FUSION OF HAMILTONIAN LOOP GROUP MANIFOLDS AND COBORDISM

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Abstract. We construct an oriented cobordism between moduli spaces of flat connections on the three holed sphere and disjoint unions of toric varieties, together with a closed two-form which restricts to the symplectic forms on the ends. As applications, we obtain formulas for mixed Pontrjagin numbers and Witten’s formulas for symplectic volumes.

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1. Introduction

Let $G$ be a compact, connected, simply connected, simple Lie group and $\Sigma$ a compact oriented 2-manifold, possibly with boundary. A standard approach to computing invariants of the moduli spaces of flat $G$-connections over $\Sigma$ is to study the behavior of these invariants when two boundary circles are glued together. Since any $\Sigma$ of genus at least two admits a pants decomposition, this approach reduces the computation of these invariants to that of the moduli spaces associated to pairs of pants (three-holed spheres).

In this paper, we develop a cobordism approach to computing invariants of these moduli spaces. We apply our technique to compute mixed Pontrjagin numbers and symplectic volumes. In a sequel to this paper, we apply our method to compute the coefficients of the fusion ring (Verlinde algebra).

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The main result is as follows. Let $T \subset G$ be a maximal torus, $\mathfrak{t}_+ \subset \mathfrak{t}$ a choice of a closed positive Weyl chamber and $\mathfrak{A} \subset \mathfrak{t}_+$ the closed fundamental Weyl alcove. We interpret $\mathfrak{A}$ as the set of conjugacy classes in $G$ since for every conjugacy class $\mathcal{C} \subset G$ there is a unique $\mu \in \mathfrak{A}$ with $\exp(\mu) \in \mathcal{C}$. For $\mu_1, \mu_2, \mu_3 \in \mathfrak{A}$ let

$$M(\Sigma_0^3, \mu_1, \mu_2, \mu_3)$$

denote the moduli space of flat $G$-connections on the three-holed sphere $\Sigma_0^3$ for which the holonomies around the three boundary components lie in the conjugacy classes labeled by the $\mu_j$. A result of L. Jeffrey \cite{Je} identifies the moduli spaces $M(\Sigma_0^3, \mu_1, \mu_2, \mu_3)$ when the holonomies are small. Use the normalized invariant inner product on $\mathfrak{g}$ to identify $\mathfrak{g} \cong \mathfrak{g}^*$ and let $\mathcal{O}_{\mu_j}$ be the coadjoint orbit through $\mu_j$, equipped with the Kirillov-Kostant-Souriau symplectic form. Let $\ast : \mathfrak{t}_+ \to \mathfrak{t}_+$ denote the involution defined by $\ast \mu = -w_0 \cdot \mu$ where $w_0$ is the longest Weyl group element. Jeffrey proves in \cite{Je} that for $\mu_1, \mu_2, \mu_3 \in \mathfrak{A}$ sufficiently small, there is a symplectomorphism

$$M(\Sigma_0^3, \mu_1, \mu_2, \mu_3) \cong (\mathcal{O}_{\ast \mu_1} \times \mathcal{O}_{\ast \mu_2} \times \mathcal{O}_{\ast \mu_3}) \, / / G$$

denote the moduli space with a symplectic reduction of a triple product of coadjoint orbits under the triagonal action. The main theorem of this paper is the following result for more general holonomies:

**Theorem 1.1.** For $\mu_1, \mu_2, \mu_3 \in \text{int}(\mathfrak{A})$ generic there is an oriented orbifold cobordism

$$M(\Sigma_0^3, \mu_1, \mu_2, \mu_3) \sim \coprod_{w \in W_+^\text{aff}} (-1)^{\text{length}(w)} (\mathcal{O}_{\ast \mu_1} \times \mathcal{O}_{\ast \mu_2} \times \mathcal{O}_{\ast \mu_3}) \, / / G.$$ 

Here the signs indicate a change in orientation relative to the symplectic orientation, and $W_+^\text{aff}$ is the set of all $w$ in the affine Weyl group $W_\text{aff}$ such that $w \mathfrak{A} \subset \mathfrak{t}_+$. The symplectic forms extend to a closed 2-form over the cobordism. For $G = \text{SU}(n)$ both sides are smooth manifolds and the cobordism is a manifold cobordism.

Cobordisms of orbifolds with closed 2-forms were introduced and studied by Ginzburg-Guillemin-Karshon in \cite{GGK}. The genericity assumption in Theorem 1.1 guarantees that both sides have at worst orbifold singularities. In fact, the cobordism can be carried further so that the moduli space is cobordant to a disjoint union of toric varieties (Theorem 4.7). An application is the computation of the symplectic volume and mixed Pontrjagin numbers of the moduli space of the three-holed sphere (Theorem 5.2). Just as for toric varieties, the mixed Pontrjagin numbers are obtained by applying differential operators to the volume functions. In \cite{GGK} we obtain Witten’s volume formulas for arbitrary compact, oriented 2-manifolds (cf. Theorem 5.3) by gluing. These formulas were proved in most cases by Witten \cite{Wi}, Alternative proofs and extensions were given by Liu \cite{Li, Li2}, Jeffrey-Kirwan \cite{JK}, and Jeffrey-Weitsman \cite{JW}.

The idea of our proof is based on the “classical analog” of the principle (proved e.g. in \cite{Wi}) that “induction from $G$-representations to representations of the loop group $LG$ takes tensor product to fusion product”. We introduce a notion of fusion product for
Hamiltonian loop group manifolds, which preserves properness and level of the moment map. The moduli space $\mathcal{M}(\Sigma^3_0, \mu_1, \mu_2, \cdot)$ of flat connections on the three-holed sphere with fixed holonomy around two boundary components is the fusion product of two coadjoint orbits of $LG$. We also define a notion of induction $\text{Ind}$ of Hamiltonian $G$-manifolds to Hamiltonian $LG$-manifolds. We show that the induction of a direct product is cobordant (by a topologically trivial cobordism) to the fusion product of the inductions. In particular, we find an equivariant topologically trivial cobordism of Hamiltonian $LG$-manifolds,

$$\mathcal{M}(\Sigma^3_0, \mu_1, \mu_2, \cdot) \sim \text{Ind}(O_{\mu_1} \times O_{\mu_2}).$$

An application of the “cobordism commutes with reduction” principle of [6] yields Theorem 1.1.

We give a brief outline of the contents. In Section 2 we review the definition of a Hamiltonian group action, and the definition of Hamiltonian cobordism according to Ginzburg-Guillemin-Karshon [6]. The orbifolds appearing in Theorem 1.1 are naturally viewed as quotients of Hamiltonian $LG$-manifolds, which are discussed in Section 3. In Section 4 we introduce the notion of fusion product of Hamiltonian $LG$-manifolds, and prove Theorem 1.1. In Section 5 we present the application of our results to mixed Pontrjagin numbers and symplectic volumes of moduli spaces. A key technical ingredient in our proof of Theorem 1.1 is a Duistermaat-Heckman principle for Hamiltonian loop group manifolds which we prove in Section 6.

2. Preliminaries

In this section we will review the material we want to generalize to the $LG$-equivariant setting.

2.1. Hamiltonian $G$-manifolds. Let $G$ be a compact Lie group acting on a manifold $M$. We denote by $T$ a choice of maximal torus, by $t^*_+ \subset t^* \subset g^*$ a choice of positive Weyl chamber and by $\Lambda^*_+ \subset t^*$ the corresponding set of dominant weights. For any $\xi$ in the Lie algebra $g$ of $G$ we have a generating vector field $\xi_M \in \text{Vect}(M)$ whose flow is given by $m \mapsto \exp(t\xi)m$. Let $\omega \in \Omega^2(M)$ be a closed $G$-invariant 2-form. The action of $G$ on $(M, \omega)$ is called Hamiltonian if there exists a moment map $\Phi : M \to g^*$ which is equivariant with respect to the coadjoint action on $g^*$, and satisfies

$$\iota(\xi_M)\omega = d(\langle \Phi, \xi \rangle).$$

Note that we do not require the two-form $\omega$ to be be non-degenerate (i.e. symplectic).

An important consequence of this definition is that for any $m \in M$, the stabilizer algebra $g_m$ annihilates the image of the tangent map $d_m\Phi$:

$$g_m \subset \text{im}(d_m\Phi)^0$$

(with equality if $M$ is symplectic). In particular, if $\mu$ is a regular value of the moment map then the action of $G_\mu$ is locally free at points of the submanifold $\Phi^{-1}(\mu)$. The
quotient (reduced space at $\mu$)
\[ M_\mu = \Phi^{-1}(\mu)/G_\mu \]
is therefore an orbifold, and carries a canonical closed 2-form $\omega_\mu$. For $\mu = 0$ we will also use the notation
\[ M_0 = M//G = \Phi^{-1}(0)/G, \]
this is convenient if there is another group action on $M$. We say that $(M, \omega, \Phi)$ is pre-quantizable if there exists a $G$-equivariant Hermitian line bundle $L$ with invariant connection $\nabla$ such that
\[ \omega = \frac{i}{2\pi} \text{curv}(\nabla), \]
(2)
\[ 2\pi i \langle \Phi, \xi \rangle = \text{Vert}(\xi_L) \in C^\infty(M, \text{End}(L)). \]
(3)
Here we have identified sections of the bundle $\text{End}(L)$ of Hermitian bundle endomorphisms with imaginary-valued functions $C^\infty(M, i\mathbb{R})$; $\xi_L$ denotes the fundamental vector on the total space of $L$ and $\text{Vert}: TL \to TL$ the vertical projection given by the connection. $(L, \nabla)$ is called a pre-quantum line bundle for $(M, \omega, \Phi)$. Clearly a necessary condition for the existence of $(L, \nabla)$ is that $\omega \in H^2(M, \mathbb{Z})$. If $G$ is connected, simply connected and compact, this condition is also sufficient. Conversely, any $G$-equivariant Hermitian line bundle with invariant connection gives $M$ the structure of a Hamiltonian $G$-manifold via the equations (2), (3).

If $0$ is a regular value of $\Phi$, the reduced space $M//G$ has pre-quantum line bundle
\[ L//G = L_0 = (L|\Phi^{-1}(0))/G \]
(in the case that the $G$-action on $\Phi^{-1}(0)$ is not free, this is an orbi-bundle). More generally, if $\mu \in \Lambda^*_+$ is a weight for $\Phi^{-1}(0)$ is not free, this is an orbi-bundle). More generally, if $\mu \in \Lambda^*_+$ is a weight for $\Phi$, let $\mathbb{C}_\mu$ denote the 1-dimensional $G_\mu$-representation defined by $\mu$. If $\mu$ is a regular value for $\Phi$, a pre-quantum (orbi)-bundle for $M_\mu$ is given by
\[ L^{\text{shift}}_\mu = ((L \otimes \mathbb{C}_\mu)|\Phi^{-1}(\mu))/G_\mu. \]

2.2. Cobordism. We recall the notion of cobordism in the category of Hamiltonian $G$-manifolds, as introduced by Guillemin-Ginzburg-Karshon. Let $G$ be a compact Lie group.

Definition 2.1 (Hamiltonian Cobordism). A cobordism between two oriented Hamiltonian $G$-manifolds $(M_1, \omega_1, \Phi_1)$ and $(M_2, \omega_2, \Phi_2)$ with proper moment maps is an oriented Hamiltonian $G$-manifold with boundary $(N, \omega, \Phi)$ with proper moment map such that $\partial N = M_1 \cup (-M_2)$ and such that $\omega$ resp. $\Phi$ pull back to $\omega_i$ resp. $\Phi_i$. We write $M_1 \sim M_2$, or sometimes $(M_1, \omega_1, \Phi_1) \sim (M_2, \omega_2, \Phi_2)$.

This notion of cobordism has the drawback that it is not well-behaved under reduction, since reduced spaces of a Hamiltonian $G$-manifold are generically orbifolds. We will use the terminology “orbifold cobordism” if $M_1$, $M_2$, $N$ in the above definition are allowed to have orbifold singularities. (See Druschel for information on oriented orbifold cobordisms.)
The following Lemma is completely obvious, but has a lot of interesting consequences as shown in [3].

**Lemma 2.2.** Suppose \((N, \omega, \Phi)\) gives a cobordism \((M_1, \omega_1, \Phi_1) \sim (M_2, \omega_2, \Phi_2)\). If \(\mu\) is a regular value of \(\Phi\), the reduced space \(N_\mu\) gives an orbifold cobordism \(((M_1)_\mu, (\omega_1)_\mu) \sim ((M_2)_\mu, (\omega_2)_\mu)\).

**Example 2.3.** Suppose that \(M_1 = M_2 = M\) are compact oriented \(G\)-manifolds and that the equivariant closed 2-forms \(\omega_i + 2\pi i \Phi_i\) are cohomologous, i.e. \(\exists \beta \in \Omega^1(M)^G\) such that

\[
\omega_2 - \omega_1 = d\beta, \quad \Phi_2 - \Phi_1 = -\beta^\sharp
\]

where \(\beta^\sharp : M \to g^*\) is defined by \(\langle \beta^\sharp, \xi \rangle = \iota(\xi_M)\beta\). Let \(N = M \times [0, 1]\) with coordinates \((m, t)\), and let

\[
\omega = \omega_1 + d(t \beta), \quad \Phi(m, t) = (1 - t)\Phi_1(m) + t\Phi_2(m).
\]

Then \((N, \omega, \Phi)\) provides a cobordism of Hamiltonian \(G\)-manifolds,

\[
(M_1, \omega_1, \Phi_1) \sim (M_2, \omega_2, \Phi_2).
\]

From this “trivial” cobordism, non-trivial examples are obtained by reduction. We remark that it is not sufficient in this example to assume properness of the moment maps \(\Phi_i\) since this does not imply properness of \(\Phi\) in general.

### 3. Hamiltonian actions of loop groups

**3.1. Loop groups.** Let \(G\) be a compact, connected, simply connected Lie group. Let the Lie algebra \(g\) be equipped with the invariant inner product, normalized by the requirement that on each simple factor, the long roots have length \(\sqrt{2}\).

We define the loop group \(LG\) as the Banach Lie group of maps \(S^1 \to G\) of some fixed Sobolev class \(s > 1/2\). \(LG\) is a semi-direct product \(LG = \Omega G \times G\) where the subgroup \(\Omega G\) of based loops is the kernel of the evaluation map \(LG \to G, g \mapsto g(1)\) and \(G\) acts on \(\Omega G\) by conjugation. Let

\[
1 \to U(1) \to \hat{L}G \to LG \to 1
\]

denote the basic central extension of \(LG\) (see [18]). The corresponding Lie algebra extension \(\hat{L}g\) is given by \(Lg \times \mathbb{R}\), with bracket

\[
[(\xi_1, t_1), (\xi_2, t_2)] = ([\xi_1, \xi_2], \int \xi_1 \cdot d\xi_2).
\]

using the normalized inner product on \(g\). The inner product on \(g\) gives defines a natural \(LG\)-invariant \(L^2\)-metric on \(Lg\).
3.1.1. The affine coadjoint action. Let $L\mathfrak{g}^*$ denote the space of $\mathfrak{g}$-valued 1-forms $\Omega^1(S^1, \mathfrak{g})$ of Sobolev class $s-1$. We consider $L\mathfrak{g}^*$ as a subset of the topological dual space to $L\mathfrak{g}$ via the natural pairing given by integration. The coadjoint action of $LG$ on $\hat{L}\mathfrak{g}^*=L\mathfrak{g}^*\times \mathbb{R}$ is given by

$$g:(\mu, \lambda) = (\text{Ad}_g(\mu) - \lambda dg^{-1}, \lambda).$$

By the affine action at level $\lambda$ we mean the $LG$-action on $L\mathfrak{g}^*$ corresponding to the identification with the hyperplane $L\mathfrak{g}^* \times \{\lambda\} \subset \hat{L}\mathfrak{g}^*$. Clearly, the actions at non-zero level are equivalent up to rescaling, so that we will always consider the level $\lambda = 1$ unless specified otherwise. We will consider $\mathfrak{g}$ as a subset of $L\mathfrak{g}^*$ by the embedding $\xi \mapsto \xi d\theta/2\pi$.

3.1.2. The holonomy map. The action of $LG$ on $L\mathfrak{g}^*$ at level 1 can be identified with the action of the gauge group of $S^1$ on connections on the trivial principal $G$-bundle over $S^1$. Taking the holonomy of such a connection around $S^1$ we obtain a smooth submersion

$$\text{Hol}: L\mathfrak{g}^* \to G$$

with the equivariance property $\text{Hol}(g \cdot \mu) = \text{Ad}_{g(1)} \text{Hol}(\mu)$. The restriction of $\text{Hol}$ to $\mathfrak{g} \subset L\mathfrak{g}^*$ is the exponential mapping. For any point $\mu \in L\mathfrak{g}^*$ the evaluation mapping $LG \to G$, $g \mapsto g(1)$ gives an isomorphism

$$\text{(LG)}_\mu \cong Z_{\text{Hol}(\mu)}$$

where $Z_{\text{Hol}(\mu)}$ is the centralizer of the holonomy of $\mu$ (see [18]). In particular, the coadjoint action of the subgroup $\Omega G$ of based loops is free, and the holonomy map may be viewed as the quotient map $L\mathfrak{g}^* \to L\mathfrak{g}^*/\Omega G = G$.

3.1.3. The inversion map. There is a natural automorphism, the inversion map

$$I^*: \widehat{LG} \to \widehat{LG}$$

which is defined on the Lie algebra level as follows. Let $I: S^1 \to S^1$ denote the map $z \mapsto z^{-1}$. The pullback action on $L\mathfrak{g}$ extends to a Lie algebra homomorphism $I^*: \widehat{L}\mathfrak{g} \to \widehat{L}\mathfrak{g}$ by $I^*(\xi, t) = (I^*\xi, -t)$. Indeed, since $I$ reverses the orientation on $S^1$ the cocycle $c(\xi_1, \xi_2) = \oint \xi_1 d\xi_2$ transforms according to $c(I^*\xi_1, I^*\xi_2) = -c(\xi_1, \xi_2)$. Note that dual action on $\hat{L}\mathfrak{g}^*$ is given by

$$I^*(\mu, \lambda) = (-I^*\mu, -\lambda)$$

which in particular changes the sign of levels.

3.1.4. The involution *. Recall that the involution $*: t_+ \to t_+$ extends to a Lie algebra homomorphism $*: \mathfrak{g} \to \mathfrak{g}$ which in turn exponentiates to a map $*: G \to G$. In a faithful matrix representation for $G$, this is the map $g \mapsto (g^{-1})^t$ where $t$ denotes “transpose”. Composition with $*: G \to G$ transforms any unitary $G$-representation into the dual representation. The induced homomorphism $*: LG \to LG$ extends to a homomorphism of $\widehat{LG}$, given on the Lie algebra by

$$*: \widehat{L}\mathfrak{g} \to \widehat{L}\mathfrak{g}, (\xi, t) \mapsto (*\xi, t).$$
Notice that the dual map $*$ on $\hat{L}_g^*$ is level preserving. Note also that if $\mu \in L_g^*$ is at level 1, $\text{Hol}(\mu^*) = \text{Hol}(\mu)$ since Hol is equivariant and restricts to the exponential map on $g \subset L_g^*$.

3.2. Hamiltonian $\hat{LG}$-manifolds. Let $M$ be a Banach manifold, together with a closed 2-form $\omega$. We call $\omega$ symplectic if $\omega$ is weakly non-degenerate, that is, if $\omega$ defines an injection $T_m M \to T_m^* M$. An action of $\hat{LG}$ on $M$ which preserves $\omega$ is called Hamiltonian if there exists a moment map $\Phi : M \to \hat{L}_g^*$ satisfying (1). Since the action of $LG$ on $\hat{L}_g^*$ has fixed point set $\{0\}$, the equivariance condition implies that $\Phi$ is unique if it exists. An $\hat{LG}$-equivariant Hermitian line bundle $L \to M$ with connection $\nabla$ satisfying (2) and (3) is called a pre-quantum line bundle.

We call $M$ a Hamiltonian $LG$-manifold at level $\lambda$ if the central U(1) acts trivially with moment map $\lambda$. If $M$ admits a pre-quantum line bundle $L$ this implies $\lambda = k \in \mathbb{Z}$, where $k$ is the weight for the action of the central U(1) on $L$.

Note that if $(M, \omega, \Phi)$ is a Hamiltonian $LG$-manifold at non-zero level $\lambda$ then $(M, \omega^\lambda, \Phi^\lambda)$ is a Hamiltonian $LG$-manifold at level 1. We will therefore always assume $\lambda = 1$ unless specified otherwise.

3.2.1. The holonomy manifold. Notice that by the equivariance condition, the action of the subgroup of based loops $\Omega G$ is free. Let $\text{Hol}(M) := M/\Omega G$ and $\text{Hol}(\Phi) : \text{Hol}(M) \to G$ the map induced by $\Phi$. We have a commutative diagram

$$
\begin{array}{ccc}
M & \xrightarrow{\Phi} & L_g^* \\
\downarrow & & \downarrow \\
\text{Hol}(M) & \xrightarrow{\text{Hol}(\Phi)} & G
\end{array}
$$

There is a unique manifold structure on the holonomy manifold $\text{Hol}(M)$ such that the quotient map $M \to \text{Hol}(M)$ is a submersion. (Existence follows e.g. from the existence of slices, or from the discussion in Section 3.6 below. For uniqueness see [12], Prop. 3.5.20.) The map $\text{Hol}(\Phi)$ is smooth and equivariant.

The $LG$-manifold $M$ together with the moment map $\Phi$ may thus be viewed as the pull-back of the universal $\Omega G$-principal bundle $E\Omega G = L_g^* \to B\Omega G = G$ under the map $\text{Hol}(\Phi)$. Note that $\text{Hol}(M)$ is compact (in particular finite-dimensional) if and only if $\Phi$ is proper. The 2-form on $M$ is not basic and therefore does not descend to $\text{Hol}(M)$. However, as shown in [11] there exists a canonical 2-form $\varpi$ on $L_g^*$ such that $\omega + \Phi^* \varpi$ is basic and descends to a 2-form on $\text{Hol}(M)$ for which $\text{Hol}(\Phi)$ may be interpreted as a group valued moment map. See [11] for a detailed discussion.

3.2.2. The involution $M \mapsto M^*$. Suppose $(M, \omega, \Phi)$ is a Hamiltonian $LG$-manifold at level 1. Define a new action on $(M, \omega)$ by composing the action map $LG \to \text{Diff}(M)$ with the involution $* : LG \to LG$. The new action is once again Hamiltonian at level 1, and the moment map is $\Phi^* := *\Phi$. We denote the resulting space by $M^*$. Clearly
Hol(M*) is just Hol(M), with new G-action defined by composing with * : G → G, and moment map Hol(Φ)* = * Hol(Φ).

3.2.3. Line bundles. An LG-equivariant line bundle L over an LG-Banach manifold M is called at level k ∈ ℤ if the central circle acts with weight k. If k = 0, this means that L is an LG-equivariant line bundle which descends to a G-equivariant line bundle Hol(L) := L/ΩG over Hol(M). It follows that the isomorphism classes of (level 0) LG-equivariant line bundles L → M are classified by the equivariant cohomology H^2_G(Hol(M), ℤ).

3.3. Coadjoint orbits. Coadjoint LG-orbits provide basic examples for Hamiltonian LG-manifolds, the moment map being simply the embedding. In particular, the coadjoint orbit LG · 0 is ΩG = LG/G. The isomorphism (1) shows that every stabilizer group (LG)_µ is compact and connected (because for a compact simply connected Lie group, the centralizer of any element is connected). Let G be a simple, simply-connected, compact, connected Lie group, with fundamental alcove ℂ. Every coadjoint LG-orbit passes through exactly one point of ℂ, considered as a subset of Lg* via the embeddings

\[ \mathfrak{A} \to \mathfrak{t} \to \mathfrak{g} \to Lg*. \]

Using the exponential mapping we can view ℂ also as a subset of T ⊂ G, and every Ad(G)-orbit passes through exactly one point of ℂ. We therefore obtain a series of identifications

\[ \mathfrak{A} \cong T/W \cong G/\text{Ad}(G) \cong \mathfrak{t}/W_{\text{aff}} \cong Lg^*/LG. \]

One also finds that for any open face σ ⊂ ℂ, the stabilizer groups (LG)_σ resp. centralizers Z_{exp µ} for all points µ ∈ σ are the same, they will therefore be denoted by (LG)_σ resp. Z_σ. The Dynkin diagrams of the semi-simple parts of these groups are obtained from the extended Dynkin diagram of G by deleting the vertices corresponding to affine simple roots not vanishing on σ. In particular, the groups (LG)_σ corresponding to the vertices of ℂ are semi-simple (but not always simply-connected). There is a natural partial ordering on the open faces of ℂ defined by inclusion: We write σ < τ if σ is properly contained in τ. Let α_1, ..., α_l be the simple roots of G, and α_0 the highest root. We fix an orientation on G, and let T have the orientation induced by the choice of positive roots.

**Lemma 3.1.** (Properties of the stabilizer groups (LG)_σ).

a. Every (LG)_σ contains the maximal torus T. Define the fundamental Weyl chamber for (LG)_σ as the cone over (ℂ − µ) where µ ∈ σ. Together with the orientation on T this induces an orientation on (LG)_σ. The simple roots for (LG)_σ are precisely those roots α in the collection \{α_1, ..., α_l, −α_0\} such that α vanishes on the span of σ − µ.

b. For σ < τ, one has (LG)_σ ⊇ (LG)_τ. The quotient (LG)_σ/(LG)_τ carries a canonical invariant complex structure.

c. If 0 ∈ σ then (LG)_σ is contained in the subgroup G ⊂ LG of constant loops.

d. For every σ, the Lie algebra (Lg)_σ of (LG)_σ has a unique (LG)_σ-invariant complement in Lg, given by the L²-orthogonal complement (Lg)_σ⊥.
If $G$ is a simply-connected, compact, connected Lie group, and $G = G_1 \times \ldots \times G_r$ its decomposition into simple factors, then coadjoint orbits of $L^G = L^G_1 \times \ldots \times L^G_r$ are products of coadjoint orbits of the $L^G_i$. We denote by $\mathfrak{A}$ the product of the fundamental alcoves $\mathfrak{A}_i$ for the $G_i$.

To describe the Kirillov-Kostant-Souriau (KKS) form on coadjoint orbits $(L^G) \cdot \mu$ let $\delta\mu$ denote the elliptic operator
\[
\delta\mu : L^\mathfrak{g}^* \to \mathfrak{g}^*, \zeta \mapsto d\zeta + [\mu, \zeta].
\]
Then $\delta\mu(\xi)$ is the value at $\mu$ for the fundamental vector field defined by $\xi$. In particular, the kernel of $\delta\mu$ is equal to $(L^\mathfrak{g})_\mu$ and the image is equal to the tangent space to the orbit $T_\mu(L^G \cdot \mu)$. The KKS-form $\nu_\mu \in \Omega^2(L^G \cdot \mu)$ is the unique invariant 2-form given on the tangent space $T_\mu(L^G \cdot \mu)$ by
\[
\nu_\mu(\delta\mu(\xi_1), \delta\mu(\xi_2)) = \langle(\mu, 1), ([\xi_1, 0], (\xi_2, 0))\rangle = \langle\mu, [\xi_1, \xi_2]\rangle + \oint \xi_1 \cdot d\xi_2
\]
This can be rewritten
\[
\nu_\mu(\delta\mu(\xi_1), \delta\mu(\xi_2)) = \oint \xi_1 \cdot \delta\mu(\xi_2).
\]

**Remark 3.2.** The coadjoint orbits $L^G \cdot \mu$ for $\mu \in \mathfrak{A}$ have a canonical invariant complex structure, making them into $L^G$-Kähler manifolds. Using the operator $\delta\mu$ the complex structure can be described as follows. Identify the tangent space $T_\mu(L^G \cdot \mu)$ with the $L^2$-orthogonal complement $\ker(\delta\mu)^\perp$. Consider $*\delta\mu$ (where $*$ is the Hodge operator) as a first order skew adjoint differential operator on $C^\infty(S^1, \mathfrak{g})$. The square $(*\delta\mu)^2$ is a non-positive operator, so that $|*\delta\mu| := \sqrt{-(\delta\mu)^2}$ is a well-defined first order pseudo-differential operator and its Greens operator $|*\delta\mu|^{-1}$, defined to be the identity on the kernel of $\delta\mu$, has order $-1$. The complex structure is given by the zeroth order pseudo-differential operator
\[
J_\mu := |*\delta\mu|^{-1} (*\delta\mu)
\]
on $\ker(\delta\mu)^\perp \subset L^\mathfrak{g}$. The proof for the fundamental homogeneous space $L^G \cdot 0 = L^G/G$ can be found in Freed \[;\] the extension to the general case is straightforward.

**Remark 3.3.** For any coadjoint orbit $L^G \cdot \mu$, we have $(L^G \cdot \mu)^* = L^G \cdot \mu$.

**Remark 3.4.** The holonomy manifold of a coadjoint orbit $L^G \cdot \mu$ is just the conjugacy class $G \cdot \exp(\mu)$, and the map $\text{Hol}(\Phi)$ is the embedding into $G$.

### 3.4. Moduli spaces of flat connections.

Let $\Sigma$ be a compact oriented 2-manifold with $b$ boundary components. We denote by $\iota : \partial \Sigma \hookrightarrow \Sigma$ the inclusion of the boundary. Fix $s > 1$, and let $\mathcal{A}(\Sigma) \cong \Omega^1(\Sigma, \mathfrak{g})$ the space of connections of Sobolev class $s - \frac{1}{2}$ in the trivial principal bundle $\Sigma \times G$, and $\mathcal{G}(\Sigma)$ the gauge group, consisting of maps $\Sigma \to G$ of Sobolev class $s + \frac{1}{2}$. Let $\iota : \partial \Sigma \hookrightarrow \Sigma$ be the inclusion of the boundary. Recall that restriction $\iota^*$ to the boundary results in the loss of half a derivative, so that there is a continuous map $\mathcal{G}(\Sigma) \to \mathcal{G}(\partial \Sigma) \cong L^G b$ which is surjective since $\pi_1(G) = 0$. The
kernel \( \mathcal{G}_\partial(\Sigma) \) consists of gauge transformations that are the identity on the boundary. According to Atiyah-Bott [2], the gauge group action on the space \( \mathcal{A}(\Sigma) \) with symplectic form

\[
\omega_A(a_1, a_2) = \int_\Sigma a_1 \wedge a_2 \quad (a_i \in T_A \mathcal{A}(\Sigma) \cong \Omega^1(\Sigma, \mathfrak{g}))
\]
is Hamiltonian, with moment map given by

\[
\Psi : \mathcal{A}(\Sigma) \to \Omega^2(\Sigma, \mathfrak{g}) \oplus \Omega^1(\partial \Sigma, \mathfrak{g}), \quad A \mapsto (\text{curv}(A), \iota^* A),
\]
that is,

\[
\langle \Psi(A), \xi \rangle = \int_\Sigma \text{curv}(A) \cdot \xi + \int_{\partial \Sigma} \iota^* (A \cdot \xi).
\]

Here \( \text{curv}(A) \) denotes the curvature of \( A \), and we have chosen the orientation on the boundary \( \partial \Sigma \) to be minus the orientation induced from \( \Sigma \). The moment map for the action of \( \mathcal{G}_\partial(\Sigma) \) on \( \mathcal{A}(\Sigma) \) can be identified with \( A \mapsto \text{curv}(A) \) and hence the symplectic quotient of \( \mathcal{A}(\Sigma) \) by \( \mathcal{G}_\partial(\Sigma) \) is

\[
\mathcal{M}(\Sigma) := \mathcal{A}_F(\Sigma) / \mathcal{G}_\partial(\Sigma)
\]
where \( \mathcal{A}_F(\Sigma) \subset \mathcal{A}(\Sigma) \) is the space of flat connections. If \( \partial \Sigma = \emptyset \) then \( \mathcal{M}(\Sigma) \) is a compact, finite dimensional stratified symplectic space (in general singular). On the other hand, if \( \partial \Sigma \neq \emptyset \) then according to Donaldson [3] \( \mathcal{M}(\Sigma) \) is a smooth infinite-dimensional symplectic manifold. It has a residual Hamiltonian action of \( \mathcal{G}(\partial \Sigma) \cong LG^b \) with moment map

\[
\Phi : \mathcal{M}(\Sigma) \to \Omega^1(\partial \Sigma, \mathfrak{g}), \quad [A] \mapsto \iota^* A
\]
and carries a natural \( \widetilde{LG}^b \)-equivariant pre-quantum line bundle \( L(\Sigma) \) with connection.

**Remark 3.5.** For any 2-manifold \( \Sigma \), the symplectomorphism \( \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma), \quad A \mapsto A^* \) induced by the involution \( * : \mathfrak{g} \to \mathfrak{g} \) preserves \( \mathcal{A}_F(\Sigma) \) and therefore descends to a symplectomorphism \( \mathcal{M}(\Sigma)^* \cong \mathcal{M}(\Sigma) \).

**Remark 3.6.** The holonomy manifold \( \text{Hol}(\mathcal{M}(\Sigma)) \) may be interpreted as the quotient of the space \( \mathcal{A}_F(\Sigma) \) of flat connections by those gauge transformations which are the identity on given base points of the boundary circles.

**Remark 3.7.** For the sake of completeness we recall from [2, 3] the holomorphic description of \( \mathcal{M}(\Sigma) \). The moduli space \( \mathcal{M}(\Sigma) \) inherits a complex structure from a choice of complex structure on \( \Sigma \). The action of \( \mathcal{G}(\partial \Sigma) = LG^b \) complexifies to an action of the \( b \)-fold complex loop group \( \mathcal{G}_C(\partial \Sigma) = LG^b_C \) and (for \( b > 0 \)) \( \mathcal{M}(\Sigma) \) is a homogeneous space

\[
\mathcal{M}(\Sigma) = \mathcal{G}_C(\partial \Sigma) / \mathcal{G}_C(\Sigma)
\]
where \( \mathcal{G}_C(\Sigma) \) is the group of holomorphic maps of \( \Sigma \) into \( G_C \).

**Example 3.8.** a. The moduli space \( \mathcal{M}(\Sigma^1_0) \) for the disk is the fundamental homogeneous space \( LG/G = \Omega G \) of based loops in \( G \).
b. The moduli space $\mathcal{M}(\Sigma^b_h)$ for the two-holed sphere is diffeomorphic to $LG \times L\mathfrak{g}^*$. Note that these two factors are of different Sobolev class! The moment map for the two $LG$-actions under this identification are $(g, \xi) \mapsto -\xi$ and $(g, \xi) \mapsto g \cdot \xi$, respectively.

Just as for the case without boundary [2], the moduli spaces $\mathcal{M}(\Sigma)$ can be described in terms of holonomies. The fundamental principle that applies here is that any two flat connections that define the same holonomy map $\pi_1(\Sigma) \to G$ are related by a gauge transformation in $\mathcal{G}(\Sigma)$. In the case of a $b$-holed sphere with $b \geq 1$, i.e. at least 1 boundary component this leads to the following description:

**Proposition 3.9.** The moduli space $\mathcal{M}(\Sigma^b_h)$ is equivariantly diffeomorphic to the smooth submanifold of $G^{2h} \times G^b \times (L\mathfrak{g}^*)^b \ni (a, c, \xi)$ defined by

$$\prod_{i=1}^b \text{Ad}_{c_i} \text{Hol}(\xi_i) = \prod_{j=1}^h [a_{2j}, a_{2j-1}]$$

where $c_1 = 1$. Here the action of $g = (g_1, \ldots, g_r) \in LG^b$ is given by

$$g \cdot a_j = \text{Ad}_{g_1(a_j)} a_j, \quad g \cdot c_j = g_1(1) c_j g_j(1)^{-1}, \quad g \cdot \xi_j = g_j \cdot \xi_j$$

and the moment map is given by projection to the $(L\mathfrak{g}^*)^b$-factor.

This description shows for example that the action of $(LG)^{b-1} \subset LG^b$ on $\mathcal{M}(\Sigma^b_h)$ is free. Setting $d_j = \text{Hol}(\xi_j)$ and eliminating $d_1$ one obtains a description of the holonomy manifold:

**Corollary 3.10.** The holonomy manifold $\text{Hol}(\mathcal{M}(\Sigma^b_h))$ is $G^{2h+2(b-1)} \ni (a, c, d)$, with action of $h = (h_1, \ldots, h_b) \in G^b$ given by

$$h \cdot a_j = \text{Ad}_{h_1(a_j)} a_j, \quad h \cdot c_i = h_1 c_i h_1^{-1}, \quad h \cdot d_i = \text{Ad}_{h_i} d_i \quad 2 \leq i \leq b$$

The components of the map $\text{Hol}(\Phi)$ are $\text{Hol}(\Phi)_j(a, c, d) = d_j$ for $j \geq 2$ and

$$\text{Hol}(\Phi)_1(a, c, d) = \prod_{j=1}^h [a_{2j}, a_{2j-1}] \left( \prod_{i=2}^b \text{Ad}_{c_i} d_i \right)^{-1}.$$ 

As an application, we have:

**Proposition 3.11.** There is a ring isomorphism in equivariant cohomology (with coefficients in $\mathbb{Z}$), $H^*_G(\mathcal{M}(\Sigma^b_h)) = H^*_G(G^{2h+b-1})$ where $G$ acts by conjugation.

**Proof.** In the description of $\text{Hol}(\mathcal{M}(\Sigma^b_h)) \cong G^{2h+2(b-1)}$ given in the Proposition, the action of $G^{b-1} \subset G^b$ is free, and the quotient is just $G^{2h+b-1}$ with $G$ acting by the adjoint action. The proposition follows. \qed

**Corollary 3.12.** Any $LG^b$-equivariant line bundle over $\mathcal{M}(\Sigma^b_h)$ for $b > 0$ at level $k$ is equivariantly isomorphic to the $k$th tensor power of the pre-quantum line bundle $L(\Sigma^b_h)$. 

Proof. It suffices to show that every $\hat{L}G^b$-equivariant line bundle $L \to M(\Sigma^n_b)$ at level 0 is equivariantly trivial. By Theorem 3.11 and since $H^1_G(G^{2h+b-1}) = H^2_G(G^{2h+b-1}) = \{0\}$, it follows that $L/\Omega G^b$ is $G$-equivariantly trivial. Consequently, $L$ is $LG^b$-equivariantly trivial. □

Remark 3.13. It is a much deeper fact (see e.g. [2, 12]) that in the case $b = 0$ every line bundle is a tensor power of the pre-quantum bundle.

3.5. Reduction. Let $G$ be a compact, connected, simply-connected Lie group and let $M$ is a Hamiltonian $L(G \times G)$-manifold with proper moment map $(\Phi_+, \Phi_-)$. Let $LG$ act on $M$ by the the embedding

$$\text{diag} : LG \to LG \times LG, \ g \mapsto (g, I^*g).$$

This action has a moment map at level 0, $\Phi = \Phi_+ - I^*\Phi_-$. Notice that $\Phi$ is not proper. In [16] it is shown that if 0 is a regular value of $\Phi$ the reduced space $M/\text{diag}(LG) := \Phi^{-1}(0)/\text{diag}(LG)$ is a compact orbifold with a naturally induced closed two-form $\omega/\text{diag}(LG)$. One way to see that $M/\text{diag}(LG)$ is finite dimensional and compact is to note that it may also be obtained from the holonomy manifold $\text{Hol}(M)$. Indeed, let $\text{diag} : G \to G \times G$ be the diagonal embedding and $\text{Hol}(\Phi) = \text{Hol}(\Phi_+) \cdot \text{Hol}(\Phi_-)$, and define $\text{Hol}(M/\text{diag}(G) = \text{Hol}(\Phi)^{-1}(e)/\text{diag}(G)$.

Proposition 3.14. Suppose 0 is a regular value of $\Phi$. The holonomy map $\text{Hol} : M \to \text{Hol}(M)$ descends to a diffeomorphism $M/\text{diag}(LG) \to \text{Hol}(M)/\text{diag}(G)$.

Proof. The condition $m \in \Phi^{-1}(0)$ means $\Phi_+(m) = I^*\Phi_-(m)$, hence

$$\text{Hol}(\Phi_+(m)) = \text{Hol}(I^*\Phi_-(m)) = \text{Hol}(\Phi_-(m))^{-1}.$$

Moreover $(\Omega G^2 \cdot m) \cap \Phi^{-1}(0) = (\text{diag}(\Omega G) \cdot m) \cap \Phi^{-1}(0)$. Consequently $\Phi^{-1}(0)/\text{diag}(\Omega G) \cong \text{Hol}(\Phi)^{-1}(e)$ and therefore

$$\Phi^{-1}(0)/\text{diag}(\Omega G \times G) \cong \text{Hol}(\Phi)^{-1}(e)/\text{diag}(G).$$

□

Suppose more generally that $H$ is another simply connected compact, connected Lie group and that $M$ is a Hamiltonian $L(H \times G \times G)$-manifold with proper moment map $(\Psi, \Phi_+, \Phi_-)$. Assume that 0 is a regular value of $\Phi = \Phi_+ - I^*\Phi_-$ and that the action of $\text{diag}(LG)$ on the zero level set is free. Then the reduced space $M/\text{diag}(LG)$ is a Hamiltonian $LH$-Banach manifold with proper moment map [16]. (Dropping the freeness assumption would lead to “Banach orbifolds.”)

If $L \to M$ is a complex $L(H \times G^2)$-equivariant line bundle, we can define a reduced bundle

$$L/\text{diag}(LG) = (L|\Phi^{-1}(0))/\text{diag}(LG).$$
which is again at level $k$. (Note that the diagonal embedding $\text{diag}: L G \to L(H \times G^2)$ lifts to an embedding into $L(\tilde{H} \times G^2)$ so that $\text{diag}(L G)$ acts on $L$.) If $L$ is a pre-quantum bundle then $L // \text{diag}(L G)$ is pre-quantum.

As an example of reduction we have the following result which we learned from S. Martin (for a proof, see [16]):

**Theorem 3.15.** *(Gluing equals reduction)* Suppose $\Sigma$ is a compact, oriented 2-manifold that is obtained from a second 2-manifold $\hat{\Sigma}$ (possibly disconnected) by gluing two boundary components $B_+ \subset \partial \hat{\Sigma}$. Let $L G \to \mathcal{G}(\partial \Sigma)$ be the embedding induced by $S^1 \hookrightarrow B_+ \times B_-, z \mapsto (z, z^{-1})$. Then $M(\Sigma)$ is the Hamiltonian quotient

$$M(\Sigma) = M(\hat{\Sigma}) // \text{diag}(L G).$$

Because every compact oriented 2-manifold $\Sigma$ of genus at least 2 admits a pants decomposition, Theorem 3.15 shows that the moduli space $M(\Sigma)$ can be obtained from products of $M(\Sigma_0^2)$ by iterated symplectic reductions.

**Example 3.16.** It follows from the description in Example 3.8 that for any Hamiltonian $LG$-manifold $M$ there is an equivariant symplectomorphism

$$M \times M(\Sigma_b^0) // \text{diag}(L G) \cong M.$$

As in the finite dimensional case, reductions with respect to coadjoint orbits are particularly important:

**Definition 3.17.** Let $(M, \omega, (\Psi, \Phi))$ be a Hamiltonian $L H \times LG$-manifold with proper moment map. Let $\mathfrak{A}$ be the alcove for the $G$ and $\mu \in \mathfrak{A}$. If $\mu$ is a regular value of $\Phi$ and the action of $(L G)_\mu$ on $\Phi^{-1}(\mu)$ is free then 0 is a regular value for the diagonal action on $M \times LG \cdot (*\mu)$ and the action of $\text{diag}(L G)$ on the zero level set is also free. We define the reduced space at $\mu$ by

$$M_\mu := (M \times LG \cdot (*\mu)) // \text{diag}(L G) \cong \Phi^{-1}(\mu) / (L G)_\mu.$$

For example, the reductions $M(\Sigma^b_{\mu_1, \ldots, \mu_b})$ may be viewed as moduli spaces of flat connections where the holonomy around the $j$th boundary component is conjugate to $\text{Hol}(\mu_j)$. The holonomy description of these spaces follows from Proposition 3.9:

$$(6) \quad M(\Sigma^b_{\mu_1, \ldots, \mu_b}) = \{(d_1, \ldots, d_b) \in C_{\mu_1} \times \ldots \times C_{\mu_b} | \prod_j d_j = e\} / G$$

where $C_{\mu_j} = G \cdot \exp(\mu_j)$ is the conjugacy class corresponding of $\exp(\mu_j)$ and where the action is the diagonal action. For the sake of comparison, note that

$$O_{\mu_1} \times \ldots \times O_{\mu_b} // G = \{ (\zeta_1, \ldots, \zeta_b) \in O_{\mu_1} \times \ldots \times O_{\mu_b} | \sum_j \zeta_j = 0 \} / G,$$

In fact the two spaces are $LG^b$-symplectomorphic, if the $\mu_i$’s are sufficiently small. This was noticed by L. Jeffrey, and also follows from the construction of symplectic cross-sections in the following section.
3.6. Cross-sections. Hamiltonian $LG$-manifolds with proper moment maps behave in many respects like compact Hamiltonian spaces for compact Lie groups. This is due to the existence of finite dimensional cross-sections for the $LG$-action. The finite collection of cross-sections behaves like an atlas for a compact, finite dimensional manifold. For every open face $\sigma$ of the fundamental alcove $A$, we define

$$A_\sigma := \bigcup_{\tau \succeq \sigma} \tau.$$

Thus $A_\sigma$ is the complement in $A$ of the closure of the face opposite to $\sigma$. The flow-out

$$U_\sigma := (LG)_\sigma \cdot A_\sigma \subset Lg^*$$

is a slice for the affine $LG$-action; that is, $LG \cdot U_\sigma \cong LG \times (LG)_\sigma U_\sigma$. There is a natural $(LG)_\sigma$-invariant decomposition

$$TLg^*|U_\sigma = U_\sigma \times Lg^* = Tu_\sigma \oplus U_\sigma \times (Lg)_\sigma^\perp.$$

We denote by $(\hat{LG})_\sigma$ the restriction to $(LG)_\sigma$ of the central extension $\hat{LG}$.

Proposition 3.18. Let $(M, \omega)$ be a connected Hamiltonian $\hat{LG}$-manifold with moment map $(\Phi, \phi) : M \to \hat{LG}^* = Lg^* \times \mathbb{R}$ such that $0 \notin \phi(M)$. Let $\sigma \subset A$ be a face of the alcove.

a. The cross-section $Y_\sigma := (\Phi/\phi)^{-1}(U_\sigma)$ is a smooth Hamiltonian $(\hat{LG})_\sigma$-manifold, with the restriction $(\Phi, \phi)|Y_\sigma$ as a moment map. There is a natural $(\hat{LG})_\sigma$-invariant decomposition $TM|Y_\sigma = TY_\sigma \oplus (Lg)_\sigma^\perp$.

b. There is a unique $\hat{LG}$-invariant closed 2-form and moment map on the associated bundle $\hat{LG} \times (\hat{LG})_\sigma Y_\sigma$ which restrict to the given form and moment map on $Y_\sigma$. They are obtained by pulling back $\omega$ resp. $(\Phi, \phi)$ via the embedding $\hat{LG} \times (\hat{LG})_\sigma Y_\sigma \to \hat{LG} \cdot Y_\sigma \subset M$. For each $\tau \prec \sigma$, one has canonical Hamiltonian embeddings

$$(\hat{LG})_\tau \times (\hat{LG})_\sigma Y_\sigma \to Y_\tau.$$

c. If $(\Phi, \phi)$ is proper as a map into $Lg^* \times \mathbb{R}\{0\}$ then the cross-section $Y_\sigma$ is finite dimensional.

d. If in addition $\omega$ is symplectic, then $Y_\sigma$ is a symplectic submanifold, and the flow-outs $LG \cdot Y_\sigma$ are either empty or connected and dense in $M$.

For a proof of these assertions in the case of Hamiltonian $LG$-manifolds, see [16]. The extensions to the case of $\hat{LG}$-actions are immediate.

Remark 3.19. Since the loop group $LG$ is a Hilbert manifold, Proposition 3.18 shows that every Hamiltonian $LG$-manifold with proper moment map is a Hilbert manifold.

One can reconstruct a Hamiltonian $\hat{LG}$-manifold $M$ from the collection of $\{Y_\sigma\}$ and the inclusions $Y_\sigma \hookrightarrow Y_\tau$ for $\tau \prec \sigma$. That is, Hamiltonian $\hat{LG}$-manifolds with proper moment maps are equivalent to a collection of finite dimensional Hamiltonian manifolds with certain compatibility relations. It is in principle possible to abandon the infinite
dimensional picture and work only with cross-sections; the space $M$ itself serves mainly as a bookkeeping device for the relations between the cross-sections. Note in particular that in the case of moduli spaces of flat connections the cross-sections are independent of the choice of Sobolev class.

There are some limitations to this point of view - for example a given invariant almost complex structure on $M$ does not in general preserve the cross-sections. Similarly an invariant connection on a given $\hat{LG}$-equivariant line bundle is not determined by its restrictions to the cross-sections.

For the rest of this section we consider only Hamiltonian $LG$-manifolds $M$ with proper moment map at level 1.

**Example 3.20.**

a. For any $\mu \in A_{\sigma}$, the cross-section $Y_{\sigma}$ for $M = (LG) \cdot \mu$ is the coadjoint orbit for the compact group, $(LG)_{\sigma} \cdot \mu$.

b. The symplectic cross-sections $Y_{\sigma}$ for the moduli space $M(\Sigma_{b}^{0})$ is given in the holonomy picture, Proposition 3.9, by imposing the extra condition $(\xi_{1}, \ldots, \xi_{b}) \in U_{\sigma}$. These are the “twisted extended moduli spaces” due to Jeffrey.

The cross-sections $Y_{\sigma}$ for a Hamiltonian $LG$-manifold can also be viewed as submanifolds of the holonomy manifold. Indeed, since the intersection $(LG)_{\sigma} \cap \Omega G$ is trivial the composition of the horizontal maps in the commutative diagram

$$
\begin{array}{ccc}
Y_{\sigma} & \hookrightarrow & M & \rightarrow & \text{Hol}(M) \\
\downarrow & & \downarrow & & \downarrow \\
U_{\sigma} & \hookrightarrow & LG^{*} & \rightarrow & G
\end{array}
$$

are equivariant embeddings. Moreover the associated bundles

$$G \times_{Z_{\sigma}} Y_{\sigma} \rightarrow \text{Hol}(M)$$

are open submanifolds. (In fact, one can use these embeddings to define the manifold structure on Hol($M$).)

Considering the cross-sections as submanifolds of Hol($M$) is useful, for example, in order to define orientations: Since $G$ and $Z_{\sigma} \cong (LG)_{\sigma}$ are oriented it follows that any orientation on Hol($M$) will induce orientations on the cross-sections $Y_{\sigma}$. Moreover, these orientations are compatible in the sense that the natural maps $Z_{\sigma} \times_{Z_{\tau}} Y_{\tau} \rightarrow Y_{\sigma}$ are orientation preserving. Conversely, compatible orientations on the cross-sections induce an orientation on Hol($M$). For example, if $M$ is a symplectic Hamiltonian $LG$-manifold with proper moment map $\Phi$ the symplectic orientations on the cross-sections are compatible.

### 3.7. Induction

Any compact Hamiltonian $G$-manifold $(M, \omega, \Phi)$ gives rise to a Hamiltonian $LG$-manifold $(\text{Ind}(M), \text{Ind}(\omega), \text{Ind}(\Phi))$ with proper moment map called its induction.

**Proposition 3.21.** Let $(M, \omega, \Phi)$ be a Hamiltonian $G$-manifold. There exists a unique closed two-form $\text{Ind}(\omega)$ on the associated bundle

$$\text{Ind}(M) := LG \times_{G} M$$
such that the induced $LG$-action on $\text{Ind}(M)$ is Hamiltonian with moment map $\text{Ind}(\Phi)$ at level 1, and such that $\text{Ind}(\Phi)$ and $\text{Ind}(\omega)$ pull back to the given moment map and symplectic form on $M \subset \text{Ind}(M)$. If $M$ is symplectic then the cross-section $Y_{\text{int}(\mathfrak{a})}$ is symplectic. If moreover $\Phi(M) \subset U_{\{0\}}$, then $\text{Ind}(M)$ is symplectic.

For a proof, see [10].

Example 3.22. As before, we denote by $O_\mu$ the coadjoint $G$-orbit through a point $\mu \in \mathfrak{a}$. If $\mu \in \mathfrak{a}_{\{0\}}$, the induction $\text{Ind}(O_\mu)$ is just the coadjoint $LG$-orbit through $\mu$. For $\mu \not\in \mathfrak{a}_{\{0\}}$, the induction $\text{Ind}(O_\mu)$ is not symplectic.

Note that since $LG = \Omega G \rtimes G$ the associated bundle (7) is simply the trivial bundle $\Omega G \times M$. Thus, the holonomy manifold $\text{Hol}(\text{Ind}(M))$ is just $M$ itself, and $\text{Hol}(\text{Ind}(\Phi))$ is just the composition $\exp \circ \Phi$. In particular, it follows from the discussion in section 3.6 that if $M$ is oriented then all cross-sections $Y_\sigma$ for $\text{Ind}(M)$ are oriented. We are mainly interested in the case $\sigma = \text{int}(\mathfrak{a})$ where the cross-sections and their orientations are given as follows.

Proposition 3.23. Let $M$ be an oriented Hamiltonian $G$-manifold, with proper moment map $\Phi$. For $\sigma = \text{int}(\mathfrak{a})$, the cross-section $Y_\sigma$ for the induction $\text{Ind}(M)$ is a disjoint union

$$Y_{\text{int}(\mathfrak{a})} = \bigsqcup_{w \in W_{\text{aff}}^+} g_w^{-1} \cdot \Phi^{-1}(w \cdot \text{int} \mathfrak{a})$$

where $g_w \in LG$ is any element representing $w$. The orientation on $\Phi^{-1}(w \cdot \text{int} \mathfrak{a}) \subset Y_\sigma$ is given by $(-1)^{\text{length}(w)}$ times the orientation as a subset of the cross-section $\Phi^{-1}(\text{int} t_+^{\ast})$ of $M$.

Corollary 3.24. Let $M$ be an oriented Hamiltonian $G$-manifold, with proper moment map $\Phi$, and $\mu \in \text{int}(\mathfrak{a})$. Then $\mu$ is a regular value (or not in the image) of $\text{Ind}(\Phi)$ if and only if $w \cdot \mu$ is a regular value (or not in the image) of $\Phi$, for every $w \in W_{\text{aff}}^+$. Furthermore,

$$\text{Ind}(M)_\mu = \bigsqcup_{w \in W_{\text{aff}}^+} (-1)^{\text{length}(w)} M_{w \cdot \mu}$$

Proof. This follows immediately from the proposition since

$$\text{Ind}(M)_\mu = (Y_\sigma)_\mu$$

for $\sigma = \text{int}(\mathfrak{a})$.

Instead of proving the proposition in this special case, we will work out the cross-sections and their orientations in general for arbitrary $\sigma$. For this we need to consider the set of all images $w \sigma$ ($w \in W_{\text{aff}}$) which are contained in the positive Weyl chamber $t_+$. This set is labeled by the double coset space $W \backslash W_{\text{aff}}/W_\sigma$ where $W_\sigma$ is the stabilizer group of $\sigma$ in $W_{\text{aff}}$. The stabilizer group of $w \sigma$ in $LG$ is given by $(LG)_{w \sigma} = g (LG)_\sigma g^{-1}$,
where \( g \in LG \) is any element with \( g \cdot \sigma = w\sigma \). For any such \( g \), the image \( U_{w\sigma} := g \cdot U_\sigma \) is a slice at \( w\sigma \) for the action of \( LG \). The subset of \( \text{Ind}(M) \) given by

\[
Y^{[w]}_\sigma := LG_\sigma \cdot g^{-1} \cdot \Phi^{-1}(U_{w\sigma} \cap g^*) = g^{-1} LG_{w\sigma} \cdot \Phi^{-1}(U_{w\sigma} \cap g^*)
\]

depends only on \( w\sigma \) and not on the choice of \( g \).

**Proposition 3.25.** Suppose that \( M \) is an oriented Hamiltonian \( G \)-manifold with equivariant map \( \Phi : M \to g^* \). The cross-section \( Y_\sigma \) for the Hamiltonian \( LG \)-manifold \( \text{Ind}(M) \) is a disjoint union

\[
Y_\sigma = \bigsqcup_{[w] \in W/W_\sigma} Y^{[w]}_\sigma.
\]

**Proof.** We first show the inclusion "\( \supset \)". For any \( [w] \in W/W_\sigma \) and \( g \in LG \) such that \( g \cdot \sigma = w\sigma \), we have

\[
\text{Ind}(\Phi)(Y^{[w]}_\sigma) = \text{Ind}(\Phi)(LG_\sigma \cdot g^{-1} \cdot \Phi^{-1}(U_{w\sigma} \cap g^*)) \subseteq (LG)_\sigma \cdot g^{-1} \cdot U_{w\sigma} = (LG)_\sigma \cdot U_\sigma = U_\sigma.
\]

For the opposite inclusion, suppose we are given \( x \in Y_\sigma \). Thus \( x = h \cdot m \) for some \( m \in M \), \( h \in LG \) with \( h \cdot \Phi(m) \in U_\sigma \). We may assume with no loss of generality that \( \mu := \Phi(m) \in t_+ \). Choose \( w \in W_\text{aff} \) such that \( \nu := w^{-1} \mu \in \mathfrak{A}_\sigma \), and choose \( g \in LG \) with \( g \cdot \sigma = w \cdot \sigma \). Then

\[
\mu \in w\mathfrak{A}_\sigma \subset U_{w\sigma} \cap g^*,
\]

so that \( m \in \Phi^{-1}(U_{w\sigma} \cap g^*) \). Moreover,

\[
gh \cdot \mu \in g \cdot U_\sigma = U_{w\sigma},
\]

which implies \( gh \in (LG)_{w\sigma} \) since \( U_{w\sigma} \) is a slice.

We now consider orientations. Note first that every \( w\sigma \) is contained in some fixed open face of the positive Weyl chamber \( t_+ \). Therefore all points in \( w\sigma \) have the same stabilizer group \( G_{w\sigma} = G \cap (LG)_{w\sigma} \) with respect to the coadjoint \( G \)-action, equal to the stabilizer group of that face of \( t_+ \). Observe next that

\[
U_{w\sigma} \cap g^* = LG_{w\sigma} \cdot w\mathfrak{A}_\sigma \cap g^* = G_{w\sigma} \cdot (w W_\sigma \mathfrak{A}_\sigma \cap t_+)
\]

is a slice for the coadjoint \( G \)-action. Hence \( \Phi^{-1}(U_{w\sigma} \cap g^*) \subset M \) is an open subset of the cross-section in \( M \) corresponding to \( G_{w\sigma} \). We can write \( Y^{[w]}_\sigma \) as an associated bundle

\[
(8)\quad Y^{[w]}_\sigma = g^{-1} \cdot (LG)_{w\sigma} \times_{G_{w\sigma}} \Phi^{-1}(U_{w\sigma} \cap g^*).
\]

Both \( \Phi^{-1}(U_{w\sigma} \cap g^*) \), being an open subset of a cross-section of \( M \), and \( G_{w\sigma} \), being the stabilizer of a face of \( t_+ \), have canonical orientations. Since \( g \) is uniquely determined up to right-multiplication by an element of \((LG)_\sigma\), which is connected, the group \((LG)_{w\sigma} = g(LG)_\sigma g^{-1}\) inherits an orientation from the orientation of \((LG)_\sigma\), and finally the associated bundle \( (8) \) obtains a canonical orientation.
Remark 3.26. The orientation of \((LG)_{w\sigma}\) depends on \(w\sigma\). For example, if \(\sigma = \text{int} A\) we have \((LG)_{w\sigma} = G_{w\sigma} = T\) for any \(w \in W_{\text{aff}}\). The orientation on \((LG)_{w\sigma}\) agrees with the given orientation on \(T\) (corresponding to the choice of \(t_\mu\)) if and only if \(\text{length}(w)\) is even. On the other hand the orientation of \(G_{w\sigma}\) agrees with the orientation on \(T\) for any \(w \in W_{\text{aff}}^+\).

As we explained in the previous section, the cross-sections \(Y_\sigma\) also acquire orientations from the orientation on the holonomy manifold \(\text{Hol}(\text{Ind}(M)) \cong M\).

Lemma 3.27. The orientations on \(Y_\sigma\) just described are identical with the orientations induced from \(\text{Hol}(\text{Ind}(M)) \cong M\).

Proof. To see this, we describe the image \(\text{Hol}(Y_\sigma) \subset \text{Hol}((\text{Ind}(M)) = M\). The evaluation map \(LG \to G\) gives an embedding \((LG)_{w\sigma} \to G_{w\sigma}\). The image of this embedding is given by \(Z_{w\sigma} := \text{Ad}_g(1) Z\sigma\) and contains \(G_{w\sigma}\). By (8),

\[
\text{Hol}(Y_\sigma^{[w]}) = g(1)^{-1} \cdot Z_{w\sigma} \times G_{w\sigma} \Phi^{-1}(U_{w\sigma} \cap g^*),
\]

with an orientation coming from the orientations on \(Z_{w\sigma}, G_{w\sigma}\) and \(\Phi^{-1}(U_{w\sigma} \cap g^*)\). Since the natural isomorphisms

\[
G \times Z_\sigma \text{ Hol}(Y_\sigma^{[w]}) \cong G \times Z_{w\sigma} \left( g(1) \cdot \text{Hol}(Y_\sigma^{[w]}) \right) \\
\cong G \times Z_{w\sigma} \left( Z_{w\sigma} \times G_{w\sigma} \Phi^{-1}(U_{w\sigma} \cap g^*) \right) \\
\cong G \times Z_{w\sigma} \Phi^{-1}(U_{w\sigma} \cap g^*)
\]

are orientation preserving, the orientation on \(Y_\sigma^{[w]} \cong \text{Hol}(Y_\sigma^{[w]})\) just described agrees with the orientation on \(\text{Hol}(Y_\sigma^{[w]})\) induced from the embedding \(G \times Z_\sigma \text{ Hol}(Y_\sigma^{[w]}) \to M\) as an open subset. □

4. Construction of the cobordism

4.1. The fusion product of Hamiltonian LG-manifolds. As we mentioned earlier, Hamiltonian \(LG\)-manifolds with proper moment maps behave in many respects like compact Hamiltonian spaces for compact Lie groups. Now it is well-known that the classical analog of taking the tensor product of representations is taking the direct product of Hamiltonian spaces with the diagonal \(G\)-action. However, the direct product \(M_1 \times M_2\) of two Hamiltonian \(LG\)-manifolds (at level 1) with proper moment maps has a non-proper moment map (at level 2). There is a different product operation that preserves the level and also properness of the moment map.

Definition 4.1. Let \(M_1, M_2\) be Hamiltonian \(LG\)-manifolds with proper moment maps \(\Phi_1, \Phi_2\). Let \(M(\Sigma_3)\) be the moduli space for the three-holed sphere. The fusion product \(M_1 \circledast M_2\) is the Hamiltonian \(LG\)-manifold obtained as the Hamiltonian quotient

\[
M_1 \circledast M_2 := (M_1 \times M_2 \times \mathcal{M}(\Sigma_3)) / \text{diag}(LG^2)
\]

under the diagonal \(LG^2\)-action. We denote the resulting moment map by \(\Phi_1 \circledast \Phi_2 : M_1 \circledast M_2 \to \mathfrak{g}^*\).
Observe that $M_1 \otimes M_2$ is a smooth Banach manifold. Indeed, since the $LG^2 \subset LG^3$-action on $\mathcal{M}(\Sigma^3_0)$ is free, the corresponding moment map $\mathcal{M}(\Sigma^3_0) \to (Lg)^2$ is a submersion. This implies that 0 is a regular value of the moment map for the $\text{diag}(LG^2)$-action on $M_1 \times M_2 \times \mathcal{M}(\Sigma^3_0)$, and that $\text{diag}(LG^2)$ acts freely on the zero level set. From general properties of symplectic reduction, it follows that $\Phi_1 \otimes \Phi_2$ is again proper, and that the fusion product of symplectic Hamiltonian $LG$-manifolds is symplectic.

In the case of moduli spaces of flat connections we have

$$\mathcal{M}(\Sigma^1_{g_1}) \otimes \mathcal{M}(\Sigma^1_{g_2}) = \mathcal{M}(\Sigma^1_{g_1+g_2}).$$

This follows immediately from the definition and Theorem 3.15.

The category of Hamiltonian $LG$-manifolds with proper moment maps with product operation $\otimes$ may be considered the classical analog to the fusion ring of $LG$-representations at a given level.

**Proposition 4.2. (The classical fusion ring)**

a. For any Hamiltonian $LG$-manifold $M$, there is an equivariant symplectomorphism $M \otimes \Omega G \cong M$.

b. Let $M_1$, $M_2$, $M_3$ be Hamiltonian $LG$-manifolds with proper moment maps. There are equivariant symplectomorphisms

$$M_1 \otimes M_2 \cong M_2 \otimes M_1,$$

$$M_1^* \otimes M_2^* \cong (M_1 \otimes M_2)^*.$$

$$(M_1 \otimes M_2) \otimes M_3 \cong M_1 \otimes (M_2 \otimes M_3).$$

c. If $LG \cdot \mu$, $LG \cdot \nu$ are coadjoint orbits through $\mu, \nu \in A$ then $(LG \cdot \mu \otimes LG \cdot \nu)_0$ is a point for $\nu = *\mu$ and empty otherwise.

**Proof.** Assertion (a) follows from $M \otimes \Omega G \cong M \times \mathcal{M}(\Sigma^1_0) \times \mathcal{M}(\Sigma^3_0) \sqcup \text{diag}(LG)^2$

$$\cong M \times \mathcal{M}(\Sigma^2_0) \sqcup \text{diag}(LG) \cong M$$

by Example 3.16 and Theorem 3.15.

For any orientation preserving diffeomorphism of $\Sigma^3_0$ the pull-back map on $\mathcal{A}(\Sigma)$ induces an equivariant symplectomorphism of $\mathcal{M}(\Sigma^3_0)$. Therefore, the first assertion in (b) follows from the existence of an orientation preserving diffeomorphism transposing two boundary components. The second assertion follows from Remark 3.3. Associativity follows from Theorem 3.15, using two different ways of cutting a 4-holed sphere into two 3-holed spheres:

$$(M_1 \otimes M_2) \otimes M_3 \cong \mathcal{M}(\Sigma^3_0) \times \mathcal{M}(\Sigma^3_0) \times M_1 \times M_2 \times M_3 \sqcup \text{diag}(LG)^4$$

$$\cong \mathcal{M}(\Sigma^3_0) \times M_1 \times M_2 \times M_3 \sqcup \text{diag}(LG)^3 \cong M_1 \otimes (M_2 \otimes M_3).$$

Part (c) follows from

$$(LG \cdot \mu \otimes LG \cdot \nu)_0 \cong \mathcal{M}(\Sigma^3_0)_{\mu,\nu,0} \cong \mathcal{M}(\Sigma^2_0)_{\mu,\nu}.$$
which is a point if $\mu = *\nu$ and empty otherwise, by the holonomy description of $\mathcal{M}(\Sigma_0^2)_{\mu,\nu}$.

\[\square\]

4.2. **Fusion product of holonomy manifolds.** What is the holonomy manifold of a fusion product of Hamiltonian $LG$-manifolds? Consider the category of isomorphism classes of pairs $(Q, \Psi)$ where $Q$ is a $G$-manifold and $\Psi : Q \to G$ an equivariant map. The involution $*$ defines an involution on this category. Define a fusion product $(Q_1 \otimes Q_2, \Psi_1 \otimes \Psi_2)$ by setting $Q_1 \otimes Q_2 = Q_1 \times Q_2$ with diagonal $G$-action and $\Psi_1 \otimes \Psi_2 = \Psi_1 \cdot \Psi_2$. This fusion product satisfies axioms analogous to 4.2. For example, commutativity follows by considering the map

$$\tau : Q_1 \times Q_2 \to Q_1 \times Q_2, \ (q_1, q_2) \mapsto (q_1, \Psi_1(q_1)^{-1} \cdot q_2)$$

since $(\Psi_1 \otimes \Psi_2) \circ \tau = \Psi_2 \cdot \Psi_1$. The analog to coadjoint orbits $LG \cdot \mu$ ($\mu \in \mathcal{A}$) are the conjugacy classes $C_\mu = \text{Ad}(G) \cdot \exp(\mu)$.

**Lemma 4.3.** Let $(M_i, \omega_i, \Phi_i)$ be Hamiltonian $LG$-manifolds with proper moment maps. Then

$$\text{Hol}(M_1 \otimes M_2) = \text{Hol}(M_1) \otimes \text{Hol}(M_2).$$

**Proof.** The claim follows from the identities

$$\text{Hol}(M_1 \otimes M_2) = \text{Hol}(M_1 \times M_2 \times \mathcal{M}(\Sigma_0^2) \// \text{diag}(LG)^2)$$

$$= \text{Hol}(M_1) \times \text{Hol}(M_2) \times \text{Hol}(\mathcal{M}(\Sigma_0^2)) \// G^2$$

and the holonomy description given in Proposition 3.9. \[\square\]

Note that we also have $\text{Hol}(LG \cdot \mu) = G \cdot \exp(\mu)$ and $\text{Hol}(M^*) = \text{Hol}(M)^*$, so that in this sense the holonomy map is a homomorphism of classical fusion rings.

**Example 4.4.** Let $\mu_1, \ldots, \mu_b \in \mathcal{A}$ and $C_{\mu_j} = G \cdot \exp(\mu_j)$ their conjugacy classes. Then (9) may be re-written

$$\mathcal{M}(\Sigma_0^b)_{\mu_1, \ldots, \mu_b} = (C_{\mu_1} \otimes \ldots \otimes C_{\mu_b}) \// G.$$

4.3. **Induction and fusion product.** The main result of this section is the following relation between induction and the fusion product:

**Theorem 4.5.** Let $(M_i, \omega_i, \Phi_i)$ ($i = 1, 2$) be compact Hamiltonian $G$-manifolds. Then there exists an $LG$-equivariant diffeomorphism

$$\phi : \text{Ind}(M_1) \otimes \text{Ind}(M_2) \to \text{Ind}(M_1 \times M_2)$$

such that the equivariantly closed 2-forms on both sides are cohomologous. That is, there exists an $LG$-invariant 1-form $\beta \in \Omega^1_{LG}(\text{Ind}(M_1) \otimes \text{Ind}(M_2))$ such that

$$\text{Ind}(\omega_1) \otimes \text{Ind}(\omega_2) = \phi^* \text{Ind}(\omega_1 + \omega_2) + d\beta$$

$$\text{Ind}(\Phi_1) \otimes \text{Ind}(\Phi_2) = \phi^* \text{Ind}(\Phi_1 + \Phi_2) - \beta^g$$

where $\beta^g : \text{Ind}(M_1) \otimes \text{Ind}(M_2) \to LG^*$ is defined by $(\beta^g, \xi) = \iota(\xi)\beta$. 

Proof. Let $Y_{0,0,0} \subset \mathcal{M}(\Sigma_3^3)$ denote the cross-section at $(0,0,0) \in \mathfrak{a} \times \mathfrak{a} \times \mathfrak{a}$ given in the holonomy description Proposition 5.9 by

\begin{equation}
Y_{0,0,0} = \{(c, \eta) \in G^2 \times \mathfrak{g}^3 | \eta_i \in U_{0i} \text{ and } \exp(\eta_1) \Ad_{c_1} \exp(\eta_2) \Ad_{c_2} \exp(\eta_3) = 1\}.
\end{equation}

Let $\Psi : \mathcal{M}(\Sigma_3^3) \to (L\mathfrak{g}^*)^3$ be the moment map for $\mathcal{M}(\Sigma_0^3)$. The zero level set is $\Psi^{-1}(0) \cong G^2$, with two copies of $(L\mathfrak{g})_0 \cong G$ acting from the right and one copy acting diagonally from the left.

Note that $\Psi^{-1}(0)$ is also the zero-level set for the $G^2 \subset G^3$-action on the cross-section $Y_{0,0,0}$. By the equivariant symplectic normal form theorem, it follows that a neighborhood of $\Psi^{-1}(0)$ inside $Y_{0,0,0}$ is modeled by a neighborhood of the zero section of the cotangent bundle $T^*G^2$, with two $G$-copies acting by the cotangent lift of the right action and one by the cotangent lift of the left action. More precisely, there exists a $G^2$-invariant neighborhood $V \subset (U_{01})^2 \subset (L\mathfrak{g})^2$ of 0 such that $\Psi^{-1}(V \times U_{01})$ is $G^3$-equivariantly symplectomorphic to a neighborhood $U$ of the zero section in $T^*G^2$.

In the special case that $\Phi_1(M_1) \times \Phi_2(M_2) \subset V$ then

\[
\text{Ind}(M_1) \otimes \text{Ind}(M_2) = \text{Ind}(M_1) \times \text{Ind}(M_2) \times \mathcal{M}(\Sigma_0^3)/\!\!/LG^2 = \text{Ind}(M_1) \times \text{Ind}(M_2) \times \text{Ind}(U) /\!\!/LG^2 = \text{Ind}(M_1 \times M_2 \times U /\!\!/G^2) = \text{Ind}(M_1 \times M_2)
\]

as claimed.

To reduce the general case to the case that $\Phi_1(M_1) \times \Phi_2(M_2) \subset V$ we apply a generalization of the Duistermaat-Heckman principle. Let $M_j^{(a)}$ denote $M_j$, with symplectic 2-form multiplied by a factor $a > 0$. By Theorem 5.8 below, there exists a family $\varphi_a$ of $LG$-equivariantly diffeomorphisms $\text{Ind}(M_1) \otimes \text{Ind}(M_2) \cong \text{Ind}(M_1^{(a)}) \otimes \text{Ind}(M_2^{(a)})$ such that the equivariant cohomology class of the equivariant closed 2-form $\text{Ind}(a\omega_1) \otimes \text{Ind}(a\omega_2)$ varies linearly in $a$. That is, there exists a closed $LG$-invariant 2-form $\gamma \in \Omega^2(\text{Ind}(M_1) \otimes \text{Ind}(M_2))$ with moment map $\Psi$ at level 1, and a family of $LG$-invariant 1-forms $\beta_a \in \Omega^1(\text{Ind}(M_1) \otimes \text{Ind}(M_2))$ such that

\[
\varphi_a^*(\text{Ind}(a\omega_1) \otimes \text{Ind}(a\omega_2)) = \text{Ind}(\omega_1) \otimes \text{Ind}(\omega_2) + (a - 1)\gamma + d\beta_a,
\]

\[
\varphi_a^*(\text{Ind}(a\Phi_1) \otimes \text{Ind}(a\Phi_2)) = \text{Ind}(\Phi_1) \otimes \text{Ind}(\Phi_2) + (a - 1)\Psi - \beta_a^2
\]

where $\langle \beta_a^2, \xi \rangle = \iota(\xi_M)\beta_a$.

The inductions $\text{Ind}(M_1^{(a)} \times M_2^{(a)})$ satisfy an analogous property; in fact, in this case the 2-form and moment map depend linearly on $a$. For $a$ small enough, $\Phi_1(M_1^{(a)}) \times \Phi_2(M_2^{(a)}) = a\Phi_1(M_1) \times a\Phi_2(M_2) \subset V$, so that $\text{Ind}(M_1^{(a)}) \otimes \text{Ind}(M_2^{(a)})$ and $\text{Ind}(M_1^{(a)} \times M_2^{(a)})$ are equivariantly symplectomorphic. In particular, the slope for the change of cohomology class of the closed 2-form is the same for both spaces, which proves the theorem. \qed
An application of Example 2.3 (which generalizes immediately to the \(LG\)-equivariant setting) shows that \(\text{Ind}(M_1) \otimes \text{Ind}(M_2)\) and \(\text{Ind}(M_1 \times M_2)\) are cobordant as Hamiltonian \(LG\)-manifolds with proper moment maps. (Properness for the moment map \(\Phi_N\) for the cobordism \(N\) follows from compactness of \(\text{Hol}(N)\).) If \(\tau \in A\) is a regular value for \(\Phi_N\) this gives a cobordism of compact orbifolds

\[
(\text{Ind}(M_1) \otimes \text{Ind}(M_2))_{\tau} \sim \text{Ind}(M_1 \times M_2)_{\tau}.
\]

If \(\tau \in \text{int} A\) the reduction on the right hand side is given by Corollary 3.24, and we obtain an oriented cobordism

\[
(\text{Ind}(M_1) \otimes \text{Ind}(M_2))_{\tau} \sim \bigsqcup_{w \in W^+_{\text{aff}}} (-1)^{\text{length}(w)}(M_1 \times M_2)_{w\tau}.
\]

Thinking of the cobordism in (10) as a quotient of the finite-dimensional cross-section \(\Phi_N^{-1}(\text{int}(A))\), the perturbation argument of Ginzburg-Guillemin-Karshon [6] shows that it suffices to assume that \(\tau\) is a regular value for moment maps of the \(LG\)-actions on the ends of the cobordism.

We are particularly interested in the case where \(M_1 = O^*_{\mu}\) and \(M_2 = O^*_{\nu}\) are coadjoint orbits through \(*_{\mu}, *_{\nu} \in A\) of.

**Theorem 4.6.** Let \(\mu, \nu \in A_0\) and \(\tau \in \text{int}(A)\) such that for every \(w \in W^+_{\text{aff}}\), \(w \cdot \tau\) is a regular value (or not in the image) of the moment map for the diagonal action of \(G\) on \(O^*_{\mu} \times O^*_{\nu}\), and \(\tau\) is a regular value (or not in the image) of the moment map for the action of \(LG\) on \(M(\Sigma^3)_{\mu, \nu}\). Then there is an oriented orbifold cobordism

\[
M(\Sigma^3)_{\mu, \nu, \tau} \sim \bigsqcup_{w \in W^+_{\text{aff}}} (-1)^{\text{length}(w)}(O^*_{\mu} \times O^*_{\nu} \times O_{w(\tau)})/G
\]

where the symplectic forms extend to a closed two-form on the cobordism. In the case of \(G = SU(n)\), the reduced spaces on both sides are smooth manifolds and the cobordism is a cobordism of smooth manifolds.

**Proof.** The induced spaces \(\text{Ind}(O^*_{\mu})\) and \(\text{Ind}(O^*_{\nu})\) are simply the coadjoint \(LG\)-orbits through \(*_{\mu}, *_{\nu}\), hence \((\text{Ind}(O^*_{\mu}) \otimes \text{Ind}(O^*_{\nu}))_{\tau}\) is the moduli space \(M(\Sigma^3)_{\mu, \nu, \tau}\). Therefore the cobordism is a special case of (10).

In the case \(G = SU(n)\), every discrete stabilizer for the \(LG\)-action on \(M(\Sigma^3)_{\mu, \nu}\) is equal to the center \(Z(G)\). This follows from the holonomy description, as explained in [17], Remark 4.3. Consequently the action of \(T/Z(G)\) on \(\Phi_N^{-1}(\tau)\) is free in this case and \(N_{\tau}\) is smooth.

The cobordism given in this Theorem can be carried further. Consider \(g/t\) as a Hamiltonian \(T\)-manifold, with symplectic structure corresponding to the \(T\)-invariant Hermitian structure defined by the metric and the choice of \(t_{\ast}\). By [17], Section 2 there are cobordisms

\[
(O^*_{\mu} \times O^*_{\nu})_{w\tau} \sim \bigsqcup_{w_1, w_2 \in W} (-1)^{\text{length}(w_1w_2)}(g/t)_{-w_1w_2, w_1w_2, w_{1\ast}w_{2\ast}w_{1\ast}}.
\]
Combining this with (11) gives:

**Theorem 4.7.** Let $\mu, \nu \in \mathfrak{A}_{(0)}$ and $\tau \in \text{int}(\mathfrak{A})$ such that $\tau$ is a regular value (or not in the image) of the action of $LG$ on $\mathcal{M}(\Sigma_0^3)_{\mu, \nu}$ and for all $w \in W_{\text{aff}}^+$ and $w_1, w_2 \in W$, $-w_1^* \mu - w_2^* \nu + w\tau$ is a regular value (or not in the image) of the $T$-action on $\mathfrak{g}/\mathfrak{t}$. Then $\mathcal{M}(\Sigma_0^3)_{\mu, \nu, \tau}$ is cobordant to a disjoint union of toric varieties

$$\mathcal{M}(\Sigma_0^3)_{\mu, \nu, \tau} \sim \coprod_{w \in W^+_{\text{aff}}, w_1, w_2 \in W} (-1)^{\text{length}(w)}(w)_{-w_1^* \mu - w_2^* \nu + w\tau}.$$  

**Remark 4.8.** Theorem 4.7 and the following discussion generalize easily to the case of more than 3 markings. Given $\mu_1, \ldots, \mu_{b-1} \in \mathfrak{A}_{(0)}, \mu_b \in \text{int}(\mathfrak{A})$ we have

$$\mathcal{M}(\Sigma_0^3)_{\mu_1, \ldots, \mu_b} \sim \coprod_{w \in W_{\text{aff}}^+} (-1)^{\text{length}(w)}(\mathcal{O}_{\ast \mu_1} \times \cdots \times \mathcal{O}_{\ast \mu_{b-1}} \times \mathcal{O}_{t \mu_b})/G.$$  

5. **Applications**

5.1. **Mixed Pontrjagin numbers for $\mathcal{M}(\Sigma_0^3)_{\mu_1, \mu_2, \mu_3}$.** In this section we use Proposition 4.7 to compute the mixed Pontrjagin numbers for the moduli space of the three-holed sphere. Recall that for regular $\mu = (\mu_1, \mu_2, \mu_3) \in \mathfrak{A}^3$, the moduli space $\mathcal{M}(\Sigma_0^3)_{\mu_1, \mu_2, \mu_3}$ has complex dimension equal to $k = \frac{1}{2}(\text{dim } G - 3 \text{dim } T)$. Given an invariant polynomial

$$p \in S(u(k)^* U(k) \cong \mathbb{C}[x_1, \ldots, x_k]^S_k$$

we can define the corresponding Chern class

$$p(\mathcal{M}(\Sigma_0^3)_{\mu_1, \mu_2, \mu_3}) \in H^*(\mathcal{M}(\Sigma_0^3)_{\mu_1, \mu_2, \mu_3})$$

and the mixed Chern number

$$\int_{\mathcal{M}(\Sigma_0^3)_{\mu_1, \mu_2, \mu_3}} p(\mathcal{M}(\Sigma_0^3)_{\mu_1, \mu_2, \mu_3}) \exp(\omega_{\mu_1, \mu_2, \mu_3})$$

where $\omega_{\mu_1, \mu_2, \mu_3}$ is the symplectic form on $\mathcal{M}(\Sigma_0^3)_{\mu_1, \mu_2, \mu_3}$.

The Pontrjagin ring of $\mathcal{M}(\Sigma_0^3)_{\mu_1, \mu_2, \mu_3}$ is the subring generated by polynomials

$$p \in S(\mathfrak{o}(2k)^* O(2k) \cong \mathbb{C}[x_1^2, \ldots, x_k^2]^S_k \subset \mathbb{C}[x_1, \ldots, x_k]^S_k$$

The **mixed Pontrjagin numbers** corresponding to these polynomials are invariants under oriented cobordism of orbifolds with closed 2-forms. Proposition 4.7 reduces the computation of the Pontrjagin numbers of $\mathcal{M}(\Sigma_0^3)_{\mu_1, \mu_2, \mu_3}$ to that for the reduced spaces of $\mathfrak{g}/\mathfrak{t}$, which are toric varieties. To any symmetric polynomial $p \in \mathbb{C}[x_1, \ldots, x_k]^S_k$ we associate a polynomial on $\mathfrak{t}$ as follows. Let $n = \frac{1}{2} \text{dim} (G/T)$. By the canonical extension map

$$\mathbb{C}[x_1, \ldots, x_k]^S_k \to \mathbb{C}[x_1, \ldots, x_n]^S_n$$

(sending the $l$th elementary symmetric polynomial in $k$ variables to that in $n$ variables) we can view $p$ as a polynomial on $\mathbb{R}^n$. Choose an ordering $\alpha_1, \ldots, \alpha_n$ of the positive roots of $G$ to identify $\bigoplus_{\alpha \in \mathfrak{a}_+} \mathbb{R} \cong \mathbb{R}^n$ and $\mathfrak{g}/\mathfrak{t} \cong \mathbb{C}^n$. The action of $T$ on $\mathfrak{g}/\mathfrak{t}$ gives a
map $T \to U(1)^n$. We define the polynomial $\phi^* p \in S(t^*)$ to be the pull-back of $p$ by the tangent map $\phi : t \to u(1)^n = \mathbb{R}^n$. The polynomial $\phi^* p$ defines a constant coefficient differential operator $(\phi^* p)(\frac{\partial}{\partial \mu})$ on $t^*$.

Let $\kappa$ be the push-forward of the characteristic measure on the positive orthant $\mathbb{R}_+^n$ under the map $\phi^* : \mathbb{R}^n \to t^*$ dual to $\phi$. By comparing with the given Lebesgue measure on $t^*$ we can consider $\kappa$ as a (piecewise polynomial) function on $t^*$.

**Lemma 5.1.** The symplectic volume of any reduced space $(\mathfrak{g}/t)_\mu$ at a regular value $\mu \in t^*$ is given by

$$\text{Vol}((\mathfrak{g}/t)_\mu) = \int_{(\mathfrak{g}/t)_\mu} \exp(\omega_\mu) = \frac{\# Z(G)}{\text{Vol}(T)} \kappa(-\mu).$$

The mixed Chern number corresponding to an invariant polynomial $p$ is given by application of the differential operator $\phi^* p(\frac{\partial}{\partial \mu})$ to the volume function:

$$\int_{(\mathfrak{g}/t)_\mu} p((\mathfrak{g}/t)_\mu) \exp(\omega_\mu) = (\phi^* p)(\frac{\partial}{\partial \mu}) \text{Vol}((\mathfrak{g}/t)_\mu).$$

**Proof.** The proof of this result can be found e.g. in [7] or [8]. We recall the argument for convenience of the reader.

The Duistermaat-Heckman measure for the standard $U(1)^n$-action on $\mathfrak{g}/t$ is given by the characteristic function of the negative orthant $-\mathbb{R}_+^n \subset \mathbb{R}^n$. Therefore the Duistermaat-Heckman measure for the $T$-action is $\mu \mapsto \kappa(-\mu)$, which proves the volume formula. The factor $\# Z(G)$ is due to the fact that the generic stabilizer for the $T$-action on $\mathfrak{g}/t$ is equal to $Z(G)$.

To prove (12) let $\Phi : \mathfrak{g}/t \to t^*$ denote the moment map for the $T$-action and $\pi : \Phi^{-1}(\mu) \to (\mathfrak{g}/t)_\mu$ and $\iota : \Phi^{-1}(\mu) \to \mathfrak{g}/t$ the projection and inclusion. Then $\pi^* T(\mathfrak{g}/t)_\mu \oplus t_C = t^* \mathfrak{g}/t$ and therefore

$$T(\mathfrak{g}/t)_\mu \oplus t_C = \bigoplus_{\alpha \in \mathfrak{a}^*_C} L_\alpha$$

where $L_\alpha = Z \times_T \mathbb{C}_\alpha$ (see e.g. [7] p. 59)). The Chern classes of $(\mathfrak{g}/t)_\mu$ are symmetric polynomials in the first Chern classes $c_\alpha = c_1(L_\alpha)$. The classes $c_\alpha$ are related to the first Chern class $c \in \Omega^2((\mathfrak{g}/t)_\mu, t)$ of the torus bundle $Z \to (\mathfrak{g}/t)_\mu$ by $(c_{\alpha_1}, \ldots, c_{\alpha_n}) = \phi \circ c$. Thus $(\phi^* p)(c) = p(c_{\alpha_1}, \ldots, c_{\alpha_n}) = p((\mathfrak{g}/t)_\mu)$. This proves the Theorem since by the Duistermaat-Heckman Theorem,

$$\phi^* p(\frac{\partial}{\partial \mu}) \text{Vol}((\mathfrak{g}/t)_\mu) = \int_{(\mathfrak{g}/t)_\mu} \phi^* p(c) \exp(\omega_\mu).$$

$\square$
Theorem 5.2. For \( \mu = (\mu_1, \mu_2, \mu_3) \in \text{int} (\mathfrak{A}^3 \cap \Phi(M(\Sigma_0^3))) \), the volume of the moduli space \( M(\Sigma_0^3)_{\mu_1, \mu_2, \mu_3} \) is given by

\[
(13) \quad (-1)^{\frac{1}{2} \dim G/T} \frac{Z(G)}{\text{Vol}(T)} \sum_{l \in \Lambda} \sum_{w_1, w_2 \in W} (-1)^{\text{length}(w_1 w_2)} \kappa(w_1 \mu_1 + w_2 \mu_2 + \mu_3 + l).
\]

If in addition \( \mu \) is a regular value for the moment map then the mixed Pontrjagin number corresponding to an invariant polynomial \( p \in S(\mathfrak{a}(2k))^{G(2k)} \) (where \( k = \frac{1}{2}(\dim G - 3 \dim T) \)) is given by application of the differential operator \( \phi^* p(\frac{\partial}{\partial \mu_k}) \) to the volume function:

\[
\int_{M(\Sigma_0^3)_{\mu_1, \mu_2, \mu_3}} p(M(\Sigma_0^3)_{\mu_1, \mu_2, \mu_3}) \exp(\omega_{\mu_1, \mu_2, \mu_3}) = (\phi^* p)(\frac{\partial}{\partial \mu_3}) \text{Vol}(M(\Sigma_0^3)_{\mu_1, \mu_2, \mu_3})
\]

Proof. Using Proposition [17], Lemma [7] and the invariance of mixed Pontrjagin numbers under oriented cobordism, the mixed Pontrjagin numbers are given by applying \( (\phi^* p)(\frac{\partial}{\partial \mu_3}) \) to

\[
\text{Vol}(M(\Sigma_0^3)_{\mu_1, \mu_2, \mu_3}) = \frac{Z(G)}{\text{Vol}(T)} \sum_{w \in W_0} (-1)^{\text{length}(w)} \kappa(w \mu_1 + w_2 \mu_2 - w \ast \mu_3)
\]

using the \( \ast \)-invariance of \( \kappa \). The sum over \( W_0 = W_{\text{aff}}/W \cong \Lambda \) can be replaced by a sum over the integral lattice \( \Lambda \subset W_{\text{aff}} \), using the \( W \)-invariance of the function

\[
\nu \mapsto \sum_{w \in W} (-1)^{\text{length}(w)} \kappa(w \mu - \nu).
\]

Using \( -\ast \mu = w_0 \mu \) where \( \text{length}(w_0) = \dim(G/T) \) gives (13). By continuity the volume formula also holds at non-regular values \( \mu \in \text{int} (\mathfrak{A}^3 \cap \Phi(M(\Sigma_0^3))) \). \( \square \)

5.2. Witten’s volume formulas. In this section we briefly outline how to obtain Witten’s formulas for the symplectic volumes of the moduli spaces. Further details can be found in [13]. We label the irreducible \( G \)-representations by their dominant weights \( \lambda \in \Lambda^*_+ := \Lambda^* \cap t_+ \) and let \( \chi_{\lambda} : G \rightarrow \mathbb{C} \) denote the character and \( d_{\lambda} = \chi_{\lambda}(e) \) the dimension. Let \( \text{Vol}(G) \) be the Riemannian volumes of \( G \) with respect to the normalized inner product on \( \mathfrak{g} \). For any \( \mu \in \mathfrak{A} \) let \( C_\mu \subset G \) the conjugacy class parametrized of \( \exp(\mu) \) and \( \text{Vol}(C_\mu) \) its Riemannian volume.

Theorem 5.3 (Witten Formula). Suppose \( 2h + b \geq 3 \). Let \( \mu = (\mu_1, \ldots, \mu_b) \in \mathfrak{A}^b \) be such that the level set \( \Phi^{-1}(\mu) \) contains a connection with stabilizer \( Z(G) \). Then the volume \( \text{Vol} \left( M(\Sigma_{h_1}^b, \mu_1, \ldots, \mu_b) \right) \) of the moduli space of the 2-manifold \( \Sigma_{h_1}^b \) with fixed holonomies \( \mu_1, \ldots, \mu_b \) is given by the formula

\[
(14) \quad \#Z(G) \text{Vol}(G)^{2h-2} \prod_{j=1}^b \left( \text{Vol}(C_{\mu_j}) \prod_{\alpha \in \Phi_+^{\mu_j}} 2 \sin(\pi \langle \alpha, \mu_j \rangle) \right) \sum_{\lambda \in \Lambda^*_+} \frac{1}{d_{\lambda}^{2h-2+b}} \prod_{j=1}^b \chi_{\lambda}(e^{\mu_j}).
\]
In particular, if $\Sigma_0$ has no boundary:

$$\text{Vol}(\mathcal{M}(\Sigma_0^0)) = \# Z(G) \text{Vol}(G)^{2h-2} \sum_{\lambda \in \Lambda^+} \frac{1}{d^2 \lambda - 2^2}. $$

We sketch the main ideas, referring to [16] for details. Witten’s formula for $\mathcal{M}(\Sigma_0^3)$

$$\text{Vol}(\mathcal{M}(\Sigma_0^3)) = \# Z(G) \text{Vol}(G)^{2h-2} \sum_{\lambda \in \Lambda^+} \frac{1}{d^2 \lambda - 2^2} \lambda \in \Lambda^+, \prod_{\alpha \in \mathfrak{A}^+} 2 \sin(\pi \langle \alpha, \xi \rangle). $$

(15)

can be proved by applying the the Poisson summation and Weyl character formulas to (13). Formulas for the general case are obtained by gluing. Let $\Sigma = \Sigma_b$ be obtained from a possibly disconnected 2-manifold $\hat{\Sigma}$ by gluing two boundary components $B_\pm \subset \partial \Sigma$, and $\mu_1, \ldots, \mu_b \in \mathfrak{A}$ such that $\mathcal{M}(\Sigma, \mu_1, \ldots, \mu_b)$ contains at least one connection with stabilizer $Z(G)$. Then

$$\text{Vol}(\mathcal{M}(\Sigma, \mu_1, \ldots, \mu_b)) = \frac{1}{k} \int_{\mathfrak{A}} \text{Vol}(\mathcal{M}(\hat{\Sigma}, \mu_1, \ldots, \mu_b, \nu, \ast \nu))|d\nu|$$

(see e.g. Jeffrey-Weitsman [11]). Here the measure $|d\nu|$ on $\mathfrak{A}$ is the normalized measure for which $\mathfrak{t}/\Lambda^*$ has measure 1, and $k = 1$ if $\hat{\Sigma}$ is connected and equal to $\# Z(G)$ if $\hat{\Sigma}$ is disconnected. Choosing a pants decomposition for $\Sigma$ and using the orthogonality relations of the characters $\chi_\lambda$ and the Weyl integration formula, carrying out the integrations gives the formula (14) for $\mu = (\mu_1, \ldots, \mu_b) \in \text{int}(\mathfrak{A})^b$. The volumes for arbitrary $\mu$ are computed by studying the limit as $\mu$ approaches the boundary of $\mathfrak{A}$.

The formula (14) was proved in most cases by Witten [22]. Alternative proofs and extensions were given in [13, 11, 10, 11].

6. A Duistermaat-Heckman principle

In this section we prove the Duistermaat-Heckman result used in the proof of Theorem 4.3. We follow the strategy from Section 3.6, carrying out all constructions in finite-dimensional cross-sections. As a first step we construct invariant tubular neighborhoods which are suitably adapted to the cross-sections. Let $(\mathcal{M}, \omega)$ be a Hamiltonian $\hat{\mathcal{L}}\mathcal{G}$-manifold, with proper moment map $(\Phi, \phi) : \mathcal{M} \to \mathfrak{L}\mathfrak{g}^* \times \mathbb{R} \setminus \{0\}$. Let $\sigma$ be a face of $\mathfrak{A}$ and $\mathcal{Y}_\sigma$ the corresponding cross-section. Recall from Proposition 3.18 that there is a canonical isomorphism

$$TM|\mathcal{Y}_\sigma \cong TY_\sigma \oplus (\mathfrak{L}\mathfrak{g})_\sigma^\perp.$$ 

(16)

Definition 6.1. Let $A \subset Y_\sigma$ be a subset.
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a. An \( \hat{\mathcal{LG}} \)-invariant metric \( g \) on \( M \) is called adapted to \( Y_{\sigma} \) over \( A \) if there is an open neighborhood \( O \subset Y_{\sigma} \) of \( A \) such that the splitting (16) is orthogonal over \( T M|O \), and the induced metric on the trivial bundle \( O \times (Lg)|_{\sigma} \) agrees with the \( L^2 \)-metric on \( (Lg)|_{\sigma} \subset Lg \).

b. An \( \hat{\mathcal{LG}} \)-invariant differential form \( \alpha \) on \( M \) is called adapted to \( Y_{\sigma} \) over \( A \) if there is an open neighborhood \( O \subset Y_{\sigma} \) of \( A \) such that for all \( x \in O \), the subspace \( (Lg)|_{\sigma} \subset T_x M \) is contained in the kernel of \( \alpha \).

Observe that adapted metrics or differential forms can be reconstructed over \( \hat{\mathcal{LG}} \cdot O \) from their restrictions to \( O \). If a metric or differential form is adapted to \( Y_{\sigma} \) over \( A \subset Y_{\sigma} \) and if \( \tau \preceq \sigma \), then it is also adapted to \( Y_{\tau} \) over \( (\hat{\mathcal{LG}})_{\tau} \cdot A \subset Y_{\tau} \).

It is in general impossible in general to choose a metric \( g \) on \( M \) which is adapted to all cross-sections \( Y_{\sigma} \) over all of \( Y_{\sigma} \). For this it is necessary to choose smaller subsets \( A \subset Y_{\sigma} \). We first choose a subdivision \( \mathfrak{A} = \bigcup_{\sigma} \mathfrak{A}_{\sigma} \) as indicated in Figure 1. The polytopes \( \mathfrak{A}_{\sigma} \) are given explicitly as follows. For any face \( \sigma \) of \( \mathfrak{A} \) let \( C_{\sigma} \subset \mathfrak{t} \) be the cone generated by outward-pointing normal vectors to facets containing \( \sigma \). Choose \( \epsilon \in (0,1) \) and \( \mu \in \text{int}(\mathfrak{A}) \), and define \( \mathfrak{A}_{\sigma} \) as the intersection
\[
\mathfrak{A}_{\sigma} = \mathfrak{A} \cap (\epsilon \mu + (1 - \epsilon)(\overline{\sigma} - \mu) + C_{\sigma}).
\]

Figure 1. Subdivision of the fundamental alcove for \( G = SU(3) \)

Set
\[
Y_{\sigma}' := (\Phi/\phi)^{-1}(\mathfrak{A}_{\sigma}).
\]

We will construct a metric \( g \) on \( M \) which is adapted to every \( Y_{\sigma} \) over the subset \( \bigcup_{\tau \preceq \sigma} Y_{\tau}' \).

Lemma 6.2. There exists an \( G \)-invariant partition of unity \( \{ \rho_{\sigma} \} \) on \( G \), subordinate to the cover \( G \cdot \mathfrak{A}_{\sigma} \) such that for each face \( \sigma \),
\[
\sum_{\tau \preceq \sigma} \rho_{\tau} = 1
\]
on an open neighborhood of \( G \cdot \bigcup_{\tau \geq \sigma} \mathcal{A}_\tau' \).

**Proof.** Observe first that the conditions (18) imply that for every \( \sigma \),

\[
\text{supp}(\rho_\sigma) \subset \bigcup_{\tau \leq \sigma} G \cdot \mathcal{A}_\tau.
\]

The proof of Lemma 6.2 is by induction, starting from \( \sigma = \text{int}(\mathcal{A}) \). Suppose by induction that we have constructed a collection \( \{\rho_\sigma\}_{\dim \sigma < k} \) of non-negative invariant functions with \( \text{supp}(\rho_\sigma) \subset G \cdot \mathcal{A}_\sigma \) such that \( \sum_{\dim \sigma < k} \rho_\sigma \leq 1 \) and

a. \( \text{supp}(\rho_\sigma) \subset \bigcup_{\tau \leq \sigma} G \cdot \mathcal{A}_\tau \) and

b. \( \sum_{\tau \leq \sigma} \rho_\tau = 1 \) on an open neighborhood of \( G \cdot \bigcup_{\tau \geq \sigma} \mathcal{A}_\tau \).

Let \( V_k \) be an invariant open neighborhood of \( G \cdot \bigcup_{\dim \tau > k} \mathcal{A}_\tau \) with \( (\sum_{\dim \tau > k} \rho_\tau)|V_k| = 1 \). If \( \sigma_1, \sigma_2 \) are disjoint faces of dimension \( k \), we have by construction

\[
(G \cdot \mathcal{A}_{\sigma_1}' \setminus V_k) \cap (G \cdot \mathcal{A}_{\sigma_2}' \setminus V_k) = \emptyset.
\]

Therefore we can choose non-negative invariant functions \( f_\sigma \leq 1 \) for every \( k \)-dimensional face \( \sigma \) with

\[
\text{supp}(f_\sigma) \subset \bigcup_{\tau \leq \sigma} G \cdot \mathcal{A}_\tau,
\]

such that the supports of the various \( f_\sigma \)'s are disjoint and such that \( f_\sigma = 1 \) on a neighborhood of \( G \cdot \mathcal{A}_\sigma \setminus V_k \). Then \( \rho_\sigma = f_\sigma (1 - \sum_{\tau > \sigma} \rho_\tau) \) satisfies Condition a and b for all faces \( \sigma \) with \( \dim \sigma \geq k \), and \( \sum_{\dim \tau > k} \rho_\tau \leq 1 \).

**Lemma 6.3.** There exists an \( \hat{\mathcal{L}}G \)-invariant Riemannian metric on \( M \) such that for each face \( \sigma \) of \( \mathcal{A} \) the canonical splitting (14) is adapted to \( Y_\sigma \) over \( \bigcup_{\tau \geq \sigma} Y_\tau' \).

**Proof.** For each \( \sigma \) choose an \( (\hat{\mathcal{L}}G)_\sigma \)-invariant Riemannian metric on \( Y_\sigma \) and let \( g_\sigma \) be the corresponding adapted metric on \( \mathcal{L}G \cdot Y_\sigma \). Let \( \{\rho_\sigma\} \) be a partition of unity as in Lemma 6.2. Then

\[
g = \sum_{\sigma} ((\text{Hol} \circ \Phi)^* \rho_\sigma) g_\sigma
\]

has the required property. \( \square \)

Using adapted metrics we can prove the tubular neighborhood theorem.

**Lemma 6.4.** Let \( (X, \omega) \) be a Hamiltonian \( \hat{\mathcal{L}}G \)-manifold, with proper moment map \( (\Phi, \phi) : X \rightarrow \mathcal{L}g^* \times \mathbb{R} \setminus \{0\} \), and suppose \( s \in \mathbb{R} \setminus \{0\} \) is a regular value of \( \phi \). Then for \( \epsilon \) sufficiently small there exists an equivariant diffeomorphism

\[
F : \phi^{-1}(s) \times (s - \epsilon, s + \epsilon) \rightarrow \phi^{-1}((s - \epsilon, s + \epsilon))
\]

such that \( (F^* \phi)(x, t) = t \), and \( F \) is adapted to the cross-sections in the sense that for each face \( \sigma \subset \mathcal{A} \), there is an open neighborhood \( V_\sigma \subset Y_\sigma \cap \phi^{-1}(s) \) of \( Y'_\sigma \cap \phi^{-1}(s) \) with

\[
F(V_\sigma \times (s - \epsilon, s + \epsilon)) \subset Y_\sigma.
\]

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Proof. Let $g$ be a Riemannian metric as in Lemma 6.3. For $\epsilon$ sufficiently small and for sufficiently small open neighborhoods $V_\sigma \subset Y_\sigma \cap \phi^{-1}(s)$ of $Y_\sigma' \cap \phi^{-1}(s)$, geodesic flow on $Y_\sigma$ in the normal direction to $Y_\sigma \cap \phi^{-1}(s)$ defines an $(\widehat{LG})_\sigma$-equivariant diffeomorphism

$$\widehat{F}_\sigma : V_\sigma \times (s - \epsilon, s + \epsilon) \rightarrow Y_\sigma.$$  

Since $\frac{\partial}{\partial t} \widehat{F}_\sigma^* \phi(x, t) > 0$ there is a unique reparametrization $\varphi_\sigma(x, t)$, defined for $t$ sufficiently close to $s$, such that $F_\sigma(x, t) = \widehat{F}_\sigma(x, \varphi_\sigma(x, t))$ satisfies $F_\sigma^* \phi(x, t) = t$. By construction $F_\sigma$ is $(\widehat{LG})_\sigma$-equivariant and for $\epsilon$ sufficiently small extends to an $\widehat{LG}$-equivariant diffeomorphism

$$F_\sigma : \widehat{LG} \cdot V_\sigma \times (s - \epsilon, s + \epsilon) \rightarrow \widehat{LG} \cdot Y_\sigma.$$  

Choosing $\epsilon$ and $V_\sigma$ smaller if necessary we can assume that for every $\sigma$, the metric $g$ is adapted to $Y_\sigma$ over the image of $F_\sigma$, hence over $\bigcup_{\tau \geq \sigma} \text{im}(F_\tau)$. This means that for $\tau \leq \sigma$, the image $\text{im}(F_\sigma)$ is a totally geodesic submanifold of $(\widehat{LG})_\tau \cdot \text{im}(F_\sigma) \subset Y_\tau$, and the above geodesic flows and reparametrizations coincide. Consequently the maps $F_\sigma$ patch together to a diffeomorphism $F$ with the required properties. \hfill \square

There are a number of forms of the Duistermaat-Heckman principle which hold in the $LG$-equivariant setting. Here we need only the following special case.

**Proposition 6.5.** Let $(X, \omega, (\Phi, \phi))$ be a Hamiltonian $\widehat{LG}$-manifold as in Lemma 6.4. Suppose that $s$ is a regular value for $\phi$ and that the central $U(1)$ acts freely on $\phi^{-1}(s)$. Let $F$ and $\epsilon$ be as in Lemma 6.4. For $t \in (s - \epsilon, s + \epsilon)$ let $(X_t, \omega_t, \Phi_t)$ be the reduced space for the central $U(1)$ and $f_t : X_s \rightarrow X_t$ the diffeomorphisms induced by $F(\cdot, t)$. Then $f_t^* \omega_t + 2\pi i f_t^* \Phi_t$ varies linearly in cohomology, with slope $\gamma + 2\pi i \Psi$. That is, there exists an $LG$-invariant closed 2-form $\gamma \in \Omega^2(X_s)$ with moment map at level $s$, $\Psi : M \rightarrow LG^*$, and a family of $LG$-invariant 1-forms $\beta_t \in \Omega^1(X_s)$ such that

$$f_t^* \omega_t = \omega_s + (t - s)\gamma + d\beta_t,$$

$$f_t^* \Phi_t = \Phi_s + (t - s)\Psi - \beta_t^2$$

The form $\gamma$ is the curvature of an $LG$-invariant connection on $\phi^{-1}(s)$.

**Proof.** For $t \in (s - \epsilon, s + \epsilon)$ let $F_t = F(t, \cdot) : \phi^{-1}(s) \rightarrow M$ be the family of equivariant embeddings defined by the tubular neighborhood. Since the moment map for the $U(1)$-action is given in this model by $\phi(x, t) = t$, the 1-form

$$\alpha := -i (dF(\frac{\partial}{\partial t})) \omega$$

is a principal connection for the $U(1)$-action. Moreover $\alpha$ is adapted to $Y_s$ over $F(Y_s' \times (s - \epsilon, s + \epsilon))$.

The integral

$$\rho_t := \int_s^t F_u^* \alpha \ du \in \Omega^1(\phi^{-1}(s))$$
is well-defined for $t \in (s - \epsilon, s + \epsilon)$, and is adapted to $Y_\sigma$ over $F(Y'_\sigma \times (s - \epsilon, s + \epsilon))$. The form $\rho_t$ satisfies
\[
F^*_t \omega - F^*_s \omega = d\rho_t, \quad F^*_t \Phi - F^*_s \Phi = -\rho^2_t.
\]
Since $\alpha$ is a principal connection on the tubular neighborhood, $\rho_t/(t - s)$ is a principal connection for the U(1)-action on $\phi^{-1}(s)$, and the equivariant 2-form
\[
\frac{d\rho_t - 2\pi i \rho^2_t}{t - s}
\]
represents its equivariant curvature. In particular it is basic, i.e descends to a closed equivariant 2-form $\gamma_t + 2\pi i \Psi_t$ on $X_s$, with
\[
\begin{align*}
    f^*_t \omega_t &= \omega_s + (t - s)\gamma_s, \\
    f^*_t \Phi_t &= \Phi_s + (t - s)\Psi_s.
\end{align*}
\]
Since the cohomology class of $\gamma_t$ does not depend on $t$, we have $\gamma_s = \gamma_t + d\beta_t$, $\Psi_s = \Psi_t - \beta_t^2$ where $\beta_t$ is given by the transgression formula.

We now apply this result to the setting of Theorem 4.3.

**Theorem 6.6.** Let $(M_1, \omega_1)$ and $(M_2, \omega_2)$ be compact Hamiltonian $G$-manifolds. For $a > 0$ let $M_i^{(a)}$ denote the same $G$-manifolds with re-scaled 2-forms $\omega_i^{(a)} = a\omega_i$. Then there exists a family of equivariant diffeomorphisms, continuous and piecewise smooth in $a$,
\[
\phi_a : \text{Ind}(M_1) \otimes \text{Ind}(M_2) \to \text{Ind}(M_1^{(a)}) \otimes \text{Ind}(M_2^{(a)})
\]
such that $\phi_a^* \left( \text{Ind}(\omega_1^{(a)}) \otimes \text{Ind}(\omega_2^{(a)}) + 2\pi i \text{Ind}(\Phi_1^{(a)}) \otimes \text{Ind}(\Phi_2^{(a)}) \right)$ varies linearly in cohomology (in the sense of Proposition 6.5).

**Proof.** For every $s = a^{-1} > 0$ we define
\[
\text{Ind}_s(M_i) := \text{Ind}(M_i^{(1/s)})^{(s)}
\]
which is a Hamiltonian $LG$-manifold at level $s$. Equivalently, $\text{Ind}_s(M_i) \cong LG \times_G M_i$ is the unique Hamiltonian $LG$-manifold with moment map at level $s$ for which the 2-form and moment map restrict to the given ones on $M_i$. The 2-form on $\text{Ind}_s(M_i)$ differs from the 2-form on $\text{Ind}(M_i)$ by $(s - 1)$ times the pull-back of the symplectic form on $LG/G$.

Note that $\text{Ind}(M_1^{(a)}) \otimes \text{Ind}(M_2^{(a)})$ is equivariantly diffeomorphic to
\[
\text{Ind}_s(M_1) \otimes \text{Ind}_s(M_2) := \mathcal{M}(\Sigma^3_0)^{(s)} \times \text{Ind}_s(M_1) \times \text{Ind}_s(M_2) \cong (\text{diag}(LG^2)
\]
with symplectic forms equal after scalar multiplication by $s$. We show that there exists a Hamiltonian $\hat{LG}$-manifold $W$ such that the quotients $W_s$ with respect to the central $U(1)$ are $\text{Ind}_s(M_1) \otimes \text{Ind}_s(M_2)$, which proves the Theorem by Proposition 6.5.

To construct $W$ let $L \to \mathcal{M}(\Sigma^3_0) \times \text{Ind}(M_1) \times \text{Ind}(M_2)$ be the pullback of the prequantum bundle on the base,
\[
L(\Sigma^3_0) \boxtimes L(\Sigma^1_0) \boxtimes L(\Sigma^1_0).
\]
equipped with the pre-quantum connection. Let $\alpha$ be the corresponding principal connection on the unit circle bundle $U(L)$ and

$$X = U(L) \times \mathbb{R}, \quad \omega_X = \pi^* \omega + d(\phi - 1, \alpha)$$

where $\omega$ is the closed two-form on $\mathcal{M}(\Sigma^3) \times \text{Ind}(M_1) \times \text{Ind}(M_2)$ and $\phi$ is the coordinate on $\mathbb{R}$. The quotient $X_s = \phi^{-1}(s)/U(1)$ is given by

$$X_s \cong \mathcal{M}^*(\Sigma^3) \times \text{Ind}_s(M_1) \times \text{Ind}_s(M_2).$$

Therefore if we set $W = X/\text{diag}(LG^2)$ then the central $U(1)$ acts freely, and

$$W_s = \text{Ind}_s(M_1) \otimes \text{Ind}_s(M_2)$$

as claimed. \qed

7. Line Bundles

We now discuss extensions of the above results to include line bundles, which will be required in the sequel [19] to this paper.

Let $M_1$ and $M_2$ be Hamiltonian $LG$-manifolds with proper moment maps. Given $\hat{LG}$-equivariant line bundles $L_1 \to M_1$, $L_2 \to M_2$ at some level $k \in \mathbb{Z}$, we define

$$L_1 \otimes L_2 := (L_1 \boxtimes L_2 \boxtimes L(\Sigma^3)\otimes)^/ \text{diag}(LG^2) \to M_1 \otimes M_2.$$

If $L_1$ and $L_2$ are pre-quantum line bundles, then $L_1 \otimes L_2$ is a pre-quantum bundle for $M_1 \otimes M_2$.

If $M$ is a Hamiltonian $G$-manifold with $G$-equivariant line bundle $L$, we define an $\hat{LG}$-equivariant line bundle at level $k$ by

$$\text{Ind}_k(L) := \hat{LG} \times_G L \to \text{Ind}(M)$$

where the central $U(1)$ acts on $L$ with fiber-weight $k$. If $L$ is a pre-quantum line bundle for $M$ then $\text{Ind}_k(L)$ is a pre-quantum line bundle for $\text{Ind}_k(M) = \text{Ind}(M(k)^{1/k}(k))$. Note that the central extension $\hat{G} = \hat{LG}_{\{0\}}$ splits $\hat{G} = G \times U(1)$ because the restriction to $\mathfrak{g}$ of the defining cocycle is zero.

**Theorem 7.1.** Let $(M_i, \omega_i, \Phi_i) \ (i = 1, 2)$ be compact Hamiltonian $G$-manifolds with $G$-equivariant line bundles $L_i \to M_i$. Then there exists an $LG$-equivariant diffeomorphism

$$\phi : \text{Ind}(M_1) \otimes \text{Ind}(M_2) \to \text{Ind}(M_1 \times M_2)$$

such that the equivariant 2-forms on both sides are cohomologous (as in Theorem 4.3) and such that $\phi$ lifts to an equivariant isomorphism of line bundles:

$$\hat{\phi} : \text{Ind}_k(L_1) \otimes \text{Ind}_k(L_2) \to \text{Ind}_k(L_1 \boxtimes L_2).$$

**Proof.** Let $\Psi : \mathcal{M}(\Sigma^3) \to (Lg^*)^3$ be the moment map for $\mathcal{M}(\Sigma^3)$. Recall from the proof of Theorem 4.3 that there exists a $G^2$-invariant neighborhood $V \subset (U_{\{0\}})^2 \subset (Lg^*)^2$ of 0 such that $\Psi^{-1}(V \times U_{\{0\}})$ is $G^3$-equivariantly symplectomorphic to a neighborhood $U$ of the zero section in $T^*G^2$. First consider the special case when $\Phi_1(M_1) \times \Phi_2(M_2) \subset V$.
Since equivariant line bundles over cotangent bundles are trivial, the pullback of $L(\Sigma^3_0)$ to $U$ is equivariantly isomorphic to the trivial bundle $U \times \mathbb{C}$. Therefore

$$\text{Ind}_k(L_1) \otimes \text{Ind}_k(L_2) = \text{Ind}_k(L_1) \boxtimes \text{Ind}_k(L_2) \boxtimes L(\Sigma^3_0)^{\otimes k} / / LG^2$$

$$= \text{Ind}_k(L_1) \boxtimes \text{Ind}_k(L_2) \boxtimes \text{Ind}_k(U \times \mathbb{C}) / / LG^2$$

$$= \text{Ind}_k(L_1 \otimes L_2 \boxtimes (U \times \mathbb{C}) / / G^2)$$

$$= \text{Ind}_k(L_1 \otimes L_2).$$

To reduce the general case to this special case, we rescale the 2-forms on $M_1, M_2$ by a factor $a > 0$. Taking fusion products we obtain a family of $LG$-equivariant line bundles $R(a) = \text{Ind}_k(L_1) \otimes \text{Ind}_k(L_2) \rightarrow \text{Ind}(M_1^{(a)}) \otimes \text{Ind}(M_2^{(a)}).$

For $a$ small enough we are in the above situation and thus have an isomorphism $R(a) \cong \text{Ind}_k(L_1 \otimes L_2)$. It remains to show that the (continuous, piecewise smooth) family of diffeomorphisms $\varphi_a$ given by Theorem 6.6 lifts to an equivariant identification of line bundles $R(a)$. By continuity the equivariant isomorphism class of the line bundle $\varphi_a^* R(a)$ does not change, i.e. there are $LG$-equivariant isomorphisms $\varphi_a^* R(a) \cong \text{Ind}_k(L_1) \otimes \text{Ind}_k(L_2)$ covering the identity map on $\text{Ind}(M_1) \otimes \text{Ind}(M_2)$. This follows from the existence of an $LG$-invariant connection on the family. Such a connection may be constructed using cross-sections and a partition of unity as in Lemma 6.2. Alternatively, the claim follows because the family of line bundles $(\varphi_a^* R(a)) \otimes (\varphi_1^* R^{(1)})^{-1}$ descends to a family of $G$-equivariant line bundles on the holonomy manifold $\text{Hol}(\text{Ind}(M_1) \otimes \text{Ind}(M_2)) \cong M_1 \times M_2$, equal to the trivial bundle for $a = 1$.

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