On Exceptional ’t Hooft Lines in 4D-Chern-Simons Theory

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Abstract

We study ’t Hooft lines and the associated $L$-operators in topological 4D Chern-Simons theory with gauge symmetry given by the exceptional groups $E_6$ and $E_7$. We give their oscillator realisations and propose topological gauge quivers encoding the properties of these topological lines where Darboux coordinates are interpreted in terms of topological fundamental matter. Other related aspects are also described.

Keywords: 4D Chern-Simons theory, Wilson and ’t Hooft lines, Topological quivers.

1 Introduction

The discovery of four dimensional Chern-Simons theory [1] has given a great impulse towards deep understanding of quantum integrability in 2D field theory [2] and integrable spin models [3, 4, 5] in lower dimensions. While standard Chern-Simons (CS) gauge theories are topological theories involving hermitian gauge fields in odd spacetime [6], the Costello-Witten Yamazaki (CWY) theory lives in 4D space and goes beyond the hermiticity property. This non unitary feature allowed to extend the application of methods of standard QFT to complexified gauge fields living on complex manifolds [1, 2, 8, 9, 10]. The 4D- CS theory gives a new approach to describe major elements and phenomena of 2D integrable systems in terms of a complex CS gauge potential living on $\Sigma \times \mathbb{C}$ and valued in a complex Lie algebra. Here, the $\Sigma$ is a real 2D topological surface and $\mathbb{C}$ the complex holomorphic line [1]. Roughly speaking, the CWY theory permits to represent the worldline $\gamma_z$ of a particle in $\Sigma$ by a Wilson line operator $W_{\gamma_z}$ characterized by: (i) a complex spectral parameter $z \in \mathbb{C}$, interpreted in 2D as rapidity of the particle [11]; and (ii) a representation space

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V of the complex gauge group $G$ describing intrinsic degrees of freedom of the particle. In this framework, the crossing of two Wilson lines in 4D is expressed by the famous R-matrix $\mathcal{R}(z): V \otimes V \rightarrow V \otimes V$ that verifies the Yang-Baxter equation of integrability. The 4D topological CWY approach was also shown to describe other topological objects like the ones we are interested in this study namely ’t Hooft lines [12]. They correspond to the Baxter Q-operator of integrable spin chains [13, 14, 15] and have an interesting interpretation in 4D CS theory. Their coupling to $W_{\gamma_z}$ is given by the so-called $\mathcal{L}$-operator [12, 16, 17], playing a quite similar role as the R-matrix [1, 2]. This gauge invariant quantity was recently interpreted as an observable measuring the parallel transport of the gauge field sourced by the ’t Hooft line and was realized for A-type and D-type gauge symmetries using symplectic oscillators [12, 16].

In this paper, we contribute to 4D- CS theory with miniscule ’t Hooft lines by working out the oscillator realisation of the $\mathcal{L}$-operator for the exceptional gauge symmetries; thus completing results obtained in [12, 3, 18]. First, we revisit useful aspects on ’t Hooft lines in the 4D- CS theory for generic gauge symmetries $G$ while illustrating the construction for $G = SL(N)$. Then, we focus on the topological $E_6$ and $E_7$ theories and give the missing oscillator representation of the $\mathcal{L}$-operators associated with these gauge symmetries. Our $\mathcal{L}_{E_6}$ and $\mathcal{L}_{E_7}$ can be also viewed as a generalization of the results regarding A-type [18] and D-type [3] gauge groups. After that, we borrow ideas from supersymmetric quiver gauge theories to propose a quiver representation of the ’t Hooft line operators. In this view, the $\mathcal{L}_{E_6}$'s are represented by topological gauge quivers $Q_{E_6}$ where nodes and links are interpreted in terms of topological gauge matter. To fix ideas on the shape of these quivers, see the Figures 4 and 7. In these topological $Q_{E_6}$’s, the Darboux coordinates of the phase space of the $\mathcal{L}_{E_6}$’s are interpreted in terms of topological fundamental matter and nodes as self-dual matter.

The organisation of this study is as follows. In section 2, we introduce the minuscule ’t Hooft lines in 4D CS theory and describe some of their representations as well as their implementation in the CWY theory. In sections 3 and 4, we study the 4D- CS theory with $E_6$ gauge symmetry. First, we build the $\mathcal{L}_{E_6}$ operator using 16 oscillators. Then, we give our proposal regarding the topological gauge quiver $Q_{E_6}$. In section 5, we do the same thing for the the 4D- CS with gauge symmetry $E_7$. Section 6 is devoted to conclusion and perspectives. The last section is an appendix detailing technical steps in the calculation of the Lax operator for the $E_7$ theory.

## 2 ’t Hooft lines in 4D Chern-Simons theory

In this section, we revisit some basic ingredients regarding classical ’t Hooft lines $\gamma_z$ in 4D-CS theory with a rank r ADE gauge symmetry $G$. First, we introduce these magnetically charged lines with some charge $\mu$ to be specified later. For convenience, we refer to these lines like $\text{tH}_z^\mu$. Then, we describe their realisation in 4D topological CS theory by following
the Costello- Gaiotto- Yagi representation given in [12]. We refer to this realisation as the CGY \( \mathcal{L} \)-operator. We end this review by describing the phase space of this gauge invariant observable formulated as an "RTT" equation and its relationship with Darboux coordinates of symplectic geometry.

### 2.1 Minuscule ’t Hooft lines and CGY observable

Generally speaking, the ’t Hooft lines are a dual version of Wilson lines introduced to represent the worldline of an infinitely heavy magnetic monopole in spacetime [19] [20] [21]. In the 4D CS theory on \( \Sigma \times \mathbb{C} \) with gauge symmetry \( G \), a ’t Hooft line \( \text{tH}^\mu_{\gamma_z} \) is a 1D topological defect \( \gamma_z \) that lives in the topological surface \( \Sigma \), taken below as the plane \( \mathbb{R}^2 \). As for the electrically charged Wilson lines \( W^{q_e}_{\xi_w} \), the magnetically charged \( \text{tH}^{q_\mu}_{\gamma_z} \) is also characterised by a spectral parameter \( z \) interpreted in the 4D CS theory as the position of the line defect in the holomorphic sector \( \mathbb{C} \). An interesting family of the \( \text{tH}^{q_\mu}_{\gamma_z} \) lines is given by the so-called *minuscule ’t Hooft lines* \( \text{tH}^{\mu}_{\gamma_z} \) on which we will be focussing on in this study. They are characterised by minuscule coweights \( \mu \) of the gauge symmetry \( G \) [22], and have nice realisations in 4D Chern-Simons theory living on \( \mathbb{R}^2 \times \mathbb{C} \) involving the compact complex holomorphic line \( \mathbb{C} \mathbb{P}^1 \). In ref. [12], the \( \text{tH}^{\mu}_{\gamma_z} \) lines were implemented in the 4D CS theory through their coupling with a Wilson line \( W^{q_e}_{\xi_w} \) in some representation \( R \) of \( G \). This \( \text{tH}^{\mu}_{\gamma_z} - W^{q_e}_{\xi_w} \) coupling is described by their crossing as depicted by the Figure 1a; see also [21]. They are modeled by a holomorphic matrix \( \mathcal{L}(z) \) valued in the algebra \( \mathfrak{A} \) of functions on the phase space of \( \text{tH}^{\mu}_{\gamma_z} \); i.e: \( \mathcal{L}(z) \in \mathfrak{A} \otimes \text{End}(R) \). Following [12], minuscule ’t Hooft lines, carrying a minuscule magnetic charge \( \mu \), can be remarkably described in 4D CS on \( \mathbb{R}^2 \times \mathbb{C} \mathbb{P}^1 \) with simply connected group \( G \). Here, the real \((x, y)\) are local coordinates of \( \mathbb{R}^2 \), and the complex \( z = Z/Z' \) is a local variable in \( \mathbb{C} \mathbb{P}^1 \) with homogeneous coordinates \((Z_1, Z_2) \neq (0, 0)\). In this 4D topological theory with field action \( \int_{M_4} dz \wedge \Omega_3(A) \) where \( \Omega_3 \) is the CS 3-form, the ’t Hooft lines are thought of as line defects in \( M_4 \) wrapping a real curve \( \gamma_z \) in \( \mathbb{R}^2 \). It is convenient to take \( \gamma_z \) as the horizontal x-axis \( \{y = 0, z = 0\} \) which is interesting to imagine as given by \( \gamma_0 = \mathcal{O}^-_0 \cap \mathcal{O}^0_+ \) with \( \mathbb{R}^2 = \mathcal{O}^0_+ \cup \mathcal{O}^0_- \); that is the intersection of two 2D patches \( \mathcal{O}^+_z \) and \( \mathcal{O}^-_z \) in \( \mathbb{R}^2 \times \mathbb{C} \mathbb{P}^1 \) like

\[
\mathcal{O}^-_0 = \left\{ (x, y; z) \mid y \leq 0 \quad z = 0 \right\}, \quad \mathcal{O}^0_+ = \left\{ (x, y; z) \mid y \geq 0 \quad z = 0 \right\}
\]

(2.1)

These ’t Hooft line defects can be also viewed as a pair \( \text{tH}^{\pm \mu}_{\gamma_z} \) living at the ends of a Dirac string at \( y = 0 \) stretched between \( z = 0 \) and \( z = \infty \). In this picture, the line defect \( \text{tH}^{\pm \mu}_{\gamma_z} \) at \( z = 0 \) has a magnetic charge \( \pm \mu \) and the line \( \text{tH}^{- \mu}_{\gamma_z} \) at infinity has a magnetic charge \( -\mu \). Other interesting representations of \( \text{tH}^{\pm \mu}_{\gamma_z} \) in integrable field theory and spin chains can be found in [12] and refs therein. For ADE gauge symmetries \( G \) with generic rank \( r \) given by simply connected groups like the exceptional \( E_6 \) and \( E_7 \) we are considering in this study, the \( G \)-bundle on the complex projective line is trivial. This remarkable property is because of the trivial behavior of the gauge potential at \( z = \infty \) and which extends to all points in
This triviality feature on \( \mathbb{CP}^1 \) applies to the 2D patches \( \mathcal{O}_x^- \) and \( \mathcal{O}_x^+ \) of eq(2.1) with \( z \sim 0 \); and has been used by Costello- Gaiotto- Yagi to propose a classical gauge invariant observable \( \mathcal{L}(z) \) given by the path ordered quantity \( P \exp \int_y A(z) \) to measure the parallel transport of the gauge field from the patch \( \mathcal{O}_y^- \) to the patch \( \mathcal{O}_y^+ \). It is a function of \( z \); and belongs to the loop group \( G((z)) \) of analytic functions valued in \( G \). The CGY observable has interesting properties described in [12]; in particular the following ones that are useful for this study: (1) As noticed before, the \( \mathcal{L}(z) \) describes the crossing of 't Hooft lines with a Wilson line as shown by the Figure [1]. It plays a quite similar role as the R-matrix of Wilson lines; and is also interpreted in terms of the Lax operator of 2D integrable systems with spectral parameter \( z \) [21,25]. (2) The \( \mathcal{L}(z) \) has poles and zeroes at \( z = 0 \) and \( z = \infty \) arising from 't Hooft lines \( \text{tH}_{\gamma_z} \) at these particular points in \( \mathbb{CP}^1 \). Near the singularity at \( z = 0 \), the CGY observable can be factorised like \( \mathcal{L}(z) = A(z) z^\mu B(z) \) where \( A(z) \) and \( B(z) \) are regular functions at \( z = 0 \). The operator \( z^\mu \) carries the Dirac monopole singularity with minuscule coweight action given by the adjoint form of \( \mu \) to be described later. A quite similar factorisation of \( \mathcal{L}(z) \) exists near the singularity at \( z = \infty \). It reads also as \( A(z) z^\mu \tilde{B}(z) \); but with \( A(z) \) and \( \tilde{B}(z) \) going to identity for \( z \) approaching infinity. (3) By imposing regularity conditions in bulk and boundary; the \( \mathcal{L}(z) \) can be brought to the following interesting form

\[
\mathcal{L}^{(\mu)}(z) = e^X z^\mu e^Y
\]  (2.2)

where \( X \) and \( Y \) are globally defined on \( \mathbb{CP}^1 \); i.e independent of \( z \). In this factorisation, the \( X \) and \( Y \) are matrix operators valued in the nilpotent subalgebras \( \mathfrak{n}_+ \) and \( \mathfrak{n}_- \) of the Levi- decomposition with respect to \( \mu \) of the Lie algebra \( g \) of the gauge symmetry \( G \) of the 4D CS theory. Notice that given a Lie algebra \( g \), one may have different minuscule coweights \( \mu_a \) and consequently various \( \text{tH}_{\gamma_z}^{\mu_a} \) different CGY observables \( \mathcal{L}^{(\mu_a)} \). As an illustration, we describe rapidly below the instructive example of 4D Chern-Simons theory with gauge symmetry \( G = SL(N) \). The gauge group of this family has \( N-1 \) minuscule coweights \( \mu_a \); and so \( N-1 \) types of minuscule 't Hooft lines \( \text{tH}_{\gamma_z}^{\mu_a} \). This property can be explicitly formulated by using the canonical vector basis \( \{ e_i = |i\rangle \} \) of the ambient space \( \mathbb{R}^N \) to express the content of the root system \( \Phi_{SL(N)} \) of \( SL(N) \) and its \( N-1 \) minuscule coweights \( \mu_1, ..., \mu_{N-1} \). Recall that the Lie algebra \( sl_N \) underlying the \( SL(N) \) gauge symmetry has \( N(N-1) \) roots \( \alpha = n_a \alpha_a \) generated by N-1 simple roots \( \alpha_a = e_a - e_{a+1} \). It also has N-1 minuscule coweights obeying \( \mu_a \cdot \alpha_b = \delta_{ab} \) and expressed in terms of the \( e_i \)'s like \( N^{-a}(e_1 + ... + e_a) - e_a + ... + e_N) \). The adjoint form of the minuscule coweights used in \( \mathcal{L}^{(\mu_a)} \) reads as \( \mu_i^a |i\rangle \langle i| \) with \( \mu_i^a = 1 - a/N \) for \( 1 \leq i \leq a \) and \( -a/N \) for \( a+1 \leq i \leq N \). It happens that for the \( SL(N) \) family, the matrix operators \( X \) and \( Y \) in (2.2) obey the nilpotency relations \( X^2 = Y^2 = 0 \); then the \( \mathcal{L} \)-operator for \( SL(N) \) reduces to the following particular form

\[
\mathcal{L}_{sl_N}(z) = (I + X) z^\mu (I + Y)
\]  (2.3)

Notice that in the above \( \mathcal{L}_{sl_N}(z) \), the higher monomial in the nilpotent matrix operators is \( X z^\mu Y \). This is a specific property of \( sl_N \). Later on, we will study the extension of this
construction to the 4D Chern-Simons gauge theories with exceptional $E_6$ and $E_7$ gauge symmetries. Then, we will derive new aspects regarding the structure of the topological $\mathcal{L}_{E_6}$ and $\mathcal{L}_{E_7}$.

### 2.2 Phase space of ’t Hooft lines

Here, we describe the classical phase space $E_{ph}[L(z)]$ of the minuscule ’t Hooft lines in 4D CS theory and its parametrisation using Darboux coordinates. An interesting way to deal with the properties of $E_{ph}$ is to consider the CGY observable $L(z)$ and use the graphic representation depicted by the Figures 1. In this representation, we think of the CGY observable as a matrix operator $\langle i | L(z) | j \rangle$ describing the crossing of a tH$_{\gamma_0}$ with a Wilson line $W_{q_\xi}$ on which a set of $|i\rangle$-states propagate. In this picture, the ’t Hooft line is materialized by the horizontal x-axis of the plane $\mathbb{R}^2$ with spectral parameter $z = 0$; and the Wilson line is given by the vertical y-axis with a generic $z$ in $\mathbb{C}P^1$. The charge of the tH$_{\gamma_0}$ is given by the minuscule coweight $\mu$ of the gauge symmetry $G$ and the Wilson line is characterised by some representation $R$ of $G$ with incoming states $|i\rangle$ and outgoing $|j\rangle$. As such, the $L(z)$ combines topological data from the tH$_{\gamma_0}$ and the Wilson $W_{q_\xi}$ encoded

![Figure 1](image.png)

**Figure 1:** (a) The operator $L(z)$ encoding the coupling between a ’t Hooft line at $z=0$ (in red) and a Wilson line at $z$ (in blue) with incoming $\langle i \rangle$ and outgoing $\langle j \rangle$ states. (b) RLL relations encoding the commutation relations between two L-operators at $z$ and $z'$.

in the $L_j^i(z)$ matrix entries of $L$. In this view, the symplectic structure of two operators $L_j^r(z)$ and $L_i^s(z')$ is given by the RLL relations shown in the Figure 1(b). These relations are due to the topological invariance of the CS theory and read explicitly as follows [3],

$$R_{rs}^{ik}(z - z') L_j^r(z) L_i^s(z') = L_j^i(z') L_i^k(z) R_{js}^{rs}(z - z')$$

(2.4)

where the tensor $R_{rs}^{ik}(z - z')$ is the usual R-operator used in the study of Yang-Baxter equation (YBE). Though looking cumbersome, the RLL relations (2.4) were shown to be equivalent, at the leading order in the $\hbar$-expansion of the R matrix —see (2.7) given below—,
to the usual Poisson bracket \( \{ b^\alpha, c_\beta \}_{PB} = \delta_\beta^\alpha \) of symplectic geometry with Darboux coordinates \((b, c)\) imagined in this limit as classical oscillators. The equivalence between \( L_j^x \) and \((b^\alpha, c_\beta)\) is established using the Levi- decomposition of the Lie algebra \( g \) of the gauge symmetry \( G \). Indeed, given a minuscule coweight \( \mu \) of \( g \), we have the following Levi-decomposition \[23\]

\[
g = l_\mu \oplus n_+ \oplus n_- = p \oplus n_-
\]

where \( l_\mu \) is the Levi- factor and the \( n_\pm \) are nilpotent subalgebras of \( g \). In the second equality of \((2.5)\), we have used the short splitting \( p = l_\mu \oplus n_+ \) with \( p \) standing for a parabolic subalgebra of \( g \). By using Dirac singularity at \( z = 0 \) and \( z = \infty \) and following \[12\], we can first factorise the \( L(z) \) operator as in eq\((2.2)\) where \( X \) and \( Y \) are nilpotent operators respectively valued in the subalgebras \( n_+ \) and \( n_- \). These nilpotent operators do not depend on the spectral parameter \( z \); they are globally defined on \( \mathbb{C}P^1 \). Moreover denoting by \( X_\alpha \) the generators of \( n_+ \) and by \( Y_\beta \) the generators of \( n_- \), then substituting the expansions \( X = b^\alpha X_\alpha \) and \( Y = c_\beta Y_\beta \) into \[24\] with \[22\] and replacing \( R_{ik}^j \) by its expression in terms of the double Casimir, we end up with the wanted Poisson bracket

\[
\{ b^\alpha, c_\beta \}_{PB} = \delta_\beta^\alpha
\]

showing that \( b^\alpha \) and \( c_\beta \) are indeed Darboux-like complex coordinates of the classical phase space \( \mathcal{E}_{ph} [ L (z) ] \) of the minuscule \('t Hooft lines in 4D CS gauge theory. Here, we used the expression of the R-matrix as a rational solution of the YBE at the leading order in \( \hbar \) (semi-classical) \[1 \ 2\]

\[
R_{ik}^j (z) = \delta_i^j \delta_k^l + \frac{\hbar}{z} \delta_i^j c_{jl}^k + O ( \hbar^2 )
\]

where \( c_{ik}^j \) stands for the double Casimir of the gauge symmetry; its value for \( sl_N \) is \( \delta_i^j c_{ik}^j \). Notice as well that at the quantum level, the \( b^\alpha \) and \( c_\beta \) are promoted to operators and eq\((2.6)\) is replaced by the commutator \([ \hat{c}_\beta, \hat{b}^\alpha ] \sim \hbar \delta_\beta^\alpha \) where \( \hat{c}_\beta \) and \( \hat{b}^\alpha \) respectively interpreted as annihilation and creation operators. To get more insight into this promotion, note that by substituting \( \hat{X} = \hat{b}^\alpha X_\alpha \) and \( \hat{Y} = \hat{c}_\beta Y_\beta \) into \[22\], we obtain \( e^{\hat{b}^\alpha X_\alpha z^\mu} e^{\hat{c}_\beta Y_\beta} \). In this relation, the creators \( e^{\hat{b}^\alpha X_\alpha} \) are put on the left and the annihilators \( e^{\hat{c}_\beta Y_\beta} \) are put on the right; thus indicating that quantum mechanically speaking, the \( L \)-operator given by eq\((2.2)\) is non ambiguous as it is normal ordered; that is \( \hat{L} (\mu ) := e^{\hat{X} z^\mu} e^{\hat{Y}} \). Notice moreover that for the case of the Levi- decomposition \( sl_N \rightarrow l_{\mu_1} \oplus n_+ \oplus n_- \) with \( l_{\mu_1} = sl_{N-1} \oplus \mathbb{C} \mu_1 \) and \( n_\pm = F \) with \( sl_{N-1} \) fundamentals \( F = N - 1 \) and its dual, the classical \( \mathcal{L} \)- operator reads as follows

\[
\mathcal{L}_{sl_N} = \left( \begin{array}{ccc}
\frac{N+1}{2} & b^T c & b^T \\
\frac{1}{2} & c & z^{-\frac{1}{2}} I_{N-1}
\end{array} \right)
\]

with \( b^T = (b_1, ..., b_{N-1}) \) and \( c = (c_1, ..., c_{N-1})^T \). Having revisited useful aspects on the CGY observable in 4D CS theory on \( \Sigma \times \mathbb{C}P^1 \); we turn now to represent our contribution by focussing first on the \('t H\) \( \mu \) in the \( E_6 \) CS gauge theory and then on the case of the \( E_7 \) theory.
3 Exceptional E\(_6\) minuscule ’t Hooft line

In this section, we study the exceptional minuscule ’t Hooft line operators living in the 4D CS theory with gauge symmetry E\(_6\). From the analysis of the previous section, the phase space \(\mathcal{E}^{E_6}_{ph}\) of the operator \(\mathcal{L}_{E_6}\) is determined by using the Levi- decomposition of the gauge symmetry with respect to a minuscule coweight \(\mu_{E_6}\) of the Lie algebra \(e_6\) of the Lie group \(E_6\). Before constructing \(\mathcal{L}_{E_6}(z)\), let us start by giving some useful tools regarding \(E_6\), its Levi- decomposition and the splitting of its fundamental representation \(27\).

3.1 Minuscule coweights and Levi subalgebra of \(E_6\)

The exceptional Lie algebra \(e_6\) has six Cartan type generators \(H_{\alpha}\), and 72 step operators \(Z_{\pm\alpha}\) labeled by the \(\pm\alpha\) roots of \(E_6\) with length \(\alpha^2 = 2\). The 72 roots of the root system \(\Phi_{E_6}\) are generated by six simple roots \(\{\alpha_i\}_{1 \leq i \leq 6}\) with intersection \(K_{ij}\) given by the Cartan matrix \(K_{E_6}\). We realise these roots in \(\mathbb{R}^8\) as follows

\[
\alpha_1 = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 - \epsilon_5 - \epsilon_6 - \epsilon_7 + \epsilon_8) \tag{3.1}
\]

and \(\alpha_i = \epsilon_i - \epsilon_{i-1}\) for \(i \neq 1, 6\) as well as \(\alpha_6 = \epsilon_1 + \epsilon_2\). The 72 roots \(\alpha\) of \(E_6\) can be organised into two subsystems. (1) a subset of 40 positive \(\pm(\epsilon_i \pm \epsilon_j)\) with \(1 \leq j < i \leq 5\); and (2) a subset of 32 roots \(\pm \frac{1}{2}(q_i \epsilon_i - \epsilon_6 - \epsilon_7 + \epsilon_8)\) where the five \(q_i\) take \(\pm 1\) with \(\Pi_{i=1}^5 q_i = 1\).

Concerning the representations of the Lie algebra of \(E_6\), they can be built out of its six fundamental representations. Here, we will be particularly interested into \(78_0\), associated with the simple root \(\alpha_6\) as depicted by the the Figure 21 and into the \(27_\pm\) associated with \(\alpha_1\) and \(\alpha_5\). The \(E_6\) has two minuscule coweights \(\mu_1\) and \(\mu_5\) dual to \(\alpha_1\) and \(\alpha_5\); they respectively correspond to the fundamentals \(27_+\) and \(27_-\). Taking as a minuscule \(\mu\) for our \(E_6\) gauge theory the \(\mu_1\) coweight, it follows that the Levi- subalgebra \(l_\mu\) of the exceptional \(E_6\) is given by \(so(2) \oplus so(10)\). Thus, the 40 roots of \(so(10)\) is a subset of the root system of \(E_6\); it is easily read from the \(E_6\) system by omitting the 32 roots containing the spinorial-like root \(\alpha_1\). In this view, the simple roots of the \(so(10)\) subalgebra of \(e_6\) are given the five \(\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\) and the Levi- decomposition \(e_6 = l_\mu \oplus n_+ \oplus n_-\) reads as follows

\[
e_6 \rightarrow so(2) \oplus so(10) \oplus 16_+ \oplus 16_- \tag{3.2}
\]

This splitting distributes the 78 dimensions of \(e_6\) like \(1 + 45 + 16_+ + 16_-\). Eq(3.2) can be also read at the level of the Dynkin diagram of \(e_6\) given by the Figure 22. By cutting the node \(\alpha_1\), associated with the minuscule coweight \(\mu_1\), we recover the Dynkin diagram of \(so(10)\) and its spinor representations \(16_+\) and \(16_-\) charged under \(so(2)\). Notice that the \(36+36\) step operators of \(e_6\) are split in the Levi- decomposition as \(20+20\) step operators \(Z_{\pm\alpha}\) generating \(so(10)\), 16 step operators \(X_{+\beta}\) generating the nilpotent subalgebra \(16_+\) and 16 other step operators \(X_{-\beta} = Y^\beta\) generating \(16_-\). Notice also that under the Levi-decomposition, representations \(R_{E_6}\) of the exceptional symmetry \(E_6\) reduce as direct sums
Figure 2: Dynkin Diagram of E6 having six nodes labeled by the simple roots $\alpha_i$. The cross ($\times$) indicates the roots used in the Levi decomposition with Levi subalgebra $so(10) \oplus so(2)$.

$$\sum (R_i^{so_{10}}, R_i^{so_2})$$ of representations of $so(10) \oplus so(2)$. For the example of the fundamental of E6, we have the following splitting [29],

$$27 = (1, -\frac{4}{3}) + (10, \frac{2}{3}) + (16, -\frac{1}{3})$$ (3.3)

3.2 Minuscule CGY observable $\mathcal{L}_{E_6}$

To construct the 't Hooft line operator of the exceptional E6 CS theory, notice that $\mathcal{L}_{E_6}^{(\mu)}$ is characterised by the representation $R$ of the Wilson line and by the minuscule coweight $\mu$ with Levi-quantum numbers as in (3.3). This $\mathcal{L}_{E_6}^{(\mu)}$ operator is given by the Levi-factorisation $e^{X} e^{\mu} e^{Y}$ with $X = \sum_{\beta=1}^{16} b^\beta X^\beta$ and $Y = \sum_{\beta=1}^{16} c^\beta Y^\beta$. In these expansions, the $b^\beta$ and $c^\beta$ stand for the 16+16 Darboux coordinates satisfying the Poisson bracket $\{b^\gamma, c^\beta\} = \delta_{\gamma}^\beta$ and $X^\beta$ and $Y^\beta$ are the generators of the nilpotent subalgebras $16_\pm$. The charge operator $\mu$ associated with the minuscule coweight is given by

$$\mu = -\frac{4}{3} \varrho_{1} + \frac{2}{3} \varrho_{10} - \frac{1}{3} \varrho_{16}$$ (3.4)

where $\varrho_{1}$, $\varrho_{10}$ and $\varrho_{16}$ are projectors on the $so(10) \oplus so(2)$ representation spaces of (3.3). If we choose to denote the 27 states of the fundamental representation of E6 by the basis vector kets $|\xi\rangle$ with $\xi = 0, 1, ... 26$ as formally depicted by the Figure 3, then the projectors are given by

$$\varrho_{1} = |0\rangle \langle 0|$$

$$\varrho_{10} = \sum_{l=1}^{10} |v_l\rangle \langle v^l|$$ as well as

$$\varrho_{16} = \sum_{\beta=1}^{16} |s_{\beta}\rangle \langle s^\beta|.$$ Using the basis state kets $|0\rangle$, $|v_l\rangle$ and $|s_{\beta}\rangle$ satisfying the orthogonality properties $\langle 0|v_l\rangle = \langle 0|s_{\beta}\rangle = \langle v_l|s_{\beta}\rangle = 0$, we can explicitly realise the generators $X^\beta$ and $Y^\beta$ in the phase space of the E6 't Hooft line operator. We find

$$X^\beta = (\Gamma^i)_{\alpha\gamma}^\beta |v_i\rangle \langle s^\gamma| + |s_{\beta}\rangle \langle 0|$$

$$Y^\beta = |0\rangle \langle s^\beta| + (\Gamma_i)_{\beta\gamma}^\beta |s_{\gamma}\rangle \langle v^i|$$ (3.5)

where the $\Gamma_i$’s are the ten-dimensional Gamma matrices satisfying the usual Clifford algebra $\Gamma_i \Gamma_j + \Gamma_j \Gamma_i = 2\delta_{ij}$. Notice that one can also construct operators of $so(10) \oplus so(2)$ from

Figure 3: A formal graphic illustrating Levi-decomposition of the representation 27 of E6 in terms of representations of SO(10). Here, we have $27 = 1 + 16 + 10$. 

by $\varrho_{1} = |0\rangle \langle 0|$ and $\varrho_{10} = \sum_{l=1}^{10} |v_l\rangle \langle v^l|$ as well as $\varrho_{16} = \sum_{\beta=1}^{16} |s_{\beta}\rangle \langle s^\beta|$. Using the basis state kets $|0\rangle$, $|v_l\rangle$ and $|s_{\beta}\rangle$ satisfying the orthogonality properties $\langle 0|v_l\rangle = \langle 0|s_{\beta}\rangle = \langle v_l|s_{\beta}\rangle = 0$, we can explicitly realise the generators $X^\beta$ and $Y^\beta$ in the phase space of the E6 't Hooft line operator. We find

$$X^\beta = (\Gamma^i)_{\alpha\gamma}^\beta |v_i\rangle \langle s^\gamma| + |s_{\beta}\rangle \langle 0|$$

$$Y^\beta = |0\rangle \langle s^\beta| + (\Gamma_i)_{\beta\gamma}^\beta |s_{\gamma}\rangle \langle v^i|$$ (3.5)

where the $\Gamma_i$’s are the ten-dimensional Gamma matrices satisfying the usual Clifford algebra $\Gamma_i \Gamma_j + \Gamma_j \Gamma_i = 2\delta_{ij}$. Notice that one can also construct operators of $so(10) \oplus so(2)$ from
these $|0\rangle$, $|v_i\rangle$ and $|s\beta\rangle$ states. An interesting operator is the generator of $so(2)$ which is nothing but (3.4); it acts on $X_\beta$ and $Y^\beta$ like $[\mu, X_\beta] = X_\beta$ and $[\mu, Y^\beta] = -Y^\beta$ as required by the Levi-decomposition. As a direct check, we can calculate the first commutator as

$$[\mu, X_\beta] = \left(-\frac{1}{3} + \frac{4}{3}\right)|s\beta\rangle\langle 0| + \left(\frac{2}{3} + \frac{1}{3}\right)(\Gamma^i)_{\beta s}|v_i\rangle \langle s^\gamma|$$  \hspace{1cm}  (3.6)

which is equal to $X_\beta$. Using the realisation (3.5), we can work out explicit calculations regarding the CGY observable $L_{E_6}$. From (3.5), we deduce the action of the $X_\beta$ and $Y^\beta$ generators of the $16_+$ and $16_-$ blocks in (3.2) on the states $|0\rangle$, $|v^i\rangle$ and $|s\beta\rangle$. As such, we can write their explicit matrix realisations $\langle A|X_\beta|B\rangle$ and $\langle A|Y^\beta|B\rangle$ with labels $A, B = 0, l, \beta$. We can also compute the powers of the operators $X = b^\beta X_\beta$ and $Y = c^\beta Y^\beta$ involved in the calculation of $e^X z^\mu e^Y$. We find that $X^3 = Y^3 = 0$ and

$$X^2 = 2V^i |v_i\rangle \langle 0|, \quad Y^2 = 2W_i |0\rangle \langle v^i|$$  \hspace{1cm}  (3.7)

where we have set $V^i = \frac{1}{2}b^{\alpha} (\Gamma^i)_{\alpha\beta} b^\beta$ and $W_i = \frac{1}{2}c_\alpha (\Gamma_i)^{\beta\alpha} c_\beta$. The nilpotency feature of the $X$ and $Y$ operators leads to the finite expansion $e^X = I + X + \frac{1}{2}X^2$ and the same goes for $e^Y$. Substituting (3.5) and (3.7) into these expansions, we obtain

$$e^X = I + b^\beta (\Gamma^i)_{\beta\gamma} |v_i\rangle \langle s^\gamma| + |s\beta\rangle \langle 0| + V^i |v_i\rangle \langle 0|$$

$$e^Y = I + c_\beta \left(|0\rangle \langle s^\beta| + (\Gamma^i)_{\gamma\beta} (\Gamma_i)^{\alpha\gamma} |v^i\rangle \langle s^\gamma|\right) + W_i |0\rangle \langle v^i|$$  \hspace{1cm}  (3.8)

reading in matrix notation $M^A_B = \langle \xi_B|M|\xi^A\rangle$ in the basis ordered like $|\xi_B\rangle = |0\rangle$, $|v_j\rangle$, $|s\beta\rangle$ as follows

$$(e^X)^A_B = \begin{pmatrix} 1 & 0 & 0 \\ V^i & \delta^i_j & B^i_\beta \\ b^\alpha & 0 & \delta^\alpha_\beta \end{pmatrix}, \quad (z^\mu e^Y)^A_B = \begin{pmatrix} z^{-\frac{4}{3}} & z^{-\frac{4}{3}} W_j & z^{-\frac{4}{3}} c_\beta \\ 0 & z^{\frac{2}{3}} \delta^i_j & 0 \\ 0 & z^{-\frac{1}{3}} c_\alpha & z^{-\frac{1}{3}} \delta^\alpha_\beta \end{pmatrix}$$  \hspace{1cm}  (3.9)

where we have set $B^i_\beta = b^\gamma \Gamma^i_{\gamma\beta}$ and $C^\alpha_j = c_\gamma \Gamma^\alpha_{\gamma\beta}$. Substituting, we end up with the expression of the CGY- observable given by

$$L_{E_6} = \begin{pmatrix} z^{-\frac{4}{3}} & z^{-\frac{4}{3}} W_j \\ z^{-\frac{4}{3}} V^i & z^{\frac{2}{3}} \delta^i_j + z^{-\frac{4}{3}} V^i W_j + z^{-\frac{4}{3}} B^i_\alpha C^\alpha_j \\ z^{-\frac{4}{3}} b^\alpha & z^{-\frac{1}{3}} b^\alpha W_j + z^{-\frac{4}{3}} C^\alpha_j \\ z^{-\frac{1}{3}} c_\beta & z^{-\frac{1}{3}} \delta^\alpha_\beta + z^{-\frac{4}{3}} b^\alpha c_\beta \end{pmatrix}$$  \hspace{1cm}  (3.10)

We end this subsection by mentioning that if instead of cutting the simple root $\alpha_1$ in the Figure 2, we omit the simple root $\alpha_5$, we follow an analogous analysis to the one represented here. In the $\alpha_5$- dual description, we obtain quite similar results as $L^\alpha_{E_6} \{z\}$; but with replacing the spectral parameter $z$ with $w = \frac{1}{z}$, that is

$$L^\alpha_{E_6} \{z\} = L^\alpha_{E_6} \{1/z\}$$  \hspace{1cm}  (3.11)

This feature may be nicely viewed from the correspondence between the simple roots $\alpha_5/\alpha_1$ and the fundamental coweights $\mu_5/\mu_1$. In this regard, recall that $\mu_5$ is the highest coweight of the fundamental representation $27_-$ just like $\mu_1$ is the highest coweight of the fundamental representation $27_+$ given by eq. (3.3). Since the two coweights are related as $\mu_5 = -\mu_1$, it follows that (3.4) gets modified as $\bar{\mu} = \frac{4}{3} \bar{\theta}_1 - \frac{2}{3} \bar{\theta}^{10} + \frac{1}{3} \bar{\theta}^{16}$. 

9
4 Quiver representation of $\mathcal{L}_{E_6}$

In this section, we develop a graphic representation of the CGY- operator $\mathcal{L}_{E_6}$ and use it to comment on the topological structure of the $L^C_{B}$ entries of (3.10). We denote this graph as $Q_{E_6}$ and we refer to it as the topological gauge quiver associated with the line operator $\mathcal{L}_{E_6}$. This denomination is borrowed from supersymmetric quiver gauge theory with gauge symmetry $G = \Pi_i G_i$ where the nodes of the supersymmetric gauge quiver represent the gauge factors $G_i$ and its links $R_{ij} \sim R_i \times R_j$ describe the bi-fundamental matter in $(G_i, G_j)$ [26, 27]. Concerning the topological quiver $Q_{E_6}$ we are interested in here, it is motivated amongst others (see below) by (i) the Levi- decomposition $N_- \times \mathbf{L}_\mu \times N_+$ of the gauge symmetry $E_6$, and (ii) the topological aspect of the 4D CS theory.

A way to introduce the topological quiver $Q_{E_6}$ of the CGY operator $\mathcal{L}_{E_6}$ is to use the phase space bracket (2.4) solved by eq(2.6). This last relation indicates that the Darboux coordinates $(b^\beta, c_\gamma)$ are fundamental objects in dealing with $\mathcal{L}_{E_6}$. As such, they play a quite similar role as the (bi-) fundamental matter $R_{ij}$ in supersymmetric quiver gauge theories. So, it is interesting to use this property of $b^\beta$ and $c_\gamma$ to encode the internal structure of $\mathcal{L}_{E_6}$ into a diagram $Q_{E_6}$ with three nodes $N_1$, $N_2$, $N_3$ related to each other by 3+3 links $L_{ij}$ and $L_{ji}$ with $1 \leq i < j \leq 3$ as depicted by the Figure 4. The nodes and the links of

![Figure 4: $\mathcal{L}_{E_6}$ as a topological quiver with 3 nodes $N_i$ and 6 links $L_{ij}$](image)

The nodes are given by the self-dual $R_i \otimes \bar{R}_i$ and the links by bi-matter $R_i \otimes \bar{R}_j$. In addition to SO(10) representations, the Darboux coordinates $b^\alpha$, $c_\alpha$ carry an SO(2) charge $q = -1, +1$. The fundamental vector like matter $V^i$ and $W_i$ carry $-2$ and $+2$. The quiver $Q_{E_6}$ are determined from the matrix entries $L^C_{B}$ of the CGY- observable $\mathcal{L}_{E_6}$ and have an interpretation in terms of representations of the Levi- subalgebra of $e_6$. Indeed, from the decomposition $e_6 \rightarrow 16_- \oplus \mathbf{1}_\mu \oplus 16_+$ with $\mathbf{1}_\mu = so(2) \oplus so(10)$ and the expansions $X = b^\beta X_\beta$ and $Y = c_\gamma Y_\beta$, we see that the Darboux coordinates transform under $\mathbf{1}_\mu$ like $b^\beta \sim 16_-$ and $c_\gamma \sim 16_+$. So, we can think of the $b^\beta$ and the $c_\gamma$ in terms of (topological) bi-fundamental matter of $SO(2) \times SO(10)$ as described below. First, we observe that the operator $\mathcal{L}_{E_6} = e^X z^\mu e^Y$ is a $27 \times 27^t$ matrix with entries $L^C_{B}$ given by eq(3.10). Second, we
split the \( L_B^C \) like

\[
\mathcal{L}_{E_6} = \begin{pmatrix}
L_0^0 & L_0^i & L_0^j \\
L_i^0 & L_i^j & L_j^j \\
L_j^0 & L_j^i & L_j^j
\end{pmatrix}
\]  (4.1)

and look for the algebraic structure of the sub-block entries. Obviously the entries \( L_B^C \) are given by particular polynomial functions \( f(b, c) \) of the 16+16 Darboux coordinates \( b^\beta \) and \( c_\gamma \).

In writing (4.1), we have decomposed the indices \( B \) and \( C \) in terms of three labels like \( B = (0, j, \beta) \) and \( C = (0, i, \gamma) \) corresponding to the decomposition \( 27 = 1_{-4/3} + 10_{+2/3} + 16_{-1/3} \) of the fundamental representation of \( E_6 \). As such, the operator \( \mathcal{L}_{E_6} \) has, generally speaking, 729 components \( L_B^C \) that can be decomposed as \( 1 + 78 + 650 \). In terms of tensor products of \( SO(10) \) representations, we also have

\[
27 \times \overline{27} = 1 \times \overline{1} \oplus 1 \times \overline{10} \oplus 1 \times \overline{16} \oplus 16 \times \overline{1} \oplus 16 \times \overline{10} \oplus 16 \times \overline{16}
\]  (4.2)

Notice that products like \( 10 \times \overline{10}, 10 \times \overline{16} \) and so on can be also reduced to sums of irreducible representation of \( SO(10) \).

Thinking of the diagonal sub-blocks \( L_0^0 = \mathcal{N}_{1 \times \mathbf{T}}, L_j^k = \mathcal{N}_{10 \times \overline{10}} \) and \( L_j^j = \mathcal{N}_{16 \times \overline{16}} \) in eq(4.1) as \( R \otimes \overline{R} \) building blocks of \( SO(2) \times SO(10) \), we can represent the operator \( \mathcal{L}_{E_6} \) in terms of the topological quiver \( Q_{E_6} \) of the Figure 4. It has three nodes \( N_i \) and 3+3 oriented links \( L_{ij} \) as depicted by the figure. The topological states propagating in the quiver \( Q_{E_6} \) are \textit{massless} as required by the \( E_6 \) gauge symmetry of the topological 4D CS theory which has \( SO(2) \times SO(10) \) as a subsymmetry. Notice as well that all states in the topological quiver of the Figure 4 are in bi-representations \( R_i \otimes \overline{R_j} \) of \( SO(2) \times SO(10) \). The three nodes \( N_i \) are in self-dual representations in the sense that \( R_i \) and its transpose \( \overline{R_j} \) have the same dimension and are related by representation-duality. So, the nodes can be imagined as describing self-dual topological matter of \( SO(2) \times SO(10) \). However, the links \( L_{ij} \), though also massless, are in oriented bi-representations \( R_i \otimes \overline{R_j} \) with \( R_i \) and \( \overline{R_j} \) having different dimensions. In the Figure 4, the 6 oriented links \( L_{ij} \) and \( L_{ji} \) between the three nodes are as follows: (a) the topological bi-matter \( B_{1 \times \mathbf{T}} \) and \( B_{10 \times \mathbf{T}} \) corresponding to \( R_1 \) and \( R_2 \) are given by the \( SO(2) \times SO(10) \) representations \( 1_{-4/3} \) and \( 10_{+2/3} \). (b) the topological bi-matter \( B_{1 \times \overline{10}} \) and \( B_{16 \times \mathbf{T}} \) with \( R_1 \) and \( R_2 \) are given by the \( SO(10) \) representations \( 1_{-4/3} \) and \( 16_{-1/3} \). (c) the bi-matter \( B_{10 \times \overline{16}} \) and \( B_{16 \times \overline{10}} \) with \( R_1 \) and \( R_2 \) are given by the representation \( 10_{+2/3} \) and \( 16_{-1/3} \).

From (4.1), we also learn the two following features descending from the decomposition (4.2): (i) the bi-matters \( B_{1 \times \mathbf{T}} \) and \( B_{16 \times \mathbf{T}} \) between the nodes \( \mathcal{N}_{1 \times \mathbf{T}} \) and \( \mathcal{N}_{16 \times \mathbf{T}} \) can be interpreted as fundamental (spinorial like) topological matter. They are expressed in terms of Darboux coordinates \( c_\gamma \) and \( b^\beta \) that satisfy the non trivial Poisson bracket

\[
\left\{ z^{-\frac{4}{3}} b^\beta, z^{-\frac{4}{3}} c_\gamma \right\} = z^{-\frac{4}{3}} \delta_\gamma^\beta
\]  (4.3)

(ii) the bi-matters \( B_{1 \times \overline{10}} \) and \( B_{10 \times \mathbf{T}} \) linking the nodes \( \mathcal{N}_{1 \times \mathbf{T}} \) and \( \mathcal{N}_{10 \times \overline{10}} \) can also be interpreted as fundamental (but vector like) topological matter. They are given by two \( SO(10) \)
vectors, namely $V^i = \frac{1}{2} b^\alpha (\Gamma^i)^{\alpha\beta} b^\beta$ and $W_i = \frac{1}{2} c^\alpha (\Gamma_i)^{\alpha\beta} c_\beta$. These vectors are quadratic in the Darboux coordinates and obey the non trivial Poisson bracket

$$\{z^{-\frac{8}{3}} V^i, z^{-\frac{8}{3}} W_j\} = \frac{1}{2} \delta^i_j z^{-\frac{8}{3}} T + z^{-\frac{8}{3}} Z^i_j$$

(4.4)

where $T$ and $Z^i_j$ are quadratic in the Darboux coordinates, they are given by the SO(2) × SO(10) scalar $T = b^\alpha c_\alpha$ and the operator $Z^i_j = b^\beta \Omega^{i\alpha}_{j\beta} c_\alpha$ with

$$\Omega^{i\alpha}_{j\beta} = \frac{1}{2} (\tilde{\Gamma}^i \Gamma_j + \Gamma_j \tilde{\Gamma}^i)_\beta$$

(4.5)

and $\tilde{\Gamma}^i$ referring to the transpose of the Gamma matrix $\Gamma^i$. The underlying properties of the Poisson brackets (4.3-4.4) and other aspects will be reported in future occasion.

## 5 Minuscule CGY operator $\mathcal{L}_{E_7}$

In this section, we construct the ’t Hooft line operator $\mathcal{L}_{E_7}$ of the 4D CS theory with $E_7$ gauge symmetry and underlying Lie algebra $e_7$. We also give the topological gauge quiver $Q_{E_7}$ associated with the $\mathcal{L}_{E_7}$ by following the approach used above for the topological E_6 theory.

### 5.1 Levi subalgebra and weights of the 56_{e_7}

We begin by giving some useful ingredients concerning the exceptional Lie algebra $e_7$. First, the root system $\Phi_{e_7}$ of this Lie algebra contains 126 roots; half of them are positive and the others are negative. This system is a subset of the $\Phi_{e_8}$ system of the exceptional Lie algebra $e_8$ containing 240 roots; 126 of them sit in its subset $\Phi_{e_7}$ and the rest in its complement $\Phi_{e_8} \setminus \Phi_{e_7}$. This feature implies that $\Phi_{e_7}$ can be derived from $\Phi_{e_8}$ which is built by using an eight dimensional vector basis $\{\epsilon_1, ..., \epsilon_8\}$ of $\mathbb{R}^8$ with $\epsilon_i \epsilon_j = \delta_{ij}$. Within this view, the seven simple roots can be taken as follows: the first simple root is equivalent to the one in (3.1)

$$E_7 : \alpha_1 = \frac{1}{2} (\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 - \epsilon_5 - \epsilon_6 - \epsilon_7 + \epsilon_8)$$

(5.1)

with five roots defined as $\alpha_i = \epsilon_i - \epsilon_{i-1}$ for $2 \leq i \leq 6$ and the seventh one as $\alpha_7 = \epsilon_1 + \epsilon_2$. The Dynkin diagram associated with these roots is given by the Figure 5, from which we can determine the Cartan matrix $K_{e_7}$. The 72 roots of $e_6$ split into 36 positive roots and 36 negative ones. The 36 positive roots split in turn as 20+16. The 20 positive roots are given by $\epsilon_i \pm \epsilon_j$ with $1 \leq j < i \leq 5$ and the 16 ones read as $\frac{1}{2} q_i \epsilon_i + \epsilon_6 - \epsilon_7 + \epsilon_8$ with the five $q_i = \pm 1$ constrained as $\Pi q_i = -1$.

As far as the root system of $e_7$ is concerned, notice that contrary to the six simple roots $\alpha_2, ..., \alpha_7$ constructed out of the six $\epsilon_1, ..., \epsilon_6$, the simple $\alpha_1$ has a spinorial-like nature, it moreover depends on two extra dimensions generated by $\epsilon_7$ and $\epsilon_8$. This extra dependence is carried by the particular combination $\beta_{\text{max}} = \epsilon_8 - \epsilon_7$ that is expressed like

$$\beta_{\text{max}} = 2 \alpha_1 + 3 \alpha_2 + 4 \alpha_3 + 3 \alpha_4 + 2 \alpha_5 + \alpha_6 + 2 \alpha_7$$

(5.2)
with length $\beta_{\text{max}}^2 = 2$. This is the highest positive root, and it plays an important role in the breaking $E_6 \rightarrow SO(2) \times SO(10)$. Regarding the Levi-decomposition $e_7 = n_- \oplus l_\mu \oplus n_+$, recall that the exceptional $e_7$ has one minuscule coweight $\mu$ given by $\lambda_6 = \epsilon_6 + \beta_{\text{max}}/2$ obeying $\lambda_6, \alpha_i = \delta_i$. It corresponds to the fundamental representation $56$ of $e_7$ which is self dual and pseudo-real [28]. Recall also that the exceptional $e_7$ has one minuscule coweight $\mu$ given by $l_\mu = so(2) \oplus e_6$ and the nilpotents are $n_+ = 27_+$ and $n_- = 27_-$. So, the Levi decomposition of the Lie algebra $e_7$ dispatches its 133 dimensions in terms of representations of $e_6$ like $1 + 78 + 27_+ + 27_-$. Similarly, the fundamental $56$ representation of $e_7$, characterising the 't Hooft line operator of the topological $E_7$ theory, decomposes with respect to $so(2) \oplus e_6$ as follows [29]

$$56 = 28_{+q} \oplus 28_{-q} \quad , \quad 28_{\pm q} = 1_{\pm q} \oplus 27_{\pm q} \quad (5.3)$$

This representation of $E_7$ is made of four $E_6$- representations: two singlets $1_{\pm 3/2}$ with $so(2)$ charges $\pm 3/2$; and two fundamentals $27_{\pm 1/2}$ with $so(2)$ charges $\pm 1/2$. To deal with the 56 states $\{|\omega_i\rangle\}_{0 \leq i \leq 55}$ of this fundamental representation, we use the diagram of the Figure 6 to label the 28 weights of $28_{+q}$ by the subset $W_+ = \{|\omega_i\rangle\}_{0 \leq i \leq 27}$; and the 28 weights of the $28_{-q}$ by $W_- = \{|\omega_i\rangle\}_{28 \leq i \leq 55}$. Generic weights $\omega_i$ in $W_+ \cup W_-$ obey some special features

Figure 6: The decomposition of the $56$ representation of $E_7$ in terms of representations of $E_6$. We have $56 = 28_{+q} \oplus 28_{-q}$ with $28_{\pm q}$ reducible like $1_{\pm 3/2} \oplus 27_{\pm 1/2}$.

that are useful for the construction of the operator $\mathcal{L}_{E_7}$, we give some of them here. First, we have $\omega_{27} = \omega_0 - \beta_{\text{max}}$ and $\omega_{28} = \omega_{55} + \beta_{\text{max}}$, from which we learn the interesting relation $\omega_{27} + \omega_{28} = \omega_0 + \omega_{55}$ relating "boundary" weights in $W_\pm$. This feature is in fact a general property of weights $\omega_i$ in $56$, it extends like $\omega_i + \omega_{55-i} = \omega_0 + \omega_{55}$ for the label $i$ ranging the full interval $0 \leq i \leq 27$. We also have the relations $\omega_i = \omega_0 - \gamma_i$ and $\omega_{55-i} = \omega_{55} + \gamma_i$ for generic roots $\gamma_i$ in the nilpotent $27_+$. In these regards, recall that the highest weight state of the representation $56$ is given by $|\omega_0\rangle = |\lambda_6\rangle$ and its lowest weight state is given by $|\omega_{55}\rangle = |0\rangle$. As such, we have the following additional properties: (a) The lowest and the highest boundary weights obey $\omega_{55} + \omega_0 = 0$ and $\omega_0 - \omega_{55} = 2\lambda_6$. (b) For generic weights,
we have the relations $\omega_{55-i} = \omega_{55} + \gamma_i$ and so $\omega_i - \omega_{55-i} = 2\lambda_6 - 2\gamma_i$ for $0 \leq i \leq 27$ with $\gamma_0 \equiv 0$. For convenience, we use the following simple notations:

1. the weights $|\omega_l\rangle$ in $W_+$ are denoted like $|l_+\rangle$ with $0_+ \leq l_+ \leq 27_+$.
2. the weights $|\omega_l\rangle$ in $W_-$ with $28 \leq l \leq 55$ are denoted like $|l_-\rangle$ with $27_- \leq l_- \leq 0_-$ where $|\omega_{55}\rangle = |0_-\rangle$.
3. For a positive root $\beta_s$ belonging to $\Phi_{e_7} \setminus \Phi_{e_6}$, the states $X_{\beta_s} |\omega_0\rangle = |\omega_0 - \beta_s\rangle$ and $Y^{\beta_s} |\omega_{55}\rangle = |\omega_{55} + \beta_s\rangle$ are denoted like $X_s |0\rangle = |s_+\rangle$ and $Y^s |0_-\rangle = |s_-\rangle$ with integer $0 \leq s \leq 27$.

### 5.2 Constructing the $L_{E_7}$-operator and its $Q_{E_7}$

The construction of the operator $L_{E_7}$ follows the same method we have used for the derivation of $L_{E_6}$. However, contrary to the $E_6$ topological theory, the $L_{E_7}$ of the 4D Chern-Simons theory with $E_7$ gauge symmetry has somehow specific features. This is because of the self-duality and the pseudo-reality of the fundamental 56 representation of $E_7$ encoded in the splitting $(28_+, 28_-)$. These particular properties lead to a remarkable expression of the operator $L_{E_7}$. Below, we describe the main lines for the derivation of the explicit value of $L_{E_7}$ and the associated topological gauge quiver $Q_{E_7}$. Further details are reported in the appendix. To fix the ideas, we think it is interesting to anticipate the study of $L_{E_7}$ by giving first the structure of the obtained quiver $Q_{E_7}$ and turn later to comment it. It

![Figure 7: The topological quiver $Q_{E_7}$ representing $L_{E_7}$. It has 4 nodes and 12 links. The nodes describe self-dual topological matter uncharged under $SO(2)$. The links describe bi-matter in $(\mathbb{R}_i, \bar{\mathbb{R}}_j)$ of $E_6$ charged under $SO(2)$ with charges $\pm 1, \pm 2, \pm 3$.](image)

has four nodes describing and 6+6 oriented links as depicted by the Figure [7]. It has an outer-automorphism symmetry exchanging the nodes $1_{3/2}$ and $1_{-3/2}$ as well as the $27_{1/2}$ and $27_{-1/2}$. Recall that the $L_{E_7}$ we want to build is given by $e^{X_2^\mu} e^Y$ with diagonal $\mu$ sitting in the Levi- subalgebra $so(2) \oplus e_6$. The $X = b^\beta X_\beta$ is valued in the nilpotent $27_+$ and the $Y = c_\beta Y^\beta$ in the $27_-$ with positive root $\beta$ belonging to the root subsystem $\Phi_{e_7} \setminus \Phi_{e_6}$. The generators obey the commutation relations $[\mu, X_\beta] = X_\beta$ and $[\mu, Y^\beta] = -Y^\beta$ from which we deduce

$$[\mu, X] = X, \quad [\mu, Y] = -Y$$

(5.4)
So, the $\mu$ is a diagonal charge operator; it can be decomposed like $\mu = \mu_{1+} + \mu_{27+} + \mu_{27-} + \mu_{1-}$ with $\mu_{1+} = \mu_{1+}^0 \omega_0$, where $\omega_0 = |\omega_0\rangle \langle \omega_0|$ and $\mu_{1+}^0 = 3q/2$. Similar expressions can be written down for $\mu_{27\pm}$ where we have $\mu_{27+} = \mu_{27+}^0 \omega_{aq}$ such that $\omega_{aq} = |\omega_{aq}\rangle \langle \omega_{aq}|$ and $\mu_{27+}^0 = q/2$ with $q = \pm 1$. Formally, we have [30]

$$\mu = \frac{3}{2} \omega_{1+} + \frac{1}{2} \omega_{27+} - \frac{1}{2} \omega_{27-} - \frac{3}{2} \omega_{1-}$$

where the charges $\pm 3/2$ and $\pm 1/2$ are associated to the four nodes of the quiver $Q_E$. For a given positive root $\beta$ in $E_6, \Phi_\delta$, the generators $X_\beta$ and $Y_\beta$ solving (5.4) are as follows

$$X_\beta = \langle \omega_{0+} \rangle \langle \omega_{\beta+} \rangle + \langle \omega_{\beta+} \rangle \langle \omega_{\gamma-} \rangle + \langle \omega_{\gamma-} \rangle \langle \omega_{0-} \rangle$$

$$Y_\beta = \langle \omega_{0-} \rangle \langle \omega_{\alpha+} \rangle + \langle \omega_{\alpha+} \rangle \langle \omega_{\gamma-} \rangle + \langle \omega_{\gamma-} \rangle \langle \omega_{0+} \rangle$$

where $\Gamma^\beta_{\gamma-}$ and $\Gamma^\beta_{\gamma+}$ are coupling tensors of three $27$ representations of $E_6$. They are respectively given by $\langle \omega_{\beta+} \rangle |X_\beta| \langle \omega_{\gamma-} \rangle$ and $\langle \omega_{\gamma-} \rangle |Y_\beta| \langle \omega_{\beta+} \rangle$. By using these expressions, we determine the corresponding Cartan charge operator $\frac{1}{2} H_\beta = [X_\beta, Y_\beta]$ from which we deduce the relation $\Gamma^\beta_{\beta+} - \Gamma^\beta_{\beta-} = \frac{4}{3} \delta^\beta_{\beta+} \delta^\beta_{\beta-}$. Multiplying (5.6) respectively by Darboux coordinates $b_\beta$ and $c_\beta$, we deduce the expressions of the matrix operators $X = b_\beta X_\beta$ and $Y = c_\beta Y_\beta$ needed for the calculation of $e^X z^{\mu e^Y}$. Using these expressions, we compute the powers $X^n$ and $Y^n$; we find that $X^4 = Y^4 = 0$ and (see Appendix for more details),

$$X^2 = 2 S^\beta- \langle \omega_{0+} \rangle \langle \omega_{\beta+} \rangle + 2 S^\beta_+ \langle \omega_{\beta+} \rangle \langle \omega_{0-} \rangle, \quad X^3 = 6 E \langle \omega_{0+} \rangle \langle \omega_{0-} \rangle$$

$$Y^2 = 2 R_{\alpha+} \langle \omega_{0-} \rangle \langle \omega_{\alpha+} \rangle + 2 R_{\alpha+} \langle \omega_{\alpha+} \rangle \langle \omega_{0-} \rangle, \quad Y^3 = 6 F \langle \omega_{0-} \rangle \langle \omega_{0+} \rangle$$

where for convenience we have set

$$S^\beta- = \frac{1}{2} b_\beta \Gamma^\beta_{\beta+} b_\beta, \quad R_{\alpha-} = \frac{1}{2} c_\alpha \Gamma^\alpha_{\alpha+} c_\alpha, \quad E = \frac{1}{3} b_\beta S^\beta+, \quad F = \frac{1}{3} R_{\alpha-} R_{\alpha+}$$

Notice as well, the following features regarding the quantities (5.8) which are involved in $e^X$ and $z^{\mu e^Y}$ and which will appear in $\mathcal{L}_E$, and its quiver $Q_E$: (i) The $S^\pm$’s are quadratic in $b_\beta$ and carry a charge $-2$ under SO(2). This is because the $b_\beta$’s carry a charge $-1$ and the $c_\alpha$’s carry a charge $1$. (ii) The $R_{\alpha-}$’s are quadratic in $c_\alpha$ and carry a charge $+2$ under SO(2). (iii) The $E$ and the $F$ are respectively cubic in $b_\beta$ and $c_\alpha$; they carry charges $-3$ and $+3$ under SO(2). In this regard, observe that $E$ and $F$ are scalars of $e_6$ and can be put in the form $E = \frac{1}{6} \Gamma_\alpha \beta_\gamma b_\alpha b_\beta b_\gamma$ and $F = \frac{1}{6} \Gamma_\alpha \beta_\gamma c_\alpha c_\beta c_\gamma$ as shown by eqs (7.12, 7.13). Using these relations and setting $T^\alpha_{\beta+} = b_\alpha \Gamma^\beta_{\gamma+}$ and $J^\alpha_{\beta+} = c_\alpha \Gamma^\beta_{\gamma+}$, we can determine the explicit expressions of the exponentials $e^X$ and $z^{\mu e^Y}$ in the vector basis of the 56 weights ordered like $\{ |\omega_{0+}\rangle, |\omega_{\beta+}\rangle, |\omega_{\beta-}\rangle, |\omega_{0-}\rangle \}$. We find that $e^X$ and $z^{\mu e^Y}$ are respectively equal to

$$e^X = \begin{pmatrix} 1 & b_{\beta+}^\alpha & S_{\beta-}^\alpha & E \\ 0 & \delta^\beta_{\alpha+} & T_{\alpha+}^\beta & S_{\alpha+} \\ 0 & 0 & \delta_{\alpha-}^\beta & b_{\alpha-} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad z^{\mu e^Y} = \begin{pmatrix} z^{\frac{3}{2}} & 0 & 0 & 0 \\ z^{\frac{3}{2}} c_{\alpha+} & z^{\frac{3}{2}} \delta_{\alpha+}^\beta & 0 & 0 \\ z^{-\frac{3}{2}} c_{\alpha-} & z^{-\frac{3}{2}} J_{\alpha+}^\beta & z^{-\frac{3}{2}} \delta_{\alpha-}^\beta & 0 \\ z^{-\frac{3}{2}} F & z^{-\frac{3}{2}} R_{\beta+} & z^{-\frac{3}{2}} c_{\beta-} & z^{-\frac{3}{2}} \end{pmatrix}$$
Putting these relations into $e^{X z^\mu e^Y}$, we obtain the explicit expression of the 't Hooft line operator $\mathcal{L}_{E_7}$ that we present as follows

$$
\mathcal{L}_{E_7} = \begin{pmatrix}
L_{0_+}^{0} & L_{0_+}^{\beta_+} & L_{0_+}^{\beta_-} & z^{-\frac{3}{2}} S \\
L_{0_+}^{0} & L_{0_+}^{\beta_+} & L_{0_+}^{\beta_-} & z^{-\frac{1}{2}} S \\
L_{0_+}^{0} & L_{0_+}^{\beta_+} & L_{0_+}^{\beta_-} & z^{-\frac{1}{2}} b \\
z^{-\frac{3}{2}} F & z^{-\frac{1}{2}} R & z^{-\frac{1}{2}} c & z^{-\frac{1}{2}}
\end{pmatrix}
$$

(5.10)

The four diagonal sub-blocks $L_{0_+}^{0}$, $L_{0_+}^{\beta_+}$, $L_{0_+}^{\beta_-}$ and $L_{0_+}^{0}$ are given by

$$
L_{0_+}^{\beta_+} = z^\frac{3}{2} + z^\frac{1}{2} b + z^{-\frac{3}{2}} S \alpha - R + z^{-\frac{1}{2}} F, \\
L_{0_+}^{\beta_-} = z^\frac{1}{2} \delta + z^{-\frac{1}{2}} S \alpha + R + z^{-\frac{1}{2}} F, \\
L_{0_+}^{0} = z^\frac{1}{2} + z^{-\frac{1}{2}} b + z^{-\frac{1}{2}} c
$$

(5.11)

and the off diagonal terms by

$$
L_{0_+}^{\beta_+} = z^\frac{1}{2} b + z^{-\frac{3}{2}} S \alpha - J + z^{-\frac{1}{2}} R \beta, \\
L_{0_+}^{\beta_-} = z^{-\frac{1}{2}} S \alpha - R + z^{-\frac{3}{2}} F, \\
L_{0_+}^{0} = z^\frac{1}{2} c + z^{-\frac{3}{2}} T \gamma + z^{-\frac{1}{2}} F, \\
L_{0_+}^{\beta_-} = z^{-\frac{1}{2}} \beta + z^{-\frac{3}{2}} b - R
$$

(5.12)

For more details concerning these calculations, see the appendix. We end this description by noticing that the topological gauge quiver $Q_{E_7}$ representing the 't Hooft line operator (5.10) is given by the Figure 7. The construction of this topological quiver is obtained by following the same method we have used for the building of $Q_{E_6}$ of the Figure 4. For the case of $Q_{E_7}$, the Darboux coordinates $(b^\alpha, c_\alpha)$ also have an interpretation in terms of fundamental matter carrying a unit charge under SO(2). We also find that $S_\alpha$ and $R_\alpha$ describe fundamental matter; but with SO(2) charges $q = \pm 2$; see the blue links in the quiver $Q_{E_7}$ given by the Figure 7.

6 Conclusion

Two-dimensional integrable field theories and integrable spin models represent a significant area in classical and quantum physics including 2D critical phenomena. They still bear several open questions intending to explicitly describe the interactions between fundamental particles and topological lines. In this paper, we have contributed in this matter by considering topological 4D CS theory in presence of exceptional minuscule 't Hooft lines. This particular topological theory has gauge symmetries given by the $E_6$ and $E_7$ groups with 't Hooft lines described by the complex representation 27 of $E_6$ and the self-dual 56 representation of $E_7$. To undertake this study, we first revisited useful aspects on the $\mathcal{L}_G$-operators in topological 4D CS theory with gauge symmetry G by following the approach of [12, 11, 2]. Then, we focused on the $E_6$ and $E_7$ theories and derived the explicit oscillator realisation of the corresponding $\mathcal{L}_{E_6}$- and $\mathcal{L}_{E_7}$- operators where specific properties for exceptional groups have been found. The order $\eta$ of nilpotency of the $X^\eta$ and $Y^\eta$ matrices in
is equal to 3 for $E_6$ versus $\eta = 2$ for $A$-theory. It is equal to 4 for the case of $E_7$-theory. The oscillator realisation of the $L$-operator of the $E_6$ gauge symmetry is given by eq(3.10), and the representation of the $L_{E_6}$ is given by eqs(5.10-5.12). We also proposed a graph to represent the operators $L_{E_6}$ and $L_{E_7}$ using topological gauge quivers $Q_{E_6}$ and $Q_{E_7}$ given by the Figures 4 and 7. In this diagrammatic representation, the Darboux coordinates have been interpreted as topological fundamental matter and the nodes as topological self-dual matter. General aspects of the topological gauge quivers $Q_G$ in 4D- CS theory with generic gauge symmetry $G$ and the underlying algebra of its nodes and links will be reported in a future occasion. It will be also interesting to derive the Lax operators for the B- and C-types spin chains obtained in [31] from 4D CS theory. An explicit investigation regarding the derivation of Lax operators of A-, D- and E-types from 4D Chern-Simons theory using topological quivers is given in [32].

7 Appendix

In this appendix, we give details regarding the calculation of the Lax operator (5.10-5.12) derived from the 4D Chern-Simons theory with gauge symmetry $E_7$. This operator $L_{E_7}$ is given by the formula $e^X z^\mu e^Y$ with $X$ and $Y$ valued in the nilpotent subalgebras $n_+ = 27_+$ and $n_- = 27_-$ of the Levi-decomposition of the $e_7$ Lie algebra underlying the $E_7$ theory. Recall that in this decomposition, the Levi-subalgebra reads as $l_\mu = so(2) \oplus e_6$ and the associated nilpotents are given by $n_\pm = 27_\pm$. To determine the explicit expression of $L_{E_7}$, we have to calculate $e^X z^\mu e^Y$. This is done in three steps: (i) We have to work out the expression of the adjoint action of the minuscule coweight $\mu$, it is given by (5.5) which is equal to $\frac{3}{2}e_{1_+} + \frac{1}{2}e_{27_+} - \frac{1}{2}e_{27_-} - \frac{3}{2}e_{1_-}$. The subscripts $R$ appearing in the projectors $\varrho_R$ refer to the decomposition (5.3) of the fundamental representation of $e_7$ with respect to the representations of $so(2) \oplus e_6$, namely

$$56 = 1_{3/2} \oplus 27_{+1/2} \oplus 27_{-1/2} \oplus 1_{-3/2}$$

(ii) We have to determine the expression of $X$ and $Y$ solving the Levi-condition (5.4) and then calculate the exponentials $e^X$ and $e^Y$ using the expansion $e^Z = \sum Z^n/n!$. (iii) Once $\mu$, $e^X$ and $e^Y$ are known, we substitute into $e^X z^\mu e^Y$ and look for the explicit expression of $L_{E_7}$. Because of the special properties of the $E_7$ symmetry and its fundamental representation 56, these calculations are somehow technical in the sense that we need to exhibit features of the root system $\Phi_{e_7}$ of $e_7$ and the weight vectors of its fundamental 56. Those useful tools for the calculation of $L_{E_7}$ were reported in the main text; see subsection 5.1. Nevertheless we think it interesting to give extra details lightening the explicit computations like aspects concerning the Levi-decomposition and related things for their role in the determination of $L_{E_7}$. The 133 generators of $e_7$, its seven diagonal charge operators (rank) and its 126 roots...
split under Levi-decomposition as collected in the following table

| algebra | $e_7$ | $so_2$ | $e_6$ | $n_+$ | $n_-$ |
|---------|---------|---------|---------|--------|--------|
| dim     | 133     | 1       | 78      | 27     | 27     |
| rank    | 7       | 1       | 6       | 0      | 0      |
| roots   | 126     | 0       | 72      | 27     | 27     |
| Cartan H's | 7 $H_1$ | 1 $H = \mu$ | 6 $H_1$ | 0 | 0 |
| Step E's | 126 $E_{\pm \alpha}$ | 0 | 72 $E_{\pm \alpha}$ | 27 $X_{\gamma} + \beta$ | 27 $Y_{-\gamma}$ |

(7.2)

In this table, the n $H_1$ refers to the number n of Cartan charge operators of the corresponding Lie algebra. Here the splitting of the rank is given by $7 = 1 + 6$. The n $E_{\pm \alpha}$ represents the number n of step operators associated with the roots $\pm \alpha$ of the Lie algebra. The splitting of the total roots is given by $126 = 72 + (27 + 27)$. Regarding the technical details, notice the two following: (1) We have denoted the 27+27 generators of the nilpotent algebras $n_+$ and $n_-$ respectively by $X_\beta$ and $Y_\beta$ instead of the conventional $E^{+\beta}$ and $E^{-\beta}$. For these generators, the roots $\beta$ belong to the subset of positive roots $\Phi_{e_7}^+ \setminus \Phi_{e_6}^+$ introduced in the main text; that is $\beta \in \Phi_{e_7}^+$ but $\beta \notin \Phi_{e_6}^+$; see subsection 5.1. This discrimination in the notation is (a) because we have used it in the main text as given by the expansions $X = b^\beta X_\beta$, $Y = c^\beta Y_\beta$ where $\beta$ is a positive root; and (b) in order to give an interpretation of the $\Gamma_{\gamma^{+\delta^-}}$ used in our calculations in terms of a 3-coupling of three $e_6$ representations namely $27 \times 27 \times 27$. (2) The entries of the $X_\beta$ and $Y_\beta$ matrices namely $\langle \omega_{\gamma^+} | X_\beta | \omega_{\delta^-} \rangle$ and $\langle \omega_{\delta^-} | Y_\beta | \omega_{\gamma^+} \rangle$, with kets $|\omega_{\gamma^+}\rangle$ and bras $\langle \omega_{\delta^-}\rangle$ being 27 weight vector states in (7.1) and their 27 duals, have been denoted in the text as $\Gamma_{\gamma^+ \delta^-}$ and $\Gamma_{\delta^- \gamma^+}$. Here also we have accommodated the labels of $\Gamma_{\gamma^+ \delta^-}$ and $\Gamma_{\delta^- \gamma^+}$ in order to make the result more accessible for the reader. The point is that the matrix elements $\langle \omega_{\gamma^+} | X_\beta | \omega_{\delta^-} \rangle$ and $\langle \omega_{\delta^-} | Y_\beta | \omega_{\gamma^+} \rangle$ should respectively be written as $(X_\beta)_{\gamma^+}$ and $(Y_\beta)_{\delta^-}$; for convenience, we have used $\Gamma_{\gamma^+ \delta^-}$ for $(X_\beta)_{\gamma^+}$ and $\Gamma_{\delta^- \gamma^+}$ for $(Y_\beta)_{\delta^-}$. This technical detail is not very important as the calculations are covariant. Notice that the label $\beta$ takes integer values from 1 to 27 as shown on the two last columns of the table (7.2). Notice moreover that the two other labels $\gamma^+$ and $\delta^-$ of the weights $\omega_{\gamma^+}$ and $\omega_{\delta^-}$ take also integer values from 1 to 27 exactly like $\beta$. This is not a coincidence, the point is that the set $\{\lambda\}_{e_7}$ of weights of the adjoint representation $\text{adj} e_7$ are precisely given by the set $\Phi_{e_7} = \{\alpha\}_{e_7}$ of roots of $e_7$; that is $\{\lambda\}_{e_7} = \{\alpha\}_{e_7}$. Under the Levi-decomposition, we have,

$$\{\lambda\}_{\text{adj} e_7} = \{\lambda\}_{\text{adj} so_2} \cup \{\lambda\}_{\text{adj} e_6} \cup \{\lambda\}_{27} \cup \{\lambda\}_{27}$$

(7.3)

with the subset $\{\lambda\}_{27} = \Phi_{e_7}^+ \setminus \Phi_{e_6}^+$ and the subset $\{\lambda\}_{27} = \Phi_{e_7}^- \setminus \Phi_{e_6}^-$. In other words, the 27 weights $\gamma^+ = +\gamma$ belong to $\Phi_{e_7}^+ \setminus \Phi_{e_6}^+$ and the 27 weights $\delta^- = -\delta$ sit in $\Phi_{e_7}^- \setminus \Phi_{e_6}^-$. This feature teaches us that $\gamma$ and $\delta$ belong also to $\Phi_{e_7}^+ \setminus \Phi_{e_6}^+$ exactly as $\beta$.

After this general description concerning technicalities, we now turn to the explicit computations of $L_{E_7}$ by using the fundamental coweight $\mu$ given above. We start from the expansions $X = b^\beta X_\beta$ and $Y = c^\beta Y_\beta$ where $X_\beta$ and $Y_\beta$ are the generators of the nilpotent.
algebras in the Levi decomposition \( e_7 \rightarrow so_2 \oplus e_6 \oplus 27_+ \oplus 27_- \). These generators solving the Levi-conditions are realised in the weight basis \( \{ |\omega_{0+}>, |\omega_{\delta+}>, |\omega_{\beta-}>, |\omega_{0-} > \} \) of the representation 56 of \( e_7 \) as follows

\[
X_\beta = |\omega_{0+} > \langle \omega_{\beta+} | + |\omega_{\delta+} > \Gamma_\beta^{\delta+\gamma-} \langle \omega_{\gamma-} | + |\omega_{\beta-} > \langle \omega_{0-} |
\]
\[
Y_\beta = |\omega_{0-} > \langle \omega_{\beta-} | + |\omega_{\gamma-} > \Gamma_{\gamma-\delta+}^\beta \langle \omega_{\delta+} | + |\omega_{\beta+} > \langle \omega_{0+} | \quad \text{(7.4)}
\]

Here \( |\omega_{\pm} > \) refer to the singlets \( 1_{\pm 3/2} \) in the decomposition (7.1) and \( |\omega_{\delta+} > \) to the \( 27_{\pm 1/2} \). The full list of the 56 weight vectors \( |\omega_{0+} >, |\omega_{\delta+} >, |\omega_{\beta-} >, |\omega_{0-} > \) is known in the literature, it has been omitted here for the simplicity of the presentation. However, we refer to [28] for readers interested in this list. As described before, notice that \( \Gamma_\beta^{\delta+\gamma-} \) in (7.4) is given by \( \langle \omega_{\delta+} | X_\beta | \omega_{\gamma-} > \) and a similar expression can be written for \( \Gamma_{\gamma-\delta+}^\beta \). The tri-coupling \( \Gamma_\beta^{\delta+\gamma-} \) and \( \Gamma_{\gamma-\delta+}^\beta \) are tensors with three labels \( (\beta, \gamma, \delta) \) all of them take integer values from 1 to 27. They have an interpretation in terms of coupling three 27 (resp. \( \bar{27} \)) representations of \( e_6 \). In these regards, we recall that the decomposition of the complex tensor product \( 27 \times 27 \times 27 \) contains the identity, and a similar property is valid for \( \bar{27} \times \bar{27} \times \bar{27} \). Focussing on \( 27 \times 27 \times 27 \), this can be seen by first calculating the product \( 27 \times 27 = 729 \); it decomposes like \( 729 = \bar{27} \times 351 \times 351 \). By putting into \( 27 \times 27 \times 27 \), we have \( 27 \times \bar{27} \times \bar{27} \times 7 \times 7 \times 7 \). Here, the hermitian product \( 27 \times \bar{27} \) decomposes like \( 1 + 78 + 650 \) showing that the product of three 27s contains indeed the desired identity.

Using the expressions given above, we can calculate the powers \( X^n \) and \( Y^n \). We give below the calculations of \( X^n \). Similar computations are valid for \( Y^n \).

- Calculations of \( X^2 \) and results for \( Y^2 \): By substituting the expansion \( X = b^\beta X_\beta \) into \( X^2 \), we have \( X^2 = b^\alpha b^\beta X_\alpha X_\beta \) with

\[
X_\alpha X_\beta = |\omega_{0+} > \langle \omega_{\alpha+} | (|\omega_{0+} > \langle \omega_{\beta+} | + |\omega_{\delta+} > \Gamma_\beta^{\delta+\gamma-} \langle \omega_{\gamma-} | + |\omega_{\beta-} > \langle \omega_{0-} |)
\]
\[\begin{align*}
+ |\omega_{\eta+} > \Gamma_\alpha^{\eta+\xi-} \langle \omega_{\xi-} | (|\omega_{\eta+} > \langle \omega_{\beta+} | + |\omega_{\delta+} > \Gamma_\beta^{\delta+\gamma-} \langle \omega_{\gamma-} | + |\omega_{\beta-} > \langle \omega_{0-} |)
\end{align*}
\]
\[\begin{align*}
+ |\omega_{\alpha-} > \langle \omega_{0-} | (|\omega_{\alpha-} > \langle \omega_{\beta+} | + |\omega_{\delta+} > \Gamma_\beta^{\delta+\gamma-} \langle \omega_{\gamma-} | + |\omega_{\beta-} > \langle \omega_{0-} |)
\end{align*}
\] \quad (7.5)

Moreover, using properties of the weight vector states, in particular the orthogonality feature exhibited by (7.1), we can bring the above expression into the following simple form

\[
X_\alpha X_\beta = |\omega_{0+} > \langle \omega_{\alpha+} | |\omega_{\delta+} > \Gamma_\beta^{\delta+\gamma-} \langle \omega_{\gamma-} | + |\omega_{\eta+} > \Gamma_\alpha^{\eta+\xi-} \langle \omega_{\xi-} | |\omega_{\beta-} > \langle \omega_{0-} | \quad \text{(7.6)}
\]

To make the calculation covariant, it is interesting to use the metric of the representation \( 27_{+1/2} \) to set \( \langle \omega_{\alpha+} | \omega_{\delta+} > \Gamma_\beta^{\delta+\gamma-} = \Gamma_{\alpha+\beta}^{\gamma-} \) and also use the metric of the representation \( 27_{-1/2} \) to do the same thing for \( \Gamma_\alpha^{\eta+\xi-} \langle \omega_{\xi-} | \omega_{\beta-} > = \Gamma_{\alpha\beta-}^{\eta+} \). In doing so, the above \( X_\alpha X_\beta \) is further reduced as follows \( X_\alpha X_\beta = |\omega_{0+} > \Gamma_{\alpha+\beta}^{\gamma-} \langle \omega_{\gamma-} | + |\omega_{\eta+} > \Gamma_{\alpha\beta-}^{\eta+} \langle \omega_{0-} | \). Substituting the obtained expression of \( X_\alpha X_\beta \) into the expansion of \( X^2 \) namely \( b^\alpha b^\beta X_\alpha X_\beta \), we end up with the following result

\[
X^2 = 2S^{\gamma-} |\omega_{0+} > \langle \omega_{\gamma-} | + 2S^{\eta+} |\omega_{\eta+} > \langle \omega_{0-} | \quad \text{(7.7)}
\]
where for commodity we have set
\[ S^{\gamma -} = \frac{1}{2} b^\alpha \Gamma^{\gamma -}_{\alpha + \beta} b^\beta, \quad S^{\eta +} = \frac{1}{2} b^\alpha \Gamma^{\eta +}_{\alpha \beta -} b^\beta \] (7.8)
which are quadratic into the \( b^\alpha \)'s. The \( S^{\gamma -} \) couples the \( \omega_{0+} \) with \( \omega_{\gamma -} \) whereas the \( S^{\eta +} \) couples the \( \omega_{\eta +} \) and \( \omega_{0-} \). Here an interesting question emerges, it concerns the proof of the equality of \( S^{\gamma -} \) and \( S^{\eta +} \) as suggested by physical and representation theory arguments; see also Figure 6. We will turn to answering positively this question later on. Before that notice that similar analysis can be done for the calculation of \( Y^2 = Y^\alpha Y^\beta c_\alpha c_\beta \) with \( Y^\alpha \) given by (7.4). We find
\[ Y^2 = 2R_{\alpha +} |\omega_{0-}\rangle \langle \omega_{\alpha +}| + 2R_{\eta -} |\omega_{\eta -}\rangle \langle \omega_{0-}| \] (7.9)
where we have set
\[ R_{\alpha -} = \frac{1}{2} c_\gamma \bar{\Gamma}^{\gamma -}_{\alpha + \delta} c_\delta, \quad R_{\alpha +} = \frac{1}{2} c_\gamma \bar{\Gamma}^{\gamma -}_{\alpha + \delta} c_\delta \] (7.10)
These \( R_{\alpha \gamma} \)'s are quadratic in the \( c \)'s. The \( R_{\alpha +} \) couples \( \omega_{0-} \) and the \( \omega_{\alpha +} \) while the \( R_{\alpha -} \) couples \( \omega_{\eta -} \) and \( \omega_{0-} \). Here also the same question asked for the two equalities in (7.8) can be asked for (7.10). Are the \( R_{\alpha -} \) and the \( R_{\alpha +} \) equal? The answer is affirmative; it is demonstrated below by considering the associative property \( X^2.X = X.X^2 \).

- **Calculation of \( X^3 \) and \( Y^3 \):** To perform the calculation of \( X^3 \), we can decompose it either like \( X.X^2 \) or as \( X^2.X \); thanks to associativity. By using \( X.X^2 \) and the above results for \( X \) and \( X^2 \), we can determine the explicit expression of \( X^3 \). First, we have
\[ X^3 = b^\beta \left( |\omega_{0+}\rangle \langle \omega_{\beta +}| \right) (2S^{\alpha -}|\omega_{0+}\rangle \langle \omega_{\alpha -}| + 2S^{\eta +}|\omega_{\eta +}\rangle \langle \omega_{0-}|) + b^\beta \left( |\omega_{\beta -}\rangle \bar{\Gamma}^{\beta -}_{\gamma + \gamma } \langle \omega_{\gamma -}| \right) (2S^{\alpha -}|\omega_{0+}\rangle \langle \omega_{\alpha -}| + 2S^{\eta +}|\omega_{\eta +}\rangle \langle \omega_{0-}|) + b^\beta \left( |\omega_{\eta -}\rangle \langle \omega_{\eta +}| \right) (2S^{\alpha -}|\omega_{0+}\rangle \langle \omega_{\alpha -}| + 2S^{\eta +}|\omega_{\eta +}\rangle \langle \omega_{0-}|) \] (7.11)
Then, using the same properties mentioned previously, we can bring the above relation to a simple form as follows \( X^3 = 2b^\beta g_{\beta \gamma \eta +} S^{\eta +} |\omega_{0+}\rangle \langle \omega_{\eta -}| \) where we have set \( g_{\beta \gamma \eta +} = \langle \omega_{\beta +} | \omega_{\eta +} \rangle \).
By putting \( b_{\eta +} = b^\beta g_{\beta \gamma \eta +} \) or equivalently \( S_{\beta +} = g_{\beta \gamma \eta +} S^{\eta +} \), we can reduce the above relation down to
\[ X^3 = 6\mathcal{E} |\omega_{0+}\rangle \langle \omega_{\eta -}|, \quad \mathcal{E} = \frac{1}{6} \Gamma_{\alpha \beta \gamma} b^\alpha b^\beta b^\gamma \] (7.12)
where we have set \( \mathcal{E} = \frac{1}{6} b_{\eta +} S^{\eta +} \) which can be also presented like \( \mathcal{E} = \frac{1}{6} b^{\eta +} S_{\eta +} \). To get more insight into the algebraic property of the scalar \( \mathcal{E} \), we re-calculate \( X^3 \) by using the factorisation \( X^2.X \) and compare it with the result obtained using \( X.X^2 \). We find \( \mathcal{E} = \frac{1}{6} b_{\eta -} S^{\eta -} \) or equivalently \( \mathcal{E} = \frac{1}{6} b^{\eta -} S_{\eta -} \). Comparing with the expression obtained before namely \( \mathcal{E} = \frac{1}{6} b_{\eta +} S^{\eta +} \), we end up with the equality \( S^{\eta +} = S^{\eta -} \). In eq(7.12), the \( \mathcal{E} \) couples \( \omega_{0+} \) and \( \omega_{\eta -} \) and is cubic into the \( b \)'s. To exhibit manifestly this cubic dependence, we substitute \( S^{\eta +} \) by its value (7.8). We end up with the following expression \( \mathcal{E} = \frac{1}{6} \Gamma_{\alpha \beta \gamma} b^\alpha b^\beta b^\gamma \) which can be interpreted in terms of the trace of the tensor product \( 27 \times 27 \times 27 \). Similar calculations for \( Y^3 \) lead to
\[ Y^3 = 6\mathcal{F} |\omega_{0-}\rangle \langle \omega_{0+}|, \quad \mathcal{F} = \frac{1}{6} \Gamma_{\alpha \beta \gamma} c_\alpha c_\beta c_\gamma \] (7.13)
with metric $g_{\beta\eta} = \langle \omega_{\beta} | \omega_{\eta} \rangle$ and $\mathcal{F} = \frac{1}{4} e^{\eta - R_{\eta}}$ where $R_{\eta}$ is given by (7.10). Here also, we have $R_{\alpha} = R_{\alpha}$. By substituting $R_{\eta}$ by its value, we end up with $\mathcal{F} = \frac{1}{6} \Gamma^{\alpha\beta\gamma} c_\alpha c_\beta c_\gamma$ corresponding to the trace of the tensor product $27 \times 27 \times 27$. From the expressions of (7.12) and (7.13), we learn that $X^4 = Y^4 = 0$ due to the decomposition (7.1). So the expansion of the exponentials $e^X$ and $e^Y$ terminates at the fourth order. This is a property of the $E_7$ theory.

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