MULTI-ANISOTROPIC GEVREY REGULARITY
OF HYPOELLIPTIC OPERATORS

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Abstract. We show a multi-anisotropic Gevrey regularity of solutions of hypoelliptic equations. This result is a precision of a classical result of Hörmander

1. Introduction

An important problem among others of linear partial differential equations is the $C^\infty$ or Gevrey regularity of solutions of these equations. L. Hörmander has completely characterized the $C^\infty$ regularity (hypoellipticity) of linear partial differential operators with complex constant coefficients, see [7]. An another fundamental result obtained by L. Hörmander says that every hypoelliptic differential operator $P(D)$ is anisotropic Gevrey hypoelliptic, i.e. $\exists \varrho = (\varrho_1, \ldots, \varrho_n) \in \mathbb{R}_+^n$ such that

$$u \in \mathcal{D}'(\Omega) \text{ and } P(D)u = 0 \implies u \in G^\varrho(\Omega),$$

where $G^\varrho(\Omega)$ is an anisotropic Gevrey space associated with $P(D)$.

A large class of hypoelliptic differential operators is the class of multi-quasielliptic differential operators, see V. P. Mikhaïlov [8], J. Friberg [3] and S. G. Gindikin, L. R. Volevich [4].

L. Zanghirati [10], proved that multi-quasielliptic differential operators are multi-anisotropic Gevrey hypoelliptic, i.e.

$$u \in \mathcal{D}'(\Omega) \text{ and } P(D)u \in G^{s, \Gamma}(\Omega) \implies u \in G^{s, \Gamma}(\Omega),$$

where $G^{s, \Gamma}(\Omega)$ is a Gevrey multi-anisotropic space associated with $P(D)$. This result clarifies the classical result of Hörmander in the case of multi-quasielliptic operators. The result of L. Zanghirati has been extended by C. Bouzar and R. Chaïli in [1] to multi-quasielliptic systems of differential operators.

The aim of this paper is to prove the multi-anisotropic Gevrey regularity of hypoelliptic linear differential operators with complex constant coefficients, and consequently we precise the result of Hörmander and extend the result of Zanghirati.

2. Multi-quasiellipticity

Let $\Omega$ be an open subset of $\mathbb{R}_+^n$, if $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, $q = (q_1, \ldots, q_n) \in \mathbb{R}_+^n$ and $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}_+^n$, we set

$$|\alpha| = \alpha_1 + \ldots + \alpha_n$$

$$< \alpha, q > = \sum_{j=1}^n \alpha_j q_j$$

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\[ \xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} \]
\[ D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}, D_j = \frac{1}{i} \partial_{\xi_j}, \ j = 1, \ldots, n. \]
\[ \mathbb{R}^n_+ = \{ \xi \in \mathbb{R}^n : \xi_j > 0, \ j = 1, \ldots, n \} \]

The space \( \mathcal{C}_0^\infty (\Omega) \) is the space of functions \( u \in \mathcal{C}_0^\infty \) with compact support in \( \Omega \). The space of distributions on \( \Omega \) is denoted \( \mathcal{D}' (\Omega) \).

**Definition 1.** Let \( A \) be a finite subset of \( \mathbb{R}^n_+ \), the Newton's polyhedron of \( A \), denoted \( \Gamma (A) \), is the convex hull of \( \{ 0 \} \cup A \).

A Newton's polyhedron \( \Gamma \) is always characterized by
\[
\Gamma = \bigcap_{q \in A (\Gamma)} \{ \alpha \in \mathbb{R}^n_+, \langle q, \alpha \rangle \leq 1 \},
\]
where \( A (\Gamma) \) is a finite subset of \( \mathbb{R}^n \ \{ 0 \} \).

**Definition 2.** Let \( \Gamma = \bigcap_{q \in A (\Gamma)} \{ \alpha \in \mathbb{R}^n_+, \langle q, \alpha \rangle \leq 1 \} \) be a Newton's polyhedron, \( \Gamma \) is said to be regular, if
\[
q_j > 0, \ \forall j = 1, \ldots, n; \ \forall q = (q_1, \ldots, q_n) \in A (\Gamma)
\]
We associate with a regular Newton's polyhedron \( \Gamma \) the following elements
\[
\mathcal{V} (\Gamma) = \{ s^0 = 0, s^1, \ldots, s^{\alpha (\Gamma)} \} \text{ the set of vertices of } \Gamma
\]
\[
|\xi|_\Gamma = \sum_{\nu \in \mathcal{V} (\Gamma)} |\xi^{\nu}|, \ \xi \in \mathbb{R}^n, \ \text{where } |\xi|^\nu = |\xi_1|^{|\nu_1|} \cdots |\xi_n|^{|\nu_n|}
\]
\[
k (\alpha, \Gamma) = \inf \{ t > 0, t^{-1} \alpha \in \Gamma \} = \max_{q \in A (\Gamma)} \langle \alpha, q \rangle, \ \alpha \in \mathbb{R}^n_+
\]
\[
\mu (\Gamma) = \max_{q \in A (\Gamma)} q_j^{-1} \text{ called the formal order of } \Gamma
\]
\[ 1 \leq j \leq n \]

A differential operators with complex constant coefficients
\[ P(D) = \sum_{\alpha} a_\alpha D^\alpha \]
has its complete symbol
\[ P(\xi) = \sum_{\alpha} a_\alpha \xi^\alpha \]

**Definition 3.** The Newton's polyhedron of \( P \), denoted \( \Gamma (P) \), is the convex hull of the set
\[ \{ 0 \} \cup \{ \alpha \in \mathbb{Z}^n_+ : a_\alpha \neq 0 \} \]
Define the weight function
\[ |\xi|_P = \sum_{\alpha \in \mathcal{V} (P)} |\xi_\alpha|, \ \forall \xi \in \mathbb{R}^n, \]
where \( \mathcal{V} (P) = \mathcal{V} (\Gamma (P)) \) is the set of vertices of \( \Gamma (P) \). Recall
\[ d(\xi) := \text{dist}(\xi, N(P)), \ \text{where } N(P) := \{ \zeta \in \mathbb{C}^n : P(\zeta) = 0 \} \]
Definition 4. The differential operator $P(D)$ is said hypoelliptic in $\Omega$, if

$$\text{singsupp} P(D)u = \text{singsupp} u, \forall u \in \mathcal{D}'(\Omega)$$

The characterization of hypoelliptic differential operators with constant coefficients is due to L. Hörmander. The following result, see the theorem 4.1.3 of [7], gives some characterizations of the hypoellipticity.

Theorem 1. Let $P(D)$ be a differential operator with constant coefficients, the following properties are equivalent:

i) The operator $P(D)$ is hypoelliptic.

ii) $\exists C > 0, \exists d > 0, |\xi|^d \leq Cd(\xi), \forall \xi \in \mathbb{R}^n, |\xi| \text{ large}.$

iii) If $\xi \in \mathbb{R}^n, |\xi| \to +\infty$, then $\frac{|D^\alpha P(\xi)|}{|P(\xi)|} \to 0, \forall \alpha \neq 0.$

iv) $\exists C > 0, \exists \rho > 0, \frac{|D^\alpha P(\xi)|}{|P(\xi)|} \leq C |\xi|^{-\rho|\alpha|}, \forall \xi \in \mathbb{R}^n, |\xi| \text{ large}.$

The connection between an hypoelliptic operator and its Newton’s polyhedron is given by the following proposition.

Proposition 2. The Newton’s polyhedron of an hypoelliptic differential operator is regular.

Proof. See [3].

Remark 1. The converse is not true, $\Box = D_x^2 - D_y^2$ has a regular Newton’s polyhedron with vertices $\{(0,0), (2,0), (0,2)\}$, but the operator $\Box$ is not hypoelliptic.

We introduce multi-quasielliptic polynomials which are a natural generalization of the classical quasi-elliptic operators. These operators were characterized first by V. P. Mikhaïlov [8], then studied by J. Friberg [3] and finally far developed by S. G. Gindikin and L. R. Volevich [4].

Definition 5. The polynomial $P(\xi) = \sum a_\alpha \xi^\alpha$ is said to be multi-quasielliptic, if

i) its Newton’s polyhedron $\Gamma(P)$ is regular.

ii) $\exists C > 0$ such that $|\xi|_p \leq C(1 + |P(\xi)|), \forall \xi \in \mathbb{R}^n$

Proposition 3. A multi-quasielliptic operator $P(D)$ is hypoelliptic.

Proof. See [3] or [4].

Remark 2. The converse is not true. Indeed, consider the following polynomial

$$P(\xi, \eta) = i\xi^5 + i\xi\eta^4 - 4i\xi^4\eta - 4i\xi^2\eta^3 + 6i\xi^3\eta^2 + i\xi^3 + i\xi\eta^2 + \xi^4\eta^2 + \eta^6 - 4\xi^3\eta^3 - 4\xi\eta^5 + 6\xi^2\eta^4 + \eta^2\xi^2 + \eta^4,$$

which is hypoelliptic thanks to the theorem 4.1.9 of [7]. We have

$$P_{(1,1)}(\xi, \eta) = \eta^2 \left( \xi^4 + \eta^4 - 4\xi^3\eta - 4\xi\eta^3 + 6\xi^2\eta^2 \right) = \eta^2 (\xi - \eta)^4$$

The $q = (1,1)$-quasiprincipal part of $P(\xi, \eta)$ degenerates on the straight $\xi = \eta$, hence the polynomial $P(\xi, \eta)$ is not multi-quasielliptic, see [4].
3. Multi-anisotropic Gevrey vectors

The multi-anisotropic Gevrey spaces were explicitly defined by L. Zanghirati in [10] for studying the multi-anistropic Gevrey regularity of multi-quasielliptic differential operators by the method of elliptic iterates.

Definition 6. Let $\Omega$ be an open subset of $\mathbb{R}^n$, $\Gamma$ a regular Newton’s polyhedron and $s \geq 1$. Denote $G^{s, \Gamma}(\Omega)$ the space of functions $u \in C^\infty(\Omega)$ such that $\forall K \subset \Omega, \exists C > 0, \forall \alpha \in \mathbb{Z}^n_+$,
\[
\sup_{x \in K} |D^\alpha u(x)| \leq C^{s|\alpha|+1}(\alpha, \Gamma)^{s\mu(\alpha, \Gamma)}
\]

Example 1. If $\Gamma$ is the regular Newton’s polyhedron defined by
\[
\Gamma = \left\{ \alpha \in \mathbb{R}^n_+ : \sum_{j=1}^n m_j^{-1}\alpha_j \leq 1, m_j \in \mathbb{R}_+ \right\},
\]
then
\[
G^{s, \Gamma}(\Omega) = \left\{ u \in C^\infty(\Omega), \forall K \subset \Omega, \exists C > 0, \forall \alpha \in \mathbb{Z}^n_+ \right\},
\]
where $q := \left( \frac{m_1}{m_1}, ..., \frac{m_n}{m_n} \right)$ and $m := \max_j m_j$, i.e. $G^{s, \Gamma}(\Omega)$ is the classical anisotropic Gevrey space $G^{s, q}(\Omega)$. If $m_1 = m_2 = ... = m_n$, we obtain the classical isotropic Gevrey space $G^s(\Omega)$.

Definition 7. Let $\Gamma$ be the regular Newton’s polyhedron of $P(D)$ and $s \geq 1$, the space of Gevrey vectors of $P(D)$, denoted $G^s(\Omega, P)$, is the space of $u \in C^\infty(\Omega)$ such that, $\forall K$ compact of $\Omega, \exists C > 0, \forall l \in \mathbb{N}$,
\[
\|P^l u\|_{L^\infty(K)} \leq C^{l+1}(l!)^{s\mu(\Gamma)}
\]

Remark 3. We can take $t^{s\mu(\Gamma)}$ instead of $(l!)^{s\mu(\Gamma)}$.

We recall a result of L. Zanghirati [10] and C. Bouzar and R. Chaïli [1] which gives the multi-anisotropic Gevrey regularity of Gevrey vectors of multi-quasielliptic operators.

Theorem 4. Let $\Omega$ be an open subset of $\mathbb{R}^n$, $s > 1$ and $P$ a linear differential operator with complex constant coefficients with regular Newton’s polyhedron $\Gamma$. Then the following assertions are equivalent:

i) $P$ is multi-quasielliptic in $\Omega$

ii) $G^s(\Omega, P) = G^{s, \Gamma}(\Omega)$

4. Multi-anisotropic Gevrey hypoellipticity of hypoelliptic operators

In this section, $P = \sum a_\alpha D^\alpha$ is an hypoelliptic differential operator with complex constant coefficients.

Definition 8. A finite set $H \subset \mathbb{R}^n_+$ is said a polyhedron of hypoellipticity of $P$, if

1. $\forall \nu \in H, \exists C > 0, \forall \xi \in \mathbb{R}^n, |\xi|^{l'} \leq C (1 + d(\xi))$

2. $H$ has vertices with rational components.

3. $H$ is regular.
Remark 4. If \( \nu \) belongs to the convex hull of \( \mathcal{H} \), i.e. \( \nu = \sum_{i \in I} \lambda_i \beta_i \), where \( \beta_i \in \mathcal{H} \) and \( \sum_{i \in I} \lambda_i = 1, \lambda_i \geq 0 \), then \( |\xi|^\nu \leq C (1 + d(\xi)) \), \( \forall \xi \in \mathbb{R}^n \), therefore it is natural to assume that \( \mathcal{H} \) is convex.

Remark 5. The set \( \mathcal{H} \) is never empty, as an hypoelliptic operator satisfies: \( \exists C > 0, \exists d > 0, |\xi|^d \leq C (1 + d |\xi|) \), \( \forall \xi \in \mathbb{R}^n \).

Definition 9. Denote \( \sigma \) be the smallest natural integer such that \( \sigma \mathcal{V}(\mathcal{H}) \subset 2\mathbb{N}_0^n \), and define the differential operator \( Q_{\mathcal{H}}(D) \), by

\[
Q_{\mathcal{H}}(D) = \sum_{\alpha \in \mathcal{V}(\mathcal{H})} D^{\sigma \alpha}
\]

Proposition 5. The operator \( Q_{\mathcal{H}}(D) \) is multi-quasielliptic.

Proof. The Newton’s polyhedron of the differential operator \( Q_{\mathcal{H}} \) has vertices with even positive integer components. Then

\[
|Q_{\mathcal{H}}(\xi)| = \sum_{\alpha \in \mathcal{V}(\mathcal{H})} |\xi^{\sigma \alpha}| = |\xi|_{Q_{\mathcal{H}}},
\]

hence

\[
1 + |\xi|_{Q_{\mathcal{H}}} \leq (1 + |Q_{\mathcal{H}}(\xi)|), \forall \xi \in \mathbb{R}^n
\]

Lemma 6. Let \( u \) be a solution of the equation \( Pu = 0 \) defined in the ball \( B_\varepsilon = \{ x \in \mathbb{R}^n : |x| < \varepsilon \} \), and let \( \varphi \in C_0^\infty(B_1) \) and the integer \( s \geq 1 \). Then

\[
\sum_{\alpha \neq 0} \varepsilon^{-2|\alpha|} \left( \int_{B_\varepsilon} |P^{(\alpha)}(D)(\varphi^\varepsilon u)|^2 \right)^{1/2} \leq C \sum_{\alpha \neq 0} \varepsilon^{-2|\alpha|} \left( \int_{B_\varepsilon} |P^{(\alpha)}(D)u|^2 \right)^{1/2} dx,
\]

where \( C \) is independent of \( \varepsilon \) and \( u \).

Remark 6. In the lemma \( \varphi^\varepsilon \) denotes \( \varphi^\varepsilon(x) := \varphi(\frac{x}{\varepsilon}) \).

Thanks to this lemma, we obtain the following result.

Lemma 7. Let \( \beta \in \mathbb{Z}_n^+ \cap \sigma \mathcal{H} \), then there exists a constant \( C > 0 \), such that for every solution \( u \) of \( Pu = 0 \) in \( B_\varepsilon \) and \( \varepsilon \in [0,1] \), we have

\[
\varepsilon^{2\alpha} \sum_{\alpha \neq 0} \varepsilon^{-2|\alpha|} \left( \int_{B_\varepsilon^2} |P^{(\alpha)}(D)D^\beta u|^2 \right)^{1/2} dx \leq C \sum_{\alpha \neq 0} \varepsilon^{-2|\alpha|} \left( \int_{B_\varepsilon} |P^{(\alpha)}(D)u|^2 \right)^{1/2} dx
\]
Proof. Let $\beta \in \mathbb{Z}^n_+ \cap \sigma \mathcal{H}$, from (1) of definition 4.1, we have
\[ |\xi^\beta| \leq C^\sigma (1 + d(\xi))^\sigma, \]
hence $\exists C > 0, \forall \xi \in ]0, 1[, \forall \xi \in \mathbb{R}^n$,
\[ (3) \quad \varepsilon^\sigma |\xi^\beta| \leq C^\sigma d_{\sigma, \varepsilon}(\xi) \]
Multiplying (4.2) by $(2\pi)^{-n} |\hat{v}(\xi)|$ and integrating with respect to $\xi$, we obtain
\[ (4) \quad \varepsilon^{2\alpha} \int |D^\beta v|^2 \ dx \leq C^2 ||v||^2_{\sigma, \varepsilon} \]
Let $\varphi \in C^{\infty}_0(B_1)$ equals 1 in $B_{\frac{1}{2}}$ and apply the estimate (4.3) to $v = P(\alpha) (D) (\varphi^\varepsilon u)$, then
\[ \varepsilon^{2\alpha} \int |P(\alpha) (D) D^\beta (\varphi^\varepsilon u)|^2 \ dx \leq C^2 ||P(\alpha) (D) (\varphi^\varepsilon u)||^2_{\sigma, \varepsilon} \]
\[ \varepsilon^{2\alpha} \sum_{\alpha \neq 0} \varepsilon^{-2|\alpha|} \int |P(\alpha) (D) D^\beta (\varphi^\varepsilon u)|^2 \ dx \leq C^2 \sum_{\alpha \neq 0} \varepsilon^{-2|\alpha|} ||P(\alpha) (D) (\varphi^\varepsilon u)||^2_{\sigma, \varepsilon} \]
consequently lemma 4.6 gives
\[ \varepsilon^{2\alpha} \sum_{\alpha \neq 0} \varepsilon^{-2|\alpha|} \int |P(\alpha) (D) D^\beta (\varphi^\varepsilon u)|^2 \ dx \leq C^2 \sum_{\alpha \neq 0} \varepsilon^{-2|\alpha|} \int |P(\alpha) (D) (u)|^2 \ dx \]
As $\varphi^\varepsilon(x) = \varphi\left(\frac{x}{\varepsilon}\right) = 1$ in $B_{\frac{1}{2}}$, then
\[ \varepsilon^{2\alpha} \sum_{\alpha \neq 0} \varepsilon^{-2|\alpha|} \int_{B_{\frac{1}{2}}} |P(\alpha) (D) D^\beta u|^2 \ dx \leq C \sum_{\alpha \neq 0} \varepsilon^{-2|\alpha|} \int_{B_1} |P(\alpha) (D) (u)|^2 \ dx \]
\[ \square \]

**Proposition 8.** Let $\Omega$ be a bounded open set in $\mathbb{R}^n$ and $\beta \in \mathbb{Z}^n_+ \cap \sigma \mathcal{H}$, then there exists a constant $C > 0$, such that for every $u$ solution of $Pu = 0$ in $\Omega$ and $\delta \in ]0, 1[$, we have
\[ \sum_{\alpha \neq 0} \delta^{-2|\alpha|} \int_{\Omega_\delta} |P(\alpha) (D) D^\beta u|^2 \ dx \leq C \delta^{-2\sigma} \sum_{\alpha \neq 0} \delta^{-2|\alpha|} \int_{\Omega} |P(\alpha) (D) (u)|^2 \ dx, \]
where
\[ \Omega_\delta = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \delta \} \]
Proof. The proof is obtained from the precedent lemma and follows the same reasoning as the proof of theorem 4.4.2 of [7].

**Corollary 9.** Let $P(D)$ an hypoelliptic operator, then $\exists C > 0$ such that for every solution of $P(D) u = 0$ in $\Omega, \forall \varepsilon \in ]0, 1[, \forall j = 1, 2, \ldots$, we have
\[ \varepsilon^{2\alpha} \sum_{\theta \neq \alpha \in \mathbb{N}_0^n} \varepsilon^{-2|\alpha|} \left\| Q_H(D) P(\alpha) (D) u \right\|^2_{L^2(\Omega_{\varepsilon(j-1)})} \leq C \sum_{\theta \neq \alpha \in \mathbb{N}_0^n} \varepsilon^{-2|\alpha|} \left\| P(\alpha) (D) u \right\|^2_{L^2(\Omega_{\varepsilon(j-1)})} \]
The principal result of this section is the following theorem.

**Theorem 10.** Let $u$ be a solution of the hypoelliptic equation $P(D) u = 0$ in $\Omega$, then for every $\omega \subset \subset \Omega$, there is a constant $C > 0$, such that $\forall j \in \mathbb{N}$, we have
\[ \left\| Q_H^j(D) u \right\|_{L^2(\omega)} \leq C^{(j+1)} j^{\sigma j} \]
Proof. Since $\rho = \rho(\omega, \partial \Omega) > 0$, then there exists $\delta \in ]0, \rho[$ such that $\omega \subset \Omega_\delta \subset \Omega$. Take $\varepsilon = \frac{\delta}{j}$, $j \in \mathbb{N}$, and let us show by induction on $j$ the following estimate

$$
\varepsilon^{2j+2m} \sum_{0 \neq \alpha \in \mathbb{N}_0^n} \varepsilon^{-2|\alpha|} \left\| (Q^j_H(D) P^{(\alpha)}(D) u) \right\|^2_{L^2(\Omega_\varepsilon)} < C^{2(j+1)},
$$

where $m$ is the order of $P$.

As every solution $u$ of an hypoelliptic equation is $C^{\infty}$, then there exists $C > 0$ such that (4.6) is satisfied for $j = 0$. Suppose that (4.6) is true for $j \leq l$ ($l \geq 0$), we have to prove that it remains true for $j = l + 1$. Since $v = Q^l_H(D) u$ is also a solution of equation $P(D) u = 0$, then from corollary 4.10, we obtain

$$
\varepsilon^{2(l+1)+2m} \sum_{0 \neq \alpha} \varepsilon^{-2|\alpha|} \left\| (Q^{l+1}_H(D) P^{(\alpha)}(D) u) \right\|^2_{L^2(\Omega_{\varepsilon(l+1)})} \leq C^{2(l+1)}.
$$

By the induction hypothesis, we have

$$
\varepsilon^{2l+2m} \sum_{0 \neq \alpha} \varepsilon^{-2|\alpha|} \left\| P^{(\alpha)}(D) Q^l_H(D) u \right\|^2_{L^2(\Omega_{\varepsilon l})} \leq C_1^{2(l+1)},
$$

consequently, we obtain

$$
\varepsilon^{2\sigma(l+1)+2m} \sum_{0 \neq \alpha} \varepsilon^{-2|\alpha|} \left\| (Q^{l+1}_H(D) P^{(\alpha)}(D) u) \right\|^2_{L^2(\Omega_{\varepsilon(l+1)})} \leq C_2^{2(l+2)},
$$

hence $\forall j \in \mathbb{N}$, we have

$$
\varepsilon^{2\sigma j+2m} \sum_{0 \neq \alpha} \varepsilon^{-2|\alpha|} \left\| P^{(\alpha)}(D) Q^j_H(D) u \right\|^2_{L^2(\Omega_{\varepsilon j})} \leq C_2^{2(j+1)}.
$$

The estimate (4.8) with $|\alpha| = m$ gives $\forall j \in \mathbb{N},$

$$
\left\| Q^j_H(D) u \right\|^2_{L^2(\Omega_{\varepsilon j})} \leq \varepsilon^{-2\sigma j} C_2^{2(j+1)},
$$

as $\varepsilon = \frac{\delta}{j}$, then

$$
\left\| Q^j_H(D) u \right\|^2_{L^2(\Omega_{\varepsilon j})} \leq \left( \frac{j}{\delta} \right)^{2\sigma j} C_2^{2(j+1)} \leq C(j+1)^{2\sigma j},
$$

hence

$$
\left\| Q^j_H(D) u \right\|_{L^2(\Omega_{\varepsilon j})} \leq C(j+1)^{\sigma j}.
$$

We denote $G^{\mu, \mathcal{H}}(\Omega)$ the multi-anisotropic Gevrey space associated with $\mathcal{H}$ and by $\mu_\mathcal{H}$ and $\mu_Q$ the respective formal orders of the Newton’s polyhedrons $\mathcal{H}$ and $\Gamma(Q_\mathcal{H})$, then we have the following relations

$$
\Gamma(Q_\mathcal{H}) = \sigma \mathcal{H} \quad \text{and} \quad \mu_Q = \sigma \mu_\mathcal{H}.
$$

The principal result of this paper is the following theorem.

**Theorem 11.** Every solution $u \in \mathcal{D}'(\Omega)$ of the hypoelliptic equation $P(D) u = 0$ is a function of $G^{\mu_\mathcal{H}, \mathcal{H}}(\Omega)$. 
Proof. The theorem 4.11 says that every $u$ solution of the hypoelliptic equation $Pu = 0$ is a Gevrey vector of the operator $Q_{\mathcal{H}}$, i.e. we have $u \in G^{s\sigma}_{\mathcal{H}}(\Omega, Q_{\mathcal{H}})$. From theorem 3.4 and as the operator $Q_{\mathcal{H}}$ is multi-quasielliptic, then we have $u \in G^{s\sigma}_{\mathcal{H}}(\Omega)$, and consequently $u \in G^{s\sigma}_{\mathcal{H}}(\Omega)$. A simple computation shows that in general $G^{s, \sigma}(\Omega) = G^{ss, \mathcal{H}}(\Omega)$, hence $u \in G^{ss, \mathcal{H}}(\Omega)$. □

Remark 7. It is interesting to compare the result of the theorem with the microlocal Gevrey regularity result obtained in [2].

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