UNIFIED THEORY OF FIELD WITH MODUL OF SQUARED CURVATURE AS LAGRANGIAN

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Abstract

The 4-D theory with connection components $\Gamma^k_{mn}$ as field variables and module of squared curvature $|R^k_{lmn}R^l_{mnr}|$ as Lagrangian is described. The Maxwell equations, the Lorentz condition and the gravity field equation, that agrees with Newton’s theory, result from equations of motion.

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1 INTRODUCTION

We shall describe three initial assumptions in that order, as they had appeared in our investigation.

ASSUMPTION 1. If we assume that, according to general relativity theory (GRT) around a mass point the space-time is central symmetrically << compressed >>, then we must consider for a charge point field a picture that is also central symmetrical, but orthogonal to previous one. We could imagine only the central symmetrical torsion. It means that by moving from the origin of quasi-Galilean coordinates \( x^0 = ct, x^1, x^2, x^3 \), where the charge is, along say the axis \( x^1 \) it must be \( \Gamma_{21}^3 = -\Gamma_{31}^2 \). But this equation will be Lorentz–invariant only if it is complemented to

\[
\begin{align*}
\Gamma_{23}^3 &= \Gamma_{31}^2 = -\Gamma_{12}^3 = -\Gamma_{13}^2 = A_0 \\
\Gamma_{30}^2 &= \Gamma_{02}^3 = -\Gamma_{23}^0 = -\Gamma_{20}^3 = -\Gamma_{03}^2 = A_1 \\
\end{align*}
\]

(the dotted line represents equations that result from previous by circular permutation of indices 1, 2, 3). Indeed (1) and (3) define a completely antisymmetrical tensor \( \Gamma^{kmn} \) and \( A_s \) is the dual to it covector. \( A_s \) seems to be alike an electromagnetic 4-potential.

ASSUMPTION 2. Conditions for connection of a gravity field. choose we as follows:

\[
\begin{align*}
\Gamma_{00}^0 &= \Gamma_{10}^1 = \Gamma_{20}^2 = \Gamma_{30}^3 = \Gamma_{01}^0 = \Gamma_{11}^1 = \Gamma_{21}^2 = \Gamma_{31}^3 = \Gamma_{02}^0 = \Gamma_{12}^1 = \Gamma_{22}^2 = \Gamma_{32}^3 = f_0 \\
\Gamma_{11}^0 &= \Gamma_{01}^1 = \Gamma_{21}^2 = \Gamma_{31}^3 = \Gamma_{02}^1 = \Gamma_{12}^2 = \Gamma_{22}^3 = \Gamma_{32}^0 = -\Gamma_{23}^2 = -\Gamma_{33}^2 = f_1 \\
\end{align*}
\]

In quasi-Galilean coordinates the connection (2):

a) is complementary to (1) with staff of nonzero components \( \Gamma^{knm} \)
b) is symmetrical in lower indices;
c) like (1), obeys the law: \( \Gamma^{knm} = \Gamma^{mkn} \) if \( k = 0 \) or \( m = 0 \) and \( \Gamma^{knm} = -\Gamma^{mkn} \) otherwise

Connection (2) is of a conformal flat space with metrics

\[
g_{mn} = \exp 2f \delta^{(0)}_{mn}
\]

where \( \delta^{(0)}_{mn} \) is metrics of flat space, so that \( f_k = f_{,k} = \partial f / \partial x^k \)

ASSUMPTION 3. The sum of all \( R^{(k)}_{lmn} = \Gamma^{knm}_l = 0 \) is the most exact indicator of curvature of space. It vanishes only if all \( R^{(k)}_{lmn} = 0 \). But it is not a scalar. The quantity

\[
R^{k}_{lmn} R^{(l)}_{mn} = L
\]

is scalar but it is not positive-defined. We choose \( |L| \) as Lagrangian.
2 EXPOSITION OF THE THEORY

Let $M$ be a 4-D space-time with metrics (3) and connection, restricted by (1) and (2). In $M$ (see Appendix 1)

\[ L = 4[g^{kk}(f_{k,k} + f_k^2 - A_k^2)]^2 + 8g^{kk}g^{mm}f_{k,k} \times \]
\[ (f_m^2 - A_m^2) - 4[g^{kk}(A_{k,k} + 2f_kA_k)]^2 - 16g^{kk} \times \]
\[ g^{mm}A_{k,k}f_mA_m - 8g^{kk}g^{mm}(A_{k,m} - A_kf_m - A_mf_k)^2 + \]
\[ 8g^{kk}g^{mm}(f_{k,m} - f_kf_m + A_kA_m)^2 \]

or in covariant form (see Appendix 2):

\[ L = 8f_{k;m}f^{k;m} - 8A_{k;m}A^{k;m} + 4(f_k^2)^2 - 4(A_k^2)^2 + \]
\[ 16f_{k;m}(f_k^m + A_k^m) - 16f_k^m(f_m^m + A_nA^n) + \]
\[ 12(f_k^k)^2 + 12(A_kA^k)^2 + 24f_k^kA_mA^m \]

This expression is wonderful symmetric in $f_k$ and $A_k$ what indicates the good choice of conditions (2). (Note that in general $R_{lmnk}^kR^k_{lmnk}$ includes 5280 summands (the sum of 96 different squared components $R^k_{lmnk}$ with 55 summands in each square). It is simple as (6) only if the triad (1), (2), (4) is used).

By (2) and $A_k \equiv 0; k = 0, 1, 2, 3$ the Ricci scalar

\[ R = g^{mn}R_{mnk}^k = 6(g^{kk}f_{k,k} + f_mf^m) = 6(f_k^k - f_mf^m) \]

We shall minimize the integral

\[ J = \int |L|\sqrt{-g}d^4x = \int L'd^4x \]

The corresponding equations of motion are (see Appendix 3):

\[ \begin{cases} -24(f_m^m - f_mf^m);k - 48(A_m^m)A_k + 24(A_mA^m);k = 0 \\ 16(A_{k;m} - A^{m;k})_{m} - 48(f_m^m - f_mf^m) \times \\ A_k + 24(A_m^m);k + 48(A_mA^m)A^k = 0 \end{cases} \]

If we suppose that $A_m$ is in proportion to electromagnetic 4-potential with small coefficient and don’t take into account $A_mA^m$ as infinitesimals, then we obtain from (9) at once the Maxwell equations in vacuum

\[ (A^{k;m} - A^{m;k})_{m} = 0 \]

the Lorentz condition

\[ A^m_{;m} = 0 \]

and, as we suppose, the equation of gravitation field in vacuum

\[ f^m_{;m} - f_mf^m = 0. \]
The central symmetrical solution of (12) is

\[ f = \ln(1 - \gamma m/c^2 r) \approx -\frac{\gamma m}{c^2 r}, \]

where \( \gamma \) is the gravity constant and \( m \) is mass.

The equations (12), (7) and (2) formally lead to Nordstrom’s theory of gravitation, that affirms:

a) geodesics of nonzero length are conics in accordance with Newton’s theory;

b) gravitational displacement of spectral lines is the same as in GRT (the other two consequences see lower).

EXPERIMENTALLY this theory can be proofed by detecting the light polarization rotation in electromagnetic field in vacuum in accordance with (1).

3 DIFFICULTIES OF THE THEORY.

In addition to (a) and (b), Nordstrom’s theory affirms:

c) displacement of planet perihelion is less then it in GRT and has the other sign;

d) light rays don’t bend in gravitation fields.

It seems to be at variance with experiments, therefore Nordstrom’s theory was rejected. But now it proves to be tight connected with electromagnetic theory. This advantage may be realized if those two effects will be explained without geometry.

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APPENDIX 1.

Conditions (1) and (2) distribute the nonzero components \( R_{klm}^k \) to seven groups (we don’t write components, result by permutation of the last indices, but multiply the sum by 2):

(A) \[ R_{klm}^k = \Gamma_{km,l}^k - \Gamma_{kl,m}^k + \Gamma_{lp}^k\Gamma_{km}^p - \Gamma_{pm}^k\Gamma_{kl}^p = 4(f_{m,l} - f_{l,m}) = 0; \]

(B) \[ R_{110}^0 = R_{010}^1 = f_{1,1} - f_{0,0} - A_2^2 - A_3^2 + f_2^2 + f_3^2; \]

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\[ R_{332}^2 = R_{223}^3 = f_{2,2} + f_{3,3} - A_1^2 + A_0^2 + f_1^2 - f_0^2; \]

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(C) \[ R_{123}^0 = R_{023}^1 = A_{3,3} + A_{2,2} + 2(f_1 A_1 - f_0 A_0); \]
\begin{align*}
R_{201}^3 &= R_{310}^2 = A_{1,1} - A_{0,0} + 2(A_3A_5 + f_2A_2); \\
&

(D) \quad R_{302}^0 = R_{002}^3 = (A_{1,0} - A_1f_0 - A_0f_1) - (f_{3,2} - f_3f_2 + A_3A_2); \\
&

R_{203}^0 = R_{003}^2 = -(A_{1,0} - A_1f_0 - A_0f_1) - (f_{2,3} - f_2f_3 + A_2A_3); \\
&

(E) \quad R_{213}^1 = R_{131}^2 = (A_{0,1} - A_0f_1 - A_1f_0) - (f_{2,3} - f_2f_3 + A_2A_3); \\
&

R_{312}^1 = R_{121}^3 = -(A_{0,1} - A_0f_1 - A_1f_0) - (f_{3,2} - f_3f_2 + A_3A_2); \\
&

(F) \quad R_{031}^3 = R_{331}^0 = -(A_{2,3} - A_2f_3 - A_3f_2) - (f_{0,1} - f_0f_1 + A_0A_1); \\
&

R_{130}^1 = R_{303}^2 = (A_{2,3} - A_2f_3 - A_3f_2) - (f_{1,0} - f_1f_0 + A_1A_0); \\
&

(G) \quad R_{120}^2 = R_{202}^1 = -(A_{3,2} - A_3f_2 - A_2f_3) - (f_{1,0} - f_1f_0 + A_1A_0); \\
&

R_{021}^3 = R_{221}^0 = (A_{3,2} - A_3f_2 - A_2f_3) - (f_{0,1} - f_0f_1 + A_0A_1).
\end{align*}

By squaring, multiplying by \(\exp^{-4f}\) or \(-\exp^{-4f}\) depending on is the number of zeros among indices \(k,l,m,n\) even or odd, and adding, we obtain (5). (The first, the second and part of the sixth summand in (5) result from group (B), the next two and part of the fifth — from group (C) and all other — from all groups together.)
APPENDIX 2.

a) 
\[ f^k_k = f^k_k + \Gamma^k_{kp} p^p = (g^{kk} f^k_k)_k + 4f_p f^p = g^{kk} f^k_k + 2f_m f^m \]  
(14)

Therefore the first and the second summands in (5) in sum are:

\[ 4(f^k_k)^2 - 16 f^k_k A_n A^p - 12(f_k f^k)^2 + 4(A_k A^k)^2 + 24(f_k f^k) (A_n A^m). \]

b) Similarly

\[ A^k_{mk} = g^{kk} A_{k,k} + 2f_m A^m \]  
(15)

By \( k \neq m \)

\[ A_{k,m} - A_k f_m - A_m f_k = A_{k,m} - \Gamma^p_{km} A_p = A_{k,m} \]

By \( k = m \)

\[ (g^{kk})(A_k,k - 2f_k A_k)^2 = (g^{kk})(A_k,k + \Gamma^p_{km} A_p - 2f_k A_k)^2 = \]

\[ \exp -4f \left\{ (A_{0,0} - (f_0 A_0 - f_1 A_1 - \cdots - f_3 A_3))^2 + [A_{1,1} + (f_0 A_0 - f_1 A_1 - \cdots - f_3 A_3)]^2 \right\} = A_{k,k} A^{k,k} - 2A_{k,m} A^m + 4(f_m A^m)^2 \]

so that

\[ -8 g^{kk} g^{mm} (A_{k,m} - A_k f_m - A_m f_k)^2 = -8 A_{k,m} A^{k,m} + 16 A_{k,m} A^m - 32(f_m A^m)^2 \]

and the third, the forth and the fifth summands in (5) in sum are

\[ -4(A^k_{mk})^2 - 8A_{k,m} A^{k,m}. \]

e) Since for example

\[ f_{0,1} = f_{0,1} - \Gamma^p_{01} f_p = f_{0,1} - 2f_0 f_1 - A_3 f_2 + A_2 f_3; \]

\[ f_{1,0} = f_{1,0} - 2f_1 f_0 + A_3 f_2 - A_2 f_3 \]

etc, then

\[ (f_{0,1} - f_0 f_1 + A_0 A_1)^2 + (f_{1,0} - f_1 f_0 + A_1 A_0)^2 = \]

\[ (f_{0,1} + f_0 f_1 + A_0 A_1)^2 + (f_{1,0} + f_1 f_0 + A_0 A_1)^2 + 2(A_3 f_2 - A_2 f_3)^2 - \]

\[ 2(A_3 f_2 - A_2 f_3)(f_{1,0} - f_{0,1}) = (f_{0,1} + f_0 f_1 + A_0 A_1)^2 + \]

\[ (f_{1,0} + f_1 f_0 + A_1 A_0)^2 - 2(A_3 f_2 - A_2 f_3)^2 \]

etc and by \( k \neq m \)

\[ 8 \sum_{k \neq m} g^{kk} g^{mm} (f_{k,m} - f_k f_m + A_k A_m)^2 = 8 \sum_{k \neq m} g^{kk} g^{mm} \times \]

\[ (f_{k,m} + f_k f_m + A_k A_m)^2 + 8 g^{kk} g^{mm} (A_k f_m - A_m f_k)^2. \]

The last sum is

\[ 16 f_k f^k A_m A^m - 16(f_m A^m)^2. \]
Its sign is plus because \( q^{22} g^{33} = -q^{00} g^{11} \) etc. By \( k = m \)

\[
(g^{kk})^2 (f_{k,k} - f_k^2 + A_k^2)^2 = (g^{kk})^2 (f_{k,k} + \Gamma_{kk}^p f_p - \frac{\partial L}{\partial A} + f_k^2 + A_k^2)^2 = \exp[-4f] \left\{ [f_{0,0} + f_0^2 + A_0^2 - (f_0^2 - f_1^2 - \cdots - f_3^2)]^2 + [f_{1,1} + f_1^2 + A_1^2 + (f_1^2 - f_2^2 - \cdots - f_3^2)]^2 + \cdots \right\},
\]

therefore

\[
8(g^{kk})^2 (f_{k,k} - f_k^2 + A_k^2)^2 = 8(f_k^2 + f_m f^n + A_m A_n)^2 - 16(f_k^2 + f_m f^n + A_m A_n) f_p f^p + 32(f_m f^n)^2
\]

and the last summand in (5) is equal

\[
8 f_{k,m} f^k m + 16 f_{k,m} (f^k f^n + A^k A^n) - 16 f_k^2 f_m f^n + 24(f_k f^n)^2 + 8(A_k A^n)^2,
\]

so that Lagrangian has value (6).

**APPENDIX 3.**

Let at first \(|L| = L\). Since \( \sqrt{-g} = \exp 4f \), we can obtain \( L' \) by substitution in

(5) \( g_{mm} \) for \( g^{(0)}_{mm} \). Therefore

\( a) \quad \partial L'/\partial f_k = g^{(0)kk} g^{(0)mm} (32 f_{m,m} f_k + 48 f_m^2 f_k - 48 A_m f_k - 32 A_{m,m} f_k - 96 f_m A_k + 16 A_{k,m} A_m + 16 A_{m,k} A_m - 32 f_{k,m} f_m); \)

\( b) \quad (\partial L'/\partial f_{k,m})_{,m} = g^{(0)kk} g^{(0)mm} (8 f_{m,m} f_k + 16 f_m^2 - 16 A_m^2)_{,k} + 16(f_{k,m} f_k - f_k f_m + A_{k,m} A_m)_{,m} = g^{(0)kk} g^{(0)mm} (24 f_{m,m,k} + 16 f_m f_{m,k} - 32 A_{m,k} A_m + 16 A_{k,m} A_m - 16 f_m f_k + 16 A_{m,k} A_k); \)

\( c) \quad \partial L'/\partial A_k = g^{(0)kk} g^{(0)mm} (-32 f_{m,m} A_k - 48 f_m^2 A_k + 48 A_m^2 A_k - 32 A_{m,m} f_k - 96 f_m A_k f_k + 16 A_{k,m} f_m + 16 A_{m,k} f_m + 32 f_{k,m} A_m); \)

\( d) \quad (\partial L'/\partial A_{k,m})_{,m} = g^{(0)kk} g^{(0)mm} (-8 A_{m,m} - 32 f_m A_m)_{,k} - 16 f_{k,m} A_m - A_k f_{m,k} - A_m f_{k,m} = g^{(0)kk} g^{(0)mm} (-8 A_{m,m,k} - 16 A_{k,m,m} - 16 f_{m,k} f_m + 16 A_{k,m} f_k + 16 f_k A_{m,m}) \)

and equations of motion are (we multiply them by \( e^{-2f} \) or \( e^{-4f} \)):

\[
(16) \begin{cases} 
  g_{mm}[-24(f_{m,m} + f_m^2)_{,k} + 48(f_{m,m} + f_m^2) f_k - 48(A_{m,m} + 2 f_m A_m) A_k + 48(A_{m,k} A_m - A^2_{m} f_k)] = 0; \\
  g^{kk} g^{mm}[16(A_{k,m} - A_{m,k})_{,m} - 48(f_{m,m} + f_m^2) A_k + 24(A_{m,m} + 2 f_m A_m) A_k - 48(A_{m,m} + 2 A_{m,k} f_m) f_k + 48 A^2_{m} A_k] = 0
\end{cases}
\]

According to (14), (15), to

\[ g_{mm}(W_{mm})_{,k} - 2 f_k g^{mm} W_{mm} = (g^{mm} W_{mm})_{,k}, \]
where $W_{mm}$ is any tensor, and to

$$g^{kk}g^{mm}(A_{k;m} - A_{m;k}),_m = (A^{k;m} - A^{m;k}),_m + 4f_m(A^{k;m} - A^{m;k}) = (A^{k;m} - A^{m;k}),_m$$

we obtain from (16) the equations (9). If $|L| = -L$ the equations of motion and all other remain the same.

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