A SHORTER PROOF ON RECENT ITERATIVE ALGORITHMS CONSTRUCTED BY THE RELAXED \((u, v)\)-COCOERCIVE MAPPINGS AND A SIMILAR CASE FOR INVERSE-STRONGLY MONOTONE MAPPINGS

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Abstract. In this short note, using the class of the relaxed \((u, v)\)-cocoercive mappings and \(\alpha\)-inverse strongly monotone mappings, we prove that if an important condition holds then we can prove the convergence of the proposed algorithm, more shorter than the original proof.

1. Introduction and preliminaries

Let \(C\) be a nonempty closed convex subset of a real Hilbert space \(H\). Recall the following well known definitions:

1. a mapping \(A : C \to H\) is said to be inverse-strongly monotone, if there exist \(\alpha > 0\) such that

\[
\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2,
\]

for all \(x, y \in C\). Such a mapping \(A\) is also called \(\alpha\)-inverse-strongly monotone.

2. a mapping \(A : C \to H\) is said to be strongly monotone, if there exists a constant \(\alpha > 0\) such that

\[
\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2.
\]

3. let \(C\) be a nonempty closed convex subset of a real Hilbert space \(H\). Suppose that \(B : C \to H\) is a nonlinear map. \(B\) is said to be relaxed \((u, v)\)-cocoercive, if there exist two constants \(u, v > 0\) such that

\[
\langle Bx - By, x - y \rangle \geq (-u)\|Bx - By\|^2 + v\|x - y\|^2,
\]

for all \(x, y \in C\). For \(u = 0\), \(B\) is \(v\)-strongly monotone. Clearly, every \(v\)-strongly monotone map is a relaxed \((u, v)\)-cocoercive map.

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In this paper, using the relaxed \((u, v)\)-cocoercive mappings, we make the proof of some recent iterative algorithms more shorter than the original one. Then we find a similar case for the inverse-strongly monotone mappings.

2. The comments for the relaxed \((u, v)\)-cocoercive mappings

Let \(C\) be a nonempty closed convex subset of a real Hilbert space \(H\).

G. Cai and S. Bu [1], W. Chantarangs and C. Jaiboon [2], S. Imnang [5], C. Jaiboon and P. Kumam [11], P. Kumam and C. Jaiboon [15], X. Qin, M. Shang and H. Zhou [19], X. Qin, M. Shang and Y. Su [22], X. Qin, M. Shang and Y. Su [23], considered some iterative algorithm for finding a common element of the set of the fixed points of nonexpansive mapping and the set of the solution of a variational inequality \(VI(C, A)\), where \(A\) is a relaxed \((u, v)\)-cocoercive mapping of \(C\) into \(H\). But, there is a condition that makes the proof more shorter than the original one.

The important condition is \(\alpha\)-expansiveness:

**Condition 1.** Let \(B\) be a self mapping on \(C\). Suppose that there exists a positive integer \(\alpha\) such that:

\[
\|Bx - By\| \geq \alpha\|x - y\|.
\]

for all \(x, y \in C\).

**Lemma 2.1.** Let \(A\) be a relaxed \((m, v)\)-cocoercive mapping and \(\epsilon\)-Lipschitz continuous such that \(v - m\epsilon^2 > 0\) and \(VI(C, A) \neq \emptyset\). Then \(A\) is an \((v - m\epsilon^2)\)-expansive mapping and \(VI(C, A)\) is singleton.

**Proof.** Let \(x_1, x_2 \in VI(C, A)\). Then

\[
\langle Ax_1, y - x_1 \rangle \geq 0,
\]

for each \(y \in C\), and

\[
\langle Ax_2, y - x_2 \rangle \geq 0,
\]

for each \(y \in C\).

Substituting \(x_1\) in (3) and \(x_2\) in (2), we have \(\langle Ax_1, x_2 - x_1 \rangle \geq 0\) and \(\langle Ax_2, x_1 - x_2 \rangle \geq 0\). Adding them, we have

\[
\langle Ax_2 - Ax_1, x_2 - x_1 \rangle \leq 0.
\]
But we know that $A$ is monotone. Indeed since $A$ is $\varepsilon$-Lipschitz continuous for each $x, y \in C$ we have

$$
\langle Ax - Ay, x - y \rangle \geq (-m)\|Ax - Ay\| + v\|x - y\|
$$

$$
\geq (-me^2)\|x - y\| + v\|x - y\|
$$

$$
= (v - me^2)\|x - y\| \geq 0,
$$

then we have $\langle Ax_2 - Ax_1, x_2 - x_1 \rangle \geq 0$, hence from (4), we have

$$
\langle Ax_2 - Ax_1, x_2 - x_1 \rangle = 0.
$$

But if $x_1 \neq x_2$, from (5) we have

$$
\langle Ax_2 - Ax_1, x_2 - x_1 \rangle \geq (v - me^2)\|x_2 - x_1\| > 0,
$$

that is contradiction with (14). Hence, $VI(C, A)$ is singleton. From (5), we have

$$
\|Ax - Ay\| \geq (v - me^2)\|x - y\|,
$$

then $A$ is $(v - me^2)$-expansive.

Now, we are ready to investigate [15] for example, in the following comment and we will bring a shorter proof for Theorem 3.1 in [15] for the relaxed $(u, v)$-cocoercive mappings:

**Comment 2.2.** Consider Theorem 3.1 in [15] and the $\xi$-Lipschitz continuous and relaxed $(m, v)$-cocoercive mapping $B$ in Theorem 3.1. From Lemma 2.1 and the condition (C7) of Theorem 3.1 in [15] and the condition $\xi$-Lipschitz continuous on $B$ we have that $VI(E, B)$ is singleton, that is, there exists an element $p \in E$ such that $VI(E, B) = \{p\}$ hence $\Gamma = \{p\}$ in Theorem 3.1. From the condition (C7) of Theorem 3.1 in [15], we may assume that $(v - m\xi^2) > 0$, hence from Lemma 2.1, $B$ is $(v - me^2)$-expansive, i.e,

$$
\|Bx - By\| \geq (v - m\xi^2)\|x - y\|,
$$

then the condition [1] is valid for $B$. The authors have also proved in the relation (3.25) in the page 521 that

$$
\lim_n \|Bz_n - Bp\| = 0.
$$

Now, putting $x = z_n$ and $y = p$ in (7), from (7) and (8), we have

$$
\lim_n \|z_n - p\| = 0,
$$

hence, $z_n \to p$, therefore here we get one of the main claims of Theorem 3.1. Note that since $\Gamma = VI(E, B) = \{p\}$ hence obviously $p$ belongs to other intersections in $\Gamma$ in Theorem 3.1. Then we can remove all parts of the proof from the relation (3.33) in page 524 to end of the proof of theorem 3.1 in [15] and another extra part of the proof.
In the following comment we introduce some more similar results.

**Comment 2.3.** Refer the readers to \[1, 2, 5, 11, 15, 19, 22, 23\] and similar results, to see more similar works. The authors of such these articles, using the relation

$$\lim_{n} \|Bx_n - Bp\| = 0,$$

for relaxed \((u, v)\)-cocoercive mapping \(B\), have proved that \(\{x_n\}\) converges to a unique element of the set \(\mathbb{F}\) which is for some conditions and situations similar to comment 2.2 about \[15\], so we can make their proofs shorter than the original proofs as in the comment 3.2.

### 3. The Comments for the Inverse Strongly Monotone Mappings

J. Chen, L. Zhang, T. Fan [3], S. Takahashia and W. Takahashi [27], H. Iiduka and W. Takahashi [4], K. R. Kazmi, Rehan Ali and Mohd Furkan [14], X. Qin and M. Shang [22], M. Zhang [30], S. Shan and N. Huang [25], T. Jitpeera and P. Kumam [7], S. Peathanom and W. Phuengrattana [17], Piri [18], X. Qin, M. Shang and Y. Su [23] and M. Lashkarizadeh Bami and E. Soori [16] considered some iterative methods for finding a common element of the set of the fixed points of nonexpansive mapping and the set of the solution of a variational inequality \(VI(C, A)\), where \(A\) is an \(\alpha\)-inverse strongly monotone mapping of \(C\) into \(H\). But, there is a case that is similar to \((m, v)\)-cocoercive mappings. First we see the following Lemma for \(\alpha\)-inverse strongly monotone mappings.

**Lemma 3.1.** Let \(A\) be an \(\alpha\)-inverse strongly monotone mapping and \(VI(C, A) \neq \emptyset\). Suppose that \(A\) is also an \(\gamma\)-expansive mapping. Then \(VI(C, A)\) is singleton.

**Proof.** Since \(A\) is an \(\alpha\)-inverse strongly monotone mapping, we have

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2 \geq 0,$$

then \(A\) is monotone. Since \(A\) is \(\gamma\)-expansive, we have

$$\|Ax - Ay\| \geq \gamma \|x - y\|.$$

Therefore, we conclude that \(A\) is one to one, because if \(Ax = Ay\), then from \((10)\), \(\|x - y\| = 0\), hence \(x = y\).

Let \(x_1, x_2 \in VI(C, A)\). Then

$$\langle Ax_1, y-x_1 \rangle \geq 0,$$

for each \(y \in C\), and

$$\langle Ax_2, y-x_2 \rangle \geq 0,$$
for each $y \in C$. Substituting $x_1$ in (12) and $x_2$ in (11), we have $\langle Ax_1, x_2 - x_1 \rangle \geq 0$ and $\langle Ax_2, x_1 - x_2 \rangle \geq 0$. Adding them, we have

$$\langle Ax_2 - Ax_1, x_2 - x_1 \rangle \leq 0. \quad (13)$$

Since $A$ is monotone, then we have $\langle Ax_2 - Ax_1, x_2 - x_1 \rangle \geq 0$, hence from (13), we have

$$\langle Ax_2 - Ax_1, x_2 - x_1 \rangle = 0. \quad (14)$$

Then from (9) we have $Ax_2 = Ax_1$, since $A$ is one to one, we get $x_2 = x_1$. Then $VI(C, A)$ is singleton. \qed

Now as for $(m, \sigma)$-cocoercive mappings, if $A$ is $\alpha$-strongly monotone mapping and $\gamma$-expansive, i.e, the condition [1] holds for $A$, we can make short the proof of the algorithms. For clearing our discussion, we investigate [4] for example, in the following comment:

**Comment 3.2.** Consider Theorem 3.1 in [4]. Since $A$ is an $\alpha$-strongly monotone mapping, if we consider the extra condition $\gamma$-expansiveness in theorem 3.1 in [4], i.e,

$$\|Ax - Ay\| \geq \gamma \|x - y\|. \quad (15)$$

for all $x, y \in C$, then from Lemma 3.1, we have $VI(C, A)$ is singleton and hence $F(S) \cap VI(C, A)$ is singleton, i.e, for example $F(S) \cap VI(C, A) = VI(C, A) = \{u\}$ for an element $u \in C$. The authors have also proved in the line 8 from below in the page 5 that

$$\lim_n \|Ax_n - Au\| = 0, \quad (16)$$

Now, put $x = x_n$ and $y = u$ then from (16) and (15), we have

$$\lim_n \|x_n - u\| = 0,$$

hence, $x_n \rightarrow u$ therefore in the line 8 from below in the page 5, we get the main claims of Theorem 3.1. Note that since $F(S) \cap VI(C, A) = VI(E, B) = \{u\}$ hence obviously $u \in F(S)$. Then we can remove all parts of the proof from the line 8 from below in the page 345 to end of the proof of theorem 3.1 in [4] and another extra part of the proof.

In the following comment we introduce some more similar results.

**Comment 3.3.** The structure of main results in [3, 4, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18, 22, 23, 24, 25, 26, 27, 28, 29, 30] is similar to [4]. The authors of such these articles, using the relation

$$\lim_n \|Ax_n - Ap\| = 0,$$
for $\alpha$-inverse strongly monotone mapping $A$, have proved that $\{x_n\}$ converges to a unique element of the set $\mathcal{F}$ which is for some conditions and situations similar to comment 3.2 about [4].

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References

[1] G. Cai and Shangquan Bu, Strong convergence theorems based on a new modified extragradient method for variational inequality problems and fixed point problems in Banach spaces, Computers and Mathematics with Applications 62 (2011) 2567-2579.
[2] W. Chantarangs, C. Jaiboon, and P. Kumam, A Viscosity Hybrid Steepest Descent Method for Generalized Mixed Equilibrium Problems and Variational Inequalities for Relaxed Cocoercive Mapping in Hilbert Spaces, Hindawi Publishing Corporation Abstract and Applied Analysis Volume 2010, Article ID 390972, 39 pages doi:10.1155/2010/390972.
[3] J. Chen, L. Zhang, T. Fan, Viscosity approximation methods for nonexpansive mappings and monotone mappings, J. Math. Anal. Appl. 334 (2007) 1450-1461.
[4] H. Iiduka, W. Takahashi, Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings, Nonlinear Analysis 61 (2005) 341-350.
[5] S. Imnang, Viscosity iterative method for a new general system of variational inequalities in Banach spaces, Journal of Inequalities and Applications 2013, 2013:249.
[6] C. Jaiboon, P. Kumam, Strong Convergence for Generalized Equilibrium Problems, Fixed Point Problems and Relaxed Cocoercive Variational Inequalities, Journal of Inequalities and Applications 2010, Article ID 728028, 43 pages, doi:10.1155/2010/728028.
[7] T. Jitpeera, P. Kumam, A new hybrid algorithm for a system of mixed equilibrium problems, fixed point problems for nonexpansive semigroup, and variational inclusion problem, Fixed Point Theory and Applications, doi:10.1155/2011/217407.
[8] T. Jitpeera, P. Kumam, A new hybrid algorithm for a system of equilibrium problems and variational inclusion, Ann Univ Ferrara (2011) 57:89108, doi: 10.1007/s11565-010-0110-4.
[9] T. Jitpeera, P. Kumam, hybrid algorithms for minimization problems over the solutions of generalized mixed equilibrium and variational inclusion problems, Mathematical Problems in Engineering Volume 2011, doi:10.1155/2011/648617.
[10] T. Jitpeera, P. Kumam, The Shrinking projection method for common solutions of generalized mixed equilibrium problems and fixed point problems for strictly pseudocontractive mappings, Journal of Inequalities and Applications 2011, doi:10.1155/2011/840319.
[11] J.S. Jung, A general composite iterative method for generalized mixed equilibrium problems, variational inequality problems and optimization problems, Journal of Inequalities and Applications 2011, 2011:51.
[12] A. Kangtunyakarn, Iterative methods for finding common solution of generalized equilibrium problems and variational inequality problems and fixed point problems of a finite family of nonexpansive mappings, Fixed Point Theory and Applications, 2010, doi:10.1155/2010/836714.

[13] K.R. Kazmi, S.H. Rizvi, A hybrid extragradient method for approximating the common solutions of a variational inequality, a system of variational inequalities, a mixed equilibrium problem and a fixed point problem, Applied Mathematics and Computation 218 (2012), 5439-5452.

[14] K. R. Kazmi, Rehan Ali and Mohd Furkan, Hybrid iterative method for split monotone variational inclusion problem and hierarchical fixed point problem for a finite family of nonexpansive mappings, Numer Algor, (2017), doi.org/10.1007/s11075-017-0448-0.

[15] P. Kumam, C. Jaiboon, A new hybrid iterative method for mixed equilibrium problems and variational inequality problem for relaxed cocoercive mappings with application to optimization problems, Nonlinear Analysis: Hybrid Systems 3 (2009) 510-530.

[16] M. Lashkarizadeh Bami and E. Soori, Strong convergence of a general implicit algorithm for variational inequality problems and equilibrium problems and a continuous representation of nonexpansive mappings, B Iran Math Soc, 40 (2014), 4, 977-1001.

[17] S. Peathanom, W. Phuengrattana, A Hybrid Method for Generalized Equilibrium, Variational Inequality and Fixed Point Problems of Finite Family of Nonexpansive Mappings, Thai Journal of Mathematics, 9 (2011), 95-119.

[18] H. Piri, A general iterative method for finding common solutions of system of equilibrium problems, system of variational inequalities and fixed point problems, Math. Comput. Model. 55(2012), 1622-1638.

[19] X. Qin, M. Shang and H. Zhou, Strong convergence of a general iterative method for variational inequality problems and fixed point problems in Hilbert spaces, Applied Mathematics and Computation, 200(2008), 242-253.

[20] X. Qin, M. Shang, Y. Su, A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces, Nonlinear Analysis 69 (2008) 3897-3909.

[21] X. Qin, M. Shang, Y. Su, Strong convergence of a general iterative algorithm for equilibrium problems and variational inequality problems, Mathematical and Computer Modelling 48 (2008) 1033-1046.

[22] X. Qin, M. Shang, Y. Su, A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces, Nonlinear Analysis 69 (2008), 3897-3909.

[23] X. Qin, M. Shang, Y. Su, Strong convergence of a general iterative algorithm for equilibrium problems and variational inequality problems, Mathematical and Computer Modelling 48 (2008), 1033-1046.

[24] S. Saewan, P. Kumam, The shrinking projection method for solving generalized equilibrium problems and common fixed points for asymptotically quasi-φ nonexpansive mappings, Fixed Point Theory and Applications 2011, 2011:9.

[25] S. Shan, N. Huang, An iterative method for generalized mixed vector equilibrium problems and fixed point of nonexpansive mappings and variational inequalities, Taiwanese Journal of Mathematics, 16(2012), 1681-1705.
[26] Lijuan Sun, Hybrid methods for common solutions in Hilbert spaces with applications, Journal of Inequalities and Applications 2014, 2014:183, doi:10.1186/1029-242X-2014-183.

[27] S. Takahashi, W. Takahashi, Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space, Nonlinear Anal. 69 (2008), 1025-1033.

[28] N. O. uea, C. Jaiboon, P. Kumam, U. W. Humphries, Convergence of iterative sequences for fixed points of an infinite family of nonexpansive mappings based on a hybrid steepest descent methods, Journal of Inequalities and Applications 2012, 2012:101.

[29] Y. Yao, Y. Liou, S. Kang, Approach to common elements of variational inequality problems and fixed point problems via a relaxed extragradient method, Computers and Mathematics with Applications 59 (2010), 3472-3480.

[30] M. Zhang, Iterative algorithms for common elements in fixed point sets and zero point sets with applications, Fixed Point Theory and Applications 2012, 2012:21.

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