A CONCENTRATION PHENOMENON OF THE LEAST ENERGY SOLUTION TO NON-AUTONOMOUS ELLIPTIC PROBLEMS WITH A TOTALLY DEGENERATE POTENTIAL

SHUN KODAMA
2-32-2, Kami-takada, Nakano, Tokyo 164-0002, Japan

(Communicated by Jaeyoung Byeon)

Abstract. In this paper we study the following non-autonomous singularly perturbed Dirichlet problem:

$$
\varepsilon^2 \Delta u - u + K(x)f(u) = 0, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega,
$$

for a totally degenerate potential \( K \). Here \( \varepsilon > 0 \) is a small parameter, \( \Omega \subset \mathbb{R}^N \) is a bounded domain with a smooth boundary, and \( f \) is an appropriate super-linear subcritical function. In particular, \( f \) satisfies \( 0 < \liminf_{t \to 0^+} f(t)/t^q \leq \limsup_{t \to 0^+} f(t)/t^q < +\infty \) for some \( 1 < q < +\infty \). We show that the least energy solutions concentrate at the maximal point of the modified distance function \( D(x) = \min \{ (q+1)d(x, \partial A), 2d(x, \partial \Omega) \} \), where \( A = \{ x \in \bar{\Omega} | K(x) = \max_{y \in \bar{\Omega}} K(y) \} \) is assumed to be a totally degenerate set satisfying \( A^\circ \neq \emptyset \).

1. Introduction and main results. We consider the following non-autonomous singularly perturbed Dirichlet problem:

$$
\begin{aligned}
-\varepsilon^2 \Delta u(x) + u(x) &= K(x)f(u(x)), \quad x \in \Omega, \\
u(x) &= 0, \quad x \in \partial \Omega, \\
u(x) &= 0, \quad x \in \Omega,
\end{aligned}
$$

where \( \varepsilon \) is a positive constant, \( \Delta \) is the Laplace operator, \( \Omega \) is a bounded domain with a smooth boundary in \( \mathbb{R}^N \) \( (N \geq 1) \), and functions \( K \) and \( f \) satisfy the following conditions:

(K0) \( K \in C^\alpha(\bar{\Omega}; \mathbb{R}) \) for some \( 1 > \alpha > 0 \).

(K1) \( m := \min_{x \in \bar{\Omega}} K(x) > 0 \).

(K2) Let \( A := \{ x \in \bar{\Omega} | K(x) = \max_{y \in \bar{\Omega}} K(y) =: M \} \). Then it holds that \( A^\circ \neq \emptyset \) and \( A \neq \Omega \).

(f0) \( f \in C^1(\mathbb{R}; \mathbb{R}) \).

(f1) \( f(t) \equiv 0 \) for \( t \leq 0 \) and \( f(t) = o(t) \) near \( t = 0 \).

(f2) \( f'(t) = O(t^{p-1}) \) as \( t \to +\infty \) for some \( 1 < p < (N+2)/(N-2) \) if \( N \geq 3 \), and \( p > 1 \) if \( N = 1, 2 \).

(f3) There exists a constant \( \theta > 2 \) such that \( \theta F(t) \leq tf(t) \) for \( t \geq 0 \), in which

\[
F(t) = \int_0^t f(s) \, ds.
\]

(f4) \( f(t) < tf'(t) \) for \( t > 0 \).

2000 Mathematics Subject Classification. Primary: 35B25, 35B40; Secondary: 35J20.

Key words and phrases. Non-autonomous, least energy solution, concentration phenomenon, singular perturbation, elliptic problem.
solutions concentrates at the maximal point of $K$ under the Neumann boundary condition arises in the following Keller-Segel system:

\begin{align}
\frac{\partial u_1}{\partial t} &= D_1 \Delta u_1 - \chi \nabla (u_1 \nabla \phi(u_2)), \quad \text{in } \Omega \times [0,T), \\
\frac{\partial u_2}{\partial t} &= D_2 \Delta u_2 + k(x, u_1, u_2), \quad \text{in } \Omega \times [0,T), \\
\frac{\partial u_1}{\partial \nu} &= \frac{\partial u_2}{\partial \nu} = 0, \quad \text{on } \partial \Omega \times [0,T), \\
u_1(x,0) &= u_1^0(x) > 0, \quad u_2(x,0) = u_2^0(x) > 0, \quad \text{on } \Omega, 
\end{align}

where $u_1(x,t)$ is the population of amoebae at place $x$ and time $t$ and $u_2(x,t)$ is the concentration of the chemical. This model was proposed by Keller and Segel in [11]. If we take $\phi(u_2) = \log u_2$ and $k(x,u_1,u_2) = -u_2 + K(x)u_1$ with $K(x) > 0$ on $\bar{\Omega}$, and consider steady states of the above system, then the system for steady states is reduced to (1) with the Neumann boundary problem and $f(u) = u^p$ for some $p > 0$. We refer to Lin, Ni and Takagi [15] for the details for the derivation.

In this paper, we consider the problem with the Dirichlet boundary condition. The autonomous case of (1), that is, $K(x) \equiv 1$, has been widely studied for a long time. In [17], Ni and Wei showed that the least energy solution concentrates at the most centered point of $\Omega$ when $f$ satisfies a uniqueness-nondegeneracy assumption on the limiting equation. Furthermore, in [7], del Pino and Felmer showed that the same results holds by a simpler approach without a uniqueness-nondegeneracy assumption for $f$. Also, in [4, 5], Byeon studied the optimal conditions for $f$ for the existence of the concentrating solution at the most centered point of $\Omega$. On the other hand, there are many studies on the existence and the properties of higher energy solutions. Especially, many researchers showed there exist positive solutions with single and multiple peaks (see [6, 21, 14, 8] and the references therein).

The non-autonomous case of (1), that is, $K(x) \not\equiv 1$, has also been studied by many researchers when $f(u) = u^p$. In [19], Ren showed that the least energy solutions concentrates at the maximal point of $K(x)$ on $\bar{\Omega}$. In [18], Qiao and Wang studied the multiplicity of positive solutions to (1). In [22], Zhao studied on the number of interior peaks of solutions to (1) with the Neumann boundary condition.

We define the energy functional associated to (1) as

$$I_{\varepsilon,\Omega}[u;K] := \frac{1}{2} \int_{\Omega} \varepsilon^2 |\nabla u(x)|^2 + u(x)^2 \, dx - \int_{\Omega} K(x)F(u(x)) \, dx, \quad u \in H^1_0(\Omega),$$

and the least energy associated to (1) as

$$e_{\varepsilon,\Omega}(K) := \inf_{u \in H^1_0(\Omega) \setminus \{0\}} \sup_{t > 0} I_{\varepsilon,\Omega}[tu;K].$$

Moreover, we call the solution $u_{\varepsilon}$ of (1) the least energy solution if it holds that

$$I_{\varepsilon,\Omega}[u_{\varepsilon};K] = e_{\varepsilon,\Omega}(K).$$

Similarly, we define the energy functional associated to the following problem:

\(- \Delta w + w = Mf(w), \quad w > 0, \quad \text{in } \mathbb{R}^N, \quad \max_{x \in \mathbb{R}^N} w(x) = w(0), \quad (3)\)
as
\[ I[w] := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla w(x)|^2 + w(x)^2 \, dx - M \int_{\mathbb{R}^N} F(w(x)) \, dx, \quad w \in H^1(\mathbb{R}^N), \]
and the least energy associated to (3) as
\[ c_* := \inf_{w \in H^1(\mathbb{R}^N) \setminus \{0\}} \sup_{t>0} I[tw]. \]

First, we state the existence and the basic properties of the least energy solution to (1) as follows.

Proposition 1. We assume that a function \( K \) satisfies conditions (K0)-(K1) and a function \( f \) satisfies conditions (f0)-(f4). Then, for any \( \varepsilon > 0 \) there exists a positive least energy solution \( u_\varepsilon \in C^2(\bar{\Omega}) \) to (1). Furthermore, the following statements hold:

(i): For \( \varepsilon \) small enough \( u_\varepsilon \) has only one local maximum point \( x_\varepsilon \) with
\[ \lim_{\varepsilon \to 0} \frac{d(x_\varepsilon, \partial \Omega)}{\varepsilon} = \infty. \]

(ii): If \( x_0 \) is a limit point of \( \{x_\varepsilon\} \) as \( \varepsilon \to 0 \), then
\[ K(x_0) = \max_{x \in \Omega} K(x). \]

(iii): There exist constants \( C_1 \) and \( C_2 \) independent of \( \varepsilon \) such that
\[ C_1 \leq \|u_\varepsilon\|_{L^\infty(\Omega)} \leq C_2. \]

(iv): Passing to a subsequence, we have
\[ v_{\varepsilon_j} \to w \quad \text{in} \quad C^2_{\text{loc}}(\mathbb{R}^N) \cap H^1(\mathbb{R}^N), \]
where \( v_\varepsilon(y) := u_\varepsilon(\varepsilon y + x_\varepsilon) \) and \( w \) is a least energy solution to (3).

When \( f(u) = u^p \), Proposition 1 has been proved by Ren ([19]). We can prove Proposition 1 by almost the same argument as in [19] for general nonlinearities \( f(u) \) satisfying (f0)-(f4). So, we omit the details. For example, refer to [13, Appendix].

For the precise asymptotic location of the maximum point \( x_\varepsilon \) and the precise asymptotic expansion of the least energy \( e_{\varepsilon, \Omega}(K) \), we show the following theorem, which is the main result of this paper. We will use the next notation:
\[ D(x) = D_{q,A,\Omega}(x) := \min\{(q+1)d(x, \partial A), 2d(x, \partial \Omega)\} \quad \text{for} \ x \in \bar{\Omega}. \]

Theorem 1.1. We assume that a function \( K \) satisfies conditions (K0)-(K2) and a function \( f \) satisfies conditions (f0)-(f5). Let \( u_\varepsilon \) be a least energy solution of (1) and \( x_\varepsilon \) be a maximum point of \( u_\varepsilon \). Then the following statements hold:

(i): For \( \varepsilon \) sufficiently small, we have \( x_\varepsilon \in A^o \).

(ii):
\[ D(x_\varepsilon) \to \max_{x \in A} D(x) \quad \text{as} \ \varepsilon \to 0. \]

(iii): Passing to a subsequence, we have
\[ e_{\varepsilon_j, \Omega}(K) = \varepsilon_j^N \left[ c_* + \exp \left( -\frac{1}{\varepsilon_j} \left( \max_{x \in A} D(x) + o(1) \right) \right) \right] \quad \text{as} \ j \to \infty. \]
Our proof of Theorem 1.1 is based on the modified argument of [7] employing the rearrangement technique and the precise asymptotic behavior of solutions to several auxiliary problems (lemma 3.2 and 3.3). Especially, we do not need to assume the uniqueness-nondegeneracy condition for the nonlinearity \( f(u) \).

Another similar problem to (1) is the nonlinear Schrödinger equation, that is,
\[
\varepsilon^2 \Delta u - V(x)u + f(u) = 0, \quad u > 0, \quad \text{in} \ \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N),
\]
for a potential \( V \) which satisfies \( B^c \neq \emptyset \), where
\[
B := \{ x \in \mathbb{R}^N | V(x) = \min_{y \in \mathbb{R}^N} V(y) \}.
\]

In [16], Lu and Wei showed that the least energy solution of the problem (5) concentrates at the maximal point of \( d(x, \partial B) \) on \( B \) when \( f(u) = u^p \) and \( V \) satisfies the assumptions that \( B^c \) is connected and \( V(x) = O(d(x, \partial B)^2) \) near \( \partial B \). Recently, in [13], the author extended the results to the more general \( f(u) \) and \( V(x) \) by using the modified argument of [7] employing the rearrangement technique.

This paper is organized as follows: In section 2, we prepare some facts, that is, the properties of the least energy solutions to (3), the modified Bessel functions, and the Schwarz symmetrization. In section 3, we show that the asymptotic expansion of the least energy in the special case that \( \Omega \) is a ball and a function \( K \) is radially symmetric. In section 4, we show the main result, that is, Theorem 1.1. In section 5, we give some concrete examples to illustrate the effect of the function \( D(x) \). We always assume that \( B(x; r) \) denotes an open ball with center \( x \), radius \( r > 0 \), \( P^c \) denotes the interior of \( P \), \( P^c \) denotes the complement of \( P \) and \( d(x, P) \) denotes the distance from \( x \) to \( P \). We also use the notation \( B_r(x) = B(x; r) \) and \( B_r = B_r(0) \).

2. Preliminaries.

2.1. The basic properties of the least energy solutions to (3). First, we remark the next fact on the problem (3), which is well known, for example, see [2, 3, 9].

Proposition 2. We assume that a function \( f \) satisfies conditions (f0)-(f2), (f4). Then there exists a solution \( w \in C^2(\mathbb{R}^N) \cap H^1(\mathbb{R}^N) \) of (3) and \( w \) satisfies that

(i): \( w \) is decreasing and radially symmetric,
(ii): \( I[w] = c_\ast \),
(iii): \( |D^\alpha w(x)| \leq C e^{-|x|} \) for \( |\alpha| \leq 1 \).

2.2. Modified Bessel function. We will need the asymptotic properties of the modified Bessel functions to show the asymptotic expansion of the least energy. The next definition is taken from [20].

Definition 2.1. We define the modified Bessel function of first kind \( I_\nu \) by
\[
I_\nu(x) := \sum_{m=0}^\infty \frac{\left(\frac{x}{2}\right)^{\nu+2m}}{m!\Gamma(\nu + m + 1)} \quad \text{for} \quad \nu \in \mathbb{R}, \ x > 0.
\]

Also, we define the modified Bessel function of second kind \( K_\nu \) by
\[
K_\nu(x) := \frac{1}{2\pi} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu \pi} \quad \text{for} \quad \nu \in \mathbb{R} \setminus \mathbb{Z}, \ x > 0,
\]
and
\[
K_n(x) := \lim_{\nu \to n} K_\nu(x) \quad \text{for} \quad n \in \mathbb{Z}, \ x > 0.
\]
The next results are basic properties of the modified Bessel functions $I_\nu(x)$ and $K_\nu(x)$. For the proof, see [20, p.77-80, p.202-203].

**Proposition 3.** For all $\nu \in \mathbb{R}$, $x > 0$, the following statements hold:

(i): $I_{\nu-1}(x) + I_{\nu+1}(x) = 2I_\nu(x)$, $K_{\nu-1}(x) + K_{\nu+1}(x) = -2K_\nu(x)$.

(ii): Both $I_\nu$ and $K_\nu$ are solutions of the equation

$$x^2w''(x) + xw'(x) - (x^2 + \nu^2)w(x) = 0.$$ 

(iii): $W[I_\nu(x), K_\nu(x)] = -1/x$, where $W[I_\nu(x), K_\nu(x)]$ denotes the Wronskian of $I_\nu(x)$ and $K_\nu(x)$. In particular, $I_\nu$ and $K_\nu$ are linearly independent.

(iv): As $x \to +\infty$, we have

$$I_\nu(x) \sim \frac{1}{\sqrt{2\pi x}}e^x, \quad K_\nu(x) \sim \sqrt{\frac{\pi}{2x}}e^{-x}.$$ 

Proposition 3 yields the next results.

**Lemma 2.2.** Let $r > 0$. The following statements hold:

(i): $I_\nu(r\zeta) = e^{\zeta(r+o(1))}$, $K_\nu(r\zeta) = e^{-\zeta(r+o(1))}$ as $\zeta \to +\infty$.

(ii): $I'_\nu(r\zeta) = e^{\zeta(r+o(1))}$, $K'_\nu(r\zeta) = -e^{-\zeta(r+o(1))}$ as $\zeta \to +\infty$.

**Proof.** By Proposition 3 (v), we have

$$I_\nu(r\zeta) = \frac{1}{\sqrt{2\pi r\zeta}} \exp(r\zeta)(1 + o(1))$$

$$= \exp(r\zeta) \exp\left(\log \left(\frac{1}{\sqrt{2\pi r\zeta}}(1 + o(1))\right)\right)$$

$$= \exp\left(r\zeta \left\{1 + \frac{\log\left(\frac{1}{\sqrt{2\pi r\zeta}}(1 + o(1))\right)}{r\zeta}\right\}\right)$$

$$= \exp(r\zeta \{1 + o(1)\})$$

$$= \exp(\zeta\{r + o(1)\}),$$

as $\zeta \to +\infty$. Similarly, we have

$$K_\nu(r\zeta) = \sqrt{\frac{\pi}{2r\zeta}} \exp(-r\zeta)(1 + o(1))$$

$$= \exp(-r\zeta) \exp\left(\log \left(\frac{\pi}{2r\zeta}(1 + o(1))\right)\right)$$

$$= \exp\left(-r\zeta \left\{1 - \frac{\log\left(\frac{\pi}{2r\zeta}(1 + o(1))\right)}{r\zeta}\right\}\right)$$

$$= \exp(-r\zeta \{1 + o(1)\})$$

$$= \exp(-\zeta\{r + o(1)\}),$$

as $\zeta \to +\infty$. Therefore, we have proved (i).

Next, we prove (ii). By Proposition 3 (ii) and Lemma 2.2 (i), we may calculate

$$I'_\nu(r\zeta) = \frac{1}{2} (I_{\nu+1}(r\zeta) + I_{\nu-1}(r\zeta)) = \frac{1}{2} \left(e^{\zeta(r+o(1))} + e^{\zeta(r+o(1))}\right) = e^{\zeta(r+o(1))},$$

as $\zeta \to +\infty$.
as $\zeta \to +\infty$. Similarly, we have
\[
K'_\nu(r\zeta) = -\frac{1}{2} (K_{\nu+1}(r\zeta) + K_{\nu-1}(r\zeta)) = -e^{-\zeta(r+o(1))},
\]
as $\zeta \to +\infty$. Hence, we have proved (ii).

2.3. **Schwarz symmetrization.** The Schwarz symmetrization plays important roles in the proof of Theorem 1.1. We just recall the definition and basic properties.

**Definition 2.3** ([12, Definition 1.1.1, Definition 1.3.1]). Let $E \subset \mathbb{R}^N$ be bounded and $u : E \to \mathbb{R}$ be a measurable function. Then the unidimensional decreasing rearrangement $u^\#$ of $u$ is defined on $[0, |E|]$ by
\[
u^\#(s) := \begin{cases} 
\text{ess. sup}(u), & s = 0, \\
\inf \{t \mid |\{x \in E \mid u(x) > t\}| < s\}, & s > 0.
\end{cases}
\]
Moreover, we define the Schwarz symmetrization $u^* : E^* \to \mathbb{R}$ of $u$ by
\[
u^*(x) := \nu^\#(|B_1(0)||x|^N), \quad x \in E^*,
\]
where $E^*$ denotes the open ball centered at the origin and having the same measure as $E$, i.e. $|E^*| = |E|$.

The Schwarz symmetrization has the next basic properties.

**Proposition 4** ([12, p.14]). Let $E \subset \mathbb{R}^N$ be a bounded measurable set and $u : E \to \mathbb{R}$ be a measurable function. Then, the following statements hold:

(i): $u$ and $u^*$ are equimeasurable.

(ii): If $F : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function such that either $F \geq 0$ or $F(u) \in L^1(E)$, then
\[
\int_{E^*} F(u^*(x)) \, dx = \int_E F(u(x)) \, dx.
\]

(iii): If $\psi : \mathbb{R} \to \mathbb{R}$ is a non-decreasing function, then
\[
(\psi(u))^* = \psi(u^*).
\]

The next result is very important for the proof of Theorem 1.1. For the proof, see [12, Theorem 1.2.2].

**Proposition 5** (Hardy-Littlewood). Let $E \subset \mathbb{R}^N$ be a bounded measurable set and $1 \leq p \leq \infty$. If $f \in L^p(E)$ and $g \in L^q(E)$ where $(1/p) + (1/q) = 1$, then
\[
\int_E f(x)g(x) \, dx \leq \int_{E^*} f^*(x)g^*(x) \, dx.
\]

The next result is well known. See, [12, Theorem 2.3.1].

**Proposition 6** (Pólya-Szegö). Let $1 \leq p < \infty$. Let $E \subset \mathbb{R}^N$ be a bounded domain and let $u \in W^{1,p}_0(E)$ be such that $u \geq 0$. Then
\[
\int_{E^*} |\nabla u^*|^p \, dx \leq \int_E |\nabla u|^p \, dx.
\]
In particular, $u^* \in W^{1,p}_0(E^*)$. 

\[\]
3. **In the case** $A$ **is a ball.** In this section, we prove the expansion of $e_{\varepsilon, \Omega}(K)$ in the special case.

Let $r > 1$. We assume that a Borel measurable function $K : \overline{B}(0; r) \to \mathbb{R}$ satisfies the following:

**H1:** $m := \inf_{x \in \overline{B}(0; r)} K(x) > 0$ and $M := \sup_{x \in \overline{B}(0; r)} K(x) < +\infty$.

**H2:** $\{x \in \overline{B}(0; r) \mid K(x) = M\} = \overline{B}(0; 1)$.

For $\rho > 0$, we define the energy functional as

$$ J_{\rho, r}[u; K] := \frac{1}{2} \int_{B_{r\rho}} |\nabla v|^2 + v^2 \, dy - \int_{B_{r\rho}} K\left(\frac{y}{\rho}\right) F(v) \, dy, \quad v \in H^1_0(B_{r\rho}). $$

Moreover, we define the least energy as

$$ C_{\rho, r}(K) := \inf_{v \in H^1_0(B_{r\rho}) \setminus \{0\}} \sup_{t > 0} J_{\rho, r}[tv; K]. $$

In this section, it is our goal to show the next Theorem.

**Theorem 3.1.** We assume that a function $K$ satisfies conditions (H1)-(H2) and a function $f$ satisfies conditions (H0)-(H5). Moreover, we assume $K^* = K$ holds, where $K^*$ is the Schwarz symmetrization of $K$. Then the following statement holds:

$$ C_{\rho, r}(K) = e_{\varepsilon}^M + e^{-\rho\left(\min\{q + 1, 2r\} + o(1)\right)} \quad \text{as} \quad \rho \to +\infty. $$

**Remark 1.** By Theorem 3.1, we obtain Theorem 1.1 (iii) in the case $\Omega = B(0; r)$ and $A = \overline{B}(0; 1)$ for $r > 1$, that is,

$$ e_{\varepsilon, \Omega}(K) = e^N C_{\rho, r}(K) = e^N \left[e_{\varepsilon}^M + \exp\left(-\rho \max_{x \in A} D(x) + o(1)\right)\right], $$

where $\rho = 1/\varepsilon$. Here remark that

$$ \max_{x \in A} D(x) = D(0) = \min\{q + 1, 2r\}. $$

3.1. **Upper bound of $C_{\rho, r}(K)$.** The next two Lemmas are crucial in the proof of the upper bound of $C_{\rho, r}(K)$.

**Lemma 3.2.** Assume that $\rho > 1$ and $r > 1$. Let $u_{\rho} \in C^2(\rho - 1, r\rho) \cap C([\rho - 1, r\rho])$ be a solution of the equation

$$ \begin{cases} -u''_{\rho}(s) - \frac{N - 1}{s} u'_{\rho}(s) + a_{\rho} u_{\rho}(s) = 0, & \rho - 1 \leq s \leq r\rho, \\ u_{\rho}(\rho - 1) = 1, & u_{\rho}(r\rho) = 0, \end{cases} \quad (9) $$

where $a_{\rho} \in \mathbb{R}$ and $\lim_{\rho \to +\infty} a_{\rho} = 1$. Then, it holds that

$$ -u'_{\rho}(\rho - 1) = \frac{N - 2}{2} (\rho - 1)^{-1} - \sqrt{a_{\rho}} K_{\nu}'(\sqrt{a_{\rho}}(\rho - 1)) - e^{-2\rho(\nu + o(1))}, $$

as $\rho \to +\infty$.

**Proof.** Let $\nu := (N - 2)/2$. Recall that $I_{\nu}$ and $K_{\nu}$ are linearly independent solutions of the equation

$$ s^2 y''(s) + sy'(s) - (s^2 + \nu^2)y(s) = 0. $$

We define $w_{\rho}$ as

$$ w_{\rho}(t) := \left(\frac{t}{\sqrt{a_{\rho}}}\right)^{\nu} u_{\rho}(t). $$
Then, \( w_\rho \) satisfy
\[
\begin{cases}
t^2 w''_\rho(t) + t w'_\rho(t) - (t^2 + \nu^2) w_\rho(t) = 0, \quad \sqrt{a_\rho}(\rho - 1) \leq t \leq \sqrt{a_\rho} r \\
w_\rho(\sqrt{a_\rho}(\rho - 1)) = (\rho - 1)^\nu, \quad w_\rho(\sqrt{a_\rho} r) = 0.
\end{cases}
\] (10)

By the uniqueness of the solution of (10), there exist unique \( \alpha_\rho \) and \( \beta_\rho \) such that
\[
w_\rho(t) = \alpha_\rho I_\nu(t) + \beta_\rho K_\nu(t).
\] (11)

Note that it holds that
\[
\begin{align*}
\alpha_\rho I_\nu(\sqrt{a_\rho}(\rho - 1)) + \beta_\rho K_\nu(\sqrt{a_\rho}(\rho - 1)) &= (\sqrt{a_\rho}(\rho - 1))^\nu \\
\alpha_\rho I_\nu(\sqrt{a_\rho} r) + \beta_\rho K_\nu(\sqrt{a_\rho} r) &= 0.
\end{align*}
\] (12)

In particular,
\[
\alpha_\rho = \frac{(\rho - 1)^\nu K_\nu(\sqrt{a_\rho} r)}{I_\nu(\sqrt{a_\rho}(\rho - 1)) K_\nu(\sqrt{a_\rho} r) - I_\nu(\sqrt{a_\rho} r) K_\nu(\sqrt{a_\rho}(\rho - 1))}.
\] (13)

Claim 1.
\[
\alpha_\rho = -e^{-\rho(2r-1+o(1))} \quad \text{as} \quad \rho \to +\infty.
\]

Proof of Claim 1. By Lemma 2.2, we may calculate
\[
\begin{align*}
I_\nu(\sqrt{a_\rho}(\rho - 1)) K_\nu(\sqrt{a_\rho} r) - I_\nu(\sqrt{a_\rho} r) K_\nu(\sqrt{a_\rho}(\rho - 1)) &= e^{\rho(1+o(1))} e^{-\rho r+o(1)} - e^{\rho(r+o(1))} e^{-\rho(1+o(1))} \\
&= e^{\rho(r-1+o(1))} \left( e^{-2\rho(r-1+o(1))} - 1 \right) \\
&= -e^{\rho(r-1+o(1))}
\end{align*}
\]
as \( \rho \to +\infty \). Also, by Lemma 2.2, we have
\[
K_\nu(\sqrt{a_\rho} r) = e^{-\rho(r+o(1))}.
\]

Hence, we have
\[
\alpha_\rho = -e^{-\rho(2r-1+o(1))} \quad \text{as} \quad \rho \to +\infty.
\]

By an elementary calculation, we have
\[
u'_\rho(\rho - 1) = -\nu(\rho - 1)^{-1} + (\rho - 1)^{-\nu} \sqrt{a_\rho} w'_\rho(\sqrt{a_\rho}(\rho - 1)).
\] (14)
By (11) and (12), we may calculate
\[
\begin{align*}
& w'_{\rho}(\sqrt{a_{\rho}(\rho - 1)}) \\
& = \alpha_{\rho} I'_{\nu}(\sqrt{a_{\rho}(\rho - 1)}) + \beta_{\rho} K'_{\nu}(\sqrt{a_{\rho}(\rho - 1)}) \\
& = \alpha_{\rho} I'_{\nu}(\sqrt{a_{\rho}(\rho - 1)}) \\
& + \left[ \frac{(\rho - 1)^{\nu}}{K_{\nu}(\sqrt{a_{\rho}(\rho - 1)})} - \frac{\alpha_{\rho} I_{\nu}(\sqrt{a_{\rho}(\rho - 1)})}{K_{\nu}(\sqrt{a_{\rho}(\rho - 1)})} \right] K'_{\nu}(\sqrt{a_{\rho}(\rho - 1)}) \\
& = \frac{(\rho - 1)^{\nu} K'_{\nu}(\sqrt{a_{\rho}(\rho - 1)})}{K_{\nu}(\sqrt{a_{\rho}(\rho - 1)})} \\
& + \alpha_{\rho} I'_{\nu}(\sqrt{a_{\rho}(\rho - 1)}) - \frac{I_{\nu}(\sqrt{a_{\rho}(\rho - 1)}) K'_{\nu}(\sqrt{a_{\rho}(\rho - 1)})}{K_{\nu}(\sqrt{a_{\rho}(\rho - 1)})} \\
& = \frac{(\rho - 1)^{\nu} K'_{\nu}(\sqrt{a_{\rho}(\rho - 1)})}{K_{\nu}(\sqrt{a_{\rho}(\rho - 1)})} - e^{-\rho(2r-1+o(1))} \left[ e^{\rho(1+o(1))} + e^{\rho(1+o(1))} \right] \\
& = \frac{(\rho - 1)^{\nu} K'_{\nu}(\sqrt{a_{\rho}(\rho - 1)})}{K_{\nu}(\sqrt{a_{\rho}(\rho - 1)})} - e^{-2\rho(r-1+o(1))},
\end{align*}
\]
where the second equality from the bottom follows by Claim 1. Hence, by (14), we conclude that
\[
w'_{\rho}(\rho - 1) = -\nu(\rho - 1)^{-1} + \frac{K'_{\nu}(\sqrt{a_{\rho}(\rho - 1)})}{K_{\nu}(\sqrt{a_{\rho}(\rho - 1)})} - e^{-2\rho(r-1+o(1))}.
\]

\[\square\]

**Lemma 3.3.** Assume that \( \rho > 1 \). Let \( v_{\rho} \in C^{2}(\rho - 1, +\infty) \cap C([\rho - 1, +\infty)) \) be a solution of the equation
\[
\begin{align*}
& -v''_{\rho}(s) - \frac{N - 1}{s} v'_{\rho}(s) + a_{\rho} v_{\rho}(s) = 0, \quad \rho - 1 \leq s < +\infty \\
& v_{\rho}(\rho - 1) = 1, \quad v_{\rho}(+\infty) = 0,
\end{align*}
\]
where \( a_{\rho} \in \mathbb{R} \) and \( \lim_{\rho \to +\infty} a_{\rho} = 1 \). Then, it holds that
\[
v'_{\rho}(\rho - 1) = -\frac{N - 2}{2} (\rho - 1)^{-1} + \frac{K'_{\nu}(\sqrt{a_{\rho}(\rho - 1)})}{K_{\nu}(\sqrt{a_{\rho}(\rho - 1)})}
\]
as \( \rho \to +\infty \).

**Proof.** We will use the same argument as in the proof of Lemma 3.2. Let \( \nu := (N - 2)/2 \). Define \( z_{\rho} \) by
\[
\begin{align*}
z_{\rho}(t) &= \left( \frac{t}{\sqrt{a_{\rho}}} \right)^{\nu} v_{\rho} \left( \frac{t}{\sqrt{a_{\rho}}} \right).
\end{align*}
\]
Then, \( z_{\rho} \) satisfies that
\[
\begin{align*}
& -t^2 z''_{\rho}(t) - t z'_{\rho}(t) + (t^2 + \nu^2) z_{\rho}(t) = 0, \quad \sqrt{a_{\rho}(\rho - 1)} \leq t < +\infty \\
& z_{\rho}(\sqrt{a_{\rho}(\rho - 1)}) = (\rho - 1)^{\nu}, \quad z_{\rho}(+\infty) = 0,
\end{align*}
\]
where \( z_{\rho}(+\infty) = 0 \) follows by the exponential decay of \( v_{\rho} \). In fact, by using the comparison principle for \( v_{\rho}(s) \) and \( w_{\rho}(s) := \exp(\sqrt{a_{\rho}(\rho - 1)}) \exp(-\sqrt{a_{\rho}}s) \), we have...
By the uniqueness of the solution of (16), there exists a unique $\alpha$ such that
\[ z_\rho(t) = \alpha K_\nu(t). \] (17)

Note that it holds that
\[ \alpha K_\nu(\sqrt{a_\rho}(\rho - 1)) = (\rho - 1)^\nu. \] (18)

By the elementary calculation, we have
\[ v'(\rho - 1) = -\nu(\rho - 1)^{-1} + (\rho - 1)^{-\nu} \sqrt{a_\rho} z'_\rho(\sqrt{a_\rho}(\rho - 1)). \] (19)

We will calculate the upper bound of $z'_\rho(\sqrt{a_\rho}(\rho - 1))$. By (17) and (18), we have
\[ z'_\rho(\sqrt{a_\rho}(\rho - 1)) = \alpha K'_\nu(\sqrt{a_\rho}(\rho - 1)) = (\rho - 1)^\nu K'_\nu(\sqrt{a_\rho}(\rho - 1)). \]

The above equality and (19) yield that
\[ v'(\rho - 1) = -\frac{N - 2}{2} (\rho - 1)^{-1} + \sqrt{a_\rho} \frac{K'_\nu(\sqrt{a_\rho}(\rho - 1))}{K_\nu(\sqrt{a_\rho}(\rho - 1))} \]
as $\rho \to +\infty$.

Now, we shall show the upper bound of $C_{\rho,r}(K)$.

**Proposition 7.** We assume that a function $K$ satisfies conditions (H1)-(H2) and a function $f$ satisfies conditions (f0)-(f5). Then, it follows that
\[ C_{\rho,r}(K) \leq c_*^M + e^{-\rho(\min\{2r,q+1\}+o(1))} \] as $\rho \to +\infty$.

**Proof of Proposition 7.** Assume that $r > 1$. Let $w \in H^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$ be a least energy solution to (3). Moreover, let $w_\rho \in H^1(B_{\rho r} \setminus B_{\rho - 1}) \cap C^2(\overline{B_{\rho r}} \setminus B_{\rho - 1})$ be a unique radially symmetric solution of the equation
\[
\begin{cases}
-\Delta w_\rho + a_\rho w_\rho = 0 & \text{in } B_{\rho r} \setminus B_{\rho - 1}, \\
w_\rho(\rho - 1) = w(\rho - 1), \ w_\rho(\rho r) = 0,
\end{cases}
\] (20)
where
\[ a_\rho := 1 - \frac{M f(w(\rho - 1))}{w(\rho - 1)}. \]

Note that $w_\rho(s) > 0$ for $\rho - 1 < s < \rho r$ by the strong maximum principle. Define $\overline{w_\rho}$ as
\[
\overline{w_\rho}(s) := \begin{cases}
w(s) & 0 \leq s \leq \rho - 1 \\
w_\rho(s) & \rho - 1 \leq s \leq \rho r.
\end{cases}
\] (21)

\[
C_{\rho,r}(K) = \inf_{v \in H^1_0(B_{\rho r})} \sup_{t > 0} J_{\rho,r}[tv; K] \leq \sup_{t > 0} J_{\rho,r}[t \overline{w_\rho}; K] = J_{\rho,r}[t \overline{w_\rho}; K],
\] (22)
where $t_\rho$ is a unique positive constant such that the last equality holds.

**Claim 1.** $t_\rho \leq 2$ for large $\rho$. 

Proof of Claim 1. Assume that there exists a subsequence \( \{ \rho_j \} \subset \{ \rho \} \) such that
\[
 t_j > 2, \quad \rho_j \to +\infty,
\]
where \( t_j := t_{\rho_j} \). Then, by the definition of \( t_j \), we may estimate
\[
 \int_{B_{r\rho_j}} |\nabla \omega_{\rho_j}|^2 + \omega_{\rho_j}^2 \, dy = \frac{1}{t_j} \int_{B_{r\rho_j}} K(\frac{y}{\rho_j})f(t_j \omega_{\rho_j}) \omega_{\rho_j} \, dy
\]
\[
 > \frac{M}{2} \int_{B_{r\rho_j-1}} f(2w) \, dy.
\]

Integrating by parts, we obtain
\[
 E_1 = \int_{B_{r\rho_j} \setminus B_{r\rho_j-1}} (\Delta w_{\rho_j} + w_{\rho_j}) w_{\rho_j} \, dy + \int_{\partial B_{r\rho_j-1}} \nabla w_{\rho_j} \cdot \frac{-y}{\rho_j} - 1 \, w_{\rho_j} \, dS
\]
\[
 \leq |B_{r\rho_j}|(1 - a_{\rho_j})w(\rho_j - 1)^2 - |\partial B_{r\rho_j-1}|w_{\rho_j}(\rho_j - 1)w(\rho_j - 1)
\]
\[
 \leq \exp(-2\rho_j(1 + o(1))",
\]
where the first inequality follows from \( w_{\rho_j}(y) \leq w(\rho_j - 1) \) and the second inequality follows from \( w(\rho_j - 1) \leq C \exp(-\rho_j + 1) \) and Lemma 3.2. Hence, letting \( j \to \infty \) in (23), we have
\[
 \int_{\mathbb{R}^N} |\nabla w|^2 + w^2 \, dy \geq \frac{M}{2} \int_{\mathbb{R}^N} f(2w) \, dy > M \int_{\mathbb{R}^N} f(w) \, dy.
\]
This is impossible since \( w \) is a solution to (3). \( \square \)

We return to the proof of the upper bound of \( C_{\rho,r}(K) \). By (21), we may decompose \( J_{\rho,r}(K) \) into the following three parts:
\[
 J_{\rho,r}[t_{\rho \omega_{\rho}}; K] = \frac{t_{\rho}^2}{2} \int_{B_{r\rho-1}} |\nabla w|^2 + w^2 \, dy - M \int_{B_{r\rho-1}} F(t_{\rho}w) \, dy
\]
\[
 + \frac{t_{\rho}^2}{2} \int_{B_{r\rho \setminus B_{r\rho-1}}} |\nabla w|^2 + w^2 \, dy - \int_{B_{r\rho \setminus B_{r\rho-1}}} K(\frac{y}{\rho})F(t_{\rho}w_{\rho}) \, dy
\]
\[
 = \left[ \frac{t_{\rho}^2}{2} \int_{B_{r\rho}} |\nabla w|^2 + w^2 \, dy - M \int_{B_{r\rho}} F(t_{\rho}w) \, dy \right]
\]
\[
 - \left[ \frac{t_{\rho}^2}{2} \int_{B_{r\rho \setminus B_{r\rho-1}}} |\nabla w|^2 + w^2 \, dy - M \int_{B_{r\rho \setminus B_{r\rho-1}}} F(t_{\rho}w) \, dy \right]
\]
\[
 + \left[ \frac{t_{\rho}^2}{2} \int_{B_{r\rho \setminus B_{r\rho-1}}} |w_{\rho}|^2 + w_{\rho}^2 \, dy - \int_{B_{r\rho \setminus B_{r\rho-1}}} K(\frac{y}{\rho})F(t_{\rho}w_{\rho}) \, dy \right]
\]
\[
 =: A - B + D.
\]
First, we shall estimate $A$. By Proposition 2 (ii), (iii), we may estimate

$$A = \frac{t_p^2}{2} \int_{B_{R_0}} |\nabla w|^2 + w^2 \, dy - M \int_{B_{R_0}} F(t_p w) \, dy$$

$$= \frac{t_p^2}{2} \int_{B_{R_0}} |\nabla w|^2 + w^2 \left[1 - 2Mw^2 \frac{F(t_p w)}{t_p^2 w^2} \right] \, dy$$

$$\leq \frac{t_p^2}{2} \int_{\mathbb{R}^N} |\nabla w|^2 + w^2 \left[1 - 2Mw^2 \frac{F(t_p w)}{t_p^2 w^2} \right] \, dy$$

$$= I[t_p w] \leq \sup_{t>0} I[tw] = I[w] = c_a.$$

Next, we shall estimate $B$. By integration by parts, we may estimate

$$-B = -\frac{t_p^2}{2} \int_{B_{R_0} \setminus B_{\rho-1}} |\nabla w|^2 + w^2 \, dy + M \int_{B_{R_0} \setminus B_{\rho-1}} F(t_p w) \, dy$$

$$= -\frac{t_p^2}{2} \int_{B_{R_0} \setminus B_{\rho-1}} (-\Delta w + w) w \, dy$$

$$- \frac{t_p^2}{2} \int_{\partial B_{R_0} \setminus B_{\rho-1}} \nabla w \cdot \frac{y}{|y|} w \, dS + \frac{t_p^2}{2} \int_{\partial B_{\rho-1}} \nabla w \cdot \frac{y}{|y|} w \, dS$$

$$+ M \int_{B_{R_0} \setminus B_{\rho-1}} F(t_p w) \, dy$$

$$= -\frac{t_p^2}{2} |\partial B_1| \rho^{N-1} w'(\rho) w(\rho) + \frac{t_p^2}{2} |\partial B_1| (\rho - 1)^{N-1} w'(\rho - 1) w(\rho - 1)$$

$$+ M \int_{B_{R_0} \setminus B_{\rho-1}} F(t_p w) \, dy$$

$$=: B_1 + B_2 + B_3.$$

By $\sup_{x \in \mathbb{R}^N} |\nabla w(x)| < +\infty$ and $w(r) \leq Ce^{-r}$, we obtain that

$$B_1 \leq e^{-\rho(R+o(1))}.$$

Next, we will estimate $B_2$. Let $v_\rho \in H^1(\mathbb{R}^N \setminus B_{\rho-1}) \cap C^2(\mathbb{R}^N \setminus B_{\rho-1})$ be a radial solution of the equation

$$\begin{cases}
-\Delta v_\rho + a_\rho v_\rho = 0 \quad \text{in } \mathbb{R}^N \setminus B_{\rho-1} \\
v_\rho(\rho - 1) = w(\rho - 1), \ v_\rho(+\infty) = 0.
\end{cases}$$

Then, we obtain that $w(x) \leq v_\rho(x)$, hence $w'(\rho - 1) \leq v_\rho'(\rho - 1)$. By Lemma 3.3, we have

$$B_2 \leq \frac{t_p^2}{2} |\partial B_1| (\rho - 1)^{N-1} w(\rho - 1)^2 \times \left[-\frac{N-2}{2} (\rho - 1)^{-1} + \sqrt{a_\rho} K_{\nu}'(\sqrt{a_\rho}(\rho - 1)) \frac{K_{\nu}(\sqrt{a_\rho}(\rho - 1))}{K_{\nu}(\sqrt{a_\rho}(\rho - 1))} \right].$$

Also, by Proposition 3 (iii), Claim 1 and (f5), it holds that

$$B_3 \leq e^{-\rho(q+1+o(1))}.$$
Finally, we shall estimate \( D \). By integration by parts, we obtain that
\[
D = \frac{t^2}{2} \int_{B_{r^2} \setminus B_{r^2-1}} |\nabla w|^2 + w^2 \, dy - \int_{B_{r^2} \setminus B_{r^2-1}} K(y)F(t_y w) \, dy
\]
\[
\leq \frac{t^2}{2} \int_{B_{r^2} \setminus B_{r^2-1}} |\nabla w|^2 + w^2 \, dy
\]
\[
= \frac{t^2}{2} \int_{B_{r^2} \setminus B_{r^2-1}} (-\Delta w + w)w \, dy - \frac{1}{2} \int_{\partial B_1} |(\rho - 1)^{N-1} w'(\rho - 1) |
\]
\[=: D_1 + D_2. \]

Since \( w \) is the solution to (20), we may estimate
\[
D_1 = \frac{t^2}{2} \int_{B_{r^2} \setminus B_{r^2-1}} (1 - a_\rho)w^2 \, dy \leq \frac{M^2}{2} |B_{r^2}| w(\rho - 1)f(w(\rho - 1)) \leq e^{\rho(q+1+o(1))},
\]
where the first inequality follows from \( w(y) \leq w(\rho - 1) \) and the second inequality follows from (f5) and Proposition 2 (iii). By Lemma 3.2, we have
\[
D_2 \leq \frac{t^2}{2} |\partial B_1| (\rho - 1)^{N-1} w(\rho - 1)^2 \times \left[ \frac{N-2}{2} (\rho - 1)^{-1} - \sqrt{\alpha_\rho} K'(\sqrt{\alpha_\rho}(\rho - 1)) + e^{-2\rho(r-1+o(1))} \right].
\]
Hence, we have
\[
C_{\rho,r}(K) \leq c_M + e^{-\rho(R+o(1))} + e^{-\rho(q+1+o(1))}
\]
\[+ \frac{t^2}{2} |\partial B_1| (\rho - 1)^{N-1} w(\rho - 1)^2 e^{-2\rho(r-1+o(1))}. \]

For \( R \) sufficiently large, it follows that
\[
C_{\rho,r}(K) \leq c_M + e^{-\rho(q+1+o(1))} + e^{-2\rho(r+o(1))}.
\]

The next corollary is convenient when we show Theorem 1.1.

**Corollary 1.** Let \( r > a > 0 \). We assume that \( \Omega = B(0; r), A = B(0; a) \). Moreover, we assume that a function \( K \) satisfies conditions (K1)-(K2) and a function \( f \) satisfies conditions (f0)-(f5). Then it follows that
\[
e_{\epsilon, \Omega}(K) \leq e^N \left[ c_M + e^{-\frac{1}{2}(\min(q+1)\alpha, 2r) + o(1))} \right] \quad \text{as} \ \epsilon \to 0.
\]

3.2. **Lower bound of \( C_{\rho,r}(K) \).** First, we note the following result which gives the information of the least energy solution when \( K \) and \( \Omega \) are radially symmetric.

**Proposition 8.** Let \( r > 0 \). We assume that a function \( K \) satisfies conditions (H1) and \( K = K^* \), and a function \( f \) satisfies conditions (f0)-(f4). Then for all \( \epsilon > 0 \) there exists a positive least energy solution \( u_\epsilon \in H_0^1(B(0; r)) \cap C^2(B(0; r)) \) of the equation
\[
\begin{cases}
-\epsilon^2 \Delta u + u = K(x) f(u), & \text{in} \ B(0; r), \\
u = 0, & \text{on} \ \partial B(0; r),
\end{cases}
\]
which satisfies \( u_\varepsilon^* = u_\varepsilon \). Moreover, passing to a subsequence, we obtain a positive least energy solution \( w \in H^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N) \) of (3) such that
\[
v_{\varepsilon_j} \to w \text{ in } H^1(\mathbb{R}^N) \cap C^1_{loc}(\mathbb{R}^N),
\]
where \( v_\varepsilon(y) := u_\varepsilon(\varepsilon y) \).

We can show Proposition 8 by combining the well known argument with the rearrangement technique. So we omit the proof of Proposition 8. For example, see [13].

**Lemma 3.4.** Let \( r > 1 \). Under the assumption of Proposition 8 and (H2), the following statement hold: For any \( \delta > 0 \) there exists a subsequence \( \{\varepsilon_j\} \subset \{\varepsilon\} \) such that
\[
u_{\varepsilon_j}(1 + \delta) = v_{\varepsilon_j}((1 + \delta)/\varepsilon_j) \geq e^{-\frac{1}{\varepsilon_j}(1+\delta+o(1))} \quad \text{as } j \to \infty.
\]

**Proof.** Take any \( R > 0 \). We write \( \rho := 1/\varepsilon \). Let \( z(r) \) be a solution of the equation
\[
\begin{align*}
z''(r) + \frac{N-1}{R} z'(r) - z(r) &= 0, \quad R \leq r \leq (1 + 2\delta)\rho, \\
z(0) &= 1, \quad z((1 + 2\delta)\rho) = 0.
\end{align*}
\]
Let \( \tilde{z}(r) \) a solution of the equaiton
\[
\begin{align*}
\tilde{z}''(r) + \frac{N-1}{R} \tilde{z}'(r) - \tilde{z}(r) &= 0, \quad 0 \leq r \leq (1 + 2\delta)\rho - R, \\
\tilde{z}(0) &= 1, \quad \tilde{z}((1 + 2\delta)\rho - R) = 0.
\end{align*}
\]
Note that \( z(r) = \tilde{z}(r - R)v_\varepsilon(R) \). Remarking \( z'(r) < 0 \), we may estimate
\[
-\Delta(z - v_\varepsilon)(y) + (z - v_\varepsilon)(y)
\leq -z''(|y|) - \frac{N-1}{|y|} z'(|y|) + z(|y|)
\leq -z''(|y|) - \frac{N-1}{R} z'(|y|) + z(|y|) = 0 \quad \text{for } |y| \in (R, (1 + 2\delta)\rho).
\]

By the weak maximum principle, we have \( z(r) \leq v_\varepsilon(r) \) for \( R \leq r \leq (1 + 2\delta)\rho \). In particular,
\[
\tilde{z}(r - R)v_\varepsilon(R) = z((1 + \delta)\rho) \leq v_\varepsilon((1 + \delta)\rho).
\]

**Step 1:** We show the estimate of \( \tilde{z}((1 + \delta)\rho - R) \).

Let \( \lambda_1(R) < 0 < \lambda_2(R) \) be solutions of the equation
\[
\lambda^2 + \frac{N-1}{R} \lambda - 1 = 0.
\]
Then, we see that \( \lambda_1(R) \to -1, \lambda_2(R) \to 1 \) as \( R \to \infty \). Moreover, it follows that
\[
\tilde{z}(r) = \alpha(\rho) \exp(\lambda_1(R)r) + \beta(\rho) \exp(\lambda_2(R)r),
\]
where \( \alpha(\rho) \) and \( \beta(\rho) \) satisfy
\[
\begin{align*}
1 &= \alpha(\rho) + \beta(\rho), \\
0 &= \alpha(\rho) \exp(\lambda_1(R)((1 + 2\delta)\rho - R)) + \beta(\rho) \exp(\lambda_2(R)((1 + 2\delta)\rho - R)).
\end{align*}
\]
Hence we have
\[
\tilde{z}((1 + \delta)\rho - R) = e^{\lambda_1(R)((1 + \delta)\rho - R)} \left[ \alpha(\rho) + \beta(\rho)e^{(\lambda_2(R) - \lambda_1(R))((1 + \delta)\rho - R)} \right].
\]
We will prove that $\alpha(\rho) \to 1$ as $\rho \to \infty$ and $\beta(\rho)e^{(\lambda_2-\lambda_1)((1+\delta)\rho-R)} \to 0$ as $\rho \to \infty$.

By the elementary calculation, we have

$$\beta(\rho)e^{(\lambda_2(R)-\lambda_1(R))((1+\delta)\rho-R)} = \frac{1}{e^{\lambda_1(R)\delta}} e^{(\lambda_2(R)-\lambda_1(R))\delta \rho}.$$ 

Hence we see that $\beta(\rho)e^{(\lambda_2-\lambda_1)((1+\delta)\rho-R)} \to 0$ as $\rho \to \infty$. Furthermore, by $\beta(\rho) \to 0$ as $\rho \to \infty$, we have $\alpha(\rho) \to 1$ as $\rho \to \infty$. By (29), for $\rho$ sufficiently large, it follows that

$$\exists((1+\delta)\rho-R) \geq \frac{1}{2}e^{-\rho\{-(\lambda_1(R)(1+\delta)-\frac{\beta}{\delta})\}}. \tag{30}$$

Step 2: We prove the estimate of $v_\varepsilon(R)$.

By Proposition 8, passing to a subsequence, we obtain that $v_{\varepsilon_j} \to w$ in $C^1_{\text{loc}}(\mathbb{R}^N)$. In particular, we may assume $v_{\varepsilon_j}(R) \to w(R)$ as $j \to \infty$. Then for $j$ sufficiently large, we have

$$v_{\varepsilon_j}(R) \geq \frac{1}{2}w(R) > 0. \tag{31}$$

(26), (30) and (31) yield that

$$v_{\varepsilon_j}((1+\delta)\rho_j) \geq \exp\left(-\rho_j \{-\lambda_1(R)(1+\delta) + o(1)\}\right) \quad \text{as } j \to \infty,$$

where $\rho_j := 1/\varepsilon_j$. By $\lambda_1(R) \to 1$ as $R \to \infty$, passing to a subsequence, we obtain that

$$v_{\varepsilon_j}((1+\delta)\rho_j) \geq \exp\left(-\rho_j \{(1+\delta) + o(1)\}\right) \quad \text{as } j \to \infty.$$ 

\hfill \qed

Proposition 9. Let $r > 1$. We assume that a function $K$ satisfies conditions (H1)-(H2) and $K = K^*$, and a function $f$ satisfies conditions (f0)-(f5). Then, passing to a subsequence, we obtain that

$$C_{\rho,r}(K) \geq c^M_* + e^{-\rho_j(\min\{q+1,2r\})+o(1)} \quad \text{as } j \to \infty.$$ 

Proof of Proposition 9. By Proposition 8, we obtain a positive radially symmetric least energy solution $v_\rho \in H^1_0(B_{r\rho})$ to the equation

$$\begin{cases}
-\Delta v_\rho + v_\rho = K(\frac{y}{\rho})f(v_\rho), & \text{in } B_{r\rho}, \\
v_\rho(r\rho) = 0, & v_\rho = v^*_\rho.
\end{cases}$$

For any $1/2 \leq t \leq 2$, we may estimate that

$$C_{\rho,r}(K) = J_{\rho,r}[v_\rho; K]$$

$$\geq \frac{t^2}{2} \int_{B_{r\rho}} |\nabla v_\rho|^2 + v_\rho^2 \, dy - \int_{B_{r\rho}} K(\frac{y}{\rho})F(tv_\rho) \, dy$$

$$= \frac{t^2}{2} \int_{B_{r\rho}} |\nabla v_\rho|^2 + v_\rho^2 \, dy - M \int_{B_{r\rho}} F(tv_\rho) \, dy + \int_{B_{r\rho}} \left[M - K(\frac{y}{\rho})\right] F(tv_\rho) \, dy$$

$$=: J_{\rho,r}[tv_\rho; M] + A.$$ 

Choose $t_\rho > 0$ such that $J_{\rho,r}[t_\rho v_\rho; M] = \sup_{t > 0} J_{\rho,r}[tv_\rho; M]$.

Claim 1. $t_\rho \to 1$ as $\rho \to +\infty$. 

Corollary 2. Let \( r > a > 0 \). Assume that \( \Omega = B(0; r), A = \bar{B}(0; a) \). Moreover, we assume that a function \( K \) satisfies conditions (H1) and \( K = K^* \), and a function \( f \) satisfies conditions (f0)-(f5). Then, passing to a subsequence, one deduces

\[
e_{\varepsilon, \Omega}(K) \geq \varepsilon^N \left[ c^M + e^{-\frac{1}{7}(\min(q+1,2r)+o(1))} \right] \quad \text{as} \; j \to +\infty.
\]
Consequently, (35), (38) and (39) yield that modify the argument of [7].

lower bound of $\varepsilon K$ since the problem which is considered in [7] has not the potential $\varepsilon$ reduce the energy show (i) and (ii) of Theorem 1.1 by (iii) of Theorem 1.1 easily as mentioned later.

4. Proof of Theorem 1.1. First, we prove (iii) of Theorem 1.1, and then we can show (i) and (ii) of Theorem 1.1 by (iii) of Theorem 1.1 easily as mentioned later.

We show (iii) of Theorem 1.1 by using the argument of [7]. Specifically, we reduce the energy $e_{\varepsilon, \Omega}(K)$ to the case $\Omega$ and $A$ are balls, and derive the upper and lower bound of $e_{\varepsilon, \Omega}(K)$ by using the inequalities which are proved in section 3. But, since the problem which is considered in [7] has not the potential $K$, we have to modify the argument of [7].
4.1. **Proof of the upper bound of** $e_{\epsilon,\Omega}(K)$. Choose $x_0 \in A$ so that

$$\min\{(q+1)d(x_0, \partial A), 2d(x_0, \partial \Omega)\} = \max_{x \in A} \min\{(q+1)d(x, \partial A), 2d(x, \partial \Omega)\}. \quad (40)$$

Let $r := d(x_0, \partial A)$ and $R := d(x_0, \partial \Omega)$. Choose a smooth function $\eta$ such that

$$\eta \equiv 0 \text{ in } \bar{B}(x_0; r), \quad 0 < \eta \leq \frac{m^2}{2} \text{ in } \mathbb{R}^N \setminus \bar{B}(x_0; r).$$

Define $\tilde{K}$ as

$$\tilde{K}(x) := K(x) - \eta(x), \quad x \in \Omega.$$

Remark that

$$\{x \in \Omega \mid \tilde{K}(x) = M\} = \bar{B}(x_0; r).$$

Since $\tilde{K}(x) \leq K(x)$ and $H_0^1(B(x_0; R)) \subset H_0^1(\Omega)$, we obtain that

$$e_{\epsilon,\Omega}(K) \leq \sup_{t > 0} I_{\epsilon,\Omega}(tu; K) \leq \sup_{t > 0} I_{\epsilon,B(x_0;R)}[tu; \tilde{K}],$$

for any $u \in H_0^1(B(x_0; R)) \setminus \{0\}$. Hence we have

$$e_{\epsilon,\Omega}(K) \leq e_{\epsilon,B(x_0;R)}(\tilde{K}).$$

By the translation and Corollary 1, we have

$$e_{\epsilon,B(x_0;R)}(\tilde{K}) \leq \epsilon^N \left[ c_\ast + e^{-\frac{1}{4}(\min((q+1)r,2R)+o(1))} \right], \quad (41)$$

as $\epsilon \to 0$. Therefore, we have proved the upper bound of $e_{\epsilon,\Omega}(K)$.

4.2. **Proof of the lower bound of** $e_{\epsilon,\Omega}(K)$. Our main idea to prove the lower bound of $e_{\epsilon,\Omega}(K)$ is to derive two inequalities. One represents effectiveness of the potential $K$ and the other represents effectiveness of the Dirichlet boundary condition. We prove the former in Proposition 10 and the latter in Proposition 11. After that, we prove the lower bound of $e_{\epsilon,\Omega}(K)$ by using them. While we can show Proposition 10 to use almost the same as the argument of [7], we need to modify the argument of [7] to prove Proposition 11.

First, we remark some important Lemmas to prove Propositions 10 and 11.

**Lemma 4.1.** Under the same assumption of Proposition 8. Let $u_\epsilon$ be a positive least energy solution of (1). Furthermore, let $x_\epsilon$ be a maximum point of $u_\epsilon$ and $x_1 := \lim_{\epsilon \to 0} x_\epsilon$. Then passing to a subsequence, we obtain

$$u_\epsilon(x) \leq e^{-\frac{1}{4}(|x-x_1|+o(1))},$$

$$|\nabla u_\epsilon(x)| \leq e^{-\frac{1}{4}(|x-x_1|+o(1))} \quad \text{for } x \in \Omega.$$

**Proof.** Let $v_\epsilon(y) := u_\epsilon(\epsilon y + x_\epsilon)$. Then $v_\epsilon$ satisfies $v_\epsilon \in C^2(\overline{\Omega_\epsilon})$ and

$$\begin{cases} -\Delta v_\epsilon(y) + v_\epsilon(y) = K(\epsilon y + x_\epsilon)f(v_\epsilon(y)), & y \in \Omega_\epsilon, \\ v_\epsilon(y) = 0, & y \in \partial \Omega_\epsilon, \quad v_\epsilon(y) > 0, & y \in \Omega_\epsilon, \end{cases} \quad (42)$$

where $\Omega_\epsilon := (1/\epsilon)(\Omega - \{x_\epsilon\})$. Moreover, by the zero extension, we obtain that

$$v_\epsilon \in H^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$$

and

$$-\Delta v_\epsilon(y) + v_\epsilon(y) = K(\epsilon y + x_\epsilon)f(v_\epsilon(y)), \quad y \in \mathbb{R}^N. \quad (43)$$

**Claim 1.** For any $\delta > 0$ there exist constants $\epsilon(\delta) > 0$ and $R(\delta) > 0$ which depend on $\delta$ such that

$$v_\epsilon(y) \leq \delta \quad \text{for } |y| \geq R(\delta), \quad 0 < \epsilon < \epsilon(\delta).$$


Proof of Claim 1. Fix any $\delta > 0$. By Proposition 2, we have $w(y) \leq Ce^{-|y|}$. Thus there exists $R(\delta) > 0$ such that $w(R(\delta)) \leq \delta/2$. Since $v_\varepsilon \to w$ in $C^2_{\text{loc}}(\mathbb{R}^N)$, it follows that there exists $\varepsilon(\delta) > 0$ such that

$$
\|v_\varepsilon - w\|_{C^2(B(0,R(\delta)))} \leq \frac{\delta}{2} \quad \text{for } 0 < \varepsilon < \varepsilon(\delta).
$$

Hence we have

$$
v_\varepsilon(R(\delta)) \leq \frac{\delta}{2} + w(R(\delta)) \leq \delta \quad \text{for } 0 < \varepsilon < \varepsilon(\delta).
$$

Since $y = 0$ is a maximum point of $v_\varepsilon(y)$, $v_\varepsilon(0) = 0$ for $y \in \mathbb{R}^N \setminus \Omega_\varepsilon$, and the local maximum point of $v_\varepsilon$ is unique, we obtain that

$$
v_\varepsilon(y) \leq \delta \quad \text{for } |y| \geq R(\delta), \ 0 < \varepsilon < \varepsilon(\delta).
$$

Claim 1 yields that for any $1 > \delta > 0$ there exist constants $R(\delta) > 0$ and $\varepsilon(\delta) > 0$ such that

$$
\max \left( v_\varepsilon(y), K(\varepsilon y + x_\varepsilon) \frac{f(v_\varepsilon(y))}{v_\varepsilon(y)} \right) \leq \delta \quad \text{for } |y| \geq R(\delta), \ 0 < \varepsilon < \varepsilon(\delta). \quad (44)
$$

Let $v$ be a solution of the equation

$$
\begin{cases}
v''(r) - (1 - \delta)v(r) = 0 & R(\delta) \leq r < +\infty, \\
v(R(\delta)) = 1, \ v(+\infty) = 0.
\end{cases}
$$

Then we may estimate that for $|y| \geq R(\delta)$,

$$
-\Delta(v_\varepsilon(y) - v(|y|)) + (1 - \delta)(v_\varepsilon(y) - v(|y|)) = K(\varepsilon y + x_\varepsilon)f(v_\varepsilon(y)) - \delta v_\varepsilon(y) + \Delta v(|y|) - (1 - \delta)v(|y|) \leq 0,
$$

where the inequality holds from (44) and $v'(|y|) \leq 0$. By the maximum principle and (44), we obtain that

$$
v_\varepsilon(y) \leq v(|y|) \quad \text{for } |y| \geq R(\delta).
$$

Now, by a simple calculation, we have

$$
v(|y|) = e^{-\sqrt{1-\delta}|y|-R(\delta)}.
$$

Hence, remarking that $\|v_\varepsilon\|_{L^\infty(\mathbb{R}^N)}$ is bounded uniformly over $\varepsilon$, we have

$$
v_\varepsilon(y) \leq C(\delta)e^{-\sqrt{1-\delta}|y|-R(\delta)} \quad \text{for } y \in \mathbb{R}^N, \ 0 < \varepsilon < \varepsilon(\delta). \quad (45)
$$

By the translation, it holds that

$$
u_\varepsilon(x) \leq C(\delta)e^{-\frac{1}{\varepsilon}(\sqrt{1-\delta}|x-x_\varepsilon|)} \quad \text{for } x \in \Omega, \ 0 < \varepsilon < \varepsilon(\delta).
$$

Since $\Omega$ is bounded, passing to a subsequence, we obtain that

$$
u_\varepsilon(x) \leq e^{-\frac{1}{\varepsilon}(|x-x_1|+o(1))} \quad \text{for } x \in \Omega.
$$

Next, we shall estimate $|\nabla u_\varepsilon(x)|$. We will use the following two Lemmas.
Lemma 4.2. [10, (3.17)] Let \( z \in \mathbb{R}^N \) and \( u \in H^1(B_1(z)) \) be a weak solution of
\[
-\Delta u = g, \quad \text{in } B_1(z),
\]
where \( g \in L^q(B_1(z)) \) for \( q > N \). Then, there exists a positive constant \( C \) which is independent of \( z \) such that
\[
\sup_{B_{3/4}(z)} |\nabla u|^2 \leq C \left( \|g\|_{L^q(B_1(z))}^2 + \|\nabla u\|_{L^2(B_1(z))}^2 \right).
\]

Lemma 4.3. Let \( z \in \mathbb{R}^N \) and \( u \in H^1(B(z;2)) \) be a weak solution of
\[
-\Delta u = g, \quad \text{in } B(z;2),
\]
where \( g \in L^q(B(z;2)) \) for \( q > N \). Then, there exists a positive constant \( C \) which is independent of \( z \) such that
\[
\int_{B(z;1)} |\nabla u(x)|^2 \, dx \leq C \left( \int_{B(z;2)} u(x)^2 \, dx + \int_{B(z;2)} |g(x)||u(x)| \, dx \right).
\]

Proof of Lemma 4.3. Choose a smooth function \( \eta \) such that
\[
\eta(x) \equiv 1 \text{ on } |x-z| \leq 1, \quad \eta(x) \equiv 0 \text{ on } |x-z| \geq 3/2, \quad 0 \leq \eta(x) \leq 1, \quad |\nabla \eta(x)| \leq C,
\]
where \( C \) is a constant which is independent of \( z \). Since \( u \) is a weak solution of \( -\Delta u = g \) in \( B(z;2) \), taking \( \eta^2 u \) as a test function, we have
\[
\int_{B(z;2)} \nabla u(x) \cdot \nabla (\eta^2 u(x)) \, dx = \int_{B(z;2)} g(x)\eta^2 u(x) \, dx. \tag{46}
\]
By \( \nabla (\eta^2 u(x)) = 2\nabla \eta \cdot \eta u(x) + \eta^2 \nabla u(x) \), we obtain that
\[
\int_{B(z;2)} \nabla u(x) \cdot \nabla (\eta^2 u(x)) \, dx \\
\geq -2C \int_{B(z;2)} \eta(x)|\nabla u(x)||u(x)| \, dx + \int_{B(z;2)} \eta(x)^2|\nabla u(x)|^2 \, dx \\
\geq -\frac{1}{2} \int_{B(z;2)} |\eta(x)|^2|\nabla u(x)|^2 \, dx - C \int_{B(z;2)} u(x)^2 \, dx + \int_{B(z;2)} \eta(x)^2|\nabla u(x)|^2 \, dx \\
\geq \frac{1}{2} \int_{B(z;1)} |\nabla u(x)|^2 \, dx - C \int_{B(z;2)} u(x)^2 \, dx. \tag{47}
\]
(46) and (47) yield that
\[
\int_{B(z;1)} |\nabla u(x)|^2 \, dx \leq C \left( \int_{B(z;2)} |g(x)||u(x)| \, dx + \int_{B(z;2)} u(x)^2 \, dx \right),
\]
where \( C \) is a positive constant which is independent of \( z \). \( \square \)

Now, since \( v_z \) satisfies
\[
-\Delta v_z(y) = K(\varepsilon y + x_z)f(v_z(y)) - v_z(y) =: g(y), \quad y \in \mathbb{R}^N,
\]
Lemma 4.2 yields that
\[
\sup_{y \in B(z;3/4)} |\nabla v_z(y)|^2 \leq C \left( \|\nabla v_z\|_{L^2(B(z;1))}^2 + \|g\|_{L^q(B(z;1))}^2 \right),
\]
where \( q := 2N/(\lfloor (N-2)/p \rfloor > N) \). Moreover, by Lemma 4.3, we have
\[
\sup_{y \in B(z;3/4)} |\nabla v_z(y)|^2 \leq C \left( \|v_z\|_{L^2(B(z;2))}^2 + \|g v_z\|_{L^1(B(z;2))} + \|g\|_{L^q(B(z;2))}^2 \right).
By (45), it holds that for any $\delta > 0$ there exists $\varepsilon(\delta) > 0$ such that

$$\sup_{y \in B(z;3/4)} |\nabla v_\varepsilon(y)|^2 \leq C(\delta) e^{-2\sqrt{1-\delta}|z|},$$

for $0 < \varepsilon < \varepsilon(\delta)$, $z \in \mathbb{R}^N \setminus \bar{B}(0;2)$. Remarking that $|\nabla v_\varepsilon|_{L^\infty(\mathbb{R}^N)}$ is bounded uniformly for $\varepsilon$, we obtain that

$$|\nabla v_\varepsilon(z)| \leq C(\delta) e^{-\sqrt{1-\delta}|z|}, \quad \text{for } 0 < \varepsilon < \varepsilon(\delta), \quad z \in \mathbb{R}^N.$$

Hence, since $\Omega$ is bounded, passing to a subsequence, we have

$$|\nabla u_\varepsilon(x)| \leq e^{-\frac{1}{\delta}|x-x_1|+o(1)} \quad \text{for } x \in \Omega.$$

\[\square\]

**Proposition 10.** Under the same assumption of Theorem 1.1, passing to a subsequence, we have

$$e_{\varepsilon_j,\Omega}(K) \geq \varepsilon_j^N \left[ c_\ast + \exp \left( -\frac{1}{\varepsilon_j} \{(q+1)d(x_1,\partial A) + o(1)\} \right) \right] \quad \text{as } j \to \infty,$$

where $x_{\varepsilon_j} \to x_1$.

We show Proposition 10 by using the modified argument of [7].

**Proof.** Let $d_0 := d(x_1, \partial A)$, and let $\delta$ be a fixed positive number. Moreover, let $\delta' := \delta/2$. Choose $d_0' \geq 0$ such that

$$|B(x_1; d_0)| = |A \cap B(x_1; d_0 + \delta')|.$$

Note that $d_0 + \delta' > d_0$. In fact, if $d_0 + \delta' = d_0'$, then we have $|B(x_1; d_0 + \delta') \setminus A| = 0$. On the other hand, since $B(x_1; d_0 + \delta') \setminus A \neq \emptyset$ and $B(x_1; d_0 + \delta') \setminus A$ is an open set, there exists an open ball $B$ such that $B(x_1; d_0 + \delta') \setminus A \subset B$. Hence, $0 < |B| \leq |B(x_1; d_0 + \delta') \setminus A| = 0$. This is impossible.

Choose a smooth function $\tilde{\eta}$ such that

$$\tilde{\eta} \equiv 0 \text{ on } \bar{B}(x_1; d_0 + \delta'), \quad 0 < \tilde{\eta} \leq \frac{m}{2} \text{ on } \mathbb{R}^N \setminus \bar{B}(x_1; d_0 + \delta').$$

Let

$$\bar{K}(x) := K(x) - \tilde{\eta}(x) \quad x \in \Omega.$$

Note that

$$\{x \in \bar{\Omega} \mid \bar{K}(x) = M\} = A \cap \bar{B}(x_1; d_0 + \delta').$$

For any $0 < t \leq 2$, we may estimate

$$e_{\varepsilon,\Omega}(K) = I_{\varepsilon,\Omega}[u_\varepsilon; K] \geq I_{\varepsilon,\Omega}[tu_\varepsilon; K]$$

$$= \frac{t^2}{2} \int_{\Omega} \varepsilon^2 |\nabla u_\varepsilon|^2 + u_\varepsilon^2 \ dx - \int_{\Omega} K(x) F(tu_\varepsilon) \ dx$$

$$= \frac{t^2}{2} \int_{\Omega} \varepsilon^2 |\nabla u_\varepsilon|^2 + u_\varepsilon^2 \ dx - \int_{\Omega} \bar{K}(x) F(tu_\varepsilon) \ dx - \int_{\Omega} \tilde{\eta}(x) F(tu_\varepsilon) \ dx$$

$$=: J_{\varepsilon,\Omega}[tu_\varepsilon; \bar{K}] + D.$$

Since

$$u_\varepsilon(x) \leq e^{-\frac{|x-x_1|+o(1)}{\varepsilon}} \quad \text{for } x \in \Omega,$$

we have

$$D \geq -e^{-\frac{|x-x_1|+o(1)}{\varepsilon}} (d_0 + \delta' + o(1)),$$
Choose $R > 0$ such that $\Omega^* = B(0; R)$. By Propositions 4-6, we obtain that

$J_{\varepsilon, \Omega^*}[tu^*_\varepsilon; K] \geq J_{\varepsilon, \Omega^*}[tu^*_\varepsilon; K^*].$

Choose $t_\varepsilon > 0$ such that

$$\sup_{t > 0} J_{\varepsilon, \Omega^*}[tu^*_\varepsilon; K^*] = J_{\varepsilon, \Omega^*}[t_\varepsilon u^*_\varepsilon; K^*].$$

**Claim 1.** $t_\varepsilon \leq 2$ when $\varepsilon$ is small enough.

**Proof.** Assume that there exists a subsequence $\{\varepsilon_j\} \subset \{\varepsilon\}$ such that

$t_{\varepsilon_j} > 2$, $\varepsilon_j \to 0.$

Then

$$P_j := \int_{\frac{1}{t_{\varepsilon_j}} \Omega^*} |\nabla (u^*_\varepsilon_j (\varepsilon_j y))|^2 + u^*_\varepsilon_j (\varepsilon_j y)^2 \, dy$$

$$= \frac{1}{t_{\varepsilon_j}} \int_{\frac{1}{t_{\varepsilon_j}} \Omega^*} \bar{K}^* (\varepsilon_j y) f(t_{\varepsilon_j} u^*_\varepsilon_j (\varepsilon_j y))u^*_\varepsilon_j (\varepsilon_j y) \, dy$$

$$> \frac{1}{2} \int_{\frac{1}{t_{\varepsilon_j}} \Omega} \bar{K} (\varepsilon_j y) f(2u_{\varepsilon_j} (\varepsilon_j y))u_{\varepsilon_j} (\varepsilon_j y) \, dy$$

$$= \frac{1}{2} \int_{\frac{1}{t_{\varepsilon_j}} (\Omega - \{x_{\varepsilon_j}\})} \bar{K}(\varepsilon_j z + x_{\varepsilon_j}) f(2u_{\varepsilon_j} (\varepsilon_j z + x_{\varepsilon_j}))u_{\varepsilon_j} (\varepsilon_j z + x_{\varepsilon_j}) \, dz$$

$$=: Q_j.$$

By Propositions 1, 4 and 6, we obtain that

$$P_j \leq \int_{\frac{1}{t_{\varepsilon_j}} (\Omega - \{x_{\varepsilon_j}\})} |\nabla u_{\varepsilon_j} (\varepsilon_j z + x_{\varepsilon_j})|^2 + u_{\varepsilon_j} (\varepsilon z + x_{\varepsilon_j})^2 \, dz$$

$$\to \int_{\mathbb{R}^N} |\nabla w(z)|^2 + w(z)^2 \, dz \quad \text{as} \quad j \to +\infty.$$

Let $R$ be a fixed positive number. Then, we may estimate

$$Q_j \geq \frac{1}{2} \int_{B(0; R)} \bar{K}(\varepsilon_j z + x_{\varepsilon_j}) f(2u_{\varepsilon_j} (\varepsilon_j z + x_{\varepsilon_j}))u_{\varepsilon_j} (\varepsilon_j z + x_{\varepsilon_j}) \, dz$$

$$\to \frac{M}{2} \int_{B_R(0)} f(2w(z))w(z) \, dz.$$

Hence, we have

$$\int_{\mathbb{R}^N} |\nabla w|^2 + w^2 \, dz \geq \frac{M}{2} \int_{B_R(0)} f(2w)w \, dz$$

for any $R > 0$. Letting $R \to +\infty$, we obtain that

$$\int_{\mathbb{R}^N} |\nabla w|^2 + w^2 \, dz \geq \frac{M}{2} \int_{\mathbb{R}^N} f(2w)w \, dz > M \int_{\mathbb{R}^N} f(w)w \, dz.$$

This is impossible since $w$ is a solution of (3). \qed

In summary,

$$e_{\varepsilon_j, \Omega^*}(K) \geq \inf_{u \in H^1_0(B(0; R)) \setminus \{0\}} \sup_{t > 0} I_{\varepsilon_j, \Omega^*}[tu; K^*] - e^{-\frac{2+1}{\varepsilon_j}(d_0 + \delta + o(1))}. \quad (48)$$
By $|A| < |\Omega|$, 
$$
|B(0; d_0)| \leq |A| < |\Omega| = |B(0; R)|,
$$
hence, $d_0 < R$. Now, we consider two cases to use Corollary 2 or Lemma 3.5.

**Case 1:** $d_0^* = 0$ for some $\delta > 0$.

In this case, since we can use Lemma 3.5, we have
$$
\inf_{u \in H^1_0(B(0; R)) \setminus \{0\}} \sup_{t > 0} J_{\varepsilon_{j, \Omega^*}}[tu; \bar{K}^*] \geq \varepsilon_j^N \left[ c_\ast + e^{\frac{q+1}{\varepsilon_j}(d_0 + o(1))} \right],
$$
for any $a > 0$. By (48), we have
$$
e_{\varepsilon_j, \Omega}(K) \geq \varepsilon_j^N \left[ c_\ast + e^{\frac{q+1}{\varepsilon_j}(a + o(1))} \right],
$$
(49) for any $a > 0$. On the other hand, by subsection 4.1, we have
$$
e_{\varepsilon_j, \Omega}(K) \leq e^N \left[ c_\ast + e^{\frac{q+1}{\varepsilon_j}(\max_{x \in A} \min \{2d(x, \partial \Omega), (q+1)d(x, \partial A)\}) + o(1)} \right].
$$
(50) Hence, by (49) and (50), we obtain that
$$
\max_{x \in A} \min \{2d(x, \partial \Omega), (q + 1)d(x, \partial A)\} \leq a,
$$
for any $a > 0$. This is impossible.

**Case 2:** $d_0^* > 0$ for any $\delta > 0$.

By Corollary 2, we have
$$
\inf_{u \in H^1_0(B(0; R)) \setminus \{0\}} \sup_{t > 0} J_{\varepsilon_{j, \Omega^*}}[tu; \bar{K}^*] \geq \varepsilon_j^N \left[ c_\ast + e^{\frac{q+1}{\varepsilon_j}(d_0 + o(1))} \right].
$$
In summary,
$$
e_{\varepsilon_j, \Omega}(K) \geq \varepsilon_j^N \left[ c_\ast + e^{\frac{q+1}{\varepsilon_j}(d_0 + o(1))} - e^{\frac{q+1}{\varepsilon_j}(d_0 + \delta')} - o(1) \right].
$$
By $d_0^* < d_0 + \delta' < d_0 + \delta$, we obtain that
$$
e_{\varepsilon_j, \Omega}(K) \geq \varepsilon_j^N \left[ c_\ast + e^{\frac{q+1}{\varepsilon_j}(d_0 + \delta + o(1))} \right].
$$
Passing to a subsequence, we have
$$
e_{\varepsilon_j, \Omega}(K) \geq \varepsilon_j^N \left[ c_\ast + e^{\frac{q+1}{\varepsilon_j}(d_0 + o(1))} \right].
$$
\[\square\]

**Proposition 11.** Under the same assumption of Theorem 1.1, passing to a subsequence, we obtain that
$$
e_{\varepsilon_j, \Omega}(K) \geq \varepsilon_j^N \left[ c_\ast + \exp \left( -\frac{1}{\varepsilon_j}(2d(x_1, \partial \Omega) + o(1)) \right) \right] \text{ as } j \to \infty,
$$
where $x_{\varepsilon_j} \to x_1$.

We show Proposition 11 by using almost the same as the argument of [7].
Proof. Remark that $x_1 \in A = \{x \in \Omega \mid K(x) = \max_{y \in \bar{\Omega}} K(y) = M\}$. Let $d_0 := d(x_1, \partial \Omega), d_x := d(x_x, \partial \Omega)$ and $\delta$ be a fixed positive number. Choose $d'_0 \geq 0$ such that
\[ |B(x_1; d'_0)| = |\Omega \cap B(x_1; d_0 + \delta)|, \]
and $d'_x \geq 0$ such that
\[ |B(x_x; d'_x)| = |\Omega \cap B(x_x; d_x + \delta)|. \]
Note that $d_0 + \delta > d'_0$, and $d'_x \to d'_0$. Moreover, since $\Omega$ is smooth, it follows that $d'_0 > 0$. Choose $\delta > \delta' > 0$ so that $d_0 + \delta' > d'_0$. Take a smooth function $\zeta_\epsilon$ such that
\[ \zeta_\epsilon \equiv 1 \text{ on } \bar{B}(x_x; d_x + \delta'), \quad \zeta_\epsilon \equiv 0 \text{ on } \mathbb{R}^N \setminus B(x_x; d_x + \delta), \quad |\nabla \zeta_\epsilon| \leq C, \]
where $C$ is a positive constant which is independent of $\epsilon$. Let
\[ \bar{u}_\epsilon(x) := u_\epsilon(x) \zeta_\epsilon(x). \]
For any $0 < t \leq 2$, we may estimate
\[
e_{\epsilon, \Omega}(K) = I_{\epsilon, \Omega}[u_\epsilon; K] \geq I_{\epsilon, \Omega}[\bar{u}_\epsilon; K] \]
\[
\geq \frac{t^2}{2} \int \int_{\Omega} \varepsilon^2 |\nabla u_\epsilon|^2 + u_\epsilon^2 d x - M \int \int_{\Omega} F(tu_\epsilon) d x \]
\[
= \frac{t^2}{2} \int \int_{\Omega} \varepsilon^2 |\nabla \bar{u}_\epsilon|^2 + \bar{u}_\epsilon^2 d x - M \int \int_{\Omega} F(t\bar{u}_\epsilon) d x \]
\[
+ \frac{t^2}{2} \int \int_{\Omega} \varepsilon^2 (|\nabla u_\epsilon|^2 - |\nabla \bar{u}_\epsilon|^2) + (1 - \zeta_\epsilon^2) \bar{u}_\epsilon^2 d x - M \int \int_{\Omega} (1 - \zeta_\epsilon^{p+1}) F(t\bar{u}_\epsilon) d x \]
\[
=: I_{\epsilon, \Omega}[\bar{u}_\epsilon; M] + F + G. \]
First, we will estimate $F$. By
\[ |\nabla \bar{u}_\epsilon|^2 = |\nabla u_\epsilon \cdot \zeta_\epsilon + u_\epsilon \cdot \nabla \zeta_\epsilon|^2 \leq C \left(|\nabla u_\epsilon|^2 + u_\epsilon^2\right), \]
where $C$ is a positive constant which is independent of $\epsilon$, we have
\[ F \geq -C \int \int_{\Omega \setminus B(x_x; d_x + \delta)} |\nabla u_\epsilon|^2 + u_\epsilon^2 d x. \]
Since
\[ |\nabla u_\epsilon(x)| \leq e^{-\frac{|x-x_\epsilon|+o(1)}{\varepsilon}}, \quad u_\epsilon(x) \leq e^{-\frac{|x-x_\epsilon|+o(1)}{\varepsilon}}, \]
we have
\[ F \geq -e^{-\frac{5}{2}(d_0 + \delta' + o(1))}. \]
Similarly,
\[ G \geq -e^{-\frac{5}{2}(d_0 + \delta' + o(1))} \geq -e^{-\frac{5}{2}(d_0 + \delta' + o(1))}. \]
Next, we will estimate $I_{\epsilon, \Omega}[\bar{u}_\epsilon; M]$.
\[ I_{\epsilon, \Omega}[\bar{u}_\epsilon; M] \geq \frac{t^2}{2} \int \int_{B(0; d'_x)} \varepsilon^2 |\nabla \bar{u}_\epsilon|^2 + (\bar{u}_\epsilon)^2 d x - M \int \int_{B(0; d'_x)} F(t\bar{u}_\epsilon) d x \]
\[ = I_{\epsilon, B(0; d'_x)}[\bar{u}_\epsilon; M]. \]
Choose $\bar{t}_\epsilon > 0$ such that
\[ \sup_{t \geq 0} I_{\epsilon, B(0; d'_x)}[\bar{u}_\epsilon; M] = I_{\epsilon, B(0; d'_x)}[\bar{t}_\epsilon \bar{u}_\epsilon^*; M]. \]
Claim 2. $\bar{t}_\epsilon \leq 2$ as long as $\epsilon$ is slightly larger than 0.
Proof. Assume that there exists a subsequence \( \{ \varepsilon_j \} \subset \{ \varepsilon \} \) such that

\[
I_{\varepsilon_j} > 2, \quad \varepsilon_j \to 0.
\]

Then we may estimate

\[
P_j := \int_{B(x, d_j)} \varepsilon_j^2 |\nabla (\tilde{u}_{\varepsilon_j})^*|^2 + ((\tilde{u}_{\varepsilon_j})^*)^2 \, dx = \frac{M}{I_{\varepsilon_j}} \int_{B(0; d_j')} f(\tilde{u}_{\varepsilon_j}^* \tilde{u}_{\varepsilon_j}^* \tilde{u}_{\varepsilon_j}) \, dx
\]

\[
> \frac{M}{2} \int_{B(0; d_j')} f(2(\tilde{u}_{\varepsilon_j})^*) \tilde{u}_{\varepsilon_j} \, dx =: Q_j.
\]

\[
P_j \leq \varepsilon_j^2 \int_{\Omega} |\nabla u_{\varepsilon_j}|^2 + u_{\varepsilon_j}^2 \, dx + o(\varepsilon_j^N)
\]

On the other hand,

\[
Q_j = \frac{M}{2} \int_{\Omega \cap B(x_j, d_j + \delta)} f(2u_{\varepsilon_j}) u_{\varepsilon_j} \, dx
great \geq \frac{M}{2} \int_{B(x_j; d_j)} f(2u_{\varepsilon_j}) u_{\varepsilon_j} \, dx = \frac{\varepsilon_j^N M}{2} \int_{B(0; \varepsilon_j)} f(2u_{\varepsilon_j}) v_{\varepsilon_j} \, dy.
\]

By Proposition 2 (iii) and Proposition 1 (i) and (iv), we have

\[
\int_{\mathbb{R}^n} |\nabla w|^2 + w^2 \, dy \geq \frac{M}{2} \int_{\mathbb{R}^n} f(2w) w \, dy > M \int_{\mathbb{R}^n} f(w) w \, dy.
\]

This is impossible since \( w \) is a solution to (3).

By [7, Lemma 2.1], we may estimate

\[
\sup_{t > 0} I_{t \varepsilon, B(0; d_j')} [tu_{\varepsilon_j}^*] \geq \inf_{u \in H^1_0(B(0; d_j'))} \sup_{t > 0} I_{t \varepsilon, B(0; d_j')} [tu_{\varepsilon_j}^*] \geq \varepsilon_j^N \left[ c_s + e^{-\frac{\pi}{2} (d_0 + o(1))} \right]
\]

In summary, it follows that

\[
e_{\varepsilon, \Omega}(K) \geq \varepsilon_j^N \left[ c_s + e^{-\frac{\pi}{2} (d_0 + o(1))} - e^{-\frac{\pi}{2} (d_0 + \delta + o(1))} \right]
\]

By \( d_0' < d_0 + \delta' < d_0 + \delta \), we have

\[
e_{\varepsilon, \Omega}(K) \geq \varepsilon_j^N \left[ c_s + e^{-\frac{\pi}{2} (d_0 + o(1))} - e^{-\frac{\pi}{2} (d_0 + \delta + o(1))} \right]
\]

Passing to a subsequence, we obtain

\[
e_{\varepsilon, \Omega}(K) \geq \varepsilon_j^N \left[ c_s + e^{-\frac{\pi}{2} (d_0 + o(1))} \right].
\]

By Propositions 10 and 11, passing to a subsequence, we obtain that

\[
e_{\varepsilon_j, \Omega}(K) \geq \varepsilon_j^N \left[ c_s + \exp \left( -\frac{1}{\varepsilon_j} (D(x_{\varepsilon_j}) + o(1)) \right) \right] \text{ as } j \to \infty.
\]

Therefore, we have proved the lower bound of \( e_{\varepsilon, \Omega}(K) \).
4.3. **Proof of (i) and (ii).** We can prove Theorem 1.1 (i) and (ii) by Theorem 1.1 (iii) easily. So, we omit details.

5. **Example.** In this section, we will give some examples for Theorem 1.1.

**Example 1.** Let \( \Omega = B(0; 1) \) and \( A = \bar{B}(0; r) \) for \( 0 < r < 1 \). Then we have

\[
\max_{x \in A} D(x) = \min \{(q+1)r, 2\}.
\]

Moreover, by \( \{x \in A \mid D(x) = \max_{y \in A} D(y)\} = \{0\} \), Theorem 1.1 yields that the least energy solution \( u_\epsilon \) concentrates at \( x = 0 \).

**Example 2.** Let \( \Omega = B(0; 1) \) and \( A = \bar{B}(0; R) \setminus B(0; r) \) for \( 0 < r < R < 1 \). Let

\[
P := \{x \in A \mid (q + 1)d(x, \partial A) \leq 2d(x, \partial \Omega)\}
\] and

\[
Q := A \setminus P^o.
\]

Then the following statements hold:

(i) If \( A = P \), we have

\[
\{x \in A \mid D(x) = \max_{y \in A} D(y)\} = \{x \in A \mid 2|x| = r + R\}.
\] (51)

Hence, Theorem 1.1 yields that the least energy solution \( u_\epsilon \) concentrates on the point \( x_0 \) which satisfies \( 2|x_0| = r + R \).

(ii) If \( Q^c \neq \emptyset \), we have

\[
\{x \in A \mid D(x) = \max_{y \in A} D(y)\} = \{x \in A \mid 2(1 - |x|) = (q + 1)(|x| - r)\}.
\] (52)

Hence, Theorem 1.1 yields that the least energy solution \( u_\epsilon \) concentrates on the point \( x_0 \) which satisfies \( 2(1 - |x_0|) = (q + 1)(|x_0| - r) \).

Figure 1. Example 2 (i)

**Acknowledgments.** I got many insightful comments and advice for this research from Professor Kazuhiro Kurata. Without his help this paper would not have been written. I am deeply grateful to him. Allow me to use this opportunity to offer my thanks to him. Also, I thank the referee for careful reading my manuscript and for giving useful comments.
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Received April 2016; revised August 2016.

E-mail address: kodamashun0119@gmail.com