FREE ERGODIC $\mathbb{Z}^2$-SYSTEMS AND COMPLEXITY

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Abstract. Using results relating the complexity of a two dimensional subshift to its periodicity, we obtain an application to the well-known conjecture of Furstenberg on a Borel probability measure on $[0,1)$ which is invariant under both $x \mapsto px \pmod{1}$ and $x \mapsto qx \pmod{1}$, showing that any potential counterexample has a nontrivial lower bound on its complexity.

1. Introduction

1.1. Complexity and periodicity. For a one dimensional symbolic system $(X, \sigma)$, meaning that $X \subset \mathcal{A}^\mathbb{Z}$ is a closed set, where $\mathcal{A}$ is a finite alphabet, that is closed under the left shift $\sigma: \mathcal{A}^\mathbb{Z} \to \mathcal{A}^\mathbb{Z}$, the Morse-Hedlund Theorem gives a simple relation between the complexity of the system and periodicity. Namely, if $P_X(n)$ denotes the complexity function, which counts the number of nonempty cylinder sets of length $n$ in $X$, then $(X, \sigma)$ is periodic if and only if there exists $n \in \mathbb{N}$ such that $P_X(n) \leq n$. Both periodicity and complexity have natural generalizations to higher dimensional systems. For example, for a two dimensional system $(X, \sigma, \tau)$, meaning that $X \subset \mathcal{A}^{\mathbb{Z}^2}$ is a closed set that is invariant under the left and down shifts $\sigma, \tau: \mathcal{A}^{\mathbb{Z}^2} \to \mathcal{A}^{\mathbb{Z}^2}$, the two dimensional complexity $P_X(n,k)$ is the number of nonempty $n$ by $k$ cylinder sets. In a partial solution to Nivat’s Conjecture [11], the authors [2] showed that if $(X, \sigma, \tau)$ is a transitive $\mathbb{Z}^2$-subshift and there exist $n,k \in \mathbb{N}$ such that $P_X(n,k) \leq nk/2$, then there exists $(i,j) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ such that $\sigma^i \tau^j x = x$ for all $x \in X$. In this note, we give an application of this theorem to Furstenberg’s well-known “$\times p, \times q$ problem.”

1.2. The $\times p, \times q$ problem. Let $S,T: [0,1) \to [0,1)$ denote the maps $Sx := px \pmod{1}$ and $Tx := qx \pmod{1}$, where $p, q \geq 1$ are multiplicatively independent integers (meaning that $p$ and $q$ are not both powers of the same integer). In the 1960’s, Furstenberg [4] proved that any closed subset of $[0,1)$ that is invariant under both $S$ and $T$ is either all of $[0,1)$ or is finite. He asked whether a similar statement holds for measures:

Conjecture 1.1 (Furstenberg). Let $\mu$ be a Borel probability measure on $[0,1)$ that is invariant under both $S$ and $T$ and is ergodic for the joint action of $S$ and $T$. Then either $\mu$ is Lebesgue measure or $\mu$ is atomic.

Progress was made in the 1980’s with the work of Lyons [9], followed soon thereafter by Rudolph’s proof that positive entropy $h_\mu(\cdot)$ of the measure $\mu$ with respect to one of the transformations implies the result for relatively prime $p$ and $q$. This was generalized to multiplicatively independent integers by Johnson:

The second author was partially supported by NSF grant 1500670.
Theorem 1.2 (Rudolph [16] and Johnson [6]). Let $\mu$ be a Borel probability measure on $[0,1]$ that is invariant under both $S$ and $T$ and is ergodic for the joint action of $S$ and $T$. If $h_\mu(S) > 0$ (or equivalently $h_\mu(T) > 0$), then $\mu$ is Lebesgue measure.

One way to interpret this theorem is that the set of $(S,T)$-ergodic measures experiences an entropy gap with respect to the one-dimensional action generated by $S$ (or equivalently by $T$). Informally, if $\mu$ has high entropy (in this case meaning that $h_\mu(S) > 0$), then its entropy with respect to $S$ is actually $\log p$ and $\mu$ is Lebesgue measure. Our main theorem is that the set of $(S,T)$-ergodic measures also experiences a complexity gap, in a sense we make precise. We show (Theorem 1.3) that if $\mu$ has low complexity (meaning that a certain function grows subquadratically), then it actually has bounded complexity (meaning that this function is bounded) and $\mu$ is atomic. Moreover, all atomic measures have bounded complexity.

1.3. Rephrasing $\times p, \times q$ in symbolic terms. We begin by recasting Furstenberg’s Conjecture and the Rudolph-Johnson Theorem as statements about symbolic dynamical systems. We start by setting some terminology and notation.

A $(\text{measure preserving})$ system $(X, \mathcal{X}, \mu, G)$ is a measure space $X$ with an associated $\sigma$-algebra $\mathcal{X}$, probability measure $\mu$, and an abelian group $G$ of measurable, measure preserving transformations. If the context is clear, we omit the $\sigma$-algebra from the notation, writing $(X, \mu, G)$, and call it a system. The system $(X, \mu, G)$ is free if the set $\{x \in X : gx = x\}$ has measure 0 for every $g \in G$ and the system is ergodic if the only sets invariant under the action of $G$ have either trivial or full measure. It follows that if $(X, \mu, G)$ is an ergodic system with an abelian group $G$ of transformations, then the action of $G$ is free if $g_1^{n_1} \circ \cdots \circ g_k^{n_k} \neq \text{Id}$ for any $g_1, \ldots, g_k \in G$ and $(n_1, \ldots, n_k) \neq (0, \ldots, 0)$.

Two systems $(X_1, \mathcal{X}_1, \mu_1, G)$ and $(X_2, \mathcal{X}_2, \mu_2, G)$ are $(\text{measure theoretically})$ isomorphic if there exist $X'_1 \subset X_1$ and $X'_2 \subset X_2$ with $\mu_1(X'_1) = \mu_2(X'_2) = 1$ such that $gX'_1 \subset X'_1$ for all $g \in G$ and $gX'_2 \subset X'_2$ for all $g \in G$, and there is an invertible bimeasurable transformation $\pi : X'_1 \to X'_2$ such that $\pi_* \mu_1 = \mu_2$ and $\pi g(x) = g \pi(x)$ for all $x \in X'_1$, $g \in G$.

We are particularly interested in the $\mathbb{Z}^2$-system generated by the two commuting, measure preserving transformations $S$ and $T$. In this case, we write $(X, \mathcal{X}, \mu, S, T)$ for the $\mathbb{Z}^2$-system.

A $(\text{topological})$ system $(X, G)$, is a compact metric space $X$ and a group $G$ of homeomorphisms mapping $X$ to itself. If it is clear from the context that we are referring to a topological system, we call $(X, G)$ a system. A system is said to be minimal if for any $x \in X$, the orbit $\{gx : g \in G\}$ is dense in $X$. By the Krylov-Bogolyubov Theorem, every system $(X, G)$ admits an invariant Borel probability measure and if this measure is unique, we say that $(X, G)$ is uniquely ergodic. A system $(X, G)$ is strictly ergodic if it is both minimal and uniquely ergodic.

Let $\mathcal{A}$ denote a finite alphabet and let $\mathcal{A}^{\mathbb{Z}^2}$ be the set of $\mathcal{A}$-colorings of $\mathbb{Z}^2$. For $x \in \mathcal{A}^{\mathbb{Z}^2}$ and $\bar{u} \in \mathbb{Z}^2$, we denote the element of $\mathcal{A}$ that $x$ assigns to $\bar{u}$ by $x(\bar{u})$. With respect to the metric

$$d(x, y) := 2^{-\inf\{|\bar{u}| : x(\bar{u}) \neq y(\bar{u})\}},$$
$\mathcal{A}^\mathbb{Z}_2$ is compact and the leftward and downward shift maps $\sigma, \tau: \mathcal{A}^\mathbb{Z}_2 \to \mathcal{A}^\mathbb{Z}_2$ given by
\begin{align*}
(\sigma x)(i,j) &:= x(i+1,j), \\
(\tau x)(i,j) &:= x(i,j+1)
\end{align*}
are homeomorphisms. A closed set $X \subset \mathcal{A}^\mathbb{Z}_2$ which is invariant under the joint action of $\langle \sigma, \tau \rangle$ is called a $\mathbb{Z}_2$-subshift. (The analogous definitions hold for $\mathbb{Z}^d$-subshifts.)

A uniquely ergodic topological system $(\hat{X}, \nu, G)$ is said to be a topological model for the measure preserving system $(X, \mathcal{X}, \mu, G)$ if there exists a measure theoretic isomorphism between $(\hat{X}, \nu, G)$ and $(X, \mu, G)$. Again, we are mainly interested in topological systems generated by two transformations, and in this case we denote the topological system by $(\hat{X}, \sigma, \tau)$.

The Jewett-Krieger Theorem [5, 8] states that any ergodic $\mathbb{Z}$-system has a strictly ergodic topological model, meaning that the system is measure theoretically isomorphic to a minimal, uniquely ergodic topological system. This was generalized to cover ergodic $\mathbb{Z}^d$-systems by Weiss [18], and further refined by Rosenthal (we only state it for $\mathbb{Z}_2$, as this is the only case relevant for our purposes):

**Theorem 1.3 (Rosenthal [15]).** Let $(X, \mathcal{X}, \mu, S, T)$ be an ergodic, free $\mathbb{Z}^2$-system with entropy less than $\log k$. Then there exists a minimal, uniquely ergodic subshift $\hat{X} \subset \{1, \ldots, k\}^\mathbb{Z}_2$ such that if $\sigma, \tau: \hat{X} \to \hat{X}$ denote the horizontal and vertical shifts (respectively) and if $\nu$ is the unique invariant Borel probability on $\hat{X}$ and $\mathcal{B}$ denotes the Borel $\sigma$-algebra, then $(\hat{X}, \mathcal{B}, \nu, \sigma, \tau)$ is a topological model for $(X, \mathcal{X}, \mu, S, T)$.

We note that in [15], the proof given shows that $\hat{X} \subset \{1, \ldots, k + 1\}^\mathbb{Z}_2$ and the result that the shift alphabet can be taken to have only $k$ letters is stated without proof. However, the size of the alphabet is not relevant for our purposes, other than the fact that it is a finite number.

The subshift $\hat{X} \subset \{1, \ldots, k\}^\mathbb{Z}_2$ in the conclusion of Theorem 1.3 is not uniquely defined, and so we make the following definition:

**Definition 1.4.** Let $(X, \mathcal{X}, \mu, S, T)$ be an ergodic $\mathbb{Z}^2$-system. A minimal, uniquely ergodic $\mathbb{Z}_2$-subshift that is measure theoretically isomorphic to $(X, \mathcal{X}, \mu, S, T)$ is called a Jewett-Krieger model for $(X, \mathcal{X}, \mu, S, T)$.

Theorem 1.3 guarantees that any free ergodic $\mathbb{Z}_2$ system of finite entropy has a Jewett-Krieger model. However the definition is still valid for non-free, ergodic $\mathbb{Z}_2$ systems; the only difference is that Rosenthal’s Theorem no longer guarantees that such a model exists. For the case of interest to us, we show (in the proof of Theorem 1.3) that if $\mu$ is $(S, T)$-ergodic, then either $\mu$ is atomic or the action of $(S, T)$ is free. This motives us to make the following observation: a finite, ergodic $\mathbb{Z}_2$-system cannot be free, but it has a Jewett-Krieger model in a trivial way, obtained by partitioning the system into individual points.

Using this language, we can rephrase Furstenberg’s Conjecture and the Rudolph-Johnson Theorem as equivalent statements about Jewett-Krieger models. Fix the transformations $S, T: [0,1) \to [0,1)$ to be the maps $Sx := px \pmod{1}$ and $Tx := qx \pmod{1}$, where $p, q \geq 1$ are multiplicatively independent integers. By the natural extension, we mean the invertible cover (see Section 2.1).
Conjecture 1.5 (Symbolic Furstenberg Conjecture). Let \( \mu \) be a Borel probability measure on \([0, 1]\) with Borel \( \sigma \)-algebra \( B \) that is invariant under both \( S \) and \( T \) and ergodic for the joint action. If \( \hat{X} \subset \{0, 1\}^{\mathbb{Z}^2} \) is a Jewett-Krieger model for the natural extension of \(((0, 1), B, \mu, S, T)\), then either \( \hat{X} \) is finite or \( \mu \) is Lebesgue measure.

Theorem 1.6 (Symbolic Rudolph-Johnson Theorem). Let \( \mu \) be a Borel probability measure on \([0, 1]\) with Borel \( \sigma \)-algebra \( B \) which is invariant under both \( S \) and \( T \) and is ergodic for the joint action. Let \( \hat{X} \subset \{0, 1\}^{\mathbb{Z}^2} \) be a Jewett-Krieger model for the natural extension of \(((0, 1), B, \mu, S, T)\) and let \( \sigma, \tau : \hat{X} \to \hat{X} \) denote the horizontal and vertical shifts (respectively). If either \( h_\nu(\sigma) > 0 \) or \( h_\nu(\tau) > 0 \), then \( \mu \) is Lebesgue measure.

Proof. An isomorphism of the \( \mathbb{Z}^2 \)-systems \((X, X, \mu, S, T)\) and \((\hat{X}, B, \nu, \sigma, \tau)\) restricts to an isomorphism of the \( \mathbb{Z} \)-systems \((X \times \mathbb{Z}, \mu, S)\) and \((\hat{X}, B, \nu, \sigma)\), and so \( h_\mu(S) = h_\nu(\sigma) \). Similarly \( h_\mu(T) = h_\nu(\tau) \). The statement then follows immediately from the Rudolph-Johnson Theorem. \( \square \)

1.4. Combinatorial rephrasing of measure theoretic entropy. The appeal of Theorem [LJG] is that the hypothesis that \( h_\nu(\sigma) > 0 \) (or equivalently that \( h_\nu(\tau) > 0 \)) can be phrased purely as a combinatorial statement about the frequency with which words in the language of \( \hat{X} \) occur in larger words in the language of \( X \). To explain this, we start with some definitions.

If \( X \subset A^2 \) is a subshift over the finite alphabet \( A \), we write \( x = (x(n): n \in \mathbb{Z}) \). A word is a defined to be a finite sequence of symbols contained consecutively in some \( x \) and we let \( |w| \) denote the number of symbols in \( w \) (it may be finite or infinite). A word \( w \) is a subword of a word \( u \) if the symbols in the word \( w \) occur somewhere in \( u \) as consecutive symbols. The language \( \mathcal{L} = \mathcal{L}(X) \) of \( X \) is defined to be the collection of all finite subwords that arise in elements of \( X \). If \( w \in \mathcal{L}(X) \), let \( [w] \) denote the cylinder set it determines, meaning that

\[ [w] = \{ u \in \mathcal{L} : u(n) = w(n) \text{ for } 1 \leq n \leq |w| \}. \]

These definitions naturally generalize to a two dimensional subshift \( X \subset \mathbb{A}^{\mathbb{Z}^2} \), and for \( x \in \mathbb{A}^{\mathbb{Z}^2} \) we write \( x = (x(\bar{u}) : \bar{u} \in \mathbb{Z}^2) \). A word is a finite, two dimensional configuration that is convex and connected (as a subset of \( \mathbb{Z}^2 \)), and a subword is a configuration contained in another word. If \( F \subset \mathbb{Z}^2 \) is finite and \( \beta \in \mathbb{A}^F \), then the cylinder set of shape \( F \) determined by \( \beta \) is defined to be the set

\[ [F; \beta] := \{ x \in \mathbb{A}^{\mathbb{Z}^2} : x(\bar{u}) = \beta(\bar{u}) \text{ for all } \bar{u} \in F \} \]

Lemma 1.7. Let \((\hat{X}, B, \nu, \sigma, \tau)\) be a strictly ergodic \( \mathbb{Z}^2 \)-subshift. Let \( w \) be a \((2n + 1) \times (2n + 1)\) word in the language of \( \hat{X} \) and let \([w]\) denote the cylinder set determined by placing the word \( w \) centered at \((0, 0)\). Let \( u_1, u_2, u_3, \ldots \) be words in the language of \( \hat{X} \) such that \( u_i \) is a square of size \((2n + 2i + 1) \times (2n + 2i + 1)\). If \( N(w, u_i) \) denotes the number of times \( w \) occurs as a subword of \( u_i \), then

\[ \nu[w] = \lim_{i \to \infty} N(w, u_i)/(2i + 1)^2. \]

Proof. By unique ergodicity, the Birkhoff averages of a continuous function converge uniformly to the integral of the function. In particular, this applies to the continuous function \( 1_{[w]} \), so the limit exists and is independent of the sequence \( \{u_i\}_{i=1}^\infty \). \( \square \)
For $m,n \in \mathbb{N}$, let $\mathcal{P}(m,n)$ be the partition of $\hat{X}$ according to cylinder sets of shape $[0,m-1] \times [-n+1,n-1]$. Observe that (recall that $\sigma$, as defined in \cite{10}, denotes the left shift)
$$
\mathcal{P}(m,n) = \bigvee_{i=0}^{k} \sigma^{-i} \mathcal{P}(1,n)
$$
and that $\bigvee_{i=-k}^{k} \sigma^{i} \mathcal{P}(1,n)$ is the partition of $\hat{X}$ into symmetric $(2m+1) \times (2n+1)$-cylinders centered at the origin. Therefore, $\{\mathcal{P}(1,n)\}_{n=1}^{\infty}$ is a sufficient (in the sense of Definition 4.3.11 in \cite{7}) family of partitions to generate the Borel $\sigma$-algebra of the system $(\hat{X}, \mathcal{B}, \nu, \sigma)$, where we view this as a $\mathbb{Z}$-system with respect to the horizontal shift $\sigma$. Let $h_{\nu}(\sigma, \mathcal{Q})$ denote the measure theoretic entropy of the system $(\hat{X}, \mathcal{B}, \nu, \sigma)$ with respect to the partition $\mathcal{Q}$ and let $h_{\nu}(\sigma)$ denote the measure theoretic entropy of the system. It follows that
$$
h_{\nu}(\sigma) = \sup_{n} h_{\nu}(\sigma, \mathcal{P}(1,n)) = \lim_{n \to \infty} h_{\nu}(\sigma, \mathcal{P}(1,n)) = - \lim_{n \to \infty} \lim_{m \to \infty} \frac{1}{m} \sum_{w \in \mathcal{P}(m,n)} \nu[w] \log \nu[w] = - \lim_{n \to \infty} \lim_{m \to \infty} \frac{1}{m} \sum_{w \in \mathcal{P}(m,n)} \frac{N(w, u_i)}{(2i+1)^2} \log \frac{N(w, u_i)}{(2i+1)^2}
$$
by Lemma \cite{17}. In other words, the Rudolph-Johnson Theorem is equivalent to:

**Theorem 1.8** (Combinatorial Rudolph-Johnson Theorem). Let $\mu$ be a Borel probability measure on $[0,1)$ with Borel $\sigma$-algebra $\mathcal{B}$ and assume that $\mu$ is invariant under both $S$ and $T$, and ergodic for the joint action. Let $\hat{X}$ be a Jewett-Krieger model for the natural extension of $([0,1), \mathcal{B}, \mu, S, T)$ and without loss suppose the horizontal shift on $\hat{X}$ is intertwined with $S$ under this isomorphism. If
$$
- \lim_{n \to \infty} \lim_{m \to \infty} \frac{1}{m} \sum_{w \in \mathcal{P}(m,n)} \frac{N(w, u_i)}{(2i+1)^2} \log \frac{N(w, u_i)}{(2i+1)^2} > 0,
$$
then the value of this limit is $\log p$ and $\mu$ is Lebesgue measure.

**1.5. Complexity of subshifts.** If $X \subset \mathcal{A}^{\mathbb{Z}^2}$ is a nonempty subshift, then its complexity function is the function $P_X : \text{finite subsets of } \mathbb{Z}^2 \to \mathbb{N}$ given by
$$
P_X(F) := |\{ \beta \in \mathcal{A}^F : [F; \beta] \cap X \neq \emptyset \}|.
$$
Let $R_n := \{(i,j) \in \mathbb{Z}^2 : 1 \leq i,j \leq n\}$ denote the $n \times n$ rectangle in $\mathbb{Z}^2$. A standard notion of the complexity of a subshift $X \subset \mathcal{A}^{\mathbb{Z}^2}$ is the asymptotic growth rate of $P_X(R_n)$. Observe that $P_X(R_n)$ is bounded (in $n$) if and only if $X$ is finite. Moreover, $P_X(R_n)$ grows exponentially if and only if $(X, \sigma, \tau)$ has positive topological entropy.

We are now in a position to state our main technical result.

**Theorem 1.9.** Let $\mu$ be a Borel probability measure on $[0,1)$ with Borel $\sigma$-algebra $\mathcal{B}$. Assume that $\mu$ is invariant under both $S$ and $T$ and ergodic for the joint action, and let $\hat{X} \subset \{0,1\}^{\mathbb{Z}^2}$ be a Jewett-Krieger model for the natural extension of $([0,1), \mathcal{B}, \mu, S, T)$. If there exists $n \in \mathbb{N}$ such that $P_X(R_n) \leq \frac{1}{n} n^2$, then $P_X(R_n)$ is bounded (independent of $n$) and $\hat{X}$ is finite. In particular, $\mu$ is atomic.
This gives a nontrivial complexity gap for the set of $\langle S, T \rangle$-ergodic probability measures, which is our main result:

**Corollary 1.10 (Complexity gap).** Let $\mu$ be a Borel probability measure on $[0, 1)$ which is invariant under both $S$ and $T$ and ergodic for the joint action, and let $\hat{X} \subset \{0, 1\}^{\mathbb{Z}^2}$ be a Jewett-Krieger model for the natural extension of $([0, 1), \mathcal{B}, \mu, S, T)$. Then either $P_{\hat{X}}(R_n)$ is bounded (and $\mu$ is atomic) or

$$\liminf_{n \to \infty} \frac{P_{\hat{X}}(R_n)}{n^2} \geq \frac{1}{2}.$$ 

This gap is nontrivial in the following sense: there exist aperiodic, strictly ergodic $\mathbb{Z}^2$-subshifts whose complexity function grows subquadratically. The statement made by Corollary 1.10 is that any such subshift cannot be a Jewett-Krieger model of any $\langle x_p, x_q \rangle$-ergodic measure on $[0, 1)$.

**Example 1.11.** Let $X \subset \{0, 1\}^{\mathbb{Z}^2}$ be a Sturmian shift (see [10] for the definition). Then $X$ is strictly ergodic and $P_X(n) = n + 1$ for all $n \in \mathbb{N}$. Let $Y \subset \{0, 1\}^{\mathbb{Z}^2}$ be the subshift whose points are obtained by placing each $x \in X$ along the $x$-axis in $\mathbb{Z}^2$ and then copying vertically (i.e. each point in $Y$ is vertically constant and its restriction to the $x$-axis is an element of $X$). It follows that $Y$ is strictly ergodic and that $P_Y(R_n) = n + 1$ for all $n \in \mathbb{N}$. Corollary 1.10 shows that $Y$ is not a Jewett-Krieger model for any $\langle x_p, x_q \rangle$-ergodic measure on $[0, 1)$.

### 1.6. Remarks on complexity growth.

We conclude our introduction with a few brief remarks on Theorem 1.9 and Corollary 1.10. We show (Lemma 2.1) that any Jewett-Krieger model $\hat{X}$ for an atomic $\langle S, T \rangle$-ergodic measure is a strictly ergodic $\mathbb{Z}^2$-subshift containing only doubly periodic $\mathbb{Z}^2$-colorings, meaning that there are only finitely many points in $\hat{X}$. From this, it is easy to deduce that $P_{\hat{X}}(R_n)$ is bounded independently of $n$ (by the number of points in $\hat{X}$). Moreover, we show that if $\hat{X}$ is a Jewett-Krieger model for $\mu$ and if $\hat{X}$ contains only doubly periodic $\mathbb{Z}^2$-colorings, then $\mu$ is atomic.

A strategy for proving Theorem 1.9 is therefore to find a nontrivial growth rate of $P_{\hat{X}}(R_n)$ which implies that $\hat{X}$ contains only doubly periodic $\mathbb{Z}^2$-colorings. A simple example of such a rate follows from the classical Morse-Hedlund Theorem [10]: if there exists $n \in \mathbb{N}$ such that $P_{\hat{X}}(R_n) \leq n$, then $\hat{X}$ contains only doubly periodic $\mathbb{Z}^2$-colorings (see e.g. the proof of Theorem 1.2 in [13]). In fact this bound is sharp: there exist $\mathbb{Z}^2$-colorings that are not doubly periodic and yet satisfy $P_{\hat{X}}(R_n) = n + 1$ for all $n \in \mathbb{N}$. Many other subquadratic growth rates can also be realized by strictly ergodic $\mathbb{Z}^2$-subshifts that do not contain doubly periodic points (see, for example, [12]). Therefore, a weak version of Theorem 1.9 that replaces the assumption that there exists $n \in \mathbb{N}$ such that $P_{\hat{X}}(R_n) \leq \frac{1}{2} \cdot n^2$ with the stronger assumption that there exists $n \in \mathbb{N}$ such that $P_{\hat{X}}(R_n) \leq n$, follows from the Morse-Hedlund Theorem. However, this weak theorem relies on the fact that there are simply no strictly ergodic $\mathbb{Z}^2$-subshifts for which $P_{\hat{X}}(R_n)$ is unbounded but for which $P_{\hat{X}}(R_n) \leq n$ (for some $n$). The complexity gap provided by this weak theorem is therefore trivial in the sense that there are no strictly ergodic $\mathbb{Z}^2$-subshifts whose complexity function lies in this gap.

On the other hand, there do exist strictly ergodic $\mathbb{Z}^2$-subshifts with unbounded complexity and such that $P_{\hat{X}}(R_n) < \frac{1}{2} \cdot n^2$. This is the interest in Theorem 1.9 and Corollary 1.10. The content of the theorem is that although such $\mathbb{Z}^2$-systems
exist, they cannot not be Jewett-Krieger models of \((S, T)\)-ergodic measures on \([0, 1]\). This is analogous to Theorem 1.8 which says that although there are strictly ergodic \(\mathbb{Z}^2\)-subshifts that have small but positive entropy, they are not Jewett-Krieger models of \((S, T)\)-ergodic measures on \([0, 1]\). Moreover, analogous to the hypothesis of Corollary 1.10 which relies on the growth rate of \(P_X(-)\), the hypothesis of Theorem 1.8 is a condition on the growth rate of the relative complexity function \(N(\cdot, \cdot)\) of Lemma 1.7 with respect to the action of the horizontal shift (a similar statement holds for the vertical shift).

2. Proof of Theorem 1.9

Throughout this section, we assume that \(p, q \geq 2\) are multiplicatively independent integers and that \(\mu\) is a Borel probability measure on \([0, 1]\) which is invariant under both
\[
Sx := px \pmod{1}; \\
Tx := qx \pmod{1}
\]
and is ergodic with respect to the joint action \((S, T)\). Let \(\mathcal{B}\) denote the associated Borel \(\sigma\)-algebra on \([0, 1]\).

2.1. The natural extension. Let \(X\) be the natural extension of the \(\mathbb{N}^2\)-system \(((0, 1), \mathcal{B}, \mu, S, T)\). Specifically (following [14]), let
\[
X := \left\{ y \in [0, 1)^{\mathbb{Z}^2} : y(i + 1, j) = Sy(i, j) \text{ and } y(i, j + 1) = Ty(i, j) \text{ for all } i, j \in \mathbb{Z} \right\},
\]
and for \((i, j) \in \mathbb{Z}^2\) let \(\pi_{(i, j)} : X \to [0, 1)\) be the map \(\pi_{(i, j)}(y) = y(i, j)\). Define a countably additive measure \(\mu_X\) on the \(\sigma\)-algebra
\[
\bigcup_{i=0}^{\infty} \pi_{(-i,-i)}^{-1} \mathcal{B}
\]
by setting \(\mu_X(\pi_{(-i,-i)}^{-1} A) := \mu(A)\). Let \(X'\) be the completion of this \(\sigma\)-algebra with respect to \(\mu_X\). Let \(S_X, T_X : X \to X\) be the left shift and the down shift, respectively. Thus \(\pi_{(0, 0)}\) defines a measure theoretic factor map from \((X, X', \mu_X, S_X, T_X)\) to \(((0, 1), \mathcal{B}, \mu, S, T)\). Moreover, \(\mu_X\) is ergodic if and only if \(\mu\) is ergodic. By construction, \(h_\mu(S) = h_{\mu_X}(S_X), h_\mu(T) = h_{\mu_X}(T_X)\), and \(h_\mu((S, T)) = h_{\mu_X}((S_X, T_X))\).

The advantage of working with \((X, X', \mu_X, S_X, T_X)\) instead of the original system is that the natural extension is an ergodic \(\mathbb{Z}^2\)-system.

2.2. Jewett-Krieger models and periodicity. If the two dimensional entropy of a system is positive, then the entropy of every one dimensional subsystem is infinite (for a proof, see, for example, [14]). In our setting, since \(h_\mu(S) \leq h_{\text{top}}(S) = \log(p)\) (and \(h_\mu(T) \leq h_{\text{top}}(T) = \log(q)\)), it follows that the measure theoretic entropy of the joint action generated by \((S, T)\) on \([0, 1]\) with respect to \(\mu\) is also zero. It follows that the measure theoretic entropy with respect to \(\mu_X\) of the joint action on \(X\) generated by \((S_X, T_X)\) is zero. Therefore, by Theorem 1.8 there exists a strictly ergodic subshift \(\hat{X} \subset \{0, 1\}^{\mathbb{Z}^2}\) such that \((X, X', \mu_X, S_X, T_X)\) is measure theoretically isomorphic to \((\hat{X}, \hat{X}, \nu, \sigma, \tau)\), where \(\hat{X}\) is the Borel \(\sigma\)-algebra on \(\hat{X}\), \(\sigma, \tau : \hat{X} \to \hat{X}\) denote the left shift and down shift (respectively), and \(\nu\) is the unique \((\sigma, \tau)\)-invariant Borel probability measure. Note that the choice of \(\hat{X}\) is not necessarily unique.
Lemma 2.1. If $(X, \mathcal{X}, \mu_X, S_X, T_X)$ is an atomic system, then any Jewett-Krieger model $(\hat{X}, \hat{\mathcal{X}}, \nu, \sigma, \tau)$ for $(X, \mathcal{X}, \mu_X, S_X, T_X)$ is finite.

Proof. Let $\pi: (\hat{X}, \hat{\mathcal{X}}, \nu, \sigma, \tau) \to (X, \mathcal{X}, \mu_X, S_X, T_X)$ be an isomorphism and let $x \in X$ be an atom. Then there exist full measure sets $\hat{X}_1 \subset \hat{X}$ and $X_1 \subset X$ such that $\pi: \hat{X}_1 \to X_1$ is a bijection which intertwines the $\mathbb{Z}^2$ actions. Every atom in $X$ is contained in $X_1$, and if $x \in X_1$ is an atom then there exists unique $y \in \hat{X}_1$ such that $\pi(y) = x$. It follows that $\nu(\{y\}) = \mu_X(\{x\}) > 0$ and so $y$ is an atom in $\hat{X}$.

By the Poincaré Recurrence Theorem, there exists $(i, j) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ such that $S_X^i T_X^j y = y$. Let $\nu_y := \{(i,j) \in \mathbb{Z}^2 \setminus \{(0,0)\} : S_X^i T_X^j y = y\}$ be the (nonempty) set of nontrivial period vectors for $y$. If $\dim(\text{Span}(\nu_y)) = 1$, then

$$\lim_{N \to \infty} \frac{1}{(2N + 1)^2} \sum_{-N \leq i, j \leq N} 1_{\{y\}}(S_X^i T_X^j y) = 0 < \nu(\{y\}),$$

which contradicts the pointwise ergodic theorem. Therefore $\dim(\text{Span}(\nu_y)) = 2$ and $y \in \mathcal{A}^{\mathbb{Z}^2}$ is doubly periodic. Moreover, for $\nu$-a.e. $z \in \hat{X}$ we have $S_X^i T_X^j z = y$ for some $(i,j) \in \mathbb{Z}^2$ and so $z$ is also doubly periodic (with periods equal to those of $y$). Thus there are only finitely many points $z \in \hat{X}$. □

Since $\hat{X}$ is minimal, and hence transitive, we can use the following tool for studying the dynamics of $(X, \mathcal{X}, \mu_X, S_X, T_X)$:

Theorem 2.2 (Cyr & Kra [2]). If $(X, \sigma, \tau)$ is a transitive $\mathbb{Z}^2$-subshift and there exist $n, k \in \mathbb{N}$ such that $P_X(n,k) \leq nk/2$, then there exists $(i,j) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ such that $\sigma^i \tau^j x = x$ for all $x \in X$.

Lemma 2.3. If there exists $(i,j) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ such that $\sigma^i \tau^j x = x$ for every $x \in \hat{X}$, then $S_X^i T_X^j x = x$ for $\mu$-almost every $x \in X$.

Proof. Let $\psi: \hat{X} \to X$ be an isomorphism. Thus there exist $\hat{X}_1 \subset \hat{X}$ and $X_1 \subset X$ such that $\nu(\hat{X}_1) = \mu_X(X_1) = 1$, $\psi: \hat{X}_1 \to X_1$ is a bi-measurable bijection, $\nu = \mu_X$, $\psi \circ \sigma = S_X \circ \psi$, and $\psi \circ \tau = T_X \circ \psi$. Let $E = \{x \in X_1 : S_X^i T_X^j x \neq x\}$. Since $\psi^{-1}(E) = \{y \in \hat{X}_1 : \sigma^i \tau^j y \neq y\}$, it follows that $\mu_X(E) = \nu(\psi^{-1}(E)) = 0$. □

Theorem 2.4. If there exist $n, k \in \mathbb{N}$ such that $P_X(n,k) \leq nk/2$, then $\mu$ is atomic. Moreover, if $\hat{Y}$ is any other Jewett-Krieger model for $([0,1], \mathcal{B}, \mu, S, T)$, then $P_{\hat{Y}}(n,k)$ is bounded independent of $n, k \in \mathbb{N}$.

Proof. Combining Theorem 2.2 and Lemma 2.3 there exist $(i,j) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ such that $S_X^i T_X^j x = x$ for $\mu_X$-a.e. $x \in X$. Therefore $(S_X^i T_X^j x)(0,0) = x(0,0)$ for $\mu_X$-a.e. $x \in X$. It is immediate that we also have $(S_X^i T_X^j x)(0,0) = x(0,0)$ for $\mu_X$-a.e. $x \in X$. So there are two cases to consider, depending on the the sign of $i \cdot j$.

Case 1. Suppose $i \cdot j \geq 0$. Then, replacing by $-i$ and $-j$ if necessary, we can assume that both $i$ and $j$ are nonnegative. Set $E := \{y \in [0,1) : S^i T^j y \neq y\}$ and let $y \in E$. Then if $x \in \pi^{-1}(y)$, we have that $S_X^i T_X^j x \neq x$. Thus $\mu(E) = \mu_X(\pi^{-1}(E)) = 0$ and so $S^i T^j y = y$ for $\mu$-a.e. $y \in [0,1)$.

Now observe that $S^i T^j y = y$ is equivalent to the statement that

$$p^i q^j y = y \pmod{1},$$

which only has finitely many solutions in the interval $[0,1)$. Therefore, $\mu$ is supported on a finite set. Since $\mu$ is $(\mathcal{S},\mathcal{T})$-invariant, this set must be $\mathcal{S}$- and $\mathcal{T}$-invariant. Therefore there exist $a, b \in \mathbb{N}$ such that $\mathcal{S}^a$ and $\mathcal{T}^b$ are both equal to the identity $\mu$-almost everywhere.

Case 2. Suppose $i \cdot j < 0$. Again, replacing by $-i$ and $-j$ if necessary, we can assume that $i < 0$ and $j > 0$. Now set $E := \{y \in [0,1): \mathcal{S}^{i|y} \neq \mathcal{T}^{j} y\}$. Thus if $y \in E$ and $x \in \pi^{-1}(y)$, then $x(-i, j) \neq x(0,0) = y$ as $\mathcal{S}^{i|y}(x(-i,j)) = x(0, j) = \mathcal{T}^{j}(x(0,0))$ by construction. Therefore $\mathcal{S}^{i} \mathcal{T}^{j} x \neq x$ and so $\mu(E) = \mu_{\mathcal{X}}(\pi^{-1}(E)) = 0$. It follows that $\mathcal{S}^{i|y} = \mathcal{T}^{j} y$ for $\mu$-a.e. $y \in [0,1)$.

Finally observe that $\mathcal{S}^{i|y} = \mathcal{T}^{j} y$ is equivalent to $y^{i|y} = q^{j} y \pmod{1}$.

As $p$ and $q$ are multiplicatively independent, there are only finitely many solutions in the interval $[0,1)$. Therefore, again, $\mu$ is supported on a finite set and there exist $a, b \in \mathbb{N}$ such that $\mathcal{S}^a$ and $\mathcal{T}^b$ are both equal to the identity $\mu$-almost everywhere.

This establishes the first claim of the theorem. By Lemma 2.1 any Jewett-Krieger model of an atomic system is finite, and the second statement follows. □

We use this to complete the proof of Theorem 1.9.

Proof of Theorem 1.9. Let $\mu$ be a Borel probability measure on $[0,1)$ that is $(\mathcal{S},\mathcal{T})$-ergodic. If this two dimensional action is not free, arguing as in the proof of Theorem 2.4 that $\mu$ is an atomic measure, we are done. Thus we can assume that the action is free, and similarly the action for the natural extension is also free.

Let $(\hat{\mathcal{X}}, \hat{\mathcal{X}}, \nu, \sigma, \tau)$ be a Jewett-Krieger model for the natural extension of the system $([0,1), \mathcal{B}, \mu, \mathcal{S}, \mathcal{T})$. If there is no such model satisfying the additional property that there exist $n, k \in \mathbb{N}$ satisfying $P_{\hat{\mathcal{X}}}(n, k) \leq nk/2$, then the conclusion of the theorem holds vacuously. Thus it suffices to assume that there exists a Jewett-Krieger model $(\hat{\mathcal{X}}, \hat{\mathcal{X}}, \nu, \sigma, \tau)$ with the property that there exist $n, k \in \mathbb{N}$ satisfying $P_{\hat{\mathcal{X}}}(n, k) \leq nk/2$. By Theorem 2.4 $([0,1), \mathcal{B}, \mu, \mathcal{S}, \mathcal{T})$ is atomic. □

3. Higher dimensions

Theorem 1.9 shows that if $\mu$ is any nonatomic $\times p, \times q$ ergodic measure then the natural extension of $([0,1), \mathcal{X}, \mu, \mathcal{S}, \mathcal{T})$ cannot be measurably isomorphic to any $\mathbb{Z}^2$-subshift of whose complexity function satisfies $P_{\mathcal{X}}(n, n) = o(n^2)$. It is natural to ask whether this result can be generalized to higher dimensions. In particular, if $p_1, \ldots, p_d$ are a multiplicatively independent set of integers and $\mu$ is a nonatomic $\times p_1, \ldots, \times p_d$ ergodic measure, we can ask if the natural extension of $(\mathcal{X}, \mathcal{X}, \mu, \times p_1, \ldots, \times p_d)$ could have a topological model whose complexity function is $o(n^d)$.

The same method used in the two dimensional case suggests a path to proving this result. If one could show that any free, strictly ergodic $\mathbb{Z}^2$-subshift whose complexity function is $o(n^d)$ is periodic, then it would follow that no such topological model for $\mu$ exists. However, the analog of Theorem 2.2 in dimension $d > 2$ is false. Julien Cassaigne [1] has shown that for $d > 2$, there exists a minimal $\mathbb{Z}^d$-subshift $\mathcal{X}$ whose elements are not periodic in any direction, and is such that for any $\varepsilon > 0$ we have $P_{\mathcal{X}}(n, n, \ldots, n) = o(n^{2+\varepsilon})$. On the other hand, the authors have recently shown [3] that the analog of Theorem 2.2 does hold for dimension $d > 2$ if a certain expansiveness assumption is imposed on the subshift.
If $Y \subset \mathcal{A}^{Z^d}$ is a subshift, then we say that the $x$-axis in $Z^d$ is **strongly expansive** if whenever $x, y \in X$ have the same restriction to the $x$-axis, we have $x = y$.

In this case, if $X \subset \mathcal{A}^Z$ is the subshift obtained by restricting elements of $Y$ to the $x$-axis, then there exist homeomorphisms $\tau_1, \ldots, \tau_{d-1} : X \to X$ which commute pairwise and with the shift $\sigma$ and are such that for any $y \in Y$ we have $y(i_1, i_2, \ldots, i_d) = \left( \tau_1^{i_1} \tau_2^{i_2} \cdots \tau_{d-1}^{i_d-1} \sigma^{i_d} \pi_X(y) \right)(0)$ for all $i_1, \ldots, i_d \in Z^d$, where $\pi_X(y)$ denotes the restriction of $y$ to the $x$-axis. In previous work, we have shown that:

**Theorem 3.1** (Cyr & Kra [3]). Let $X \subset \mathcal{A}^Z$ be a minimal subshift and let $\tau_1, \ldots, \tau_{d-1} : X \to X$ be homeomorphisms of $X$ that commute with the shift $\sigma$. If $\langle \sigma, \tau_1, \ldots, \tau_{d-1} \rangle \cong Z^d$, then $\liminf_{n \to \infty} P_n(X)/n^d > 0$.

With some additional effort, the same result can be shown if the assumption that $(X, \sigma)$ is minimal (as a $Z$-system) is relaxed to only require that $(X, \sigma, \tau_1, \ldots, \tau_d)$ is minimal (as a $Z^d$-system). Thus, the only obstruction to generalizing Theorem 1.9 to the higher dimensional setting is the following:

**Conjecture 3.2.** For every nonatomic Borel probability $\mu$ on $[0, 1)$ which is ergodic for the joint action of $\times p_1, \ldots, \times p_d$, there is a strongly expansive, minimal topological model for $(X, X, \mu, \times p_1, \ldots, \times p_d)$.

If this conjecture holds, then it follows that any such system is measurably isomorphic to a subshift whose complexity function grows on the order of $n^d$.

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