A bound on the irregularity of abelian scrolls in projective space

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Dedicated to Professor Hans Grauert

Abstract. We prove that the irregularity of a smooth abelian scroll whose dimension is at least half of that of the surrounding projective space is bounded by 2. We also discuss some existence results and open problems.

0 Introduction

Let $A \subset \mathbb{P}^{l-1}$ be an $n$-dimensional abelian variety and let $G \subset A$ be a subgroup of order $k$. Then one can define for every point $P \in A$ the linear space

$$S^G(P) := \text{Span}(P + \rho; \rho \in G).$$

In general one expects $S^G(P)$ to have dimension $k-1$ and the union of these spaces

$$Y = \bigcup_{P \in A} S^G(P)$$

is then a (possibly singular) scroll of expected dimension $n + k - 1$. The classical example of this construction is the case where $E$ is an elliptic quintic normal curve in $\mathbb{P}^4$ and $G = \mathbb{Z}_2$. This leads to the quintic elliptic scroll, the only smooth irregular scroll in $\mathbb{P}^4$.

We shall refer to a smooth scroll $Y$ constructed as above and having the expected dimension as the abelian scroll determined by the pair $(A, G)$. In particular, if $\rho \in A_{(k)}$ is a point of order $k$, we can consider the cyclic subgroup $G \simeq \mathbb{Z}_k$ generated by $\rho$. We call the corresponding abelian scroll the cyclic scroll determined by the pair $(A, \rho)$.

Observe that for smooth abelian scrolls $Y$ we have $2\dim Y \leq l - 1$. This follows immediately from Barth’s theorem which says that any smooth subvariety $Y \subset \mathbb{P}^{l-1}$ with $2\dim Y > l - 1$ has irregularity $q(Y) = 0$. The main result of this paper is the following:

**Theorem 0.1** Let $Y \subset \mathbb{P}^{l-1}$ be a smooth abelian scroll with $2\dim Y = l - 1$. Then $q(Y) \leq 2$, i.e. $A$ is either an elliptic curve or an abelian surface. Moreover, in this case $Y$ (or equivalently $A$) is linearly normally embedded.

This non-existence result can be seen as an extension of a result of Van de Ven [V] which says that an abelian variety $A$ of dimension $n$ can be embedded in $\mathbb{P}^{2n}$ only if $n = 1$ or 2. We shall briefly discuss the existence of abelian scrolls and some open problems.
As we indicate at the end of section 2, the question considered here is related to the general interesting problem of classifying smooth projective varieties whose dimension equals their codimension, a problem motivated by questions of Griffiths and van de Ven concerning smooth surfaces in $\mathbb{P}^4$.

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1 Non-existence of scrolls

This section is devoted to the proof of the non-existence theorem (0.1). The main ingredients in the proof are the double point formula and some estimates involving binomial coefficients.

Before we can prove our result we need some preparations. Consider the étale $k : 1$ quotient

$$\tilde{\pi} : A \to \tilde{A} := A/G$$

and let $\mathcal{L}$ be the line bundle on $A$ which defines the embedding into $\mathbb{P}^{d-1}$. Then

$$\mathcal{E} := \tilde{\pi}_* \mathcal{L}$$

is a rank $k$ vector bundle on $\tilde{A}$ and we set

$$X := \mathbb{P}(\mathcal{E}).$$

The natural map $\tilde{\pi}^* \mathcal{E} \to \mathcal{L}$ induces an inclusion of $A$ into $X$ as a multi-section and the linear system $|\mathcal{O}_X(1)| = |\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$ induces the complete linear system $|\mathcal{L}|$ on $A$ and maps $X$ onto a scroll $Z$ (see [CH, Lemma 2.1]).

If $Y$ is embedded by a complete linear system, then $Z = Y$, otherwise $Y$ is a projection of $Z$. Also as in [CH, p. 360] we have

$$\tilde{\pi}_* \mathcal{E} = \mathcal{L} \oplus t_{\rho_1}^* \mathcal{L} \oplus \ldots \oplus t_{\rho_{k-1}}^* \mathcal{L}$$

where $G = \{1, \rho_1, \ldots, \rho_{k-1}\}$. Note that topologically $\tilde{X} := \mathbb{P}(\tilde{\pi}^* \mathcal{E}) \cong A \times \mathbb{P}^{k-1}$ is trivial. We can also choose another covering $\tilde{\pi} : \tilde{A} \to A$ of order $h$ where $h$ divides $k$ such that $\tilde{X} := \mathbb{P}(\tilde{\pi}^* \tilde{\pi}^* \mathcal{E}) \cong A \times \mathbb{P}^{k-1}$ as algebraic varieties.

**Proof of theorem 0.1** Let $m = \dim Y = n + k - 1$. Then the double point formula (see [E, Theorem (9.9.3)]) reads

$$[D] = \varphi^* \varphi_* [X] - c_m \left(N_{\phi} \cap [X] \right)$$
where $\varphi : X \to Y$ is the map given by $|O_X(1)|$, possibly followed by a projection. We first determine the contribution $\varphi^*\varphi_*[X]$. We have already observed that topologically $\tilde{X}$ is a product and we can write for the hyperplane section on $\tilde{X}$:

$$\tilde{H} = c + h$$

where $c = c_1(\mathcal{L}) \in H^2(A, \mathbb{Z})$ and $h$ is the positive generator of $H^2(\mathbb{P}^{k-1}, \mathbb{Z})$. Hence

$$\tilde{H}^m = \tilde{H}^{n+k-1} = (c + h)^{n+k-1} = \binom{n+k-1}{k-1} c^n.$$

Since $\tilde{X} \to X$ has degree $k$ we find that

$$\varphi^*\varphi_*[X] = \frac{1}{k^2} \binom{n+k-1}{k-1}^2 (c^n)^2.$$

Instead of the normal bundle of $\varphi$ we compute the normal bundle of $\tilde{\varphi} := \varphi \circ f$ where $f : \tilde{X} \to X$ is the étale $k : 1$ map induced from $\tilde{\pi} : A \to \tilde{A}$. From the exact sequence

$$0 \to T_{\tilde{X}} \to \tilde{\varphi}^*T_{\mathbb{P}^{k-1}} \to N_{\tilde{\varphi}} \to 0$$

we find

$$c(N_{\tilde{\varphi}}) = c(\tilde{\varphi}^*T_{\mathbb{P}^{k-1}})c(T_{\tilde{X}})^{-1}$$

$$= (1 + c + h)^l (1 + h)^{-k}$$

where we have again used that topologically $\tilde{X} = A \times \mathbb{P}^{k-1}$. Now we notice that $c^{n+1} = h^k = 0$ and, therefore, the top term of $c(N_{\tilde{\varphi}})$ is just the coefficient of the monomial $c^n h^{k-1}$, which is the same as the coefficient of the same monomial in

$$\binom{l}{n} c^n (1 + h)^{l-n} (1 + h)^{-k} = \binom{l}{n} c^n (1 + h)^{l-n-k}.$$

Since $l = 2n + 2k - 1$ we get

$$c_m(N_{\tilde{\varphi}}) = c^n \binom{n+k-1}{n} \binom{2n+2k-1}{n}.$$

Altogether we find for the class of the double locus $\mathbb{D}$ that

$$[\mathbb{D}] = \frac{1}{k} c^n \left[ \frac{1}{k} \binom{n+k-1}{k-1}^2 c^n - \binom{n+k-1}{n} \binom{2n+2k-1}{n} \right].$$

Using Riemann-Roch on the abelian variety $A$ we find

$$c^n \geq n!(2n + 2k - 1)$$

with equality if and only if the linear system is complete. Then the assertion of the theorem follows if one shows that

$$\binom{n+k-1}{k-1} (2n+2k-1)n! \geq k \binom{2n+2k-1}{n}.$$
where equality holds if and only if \( n = 1 \) or \( 2 \). Indeed, it is a straightforward calculation to check equality for \( n = 1 \) and \( n = 2 \). Now assume \( n \geq 3 \). Evaluating the binomial coefficients and cancelling terms, the above inequality becomes

\[
\frac{n!}{n} \prod_{l=2}^{n} (n + k - l + 1) \geq \prod_{l=2}^{n} (2n + 2k - l)
\]

which can be rewritten as

\[
\prod_{l=2}^{n} (n + k - l + 1) l \geq \prod_{l=2}^{n} (2n + 2k - l).
\]

To check this inequality it is enough to check it termwise. The inequality

\[
(n + k - l + 1) l \geq 2n + 2k - l \quad \text{for} \quad l = 2, \ldots, n
\]

is equivalent to

\[
n + k \geq l \quad \text{for} \quad l = 2, \ldots, n
\]

which is trivially true. ✷

2 Existence of scrolls

In this section we shall discuss conditions under which a linearly normal abelian scroll \( Y \) determined by a pair \((A, G)\) is smooth. We shall keep the notation as in section 1 and we shall start with the scroll \( X = \mathbb{P}(E) \). Recall that \( A \) is contained in \( X \) as a \( k \)-section and that \( \mathcal{O}_X(1)|_A \cong \mathcal{L} \), essentially by the definition of \( \mathcal{O}_X(1) \). We shall always assume that \( \mathcal{L} \) is very ample on \( A \). The line bundle \( \mathcal{L} \) is called \((m-1)\)-very ample, if for every cluster (i.e. 0-dimensional subscheme) \( \zeta \subset A \) of length \( \leq m \) the restriction map \( H^0(A, \mathcal{L}) \to H^0(\zeta, \mathcal{L} \otimes \mathcal{O}_\zeta) \) is surjective and hence \( \varphi|_{\zeta}(\zeta) \) spans a \( \mathbb{P}^{m-1} \).

**Proposition 2.1**

(i) If \( \mathcal{L} \) is \((k-1)\)-very ample, then \( |\mathcal{O}_X(1)| \) is base point free and \( \varphi : X \to Y \) is a finite map.

(ii) If \( \mathcal{L} \) is \( k \)-ample, then \( \varphi : X \to Y \) is birational and an embedding near \( A \).

**Proof.** (i) \( A \) intersects each fibre of \( X \) in \( k \) independent points. This follows since by construction of \( X \) these \( k \) points are the image of points in \( \tilde{X} \) given by the sections which come from the splitting of \( \tilde{\pi}^*E \). Since these points are mapped to independent points in \( \mathbb{P}^{l-1} \) it follows that \( |\mathcal{O}_X(1)| \) restricted to a fibre of \( X \) cannot have base points. In order to show that \( \varphi \) is finite, it is enough to prove that \( \mathcal{O}_X(1) \) has positive degree on each curve \( C \) in \( X \). This can be checked by pulling back to the product \( \tilde{X} = \tilde{A} \times \mathbb{P}^{k-1} \) where we have a product polarization.
(ii) It follows immediately from our assumption that every $\mathbb{P}^{k-1}$ of the scroll $Y$ meets $A$ only in the $k$ points which span it. Hence for every point $P \in A$ we have that $\varphi^{-1}(P)$ consists of only one point. Similarly we prove that $d\varphi$ is injective along $A$. Now assume that $\varphi : X \to Y$ has degree $d \geq 2$. Over each point of $A \subset Y$ we have only one preimage. Hence $\varphi$ is ramified along $A$, contradicting what we have just proved.

Proposition 2.2 If $\mathcal{L}$ is $(2k-1)$-very ample, then $\varphi : X \to Y$ is an isomorphism and, in particular, $Y$ is smooth.

Proof. The assumption gives immediately that, for two points $P$ and $Q$ whose difference on $A$ lies in the group $G$, we have $S^{k-1}(P) \cap S^{k-1}(Q) = \emptyset$. Hence the map $\varphi$ is injective. It remains to prove that the differential $d\varphi$ is injective everywhere. The map $\varphi : X \to Y \subset \mathbb{P}^{l-1}$ embeds every fibre $\pi_x$ of $\pi : X \to A$ through a point $x \in X$ as a $(k-1)$-dimensional subspace which, by abuse of notation, we also denote by $\pi_x$. We consider the map

$$\gamma : A \to G(k-1, l-1) =: \text{Gr}$$

$$x \mapsto \pi_x,$$

where $\pi_x$ is the unique fibre of $X$ containing the point $x$. This map factors through $A$. Taking the projectivised differential of this map gives us a map

$$\omega = \mathbb{P}(d\gamma) : \mathbb{P}(T_{A,x}) \to \mathbb{P}(T_{\text{Gr}, \pi_x}) = \mathbb{P}(\pi_x, \pi^0)$$

$$t \mapsto \omega_t,$$

where $\pi^0$ is a complementary subspace of $\pi_x$ and $\text{Pr}(\pi_x, \pi^0)$ is the projective space of projective transformations of $\pi_x$ to $\pi^0$. (If $\pi_x = \mathbb{P}(U)$ and $\pi^0 = \mathbb{P}(W)$ then $\text{Pr}(\pi_x, \pi^0) = \mathbb{P}(\text{Hom}(U, W))$.) Here the identification $\mathbb{P}(T_{\text{Gr}, \pi_x}) = \text{Pr}(\pi_x, \pi^0)$ comes from the fact that there is a canonical isomorphism $T_{\text{Gr}} = \text{Hom}(\mathcal{U}, \mathcal{V})$ where $\mathcal{U}$ and $\mathcal{V}$ are the canonical sub bundle resp. the quotient bundle. We also have the Gauss map

$$\Gamma : X \to G(n+k-1, l-1)$$

$$P \mapsto T_{X,P},$$

where $T_{X,P}$ is the projective closure of the image of the differential of $\varphi$ at $P$, considered as a subspace through the point $P$. The relation between $\gamma$ and $\Gamma$ is the following. Let $P \in X$ and let $x \in \bar{\pi}^{-1}(\pi(P))$. Then

$$T_{X,P} = \langle \pi_x, \omega_t(P) ; \ t \in \mathbb{P}(T_{A,x}) \rangle.$$

In order to check that $d\varphi$ is injective at $P$ we have to prove that the projective map

$$\omega(P) : \mathbb{P}(T_{A,x}) \to \pi^0$$

$$t \mapsto \omega_t(P)$$

is...
is injective, i.e. is well defined. If this is not the case, then we have a tangent direction \( t \in \mathbb{P}(T_{A,x}) \) such that the linear map associated to the map
\[
\omega_t : \pi \xrightarrow{\omega} \pi^0
\]
has a kernel. Assume that this is the case. Consider the germ of a holomorphic curve \((\mathbb{C}, 0) \to (A, x)\) which represents the tangent direction \( t \) at \( x \). This determines a family of \((\mathbb{P}^{k-1})'s\) given by
\[
\pi_s = \langle z_0(s), \ldots, z_{k-1}(s) \rangle.
\]
Choosing suitable coordinates in \( \mathbb{P}^{l-1} \) we can assume that
\[
z_i(0) = e_i; \quad i = 0, \ldots, k - 1
\]
with \( e_0, \ldots, e_{l-1} \) the standard basis. For the complementary space \( \pi^0 \) we can choose
\[
\pi^0 = \langle e_k, \ldots, e_{l-1} \rangle.
\]
It is an easy local computation to check that with respect to the coordinates on \( \pi \) and \( \pi^0 \) given by \( e_0, \ldots, e_{k-1} \) and \( e_k, \ldots, e_{l-1} \) the linear map associated to the map
\[
\omega_0 : \pi \to \pi^0
\]
is given by the matrix
\[
M = \begin{pmatrix}
z'_0 \kappa & z'_{1k} & \cdots & z'_{k-1,k} \\
\vdots & \ddots & & \vdots \\
z'_{0,l-1} & z'_{1,l-1} & \cdots & z'_{k-1,l-1}
\end{pmatrix}
\]
(\( s = 0 \)).

Hence \( \omega_0 \) has a kernel if and only if rank \( M < k \). But this is equivalent to the assertion that \( z_0(0), \ldots, z_{k-1}(0), z'_0(0), \ldots, z'_{k-1}(0) \) are dependent. In particular there is a cluster of length \( 2k \) on \( A \) which contradicts \((2k - 1)\)-very ampleness of \( L \).

We can apply this result immediately to elliptic normal curves \( E \subset \mathbb{P}^{m-1} \). Recall that any \( m - 1 \) points on \( E \) (possibly infinitely near) are linearly independent (otherwise we could find a hyperplane through these \( m - 1 \) points and any other point on \( E \)).

**Corollary 2.3** Let \( m = 2n + 1 \) be an odd integer. Then there exists an \( n \)-dimensional elliptic curve scroll \( Y \) in \( \mathbb{P}^{2n} \).

**Proof.** Let \( E \subset \mathbb{P}^{m-1} = \mathbb{P}^{2n} \) be an elliptic normal curve of degree \( m \) and choose a subgroup \( G \) of order \( n \) of \( E \), e.g. a cyclic subgroup. We want to prove that the abelian scroll \( Y \) determined by the pair \( (E, G) \) is smooth. This follows from Proposition 2.2 since the embedding line bundle \( L \) is \((m - 2 = 2n - 1)\)-very ample. \( \square \)

Conversely we can also use our results to bound the very-ampleness of line bundles on abelian varieties.
Corollary 2.4 Let $L$ be a line bundle on an abelian variety $A$ of dimension $\dim A = n \geq 3$ with $h^0(A, L) = l$. Assume that $L$ is $k$-very ample for some odd integer $k$. Then $k < l - 2n$.

Proof. For $k = 1$ this is Van de Ven’s result [V]. If $k \geq 3$ we can write $k = 2s - 1$ for some $s \geq 2$. The assertion then follows by combining Theorem (0.1) and Proposition (2.2), applied to the abelian scroll determined by a pair $(A, G)$, where $G$ can be chosen as any subgroup of order $k$ of $A$, e.g. a cyclic subgroup. $\blacksquare$

The crucial open question which remains is the existence of abelian surface scrolls $Y \subset \mathbb{P}^{l-1}$ with $2 \dim Y = l - 1$. In [CH] we constructed one non-trivial such example, namely the following.

Example. Let $A \subset \mathbb{P}^6$ be a general (i.e. rank $\text{NS}(A) = 1$) abelian surface embedded by a line bundle $L$ of type $(1, 7)$ and let $\varepsilon \in A(2)$ be a non-zero $2$-torsion point. Then the cyclic scroll determined by the pair $(A, \varepsilon)$ is smooth of dimension 3 and has degree 21.

The next step would be to investigate cyclic $\mathbb{P}^2$-scrolls determined by pairs $(A, \rho)$ where $A \subset \mathbb{P}^8$ is an abelian surface of degree 18, embedded by a complete linear system of type $(1, 9)$ and $\rho \in A(3)$ is a 3-torsion point. More generally we pose the

Problem. Let $A \subset \mathbb{P}^{2n}$, $n \geq 4$ be a general abelian surface of degree $4n$ and let $\rho \in A_{(n-1)}$ be a non-zero $(n-1)$-torsion point. Is the cyclic scroll defined by the pair $(A, \rho)$ smooth?

A positive answer to this question would provide us with an infinite series of smooth scrolls $Y \subset \mathbb{P}^{2n}$ whose dimension is half that of the surrounding space and with irregularity 2. Van de Ven has asked the question whether the irregularity of smooth surfaces in $\mathbb{P}^4$ is bounded by 2. As far as we know, this question is still open. In fact, we do not know of any smooth subvariety $Y \subset \mathbb{P}^{2n}$ with $\dim Y = n \geq 2$ and $q \geq 3$. Let us, therefore, pose the

Problem. Give examples of smooth subvarieties $Y \subset \mathbb{P}^{2n}$ with $\dim Y = n \geq 2$ and $q \geq 3$.

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