A LOOP TYPE COMPONENT IN THE NON-NEGATIVE SOLUTIONS SET OF AN INDEFINITE ELLIPTIC PROBLEM

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Abstract. We prove the existence of a loop type component of non-negative solutions for an indefinite elliptic equation with a homogeneous Neumann boundary condition. This result complements our previous results obtained in [12], where the existence of another loop type component was established in a different situation. Our proof combines local and global bifurcation theory, rescaling and regularizing arguments, a priori bounds, and Whyburn’s topological method. A further investigation of the loop type component established in [12] is also provided.

1. Introduction. Let Ω be a smooth bounded domain of \( \mathbb{R}^N \), \( N \geq 1 \). This article is devoted to the problem

\[
(P_\lambda) \quad \begin{cases} 
-\Delta u = \lambda b(x)u^{q-1} + a(x)u^{p-1} & \text{in } \Omega, \\
\frac{\partial u}{\partial n} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where

- \( \lambda \in \mathbb{R} \);
- \( 1 < q < 2 < p \);
- \( a, b \in C^\alpha(\overline{\Omega}) \), \( \alpha \in (0, 1) \);
- \( a \not\equiv 0 \) and \( b \) changes sign;
- \( n \) is the unit outer normal to \( \partial \Omega \).

By a solution of \((P_\lambda)\), we mean a classical solution. A solution \( u \) of \((P_\lambda)\) is said to be nontrivial and non-negative if it satisfies \( u \geq 0 \) on \( \overline{\Omega} \) and \( u \not\equiv 0 \), whereas it is said to be positive if it satisfies \( u > 0 \) on \( \overline{\Omega} \). Note that since \( b \) changes sign and \( 1 < q < 2 \), the strong maximum principle does not apply and, as a consequence, we cannot deduce that nontrivial non-negative solutions of \((P_\lambda)\) are actually positive solutions, unlike the case \( q \geq 2 \).

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In [12] we have investigated existence, non-existence, and multiplicity of non-negative solutions as well as their asymptotic behavior as \( \lambda \to 0 \). These results led us to analyse the structure of the set of non-negative solutions of \((P_\lambda)\). In particular, we have proved the existence of a loop type component in this set, under the following conditions (see [12, Theorem 1.6] and Figure 1(a)):

\[
a, b \text{ change sign, } \int_\Omega b \leq 0, \quad \text{and} \quad \int_\Omega a < 0.
\]

We shall assume that \( a \) and \( b \) are positive in some open ball (see \((H_0)\) below), in which case the nonlinearity in \((P_\lambda)\) has (locally) a concave-convex nature. We refer the reader to [12] for a more general discussion on \((P_\lambda)\) and related concave-convex problems.

Our purpose is to go further in this investigation, focusing now mostly on the case

\[
\int_\Omega b < 0 \leq \int_\Omega a.
\]

Before stating our result, let us set

\[
\Omega^a_\pm = \{x \in \Omega: a \gtrless 0\}, \quad \Omega^b_\pm = \{x \in \Omega: b \gtrless 0\}.
\]

The following conditions shall be assumed in our main result:

\( (H_0) \) \( a(x_0), b(x_0) > 0 \) for some \( x_0 \in \Omega \);

\( (H_1) \) \( \Omega^+_\pm \) and \( \Omega' := \Omega \setminus \Omega^+_\pm \) are subdomains of \( \Omega \) with smooth boundaries, and satisfy either \( \partial \Omega^+_\pm \subset \Omega \) or \( \partial \Omega' \subset \Omega \);

\( (H_2) \) There exist \( \gamma > 0 \) and a function \( \alpha^+ \) which is continuous, positive, and bounded away from zero in a tubular neighborhood \( U \) of \( \partial \Omega^+_\pm \) in \( \Omega^+_\pm \), such that

\[
a^+(x) = \alpha^+(x) \text{dist}(x, \partial \Omega^+_\pm)^\gamma, \quad x \in U,
\]

\[
2 < p < \min \left\{ \frac{2N}{N-2}, \frac{2N + \gamma}{N-1} \right\} \quad \text{if} \quad N > 2;
\]

\( (H_3) \) \( \Omega^b_{\pm} \) are subdomains of \( \Omega \).

These conditions guarantee some \textit{a priori} bounds in \((0, \infty) \times C(\overline{\Omega})\) for non-negative solutions. More precisely, \((H_0)\) implies that \((Q_{\mu, \epsilon})\), a rescaled and regularized version of \((P_\lambda)\), has no positive solutions for \( \mu \) sufficiently large, similarly as [12, Proposition 6.1]. On the other hand, \((H_1)\) and \((H_2)\) provide us with an \textit{a priori} bound on \( \|u\|_{C(\overline{\Omega})} \) for any non-negative solution \( u \) of \((Q_{\mu, \epsilon})\), see [12, Proposition 6.5]. Let us mention that \((H_2)\) goes back to Amann and López-Gómez [2], where the authors have established \textit{a priori} bounds for positive solutions of indefinite elliptic problems. Finally, \((H_3)\) is employed to show that bifurcation from zero for nontrivial non-negative solutions of \((P_\lambda)\) does not occur at \( \lambda \neq 0 \), as in [12, Proposition 3.3].

We state now our main result, which gives a positive answer to the open question raised in Subsection 6.1 of [12]. Note that, in contrast with [12, Theorem 1.6], \( a \) may be non-negative.

**Theorem 1.** Assume \((2)\), \((H_0)\) and \((H_3)\). In addition, assume one of the following conditions:

\( (a) \) \( a > 0 \) on \( \overline{\Omega} \), and \( 2 < p < \frac{2N}{N-1} \) if \( N > 2 \);

\( (b) \) \((H_1)\) and \((H_2)\) hold, and \( (\Omega' \setminus \Omega^a_\pm) \subset \Omega^b_\pm \) if \( \Omega' \neq \Omega^a_\pm \).
Then \((P_\lambda)\) has a bounded component (non-empty, closed, and connected subset in \(\mathbb{R} \times C(\Omega)\)) of non-negative solutions \(C_0 = \{(\lambda, u)\}\). In addition, \(C_0\) is of loop type, i.e., it is a bounded component that meets a single point on a trivial solution line and joins this point to itself. More precisely, \(C_0\) starts and ends at \((0, 0)\), and has the following properties (see Figure 1(b)):

(i) \(C_0 \cap \{(\lambda, 0) : \lambda \neq 0\} = \emptyset\). Consequently, \(C_0 \setminus \{(0, 0)\}\) consists of nontrivial non-negative solutions.

(ii) \((P_\lambda)\) has no nontrivial non-negative solution for \(\lambda = 0\), so that \(u \equiv 0\) if \((0, u) \in C_0\).

(iii) There is no \((\lambda, u) \in C_0\) with \(\lambda < 0\), i.e., \(C_0\) bifurcates to the region \(\lambda > 0\) at \((0, 0)\).

(iv) There exist at least two nontrivial non-negative solutions \((\lambda, u_{1,\lambda}), (\lambda, u_{2,\lambda}) \in C_0\), for \(\lambda > 0\) sufficiently small.

Remark 2.

(i) Condition (a) can be understood as \((H_2)\) with \(\gamma = 0\).

(ii) Condition (b) includes the following cases:

1. \(a(x) > 0\) in \(\Omega\), and \(a(x)\) vanishes on \(\partial \Omega\). This situation is understood as \(\Omega' = \emptyset\).

2. \(\{x \in \Omega' : a(x) = 0\} \neq \emptyset\). In particular, it includes the case that \(a(x)\) changes sign, as well as the case that \(a(x) \geq 0\) in \(\Omega\). In both cases we need that \(b(x) > 0\) in \(\{x \in \Omega' : a(x) = 0\}\).

Remark 3. The existence of the loop type component \(C_0\) provided by Theorem 1 is consistent with [12, Theorem 1.1]. As a matter of fact, in [12, Theorem 1.1] it is proved that if (2) holds then \((P_\lambda)\) has two nontrivial non-negative solutions for \(\lambda > 0\) sufficiently small. Moreover, these solutions converge both to 0 in \(C^2(\Omega)\) as \(\lambda \to 0^+\). We believe that these solutions correspond to the upper and lower branches of \(C_0\).

\[
\|u\|_{C(\Omega)} \quad \|u\|_{C(\Omega)}
\]

\[C_0\]

\[
\lambda \quad \lambda
\]

\[O\]

(a) Minimal possibility for \(C_0\) when \(a, b\) change sign, \(\int_{\Omega} b \leq 0\), and \(\int_{\Omega} a < 0\).

(b) Minimal possibility for \(C_0\) when \(a \neq 0\), \(b\) changes sign, \(\int_{\Omega} b < 0\), and \(\int_{\Omega} a \geq 0\).

Figure 1. Loop type components of nontrivial non-negative solutions of \((P_\lambda)\).

The existence of bounded (or compact) components in the solution set of nonlinear problems has been investigated by Cingolani and Gámez [6], Cano-Casanova
A mushroom, i.e. a component connecting two simple eigenvalues of the linearized eigenvalue problem at the trivial solution \( u = 0 \), was obtained by Cingolani and Gámez for both the Dirichlet case and \( \Omega = \mathbb{R}^N \), and by Cano-Casanova for a mixed linear boundary condition. In addition to the existence of a mushroom, López-Gómez and Molina-Meyer (for the Dirichlet case) and Brown (for the Neumann case) obtained a loop, i.e. a component that meets a single point on the trivial solution line. Moreover, López-Gómez and Molina-Meyer also proved the existence of an isola, i.e. a component that does not touch the trivial solution line. Finally, we refer to [12, Theorem 1.6], where the existence of a loop type component for \((P_\lambda)\) was proved in case (1).

Let us remark that the nonlinearities in [6, 5, 10, 3] are \( C^1 \) at \( u = 0 \), which is not the case for \((P_\lambda)\). Therefore the standard global bifurcation theory of Rabinowitz [11] (see also López-Gómez [9]) does not apply straightforwardly to \((P_\lambda)\). To overcome this difficulty, we employ a regularization method around the trivial solution and develop Whyburn’s topological analysis [15, (9.12)Theorem] to convert the bifurcation results obtained for the regularized problem to the original problem. In fact, before considering the regularization, we carry out a scaling argument to overcome a difficulty which appears in case (2). Unlike in case (1), it is difficult to study directly \((P_\lambda)\) and its regularization under (2), since these problems have no positive solutions for \( \lambda = 0 \) [12, Lemma 6.8(1)]. Even if we can prove the existence of a component of positive solutions for the regularized problem, the non-existence result for \( \lambda = 0 \) may cause the shrinking of the component into the set of trivial solutions when the topological method is employed. It should be emphasized that in order to obtain the loop in case (1) as in Figure 1(a), the following fact was crucial: a component of positive solutions for the \( \epsilon \)-regularized problem of \((P_\lambda)\) does cut the vertical axis \( \lambda = 0 \), at some point that does not shrink to \((0, 0)\) as \( \epsilon \to 0 \).

In order to verify that a component of non-negative solutions of \((P_\lambda)\) is bounded in \((0, \infty) \times C(\overline{\Omega})\), we shall make good use of \textit{a priori} bounds for non-negative solutions of \((P_\lambda)\), as well as for non-negative solutions of \((Q_\mu)\) and \((Q_{\mu, \epsilon})\) below. We obtain these \textit{a priori} bounds under either conditions (a) or (b) in Theorem 1, proceeding in the same way just as in [12, Proposition 6.5].

The rest of this article is organized as follows. In Section 2, by the change of variables \( \mu = \lambda^{\frac{p-2}{2}} \) and \( v = \lambda^{-\frac{1}{p-q}} u \), we transform \((P_\lambda)\) into \((Q_\mu)\), and consider a \( \epsilon \)-regularized version of \((Q_\mu)\), i.e. \((Q_{\mu, \epsilon})\). This regularization scheme enables us to apply the local and global bifurcation theory from simple eigenvalues. We deduce then the existence of a component of bifurcating positive solutions of \((Q_{\mu, \epsilon})\) from \( \{(\mu, 0)\} \). Section 3 is devoted to the proof of our main result, Theorem 1. Using Whyburn’s topological method, we establish the limiting behavior of the component of \((Q_{\mu, \epsilon})\) obtained in Section 2 as \( \epsilon \to 0^+ \), and obtain a component of nontrivial nonnegative solutions of \((Q_\mu)\) which bifurcates from \((0,0)\) into the region \( \mu > 0 \). Finally, by the scaling, we go back to \((P_\lambda)\), and obtain thus a bounded component of nontrivial nonnegative solutions which is of loop type, joins \((0, 0)\) to itself, and lies in the region \( \lambda > 0 \), as shown in Figure 1(b). In Section 4, we carry out a further analysis for the loop of nontrivial nonnegative solutions of \((P_\lambda)\) obtained in the case (1) by [12]. The analysis concentrates on the direction of the bifurcation point from which the loop emanates. The main result of this section is Theorem 8.
2. Scaling and regularization schemes. We set \( v = \lambda^{-\frac{1}{q-1}} u \) and \( \mu = \lambda^{\frac{q-2}{q}} \) for \( \lambda > 0 \), so that \((P_\lambda)\) is transformed into

\[
\begin{cases}
-\Delta v = \mu \left( b(x) v^{q-1} + a(x) v^{p-1} \right) & \text{in } \Omega, \\
\frac{\partial v}{\partial n} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( \mu \geq 0 \). We note that the nonlinearity in \((Q_\mu)\) is not differentiable at \( v = 0 \), so that the local and global bifurcation theory from simple eigenvalues on the trivial line

\[ \Gamma_0 = \{ (\mu, 0) : \mu \geq 0 \} \]

can not be directly applied to \((Q_\mu)\). To overcome this difficulty, we shall consider the following regularized version of \((Q_\mu)\), where \( \epsilon \in (0, 1] \) is fixed:

\[
\begin{cases}
-\Delta v = \mu \left( b(x) (v + \epsilon)^{q-2} v + a(x) v^{p-1} \right) & \text{in } \Omega, \\
\frac{\partial v}{\partial n} = 0 & \text{on } \partial \Omega.
\end{cases}
\]

It is understood that \((Q_{\mu, \epsilon}) = (Q_\mu)\) when \( \epsilon = 0 \). It is clear that, in addition to \( \Gamma_0 \), \((Q_{\mu, \epsilon})\) has the trivial line of positive solutions

\[ \Gamma_{00} = \{ (0, c) : c \text{ is a non-negative constant} \} \]

Furthermore, by the strong maximum principle and the boundary point lemma, any nontrivial non-negative solution of \((Q_{\mu, \epsilon})\) is positive.

First, we discuss bifurcation from \( \Gamma_{00} \).

\textbf{Lemma 4.} Let \( \epsilon \in (0, 1] \). Under the conditions of Theorem 1, we have the following:

(i) Assume that \( \int_\Omega a > 0 \). Let \( (\mu_n, v_n) \) be positive solutions of \((Q_{\mu_n, \epsilon})\) with \( \mu_n > 0 \) and \( (\mu_n, v_n) \) converging to \( (0, c) \) in \( \mathbb{R} \times C(\Omega) \) for some positive constant \( c \).

Then \( c = c^*_\epsilon \), where \( c^*_\epsilon \) is the unique positive solution of the equation

\[
c^{p-2} (c + \epsilon)^{2-q} = -\frac{\int_\Omega b}{\int_\Omega a}.
\]

Moreover,

\[
c^*_\epsilon \to c^*_0 \quad \text{as } \epsilon \to 0^+.
\]

(ii) Assume that \( \int_\Omega a = 0 \). Then, there are no positive solutions \( v_n \) of \((Q_{\mu_n, \epsilon})\) such that \( \mu_n \to 0^+ \), and \( v_n \to c \) in \( C(\Omega) \) for some constant \( c > 0 \).

\textbf{Proof.}

(i) The divergence theorem shows that

\[
0 = \frac{1}{\mu_n} \int_{\partial \Omega} \left( -\frac{\partial v_n}{\partial n} \right) = \frac{1}{\mu_n} \int_\Omega (-\Delta v_n) = \int_\Omega \left\{ b(v_n + \epsilon)^{q-2} v_n + a v_n^{p-1} \right\}.
\]

By passing to the limit as \( n \to \infty \), it follows that \( c \) satisfies (3). Thus, \( c = c^*_0 \).

Moreover, assertion (4) is verified in a trivial way.

(ii) If not, then, in the same way as in (5), we deduce that \( 0 = c(c + \epsilon)^{q-2} \int_\Omega b < 0 \), a contradiction.

\( \square \)

\textbf{Remark 5.} The assertions of Lemma 4 except (4) are also valid for \( \epsilon = 0 \).
Now, using [7, Theorem 1.7], we carry out a local bifurcation analysis for \((Q_{\mu,\epsilon})\) with \(\epsilon > 0\) on \(\Gamma_0\), where \(\mu\) is the bifurcation parameter. To this end, we reduce \((Q_{\mu,\epsilon})\) to an operator equation in \(C(\Omega)\). Let \(M > 0\) be fixed. Given \(f \in C^\theta(\Omega)\), \(\theta \in (0, 1)\), let \(v \in C^{2+\theta}(\Omega)\) be the solution of

\[
\begin{align*}
(-\Delta + M)v &= f(x) \quad \text{in } \Omega, \\
\frac{\partial v}{\partial n} &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

We introduce the resolvent \(K : C^\theta(\Omega) \to C^{2+\theta}(\Omega) := \{v \in C^{2+\theta}(\Omega) : \frac{\partial v}{\partial n} = 0 \text{ on } \partial \Omega\}\) for (6), i.e., \(Kf = v\). It is well known (cf. [8]) that \(K\) is bijective and homeomorphic. It is also well known (cf. [1]) that \(K\) is extended to a compact linear mapping from \(C(\Omega)\) into \(C^1(\Omega)\). In this way, as far as non-negative solutions are concerned, \((Q_{\mu,\epsilon})\) is reduced to

\[\mathcal{F}(\mu, v) := v - K \left( Mv + \mu(b\epsilon^{q-2}v + g(x, v)) \right) = 0 \quad \text{in } C(\Omega),\]

where \(g(x, s) = b(x)(s + \epsilon)q^{-2}s - \epsilon^q s^2 + a(x)s^\theta\). We note that \(g(x, \cdot)\) is \(C^1\) at \(s = 0\), and \(\frac{\partial g}{\partial x}(x, 0) = 0\), so that \(\mathcal{F}\) has Fréchet derivatives \(\mathcal{F}_v(\mu, 0)\) and \(\mathcal{F}_{\mu v}(\mu, 0)\) given, respectively, by

\[
\begin{align*}
\mathcal{F}_v(\mu, 0) \varphi &= \varphi - K(M\varphi + \mu \epsilon^{q-2}b\varphi), \\
\mathcal{F}_{\mu v}(\mu, 0) \varphi &= -K(\epsilon^{q-2}b\varphi).
\end{align*}
\]

We consider the eigenvalue problem

\[
\begin{align*}
-\Delta \varphi &= \mu \epsilon^{q-2}b(x)\varphi \quad \text{in } \Omega, \\
\frac{\partial \varphi}{\partial n} &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where \(\epsilon\) is fixed, and \(\mu\) is the eigenvalue parameter. Since \(b\) changes sign and \(\int_{\Omega} b < 0\), this problem has exactly two principal eigenvalues, \(\mu = 0\), and \(\mu = \mu_\epsilon > 0\) (cf. [4]), which are both simple and possess positive eigenfunctions \(\varphi = 1\) and \(\varphi = \varphi_\epsilon\), respectively, both satisfying \(\|\varphi\|_\infty = 1\). Hence, the principal eigenvalues \(\mu = 0\), and \(\mu = \mu_\epsilon\) satisfy

\[\mathcal{N}(\mathcal{F}_v(0, 0)) = (1), \quad \mathcal{N}(\mathcal{F}_{\mu v}(\mu_\epsilon, 0)) = (\varphi_\epsilon),\]

and

\[\mathcal{F}_{\mu v}(0, 0) \not\in \mathcal{R}(\mathcal{F}_v(0, 0)), \quad \mathcal{F}_{\mu v}(\mu_\epsilon, 0) \varphi_\epsilon \not\in \mathcal{R}(\mathcal{F}_v(\mu_\epsilon, 0)),\]

where \(\mathcal{N}(\cdot)\) and \(\mathcal{R}(\cdot)\) denote the kernel and range of mappings, respectively. Indeed, the latter two assertions are verified using the conditions that \(\int_{\Omega} b < 0\) and \(\int_{\Omega} b\varphi_\epsilon^2 > 0\), respectively. The local bifurcation theory [7, Theorem 1.7] can now be applied. Moreover, by the unilateral global bifurcation theory ([14, Theorem 1.1], see also Lopez-Gomez [9, Theorem 6.4.3]), we infer that \((Q_{\mu,\epsilon})\) has a component \(C_\epsilon = \{(\mu, u)\}\) of non-negative solutions bifurcating at \((\mu_\epsilon, 0)\) in \(\mathbb{R} \times C(\Omega)\), and there are no positive solutions of \((Q_{\mu,\epsilon})\) bifurcating at \((0, 0)\), except \(\Gamma_{00}\). Moreover, \(C_\epsilon\) has the following properties:

**Lemma 6.** Let \(\epsilon \in (0, 1]\). Under the conditions in Theorem 1, we have the following:

(i) There exists \(\Lambda > 0\) such that \((Q_{\mu,\epsilon})\) has no positive solution for \(\mu > \frac{\Lambda}{2}\). Here, \(\Lambda\) does not depend on \(\epsilon \in (0, 1]\).
Proof. 

(i) The proof is similar to the one of [12, Proposition 6.1], so we provide an outline of it. By $(H_0)$, we can choose an open ball $B$ centered at $x_0$ such that $\overline{B} \subset \Omega^+ \cap \Omega^-$. We consider the Dirichlet eigenvalue problem

\[
\begin{cases}
-\Delta w = \mu a(x)w & \text{in } B, \\
w = 0 & \text{on } \partial B.
\end{cases}
\] (8)

Let $w_D \in C^2(\overline{B})$ be a positive eigenfunction associated with the first eigenvalue $\mu_D > 0$ of (8). We extend $w_D$ to the $\Omega$ by setting $w_D = 0$ in $\Omega \setminus \overline{B}$. Then, $w_D \in H^1(\Omega)$.

Let $v$ be a positive solution of $(Q_{\mu, \epsilon})$. By the divergence theorem, we deduce that $\int_B \nabla \cdot (v \nabla w_D) = \int_{\partial B} v \frac{\partial w_D}{\partial n} < 0$. It follows that

\[
\int_B \nabla v \nabla w_D - \mu_D \int_B avw_D < 0.
\]

On the other hand, by the definition of $v$, we see that

\[
\int_B \nabla v \nabla w_D = \mu \int_B av^{p-1}w_D + \mu \int_B b(v + \epsilon)^{q-2}vw_D.
\]

If $\mu \geq 1$, then we deduce that

\[
0 > \int_B v^{q-1}w_D \left\{ \mu av^{p-q} + \mu b \left( \frac{v}{v + \epsilon} \right)^{2-q} \right\} - \mu_D av^{2-q}.
\]

\[
\geq \int_B v^{q-1}w_D \left\{ av^{p-q} + \mu b \left( \frac{v}{v + \epsilon} \right)^{2-q} \right\} - \mu_D av^{2-q}.
\]

The rest of the proof follows as in [12, Proposition 6.1]. Indeed, we can show that there exists $\overline{\mu} > 0$ such that if $\mu \geq \overline{\mu}$, $\epsilon \in (0, 1)$, $x \in B$ and $s \geq 0$, then

\[
a(x)s^{p-q} + \mu b(x) \left( \frac{s}{s + \epsilon} \right)^{2-q} - \mu_D a(x)s^{2-q} \geq 0.
\]

Consequently, $\mu$ is bounded from above, uniformly in $\epsilon \in (0, 1]$.

(ii) If $\int_\Omega a > 0$, then, thanks to the previous item, we infer by Lemma 4 (i) that $C_\epsilon$ bifurcates from infinity if it does not meet $(0, c^*_\epsilon)$, so assertion (ii)(a) follows. Similarly, if $\int_\Omega a = 0$, then the previous item and Lemma 4 (ii) yield that $C_\epsilon$ bifurcates from infinity, so assertion (ii)(b) follows.

(iii) This assertion is deduced from the fact that (7) has exactly two principal eigenvalues $\mu = 0$ and $\mu = \mu_\epsilon$. 

(ii) (a) Assume that $\int_\Omega a > 0$. Then either $C_\epsilon$ meets $\Gamma_{00}$ at $(0, c^*_\epsilon)$ or it does not meet $\Gamma_{00}$. In the latter case, $C_\epsilon$ bifurcates from infinity. Moreover, $C_\epsilon$ does not meet any $(0, c)$, except if $c = c^*_\epsilon$.

(b) Assume that $\int_\Omega a = 0$. Then $C_\epsilon$ does not meet $\Gamma_{00}$. Consequently, $C_\epsilon$ bifurcates from infinity.

(iii) $C_\epsilon$ does not meet any $\mu, 0$, except if $\mu = 0$ or $\mu = \mu_\epsilon$. In particular, $C_\epsilon$ meets $(0, 0)$ if and only if $C_\epsilon$ meets $(0, c^*_\epsilon)$. Consequently, $C_\epsilon \setminus \{(0, 0), (\mu, 0)\}$ is composed by positive solutions of $(Q_{\mu, \epsilon})$.

Possible bifurcation diagrams for $(Q_{\mu, \epsilon})$ are depicted in Figures 2 and 3 for the cases $\int_\Omega a > 0$ and $\int_\Omega a = 0$, respectively.
\[ \|v\|_{C(\overline{\Omega})} \]

\begin{align*}
\varepsilon_*^c & \quad \varepsilon_*^c \\
C_\varepsilon & \quad C_\varepsilon \\
\Gamma_{00} & \quad \Gamma_{00} \\
\text{O} & \quad \text{O} \\
\mu & \quad \mu \\
\Lambda & \quad \Lambda
\end{align*}

(a) \( C_\varepsilon \) meets \( \Gamma_{00} \) at \((0,c^*_\varepsilon)\), and \( C_\varepsilon \setminus (\Gamma_0 \cup \Gamma_{00}) \) is bounded.

(b) \( C_\varepsilon \) meets \( \Gamma_{00} \) at \( c = c^*_\varepsilon \), and bifurcation from infinity occurs.

(c) \( C_\varepsilon \) does not meet \( \Gamma_{00} \), but bifurcation from infinity occurs.

**Figure 2.** Possible bifurcation diagrams for \( C_\varepsilon \): the case \( \int_\Omega a > 0 \).

\[ \|v\|_{C(\overline{\Omega})} \]

\begin{align*}
\varepsilon_*^c & \quad \varepsilon_*^c \\
C_\varepsilon & \quad C_\varepsilon \\
\Gamma_{00} & \quad \Gamma_{00} \\
\text{O} & \quad \text{O} \\
\mu & \quad \mu \\
\Lambda & \quad \Lambda
\end{align*}

**Figure 3.** Possible bifurcation diagram for \( C_\varepsilon \): the case \( \int_\Omega a = 0 \).
3. Proof of Theorem 1. In the sequel we study the limiting behavior of $C_\varepsilon$ as $\varepsilon \to 0^+$. To this end, we recall some definitions. Let $X$ be a complete metric space. Given $E_n \subset X$, $n \geq 1$, we set
\[
\liminf_{n \to \infty} E_n := \{ x \in X : \lim_{n \to \infty} \text{dist}(x, E_n) = 0 \},
\]
\[
\limsup_{n \to \infty} E_n := \{ x \in X : \liminf_{n \to \infty} \text{dist}(x, E_n) = 0 \},
\]
where dist$(x, A)$ is the usual distance function for a set $A$. It is well known from Whyburn [15, (9.12)Theorem] that if $\{E_n\}$ is a sequence of connected sets in $X$ satisfying
\[
\liminf_{n \to \infty} E_n \neq \emptyset,
\]
\[
\bigcup_{n \geq 1} E_n \text{ is precompact in } X,
\]
then $\limsup_{n \to \infty} E_n$ is nonempty, closed, and connected.

Let $\rho > 0$, and set
\[
C_{\varepsilon, \rho} := \overline{C_\varepsilon \cap ((0, \Lambda) \times B_\rho)},
\]
where $B_\rho := \{ u \in C(\overline{\Omega}) : \|u\|_{C(\overline{\Omega})} \leq \rho \}$ is a closed ball in $C(\overline{\Omega})$, and $\Lambda$ is the positive constant provided by Lemma 6(i). Then, $C_{\varepsilon, \rho}$ is a bounded component satisfying (see Figure 4):
\begin{itemize}
  \item $C_{\varepsilon, \rho}$ contains only $(\mu, 0)$ on $\Gamma_0$ (by Lemma 6(iii));
  \item $C_{\varepsilon, \rho}$ does not meet $\mu = \Lambda$ (by Lemma 6(i));
  \item $C_{\varepsilon, \rho}$ does not meet $\Gamma_{00}$, except at $(0, c')$ (by Lemma 6(ii));
  \item $C_{\varepsilon, \rho}$ contains either $(0, c')$ or some $(\mu, v)$ such that $\mu \in (0, \Lambda)$ and $\|v\|_{C^1(\overline{\Omega})} = \rho$.
\end{itemize}

Letting $X = [0, \Lambda] \times B_\rho$, $\varepsilon_n \to 0^+$, and $E_n = C_{\varepsilon_n, \rho}$, we shall verify (9) and (10).

We note from (7) that $\mu_n = \mu_{\varepsilon_n} c_n q^{-2}$, so that $\mu_{\varepsilon_n} \to 0$. It follows that $(0, 0) \in \liminf_{n \to \infty} C_{\varepsilon_n, \rho}$, since $(\mu_{\varepsilon_n}, 0) \in C_{\varepsilon_n, \rho}$. In particular, we obtain assertion (9).

The boundedness of $C_{\varepsilon_n, \rho}$ implies that $\bigcup C_{\varepsilon_n, \rho}$ is precompact. Indeed, for any $\{(\mu_k, v_k)\} \subset \bigcup C_{\varepsilon_n, \rho}$, the sequence $\varepsilon_n$ has a subsequence $\varepsilon_{n_k}$ such that $(\mu_k, v_k) \in C_{\varepsilon_{n_k}, \rho}$, where $\varepsilon_{n_k} \in (0, 1]$. Then, by elliptic regularity, we deduce that $v_k \in C^2(\overline{\Omega})$, $\|v_k\|_{C^1(\overline{\Omega})}$ is bounded, and
\[
\begin{cases}
-\Delta v_k = \mu_k \left( b(x)(v + \varepsilon_{n_k})^{q-2} v_k + a(x) v_k^{p-1} \right) & \text{in } \Omega, \\
\frac{\partial v_k}{\partial n} = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Since $(\mu_k, v_k)$ and $\varepsilon_{n_k}$ are bounded, using the compact embedding $C^1(\overline{\Omega}) \subset C(\overline{\Omega})$ we deduce that $\{(\mu_k, v_k)\}$ has a convergent subsequence in $[0, \Lambda] \times B_\rho$. Thus, assertion (10) is verified.

We may now apply Whyburn’s result, so that [15, (9.12)Theorem] implies that
\[
C_{0, \rho} := \lim_{n \to \infty} C_{\varepsilon_n, \rho}
\]
is nonempty, closed and connected, i.e., it is a nonempty component in $[0, \Lambda] \times B_\rho$. Moreover, we shall show that $C_{0, \rho}$ consists of non-negative solutions of $(Q_\mu)$, and
\[
(0, 0) \in \liminf_{n \to \infty} C_{\varepsilon_n, \rho} \subset C_{0, \rho}.
\]

The proof of these facts is similar to the verification of the precompactness of $\bigcup C_{\varepsilon_n, \rho}$. Indeed, given $(\mu, v) \in C_{0, \rho}$, the sequence $\varepsilon_n \to 0^+$ has a subsequence,
\[ \|v\|_{C(\Omega)} = \rho \]

(a) \( C_{\mu,\rho} \) meets \( \Gamma_{00} \) at \( (0, c_{*}^{\mu}) \) but does not meet \( \|v\|_{C(\Omega)} = \rho \).

(b) \( C_{\mu,\rho} \) meets both \( \Gamma_{00} \) at \( c = c_{*}^{\mu} \) and \( \|v\|_{C(\Omega)} = \rho \).

(c) \( C_{\mu,\rho} \) does not meet \( \Gamma_{00} \) but does meet \( \|v\|_{C(\Omega)} = \rho \).

\[ \begin{array}{c|c|c}
\mu & \mu_{\epsilon} & \Lambda \\
\hline
O & \mu_{\epsilon} & \Lambda \\
\end{array} \]

\[ \begin{array}{c|c|c}
\mu & \mu_{\epsilon} & \Lambda \\
\hline
O & \mu_{\epsilon} & \Lambda \\
\end{array} \]

\[ \begin{array}{c|c|c}
\mu & \mu_{\epsilon} & \Lambda \\
\hline
O & \mu_{\epsilon} & \Lambda \\
\end{array} \]

Figure 4. Three possibilities for the bounded component \( C_{\epsilon,\rho} \).

still denoted by the same notation, such that there exist \((\mu_{n}, v_{n}) \in C_{\epsilon_{n},\rho} \) satisfying \((\mu_{n}, v_{n}) \to (\mu, v) \) in \( \mathbb{R} \times C(\Omega) \). It follows, by a bootstrap argument based on elliptic regularity, that \( v_{n} \to v \) in \( C^{1}(\Omega) \), so that \( v \) is a non-negative weak solution of \((Q_{\mu})\) (see (11)), and eventually, a non-negative solution in \( C^{2+\theta}(\Omega) \) for some \( \theta \in (0,1) \), by elliptic regularity.

Next, we shall prove that \( C_{0,\rho} \) is nontrivial, i.e., we exclude the possibility that \( C_{0,\rho} \subset \Gamma_{0} \cup \Gamma_{00} \). Let \( \rho, M \) be such that \( 0 < M < c_{0}^{\mu} < \rho \). Then, we find from (4) and (13) that \( C_{0,\rho} \) joins \((0,0)\) to either \((0, c_{0}^{\mu})\) or \((\mu, v) \in [0,\Lambda] \times B_{\rho} \). Since \( C_{0,\rho} \) is connected, the intermediate value theorem shows the existence of \((\mu_{0}, v_{0}) \in C_{0,\rho} \) such that \( \|v_{0}\|_{C(\Omega)} = M \). By definition, the sequence \( \epsilon_{n} \to 0^{+} \) has a subsequence, still denoted by the same notation, such that there exist \((\mu_{n}, v_{n}) \in C_{\epsilon_{n},\rho} \) with \( 0 < \mu_{n} \to \mu_{0} \) and \( v_{n} \to v_{0} \) in \( C(\Omega) \). Assume by contradiction that \( \mu_{0} > 0 \). Then, \( v_{0} = M \), so that \( v_{n} \to M \) in \( C(\Omega) \). However, applying the divergence theorem to the solution \( v_{n} \) of \((Q_{\mu_{n},\epsilon_{n}})\), we obtain

\[ 0 = \mu_{n}^{-1} \int_{\Omega} (-\Delta v_{n}) = \int_{\Omega} \left\{ b(x)(v_{n} + \epsilon_{n})^{q-2}v_{n} + a(x)v_{n}^{p-1} \right\}, \]

so that passing to the limit, we deduce that

\[ 0 = M^{q-1} \int_{\Omega} b + M^{p-1} \int_{\Omega} a, \] (14)
and thus that \( M = c^*_0 \), which is impossible. Consequently, \( \mu_0 > 0 \), and thus, \( C_{0,\rho} \) is nontrivial.

Using \((H_3)\) and \([12, \text{Proposition 3.3}]\), we infer that the nontrivial non-negative solutions set of \((P_\lambda)\) does not meet \( \Gamma_0 \) at any \( \lambda \neq 0 \), so that neither does the one of \((Q_\mu)\). Similarly to Lemma 6(ii) and (iii), we see from Lemma 4 (see Remark 5) that the nontrivial non-negative solutions set of \((Q_\mu)\) does not meet \( \Gamma_0 \) at any \( c \neq c^*_0 \), and \((13)\) ensures that \( C_{0,\rho} \) joins \((0,0)\) to either \((0,c^*_0)\) or some \((\mu_1, v_1)\) such that \( \mu_1 \in (0, \Lambda) \) and \( \|v_1\|_{C(\Omega)} = \rho \). Since \( \rho \) is arbitrary, we obtain a component \( C'_0 \) of non-negative solutions of \((Q_\mu)\) such that (see Figure 5):

\[ \begin{align*}
(1) \quad & C'_0 \setminus \{(0,0), (0,c^*_0)\} \text{ consists of nontrivial non-negative solutions; } \\
(2) \quad & C'_0 \text{ joins } (0,0) \text{ to either } (0,c^*_0) \text{ or } (0,\infty). \\
(3) \quad & \text{If } (\mu, v) \in C'_0, \text{ then } \mu \geq 0.
\end{align*} \]

Note that the second possibility in (c2) follows from the following \textit{a priori} upper bound for positive solutions of \((Q_\mu, \epsilon)\): given \( \mu \in (0,1) \), there exists \( \rho > 0 \) such that \( v \leq C_\rho \) on \( \Omega \) for any positive solution \( v \) of \((Q_\mu, \epsilon)\) with \( \mu \in [\mu, \mu - 1] \) and \( \epsilon \in (0,1] \) (cf. [12, Proposition 6.5]).

\[ \begin{align*}
\|v\|_{C(\Omega)} & \quad \|v\|_{C(\Omega)} \\
c^*_0 & \quad c^*_0 \\
C'_0 & \quad C'_0 \\
O & \quad O \\
\Lambda & \quad \Lambda
\end{align*} \]

(a) \( C'_0 \) meets \( \Gamma_0 \) at \( c = c^*_0 \), and \( C'_0 \setminus (\Gamma_0 \cup \Gamma_0^0) \) is bounded. This is possible when \( \int_\Omega a > 0 \).

(b) \( c^*_0 \) meets \( \Gamma_0 \) at \( c = c^*_0 \), and bifurcation from infinity occurs. This is possible when \( \int_\Omega a > 0 \).

(c) \( C'_0 \) does not meet \( \Gamma_0^0 \setminus \{(0,0)\} \), and bifurcation from infinity occurs. This is possible when \( \int_\Omega a \geq 0 \).

\[ \begin{align*}
\|v\|_{C(\Omega)} & \\
C'_0 & \\
O & \quad O \\
\Lambda & \quad \Lambda
\end{align*} \]

\[ \begin{align*}
(1) \quad & C'_0 \setminus \{(0,0), (0,c^*_0)\} \text{ consists of nontrivial non-negative solutions; } \\
(2) \quad & C'_0 \text{ joins } (0,0) \text{ to either } (0,c^*_0) \text{ or } (0,\infty). \\
(3) \quad & \text{If } (\mu, v) \in C'_0, \text{ then } \mu \geq 0.
\end{align*} \]

Figure 5. Possible bifurcation diagrams for \( C'_0 \) when \( \int_\Omega a \geq 0 \).

We conclude now the proof of Theorem 1. By the rescaling \( u = \lambda^{\frac{1}{p-q}} v \), we transform the component \( C'_0 \) for \((Q_\mu)\) into a component of non-negative solutions
Remark 7. When \( \int_{\Omega} b = 0 < \int_{\Omega} a \), we may consider, instead of \((Q_{\mu, \epsilon})\), the problem
\[
\begin{cases}
-\Delta v = \mu ((b(x) - \epsilon)(v + \epsilon)^{q-2}v + a(x)v^{p-1}) & \text{in } \Omega, \\
\frac{\partial v}{\partial \eta} = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where \( b(x) - \epsilon \) changes sign if \( \epsilon > 0 \) is small enough. In fact, the eigenvalue problem associated with \((15)\), as introduced in \((7)\), possesses exactly two principal eigenvalues \( 0, \mu_* \), with \( \mu_* > 0 \), and \( \mu_e \to 0 \) as \( \epsilon \to 0^+ \) (see \[12, \text{Lemma 6.6}\]). However, since \( c_\epsilon^* \to 0 \) as \( \epsilon \to 0^+ \), we can not exclude the possibility that \( C_\epsilon \) shrinks to \( \{(0, 0)\} \) as \( \epsilon \to 0^+ \) when \( C_\epsilon \setminus (\Gamma_0 \cup \Gamma_0) \) is bounded, see Figure 2(a).

Finally, in the case \( \int_\Omega a = \int_\Omega b = 0 \), \( C_\epsilon \) is provided from \((15)\) as in Figure 3. Indeed, letting \( v_n \) be a positive solution of \((15)\) for \( \mu = \mu_n > 0 \), it is not possible that \( (\mu_n, v_n) \to (0, c) \) in \( \mathbb{R} \times C(\overline{\Omega}) \) for some constant \( c \geq 0 \). However, we can not exclude the possibility that \( C_{0, \rho} \subset \Gamma_0 \cup \Gamma_0 \), since \((14)\) holds for any positive constant \( M \) in this case.

\textit{Note added in proof.} (i) Regarding \((H_3)\), the condition that \( \Omega_\rho^b \) is a subdomain can be removed from Theorem 1. Indeed, although this condition is needed to verify the non-existence of nontrivial non-negative solutions of \((P_\lambda)\) bifurcating from \( \{(\lambda, 0)\} \) for \( \lambda < 0 \), such verification is required only for \( \lambda > 0 \) in Theorem 1.

(ii) Let \( C_0 \) be a maximal component of nonnegative solutions of \((P_\lambda)\) that includes the loop type component \( C_0 \) provided by Theorem 1 and such that \( C_0 \setminus \{(0, 0)\} \) consists of nontrivial non-negative solutions. As a further result for Theorem 1, we obtain that \( C_0 \) is bounded in \( \mathbb{R} \times C(\overline{\Omega}) \) if, in addition to the hypotheses in Theorem 1, one of the following conditions is assumed.

(a) \( \overline{\Omega_\rho}^b \subset \Omega \), \( \Omega':=\Omega \setminus \overline{\Omega_\rho}^b \) is a subdomain, and \( \Omega_\rho^b \subset \Omega_\rho^b \).

(b) \( \Omega_\rho^b \) contains a tubular neighborhood of \( \partial \Omega \), \( \Omega_\rho^b \) is a subdomain, and \( \Omega_\rho^b \subset \Omega_\rho^b \).

Indeed, under the additional condition, the strong maximum principle and boundary point lemma show that any nontrivial non-negative solution of \((Q_{\mu, \epsilon})\) is positive in \( \Omega_\rho^b \). Consequently, Lemma 6 (i) is also valid for nontrivial non-negative solutions of \((Q_{\mu, \epsilon})\), and the desired conclusion follows.

4. A further analysis for the case \( \int_\Omega a < 0 \). Let us assume now condition \((1)\). In addition, we assume \((H_0), (H_3)\), and condition (b) from Theorem 1 with \( \Omega' = \Omega_\rho^b \neq \emptyset \). Then \((P_\lambda)\) has a bounded loop type component of non-negative solutions \( C_0 \) in \( \mathbb{R} \times C(\overline{\Omega}) \), satisfying the following properties (see \[12, \text{Theorem 1.6}\] and Figure 1(a)):

(i) \( C_0 \) bifurcates at \((0, 0)\) and joins \((0, 0)\) to itself;

(ii) \( C_0 \) is non-trivial, i.e., \( C_0 \neq \{(0, 0)\} \). More precisely, \( C_0 \) contains a positive solution \( u_0 \) of \((P_\lambda)\) with \( \lambda = 0 \);
(iii) The only trivial solution contained in $C_0$ is $(\lambda, u) = (0, 0)$, i.e., $C_0$ does not contain any $(\lambda, 0)$ with $\lambda \neq 0$.

(iv) There exists $\delta > 0$ such that $C_0$ does not contain any positive solution $u$ of $(P_\lambda)$ with $\lambda = 0$ satisfying $\|u\|_{C(\overline{\Omega})} \leq \delta$.

According to the arguments developed in [12], this existence result can be verified by considering the regularized version of $(P_\lambda)$ for $u^{q-1}$ at $u = 0$:

$$(P_{\lambda, \epsilon}) \begin{cases} -\Delta u = \lambda(b(x) - \epsilon)(u + \epsilon)^{q-2}u + a(x)u^{p-1} & \text{in } \Omega, \\
\frac{\partial u}{\partial n} = 0 & \text{on } \partial \Omega, \end{cases}$$

where $\epsilon > 0$ and $\Omega_{+}^{\lambda_0 - \epsilon} \neq \emptyset$. Then $(P_{\lambda, \epsilon})$ is regular, so that the unilateral global bifurcation theorem by López-Gómez [9, Theorem 6.4.3] may be applied. To this end, we consider the linearized problem at $u = 0$:

$$\begin{cases} -\Delta \varphi = \lambda(b - \epsilon)\epsilon^{q-2}\varphi & \text{in } \Omega, \\
\frac{\partial \varphi}{\partial n} = 0 & \text{on } \partial \Omega. \end{cases}$$

Since $b - \epsilon$ changes sign and $\int_\Omega (b - \epsilon) < 0$, this eigenvalue problem has exactly two principal eigenvalues, $\lambda = 0$ and $\lambda = \lambda_\epsilon > 0$, which are both simple. We use now the unilateral global bifurcation theory to obtain two components $C_{0, \epsilon}$ and $C_{1, \epsilon}$ of positive solutions of $(P_{\lambda, \epsilon})$, bifurcating from $(0, 0)$ and $(\lambda_\epsilon, 0)$, respectively. Moreover, we can analyze the local nature of these components at the bifurcation points by using the local bifurcation theory proposed by Crandall and Rabinowitz.

We can also analyze the global nature of these components by making good use of an a priori bound in $\mathbb{R} \times C(\overline{\Omega})$ for positive solutions $(\lambda, u)$ of $(P_{\lambda, \epsilon})$. Consequently, $C_{0, \epsilon}$ and $C_{1, \epsilon}$ are both bounded, so that $C_{0, \epsilon} = C_{1, \epsilon} (=: C_\epsilon)$, i.e., $C_\epsilon$ is a mushroom. Finally, based on the fact that $\lambda_\epsilon \to 0$ as $\epsilon \to 0^+$ (see [12, Lemma 6.6]), we may apply Whyburn’s topological method to infer that $C_0 = \limsup_{\epsilon \to 0^+} C_\epsilon$ is a non-empty component of non-negative solutions $(\lambda, u)$ of $(P_\lambda)$. The limiting features of $C_0$ mentioned above follow by using some additional results on the set of non-negative solution of $(P_\lambda)$.

In addition to these features, we shall provide a further result on the direction of bifurcation at $(0, 0)$ for $C_0$, using Whyburn’s topological method again. We remark that, although properties (i)-(iv) above provide that $C_0$ is a loop, i.e., $C_0$ joins $(0, 0)$ to itself passing by $(0, u_0)$ for some positive solution $u_0$ of $(P_\lambda)$ with $\lambda = 0$, Theorem 8 confirms additionally the loop property of $C_0$, as follows:

**Theorem 8.** Under the conditions stated above, $C_0$ contains closed connected sets $C_0^{\pm}$ such that $(0, 0) \in C_0^+$ and $C_0^- \neq \{(0, 0)\}$. Moreover, if $(\lambda, u) \in C_0^\pm \setminus \{(0, 0)\}$ then $\lambda \geq 0$, i.e. $C_0$ bifurcates both to the left and to the right at $(0, 0)$, see Figure 6.

**Proof.** Let $\Sigma_+^\epsilon$ and $\Sigma_-^\epsilon$ be closed connected subsets of $\{(\lambda, u) \in C_\epsilon : \lambda \geq 0\}$ and $\{(\lambda, u) \in C_\epsilon : \lambda \leq 0\}$, respectively, such that $(0, 0), (0, u_-^\epsilon) \in \Sigma_-^\epsilon$, and $(\lambda_\epsilon, 0), (0, u_+^\epsilon) \in \Sigma_+^\epsilon$ for some positive solutions $u^{\pm}_\epsilon$ of $(P_\lambda)$ with $\lambda = 0$, see Figure 7. $\Sigma_+^\epsilon$ and $\Sigma_-^\epsilon$ are well defined by virtue of the following facts, see [12, Sections 5 and 6].

- $C_\epsilon = \{(\lambda, u)\}$ is continuously parametrized by $\lambda = \lambda_k(s), u = u_k(s)$ for $s \in [0, s_0), k = 0, 1$, in certain neighborhoods of the bifurcation points $(0, 0)$ and
\[ \|u\|_{C(\overline{\Omega})} \]

\[ C_0^+ \quad C_0^- \]

\[ O \lambda \]

**Figure 6.** A bifurcation diagram for \( C_0 \) at \((0,0)\): the case \( \int_\Omega a < 0 \).

\[ \|u\|_{C(\overline{\Omega})} \]

\[ (0, u_\epsilon^+) \]

\[ \Sigma^-_\epsilon \quad (0, u^-_\epsilon) \quad \Sigma^+_\epsilon \]

\[ O \lambda_\epsilon \]

**Figure 7.** The behaviors of \( \Sigma^\pm_\epsilon \).

\((\lambda, 0)\). In addition, \( \lambda_\epsilon \) and \( u_\epsilon \) satisfy \((\lambda_0(0), u_0(0)) = (0, 0), (\lambda_1(0), u_1(0)) = (\lambda_\epsilon, 0)\), respectively;

- \( C_\epsilon \) bifurcates to the region \( \lambda < 0 \) at \((0,0)\) under condition (1);
- \( C_\epsilon \) contains a positive solution \( w_0 \) of \((P_\lambda)\) with \( \lambda = 0 \);
- \( C_\epsilon \) does not contain any point \((\lambda, 0)\) with \( \lambda \neq 0, \lambda_\epsilon \);
- There exists \( \delta > 0 \), independent of \( \epsilon \), such that \( C_\epsilon \) does not contain any positive solution \( u \) of \((P_\lambda, \epsilon)\) with \( \lambda = 0 \) satisfying \( \|u\|_{C(\overline{\Omega})} \leq \delta \). Consequently, for the positive solution \( w_0 \) above, we have that \( \|w_0\|_{C(\overline{\Omega})} \) is bounded below by some positive constant independent of \( \epsilon \).

Since \( \Sigma^\pm_\epsilon \subset C_\epsilon \), we observe that

\[ \Sigma^\pm_{\epsilon_0} := \limsup_{\epsilon \to 0^+} \Sigma^\pm_\epsilon \subset \limsup_{\epsilon \to 0^+} C_\epsilon = C_0. \]

Repeating the argument above, Whyburn’s topological approach yields that \( \Sigma^\pm_{\epsilon_0} \) are non-empty, closed and connected sets consisting of non-negative solutions of \((P_\lambda)\) and such that \((0,0) \in \liminf_{\epsilon \to 0^+} \Sigma^\pm_\epsilon \subset \Sigma^\pm_{\epsilon_0} \). By property (iv), we have \((0, u^+_0) \in \Sigma^+_0\) for some positive solutions \( u^+_0 \) of \((P_\lambda)\) with \( \lambda = 0 \). It follows that \( \Sigma^+_0 \neq \{(0,0)\} \), and moreover, by property (iii), that \( \Sigma^-_0 \setminus \{(0,0)\} \) consists of nontrivial non-negative solutions of \((P_\lambda)\).

Now, by definition, we see that \((\lambda, u) \in \Sigma^+_0 \) (respect. \(\Sigma^-_0\)) implies \( \lambda \geq 0 \) (respect. \( \lambda \leq 0 \)). Lastly, by using property (iv) again, there exists \( \rho > 0 \) small enough such
that $\Sigma_{0,\rho}^\pm := \Sigma_0^\pm \cap B_\rho((0,0))$ is closed and connected, and if $(\lambda, u) \in \Sigma_{0,\rho}^\pm \setminus \{(0,0)\}$ then $\lambda \gg 0$. Therefore $\Sigma_0^\pm := \Sigma_{0,\rho}^\pm$ have the desired properties.

REFERENCES

[1] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM Rev.*, 18 (1976), 620–709.
[2] H. Amann and J. López-Gómez, A priori bounds and multiple solutions for superlinear indefinite elliptic problems, *J. Differential Equations*, 146 (1998), 336–374.
[3] K. J. Brown, Local and global bifurcation results for a semilinear boundary value problem, *J. Differential Equations*, 239 (2007), 296–310.
[4] K. J. Brown and S. S. Lin, On the existence of positive eigenfunctions for an eigenvalue problem with indefinite weight function, *J. Math. Anal. Appl.*, 75 (1980), 112–120.
[5] S. Cano-Casanova, Compact components of positive solutions for superlinear indefinite elliptic problems of mixed type, *Topol. Methods Nonlinear Anal.*, 23 (2004), 45–72.
[6] S. Cingolani and J. L. Gómez, Positive solutions of a semilinear elliptic equation on $\mathbb{R}^N$ with indefinite nonlinearity, *Adv. Differential Equations*, 1 (1996), 773–791.
[7] M. G. Crandall and P. H. Rabinowitz, Bifurcation from simple eigenvalues, *J. Functional Analysis*, 8 (1971), 321–340.
[8] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Reprint of the 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2001.
[9] J. López-Gómez, *Spectral Theory and Nonlinear Functional Analysis*, Research Notes in Mathematics 426, Chapman & Hall/CRC, Boca Raton, FL, 2001.
[10] J. López-Gómez and M. Molina-Meyer, Bounded components of positive solutions of abstract fixed point equations: mushrooms, loops and isolas, *J. Differential Equations*, 209 (2005), 416–441.
[11] P. H. Rabinowitz, Some global results for nonlinear eigenvalue problems, *J. Functional Analysis*, 7 (1971), 487–513.
[12] H. Ramos Quoirin and K. Umezu, An indefinite concave-convex equation under a Neumann boundary condition I, *Israel J. Math.*, 220 (2017), 103–160.
[13] H. Ramos Quoirin and K. Umezu, An indefinite concave-convex equation under a Neumann boundary condition II, *Topol. Methods Nonlinear Anal.*, 49 (2017), 739–756.
[14] K. Umezu, Global bifurcation results for semilinear elliptic boundary value problems with indefinite weights and nonlinear boundary conditions, *Nonlinear Differential Equations Appl. NoDEA*, 17 (2010), 323–336.
[15] G. T. Whyburn, *Topological Analysis*, Second, revised edition, Princeton Mathematical Series, No. 23, Princeton University Press, Princeton, N.J., 1964.

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