Loxodromic Möbius Transformations with Disjoint Axes

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Abstract

This paper is concerned with Loxodromic Möbius Transformations with disjoint Axes in a Kleinian Group. We study mainly the distance between their axes, and give some estimates about their translation lengths.

1. Introduction

Let the set

\[ \mathcal{H}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\} \]

be the hyperbolic 3-space, and its metric is the complete Riemannian metric \( ds = |dx|/x_3 \). Let \( \mathcal{M} \) denote the group of all Möbius Transformations of extended complex plane \( \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \). A Fuchsian group \( G \) is a discrete subgroup of \( \mathcal{M} \) with an invariant disc \( D \in \hat{\mathbb{C}} \). The group of orientation preserving isometries is denoted by \( Isom^+(\mathcal{H}^3) \), then a Kleinian group \( G \) is a discrete nonelementary subgroup of \( Isom^+(\mathcal{H}^3) \). So it is easy to see, this kind of Kleinian group \( G \) is not virtually abelian, and it is also can be regarded as the extension of a nonelementary Fuchsian group acting in \( \mathcal{H}^3 \). This paper is concerned with Loxodromic Möbius Transformations with disjoint Axes in a Kleinian group, and we will give some estimates about their translation lengths. For each Möbius Transformation

\[ f = \frac{az + b}{cz + d} \in \mathcal{M}, \quad ad - bc = 1, \]

the \( 2 \times 2 \) complex matrix

\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}) \]

induces \( f \), and set \( \text{tr}(f)^2 = \text{tr}(A)^2 \), where \( \text{tr}(A) = a + d \), denotes the trace of the matrix \( A \). And for each \( f \) and \( g \) in \( \mathcal{M} \), the commutator \([f, g]\) of \( f \) and \( g \) is \( f g f^{-1} g^{-1} \). We call the three complex numbers

\[ \gamma(f, g) = \text{tr}([f, g]) - 2, \quad \beta(f) = \text{tr}^2(f) - 4, \quad \beta(g) = \text{tr}^2(g) - 4 \]
the parameters of the two-generator group $< f, g >$. These parameters are independent of the choice of matrix representations for $f$ and $g$ in $SL(2, \mathbb{C})$, and they determine uniquely up to conjugacy whenever $\gamma(f, g) \neq 0$.

Let $f \in \mathcal{M}$ not be the identity, then

1. $f$ is parabolic if and only if $\beta(f) = 0$, and then $f$ is conjugate to $z \to z + 1$.  
2. $f$ is elliptic if and only if $\beta(f) \in [-4, 0)$, and then $f$ is conjugate to $z \to \mu z$, where $|\mu| = 1$.  
3. $f$ is loxodromic if and only if $\beta(f) \not\in [-4, 0]$, and then $f$ is conjugate to $z \to \mu z$, where $|\mu| > 1$; and $f$ is hyperbolic if $\mu > 0$, $f$ is strictly loxodromic if $\mu < 0$ or $\mu$ is not real. Furthermore, $\beta(f) = \mu - 2 + \mu^{-1}$.

A parabolic or hyperbolic element $g$ of a Fuchsian group $G$ is said to be primitive if and only if $g$ generates the stabilizer of each of its fixed points. If $g$ is elliptic, it is primitive when it generates the stabilizer and has an angle of rotation of the form $2\pi/n$. And each Möbius transformation of $\hat{C} = \partial H^3$ has a natural extension uniquely via the Poincaré’s way to an orientation-preserving isometry of hyperbolic 3-space $H^3$, see [1]. Then Kleinian groups equal to discrete Möbius groups.

If $f \in \mathcal{M}$ is nonparabolic, then $f$ has two fixed points in $\hat{C}$ and the hyperbolic line (geodesic) joining these two fixed points in $H$ is called the axis of $f$, denoted by $\text{ax}(f)$. In this case $f$ translates along $\text{ax}(f)$ by an amount $t(f) \geq 0$, and $t(f)$ is called the translation length of $f$. $f$ rotates about $\text{ax}(f)$ by an angle $\theta(f) \in (-\pi, \pi]$, and

$$\beta(f) = 4\sinh^2\left(\frac{t(f) + i\theta(f)}{2}\right).$$  

(1.1) It then follows from (1.1) that

$$\cosh(t(f)) = \frac{|\beta(f) + 4| + |\beta(f)|}{4}$$  
and

$$\cosh(\theta(f)) = \frac{|\beta(f) + 4| - |\beta(f)|}{4}$$  
(cf. (15), (17) and (18) in [3]).

If $f, g \in \mathcal{M}$ are nonparabolic and if $\alpha$ is the hyperbolic line in $H^3$ that is orthogonal to $\text{ax}(f)$ and $\text{ax}(g)$, then

$$\frac{4\gamma(f, g)}{\beta(f)\beta(g)} = \sinh^2(\delta + i\phi),$$  

(1.2)
where $\delta$ is the hyperbolic distance between $\text{ax}(f)$ and $\text{ax}(g)$ and $\phi \in [0, \pi]$ is the angle between the hyperplanes in $H^3$ that contain $\text{ax}(f) \cup \alpha$ and $\text{ax}(g) \cup \alpha$ respectively (see Lemma 4.2 in [4]). In particular if $\text{ax}(f)$ and $\text{ax}(g)$ are in the same hyperplane then

$$\frac{4\gamma(f, g)}{\beta(f)\beta(g)} = \sinh^2(\delta). \quad (1.3)$$

Finally, there is a definition of triangle groups. A group $G$ of isometries of the hyperbolic plane is said to be of type $(\alpha, \beta, \gamma)$ if and only if $G$ is generated by the reflections across the sides of some triangle with angles $\alpha, \beta$ and $\gamma$. A group $G$ is a $(p, q, r)$-Triangle group if and only if $G$ is a conformal group of type $(\pi/p, \pi/q, \pi/r)$. We call $G$ a Triangle group if it is a $(p, q, r)$-Triangle group for some integers $p, q$ and $r$. Triangle groups are an important class of Fuchsian groups. Roughly speaking, these are the discrete groups with the more closely packed orbits and the smallest fundamental regions, especially the $(2, 3, 7)$ triangle group, which is a simple but powerful example for kinds of extremal conditions. It also can deduce the following two numbers that occur frequently in this paper:

$$c = 2(\cos(2\pi/7) + \cos(\pi/7) - 1) = 1.048..., \quad d = 2(1 - \cos(\pi/7)) = 0.198....$$

F. W. Gehring and G. J. Martin have given some similar results, see [2] in details. Their research is mainly in the condition of intersecting axes. There are two Theorems form theirs.

**THEOREM A.** If $< f, g >$ is a Kleinian group, if $f$ and $g$ are hyperbolics, and if $\text{ax}(f)$ and $\text{ax}(g)$ intersect at an angle $0 < \phi < \pi$, then

$$\sinh(t(f)/2)\sinh(t(g)/2)\sin(\phi) \geq \lambda^2,$$

where $\lambda = 0.471....$ The constant $\lambda$ is sharp and the exponent of $\sin(\phi)$ can not be replaced by a constant greater than 1.

**THEOREM B.** If $< f, g >$ is a Kleinian group, if $f$ and $g$ are loxodromics with axes that intersect at an angle $0 < \phi < \pi$, then

$$| \beta(f)\beta(g) | \sin^{4/3}(\phi) \geq b,$$

where $0.777 \geq b \geq 0.884.$
The main theorems of this paper are follows.

**THEOREM 1.** If \(< f, g >\) is a Kleinian group, \(f\) and \(g\) are hyperbolic, and if \(\text{ax}(f)\) and \(\text{ax}(g)\) do not intersect then

\[
\sinh(t(f)/2)\sinh(t(g)/2)\sinh(\delta) \geq \sqrt{d}/2, \tag{1.4}
\]

where \(\delta\) is the distance between \(\text{ax}(f)\) and \(\text{ax}(g)\).

**THEOREM 2.** If \(< f, g >\) is discrete, \(f\) and \(g\) are loxodromics, if \(\text{ax}(f)\) and \(\text{ax}(g)\) do not intersect, and if \(\sinh(\delta) \leq 1\), then

\[
|\beta(f)\beta(g)| \sinh^{4/3}(\delta) \geq 4d, \tag{1.5}
\]

where \(\delta\) is the hyperbolic distance between \(\text{ax}(f)\) and \(\text{ax}(g)\).

**THEOREM 3.** If \(< f, g >\) is discrete, if \(f\) and \(g\) are loxodromics, \(\text{ax}(f)\) and \(\text{ax}(g)\) do not intersect, and \(\sinh(\delta) \leq 1\), then

\[
\sinh(t(f))\sinh(t(g))\sinh^{4/3}(\delta) \geq \frac{3b}{16\pi^2}, \tag{1.6}
\]

where \(\delta\) is the hyperbolic distance between \(\text{ax}(f)\) and \(\text{ax}(g)\).

**THEOREM 4.** If \(< f, g >\) is discrete, if \(f\) and \(g\) are loxodromics with \(\beta(f) = \beta(g)\), and if \(\text{ax}(f)\) and \(\text{ax}(g)\) do not intersect, then

\[
\sinh(t(f))\sinh(\delta) \geq \frac{\sqrt{3d}}{2\pi}, \tag{1.7}
\]

where \(\delta\) is the hyperbolic distance between \(\text{ax}(f)\) and \(\text{ax}(g)\).

**THEOREM 5.** If \(< f, g >\) is discrete, if \(f\) is loxodromic and \(g\) is elliptic of order \(n \geq 3\), and if \(\text{ax}(f)\) and \(\text{ax}(g)\) do not intersect, then

\[
\sinh(t(f))\sin^2(\pi/n)\sinh^2(\delta) \geq \sqrt{3a(n)/4\pi}, \tag{1.8}
\]

where \(\delta\) is the hyperbolic distance between \(\text{ax}(f)\) and \(\text{ax}(g)\). When \(n \geq 5\), we also have

\[
\sinh(t(f))\sin^2(\pi/n)\sinh^2(\delta) \geq \sqrt{3\cos(2\pi/n)/2\pi}.
\]

2. Proofs of Theorems
These lemmas are to be used.

**LEMMA 1.** If \(< f, g >\) is a Fuchsian group, then

\[
|\gamma(f, g)| \geq d
\]

(see [5]).

The following result is established in [7] and [8], and then is sharpened by Cao in [9].

**LEMMA 2.** If \(< f, g >\) is a Kleinian group and if either

\[
|\beta(f)| \leq c \text{ or } \beta(f) = \beta(g),
\]

then

\[
|\gamma(f, g)| \geq d.
\]

This result is sharp under either assumption in (2.1).

**LEMMA 3.** For each loxodromic Möbius transformation \(f\) there exists an integer \(m \geq 1\) such that

\[
|\beta(f^m)| \leq \frac{4\pi}{\sqrt{3}} \sinh(t(f))
\]

(see [10]).

**LEMMA 4.** If \(< f, g >\) is a Kleinian group, \(f\) is elliptic of order \(n \geq 3\), and \(g\) is not of order 2, then

\[
|\gamma(f, g)| \geq a(n)
\]

where

\[
a(n) = \begin{cases} 
2\cos(2\pi/7) - 1 & \text{if } n = 3, \\
2\cos(2\pi/5) & \text{if } n = 4, 5, \\
2\cos(2\pi/6) & \text{if } n = 6, \\
2\cos(2\pi/n) - 1 & \text{if } n \geq 7
\end{cases}
\]

(See [4]).

**PROOF OF THEOREM 1.** Let \(S\) be the hyperplane in \(H^3\) determined by \(ax(f)\) and \(ax(g)\), then \(S\) is invariant under \(G\), \(F = G\)\rceil S\) is conjugate to a Fuchsian group and

\[
|\gamma(f, g)| \geq d
\]

by Lemma 1.

Next since \(f\) and \(g\) are hyperbolic, \(\theta(f) = \theta(g) = 0\) and

\[
|\beta(f)| = 4\sinh^2(t(f)/2), \quad |\beta(g)| = 4\sinh^2(t(g)/2)
\]

(2.6)
by (1.1). Thus
\[16 \sinh^2(t(f)/2) \sinh^2(t(g)/2) \sinh^2(\delta) = |\beta(f)||\beta(g)| \sinh^2(\delta)\]
\[= 4|\gamma(f,g)|\]
\[\geq 4d\]
by (1.3), (2.5) and (2.6), then we obtain (1.4). \(\square\)

An example due to Jørgensen [6] shows that there exists no absolute lower bound for \(|\gamma(f,g)|\) when \(<f,g>\) is a Kleinian group.

PROOF OF THEOREM 2. By (1.2),
\[|\beta(f)\beta(g)| \sinh^2(\delta) = |4\gamma(f,g)|.\] \hspace{1cm} (2.7)

We want to find a lower bound for
\[u = |\beta(f)\beta(g)| \sinh^{4/3}(\delta).\]
By relabeling, we may assume that \(|\beta(f)| \leq |\beta(g)|\).

If \(|\beta(f)| \leq c = 1.048...\), then \(\gamma(f,g) \neq 0\), by (2.7), and \(<f,g>\) is a Kleinian group.
Thus \(|\gamma(f,g)| \geq d\) by Lemma 2, and
\[u \geq |\beta(f)\beta(g)| \sinh^2(\delta) = |4\gamma(f,g)| \geq 4d = 0.792...\]

Next, if \(|\beta(f)| \geq c\), then
\[c^{-2}u^3 + 4u^{3/2} = |\beta(f)\beta(g)|^3 c^{-2} \sinh^4(\delta) + 4|\beta(f)\beta(g)|^{3/2} \sinh^2(\delta)\]
\[\geq |\beta(f)\beta(g)|^2 \sinh^4(\delta) + 4|\beta(f)\beta(g)| \sinh^2(\delta) |\beta(f)|\]
\[\geq 16|\gamma(f,g)|^2 + 16|\gamma(f,g)||\beta(f)|\]
\[\geq 16dc = 3.320...\]
by (2.8) and Lemma 2, and we obtain
\[u > 0.798...\]

Thus (1.5) follows. \(\square\)

The fact that \(\sinh(\delta) \leq 1\) is necessary. When \(\sinh(\delta) > 1\), \(u\) has no uniform lower bound. There is a counterexample. For real \(\lambda, \mu\) and \(\delta\), define
\[
f = \left(\begin{array}{cc}
cosh(\lambda) & e^\delta \sinh(\lambda) \\
e^{-\delta} \sinh(\lambda) & \cosh(\lambda)
\end{array}\right), \quad g = \left(\begin{array}{cc}
cosh(\mu) & \sinh(\mu) \\
\sinh(\mu) & \cosh(\mu)
\end{array}\right).
\]
It is clear that \( f \) and \( g \) are both loxodromic. The axis of \( f \) has endpoints \( \pm e^\delta \) and the axis of \( g \) has endpoints \( \pm 1 \). By Reference [11], their axes are disjoint and the distance between the two axes is \( \delta \). Moreover, \( \beta(f) = 4 \sinh^2(\lambda) \) and \( \beta(g) = 4 \sinh^2(\mu) \).

We suppose that \( \text{tr}(fg^{-1}) = -2 \), so \( fg^{-1} \) is parabolic in \( PSL(2, \mathbb{C}) \). For all \( \lambda > 0 \) and all \( \mu > 0 \) the group \( < f, g > \) is clearly discrete and non-elementary. Indeed it is a free group. Moreover,

\[
-2 = \text{tr}(fg^{-1}) = 2 - 2 \sinh(\delta) \sinh(\lambda) \sinh(\mu).
\]

Therefore,

\[
\sinh(\delta) = \frac{2}{\sinh(\lambda) \sinh(\mu)}.
\]

We have

\[
u = |\beta(f) \beta(g)| \sinh^{4/3}(\delta)
= 16 \sinh^2(\lambda) \sinh^2(\mu) (2/(\sinh(\lambda) \sinh(\mu)))^{4/3}
= 2^{16/3} \sinh^{2/3}(\lambda) \sinh^{2/3}(\mu).
\]

Fixing \( \mu \) and letting \( \lambda \) tend to zero shows that there is no uniform lower bound on \( u \).

**Proof of Theorem 3.** By Lemma 3 we can choose integers \( m, n \geq 1 \) such that

\[
|\beta(f^m)| \leq \frac{4\pi}{\sqrt{3}} \sinh(t(f)), \quad |\beta(g^n)| \leq \frac{4\pi}{\sqrt{3}} \sinh(t(g)).
\]  

(2.8)

Then \( < f^m, g^n > \) is Kleinian and we obtain

\[
\left( \frac{4\pi}{\sqrt{3}} \right)^2 \sinh(t(f)) \sinh(t(g)) \sinh^{4/3}(\delta) \geq |\beta(f^m) \beta(g^n)| \sinh^{4/3}(\delta) \geq b
\]

by (2.8) and (1.5). This implies (1.6). \( \square \)

We see immediately there is not a lower bound for \( \max(t(f), t(g)) \).

**Proof of Theorem 4.** By Lemma 3 we can choose an integer \( m \geq 1 \) such that

\[
|\beta(f^m)| \leq \frac{4\pi}{\sqrt{3}} \sinh(t(f)).
\]

Then \( < f^m, g^n > \) is Kleinian with \( \beta(f^m) = \beta(g^m) \) and we obtain

\[
\frac{4\pi}{\sqrt{3}} \sinh(t(f)) \sinh(\delta) \geq (|\beta(f^m) \beta(g^m)| \sinh^2(\delta))^{1/2}
= (4 |\gamma(f^m, g^n)|)^{1/2} \geq 2\sqrt{d}
\]

from (2.3), (1.3), and (2.2). This implies (1.7). \( \square \)
PROOF OF THEOREM 5. We may assume without loss of generality that $g$ is a primitive elliptic. Next, by Lemma 3 we can choose an integer $m \geq 1$ such that

$$\beta(f^m) \leq \frac{4\pi}{\sqrt{3}} \sinh(t(f)).$$

Then $\langle f^m, g \rangle$ is Kleinian and $f^m$ is not of order 2, so we obtain

$$\frac{4\pi}{\sqrt{3}} \sinh(t(f)) \sin^2(\pi/n) \sinh^2(\delta) \geq |\gamma(f^m, g)| \geq a(n)$$

from (1.3) and Lemma 4, where $a(n)$ is as in (2.4). Then (2.9) yields (1.8). □

3. Remarks

For elliptic transformations with disjoint axes whenever intersect, there are also similar results as Theorem 1 and Theorem 2. Theorem A holds with equality if $f$ and $g$ are hyperbolic generators for the $(2,3,7)$–triangle group with

$$\text{par}(<f, g>) = (-d, c, c) = (-0.198\ldots, 1.048\ldots, 1.048\ldots).$$

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