Quantaloidal approach to constraint satisfaction

Soichiro Fujii, Yuni Iwamasa and Kei Kimura

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Quantaloids

= \{\text{complete join-semilattices}\}-enriched categories

Quantaloidal approach to constraint satisfaction

Constraint satisfaction problem (CSP):
general framework for computational problems
including $k$-SAT, graph $k$-colouring, ...

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Special case

TVCSP (Optimisation problem)

Quantaloids

\( \mathcal{P}\text{FinSet} \)

\( \mathcal{Q}\text{FinSet} \)

\( \mathcal{Q}: \text{quantale} \)

\( \mathcal{R}\text{FinSet} \)

quantale

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\(\overline{\mathcal{R}\text{FinSet}}\)

Special case
Graph $k$-colouring ($k \in \mathbb{N}$)

$\exists s: \{v_1, \ldots, v_5\} \rightarrow \{1, \ldots, k\}$ s.t. $\forall$ edge $(v_i, v_j)$, $s(v_i) \neq s(v_j)$?

Ex. $k = 3$ \{ \, \, \, \, \, \, \, \}
A CSP instance $I = (V, D, \mathcal{C})$ consists of:
- $V$: finite set of variables
- $D$: finite set called the domain
- $\mathcal{C}$: finite set of “constraints”

A constraint is $(k, x, \rho)$ where
- $k \in \mathbb{N}$, $x \in V^k$, $\rho \subseteq D^k$.

A function $s : V \rightarrow D$ satisfies the constraint $(k, x = (x_1, \ldots, x_k), \rho)$ if $(s(x_1), \ldots, s(x_k)) \in \rho$.

A solution of $I = (V, D, \mathcal{C})$ is a function $s : V \rightarrow D$ satisfying every constraint in $\mathcal{C}$.

$\mathcal{S}(I) = \{\text{solutions of } I\} \subseteq [V, D]$
Ex. Graph $k$-colouring

$V = \{v_1, \ldots, v_5\}$
$D = \{1, \ldots, k\}$
$C = \{(2, (v_i, v_j), \neq \subseteq D^2) \mid (v_i, v_j): \text{edge}\}$

$\exists s: \{v_1, \ldots, v_5\} \rightarrow \{1, \ldots, k\}$ s.t. $\forall \text{edge} (v_i, v_j), s(v_i) \neq s(v_j)$?

A function $s: V \rightarrow D$ satisfies the constraint $(k', x = (x_1, \ldots, x_{k'}), \rho)$ if $(s(x_1), \ldots, s(x_{k'})) \in \rho$. 
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Finite Set

Special case

Finite Set

Special case

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The 2-category $\mathcal{P}\text{FinSet}$:

| Obj. | Finite sets |
|------|-------------|
| Mor. | $A \xrightarrow{\varphi} B$ |

$\varphi \subseteq [A, B]$

| Comp. | $A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ |
|-------|--------------------------------------------------|
| $\psi \circ \varphi = \{ g \circ f \mid g \in \psi, f \in \varphi \}$ |

| Id. | $A \xrightarrow{\text{id}_A} A$ |

| 2-cell | $A \xrightarrow{\varphi} B$ |
|-------|-----------------------------|
| $\varphi' \subseteq \varphi$ |

$\mathcal{P}\text{FinSet}$ is a quantaloid (the free quantaloid over $\text{FinSet}$):

- $\forall A, B \in \mathcal{P}\text{FinSet}$, $\mathcal{P}\text{FinSet}(A, B) = (\mathcal{P}[A, B], \subseteq)$ is a complete lattice.

- $\forall A, B, C \in \mathcal{P}\text{FinSet}$,

  $\mathcal{P}\text{FinSet}(B, C) \times \mathcal{P}\text{FinSet}(A, B) \xrightarrow{\circ} \mathcal{P}\text{FinSet}(A, C)$

  preserves arbitrary joins in each variable:

  $\psi \circ \left( \bigvee_{i \in I} \varphi_i \right) = \bigvee_{i \in I} (\psi \circ \varphi_i)$

  $(\bigvee_{i \in I} \psi_i) \circ \varphi = \bigvee_{i \in I} (\psi_i \circ \varphi)$
In particular,

- \( \forall A \xrightarrow{\varphi} B, C \in \mathcal{P}\text{FinSet}, \)

\[
\mathcal{P}\text{FinSet}(\varphi, C) : \mathcal{P}\text{FinSet}(B, C) \longrightarrow \mathcal{P}\text{FinSet}(A, C)
\]

preserves arbitrary joins.

\( \iff \) \( \mathcal{P}\text{FinSet}(\varphi, C) \) has a right adjoint

\[
\psi \downarrow \varphi : \mathcal{P}\text{FinSet}(A, C) \longrightarrow \mathcal{P}\text{FinSet}(B, C)
\]
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\( \mathcal{Q} \text{FinSet} \)

\( \mathcal{Q} : \text{quantale} \)

\( \mathcal{R} \text{FinSet} \)

Special case

Special case

Quantaloids

Special case
A **CSP instance** $I = (V, D, \mathcal{C})$ consists of:
- $V$: finite set of **variables**
- $D$: finite set called the **domain**
- $\mathcal{C}$: finite set of “constraints”

A constraint is $(k, x, \rho)$ where
- $k \in \mathbb{N}$, $x \in V^k$, $\rho \subseteq D^k$.

A function $s : V \rightarrow D$ **satisfies** the constraint $(k, x = (x_1, \ldots, x_k), \rho)$ if $(s(x_1), \ldots, s(x_k)) \in \rho$.

A **solution** of $I = (V, D, \mathcal{C})$ is a function $s : V \rightarrow D$ satisfying every constraint in $\mathcal{C}$.

$s(I) = \{ \text{solutions of } I \} \subseteq [V, D]$
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\(\mathcal{Q}\)FinSet
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Polymorphisms

Q-valued polymorphisms

Q: quantale

\( \mathcal{Q} \text{FinSet} \)

R-valued polymorphisms

\( \mathcal{R} \text{FinSet} \)

TV CSP (Optimisation problem)
Overview

CSP —— Quantaloidal CSP —— Polymorphisms —— TVCSP (Optimisation problem) —— TVCSP

(Computational) problems

Quantaloids

\( \mathcal{P} \text{FinSet} \)

\( \mathcal{Q} \text{FinSet} \)

\( \mathcal{Q} \): quantale

\( \mathcal{R} \text{FinSet} \)

\( \mathcal{R} \)-valued polymorphisms

Quantaloidal CSP

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\( \mathcal{Q} \)-valued polymorphisms

\( \mathcal{Q} \)-valued polymorphisms

\( \mathcal{R} \)-valued polymorphisms
Dichotomy theorem. [Bulatov 2017, Zhuk 2020]
For each “constraint language” $\mathcal{D}$, 
CSP($\mathcal{D}$) is either in P or is NP-complete.

A constraint language $\mathcal{D}$ consists of
- $D$: finite set
- $(\rho_i \subseteq D^k)_{i \in I}$: finite family of relations on $D$.

$\mathcal{D} = (D, (\rho_i)_{i \in I})$: constraint language
CSP($\mathcal{D}$): set of CSP instances defined by

$I = (V, D', \mathcal{C}) \in \text{CSP}(\mathcal{D}) \iff D' = D \text{ and } \forall (k, x, \rho) \in \mathcal{C}, \rho \in \mathcal{D}$
When is $\text{CSP}(\mathcal{D})$ easy to solve?
- $\text{CSP}(\mathcal{D})$ is in $\mathsf{P}$ if $\mathcal{D}$ admits enough “symmetry”
- $\text{CSP}(\mathcal{D})$ is $\mathsf{NP}$-complete otherwise

The relevant “symmetry” of $\mathcal{D}$ is captured by polymorphisms of $\mathcal{D}$

$\quad= \text{homomorphisms (of relational structures)} \quad \mathcal{D}^n \to \mathcal{D}.$

**Dichotomy theorem.** [Bulatov 2017, Zhuk 2020]

$\mathcal{D}$: constraint language

$\forall x, y, z \in D. \ f(y, x, y, z) = f(x, y, z, x)$

- $\text{CSP}(\mathcal{D})$ is in $\mathsf{P}$ if $\mathcal{D}$ admits a Siggers operation $f: D^4 \to D$ as a polymorphism
- $\text{CSP}(\mathcal{D})$ is $\mathsf{NP}$-complete otherwise.
$\mathcal{D} = (D, (\rho_i)_{i \in I})$: constraint language

$\forall n \in \mathbb{N}$, let $\text{Pol}(\mathcal{D})_n = \{n\text{-ary polymorphisms of } \mathcal{D}\}$

$= \{\text{homomorphisms } \mathcal{D}^n \rightarrow \mathcal{D}\}$

Assume $I$: singleton, so that $\mathcal{D} = (D, \rho \subseteq D^k)$.

Then $\text{Pol}(\mathcal{D})_n : D^n \rightarrow D$ is given by:
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\( \mathbb{Q} \)-valued polymorphisms

\( \mathbb{R} \)-valued polymorphisms

\( \mathbb{Q} : \text{quantale} \)
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Quantaloidal CSP

\(\mathcal{Q}\text{FinSet}\)

\(\mathcal{Q}:\text{quantale}\)

\(\text{FinSet}\)

\(\mathbb{R}\)-valued polymorphisms

\(\mathcal{Q}\)-valued polymorphisms

TVCSP

(Optimisation problem)

\(\mathbb{R}\)-valued polymorphisms
A quantale is a one-object quantaloid.

Explicitly,
\( \mathcal{Q} = (Q, \leq, e, \otimes) \) is a quantale if
• \((Q, \leq)\): complete lattice
• \((Q, e, \otimes)\): monoid
satisfying:
\[
\alpha \otimes (\bigvee_{i \in I} \beta_i) = \bigvee_{i \in I} (\alpha \otimes \beta_i)
\]
\[
(\bigvee_{i \in I} \alpha_i) \otimes \beta = \bigvee_{i \in I} (\alpha_i \otimes \beta)
\]
\( Q = (Q, \leq, e, \otimes) \): quantale

The quantaloid \( \mathcal{Q} \text{FinSet} \):

**Obj.** Finite sets

**Mor.**

\[
\frac{A \overset{\varphi}{\longrightarrow} B}{\varphi : [A, B] \to Q}
\]

(\(\psi \circ \varphi\))(h) = \(\bigvee\{\psi(g) \otimes \varphi(f) \mid f: A \to B, g: B \to C, g \circ f = h\}\)

"Singleton" morphism

\[
\frac{A \overset{f}{\longrightarrow} B}{\{f\} : [A, B] \to Q}
\]

\[
g \longmapsto \begin{cases} e & \text{if } g = f \\ \bot_Q & \text{otherwise} \end{cases}
\]

**Comp.**

\[
A \overset{\varphi}{\longrightarrow} B \overset{\psi}{\longrightarrow} C
\]

**Id.**

\[
\{\text{id}_A\}
\]

2-cell

\[
\frac{A \downarrow \varphi \quad \varphi' \downarrow B}{\varphi \leq \varphi'}
\]
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\(\mathcal{P}\text{FinSet} \uparrow \mathcal{Q} = 2\)

\(\mathcal{Q}\text{FinSet} \downarrow \mathcal{Q}: \text{quantale}\)

\(\mathcal{R}\text{-valued polymorphisms}\)

\(\mathcal{R}\text{FinSet} \Downarrow \mathcal{R}\text{-valued polymorphisms}\)
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\(\mathcal{P}\text{FinSet}\)

\(\mathcal{Q}\text{FinSet}\)

\(\mathcal{Q}:\text{quantale}\)

\(\uparrow\mathcal{Q} = 2\)

Special case

\(\mathcal{Q}\text{-valued polymorphisms}\)

\(\mathcal{R}\text{-valued polymorphisms}\)

(Computational) problems

(Optimisation problem)
A \( Q \)-valued CSP instance \( I = (V, D, \mathcal{C}) \) consists of:

- \( V \): finite set of variables
- \( D \): finite set called the domain
- \( \mathcal{C} \): finite set of \( \langle Q \text{-valued} \rangle \) constraints

\( A \) \( Q \)-valued constraint is \( (k, x, \rho) \) where

- \( k \in \mathbb{N}, \quad x \in V^k, \quad \rho \subseteq D^k \).

Each \( Q \)-valued constraint \( (k, x, \rho) \) yields

\[
\mathcal{S}(I) = \bigwedge_{(k,x,\rho) \in \mathcal{C}} \rho \not \subseteq \{x\} : V \to D
\]

\[
\mathcal{S}(I) : [V, D] \to Q
\]
A \( \mathcal{Q} \)-valued constraint language \( \mathcal{D} \) consists of

- \( D \): finite set
- \( (\rho_i \subseteq D^{k_i})_{i \in I} \): finite family of relations on \( D \)
- \( (\rho_i : [k_i] \rightarrow D)_{i \in I} \): finite family of morphisms in \( \mathcal{Q} \text{FinSet} \)

Assume \( I \): singleton, so that \( \mathcal{D} = (D, \rho : [k] \rightarrow D) \).

Then \( \text{Pol}(\mathcal{D})_n : D^n \rightarrow D \) is given by:

\[
\begin{align*}
\text{Pol}(\mathcal{D})_n & : [D^n, D] \rightarrow Q \\
\text{Pol}(\mathcal{D})_n(f) \in Q & : \text{the “degree” to which } f : D^n \rightarrow D \text{ is a polymorphism of } \mathcal{D}
\end{align*}
\]
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$\mathcal{Q} \text{FinSet}$

$\mathcal{Q} = 2$

$\mathcal{Q}: \text{quantale}$

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$\overline{\mathcal{R}} \text{FinSet}$
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$\mathcal{P}\FinSet$

$\mathcal{Q} = 2$

$\mathcal{Q}\FinSet$

$\mathcal{Q} : \text{quantale}$

$\mathcal{R}\FinSet$

$\mathcal{Q} = \mathbb{R}$
Letting \( Q = \overline{\mathbb{R}} = (\mathbb{R} \cup \{ \pm \infty \}, \geq, 0, +) \) (cf. [Lawvere 1973]), we obtain a class of optimisation problems:

\[
\inf_{s: V \rightarrow D} \sup_{(k, x, \rho) \in \mathcal{C}} \rho(s(x_1), \ldots, s(x_k))
\]

which we call “tropical valued CSPs”.

**Dichotomy theorem for TVCSPs.**

\( \mathcal{D} : \overline{\mathbb{R}} \)-valued constraint language

- TVCSP(\( \mathcal{D} \)) is in P if there exists a Siggers operation \( f: D^4 \rightarrow D \) with \( 0 \geq \text{Pol}(f)_4 \).
- TVCSP(\( \mathcal{D} \)) is NP-hard otherwise.

* For a slightly more expressive version of TVCSPs.
Summary

CSP

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\( Q = 2 \)

\( \mathcal{Q}\text{-valued polymorphisms} \)

\( Q: \text{quantale} \)

\( Q = \overline{\mathbb{R}} \)

(Computational) problems

Dichotomy theorem