Convexity of the Exercise Boundary of the American Put Option

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Abstract

This paper studies the parabolic free boundary problem arising from pricing American-style put options on an asset whose index follows a geometric Brownian motion process. The contribution is to propose a condition for that the early exercise boundary is a convex function.

Keywords: American-style put, convexity, free boundary problem, early exercise boundary

1 Introduction

From a theoretical as well as practical point of view, the valuation of American-style options has attracted considerable attention in the field of financial mathematics. Under the Black-Scholes (BS) framework [3], Merton [24] presented the price of American options in conjunction with an early exercise boundary as a solution to the free boundary problem in the BS equation. Since that time, considerable effort has been made to solve the free boundary problem associated with the pricing of American options [2, 4, 5, 6, 7, 9, 10, 16, 17, 18, 20].

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Nonetheless, an entirely satisfactory analytic solution has not been found. Several researchers have concentrated on finding more accurately expansions or simulations for the early exercise boundary, such as [4], [5], [10], [16], [17], [20]. An overview of their results indicates that the early exercise boundary of American put options is a convex function when the dividend rate is less than the risk-free rate and that the convexity may break down when the dividend rate exceeds the risk-free rate [7]. Chen et al. [6] and Ekström [9] proposed a rigorous verification of the supposition that the early exercise boundary is convex when a stock does not pay dividends. Chen et al. [7] demonstrated a proof for that the early exercise boundary is not convex when the dividend rate exceeds the risk-free rate. Currently, the convexity of the early exercise boundary remains an open problem when the dividend rate is non-zero [7].

The contribution of this paper is to examine the convexity of the exercise boundary of the American put option. we show that the early exercise boundary \( X_f(T) \) is a strictly decreasing convex function if \( q + \frac{\sigma^2}{2} \leq r \).

In summary, the following results have been provided for the convexity of the early exercise boundary of an American put option.

(a) The early exercise boundary is convex when \( q = 0 \) [6] [9].

(b) The early exercise boundary is not convex when \( r < q \) [7].

(c) We show that the early exercise boundary is convex when \( q + \frac{\sigma^2}{2} \leq r \).

Therefore, the convexity of the early exercise boundary remains an open problem when \( 0 < q < r < q + \frac{\sigma^2}{2} \).

This paper is organized as follows. In Section 2, we demonstrate properties of the solution \( u(s,t) \) as well as the early exercise boundary \( s(t) \). In Section 3, we present a proof of the convexity for the early exercise boundary.

## 2 Problem statement

Let \( S_T \) denote the stock price at time \( T \). We assume that the stock price satisfies the geometric Brownian motion. A standard argument explains that
the expectation

\[ P(S, T) = \mathbb{E}_x[e^{-r(T_F - T)} \max\{0, K - ST_F\}] \]

solves a parabolic equation, where \( r > 0 \) is the interest rate, \( T_F \) is the expiration date and \( \psi(S) = \max\{0, K - S\} \) is the payoff function of a put option. The parabolic equation is expressed as the form:

\[ \mathcal{L}_{BS} P = 0, \quad (1) \]

with the terminal condition \( P(S, T_F) = \max\{0, K - S\} \), where the Black-Scholes operator \( \mathcal{L}_{BS} \) is defined as

\[ \mathcal{L}_S \equiv \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - q)S \frac{\partial}{\partial S} - r + \frac{\partial}{\partial T}. \]

The solution of [1] provides a formula for valuing a European put option. For the American counterpart, the price satisfies the following optimal stopping problem

\[ P(S, T) = \text{ess sup}_{\tau \in \mathcal{T}_{T,F}} \mathbb{E}_x\left[ e^{-r\tau} \psi(S_\tau) \right], \]

where \( \mathcal{T} \) is the set of all stopping times and \( \mathcal{T}_{T,F} = \{ \tau \in \mathcal{T} | \mathbb{P}(\tau \in [T, T_F]) = 1 \} \), \( 0 \leq T \leq T_F < \infty \). The details of the optimal stopping problem for arbitrary diffusion processes can be found in Dayanik [8] and Lamberton [22]. The connection between the free boundary and the optimal stopping problem for the diffusion process was discussed by Kotlow [19] and Lamberton [22].

We examine the following one-dimensional free boundary problem for linear parabolic equations arising from the problem of valuing an American put option.

**Problem (BS)**

\[ \mathcal{L}_{BS} P = 0 \quad X_f(T) < S < \infty, \ 0 < T < T_F, \quad (2) \]

\[ P(S, T_F) = \max\{0, K - S\} \quad 0 \leq S < \infty, \quad (3) \]

\[ P(S, T) > \max\{0, K - S\} \quad X_f(T) < S < \infty, \ 0 < T < T_F, \quad (4) \]

\[ \lim_{S \to \infty} P(S, T) = 0 \quad 0 < T < T_F, \quad (5) \]

\[ P(X_f(T), T) = K - X_f(T) \quad 0 < T < \infty, \quad (6) \]

\[ \frac{\partial P}{\partial S}(X_f(T), T) = -1 \quad 0 < T < T_F. \quad (7) \]
The far-field condition (5) states that an American put option becomes worthless when the stock price becomes very large. This is because there is no possibility of exercising the option early. The condition (6) states that the American put option should be exercised to maximize the expected income when the price $S$ at time $T$ falls to the value of $X_f(T)$. The smooth-pasting condition (7) holds when the hedging ratio remains continuous across the early exercise boundary (see Kwok [21]).

The following properties for $P(S,T)$ and $X_f(T)$ are known to be valid (see [25] and [26]).

Theorem 2.1 Let \{X_f, P\} be a solution of Problem (BS). Then

(a) $X_f(T)$ is a strictly increasing function with $X_f(T_F) = \min\{K, \frac{r}{q}K\}$.

(b) $P(S,T)$ is a convex decreasing function of the stock price $S$ with $P_S \in [-1,0]$ for $X_f(T) < S < \infty$ and $0 < T < T_F$.

(c) $P(S,T)$ is a decreasing function of the time $T$ for $X_f(T) < S < \infty$ and $0 < T < T_F$.

Evens et al. [10] provided the following estimate for the early exercise boundary $X_f(T)$ (see also Barle et al. [1] and Lamberton [23]).

Theorem 2.2 Let \{X_f, P\} be a solution of Problem (BS). The asymptotic expansion of $X_f(T)$ as $T \to T_F$ takes the following form [10]

\[ 0 \leq q < r \]

\[ X_f(T) \sim K - \sigma K \sqrt{(T_F - T) \log \left( \frac{\sigma^2}{8\pi(r-q)^2(T_F-T)} \right)} \]

\[ q = r \]

\[ X_f(T) \sim K - \sigma K \sqrt{(T_F - T) \log \left( \frac{1}{4\sqrt{\pi q(T_F-T)}} \right)} \]

This implies that the early exercise boundary $X_f(T)$ is convex near the maturity for the case of $0 \leq q \leq r$. 
The numerical results demonstrated that the early exercise boundary of the American put option is a convex function when \( r > q \) and that the convexity may break down when \( r < q \). Chen et al. [6] and Ekström [9] verified that the early exercise boundary is convex when \( q = 0 \). Recently, Chen et al. [7] showed that the early exercise boundary is not convex when \( r < q \).

In the following, we demonstrate that the early exercise boundary \( X_f(T) \) of an American put option is convex if \( q + \frac{\sigma^2}{2} \leq r \).

**Theorem 2.3** Suppose that the process of the stock price satisfies the geometric Brownian motion process. The free boundary \( X_f(T) \) of an American put option is a convex function when \( q + \frac{\sigma^2}{2} \leq r \).

The proof of this theorem is provided in the next section.

### 3 A proof for Theorem 2.3

To verify the convexity of \( X_f(T) \), we change the operator \( L_{BS} \) to an operator with constant coefficients by

\[
S = e^x, \quad T = T_F - 2t/\sigma^2, \quad P(S, T) = u(x, t), \quad X_f(T) = e^{s(t)}. \tag{8}
\]

Then Problem (BS) becomes

**Problem (P)**

\[
\begin{align*}
L u &= 0 & s(t) < x < \infty, 0 < t < \infty, \\
u(x, 0) &= \max(0, K - e^x), & -\infty < x < \infty, \\
u(x, t) &= \max(0, K - e^x) & s(t) < x < \infty, 0 < t < \infty, \\
\lim_{x \to \infty} u(x, t) &= 0 & 0 < t < \infty, \\
u(s(t), t) &= K - e^{s(t)} & 0 < t < \infty, \\
\frac{\partial u}{\partial x}(s(t), t) &= -e^{s(t)} & 0 < t < \infty,
\end{align*}
\tag{9}
\]

where \( k = \frac{2r}{\sigma^2} \), \( h = \frac{2q}{\sigma^2} \) and the operator \( L \) is defined as \( \mathcal{L} = \mathcal{L}_0 - \frac{\partial}{\partial x} \) and

\[
\mathcal{L}_0 = \frac{\partial^2}{\partial x^2} + (k - h - 1) \frac{\partial}{\partial x} - k.
\]
Let \( \{s, u\} \) be the solution to (P). We introduce two sets:

\[
C = \{(x, t) \in \mathbb{R}^+ \times [0, \infty) | u(x, t) > \max(K - e^x, 0)\},
\]

\[
S = \{(x, t) \in \mathbb{R}^+ \times [0, \infty) | u(x, t) = \max(K - e^x, 0)\}.
\]

The set \( C \) is called the continuation region and the set \( S \) is the early exercise region.

**Definition 3.1** Given \( t \in [0, \infty) \), the \( t \)-section of \( S \) is defined as

\[
S_t = \{x \in \mathbb{R}^+ | u(x, t) = \max(K - e^x, 0)\}.
\]

(15)

Clearly, we have

\[
S = \bigcup_{t<\infty} (S_t \times \{t\})
\]

and

\[
s(t) = \sup\{x | x \in S_t\}.
\]

(16)

The continuation region is then represented as

\[
C = \{(x, t); s(t) < x < \infty, 0 < t < \infty\}.
\]

(17)

According to Theorem 2.1, we obtain the following properties for the solution of Problem (P) directly.

**Theorem 3.2** Let \( \{s, u\} \) be a solution of (P). Then

(a) \( s(t) \) is a strictly decreasing function with \( s(0) = \min\{\log K, \log(\frac{1}{h}K)\} \).

(b) \( u_x(x, t) < 0 \) for \( (x, t) \in C \).

(c) \( u_x(x, t) > -e^x \) for \( (x, t) \in C \).

Since \( s(t) \) is not convex when \( q > r \), we consider the convexity of \( s(t) \) for \( k \geq h \) (ie. \( r \geq q \)) and define \( d = \log K \). Since \( s(t) \) is a decreasing function with \( s(0) = d \) and \( w(x, t) = u(x, t) - (K - e^x) \) for \( x < d \), we have \( w(s(t), t) = 0, \quad w_x(s(t), t) = 0, \quad w_t(s(t), t) = 0, \quad w_{xx}(s(t), t) = Kk - he^{s(t)} > 0 \). Differentiating the equality \( w_x(s(t), t) = 0 \) with respect to \( t \) yields \( w_{xx}s'(t) + w_{xt} = 0 \). Hence we have
\( \frac{w_{xx}}{w_{xt}} = -s'(t) \) at \( x = s(t) \). Moreover, differentiating the equality \( w_{xx}(s(t), t) = Kk - he^{s(t)} \) with respect to \( t \) yields \( w_{xxx}s'(t) + w_{xxt} = -hs'(t)e^{s(t)} > 0 \) since \( s'(t) < 0 \).

**Remark 3.3** By the interior regular theorem of Friedman [11], the derivatives \( u_{xt}, u_{xxt} \) and \( u_{xxx} \) exist and are Hölder continuous in \( C \).

Let

\[
    v = \begin{cases} 
        \frac{w_{xx}}{w_{xt}} & \text{if } (x, t) \in C_d, \\
        -s'(t) & \text{if } x = s(t), 
    \end{cases}
\]

which is well-defined on \( \tilde{C}_d = \{(x, t) \in \mathbb{R}^2 | s(t) \leq x \leq d, 0 < t < \infty \} \). Applying the differential operator \( L \) to equality \( vw_{xx} = w_{xt} \), we determine that \( v \) satisfies the following equation

\[
    v_{xx} + ((k - h - 1) + 2\frac{w_{xxx}}{w_{xx}})v_x + \frac{Lw_{xx}}{w_{xx}}v - v_t = 0 \tag{19}
\]
on \( C_d = \{(x, t) \in \mathbb{R}^2 | s(t) < x < d, 0 < t < \infty \} \).

Since \( u_x < 0, u > 0, u_t > 0 \) by (8) on \( C_d \) and \( Lw = Kk - he^x \), we have

\[
    w_{xx} = -(k - h - 1)u_x + ku + u_t + Kk - he^x > 0 \quad \text{on } C_d
\]
if \( k - h - 1 \geq 0 \). Since \( w(x, t) = u(x, t) - (K - e^x) \) on \( C_d \), we have \( w_{xx} = u_{xx} + e^x \).

Applying the constant coefficients operator \( L \) to \( w_{xx} \) yields

\[
    Lw_{xx} = L(u_{xx} + e^x) = 2 \frac{\partial^2}{\partial x^2}Lu + Le^x = Le^x = -he^x < 0.
\]
We also have \( w_{xx}(s(t), t) > 0 \). Therefore, the equation (19) is a parabolic equation with bounded coefficients if \( k - h - 1 \geq 0 \).

Friedman [13] defined the lower \( \Omega \)-neighborhood as follows.

**Definition 3.4** An \( \Omega \)-neighborhood of a point \( (x_0, t_0) \) is the intersection of a neighborhood of \( (x_0, t_0) \) with \( \Omega \). A lower \( \Omega \)-neighborhood of a point \( (x_0, t_0) \) is the intersection of an \( \Omega \)-neighborhood of \( (x_0, t_0) \) with the half space \( t \leq t_0 \).

To show the convexity of \( X_f(T) \), it suffices to show that \( s(t) \) is a convex function. Now, we provide a proof of the main contribution in this paper.
Proof of Theorem 2.3  We have determined that \(s(t)\) is a strictly decreasing function. Suppose that there is a closed interval \(I\) such that \(s(t)\) is a concave function on the interval \(I = [a, b]\). According to the estimate of \(X_f(t)\) near the maturity in Theorem 2.2, we known that \(s(t)\) is convex near 0. Thus \(0 \neq a\).

Suppose that there exists a \(t_0 \in I\) with \(s'(t_0) = m < 0\) because \(s(t)\) is strictly decreasing and is differentiable almost everywhere. Then \(s'(t) \leq m\) for almost every \(t > t_0\) in \(I\).

When \(s(t)\) is assumed to be a concave function on \(I\), we consider the following two lemmas for the level curve \(\Gamma_\alpha = \{(x, t) \in C_d | v(x, t) = \alpha\}\).

Lemma 3.5 Let \(v\) be a solution of (19). If \(s(t)\) is a concave function on an interval \(I\), then for any \(t_0 \in I\) \(v\) cannot attain an extremum at \((s(t_0), t_0)\) with respect to any lower \(\bar{\Omega}\)-neighborhood of \((s(t_0), t_0)\).

Proof. Since \(s(t)\) is a concave function on the interval \(I\), then \(s''(t) < 0\); this implies that \(-s'(t)\) is an increasing function on \(I\). Since \(v(s(t), t) = -s'(t)\), we conclude that \(v\) cannot attain a minimum at \((s(t_0), t_0)\) with respect to any lower \(\bar{\Omega}\)-neighborhood of \((s(t_0), t_0)\) on \(I\).

Suppose that \(v\) attains a maximum at \((s(t_0), t_0)\) on \(I\). Then

\[
v_x(s(t_0), t_0) \leq 0. \tag{20}\]

However, at \((s(t_0), t_0)\),

\[
v_x = \left(\frac{w_xt}{w_xx}\right)_x = \frac{w_{xxt}w_{xx} - w_{xxxx}w_{x}}{w_{xx}^2} = \frac{w_{xxt} - w_{xxxx}v}{w_{xx}} = \frac{-hs'(t)e^{s(t)}}{w_{xx}} > 0.
\]

Thus contradicting to (20).

Lemma 3.6 Let \(\Gamma_\alpha\) be the level curves on which \(v = \alpha\). If \(s(t)\) is a concave function on an interval \(I\), then, for each \(\alpha\) there exists a \(g_\alpha(t)\) such that

\[
\Gamma_\alpha = \{(g_\alpha(t), t) | v(g_\alpha(t), t) = \alpha, \ t > 0\}.
\]

Proof. Since \(w_{xx} > 0, \mathcal{L}w_{xx} < 0\) and \(v\) satisfy the parabolic equation (19), the \(t\)-coordinate along \(\Gamma_\alpha\) can not be (i) first decreasing and then increasing.
and (ii) first increasing and then decreasing. For (i), a region would exist in which the parabolic boundary is a part of $\Gamma_\alpha$; consequently $v \equiv \alpha$ in this region and $v \equiv \alpha$ in $C_d$. For (ii), there would be a region with parabolic boundary consisting of a part of $\Gamma_\alpha$ and a part of $\{(s(t), t) | 0 < t \leq t_0\}$ which implies that an extremum exists at $(s(t_0), t_0)$ with respect to the lower $\Omega$-neighborhood of $(s(t_0), t_0)$. Employing Lemma 3.5, we have that the extremum can not appear at $v(s(t_0), t_0)$. Therefore, we conclude that the level curve $\Gamma_\alpha$ can not first increasing.

The idea of this proof is same as 13 (seeing Page 4 of 13 for the details).

Since $I = [a, b]$ and $a > 0$, there is a point $t_0 \in I$ with $v(s(t_0), t_0) = -s'(t_0) = -m$ such that the line

$$y(t) = m(t - t_0) + s(t_0), \quad t > 0$$

intersects $s(t)$ at $t_2 < t_0$ and $t_0$; that is $t_2 = \inf \{t | (y(t), t) \in C_d\}$ with $y(t_2) = s(t_2)$ and $y(t_0) = s(t_0)$. Since $t_0 \in I$, we have $v(s(t), t) = -s'(t) \geq -m$ for $t > t_0$ in $I$. Since $s(t)$ is bounded below and $m < 0$, there must exist another point $t_1 > t_0$ such that $y(t_1) = s(t_1)$. Now, we have $s(t_i) = y(t_i), i = 0, 1, 2.$

We also have $w_x = u_x + e^x > 0$ on $C_d$ according to (c) in Theorem 3.2. Let $f(t) = w_x(y(t), t) = u_x(y(t), t) + e^{y(t)} > 0$ for some $t > t_2$. Thus, we derive that

$$f'(t) = mw_{xx}(y(t), t) + w_{xt}(y(t), t)$$

$$= w_{xx}(y(t), t)(m + v(y(t), t))$$

for $t > t_2$. Since $y(t_0) = s(t_0)$ and $v(s(t_0), t_0) = -s'(t_0) = -m$, we obtain

$$f'(t_0) = w_{xx}(y(t_0), t_0)(m + v(y(t_0), t_0))$$

$$= w_{xx}(s(t_0), t)(m + v(s(t_0), t)) = 0,$$

We also have $w_x(s(t), t) = 0$ by 14. Since $y(t_i) = s(t_i), i = 0, 1, 2$ and $w_x(x, t) > 0$ for $(x, t) \in C_d$, we also have $f(t_i) = w_x(y(t_i), t_i) = w_x(s(t_i), t_i) = 0, i = 0, 1, 2$ and $(y(t), t) \in C_d$ for $t \in (t_2, t_1)$. Thus, a local maximum of $f$ exists in $(t_0, t_1)$ and $(t_2, t_0)$, namely $f(t_3)$ and $f(\bar{t}_3)$ where $t_3 \in (t_0, t_1)$ and $\bar{t}_3 \in (t_2, t_0)$. This implies that $f'(t_3) = 0$ and $f'(\bar{t}_3) = 0$. Since $w_x = u_x + e^x$ is a solution of parabolic equation and $f(t) = w_x(y(t), t)$, which does not oscillate as $t \to t_0$. 

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This implies that $f(t)$ do not produce an infinite sequence of local maximum, the locations of which tends to $t_0$. We can therefore assume that $t_3$ and $\tilde{t}_3$ are the first maximum from $t_0$ and no local maximum exists between $t_0$ and $t_3$ and between $\tilde{t}_3$ and $t_0$. By the same reason, there also exists a point $\tilde{t}_3 \in (t_2,t_0)$ such that $f'(\tilde{t}_3) = 0$. Since $f(t_0) = f(t_1) = 0$, $f(t_3) > 0$, and $f'(t_i) = 0$, $i = 0, 3$, we have

$$f'(t) > 0 \text{ for } t \in (t_0, t_3)$$

(22)

and

$$f'(t) < 0 \text{ for } t \in (t_3, t_4),$$

(23)

where $t_3 < t_4 \leq t_1$.

Let $\Gamma_{-m}$ be the level curves on which $v = -m$. According to Lemma 3.6 there exists the $g_{-m}(t)$ such that

$$\Gamma_{-m} = \{(g_{-m}(t),t)|v(g_{-m}(t),t) = -m, t > 0\}.$$ 

Since $f'(t_i) = 0$, $i = 0, 3$ and $f'(t) = w_{xx}(y(t),t)(m + v(y(t),t))$, we have $v(y(t_i),t_i) = -m$, $i = 0, 3$, which implies that $(y(t_i),t_i) \in \Gamma_{-m}$, $i = 0, 3$. Next, we consider the function $g_{-m}(t)$. Since $(y(t_i),t_i) \in \Gamma_{-m}$, that is

$$v(y(t_i),t_i) = -m, \ i = 0, 3,$$

(24)

we have $y(t_i) = g_{-m}(t_i)$, $i = 0, 3$. Since $f'(t) = w_{xx}(y(t),t)(m + v(y(t),t)) > 0$ for $t \in (t_0, t_3)$ by (22) and (21) and $w_{xx}(y(t),t) > 0$ by the assumption, this implies that

$$v(y(t),t) > -m, \text{ for } t \in (t_0, t_3).$$

(25)

Since $g_{-m}(t)$ is continuous on $(t_2, t_1)$, we have only the following two cases: (1) $y(t) > g_{-m}(t)$ for $t \in (t_0, t_3)$, and (2) $y(t) < g_{-m}(t)$ for $t \in (t_0, t_3)$.

We first consider case (1). Since $g_{-m}(t_0) = y(t_0) = s(t_0)$ and $y(t) > g_{-m}(t) > s(t)$ for $t \in (t_0, t_3)$, there is a $\delta > 0$ such that $y'(t) > g'(t) > s'(t)$ for $t \in (t_0, t_0 + \delta)$. Since $y'(t) = m$, we have $v(s(t),t) = -s'(t) > -y'(t) = -m$ for $t \in (t_0, t_0 + \delta)$. Let $\Omega = \{(x,t)|s(t) \leq x \leq y(t), \ t_0 \leq t \leq t_0 + \delta\}$. On $\Omega$, we have $t', t'' \in (t_0, t_0 + \delta)$ such that $v(s(t'),t') = v(y(t''),t'') = \beta > -m$,
but \(v(g_{-m}(t), t) = -m\) for all \(t \in (t_0, t_0 + \delta)\). This implies that there exists a level curve, say \(\Gamma_\beta\), crosses \(g_{-m}(t)\) connected \(s(t')\) and \(y(t'')\). This contracts to \(\Gamma_\beta \cap \Gamma_{-m} \neq \emptyset, \beta \neq -m\). Therefore, case (1) does not hold.

Next, we consider case (2). We know that the level curves \(\Gamma_\alpha\) of a parabolic equation are continuous. Since \(f'(\bar{t}_3) = w_{xx}(m + v(y(\bar{t}_3), \bar{t}_3)) = 0\), we also have \(v(y(\bar{t}_3), \bar{t}_3) = -m\); that is \((y(\bar{t}_3), \bar{t}_3) \in \Gamma_{-m}\). Consider the line \(y(t)\) for \(t \in (t_2, t_0) \cup (t_0, t_3)\). In (2.1), we have \(v(y(t), t) > -m\) for \(t \in (t_0, t_3)\). We also have \(f(t_0) = 0\) and \(f(t) = w_x(y(t), t) > 0\) for \(t \in (t_2, t_0)\). This implies that there is a \(\delta_t > 0\) such that \(f'(t) < 0\) for \(t \in (t_0 - \delta_t, t_0)\). Since \(f'(t) = w_{xx}(y(t), t)(m + v(y(t), t))\) and \(f'(t) < 0\) for \(t \in (t_0 - \delta_t, t_0)\) and \(w_{xx} > 0\) for \((x, t) \in C_d\), we obtain

\[
v(y(t), t) < -m \tag{26}
\]

for \(t \in (t_0 - \delta_t, t_0)\). Now, we have only the following two subcases for case (2):

(2.1) \(g_{-m}(t) > y(t)\) for \(t \in (t_0 - \delta_t, t_0)\) and (2.2) \(g_{-m}(t) < y(t)\) for \(t \in (t_0 - \delta_t, t_0)\).

For case (2.1), we can select a suitable \(\delta > 0\) such that \(v(y(t), t) < -m\) for \(t \in (t_0 - \delta, t_0) \cup (t_3, t_3 + \delta)\), \(t_3 + \delta < t_4\) by (2.1). Since \(v(y(t_0), t_0) = -m = v(y(t_3), t_3)\) by (2.1) and \(v(y(t), t) < -m\) for \(t \in (t_0 - \delta, t_0) \cup (t_3, t_3 + \delta)\), there exists a \(t' \in (t_0 - \delta, t_0)\) and a \(t'' \in (t_3, t_3 + \delta)\) such that

\[
v(y(t'), t') = \beta = v(y(t''), t''),\text{ for some } \beta < -m.
\]

Since the level curves of a parabolic equation are continuous, there exists a level curve \(\Gamma_\beta\) connecting \((y(t'), t')\) and \((y(t''), t'')\). There is an intersection of \(\Gamma_{-m}\) and \(\Gamma_\beta\) on \((t_0 - \delta, t_0)\). This contradicts to \(\Gamma_{-m} \cap \Gamma_\beta \neq \emptyset\).

For case (2.2), we have \(v(y(t), t) < -m\) for \(t \in (t_0 - \delta, t_0)\) by (2.1) and \(v(g_{-m}(t), t) = -m\) for \(t \in (t_0 - \delta, t_0)\). If \(v(s(t), t) < -m\) for \(t \in (t_0 - \delta, t_0)\), there exists a level curve, say \(\Gamma_\alpha\), crosses over \(g_{-m}(t)\) connected \(s(t)\) and \(y(t)\). This contradicts to \(\Gamma_\alpha \cap \Gamma_{-m} \neq \emptyset, \alpha \neq -m\). If \(v(s(t), t) > -m\) for \(t \in (t_0 - \delta, t_0)\), we have \(v(y(t), t) > -m\) on \((t_0, t_3)\) by (2.1) and \(v(y(t_0), t_0) = v(s(t_0), t_0) = -m\).

This implies that there exists a \(t' \in (t_0 - \delta, t_0)\) and a \(t'' \in (t_0, t_3)\) such that

\[
v(s(t'), t') = \beta = v(y(t''), t''),\text{ for some } \beta > -m.
\]
Since the level curves of a parabolic equation are continuous, there exists a level curve $\Gamma_\beta$ connecting $(y'(t), t')$ and $(y(t''), t'')$. This contradicts $\Gamma_m \cap \Gamma_\beta \neq \emptyset$. Therefore, case (2) does not hold.

Both case (1) and case (2) do not hold; therefore we conclude that $s(t)$ cannot be a concave function in any interval. Thus, $s(t)$ is a convex function.

**Remark 3.7** Given $\alpha \in \mathbb{R}$ and $g_\alpha(t)$ as the function, such that
\[
v(g_\alpha(t), t) = \alpha
\]
with $g_\alpha(t_0) = s(t_0)$, where $v(s(t_0), t_0) = \alpha$. Then
\[
\frac{dv}{dt} = v_x \frac{dg_\alpha(t)}{dt} + v_t = 0.
\]
According to Sard’s lemma, the set of $v_x(x, t) = 0$ is measure zero. Thus, $\frac{dg_\alpha}{dt}$ is defined for almost every point on $\Omega$. We consider the following IVP
\[
\frac{dg_\alpha(t)}{dt} = -\frac{v_t}{v_x} \quad (a.e.)
\]
with $g_\alpha(t_0) = s(t_0)$. Indeed, the weak solution for (27) exists. Therefore $g_\alpha(t)$ is continuous for all $t$ with $v(g_\alpha(t), t) = \alpha$.

**References**

[1] G. Barles, J. Burdeau, M. Romano and N. Sansoen, Critical Stock Price Near Expiration. Math. Finance 5 (1995) 77-95.

[2] G. Barone-Adesi and R. E. Whaley, Efficient Analytic Approximation of American Option Values, J. Finance 42 (1987) 301-320.

[3] F. Black and M. Scholes, The Pricing of Options and Corporate Liabilities, J. Polit. Econ. 81 (1973) 637-654.

[4] P. Carr, R. Jarrow, and R. Myneni, Alternative Characterizations of American Put Options, Math. Finance 2 (1992) 87-106.
[5] X. Chen and J. Chadam, A Mathematical Analysis for the Optimal Exercise Boundary of American Put Options, Working paper, University of Pittsburgh (2000).

[6] X. Chen, J. Chadam, L. Jiang, and W. Zheng, Convexity of the Exercise Boundary of the American Put on a Zero Dividend Asset, Math. Finance 18 (2008) 185-197.

[7] X. Chen, H. Chen, and J. Chadam, Nonconvexity of the Optimal Boundary of an American Put Option On a Dividend-paying Asset, Math. Finance 23 (2013) 169-185.

[8] S. Dayanik and I. Karatzas, On the optimal stopping problem for one-dimensional diffusions, Stochastic Process. Appl. 107 (2003) 173212.

[9] E. Ekström, Convexity of the Optimal Stopping Boundary for the American Put Option, J. Math. Anal. Appl. 299 (2004) 147-156.

[10] J. D. Evans, R. Kuske, and J. B. Keller, American Options with Dividends Near Expiry, Math. Finance 12 (2002) 219-237.

[11] A. Friedman, Interior Estimates for Parabolic Systems of Partial Differential Equations, J. Math. Mech. 7 (1958) 393-417.

[12] A. Friedman, Partial Differential Equations of Parabolic Type, Prentice-Hall Inc, 1964.

[13] A. Friedman and R. Jensen, Convexity of the Free Boundary in the Stefan Problem and in the Dam Problem, Arch. Rational Mech. Anal. 67 (1977) 1-24.

[14] A. Friedman, Parabolic variational inequalities in one space dimension and smoothness of the free boundary, J. Funct Anal. 18 (1975) 151-176.

[15] A. Friedman, Variational principles and free-boundary problems, 2nd ed., Robert E. Krieger Publishing Co. Inc., Malabar, FL, 1988.
[16] R. Geske and H. E. Johnson, The American Put Option Valued Analytically, J. Finance 39 (1984) 1511-1524.

[17] S. D. Jacka, Optimal Stopping and the American Put, Math. Finance 1 (1992) 1-14.

[18] I. Karatzas, On the Pricing of American Option. Appl. Math. Optim. 17 (1988) 37-60.

[19] D. B. Kotlow, A Free Boundary Problem Connected with Optimal Stopping Problem for Diffusion Processes, Trans. Amer. Math. Soc. 184 (1973) 457-478.

[20] R. A. Kuske and J. B. Keller, Optimal Exercise Boundary for an American Put Option, Appl. Math. Finance 5 (1998) 107-116.

[21] Y.-K. Kwok, Mathematical Models of Financial Derivatives, Springer Inc, 2008.

[22] D. Lamberton, Optimal stopping and American options. Lecture notes, Ljubljana Summer School on Financial Mathematics, 2009.

[23] D. Lamberton and S. Villeneuve, Critical Price Near Maturity for an American option on a Dividend-Paying Stock. Ann. Appl. Probab. 13 (2003) 800-815.

[24] R. Merton, The Theory of Rational Option Pricing, Bell J. Econ. Management Sci. 4 (1973) 141-183.

[25] R. Myneni, The Pricing of the American Option, Ann. Appl. Probab. 2 (1992) 1-23.

[26] G. Peskir, On the American Option Problem, Math. Finance 15 (2005) 169-181.