Abstract. We investigate the existence of least energy solutions and infinitely many solutions for the following nonlinear fractional equation
\[ (-\Delta)^s u = g(u) \text{ in } \mathbb{R}^N, \]
where \( s \in (0,1), \ N \geq 2, \ (-\Delta)^s \) is the fractional Laplacian and \( g : \mathbb{R} \to \mathbb{R} \) is an odd \( C^{1,\alpha} \) function satisfying Berestycki-Lions type assumptions. The proof is based on the symmetric mountain pass approach developed by Hirata, Ikoma and Tanaka in [32]. Moreover, by combining the mountain pass approach and an approximation argument, we also prove the existence of a positive radially symmetric solution for the above problem when \( g \) satisfies suitable growth conditions which make our problem fall in the so called “zero mass” case.

1. Introduction

In this paper we deal with the following fractional nonlinear scalar field equation with fractional diffusion
\[ (-\Delta)^s u = g(u) \text{ in } \mathbb{R}^N \tag{1.1} \]
with \( s \in (0,1), \ N \geq 2, \) and \( (-\Delta)^s \) is the fractional Laplacian which may be defined along a function \( u \) belonging to the Schwartz space \( \mathcal{S}(\mathbb{R}^N) \) of rapidly decaying functions as
\[ (-\Delta)^s u(x) = C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \, dy \quad (x \in \mathbb{R}^N). \]
The symbol \( P.V. \) stands for the Cauchy principal value and \( C_{N,s} \) is a normalizing constant whose value can be found in [23].

In the last decade, a great attention has been devoted to the study of elliptic equations involving fractional powers of the Laplacian. This type of operators appears in a quite natural way in many different contexts, such as, the thin obstacle problem, optimization, finance, phase transitions, anomalous diffusion, crystal dislocation, soft thin films, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, minimal surfaces, materials science and water waves. Since we cannot review the huge literature here, we refer the interested reader to [23, 43], where a more extensive bibliography and an introduction to this topic are given.

Equation (1.1) arises in the study of standing wave solutions for the fractional Schrödinger equation
\[ \frac{\partial \psi}{\partial t} = (-\Delta)^s \psi(t, x) + g(\psi) \text{ in } \mathbb{R}^N \times \mathbb{R}. \]
when looking for standing waves, that is, solutions having the form \( \psi(x, t) = e^{-\omega t}u(x) \), where \( \omega \) is a constant. Such equation is fundamental in fractional quantum mechanics, and it has been introduced by Laskin [38, 39] as a result of expanding the Feynman path integral, from the Brownian like to the Lévy like quantum mechanical paths.

Recently, the study of the fractional Schrödinger equation has attracted a lot of interest from many mathematicians. Felmer et al. [26] studied the existence, regularity and qualitative properties of positive solutions to \((-\Delta)^su + u = f(x, u)\) when \( f \) has a subcritical growth and satisfies the Ambrosetti-Rabinowitz condition; see also [24]. Coti Zelati and Nolasco [19, 20] investigated positive stationary solutions for a class of nonlinear pseudo-relativistic Schrödinger equations involving the operator \( \sqrt{-\Delta + m^2} \) with \( m > 0 \); see also [5, 6, 25, 33] for related models. Secchi [46, 47] proved several existence results for some nonlinear fractional Schrödinger equations under the assumptions that the nonlinearity is either of perturbative type or satisfies the Ambrosetti-Rabinowitz condition. Molica Bisci and Rădulescu [42] obtained the existence of multiple ground state solutions for a class of parametric fractional Schrödinger equations. Frank et al. [29] established uniqueness and nondegeneracy of ground state solutions to \((-\Delta)^s u + u = |u|^\alpha u\) for all \( H^\alpha \)-admissible powers \( \alpha \in (0, \alpha^*) \). Pucci et al. [44] dealt with the existence of multiple solutions for the nonhomogeneous fractional \( p \)-Laplacian equations of Schrödinger-Kirchhoff type, by using the Ekeland variational principle and the Mountain Pass theorem. Figueiredo and Siciliano [28] investigated existence and multiplicity of positive solutions for a class of fractional Schrödinger equation by means of the Ljusternick-Schnirelmann and Morse theory.

In the fundamental papers [12, 13], Berestycki and Lions studied the existence and the multiplicity of nontrivial solutions to (1.1) when \( s = 1 \), that is

\[-\Delta u = g(u) \quad \text{in} \quad \mathbb{R}^N,
\]

where \( N \geq 3 \) and \( g : \mathbb{R} \to \mathbb{R} \) is an odd continuous function such that

(BL1) \(-\infty < \liminf_{t \to 0^+} \frac{g(t)}{t} \leq \limsup_{t \to 0^+} \frac{g(t)}{t} = -m < 0\);

(BL2) \(-\infty < \limsup_{t \to +\infty} \frac{g(t)}{t^{N+2}} \leq 0\);

(BL3) There exists \( \xi_0 > 0 \) such that \( G(\xi_0) = \int_0^{\xi_0} g(\tau)d\tau > 0 \).

To obtain the existence of a positive solution to (1.2), the authors in [12] developed a subtle Lagrange multiplier procedure which ultimately relies on the Pohozaev’s identity for (1.2). The lack of compactness due to the translational invariance of (1.2) is regained working in the subspace of \( H^1(\mathbb{R}^N) \) of radially symmetric functions. They also proved the existence of a positive solution to (1.2) when \( m = 0 \), the so called zero mass case.

In [13] the authors showed that (1.2) possesses infinitely many distinct bound states by applying the genus theory to the even functional \( V(u) = \int_{\mathbb{R}^N} G(u) \, dx \) defined on the symmetric manifold \( M = \{ u \in H^1_{\text{rad}}(\mathbb{R}^N) : \| \nabla u \|_{L^\infty(\mathbb{R}^N)} = 1 \} \). Subsequently, the existence and multiplicity of solutions of closely related problems to (1.2) have been extensively studied by many authors; see for instance [2, 9, 14, 36, 37].

The question that naturally arises is whether or not the above classical existence and multiplicity results for equation (1.2) still hold in the nonlocal framework of (1.1). Chang and Wang [18] obtained the existence of a positive solution to (1.1) by combining the monotonicity trick of Struwe-Jeanjean [35, 48] and the Pohozaev identity for the fractional Laplacian.
The first aim of this paper is to answer the question with respect to the existence of infinitely many solutions for the equation (1.1). We also provide a mountain-pass characterization of least energy solutions to (1.1) in the spirit of the result obtained in [37]. In this way, we are able to complement and improve the result proved in [18]. We recall that $u$ is a least energy solution to (1.1) if and only if

$$ I(u) = m$$

and $m = \inf \left\{ I(u); u \in H^s(\mathbb{R}^N) \setminus \{0\} \text{ and } I'(u) = 0 \right\}$, where $I : H^s(\mathbb{R}^N) \to \mathbb{R}$ is the energy functional associated to (1.1), that is

$$ I(u) = \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx dy - \int_{\mathbb{R}^N} G(u) \, dx. $$

Now we state the main assumptions on the nonlinearity $g$. We will assume that $g \in C^{1,\alpha}(\mathbb{R}, \mathbb{R})$, with $\alpha > \max\{0, 1 - 2s\}$, is odd and satisfies the following properties

$(g1)$ $-\infty < \liminf_{t \to 0^+} \frac{g(t)}{t} \leq \limsup_{t \to 0^+} \frac{g(t)}{t} = -m < 0$;

$(g2)$ $-\infty < \limsup_{t \to +\infty} \frac{g(t)}{t^{2s-1}} \leq 0$, where $2^*_s = \frac{2N}{N - 2s}$;

$(g3)$ There exists $\xi_0 > 0$ such that $G(\xi_0) = \int_0^{\xi_0} g(\tau) \, d\tau > 0$.

We note that differently from [12, 13], we don’t require that $g$ is simply continuous, but a higher regularity is needed to obtain a classical solution of (1.1); see [16].

Under the assumptions (g1)-(g3), problem (1.1) has a variational nature, so its weak solutions can be found as critical points of the energy functional $I(u)$. We recall that for a weak solution of problem (1.1), we mean a function $u \in H^s(\mathbb{R}^N)$ such that

$$ \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))}{|x-y|^{N+2s}} (\varphi(x) - \varphi(y)) \, dx dy = \int_{\mathbb{R}^N} g(u) \varphi \, dx $$

for any $\varphi \in H^s(\mathbb{R}^N)$.

Our first main result is the following:

**Theorem 1.** Let $s \in (0, 1)$ and $N \geq 2$ and $g \in C^{1,\alpha}(\mathbb{R}, \mathbb{R})$, with $\alpha > \max\{0, 1 - 2s\}$, be an odd function satisfying (g1)-(g3). Then (1.1) possesses a positive least energy solution and infinitely many (possibly sign-changing) radially symmetric solutions $(u_n)_{n \in \mathbb{N}}$ such that $I(u_n) \to \infty$ as $n \to \infty$. Moreover, these solutions are characterized by a mountain pass and symmetric mountain-pass arguments respectively.

It is worth noting that, a common approach to deal with fractional nonlocal problems, is to use the Caffarelli-Silvestre extension [17], which consists to realize a given nonlocal problem through a degenerate local problem in one more dimension by a Dirichlet to Neumann map. Anyway, in this work, we prefer to investigate the problem directly in $H^s(\mathbb{R}^N)$ in order to adapt some useful methods and arguments developed in [15, 32, 37].

We would like to observe that our result is in clear accordance with that for the classical local counterpart. For this reason, Theorem 1 can be seen as the fractional version of the existence and multiplicity results given in [12, 13, 32].

In the second part of this paper, we use the mountain pass approach in [32, 37], to prove the existence of a positive solution to (1.1) in the null mass case, that is when $g$ satisfies the following properties
\((h_1)\) \(\limsup_{t \to 0^+} \frac{g(t)}{t^{2s-1}} \leq 0\) where \(2^*_s = \frac{2N}{N-2s}\);

\((h_2)\) \(\lim_{t \to +\infty} \frac{g(t)}{t^{2s-1}} = 0\);

\((h_3)\) There exists \(\xi_0 > 0\) such that \(G(\xi_0) = \int_0^{\xi_0} g(\tau) d\tau > 0\).

We point out that the main difficulty related to the zero mass case is due to the fact that the energy of solutions of (1.1) can be infinite. In the local setting, that is \(s = 1\), several results for zero mass problems have been established in [3, 4, 8, 10, 11, 12, 48]. We also recall that from a physical point of view, this type of problem is related to Yang-Mills equations, see for instance [30, 31]. However, differently from the classic literature, as far as we know, there is only one work [7] concerning with zero-mass problems in non-local setting. Motivated by it, here we would like to go further in this direction, by studying (1.1) when \(g\) is a general nonlinearity such that \(g'(0) = 0\).

Our main second result can be stated as follows:

**Theorem 2.** Let \(s \in (0, 1)\) and \(N \geq 2\) and \(g \in C^{1,\alpha}(\mathbb{R}, \mathbb{R})\) be an odd function satisfying \((h_1)-(h_3)\). Then (1.1) possesses a positive radially decreasing solution.

The proof of the above Theorem 2, is obtained by combining the mountain pass approach and an approximation argument. Indeed, we show that a solution of (1.1) can be approximated by a sequence of positive radially symmetric solutions \((u_\varepsilon)\) in \(H^s(\mathbb{R}^N)\), each one solves an approximate “positive mass” problem \((-\Delta)^s u = g(u) - \varepsilon u\) in \(\mathbb{R}^N\). Taking into account the mountain-pass characterization of least energy solution of (1.1) given in Theorem 1, we are capable to obtain lower and upper bounds for the mountain pass critical levels \(b^\varepsilon_{mp}\), which can be estimated independently from \(\varepsilon\), when \(\varepsilon\) is sufficiently small. This allows us to pass to the limit in \((-\Delta)^s u_\varepsilon = g(u_\varepsilon) - \varepsilon u_\varepsilon\) in \(\mathbb{R}^N\) as \(\varepsilon \to 0\), and to find a nontrivial solution \(u\) to (1.1). It is worth to point out that our proof is different from the ones given in [12], and that it works even when \(s = 1\).

To our knowledge all results presented here are new. The paper is organized as follows. Section 2 contains some preliminary results about fractional Sobolev spaces and functional setting. In Section 3 we prove the existence of infinitely many solutions to (1.1) by using an auxiliary functional \(\tilde{I}(\theta, u)\) on the augmented space \(\mathbb{R} \times H^s_{rad}(\mathbb{R}^N)\). In Section 4 we obtain the existence of a least energy solution to (1.1) characterized by a mountain-pass argument, and we investigate regularity and symmetry of this solution. Finally, by using an approximation argument, we get a positive radially symmetric positive solution to (1.1) in the null mass case.

2. Preliminaries and functional setting

In this section we collect some preliminary results which will be useful along the paper.

2.1. Fractional Sobolev spaces. We recall some useful facts of the fractional Sobolev spaces; for more details we refer to [23].

For any \(s \in (0, 1)\) we denote by \(\mathcal{D}^{s,2}(\mathbb{R}^N)\) the completion of \(C^\infty_0(\mathbb{R}^N)\) with respect to the so-called Gagliardo seminorm

\[ [u]^{2}_{H^s(\mathbb{R}^N)} = \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy. \]
That is
\[ \mathcal{D}^{s,2}(\mathbb{R}^N) = \{ u \in L^{2^*_s}(\mathbb{R}^N) : [u]_{H^s(\mathbb{R}^N)} < \infty \}, \]
where
\[ 2^*_s = \frac{2N}{N-2s} \]
is the critical Sobolev exponent. We also use the notation
\[ \langle u, v \rangle_{\mathcal{D}^{s,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} \, dx \, dy \]
for all \( u, v \in \mathcal{D}^{s,2}(\mathbb{R}^N) \). Now, we define the standard fractional Sobolev space
\[ H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \frac{|u(x) - u(y)|}{|x-y|^{N+2s}} \in L^2(\mathbb{R}^{2N}) \right\} \]
endowed with the natural norm
\[ \| u \|_{H^s(\mathbb{R}^N)} = \sqrt{[u]_{H^s(\mathbb{R}^N)}^2 + \| u \|_{L^2(\mathbb{R}^N)}^2} \]
Let us denote by
\[ \langle u, v \rangle_{H^s(\mathbb{R}^N)} = \langle u, v \rangle_{\mathcal{D}^{s,2}(\mathbb{R}^N)} + \int_{\mathbb{R}^N} uv \, dx \]
for any \( u, v \in H^s(\mathbb{R}^N) \). For the reader’s convenience, we review the main embedding result for this class of fractional Sobolev spaces.

**Theorem 3.** Let \( s \in (0, 1) \) and \( N > 2s \). Then \( H^s(\mathbb{R}^N) \) is continuously embedded in \( L^q(\mathbb{R}^N) \) for any \( q \in [2, 2^*_s] \) and compactly in \( L^q_{\text{loc}}(\mathbb{R}^N) \) for any \( q \in [2, 2^*_s) \).

Now we introduce
\[ H^s_{\text{rad}}(\mathbb{R}^N) = \{ u \in H^s(\mathbb{R}^N) : u(x) = u(|x|) \} \]
the space of radial functions in \( H^s(\mathbb{R}^N) \). We recall the following compactness result due to Lions [41]:

**Theorem 4.** [41] Let \( s \in (0, 1) \) and \( N \geq 2 \). Then \( H^s_{\text{rad}}(\mathbb{R}^N) \) is compactly embedded in \( L^q(\mathbb{R}^N) \) for any \( q \in (2, 2^*_s) \).

Finally, we recall the following useful lemmas:

**Lemma 1.** [12] Let \( u \in L^t(\mathbb{R}^N) \), \( 1 \leq t < \infty \) be a nonnegative radially decreasing function (that is \( 0 \leq u(x) \leq u(y) \) if \( |x| \geq |y| \)). Then
\[ |u(x)| \leq \left( \frac{N}{\omega_{N-1}} \right)^{\frac{1}{t}} |x|^{-\frac{N}{t}} \| u \|_{L^t(\mathbb{R}^N)} \text{ for any } x \in \mathbb{R}^N \setminus \{0\}, \quad (2.1) \]
where \( \omega_{N-1} \) is the Lebesgue measure of the unit sphere in \( \mathbb{R}^N \).

**Lemma 2.** [9, 12] Let \( P \) and \( Q : \mathbb{R} \to \mathbb{R} \) be a continuous functions satisfying
\[ \lim_{t \to \pm \infty} \frac{P(t)}{Q(t)} = 0. \]
Let \((v_k), v\) and \(w\) be measurable functions from \(\mathbb{R}^N\) to \(\mathbb{R}\), with \(w\) bounded, such that
\[
\sup_k \int_{\mathbb{R}^N} |Q(v_k(x))w| \, dx < +\infty,
\]
\[P(v_k(x)) \to v(x) \text{ a.e. in } \mathbb{R}^N.
\]
Then \(\|(P(v_k) - v)w\|_{L^1(\mathcal{B})} \to 0\), for any bounded Borel set \(\mathcal{B}\).
Moreover, if we have also
\[
\lim_{t \to 0} P(t) = 0,
\]
and
\[
\lim_{|x| \to \infty} \sup_{k \in \mathbb{N}} |v_k(x)| = 0,
\]
then \(\|(P(v_k) - v)w\|_{L^1(\mathbb{R}^N)} \to 0\).

**Lemma 3.** [18] Let \((X, \| \cdot \|)\) be a Banach space such that \(X\) is embedded continuously and compactly into \(L^q(\mathbb{R}^N)\) for \(q \in [q_1, q_2]\) and \(q \in (q_1, q_2)\) respectively, where \(q_1, q_2 \in (0, \infty)\).
Assume that \((u_k) \subset X, u : \mathbb{R}^N \to \mathbb{R}\) is a measurable function and \(P \in C(\mathbb{R}, \mathbb{R})\) is such that
\[
(i) \lim_{|t| \to 0} \frac{P(t)}{|t|^{q_1}} = 0,
\]
\[
(ii) \lim_{|t| \to \infty} \frac{P(t)}{|t|^{q_2}} = 0,
\]
\[
(iii) \sup_{k \in \mathbb{N}} \|u_k\| < \infty,
\]
\[
(iv) \lim_{k \to \infty} P(u_k(x)) = u(x) \text{ for a.e. } x \in \mathbb{R}^N.
\]
Then, up to a subsequence, we have
\[
\lim_{k \to \infty} P(u_k) - u\|_{L^1(\mathbb{R}^N)} = 0.
\]

### 3. Infinitely many solutions to (1.1)

#### 3.1. The modification of \(g\) and introduction of a penalty function.
In order to give the proof of Theorem 1, we redefine the nonlinearity \(g\) as follows:

(i) If \(g(t) > 0\) for all \(t \geq \xi_0\), we simply extend \(g\) to the negative axis:
\[
\tilde{g}(t) = \begin{cases} 
g(t) & \text{for } t \geq 0 \\
-g(-t) & \text{for } t < 0;
\end{cases}
\]

(ii) If there exists \(t_0 > \xi_0\) such that \(g(t_0) = 0\), we put
\[
\tilde{g}(t) = \begin{cases} 
g(t) & \text{for } t \in [0, t_0] \\
0 & \text{for } t > t_0 \\
-g(-t) & \text{for } t < 0.
\end{cases}
\]

Then \(\tilde{g}\) satisfies \((g1), (g3)\) and
\[
\lim_{t \to \infty} \frac{\tilde{g}(t)}{|t|^\frac{N+2}{N-2s}} = 0. \tag{g2'}
\]

Moreover, by the weak maximum principle [16], any solution to \((-\Delta)^s u = \tilde{g}(u)\) in \(\mathbb{R}^N\) is also a solution to (1.1). Indeed, in the case (ii) above, any solution to \((-\Delta)^s u = \tilde{g}(u)\) in \(\mathbb{R}^N\)
satisfies \(-t_0 \leq u(x) \leq t_0\) for any \(x \in \mathbb{R}^N\). To prove this, we use \((u - t_0)^+ \in H^s(\mathbb{R}^N)\) as test function in the weak formulation of \((-\Delta)^s u = \tilde{g}(u)\) in \(\mathbb{R}^N\), and we get

\[
\iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))}{|x - y|^{N+2s}}((u(x) - t_0)^+ - (u(y) - t_0)^+) \, dx \, dy = \int_{\mathbb{R}^N} \tilde{g}(u)(u(x) - t_0)^+ \, dx.
\]

From the definition of \(\tilde{g}\), it follows that \(\int_{\mathbb{R}^N} \tilde{g}(u)(u(x) - t_0)^+ \, dx \leq 0\). Then, recalling that \((x-y)(x^+-y^+) \geq |x^+-y^+|^2\) for all \(x,y \in \mathbb{R}\), we can see that

\[
\iint_{\mathbb{R}^{2N}} \frac{|(u(x) - t_0)^+ - (u(y) - t_0)^+|^2}{|x - y|^{N+2s}} \, dx \, dy \leq 0,
\]

which implies that \([((u - t_0)^+)_H^s(\mathbb{R}^N) \leq 0\). Since \(u \in H^s(\mathbb{R}^N)\), we deduce that \(u \leq t_0\) in \(\mathbb{R}^N\). The other inequality is obtained in similar way by using \((u + t_0)^-\) as test function. Here we used the notations \(x^+ = \max\{x,0\}\) and \(x^- = \min\{x,0\}\).

Therefore, from now on, we will tacitly write \(g\) instead of \(\tilde{g}\), and we will assume that \(g\) satisfies \((g1)\), \((g2')\) and \((g3)\).

Now we introduce a penalty function to construct an auxiliary function. For \(t \geq 0\) we define

\[
f(t) = \max\left\{0, \frac{1}{2}mt + g(t)\right\}
\]

and

\[
h(t) = t^p \sup_{0 < \tau \leq t} \frac{f(\tau)}{\tau^p}
\]

where \(p\) is a positive number such that \(1 < p < \frac{N+2s}{N-2s}\).

Note that \(f\) and \(h\) are well defined in view of \((g1)\). We extend \(h\) as an odd function on \(\mathbb{R}\) and we set

\[
H(t) = \int_0^t h(\tau) \, d\tau.
\]

Next, we prove the following result whose proof follows the lines in [32]. For reader’s convenience we give the proof here.

**Lemma 4.** The above function \(h\) satisfies the following properties:

\((h1)\) \(h \in C(\mathbb{R}, \mathbb{R})\), \(h(t) \geq 0\) and \(h(-t) = -h(t)\) for all \(t \geq 0\).

\((h2)\) There exists \(\beta > 0\) such that \(h = 0 = H\) on \([-\beta, \beta]\).

\((h3)\) For all \(t \in \mathbb{R}\)

\[
\frac{1}{2}mt^2 + g(t)t \leq h(t)t \quad \text{and} \quad \frac{1}{4}mt^2 + G(t) \leq H(t).
\]

\((h4)\) \(\lim_{|t| \to \infty} \frac{h(t)}{t^{N+2s}} = 0\).

\((h5)\) \(h\) satisfies the following Ambrosetti-Rabinowitz condition:

\[
0 \leq (p + 1)H(t) \leq h(t)t \quad \text{for all} \ t \in \mathbb{R}.
\]

\((h6)\) If \((u_k)\) is a bounded sequence in \(H^s_{rad}(\mathbb{R}^N)\) then

\[
\lim_{k \to \infty} \int_{\mathbb{R}^N} h(u_k)u_k \, dx = \int_{\mathbb{R}^N} h(u)u \, dx.
\]
Proof. Let us observe that (h1), (h2) and (h3) follow by (g1) and the definition of \( f, h \) and \( H \).

(h4) Firstly, we remark that

\[
\frac{h(t)}{t^{2s-1}} = t^{-2s+1-p} \sup_{0<\tau \leq t} \frac{f(\tau)}{\tau^p} = \sup_{0<\tau \leq t} \frac{f(\tau)}{\tau^{2s-1-p}}
\]

Since \( f \) satisfies \((g2')\), for any \( \varepsilon > 0 \) there exists \( t_\varepsilon > 0 \) such that

\[
\frac{|f(\tau)|}{\tau^{2s-1}} < \varepsilon \text{ for all } \tau \geq t_\varepsilon.
\]

Set \( C_\varepsilon = \sup_{0<\tau \leq t_\varepsilon} \frac{|f(\tau)|}{\tau^{2s-1}} \). Then we have

\[
\frac{h(t)}{t^{2s-1}} \leq \max \left\{ \sup_{0<\tau \leq t_\varepsilon} \frac{|f(\tau)|}{\tau^{2s-1}}, \sup_{t_\varepsilon \leq \tau \leq t} \frac{|f(\tau)|}{\tau^{2s-1}} \right\}
\]

so we deduce that

\[
0 \leq \limsup_{t \to +\infty} \frac{h(t)}{t^{2s-1}} \leq \varepsilon.
\]

(h5) By the definition of \( h \) and \( H \) we have

\[
(p+1)H(t) - h(t) = \int_0^t \left[ (p+1)h(\tau) - h(t) \right] d\tau
\]

\[
= \int_0^t \left[ (p+1)\tau^p \sup_{0<\xi \leq \tau} \frac{f(\xi)}{\xi^p} - t^p \sup_{0<\xi \leq t} \frac{f(\xi)}{\xi^p} \right] d\tau
\]

\[
\leq \sup_{0<\xi \leq t} \frac{f(\xi)}{\xi^p} \int_0^t (p+1)\tau^p - t^p d\tau = 0.
\]

Now we prove (h6). Let \((u_k)\) be a bounded sequence in \( H^s_{rad}(\mathbb{R}^N) \). By using (h2) and (h4) we know that

\[
\lim_{|t| \to 0} \frac{h(t)t}{|t|^{2s}} = 0
\]

and

\[
\lim_{|t| \to \infty} \frac{h(t)t}{|t|^{2s}} = 0.
\]

Then we can apply Lemma 3 with \( P(t) = h(t)t, q_1 = 2 \) and \( q_2 = 2^*_s \), to deduce that

\[
\lim_{k \to \infty} \int_{\mathbb{R}^N} h(u_k)u_k \, dx = \int_{\mathbb{R}^N} h(u)u \, dx.
\]

\[\square\]
3.2. Comparison between functionals. Let us consider the following norm on $H^s(\mathbb{R}^N)$

$$
\|u\|^2 = [u]^2_{H^s(\mathbb{R}^N)} + \frac{m}{2} [u]^2_{L^2(\mathbb{R}^N)}
$$

which is equivalent to the standard norm $\| \cdot \|_{H^s(\mathbb{R}^N)}$ defined in Section 2.

Let us introduce the following functionals $I : H^s_{rad}(\mathbb{R}^N) \to \mathbb{R}$ and $J : H^s_{rad}(\mathbb{R}^N) \to \mathbb{R}$, by setting

$$
I(u) = \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}^N} \frac{m}{4} u^2 + G(u) \, dx
$$

and

$$
J(u) = \frac{1}{2} \|u\|^2 - 2 \int_{\mathbb{R}^N} H(u) \, dx.
$$

By the growth assumptions on $g$, it is easy to check that $I$ and $J$ are well defined and that they are $C^1(H^s_{rad}(\mathbb{R}^N))$-functionals. Clearly, critical points of $I$ and $J$ are weak solutions to (1.1) and $(-\Delta)^s u + \frac{m}{2} u = 2h(u)$ in $\mathbb{R}^N$, respectively.

In what follows, we show that $I$ and $J$ have a symmetric mountain pass geometry:

**Lemma 5.**

(i) $I(u) \geq J(u)$ for all $u \in H^s_{rad}(\mathbb{R}^N)$;

(ii) There are $\delta > 0$ and $\rho > 0$ such that

$$
I(u), J(u) \geq \delta > 0 \text{ for } \|u\| = \rho
$$

$$
I(u), J(u) \geq 0 \text{ for } \|u\| \leq \rho;
$$

(iii) For any $n \in \mathbb{N}$ there exists an odd continuous map $\gamma_n : \mathbb{S}^{n-1} \to H^s_{rad}(\mathbb{R}^N)$ such that

$$
I(\gamma_n(\sigma)), J(\gamma_n(\sigma)) < 0 \text{ for all } \sigma \in \mathbb{S}^{n-1}.
$$

**Proof.** (i) follows by (h3) of Lemma 4.

(ii) By Lemma 4 there exists $C > 0$ such that

$$
H(t) \leq C|t|^{2^*_s} \text{ for all } t \in \mathbb{R}.
$$

Then, by using Sobolev’s embedding, we can see that

$$
J(u) \geq \frac{1}{2} \|u\|^2 - C \|u\|^{2^*_s}_{L^{2^*_s}(\mathbb{R}^N)} \geq \|u\|^2 \left[ \frac{1}{2} - C \|u\|^{2^*_s-2} \right].
$$

Since $2^*_s > 2$, we can find $\delta, \rho > 0$ such that $J(u) \geq \delta$ for $\|u\| = \rho$, and $J(u) \geq 0$ if $\|u\| \leq \rho$. From (i), we deduce that (ii) holds.

(iii) For $n \in \mathbb{N}$ we consider the polyhedron in $\mathbb{R}^n$ defined by

$$
\mathbb{S}^{n-1} = \left\{ \sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^n : \sum_{j=1}^n |\sigma_j| = 1 \right\}.
$$

By using Theorem 10 in [13], for any $n \in \mathbb{N}$, there exists an odd continuous map $\pi_n : \mathbb{S}^{n-1} \to H^1(\mathbb{R}^N)$ such that:

(i) $\pi_n(\sigma)$ is radial for all $\sigma \in \mathbb{S}^{n-1}$;

(ii) $0 \notin \pi_n(\mathbb{S}^{n-1})$;

(iii) $\int_{\mathbb{R}^N} G(\pi_n(\sigma)) \, dx \geq 1$ for any $\sigma \in \mathbb{S}^{n-1}$.
Since \( H^1(\mathbb{R}^N) \subset H^s(\mathbb{R}^N) \) and \( \pi_n(\mathbb{S}^{n-1}) \) is compact, there exists \( M > 0 \) such that
\[
\| \pi_n(\sigma) \|_{H^s(\mathbb{R}^N)} \leq M \text{ for any } \sigma \in \mathbb{S}^{n-1}.
\]
Now, let us define \( \psi_n^t(\sigma)(x) = \pi_n(\sigma)(\frac{x}{t}) \) with \( t \geq 1 \).
Then we have
\[
I(\psi_n^t(\sigma)) = t^{N-2s} \left[ \frac{1}{2} \pi_n(\sigma)^2 \right]_{H^s(\mathbb{R}^N)} - t^N \int_{\mathbb{R}^N} G(\pi_n(\sigma)) dx
\leq t^{N-2s} \left[ \frac{M}{2} - t^{2s} \right] \rightarrow -\infty \text{ as } t \rightarrow +\infty.
\]
Therefore we can chose \( t \) such that \( I(\psi_n^t(\sigma)) < 0 \) for all \( \sigma \in \mathbb{S}^{n-1} \), and by setting \( \gamma_n(\sigma)(x) := \psi_n^t(\sigma)(x) \), we can infer that \( \gamma_n \) satisfies the required properties for \( I \). Taking into account (i), we can conclude the proof.

Differently from \( I(u) \), the comparison functional \( J(u) \) satisfies the following compactness property:

**Theorem 5.** The functional \( J \) satisfies the Palais-Smale condition.

**Proof.** Let \( c \in \mathbb{R} \) and let \( (u_k) \subset H^s_{\text{rad}}(\mathbb{R}^N) \) such that
\[
J(u_k) \rightarrow c \text{ and } J'(u_k) \rightarrow 0.
\]
By using (h5) of Lemma 4, we can see
\[
J(u_k) - \frac{J'(u_k)u_k}{p+1} = \left( \frac{1}{2} - \frac{1}{p+1} \right) \| u_k \|^2 - 2 \int_{\mathbb{R}^N} \left[ H(u_k) - \frac{1}{p+1} h(u_k) u_k \right] dx
\geq \left( \frac{1}{2} - \frac{1}{p+1} \right) \| u_k \|^2
\]
so we can deduce that \( (u_k) \) is bounded in \( H^s_{\text{rad}}(\mathbb{R}^N) \).
Then, by using Lemma 4, we may assume, up to a subsequence, that
\[
u_k \rightharpoonup u \text{ in } H^s_{\text{rad}}(\mathbb{R}^N),
\]
\[
u_k \rightarrow u \text{ in } L^q(\mathbb{R}^N) \quad \forall q \in (2, 2^*_s),
\]
\[
u_k \rightarrow u \text{ a.e. } \mathbb{R}^N.
\]
Now, by using (h1) and (h4) in Lemma 4 and (3.2), we can apply the first part of Lemma 2 with \( P(t) = h(t) \) and \( Q(t) = |t|^{2^*_s-1} \), to infer that for any \( \varphi \in C^\infty_0(\mathbb{R}^N) \)
\[
\lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} h(u_k(x)) \varphi(x) dx = \int_{\mathbb{R}^N} h(u(x)) \varphi(x) dx.
\]
Putting together (3.1), (3.2) and (3.3) we obtain
\[
J'(u) \varphi = J'(u_k) \varphi - \left[ \langle u_k - u, \varphi \rangle_{H^s(\mathbb{R}^N)} - 2 \int_{\mathbb{R}^N} (h(u_k) - h(u)) \varphi dx \right] \rightarrow 0
\]
for any \( \varphi \in C^\infty_0(\mathbb{R}^N) \). Since \( C^\infty_0(\mathbb{R}^N) \) is dense in \( H^s_{\text{rad}}(\mathbb{R}^N) \), it follows that
\[
J'(u) \varphi = 0 \text{ for all } \varphi \in H^s_{\text{rad}}(\mathbb{R}^N).
\]
Moreover, \( J'(u)u = 0 \), which implies that \( \|u\|^2 = 2 \int_{\mathbb{R}^N} h(u)u \, dx \).

By using (h6) of Lemma 4, we know that

\[
\lim_{k \to \infty} \int_{\mathbb{R}^N} h(u_k)u_k \, dx = \int_{\mathbb{R}^N} h(u)u \, dx. \tag{3.4}
\]

Taking into account the boundedness of \((u_k)\) and (3.1), we get \( J'(u_k)u_k \to 0 \), which together with (3.4) implies that \( \|u_k\| \to \|u\| \) as \( k \to \infty \). Therefore, we can conclude that \( u_k \to u \) in \( H^s_{\text{rad}}(\mathbb{R}^N) \) as \( k \to \infty \).

Now we define minimax values of \( I \) and \( J \) by using maps \((\gamma_n)\) in Lemma 5. For any \( n \in \mathbb{N} \), we define \( b_n \) and \( c_n \) as follows:

\[
b_n = \inf_{\gamma \in \Gamma_n} \max_{\sigma \in D_n} I(\gamma(\sigma)), \quad c_n = \inf_{\gamma \in \Gamma_n} \max_{\sigma \in D_n} J(\gamma(\sigma))
\]

where \( D_n = \{ \sigma \in \mathbb{R}^n : |\sigma| \leq 1 \} \) and

\[
\Gamma_n = \{ \gamma \in C(D_n; H^s_{\text{rad}}(\mathbb{R}^N)) : \gamma \text{ is odd and } \gamma = \gamma_n \text{ on } S^{n-1} \}.
\]

The values \( b_n \) and \( c_n \) satisfy the following properties.

**Lemma 6.**

(i) \( \Gamma_n \neq \emptyset \) for any \( n \in \mathbb{N} \);

(ii) \( 0 < \delta \leq c_n \leq b_n \) for any \( n \in \mathbb{N} \), where \( \delta \) appears in Lemma 5.

**Proof.** (i) Let us define

\[
\tilde{\gamma}_n(\sigma) := \begin{cases} 
|\sigma| \gamma_n(\sigma/|\sigma|) & \text{if } \sigma \in D_n \setminus \{0\} \\
0 & \text{for } \sigma = 0.
\end{cases}
\]

Then, it is clear that \( \tilde{\gamma}_n \in \Gamma_n \).

(ii) Since \( I(u) \geq J(u) \) in view of Lemma 5, it holds \( c_n \leq b_n \). The property \( \delta \leq c_n \) follows from the fact that

\[
\{ u \in H^s_{\text{rad}}(\mathbb{R}^N) : \|u\| = \rho \} \cap \gamma(D_n) \neq \emptyset \text{ for all } \gamma \in \Gamma_n.
\]

**Lemma 7.**

(i) The value \( c_n \) is a critical value of \( J \).

(ii) \( c_n \to \infty \) as \( n \to \infty \).

**Proof.** (i) follows by Lemma 5.

(ii) Set

\[
\Sigma_n = \left\{ h \in C(D_m \setminus \overline{Y}) : h \in \Gamma_m, m \geq n, Y \in \mathcal{E}_m \text{ and genus}(Y) \leq m - n \right\}
\]

where \( \mathcal{E}_m \) is the family of closed sets \( A \subset \mathbb{R}^m \setminus \{0\} \) such that \(-A = A\) and \( \text{genus}(A) \) is the Krasnoselski genus of \( A \). Now we define another sequence of minimax values by setting

\[
d_n = \inf_{A \in \Sigma_n} \max_{u \in A} J(u).
\]
Then \( d_n \leq d_{n+1} \) for all \( n \in \mathbb{N} \), and \( d_n \leq c_n \) for all \( n \in \mathbb{N} \). Since \( J \) satisfies the Palais-Smale condition, we can proceed as in the proof of Proposition 9.33 in [45] to infer that \( d_n \to +\infty \) as \( n \to \infty \).

Let us observe that (ii) of Lemma 6 and (ii) of Lemma 7 yield

\[ b_n > 0 \text{ for all } n \in \mathbb{N} \text{ and } \lim_{n \to \infty} b_n = \infty. \] (3.5)

In the next section we will prove that \( b_n \) are critical values of \( I(u) \).

3.3. An auxiliary functional on augmented space. Let us introduce the following functional

\[ \tilde{I}(\theta, u) = \frac{1}{2} e^{(N-2)s} [u]_{H^s(\mathbb{R}^N)}^2 - e^{N\theta} \int_{\mathbb{R}^N} G(u) \, dx \]

for \((\theta, u) \in \mathbb{R} \times H^s_{rad}(\mathbb{R}^N)\).

We endow \( \mathbb{R} \times H^s_{rad}(\mathbb{R}^N) \) with the norm

\[ \| (\theta, u) \|_{\mathbb{R} \times H^s(\mathbb{R}^N)} = \sqrt{|\theta|^2 + \|u\|^2}. \]

Let us point out that \( \tilde{I} \in C^1(\mathbb{R} \times H^s_{rad}(\mathbb{R}^N), \mathbb{R}) \), \( \tilde{I}(0, u) = I(u) \) and being

\[ \int_{\mathbb{R}^N} \frac{|u(e^{-\theta}x) - u(e^{-\theta}y)|^2}{|x-y|^{N+2s}} \, dxdy = e^{(N-2)s} [u]_{H^s(\mathbb{R}^N)}^2 \]

\[ \int_{\mathbb{R}^N} G(u(e^{-\theta}x)) \, dx = e^{N\theta} \int_{\mathbb{R}^N} G(u(x)) \, dx \]

we have

\[ \tilde{I}(\theta, u(x)) = I(u(e^{-\theta}x)) \text{ for all } \theta \in \mathbb{R}, u \in H^s_{rad}(\mathbb{R}^N). \] (3.6)

We also define minimax values \( \tilde{b}_n \) for \( \tilde{I}(\theta, u) \) by

\[ \tilde{b}_n = \inf_{\tilde{\gamma} \in \tilde{\Gamma}_n} \max_{\sigma \in D_n} \tilde{I}(\tilde{\gamma}(\sigma)), \]

where

\[ \tilde{\Gamma}_n = \{ \tilde{\gamma} \in C(D_n, \mathbb{R} \times H^s_{rad}(\mathbb{R}^N)) : \tilde{\gamma}(\sigma) = (\theta(\sigma), \eta(\sigma)) \text{ satisfies} \]

\( (\theta(-\sigma), \eta(-\sigma)) = (\theta(\sigma), -\eta(\sigma)) \) for all \( \sigma \in D_n \)

\( (\theta(\sigma), \eta(\sigma)) = (0, \gamma_\Sigma(\sigma)) \) for all \( \sigma \in S^{n-1} \} \).

Through the modified functional \( \tilde{I}(\theta, u) \) we aim to show that \( b_n \) are critical values for \( I(u) \).

We begin proving that

**Lemma 8.** \( \tilde{b}_n = b_n \) for all \( n \in \mathbb{N} \).

**Proof.** Let us observe that \((0, \gamma(\sigma)) \in \tilde{\Gamma}_n \) for any \( \gamma \in \Gamma_n \), so we can see that \( \Gamma_n \subset \tilde{\Gamma}_n \). Since \( \tilde{I}(0, u) = I(u) \), by the definitions of \( b_n \) and \( \tilde{b}_n \), it follows that \( \tilde{b}_n \leq b_n \) for all \( n \in \mathbb{N} \).

Now, we take \( \tilde{\gamma}(\sigma) = (\theta(\sigma), \gamma(\sigma)) \in \tilde{\Gamma}_n \) and we put \( \gamma(\sigma) = \eta(\sigma)(e^{-\theta(\sigma)}x) \). Then it is easy to see that \( \gamma \in \Gamma_n \), and by using (3.6), \( I(\gamma(\sigma)) = \tilde{I}(\tilde{\gamma}(\sigma)) \) for all \( \sigma \in D_n \). As a consequence we have \( \tilde{b}_n \geq b_n \) for all \( n \in \mathbb{N} \).
At this point, we show that the functional $\bar{I}(\theta, u)$ admits a Palais-Smale sequence in $\mathbb{R} \times H^s_{rad}(\mathbb{R}^N)$ with a property related to the fractional Pohozaev identity [18]:

$$\frac{N - 2s}{2} \int_{\mathbb{R}^N} |(\Delta)^{\frac{s}{2}} u|^2 \, dx = N \int_{\mathbb{R}^N} G(u) \, dx. \quad (3.7)$$

Firstly we give a version of Ekeland’s principle:

**Lemma 9.** Let $n \in \mathbb{N}$ and $\varepsilon > 0$. Assume that $\tilde{\gamma}_n \in \tilde{\Gamma}_n$ satisfies

$$\max_{\sigma \in D_n} \bar{I}(\tilde{\gamma}(\sigma)) \leq \tilde{b}_n + \varepsilon.$$

Then there exists $(\theta, u) \subset \mathbb{R} \times H^s_{rad}(\mathbb{R}^N)$ such that:

(i) $\text{dist}_{\mathbb{R} \times H^s_{rad}(\mathbb{R}^N)}((\theta, u), \tilde{\gamma}(D_n)) \leq 2\sqrt{\varepsilon}$;

(ii) $\bar{I}(\theta, u) \in [b_n - \varepsilon, b_n + \varepsilon]$;

(iii) $\|\nabla \bar{I}(\theta, u)\|_{\mathbb{R} \times (H^s_{rad}(\mathbb{R}^N))^*} \leq 2\sqrt{\varepsilon}$.

Here we have used the notations

$$\text{dist}_{\mathbb{R} \times H^s_{rad}(\mathbb{R}^N)}((\theta, u), A) = \inf_{(t, v) \in A} \sqrt{|\theta - t|^2 + \|u - v\|^2}$$

for $A \subset \mathbb{R} \times H^s_{rad}(\mathbb{R}^N)$, and

$$\nabla \bar{I}(\theta, u) := \left( \frac{\partial}{\partial \theta} \bar{I}(\theta, u), \bar{I}'(\theta, u) \right).$$

**Proof.** Since $\bar{I}(\theta, -u) = \bar{I}(\theta, u)$ for all $(\theta, u) \in \mathbb{R} \times H^s_{rad}(\mathbb{R}^N)$, we can deduce that $\tilde{\Gamma}_n$ is stable under the pseudo-deformation flow generated by $\bar{I}(\theta, u)$. Taking into account $\tilde{b}_n = b_n > 0$, $I(0) = 0$ and $\max_{\sigma \in D_n} \bar{I}(0, \gamma_n(\sigma)) < 0$, the proof of Lemma goes as in [34].

Thus we can deduce the following result:

**Theorem 6.** For any $n \in \mathbb{N}$ there exists a sequence $(\theta_k, u_k) \subset \mathbb{R} \times H^s_{rad}(\mathbb{R}^N)$ such that

(i) $\theta_k \to 0$;

(ii) $I(\theta_k, u_k) \to b_n$;

(iii) $\bar{I}(\theta_k, u_k) \to 0$ strongly in $(H^s_{rad}(\mathbb{R}^N))^*$;

(iv) $\frac{\partial}{\partial \theta} \bar{I}(\theta_k, u_k) \to 0$.

**Proof.** For any $k \in \mathbb{N}$ there exists $\gamma_k \in \Gamma_n$ such that

$$\max_{\sigma \in D_n} \bar{I}(\gamma_k(\sigma)) \leq b_n + \frac{1}{k}.$$

Since $\tilde{\gamma}_n(\sigma) = (0, \gamma_k(\sigma)) \in \tilde{\Gamma}_n$ and $\tilde{b}_n = b_n$, we have

$$\max_{\sigma \in D_n} \bar{I}(\tilde{\gamma}_n(\sigma)) \leq \tilde{b}_n + \frac{1}{k}.$$

By using Lemma 9, it follows the existence of $(\theta_k, u_k) \subset \mathbb{R} \times H^s_{rad}(\mathbb{R}^N)$ such that

$$\text{dist}_{\mathbb{R} \times H^s_{rad}(\mathbb{R}^N)}((\theta_k, u_k), \tilde{\gamma}_k(D_n)) \leq \frac{2}{\sqrt{k}} \quad (3.8)$$

$$\bar{I}(\theta_k, u_k) \in [b_n - \frac{1}{k}, b_n + \frac{1}{k}] \quad (3.9)$$

$$\|\nabla \bar{I}(\theta_k, u_k)\|_{\mathbb{R} \times (H^s_{rad}(\mathbb{R}^N))^*} \leq \frac{2}{\sqrt{k}}. \quad (3.10)$$
Then, (3.8) and \( \tilde{\gamma}_k(D_n) \subset \{0\} \times H^s_{\text{rad}}(\mathbb{R}^N) \) yields (i). Clearly, (ii) follows by (3.9), and (iii) and (iv) are obtained as consequence of (3.10).

Now, we investigate the boundedness and the compactness properties of the sequence \( (\theta_k, u_k) \subset \mathbb{R} \times H^s_{\text{rad}}(\mathbb{R}^N) \) obtained in Theorem 6. More precisely, we are able to prove that

**Theorem 7.** Let \( (\theta_k, u_k) \subset \mathbb{R} \times H^s_{\text{rad}}(\mathbb{R}^N) \) be a sequence satisfying (i)-(iv) of Theorem 6. Then we have

(a) \( (u_k) \) is bounded in \( H^s_{\text{rad}}(\mathbb{R}^N) \);
(b) \( (\theta_k, u_k) \) has a strongly convergent subsequence in \( \mathbb{R} \times H^s_{\text{rad}}(\mathbb{R}^N) \).

**Proof.** (a) By using (ii) and (iv) of Theorem 6, we can see that

\[
\frac{1}{2} e^{(N-2s)\theta_k} [u_k]_{H^s(\mathbb{R}^N)}^2 - e^{N\theta_k} \int_{\mathbb{R}^N} G(u_k) \, dx \to b_n
\]

and

\[
\frac{N - 2s}{2} e^{(N-2s)\theta_k} [u_k]_{H^s(\mathbb{R}^N)}^2 - Ne^{N\theta_k} \int_{\mathbb{R}^N} G(u_k) \, dx \to 0
\]

as \( k \to \infty \). Therefore we deduce that

\[
[u_k]_{H^s(\mathbb{R}^N)}^2 \to \frac{Nb_n}{s} \tag{3.11}
\]

and

\[
\int_{\mathbb{R}^N} G(u_k) \, dx \to \frac{N - 2s}{2} b_n
\]

as \( k \to \infty \).

By Lemma 4, there exists \( C > 0 \) such that

\[
|h(t)| \leq C|t|^{2^*_s - 1} \text{ for all } t \in \mathbb{R}. \tag{3.12}
\]

Set

\[
\varepsilon_k = \| \bar{\varphi}'(\theta_k, u_k) \|_{(H^s_{\text{rad}}(\mathbb{R}^N))^*}.
\]

By (iii) of Theorem 6, we deduce that \( \varepsilon_k \to 0 \) as \( k \to \infty \), so we get

\[
|\bar{\varphi}'(\theta_k, u_k) u_k| \leq \varepsilon_k \| u_k \|. \tag{3.13}
\]

Taking into account (3.11), (3.12), (3.13), (h3) of Lemma 4 and by using the Sobolev inequality we have

\[
e^{(N-2s)\theta_k} [u_k]_{H^s(\mathbb{R}^N)}^2 + \frac{m}{2} e^{N\theta_k} \| u_k \|_{L^2(\mathbb{R}^N)}^2
\]

\[
\leq e^{N\theta_k} \int_{\mathbb{R}^N} \frac{m}{2} u_k^2 + g(u_k) u_k \, dx + \varepsilon_k \| u_k \|
\]

\[
\leq e^{N\theta_k} \int_{\mathbb{R}^N} h(u_k) u_k \, dx + \varepsilon_k \| u_k \|
\]

\[
\leq Ce^{N\theta_k} \| u_k \|_{L^2(\mathbb{R}^N)}^{2^*_s} + \varepsilon_k \| u_k \|
\]

\[
\leq CC_e e^{N\theta_k} [u_k]_{H^s(\mathbb{R}^N)}^2 + \varepsilon_k \| u_k \|
\]

\[
\leq CC'e e^{N\theta_k} + \varepsilon_k \| u_k \|
\]

from which follows the boundedness of \( \| u_k \|_{L^2(\mathbb{R}^N)} \).
Therefore, recalling (3.11), we can deduce that \((u_k)\) is bounded in \(H^{s}_{rad}(\mathbb{R}^N)\).

\((b)\) By \((a)\) we may assume that, up to a subsequence,

\[
\begin{align*}
  &u_k & \rightharpoonup & u \text{ in } H^{s}_{rad}(\mathbb{R}^N), \\
  &u_k & \rightarrow & u \text{ in } L^q(\mathbb{R}^N) \quad \forall q \in (2, 2^*_s), \\
  &u_k & \rightarrow & u \text{ a.e. } \mathbb{R}^N.
\end{align*}
\]

(3.14)

By using \((iii)\) of Theorem 6, we can see that

\[
e^{(N-2s)\theta_k} \int_{\mathbb{R}^N} \frac{(u_k(x) - u_k(y))}{|x-y|^{N+2s}} (\varphi(x) - \varphi(y)) \, dx dy - e^{N\theta_k} \int_{\mathbb{R}^N} g(u_k)\varphi \, dx \rightarrow 0
\]

for any \(\varphi \in C_0^\infty(\mathbb{R}^N)\).

Then \(I'(u)\varphi = 0\) for any \(\varphi \in C_0^\infty(\mathbb{R}^N)\), and by the density of \(C_0^\infty(\mathbb{R}^N)\) in \(H^{s}_{rad}(\mathbb{R}^N)\), we get \(I'(u)\varphi = 0\) for any \(\varphi \in H^{s}_{rad}(\mathbb{R}^N)\). Moreover, we get

\[
[u]_{H^{s}(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} g(u)u \, dx.
\]

(3.16)

Now, we observe that (3.15) holds for any \(\varphi \in H^{s}_{rad}(\mathbb{R}^N)\) and \((u_k)\) is bounded in \(H^1(\mathbb{R}^N)\).

Therefore, taking \(\varphi = u_k\) in (3.15), we have

\[
e^{(N-2s)\theta_k} [u_k]_{H^1(\mathbb{R}^N)}^2 - e^{N\theta_k} \int_{\mathbb{R}^N} g(u_k)u_k \, dx = o(1) \text{ as } k \to \infty.
\]

Hence we obtain

\[
\begin{align*}
  e^{(N-2s)\theta_k} [u_k]_{H^1(\mathbb{R}^N)}^2 & + \frac{m}{2} e^{N\theta_k} \|u_k\|^2_{L^2(\mathbb{R}^N)} \\
  = & e^{N\theta_k} \int_{\mathbb{R}^N} h(u_k)u_k dx + e^{N\theta_k} \int_{\mathbb{R}^N} [h(u_k)u_k - \frac{m}{2} u_k^2 - g(u_k)u_k] \, dx + o(1) \\
  = & e^{N\theta_k} A_k - e^{N\theta_k} B_k + o(1).
\end{align*}
\]

(3.17)

Now, by using \((h6)\) of Lemma 4, we can see that

\[
\lim_{k \to \infty} A_k = \int_{\mathbb{R}^N} h(u)u \, dx,
\]

(3.18)

and by \((h3)\) of Lemma 4 and Fatou’s Lemma, we get

\[
\liminf_{k \to \infty} B_k \geq \int_{\mathbb{R}^N} \left[ h(u)u - \frac{m}{2} u^2 - g(u)u \right] \, dx.
\]

(3.19)

Taking into account (3.14), (3.16), (3.17), (3.18) and (3.19), we can infer that

\[
\|u\|^2 \leq \limsup_{k \to \infty} \|u_k\|^2 = \limsup_{k \to \infty} \left[ e^{(N-2s)\theta_k} [u_k]_{H^1(\mathbb{R}^N)}^2 + \frac{m}{2} e^{N\theta_k} \|u_k\|^2_{L^2(\mathbb{R}^N)} \right] \\
\leq \int_{\mathbb{R}^N} \left[ \frac{m}{2} u^2 + g(u)u \right] \, dx \\
= \|u\|^2
\]

which gives that \(u_k \rightharpoonup u\) in \(H^{s}_{rad}(\mathbb{R}^N)\).
Putting together Theorem 7, Lemma 8 and (3.5), we can provide the following multiplicity result.

**Theorem 8.** Under the assumptions of Theorem 1, there exist infinitely many solutions to (1.1).

**Proof.** Fix $n \in \mathbb{N}$, and let $(\theta_k, u_k) \subset \mathbb{R} \times H^s_{rad}(\mathbb{R}^N)$ be a sequence satisfying (i)-(iv) of Theorem 6. By using Theorem 7, we know that there exists $u_n \in H^s_{rad}(\mathbb{R}^N)$ such that $u_k \to u_n$ in $H^s_{rad}(\mathbb{R}^N)$ as $k \to \infty$. Then $u_n$ satisfies

$$I(u_n) = \tilde{I}(0, u_n) = b_n \quad \text{and} \quad I'(u_n) = \tilde{I}'(0, u_n) = 0.$$  

Since $b_n \to \infty$ as $n \to \infty$, we can conclude that (1.1) admits infinitely many solutions characterized by a mountain-pass argument in $H^s_{rad}(\mathbb{R}^N)$. $\square$

4. Mountain pass value gives the least energy level

In this section we prove the existence of a positive solution to (1.1) by using the mountain pass approach developed in the previous section. We recall that the existence of a positive ground state solution to (1.1) has been obtained in [18]. Here, we give an alternative proof of this fact, and in addition, we show that the least energy solution of (1.1) coincides with the mountain-pass value. This is in clear accordance with the result obtained in the classic framework by Jeanjean and Tanaka in [37]. Moreover, we investigate the symmetry of solutions to (1.1) by using the moving-plane method.

The main result of this section can be stated as follows.

**Theorem 9.** Under the assumptions (g1)-(g3) there exists a classical positive solution $u$ of (1.1). Moreover $u$ is radially decreasing and $u$ can be characterized by a mountain pass argument in $H^s_{rad}(\mathbb{R}^N)$.

Let us consider the functionals $I : H^s_{rad}(\mathbb{R}^N) \to \mathbb{R}$ and $\tilde{I} : \mathbb{R} \times H^s_{rad}(\mathbb{R}^N) \to \mathbb{R}$ introduced in Section 3:

$$I(u) = \frac{1}{2}[u]_{H^s(\mathbb{R}^N)}^2 - \int_{\mathbb{R}^N} G(u) \, dx$$

and

$$\tilde{I}(\theta, u) = \frac{1}{2}e^{(N-2s)\theta}[u]_{H^s(\mathbb{R}^N)}^2 - e^{N\theta} \int_{\mathbb{R}^N} G(u) \, dx.$$ 

We define the following minimax values:

$$b_{mp,r} = \inf_{\gamma \in \Gamma_r} \max_{t \in [0,1]} I(\gamma(t)),$$

$$b_{mp} = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$

$$\tilde{b}_{mp,r} = \inf_{\tilde{\gamma} \in \tilde{\Gamma}_r} \max_{t \in [0,1]} \tilde{I}(\tilde{\gamma}(t)),$$

where

$$\Gamma_r = \{ \gamma \in C([0,1], H^s_{rad}(\mathbb{R}^N)) : \gamma(0) = 0 \text{ and } I(\gamma(1)) < 0 \},$$

$$\Gamma = \{ \gamma \in C([0,1], H^s(\mathbb{R}^N)) : \gamma(0) = 0 \text{ and } I(\gamma(1)) < 0 \},$$

$$\tilde{\Gamma}_r = \{ \tilde{\gamma} = (\theta, \gamma) \in C([0,1], \mathbb{R} \times H^s_{rad}(\mathbb{R}^N)) : \gamma \in \Gamma_r \text{ and } \theta(0) = 0 = \theta(1) \}.$$ 

Then, we can see that the following result holds:
Lemma 10.

\[ b_{mp} = b_{mp,r} = b_1 = \bar{b}_{mp,r}. \]

Proof. Arguing as in the proof of Lemma 8, we have \( b_{mp,r} = \bar{b}_{mp,r} \). Furthermore, it follows by the above definitions that \( b_{mp} \leq b_{mp,r} \leq b_1 \). In order to conclude the proof of Lemma, it is enough to show that \( b_1 \leq b_{mp} \).

Firstly, we show that \( b_{mp} = \bar{b} \),

where

\[ \bar{b} = \inf \max_{\gamma \in \Gamma, t \in [0,1]} I(\gamma(t)) \]

and

\[ \bar{\Gamma} = \{ \gamma \in C([0,1], H^s(\mathbb{R}^N)) : \gamma(0) = 0 \text{ and } \gamma(1) = \gamma_1(1) \}. \]

Here \( \gamma_1 \) is the path appearing in Lemma 5 (with \( n = 1 \)). As observed in [12, 13], we may assume that \( \gamma_1 \) satisfies the following properties: \( \gamma_1(x) \geq 0 \) for any \( x \in \mathbb{R}^N \), \( \gamma_1(|x|) = \gamma(1)(x) \) and \( r \mapsto \gamma_1(1)(r) \) is piecewise linear and non increasing.

Since \( I(\gamma_1(1)) < 0 \), it is clear that \( \bar{\Gamma} \subset \Gamma \) which gives \( b_{mp} \leq \bar{b} \). Now, we show that \( b_{mp} \geq \bar{b} \).

For this purpose, it is suffices to prove that \( B = \{ u \in H^s(\mathbb{R}^N) : I(u) < 0 \} \) is path connected.

We proceed as in [15]. Take \( u_1, u_2 \in B \) and assume that \( \text{supp}(u_1) \cup \text{supp}(u_2) \subset B_R \) for some \( R > 0 \). In fact, if \( \phi \in C^\infty_0(\mathbb{R}^N) \) is such that \( \phi(x) = 0 \) for \( |x| \geq 2 \) and \( \phi(x) = 1 \) for \( |x| \leq 1 \), denoted by \( \phi_i(x) = \phi(tx) \) for \( t > 0 \), we can see that there exists \( t_0 > 0 \) such that

\[ I(tu_i + (1-t)\phi_{t_0}u_i) < 0 \]

for any \( t \in [0,1] \) and \( i = 1, 2 \).

Now, let us denote by \( u^t_i(x) = u_i(t\frac{x}{t}) \) for \( t > 0 \) and \( i = 1, 2 \).

Then

\[ I(u^t_i) = t^{N-2s} \frac{2}{2}[u_i]^2_{H^s(\mathbb{R}^N)} - t^N \int_{\mathbb{R}^N} G(u_i) \, dx \]

so we get

\[ \frac{d}{dt} I(u^t_i) = Nt^{N-1}\left(t^{-2s} \frac{1}{2}[u_i]^2_{H^s(\mathbb{R}^N)} - \int_{\mathbb{R}^N} G(u_i) \, dx \right) - st^{N-2s-1}[u_i]^2_{H^s(\mathbb{R}^N)}. \]

In particular, for \( t \geq 1 \) we have

\[ \frac{d}{dt} I(u^t_i) \leq Nt^{N-1} I(u_i) - st^{N-2s-1}[u_i]^2_{H^s(\mathbb{R}^N)} < 0. \]

Since \( u_i \in B \), we deduce that \( \int_{\mathbb{R}^N} G(u_i) \, dx > 0 \), so we can see that \( I(u^t_i) \to -\infty \) as \( t \to \infty \).

Hence, we can choose \( t_1 > 1 \) and \( t_2 > 1 \) such that

\[ I(u^{t_1}_1) \leq -1 - \max_{t \in [0,1]} |I(tu_2)| \quad (4.2) \]

and

\[ I(u^{t_2}_2) \leq -1 - \max_{t \in [0,1]} |I(tu_1^{t_1})|. \quad (4.3) \]

Now, we define \( u^\theta(x) = u^{t_1}_1(x + 2\theta R(t_1 + t_2)e_1) \) for \( \theta \in [0,1] \), where \( e_1 = (1,0,\ldots,0) \). It is clear that for all \( \theta \in [0,1] \)

\[ I(u^\theta_3) = I(u^0_3) = I(u^{t_1}_1) < 0. \]
Let us observe that $\text{supp}(u_3^1) \cap \text{supp}(u_2^1) = \emptyset$ for all $t \in [1, t_2]$, so by using (4.2), we deduce that
\[ I(u_3^1 + \theta u_2) = I(u_3^1) + I(\theta u_2) < 0 \]
for all $\theta \in [0, 1]$. We also note that $I(u_3^1 + u_2^1) = I(u_3^1) + I(u_2^1)$ for $t \in [1, t_2]$, $I(u_2^2) < 0$ for all $t \in [1, t_2]$, and $I(\theta u_3^1 + u_2^2) = I(\theta u_1^1) + I(u_2^2) < 0$ for $\theta \in [0, 1]$ in view of (4.3). Then, considering the following paths
\[
\begin{align*}
\gamma_1(\theta) &= u_1^\theta \text{ for } \theta \in [1, t_1], \\
\gamma_2(\theta) &= u_3^\theta \text{ for } \theta \in [0, 1], \\
\gamma_3(\theta) &= u_3^1 + \theta u_2 \text{ for } \theta \in [0, 1], \\
\gamma_4(\theta) &= u_3^1 + u_2^\theta \text{ for } \theta \in [1, t_2], \\
\gamma_5(\theta) &= \theta u_3^1 + u_2^2 \text{ for } \theta \in [0, 1], \\
\gamma_6(\theta) &= u_2^\theta \text{ for } \theta \in [1, t_2]
\end{align*}
\]
we can find a path connecting $u_1$ and $u_2$ on which $I$ is negative. Therefore, $B$ is path connected and $b_{mp} \geq \bar{b}$. This implies that (4.1) holds.

Now, observing that $I(u) = I(-u)$, $\gamma_1(1) \in H^s_{rad}(\mathbb{R}^N)$ and $\gamma(t) = -\gamma(t)$ for any $\gamma \in \Gamma_1$, we aim to show that
\[
\bar{b} = \inf_{\gamma \in \bar{\Gamma}_r} \max_{t \in [0, 1]} I(\gamma(t)) = b_1 (4.4)
\]
where
\[
\bar{\Gamma}_r = \{ \gamma \in \mathcal{C}([0, 1], H^s_{rad}(\mathbb{R}^N)) : \gamma(0) = 0 \text{ and } \gamma(1) = \gamma_1(1) \}.
\]
It is clear that by definition $\bar{b} \leq b_1$. In what follows, we show that $\bar{b} \geq b_1$.

Take $\eta \in \bar{\Gamma}$ and we set $\gamma(t) := |\eta(t)|$. Clearly, $\gamma \in \mathcal{C}([0, 1], H^s(\mathbb{R}^N))$. Moreover, recalling that $G$ is even and by using the fact that $[|u|]_{H^s(\mathbb{R}^N)} \leq [|u|]_{H^s(\mathbb{R}^N)}$ for any $u \in H^s(\mathbb{R}^N)$ (this inequality follows by $||x| - |y|| \leq |x - y|$ for any $x, y \in \mathbb{R}$), we can see that for any $t \in [0, 1]$ it holds
\[
I(\gamma(t)) = \frac{1}{2} [\eta(t)]^2_{H^s(\mathbb{R}^N)} - \int_{\mathbb{R}^N} G(|\eta(t)|) \, dx \\
\leq \frac{1}{2} [\eta(t)]^2_{H^s(\mathbb{R}^N)} - \int_{\mathbb{R}^N} G(\eta(t)) \, dx \\
\leq \frac{1}{2} [\eta(t)]^2_{H^s(\mathbb{R}^N)} - \int_{\mathbb{R}^N} G(\eta(t)) \, dx = I(\eta(t)).
\]
Moreover, by using $\gamma_1(1) \geq 0$, we can see that $\gamma(1) = |\eta(1)| = |\gamma_1(1)| = \gamma_1(1)$, so that $\gamma \in \bar{\Gamma}$. Now, we denote by $\gamma^*(t)$ the Schwartz symmetrization $\gamma(t)$. Then, $\gamma^*(t) \in H^s_{rad}(\mathbb{R}^N)$, and by using the continuity of $G$ and the fractional Polya-Szegő inequality $[u^*]_{H^s(\mathbb{R}^N)} \leq [|u|]_{H^s(\mathbb{R}^N)}$ (see [1]), we can deduce that $I(\gamma^*(t)) \leq I(\gamma(t))$ for any $t \in [0, 1]$. Since the rearrangement map is continuous everywhere on $H^s(\mathbb{R}^N)$ (see [1]), we have $\gamma^* \in \mathcal{C}([0, 1], H^s_{rad}(\mathbb{R}^N))$, and by using the fact that $(\gamma_1(1))^* = \gamma_1(1)$, we can deduce that $\gamma^* \in \bar{\Gamma}_r$.

Therefore, in view of (4.5), we have
\[
b_1 \leq \max_{t \in [0, 1]} I(\gamma^*(t)) \leq \max_{t \in [0, 1]} I(\gamma(t)) \leq \max_{t \in [0, 1]} I(\eta(t)),
\]
which implies that \( b_1 \leq \bar{b} \). This shows that (4.4) is satisfied. Putting together (4.1) and (4.4), we can infer that \( b_{mp} = \bar{b} = b_1 \). This ends the proof of lemma.

Then, we are able to prove the following result.

**Theorem 10.** There exists a positive solution to (1.1) such that \( I(u) = b_{mp,r} \). Moreover, for any non-trivial solution \( v \) to (1.1), we have \( b_{mp,r} \leq I(v) \). This means that \( u \) is the least energy solution to (1.1) and that \( b_{mp,r} \) is the least energy level.

**Proof.** We argue as in Section 3. Let \( (\gamma_k) \subset \Gamma_r \) be a sequence such that

\[
\max_{t \in [0,1]} I(\gamma_k(t)) \leq b_{mp,r} + \frac{1}{k};
\]

(4.6)

Since \( ||x| - |y|| \leq |x - y| \) for any \( x, y \in \mathbb{R} \) and \( G \) is even, we can see that \( I(|u|) \leq I(u) \). Then, we may assume that \( \gamma_k \in \Gamma_r \) in (4.6) satisfies

\[
\gamma_k(t)(x) \geq 0 \quad \text{for all} \quad t \in \mathbb{R}, x \in \mathbb{R}^N.
\]

In view of Lemma 10, we can deduce that

\[
\max_{t \in [0,1]} \bar{I}(0, \gamma_k(t)) \leq \bar{b}_{mp,r} + \frac{1}{k} = b_1 + \frac{1}{k}.
\]

(4.7)

Therefore, by applying Lemma 9 to \( \bar{I} \) and \( (0, \gamma_k) \), we can find \( (\theta_k, u_k) \subset \mathbb{R} \times H^r_{rad}(\mathbb{R}^N) \) such that as \( k \to \infty \)

(i) \( \text{dist}_{\mathbb{R} \times H^r_{rad}(\mathbb{R}^N)}((\theta_k, u_k), \{0\} \times \gamma_k([0,1])) \to 0 \);

(ii) \( \bar{I}(\theta_k, u_k) \to b_1 \);

(iii) \( \|\nabla \bar{I}(\theta_k, u_k)\|_{\mathbb{R} \times H^r_{rad}(\mathbb{R}^N)^*} \to 0 \).

Taking into account (ii), we can see that \( \theta_k \to 0 \) as \( k \to \infty \), and

\[
\|u_k\|_{H^*(\mathbb{R}^N)} \leq \text{dist}_{\mathbb{R} \times H^r_{rad}(\mathbb{R}^N)}((\theta_k, u_k), \{0\} \times \gamma_k([0,1])) \to 0.
\]

Proceeding as in the proof of Theorem 7, we can prove that \( u_k \to u \) in \( H^*(\mathbb{R}^N) \), \( I'(u) = 0 \) and \( I(u) = b_1 \), for some \( u \in H^r_{rad}(\mathbb{R}^N) \) such that \( u \geq 0, u \neq 0 \). Since \( u \in C^{0,\beta}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) (see Lemma 11 before), we can apply the Harnack inequality [16, 25] to obtain \( u > 0 \) in \( \mathbb{R}^N \). This ends the proof of the first statement of theorem.

Now, let \( v \) is a nontrivial solution to (1.1) and we define

\[
\gamma(t)(x) = \begin{cases} v\left(\frac{x}{t}\right) & \text{for} \ t > 0 \\ 0 & \text{for} \ t = 0. \end{cases}
\]

Then we can see

\[
\|\gamma(t)\|_{H^*(\mathbb{R}^N)}^2 = t^{N-2s}[v]_{H^s(\mathbb{R}^N)}^2 + t^N\|v\|_{L^2}^2
\]

\[
I(\gamma(t)) = \frac{t^{N-2s}}{2}[v]_{H^s(\mathbb{R}^N)}^2 - t^N\int_{\mathbb{R}^N} G(v) \, dx.
\]

It is clear that \( \gamma \in C([0, \infty), H^s_{rad}(\mathbb{R}^N)) \).

Since every weak solution of (1.1) satisfies the Pohozaev Identity [18], we have

\[
\int_{\mathbb{R}^N} G(v) \, dx = \frac{N - 2s}{2N}[v]_{H^s(\mathbb{R}^N)}^2 > 0,
\]
so we can infer that 
\[ \frac{d}{dt} I(\gamma(t)) > 0 \text{ for } t \in (0, 1) \text{ and } \frac{d}{dt} I(\gamma(t)) < 0 \text{ for } t > 1. \]

Thus, after a suitable scale change in \( t \), there exists a path \( \gamma(t) : [0, 1] \to H^s_{rad}(\mathbb{R}^N) \) such that \( \gamma(0) = 0, I(\gamma(1)) < 0, v \in \gamma([0, 1]) \) and

\[ \max_{t \in [0, 1]} I(\gamma(t)) = I(v). \]

Therefore, \( b_{m_p,r} = b_1 \) is corresponding to a positive least energy solution to (1.1).

Now, we study regularity and symmetry of solutions of (1.1) in the case of “positive mass”.

**Lemma 11.** Let \( u \) be a positive solution to \( (-\Delta)^s u = g(u) \) in \( \mathbb{R}^N \), where \( g \in \mathcal{C}^{1,\alpha}(\mathbb{R}) \) is such that \( g'(0) < 0 \) and \( \lim_{|t| \to \infty} \frac{g(t)}{|t|^{2s-1}} = 0 \). Then \( u \in \mathcal{C}^{2,\beta}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) is radially symmetric and strictly decreasing about some point in \( \mathbb{R}^N \).

**Proof.** The proof of regularity of \( u \) is given in [18]. However, for reader’s convenience, we give a proof of it here. Since \( |g(t)| \leq C(1 + |t|^{2s-1}) \) for any \( t \in \mathbb{R} \) and \( u \geq 0 \), we can apply Proposition 5.1.1 in [21], to deduce that \( u \in L^\infty(\mathbb{R}^N) \). Taking into account \( g \in \mathcal{C}^{1,\alpha}(\mathbb{R}) \), we can use Lemma 4.4 in [16] to infer that \( u \in \mathcal{C}^{2,\beta}(\mathbb{R}^N) \) for some \( \beta \in (0, 1) \) depending only on \( s \) and \( \alpha \). Observing that \( u \in \mathcal{C}^{0,\beta}(\mathbb{R}^N) \cap L^{2s}(\mathbb{R}^N) \), we have \( u(x) \to 0 \) as \( |x| \to \infty \).

Now, we show that \( u \) is radially symmetric and strictly decreasing about some point in \( \mathbb{R}^N \). In order to achieve our aim, we follow some ideas developed in [22, 27].

For \( \lambda \in \mathbb{R} \), we define the following sets

\[ \Sigma_\lambda = \{(x_1, x') \in \mathbb{R}^N : x_1 > \lambda \} \]

and

\[ T_\lambda = \{(x_1, x') \in \mathbb{R}^N : x_1 = \lambda \} \]

and we denote by \( u_\lambda(x) = u(x^\lambda) \), where \( x^\lambda = (2\lambda - x_1, x') \).

We divide the proof into three steps.

Step 1: We show that \( \lambda_0 = \sup \{\lambda : u_\lambda \leq u \text{ in } \Sigma_\lambda\} \) is finite.

Let us define

\[ w(x) := \begin{cases} (u_\lambda - u)^+(x) & \text{if } x \in \Sigma_\lambda \\ (u_\lambda - u)^-(x) & \text{if } x \in \Sigma_\lambda^c. \end{cases} \]

Since \( w(x) = -w(x_\lambda) \) for any \( x \in \mathbb{R}^N \), we get \( \int_{\mathbb{R}^N} |w|^2 \, dx = 2 \int_{\Sigma_\lambda} |w|^2 \, dx \). By using the fact that if \( \varphi \in \mathcal{D}^{s,2}(\mathbb{R}^N) \) then \( \varphi_\lambda \in \mathcal{D}^{s,2}(\mathbb{R}^N) \), we can deduce that \( w \in H^s(\mathbb{R}^N) \).

Hence, we can use \( w \) as test function in the weak formulations of (1.1) and \( (-\Delta)^s u_\lambda = g(u_\lambda) \) in \( \mathbb{R}^N \), and subtracting them, we obtain

\[ \int_{\mathbb{R}^N} \frac{[(u_\lambda(x) - u(x)) - (u_\lambda(y) - u(y))]}{|x - y|^{N+2s}} (w(x) - w(y)) \, dx dy = \int_{\mathbb{R}^N} (g(u_\lambda) - g(u)) w \, dx. \]

Recalling that \( x^- = -(-x)^+ \) for any \( x \in \mathbb{R} \), we can note that

\[ \int_{\mathbb{R}^N} (g(u_\lambda) - g(u)) w \, dx = \int_{\Sigma_\lambda} (g(u_\lambda) - g(u))(u_\lambda - u)^+ \, dx + \int_{\Sigma_\lambda^c} (g(u_\lambda) - g(u))(u_\lambda - u)^- \, dx \]

\[ = 2 \int_{\Sigma_\lambda} (g(u_\lambda) - g(u))(u_\lambda - u)^+ \, dx. \]
Therefore, we have
\[
\iint_{\mathbb{R}^{2N}} \frac{[(u_{\lambda}(x) - u(x)) - (u_{\lambda}(y) - u(y))]}{|x - y|^{N+2s}} \,(w(x) - w(y)) \,dxdy = 2 \int_{\Sigma_{\lambda}} (g(u_{\lambda}) - g(u))(u_{\lambda} - u)^+ \,dx. \tag{4.8}
\]

Let us observe that
\[
\iint_{\mathbb{R}^{2N}} \frac{[(u_{\lambda}(x) - u(x)) - (u_{\lambda}(y) - u(y))]}{|x - y|^{N+2s}} \,(w(x) - w(y)) \,dxdy
= \iint_{\mathbb{R}^{2N}} \frac{|w(x) - w(y)|^2}{|x - y|^{N+2s}} \,dxdy + \iint_{\mathbb{R}^{2N}} \frac{G(x, y)}{|x - y|^{N+2s}} \,dxdy \tag{4.9}
\]
where
\[
G(x, y) := ((u_{\lambda}(x) - u(x)) - (u_{\lambda}(y) - u(y)) - (w(x) - w(y)))(w(x) - w(y)).
\]

Arguing as in the proof of Theorem 1.6 in [22] (see formula (3.29) there), we can show that
\[
\iint_{\mathbb{R}^{2N}} \frac{G(x, y)}{|x - y|^{N+2s}} \,dxdy \geq 0,
\]
which together with (4.8) and (4.9) yields
\[
\iint_{\mathbb{R}^{2N}} \frac{|w(x) - w(y)|^2}{|x - y|^{N+2s}} \,dxdy \leq 2 \int_{\Sigma_{\lambda}} (g(u_{\lambda}) - g(u))(u_{\lambda} - u)^+ \,dx. \tag{4.10}
\]

Now, we know that \(g'(0) < 0\), so we can find \(\delta > 0\) such that \(g'(t) < 0\) for all \(|t| < \delta\). Since \(u(x) \to 0\) as \(|x| \to \infty\), there exists \(R > 0\) such that \(0 < u(x) < \delta\) for all \(|x| > R\). Then, if \(x \in \Sigma_{\lambda}\) and \(\lambda < -R\), we deduce that \(u_{\lambda}(x) < \delta\), so we get
\[
\int_{\Sigma_{\lambda}} (g(u_{\lambda}) - g(u))(u_{\lambda} - u)^+ \,dx \leq 0 \tag{4.11}
\]
for all \(\lambda < -R\).

Putting together (4.10) and (4.11) we deduce that \(w\) is constant on \(\mathbb{R}^N\). Since \(w = 0\) on \(T_{\lambda}\), we can infer that \(u_{\lambda} \leq u\) in \(\Sigma_{\lambda}\) for all \(\lambda < -R\). This implies that \(\lambda_0 \geq -R\). Since \(u\) decays at infinity, there exists \(\lambda_1\) such that \(u(x) < u_{\lambda_1}(x)\) for some \(x \in \Sigma_{\lambda_1}\). Hence \(\lambda_0\) is finite.

Step 2: We prove that \(u = u_{\lambda_0}\) in \(\Sigma_{\lambda_0}\). We assume by contradiction that \(u \neq u_{\lambda_0}\) and \(u \geq u_{\lambda_0}\) in \(\Sigma_{\lambda_0}\). Suppose that there exists \(x_0 \in \Sigma_{\lambda_0}\) such that \(u_{\lambda_0}(x_0) = u(x_0)\) Then we can see that
\[
(-\Delta)^s u_{\lambda_0}(x_0) - (-\Delta)^s u(x_0) = g(u_{\lambda_0}(x_0)) - g(u(x_0)) = 0. \tag{4.12}
\]

On the other hand, observing that \(|x_0 - y| < |x_0 - y_{\lambda_0}|\) for all \(y \in \Sigma_{\lambda_0}\) and recalling that \(u \neq u_{\lambda_0}\), we get
\[
(-\Delta)^s u_{\lambda_0}(x_0) - (-\Delta)^s u(x_0) = - \int_{\mathbb{R}^N} \frac{u_{\lambda_0}(y) - u(y)}{|x_0 - y|^{N+2s}} \,dy
= - \int_{\Sigma_{\lambda_0}} (u_{\lambda_0}(y) - u(y)) \left( \frac{1}{|x_0 - y|^{N+2s}} - \frac{1}{|x_0 - y_{\lambda_0}|^{N+2s}} \right) \,dy > 0,
\]
which contradicts (4.12). Therefore \(u > u_{\lambda_0}\) in \(\Sigma_{\lambda_0}\).

In order to complete Step 2, we only need to prove that \(u \geq u_{\lambda}\) in \(\Sigma_{\lambda}\) continues to hold when \(\lambda > \lambda_0\) is close to \(\lambda_0\).

Fix \(\varepsilon > 0\) whose value will be chosen later, and take \(\lambda \in (\lambda_0, \lambda_0 + \varepsilon)\). Let \(P = (\lambda, 0)\) and
$B(P, R)$ be the ball centered at $P$ and with radius $R > 1$ to be chosen later. 
Set $\tilde{B} = \Sigma_\lambda \cap B(P, R)$. By using $w$ as test function in the equations for $u$ and $u_\lambda$, we have
\[
\iint_{\mathbb{R}^{2N}} \frac{[(u_\lambda(x) - u(x)) - (u_\lambda(y) - u(y))]}{|x - y|^{N+2s}}(w(x) - w(y)) \, dx \, dy \\
= 2 \iint_{\Sigma_\lambda} (g(u_\lambda) - g(u))(u_\lambda - u)^+ \, dx \\
= 2 \iint_{\tilde{B}} (g(u_\lambda) - g(u))(u_\lambda - u)^+ \, dx + 2 \iint_{\Sigma_\lambda \setminus \tilde{B}} (g(u_\lambda) - g(u))(u_\lambda - u)^+ \, dx. \tag{4.13}
\]
Since $u$ goes to zero at infinity, there exists $R_0 > 0$ such that $u(x) < \delta$ for all $x \in B_{R_0}^c(0)$. Choose $R > 0$ such that $\Sigma_\lambda \setminus \tilde{B} \subset B^c(P, R) \subset B_{R_0}^c(0)$. Therefore, $u_\lambda(x) < \delta$ for all $x \in \Sigma_\lambda \setminus \tilde{B}$, and we can see that
\[
\iint_{\Sigma_\lambda \setminus \tilde{B}} (g(u_\lambda) - g(u))(u_\lambda - u)^+ \, dx \leq 0.
\]
This and (4.13) yield
\[
\iint_{\mathbb{R}^{2N}} \frac{[(u_\lambda(x) - u(x)) - (u_\lambda(y) - u(y))]}{|x - y|^{N+2s}}(w(x) - w(y)) \, dx \, dy \leq 2 \iint_{\tilde{B}} (g(u_\lambda) - g(u))(u_\lambda - u)^+ \, dx.
\]
Now, arguing as in Step 1, and by using Theorem 3, we can find a positive constant $C_0 > 0$ such that
\[
\iint_{\mathbb{R}^{2N}} \frac{[(u_\lambda(x) - u(x)) - (u_\lambda(y) - u(y))]}{|x - y|^{N+2s}}(w(x) - w(y)) \, dx \, dy \geq \iint_{\mathbb{R}^{2N}} \frac{|w(x) - w(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \geq C_0 ||w||^2_{L^2_s(\Sigma_\lambda)}.
\]
Then, we deduce that
\[
||w||^2_{L^2_s(\Sigma_\lambda)} \leq 2C_0^{-1} \iint_{\tilde{B}} (g(u_\lambda) - g(u))(u_\lambda - u)^+ \, dx. \tag{4.14}
\]
By using Hölder inequality, we can see that
\[
2C_0^{-1} \iint_{\tilde{B}} (g(u_\lambda) - g(u))(u_\lambda - u)^+ \, dx \leq C \int_{\tilde{B}} |w|^2 \chi_{\text{supp}(u_\lambda - u)^+} \, dx \\
= C |\tilde{B} \cap \text{supp}(u_\lambda - u)^+|^{\frac{2s}{N}} \left( \int_{\tilde{B}} |w|^{\frac{2N}{N-2s}} \, dx \right)^{\frac{N-2s}{N}}. \tag{4.15}
\]
Taking into account $u > u_{\lambda_0}$ in $\Sigma_{\lambda_0}$ and the continuity of $u$, we can see that $u > u_{\lambda}$ on any compact $K \subset \Sigma_\lambda$, for $\lambda$ close to $\lambda_0$. This means that $\text{supp}(u_\lambda - u)^+$ is small in $\tilde{B}$ for $\lambda$ close to $\lambda_0$, so we can choose $\varepsilon > 0$ such that
\[
C |\tilde{B} \cap \text{supp}(u_\lambda - u)^+|^{\frac{2s}{N}} < \frac{1}{2}. \tag{4.16}
\]
Putting together (4.14), (4.15) and (4.16) we deduce that $w = 0$ in $\Sigma_\lambda$, which gives a contradiction. This concludes the proof of Step 2.

Step 3: By translation, we may say that $\lambda_0 = 0$. Thus, $u(-x_1, x') \geq u(x_1, x')$ for $x_1 \geq 0$. A similar argument in $x_1 < 0$ shows that $u(-x_1, x') \leq u(x_1, x')$ for $x_1 \geq 0$, so we have
and Lemma 10.1.1. Proof of Theorem 10. It follows by Theorem 10 and Lemma 11.

Proof of Theorem 1. Putting together Theorem 8, Theorem 9 and Lemma 11, we obtain the desired result.

5. The zero mass case: Proof of Theorem 2

This last section is devoted to the proof of the existence of a nontrivial solution of (1.1) in the zero mass case. Let \( \varepsilon_0 = \frac{G(0)}{\Pi^2} > 0 \) and let us define \( g_\varepsilon(t) = g(t) - \varepsilon t \) with \( \varepsilon \in (0, \varepsilon_0] \).

Then \( g_\varepsilon \) satisfies the assumption of Theorem 9, so we know that for any \( \varepsilon \in (0, \varepsilon_0] \) there exists \( u_\varepsilon \in H^s_\text{rad}(\mathbb{R}^N) \) positive and radially decreasing in \( r = |x| \), such that \( I_\varepsilon(u_\varepsilon) = b_\varepsilon \) and \( I_\varepsilon'(u_\varepsilon) = 0 \), where the mountain pass minimax value \( b_\varepsilon \) is defined as \( b_\varepsilon = \inf_{t \in [0,1]} I_\varepsilon(\gamma(t)) \) with \( \Gamma_\varepsilon = \{ \gamma \in C([0,1], H^s_\text{rad}(\mathbb{R}^N)) : \gamma(0) = 0, I_\varepsilon(\gamma(1)) < 0 \} \). Since \( u_\varepsilon \) satisfies the Pohozaev identity (3.7), we can see that \( \frac{\varepsilon}{2} [u_\varepsilon]_{H^s(\mathbb{R}^N)}^2 = b_\varepsilon \). Now we prove that it is possible to estimate \( b_\varepsilon \) from above independently of \( \varepsilon \):

Lemma 12. There exists \( b_0 > 0 \) such that \( 0 < b_\varepsilon \leq b_0 \) for all \( \varepsilon \in (0, \varepsilon_0] \).

Proof. For \( R > 1 \), define

\[
\begin{align*}
w_R(x) &= \begin{cases} 
\xi_0 & \text{for } |x| \leq R \\
\xi_0 (R + 1 - r) & \text{for } r = |x| \in [R, R + 1] \\
0 & \text{for } |x| \geq R + 1.
\end{cases}
\end{align*}
\]

It is clear that \( w_R \in H^s(\mathbb{R}^N) \). By the definition of \( w_R \) and \( g_\varepsilon(t) = g(t) - \varepsilon t \), we deduce that for all \( \varepsilon \in (0, \varepsilon_0] \)

\[
\begin{align*}
\int_{\mathbb{R}^N} G_\varepsilon(w_R) \, dx & \geq \int_{\mathbb{R}^N} G_{\varepsilon_0}(w_R) \, dx \\
& = G_{\varepsilon_0}(\xi_0) |B_R| + \int_{\{r \leq |x| \leq r + 1\}} G_{\varepsilon_0}(\xi_0 (R + 1 - |x|)) \, dx \\
& \geq G_{\varepsilon_0}(\xi_0) |B_R| - |B_{R+1} - B_R| \max_{t \in [0, \xi_0]} |G_{\varepsilon_0}(t)| \\
& = \frac{\pi^N}{\Gamma(\frac{N}{2} + 1)} [G_{\varepsilon_0}(\xi_0) R^N - \max_{t \in [0, \xi_0]} |G_{\varepsilon_0}(t)| ((R + 1)^N - R^N)] \\
& \geq \frac{\pi^N}{\Gamma(\frac{N}{2} + 1)} [G_{\varepsilon_0}(\xi_0) - \max_{t \in [0, \xi_0]} |G_{\varepsilon_0}(t)| ((1 + \frac{1}{R})^N - 1)] R^N.
\end{align*}
\]
so there exists $\bar{R} > 0$ (independent of $\varepsilon$) such that $\int_{\mathbb{R}^N} G_\varepsilon (w_R) \, dx > 0$ for all $R \geq \bar{R}$ and $\varepsilon \in (0, \varepsilon_0]$. Then, by setting $w_t(x) = w_R(x/t)$ we have for any $\varepsilon \in (0, \varepsilon_0]$

$$I_\varepsilon (w_t) = \frac{t^{N-2s}}{2} [w_R]_{H^s (\mathbb{R}^N)}^2 - t^N \int_{\mathbb{R}^N} G_\varepsilon (w_R) \, dx \to -\infty$$  \hspace{1cm} (5.1)

as $t \to \infty$. Hence there exists $\tau > 0$ such that $I_\varepsilon (w_\tau) < 0$ for any $\varepsilon \in (0, \varepsilon_0]$. We put $\bar{e}(x) = w_\tau (x)$. Therefore,

$$I_\varepsilon (u_\varepsilon) = b_\varepsilon \leq \sup_{\varepsilon \in (0, \varepsilon_0]} \max_{t \in [0, 1]} I_\varepsilon (t \bar{e}) =: b_0$$

for any $\varepsilon \in (0, \varepsilon_0]$.

In view of Lemma 12, we can infer that

$$[u_\varepsilon]_{H^s (\mathbb{R}^N)}^2 = \frac{N}{s} b_\varepsilon \leq \frac{N}{s} b_0 \text{ for all } \varepsilon \in (0, \varepsilon_0],$$  \hspace{1cm} (5.2)

and as a consequence, we can assume that as $\varepsilon \to 0$

$$u_\varepsilon \rightharpoonup u \text{ in } \mathcal{D}^{s,2} (\mathbb{R}^N)$$
$$u_\varepsilon \to u \text{ in } L^q_{loc} (\mathbb{R}^N), \text{ for any } q \in [2, 2^*_s)$$
$$u_\varepsilon \to u \text{ a.e. in } \mathbb{R}^N.$$  \hspace{1cm} (5.3)

Since $u_\varepsilon$ is a weak solution to (1.1), we know that

$$\langle u_\varepsilon, \varphi \rangle_{\mathcal{D}^{s,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} [g(u_\varepsilon) - \varepsilon u_\varepsilon] \varphi \, dx,$$  \hspace{1cm} (5.4)

for any $\varphi \in C_0^\infty (\mathbb{R}^N)$. Taking into account (5.2), (5.3), (h1)-(h2), $\mathcal{D}^{s,2} (\mathbb{R}^N) \subset L^{2^*_s} (\mathbb{R}^N)$ we can apply the first part of Lemma 2 with $P(t) = g(t)$, $Q(t) = |t|^{2^*_s - 1}$, $v_\varepsilon = u_\varepsilon$, $v = u$ and $w = \varphi$ to pass to the limit in (5.4) as $\varepsilon \to 0$. Then we obtain

$$\langle u, \varphi \rangle_{\mathcal{D}^{s,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} g(u) \varphi \, dx,$$  for any $\varphi \in C_0^\infty (\mathbb{R}^N)$. This means that $u$ is a weak solution to $(-\Delta)^s u = g(u)$ in $\mathbb{R}^N$. Now, we only need to prove that $u \not\equiv 0$.

We recall that $u_\varepsilon > 0$ and satisfies the Pohozaev identity (3.7) with $G$ replaced by $G_\varepsilon$, that is

$$\frac{N - 2s}{2} [u_\varepsilon]_{H^s (\mathbb{R}^N)}^2 = N \int_{\mathbb{R}^N} \left( G(u_\varepsilon) - \frac{\varepsilon}{2} w_\varepsilon^2 \right) \, dx.$$  \hspace{1cm} (5.5)

Now, by using (h1)-(h2), there exists $c > 0$ such that

$$G(t) \leq c |t|^{2^*_s} \text{ for all } t \in \mathbb{R}.$$  \hspace{1cm} (5.6)
Then, in view of (5.5) and (5.6), we get
\[
[u_\varepsilon]^2_{H^s(\mathbb{R}^N)} \leq \frac{2N}{N-2s} \varepsilon \| u_\varepsilon \|_{L^2(\mathbb{R}^N)}^2 + [u_\varepsilon]^2_{H^s(\mathbb{R}^N)} \\
= \frac{2N}{N-2s} \int_{\mathbb{R}^N} G(u_\varepsilon) \, dx \\
\leq \frac{2N}{N-2s} c \int_{\mathbb{R}^N} |u_\varepsilon|^{2^*_s} \, dx \\
\leq C[u_\varepsilon]^{2^*_s}_{H^s(\mathbb{R}^N)}
\]
so we can see that
\[
[u_\varepsilon]_{H^s(\mathbb{R}^N)} \geq c > 0, \quad (5.7)
\]
for some $c$ independent of $\varepsilon$.

Next, we argue by contradiction, and we assume that $u = 0$. We begin proving that
\[
\int_{\mathbb{R}^N} G^+(u_\varepsilon) \, dx \to 0 \text{ as } \varepsilon \to 0. \quad (5.8)
\]
By using Lemma 1 with $t = 2^*_s$, Theorem 3 and (5.2), we can see that, for any $x \in \mathbb{R}^N \setminus \{0\}$ and $\varepsilon \in (0, \varepsilon_0]$
\[
|u_\varepsilon(x)| \leq \left( \frac{N}{\omega_{N-1}} \right)^{\frac{1}{2^*_s}} |x|^{-\frac{N}{2^*_s}} \| u_\varepsilon \|_{L^{2^*_s}(\mathbb{R}^N)} \\
\leq \left( \frac{N}{\omega_{N-1}} \right)^{\frac{1}{2^*_s}} |x|^{-\frac{N}{2^*_s}} S_s[u_\varepsilon]_{H^s(\mathbb{R}^N)} \\
\leq \left( \frac{N}{\omega_{N-1}} \right)^{\frac{1}{2^*_s}} |x|^{-\frac{N}{2^*_s}} S_s \sqrt{\frac{N}{s} b_0} = C|x|^{-\frac{N}{2^*_s}}, \quad (5.9)
\]
where $C$ is independent of $\varepsilon$. By (5.9), for any $\delta > 0$ there exists $R > 0$ such that $|u_\varepsilon(x)| \leq \delta$ for $|x| \geq R$. Moreover by assumption $(h_1)$, for any $\eta > 0$ there exists $\delta > 0$ such that
\[
G^+(t) \leq \eta |t|^{2^*_s} \text{ for } |t| \leq \delta.
\]
Hence for large $R$ we obtain the following estimate
\[
\int_{\mathbb{R}^N \setminus B_R} G^+(u_\varepsilon) \, dx \leq \eta \int_{\mathbb{R}^N \setminus B_R} |u_\varepsilon|^{2^*_s} \, dx \\
\leq C\eta[u_\varepsilon]^{2^*_s}_{H^s(\mathbb{R}^N)} \leq C\eta,
\]
uniformly in $\varepsilon \in (0, \varepsilon_0]$. On the other hand, by assumption $(h_2)$, for any $\eta > 0$ there exists a constant $C_\eta$ such that
\[
G^+(t) \leq \eta |t|^{2^*_s} + C_\eta \text{ for all } t \in \mathbb{R}.
\]
Now, we fix $\Omega \subset \mathbb{R}^N$ with sufficiently small measure $|\Omega| < \frac{\eta}{C_\eta}$. Then, for any $\varepsilon \in (0, \varepsilon_0]$
\[
\int_{\Omega} G^+(u_\varepsilon) \, dx \leq \eta \int_{\Omega} |u_\varepsilon|^{2^*_s} \, dx + C_\eta |\Omega| \leq C\eta.
\]
By applying Vitali’s convergence Theorem, we can deduce that (5.8) is satisfied. Therefore, by using the Pohozaev Identity (5.5) and the facts $G = G^+ + G^-$ and $G^- \leq 0$, we have

$$[u_\varepsilon]^2_{H^s(\mathbb{R}^N)} \leq \frac{2N}{N-2s} \int_{\mathbb{R}^N} G^-(u_\varepsilon) \, dx + \frac{2N}{N-2s} \int_{\mathbb{R}^N} \frac{\varepsilon}{2} |u_\varepsilon|^2 \, dx$$

$$= \frac{2N}{N-2s} \int_{\mathbb{R}^N} G^+(u_\varepsilon) \, dx$$

which together with (5.8) yields $0 < c \leq \limsup_{\varepsilon \to 0} [u_\varepsilon]^2_{H^s(\mathbb{R}^N)} = 0$. This gives a contradiction in view of (5.7). Then $u \in \mathcal{D}^{s,2}_{rad}(\mathbb{R}^N)$ is a nontrivial weak solution to (1.1). Arguing as in the proof of Lemma 11, we can see that $u \in C^{0,\beta}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. From the Harnack inequality [16, 25] we conclude that $u > 0$ in $\mathbb{R}^N$.

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Dipartimento di Scienze Pure e Applicate (DiSPeA), Università degli Studi di Urbino ‘Carlo Bo’ Piazza della Repubblica, 13 61029 Urbino (Pesaro e Urbino, Italy)

E-mail address: vincenzo.ambrosio@uniurb.it