Abstract

A systematic method is presented for the construction and classification of algebras of gauge transformations for arbitrary high rank tensor gauge fields. For every tensor gauge field of a given rank, the gauge transformation will be stated, in a generic way, via an ansatz that contains all the possible terms, with arbitrary coefficients and the maximum number of tensor gauge functions. The requirement for the closure of the algebra will prove to be restrictive, but, nevertheless, leave a variety of choices. Properly adjusting the values of the initial coefficients and imposing restrictions on the gauge functions, one can, one the one hand, recover all the, so far, analysed algebras and on the other, construct new ones. The presentation of a brand new algebra for tensor gauge transformations is the central result of this article.
1 Introduction

The investigation on theories describing fields of arbitrary high spin dates back to 1936, when P. Dirac formulated the Relativistic Wave Equations for massless and massive fermions of spin greater that $1/2$ [1]. His work was extended by M. Fierz and W. Pauli, who followed a field theoretical approach to include bosons of higher spin [2], and tuned a few months later [4]. All the attempts gave elegant results for the free particle case, but did not succeed to include the electromagnetic interactions in a satisfactory and free of ambiguities unified framework. So far, the physical requirements for constructing higher spin theories was the invariance under Poincaré transformations and the positivity of the energy after quantisation [8]. The revolutionary, group theoretical approach carried out in [3, 5], together with the establishment of the Principle of Invariance under Gauge Transformations [6], opened the way for the introduction of new models for the description of higher spin fields with less restrictive principal requirements [7, 8, 9]. Nevertheless, a satisfactory and self contained, interacting higher spin field theory was yet to be formulated both for the abelian (electromagnetic) and non-abelian interactions (weak, strong).

On 2005 it was proposed that the description of higher spin particles can be carried out following a natural extension of the Yang-Mills (Y-M) principle to include higher rank bosonic and fermionic fields [10]. The theory was developed quite extensively under the name Tensor Gauge Field Theory, has been theoretically tested with regards to self consistency issues [12, 15], properly analysed and extrapolated to provide both phenomenological and theoretical predictions [13, 17, 18, 19, 23, 24, 25] and served as a basis for further developments [26].

As far as one departs from the Y-M case, where the mediators of the forces are described by vector gauge fields, one faces the challenging problem of defining new algebras for tensor gauge transformations. Under the light and guidance of the Y-M principle, the starting point for the formulation of a consistent higher spin theory is the extension of gauge transformations to higher rank tensor fields. Fortunately, this investigation is facilitated by the requirement that new proposed transformations are legitimate candidates as long as they form a closed algebraic structure and hence belong to the class of Lie algebras. Quantitatively this is translated to the imperative that the commutator of two infinitesimal successive transformations belongs to the same set of transformations.
In the following section, we give a brief review of all closed algebras that have been proposed under the framework of Tensor Gauge Field Theory and make some remarks concerning their internal structure. In section 3, we present a general method that will, one the one hand, allow the embedding of the previously mentioned algebras in a generalised framework and on the other, facilitate the investigation of new ones. Finally, we conclude with a review of the new algebra constructed by the introduced method and comment on the potentials of its usage.

2 Known Closed Algebras

In recent papers, \[10, 11, 14, 16, 20\], a number of gauge transformations for bosonic tensor fields of arbitrary high rank have been constructed and proven to form a closed algebraic structure. Each of these algebras was constructed to serve a particular purpose. The first that appeared historically, under the framework of Tensor Gauge Field Theory, served as the building block for the construction of field strength tensors, by the aid of which gauge invariant Langrangians for higher rank fields were presented. We will call these transformations the standard ones, to discriminate from the others that followed. The dual gauge transformations, appeared in two versions, and elucidated the fact that two seemingly different algebras can be equivalent and thus related by a similarity transformation. The symmetric gauge Transformations was part of an attempt to formulate a theory solely for the irreducible representations of the Poincaré group which proved to be technically impossible. Finally, the algebra of the gauge transformations used for the construction of new topological invariants in higher dimensions is our last example of extended algebras. We will call these transformations, conjugate, for classification purposes.

2.1 Standard Extended Gauge Transformations

The algebra of standard extended gauge Transformations appeared for the first time in \[10\], as the fundamental building block of non-Abelian Tensor Gauge Field Theory \[11, 12\]. It concerns higher rank bosonic fields, \(A_{\mu \lambda_1 \lambda_2 ... \lambda_s}\), which are by construction symmetric under the permutation of their lambda indices, but bear no symmetry with respect to the index \(\mu\). Besides, the tensor gauge functions, \(\xi_{\lambda_1 \lambda_2 ... \lambda_s}\), needed for the definition of the gauge function.
transformations of $A_{\mu \lambda_1 \lambda_2 \ldots \lambda_s}$, are totally symmetric\(^2\).

For successively higher rank fields, the gauge transformations are given below,

\[
\begin{align*}
\delta \xi A_\mu &= \partial_\mu \xi - ig [A_\mu, \xi] \\
\delta \xi A_{\mu \lambda} &= \partial_\mu \xi_\lambda - ig ([A_\mu, \xi_\lambda] + [A_{\mu \lambda}, \xi]) \\
\delta \xi A_{\mu \lambda_1 \lambda_2} &= \partial_\mu \xi_{\lambda_1 \lambda_2} - ig ([A_\mu, \xi_{\lambda_1 \lambda_2}] + [A_{\mu \lambda_1}, \xi_{\lambda_2}] + [A_{\mu \lambda_2}, \xi_{\lambda_1}] + [A_{\mu \lambda_1 \lambda_2}, \xi]) \\
& \quad \ldots \\
\delta \xi A_{\mu \lambda_1 \ldots \lambda_s} &= \partial_\mu \xi_{\lambda_1 \ldots \lambda_s} - ig \sum_{i=0}^{s} \sum_{P_s} [A_{\mu \lambda_1 \ldots \lambda_i}, \xi_{\lambda_{i+1} \ldots \lambda_s}].
\end{align*}
\]

(2.1)

The first is the well known $Y-M$ gauge transformation for the vector field, while the rest define the way the higher rank bosonic tensors transform. In particular, the non-homogenous part of the transformations involves differentiation in terms of the first index $\mu$. As one can show, they define an infinite dimensional gauge group, $G$, with a closed algebraic structure [11],

\[
[\delta \xi, \delta \eta] A_{\mu \lambda_1 \lambda_2 \ldots \lambda_s} = -ig \delta \xi A_{\mu \lambda_1 \lambda_2 \ldots \lambda_s}
\]

(2.2)

where,

\[
\begin{align*}
\zeta &= [\xi, \eta] \\
\zeta_\lambda &= [\xi, \eta_\lambda] + [\xi_\lambda, \eta] \\
\zeta_{\lambda_1 \lambda_2} &= [\xi, \eta_{\lambda_1 \lambda_2}] + [\xi_{\lambda_1}, \eta_{\lambda_2}] + [\xi_{\lambda_2}, \eta_{\lambda_1}] + [\xi_{\lambda_1 \lambda_2}, \eta] \\
& \quad \ldots \\
\zeta_{\lambda_1 \ldots \lambda_s} &= [\xi, \eta_{\lambda_1 \ldots \lambda_s}] + \sum_{i=1}^{s} [\xi_{\lambda_i}, \eta_{\lambda_1 \ldots \lambda_{i-1} \lambda_{i+1} \ldots \lambda_s}] + \cdots + [\xi_{\lambda_1 \ldots \lambda_s}, \eta]
\end{align*}
\]

(2.3)

The above algebra allowed for the definition of consistent field strength tenors which gauge transform homogeneously, for the construction of two classes of gauge invariant *Lagrangians* for each rank of tensor gauge fields, and for a fundamental extension of the *Poincaré* group [21].

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\(^2\)The tensor gauge fields and the tensor gauge functions carry a colour index which enumerates the independent generators of the underlying $SU(N)$ Lie algebra. Hence, both the quantities are summation shortcuts over the fundamental representation matrices of the generators of the *Lie* algebra, $A_{\mu \lambda_1 \lambda_2 \ldots \lambda_s} = A^{a}_{\mu \lambda_1 \lambda_2 \ldots \lambda_s} L^a$ and $\xi_{\lambda_1 \ldots \lambda_s} = \xi^{a}_{\lambda_1 \ldots \lambda_s} L^a$, where $[L^a, L^b] = if^{abc} L^c$.  

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3
2.2 Dual Extended Gauge Transformations

Parallely to the previous transformations, one can define complementary ones which ended up to be equivalent with the former. They have been formulated in two different versions [14, 16], which we will shortly review.

2.2.1 1st Version

In the first version [14], the roles of the \( \mu \) and the \( \lambda \) indices are properly interchanged so that the inhomogeneous terms entail derivatives with respect to each of the latter indices. Up to the tensor of third rank, the transformations are listed below,

\[
\begin{align*}
\tilde{\delta}_\xi A_\mu &= \partial_\mu \xi - ig[A_\mu, \xi] \\
\tilde{\delta}_\xi A_{\mu\lambda} &= \partial_\lambda \xi_\mu - ig ([A_\lambda, \xi_\mu] + [A_{\mu\lambda}, \xi]) \\
\tilde{\delta}_\xi A_{\mu\lambda_1\lambda_2} &= \partial_{\lambda_1} \xi_{\mu\lambda_2} + \partial_{\lambda_2} \xi_{\mu\lambda_1} - ig ([A_{\lambda_1}, \xi_{\mu\lambda_2}] + [A_{\lambda_2}, \xi_{\mu\lambda_1}] + [A_{\mu\lambda_1}, \xi_{\lambda_2}] + [A_{\mu\lambda_2}, \xi_{\lambda_1}] + [A_{\lambda_1\lambda_2}, \xi_\mu] + [A_{\lambda_2\lambda_1}, \xi_\mu] + [A_{\mu\lambda_1\lambda_2}, \xi]) \\
\end{align*}
\]

(2.4)

It has been proved [14], that the above transformations form a closed algebraic structure. Indeed, the commutator of two successive transformations leads to a third one,

\[
[\tilde{\delta}_\xi, \tilde{\delta}_\eta] A_{\mu_1...\mu_s} = -ig\tilde{\delta}_\zeta A_{\mu_1...\mu_s},
\]

(2.5)

with

\[
\begin{align*}
\zeta &= [\xi, \eta] \\
\zeta_\lambda &= [\xi, \eta_\lambda] + [\xi_\lambda, \eta] \\
\zeta_{\lambda_1\lambda_2} &= [\xi, \eta_{\lambda_1\lambda_2}] + [\xi_{\lambda_1}, \eta_{\lambda_2}] + [\xi_{\lambda_2}, \eta_{\lambda_1}] + [\xi_{\lambda_1\lambda_2}, \eta] \\
\zeta_{\lambda_1...\lambda_s} &= [\xi, \eta_{\lambda_1...\lambda_s}] + \sum_{i=1}^{s} [\xi_{\lambda_i}, \eta_{\lambda_1...\lambda_{i-1}\lambda_{i+1}...\lambda_s}] + \cdots + [\xi_{\lambda_1...\lambda_s}, \eta]
\end{align*}
\]

(2.6)
Besides this, in the same article, it has been shown that the introduction of the transpose tensor gauge fields,

\[
\tilde{A}_{\mu} = A_{\mu}
\]
\[
\tilde{A}_{\mu\lambda\lambda_2} = \frac{1}{2} (A_{\lambda_1\mu\lambda_2} + A_{\lambda_2\mu\lambda_1}) - \frac{1}{2} A_{\mu\lambda_1\lambda_2}
\]
\[
\tilde{\ldots}
\]
\[
\tilde{A}_{\mu\lambda_1...\lambda_s} = \frac{1}{s} \sum_{i=1}^{s} (A_{\lambda_i\mu\lambda_1...\lambda_{i-1}\lambda_{i+1}...\lambda_s}) - \frac{s}{s-1} A_{\mu\lambda_1...\lambda_s}, \quad (2.7)
\]

illuminates the fact that the duality map serves as a similarity transformation between two representations of the same algebra [16]. Nevertheless, a bug in the definition of the transposition operator which transforms the fields \(A_{\mu\lambda_1...\lambda_s}\) into the fields \(\tilde{A}_{\mu\lambda_1...\lambda_s}\) and in particular the fact that the former is not an idempotent, resulted in the necessity for a second version of the dual gauge transformations which would resolve this bug.

2.2.2 2nd Version

In the article [16] it has been shown that if, instead of (2.8), we use the following definition for the transposed fields,

\[
\tilde{\bar{A}}_{\mu} = A_{\mu}
\]
\[
\tilde{\bar{A}}_{\mu\lambda\lambda_2} = \frac{2}{3} (A_{\lambda_1\mu\lambda_2} + A_{\lambda_2\mu\lambda_1}) - \frac{1}{3} A_{\mu\lambda_1\lambda_2}
\]
\[
\tilde{\ldots}
\]
\[
\tilde{\bar{A}}_{\mu\lambda_1...\lambda_s} = \frac{2}{s+1} \sum_{i=1}^{s} (A_{\lambda_i\mu\lambda_1...\lambda_{i-1}\lambda_{i+1}...\lambda_s}) - \frac{s}{s+1} A_{\mu\lambda_1...\lambda_s}, \quad (2.8)
\]

we see that the transposition operator becomes an idempotent, as it should be. However, the change in the definition of the transposition operator reflects itself upon the definition of the gauge transformations, which now acquire the form [16],

\[
\tilde{\delta}_\xi A_{\mu} = \partial_{\mu} \xi - ig[A_{\mu}, \xi]
\]
\[
\tilde{\delta}_\xi A_{\mu\lambda} = \partial_{\lambda} \xi_{\mu} - ig[A_{\lambda}, \xi_{\mu}] - ig[A_{\mu\lambda}, \xi]
\]
\[
\tilde{\delta}_\xi A_{\mu\lambda\lambda_2} = \frac{2}{3} (\partial_{\lambda_1} \xi_{\mu\lambda_2} - ig[A_{\lambda_1}, \xi_{\mu\lambda_2}] + \partial_{\lambda_2} \xi_{\mu\lambda_1} - ig[A_{\lambda_2}, \xi_{\mu\lambda_1}]) - \frac{1}{3} (\partial_{\mu} \xi_{\lambda_1\lambda_2} - ig[A_{\mu}, \xi_{\lambda_1\lambda_2}])
\]
\[
-ig\frac{2}{3}[A_{\mu\lambda_1}, \xi_{\lambda_2}] - ig\frac{2}{3}[A_{\lambda_1\lambda_2}, \xi_{\mu}] + ig\frac{1}{3}[A_{\lambda_1\mu}, \xi_{\lambda_2}]
\]
\[
-ig\frac{2}{3}[A_{\mu\lambda_2}, \xi_{\lambda_1}] - ig\frac{2}{3}[A_{\lambda_2\lambda_1}, \xi_{\mu}] + ig\frac{1}{3}[A_{\lambda_2\mu}, \xi_{\lambda_1}] - ig[A_{\mu\lambda_1\lambda_2}, \xi]
\]
\[
\tilde{\ldots}
\]
\[
(2.9)
\]
and has been proven to form a closed algebraic structure. We see here that the price we paid for casting the transposition operator an idempotent, was to include, in the transformations of higher rank fields, derivatives over the index $\mu$.

### 2.3 Symmetrized Extended Gauge Transformations

According to E. Wigner, one particle states fall into irreducible representations of the Poincaré group [3, 5]. Since the irreducible component of a tensor, that describe the physical propagating modes, is its symmetric component, it sounds reasonable to specialise the gauge transformations for symmetric, over all their indices, tensor gauge fields [11]. To simplify the notation, we replace the index $\mu$ with the index $\lambda_1$, so that the symmetry properties of the tensor become more apparent. The totally symmetric version of the transformations (2.1) is given below,

$$
\delta_A^{S\lambda_1\lambda_2\lambda_3} = \partial_{\lambda_1}\xi_{\lambda_2} + \partial_{\lambda_2}\xi_{\lambda_1} - ig ([A_{\lambda_1}, \xi_{\lambda_2}] + [A_{\lambda_2}, \xi_{\lambda_1}] + [A_{\lambda_1\lambda_2}, \xi])
$$

$$
\delta_A^{S\lambda_1\lambda_2\lambda_3} = \partial_{\lambda_1}\xi_2 + \partial_{\lambda_2}\xi_{\lambda_1} + \partial_{\lambda_3}\xi_{\lambda_1\lambda_2} - ig ([A_{\lambda_1}, \xi_{\lambda_2\lambda_3}] + [A_{\lambda_2}, \xi_{\lambda_3\lambda_1}] + [A_{\lambda_3}, \xi_{\lambda_1\lambda_2}] +
$$

$$
+ [A_{\lambda_2\lambda_3}, \xi_{\lambda_1}] + [A_{\lambda_3\lambda_1}, \xi_{\lambda_2}] + [A_{\lambda_1\lambda_2}, \xi_{\lambda_3}] + [A_{\lambda_1\lambda_2\lambda_3}, \xi])
$$

$$
\delta_A^{S\lambda_1...\lambda_s} = \sum_{i=1}^{s} \partial_{\lambda_i}\xi_{\lambda_1...\lambda_{i-1}\lambda_{i+1}...\lambda_s} - ig \left( \sum_{i=1}^{s} [A_{\lambda_i}, \xi_{\lambda_1...\lambda_{i-1}\lambda_{i+1}...\lambda_s}] + \right.
$$

$$
\left. + \sum_{i<j} [A_{\lambda_i\lambda_j}, \xi_{\lambda_1...\lambda_{i-1}\lambda_{i+1}...\lambda_{j-1}\lambda_{j+1}...\lambda_s}] + \cdots + [A_{\lambda_1...\lambda_s}, \xi] \right) \quad (2.10)
$$

The above transformations have been proven to form another set of a closed algebra [11],

$$
[\delta_\zeta, \delta_\eta]A_{\lambda_1\lambda_2...\lambda_s}^S = -ig\delta_\zeta A_{\lambda_1\lambda_2...\lambda_s}^S, \quad (2.11)
$$

where

$$
\zeta = [\xi, \eta]
$$

$$
\zeta_\lambda = [\xi, \eta_\lambda] + [\xi_\lambda, \eta]
$$

$$
\zeta_{\lambda_1\lambda_2} = [\xi, \eta_{\lambda_1\lambda_2}] + [\xi_{\lambda_1}, \eta_{\lambda_2}] + [\xi_{\lambda_2}, \eta_{\lambda_1}] + [\xi_{\lambda_1\lambda_2}, \eta]
$$

$$
\cdots
$$

$$
\zeta_{\lambda_1...\lambda_s} = [\xi, \eta_{\lambda_1...\lambda_s}] + \sum_{i=1}^{s} [\xi_{\lambda_i}, \eta_{\lambda_1...\lambda_{i-1}\lambda_{i+1}...\lambda_s}] + \cdots + [\xi_{\lambda_1...\lambda_s}, \eta] \quad (2.12)
$$
Defining,
\[
A^S_{\mu\lambda_1\ldots\lambda_s} \equiv \sum_{i=1}^{s} A_{\lambda_i\lambda_1\ldots\lambda_{i-1}\mu\lambda_{i+1}\ldots\lambda_s},
\]
we observe\(^3\) that \(^1\(^4\),
\[
A_{\mu\lambda_1\ldots\lambda_s} + \tilde{A}_{\mu\lambda_1\ldots\lambda_s} = \frac{1}{s} A^S_{\mu\lambda_1\ldots\lambda_s}
\]
Nevertheless, from the above transformations, no homogeneously gauge transformed field strength tensor can be defined and hence, no gauge invariant \textit{Lagrangian} that consists solely of symmetric tensor gauge fields can be constructed. This is probably a hint that the construction of a higher spin theory dictates for a further departure of the common practices which focus primarily on irreducible representations of the \textit{Poincaré} group and hence, on one particle states. This is, however, the case with the \textit{Dirac} equation which, inevitably describes both the particle and antiparticle in the same 4-spinor, reducible representation. The above observation underlines the central role played by the index \(\mu\) in the standard transformations, without which the construction of a gauge invariant \textit{Lagrangian} would be impossible. The further relaxation of the symmetry of the indices and in particular the formulation of the theory without any symmetry properties between them was the primary motivation for the current study.

### 2.4 Conjugate Extended Gauge Transformations

Here, we review the algebra found in \[^{20}\] and served as the basis for the construction of new topological invariants and \textit{Chern-Simons} forms in higher dimensions \[^{22}\] \[^{23}\]. The higher rank tensors fields and tensor gauge functions are antisymmetric under permutations of indices of the same letter, e.g. \((\sigma_1 \leftrightarrow \sigma_2)\) and symmetric under permutations of pairs of indices of different letters (e.g. \((\sigma_1\sigma_2) \leftrightarrow (\rho_1\rho_2)\)). There is no symmetry with respect to the index \(\mu\).

\[
\begin{align*}
\delta A_{\mu} &= \partial_{\mu} \xi - ig[A_{\mu}, \xi] \\
\delta A_{\mu\sigma_1\sigma_2} &= \partial_{\mu} \xi_{\sigma_1\sigma_2} - ig([A_{\mu}, \xi_{\sigma_1\sigma_2}] + [A_{\mu\sigma_1\sigma_2}, \xi]) \\
\delta A_{\mu\sigma_1\sigma_2\rho_1\rho_2} &= \partial_{\mu} \xi_{\sigma_1\sigma_2\rho_1\rho_2} - ig([A_{\mu}, \xi_{\sigma_1\sigma_2\rho_1\rho_2}] + [A_{\mu\sigma_1\sigma_2}, \xi_{\rho_1\rho_2}] + [A_{\mu\rho_1\rho_2}, \xi_{\sigma_1\sigma_2}] + [A_{\mu\sigma_1\sigma_2\rho_1\rho_2}, \xi]) \\
&\quad + \ldots
\end{align*}
\]

\(^3\)Here we follow the definition of the transpose tensor gauge fields as given in the 1st version.
The above transformations follow the spirit of the standard ones. Indeed, if we treat the additional pairs of indices, as one index, \( \sigma_1 \sigma_2 = \hat{\sigma} \), the two algebras coincide. By construction, only tensor fields of odd number of indices participate in the algebra. This, as it might seems restrictive, has been proven sufficient for the construction of new topological invariants \cite{22, 23}.

3 A Systematic Treatment of Extended Gauge Algebras

In this section we present a systematic method for constructing algebras of gauge transformations for arbitrary high rank tensor gauge fields. For every tensor field of a given rank, the gauge transformation will be stated in a generic way via an ansatz that contains all the possible terms, with arbitrary coefficients, and the maximum number of tensor gauge functions. The requirement for the closure of the algebra will prove to be restrictive, but, nevertheless, leave a variety of choices. Properly adjusting the values of the initial coefficients and imposing restrictions on the gauge functions, one can, on the other hand, recover all the, so far, analysed algebras and, on the other, construct new ones.

3.1 Vector Gauge Fields

We begin with the fundamental building block of the Standard Model, the \( Y-M \) vector gauge transformations. Since we know that the group of gauge transformations is uniquely defined, we will suffice to present a proof for the closure of the algebra, just to clarify the details of the formalism that will be adopted throughout the article. The infinitesimal, non-abelian gauge transformations are known to be,

\[
\delta_\xi A_\mu = \partial_\mu \xi - ig[A_\mu, \xi]
\] (3.1)

In order to check that the algebra of these transformations has a closed structure, let us calculate the commutator of two successive operations on the vector field \( A_\mu \). During the process, we take into account that the operators \( \delta_\xi \) and \( \delta_\eta \), act only on \( A_\mu \), since gauge
functions are not themselves gauge transformed. We have,
\[ \frac{i}{g}[\delta \xi, \delta \eta]A_\mu = \frac{i}{g}(\delta \xi (\delta \eta A_\mu) - \delta \eta (\delta \xi A_\mu)) = [\delta \xi A_\mu, \eta] - [\delta \eta A_\mu, \xi] \]
\[ = [\partial_\mu \xi, \eta] - [\partial_\mu \eta, \xi] - ig([A_\mu, \xi], \eta) - [A_\mu, \eta], \xi) \]
\[ = \partial_\mu [\xi, \eta] - ig[A_\mu, [\xi, \eta]], \]
where in the last step we used the Jacobi Identity. We conclude that,
\[ [\delta \xi, \delta \eta]A_\mu = -ig\delta \xi A_\mu, \]
where \( \zeta = [\xi, \eta] \).

This shows that the commutator of two successive \( Y-M \) gauge transformations results in a third one, thus proving that they form a closed algebraic structure.

### 3.2 Second Rank Tensor Gauge Fields

We wish to follow an analogous procedure to examine the structure of the extended non-abelian gauge transformations. In order to begin with the most general transformations for 2nd rank tensor gauge fields, we introduce two vector gauge functions, \( \xi^1_\mu, \xi^2_\mu \), four coefficients, \( c_i \), and begin with the ansatz,
\[ \delta \xi^1 A_{\mu \nu} = \partial_\mu \xi^1_\nu + \partial_\nu \xi^2_\mu - ig ([A_\mu, c_1 \xi^1_\nu + c_2 \xi^2_\mu] + [A_\nu, c_3 \xi^1_\mu + c_4 \xi^2_\mu] + [A_{\mu \nu}, \xi]). \] 
(3.4)

We shall attempt a determination of the coefficients under the restriction of the closure of the algebra. The commutator of two successive operations on the tensor field \( A_{\mu \nu} \) is,
\[ \frac{i}{g}[\delta \xi, \delta \eta]A_{\mu \nu} = \frac{i}{g}(\delta \xi (\delta \eta A_{\mu \nu}) - \delta \eta (\delta \xi A_{\mu \nu})) = \]
\[ = [\delta \xi A_{\mu \nu}, c_1 \eta^1_\nu + c_2 \eta^2_\mu] + [\delta \xi A_{\nu \nu}, c_3 \eta^1_\mu + c_4 \eta^2_\mu] + [\delta \xi A_{\mu \nu}, \eta] - [\delta \eta A_{\mu \nu}, c_1 \xi^1_\nu + c_2 \xi^2_\mu] - [\delta \eta A_{\nu \nu}, c_3 \xi^1_\mu + c_4 \xi^2_\mu] - [\delta \eta A_{\mu \nu}, \xi] \]
(3.5)

To determine the coefficients let us first examine the structure of the zeroth order terms over the gauge coupling \( g \). Substituting (3.4), we get,
\[ \frac{i}{g}[\delta \xi, \delta \eta]A_{\mu \nu} = \partial_\mu [\xi^1_\nu + c_2 \eta^2_\mu] + [\partial_\nu [\xi^1_\nu + c_3 \eta^2_\mu] + [\xi, \partial_\eta \eta^1_\nu + \partial_\nu \eta^2_\mu] - [\partial_\eta \eta^1_\nu + \partial_\nu \eta^2_\mu] - [\partial_\nu \eta^1_\nu + \partial_\nu \eta^2_\mu] - \eta [\partial_\mu \xi^1_\nu + c_2 \xi^2_\mu] + [\partial_\nu \eta^1_\mu + \partial_\nu \eta^2_\mu] - [\eta, \partial_\mu \xi^1_\nu] + \partial_\nu \xi^2_\mu] + O(g) = \]
\[ = \partial_\mu ([\xi^1_\nu - \eta^1_\nu]) + \partial_\nu ([\xi^2_\mu - \eta^2_\mu]) + [\partial_\mu \xi, c_2 \eta^2_\nu + (c_1 - 1) \eta^1_\mu] - [\partial_\mu \eta, c_2 \xi^2_\nu + (c_1 - 1) \xi^1_\mu] + [\partial_\nu \eta, c_3 \xi^1_\mu + (c_4 - 1) \xi^2_\mu] + O(g), \]
(3.6)
where,

\[ \mathcal{O}(g) = -ig \left\{ \left[ [A_\mu, \xi], c_1 \eta_\nu + c_2 \eta_\nu^2 \right] + \left[ [A_\nu, \xi], c_3 \eta_\mu + c_4 \eta_\mu^2 \right] - \left[ [A_\mu, \eta], c_1 \xi_\nu + c_2 \xi_\nu^2 \right] - \left[ [A_\nu, \eta], c_3 \xi_\mu + c_4 \xi_\mu^2 \right] + \left[ [A_\mu, c_1 \xi_\nu + c_2 \xi_\nu^2], \eta \right] + \left[ [A_\nu, c_3 \xi_\mu + c_4 \xi_\mu^2], \eta \right] + \left[ [A_\mu, \xi_\nu], c_1 \eta_\mu + c_2 \eta_\mu^2 \right] - \left[ [A_\nu, \xi_\mu], c_3 \eta_\nu + c_4 \eta_\nu^2 \right] + \left[ [A_\mu, \xi_\nu], \eta \right] + \left[ [A_\nu, \xi_\mu], \eta \right] + \left[ [A_\mu, \xi_\nu], \xi_\mu \right] + \left[ [A_\nu, \xi_\mu], \xi_\nu \right] \right\} \]  \quad (3.7)

What is obvious from (3.4) is that if the commutator of two successive transformations is to represent another transformation, the zeroth order terms in \( g \) must be written solely as partial derivatives over the two indices of the 2nd rank tensor. Hence, the only choices for the terms of the last two lines of (3.6) are either to eliminate them by properly choosing the coefficients \( c_i \), or to absorb them in the terms of the first two lines, by imposing suitable symmetry properties and restricting the vector gauge functions \( \xi_i^\mu \).

Following the first option, we keep the two vector gauge functions \( \xi_i^\mu \), \( i = 1, 2 \), independent and eliminate the last four terms of (3.6) by setting

\[ c_1 = c_4 = 1 \quad , \quad c_2 = c_3 = 0 \]

Then we get,

\[ \frac{i}{g} \left[ \delta_\xi, \delta_\eta \right] A_{\mu\nu} = \partial_\mu \zeta_\nu^1 + \partial_\nu \zeta_\mu^2 + \mathcal{O}(g) \]  \quad (3.8)

where,

\[ \zeta_\mu^i = [\xi, \eta_\mu^i] + [\xi^i_\mu, \eta], \quad i = 1, 2 \]  \quad (3.9)

In order to guarantee that the algebra closes for this specific choice of the \( c_i \)'s and the new gauge functions \( \zeta_\mu^1, \zeta_\mu^2 \), and furthermore to determine \( \zeta \), we need to examine the \( \mathcal{O}(g) \) terms. We have,

\[ \frac{i}{g} \mathcal{O}(g) = \left[ [A_\mu, \xi], \eta_\nu \right] + \left[ [\eta_\nu, A_\mu], \xi \right] + \left[ [A_\nu, \xi], \eta_\mu \right] + \left[ [\eta_\mu, A_\nu], \xi \right] + \left[ [A_\mu, \xi_\nu^1], \eta \right] + \left[ [\eta, A_\mu], \xi^1_\nu \right] + \left[ [A_\nu, \xi_\mu^2], \eta \right] + \left[ [\eta, A_\nu], \xi^2_\mu \right] + \left[ [A_\mu, \xi_\nu], \eta \right] + \left[ [\eta, A_\mu], \xi \right] = \left[ A_\mu, [\xi, \eta_\nu^1] - [\eta, \xi_\nu^1] \right] + \left[ A_\nu, [\xi, \eta_\mu^2] - [\eta, \xi_\mu^2] \right] + [A_{\mu\nu}, [\xi, \eta]]. \]  \quad (3.10)

where in the last step we employed *Jacobi Identity*, once for each pair of adjacent terms. We conclude that,

\[ \mathcal{O}(g) = -ig \left\{ [A_\mu, \zeta_\nu^1] + [A_\nu, \zeta_\mu^2] + [A_{\mu\nu}, \zeta] \right\}, \]  \quad (3.11)
with $\zeta = [\xi, \eta]$. Hence, the gauge transformations,

$$
\delta_\xi A_{\mu\nu} = \partial_\mu \xi_1^\nu + \partial_\nu \xi_2^\mu - ig \left( [A_\mu, \xi_1^\nu] + [A_\nu, \xi_2^\mu] + [A_{\mu\nu}, \xi] \right),
$$

(3.12)

form a closed group algebra with

$$
[\delta_\xi, \delta_\eta] A_{\mu\nu} = -ig \delta_\zeta A_{\mu\nu}
$$

(3.13)

and

$$
\zeta = [\xi, \eta]
$$

$$
\zeta_1^\mu = [\xi, \eta_1^\mu] + [\xi_1^\mu, \eta]
$$

$$
\zeta_2^\mu = [\xi, \eta_2^\mu] + [\xi_2^\mu, \eta]
$$

(3.14)

What we observe from (3.12) is that the vector field, $A_\mu$, participates in the gauge transformation of $A_{\mu\nu}$ only as part of the covariant derivative over the two gauge functions.

Thus far, we have managed to merge the two independent, standard and dual, extended gauge transformations into a more general one, indicating a higher symmetry for the system of 2nd rank tensor gauge fields. Before we continue the same procedure for the 3rd rank tensor gauge fields, let us recover the algebras of the standard (2.1) and dual (2.4) transformations for the 2nd rank field.

Let us return to (3.6) and check if there is an alternative way to nullify the terms that cannot be written as partial derivatives with respect to the indices $\mu$ and $\nu$. If we set the second vector gauge function equal to zero, we see that the algebra closes if and only if, $c_1 = 1$, $c_3 = 0$. This, obviously, recovers the standard transformations for the 2nd rank field (2.1). If, on the other hand, we set the first vector gauge function to equal to zero, the algebra closes provided, $c_2 = 0$, $c_4 = 1$. This recovers the dual transformations for the 2nd rank field (2.4). Lastly, setting in the two vector gauge functions equal, and symmetrising the tensor gauge field over its two indices, we recover the first equation of the symmetrized transformations (2.10).

---

4The standard geometrical interpretation of $A_\mu$, as a principal bundle connection is not directly extrapolated for analogous interpretations of the higher rank gauge fields. This will be examined in subsequent studies.
3.3 Third Rank Tensor Gauge Fields

In an analogous way followed in the case of the 2nd rank tensor gauge fields, let us introduce three 2nd rank gauge functions $\xi^i_{\mu\nu}, i = 1, 2, 3,$ and begin with the ansatz

$$\delta \xi A_{\mu\nu\lambda} = \partial_\mu \xi^1_{\nu\lambda} + \partial_\nu \xi^2_{\lambda\mu} + \partial_\lambda \xi^3_{\mu\nu} - ig \left( [A_\mu, \xi^1_{\nu\lambda}] + [A_\nu, \xi^2_{\lambda\mu}] + [A_\lambda, \xi^3_{\mu\nu}] + [A_{\mu\nu}, c_1 \xi^1_{\lambda\nu} + c_2 \xi^2_{\lambda\mu}] + [A_{\nu\lambda}, c_3 \xi^1_{\mu\nu} + c_4 \xi^2_{\mu\nu}] + [A_{\lambda\mu}, c_5 \xi^1_{\nu\lambda} + c_6 \xi^2_{\nu\lambda}] \right)$$

(3.15)

We will try to determine the 6 coefficients $c_i$ so that the algebra of the gauge transformations forms a closed structure. We need to underline that the 3rd rank tensor gauge field, $A_{\mu\nu\lambda}$, contrary to its standard definition [10], bares no symmetry under the permutation of its last two indices. The same holds for the three tensor gauge functions, $\xi^i_{\mu\nu}$ which are no longer symmetric. This is a significant departure which will hopefully lead to an enhancing of the symmetry of the theory of non-abelian tensor gauge fields.

The commutator of two successive gauge transformations on the field $A_{\mu\nu\lambda}$ gives,

$$\frac{i}{g} [\delta \xi, \delta \eta] A_{\mu\nu\lambda} = \frac{i}{g} \left( \delta \xi (\delta \eta A_{\mu\nu\lambda}) - \delta \eta (\delta \xi A_{\mu\nu\lambda}) \right) =$$

$$= [\delta \xi A_\mu, \eta^1_{\nu\lambda}] + [\delta \xi A_\nu, \eta^2_{\lambda\mu}] + [\delta \xi A_\lambda, \eta^3_{\mu\nu}] - [\delta \eta A_\mu, \xi^1_{\nu\lambda}] - [\delta \eta A_\nu, \xi^2_{\lambda\mu}] - [\delta \eta A_\lambda, \xi^3_{\mu\nu}] +$$

$$+ [\delta \xi A_{\mu\nu}, c_1 \eta^1_{\lambda\nu} + c_2 \eta^2_{\lambda\mu}] + [\delta \xi A_{\nu\lambda}, c_3 \eta^1_{\mu\nu} + c_4 \eta^2_{\mu\nu}] + [\delta \xi A_{\lambda\mu}, c_5 \eta^1_{\nu\lambda} + c_6 \eta^2_{\nu\lambda}] -$$

$$- [\delta \eta A_{\mu\nu}, c_1 \xi^1_{\lambda\nu} + c_2 \xi^2_{\lambda\mu}] - [\delta \eta A_{\nu\lambda}, c_3 \xi^1_{\mu\nu} + c_4 \xi^2_{\mu\nu}] - [\delta \eta A_{\lambda\mu}, c_5 \xi^1_{\nu\lambda} + c_6 \xi^2_{\nu\lambda}] +$$

$$+ [\delta \xi A_{\mu\nu\lambda}, \eta] - [\delta \eta A_{\mu\nu\lambda}, \xi]$$

(3.16)

To determine the coefficients for the gauge transformation (3.15), we will first examine the structure of the zeroth order terms over the gauge coupling $g$. As is indicated in (3.15) a necessary condition for the closure of the algebra is that the zeroth order terms must be written as partial derivatives over the three indices $\mu, \nu$ and $\lambda$. Focusing on the zeroth order terms over the coupling $g$, and substituting (3.1), (3.12) and (3.15) in the above

5Directed from the case of the 2nd rank field, the non-homogenous, derivative terms combine with the first $O(g)$ terms to give the covariant derivative of the higher rank gauge functions. With this in mind, we do not add coefficients on the first three $O(g)$ terms.
one gets,

\[ \frac{i}{g}[\delta_{\xi}, \delta_{\eta}]A_{\mu\nu\lambda} = \left[ \partial_{\mu}\xi_{1} - \partial_{\lambda}\eta_{1\mu} - \partial_{\nu}\eta_{1\mu} + \partial_{\nu}\eta_{1\nu} - \partial_{\nu}\eta_{1\nu} - \partial_{\nu}\eta_{1\nu} \right] + \left[ \partial_{\nu}\eta_{1\nu} - \partial_{\nu}\eta_{1\nu} - \partial_{\nu}\eta_{1\nu} - \partial_{\nu}\eta_{1\nu} - \partial_{\nu}\eta_{1\nu} - \partial_{\nu}\eta_{1\nu} \right] + \left[ \partial_{\nu}\eta_{2\nu} - \partial_{\nu}\eta_{2\nu} - \partial_{\nu}\eta_{2\nu} - \partial_{\nu}\eta_{2\nu} - \partial_{\nu}\eta_{2\nu} - \partial_{\nu}\eta_{2\nu} \right] + \left[ \partial_{\nu}\eta_{3\nu} - \partial_{\nu}\eta_{3\nu} - \partial_{\nu}\eta_{3\nu} - \partial_{\nu}\eta_{3\nu} - \partial_{\nu}\eta_{3\nu} - \partial_{\nu}\eta_{3\nu} \right] \]

\[ + \left[ \partial_{\nu}\eta_{3\nu} - \partial_{\nu}\eta_{3\nu} - \partial_{\nu}\eta_{3\nu} - \partial_{\nu}\eta_{3\nu} - \partial_{\nu}\eta_{3\nu} - \partial_{\nu}\eta_{3\nu} \right] + \left[ \partial_{\nu}\eta_{3\nu} - \partial_{\nu}\eta_{3\nu} - \partial_{\nu}\eta_{3\nu} - \partial_{\nu}\eta_{3\nu} - \partial_{\nu}\eta_{3\nu} - \partial_{\nu}\eta_{3\nu} \right] + \left[ \partial_{\nu}\eta_{3\nu} - \partial_{\nu}\eta_{3\nu} - \partial_{\nu}\eta_{3\nu} - \partial_{\nu}\eta_{3\nu} - \partial_{\nu}\eta_{3\nu} - \partial_{\nu}\eta_{3\nu} \right] + \left[ \partial_{\nu}\eta_{3\nu} - \partial_{\nu}\eta_{3\nu} - \partial_{\nu}\eta_{3\nu} - \partial_{\nu}\eta_{3\nu} - \partial_{\nu}\eta_{3\nu} - \partial_{\nu}\eta_{3\nu} \right] \]

Up to zeroth order, over the gauge coupling, we have nine terms in total. The first three have exactly the desired property for the closure of the algebra. We can isolate the parts of the last six terms, that contain commutators of both the vector gauge functions and can be written as partial differentials over the three indices, and insert them in the first three

\[ (3.17) \]

The substitution of (3.12) does not harm generality since it is the most general transformation for 2nd rank fields which is compatible with the closure of the algebra.
\[ i g [\delta_\xi, \delta_\eta] A_{\mu\nu\lambda} = \partial_\mu \left( [\xi, \eta^1_{\nu\lambda}] - [\eta, \xi^1_{\nu\lambda}] + c_2 ([\xi^1_{\nu}, \eta^2_{\lambda}] - [\eta^1_{\nu}, \xi^2_{\lambda}]) \right) + \\
\quad + \partial_\nu \left( [\xi, \eta^2_{\alpha\mu}] - [\eta, \xi^2_{\alpha\mu}] + c_1 ([\xi^1_{\alpha}, \eta^1_{\mu}] - [\eta^1_{\alpha}, \xi^1_{\mu}]) \right) + \\
\quad + \partial_\lambda \left( [\xi, \eta^3_{\beta\mu}] - [\eta, \xi^3_{\beta\mu}] + c_3 ([\xi^1_{\beta}, \eta^2_{\mu}] - [\eta^1_{\beta}, \xi^2_{\mu}]) \right) + \\
\quad + c_4 ([\partial_\mu \xi^1_{\nu}, \eta^1_{\lambda}] - [\partial_\mu \eta^1_{\nu}, \xi^1_{\lambda}]) + c_5 ([\partial_\lambda \xi^2_{\mu}, \eta^1_{\nu}] - [\partial_\lambda \eta^2_{\mu}, \xi^1_{\nu}]) + \\
\quad + c_6 ([\partial_\nu \xi^1_{\lambda}, \eta^1_{\mu}] - [\partial_\nu \eta^1_{\lambda}, \xi^1_{\mu}]) + c_6 ([\partial_\mu \xi^2_{\lambda}, \eta^2_{\nu}] - [\partial_\mu \eta^2_{\lambda}, \xi^2_{\nu}]) + \\
\quad + (c_4 - c_1) ([\xi^2_{\mu}, \partial_\nu \eta^1_{\lambda}] - [\eta^2_{\mu}, \partial_\nu \xi^1_{\lambda}]) + \\
\quad + (c_5 - c_2) ([\xi^2_{\nu}, \partial_\mu \eta^1_{\lambda}] - [\eta^2_{\nu}, \partial_\mu \xi^1_{\lambda}]) + \\
\quad + (c_6 - c_3) ([\xi^2_{\nu}, \partial_\lambda \eta^1_{\mu}] - [\eta^2_{\nu}, \partial_\lambda \xi^1_{\mu}]) + O(g) \] (3.18)

Up to zeroth order, over the gauge coupling, we end up with twelve terms. Further, one can see that the symmetric parts of the fourth to the ninth terms can also be isolated and written as partial derivatives over the three indices. For example, one can easily show that the symmetric part, over the indices \( \nu \) and \( \lambda \), of the fourth term can be written as a partial derivative over the index \( \mu \),

\[ [\partial_\mu \xi^1_{(\nu), \eta^1_{\lambda}]} - [\partial_\mu \eta^1_{(\nu), \xi^1_{\lambda}}] = \frac{1}{2} \partial_\mu ([\xi^1_{\nu}, \eta^1_{\lambda}] + [\xi^1_{\lambda}, \eta^1_{\nu}]) \] (3.19)

Such is the case for the remaining terms, symmetrized over the proper indices. With these

\(^7\)This will simplify the subsequent calculations and especially the cases where one of the vector gauge functions is neglected
in mind we get,

\[ \frac{i}{g}[\delta_\xi, \delta_\eta]A_{\mu\nu\lambda} = \partial_\mu(\xi, \eta_{\mu\lambda}^1 - [\eta, \xi_{\mu\lambda}]) + \partial_\nu(\xi, \eta_{\nu\lambda}^1 - [\eta, \xi_{\nu\lambda}]) + \partial_\lambda(\xi, \eta_{\lambda\mu}^1 - [\eta, \xi_{\lambda\mu}]) + \frac{c_1}{2}(\xi_{\mu\lambda}^1 + \xi_{\nu\lambda}^1) + \frac{c_6}{2}(\xi_{\lambda\mu}^2 + \xi_{\lambda\nu}^2) + \frac{c_2}{2}(\xi_{\mu\lambda}^2 + \xi_{\nu\lambda}^2) + \frac{c_3}{2}(\xi_{\lambda\mu}^1 + \xi_{\lambda\nu}^1) + \frac{c_4}{2}(\xi_{\lambda\mu}^2 + \xi_{\lambda\nu}^2) + \frac{c_5}{2}(\xi_{\lambda\mu}^1 + \xi_{\lambda\nu}^1) + \frac{c_6}{2}(\xi_{\lambda\mu}^2 + \xi_{\lambda\nu}^2) + \frac{c_2}{2}(\xi_{\lambda\mu}^2 + \xi_{\lambda\nu}^2) + \frac{c_3}{2}(\xi_{\lambda\mu}^1 + \xi_{\lambda\nu}^1) + \frac{c_4}{2}(\xi_{\lambda\mu}^2 + \xi_{\lambda\nu}^2) + \frac{c_5}{2}(\xi_{\lambda\mu}^1 + \xi_{\lambda\nu}^1) + \frac{c_6}{2}(\xi_{\lambda\mu}^2 + \xi_{\lambda\nu}^2) + O(g) \]  

(3.20)

Now we have to nullify the last 9 terms and there are many ways to do this. The forth till ninth terms can be omitted, either by nullifying the respective coefficients, or by imposing suitable symmetric properties over the indices of the 3rd rank tensor gauge field or by restricting one vector gauge function (neglecting it, equating it with the other etc.). The last three terms vanish either by equating the suitable pairs of \( c_i s \), or by equating the two vector gauge functions and imposing proper symmetries on their indices, or by neglecting one of them. Let us examine the different cases.

If we keep all the tensor gauge functions independent, and do not assume any symmetry properties on the indices of the tensor gauge fields, then, for the closure of the algebra, it is necessary to set,

\[ c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 0 \]  

(3.21)

Then we get,

\[ \frac{i}{g}[\delta_\xi, \delta_\eta]A_{\mu\nu\lambda} = \partial_\mu(\xi, \eta_{\mu\lambda}^1 - [\eta, \xi_{\mu\lambda}]) + \partial_\nu(\xi, \eta_{\nu\lambda}^1 - [\eta, \xi_{\nu\lambda}]) + \partial_\lambda(\xi, \eta_{\lambda\mu}^1 - [\eta, \xi_{\lambda\mu}]) \]  

(3.22)

Let us now examine the closure of the full algebra of the transformations,

\[ \delta A_{\mu\nu\lambda} = \partial_\mu\xi_{\nu\lambda}^1 + \partial_\nu\xi_{\lambda\mu}^2 + \partial_\lambda\xi_{\mu\nu}^3 - ig([A_{\mu\nu}, \xi_{\lambda\mu}^1] + [A_{\nu\lambda}, \xi_{\mu\nu}^2] + [A_{\lambda\mu}, \xi_{\mu\nu}^3] + [A_{\mu\nu\lambda}, \xi]) \]  

(3.23)
taking into consideration all the higher order terms in $g$. We get,

$$\frac{i}{g}[\delta_\xi, \delta_\eta]A_{\mu\nu\lambda} = [\delta_\xi A_\mu, \eta^\lambda_{\nu}] + [\delta_\xi A_\nu, \eta^\mu_{\lambda}] + [\delta_\xi A_\lambda, \eta^\alpha_{\mu\nu}] + [\delta_\xi A_{\mu\nu\lambda}, \eta] -$$

$$- [\delta_\eta A_\mu, \xi^{\nu}_{\lambda}] - [\delta_\eta A_\nu, \xi^{\mu}_{\lambda}] - [\delta_\eta A_\lambda, \xi^3_{\mu\nu}] - [\delta_\eta A_{\mu\nu\lambda}, \xi] =$$

$$= [\partial_\mu \xi - ig[A_\mu, \xi], \eta^1_{\nu\lambda}] + [\partial_\nu \xi - ig[A_\nu, \xi], \eta^2_{\mu\lambda}] + [\partial_\lambda \xi - ig[A_\lambda, \xi], \eta^3_{\mu\nu}] +$$

$$+ \left[ \partial_\mu \xi^{1}_{\nu\lambda} + \partial_\nu \xi^{2}_{\mu\lambda} + \partial_\lambda \xi^{3}_{\mu\nu} - ig\left( [A_\mu, \xi^{1}_{\nu\lambda}] + [A_\nu, \xi^{2}_{\mu\lambda}] + [A_\lambda, \xi^{3}_{\mu\nu}] + [A_{\mu\nu\lambda}, \xi] \right) \right] -$$

$$- [\partial_\mu \eta - ig[A_\mu, \eta], \xi^1_{\nu\lambda}] - [\partial_\nu \eta - ig[A_\nu, \eta], \xi^2_{\mu\lambda}] - [\partial_\lambda \eta - ig[A_\lambda, \eta], \xi^3_{\mu\nu}] -$$

$$- \left[ \partial_\mu \eta^{1}_{\nu\lambda} + \partial_\nu \eta^{2}_{\mu\lambda} + \partial_\lambda \eta^{3}_{\mu\nu} - ig\left( [A_\mu, \eta^{1}_{\nu\lambda}] + [A_\nu, \eta^{2}_{\mu\lambda}] + [A_\lambda, \eta^{3}_{\mu\nu}] + [A_{\mu\nu\lambda}, \eta] \right) \right] \xi =$$

$$= \partial_\mu \left( [\xi, \eta^1_{\nu\lambda}] - [\eta, \xi^1_{\nu\lambda}] \right) + \partial_\nu \left( [\xi, \eta^2_{\mu\lambda}] - [\eta, \xi^2_{\mu\lambda}] \right) + \partial_\lambda \left( [\xi, \eta^3_{\mu\nu}] - [\eta, \xi^3_{\mu\nu}] \right) -$$

$$- ig\left( [A_\mu, [\xi, \eta^1_{\nu\lambda}] - [\eta, \xi^1_{\nu\lambda}] \right] + [A_\nu, [\xi, \eta^2_{\mu\lambda}] - [\eta, \xi^2_{\mu\lambda}] \right] + [A_\lambda, [\xi, \eta^3_{\mu\nu}] - [\eta, \xi^3_{\mu\nu}] \right] +$$

$$+ [A_{\mu\nu\lambda}, [\xi, \eta]],$$

(3.24)

where in the second step we substituted $321$, $322$, and in last we employed the Jacobi Identity. We conclude that,

$$[\delta_\xi, \delta_\eta]A_{\mu\nu\lambda} = -ig\delta_\xi A_{\mu\nu\lambda},$$

(3.25)

with,

$$\zeta^i_{\mu\nu} = [\xi, \eta^i_{\nu\lambda}] + [\xi^i_{\nu\lambda}, \eta] \quad , \quad i = 1, 2, 3$$

$$\zeta = [\xi, \eta],$$

(3.26)

which proves that the transformations $323$ form a closed algebraic structure. As in the case of the 2nd rank tensor field, the vector field, $A_\mu$, participates in the gauge transformation of $A_{\mu\nu\lambda}$ only as part of the covariant derivative over the three gauge functions.

Now, let us explore the case where the second vector gauge function and the second and third 2nd rank tensor gauge functions are set to zero,

$$\xi^2_{\mu} = \xi^2_{\nu\lambda} = \xi^3_{\mu\nu} = 0$$

(3.27)

Also, let us symmetrize the 3rd rank tensor gauge field over its last two indices, $\nu$ and $\lambda$. The last three, together with the fifth, sixth and ninth terms of $320$, vanish identically because all of them contain the second vector gauge function ($\xi^2_{\mu}$ or $\eta^2_{\mu}$). The fourth term vanishes because of the symmetrization over the indices $\nu$ and $\lambda$. In order to get rid of the seventh and eighth terms, it is sufficient to set,

$$c_3 = c_5$$

(3.28)
Then, taking the only surviving 2nd rank tensor gauge function $\xi_{\mu\nu}$ symmetric, the gauge transformation (3.15) simplifies to,

$$
\delta A_{\mu\lambda_1\lambda_2} = \partial_\mu \xi_{\lambda_1\lambda_2} - ig \left( [A_\mu, \xi_{\lambda_1\lambda_2}] + c_1 ([A_{\mu\lambda_1}, \xi_{\lambda_2}] + [A_{\mu\lambda_2}, \xi_{\lambda_1}]) + 
+ c_3 ([A_{\lambda_1\mu}, \xi_{\lambda_2}] + [A_{\lambda_2\mu}, \xi_{\lambda_2}] + [A_{\lambda_1\lambda_2}, \xi_{\mu}] + [A_{\lambda_2\lambda_1}, \xi_{\mu}]) + [A_{\mu\lambda_1\lambda_2}, \xi]\right)
$$

(3.29)

which for $c_1 = 1$ and $c_3 = 0$, coincides with the standard gauge transformation (2.1).

In an analogous way, the 1st version of the dual transformations for the 3rd rank field can be recovered by setting $\xi^1_{\mu\nu} = \xi^2_{\mu\nu} = \xi^1_\mu = 0$, symmetrising the third tensor gauge function, $\xi^3_{\mu\nu}$, and 3rd rank tensor gauge field over its last two indices. Then, the closure of the algebra forces,

$$
c_2 = c_4, \quad (3.30)
$$

so that the gauge transformation becomes,

$$
\delta \xi A_{\mu\lambda_1\lambda_2} = \partial_\mu \xi_{\lambda_1\lambda_2} + \partial_\lambda_2 \xi_{\mu\lambda_1} - ig \left( [A_{\mu\lambda_1}, \xi_{\mu\lambda_2}] + [A_{\lambda_2\mu}, \xi_{\mu\lambda_1}] + c_2 ([A_{\lambda_1\mu}, \xi_{\lambda_2}] + [A_{\lambda_2\mu}, \xi_{\lambda_1}]) + 
+ [A_{\lambda_1\lambda_2}, \xi_{\mu}] + [A_{\lambda_2\lambda_1}, \xi_{\mu}]) + c_6 ([A_{\mu\lambda_1\lambda_2}, \xi_{\mu}] + [A_{\lambda_2\mu\lambda_1}, \xi_{\mu}]) \right), \quad (3.31)
$$

which for the particular choice $c_2 = 1$, $c_6 = 0$, coincides with (2.4).

It is not hard to see that the 2nd version of the dual transformation is recovered if we set,

$$
\xi^3_{\mu\nu} = 0 \ , \ \xi^2_{\mu\nu} = -4\xi^1_{\mu\nu} \ , \ \xi^1_\mu = 0 \ , \ c_2 = c_4 = 1 \ , \ c_6 = -1/2, \quad (3.32)
$$

and symmetrise and the 3rd rank tensor gauge field over its last two indices together with the surviving 2nd rank tensor gauge functions. Then the exact form of (2.9) is recovered if we normalise

$$
\xi^1_{\mu\nu} = -\frac{1}{3}\xi_{\mu\nu} \ , \ \xi^2_{\mu\nu} = \frac{4}{3}\xi_{\mu\nu} \quad (3.33)
$$

Finally, let us examine the case where the 3rd rank tensor is antisymmetric over its last two indices. If we also set,

$$
\xi^2_{\mu\nu} = \xi^3_{\mu\nu} = 0 \quad (3.34)
$$

---

8 We renamed the indices $\nu$, $\lambda$ to $\lambda_1$, $\lambda_2$ respectively so that the their permuting property is visible.
it is easy to see from (3.18) that the nullification of the terms that cannot be written as partial differentials is achieved only if we set,

\[ c_1 = 0, \quad c_5 = -c_3 \]  

(3.35)

Then (3.15) reduces to,

\[
\delta \xi A_{\mu \sigma_1 \sigma_2} = \partial_\mu \xi_{\sigma_1 \sigma_2} - ig \left( [A_\mu, \xi_{\sigma_1 \sigma_2}] + \frac{c_3}{2} \left( [A_{\sigma_2}, \xi_\mu] - [A_{\sigma_1}, \xi_{\sigma_2}] + [A_{\sigma_2}, \xi_{\sigma_1}] \right) + [A_{\mu \sigma_1 \sigma_2}, \xi] \right),
\]

(3.36)

where the tensor gauge function, \( \xi_{\mu \nu} \), is antisymmetric under the permutation of its two indices. Now the commutator of two successive transformations (3.18) reduces to,

\[
i \frac{g}{i} [\delta \xi, \delta \eta] A_{\mu \sigma_1 \sigma_2} = \partial_\mu \left( [\xi, \eta_{\sigma_1 \sigma_2}] + [\xi_{\sigma_1 \sigma_2}, \eta] \right) + c_3 \left( \partial_{\sigma_1} \left( [\xi_{\sigma_2}, \eta_\mu] + [\xi_\mu, \eta_{\sigma_2}] \right) - \partial_{\sigma_2} \left( [\xi_{\sigma_1}, \eta_\mu] + [\xi_\mu, \eta_{\sigma_1}] \right) \right) + O(g)
\]

(3.37)

Since the closure of the algebra requires that partial derivatives in terms of the indices \( \sigma_1 \) and \( \sigma_2 \) should be absent when we ignore the gauge functions \( \xi^2_{\mu \nu}, \xi^3_{\mu \nu} \), the only legitimate possibility is to set \( c_3 = 0 \). Thus, we recover the algebra (2.15) which has been proven to form a closed structure [20].

### 3.4 The General Case

The method we developed seems to be sufficiently generic to accomplish two aims. On the one hand to recover all the existing closed algebras of higher rank tensor gauge fields and thus to embed them in a more general framework, on the other to provide a tool for the investigation of new ones. Implementing it as a tool, the existence of a brand new closed algebra for higher rank tensor gauge fields became apparent. Hence, we have proved that up to the tensor of the third rank, the following transformations provide a closed algebraic structure,

\[
\begin{align*}
\delta \xi A_\mu &= \partial_\mu \xi - ig[A_\mu, \xi] \\
\delta \xi A_{\mu \nu} &= \partial_\mu \xi^1_\nu + \partial_\nu \xi^2_\mu - ig \left( [A_\mu, \xi^1_\nu] + [A_{\nu}, \xi^2_\mu] + [A_{\mu \nu}, \xi] \right) \\
\delta \xi A_{\mu \nu \lambda} &= \partial_\mu \xi^1_{\nu \lambda} + \partial_\nu \xi^2_{\mu \lambda} + \partial_\lambda \xi^3_{\mu \nu} - ig \left( [A_\mu, \xi^1_{\nu \lambda}] + [A_{\nu}, \xi^2_{\mu \lambda}] + [A_{\lambda}, \xi^3_{\mu \nu}] + [A_{\mu \nu \lambda}, \xi] \right)
\end{align*}
\]

(3.38)
It seems natural to postulate that the gauge transformations for the general case of an \( r \)-th rank gauge field is given by,

\[
\delta_{\xi} A_{\mu_1 \ldots \mu_r} = \sum_{i=1}^{r} \partial_{\mu_i} \xi^{i}_{\mu_i+1 \ldots \mu_r \mu_{i-1}} - i g \left( \sum_{i=1}^{r} [A_{\mu_i}, \xi^{i}_{\mu_i+1 \ldots \mu_r \mu_{i-1}}] + [A_{\mu_1 \ldots \mu_r}, \xi] \right). \tag{3.39}
\]

Let us prove that the postulated generalisation forms a closed algebra. The commutator of two successive transformations gives,

\[
\frac{i}{g} [\delta_{\xi}, \delta_{\eta}] A_{1 \ldots r} = \frac{i}{g} \left( \delta_{\xi} (\delta_{\eta} A_{1 \ldots r}) - \delta_{\eta} (\delta_{\xi} A_{1 \ldots r}) \right)
\]

\[
= \sum_{i=1}^{r} [\delta_{\xi} A_{\mu_i}, \eta^{i}_{\mu_i+1 \ldots \mu_r \mu_{i-1}}] - \sum_{i=1}^{r} [\delta_{\eta} A_{\mu_i}, \xi^{i}_{\mu_i+1 \ldots \mu_r \mu_{i-1}}] + \sum_{i=1}^{r} \left[ \partial_{\mu_i} \xi - i g [A_{\mu_i}, \xi], \eta^{i}_{\mu_i+1 \ldots \mu_r \mu_{i-1}} \right] - \sum_{i=1}^{r} \left[ \partial_{\mu_i} \eta - i g [A_{\mu_i}, \eta], \xi^{i}_{\mu_i+1 \ldots \mu_r \mu_{i-1}} \right]
\]

\[
+ \sum_{i=1}^{r} \left[ \partial_{\mu_i} \xi^{i}_{\mu_i+1 \ldots \mu_r \mu_{i-1}} - i g [A_{\mu_i}, \xi^{i}_{\mu_i+1 \ldots \mu_r \mu_{i-1}}], \eta \right] - i g [A_{\mu_1 \ldots \mu_r}, \xi] - i g [A_{\mu_1 \ldots \mu_r}, \xi]\]

\[
= \sum_{i=1}^{r} \partial_{\mu_i} \left( \left[ \xi, \eta^{i}_{\mu_i+1 \ldots \mu_r \mu_{i-1}} \right] + \left[ \xi^{i}_{\mu_i+1 \ldots \mu_r \mu_{i-1}}, \eta \right] \right) - i g \left\{ \sum_{i=1}^{r} \left[ A_{\mu_i}, \left[ \xi, \eta^{i}_{\mu_i+1 \ldots \mu_r \mu_{i-1}} \right] + \left[ \xi^{i}_{\mu_i+1 \ldots \mu_r \mu_{i-1}}, \eta \right] \right] + [A_{\mu_1 \ldots \mu_r}, \xi, \eta] \right\} \tag{3.40}
\]

In the third step we substituted (3.41) and (3.39) and in the last, we employed the \textit{Jacobi Identity}, where needed. We conclude that,

\[
[\delta_{\xi}, \delta_{\eta}] A_{1 \ldots r} = - i g \delta_{\xi} A_{1 \ldots r}, \tag{3.41}
\]

with

\[
\zeta = [\xi, \eta]
\]

\[
\zeta^{i}_{\mu_i+1 \ldots \mu_r \mu_{i-1}} = \left[ \xi, \eta^{i}_{\mu_i+1 \ldots \mu_r \mu_{i-1}} \right] + \left[ \xi^{i}_{\mu_i+1 \ldots \mu_r \mu_{i-1}}, \eta \right], \tag{3.42}
\]

hence that the general gauge transformation (3.39) forms a closed algebraic structure.
4 Conclusions

We presented a general method for constructing extended gauge transformations that include bosonic fields of arbitrary high spin under the requirement that they should form a closed algebraic structure. After we having recovered the known closed algebras of extended gauge transformations \((2.1), (2.4), (2.9), (2.10)\) and \((2.15)\), by properly adjusting the initial coefficients and restricting the tensor gauge functions of the transformations, we advocated for the existence of the following new algebra of extended gauge transformations,

\[
\delta \xi_A^\mu = \partial^\mu \xi - ig [A^\mu, \xi]
\]
\[
\delta \xi_{A^\mu^\nu} = \partial^\mu \xi^\nu + \partial^\nu \xi^\mu - ig \left( [A^\mu, \xi^\nu] + [A^\nu, \xi^\mu] + [A^\mu^\nu, \xi] \right)
\]
\[
\delta \xi_{A^\mu^\nu^\lambda} = \partial^\mu \xi^\nu^\lambda + \partial^\nu \xi^\mu^\lambda + \partial^\lambda \xi^\mu^\nu - ig \left( [A^\mu, \xi^\nu^\lambda] + [A^\nu, \xi^\mu^\lambda] + [A^\lambda, \xi^\mu^\nu] + [A^\mu^\nu^\lambda, \xi] \right)
\]

\[
\delta \xi_{A^\mu_1...\mu_r} = \sum_{i=1}^r \partial^\mu \xi^i_{\mu_{i+1}...\mu_r\mu_1...\mu_{i-1}} - ig \left( \sum_{i=1}^r [A^\mu_i, \xi^i_{\mu_{i+1}...\mu_r\mu_1...\mu_{i-1}}] + [A^\mu_1...\mu_r, \xi] \right),
\]

which was proven to be closed under the commutator,

\[
[\delta \xi, \delta \eta] A^\mu_1...\mu_r = -i g \delta \xi A^\mu_1...\mu_r, \quad (4.2)
\]

with,

\[
\xi = [\xi, \eta]
\]
\[
\xi^i_{\mu_{i+1}...\mu_r\mu_1...\mu_{i-1}} = \left[ \xi, \eta^i_{\mu_{i+1}...\mu_r\mu_1...\mu_{i-1}} \right] + \left[ \xi^i_{\mu_{i+1}...\mu_r\mu_1...\mu_{i-1}}, \eta \right] \quad (4.3)
\]

The potential usage of the new algebra is still under investigation. The fact that it introduces the same number of gauge functions as the rank of the tensor gauge field indicates a higher symmetry for the system and sounds promising as regards the cancellation of non-propagating degrees of freedom. It is worth to mention that at each step of the transformations, apart from the field transformed, no lower rank tensor fields participate other than the \(Y-M\) field. The latter participate exactly in the way to define the covariant derivative on each of the tensor gauge functions,

\[
\nabla^\mu_i \xi^i_{\mu_{i+1}...\mu_r\mu_1...\mu_{i-1}} \equiv \partial^\mu_i \xi^i_{\mu_{i+1}...\mu_r\mu_1...\mu_{i-1}} - ig [A^\mu_i, \xi^i_{\mu_{i+1}...\mu_r\mu_1...\mu_{i-1}}],
\]

hence playing its custom, geometrical role as a principal bundle connection.
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