Open Inflation, the Four Form and the Cosmological Constant

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Abstract

Fundamental theories of quantum gravity such as supergravity include a four form field strength which contributes to the cosmological constant. The inclusion of such a field into our theory of open inflation [1] allows an anthropic solution to the cosmological constant problem in which cosmological constant gives a small but non-negligible contribution to the density of today’s universe. We include a discussion of the role of the singularity in our solution and a reply to Vilenkin’s recent criticism.

I. INTRODUCTION

Inflationary theory has for some time had two skeletons in its cupboard. The first has been the question of the pre-inflationary initial conditions. The problem is to explain why the scalar field driving inflation was initially displaced from the true minimum of its effective potential. One possibility is that this happened through a supercooled phase transition, with the field being shifted away from its true minimum by thermal couplings. Another possibility is that the field became trapped in a ‘false vacuum’, a metastable minimum of the potential. But both of these scenarios are hard to reconcile with the very flat potential and weak self-couplings required to suppress the inflationary quantum fluctuations to an acceptable level. Most commonly, people have simply placed the field driving inflation high up its potential by hand in order to get inflation going. The problem here is that these initial conditions may be very unlikely. The only proposed measure on the space of initial conditions with some pretensions to completeness, the Hartle-Hawking prescription for the Euclidean path integral [2], predicts that inflationary initial conditions are exponentially improbable.

The second problem for inflation is the cosmological constant. The effective cosmological constant is what drives inflation, so it must be large during inflation. But it must also be cancelled to extreme accuracy after inflation to allow the usual radiation and matter dominated eras. With no explanation of how this cancellation could occur, the practice has been to simply set the minimum of the effective potential to be zero, or very nearly zero.

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This is a terrible fine tuning problem leading one to suspect that some important physics is missing.

In this paper we propose a solution to the cosmological constant problem, extending our recent paper on open inflation, where we calculated the Euclidean path integral with the Hartle-Hawking prescription using a new family of singular but finite action instanton solutions. We found that in this approach the simplest inflationary models with a single scalar field coupled to gravity gave the unfortunate prediction that the most likely open universes were nearly empty. We were forced to invoke the anthropic principle to determine the value of $\Omega_0$. Imposing the minimal requirement that our galaxy formed led to the most probable value for $\Omega_0$ being 0.01. This is far too low to fit current observations, although the issue is not completely straightforward because the region of gravitationally condensed matter our galaxy would be in would necessarily be large, and would contain many other galaxies.

In this paper we extend the simplest scalar field models by including a four form field, a natural addition to the Lagrangian which occurs automatically in supergravity. The four form field’s peculiar properties have been known for some time: it provides a contribution to the cosmological constant whose magnitude is not determined by the field equations. This property was exploited before by one of us in an attempt to explain why the present cosmological constant might be zero. A subtlety in the calculation with the four form was later pointed out by Duff, who showed that the Euclidean path integral actually gave $\Lambda = 0$ as the most unlikely possibility. Here we shall perform the calculation appropriate to an anthropic constraint on $\Lambda$ at late times. We shall show that in this context the four form allows an anthropic solution of the cosmological constant problem in which the prior probability for $\Lambda$ is very nearly flat, and the actual value of $\Lambda$ today is then determined by considerations of galaxy formation alone.

An earlier version of this paper incorrectly claimed that Duff’s calculation solved the empty universe problem. Bousso and Linde (private communication) pointed out that the action we computed for the four form field was not proportional to the geometric entropy. This prompted us to reconsider the calculation, and when we did so we discovered an error. The problem with the calculation was that we used the action appropriate for computing the wavefunction in the coordinate representation, whereas the anthropic constraint on $\Lambda$ is a constraint on the momentum of the three form gauge potential. One therefore needs to compute the path integral for the wavefunction in the momentum representation, and this turns out to restore the validity of Hawking’s original result for the prior probability for $\Lambda$. The empty universe problem remains, though there may be other solutions as were mentioned in [1], and will be discussed below.

In [1] we introduced a new family of singular but finite action instantons which describe the beginning of inflationary universes. Prior to our work the only known finite action instantons were those which occurred when the scalar field potential had a positive extremum or a sharp false vacuum. In contrast, the family of instantons we found exists for essentially any scalar field potential. When analytically continued to the Lorentzian region, the instantons describe infinite, open inflationary universes. Several subsequent papers have appeared, making various criticisms of these instantons, and of our interpretation of them. Linde [7] has made general arguments against the Hartle-Hawking prescription, to which we have replied in [8]. Vilenkin [4] argues that singular instantons must be forbidden or
else they would lead to an instability of Minkowski space. We respond to this criticism in Section III below. Unruh [10] has explored some of the properties of our solutions and interpreted them in terms of a closed universe including an ever growing region of an infinite open universe. Finally, Wu [12] has discussed interpreting instantons we use as ‘constrained’

The family of instantons we study allows one to compute the theoretical prior probabilities for cosmological parameters such as the density parameter $\Omega_0$ and the cosmological constant $\Lambda$ (where that is a free parameter, as it will be here) directly from the path integral for quantum gravity. An interesting consequence of our calculations is that over the range of values for the cosmological constant allowed by the anthropic principle, the theoretical prior probability for $\Omega_\Lambda$ is very nearly flat. Thus there is a high probability that $\Omega_\Lambda$ is non-negligible in today’s universe.

II. THE FOUR FORM AND THE EUCLIDEAN ACTION

The Euclidean action for the theory we consider is:

$$S_E = \int d^4x \sqrt{g} \left( -\frac{1}{16\pi G} R + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) - \frac{1}{48} F_{\mu\nu\rho\lambda} F^{\mu\nu\rho\lambda} \right) + \sum_i B_i$$

where the sum includes surface terms which do not contribute to the equations of motion, but are needed for the reasons to be explained. We use conventions where the Ricci scalar $R$ is positive for positively curved manifolds. The inflaton field is $\phi$ and $V(\phi)$ is its scalar potential. The negative sign of the $F^2$ term in the Euclidean action looks strange, but is actually implied by eleven dimensional supergravity compactified on a seven sphere as described by Freund and Rubin [13]. The minus sign is needed to reproduce the correct four dimensional field equations. The point is that the seven dimensional Ricci scalar contributes to the four dimensional Einstein equations, with the contribution being proportional to the square of the four form field strength $F^2$, which determines the size of the seven sphere.

The first surface term (which was neglected in [1]) occurs because we wish to compute the path integral for the wavefunction of the three-metric in the coordinate representation. The Ricci scalar contains terms involving second derivatives of the metric, which are undesirable because when the action is varied and one integrates by parts, they lead to surface terms involving normal derivatives of the metric variation on the boundary. But the action we want is that relevant for computing the wavefunction in the coordinate representation, and that should be stationary for arbitrary variations of the metric which vanish on the boundary.

The second derivative terms can be eliminated by integrating by parts, and the boundary term turns out to be

$$B_1 = \int d^3x \sqrt{h} K / (8\pi G)$$

where $K = h^{ij} K_{ij}$ is the trace of the second fundamental form, calculated using the induced metric $h_{ij}$ on the boundary [11].

The four form field strength $F_{\mu\nu\rho\lambda}$ is expressed in terms of its three-form potential as

$$F_{\mu\nu\rho\lambda} = \partial_{[\mu} A_{\nu\rho\lambda]}.$$
The field equations for $F$, obtained by setting $\delta S/\delta A_{\nu\rho\lambda} = 0$, are
\[ D_\mu F^{\mu\nu\rho\lambda} = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} F^{\mu\nu\rho\lambda}) = 0. \] (4)

The general solution is
\[ F^{\mu\nu\rho\lambda} = \frac{c}{\sqrt{g}} \epsilon^{\mu\nu\rho\lambda} \] (5)
with $c$ an arbitrary constant, and where we have inserted a factor of $i$ so that the four form will be real in the Lorentzian region.

The quantity $\sqrt{g}F^{0123}$ is the canonical momentum conjugate to the three form potential $A_{123}$. The four form theory has no propagating degrees of freedom: its only degree of freedom is the constant $c$ which corresponds to the momentum $p$ of a free particle in one dimension. As we shall see below, the constant $c$ is what determines the cosmological constant today, and we shall be imposing an anthropic constraint on that. So we want to compute the wavefunction as a function of the canonical momentum $\sqrt{g}F^{0123}$, not the coordinate $A_{123}$. (There was an error in the earlier version of this paper on this point - for analogous considerations regarding black hole duality see [13]). The action relevant for computing the wavefunction in the momentum representation should be stationary under arbitrary variations which leave the momentum $F_{0123}$ unchanged on the boundary. This action is obtained by adding a boundary term which cancels the dependence on the variation of the gauge field $\delta A_{\nu\rho\lambda}$ on the boundary. The variation of the modified action then equals a term involving the the equations of motion plus a term proportional to $\delta F_{0123}$ evaluated on the boundary, which is zero. The required boundary term is
\[ B_2 = - \int d^3x \sqrt{h} \frac{1}{24} F^{\mu\nu\rho\lambda} A_{\nu\rho\lambda} n_\mu \] (6)
where $n^\mu$ is the unit vector normal to the boundary. This term may be rewritten as the integral of a total divergence:
\[ B_2 = - \int d^4x \frac{1}{24} \partial_\mu \left( \sqrt{g} F^{\mu\nu\rho\lambda} A_{\nu\rho\lambda} \right). \] (7)

When this term is evaluated on a solution to the field equations (4), it equals precisely minus twice the original $\int \sqrt{g} F^2$ term.

In the Lorentzian region (where $g$ is negative) this solution continues to
\[ F^{\mu\nu\rho\lambda} = \frac{c}{\sqrt{-g}} \epsilon^{\mu\nu\rho\lambda} \] (8)
which is real for real $c$. Note that the quantity
\[ F^2 = F^{\mu\nu\rho\lambda} F_{\mu\nu\rho\lambda} = -24c^2 \] (9)
is constant and real in both the Euclidean and Lorentzian regions.

The Einstein equations, given by setting $\delta S/\delta g_{\mu\nu} = 0$, are
\[ G_{\mu\nu} = 8\pi G \left[ T^\phi_{\mu\nu} - \frac{1}{6} \left( F_{\mu\alpha\beta\gamma} F_{\nu}^{\alpha\beta\gamma} - \frac{1}{8} g_{\mu\nu} F^{\alpha\beta\gamma\delta} F_{\alpha\beta\gamma\delta} \right) \right], \]  

(10)

with \( T^\phi_{\mu\nu} \) the stress energy of the scalar field. Taking the trace of this equation one finds

\[ R = 8\pi G \left( (\partial \phi)^2 + 4V(\phi) + \frac{1}{12} F^2 \right), \]  

(11)

so that from (1), (2) and (7) the Euclidean action is just

\[ S_E = -\int d^4x \sqrt{g} \left( V(\phi) + \frac{1}{48} F^2 \right) + \frac{1}{8\pi G} \int d^3x \sqrt{h} K. \]  

(12)

Now we follow our previous work in looking for \( O(4) \) invariant solutions to the Euclidean field equations. The four form field does not contribute to the scalar field equations of motion, so the solutions are just those we found before [1], but with the constant term \( \frac{1}{48} F^2 \) added to the scalar field potential in the Einstein equations.

The instanton metric is given in the Euclidean region by

\[ ds^2 = d\sigma^2 + b^2(\sigma) d\Omega_3^2 \]  

(13)

with \( d\Omega_3^2 \) the metric for the three sphere, and \( b(\sigma) \) the radius of the three sphere. The field equation for the scalar field is

\[ \phi'' + \frac{3b'}{b} \phi' = V_{,\phi}, \]  

(14)

and the Einstein constraint equation is

\[ \left( \frac{b'}{b} \right)^2 = \frac{1}{3M_{Pl}^2} \left( \frac{1}{2} \phi'^2 - V_F \right) + \frac{1}{b^2} \]  

(15)

where \( V_F = V + \frac{1}{48} F^2 \) and primes denote derivatives with respect to \( \sigma \). The instantons discussed in [1] are solutions to these equations in which \( b = \sigma + o(\sigma^3) \) and \( \phi = \phi_0 + o(\sigma^2) \) near \( \sigma = 0 \). As \( \sigma \) increases there is a singularity, where \( b \) vanishes as \( (\sigma_f - \sigma)^{\frac{1}{3}} \), and \( \phi \) diverges logarithmically. The Ricci scalar diverges at the singularity as \( \frac{2}{3}(\sigma_f - \sigma)^{-2} \).

The presence of the singularity at the south pole of the deformed four sphere means that to evaluate the instanton action we have to include the surface term evaluated on a small three sphere around the south pole. The surface term in the action is calculated by noting that the action density involves \( \sqrt{g} R = -6(b'b + b'^2 - 1)b \). The second derivative term can be integrated by parts to produce an action with first derivatives only. Doing so produces a surface term which must be cancelled by the boundary term above. The required boundary term is thus

\[ \frac{1}{8\pi G} \int d^3x \sqrt{h} K = -\frac{1}{8\pi G}(b^3)' \int d\Omega^3 \]  

(16)

where \( \int d\Omega^3 = \pi^2 \) is half the volume of the three sphere.

The complete Euclidean instanton action is given by
\[ S_E = -\pi^2 \int_0^{\sigma_f} d\sigma b^3(\sigma) V_F(\phi) - \pi^2 M_{Pl}^2 (b^3)'(\sigma_f) \]  

(17)

with \( M_{Pl} = (8\pi G)^{-\frac{1}{2}} \) the reduced Planck mass.

For the flat potentials of interest, a good approximation to the volume term is obtained by treating \( V(\phi) \) as constant over most of the instanton. The surface term can be rewritten as a volume integral over \( V_\phi \) as follows. Near the boundary of the instanton, the gradient term \( \phi^2 \) dominates over the potential and the Einstein constraint equation (15) yields \( b' \approx \phi' b / (\sqrt{6} M_{Pl}) \). We then rewrite the surface term (16) as

\[ M_{Pl}^2 (b^3)'(\sigma_f) = 3 M_{Pl}^2 b^2 b'(\sigma_f) \approx 3 \int_0^{\sigma_f} d\sigma b^3(\sigma) M_{Pl} V_\phi. \]  

(18)

where we used the scalar field equation (14) in the last step. We perform the integral by treating \( V_\phi \) as constant. The integral is performed using the approximate solution \( b(\sigma) \approx H^{-1} \sin(H\sigma) \), where \( H^2 = V_F / (3 M_{Pl}^2) \). One finds \( \int_0^\pi d\sigma b^3(\sigma) \approx \frac{1}{3} H^{-4} = 12 M_{Pl}^4 / V_F^2 \).

With these approximations the Euclidean action (12) is given by

\[ S_E \approx -12 \pi^2 M_{Pl}^4 \left[ \frac{1}{V_F(\phi_0)} - \sqrt{\frac{3}{2}} M_{Pl} V_\phi(\phi_0) / V_F^2(\phi_0) \right] \]  

(19)

where \( \phi_0 \) is the initial scalar field value, and the term containing \( V_\phi(\phi_0) \) is the surface contribution.

Before continuing, we must deal with the issues of principle raised by the existence of the singularity.

### III. AVOIDING THE SINGULARITY

One might worry that the presence of a singularity meant that one could not use the instanton to make sensible physical predictions [9] but this is not the case. The important point is that to calculate a wave function one only needs half an instanton [12]. In other words, the wave function \( \Psi[h_{ij}, \phi] \) for a metric \( h_{ij} \) and matter fields \( \phi \) on a three surface \( \Sigma \) is given by a path integral over metrics and matter fields on a four manifold \( B \) whose only boundary is \( \Sigma \). We shall assume that the dominant contribution to this path integral comes from a non-singular solution of the field equations on \( B \). Then the probability of finding \( h_{ij} \) and \( \phi \) on \( \Sigma \) is

\[ |\Psi|^2 \]  

(20)

This can be represented by the double of \( B \), that is, two copies of \( B \) joined along \( \Sigma \). Only in exceptional cases will the double be smooth on \( \Sigma \). In general if one analytically continues the solution on one \( B \) onto the other it will have singularities.

Because one is interested in the probabilities for Lorentzian spacetimes, one has to impose the Lorentzian condition [14]

\[ Re(\pi^{ij}) = 0 \]  

(21)
where $\pi^{ij}$ is the Euclidean momentum conjugate to $h_{ij}$. This condition ensures that the second fundamental form of $\Sigma$ is imaginary, that is, Lorentzian. One way of satisfying this condition in the solution considered in [1] is to continue the coordinate $\sigma$ as $\sigma = \sigma_e + it$ where $\sigma_e$ is the value at the equator where the radius $b(\sigma)$ of the three spheres is maximal. This gives the wave function for a closed homogeneous and isotropic universe. In this case $B$ can be taken to be the Euclidean region from the north pole to the equator plus this Lorentzian continuation in imaginary $\sigma$. Clearly this is non singular since it doesn’t include the south pole.

There is another way of slicing our $O(4)$ solution with a three surface $\Sigma$ of zero second fundamental form: a great circle through the north and south poles. Let $\chi$ be a coordinate on the instanton which is zero on the great circle but with non zero derivative. Then $t = i\chi$ will be a Lorentzian time and the surfaces of constant $t$ will be inhomogeneous three spheres that sweep out a deformed de Sitter like solution. The light cone of the north pole of the $t = 0$ surface will contain the open inflationary universe and there will be a time like singularity running through the south pole. One might think this singularity would destroy one’s ability to predict because the Einstein equations do not hold there. However one can deform $\Sigma$ in a small half three sphere on one side of the singularity at the south pole and take $B$ to be the region on the non singular side of $\Sigma$. The deformation of $\Sigma$ near the south pole means that the Lorentzian condition will not be satisfied there. However this does not matter because this is not in the open universe region where observations of the Lorentzian condition are made. This is the important difference with the asymptotically flat singular instantons considered by Vilenkin [8] in which the singularity expands to infinity and would be in the region of observation. The double of $B$ will be the whole $O(4)$ solution apart from a small region round the south pole. One therefore has to include a surface term at the south pole, as we have done above.

**IV. THE VALUE OF $\Lambda$ AND $\Omega_0$**

Let us consider a scalar field potential

$$V(\phi) = V_0 + V_1(\phi); \quad \min V_1(\phi) \equiv 0.$$  \hspace{1cm} (22)

so that $V_0$ represents the minimum potential energy. We shall assume that $V_1$ is monotonically increasing over the range of initial fields $\phi_0$ of interest. In most inflationary models $V_0$ is simply set to zero by hand. Here the $F$ field can be chosen to cancel the ‘bare’ cosmological constant. This could occur for some symmetry or dynamical reason which we do not yet understand, or for anthropic reasons as we discuss below.

For the moment let us just assume that the $F$ field is chosen such that the effective cosmological constant today vanishes. This condition reads

$$\Lambda = V_0 + \frac{1}{48} F^2 = 0.$$  \hspace{1cm} (23)

If $V_0$ is positive this requires real $F$ in the Lorentzian region, and imaginary $F$ in the Euclidean region. From the point of view of eleven dimensional supergravity, including a positive $V_0$ cancels the negative four dimensional cosmological constant of the Freund-Rubin solution, allowing a four dimensional universe with zero cosmological constant. (The
Freund-Rubin solution gives four dimensional anti-De Sitter space cross a seven sphere. The condition that \( V_0 \) be positive is very interesting in the light of the well known fact that this is a requirement for supersymmetry breaking. Another implication of (23) is that the radius of the seven dimensional sphere is \( R \sim M_{Pl}/V_0^{1/2} \).

Substituting (23) back into the Euclidean action, we find
\[
S_E \approx -12\pi^2 M_{Pl}^4 \left( \frac{1}{V_1(\phi_0)} - \frac{\sqrt{3/2} M_{Pl} V_{1,\phi}(\phi_0)}{V_1(\phi_0)^2} \right)
\]
where we now have terms of opposite sign contributing to \( S_E \). For example if \( V_1(\phi) \propto \phi^2 \), the first term goes \(-\phi_0^2\) whereas the second goes as \(+\phi_0^{-3}\). So the minimum Euclidean action occurs at some nonzero value of \( \phi_0 \), just what we need for inflation [16]. However for general polynomial potentials it is straightforward to check that this effect is not enough to give much inflation [16].

However, for a potential with a local maximum, such as \( V_1 = \mu^4(1-\cos(\phi/v)) \), one obtains a second local minimum of the Euclidean action at the maximum of the potential. The point is that if we expand about the maximum, in this case \( \phi_0 = v(\pi - \delta) \) with \( \delta \) small, then the \( V_{1,\phi} \) contribution to the Euclidean action increases linearly with \( \delta \), whereas \( V_1 \) itself includes only quadratic corrections in \( \delta \). Therefore \( \delta = 0 \) is a local minimum of the Euclidean action. Consider the case \( v/M_{Pl} >> 1, \mu << M_{Pl} \), so that the potential is very flat. As \( \delta \) increases away from zero, \( V_1 \) decreases and the action turns over, becoming smaller than the value at \( \delta = 0 \) when \( \delta \sim \sqrt{6} M_{Pl}/v \). Universes with \( \delta \) larger than this have a larger prior probability. But the number of inflationary efoldings \( N \approx M_{Pl}^2 \int_0^{\phi_0} d\phi (V_1/V_{1,\phi}) \approx 2(v/M_{Pl})^2\log(1/\delta) \).

For example if \( v^2/M_{Pl}^2 \approx 10 \), the number of efoldings corresponding to \( \delta \) would be small, and the corresponding universes would be much too open to allow galaxy formation. So one can concentrate on the region around \( \delta = 0 \). The problem with very small \( \delta \) is that the density perturbation amplitude \( \Delta^2 = V_1^3/(M_{Pl}^6 V_{1,\phi}^2) \approx 8\mu^4 v^2/(M_{Pl}^6 \delta^2) \) is very large. Such universes might also be ruled out by anthropic considerations, for a recent discussion see [17]. The latter authors argue that if \( \Delta^2 \) is only modestly larger than the value set by normalising to COBE, one would form galaxies so dense that planetary systems would be impossible. This consideration disfAVours \( \delta \) being too small. Whether the anthropic effect is strong enough to counteract the Euclidean action remains to be seen.

V. THE ANTHROPIC FIX FOR \( \Lambda \)

Now let us return to the cosmological constant. Since we do not at present have any physics reason for the \( F \) field to cancel the bare cosmological constant, we resort to an anthropic argument. As Weinberg [20] points out, anthropic arguments are particularly powerful when applied to the cosmological constant, because there is a convincing case that unless the cosmological constant today is extremely small in Planck units, the formation of life would have been impossible. A very important and perhaps even compelling feature of the anthropic argument is that it applies to the full cosmological constant, after all the contributions from electroweak symmetry breaking, confinement and chiral symmetry breaking have been taken into account.
The expression (19) gives us the theoretical prior probability $P(\phi_0, F^2) \sim e^{-2S_E(\phi_0, F^2)}$ for the four form $F^2$ and the initial scalar field $\phi_0$. But most of the possible universes have large positive or negative cosmological constants, and life would be impossible in them. Following [1], we shall assume what seems the minimal conditions needed for our existence, namely that our galaxy formed and lasted long enough for life to evolve. The latter condition eliminates large negative values of $\Lambda$, since the universe would have recollapsed too soon. Large positive values for $\Lambda$ are excluded because $\Lambda$ domination would occur during the radiation epoch, before the galaxy scale could re-enter the Hubble radius. This would drive a second phase of inflation, which would never end. These two conditions alone force $\Lambda$ to be very small in Planck units. Note that since the fluctuations are approximately scale invariant in the theories of interest, the precise definition of a ‘galaxy’ is unimportant. The broad conclusions we reach here would apply even if we took the ‘galaxy’ mass scale to be as small as a solar mass.

We implement the anthropic principle via Bayes theorem, which tells us that the posterior probability for $\phi_0$ and $F^2$ is given by

$$P(\phi_0, F^2|\text{gal}) \propto P(\text{gal}|\phi_0, F^2)P(\phi_0, F^2)$$

(25)

where first factor represents the probability that a galaxy sized region about us underwent gravitational collapse, given $\phi_0$ and $F^2$, and the second is the theoretical prior probability $P(\phi_0, F^2) \sim e^{-2S_E(\phi_0, F^2)}$. We want to maximise (25) as a function of the initial field $\phi_0$ and the four form field $F^2$, or equivalently of $\Omega_0 = \Omega_M + \Omega_\Lambda$ and $\Omega_\Lambda$.

Consider the $\Omega_\Lambda$ dependence of (19) first. The galaxy formation probability $P(\text{gal}|\phi_0, F^2)$ is negligible unless $\Lambda$ domination happened after the galaxy scale re-entered the Hubble radius, at $t \sim 10^9$ seconds. We re-express $\Lambda$ as $\Lambda = \Omega_\Lambda \rho_c$ where $\rho_c = 3H_0^2/(8\pi G) = 3H_0^2M_P^2$ is the critical density. The condition that $\Lambda$ domination happened later than $10^9$ seconds after the big bang reads $|\Omega_\Lambda| < 10^{17}$, a mild constraint but strong enough for us to draw an important conclusion. We expand the Euclidean action in $\Omega_\Lambda$ to obtain

$$S_E = 12\pi^2 M_P^4 \left[ -\frac{1}{V_1} \left( 1 - 6\frac{\Omega_\Lambda M_P^2 H_0^2}{V_1} \right) - \frac{9\Omega_\Lambda M_P^2 H_0^2}{V_1^2} + \ldots \right].$$

(26)

The point is that the present Hubble constant $H_0$ is tiny compared to $V_1$: in the example above we had $V_1(\phi_0) \approx 120M_P^2m^2$, and normalising to COBE requires $m^2 \approx 10^{-11}M_P^2$. But today’s Hubble constant is $H_0 \sim 10^{-60}M_P$, so that even the above very minimal bound on $\Omega_\Lambda$ means that the quantity we are expanding in, $H_0^2M_P^2\Omega_\Lambda/V_1 < 10^{-94}$! Thus over the range of values of $F^2$ such that we can even discuss the possibility of galaxies existing, the dependence of the Euclidean action on $\Omega_\Lambda$ is completely negligible.

Likewise, if a physical mechanism such as the cosine potential described above increases $\phi_0$ so that we get an acceptable value $0.1 < \Omega_0 < 1.0$ today, the $\phi_0$ dependence of the prior probability is likely to massively outweigh that of the galaxy formation probability. The reason for this is the Euclidean action depends inversely on $V_1(\phi_0)$. If we are to match COBE, $V_1(\phi_0)$ has to be much smaller than the Planck density and the Euclidean action is enormous. However, if we normalise to COBE and $\Omega_0$ is not far from unity, the galaxy formation probability is a function of $\Omega_0$ containing no large dimensionless number. So the problem of maximising the joint probability factorises. The anthropic principle fixes $\Lambda$ to be small, and the Euclidean action (or prior probability) then fixes $\Omega_0$. 

One can also consider the posterior probability for $\Lambda$ within this framework. As we have argued, the posterior probability is to a good approximation completely determined by the galaxy formation probability alone. The possibility that this might be the case was anticipated by Weinberg [20] and Efstathiou [19].

Let us briefly review the effect on galaxy formation of varying $\Lambda$, for modest values of $\Omega_\Lambda$ today. In (25), we should use

$$\mathcal{P}(\text{gal}|\phi_0, F^2) \sim \text{erfc}(\delta_c/\sigma_{gal})$$

(27)

where we assume Gaussian statistics. Here, $\delta_c$ is the value of the linear density perturbation required for gravitational collapse, usually taken to be that in the spherical collapse model, $\delta_c = 1.68$. The amplitude of density perturbations on the galaxy scale in today’s universe, $\sigma_{gal}$ is given roughly by

$$\sigma_{gal} \approx \Delta(\phi_{gal})G(\Omega_M, \Omega_\Lambda)$$

(28)

where $\Delta(\phi_{gal}) \sim 3 \times 10^{-4}$ is the amplitude of perturbations at horizon crossing, fixed by normalising to COBE, and $G(\Omega_M, \Omega_\Lambda)$ is the growth factor for density perturbations in the matter era. The latter varies strongly with $\Omega_M$: for example in a flat universe, with $\Omega_\Lambda = 1 - \Omega_M$, we have $G \propto \Omega_M^{\frac{1}{3}}$ at small $\Omega_M$ [18], whereas in an open universe with small $\Omega_\Lambda$ we have $G \propto \Omega_\Lambda^{\frac{2}{3}} \approx (\Omega_0 - \Omega_\Lambda)^2$. One factor of $\Omega_M$ occurs because of the change in the redshift of matter-radiation equality, and the remaining dependence is due to the loss of growth at late times. In any case, for fixed $\phi_0$ and therefore fixed total density $\Omega_M + \Omega_\Lambda$, reducing $\Omega_\Lambda$ increases the probability of galaxy formation. So for fixed $T_0$ and $H_0$ the most probable value of $\Lambda$ is zero, but there is a high probability for non-negligible $\Omega_\Lambda$. Detailed computations of the posterior probability for $\Omega_\Lambda$ have been carried out by Efstathiou [19] and Martel et al. [21]. It would be interesting to generalise these to the open universes discussed here.

VI. CONCLUSIONS

We have reached the somewhat surprising conclusion that the universe most favoured by simple inflationary models with a four form field is open and with a small but non-negligible cosmological constant today. Our use of the anthropic argument to fix $\Lambda$ is not new, and the possibility that the theoretical prior probability might be a very flat function of $\Omega_\Lambda$ was anticipated. However it is an important advance that we can actually calculate the prior probability from first principles.

Finally, we emphasise that the problem of explaining why $\Omega_0 > 0.01$ today remains, although we have noticed some promising aspects of potentials with local maxima in this regard. As we have mentioned, in that case the problem is to understand whether anthropic considerations disfavour very large perturbation amplitudes as strongly as the Euclidean action favours the initial field starting near the potential maximum.

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