A NOTE ON THE INFINITUDE OF PRIME IDEALS IN DEDEKIND DOMAINS

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Abstract. Let \( R \) be an infinite Dedekind domain with at most finitely many units, and let \( K \) denote its field of fractions. We prove the following statement. If \( L/K \) is a finite Galois extension of fields and \( O \) is the integral closure of \( R \) in \( L \), then \( O \) contains infinitely many prime ideals. In particular, if \( O \) is further a unique factorization domain, then \( O \) contains infinitely many non-associate prime elements.

1. Introduction

Let \( R \) be a integral domain that has at most finitely many units. We denote by \( R^\times \) the set of all units of \( R \), i.e., \( u \in R^\times \) if and only if there exists \( v \in R \) such that \( uv = 1 \). We denote by \( R[x] \) the ring of all polynomials in the variable \( x \) with coefficients in \( R \). Let \( a \) and \( b \) be arbitrary elements of \( R \). We say that \( a \) divides \( b \) in \( R \) and write \( a \mid b \), provided that there exits an element \( c \) in \( R \) such that \( b = ac \). We say that \( a \) and \( b \) are associate provided that there exists \( u \) in \( R^\times \) such that \( a = ub \). Let \( p \) be an element that is neither zero nor a unit in \( R \). Recall that \( p \) is said to be prime if for all elements \( r \) and \( s \) in \( R \) such that \( p \mid rs \) then either \( p \mid r \) or \( p \mid s \). We denote by \( K \) the field of fractions of \( R \), i.e., \( K = \{ p/q : p, q \in R \text{ and } q \neq 0 \} \). Recall that an ideal \( p \) of \( R \) is said to be prime, provided that for all elements \( a, b \in R \), if \( ab \in p \), then either \( a \in p \) or \( b \in p \). It follows from \([5, \text{Exercise \S 1-1.8}]\) that \( R \) contains infinitely many prime ideals. If \( R \) is further a unique factorization domain, then it follows from \([3, \text{Proposition 2.2 (ii)}]\) that \( R \) contains infinitely many non-associate prime elements. Assume that \( R \) is a subring of a ring \( L \). Recall that an element \( \alpha \in L \) is integral over \( R \) if there exists a monic polynomial \( f \in R[x] \) such that \( f(\alpha) = 0 \). In particular, when \( R = \mathbb{Z} \), the element \( \alpha \) is said to be an algebraic integer in \( L \). It is a well-known result that the set \( B \) consisting of all the elements that are integral over \( R \) is a ring, which is called the integral closure of \( R \) in \( L \) (see e.g. \([2, \text{Corollary 5.3}]\)). In particular, if \( R = \mathbb{Z} \) and \( L \) is a field containing \( \mathbb{Z} \), the integral closure of \( \mathbb{Z} \) in \( L \) is called the ring of integers of \( L \), and we denote this ring by \( O_L \). For example, let \( d \) be a square-free integer and consider \( \mathbb{Q}(\sqrt{d}) = \{ a + b\sqrt{d} : a, b \in \mathbb{Q} \} \). The ring of integers in \( \mathbb{Q}(\sqrt{d}) \) is

\[
O_{\mathbb{Q}(\sqrt{d})} = \mathbb{Z}[\omega] = \{ a + b\omega : a, b \in \mathbb{Z} \},
\]

where

\[
\omega = \begin{cases} 
\sqrt{d}, & \text{if } d \equiv 2, 3 \mod 4 \\
1 + \sqrt{d}, & \text{if } d \equiv 1 \mod 4 
\end{cases}
\]

We say that \( R \) is integrally closed if \( R \) is equal to its integral closure in its field of fractions. In particular, \( \mathbb{Z} \) is integrally closed.
For example, let \( d \in \{-1, -2, -3, -7, -11, -19, -43, -67, -163\} \) and consider the ring of integers \( \mathcal{O}_{\mathbb{Q}(\sqrt{d})} \) of \( \mathbb{Q}(\sqrt{d}) \). By [1, Theorem 13.2.5], \( \mathcal{O}_{\mathbb{Q}(\sqrt{d})} \) is an infinite unique factorization domain with finitely many units, which implies that there are infinitely many non-associative prime elements in \( \mathcal{O}_{\mathbb{Q}(\sqrt{d})} \). This gives rise to the discussion of the infinitude of prime elements in rings such as \( \mathcal{O}_{\mathbb{Q}(\sqrt{2})} = \mathbb{Z}[\sqrt{2}] \).

Note that \( \mathcal{O}_{\mathbb{Q}(\sqrt{2})} \) is a unique factorization domain that contains infinitely many units, for if \( \alpha = 1 + \sqrt{2} \), then for all integers \( n \geq 0 \), \( \alpha^n \) is a unit in \( \mathcal{O}_{\mathbb{Q}(\sqrt{2})} \) (see e.g. [1, §13.9]). In this note, we prove the infinitude of prime elements in \( \mathcal{O}_{\mathbb{Q}(\sqrt{2})} \) by proving the following result.

**Theorem 1.1.** Assume that \( R \) is a Dedekind domain. If \( L/K \) is a finite Galois extension of fields and \( \mathcal{O} \) is the integral closure of \( R \) in \( L \), then \( \mathcal{O} \) contains infinitely many prime ideals.

Recall that \( R \) is a Dedekind domain if \( R \) is Noetherian, 1-dimensional and integrally closed. If \((R, K, \mathcal{O}, L)\) as in Theorem 1.1, then \( \mathcal{O} \) is also a Dedekind domain (see e.g. [6, Thm. I.6.2]), and \( R \) satisfies the property of unique factorization of ideals, i.e., every non-trivial ideal \( a \) of \( R \) can be written as \( a = p_1 \cdots p_s \), where for all \( i \in \{1, \ldots, s\} \), \( p_i \) is a prime ideal of \( R \), and this factorization is unique up to the order of the factors (see e.g. [6, Thm. 6.3]).

Since the ring \( \mathbb{Z} \) is an infinite Dedekind domain that has finitely many units, and since a Dedekind domain is a unique factorization domain if and only if it is a principal ideal domain, we get the following immediate consequence of Theorem 1.1.

**Corollary 1.2.** Assume that \( L/\mathbb{Q} \) is a finite Galois extension of fields. Then the ring of integers \( \mathcal{O}_L \) contains infinitely many prime ideals. In particular, if \( \mathcal{O}_L \) is a unique factorization domain, then \( \mathcal{O}_L \) contains infinitely many non-associate prime elements.

It follows from Corollary 1.2 that \( \mathcal{O}_{\mathbb{Q}(\sqrt{2})} \) contains infinitely many non-associate prime elements.

For further details concerning integral closures and Dedekind domains, see e.g. [2, Chapter 5], [3, §VIII.5] and [6, Chapters I, III & IV].

2. Proof of the main result

Let \((R, K, \mathcal{O}, L)\) be as in Theorem 1.1. For all ideals \( \mathfrak{A} \) of \( \mathcal{O} \), define

\[
N_{\mathcal{O}/R}(\mathfrak{A}) := \left( \prod_{\sigma \in \text{Gal}(L/K)} \sigma(\mathfrak{A}) \right) \cap R.
\]

It follows from [3, Lemma IV.6.4] that if \( \mathfrak{P} \) is a maximal ideal of \( \mathcal{O} \), then \( N_{\mathcal{O}/R}(\mathfrak{P}) = p^{f_{\mathfrak{P}} / p} \), where \( p = \mathfrak{P} \cap R \) and \( f_{\mathfrak{P}} / p = [\mathcal{O}/\mathfrak{P} : R/p] \). Moreover, if \( \mathfrak{A} = \mathfrak{P}_1^{e_1} \mathfrak{P}_2^{e_2} \cdots \mathfrak{P}_k^{e_k} \) is a product of maximal ideals of \( \mathcal{O} \), then

\[
N_{\mathcal{O}/R}(\mathfrak{A}) = \prod_{i=1}^k N_{\mathcal{O}/R}(\mathfrak{P}_i)^{e_i}.
\]

Consider the inclusion map

\[
\iota_{\mathcal{O}/R} : R \to \mathcal{O}.
\]
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It follows that \( \iota_{O/R} \) induces an injective map between the set \( \mathcal{I}(R) \) of all ideals of \( R \) to the set \( \mathcal{I}(O) \), which we also denote by \( \iota_{O/R} \) and which is defined as \( \iota_{O/R}(a) = aO \) for all ideals \( a \) of \( R \). It follows from [6, Lemma IV.6.7] that if \( n = |\text{Gal}(L/K)| \), then for all ideals \( a \) of \( R \),

\[
N_{O/R}(\iota_{O/R}(a)) = a^n.
\]

The following well-known result is an exercise in [5, Exercise §1-1.8] (for its proof, see e.g. [3, Lemma 2.1]).

**Lemma 2.1.** Assume that \( A \) is infinite integral domain with at most finitely many units. Then \( A \) has infinitely many maximal ideals. In particular, \( A \) has infinitely many prime ideals.

We conventionally assume that for all non-zero ideals \( a \) of \( R \), \( a^0 = R \).

**Proof of Theorem 1.1.** Assume that \( O \) contains at most finitely many prime ideals, i.e., assume that \( \mathfrak{N}_1, \ldots, \mathfrak{N}_k \) is a complete list of all distinct prime ideals of \( O \). Let \( m \) be a maximal ideal of \( R \). Since \( R \) is infinite and \( R \) contains at most finitely many units, it follows that \( m \) is non-zero. Since \( \iota_{O/R}(m) \) is an ideal of \( O \), it follows that there exist non-negative integers \( s_1, \ldots, s_k \) such that

\[
\iota_{O/R}(m) = \mathfrak{N}_1^{s_1} \cdots \mathfrak{N}_k^{s_k}.
\]

If \( n = |\text{Gal}(L/K)| \), then after applying \( N_{O/R} \) to (2.5), and using (2.2) and (2.4), we get

\[
m^n = N_{O/R}(\iota_{O/R}(m)) = N_{O/R}(\mathfrak{N}_1^{s_1} \cdots \mathfrak{N}_k^{s_k}) = N_{O/R}(\mathfrak{N}_1)^{s_1} \cdots N_{O/R}(\mathfrak{N}_k)^{s_k} = n_1^{s_1/n_1} \cdots n_k^{s_k/n_k},
\]

where for all \( i \in \{1, \ldots, k\} \), \( n_i = \mathfrak{N}_i \cap R \). Since \( R \) is a Dedekind domain, and since for all \( i \in \{1, \ldots, k\} \), \( \mathfrak{N}_i \) is a prime ideal of \( R \), it follows that there exist suitable non-negative integers \( r_1, \ldots, r_k \) such that

\[
m = n_1^{r_1} \cdots n_k^{r_k}.
\]

Since \( m \) is a maximal ideal of \( R \), there exists \( i_0 \in \{1, \ldots, k\} \) such that \( r_{i_0} > 0 \). It follows that

\[
m \subseteq n_{i_0}^{r_{i_0}} \subseteq n_{i_0}.
\]

It follows that from maximality of \( m \) that \( m = n_{i_0} \). Therefore, the set \( \text{Max}(R) \) consisting of all maximal ideals of \( R \) is contained in the set \( \{n_1, \ldots, n_k\} \), which implies that \( \text{Max}(R) \) is finite. However, Lemma 2.1 implies that \( \text{Max}(R) \) is infinite, which leads to a contradiction. Hence \( O \) contains infinitely many prime ideals. □

From the proof of Theorem 1.1, we obtain the following result.

**Corollary 2.2.** Let \( A \) be a Dedekind domain, and let \( Q \) be its field of fractions. Assume that \( F/Q \) is a finite Galois extension, and let \( B \) the integral closure of \( A \) in \( F \). If \( B \) contains at most finitely many prime ideals, then so does \( A \).
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