Faster parameterized algorithm for pumpkin vertex deletion set

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Abstract

A directed graph $G$ is called a pumpkin if $G$ is a union of induced paths with a common start vertex $s$ and a common end vertex $t$, and the internal vertices of every two paths are disjoint. We give an algorithm that given a directed graph $G$ and an integer $k$, decides whether a pumpkin can be obtained from $G$ by deleting at most $k$ vertices. The algorithm runs in $O^*(2^k)$ time.

Keywords graph algorithms, parameterized complexity.

1 Introduction

A directed graph $G$ is called a pumpkin if $G$ is a union of induced paths with a common start vertex $s$ and a common end vertex $t$, and the internal vertices of every two paths are disjoint. The vertices $s$ and $t$ are called the source and sink of the pumpkin. In the Pumpkin Vertex Deletion Set problem (PVDS), the input is a directed graph $G$ and an integer $k$, and the goal is to decide whether there is a set of vertices $S$ of size at most $k$ such that the graph obtained from $G$ by deleting the vertices of $S$ is a pumpkin. The PVDS problem is NP-hard [4]. Agrawal et al. [2] gave an $O^*(2.562^k)$-time algorithm for PVDS. Polynomial kernels for this problem were given in [1,4].

In this paper, we give an $O^*(2^k)$-time algorithm for PVDS.

2 Preliminaries

For a vertex $v$ in a directed graph $G$, $N^-_G(v)$ and $N^+_G(v)$ denote the set of in-neighbors and out-neighbors of $v$. Additionally, $d^-_G(v) = |N^-_G(v)|$ and $d^+_G(v) = |N^+_G(v)|$. We will sometimes omit the subscript $G$ if the graph $G$ is clear from the context.

For set of vertices $S$, $G - S$ is the graph obtained from $G$ by deleting the vertices of $S$ (and incident edges).

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3 The algorithm

In the Restricted Pumpkin Vertex Deletion Set problem (RPVDS), the input is a directed graph $G$, an integer $k$ and vertices $s, t$. The goal is to decide whether there is a set of vertices $S$ of size at most $k$ such that the graph obtained from $G$ by deleting the vertices of $S$ is a pumpkin with source $s$ and sink $t$. An $O^*(c^k)$-time algorithm for RPVDS implies an $O^*(c^k)$-time algorithm for PVDS: Given an input $(G, k)$ to PVDS, go over all possible ways to choose $s$ and $t$, and for each choice run the RPVDS algorithm on $(G, k, s, t)$. The time complexity is $O^*(n^2 \cdot c^k) = O^*(c^k)$.

In the sequel, we describe an $O^*(2^k)$-time algorithm for RPVDS.

Our algorithm is a branching algorithm (cf. [3]). Given an instance $(G, k, s, t)$, the algorithm applies the first applicable reduction rule from the rules below, and if no reduction rule is applicable, the algorithm applies the first applicable branching rule from the rules below.

The algorithm uses the following reduction rules from Agrawal et al. [2] (we present the rules in slightly simplified form).

(R1) If $k < 0$ return ‘no’.
(R2) If $k = 0$ and $G$ is not a pumpkin with source $s$ and sink $t$ return ‘no’.
(R3) If $G$ is a pumpkin with source $s$ and sink $t$ return ‘yes’.
(R4) If there is a vertex $v$ that is not reachable from $s$, delete $v$ from $G$ and decrease $k$ by 1.
(R5) If there is a vertex $v \neq t$ such that $t$ is not reachable from $v$, delete $v$ from $G$ and decrease $k$ by 1.
(R6) If there is a vertex $v$ such that $s \in N^+(v)$, delete $v$ from $G$ and decrease $k$ by 1.
(R7) If there is a vertex $v$ such that $t \in N^-(v)$, delete $v$ from $G$ and decrease $k$ by 1.
(R8) If $t$ is not reachable from $s$, return ‘no’.

We now describe the branching rules of the algorithm. When we say that the algorithm branches on sets $S_1, \ldots, S_p$, we mean that the algorithm is called recursively on the instances $(G - S_1, k - |S_1|, s, t), \ldots, (G - S_p, k - |S_p|, s, t)$. For each Rule $(i)$ below, except Rule (1), there is also a symmetric rule, denoted Rule $(i)'$, in which the roles of in-neighbors and out-neighbors are reversed. For example, Rule (2) is:

If there is a vertex $v \neq s, t$ such that $d^-(v) \geq 2$ and $s \in N^-(v)$, branch on $\{v\}$ and $N^-(v) \setminus \{s\}$. The order in which the branching rules are considered is (1), (2), (2'), (3), (3') and so on.

(1) If there are vertices $u$ and $v$ such that $(u, v) \in E(G)$ and $(v, u) \in E(G)$, branch on $\{u\}$ and $\{v\}$.

The correctness of Rule (1) is trivial. The branching vector of Rule (1) is $(1, 1)$.

(2) If there is a vertex $v \neq s, t$ such that $d^+(v) \geq 2$ and $t \in N^+(v)$, branch on $\{v\}$ and $N^+(v) \setminus \{t\}$.

To prove the correctness of Rule (2), note that if $S$ is a solution to the instance $(G, k, s, t)$ and $v \notin S$ then $N^+(v) \setminus \{t\} \subseteq S$, otherwise $d^+_{G-S}(v) \geq 2$ contradicting
the assumption that $S$ is a solution. The branching vector of Rule (2) is at least $(1, 1)$.

(3) If there is a vertex $v \neq s, t$ such that $d^+(v) \geq 4$, arbitrarily choose distinct $w_1, w_2, w_3, w_4 \in N^+(v)$. Branch on $\{v\}$ and on $\{w_1, w_2, w_3, w_4\} \setminus \{v\}$ for every $i \leq 4$.

The correctness of Rule (3) is obvious: If $S$ is a solution to the instance $(G, k, s, t)$ and $v \notin S$ then $d^+_{G\setminus S}(v) = 1$. Therefore, $S$ contains at least 3 vertices of $\{w_1, w_2, w_3, w_4\}$. The branching vector of Rule (3) is $(1, 3, 3, 3, 3)$.

(4) If there is a vertex $v \neq s, t$ such that $d^+(v) \geq 2$ and there is a vertex $w \in N^+(v)$ for which $d^-(w) = 1$, branch on $\{w\}$ and $N^+(v) \setminus \{w\}$.

We now prove the correctness of Rule (4). Note that $w \neq s$ otherwise Rule (R6) can be applied on $v$, a contradiction. Let $S$ be a solution to $(G, k, s, t)$. If $w \notin S$ then we also have $v \notin S$ (otherwise $d^+_{G\setminus S}(w) = 0$ and since $w \neq s$ this is a contradiction to the assumption that $S$ is a solution). Therefore, $S$ contains all the vertices in $N^+(v) \setminus \{w\}$ (otherwise $d^+_{G\setminus S}(v) \geq 2$). The branching vector of Rule (4) is at least $(1, 1)$.

(5) If there is a vertex $v \neq s, t$ such that there are $w_1, w_2 \in N^+(v)$ for which $(w_1, w_2) \in E(G)$ and $N^+(w_1) = \{w_2\}$, branch on $\{v\}$ and $\{w_1\}$.

To prove the correctness of Rule (5), suppose that $S$ is a solution to $(G, k, s, t)$ and $v \notin S$. We claim that $w_1 \in S$. Suppose conversely that $w_1 \notin S$. Note that $w_1 \neq t$ otherwise Rule (2) can be applied on $v$, a contradiction. Therefore, $w_2 \notin S$ (otherwise $d^+_{G\setminus S}(w_1) = 0$ and since $w_1 \neq t$ we obtain a contradiction). Since $w_1, w_2 \in N^+(v)$, it follows that $d^+_{G\setminus S}(v) \geq 2$, a contradiction. The branching vector of Rule (5) is $(1, 1)$.

(6) If there is a vertex $v \neq s, t$ such that $d^+(v) = 3$ and there are at least two vertices $w_1, w_2 \in N^+(v)$ for which $N^-(w_i) \setminus \{v\} \subseteq N^+(v)$ for $i = 1, 2$, denote by $w_3$ the third vertex in $N^+(v)$. Branch on $\{v\}$, $\{w_2, w_3\} \cup (N^-(w_1) \setminus \{v\})$, $\{w_1, w_3\} \cup (N^-(w_2) \setminus \{v\})$ and $\{w_1, w_2\}$.

If $S$ is a solution to $(G, k, s, t)$ and $v \notin S$ then $S \cap \{w_1, w_2, w_3\}$ is either $\{w_2, w_3\}$, $\{w_1, w_3\}$, or $\{w_1, w_2\}$ (otherwise $d^+_{G\setminus S}(v) \neq 1$). In the first case we have that $N^-(w_1) \setminus \{v\} \subseteq S$ (otherwise $d^+_{G\setminus S}(w_1) \geq 2$) and in the second case $N^-(w_2) \setminus \{v\} \subseteq S$. The branching vector of Rule (6) is at least $(1, 3, 3, 2)$.

(7) If there is a vertex $v \neq s, t$ such that $d^+(v) = 3$, denote $N^+(v) = \{w_1, w_2, w_3\}$ where $N^-(w_2) = \{v, w_1\}$ (we will show below that there is always such numbering of the vertices in $N^+(v)$). Branch on $\{w_2\}$, $\{w_1, w_3\}$, and $\{v\} \cup (N^+(w_1) \setminus \{w_2\})$.

Since Rule (R) cannot be applied, there are two vertices $w_2, w_3$ in $N^+(v)$ for which $N^-(w_i) \setminus \{v\} \subseteq N^+(v)$ for $i = 2, 3$. Let $w_1$ be the third vertex in $N^+(v)$. Since there is at most one edge whose endpoints are $w_2$ and $w_3$, without loss of generality $(w_2, w_3) \notin E(G)$. Since Rule (H) cannot be applied, $d^-(w_2) \geq 2$. It follows that $N^-(w_2) = \{v, w_1\}$.

Let $S$ be a solution to $(G, k, s, t)$ and suppose that $w_2 \notin S$. Since $d^+_{G\setminus S}(w_2) = 1$, either $v \notin S$ or $w_1 \notin S$. In the former case $w_1, w_3 \in S$ (otherwise $d^+_{G\setminus S}(v) \geq 2$). In the latter case $v \in S$ (otherwise $d^+_{G\setminus S}(v) \geq 2$ and $N^+(w_1) \setminus \{w_2\} \subseteq S$ (otherwise $d^+_{G\setminus S}(w_1) \geq 2$). Note that $N^+(w_1) \setminus \{w_2\} \neq \emptyset$ otherwise Rule (E) can be applied, a contradiction. The branching vector of Rule (7) is at least $(1, 2, 2)$.  

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Note that if Rules (R1)–(R8), (1)–(7), and (2′)–(7′) cannot be applied then \( d^-(v) \in \{1, 2\} \) and \( d^+(v) \in \{1, 2\} \) for every vertex \( v \neq s, t \).

(8) If there is a vertex \( v \neq s, t \) such that \( d^+(v) = 2 \), let \( v \) be such vertex whose distance from \( s \) is minimal. Denote \( N^+(v) = \{w_1, w_2\} \) such that \( (w_1, w_2) \notin E(G) \). We have that \( d^-(w_2) = 2 \) (otherwise Rule (4) can be applied on \( v \)), and let \( x \) be the single vertex in \( N^-(w_2) \setminus \{v\} \). Moreover, \( d^+(x) = 2 \) (otherwise Rule (4) can be applied on \( w_2 \)). Let \( y \) be the single vertex in \( N^+(x) \setminus \{w_2\} \). branch on \( \{w_2\}, \{w_1, x\}, \) and \( \{v, y\} \).

Note that \( x \neq s, t \) (due to Rule (2) and Rule (R7)). Additionally, \( x \neq w_1 \) due to the assumption that \( (w_1, w_2) \notin E(G) \). If \( S \) is a solution to \( (G, k, s, t) \) and \( w_2 \notin S \) then either \( v \notin S \) or \( x \notin S \). In the former case \( w_1 \in S \) (otherwise \( d^+_{G-S}(v) \geq 2 \)) and \( x \in S \) (otherwise \( d^+_{G-S}(w_2) \geq 2 \)). In the latter case \( v \in S \) (otherwise \( d^+_{G-S}(w_2) \geq 2 \)) and \( y \in S \) (otherwise \( d^+_{G-S}(x) \geq 2 \)).

To get the desired branching vector, we show that \( y \neq v \). Suppose conversely that \( y = v \). Therefore, \( d^-(v) = 2 \) (otherwise Rule (4) can be applied on \( x \)). Let \( z \) be the single vertex in \( N^-(v) \setminus \{x\} \). We have that \( d^+(z) = 2 \) (otherwise Rule (4) can be applied on \( v \)). By definition, \( N^-(v) = \{x, z\} \). Therefore, at least one of \( x \) and \( z \) have smaller distance from \( s \) than \( v \). Since \( d^+(x) = d^+(z) = 2 \), we obtain a contradiction to the definition of \( v \). Therefore, \( y \neq v \), and thus the branching vector of Rule (8) is \( (1, 2, 2) \).

The branching vectors of the branching rules of the algorithm are \( (1, 1), (1, 2, 2), \) \( (1, 3, 3, 2) \), and \( (1, 3, 3, 3) \) (in the worst cases). All these vectors have branching number 2. Therefore, the running time of the algorithm is \( O^*(2^k) \).

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