Planckian $AdS_2 \times S_2$ space is an exact solution of the semiclassical Einstein equations

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Abstract

The product space configuration $AdS_2 \times S_2$ (with $l$ and $r$ being radiiuses of the components) carrying the electric charge $Q$ is demonstrated to be an exact solution of the semiclassical Einstein equations in presence of the Maxwell field. If the logarithmic UV divergences are absent in the four-dimensional theory the solution we find is identical to the classical Bertotti-Robinson space ($r = l = Q$) with no quantum corrections added. In general, the analysis involves the quadratic curvature coupling $\lambda$ appearing in the effective action. The solutions we find are of the following types: i) (for arbitrary $\lambda$) charged configuration which is quantum deformation of the Bertotti-Robinson space; ii) ($\lambda > \lambda_{cr}$) $Q = 0$ configuration with $l$ and $r$ being of the Planck order; iii) ($\lambda < \lambda_{cr}$) $Q \neq 0$ configuration ($l$ and $r$ are of the Planck order) not connected analytically to the Bertotti-Robinson space. The interpretation of the solutions obtained and an indication on the internal structure of the Schwarzschild black hole are discussed.
It is an interesting and important problem as how the known solutions of classical Einstein equations are modified in the quantum domain. It is believed that the quantum corrections become essential when the space-time curvature is of the Planck order and they may change drastically the space-time structure appearing in the classical theory \[1\], \[2\], \[3\]. Particularly, the black hole geometry may be corrected at the Planck distances in the way that the singularity at \(r = 0\) is replaced by a regular manifold. On the other hand, the quantum corrections may occur to be important on the last stage of the gravitational collapse and be highly influential on the ultimate fate of the black hole. However, the current status of the theory does not allow us to make any definite conclusion about realization of these expectations. The main problem is that we still do not have the consistent theory of quantum gravity in the framework of which all these questions might be answered consistently. The best we can do is to develop the semiclassical approximation when the gravitational field (metric \(g_{\mu\nu}\) on space-time) is still treated as a classical object while the matter fields are quantized on its background. In this case configurations of the gravitational field are governed by the “quantum” gravitational action

\[
W_Q[g_{\mu\nu}] = W_{cl} + \Gamma[g_{\mu\nu}],
\]

where \(W_{cl}\) is the classical action. For the gravity coupled with the Maxwell field we have

\[
W_{cl} = -\frac{1}{16\pi G} \left( \int_{M^4} R(4) + 2 \int_{\partial M^4} K(4) + \int_{M^4} F_{\mu\nu}^2 \right),
\]

on the 4-dimensional space \(M^4\), \(R(4)\) is the four-dimensional scalar curvature and \(K(4)\) is the trace of the extrinsic curvature of the boundary \(\partial M^4\).

The term \(\Gamma[g_{\mu\nu}]\) in (1) is due to the quantum matter fields. It is highly non-local functional of \(g_{\mu\nu}\) and its form is not known in general. This fact makes the analysis of the semiclassical Einstein equations

\[
\frac{\delta W_Q}{\delta g_{\mu\nu}} = 0
\]

possible only in some approximation \[4\]. Note, that the functional \(\Gamma[g_{\mu\nu}]\) is better studied in two dimensions that makes the study of two-dimensional models \[5\] so attractive. The efficiency, however, of the approximate methods and toy models would be considerably supported by finding at least one configuration which is a guaranteed exact solution of

\[
\]
the equations (3). In this note we find such a solution. It is the direct product of two-dimensional Anti-de Sitter space $\text{AdS}_2$ and two-dimensional sphere $\text{S}_2$ with respectively radius $l$ and $r$ of each component.

An indirect indication of existence of this solution comes from the analysis of 2d quantum models. It was found that there always appears a solution of the quantum gravitational equations describing 2d space with constant curvature and constant value of the dilaton field. First this fact was observed in [4] for the RST model and later found to be a feature of more general class of 2d models [7]. Translating this into the 4-dimensional language the dilaton field should be identified with the radius $r$ of the spheri-symmetric metric

$$ds^2 = \gamma_{\alpha\beta}(z)dz^\alpha dz^\beta + r^2(z)(d\theta^2 + \sin^2 \theta d\phi^2)$$

and the corresponding four-dimensional configuration would be $\text{AdS}_2 \times \text{S}_2$. In what follows we give proof both in terms of the 2d model and the four-dimensional theory (1) that the space $\text{AdS}_2 \times \text{S}_2$ is indeed an exact solution of the semiclassical gravitational equations.

Before proceeding we pause for a few remarks. The space $\text{AdS}_2 \times \text{S}_2$ with

$$l = r = Q ,$$

where $Q$ is the electric charge, is so-called the Bertotti-Robinson space. It is known to be a solution of the classical Einstein equations in presence of the Maxwell field and can be viewed as near-horizon geometry in the extreme limit ($M \to Q$) of the charged black hole metric. For $Q \neq 0$ the solution we find below is the quantum deformation of this classical solution. Remarkably, (and it is one of our main points) the space of this kind remains a solution of eq.(3) even for $Q = 0$. The radiuses $l$ and $r$ then are of the Planck order. Actually, an $\text{AdS}_2 \times \text{S}_2$ space (with arbitrary $r$ and $l$) appears to describe universally the extreme limit [8] of a black hole configuration both in the classical and semiclassical theories. The quantum black hole entropy then takes an universal form [9] (dependent on $r$ and $l$) when the extreme limit is approached. The product space $\text{AdS}_2 \times \text{S}_2$ has also become recently a subject of intensive study from different point of view in [10]. It would be interesting to exploit our results in the context of the study made in [10].

In our study of the equations (3) we are interested in a solution from the class of
the spheri-symmetric metrics (4). Any such metric is completely determined by fixing the two-dimensional metric $\gamma_{\alpha\beta}$ and the “dilaton” field $r(z)$. Being considered on the class of the four-dimensional metrics (4) the classical action (2) takes the form of the two-dimensional theory of gravity†

$$W_{cl}[\gamma_{\alpha\beta}, r] = -\frac{1}{4G} \int_{M^2} \left(r^2 R + 2(\nabla r)^2 + 2U(r)\right),$$

where $M^2$ is 2d space with coordinates $\{z^\alpha\}$, $R$ is 2d scalar curvature and $U(r)$ is the dilaton potential. When electric charge is zero we have $U(r) = 1$. Otherwise, it is $U(r) = 1 - \frac{Q^2}{r^2}$. It is easy to see that the configuration (5) is a solution of field equations obtained from the action (3). Indeed, variation with respect to dilaton $r$ (for a configuration with constant $R$ and $r$) gives $R = -r^{-1}U'(r) = -2\frac{Q^2}{r^4}$ while the variation with respect to metric results in vanishing the potential $U(r)$. Altogether, both conditions lead to (5).

Now we have to find the form of the effective action $\Gamma[g_{\mu\nu}]$ in (1). Considered on the class of spheri-symmetric metrics (4) it becomes a functional $\Gamma[R, r, \nabla R, \nabla r]$ of two-dimensional scalar curvature $R$, dilaton $r$ and their derivatives $\nabla R, \nabla r$. Just for the illustration we first consider the functional $\Gamma$ arising in two-dimensional case when the 2d massless fields couple to the dilaton $r$. In two dimensions, when $r$ and $R$ are constant $\Gamma$ changes as $\Gamma \rightarrow \Gamma + c \int_{M^2} R \ln \alpha$ under the scaling transformation $r \rightarrow r\alpha$. Therefore, the form of $\Gamma$ in two dimensions is the following (see [3] and [11])

$$\Gamma[\gamma_{\alpha\beta}, r] = A \int_{M^2} R \square^{-1} R + B \int_{M^2} R \ln r + w[\nabla r, \nabla R],$$

where $A$ and $B$ are constants, $w[\nabla r, \nabla R]$ is the functional which vanishes when $r$ and $R$ are constant. It is important for the analysis we are carrying on that variation of the term $w[\nabla r, \nabla R]$ with respect to metric $\gamma_{\alpha\beta}$ or dilaton $r$ vanishes for a configuration with constant $R$ and $r$. Therefore, we may ignore the term $w[\nabla r, \nabla R]$ when looking for a solution of the semiclassical equations that describes constant curvature 2d space-time with constant value of the dilaton field $r$. Variation of $W_Q[\gamma_{\alpha\beta}, r]$ (1) then gives rise to the equations

$$rR + U'(r) = 2GBr \ ,$$

$$U(r) = -4AGR .$$

† We use the Euclidean signature of the metric.
where \( U'(r) = \partial_r U(r) \). The analysis of the eqs. (8) goes for arbitrary \( A \) and \( B \). For simplicity, however, we assume that \( B = 4A \). Then, defying the radius \( l \) of the 2d space as \( R = -\frac{2}{l^2} \) we find from (8) that

\[
\begin{align*}
    r^2 = l^2 &= Q^2 + 2Al^2_{pl} \\
\end{align*}
\]  

(9)

where the Planck length \( l_{pl} \) is defined as \( l^2_{pl} = 4G \). The solution (9) is a quantum deformation (governed by the Planck length \( l_{pl} \)) of the classical solution (3) describing the near horizon geometry of the extreme limit of the charged black hole. However, the solution (9) has a new feature absent in the classical case. Namely, it is still valid even if the electric charge vanishes (\( Q = 0 \)). The four-dimensional space then is \( AdS_2 \times S_2 \) with each component having the Planck order radius \( \sqrt{2Al}_{pl} \). It is worth noting that the coupling to dilaton is important for the existence of the solution of the equations (8) with \( Q = 0 \). Otherwise, if \( B = 0 \) (as it is in the 2d model considered in [5]) in (7) one finds from (8) that

\[
\begin{align*}
    r^2 = lQ , \quad l^2 - r^2 &= 8AG = 2Al^2_{pl} . \\
\end{align*}
\]  

(10)

In the limit \( Q \to 0 \) it describes a singular (in the four-dimensional picture) configuration

\[
\begin{align*}
    r = 0 , \quad l = \sqrt{2Al}_{pl} \\
\end{align*}
\]  

(11)

with vanishing size of the spheric component \( S_2 \).

We now want to extend this analysis to the four-dimensional case. This requires the knowledge of the structure of the four-dimensional effective action \( \Gamma[r, R, \nabla r, \nabla R] \) which as we have already mentioned is very complicated and not known in general. For our purposes, however, we need to know only how \( \Gamma \) depends on \( r \) and \( R \) ignoring gradients \( \nabla r \) and \( \nabla R \). In general, the functional \( \Gamma \) can be represented in the form

\[
\Gamma[r, R, \nabla r, \nabla R] = \Gamma_0[r, R] + w[\nabla r, \nabla R] 
\]  

(12)

where \( \Gamma_0[r, R] \) is a functional of \( r \) and \( R \) but not their derivatives and \( w[\nabla r, \nabla R] \) is the functional vanishing when \( \nabla r = \nabla R = 0 \), it can be expended in powers of \( \nabla r \) and \( \nabla R \). As in the 2d case considered above we may ignore variation of \( w[\nabla r, \nabla R] \) when looking for a solution with constant \( r \) and \( R \). Thus, only variation of \( \Gamma_0[r, R] \) produces the essential
for our purposes part of the semiclassical Einstein equations. The structure of $\Gamma_0[r, R]$ can be obtained by quantizing the matter fields on the background space $AdS_2 \times S_2$ with arbitrary radius $l \ (r)$ of the component $AdS_2 \ (S_2)$. This can be carried out by, for example, the zeta-function method

$$\Gamma_0 = -\frac{1}{2}(\zeta'(0) - \zeta(0) \ln \mu^2) ,$$

where $\mu$ is an arbitrary length scale and $\zeta$-function on the product space $AdS_2 \times S_2$ is defined as

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} dt t^{s-1} Tr K_{H_2} Tr K_{S_2} ,$$

(13)

where $K_{H_2}(t) \ (K_{S_2}(t))$ is the heat kernel on $AdS_2 \ (S_2)$ space.

For concreteness we consider a four-dimensional scalar field with zero mass. The heat kernels on spaces $AdS_2$ and $S_2$ are known explicitly,

$$Tr K_{H_2} = \frac{V_{H_2}}{4l^2}, \quad Tr K_{S_2} = \Theta\left(\frac{t}{r^2}\right),$$

and the corresponding expressions for $k(t/l^2)$ and $\Theta(t/r^2)$ can be found in [12]. It is important that the heat kernels on sphere and on Anti-de Sitter space are related by an analytical continuation $l^2 \to -r^2$. In particular, it is manifested in the small $t$ expansion

$$k(t/l^2) \simeq \frac{l^2}{t} (1 - \frac{t}{3l^2} + \frac{t^2}{15l^4})$$

$$\Theta(t/r^2) \simeq \frac{r^2}{t} (1 + \frac{t}{3r^2} + \frac{t^2}{15r^4})$$

(14)

of the heat kernels.

In our case (13) is function $\zeta(s, r^2, l^2)$ of the parameter $s$ and the radiuses $r$ and $l$ of the space $S_2$ and $AdS_2$ respectively. By means of the known scaling property $\zeta(s, r^2, l^2) = r^{2s}\zeta(s, 1, l^2)$ we find that

$$\zeta'(0, r^2, l^2) = \zeta'(0, 1, l^2) + \zeta(0, 1, l^2) \ln r^2 .$$

(15)

The effective action then takes the form

$$\Gamma_0[r, l] = -\frac{V_{H_2}}{2l^2} \left(C\left(\frac{r^2}{l^2}\right) \ln \frac{r^2}{G} + \lambda(\mu)C\left(\frac{r^2}{l^2}\right) + F\left(\frac{r^2}{l^2}\right)\right) ,$$

(16)

where $V_{H_2}$ is volume of $AdS_2$ space, note that $V_{H_2} = l^2v$ where $v$ is the divergent dimensionless quantity. We denoted $F\left(\frac{r^2}{l^2}\right) = \zeta'(0, 1, l^2)$ and $C\left(\frac{r^2}{l^2}\right) = \zeta(0, 1, l^2)$ and introduced the $\mu$-dependent constant $\lambda(\mu) = \ln \frac{G}{\mu^2}$ when derived (14). It is known that
\( \zeta(0) = a_2 \), where \( a_2 \) is a coefficient in the small \( t \) expansion

\[
T_r K_{M^4}(t) = \sum_{n=0}^{\infty} a_\frac{n}{2} t^{(n-4)/2}
\]

of the heat kernel. The coefficient \( a_2 \) is responsible for the logarithmic UV divergences of the effective action. Note, that the \( \zeta \)-function method provides us with already regularized expression for the effective action. The function \( C(x) \) in (16), thus, represents the local quadratic in curvature terms with \( \lambda(\mu) \) being the corresponding renormalized coupling in the four-dimensional effective action. On the other hand, the function \( F(x) \) represents the non-local part of the action. For the massless scalar field we find using the expansion (14) that

\[
C(x) = \frac{1}{60\pi} \left( x + \frac{1}{x} - \frac{5}{3} \right),
\]

where \( x = \frac{r^2}{\ell^2} \). Note, that the function \( C(x) \) is positive for \( x > 0 \) and has minimum at \( x = 1 \) \( (C''(1) = 0) \).

Knowing the explicit form of the heat kernels entering the formula (13) one can, in principle, directly calculate \( \zeta'(0) \) and find the function \( F(x) \) entering eq.(16). In reality, however, it is technically difficult because we know only integral representation for the heat kernel on \( AdS_2 \) and integral-infinite sum form for the heat kernel on \( S_2 \). Below we therefore find an approximate form of the function \( F(x) \) that is enough for our purposes.

In the regime of large or small \( x \) we may find the asymptotic behavior of the function \( F(x) \). In the limit \( x \to \infty \) \((r^2 \to \infty)\) the heat kernel on sphere can be approximated by the heat kernel on flat space \( R_2 \) which is the first term in the expansion (14). The function \( \zeta(s, r^2, l^2) \) then is approximated by \( \zeta \)-function \( \zeta_0(s, r^2, l^2) \) on the product space \( R_2 \times H_2 \). Since we have that \( \zeta_0(s, r^2, l^2) = \frac{r^2 l^2}{4\pi} \zeta_0(s, 1, 1) \) the calculation of the effective action is straightforward and we find that

\[
F(x) = -\frac{1}{60\pi} x \ln x + \zeta_0'(0, 1, 1)x
\]

for large \( x \). The similar analysis for \( x \to 0 \) gives us that \( F(x) = O(1/x) \).

An important property of the functions \( F(x) \) and \( C(x) \) is how they transform under the inverse transformation \( x \to \frac{1}{x} \). The analysis based on the scaling property (15) of \( \zeta \)-function and the analyticity between the heat kernels on \( S_2 \) and \( AdS_2 \) shows that

\[
C\left( \frac{1}{x} \right) = C(x),
\]

\[
F\left( \frac{1}{x} \right) = F(x) + C(x) \ln x,
\]

(19)
where \( x = \frac{r^2}{l^2} \). Taking the function \( C(x) \) in the form \((17)\) one solves the eq.\((19)\) explicitly as follows

\[
F(x) = -\frac{1}{60\pi}(x - \frac{5}{6})\ln x + F_0(x) ,
\]

where \( F_0(x) = F_0(1/x) \). It follows that \( F'_0(1) = 0 \) and hence \( F_0(x) \sim (x - 1)^2 \) near \( x = 1 \).

Comparing \((20)\) and \((18)\) we find that the leading term in \( F_0(x) \)

\[
F_0(x) \simeq \zeta'_0(0,1,1)(x + 1) \quad \text{near} \quad x = 1.
\]

Note the difference of \((22)\) and the 2d effective action \((7)\) in the term proportional to \( \ln r \). This is due to different structure of the logarithmic UV divergences in two and four dimensions.

Now we may find the desired equations by varying the functional \((1)\), \((12)\), \((22)\) with respect to \( r \) and \( \gamma_{\alpha\beta} \) ignoring the variation of the term \( w[\nabla r, \nabla R] \) in \((12)\). An equivalent but simpler way is to consider the functional \( W_Q \) \((1)\) on a space \( AdS_2 \times S_2 \) with arbitrary radiuses \( l \) and \( r \). Variables \( l \) and \( r \) then become the only gravitational degrees of freedom left. Variation with respect to \( l \) and \( r \) gives us the equations we are looking for. Following this way we find

\[
W_Q[l, r] = \left( \frac{1}{2G}(r^2 - l^2 U(r)) - \frac{1}{2}C(r^2)\ln \frac{r^2}{G} - \lambda \frac{1}{2}C(r^2) - \frac{1}{2}F(r^2) \right) v
\]

for the “quantum” action functional. It gives us the equations

\[
r^2 - \frac{l^2}{2}r''(r) = G \left( C(x) + xC'(x)\ln \frac{r^2}{G} + \lambda xC'(x) + xF'(x) \right)
\]

\[
l^2 U(r) = G \left( xC'(x)\ln \frac{r^2}{G} + \lambda xC'(x) + xF'(x) \right) ,
\]

\( (24) \)
where $x = \frac{r^2}{r^2}$. For the potential $U(r) = 1 - \frac{Q^2}{r}$ we find that eqs.\((24)\) take the form
\begin{align*}
    r^2 - l^2 &= GC(x) \ , \\
    r^2 &= \frac{Q^2}{x} + G \left( C(x) + xC'(x) \ln \frac{r^2}{G} + \lambda xC''(x) + xF'(x) \right) .
\end{align*}
\((25)\)

Analyzing these equations we first observe that when the logarithmic UV divergences are absent in the four-dimensional theory, i.e. $C(x) \equiv 0$, the solution of the eqs.\((25)\) is especially simple
\begin{equation}
    l^2 = r^2 = Q^2 ,
\end{equation}
\((26)\)
where we took into account that in this case $F(x) = F_0(x)$ and $F_0'(1) = 0$. It is exactly the Bertotti-Robinson solution \((5)\) of the classical Einstein equations. So this solution is not deformed by any quantum corrections if $C(x) \equiv 0$. This observation seems to be in agreement with the non-renormalization theorem proven in \([13]\). Though there may appear corrections which depend strongly on the amount of unbroken supersymmetry and correspond to a renormalization of the radius of the sphere, they are shown to absent for the maximally supersymmetric case.

If the function $C(x)$ is not identically zero or vanishing at $x = 1$ the solution of the equations \((25)\) reads
\begin{equation}
    \frac{l^2}{G} = \frac{C(x)}{x - 1} , \quad \frac{r^2}{G} = \frac{xC(x)}{x - 1} ,
\end{equation}
\((27)\)
where the ratio $x = \frac{r^2}{l^2}$ should be found from the equation
\begin{align*}
    M(x) &= \frac{Q^2}{Gx} , \\
    M(x) &\equiv \frac{C(x)}{(x - 1)} - xC'(x) \ln \left( \frac{xC(x)}{x - 1} \right) - \lambda xC''(x) - xF'(x) .
\end{align*}
\((28)\)
Since the eqs.\((27)\), \((28)\) are defined for $x > 1$ the radius of the spheric component is always bigger than the radius of the Anti-de Sitter space. For $x \to 1$ the function $M(x)$ goes to infinity as follows
\begin{equation}
    M(x) \simeq \frac{1}{180 \pi} \frac{1}{(x - 1)}
\end{equation}
\((29)\)
while for $x \to +\infty$ the asymptote is
\begin{equation}
    M(x) \simeq \frac{1}{60 \pi} (1 + \ln 60 \pi - \lambda)x + \frac{1}{72 \pi} + O\left( \frac{1}{x \ln x} \right) .
\end{equation}
\((30)\)

\footnote{According to \([3]\) the potential $U(r)$ in \((7)\) is modified by the quantum corrections. We do not consider this possibility here though the eqs.\((24)\) can be analyzed for arbitrary $U(r)$.}
Analyzing behavior of the function $M(x)$ we find that it depends crucially on value of the coupling $\lambda$. For large positive $\lambda$ it monotonically decreases and goes from $+\infty$ at $x = 1$ to $-\infty$ at $x = +\infty$ and vanishes at one point $x = x_0$. The root $x_0$ is very close to 1 and the difference $(x_0 - 1)$ vanishes for large $\lambda$. At $\lambda \simeq 6.3$ the monotonic behavior of $M(x)$ changes and for $\lambda < 6.3$ the function develops local minimum and maximum. The local minimum becomes lower when $\lambda$ decreases but it never hits zero. As it is seen from the asymptote (30), at value $\lambda_{cr} = 1 + \ln 60\pi \simeq 6.239$ the behavior of the function $M(x)$ changes drastically: it increases linearly at large $x$ for $\lambda < \lambda_{cr}$. Though $M(x)$ still develops a minimum at some point it never vanishes for $\lambda < \lambda_{cr}$. The form of the function $M(x)$ for different $\lambda$ is shown on Figure 1.

Roots of the equation $M(x_0) = 0$ describe configurations with zero charge $Q$. The above analysis shows that $x_0$ is close to 1 for large positive $\lambda$ and we find from (27) that

$$l^2 \simeq \frac{G}{60\pi} \left( \frac{1}{3(x_0 - 1)} + (x_0 - 1)^2 \right),$$

$$r^2 \simeq \frac{G}{60\pi} \left( \frac{1}{3(x_0 - 1)} + \frac{1}{3} + x_0(x_0 - 1) \right).$$

(31)

On the other hand, $x_0$ becomes infinitely large when $\lambda$ approaches the critical value $\lambda_{cr}$.
From eq. (27) we find that in this case
\[ l^2 \approx \frac{G}{60\pi} (1 - \frac{2}{3x_0(\lambda)}) , \quad r^2 \approx \frac{G}{60\pi} x_0(\lambda) . \]  (32)

Note, that there is no configuration with zero charge if \( \lambda < \lambda_{cr} \).

In the charged case we are looking for the roots of the equation \( M(x) = \frac{Q^2}{G^2} \). The analysis above shows that for any charge \( Q^2 > 0 \) there always exists such a root \( x_1 \approx 1 \), the difference \( (x_1 - 1) \) vanishes when charge \( Q \) becomes infinitely large. For large positive \( \lambda \) there is only this solution of the equation (28). It is different for \( \lambda < \lambda_{cr} \). Then there appears additional root \( x_2 \) which grows to infinity when charge \( Q \) becomes large. Using the asymptotes (29) and (30) we find in the limit of large \( Q \)
\[ x_1 \approx 1 + \frac{G}{180\pi Q^2} \]  (33)
for the first root and
\[ x_2 \approx \frac{Q}{\sqrt{G\sqrt{\gamma}}} , \]  (34)
where \( \gamma = \frac{1}{60\pi} (1 + \ln 60\pi - \lambda) \), for the second root. The first root exists for any \( \lambda \) and describes the configuration \( AdS_2 \times S_2 \) with radiuses
\[ l^2_1 \approx Q^2 + \frac{1}{10800\pi^2} \frac{G^2}{Q^2} , \]
\[ r^2_1 \approx Q^2 + \frac{G}{180\pi} + \frac{1}{10800\pi^2} \frac{G^2}{Q^2} \]  (35)
and is a quantum deformation of the classical solution (5) analytically governed by Newton’s constant \( G(\sim l_{Pl}^2) \). The second root (34) appears if \( \lambda < \lambda_{cr} \) and corresponds to the configuration with the radiuses
\[ l^2_2 \approx \frac{G}{60\pi} - \frac{1}{90\pi} \frac{\sqrt{\gamma}G^{3/2}}{Q} , \quad r^2_2 \approx \frac{G^{1/2}Q}{60\pi \sqrt{\gamma}} , \]  (36)
which are not analytical with respect to \( G \). It is an interesting feature of the solution (36) that when charge \( Q \) grows the radius of the Anti-de Sitter component tends to the fixed (\( \sim l_{Pl} \)) value while the radius of the spheric component grows as \( Q^{1/2} \). So, for large values of \( Q \) the spheric component may have macroscopic size while the size of the hyperbolic component remains Planckian.
Thus, we have found that i) for any value of $\lambda$ the classical solution (3) is modified by quantum corrections according to (35); ii) if $\lambda > \lambda_{cr}$ there exists also uncharged configuration with radiiuses $r$ and $l$ determined by (31) for large $\lambda$ or (32) for $\lambda$ approaching $\lambda_{cr}$; iii) if $\lambda < \lambda_{cr}$ there does not exist anymore the $Q = 0$ configuration but appears new charged configuration (36) with radiiuses of the Planck order. The configurations (31), (32) and (36) are absent in the classical theory and lie completely in the Planckian domain.

The analysis we present can be easily generalized for different quantum fields including gravitons. The only information one should know about the effective action is how it behaves for large values of the ratio $x = \frac{r^2}{l^2}$. If the corresponding function $M(x)$ (28) decreases to $-\infty$ for large $x$ then we have $Q = 0$ solution of the type ii). Otherwise, if $M(x)$ grows to $+\infty$ there exists the $Q \neq 0$ configuration of the type iii). The solution of the type i) exists in any case.

What is the physical meaning of the Planckian AdS$_2 \times$ S$_2$ spaces we have found? In particular, what is the interpretation of the configuration with zero electric charge? In order to answer these questions it is useful to start with the classical solution (5). Usually, it is interpreted as describing the geometry near horizon of the charged black hole in the extreme limit when inner and outer horizons merge. In the classical Einstein gravity the uncharged (Schwarzschild) black hole has only outer horizon. Therefore there is no such a phenomenon as extreme black hole if the electric charge vanishes. In the semiclassical theory the situation may be different and the uncharged black hole may have an inner horizon staying at the Planck distance from the origin. Then the space AdS$_2 \times$ S$_2$ with zero charge $Q$ may be interpreted as describing the extreme limit of the semiclassical Schwarzschild black hole. As we have seen, the $Q = 0$ configuration (31) or (32) is an analytical continuation of $Q \neq 0$ solution and should have the similar interpretation. Thus, our result is an indication that the Schwarzschild black hole has an inner Planck size horizon in the quantum case§. Note, that in our consideration it is valid only for certain (though wide) range of values of the higher curvature coupling $\lambda$.

The interpretation of the charged solution additional to the (quantum deformed)

\footnote{Note, that the existence of the inner (with the size of the Planck order) horizon in the semiclassical Schwarzschild black hole is an important ingredient in the scenario of the gravitational collapse without information loss proposed in \cite{2}.}
Bertotti-Robinson space is less straightforward. Existence of this solution likely means that there may appear different black hole configurations with the same value of the charge $Q$ (indicating, thus, that Birkhoff’s Theorem is not valid in the semiclassical case). So, for large $Q$ there may exist a near-extreme black hole configuration with macroscopic size $\sim Q^{1/2}$ but Planckian curvature. This configuration is absent classically and is additional to the usual (quantum deformed) Reissner-Nordstrom configuration having macroscopic size proportional to $Q$ and the curvature invariants bounded by $1/Q^2$.

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Note added:
It is interesting to note that our analysis can be extended to describe the configurations of the type $S_2 \times S_2$ (in the Minkowskian signature it is space $dS_2 \times S_2$). One should just make the substitution $l^2 \rightarrow -l^2$ ($x \rightarrow -x$), $C(x) \rightarrow \hat{C}(x) = -C(-x)$ in the equations (24), the parameter $l$ now is the radius of the second sphere. In result we find $r^2 = G\frac{x}{x+1}\hat{C}(x)$, $l^2 = G\frac{1}{x+1}\hat{C}(x)$ for the radiuses where $x$ should be determined from the equation

$$\hat{M}(x) = \frac{Q^2}{Gx},$$

$$\hat{M}(x) = \frac{1}{x+1}\hat{C}(x) + x\hat{C}'(x)\ln\left(\frac{x\hat{C}}{x+1}\right) + \lambda x\hat{C}''(x) + x\hat{F}'(x).$$

For the massless field with $C(x)$ in the form (17) we have $\hat{C}'(x) = \frac{1}{60\pi}(x + \frac{1}{x} + \frac{5}{3})$ and $\hat{F}(x) = -\frac{1}{60\pi}(x + \frac{5}{6})\ln x + \hat{F}_0(x)$, $\hat{F}_0(1/x) = \hat{F}_0(x)$.

The function $\hat{M}(x)$ is monotonic with asymptotes $\hat{M}(x) = \frac{1}{60\pi}(\lambda - \lambda_{cr})x + \frac{1}{x^2\pi} + O(1/x)$ for $x \rightarrow \infty$ and $\hat{M}(x) = -\frac{1}{60\pi}(\lambda - \lambda_{cr})x^{-1} - \frac{1}{x^2\pi} + O(x)$ for $x \rightarrow 0$ ($\lambda_{cr}$ is the same as in the text). It follows that for small $x$ we have solution with $\frac{x}{x^2\pi} = \frac{\lambda_{cr} - \lambda}{60\pi} - \frac{Q^2}{G}$ and, in particular, for $Q^2 = \frac{G}{60\pi}(\lambda_{cr} - \lambda)$ ($\lambda < \lambda_{cr}$) we have $x = 0$. The corresponding configuration is $R_2 \times S_2$ with $l^2 = \infty$ and $r^2 = \frac{G}{120\pi}$. On the other hand, for large $x$ we have $x^2 = \frac{Q^2}{G}60\pi(\lambda - \lambda_{cr})$ ($\lambda > \lambda_{cr}$) and $r^2 = \frac{x}{60\pi}$, $l^2 = \frac{G}{60\pi}$ that is similar to the $AdS_2 \times S_2$ configuration (36).
It follows from the above asymptotes that the function $\hat{M}(x)$ should vanish at some point. Remarkably, it happens exactly at $x = 1$. This fact does not depend on the type of the quantum massless field and is a simple consequence of the identities (19) (indeed, we have then $\hat{C}'(1) = 0$ and $\hat{F}'(1) = -\frac{1}{2}\hat{C}(1)$ that results in $\hat{M}(1) = 0$). The corresponding (uncharged) configuration is $S_2 \times S_2$ with $r^2 = l^2 = G_1 \hat{C}(1)$. Configurations of this type are also a subject of the consideration in [14].

References

[1] V. P. Frolov, M. Markov and V. Mukhanov, Phys. Rev. D41 (1990), 383.

[2] V. P. Frolov and G. A. Vilkoviskii, Phys. Lett. B106 (1981), 307.

[3] D. I. Kazakov and S. N. Solodukhin, Nucl. Phys. B429 (1994), 153.

[4] J. W. York, Phys. Rev. D31 (1985), 775.

[5] V. P. Frolov, W. Israel and S. N. Solodukhin, Phys.Rev. D54, (1996), 2732.

[6] S. N. Solodukhin, Phys. Rev. D53 (1996), 824.

[7] O. B. Zaslavskii, Phys. Lett. B242 (1998), 271.

[8] O. B. Zaslavskii, Phys. Rev. D56 (1997), 2188;

[9] R.B. Mann and S.N. Solodukhin, Nucl.Phys. B253 (1998), 293.

[10] J. Maldacena, The large $N$ limit of superconformal field theories and supergravity, [hep-th/9711200]. S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Gauge theory correlators from noncritical string theory, [hep-th/9802109]; E. Witten, Anti-de Sitter space and holography, [hep-th/9802150].

[11] V. Mukhanov, A. Wipf and A. Zelnikov, Phys. Lett. B332 (1994), 283; R. Bousso and S.W. Hawking, Phys. Rev. D56 (1997), 7788; W. Kummer, H. Liebl and D.V. Vassilevich, Mod. Phys. Lett A12 (1997), 2683; S. Nojiri and S.D. Odintsov, Phys. Rev. D57 (1998), 4847; J. S. Dowker, Class. Quant. Grav. 15 (1998), 1881; A. Mikovic and
V. Radovanovic, Nucl. Phys. B504 (1997), 511; R. Balbinot and A. Fabbri, Hawking radiation by effective two-dimensional theories, gr-qc/9807123.

[12] R. Camporesi, Phys. Rep. 196 (1990), 1.

[13] R. Kallosh, Phys. Lett. B282 (1992), 80.

[14] O. B. Zaslavskii, Geometry and thermodynamics of quantum-corrected acceleration horizons, gr-qc/9812052.