SURGERY DESCRIPTIONS AND VOLUMES OF BERGE KNOTS
II: DESCRIPTIONS ON THE MINIMALLY TWISTED FIVE
CHAIN LINK

KENNETH L. BAKER

ABSTRACT. Using Kirby Calculus, we explicitly pass from Berge’s R-R descriptions of ten families of knots with lens space surgeries to surgery descriptions on the minimally twisted five chain link (MT5C). Since the MT5C admits a strong involution, we also give the corresponding tangle descriptions.

1. Introduction

In [2] Berge describes twelve families of knots that admit lens space surgeries. These knots are referred to as Berge knots and appear to comprise all knots in $S^3$ known to have lens space surgeries. Ten of these families, (I)-(VI) being the knots in solid tori with surgeries yielding solid tori and (IX)-(XII) being the ‘sporadic’ knots, are described via R-R diagrams.

**Theorem.** The knots in Berge’s families (I)-(VI) and (IX)-(XII) admit surgery descriptions on the minimally twisted five chain link.

Figure 1. The minimally twisted five chain link, MT5C, with an axis of strong involution.

Figure 1 shows the minimally twisted five chain link, MT5C for short. The proof of the theorem is given in [2]. We use Kirby Calculus (see e.g. [3] or [4]) to pass from these R-R diagrams to surgery descriptions on the MT5C. Since the MT5C is strongly invertible, we give the corresponding tangle descriptions of these surgeries in [3].

2000 Mathematics Subject Classification. Primary 57M50, Secondary 57M25.

Key words and phrases. Berge knots, lens space, surgery description, chain link.

This work was partially supported by a graduate traineeship from the VIGRE Award at the University of Texas at Austin and a VIGRE postdoc under NSF grant number DMS-0089927 to the University of Georgia at Athens.
As an immediate consequence of Thurston’s Hyperbolic Dehn Surgery Theorem [4] and the fact that the MT5C is a hyperbolic link (e.g. Theorem 5.1 (ii) of [4]), our theorem has the following corollary.

**Corollary.** Volumes of hyperbolic knots in Berge’s families (I)-(VI) and (IX)-(XII) are bounded above by the volume of the MT5C.

Families (VII) and (VIII) are the knots which lie as essential simple closed curves on the fiber of the trefoil and figure eight knot respectively. In the prequel [1] we show that they contain hyperbolic knots of arbitrarily large volume. Consequentially, they cannot all be all be described by surgery on the MT5C. Nevertheless, each of them admits a surgery description on some minimally twisted chain link.

1.1. **Acknowledgements.** The author wishes to thank both John Luecke for his direction and many useful conversations and Yuichi Yamada for his comments.

2. **Proof of Theorem**

We prove the theorem by exhibiting a passage from Berge’s R-R diagrams to surgery descriptions on the MT5C, or its reflection. Please refer to [2] for Berge’s original descriptions of these knots and the notation conventions. The first step in this passage translates the R-R diagram to a knot on a genus 2 Heegaard surface for $S^3$ together with surgery instructions on a surrounding link. Half of the correspondence is shown in Figure 2. The other half is obtained by rotating the pictures $180^\circ$ in the plane of the page.

![Figure 2. Passing from a R-R diagrams to a surgery description.](image)

The first six families of Berge knots arise from knots in solid tori with surgeries yielding solid tori. Where relevant we take $n, p, q, r, s, K \in \mathbb{Z}, |ps - qr| = 1$ and $\epsilon = \pm 1$.

2.1. **Type (I), torus knots.** We pass from Berge’s diagram (Figure 3) to a corresponding realization of these knots $k$ on a genus 2 Heegaard surface via surgeries (Figure 4). In Figure 4 $x = \frac{1}{6}$ corresponds to the meridional filling of $k$ while $x = -\frac{1}{6}$ corresponds to the lens space surgery of $k$.

After dropping the Heegaard surface from the picture, we add components with meridional framings, perform isotopies, and do Kirby Calculus as shown in Figure 5 to get a description of $k$ as surgery on the MT5C.
2.2. Type (II), cables about torus knots. We pass from Berge’s diagram (Figure 6) to a corresponding realization of these knots \( k \) on a genus 2 Heegaard surface via surgeries (Figure 7). In Figure 7, \( \frac{\bar{x}}{y} = \frac{1}{0} \) corresponds to the meridional filling of \( k \) while \( \frac{\bar{x}}{y} = -\frac{1}{1} \) corresponds to the lens space surgery of \( k \).

After dropping the Heegaard surface from the picture, we add components with meridional framings, perform isotopies, and do Kirby Calculus as shown in Figure 8 to get a description of \( k \) as surgery on the MT5C. Notice that the surgery of \( +\frac{1}{1} \) on the component corresponding to \( k \) is the lens space surgery. The trivial (i.e. \( S^3 \)) surgery is still \( \frac{1}{0} \).

2.3. Type (III). We pass from Berge’s diagram (Figure 9) to a corresponding realization of these knots \( k \) on a genus 2 Heegaard surface via surgeries (Figure 10). In Figure 10, \( \frac{\bar{x}}{y} = \frac{1}{0} \) corresponds to the meridional filling of \( k \) while \( \frac{\bar{x}}{y} = -\frac{1}{1} \) corresponds to the lens space surgery of \( k \). Also, \( \frac{\bar{x}}{s} = \frac{2\pi + (2p + r)K}{r + pK} \).
We drop the Heegaard surface to get the first link of Figure 11. The surgeries on the two parallel components in the first link of Figure 11 can be amalgamated into a surgery on a single component. From here we perform isotopies and Kirby Calculus to arrive at the MT5C. Notice that the surgery of $+\frac{1}{2}$ on the component corresponding to $k$ on the MT5C is the lens space surgery. The trivial (i.e. $S^3$) surgery is still $\frac{1}{0}$. 

Figure 6. Berge's R-R diagram for type (II) knots (cables about torus knots).

Figure 7. Type (II) knots on Heegaard surface via surgeries.

Figure 8. Type (II) knots. Isotopies and Kirby Calculus moves ending in the MT5C.
2.4. **Type (IV).** We pass from Berge’s diagram (Figure 12) to a corresponding realization of these knots $k$ on a genus 2 Heegaard surface via surgeries (Figure 13). In Figure 13 $\frac{x}{y} = \frac{5}{6}$ corresponds to the meridional filling of $k$ while $\frac{x}{y} = -\frac{1}{2}$ corresponds to the lens space surgery of $k$. Also, $\frac{r}{s} = \frac{2r+e}{r+e+pK}$. 

![Figure 9. Berge’s R-R diagram for type (III) knots.](image)

![Figure 10. Type (III) knots on Heegaard surface via surgeries.](image)

![Figure 11. Type (III) knots. Isotopies and Kirby Calculus moves ending in the MT5C.](image)
After dropping the Heegaard surface from the picture, we add components with meridional framings, perform isotopies, and do Kirby Calculus as shown in Figure 14 to get a description of $k$ as surgery on the MT5C. Notice that the surgery of $-\frac{1}{n}$ on the component corresponding to $k$ is the lens space surgery. The trivial (i.e. $S^3$) surgery is still $\frac{1}{n}$.

Figure 12. Berge’s R-R diagram for type (IV) knots.

Figure 13. Type (IV) knots on Heegaard surface via surgeries.

Figure 14. Type (IV) knots. Isotopies and Kirby Calculus moves ending in the MT5C.
2.5. **Type (V).** We pass from Berge’s diagram (Figure 15) to a corresponding realization of these knots $k$ on a genus 2 Heegaard surface via surgeries (Figure 16). In Figure 16 $x = \frac{1}{6}$ corresponds to the meridional filling of $k$ while $x = -\frac{5}{1}$ corresponds to the lens space surgery of $k$. Also, $x = \frac{2+2p+\epsilon}{\epsilon+pK}$.

![Figure 15. Berge’s R-R diagram for type (V) knots.](image)

After dropping the Heegaard surface from the picture, we amalgamate two parallel components and do Kirby Calculus as shown in Figure 17 to get a description of $k$ as surgery on the MT5C. Notice that the surgery of $+\frac{1}{1}$ on the component corresponding to $k$ is the lens space surgery. The trivial (i.e. $S^3$) surgery is still $\frac{1}{6}$.

2.6. **Type (VI).** We pass from Berge’s diagram (Figure 18) to a corresponding realization of these knots $k$ on a genus 2 Heegaard surface via surgeries (Figure 19). In Figure 19 $x = \frac{1}{6}$ corresponds to the meridional filling of $k$ while $x = -\frac{5}{1}$ corresponds to the lens space surgery of $k$.

After dropping the Heegaard surface from the picture, we amalgamate the surgeries on two parallel components in the first step of Figure 20. We then continue with Kirby Calculus to get a description of $k$ as surgery on the MT5C. Notice that the surgery of $-\frac{1}{1}$ on the component corresponding to $k$ is the lens space surgery. The trivial (i.e. $S^3$) surgery is still $\frac{1}{6}$.

![Figure 16. Type (V) knots on Heegaard surface via surgeries.](image)
2.7. Sporadic knots type a) and b). We pass from Berge’s diagram (Figure 21) to a corresponding realization of these knots $k$ on a genus 2 Heegaard surface via surgeries (Figure 22). In Figure 22 $\frac{x}{y} = \frac{1}{0}$ corresponds to the meridional filling of $k$ while $\frac{x}{y} = \frac{1}{n}$ corresponds to the lens space surgery of $k$. Here $n \in \mathbb{Z}$. For type a) knots $(p, p', m, m') = (1, 1, 2, 3)$. For type b) knots $(p, p', m, m') = (2, 1, 3, 2)$.
As shown in Figure 23, one component bounds a Möbius band. The framing on that component induced by the Möbius band agrees with the framing from the surface.

In Figure 24 we replace the component that bounds the Möbius band with the core curve of the Möbius band and the corresponding surgery instructions. We continue, after an isotopy, with Kirby Calculus in Figure 25 to get a description of $k$ as surgery on the MT5C.
2.8. **Sporadic knots type c) and d).** We pass from Berge's diagram (Figure 26) to a corresponding realization of these knots $k$ on a genus 2 Heegaard surface via surgeries (Figure 27). In Figure 27, $\frac{2}{p} = 1$ corresponds to the meridional filling of $k$ while $\frac{2}{p} = +\frac{1}{1}$ corresponds to the lens space surgery of $k$. Here $n \in \mathbb{Z}$. For type c) knots $(p, p', m, m') = (4, -3, 3, -2)$. For type d) knots $(p, p', m, m') = (3, -5, 2, -3)$.

As shown in Figure 28, one component bounds a Möbius band. The framing on that component induced by the Möbius band agrees with the framing from the surface.
Figure 25. Sporadic types a) and b). Isotopies and Kirby Calculus moves ending in the MT5C.

Figure 26. Berge’s sporadic knot types c) and d) diagram.

Figure 27. Sporadic knot types c) and d) on Heegaard surface via surgeries.

In Figure 26, we replace the component that bounds the Möbius band with the core curve of the Möbius band and the corresponding surgery instructions. We continue, after an isotopy, with Kirby Calculus in Figure 27 to get a description of $k$ as surgery on the MT5C.
2.9. **Summary of Berge knots as surgeries on the MT5C.** We conclude the proof of the theorem by summarizing the results above. Figure 31 shows Berge knot types (I) — (VI) as surgeries on the minimally twisted five chain link or its mirror. Figure 32 shows Berge sporadic knot types a) — d) as surgeries on the the minimally twisted five chain link or its mirror. The component corresponding to the Berge knot is shown with a pair of surgeries $(\rho_{S^3}, \rho_{\text{LensSp}})$ where $\rho_{S^3}$ is the surgery slope yielding $S^3$ and $\rho_{\text{LensSp}}$ is the surgery slope yielding the lens space. Recall also that $n, p, q, r, s, \in \mathbb{Z}$, $|ps - qr| = 1$ and $\epsilon = \pm 1$. □
3. Tangles

Because the MT5C is strongly invertible, we may rephrase the minimally twisted chain link surgery descriptions of Berge’s families of knots as tangle descriptions. We include these tangle descriptions as they may be of interest to others and provide an alternate means of verification of the above surgery descriptions.

On the remaining boundary component of each of the the tangles below, a choice of rational tangles to be inserted is indicated. Inserting the first rational tangle (which is the $\infty$ tangle in each case) renders the tangle into the unknot. As the double branched cover of the unknot in $S^3$ is $S^3$ again, this corresponds to the trivial surgery on the Berge knot. Inserting the second rational tangle yields a two-bridge link, a link that may be decomposed into two rational tangles. As the double branched cover of a two-bridge knot is a lens space, this corresponds to the lens space surgery on the Berge knot.

3.1. Conventions. A tangle $(B, t)$ is a pair consisting of a punctured 3-sphere $B$ and a properly embedded collection $t$ of disjoint arcs and simple closed curves. Two tangles $(B_1, t_1)$ and $(B_2, t_2)$ are homeomorphic if there is a homeomorphism of pairs $h: (B_1, t_1) \rightarrow (B_2, t_2)$. 

![Figure 31. Berge knot types (I)–(VI)](image)

![Figure 32. Sporadic Berge knot types a)–d)](image)
Figure 33. The correspondence of bases and framings.

A boundary component \((\partial B_0, t \cap \partial B_0)\) of a tangle \((B, t)\) is a sphere \(\partial B_0\) together with some finite set of points \(p = t \cap \partial B_0\). Here we consider the situation where \(p\) consists of just four points.

Given a sphere \(S\) and set \(p\) of four distinct points on \(S\), a framing of \((S, p)\) is an ordered pair of (unoriented) simple closed curves \((\hat{m}, \hat{l})\) on \(S - N(p)\) such that each curve separates different pairs of points of \(p\).

Let \((S, p)\) be a sphere with four points with framing \((\hat{m}, \hat{l})\). The double cover of \(S\) branched over \(p\) is a torus. Single components, say \(m\) and \(l\), of the lifts of the framing curves \(\hat{m}\) and \(\hat{l}\) when oriented so that \(m \cdot l = +1\) (with respect to the orientation of the torus) form a basis for the torus. Similarly, a basis on a torus induces a framing on the sphere with four points obtained by the quotient of an involution that fixes four points on the torus. See Figure 33.

This implies the correspondence between inserting into a boundary component of a tangle the rational tangle \(p/q\) and the \(p/q\) Dehn filling of a torus boundary component in the double branched cover of the tangle.

3.2. Tangle descriptions. Please refer to [1] for an illustration of the passage between a surgery description on the MT5C to a tangle description. We now present the tangle descriptions obtained from the surgery descriptions in §2.9 above using the tangle conventions in §3.1. Note that some of the surgery descriptions in §2.9 are on the mirror of the MT5C and these tangle descriptions reflect that accordingly.

In each tangle below, where presented with the choice \((\infty, \delta)\) for \(\delta \in \{-1, 0, +1\}\) inserting the rational tangle \(\infty\) yields the unknot and inserting the rational tangle \(\delta\) yields a 2–bridge link.
\[ \infty = ) \quad 1/n = [n] \quad n = \begin{cases} \text{n-half twists for } & n > 0 \\ \text{n-half twists for } & n < 0 \end{cases} \]

\[ 0 = \quad -n = [0, n] \quad n = \begin{cases} \text{n-half twists for } & n > 0 \\ \text{n-half twists for } & n < 0 \end{cases} \]

\[ [r_n, r_{n-1}, \ldots, r_1] \]

Figure 34. Tangle legend

Figure 35. Tangle description of Berge knot type (I), torus knots.
FIGURE 36. Tangle description of Berge knot type (II), cables about torus knots.

FIGURE 37. Tangle description of Berge knot type (III).

FIGURE 38. Tangle description of Berge knot type (IV).
Figure 39. Tangle description of Berge knot type (V).

Figure 40. Tangle description of Berge knot type (VI).
Figure 41. Tangle description of Berge knot types (IX) and (X), Sporadics a) and b).

Figure 42. Tangle description of Berge knot type (XI) and (XII), Sporadics c) and d).
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Department of Mathematics, University of Georgia, Athens, Georgia 30602
E-mail address: kb@math.uga.edu