Ando-Hiai and Golden-Thompson inequalities

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Abstract
Our main target in this article is to present complementary inequalities of Golden-Thompson and Ando-Hiai types. For example, under mild conditions on the positive definite matrices $A, B$, we show the complementary Golden-Thompson inequality

$$
\left\| \left( e^{PA} \sigma_v e^{PB} \right)^{\frac{1}{p}} \right\| \leq \theta \left\| e^{A \nabla_v B} \right\|, \quad 0 \leq v \leq 1,
$$

for any matrix mean $\sigma_v$ between the weighted harmonic mean $!_v$ and the weighted arithmetic mean $\nabla_v$, where $\| \cdot \|$ is an arbitrary unitarily invariant norm and $\theta$ being a constant depending on $p > 1$ and $v$. Further inequalities of Golden-Thompson and Ando-Hiai types will be presented for the domain $v \notin [0, 1]$ and for means other than the geometric mean.

Keywords Ando-Hiai inequality · Golden-Thompson inequality · Pólya-Szegö inequality · Operator monotone function

Mathematics Subject Classification Primary 47A63; Secondary 46L05 · 47A60

1 Introduction

For two complex $n \times n$ Hermitian matrices $A$ and $B$, the Golden-Thompson inequality states that $[7,17]$:

$$
\text{tr} \left( e^{A+B} \right) \leq \text{tr} \left( e^A e^B \right).
$$

(1.1)

as a non-commutative version of the scalar identity $e^{a+b} = e^a e^b$. The inequality (1.1) has its application in statistical mechanics and random matrix theory. In an unpublished work, Dyson proved (1.1) when he was studying random matrix theory and its application to nuclear physics. Thus, this inequality is expected to have future application in this direction. Further applications of this inequality can be found in [4].
The inequality (1.1) has been extended in various forms, among which we are interested in the following unitarily invariant norm version [1]
\[
\left\| \left( e^{pA} \varpi_v e^{pB} \right)^{\frac{1}{p}} \right\| \leq \left\| e^{p \nabla_v B} \right\|, \quad 0 \leq v \leq 1, \quad p > 0
\]
(1.2)
where \( A, B \) are Hermitian, \( \varpi_v \) is the weighted geometric mean, \( \nabla_v \) is the weighted arithmetic mean and \( \| \cdot \| \) is any unitarily invariant norm.

At this point, we recall some notions from matrix theory. The algebra of all \( n \times n \) complex matrices will be denoted by \( \mathcal{M}_n \). If \( A \in \mathcal{M}_n \) is such that \( \langle Ax, x \rangle \geq 0 \) for all \( x \in \mathbb{C}^n \), the matrix \( A \) will be called a positive semi-definite matrix (denoted \( A \geq 0 \)). If in addition, \( A \) is invertible, it will be called a positive definite matrix (denoted \( A > 0 \)). In the sequel, the notation \( \| \cdot \|_\infty \) will denote the usual operator norm, while \( \| \cdot \| \) will stand for an arbitrary unitarily invariant norm.

If \( f : J \to \mathbb{R} \) is a real valued function defined on an interval \( J \) containing the spectrum of a Hermitian matrix \( A \), the new matrix \( f(A) \) is simply defined by
\[
f(A) = U \text{diag}(f(\lambda_i)) U^*,
\]
where \( U \) is a unitary matrix in the spectral decomposition \( A = U \text{diag}(\lambda_i) U^* \) of \( A \). An important class of functions related to matrix theory is the so called operator monotone functions. Recall that a function \( f : J \to \mathbb{R} \) is said to be operator monotone if \( f(A) \leq f(B) \) for all Hermitian matrices \( A, B \) with spectra in \( J \) and satisfying \( A \leq B \). In this context, the notation \( A \leq B \) means that \( B - A \) is positive semi-definite. A linear mapping \( \Phi : \mathcal{M}_n \to \mathcal{M}_n \) is said to be positive if \( A \geq 0 \) implies \( \Phi(A) \geq 0 \). Further, it is called normalized if \( \Phi(I) = I \), where \( I \) is the identity matrix.

For two positive definite matrices \( A, B \), the matrix \( A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \) is also positive definite. When \( f : (0, \infty) \to (0, \infty) \) is operator monotone, \( f \) induces a mean between positive definite matrices, via the equation
\[
A \sigma_f B = A^\frac{1}{2} f \left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^\frac{1}{2}.
\]
Here \( \sigma_f \) means that the mean \( \sigma \) is characterized by the operator monotone function \( f : (0, \infty) \to (0, \infty) \). We refer the reader to [13] for further details on this concept.

The weighted harmonic, geometric and arithmetic means are examples of operator means. These means are defined respectively by
\[
A \varpi_v B = \left( (1 - v)A^{-1} + vB^{-1} \right)^{-1}, \quad A \nabla_v B = A^\frac{1}{2} \left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^v A^\frac{1}{2}
\]
and
\[
A \nabla_v B = (1 - v)A + vB; \quad 0 \leq v \leq 1.
\]

These means correspond to the operator monotone functions
\[
f(x) = ((1 - v) + vx^{-1})^{-1}, \quad f(x) = x^v, \quad f(x) = (1 - v) + vx, \quad 0 \leq v \leq 1
\]
respectively.

It is clear that when \( v \not\in [0, 1] \) none of the three above functions is operator monotone. Therefore, when \( v \not\in [0, 1] \), the quantities
\[
\left( (1 - v)A^{-1} + vB^{-1} \right)^{-1}, \quad A^\frac{1}{2} \left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^v A^\frac{1}{2}, \quad (1 - v)A + vB
\]
are not operator means. Yet, we will use the same notations $!, \varpi_v, \nabla_v$ to denote these quantities, but we will not refer to them as means in this article.

The inequality (1.2) has been reversed in [16, Theorem 3.4] using the Specht ratio. Very recently, the authors in [8, Corollary 2.7] have shown a stronger reverse. However, all these results treat $\varpi_v$ for $0 \leq v \leq 1$.

One target of the current paper is to prove variants of (1.2). These new versions extend the domain of $v$ to values outside the interval $[0, 1]$ and also extend the treatment of the weighted geometric mean $\varpi_v$ to any operator mean between the weighted harmonic mean $!_v$ and the weighted arithmetic mean $\nabla_v$. However, these extensions will be at the cost of an additional constant. For example, we prove that, under mild conditions on $A, B$,

$$\left\| e^{pA \varpi_v B} \right\|_p^{\frac{1}{p}} \leq \gamma_{v, p} \left\| e^{A \nabla_v B} \right\|_p, \quad v > 1$$

where $\gamma_{v, p}$ is a certain constant and $\| \cdot \|$ is an arbitrary unitarily invariant norm. Moreover, we prove that, for some constant $\theta$, depending on $v, p$,

$$\left\| e^{pA \sigma_v B} \right\|_p^{\frac{1}{p}} \leq \theta_{v, p} \left\| e^{A \nabla_v B} \right\|_p, \quad 0 \leq v \leq 1,$$

for any operator mean $\sigma_v$ between $!_v$ and $\nabla_v$, where $\| \cdot \|$ is any unitarily invariant norm on $\mathcal{M}_n$. To the best of our knowledge, such extensions have not been considered earlier in the literature.

Our methods for proving such results allow us, also, to obtain variants of the well known Ando-Hiai inequality, which asserts that if $A$ and $B$ are positive definite matrices, then for any $v \in [0, 1]$, we have [1]

$$\left\| A^{\sigma_v B} \right\|_\infty \leq \left\| A^{\varpi_v B} \right\|_p^p \quad \text{for all } p > 1,$$

where $\| \cdot \|_\infty$ is the usual operator norm. The inequality (1.3) is equivalent to

$$A^{\varpi_v B} \leq I \quad \Rightarrow \quad A^{p \varpi_v B} \leq I \quad \text{for all } p > 1.$$

A counterpart to the Ando–Hiai inequality (1.3) has been presented by Nakamoto and Seo [14, Theorem 4] as follows

$$\| A^{\varpi_v B} \|_p^p \leq \frac{1}{K(h^2 p, v)} \left( A^{p \varpi_v B} \right) \quad \text{for all } p > 1, \quad 0 \leq v \leq 1$$

whenever $mI \leq A, B \leq M I$ for some scalars $0 < m < M$, $h = \frac{M}{m}$, and

$$K(h, v) \equiv \frac{h^v - h}{(v - 1)(h - 1)} \left( \frac{v - 1}{h^v - h} \right)^v$$

is a generalized Kantorovich constant.

Our second target in this article is to extend the domain of the Ando-Hiai inequality to $v \notin [0, 1]$ and to extend it to arbitrary means, rather than the geometric mean. For example, we show that when $0 < mI \leq A, B \leq M I$ and $\sigma_v, \tau_v$ are arbitrary operator means between $!_v$ and $\nabla_v$,

$$\| A^{\sigma_v B} \|_\infty \leq \theta_v (m^p, M^p) \left( A^{p \tau_v B} \right) \quad \text{for all } p > 1$$

where $\theta_v (m^p, M^p)$ is a certain constant, see Theorem 3.1 below.
The above extensions we prove will follow as special cases of a more general treatment of operator monotone functions. This treatment, of operator monotone functions, will imply the Pólya-type inequality
\[ f(\Phi(A\sigma_v B)) \leq \xi \psi (f(\Phi(A))) \tau_v f(\Phi(B)), \]
where \( f \) is an operator monotone function, \( A, B \) are positive definite matrices related via a sandwich condition, \( \xi, \psi \) are certain constants and \( \Phi \) is a normalized positive linear map.

2 Extending the domain of the Ando-Hiai and Golden-Thompson inequalities

In this section, we present Ando-Hiai and complementary Golden-Thompson inequalities for \( v \notin [0, 1] \).

The following inequalities were pointed out in [5]. For completeness, we present a simple proof.

**Lemma 2.1** Let \( A, B \in M_n \) be positive definite matrices such that \( m_2 I \leq A \leq m_1 I < M_1 I \leq B \leq M_2 I \) for some positive numbers \( m_i, M_i \) and \( v > 1 \). Then
\[
\frac{m_1^{1/v}M_1}{m_1^{1/v}M_1} A^{1/v} \leq A^{1/v}B \leq \frac{m_2^{1/v}M_2}{m_2^{1/v}M_2} A^{1/v}B
\]
and
\[
\frac{m_1^{1/v}M_1}{m_1^{1/v}M_1} A^{1/v} \leq A^{1/v}B \leq \frac{m_2^{1/v}M_2}{m_2^{1/v}M_2} A^{1/v}B.
\]

**Proof** Let, for \( v > 1 \),
\[
f(x) = \frac{1 - v + v x}{x^v}, \quad 1 < \frac{M_1}{m_1} \leq x \leq \frac{M_2}{m_2}.
\]
Then
\[
f'(x) = v(1 - v)(x - 1)/x^{1+v}.
\]
That is, \( f \) is decreasing for \( x > 1 \), since \( v > 1 \). Therefore,
\[
\frac{M_1}{m_1} \leq x \leq \frac{M_2}{m_2} \Rightarrow f\left(\frac{M_2}{m_2}\right) \leq f(x) \leq f\left(\frac{M_1}{m_1}\right).
\]
Since \( m_2 I \leq A \leq m_1 I < M_1 I \leq B \leq M_2 I \), it follows that \( \frac{M_1}{m_1} A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \leq \frac{M_2}{m_2} \) and \( A^{1/v}B = A + v(B - A) \geq 0 \) for \( v > 1 \). Then the first two desired inequalities follow by applying a standard functional calculus argument using \( X = A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \) in (2.1). The other inequalities involving the harmonic mean follow similarly. \( \square \)

**Remark 2.1** It should be noted that the assumption \( m_2 I \leq A \leq m_1 I < M_1 I \leq B \leq M_2 I \) in Lemma 2.1 can be relaxed to the easier assumption that \( m \leq A, B \leq M \) for some positive scalars \( m, M \). In this case we get, for example,
\[
\frac{M^{1/m} m}{M^{1/m} m} A^{1/m} \leq A^{1/m}B \leq \frac{m^{1/m} M}{m^{1/m} M} A^{1/m}B,
\]
for the first inequality in Lemma 2.1. Although this condition is easier than the one stated in the lemma, we prefer to keep the stated condition as it is, because then the bounds are sharper.
As a corollary, we have the following auxiliary inequality that we will use to prove the desired Ando-Hiai and Golden-Thompson inequalities. Notice first that when \( f : (0, \infty) \to (0, \infty) \) is operator monotone, it is operator concave [2]. Therefore, when \( v \geq 1 \) and \( 0 < A < B \), we have [15]

\[
f(A \nabla_v B) \leq f(A) \nabla_v f(B).
\]

Further, the arithmetic-geometric mean inequality states that for \( 0 \leq v \leq 1 \), one has \( A_v^\# B \leq A \nabla_v B \). However, when \( v \notin [0, 1] \), the inequality is reversed. We remark here that our functions \( f \) will be defined on \((0, \infty)\). This is the main reason we consider \( v \geq 1 \).

For example, we will be dealing with the quantity \( m_2^v \nabla_v M_2 \), for \( m_2^v < M_2 \).

Notice that when \( v \geq 1 \), we have \( m_2^v \nabla_v M_2 \geq M_2 > 0 \). However, if \( v < 0 \), we do not guarantee positivity of \( m_2^v \nabla_v M_2 \). Similar argument applies for \( A \nabla_v B \); where we need the condition \( A < B \) to guarantee positivity of this matrix quantity.

**Corollary 2.1** Let \( A, B \in \mathcal{M}_n \) be positive definite matrices such that \( m_2 I \leq A \leq m_1 I < M_1 I \leq B \leq M_2 I \) for some positive numbers \( m_i, M_i \) and \( v \geq 1 \). Then for an operator monotone function \( f : (0, \infty) \to (0, \infty) \),

\[
f(A_v^\# B) \leq f(c_v A \nabla_v f(c_v B)),
\]

where \( c_v = \frac{m_2^v M_2}{m_2^v \nabla_v M_2} > 0 \).

**Proof** Noting that \( f \) is operator concave and using Lemma 2.1, we obtain

\[
\begin{align*}
f(A_v^\# B) & \leq f(c_v (A \nabla_v B)) \\
& = f((c_v A) \nabla_v (c_v B)) \\
& \leq f(c_v A) \nabla_v f(c_v B) \quad (f \text{ being operator concave and } v \geq 1) \\
& \leq f(c_v A)_v^\# f(c_v B) \quad (\text{since } v \geq 1).
\end{align*}
\]

This completes the proof. \( \square \)

Now we are ready to present the Ando-Hiai and the complementary Golden-Thompson inequalities for \( v \geq 1 \).

**Theorem 2.1** Let \( A, B \in \mathcal{M}_n \) be positive definite matrices such that \( m_2 I \leq A \leq m_1 I < M_1 I \leq B \leq M_2 I \) for some positive numbers \( m_i, M_i \), \( v \geq 1 \) and \( \| \cdot \| \) be an arbitrary unitarily invariant norm.

1. **(Ando-Hiai inequality)** If \( p > 1 \) then

\[
\| A^p_v B^p \| \leq c_{v, p} \| (A^\# v B)^p \|,
\]

where \( c_{v, p} = \frac{m_2^p \nabla_v M_2^p}{m_2^p \nabla_v M_2^p} \).

Equivalently,

\[
A^\# v B \leq I \Rightarrow A^p_v B^p \leq c_{v, p} I.
\]

2. **(Complementary Golden-Thompson inequality)** If \( p > 0 \), then

\[
\left\| (e^{p A^\# v} e^{p B})^\frac{1}{p} \right\| \leq \gamma_{v, p} \left\| e^{A \nabla_v B} \right\|,
\]

where \( \gamma_{v, p} = \frac{e^{m_2^p \nabla_v M_2^p}}{e^{m_2^p \nabla_v M_2^p}} \).
Proof (1) For $p > 1$, let $f(t) = t^\frac{1}{p}$. Then $f$ is operator monotone. Therefore, replacing $A$ and $B$ by $A^p$ and $B^p$ in (2.2), we obtain

$$(A^p v B^p)^\frac{1}{p} \leq (c_{v, p} A^p)^\frac{1}{p} V_v (c_{v, p} B^p)^\frac{1}{p}; \quad \text{where } c_{v, p} = \frac{m^p_2 V_v M^p_2}{m^p_2 V_v M^p_2} = c_{v, p} (A^p v B).$$

Then a unitary matrix $U$ exists such that

$$A^p v B^p \leq c_{v, p} U (A^p v B) U^*.$$ 

This implies the Ando-Hiai inequality

$$\|A^p v B^p\| \leq c_{v, p} \|A^p v B\|^p, \quad \text{where } p > 1 \text{ and } v \geq 1.$$ 

(2) If $0 < q < p$, let $f(t) = t^\frac{q}{p}$ in (2.2) and replace $(A, B)$ by $(e^{pA}, e^{pB})$. Then

$$\left( e^{pA^p v} e^{pB} \right)^\frac{q}{p} \leq \gamma_{v, p} \left( e^{qA^p v} e^{qB} \right), \quad \text{where } \gamma_{v, p} = \frac{e^{pm^p_2 V_v e^{pM^p_2}}}{e^{pm^p_2 V_v e^{pM^p_2}}}.$$ 

Consequently, if $\| \|$ is a unitarily invariant norm, then we have

$$\left\| \left( e^{pA^p v} e^{pB} \right)^\frac{q}{p} \right\| \leq \left\| \gamma_{v, p} \left( e^{qA^p v} e^{qB} \right) \right\|.$$ 

In particular, if $0 < q < 1$ and $\| \|$ is a given unitarily invariant norm, then $\| \|_q$ defined by $\|X\|_q = \|X^{\frac{1}{q}}\|$ is a unitarily invariant norm, see [3, Page 95] or [10, Lemma 2.13]. Therefore,

$$\left\| \left( e^{pA^p v} e^{pB} \right)^\frac{q}{p} \right\|_q \leq \left\| \gamma_{v, p} \left( e^{qA^p v} e^{qB} \right) \right\|_q \Rightarrow \left\| \left( e^{pA^p v} e^{pB} \right)^{\frac{1}{p}} \right\| \leq \left\| \gamma_{v, p} \left( e^{qA^p v} e^{qB} \right)^\frac{1}{q} \right\|.$$ 

Letting $q \to 0^+$, we obtain

$$\left\| \left( e^{pA^p v} e^{pB} \right)^{\frac{1}{p}} \right\| \leq \left\| \gamma_{v, p} \right\| e^{A V_v B}.$$ 

We remark here that in [11] the limit

$$\lim_{q \to 0^+} \left( e^{qA^p v} e^{qB} \right)^\frac{1}{q} = e^{A V_v B}$$ 

was shown for $0 \leq v \leq 1$. As pointed out in [14], the same proof applies for $v \notin [0, 1]$.

\[\qed\]

Remark 2.2 (1) In Theorem 2.1, we obtained a variant of the Ando-Hiai inequality when $v \geq 1$. It should be pointed out that a known variant is the following inequality

$$\|A^p v B^p\| \leq K(h, p)^v K(h^{2p}, v) \|(A^p v B)^p\|; \quad v, p > 1, \quad (2.3)$$ 

where $A, B \in M_n$ are positive definite such that $mI \leq A, B \leq MI$ for some positive scalars $m, M$ and $h = m/M$.

We notice first that Theorem 2.1 presents an inequality under a different condition that imposes an ordering between $A$ and $B$. Namely, $m_2 I \leq A \leq m_1 I < M_1 I \leq B \leq M_2 I$. 

\[\end{document}\]
Notice that for this latter condition, we can apply (2.3) with $m = m_2$ and $M = M_2$. Numerical examples show that the constant $c_{v, p}$ obtained in 2.1 is much better than the constant $K(h, p)^v K(h^2, p, v)$ appearing in (2.3). While this gives a significant advantage to Theorem 2.1 over (2.3), we notice that (2.3) is more general since there is no ordering assumption between $A$ and $B$.

(2) It should be remarked that the second part of Theorem 2.1 (i.e., the complementary Golden-Thompson inequality) holds true for any Hermitian matrices $A, B$ such that $m_2 I \leq A \leq m_1 I < M_1 I \leq B \leq M_2 I$, where $m_i, M_i \in \mathbb{R}$. This is justified because $e^A$ is positive definite for any Hermitian matrix $A$.

3 Ando-Hiai and Golden-Thompson inequalities for arbitrary means

The original Ando-Hiai and the complementary Golden-Thompson inequalities and their extensions treat the geometric mean $\gamma_v$. In this section, we present variants of these important inequalities for arbitrary means.

In our recent work [5, Theorem A], we showed that if $A, B \in \mathcal{M}_n$ are positive definite such that $0 < s A \leq B \leq t A$, then for any $v \in [0, 1]$

$$\frac{1}{\xi} A \nabla_v B \leq A \nabla_v B \leq \psi A \nabla_v B \tag{3.1}$$

where $\xi = \max \left\{ \frac{(1-v)+v}{s}, \frac{(1-v)+v}{t} \right\}$ and $\psi = \max \left\{ s^v ((1-v) + \frac{v}{s}), t^v ((1-v) + \frac{v}{t}) \right\}$. The following few inequalities will be needed to prove the next main result.

The following Lemma is shown in [12, Lemma 3.3]. But, we give a proof for readers’ convenience.

**Lemma 3.1** Let $A, B \in \mathcal{M}_n$ be positive definite and let $f : (0, \infty) \to (0, \infty)$ be an operator monotone function. Then for any $v \in [0, 1]$

$$f(A \nabla_v B) \leq f(A) \nabla_v f(B).$$

**Proof** Notice that operator monotonicity of $f$ implies operator concavity, and hence

$$f \left( A^{-1} \nabla_v B^{-1} \right) \geq f \left( A^{-1} \right) \nabla_v f \left( B^{-1} \right). \tag{3.2}$$

Moreover, operator monotonicity of $f(t)$ implies operator monotonicity of $f(t^{-1})$. Now, if we rewrite (3.2) for the function $f(t^{-1})$, we get

$$f(A \nabla_v B)^{-1} = f \left( (A^{-1} \nabla_v B^{-1})^{-1} \right)^{-1} \geq f(A) \nabla_v f(B)^{-1}. \tag{3.3}$$

By taking the inverses for both sides we infer $f \left( A \nabla_v B \right) \leq f \left( A \right) \nabla_v f \left( B \right)$.(Of course, if $f$ is operator monotone decreasing then $f \left( A \nabla_v B \right) \geq f \left( A \right) \nabla_v f \left( B \right)$.) \hfill \Box

**Lemma 3.2** Let $A, B \in \mathcal{M}_n$ be positive definite such that $0 < s A \leq B \leq t A$. Assume $v \in [0, 1]$ and let $\tau_v, \sigma_v$ be two arbitrary operator means between $\gamma_v$ and $\nabla_v$. If $f : (0, \infty) \to (0, \infty)$ is an operator monotone function, then

$$f(A) \sigma_v f(B) \leq f\left( (\xi / \psi) A \tau_v B \right), \tag{3.4}$$

and

$$f \left( \frac{1}{\xi / \psi} (A \sigma_v B) \right) \leq f(A) \tau_v f(B), \tag{3.5}$$
where $\xi$ and $\psi$ as in (3.1). The reverse of each of the above inequalities holds when $f$ is operator decreasing.

**Proof** The inequality (3.3) follows from a more general result of [5, Theorem B]. We prove (3.4). Direct calculations show that

$$f \left( \frac{1}{\xi \psi} (A \sigma_v B) \right) \leq f \left( \frac{1}{\xi \psi} (A \nabla_v B) \right) \leq f (A !_v B) \quad \text{(by (3.1))}$$

$$\leq f (A) !_v f (B) \quad \text{(by Lemma 3.1)}$$

$$\leq f (A) \tau_v f (B) \quad \text{(since !}_v \leq \tau_v).$$

$\Box$

To prove the next result, we need the following inequality [5, Corollary 2.1]:

$$\frac{m \nabla_\lambda M}{m \nabla_\lambda M} A \nabla_v B \leq A \tau_v B \leq \frac{m \nabla_\mu M}{m \nabla_\mu M} A !_v B,$$

(3.5)

where $A, B \in \mathcal{M}_n$ are positive definite matrices such that $0 < m I \leq A, B \leq M I, 0 \leq v \leq 1, \lambda = \min\{v, 1 - v\}$ and $\mu = \max\{v, 1 - v\}$.

Let $v \in [0, 1], \lambda = \min\{v, 1 - v\}$ and $\mu = \max\{v, 1 - v\}$. If $m, M > 0$, the following notation will be adopted in the sequel.

$$\theta_v (m, M) \equiv \frac{(m \nabla_\lambda M)(m \nabla_\mu M)}{(m \nabla_\lambda M)(m \nabla_\mu M)}.$$  (3.6)

**Corollary 3.1** Let $A, B \in \mathcal{M}_n$ be such that $m I \leq A, B \leq M I$ for some scalars $0 < m < M$. Assume $v \in [0, 1]$ and let $\tau_v, \sigma_v$ be two arbitrary operator means between $!_v$ and $\nabla_v$. If $f : (0, \infty) \rightarrow (0, \infty)$ is an operator monotone function, then

$$f (A) \sigma_v f (B) \leq f (\theta_v (m, M) (A \tau_v B)),$$

(3.7)

and

$$f \left( \frac{1}{\theta_v (m, M)} (A \sigma_v B) \right) \leq f (A) \tau_v f (B),$$

(3.8)

where $\theta_v (m, M)$ is as in (3.6).

**Proof** We only prove (3.7). We have

$$f (A) \sigma_v f (B) \leq f (A) \nabla_v f (B) \quad \text{(since $\sigma_v \leq \nabla_v$)}$$

$$\leq f (A \nabla_v B) \quad \text{(since $f$ is operator concave)}$$

$$\leq f \left( \frac{m \nabla_\lambda M}{m \nabla_\lambda M} A !_v B \right) \quad \text{(since $!}_v \leq \tau_v)}$$

$$\leq f (\theta_v (m, M) A \tau_v B) \quad \text{(since $!}_v \leq \tau_v).$$

This completes the proof. $\Box$

Now we are ready to present the inequalities extending and reversing the Ando-Hiai inequalities (1.3) and (1.4).
Theorem 3.1 Let $A, B \in \mathcal{M}_n$ be such that $m I \leq A, B \leq M I$ for some scalars $0 < m < M$. Assume $v \in [0, 1]$ and let $\tau_v, \sigma_v$ be two arbitrary operator means between $!_v$ and $\nabla_v$. Then

\[
\|A \sigma_v B\|_\infty^p \leq \theta_v(m^p, M^p) \left\|A^p \tau_v B^p\right\|_\infty \text{ for all } p > 1 \tag{3.9}
\]

\[
\left\|A^p \sigma_v B^p\right\|_\infty \leq \theta_v(m^p, M^p) \left\|A \tau_v B\right\|_\infty^p \text{ for all } p > 1, \tag{3.10}
\]

where $\theta_v(m^p, M^p)$ is as in (3.6).

Proof It follows from the inequality (3.7) that

\[
A^{\frac{1}{p}} \sigma_v B^{\frac{1}{p}} \leq \theta_v(m, M) \left(A \tau_v B\right)^{\frac{1}{p}}
\]

where $p > 1$. By replacing $A$ by $A^p$ and $B$ by $B^p$, we get

\[
A \sigma_v B \leq \theta_v(m^p, M^p) \left(A^p \tau_v B^p\right)^{\frac{1}{p}}.
\]

This implies that

\[
\left\|A \sigma_v B\right\|_\infty \leq \theta_v(m^p, M^p) \left\|A^p \tau_v B^p\right\|_\infty^{\frac{1}{p}},
\]

which is equivalent to (3.9).

Employing inequality (3.8), and by the same method as in the proof of inequality (3.9) we get the desired inequality (3.10). \qed

Another interesting application of Corollary 3.1 is the following extension of the Golden-Thompson inequality to arbitrary means.

Corollary 3.2 Let $A, B \in \mathcal{M}_n$ be Hermitian such that $m I \leq A, B \leq M I$ for some real scalars $m, M$. Assume $v \in [0, 1]$, $p > 0$ and let $\sigma_v$ be an arbitrary operator mean between $!_v$ and $\nabla_v$. Then for any unitarily invariant norm $\left\| \right\|$, $\left\| \left( e^{pA} \sigma_v e^{pB} \right)^{\frac{1}{p}} \right\| \leq \theta_v^p \left\| e^{A \nabla_v B} \right\|$ and

\[
\left\| e^{A \nabla_v B} \right\| \leq \theta_v^p \left\| \left( e^{pA} \sigma_v e^{pB} \right)^{\frac{1}{p}} \right\|,
\]

where $\theta_v(e^{pm}, e^{pM})$ is as in (3.6)

Proof In (3.8), let $f(t) = t^{\frac{q}{p}}$, where $0 < q < p$ and replace $(A, B)$ with $(e^{pA}, e^{pB})$. Then

\[
\left( e^{pA} \sigma_v e^{pB} \right)^{\frac{q}{p}} \leq \theta \left( e^{qA} \sigma_v e^{qB} \right) \Rightarrow \left\| \left( e^{pA} \sigma_v e^{pB} \right)^{\frac{q}{p}} \right\| \leq \theta \left\| \left( e^{qA} \sigma_v e^{qB} \right)^{\frac{q}{p}} \right\|.
\]

Again, since this is valid for any unitarily invariant norm, it is still true for the norm $\left\| X \right\|_q := \left\| X^{\frac{1}{q}} \right\|^q$ provided that $q < 1$, see [3, Page 95] or [10, Lemma 2.13]. That is,

\[
\left\| \left( e^{pA} \sigma_v e^{pB} \right)^{\frac{1}{p}} \right\| \leq \theta \left\| \left( e^{qA} \sigma_v e^{qB} \right)^{\frac{1}{q}} \right\|^q, \quad p > q, 0 < q < 1.
\]

Then letting $q \to 0^+$ implies the first desired inequality.

The second desired inequality follows similarly from (3.7). This completes the proof. \qed

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Now we prove the following norm version of Lie-Trotter formula.

**Corollary 3.3** Under the assumptions of Corollary 3.2, the following identity holds

\[
\lim_{p \to 0^+} \left\| \left( e^{pA\sigma_v} e^{pB} \right)^{\frac{1}{p}} \right\| = \left\| e^{AV_v B} \right\|.
\]

**Proof** From Corollary 3.2, we have

\[
\frac{1}{\theta} \left\| e^{AV_v B} \right\| \leq \left\| \left( e^{pA\sigma_v} e^{pB} \right)^{\frac{1}{p}} \right\| \leq \theta \left\| e^{AV_v B} \right\|,
\]

where

\[
\theta = \frac{\left( e^{pm} \nabla \lambda e^{pM} \right) \left( e^{pm} \mu e^{pM} \right)}{\left( e^{pm} \lambda e^{pM} \right) \left( e^{pm} \mu e^{pM} \right)};
\]

\[
\lambda = \min\{v, 1 - v\}, \quad \mu = 1 - \lambda.
\]

Direct computations show that \( \lim_{p \to 0^+} \frac{1}{\theta} = 1 \). Then the desired limit follows from (3.11) by Squeeze theorem.

**Remark 3.1** It should be pointed out that a stronger version of Corollary 3.3 has been shown recently in [9, Theorem 5.1] using different approaches and techniques.

On account of (3.4) and the fact that for any operator monotone function \( f \) and \( 0 < \alpha \leq 1, \ \alpha f (t) \leq f (\alpha t) \), and using Ando’s inequality [6, Theorem 5.8] stating that \( \Phi (A \sigma B) \leq \Phi (A) \sigma \Phi (B) \) for the positive definite matrices \( A, B \), the normalized positive linear mapping \( \Phi \) and the operator mean \( \sigma \), we get

\[
f (\Phi (A \sigma_v B)) \leq \xi \psi f (\Phi (A)) \tau_v f (\Phi (B)).
\]

(3.12)

The above inequality can be regarded as a reverse of [5, Theorem B]. We conclude this paper by the following proposition; where the generalized Kantorovich constant is involved instead of \( \xi \psi \). Such inequalities are usually referred to as Pólya-type inequalities.

In this proposition, we use the notations

\[
a_f \equiv \frac{f (M) - f (m)}{M - m}, \quad b_f \equiv \frac{M f (m) - m f (M)}{M - m},
\]

so that \( a_f t + b_f \) represents the secant line of \( f \) at \( (m, f (m)) \) and \( (M, f (M)) \).

**Proposition 3.1** Let \( A, B \in \mathcal{M}_n \) be such that \( m I \leq A, B \leq M I \) for some scalars \( 0 < m < M \). Assume \( v \in [0, 1] \) and let \( \tau_v, \sigma_v \) be two arbitrary operator means between \( !_v \) and \( \nabla_v \). If \( f : (0, \infty) \to (0, \infty) \) is an operator monotone decreasing function, then

\[
f (\Phi (A)) \tau_v f (\Phi (B)) \leq K (m, M, f) f (\Phi (A \sigma_v B)),
\]

(3.13)

where \( K (m, M, f) \equiv \max \left\{ \frac{a_f t + b_f}{f(t)} : t \in [m, M] \right\}. \) On the other hand, if \( f : (0, \infty) \to (0, \infty) \) is operator monotone increasing, then

\[
f (\Phi (A \sigma_v B)) \leq K \left( m, M, \frac{1}{f} \right) (f (\Phi (A)) \tau_v f (\Phi (B))).
\]

(3.14)
Proof On account of [2, Theorem 2.1], operator monotone decreasing implies operator convexity (and of course, convexity). Therefore,

\[ f(t) \leq a_f t + b_f, \quad t \in [m, M]. \]

By the assumption \( mI \leq A, B \leq MI \), we can write

\[ f(A) \leq a_f A + b_f I \quad \text{and} \quad f(B) \leq a_f B + b_f I. \]

Whence

\[ f(A) \nabla_v f(B) \leq a_f (A \nabla_v B) + b_f I. \]

Now, by applying the Mond–Pečarić method (see [6, Section 1.5]) we have for a given \( \alpha > 0 \),

\[ f(A) \nabla_v f(B) - \alpha f(A \sigma_v B) \leq a_f (A \nabla_v B) + b_f I - \alpha f(A \sigma_v B) \]

where in the second inequality we used \( \sigma_v \leq \nabla_v \) and \( a_f \leq 0 \). Consequently,

\[ f(A) \nabla_v f(B) \leq \beta I + \alpha f(A \sigma_v B). \] (3.15)

Now, by replacing \( A, B \) by \( \Phi(A), \Phi(B) \), respectively, and applying Ando’s inequality, we get

\[ f(\Phi(A)) \nabla_v f(\Phi(B)) \leq \beta I + \alpha f(A \sigma_v B). \] (3.16)

Since \( \alpha > 0 \) was arbitrary, we may set \( \alpha = K(m, M, f) \) in (3.16). Then \( \beta = 0 \). With these values, (3.16) reduces to

\[ f(\Phi(A)) \nabla_v f(\Phi(B)) \leq K(m, M, f) f(A \sigma_v B), \] (3.17)

whenever \( f \) is an operator monotone decreasing. On the other hand, we know that if \( f \) is operator monotone (increasing) on \((0, \infty)\), then \( 1/f \) is operator monotone decreasing on \((0, \infty)\). It follows from the inequality (3.17) that

\[ f(\Phi(A))^{-1} \nabla_v f(\Phi(B))^{-1} \leq K(m, M, \frac{1}{f}) f(A \sigma_v B)^{-1}. \] (3.18)

Taking inverse from inequality (3.18), we have

\[ f(\Phi(A \sigma_v B)) \leq K(m, M, \frac{1}{f}) (f(\Phi(A))^{-1} \nabla_v f(\Phi(B))^{-1})^{-1} = K(m, M, \frac{1}{f}) (f(\Phi(A)) \nabla_v f(\Phi(B))). \]

where \( \tau_v^* \) is the adjoint of \( \tau_v \). Now, since \( \tau_v \) is arbitrary, by replacing \( \tau_v^* \) by \( \tau_v \) we get the desired inequality (3.14).
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