Mixing Matrix of Quarks Having Natural Twisting in $E_6$ Grand Unified Model

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Abstract

Mass matrix of quarks is studied in the Supersymmetric $E_6$ Grand Unified Theory (GUT). The fundamental representation $27$ in $E_6$ which corresponds to one generation contains two sets of $5^*$’s in SU(5), so that there are six flavors of lepton doublets and right-handed down-quark triplets. It is known that the twisting ( interchange) among the $5^*$ representations may reproduce the observed quark and lepton mixing matrices. If $E_6$ is directly broken down to $SU(3)_C \times SU(2)_L \times U(1)_Y$ and two extra U(1)’s, the extra 3 sets of down-type quarks do not necessarily decouple from the quark mass matrix, under the appropriate choice of the U(1) flavor charges on the quark and Higgs fields. Then, by diagonalizing the $6 \times 6$ down-quark mass matrix, we find a certain set of parameters, in which the twisting between a pair of the right-handed down-quarks occurs naturally, reproducing the reasonable values for the $d$-, $s$- and $b$-quark masses, and for $V_{ud}$, $V_{us}$, $V_{cd}$, $V_{cs}$ and $V_{tb}$ of the CKM matrix elements. As a by-product, one vector-like down-quark appears at the experimentally accessible TeV scale.
1 Introduction

The discovery of the neutrino masses and large mixing angles at Super-Kamiokande \[1\] triggers the burst of investigations on the masses and mixing matrices of quarks and leptons. From a view of the grand unified theory (GUT), one of the most important tasks is to understand the different structure of the flavor mixing matrix between quarks and leptons. It is known that the quark mixing, which is described by the Cabibbo-Kobayashi-Maskawa (CKM) matrix \( V \)[2], is small, i.e., \( V_{12} \approx \lambda \approx 0.22 \) (Cabibbo angle), \( V_{23} = {\cal O}(\lambda^2) \), and \( V_{13} = {\cal O}(\lambda^3) \). On the other hand, the neutrino oscillation experiments tell us that the lepton flavor mixing which is expressed by the Maki-Nakagawa-Sakata (MNS) matrix \( U \)[3] may be large, probably being “doubly maximal”: \( U_{12} \approx U_{23} \approx 1/\sqrt{2} \)[4].

Bando and Kugo have proposed an idea which is possible to explain the flavor mixing matrices based on the Supersymmetric (SUSY) \( E_6 \) GUT, in which one family of quarks and leptons belongs to a fundamental representation \( 27 \) of \( E_6 \)[5].

The \( 27 \) representation is decomposed in terms of \( \text{SO}(10) \) and \( \text{SU}(5) \) as follows:

\[
27 = 16 + 10 + 1, \quad \text{[SO}(10)]
\]
\[
= (10 + 5^\ast + 1) + (5 + 5^\ast) + 1, \quad \text{[SU}(5)]
\]

As is well known, \( 10 \) of \( \text{SU}(5) \) includes the left-handed down-type quarks \( d_L \) as well as the left- and right-handed up-type quarks \( u_L \) and \( u_R \), while the left-handed leptons and the right-handed down-type quarks are usually assigned to \( 5^\ast \). It is interesting that, in \( E_6 \) GUT, there are two sets of \( 5^\ast \) — \( (16, 5^\ast) \) and \( (10, 5^\ast) \), where \( 16 \) and \( 10 \) denote the representations in \( \text{SO}(10) \). The representation \( (16, 5^\ast) \) is “usually” identified as the \( 5^\ast \) representation in the ordinary \( \text{SU}(5) \) GUT, which consists of \( (d_R^c, e_L, \nu_L) \). On the other hand, the representation \( (10, 5^\ast) \), which consists of \( (D_R, E_L, N_L) \), is “usually” expected to decouple from the \( \text{SU}(5) \) GUT since it has a gauge invariant mass term associated with the \( (10, 5) \) representation, \( (D_R, E_L^c, N_L^c) \). But, from the viewpoint of the Standard Model (SM), both \( d_R^c \) in \( (16, 5^\ast) \) and \( D_R \) in \( (10, 5^\ast) \) could be candidates of the light down-type quarks because they have the same quantum number under the SM gauge group, \( \text{SU}(3)_C \times \text{SU}(2)_L \times \text{U}(1)_Y \). Then, there is a freedom to interchange (called “twisting”) between two \( 5^\ast \) representations if \( (D_R, E_L, N_L) \) is not much heavier than \( (d_R^c, e_L, \nu_L) \). The consequence of the twisting in the 3rd generation has been studied in ref. \[5\].

The flavor mixing matrices of quarks and leptons are defined from the charged current interactions in their mass eigenstates. Then, the twisting between \( d_R^c \) and
$D_R$ does not affect the quark flavor mixing because they do not couple to the $W$ boson. However, since the leptons embedded in $5^*$ are left-handed, the twisting between $(\tau_L, \nu_{\tau L})$ and $(E_L, N_L)_3$ affects the charged current interactions. Then, the mixing matrix of leptons, especially the related part with the 3rd generation, can be drastically changed. As a consequence, one of the double maximal mixings, $U_{23}$, is naturally derived [5]. The other interesting twisting mechanisms have been proposed some time ago [6].

In ref. [5], the Froggatt-Nielsen mechanism [8] is used to give the hierarchical structure of the down-quark mass matrix. As pointed out by Froggatt and Nielsen [8], the renormalization group equations of the Yukawa couplings for up- and down-quarks do not differ so much. Hence, it may be difficult to explain the hierarchical structure of the mass matrices in this way unless using the extra $U(1)_F$ flavor symmetry. From the $U(1)_F$ invariance, for example, the mass term of up-quarks takes the following form:

$$- \mathcal{L}_{\text{mass}} = y_{ij} \bar{\psi}_i \psi_j H \left( \frac{\Theta}{M_P} \right)^{f_i + f_j + x},$$

where $\psi_i$ and $H$ are the quark and Higgs fields, respectively. The indices $i$ and $j$ denote the generation, and $y_{ij}$ is the Yukawa coupling whose magnitude is assumed to be order one. The field $\Theta$ is an another Higgs field which is responsible for the $U(1)_F$ symmetry breaking, and $M_P$ is the Plank mass. We fix the $U(1)_F$ charge of the quark field $\psi_i$, which is denoted by $f_i$ as follows:

$$(f_1, f_2, f_3) = (3, 2, 0),$$

while those of $H$ and $\Theta$ are taken as $x$ and $-1$, respectively [5]. If the $U(1)_F$ symmetry breaking scale is given by $\langle \Theta \rangle / M_P \approx \lambda$, the mass matrix of up-quarks is given by

$$M_{ij} = y_{ij} \langle H \rangle \lambda^{f_i + f_j + x}.$$  

The $U(1)_F$ charge $x$ of the Higgs field $H$ may fix the overall magnitude of the mass matrix. If one takes $x = -4$, the power of the superfield $\Theta$ becomes negative for $i = 3$ or $j = 3$, so that the corresponding elements in the mass matrix are prohibited and set to be zeros, which is called the SUSY zeros [4]. The presence of zero entries in the mass matrix owing to the SUSY zeros leads to the twisting among $5^*$ representations [5]. Although it should be dynamically clarified whether this turning around mechanism of hierarchy is consistent with the Froggatt-Nielsen mechanism, so far we do not understand so well the dynamics of how and why the
Froggatt-Nielsen mechanism works. The mechanism itself is, however, what we want to have, so that it seems to be inevitable in the study of mass matrices.

In these circumstances, we study the possibility of the twisting among the down-type quarks in the SUSY $E_6$ GUT without using the SUSY zeros. In the following, we restrict ourselves to the case of $x = 0$ so that there is no entry in the mass matrix which is negative power of $\lambda$. Then we can “naively” use the Froggatt-Nielsen mechanism in order to generate the hierarchical structure of the mass matrix. We will diagonalize the $6 \times 6$ mass matrix of the down-quarks which consists of 3 generations of $d_R^c$ in $(16, 5^*)$ and $D_R^c$ in $(10, 5^*)$, and study if the appropriate quark masses and mixings could be obtained due to the twisting among them.

Let us briefly review our scenario. We prepare 3 superfields, $\Psi_i(27)(i = 1, 2, 3)$, corresponding to the 3 generations of quarks and leptons. We also introduce two Higgs fields, $H(27)$ and $\phi(78)$. In $78$, there are $8_L$ and $8_R$ under the decomposition of $E_6$ into $SU(3)_C \times SU(3)_L \times SU(3)_R$. We assume that the components of $8_L$ and $8_R$ develop the vacuum expectation values (VEVs) of the GUT scale ($\sim O(10^{16}\text{GeV})$) so that $E_6$ is broken down to $SU(3)_C \times SU(2)_L \times U(1)_{\psi} \times U(1)_{\chi}$ directly. Then, in general, the baryon number violating operators in the superpotential are allowed from the gauge invariance and the proton stability cannot be preserved. We, therefore, introduce a discrete symmetry which is assigned to be odd for $\Psi_i(27)$ and even for $H(27)$ [5]. Another source of the proton decay is some Higgs fields in $5^*$ and $10$, which carry both the weak and color charges, and they are assumed to be sufficiently heavy owing to some unknown mechanisms.

There are two extra $U(1)$ symmetries besides the SM gauge symmetry. As is studied in the superstring inspired $E_6$ model, the breaking scale of the extra $U(1)$ symmetries could be lowered to $O(\text{TeV})$ without conflicting the electroweak precision measurements at LEP1 and SLC [9]. We assume that the component fields of $H(27)$, which are singlets under the SM gauge group, play role to break the extra $U(1)$ symmetries, i.e., we suppose that the VEV $\langle H(16, 1) \rangle$ breaks $U(1)_\psi \times U(1)_\chi$ down to $U(1)'_\psi$ and $U(1)'_\chi$. These VEVs may appear in the mass matrices of quarks and leptons, if their scale is around $O(\text{TeV})$, which may be attainable in the future colliders.

In practice, we examine the mass and mixing matrices of down-type quarks employing the perturbation theory. Although it is rather hard to calculate the realistic mass and mixing values for quarks based on the perturbative treatment,
it is worth to use this method to clarify the characteristic features of the mass and mixing matrices in our model. We show that, in a certain parameter set, the hierarchical structure of the CKM matrix can be found due to the twisting. As a by-product, there is a light vector-like down-quark whose mass is around $O(\text{TeV})$.

The twisting which we found also lead to the large mixing in the leptonic sector and the result will be reported elsewhere [10]. It is worth to mention that the $6 \times 6$ down-quark mass matrix has been studied in the superstring inspired SU(6) × SU(2)$_R$ model (Gepner model) [11] in which some elements of the mass matrix vanish from the gauge symmetry.

This paper is organized as follows: the mass and mixing matrices of quarks in our model are examined in Sec. 2. The quantitative estimations of mass and mixing matrices will be done in Sec. 3. Sec. 4 is devoted to summary and discussion. Some formulae used in the calculation based on the perturbative method are given in detail in Appendix.

\section{Mass and Flavor Mixing Matrices}

\subsection{Superpotential}

We first review the superpotentials in SUSY $E_6$ GUT following ref. [5]. As stated in the previous section, we introduce the flavor symmetry U(1)$_F$ and assume that the Froggatt-Nielsen mechanism [8] works on. Then, the superpotential relevant to the Higgs superfield $H(27)$ is given by

$$W_H = y_{ij} \Psi_i(27) \Psi_j(27) H(27) \left( \frac{\Theta}{M_P} \right)^{f_i+f_j} ,$$

(2.1)

where $f_i$ are the U(1)$_F$ charges of the $i$-th generation quarks and leptons, and the Yukawa coupling constant $y_{ij}$ is assumed to be of the order one.

The superpotential relevant to the adjoint Higgs superfield $\phi(78)$ is given by the higher dimensional operators:

$$W_\phi = \sum_{i,j} s_{ij} M_P^{-1} \Psi_i(27) \Psi_j(27) (\phi(78) H(27))_{27} \left( \frac{\Theta}{M_P} \right)^{f_i+f_j}$$

$$+ \sum_{i,j} a_{ij} M_P^{-1} (\phi(78) \Psi_i(27))_{27} \Psi_j(27) H(27) \left( \frac{\Theta}{M_P} \right)^{f_i+f_j} ,$$

(2.2)

where $s_{ij}$ and $a_{ij}$ are symmetric and anti-symmetric tensors with respect to the generation indices, respectively. Here we take the U(1)$_F$ charges of Higgs fields to
be zero according to the discussion in the previous section. The coupling constants \( s_{ij} \) and \( a_{ij} \) are also assumed to be of the order one. In eq. (2.2), two Higgs fields are multiplied such as \((\phi(78)H(27))_{27}\), which denotes the infinitesimal transformation of the fundamental representation 27 by the adjoint representation 78. Therefore, \( M_P^{-1} \) has to be introduced to modify the dimensionality.

The adjoint Higgs \( \phi(78) \) can be decomposed under the subgroup \( \text{SU}(3)_L \times \text{SU}(3)_R \times \text{SU}(3)_C \subset E_6 \) as follows:

\[
78 = 8_L + 8_R + 8_C + (3, 3, 3) + (3^*, 3^*, 3^*). 
\]

Suppose that the components of \( 8_R \) and \( 8_L \) develop the VEVs as follows:

\[
\frac{\langle \phi(8_R) \rangle}{M_P} = \begin{pmatrix}
\omega + \chi_R & 0 & 0 \\
0 & -\omega + \chi_R & 0 \\
0 & 0 & -2\chi_R
\end{pmatrix}, 
\]

(2.4a)

\[
\frac{\langle \phi(8_L) \rangle}{M_P} = \begin{pmatrix}
\chi_L & 0 & 0 \\
0 & \chi_L & 0 \\
0 & 0 & -2\chi_L
\end{pmatrix}, 
\]

(2.4b)

then \( E_6 \) is broken down to \( \text{SU}(3)_C \times \text{SU}(2)_L \times \text{U}(1)_Y \times \text{U}(1)_\psi \times \text{U}(1)_\chi \). As a consequence, the mass matrices for the down-quarks and charged-leptons can be different. Even if the breaking scale \( \langle \phi(78) \rangle \) is the GUT scale, \( \sim \mathcal{O}(10^{16}\text{GeV}) \), the modification for the Yukawa couplings in eq. (2.2) is expressed by the ratio \( \langle \phi(78) \rangle/M_P \) so that it is reasonably small.

### 2.2 Up-quark sector

Owing to the \( \text{U}(1)_F \) charge assignment (1.3) and the superpotentials (2.1) and (2.2), the Yukawa matrix for the up-quark sector has the hierarchical structure which is phenomenologically acceptable:

\[
Y_u \equiv \begin{pmatrix}
Y_{11}\lambda^6 & Y_{12}\lambda^5 & Y_{13}\lambda^3 \\
Y_{21}\lambda^5 & Y_{22}\lambda^4 & Y_{23}\lambda^2 \\
Y_{31}\lambda^3 & Y_{32}\lambda^2 & Y_{33}
\end{pmatrix},
\]

(2.5)

where

\[
Y_{ij} = y_{ij} + (\chi_R - \chi_L + \omega)s_{ij} + \frac{1}{2}(\chi_R + \chi_L + \omega)a_{ij}. 
\]

(2.6)

The Higgs field, \( H(27) \), is decomposed following eq. (1.1). Then, \( H(10, 5) \) couples to the up-type quarks and its VEV gives the mass matrix as follows:

\[
M_u \equiv \begin{pmatrix}
Y_{11}\lambda^6 & Y_{12}\lambda^5 & Y_{13}\lambda^3 \\
Y_{21}\lambda^5 & Y_{22}\lambda^4 & Y_{23}\lambda^2 \\
Y_{31}\lambda^3 & Y_{32}\lambda^2 & Y_{33}
\end{pmatrix} v \sin \beta, 
\]

(2.7)
where \( \langle H(10, 5) \rangle = v \sin \beta \equiv v_u \) is the VEV of \( H(10, 5) \). The VEV \( v_u \) is normalized as \( v^2 = v_u^2 + v_d^2 = (v \sin \beta)^2 + (v \cos \beta)^2 \simeq (174 \text{GeV})^2 \), where \( v \cos \beta \equiv v_d \) provides the mass to the down-type quarks. The mass matrix \( M_u \) can be diagonalized by using the unitary matrix \( U_{uL} \) as follows:

\[
U_{uL} (M_u M_u^\dagger) U_{uL}^\dagger = \text{diag}(m_u^2, m_c^2, m_t^2),
\]

where the mass eigenvalues are given by

\[
m_u^2 = f_u^2 v^2 \sin^2 \beta, \\
m_c^2 = f_c^2 v^2 \sin^2 \beta, \\
m_t^2 = f_t^2 v^2 \sin^2 \beta.
\]

In eq. (2.9), \( f_u^2, f_c^2 \) and \( f_t^2 \) represent the eigenvalues for the Yukawa matrix, whose magnitudes are given as follows [5]:

\[
f_u^2 \sim \mathcal{O}((\lambda^6)^2), \\
f_c^2 \sim \mathcal{O}((\lambda^4)^2), \\
f_t^2 \sim \mathcal{O}(1).
\]

### 2.3 Down-quark sector

We study the mass matrix for the down-quark sector, having totally 6 flavors, \( d_i \) and \( D_i (i = 1, 2, 3) \). Then, the \( 6 \times 6 \) Yukawa matrix for the down-quarks is expressed by the four blocks of the \( 3 \times 3 \) matrices,

\[
Y_d \equiv \begin{pmatrix}
Y_u + \alpha \bar{s} & Y_u + \epsilon \bar{s} \\
Y_u + \alpha \bar{s} + \gamma \bar{s}^T & Y_u + \epsilon \bar{s} + \gamma \bar{s}^T
\end{pmatrix},
\]

where we define

\[
\alpha \equiv -2\omega, \\
\epsilon \equiv -(\omega + 3\chi_R), \\
\gamma \equiv 3\chi_L, \\
\bar{s} \equiv \left(s_{ij} + \frac{1}{2} a_{ij}\right) \chi_{i + j}.
\]

As shown in eq. (2.4), their order is \( \mathcal{O}(\phi(78)/M_P) \). Parametrizing the VEVs of the Higgs fields which couple to \( d_i \) and \( D_i \) as

\[
\langle H(10, 5^*) \rangle = v_d \cos \theta,
\]
the mass matrix for the down-quarks can be expressed as follows:

\[ M_d = \begin{pmatrix} (Y_u + \alpha \tilde{s})v_d \cos \theta & (Y_u + \epsilon \tilde{s})v_d \sin \theta \\ (Y_u + \alpha \tilde{s} + \gamma \tilde{s}^T)v_D \cos \varphi & (Y_u + \epsilon \tilde{s} + \gamma \tilde{s}^T)v_D \sin \varphi \end{pmatrix} \]  

(2.14)

It is easy to see that each block in eq. (2.14) includes the Yukawa matrix of the up-quarks. Let us take \( \alpha, \epsilon, \gamma \) as the parameters of perturbation by assuming

\[ \alpha, \epsilon, \gamma \sim \mathcal{O}(\langle \phi(78) \rangle / M_P) \sim \lambda^4. \]  

(2.15)

Then we diagonalize the mass matrix using the following decomposition:

\[ M_d \equiv M^{(0)} + M^{(1)}, \]  

(2.16)

where \( M^{(0)} \) is the unperturbed mass matrix and \( M^{(1)} \) is its perturbation. Their explicit forms are given by

\[ M^{(0)} = Y_u \otimes \begin{pmatrix} v_d \cos \theta & v_d \sin \theta \\ v_D \cos \varphi & v_D \sin \varphi \end{pmatrix}, \]  

(2.17a)

\[ M^{(1)} = \tilde{s} \otimes \begin{pmatrix} \alpha v_d \cos \theta & \epsilon v_d \sin \theta \\ \alpha v_D \cos \varphi & \epsilon v_D \sin \varphi \end{pmatrix} + \tilde{s}^T \otimes \begin{pmatrix} 0 & 0 \\ \gamma v_D \cos \varphi & \gamma v_D \sin \varphi \end{pmatrix}. \]  

(2.17b)

The mass matrix \( M_d \) is diagonalized by using the unitary matrix \( U_{dL} \) as

\[ U_{dL} (M_d M_d^\dagger) U_{dL}^\dagger = \text{diag}(m^2_{d_1}, m^2_{d_2}, m^2_{d_3}, m^2_{D_1}, m^2_{D_2}, m^2_{D_3}), \]  

(2.18)

where \( M_d M_d^\dagger \) is expressed by using (2.16) as

\[ M_d M_d^\dagger = M^{(0)} M^{(0)\dagger} + \delta M^2, \]  

(2.19a)

\[ \delta M^2 \equiv M^{(0)} M^{(1)\dagger} + M^{(1)} M^{(0)\dagger} + M^{(1)} M^{(1)\dagger}. \]  

(2.19b)

Correspondingly, the unitary matrix \( U_{dL} \) is expanded as

\[ U_{dL} \equiv U_{dL}^{(0)} + U_{dL}^{(1)}. \]  

(2.20)

The mass eigenvalues at the leading order are given by

\[ U_{dL}^{(0)} (M^{(0)} M^{(0)\dagger}) U_{dL}^{(0)\dagger} = \text{diag}\left( (m^2_{d_1})^{\delta_1}, \cdots, (m^2_{d_3})^{\delta_3}, (m^2_{D_1})^{\delta_1}, \cdots, (m^2_{D_3})^{\delta_3}\right). \]  

(2.21)
The correction of the mass eigenvalue in the lowest order of the perturbation $\Delta_n^{(1)}$ is given by

$$\Delta_n^{(1)} = \langle n^{(0)} | \delta M^2 | n^{(0)} \rangle,$$

and the lowest order correction for the mass eigenstate $|n^{(1)}\rangle$ is given by

$$|n^{(1)}\rangle = \sum_{n \neq k} \frac{|k^{(0)}\rangle \langle k^{(0)} | \delta M^2 | n^{(0)} \rangle}{(m_n^{(0)})^2 - (m_k^{(0)})^2}.$$  

(2.22)

where $n, k = d_1, d_2, d_3, D_1, D_2, D_3$. The generic form of $\langle k^{(0)} | \delta M^2 | n^{(0)} \rangle$ can be found in Appendix.

If we assume $v_d \ll v_D$ as a natural consequence of our scenario, we find the mass eigenvalues (2.21) and the unitary matrix $U_{dL}^{(0)}$ at the leading order as follows:

$$U_{dL}^{(0)} (M^{(0)} M^{(0)\dagger}) U_{dL}^{(0)\dagger} = \left( \begin{array}{ccc} f_u^2 & 0 & 0 \\ 0 & f_c^2 & 0 \\ 0 & 0 & f_t^2 \end{array} \right) \otimes \left( \begin{array}{cc} v_d^2 \sin^2(\theta - \varphi) & 0 \\ 0 & v_D^2 \end{array} \right),$$

(2.24a)

$$U_{dL}^{(0)} = U_{uL} \otimes \left( \begin{array}{cc} 1 & -\frac{v_d}{v_D} \cos(\theta - \varphi) \\ \frac{v_d}{v_D} \cos(\theta - \varphi) & 1 \end{array} \right).$$

(2.24b)

Then the mass eigenstates for leading order $|n^{(0)}\rangle$'s correspond to the six columns of the matrix $U_{dL}^{(0)\dagger}$. So, at the lowest order of the perturbation, the mass eigenvalues are given as follows:

$$m_{d_1}^2 \simeq \frac{m_u^2}{\tan^2 \beta} \sin^2(\theta - \varphi) + \Delta_{d_1}^{(1)},$$

(2.25a)

$$m_{d_2}^2 \simeq \frac{m_c^2}{\tan^2 \beta} \sin^2(\theta - \varphi) + \Delta_{d_2}^{(1)},$$

(2.25b)

$$m_{d_3}^2 \simeq \frac{m_t^2}{\tan^2 \beta} \sin^2(\theta - \varphi) + \Delta_{d_3}^{(1)},$$

(2.25c)

$$m_{D_1}^2 \simeq v_D^2 f_u^2 + \Delta_{D_1}^{(1)},$$

(2.25d)

$$m_{D_2}^2 \simeq v_D^2 f_c^2 + \Delta_{D_2}^{(1)},$$

(2.25e)

$$m_{D_3}^2 \simeq v_D^2 f_t^2 + \Delta_{D_3}^{(1)}.$$  

(2.25f)

On the other hand the correction of the mass eigenstate for the lowest order $|n^{(1)}\rangle$ corresponds to each column of the unitary matrix $U_{dL}^{(1)\dagger}$.

From eq. (2.24a), it is clear that $m_{d_1} < m_{d_2} < m_{d_3}$ and $m_{D_1} < m_{D_2} < m_{D_3}$, but $m_{d_3} < m_{D_1}$ may not hold in general. Even under the assumption $v_d \ll v_D$, for
example, $m_{d_3}$ could be heavier than $m_{D_1}$ and $m_{D_2}$ in a certain parameter space. But this possibility is unacceptable. Let us recall that the CKM matrix has the small off-diagonal elements. In our model, it could be explained by the structure of the unitary matrix $U_{d_L}$. Since the leading order of the matrix $U_{d_L}$ (2.24f) is proportional to the matrix $U_{u_L}$, the CKM matrix is given as the unit matrix at the leading order only if the $(1,1)$ block of r.h.s. in eq. (2.24b) is identified as the light down-quarks. The non-vanishing off-diagonal elements of the CKM matrix are derived from the lowest order of the perturbation, $U_{d_L}$, so that they become small. One may consider the $(2,2)$ block of r.h.s. in eq. (2.24b) is also another candidate of the light down-quarks, however, the condition $v_d \ll v_D$ does not allow this possibility. We, therefore, identify $d_1, d_2$ and $d_3$ as the ordinary light down-quarks, $d, s$ and $b$.

Now the ordinary $3 \times 3$ CKM matrix can be defined by

$$V_{ij} \equiv \sum_{\alpha=1}^{3}(U_{u_L})_{i\alpha}(U_{d_L}^\dagger)_{\alpha j},$$

(2.26)

where the indices $i$ and $j$ denote $u, c, t$ and $d, s, b$, respectively. Using new symbols $m_+^2 \equiv v_D^2$ and $m_-^2 \equiv v_D^2 \sin^2(\theta - \varphi)$, we explicitly write down the CKM matrix elements as follows:

$$V_{ud} \simeq 1 + \frac{v_d}{v_D} \cos(\theta - \varphi) \frac{\langle D_1 | \delta \mathcal{M}^2 | d_1 \rangle}{f_\alpha^2 (m_-^2 - m_+^2)},$$

(2.27a)

$$V_{us} \simeq \frac{\langle d_1 | \delta \mathcal{M}^2 | d_2 \rangle}{m_-^2 (f_c^2 - f_u^2)} + \frac{v_d}{v_D} \cos(\theta - \varphi) \frac{\langle D_1 | \delta \mathcal{M}^2 | d_2 \rangle}{m_-^2 f_c^2 - m_+^2 f_u^2},$$

(2.27b)

$$V_{ub} \simeq \frac{\langle d_1 | \delta \mathcal{M}^2 | d_3 \rangle}{m_-^2 (f_t^2 - f_u^2)} + \frac{v_d}{v_D} \cos(\theta - \varphi) \frac{\langle D_1 | \delta \mathcal{M}^2 | d_3 \rangle}{m_-^2 f_t^2 - m_+^2 f_u^2},$$

(2.27c)

$$V_{cd} \simeq \frac{\langle d_2 | \delta \mathcal{M}^2 | d_1 \rangle}{m_-^2 (f_u^2 - f_c^2)} + \frac{v_d}{v_D} \cos(\theta - \varphi) \frac{\langle D_2 | \delta \mathcal{M}^2 | d_1 \rangle}{m_-^2 f_u^2 - m_+^2 f_c^2},$$

(2.27d)

$$V_{cs} \simeq 1 + \frac{v_d}{v_D} \cos(\theta - \varphi) \frac{\langle D_2 | \delta \mathcal{M}^2 | d_2 \rangle}{f_\alpha^2 (m_-^2 - m_+^2)},$$

(2.27e)

$$V_{cb} \simeq \frac{\langle d_2 | \delta \mathcal{M}^2 | d_3 \rangle}{m_-^2 (f_t^2 - f_u^2)} + \frac{v_d}{v_D} \cos(\theta - \varphi) \frac{\langle D_2 | \delta \mathcal{M}^2 | d_3 \rangle}{m_-^2 f_t^2 - m_+^2 f_u^2},$$

(2.27f)

$$V_{td} \simeq \frac{\langle d_3 | \delta \mathcal{M}^2 | d_1 \rangle}{m_-^2 (f_u^2 - f_c^2)} + \frac{v_d}{v_D} \cos(\theta - \varphi) \frac{\langle D_3 | \delta \mathcal{M}^2 | d_1 \rangle}{m_-^2 f_u^2 - m_+^2 f_c^2},$$

(2.27g)

$$V_{ts} \simeq \frac{\langle d_3 | \delta \mathcal{M}^2 | d_2 \rangle}{m_-^2 (f_c^2 - f_t^2)} + \frac{v_d}{v_D} \cos(\theta - \varphi) \frac{\langle D_3 | \delta \mathcal{M}^2 | d_2 \rangle}{m_-^2 f_c^2 - m_+^2 f_t^2},$$

(2.27h)
\[ V_{tb} \simeq 1 + \frac{v_d}{v_D} \cos(\theta - \varphi) \frac{\langle D_3 | \delta M^2 | d_3 \rangle}{f_\ell^2 (m_\pi^2 - m_\pi^2)}. \]  

(2.27i)

## 3 Estimation of Masses and Mixings

In this section, we estimate the mass and mixing matrices quantitatively in a certain set of the parameters. In the case of \( \alpha = \gamma = 0 \) and \( \epsilon \neq 0 \), that is, \( \chi_L = \omega = 0 \) and \( \chi_R \neq 0 \), the mass eigenvalues eqs. (2.25a) \( \sim \) (2.25i) are given as follows:

\[
\begin{align*}
    m_i^2 &\sim \frac{m_i^2}{\tan^2 \beta} \sin^2(\theta - \varphi) + \epsilon v_d^2 \lambda^6 \sin(\theta - \cos(\theta - \varphi))^2, \\
    m_i^2 &\sim \frac{m_i^2}{\tan^2 \beta} \sin^2(\theta - \varphi) + \epsilon v_d^2 \lambda^4 \sin(\theta - \cos(\theta - \varphi))^2, \\
    m_i^2 &\sim \frac{m_i^2}{\tan^2 \beta} \sin^2(\theta - \varphi) + \epsilon v_d^2 (\sin(\theta - \cos(\theta - \varphi))^2, \\
    m_i^2 &\sim v_i^2 f_{i_u}^2 + \epsilon v_i^2 \lambda^6 \sin^2 \varphi, \\
    m_i^2 &\sim v_i^2 f_{i_c}^2 + \epsilon v_i^2 \lambda^4 \sin^2 \varphi, \\
    m_i^2 &\sim v_i^2 f_{i_t}^2 + \epsilon v_i^2 \sin^2 \varphi,
\end{align*}
\]

(3.1a) (3.1b) (3.1c) (3.1d) (3.1e) (3.1f)

and the CKM matrix elements eqs. (2.27a) \( \sim \) (2.27i) are given by

\[
\begin{align*}
    V_{ud} &\sim 1, \\
    V_{us} &\sim -\epsilon \lambda^5 \sin(\theta - \cos(\theta - \varphi) \sin \varphi)^2, \\
    V_{ub} &\sim -\epsilon \lambda^3 \sin(\theta - \cos(\theta - \varphi) \sin \varphi)^2, \\
    V_{cd} &\sim -V_{us}, \\
    V_{cs} &\sim 1, \\
    V_{cb} &\sim -\epsilon \lambda^2 \sin(\theta - \cos(\theta - \varphi) \sin \varphi)^2, \\
    V_{ld} &\sim -V_{ub}, \\
    V_{ls} &\sim -V_{cb}, \\
    V_{tb} &\sim 1.
\end{align*}
\]

(3.2a) (3.2b) (3.2c) (3.2d) (3.2e) (3.2f) (3.2g) (3.2h) (3.2i)

We examine the masses and mixing angles at the GUT scale with the following inputs:
(i) $v_D \sim 10^4 \text{TeV}$ so that the heavy down-quarks ($D_i$) do not decouple from the mass matrix,

(ii) $\sin^2(\theta - \varphi)/\tan^2 \beta \sim (1/100)^2$ and $\tan \beta = 40$ to reproduce the bottom quark mass ($m_b \sim 1 \text{GeV}$) in eq. (3.1c),

(iii) $\sin \theta \sim 0.4$, $\cos \theta \sim 0.9$, $\sin \varphi \sim \lambda^8$ and $\cos \varphi \sim 1$ so that $V_{us}$ in eq. (3.2b) is approximately equal to $\lambda$, taking account of eqs. (2.10b) and (2.15).

From our inputs (i) $\sim$ (iii), we find the mass eigenvalues of down-quarks as

\begin{align*}
  m_d &\sim 0.2 \text{ MeV}, & (3.3a) \\
  m_s &\sim 4 \text{ MeV}. & (3.3b)
\end{align*}

The obtained $d$- and $s$-quark masses (3.3) are a bit small compared with those in the MSSM at the GUT scale given in ref. [12]. Then, the CKM matrix elements are obtained as

\[ V \sim \begin{pmatrix} 1 & \lambda & \lambda^7 \\ \lambda & 1 & \lambda^6 \\ \lambda^7 & \lambda^6 & 1 \end{pmatrix}. \]  

(3.4)

Therefore, the values for $V_{us}$ and $V_{cd}$ can be reasonably reproduced in our model, but the other off-diagonal elements of the CKM matrix are much smaller than those expected at the GUT scale [12]. The smallness of $V_{ub}$ and $V_{cb}$ is caused by the hierarchical structure of the Yukawa couplings in eq. (2.10). (See (3.2b), (3.2c) and (3.2f)).

Next let us see $U_{dR}$, which diagonalizes the mass matrix $M_d^T M_d$ in the same manner as in Sec. 2.3, in order to see that if the twisting among the down-type quarks occurs or not. After the tedious calculations, we find

\begin{align*}
  |d_R\rangle &= \mathcal{O}(\lambda^6) |d'_{1R}\rangle + \mathcal{O}(\lambda^7) |d'_{2R}\rangle + \mathcal{O}(\lambda^9) |d'_{3R}\rangle \\
  &\quad + \mathcal{O}(1)|D'_{1R}\rangle + \mathcal{O}(\lambda)|D'_{2R}\rangle + \mathcal{O}(\lambda^3)|D'_{3R}\rangle, \quad (3.5a) \\
  |s_R\rangle &= \mathcal{O}(\lambda^5) |d'_{1R}\rangle + \mathcal{O}(\lambda^6) |d'_{2R}\rangle + \mathcal{O}(\lambda^8) |d'_{3R}\rangle \\
  &\quad + \mathcal{O}(\lambda)|D'_{1R}\rangle + \mathcal{O}(1)|D'_{2R}\rangle + \mathcal{O}(\lambda^2)|D'_{3R}\rangle, \quad (3.5b) \\
  |b_R\rangle &= \mathcal{O}(\lambda^3) |d'_{1R}\rangle + \mathcal{O}(\lambda^4) |d'_{2R}\rangle + \mathcal{O}(\lambda^6) |d'_{3R}\rangle \\
  &\quad + \mathcal{O}(\lambda^3)|D'_{1R}\rangle + \mathcal{O}(\lambda^2)|D'_{2R}\rangle + \mathcal{O}(1)|D'_{3R}\rangle. \quad (3.5c)
\end{align*}
where the l.h.s. and the r.h.s. denote the mass and the current eigenstates, respectively. The state with the underline in the r.h.s. is the dominant component in the mass eigenstate. From eq. (3.5), we find that the mass eigenstates for the right-handed down-type quarks, \(|d_R\rangle, |s_R\rangle\) and \(|b_R\rangle\) are mainly dominated by the current eigenstates, \(|D'_{1R}\rangle, |D'_{2R}\rangle\) and \(|D'_{3R}\rangle\) in \((10, 5^*)\), respectively. This means that the twisting occurs between \((10, 5^*)\) and \((16, 5^*)\). As a complement of the above estimation based on the perturbation, we examined the mass and mixing matrices numerically and confirmed that the twisting occurs in the same parameter space.

4 Summary

In this paper we have studied the mass and mixing matrices of the supersymmetric \(E_6\) GUT model, in which \(E_6\) is assumed to be broken down to \(SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_\psi \times U(1)_\chi\) by the VEVs of the 78 Higgs scalar at GUT scale, while the extra U(1) symmetries are assumed to be broken by the VEVs of the component fields of the 27 Higgs scalar taking around the energy scale, \(\mathcal{O}(10^2\text{GeV})–\mathcal{O}(10^4\text{TeV})\). Quarks and leptons belong to the fundamental representation 27 which contains two 5*’s — \((16, 5^*)\) and \((10, 5^*)\). Then, we have 6 flavors of down-type quarks. By diagonalizing the 6 × 6 mass matrix, we find the parameter regions in which the twisting “naturally” occurs, having reasonable values for the CKM matrix elements, \(V_{ud}, V_{cs}, V_{tb} \sim 1\) and \(V_{us}, V_{cd} \sim \lambda\), and the reasonable values for the down-quark masses, \(m_d \sim 0.2\text{MeV}, m_s \sim 4\text{MeV}\) and \(m_b \sim 1\text{GeV}\). As a by-product, one vector-like down-quark is produced at TeV scale with \(v_D \sim 10^4\text{TeV}\). We obtain, however, the rather small values for \(V_{ub}\) and \(V_{cb}\) because of the hierarchical structure of the Yukawa couplings.

In the derivation we employed the perturbation theory, in which the VEVs of the 78 Higgs scalar, \(\langle \phi(78) \rangle/M_P\), are taken as the perturbations to those of 27 Higgs scalar, \(\langle H(27) \rangle/v\). Since the 5* multiplets consist of the right-handed down quarks and the left-handed leptons, the natural twisting found in this paper in the right-handed down-quark sector leads to the natural twisting in the left-handed lepton sector. Further study may lead to the understanding of the large neutrino mixings, hopefully the double maximal ones \([10]\).
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Appendix

In this Appendix, we present the generic form of $\langle k^{(0)} | \delta M^2 | n^{(0)} \rangle$, which has appeared in eq. (2.23). Here, for later convenience, we rewrite $d_1, \ldots, d_3, D_1, \ldots, D_3$ as follows: $d_1 \rightarrow 1, \ldots, d_3 \rightarrow 3, D_1 \rightarrow 4, \ldots, D_3 \rightarrow 6$.

The perturbation term is composed of six parts:

$$
\delta M^2 = \delta M^2(\alpha, \alpha^2) + \delta M^2(\epsilon, \epsilon^2) + \delta M^2(\gamma, \gamma^2)
$$

$$
+ \delta M^2(\alpha \epsilon) + \delta M^2(\epsilon \gamma) + \delta M^2(\alpha \gamma), \quad (A1)
$$

where the arguments $\alpha, \epsilon$ and $\gamma$ are the parameters of the perturbation and defined in eqs. (2.12a) $\sim$ (2.12d).

In the case of six flavors, the generic form of $\langle k^{(0)} | \delta M^2 | n^{(0)} \rangle$ is given as:

for $k, n = 1, 2, 3$,

$$
\langle k^{(0)} | \delta M^2 | n^{(0)} \rangle = \alpha v_d^2 Z_{\kappa n} I_C(\theta, \varphi)^2 + \alpha^2 v_d^2 S_{\kappa n} I_C(\theta, \varphi)^2
$$

$$
+ \epsilon v_d^2 Z_{\kappa n} I_S(\theta, \varphi)^2 + \epsilon^2 v_d^2 S_{\kappa n} I_S(\theta, \varphi)^2
$$

$$
+ \gamma^2 v_d^2 S_{\kappa n} \cos(\theta - \varphi)^2, \quad (A2)
$$

for $k, n = 4, 5, 6$,

$$
\langle k^{(0)} | \delta M^2 | n^{(0)} \rangle = \alpha v_d^2 Z_{\kappa-3 n-3} \cos^2 \varphi + \alpha^2 v_d^2 S_{\kappa-3 n-3} \cos^2 \varphi
$$

$$
+ \epsilon v_d^2 Z_{\kappa-3 n-3} \sin^2 \varphi + \epsilon^2 v_d^2 S_{\kappa-3 n-3} \sin^2 \varphi
$$

$$
+ \gamma v_d^2 T_{\kappa-3 n-3} + \gamma^2 v_d^2 S_{\kappa-3 n-3}, \quad (A3)
$$

for $k = 4, 5, 6, n = 1, 2, 3$,

$$
\langle k^{(0)} | \delta M^2 | n^{(0)} \rangle = \alpha v_d v_d Z_{\kappa-3 n} I_C(\theta, \varphi) \cos \varphi + \alpha^2 v_d v_d S_{\kappa-3 n} I_C(\theta, \varphi) \cos \varphi
$$

$$
+ \epsilon v_d v_d Z_{\kappa-3 n} I_S(\theta, \varphi) \sin \varphi + \epsilon^2 v_d v_d S_{\kappa-3 n} I_S(\theta, \varphi) \sin \varphi
$$

$$
- \gamma v_d v_d Q_{\kappa-3 n} \cos(\theta - \varphi) - \gamma^2 v_d v_d S_{\kappa-3 n} \cos(\theta - \varphi), \quad (A4)
$$
and for $k = 1, 2, 3, n = 4, 5, 6,$

$$
\langle k^{(0)} | \delta M^2 | n^{(0)} \rangle = \alpha v_d v_D Z_{k n-3} I_C(\theta, \varphi) \cos \varphi + \alpha^2 v_d v_D S_{k n-3} I_C(\theta, \varphi) \cos \varphi
+ \epsilon v_d v_D Z_{k n-3} I_S(\theta, \varphi) \sin \varphi + \epsilon^2 v_d v_D S_{k n-3} I_S(\theta, \varphi) \sin \varphi
- \gamma v_d v_D (Q_{n-3 k})^T \cos(\theta - \varphi) - \gamma^2 v_d v_D S_{k n-3} \cos(\theta - \varphi). \quad (A5)
$$

In the above expressions, we define the various functions and coefficients as

$$
I_C(\theta, \varphi) \equiv \cos \theta - \cos(\theta - \varphi) \cos \varphi, \quad (A6)
$$

$$
I_S(\theta, \varphi) \equiv \sin \theta - \cos(\theta - \varphi) \sin \varphi, \quad (A7)
$$

and

$$
Q_{kn} \equiv u_k Y_u \tilde{s} u_n^T, \quad (A8)
$$

$$
S_{kn} \equiv u_k \tilde{s}^T \tilde{s} u_n^T, \quad (A9)
$$

$$
T_{kn} \equiv u_k (Y_u \tilde{s} + \tilde{s}^T Y_u^T) u_n^T, \quad (A10)
$$

$$
Z_{kn} \equiv u_k (Y_u \tilde{s}^T + \tilde{s} Y_u^T) u_n^T, \quad (A11)
$$

where we define the three vectors $u_i$'s ($i = 1, 2, 3$) as the three rows of the unitary matrix $U_L$. We mention that $S_{kn}, T_{kn}$ and $Z_{kn}$ are symmetric matrix, but $Q_{kn}$ is not symmetric. The generic form of $\Delta^{(1)}_n (n = 1, 2, 3)$ is the same as the case of $n = k$ in eq. (A2), and that of $\Delta^{(1)}_n (n = 4, 5, 6)$ is the same as the case of $n = k$ in eq. (A3).

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