INTERVAL TOPOLOGY IN CONTACT GEOMETRY

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Abstract. A topology is introduced on spaces of Legendrian submanifolds and groups of contactomorphisms. The definition is motivated by the Alexandrov topology in Lorentz geometry.

1. Introduction

Let \( (Y, \xi) \) be a contact manifold with a co-oriented contact structure. On every connected component \( \mathcal{C} \) of the group of contactomorphisms of \( Y \) or of the space of Legendrian submanifolds in \( Y \), there is a binary partial relation defined by setting

\[
a \ll b
\]

if there is a positive isotopy from \( a \) to \( b \). The family of intervals

\[
(a, b) := \{ z \in \mathcal{C} \mid a \ll z \ll b \}
\]

with respect to this relation generates the interval topology on \( \mathcal{C} \).

In Lorentz geometry, the topology generated by the family of intervals with respect to the chronology relation \( \ll \) on a spacetime \( X \) was considered in [27, 19] and became known as the Alexandrov topology. For a sufficiently nice spacetime, the Alexandrov topology is the pull-back of the interval topology on Legendrian spheres in the contact manifold of null geodesics \( \mathfrak{N}_X \) by the Penrose–Low twistor map [21, 22] sending a point \( x \in X \) to its celestial sphere \( \mathcal{S}_x \subset \mathfrak{N}_X \), see §4.2.

A basic question about the interval topology on \( \mathcal{C} \) is whether or not it is Hausdorff. It is similar (to an extent) to the non-degeneracy question for the Hofer distance [18] and its descendants [17, 29, 28].

An immediate observation is that the Hausdorff axiom is not satisfied if there exists a positive loop in \( \mathcal{C} \). If that loop is contractible, the issue persists even after passing to the universal cover of \( \mathcal{C} \). For instance, it follows from [20] that the interval topology can never be Hausdorff on the universal cover of the Legendrian isotopy class of a loose Legendrian submanifold. Positive loops are known to be the only obstruction to orderability [14, 10], which suggests the following problem:

**Question.** Suppose that \( \mathcal{C} \) is (universally) orderable (see §2.2). Is the interval topology Hausdorff on (the universal cover of) \( \mathcal{C} \)?

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The answer remains unknown in general. Using generating functions methods \cite{32,8,12}, we check that the interval topology is Hausdorff on the class of the zero section of the 1-jet bundle of a closed manifold and on the class of the fibre of the spherical cotangent bundle of a manifold covered by an open subset of $\mathbb{R}^n$. The latter case is relevant for Lorentz geometry as it leads to a new causal completion for some globally hyperbolic spacetimes.

**Organisation of the paper.** Section 2 introduces the interval topology and discusses its general properties. The case of 1-jet bundles and (certain) spherical cotangent bundles is dealt with in Section 3. The last section explores the relation to the Alexandrov topology on spacetimes.

**Conventions.** All manifolds and maps are taken to be $C^\infty$-smooth. Contactomorphisms of co-oriented contact structures are assumed to be co-orientation preserving.

## 2. Interval Topology and Orderability

### 2.1. Positive and non-negative isotopies in contact geometry.

Let $(Y, \ker \alpha)$ be a contact manifold with a co-oriented contact structure. A Legendrian isotopy $\{L_t\}_{t \in [0,1]}$ in $(Y, \ker \alpha)$ is called *non-negative* if it has a parametrisation $\ell_t : L_0 \to L_t$ such that

$$\alpha \left( \frac{d}{dt}\ell_t(x) \right) \geq 0$$

for all $t \in [0,1]$ and $x \in L_0$. If the inequality is strict, the isotopy is said to be *positive*. It is clear from the definition that the property to be positive or non-negative does not depend on the choice of a parametrisation of the isotopy and of a contact form defining the (co-oriented) contact structure. This property is also obviously preserved by (co-orientation preserving) contactomorphisms of $(Y, \ker \alpha)$.

**Example 2.1.** A neighbourhood of a Legendrian submanifold $L \subset Y$ is contactomorphic to a neighbourhood of the zero section of the 1-jet bundle $J^1(L)$ with its canonical contact form, see Subsec. 3.1. Every Legendrian that is sufficiently $C^1$-close to $L$ corresponds to the graph of the 1-jet of a smooth function on $L$. A small Legendrian isotopy of $L$ is non-negative (or positive) if and only if the corresponding family of functions on $L$ is pointwise non-decreasing (or increasing).

Similarly, an isotopy of contactomorphisms $\{\varphi_t\}_{t \in [0,1]}$ is called non-negative if its contact Hamiltonian

$$H(\varphi_t(x), t) := \alpha \left( \frac{d}{dt}\varphi_t(x) \right) \geq 0$$

for all $t \in [0,1]$ and $x \in Y$. If the inequality is strict, the isotopy is called positive. This property is invariant with respect to the left and
right actions of the contactomorphism group on itself. Note also that if \( L \subset Y \) is a Legendrian submanifold and \( \{ \varphi_t \} \) is a non-negative (or positive) contact isotopy, then \( \{ \varphi_t(L) \} \) is a non-negative (or positive) Legendrian isotopy.

**Example 2.2.** The Reeb flow of any contact form is a positive contact isotopy. (Its contact Hamiltonian with respect to that contact form is identically equal to one.)

**Remark 2.3.** For a contactomorphism \( \varphi \) with \( \varphi^* \alpha = e^f \alpha \), let
\[
\Gamma_\varphi = \{(x, \varphi(x), -f(x)) \in Y \times Y \times \mathbb{R} \mid x \in Y\}
\]
be its Legendrian graph. It is a Legendrian submanifold in the contact manifold \((Y \times Y \times \mathbb{R}, \ker(e^u \pi_2^* \alpha - \pi_1^* \alpha))\), where \( \pi_1 \) and \( \pi_2 \) are the projections to the first and second factors and \( u \) is the coordinate on \( \mathbb{R} \). A contact isotopy \( \{ \varphi_t \} \) is non-negative or positive if and only if the Legendrian isotopy \( \{ \Gamma_{\varphi_t} \} \) is non-negative or positive in \( Y \times Y \times \mathbb{R} \).

### 2.2. Partial (pre-)orders

Given two Legendrian submanifolds \( L_1, L_2 \) in a co-oriented contact manifold \((Y, \ker \alpha)\), we write
\[
L_1 \preceq L_2
\]
and
\[
L_1 \ll L_2
\]
if there is a non-negative and, respectively, positive Legendrian isotopy from \( L_1 \) to \( L_2 \).

For a pair of contactomorphisms \( \varphi_1, \varphi_2 \in \text{Cont}(Y, \ker \alpha) \), we write
\[
\varphi_1 \preceq \varphi_2
\]
and
\[
\varphi_1 \ll \varphi_2
\]
if there is a non-negative and, respectively, positive contact isotopy from \( \varphi_1 \) to \( \varphi_2 \).

If \( \mathcal{C} \) is a connected component of the space of Legendrians or of the contactomorphism group equipped with the usual \( C^\infty \)-topology, then the relations \( \preceq \) and \( \ll \) admit obvious lifts to the universal cover \( \tilde{\Pi} : \tilde{\mathcal{C}} \to \mathcal{C} \). Namely, \( a \preceq b \) and, respectively, \( a \ll b \) for two elements \( a, b \in \tilde{\mathcal{C}} \) if there is a path connecting \( a \) to \( b \) such that its projection to \( \mathcal{C} \) is a non-negative and, respectively, positive isotopy. (Note that \( \preceq \) was used to denote the lift of \( \preceq \) in \([10]\).)

The relations \( \preceq \) and \( \ll \) are transitive. (This is obvious for \( \preceq \) and requires an easy interpolation argument for \( \ll \), cf. e.g. \([9] \) Lemma 2.2 or \([10] \) Proof of Lemma 4.4.) It is also clear that \( \preceq \) is reflexive because a constant isotopy is non-negative.

**Definition 2.4.** \( \mathcal{C} \) is said to be *orderable* if \( \preceq \) is a partial order (i.e. if \( \preceq \) is antisymmetric) and *universally orderable* if the lift of \( \preceq \) is a partial order on its universal cover \( \tilde{\mathcal{C}} \).
Remark 2.5. The notion of orderability in contact geometry was introduced by Eliashberg–Polterovich [14] and Bhupal [5]. Varied terminology has been in use since then. For instance, a closed contact manifold is orderable in the sense of [14] if the identity component of its contactomorphism group is universally orderable.

2.3. Interval topology on Legendrians. Let \( \mathcal{L} \) be an isotopy class of closed Legendrian submanifolds in a contact manifold \( (Y, \ker \alpha) \). The interval topology on \( \mathcal{L} \) is defined by the family of intervals

\[
I_{a,b} := (a, b) := \{ z \in \mathcal{L} \mid a \ll z \ll b \}, \quad a, b \in \mathcal{L}.
\]

The interval topology on the universal cover of \( \mathcal{L} \) is defined in the same way using the lift of \( \ll \).

Note that intervals form a base for a topology. Indeed, every point \( L \in \mathcal{L} \) lies in an interval between two \( C^\infty \)-close Legendrians. (For instance, one can take shifts of \( L \) by the Reeb flow of a contact form.) Furthermore, it follows from Example 2.1 that if \( L \in I_1 \cap I_2 \), then there is an interval of this type contained in \( I_1 \cap I_2 \).

The interval topology is obviously invariant with respect to the action of all contactomorphisms (i.e. not necessarily co-orientation preserving, as reversing \( \ll \) does not affect intervals).

It is easy to see from Example 2.1 that the interval topology is rougher than the \( C^k \)-topology for every \( k \geq 1 \) in the sense that its open sets are open in any smooth topology. Proposition 3.6 implies that restricting the definition of the interval topology to a \( C^0 \)-neighbourhood of a Legendrian submanifold equips its \( C^1 \)-neighbourhood with a topology that is strictly rougher than the \( C^0 \)-topology on Legendrians.

2.4. Hausdorff-ness and orderability. The property of the interval topology to be Hausdorff appears to be rather similar to orderability. There are two subtle points, however. First, the interval topology is defined in terms of \( \ll \) whereas orderability is a property of \( \preceq \). This difficulty has been already addressed in [14] and [10], which we are going to use now and in \( \S 2.5 \). Secondly, the failure of the Hausdorff axiom does not formally imply the existence of non-negative loops. We could only find a partial solution to this problem in Proposition 2.8.

Proposition 2.6. Let \( \mathcal{L} \) be an isotopy class of closed Legendrian submanifolds. Suppose that the interval topology on the (universal cover) of \( \mathcal{L} \) is Hausdorff. Then \( \mathcal{L} \) is (universally) orderable.

Proof. If \( \mathcal{L} \) is not (universally) orderable, then it contains a (contractible) positive loop by [10, Proposition 4.7]. Any two elements on (the lift of) such a loop cannot have disjoint neighbourhoods in the interval topology. \( \square \)
Lemma 2.2]. Assume that \( L, L' \subseteq Y \) are two closed Legendrian submanifolds such that \( L \cap L' = \emptyset \). Then for any Legendrian \( L'' \) sufficiently close to \( L' \) in the \( C^1 \)-topology, there exists a contactomorphism \( \varphi \in \text{Cont}_0(Y) \) such that \( \varphi(L'') = L' \) and \( \text{supp}(\varphi) \cap L = \emptyset \).

Suppose now that \( L_1 \) and \( L_2 \) are two Legendrians that do not have disjoint interval neighbourhoods. Let \( L_1^+ \) be any two Legendrians such that \( L_1^- \ll L_1 < L_1^+ \) and \( L_1^+ \cap L_2 = \emptyset \).

For every \( \varepsilon > 0 \), the interval \( I := (L_1^- \cap L_1^+) \) must intersect the interval \( (\tau_{-\varepsilon}(L_2), \tau_{\varepsilon}(L_2)) \), where \( \tau_t, t \in \mathbb{R} \), is the Reeb flow of any contact form. Hence, \( L_1^- \ll \tau_{\varepsilon}(L_2) \) and \( \tau_{-\varepsilon}(L_2) \ll L_1^+ \).

If \( \varepsilon > 0 \) is small enough, we can find contactomorphisms \( \varphi \pm \) such that \( \varphi_{\pm}(\tau_{\pm\varepsilon}(L_2)) = L_2 \) and \( \text{supp}(\varphi) \cap L_1 = \emptyset \). It follows from the invariance of \( < \) that \( L_1^- \ll L_2 \ll L_1^+ \) and therefore \( L_2 \in I \).

The Legendrians \( L_1^+ \) disjoint from \( L_2 \) can be chosen as close to \( L_1 \) in the \( C^\infty \)-topology as we wish by Example 2.1 and a general position argument. Thus, \( I \) can be made arbitrarily small in the interval topology and therefore \( L_2 \) lies in every interval neighbourhood of \( L_1 \).

The non-distinguishable Legendrians obtained from a positive loop in a non-orderable isotopy class can be chosen disjoint by Example 2.1.

As a partial converse to Proposition 2.6, we use an argument very similar to the one in the proof of Proposition 2.7 to show that disjoint Legendrians in an orderable class are always separated by intervals.

**Proposition 2.8.** Let \( \mathcal{L} \) be an orderable Legendrian isotopy class. If \( L_1, L_2 \in \mathcal{L} \) are closed Legendrian submanifolds that are disjoint as sets, then they have disjoint neighbourhoods for the interval topology on \( \mathcal{L} \).

Let us emphasise that this falls short of proving that ‘orderable’ implies ‘Hausdorff’ because of the condition that \( L_1 \cap L_2 = \emptyset \).

**Proof.** Suppose that \( L_1 \) and \( L_2 \) do not have disjoint interval neighbourhoods. In particular, for every \( \varepsilon > 0 \), there exists a Legendrian \( L \in \mathcal{L} \) such that

\[
\tau_{-\varepsilon}(L_j) \ll L \ll \tau_{\varepsilon}(L_j), \quad j = 1, 2,
\]

where \( \tau_t, t \in \mathbb{R} \), is the Reeb flow of a contact form. Hence, there exist positive Legendrian isotopies connecting \( \tau_{-\varepsilon}(L_1) \) to \( \tau_{\varepsilon}(L_2) \) and \( \tau_{-\varepsilon}(L_2) \) to \( \tau_{\varepsilon}(L_1) \). Since \( L_1 \cap L_2 = \emptyset \), it follows that for a sufficiently small
\( \varepsilon > 0 \), we can apply contactomorphisms supported near \( L_1 \) and \( L_2 \) to these isotopies and obtain positive Legendrian isotopies connecting \( L_1 \) to \( L_2 \) and \( L_2 \) to \( L_1 \). Hence, \( L_1 \ll L_2 \) and \( L_2 \ll L_1 \), which is impossible because \( \mathcal{L} \) is orderable. \( \square \)

**Remark 2.9.** The same argument shows that if \( L_2 \) does not intersect a given neighbourhood of \( L_1 \) in the contact manifold, then it cannot lie in the interval \( (\tau_{-\varepsilon}(L_1), \tau_{\varepsilon}(L_1)) \) for any sufficiently small \( \varepsilon > 0 \). (Assuming, of course, that the Legendrian isotopy class of \( L_1 \) is orderable.)

### 2.5. Interval topology on contactomorphisms.

Let \( C \) be a connected component of the contactomorphism group of a contact manifold \((Y, \ker \alpha)\). The **interval topology** on \( C \) is defined by the family of intervals

\[
I_{a,b} := (a, b) := \{ z \in C \mid a \ll z \ll b \}, \quad a, b \in C.
\]

The interval topology on the universal cover of \( C \) is defined in the same way using the lifted relation.

This topology is invariant with respect to the left and right action of \( \text{Cont}_0(Y, \ker \alpha) \) as well as conjugation by arbitrary contactomorphisms (whenever it preserves \( C \)).

**Proposition 2.10.** Let \( C \) be a connected component of the contactomorphism group of a closed contact manifold. Suppose that the interval topology on the (universal cover) of \( C \) is Hausdorff. Then \( C \) is (universally) orderable.

**Proof.** If \( C \) is not (universally) orderable, then it contains a (contractible) positive loop of contactomorphisms by [14, Criterion 1.2.C]. Any two elements on (the lift of) such a loop cannot have disjoint neighbourhoods in the interval topology. \( \square \)

The Hausdorff property is inherited by contactomorphisms from Legendrians in the same way as orderability.

**Proposition 2.11.** Suppose that a contact manifold \((Y, \ker \alpha)\) contains an isotopy class of closed Legendrians on (the universal cover of) which the interval topology is Hausdorff. Then the interval topology is Hausdorff on (the universal cover of) \( \text{Cont}_0(Y, \ker \alpha) \).

**Proof.** Let us show that if \( \psi \in \text{Cont}_0(Y, \ker \alpha) \) cannot be separated from the identity \( \text{id} \in \text{Cont}_0(Y, \ker \alpha) \), then \( \psi(L) \) cannot be separated from \( L \) for any closed Legendrian \( L \subset Y \). Indeed, for any intervals \( I_1, I_2 \subset \text{Leg}(L) \) such that \( L \in I_1 \) and \( \psi(L) \in I_2 \), there exists a small \( \varepsilon > 0 \) such that \( \tau_{\pm \varepsilon}(L) \in I_1 \) and \( \tau_{\pm \varepsilon}(\psi(L)) \in I_2 \), where \( \tau_t, t \in \mathbb{R} \), is the Reeb flow of the contact form \( \alpha \). Hence, \( I_1 \supset (\tau_{-\varepsilon}(L), \tau_{\varepsilon}(L)) \) and \( I_2 \supset (\tau_{-\varepsilon}(\psi(L)), \tau_{\varepsilon}(\psi(L))) \) by the transitivity of \( \ll \).

Now take \( \varphi \in \text{Cont}_0(Y, \ker \alpha) \) from the intersection of the intervals \((\tau_{-\varepsilon}, \tau_{\varepsilon}) \ni \text{id} \) and \((\tau_{-\varepsilon} \circ \psi, \tau_{\varepsilon} \circ \psi) \ni \psi \). Then \( \varphi(L) \in I_1 \cap I_2 \). The same argument works *mutatis mutandis* for the universal covers. \( \square \)
Remark 2.12. There is an a priori different way to introduce a topology of this type on contactomorphisms. Namely, if \( C \) is a connected component of the contactomorphism group, the map \( \varphi \mapsto \Gamma_{\varphi} \) from Remark 2.3 embeds it into a Legendrian isotopy class in \( Y \times Y \times \mathbb{R} \). The topology induced from the interval topology on Legendrians by this embedding is formally rougher than the interval topology on \( C \).

3. 1-JET BUNDLES AND SPHERICAL COTANGENT BUNDLES

3.1. Minimax invariants from generating functions. Let \( \mathcal{J}^1(L) \) denote the 1-jet bundle of a closed connected manifold \( L \) equipped with the standard contact form \( du - \lambda_{\text{can}} \), where \( u \) is the fibre coordinate in \( \mathcal{J}^0(L) \) and \( \lambda_{\text{can}} \) is the Liouville form on \( T^*L \).

Let \( \Lambda \subset \mathcal{J}^1(L) \) be a Legendrian submanifold. A function \( S = S(q, \xi) : L \times \mathbb{R}^N \to \mathbb{R} \) is a generating function for \( \Lambda \) if zero is a regular value of the partial differential \( d_\xi S \) and the map

\[
\{d_\xi S(q, \xi) = 0\} \ni (q, \xi) \mapsto (q, d_\xi S(q, \xi), S(q, \xi)) \in \mathcal{J}^1(L)
\]

is a diffeomorphism onto \( \Lambda \). A generating function is said to be quadratic at infinity if \( S(q, \xi) = Q(\xi) + \sigma(q, \xi) \), where \( \sigma \) has compact support and \( Q(\cdot) \) is a non-degenerate quadratic form in the variable \( \xi \).

For a quadratic at infinity function \( S : L \times \mathbb{R}^N \to \mathbb{R} \), let \( S_c : = \{(q, \xi) \in L \times \mathbb{R}^N \mid S(q, \xi) \leq c\} \) be its sublevel sets and denote by \( S^{-\infty} \) the set \( S_c \) for a sufficiently negative \( c \ll 0 \). Following Viterbo \[32, \S2\], one can use homology relative to \( S^{-\infty} \) to select special critical values \( c_{\pm}(S) \) of \( S \).

Let \( \mathbb{R}^N = V_+ \times V_- \) be a decomposition into linear subspaces such that \( Q \) is positive definite on \( V_+ \) and negative definite on \( V_- \). Consider the relative \( \mathbb{Z}/2 \)-homology classes

\[
[L \times V_-] \in H_{\nu + \dim L}(L \times \mathbb{R}^N, S^{-\infty}; \mathbb{Z}/2)
\]

and

\[
\{(q_0) \times V_-\} \in H_\nu(L \times \mathbb{R}^N, S^{-\infty}; \mathbb{Z}/2),
\]

where \( \nu = \dim V_- \) and \( q_0 \) is any point in \( L \). Define

\[
c_+(S) : = \inf \{c \in \mathbb{R} \mid [L \times V_-] \in \iota_* H_{\nu + \dim L}(S^c, S^{-\infty}; \mathbb{Z}/2)\}
\]

and

\[
c_-(S) : = \inf \{c \in \mathbb{R} \mid \{(q_0) \times V_-\} \in \iota_* H_\nu(S^c, S^{-\infty}; \mathbb{Z}/2)\},
\]

where the map \( \iota_* : H_\nu(S^c, S^{-\infty}; \mathbb{Z}/2) \to H_\nu(L \times \mathbb{R}^N, S^{-\infty}; \mathbb{Z}/2) \) of relative homology groups with \( \mathbb{Z}/2 \) coefficients is induced by the inclusion \( \iota : S^c \to L \times \mathbb{R}^N \).
It follows from the definitions that \( c_\pm(S) \) may also be defined by minimax. Namely,
\[
c_+(S) = \min_{V_+} \max_{L \times \{v_+\} \times V_-} S(q, v_+, v_-)
\]
and
\[
c_-(S) = \min_{L \times V_+} \max_{\{q\} \times \{v_+\} \times V_-} S(q, v_+, v_-).
\]

**Example 3.1.** Let \( \Lambda^f := \{(q, df(q), f(q)) \mid q \in L\} \subset J^1(L) \) be the graph of the 1-jet of a smooth function \( f : L \to \mathbb{R} \). Then
\[
c_-(S) = \min_L f \quad \text{and} \quad c_+(S) = \max_L f
\]
for any quadratic at infinity generating function \( S : L \times \mathbb{R}^N \to \mathbb{R} \) of the Legendrian submanifold \( \Lambda^f \subset J^1(L) \). In particular, \( c_+(S) = c_-(S) = c \) if \( S \) generates the graph of the 1-jet of the constant function \( f \equiv c \).

**Lemma 3.2** (cf. [32, Corollary 2.3]). Let \( S \) be a quadratic at infinity generating function of a closed connected Legendrian submanifold \( \Lambda \).

If \( c_+(S) = c_-(S) = c \), then \( \Lambda = \Lambda^c = \{(q, 0, c) \mid q \in L\} \) is the graph of the 1-jet of the constant function \( f \equiv c \).

**Proof.** For every \( q \in L \), it follows from the assumption and the minimax characterisation of \( c_\pm \) that
\[
c = c_+(S) \geq \min_{V_+} \max_{\{q\} \times \{v_+\} \times V_-} S(q, v_+, v_-) \geq c_-(S) = c.
\]
Thus, \( c \) is a critical value of the restriction of \( S \) to each fibre \( \{q\} \times \mathbb{R}^N \).

A point at which it is attained is a critical point of \( S \). Hence, \( \Lambda \supseteq \Lambda^c \) by formula (3.1). Since \( \Lambda \) and \( \Lambda^c \) are closed connected submanifolds of the same dimension, this implies that \( \Lambda = \Lambda^c \). \( \square \)

By Chekanov’s theorem [9], for any Legendrian isotopy \( \{\Lambda_t\}_{t \in [0,1]} \) of the zero section in \( J^1(L) \), there exists a smooth family of quadratic at infinity generating functions \( S_t : L \times \mathbb{R}^N \to \mathbb{R} \) for \( \Lambda_t \). Furthermore, this family is unique up to stabilisations and fibrewise diffeomorphisms by the Viterbo–Théret theorem [32, 30, 31]. Therefore, for a Legendrian submanifold \( \Lambda \) in the Legendrian isotopy class of the zero section, one can define
\[
c_\pm(\Lambda) := c_\pm(S)
\]
for any quadratic at infinity generating function of \( \Lambda \) obtained by Chekanov’s theorem.

**3.2. INTERVAL TOPOLOGY ON** \( \text{Leg}(\Lambda^0) \). On the 1-jet bundle \( J^1(L) \) of a closed connected manifold \( L \), let
\[
\tau_r(q, p, u) := (q, p, u + r)
\]
be the shift by \( r \in \mathbb{R} \) in the \( u \)-direction, i.e. the time-\( r \) map of the Reeb flow for the standard contact form. Every Legendrian submanifold \( \Lambda \subset J^1(L) \) is obviously contained in the interval between \( \tau_{-r}(\Lambda) \) and
τ_r(Λ) for any r > 0. Another trivial observation is that if S is a generating function for Λ, then S + r is a generating function for τ_r(Λ). In particular, we have c_±(τ_r(Λ)) = c_±(Λ) + r for any Λ ∈ Leg(Λ^0).

**Theorem 3.3.** The interval topology is Hausdorff on the Legendrian isotopy class of the zero section in J^1(L).

**Proof.** Suppose that Λ_1, Λ_2 ∈ Leg(Λ^0) do not have disjoint open neighbourhoods for the interval topology. Applying a global contact isotopy, we may assume that Λ_1 is the zero section itself. For every ε > 0, there must then exist a Legendrian Λ ∈ Leg(Λ^0) such that

\[ τ_{-ε}(Λ_0) \ll Λ \ll τ_ε(Λ_0) \]

and

\[ τ_{-ε}(Λ_2) \ll Λ \ll τ_ε(Λ_2). \]

By [8, Lemma 5.2], the minimax invariants c_± are non-decreasing along a non-negative Legendrian isotopy. Hence,

\[ -ε \leq c_±(Λ) \leq ε \]

and

\[ c_±(Λ_2) - ε \leq c_±(Λ) \leq c_±(Λ_2) + ε. \]

It follows that

\[ -2ε \leq c_±(Λ_2) \leq 2ε \]

for all ε > 0. Thus,

\[ c_±(Λ_2) = 0 \]

and Λ_2 coincides with the zero section by Lemma 3.2. □

**Corollary 3.4.** The interval topology is Hausdorff on the identity component Cont_0(J^1(L), ker(du − λ_can)) for every closed manifold L.

**Proof.** Follows from Theorem 3.3 and Proposition 2.11. □

**Example 3.5.** (i) Let Y ∼= T^*L × S^1 be the quotient of J^1(L) by the Z-action generated by τ_1 with the contact form induced by du − λ_can. The Reeb flow of this form on Y is periodic. Hence, the interval topology is not Hausdorff on Cont_0(Y). However, it is not hard to deduce from Theorem 3.3 and the covering homotopy theorem that the interval topology is Hausdorff on the universal cover of the Legendrian isotopy class of the projection of the zero section to Y and therefore on Cont_0(Y).

(ii) If one is willing to consider one-dimensional contact manifolds, it is possible to take L = {pt} in (i). Then J^1(L) = R and Y = S^1. A co-oriented contact structure in this dimension is just an orientation. A connected Legendrian submanifold is a point and its Legendrian isotopy class is the connected component of the ambient manifold. The relation ∨ defines the usual order on R and the cyclic order on S^1. Hence, the interval topology on S^1 is clearly non-Hausdorff but it becomes Hausdorff on the universal cover.
Generating functions methods developed in [32] and adapted to the contact case in [8] may also be used to show that the interval topology extends the topology of uniform convergence on ‘potentials’, i.e. on smooth functions corresponding to Legendrian graphs in \( J^1(L) \).

**Proposition 3.6.** The interval topology on \( \text{Leg}(\Lambda^0) \) induces the topology of uniform convergence on the space of smooth functions on \( L \) via the embedding 

\[
C^\infty(L) \ni f \mapsto \Lambda^f \in \text{Leg}(\Lambda^0).
\]

**Proof.** If \( f, g \in C^\infty(L) \), then \( f \leq g \) pointwise on \( L \) if (and, obviously, only if) \( \Lambda^f \prec \Lambda^g \) by [8, Corollary 5.4]. Hence, \( |f - g| < \varepsilon \) on \( L \) if and only if \( \tau_{-\varepsilon}(\Lambda^g) \ll \Lambda^f \ll \tau_{\varepsilon}(\Lambda^g) \), and the result follows. \( \square \)

**Remark 3.7.** Shelukhin [29] used the Hofer distance functional [18] to define a norm on \( \text{Cont}_0(Y, \ker \alpha) \). This norm is not conjugation invariant (in that case it would have to be discrete [15]) but it defines an invariant topology on \( \text{Cont}_0(Y, \ker \alpha) \) by [29, Lemma 10]. The associated analogue of Chekanov’s metric [7] considered by Rosen and Zhang [28] defines a \( \text{Cont}_0(Y, \ker \alpha) \)-invariant topology on Legendrian isotopy classes in \( Y \). Proposition 3.6 implies that those topologies coincide with the interval topology ‘infinitesimally’. However, their global behaviour seems to be different. For instance, Shelukhin’s norm is non-degenerate for every closed contact manifold and therefore the topology defined by it is always Hausdorff.

### 3.3. Interval topology on \( \text{Leg}(ST^*_\{pt\} M) \)

As in [8, §6] and [12], let us combine the results of §3.2 for \( L = S^{n-1} \) with the ‘hodograph’ contactomorphism

\[
J^1(S^{n-1}) \overset{\cong}{\to} ST^*\mathbb{R}^n
\]

taking the zero section to a fibre.

**Corollary 3.8.** The interval topology is Hausdorff on the Legendrian isotopy class of the fibre of \( ST^*\mathbb{R}^n \).

**Lemma 3.9.** Let \( \widetilde{M} \) be a connected smooth cover of a manifold \( M \) with \( \dim M \geq 2 \). Assume that the interval topology is Hausdorff on the Legendrian isotopy class of the fibre of \( ST^*M \). Then the same holds for \( ST^*\widetilde{M} \).

**Remark 3.10.** Taking \( M = S^1 \) and \( \widetilde{M} = \mathbb{R} \) shows that assuming \( \dim M \geq 2 \) is necessary, cf. Example 3.5(ii).

**Proof of the lemma.** Let \( p : \widetilde{M} \to M \) be the covering map and \( P : ST^*\widetilde{M} \to ST^*M \) the induced projection of the spherical cotangent bundles. Fix a contact form \( \alpha \) defining the standard contact structure on \( ST^*M \). Then \( \tilde{\alpha} = P^*\alpha \) is a contact form defining the standard contact structure on \( ST^*\widetilde{M} \). The Reeb flows \( \tau_s \) and \( \widetilde{\tau}_s \) associated to \( \alpha \) and \( \tilde{\alpha} \) satisfy \( P \circ \widetilde{\tau}_s = \tau_s, s \in \mathbb{R} \).
Pick a point \( x \in M \) and let \( \tilde{x}_j, j \geq 1, \) be its pre-images in \( \tilde{M} \). Set \( F = ST^*_j M \) and \( \tilde{F}_j = ST^*_j \tilde{M} \). Since \( \text{Leg}(\tilde{F}_1) \) is orderable by Proposition \[2.6\], there does not exist a non-constant non-negative Legendrian loop based at \( \tilde{F}_1 \). Furthermore, there does not exist a non-negative isotopy from \( \tilde{F}_j \) to \( \tilde{F}_k \) in \( ST^* \tilde{M} \) for \( j \neq k \) because a contactomorphism interchanging these two fibres (induced by a diffeomorphism of \( \tilde{M} \) interchanging their base points) would map it to a non-negative isotopy from \( \tilde{F}_k \) to \( \tilde{F}_j \) and the concatenation of the two isotopies would be a non-constant non-negative loop in the fibre class.

Suppose now that the interval topology is not Hausdorff on \( \text{Leg}(F) \). By Proposition \[2.7\] there exists a Legendrian \( L \neq F \in \text{Leg}(F) \) contained in every interval of the form \((\tau_{-\varepsilon}(F), \tau_{\varepsilon}(F))\), \( \varepsilon > 0 \). In other words, for every \( \varepsilon > 0 \), there exists a positive Legendrian isotopy from \( \tau_{-\varepsilon}(F) \) to \( \tau_{\varepsilon}(F) \) passing through \( L \). By the covering homotopy theorem, this isotopy lifts to a positive Legendrian isotopy from \( \tilde{\tau}_{-\varepsilon}(\tilde{F}_1) \) to \( \tilde{\tau}_{\varepsilon}(\tilde{F}_k) \) for some \( k \geq 1 \) passing through a Legendrian lift \( \tilde{L}_\varepsilon \) of \( L \).

Note first that if \( k \neq 1 \) for small enough \( \varepsilon \), then since \( \tilde{F}_k \cap \tilde{F}_1 = \emptyset \), it would follow from the argument in the proof of Proposition \[2.7\] that there is a positive isotopy from \( \tilde{F}_1 \) to \( \tilde{F}_k \), which is impossible. Hence, \( \tilde{L}_\varepsilon \in (\tilde{\tau}_{-\varepsilon}(\tilde{F}_1), \tilde{\tau}_{\varepsilon}(\tilde{F}_1)) \) for all sufficiently small \( \varepsilon \). Using Remark \[2.9\] we see that \( \tilde{L}_\varepsilon \) must then intersect a fixed small neighbourhood of \( \tilde{F}_1 \) in \( ST^* \tilde{M} \). However, there are only finitely many lifts of \( L \) to \( ST^* \tilde{M} \) with that property. Thus, at least one such lift is contained in arbitrarily small interval neighbourhoods of the fibre \( \tilde{F}_1 \), which contradicts the assumption of the lemma. \( \square \)

From Lemma \[3.9\] and Corollary \[3.8\] we obtain the following (potential) improvement of the orderability result in \[8\] Corollary 6.2.

**Theorem 3.11.** The interval topology is Hausdorff on the Legendrian isotopy class of the fibre of \( ST^* M \) for any manifold \( M \) smoothly covered by an open subset of \( \mathbb{R}^n \), \( n \geq 2 \).

**Remark 3.12.** The theorem applies to every surface other than \( S^2 \) and \( \mathbb{R} \mathbb{P}^2 \) and to every compact three-manifold other than a quotient of \( S^3 \) by a finite group of isometries of the standard round metric, see the discussion starting at the bottom of p. 1321 in \[8\].

Proposition \[2.11\] shows now that the interval topology on contactomorphisms is Hausdorff for another class of contact manifolds.

**Corollary 3.13.** The interval topology is Hausdorff on \( \text{Cont}_0(ST^* M) \) for any manifold \( M \) smoothly covered by an open subset of \( \mathbb{R}^n \), \( n \geq 2 \).
4. Relation to Lorentz geometry

4.1. Causality and Alexandrov topology. A spacetime is a connected Lorentz manifold \((\mathcal{X}, \langle , \rangle)\) equipped with a time-orientation, that is, a continuous choice of the future hemicone

\[ C^+_x \subset \{ v \in T_x \mathcal{X} | \langle v, v \rangle \geq 0, v \neq 0 \} \]

in the cone of non-spacelike vectors at each point \(x \in \mathcal{X}\). (We are assuming that the Lorentz metric has signature \((+, -, \ldots, -)\) so that \(\langle v, v \rangle > 0\) for timelike vectors and \(\langle v, v \rangle < 0\) for spacelike vectors.) The vectors in \(C^+_x\) are called future-pointing. A piecewise smooth curve in \(\mathcal{X}\) is future-directed if all its tangent vectors are future-pointing.

The causality relation \(\leq\) on \(\mathcal{X}\) is defined by setting \(x \leq y\) if either \(x = y\) or there is a future-directed curve connecting \(x\) to \(y\). The chronology relation \(\ll\) is defined similarly by writing \(x \ll y\) if there is a future-directed timelike curve connecting \(x\) to \(y\).

\(\mathcal{X}\) is called causal if it does not contain closed future-directed curves. (This is equivalent to requiring that \(\leq\) is a partial order.) \(\mathcal{X}\) is strongly causal if every point has an arbitrarily small neighbourhood such that every future-directed curve intersects it at most once.

The Alexandrov topology on \(\mathcal{X}\) is the interval topology associated to the chronology relation. (It is named after Alexander D. Alexandrov and must not be confused with the Alexandrov topology on posets named after Pavel S. Alexandrov.) This topology was introduced by Kronheimer and Penrose who proved the following result (see [26, Theorem 4.24] or [2, Proposition 3.11]).

**Proposition 4.1.** The Alexandrov topology on a spacetime \(\mathcal{X}\) is Hausdorff if and only if \(\mathcal{X}\) is strongly causal. In that case the Alexandrov topology coincides with the manifold topology on \(\mathcal{X}\).

Let us point out that assuming strong causality is important here. For instance, the Alexandrov topology is not Hausdorff on the causal spacetime shown in [17, Figure 38]. (The points on the dashed null geodesic do not have disjoint interval neighbourhoods.) However, in contrast to Proposition 2.7, the Alexandrov topology in that example is nevertheless T0 and even T1. At the same time, causality may be equivalent to strong causality under additional compactness assumptions, see e.g. [4].

**Remark 4.2.** It is unclear whether a useful analogue of strong causality can be defined for a Legendrian isotopy class \(\mathcal{L}\). Such a definition would require a background topology on \(\mathcal{L}\) playing the role of the manifold topology on \(\mathcal{X}\). That topology should be Hausdorff but not ‘too fine’ compared to the interval topology, lest the notion become vacuous. For instance, it is easy to see (e.g. by considering wavefronts with swallowtails) that no Legendrian isotopy class can be ‘strongly orderable’ with respect to the \(C^k\)-topology for any \(k \geq 0\).
4.2. **Null geodesics, skies, and contact geometry.** Suppose now that $\mathcal{X}$ is a *globally hyperbolic* spacetime, i.e. it is strongly causal and the causal segments $\{z \in \mathcal{X} \mid x \leq z \leq y\}$ are compact for all $x, y \in \mathcal{X}$. By the Bernal–Sánchez smooth splitting theorem [3], a globally hyperbolic spacetime is foliated by smooth spacelike Cauchy (hyper)surfaces, where a *Cauchy surface* is a subset of a spacetime such that every inextendible future-directed curve intersects it exactly once.

The **space of null geodesics** of $\mathcal{X}$ is the set $\mathfrak{N}_\mathcal{X}$ of equivalence classes of inextendible future-directed null geodesics up to an orientation preserving affine reparametrisation. This space carries a canonical structure of a contact manifold contactomorphic to the spherical cotangent bundle of a Cauchy surface in $\mathcal{X}$, see e.g. [25, pp. 252–253] or [11, §§1-2].

The set $S_x \subset \mathfrak{N}_\mathcal{X}$ of all null geodesics passing through a point $x \in \mathcal{X}$ is a Legendrian sphere in $\mathfrak{N}_\mathcal{X}$ called the *sky* (or the *celestial sphere*) of that point, see [8, §4]. Since $\mathcal{X}$ is connected, all skies lie in the same Legendrian isotopy class. For any Cauchy surface $M \subset \mathcal{X}$, the associated contactomorphism $\rho_M : \mathfrak{N}_\mathcal{X} \xrightarrow{\sim} ST^*_x M$ takes the sky of a point $x \in M$ to the fibre $ST^*_x M$ and so maps the Legendrian isotopy class of skies to the Legendrian isotopy class of the fibre.

A conformal (or, equivalently, causal [24]) isomorphism $f : \mathcal{X} \to \mathcal{X}'$ maps null pregeodesics to null pregeodesics [2, Lemma 9.17]. Hence, it induces a contactomorphism $f_* : \mathfrak{N}_\mathcal{X} \to \mathfrak{N}_{\mathcal{X}'}$ such that $f_*(S_x) = S_{f(x)}$.

The map $x \mapsto S_x$ is compatible with the relations $\ll$ and $\leq$ on $\mathcal{X}$ and $\ll$ and $\preceq$ on the Legendrian isotopy class of skies. For $\leq$ and $\preceq$, this was pointed out in [8] and elaborated upon in [1].

**Proposition 4.3.** The Legendrian isotopy $S_{\beta(t)}$ corresponding to a curve $\beta : (a, b) \to \mathcal{X}$ is non-negative (respectively, positive) if and only if that curve is future-directed (respectively, future-directed timelike).

**Proof.** For the sake of completeness, we give a short proof by computation using the notation and formulas from [11]. (See [8, §4] for a geometric argument and [11, Corollary 3] for another computation. Note that both references used the opposite convention for the signature of the Lorentz metric, which ‘reversed’ the relations.)

Let $\ell_t : S \to \mathfrak{N}_\mathcal{X}$, $t \in (a, b)$, be a parametrisation of the Legendrian isotopy $S_{\beta(t)}$. (Here $S$ denotes the sphere of dimension $\dim \mathcal{X} - 2$.) Given $(t_0, \xi) \in (a, b) \times S$, let

$$v = v(t_0, \xi) := \frac{d}{dt}igr|_{t=t_0} \ell_t(\xi) \in T\mathfrak{N}_\mathcal{X}$$

and choose a family of future-directed null geodesics $\gamma_s : (-1, 1) \to \mathcal{X}$, $s \in (-\varepsilon, \varepsilon)$, so that $\gamma_s(0) = \beta(t_0 + s)$ and the maximal extension of $\gamma_s$ represents the equivalence class $\ell_{t_0 + s}(\xi)$ in $\mathfrak{N}_\mathcal{X}$. The Jacobi vector field
of this family at the point \( x = \beta(t_0) = \gamma_0(0) \) is then
\[
J(x) = \frac{d}{ds} \bigg|_{s=0} \gamma_s(0) = \dot{\beta}(t_0).
\]
(4.1)

Take a spacelike Cauchy surface \( M \) passing through the point \( x = \beta(t_0) \) and let \( \alpha_M \) be the associated contact form on \( \mathcal{N}_X \). Plugging (4.1) into the formula for \( \alpha_M(v) \) on p. 381 in [11], we obtain
\[
\alpha_M(v) = \frac{\langle \dot{\gamma}_0(0), \dot{\beta}(t_0) \rangle}{\langle \gamma_0(0), n_M(x) \rangle},
\]
(4.2)

where \( n_M \) is the future-pointing unit normal vector to \( M \).

A vector in a time-oriented Lorentz vector space is future-pointing (respectively, future-pointing timelike) if and only if its scalar product with every future-pointing null vector is non-negative (respectively, positive). Thus, the denominator in (4.2) is positive. Furthermore, since \( \ell_{t_0} : S \to \mathcal{N}_X \) parametrises the sky of \( x = \beta(t_0) \), the tangent vector \( \dot{\gamma}_0(0) \) runs through all null directions in \( C^*_x \) as \( \zeta \in S \) varies. Hence, \( \alpha_M(v) \geq 0 \) (respectively, \( > 0 \)) for all \( v = v(t_0, \zeta) \) if and only if \( \dot{\beta}(t_0) \) is future-pointing (respectively, future-pointing timelike).

\[\square\]

**Corollary 4.4.** \( x \leq y \implies \mathcal{S}_x \preceq \mathcal{S}_y \) and \( x \preceq y \implies \mathcal{S}_x \ll \mathcal{S}_y \)

The converse implications do not hold for certain (somewhat special) spacetimes, see e.g. [9, Example 10.5]. This problem does not occur if the Legendrian isotopy class of skies is orderable.

**Proposition 4.5.** Suppose that the Legendrian isotopy class of skies in \( \mathcal{N}_X \) is orderable. Then \( \mathcal{S}_x \preceq \mathcal{S}_y \implies x \leq y \) and \( \mathcal{S}_x \ll \mathcal{S}_y \implies x \ll y \).

**Proof.** The claim about \( \preceq \) and \( \ll \) was proved in [10, Proposition 1.3]. If \( \mathcal{S}_x \ll \mathcal{S}_y \), then \( \mathcal{S}_{x'} \ll \mathcal{S}_y \) for all points \( x' \) in a small neighbourhood of \( x \) in \( \mathcal{X} \) because the sky depends smoothly on the point and \( \ll \) is open in the smooth topology. So \( x' \leq y \) for all such points by the previous case. Hence, we can pick \( x' \) so that \( x \ll x' \) and \( x' \leq y \). This implies that \( x \ll y \) by [26, Proposition 2.18].

\[\square\]

**Corollary 4.6.** Suppose that the Legendrian isotopy class of skies in \( \mathcal{N}_X \) is orderable. The interval topology on this Legendrian isotopy class induces the usual manifold topology on \( \mathcal{X} \) via the embedding \( x \mapsto \mathcal{S}_x \).

**Proof.** It follows from Corollary 4.4 and Proposition 4.5 that the interval topology induces the Alexandrov topology on \( \mathcal{X} \). A globally hyperbolic spacetime is strongly causal, so the Alexandrov topology coincides with the manifold topology by Proposition 4.4.

\[\square\]

**Remark 4.7.** The Legendrian isotopy class of skies or, equivalently, the Legendrian isotopy class of the fibre of \( ST^*M \) for a Cauchy surface \( M \subset \mathcal{X} \) is orderable if the universal cover \( \tilde{M} \) is non-compact by [9, Remark 8.2] or the integral cohomology ring of \( \tilde{M} \) is not isomorphic
to that of a compact rank one symmetric space by [16, Theorem 1.2] combined with [10, Proposition 4.7]. In the remaining cases, one can use the fact that this Legendrian isotopy class is always universally orderable by [10, Theorem 1.1] and obtain a substitute for Corollary 4.6 by considering the map $\tilde{x} \mapsto \tilde{S}_{\tilde{x}}$ from the (finite) universal cover $\tilde{X}$ of the spacetime $X$ to the universal cover of the Legendrian isotopy class of skies in $\mathcal{R}_X$, cf. [10, §1.2].

4.3. Interval completion of a globally hyperbolic spacetime.
Let now $\mathcal{X}$ be a globally hyperbolic spacetime such that

\[ \text{(*) the interval topology is Hausdorff on the Legendrian isotopy class in } \mathcal{R}_X \text{ containing the skies of points in } X. \]

In particular, this Legendrian isotopy class is orderable by Proposition 4.6. Theorem 3.11 guarantees that condition (*) is satisfied if a smooth spacelike Cauchy surface $M \subset X$ is smoothly covered by an open subset of $\mathbb{R}^n$, and Remark 3.12 shows that this assumption is not too restrictive for $(2+1)$- and $(3+1)$-dimensional spacetimes.

Let us define the \textit{interval completion} of $X$ by setting

\[ \hat{X} := \{ \mathcal{S}_x \mid x \in X \}, \]

where the closure is taken with respect to the interval topology on the Legendrian isotopy class of skies in $\mathcal{R}_X$. In other words, a point in $\hat{X}$ is a Legendrian sphere in the space of null geodesics such that there is a sequence of skies of points in $X$ converging to it in the interval topology. The \textit{interval boundary} of $X$ is the difference

\[ \partial X := \hat{X} - \{ \mathcal{S}_x \mid x \in X \}. \]

The definition of $\hat{X}$ and the results collected in the previous subsection have the following immediate consequences:

1. $\hat{X}$ is a Hausdorff topological space.
2. The map $x \mapsto \mathcal{S}_x$ is an open embedding of $X$ into $\hat{X}$.
3. The relations $\ll$ and $\preceq$ on $\hat{X}$ restrict to $\ll$ and $\preceq$ on $X$.
4. Every causal isomorphism $f : \mathcal{X} \to \mathcal{X}'$ extends to a causal isomorphism $\hat{f} : \hat{X} \to \hat{X}'$.

Another basic corollary is that a point of $\hat{X}$ lies in $\partial X$ if (and, obviously, only if) it doesn’t have either a past or a future in $X$.

\textbf{Proposition 4.8.} If $L \in \partial X$, then at most one of the sets

\[ I^+_X(L) := \{ x \in \mathcal{X} \mid L \ll \mathcal{S}_x \} \quad \text{and} \quad I^-_X(L) := \{ x \in \mathcal{X} \mid \mathcal{S}_x \ll L \} \]

is nonempty.

\textit{Proof.} Suppose that there exist $x_+ \in I^+_X(L)$. Let $x_n, n \to \infty$, be a sequence of points in $\mathcal{X}$ such that their skies converge to $L$ in the interval topology. Then $\mathcal{S}_{x_-} \ll \mathcal{S}_{x_n} \ll \mathcal{S}_{x_+}$ for all large $n$. Hence, $x_- \ll x_n \ll x_+$ by Proposition 4.5 and therefore $x_n$ is contained in
the compact causal segment \( \{ z \in \mathcal{X} \mid x_- \leq z \leq x_+ \} \). Thus, there is a subsequence converging to a point \( x \in \mathcal{X} \). The corresponding sequence of skies will converge to \( \mathcal{S}_x \neq L \), which contradicts the uniqueness of limit in a Hausdorff space.

Finally, let us show that the interval boundary differs from the classical causal boundary [17 §6.8] and from the boundary defined by Low [23 §6] already in the simplest example.

**Proposition 4.9.** Let \( \mathcal{X} = \mathbb{R}^{1,n} \) be the flat Minkowski spacetime. Then \( \mathcal{X} = \mathcal{X} \) and \( \partial \mathcal{X} = \emptyset \).

**Proof.** Consider the Cauchy surface \( M = \{0\} \times \mathbb{R}^n \subset \mathbb{R}^{1,n} \) and the associated contactomorphism \( \rho_M : \mathcal{N}_{\mathbb{R}^{1,n}} \xrightarrow{\cong} ST^*\mathbb{R}^n \).

By definition, \( \rho_M \) maps the sky of a point \( (t, y) \in \mathbb{R}^{1,n} \) to the fibre \( ST^*_y\mathbb{R}^n \) for \( t = 0 \) and to the Legendrian lift of the \((n - 1)\)-sphere \( S(y, |t|) = \{ y' \in \mathbb{R}^n \mid \|y' - y\| = |t| \} \) co-oriented inwards for \( t < 0 \) and outwards for \( t > 0 \). If \( (t, y) \to \infty \) in \( \mathbb{R}^{1,n} \), then a straightforward computation shows that the minimax invariants \( c_{\pm} \) of the image of \( \rho_M(\mathcal{S}_{(t,y)}) \) under the hodograph contactomorphism

\[
ST^*\mathbb{R}^n \xrightarrow{\cong} J^1(S^{n-1})
\]

satisfy \( |c_+| + |c_-| \to \infty \), see [8 §6]. Hence, the skies of such points cannot be contained in any fixed interval by the monotonicity of \( c_{\pm} \). \( \square \)

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