ON RANK OF THE JOIN OF TWO SUBGROUPS IN A FREE GROUP

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Abstract. Let \( H, K \) be two finitely generated subgroups of a free group, let \( \langle H, K \rangle \) denote the subgroup generated by \( H, K \), called the join of \( H, K \), and let neither of \( H, K \) have finite index in \( \langle H, K \rangle \). We prove the existence of an epimorphism \( \zeta : \langle H, K \rangle \to F_2 \), where \( F_2 \) is a free group of rank 2, such that the restriction of \( \zeta \) on both \( H \) and \( K \) is injective and the restriction \( \zeta_0 : H \cap K \to \zeta(H) \cap \zeta(K) \) of \( \zeta \) on \( H \cap K \) to \( \zeta(H) \cap \zeta(K) \) is surjective. This is obtained as a corollary of an analogous result on rank of the generalized join of two finitely generated subgroups in a free group.

1. Introduction

It is a recurring theme in group theory to embed a countable group into a 2-generated group often with some additional properties, see [1], [3], [4], [7], [11]. In this article we will look at two finitely generated subgroups of a free group from a similar perspective.

Let \( H, K \) be two finitely generated subgroups of a free group \( F \). Let \( S(H, K) \) denote a set of representatives of those double cosets \( HgK, g \in F \), for which the intersection \( H \cap gKg^{-1} \) is nontrivial.

According to Walter Neumann [9], the set \( S(H, K) \) is finite and a formal disjoint union \( \bigsqcup_{s \in S(H, K)} H \cap sKs^{-1} \) of subgroups \( H \cap sKs^{-1}, s \in S(H, K) \), could be considered as a generalized intersection of \( H \) and \( K \). Let \( \bar{r}(F) \) denote the rank of a free group \( F \) and let \( \bar{r}(F) \) denote the reduced rank of \( F \).

Theorem 1.1. Suppose that \( H, K \) are two finitely generated subgroups of a free group \( F, L = \langle H, K, S(H, K) \rangle \) denotes the subgroup generated by \( H, K, S(H, K) \), and neither \( H \) nor \( K \) has finite index in \( L \). Then there is an epimorphism \( \eta : L \to F_2 \), where \( F_2 \) is a free group of rank 2, such that the restriction of \( \eta \) on \( H \) and \( K \) is injective, a set \( S(\eta(H), \eta(K)) \) for subgroups \( \eta(H), \eta(K) \) of \( F_2 \) can be taken to be \( \eta(S(H, K)) \), and, for every \( s \in S(H, K) \), the restriction

\[
\eta_s : H \cap sKs^{-1} \to \eta(H) \cap \eta(s) \eta(K) \eta(s)^{-1}
\]

of \( \eta \) on \( H \cap sKs^{-1} \) to \( \eta(H) \cap \eta(s) \eta(K) \eta(s)^{-1} \) is surjective.

Informally, we can say that, when given a generalized join

\[
L = \langle H, K, S(H, K) \rangle
\]

of two finitely generated subgroups \( H, K \) of a free group \( F \) such that neither of \( H, K \) has finite index in \( L \), it is always possible to assume that \( r(L) = 2 \), i.e., the subgroup \( \langle H, K, S(H, K) \rangle \) is 2-generated. Since a free group of an arbitrary finite
rank is isomorphic to a subgroup of a free group of rank 2, it is easy to obtain the equality \( r(F) = 2 \) and our goal here will be to attain the equality \( r(L) = 2 \).

We mention an easy corollary of Theorem 1.1 for a “pure” intersection case.

**Corollary 1.2.** Suppose that \( H, K \) are two finitely generated subgroups of a free group \( F \), \( \langle H, K \rangle \) denotes the subgroup generated by \( H, K \), and neither \( H \) nor \( K \) has finite index in their join \( \langle H, K \rangle \). Then there is an epimorphism \( \zeta : \langle H, K \rangle \to F_2 \), where \( F_2 \) is a free group of rank 2, such that the restriction of \( \zeta \) on \( H \) and \( K \) is injective and the restriction \( \zeta_0 : H \cap K \to \zeta(H) \cap \zeta(K) \) of \( \zeta \) on \( H \cap K \) to \( \zeta(H) \cap \zeta(K) \) is surjective.

It is worthwhile to mention that Theorem 1.1 is false in case when one of subgroups \( H, K \) has finite index in \( L = \langle H, K, S(H, K) \rangle \). Indeed, if \( H \) has finite index \( j \) in \( L \) and \( r(L) = n > 2 \) then, according to the Schreier’s formula, we have 
\[
\text{r}(H) = (n - 1)j + 1, \quad \text{see } [5].
\]
Since \( \eta \) is an epimorphism, the subgroup \( \eta(H) \) has finite index \( j' \leq j \) in \( F_2 = \eta(L) \) and it follows from the Schreier’s formula that 
\[
\text{r}(\eta(H)) = j' + 1.
\]
Hence, the equality \( \text{r}(H) = \text{r}(\eta(H)) \) is impossible as \( (n - 1)j > j' \) and the restriction of \( \eta \) may not be injective on \( H \).

We also remark that Theorem 1.1 is motivated by the author’s article [5] in which certain modifications of Stallings graphs of subgroups \( H, K \), that do not change the ranks \( \text{r}(H), \text{r}(K), \text{r}(H \cap sKs^{-1}) \) for every \( s \in S(H, K) \), are used to achieve some desired properties of the Stallings graph of \( L \) whose rank \( r(L) \), however, might increase under carried out modifications. In this article, we make different modifications, in somewhat opposite direction, that decrease the rank \( r(L) \) down to 2 while the ranks \( \text{r}(H), \text{r}(K), \text{r}(H \cap sKs^{-1}) \) for every \( s \in S(H, K) \), and the cardinality \(|S(H, K)|\) are kept fixed.

## 2. Preliminaries

Suppose that \( Q \) is a graph. Let \( VQ \) denote the set of vertices of \( Q \) and let \( EQ \) denote the set of oriented edges of \( Q \). If \( e \in EQ \) then \( e^{-1} \) denotes the edge with the opposite to \( e \) orientation, \( e^{-1} \neq e \).

For \( e \in EQ \), let \( e_- \) and \( e_+ \) denote the initial and terminal, respectively, vertices of \( e \). A path \( p = e_i \ldots e_k \), where \( e_i \in EQ \), \( (e_i)_+ = (e_{i+1})_- \), \( i = 1, \ldots, k - 1 \), is called reduced if, for every \( i = 1, \ldots, k - 1 \), \( e_i \neq e_{i+1}^- \). The length of \( p \) is \( k \), denoted \(|p| = k \). The initial vertex of \( p \) is \( p_- = (e_1)_- \) and the terminal vertex of \( p \) is \( p_+ = (e_k)_+ \). A path \( p \) is closed if \( p_- = p_+ \). If \( p = e_1 \ldots e_k \) is a closed path then a cyclic permutation \( \bar{p} \) of \( p \) is a path of the form \( e_{1+i}e_{2+i} \ldots e_{k+i} \), where \( i = 1, \ldots, k \) and the indices are considered \( \text{mod } k \).

The subgraph of \( Q \) that consists of edges of all closed paths \( p \) of \( Q \) such that \(|p| > 0 \) and any cyclic permutation of \( p \) is reduced is called the core of \( X \), denoted \( \text{core}(X) \).

Let \( F \) be a free group of finite rank \( r(F) > 1 \). We consider \( F \) as the fundamental group \( \pi_1(U) \) where \( U \) is a bouquet of \( r(F) \) circles.

Following Stallings [10], see also [2], [6], with every (finitely generated) subgroup \( H \) of \( F = \pi_1(U) \), we can associate a connected (resp. finite) graph \( X = X(H) \) with a distinguished vertex \( o \in VX \) and a locally injective map \( \varphi : X \to U \) of graphs so that \( H \) is isomorphic to \( \pi_1(X, o) \). Such a graph \( X \) of \( H \) is called a Stallings graph of \( H \) and the map \( \varphi \) is called a canonical immersion.
Consider two finitely generated subgroups $H, K$ of the free group $F$. Pick a set $S(H, K)$ of representatives of those double cosets $HgK$, $g \in F$, for which the intersection $H \cap gKg^{-1}$ is nontrivial.

Let $X, Y$ be finite Stallings graphs of the subgroups $H, K$, resp., and let $X \times_U Y$ denote the pullback of canonical immersions

$$\varphi_X : X \to U, \quad \varphi_Y : Y \to U.$$ (2.1)

According to Walter Neumann [9], the set $S(H, K)$ is finite and the nontrivial intersections $H \cap sKs^{-1}$, where $s \in S(H, K)$, are in bijective correspondence with connected components $W_s$ of the core $W := \text{core}(X \times_U Y)$.

Moreover, for every $s \in S(H, K)$, we have

$$\bar{r}(H \cap sKs^{-1}) = \frac{1}{2}|EW_s| - |VW_s|,$$

where $\bar{r}(F) = \max(r(F) - 1, 0)$ is the reduced rank of a free group $F$ and $|T|$ is the cardinality of a finite set $T$. Recall that, according to our notation, the number of nonoriented edges of $W_s$ is $\frac{1}{2}|EW_s|$.

For a finite graph $Q$, denote

$$\bar{r}(Q) := \frac{1}{2}|EQ| - |VQ|,$$

therefore, $\bar{r}(Q)$ is the negative Euler characteristic of $Q$.

In particular, $\bar{r}(W_s) = \bar{r}(H \cap sKs^{-1})$ and

$$\sum_{s \in S(H, K)} \bar{r}(H \cap sKs^{-1}) = \bar{r}(W) = \frac{1}{2}|EW| - |VW|.$$ (2.2)

Let $\alpha'_X, \alpha'_Y$ denote the projection maps $X \times_U Y \to X, X \times_U Y \to Y$, resp. Restricting $\alpha'_X, \alpha'_Y$ to $W \subseteq X \times_U Y$, we obtain immersions

$$\alpha_X : W \to X, \quad \alpha_Y : W \to Y.$$ (2.3)

We also consider the subgroup $L = \langle H, K, S(H, K) \rangle$ of $F$. Let $Z$ denote a Stallings graph of $\langle H, K, S(H, K) \rangle$ and let $\gamma : Z \to U$ denote a canonical immersion.

Let $\beta_X : X \to Z, \beta_Y : Y \to Z$ be graph maps that satisfy the equalities $\varphi_X = \gamma \beta_X, \varphi_Y = \gamma \beta_Y$, see Fig. 1 and (2.4) and Fig. 1. Clearly, $\beta_X$ and $\beta_Y$ are immersions.

It follows from the definitions that, for every $Q \in \{U, W, X, Y, Z\}$, there is a canonical immersion $\varphi : Q \to U$, where $\varphi = \varphi_Q$ if $Q = X$ or $Q = Y$, $\varphi = \text{id}_U$ if $Q = U$, $\varphi = \gamma \beta_X \alpha_X = \gamma \beta_Y \alpha_Y$ if $Q = W$, and $\varphi = \gamma$ if $Q = Z$, see Fig. 1.

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha_X} & W \\
& \beta_X \varphi_X & \downarrow \gamma \\
& & Z \\
& \alpha_Y & \\
Y & \xrightarrow{\beta_Y \varphi_Y} & U \\
\end{array}
\]

Fig. 1
It is not difficult to see that, up to a suitable conjugation of the ambient free group \( F \), we may assume that the graphs \( X, Y \) coincide with their cores. Clearly, the same property holds for graphs \( U, W, Z \) as well.

Without loss of generality, we may also assume that \( Z = U \) and \( \gamma = \text{id}_Z \). For this reason, we will disregard \( U \) and \( \gamma \) in subsequent arguments.

3. Five Lemmas

Let \( A = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1} \} \) be an alphabet and let \( F(A) = \langle a_1, \ldots, a_n \rangle \) be a free group with free generators \( a_1, \ldots, a_n \). It will be convenient to consider elements of the free group \( F(A) \) as words over the alphabet \( A = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1} \} \). A letter-by-letter equality of words \( u, w \) over \( A \) is denoted \( u \equiv v \). Suppose \( w \equiv c_1 \ldots c_\ell \) is a word over \( A \), where \( c_1, \ldots, c_\ell \in A \) are letters. The length of \( w \) is denoted \( |w| = \ell \). We say that a word \( w \equiv c_1 \ldots c_\ell \) is reduced if \( |w| > 0 \) and \( c_i \neq c_{i+1}^{-1} \) for every \( i = 1, \ldots, \ell - 1 \).

A finite graph \( B \) is called a labeled \( A \)-graph, or just an \( A \)-graph, if \( B \) is equipped with a function \( \varphi : EB \to A \) so that, for every \( e \in EB \), we have \( \varphi(e^{-1}) = \varphi(e)^{-1} \). An \( A \)-graph \( B \) is called irreducible if, for every pair \( e_1, e_2 \in EB \), the equalities \( \varphi(e_1) = \varphi(e_2) \) and \( (e_1)_- = (e_2)_- \) imply that \( e_1 = e_2 \). Note that an irreducible \( A \)-graph need not be connected and may contain vertices of degree < 2.

It is easy to see that \( B \) is an irreducible \( A \)-graph with a labeling function \( \varphi \) if and only if there is an immersion \( \varphi_0 : B \to A_0 \), where \( A_0 \) is a bouquet of \( n \) oriented circles \( a_1, \ldots, a_n \) so that the restriction of \( \varphi_0 \) on \( EB \) is \( \varphi \).

We now discuss some operations over irreducible \( A \)-graphs. Let \( B \) be a finite irreducible \( A \)-graph. A connected component \( C \) of \( B \) is called \( A \)-complete if every vertex of \( C \) has degree \( |A| = 2n \). Equivalently, the restriction of \( \varphi_0 \) on \( C \), \( \varphi_{0,C} : C \to A_0 \), is a covering. If a connected component \( C \) of \( B \) is not \( A \)-complete, we will say that \( C \) is \( A \)-incomplete.

Suppose \( V_1 \subseteq VB \) is a subset of vertices of \( B \), \( w \) is a reduced word over \( A \), \( w \equiv c_1 \ldots c_\ell \), where \( c_1, \ldots, c_\ell \in A \) are letters. For every \( v \in V_1 \), we consider a new graph which is a path \( p(v) = e_1(v) \ldots e_\ell(v) \), consisting of edges \( e_1(v), \ldots, e_\ell(v) \) labeled by the letters \( c_1, \ldots, c_\ell \), resp., so \( \varphi(p(v)) \equiv w \). For every \( v \in V_1 \), we attach the path \( p(v) \) to \( B \) by identifying the vertices \( p(v)_- \) and \( v \). This way we obtain a labeled \( A \)-graph \( B'(V_1, w) \). We will then apply a folding process to \( B'(V_1, w) \) that inductively identifies edges \( e \) and \( e' \) whenever \( \varphi(e) = \varphi(e') \) and \( e_- = e'_- \).

As a result, we obtain an irreducible \( A \)-graph \( B(V_1, w) \) that contains the original \( A \)-graph \( B \), \( B \subseteq B(V_1, w) \), and has the following properties: \( \bar{\imath}(B(V_1, w)) = \bar{\imath}(B) \) and, for every \( v \in V_1 \), there is a unique path \( p(v) \) in \( B(V_1, w) \) such that \( p(v)_- = v \) and \( \varphi(p(v)) \equiv w \).

We also observe that \( B(V_1, w) = B \cup F(V_1, w) \), where \( F(V_1, w) \) is a forest, i.e., a disjoint collection of trees, and \( B \cap F(V_1, w) = V_1^* \), where \( V_1^* \subseteq VB \).

Lemma 3.1. Suppose that \( B \) is a finite irreducible \( A \)-graph and no connected component of \( B \) is \( A \)-complete. Then, for every subset \( V_1 \subseteq VB \), there is a reduced word \( w = w(V_1) \) over \( A \) and there is an irreducible \( A \)-graph \( B(V_1, w) \) that has the following properties. The graph \( B(V_1, w) \) contains \( B \) as a subgraph, \( B(V_1, w) = B \cup F(V_1, w) \), where \( F(V_1, w) \) is a forest, \( B \cap F(V_1, w) = V_1^* \), \( V_1^* \subseteq VB \), and \( \bar{\imath}(B(V_1, w)) = \bar{\imath}(B) \).

Furthermore, for every vertex \( v \in V_1 \), there is a unique path \( p(v) \) in \( B(V_1, w) \) such that \( p(v)_- = v \), \( \varphi(p(v)) \equiv w \), \( p(v)_+ \) has degree one, and \( p(v)_+ \notin B \).
Proof. We prove this Lemma by induction on $|V_1| \geq 1$. To make the basis step, we let $V_1 := \{ v \}$. By the hypothesis of the Lemma, there is a path $q$ in $B$ such that $\varphi(q)$ is reduced, $q_\infty = v$ and $q_- = u$, where $u$ is a vertex of $B$ with $\deg(u) < 2n$. Then there exists a letter $c \in A$ such that there is no edge in $B$ with $c_- = u$ and $\varphi(e) = c$. Taking the word $w = \varphi(q)c$, we will obtain the desired result.

To make the induction step, assume that our claim holds for a set $V_1 = \{ v_1, \ldots, v_k \}$ with $|V_1| = k$ and $V_2 = \{ v_1, \ldots, v_k, v_{k+1} \}$, where $v_{k+1} \notin V_1$. By the induction hypothesis, there exits a word $w_1 = c_1 \ldots c_l$ and an irreducible $A$-graph $B(V_1, w_1)$ with the properties stated in Lemma. Let $m$ denote the maximum of distances between vertices in the same connected component of $B$ over all components of $B$. We replace $w_1$ by the reduced word $w_2 = w_1c_{k+1}^{m+2}$ and construct the graph $B(V_1, w_2)$ for this new word $w_2$. It is clear that $B(V_1, w_2)$ has all of the properties of $B(V_1, w_1)$. In addition, for every path $p(v_i)$ in $B(V_1, w_2)$, there is a factorization $p(v_i) = p_1(v_i)p_2(v_i)$ so that $\varphi(p_2(v_i)) = c_{k+1}^{m+2}$, all the vertices of $p_2(v_i)$, except for $p_2(v_i)_+$, have degree 2 and $\deg(p_2(v_i)_+) = 1$.

Now we take a new graph which is a path $p(v_{k+1})$ with $\varphi(p(v_{k+1})) = w_2$ and attach it to $B(V_1, w_2)$ so that $p(v_{k+1}) = v_{k+1}$. Then we do foldings to produce an irreducible graph $B(V_2, w_2)$ in which there is a path that starts at $v_{k+1}$ and is labeled by $w_2$. We will denote this path by $p(v_{k+1})$. Note $B(V_1, w_2)$ is a subgraph of $B(V_2, w_2)$. Moreover, if $B(V_1, w_2) \subseteq B(V_2, w_2)$, then $\deg(p(v_{k+1})_+) = 1$ and $p(v_{k+1})_+ \notin B$, hence $B(V_2, w_2)$ has all of the desired properties and the induction step is complete. Otherwise, we may assume that $B(V_1, w_2) = B(V_2, w_2)$ and we consider two cases depending on whether or not $p(v_{k+1})_+ \in B \subseteq B(V_2, w_2)$.

First assume that $p(v_{k+1})_+ \notin B \subseteq B(V_1, w_2)$, that is, the path $p(v_{k+1})$ ends in a vertex of a tree $T \subseteq F(V_1, w_2)$, where

$$B(V_1, w_2) = B \cup F(V_1, w_2), \quad B \cap F(V_1, w_2) = V_1^*, \quad V_1^* \subseteq VB,$$

and $F(V_1, w_2)$ is a forest. Let $q_T$ be a shortest path in $T$ so that $(q_T)_- = p(v_{k+1})_+$ and $(q_T)_+ \notin B$, $\deg(q_T)_+ = 1$. We remark that $|q_T| > 0$ because, otherwise, $p(v_{k+1}) = p(v_i)$ for some $i, i \leq k$, whence $v_{k+1} = v_i$, contrary to $v_{k+1} \notin V_k$. Let $b \in A$ be the first letter of $\varphi(q_T)$. Since $w_2$ ends in $c_\ell$, it follows that $b \neq (c_\ell)^{-1}$. Let $b' \in A$ be different from the letters $b, (c_\ell)^{-1}$. Then it follows from the definitions that the word

$$w_3 \equiv w_2\varphi(q_T)b' \equiv w_1c_{k+1}^{m+2}\varphi(q_T)b'$$

is reduced and can be used as a desired word $w$ for the set $V_{k+1} = V_k \cup \{ v_{k+1} \}$.

Now consider the case when $p(v_{k+1})_+ \in B \subseteq B(V_1, w_2)$. By the definition of $m$, the vertex $p(v_{k+1})_+$ can be joined in $B(V_1, w_2)$ with a vertex $u$ of a tree $T'$, $T' \subseteq F(V_1, w_2)$, by a path $q$ so that $|q| \leq m+1$, where $u \notin B$, and $u$ is connected to a vertex in $B$ by an edge of $T'$. Consider a reduced word $w_4$ equal in $F(A)$ to $w_1c_{k+1}^{m+2}\varphi(q)$ which we write in the form

$$w_4 \equiv w_1c_{k+1}^{m+2}\varphi(q).$$

Note that at most $|q|$ letters of the reduced word $w_1c_{k+1}^{m+2}$ could cancel with those of $\varphi(q)$. Since $|q| \leq m+1$, the word $w_4$ can still be used for the set $V_1$ in place of $w_2$. As above, we construct an irreducible $A$-graph $B(V_2, w_4)$ with $\varphi(p(v_i)) = w_4$ for every $i = 1, \ldots, k+1$. The path $p(v_{k+1})$ with $p(v_{k+1})_+ = v_{k+1}$ and $\varphi(p(v_{k+1})) = z_4$ will have its terminal vertex $p(v_{k+1})_+$ on a tree $T$ of $F(V_1, w_4)$. Now we can argue as
in the foregoing case. We pick a shortest path \( q_T \) in \( T \) such that \((q_T)_- = p(v_{k+1})_+\), \((q_T)_+ \notin B\), \(\deg(q_T)_+ = 1\), and consider a reduced word \( w_5 \) such that

\[
w_5 = w_4 \varphi(q_T) c' \equiv w_4 c''_1 \cdots c''_{q+2} \varphi(q) \varphi(q_T) c',
\]

where \( c' \in A \) is now chosen so that \( c' \) is different from the first letter of \( \varphi(q_T) \) and from \( b^{-1} \), where \( b \) is the last letter of \( w_4 \). As above, we see that \(|q_T| > 0\). It is straightforward to verify that the graph \( B(V_2, w_5) \) has all of the required properties and Lemma 3.1 is proven.

We now generalize Lemma 3.1 to the situation with two subsets \( V_1, V_2 \subseteq VB \).

**Lemma 3.2.** Suppose that \( B \) is a finite irreducible \( A \)-graph and no connected component of \( B \) is \( A \)-complete. Then, for every pair of subsets \( V_1, V_2 \subseteq VB \), there are nonempty reduced words \( w_1 = w_1(V_1, V_2) \), \( w_2 = w_2(V_1, V_2) \) over \( A \) and there is an irreducible \( A \)-graph \( B(V_1, w_1; V_2, w_2) \) such that \( B(V_1, w_1; V_2, w_2) \) contains \( B \), \( B(V_1, w_1; V_2, w_2) = B \cup F(V_1, w_1; V_2, w_2) \), where \( F(V_1, w_1; V_2, w_2) \) is a forest, \( B \cap F(V_1, w_1; V_2, w_2) = V_1^* \), where \( V_1^* \subseteq VB \), and \( \bar{r}(B(V_1, w_1; V_2, w_2)) = \bar{r}(B) \).

Furthermore, for every vertex \( v_i \in V_i, i = 1, 2 \), there is a unique path \( p(v_i) \) in \( B(V_1, w_1; V_2, w_2) \) such that \( p(v_i)_- = v_i, \varphi(p(v_i)) = v_i, p(v_i)_+ \) has degree one, \( p(v_i)_+ \notin B \), and the sets \( \{p(v_i)_+ \mid v_i \in V_i\} \) are disjoint.

**Proof.** First we apply Lemma 3.1 to the set \( V_1 \) to obtain a word \( w'_1 \in F(A) \) and a graph \( B(V_1, w'_1) \) with properties of Lemma 3.1. We also apply Lemma 3.1 to the set \( V_2 \) to obtain a word \( w'_2 \in F(A) \) and a graph \( B(V_2, w'_2) \) with properties of Lemma 3.1. Since \(|A| \geq 4\), it follows that there are letters \( b_1, b_2 \in A \) so that the words \( w'_1 b_1, w'_2 b_2 \) are reduced and \( b_1 \notin \{b_2, (b_2)^{-1}\} \). Let \( k_2 = |w'_1| + |w'_2| \). Define the words \( w_1 = w'_1 b_1^k, \ w_2 = w'_2 b_2^k \). Now we attach paths \( p(v_i) \) with \( \varphi(p(v_i)) = w_i \), over all \( v_i \in V_i, i = 1, 2 \), to the graph \( B \) by identifying the vertices \( p(v_i)_- \) and \( v_i \in V_i \subseteq B \). Making foldings, we obtain an irreducible \( A \)-graph \( B(V_1, w_1; V_2, w_2) \). It is easy to see that the graph \( B(V_1, w_1; V_2, w_2) \) has all of the properties stated in Lemma 3.2.

As above, let \( B \) be a finite irreducible \( A \)-graph and \( f \in A \). Let

\[
E_f B := \{e \mid e \in EB, \varphi(e) = f\}
\]
denote the set of all edges \( e \in EB \) such that \( \varphi(e) = f \) and, analogously, denote

\[
E_{f^{-1}} B := \{e \mid e \in EB, \varphi(e) = f^{-1}\}.
\]

Let \( B_f \) denote the graph \( B \setminus (E_f B \lor E_{f^{-1}} B) \),

\[
B_f := B \setminus (E_f B \lor E_{f^{-1}} B),
\]

and, similarly, denote

\[
A_f := A \setminus \{f, f^{-1}\}.
\]

The restriction of the immersion \( \varphi_0 : B \to A_0 \) to \( B_f \) denote \( \varphi_{0,f} : B_f \to A_{0,f} \), where \( VA_{0,f} = \{0_f\} \) and \( E\!A_{0,f} = A_f \). Denote

\[
\begin{align*}
(E_f B)_- & := \{e_- \mid e \in E_f B\}, \\
(E_f B)_+ & := \{e_+ \mid e \in E_f B\}.
\end{align*}
\]

A connected component of the graph \( B_f \) is called \( A_f \)-**complete** if the degree of its every vertex is equal to \(|A_f| = 2n - 2\). Otherwise, a connected component of the graph \( B_f \) is called \( A_f \)-**incomplete**.
Let \( B^\text{com}_f \) denote the set of all \( A_f \)-complete connected components of \( B_f \) and \( B^\text{inc}_f \) denote the set of all \( A_f \)-incomplete connected components of \( B_f \). We also denote
\[
V^\text{inc}_1(f_-) := B^\text{inc}_f \cap (E_f B)_-, \quad V^\text{inc}_2(f_+) := B^\text{inc}_f \cap (E_f B)_+.
\]

Let us apply Lemma 3.2 to the irreducible \( A_f \)-graph \( B^\text{inc}_f \) and its sets of vertices \( V^\text{inc}_1(f_-) \) and \( V^\text{inc}_2(f_+) \). As a result, we obtain two reduced words \( w_1, w_2 \) over \( A_f \) and an irreducible \( A_f \)-graph
\[
B^\text{inc}_f (V^\text{inc}_1(f_-), w_1; V^\text{inc}_2(f_+), w_2)
\]
with the properties of Lemma 3.2.

Now we construct more labeled \( A \)-graphs in the following fashion. For every edge \( e \in E_f B \), we consider a new graph which is a path \( p_1(e) e p_2(e)^{-1} \) so that \( \varphi(p_1(e)) \equiv w_1, \varphi(e) = f, \varphi(p_2(e)) \equiv w_2 \). We replace every \( e \in E_f B \) in \( B \) by such a path \( p_1(e) e p_2(e)^{-1} \) and then do edge foldings to get an irreducible \( A \)-graph which we denote \( B[f] \). We will call this graph \( B[f] \), obtained by the described application of Lemma 3.2 an \( f \)-treatment of \( B \). It is clear that the graph \( B[f] \) can be thought of as the image of \( B \) under the automorphism of the free group \( F(A) \) that takes \( f \) to \( w_1 f w_2^{-1} \) and takes \( a \) to \( a \) if \( a \in A_f \). We also observe that the connected components of \( B[f] \) and those of \( B[f]_f \) are in the natural bijective correspondence. Moreover, it follows from the definitions and Lemma 3.2 that a connected component of \( B[f] \) is \( A_f \)-complete if and only if its image in \( B[f]_f \) is \( A_f \)-complete and, if they are both \( A_f \)-complete, then they are identical.

Suppose \( c \in A, c \not\in \{ f, f^{-1} \} \). Using the notation analogous to what we introduced above for \( f \), we consider the graph
\[
B[f]_c := B[f] \setminus (E_c B[f] \vee E_{c^{-1}} B[f]).
\]
As above, we consider a partition
\[
B[f]_c = B[f]_c^\text{inc} \cup B[f]_c^\text{com},
\]
where \( B[f]_c^\text{inc} \) is the union of \( A_c \)-incomplete connected components of \( B[f]_c \) and \( B[f]_c^\text{com} \) is the union of \( A_c \)-complete connected components of \( B[f]_c \).

**Lemma 3.3.** Suppose \( B \) is a finite irreducible \( A \)-graph, \( |A| \geq 6 \), no connected component of \( B \) is a complete \( A \)-graph, \( f \in A \), and \( B[f] \) is an \( f \)-treatment of \( B \). Then, for every \( c \in A \setminus \{ f, f^{-1} \} \), considering the sets \( V(B[f]_c^\text{com}), V(B[f]_f^\text{com}) \) as subsets of \( V B[f] \), one has that
\[
V(B[f]_c^\text{com}) \subseteq V(B[f]_f^\text{com})
\]
and either \( V(B[f]_c^\text{com}) = V(B[f]_f^\text{com}) = \emptyset \) or \( V(B[f]_c^\text{com}) \neq V(B[f]_f^\text{com}) \). Moreover,
\[
|V(B[f]_c^\text{com})| = |V(B[f]_f^\text{com})|.
\]

**Proof.** As was observed above, connected components of \( B_f \) and those of \( B[f]_f \) are in the natural bijective correspondence, moreover, a connected component \( C \) of \( B_f \) is \( A_f \)-complete if and only if its image in \( B[f]_f \) is \( A_f \)-complete and, if, they are both \( A_f \)-complete, then they are identical. Hence, the \( A_f \)-graphs \( B^\text{com}_f \) and \( B[f]_f^\text{com} \) are isomorphic and \( |V(B[f]_f^\text{com})| = |V(B^\text{com}_f)| \).

Now suppose \( v \in V(B[f]_c^\text{com}) \). Considering \( v \) as a vertex of \( B[f] \), we see that there are edges \( e_1, e_2 \) in \( B[f] \) starting at \( v \) such that \( \varphi(e_1) = \varphi(e_2^{-1}) = f \). However, it follows from Lemma 3.2 that no vertex \( u \) of \( B[f] \) has distinct edges \( e_1, e_2 \) starting
Hence, it follows from these equalities and inequalities that
\[ e \in V(B[f]^\text{com}) \] and, hence, \( B \) contains an A-complete connected component. This contradiction to the assumption that \( B \) is empty proves that \( V(B[f]^\text{com}) \subset V(B[f]^\text{com}) \) whenever \( V(B[f]^\text{com}) \neq \emptyset \), as desired. 

Lemma 3.4. Suppose \( B \) is a finite irreducible A-graph, \( |A| \geq 6 \), and no connected component of \( B \) is a complete A-graph. Then there exists a finite sequence of \( f_1, \ldots, f_{t+1} \)-treatments of the graph \( B \) so that the resulting graph, denoted \( B[f_1, \ldots, f_{t+1}] \), has the following property. For \( f \in A \), denote \( B[f_1, \ldots, f_{t+1}]_f := B[f_1, \ldots, f_{t+1}] \setminus (E_f(B[f_1, \ldots, f_{t+1}]) \cup E_{f-1}(B[f_1, \ldots, f_{t+1}])) \).

For each \( f \in A \), every connected component of \( B[f_1, \ldots, f_{t+1}]_f \) is A-f-incomplete. In addition, for every edge \( e \in E(B[f_1, \ldots, f_{t+1}]) \) such that \( \varphi(e) = e_{t+1} \), one has \( \deg_e = \deg_{e+} = 2 \) and, if \( h_1(e)h_2(e) \) is a reduced path in \( B[f_1, \ldots, f_{t+1}] \), where \( h_1(e), h_2(e) \) are edges, then the labels \( \varphi(h_1(e)), \varphi(h_2(e)) \) are independent of \( e \).

Proof. Pick an arbitrary letter \( f_1 \in A \) and do an \( f_1 \)-treatment of the graph \( B \). Assuming that \( B[f_1]^\text{com} \neq \emptyset \), we can see from Lemma 3.3 that the result is a new irreducible A-graph \( B[f_1] \) such that, for every \( c \in A \), where \( c \notin \{f_1, f_{-1}\} \), we have
\[ V(B[f_1]^\text{com}) \subset V(B[f_1]^\text{com}) \quad |V(B[f_1]^\text{com})| = |V(B[f_1]^\text{com})| \]

Picking an edge \( f_2 \) such that \( f_2 \in A \), \( f_2 \notin \{f_1, f_{-1}\} \), we will do a second \( f_2 \)-treatment of the graph \( B[f_1] \) and get a new irreducible A-graph \( B[f_1, f_2] := B[f_1][f_2] \).

Assuming that \( B[f_1, f_2]^\text{com} \neq \emptyset \), we obtain from Lemma 3.3 that, for every \( c \in A \) such that \( c \notin \{f_2, f_{-1}\} \), we have
\[ V(B[f_1, f_2]^\text{com}) \subset V(B[f_1, f_2]^\text{com}) \quad |V(B[f_1, f_2]^\text{com})| = |V(B[f_1, f_2]^\text{com})| \]

Then we pick an edge \( f_3 \), where \( f_3 \notin A \), \( f_3 \notin \{f_2, f_{-1}\} \), and do a third \( f_3 \)-treatment of the graph \( B[f_1, f_2] \) to construct an irreducible A-graph \( B[f_1, f_2, f_3] = B[f_1][f_2][f_3] \) with further decreased \( |V(B[f_1, f_2, f_3]^\text{com})| \) and so on. Assuming for each \( j = 1, \ldots, i \) that \( B[f_1, \ldots, f_i]^\text{com} \neq \emptyset \), we obtain from Lemma 3.3 that, for every \( c \in A \) such that \( c \notin \{f_j, f_{j-1}\} \), it is true that
\[ V(B[f_1, \ldots, f_i]^\text{com}) \subset V(B[f_1, \ldots, f_i]^\text{com}) \quad |V(B[f_1, \ldots, f_i]^\text{com})| < |V(B[f_1, \ldots, f_{i-1}]^\text{com})| \quad |V(B[f_1, \ldots, f_i]^\text{com})| = |V(B[f_1, \ldots, f_{i-1}]^\text{com})| \]

Hence, it follows from these equalities and inequalities that
\[ |V(B[f_1, \ldots, f_{i+1}]^\text{com})| < \cdots < |V(B[f_1, \ldots, f_j]^\text{com})| < \cdots < |V(B[f_1, f_2]^\text{com})| < |V(B[f_1]^\text{com})|. \]
Therefore, the graph $B[f_1, \ldots, f_{\ell+1}]^\text{com}$ will eventually become empty and then $B[f_1, \ldots, f_{\ell+1}]^\text{com}$ will be empty for every $c \in A$. Thus we may suppose that $B[f_1, \ldots, f_{\ell+1}]^\text{com}$ is empty for some $\ell \geq 1$.

We will do one more $f_{\ell+1}$-treatment of the graph $B[f_1, \ldots, f_{\ell}]$. Pick an edge $f_{\ell+1}$ such that $f_{\ell+1} \in A$, $f_{\ell+1} \notin \{f_{\ell}, f_{\ell}^{-1}\}$, do an $f_{\ell+1}$-treatment of the graph $B[f_1, \ldots, f_{\ell}]$ and obtain an irreducible A-graph $B[f_1, \ldots, f_{\ell+1}] := B[f_1, \ldots, f_{\ell}]|_{f_{\ell+1}}$. It follows from Lemma 3.3 that, for every $c \in A$, the set $B[f_1, \ldots, f_{\ell+1}]^\text{com}$ is still empty, in particular, $B[f_1, \ldots, f_{\ell+1}]^\text{com} = \emptyset$. In addition, by the definitions and Lemma 3.2 we will also have the claimed property of edges $e \in E B[f_1, \ldots, f_{\ell+1}]$ such that $\varphi(e) = f_{\ell+1}$. □

**Lemma 3.5.** Suppose $|A| \geq 4$, $d_1, d_2 \in A$ and $w$ is a nonempty reduced word over $A$. Then there are letters $b_1, b_2 \in A$ such that the word $d_1(b_1 wb_2)^2 d_2$ is reduced. Furthermore, let $c \in A$, $c \notin \{b_2, b_2^{-1}\}$, and $n_w := |w| + 1$. Then the word

$$z_4(w) := b_1 wb_2 c_1^{n_w+1} b_2 b_1 wb_2 c_1^{n_w+2} b_2 b_1 wb_2 c_1^{n_w+3} b_2 b_1 wb_2 c_1^{n_w+4} b_2$$

(3.1)

has the following properties. If $(b_2 c_1^{n_w+i} b_2)^k$, where $k = \pm 1, i = 1, 2, 3, 4$, is a subword of $z_4(w)$, then $k = 1$ and the location of the subword $b_2 c_1^{n_w+i} b_2$ is standard, i.e., it is the suffix of length $n_w + i + 2$ of the $i$th syllabus $b_1 wb_2 c_1^{n_w+i} b_2$ of $z_4(w)$.

Moreover, the word $d_1 z_4(w) d_2$ is reduced.

**Proof.** Since $|A| \geq 4$, there are at least two letters $x \in A$ such that the word $d_1 x w$ is reduced. Also, there are at least two letters $y \in A$ such that the word $w y d_2$ is reduced. If $d_1(x w y)^2 d_2$ is not reduced then $x = y^{-1}$. For every $x$ with $d_1 x w$ being reduced, there is at least one $y$ with $w y d_2$ being reduced such that $d_1(x w y)^2 d_2$ is not reduced. Hence there exist $x$ and $y$ in $A$ such that $d_1(x w y)^2 d_2$ is reduced. Denote $b_1 := x$ and $b_2 := y$.

Since $c \notin \{b_2, b_2^{-1}\}$ and the maximal power of $c$ that can occur in $b_1 wb_2$ is $c_1^{n_w+1} = c_1^{n_w}$, it follows from the definition 3.1 of the word $z_4(w)$ that $b_2 c_1^{n_w+i} b_2$ can only occur in $z_4(w)^{\pm 1}$ as the suffix of length $n_w + i + 2$ of the $i$th syllabus $b_1 wb_2 c_1^{n_w+i} b_2$ of $z_4(w)$.

It remains to note that the word $d_1 z_4(w) d_2$ is reduced because the word $d_1(x w y)^2 d_2$ is reduced. □

4. PROOF OF THEOREM 1.1

As in Sect. 2, consider the graphs $W, X, Y, Z$.

**Lemma 4.1.** Suppose either graph $Q$, where $Q \in \{X, Y\}$, contains a path $p_Q$ so that every vertex of $p_Q$ has degree 2 in $Q$, $\beta_Q(p_Q) \equiv z_4(w)$, where $w$ is a nonempty reduced word over the alphabet $A = EZ$ and $z_4(w)$ is the word defined by (3.1). Let $z_4(w) = z_{41} z_{42}$ be a factorization of $z_4(w)$ so that

$$z_{41} \equiv b_1 wb_2 c_1^{n_w+1} b_2 b_1 wb_2 c_1^{n_w+2} b_2, \quad z_{42} \equiv b_1 wb_2 c_1^{n_w+3} b_2 b_1 wb_2 c_1^{n_w+4} b_2$$

and let $v_2(p_Q)$ be a vertex of the path $p_Q$ that defines a corresponding factorization $p_Q = p_1 p_2$ so that $\varphi(p_1) \equiv z_{41}, \varphi(p_2) \equiv z_{42}$.

Furthermore, assume that there exists a vertex $u \in VW$, where $W = \text{core}(X \times_{Z} Y)$, such that either $\alpha_X(u) = v_2(p_X)$ and $\alpha_Y(u) \in p_Y$ or $\alpha_X(u) \in p_X$ and $\alpha_Y(u) = v_2(p_Y)$. Then $\alpha_X(u) = v_2(p_X), \alpha_Y(u) = v_2(p_Y)$, and there is a path $p$ in $W$ such that $\alpha_X(p) = p_X$ and $\alpha_Y(p) = p_Y$. 
Proof. For definiteness, suppose \( u \in VW \) is such that \( \alpha_X(u) = v_2(p_X) \) and \( \alpha_Y(u) \in p_Y \).
Let \( p_Y = p_{Y1}p_{Y2} \) be the factorization of \( p_Y \) defined by the vertex \( \alpha_Y(u) \).
Pick a path \( p_{Y1}r \), \( i^* = 1, 2 \), such that \( |p_{Y1}r| \geq \frac{1}{2}|p_Y| \). Since the factorization of \( p_X \) defined by the vertex \( \alpha_X(u) = v_2(p_X) \) defines the factorization \( z_4(w) = z_{41}z_{42} \) of the word \( z_4(w) = \beta_X(p_X) \), it follows that if \( i^* = 2 \) then the word \( \beta_Y(p_{Y1}r) \) contains a subword \( b_2c^{n_2 + 3}b_2 \) or contains a subword \( b_2^{-1}c^{-n_2 - 3}b_2^{-1} \).

The second subcase, however, is impossible by Lemma 4.2. Similarly, if \( i^* = 1 \) then the word \( \gamma_Y(p_{Y1}r) \) contains a subword \( b_2c^{n_2 + 2}b_2 \) or contains a subword \( b_2^{-1}c^{-n_2 - 2}b_2^{-1} \).

The second subcase is impossible by Lemma 3.5. In either case \( i^* = 1, 2 \), we can apply Lemma 3.5 to the subword \( b_2c^{n_2 + 3}b_2 \) or the subword \( b_2^{-1}c^{-n_2 - 3}b_2^{-1} \) of \( \beta_Y(p_Y) \) and obtain the desired conclusion. \( \square \)

We now define a special type of transformations over the graphs \( W, X, Y, Z \), called \((f, p)\)-transformations.

Suppose that \( f \) is an edge of the graph \( Z \) and \( p = e_1 \ldots e_\ell \) is a path in \( Z \) such that \( p_+ = (e_1)_+ \ldots p_+ = (e_\ell)_+ = f_+ \), and there are no occurrences of \( f, f^{-1} \) among edges \( e_1, \ldots, e_\ell \).

Let \( Q \in \{W, X, Y, Z\} \) and \( \varphi : Q \to Z = U \) be a canonical immersion. Consider a set \( H_f \) of all edges \( h \) in \( EW \lor EX \lor EY \lor EZ \) that are sent by \( \varphi \) to \( f \). We replace every edge \( h \in H_f \) with a path \( p(h) = e_1(h) \ldots e_\ell(h) \) such that \( \varphi(e_i(h)) := \varphi(e_i) \) for every \( i = 1, \ldots, \ell \), and we extend every map \( \nu \in \{\alpha_X, \ldots, \beta_Y\} \) to paths \( p(h) \) so that if \( \nu : Q \to Q' \), where \( Q, Q' \in \{W, X, Y, Z\} \), and \( \nu(h) = h' \), then we extend \( \nu \) to paths \( p(h), p(h') \) by setting \( \nu(e_i(h)) := e_i(h') \), \( \ldots \), \( \nu(e_\ell(h)) := e_\ell(h') \). Thus obtained graphs we denote by \( Q_{fp} \), where \( Q \in \{W, X, Y, Z\} \), and thus obtained maps we denote by \( \alpha_{X_{fp}}, \ldots, \beta_{Y_{fp}} \).

We now perform folding process over every modified graph \( Q_{fp}, Q \in \{W, X, Y, Z\} \). Recall that this is an inductive procedure which identifies every pair of oriented edges \( g_1, g_2 \in EQ_{fp} \) whenever \((g_1)_- = (g_2)_- \) and \( g_1, g_2 \) have the same labels \( \varphi(g_1), \varphi(g_2) \).

Note that folding process decreases by one the reduced rank \( r(Z) \) because the new path \( p(f) = e_1(f) \ldots e_\ell(f) \) that replaces the edge \( f \) in \( Z \) will be attached to the path \( p = e_1 \ldots e_\ell \) in \( Z \setminus \{f, f^{-1}\} \) thus producing a graph \( \bar{Z}_{fp} \) with \( r(\bar{Z}_{fp}) = r(Z) - 1 \).

Clearly, \( \bar{Z}_{fp} \) is a bouquet of \( r(Z) - 1 \) circles and \( \text{core}(\bar{Z}_{fp}) = \bar{Z}_{fp} \).

Furthermore, when a folding process applied to \( Q_{fp}, Q \in \{W, X, Y, Z\} \), is complete and produces a graph \( \bar{Q}_{fp} \), we will take the core of \( \bar{Q}_{fp} \) thus obtaining a graph \( Q_p = \text{core}(\bar{Q}_{fp}) \). It follows from the definitions that we will have maps \( \alpha_{X_p}, \ldots, \beta_{Y_p} \) with properties of original maps \( \alpha_X, \ldots, \beta_Y \). This alteration of the graphs \( W, X, Y, Z \) by means of an edge \( f \in EZ \) and a path \( p \in Z \) will be called an \((f, p)\)-transformation over the graphs \( W, X, Y, Z \).

We will say that an \((f, p)\)-transformation is conservative if it preserves the numbers \( \bar{r}(X), \bar{r}(Y), \bar{r}(W) \) for every connected component \( W_s \) of \( W, s \in S(H, K) \), and the core \( \text{core}(X_p \times Z_p Y_p) \) of the pullback \( X_p \times Z_p Y_p \) coincides with the graph \( W_p \). Thus a conservative \((f, p)\)-transformation decreases \( \bar{r}(Z) \) by one while keeping the numbers \( \bar{r}(X), \bar{r}(Y), \bar{r}(W) \) unchanged.

Here is our principal technical result that will be used to prove Theorem 11.

**Lemma 4.2.** Suppose \( \bar{r}(Z) \geq 2, U = Z, \gamma = \text{id}_Z \), and neither of the maps \( \beta_X : X \to Z, \beta_Y : Y \to Z \) is a covering. Then there is an automorphism \( \tau \) of the
free group \(F(EZ) = \pi_1(Z)\) such that Stallings graphs of subgroups \(\tau(H), \tau(K), \tau(\bigvee s \in S(H,K)(H \cap sKs^{-1}))\), denoted \(X^\tau, Y^\tau, W^\tau\), resp., have no vertices of degree 1 and there exists a conservative \((f,p)\)-transformation over the graphs \(W^\tau, X^\tau, Y^\tau, Z^\tau = Z\).

Proof. Assume that neither of the maps \(\beta_X : X \to Z\), \(\beta_Y : Y \to Z\) is a covering. Consider the disjoint union \(X \vee Y\) of \(X, Y\) as a graph \(B\) and the set \(EZ\) as the alphabet \(A\). Let \(\varphi : EB \to A\) be defined on \(EX\) as the restriction of \(\beta_X\) and on \(EY\) as the restriction of \(\beta_Y\). We also consider \(W\) as an \(A\)-graph by using the map \(\varphi : EW \to A\), where \(\varphi\) is the restriction of \(\beta_X\alpha_X = \beta_Y\alpha_Y\).

Since neither of \(X, Y\) is \(A\)-complete, we can apply Lemma 3.4 and find a sequence of \(f_1, \ldots, f_{t+1}\)-treatments for \(B\) which transform the graph \(B = X \vee Y\) into \(B[f_1, \ldots, f_{t+1}]\) with the properties of Lemma 3.4. Note that an \(f\)-treatment of \(B = X \vee Y\) can equivalently be described as an application of a suitable automorphism \(\tau_f\) of the free group \(F(A) = \pi_1(Z)\). We specify that this automorphism \(\tau_f\) is the composition of an automorphism \(\theta_f\), given by \(\theta_f(f) = w_1f w_2^{-1}\), \(\theta_f(a) = a\) for every \(a \in A_f\), where \(w_1, w_2\) are some words over \(A_f\), and an inner automorphism of \(F(A)\) applied, if necessary, to move the base vertices of graphs \(\theta_f(X)\), \(\theta_f(Y)\), representing subgroups \(\theta_f(H), \theta_f(K)\), resp., so that the base vertices, after the move, would be in \(\text{core}(\theta_f(H)), \text{core}(\theta_f(K))\) and the correspondence between graphs and subgroups would be preserved. Therefore, the composition of these \(f_1, \ldots, f_{t+1}\)-treatments can be induced by a suitable automorphism of the free group \(F(A)\) applied to subgroups \(H, K, \bigvee \in S(H, K)(H \cap sKs^{-1})\) and to corresponding Stallings graphs \(X, Y, W\). By Lemmas 3.4, 5.2 we may assume that, for every \(c \in A\), the graph \(B[f_1, \ldots, f_{t+1}]c\) has no \(A_c\)-complete component and that every edge \(e \in EB[f_1, \ldots, f_{t+1}]c\) such that \(\varphi(e) = f_{t+1}\) is contained in a reduced path \(h_1(e) \overline{e} h_2(e)\), where \(h_1(e), h_2(e)\) are edges, so that \(\deg e_- = \deg e_+ = 2\) and the letters \(\varphi(h_1(e)), \varphi(h_2(e))\) are independent of \(e\).

Denote
\[
\varphi(h_1(e)) = d_1, \quad \varphi(h_2(e)) = d_2,
\]
where \(d_1, d_2 \in A_{f_{t+1}} = EZ \setminus \{f_{t+1}, f_{t+1}^{-1}\}\). Since \(B = X \vee Y\), we can represent the graph \(B[f_1, \ldots, f_{t+1}]\) in the form
\[
B[f_1, \ldots, f_{t+1}] = X[f_1, \ldots, f_{t+1}] \vee Y[f_1, \ldots, f_{t+1}],
\]
where \(X[f_1, \ldots, f_{t+1}] = X^\tau\) is obtained from \(X\) by these \(f_1, \ldots, f_{t+1}\)-treatments and \(Y[f_1, \ldots, f_{t+1}] = Y^\tau\) is obtained from \(Y\) by the \(f_1, \ldots, f_{t+1}\)-treatments. Similarly, let \(W[f_1, \ldots, f_{t+1}] = W^\tau\) denote the graph obtained from \(W\) by the \(f_1, \ldots, f_{t+1}\)-treatments.

Consider the irreducible \(A_{f_{t+1}}\)-graph
\[
B[f_1, \ldots, f_{t+1}]_{f_{t+1}} = X[f_1, \ldots, f_{t+1}]_{f_{t+1}} \vee Y[f_1, \ldots, f_{t+1}]_{f_{t+1}},
\]
and apply Lemma 3.1 to this graph and to the vertex set \(V_1 = VB[f_1, \ldots, f_{t+1}]_{f_{t+1}}\). According to Lemma 3.1, there exists a reduced nonempty word \(w\) over the alphabet \(A_{f_{t+1}} = EZ \setminus \{f_{t+1}, f_{t+1}^{-1}\}\) with the properties of Lemma 3.1. In particular, for every vertex \(v \in VB[f_1, \ldots, f_{t+1}]_{f_{t+1}}\), the \(A_{f_{t+1}}\)-graph \(B[f_1, \ldots, f_{t+1}]_{f_{t+1}}\) contains no path \(p\) such that \(p_- = v\) and \(\varphi(p) \equiv w\).

Since \(A = EZ\) and \(|A| \geq 6\), we have \(|A_{f_{t+1}}| \geq 4\). Hence, Lemma 3.5 applies to the word \(w\), to the alphabet \(A_{f_{t+1}}\) and to the letters \(d_1, d_2 \in A_{f_{t+1}}\) defined
by equalities \[1.1\]. By Lemma \[8.5\] there are letters \(b_1, b_2 \in A_{f_{\ell+1}}\) such that the word \(d_1(b_w b_2)^2 d_2\) is reduced. Furthermore, let \(c \in A_{f_{\ell+1}}, c \not\in \{b_2, b_2^{-1}\}\), and \(n_w = |w| + 1\). Then the word \(z_4(w)\) over \(A_{f_{\ell+1}}\), given by formula (3.1), has the properties stated in Lemma 3.5.

Now we perform an \((f_{\ell+1}, z_4(w))-\)transformation over the graphs

\[W[f_1, \ldots, f_{\ell+1}] = W^\tau, \quad X[f_1, \ldots, f_{\ell+1}] = X^\tau, \quad Y[f_1, \ldots, f_{\ell+1}] = Y^\tau, \quad Z.\]

Making this transformation turns every edge \(e\) such that \(\varphi(e) = f_{\ell+1}\) into a path \(p = p(e) = p_+ \cdots p_+ = e_+\), and \(\varphi(p) \equiv z_4(w)\). Let \(B(f_{\ell+1}, z_4), X(f_{\ell+1}, z_4), Y(f_{\ell+1}, z_4), w(f_{\ell+1}, z_4)\) denote the resulting \(A_{f_{\ell+1}}\)-graphs.

Observe that if \(\mu : F(A) \to F(A_{f_{\ell+1}})\) is the epimorphism defined by \(\mu(b) = b\) for every \(b \in A_{f_{\ell+1}}\) and \(\mu(f_{\ell+1}) = z_4(w)\) then we have

\[W(f_{\ell+1}, z_4) = \mu(W[f_1, \ldots, f_{\ell+1}]) = \mu(W^\tau),\]

\[X(f_{\ell+1}, z_4) = \mu(X[f_1, \ldots, f_{\ell+1}]) = \mu(X^\tau),\]

\[Y(f_{\ell+1}, z_4) = \mu(Y[f_1, \ldots, f_{\ell+1}]) = \mu(Y^\tau).\]

A path \(p = p(e)\) in graphs \(X(f_{\ell+1}, z_4), Y(f_{\ell+1}, z_4), W(f_{\ell+1}, z_4)\) such that \(\varphi(p) = z_4(w)\) and \(p\) results from an edge \(e\) with \(\varphi(e) = f_{\ell+1}\) will be called a standard \(z_4\)-path. Let \(p = p_1 p_2 p_3 p_4\) be a factorization of \(p\) so that \(\varphi(p_1) \equiv b_1 w b_2 e^{n_1 + b_2}\), where \(i = 1, 2, 3, 4\). Denote \(v_2(p) := (p_2)_+\).

Recall that \(Z\) has a single vertex and an \((f_{\ell+1}, z_4(w))-\)transformation converts \(Z\) into \(Z = Z(f_{\ell+1}) = Z \setminus \{f_{\ell+1}, f_{\ell+1}\}\) such that \(E_{Z(f_{\ell+1})} = A_{f_{\ell+1}}\). Let

\[\varphi_{f_{\ell+1}} : B(f_{\ell+1}, z_4) = X(f_{\ell+1}, z_4) \lor Y(f_{\ell+1}, z_4) \to Z_{f_{\ell+1}}\]

denote the corresponding immersion. Consider the pullback

\[X(f_{\ell+1}, z_4) \times Z_{f_{\ell+1}} Y(f_{\ell+1}, z_4)\]

and its core \(\overline{W} := \text{core}(X(f_{\ell+1}, z_4) \times Z_{f_{\ell+1}} Y(f_{\ell+1}, z_4))\).

Let us prove the equality \(\overline{W} = W(f_{\ell+1}, z_4)\).

Let

\[\overline{\alpha}_X : \overline{W} \to X(f_{\ell+1}, z_4), \quad \overline{\alpha}_Y : \overline{W} \to Y(f_{\ell+1}, z_4)\]

denote the projection maps. Suppose \(u \in V\overline{W}\) is a vertex such that \(\overline{\alpha}_X(u) = v_2(p_X)\), where \(p_X\) is a standard \(z_4\)-path of \(X(f_{\ell+1}, z_4)\).

First we assume that

\[\overline{\alpha}_Y(u) \in Y[f_1, \ldots, f_{\ell+1}]_{f_{\ell+1}}, \quad (4.2)\]

here the graph \(Y[f_1, \ldots, f_{\ell+1}]_{f_{\ell+1}}\) is regarded as a subgraph of \(Y(f_{\ell+1}, z_4)\).

Note that the conclusion of Lemma 6.1 holds for the graph \(B[f_1, \ldots, f_{\ell+1}]_{f_{\ell+1}}\), and for the reduced word \(w_1 = b w \) in place of \(w\). Hence, the graph \(Y[f_1, \ldots, f_{\ell+1}]_{f_{\ell+1}}\) contains no path \(r\) such that \(r\) starts at \(\overline{\alpha}_Y(u)\) and \(r\) has the label \(\varphi_{f_{\ell+1}}(r) = b_1 w\). Since there is a path \(p'\) in \(X(f_{\ell+1}, z_4)\) such that \(p'\) starts at \(\overline{\alpha}_X(u) = v_2(p_X), \varphi_{f_{\ell+1}}(p') = b_1 w b_2 e^{n_3 + b_2}\), and all vertices of \(p'\) have degree 2, it follows that there is a path \(q\) in \(Y(f_{\ell+1}, z_4)\) such that \(q\) starts at \(\overline{\alpha}_Y(u)\) and

\[\varphi_{f_{\ell+1}}(q) \equiv \varphi_{f_{\ell+1}}(p') \equiv b_1 w b_2 e^{n_3 + b_2}.

Let \(q = q_1 q_2\) be the factorization of \(q\) defined so that

\[\varphi_{f_{\ell+1}}(q_1) \equiv b_1 w, \quad \varphi_{f_{\ell+1}}(q_2) \equiv b_2 e^{n_3 + b_2}.\]
By the above observation based on Lemma 3.1, the path \( q_1 \) may not be entirely contained in \( Y[f_1, \ldots, f_{t+1}]_{f_{t+1}} \subseteq Y(f_{t+1}, z_4) \). On the other hand, if \( (q_1)_+ \in Y[f_1, \ldots, f_{t+1}]_{f_{t+1}} \subseteq Y(f_{t+1}, z_4) \), then, in view of (4.2), the path \( q_1 \) would have to contain a standard \( z_4 \)-path of \( Y(f_{t+1}, z_4) \) which is impossible for \( |q_1| < |z_4(w)| \). Therefore, \( (q_1)_+ \notin Y[f_1, \ldots, f_{t+1}]_{f_{t+1}} \subseteq Y(f_{t+1}, z_4) \) and the vertex \( (q_1)_+ \) must belong to a standard \( z_4 \)-path \( p_Y \) of \( Y(f_{t+1}, z_4) \).

Since \( |q_1| = |w| + 1, |q_2| = |w| + 6 \), and \( |p_Y| = |z_4(w)| = 8|w| + 26 \), it follows from (4.2) and from \( (q_1)_+ \in p_Y \) that \( q_2 \) is a subpath of \( p_Y^{\pm 1} \) and there is a factorization \( p_Y = p_Y \varphi p_Y \) defined by the vertex \( (q_2)_+ \in p_Y \), where a shortest path out of \( p_Y \) contains \( q_2^{\pm 1} \) and

\[
\min(|p_Y|, |p_Y|) \leq |q| = 2|w| + 7. \quad (4.3)
\]

On the other hand, if \( p_Y = p_Y \varphi p_Y = p_Y \varphi p_Y \varphi \) are factorizations of \( p_Y \) so that the words \( \varphi_{f_{t+1}}(p_Y) \), \( \varphi_{f_{t+1}}(p_Y) \) contain the standard occurrence of the word \( b_2 c^{a_\pm 3} b_2 \) in \( z_4(w) \), then \( |p_Y| \geq 6|w| + 18 \) and \( |p_Y| \geq 3|w| + 14 \). These inequalities, in view of (4.3), mean that the occurrence of

\[
(b_2 c^{a_\pm 3} b_2)^{\pm 1} = \varphi_{f_{t+1}}(q_2)^{\pm 1}
\]

in \( z_4(w) \equiv \varphi_{f_{t+1}}(p_Y) \) is not standard. This contradiction to Lemma 3.5 proves that the inclusion (4.2) is impossible.

Thus it is shown that, for every vertex \( u \in V \tilde{W} \), if \( \tilde{\alpha}_X(u) = v_2(p_X) \) for some standard \( z_4 \)-path \( p_X \) of \( X(f_{t+1}, z_4) \), then \( \tilde{\alpha}_Y(u) = p_Y \), where \( p_Y \) is a standard \( z_4 \)-path in \( Y(f_{t+1}, z_4) \). Switching the graphs \( X(f_{t+1}, z_4) \) and \( Y(f_{t+1}, z_4) \) in the above arguments, we can analogously show that, for every vertex \( u \in \tilde{W} \), if \( \tilde{\alpha}_Y(u) = v_2(p_Y) \) for some standard \( z_4 \)-path \( p_Y \) of \( Y(f_{t+1}, z_4) \), then \( \tilde{\alpha}_X(u) = p_X \), where \( p_X \) is a standard \( z_4 \)-path in \( X(f_{t+1}, z_4) \).

Now we can use Lemma 4.2 to conclude that, for every vertex \( u \in V \tilde{W} \), if \( \tilde{\alpha}_X(u) = v_2(p_X) \), where \( p_X \) is a standard \( z_4 \)-path of \( X(f_{t+1}, z_4) \), or if \( \tilde{\alpha}_Y(u) = v_2(p_Y) \), where \( p_Y \) is a standard \( z_4 \)-path of \( Y(f_{t+1}, z_4) \), then, in either case, \( \tilde{\alpha}_X(u) = v_2(p_X) \) and \( \tilde{\alpha}_Y(u) = v_2(p_Y) \), where both \( p_X, p_Y \) are standard \( z_4 \)-paths. In addition, there exists a path \( p_W \) in \( \tilde{W} \) such that \( \tilde{\alpha}_X(p_W) = p_X \) and \( \tilde{\alpha}_Y(p_W) = p_Y \). Consequently, if \( p \) is a path in \( W \) such that one of \( \tilde{\alpha}_X(p) \), \( \tilde{\alpha}_Y(p) \) is a standard \( z_4 \)-path, then both \( \tilde{\alpha}_X(p), \tilde{\alpha}_Y(p) \) must be standard \( z_4 \)-paths. Now the desired equality \( \tilde{W} = W(f_{t+1}, z_4) \) becomes apparent.

Since \( \tilde{r}(X) = \tilde{r}(Y(f_{t+1}, z_4)) = \tilde{r}(X^\tau) \) and \( \tilde{r}(Y) = \tilde{r}(Y(f_{t+1}, z_4)) = \tilde{r}(Y^\tau) \), this \( (f_{t+1}, z_4(w))-\)transformation over the graphs \( W^\tau, X^\tau, Y^\tau, Z \) is conservative, as required. It remains to note that \( A_{f_{t+1}} \)-graphs

\[
X(f_{t+1}, z_4) = \mu(X^\tau), \quad Y(f_{t+1}, z_4) = \mu(Y^\tau)
\]

are \( A_{f_{t+1}} \)-incomplete and Lemma 4.3 is proven.

Proof of Theorem 1.1 Since neither of subgroups \( H, K \) has finite index in \( L = \langle H, K, S(H, K) \rangle \), we conclude that \( r(L) \geq 2 \). There is nothing to prove if \( r(L) = 2 \). Hence, we may assume that \( r(L) = n > 2 \). It follows from Lemma 4.2 and the definitions that there is an epimorphism \( \eta_1 : L \to F_{n-1} \), where \( F_{n-1} \) is a free group of rank \( n - 1 \), with the following properties. The restriction of \( \eta_1 \) on \( H \) and on \( K \) is injective, a set \( S(\eta_1(H), \eta_1(K)) \) for subgroups \( \eta_1(H), \eta_1(K) \) of \( F_{n-1} \) can be taken
to be $\eta_1(S(H, K))$, for every $s \in S(H, K)$, the restriction
$$\eta_{1,s} : H \cap sKs^{-1} \to \eta(H) \cap \eta(s)\eta(K)\eta(s)^{-1}$$
of $\eta_1$ on $H \cap sKs^{-1}$ to $\eta(H) \cap \eta(s)\eta(K)\eta(s)^{-1}$ is surjective, and neither of $\eta_1(H), \eta_1(K)$ has finite index in the group
$$F_n-1 = \eta_1(L) = \langle \eta_1(H), \eta_1(K), \eta_1(S(H, K)) \rangle.$$ Iterating this argument, we obtain a desired epimorphism $\eta : L \to F_2$. □

Proof of Corollary 1.2. If one of subgroups $H$, $K$ has infinite index in their join $\langle H, K \rangle$ then one of $H$, $K$ has also infinite index in the generalized join $L = \langle H, K, S(H, K) \rangle$ and Theorem 1.1 applies. By Theorem 1.1 there is an epimorphism $\eta : L \to F_2$ that has the required properties of $\zeta$. □

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