On uniqueness in inverse acoustic and electromagnetic obstacle scattering problems

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Abstract. We review some of the uniqueness results in inverse acoustic and electromagnetic obstacle scattering problems. The emphasis is given to the recent progress on unique determination of general polyhedral scatterers by the far field data corresponding to a single or several incident fields.

1. Introduction

The active and intriguing field of inverse scattering contains a tremendous variety of topics, including the ones for acoustic and electromagnetic waves. In this review article, we shall focus only on the inverse obstacle scattering problems, where one utilizes the time-harmonic acoustic or electromagnetic far-field measurements to identify the inaccessible unknown impenetrable obstacles. Our main concern is on the uniqueness/identifiability issue for such inverse problems. Several review papers are available in the literature on this topic, see, e.g., \cite{4} \cite{10} \cite{13} \cite{25}. Hence, we have chosen to address some very recent progress on the unique determination of general polyhedral scatterers by the far field data corresponding to a single or several incident fields. Nevertheless, for the self-containedness, we shall also include a sketchy summary of the existing results, an outline of some mathematical tools utilized in the proofs of the uniqueness results from our perspective.

We start with a brief description of the forward scattering problems. Let $D \subset \mathbb{R}^N (N \geq 2)$ be a compact set with (unbounded) connected Lipschitz complement $G := \mathbb{R}^N \setminus D$. Suppose that the target object $D$ is an impenetrable scatterer situated in some isotropic homogeneous background medium. Assume that an incident field, which is chosen to be the time-harmonic acoustic or electromagnetic plane wave, is sent and scattered when it meets the obstacle $D$, leading to a scattered field. Now, the forward scattering problem is to study the behavior of the scattered field if it is provided with the information of the underlying scatterer, e.g., its shape, location and physical properties, etc. Next, we would formulate this scattering problem mathematically. To do so, we shall take the incident wave to be the time-harmonic (acoustic) plane wave of the form

$$u^i(x, t) = e^{ikx \cdot d}, \quad x \in \mathbb{R}^N,$$

where $k > 0$ is the wave number and $d$ is the incident direction. Let $u^s$ be the scattered field, and $u = u^s + u^i$ the total field, then the forward scattering problem can be described by the
following Helmholtz equation
\[ \Delta u + k^2 u = 0 \quad \text{in} \quad G := \mathbb{R}^N \setminus D, \] (1.2)
complemented by the Sommerfeld radiation condition
\[ \lim_{r \to \infty} r^{(N-1)/2} \left( \frac{\partial u^s}{\partial r} - ik u^s \right) = 0 \]
where \( r = |x| \) for any \( x \in \mathbb{R}^N \). For the well-posedness of the above equation, we still need to impose suitable boundary conditions on \( \partial D \). This depends on the intrinsic physical properties of the underlying scatterer. For a sound-soft obstacle, the pressure of the total wave vanishes on the boundary, and this gives the Dirichlet boundary condition \( u = 0 \) on \( \partial G \); while for a sound-hard obstacle, the normal velocity of the acoustic wave vanishes on the boundary and we have the Neumann boundary condition \( \partial u / \partial n = 0 \), where \( n \) is the unit inward normal to \( \partial G \). More generally, allowing the obstacles for which the normal velocity is proportional to the excess pressure on the boundary leads to an impedance boundary condition \( \partial u / \partial n + i \lambda u = 0 \) on \( \partial G \), with a positive continuous function \( \lambda(x) \).

If we take the incident field to be the time-harmonic electromagnetic plane wave of the form
\[ E^i(x) := \frac{i}{k} \text{curl} \ p e^{ikx \cdot d} = ik(d \times p) \times e^{ikx \cdot d}, \]
\[ H^i(x) := \text{curl} \ p e^{ikx \cdot d} = ikd \times p e^{ikx \cdot d}, \] (1.3) (1.4)
where \( p \in \mathbb{R}^3 \), \( k > 0 \) and \( d \in S^2 := \{ x \in \mathbb{R}^3; |x| = 1 \} \) represent respectively the polarization, wave number and direction of propagation, then the associated forward scattering problem can be described by the following time-harmonic Maxwell equations
\[ \text{curl} \ E - ikH = 0, \quad \text{curl} \ H + ikE = 0 \quad \text{in} \quad G := \mathbb{R}^3 \setminus D, \] (1.5)
complemented by the following radiative condition
\[ \lim_{|x| \to \infty} (H^s \times x - |x|E^s) = 0, \]
where \( E = (E_1, E_2, E_3) \) and \( H = (H_1, H_2, H_3) \) are respectively the total electric and magnetic fields formed by the incident fields \( E^i(x) \), \( H^i(x) \) and scattered fields \( E^s(x) \) and \( H^s(x) \):
\[ E(x) = E^i(x) + E^s(x), \quad H(x) = H^i(x) + H^s(x). \] (1.6)
To complete the forward scattering system, we impose either of the following two boundary conditions
\[ \nu \times E = 0 \quad \text{on} \quad \partial G; \]
\[ \nu \times \text{curl} \ E - i\lambda(\nu \times E) \times \nu = 0 \quad \text{on} \quad \partial G, \] (1.7) (1.8)
which correspond to the cases that \( D \) is a perfect and imperfect conductor respectively.

Both the forward scattering problems (1.2) and (1.5) with suitable boundary conditions have been well understood. It is known that there exists a unique solution \( u \in H_{loc}(G) \) to (1.2) whereas a unique solution \( (E, H) \in H_{loc}(\text{curl}; G) \times H_{loc}(\text{curl}; G) \) to (1.4) (see [5], [11], [12], [39]). Moreover, both solutions are (real) analytic in \( G \) and their asymptotic behaviors are respectively governed by
\[ u^* (x) = \frac{e^{i|k|x}}{|x|^{(N-1)/2}} \left\{ u_\infty (\hat{x}) + O \left( \frac{1}{|x|} \right) \right\} \quad \text{as} \quad |x| \to \infty, \quad (1.9) \]

and

\[ E^* (x; D, p, k, d) = \frac{e^{i|k|x}}{|x|} \left\{ E_\infty (\hat{x}; D, p, k, d) + O \left( \frac{1}{|x|} \right) \right\} \quad \text{as} \quad |x| \to \infty, \quad (1.10) \]

\[ H^* (x; D, p, k, d) = \frac{e^{i|k|x}}{|x|} \left\{ H_\infty (\hat{x}; D, p, k, d) + O \left( \frac{1}{|x|} \right) \right\} \quad \text{as} \quad |x| \to \infty, \quad (1.11) \]

Now we are ready to formulate the inverse scattering problems. We define the acoustic far-field operator \( \mathcal{F}_a \) which sends the underlying obstacle to the corresponding far-field data by

\[ \mathcal{F}_a (\partial \mathbf{G}) = u_\infty (\hat{x}; D, p, k, d), \quad (\hat{x}, k, d) \in \mathbb{S}^{N-1}_0 \times \mathbb{K} \times \tilde{\mathbb{S}}^{N-1}_0 \subset \mathbb{S}^{N-1} \times \mathbb{R}^+ \times \mathbb{S}^{N-1}, \quad (1.12) \]

where \( \mathbb{S}^{N-1}_0, \tilde{\mathbb{S}}^{N-1}_0 \) are the subsets of \( \mathbb{S}^{N-1} \), and \( \mathbb{K} \subset \mathbb{R}^+ \), and the electromagnetic far-field operator by

\[ \mathcal{F}_e (\partial \mathbf{G}) = E_\infty (\hat{x}; D, p, k, d), \quad (\hat{x}, p, k, d) \in \mathbb{S}^2_0 \times \mathbb{U} \times \mathbb{K} \times \tilde{\mathbb{S}}^2_0, \quad (1.13) \]

where \( \mathbb{S}^2_0, \tilde{\mathbb{S}}^2_0 \) are the subsets of \( \mathbb{S}^2, \mathbb{U} \subset \mathbb{R}^3, \mathbb{K} \subset \mathbb{R}^+ \). The inverse acoustic and electromagnetic obstacle scattering problems are, respectively, to find the inversions of the nonlinear operators \( \mathcal{F}_a \) and \( \mathcal{F}_e \). These inverse problems are central to many areas of science and technology, such as sonar, medical imaging, geophysical exploration and nondestructive testing, etc.

Like for most other inverse problems, a very important issue one would like to understand is the \textit{uniqueness} or \textit{identifiability}, i.e., whether a scatterer can be identified from a knowledge of its far-field pattern. Mathematically, the uniqueness is about the injectivity of the (nonlinear) operators \( \mathcal{F}_a \) and \( \mathcal{F}_e \), or the one-to-one correspondence between \( \partial \mathbf{D} \) and the given far-field data. It is observed that the uniqueness results also provide the practical information on how much measurement data we should use to identify the scatterer. As an important ingredient in the uniqueness study, one can assume that \( \mathbb{S}^{N-1}_0 \) (resp. \( \mathbb{S}^2_0 \)) is only an open subset of the unit sphere, since one can recover the data on the whole unit sphere by analytic continuation by knowing that \( u_\infty \) (resp. \( E_\infty \)) is analytic. In fact, all the existing results about uniqueness for the inverse obstacle scattering problems have made use of the fact that far-field pattern is given on the whole unit sphere, with the only exceptional case in [38], where only a single datum is used to uniquely determine the shape of a ball or disc. So unless otherwise stated, we will assume that \( \mathbb{S}^{N-1}_0 = \mathbb{S}^{N-1} \) (resp. \( \mathbb{S}^2_0 = \mathbb{S}^2 \)) throughout the paper.

Since the uniqueness result due to Schiffer for sound-soft general obstacles by countably many incident plane waves (see [11,29]), there has been an extensive study in this direction in the literature, see, e.g., the results on uniqueness for general domains in [14, 19, 23, 27, 43, 44, 45, 47], for polyhedral type scatterers in [1, 6, 7, 16, 17] and [31–37], for discs or balls in [30, 38, 49], and for smooth planar curves in [24, 26, 40, 46].

However, this important problem still remains largely unsolved, even up to now. It is noted that for an obstacle, the inverse obstacle scattering problem is formally determined with measurement data in every possible direction corresponding to a single incident plane wave at fixed wave number and incident direction (and fixed polarization in the electromagnetic case), since the unknown \( \partial \mathbf{G} \) depends on the same number of variables as does the measurement data. But as was pointed out in [13], although there is a widespread belief that the far field pattern for one single incident wave determines a sound-soft scatterer without any additional a prior information, a rigorous mathematical justification was still not available. Also a remark was made in [20] for the uniqueness in inverse acoustic obstacle scattering problem and it reads as follows: \textit{This is a well-known question that supposedly can be solved by elementary means.}
However, it has been open for thirty to forty years. Furthermore, for the uniqueness in the electromagnetic case with far-field data from a finite number of incident fields, it is widely recognized to be even more difficult.

In the remaining sections, we will provide a sketchy summary of the existing uniqueness results together with a brief description of the mathematical tools involved in the proofs of those results, such as spectral arguments, singular source method, unique continuation and reflection for solutions of Helmholtz & Maxwell equations etc. We shall arrange our presentation according to the geometry of the obstacle under discussion, such as general scatterers, open arcs, polyhedral or polygonal scatterers etc. Such ordering has its advantage due to the fact that the mathematical arguments are mostly similar for some fixed type of scatterers.

2. Uniqueness for General Scatterers

By a general scatterer, we mean that a bounded $C^2$-domain $D \subset \mathbb{R}^N$ with connected complement $G := \mathbb{R}^N \setminus D$. This assumption is only for the regularity of the forward scattering problems. In fact, for the uniqueness in the present section to be held, we only require that the solutions are continuous up to the boundary.

Firstly, we would like to mention the Rellich’s lemma which builds up the one-to-one correspondence between the far field $u_\infty$ and the scattered field $u_s$, and has been essentially utilized in almost all the uniqueness studies mentioned in section 1. The lemma can be found in [11] (Lemma 2.11 and Theorem 6.9 respectively for the acoustic and electromagnetic cases).

For a general sound-soft scatterer $D \subset \mathbb{R}^3$, the classical uniqueness result by Schiffer can be stated as follows:

**Theorem 2.1.** Assume that $D_1$ and $D_2$ are two general sound-soft scatterers such that their far-field patterns coincide for an infinite number of incident plane waves with (a) one fixed wave number $k_0 \in \mathbb{R}_+^*$ and distinct incident directions in $\mathbb{S}^2$ or (b) one fixed incident direction $d_0 \in \mathbb{S}^2$ and distinct wave numbers in $\mathbb{R}_+^*$ which do not accumulate at infinity. Then $D_1 = D_2$.

The proof for case (a) was presented in Theorem 5.1 [11], while the proof for case (b) in Theorem 6.3.1 [20]. The basic idea is to apply the spectral argument for $-\Delta$ in some bounded domain, say, $D^* := (\mathbb{R}^3 \setminus \Omega) \setminus \bar{D}_1$, where $\Omega$ is the unbounded connected component of $\mathbb{R}^3 \setminus (D_1 \cup \bar{D}_2)$, which can be assumed to be non-empty if $D_1 \neq D_2$. Letting $u_1$ and $u_2$ be respectively the total fields associated with $D_1$ and $D_2$, then we have $u_1 = u_2 = 0$ on $\partial D^*$. Hence, $u_1 \in H_0^1(D^*)$ is a Dirichlet eigenvalue for $-\Delta$ in $D^*$. But for the fixed eigenvalue $k_0^2$, there are only finitely many linearly independent Dirichlet eigenfunctions in $H_0^1(D^*)$, the proof is then completed by noting the fact that the total fields for distinct incoming plane waves are linearly independent. Case (b) is a weaker version of the one presented in [20].

Schiffer’s result has been modified by Colton and Sleeman to finitely many incident waves based on the use of Courant’s maximum-minimum principles for compact symmetric operators (see Theorem 4.7, [15]), which reads: the $n$-th Dirichlet eigenvalue for a ball $B$ containing the domains $D_1$ and $D_2$ is always smaller than the $n$-th Dirichlet eigenvalue for the sub-domain $D^*$, where the eigenvalues are arranged according to the increasing magnitude and taken with their respective multiplicity. In particular, the multiplicity of the eigenvalue $k_0^2$ is less than or equal to the sum of the multiplicities of all the eigenvalues for $B$ which are less than $k_0^2$. Hence, if $N(B, k_0)$ is the number of those Dirichlet eigenvalues less than $k_0^2$, then from our early discussion we know that $N(B, k_0) + 1$ distinct incident plane waves are sufficient to establish the uniqueness result. In like manner, we can establish the uniqueness result with fixed incident direction and finitely many distinct wave numbers. Moreover, let the radius of $B$ is known to be $R$, then $N(B, k_0)$ can be expressed in term of the positive zeros of the spherical Bessel functions. Particularly, it can be calculated that when $k_0 R < \pi$, $N(B, k_0) = 0$: that is, with this a prior
information, the uniqueness can be built up by only one single incident plane wave. In [19],
it is shown that, corresponding to a fixed incident plane wave \( \exp\{ikx \cdot d\} \), the total field \( u \) and its conjugate \( \bar{u} \) are linearly independent. Based on this observation, Gintides has improved
the uniqueness result of Colton and Sleeman in [14] to use only half the number of the needed
incident waves. For the case of one incident wave, the condition \( k_0R < \pi \) was also improved
to be \( k_0R < 4.939 \). Such conditions are also known in some sense to be the local uniqueness
issue, since they make the uniqueness held under the restriction that the possible scatterers do
not deviate “too much” in measure, such as, area, diameter etc. There are also local uniqueness
results concerning the volume difference [48] [19], where estimations via Poincaré inequality and
Faber-Krahn inequality are important tools. We would like to mention that based on some novel
estimation for the lower bound of the first Dirichlet eigenvalue in [18], through straightforward
calculations, one can derive some new local uniqueness concerning the diameter for some special
obstacles, such as polygons. However, we are mainly concerned with the global uniqueness in
the present paper, hence will not explore more related to local uniqueness issues. Finally, we
would like to remark that most of the aforementioned or slightly modified uniqueness results are
valid in the \( \mathbb{R}^2 \) case.

Using Schiffer’s idea, a uniqueness result has been established for a much more general type
of sound-soft scatterers in \( \mathbb{R}^3 \) by finitely many incident plane waves in [45]. The admissible
scatterer in [45] may consists of finitely many real body obstacles and crack-type obstacles,
such as surfaces, that is, the considered scatterer may have components with empty interior.
Clearly, in this case, we can not guarantee that we can find a non-empty bounded domain
\( D^* \subset \mathbb{R}^3 \setminus (D_1 \cup D_2) \) as it was argued for Theorem 2.1. Here is one way to overcome the difficulty.
By noting that \( D_1 \neq D_2 \) and both have connected complements, one can find a subset \( S \) on \( \partial \Omega \), but contained, say, in \( D_2 \setminus D_1 \). Then by Rellich’s Lemma, we have \( u_1 = u_2 = 0 \) on \( S \), so
\( S \) is contained in a nodal set of \( u_1 \). By the properties of solutions to Helmholtz equation (1.2),
we know that \( S \) contains an analytic surface \( S_0 \). Now, we consider \( S_0 \), the maximum analytic
extension of \( S_0 \), and obviously \( u_1 = 0 \) on \( S_0 \) by analyticity. The proof can be completed by
showing that \( \tilde{S}_0 \) together with some subsets on \( \partial D_1 \) forms a bounded domain \( D^* \), which plays
the same role as in Schiffer’s proof. And this is done by a careful study of the nodal sets of
solutions to the Helmholtz equation. However, for such continuation up to the boundary, we
must impose some proper topological assumptions on the scatterers, which, nevertheless, include
the scatterers considered in [14]. We would like to emphasize that, combining the results in [19]
and [45], the needed number of incident plane waves in [45] can surely be halved.

Schiffer’s proof can hardly be extended to other boundary conditions. This is due to the
fact that, generally, the space of solutions of homogeneous Neumann problem in a domain \( D^* \),
whose boundary might have arbitrary cusps, may not be finite-dimensional. Due to the same
reason and noting that the perfect and imperfect boundary conditions for Maxwell equations
involve the normal to the concerned domain, Schiffer’s proof is not applicable to the inverse
electromagnetic scattering problems.

Isakov proved in [21] the uniqueness for the inverse transmission problem by using incident
plane waves with fixed wave number and all incident directions. A contradiction is obtained by
considering a sequence of solutions with a singularity moving towards a boundary point of one
scatterer that is not contained in the other scatterer. Based on this technical idea, Kirsch and
Kress proved that [23]

**Theorem 2.2.** Assume that \( D_1 \) and \( D_2 \) are two scatterers with boundary conditions \( B_1 \) and
\( B_2 \), where \( B_i \) (\( i = 1, 2 \)) may be either of the sound-soft, sound-hard and impedance boundary
conditions. If the far-field patterns coincide for an infinite number of incident plane waves with
distinct incident directions and one fixed wave number, then \( D_1 = D_2 \) and \( B_1 = B_2 \).

We refer to [2, 3, 28]) for some other efforts that are related to Theorem 2.2. One of the
remarkable feature of this uniqueness result is that it is not necessary to need to know a priori the physical properties of the underlying obstacles, and they can still be uniquely identified. The proof for Theorem 2.2 can be extended to show a similar uniqueness result in inverse electromagnetic scattering problem. Later, both proofs are simplified significantly by the use of the point source method due to Potthast [42] and we refer to [4], [13], [9] for a review of the topic. To conclude this section, we would like to mention that the use of the singular source in the point source method due to Potthast [42] and we refer to [4], [13], [9] for a review of the electromagnetic scattering problem. Later, both proofs are simplified significantly by the use of

The proof for Theorem 2.2 can be extended to show a similar uniqueness result in inverse the physical properties of the underlying obstacles, and they can still be uniquely identified.

3. Uniqueness for Balls

It is clearly an interesting issue to look into the uniqueness for some simple obstacles such as balls or discs. Indeed for balls and discs, the uniqueness results can be established by means of some facts that the radiating solutions to the Helmholtz equation corresponding to balls or discs can be analytically extended to a solution in $\mathbb{R}^N(N = 2, 3)$ except for the center. The extension of radiating solutions for a sound-soft ball was firstly shown in [8] based on the study of Goursat problem for wave equation. Whereas for the sound-hard case, it is shown in [49] by carefully estimating the convergence radius of the expansion of the solution in terms of the spherical wave functions. As a consequence of this property, one readily finds that the location of a soft/hard ball is uniquely determined by the far-field data on $S^{N-1}$ corresponding to a single incident plane wave. So if two balls, say $B_{r_1}(x_1)$ and $B_{r_2}(x_2)$ with $x_1 \neq x_2$, has the same far field, $u_{\infty,1}(\hat{x}) = u_{\infty,1}(\hat{x})$ for $\hat{x} \in S^{N-1}$, one can easily derive from Rellich’s lemma and the aforementioned analytic extension property that $u(x) := u_1(x) = u_2(x)$ is a radiating entire solution in the whole $\mathbb{R}^N$; that is, $u = 0$ in $\mathbb{R}^N$, which is certainly not true. Then, with the asymptotic expression for Bessel or spherical Bessel functions, one can further deduce that $r_1 = r_2$; that is, the shape of the ball/disc is also uniquely determined. There is an alternative proof of the uniqueness due to Kress by using the Karp’s reciprocity relation and the uniqueness result in Theorem 2.2.

However, for a ball in $\mathbb{R}^3$ or a disc in $\mathbb{R}^2$, its radius and center suffice to uniquely identify the object. Hence, the measurement data two spots are sufficient to formally determine the obstacle. This is the same situation for some other simply-structured obstacles, such as a cube in $\mathbb{R}^3$ or a square in $\mathbb{R}^2$, then two measurement data formally determine the object; while for an ellipsoid in $\mathbb{R}^3$ or an ellipse in $\mathbb{R}^2$, they are, respectively, formally determined by four and three measurement data. This raises a natural question: could uniqueness be established in those cases mathematically? The first step towards such important direction is made in [38], where it is proved the uniqueness for (sound-soft or sound-hard) balls or discs by using the expansion in terms of wave functions and some novel properties concerning the positive zeros of Bessel and spherical Bessel functions.

4. Uniqueness for Smooth Curves

There are also uniqueness results concerning scatterers having empty interior such as a surface in $\mathbb{R}^3$ or a curve in $\mathbb{R}^2$, but all of them deal with the planar case. Kress proved in [24] that a sound-soft open $C^3$ curve in $\mathbb{R}^2$ can be uniquely determined by measuring the far-field patterns at a fixed wave number and all the possible incident directions. This result was then extended to the sound-hard case by Mönch in [40]. In [26], Kress has further proved that a sound-soft open analytic curve in $\mathbb{R}^2$ is uniquely determined by a single incident plane wave provided the wave number is sufficiently low. A more general multiple scatterers in $\mathbb{R}^2$, possibly consisting of finitely many holes or curves, was shown in [46] to be uniquely determined by measuring the far-field patterns corresponding to incident plane waves at a fixed and sufficiently low wave number and all the possible incident directions.
5. Uniqueness for Polygonal or Polyhedral Scatterers

In the past few years, significant progress has been made for the unique determination of general polyhedral scatterers by a single or several far-field measurements. This is mainly based on various reflection principles for solutions of Helmholtz and Maxwell equations. Moreover, the path arguments developed in [1] and [35] have been playing an indispensable role in the development. In [1], in order to establish the unique determination of an acoustic sound-soft polyhedral scatterer via contradiction argument, a hidden path is constructed to connect the nodal domains of Dirichlet Helmholtz equation outside the scatterer. This idea is modified and simplified considerably in [35], where a path is constructed at the very start and then one makes the continuation of the Dirichlet hyperplanes (see the descriptions given below) along this path to arrive at a contradiction. Furthermore, it turns out that the path argument in [35] are not only applicable to inverse acoustic sound-soft obstacle scattering, but also, with necessary modifications, to inverse acoustic sound-hard or impedance obstacle scattering as well as inverse electromagnetic obstacle scattering etc. In the following, we shall first summarize the reflection principles for Helmholtz equation and Maxwell equations. Then we take the inverse acoustic sound-soft obstacle scattering as a model problem to discuss the path argument in [35]. Finally, we briefly indicate how suitable modifications of the path argument together with the corresponding reflection principles yield various uniqueness results in different settings. We would like to mention here that alternative treatments other than path arguments are presented in [6], [16] and [37], where the arguments are mainly based on careful study of the behaviors of scattered waves near the boundary of the obstacle. Since the behaviors of the scattered waves would become much complicated in higher dimensions, such arguments mainly work well for 2 dimensions.

5.1. Reflection principles for Helmholtz and Maxwell equations

For the subsequent discussion of various reflection principles, we need to fix some notations. In the following, \( \Pi \) always represents a hypersurface in \( \mathbb{R}^N \). In case \( \Pi \) lies entirely on some hyperplane in \( \mathbb{R}^N \), we denote by \( \tilde{\Pi} \) the hyperplane while \( R_{\Pi} \) the reflection in \( \mathbb{R}^N \) with respect to \( \Pi \). Unless otherwise stated, \( \nu_{\Pi}, \nu_{\Gamma} \) etc. shall represent the unit normal to some specific hypersurfaces or domains. Finally, \( u(\Pi) = 0, \nu_{\Pi} \times u(\Pi) = 0, \nu_{\Pi} \times E(\Pi) = 0 \) etc. will be frequently used to denote that \( u(x) = 0, \nu_{\Pi} \times u(x) = 0, \nu_{\Pi} \times E(x) = 0 \) for \( x \in \tilde{\Pi} \) etc..

The first reflection principle is for the solutions to the Helmholtz equation or vector-valued Helmholtz equations. In the sequel, \( u \) may represent a scalar or vector-valued function at different occurrences.

**Theorem 5.1.** For a connected domain \( \Omega \) in \( \mathbb{R}^N \setminus \mathbb{D} \), let \( \Pi \) be part of the boundary \( \partial \Omega \) and lie on a hyperplane in \( \mathbb{R}^N \) such that \( u(\Pi) = 0 \) or \( \nu_{\Pi} \times u(\tilde{\Pi}) = 0 \). Furthermore, let \( \Lambda := \Omega \cup R_{\Pi}\Omega \subset \mathbb{G} \). Suppose that \( \Gamma \subset \partial \Omega \) is a different hypersurface from \( \Pi \) and on which \( u(\Gamma) = 0 \) (resp. \( \nu_{\Gamma} \times u(\Gamma) = 0 \)). Then let \( \Gamma' = R_{\Pi}\Gamma \) and we have \( u(\Gamma') = 0 \) (resp. \( \nu_{\Gamma'} \times u(\Gamma') = 0 \)). Moreover, \( u(\Pi \cap \Lambda) = 0 \) if \( u(\Pi) = 0 \), and \( \nu_{\Pi} \times u(\Pi \cap \Lambda) = 0 \) if \( \nu_{\Pi} \times u(\Pi) = 0 \).

In fact, for our uniqueness study in the inverse obstacle scattering problems, all the hypersurfaces involved in Theorem 5.1 are only required to lie entirely on some hyperplane in \( \mathbb{R}^N \), but here we prefer to a general discussion. We would also like to remark that the last statement in Theorem 5.1 is particularly useful in the path argument in the case where \( \Lambda = \Omega \cup R_{\Pi}\Omega \) is unbounded.

The proof for Theorem 5.1 is relatively simple. In fact, taking \( u(\tilde{\Pi}) = 0 \) as an example, it is directly verified that \( u(x) := u(x) + u(R_{\Pi}x) \) is still a solution to the Helmholtz equation in \( \Lambda \) due to the invariance of Laplacian under rotation, and \( u \) assumes homogeneous Cauchy data on \( \Pi \). Hence, \( u(\Lambda) = 0 \) by the Holmgren’s unique continuation theorem; namely, \( u(x) \) is odd symmetric in \( \Lambda \). Then, the theorem follows readily from the odd symmetry. Such arguments
can be easily extended to show more general reflection principles associated with impedance boundary conditions.

**Theorem 5.2.** Assume that $\Omega$, $\tilde{\Pi}$ and $\Gamma$ are the same as those stated in Theorem 5.1, and $u$ satisfies

$$\frac{\partial u}{\partial \nu} + i\lambda u = 0 \quad \text{on} \quad \Gamma,$$

(5.1)

where $\nu$ is the unit normal to $\Gamma$ directed to the interior of $\Omega$, then it holds that

$$\frac{\partial u}{\partial \nu'} + i\eta u = 0 \quad \text{on} \quad \Gamma',$$

(5.2)

where $\Gamma' = R_{\Pi}\Gamma$ and $\nu'$ is the unit normal to $\Gamma'$ directed to the interior of $R_{\Pi}\Omega$ and $\eta(x) = \lambda(R_{\Pi}x)$ for $x \in \Gamma'$.

Theorem 5.2 will be used to show the uniqueness in determining partially coated obstacles. We proceed to the reflection principles for the Maxwell equations, which were established in [33], the first time in the literature.

**Theorem 5.3.** Assume that $\Omega$, $\tilde{\Pi}$, $\Lambda$ and $\Gamma$ are the same as those stated in Theorem 5.1 in $\mathbb{R}^3$, and $\nu_1 \times E(\tilde{\Pi}) = 0$ (resp. $\nu_1 \times H(\tilde{\Pi}) = 0$), and $\nu' \times E(\Gamma) = 0$ (resp. $\nu' \times H(\Gamma) = 0$). Then for $\Gamma' = R_{\Pi}\Gamma$, it holds that $\nu' \times E(\Gamma') = 0$ (resp. $\nu' \times H(\Gamma') = 0$). Moreover, $\nu_1 \times E(\Pi \cap \Lambda) = 0$ if $\nu_1 \times E(\tilde{\Pi}) = 0$, and $\nu_1 \times H(\Pi \cap \Lambda) = 0$ if $\nu_1 \times H(\tilde{\Pi}) = 0$.

The proof of Theorem 5.3 can be found in [33] and [34]. The arguments are based on the vector-valued Helmholtz equations in [33], whereas the arguments are carried out directly from the Maxwell equations in [34]. The main idea is to first show that the theorem holds in some special case with $\Pi$ being the plane $\{(x_1, x_2, x_3) \in \mathbb{R}^3; \ x_3 = c\}$, where $c$ is a constant, then extend it to the general case by using the rotational invariance of the Maxwell equations or the vector-valued Helmholtz equations. Since the proof is much tedious and technical, we direct the readers to the aforementioned references. Similarly to Theorem 5.2, we can also show the reflection principles for the mixed boundary conditions, which shall be used to establish the unique determination of partially coated scatterers in inverse electromagnetic obstacle scattering problems.

**Theorem 5.4.** Assume that $\Omega$, $\tilde{\Pi}$, $\Lambda$ and $\Gamma$ are the same as those stated in Theorem 5.3, $\nu' \times \text{curl} \ E - i\lambda(\nu' \times E) \times \nu' = 0$ (resp. $\nu' \times \text{curl} \ H - i\lambda(\nu' \times H) \times \nu' = 0$) on $\tilde{\Pi}$, where $\nu'$ is the unit normal to $\Gamma$ directed to the interior of $\Omega$. Then for $\Gamma' = R_{\Pi}\Gamma$, it holds that $\nu' \times \text{curl} \ E - i\eta(\nu' \times E) \times \nu' = 0$ (resp. $\nu' \times \text{curl} \ H - i\eta(\nu' \times H) \times \nu' = 0$), where $\nu'_1$ is the unit normal to $\Gamma'$ directed to the interior of $R_{\Pi}\Omega$, and $\eta(x) = \lambda(R_{\Pi}x)$ for $x \in \Gamma'$.

### 5.2. Path argument for inverse acoustic sound-soft scattering

We take the inverse acoustic sound-soft scattering as an example to illustrate briefly how the path argument works to establish uniqueness in determining polyhedral type scatterers. For more details, we refer to [35]. It is remarked that all the uniqueness results presented subsequently for inverse acoustic scattering problems hold for any dimension equal to or larger than 2, but we stick to the three dimensional terminology. To ease the exposition, we let $\mathbf{D}$ be composed of finitely many compact polyhedra. For the fixed $k$ and $d$, if $u_\infty(\hat{x})$ can not uniquely determine $\mathbf{D}$, namely, there exists another polyhedral scatterer $\mathbf{D}'$ such that $u_\infty(\hat{x}; \mathbf{D}) = u_\infty(\hat{x}; \mathbf{D}')$, then with the help of the Rellich’s lemma, one can show that there holds: either an open subset $\tilde{\Pi}$ from one face of $\mathbf{D}'$ such that $u(\tilde{\Pi}; \mathbf{D}) = 0$ in $\mathbf{G} := \mathbb{R}^N \setminus \mathbf{D}$; or an open subset $\Pi'$ from one face of $\mathbf{D}$ such that $u(\Pi'; \mathbf{D}') = 0$ in $\mathbf{G}' := \mathbb{R}^N \setminus \mathbf{D}'$. Without loss of generality, let us assume the former. This subset $\tilde{\Pi}$ will be our starting point. Based on this observation, we introduce the following concept on Dirichlet set for the forward scattering problem.
Definition 5.5. Let \( u(x) \) be the total field to (1.2) associated with a single incident field \( u'(x) = \exp\{ikx \cdot d\} \) with fixed \( k \) and \( d \), then the following set

\[
\mathcal{D}_u = \{ x \in \Omega : u \big|_{\Pi \cap B_r(x) \cap \Omega} = 0 \text{ for some } r > 0 \text{ and hyperplane } \Pi \text{ passing through } x \}
\]

is called a Dirichlet set of \( u \) in \( \Omega \).

For any \( x \in \mathcal{D}_u \), let \( \Pi \) be the corresponding hyperplane involved in the definition of \( \mathcal{D}_u \) and \( \bar{\Pi} \) be the open connected component of \( \Pi \setminus \mathcal{D} \) containing \( x \), then by analytic continuation, we know \( u \big|_{\bar{\Pi}} = 0 \). And each such set \( \bar{\Pi} \) will be called a Dirichlet hyperplane in the sequel. The crucial fact about Dirichlet set and Dirichlet hyperplanes is that all of them are bounded. This can be shown by using the asymptotic behavior of the solution to the forward scattering problem.

Clearly, our contradiction argument prior to Definition 5.5 has shown that if \( \mathcal{D} \neq \mathcal{D}' \), we can assume that there exists a Dirichlet hyperplane \( \bar{\Pi}_1 \) in \( \mathcal{D} \). Now, noting that \( \partial \Omega \) are composed of finitely many Dirichlet hyperplanes, it is natural for us to implement the reflection principle in Theorem 5.1 to find another Dirichlet hyperplane starting from \( \bar{\Pi}_1 \). The general strategy can be described as follows. First note that the connected set \( \Omega \) is divided into at most two connected components, that is, either \( \Omega \setminus \bar{\Pi}_1 \) is bounded or \( \Omega \setminus \bar{\Pi}_1 \) has two connected components (one bounded and the other unbounded). In the latter case, we let \( \Lambda_1 \) denote the bounded component while \( \Lambda_2 \) the unbounded component. Then it is easy to construct a symmetric set w.r.t. \( \bar{\Pi}_1 \) as the connected component of the set \( \left( \Lambda_1 \cap R_{\Pi_1}(\Lambda_2) \right) \cup \bar{\Pi}_2 \) containing \( \bar{\Pi}_2 \). Let us denote by \( \Lambda_1 \) this connected symmetric set, which plays the same role as \( \Lambda \) in Theorem 5.1. Here, we make another important observation that \( \Lambda_1 \) is bounded due to the boundedness of \( \Lambda \). Hence, from the reflection principle in Theorem 5.1, we are led to a different Dirichlet hyperplane from the analytic extension of any face on \( \partial \Lambda_1 \) that lies in \( \Omega \). Whereas for the case that \( \Omega \setminus \bar{\Pi}_1 \) is unbounded, we take \( \Lambda_1 \) to be the connected component of \( ((\Omega \setminus \bar{\Pi}_1) \cap R_{\Pi_1}(\Omega \setminus \bar{\Pi}_1)) \cup \bar{\Pi}_1 \) containing \( \bar{\Pi}_1 \). It is noted that \( \Lambda_1 \) must also be bounded, since otherwise, by Theorem 5.1, we would have \( u(\Pi_1 \cap \Lambda_1) = 0 \). Hence, by doing this, we also find another Dirichlet hyperplane. Continuing with such reflection procedure, we can find many different Dirichlet hyperplanes. But our final goal is to find one that can be extended to infinity, which contradicts then with the boundedness of a Dirichlet set, thus completing the uniqueness proof. So, we need to find an ‘exit path’ along which our reflections are conducted. This is provided as follows and is the core of the path argument. Let \( \gamma(t) \ (t \geq 0) \) be a regular curve in \( \Omega \setminus \bar{\Pi}_1 \) which starts from an arbitrary point \( x_1 := \gamma(0) \) on \( \bar{\Pi}_1 \). Noting \( \mathcal{D} \) is a compact set while \( \gamma \) is a closed set, \( r_0 := \text{dist}(\mathcal{D}, \gamma) > 0 \) is attainable. It is easily seen that for any fixed \( t \geq 0 \), the ball \( B_{r_0/2}(\gamma(t)) \) lies entirely in \( \Omega \). Now, we look back the reflection argument made above and can see that \( \Lambda_1 \) must contain the ball \( B_{r_0/2}(x_1) \). Next, we choose \( \bar{\Pi}_2 \), the second Dirichlet hyperplane, to be the one which has intersection with \( \gamma \). Then it is easily seen that \( \bar{\Pi}_2 \) is at least \( r_0/2 \) away from \( \bar{\Pi}_1 \) along the path \( \gamma \). Continuing with such reflection procedure, starting from the Dirichlet hyperplane \( \bar{\Pi}_2 \), we can find another Dirichlet hyperplane \( \bar{\Pi}_{l+1} \), and moreover, in doing this, we step at least \( r_0/2 \) distance along the path \( \gamma \). Eventually, we are led to a Dirichlet hyperplane far away from the scatterer \( \mathcal{D} \) and thus can be extended to infinity.

Finally, we would like to mention an extension of the above uniqueness result to include more general scatterers. Let a crack be defined to be any compact subset of a hyperplane with non-empty interior. Then, by the above path argument together with some very minor modifications, one can show that a single far-field measurement can uniquely determine a sound-soft scatterer consisting of finitely many (but unknown) compact polyhedra and cracks.
5.3. Various uniqueness results for polyhedral scatterers

As can be easily seen from our exposition in Subsection 5.2, the path argument developed for inverse acoustic sound-soft scattering also applies, mutatis mutandis, to the inverse acoustic sound-hard scattering, inverse electromagnetic scattering etc. provided one has the corresponding reflection principles. This is remarkable due to the essential differences between the Dirichlet problem and the Neumann problem, between the Helmholtz equation and Maxwell equations. Next, let us give a brief review of some of the important uniqueness results.

Similar to the Dirichlet hyperplane, one can introduce the Neumann hyperplane as the open subset of some hyperplane on which \( u \) takes the homogeneous Neumann data. But in order to achieve the boundedness of the Neumann hyperplanes (and so the corresponding Neumann set, see [35]), one has to make use of \( N \) different incident plane waves at linearly independent incident directions but fixed energy. With those modifications, one can show by the path argument that a sound-hard general polyhedral scatterer can be uniquely determined by \( N \) different far-field measurements. In fact, due to the fact that the path argument is applicable to both Dirichlet and Neumann problems, one can show the unique determination of general polyhedral scatterers without knowing their \textit{a priori} physical properties, namely, they may be either sound-soft or sound-hard, or part is sound-soft and part is sound-hard (see [37]). These results are shown to be optimal in [36], where some counter examples are constructed to illustrate that one cannot uniquely determine a polyhedral scatterer by any less than \( N \) far-field measurements provided it admits the simultaneous presence of sound-hard crack component. For further consideration, one may still anticipate the uniqueness with a single far-field measurement in determining a polyhedral scatterer excluding the presence of any sound-hard crack component. However, the path argument presented in Subsection 5.2 may not work since one might encounter unbounded Neumann hyperplanes due to the use of any less than \( N \) different far-field measurements. By using different techniques, this problem is verified in [16] and [37] for the 2 dimensions and higher-dimensional cases are still left open. Whereas there is some very recent progress obtained in [17] and [31] for this problem in higher dimensions. Indeed, the uniqueness is established by \( N - 1 \) far-field measurements in [31], and by a single far-field measurement in [17] under the restriction that the component obstacles are all convex. The main idea is to construct a path which avoids the intersection with the unbounded Neumann hyperplanes.

We proceed to the inverse electromagnetic scattering problems. Let a \textit{perfect plane} be defined to be the open subset of some hyperplane on which \( E \) or \( H \) takes the homogeneous data. Clearly, the perfect plane shall play the similar role as the Dirichlet or Neumann hyperplane in the path argument. Through some rather straightforward calculations, one can show that two incident waves are needed to ensure the boundedness of any perfect plane (and so the corresponding perfect set, see [33] and [34]). Then using the reflection principle in Theorem 5.3 and appropriately devised path argument, one can show that a general polyhedral scatterer of perfect conductor can be uniquely determined by two different electric far-field measurements. In fact, similar uniqueness result can be established without knowing the \textit{a priori} physical properties of the underlying scatterer. That is, on part of the boundary of the underlying scatterer, \( E \) takes the perfect boundary condition, and on the other part of the boundary, \( H \) takes the perfect boundary condition, then both of the shape and boundary condition of the underlying scatterer can be uniquely determined by two electric (or equivalently, magnetic) far-field measurements. For further improvement with a single electric or magnetic far-field measurement, one may encounter the unbounded perfect planes as in the acoustic case with unbounded Neumann planes. By using the technique of constructing a path avoiding the intersection with unbounded perfect planes, it is shown in [31] that a perfect conducting polyhedral scatterer can be uniquely determined by a single far-field measurement.

Finally, we turn to the unique determination of partially coated scatterers. Here by ‘partial coating’, we mean that on the ‘partially coated’ part of the boundary, the total field of the
forward scattering problem takes the impedance type boundary conditions. Apparently, such partial coating or not is the intrinsic property of the underlying scatterer: there would be no impedance type boundary conditions at all if the scatterer is not coated. For such case, we would first make the crucial observation that both the Helmholtz equation and Maxwell equations don’t admit the non-null interior eigenfunctions for impedance problem. Based on this fact, it is shown in [34] and [36] that a coated polyhedral scatterer and a non-coated one cannot produce the same far-filed patterns corresponding to one or several incident fields in both acoustic and magnetic cases. That is, the partial coating properties of the underlying scatterers can be uniquely determined by a single or several far-field measurements. These results have significant implications for practical applications. Here we mention an example arising from radar technology. In order to avoid detection by radar, hostile objects are often partially coated by a material designed to reduce the radar cross section of the scattered wave. Let $D_1$ and $D_2$ be two scatterers with $D_1$ perfect conducting and $D_2$ partially coated. If $D_1$ and $D_2$ produce the same far-field data $F$, then for a radar based on the assumption that the underlying scatterer is perfecting conducting, one can uniquely determine $D_1$ with $F$. However, if the underlying object is purposely designed to be the partially coated $D_2$, using known results (see, e.g., [34]), one easily see that $\partial D_1 \neq \partial D_2$, and hence $D_2$ is invisible to the radar since it gives a fake $D_1$. Clearly, the uniqueness results mentioned earlier exclude such possibility.

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