Maxwell’s equal area law for black holes
in power Maxwell invariant

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In this paper, we consider the phase transition of black hole in power Maxwell invariant by means of Maxwell’s equal area law. First, we review and study the analogy of nonlinear charged black hole solutions with the Van der Waals gas-liquid system in the extended phase space, and obtain isothermal $P$-$v$ diagram. Then, using the Maxwell’s equal area law we study the phase transition of AdS black hole with different temperatures. Finally, we extend the method to the black hole in the canonical (grand canonical) ensemble in which charge (potential) is fixed at infinity. Interestingly, we find the phase transition occurs in the both ensembles. We also study the effect of the parameters of the black hole on the two-phase coexistence. The results show that the black hole may go through a small-large phase transition similar to those of usual non-gravity thermodynamic systems.

I. INTRODUCTION

In recent years, the cosmological constant in $n$-dimensional AdS and dS spacetime has been regarded as pressure of black hole thermodynamic system. The ($P, v$) critical behaviors in AdS and dS black holes have been extensively studied \cite{1-51}. It shows that black holes also have the standard thermodynamic quantities, such as temperature, entropy, even possess abundant phase structures like the Hawking-Page phase transition and the critical phenomena similar to ones in the ordinary thermodynamic system. What is more interesting is the research on charged, non-rotating RN-AdS black hole, which shows that there exists a phase transition similar to the van der Waals-Maxwell gas-liquid phase transition \cite{1-51}.

The isotherms in ($P, v$) diagrams of AdS black hole in Ref. \cite{1-12} show there exists thermodynamic unstable region with $\partial P/\partial v > 0$ when temperature is below critical temperature.

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and the negative pressure emerges when temperature is below a certain value. This situation also exists in van der Waals-Maxwell gas-liquid system, which has been resolved by Maxwell’s equal area law \[52, 54\]. At this point, it is worth mentioning that the Maxwell equal area construction can be equally applied in the \((P, v)\) plane at constant temperature. This has been done in \[52, 54\] with interesting results: i) the equal area law can be analytically solved; ii) the unphysical negative specific heat region is cut off; iii) a new black hole phase structure emerges; iv) the role of the Van der Waals un-shrinkable molecular volume taken by the extremal black hole configuration, thus justifying its stability. So, we hope that the Maxwell’s equal area laws can help us to find more phenomenon in the thermodynamics of black hole.

By this observations one may find it is worthwhile to study the effects of nonlinear electrodynamics (NLEDs) on phase transition of black holes in the extended phase space. In this direction, the effects of nonlinear electromagnetic field of static and rotating AdS black holes in the extended phase space have been analyzed \[19\]. In the last five years, a class of NLEDs has been introduced, the so-called power Maxwell invariant (PMI) field. The PMI field is significantly richer than that of the Maxwell field, and in the special case \((s = 1)\) it reduces to linear electromagnetic source. The black hole solutions of the Einstein-PMI theory and their interesting thermodynamics and geometric properties have been examined before.

In this paper, using the Maxwell’s equal area law, we establish a phase transition process in charged AdS black holes with PMI, where the issues about unstable states and negative pressure are resolved. By studying the phase transition process, we acquire the two-phase equilibrium properties including the \(P – v\) phase diagram. Using the Maxwell’s equal area law we study the phase transition of AdS black hole with different temperature. Finally, we extend the method to the black hole in the (grand canonical) canonical ensemble in which (potential) charge is fixed at infinity. Interestingly, we find the phase transition occurs in the both of canonical and grand canonical ensembles. We also study the effect of the parameters of the black hole on the two phases coexistence. The results show the phase transition below critical temperature is of the first order but phase transition at critical point belongs to the continuous one.

The paper is organized as follows: In Sec. II we review and consider spherically symmetric black hole solutions of Einstein gravity in the presence of the PMI source. Regarding the
cosmological constant as thermodynamic pressure, we study thermodynamic properties and obtain Smarr’s mass relation. In Sec. [III] by Maxwell’s equal area law the phase transition processes at certain temperatures are obtained and the boundary of two phase equilibrium region are depicted in $P - v$ diagram for a charged AdS black hole with PMI. Then some parameters of the black hole are analyzed to find the relevance with the two-phase equilibrium. In Sec. [IV] we consider the possibility of the phase transition in the BTZ-like black hole and the grand canonical ensemble and find that in contrast to RN black holes, the phase transition occurs. Finally, we finish this work with some concluding remarks.

II. EXTENDED PHASE-SPACE THERMODYNAMICS OF BLACK HOLES WITH PMI SOURCE

The bulk action of Einstein-PMI gravity has the following form

$$I_b = -\frac{1}{16\pi} \int_M d^{n+1}x \sqrt{-g} \left( R + \frac{n(n-1)}{l^2} + \mathcal{L}_{PMI} \right),$$

(2.1)

where $\mathcal{F} = F_{\mu\nu}F^{\mu\nu}$. Expanding the PMI Lagrangian near the linear Maxwell case ($s \to 1$), one can obtain

$$\mathcal{L}_{PMI} = (-\mathcal{F})^s \to \mathcal{L}_{Max} + o(s - 1),$$

(2.2)

where $\mathcal{L}_{Max} = -\mathcal{F}$ is the Maxwell Lagrangian. We consider a spherically symmetric space-time as

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-2}^2,$$

(2.3)

where $d\Omega_{d-2}^2$ stands for the standard element on $S^{d-2}$. Considering the field equations following from the variation of the bulk action with Eq. (2.3), one can show that the metric function $f(r)$, gauge potential one–form $A$ and electromagnetic field two–form $F$ are given by [19, 55–60]

$$f(r) = 1 + \frac{r^2}{l^2} - \frac{m}{r^{n-2}} + \frac{(2s - 1)^2 \left( \frac{(n-1)(2s-n)^2 q^2}{(n-2)(2s-1)^2} \right)^s}{(n - 1)(n - 2s)r^{2(n-3s+1)/(2s-1)}},$$

(2.4)

$$A = -\sqrt{\frac{n - 1}{2(n - 2)}} q r^{(2s-n)/(2s-1)} dt,$$

(2.5)

$$F = dA.$$

(2.6)

The power $s \neq n/2$ denotes the nonlinearity parameter of the source which is restricted to $s > 1/2$. In the above expression, $m$ appears as an integration constant and is related to the
Arnowitt-Deser-Misner (ADM) mass of the black hole. According to the definition of mass due to Abbott and Deser, the mass of the solution (2.4) is

$$ M = \frac{\omega_{n-1}}{16\pi} (n - 1)m, $$

the electric charge is

$$ Q = \frac{\sqrt{2}(2s - 1)s}{8\pi} \omega_{n-1} \left( \frac{n - 1}{n - 2} \right)^{s/2} \left( \frac{(n - 2s)q}{2s - 1} \right)^{2s - 1}, $$

where $$ \omega_{n-1} $$ represents the volume of constant curvature hypersurface described by $$ d\Omega_{n-1} $$,

$$ \omega_{n-1} = \frac{2\pi^n}{\Gamma(n/2)}. $$

The cosmological constant is related to spacetime dimension $$ n $$ by

$$ \Lambda = -\frac{n(n-1)}{2l^2}. $$

The Hawking temperature of the black hole on the outer horizon $$ r_+ $$ can be calculated using the relation

$$ T = \frac{\kappa}{2\pi} = \frac{f'(r_+)}{4\pi}, $$

where $$ \kappa $$ is the surface gravity. The, one can easily show that

$$ T = \frac{n - 2}{4\pi r_+} \left( 1 + \frac{n - 2}{n - 2} \frac{r_+^2}{l^2} - \frac{(2s - 1)}{(n - 1)(n - 2)} \left( \frac{(n - 1)(2s - n)^2q^2}{(n - 2)(2s - 1)} \right)^s \right), $$

with $$ r_+ $$ denotes the radius of the event horizon which is the largest root of $$ f(r_+) = 0 $$.

The electric potential $$ \Phi $$, measured at infinity with respect to the horizon while the black hole entropy $$ S $$, determined from the area law. It is easy to show that

$$ \Phi = \sqrt{\frac{n - 1}{2(n - 2)} \frac{q}{r_+^{(n-2s)/(2s-1)}}}, $$

$$ S = \frac{\omega_{n-1} r_+^{n-1}}{4}. $$

One may then regard the parameters $$ S $$, $$ Q $$, and $$ P $$ as a complete set of extensive parameters for the mass $$ M(S, Q, P) $$ and define the intensive parameters conjugate to $$ S $$, $$ Q $$, and $$ P $$. These quantities are the temperature, the electric potential and volume.

$$ T = \left( \frac{\partial M}{\partial S} \right)_{Q,P}, \quad U = \left( \frac{\partial M}{\partial Q} \right)_{S,P}, \quad V = \left( \frac{\partial M}{\partial P} \right)_{Q,S}, $$
where

\[ P = \frac{n(n - 1)}{16\pi l^2}, \quad V = \frac{\omega_{n-1}}{n} r^*_n. \] (2.16)

It is a matter of straightforward calculation to show that the quantities calculated by Eq. (2.16) for the temperature, and the electric potential coincide with Eqs. (2.12) and (2.14). Thus, the thermodynamics quantities satisfy the first law of thermodynamics

\[ dM = TdS + UdQ + VdP. \] (2.17)

From the above calculation, the thermodynamic quantities energy \( M \), entropy \( S \), temperature \( T \), volume \( V \), pressure \( P \), electric potential \( U \) and electric charged \( Q \) satisfy the Smarr formula:

\[ M = \frac{(n - 1)}{(n - 2)} TS + \frac{ns - 3s + 1}{s(2s - 1)(n - 2)} \Phi Q - \frac{2}{n - 2} VP. \] (2.18)

In what follows we concentrate on analyzing the phase transition of the black hole with PMI source system in the extended phase space while we treat the black hole charge \( Q \) as a fixed external parameter, or the cosmical constant is a invariable parameters, not a thermodynamic variable. We shall find that an even more remarkable coincidence with the Van der Waals fluid is realized in this case.

Using the Eqs. (2.12) and (2.16) for a fixed charge \( Q \), one may obtain the equation of state, \( P(v, T) \)

\[ P = \frac{T}{v} - \frac{(n - 2)}{\pi(n - 1)v^2} + \frac{1}{16\pi} \frac{kq^{2s}}{v^{2s(n-1)/(2s-1)}}, \] (2.19)

\[ k = \frac{4^{2s(n-1)/(2s-1)}(2s - 1)\left(\frac{(n-1)(2s-n)^2}{(n-2)(2s-1)^2}\right)^s}{(n-1)^{2s(n-1)/(2s-1)}}, \] (2.20)

where

\[ v = \frac{4}{(n - 1)} r^*_n, \] (2.21)

is specific volume.

In Fig[3] we plot the isotherms in \( P - v \) diagrams at different dimension \( n \), nonlinear parameters \( s \), charge \( q \). One can see from Fig[3] that there are thermodynamic unstable segments with \( \partial P/\partial v > 0 \) on the isotherms when temperature \( T < T_c \), where \( T_c \) is critical temperature. When the temperature \( T = T_0 \), there is a point of intersection between the isotherms and the horizontal \( v \) axis. And the negative pressure emerges when temperature is below certain value \( T_0 \). Using the above equation, \( T_0 \) and the corresponding specific volume
FIG. 1: Isotherms in $P - v$ diagrams of charged AdS black holes in PMI in AdS spacetime. The temperature of isotherms decreases from top to bottom.

$v_0$ can be derived,

$$T_0 = \frac{(n - 2)}{\pi(n - 1)v_0} + \frac{k q^{2s}}{16\pi v_0^{2s - 1}} - 1, \quad v_0 = \left(\frac{k q^{2s}(2sn - 1)(n - 1)}{16(2s - 1)(n - 2)}\right)^{\frac{1}{2s - 1}} - 2. \quad (2.22)$$

III. TWO-PHASE EQUILIBRIUM AND MAXWELL EQUAL AREA LAW

The state equation of the charged black hole with PMI is exhibited by the isotherms in Fig. 1, in which the thermodynamic unstable states with $\partial P/\partial v > 0$ will lead to the system expansion or contraction automatically and the negative pressure situation have no physical meaning. The cases occur also in van der Waals equation but they have been resolved by Maxwell equal area law.

We extend the Maxwell equal area law to $n$-dimensional charged AdS black hole with PMI to establish an phase transition process of the black hole thermodynamic system. On the isotherm with temperature $T_0$ in $P - v$ diagram, the two points $(P_0, v_1)$ and $(P_0, v_2)$ meet the Maxwell equal area law,

$$P_0(v_2 - v_1) = \int_{v_1}^{v_2} P dv, \quad (3.1)$$

which results in

$$P_0(v_2 - v_1) = T_0 \ln \left(\frac{v_2}{v_1}\right) - A \left(\frac{1}{v_1} - \frac{1}{v_2}\right) + \frac{B}{d - 1} \left(\frac{1}{v_1^{d-1}} - \frac{1}{v_2^{d-1}}\right), \quad (3.2)$$

where the two points $(P_0, v_1)$ and $(P_0, v_2)$ are seen as endpoints of isothermal phase transition. Considering

$$P_0 = \frac{T_0}{v_1} - \frac{A}{v_1^d} + \frac{B}{v_1^{d-1}}, \quad P_0 = \frac{T_0}{v_2} - \frac{A}{v_2^d} + \frac{B}{v_2^{d-1}}, \quad (3.3)$$
from the eq.\((3.3)\), we can get

\[
0 = T_0 \left( \frac{1}{v_1} - \frac{1}{v_2} \right) - A \left( \frac{1}{v_1^2} - \frac{1}{v_2^2} \right) + B \left( \frac{1}{v_1^d} - \frac{1}{v_2^d} \right),
\]

\((3.4)\)

\[
2P_0 = T_0 \left( \frac{1}{v_1} + \frac{1}{v_2} \right) - A \left( \frac{1}{v_1^2} + \frac{1}{v_2^2} \right) + B \left( \frac{1}{v_1^d} + \frac{1}{v_2^d} \right),
\]

\((3.5)\)

where

\[
A = \frac{(n - 2)}{\pi(n - 1)}, \quad B = \frac{kq^{2s}}{16\pi}, \quad d = \frac{2s(n - 1)}{2s - 1}.
\]

From the eqs.\((3.3)\), \((3.4)\) and \((3.5)\), we can obtain

\[
T_0 v_2^{d-1} = A v_2^{d-2} (1 + x) - B \frac{1 - x^d}{1 - x},
\]

\((3.6)\)

and

\[
v_2^{d-2} = \frac{B}{A} \frac{d(1 - x^{d-1})(1 - x) + (d - 1)(1 - x^d) \ln x}{x^{d-2}(d - 1)(1 - x) (2(1 - x) + (1 + x) \ln x)} = f(x),
\]

\((3.7)\)

Substituting \((3.7)\) into \((3.6)\), we can obtain

\[
\chi T_c x^{d-1} f^{(d-1)/(d-2)}(x) = A f(x) x^{d-2} (1 + x) - B \frac{1 - x^d}{1 - x},
\]

\((3.8)\)

where \(x = v_1/v_2\), \(T_0 = \chi T_c\), \(T_c\) is critical temperature. The value of \(\chi = \frac{T}{T_c}\) is form 0 to 1.

When \(x \to 1\) and \(\chi \to 1\), the corresponding state is critical state

\[
f(1) = \frac{d(d - 1)B}{2A}.
\]

\((3.9)\)

So, the critical point satisfies

\[
v_2^{d-2} = v_1^{d-2} = v_c^{d-2} = \frac{d(d - 1)B}{2A} = \frac{ks(n - 1)^2(2ns - 4s + 1)q^{2s}}{16(n - 2)(2s - 1)^2}.
\]

\((3.10)\)

Substituting eq.\((3.10)\) into the eqs.\((3.6)\) and \((3.5)\), we can obtain

\[
T_c = \frac{2A(d - 2)}{(d - 1)} \left( \frac{2A}{d(d - 1)B} \right)^{1/(d-2)}
\]

\[
= \frac{4(n - 2)(ns - 3s + 1)}{\pi(n - 1)(2ns - 4s + 1)} \left( \frac{ks(n - 1)^2(2ns - 4s + 1)q^{2s}}{16(n - 2)(2s - 1)^2} \right)^{\frac{1-2s}{2(ns-3s+1)}},
\]

\((3.11)\)

\[
P_c = \frac{A(d - 2)}{d} \left( \frac{2A}{d(d - 1)B} \right)^{2/(d-2)}
\]

\[
= \frac{(n - 2)(ns - 3s + 1)}{\pi s(n - 1)^2} \left( \frac{ks(n - 1)^2(2ns - 4s + 1)q^{2s}}{16(n - 2)(2s - 1)^2} \right)^{\frac{1-2s}{(ns-3s+1)}}.
\]
FIG. 2: The simulated isothermal phase transition by isobars and the boundary of two phase coexistence region for the charged black hole with PMI as $n = 5$, $s = 3/4$, $q = 0.5$.

Substituting the eq.(3.11) into eq.(3.8), we can obtain

$$\chi x^{d-1} f^{(d-1)/(d-2)}(x) \frac{2A(d-2)}{(d-1)} \left( \frac{2A}{d(d-1)B} \right)^{1/(d-2)} = Af(x)x^{d-2}(1 + x) - B \frac{1 - x^d}{1 - x}. \quad (3.12)$$

Because we take account of the case that the temperature $T$ below the critical temperature $T_c$, the value of $\chi = \frac{T}{T_c}$ is form 0 to 1. When $x \to 1$ and $\chi \to 1$, the corresponding state is critical state. For a fixed $\chi$, i.e. a fixed $T_0$, we can get a certain $x$ from Eq. (3.12), and then according to Eqs. (3.7) and (3.5), the $v_2$ and $P_0$ are solved.

To analyze the effect of parameters $n$ and $q$ on the phase transition processes, we take $\chi = 0.1$, 0.3, 0.5, 0.7, 0.9, and calculate the quantities $x$, $v_2$, $P_0$ as $n = 3$, 5, 6 and $q = 0.2$, 0.5, 1 when $s = 3/4$, $s = 2$, respectively. The results are shown in Table I and II.

From Table I and II, it can be seen that $x$ is unrelated to $q$, but incremental with the increase of $\chi(n)$ at certain $n(\chi)$. $v_2$ decreases with the increase $\chi$ and $n$ with $s = 3/4$. However, when $s = 2$, $v_2$ decreases with the increase $\chi$ and is nonmonotonic with $n$. $P_0$ increases with the incremental $\chi(n)$ and decreases with the increasing $q$. The doubt is whether $P_0$ is negative when the temperature is low enough.

**IV. MAXWELL EQUAL AREA LAW: SOME EXAMPLES**

In order to further study the phase transition for the charged black hole with PMI, we expand the method of Maxwell’s equal-area law to the canonical ensemble and grand
TABLE I: For $s = 3/4$, State quantities at phase transition endpoints with different parameters $q$ and spacetime dimensional $n$

|   | $n = 3$       |       | $n = 5$       |       | $n = 6$       |       |
|---|------------|-------|------------|-------|------------|-------|
|   | $q$ | $\chi$ | $x$    | $v_2$ | $P_0$  | $x$ | $v_2$ | $P_0$  | $x$ | $v_2$ | $P_0$  |
| 0.2 | 0.9 | 0.4755 | 4.1100 | 0.1120 | 0.5954 | 1.6638 | 0.1126 | 0.6267 | 1.2719 | 0.2009 |
|   | 0.7 | 0.2182 | 7.7148 | 0.0062 | 0.3240 | 2.7803 | 0.0610 | 0.3533 | 2.0815 | 0.1115 |
|   | 0.5 | 0.0859 | 17.8745| 0.0022 | 0.1491 | 5.7145 | 0.0246 | 0.1678 | 4.1802 | 0.0459 |
|   | 0.3 | 0.0166 | 86.3399| 0.0003 | 0.0378 | 21.65  | 0.0046 | 0.0449 | 15.0825| 0.0090 |
|   | 0.1 | 0.0002 | 68530.6| 1.41E-7| 0.0002 | 4269.4 | 8.54E-6| 0.0003 | 2267.96| 0.0002 |
| 0.5 | 0.9 | 0.4755 | 5.8389 | 0.0060 | 0.5954 | 1.9090 | 0.0845 | 0.6267 | 1.4137 | 0.1626 |
|   | 0.7 | 0.2182 | 10.878 | 0.0031 | 0.3240 | 3.1900 | 0.04635| 0.3533 | 2.3135 | 0.0902 |
|   | 0.5 | 0.0859 | 25.2035| 0.0011 | 0.1491 | 6.5565 | 0.0187 | 0.1678 | 4.6464 | 0.0372 |
|   | 0.3 | 0.0166 | 121.742| 0.0002 | 0.0378 | 24.816 | 0.0035 | 0.0449 | 16.7645| 0.0072 |
|   | 0.1 | 0.0002 | 96630  | 7.14E-8| 0.0002 | 4898.4 | 6.48E-6| 0.0003 | 2520.88| 0.0002 |
| 1  | 0.9 | 0.4755 | 7.5721 | 0.0035 | 0.5954 | 2.1181 | 0.0687 | 0.6267 | 1.5314 | 0.1386 |
|   | 0.7 | 0.2182 | 14.107 | 0.0018 | 0.3240 | 3.5395 | 0.0377 | 0.3533 | 2.5062 | 0.0769 |
|   | 0.5 | 0.0859 | 32.685 | 0.0007 | 0.1491 | 7.2749 | 0.0152 | 0.1678 | 5.0333 | 0.0317 |
|   | 0.3 | 0.0166 | 157.879| 0.0001 | 0.0378 | 27.5302| 0.0028 | 0.0449 | 18.16  | 0.0062 |
|   | 0.1 | 0.0002 | 125314 | 4.25E-8| 0.0002 | 5433.1 | 5.26E-6| 0.0003 | 2730.77| 0.0001 |

canonical ensemble.

A. BTZ-like black holes

one can select an ensemble in which black hole charge is fixed at infinity. Considering the fixed charge as an extensive parameter, the corresponding ensemble is called a canonical ensemble. Interestingly, for $s = n/2$, the solutions (the so-called BTZ black holes) have different properties. As we will see, for $s = n/2$ the charge term in metric function is logarithmic and the electromagnetic field is proportional to $r^{-1}$ (logarithmic gauge potential). In other words, in spite of some differences, this special higher dimensional solution has some
TABLE II: For $s = 2$, State quantities at phase transition endpoints with different parameters $q$ and spacetime dimensional $n$

| $q$ | $\chi$ | $x$  | $v_2$ | $P_0$  | $x$  | $v_2$ | $P_0$  | $x$  | $v_2$ | $P_0$  |
|-----|--------|------|-------|--------|------|-------|--------|------|-------|--------|
| 0.2 | 0.9    | 0.2817 | 0.00002 | 3.9E8  | 0.4513 | 0.1033 | 28.5   | 0.4961 | 0.3023 | 3.61469 |
|     | 0.7    | 0.0815 | 0.00005 | 1.6E8  | 0.1985 | 0.1980 | 14.4142 | 0.2355 | 0.5508 | 1.8853  |
|     | 0.5    | 0.0200 | 0.0002  | 3.9E7  | 0.0750 | 0.4737 | 5.0536 | 0.0957 | 1.2447 | 0.6968  |
|     | 0.3    | 0.0016 | 0.0018  | 2.4E6  | 0.0135 | 2.4421 | 0.6720 | 0.0195 | 5.7158 | 0.1047  |
|     | 0.1    | 2.4E-8 | 114.59  | 12.9982 | 0.00001 | 2798.52 | 0.0002 | 0.00003 | 3405.2 | 0.00006 |
| 0.5 | 0.9    | 0.2817 | 0.0045  | 6472.47 | 0.4513 | 0.3100 | 3.1617 | 0.4961 | 0.6632 | 0.7514  |
|     | 0.7    | 0.0815 | 0.0117  | 2658.28 | 0.1985 | 0.5945 | 1.5986 | 0.2355 | 1.2080 | 0.3919  |
|     | 0.5    | 0.0200 | 0.0404  | 663461 | 0.0750 | 1.4226 | 0.5605 | 0.0957 | 2.7298 | 0.1448  |
|     | 0.3    | 0.0016 | 0.4365  | 41.1060 | 0.0135 | 7.3332 | 0.0745 | 0.0195 | 12.5362 | 0.0217  |
|     | 0.1    | 2.4E-8 | 27247.8 | 0.0002  | 0.00001 | 8403.42 | 0.00002 | 0.00003 | 7469.43 | 0.00001 |
| 1   | 0.9    | 0.2817 | 0.2864  | 1.5802  | 0.4513 | 0.7122 | 0.5990 | 0.4961 | 1.2013 | 0.229   |
|     | 0.7    | 0.0815 | 0.7466  | 0.6489  | 0.1985 | 1.3658 | 0.3029 | 0.2355 | 2.1883 | 0.1194  |
|     | 0.5    | 0.0200 | 2.5863  | 0.162   | 0.0750 | 3.2682 | 0.1062 | 0.0957 | 4.9490 | 0.0441  |
|     | 0.3    | 0.0016 | 27.9351 | 0.01    | 0.0135 | 16.8472 | 0.0141 | 0.0195 | 22.7087 | 0.0066  |
|     | 0.1    | 2.4E-8 | 1.74E6  | 5.47E-8 | 0.00001 | 19306  | 4.35E-6 | 0.00003 | 13530  | 3.98E-6 |

similarity with the charged BTZ solution and reduces to the original BTZ black hole for $n = 2$ \[19\].

Considering the metric (2.3) and the field equations of the bulk action (2.1) with $s = n/2$, we can find that the metric function $f(r)$ and the gauge potential may be written as \[19\]

$$f(r) = 1 + \frac{r^2}{l^2} - \frac{m}{r^{n-2}} - \frac{2^{n/2} q^n}{r^{n-2}} \ln \left( \frac{r}{l} \right),$$  \[(4.1)\]

$$A = q \ln \left( \frac{r}{l} \right) dt,$$  \[(4.2)\]

Straightforward calculation show that BTZ-like spacetime has a curvature singularity located at $r = 0$ in which covered with an event horizon. The temperature of this black hole be
obtained as
\[
T = \frac{n - 2}{4\pi r_+} \left( 1 + \frac{n}{n - 2} \frac{r_+^2}{l^2} - \frac{2^{n/2} q^n}{(n - 2)r_+^{n-2}} \right). \tag{4.3}
\]

Substituting eq. (2.16) into eq. (4.3), we can obtain
\[
P = \frac{T}{v} - \frac{(n - 2)}{\pi (n - 1)v^2} + \frac{1}{16\pi} \frac{k' q^n}{v^n}, \tag{4.4}
\]
where
\[
k' = \frac{2^{5n/2}}{(n - 1)^{n-1}}, \quad v = \frac{4}{(n - 1)r_+}. \tag{4.5}
\]

Substituting eq. (4.4) into eq. (3.1), we can obtain
\[
T_0 v_2^{n-1} x^{n-1} = A v_2^{n-2} x^{n-2} (1 + x) - B' \frac{1 - x^n}{1 - x}, \tag{4.6}
\]
and
\[
v_2^{n-2} = B' \frac{n(1 - x^{n-1})(1 - x) + (n - 1)(1 - x^n) \ln x}{A x^{n-2}(n - 1)(1 - x) (2(1 - x) + (1 + x) \ln x)} = f_1(x), \tag{4.7}
\]
with the method which used the above section, with \(B' = \frac{k' q^n}{4\pi} \). Substituting eq. (4.7) into (4.6)
\[
\chi T_c x^{n-1} f_1^{(n-1)/(n-2)}(x) = A f_1(x) x^{n-2} (1 + x) - B' \frac{1 - x^n}{1 - x}, \tag{4.8}
\]
when \(x \to 1\), from the eq. (4.7), we can get
\[
f_1(1) = \frac{n(n - 1)B'}{2A}, \tag{4.9}
\]
So, the critical point meet with
\[
v_2^{n-2} = v_1^{n-2} = v_c^{n-2} = \frac{n(n - 1)B'}{2A} = \frac{k' n(n - 1)^2 q^n}{32(n - 2)}. \tag{4.10}
\]
Combining (4.10), (4.6) and (4.8), we can obtain
\[
T_c = \frac{2A(n - 2)}{(n - 1)} \left( \frac{2A}{n(n - 1)B'} \right)^{1/(n-2)} = \frac{2(n - 2)^2}{\pi (n - 1)^2} \left( \frac{32(n - 2)}{k' n(n - 1)^2 q^n} \right)^{1/(n-2)}, \tag{4.11}
\]
\[
P_c = \frac{A(n - 2)}{n} \left( \frac{2A}{n(n - 1)B'} \right)^{2/(n-2)} = \frac{(n - 2)^2}{\pi n(n - 1)} \left( \frac{32(n - 2)}{k' n(n - 1)^2 q^n} \right)^{2/(n-2)}. \tag{4.11}
\]
Combining (4.11) and (4.8), we can get
\[
\chi x^{n-1} f_1^{(n-1)/(d-2)}(x) \frac{2A(n - 2)}{(n - 1)} \left( \frac{2A}{n(n - 1)B'} \right)^{1/(n-2)} = A f_1(x) x^{n-2} (1 + x) - B' \frac{1 - x^n}{1 - x}, \tag{4.12}
\]
We plot the $P - T$ curves with $0 < x \leq 1$ in Fig. 3 when the parameters $n$, $s$, $q$ take different values respectively. The curves represent two-phase equilibrium condition for the charged AdS black hole and the terminal points of the curves represent corresponding critical points. From fig. 3 it can be seen that the influence of the electric charge $q$ and spacetime $n$ on the phase diagrams, however, pressure $P_0$ tends zero with decreasing temperature $T_0$ for all of the fixed $q$ and $n$ cases. The process of phase transition becomes longer as the spacetime dimensional $n$ is increase. That the pressure $P_0$ is always positive means Maxwell’s equal area law is appropriate to resolve the doubts about the negative pressure and unstable states in the phase transition of the BTZ-like black hole.

**B. Grand canonical ensemble**

In addition to canonical ensemble, one can work with a fixed electric potential at infinity. The ensemble of this fixed intensive quantity translates into the grand canonical ensemble. It is worthwhile to note that, for linear Maxwell field, the criticality cannot happen in the grand canonical ensemble [1, 62].

In this section, we study the critical behavior of charged black holes in the grand canonical (fixed $\Phi$) ensemble. We take $q = \Phi r_+^{(n-2s)/(2s-1)}$ with $v = \frac{4r_+}{n-1}$ to rewrite Eq. (2.19) in the
following form

\[ P = \frac{T}{v} - \frac{(n - 2)}{\pi(n - 1)v^2} + \frac{2s - 1}{16\pi} \left(\frac{4\sqrt{2}(n - 2s)\Phi}{(2s - 1)(n - 1)v}\right)^{2s}, \]  

(4.13)

Using the method in the above section, substituting Eq. (4.13) into Eq. (3.1), we can obtain

\[ T_0v_2^{2s-1}x^{2s-1} = Av_2^{2s-2}x^{2s-2}(1 + x) - B''\frac{1 - x^{2s}}{1 - x}, \]  

(4.14)

and

\[ v_2^{2s-2} = B'' \frac{2s(1 - x^{2s-1})(1 - x) + (2s - 1)(1 - x^{2s})\ln x}{A \ x^{2s-2}(2s - 1)(1 - x) (2(1 - x) + (1 + x)\ln x)} = f_2(x), \]  

(4.15)

where \( B'' = \frac{2s-1}{\pi} \left(\frac{4\sqrt{2}(n-2s)\Phi}{(2s-1)(n-1)}\right)^{2s} \). Substituting (4.15) into (4.14), we have

\[ \chi T x^{2s-1} f_2^{(2s-1)/(2s-2)}(x) = Af_1(x)x^{2s-2}(1 + x) - B''\frac{1 - x^{2s}}{1 - x}. \]  

(4.16)

When \( x \to 1 \), from (4.15), we can obtain

\[ f_2(1) = \frac{2s(2s - 1)B''}{2A}. \]  

(4.17)

So, the critical point satisfy

\[ v_2^{2s-2} = v_1^{2s-2} = v_c^{2s-2} = \frac{s(2s - 1)B''}{A} = \frac{4\sqrt{2}(n - 2s)}{(2s - 1)(n - 1)} \left(\frac{32s(2s - n)^2}{(n - 2)(n - 1)^2}\right)^{1/(2s-2)} \Phi^{s/(s-1)}. \]  

(4.18)

Applying Eqs. (4.14) and (4.13) to the states equation, it is easy to calculate the critical temperature and critical pressure

\[ T_c = \frac{4A(s - 1)}{(2s - 1)} \left(\frac{A}{s(2s - 1)B''}\right)^{1/(2s-2)} = \frac{(s - 1)(n - 2)}{2\pi} \frac{(n - 2)(n - 1)}{32s(n - 2s)^2} \Phi^{s/(s-1)}, \]  

\[ P_c = \frac{A(s - 1)}{s} \left(\frac{A}{s(2s - 1)B''}\right)^{1/(s-1)} = \frac{(s - 1)(2s - 1)^2}{s\pi} \frac{(n - 1)(n - 2)}{32(n - 2s)^2 s^{1/s}} \Phi^{-2s/(s-1)}. \]  

(4.19)

Combining (4.19) and (4.16) and taking \( \chi \) is constant, we find that \( x \) satisfy the equation

\[ \chi x^{2s-1} f_2^{(2s-1)/(2s-2)}(x) \frac{2A(2s - 2)}{(2s - 1)} \left(\frac{A}{s(2s - 1)B''}\right)^{1/(2s-2)} = Af_2(x)x^{2s-2}(1 + x) - B''\frac{1 - x^{2s}}{1 - x}. \]  

(4.20)

For a fixed \( \chi \), i.e. a fixed \( T_0 \), we can get a certain \( x \) from Eq. (4.20), and then according to Eqs. (4.13) and (4.15), the \( v_2 \) and \( P_0 \) are solved. The corresponding \( v_1 \) can be got
FIG. 4: \( P - v \) diagram of charged AdS black holes in PMI for \( s = \frac{6}{5} \) with \( n = 3 \) (left) and \( s = \frac{5}{4} \) with \( n = 4 \) (right). The temperature of isotherms decreases from top to bottom. The bold line is the critical isotherm diagram.

from \( x = v_1/v_2 \). Join the points \((v_1, P_0)\) and \((v_2, P_0)\) on isotherms in \( P - v \) diagram, which generate an isobar representing the process of isothermal phase transition or the two phase coexistence situation like that of van der Waals system. Fig.4 shows the isobars on the background of isotherms at different temperature and the boundary of the two-phase equilibrium region by the dot-dashed curve as \( n = 3, s = 6/5 \) and \( n = 4, s = 5/4 \), respectively. The isothermal phase transition process becomes shorter as the temperature goes up until it turns into a single point at a certain temperature, which is critical temperature, and the point corresponds to critical state of the charged AdS black hole with PMI in the grand canonical ensemble.

V. CONCLUDING REMARKS

In this paper we have extended the idea of fluid/gravity analog in order to provide a new picture of the isothermal behavior of critical charged black hole in AdS background with a nonlinear source. The results of this method is that physical black hole undergoes an isothermal transition from gas to liquid phase at constant pressure. Consequently there are neither regions with negative nor divergent specific heat. Furthermore, we were able to obtain analytic solutions the area law both in canonical(grand canonical) ensemble. We conclude that working in the \((P, v)\) plane gives non-trivial advantage with respect to the Van der Walls description in \((P, V)\) plane.

The charged AdS black hole with PMI is regarded as a thermodynamic system, and its state equation has been derived. But when temperature is below critical temperature,
thermodynamic unstable situation appears on isotherms, and when temperature reduces to a certain value the negative pressure emerges, which can be seen in Fig.1, Fig.2 and Fig.4. However, by Maxwell equal law we established an phase transition process and the problems can be resolved. The phase transition process at a defined temperature happens at a constant pressure, where the system specific volume changes along with the ratio of the two coexistent phases. According to Ehrenfest scheme the phase transition belongs to the first order one, which can be seen in Table 1 and Table 2. We draw the isothermal phase transition process and depict the boundary of two-phase coexistence region in Fig.4. The obtained $P_0 - T_0$(Fig.3) diagram for different dimensions $n$ shows that as dimensionality increases, the temperature of critical points increases which indicates the necessity of more energy for having a phase transition.

Taking black hole as an thermodynamic systems, many investigations show the phase transition of some black holes in AdS spacetime and dS spacetime is similar to that of van der Waals-Maxwell gas-liquid system [43], and the phase transition of some other AdS black hole is alike to that of multicomponent superfluid or superconducting system[6–8]. It would make sense if we can seek some observable system, such as van der Waals gas, to back analyze physical nature of black holes by their similar thermodynamic properties. That would help to further understand the thermodynamic quantities, such as entropy, temperature, heat capacity and so on, of black hole and that is significant for improving self-consistent thermodynamic theory of black holes. Also, we have applied the same procedure for the BTZ-like black holes to obtain their phase transition. Calculations showed that thermodynamic behaviors of BTZ-like black holes are the same as PMI ones. Moreover, we have studied the grand canonical ensemble in which the potential, instead of charge, should be fixed on the boundary. In contrast to the Maxwell case, here one sees a phase transition. Finally, Perhaps a holographic approach helps us to have a better understanding of this problem. We leave the study of these interesting questions for future studies.

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