Direct Cauchy Theorem and Fourier integral in Widom domains

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In the study of mathematics, there is a grave injustice: we put in so much effort, but we get such miserable results...
Larry Zalcman (from a private conversation)

Abstract

We derive Fourier integral associated to the complex Martin function in the Denjoy domain of Widom type with the Direct Cauchy Theorem (DCT). As an application we study reflectionless Weyl-Titchmarsh functions in such domains, related to them canonical systems and transfer matrices. The DCT property appears to be crucial in many aspects of the underlying theory.

1 Introduction

We develop here some specific aspects of the general de Branges theory [3], which deals with the function theory in infinitely connected domains [7] and spectral properties of random and almost-periodic operators [15].

Let \( E \) be a closed unbounded subset of the positive half axis,

\[
E = \mathbb{R}_+ \setminus \bigcup_{j \geq 1} (a_j, b_j).
\]

We assume that the domain \( \Omega = \mathbb{C} \setminus E \) is regular in the sense of the potential theory [6]. By \( \mathcal{G}(\lambda, \lambda_0) \) we denote the Green function of the domain with singularity at \( \lambda_0 \in \Omega \).

The complex Green function is defined by

\[
\Phi_{\lambda_0}(\lambda) = e^{i\theta_{\lambda_0}(\lambda)}, \quad \theta_{\lambda_0}(\lambda) = - \star \mathcal{G}(\lambda, \lambda_0) + i \mathcal{G}(\lambda, \lambda_0),
\]

where \( \star \mathcal{G}(\lambda, \lambda_0) \) is the harmonically conjugated to \( \mathcal{G}(\lambda, \lambda_0) \) function, \( \star \mathcal{G}(\lambda_*, \lambda_0) = 0 \) for a normalization point \( \lambda_* \in \mathbb{R}_- \). The complex Green function is multivalued in \( \Omega \).

Let \( \pi_1(\Omega) \) be the fundamental group of \( \Omega \). It is generated by simple loops \( \{ \gamma^{(j)} \}_{j \geq 1} \), \( \gamma^{(j)} \) starts and ends at \( \lambda_* \in \mathbb{R}_- \) and goes through the gap \( (a_j, b_j) \). To be extended by continuity along \( \gamma^{(j)} \) the complex Green function obeys the following identity

\[
\Phi_{\lambda_0}(\gamma^{(j)}(\lambda)) = e^{2\pi i \omega(\lambda_0, E_j)} \Phi_{\lambda_0}(\lambda),
\]
where $\omega(\lambda_0, E_j) = \omega(\lambda_0, E_j, \Omega)$ is the harmonic measure of the set $E_j = E \cap [0, a_j]$ computed at $\lambda_0$.

By $\pi_1(\Omega)^*$ we denote the group of characters of the group $\pi_1(\Omega)$, see e.g. 

$$\alpha : \pi_1(\Omega) \to \mathbb{R}/\mathbb{Z}, \quad \alpha(\gamma_1 \gamma_2) = \alpha(\gamma_1) + \alpha(\gamma_2), \quad \gamma_j \in \pi_1(\Omega).$$

We say that $F(\lambda)$ is character automorphic with a certain character $\alpha \in \pi_1(\Omega)^*$ if

$$F(\gamma(\lambda)) = e^{2\pi i \alpha(\gamma)} F(\lambda). \quad (1.1)$$

Note that $|F(\lambda)|$ is single valued in the domain.

For a fixed character $\alpha$ by $H^\infty_{\Omega}(\alpha)$ we denote the collections of bounded analytic multivalued functions $F(\lambda)$ such that (1.1) holds. More generally the Hardy spaces $H^p_{\Omega}(\alpha)$ are formed by functions which obeys (1.1) and $|F(\lambda)|^p$ possesses a harmonic majorant in the domain.

**Theorem 1.1** (Widom). The following two statements are equivalent

- $H^\infty_{\Omega}(\alpha)$ contains a non constant function for all $\alpha \in \pi_1(\Omega)^*$.
- Let $\{c_j\}$ be the collection of critical points of $G(\lambda, \lambda_*)$, i.e., $\nabla G(c_j, \lambda_*) = 0$. Then

$$\sum G(c_j, \lambda_*) < \infty. \quad (1.2)$$

In the Widom domain $\Omega$ the harmonic measure $\omega(\lambda_*, d\xi)$ is absolutely continuous with respect to the Lebesgue measure, moreover

$$\omega(\lambda_*, d\xi) = |\Psi_{\lambda_*}(\xi)| \frac{d\xi}{\sqrt{|\xi|}}, \quad \xi \in E, \quad (1.3)$$

where $\Psi(\lambda) = \Psi_{\lambda_*}(\lambda)$ is an outer character automorphic function, $\Psi(\gamma(\lambda)) = e^{2\pi i \beta_\Psi(\gamma)} \Psi(\lambda)$.

Using this function we can reduce Hardy spaces $H^p_{\Omega}(\alpha)$ to the Smirnov spaces $E^p_{\Omega}(\beta)$.

**Definition 1.2.** We say that $F(\lambda)$ belongs to the Smirnov class $N_+(\Omega)$ if it can be represented as a ratio of two bounded character automorphic functions with an outer denominator [5] Ch. II, Sect. 5]. We say that $F(\lambda) \in N_+(\Omega)$ belongs to the class $E^p_{\Omega}(\alpha)$ if (1.1) holds, and its boundary values ($\xi \pm i0, \xi \in \mathbb{E}$) satisfy

$$\frac{1}{2\pi} \int_\mathbb{E} |F(\xi)|^p \frac{d\xi}{\sqrt{|\xi|}} := \frac{1}{2\pi} \int_\mathbb{E} (|F(\xi + i0)|^p + |F(\xi - i0)|^p) \frac{d\xi}{\sqrt{|\xi|}} < \infty. \quad (1.4)$$

**Proposition 1.3.** $F(\lambda)$ belongs to $E^p_{\Omega}(\alpha)$ if and only if

$$\Psi_{\lambda_*}^{-1/p}(\lambda) F(\lambda) \in H^2_{\Omega}(\alpha - \beta_{\Psi_{\lambda_*}^{1/p}}), \quad \Psi_{\lambda_*}^{1/p}(\gamma(\lambda)) = \exp 2\pi i \beta_{\Psi_{\lambda_*}^{1/p}}(\gamma) \Psi_{\lambda_*}^{1/p}(\lambda).$$

Let $j$ be the character generated by the function $\sqrt{\lambda}$, i.e., $e^{2\pi ij(\gamma(m))} = -1$ for all generators $\gamma(m)$.  

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**Definition 1.4.** We say that the Direct Cauchy Theorem (DCT) holds in Ω if
\[
\frac{1}{2\pi i} \oint_{E} F(\xi) \frac{d\xi}{\sqrt{\xi}} = 0, \quad \forall F \in E_{\Omega}^1(j).
\]
(1.5)

The space \( E_{\Omega}^2(\alpha) \) possesses the reproducing kernel, which we denote by \( k_{\lambda_0}^\alpha(\lambda) = k_{\lambda_0}^\alpha(\lambda, \lambda_0) \),
\[
\langle F, k_{\lambda_0}^\alpha \rangle = \frac{1}{2\pi} \oint_{E} k_{\lambda_0}^\alpha(\xi, \lambda_0) F(\xi) \frac{d\xi}{\sqrt{\xi}}, \quad \forall F \in E_{\Omega}^2(\alpha).
\]
(1.6)

Let \( F(\lambda) \) be a measurable function on \( \partial \Omega \), i.e., \( F = \{F(\xi + i0), F(\xi - i0)\}_{\xi \in E} \). We say that \( F \in L^2_{\partial \Omega} \) if (1.4) holds.

Let \( W_\alpha(\lambda) \) be a solution of the following extremal problem
\[
W_\alpha(\lambda) = \sup \{W(\lambda) : \langle W, k_{\lambda_0}^\alpha \rangle \}
\]
(1.7)

These special functions in a Widom domain (i.e., the complex Green functions and reproducing kernels, which can be given in terms of canonical products, see Sect. 2) allow to construct intrinsic basis in the Hardy/Smirnov spaces of character automorphic functions.

**Theorem 1.5** (see [7]). In a Widom domain \( \Omega \) the following are equivalent

(i) DCT holds.

(ii) \( k_{\lambda_0}^\alpha(\lambda, \lambda_0) \) is a continuous function in \( \alpha \in \pi_1(\Omega)^* \).

(iii) \( W_\alpha(\lambda) \to 1 \) for a fixed \( \lambda \in \Omega \) and \( \alpha \to 0_{\pi_1(\Omega)^*} \).

(iv) \( F \in L^2_{\partial \Omega} \oplus E_{\Omega}^2(\alpha) \) if and only if \( \overline{F} \in E_{\Omega}^2(j - \alpha) \) for all \( \alpha \in \pi_1(\Omega)^* \).

These special functions in a Widom domain (i.e., the complex Green functions and reproducing kernels, which can be given in terms of canonical products, see Sect. 2) allow to construct intrinsic basis in the Hardy/Smirnov spaces of character automorphic functions.

**Theorem 1.6** (see [19]). Let \( \beta_0 \) be the character generated by the complex Green function \( \Phi_{\lambda_0} \). The following system of functions
\[
e_n^\alpha(\lambda) = c_n^\alpha(\lambda, \lambda_0) = \Phi_{\lambda_0}(\lambda)^n \frac{k_{\lambda_0}^{\alpha-n\beta_0}(\lambda)}{\sqrt{k_{\lambda_0}^{\alpha-n\beta_0}(\lambda_0, \lambda_0)}}
\]
(1.7)
forms an orthonormal basis in \( E_{\Omega}^2(\alpha) \). That is, for an arbitrary \( F \in E_{\Omega}^2(\alpha) \)
\[
F(\lambda) = \sum_{n \geq 0} c_n e_n^\alpha(\lambda), \quad c_n = \langle F, e_n^\alpha \rangle.
\]
(1.8)

Our first goal is to prove a continual analog of the decomposition (1.8). First of all we introduce the limit counterpart of the Green function associated to a boundary point of the domain. We choose infinity as such point (this explains why we were interested to have \( E \) as an unbounded set).
The Martin function $\mathcal{M}(\lambda)$ in $\Omega$ (with respect to $\infty$) is a positive harmonic function continuously vanishing at all boundary points of the domain except for $\infty$, especially for Denjoy domains see e.g. [10]. This function is unique up to a positive multiplier and can be obtain in the following limit procedure

$$
\mathcal{M}(\lambda) = \lim_{\lambda_0 \to -\infty} \frac{G(\lambda, \lambda_0)}{G(\lambda_*, \lambda_0)}.
$$

By $\lambda_0 \to -\infty$ we mean $\lambda_0 \in \mathbb{R} \subset \Omega, \lambda_0 \to \infty$. Evidently, in this case the Martin function meets the normalization $\mathcal{M}(\lambda_*) = 1$. Respectively the complex Martin function is given as

$$
e^{i\theta(\lambda)} = \lim_{\lambda_0 \to -\infty} e^{i\theta(\lambda_0)} G(\lambda_0)/G(\lambda_*, \lambda_0), \quad \text{Im} \theta(\lambda) = \mathcal{M}(\lambda).
$$

In this case $e^{ix\theta(\lambda)}$ is a character automorphic function for an arbitrary real $x$. We also introduce a special notation for the corresponding character

$$
e^{ix\theta(\gamma(\lambda))} = e^{2\pi i\eta(\gamma)} e^{ix\theta(\lambda)}.
$$

Therefore the system of subspaces $e^{ix\theta} E^2_\Omega(\alpha - \eta x), x \in \mathbb{R}_+$, is a natural continuous counterpart of the generating the Fourier series discrete system of subspaces $\Phi^m_{\lambda_0} E^2_\Omega(\alpha - n\beta_0), n \in \mathbb{Z}_+$. Our first main result is the following theorem.

**Theorem 1.7.** Let $\Omega$ be a Widom domain with DCT. The following limit exists

$$v_\alpha(\lambda) = v_{\alpha, \lambda_*}(\lambda) = \lim_{\lambda_0 \to -\infty} \frac{k^\alpha(\lambda, \lambda_0)}{k^\alpha(\lambda_*, \lambda_0)} \tag{1.12}$$

and represents a continuous function in $\alpha \in \pi_1(\Omega)^*$. We define a positive continuous measure by its distribution function

$$\nu_{\alpha, \lambda_*}^\alpha(x) = k^\alpha(\lambda_*, \lambda_*) - e^{-2\pi i \theta_*} k^{\alpha - \eta x}(\lambda_*, \lambda_*) , \quad \theta_* = \theta(\lambda_*).
$$

Then the following (Fourier) transform

$$(\mathcal{F}^\alpha f)(\lambda) = \int_{-\infty}^{\infty} f(x) e^{i\theta(\lambda) - \theta_*} v_{\alpha - \eta x, \lambda_*}(\lambda) dx_{\lambda_*}^\alpha(x), \quad \lambda \in \partial \Omega, \tag{1.14}$$

acts unitary from $L^2_{d\nu_{\alpha, \lambda_*}^\alpha}$ to $L^2_{\partial \Omega}$. Moreover $\mathcal{F}^\alpha L^2_{d\nu_{\alpha, \lambda_*}^\alpha} \big|_{\mathbb{R}_+} = E^2_\Omega(\alpha)$.

We apply this result to introduce and study associated to such domains reflectionless Weyl-Titchmarsh functions, canonical systems, and transfer matrix functions.

The Nevanlinna class is formed by functions $w(\lambda)$ analytic in the upper half $\mathbb{C}_+$ and having positive imaginary part, $\text{Im} w(\lambda) > 0$. Such functions possess the additive

$$w(\lambda) = a\lambda + b + \int_{\mathbb{R}} \frac{1 + \xi \lambda d\sigma(\xi)}{\xi - \lambda + 1 + \xi^2}, \quad a > 0, \ b \in \mathbb{R}, \tag{1.15}$$

where $\sigma(\xi)$ is a finite sum of Dirac delta functions.
and the multiplicative
\[ w(\lambda) = c e^{\int_{\mathbb{R}} \frac{1}{\xi - \lambda} \chi(\xi) d\xi}, \quad c > 0, \tag{1.16} \]
representations. Here \( \sigma \) is a nonnegative measure such that
\[ \int_{\mathbb{R}} \frac{d\sigma(\xi)}{1 + \xi^2} < \infty, \]
and a measurable function \( \chi(\xi) \) is such that \( \chi(\xi) \in [0, 1] \). Moreover
\[ \chi(\xi) = \frac{1}{\pi} \arg w(\xi + i0), \quad \text{a.e. } \xi \in \mathbb{R}. \]

We say that \( w \) belongs to the Stieltjes class \( S \) if the measure \( \sigma \) in the representation (1.15) is supported on the positive half axis. Such functions allow an analytic extension in the lower half plane by the symmetry principle, \( w(\lambda) = \overline{w(\lambda)} \) (the function is analytic in \( \mathbb{C} \setminus \mathbb{R}_+ \)).

By \( S_0 \) we denote the subclass \( S \) of functions \( m(\lambda) \) such that
\[ \lim_{\lambda \to -0} m(\lambda) = 0. \]
These functions possess a special additive representation
\[ m(\lambda) = a\lambda + \int_{\mathbb{R}^+} \frac{\lambda d\sigma(\xi)}{\xi - \lambda}, \quad a > 0, \quad \int_{\mathbb{R}^+} \frac{d\sigma(\xi)}{1 + \xi} < \infty, \quad \sigma\{0\} = 0. \tag{1.17} \]
Note \( S_0 \) is related to the Nevanlinna class functions \( n(\mu) \) with an associated symmetric measure in a simple way
\[ n(\mu) = \frac{1}{\mu} m(\mu^2) = a\mu + \frac{1}{2} \int_{\mathbb{R}^+} \left\{ \frac{1}{t - \mu} - \frac{1}{t + \mu} \right\} d\sigma(\mu^2), \quad \mu \in \mathbb{C}_+. \tag{1.18} \]

**Definition 1.8.** We say that \( m_+ \in S_0 \) belongs to the set \( m_0(E) \) if there exists \( m_- \) of the Stieltjes class such that
\[ m_-(\lambda) = -m_+(\lambda) \quad \text{for a.e. } \lambda \in E, \tag{1.19} \]
and the following two their symmetric combinations
\[ R_0(\lambda) = -\frac{1}{m_+(\lambda) + m_-(\lambda)}, \quad R_1(\lambda) = \frac{m_+(\lambda)m_-(-\lambda)}{m_+(\lambda) + m_-(\lambda)} \tag{1.20} \]
are holomorphic in \( \Omega = \mathbb{C} \setminus E \).

The relation (1.19) means that \( m_+(\lambda) \) has a *pseudocontinuation* \[ \text{[14, Lecture II, Sect. 1]} \] through the set \( E \). In the spectral theory it is called the *reflectionless property* \[ \text{[17, 16]} \]. Due to this property \( R_1(\lambda) \) assumes pure imaginary boundary values a.e. on \( E \) (on the negative half axis and in the gaps \((a_j, b_j)\) they are real valued by the definition).

The following proposition gives a parametric description of the class \( m_0(E) \approx \mathbb{R}_+ \times \pi_1(\Omega)^* \).
**Theorem 1.9.** Let $\Omega$ be of Widom type and DCT hold. Then $m_+ \in m_0(E)$ if and only if it is of the form

$$m_+(\lambda) = \frac{m_+(\lambda_s)}{i\sqrt{\lambda_s}} m_+^\alpha(\lambda), \quad m_+^\alpha(\lambda) := i \sqrt{\frac{v_{\alpha + i}(\lambda)}{v_\alpha(\lambda)}},$$

(1.21)

where $\alpha \in \pi_1(\Omega)^*$. We describe the collection $\{m_+^\alpha(\lambda)\}_{\alpha \in \pi_1(\Omega)^*}$ as the Weyl-Titchmarsh functions of canonical systems.

**Theorem 1.10.** The following limit exists

$$\frac{\Upsilon^\alpha(x)}{\Upsilon^\alpha(0)} := \lim_{\lambda \to 0} \frac{v_{\alpha - \eta x}(\lambda)}{v_\alpha(\lambda)} = \lim_{\lambda \to 0} \lim_{\lambda_0 \to -\infty} \frac{k^{\alpha - \eta x}(\lambda, \lambda_0)k^{\alpha}(\lambda_0, \lambda_0)}{k^{\alpha - \eta x}(\lambda, \lambda_0)k^{\alpha}(\lambda_0, \lambda_0)}$$

(1.22)

and can be given explicitly as

$$\Upsilon^\alpha(x) = \Upsilon^\alpha_{\lambda_s}(x) = \sqrt{k^{\alpha - \eta x}(\lambda_s, \lambda_s) + k^{\alpha + j - \eta x}(\lambda_s, \lambda_s)}$$

$$\times \exp \left( \frac{1}{2} \int_0^x \frac{d e^{-2\Im \theta_s} e^{2\Im \theta_s} (k^{\alpha + j - \eta x}(\lambda_s, \lambda_s) - k^{\alpha - \eta x}(\lambda_s, \lambda_s))}{\Upsilon^\alpha(\xi)} \right).$$

(1.23)

Let

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad T_\alpha(x, \lambda) = \begin{bmatrix} \tau^{\alpha+i}(x) & 0 \\ 0 & \lambda \tau^\alpha(x) \end{bmatrix}, \quad \tau^\alpha(x) = \frac{i}{\sqrt{\lambda_s}} \int_0^x \frac{e^{2\Im \theta_s} d \xi^\alpha(\xi)}{\Upsilon^\alpha(\xi)^2},$$

and $A_\alpha(\lambda, x)$ be the family of the transfer matrices of the canonical system given in the integral form

$$A_\alpha(\lambda, x)J = J - \int_0^x A_\alpha(\lambda, \xi) dT_\alpha(\lambda, \xi), \quad A(\lambda, x) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}(\lambda, x).$$

(1.24)

Then $m_+^\alpha(\lambda)$, defined in (1.21), is the corresponding Weyl-Titchmarsh function. That is, for an arbitrary $\lambda \in \Omega$, its value is the unique intersection point of the nesting Weyl circles

$$m_+^\alpha(\lambda) = \lim_{x \to \infty} \frac{a_{22}(\lambda, x)w - a_{21}(\lambda, x)}{-a_{12}(\lambda, x)w + a_{11}(\lambda, x)}, \quad w \in \mathbb{R} \cup \{\infty\}. \quad (1.25)$$

Being objects of the general de Branges theory, the transfer matrices have some standard properties. They form a monotonic family of entire matrix functions $J$-contractive in the upper half plane,

$$\frac{J - A_\alpha(\lambda, x)J A(\lambda, x)^*}{\lambda - \overline{\lambda}} \geq 0,$$

also $\overline{A(\lambda, x)} = A(\lambda, x)$, $\det A(\lambda, x) = 1$. Note that passing from the class $S_0$ to the class of symmetric Nevanlinna functions, see (1.18), we pass to the family of transfer matrices

$$B(\mu, x) = \begin{bmatrix} \mu & 0 \\ 0 & 1 \end{bmatrix} A(\mu^2, x) \begin{bmatrix} 1/\mu & 0 \\ 0 & 1 \end{bmatrix}.$$
which satisfy the canonical system with a diagonal Hamiltonian, i.e.,

\[ \mathfrak{B}_\alpha(\mu, x) \mathcal{J} = \mathcal{J} - \mu \int_0^x \mathfrak{B}_\alpha(\mu, \xi) \begin{bmatrix} d\tau^{\alpha+}(\xi) & 0 \\ 0 & d\tau^{\alpha}(\xi) \end{bmatrix}. \]

The featured properties are given in the following statement.

**Corollary 1.11.** 1. To find the argument \( x \) of the entire matrix function \( \mathfrak{A}(\lambda, x) \) we consider its entries as functions of bounded characteristic in the domain \( \Omega \), after that we compute the “exponential type” of these functions, i.e.,

\[ x = \lim_{\lambda \to -\infty} \frac{\log \| \mathfrak{A}(\lambda, x) \|}{M(\lambda)}. \]

(1.26)

2. Assume that

\[ \lim_{\lambda \to -\infty} \frac{M(\lambda)}{\sqrt{|\lambda|}} = 0. \]

(1.27)

Then, the generating canonical system measures \( \tau^\alpha \) and \( \tau^{\alpha+} \) are mutually singular.

**Remark 1.12.** A set \( E \) is called of Akhiezer-Levin type if

\[ \lim_{\lambda \to -\infty} \frac{M(\lambda)}{\sqrt{|\lambda|}} > 0. \]

Such kind of domains are widely studied in particular in a connection with 1-D Schrödinger operators (i.e. the Marchenko-Ostrovskii class [11], see also [18]). A quite general result, including the KdV hierarchy, was presented recently in [4]. That is why we are mostly interested in the case (1.27). As the simplest example, one can have in mind a set which is formed by geometric progressions

\[ a_n = \rho^n a_0, \quad b_n = \rho^n b_0, \quad n \in \mathbb{Z}, \quad 0 < a_0 < b_0 < \rho a_0. \]

**Remark 1.13.** Let us say that 0 is a regular point for \( E \) if the following limit

\[ \lim_{\lambda \to -0} \frac{v_\beta(\lambda)}{v_\alpha(\lambda)} \]

exists for all \( \alpha, \beta \in \pi_1(\Omega)^* \). Then we can define

\[ \Xi(\alpha) = \lim_{\lambda \to -0} \frac{v_\alpha(\lambda)}{v_0(\lambda)}, \quad 0 = 0_{\pi_1(\Omega)^*}, \]

and represent the limit (1.22) in terms of this function on the group of characters,

\[ \frac{\Upsilon^\alpha(x)}{\Upsilon^\alpha(0)} = \frac{\Xi(\alpha - \eta x)}{\Xi(\alpha)}. \]

Especially in the finite gap case one gets explicit formulas in terms of theta functions [12]. However, say, for a geometric progression set \( E \), 0 is not a regular point, nevertheless \( \Upsilon^\alpha(x) \) has perfect sense and represents a continuous function in \( x \). On the other hand regularity of 0 is, of course, not an extraordinary property of a set \( E \). The simplest case of regularity: \([0, \varepsilon] \subset E \) for some \( \varepsilon > 0 \).
2 Reproducing kernels and Fourier Integral

We describe reproducing kernels in terms of canonical products. First, we define the set of divisors

\[ \mathcal{D}(E) = \{ D = \{ (\lambda_j, \epsilon_j) \}_{j \geq 1} : \lambda_j \in [a_j, b_j], \epsilon_j = \pm 1 \} \]

with the identification \((a_j, 1) = (a_j, -1)\) and \((b_j, 1) = (b_j, -1)\), endowed with the product topology of circles. To \( D \in \mathcal{D}(E) \) we associate

\[ V(\lambda, D) = \left( \prod_{j \geq 1} \sqrt{\frac{(\lambda - a_j)(\lambda - b_j)}{(\lambda - a_j)(\lambda - b_j)}} \frac{1}{\lambda - \lambda_j} \Phi_{\lambda_j}(\lambda) \right)^{1/2} \prod_{j \geq 1} \Phi_{\lambda_j}(\lambda). \]  

(2.1)

We note that the product in the brackets represents an outer function, therefore the square root of this product is well defined and represents a character automorphic function. The second factor is an inner function (Blaschke product) in the given domain. Alternatively, we can write

\[ V(\lambda, D) = \sqrt{O(\lambda, D)I(\lambda, D)}, \quad I(\lambda, D) = \prod_{j \geq 1} \Phi_{\epsilon_j\lambda_j}(\lambda) \]  

(2.2)

\[ O(\lambda, D) = e^{\int (\frac{1}{\lambda - \xi} - \frac{1}{\lambda - \lambda_j}) \chi_D(\xi) d\xi}, \quad \chi_D(\xi) = \begin{cases} 1/2, & \xi \in (a_j, \lambda_j) \\ -1/2, & \xi \in (a_j, \lambda_j) \\ 0, & \text{otherwise} \end{cases} \]

(2.3)

**Definition 2.1.** We define the Abel map \( A : \mathcal{D}(E) \to \pi_1(\Omega)^* \) by

\[ A(D) = \text{character of } V(\lambda, D). \]

**Theorem 2.2** (see [13]). For a Widom domain \( \Omega \) if DCT holds, then the Abel map is a homeomorphism.

In particular, \( D = D(\alpha) \) is uniquely defined by \( A(D) = \alpha \in \pi_1(\Omega)^* \). To simplify notation we write

\[ O_\alpha(\lambda) := O(\lambda, D), \quad I_\alpha(\lambda) := I(\lambda, D), \quad V_\alpha(\lambda) := V(\lambda, D) \quad \text{for} \quad \alpha = A(D). \]

**Lemma 2.3.** Let \( D_* := \{ (\lambda_j, -\epsilon_j) \}_{j \geq 1} \). Then

\[ V(\lambda, D_*) = V_{-\alpha(D)}(\lambda), \quad V(\lambda, D) = V_{\alpha(D)}(\lambda), \quad \lambda \in \Omega, \quad \lambda \in E. \]

(2.4)

**Proof.** Note that \( O(\lambda, D) \) assumes positive values on \( E \), see (2.3), and \(|I(\lambda, D)| = 1 \) here. Therefore

\[ \overline{V(\lambda, D)V(\lambda, D)} = O(\lambda, D), \quad \lambda \in E. \]
and by (2.2)

\[
\overline{V(\lambda, D)} = \frac{O(\lambda, D)}{V(\lambda, D)} = \frac{O(\lambda, D)}{\sqrt{O(\lambda, D)I(\lambda, D)}} = \sqrt{O(\lambda, D)I(\lambda, D)^{-1}} = V(\lambda, D_+).
\]

Further, \(\sqrt{\lambda}O(\lambda, D)\) is single valued in the domain, hence \(j\) is the character of \(O(\lambda, D)\). In conjunction with the previous line this proves the first identity in (2.4).

\[\square\]

Lemma 2.4. Let \(\lambda_0 \in \mathbb{R}_-\). By \(\frac{1}{2}\beta_0\) we understand the character of the outer function \(\sqrt{\Phi_{\lambda_0}(\lambda)/(\lambda - \lambda_0)}\). Then

\[
k_{\lambda_0}^\alpha(\lambda) = \frac{\sqrt{\lambda}}{i} \frac{V_{\alpha - \frac{1}{2}\beta_0}(\lambda)}{V_{\alpha - \frac{1}{2}\beta_0}(\lambda_0)} \sqrt{\frac{\Phi_{\lambda_0}(\lambda)\Phi_{\lambda_0}'(\lambda_0)}{\lambda - \lambda_0}}.
\]  

(2.5)

Proof. By DCT and Lemma 2.3 we have

\[
\langle F, k_{\lambda_0}^\alpha \rangle = \frac{\sqrt{\lambda}}{2\pi i} \int_E \frac{V_{\alpha - \frac{1}{2}\beta_0}(\xi)}{V_{\alpha - \frac{1}{2}\beta_0}(\lambda_0)} \sqrt{\frac{\Phi_{\lambda_0}(\lambda_0)}{(\xi - \lambda_0)\Phi_{\lambda_0}(\xi)}} F(\xi) \frac{d\xi}{\sqrt{\xi}} = \langle F, (\lambda_0) \rangle.
\]

(2.6)

A proof of Theorem 1.7 is based essentially on a combination of Lemma 2.5 and Theorem 2.6 below.

Lemma 2.5. For a fixed \(\lambda \in \mathbb{C} \setminus \mathbb{R}_+\) the function \(V_\alpha(\lambda)\) is continuous on the compact Abelian group \(\pi_1(\Omega)^*\). The following limit exists uniformly in \(\alpha\).

\[
v_{\alpha, \lambda_*}(\alpha) = \lim_{\lambda_0 \to -\infty} \frac{k_\alpha(\lambda, \lambda_0)}{k_\alpha(\lambda_*, \lambda_0)} = \frac{V_\alpha(\lambda)}{V_\alpha(\lambda_*)}.
\]

Proof. We note that both functions \(O(\lambda, D)\) and \(I(\lambda, D)\) are continuous in \(D \in D(E)\) as soon as \(\lambda \in \mathbb{C} \setminus \mathbb{R}_+\). By continuity of the Abel map we have continuity of \(V_\alpha(\lambda)\). Then, we pass to the limit using the representation (2.5). Thus the first statement (1.12) of Theorem 1.7 is proved.

\[\square\]

Theorem 2.6. For a Widom domain \(\Omega\) with DCT we define a positive continuous measure \(\rho^\alpha = \rho^\alpha_{\lambda_*}\) by (1.13). Then

\[
k_\alpha(\lambda, \lambda_0) - e^{2ix(\lambda - \theta(\lambda_0))}k_\alpha(\lambda, \lambda_0) = \int_0^x f_{\alpha, \lambda_*}(\lambda_0, \xi) f_{\alpha, \lambda_*}(\lambda, \xi) d\rho^\alpha(\xi),
\]

(2.6)

where

\[
f_{\alpha, \lambda_*}(\lambda, x) = e^{ix(\theta(\lambda) - \theta(\lambda_*))} \frac{V_\alpha - \theta(\lambda) \lambda}{V_\alpha - \theta(\lambda_*) \lambda_*}.
\]

In particular,

\[
k_\alpha(\lambda, \lambda_0) = \int_{0}^{\infty} e^{ix(\theta(\lambda_0) - \theta_*)} v_{\alpha - \eta_x, \lambda_*}(\lambda_0) e^{ix(\theta(\lambda) - \theta_*)} v_{\alpha - \eta_x, \lambda_*}(\lambda) d\rho^\alpha(x).
\]

(2.7)
Proof. WLOG we set \( x = 1 \). First, we introduce a sequence of measures \( \{ \mathcal{H}_N \} \) on \([0, 1]\). By regularity for a fixed \( N \) we define \( \lambda_N < \lambda_0 \) such that \( \mathcal{G}(\lambda_N, \lambda_0) = 1/N \). We set

\[ \mathcal{H}_N \{ k/N \} = |e_k^\alpha(\lambda_*, \lambda_N)|^2 = |\Phi_\lambda(\lambda_*)|^k |k\alpha-k\beta_N(\lambda_*, \lambda_N)|^2 \]

where \( \beta_N \) is the character generated by the function \( \Phi_\lambda(\lambda) \) and \( \mathcal{H}_N \{ B \} \) is the measure of a set \( B \) (a single point in our case). By (1.7) and (1.8) we have the standard for reproducing kernels relation

\[ k^\alpha(\lambda, \lambda_0) = \sum_{n \geq 0} e_n^\alpha(\lambda_0, \lambda_N) e_n^\alpha(\lambda, \lambda_N). \] (2.8)

Therefore, the corresponding distribution function can be simplified to the form

\[ \mathcal{H}_N(x) = \sum_{k=0}^{[Nx]} |e_k^\alpha(\lambda_*, \lambda_N)|^2 = k^\alpha(\lambda_*, \lambda_*) - |\Phi_\lambda(\lambda_*)|^{2([Nx]+1)} k^{\alpha-([Nx]+1)\beta_N}(\lambda_*, \lambda_*), \]

where \( x \in (0, 1) \) is irrational.

By (1.7) and (1.8) we have

\[ \Phi_\lambda(\lambda_*)|^{[Nx]+1} = \left( e^{i\theta_\lambda(\lambda_*)/\mathcal{G}_\lambda(\lambda_*)} \right)^{[Nx]+1} \to e^{i\pi \theta(\lambda_*)}, \quad \text{as } N \to \infty. \]

For the same reason, \( e^{2\pi i N \beta_N(\gamma)} \to e^{2\pi i \eta(\gamma)} \) for all \( \gamma \in \pi_1(\Omega) \). Using continuity of the reproducing kernel we obtain

\[ \lim_{N \to \infty} \mathcal{H}_N(x) = k^\alpha(\lambda_*, \lambda_*) - e^{-2\pi i \eta(\lambda_*)} k^{\alpha-\eta}(\lambda_*, \lambda_*) = \mathcal{H}(x). \] (2.9)

Going back to the general expression (2.8) for fixed \( \lambda \) and \( \lambda_0 \) we write

\[ k^\alpha(\lambda, \lambda_0) = \Phi_\lambda(\lambda_*)^N \Phi_\lambda(\lambda)^N \mathcal{H}_N(k/N), \]

(2.10)

\[ = \sum_{n=0}^{N-1} \frac{e_n^\alpha(\lambda_0, \lambda_N)}{e_n^\alpha(\lambda_*, \lambda_N)} \frac{|e_n^\alpha(\lambda_*, \lambda_N)|^2 e_n^\alpha(\lambda, \lambda_N)}{e_n^\alpha(\lambda_*, \lambda_N)} \]

\[ = \sum_{n=0}^{N-1} \frac{e_n^\alpha(\lambda_0, \lambda_N)}{e_n^\alpha(\lambda_*, \lambda_N)} \frac{e_n^\alpha(\lambda_*, \lambda_N)}{e_n^\alpha(\lambda_*, \lambda_N)} \mathcal{H}_N(k/N). \]

In the right hand side we can pass to the limit as it was discussed above

\[ k^\alpha(\lambda, \lambda_0) - \lim_{N \to \infty} \Phi_\lambda(\lambda_*)^N \Phi_\lambda(\lambda)^N \mathcal{H}_N(k/N) = k^\alpha(\lambda, \lambda_0) - e^{i(\theta(\lambda)-\theta(\lambda_0))} k^{\alpha-\eta}(\lambda, \lambda_0). \]

Due to Lemma 2.3 we have

\[ \frac{e_k^\alpha(\lambda, \lambda_N)}{e_k^\alpha(\lambda_*, \lambda_N)} = \sqrt{1 - \lambda_*/\lambda_N} e^{ikP(\lambda_0(\lambda)-\theta_\lambda(\lambda_*)/\mathbb{G}_\lambda(\lambda_*)}/V_\lambda - \frac{k+0.5}{N} \beta_N(\lambda), \]

where

\[ V_\lambda = \frac{\frac{k+0.5}{N} \beta_N(\lambda)}{\frac{k+0.5}{N} \beta_N(\lambda_*)}, \]

\[ P(\lambda) \]
Therefore, for an arbitrary $\varepsilon > 0$, for sufficiently big $N > N_0$

$$\left| \frac{e_k^\alpha(\lambda_\ast, \lambda_N)}{e_k^\alpha(\lambda_{\ast'}, \lambda_N)} - e^{i\frac{\pi}{N}(\theta(\lambda) - \theta(\lambda_\ast))} \frac{V_{\alpha - \frac{x}{N}(\lambda)}}{V_{\alpha - \frac{x}{N}(\lambda_\ast)}} \right| \leq \varepsilon$$

holds for all $k \leq N$. Thus, the last expression in (2.10) can be substituted with a fixed error by the integral

$$\int_0^1 e^{-ix(\theta(\lambda_\ast) - \theta(\lambda_\ast))} \frac{V_{\alpha - \frac{x}{N}(\lambda)}}{V_{\alpha - \frac{x}{N}(\lambda_\ast)}} e^{i\pi(\theta(\lambda) - \theta(\lambda_\ast))} \frac{V_{\alpha - \frac{x}{N}(\lambda)}}{V_{\alpha - \frac{x}{N}(\lambda_\ast)}} \frac{d\alpha}{\alpha^2}.$$

Due to the measure convergence (2.9) we obtain (2.6). Finally, we can pass to the limit as $x \to \infty$ in the left hand side of this relation, we obtain (2.7).

**Remark 2.7.** DCT, equivalently a continuity of the reproducing kernels with respect to the character, plays the key role in the proof of Theorem 1.7. Moreover, if it fails the chain of subspaces $e^{ix}\mathcal{E}^2_\Omega(\alpha - \eta x)$ is not necessary complete for a certain $\alpha \in \pi_1(\Omega)^*$, as it was shown in [23], what contradicts to the integral representation (2.6).

**Proof of Theorem 1.7.** On the dense set in $\mathcal{E}^2_\Omega(\alpha)$ we define a map to $\chi_{\mathbb{R}^+}L^2_{d\kappa_{\alpha}}$ by

$$e^{i\pi(\theta(\lambda) - \theta(\lambda_\ast))}k_{\alpha - \eta x}(\lambda, \lambda_0) \mapsto \chi_{(x, \infty)}(\lambda) e^{i\pi(\theta(\lambda_\ast) - \theta(\lambda_\ast))} = \int_\mathbb{R}^+ \chi_{(x, \infty)}(\lambda) e^{i\pi(\theta(\lambda_\ast) - \theta(\lambda_\ast))} \frac{d\alpha}{\alpha^2}.$$

where $\chi_B$ is the characteristic function of a set $B$. By Theorem 2.6 this is an isometry. Therefore this map is well defined on $\mathcal{E}^2_\Omega(\alpha)$. Since in the image we have all functions $\{\chi_{(x, \infty)}\}_{x \in \mathbb{R}^+}$, it is dense. Indeed, assume that $f(\xi) \in \chi_{\mathbb{R}^+}L^2_{d\kappa_{\alpha}}$ is orthogonal to this collection, then

$$\int_\mathbb{R}^+ f(\xi) d\kappa_{\alpha}(\xi) = 0.$$

That is, $f(\xi) = 0$ for a.e. $\xi$ with respect to the measure $\kappa_{\alpha}$. Thus $\mathcal{F}^\alpha$ restricted to $\chi_{\mathbb{R}^+}L^2_{d\kappa_{\alpha}}$ is well defined as the inverse to the map (2.11).

In fact, for the same reason, we have that $\mathcal{F}^\alpha : \chi_{(x, \infty)}L^2_{d\kappa_{\alpha}}(\alpha - \eta x) \to e^{ix\theta}E^2_\Omega(\alpha - \eta x)$ acts unitary for all $x \in \mathbb{R}$. It remains to show that

$$\mathcal{F}^\alpha = \chi_{\mathbb{R}^+}L^2_{d\kappa_{\alpha}}(\alpha - \eta x) = L^2_{d\kappa_{\alpha}}.$$

Equivalently, by the property (iv) of Theorem 1.5 we have to show that

$$\cap_{x \in \mathbb{R}} e^{-ix\theta}E^2_\Omega(j - \alpha + \eta x) = \{0\}.$$

If $F$ belongs to the intersection, then for every $x$ there exits $G_x \in E^2_\Omega(j - \alpha + \eta x)$ such that $F = e^{-ix\theta}G_x$. In particular, $F \in E^2_\Omega(j - \alpha)$. So, it is enough to show that $F(\lambda) = 0$ for all $\lambda \in \Omega$. Note that $\|G_x\| = \|F\|$. Since

$$|G_x(\lambda)| \leq \|F\| \sqrt{k^{1-\alpha+\eta x}(\lambda, \lambda)}$$

and $C = \sup_{\beta \in \pi_1(\Omega)^*} k^\beta(\lambda, \lambda) < \infty$, $\lambda \in \Omega$. Therefore, for sufficiently big $N > N_0$
we have
\[ |F(\lambda)| \leq C\|F\| e^{x\text{Im}\theta(\lambda)} \to 0, \quad \text{as} \quad x \to -\infty. \]

\[ \square \]

**Remark 2.8.** As a byproduct, we proved that the span of \( \{e^{ix(\theta-\bar{\theta})}k_{\lambda_0}^{\alpha-\eta x}\} \) is dense in \( E_{\Omega}^2(\alpha) \) as \( x \in \mathbb{R} \) and in \( L_{\Omega}^2 \) if the parameter \( x \) runs in \( \mathbb{R} \).

### 3 Reproducing kernels and Transfer matrices

In plain domains the reproducing kernels of Hardy/Smirnov spaces have a very specific structure: a kind of resolvent expression related to the operator multiplication by the independent variable \( \lambda \).

**Proposition 3.1.** The reproducing kernel of \( E_{\Omega}^2(\alpha) \) is of the form
\[ k^\alpha(\lambda, \lambda_0) = iC(\alpha) \frac{\sqrt{\lambda}V_{\alpha+j}(\lambda)V_{\alpha}(\lambda_0) + V_{\alpha}(\lambda)\sqrt{\lambda_0}V_{\alpha+j}(\lambda_0)}{\lambda - \lambda_0}, \]  
where \( C(\alpha) = C_{\lambda_0}(\alpha) \) is given by
\[ \frac{1}{C_{\lambda_0}(\alpha)} = V_{\alpha}(\lambda_0)V_{\alpha}(\lambda) + V_{\alpha}(\lambda)V_{\alpha+j}(\lambda_0) = \frac{I_{\alpha}(\lambda_0)}{I_{\alpha+j}(\lambda_0)} + \frac{I_{\alpha+j}(\lambda_0)}{I_{\alpha}(\lambda_0)}. \]  

**Proof.** Note that \( V_{\alpha}(\lambda)/(\lambda - \lambda_0) \in L_{\Omega}^2, \lambda_0 \in \Omega, \) and even after multiplication by \( \sqrt{\lambda} \) we still have a function from \( L_{\Omega}^2 \). Since
\[ G(\lambda) = i\frac{\sqrt{\lambda}V_{\alpha+j}(\lambda)V_{\alpha}(\lambda_0) - V_{\alpha}(\lambda)\sqrt{\lambda_0}V_{\alpha+j}(\lambda_0)}{\lambda - \lambda_0} \]
is of Smirnov class \( N_+(\Omega) \), we get that \( G(\lambda) \in E_{\Omega}^2(\alpha) \). Note that by the construction \( V_{\alpha} \) is real on \( \mathbb{R}_- \) and \( \sqrt{\lambda} \) takes pure imaginary values here, so we can rewrite \( G(\lambda) \) into the form
\[ G(\lambda) = i\frac{\sqrt{\lambda}V_{\alpha+j}(\lambda)V_{\alpha}(\lambda_0) + V_{\alpha}(\lambda)\sqrt{\lambda_0}V_{\alpha+j}(\lambda_0)}{\lambda - \lambda_0}. \]  

We use Lemma 2.3 and DCT, then for an arbitrary \( F(\lambda) \in E_{\Omega}^2(\alpha) \) we have
\[ \langle F, G \rangle = \frac{1}{2\pi i} \int_{\mathbb{E}} \frac{\sqrt{\xi}V_{\alpha-j}(\xi)V_{\alpha}(\lambda_0) + V_{\alpha-j}(\xi)\sqrt{\lambda_0}V_{\alpha+j}(\lambda_0)}{\xi - \lambda_0} F(\xi) \frac{d\xi}{\sqrt{\xi}} 
= (V_{\alpha}(\lambda_0)V_{\alpha}(\lambda_0) + V_{\alpha-j}(\lambda_0)V_{\alpha+j}(\lambda_0))F(\lambda_0). \]  

In particular,
\[ \|G\|^2 = (V_{\alpha}(\lambda_0)V_{\alpha}(\lambda_0) + V_{\alpha-j}(\lambda_0)V_{\alpha+j}(\lambda_0))G(\lambda_0). \]
Since $G$ is not identically zero $\|G\|^2 > 0$. Since $G(\lambda_0)$ is a real number, we obtain that the analytic function $V_{\alpha}(\lambda_0)V_{\alpha}(\lambda_0) + V_{\beta}(\lambda_0)V_{\gamma}(\lambda_0)$ assumes only real values for all $\lambda_0 \in \Omega$. Therefore it is constant, which we denote by $1/C_{\lambda_0}(\alpha) > 0$. We have (3.2), see (2.2) and (2.3). Consequently, (3.3) and (3.4) implies (3.1).

**Definition 3.2.** In what follows, the relation

$$C(\alpha) \det \begin{bmatrix} V_{\alpha+j}(\lambda) & V_{\alpha}(\lambda) \\ -V_{\beta}(\lambda) & V_{\gamma}(\lambda) \end{bmatrix} = 1$$

(3.5)

we call the Wronskian identity.

**Corollary 3.3.** The generalized eigenfunction $v_{\alpha,\beta}(\lambda)$ possesses the following representation in terms of reproducing kernels

$$\frac{iv_{\alpha,\beta}(\lambda)}{I_{\alpha}(\lambda) + I_{\alpha+j}(\lambda)} = iC(\alpha)V_{\alpha+j}(\lambda)V_{\alpha}(\lambda) = \sqrt{\lambda}k^{\alpha+j}(\lambda, \lambda) + \sqrt{\lambda}k^{\alpha}(\lambda, \lambda).$$

(3.6)

**Proof.** Note that by (3.2) $C_{\lambda_0}(\alpha) = C_{\lambda_0}(\alpha + j)$. By (3.1) we have

$$\begin{bmatrix} k^{\alpha}(\lambda, \lambda) \\ k^{\alpha+j}(\lambda, \lambda) \end{bmatrix} = iC_{\lambda_0}(\alpha) \begin{bmatrix} \sqrt{\lambda} & -1 \\ -1 & \sqrt{\lambda} \end{bmatrix} \begin{bmatrix} V_{\alpha+j}(\lambda)V_{\alpha}(\lambda) \\ V_{\alpha}(\lambda)V_{\alpha+j}(\lambda) \end{bmatrix}.$$ 

Due to Lemma 2.5, we have (3.6).

Further, we will use a certain very general construction related to the theory of extensions of isometries [1, 2], either the functional models of contractive operators [13], or the Lax-Phillips scattering theory, or the generalized interpolation in the sense of Potapov’s approach [9] and so on...

Consider the Cayley transformation of the multiplication by $\lambda$ in $L^2_{\Omega}$,

$$v(\lambda) = \frac{\lambda - \lambda_0}{\lambda - \lambda_0} = \frac{\Phi_{\lambda_0}(\lambda)}{\Phi_{\lambda_0}(\lambda)}, \quad \text{Im} \lambda_0 > 0.$$ 

Note that both complex Green functions have the same character $\beta_0$. Evidently

$$\Phi: \Phi_{\lambda_0}E^2_{\Omega}(\alpha - \beta_0) \to \Phi_{\lambda_0}E^2_{\Omega}(\alpha - \beta_0)$$

acts unitary. From this relation, passing to orthogonal complements we obtain that the multiplication by $v(\lambda)$ acts unitary from

$$\begin{cases}
\frac{\epsilon^{\alpha + \beta_0}}{\Phi_{\lambda_0}} \\
\frac{\epsilon^{\alpha - \eta x}}{\Phi_{\lambda_0}}
\end{cases} \oplus \left( E^2_{\Omega}(\alpha) \oplus e^{ix\theta}E^2_{\Omega}(\alpha - \eta x) \right) \oplus e^{ix\theta}e^{\lambda_0 - \eta x}$$

(3.7)
to
\[
\left\{ \frac{e^{\alpha+\beta_0}}{\Phi_{\lambda_0}} \right\} \oplus \left( E_\Omega^2(\alpha) \oplus e^{ix\theta} E_\Omega^2(\alpha-\eta x) \right) \oplus e^{ix\theta} \frac{e^{\alpha-\eta x}}{\lambda_0},
\] (3.8)
where
\[ e^{\alpha}_{\lambda_0} = \frac{k^\alpha_{\lambda_0}}{\|k^\alpha_{\lambda_0}\|}. \]

Now we recall the notion of the unitary node [2, 9]. This is a unitary operator \( U \) acting from the space \( K \oplus E_1 \) to \( K \oplus E_2 \). \( K \) is called the internal space and \( E_1, E_2 \) are called the scaling spaces. Evidently, in the decompositions (3.7) and (3.8) we have the unitary node with two dimensional scaling spaces and the internal space \( K(\alpha, x) = E_\Omega^2(\alpha) \ominus e^{ix\theta} E_\Omega^2(\alpha-\eta x) \).

The scattering matrix of the unitary node is a contractive operator valued analytic function \( S(\zeta), \zeta \in \mathbb{D} \), acting from \( E_1 \) to \( E_2 \) (for a fixed \( \zeta \)) and given by
\[
S(\zeta) = P_{E_2}(I - \zeta K U)^{-1} U|_{E_1},
\]
where \( P_K, P_{E_2} \) are the orthogonal projections on the corresponding spaces. Note that as soon as we fix basis in the scaling spaces we get a scattering matrix valued analytic function.

Applying this construction in our case \( U = \nu(\lambda) \), we obtain the scattering matrix
\[
\begin{bmatrix}
\frac{e^{\alpha+\beta_0}}{\Phi_{\lambda_0}} & e^{ix\theta} \frac{e^{\alpha-\eta x}}{\lambda_0} \\
\frac{e^{\alpha-\beta_0}}{\Phi_{\lambda_0}} & e^{ix\theta} \frac{e^{\alpha-\eta x}}{\lambda_0}
\end{bmatrix} S(\zeta) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = P_{E_2}(I - \zeta K(x) U)^{-1} U \begin{bmatrix}
\frac{e^{\alpha+\beta_0}}{\Phi_{\lambda_0}} & e^{ix\theta} \frac{e^{\alpha-\eta x}}{\lambda_0} \\
\frac{e^{\alpha-\beta_0}}{\Phi_{\lambda_0}} & e^{ix\theta} \frac{e^{\alpha-\eta x}}{\lambda_0}
\end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \] (3.9)

Switching of “channels” to a more natural pairs
\[
\begin{bmatrix}
\frac{e^{\alpha+\beta_0}}{\Phi_{\lambda_0}} & \frac{e^{\alpha-\beta_0}}{\Phi_{\lambda_0}} \\
\frac{e^{\alpha-\beta_0}}{e^{\alpha+\beta_0}}
\end{bmatrix}
\]
leads to the so-called Potapov-Ginzburg transform of \( S(\zeta) \):
\[
A(\zeta) = \begin{bmatrix} S_{11}(\zeta) & S_{12}(\zeta) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ S_{21}(\zeta) & S_{22}(\zeta) \end{bmatrix}^{-1}.
\] (3.10)

\( A(\zeta) \) is called the transfer matrix and possesses a much easier chain property with respect to \( x \). The contractive property of the scattering matrix, \( I - S(\zeta)S(\zeta)^* \geq 0 \), perturbs to the \( j \)-contractive property of the transfer matrix,
\[
j - A(\zeta) j A(\zeta)^* \geq 0, \quad j = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},
\] (3.11)
see e.g. the beginning of Section 6 in [9].

**Theorem 3.4.** Let
\[
V_\alpha(\lambda) = \sqrt{C_\lambda(\alpha)} \begin{bmatrix} i\sqrt{\lambda} V_{\alpha+j}(\lambda) & V_\alpha(\lambda) \\ -i\sqrt{\lambda} V_{\alpha-j}(\lambda) & V_{\alpha-j}(\lambda) \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\] (3.12)
Then the family of transfer matrices $A_\alpha(\lambda, x)$ is given by
\[
\begin{bmatrix}
e^{ix\theta(\lambda)} & 0 \\
0 & e^{-ix\theta(\lambda)}
\end{bmatrix} V_{\alpha-\eta x}(\lambda) = V_\alpha(\lambda) A_\alpha(\lambda, x).
\] (3.13)

They form a monotonic family of $J$-contractive entire matrix functions.

**Proof.** Formulas (3.13) and (3.12) follow from (3.9) and (3.10) as soon as we take into account the representation (3.1) for reproducing kernels. We performed these computations several times, see for details e.g. [20, Appendix]. Note that the modification of the basis functions in the scaling spaces leads to another form of $J$-matrix. Respectively, (3.11) has the form
\[
J - A_\alpha(\lambda, x) J A_\alpha(\lambda, x)^* \geq 0.
\]

Particularly, for the upper corner entry we have here
\[
\left\{ V_\alpha(\lambda) J - A_\alpha(\lambda, x) J A_\alpha(\lambda, x)^* \right\}_{11} = k_\alpha(\lambda, \lambda_0) - e^{i\theta(\lambda)-\theta(\lambda_0)} k_\alpha-\eta x(\lambda, \lambda_0).
\]

The chain property
\[
A_\alpha(\lambda, x_1 + x_2) = A_\alpha(\lambda, x_1) A_{\alpha-\eta_1}(\lambda, x_2)
\]
(3.14)
follows immediately from the representation (3.13).

The fact that $A_\alpha(\lambda, x)$ is an entire matrix function requires again the DCT property. We use the following lemma, which we prove later on.

**Lemma 3.5.** If $\Omega = \mathbb{C} \setminus E$ is of Widom type and DCT holds, then $\Omega_n := \Omega \cap \mathbb{D}_{a_n+b_n}$ is of Widom type and DCT holds in it.

Due to the Wronskian identity $A_\alpha(\lambda, x)$ is holomorphic in $\Omega$. We have to consider the boundary points $\lambda = \xi \pm i0$, $\xi \in E$. Note that for such $\lambda$
\[
V_\alpha(\lambda) = \sqrt{C(\alpha)} \begin{bmatrix} -i\sqrt{\lambda} V_{\alpha-\lambda}(\lambda) & V_{\lambda-\alpha}(\lambda) \\
 i\sqrt{\lambda} V_{\lambda+1}(\lambda) & V_\alpha(\lambda) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\
1 & 0 \end{bmatrix} V_\alpha(\lambda)
\]
and $e^{ix\theta(\lambda)} = e^{-ix\theta(\lambda)}$. Therefore, $A_\alpha(x, \lambda) = A_\alpha(x, \lambda)$ and boundary values at $\lambda = \xi \pm i0$ coincide. Let us write explicitly the entries of the transfer matrix,
\[
A_\alpha(\lambda, x) = \frac{V_{\alpha-\lambda}(\lambda)}{\sqrt{C(\alpha)} C(\alpha - \eta x)} \begin{bmatrix} V_{\alpha-\lambda}(\lambda) & -V_{\alpha}(\lambda) \\
 V_{\lambda}(\lambda) & V_{\alpha+1}(\lambda) \end{bmatrix} \begin{bmatrix} \sqrt{\lambda} e^{ix\theta(\lambda)} V_{\alpha+\eta_1}(\lambda) & e^{ix\theta(\lambda)} V_{\eta_1-\alpha}(\lambda) \\
 -i\sqrt{\lambda} e^{ix\theta(\lambda)} V_{\eta_1-\alpha}(\lambda) & e^{ix\theta(\lambda)} V_{\eta_1-\alpha}(\lambda) \end{bmatrix}.
\]
It is easy to see that the entries of $\sqrt{\lambda} A_\alpha(\lambda, x)$ belong to $E^1_{1\eta_n}(\bar{j})$, $j \equiv j|\pi_1(\Omega_n)$. Applying DCT in this domain we have
\[
A_\alpha(\lambda, x) = \frac{1}{2\pi i} \int_{\partial \Omega_n} \frac{\sqrt{\xi}}{\xi - \lambda} A_\alpha(\xi, x) d\xi = \frac{1}{2\pi i} \int_{2|\xi|=a_n+b_n} A_\alpha(\xi, x) d\xi \frac{d\xi}{\xi - \lambda}.
\]
That is, $A_\alpha(\lambda, x)$ is holomorphic at an arbitrary disk $\mathbb{D}_{a_n+b_n}$. \hfill \Box
Proof of Lemma 3.5. The proof is based on the property (iii) in Theorem 1.5. It is evident that $H^\infty_{\Omega_n}(\hat{\alpha}) \subseteq H^\infty_{\Omega_n}(\alpha')$, $\hat{\alpha} = \alpha|_{\pi_1(\Omega_n)}$. Further, let $\hat{\alpha}_m \to 1_{\pi_1(\Omega_n)}$. We define $\alpha_m$ such that for all generators $\gamma_j$ of $\pi_1(\Omega)$

$$\alpha_m(\gamma_j) = \hat{\alpha}_m(\gamma_j) \text{ if } \gamma_j \in \pi_1(\Omega_n) \text{ and } \alpha_m(\gamma_j) = 1 \text{ otherwise.}$$

Evidently $\alpha_m \to 1_{\pi_1(\Omega)}$. For the minimizer $\hat{W}_{\alpha_m} \in H^\infty_{\Omega_m}$ we have

$$\hat{W}_{\alpha_m}(\lambda^*) \geq W_{\alpha_m}(\lambda^*) \to 1 \text{ as } m \to \infty.$$ 

\[\square\]

Remark 3.6. If DCT fails, then remaining singularities on $E$ for a transfer matrix of the form (3.13) are possible [23].

4 Weyl-Titchmarsh functions

Matrices

$$A_\alpha(\lambda, x)^{-1} = \begin{bmatrix} A_\alpha(\lambda, x) & C_\alpha(\lambda, x) \\ B_\alpha(\lambda, x) & D_\alpha(\lambda, x) \end{bmatrix}$$

(4.1)

which were defined in the previous section, form a monotonic family of $\mathcal{J}$ expanding matrix functions in $\mathbb{C}_+$. For an arbitrary such family the Weyl circle is formed by values

$$m = \frac{A(\lambda, x)w + B(\lambda, x)}{C(\lambda, x)w + D(\lambda, x)}, \quad w \in \mathbb{R} \cup \{\infty\},$$

where $\lambda \in \mathbb{C}_+$ and $x \in \mathbb{R}_+$ are fixed. These circles are nesting in the upper half plane as $x$ increases, and in the limit converge either to a circle or to a point. Due to the explicit formula (3.12) we have the limit point case.

Theorem 4.1. For an arbitrary Nevanlinna class function $w(\lambda)$ the following limit exists

$$m_+^\alpha(\lambda) := \lim_{x \to \infty} \frac{A_\alpha(\lambda, x)w(\lambda) + B_\alpha(\lambda, x)}{C_\alpha(\lambda, x)w(\lambda) + D_\alpha(\lambda, x)} = i\sqrt{\lambda} \frac{V_{\alpha+1}(\lambda)}{V_\alpha(\lambda)}. \quad (4.2)$$

Moreover,

$$m_-^\alpha(\lambda) = -\overline{m_+^\alpha(\lambda)} = i\sqrt{\lambda} \frac{V_{-\alpha}(\lambda)}{V_{j\alpha}(\lambda)}, \quad \text{for a.e. } \lambda \in E, \quad (4.3)$$

and

$$R^\alpha(\lambda) := \frac{1}{m_+^\alpha(\lambda) + m_-^\alpha(\lambda)} = \frac{i\lambda^{\alpha}(\alpha)}{\sqrt{\lambda}} \prod_{j \geq 1} \frac{\lambda - \lambda_j}{(\lambda_s - \lambda_j)\sqrt{(\lambda_s - a_j)(\lambda_s - b_j)}} \prod_{j \geq 1} \frac{\lambda^* - \lambda_j}{(\lambda_* - \lambda_j)\sqrt{(\lambda_* - a_j)(\lambda_* - b_j)}}, \quad (4.4)$$

where $\alpha = \mathcal{A}(D), D \in \mathcal{D}(E)$.

Note that $m_{\pm}$ belongs to the Stieltjes, we will call them Weyl-Titchmarsh functions. The property (4.3) is reflectionless (on the set $E$).
Proof of Theorem 4.1. (4.2) follows directly from the definition (3.12). (4.3) follows from Lemma 2.3. (4.4) is a consequence of the Wronskian identity and the definition of $V_{\alpha(D)}$.

In a sense we will invert Theorem 4.1.

Definition 4.2. Let $E = \mathbb{R}_+ \cup \bigcup_{j \geq 1} (a_j, b_j)$ be of positive Lebesgue measure. We say that $m_+ \in S$ belongs to the set $m(E)$ if this function is reflectionless on $E$, that is, there exists $m_- \in S$ such that $m_-(\lambda) = -m_+(\lambda)$ for a.e. $\lambda \in E$, and the both functions

$$R_0(\lambda) = -\frac{1}{m_+(\lambda) + m_-(\lambda)}, \quad R_i(\lambda) = \frac{m_+(\lambda)m_-(\lambda)}{m_+(\lambda) + m_-(\lambda)}$$

are holomorphic in $\Omega = \mathbb{C} \setminus E$, being extended in the lower half plane due to the symmetry principle $R_i(\bar{\lambda}) = R_i(\lambda)$, $\lambda \in \Omega$.

Note that a function of the Nevanlinna class is defined uniquely by its limit values on a set of positive Lebesgue measure. Thus $m_+$ defines $m_-$ uniquely.

Further, automatically, $R_i$ belongs to the Nevanlinna class. Therefore they can be restored (up to positive multipliers) by their arguments on the real axis due to (1.16). The boundary values of $R_i$ on $E$ are pure imaginary, that is, $\arg R_i(\xi) = \pi/2$, a.e. $\xi \in E$. On the complement $\mathbb{R} \setminus E$ they are real. Since $R_i$ is increasing in each gap, there is a unique point $\lambda_i^{(j)} \in [a_j, b_j]$, $j \geq 0$, such that $\arg R_i(\xi) = 0$ in $(\lambda_i^{(j)}, b_j)$ (respectively, $\arg R_i(\xi) = \pi$ in $(a_j, \lambda_i^{(j)})$; one of these sets could be empty). As the result we have

$$R_i(\lambda) = R_i(\lambda_s)\frac{\lambda - \lambda_i^{(j)}}{\lambda_s - \lambda_i^{(j)}} \sqrt[\lambda_s]{\prod_{j \geq 1} \frac{(\lambda - \lambda_i^{(j)})\sqrt{(\lambda_s - a_j)(\lambda_s - b_j)}}{(\lambda - a_j)(\lambda - b_j)}}$$

(4.6)

where $\lambda_s < 0$ is a normalization point, $\lambda_s \neq \lambda_i^{(j)}$, $-\infty \leq \lambda_i^{(j)} \leq 0$. That is, $R_i(\lambda)$ is completely defined by the collections of $\{\lambda_i^{(j)}\}_{j \geq 0}$ and $R_i(\lambda_s)$.

Definition 4.3. If $m_+ \in m(E)$ meets the additional conditions

(a) $\lambda_0^{(0)} = -\infty$ and $\lambda_0^{(1)} = 0$,

(b) along the negative half axis $\lim_{\lambda \to -0} m_+(\lambda) = 0$,

we say that $m_+ \in m_0(E)$.

Note that if (a) holds, then the increasing function $m_+$ is bounded on the positive half axis, that is, the limit exists, but not necessarily 0. Thus (b) is a certain (additive) normalization condition.

Going back to Widom domains with DCT we have the following important property.
Theorem 4.4 (see [16, 22]). Let \( \Omega = \mathbb{C} \setminus E \) be of Widom type and DCT hold. Assume that \( R_i(\lambda) \) is of the form (4.6) corresponding to an arbitrary collection of \( \lambda_j^{(i)} \in [a_j, b_j], -\infty \leq \lambda_0^{(i)} \leq 0 \). Then the measures, corresponding to the Nevanlinna functions \( \pm R_i(\lambda)^{\pm 1} \) in their integral representations (1.15), are absolutely continuous on \( E \). In particular, this implies that for \( \lambda_0^{(0)} = -\infty \)

\[
\lim_{\lambda \to -\infty} R_0(\lambda) = \infty, \quad \lim_{\lambda \to -\infty} R_0(\lambda) = 0.
\] (4.7)

Corollary 4.5. If \( \Omega \) is of Widom type and DCT holds then \( m_+ \in m_0(E) \) implies \( m_- \in m_0(E) \).

Proof. By (4.7).

Remark 4.6. Once again we note importance of DCT property. A singular component for a measure, associated to a reflectionless function, is possible if DCT fails in a Widom domain, as well as \( m_- \) not necessarily belongs to \( m_0(E) \) for a certain \( m_+ \in m_0(E) \).

Theorem 4.7. As soon as DCT holds, one can parametrize the set \( m_0(E) \) by the following collection of data \( \{R_0(\lambda_*) D\} \in \mathbb{R}_+ \times D(E) \).

Proof. First, we recall that the Nevanlinna functions \( w \), which can be extended by the symmetry through \( R^- \) in the lower half plane and such that \( \lim_{\lambda \to -0} w(\lambda) = 0 \), allows the following representation

\[
w(\lambda) = a\lambda + \int_{\mathbb{R}_+} \frac{\lambda}{\xi - \lambda} d\sigma(\xi), \quad a \geq 0, \quad \int_{\mathbb{R}_+} \frac{d\sigma(\xi)}{1 + \xi} < \infty.
\] (4.8)

Indeed, we can represent it in the form

\[
w(\lambda) = w(\lambda_0) + a(\lambda - \lambda_0) + \int_{\mathbb{R}_+} \left( \frac{1}{\xi - \lambda} - \frac{1}{\xi - \lambda_0} \right) d\sigma(\xi) \quad a \geq 0, \quad \int_{\mathbb{R}_+} \frac{d\sigma(\xi)}{1 + \xi^2} < \infty,
\]

where \( \lambda_0 < 0 \). Then, pass to the limit as \( \lambda_0 \to -0 \). We get (4.8) with \( d\sigma = \frac{1}{\xi} d\tilde{\sigma} \geq 0 \).

Let now \( m_+ \in m_0(E) \). Then we have the collection \( R_0(\lambda_*) \) and \( \lambda_j \in [a_j, b_j] \) such that

\[
R_0(\lambda) = R_0(\lambda_*) \sqrt{\lambda \prod_{j \geq 1} (\lambda - \lambda_j)(\lambda_* - \lambda_j) \sqrt{(\lambda_* - a_j)(\lambda_* - b_j)}}.
\] (4.9)

On the other hand, by (4.8)

\[
\frac{-1}{R_0(\lambda)} = a\lambda + \int_{E} \frac{\lambda}{\xi - \lambda} d\sigma_0(\xi) + \sum_{\lambda_j \in (a_j, b_j)} \lambda \delta_j(\lambda) \frac{\lambda_j}{\lambda_j - \lambda}.
\]

Since, there is no mass points on \( E \), including infinity, we have \( a = 0 \). The measure \( d\sigma_0 \) is absolutely continuous. Due to

\[
\frac{-1}{R_0(\lambda)} = m_+(\lambda) + m_-(\lambda)
\]

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Comparing (4.2) and (4.10) we get that, in the sense of Theorem 4.7, to Remark 4.8.

We have to distribute this measure as $d\sigma = d\sigma_+ + d\sigma_-$. That is, both measures are absolutely continuous and due to $\text{Im} m_+ (\xi) = \text{Im} m_- (\xi)$, $\xi \in E$, they are equal. With respect to $\sigma_j^{(0)}$, if $\sigma_j^{(0)} = \sigma_j^+ + \sigma_j^-$ and both values are positive, then $R_1 (\lambda)$ has a pole in $\lambda_j$, see (4.5). We write $\epsilon_j = \pm 1$ if $\sigma_j^{(0)} = \sigma_j^\pm$.

As the result we have

$$m_\pm (\lambda) = \frac{1}{2} \left( -\frac{1}{R_0 (\lambda)} \pm \sum_{\lambda_j \in (a_j, b_j)} \frac{\lambda \sigma_j^{(0)} \epsilon_j}{\lambda_j - \lambda} \right). \tag{4.10}$$

Vice versa, for an arbitrary collection from $R_+ \times D (E)$, we define $R_0 (\lambda)$ by (4.9) and $m_+ (\lambda)$ by (4.10). We have $m_+ (\lambda) \in m_0 (E)$. \qed

Remark 4.8. Comparing (4.2) and (4.10) we get that, in the sense of Theorem 4.7 to the function $m_+^* \varepsilon$ corresponds exactly that divisor $D$ for which $A (D) = \alpha$. Comparing (4.4) and (4.9) we have $\sqrt{\lambda_0} R_0 (\lambda_*) = i \mathcal{C} \lambda_* (\alpha)$.

Lemma 4.9. For a function $m_+ \in m_0 (E)$ there exits a unique representation

$$m_+ (\lambda) = i \sqrt{\lambda} \frac{V_2 (\lambda)}{V_1 (\lambda)}, \tag{4.11}$$

where $V_1, V_2 \in \mathcal{N}_+ (\Omega)$ with mutually simple inner parts, which obey the Wronskian identity

$$\sqrt{|\lambda_*| R_0 (\lambda_*)} \det \begin{bmatrix} V_2 (\lambda) & V_1 (\lambda) \\ -V_2 (\lambda) & V_1 (\lambda) \end{bmatrix} = 1, \quad \lambda \in E.$$

Proof. We use essentially Theorem D [19], according to which $m_+$ is of bounded characteristic in $\Omega$ and has no singular component in its inner part. For $\lambda = \xi \pm i 0$, $\xi \in E$, we have

$$\frac{m_+ (\lambda) - m_+ (\lambda)}{i} = \frac{m_+ (\lambda) + m_- (\lambda)}{i} = \frac{i}{R_0 (\lambda)}.$$

Having in mind the Wronskian identity, we get

$$-\frac{i}{R_0 (\lambda_*)} \sqrt{\lambda} \frac{1}{\lambda_*} |V_1 (\lambda)|^2 = \frac{i}{R_0 (\lambda_*)} \sqrt{\lambda} \frac{1}{\lambda_*} \prod_{j \geq 1} (\lambda_* - \lambda_j) \sqrt{(\lambda_* - a_j)(\lambda_* - b_j)}.$$

That is, $|V_1 |^2 = O (\lambda, D)$, which define uniquely the outer part of $V_1$. By (4.11) its inner part is the Blaschke product $\prod_{j \geq 1} \Phi_{\lambda_j}$. Thus, $V_1 (\lambda) = V (\lambda, D)$, see the definition (2.1), and $i \sqrt{\lambda} V_2 (\lambda) = m_+ (\lambda) V (\lambda, D)$. Moreover, since on $E$

$$m_+ (\lambda) m_- (\lambda) = - |m_+ (\lambda)|^2 = \frac{R_1 (\lambda)}{R_0 (\lambda)},$$

we have

$$\lambda |V_2 (\lambda)|^2 = \frac{m_+ (\lambda_*) m_- (\lambda_*)}{-\lambda_*} \prod_{j \geq 1} \frac{\lambda - \lambda_j^{(i)}}{\lambda_* - \lambda_j^{(i)}} \frac{\lambda - \lambda_j}{\lambda_* - \lambda_j} |V_1 (\lambda)|^2.$$

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We define $\epsilon^{(1)}_j$ such that $\prod_{j=1}^{1+(1)} \Phi^{(1)}_{\lambda_j}$ is the numerator of the inner part of $m_+(\lambda)$, then

$$V_2(\lambda) = \sqrt{\frac{m_+(\lambda_*) m_-(\lambda_*)}{-\lambda_*}} V(\lambda, D^{(1)})$$

and $A(D^{(1)}) = \alpha + j$.

\[\Box\]

5 Canonical systems

In this section we will get consequences of the Fourier representation, Theorem 1.7.

**Lemma 5.1** (Main lemma). For all $x$ the we can pass to the limits as $\lambda \to -0$ in the ratio $v_{\alpha-\eta x}(\lambda)/v_{\alpha}(\lambda)$, see (1.22), (1.23). Moreover, the following limit exists as $\lambda$ approach infinity along the negative half axis

$$\lim_{\lambda \to -\infty} \frac{1}{\theta(\lambda)} \int_0^x m_{\alpha-\eta \xi}(\lambda) \frac{e^{2\xi \text{Im} \theta_\lambda d \xi}^{\alpha + j} (\xi)}{\epsilon(\alpha + j - \eta \xi)} = x.$$  \hspace{1cm} (5.1)

**Proof.** Let

$$\epsilon(\alpha) = \epsilon(\alpha + j) = V_\alpha(\lambda_*) V_{\alpha + j}(\lambda_*) C(\alpha) = \sqrt{\frac{\lambda_*}{i}} \left(k^\alpha(\lambda_*, \lambda_*) + k^{\alpha + j}(\lambda_*, \lambda_*) \right).$$  \hspace{1cm} (5.2)

Using (3.6), from (2.6) we have an integral relation

$$i\epsilon(\alpha - \eta x) v_{\alpha-\eta x}(\lambda) e^{ix(\theta(\lambda) - \theta_\lambda)} = \sqrt{\lambda} \int_x^\infty e^{i \xi (\theta(\lambda) - \theta_\lambda)} v_{\alpha + j - \eta \xi}(\lambda) d \xi^{\alpha + j}(\xi)$$

$$+ \sqrt{\lambda_*} \int_x^\infty e^{i \xi (\theta(\lambda) - \theta_\lambda)} v_{\alpha - \eta \xi}(\lambda) d \xi^{\alpha}(\xi).$$  \hspace{1cm} (5.3)

In terms of differentials we get

$$-i d \log \left( \epsilon(\alpha - \eta x) v_{\alpha-\eta x}(\lambda) e^{ix(\theta(\lambda) - \theta_\lambda)} \right) = \sqrt{\lambda} \frac{v_{\alpha + j - \eta x}(\lambda)}{v_{\alpha - \eta x}(\lambda)} \frac{e^{2\xi \text{Im} \theta_\lambda d \xi}^{\alpha + j}(x)}{\epsilon(\alpha + j - \eta x)}$$

$$+ \sqrt{\lambda_*} \frac{e^{2\xi \text{Im} \theta_\lambda d \xi}^{\alpha}(x)}{\epsilon(\alpha - \eta x)}.$$  \hspace{1cm} (5.4)

Integrating on the interval $(0, \ell)$, we obtain

$$(\theta(\lambda) - \theta_\lambda) \ell - i \log \frac{\epsilon(\alpha - \eta \ell) v_{\alpha - \eta \ell}(\lambda)}{\epsilon(\alpha) v_{\alpha}(\lambda)}$$

$$= \int_0^\ell \sqrt{\lambda} \frac{v_{\alpha + j - \eta x}(\lambda)}{v_{\alpha - \eta x}(\lambda)} \frac{e^{2\xi \text{Im} \theta_\lambda d \xi}^{\alpha + j}(x)}{\epsilon(\alpha + j - \eta x)} + \int_0^\ell \sqrt{\lambda_*} \frac{e^{2\xi \text{Im} \theta_\lambda d \xi}^{\alpha}(x)}{\epsilon(\alpha - \eta x)}.$$  \hspace{1cm} (5.4)
For \( \lambda = \lambda_* \) we have

\[
2\Im \theta_* - \log \frac{c(\alpha - \eta \ell)}{c(\alpha)} = \sqrt{\lambda_*} \int_0^\ell \frac{e^{2\xi \Im \theta_*} d\xi^{\alpha + 1}(x)}{\xi^{\alpha + 1}(x)} + \sqrt{\lambda_*} \int_0^\ell \frac{e^{2\xi \Im \theta_*} d\xi^\alpha(x)}{\xi^\alpha(x)}.
\] (5.5)

Passing to the limit in (5.4) as \( \lambda \to 0 \), we get

\[
\lim_{\lambda \to 0} \log \frac{v_{\alpha - \eta \ell}(\lambda)}{v\alpha(\lambda)} = x \Im \theta_* - \frac{\sqrt{\lambda_*}}{i} \int_0^\ell \frac{e^{2\xi \Im \theta_*} d\xi^{\alpha + 1}(\xi)}{\xi^{\alpha + 1}(\xi)}.
\] Using (5.5), we obtain

\[
\lim_{\lambda \to 0} \log \frac{v_{\alpha - \eta \ell}(\lambda)}{v\alpha(\lambda)} = x \Im \theta_* + \int_0^\ell \frac{de^{-2\xi \Im \theta_*} k^{\alpha + 1 - \eta \ell}(\lambda_*, \lambda_*)}{e^{-2\xi \Im \theta_*} k^{\alpha + 1 - \eta \ell}(\lambda_*, \lambda_*)}.
\]

which we can bring to a more symmetric form

\[
= x \Im \theta_* + \frac{1}{2} \log e^{-2\xi \Im \theta_*} (k^{\alpha - \eta \ell}(\lambda_*, \lambda_*) + k^{\alpha + 1 - \eta \ell}(\lambda_*, \lambda_*)) |_0^\ell
\]

\[
- \frac{1}{2} \int_0^\ell \frac{d e^{-2\xi \Im \theta_*} (k^{\alpha - \eta \ell}(\lambda_*, \lambda_*) - k^{\alpha + 1 - \eta \ell}(\lambda_*, \lambda_*))}{e^{-2\xi \Im \theta_*} (k^{\alpha - \eta \ell}(\lambda_*, \lambda_*) + k^{\alpha + 1 - \eta \ell}(\lambda_*, \lambda_*))}
\]

\[
= \log \sqrt{\frac{k^{\alpha - \eta \ell}(\lambda_*, \lambda_*) + k^{\alpha + 1 - \eta \ell}(\lambda_*, \lambda_*)}{k^{\alpha}(\lambda_*, \lambda_*) + k^{\alpha + 1}(\lambda_*, \lambda_*)}}
\]

\[
+ \frac{1}{2} \int_0^\ell \frac{d e^{-2\xi \Im \theta_*} (k^{\alpha + 1 - \eta \ell}(\lambda_*, \lambda_*) - k^{\alpha - \eta \ell}(\lambda_*, \lambda_*))}{e^{-2\xi \Im \theta_*} (k^{\alpha - \eta \ell}(\lambda_*, \lambda_*) + k^{\alpha + 1 - \eta \ell}(\lambda_*, \lambda_*))}.
\]

Thus, (122), (123) are proved.

The second limit (5.1) is an essentially more delicate question. In fact, we again pass to the limit in (5.4) and claim that (for any \( \alpha \in \pi_1(\Omega) \))

\[
\lim_{\lambda \to -\infty} \frac{v\alpha(\lambda)}{\theta(\lambda)} = 0.
\]

But, this is exactly the claim of Theorem 2 and Theorem 3 in [21].

Lemma 5.2. Let

\[
\epsilon_\alpha(x) = \exp \left\{ \frac{1}{2} \int_0^\ell \frac{d e^{-2\xi \Im \theta_*} (k^{\alpha + 1 - \eta \ell}(\lambda_*, \lambda_*) - k^{\alpha - \eta \ell}(\lambda_*, \lambda_*))}{e^{-2\xi \Im \theta_*} (k^{\alpha + 1 - \eta \ell}(\lambda_*, \lambda_*) + k^{\alpha - \eta \ell}(\lambda_*, \lambda_*))} \right\}
\]

and

\[
\tilde{\epsilon}_\alpha(\lambda, x) = \epsilon_\alpha(x) \sqrt{c(\alpha - \eta x)} v_{\alpha - \eta x}(\lambda) e^{ix\theta(\lambda)}.
\] (5.6)

Then

\[
df_\alpha(\lambda, x) = i\sqrt{\lambda} \tilde{\epsilon}_\alpha(\lambda, x) d\tau^{\alpha + 1}(x), \quad d\tau^{\alpha + 1}(x) = \frac{i e^{2\xi \Im \theta_*} d\xi^{\alpha + 1}(x)}{\sqrt{\lambda} \tau^{\alpha + 1}(x)}.
\] (5.7)
Proof. We define $a_\alpha(x)$ by

$$ic(\alpha - \eta x)e^{ix(\theta_1 - \tilde{\theta}_1)}da_\alpha(x) = a_\alpha(x)\sqrt{\lambda_s}d\mathcal{X}_\alpha(\xi).$$  

(5.8)

With this multiplier, we get

$$d\left(a_\alpha(x)ic(\alpha - \eta x)v_{\alpha - \eta x}(\lambda)e^{ix(\theta_1 - \tilde{\theta}_1)}\right)$$

$$= ic(\alpha - \eta x)v_{\alpha - \eta x}(\lambda)e^{ix(\theta_1 - \tilde{\theta}_1)}da_\alpha(x) + a_\alpha(x)d\left(ic(\alpha - \eta x)v_{\alpha - \eta x}(\lambda)e^{ix(\theta_1 - \tilde{\theta}_1)}\right),$$

and by (5.3),

$$= ic(\alpha - \eta x)v_{\alpha - \eta x}(\lambda)e^{ix(\theta_1 - \tilde{\theta}_1)}da_\alpha(x) - a_\alpha(x)\sqrt{\lambda_s}e^{ix(\theta_1 - \tilde{\theta}_1)}v_{\alpha - \eta x}(\lambda)d\mathcal{X}_\alpha(x)$$

$$- a_\alpha(x)\sqrt{\lambda_s}e^{ix(\theta_1 - \tilde{\theta}_1)}v_{\alpha + ij - \eta x}(\lambda)d\mathcal{X}_{\alpha + i}(x)$$

$$= -a_\alpha(x)\sqrt{\lambda_s}e^{ix(\theta_1 - \tilde{\theta}_1)}v_{\alpha + ij - \eta x}(\lambda)d\mathcal{X}_{\alpha + i}(x).$$  

(5.9)

We integrate (5.8)

$$\log a_\alpha(x) = \frac{\sqrt{\lambda_s}}{i} \int_0^x \frac{e^{2xIm\theta_1}d\mathcal{X}_\alpha(\xi)}{c(\alpha - \eta x)}$$

$$= xIm\theta_1 - log \sqrt{\frac{k^{\alpha - \eta x}(\lambda_s,\lambda_s) + k^{\alpha + i - \eta x}(\lambda_s,\lambda_s)}{k^\alpha(\lambda_s,\lambda_s) + k^{\alpha + i}(\lambda_s,\lambda_s)}}$$

$$+ \frac{1}{2} \int_0^x \frac{d e^{-2xIm\theta_1}(k^{\alpha - \eta x}(\lambda_s,\lambda_s) - k^{\alpha - \eta x}(\lambda_s,\lambda_s))}{e^{-2xIm\theta_1}(k^{\alpha - \eta x}(\lambda_s,\lambda_s) + k^{\alpha + i - \eta x}(\lambda_s,\lambda_s))},$$

to obtain explicitly

$$a_\alpha(x) = e^{xIm\theta_1} \left(\frac{c(\alpha - \eta x)}{c(\alpha)}\right)^{-1/2} c_\alpha(x).$$

With this expression, going back to (5.9), we have

$$d\left(c_\alpha(x)\sqrt{c(\alpha - \eta x)}v_{\alpha - \eta x}(\lambda)e^{ix\theta(\lambda)}\right) = i\sqrt{\lambda}c_\alpha(x)v_{\alpha + i - \eta x}(\lambda)e^{ix\theta(\lambda)}\frac{e^{2xIm\theta_1}d\mathcal{X}_{\alpha + i}(x)}{\sqrt{c(\alpha - \eta x)}}.$$  

Now, we recall that

$$\frac{\sqrt{\lambda_s}}{i} \gamma_\alpha(x)^2 = c_\alpha(x)^2 c(\alpha - \eta x),$$

as it was defined in (1.23), see also (5.2). Thus, using the symmetry properties

$$c_{\alpha + i}(x) = \frac{1}{c_\alpha(x)}, \quad c(\alpha + i) = c(\alpha),$$

we obtain (5.7).
Proof of Theorem 1.10. We note that

\[ e_\alpha(x) \sqrt{C(x - \eta x)} v_{\alpha - \eta x}(\lambda) = \sqrt{C(\alpha - \eta x)} e_\alpha(x) \sqrt{\frac{V_{\alpha + \eta x}(\lambda_x)}{V_{\alpha - \eta x}(\lambda_x)}} \sqrt{V_{\alpha + \eta x}(\lambda_x)} V_{\alpha - \eta x}(\lambda). \]

According to this remark, we modify slightly the transfer matrix \( A_\alpha(\lambda, x) \), see (3.13), making rescaling of the basis vectors in the scaling subspaces

\[
\begin{bmatrix}
  e^{ix\theta(\lambda)} & 0 \\
  0 & e^{-ix\theta(\lambda)}
\end{bmatrix} V_{\alpha - \eta x}(\lambda) \begin{bmatrix}
  e_{\alpha + j}(x) \sqrt{\frac{V_{\alpha - \eta x}(\lambda_x)}{V_{\alpha + \eta x}(\lambda_x)}} & 0 \\
  0 & e_\alpha(x) \sqrt{\frac{V_{\alpha + \eta x}(\lambda_x)}{V_{\alpha - \eta x}(\lambda_x)}}
\end{bmatrix} = V_{\alpha}(\lambda) \begin{bmatrix}
  \sqrt{\frac{V_{\alpha}(\lambda_x)}{V_{\alpha + \eta x}(\lambda_x)}} & 0 \\
  0 & \sqrt{\frac{V_{\alpha + \eta x}(\lambda_x)}{V_{\alpha}(\lambda_x)}}
\end{bmatrix} A_\alpha(\lambda, x). \quad (5.10)
\]

In this normalization we have

\[
[i \sqrt{\lambda} f_{\alpha + j}(\lambda, x) \ f_\alpha(\lambda, x)] = [i \sqrt{\lambda} f_{\alpha + j}(\lambda, 0) \ f_\alpha(\lambda, 0)] A(\lambda, x).
\]

Due to Lemma 5.2,

\[
d [i \sqrt{\lambda} f_{\alpha + j}(\lambda, x) \ f_\alpha(\lambda, x)] = [-\lambda f_\alpha(\lambda, x) d \tau^\alpha(x) \ i \sqrt{\lambda} f_{\alpha + j}(\lambda, x) d \tau^{\alpha + j}(x)] \begin{bmatrix}
  0 & 1 \\
  -1 & 0
\end{bmatrix}
\]

\[= - [i \sqrt{\lambda} f_{\alpha + j}(\lambda, x) \ f_\alpha(\lambda, x)] \begin{bmatrix}
  d \tau^{\alpha + j}(x) & 0 \\
  0 & \lambda d \tau^\alpha(x)
\end{bmatrix}.
\]

Therefore we get

\[
d A(\lambda, x) = -A(\lambda, x) \begin{bmatrix}
  d \tau^{\alpha + j}(x) & 0 \\
  0 & \lambda d \tau^\alpha(x)
\end{bmatrix},
\]

that is, (1.24).

Respectively, for the new Weyl-Titchmarsh function we have

\[ m^\alpha_+(\lambda) = \frac{V_{\alpha}(\lambda_x)}{V_{\alpha + \eta x}(\lambda_x)} m^\alpha_+(\lambda) = i \sqrt{\lambda} \frac{v_{\alpha + j}(\lambda)}{v_\alpha(\lambda)}, \]

which, according to (4.2), means (1.25) and this is the last claim of Theorem 1.10

**Proof of Corollary 1.11.** To prove the first statement we use (5.10). We have

\[
\frac{1}{C} \| V_{\alpha}(\lambda) \|^{-1} \| V_{\alpha - \eta x}(\lambda) \|^{-1} e^{xM(\lambda)} \leq \| A_\alpha(\lambda, x) \| \leq C \| V_{\alpha - \eta x}(\lambda) \| \| V_{\alpha}(\lambda) \|^{-1} e^{xM(\lambda)}.
\]

By Theorems 2 and 3 [21] for an arbitrary \( \alpha \in \pi_1(\Omega)^* \) we have

\[
\lim_{\lambda \to -\infty} \frac{\log \| V_{\alpha}(\lambda) \|^{\pm 1}}{\mathcal{M}(\lambda)} = 0.
\]
Therefore we get (1.26).

To prove the second claim, let us rewrite (5.1) into the form

\[
\lim_{\lambda \to -\infty} \sqrt{\frac{1}{\theta(\lambda)}} \int_0^\ell \frac{v_{\alpha+i-\eta x}(\lambda)}{v_{\alpha-\eta x}(\lambda)} e^{2x\text{Im} \theta} d\alpha^{\alpha+1}(x) = 1.
\]

Recall that \( c(\alpha) = c(\alpha+j) \). Let

\[ d\kappa^\alpha(x) = \frac{e^{2x\text{Im} \theta} \alpha}{c(\alpha-\eta x)} (d\alpha^{\alpha}(x) + d\alpha^{\alpha+1}(x)) \]

and we define the densities

\[ \rho^\alpha(x) = \frac{e^{2x\text{Im} \theta} \alpha}{c(\alpha-\eta x)} d\kappa^\alpha(x), \quad \rho^{\alpha+1}(x) = \frac{e^{2x\text{Im} \theta} \alpha}{c(\alpha+j-\eta x)} d\kappa^\alpha(x). \]

Then, we have

\[
1 = \lim_{\lambda \to -\infty} \sqrt{\frac{1}{\theta(\lambda)}} \frac{1}{2\ell} \int_0^\ell \left( \frac{v_{\alpha+i-\eta x}(\lambda)}{v_{\alpha-\eta x}(\lambda)} \rho^{\alpha+1}(x) + \frac{v_{\alpha-\eta x}(\lambda)}{v_{\alpha+i-\eta x}(\lambda)} \rho^\alpha(x) \right) d\kappa^\alpha(x) \\
\geq \lim_{\lambda \to -\infty} \sqrt{\frac{1}{\theta(\lambda)}} \frac{1}{\ell} \int_0^\ell \sqrt{\rho^\alpha(x)\rho^{\alpha+1}(x)} d\kappa^\alpha(x).
\]

By (1.27)

\[
\frac{1}{\ell} \int_0^\ell \sqrt{\rho^\alpha(x)\rho^{\alpha+1}(x)} d\kappa^\alpha(x) = 0 \quad \text{and} \quad \rho^\alpha(x)\rho^{\alpha+1}(x) = 0 \text{ for a.e. } x \text{ w.r.t. } \kappa^\alpha,
\]

in other words the measures are mutually singular.

\( \square \)

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