TITCHMARSH THEOREMS ON DAMEK-RICCI SPACES VIA MODULI OF CONTINUITY OF HIGHER ORDER

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Abstract. A classical theorem of Titchmarsh relates the $L^2$-Lipschitz functions and decay of the Fourier transform of the functions. In this note, we prove the Titchmarsh theorem for Damek-Ricci space (also known as harmonic NA groups) via moduli of continuity of higher orders. We also prove an analogue of another Titchmarsh theorem which provides integrability properties of the Fourier transform for functions in the Hölder Lipschitz spaces.

1. Introduction

The classical Titchmarsh theorem [28, Theorem 85] characterizes the $L^2$-Lipschitz functions in terms of certain decay of the Fourier transform of the functions. It can be stated as follows: Let $\alpha \in (0, 1)$ and $f \in L^2(\mathbb{R})$. Then, $\|\tau_t f - f\|_2 \leq C_1 t^\alpha$ for all sufficiently small $t > 0$, if and only if

$$\int_{|\xi| > \frac{1}{t}} |\widehat{f}(\xi)|^2 d\xi \leq C_2 t^{2\alpha},$$

for all sufficiently small $t > 0$. Here, $\widehat{f}$ is the Fourier transform of $f$ and $\tau_t$ is the translation operator.

The Titchmarsh theorem has been extensively studied in many different contexts on various groups, for instances, the higher dimensional Euclidean spaces [31, 5], the Vilenkin groups [32], the special linear group of real matrices of order two $SL_2(\mathbb{R})$ [33], the rank one symmetric spaces of non-compact type [25, 15], the $p$-adic groups [26] and the compact homogeneous manifolds [12]. In terms of the moduli of continuity, the theorem has been explored on $\mathbb{R}$ [30, 13] and the rank one symmetric spaces [16]. See [6, 17, 18] for some growth properties of the Fourier transform on certain spaces via moduli of continuity.

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Damek-Ricci spaces, also known as Harmonic NA groups, are natural generalizations of the Iwasawa NA groups of the real rank one simple Lie groups. Particularly, the rank one symmetric spaces of non-compact type form a subclass of the Damek-Ricci spaces. In general, the Damek-Ricci spaces need not be symmetric spaces.

Very recently, Titchmarsh type results for Damek-Ricci spaces were explored in [23, 14]. In this note, we extend the classical Titchmarsh theorem to the setting of Damek-Ricci spaces in terms of the moduli of continuity of higher orders (see Corollary 3.5). We note here that it is new on the Damek-Ricci spaces even in the case of moduli of continuity of order one.

[18, Theorem 1.5 (A)] provides certain decay properties of the Helgason Fourier transform for functions in the generalized Besov spaces. We also prove an analogue of this theorem on Damek-Ricci spaces.

Another Titchmarsh theorem [28, Theorem 84] provides the integrability properties of the Fourier transform of functions belonging to the Hölder-Lipschitz spaces. This theorem has been studied for various groups, for examples, $SL_2(\mathbb{R})$ [33], the Euclidean space [5], the compact homogeneous manifolds [12] and the Damek-Ricci spaces [23, 14]. We also prove a generalization of this Titchmarsh theorem over Damek-Ricci spaces.

2. Preliminaries

In this section, we recall basics required about the Damek-Ricci spaces and moduli of continuity. Throughout the paper, we denote by $C, C_1, C_2, ...$ constants whose values may vary from one line to the other.

2.1. Fourier analysis on Damek-Ricci spaces. For the details about the analysis and geometry of Damek-Ricci spaces and associated Fourier analysis, one can refer to [8, 9, 10, 11, 1, 4, 7, 27, 21, 19, 2, 3, 22].

Let $\mathfrak{n}$ be a two-step real nilpotent Lie algebra, equipped with an inner product $\langle \cdot, \cdot \rangle$. Let $\mathfrak{z}$ denote the center of $\mathfrak{n}$ and let $\mathfrak{v}$ denote the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{n}$ w.r.t. $\langle \cdot, \cdot \rangle$. Suppose that the dimensions of $\mathfrak{v}$ and $\mathfrak{z}$ are denoted by $m$ and $k$ respectively as real vector spaces. The Lie algebra $\mathfrak{n}$ is said to be $H$-type algebra if for each $Z \in \mathfrak{z}$, the map $J_Z : \mathfrak{v} \to \mathfrak{v}$ given by

$$\langle J_Z X, Y \rangle = \langle Z, [X, Y] \rangle, \quad X, Y \in \mathfrak{v}, \ Z \in \mathfrak{z},$$
satisfies the condition $J_Z^2 = -\|Z\|^2 I_v$. Here, $I_v$ denotes the identity operator on $v$. Kaplan [19] proved that for $Z \in \mathfrak{z}$ with $\|Z\| = 1$ one has $J_Z^2 = -I_v$; that is, $J_Z$ induces a complex structure on $v$. Therefore, $m = \dim(v)$ is always even.

A connected and simply connected Lie group $N$ is said to be $H$-type if its Lie algebra is an $H$-type algebra. Since $n$ is nilpotent, it follows that the exponential map is a global diffeomorphism. Hence, the elements of $N = \exp n$ can be parametrized by $(X, Z)$, for $X \in v$ and $Z \in \mathfrak{z}$. From the Campbell-Baker-Hausdorff formula, the multiplication on $N$ is given by

$$(X, Z)(X', Z') = \left( X + X', Z + Z' + \frac{1}{2}[X, X'] \right).$$

Note that the group $A = \mathbb{R}^+_*$ acts on $N$ by nonisotropic dilations: $(X, Y) \mapsto (a^{\frac{1}{2}}X, aZ)$. Therefore, by setting $\dim(\mathfrak{z}) = k$, the homogeneous dimension of $N$ is given by $Q = m + k$.

Let $S = N \ltimes A$ be the semidirect product of $N$ with $A$ under the aforementioned action. Therefore, the group multiplication on $S$ is defined by

$$(X, Z, a)(X', Z', a') = \left( X + a^{\frac{1}{2}}X', Z + aZ' + \frac{1}{2}a^{\frac{1}{2}}[X, X'], aa' \right).$$

Then, $S$ is a solvable (connected and simply connected) Lie group with Lie algebra $\mathfrak{s} = \mathfrak{z} \oplus v \oplus \mathbb{R}$ and Lie bracket

$$[[X, Z, \ell], (X', Z', \ell')] = \left( \frac{1}{2}\ell X' - \frac{1}{2}\ell' X, \ell Z' - \ell' Z + [X, X'], 0 \right).$$

The group $S$ is equipped with the left-invariant Riemannian metric induced by

$$\langle (X, Z, \ell), (X', Z', \ell') \rangle = \langle X, X' \rangle + \langle Z, Z' \rangle + \ell \ell'$$

on $\mathfrak{s}$. The associated left Haar measure $dx$ on the group $S$ is given by $a^{-Q-1}dX dZ da = a^{-Q-1}dn da$, where $dX$, $dZ$ and $da$ are the Lebesgue measures on $v$, $\mathfrak{z}$ and $\mathbb{R}^+_*$, respectively. The elements of $A$ will be identified with $a_t = e^t$, $t \in \mathbb{R}$. We will also write any element $s \in S$ as $na_t$ by writing $S = NA$. In particular, any element $a_t \in A$ can be thought as an element of $S$ by writing $a_t = e_N a_t$, where $e_N$ is the identity element of $N$. The group $S$ can be realized as the unit ball $B(\mathfrak{s})$ in $\mathfrak{s}$ using the Cayley transform $C : S \to B(\mathfrak{s})$ (see [1]).

To define the Helgason Fourier transform on the group $S$, we need to describe the notion of the Poisson kernel ([4]). The Poisson Kernel $P : S \times N \to \mathbb{R}$ is defined by
The value of $C$ is appropriately chosen so that $\int_N P_a(n) dn = 1$ and $P_1(n) \leq 1$. Now, we list some useful properties of the Poisson kernel; see [21, 27, 4]. For $\lambda \in \mathbb{C}$, the complex power of the Poisson kernel is given by

$$P_\lambda(x,n) = P^1_{\lambda - \frac{Q}{4}}.$$

It is known ([27, 4]) that for each fixed $x \in S$, $P_\lambda(x,\cdot) \in L^p(N)$ for $1 \leq p \leq \infty$ if $\lambda = i\gamma_p \rho$, where $\gamma_p = \frac{2}{p} - 1$. An important feature of the Poisson kernel $P_\lambda(x,n)$ is that it is constant on the hypersurfaces $H_{n,a_t} = \{n\sigma(a_t n') : n' \in N\}$. Here, $\sigma$ denotes the geodesic inversion on $S$; see [7].

Let $\Delta_S$ denote the Laplace-Beltrami operator on $S$. Then, for every fixed $n \in N$, the function $P_\lambda(x,n)$ is an eigenfunction of $\Delta_S$ with eigenvalue $-(\lambda^2 + \frac{Q^2}{4})$; see [4]. The Helgason Fourier transform of a measurable function $f$ on $S$ is given by

$$\tilde{f}(\lambda,n) = \int_S f(x) P_\lambda(x,n) dx$$

provided that the integral exists.

For $f \in C^\infty_c(S)$, the Fourier inversion formula [4, Theorem 4.4] is given by

$$f(x) = C \int_{\mathbb{R}} \int_N \tilde{f}(\lambda,n) P_{-\lambda}(x,n) |c(\lambda)|^{-2} d\lambda dn.$$

The Plancherel theorem [4, Theorem 5.1] states that the Helgason Fourier transform extends to an isometry from $L^2(S)$ onto the space $L^2(\mathbb{R}_+ \times N, |c(\lambda)|^{-2} d\lambda dn)$. For precise value of the constants; see [4]. The function $|c(\lambda)|$ satisfies:

$$|c(\lambda)|^{-2} \asymp \begin{cases} \lambda^2 & \text{if } \lambda \in [0, 1] \\ \lambda^{d-1} & \text{if } \lambda > 1 \end{cases}.$$

(1)

See [29, Theorem 1.14] and [27, Lemma 4.8].

Let $e$ denote the identity element of $S$ and let $\bar{\mu}$ be the metric induced by the canonical left invariant Riemannian structure on $S$. A function $f$ on $S$ is said to be radial if for all $x, y \in S$, $f(x) = f(y)$ whenever $\bar{\mu}(x,e) = \bar{\mu}(y,e)$. Note that the radial functions on $S$ can be identified with the functions $f = f(r)$ of the geodesic distance $r = \bar{\mu}(x,e) \in [0, \infty)$. 

\[P(na_t, n') = P_a(n'^{-1}n), \]

where

$$P_a(n) = P_a(X,Z) = Ca^Q \left( \left( a_t + \frac{|X|^2}{4} \right)^2 + |Z|^2 \right)^{-\frac{Q}{2}}, \quad n = (X,Z) \in N \text{ and } t \in \mathbb{R}.$$
Clearly, $\mu(t,e) = |t|$ for $t \in \mathbb{R}$, where we have identified $a_t$ with $e_Na_t \in S := NA$ with $e_N$ being the identity of $N$. For any radial function $f$, sometimes we use the notation $f(a_t) = f(t)$. The elementary spherical function $\phi_\lambda(x)$ is given by

$$
\phi_\lambda(x) := \int_N p_\lambda(x,n)p_{-\lambda}(x,n)dn.
$$

Note that the function $\phi_\lambda$ is a radial eigenfunction of the Laplace-Beltrami operator $\Delta_S$ with eigenvalue $-(\lambda^2 + \frac{Q^2}{4})$; see [1, 4]. Also, $\phi_\lambda(x) = \phi_{-\lambda}(x)$, $\phi_\lambda(x) = \phi_\lambda(x^{-1})$ and $\phi_\lambda(e) = 1$. In [1], the authors proved that the radial part (in geodesic polar coordinates) of the Laplace-Beltrami operator $\Delta_S$ given by

$$
\text{rad } \Delta_S = \frac{\partial^2}{\partial t^2} + \left\{ \frac{m + l}{2} \coth \frac{t}{2} + \frac{k}{2} \tanh \frac{t}{2} \right\} \frac{\partial}{\partial t},
$$

is (by substituting $r = \frac{t}{2}$) equal to $\frac{1}{4} L_{\alpha,\beta}$ with indices $\alpha = \frac{m + l + 1}{2}$ and $\beta = \frac{l - 1}{2}$. Here, $L_{\alpha,\beta}$ is the Jacobi operator and Koornwinder [24] treated it in detail. It is worth pointing out that we are in the ideal condition of the Jacobi analysis with $\alpha > \beta > -\frac{1}{2}$. To be more precise, the Jacobi functions $\phi_\alpha^\beta$ are related to the elementary spherical functions $\phi_\lambda$ by $\phi_\lambda(t) = \phi_\alpha^\beta(\frac{t}{2})$; see [1]. Hence, we have the following important lemma; see [25].

**Lemma 2.1.** Let $t, \lambda \in \mathbb{R}_+$. Then,

- $|\phi_\lambda(t)| \leq 1$.
- $|1 - \phi_\lambda(t)| \leq \frac{\alpha^2}{2} \left( 4\lambda^2 + \frac{Q^2}{4} \right)$.
- There exists a constant $C > 0$ such that $|1 - \phi_\lambda(t)| \geq C$ for $\lambda t \geq 1$.

Let $\sigma_t$ denote the normalized surface measure induced by the left invariant Riemannian metric on the geodesic sphere $S_t = \{ y \in S : \mu(y,e) = t \}$ of radius $t$. Then, $\sigma_t$ is a nonnegative radial measure. The spherical mean operator $M_t$ is given by $M_tf := f * \sigma_t$ for a suitable function $f$ on $S$. Note that $M_tf(x) = \mathcal{R}(f^x)(t)$. Here, $f^x$ is the right translation of function $f$ by $x$ and $\mathcal{R}$ is the radialization operator given by

$$
\mathcal{R}f(x) = \int_{S_\nu} f(y) \, d\sigma_\nu(y)
$$

for a suitable function $f$ and $\nu = r(x) = \mu(C(x), 0)$, where $C$ is the Cayley transform. It is easy to check that $\mathcal{R}f$ is a radial function. Also, $\mathcal{R}f = f$ for any radial function $f$. Thus, for a radial function $f$, $M_tf$ is the usual translation of $f$ by $t$. The spherical mean
operator $M_t$ is a bounded linear operator on $L^p(S)$ and

$$\tilde{M}_t f(\lambda, n) = \tilde{f}(\lambda, n)\phi_\lambda(t)$$

(2)

for a suitable function $f$ on $S$. Further, $M_t f$ converges to $f$ as $t \to 0$. See [21].

2.2. Moduli of continuity of higher orders. Let $\omega$ be a mapping from $I \subset \mathbb{R}$ to the set $[0, \infty)$. The map $\omega$ is said to be almost increasing if there exists a constant $C \geq 1$ such that $\omega(t) \leq C\omega(s)$ whenever $t \leq s$ and $t, s \in I$. The map $\omega$ is said to be almost decreasing if there exists a constant $C \geq 1$ such that $\omega(t) \leq C\omega(s)$ whenever $t \geq s$ and $t, s \in I$.

Let $\delta_0 > 0$ and $k \in \mathbb{R}_+$. A continuous function $\omega_k : [0, \delta_0] \to \mathbb{R}_+$ is said to be a $k$th order modulus of continuity if $\omega_k(0) = 0$, $\omega_k(t)$ is almost increasing on $t \in [0, \delta_0]$ and $\frac{\omega_k(t)}{t^k}$ is almost decreasing on $t \in [0, \delta_0]$. Note that if $\omega$ is a $k$th-order modulus of continuity then $\omega$ is also an $m$th-order modulus of continuity for all $m \geq k$.

We say that the $k$th-order modulus of continuity $\omega_k$ belongs to the Zygmund class $\mathcal{Z}^0$ if there exists a constant $C > 0$ such that

$$\int_0^t \frac{\omega_k(s)}{s} ds \leq C\omega_k(t), \ t \in [0, \delta_0].$$

We say that the $k$th-order modulus of continuity $\omega_k$ belongs to the Zygmund class $\mathcal{Z}^k$ if there exists a constant $C$ such that

$$\int_{\delta_0}^t \frac{\omega_k(s)}{s^{1+k}} ds \leq C\frac{\omega_k(t)}{t^k}, \ t \in [0, \delta_0].$$

The class $\mathcal{Z}^0 \cap \mathcal{Z}^k$ is called the Zygmund-Bari-Stechkin class. Some classical examples in the Zygmund-Bari-Stechkin class are $t^\alpha$, $t^\alpha \left(\ln \frac{1}{t}\right)^\gamma$ and $t^\alpha \left(\ln \ln \frac{1}{t}\right)^\gamma$, where $\alpha \in (0, k)$ and $\gamma \in \mathbb{R}$. For more details on Zygmund classes, see [20].

The crucial behaviour of the functions discussed above is near zero. In Theorem 3.4, we also need to do certain estimations over $[\delta_0, \infty)$. Note that there is no loss of generality of the results in assuming certain restrictions, as in Theorem 3.4, on $\omega_k$ over the interval $[\delta_0, \infty)$.

3. Titchmarsh theorems on Damek-Ricci spaces

In this section, we present the main result of this paper that provides a description of generalised Lipschitz class functions in terms of the Helgason Fourier transform. We
also prove the decay properties of the Helgason Fourier transform for the functions in the generalized Besov spaces. Lastly, we discuss certain integrability properties of the Helgason Fourier transform for the functions in the Hölder Lipschitz spaces.

Now, we begin with the following definition of generalised Lipschitz class. For that, let us denote the harmonic NA group by \( S \) and the \( k \)th-order modulus of continuity by \( \omega_k \).

**Definition 3.1.** A function \( f \in L^2(S) \) is said to be in the generalised Lipschitz class \( \text{Lip}(\omega_k) \) if there exists a positive constant \( C \) such that for all sufficiently small \( t \in (0, 1) \) we have \( \| M_t f - f \|_2 \leq C \omega_k(t) \).

The following theorem provides a growth property of image of the generalised Lipschitz functions under the Helgason Fourier transform.

**Theorem 3.2.** Let \( \omega_k \) be a \( k \)th-order modulus of continuity. If \( f \in \text{Lip}(\omega_k) \) then there exists a positive constant \( C \) such that for all sufficiently small \( t \in (0, 1) \),

\[
\int_{\frac{1}{t}}^{\infty} \int_{\mathbb{N}} |\tilde{f}(\lambda, n)|^2 \, dn \, d\lambda \leq Ct^{d-1} \omega_k(t)^2.
\]

**Proof.** Using the Plancherel theorem and (2) we have

\[
\| M_t f - f \|_2^2 = \int_{\mathbb{R}^+} \int_{\mathbb{N}} |(M_t f - f)(\lambda, n)|^2 |c(\lambda)|^{-2} \, dn \, d\lambda \quad = \int_{\mathbb{R}^+} \int_{\mathbb{N}} |1 - \phi(\lambda)|^2 |\tilde{f}(\lambda, n)|^2 |c(\lambda)|^{-2} \, dn \, d\lambda.
\]

Since \( f \in \text{Lip}(\omega_k) \), it follows that

\[
\int_{\mathbb{R}^+} \int_{\mathbb{N}} |1 - \phi(a_t)|^2 |\tilde{f}(\lambda, n)|^2 |c(\lambda)|^{-2} \, dn \, d\lambda \leq C^2 \omega_k(t)^2. \quad (3)
\]

Now,

\[
\int_{\frac{1}{t}}^{\infty} \int_{\mathbb{N}} |\tilde{f}(\lambda, n)|^2 \, dn \, d\lambda = t^{d-1} \int_{\frac{1}{t}}^{\infty} \int_{\mathbb{N}} |\tilde{f}(\lambda, n)|^2 \, dn \frac{1}{t^{d-1}} \, d\lambda \leq t^{d-1} \int_{\frac{1}{t}}^{\infty} \int_{\mathbb{N}} |\tilde{f}(\lambda, n)|^2 \, dn \lambda^{d-1} \, d\lambda.
\]

Since \( \frac{1}{t} > 1 \), then using (1) we obtain

\[
\int_{\frac{1}{t}}^{\infty} \int_{\mathbb{N}} |\tilde{f}(\lambda, n)|^2 \, dn \, d\lambda \leq Ct^{d-1} \int_{\frac{1}{t}}^{\infty} \int_{\mathbb{N}} |\tilde{f}(\lambda, n)|^2 \, dn |c(\lambda)|^{-2} \, d\lambda.
\]
Using Lemma 2.1 we have

\[
\int_\frac{1}{t}^\infty \int_N |\tilde{f}(\lambda, n)|^2 \, dn \, d\lambda \leq \frac{C t^{d-1}}{C_1^2} \int_\frac{1}{t}^\infty |1 - \phi_\lambda(a_t)|^2 \int_N |\tilde{f}(\lambda, n)|^2 \, dn |c(\lambda)|^{-2} \, d\lambda
\]

\[
\leq \frac{C t^{d-1}}{C_1^2} \int_0^\infty |1 - \phi_\lambda(a_t)|^2 \int_N |\tilde{f}(\lambda, n)|^2 \, dn |c(\lambda)|^{-2} \, d\lambda.
\]

Hence, by (3) we have

\[
\int_\frac{1}{t}^\infty \int_N |\tilde{f}(\lambda, n)|^2 \, dn \, d\lambda \leq C t^{d-1} \omega_k(t)^2.
\]

We need the following lemma to prove a converse to the previous theorem.

**Lemma 3.3.** Let \( \omega_k \) belongs to the Zygmund class \( \mathcal{Z}_0 \) and \( f \in L^2(S) \). The following are equivalent.

(i) There exists a positive constant \( C \) such that

\[
\int_\frac{1}{t}^\infty \int_N |\tilde{f}(\lambda, n)|^2 \lambda^{d-1} \, dn \, d\lambda \leq C \omega_k(t)^2, \ t \in (0, 1).
\]

(ii) There exists a positive constant \( C \) such that

\[
\int_\frac{1}{t}^\frac{1}{2t} \int_N |\tilde{f}(\lambda, n)|^2 \lambda^{d-1} \, dn \, d\lambda \leq C \omega_k(t)^2, \ t \in (0, 1).
\]

**Proof.** (i) implies (ii) is clear. Now, assume that (ii) holds. Then, for \( i \geq 0 \), we get

\[
\int_\frac{1}{t}^{\frac{1}{2^i+1}t} \int_N |\tilde{f}(\lambda, n)|^2 \lambda^{d-1} \, dn \, d\lambda \leq C \omega_k \left( \frac{t}{2^i} \right)^2.
\]

Therefore,

\[
\int_\frac{1}{t}^\infty \int_N |\tilde{f}(\lambda, n)|^2 \lambda^{d-1} \, dn \, d\lambda = \sum_{i=0}^{\infty} \int_\frac{1}{2^i t}^{\frac{1}{2^{i+1}}t} \int_N |\tilde{f}(\lambda, n)|^2 \lambda^{d-1} \, dn \, d\lambda
\]

\[
\leq C \sum_{i=0}^{\infty} \omega_k \left( \frac{1}{2^i} \right)^2.
\]

Let \( \mu > 0 \) be such that \( \mu < m(\omega_k) \). Here, \( m(\omega_k) \) is the lower MO index [20, Pg. 31]. Since \( \omega_k \) belongs to the Zygmund class \( \mathcal{Z}_0 \), it follows by [20, Theorem 2.10] that \( \frac{\omega_k(t)}{t^\mu} \) is almost increasing. Since \( \frac{1}{2^i} t \leq t, \ i \geq 0 \), it follows by the definition of almost increasing that

\[
\frac{\omega_k(t)}{(\frac{1}{2^i} t)^\mu} \leq C \frac{\omega_k(t)}{t^\mu}.
\]
Therefore, we have
\[ \int_1^\infty \int_N |\tilde{f}(\lambda, n)|^2 \lambda^{d-1} \, dn \, d\lambda \leq C_2 \sum_{i=0}^{\infty} \left( \left( \frac{t}{2^i} \right)^{\mu} \frac{\omega_k(t)}{t^{\mu}} \right)^2 = C_2 \omega_k(t)^2 \sum_{i=0}^{\infty} \left( \frac{1}{2^i} \right)^{2\mu}. \]

Hence, we have
\[ \int_1^\infty \int_N |\tilde{f}(\lambda, n)|^2 \lambda^{d-1} \, dn \, d\lambda \leq C_3 \omega_k(t)^2. \]

The following theorem is a converse to Theorem 3.2 under more assumptions.

**Theorem 3.4.** Let \( k \leq 2 \) and assume that \( \omega_k \) belongs to the Zygmund classes \( \mathcal{Z}^0 \) and \( \mathcal{Z}_k \). Let \( \omega_k(t) \) be bounded below by a positive number on interval \([\delta_0, \infty)\) and let \( \frac{\omega_k(t)^2}{t^\mu} \in L^1([\delta_0, \infty))\). For a function \( f \in L^2(S) \), if there exists a positive constant \( C \) such that for all sufficiently small \( t \in (0,1) \), we have
\[ \int_1^\infty \int_N |\tilde{f}(\lambda, n)|^2 \, dn \, d\lambda \leq Ct^{d-1} \omega_k(t)^2, \tag{4} \]
then \( f \in \text{Lip}(\omega_k) \).

**Proof.** Using (4) we have
\[ \int_1^2 \int_N |\tilde{f}(\lambda, n)|^2 \lambda^{d-1} \, dn \, d\lambda \leq \left( \frac{2}{t} \right)^{d-1} \int_1^\infty \int_N |\tilde{f}(\lambda, n)|^2 \, dn \, d\lambda \leq C_1 \omega_k(t)^2. \]
Then, by Lemma 3.3 we have
\[ \int_1^\infty \int_N |\tilde{f}(\lambda, n)|^2 \lambda^{d-1} \, dn \, d\lambda \leq C_2 \omega_k(t)^2. \tag{5} \]

As in the proof of Theorem 3.2, using the Plancherel theorem and (2) we get
\[ \|M_t f - f\|_2^2 = \int_{\mathbb{R}^+} \int_N |1 - \phi_\lambda(a_t)|^2 |\tilde{f}(\lambda, n)|^2 |c(\lambda)|^{-2} \, dn \, d\lambda = I_1 + I_2. \tag{6} \]

Here,
\[ I_1 = \int_0^1 \int_N |1 - \phi_\lambda(a_t)|^2 |\tilde{f}(\lambda, n)|^2 |c(\lambda)|^{-2} \, dn \, d\lambda \]
and
\[ I_2 = \int_1^\infty \int_N |1 - \phi_\lambda(a_t)|^2 |\tilde{f}(\lambda, n)|^2 |c(\lambda)|^{-2} \, dn \, d\lambda. \]
First, we estimate $I_2$. Since $\frac{t}{4} > 1$, using (1) and Lemma 2.1 we have

$$I_2 \leq 2^2 C \int_{\frac{t}{4}}^{\infty} \int_{N} |\tilde{f}(\lambda, n)|^2 \lambda^{d-1} dn d\lambda.$$

Thus, applying (5) we get

$$I_2 \leq 4C_3 \omega_k(t)^2. \quad (7)$$

Now, we estimate $I_1$. Using Lemma 2.1 we have

$$I_1 \leq \int_{0}^{\frac{t}{4}} \int_{N} \left( \frac{t^2}{2} \left( 4\lambda^2 + \frac{Q^2}{4} \right) \right)^2 |\tilde{f}(\lambda, n)|^2 |c(\lambda)|^{-2} dn d\lambda$$

$$\leq C_4 t^4 \int_{0}^{\frac{t}{4}} \int_{N} \lambda^4 |\tilde{f}(\lambda, n)|^2 |c(\lambda)|^{-2} dn d\lambda$$

$$+ C_5 t^4 \int_{0}^{\frac{t}{4}} \int_{N} |\tilde{f}(\lambda, n)|^2 |c(\lambda)|^{-2} dn d\lambda.$$

By the Plancherel theorem in the second term, we get

$$I_1 \leq C_4 t^4 \int_{0}^{\frac{t}{4}} \int_{N} \lambda^4 |\tilde{f}(\lambda, n)|^2 |c(\lambda)|^{-2} dn d\lambda + C_6 t^4.$$

Denote $\psi(s) = \int_{s}^{\infty} \int_{N} |\tilde{f}(\lambda, n)|^2 |c(\lambda)|^{-2} dn d\lambda$, $s \in \mathbb{R}$. By the Plancherel theorem, $\psi(s)$ is bounded on $\mathbb{R}$. Then, using the assumption that $\omega_k(s)$ is bounded below by a positive number on $[\delta_0, \infty)$, and (5), we have $\psi(s) \leq C_3 \omega_k \left( \frac{1}{s} \right)^2$, $s \in \mathbb{R}$. Therefore,

$$I_1 \leq C_4 t^4 \int_{0}^{\frac{t}{4}} \lambda^4 (-\psi'(\lambda)) d\lambda + C_6 t^4$$

$$= C_4 t^4 \left( -\frac{1}{t^4} \psi \left( \frac{1}{t} \right) + \int_{0}^{\frac{t}{4}} 4\lambda^3 \psi(\lambda) d\lambda \right) + C_6 t^4$$

$$\leq C_4 t^4 \int_{0}^{\frac{t}{4}} 4\lambda^3 \psi(\lambda) d\lambda + C_6 t^4$$

$$\leq C_7 t^4 \int_{0}^{\frac{t}{4}} \lambda^3 \omega_k \left( \frac{1}{\lambda} \right)^2 d\lambda + C_6 t^4$$

$$= C_7 t^4 \int_{\frac{t}{4}}^{\infty} \frac{\omega_k(\lambda)^2}{\lambda^5} d\lambda + C_6 t^4.$$

Using the assumption $\frac{\omega_k(t)^2}{t} \in L^1([\delta_0, \infty))$, we get

$$I_1 \leq C_7 t^4 \int_{t}^{\delta_0} \frac{\omega_k(\lambda)^2}{\lambda^5} d\lambda + C_8 t^4.$$
Since $k \leq 2$ then $4 - k \geq k$. Therefore, $\frac{\omega_k(\lambda)}{\lambda^{k+1}}$ is almost decreasing. Then, we have

$$I_1 \leq C_7 t^{4-k} \int_{t}^{\delta_0} \frac{\omega_k(\lambda)}{\lambda^{k+1}} d\lambda + C_8 t^4.$$ 

Using the assumption that $\omega_k$ belongs to the Zygmund class $Z_0$ in the first term and using the fact that $k \leq 2$ implies that $w_k(t)$ is almost decreasing which implies that $t^2 \leq C \omega_k(t)$ in the second term, it follows that

$$I_1 \leq C_9 \omega_k(t)^2.$$ \hspace{1cm} (8)

Hence, the result follows form (6), (8) and (7). \hfill \Box

The following result is the main result of the paper and it is a direct consequence of Theorem 3.2 and Theorem 3.4.

**Corollary 3.5.** Let $k \leq 2$ and assume that $\omega_k$ belongs to the Zygmund classes $Z_0$ and $Z_k$. Let $\omega_k(t)$ be bounded below by a positive number on interval $[\delta_0, \infty)$ and let $\frac{\omega_k(t)}{t^2} \in L^1([\delta_0, \infty))$. A function $f \in L^2(S)$ belongs to the generalised Lipschitz class $\text{Lip}(\omega_k)$ if and only if there exists a positive constant $C$ such that for all sufficiently small $t \in (0, 1)$, we have

$$\int_{t}^{\infty} \int_{N} |\tilde{f}(\lambda, n)|^2 \, dn \, d\lambda \leq C t^{d-1} \omega_k(t)^2.$$ 

If we take $\omega_2(t) = t^\alpha (\ln \frac{1}{t})^\gamma$, where $\alpha \in (0, 2)$, $\gamma \in \mathbb{R}$ and $t \in [0, \delta_0]$, and $\omega_2$ is constant on $[\delta_0, \infty)$, then as a special case of Corollary 3.5 we obtain the following result.

**Corollary 3.6.** Let $\alpha \in (0, 2), \gamma \in \mathbb{R}$ and $f \in L^2(S)$. Then, we have

$$\|M_t f - f\|_2 \leq C_1 t^\alpha \left( \ln \frac{1}{t} \right)^\gamma,$$

for all sufficiently small $t \in (0, 1)$ if and only if

$$\int_{t}^{\infty} \int_{N} |\tilde{f}(\lambda, n)|^2 \, dn \, d\lambda \leq C_2 t^{2\alpha + d - 1} \left( \ln \frac{1}{t} \right)^{2\gamma},$$

for all sufficiently small $t \in (0, 1)$.

The following result is an analogue of [18, Theorem 1.5 (A)] which gives decay properties of the Helgason Fourier transform for functions in the generalized Besov spaces.
Theorem 3.7. Let $\alpha > 0$ and $f \in L^2(S)$. If

$$\int_0^1 \left( \frac{\|M_t f - f\|_2}{t^\alpha} \right)^2 \frac{dt}{t} < \infty, \quad (9)$$

then

$$\int_0^\infty \int_{\lambda = t}^{2t} \int_N \left( \frac{|\tilde{f}(\lambda, n)|}{\lambda^{-\alpha}} \right)^2 d\lambda \frac{dt}{t} < \infty. \quad (10)$$

Proof. Let $f \in L^2(S)$ be such that (9) holds. We decompose

$$\int_0^\infty \int_{\lambda = t}^{2t} \int_N \left( \frac{|\tilde{f}(\lambda, n)|}{\lambda^{-\alpha}} \right)^2 d\lambda \frac{dt}{t} = I_1 + I_2,$$

where

$$I_1 = \int_0^\frac{1}{2} \int_{\lambda = t}^{2t} \int_N \left( \frac{|\tilde{f}(\lambda, n)|}{\lambda^{-\alpha}} \right)^2 d\lambda \frac{dt}{t},$$

and

$$I_2 = \int_\frac{1}{2}^\infty \int_{\lambda = t}^{2t} \int_N \left( \frac{|\tilde{f}(\lambda, n)|}{\lambda^{-\alpha}} \right)^2 d\lambda \frac{dt}{t}.$$

First, we estimate $I_1$. Using the Plancherel theorem we have

$$I_1 \leq C_1 \int_0^\frac{1}{2} t^{2\alpha} \int_{\lambda = t}^{2t} |\tilde{f}(\lambda, n)|^2 d\lambda \frac{dt}{t} \leq C_2 \|f\|_2^2 \int_0^\frac{1}{2} t^{2\alpha} \frac{dt}{t} < \infty.$$

Now, we estimate $I_2$. By the change of variables $t = \frac{1}{2s}$ we obtain

$$I_2 = \int_0^1 \int_{\lambda = \frac{1}{2s}}^{\frac{1}{s}} \int_N \left( \frac{|\tilde{f}(\lambda, n)|}{\lambda^{-\alpha}} \right)^2 d\lambda \frac{ds}{s} \leq C_3 \int_0^1 \int_{\lambda = \frac{1}{2s}}^{\frac{1}{s}} \int_N |\tilde{f}(\lambda, n)|^2 d\lambda \frac{ds}{s}.$$

Since $s\lambda \leq 1$, it follows by [21, Theorem 4.7 (a)] that

$$I_2 \leq C_4 \int_0^1 \frac{1}{s^{2\alpha}} \|M_s f - f\|_2^2 \frac{ds}{s}.$$

Therefore, by (9), $I_2$ is finite. Hence (10) holds. \qed
Now, we deal with the analogue of the [28, Theorem 37]. This provides an improvement of the well known Hausdorff-Young inequality. For \( q = p' \), the next theorem is also proved in [23].

Let \( \alpha \in (0,1] \) and \( p \geq 1 \). A function \( f \in L^p(S) \) is said to be in the Hölder-Lipschitz space \( Lip(\alpha; p) \) if there exists a positive constant \( C \) such that for all sufficiently small \( t \in (0, 1) \) we have \( \|M_t f - f\|_p \leq Ct^\alpha \).

**Theorem 3.8.** Let \( \alpha \in (0, 1] \), \( p \in (1, 2] \) and \( q \in [p, p'] \). Assume that \( f \in Lip(\alpha; p) \) and define

\[
F(\lambda) := \left( \int_N |\tilde{f}(\lambda + i\gamma \rho, n)|^q \, dn \right)^{\frac{1}{q}}.
\]

Then, \( F \in L^b(\mathbb{R}_+, |c(\lambda)|^{-2} \, d\lambda) \), for any \( \beta \in \left( \frac{dp'}{d+\alpha p'}, p' \right) \).

**Proof.** Using [21, Theorem 4.7] we get

\[
\int_{\mathbb{R}} \min\{1, (\lambda t)^{2p'}\} F(\lambda)^{p'} |c(\lambda)|^{-2} \, d\lambda \leq C \|M_t f - f\|_p^{p'}.
\]

Then, for small enough \( t \) we have

\[
\int_{|\lambda| < \frac{1}{t}} \lambda^{2p'} F(\lambda)^{p'} |c(\lambda)|^{-2} \, d\lambda \leq C \left( \frac{\|M_t f - f\|_p}{t^2} \right)^{p'} \leq C_1 t^{(\alpha-2)p'}.
\]

(11)

For \( s = \frac{1}{t} \) and \( \beta < p' \), define

\[
\phi(s) := \int_1^s \lambda^{2\beta} F(\lambda)^{\beta} |c(\lambda)|^{-2} \, d\lambda.
\]

By the Hölder inequality with \( \frac{\beta}{p'} + \left(1 - \frac{\beta}{p'}\right) = 1 \) and using (1), we get

\[
\phi(s) \leq C_2 \left( \int_1^s \lambda^{2p'} F(\lambda)^{p'} |c(\lambda)|^{-2} \, d\lambda \right)^{\frac{\beta}{p'}} \left( \int_1^s |c(\lambda)|^{-2} \, d\lambda \right)^{1-\frac{\beta}{p'}}
\leq C_2 \left( \int_1^s \lambda^{2p'} F(\lambda)^{p'} |c(\lambda)|^{-2} \, d\lambda \right)^{\frac{\beta}{p'}} \left( \int_1^s \lambda^{d-1} \, d\lambda \right)^{1-\frac{\beta}{p'}}.
\]

Using (11) we have

\[
\phi(s) \leq C_2 s^{(2-\alpha)\beta + d(1-\frac{\beta}{p'})}.
\]

Hence, using the integration by parts, we obtain

\[
\int_1^s F(\lambda)^{\beta} |c(\lambda)|^{-2} \, d\lambda = \int_1^s \lambda^{-2\beta} \phi'(\lambda) \, d\lambda
\leq s^{-2\beta} \phi(s) + 2\beta \int_1^s \lambda^{-2\beta-1} \phi(\lambda) \, d\lambda
\]
\[ \leq C_2 s^{-\alpha\beta + d\left(1 - \frac{\beta}{p'}\right)} + C_3 \int_1^s \lambda^{-1-\alpha\beta + d\left(1 - \frac{\beta}{p'}\right)} d\lambda \]
\[ = O(1) \]

provided that \(-\alpha\beta + d\left(1 - \frac{\beta}{p'}\right) < 0\), that is, \(\beta > \frac{dp'}{d+\alpha p}\). \(\square\)

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