1. Introduction

The fact that symmetries which are spontaneously broken at zero temperature may be restored at a sufficiently large finite temperature is by now well-appreciated\textsuperscript{1,2,3}. The so-called Unitary Gauge Puzzle (UGP) in the context of the Abelian Higgs model arose almost immediately upon the initial realisation that symmetry restoration was possible\textsuperscript{2,3}.

A methodology was proposed for the estimation of the temperature at which symmetry restoration occurs. The transition or critical temperature, $T_c$, is characterised by the vanishing of a suitable order-parameter (it is implicitly assumed that the phase transition is of second-order). The effective mass of the Higgs field fulfills this role. In the standard method proposed for this analysis, the effective thermal mass is determined perturbatively through construction of the temperature dependent Effective Potential,

$$m_{eff}^2 = m^2 + 2 \frac{\partial^2 V(\phi)}{\partial \phi^2} \bigg|_{\phi=0}.$$  

The Effective Potential is the generator of 1PI (vertex) functions evaluated at zero external four-momentum. It is a function of $\phi$, the translationally invariant (constant) field. The $\beta = 1/T$ superscript indicates that attention is being paid to the temperature-dependent contributions; the vacuum (\textit{viz.}, $T = 0$) parts comprise the definition of $m^2$. The second derivative generates the thermal contribution to the 1PI two-point function, evaluated at zero external four-momentum, and has a natural interpretation as a correction to the mass-squared. The ultimate evaluation

\textsuperscript{*} This paper reports upon work which was undertaken in collaboration with R. Kobes and G. Kunstatter.
of this quantity is about the zero-field configuration corresponding to the symmetric (restored) minimum: \( \phi = 0 \), although the calculation takes place in the broken phase.

In the Abelian Higgs model, where the objects in the theory are a complex scalar field, \( \Phi \), and a \( U(1) \) local gauge field, \( A_\mu \), the Euclidean Lagrangian density is:

\[
L_E = (D_\mu \Phi)^\dagger (D^\mu \Phi) + \mu^2 |\Phi|^2 - \frac{\lambda}{3!} (|\Phi|^2)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + L_{\text{Gauge}} .
\]  

Note that the coefficient of the quadratic term has the appropriate (“opposite”) sign for spontaneous symmetry breaking (SSB). Implementing the programme sketched above is a straightforward task, but there remains the detail of choosing a parameterisation of the scalar field.

One natural choice is to decompose \( \Phi \) into its real and imaginary parts,

\[
\Phi = \frac{1}{\sqrt{2}} [\phi^R + i\phi^I] .
\]  

As this parameterisation maintains the manifest (power-counting) renormalisability of the model, the set of variables \( \{\phi^R, \phi^I, A_\mu\} \) has been dubbed “renormalisable gauge.” Symmetry breaking is assumed to occur such that the \( \phi^R \) field is shifted by a vacuum expectation value (VEV), while the quantum fluctuations of \( \phi^I \), and \( A_\mu \) are about zero. One-loop calculation of the order-parameter and insistence upon its vanishing at \( \beta_c \) leads to the following relation, used to estimate \( \beta_c \):

\[
0 = \mu^2 - \frac{1}{12\beta_c} \left[ 3e^2 + \frac{\lambda}{2} + \frac{\lambda}{6} \right] + O(\beta_c^{-1}) .
\]

Yet another natural choice for the parameterisation of \( \Phi \) is an expression in terms of modulus, \( \rho \), and angle \( \theta/\rho_0 \), viz.,

\[
\Phi = \frac{1}{\sqrt{2}} \rho \, e^{i\theta/\rho_0} .
\]

The angular argument is scaled by a dimensionful factor so that the fields \( \rho \) and \( \theta \) have the same canonical dimension as \( \Phi \). The most natural choice of scale is \( \rho_0 \), the zero temperature VEV of the modulus field. It so happens that the “hidden symmetry” of the system enables the Higgs mechanism to operate, and the \( \theta \)-field may be eliminated by a redefinition of the vector-gauge field \( A_\mu \). As this parameterisation makes manifest the true degrees of freedom of the model, it has been dubbed “unitary gauge”. The quantum fluctuations of the modulus field are shifted away from zero by \( \rho_0 \), while those of the combined vector-gauge-Goldstone mode are about zero. One-loop calculation of the order-parameter, and the demand that it vanishes at the critical temperature leads to an estimate of \( \beta_c \) from:

\[
0 = \mu^2 - \frac{1}{12\beta_c^2} \left[ 3e^2 + \frac{\lambda}{2} \right] + O(\beta_c^{-1}) .
\]

\[\text{We regret this usage because it confuses true gauge issues with those of parameterisation.}\]
Surprisingly, equations (4) and (6) do not agree. This is the UGP. Expression (4) is deemed to be correct.

There have been several proposed resolutions to this perplexing problem which may be assigned to two schools of thought. The first advocates some means by which the unitary calculation may be “fixed” so as to obtain the correct result. Ueda proposed to modify the method by adding and subtracting self-energy terms and evaluating these on the mass-shell as determined at tree-level. Ueda, however, was not able to provide a compelling rationale for his refinement of the method (other than that he was able to get agreement between the two parameterisations at one-loop order). Arnold, Braaten, and Vokos (ABV) suggested that the problem arose as a consequence of a breakdown of the loop-expansion and were able to obtain consistent results by an order-by-order reformulation in powers of temperature. Their analysis indicated that higher-order terms of the form \( (T/T_c)^2n \) were inevitable, and although they were able to verify explicitly that all of the \( T^4 \) terms arising from two-loop diagrams cancelled precisely, they had no reason to believe that the contributions at higher powers of \( T \) would similarly conspire to vanish.

Thus, ABV ended their analysis in the second school among those who despair of ever calculating physical quantities accurately in the unitary gauge. The common reasoning for this opinion is as follows: unitary gauge is not manifestly renormalisable. The naive power-counting arguments fail, and hence higher-order terms in the series can dominate over lower-order ones. Thus, there is an intrinsic inaccuracy when calculating in unitary gauge. However, in spite of appearances, the Abelian Higgs model is renormalisable.

It might seem then, that we are in the unpalatable situation in which the physical properties that we extract from a model depend crucially upon how we choose to parameterise the model. Fortunately, this is not the case and a more compelling resolution to the UGP is presented in the rest of this paper.

2. The Revised Programme for Estimation of the Critical Temperature

Motivated by issues of gauge invariance, Kobes, Kunstatter, and Rebhan derived a set of gauge dependence identities within the context of the Effective Action formalism. It was quickly realised that these formal identities were sufficiently general to encompass parameterisation dependence as well as gauge dependence (the two are interrelated). Their result may be paraphrased as follows: *In a self-consistent series approximation to a physical quantity, as expressed through the Effective Action/1PI formalism, the results are gauge and parameterisation independent at each order in the series when evaluated “on-shell”. The issue of self-consistency and the exact nature of “on-shell” are discussed below. There are two additional caveats which must be mentioned. First, these algebraic identities may break down when divergent quantities are encountered. Second, the identities are not a panacea to theorists: there is no guarantee of the accuracy of a calculated result, only of its gauge and parameterisation invariance.*

With these considerations in mind, we advocate a revised programme for the
estimation of $T_c$:
1). Choose as order-parameter the physical $m_{\text{eff}}^2$ defined by the pole of the propagator (i.e. “bare plus self-energy”).
2). Determine the self-energy corrections self-consistently within a suitable perturbative scheme.
3). Calculate and self-consistently put on-shell the temperature-dependent contributions. Estimate $T_c$ from the vanishing of the order-parameter.

In the present case, the loop expansion constitutes a valid series approximation. Although it is conventional to regard the loop-expansion as an expansion in powers of $\bar{h}$, we choose to introduce another dimensionless loop-counting parameter $l$. The formality of the series, and the order-by-order gauge and parameterisation invariance are thus made more manifest.

The subject of self-consistency shall now be addressed. The equation for the pole position is:

$$0 = S^{-1} = P^2 + m^2 + \Pi(P^2) .$$

(7)

The self-energy (vacuum polarisation) is expanded in loops,

$$\Pi(P^2) = \Pi(0)(P^2) + l \Pi(1)(P^2) + l^2 \Pi(2)(P^2) + \ldots ,$$

(8)

where the subscripts $(n)$ denote the loop-order of the specified self-energy contribution, and there is no tree-level term ($\Pi(0)(P^2) \equiv 0$). The value of $P^2$ which solves the pole equation, (7), may be expanded in a series,

$$P^2 = P^2_{(0)} + l P^2_{(1)} + l^2 P^2_{(2)} + \ldots ,$$

(9)

leading to a consistent iterated set of equations:

$$P^2_{(0)} = -m^2$$

(10a)

$$P^2_{(1)} = -\Pi(1)(P^2_{(0)}) = -\Pi(1)(-m^2)$$

(10b)

$$P^2_{(2)} = \ldots = -\Pi(2)(-m^2) + \frac{\partial \Pi(1)(x)}{\partial x} \bigg|_{-m^2} \Pi(1)(-m^2)$$

(10c)

\[ \ldots \text{etc.} \ldots \]

Note that all quantities are evaluated on the tree-level mass-shell, so as to be properly ordered in the perturbative expansion.

3. Application to the UGP

Let us first observe, along with many other researchers, that the essential aspects of the UGP lie entirely in the scalar sector. In the interests of clarity, we shall restrict our attention to the complex scalar $\Phi^4$ model with a global (ungauged) $U(1)$ symmetry. The transition temperature will be estimated by application of the programme discussed above. The one-loop temperature-dependent parts will
be determined in the context of the Imaginary Time Formalism (ITF)\textsuperscript{8,9}. The Lagrangian under consideration is:

\[ \mathcal{L}_E = (\partial_{\mu} \Phi)^\dagger (i \partial^\mu \Phi) + \mu^2 |\Phi|^2 - \frac{\lambda}{3!} (|\Phi|^2)^2. \]  

(11)

Note that this is not a gauge theory and that the quadratic coefficient is appropriate for SSB.

The issue of parameterisation choice arises once again. Let us adopt a one-parameter family of parameterisations of the complex scalar field:

\[ \Phi = \frac{1}{\sqrt{2}} [\phi^R + i \phi^I], \quad \Phi^R = (1 - \epsilon) \frac{1}{\sqrt{2}} \phi + \epsilon \frac{1}{\sqrt{2}} \cos(\frac{2}{\rho_0} \phi), \]

\[ \Phi^I = (1 - \epsilon) \frac{1}{\sqrt{2}} \phi + \epsilon \frac{1}{\sqrt{2}} \sin(\frac{2}{\rho_0} \phi), \]  

(12)

where, as before, the scale introduced for convenience is taken to be \( \rho_0 \), the zero temperature VEV. Scrutiny of (12) reveals that for \( \epsilon = 0 \), the cartesian (renormalisable) parameterisation is obtained; for \( \epsilon = 1 \), the polar (unitary) parameterisation results; while for \( 0 < \epsilon < 1 \), some form of interpolating parameterisation ensues.

The Lagrangian (11) may be recast in terms of \( \{1 \phi, 2 \phi\} \), with the \( \epsilon \) dependence subsumed into the coefficients. SSB is affected by giving the entire VEV to \( 1 \phi \),

\[ 1 \phi = \rho_0 + \frac{1}{\sqrt{2}} \phi, \quad 2 \phi = 0 + \frac{1}{\sqrt{2}} \phi, \]  

(13)

where \( \{\phi, \overline{\phi}\} \) are the quantum (fluctuating) fields. Writing the Lagrangian in terms of \( \{\phi, \overline{\phi}\} \) amounts to shifting the fields in the conventional manner.

Propagators and vertices (Feynman rules) may be read from \( \mathcal{L}_E \). As we are seeking to determine the one-loop self-energy, we need only keep vertices up to fourth-order in the fields. In general, for \( \epsilon \neq 0 \), vertices of arbitrarily high-order arise through the expansion of the \( \sin \) and \( \cos \) terms in (12). The details of this construction are thoroughly discussed elsewhere\textsuperscript{10}. In addition, it proves vital to include contributions from the functional measure (a jacobian term arises from the transformation of field variables from \( \{\Phi, \Phi^*\} \rightarrow \{\phi, \overline{\phi}\}) \), so as to properly dispose of spurious \( T^4 \) terms which arise at one-loop as a consequence of derivative interactions. The quartic divergences which one might expect to find in the vacuum (zero temperature) sector of the model are precisely cancelled as well\textsuperscript{2}. Renormalisation involves only the vacuum contributions to the self-energy and is affected through the addition of appropriate counterterms.

The upshot of this analysis is a rather complicated general expression for the thermal contributions to the self-energy of the Higgs field: \( \gamma \). The Goldstone boson mode, \( \gamma \), can be studied also, and consistent results are obtained. The expression for the order-parameter,

\[ m^2_{\text{eff}} = m^2_1 - \Pi^{(11)}_{\text{TOTAL \ thermal \ \ on-shell}} (P^2), \]  

(14)

can be expanded in the high temperature limit \( (\beta \rightarrow 0) \). Of course, this may only be done after analytic continuation back to real values of the external momentum.
in the ITF. In addition, there are the familiar criteria enforcing tree-level stability which relate the coupling, masses and $\rho_0$ so as to produce a zero tree-level tadpole term (equivalently, yielding a massless Goldstone boson). These may be applied, leading to the almost final (not yet put “on-shell”) result:

$$m^2_{t, \text{eff}} = 2\mu^2 - \frac{\lambda\beta}{9\beta^2}\left\{1 + \frac{1}{8}(1 + P^2/m_t^2)\left[(2\epsilon - 1/2)^2 - \frac{1}{4}\right]\right\} + O(\beta^{-1}) \quad .$$  \hspace{1cm} (15)

At first glance, this formula looks disastrous. It was our intention to show parameterisation independence, and yet our result appears to vary with $\epsilon$. Fortunately, this is not the case. Setting the result “on-shell” fixes $P^2 = -m_t^2$, causing the $\epsilon$-dependent part to appear with a vanishing coefficient, and hence the correct equation for the critical temperature,

$$0 = 2\mu^2 - \frac{\lambda\beta}{9\beta^2} + O(\beta^{-1}) \quad ,$$  \hspace{1cm} (16)

is obtained for all values of $\epsilon$, viz., for all parameterisations. Thus the UGP has been avoided through our insistence upon using a physical quantity for the order-parameter, and evaluating it self-consistently on-shell.

Equation (15) has more content than has yet been exposed. For instance, when $\epsilon$ is set to zero (i.e., the cartesian parameterisation), the term in the square brackets vanishes identically and the correct result (16) is obtained whether the on-shell condition is enforced or not. This is not an accident. It stems rather from the fact that the self-energy to one-loop in the cartesian parameterisation is independent of the external momentum as a consequence of the absence of derivative interactions. Thus, one is able to better understand the success of the Effective Potential method in capturing the correct physics in the case of the cartesian parameterisation. Astoundingly, inspection of (15) reveals another instance of this fortuitous ability to obtain the correct result both on- and off-shell, in the case of $\epsilon = 1/2$, the “midway” parameterisation. This particular feature is most unlikely to be of any genuine physical consequence. Finally, through (15), we are able to arrive at the “incorrect” result, (6), as well. We accomplish this by setting $\epsilon = 1$, selecting the polar or unitary parameterisation, and, contrary to our prescription, setting the external four-momentum to zero in the evaluation of the order-parameter. This shows unequivocally the origin and resolution of the UGP.

4. Conclusion

Application of a prescription based upon the self-consistent calculation of a physical quantity, and its subsequent on-shell evaluation, has demonstrated that the estimate of the critical temperature, so obtained, may be expected to be independent of the parameterisation and gauge used. Of course, the accuracy of such a calculation must be checked by some other means (best undertaken in a renormalisable gauge or parameterisation). Our results have solved the long-standing UGP by revealing it to be an artifact of the difficulties inherent in the extraction of physical quantities from a gauge-variant, off-shell object, such as the Effective Potential.
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