Let $V$ be an arbitrary system of weights on an open connected subset $G$ of $\mathbb{C}^N$ $(N \geq 1)$ and let $B(E)$ be the Banach algebra of all bounded linear operators on a Banach space $E$. Let $HV_b (G, E)$ and $HV_0 (G, E)$ be the weighted locally convex spaces of vector-valued analytic functions. In this survey, we present a development of the theory of multiplication operators and composition operators from classical spaces of analytic functions $H(G)$ to the weighted spaces of analytic functions $HV_b (G, E)$ and $HV_0 (G, E)$.

Copyright © 2007 J. S. Manhas. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

Multiplication operators (also known as multipliers) and composition operators, on different spaces of analytic functions, have been actively appearing in different areas of mathematical sciences like dynamical systems, theory of semigroups, isometries, and, in turn, the theory of weighted composition operators besides their role in the theory of operator algebras and operator spaces. Evard and Jafari [1] and Siskakis [2, 3] have employed these operators to make a study of weighted composition semigroups and dynamical systems on Hardy Spaces. De Leeuw et al. [4] and Nagasawa [5] have described isometries of Hardy spaces $H^1 (\mathbb{D})$ and $H^\infty (\mathbb{D})$ as a product of multiplication operators and composition operators. Isometries on $H^p$-spaces and Bergman spaces are very much related with multiplication operators and composition operators, and for details on these isometries, we refer to Forelli [6], Cambern and Jarosz [7], Kolaski [8], Mazur [9], and Lin [10]. In [11], Arveson has recently obtained Toeplitz $C^*$-algebras and operator spaces associated with these multiplication operators on Hardy Spaces.
In recent years, many authors like Attele [12], Axler [13–16], Bercovici [17], Eschmeier [18], Luecking [19], Vukotic [20], and Zhu [21] have made a study of multiplication operators on Bergman spaces, whereas Campbell and Leach [22], Feldman [23], Lin [10], and Ohno and Takagi [24] have obtained a study of these operators on Hardy spaces. On Bloch spaces, these operators are studied by Arazy [25], Axler [15] and Brown and Shields [26]. Also, Axler and Shields [16] and Stegenga [27] have explored multiplication operators on Dirichlet spaces. On BMOA, these operators are studied by Ortega and Fabregà [28]. Further, on Nevanlinna classes of analytic functions, these operators are studied by Jarchow et al. [29] and Yanagihara [30]. Besides these well-known analytic function spaces, a study of these operators on some other Banach spaces of analytic functions has also been pursued by Bonet et al. [31–34], Contreras and Hernández-Díaz [35], Ohno and Takagi [24], and Shields and Williams [36, 37].

In [38], Contreras and Hernández-Díaz have made a study of weighted composition operators on Hardy spaces, whereas Mirzakarimi and Siddighi [39] have considered these operators on Bergman and Dirichlet spaces. On Bloch and Block-type spaces, these operators are studied by MacCluer and Zhao [40], Ohno [41], Ohno and Zhao [42], and Ohno et al. [43]. In [24], Ohno and Takagi have obtained some properties of these operators on the disc algebra and the Hardy space $H^\infty(D)$. Also, recently, Montes-Rodríguez [44] and Contreras and Hernández-Díaz [35] have studied the behaviour of these operators on weighted Banach spaces of analytic functions. The applications of these operators can be found in the theory of semigroups and dynamical systems (see [2, 3, 45]). For more information on composition operators on spaces of analytic functions, we refer to three monographs (see Cowen and MacCluer [46], Shapiro [47], and Singh and Manhas [48]).

In the present survey, we report on a recent study of composition operators and multiplication operators on the weighted spaces of analytic functions.

2. Weighted spaces of analytic functions

Let $G$ be an open connected subset of $\mathbb{C}^N$ ($N \geq 1$) and let $H(G,E)$ be the space of all vector-valued analytic functions from $G$ into the Banach space $E$. Let $V$ be a set of nonnegative upper semicontinuous functions on $G$. Then $V$ is said to be directed upward, if for every pair $u_1, u_2 \in V$ and $\lambda > 0$, there exists $v \in V$ such that $\lambda u_i \leq v$ (pointwise on $G$), for $i = 1, 2$. If $V$ is directed upward and for each $z \in G$, there exists $v \in V$ such that $v(z) > 0$, then we call $V$ as an arbitrary system of weights on $G$. If $U$ and $V$ are two arbitrary systems of weights on $G$ such that for each $u \in U$, there exists $v \in V$ for which $u \leq v$, then we write $U \leq V$. If $U \leq V$ and $V \leq U$, then we write $U \equiv V$. Let $V$ be an arbitrary system of weights on $G$. Then we define

$$HV_b(G,E) = \{ f \in H(G,E) : \nu f(G) \text{ is bounded in } E, \text{ for each } \nu \in V \},$$

$$HV_0(G,E) = \{ f \in H(G,E) : \nu f \text{ vanishes at infinity on } G, \text{ for each } \nu \in V \}. \quad (2.1)$$

For $\nu \in V$ and $f \in H(G,E)$, we define

$$\|f\|_{\nu,E} = \sup \{ \nu(z) \|f(z)\| : z \in G \}. \quad (2.2)$$
Clearly, the family \( \{\|\cdot\|_{v,E} : v \in V\} \) of seminorms defines a Hausdorff locally convex topology on each of these spaces: \( HV_b(G,E) \) and \( HV_0(G,E) \). With this topology, the spaces \( HV_b(G,E) \) and \( HV_0(G,E) \) are called the weighted locally convex spaces of vector-valued analytic functions. These spaces have a basis of closed absolutely convex neighbourhoods of the form

\[
B_{v,E} = \{ f \in HV_b(G,E) \text{ (resp.,} HV_0(G,E)) : \|f\|_{v,E} \leq 1 \}.
\]  

If \( E = \mathbb{C} \), then we write \( HV_b(G,E) = HV_b(G) \), \( HV_0(G,E) = HV_0(G) \) and

\[
B_v = \{ f \in HV_b(G) \text{ (resp.,} HV_0(G)) : \|f\|_v \leq 1 \}.
\]  

Throughout the paper, we assume for each \( z \in G \), there exists \( f_z \in HV_0(G) \) such that \( f_z(z) \neq 0 \).

If \( v : \mathbb{D} \to \mathbb{R}^+ \) is a continuous weight and \( E = \mathbb{C} \), then the corresponding weighted Banach spaces of analytic functions are defined as follows:

\[
\begin{align*}
H_\infty^v(\mathbb{D}) &= \{ f \in H(\mathbb{D}) : vf(\mathbb{D}) \text{ is bounded} \}, \\
H_0^v(\mathbb{D}) &= \left\{ f \in H(\mathbb{D}) : \lim_{|z| \to 1^-} v(z) \|f(z)\| = 0 \right\}.
\end{align*}
\]  

Now, using the definitions of weights given in [32, 49–51], we give definitions of some systems of weights which are required for characterizing some results in the remaining sections.

Let \( V \) be an arbitrary system of weights on \( G \) and let \( v \in V \). Then define \( \tilde{w} : G \to \mathbb{R}^+ \) as

\[
\begin{align*}
\tilde{w}(z) &= \sup \{ |f(z)| : \|f\|_v \leq 1 \} \leq \frac{1}{v(z)}, \\
\tilde{v}(z) &= \frac{1}{\tilde{w}(z)}, \quad \text{for every } z \in G.
\end{align*}
\]  

In case \( \tilde{w}(z) \neq 0 \), \( \tilde{v} \) is an upper semicontinuous, and we call it an associated weight of \( v \). Let \( \tilde{V} \) denote the system of all associated weights of \( V \). Then an arbitrary system of weights \( V \) is called a reasonable system as it satisfies the following properties:

\[
\begin{align*}
&\text{for each } v \in V, \text{ there exists } \tilde{v} \in \tilde{V} \text{ such that } v \leq \tilde{v}; \quad (2.7a) \\
&\text{for each } v \in V, \|f\|_v \leq 1 \iff \|f\|_{\tilde{v}} \leq 1, \quad (2.7b) \\
&\text{for every } f \in HV_b(G); \\
&\text{if } v \in V, \text{ then for every } z \in G, \text{ there exists } f_z \in B_v \text{ such that } |f_z(z)| = \frac{1}{\tilde{v}(z)}. \quad (2.7c)
\end{align*}
\]  

Let \( v \in V \). Then \( v \) is called essential if there exists a constant \( \lambda > 0 \) such that \( v(z) \leq \tilde{v}(z) \leq \lambda v(z) \), for each \( z \in G \). A reasonable system of weights \( V \) is called an essential system if each \( v \in V \) is an essential weight. If \( V \) is an essential system of weights, then we have
V \equiv \tilde{V}. For example, let G = \mathbb{D}, the open unit disc, and let f \in H(\mathbb{D}) be nonzero. Then define \nu_f(z) = \left[ \sup \{|f(z)| : |z| = r\} \right]^{-1}, for every z \in \mathbb{D}. Clearly, each \nu_f is a weight satisfying \tilde{\nu}_f = \nu_f, and the family V = \{\nu_f : f \in H(\mathbb{D}), f \text{ nonzero} \} is an essential system of weights on \mathbb{D}. For more details on these weights, we refer to [50]. Let G be any balanced (i.e., \lambda z \in G, whenever z \in G and \lambda \in \mathbb{C} with |\lambda| \leq 1) open subset of \mathbb{C}^N (N \geq 1).

Then a weight \nu \in V is called radial and typical if \nu(z) = \nu(\lambda z) for all z \in G and \lambda \in \mathbb{C} with |\lambda| = 1, and vanishes at the boundary \partial G. In particular, a weight \nu on \mathbb{D} is radial and typical if \nu(z) = \nu(|z|) and \lim_{|z| \to 1} \nu(z) = 0. For instance, in [35], it is shown that \nu_p(z) = (1 - |z|^2)^p, (0 < p < \infty), for every z \in \mathbb{D}, are essential typical weights. For more details on the weighted Banach spaces of analytic functions and the weighted locally convex spaces of analytic functions associated with these weights, we refer to [32, 49–54]. For basic definitions and facts in complex analysis and functional analysis, we refer to [55–58].

Let F(G,E) be a topological vector space of vector-valued analytic functions from G into E, and let L(G,E) be the vector space of all vector-valued functions from G into E. Let B(E) be the Banach algebra of all bounded linear operators on E. Then for an operator-valued map \Psi : G \to B(E) and self-map \phi : G \to G, we define the linear map \Psi\phi : F(G,E) \to L(G,E) as \Psi\phi(f) = \Psi \cdot f \circ \phi for every f \in F(G,E), where the product \Psi \cdot f \circ \phi is defined pointwise on G as (\Psi \cdot f \circ \phi)(z) = \Psi_z(f(\phi(z))) for every z \in G. In case \Psi\phi takes F(G,E) into itself and is continuous, we call \Psi\phi, the weighted composition operator on F(G,E) induced by the symbols \Psi and \phi. If \Psi(z) = I, the identity operator on E for every z \in G, then \Psi\phi is called the composition operator induced by \phi and we denote it by C\phi. In case \phi(z) = z for every z \in G, \Psi\phi is called the multiplication operator induced by \Psi, and we denote it by M_\Psi.

3. Characterizations of multiplication operators

In this section, we give characterizations of multiplication operators on the weighted spaces of analytic functions. We begin with the following straightforward observations obtained by [31] on the weighted Banach spaces of scalar-valued analytic functions.

**Proposition 3.1.** Let \Psi : \mathbb{D} \to \mathbb{C} be analytic function. Then M_\Psi : H_\nu^\infty(\mathbb{D}) \to H_\nu^\infty(\mathbb{D}) is bounded if and only if \Psi \in H^\infty(\mathbb{D}). If H_\nu^0(\mathbb{D}) \neq \{0\}, then the previous statement is equivalent to that M_\Psi : H_\nu^0(\mathbb{D}) \to H_\nu^0(\mathbb{D}) is bounded. Also, \|M_\Psi\| = \|\Psi\|_\infty.

**Proposition 3.2.** If M_\Psi : H_\nu^0(\mathbb{D}) \to H_\nu^0(\mathbb{D}) is bounded and both \nu and w are radial weights vanishing on the boundary, then M'_\Psi = M_\Psi : H_\nu^\infty(\mathbb{D}) \to H_w^\infty(\mathbb{D}).

**Proof.** It is shown by [51, 59] that (H_\nu^0(\mathbb{D}))'' = H_\nu^\infty(\mathbb{D}) and (H_w^0(\mathbb{D}))'' = H_w^\infty(\mathbb{D}). In [32], it is observed that the evaluation functional \delta_z : H_\nu^0(\mathbb{D}) \to \mathbb{C} defined as \delta_z(f) = f(z) is also acting as the evaluation functional on H_\nu^\infty(\mathbb{D}). It is obvious that M'_\Psi(\delta_z) = \Psi(z) \delta_z. Now, for f \in H_\nu^\infty(\mathbb{D}), we have \langle M'_\Psi f, \delta_z \rangle = \langle f, \Psi(z) \delta_z \rangle = f(z) \Psi(z). \square

Now, we present the generalizations of the above characterizations to the weighted spaces of vector-valued analytic functions for general systems of weights, which was obtained by Manhas in [60].
Proposition 3.3. Let $U$ and $V$ be arbitrary systems of weights on $G$, and let $\Psi : G \to B(E)$ be an analytic map. Then $M_\Psi : HU_b(G,E) \to HV_b(G,E)$ is a multiplication operator, if for every $v \in V$, there exists $u \in U$ such that $v(z)\|\Psi(z)\| \leq u(z)$, for every $z \in G$.

Remark 3.4. Proposition 3.3 makes it clear that every bounded analytic function $\Psi : G \to B(E)$ induces the multiplication operator $M_\Psi$ on $HV_b(G,E)$, for any system of weights $V$ on $G$. Also, if $V = \{\lambda_{\chi_K} : \lambda \geq 0, K \subseteq G, K$ compact set$\}$, then every operator-valued analytic map $\Psi : G \to B(E)$ induces a multiplication operator $M_\Psi$ on $HV_b(G,E)$. This makes it clear that even unbounded analytic operator-valued mappings generate multiplication operators on some of weighted locally convex spaces $HV_b(G,E)$, whereas it is not true for other spaces of analytic functions. For instance, Arveson [11] and Axler [13] have shown that only bounded analytic functions give rise to multiplication operators on Hardy spaces and Bergman spaces, respectively. Also, the same behaviour has been observed on the weighted Banach spaces of analytic functions $H^\infty(D)$ defined by a single continuous weights (see Proposition 3.1). Thus the behaviour of the multiplication operators on the weighted locally convex spaces of analytic functions is very much influenced by different systems of weights $V$ on $G$.

Theorem 3.5. Let $V$ be an arbitrary system of weights and $U$ a reasonable system of weights on $G$. Let $\Psi : G \to B(E)$ be an analytic map. Then $M_\Psi : H\tilde{U}_b(G,E) \to HV_b(G,E)$ is a multiplication operator if and only if for every $v \in V$, there exists $u \in U$ such that $v(z)\|\Psi(z)\| \leq \tilde{u}(z)$, for every $z \in G$.

Proof. The sufficient part follows from Proposition 3.3. Conversely, suppose $M_\Psi : H\tilde{U}_b(G,E) \to HV_b(G,E)$ is a multiplication operator. Let $v \in V$. Then by the continuity of $M_\Psi$ at the origin, there exists $\tilde{u} \in \tilde{U}$ with $u \in U$ such that $u \leq \tilde{u}$ and $M_\Psi(B_{\tilde{u},E}) \subseteq B_{v,E}$. To establish the inequality $v(z)\|\Psi(z)\| \leq \tilde{u}(z)$ for every $z \in G$, it is enough to prove that $v(z)\|\Psi(z)(y)\| \leq \tilde{u}(z)\|y\|$, for every $z \in G$ and $y \in E$. Fix $z_0 \in G$ and $y_0 \in E$. Then by (2.7c), there exists $f_{z_0} \in B_u$ such that $\|f_{z_0}(z_0)\| = 1/\tilde{u}(z_0)$. Let $g_0 : G \to E$ be defined as

$$g_0(z) = \frac{1}{\|y_0\|} f_{z_0}(z) y_0, \quad \text{for every } z \in G. \quad (3.1)$$

Clearly, $g_0 \in B_{u,E}$ and $\|g_0(z_0)\| = 1/\tilde{u}(z_0).$ Also, according to (2.7b), $f_{z_0} \in B_{\tilde{u}}$ and therefore $g_0 \in B_{\tilde{u},E}$. Thus it follows that $M_\Psi(g_0) \in B_{v,E}$. That is, $v(z)\|\Psi(z)(g_0(z))\| \leq 1$, for every $z \in G$. In particular, for $z = z_0$, we have $v(z_0)\|\Psi(z_0)(y_0)\| \leq \tilde{u}(z_0)\|y_0\|$. This completes the proof of the theorem.

Corollary 3.6. Let $V$ be an arbitrary system of weights and let $U$ be an essential system of weights on $G$. Let $\Psi : G \to B(E)$ be an operator-valued analytic map. Then $M_\Psi : HU_b(G,E) \to HV_b(G,E)$ is a multiplication operator if and only if for every $v \in V$, there exists $u \in U$ such that $v(z)\|\Psi(z)\| \leq u(z)$, for every $z \in G$.

Proof. It follows from Theorem 3.5 and from the relation that $\tilde{U} \equiv U$. 

Remark 3.7. All the results proved above are also hold for the spaces $HV_0(G,E)$ and $HU_0(G,E)$.
4. Invertible multiplication operators

In this section, we present characterizations of invertible multiplication operators on the weighted spaces of analytic functions. We begin with the following characterization of invertible multiplication operators on the weighted Banach spaces of scalar-valued analytic functions [31].

**Proposition 4.1.** Let \( \Psi \in H^\infty \). Then \( M_\Psi : H^\infty_v(\mathbb{D}) \to H^\infty_v(\mathbb{D}) \) is invertible if and only if \( 1/\Psi \in H^\infty \) (or equivalently, there exists \( \epsilon > 0 \) such that \( |\Psi(z)| \geq \epsilon \), for all \( z \in \mathbb{D} \)). The same is true for \( H^\infty_v(\mathbb{D}) \) if \( H^\infty_v(\mathbb{D}) \neq \{0\} \).

**Remark 4.2.** Since \( \lambda - M_\Psi = M_{\lambda - \Psi} \), the above result shows that the spectrum of \( M_\Psi \) satisfies \( \sigma(M_\Psi) = \overline{\Psi(\mathbb{D})} \). This shows that the multiplication operator \( M_\Psi \) is not compact.

In [13], Axler has characterized the Fredholm multiplication operators on Bergman spaces. Then on the spaces \( H^\infty_v(\mathbb{D}) \), the Fredholm multiplication operators and closed range multiplication operators are characterized by [31]. Further, in [61], Cichon and Seip have proved the following theorem related to closed range multiplication operators, which was conjectured by [31].

**Theorem 4.3.** Let \( v \in C^2(\mathbb{D}) \) be a radial weight such that \( -(1 - |z|^2)^2 \Delta \log v(z) \to +\infty \) as \( ||z|| \to 1^- \), where \( \Delta \) denotes the Laplacian. Then \( M_\Psi : H^\infty_v(\mathbb{D}) \to H^\infty_v(\mathbb{D}) \) has closed range if and only if \( \Psi = h\beta \), where \( h \) is invertible in \( H^\infty \) and \( \beta \) is finite Blaschke product.

Further, Manhas has extended in [60] the characterizations of invertible multiplication operators to the weighted spaces of vector-valued analytic functions. We begin with stating an invertibility criterion on a Hausdorff topological vector space [62], which we have used for characterizing invertible multiplication operators on the spaces \( HV_b(G, E) \).

**Theorem 4.4.** Let \( E \) be a complete Hausdorff topological vector space and let \( T : E \to E \) be a continuous linear operator. Then \( T \) is invertible if and only if \( T \) is bounded below and has dense range. Or, let \( E \) be a Hausdorff topological vector space and let \( T : E \to E \) be a continuous linear operator. Then \( T \) is invertible if and only if \( T \) is bounded below and onto.

In the above invertible criterion, a generalized definition of bounded below operators on Hausdorff topological vector spaces is used. Now, we give this definition as it is needed for proving some of the results of this section. A continuous linear operator \( T \) on a Hausdorff topological vector space \( E \) is said to be bounded below if for every neighbourhood \( N \) of the origin in \( E \), there exists a neighbourhood \( M \) of the origin in \( E \) such that \( T(N^c) \subseteq M^c \), where the symbol \( c \) stands for the complement of the neighborhood in \( E \). We begin with the following proposition.

**Proposition 4.5.** Let \( V \) be an arbitrary system of weights on \( G \) and let \( \Psi : G \to B(E) \) be an analytic map such that \( M_\Psi \) is a multiplication operator on \( HV_b(G, E) \). Then \( M_\Psi \) is invertible if

(i) for each \( z \in G \), \( \Psi(z) : E \to E \) is onto;

(ii) for each \( v \in V \), there exists \( u \in V \) such that \( v(z)\|y\| \leq u(z)\|\Psi_v(y)\| \), for every \( z \in G \) and \( y \in E \).
Proof. Fix \( z_0 \in G \). Let \( v \in V \) such that \( v(z_0) > 0 \). Then by condition (ii), there exists \( u \in V \) such that \( v(z_0) \| y \| \leq u(z_0) \| \Psi_{z_0}(y) \| \), for every \( y \in E \). That is, \( \| \Psi_{z_0}(y) \| \geq \lambda_0 \| y \| \), for every \( y \in E \), where \( \lambda_0 = v(z_0)/u(z_0) > 0 \). This proves that \( \Psi(z_0) \) is bounded below on \( E \) and hence by condition (i), \( \Psi(z_0) \) is invertible in \( B(E) \). We denote the inverse of \( \Psi(z_0) \) by \( \Psi_{z_0}^{-1} \). Now, we define \( \Psi^{-1} : G \to B(E) \) as \( \Psi^{-1}(z) = \Psi_{z}^{-1} \), for every \( z \in G \). Clearly, \( \Psi^{-1} \) is an analytic map. Again, by condition (ii), it follows that

\[
v(z)\|\Psi^{-1}_z(y)\| \leq u(z)\|y\|, \quad \text{for every } z \in G, y \in E. \quad (4.1)
\]

That is, \( v(z)\|\Psi^{-1}(z)\| \leq u(z) \), for every \( z \in G \). Thus according to Proposition 3.3, \( \Psi^{-1} \) induces the multiplication operator \( M_{\Psi^{-1}} \) on \( HV_b(G,E) \) such that \( M_{\Psi}M_{\Psi^{-1}} = M_{\Psi^{-1}}M_{\Psi} = I \), the identity operator. Hence \( M_{\Psi} \) is invertible.

**Corollary 4.6.** Let \( V \) be an arbitrary system of weights on \( G \) and let \( \Psi : G \to B(E) \) be a bounded analytic map. Then \( M_{\Psi} \) is an invertible multiplication operator on \( HV_b(G,E) \) if

(i) for each \( z \in G \), \( \Psi(z) : E \to E \) is onto;
(ii) \( \Psi \) is bounded away from zero.

Proof. Since \( \Psi : G \to B(E) \) is a bounded analytic function, Proposition 3.3 implies that \( M_{\Psi} \) is a multiplication operator on \( HV_b(G,E) \), for any arbitrary system of weights \( V \) on \( G \). By condition (ii), there exists \( \lambda > 0 \) such that \( \| \Psi_z(y) \| \geq \lambda \| y \| \), for every \( z \in G \) and \( y \in E \). Let \( v \in V \). Then \( v(z)\|y\| \leq (1/\lambda)v(z)\|\Psi_z(y)\| \), for every \( z \in G \) and \( y \in E \). Further, it implies that there exists \( u \in V \) such that \( v(z)\|y\| \leq u(z)\|\Psi_z(y)\| \), for every \( z \in G \) and \( y \in E \). Thus according to Proposition 4.5, it follows that \( M_{\Psi} \) is invertible on \( HV_b(G,E) \).

The converse of the above Corollary 4.6 may not be true. That is, if an analytic map \( \Psi : G \to B(E) \) is not bounded away from zero, even then \( M_{\Psi} \) is invertible on some of the weighted spaces \( HV_b(G,E) \). This can be easily seen from the following corollary.

**Corollary 4.7.** Let \( V = \{ \lambda \chi_K : \lambda \geq 0 \text{ and } K \subseteq G, \text{ compact set} \} \). Then every analytic map \( \Psi : G \to B(E) \) induces an invertible multiplication operator \( M_{\Psi} \) on \( HV_b(G,E) \) if \( \Psi(z) \) is invertible for every \( z \in G \).

**Remark 4.8.** Let \( G = \{ z \in \mathbb{C} : z = x + iy \text{ and } x > 0 \} \) and let \( V = \{ \lambda \chi_K : \lambda \geq 0, \text{ } K \subseteq G, \text{ compact set} \} \). Let \( E = H^\infty(G) \) be the Banach space of bounded analytic functions on \( G \). We define an analytic map \( \Psi : G \to B(E) \) as \( \Psi(z) = M_z \), for \( z \in G \), where the bounded operator \( M_z : E \to E \) is defined as \( M_z f = zf \), for every \( f \in E \). Clearly, each \( \Psi(z) \) is invertible in \( B(E) \) and hence by Corollary 4.7, \( M_{\Psi} \) is an invertible multiplication operator on \( HV_b(G,E) \). But \( \Psi : G \to B(E) \) is not bounded away from zero. Also, we note that invertible multiplication operators on Bergman spaces of analytic functions [13] and weighted Banach spaces of analytic functions (see Proposition 4.1) are generated only by the functions which are bounded away from zero. Thus in general, the invertible behaviour is very much controlled by different systems of weights \( V \) on \( G \).

**Theorem 4.9.** Let \( V \) be a reasonable system of weights on \( G \) and let \( \Psi : G \to B(E) \) be an operator-valued analytic map such that each \( \Psi(z) \) is one-to-one and \( M_{\Psi} \) is a multiplication operator...
operator on $H\tilde{V}_b(G,E)$. Then $M_\Psi$ is invertible if and only if

(i) for each $z \in G$, $\Psi(z) : E \to E$ is onto;

(ii) for each $v \in V$, there exists $u \in V$ such that $\tilde{\Psi}(z)\|y\| \leq \tilde{u}(z)\|\Psi_z(y)\|$, for every $z \in G$ and $y \in E$.

**Proof.** If conditions (i) and (ii) hold, then from Proposition 4.5, it clearly follows that $M_\Psi$ is invertible.

Conversely, suppose that $M_\Psi$ is invertible on $H\tilde{V}_b(G,E)$. To establish condition (i), let $z_0 \in G$ and let $f_{z_0} \in H\tilde{V}_b(G)$ such that $f_{z_0}(z_0) = 1$. Fix $0 \neq y_0 \in E$. Define an analytic map $g_{z_0} : G \to E$ as $g_{z_0}(z) = f_{z_0}(z)y_0$, for every $z \in G$. Clearly, $g_{z_0} \in H\tilde{V}_b(G,E)$. Since $M_\Psi$ is onto, there exists $f_0 \in H\tilde{V}_b(G,E)$ such that $M_\Psi(f_0) = g_{z_0}$. That is, $\Psi_{z_0}(f_0(z_0)) = y_0$. This shows that each $\Psi(z)$ is onto. Now, to prove condition (ii), we fix $v \in V$. Then there exists $\tilde{v} \in \tilde{V}$ such that $\tilde{v} \leq \tilde{y}$. In view of Theorem 4.4, we conclude that $M_\Psi$ is bounded below and onto. Further, it implies that there exists $\tilde{u} \in \tilde{V}$ with $u \in V$ such that $M_\Psi(B_{\tilde{v}}(E) \subseteq B_{\tilde{u}}(E)$. Now, we claim that $\tilde{\Psi}(z)\|\Psi_z^{-1}(y)\| \leq \tilde{u}(z)\|y\|$, for every $z \in G$ and $y \in E$. We fix $z_0 \in G$ and $y_0 \in E$. According to (2.7c), there exists $f_{z_0} \in B_u$ such that $f_{z_0}(z_0) = 1/\tilde{u}(z_0)$. Let $g_0 : G \to E$ be defined as $g_0(z) = (1/\|y_0\|) f_{z_0}(z) y_0$, for every $z \in G$. Clearly, $g_0 \in B_{u,E}$ and $\|g_0(z_0)\| = 1/\tilde{u}(z_0)$. Also, (2.7b) implies that $f_{z_0} \in B_{\tilde{u}}$ and hence $g_0 \in B_{\tilde{u},E}$. Since $M_\Psi$ is onto, there exists $h_0 \in H\tilde{V}_b(G,E)$ such that $M_\Psi(h_0) = g_0$. That is, $\Psi_{z_0}(h_0(z_0)) = g_0(z_0)$. Since each $\Psi_{z_0}$ is invertible, we have $h_0(z_0) = \Psi_{z_0}^{-1}(y_0) f_{z_0}(z_0) 1/\|y_0\|$. Again, since $\|g_0\| \leq 1$, we conclude that $M_\Psi(h_0) \notin B_{\tilde{u},E}$. Further, it implies that $h_0 \notin B_{\tilde{u},E}$. That is, $\tilde{\Psi}(z)\|h_0(z)\| \leq 1$, for every $z \in G$. In particular, for $z = z_0$, we have $\tilde{\Psi}(z_0)\|\Psi_{z_0}^{-1}(y_0)\| \leq \tilde{u}(z_0)\|y_0\|$. This proves our claim. Since each $\Psi(z)$ is invertible, we have $\tilde{\Psi}(z)\|y\| \leq \tilde{u}(z)\|\Psi_z(y)\|$, for every $z \in G$ and $y \in E$. This proves condition (ii). With this, the proof of the theorem is complete. \qed

**Corollary 4.10.** Let $V$ be an essential system of weights on $G$ and let $\Psi : G \to B(E)$ be an analytic map such that each $\Psi(z)$ is one-to-one and $M_\Psi$ is a multiplication operator on $HV_b(G,E)$. Then $M_\Psi$ is invertible if and only if

(i) for each $z \in G$, $\Psi(z) : E \to E$ is onto;

(ii) for each $v \in V$, there exists $u \in V$ such that $v(z)\|y\| \leq u(z)\|\Psi_z(y)\|$, for every $z \in G$ and $y \in E$.

**Proof.** Follows from Theorem 4.9 since $V \subseteq \tilde{V}$. \qed

**Theorem 4.11.** Let $V = \{\lambda\chi_K : \lambda \geq 0, K \subseteq G, K$ compact set$\}$ and let $\Psi \in H(G)$ be nonzero. Then $M_\Psi$ is not a compact multiplication operator on $HV_b(G)$.

**Proof.** Suppose that $M_\Psi$ is a compact multiplication operator on $HV_b(G)$. Since Corollary 4.7 implies that $M_\Psi$ is invertible on $HV_b(G)$ if and only if $\Psi(z) \neq 0$, for every $z \in G$, we conclude that the spectrum of $M_\Psi$ satisfies $\sigma(M_\Psi) = \Psi(G)$. Now, from [57, Theorem 4], it follows that each point in the spectrum of a compact operator on a locally convex Hausdorff space is an isolated point which is a contradiction to the fact that $\sigma(M_\Psi) = \Psi(G)$ is a connected set. Thus there is no nonzero compact multiplication operator on $HV_b(G)$. \qed
Now, we will characterize quasicompact multiplication operators on weighted Banach spaces of analytic functions. For this, we need to give some definitions: a continuous linear operator $S$ on a Banach space $E$ is said to be quasicompact [63] if there exists an integer $n$ and a compact operator $K$ on $E$ such that $\|S^n - K\| < 1$. The essential norm of a continuous linear operator $S$ on a Banach space $E$ is defined by $\|S\|_c = \inf\{\|S - K\| : K \text{ compact on } E\}$. Clearly, $S$ is compact if and only if $\|S\|_c = 0$. The $n$th approximation number of $S$ is defined as $a_n(S) = \inf\{\|S - T_n\| : T_n \text{ is bounded on } E, \text{ rank } T_n \leq n\}$. Now, it readily follows that $\|S\|_c \leq a_n(S) \leq \|S\|$. For more details on the properties of the approximation numbers, we refer to [63].

**Theorem 4.12.** Let $\nu$ be a continuous weight on $\mathbb{D}$ and let $\Psi \in H^\infty(\mathbb{D})$. Then the multiplication operator $M_{\Psi}$ on $HV_b(\mathbb{D})$ is quasicompact if and only if $\|\Psi\|_\infty < 1$.

**Proof.** If $\|\Psi\|_\infty < 1$, then by choosing $K$ as the zero operator on $HV_b(\mathbb{D})$, it follows that $\|M_{\Psi} - 0\| = \|M_{\Psi^n}\| = \|\Psi\|_\infty^n < 1$. Thus $M_{\Psi}$ is quasicompact. Conversely, suppose that $M_{\Psi}$ is a quasicompact multiplication operator on $HV_b(\mathbb{D})$. Then there exists an integer $n$ and a compact operator $K$ on $HV_b(\mathbb{D})$ such that $\|M_{\Psi}^n - K\| < 1$. Now, from [31, Corollary 2.5], it follows that $\|M_{\Psi}\|_c = \|M_{\Psi}\| = \|\Psi\|_\infty = a_n(M_{\Psi})$, for all $n$. Further, it implies that $1 > \|M_{\Psi}^n - K\| \geq \|M_{\Psi}\|_c = \|\Psi\|_\infty^n$. Thus $\|\Psi\|_\infty < 1$. \[ \square \]

5. Dynamical systems and multiplication operators

Let $g \in H^\infty(G, B(E))$ and let $\|g\|_\infty = \sup\{\|g(z)\| : z \in G\}$. Then for each $t \in \mathbb{R}$, we define $\Psi_t : G \to B(E)$ as $\Psi_t(z) = e^{it\phi(z)}$, for every $z \in G$. Clearly, $\Psi_t$ is an operator-valued bounded analytic map and hence by Proposition 3.3, $M_{\Psi_t}$ is a multiplication operator on $HV_b(G, E)$, for any arbitrary system of weights $V$ on $G$.

**Theorem 5.1.** Let $V$ be an arbitrary system of weights on $G$ and let $\Pi : \mathbb{R} \times HV_b(G, E) \to H(G, E)$ be defined as $\Pi(t, f) = M_{\Psi_t} f$, for every $t \in \mathbb{R}$ and $f \in HV_b(G, E)$. Then $\Pi$ is a linear dynamical system on $HV_b(G, E)$. Moreover, if $V$ is a system of weights on $G$ such that $HV_b(G, E)$ is completely metrizable, then the family $\mathcal{M} = \{M_{\Psi_t} : t \in \mathbb{R}\}$ is locally equicontinuous $C_0$-group of multiplication operators in $B(HV_b(G, E))$.

**Proof.** We have already observed that $M_{\Psi_t}$ is a multiplication operator on $HV_b(G, E)$, for every $t \in \mathbb{R}$. Thus it follows that $\Pi(t, f) \in HV_b(G, E)$, for every $t \in \mathbb{R}$ and $f \in HV_b(G, E)$. Clearly, $\Pi$ is linear and $\Pi(0, f) = f$, for every $f \in HV_b(G, E)$. Again, it is easy to see that $\Pi(t + s, f) = \Pi(t, \Pi(s, f))$, for every $t, s \in \mathbb{R}$ and $f \in HV_b(G, E)$. To show that $\Pi$ is a dynamical system, it is sufficient to prove that $\Pi$ is jointly continuous. Let $\{((t_\alpha, f_\alpha) : \alpha \in \Delta)\}$ be a net in $\mathbb{R} \times HV_b(G, E)$ such that $(t_\alpha, f_\alpha) \to (t, f)$ in $\mathbb{R} \times HV_b(G, E)$. Let $v \in V$.

Then

$$
\|\Pi(t_\alpha, f_\alpha) - \Pi(t, f)\|_{v,E} = \|\Psi_{t_\alpha} f_\alpha - \Psi_t f\|_{v,E} = \sup\{v(z)\|\Psi_{t_\alpha}(z)f_\alpha(z) - \Psi_t(z)f(z)\| : z \in G\}
$$

$$
\leq \sup\{v(z)\|\Psi_{t_\alpha}(z) - \Psi_t(z)\| f_\alpha(z)\| : z \in G\}
+ \sup\{v(z)\|\Psi_t(z)(f_\alpha(z) - f(z))\| : z \in G\}
$$
Theorem 6.1. Let \( \nu \) and \( \omega \) be continuous bounded weights. Then the following are equivalent:

(i) \( C_\phi : H^\infty_\nu (\mathbb{D}) \to H^\infty_\omega (\mathbb{D}) \) is bounded;

(ii) \( \sup_{z \in \mathbb{D}} (w(z)/\nu(\phi(z))) < \infty \);

(iii) \( \sup_{z \in \mathbb{D}} (\tilde{w}(z)/\nu'(\phi(z))) < \infty \).

If \( \nu \) and \( \omega \) are typical weights, then the above conditions are equivalent to

(iv) \( C_\phi : H^0_\nu (\mathbb{D}) \to H^0_\omega (\mathbb{D}) \) is bounded.

Further, Garcia et al. [53] have generalized the above characterization to the weighted Banach spaces of scalar-valued analytic functions defined on the open unit ball of a Banach space. For presenting this generalization, we need to fix some definitions and notations.
Let $X$ be a complex Banach space and $B_X$ its open unit ball. Then clearly, the space $H^\omega_v(B_X)$ (defined in the same way as $H^\omega_v(\mathbb{D})$) is a Banach space. By $\mathcal{B}_v$, we denote the closed unit ball of $H^\omega_v(B_X)$. It is well known that in $H^\omega_v(B_X)$, the $\tau_v$ (norm) topology is finer than the $\tau_0$ (compact-open) topology and that $\mathcal{B}_v$ is $\tau_0$-compact [64]. A weight $v$ satisfies Condition-I if $\inf_{x \in \mathcal{B}_v} v(x) > 0$, for every $0 < r < 1$ [65]. If $v$ satisfies Condition-I, then $H^\omega_v(B_X) \subseteq H^\omega_\infty(B_X)$ [65]. If $X$ is finite dimensional, then all weights on $B_X$ satisfy Condition-I.

Now, we can present the extended version of the above theorem [53].

**Theorem 6.2.** Let $B_X$ and $B_Y$ be open unit balls of the Banach spaces $X$ and $Y$, respectively. Let $w$ and $v$ be weights on $B_X$ and $B_Y$, respectively, satisfying Condition-I. Let $\phi : B_X \to B_Y$ be a holomorphic map. Then the following are equivalent:

(i) $C_\phi : H^\omega_v(B_Y) \to H^\omega_w(B_X)$ is bounded;
(ii) $\sup_{x \in B_X} (w(x)/v(\phi(x))) < \infty$;
(iii) $\sup_{x \in B_X} (\hat{w}(\phi(x))/\tilde{v}(\phi(x))) < \infty$;
(iv) $\sup_{|\phi(x)| > r_0} (w(x)/v(\phi(x))) < \infty$, for some $0 < r_0 < 1$.

**Proof.** (iii)⇒(ii) is obvious because $w \leq \hat{w}$.

(ii)⇒(i) let $f \in H^\omega_v(B_Y)$. Then we have $w(x)|f(\phi(x))| = (w(x)/\tilde{v}(\phi(x)))\tilde{v}(\phi(x)) \cdot |f(\phi(x))| \leq M\|f\|_{\tilde{v}} = M\|f\|_v$, for all $x$. Hence $C_\phi$ is bounded.

(i)⇒(iii) if (iii) does not hold, then there exists a sequence $(x_n)_{n \in \mathbb{N}} \subseteq B_X$ such that $\lim_{n \to \infty} \hat{w}(x_n)/\tilde{v}(\phi(x_n)) = \infty$. Fix $n \in \mathbb{N}$. Let $f_n \in \mathcal{B}_v$ be such that $|f_n(\phi(x_n))| = \hat{v}(\phi(x_n)) = 1/\tilde{v}(\phi(x_n))$. Hence $f_n(\phi(x_n))|\hat{w}(x_n) - \hat{w}(x_n)/\tilde{v}(\phi(x_n))$, which is a contradiction to the fact that $C_\phi$ is bounded.

(ii)⇒(iv) is straightforward.

(iv)⇒(i) let $M = \sup_{|\phi(x)| > r_0} (w(x)/\tilde{v}(\phi(x)))$. Let $x \in X$. If $|\phi(x)| > r_0$, then we have $w(x)|f(\phi(x))| = (w(x)/\tilde{v}(\phi(x)))\tilde{v}(\phi(x))||f(\phi(x))| \leq M\|f\|_{\tilde{v}}$. If $|\phi(x)| \leq r_0$, then we have $w(x)|f(\phi(x))| \leq (\sup_{x \in B_X} w(x))(\sup_{y \in B_Y} |f(y)|)$ because $f$ is bounded in $\tau_{0\mathcal{B}_Y}$. Thus we have $\sup_{x \in B_X} w(x)|f(\phi(x))| < \infty$ and $C_\phi(f) \in H^\omega_w(B_X)$, for all $f \in H^\omega_v(B_Y)$. Hence $C_\phi$ is bounded. □

**Remark 6.3.** In the above theorem, the first three conditions are equivalent even if Condition-I does not holds. On the other hand, in [53], Garcia et al. have given an example, which shows that Condition-I is necessary to prove that (iv) implies (i).

Also, Manhas [66] has further generalized Theorem 6.1 and related results of [32] to the general weighted spaces of analytic functions, which are given below.

**Theorem 6.4.** Let $U$ and $V$ be arbitrary systems of weights on $G$. Let $\phi \in H(G)$ be such that $\phi(G) \subseteq G$. Then $C_\phi : HU_0(G) \to HV_0(G)$ is a composition operator if $V \leq U \circ \phi$.

**Remark 6.5.** The condition $V \leq U \circ \phi$ in the above theorem is not a sufficient condition for $C_\phi$ to be a composition operator from $HU_0(G) \to HV_0(G)$. For instance, let $G = \{z \in \mathbb{C} : z = x + iy, x > 0\}$ be the right half plane. Let $U = V$ be the system of constant weights on $G$. Let $\phi : G \to G$ be defined as $\phi(z) = z_0$, for every $z \in G$, where $z_0 \in G$ is fixed. Then, clearly, the inequality $V \leq U \circ \phi$ is true. But $C_\phi : HU_0(G) \to HV_0(G)$ is not even an into
map. For instance, if we take \( f(z) = 1/z \), for every \( z \in G \), then \( f \in HU_0(G) \) but \( C_\phi(f) \notin HV_0(G) \). So, in order to show that \( C_\phi : HU_0(G) \to HV_0(G) \) is a composition operator, we need an additional condition on \( \phi \). Let \( v \in V \) and \( \varepsilon > 0 \). Then consider the set \( F(v, \varepsilon) = \{ z \in G; v(z) \geq \varepsilon \} \). Clearly \( F(v, \varepsilon) \) is a closed subset of \( G \). In the next theorem, we have obtained a sufficient condition for \( C_\phi \) to be a composition operator from \( HU_0(G) \) into \( HV_0(G) \).

**Theorem 6.6.** Let \( U \) and \( V \) be arbitrary systems of weights on \( G \). Let \( \phi \in H(G) \) be such that \( \phi(G) \subseteq G \). Then \( C_\phi : HU_0(G) \to HV_0(G) \) is a composition operator if

(i) \( V \leq U \circ \phi \);

(ii) for every \( v \in V \), \( \varepsilon > 0 \), and compact set \( K \subseteq G \), the set \( \phi^{-1}(K) \cap F(v, \varepsilon) \) is compact.

**Proof.** In view of Theorem 6.4, condition (i) implies that \( C_\phi : HU_b(G) \to HV_b(G) \) is a composition operator. To show that \( C_\phi : HU_0(G) \to HV_0(G) \) is a composition operator, it is enough to prove that \( C_\phi \) is an onto map. Let \( f \in HU_0(G) \). Let \( v \in V \) and \( \varepsilon > 0 \). Then we consider the set \( K = \{ z \in G; v(z) |f(\phi(z))| \geq \varepsilon \} \). We will show that \( K \) is a compact subset of \( G \). By condition (i), there exists \( u \in U \) such that \( v(z) \leq u(\phi(z)) \), for every \( z \in G \). Let \( S = \{ z \in G; u(z) |f(\phi(z))| \geq \varepsilon \} \). Then clearly, \( S \) is a compact subset of \( G \) and \( \phi(K) \subseteq S \). Let \( M = \sup \{ |f(z)| : z \in S \} \). Then \( M > 0 \) and \( S \subseteq F(u, \varepsilon/M) \). By condition (ii), the set \( \phi^{-1}(S) \cap F(v, \varepsilon/M) \) is compact. Since \( K \) is a closed subset of the set \( \phi^{-1}(S) \cap F(v, \varepsilon/M) \), it follows that \( K \) is compact. Thus \( C_\phi(f) \in HV_0(G) \). This completes the proof. \( \square \)

**Corollary 6.7.** Let \( U \) and \( V \) be arbitrary systems of weights on \( G \). Let \( \phi \in H(G) \) be such that \( \phi(G) \subseteq G \). Then

(i) \( C_\phi : HU_b(G) \to HV_b(G) \) is a composition operator if \( V \leq U \circ \phi \);

(ii) \( C_\phi : HU_0(G) \to HV_0(G) \) is a composition operator if \( V \leq U \circ \phi \) and \( \phi \) is a conformal mapping of \( G \) onto itself.

The converse of the above corollary may not be true. That is, if \( C_\phi \) is a composition operator on \( HV_b(G) \) and \( HV_0(G) \), then \( \phi \in H(G) \) may not be conformal mapping of \( G \) onto itself. For example, let \( V = \{ \lambda \chi_K : \lambda \geq 0, K \subseteq G, K \text{ is compact} \} \), then it can be easily seen that \( C_\phi \) is a composition operator on \( HV_0(G) \) if and only if \( \phi : G \to G \) is an analytic map.

In the next theorem, Manhas [66] has obtained a necessary and sufficient condition for \( C_\phi \) to be a composition operator on \( HV_b(G) \) in terms of the inducing map \( \phi \) and the system of weights \( V \).

**Theorem 6.8.** Let \( V \) be an arbitrary system of weights on \( G \) and let \( U \) be a reasonable system of weights on \( G \). Let \( \phi \in H(G) \) be such that \( \phi(G) \subseteq G \). Then \( C_\phi : HU_b(G) \to HV_b(G) \) is a composition operator if and only if \( V \leq \tilde{U} \circ \phi \).

**Proof.** Suppose that \( C_\phi : HU_b(G) \to HV_b(G) \) is a composition operator. Let \( v \in V \). Then by the continuity of \( C_\phi \) at the origin, there exists \( u \in U \) and a neighbourhood \( B_u \) of the origin in \( HU_b(G) \) such that \( C_\phi(B_u) \subseteq B_v \). Let \( \tilde{u} \) be the associated weight of \( u \). Then \( \tilde{u} \in \tilde{U} \). Now, we claim that \( v \leq \tilde{u} \circ \phi \). Fix \( z_0 \in G \). Then by (2.7c), there exists \( f_0 \in B_u \) such that \( |f_0(\phi(z_0))| = 1/\tilde{u}(\phi(z_0)) \). Further, it implies that \( C_\phi(f_0) \in B_v \). That is, \( v(z) |f_0(\phi(z))| \leq 1 \),
for every $z \in G$. In particular, for $z = z_0$, we have $v(z_0) \leq \tilde{u}(\phi(z_0))$. This proves our claim and hence $V \leq \tilde{U} \circ \phi$.

Conversely, suppose that the condition is true. To show that $C_\phi : H U_b(G) \to H V_b(G)$ is a composition operator, it is sufficient to show that $C_\phi$ is continuous at the origin. Let $v \in V$ and $B_v$ be a neighbourhood of the origin in $H V_b(G)$. Then by the given condition, there exists $\tilde{u} \in \tilde{U}$ with $u \in U$ such that $v \leq \tilde{u} \circ \phi$. That is, $v(z) \leq \tilde{u}(\phi(z))$, for every $z \in G$. Now, we claim that $C_\phi(B_u) \subseteq B_v$. Let $f \in B_u$. Then by (2.7b), $\|f\|_u \leq 1$ if and only if $\|f\|_{\tilde{u}} \leq 1$. Now,

$$\|C_\phi f\|_v = \sup \{v(z) \mid f(\phi(z)) \mid : z \in G\} \leq \sup \{\tilde{u}(\phi(z)) \mid f(\phi(z)) \mid : z \in G\} \leq \sup \{\tilde{u}(z) \mid f(z) \mid : z \in G\} = \|f\|_{\tilde{u}} \leq 1.$$  

This proves that $C_\phi f \in B_v$ and hence $C_\phi$ is a composition operator. This completes the proof of the theorem. $\Box$

**Corollary 6.9.** Let $U$ and $V$ be reasonable systems of weights on $G$. Let $\phi \in H(G)$ be such that $\phi(G) \subseteq G$. Then the following statements are equivalent:

(i) $C_\phi : H U_b(G) \to H V_b(G)$ is a composition operator;

(ii) $V \leq \tilde{U} \circ \phi$;

(iii) $\tilde{V} \leq \tilde{U} \circ \phi$.

**Corollary 6.10.** Let $V$ be an arbitrary system of weights on $G$ and let $U$ be an essential system of weights on $G$. Let $\phi \in H(G)$ be such that $\phi(G) \subseteq G$. Then $C_\phi : H U_b(G) \to H V_b(G)$ is a composition operator if and only if $V \leq U \circ \phi$.

**Theorem 6.11.** Let $V$ be an arbitrary system of weights on $G$ and let $U$ be an essential system of weights on $G$ such that each weight of $V$ and $U$ vanishes at infinity. Let $\phi \in H(G)$ be such that $\phi(G) \subseteq G$. Then $C_\phi : H U_0(G) \to H V_0(G)$ is a composition operator if and only if $V \leq U \circ \phi$.

**Example 6.12.** Let $G = \mathbb{D}$, the open unit disc, and let $\nu$ be a weight defined as $\nu(z) = 1 - |z|^2$, for every $z \in G$. Let $V = \{\lambda v : \lambda > 0\}$. Then clearly, $V$ is an essential system of weights on $G$. Let $\phi : G \to G$ be an analytic map defined by $\phi(z) = (z + 1)/2$, for every $z \in G$. Now, by the Pick-Schwarz lemma, it follows that

$$|(1 - |z|^2)| \phi'(z)| \leq 1 - |\phi(z)|^2, \quad \text{for every } z \in G.$$  

That is, $\nu(z) \leq 2\nu(\phi(z))$, for every $z \in G$. Hence by Theorem 6.4, $C_\phi$ is a composition operator on $H V_b(G)$.

**Remark 6.13.** If $G = \mathbb{D}$ and $U$ and $V$ consist of single continuous weights only, then Corollaries 6.9, 6.10 and Theorem 6.11 reduce to the results of [32, Proposition 2.1 and Corollary 2.2].
7. Compact and weakly compact composition operators

In [67], Aron et al. have characterized the compact composition operators on the Banach algebra of bounded analytic functions. This is recorded in the following theorem.

**Theorem 7.1.** Let \( C_\phi : H^\infty(B_Y) \to H^\infty(B_X) \) be a composition operator. Then the following statements are equivalent:

(i) \( C_\phi \) is compact;
(ii) \( C_\phi \) is weakly compact and \( \phi(B_X) \) is relatively compact in \( Y \);
(iii) \( \phi(B_X) \) lies strictly inside \( B_Y \) and \( \phi(B_X) \) is relatively compact in \( Y \).

Further, Galindo et al. [68] have obtained a characterization of weakly compact composition operators in terms of the inducing map \( \phi : B_X \to B_Y \), which is stated below.

**Theorem 7.2.** The composition operator \( C_\phi : H^\infty(B_Y) \to H^\infty(B_X) \) is weakly compact if (i) \( \phi(B_X) \subset rB_Y \), for some \( 0 < r < 1 \) and (ii) \( \phi(B_X) \) is relatively compact in \((Y, \sigma(Y, P(Y)))\). The converse holds, if moreover, \( Y \) has the approximation property. (By \((Y, \sigma(Y, P(Y)))\), we mean the space \( Y \) endowed with the weakest topology, making all \( p \in P(Y) \) continuous, where \( P(Y) \) denote the algebra of all continuous polynomials on \( Y \).)

Recently, Garcia et al. [53] has obtained characterizations of compact composition operators on weighted Banach spaces of analytic functions, which generalizes the above Theorem 7.1 and [32, Theorem 3.3]. This generalization is presented in the following theorem.

**Theorem 7.3.** Let \( v \) and \( w \) be weights on \( B_Y \) and \( B_X \), respectively, with \( \lim_{\|x\| \to 1^-} w(x) = 0 \). Let \( \phi : B_X \to B_Y \) be a holomorphic map. Then \( C_\phi : H^\infty_v(B_Y) \to H^\infty_w(B_X) \) is compact if and only if

(i) \( \lim_{\|x\| \to 1^-} (w(x)/\tilde{v}(\phi(x))) = 0 \);
(ii) \( \phi(rB_X) \) is relatively compact, for every \( 0 < r < 1 \).

It has been observed in [32, 50] that many weights do not satisfy this condition on the limit. In [32], Bonet et al. have characterized compact composition operators for general weights when \( X = Y = \mathbb{C} \). This characterization is given in terms of an analytic condition (see (A) below in part (c)). Then by using a topological condition, Garcia et al. [53] have obtained a characterization of compact composition operators for general Banach spaces \( X \) and \( Y \). This is presented in the following theorem.

**Theorem 7.4.** Let \( v \) and \( w \) be weights on \( B_Y \) and \( B_X \), respectively, with Condition-I. Let \( \phi : B_X \to B_Y \) be a holomorphic map. Then the following hold.

(a) If \( C_\phi : H^\infty_v(B_Y) \to H^\infty_w(B_X) \) is compact, then \( \phi(rB_X) \) is relatively compact, for every \( 0 < r < 1 \).

(b) Suppose that \( \|\phi\|_\infty < 1 \). If \( \phi(B_X) \) is relatively compact, then \( C_\phi : H^\infty_v(B_Y) \to H^\infty_w(B_X) \) is compact.

(c) Suppose that \( \|\phi\|_\infty = 1 \).

(i) If \( C_\phi : H^\infty_v(B_Y) \to H^\infty_w(B_X) \) is compact, then

\[
\lim_{r \to 1^-} \sup_{\|\phi(x)\| > r} \frac{w(x)}{\tilde{v}(\phi(x))} = 0.
\]

(A)
(ii) If \( \phi(B_X) \cap rB_Y \) is relatively compact, for every \( 0 < r < 1 \), and
\[
\lim_{r \to 1} \sup_{\|x\| > r} \frac{w(x)}{|\phi(x)|} = 0,
\]
then \( C_\phi : H_\omega^\infty(\mathbb{D}) \to H_\omega^\infty(B_X) \) is compact.

In part (b) of the above theorem, if the space \( H_\omega^\infty(B_X) \) is replaced by \( H^\infty(B_X) \), then the improved result is as follows.

**Proposition 7.5.** Let \( \nu \) be a weight on \( Y \) and let \( \phi : B_X \to B_Y \) be a holomorphic map. Then \( C_\phi : H_\omega^\infty(B_Y) \to H^\infty(B_X) \) is compact if and only if \( \phi(B_X) \) is relatively compact and \( \| \phi \|_\infty < 1 \).

Further, if \( \nu \) is taken as a norm-radial weight (i.e., \( w(x) = w(y) \)), for every \( x, y \), such that \( \|x\| = \|y\| \), then the compactness of \( C_\phi \) is better and given in the following corollary.

**Corollary 7.6.** Let \( \nu \) and \( \nu \) be weights on \( B_Y \) and \( B_X \), respectively, such that \( \nu \) is norm-radial. Let \( \phi : B_X \to B_Y \) be a holomorphic map. Then we have the following:

(a) If \( w(x) \to 0 \) as \( \|x\| \to 1^{−} \), then \( C_\phi : H_\omega^\infty(B_Y) \to H_\omega^\infty(B_X) \) is compact if and only if \( \phi(B_X) \) is relatively compact, for every \( 0 < r < 1 \) and \( \lim_{|x| \to 1} \frac{w(x)}{|\phi(x)|} = 0 \).

(b) If \( w(x) \to 0 \) as \( \|x\| \to 1^{−} \), then \( C_\phi : H_\omega^\infty(B_Y) \to H_\omega^\infty(B_X) \) is compact if and only if \( \phi(B_X) \) is relatively compact and \( \| \phi \|_\infty < 1 \).

If \( Y \) is finite dimensional, then \( \phi(B_X) \) is always relatively compact and in this case, the following corollary reduces to [32, Theorem 3.3], whenever \( X = Y = \mathbb{C} \).

**Corollary 7.7.** Let \( Y \) be a finite dimensional Banach space and \( X \) a complex Banach space. Let \( \nu, \nu \) be weights and let \( \phi : B_X \to B_Y \) be a holomorphic map. Then we have the following.

(a) If \( \| \phi \|_\infty < 1 \), then \( C_\phi : H_\omega^\infty(B_Y) \to H_\omega^\infty(B_X) \) is compact.

(b) If \( \| \phi \|_\infty = 1 \), then \( C_\phi : H_\omega^\infty(B_Y) \to H_\omega^\infty(B_X) \) is compact if and only if
\[
\lim_{r \to 1} \sup_{\|x\| > r} \frac{w(x)}{|\phi(x)|} = 0.
\]

**Remark 7.8.** Garcia et al. [53] have given examples to show that the converse of (a), (b), and (c)(i) or (c)(ii) in Theorem 7.4 does not hold in general.

In [34], Bonet et al. have further generalized Theorem 7.2 to the Weighted Banach spaces of vector-valued analytic functions besides characterizing weakly compact composition operators on vector-valued Hardy spaces, Bergman spaces, and Bloch spaces. Here, we present a vector-valued version of Theorem 7.2.

**Theorem 7.9.** Let \( \nu \) be an essential weight. Then \( C_\phi : H_\omega^\infty(D, E) \to H_\omega^\infty(D, E) \) is weakly compact if and only if the Banach space \( E \) is reflexive and
\[
\lim_{r \to 1} \sup_{|\phi(z)| > r} \frac{\nu(z)}{|\phi(z)|} = 0 \quad \text{or} \quad \| \phi \|_\infty < 1.
\]

**Remark 7.10.** Recently in 2002, Bonet and Friz [69] have extended the above characterization of weakly compact composition operators to the weighted locally convex spaces
HV(𝔩, E) of vector-valued analytic functions, where V = {νn} is an increasing sequence of strictly positive, radial, continuous, bounded weights and E is a complete, barrelled locally convex space.

8. Composition operators and homomorphisms

In this section, we present a few results which relate homomorphisms with composition operators [66]. We will begin with a characterization of all continuous linear operators on HVb(G), which are composition operators and this parallels a standard result for functional Hilbert spaces.

For each z ∈ G, the point evaluation δz defines a continuous linear functional on HVb(G). If we put Δ(G) = {δz : z ∈ G}, then Δ(G) is a subset of the continuous dual HVb(G)*.

THEOREM 8.1. Let Φ : HVb(G) → HVb(G) be a linear transformation. Then there exists φ : G → G such that Φ = Cφ if and only if the transpose mapping Φ′ from HVb(G)* into the algebraic dual HVb(G)' leaves Δ(G) invariant. In case V is a reasonable system of bounded weights on G and Φ′(Δ(G)) ⊂ Δ(G), φ is necessarily analytic and Φ = Cφ is continuous if and only if V ≤ ˜V φ.

Proof. Suppose that Φ = Cφ, for some φ : G → G. Let z ∈ G and f ∈ HVb(G). Then

\[(Φ°δ_z)(f) = (δ_z°Φ)(f) = δ_z(Φ(f)) = δ_z(Cφ(f)) = f(Φ(z)) = Φ(δ_z)(f).\]  \hspace{1cm} (8.1)

This implies that Φ°δ_z = δ_φ(·). Conversely, let us suppose that Φ′(Δ(G)) ⊂ Δ(G). For z ∈ G if we define φ(z) to be the unique element of G such that Φ°δ_z = δ_φ(z). Let f ∈ HVb(G). Then

\[Φ(f)(z) = δ_z(Φ(f)) = (δ_z°Φ)(f) = (Φ°δ_z)(f) = δ_φ(·)(f) = f(φ(z)) = Cφ(f)(z).\]  \hspace{1cm} (8.2)

Thus Φ = Cφ. Also, since the identity function f(z) = z belongs to HVb(G) and the range of Cφ is contained in H(G), φ is necessarily an analytic map. Also, in view of Corollary 6.9, Φ = Cφ is continuous when V ≤ ˜V φ. □

THEOREM 8.2. Let G be an open connected bounded subset of ℂ and let V be a system of bounded weights on G such that V ≤ V^2. Let Φ : HVb(G) → ℂ be a nonzero multiplicative linear functional. Then there exists z_0 ∈ G such that Φ = δ_z_0.

Proof. Let λ ∈ ℂ and let K_λ denote the constant function K_λ(z) = λ, for every z ∈ G. Since each weight ν ∈ V is bounded, it follows that each constant function K_λ ∈ HVb(G). Let Φ : HVb(G) → ℂ be a nonzero multiplicative linear functional. Then we have Φ(K_1) = Φ(K_1 · K_1) = Φ(K_1)Φ(K_1). That is, Φ(K_1) is equal to zero or one. In case Φ(K_1) = 0, it follows that Φ(f) = Φ(f · K_1) = Φ(f)Φ(K_1) = 0, for every f ∈ HVb(G). Thus Φ = 0, a contradiction. This shows that Φ(K_1) = 1. Further, it implies that Φ(K_1) = Φ(K_1 · K_1) = Φ(λ · K_1) = λΦ(K_1) = λ. Let f : G → ℂ be defined as f(z) = z, for every z ∈ G. Then
clearly, \( f \in HV_b(G) \). Now, we fix \( z_0 = \Phi(f) \). We will show that \( z_0 \in G \). Then we define the function \( h_{z_0} : G \to \mathbb{C} \) as \( h_{z_0}(z) = 1/(z - z_0) \), for every \( z \in G \). Again, since each weight \( v \in G \) is bounded and \( G \) is a bounded domain, it follows that \( h_{z_0} \in HV_b(G) \). Also, from the definition of \( h_{z_0} \), we have \((z - z_0)h_{z_0}(z) = 1\), for every \( z \in G \). That is, \((f(z) - Kz)(z)h_{z_0}(z) = 1\), for every \( z \in G \). Thus \((f - Kz)h_{z_0} = K_1\). Further, it implies that \( \Phi(K_1) = \Phi(f - Kz)\Phi(h_{z_0}) = (\Phi(f) - \Phi(Kz))\Phi(h_{z_0}) = (z - z_0)\Phi(h_{z_0}) = 0\), which is a contradiction because \( \Phi(K_1) = 1\). This proves that \( z_0 \in G \). Now, let \( g \in HV_b(G) \). Then we define the function \( h : G \to \mathbb{C} \) as

\[
h(z) = h_{z_0}(z)(g(z) - g(z_0)), \quad \text{for } z \neq z_0, \quad h(z) = g'(z_0), \quad \text{for } z = z_0.
\] (8.3)

It can be easily seen that \( h \in HV_b(G) \). Now, it readily follows that \((f - Kz_0)h = g - Kg(z_0)\). Further, we have \( \Phi((f - Kz_0)h) = \Phi(g - Kg(z_0)) \). That is, \( 0 = \Phi(g) - \Phi(Kg(z_0)) \). Thus it follows that \( \Phi(g) = g(z_0) = \delta_{z_0}(g) \). This proves that \( \Phi = \delta_{z_0} \). With this, the proof of the theorem is complete.

**Theorem 8.3.** Let \( G_1 \) and \( G_2 \) be open connected bounded subsets of \( \mathbb{C} \). Let \( V \) and \( U \) be systems of bounded weights on \( G_1 \) and \( G_2 \), respectively, such that \( V \leq V^2 \) and \( U \leq U^2 \). Let \( \Phi : HV_b(G_1) \to HV_b(G_2) \) be a nonzero algebra homomorphism. Then there exists a holomorphic map \( \phi : G_2 \to G_1 \) such that \( \Phi = C_\phi \).

**Proof.** Since \( HV_b(G_1) \) and \( HV_b(G_2) \) contains constant functions, it follows that \( K_1 \in HV_b(G_1) \) and \( \Phi(K_1) = \Phi(K_1) \cdot \Phi(K_1) \). Then using connectedness of \( G_2 \), we can conclude that \( \Phi(K_1) = K_1 \). Further, it implies that \( \Phi(K_2) = K_2 \), for every \( \lambda \in \mathbb{C} \). Now, let \( z_0 \in G_2 \). Then define \( \delta_{z_0} : HV_b(G_1) \to \mathbb{C} \) as \( \delta_{z_0}(f) = (\Phi f)(z_0) \). Clearly, \( \delta_{z_0} \) is a multiplicative linear functional on \( HV_b(G_1) \). Hence by Theorem 8.2, there exists \( \alpha \in G_1 \) such that \( \delta_{z_0}(f) = \delta_{z_0}(f) = f(\alpha) \), for every \( f \in HV_b(G_1) \). Let \( g : G_1 \to \mathbb{C} \) defined as \( g(z) = z \), for every \( z \in G_1 \). Then clearly, \( g \in HV_b(G_1) \) and \( \delta_{z_0}(g) = g(\alpha) \). Thus it follows that \( (\Phi g)(z_0) = \alpha \). Let us define \( \phi = \Phi(g) \). Thus \( \phi : G_2 \to G_1 \) is an analytic map such that \( (\Phi f)(z_0) = f(\alpha) = (\Phi g)(z_0) = (f \phi)(z_0) \), \( z_0 \in G_2 \). This shows that \( \Phi(f) = C_\phi(f) \), for every \( f \in HV_b(G_1) \). Hence \( \Phi = C_\phi \). With this, the proof of the theorem is complete.

**Theorem 8.4.** Let \( G \) be an open connected bounded subset of \( \mathbb{C} \) and let \( V \) be a system of bounded weights on \( G \) such that \( V \leq V^2 \). Then a composition transformation \( C_\phi \) on \( HV_b(G) \) is invertible if and only if \( \phi : G \to G \) is a conformal mapping.

**Proof.** If \( \phi \) is a conformal mapping, then obviously, \( C_\phi \) is invertible on \( HV_b(G) \). On the other hand, suppose \( A \) is the inverse of \( C_\phi \). Then we have \( AC_\phi = C_\phi A = I \). For \( f \) and \( g \) in \( HV_b(G) \), we have \( C_\phi A(fg) = fg \). Further, it implies that \( A(fg)\phi = fg = (C_\phi A)f \cdot (C_\phi A)g = (Af)\phi \cdot (Ag)\phi = (Af \cdot Ag)\phi \). That is, \( (A(fg) - Af \cdot Ag)\phi = 0 \). Since \( C_\phi \) is invertible, \( \phi \) is nonconstant and hence the range of \( \phi \) is an open set. Thus it follows that \( A(fg) = Af \cdot Ag \). According to Theorem 8.3, there exists an analytic map \( \psi : G \to G \) such that \( A = C_\psi \). Let \( f(z) = z \), for every \( z \in G \). Then \( f \in HV_b(G) \) and we have \( (C_\psi C_\phi)(z) = (f \phi)(z) = (\phi)(z) \), for every \( z \in G \). Also, \( (C_\phi C_\psi)(z) = (f \psi \phi)(z) = (\psi \phi)(z) \), for every \( z \in G \). From this, we conclude that \( \phi \) is invertible with an analytic inverse map as \( \psi \). Hence \( \phi \) is a conformal mapping of \( G \) onto itself.
References

[1] J. C. Evard and F. Jafari, “On semigroups of operators on Hardy spaces,” preprint, 1995.
[2] A. G. Siskakis, “Weighted composition semigroups on Hardy spaces,” Linear Algebra and Its Applications, vol. 84, pp. 359–371, 1986.
[3] A. G. Siskakis, “Semigroups of composition operators on spaces of analytic functions, a review,” in Studies on Composition Operators (Laramie, WY, 1996), vol. 213 of Contemporary Mathematics, pp. 229–252, American Mathematical Society, Providence, RI, USA, 1998.
[4] K. de Leeuw, W. Rudin, and J. Wermer, “The isometries of some function spaces,” Proceedings of the American Mathematical Society, vol. 11, no. 5, pp. 694–698, 1960.
[5] M. Nagasawa, “Isomorphisms between commutative Banach algebras with an application to rings of analytic functions,” Kodai Mathematical Seminar Reports, vol. 11, pp. 182–188, 1959.
[6] F. Forelli, “The isometries of $H^p$,” Canadian Journal of Mathematics, vol. 16, pp. 721–728, 1964.
[7] M. Cambern and K. Jarosz, “The isometries of $H^p_k$,” Proceedings of the American Mathematical Society, vol. 107, no. 1, pp. 205–214, 1989.
[8] C. J. Kolaski, “Surjective isometries of weighted Bergman spaces,” Proceedings of the American Mathematical Society, vol. 105, no. 3, pp. 652–657, 1989.
[9] T. Mazur, “Canonical isometry on weighted Bergman spaces,” Pacific Journal of Mathematics, vol. 136, no. 2, pp. 303–310, 1989.
[10] P.-K. Lin, “The isometries of $H^p(E)$,” Pacific Journal of Mathematics, vol. 143, no. 1, pp. 69–77, 1990.
[11] W. Arveson, “Subalgebras of $C^*$-algebras—III: multivariable operator theory,” Acta Mathematica, vol. 181, no. 2, pp. 159–228, 1998.
[12] K. R. M. Attele, “Analytic multipliers of Bergman spaces,” The Michigan Mathematical Journal, vol. 31, no. 3, pp. 307–319, 1984.
[13] S. Axler, “Multiplication operators on Bergman spaces,” Journal für die reine und angewandte Mathematik, vol. 336, pp. 26–44, 1982.
[14] S. Axler, “Zero multipliers of Bergman spaces,” Canadian Mathematical Bulletin, vol. 28, no. 2, pp. 237–242, 1985.
[15] S. Axler and A. L. Shields, “Univalent multipliers of the Dirichlet space,” The Michigan Mathematical Journal, vol. 32, no. 1, pp. 65–80, 1985.
[16] H. Bercovici, “The algebra of multiplication operators on Bergman spaces,” Archiv der Mathematik, vol. 48, no. 2, pp. 165–174, 1987.
[17] J. Eschmeier, “Multiplication operators on Bergman spaces are reflexive,” in Linear Operators in Function Spaces (Timișoara, 1988), H. H. Helson, B. Sz-Nagy, F.-H. Vasilescu, and G. Arsene, Eds., vol. 43 of Operator Theory Adv. Appl., pp. 165–184, Birkhäuser, Basel, Switzerland, 1990.
[18] D. H. Luecking, “Multipliers of Bergman spaces into Lebesgue spaces,” Proceedings of the Edinburgh Mathematical Society, vol. 29, no. 1, pp. 125–131, 1986.
[19] D. Vukotić, “Pointwise multiplication operators between Bergman spaces on simply connected domains,” Indiana University Mathematics Journal, vol. 48, no. 3, pp. 793–803, 1999.
[20] K. Zhu, “A trace formula for multiplication operators on invariant subspaces of the Bergman space,” Integral Equations and Operator Theory, vol. 40, no. 2, pp. 244–255, 2001.
[21] D. M. Campbell and R. J. Leach, “A survey of $H^p$ multipliers as related to classical function theory,” Complex Variables, vol. 3, no. 1–3, pp. 85–111, 1984.
[22] N. S. Feldman, “Pointwise multipliers from the Hardy space to the Bergman space,” Illinois Journal of Mathematics, vol. 43, no. 2, pp. 211–221, 1999.
[24] S. Ohno and H. Takagi, “Some properties of weighted composition operators on algebras of analytic functions,” *Journal of Nonlinear and Convex Analysis*, vol. 2, no. 3, pp. 369–380, 2001.

[25] J. Arazy, *Multipliers of Bloch Functions*, vol. 54, University of Haifa Mathematics, Haifa, Israel, 1982.

[26] L. Brown and A. L. Shields, “Multipliers and cyclic vectors in the Bloch space,” *The Michigan Mathematical Journal*, vol. 38, no. 1, pp. 141–146, 1991.

[27] D. A. Stegenga, “Multipliers of the Dirichlet space,” *Illinois Journal of Mathematics*, vol. 24, no. 1, pp. 113–139, 1980.

[28] J. M. Ortega and J. Fábrega, “Pointwise multipliers and corona type decomposition in BMOA,” *Annales de l’Institut Fourier*, vol. 46, no. 1, pp. 111–137, 1996.

[29] H. Jarchow, V. Montesinos, K. J. Wirths, and J. Xiao, “Duality for some large spaces of analytic functions,” *Proceedings of the Edinburgh Mathematical Society*, vol. 44, no. 3, pp. 571–583, 2001.

[30] N. Yanagihara, “Multipliers and linear functionals for the class $N^+$,” *Transactions of the American Mathematical Society*, vol. 180, pp. 449–461, 1973.

[31] J. Bonet, P. Domaniński, and M. Lindström, “Pointwise multiplication operators on weighted Banach spaces of analytic functions,” *Studia Mathematica*, vol. 137, no. 2, pp. 177–194, 1999.

[32] J. Bonet, P. Domaniński, M. Lindström, and J. Taskinen, “Composition operators between weighted Banach spaces of analytic functions,” *Journal of Australian Mathematical Society*, vol. 64, no. 1, pp. 101–118, 1998.

[33] J. Bonet, P. Domaniński, and M. Lindström, “Essential norm and weak compactness of composition operators on weighted Banach spaces of analytic functions,” *Canadian Mathematical Bulletin*, vol. 42, no. 2, pp. 139–148, 1999.

[34] J. Bonet, P. Domaniński, and M. Lindström, “Weakly compact composition operators on analytic vector-valued function spaces,” *Annales Academiae Scientiarum Fennicae. Mathematica*, vol. 26, no. 1, pp. 233–248, 2001.

[35] M. D. Contreras and A. G. Hernández-Díaz, “Weighted composition operators in weighted Banach spaces of analytic functions,” *Journal of Australian Mathematical Society*, vol. 69, no. 1, pp. 41–60, 2000.

[36] A. L. Shields and D. L. Williams, “Bonded projections, duality, and multipliers in spaces of analytic functions,” *Transactions of the American Mathematical Society*, vol. 162, pp. 287–302, 1971.

[37] A. L. Shields and D. L. Williams, “Bounded projections, duality, and multipliers in spaces of harmonic functions,” *Journal für die reine und angewandte Mathematik*, vol. 299/300, pp. 256–279, 1978.

[38] M. D. Contreras and A. G. Hernández-Díaz, “Weighted composition operators on Hardy spaces,” *Journal of Mathematical Analysis and Applications*, vol. 263, no. 1, pp. 224–233, 2001.

[39] G. Mirzakarimi and K. Seddighi, “Weighted composition operators on Bergman and Dirichlet spaces,” *Georgian Mathematical Journal*, vol. 4, no. 4, pp. 373–383, 1997.

[40] B. D. MacCluer and R. Zhao, “Essential norms of weighted composition operators between Bloch-type spaces,” *The Rocky Mountain Journal of Mathematics*, vol. 33, no. 4, pp. 1437–1458, 2003.

[41] S. Ohno, “Weighted composition operators between $H^\infty$ and the Bloch space,” *Taiwanese Journal of Mathematics*, vol. 5, no. 3, pp. 555–563, 2001.

[42] S. Ohno and R. Zhao, “Weighted composition operators on the Bloch space,” *Bulletin of the Australian Mathematical Society*, vol. 63, no. 2, pp. 177–185, 2001.

[43] S. Ohno, K. Stroethoff, and R. Zhao, “Weighted composition operators between Bloch-type spaces,” *The Rocky Mountain Journal of Mathematics*, vol. 33, no. 1, pp. 191–215, 2003.

[44] A. Montes-Rodríguez, “Weighted composition operators on weighted Banach spaces of analytic functions,” *Journal of the London Mathematical Society*, vol. 61, no. 3, pp. 872–884, 2000.
[45] F. Jafari, T. Tonev, E. Toneva, and K. Yale, “Holomorphic flows, cocycles, and coboundaries,” The Michigan Mathematical Journal, vol. 44, no. 2, pp. 239–253, 1997.

[46] C. C. Cowen and B. D. MacCluer, Composition Operators on Spaces of Analytic Functions, Studies in Advanced Mathematics, CRC Press, Boca Raton, Fla, USA, 1995.

[47] J. H. Shapiro, Composition Operators and Classical Function Theory, Universitext: Tracts in Mathematics, Springer, New York, NY, USA, 1993.

[48] R. K. Singh and J. S. Manhas, Composition Operators on Function Spaces, vol. 179 of North-Holland Mathematics Studies, North-Holland, Amsterdam, The Netherlands, 1993.

[49] K. D. Bierstedt, J. Bonet, and A. Galbis, “Weighted spaces of holomorphic functions on balanced domains,” The Michigan Mathematical Journal, vol. 40, no. 2, pp. 271–297, 1993.

[50] K. D. Bierstedt, J. Bonet, and J. Taskinen, “Associated weights and spaces of holomorphic functions,” Studia Mathematica, vol. 127, no. 2, pp. 137–168, 1998.

[51] K. D. Bierstedt and W. H. Summers, “Biduals of weighted Banach spaces of analytic functions,” Journal of Australian Mathematical Society, vol. 54, no. 1, pp. 70–79, 1993.

[52] A. Galbis, “Weighted Banach spaces of entire functions,” Archiv der Mathematik, vol. 62, no. 1, pp. 58–64, 1994.

[53] D. García, M. Maestre, and P. Sevilla-Peris, “Composition operators between weighted spaces of holomorphic functions on Banach spaces,” Annales Academiae Scientiarum Fennicae Mathematica, vol. 29, no. 1, pp. 81–98, 2004.

[54] W. Lusky, “On weighted spaces of harmonic and holomorphic functions,” Journal of the London Mathematical Society, vol. 51, no. 2, pp. 309–320, 1995.

[55] J. Burbea and P. Masani, Banach and Hilbert Spaces of Vector-Valued Functions, vol. 90 of Research Notes in Mathematics, Pitman, Boston, Mass, USA, 1984.

[56] J. B. Garnett, Bounded Analytic Functions, vol. 96 of Pure and Applied Mathematics, Academic Press, New York, NY, USA, 1981.

[57] A. Grothendieck, Topological Vector Spaces, Gordon and Breach Science, New York, NY, USA, 1992.

[58] W. Rudin, Functional Analysis, McGraw-Hill Series in Higher Mathematics, McGraw-Hill, New York, NY, USA, 1973.

[59] L. A. Rubel and A. L. Shields, “The second duals of certain spaces of analytic functions,” Journal of Australian Mathematical Society, vol. 11, pp. 276–280, 1970.

[60] J. S. Manhas, “Multiplication operators on weighted locally convex spaces of vector-valued analytic functions,” Southeast Asian Bulletin of Mathematics, vol. 27, no. 4, pp. 649–660, 2003.

[61] K. Cichoń and K. Seip, “Weighted holomorphic spaces with trivial closed range multiplication operators,” Proceedings of the American Mathematical Society, vol. 131, no. 1, pp. 201–207, 2003.

[62] R. K. Singh and J. S. Manhas, “Invertible composition operators on weighted function spaces,” Acta Scientiarum Mathematicarum, vol. 59, no. 3–4, pp. 489–501, 1994.

[63] A. Pietsch, Operator Ideals, vol. 16 of Mathematical Monographs, VEB Deutscher Verlag der Wissenschaften, Berlin, Germany, 1978.

[64] P. Rueda, “On the Banach-Dieudonné theorem for spaces of holomorphic functions,” Quaestiones Mathematicae, vol. 19, no. 1-2, pp. 341–352, 1996.

[65] D. García, M. Maestre, and P. Rueda, “Weighted spaces of holomorphic functions on Banach spaces,” Studia Mathematica, vol. 138, no. 1, pp. 1–24, 2000.

[66] J. S. Manhas, “Homomorphisms and composition operators on weighted spaces of analytic functions,” preprint, 2006.

[67] R. Aron, P. Galindo, and M. Lindström, “Compact homomorphisms between algebras of analytic functions,” Studia Mathematica, vol. 123, no. 3, pp. 235–247, 1997.
[68] P. Galindo, M. Lindström, and R. Ryan, “Weakly compact composition operators between algebras of bounded analytic functions,” *Proceedings of the American Mathematical Society*, vol. 128, no. 1, pp. 149–155, 2000.

[69] J. Bonet and M. Friz, “Weakly compact composition operators on locally convex spaces,” *Mathematische Nachrichten*, vol. 245, no. 1, pp. 26–44, 2002.

J. S. Manhas: Department of Mathematics and Statistics, College of Science, Sultan Qaboos University, P.O. Box 36, Muscat 123, Oman

*Email address*: manhas@squ.edu.om
Special Issue on
Time-Dependent Billiards

Call for Papers

This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

To be acceptable for publication in the special issue of Mathematical Problems in Engineering, papers must make significant, original, and correct contributions to one or more of the topics above mentioned. Mathematical papers regarding the topics above are also welcome.

Authors should follow the Mathematical Problems in Engineering manuscript format described at http://www.hindawi.com/journals/mpe/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

| Period               | Date       |
|----------------------|------------|
| Manuscript Due       | March 1, 2009 |
| First Round of Reviews | June 1, 2009 |
| Publication Date     | September 1, 2009 |

Guest Editors

Edson Denis Leonel, Department of Statistics, Applied Mathematics and Computing, Institute of Geosciences and Exact Sciences, State University of São Paulo at Rio Claro, Avenida 24A, 1515 Bela Vista, 13506-700 Rio Claro, SP, Brazil; edleonel@rc.unesp.br

Alexander Loskutov, Physics Faculty, Moscow State University, Vorob’evy Gory, Moscow 119992, Russia; loskutov@chaos.phys.msu.ru