [4] Theorems 2.2 and 2.3, and Remark 2.4] together with covering and scaling arguments. Moreover, the estimate (A.3) follows from (A.2). Indeed, using the following Poincaré inequality
\[
\|u - (u)_{B_{R/2}}\|_{L^\infty(B_{R/2})} \leq CR^d \|Du\|_{L^\infty(B_{R/2})}, \tag{A.5}
\]
we have
\[
\|u\|_{L^\infty(B_{R/2})} \leq C \left( R^{-d} \|u\|_{L^1(B_{R/2})} + R \|Du\|_{L^\infty(B_{R/2})} \right),
\]
where $C = C(d)$. This inequality together with (A.2) implies (A.3). Thus, to complete the proof of the lemma, we only need to prove (A.2). By a covering argument, it suffices to show that
\[
\|Du\|_{L^\infty(B_{R/3})} + \|p\|_{L^\infty(B_{R/3})} \leq C \left( R^{-d} \left( \|Du\|_{L^{1}(B_R)} + \|p\|_{L^{1}(B_R)} \right) + R \|f\|_{L^\infty(B_R)} + |\ell| \right), \tag{A.6}
\]
where $C = C(d,\lambda,\omega_A)$. Let us fix $q > d$. By [4] Lemma 3.1, there exist functions $f_\alpha \in W^{1,q}(B_R)^d$, $\alpha \in \{1,\ldots,d\}$, such that
\[
D_\alpha f_\alpha = f \text{ in } B_R, \quad \|Df_\alpha\|_{L^q(B_R)} \leq C(d,q)\|f\|_{L^q(B_R)},
\]
which together with the Morrey inequality implies
\[
\|f_\alpha\|_{L^\infty(B_R)} + R^{1-d/q}|f_\alpha|_{C^{1-d/q}(B_R)} \leq C(d,q)R\|f\|_{L^\infty(B_R)}. \tag{A.7}
\]
Note that $(u,p)$ satisfies
\[
\begin{cases}
\mathcal{L}u + \nabla p = D_\alpha f_\alpha & \text{in } B_R, \\
\text{div} u = \ell & \text{in } B_R,
\end{cases}
\]
We now apply [4] Theorem 2.2 (a)] with the $L^\infty$-estimate [4] Eq. (4.16)] to the scaled system of (A.8). To this end, let
\[
\hat{u}(y) = u(x), \quad \hat{p}(y) = p(x), \quad \hat{A}^\alpha\beta(y) = A^\alpha\beta(x), \quad \hat{f}_\alpha(y) = f_\alpha(x),
\]
where \( y = 6x/R \), and observe that

\[
D_\alpha \left( \hat{A}^{\alpha \beta} D_\beta \hat{u} \right) + \nabla(\hat{R}\hat{p}/6) = \hat{D}_\alpha (R\hat{f}_\alpha/6) \quad \text{in } B_6, \\
\text{div} \, \hat{u} = R\hat{f}/6 \quad \text{in } B_6.
\]

(A.9)

Fix a constant \( \gamma \) satisfying \( 1 - d/q < \gamma < 1 \), and let \( \kappa = \kappa(d, \lambda, \gamma) \in (0, 1/2] \) be the constant from [11, Lemma 4.1]. For \( r \in (0, 1) \) and \( h \in L^1(B_6) \), we denote

\[
\omega_{h,y'}(r) = \sup_{y_0 \in B_4} \int_{B_r(y_0)} |h(y) - \int_{B_r'(y_0')} h(y_1, z') \, dz'| \, dy,
\]

\[
\tilde{\omega}_{h,y'}(r) = \sum_{i=1}^{\infty} \kappa^{\gamma i} (\omega_{h,y'}(\kappa^{-i}r)[\kappa^{-i}r < 1] + \omega_{h,y'}(1)[\kappa^{-i}r \geq 1]),
\]

where we used Iverson bracket notation, that is, \( [P] = 1 \) if \( P \) is true and \( [P] = 0 \) otherwise. By (A.7) and the fact that \( 1 - d/q < \gamma < 1 \), we have

\[
R\|\tilde{f}_1\|_{L^\infty(B_6)} + \int_0^1 \frac{\tilde{\omega}_{Rf_{\alpha,x'}}(t)}{t} \, dt \leq CR^2\|f\|_{L^\infty(B_R)},
\]

where \( C = C(d, q, \gamma) \). Using this inequality and [11] Eq. (4.16) and Remark 4.2 applied to (A.9), we find that

\[
\|D\hat{u}\|_{L^\infty(B_2)} + \|\hat{p}\|_{L^\infty(B_2)}
\]

\[
\leq C \left( \|D\hat{u}\|_{L^1(B_6)} + R\|\hat{p}\|_{L^1(B_6)} + R\|\tilde{f}_1\|_{L^\infty(B_6)} + R\|\ell\| + \int_0^1 \frac{\tilde{\omega}_{Rf_{\alpha,x'}}(t)}{t} \, dt \right),
\]

\[
\leq C \left( \|D\hat{u}\|_{L^1(B_6)} + R\|\hat{p}\|_{L^1(B_6)} + R^2\|f\|_{L^\infty(B_R)} + R\|\ell\| \right),
\]

where \( C = C(d, \lambda, \omega_0, q, \gamma) \). Here, \( \omega_0 : (0, 1) \to [0, \infty) \) is a function such that

\[
\int_0^r \frac{\tilde{\omega}_{A^{\alpha \beta,y'}_n}(t)}{t} \, dt \leq \int_0^r \frac{\omega_0(t)}{t} \, dt < \infty \quad \text{for all } r \in (0, 1].
\]

Therefore, from the change of variables, we get

\[
\|D\hat{u}\|_{L^\infty(B_{R/2})} + \|\hat{p}\|_{L^\infty(B_{R/2})}
\]

\[
\leq C \left( \|D\hat{u}\|_{L^1(B_{R/2})} + \|\hat{p}\|_{L^1(B_{R/2})} + R\|f\|_{L^\infty(B_{R/2})} + \|\ell\| \right),
\]

where \( C = C(d, \lambda, \omega_0, q, \gamma) = C(d, \lambda, \omega_0) \).

To complete the proof of (A.6), it remains to show that \( \omega_0 \) can be derived from \( \omega_A \). Set

\[
\tilde{\omega}_A(r) = \sum_{i=1}^{\infty} \kappa^{\gamma i} (\omega_A(\kappa^{-i}r)[\kappa^{-i}r < 1] + \omega_A(1)[\kappa^{-i}r \geq 1]),
\]

and observe that \( \tilde{\omega}_A : [0, 1] \to [0, \infty) \) is a Dini function; see [11] Lemma 1. Using the fact that

\[
\omega_{\hat{A}^{\alpha \beta,y'}}(r) \leq \omega_A(rR/6) \quad \text{for all } r \in (0, 1],
\]

we have

\[
\tilde{\omega}_{\hat{A}^{\alpha \beta,y'}}(r) \leq \sum_{i=1}^{\infty} \kappa^{\gamma i} \omega_A(\kappa^{-i}rR/6)[\kappa^{-i}r < 1] + \sum_{i=1}^{\infty} \kappa^{\gamma i} \omega_A(R/6)[\kappa^{-i}r \geq 1]
\]

\[
=: I_1(r) + I_2(r).
\]

(A.10)
Obviously, \( I_1(r) \leq \tilde{\omega}_A(rR/6) \). Since \( \omega_A \) is a Dini function, we obtain that (using \( R \leq 1 \))
\[
\omega_A(R/6) \leq C \inf_{R/6 \leq t \leq R/3} \omega_A(t) \leq C \int_{R/6}^{R/3} \frac{\omega_A(t)}{t} dt \leq C \int_0^1 \frac{\omega_A(t)}{t} dt,
\]
where \( C = C(\omega_A) \). This implies that
\[
I_2(r) \leq C \sum_{\kappa = 1}^{\infty} \kappa^\gamma \cdot \int_0^1 \frac{\omega_A(t)}{t} dt \leq C_0 r^\gamma \int_0^1 \frac{\omega_A(t)}{t} dt,
\]
where \( C_0 = C_0(\omega_A, \kappa, \gamma) = C_0(d, \lambda, \omega_A) \). From (A.10) and the estimates of \( I_1 \) and \( I_2 \), we get
\[
\tilde{\omega}_{A^\alpha, \beta, \gamma}(r) \leq \tilde{\omega}_A(rR/6) + C_0 r^\gamma \int_0^1 \frac{\omega_A(t)}{t} dt.
\]
Therefore, for any \( r \in (0, 1] \), we have
\[
\int_0^r \frac{\tilde{\omega}_{A^\alpha, \beta, \gamma}(t)}{t} dt \leq \int_0^r \frac{\tilde{\omega}_A(tR/6)}{t} dt + C_0 \int_0^r t^\gamma \frac{t}{t} dt \cdot \int_0^1 \frac{\omega_A(t)}{t} dt
\leq \int_0^r \frac{\omega_0(t)}{t} dt < \infty,
\]
where we set
\[
\omega_0(r) = \tilde{\omega}_A(r) + C_0 r^\gamma \int_0^1 \frac{\omega_A(t)}{t} dt.
\]
This completes the proof of (A.6). The lemma is proved. \( \square \)

Based on the \( C^1 \)-regularity result in [3], we obtain the following \( L^\infty \)-estimate in a domain having a \( C^{1, \text{Dini}} \) boundary.

**Lemma A.2.** Let \( \Omega \) be a (possibly unbounded) domain in \( \mathbb{R}^d \) having a \( C^{1, \text{Dini}} \) boundary as in Definition 2.2. Suppose that the coefficients \( A^\alpha, \beta \) of \( \mathcal{L} \) are of Dini mean oscillation in \( \Omega \) satisfying Definition 2.2 (b) with a Dini function \( \omega = \omega_A \). Let \( x_0 \in \Omega \) and \( R \in (0, 1] \). If \((u, p) \in W^{1,2}(\Omega_R(x_0))^d \times L^2(\Omega_R(x_0)) \) satisfies
\[
\begin{cases}
\mathcal{L}u + \nabla p = f & \text{in } \Omega_R(x_0), \\
\text{div } u = \ell & \text{in } \Omega_R(x_0), \\
u = 0 & \text{on } \partial \Omega \cap B_R(x_0),
\end{cases}
\]
where \( f \in L^\infty(\Omega_R(x_0))^d \) and \( \ell \in \mathbb{R} \), then we have
\[
(u, p) \in C^1(\Omega_{R/2}(x_0))^d \times C(\Omega_{R/2}(x_0))
\]
with the estimate
\[
R^{-1}\|u\|_{L^\infty(\Omega_{R/2}(x_0))} + \|Du\|_{L^\infty(\Omega_{R/2}(x_0))} + \|p\|_{L^\infty(\Omega_{R/2}(x_0))}
\leq CR^{-d-1}\|u\|_{L^1(\Omega_R(x_0))} + CR^{-d}\left(\|Du\|_{L^1(\Omega_R(x_0))} + \|p\|_{L^1(\Omega_R(x_0))}\right)
\]
(A.13)
\[
+ CR\|f\|_{L^\infty(\Omega_R(x_0))} + C|\ell|,
\]
where \( C = C(d, \lambda, \omega_A, R_0, \ell_0) \).

**Remark A.3.** In the proof of Lemma A.2 below, we will use the \( W^{1,q} \)-regularity result in [3] Corollary 5.3 (see also [12]) for the Stokes system. If \( \Omega \) is bounded, then under the hypothesis of Lemma A.2, the regularity result is available. For further details, see the proof of [3] Theorem 1.6.
Proof of Lemma A.2. If Ω is unbounded, then one may construct a bounded domain Ω* such that Ω* ⊂ Ω and it has the same nice properties as Ω. Thus we may assume that Ω is bounded.

Let x0 ∈ Ω and R ∈ (0, 1]. We first prove (A.12). By using localization and bootstrap arguments combined with the W1,q-regularity result in [8, Corollary 5.3], we see that

\[(u, p) ∈ W^{1,q}(Ω_r(x_0))^d × L^q(Ω_r(x_0)) \text{ for all } r ∈ (0, R) \text{ and } q ∈ (1, ∞). \quad (A.14)\]

Let η be a smooth function on Rd with a compact support in B_R(x_0). Then, the pair

\[\(v, π\) = (ηu, ηp - (ηp)Ω) ∈ W^{1,2}(Ω) × L^2(Ω) \quad (A.15)\]

satisfies

\[
\begin{align*}
Lv + ∇π &= h + D_α h_α \quad \text{in } Ω, \\
\text{div } v &= g \quad \text{in } Ω, \\
v &= 0 \quad \text{on } ∂Ω,
\end{align*}
\]

where we set

\[h = A^{αβ} D_β u D_α η + p∇η + ηf, \quad h_α = A^{αβ} D_β η u, \quad g = \nabla η · u + ℓη.\]

Obviously, (A.14) implies that \(h ∈ L^q(Ω)^d\) with \(q > d\). Moreover, since \(u\) is Hölder continuous in \(Ω \cap \supp η\), it follows from [11, Lemma 2.1] that \(h_α\) and \(g\) are of Dini mean oscillation in \(Ω\). Therefore, by [3, Theorem 1.4 and Remark 1.5], we conclude that

\[\(v, π\) ∈ C^1(Ω)^d × C(Ω),\]

which gives (A.12) if we choose the function \(η\) such that \(η ≡ 1\) on \(B_{R/2}(x_0)\).

We now turn to the proof of (A.13). By the Poincaré inequality (A.5) applied to the zero extension of \(u\), we have

\[\|u\|_{L^∞(Ω_{R/2}(x_0))} ≤ CR^{-d} \|u\|_{L^1(Ω_{R/2}(x_0))} + CR \|D_α u\|_{L^1(Ω_{R/2}(x_0))}.\]

Thus, it suffices to prove that

\[\|D_α u\|_{L^∞(Ω_{R/2}(x_0))} + \|p\|_{L^∞(Ω_{R/2}(x_0))} ≤ CR^{-d} \|u\|_{L^1(Ω_R(x_0))} + C \|f\|_{L^∞(Ω_R(x_0))} + |ℓ|.) \quad (A.17)\]

Let \(y ∈ Ω_R(x_0)\) and \(r ∈ (0, R]\) such that \(r ≤ \min\{R_1, \text{dist}(y, ∂B_R(x_0))\}\), where \(R_1 = R_1(R_0, θ_0) ∈ (0, R_0/4)\) is the constant from [3, Lemma 2.2]. We use the abbreviations

\[B_r = B_r(y) \quad \text{and} \quad Ω_r = Ω_r(y).\]

We fix \(q > d\) and choose the function \(η\) in (A.15) satisfying

\[0 ≤ η ≤ 1, \quad η ≡ 1 \text{ on } B_{r/2}, \quad \supp η ⊂ B_r, \quad r |∇ η| + r^2 |∇^2 η| ≤ C(d).\]

Note that

\[\int_{B_r} hξ_{Ω_r} dx = \int_{Ω} h dx = 0.\]

Hence, from the existence of solutions to the divergence equation in a ball, there exist \(h_α ∈ W^{1,q}(B_r)^d, \alpha ∈ \{1, …, d\}\), such that

\[D_α h_α = hξ_{Ω_r} \text{ in } B_r, \quad \|Dh_α\|_{L^q(B_r)} ≤ C(d, q) \|h\|_{L^q(Ω_r)}. \quad (A.18)\]
We extend \( \hat{h}_\alpha \) by zero on \( \Omega \setminus B_r \) to see that \((v, \pi)\) satisfies
\[
\begin{align*}
L v + \nabla \pi & = D\alpha (\hat{h}_\alpha + h_\alpha) \quad \text{in } \Omega, \\
\text{div } v & = g \quad \text{in } \Omega, \\
v & = 0 \quad \text{on } \partial \Omega.
\end{align*}
\]
Since the coefficients and data of the above system are of Dini mean oscillation in \( \Omega \), we obtain by [3] Eq. (2.27)] that
\[
\|D v\|_{L^\infty(\Omega, t^2)} + \|\pi\|_{L^\infty(\Omega, t^2)} \\
\leq C r^{-d} (\|D v\|_{L^1(\Omega, t)} + \|\pi\|_{L^1(\Omega, t)}) + C (\|\hat{h}_\alpha + h_\alpha\|_{L^\infty(\Omega, t^2)} + \mathcal{H}(r)),
\]
where \( C = C(d, \lambda, \gamma, \omega_\alpha, R, g_0) \) and
\[
\mathcal{H}(r) = \int_0^r \frac{\omega^2_h + \omega^2_h}{t} dt.
\]
Here, we use the notation (see [3] Section 2.1)
\[
\omega^2_h(\rho) = \sup_{\rho \leq \rho_0 \leq R_1} \left( \frac{\rho}{\rho_0} \right)^\gamma \tilde{\omega}_h(\rho),
\]
\[
\tilde{\omega}_h(\rho) = \sum_{i=1}^\infty \kappa_i \left( \omega_h(\kappa^{-i}\rho) + \omega_h(1) \right),
\]
\[
\omega_{f_0}(\rho) = \sup_{x \in \Omega} \int_{\Omega} |f_0 - (f_0)_{\Omega, x}| \, dy \quad \text{for } f_0 \in L^1(\Omega),
\]
where \( \gamma \in (1 - d/q, 1) \) is a fixed constant and \( \kappa = \kappa(d, \lambda, \gamma, R_0, g) \in (0, 1/8] \) is the constant from [3] Lemma 2.3. We extend \( u \) by zero on \( B_r \setminus \Omega \). Since \( D\eta u \in W_0^{1,q}(B_r)^d \), by both Morrey and Poincaré inequalities with a scaling, we have
\[
\begin{align*}
r^{-d/q + 1} [D\eta u]_{C^{1, d/q}(\overline{\Omega})} + \|D\eta u\|_{L^\infty(\Omega)} & \leq C r^{-d/q + 1} \|D(D\eta u)\|_{L^q(\Omega, t^2)} \\
& \leq C r^{-d/q} (r^{-1}\|u\|_{L^q(\Omega, t^2)} + \|Du\|_{L^q(\Omega, t^2)}) \\
& \leq C (r^{-d-1}\|u\|_{L^1(\Omega, t^2)} + r^{-d/q}\|Du\|_{L^q(\Omega, t^2)}).
\end{align*}
\]
Then for any \( 0 < \rho \leq R_1 \), we obtain that
\[
\omega_h(\rho) \leq C \left( \frac{\rho^{1-d/q}}{r^{1-d/q}} + \omega_A(\rho) \right) \|D\eta u\|_{L^\infty(\Omega)}
\]
\[
\leq C \left( \frac{\rho^{1-d/q}}{r^{1-d/q}} + \omega_A(\rho) \right) (r^{-d-1}\|u\|_{L^1(\Omega, t^2)} + r^{-d/q}\|Du\|_{L^q(\Omega, t^2)}).
\]
This together with \( 1 - d/q < \gamma < 1 \) implies
\[
\omega^2_h(\rho) \leq C \left( \frac{\rho^{1-d/q}}{r^{1-d/q}} + \omega^2_A(\rho) \right) (r^{-d-1}\|u\|_{L^1(\Omega, t^2)} + r^{-d/q}\|Du\|_{L^q(\Omega, t^2)}),
\]
where \( C = C(d, \lambda, \gamma, q) \). Similarly, from the fact that (using [A.18])
\[
[\hat{h}_\alpha]_{C^{1, d/q}(\overline{\Omega})} \leq C \|D\hat{h}_\alpha\|_{L^q(\Omega, t^2)} \leq C \|h\|_{L^q(\Omega)}
\]
\[
\leq C r^{-1} (\|Du\|_{L^q(\Omega, t^2)} + \|\pi\|_{L^q(\Omega, t^2)}) + Cr^{d/q}\|f\|_{L^\infty(\Omega, t^2)},
\]
we get
\[ \omega_{k_0}^2(\rho) \leq C \left( \frac{\rho^{1-d/q}}{r^{1-d/q}} \right) (r^{-d/q} (\|Du\|_{L^q(\Omega_r)} + \|p\|_{L^q(\Omega_r)}) + r\|f\|_{L^\infty(\Omega_r)}). \]

Combining the above inequality and (A.20), we have
\[ \mathcal{H}(r) \leq C r^{-d-1} \|u\|_{L^1(\Omega_r)} \]
\[ + C r^{-d/q} (\|Du\|_{L^q(\Omega_r)} + \|p\|_{L^q(\Omega_r)}) + C (r\|f\|_{L^\infty(\Omega_r)} + |\ell|), \]
where \( C = C(d, \lambda, \gamma, q, \omega_A) \). Therefore, it follows from (A.19) and (A.21) that
\[ \|Du\|_{L^\infty(\Omega_{r/2}(y))} + \|p\|_{L^\infty(\Omega_{r/2}(y))} \leq C r^{-d-1} \|u\|_{L^1(\Omega_{r/2}(y))} \]
\[ + C r^{-d/q} (\|Du\|_{L^q(\Omega_{r/2}(y))} + \|p\|_{L^q(\Omega_{r/2}(y))}) + C (r\|f\|_{L^\infty(\Omega_{r/2}(y))} + |\ell|) \]
for any \( y \in \Omega_R(x_0) \) and \( 0 < r \leq \min\{R_1, \text{dist}(y, \partial B_R(x_0))\} \), where the constant \( C \) depends only on \( d, \lambda, \omega_A, R_0, \) and \( g_0 \).

We now complete the proof of (A.17). Set \( U = |Du| + |p| \) and let \( R/2 < r < R \) with \( r - \rho \leq R_1 \). Then for any \( y \in \Omega_\rho(x_0) \), we obtain by (A.22) that
\[ \|U\|_{L^\infty(\Omega_{r-\rho}/2)} \leq C r^{-d} \|u\|_{L^1(\Omega_{r-\rho})} \]
\[ + C (r - \rho)^{-d/2} \|U\|_{L^q(\Omega_{r-\rho})} + (r - \rho)\|f\|_{L^\infty(\Omega_{r-\rho})} + |\ell|), \]
and thus, we get from Young’s inequality that
\[ \|U\|_{L^\infty(\Omega_{r-\rho}/2)} \leq \delta \|U\|_{L^\infty(\Omega_{r-\rho})} + C r^{-d-1} \|u\|_{L^1(\Omega_{r-\rho})} \]
\[ + C_\delta (r - \rho)^{-d} \|U\|_{L^1(\Omega_{r-\rho})} + C (r - \rho)\|f\|_{L^\infty(\Omega_{r-\rho})} + |\ell|) \]
for any \( \delta \in (0, 1] \), where \( C = C(d, \lambda, \omega_A, R_0, g_0) \) and \( C_\delta \) depends also on \( \delta \). Since \( y \) is an arbitrary point in \( \Omega_\rho(x_0) \) and \( \Omega_{r-\rho}(y) \subset \Omega_r(x) \), we have
\[ \|U\|_{L^\infty(\Omega_\rho(x))} \leq \delta \|U\|_{L^\infty(\Omega_\rho(x))} + C r^{-d-1} \|u\|_{L^1(\Omega_\rho(x))} \]
\[ + C_\delta (r - \rho)^{-d} \|U\|_{L^1(\Omega_\rho(x))} + C (r - \rho)\|f\|_{L^\infty(\Omega_\rho(x))} + |\ell| \]
for any \( R/2 < r \leq R \) with \( r - \rho \leq R_1 \). Set
\[ r_k = R \left( 1 - \frac{1}{2^k} \right), \quad k \in \{1, 2, \ldots\}, \]
and let \( k_0 \) be the smallest positive integer depending only on \( R_1 \), such that
\[ r_{k_0+1} - r_{k_0} \leq \frac{1}{2^{k_0+1}} \leq R_1. \]
By (A.23) we have for any \( k \in \{k_0, k_0 + 1, \ldots\} \),
\[ \|U\|_{L^\infty(\Omega_{r_k}(x))} \leq \delta \|U\|_{L^\infty(\Omega_{r_{k+1}}(x))} + \frac{C_\delta 2^d k}{R^{d+1}} \|u\|_{L^1(\Omega_{r_{k+1}}(x))} \]
\[ + \frac{C_\delta 2^d k}{R^{d}} \|U\|_{L^1(\Omega_{r_{k+1}}(x))} + C (R\|f\|_{L^\infty(\Omega_{r_{k+1}}(x))} + |\ell|). \]
By multiplying both sides of the above inequality by $\delta^k$ and summing the terms with respect to $k \in \{k_0, k_0 + 1, \ldots\}$, we see that
\[
\sum_{k=k_0}^{\infty} \delta^k \|U\|_{L^\infty(\Omega_{r_k}(x))} \leq \sum_{k=k_0}^{\infty} \delta^k \|U\|_{L^\infty(\Omega_{r_k}(x))} + \frac{C}{R^{d+1}} \|u\|_{L^1(\Omega_R(x))} \sum_{k=k_0}^{\infty} (2^{d+1})^k
\]
\[
+ \frac{C \delta}{R^d} \|U\|_{L^1(\Omega_R(x))} \sum_{k=k_0}^{\infty} (2^{d+1})^k + C(R\|f\|_{L^\infty(\Omega_R(x))} + |\ell|) \sum_{k=k_0}^{\infty} \delta^k,
\]
where each summation is finite upon choosing, for instance, $\delta = 2^{-(d+1)}$. Therefore, by subtracting
\[
\sum_{k=k_0}^{\infty} \delta^k \|U\|_{L^\infty(\Omega_{r_k}(x))}
\]
from both sides of the above inequality, we obtain
\[
\delta^{k_0} \|U\|_{L^\infty(\Omega_{r_0}(x))} \leq \frac{C}{R^{d+1}} \|u\|_{L^1(\Omega_R(x))}
\]
\[
+ \frac{C}{R^d} \|U\|_{L^1(\Omega_R(x))} + C(R\|f\|_{L^\infty(\Omega_R(x))} + |\ell|),
\]
which implies (A.17). The lemma is proved. \(\square\)

The following lemma is analogous to Lemma A.2.

**Lemma A.4.** Let $\Omega = \mathbb{R}^d$, $x_0 \in \Omega$, and $R \in (0, 1]$. Suppose that the coefficients $A^{\alpha \beta}$ of $\mathcal{L}$ are of partially Dini mean oscillation with respect to $x'$ in $\mathbb{R}^d$ satisfying Definition 3.11 with a Dini function $\omega = \omega_A$. If $(u, p) \in W^{1,2}(\Omega_R(x_0))^d \times L^2(\Omega_R(x_0))$ satisfies
\[
\begin{align*}
\mathcal{L}u + \nabla p &= f & \text{in } \Omega_R(x_0), \\
\text{div } u &= 0 & \text{in } \Omega_R(x_0), \\
u &= 0 & \text{on } B_R(x_0) \cap \partial \Omega,
\end{align*}
\]
where $f \in L^\infty(\Omega_R(x_0))^d$, then we have
\[
(u, p) \in W^{1,\infty}(\Omega_{R/2}(x_0))^d \times L^\infty(\Omega_{R/2}(x_0))
\]
with the estimates
\[
\|Du\|_{L^\infty(\Omega_{R/2}(x_0))} + \|p\|_{L^\infty(\Omega_{R/2}(x_0))}
\]
\[
\leq CR^{-d} \left( \|Du\|_{L^1(\Omega_R(x_0))} + \|p\|_{L^1(\Omega_R(x_0))} \right) + CR \|f\|_{L^\infty(\Omega_R(x_0))}, \tag{A.24}
\]
and
\[
\|u\|_{L^\infty(\Omega_{R/2}(x_0))} \leq CR^{-d} \|u\|_{L^1(\Omega_{R/2}(x_0))}
\]
\[
+ CR^{1-d} \left( \|Du\|_{L^1(\Omega_R(x_0))} + \|p\|_{L^1(\Omega_R(x_0))} \right) + CR^2 \|f\|_{L^\infty(\Omega_R(x_0))},
\]
where $C = C(d, \lambda, \omega_A)$.

**Proof.** With a standard covering argument, we only need to prove the desired estimates with $R/36$ in place of $R/2$ on the left-hand sides. In this proof, we denote $B_r^+(x) = \Omega_r(x)$ and $B_r^+ = B_r^+(0)$ for any $x \in \mathbb{R}^d$ and $r > 0$. By the Poincaré inequality (A.15) with the zero extension of $u$, we only need to prove (A.24). Let $z = (z_1, z') \in B_{R/2}^+(x_0)$. We consider the following two cases:
\[
dist(z, \partial \mathbb{R}^d) > R/18, \quad \dist(z, \partial \mathbb{R}^d) \leq R/18.
\]
i. \( \text{dist}(z, \partial R^d_+) > R/18 \): In this case, since it holds that
\[
B^+_{R/18}(z) = B_{R/18}(z) \subset B^+_{R}(x_0),
\]
we get from Lemma [A.1] that
\[
\|D u\|_{L^\infty(B^+_{R/36}(z))} + \|p\|_{L^\infty(B^+_{R/36}(z))} \leq CR^{-d} \left( \|D u\|_{L^1(B^+_{R/4}(x_0))} + \|p\|_{L^1(B^+_{R/4}(x_0))} \right) + CR\|f\|_{L^\infty(B^+_{R}(x_0))},
\]
where \( C = C(d, \lambda, \omega_A) \).

ii. \( \text{dist}(z, \partial R^d_+) \leq R/18 \): Without loss of generality, we assume that \( z' = 0' \). To complete the proof of the lemma, it suffices to prove that
\[
\|D u\|_{L^\infty(B^+_{R/12})} + \|p\|_{L^\infty(B^+_{R/12})} \leq CR^{-d} \left( \|D u\|_{L^1(B^+_{R/4})} + \|p\|_{L^1(B^+_{R/4})} \right) + CR\|f\|_{L^\infty(B^+_{R}(x_0))},
\]
where \( C = C(d, \lambda, \omega_A) \), because (A.21) follows from (A.25), (A.26), and the fact that
\[
B^+_{R/36}(z) \subset B^+_{R/12} \subset B^+_{R/4} \subset B^+_{R}(x_0).
\]
Let us set \( R_1 = R/4 \) and fix \( q > d \). By the same reasoning as in (A.3), \((u, p)\) satisfies
\[
\begin{cases}
Lu + \nabla p = D_\alpha f_\alpha & \text{in } B^+_{R_1}, \\
\text{div } u = 0 & \text{in } B^+_{R_1}, \\
u = 0 & \text{on } B_{R_1} \cap \partial R^d_+,
\end{cases}
\]
where \( f_\alpha \in \tilde{W}^{1,q}(B^+_{R_1}) \) satisfy
\[
\|f_\alpha\|_{L^\infty(B^+_{R_1})} + R_1^{-d/q} |f_\alpha|_{C^{1-\delta/q}(B^+_{R_1})} \leq C(d, q) R_1 \|f\|_{L^\infty(B^+_{R_1})}.
\]
We denote
\[
\hat{u}(y) = u(x), \quad \hat{p}(y) = p(x), \quad \hat{A}^{\alpha\beta}(y) = A^{\alpha\beta}(x), \quad \hat{f}_\alpha(y) = f_\alpha(x),
\]
where \( y = 6x/R_1 \), and observe that
\[
\begin{cases}
D_\alpha (\hat{A}^{\alpha\beta} D_\beta \hat{u}) + \nabla (R_1 \hat{p}/6) = D_\alpha (R_1 \hat{f}_\alpha / 6) & \text{in } B^+_6, \\
\text{div } \hat{u} = 0 & \text{in } B^+_6, \\
\hat{u} = 0 & \text{on } B_6 \cap \partial R^d_+.
\end{cases}
\]
Fix a constant \( \gamma \) satisfying \( 1 - d/q < \gamma < 1 \), and let \( \kappa = \kappa(d, \lambda, \gamma) \in (0, 1/2) \) be the constant from [3] Lemma 7.1. For \( r \in (0, 1] \) and \( h \in L^1(B^+_6) \), we denote
\[
\omega_{h, y'}(r) = \sup_{y_0 \in B^+_6} \int_{B^+_6(y_0)} \left| h(y) - \int_{B^+_6(y_0)} h(y_1, z') \, dz' \right| \, dy,
\]
\[
\tilde{\omega}_{h, y'}(r) = \sum_{i=1}^\infty \kappa^{ri} \left( \omega_{h, y'}(\kappa^{-i} r) |\kappa^{-i} r < 1| + \omega_{h, y'}(1) |\kappa^{-i} r \geq 1| \right),
\]
\[
\omega_{h, y'}^\gamma(r) = \sup_{r \leq r_0 \leq 1} \left( \frac{r}{r_0} \right) \gamma \tilde{\omega}_{h, y'}(r_0).
\]
By (A.27), we obtain that (using \(1 - d/q < \gamma < 1\))

\[
R_1 \| \hat{f} \|_{L^\infty(B_1^+)} + \int_0^1 \frac{\omega_{R_1, f_{t, y'}}(t)}{t} \, dt \leq C R_1^2 \| f \|_{L^\infty(B_{R_1}^+)},
\]

(A.29)

where \(C = C(d, q, \gamma)\). Since \((\hat{u}, \hat{p})\) satisfies (A.28), we obtain by [4, Eq. (7.6)] that

\[
\|[D\hat{u}]\|_{L^\infty(B_1^+)} + R_1 \| \hat{p} \|_{L^\infty(B_1^+)}
\]

\[
\leq C \left( \|[D\hat{u}]\|_{L^1(B_1^+)} + R_1 \| \hat{p} \|_{L^1(B_1^+)} + R_1 \| \hat{f} \|_{L^\infty(B_1^+)} + \int_0^1 \frac{\omega_{R_1, f_{t, y'}}(t)}{t} \, dt \right),
\]

where \(C = C(d, \lambda, \omega_0, q, \gamma)\) and \(\omega_0 : (0, 1] \rightarrow [0, \infty)\) is a function such that

\[
\int_0^r \frac{\omega_{R_1, f_{t, y'}}(t)}{t} \, dt \leq \int_0^r \frac{\omega_0(t)}{t} \, dt < \infty \quad \text{for all } r \in (0, 1].
\]

Therefore, by (A.29) and the change of variables, we get

\[
\|[D\hat{u}]\|_{L^\infty(B_{R_1/3}^+)} + \|p\|_{L^\infty(B_{R_1/3}^+)}
\]

\[
\leq C R_1^{-\alpha} \left( \|[D\hat{u}]\|_{L^1(B_{R_1/3}^+)} + \|p\|_{L^1(B_{R_1/3}^+)} + C R_1 \|f\|_{L^\infty(B_{R_1}^+)},
\]

where \(C = C(d, \lambda, \omega_0, q, \gamma) = C(d, \lambda, \omega_A)\).

To complete the proof of (A.26), it remains to show that \(\omega_0\) can be derived from \(\omega_A\). We set

\[
\tilde{\omega}_A(r) = \sum_{i=1}^{\infty} \kappa^i \left( \omega_A(\kappa^{-i} r) [\kappa^i r < 1] + \omega_A(1) [\kappa^i r \geq 1] \right),
\]

\[
\omega_A^\delta(r) = \sup_{r \leq r_0 \leq 1} \left( \frac{r}{r_0} \right)^{\gamma} \omega_A(r_0),
\]

and observe that (see (A.11))

\[
\tilde{\omega}_{A^\delta, y'}(r) \leq \tilde{\omega}_A(r R_1/6) + C_0 r^\gamma \int_0^1 \frac{\omega_A(t)}{t} \, dt,
\]

where \(C = C(d, \lambda, \omega_A)\). From the above inequality it follows that

\[
\omega_{A^\delta, y'}^\delta(r) \leq \omega_{A^\delta, y'}^\delta(r R_1/6) + C_0 r^\gamma \int_0^1 \frac{\omega_A(t)}{t} \, dt.
\]

Therefore, for any \(r \in (0, 1]\), we have

\[
\int_0^r \frac{\omega_{A^\delta, y'}(t)}{t} \, dt \leq \int_0^r \frac{\omega_{A}(t R_1/6)}{t} \, dt + C_0 \int_0^r \frac{t^\gamma}{t} \, dt \cdot \int_0^1 \frac{\omega_A(t)}{t} \, dt
\]

\[
\leq \int_0^r \frac{\omega_0(t)}{t} \, dt < \infty,
\]

where we set

\[
\omega_0(r) = \omega_A^\delta(r) + C_0 r^\gamma \int_0^1 \frac{\omega_A(t)}{t} \, dt.
\]

The lemma is proved. \(\square\)
REFERENCES

[1] Hammadi Abidi, Guilong Gui, and Ping Zhang. On the decay and stability of global solutions to the 3D inhomogeneous Navier-Stokes equations. *Comm. Pure Appl. Math.*, 64(6):832–881, 2011.

[2] Gabriel Acosta, Ricardo G. Durán, and María A. Muschietti. Solutions of the divergence operator on John domains. *Adv. Math.*, 206(2):373–401, 2006.

[3] Jongkeun Choi and Hongjie Dong. Gradient estimates for Stokes systems in domains. *Dyn. Partial Differ. Equ.*, 16(1):1–24, 2019.

[4] Jongkeun Choi and Hongjie Dong. Gradient estimates for Stokes systems with Dini mean oscillation coefficients. *J. Differential Equations*, 266(8):4451–4509, 2019.

[5] Jongkeun Choi, Hongjie Dong, and Doyoon Kim. Green functions of conormal derivative problems for Stokes system. *J. Math. Fluid Mech.*, 20(4):1745–1769, 2018.

[6] Jongkeun Choi, Hongjie Dong, and Doyoon Kim. Conormal derivative problems for stationary Stokes system in Sobolev spaces. *Discrete Contin. Dyn. Syst.*, 38(5):2349–2374, 2018.

[7] Jongkeun Choi and Doyoon Kim. Estimates for Green functions of Stokes systems in two dimensional domains. *J. Math. Anal. Appl.*, 471(1-2):102–125, 2019.

[8] Jongkeun Choi and Ki-Ahm Lee. The Green function for the Stokes system with measurable coefficients. *Commun. Pure Appl. Anal.*, 16(6):1989–2022, 2017.

[9] Jongkeun Choi and Minsuk Yang. Fundamental solutions for stationary Stokes systems with measurable coefficients. *J. Differential Equations*, 263(7):3854–3893, 2017.

[10] Hongjie Dong. Gradient estimates for parabolic and elliptic systems from linear laminates. *Arch. Ration. Mech. Anal.*, 205(1):119–149, 2012.

[11] Hongjie Dong, Luis Escauriaza, and Seick Kim. On $C^1, C^2$, and weak type-(1,1) estimates for linear elliptic operators: part II. *Math. Ann.*, 370(1-2):447–489, 2018.

[12] Hongjie Dong and Doyoon Kim. Weighted $L_q$-estimates for stationary Stokes system with partially BMO coefficients. *J. Differential Equations*, 264(7):4603–4649, 2018.

[13] Hongjie Dong and Doyoon Kim. $L_q$-estimates for stationary Stokes system with coefficients measurable in one direction. *Bull. Math. Sci.*, (to appear), https://doi.org/10.1007/s13373-018-0120-6.

[14] Giovanni Paolo Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations*. Springer Monographs in Mathematics. Springer, New York, second edition, 2011.

[15] Mariano Giaquinta. *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, volume 105 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1983.

[16] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, Reprint of the 1998 edition, 2001.

[17] Shu Gu and Jinping Zhuge. Periodic homogenization of Green’s functions for Stokes systems. arXiv:1710.05383.

[18] Steve Hofmann and Seick Kim. The Green function estimates for strongly elliptic systems of second order. *Manuscripta math.*, 124(2):139–172, 2007.

[19] Kyungsun Kang and Seick Kim. Global pointwise estimates for Green’s matrix of second order elliptic systems. *J. Differential Equations*, 249(11):2643–2662, 2010.

[20] O’ga Aleksandrovna Ladyženskaja and Vsevolod Alekseevich Solonnikov. The unique solvability of an initial-boundary value problem for viscous incompressible inhomogeneous fluids. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 52:52–109, 218–219, 1975.

[21] O’ga Aleksandrovna Ladyzhenskaya. *The mathematical theory of viscous incompressible flow*. Second English edition, revised and enlarged. Translated from the Russian by Richard A. Silverman and John Chu. Mathematics and its Applications, Vol. 2. Gordon and Breach, Science Publishers, New York-London-Paris, 1969.

[22] Pierre-Louis Lions. *Mathematical topics in fluid mechanics. Vol. 1*, volume 3 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, New York, 1996.

[23] Jan Malý and William P. Ziemer. *Fine regularity of solutions of elliptic partial differential equations*, volume 51 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997.
[24] Vladimir Gilelevich Maz’ya and Boris Plamenevskii. The first boundary value problem for classical equations of mathematical physics in domains with piecewise-smooth boundaries. I. Z. Anal. Anwendungen, 2(4):335–359, 1983.

[25] Vladimir Gilelevich Maz’ya and Boris Plamenevskii. The first boundary value problem for classical equations of mathematical physics in domains with piecewise smooth boundaries. II. Z. Anal. Anwendungen, 2(6):523–551, 1983.

[26] Vladimir Gilelevich Maz’ya and Jürgen Rossmann. Pointwise estimates for Green’s kernel of a mixed boundary value problem to the Stokes system in a polyhedral cone. Math. Nachr., 278(2005):1766–1810, 2005.

[27] Dorina Mitrea and Irina Mitrea. On the regularity of Green functions in Lipschitz domains. Comm. Partial Differential Equations, 36(2):304–327, 2011.

[28] Katharine A. Ott, Seick Kim, and Russell Murray Brown. The Green function for the mixed problem for the linear Stokes system in domains in the plane. Math. Nachr., 288(4):452–464, 2015.

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