ASSOCIATED CYCLES OF LOCAL THETA LIFTS OF UNITARY CHARACTERS AND UNITARY LOWEST WEIGHT MODULES

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Abstract. In this paper we first construct natural filtrations on the full theta lifts for any real reductive dual pairs. We will use these filtrations to calculate the associated cycles and therefore the associated varieties of Harish-Chandra modules of the indefinite orthogonal groups which are theta lifts of unitary lowest weight modules of the metaplectic double covers of the real symplectic groups. We will show that some of these representations are special unipotent and satisfy a $K$-type formula in a conjecture of Vogan.

1. Introduction

In this paper, we first study natural filtrations on the full theta lifts for any real reductive dual pairs. We will use these filtrations to calculate the associated cycles and therefore the associated varieties of Harish-Chandra modules of the indefinite orthogonal groups which are theta lifts of unitary lowest weight modules of the metaplectic double covers of the real symplectic groups. We will show that some of these representations are special unipotent and satisfy a $K$-type formula in a conjecture of Vogan in [V3].

1.1. Let $W_R$ be a real symplectic space and $\tilde{\text{Sp}}(W_R)$ be the metaplectic double cover of the symplectic group $\text{Sp}(W_R)$. For every subgroup $E$ of $\text{Sp}(W_R)$, we let $\tilde{E}$ denote its inverse image in $\tilde{\text{Sp}}(W_R)$. We fix a maximal Cartan subgroup $U$ of $\text{Sp}(W_R)$ such that $K = G \cap U$ and $K' = G' \cap U$ are maximal compact subgroups of $G$ and $G'$ respectively.

Let $g = \mathfrak{k} \oplus \mathfrak{p}$ and $g' = \mathfrak{k}' \oplus \mathfrak{p}'$ denote the complexified Cartan decompositions of the complex Lie algebras of $G$ and $G'$ respectively. Let $\rho'$ be an irreducible $(\mathfrak{g}', \tilde{K}')$-module. We will recall the definition of its full theta lift $\Theta(\rho')$ in (2). We suppose that $\Theta(\rho')$ is nonzero. It is a $(\mathfrak{g}, \tilde{K})$-module of finite length.

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smaller member. The proof uses a natural filtration on \( \Theta(\rho') \) and assumes some of its properties. The first objective of this paper is to provide the proofs of these properties. See Section \( \text{[2]} \).

The second objective of this paper is to provide evidence that the identity \( \text{(1)} \) extends beyond the stable range. We will work with the dual pair \((G,G') = (O(p,q), \text{Sp}(2n, \mathbb{R}))\). Using a more detail analysis of the geometry of the null cones and moment maps, we are able to prove in this paper that for certain range outside the stable range, \( \text{(1)} \) continues to hold if \( \rho' \) is a unitary lowest weight module. See Theorem \( \text{[3]} \). This extends and gives a shorter proof to a main result of \([NZ]\). We can also compute the associated cycles of certain \( \Theta(\rho') \) when \( \text{(1)} \) fails. See Theorem \( \text{[4]} \).

Although we only work with the orthogonal symplectic dual pairs, most of our results extends to the dual pairs \((U(p,q), U(n_1, n_2))\) and \((\text{Sp}(2p, 2q), O^*(2n))\) too. See Section \( \text{[7]} \). We hope that our investigation would shed light on how to extend \( \text{(1)} \) in general beyond the stable range.

For the rest of this section, we will describe and state our main theorems.

1.2. Associated varieties and associated cycles. First we briefly review the definitions of associated varieties, associated cycles and other related invariants of a \((g,K)\)-module. See Section 2 in \([V3]\) for details.

Let \( \mathfrak{g} \) be a \((g,K)\)-module of finite length and let \( 0 \subset \mathfrak{g}_0 \subset \cdots \subset \mathfrak{g}_j \subset \mathfrak{g}_{j+1} \subset \cdots \) be a good filtration of \( \mathfrak{g} \). Then \( \text{Gr}_{\mathfrak{g}} = \bigoplus \mathfrak{g}_j/\mathfrak{g}_{j-1} \) is a finitely generated \((S(p), K_C)\)-module where \( S(p) \) is the symmetric algebra on \( p \) and \( K_C \) is a complexification of \( K \).

Let \( \mathscr{A} \) be the associated \( K_C \)-equivariant coherent sheaf of \( \text{Gr}_{\mathfrak{g}} \) on \( p^* = \text{Spec}(S(p)) \). The associated variety of \( \mathfrak{g} \) is defined to be \( \text{AV}(\mathfrak{g}) := \text{Supp}(\mathscr{A}) \) in \( p^* \). Its dimension is called the Gelfand-Kirillov dimension of \( \mathfrak{g} \). Let \( N(p^*) := \{ x \in p^* \mid 0 \in K_C \cdot x \} \) be the nilpotent cone in \( p^* \). Alternatively, we may identify \( p^* \simeq p \) using the Killing form and \( N(p^*) \) is defined as the subset of \( p^* \) which corresponds to the set of nilpotent elements \( N(p) \) in \( p \). It is well known that \( \text{AV}(\mathfrak{g}) \) is a closed \( K_C \)-invariant subset of \( N(p^*) \).

Let \( \text{AV}(\mathfrak{g}) = \bigcup_{j=1}^\infty O_j \) such that \( O_j \) are distinct open \( K_C \)-orbits in \( \text{AV}(\mathfrak{g}) \). By Lemma 2.11 in \([V3]\), there is a finite \((S(p), K_C)\)-invariant filtration \( 0 = \mathscr{A}_0 \subset \cdots \subset \mathscr{A}_1 \subset \cdots \subset \mathscr{A}_n = \mathscr{A} \) of \( \mathscr{A} \) such that \( \mathscr{A}_i / \mathscr{A}_{i-1} \) is generically reduced on each \( O_j \). For a closed point \( x_j \in O_j \), let \( i_{x_j} : \{ x_j \} \hookrightarrow O_j \) be the natural inclusion map and let \( K_{x_j} = \text{Stab}_{K_C}(x_j) \) be the stabilizer of \( x_j \) in \( K_C \).

Now

\[
\chi_{x_j} := \bigoplus_i (i_{x_j})^* (\mathscr{A}_i / \mathscr{A}_{i-1})
\]

is a finite dimensional rational representation of \( K_{x_j} \). We call \( \chi_{x_j} \) an isotropy representation of \( \mathfrak{g} \) at \( x_j \). Its image \([\chi_{x_j}]\) in the Grothendieck group of finite dimensional rational \( K_C \)-modules is called the isotropy character of \( \mathfrak{g} \) at \( x_j \). The isotropy representation depends on the filtration. On the other hand the isotropy character is an invariant, i.e. it is independent of the filtration.

We define the multiplicity of \( \mathfrak{g} \) at \( O_j \) to be

\[
m(O_j, \mathfrak{g}) = \dim_{\mathbb{C}} \chi_{x_j}
\]

and the associated cycle of \( \mathfrak{g} \) to be

\[
\text{AC}(\mathfrak{g}) = \sum_{j=1}^r m(O_j, \mathfrak{g})[O_j].
\]

In Section \( \text{[2]} \), we will study the filtrations of local theta lifts generated by the joint harmonics.
1.3. **Local theta correspondence.** Let \((G, G')\) be a real reductive dual pair in \(\mathrm{Sp}(W_\mathbb{R})\). We recall some basic facts of theta correspondences. Let \(\mathfrak{sp}(W_\mathbb{R}) \otimes \mathbb{C}\) denote the complex Lie algebra of \(\mathrm{Sp}(W_\mathbb{R})\) and let \(\mathcal{Y}\) be the Fock model (i.e. \((\mathfrak{sp}(W_\mathbb{R}) \otimes \mathbb{C}, U)\)-module) of the oscillator representation. Let \(\rho'\) be a genuine \((g', \tilde{K}')\)-module. By \([H2]\),

\[
\mathcal{Y} / (\bigcap_{T \in \text{Hom}_{g', \tilde{K}}(\mathcal{Y}, \rho')} \ker T) \simeq \rho' \otimes \Theta(\rho')
\]

where \(\Theta(\rho')\) is a \((g, \tilde{K})\)-module called the full (local) theta lift of \(\rho'\). Theorem 2.1 in \([H2]\) states that if \(\Theta(\rho') \neq 0\), then \(\Theta(\rho')\) is a \((g, \tilde{K})\)-module of finite length with an infinitesimal character and it has an unique irreducible quotient \(\theta(\rho')\) called the (local) theta lift of \(\rho'\). We set \(\theta(\rho') = 0\) if \(\Theta(\rho') = 0\). Let \(\mathcal{R}(g', \tilde{K}'; \mathcal{Y})\) denote the set of irreducible \((g', \tilde{K}')\)-modules such that \(\Theta(\rho') \neq 0\). Then \(\rho' \mapsto \theta(\rho')\) is bijection from \(\mathcal{R}(g', \tilde{K}'; \mathcal{Y})\) to \(\mathcal{R}(g, \tilde{K}; \mathcal{Y})\). Similarly, we could define the theta lifting from \(\mathcal{R}(g, \tilde{K}; \mathcal{Y})\) to \(\mathcal{R}(g', \tilde{K}'; \mathcal{Y})\).

1.4. **Theta lifts of orbits and cycles.** We recall the complexified Cartan decompositions \(g = \mathfrak{t} \oplus \mathfrak{p}\) and \(g' = \mathfrak{t}' \oplus \mathfrak{p}'\). There are two moment maps

\[
p^* \xrightarrow{\psi} W \xrightarrow{\phi} p^*.
\]

We also recall the set of nilpotent elements \(N(p^*)\) in \(p^*\) and the set of nilpotent \(K_C\)-orbits \(\mathcal{R}_{K_C}(p^*)\) in \(p^*\). Similarly we have \(N(p'^*)\) and \(\mathcal{R}_{K'_C}(p'^*)\).

**Definition 1.5.**

(i) For a \(K'_C\)-invariant closed subset \(S'\) of \(p'^*\), we define the theta lift of \(S'\) to be

\[
\theta(S') = \theta(S'; \rho', G) := \phi(\psi^{-1}(S')).
\]

It is a \(K'_C\)-invariant closed subset of \(p^*\). If \(S' \subseteq N(p'^*)\), then \(\theta(S') \subseteq N(p^*)\).

(ii) If \(S' = \bar{O}' \subseteq N(p'^*)\) is the closure of a \(K'_C\)-orbit \(O'\) and \(\theta(S') = \bar{O}\) is the closure of a \(K_C\)-orbit \(O\), then we denote \(O\) by \(\theta(O') = \theta(O'; \rho', G, G)\).

Conversely for a closed \(K_C\)-invariant subset \(S\) of \(N(p^*)\), we define \(\theta(S) = \theta(S; \rho, G, G') := \psi(\tilde{\phi}^{-1}(S))\) which is a closed \(K'_C\)-invariant subset of \(N(p'^*)\). When \(\theta(O) = \bar{O}'\), we define \(\theta(O) = O'\).

(iii) We extend theta lifts of nilpotent orbits to cycles linearly. More precisely, we define \(\theta(\sum_j m_j(\bar{O}_j')) = \sum_j m_j(\theta(\bar{O}_j'))\) if every \(\bar{O}_j\) admits a theta lift.

1.6. From Section 3 onwards, we specialize to the dual pair \((G^{p,q}, G') = (O(p, q), \mathrm{Sp}(2n, \mathbb{R}))\) in \(\mathrm{Sp}(W_\mathbb{R}) = \mathrm{Sp}(2n(p+q), \mathbb{R})\). Choosing a maximal compact subgroup \(U\) as in Section 1.1, we set \(K^{p,q} = G^{p,q} \cap U \cong O(p) \times O(q)\) and \(K^n = G' \cap U \cong U(n)\) to be the maximal compact subgroup of \(G\) and \(G'\) respectively. We also denote \(O(p)\) by \(K^p\).

A Harish-Chandra module of \(\tilde{G}\) is called genuine if it does not descend to a Harish-Chandra module of \(G\). We will introduce some genuine Harish-Chandra modules in this paper.

(a) \(\Theta^{p,q}(\sigma')\) : Let \(\sigma'\) be the genuine one dimensional character of \(\tilde{G}' = \tilde{\mathrm{Sp}}(2n, \mathbb{R})\) such that \(\sigma'|_p\) is trivial. It exists if and only if \(\tilde{G}'\) is a split double cover of \(G'\), i.e. \(p + q\) is even. We say that the dual pair \((G^{p,q}, G')\) is under the stable range if:

\[
\min(p, q) \geq 2n \quad \text{and} \quad \max(p, q) > 2n.
\]

Let \(\theta^{p,q}(\sigma') := \theta(\sigma')\). Then, in stable range, it is a nonzero unitarizable genuine Harish-Chandra module of \(\tilde{G}^{p,q}\) (c.f. \([Lo, ZH]\)).
(b) \( L(\mu') \): Let \( \mu \) be a genuine \( \tilde{G}^{p,0} \)-module. Let \( L(\mu') := \theta(\mu) \) be the \((\mathfrak{g}', \tilde{K}')\)-module, which is the theta lift of \( \mu \). It is well known that \( L(\mu') \) is a unitary lowest module and it is also the full theta lift of \( \mu \). Here \( \mu' \) denotes the lowest \( \tilde{K}' \)-type.

(c) \( \theta^{p,q}(L(\mu')) \): We suppose that \( p+q+t \) is an even integer. Then the double cover \( \tilde{G}' \) for the dual pairs \((G^{p,q}, G')\) and \((G^{p,0}, G')\) are isomorphic. Let \( \theta^{p,q}(L(\mu')) \) be the \((\mathfrak{g}, \tilde{K}'^{p,q})\)-module which is the theta lift of the \( L(\mu') \).

1.7. Before we state our main results, we have to describe the theta lifts of certain orbits. Since all groups appearing here are classical, we will use signed Young diagram to parametrize nilpotent \( K_\mathbb{C} \)-orbits in \( \mathfrak{N}_{K_\mathbb{C}}(\mathfrak{p}^*) \). More precisely, let \( \mathfrak{g}_0 \) denote the real Lie algebra of a classical group \( G \) and let \( \mathfrak{N}_{G}(\mathfrak{g}_0) \) denote the set of nilpotent \( G \)-orbits in \( \mathfrak{g}_0 \). Then \( \mathfrak{N}_{G}(\mathfrak{g}_0) \) is parametrized by signed Young diagrams or signed partitions(c.f. Section 9.3 [CM]). As before we identify \( \mathfrak{p}^* \simeq \mathfrak{p} \) using the Killing form. Then the Kostant-Sekiguchi correspondence identifies \( \mathfrak{N}_{G}(\mathfrak{g}_0) \) with the set of nilpotent \( K_\mathbb{C} \)-orbits \( \mathfrak{N}_{K_\mathbb{C}}(\mathfrak{p}) \simeq \mathfrak{N}_{K_\mathbb{C}}(\mathfrak{p}^*) \), (c.f. Theorem 9.5.1 [CM]).

(a) \( \mathcal{O}_d \): We consider the compact dual pair \( G^{0,d} \times G' \). Let \( 0 \) denote the zero orbit of \( \mathfrak{p} = 0 \). Let \( \mathcal{O}'_d := \theta(0; G^{0,d}, G') \) where \( d = \min \{ t, n \} \). Here \( \mathcal{O}'_d \) only depends on \( d \). Indeed let \( \mathfrak{g}' = \mathfrak{t}' \oplus \mathfrak{p}'^{+} \oplus \mathfrak{p}'^{-} \) denote the complexified Cartan decomposition for the Lie algebra of the Hermitian symmetric group \( G' \). Then \( \mathcal{O}'_d \) is the \( K'_\mathbb{C} \)-orbit in \( \mathfrak{p}'^{-} \) generated by a sum of \( d \) strongly orthogonal non-compact long roots in \( \mathfrak{p}'^{-} \). In terms of partitions, we have

\[
\mathcal{O}'_d = 2^d_+ 1^n_{-} - n_d.
\]

(b) \( \mathcal{O}_{p,q,t} \): Suppose \( 2n \leq \min \{ p, q+t \} \). Let \( d = \min(t, n) \). By [Oh] or [DKP], \( \theta(\mathcal{O}'_d; G', G^{p,q}) \) is the closure of a single \( K_\mathbb{C} \)-orbit \( \mathcal{O}_{p,q,t} \). In terms of partitions,

\[
\mathcal{O}_{p,q,t} = \theta(\mathcal{O}'_d; G', G^{p,q}) = \theta(\theta(0)) = 3^d_+ 2^n_{-} - 2n_d 1^{p-2n}_+ 1^{q+d-2n}_-.
\]

Note that the Young diagram of \( \mathcal{O}_{p,q,t} \) is obtained by adding a column to the left of the Young diagram of \( \mathcal{O}'_d \).

(c) \( \mathcal{O}_{p,q} \). Suppose \( t = 0 \) in the definition of \( \mathcal{O}_{p,q,t} \), we set

\[
\mathcal{O}_{p,q} := \mathcal{O}_{p,q,0} = \theta(0; G', G^{p,q}) = 2^n_+ 2^{q-2n}_- 1^{q-2n}_-.
\]

In [CM], the above orbits are denoted by \( \mathcal{O}_d = 2^d_+ 1^n_{-} - 2d_+ - 2d_+ 1^{p-2n}_+ 1^{q+t+d-2n}_- \).

1.8. Theta lifts of unitary characters. We first recover the following theorem which is known to the experts.

**Theorem A.** Let \((G, G') = (G^{p,q}, G') = (O(p,q), Sp(2n, \mathbb{R}))\) be the dual pair in the stable range defined by (4) and \( p+q \) is even integer. Let \( \sigma' \) be the genuine unitary character of \( G' \). Let \( \mathcal{O}_{p,q} = \theta(0; G', G^{p,q}) = 2^n_+ 2^{q-2n}_- 1^{q-2n}_- \) as in (6).

(i) Then \( AV(\theta(\sigma')) = \underline{\mathcal{O}}_{p,q} \) and \( AC(\theta(\sigma')) = 1 \underline{\mathcal{O}}_{p,q} \).

(ii) Let \( x \in \mathcal{O}_{p,q} \) and let \( K_x = Stab_{K_\mathbb{C}}(x) \) be the stabilizer of \( x \) in \( K_\mathbb{C} \). Then

\[
K_x \simeq \{ (p_1, p_2) \in P_{p,n} \times P_{q,n} \mid \beta_{p,n}(p_1) = \beta_{q,n}(p_2) \}.
\]

Here \( P_{p,n} = (GL(n, \mathbb{C}) \times O(p-2n, \mathbb{C})) \times N \) denote the maximal parabolic subgroup of \( K^p_{\mathbb{C}} \) which stabilizes an \( n \)-dimensional isotropic subspace of \( \mathbb{C}^p \) and \( \beta_{p,n} : P_{p,n} \to GL(n, \mathbb{C}) \) denotes the quotien map.
(iii) Let $\chi_x: K_x \to \mathbb{C}$ denote the character $(p_1, p_2) \mapsto \det(\beta_{p,n}(p_1))^{\frac{p-n}{2}}$. Then the isotropy representation of $\theta(\rho)$ at $x$ is $\tilde{\chi}_x = \mid \tilde{K} \otimes \chi_x$ where $\xi$ is the minimal U-type of the oscillator representation $\mathcal{Y}$ (see Section 2.4). Moreover, we have

$$\theta(\sigma'|_K) = \xi|_K \otimes \text{Ind}_{K_x}^{K_C} \chi_x = \text{Ind}_{K_x}^{K_C} \tilde{\chi}_x.$$  

Equivalently we have

\begin{equation}
\chi_{p,q} = \min\{p+q+2, \max(p+q+2)\} \geq 2n \quad \text{and} \quad \max(p+q+2) > 2n.
\end{equation}

Let $L(\mu') = \theta(\mu)$ denote the unitary lowest module of $\tilde{G}'$ which is the theta lift of $\mu \in R(G'; \mathcal{Y})$ as in Section 1.6. We denote the dual Harish-Chandra module of $L(\mu)$ by $L(\mu)^*$. By [Ya], $\text{AV}(L(\mu')^*) = \overline{O}_d = \theta(0, G'^{t}, G')$ where $d = \min\{t, n\}$.

In order to describe the associated cycles of $\theta^{p,q}(L(\mu'))$, we have to divide into two cases: When $q \geq n$, we will denote this as Case I. When $q < n$, we will denote this as Case II. The geometries of the moment maps in these two cases differ significantly.

1.9. We assume that $(G^{p,q+t}, G')$ is in the stable range and $p + q + t$ is an even integer. Equivalently we have

\begin{equation}
p + q + t \text{ is even, } \min(p, q + t) \geq 2n \quad \text{and} \quad \max(p, q + t) > 2n.
\end{equation}

First we describe the main result for Case I.

**Theorem B.** Suppose that $(G^{p,q+t}, G')$ is in the stable range satisfying (8) and $q \geq n$. Let $L(\mu') = \theta(\mu)$ be a non-zero irreducible unitarizable lowest module of $\tilde{G}'$.

(i) The theta lift $\theta^{p,q}(L(\mu'))$ is nonzero.

(ii) Let $d = \min\{t, n\}$. Then

$$\text{AV}(\theta^{p,q}(L(\mu'))) = \overline{O}_{p,q,t} = \theta(\overline{O}_d) = \theta(\text{AV}(L(\mu')^*))$$

(iii) We fix a closed point $x \in O_{p,q,t}$ and a closed point $x' \in O'_d$. Let $\tilde{\chi}_x$ be the isotropy representation of $L(\mu')^*$ at $x' \in O'_d$ (see Section 4.2). Then there is a map $\beta: K_x \to K_{x'}$ such that the isotropy representation of $\text{Gr}(\theta(\mu'))$ at $x$ is $\tilde{\chi}_x = \xi|_K \otimes (\xi|_{K'} \otimes \tilde{\chi}_{x'}) \circ \beta$.

(iv) We have

\begin{equation}
\text{AC}(\theta(L(\mu'))) = (\dim \tilde{\chi}_x)|\overline{O}_{p,q,t}| = (\dim \tilde{\chi}_{x'})|\theta(\overline{O}_d)| = \theta(\text{AC}(L(\mu')^*)�).
\end{equation}

The proof of the above theorem is given in Section 5.7. Part (iv) extends the main result in [NZ] where $p, q > 2n$ is considered.
1.9.2. Next we describe the main result for Case II.

**Theorem C.** Suppose that $(G^{p,q+t}, G')$ is in the stable range satisfying (S) and $q < n$. Let $L(\mu') = \theta(\mu)$ be an irreducible unitarizable lowest module of $\tilde{G}'$. We fix a closed point $x \in O_{p,q,t}$ and a closed point $x' \in O'_d$ where $d = \min \{ t, n \}$.

(i) Let $L$ be the Levi part of a Levi decomposition of $K_x$, then $L \cong K^q_C \times K^{q-2q}_C$.

(ii) Let $\tilde{x}_x$ be the isotropy representation of $\theta^{p,q}(L(\mu')^\ast)$ at $x \in O_{p,q,t}$. Then

$$\tilde{x}_x|_L = (\theta^{p-2q,t,q}(\sigma'))|_{\tilde{K}_C^{q-2q} \times \tilde{K}} \otimes \mu|_{\tilde{K}_C^{q-2q} \times \tilde{K}}) K^{q-2q}_C \otimes \mathbb{C}[\tilde{K}].$$

(iii) The lift $\theta^{p,q}(L(\mu'))$ is nonzero if and only if $\tilde{x}_x$ is nonzero.

(iv) If $\theta^{p,q}(L(\mu')) \neq 0$, then

$$\text{AV}(\theta^{p,q}(L(\mu'))) = \overline{O_{p,q,t}} = \theta(\overline{O_d}) = \theta(\text{AV}(L(\mu')^\ast))$$

and

$$\text{AC}(\theta^{p,q}(L(\mu'))) = (\dim \tilde{x}_x)[O].$$

The proof of the above theorem is given in Section 5.3. In general dim $\tilde{x}_x$ is not equal to the dimension of the isotropy character $\chi_{x'}$ of $G_r(L(\mu'))$ so (B) is usually invalid for Case II.

1.10. Let $g^{p,q+t} = \mathfrak{k}_H \oplus \mathfrak{p}_H$ denote the complexified Cartan decomposition. The inclusion $g \to g^{p,q+t}$ includes a projection map $\text{pr}_H : \mathfrak{p}_H^* \to \mathfrak{p}^*$. By Proposition 5.14, $\theta^{p,q}(L(\mu'))$ occurs discretely as a submodule in $\theta^{p,q+t}(\sigma')$. A general theory of [K] gives $\text{pr}_H(\text{AV}(\theta^{p,q+t}(\sigma')) \supseteq \text{AV}(\theta^{p,q}(L(\mu')))$. For the representations considered in Theorems [B] and [C] the containment is in fact an equality. See Lemma 5.4.

1.11. In both Cases I and II above, we have the following theorem on the $K$-spectrum of $\theta^{p,q}(L(\mu'))$.

**Theorem D.** Suppose $p,q,t$ satisfies (S) and $\min \{ q, t \} < n$, then

$$\theta^{p,q}(L(\mu'))|_{\tilde{K}} = \text{Ind}_{\tilde{K}_C^{q-2q}}^{\tilde{K}_C} \chi_{x}.$$ 

The proof is given in Section 5.10 and it is a consequence of Proposition 5.11.

Motivated by geometric quantization of orbits, Vogan defined an admissible isotropy representation in Definition 7.13 in [V3]. It is not difficult to see that $\theta^{p,q}(L(\mu'))$ for $q \geq n \geq t$ and $\dim \mu = 1$ has admissible isotropy representation. Then by Theorems A(iii) and D these representations satisfy Vogan’s Conjecture 12.1 in [V3]. Such modules are candidates for the conjectured unipotent $(\mathfrak{g}, \tilde{K})$-modules attached to the orbits $O_{p,q,t}$. In Section 6, we will show that $\theta^{p,q}(L(\mu'))$ is a special unipotent representation in the sense of [BV].

A first draft of this paper was written before [LM]. The current paper is a major revision where we incorporate ideas from [LM], bypass the $K$-types and asymptote calculations, and give more geometric and conceptual proofs to Theorems A to D above.

**Notation.** In this paper, all varieties and schemes are defined over $\mathbb{C}$. We will denote the ring of regular functions on a variety or scheme $X$ by $\mathbb{C}[X]$. For a real Lie group $K$, its complexification is denoted by $K_C$.

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2. Natural filtrations of theta lifts

In this section we will construct good filtrations using local theta correspondences. Unless otherwise stated, all Lie algebras are complex Lie algebras.

We let \((G, G')\) be an arbitrary reductive dual pair in \(\text{Sp}(W_\mathbb{R})\). We do not assume that they are in stable range. Let \(K\) and \(K'\) denote the maximal compact subgroups of \(G\) and \(G'\) respectively. We set \(\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}\) and \(\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'\) to be the complexified Cartan decomposition of the Lie algebras of \(G\) and \(G'\) respectively. A tilde above a group will denote an appropriate double cover which is usually clear from the content.

First we review Section 3.3 in Nishiyama and Zhu \([NZ]\).

We recall that the Fock model of the oscillator representation of \(\widetilde{\text{Sp}}(W_\mathbb{R})\) is realized on the Fock space \(\mathcal{Y} \cong \mathbb{C}[W]\) of complex polynomials on the complex vector space \(W\). We follow Howe’s notation \([H2]\) about diamond dual pairs. Let \((M, K')\) be the dual pair in \(\text{Sp}(W_\mathbb{R}, \mathbb{R})\). Let \(\mathfrak{t}_M \oplus (m^{(2,0)} \oplus m^{(0,2)})\) denote the Cartan decomposition of the complexified Lie algebra of \(M\) such that \(m^{(2,0)}\) acts by multiplication of \(K'\)-invariant quadratic polynomials on \(W\) and \(m^{(0,2)}\) acts by degree two \(K'\)-invariant differential operators.

Fact 3 in Howe’s paper \([H2]\) states that in \(m\), we have

\[
m^{(2,0)} \oplus m^{(0,2)} = \mathfrak{p} \oplus m^{(0,2)}.
\]

The projection of \(\mathfrak{p}\) to \(m^{(2,0)}\) under the decomposition of the left hand side of (10) is a \(K\)-isomorphism. We identify \(\mathfrak{p}\) as \(m^{(2,0)}\) via this projection.

We also have the compact dual pair \((M', K')\) in \(\text{Sp}(W_\mathbb{R})\). In a similar fashion, we have a Cartan decomposition

\[
m' = \mathfrak{t}_{M'} \oplus (m'^{(2,0)} \oplus m'^{(0,2)}) \quad \text{and} \quad \mathfrak{p}' \cong m'^{(2,0)}
\]

where \(\mathfrak{p}'\) is the non-compact part of \(\mathfrak{g}'\).

Let \(\mathcal{Y}_b\) be the subspace of complex polynomials in \(\mathbb{C}[W]\) of degree not greater than \(b\). Then \(\mathcal{Y} = \cup \mathcal{Y}_b\) and gives a filtration of the Fock model \(\mathcal{Y}\). Let \((\rho, V_\rho)\) be the full theta lift of an irreducible \((\mathfrak{g}', \tilde{K}')\)-module \((\rho', V_{\rho'})\). Let \(\pi : \mathcal{Y} \to V_\rho \otimes V_{\rho'}\) be the natural quotient map. Let \((\tau, V_\tau)\) be a lowest degree \(\tilde{K}'\)-type of \((\rho, V_\rho)\) of degree \(j_0\). Let \(V_\tau \otimes V_{\tau'}\) be the image of the joint harmonics. We define filtrations on \(V_\rho\) and \(V_{\rho'}\) by \(V_j = \mathcal{U}_j(\mathfrak{g})V_\tau\) and \(V^\sigma_j := \mathcal{U}_j(\mathfrak{g'})V_{\tau'}\) respectively. The filtration \(V^\sigma_j\) is a good filtration of \(V_{\tau'}\) since \(\mathfrak{p}'\) is irreducible, and \(V_j\) is a good filtration of \(V_\rho\) because \(V_{\rho'} = \mathcal{U}(\mathfrak{g})V_\rho\) due to \([H2]\).

We view \(V_{\rho'^*}\) as \(\text{Hom}_\mathbb{C}(V_{\rho'}, \mathbb{C})_{\tilde{K}'-\text{finite}}\). Let \(V_{\rho'^*} \subset V_{\rho'^*}\) be an irreducible \(\tilde{K}'\)-submodule with type \(\tau'^*\) which pairs perfectly with \(V_\tau\). By Theorem 13 (5) in \([H2]\), the lowest degree \(\tilde{K}'\)-type \(V_\tau\) has multiplicity one in \(\rho'\). Hence \(V_\tau\) and \(V_{\rho'^*}\) are well defined subspaces in \(V_{\rho'}\) and \(V_{\rho'^*}\) respectively. We define a good filtration \(V^\sigma_{\rho'^*} := \mathcal{U}_j(\mathfrak{g'})V_{\rho'^*}\) on \(V_{\rho'^*}\).

Let \(l \in V_{\rho'^*}\) be a nonzero linear functional on \(V_{\rho'}\). We consider the composite map

\[
\nu : \mathcal{Y} \xrightarrow{\pi} V_\rho \otimes V_{\rho'} \xrightarrow{\text{id} \otimes \tilde{l}} V_\rho.
\]

We define \(F_j = \nu(\mathcal{Y}_j)\). Since \(\nu(\mathcal{Y}_{j_0}) = V_\tau\), we also have \(V_j = \mathcal{U}_j(\mathfrak{g})\nu(\mathcal{Y}_{j_0})\).

**Lemma 2.1.** We have \(F_{2j+j_0+1} = F_{2j+j_0}\) and \(F_{2j+j_0} = V_j\).

The proof is given in Appendix A.1. We remark that the first equality of the above lemma was proved in \([NZ]\). The second equality was assumed without proof in that paper.

Lemma 2.1 suggests that \(\{V_j\}\) is a natural choice of filtration. We define \(\text{Gr} V_\rho = \bigoplus_{j=0}^\infty V_j/V_{j-1}\) to be the corresponding graded module of \(\rho\).
2.2. For any \((g', K')\)-module \(V\) and \(s' \subset g'\), we set \(V_{s'} = V/V(s')\) and \(V_{s',K'} = V/V(s',K')\) where \(V(s') = \{ X'v \mid v \in V, X' \in s' \}\) and
\[
V(s', K') = \{ X'v, k'v - v \mid v \in E, X \in g', k \in K' \}.
\]

The next proposition gives another realization of \(\rho = \Theta(\rho')\) which is crucial for us.

**Proposition 2.3.** We have
\[
\Theta(\rho') \cong (V_{\rho'} \otimes \mathcal{Y})_{g',K'} \cong ((V_{\rho'} \otimes \mathcal{Y})_{\rho'})^{K'}.
\]

**Proof.** Since \(K'\) is compact, the last equality follows if we identify \(K'\)-invariant quotient as \(K'\)-invariant subspace.

Now we prove the first identity. Let \(\mathcal{N} = \bigcap_{\psi \in \text{Hom}_{g',K'}(\mathcal{Y},\rho')} \ker \psi\) as in (2). We have
\[
\begin{align*}
\text{Hom}_C((V_{\rho'} \otimes \mathcal{Y})_{g',K'}, \mathbb{C}) &= \text{Hom}_{g',K'}((V_{\rho'} \otimes \mathcal{Y})_{g',K'}, \mathbb{C}) \\
(12) &= \text{Hom}_{g',K'}(\mathcal{Y}) = \text{Hom}_{g',K'}(\mathcal{Y}) \circ \text{Hom}_C(V_{\rho'}, \mathbb{C}) = \text{Hom}_{g',K'}(\mathcal{Y}, V_{\rho'}) \\
(13) &= \text{Hom}_{g',K'}(\mathcal{Y}/\mathcal{N}, V_{\rho'})\end{align*}
\]

The last equality in (12) follows from the fact that \(\mathcal{Y}\) is \(\tilde{K}'\)-finite and \(\text{Hom}_C(V_{\rho'}, \mathbb{C})_{\tilde{K}'-finite} = V_{\rho'}\). Starting from (13), we reverse the steps by replacing \(\mathcal{Y}\) with \(\mathcal{Y}/\mathcal{N}\) in (2) and we get
\[
\text{Hom}_C((V_{\rho'} \otimes \mathcal{Y})_{g',K'}, \mathbb{C}) \cong \text{Hom}_C((V_{\rho'} \otimes \mathcal{Y}/\mathcal{N})_{g',K'}, \mathbb{C}) \cong \text{Hom}_C(V_{\rho}, \mathbb{C})\]

This proves the proposition. \(\square\)

2.4. Let \(E = V_{\rho'} \otimes \mathcal{Y}\). We summarize Proposition 2.3 in the following diagram

\[
E = V_{\rho'} \otimes \mathcal{Y} \xrightarrow{\text{pr}_{g'}} (V_{\rho'} \otimes \mathcal{Y})_{\rho'} \xrightarrow{\text{pr}_{K'}} ((V_{\rho'} \otimes \mathcal{Y})_{\rho'})^{K'} = V_{\rho}.\]

where \(\text{pr}_{g'}\) is the projection map to the \(p'\)-coinvariant quotient space, \(\text{pr}_{K'}\) is the projection map to the \(K'\)-invariant subspace and \(\eta: E \to E_{g',K'} = V_{\rho}\) is the natural quotient map.

We define a filtration on \(E\) by
\[
E_j = \sum_{2a+b=j} V_a^{*s} \otimes \mathcal{Y}_b
\]
and a filtration on \(V_{\rho}\) by \(E_j = \eta(E_{j+2j}) = \eta(E_{j+2j+1})\).

**Lemma 2.5.** The filtrations \(E_j\) and \(V_j = U_j(g)V_{\rho}\) on \((\rho, V_{\rho})\) are the same.

**Proof.** Since \(\nu(\mathcal{Y}_{j+2j}) = V_{j+2j}\), we have \(\eta(V_{\rho^{*s}} \otimes \mathcal{Y}_{j+2j}) = V_j\). Hence \(V_j \subseteq E_j\). On the other hand, for \(a + [b/2] = j\),
\[
\eta(V_a^{*s} \otimes \mathcal{Y}_{j+2j}) = \eta(U_a(g)V_{\rho^{*s}} \otimes \mathcal{Y}_{j+2j}) = \eta(V_{\rho^{*s}} \otimes U_a(g')\mathcal{Y}_{j+2j}) \quad (\text{Since } g' \text{ acts trivially on the image})
\]
\[
\subseteq \eta(V_{\rho^{*s}} \otimes \mathcal{Y}_{j+2j+2a}) \subseteq V_j.
\]

By (14), \(E_j \subseteq V_j\) and this proves the lemma. \(\square\)

Taking the graded module, \(\eta\) induces a map
\[
\text{Gr}\rho^{*s} \otimes_{S(p)} \text{Gr}\mathcal{Y} \rightarrow \text{Gr}(\text{Gr}_{p}(E)) \xrightarrow{\text{Gr}_{p_{K'}}} \text{Gr}\rho.
\]

We will study (15) more thoroughly in \([LM]\).
2.6. We recall that the unitary group $U = U(W)$ is a maximal compact subgroup of $\text{Sp}(W_\mathbb{R})$. Let $\mathfrak{sp}(W_\mathbb{R}) \otimes \mathbb{C} = \mathfrak{u} \oplus \mathfrak{s}$ denote the complexified Cartan decomposition. Let $\varsigma$ be the minimal one dimensional $\mathbb{U}$-type of the Fock model $\mathcal{Y}$. For $k \in U$, $\varsigma^{-2}(k)$ is equal to the determinant of the $k$ action on $W$. We extend $\varsigma$ to an $(S(\mathfrak{s}), \mathbb{U})$-module where $\mathfrak{s}$ acts trivially. We will continue to denote this one dimensional module by $\varsigma$. In this way, $\text{Gr } \mathcal{Y} = \oplus (\mathcal{Y}_{n+1}/\mathcal{Y}_n) \simeq \varsigma \otimes \mathbb{C}[W]$ where $\mathbb{U}$ acts geometrically on $\mathbb{C}[W]$. Since $(G, G')$ is a reductive dual pair in $\text{Sp}(W_\mathbb{R})$, we denote the restriction of $\varsigma$ as a $(S(\mathfrak{p}'), \widetilde{K})$-module by $\varsigma|_{\widetilde{K}}$. Similarly we get a one dimensional $(S(\mathfrak{p}', \widetilde{K}'))$-module $\varsigma|_{\widetilde{K}'}$.

Let $A = \varsigma|_{\widetilde{K}} \otimes \text{Gr } V_{\rho'}$ and $B = \varsigma|_{\widetilde{K}} \otimes \text{Gr } V_{\rho}$. Since $\rho'$ is a genuine Harish-Chandra module of $\widetilde{G}$, $A$ is an $(S(\mathfrak{p}'), K'_C)$-module. Similarly $B$ is an $(S(\mathfrak{p}), K_C)$-module.

We take $\varsigma$ into account and the fact that $K'$ acts on $A \otimes \mathbb{C}[W]$ reductively and preserves the degrees. Then (15) gives the following $(S(\mathfrak{p}), K_C)$-module morphisms

\[
A \otimes_{S(\mathfrak{p}')} \mathbb{C}[W] \xrightarrow{\eta_0} (A \otimes_{S(\mathfrak{p}')} \mathbb{C}[W])^{|\widetilde{K}'|} \xrightarrow{m} B.
\]

The merit of introducing $\varsigma$ is that the $\widetilde{K} \cdot \widetilde{K}'$ action on $\text{Gr } \mathcal{Y}$ descends to a geometric $K_C \times K'_C$ action on $\mathbb{C}[W]$.

**Proposition 2.7.** Suppose $(G, G')$ is in the stable range where $G'$ is the smaller member. Then $\eta_0 : (A \otimes_{S(\mathfrak{p}')} \mathbb{C}[W])^{|\widetilde{K}'|} \xrightarrow{\sim} B$ in (16) is an isomorphism of $(S(\mathfrak{p}), K_C)$-modules.

We refer to [LM] for a proof.

3. Local theta lifts of unitary characters

Throughout this section, we let $(G, G') = (G^{p,q}, G') = (O(p, q), \text{Sp}(2n, \mathbb{R}))$ be a dual pair in stable range (see (1)). We consider the theta lift $\theta(\sigma')$ of the genuine unitary character $\sigma'$ of $\widetilde{G}$. We will discuss the associated cycle and the isotropy representation of $\theta(\sigma')$.

3.1. Let $\sigma'$ be the genuine unitary character of $\widetilde{G}'$. It exists if and only if the double cover $\widetilde{G}'$ splits over $G'$, i.e. $p + q$ is even. First we recall some facts in [Lo] about the local theta lift of $\sigma'$ to $\widetilde{G}$. Also see [KO] when $n = 1$. Let $\mathfrak{g}$ denote the complexified Lie algebra of $G$. Let $K = K^{p,q} = G^{p,0} \times G^{0,q}$ be the maximal compact subgroup of $G$.

**Proposition 3.2.** Suppose $(G^{p,q}, G')$ is in stable range where $G'$ is the smaller member satisfying (4) and $p + q$ is even. Let $\sigma'$ be the genuine character of $G'$. Then $\Theta(\sigma')$ is a nonzero, irreducible and unitarizable $(\mathfrak{g}, \widetilde{K})$-module. In particular $\Theta(\sigma') = \theta(\sigma')$.

**Proof.** The fact that $\Theta(\sigma')$ is irreducible follows from [ZH] or [Lo]. It is also a special case of Theorem A in [LM]. The fact that $\theta(\sigma')$ is unitarizable follows from [Li].

The $\widetilde{K}$-types of $\theta(\sigma') = \Theta(\sigma')$ are well known. For example see [Lo] and [Z]. It is also a special case of Propositions 2.2 and 3.2 in [LM]. We state it as a proposition below.

**Proposition 3.3.** Let $\mathcal{H}$ denote the $\widetilde{K}$-harmonics in the Fock model $\mathcal{Y}$ for the dual pair $(G, G')$ in stable range. Then as $\widetilde{K}$-modules,

\[
\theta(\sigma')|_{\widetilde{K}} = \left( \mathcal{H} \otimes \sigma'' \right)^{|\widetilde{K}'|}.
\]
3.4. We refer to Proposition 2.7. If we set \( \rho' = \sigma' \) to be the genuine character of \( \check{G}' \), then \( A = \zeta|_{\check{K}} \otimes \sigma' \) is a one-dimensional trivial \( S(\mathbf{p}') \)-module. As a representation of \( K' \cong U(n) \), \( A = \det_{\check{K}} \). Let \( B = \zeta|_{\check{K}} \otimes \text{Gr}(\theta(\sigma')) \). We note that in this special case, Proposition 2.7 is a direct consequence of Proposition 3.3.

We recall the moment maps \( \phi: W \rightarrow \mathbf{p}' \) and \( \psi: W \rightarrow \mathbf{p}'^{*} \) in (3). We set \( \mathcal{N} = \psi^{-1}(0') \) and it is called the null cone in \( W \). Let \( \mathcal{O} = \mathcal{O}_{p,q} = \theta(0'; G', G) \) be the \( K_{C} \)-orbit in (5). Then \( \mathcal{N} = \phi^{-1}(\mathcal{O}) \) is an open \( K_{C} \)-orbit in \( \mathcal{N} \) which we call the open null cone. Furthermore, \( \psi^{*}(\mathbf{p}^{*})C[W] \) is precisely the radical ideal \( I(\mathcal{N}) \) of \( C[W] \) (see [GW, H3]). Let \( I(\overline{\mathcal{O}}) \) be the radical ideal of \( \overline{\mathcal{O}} \) in \( S(\mathbf{p}) \). We state a corollary of Proposition 2.7.

**Corollary 3.5.** We have an \( (S(\mathbf{p}), K_{C}) \)-module isomorphism

\[
(C[\mathcal{N}] \otimes A)^{K_{C}} \cong B.
\]

Furthermore \( I(\overline{\mathcal{O}}) \subset \text{Ann}_{S(\mathbf{p})}B \) and \( B \) is a \( (C[\overline{\mathcal{O}}], K_{C}) \)-module.

**Proof.** If we regard \( A \) as the trivial \( S(\mathbf{p}') \)-module, then \( C[\mathcal{N}] = C[W]/I(\mathcal{N}) = C[W] \otimes_{S(\mathbf{p}')} A \). The identity in the corollary follows from Proposition 2.7. We have \( \phi^{*}(I(\overline{\mathcal{O}})) \subset I(\mathcal{N}) \) so \( I(\overline{\mathcal{O}}) \subset \text{Ann}_{S(\mathbf{p})}B \). \( \square \)

**Proposition 3.6.** The natural inclusion \( C[\mathcal{N}] \rightarrow C[\mathcal{N}] \) induces an isomorphism of \( (S(\mathbf{p}), K_{C}) \)-modules

\[
B \cong (C[\mathcal{N}] \otimes A)^{K_{C}}.
\]

**Proof.** It suffices to prove that \( (C[\mathcal{N}] \otimes A)^{K_{C}} \cong (C[\mathcal{N}] \otimes A)^{K_{C}} \) as admissible \( K_{C} \)-modules. This is verified in [Yn]. \( \square \)

We remark that if \( \min(p, q) > 2n \), then the above lemma follows immediately from the fact that \( C[\mathcal{N}] \cong C[\mathcal{N}] \). Indeed, in these cases, \( \mathcal{N} \) is a normal variety by [Ko, NOZ] and \( \partial \mathcal{N} = \mathcal{N} - \mathcal{N} \) has codimension at least 2 in \( \mathcal{N} \).

3.7. Let \( \mathcal{B} \) be the coherent sheaf associated to the module \( B \) on \( \mathbf{p}' \). Clearly \( AV(\theta(\sigma')) = \text{Supp} \mathcal{B} \subset \phi(\mathcal{N}) = \overline{\mathcal{O}} \). In order to calculate the isotropy representation of \( \text{Gr}(\theta(\sigma')) \), we first recall a special case of Corollary A.5 in [LM].

**Lemma 3.8.** Fix a \( w \in \mathcal{N} \) such that \( \phi(w) = x \in \mathcal{O} \). Let \( K_{x} \) be the stabilizer of \( x \) in \( K_{C} \). Then the fiber \( F_{x} = \phi^{-1}(x) \cap \mathcal{N} \) in \( \mathcal{N} \) is a single \( K_{C}' \)-orbit where \( K_{C}' \) acts freely. Moreover, there is a (unique) surjective homomorphism \( \alpha: K_{x} \rightarrow K_{C}' \) such that

\[
S_{w} = K_{x} \times_{\alpha} K_{C}' = \{ (k, \alpha(x)) \in K_{x} \times K_{C}' \}.
\]

For \( k \in K_{x} \), \( \alpha(k) \) is the unique element in \( K_{C}' \) such that \( (k, \alpha(k)) \cdot w = w \). The above definitions are summarized in the diagram (17) below.

\[
(\mathbf{K}_{x} \times K_{C}')/S_{w} \xrightarrow{\phi_{w}} \mathcal{N}^{c} \xrightarrow{\phi} \mathcal{N} \xrightarrow{\phi} W \xrightarrow{\phi} \mathcal{O} \xrightarrow{\iota_{\mathcal{O}}} \mathcal{O} \xrightarrow{\mathcal{O}^{*}} \mathbf{p}^{*}
\]

(17)
4.1. The group $G'$ is Hermitian symmetric so its has complexified Cartan decomposition $g' = t' \oplus p'$ and $p' = p'^+ \oplus p'^-$ where $p'^\pm$ are $K'$-invariant abelian Lie subalgebras of $g'$. Let $K'^L = G^{t,0} \cong O(t)$. Let $\mathcal{Y}_2$ be the Fock model of the oscillator representation for compact dual pair $(K'^L, G')$. Let $\mu \in \mathcal{R}(K'^L; \mathcal{Y}_2)$. We define
\[
L(\mu^t) = \theta(\mu^t) = (\mathcal{Y}_2 \otimes \mu^*)^{K'^L}
\]
and $L(\mu'^* \mu^t) = (\mathcal{Y}_2^* \otimes \mu^*)^{K'^L}$. Here $L(\mu^t)$ is a lowest weight module with lowest $K'^L$-type $\mu^t$, and $L(\mu'^* \mu^t)$ is its contragredient module.

It is a well known result of [EHW] and [DER] that all unitarizable lowest weight modules of $G'$ up to unitary characters are obtained from compact dual pair correspondences.

Let $W_2$ be the complex space with Hermitian form compatible with $(G^{t,0}, G')$. Its Fock model is $\mathbb{C}[W_2] \cong \mathcal{Y}_2^*$. The $p'^-$ action on $\mathcal{Y}_2^*$ is by multiplying degree two $K'^L$-invariant
polynomials. It gives an algebra homomorphism \( \psi^*_2 : S(p^-) \to \mathbb{C}[W_2]^K \) which in turn defines the moment map
\[
\psi_2 : W_2 \to (p^-)^* = \text{Spec} S(p^-).
\]

4.2. Associated cycles. In Section 2, we have a filtration on \( L(\mu')^* \) which gives a graded module \( \text{Gr}(L(\mu')^*) \). Despite the fact that the dual pair is not in stable range, it is well known that Proposition 2.7 extends to the graded module (for example see [KV] and [Ya]) and we have
\[
B' := \mathfrak{g}_2 |_{\widetilde{K}} \otimes \text{Gr}(L(\mu')^*) = \mathfrak{g}_2 |_{\widetilde{K}} \otimes (\text{Gr} \mathcal{Y}_2^* \otimes \mu)^{K^t} = (\mathbb{C}[W_2] \otimes \tau)^{K^t}
\]
where \( \mathfrak{g}_2 \) is the minimal \( \widetilde{U}(W_2) \)-type of \( \mathcal{Y}_2^* \) and
\[
(18) \quad \tau = \mathfrak{g}_2 |_{\widetilde{K}} \otimes \mu
\]

We recall \( \mathcal{O} = \mathcal{O}_d = \psi_2(W_2) \) where \( d = \min \{ t, n \} \). Hence the graded module \( B' \) is a finitely generated \( \mathbb{C}[\mathcal{O}] \)-module.

Let \( x' \in \mathcal{O}' \). We consider following diagram
\[
\begin{array}{ccc}
\psi_2^{-1}(x') & \xrightarrow{i_{x'}} & \mathcal{O}' \\
\downarrow & & \downarrow \psi_2 \\
\{ x' \} & \to & (p^-)^*.
\end{array}
\]

Let \( \mathcal{B}' \) be the coherent sheaf on \( \mathcal{O}' \) associated with the module \( B' \). Let \( K'_{x'} \) be the stabilizer of \( x' \) in \( K'_{\mathcal{O}'} \). Then the isotropy representation of \( \mathcal{B}' \) and \( \text{Gr}(L(\mu')^*) \) at \( x' \) are
\[
(19) \quad \chi_{x'} = i_{x'}^* \mathcal{B}' = (\mathbb{C}[\psi_2^{-1}(x')] \otimes \tau)^{K^t} \quad \text{and} \quad \tilde{\chi}_{x'} = \mathfrak{g}_2 |_{\widetilde{K}} \otimes \chi_{x'}
\]
respectively. The representation \( \chi_{x'} = i_{x'}^* \mathcal{B}' \) is calculated in [Ya]. We state the result for pair \( (K', G') = (O(t), \text{Sp}(2n, \mathbb{R})) \).

**Theorem 4.3 ([Ya]).** (i) The module \( L(\mu') = \theta(\mu) \) is nonzero if and only if \( \chi_{x'} \) is nonzero.

(ii) Suppose \( t \leq n \). Then \( K'_{x'} \cong (O(t, \mathbb{C}) \times \text{GL}(n-t, \mathbb{C})) \ltimes N \) where \( N \) is a nilpotent subgroup. The isotropy representation is \( \chi_{x'} \cong \tau \) as an \( O(t, \mathbb{C}) \)-module and the other subgroups of \( K'_{x'} \) act trivially on it.

(iii) Suppose \( t > n \), then we have \( K'_{x'} \cong O(n, \mathbb{C}) \). The isotropy representation is \( \chi_{x'} \cong \tau^{O(t-n)} \) as an \( O(n, \mathbb{C}) \)-module.

The next theorem follows immediately from the definitions in Section 1.2 and the above theorem on the isotropy representations.

**Theorem 4.4 ([NOT] [Ya]).** We have \( \text{AV}(L(\mu')^*) = \mathcal{O}_d \) and
\[
\text{AC}(L(\mu')^*) = (\dim \mathbb{C} i_{x'}^* \mathcal{B}')[\mathcal{O}_d] = \begin{cases} 
(\dim \mathbb{C} \mu)[\mathcal{O}_t] & \text{if } t \leq n \\
(\dim \mathbb{C} (\mathfrak{g}_2 |_{\widetilde{K}} \otimes \mu)^{O(t-n)})[\mathcal{O}_n] & \text{if } t > n \quad \square
\end{cases}
\]

4.5. Theta lift of \( L(\mu') \). We consider the dual pairs \( (G, G') = (G^{p,q}, G') \) and \( (K^t, G') = (G^{t,0}, G') \). Let \( (H, G') = (G^{p,q+t}, G') \). For reasons which will be clear later (see Proposition 4.3 (ii)), we assume [N]. Then \( H \) contains \( G \times K^t = G^{p,q} \times G^{0,t} \). We note that \( (H, G') \) is in the stable range but \( (G, G') \) could be outside the stable range. Let \( K_H = K^{p,q+t} \) denote a maximal compact subgroup of \( H \) compatible with \( G \) and \( K^t \). Let \( \mathcal{B}_H \) (resp. \( \mathcal{Y}, \mathcal{Y}_2 \)) be Fock model of the oscillator representation associated to the dual pair \( (H, G'_0) \)
(resp. $(G, G_1^1), (K^t, G_2^t)$) where $G_0' = G_1' = G_2 = G'$. Then as an infinitesimal module of 
$(\tilde{G} \times \tilde{G}_1^1) \times (\tilde{K}^t \times \tilde{G}_2^t)$, 
\begin{equation}
\mathcal{Y}_H \cong \mathcal{Y} \otimes \mathcal{Y}_2^*.
\end{equation}
We note that $\tilde{G}_1'$ is a split double cover of $G'$. On the other hand $\tilde{G}_1' \cong \tilde{G}_2'$ and it is a 
split double cover of $G'$ if and only if $t$ is even. Without fear of confusion, we will denote 
all the three $G_i$'s by $\tilde{G}^t$. 

Let $\theta^{p,q+t} : \mathcal{R}(\tilde{G}^t; \mathcal{Y}_H) \to \mathcal{R}(\tilde{G}_1'; \mathcal{Y}_2)$ and $\theta^{p,q} : \mathcal{R}(\tilde{G}^t; \mathcal{Y}) \to \mathcal{R}(\tilde{G}_1'; \mathcal{Y})$ be the theta 
lifting maps. For $\mu \in \mathcal{R}(\tilde{K}^t, \mathcal{Y}_2)$, we set $L(\mu') = \theta(\mu)$ to be the unitary lowest module as 
defined in Section 4.1. Let $\mathfrak{g}^{p,q}$ denote the complex Lie algebra of $G^{p,q}$.

**Proposition 4.6.** (i) For $\mu \in \mathcal{R}(\tilde{K}^t, \mathcal{Y}_2)$, we have 
\begin{equation}
\Theta^{p,q}(L(\mu')) \cong (\Theta^{p,q+t}(\sigma') \otimes \mu)^{K^t}
\end{equation}
as (possibly zero) $(\mathfrak{g}^{p,q}, \tilde{K}^{p,q})$-modules. 
(ii) Suppose $p, q, t$ satisfies \((\star)\) so that $\Theta^{p,q+t}(\sigma')$ is a nonzero and unitarizable 
$(\mathfrak{g}^{p,q+t}, \tilde{K}^{p,q+t})$-module by Proposition 3.2. If $\Theta^{p,q}(L(\mu'))$ is nonzero, then it is a unitarizable 
and irreducible $(\mathfrak{g}^{p,q}, \tilde{K}^{p,q})$-module. In particular $\Theta^{p,q}(L(\mu')) = \Theta^{p,q}(\sigma')$.

**Proof.** Part (i) is proved in \[Lo\] as a consequence of a see-saw pair argument. In (ii), 
$\Theta(L(\mu'))$ is unitarizable because it is a submodule of the unitarizable module $\Theta(\sigma') \otimes \mu$. 
In particular, $\Theta(L(\mu'))$ is a direct sum of its irreducible submodules. On the other hand 
$\Theta(L(\mu'))$ is a full theta lift so it has a unique irreducible quotient module. This proves 
that $\Theta(L(\mu'))$ is irreducible. \hfill $\square$

4.7. **Outside stable range.** In Propositions 3.3 and 4.6, we have assumed \(\boxed{11}\) and \(\boxed{8}\). 
One could easily extend the definitions of $\Theta(\sigma')$ and $\Theta(L(\mu'))$ beyond these assumptions. 
Equation (21) continues to hold. However both $\Theta(\sigma')$ and $\Theta(L(\mu'))$ are not necessarily 
nonzero or irreducible. We will briefly discuss below. We will denote $\theta^{p,q+t}(\sigma')$ by $\theta_n^{p,q+t}(\sigma')$ 
and $\theta^{p,q}(L(\mu'))$ by $\theta_n^{p,q}(L(\mu'))$. We refer the reader to \[Lo\] and \[LMT\] for more details.

Outside the stable range, $\theta_n^{p,q}(\sigma')$ is nonzero if and only if one of the following situation holds:

(A) We have $p = q + t \leq 2n$. If $n \leq p - 1$, then $\theta_n^{p,q}(\sigma') = \theta_{p-1-n}^{p,q}(\sigma')$ which is in the stable 
range. By \[21\], if $\theta_{p-1-n}^{p,q}(L(\mu')) \neq 0$, then $\theta_n^{p,q}(L(\mu')) = \theta_{p-1-n}^{p,q}(L(\mu'))$. If $n \geq p$, then 
$\theta_n^{p,q}(\sigma')$ is finite dimensional and its associated variety is the zero orbit.

(B) We have $n \leq p \leq 2n - 1$ and $q + t = p + 2$. Let $\delta$ denote the one dimensional character 
of $O(p, p + 2)$ which is $\det_{O(p)}$ on $O(p)$ and trivial on $O(p + 2)$. Then $\delta \theta_n^{p+2,q}(\sigma') = 
\theta_{p-2}^{p,q}(\sigma')$ and we are back to the stable range. By \[21\], $\delta \theta_n^{p,q}(L(\mu')) = \theta_{p-2}^{p,q}(L(\mu'))$.

Finally in Case (A), it is possible that $\theta_n^{p,q}(L(\mu')) \neq 0$ but $\theta_{p-1-n}^{p,q}(L(\mu')) = 0$. Most of 
these lifts $\theta_n^{p,q}(L(\mu'))$'s are non-unitarizable. This situation arises because the maximal 
Howe quotient $\Theta(\sigma')$ is reducible. It is possible to analyze of the $K$-types of $\Theta(\sigma')$ as in \[Lo\] 
and compute its associated cycles. This is tedious so we will omit this case.

5. **Associated cycles of theta lifts of unitary lowest weight modules**

5.1. In this section, we assume the notation in Section 4.1: $G = Sp(2n, \mathbb{R})$, $H = G^{p,q+t}$ 
contains $G \times K^t = G^{p,q} \times G^{0,t}$ and $p, q, t$ satisfies \(\check{\star}\). We pick a $\mu \in \mathcal{R}(\tilde{K}^t, \mathcal{Y}_2)$ and we 
let $L(\mu') = \theta(\mu)$ be the lowest weight $(g', \tilde{K}^t)$-module. By Proposition 4.6, $\theta^{p,q}(L(\mu'))$ is 
the full theta lift.
In Lemma \ref{lemma:graded_module} we define the grade module $\text{Gr} \theta^{p,q}(L(\mu'))$ via a natural filtration $V_j = \mathcal{U}_j(g)V_0$ on $\theta^{p,q}(L(\sigma'))$ where $V_0$ is the lowest degree $K$-type. Similarly we define the graded module $\text{Gr} \theta^{p,q+t}(\sigma')$ via a natural filtration $E_j = \mathcal{U}_j(h)E_0$ on $\theta^{p,q+t}(\sigma')$ where $E_0$ is the lowest degree $K_H$-type. The following lemma is a commutative version of Proposition \ref{proposition:moment_maps}.

**Lemma 5.2.** As $(\mathcal{S}(p), \tilde{K}^{p,q})$-modules,

$$\text{Gr} \theta^{p,q}(L(\mu')) \cong (\text{Gr} \theta^{p,q+t}(\sigma') \otimes \mu)^{K'}.$$  

The proof is given in Appendix \ref{appendix:A.2}.  

### 5.3. The moment maps

We will describe some moment maps. These maps are given explicitly in terms of complex matrices in Appendix \ref{appendix:B}.

With reference to \ref{definition:moment_map}, we denote the following moment maps with respect to the dual pairs:

$$
\begin{array}{c|c|c|c}
(G, G') & (G^{p,0}, G') & (G^{0,q}, G') \\
\psi: W \to p^* & \psi^+: W^+ \to (p^+)^* & \psi^-: W^- \to (p^-)^* \\
\end{array}
$$

We have the decomposition $W = W^+ \oplus W^-$, $p' = p^+ \oplus p^-$ and $\psi = \psi^+ \oplus \psi^-$. The containment $H = G^{p,q+t} \supset G \times K'$ gives a decomposition $W_H = W \oplus W_2 = (W^+ \oplus W^-) \oplus W_2$. If we replace $G$ by $H$ in the above table, then we have the moment maps $\psi_H = \psi^+ \oplus \psi_H^-$ where $\psi_H^-$: $W_H^- = W^- \oplus W_2 \to (p^-)^*$ is the moment map for pair $(G^{0,q+t}, G')$. With respect to dual pair $(K', G')$, we have

$$\psi_2: W_2 \to (p^-)^* = 0 \oplus (p^-)^* \subset p^*.$$

Finally we get $\psi_H^- = \psi^- \oplus \psi_2: W_H^- = W^- \oplus W_2 \to (p^-)^*$.

On the other side of \ref{definition:moment_map}, we have $\phi_H: W_H \to p_H^*$ and $\phi: W \to p^*$. Let $\text{pr}: W \oplus W_2 \to W$ be the natural projection and $\text{pr}_H: p_H^* \to p^*$ be the projection induced from $p \to p_H$. They form a commutative diagram:

$$
\begin{array}{ccc}
W_H = W \oplus W_2 & \xrightarrow{\text{pr}} & W \\
\phi_H \downarrow & & \phi \downarrow \\
p_H^* & \xrightarrow{\text{pr}_H} & p^*. \\
\end{array}
$$

For the rest of this paper, we refer to the orbits in Section \ref{section:orbits} and we set (i) $\mathcal{O}_H = \mathcal{O}_{p,q,t} = \theta(0, G', H)$ in $p_H^*$, (ii) $\mathcal{O}' = \mathcal{O}'_d$ in $(p^-)^*$, (iii) $\mathcal{O} = \mathcal{O}_{p,q,t} = \theta(\mathcal{O}', G', G)$ in $p^*$ and (iv) $\mathcal{N} = \mathcal{N}^+ \times \mathcal{N}^- = (\psi^+)^{-1}(0) \times (\psi_H^-)^{-1}(0)$ to be the null cone corresponding to the dual pair $(H, G')$.

**Lemma 5.4.** We have $\text{pr}_H(\overline{\mathcal{O}_H}) = \theta(\overline{\mathcal{O}}; G', G)$ which is the Zariski closure of the $K_C$-orbit $\mathcal{O}$.

**Proof.** We have $\overline{\mathcal{O}_H} = \theta(0; G', H) = \phi_H(\mathcal{N})$. By \ref{proposition:projection}, $\text{pr}_H(\overline{\mathcal{O}_H}) = \text{pr}_H \circ \phi_H(\mathcal{N}) = \phi \circ \text{pr}(\mathcal{N})$ so it suffices to show that $\text{pr}(\mathcal{N}) = \psi^{-1}(\overline{\mathcal{O}})$.

Let $(w, w_2) \in W^- \oplus W_2$. Then $(w, w_2) \in \mathcal{N}^-$ if and only if $\psi^-(w) + \psi_2(w_2) = 0$. Hence $(w^+, w) \in \text{pr}(\mathcal{N})$ if and only if $w \in (\psi^+)^{-1}(\psi_2(W_2))$ and $w^+ \in \mathcal{N}^+$. Since $\overline{\mathcal{O}'} = \psi_2(W_2)$, we have

$$\text{pr}(\mathcal{N}) = \psi^+(0) \oplus (\psi^-)^{-1}(\psi_2(W_2)) = \psi^{-1}(\psi_2(W_2)) = \psi^{-1}(\overline{\mathcal{O}})$$

as required. This proves the lemma. \qed
5.5. Let $\zeta_H$ denote the lowest $\tilde{U}(W_H)$-type of $\mathcal{Y}_H$. We recall $B_H = \zeta_H|_{\tilde{K}_H} \otimes \text{Gr}(\theta^{p,q+1}(\sigma'))$. Let $B = \zeta_H|_{\tilde{K}} \otimes \text{Gr}(\theta^{p,q}(L(\mu')))$. We set $\tau := \zeta_H|_{\tilde{K}} \otimes \mu$ as in \cite{[18]}. Then by Lemma 5.2, we have
\begin{equation}
B = (B_H \otimes \tau)^{K^t}.
\end{equation}
Since $B_H$ is a $(\mathbb{C}[\mathcal{O}_H], K_H)$-module, applying Lemma 5.4 to (23) shows that $B$ is a $(\mathbb{C}[\mathcal{O}], K_C)$-module. Let $\mathcal{B}$ be the quasi-coherent sheaf on $\mathcal{O}$ associated with $B$. In particular, $\text{Supp}(\mathcal{B}) \subseteq \mathcal{O}$.

Fix $x \in \mathcal{O}$ and let $i_x : \{ x \} \to \mathcal{O}$ be the inclusion map. Let $\mathcal{I}_x$ be the maximal ideal in $\mathcal{S}(\mathfrak{p})$ defining $x$. Let $q_H := \text{pr}_H \circ \phi_H : W_H \to p^*$ and let $Z = q_H^{-1}(x) \cap \mathcal{N}$ be the set theoretic fiber.

**Lemma 5.6.** We have $C[Z] = C[\mathcal{N}] / \mathcal{I}_x C[\mathcal{N}] = C[x] \otimes_{\mathbb{C}[\mathcal{O}]} C[\mathcal{N}]$.

**Proof.** Let $\mathcal{N}_x$ be the scheme theoretic fiber of $x$. We claim that $\mathcal{N}_x$ is reduced and thus equals to $Z$. Indeed in characteristic zero, a generic scheme theoretic fiber is reduced. Since $x$ generates the dense open orbit $\mathcal{O}$ in $\mathcal{O}$, $\mathcal{N}_x$ is reduced and the claim follows. Taking regular functions of $Z = \mathcal{N}_x$ gives the lemma. $\square$

By Proposition 3.6, $B_H = (\mathbb{C}[\mathcal{N}] \otimes A)^{K^t}$ where $A = \zeta_H|_{\tilde{K}} \otimes \sigma^*$. Since $\mathcal{S}(\mathfrak{p})$ is $K'_C \times K^t_{\mathfrak{p}}$-invariant, (23) and Lemma 5.6 give
\begin{equation}
i_x^* \mathcal{B} = B / \mathcal{I}_x B = (B_H / \mathcal{I}_x B_H \otimes \tau)^{K^t} = ((\mathbb{C}[\mathcal{N}] / \mathcal{I}_x C[\mathcal{N}] \otimes A)^{K^t} \otimes \tau)^{K^t}
=(C[Z] \otimes A \otimes \tau)^{K^t_{\mathfrak{p}} \times K^t_{\mathfrak{p}}}.
\end{equation}

In Appendix B, we see that $\psi^* : W^- \to (p')^*$ is surjective if and only if $q \geq n$. From now on we split out calculations into two cases, depending on whether $\psi^*$ is surjective or not surjective.

5.7. **Case I.** We assume that $\psi^* : W^- \to (p')^*$ is surjective, i.e. $q \geq n$.

Let $Y = \text{pr}(Z) = \phi^{-1}(x) \cap \mathcal{N}$. Fix $y = (y^+, y^-) \in Y \subset W^+ \oplus W^-$. Let $Z_y = \text{pr}^{-1}(y) \cap Z$. Let $x' = -\psi^*(y^-) \in (p')^*$ and let $Z_{x'} = (\psi_2)^{-1}(x')$ in $W_2$. We consider the diagram:
\begin{equation}
\begin{array}{ccc}
Z_{x'} & \xrightarrow{T} & Z \\
\downarrow{\text{pr}} & & \downarrow{\text{pr}} \\
\{ y \} & \xrightarrow{i_y} & Y.
\end{array}
\end{equation}
The map $T$ will be given in Lemma 5.8 (iii) below. Let $S_y = \text{Stab}_{K_C \times K'_{\mathfrak{p}}}(y)$ and $K_{x'} = \text{Stab}_{K^t_{x'}}(x')$. We now state our key geometric Lemma 5.8. Its proof is given in Appendix B.3.

**Lemma 5.8.** Suppose $\psi^* : W^- \to (p')^*$ is a surjection.

(i) Then $Y$ is a single $K_x \times K'_{x'}$-orbit generated by $y$ so $Y \simeq (K_x \times K'_{x'}) / S_y$.

(ii) There is group homomorphism $\beta : K_x \to K'_{x'}$ such that
$$S_y = \{ (k, \beta(k)) \mid k \in K_x \}.$$ We denote the right hand side by $K_x \times_{\beta} K'_{x'}$. 

(iii) There is a bijection $T : Z_y \to Z_{x'}$ such that $T$ commutes with the actions of $K_C'$ and
\begin{equation}
T((k, \beta(k))z) = \beta(k)z
\end{equation}
for all $z \in Z_y$ and $(k, \beta(k)) \in S_y = K_x \times_\beta K_{x'}$.

**Proof of Theorem B.** Let $\mathcal{O}_Z$ denote the structure sheaf of $Z$. Clearly $\mathbb{C}[Z] = ((\text{pr}_Z)_*, \mathcal{O}_Z)(Y)$. By Lemma 5.8(i) above, $Y$ is a single $K_x \times K_C'$-orbit. Let $\mathcal{I}_y$ be the ideal of $y$ in $\mathbb{C}[Y]$. We recall that $Y$ is affine. Again by the generic reduceness of scheme theoretical fiber in characteristic 0, $\mathbb{C}[Z_y] = \mathbb{C}[Z]/\mathcal{I}_y \mathbb{C}[Z]$. Therefore by [CPS],
\begin{equation}
\mathbb{C}[Z] = \text{Ind}_{S_y}^{K_x \times K_C'}((\text{pr}_Z)_* \mathcal{O}_Z) = \text{Ind}_{S_y}^{K_x \times K_C'}(\mathbb{C}[Z]/\mathcal{I}_y \mathbb{C}[Z]) = \text{Ind}_{S_y}^{K_x \times K_C'} \mathbb{C}[Z_y].
\end{equation}
Putting (27) into (24), we have
\begin{align*}
(i_x^* \mathcal{B}) &= (\mathbb{C}[Z] \otimes A \otimes \tau)^{K_C' \times K_C'} = \left(\text{Ind}_{S_y}^{K_x \times K_C'}(\mathbb{C}[Z_y] \otimes A \otimes \tau)\right)^{K_C' \times K_C'} \\
&= \left(\text{Ind}_{S_y}^{K_x \times K_C'}(\mathbb{C}[Z_y] \otimes \tau)\right)^{K_C'} \otimes A.
\end{align*}
By Lemma 5.8(iii), $T$ induces an isomorphism $(\mathbb{C}[Z_y] \otimes \tau)^{K_C'} = (\mathbb{C}[Z_{x'}] \otimes \tau)^{K_C'}$ of $S_y$-modules through $\beta$.

By (19), $(\mathbb{C}[Z_{x'}] \otimes \tau)^{K_C'} \simeq \chi_{x'}$. Moreover, $A = \varsigma_{\mathfrak{H}}|_{\mathfrak{K}_x} \otimes \sigma^x \simeq \varsigma_{\mathfrak{K}_x} \otimes \varsigma_{\mathfrak{K}_{x'}}$. Hence
\begin{equation}
\chi_x = i_x^* \mathcal{B} = \left(\text{Ind}_{S_y}^{K_x \times K_C'}(\chi_{x'} \otimes A)\right)^{K_C'} \simeq (\chi_{x'} \otimes A) \circ \beta = (\tilde{x}_{x'} \otimes \varsigma_{\mathfrak{K}_{x'}}) \circ \beta
\end{equation}
as a representation of $K_x$. The isotropy representation of $\text{Gr} \theta(L(\mu'))$ at $x$ is
\[\tilde{x}_x = \varsigma_{\mathfrak{K}_x} \otimes i_x^* \mathcal{B} = \varsigma_{\mathfrak{K}_x} \otimes (\tilde{x}_{x'} \otimes \varsigma_{\mathfrak{K}_{x'}}) \circ \beta.\]
This proves (iii).

Suppose that $L(\mu') \neq 0$. By Theorem 4.3, $\tilde{x}_{x'} \neq 0$ so $\tilde{x}_x \neq 0$. This implies that $\theta^{p,q}(L(\mu')) \neq 0$ and proves (i). We have seen before that $\text{Supp} \mathcal{B} \subseteq \overline{\mathcal{O}}$. Since $\chi_x \neq 0$, $x \in \text{Supp} \mathcal{B} \subseteq \overline{\mathcal{O}}$ and $K_C \cdot x = \mathcal{O}$. Hence $AV(\theta^{p,q}(L(\mu')) = \overline{\mathcal{O}}$. This proves (ii). Part (iv) is an immediate consequence of (ii) and (iii). This completes the proof of Theorem B. $\square$

5.9. **Case II.** We assume that $\psi^- : W^- \to (p^-)^*$ is not surjective, i.e. $t > n > q$. Then $K_n$ has a Levi decomposition $K_n = L \times N$ (see p. 184 in [Hul]) where $L$ is the Levi part and $N$ is the unipotent part. By the calculation in Appendix B.1.2, we have $L \cong \triangle \mathbb{C}^q \times K_{C}^{p-2q}$.

We would like to mimic (25) in Case I which constructs an orbit $Y$. More precisely we will construct in Appendix B.1.2 an (affine) algebraic set $\mathcal{M}$ and a surjective algebraic morphism $\pi : Z \to \mathcal{M}$ with the following properties:

(A) Let $Q = L \times K_C^t \times K_C \cong \triangle K_C^q \times K_{C}^{p-2q} \times K_C^t \times K_C$. There is a $Q$-action on $\mathcal{M}$ such that $\pi$ is $Q$-equivariant.

(B) The set $\mathcal{M}$ is an $Q$-orbit. Fix $m \in \mathcal{M}$. We form the following set theoretical fiber:
\[
\begin{array}{ccc}
\mathcal{N}_m & \xrightarrow{T} & Z_m \\
\downarrow & & \downarrow \pi \\
\{m\} & \subseteq & \mathcal{M}
\end{array}
\]

By the same argument as in the proof of Lemma 5.6, $Z_m$ is equal to the scheme theoretical fiber $Z \times_{\mathcal{M}} \{m\}$ because the latter is reduced.
(C) Let $Q_m$ be the stabilizer of $m$ in $Q$. We set $K_s = K^{p-2q} \times K^{t-q}$ and $K'_s \equiv U(n-q)$. Then
\[
Q_m = \triangle' K^q_s \times K_C^{p-2q} \times K_C^{t-q} \times K_C^{m-q} \\
\cong \triangle' K^q_s \times K_s \times K'_s \cong L \times \gamma K_s \times K'_s
\]
Here $\triangle' K^q_s$ embeds diagonally into $\triangle K^q_s \times K'_s \times K^t_s$, $K_C^{L-q}$ lies in $K'_s$ and
\[\gamma' : L = \triangle K^q_s \times K_C^{p-2q} \longrightarrow K_C^{p-2q}\subset K_s \]
is the natural projection.

(D) Let $(G_s, G'_s) = (O(p-2q, t-q), Sp(2n-2q))$ be the dual pair with maximal compact subgroup $(K_s, K'_s)$ as in (C). Then $(G_s, G'_s)$ is in the stable range (c.f. (4)). We put subscript $s$ for all objects with respect to this pair. For example, $\sigma_s'$ denotes the genuine character of $G'_s$. Correspondingly we have the closed null cone $N_s = \psi_s^{-1}(0)$.

(E) Let $Q_m$ act on $\overline{V_s}$ with trivial $\triangle' K^q_s$-action. Then there is a $Q_m$-equivariant bijection $T : Z_m \rightarrow \overline{V_s}$.

(F) By (B), we have $M \cong Q/Q_m$ and
\[
\mathcal{C}[Z] = \text{Ind}_{Q_m}^Q \mathcal{C}[Z_m] = \text{Ind}_{Q_m}^Q \mathcal{C}[\overline{V_s}].
\]
The proofs of (A) to (E) are given in Appendix B.1.2. Part (F) follows from [CPS] using a similar argument as (27).

Proof of Theorem C. Putting (29) into (24), the isotropy representation of $B$ at $x$ is
\[
\chi(x|L) = i^*_x \mathcal{B} |L \left( \text{Ind}_{Q_m}^{L \times K'_s \times K'_s} \mathcal{C} [\overline{V_s}] \otimes A \otimes \tau \right) K'_s \times K'_s \\
= \left( (\mathcal{C} [\overline{V_s}] \otimes A |K'_s) K'_s \times \tau \right) K_s \times K_s.
\]
Note that $A |K'_s \cong \varsigma \otimes \sigma'_s$, $\varsigma |K_s = \varsigma$, $\mu |K^{t-q} \times K_s = \varsigma |K^{t-q} \times K_s \otimes \tau |K_s \times K_q$. By Corollary 3.3, $(\mathcal{C} [\overline{V_s}] \otimes A |K'_s) K'_s = B_s$. Finally, we get
\[
\chi(x|L) = \varsigma |K^{p-2q} \times K^{t-q} \otimes \theta^{p-2q,t-q} (\sigma'_s) \otimes \tau |K^{t-q} \times K_q
\]
as a representation of $\tilde{K}^q \times \tilde{K}^{p-2q} \rightarrow \tilde{L}$. This proves (ii).

By (30), $\theta^{p-2q} (L(\mu')) = 0$ implies that $\tilde{\chi}_x = 0$. Conversely if $\theta^{p-2q} (L(\mu'))$ is non-zero then by Proposition 5.11 below (whose proof does not depend on the result of this subsection) $\tilde{\chi}_x$ is non-zero. This proves (iii).

The proof of (iv) is the same as that of Theorem B (ii) and (iv). We have $\text{Supp} (\mathcal{B}) \subseteq \mathcal{O}$. If $\theta^{p-2q} (L(\mu')) \neq 0$, then $\tilde{\chi}_x \neq 0$ so $x \in \text{Supp} (\mathcal{B}) \subseteq \mathcal{O} \text{ and } K \cdot x = \mathcal{O}$. Hence $\text{AV}(\theta^{p-2q} (L(\mu'))) = \mathcal{O}$. This proves (iv). This completes the proof of Theorem C.

5.10. Proof of Theorem D. In this section, we assume the notation of the Sections 5.7 and 5.9. $x \in \mathcal{O} = \mathcal{O}_{p,q,t} = \theta (\mathcal{O}', G', G)$ and $B = \varsigma |K \otimes \text{Gr} (\theta (L(\mu')))$. Since $\theta^{p-2q} (L(\mu')) \cong \varsigma |K \otimes B$, Theorem D follows from Proposition 5.11(ii) below. We emphasize that the proof of the proposition is independent of the calculations of $\chi_x$ in Section 5.7 and Section 5.9.

Proposition 5.11. Suppose that $p,q,t$ satisfies (8) and $n > \min \{ q, t \}$. Let $\chi_x$ be the $K_x$-module calculated in (28) and (30).
(i) We have $\mathcal{B} = (i_\mathcal{O})_* \mathcal{L}$ as $(\mathcal{O}_\Sigma, K_\Sigma)$-modules where $i_\mathcal{O} : \mathcal{O} \hookrightarrow \overline{\mathcal{O}}$ is the natural open embedding and $\mathcal{L}$ is the $K_\Sigma$-equivariant coherent sheaf with fiber $\chi_x$ at $x$ in sense of [CPS].

(ii) As $K_\Sigma$-modules, $$B = \text{Ind}^{K_\Sigma}_{K_x} \chi_x.$$ In particular, $B \neq 0$ if and only if $\chi_x \neq 0$.

Proof. By definition $\mathcal{B}(\overline{\mathcal{O}}) = B = \phi^*(\mathcal{O}) \otimes \theta(L(\mu'))$. On the other hand, $i_\mathcal{O}^* \mathcal{B} \cong \mathcal{L}$ and $\mathcal{B}(\mathcal{O}) = (i_\mathcal{O})_* \mathcal{L}(\mathcal{O}) = \text{Ind}^{K_\Sigma}_{K_x} \chi_x$ by [CPS]. We will show (ii), i.e. $\mathcal{B}(\overline{\mathcal{O}}) = \mathcal{B}(\mathcal{O})$ under the restriction map. Then (i) follows because both $\mathcal{B}$ and $(i_\mathcal{O})_* \mathcal{L}$ are quasi-coherent sheaves over an affine scheme with the same space of global sections.

By Lemma 5.2 and Proposition 3.6

$$\mathcal{B}(\overline{\mathcal{O}}) = (\mathbb{C}[\mathcal{N}] \otimes A \otimes \tau)^{K_\Sigma \times K_x} = (\mathbb{C}[\mathcal{N}] \otimes A \otimes \tau)^{K_\Sigma \times K_\Sigma}.$$ On the other hand, since $p$ is $K_\Sigma \times K_\Sigma$-invariant, localization commutes with taking $K_\Sigma \times K_\Sigma$-invariants. Let $D = (\phi \circ \text{pr})^{-1}(\mathcal{O}) \cap \mathcal{N}$ and we consider

$$\begin{array}{c}
\mathcal{D}^\prime \\
\downarrow \phi \circ \text{pr}
\end{array} \mathcal{N} \quad \begin{array}{c}
\downarrow \phi \circ \text{pr}
\end{array} \begin{array}{c}
\mathcal{O} \\
\downarrow i_\mathcal{O}
\end{array} \overline{\mathcal{O}}.$$ Since $i_\mathcal{O}$ is an open embedding, it is flat and we have (c.f. Corollary 9.4 in [Ha])

$$\Gamma(\mathcal{O}, i_\mathcal{O}^*(\phi \circ \text{pr})_* \mathcal{O}_\mathcal{N}) = \Gamma(\mathcal{O}, (\phi \circ \text{pr})_* \mathcal{O}_\mathcal{D}) = \Gamma(\mathcal{D}, \mathcal{O}_\mathcal{D}) = \mathbb{C}[\mathcal{D}].$$

This gives $\mathcal{B}(\mathcal{O}) = (\mathbb{C}[\mathcal{D}] \otimes A \otimes \tau)^{K_\Sigma \times K_\Sigma}$. Therefore, it suffices to show that

$$(31) \quad H^0(\mathcal{N}, \mathcal{O}_\mathcal{N}) = \mathbb{C}[\mathcal{N}] \to \mathbb{C}[\mathcal{D}] = H^0(\mathcal{D}, \mathcal{O}_\mathcal{D})$$

is an isomorphism.

Lemma 5.12. Suppose $n > \min \{ q, t \}$, then $D$ is a $K_\Sigma \times K_\Sigma$-orbit and $\partial D = \mathcal{N} - D$ has codimension at least 2 in $\mathcal{N}$.

The proof of the lemma is given in Appendix B.5.

We continue with the proof of the theorem. We note that $\mathcal{N}$ is a $(K_H)_\mathcal{N} \times K_\Sigma$-orbit and $\mathcal{D}$ is a $K_\Sigma \times K_\Sigma \times K_\Sigma$-orbit. Now (31) follows from Theorem 4.4 in [CPS]. This proves Proposition 5.11.

6. Special unipotent representations

We will briefly review special unipotent primitive ideals and representations in Chapter 12 in [V2]. Also see Section 2.2 in [I]. We will apply these to $\theta^{(q)}(L(\mu'))$.

Let $\mathfrak{g} = \mathfrak{so}(p + q, \mathbb{C})$. We consider $\theta^{(q)}(L(\mu'))$ where $q > n > t$, $q + t \geq 2n$, $\text{dim} \, \mu = 1$ and $L(\mu') = \theta(\mu)$. The infinitesimal character of $\theta^{(q)}(L(\mu'))$ corresponds to the weight $\lambda = (\delta_t, \delta_{p+q-2n}, \delta_{2n-t+2})$ under the Harish-Chandra parametrization [L]. Here $\delta_N = (\frac{N}{2} - 1, \frac{N}{2} - 2, \ldots )$ denotes the half sum of the positive roots of $\mathfrak{so}(N)$, and we insert or remove zeros from $\lambda$ if the string of numbers is too short or too long. The restriction of $\theta(L(\mu'))$ to $(\mathfrak{g}, (\bar{K}^{p,q})^0)$ decomposes into a finite number of irreducible $(\mathfrak{g}, (\bar{K}^{p,q})^0)$-submodules. Let $\theta(L(\mu'))^0$ denote one of these irreducible $(\mathfrak{g}, (\bar{K}^{p,q})^0)$-submodules. Since $\mathcal{O} = \mathcal{O}_{p,q,t}$ is also a $(\bar{K}^{p,q})^0$-orbit, $\theta(L(\mu'))^0$ also have associated variety $\mathcal{O}$.

We claim that the weight $\lambda$ also represents the infinitesimal character of $\theta(L(\mu'))^0$ as a $(\mathfrak{g}, (\bar{K}^{p,q})^0)$-module. Indeed we may have an ambiguity only if $p + q$ is even where the
infinitesimal character is either $\lambda$ or $s(\lambda)$. Here $s$ is the involution induced by the outer automorphism of $\mathfrak{g}$. In this case $t$ is even and the weight $\lambda$ contains a zero in the string of numbers. Hence $\lambda$ and $s(\lambda)$ represent the same infinitesimal character and proves our claim.

Under the Kostant-Sekiguchi correspondence, $\mathcal{O}$ generates a nilpotent $SO(p+q,\mathbb{C})$-orbit $C$ in $\mathfrak{g}^*$. The orbit $C$ has the same Young diagram as that of $\mathcal{O}$ in (5) less the plus and minus signs. Let $J$ denote the primitive ideal of $\theta(L(\mu'))^0$ in $U(\mathfrak{g})$. The ideal $J$ has a filtration $\{J_t = U_*(\mathfrak{g}) \cap J : s \in \mathbb{N}\}$. By [Y], $Gr(J)$ cuts out the variety $\overline{\mathcal{C}}$ in $\mathfrak{g}^*$.

Let $\Phi$ and $R$ be the roots and the root lattice of $\mathfrak{g}$. Let $\mathfrak{g}^\vee$ denote the simple Lie algebra with $\Phi$ as coroots. In particular $\mathfrak{g}^\vee = so(p+q,\mathbb{C})$ if $p+q$ is even and $\mathfrak{g}^\vee = sp(p+q-1,\mathbb{C})$ if $p+q$ is odd. We refer to [12] on the order reversing map $d$ (resp. $d^\vee$) from the set of complex nilpotent orbits in $\mathfrak{g}^*$ (resp. $(\mathfrak{g}^\vee)^\ast$) to the complex nilpotent orbits in $(\mathfrak{g}^\vee)^\ast$ (resp. $\mathfrak{g}^\ast$). The orbit $C$ is called special if it is in the image of $d^\vee$ and for a special orbit $C$, we have $d^\vee(d(C)) = C$.

Suppose $C$ is a special orbit. By the Jacobson-Morozov theorem, let $\{X^\vee, H^\vee, Y^\vee\}$ be the $\mathfrak{sl}_2$-triple such that $X^\vee \in d(C)$. We may assume that $\frac{1}{2}H^\vee$ lies in $\mathbb{C} \otimes_\mathbb{R} R$. In this way $\frac{1}{2}H^\vee$ defines an infinitesimal character of $U(\mathfrak{g})$ via the Harish-Chandra homomorphism. If $d(C)$ has Young diagram $(a_1, a_2, \ldots, a_s)$ then $\frac{1}{2}H^\vee = (\delta_{a_1+1}, \delta_{a_2+1}, \ldots, \delta_{a_s+1})$.

Suppose $\frac{1}{2}H^\vee$ gives the same infinitesimal character $\lambda$ as that of $\theta(L(\mu'))^0$. Let $J_\lambda$ denote the unique maximal primitive ideal of $U(\mathfrak{g})$ with infinitesimal character $\frac{1}{2}H^\vee$. We will call $J_\lambda$ a special unipotent primitive ideal. By Corollary A3 in [BV], the variety cut out by $Gr(J_\lambda)$ in $\mathfrak{g}^\ast$ is the same as that of $Gr(J)$, namely $\overline{\mathcal{C}}$. By Corollary 4.7 in [BK], $J_\lambda = J$. We say that $\theta(L(\mu'))^0$ is a special unipotent representation.

**Proposition 6.1.** Suppose $q + t \geq 2n$, $q \geq n \geq t$ and $\dim \mu = 1$. Then $C$ is a special orbit and $\theta(L(\mu'))^0$ is a special unipotent representation.

**Proof.** The orbit is special could be read off from page 100 in [CM]. Indeed $d(C) = (p+q-2n-1, 2n-t+1, t-1, \epsilon)$ where $\epsilon = 0$ if $t$ is odd (i.e. $\mathfrak{g}^\vee$ of type $C$) and $\epsilon = 1$ if $t$ is even (i.e. $\mathfrak{g}^\vee$ of type $D$). Furthermore $d^\vee d(C) = C$. We have $\frac{1}{2}H^\vee = (\delta_t, \delta_{p+q-2n}, \delta_{2n-t+2}) = \lambda$ which is the infinitesimal character of $\theta(L(\mu'))^0$. The conclusion that $\theta(L(\mu'))^0$ is special unipotent follows from the discussion prior to the proposition.

The above argument applies to $\theta(\sigma')$ too by setting $t = 0$. It is easier and we leave the details to the reader.

## 7. Other dual pairs

Most of our methods and results extend to the two dual pairs $(G^{p,q},G')$ in Table 1 below. We have omitted them in the main body of this paper in order to keep this paper simple. In this section, we will briefly describe these two dual pairs.

| $G^{p,q}$ | $G'$ | Stable range | Case I | Case II | codim $\partial D \geq 2$ |
|-----------|------|--------------|--------|---------|-------------------------|
| $U(p,q)$  | $U(n_1,n_2)$ | $p,q \geq n_1 + n_2$ | $q \geq n_1,n_2$ | $\max\{n_1,n_2\} > q$ | $\max\{n_1,n_2\} > \min\{t,n_1,n_2\}$ |
| $Sp(2p,2q)$ | $O^*(2n)$ | $p,q \geq n$ | $2q \geq n$ | $2q < n$ | $n > 2t$ or $n$ is odd |

**Table 1.** List of dual pairs

There is also a notion of theta lifts of $K_C'$-orbits on $p'^*$ to $K_C$-orbits on $p^*$, and conversely.
First we suppose \((G^{p,q}, G')\) is in stable range where \(G'\) is the smaller member. This condition is given in the second column of Table [1]. Let \(\sigma'\) be a genuine unitary character of \(G'\). For the dual pair \((\text{Sp}(2p, 2q), \text{O}^*(2n))\), \(\text{O}^*(2n)\) always splits and \(\sigma'\) is unique. The local theta lift \(\theta^{p,q}(\sigma')\) is nonzero and unitarizable, and it also the full theta lift (c.f. Proposition [22]). By an almost identical proof as that of Theorem A, one shows that \(\text{AC}(\vartheta(\sigma')) = 1[O_{p,q}]\) where \(O_{p,q} = \theta(0; G', G'^{p,q})\). Furthermore we have \(\vartheta(\sigma')|_{\tilde{K}} = \text{Ind}_{K_x}^{K_{\text{Rc}}} \chi_x\) where \(\tilde{\chi}_x\) is the isotropy representation \([Y_n]\).

Next we consider the dual pair \((G'^{0}, G')\). Let \(\mu\) be an irreducible genuine representation of \(G'^{0}\) such that \(L(\mu') = \theta(\mu)\) is nonzero. Then \(L(\mu')\) is a unitary lowest weight Harish-Chandra module of \(G'\). By \([Y_n]\), \(\text{AV}(L(\mu'^*))\) is the Zariski closure of an orbit \(O'\) in \(p^\circ\). The isotropy representation \(\tilde{\chi}_{x'}\) of \(L(\mu'^*)\) at a closed point \(x' \in O'\) is also computed explicitly in \([Y_n]\).

Let \(\sigma'\) be a genuine unitary character of \(G'\) for the dual pair \((G^{p,q+1}, G')\) in stable range. The local theta lift \(\vartheta := \theta^{p,q}(\sigma'^{*} L(\mu'))\) to \(G'^{p,q}\) is also the full theta lift (c.f. Proposition [14]). Again we have to divide into Cases I and II as in Theorems [3] and [4]. The conditions for Cases I and II are given in the third and fourth columns of Table [1] respectively.

In Case I, we have results similar to that of Theorem [3]. More precisely, \(\vartheta = \theta^{p,q}(\sigma'^{*} L(\mu'))\) is nonzero. Its associated variety is
\[
\text{AV}(\vartheta) = \theta(O; G', G) = \theta(\text{AV}(L(\mu'^*)))
\]
and it is the Zariski closure of single \(K'_{\text{Rc}}\)-orbit \(O\). We fix a closed point \(x \in O\). Let \(K_x\) and \(K_{x'}\) be the stabilizers of \(x\) and \(x'\) in \(K_{\text{C}}\) and \(K_{\text{C}}'\) respectively. Then there is a group homomorphism \(\beta: K_x \to K_{x'}\) such that the isotropy representation of \(\text{Gr} \vartheta\) at \(x\) is \(\tilde{\chi}_x = \varsigma|_{\tilde{K}} \otimes (\varsigma)|_{K_{x'}} \otimes \sigma'^* \otimes \tilde{\chi}_{x'} \circ \beta\). Therefore \(\vartheta\) satisfies \([1]\), i.e.
\[
\text{AC}(\vartheta) = (\text{dim } \tilde{\chi}_x)[O] = (\text{dim } \tilde{\chi}_{x'})[\theta(O')]) = \theta(\text{AC}(L(\mu'^*))).
\]
The last column lists the conditions in Case I such that (c.f. Theorem [D])
\[
\vartheta|_{\tilde{K}} = \text{Ind}_{K_{x'}}^{K_{\text{Rc}}} \tilde{\chi}_x.
\]
For Case II, the situation is more complicated. Equation (32) continues to hold but (33) fails in general.

**APPENDIX A. NATURAL FILTRATIONS**

**A.1. Proof of Lemma [2.1]** The map \(\nu\) in \([11]\) factors through the \(K'_{\text{Rc}}\)-covariant subspace \(\mathcal{Y}_{\tau'}\) of type \(\tau'\) of \(\mathcal{Y}\). Let \(\mathcal{Y}(\tau')\), \(\mathcal{Y}^d(\tau')\) and \(\mathcal{Y}_j(\tau') = \oplus_{d \leq j} \mathcal{Y}^d(\tau')\) denote the \(\tau'\)-isotypic components of \(\mathcal{Y}\), \(\mathcal{Y}^d\) and \(\mathcal{Y}_j\) respectively. Since \(K'\) has reductive action on \(\mathcal{Y}\) and preserves degrees, \(\mathcal{Y}(\tau')\) maps bijectively onto the covariant \(\mathcal{H}_{\tau'}\). Moreover \(\mathcal{Y}^d(\tau') = \mathcal{Y}^d \cap \mathcal{Y}(\tau')\) and \(\mathcal{Y}_j(\tau') = \mathcal{Y}_j \cap \mathcal{Y}(\tau')\). Hence
\[
\nu(\mathcal{Y}_j(\tau')) = F_j.
\]

Let \(\mathcal{H}(\tau')\) denote the \(\tau'\)-isotypic component in the harmonic subspace \(\mathcal{H}(K'_{\text{Rc}})\) of \(\mathcal{C}[W]\) for \(K'\). By \([12]\), we have \(\mathcal{H}(\tau') \subset \mathcal{Y}_{j_0}\) and by \([11]\) we have
\[
\mathcal{Y}(\tau') = \mathcal{U}(\mathfrak{m}^{(2,0)}) \mathcal{H}(\tau').
\]

Since \(\mathfrak{m}^{(2,0)}\) acts by degree two polynomials, \(\mathcal{Y}_{j}(\tau') = 0\) if \(j \neq j_0\) (mod 2) and \(\mathcal{Y}_{j_0}(\tau') = \mathcal{U}_j(\mathfrak{m}^{(2,0)}) \mathcal{H}(\tau')\). It follows from (34) that \(\nu(\mathcal{U}_j(\mathfrak{m}^{(2,0)}) \mathcal{H}(\tau')) = F_{2j_0} = F_{2j_0+1}.\)
We will prove $V_j = F_{2j+j_0}$ by induction. First we have $V_0 = F_{j_0}$. Suppose $V_j = F_{2j+j_0} = \nu(Y_j)$ where $Y_j = \mathcal{Y}_{2j+j_0}(\tau')$. Since $V_{j+1} = \nu(U_{j+1}(g)\mathcal{Y}_{j_0}) \subseteq \nu(\mathcal{Y}_{2(j+1)+j_0}) = F_{2(j+1)+j_0}$, it suffices to show that $F_{2(j+1)+j_0} \subseteq V_{j+1}$. By (10), $Y_j + m^{(2,0)}Y_j = Y_j + pY_j$. Hence

$$F_{2(j+1)+j_0} = \nu(U_{j+1}(m^{(2,0)})H(\tau')) \subseteq \nu(Y_j + m^{(2,0)}Y_j) = \nu(Y_j + pY_j) = \nu(Y_j) + p\nu(Y_j) = V_j + pV_j = V_{j+1}.$$

This shows that $F_{2(j+1)+j_0} = V_{j+1}$ and completes the proof of the lemma. \hfill \Box

A.2. Proof of Lemma 5.2. By Proposition 4.6 $\theta^{p,q}(L(\mu')) \cong (\theta^{p,q+(s') \otimes \mu})^{K^t}$. This defines another filtration $E'_j := (E_j \otimes \mu)^{K^t}$ on $\theta^{p,q}(L(\mu'))$. Let $v_0$ be the degree of the lowest degree $K$-type $V_0$ and $e_0$ be the degree of the lowest degree $\bar{K}$-type $\nu$. Let $j_0$ be the smallest integer such that $E'_j \neq 0$. In order to prove Lemma 5.2, it suffices to prove that

$$V_j = E'_{j+j_0}. \tag{35}$$

Indeed, by Lemma 2.5 and (20), we have a surjection

$$\eta_j : \sum_{a+b=2j+e_0} \mathcal{Y}_a \otimes (\mathcal{Y}_2^*)_b \longrightarrow E_j.$$

Let $l_0$ be the degree of $\mu'$. Then $L_l := ((\mathcal{Y}_2^*)_2l \otimes \mu)^{K^t} = ((\mathcal{Y}_2^*)_2l_0+1 \otimes \mu)^{K^t}$ is the natural filtration on $L(\mu')^*$.

Taking the $\mu$-coinvariant of $\eta_j$ gives a surjection

$$\eta'_j : \sum_{a+2l=2j+e_0-e_0} \mathcal{Y}_a \otimes L_l = \sum_{a+b=2j+e_0} \mathcal{Y}_a \otimes ((\mathcal{Y}_2^*)_b \otimes \mu)^{K^t} \longrightarrow (E_j \otimes \mu)^{K^t} = E'_j.$$

Hence the image $E'_j$ of $\eta'_j$ is also the filtration for $\theta^{p,q}(L(\mu'))$ defined before Lemma 2.5 up to a degree shifting. Now (35) follows from Lemma 2.5. \hfill \Box

APPENDIX B. GEOMETRY

B.1. Explicit moment maps. In this section, we will denote the space of $p$ by $n$ complex matrices (resp. symmetric matrices) by $M_{p,n}$ (resp. $\text{Sym}^n$). Let $I_{p,n} = (a_{ij})$ denote the matrix in $M_{p,n}$ such that $a_{ij} = \delta_{ij}$.

We identify

$$W_H = W^+ \oplus W^\perp = W^+ \oplus W^- \oplus W_2$$

with the set of complex matrices

$$M_{p+q+t,n} = M_{p,n} \oplus M_{q+t,n} = M_{p,n} \oplus M_{q,n} \oplus M_{t,n}.$$

We will denote an element in $W_H$ by $w = (w^w; w^-) = (w^w; w_1, w_2)$ and an element in $W = W^+ \oplus W^-$ by $(w^+; w_1)$. The projection map $\text{pr} : W_H \rightarrow W$ is given by $\text{pr}(w^w; w_1, w_2) = (w^w; w_1)$ and $\psi^- : W^- \rightarrow p^w = \text{Sym}^n$ is given by $\psi^-(w_1) = (w_1)^T w_1 \in \text{Sym}^n$. In particular $\psi^-$ is surjective if and only if $q \geq n$.

As always we will denote $O(p, \mathbb{C})$ by $K_C^p$. An element $(o_p, o_q, g) \in K_C^p \times K_C^q \times K_C^r$ acts on $W$ by $(o_p, o_q, g) \cdot (w^+, w_1) = (o_p w^g, o_q w_{1g})$. Let $E_{p,n} = \left( \begin{smallmatrix} I_{p} \\ 0 \\
_{n-p} \end{smallmatrix} \right)$ be the $p$ by $n$ matrix with $n$-linearly independent column vectors whose column space is isotropic.

Let $P_{p,n} \subseteq K_C^p$ be the stabilizer of the isotropic subspace spanned by the columns of $E_{p,n}$. It is a maximal parabolic subgroup of $K_C^p$. Then the column space of the complex
conjugation $E_{p,n}$ is an isotropic subspace dual to the column space of $E_{p,n}$. This gives a Levi decomposition

$$P_{p,n} \cong (GL(n, \mathbb{C}) \times K_C^{p-2n}) \ltimes N_{p,n}$$

with $N_{p,n}$ its unipotent radical. Let

$$\beta_{p,n}: P_{p,n} \to GL(n, \mathbb{C})$$

be the group homomorphism defined via quotient by $K_C^{p-2n} \ltimes N_{p,n}$.

B.1.1. *Case I: $q \geq n$. We refer to (25).* We set $z_0 = (z_0^+; z_0^-) = (E_{p,n}; E_{q,t,n}) \in W^+ \oplus W_H^-$. Let $(y^+; y^-) = pr(z_0)$. We also set $x = \phi(y^+, y^-) = y^+(y^-)^T$. Then $y^-$ has full rank $n$.

B.1.2. *Case II: $q < n$. We will change the basis in $\mathbb{C}^p$ such that the first $r$-coordinates are isotropic and dual to the last $r$-coordinates.*

Let $z_0 = (z_0^+; z_0^-)$ with $z_0^+ = I_{p,n}$ and $z_0^- = \begin{pmatrix} I_q & 0 \\ 0 & 0 \\ E_{q-2q,n-q} \end{pmatrix}$. Then $x = \phi(pr(z_0)) = I_{p,q} \in \mathfrak{p}^* = M_{p.q}$. Its stabilizer in $K_C^{p,q}$ is

$$K_x = \{ (o_1, o_2) \in P_{p,q} \times K_C^q \mid \beta_{p,q}(o_1) = o_2 \in O(q, \mathbb{C}) \}$$

with Levi subgroup

$$L = \{ ((k_q, k_{p-2q}), k_q) \in K_C^{p,q} \mid k_q \in K_C^q, k_{p-2q} \in K_C^{p-2q} \} \cong \Delta K_C^q \times K_C^{p-2q}.$$

We recall that $Q = L \times K_C^p \times K_C^q$.

Given $(z^+; z^-) \in Z$, i.e. $\phi(pr(z^+, z^-)) = x = I_{p,q}$. Then one can show that up to an action of $Q$,

$$\begin{pmatrix} I_q & 0 \\ 0 & A_s \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} I_q & 0 \\ 0 & iI_q \\ 0 & B_s \end{pmatrix} \in \overline{N}^+ \times \overline{N}^-$$

where $A_s \in M_{p-2q,n-q}$ and $B_s \in M_{q,n-2n}$. Let $\mathcal{M} = M_{q,n} \times M_{q,n} \times M_{q,t}$. We define an action of $Q$ on $\mathcal{M}$ by $(r_1, r_2, g)(m_1, m_2, m_3) = (r_1 m_1 g^{-1}, r_1 m_2 g^2 r_1 m_3)$ where $r_1 \in K_C^p, r_2 \in K_C^q, g \in K_C^t$, and $(m_1, m_2, m_3) \in M_{q,n} \times M_{q,n} \times M_{q,t}$. The subgroup $K_C^{p-2q}$ acts trivially. We define $\pi: \overline{N} \to \mathcal{M}$ by

$$M_{p,n} \times M_{q,n} \ni (A_1, B_1, A_1 B_2^T) \mapsto (A_1, B_1, A_1 B_2^T).$$

Here $A_1, B_1 \in M_{q,n}, B_2 \in M_{n,n}$. Note that $\pi$ commutes with the action of $Q$.

Let $m = (I_{q,n}, I_{q,n}, iI_{q,t})$. Let $\mathcal{M} = \pi(Z)$ in $\mathcal{M}$. Using (38), we deduce that $\mathcal{M}$ is an $Q$-orbit generated by $m$. One can also check that

$$Z_m = \pi^{-1}(m) = \left\{ \begin{pmatrix} I_q & 0 \\ 0 & A_s \\ 0 & 0 \end{pmatrix}; \begin{pmatrix} I_q & 0 \\ 0 & iI_q \\ 0 & B_s \end{pmatrix} \bigg| (A_s, B_s) \in \overline{N}_s \right\} \to \overline{N}_s$$

where $T$ is a bijection which maps the above element in the parathesis to $(A_s, B_s)$ and $\overline{N}_s$ is the null cone for pair

$$(G_s, G_s') = (G^{p-2q,t-q}, G^{q_2(n-q)}) = (O(p - 2q, t - q), Sp(2(n - q), \mathbb{R})).$$
Finally, the stabilizer of $m$ in $Q$ is

$$Q_m \cong \Delta K_C^n \times K_C^{e-2q} \times K_C^{l-q} \times K_C^{m-q}$$

$$\cong \Delta O(q, C) \times O(p - 2q, C) \times GL(n - q, C) \times O(t - q, C)$$

$$\cong L \times GL(n - q, C) \times O(t - q, C).$$

The subgroup $\Delta K_C^n = \Delta O(q, C)$ acts trivially on $\mathcal{N}_s$. This proves (A), (B), (C) and (E) of Section 5.9.

B.2. We refer to the notation in Section B. We recall $D = (\phi \circ \text{pr})^{-1}(O) \cap N$.

We recall $z_0 = (z_0^+, z_0^-) = (E_{p,n}, E_{q+t,n}) \in N$ in Appendices B.1.1 and B.1.2. We write $z_0 = (z_0^+; z_1, z_2)$ and $E = \text{pr}(D)$.

**Lemma B.3.** In both Cases I and II, we have the following statements.

(i) The set $D$ is a non-empty open dense in $N$.

(ii) The set $D$ is a single $K_C^{p,q} \times K_C^l$-orbit generated by $z_0$.

(iii) The set $E$ is dense in $\text{pr}(\mathcal{N})$ and it is a single $K_C^{p,q} \times K_C^l$-orbit generated by $\text{pr}(z_0)$.

**Sketch of the proof.** (i) Clearly $z_0 \in D$ so $D$ is non-empty and open in $N$. If $N$ is irreducible, then $D$ is open dense in $N$. If $N$ is not irreducible, then $D$ is open dense because $K_C^{p,q} \times K_C^l$ permutes the irreducible components.

(ii) Let $w = (z^+, z^-) \in D$. By the action of $K_C^n$, we may assume that $z^+ = z_0^+$. Now (ii) would follow from the claim that $z^-$ is in the $K_C^{n} \times K_C^l -$orbit of $z_0^-$. Indeed it is a case by elementary computation to show that

$$(39) \quad D = \{ w = (w^+; w_1, w_2) \in M_{p,n} \times M_{q,n} \times M_{t,n} \mid w \in N, \text{rank}(w_1) = \min(q,n), \text{rank}(w^T w_1) = \min(n,t) \}. $$

Our claim is an application of the Witt’s theorem to (39). We will leave the details to the reader.

(iii) This follows from (i), (ii) and the definition $E = \text{pr}(D)$.

\[\square\]

B.4. Proof of Lemma B.3  

(i) Without loss of generality, we set $y = \text{pr}(z_0) = (y^+; y^-)$ and $x = \phi(y) \in O$. We claim that $Y = \phi^{-1}(x) \cap \text{pr}(\mathcal{N}) \subset E$. Let $\tilde{y} = (y^+; \tilde{y}^-) \in Y$. Since $x$ has rank $n$, $y^+$ and $\tilde{y}^+$ has rank $n$.

Note that the column spaces of $x, y^+, \tilde{y}^+$ are the same. Hence there is $k' \in GL(n, C) = K_C^l$ such that $\tilde{y}^+ = y^+ k'$, i.e. $\tilde{y}^+ \in \mathcal{N}$. Now $y^+(\tilde{y}^- k'^T) = \tilde{y}^+ (\tilde{y}^-)^T = x = y^+ y^-$. Viewing $y^+$ as an injection, we get $y^- = \tilde{y}^- k'^T$. Therefore $(\tilde{y}^+, \tilde{y}^-) \in E$ by (39) which proves the claim.

By Lemma B.3 (iii), $E$ is a single $K_C \times K_C^l$-orbit so $\tilde{y} = (k, k') \cdot y \in Y$ for some $k \in K_C$ and $k' \in K_C^l$. We have $x = \phi((k, k') \cdot y) = k \cdot \phi(\tilde{y}) = k \cdot x$ so $k \in K_x$. Hence $Y$ is a single orbit of $K_C \times K_C^l$ and this proves (i).

(ii) We claim that for every $k = (k^p, k^q) \in K_x \subset K_C^{p,q}$, there is a unique $k' \in K_C^l$, denoted by $\beta(k)$, such that $(k, k') \in S_y$. It is easy to see that $k \mapsto \beta(k)$ is a group homomorphism $\beta: K_x \rightarrow K_C^l$. The image of $\beta$ is contained in $K_C^l$, since $\beta(k)$ stabilizes $\psi(y) = x'$.

We recall (30) that $P_{p,n}$ is the parabolic subgroup in $K_C^n$ which stabilizes column space of $y^+$. We also recall (31) the projection $\beta_{p,n}: P_{p,n} \rightarrow GL(n, C) = K_C^l$. It satisfies $k^p \cdot y^+ = y^+ \beta_{p,n}(k^p) = \beta_{p,n}(k^p)^{-1} \cdot y^+$.

We now prove our claim. We define $\beta(k) = \beta_{p,n}(k^p)$. Then $(k, \beta(k))$ stabilizes $y^+$. Now

$$y^+(y^-)^T = x = \phi(y) = \phi((k, \beta(k)) \cdot y) = \phi(y^+, (k^q, \beta(k)) \cdot y^-) = y^+(k^q y^- \beta(k)^T)^T$$
as matrices. Since \( y^+ \) is an injective linear transformation, we have \( k^a y^- \beta(k)^T = y^- \), i.e. \((k, \beta(k)) \in S_y\).

Next we prove the uniqueness of \( k' \) in our claim. Indeed if \((k, a), (k, b) \in S_y\) where \(a, b \in K'_C = GL(n, \mathbb{C})\). Then \( k^p y^+(k^a y^- a^T)^T = k^p y^+(k^a y^- b^T)^T \) as matrices so \( y^- a^T = y^- b^T \). Since \( y^- \) has rank \( n \), \( a = b \). This proves our claim and (ii).

(iii) An element of \( Z_y \) is of the form \((y^+, y^-; y_2)\). Since \((y^-, y_2)\) is in the null cone \( \overline{N}^- \), \( \psi_2(y_2) = -\psi_2(y^-) = x', \) i.e. \( y_2 \in \psi_2^{-1}(x') \). On the other hand, for any \( y_2 \in \psi_2^{-1}(x') \), \((y^-, y_2) \in N^- \) since \( y^- \) already has full rank. We define \( T : Z_y \rightarrow \mathbb{Z}_p \) by \((y^+; y^-; y_2) \mapsto y_2\). This is a bijection which satisfies (iii). \( \square \)

B.5. Proof of Lemma 5.13. The proof of the lemma involves some elementary but tedious case by case consideration. The case \( n > t \) and \( n > q \) are symmetric, so we only sketch the proof for \( q > n > t \).

By Lemma B.3(ii), \( D \) is a single \( K_{p, q}^C \times K_p^C \times K_{q, p}^C \)-orbit. Let \( D^- = \{(w_1, w_2) | (w^+; w_1, w_2) \in D \} \). Then \( D^- \) is the \( K_{p, q}^C \times K_p^C \times K_{q, p}^C \)-orbit of \( z_0^* = E_{q+t,n} \) and \( D = N^+ \times D^- \).

Let \( c \) denote the co-dimension of \( N^- \) in \( N \). It is also the co-dimension of \( N^- \times D^- \) in \( N^- \). We need to show that \( c \geq 2 \). If \( q + t > 2n \) so there exists a row of zeros in \( E_{q+t,n} \) and we set \( E^* \) to be the \( q + t \) by \( n \) matrix obtained from the matrix \( E_{q+t,n} \) by interchanging a zero row and with the last row. If \( q + t = 2n \), we interchange the \( n \)-th row with the \((q + t - 1)\)-th row. In both cases, let \( N^* \) denote the \( K_{p, q}^C \times K_p^C \times K_{q, p}^C \)-orbit generated by \( E^* \).

We observe that \( N^* \) is open dense in \( N^- \times D^- \).

The codimension \( c = \dim N^- - \dim N^* = \dim S_0^* - \dim S_0 \) where \( S_0 \) and \( S_0^* \) are the stabilizers of \( E_{q+t,n} \) and \( E^* \) respectively in \( K_{p, q}^C \times K_p^C \times K_{q, p}^C \). One may compute that

\[
\dim S_0 = \frac{1}{2} \left( n^2 - n + q^2 - q + t^2 - t \right) + n(n - q - t + 1).
\]

Similarly \( \dim S_0^* \) has the same formula as \( \dim S_0 \) above except that we have to reduce \( t \) by 1. With these, we compute that \( \dim S_0^* - \dim S_0 = 1 + n - t \geq 2 \) as required. \( \square \)

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