A nonperturbative regularization of the supersymmetric Schwinger model

C. Klimčík
Institute de mathématiques de Luminy,
163, Avenue de Luminy, 13288 Marseille, France

Abstract

It is shown that noncommutative geometry is a nonperturbative regulator which can manifestly preserve a space supersymmetry and a supergauge symmetry while keeping only finite number of degrees of freedom in the theory. The simplest $N = 1$ case of the $U(1)$ supergauge theory on the sphere is worked out in detail.
1 Introduction

There is a widespread belief that ”it is unlikely having any regulated form of the theory with the supersymmetry besides string theory” [1]. Nevertheless, we wish to show here, though only in the simplest possible case of the supersymmetric Schwinger model in two dimensions, that there exists a regulator reconciling the supersymmetry with the short distance cut-off. Moreover, this regulator has two advantages with respect to strings. First of all, it is genuinely nonperturbative and, secondly, it is more economic because it requires only finite number of degrees of freedom in game.

The mathematical structure which is behind this regulator is noncommutative geometry. It is a discipline largely developed by A. Connes [2] and besides applications in the physics of the standard model it is expected to have an important impact on a general structure of quantum field theory [3]. It was actually from the noncommutative rather than from the supersymmetric side where the motivation for this work came from. One could witness in the last few years a lot of activity [4, 5, 6, 7, 8, 9, 10, 11] concerning model building on the so-called fuzzy sphere. The latter concept was probably invented by Berezin [12], but the idea to use it for regularization of scalar field theories was independently advocated firstly in [13] and [14].

We should perhaps mention, in order to avoid a misunderstanding, that all quoted references do not study the fuzzy sphere as such but rather field theories on it. In the same spirit, we may say that people working on quantum inverse scattering method in two dimensions have not been studying the two dimensional Minkowski space but the immensely rich world of two-dimensional field theoretical models living on it. So a question why to devote so much time for studying the fuzzy sphere is not well posed; one’d better ask: ”Which theories one can formulate on it?”

There is an encouraging experience so far that all interesting field theoretical structures can be formulated on the fuzzy sphere. We mean here theories including fermions [1, 4, 7, 9, 10], gauge fields [8, 5, 10], topologically nontrivial field configurations [4, 10] and even those incorporating superscalar fields like supersymmetric nonlinear $\sigma$-models [4]. A next step to undertake is to formulate supersymmetric gauge theories. We shall do it in this paper.

The construction of supersymmetric gauge theories is distinguished in comparison with the previously studied models in several aspects. First
of all, there is more structure to reconcile with the fuzziness of the space; besides the supersymmetry, there is the supergauge symmetry and a need to build up a complex of superdifferential forms on the noncommutative sphere. Secondly, the structure appears very rigid; while in the nonsupersymmetric context one usually might advocate several nonequivalent constructions, in the supersymmetric context there seems to be a little room for ambiguities. Thirdly, but not less importantly, it is quite a technical task to study the supergauge theories. A field content of these theories is rich and much work is needed for disentangling good theories from pathological ones by imposing suitable constraints on the superfields.

As it is well-known [15], the language of differential forms is the most natural one for building up the supersymmetric gauge theories. However, the field strength obtained by applying the coboundary of the supersymmetric complex on the gauge field (a 1-form) cannot be arbitrary but must be, in general, constrained in a way compatible with supersymmetry. Without those constraints, the inconvenience would be much stronger than just an abundance of additional propagating fields in the theory. One risks rather the violation of the spin-statistics theorem, presence of terms containing four bosonic derivatives and similar serious pathologies.

Unfortunately, the understanding that the constraints are indispensable is not sufficient for finding them. Hitherto there was not found a method which would lead to an algorithmic identification of the constraints yielding a good supersymmetric theory. One therefore has to combine an intuition and a lot of calculation to work out the content of the theory for a given set of guessed constraints. As we have alluded already above, in our case additional complications enter the story. Indeed, we need to work always with global description of all involved structures in order to ensure a possibility of a noncommutative generalization (non-commutative geometry does not know local charts). Thus, we have to find a suitable globally defined differential complex and a set of constraints which would give a good theory.

The guiding principle for seeking the complex which would underlie supersymmetric gauge theories on the sphere seems clear. One should covariantize the superderivatives present in an matter action possessing a global symmetry to be gauged. If we take the standard superscalar matter [4], those superderivatives turn out to be a generators of certain superalgebra which is not $\mathfrak{osp}(2,1)$ as one may have expected. What happens is that, though the resulting supersymmetric theory does turn out to have the $\mathfrak{osp}(2,1)$ su-
persymmetry, one needs an \( osp(2,2) \) covariant structure to uncover it. The reason of this can be seen already in the standard case of the flat space (super-Poincaré) supersymmetry where one has to add to the generators of supersymmetry \((Q, \bar{Q})\) algebra also the so-called supersymmetric covariant derivatives \((D, \bar{D})\). In the flat case the derivatives \((D, \bar{D})\) are detached from the superalgebra (they anticommute with the \(Q\)-generators) and it is of little usefulness to remark that the \(Q\)'s and \(D\)'s form together a \(N = 2\) superalgebra. However, when we move from the flat space to the sphere, it can be seen (\[4\]) that the supersymmetric covariant derivatives added to the superalgebra \(osp(2, 1)\) entail even an introduction of another bosonic generator which completes the structure to that of the \(osp(2, 2)\) superalgebra. Thus the supersymmetric covariant derivatives are not detached from the supersymmetry algebra and it is natural and, in fact, inevitable to consider the bigger structure which involve them.

In what follows, we shall first introduce an algebra of superfunctions on the supersphere following \[8\] and then a differential complex which will underlie the notion of the supergauge field. Since the theory of representations of \(osp(2, 2)\) is not so notoriously known as that of \(su(2)\) we shall review its basic elements. In section 3, we first describe the construction of the supersymmetric Schwinger model on the ordinary sphere in a way which is most probably also original. Then we shall give an invariant description of the complex of the differential superforms. This invariant description will not only render quite transparent the logic of the construction but it will prove technically efficient in formulating the supersymmetric Schwinger model on the noncommutative sphere. In fact, formulae which would appear in the noninvariant formulation in the noncommutative case would be exceedingly cumbersome. We shall finish with a brief outlook.

2 A differential complex on the supersphere

2.1 Superfunctions on the supersphere

Consider the algebra of functions on the complex \(C^{2,1}\) superplane, i.e. algebra generated by bosonic variables \(\bar{\chi}^\alpha, \chi^\alpha, \alpha = 1, 2\) and by fermionic ones \(\bar{a}, a\). The algebra is equipped with the graded involution

\[
(\chi^\alpha)^\dagger = \bar{\chi}^\alpha, \quad (\bar{\chi}^\alpha)^\dagger = \chi^\alpha, \quad a^\dagger = \bar{a}, \quad \bar{a}^\dagger = -a,
\]  

(1)
satisfying the following properties

\[(AB)^\dagger = (-1)^{AB}B^\dagger A^\dagger, \quad (A^\dagger)^\dagger = (-1)^AA,\] 

and with the super-Poisson bracket

\[\{f, g\} = \partial_{\chi^a} f \partial_{\bar{\chi}^a} g - \partial_{\chi^a} f \partial_{\bar{\chi}^a} g + (-1)^{f+1}[\partial_a f \partial_a g + \partial_a f \partial_a g]. \] 

We can now apply the (super)symplectic reduction with respect to a moment map \(\bar{\chi}^i\chi^i + \bar{a}a - 1\). The result is a smaller algebra \(A_\infty\), that by definition consists of all functions \(f\) with the property

\[\{f, \bar{\chi}^i\chi^i + \bar{a}a - 1\} = 0.\] 

Moreover, two functions obeying (4) are considered to be equivalent if they differ just by a product of \((\bar{\chi}^i\chi^i + \bar{a}a - 1)\) with some other such function. The smaller algebra \(A_\infty\) (the reason for using of the subscript \(\infty\) will become clear soon) is referred to as the algebra of superfunctions on the supersphere \([8]\). It is sometimes more convenient to work with a different parametrization of \(A_\infty\), using rather the following coordinates

\[z = \frac{\chi^1}{\chi^2}, \quad \bar{z} = \frac{\bar{\chi}^1}{\bar{\chi}^2}, \quad b = \frac{a}{\chi^2}, \quad \bar{b} = \frac{\bar{a}}{\bar{\chi}^2}.\] 

The Poisson bracket (3) then becomes

\[\{f, g\} = (1 + \bar{z}z)(1 + \bar{z}z + \bar{b}b)(\partial_z f \partial_{\bar{z}}g - \partial_z f \partial_{\bar{z}}g)\]
\[+ (1 + \bar{z}z)\bar{b}z((-1)^f \partial_z f \partial_{\bar{b}}g - \partial_z f \partial_{\bar{b}}g) + (1 + \bar{z}z)b\bar{z}(\partial_{\bar{b}} f \partial_z g - (-1)^f \partial_z f \partial_{\bar{b}}g)\]
\[+ (-1)^{(f+1)}(1 + \bar{z}z - \bar{b}bz)(\partial_{\bar{b}} f \partial_z b + \partial_z f \partial_{\bar{b}}g).\] 

A natural Berezin integral on \(A_\infty\) can be written as

\[I[f] = -\frac{1}{4\pi^2} \int d\bar{\chi}^1 \wedge d\chi^1 \wedge d\bar{\chi}^2 \wedge d\chi^2 \wedge d\bar{a} \wedge da \delta(\bar{\chi}^i\chi^i + \bar{a}a - 1) f.\] 

It can be rewritten as

\[I[f] \equiv -\frac{i}{2\pi} \int \frac{dz \wedge d\bar{z} \wedge d\bar{b} \wedge db}{1 + \bar{z}z + \bar{b}b} f.\]
Now we are ready to quantize the infinitely dimensional algebra \( A_\infty \) with the goal of obtaining its (noncommutative) finite dimensional deformation. The quantization was actually performed in [4] using the representation theory of \( osp(2,2) \) superalgebra. Here we adopt a different procedure, namely the quantum symplectic reduction (or, in other words, quantization with constraints). We start with the well-known quantization of the complex plane \( \mathbb{C}^2 \). The generators \( \bar{\chi}^\alpha, \chi^\alpha, \bar{a}, a \) become creation and annihilation operators on the Fock space whose commutation relations are given by the standard replacement
\[
\{.,.\} \rightarrow \frac{1}{\hbar} [.,.].
\] (9)
Here \( \hbar \) is a real parameter (we have absorbed the imaginary unit into the definition of the Poisson bracket) referred to as the "Planck constant". Explicitly
\[
[\chi^\alpha, \bar{\chi}^\beta]_- = \hbar \delta^{\alpha\beta}, \quad [a, \bar{a}]_+ = \hbar
\] (10)
and all remaining graded commutators vanish. The Fock space is built up as usual, applying the creation operators \( \bar{\chi}^\alpha, \bar{a} \) on the vacuum \( |0\rangle \), which is in turn annihilated by the annihilation operators \( \chi^\alpha, a \). We use here the same symbols for the classical and quantum quantities with a hope that it will be always clear from the context which usage we have in mind.

Now we perform the quantum symplectic reduction with the moment map \( (\bar{\chi}^\alpha \chi^\alpha + \bar{a}a) \). First we restrict the Hilbert space only to the vectors \( \psi \) satisfying the constraint
\[
(\bar{\chi}^\alpha \chi^\alpha + \bar{a}a - 1)\psi = 0.
\] (11)
Hence operators \( \hat{f} \) acting on this restricted space which fulfil
\[
[\hat{f}, (\bar{\chi}^\alpha \chi^\alpha + \bar{a}a - 1)] = 0
\] (12)
form our deformed version of \( A_\infty \).

The spectrum of the operator \( (\bar{\chi}^\alpha \chi^\alpha + \bar{a}a - 1) \) in the Fock space is given by a sequence \( Nh - 1 \), where \( N \)'s are integers. In order to fulfil (11) for a non-vanishing \( \psi \), we observe that the inverse Planck constant \( 1/\hbar \) must be an integer \( N \). The constraint \( (xy) \) then selects only \( \psi \)'s living in the eigenspace \( H_N \) of the operator \( (\bar{\chi}^\alpha \chi^\alpha + \bar{a}a - 1) \) with the eigenvalue \( 0 \). This subspace of
the Fock space has the dimension $2N + 1$ and the algebra $A_N$ of operators (i.e. supermatrices) $\hat{f}$ acting on it is $(2N + 1)^2$-dimensional.

When $N \to \infty$ (the dimension $(2N + 1)^2$ then also diverges) we have the Planck constant approaching 0 and the algebras $A_N$ tend to the classical limit $A_{\infty}$ \[2\].

The Hilbert space $H_N$ is naturally graded. The even subspace $H_{eN}$ is created from the Fock vacuum by applying only the bosonic creation operators:

$$(\bar{\chi}^1)^{n_1} (\bar{\chi}^2)^{n_2} |0\rangle, \quad n_1 + n_2 = N,$$

while the odd one $H_{oN}$ by applying both bosonic and fermionic creation operators:

$$(\bar{\chi}^1)^{n_1} (\chi^2)^{n_2} |\bar{a}\rangle, \quad n_1 + n_2 = N - 1.$$ \[14\]

At the level of supercomplex plane $C^{2,1}$, it is the textbook fact from quantum mechanics that the integral $\int d\bar{\chi}^\alpha d\chi^\alpha d\bar{a} d\bar{a}$ (this is the Liouville integral over the superphase space) is replaced under the quantization procedure by the supertrace in the Fock space. (The supertrace is the trace over the indices of the zero-fermion states minus the trace over the one-fermion states). The $\delta$ function of the operator $(\bar{\chi}^\alpha \chi^\alpha + \bar{a} a - 1)$ just restrict the supertrace to the trace over the indices of $H_{eN}$ minus the trace over the indices of $H_{oN}$. Hence an integration in $A_N$ is given by the formula

$$I[\hat{f}] \equiv \text{STr}[\hat{f}], \quad \hat{f} \in A_N.$$

The graded involution $\check{}$ in the noncommutative algebra $A_N$ is defined exactly as in (1).

2.2 $osp(2, 1)$ and $osp(2, 2)$ superalgebras and their representations

The $osp(2, 2)$ superalgebra has a convenient basis of even generators $R_\pm, R_3, \Gamma$ and the odd ones $V_\pm, D_\pm$, satisfying the following (anti)commutation relations (see also \[16\]):

$$[R_3, R_\pm] = \pm R_\pm, \quad [R_+, R_-] = 2R_3, \quad [R_i, \Gamma] = 0; \quad [D_\pm, V_\pm] = 0, \quad [D_\pm, V_\mp] = \pm \frac{1}{4} \Gamma;$$ \[16\] \[17\]
Here and it what follows the commutators are denoted as $[.,.]$ but the anticommutators have a subscript $[.,.]_\pm$. We reserve the notation $\{.,.\}$ for Poisson brackets and their noncommutative generalizations (see section 3.2.). If we take a Poisson bracket of two odd elements of $\mathcal{A}_\infty$ we write $\{.,.\}_\pm$.

The superalgebra $osp(2,1)$ is a subsuperalgebra of $osp(2,2)$ generated by $R_i, V_\pm$. The irreducible representations of $osp(2,1)$ are classified by one parameter, which may be a positive integer or a positive half-integer $j$ and is referred to as the superspin \[ \Gamma \]. Every irreducible $j$-representation is of course a (reducible) representation of the $su(2)$-subalgebra of $osp(2,1)$. Its decomposition into the irreducible components from the $su(2)$ point of view is given by

\[
\hat{j}_{osp(2,1)} = \hat{j}_{su(2)} \oplus (j - \frac{1}{2})_{su(2)},
\]

where $\hat{j}_{su(2)}$ means obviously the standard $su(2)$ spin. The only exception from the rule (23) is a trivial superspin zero representation.

The classification of irreducible representations of $osp(2,2)$ is more involved \cite{15}. There exist two types of them: the typical and the non-typical ones. The former are characterized by the property that they are reducible from the point of view of the $osp(2,1)$ superalgebra, while the latter are irreducible. The typical representation is characterized by one positive integer or half-integer $\hat{j}_{osp(2,2)} \geq 1$ called the $osp(2,2)$ superspin and by an arbitrary complex number $\gamma \neq \pm 2j$ which is related to $\Gamma$ and may be called a $\Gamma$-spin. The typical representations considered in this paper will have always the $\Gamma$-spin equal to zero. They are $8j_{osp(2,2)}$ dimensional and they have the following $osp(2,1)$ content

\[
\hat{j}_{osp(2,2)} = \hat{j}_{osp(2,1)} \oplus (j - 1/2)_{osp(2,1)},
\]
hence the following $su(2)$ content

$$j_{osp(2,2)} = j_{su(2)} \oplus (j - 1/2)_{su(2)} \oplus (j - 1/2)_{su(2)} \oplus (j - 1)_{su(2)}.$$  \hspace{1cm} (25)

The Lie superalgebra $osp(2, 2)$ is naturally represented on the (graded) commutative associative superalgebra $A_\infty$ and also on its noncommutative deformations $A_N$. In the commutative case, this representation can be called Hamiltonian since it is generated via the super-Poisson bracket (3) by the following charges

$$r_+ = \bar{\chi}^1 \chi^2, \quad r_- = \bar{\chi}^2 \chi^1, \quad r_3 = \frac{1}{2}(\bar{\chi}^1 \chi^1 - \bar{\chi}^2 \chi^2) \quad \gamma = \bar{a}a + 1 \hspace{1cm} (26)$$

$$2v_+ = \bar{\chi}^1 a + \bar{\alpha} \chi^1, \quad 2v_- = \bar{\chi}^2 a - \bar{\alpha} \chi^1, \quad 2d_+ = \bar{\alpha} \chi^2 - \bar{\chi}^1 a, \quad 2d_- = -\bar{\chi}^2 a - \bar{\alpha} \chi^1. \hspace{1cm} (27)$$

This means that, for instance, $V_+$ acts on an (even) element $f \in A_\infty$ as

$$V_+ f = \{v_+, f\} \hspace{1cm} (28)$$

and so on for every generator of $osp(2, 2)$.

In the non-commutative case, the representation of $osp(2, 2)$ is defined by the same charges (26) and (27) but now thought as the operators acting on $H_N$ via scaled (graded) commutators; for instance, for even $f$,

$$V_+ f = N[v_+, f], \quad f \in A_N. \hspace{1cm} (29)$$

The explicit form of the supermatrices $r_i, v_\alpha, d_\alpha$ and $\gamma$ was given in [4] (eqs. (83)-(91)).

Both the commutative algebra $A_\infty$ and its noncommutative deformations $A_N$ are completely reducible with respect of the $osp(2, 2)$ action described above. Their decompositions into irreducible components involve only the typical representations and they are given explicitly as follows [4]

$$A_N = \bigoplus_{j=0}^N j, \quad A_\infty = \bigoplus_{j=0}^\infty j, \hspace{1cm} (30)$$

where $j$ stands for the $osp(2, 2)$ superspin and $j = 0$ means the trivial representation. The representation space of the latter consists of the constant elements of $A$ and of the constant multiples of the unit supermatrix in the case of $A_N$.  

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The description of the typical multiplet in $A_\infty$ or in $A_N$ with the $osp(2,2)$ superspin equal to 1 is easy. In both cases, the commutative and the non-commutative one, the representation space of this superspin 1 representation is spanned by the charges $r_i, \gamma, v_\pm, d_\pm$ which are considered as the elements of $A_\infty$ and $A_N$, respectively. Evidently, the $osp(2,2)$-superspin 1 representation coincides with the adjoint representation and its dimension is 8. This fact has an interesting consequence, namely that the $A_N$ valued charges provide also the $osp(2,2)$ representation with the representation space being $H_N$. This representation can be showed to be the non – typical irreducible representation of $osp(2,2)$ and, as such, it is also the irreducible representation of $osp(2,1)$. Its $osp(2,1)$ superspin is given by $N/2$.

2.3 Seeking the complex

We could shortly define the differential complex on the supersphere and then construct the supergauge theories based on it, but before doing that it is perhaps desirable to indicate the way how this complex was invented. Without those indications, the interested reader could check that the construction gives correct results but possibly he would not be convinced that it is somehow unique.

Suppose therefore that we want to construct supergauge theories with the underlying superalgebra being $osp(2,1)$. A natural way to do it consists in trying to ”covariantize” the derivatives which appear in the action of charged scalar superfield (we are in two dimension). This action was constructed in [4] and is explicitly given by

$$S = I[D_+ \Phi^\dagger D_- \Phi - D_- \Phi^\dagger D_+ \Phi + (1/4)\Gamma \Phi^\dagger \Gamma \Phi],$$

(31)

where $\Phi \in A_\infty$ is a complex scalar superfield on the sphere, $I$ is the integral over $A_\infty$ defined in (7) and the derivatives $D_\pm, \Gamma$ were defined via the Poisson bracket (see (29)). If we add those derivatives to the $osp(2,1)$ superalgebra (which acts by means of $R_i, V_\pm$) we obtain the $osp(2,2)$ superalgebra. As was explained in [4], the using of sole $osp(2,1)$ generators was insufficient for constructing a theory respecting the spin-statistics theorem.

It seems that our supergauge field multiplet is composed of three superfields $A_\pm, W$ which one has to add respectively to three derivatives $D_\pm, \Gamma$ in order to covariantize them. The action (31) would then become

$$S = I[(D_+ - A_+) \Phi^\dagger (D_- + A_-) \Phi - (D_- - A_-) \Phi^\dagger (D_+ + A_+ ) \Phi$$
+ (1/4)(\Gamma - W)\Phi^4(\Gamma + W)\Phi].
\tag{32}

However, we shall encounter a big trouble in trying to find a $osp(2,1)$ invariant field strength corresponding to gauge multiplet $A_\pm, W$. It seems that this field strength should be given by an expression that contains only the first derivatives $D_\pm, \Gamma$ of the multiplet $A_\pm, W$, for example something like:

$$F = D_+ A_- - D_- A_+ + (1/4)\Gamma W.$$  \tag{33}

$F$ defined in this way seems to be nice, since it is indeed $osp(2,1)$ invariant (for consistency, the multiplet $A_\pm, W$ has to transform under the $osp(2,1)$ action in the same way as the derivatives $D_\pm, \Gamma$ which are transformed according to (16),(17), (21) and (22)). However, the strength so defined is not gauge invariant if we impose the evident gauge transformation rule

$$A_\pm \rightarrow A_\pm + iD_\pm \Lambda, \quad W \rightarrow W + i\Gamma \Lambda,$$  \tag{34}

where $\Lambda$ is a real scalar superfield.

Let’s continue our search and suppose that the needed field strength is not a $osp(2,1)$ singlet but it is some multiplet with a higher $osp(2,1)$ superspin. However, it is not difficult to show, that no such multiplet exist which would be linear in the derivatives $D_\pm, \Gamma$ and would respect the gauge transformation (34). The same no go theorem can be proved if we add into the game the derivatives $R_i, V_\pm$ acting on $A_\pm, W$.

Let us therefore add to the superspin 1/2 multiplet $A_\pm, W$ a superspin 1 multiplet of superfields $C_i, B_\pm$ whose gauge transformations are defined as follows

$$C_i \rightarrow C_i + iR_i \Lambda, \quad B_\pm \rightarrow B_\pm + iV_\pm \Lambda.$$  \tag{35}

Now it turns out that it exists an $osp(2,1)$ covariant multiplet of gauge invariant field strengths that is linear in the derivatives $R_i, V_\pm, D_\pm, \Gamma$ and in the superfields $C_i, B_\pm, A_\pm, W$. However, the trouble reappears: firstly, it seems to be unnatural to have an abundance of new gauge superfields in game which even do not interact with the matter field and which enter only the pure gauge field sector of the Lagrangian. Secondly, and even more importantly, even if we accept that abundance of fields, any polynomial $osp(2,1)$ invariant Lagrangian built up of that field strength leads to a pathological theory (higher derivatives, violation of spin-statistics etc.).
It turns out, however, that even this difficulty can be circumvented by imposing suitable constraints on the supergauge field multiplet $A_\pm, W, C_i, B_\pm$ which would eliminate the unwanted superfields. This strategy is, of course, standard in the superworld but not necessarily easy. The subtlety consists in ensuring that differential constraints in the superspace do not generate differential constraints in the bosonic variables on the fields which remain in the Lagrangian. At the same time one has to ensure that the constraints are compatible with the $osp(2,1)$ supersymmetry and the supergauge transformations (34) and (35). All these conditions are quite stringent and the fact that a solution exists even in the noncommutative case indicates the naturaleness of the compatibility of noncommutative geometry and supersymmetry.

For seeking the good constraints, we adopt a natural assumption that the constraints are linear both in the derivatives $R_i, V_\pm, D_\pm, \Gamma$ and in the superfields $C_i, B_\pm, A_\pm, W$. The condition of the compatibility with the gauge transformations (34) and (35) selects 32 constraints of that type which fall into six $osp(2,1)$ supermultiplets. One of these multiplet has the superspin $1/2$, three of them the superspin 1 and two of them the superspin $3/2$. The superspin $1/2$ multiplet is nothing but the field strength mentioned above. It reads

$$F_\pm = (\Gamma B_\pm - V_\pm W - 2R_\pm A_\mp + 2D_\pm C_\pm \mp 2R_3 A_\pm \pm 2D_\pm C_3 \pm 2A_\pm); \quad (36)$$

$$f = 4V_+ A_+ - 4V_- A_+ + 4D_- B_+ - 4D_+ B_- + 2W. \quad (37)$$

A tedious (though straightforward) inspection shows that the only viable constraint is given by the following superspin 1 multiplet, i.e.:

$$\pm 4D_+ A_+ + C_\pm = 0, \quad C_3 - 2D_- A_+ - 2D_+ A_- = 0; \quad (38)$$

$$B_\pm + D_\pm W - \Gamma A_\pm = 0. \quad (39)$$

The explicite formulas for the field strength and the correct constraint will reappear in the following subsection in terms of the structures of the differential complex alluded in the introduction. It should be clear that the structure of complex that we are going to construct is implied by our previous discussion. In other words, we grasp and formalize our search of the supersymmetric field strength and the supersymmetric constraints in terms of that complex.
2.4 The complex

We shall describe the differential complex over the supersphere by working with the commutative and noncommutative case at the same time. In fact, whenever we shall consider the "commutator" in the algebra of $A_N$ we shall have in mind the commutator multiplied by $N$ (for $N$ finite) and the Poisson bracket (3) for $N = \infty$.

We denote the complex by $\Xi_N$ and we define it as follows

$$\Xi_N = \bigoplus_{j=0}^{3}(\Xi_N)_j. \quad (40)$$

As usual, the elements of $(\Xi_N)_j$ will be called the $j$-forms. The spaces $(\Xi_N)_j$ have the following structure

$$(\Xi_N)_0 = (\Xi_N)_3 = A_N; \quad (41)$$

$$(\Xi_N)_1 = (\Xi_N)_2 = \bigoplus_{i=1}^{8}(A_N)_i, \quad (42)$$

where $(A_N)_i = A_N$ for every value of the index $i$.

Generically, we shall denote by small (capital) Greek characters the 0-forms (3-forms) and by capital (small) Latin characters the 1-forms (2-forms). Then the associative product $*$ is given by the rules

$$\phi * \psi = \phi \psi, \quad \phi * (A_{\pm}, W, C_i, B_{\pm}) = (\phi A_{\pm}, \phi W, \phi C_i, \phi B_{\pm}); \quad (43)$$

$$\phi * (a_{\pm}, w, c_i, b_{\pm}) = (\phi a_{\pm}, \phi w, \phi c_i, \phi b_{\pm}), \quad \phi * \Psi = \phi \Psi; \quad (44)$$

$$(A_{\pm}^1, W^1, C_i^1, B_{\pm}^1) * (A_{\pm}^2, W^2, C_i^2, B_{\pm}^2) =$$

$$(W^1 B_{\pm}^2 - W^2 B_{\pm}^1 - 2C_3 A_{\pm}^2 + 2C_3^2 A_{\pm}^1 - 2C_3 A_{\pm}^1 + 2C_3^2 A_{\pm}^1, \quad (45)$$

$$(W^1 B_{\pm}^2 - W^2 B_{\pm}^1 - 2C_3 A_{\pm}^2 + 2C_3^2 A_{\pm}^1 - 2C_3 A_{\pm}^1 + 2C_3^2 A_{\pm}^1, \quad W^1 A_{\pm}^1 - W^2 A_{\pm}^1, W^1 A_{\pm}^2 - W^2 A_{\pm}^1);$$

$$(A_{\pm}, W, C_i, B_{\pm}) * (a_{\pm}, w, c_i, b_{\pm}) =$$

$$A_{\pm} a_{\pm} - A_{-} a_{+} + \frac{1}{4} W w - \frac{1}{2} C_{+} c_{-} - \frac{1}{2} C_{-} c_{+} - C_3 c_3 - B_{+} b_{-} + B_{-} b_{+}. \quad (46)$$
The multiplication of forms by the scalars from the right is defined as in (43) and (44) but with \( \phi \) standing from the right. The product of a two form with a one-form is given as in (46) but with reversed order of the small and the capital characters. Finally, all other products are defined to be zero.

It is important to notice, that \( A, B, C, W \) are understood as odd elements of \( A_N \) while \( C_i, W \) are even (we did not indicate it in (42) in order to avoid too cumbersome notation). The same is true for \( a, b, c, w \) respectively. \( \Phi \) and \( \psi \) are even. Of course, these facts play an important role in checking that the product (43)-(46) is associative and, in case of \( N = \infty \), also graded commutative.

The coboundary operator \( \delta \) is given by the rules

\[
\delta \Phi = (D_\Phi, \Gamma \Phi, R_\Phi, V_\Phi); \tag{47}
\]

\[
\delta(A_\pm, W, C_i, B_\pm) =
(\Gamma B_\pm - V_\pm W - 2R_\pm A_\mp + 2D_\pm C_\mp \mp 2R_3 A_\pm \pm 2D_\pm C_3 + 2A_\pm, \\
4V_+ A_- - 4V_- A_+ + 4D_+ B_+ - 4D_- B_- + 2W, \\
-4D_+ A_+ - C_3 + 2D_- A_+ - 2D_+ A_- + 4D_+ A_- - C_3 - B_\pm - D_\pm W + \Gamma A_\pm); \tag{48}
\]

\[
\delta(a_\pm, w, c_\pm, b_\pm) =
D_+ a_- - D_- a_+ + \frac{1}{4} \Gamma w - \frac{1}{2} R_+ c_- - \frac{1}{2} R_- c_+ - R_3 c_3 - V_+ b_- + V_- b_+; \tag{49}
\]

\[
\delta \psi = 0. \tag{50}
\]

The operators \( R_\pm, V_\pm, D_\pm, \Gamma \) in (47)-(49) act via the scaled commutators (29) for \( N \) finite or via the Poisson brackets (3) for \( N \) infinite.

One easily checks that the coboundary \( \delta \) is nilpotent

\[
\delta^2 = 0 \tag{51}
\]

and that the \( \delta \) does verify the graded Leibniz rule

\[
\delta(\alpha \ast \beta) = \delta \alpha \ast \beta + (-1)^{\alpha} \alpha \ast \delta \beta \tag{52}
\]

in both commutative (infinite \( N \)) and noncommutative (finite \( N \)) cases.

Now we shall give the action of the \( osp(2,1) \) superalgebra on the complex \( \Xi_N \). The \( osp(2,1) \) action on the 0-forms is given basically in terms of the odd
generators $V_\pm$. The action of the bosonic generators $R_i$ can be then derived in terms of the anticommutators (19) of the odd transformations. On the 0-forms (and 3-forms), we have the following action:

$$\Delta \Phi = (\epsilon_+ V_+ + \epsilon_- V_-) \phi \equiv (\epsilon V) \Phi. \quad (53)$$

Here $\epsilon_\pm$ are constant Grassmann parameters and $\Delta$ stands for an infinitesimal variation. The parameters $\epsilon_\pm$ behave with respect to the graded involution as follows

$$\epsilon_+^\dagger = \epsilon_- , \quad \epsilon_-^\dagger = -\epsilon_+ . \quad (54)$$

Note that this is the choice of a real form of $osp(2,1)$ with respect to the graded involution and it is dictated by the fact that a 1-form $(A_\pm, W, C_i, B_\pm)$ must fulfil the following "reality" conditions

$$A_+^\dagger = A_-, \quad A_-^\dagger = -A_+, \quad B_+^\dagger = -B_-, \quad B_-^\dagger = B_+; \quad (55)$$

$$C_+^\dagger = C_-, \quad C_-^\dagger = C_+, \quad C_3^\dagger = C_3, \quad W^\dagger = W, \quad (56)$$

in order to yield the correct degrees of freedom of a (super)gauge field.

The action on the 1-forms (and also on the 2-forms) is given as follows

$$\Delta (A_\pm, W, C_i, B_\pm) =$$

$$((\epsilon V) A_+ - \frac{1}{4} \epsilon_- W, (\epsilon V) A_- + \frac{1}{4} \epsilon_+ W, (\epsilon V) W + \epsilon_+ A_+ + \epsilon_- A_-,$$

$$(\epsilon V) C_+ + \epsilon_- B_+, (\epsilon V) C_- + \epsilon_+ B_-, (\epsilon V) C_3 + \frac{1}{2} \epsilon_+ B_+ - \frac{1}{2} \epsilon_- B_-,$$

$$(\epsilon V) B_+ - \frac{1}{2} \epsilon_+ C_+ + \frac{1}{2} \epsilon_- C_3, (\epsilon V) B_- + \frac{1}{2} \epsilon_+ C_3 + \frac{1}{2} \epsilon_- C_-). \quad (57)$$

It can be easily checked that the coboundary $\delta$ is $osp(2,1)$ invariant, i.e.

$$\Delta \delta \omega = \delta \Delta \omega, \quad \omega \in \Xi_N. \quad (58)$$

In fact, Eq.(58) becomes evident when we introduce the invariant description of the complex in section 3.2 because the coboundary $\delta$ will be given in terms of invariant operators of $osp(2,1)$. Another important property of $\Delta$ is that it verifies the Leibniz rule in the complex $\Xi_N$, in other words

$$\Delta(\omega_1 * \omega_2) = (\Delta \omega_1) * \omega_2 + \omega_1 * (\Delta \omega_2). \quad (59)$$
This property enables us to construct the $osp(2, 1)$ invariants. For example, the product of a 1-form with a 2-form, given by the formula (46), is the $osp(2, 1)$ scalar. Note, however, that the expressions

$$A_+a_- - A_-a_+ + \frac{1}{4}Ww$$

and

$$\frac{1}{2}C_+c_- + \frac{1}{2}C_-c_+ + C_3^2 + B_+b_- - B_-b_+,$$

are separately $osp(2, 1)$ invariant.

The last things we shall need are the "Hodge triangle" $\triangledown$ and the theory of integration on $\Xi_N$. The Hodge triangle converts an $i$-form in to $(3-i)$-form; it is defined simply as the identity map between $\Xi_0$ and $\Xi_3$, and $\Xi_1$ and $\Xi_2$, respectively. The integral $I$ will be defined only on the 3-forms and will be given by (15) for $N$ finite and by (7) for $N$ infinite.

3 Field theories

3.1 The commutative case

Let us consider first the pure gauge theories in the commutative case. The gauge field will be a 1-form $V \in \Xi_N$ subject to the following constraint

$$F \equiv \delta V = (.,.,0,0,0,0,0),$$

in words, the first three components of the coboundary $\delta V$ are unconstrained but the remaining five have to vanish. The reader might have noticed that under the action of the subalgebra $osp(2, 1)$ a generic 1-form (2-form) decomposes into two multiplets. The first three components form a multiplet with the $osp(2, 1)$-superspin 1/2 and the remaining five with superspin 1. Since the coboundary $\delta$ commutes with the action $\Delta$ of $osp(2, 1)$, it is evident that our constraint (62), does respect the $osp(2, 1)$ supersymmetry.

The gauge symmetry of the constraint (62) is also obvious due to the nilpotency of $\delta$. It remains to see whether the constraint involves some unwanted space derivatives. Fortunately, it is not the case. Indeed, looking at (48) it is immediately evident that the constraint is resolved with respect to the "additional" superfields $C_{\pm}, C_3, B_{\pm}$:

$$C_{\pm} = \mp D_{\pm}A_\pm, \quad C_3 = 2D_-A_+ + 2D_+A_-, \quad B_{\pm} = -D_{\pm}W + \Gamma A_{\pm}.$$
If the field $V$ satisfies the constraint (62) then the three nonzero components of the field strength $\delta V$ will be second order expressions in the derivatives $R_i, V_\pm, D_\pm, \Gamma$ acting on the superspin 1/2 multiplet $A_\pm, W$. This may seem awkward, since a Langrangian quadratic in the field strength will contain terms with four derivatives. It turns out, however, that working out the action in components of the superfields $A_\pm, W$ will give a non-pathological action. The same phenomenon takes place in the standard (super-Poincaré) supersymmetric electrodynamics in two dimensional flat space [17] where the kinetic term of the gauge field also contains expressions quartic in the supersymmetric covariant derivatives nevertheless the action in components is the standard second-order one.

Let us write a pure gauge field action on the commutative supersphere as follows

$$S_\infty(V) = I[\alpha' \delta V \star \delta V + \beta' V \star \delta V], \quad V \in \Xi_N.$$  \hspace{1cm} (64)

Where $\alpha'$ and $\beta'$ are real parameters and the components of $V = (A_\pm, W, C_i, B_\pm)$ are supposed to satisfy the reality conditions (55) and (56). It is evident that the action is gauge invariant with respect to the transformation

$$V \rightarrow V + i \delta \Lambda,$$  \hspace{1cm} (65)

where $\Lambda$ is a 0-form. Of course, $V$ is to satisfy the constraint (62), hence the components $C_i, B_\pm$ are given by (63). Having in mind this, we can evaluate the coboundary of $V$:

$$\delta V = (F_+, F_-, f, 0, 0, 0, 0, 0),$$  \hspace{1cm} (66)

where

$$F_\pm = (\Gamma^2 + 2)A_\pm - (\Gamma D_\pm + V_\pm)W \mp 12D_\pm D_\pm A_\pm \pm 12D^2_\pm A_\pm;$$

$$f = 2W + 4(D_+ D_- - D_- D_+)W + 4(V_+ - D_+ \Gamma)A_- - 4(V_- - D_- \Gamma)A_+.$$  \hspace{1cm} (67)

Before giving a noncommutative version of the action (64), let us write its content in a more familiar parametrization. Set

$$A_+ = \frac{1}{2}(A - \bar{z} \bar{A}) + \frac{1}{2} d_+ K, \quad A_- = -\frac{1}{2}(z A + \bar{A}) + \frac{1}{2} d_- K, \quad W = \bar{b} \bar{A} - b A + \gamma K,$$  \hspace{1cm} (68)
where \(d_\pm, \gamma\) were defined in (26) and (27). The components \(F_\pm, f\) then become

\[
F_+ = -\frac{3}{2} n [\bar{D}D A + \bar{z}D D \bar{A} + \bar{b}(D \bar{A} + \bar{D}A) + D^2 \bar{A} + \bar{z} \bar{D}^2 A]
+ 2d_+ n(\bar{D}A + D \bar{A}) - 4d_+ K + 2nd_+ \bar{D}D K - 2(D + \bar{z} \bar{D})K; \quad (69)
\]

\[
F_- = \frac{3}{2} n [-z\bar{D}D A - \bar{D}D \bar{A} + b(D \bar{A} + \bar{D}A) + zD^2 \bar{A} - \bar{D}^2 A]
+ 2d_- n(\bar{D}A + D \bar{A}) - 4d_- K + 2nd_- \bar{D}D K - 2(\bar{D} - zD)K; \quad (70)
\]

\[
f = 3n[\bar{b}(\bar{D} \bar{D} A) - b(\bar{D}DA) - 2(D \bar{A} + \bar{D}A) + bD^2 \bar{A} + \bar{b} \bar{D}^2 A]
2\gamma n(\bar{D}A + D \bar{A}) + 2\gamma n \bar{D}D K + 4(\bar{b} \bar{D} + bD)K - 4\gamma K. \quad (71)
\]

Here \(n = 1 + \bar{z}z + \bar{b}b\) and the operators \(D, \bar{D}\) are the standard supersymmetric covariant derivatives in two dimensions, i.e.

\[
D = \partial_b + b \partial_{\bar{z}}, \quad \bar{D} = \partial_{\bar{b}} + \bar{b} \partial_z. \quad (72)
\]

It is perhaps worth giving formulae that express the derivatives \(R_i, \Gamma, D_\pm, V_\pm\) in terms of the derivatives \(D, \bar{D}\) and \(Q, \bar{Q}\), where

\[
Q = \partial_b - b \partial_{\bar{z}}, \quad \bar{Q} = \partial_{\bar{b}} - \bar{b} \partial_z. \quad (73)
\]

Here they are:

\[
D_+ = \frac{1}{2}(D - \bar{z} \bar{D}), \quad D_- = -\frac{1}{2}(\bar{D} + zD); \quad (74)
\]

\[
V_+ = \frac{1}{2}(Q + \bar{z} \bar{Q}), \quad V_- = \frac{1}{2}(\bar{Q} - zQ); \quad (75)
\]

\[
\Gamma = \bar{b} \partial_b - b \partial_{\bar{b}}, \quad R_3 = \bar{z} \partial_z - z \partial_{\bar{z}} + \frac{1}{2} \bar{b} \partial_{\bar{b}} - \frac{1}{2} b \partial_b; \quad (76)
\]

\[
R_+ = -\partial_z - \bar{z}^2 \partial_{\bar{z}} - \bar{z} \bar{b} \partial_{\bar{b}}, \quad R_- = \partial_{\bar{z}} + z^2 \partial_z + z b \partial_b. \quad (77)
\]

Using the formulae (46),(68) and (69)-(71), we obtain the action (64) in terms of \(A, \bar{A}, K\). The explicit formula is somewhat cumbersome and we do not list it here since it is pathological! The trouble is caused by the superfield \(K\) that is contained in the action in the form \((\bar{D}D K)^2\). This gives the bosonic derivatives of fourth order. Fortunately, there is a manifestly supersymmetric
and gauge invariant way for getting rid of the unwanted field $K$. Indeed, a constraint
\begin{equation}
K = 0
\end{equation}
(simply does the job. Having in mind the noncommutative generalization, it is desirable to formulate this additional constraint in terms of the fields $A_{\pm}, W$. It reads
\begin{equation}
d_+A_- - d_-A_+ + \frac{1}{4}\gamma W = 0.
\end{equation}
It is not difficult to verify the gauge symmetry (78) or (79) and the $osp(2,1)$ supersymmetry of (79).

After imposing the additional constraint (79), the action (64) becomes
\begin{equation}
S_{\infty} = \frac{1}{2\pi i} \int d\bar{z}dzbdb\{\alpha D(n\omega)D(n\omega) + \beta n\omega^2\},
\end{equation}
where
\begin{equation}
\omega = \bar{D}A + D\bar{A}, \quad n = 1 + \bar{z}z + \bar{b}b
\end{equation}
and the parameters $\alpha, \beta$ are linear combinations of $\alpha', \beta'$.

It is instructive to see how the $osp(2,1)$ superinvariance is realized in this parametrization of the gauge field multiplet. We have
\begin{align}
\Delta A &= (\epsilon_+ V_+ + \epsilon_- V_-)A + \frac{1}{2}\epsilon_- bA; \quad (82) \\
\Delta \bar{A} &= (\epsilon_+ V_+ + \epsilon_- V_-)\bar{A} - \frac{1}{2}\epsilon_+ b\bar{A} \quad (83)
\end{align}
and therefore
\begin{equation}
\Delta \omega = (\epsilon_+ V_+ + \epsilon_- V_-)\omega + \frac{1}{2}(\epsilon_- b - \epsilon_+ \bar{b})\omega, \quad (84)
\end{equation}
or, equivalently,
\begin{equation}
\Delta(n\omega) = (\epsilon_+ V_+ + \epsilon_- V_-)(n\omega). \quad (85)
\end{equation}
From the last formula (85), the $osp(2,1)$ supersymmetry of the action (80) immediately follows. The gauge symmetry $A \to A + iD\Lambda, \bar{A} \to \bar{A} + i\bar{D}\Lambda$ is also evident.

We are now ready to cast the action (80) in components. Set
\begin{equation}
iA = \zeta + bv + \frac{1}{2}b\frac{(w + iu)}{1 + \bar{z}z} + \bar{b}b\frac{\eta}{1 + \bar{z}z} - \partial_z\zeta; \quad (86)
\end{equation}
\[ i\dot{A} = \bar{\zeta} - \frac{1}{2} \frac{(w - iu)}{1 + \bar{z} z} + \bar{b} \bar{v} + \bar{b}b \left( \frac{\bar{\eta}}{1 + \bar{z} z} - \partial_{\bar{z}} \bar{\zeta} \right). \] (87)

Here \( u, w \) are real, \( v \) and \( \bar{v} \) mutually complex conjugate and \( \eta^\dagger = \bar{\eta}, \bar{\zeta}^\dagger = \bar{\zeta} \).

We calculate
\[ \text{in}(\bar{D}A + D\bar{A}) = iu + b\eta - \bar{b}\bar{\eta} + \bar{b}b [(1 + \bar{z} z)(\partial_z v - \partial_{\bar{z}} \bar{v}) + \frac{i u}{1 + \bar{z} z}]. \] (88)

Thus the fields \( \zeta, \bar{\zeta} \) and \( w \) have dropped out.

It turns out that an avoiding a mutual coupling of the fields \( u \) and \( v \) in the action requires setting \( \beta = -2\alpha \) in the action (80). Thus we respect this most natural choice and write finally
\[ S_\infty = \frac{-\alpha}{2\pi i} \int d\bar{z}dz \{ -(1 + \bar{z} z)^2 (\partial_z v - \partial_{\bar{z}} \bar{v})^2 + \partial_z u \partial_{\bar{z}} u + \frac{u^2}{(1 + \bar{z} z)^2} \]
\[ + \eta \partial_z \eta + \bar{\eta} \partial_{\bar{z}} \bar{\eta} + 4 \frac{\bar{\eta} \eta}{(1 + \bar{z} z)} \}. \] (89)

Note that the action (89) differs from that of the standard free supersymmetric electrodynamics in the flat Euclidean space [17] by the presence of the mass term for the dynamical fields \( u, \eta, \bar{\eta} \). The reader may say that it is not surprising that theories on different manifolds have different actions. Note, however, that the matter action (32), rewritten in terms of the superfields \( A, \bar{A} \) and a complex matter superfield \( \Phi \), is exactly of the same form as that of the matter sector of supersymmetric Schwinger model in the flat two-dimensional space [17]:
\[ S_{\text{matter}} = \frac{1}{2\pi i} \int d\bar{z}dzd\bar{b}db [(\bar{D} - \bar{A}) \Phi^\dagger (D + A) \Phi + (\bar{D} + \bar{A}) \Phi (D - A) \Phi^\dagger]. \] (90)

Of course, this coincidence (due to superconformal invariance of the massless superscalar matter in two dimensions) is only formal because, in spite of the same form of the action, the superfields belong to different algebras! The matter superfield \( \Phi \) is an element of the algebra of the superfunctions on the supersphere while in the flat case it would be an element of the algebra of superfunctions on the flat Euclidean superspace.
3.2 An invariant description of the complex

The complex constructed in section 2.4 looks somewhat cumbersome and we may wonder whether it exists its description which would be more elegant and natural. It turns out, that the answer to this question is positive; in fact, we shall present the complex in a way which does not need to choose a basis in the superalgebra $osp(2, 2)$ or a basis in the space of 1-forms. The invariant picture is entirely based on standard structures of super-Lie algebras and it is not only more esthetic but also it is efficient for technical purposes. Indeed, the noncommutative generalization of the commutative theories of the previous subsection requires adding a quadratic term in the definition of the field strength and cubic and quartic terms in the action functional of the field theory. Consulting the formula (45) the reader may easily convince himself that working with the product of forms in the previous non-invariant description would be very messy. However, the invariant picture will enable us readily to formulate the constraint (62) and even to solve it explicitly! What is even more remarkable in the story that the complex can be constructed for a much general class of Lie super-algebras than just $osp(2, 2)$. Here are the details:

**Definition 1**: A super-Poisson algebra $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ is a $\mathbb{Z}_2$-graded associative algebra over the field of complex numbers $\mathbb{C}$ equipped with the structure of a super-Lie algebra with an even (super-Poisson) bracket $\{ \ldots \}$ compatible with the associative multiplication $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$; i.e.

$$\{ X, Y Z \} = \{ X, Y \} Z + (-1)^{XY} Y \{ X, Z \}. \quad (91)$$

Moreover, it is required that $\mathcal{A}$ possess an even unit element $e$ such that $eX = Xe = X$ and $\{ e, X \} = 0$ for all $X \in \mathcal{A}$. Finally, $\mathcal{A}$ is equipped with a linear supertrace $\text{STr} : \mathcal{A} \to \mathbb{C}$, in particular $\text{STr}(e) = 1$. The supertrace is supposed to vanish for Poisson brackets: $\text{STr}\{X, Y\} = 0; X, Y \in \mathcal{A}$ and also for odd elements: $\text{STr}(\mathcal{A}_1) = 0$.

Let us take as an example of $\mathcal{A}$ the algebra $\mathcal{A}_\infty$ of the superfunctions on the supersphere with the super-Poisson bracket given by (3) or (6) and the supertrace given by the integral (7). Another example is the noncommutative algebra of $(2N + 1) \otimes (2N + 1)$-supermatrices, denoted as $\mathcal{A}_N$ in section 2.1. This algebra defines the fuzzy supersphere \cite{[H]} and it is the noncommutative
deformation (or Berezin quantization) of the algebra $\mathcal{A}_\infty$ with the value of the "Planck constant" $\hbar = 1/N$. It should be therefore no surprise that the Lie bracket in $\mathcal{A}_N$ is not just the ordinary commutator inherited from the associative multiplication in $\mathcal{A}_N$ but the commutator multiplied by $N$ (playing role of the inverse Planck constant):

$$\{X,Y\} = Nm(X \otimes Y) - Nm(Y \otimes X) = N [X,Y], \quad X, Y \in \mathcal{A}_N.$$  \hfill (92)

The definition of the bracket with this normalization is crucial for verifying normalization of all formulae in this paper, in particular the important constrain (113). Note that we denote the super-Lie bracket in $\mathcal{A}_\infty$ by the same symbol as in $\mathcal{A}_N$. The reason is that the former gives the latter in the commutative limit $N \to \infty$. Finally, the supertrace $\text{STr}$ in $\mathcal{A}_N$ is nothing but the standard supertrace over $(2N + 1) \otimes (2N + 1)$-supermatrices. It is evident from (13) and (14), that $\text{STr}$ is correctly normalized, which means that the $\text{STr}$ of the unit supermatrix is equal to 1.

**Definition 2**: We say that $(\mathcal{A}, \mathcal{G})$ is a supersymmetric double over a super-Poisson algebra $\mathcal{A}$, if $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ is a super-Lie subalgebra of $\mathcal{A}$ (but not necessarily the associative subalgebra of $\mathcal{A}$!) and a bilinear form $\text{STr} \circ m$ restricted to $\mathcal{G}$ is non-degenerate. In this case the bilinear form $\text{STr} \circ m$ determines an element $C_G \in \mathcal{G} \otimes \mathcal{G}$ called a quadratic Casimir element of the double $(\mathcal{A}, \mathcal{G})$.

Now we construct a canonical complex $\Omega(\mathcal{A}, \mathcal{G})$ over the double $(\mathcal{A}, \mathcal{G})$ as follows

$$\Omega(\mathcal{A}, \mathcal{G}) = \bigoplus_{i=0}^{3} \Omega_i(\mathcal{A}, \mathcal{G}),$$  \hfill (93)

where

$$\Omega_0(\mathcal{A}, \mathcal{G}) = \Omega_3(\mathcal{A}, \mathcal{G}) = (\mathcal{A}_P)_0 \equiv e \otimes (\mathcal{A}_P)_0,$$

$$\Omega_1(\mathcal{A}, \mathcal{G}) = \Omega_2(\mathcal{A}, \mathcal{G}) = (\mathcal{G}_0 \otimes (\mathcal{A}_P)_0) \oplus (\mathcal{G}_1 \otimes (\mathcal{A}_P)_1).$$  \hfill (94)

Here the notation $\mathcal{A}_P$ means that one does not consider the algebra $\mathcal{A}$ over the field $\mathbb{C}$ but one replaces $\mathbb{C}$ by a graded commutative algebra $\mathcal{P}$. The subscript 0 in $(\mathcal{A}_P)_0$ (1 in $(\mathcal{A}_P)_1$) then means that one considers only a subspace of $\mathcal{A}_P$ which consists of all elements of $\mathcal{A}_P$ even (odd) with respect to the sum of gradings of $\mathcal{A}$ and of $\mathcal{P}$. 

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Note: Considering the algebra $A$ over $P$ instead over the field of complex numbers $C$ is not an useless complication, but it is dictated by the field theoretical applications. For instance, any element $f(z, \bar{z}, b, \bar{b}) \in A_\infty$ can be expanded in the Taylor series in $b, \bar{b}$. The term proportional to $b$ is of the form $\psi(z, \bar{z})b$. But $\psi(z, \bar{z})$, being a fermion, is not a $C$-valued function! It should be Grassmann-valued which means that it should be valued in odd part $P_1$ of some $Z_2$-graded commutative algebra $P$. All this is the standard supersymmetric story so we do not give more details here.

In order $\Omega(A, G)$ be a complex we need to introduce a coboundary operator $\delta^G : \Omega_i(A, G) \to \Omega_{i+1}(A, G)$ such that $(\delta^G)^2 = 0$ and an associative product $*^G : \Omega_i(A, G) \otimes \Omega_j(A, G) \to \Omega_{i+j}(A, G)$ compatible with $\delta^G$. By the compatibility we mean, of course, the Leibniz rule

$$\delta^G(X^{i} \ast^G Y^{j}) = \delta^G X^{i} \ast^G Y^{j} + (-1)^i X^{i} \ast^G \delta^G Y^{j}, \quad X^{i} \in \Omega_i(A, G), Y^{j} \in \Omega_j(A, G).$$

(95)

In order to give a simple description of $\delta^G$ and $*^G$, we note that $A \otimes A$ naturally acts on itself in one of four ways: $m \otimes m, m \otimes ad, ad \otimes m$ and $ad \otimes ad$; e.g. for $X, Y \in A$ we have

$$X_{m \otimes ad}(Y) \equiv (-1)^{Y^{(1)}} X^{(1)} X^{(1)} Y^{(1)} \otimes \{X^{(2)}, Y^{(2)}\},$$

(96)

where $X = X^{(1)} \otimes X^{(2)}$ and $Y = Y^{(1)} \otimes Y^{(2)}$. Now we define $\delta^G$ as follows,

$$\delta^G X^{0} = C^G_{m \otimes ad} X^{0};$$

(97)

$$\delta^G X^{1} = C^G_{ad \otimes ad} X^{1} + \frac{1}{2} d^G X^{1};$$

(98)

$$\delta^G X^{2} = ad X^{2} \equiv \{X^{(2)}, X^{(2)}\};$$

(99)

$$\delta^G X^{3} = 0,$$

(100)

where $X^{i} \in \Omega_i(A, G)$ and $d^G$ is a "Dynkin" number which can be defined by the relation

$$\text{STr}(XY) = \frac{1}{d^G} \text{sTr}(adXadY).$$

(101)
Here $s\text{Tr}$ is the supertrace over supermatrices in adjoint representation of the Lie superalgebra $\mathcal{G}$. The multiplication $*_{\mathcal{G}}$ is given by the following table

$$
\begin{pmatrix}
X^i *_{\mathcal{G}} Y^j & Y^0 & Y^1 & Y^2 & Y^3 \\
X^0 & m \otimes m & m \otimes m & m \otimes m & m \otimes m \\
X^1 & m \otimes m & ad \otimes m & (\text{STr} \otimes \text{Id})(m \otimes m) & 0 \\
x^2 & m \otimes m & (\text{STr} \otimes \text{Id})(m \otimes m) & 0 & 0 \\
x^3 & m \otimes m & 0 & 0 & 0
\end{pmatrix}.
$$

(102)

For example,

$$X^1 *_{\mathcal{G}} Y^1 = X^1_{ad \otimes m}(Y^1),$$

(103)

or

$$X^1 *_{\mathcal{G}} Y^2 = (-1)^{Y(1)X(2)}X^1(2)Y^2(2)\text{STr}(X^1(1)Y^2(1)).$$

(104)

**Definition 3:** We say that $(\mathcal{A}, \mathcal{G}, \mathcal{H})$ is a supersymmetric triple, if it exists a subspace $\mathcal{H}$ of $\mathcal{A}$ such that

1) $\mathcal{H}$ is a super-Lie subalgebra of $\mathcal{G}$;

2) $(\mathcal{A}, \mathcal{H})$ is the supersymmetric double with the Casimir element $C^H \in \mathcal{H} \otimes \mathcal{H}$, coboundary $\delta^H$ and product $*_{\mathcal{H}}$;

3) An element $C \equiv C^G - C^H$ fulfils $m(C) \in \mathcal{C}e$;

4) $ad(\mathcal{H}^\perp \otimes \mathcal{H}^\perp) \subset \mathcal{H}$, where $\mathcal{H}^\perp$ is an orthogonal complement of $\mathcal{H}$ in $\mathcal{G}$ with respect to $s\text{Tr} \circ m$.

Now we construct a canonical complex $\Omega(\mathcal{A}, \mathcal{G}, \mathcal{H})$ over the triple $(\mathcal{A}, \mathcal{G}, \mathcal{H})$ as follows

$$\Omega(\mathcal{A}, \mathcal{G}, \mathcal{H}) = \oplus_{i=0}^{3} \Omega_i(\mathcal{A}, \mathcal{G}, \mathcal{H}),$$

(105)

where

$$\Omega_0(\mathcal{A}, \mathcal{G}, \mathcal{H}) = \Omega_3(\mathcal{A}, \mathcal{G}, \mathcal{H}) = (\mathcal{A}_P)_0 \equiv e \otimes (\mathcal{A}_P)_0,$$

$$\Omega_1(\mathcal{A}, \mathcal{G}, \mathcal{H}) = \Omega_2(\mathcal{A}, \mathcal{G}, \mathcal{H}) = (\mathcal{G}_0 \otimes (\mathcal{A}_P)_0) \oplus (\mathcal{G}_1 \otimes (\mathcal{A}_P)_1).$$

(106)

We define the exterior derivative $\delta$ on $\Omega(\mathcal{A}, \mathcal{G}, \mathcal{H})$ as follows,

$$\delta X^0 = \delta^G X^0, \quad \delta X^2 = \delta^G X^2, \quad \delta X^3 = \delta^G X^3;$$

$$\delta X^1 = \delta^G X^1 - \delta^H X^1_H,$$

(107)

(108)

where $X^i \in \Omega_i(\mathcal{A}, \mathcal{G}, \mathcal{H})$ and $X^1_H$ means the orthogonal projection of $X^1$ from $\mathcal{G} \otimes \mathcal{A}$ into $\mathcal{H} \otimes \mathcal{A}$.  

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A product $*$ in $\Omega(\mathcal{A}, \mathcal{G}, \mathcal{H})$ is defined by the same table as the product $*_{\mathcal{G}}$ in $\Omega(\mathcal{A}, \mathcal{G})$, except of the multiplication of 1-forms where we have

$$X^1 * Y^1 = X^1 *_{\mathcal{G}} Y^1 - X^1_{\mathcal{H}} *_{\mathcal{H}} Y^1_{\mathcal{H}}.$$  \hspace{1cm} (109)

It is straightforward exercise to check that the product $*$ and the coboundary $\delta$ verify the Leibniz rule.

3.3 Noncommutative supersymmetric electrodynamics

Suppose that the super-Poisson algebra $\mathcal{A}$ is such that its Lie bracket $\{\ldots, \}$ is derived from its associative multiplication i.e., as an $N$-multiple of its commutator like in (92). Then a noncommutative pure supersymmetric electrodynamics over $\mathcal{A}$ is a theory of 1-forms in the triple complex $\Omega(\mathcal{A}, \mathcal{G}, \mathcal{H})$ defined by an action

$$S = \frac{1}{g^2} \text{STr}[\alpha' \triangleleft F * F + \beta' (V * \delta V + \frac{2}{3} V * V * V)],$$  \hspace{1cm} (110)

where

$$F = \delta V + V * V$$  \hspace{1cm} (111)

is the field strength of $V$, $\alpha', \beta'$ two real parameters (cf. (64)), $g$ a coupling constant and the Hodge triangle $\triangleleft$ is the identity map between $\Omega_1(\mathcal{A}, \mathcal{G}, \mathcal{H})$ and $\Omega_2(\mathcal{A}, \mathcal{G}, \mathcal{H})$. Moreover, $V$ is considered to be a real 1-form $V^\perp = V$ subject to two constraints

$$(\delta V + V * V)_{\mathcal{H}} = 0$$  \hspace{1cm} (112)

and

$$C * V_{\mathcal{H}} + V_{\mathcal{H}} * C + \frac{1}{N} \triangleleft V_{\mathcal{H}} * V_{\mathcal{H}} = 0.$$  \hspace{1cm} (113)

Here $V_{\mathcal{H}}$ means the orthogonal projection of $V$ to $\mathcal{H}$ and $C$ is viewed as a 2-form. The graded star $\dagger$ on the complex is defined by means of the graded star $\dagger$ on the algebra $\mathcal{A}$. For example,

$$(X^1)^\dagger = (X^{1(1)})^\dagger \otimes (X^{1(2)})^\dagger, \quad X^1 \in \Omega_1(\mathcal{A}, \mathcal{G}, \mathcal{H}).$$  \hspace{1cm} (114)

The action $S$ and both constraints are invariant with respect to
1) gauge transformations

\[ V \to U V U^{-1} - (\delta U) U^{-1}, \quad U^{-1} = U^4; \]  

(115)

2) \( \mathcal{H} \)-supersymmetry\(^1\), where \( h \in \mathcal{H} \) acts on \( \Omega(\mathcal{A}, \mathcal{G}, \mathcal{H}) \) as follows

\[ h(X^{0,3}) = \text{ad}(h \otimes X^{0,3}); \]  

(116)

\[ h(X^{1,2}) = (e \otimes h)_{m \otimes \text{ad}} X^{1,2} + (h \otimes e)_{ad \otimes m} X^{1,2}. \]  

(117)

The constraint \( (\delta V + V \ast V)_\mathcal{H} = 0 \) can be solved explicitly, thanks to the assumption (4) in the definition of the supersymmetric triple \( (\mathcal{A}, \mathcal{G}, \mathcal{H}) \) (the assumption (3) is needed for the gauge invariance of the constraint (113)). The solution reads

\[ V_\mathcal{H} = \frac{2}{d_\mathcal{H} - d_\mathcal{G}} (C_{ad \otimes ad} V_{\mathcal{H} \perp} + V_{\mathcal{H} \perp} \ast V_{\mathcal{H} \perp}). \]  

(118)

Thus we insert \( V_\mathcal{H} \) from (118) into (110) and we obtain a theory containing only 1-forms \( V_{\mathcal{H} \perp} \).

An interaction with matter can also be expressed in terms of the complex \( \Omega(\mathcal{A}, \mathcal{G}, \mathcal{H}) \). Let \( \Phi \) be a complex 0-form, then

\[ S_{\text{matter}} = \text{STr}[(\delta^\mathcal{G} - \delta^\mathcal{H} + V_{\mathcal{H} \perp}) \Phi]^\dagger \ast \langle \sigma(\delta^\mathcal{G} - \delta^\mathcal{H} + V_{\mathcal{H} \perp}) \Phi]. \]  

(119)

If we add \( S_{\text{matter}} \) to \( S \) in (110) we obtain the \( \mathcal{H} \)-supersymmetric Schwinger model over the Poisson algebra \( \mathcal{A} \).

For the commutative resp. noncommutative fuzzy supersphere \( (\mathcal{A} = \mathcal{A}_\infty \) resp. \( \mathcal{A} = \mathcal{A}_N \)) and the superalgebras \( \mathcal{H} = \text{osp}(2,1) \) and \( \mathcal{G} = \text{osp}(2,2) \), the complex \( \Omega(\mathcal{A}, \mathcal{G}, \mathcal{H}) \) is isomorphic to the complex \( \Xi_\infty \) resp. \( \Xi_N \) of the section 2.4. We present few formulae connecting the two presentations of the same complex. First of all the algebras \( \mathcal{G} \) and \( \mathcal{H} \) are generated by the Hamiltonians (26) and (27) in both commutative and noncommutative cases. In what follows, the parameter \( N \) will stand for either a finite integer (the noncommutative case) or \( \infty \) (the commutative one). Thus

\[ V_{\mathcal{H} \perp} = +2d_- \otimes A_+ - 2d_+ \otimes A_- - \frac{1}{2} \gamma \otimes W \equiv (A_+, A_-, W, 0, 0, 0, 0, 0); \]  

(120)

\(^1\)We mean here the real form of \( \mathcal{H} \)-superalgebra which respects the reality of the 1-form \( V \); cf. (54).
\[
F = +2d_- \otimes F_+ - 2d_+ \otimes F_- - \frac{1}{2} \gamma \otimes f \equiv (F_+, F_-, f, 0, 0, 0, 0, 0); 
\]
\[
C = +2d_- \otimes d_+ - 2d_+ \otimes d_- - \frac{1}{2} \gamma \otimes \gamma; 
\]
\[
d_G = \frac{4}{1+1/N}, \quad d_H = \frac{6}{1+1/N}. 
\]

Note also that \( C^\dagger = C \) and the condition \( V_{H^\perp}^\dagger = V_{H^\perp} \) does indeed reproduce the reality conditions (55) and (56). It is trivial to check that the conditions 3) and 4) of the definition 3 are indeed fulfilled. For this, one uses the relations (26)-(27), (122) and (16)-(22).

The main message of this paper is that the Schwinger model formulated in this section for finite \( N \) becomes in the limit \( N \to \infty \) the standard commutative supersymmetric theory of section 3.1. In particular, the actions (110) and (119) become (64) and (90) respectively and the constraints (112) and (113) become (62) and (79) in this limit. Moreover, the gauge transformation (115) becomes the gauge transformation (34) and (35) and the supersymmetry transformations (116) and (117) become the transformations (53) and (57). It is in this sense that we consider the theory for finite \( N \) as the nonperturbative regularization of the standard commutative theory.

Note the presence of the expressions like \( \delta V + V^* V \) in our theory which are characteristic for non-abelian gauge theories. They appear because of the noncommutativity of the fuzzy sphere, but in the commutative limit the terms like \( V^* V \) disappear and we are left with the abelian theory. In fact, the coboundary \( \delta \) commute only with one parametric subgroup of the gauge group, consisting of the elements of the form \( U = \exp (i \alpha) e \) where \( e \) is the unit element of \( A \). This is another sign that we deal with the noncommutative deformation of an \( U(1) \) gauge theory. The reader might appreciate also the economy of using the invariant formulation for writing the action functionals. In fact, the already cumbersome formulae of section 3.1, which are written in the noninvariant language, would be even much more cumbersome in the noncommutative case due to the presence of the \( V^* V \) terms.

4 Conclusions and outlook

We have constructed the supersymmetric Schwinger model on the noncommutative sphere. The theory possess only finite number of degrees of freedom
nevertheless it is manifestly supersymmetric and supergauge invariant. The basic structural ingredient of the construction of the model is the complex \( \Omega(\mathcal{A}, \mathcal{G}, \mathcal{H}) \) for \( \mathcal{G} = \text{osp}(2, 2) \) and \( \mathcal{H} = \text{osp}(2, 1) \). It is remarkable that the complex can be constructed for large class of superalgebras hence we expect that supergauge theories could be in a similar fashion constructed for higher dimensional projective spaces. It is also not difficult to suggest a generalization to more general gauge groups than \( U(1) \), however, due to the amount of work necessary for doing it we prefer to postpone it for a later publication.

Note finally that 1-forms in \( \Omega(\mathcal{A}, \mathcal{G}) \) can be interpreted as 1-cochains in the Lie superalgebra cohomology but 1-forms in \( \Omega(\mathcal{A}, \mathcal{G}, \mathcal{H}) \) are not relative cochains modulo \( \mathcal{H} \). This suggests that it may exists an interesting variant of the Lie superalgebra cohomology of \( \mathcal{G} \) with respect to \( \mathcal{H} \).

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