Abstract    We deal with the problem of estimating the volume of inclusions using a small number of boundary measurements in electrical impedance tomography. We derive upper and lower bounds on the volume fractions of inclusions, or more generally two phase mixtures, using two boundary measurements in two dimensions. These bounds are optimal in the sense that they are attained by certain configurations with some boundary data. We derive the bounds using the translation method which uses classical variational principles with a null Lagrangian. We then obtain necessary conditions for the bounds to be attained and prove that these bounds are attained by inclusions inside which the field is uniform. When special boundary conditions are imposed the bounds reduce to those obtained by Milton and these in turn are shown here to reduce to those of Capdeboscq–Vogelius in the limit when the volume fraction tends to zero. The bounds of this article, and those of Milton, work for inclusions of arbitrary volume fractions. We then perform some numerical experiments to demonstrate how good these bounds are.

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1 Introduction

One of the central problems of the theory and practice of electrical impedance tomography is the problem of estimating the volume of the inclusions in terms of boundary measurements, either voltage measurements when currents are applied around the boundary of the body or current measurements when voltages are applied. The problem can be described in rigorous terms as follows: Let $D$ be an inclusion inside a body $\Omega_1$, and suppose that the conductivities of $D$ and $\Omega_1 \setminus D$ are $\sigma_1$ and $\sigma_2$ ($\sigma_1 \neq \sigma_2$), respectively. Let $\sigma = \sigma_1 \chi(D) + \sigma_2 \chi(\Omega_1 \setminus D)$ where $\chi(D)$ is the characteristic function of $D$ and the potential $V$ be the solution to

$$\begin{cases} \nabla \cdot \sigma \nabla V = 0 & \text{in } \Omega, \\ V = V^0 & \text{on } \partial \Omega \end{cases}$$

(1.1)

for some Dirichlet data (voltage) $V^0$ on $\partial \Omega$. Then the measurement of current (the Neumann data) is $q := \sigma \partial V / \partial n$ on $\partial \Omega$. (Throughout this article $\partial / \partial n$ denotes the normal derivative.) The problem is to estimate the volume $|D|$ of the inclusion using the boundary data $(V^0, q)$ for a small number of voltages, say $V^0 = V^0_1, \ldots, V^0_n$. If the Neumann boundary condition $\sigma \partial V / \partial n = q$ is prescribed on $\partial \Omega$ instead of the Dirichlet condition, then the measurement is $V^0 := V|_{\partial \Omega}$.

The purpose of this article is to consider this problem and derive optimal upper and lower bounds for the volume fraction of inclusions in two dimensions. In fact, we deal with a more general situation where $\Omega_1$ is a two phase mixture in which the phase 1 has conductivity $\sigma_1$ and the phase 2 has conductivity $\sigma_2$ ($\sigma_1 > \sigma_2$) so that the conductivity distribution $\sigma$ of $\Omega_1$ is given by $\sigma(x) = \sigma_1 \chi_1(x) + \sigma_2 \chi_2(x)$ where $\chi_j$ is the characteristic function of phase $j$ for $j = 1, 2$, i.e.,

$$\chi_1(x) = 1 - \chi_2(x) = \begin{cases} 1 & \text{in phase 1,} \\ 0 & \text{in phase 2.} \end{cases}$$

(1.2)

We derive optimal upper and lower bounds for the volume fraction $f_1$ of phase 1 ($f_1 = \frac{1}{|\Omega_1|} \int_\Omega \chi_1(x)$) using boundary measurements corresponding to either a pair of Dirichlet data $(V^0_1$ and $V^0_2$) or a pair of Neumann data $(q_1$ and $q_2)$ on $\Omega$. The bounds are optimal in the sense that they are attained by some inclusions or configurations. The bounds can be easily computed from the boundary measurements. In fact, they are given by two quantities: the measurement (or response) matrix $A = (a_{ij})_{i,j=1,2}$ where

$$a_{ij} := \frac{1}{|\Omega_1|} \int_{\partial \Omega} V^0_i q_j$$

(1.3)

and

$$b_D := \frac{1}{|\Omega_1|} \int_{\partial \Omega} V^0_1 \frac{\partial V^0_2}{\partial t}$$

(1.4)

if the Dirichlet data are used. Here and throughout this article, $\frac{\partial}{\partial t}$ denotes the tangential derivative along $\partial \Omega$ in the positive orientation. If the Neumann data are used, then $b_D$ is replaced with

$$b_N := \frac{1}{|\Omega_1|} \int_{\partial \Omega} q_1(x) \left( \int_{x_0}^x q_2 \right)$$

(1.5)

where the $x_0 \in \partial \Omega$ and the last integral is on the surface $\partial \Omega$. See Theorems 2.1 and 2.2.