Deciding Probabilistic Program Equivalence in NetKAT

STEFFEN SMOLKA, Cornell University
DAVID KAHN, Cornell University
PRAVEEN KUMAR, Cornell University
NATE FOSTER, Cornell University
DEXTER KOZEN, Cornell University
ALEXANDRA SILVA, University College London

We tackle the problem of deciding whether a pair of probabilistic programs are equivalent in the context of Probabilistic NetKAT, a formal language for reasoning about the behavior of packet-switched networks. We show that the problem is decidable for the history-free fragment of the language. The main challenge lies in reasoning about iteration, which we address by a reduction to finite-state absorbing Markov chains.

This approach naturally leads to an effective decision procedure based on stochastic matrices that we have implemented in an OCaml prototype. We demonstrate how to use this prototype to reason about probabilistic network programs.

1 INTRODUCTION

Program equivalence is one of the most fundamental problems in Computer Science: given a pair of programs, do they describe the same computation? The problem is undecidable in the general case, but it can often be solved in the context of domain-specific languages based on restricted computational models. For example, a classical approach for deciding whether a pair of regular expressions denote the same language is to first convert the expressions to deterministic finite automata, which then admit an equivalence check in almost linear time [27]. In addition to the obvious theoretical motivation, there is also an important practical reason to study program equivalence: it is a powerful tool that can be used to solve a wide range of problems in verification, compilation, and synthesis.

This paper tackles the problem of deciding equivalence in Probabilistic NetKAT (ProbNetKAT), a language for modeling and reasoning about the behavior of packet-switched networks. As its name suggests, ProbNetKAT is based on NetKAT [1, 5, 25], which is in turn based on Kleene algebra with tests (KAT), an algebraic system obtained by combining Boolean predicates and regular expressions. ProbNetKAT extends NetKAT with a random choice operator and a semantics based on Markov kernels [26]. The framework can be used to encode and reason about the behavior of randomized protocols (e.g., a routing scheme that uses random paths to forward packets to balance load [28]); uncertainty about traffic demands (e.g., the diurnal/nocturnal fluctuation in access patterns commonly seen in networks for large content providers [21]); and failures (e.g., switches or links that are known to fail with some probability [6]).

The semantics of ProbNetKAT is surprisingly subtle. In particular, because the language provides an iteration operator, it is possible to write programs that generate continuous distributions over the uncountable space of history sets [4, Theorem 3]. This makes reasoning about convergence non-trivial, and raises the issue of representing infinitary objects in an implementation. To address these issues, prior work [26] developed a domain-theoretic characterization of ProbNetKAT that provides notions of approximation and continuity, which can be used to reason about programs using only discrete distributions. However, that work left the decidability of program
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equivalence as an open problem. In this paper, we settle this question positively for the history-free fragment of the language. This is a subtle and challenging problem, as many problems in probabilistic extensions of regular languages turn out to be undecidable—e.g., emptiness of probabilistic automata or, more generally, the threshold problem (i.e., is some word accepted with probability at least $p$?) Hence, the problem we tackle in this paper lies at the edge of decidability and requires care in its formulation.

At a technical level, our decision procedure for history-free ProbNetKAT follows a general approach: we transform programs into canonical representations for which checking equivalence is straightforward. Specifically, we define a big-step semantics that interprets each program as a finite stochastic matrix—equivalently, a Markov chain that transitions from input to output in a single step. Equivalence is trivially decidable on this representation, but the challenge lies in computing the big-step matrix in the case of iteration. Intuitively, the matrix needs to capture the result of an infinite stochastic process. We address this by embedding the system in a second Markov chain with a larger state space that models iteration in the spirit of a small-step semantics. With some care, this chain can be transformed to an absorbing Markov chain, which admits a closed form analytic solution using elementary matrix operations that represents the limit of the iteration. We prove the soundness of this approach.

Although the history-free fragment of ProbNetKAT is a restriction of the general language, it captures the “input-output” behavior of a network and is still expressive enough to handle a wide range of practical problems of interest. Many practical problems in networking are concerned with end-to-end behavior and do not require knowledge of specific routes, such as reachability, loop freedom, and isolation, and several other contemporary network verification tools including Anteater [19], Header Space Analysis [12], and Veriflow [14] are also limited to a history-free model. In ProbNetKAT, the main advantage of the restriction is that it lowers the complexity of the implementation by an exponential factor. This is critical for a tool that tracks probabilities in addition to packet-forwarding behavior.

Readers familiar with prior work on probabilistic automata might wonder if we could directly apply known results on (un)decidability of probabilistic rational languages. This is not the case—probabilistic automata accept distributions over words, while ProbNetKAT programs encode distributions over languages. Hence, we believe that having a domain-specific tool for deciding equivalence of probabilistic network programs is of value. Similarly, probabilistic programming languages, which have gained popularity in the last decade motivated by applications in machine learning, focus largely on Bayesian inference. They typically come equipped with a primitive for probabilistic conditioning and often have a semantics based on sampling. ProbNetKAT is somewhat different in that it focuses on verification. Thus, having a precise and complete semantics—given by a denotational model that interprets programs as functions mapping sets of input packet histories to distributions over sets of output histories—is crucial.

We have built a prototype implementation of our approach in OCaml. It leverages Eigen and BLAS as back-end libraries for representing and transforming matrices and incorporates a number of optimizations to improve performance. Although building a scalable implementation would require much more engineering (and is not the primary focus of this paper), our prototype is already able to handle inputs of moderate size. We have used it to carry out several case studies, including one based on modeling and verifying load (im)balance in data centers. Importantly, unlike an earlier implementation of ProbNetKAT [26], which implemented iteration through an infinite convergent sequence of approximations with no guaranteed bounds on the rate of convergence, our new implementation computes fixpoints directly.
Summary of contributions. In brief, the main contribution of this paper is the development of a decision procedure for the history-free fragment of ProbNetKAT. We develop a new semantics for this sublanguage in terms of stochastic matrices in two steps, we establish the soundness of the semantics with respect to ProbNetKAT’s original denotational model, and we use the semantics as the basis for building a prototype implementation and implementing several case studies.

(1) In Section 4, we provide a semantics for dup-free programs based on finite stochastic matrices and show that it fully characterizes the behavior of dup-free programs on packets (Theorem 4.1). This yields a big-step semantics in which these matrices can be understood as Markov chains over the state space \( 2^{P_k} \). A single step of the chain models the entire execution of a program, going directly from an initial state corresponding to the set of input packets to a final state corresponding to the set of output packets. Although this reduces the equivalence problem on programs to checking equality of finite matrices, we still need to provide a way to explicitly compute the matrices. This requires the computation of a limit in the case of iteration.

(2) In Section 5, we derive a closed form for the big-step matrix associated with \( p^* \). This gives an explicit representation of the big-step semantics. It is important to note that this is not simply the calculation of the stationary distribution of a Markov chain, as the calculation of \( p^* \) requires additional state. To this end, we define a small-step semantics, a second Markov chain with a larger state space such that one transition models one iteration of \( p^* \). We then show how to transform it to an absorbing Markov chain, which admits a closed form solution for its limiting distribution.

The development of the big and small-step semantics enables us to compute fixpoints analytically, thus providing an effective decision procedure for dup-free programs (Corollary 5.8). This is in contrast with the previous semantics [4], which merely provided an approximation theorem for the semantics of iteration \( p^* \) and did not give a decision procedure. It is worth noting that this approach is not an immediate application of classical probabilistic automata or the theory of Markov chains, although the theory of Markov chains figures prominently in the development. There are several additional subtleties that need to be resolved, which will be clear in the details of the technical development.

(3) As a direct consequence of our decidability result, we can also analyze termination of while loops. For example, the fact that the while loop below terminates with probability 1 reduces to a question about equivalence:

\[
\text{while } \neg f = 0 \text{ do } (1 \oplus r, f \leftarrow 0) \equiv f \leftarrow 0
\]
(4) Although we currently do not have a decision procedure for the full language, in Section 7 we make an important step towards this goal. We show that the meaning of a general program (not necessarily dup-free) is completely determined by its value on the single input $\{\pi \pi \mid \pi \in P_k\}$. This is a remarkable result that reduces the equivalence of two programs to their equivalence on a single input. It is a first step towards the development of a language model in the spirit of the one for NetKAT \[5\], which is a key factor in the decision procedure for that system, although in the probabilistic case this is far from trivial and not yet fully understood.

2 OVERVIEW

This section presents an example of a probabilistic network program, which serves both to introduce the features of ProbNetKAT and to motivate our approach for deciding equivalence.

**Example: Load balancing.** The networks that underpin modern services such as Facebook, Google, Netflix, etc. are expected to handle massive amount of traffic—both requests from billions of users around the world as well as internal traffic. One way to achieve good performance at scale—perhaps the only way!—is to design the network topology with multiple, redundant paths between hosts, and somehow balance incoming traffic flows onto those paths to use the full capacity of the network and avoid bottlenecks.

Consider the network shown in Figure 2. It depicts a simple tree-structured topology with 3 tiers of switches, each with a fan-out of 2. Although this network only contains 8 hosts, it loosely resembles the topologies used in real-world data centers with edge, aggregation, and core layers. Suppose that incoming requests arrive at the core switch, which directs them to one of the hosts. The hosts each implement the same service and can respond to any request. The objective is to balance the offered load among the servers without overloading any single server.

There are a number of different algorithms we could use to route incoming traffic from the core switch to the servers. In this section, we will consider just four representative algorithms:

- **Random Path:** The core switch picks a server at random, rewrites the destination address of incoming packets to the address of the selected server, and forwards it through the correct port. The aggregation and edge switches are configured to forward packets along the unique path to each destination.
- **Random Hops:** All switches make a random decision to send the packet to the left or the right with equal probability. Hence, each packet makes three random choices when going from the core switch to a server.
- **Random Walk:** This is similar to Random Hops, except that the edge and aggregation switches can also forward packets up in the topology, away from the servers. Thus, the number of random choices made by a packet is unbounded.
- **Hashed Hops:** Each switch hashes certain specified fields in the packet header, and either forwards to the left or the right depending on the parity of the hash. Specifically, the core
switch hashes the packet’s source address, the aggregation switches hash the source port, and the edge switches hash the destination port. This scheme is a simplified version of Equal Cost Multipath (ECMP) routing, which is widely deployed in data centers today.

It is natural to ask whether these approaches implement the same behavior even though they are based on different algorithms. Today, network operators must answer such questions empirically, by collecting and analyzing detailed traffic traces to detect anomalies such as load imbalance [18]. With ProbNetKAT we can answer these questions analytically.

**Network model.** Modeling a network in ProbNetKAT requires encoding two components: the program executing on each switch and the program that encodes the behavior of the topology that connects them. A topology can be viewed as a collection of links, where each link is responsible for modifying a packet’s location from one of its endpoints to the other. For example, we can model the link between core switch $C_1$’s port 1 and aggregation switch $A_1$’s port 3 in the example topology as:

$$l_{C_1,A_1} \triangleq (sw=C_1; pt=1; sw\leftarrow A_1; pt\leftarrow 3)$$

$$& (sw=A_1; pt=3; sw\leftarrow C_1; pt\leftarrow 1)$$

This program consists of smaller programs that are composed together using the union ($\&$) and sequential composition ($;$) operators. The sub-program on the first line matches on the packet’s location (switch = $C_1$, port = 1) and then modifies it to a new value (switch = $A_1$, port = 3). The overall effect is to transport packets from $C_1$ to $A_1$. The sub-program on the second line is similar, and transports packets in the opposite direction along the same link. We can model the entire topology as the union of all links in the network. Thus, the following ProbNetKAT program encodes the topology shown in Figure 2:

$$t \triangleq l_{C_1,A_1} \& l_{C_1,A_2} \& \cdots \& l_{E_4,h_7} \& l_{E_4,h_8}$$

To encode the programs that execute on each switch, we can use the same operators to match and modify packets, transforming the values of header fields and also moving around packets between the physical ports on the switch. Such programs can be compiled into routing tables that can be implemented efficiently in hardware [25]. For instance, suppose that the edge and aggregation switches are programmed to perform destination-based forwarding, as in the first scheme based on random paths. The program for edge switch $E_1$ forwards packets with destination address $h_1$ and $h_2$ through ports 1 and 2, and forwards all other packets to aggregation switch $A_1$ through port 3:

$$p_{E_1} \triangleq (dst=h_1; pt\leftarrow 1)$$

$$& (dst=h_2; pt\leftarrow 2)$$

$$& (\neg (dst=h_1 \& dst=h_2); pt\leftarrow 3)$$

We can write similar programs for the other switches in the topology. Then, to encode the forwarding logic for all switches into a single program, we take the union of their individual programs, after guarding the policy for each switch with a test that matches packets at that switch:

$$p \triangleq (sw=C_1; p_{C_1}) \& \cdots \& (sw=E_4; p_{E_4})$$

The behavior of the entire network is captured by a program that interleaves steps of processing by the switch program and the topology program until the packet reaches the egress of the network. In our running example, the egress of the network consists of the links attached to servers, as captured in the following program:

$$egress \triangleq (sw=h_1) \& \cdots \& (sw=h_8)$$
Hence, the behavior of the entire network can be captured as the following program:

\[
\text{while } \neg \text{egress do } (p \leftarrow t)
\]

Returning to our running example, we can implement each of the load balancing algorithms described above in ProbNetKAT as follows:

- **Random Path:** The core switch sets the destination address for incoming packets. The other switches implement standard destination-based forwarding:

  \[
p_{C1} \triangleq ((\text{dst} \leftarrow h1) \oplus_{1/8} \cdots \oplus_{1/8} (\text{dst} \leftarrow h8));
\]

  \[
  \text{set destination address} \\
  ([(\text{dst}=h1 : pt \leftarrow 1) \& \cdots \& (\text{dst}=h8 : pt \leftarrow 2))
  \quad \text{destination-based forwarding}
\]

  The program for the core switch uses ProbNetKAT’s choice operator (⊕) to randomly set the destination address to one of the 8 servers with equal probability.

- **Random Hops:** Every switch \( S \) in the network executes the same program that randomly selects one of the ports on the paths toward the servers:

  \[
p_S \triangleq (pt \leftarrow 1) \oplus_{1/2} (pt \leftarrow 2)
\]

- **Random Walk:** The program for the core switch remains the same as above. But, every edge and aggregation switch \( S \) executes the following program that randomly selects one of the ports:

  \[
p_S \triangleq (pt \leftarrow 1) \oplus_{1/3} (pt \leftarrow 2) \oplus_{1/3} (pt \leftarrow 3)
\]

- **Hashed Hops:** Each switch in the same layer executes the same program. As hash functions are usually deterministic, we model the value of hashed fields as additional packet fields, writing \( \hat{f} \) for \( \text{hash}(f) \mod 2 \) of field \( f \). The programs executed by switches at different layers are:

  \[
p_{C_i} \triangleq (\text{src} = 0 : pt \leftarrow 1) \& (\text{src} = 1 : pt \leftarrow 2)
\]

  \[
p_{C_i} \triangleq (\text{sp} = 0 : pt \leftarrow 1) \& (\text{sp} = 1 : pt \leftarrow 2)
\]

  \[
p_{E_i} \triangleq (\text{dpt} = 0 : pt \leftarrow 1) \& (\text{dpt} = 1 : pt \leftarrow 2)
\]

where \( \text{src}, \text{sp}, \text{dpt} \) refer to source address, source port and destination port, respectively.

**Analysis.** Having encoded these algorithms in ProbNetKAT, we can use the semantics of the language to reason analytically about their behavior. In this paper, we develop a semantics based on **stochastic matrices**—i.e., matrices in which the rows correspond to input packets, the columns correspond to output packets, and the \((i, j)\) entry represents the probability that the program transforms the input packet corresponding to row \( i \) into the output packet corresponding to row \( j \). Each of the matrices depicted in this section were generated using our OCaml prototype.

The first question we might ask is whether the load balancing algorithms are equivalent to each other. It is not hard to see that the random-path and random-hops algorithms are equivalent—even though one makes a single random choice while the other makes a series of local choices, they both map incoming packets to servers uniformly at random. While the random-walk does not converge in a bounded number of steps, we find that it delivers packets with probability 1 eventually, reaching each server with equal probability. Thus it is also equivalent to the previous two algorithms, and the stochastic matrices for all three programs are identical:
Because the length of the path taken by a packet using random-walk is unbounded, an implementation based on a naive fixpoint computation would not converge within any finite number of iterations and thus would not be sufficient to show this equivalence [26].

However, the hashed-hops algorithm is not equivalent. Specifically, if we label the rows using the following equivalence classes on packets, (\(\hat{\text{src}}\), \(\hat{\text{spt}}\), \(\hat{\text{dpt}}\)) we can represent the matrix as follows:

\[
\begin{bmatrix}
1/8 & 1/8 & 1/8 & 1/8 & 1/8 & 1/8 & 1/8 & 1/8 \\
\end{bmatrix}
\]

Upon inspection, it’s clear that this algorithm is actually deterministic—packets belonging to the same equivalence class are always directed to the same server.

A desirable property of any load balancing algorithm is that the load should be shared equally among all servers under any incoming traffic pattern. In cases where the incoming traffic is uniformly distributed, we can compute from the matrices that every server is equally loaded with any of the four algorithms. However, in cases where the incoming traffic is skewed, the hashed-hops algorithm overloads some servers and leaves others unused. For example, suppose that the network receives a burst of packets in which the hash of the source port is always even (\(\hat{\text{spt}} = 0\)). Using hashed-hops, all of the traffic will be directed to only four hosts. We can model this analytically in ProbNetKAT by writing a program that models this skewed distribution and pre-composing it with the main program, yielding the following matrix, which illustrates the anomalous behavior:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

Although this example has been simplified for the purpose of illustration, load imbalance using protocols such as ECMP is a common problem and a frequent cause of performance degradation in real-world networks [23]. Hence, having a tool for modeling and detecting such situations automatically would be of significant value for network operators.

3 BACKGROUND ON PROBABILISTIC NETKAT

This section reviews the syntax and semantics (Figure 3) of probabilistic NetKAT [4, 26] and proves some basic properties along the way.

3.1 Syntax

A packet \(\pi\) is a record mapping a finite set of fields \(f_1, f_2, \ldots, f_k\) to bounded integers \(n\). Fields include standard headers such as source (src) and destination (dst) addresses and two logical fields for the switch (sw) and port (pt) that record the current location of the packet in the network.
The logical fields are not present in a physical network packet, but it is convenient to model them just like proper header fields. We write $\pi.f$ to denote the value of field $f$ of $\pi$ and $\pi[f:=n]$ for the packet obtained from $\pi$ by updating field $f$ to $n$. We let $P_k$ denote the (finite) set of all packets.

A history $h = \pi::h$ is a non-empty list of packets with head packet $\pi$ and (possibly empty) tail $h$. The head packet models the packet’s current state. The tail contains its prior states, which capture the trajectory of the packet through the network. Operationally, only the head packet exists, but semantically, histories allow us to record the trajectories of packets through the network. We write $\mathcal{H}$ to denote the (countable) set of all histories.

We differentiate between predicates ($t, u, \ldots$) and programs ($p, q, \ldots$). Predicates include tests ($f=n$), the Boolean primitives $false$ (0) and $true$ (1), and the Boolean connectives—disjunction ($t \& u$), conjunction ($t ; u$), and negation ($\neg t$). Programs include predicates ($t$) and modifications ($f \leftarrow n$) as primitives, and the operators parallel composition ($p \& q$), sequential composition ($p ; q$), and iteration ($p^\ast$). The primitive dup takes a snapshot of the current packet state and saves it to the history. Intuitively, we can think of a history as a (partial) log of a packet’s activity and of dup as the logging command. Finally, choice $p \oplus r$, $q$ executes $p$ with probability $r$ or $q$ with probability $1 - r$. For computational reasons, we require that $r$ be rational. We often use an $n$-ary version of probabilistic choice in examples, which can be desugared into the binary version.

Conjunction of predicates and sequential composition of programs use the same syntax ($t ; u$), as their semantics coincide. The same is true for disjunction of predicates and parallel composition of programs ($t \& u$). The distinction between predicates and programs allows us to restrict negation to predicates and rule out nonsensical expressions like $\neg p^\ast$.

The language as presented in Figure 3 only includes core primitives. It is worth noting that many useful constructs can be derived. In particular, it is straightforward to encode conditionals and while loops:

$$\begin{align*}
\text{if } t \text{ then } p \text{ else } q & \triangleq t ; p \& \neg t ; q \\
\text{while } t \text{ do } p & \triangleq (t ; p)^\ast ; \neg t
\end{align*}$$

(1)

These encodings are well-known from KAT [16].

### 3.2 Semantics

ProbNetKAT’s primitives and operators have an intuitive operational interpretation. However, some care is needed to give a formal denotational semantics that matches these operational intuitions. The main challenge is that the space $2^\mathcal{H}$ of history sets is uncountable, so there exist distributions over $2^\mathcal{H}$ that assign probability 0 to any particular outcome $a \in 2^\mathcal{H}$. ProbNetKAT can in fact generate such continuous distributions [4], so we must resort to measure theory to obtain a mathematically sound semantics. We represent distributions on $2^\mathcal{H}$ as probability measures over the measurable space $(2^\mathcal{H}, \mathcal{B})$, where $\mathcal{B}$ is the family of Borel sets of $2^\mathcal{H}$. We discuss this in more detail in § 3.3.

Programs are interpreted as Markov kernels on the space $(2^\mathcal{H}, \mathcal{B})$. A Markov kernel is a function $2^\mathcal{H} \to \mathcal{D}(2^\mathcal{H})$ in the probability (or Giry) monad $\mathcal{D}$ [7, 15]. Thus, a program $p$ maps an input set of histories $a \in 2^\mathcal{H}$ to a distribution $\|p\|(a) \in \mathcal{D}(2^\mathcal{H})$ over output sets of histories. Formally, $\mathcal{D}$ is an endofunctor in the category of measurable spaces $\mathcal{M}eas$, but for our purposes only the following facts will be important:

- For a measurable space $(X, \Sigma_X)$, $\mathcal{D}(X)$ denotes the set of probability measures over $X$; that is, the set of countably additive functions $\mu : \Sigma_X \to [0, 1]$ with $\mu(X) = 1$.
- For a measurable function $f : X \to Y$, $\mathcal{D}(f) : \mathcal{D}(X) \to \mathcal{D}(Y)$ denotes the pushforward along $f$; that is, the function that maps a measure $\mu$ on $X$ to its pushforward measure on $Y$ given by $\mathcal{D}(f)(\mu) \triangleq \mu \circ f^{-1} = \lambda A \in \Sigma_Y. \mu(\{x \in X \mid f(x) \in A\})$. 

### Syntax

| Naturals | $n ::= 0 | 1 | 2 | \ldots$ |
| Fields   | $f ::= f_1 | \ldots | f_k$ |
| Packets  | $Pk \ni \pi ::= \{ f_i = n_1, \ldots, f_k = n_k \}$ |
| Histories| $H \ni h ::= \pi::h$ |
| Probabilities | $r \in [0,1] \cap \mathbb{Q}$ |
| Predicates| $t, u ::= 0 \quad \text{False/Drop}$ |
| Programs | $p, q ::= t \quad \text{Filter}$ |

### Semantics

$\{p\} \in 2^H \rightarrow \mathcal{D}(2^H)$

| $\{0\}(a)$ | $\triangleq \eta(\emptyset)$ |
| $\{1\}(a)$ | $\triangleq \eta(a)$ |
| $\{f=n\}(a)$ | $\triangleq \eta(\{ \pi::h \in a | \pi.f = n \})$ |
| $\{f\leftrightarrow n\}(a)$ | $\triangleq \eta(\{ \pi|f::n::h | \pi::h \in a \})$ |
| $\{\text{dup}\}(a)$ | $\triangleq \eta(\{ \pi::h | \pi::h \in a \})$ |
| $\neg t(a)$ | $\triangleq D(\lambda b.a - b)(\{ t\}(a))$ |
| $p \land q(a)$ | $\triangleq D(\cup)(\{ p\}(a) \times \{ q\}(a))$ |
| $p; q(a)$ | $\triangleq \{ p\}(a) \times \{ q\}(a)$ |
| $p \lor q(a)$ | $\triangleq r \cdot \{ p\}(a) + (1 - r) \cdot \{ q\}(a)$ |
| $p^*(a)$ | $\triangleq \bigcup_{n \in \mathbb{N}} \{ p\}^n(a)$ |

where $p^{(0)} \triangleq 1$ and $p^{(n+1)} \triangleq 1 \land p; p^n$.

### Probability Monad

$\mathcal{D} : \mathcal{Meas} \rightarrow \mathcal{Meas}$

| Unit | $\eta : X \rightarrow \mathcal{D}(X)$, $\eta(x) \triangleq \delta_x$ |
| Bind | $\triangledown : (X \rightarrow \mathcal{D}(Y)) \rightarrow \mathcal{D}(X) \rightarrow \mathcal{D}(Y)$ |

$f^\triangledown(\mu)(A) \triangleq \int_{x \in X} f(x)(A) \cdot \mu(dx)$

- The unit $\eta : X \rightarrow \mathcal{D}(X)$ of the monad maps a point $x \in X$ to the point mass (or Dirac measure) $\delta_x \in \mathcal{D}(X)$. The Dirac measure is given by $\delta_x(A) \triangleq 1[x \in A]$, i.e. it is 1 if $x \in A$ and 0 otherwise.
- The bind operation $\triangledown : (X \rightarrow \mathcal{D}(Y)) \rightarrow \mathcal{D}(X) \rightarrow \mathcal{D}(Y)$ of the monad lifts a functions $f : X \rightarrow \mathcal{D}(Y)$ with deterministic inputs to a function $f^\triangledown : \mathcal{D}(X) \rightarrow \mathcal{D}(Y)$ that takes random inputs. Intuitively, this is achieved by averaging the output of $f$ when the inputs are randomly distributed according to $\mu$. Formally,

$$f^\triangledown(\mu)(A) \triangleq \int_{x \in X} f(x)(A) \cdot \mu(dx).$$

- Given two measures $\mu \in \mathcal{D}(X)$ and $\nu \in \mathcal{D}(Y)$, $\mu \times \nu \in \mathcal{D}(X \times Y)$ denotes their product measure. This is the unique measure satisfying $(\mu \times \nu)(A \times B) = \mu(A) \cdot \nu(B)$. Intuitively, it models distributions over pairs of independent values.

With these denotational tools at hand, we can now make our operational intuitions precise. Formal definitions are given in Figure 3. A predicate $t$ maps (with probability 1) the set of input histories $a \in 2^H$ to the subset of histories $b \subseteq a$ satisfying the predicate. In particular, the false or drop primitive 0 simply drops all packets (that is, it returns the empty set with probability 1), and the true or skip primitive 1 simply keeps all packets (that is, it returns the input set with probability 1). The test $f=n$ returns the subset of input histories whose head packets’ $f$-field contains $n$. Negation $\neg t$ filters out the histories returned by $t$.

Union (or parallel composition) $p \land q$ executes $p$ and $q$ independently on the input set, then returns the union of their results. Note that history sets do not model nondeterminism! Rather, they model collections of in-flight packets that traverse the network simultaneously. Probabilistic
choice $p \oplus_r q$ feeds the input to both $p$ and $q$ and returns a convex combination of the output distributions according to $r$. Sequential composition $p ; q$ can be thought of as a two-stage probabilistic experiment: it first executes $p$ on the input set to obtain a random intermediate result, then feeds that into $q$ to obtain the final distribution over outputs. The outcome of $q$ needs to be averaged over the distribution of intermediate results produced by $p$. (Think of summing over the paths in a probabilistic tree diagram and multiplying the probabilities along each path. The continuous analog of this calculation is obtained through integration.)

We say that two programs are equivalent, denoted $p \equiv q$, if they denote the same Markov kernel, i.e. if $[p] = [q]$. As usual, we expect Kleene star $p^*$ to satisfy the characteristic fixpoint equation $p^* \equiv 1 \uplus p ; p^*$, which allows it to be unrolled ad infinitum. Thus we define it as the last upper bound (or supremum) of its finite unrollings $p^{(n)}$, see Figure 3. This supremum is taken in a CPO $(\mathcal{D}(2^H), \sqsubseteq)$ of distributions that is described in more detail in § 3.3.

A fact that should be intuitively clear, although it is somewhat hidden in our presentation of the denotational semantics, is that the predicates form a Boolean algebra:

**Lemma 3.1.** Every predicate $t$ satisfies $[t](a) = \delta_{a \sqsupseteq b_t}$ for a certain history set $b_t \subseteq H$, where $b_0 = \emptyset$, $b_1 = H$, $b_{(f=n)} = \{ \pi:: h \in H \mid \pi.f = n \}$, $b_t \uplus t = H - b_t$, $b_{t \uplus u} = b_t \cup b_u$, and $b_{t ; u} = b_t \cap b_u$.

**Proof.** For $0$, $1$, and $f=n$, the claim holds trivially. For $\neg t$, $t \uplus u$, and $t ; u$, the claim follows inductively, using that $\mathcal{D}(f)(\delta_b) = \delta_{f(b)}$, $\delta_b \times \delta_c = \delta_{(b,c)}$, and that $f^{\uplus}(\delta_b) = f(b)$. The first and last equations hold because $(\mathcal{D}, \eta, \uplus)$ is a monad. \qed

### 3.3 The measurable space $(2^H, \mathcal{B})$ and the CPO $(\mathcal{D}(2^H), \sqsubseteq)$

The measurable space of history sets and the CPO of its distributions are characterized in detail in [26]. We briefly review the basics here.

The basic building blocks of the measurable space $(2^H, \mathcal{B})$ are the upward-closed events

$$B_b \triangleq \{ b \uplus \} = \{ c \in 2^H \mid c \supseteq b \} \quad \text{for finite } b \in 2^H.$$ 

The event $B_b$ comprises all sets that contain at least $b$, and so $[p](a)(B_b)$ is the probability that program $p$ outputs at least $b$ on input $a$. The $\sigma$-algebra of measurable sets $\mathcal{B}$ can be characterized as being built from these basic sets by taking complements, countable unions, and countable intersections. Formally,\footnote{\textbf{For readers familiar with topology we note that the sets $B_b$ form a basis of the Scott-topology of the CPO $(2^H, \sqsubseteq)$, and that $\mathcal{B}$ is the Borel $\sigma$-algebra of that space.}}

$$\mathcal{B} = \sigma(\{ B_b \mid b \in \wp_\omega(H) \})$$

ProbNetKAT programs satisfy a natural monotonicity property: if we increase the set of input histories $a$, then the probability of observing some set of histories $b$ in the output can only increase. Formally,

$$a \subseteq a' \implies [p](a)(B_b) \leq [p](a')(B_b) \quad (2)$$

A similar monotonicity property holds also for the unrollings $p^{(n)}$ of the iterate $p^*$: if we increase the number of iterations, then the probability of observing some set of histories $b$ in the output can only increase:

$$n \leq m \implies [p^{(n)}](a)(B_b) \leq [p^{(m)}](a)(B_b) \quad (3)$$

In fact these two implications hold more generally for arbitrary unions of basic sets, and we let $O$ denote the collection of all such unions. (The notation $O$ derives from the fact that these are the
Scott-open sets of the CPO \( (2^{H}, \subseteq) \). These observations motivate the following definition of an order \( \subseteq \) on distributions \( \mu, \nu \in \mathcal{D}(2^{H}) \):

\[
\mu \subseteq \nu \iff \mu(A) \leq \nu(A), \forall A \in O
\]

Implications (2) and (3) can then be rephrased as

\[
a \subseteq a' \implies \|p\|\mu(a) \subseteq \|p\|\mu(a') \quad \text{and} \quad n \leq m \implies \|p\|\mu(n)(a) \subseteq \|p\|\mu(m)(a)
\]

The distributions \( \mathcal{D}(2^{H}) \) ordered by \( \subseteq \) form a CPO, as was first observed by Saheb-Djahromi [22]: reflexivity and transitivity of \( \subseteq \) are immediate; \( \subseteq \) is antisymmetric because distributions over \( 2^{H} \) are already fully determined by the probabilities they assign to the basic open sets:

\[
\mu = \nu \iff \mu(B_{a}) = \nu(B_{a}), \forall a \in \wp_{\omega}(H)
\]

and the supremum of an increasing chain \( \mu_{1} \subseteq \mu_{2} \subseteq \ldots \) of measures is the measure \( \bigsqcup \mu_{n} \) satisfying:

\[
\left( \bigsqcup_{n} \mu_{n} \right)(A) = \sup_{n} \mu_{n}(A), \forall A \in O
\]

which is unique by (6). Since \( \mu_{n} \triangleq \|p\|\mu(n)(a) \) defines such an increasing sequence of measures by (5), the semantics of \( p^{*} \) given in Figure 3 is well-defined, and in fact the measures \( \|p\|\mu(n)(a) \) converge pointwise (7) and monotonically from below (4) to \( \|p\|\mu^{*}(a) \) on the open sets. Unfortunately, pointwise convergence fails in general, even for deterministic programs. For instance, if we fix some packet \( \pi \) and consider the history set \( a_{n} \triangleq \{ \pi^{k} \mid 0 < k < n \} \), then we have

\[
\|p\|\mu^{(n)}(\{\pi\}) = \delta_{a_{n+1}}, \forall n \quad \text{and} \quad \|p\|\mu^{*}(\{\pi\}) = \delta_{a_{\infty}}
\]

and in particular, there is no convergence on the (non-open) event \( A \triangleq \bigcap_{n \in \mathbb{N}} B_{a_{n}} \in \mathcal{B} - O \):

\[
\|p\|\mu^{(n)}(\{\pi\})(A) = 0, \forall n \quad \text{but} \quad \|p\|\mu^{*}(\{\pi\})(A) = 1
\]

Luckily, pointwise (albeit not necessarily monotone) convergence does extend to finite boolean combinations of basic open sets:

**Lemma 3.2.** Let \( A \) be a finite boolean combination of basic open sets, i.e. sets of the form \( B_{a} = \{ a \} \uparrow \) for \( a \in \wp_{\omega}(H) \). Then for all programs \( p \) and inputs \( a \in 2^{H} \),

\[
\|p\|\mu^{*}(a)(A) = \lim_{n \to \infty} \|p\|\mu^{(n)}(a)(A)
\]

**Proof Sketch.** The proof applies the inclusion-exclusion principle to express the probability of \( A \) as a finite sum of probabilities of basic open sets, for which we have convergence by Equation (7). The details are given in Appendix A. \( \square \)

This turns out to be the crucial insight that allows to link the denotational semantics, defined over the uncountable state space of history sets, with the much simpler semantics in the next section, defined over the finite state space of packet sets.

---

\(^{2}\)See Lemma 7 in [26].

\(^{3}\)See the proof of Theorem 11 in the full version of [26].
\( \mathcal{B}[p] \in \mathcal{S}(2^{pk}) \)

\[
\begin{align*}
\mathcal{B}[0]_{a,b} & \triangleq 1[b = \emptyset] \\
\mathcal{B}[1]_{a,b} & \triangleq 1[a = b] \\
\mathcal{B}[f = n]_{a,b} & \triangleq 1[b = \{ \pi \in a \mid \pi.f = n \}] \\
\mathcal{B}[\neg t]_{a,b} & \triangleq 1[b \subseteq a] \cdot \mathcal{B}[t]_{a,a-b} \\
\mathcal{B}[f \leftarrow n]_{a,b} & \triangleq 1[b = \{ \pi[f := n] \mid \pi \in a \}]
\end{align*}
\]

\[
\begin{align*}
\mathcal{B}[p \circ q]_{a,b} & \triangleq \sum_{c,d} 1[c \cup d = b] \cdot \mathcal{B}[p]_{a,c} \cdot \mathcal{B}[q]_{a,d} \\
\mathcal{B}[p ; q] & \triangleq \mathcal{B}[p] \cdot \mathcal{B}[q] \\
\mathcal{B}[p \oplus_r q] & \triangleq r \cdot \mathcal{B}[p] + (1 - r) \cdot \mathcal{B}[q] \\
\mathcal{B}[p^*_k]_{a,b} & \triangleq \lim_{n \to \infty} \mathcal{B}[p^{(n)}]_{a,b}
\end{align*}
\]

Fig. 4. Big-Step Semantics: \( \mathcal{B}[p]_{a,b} \) denotes the probability that program \( p \) produces output \( b \) on input \( a \).

## 4 BIG-STEP SEMANTICS

This section shows that the behavior of a dup-free program on history-free inputs can be fully captured by a finite stochastic matrix. This reduces the equivalence problem on programs to checking equality of finite matrices.

The denotational semantic of ProbNetKAT, described in the previous section, interprets programs as Markov kernels \( 2^H \to D(2^H) \). The semantics requires some heavy machinery: we have to invoke measure theory on a measurable space \( (2^H, \mathcal{B}) \) to deal with potentially continuous distributions; and we have to invoke domain theory on a CPO \( (D(2^H), \sqsubseteq) \) of distributions to characterize unbounded iteration. The root cause for this complexity lies in the cardinality of \( 2^H \): because there are as many histories as there are natural numbers, the powerset \( 2^H \) is as large as the set of real numbers.

Unsurprisingly, then, we obtain a much simpler system by discarding the notion of histories and focusing solely on packets (or, singleton histories), of which there are only finitely many. Formally, we identify \( \pi \in Pk \) and \( \langle \pi \rangle \in H \) and work with the subset \( Pk \subseteq H \). We discard the program primitive \( \text{dup} \), which no longer serves a purpose. Since the set of packets \( Pk \) is finite, so is its powerset \( 2^{pk} \). Thus any distribution over packet sets is discrete and can be characterized by a so called probability mass function, i.e. a function

\[
f : 2^{pk} \to [0, 1], \quad \sum_{a \subseteq Pk} f(a) = 1
\]

It is convenient to organize \( f \) in a so called stochastic vector, i.e. a vector \( v \) of non-negative entries that sums up to 1. The vector is indexed by packet sets \( a \subseteq Pk \), with the \( a \)th component of \( v \) giving the probability \( f(a) \). A program, being a function that maps inputs to distributions over outputs, can then be organized as a square matrix by stacking these stochastic vectors, one for each input, as rows on top of each other.

More formally, we will interpret a program \( p \) as a matrix \( \mathcal{B}[p] \in [0, 1]^{2^{pk} \times 2^{pk}} \) indexed by packet sets, where the matrix entry \( \mathcal{B}[p]_{a,b} \) denotes the probability that program \( p \) produces output \( b \in 2^{pk} \) on input \( a \in 2^{pk} \). The rows of \( \mathcal{B}[p] \) are stochastic vectors, each encoding the output distribution corresponding to a particular input set \( a \). Such a matrix is called (left-)stochastic, and we let \( \mathcal{S}(2^{pk}) \) denote the set of stochastic square matrices with indices in \( 2^{pk} \).

The interpretation of programs as stochastic matrices is mostly straightforward and given formally in Figure 4. At a high-level, deterministic program primitives map to simple \((0, 1)\)-matrices, and program operators map to operations on matrices. For example, the program primitive \( 0 \) is
interpreted as the matrix

\[
\begin{bmatrix}
\emptyset & b_2 & \ldots & b_n \\
1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
a_0 & 1 & 0 & \cdots & 0
\end{bmatrix}
\]

that puts all probability mass in the \(\emptyset\)-column; and the primitive \(\mathbf{1}\) is the identity matrix. The formal definitions are given using Iverson brackets: \(1[\varphi]\) is defined to be 1 if \(\varphi\) is true, or 0 otherwise.

As suggested by the picture in (9), a stochastic matrix \(B \in \mathcal{S}(2^{PK})\) can be interpreted as a so-called Markov chain (MC): a probabilistic transition system with state space \(2^{PK}\) that makes a random transition between states at each time step. The matrix entry \(B_{a,b}\) gives the probability that the system transitions from state \(a\) to state \(b\). The name “big-step semantics” derives from this interpretation: \(B[p]\) models \(p\) as a Markov chain that transitions from a state \(a\) (modeling the input set) in a single step straight to the output set \(b\). This interpretation also explains why sequential composition is interpreted as matrix product: a step from \(a\) to some intermediate state \(c\) in \(B[p]\) and a step from \(c\) to the final state \(b\) in \(B[q]\); it thus occurs with probability

\[
B[p; q]_{a,b} = \sum_c B[p]_{a,c} \cdot B[q]_{c,b}
\]

### 4.1 Soundness

The main theoretical result of this section is that the finite matrix \(B[p]\) fully characterizes the behavior of a dup-free program \(p\) on packets:

**Theorem 4.1 (Soundness).** For any dup-free program \(p\) and any sets \(a, b \subseteq 2^{PK}\), \(B[p^*]\) is well-defined, \(B[p]\) is a stochastic matrix, and \(B[p]_{a,b} = \llbracket p \rrbracket(a)\{b\}\).

The proof of this theorem relies on two key insights about the original semantics \(\llbracket - \rrbracket\). If \(p\) is dup-free and \(a\) is a set of packets, then

1. \(\llbracket p \rrbracket(a)\{b\}\) is a discrete distribution on packet sets (Lemma 4.2); and
2. the measures \(\llbracket p^{(n)} \rrbracket(a)\{b\}\) converges pointwise to \(\llbracket p^* \rrbracket(a)\{b\}\) as \(n \to \infty\) (Lemma 4.3).

The first fact is intuitively obvious, but the proof requires some care in the case of \(p^*\):

**Lemma 4.2.** Let \(p\) be dup-free and \(a \subseteq PK\). Then \(\llbracket p \rrbracket(a)(2^{PK}) = 1\).

**Proof Sketch.** By structural induction on \(p\) (see Appendix A) – here we will only discuss the case of \(p^*\). We let \(\mu_n \triangleq \llbracket p^{(n)} \rrbracket(a)\) and \(\mu \triangleq \llbracket p^* \rrbracket(a)\) and assume \(\mu_n(2^{PK}) = 1\). The key is to notice that the complement of \(2^{PK}\) is a finite boolean combination of basic open sets: \(2^{H} - 2^{PK} = \bigcap_{\pi \in PK} B(\pi)\). For such sets, we are guaranteed pointwise convergence of \(\mu_n\) to \(\mu\) (Lemma 3.2), and so \(\mu(2^{H} - 2^{PK}) = \lim_{n \to \infty} \mu_n(2^{H} - 2^{PK}) = \lim_{n \to \infty} 0 = 0\).

**A useful induction principle.** When proving an hypothesis \(H(p)\) about ProbNetKAT by induction on \(p\), we often assume \(H(p^{(n)})\) when showing \(H(p^*)\). Formally, this is justified by well-founded induction on the smallest relation \(\prec\) on terms satisfying \(p^{(n)} \prec p^*\) for all \(n\), and \(p < q\) whenever \(p\) is a proper subterm of \(q\). Any chain \(p_0 > p_1 > p_2 > \ldots\) decreases lexicographically in (largest \(^*-ed\) term in \(p\), size of \(p\)) and must thus terminate.

**Lemma 4.3.** Let \(p\) be dup-free and \(a \subseteq PK\). Then \(\llbracket p^* \rrbracket(A) = \lim_{n \to \infty} \llbracket p^{(n)} \rrbracket(A)\) for all \(A \in \mathcal{B}\).
Proof. This lemma crucially relies on the assumption that \( p \) is dup-free. (Recall from (8) in § 3.3 that convergence fails, for example, for \( p = \text{dup} \).) This forces \( p^* \) to distribute all probability mass over a finite set of points (by Lemma 4.2), and so no mass can escape to infinity.

Let \( \mu_n \triangleq \| p^{(n)} \| (a) \) and \( \mu \triangleq \| p^* \| (a) \). Because these measures are discrete, it suffices to show

\[
\mu(\{b\}) = \lim_{n \to \infty} \mu_n(\{b\}) \quad \forall b \subseteq \text{P}k
\]

But this fact is guaranteed by Lemma 3.2, because the finite boolean combination of open sets \( B \triangleq B_b \cap \bigcap_{\pi \in \text{P}k-b} \overline{B_{\pi}} \) decomposes disjointly as \( B = \{b\} \cup (B - 2^\text{P}k) \) and thus with Lemma 4.2

\[
\mu(\{b\}) = \mu(B) = \lim_{n \to \infty} \mu_n(B) = \lim_{n \to \infty} \mu_n(\{b\}) \quad \square
\]

With these preparations, the proof of the main result is now a routine induction on \( p \):

Proof of Theorem 4.1. It’s sufficient to show the equality \( \mathcal{B}[\| p \|_{a,b}] = \| p \|_{\{b\}} \); the remaining claims then follow by well-definedness of \( \| - \| \) and by Lemma 4.2. By structural induction on \( p \):

For \( p = 0, 1, f = n, f \leftarrow n \) we have

\[
\| p \|_{a,\{b\}} = \delta_c(\{b\}) = 1[b = c] = \mathcal{B}[\| p \|_{a,b}]
\]

for \( c = \emptyset, a, \{\pi \in a \mid \pi.f = n\}, \{\pi[f := n] \mid \pi \in a\} \), respectively.

For \( \neg t \),

\[
\mathcal{B}[\neg t]_{a,b} = 1[b \subseteq a] \cdot \mathcal{B}[t]_{a,a-b} \\
= 1[b \subseteq a] \cdot \| t \|_{a,\{a - b\}} \quad \text{(IH)} \\
= 1[b \subseteq a] \cdot 1[a - b = a \cap b_T] \quad \text{(Lemma 3.1)} \\
= 1[b \subseteq a] \cdot 1[a - b = a - (H - b_T)] \\
= 1[b = a \cap (H - b_T)] \\
= \| \neg t \|_{a,b} \quad \text{(Lemma 3.1)}
\]

For \( p \& q \), letting \( \mu = \| p \|_{a} \) and \( v = \| q \|_{a} \) we have

\[
\| p \& q \|_{a,\{b\}} = (\mu \times v)((\{b_1, b_2\} \mid b_1 \cup b_2 = b)) \\
= \sum_{b_1, b_2} 1[b_1 \cup b_2 = b] \cdot (\mu \times v)((\{b_1, b_2\})) \quad \text{(\( \omega \)-additivity)} \\
= \sum_{b_1, b_2} 1[b_1 \cup b_2 = b] \cdot \mu(\{b_1\}) \cdot v(\{b_2\}) \\
= \sum_{b_1, b_2} 1[b_1 \cup b_2 = b] \cdot \mathcal{B}[\| p \|_{a,b_1} \cdot \mathcal{B}[q]_{a,b_2}] \quad \text{(IH)} \\
= \mathcal{B}[p \& q]_{a,b}
\]

where we use in the second step that \( b \subseteq \text{P}k \) is finite and thus \( \{(b_1, b_2) \mid b_1 \cup b_2 = b\} \) is finite.
For $p; q$, let $\mu = \llbracket p\rrbracket(a)$ and $\nu_c = \llbracket q\rrbracket(c)$ and recall that $\mu$ is a discrete distribution on $2^{Pk}$ by Lemma 4.2. Thus

$$\llbracket p; q\rrbracket(a, \{b\}) = \int_{c \in 2^k} \nu_c(\{b\}) \cdot \mu(dc)$$

$$= \int_{c \in 2^k} \nu_c(\{b\}) \cdot \mu(dc)$$

$$= \sum_{c \in 2^k} \nu_c(\{b\}) \cdot \mu(\{c\})$$

$$= \sum_{c \in 2^k} \mathcal{B}[q]_{c, b} \cdot \mathcal{B}[p]_{a, c}$$

$$= \mathcal{B}[p; q]_{a, b}$$

For $p \oplus_r q$, the claim follows directly from the induction hypotheses.

For $p^*$, we know that $\mathcal{B}[p^{(n)}]_{a, b} = \llbracket p^{(n)}\rrbracket(a, \{b\})$ by induction hypothesis. The key to proving the claim is Lemma 4.3, which allows us to take the limit on both sides of this equation and deduce

$$\mathcal{B}[p^*]_{a, b} = \lim_{n \to \infty} \mathcal{B}[p^{(n)}]_{a, b} = \lim_{n \to \infty} \llbracket p^{(n)}\rrbracket(a, \{b\}) = \llbracket p^*\rrbracket(a, \{b\})$$

Together, these results reduce the problem of checking the equivalence of two dup-free programs $p$ and $q$ to checking equality of their corresponding big-step matrices $\mathcal{B}[p]$ and $\mathcal{B}[q]$.

**Corollary 4.4.** For dup-free programs $p$ and $q$, $\llbracket p\rrbracket = \llbracket q\rrbracket$ on $2^{Pk}$ if and only if $\mathcal{B}[p] = \mathcal{B}[q]$.

**Proof.** Follows directly from Theorem 4.1 and Lemma 4.2.

Unfortunately, $\mathcal{B}[p^*]$ is defined in terms of a limit. Thus, it is not obvious how to compute the big-step matrix in general. The next section is concerned with finding a closed form for the limit, resulting in an effective decision procedure.

## 5 SMALL-STEP SEMANTICS

This section derives a closed form for $\mathcal{B}[p^*]$, allowing to compute $\mathcal{B}[\_\_\_\_]$ explicitly. This yields an effective mechanism for checking program equivalence on packets.

In the previous section we gave a “big-step” semantics for ProbNetKAT: programs were interpreted as Markov chains over the state space $2^{Pk}$, such that a single step of the chain modeled the entire execution of a program, going directly from some initial state $a$ (corresponding to the set of input packets) to the final state $b$ (corresponding to the set of output packets). Here we will instead take a “small-step” approach and design a Markov chain such that one transition models one iteration of $p^*$.

To a first approximation, the states (or configurations) of our probabilistic transition system are triples $\langle p, a, b \rangle$, consisting of the program $p$ we mean to execute, the current set of (input) packets $a$, and an accumulator set $b$ of packets output so far. The execution of $p^*$ on input $a \subseteq Pk$ starts from the initial state $\langle p^*, a, \emptyset \rangle$. It proceeds by unrolling $p^*$ according to the characteristic equation $p^* \equiv 1 \& p; p^*$ with probability 1:

$$\langle p^*, a, \emptyset \rangle \xrightarrow{1} \langle 1 \& p; p^*, a, \emptyset \rangle$$

To execute a union of programs, we must execute both programs on the input set and take the union of their results. In the particular case of $1 \& p; p^*$, we can immediately execute $1$ by outputting the
input set with probability 1, after which we still have to execute the right hand side of the union:

\[ \langle 1 \cup p ; p^*, a, \emptyset \rangle \xrightarrow{1} \langle p ; p^*, a, a \rangle \]

To execute the sequence \( p ; p^* \), we first execute \( p \) and then feed its (random) output into \( p^* \):

\[ \forall a' : \langle p ; p^*, a, a \rangle \xrightarrow{B[p]_{a,a'}} \langle p^*, a', a \rangle \]

At this point the cycle closes and we are back to executing \( p^* \), albeit with a different input set \( a' \) and some outputs already accumulated. The structure of the resulting Markov chain is summarized in Figure 5.

At this point we notice that the first two steps of execution are deterministic, and so we can collapse all three steps into a single one, as illustrated in Figure 5. After this simplification, the program component of the states is rendered obsolete since it remains constant across transitions. Thus we can eliminate it, resulting in a Markov chain over the state space \( 2^{pk} \times 2^{pk} \). Formally, it can be defined concisely as

\[
S[p] \in S(2^{pk} \times 2^{pk}) \\
\chi_{(a,b),(a',b')} \triangleq 1[b' = b \cup a] \cdot B[p]_{a,a'}
\]

As a first sanity check, we verify that the matrix \( S[p] \) defines indeed a Markov chain:

**Lemma 5.1.** \( S[p] \) is stochastic.

**Proof.** For arbitrary \( a, b \subseteq pk \), we have

\[
\sum_{a', b'} S[p]_{(a,b),(a',b')} = \sum_{a'} \sum_{b'} 1[b' = a \cup b] \cdot B[p]_{a,a'} \\
= \sum_{a'} \left( \sum_{b'} 1[b' = a \cup b] \right) \cdot B[p]_{a,a'} \\
= \sum_{a'} B[p]_{a,a'} = 1
\]

where, in the last step, we use that \( B[p] \) is stochastic (Theorem 4.1).

Next, we show that \( n \) steps in \( S[p] \) model indeed \( n \) iterations of \( p^* \). More formally, the \((n + 1)\)-step behavior of \( S[p] \) is equivalent to the big-step behavior of the \( n \)-th unrolling of \( p^* \) in the following sense:

**Proposition 5.2.** \( B[p^{(n)}]_{a,b} = \sum_{a'} S[p]^{n+1}_{(a,\emptyset),(a',b)} \)
Proof. Naive induction on the number of steps \( n \geq 0 \) fails, because the hypothesis is too weak. We must first generalize it to apply to arbitrary start states in \( S[p] \), not only those with empty accumulator. The appropriate generalization of the claim turns out to be:

**Lemma 5.3.** Let \( p \) be dup-free. Then for all \( n \in \mathbb{N} \) and \( a, b, b' \subseteq P_k \),

\[
\sum_{a'} 1[b' = a' \cup b] \cdot B[p^{(n)}]_{a, a'} = \sum_{a'} S[p]_{(a, b), (a', b')}^{n+1}
\]

**Proof.** By induction on \( n \geq 0 \). For \( n = 0 \), we have

\[
\sum_{a'} 1[b' = a' \cup b] \cdot B[p^{(0)}]_{a, a'} = \sum_{a'} 1[b' = a' \cup b] \cdot B[1]_{a, a'}
\]

\[
= \sum_{a'} 1[b' = a' \cup b] \cdot 1[a = a']
\]

\[
= 1[b' = a \cup b]
\]

\[
= 1[b' = a \cup b] \cdot \sum_{a'} B[p]_{a, a'}
\]

\[
= \sum_{a'} S[p]_{(a, b), (a', b')}^{1}
\]

In the induction step \( (n > 0) \),

\[
\sum_{a'} 1[b' = a' \cup b] \cdot B[p^{(n)}]_{a, a'}
\]

\[
= \sum_{a'} 1[b' = a' \cup b] \cdot B[1 \& p ; p^{(n-1)}]_{a, a'}
\]

\[
= \sum_{a'} 1[b' = a' \cup b] \cdot \sum_{c} 1[a' = a \cup c] \cdot B[p ; p^{(n-1)}]_{a, c}
\]

\[
= \sum_{c, k} \left( \sum_{a'} 1[b' = a' \cup b] \cdot 1[a' = a \cup c] \right) \cdot \sum_{k} B[p]_{a, k} \cdot B[p^{(n-1)}]_{k, c}
\]

\[
= \sum_{c, k} 1[b' = a \cup c \cup b] \cdot B[p]_{a, k} \cdot B[p^{(n-1)}]_{k, c}
\]

\[
= \sum_{k} B[p]_{a, k} \cdot \sum_{a'} 1[b' = a' \cup (a \cup b)] \cdot B[p^{(n-1)}]_{k, a'}
\]

\[
= \sum_{k} B[p]_{a, k} \cdot \sum_{a'} S[p]_{(k, a \cup b), (a', b')}^{n}
\]

\[
= \sum_{a'} \sum_{k_1, k_2} 1[k_2 = a \cup b] \cdot B[p]_{a, k_1} \cdot S[p]_{(k_1, k_2), (a', b')}^{n}
\]

\[
= \sum_{a'} \sum_{k_1, k_2} S[p]_{(a, b), (k_1, k_2)} \cdot S[p]_{(k_1, k_2), (a', b')}^{n}
\]

\[
= \sum_{a'} S[p]_{(a, b), (a', b')}^{n+1}
\]

Proposition 5.2 then follows by instantiating Lemma 5.3 with \( b = \varnothing \). ☐

### 5.1 Closed form

Let \((a_n, b_n)\) denote the random state of the Markov chain \( S[p] \) after taking \( n \) steps starting from \((a, \varnothing)\). We are interested in the distribution of \( b_n \) for \( n \to \infty \), since this is exactly the distribution
of outputs generated by \( p' \) on input \( a \) (by Proposition 5.2 and the definition of \( \mathcal{B}[p'] \)). Intuitively, the \( \infty \)-step behavior of \( \mathcal{S}[p] \) is equivalent to the big-step behavior of \( p' \). The limiting behavior of finite state Markov chains has been well-studied in the literature (e.g., see [13]), and we can exploit these results to obtain a closed form by massaging \( \mathcal{S}[p] \) into a so called absorbing Markov chain.

A state \( s \) of a Markov chain \( T \) is called absorbing if it transitions to itself with probability 1:

\[
\begin{array}{c}
\text{S} \rightarrow 1 \\
\end{array}
\]

(formally: \( T_{s,s'} = 1[s = s'] \))

A Markov chain \( T \in \mathcal{S}(S) \) is called absorbing if each state can reach an absorbing state:

\[
\forall s \in S. \exists s' \in S, n \geq 0. \ T_{s,s'}^n > 0 \text{ and } T_{s,s'}^n = 1
\]

The non-absorbing states of an absorbing MC are called transient. Assume \( T \) is absorbing with \( n_t \) transient states and \( n_a \) absorbing states. After reordering the states so that absorbing states appear before transient states, \( T \) has the form

\[
T = \begin{bmatrix}
I & 0 \\
R & Q
\end{bmatrix}
\]

where \( I \) is the \( n_a \times n_a \) identity matrix, \( R \) is an \( n_t \times n_a \) matrix giving the probabilities of transient states transitioning to absorbing states, and \( Q \) is an \( n_t \times n_t \) square matrix specifying the probabilities of transient states transitioning to transient states. Absorbing states never transition to transient states, thus the \( n_a \times n_t \) zero matrix in the upper right corner.

No matter the start state, a finite state absorbing MC always ends up in an absorbing state eventually, i.e. the limit \( T^\infty \triangleq \lim_{n \to \infty} T^n \) exists and has the form

\[
T^\infty = \begin{bmatrix}
I & 0 \\
A & 0
\end{bmatrix}
\]

for an \( n_t \times n_t \) matrix \( A \) of so called absorption probabilities. Indeed, \( A \) can be given in closed form. It satisfies

\[
A = (I + Q + Q^2 + \ldots) R
\]

That is, to transition from a transient state to an absorbing state, the MC can first take an arbitrary number of steps between transient states, before taking a single and final step into an absorbing state. The infinite sum \( X \triangleq \sum_{n \geq 0} Q^n \) satisfies \( X = I + QX \), and solving for \( X \) we get

\[
X = (I - Q)^{-1} \quad\text{and}\quad A = (I - Q)^{-1} R \quad (10)
\]

(We refer the reader to [13] or Lemma A.1 in Appendix A for the proof that the inverse must exist.)

Before we apply this theory to the small-step semantics \( \mathcal{S}[-] \), it will be useful to introduce some MC-specific notation. Let \( T \) be an MC. We write \( s \xrightarrow{T} \ n \ s' \) if \( s \) can reach \( s' \) in precisely \( n \) steps, i.e. if \( T_{s,s'}^n > 0 \); and we write \( s \xrightarrow{T} s' \) if \( s \) can reach \( s' \) in any number of steps, i.e. if \( T_{s,s'}^n > 0 \) for any \( n \geq 0 \). Two states are said to communicate, denoted \( s \xleftarrow{T} s' \), if \( s \xrightarrow{T} s' \) and \( s' \xrightarrow{T} s \). The relation \( \xleftarrow{T} \) is an equivalence relation, and its equivalence classes are called communication classes. A communication class is called absorbing if it cannot reach any states outside the class. We sometimes write \( \Pr[s \xrightarrow{T} \ n \ s'] \) to denote the probability \( T_{s,s'}^n \). For the rest of the section, we fix a dup-free program \( p \) and abbreviate \( \mathcal{B}[p] \) as \( B \) and \( \mathcal{S}[p] \) as \( S \).

Of central importance are what we will call the saturated states of \( S \):

**Definition 5.4.** A state \( (a,b) \) of \( S \) is called saturated if the accumulator \( b \) has reached its final value, i.e. if \( (a,b) \xrightarrow{S} (a',b') \) implies \( b' = b \).
Once we have reached a saturated state, the output of $p^*$ is determined. The probability of ending up in a saturated state with accumulator $b$, starting from an initial state $(a, \emptyset)$, is

$$\lim_{n \to \infty} \sum_b S^n(a, \emptyset, (a', b))$$

and indeed this is the probability that $p^*$ outputs $b$ on input $a$ by Proposition 5.2. Unfortunately, a saturated state is not necessarily absorbing. To see this, assume there exists only a single field $f$ ranging over $\{0, 1\}$ and consider the program $p^* = (f \leftarrow 0 \oplus \frac{1}{2} f \leftarrow 1)$. Then $S$ has the form

where all edges are implicitly labeled with $\frac{1}{2}$, 0 denotes the packet with $f$ set to 0 and 1 denotes the packet with $f$ set to 1, and we omit states not reachable from $(0, \emptyset)$. The two right most states are saturated; but they communicate and are thus not absorbing.

We can fix this by defining the auxiliary matrix $U \in S(2^{P_k} \times 2^{P_k})$ as

$$U_{(a, b), (a', b')} \triangleq 1[b' = b] \cdot \begin{cases} 1[a' = \emptyset] & \text{if } (a, b) \text{ is saturated} \\ 1[a' = a] & \text{else} \end{cases}$$

It sends a saturated state $(a, b)$ to the canonical saturated state $(\emptyset, b)$, which is always absorbing; and it acts as the identity on all other states. In our example, the modified chain $SU$ looks as follows:

To show that $SU$ is always an absorbing MC, we first observe:

**Lemma 5.5.** $S$, $U$, and $SU$ are monotone in the following sense: $(a, b) \xrightarrow{S} (a', b')$ implies $b \subseteq b'$ (and similar for $U$ and $SU$).

**Proof:** For $S$ and $U$ the claim follows directly from their definitions. For $SU$ the claim then follows compositionally. \qed

Now we can show:

**Proposition 5.6.** Let $n \geq 1$.

1. $(SU)^n = S^n U$
2. $SU$ is an absorbing MC with absorbing states $\{(\emptyset, b) \mid b \subseteq P_k\}$.

**Proof:**

1. It suffices to show that $USU = SU$. Suppose that

$$\Pr[(a, b) \xrightarrow{SU} (a', b')] = p > 0.$$

It suffices to show that this implies

$$\Pr[(a, b) \xrightarrow{USU} (a', b')] = p.$$
If \((a, b)\) is saturated, then we must have \((a', b') = (\emptyset, b)\) and

\[
\Pr[(a, b) \xrightarrow{SU} (\emptyset, b)] = 1 = \Pr[(a, b) \xrightarrow{SU} (\emptyset, b)]
\]

If \((a, b)\) is not saturated, then \((a, b) \xrightarrow{U} (a, b)\) with probability 1 and therefore

\[
\Pr[(a, b) \xrightarrow{SU} (a', b')] = \Pr[(a, b) \xrightarrow{SU} (a', b')]
\]

(2) Since \(S\) and \(U\) are stochastic, clearly \(SU\) is a MC. Since \(SU\) is finite state, any state can reach an absorbing communication class. (To see this, note that the reachability relation \(\xrightarrow{SU}\) induces a partial order on the communication classes of \(SU\). Its maximal elements are necessarily absorbing, and they must exist because the state space is finite.) It thus suffices to show that a state set \(C \subseteq 2^\mathbb{P}_k \times 2^\mathbb{P}_k\) in \(SU\) is an absorbing communication class iff \(C = \{(\emptyset, b)\}\) for some \(b \subseteq \mathbb{P}_k\).

\(\Leftarrow\): Observe that \(\emptyset \xrightarrow{B} a'\) iff \(a' = \emptyset\). Thus \((\emptyset, b) \xrightarrow{SU} (a', b')\) iff \(a' = \emptyset\) and \(b' = b\), and likewise \((\emptyset, b) \xrightarrow{SU} (a', b')\) iff \(a' = \emptyset\) and \(b' = b\). Thus \((\emptyset, b)\) is an absorbing state in \(SU\) as required.

\(\Rightarrow\): First observe that by monotonicity of \(SU\) (Lemma 5.5), we have \(b = b'\) whenever \((a, b) \xrightarrow{SU} (a', b')\); thus there exists a fixed \(b_C\) such that \((a, b) \in C\) implies \(b = b_C\).

Now pick an arbitrary state \((a, b_C) \in C\). It suffices to show that \((a, b_C) \xrightarrow{SU} (\emptyset, b_C)\), because that implies \((a, b_C) \xrightarrow{SU} (\emptyset, b_C)\), which in turn implies \(a = \emptyset\). But the choice of \((a, b_C) \in C\) was arbitrary, so that would mean \(C = \{(\emptyset, b_C)\}\) as claimed.

To show that \((a, b_C) \xrightarrow{SU} (\emptyset, b_C)\), pick arbitrary states such that

\[
(a, b_C) \xrightarrow{S} (a', b') \xrightarrow{U} (a'', b'')
\]

and recall that this implies \((a, b_C) \xrightarrow{SU} (a'', b'')\) by claim (1). Then \((a'', b'') \xrightarrow{SU} (a, b_C)\) because \(C\) is absorbing, and thus \(b_C = b' = b''\) by monotonicity of \(S, U\), and \(SU\). But \((a', b')\) was chosen as an arbitrary state \(S\)-reachable from \((a, b_C)\), so \((a, b)\) and by transitivity \((a', b')\) must be saturated. Thus \(a'' = \emptyset\) by the definition of \(U\).

Arranging the states \((a, b)\) in lexicographically ascending order according to \(\subseteq\) and letting \(n = |2^\mathbb{P}_k|\), it then follows from Proposition 5.6.2 that \(SU\) has the form

\[
SU = \begin{bmatrix} I_n & 0 \\ R & Q \end{bmatrix}
\]

where for \(a \neq \emptyset\)

\[
(SU)_{(a, b), (a', b')} = \begin{bmatrix} R & Q \end{bmatrix}_{(a, b), (a', b')}
\]

Moreover, \(SU\) converges and its limit is given by

\[
(SU) = \lim_{n \to \infty} (SU)^n = \begin{bmatrix} I_n & 0 \\ (I - Q)^{-1}R & 0 \end{bmatrix}
\]

We can use the modified Markov chain \(SU\) to compute the limit of \(S\):

**Theorem 5.7 (Closed Form).** Let \(a, b, b' \subseteq \mathbb{P}_k\). Then

\[
\lim_{n \to \infty} \sum_{a'} S^n_{(a, b), (a', b')} = (SU)^{\infty}_{(a, b), (\emptyset, b')}
\]
or, using matrix notation,
\[
\lim_{n \to \infty} \sum_{a'} S^n_{(-,-),(a',-)} = \left[ I_n - (I - Q)^{-1} R \right] \in [0,1]^{2^{|p_k|} \times 2^{|p_k|}} \tag{13}
\]

In particular, the limit in (12) exists its analytical value is computable.

**Proof.** Using Proposition 5.6.1 in the second step and equation (11) in the last step,
\[
\lim_{n \to \infty} \sum_{a'} S^n_{(a,b),(a',b')} = \lim_{n \to \infty} \sum_{a'} (S^n U)_{(a,b),(a',b')} = \lim_{n \to \infty} \sum_{a'} (SU)^n_{(a,b),(a',b')} = \sum_{a'} (SU)_{(a,b),(a',b')} = (SU)^\infty_{(a,b),(\emptyset,b')}
\]

$(SU)^\infty$ is computable because $S$ and $U$ are matrices over $\mathbb{Q}$ and hence so is $(I - Q)^{-1} R$. \hfill \Box

**Corollary 5.8.** For $\mathit{dup}$-free programs $p$ and $q$, it is decidable whether $\|p\|_\mathcal{B}(a) = \|q\|_\mathcal{B}(a)$ for all $a \subseteq P_k$.

**Proof.** Recall from Cor 4.4 that it suffices to compute the finite rational matrices $\mathcal{B}[p]$ and $\mathcal{B}[q]$ and check them for equality. But Theorem 5.7 together with Proposition 5.2 gives us an effective mechanism to compute $\mathcal{B}[-]$ in the case of Kleene star, and $\mathcal{B}[-]$ is straightforward to compute in all other cases.

To summarize, we repeat the full chain of equalities we have deduced:
\[
\|p^*(a,\{b\}) = \mathcal{B}[p^*]_{a,b} = \lim_{n \to \infty} \mathcal{B}[p^{(n)}] = \lim_{n \to \infty} \sum_{a'} S[p]_{(a,\emptyset),(a',b')} = (SU)^\infty_{(a,\emptyset),(\emptyset,b')}
\]

(From left to right: Theorem 4.1, Definition of $\mathcal{B}[-]$, Proposition 5.2, and Theorem 5.7.) \hfill \Box

### 6 IMPLEMENTATION

We have built a simple OCaml prototype that—given a $\mathit{dup}$-free program $p$—computes the big-step matrix $\mathcal{B}[p]$, using the closed form (13) from § 5.1 to handle unbounded iteration. The resulting matrices can then be compared for equality to check program equivalence (recall Corollary 5.8) or they can be analyzed otherwise, e.g. to compute probabilities or expected values.

Although our implementation is naive and largely unoptimized, it employs a few key ideas to translate our theoretical results into more practical algorithms:

- **Up-to equivalence**: Although the assumption of a finite packet space $P_k$ is accurate—e.g., an IPv4 header is limited to 20 bytes, so there are "only" $2^{160}$ different header instances—explicitly expressing all packets is infeasible. In practice, networks typically make forwarding decision based on a few bits only, and we can consider the much smaller space of packet equivalence classes $P_k / \sim$ induced by the particular network model.
- **No multicast**: To reduce the state space further, we restrict the language in a way that guarantees no proper packet sets will be generated and consider singleton sets (and $\emptyset$) only.
- **Sparsity**: We use a sparse matrix encoding, exploiting that the uncertainty in networks is typically limited to a few possible outcomes. (For comparison, we also implemented a dense version.)
- **Optimized back-ends**: We use the linear algebra libraries Eigen [9] and LAPACK/BLAS [2] to efficiently perform matrix operations on sparse and dense matrices, respectively.
Restricted Language. Our implementation supports only a restricted version of ProbNetKAT in which union \((p \& q)\) and iteration \((p^\ast)\) are replaced with “guarded union” (if \(t\) then \(p\) else \(q\)) and “guarded iteration” (while \(t\) do \(p\)):

\[
\text{if } t \text{ then } p \text{ else } q \triangleq t ; p \& \neg t ; q \quad \text{while } t \text{ do } p \triangleq (t ; p)^\ast ; \neg t
\]

This restriction guarantees that programs can output at most a single packet (assuming a single input). We sacrifice the ability to model multicast but in return can work over the exponentially smaller state space \(P_k \cup \{\emptyset\}\) rather than \(\mathcal{P}^k\).

Up-to Equivalence. The idea of partitioning the packet space into equivalence classes is common in network verification (see for example \([12, 14]\)). In NetKAT, we can characterize a suitable equivalence relation on packets syntactically. For a program \(p\), let \(f_1, \ldots, f_k\) denote the set of fields that appear in \(p\). Likewise, let \(V_i\) denote the set of values that appear with field \(f_i\) in \(p\), either in form of a test \(f_i = v\) or an assignment \(f_i \leftarrow v\). Then we use tuples \(\pi \in (V_1 \cup \{\ast\}) \times \cdots \times (V_k \cup \{\ast\})\) to represent classes of packets that the program \(p\) cannot distinguish. For example, if \(k = 2\) then the tuple \(\pi = (4, \ast)\) represents the set of packets \(\pi\) satisfying \(\pi.f_1 = 4\) and \(\pi.f_2 \neq v\) for all \(v \in V_2\). This approach was pioneered by Yang and Lam \([29]\). A similar idea was also employed in the decision procedure for deterministic NetKAT \([5]\).

For efficient indexing, we map the symbolic packets \(\widetilde{\pi}\) to distinct consecutive integers. This can be done in two simple steps: first, we apply a bijective mapping into the hypercube \([1, \ldots, |V_i|]\) of consecutive integers. Then, we apply a bijective mapping from the hypercube into \([0, \ldots, n-1]\), where \(n\) is the number of distinct symbolic packets given by \(n \triangleq |V_1|(|V_1| + 1)\). The two bijections and their inverses are easy to implement and compute. We can thus represent the big-step matrix as a \(n \times n\) square matrix of 64-bit floating point numbers, using a sparse encoding in the Eigen-based implementation and a dense encoding in the BLAS-based version. Note that the state space does not include the empty set explicitly: we use sub-stochastic matrices in which the probability of \(\emptyset\) is implicitly given by 1 minus the sum of the explicit probabilities. In particular, \(B[0]\) is represented simply as the \(n \times n\) zero matrix.

Small-Step Semantics. The restricted language admits an especially elegant and efficient small-step semantics to compute \(B[\text{while } t \text{ do } p]\). We have\(^4\)

\[
\text{while } t \text{ do } p \equiv \text{if } t \text{ then } (p ; \text{while } t \text{ do } p) \text{ else } 1 \\
\equiv \text{if } t \text{ then } (p ; \text{while } t \text{ do } p) \text{ else } (\text{while } t \text{ do } p) \\
\equiv (\text{if } t \text{ then } p \text{ else } 1) ; (\text{while } t \text{ do } p) \\
\equiv (\text{if } t \text{ then } p \text{ else } 1) ; (\text{if } t \text{ then } p \text{ else } 1) ; (\text{if } t \text{ then } p \text{ else } 1) ; \ldots
\]

Thus, \(B[\text{while } t \text{ do } p] = \lim_{n \to \infty} B[\text{if } t \text{ then } p \text{ else } 1]^n\). When restricted to the state space of packets (rather than packet sets), the Markov chain on the right turns out to be absorbing, with the absorbing states being the empty set and all packets that do not satisfy the loop condition. Thus we can directly apply the closed form from Section 5 to compute the limit.

Complexity. As above, let \(n\) denote the number of symbolic packets, and let \(|p|\) denote the size of the program. To compute \(B[|p|]\), our implementation requires \(|p|\) computations involving matrices of dimension \(n \times n\). Thus, the worst case complexity is bounded by \(O(|p|n^\alpha)\), where the constant \(2 \leq \alpha < 3\) is such that \(O(n^\alpha)\) denotes the complexity of matrix multiplication/inversion, which dominate all other operations. While the algorithm is only linear in the program size, it suffers from the large "constant" \(n\), which grows exponentially in the number of fields (but only linearly

\(^4\)Note that unlike deterministic NetKAT, probabilistic NetKAT does not satisfy the unguarded analog \(p^\ast \equiv (p \& 1) ; (p \& 1) ; (p \& 1) ; \ldots\) of this equivalence in general.
in the number of a field’s values). A naive implementation of the decision procedure for the full dup-free language would require construction and inversion of an \( O((2^n)^2) = O(2^{2n}) \) matrix in the case of \( p^* \), since the small-step matrix is defined over the state space \( 2^{pk} \times 2^{pk} \). The worst case complexity of the algorithm is thus bounded by \( O(|p| \cdot 2^{2n\alpha}) \). A more detailed discussion can be found in Appendix B.

Floating point arithmetic. The decision procedure developed in the previous sections is correct under exact rational arithmetic. In our implementation, we use exact arithmetic in the frontend, but revert to 64-bit floating point arithmetic in the backend. Checking the resulting matrices for equality may still be sound, but formally would have to be justified by a numerical argument showing that there is no loss in precision. Such an analysis is beyond the scope of this paper. In practice, we have observed no numerical instabilities.

7 TOWARDS A FULL DECISION PROCEDURE

We have shown that the equivalence of dup-free programs is decidable. The decision problem for arbitrary programs has so far resisted our efforts and remains open. Indeed, Kahn [11] has shown that certain closely related decision problems in ProbNetKAT are undecidable. However, we have made some initial progress in this direction, including an intriguing reduction that shows that two programs are equivalent iff they agree on sets of histories of length two (histories of length one do not suffice). In this section, we share these insights.

In the absence of probabilistic choice \( \oplus \), programs are uniquely determined by their action on single packets. In fact, in non-probabilistic NetKAT [1], programs are defined as functions of type \( \text{Pk} \rightarrow 2^\text{H} \) that are then lifted to \( \text{H} \rightarrow 2^\text{H} \) and \( 2^\text{H} \rightarrow 2^\text{H} \). In ProbNetKAT, deterministic programs can be similarly defined as functions of type \( \text{Pk} \rightarrow 2^\text{H} \) that are embedded into \( 2^\text{H} \rightarrow \mathcal{D}(2^\text{H}) \) by first lifting them to \( \text{H} \rightarrow 2^\text{H} \), then to \( 2^\text{H} \rightarrow 2^\text{H} \), and finally to \( 2^\text{H} \rightarrow \mathcal{D}(2^\text{H}) \).

However, in the presence of \( \oplus \), programs are no longer uniquely determined by their action on subsets of \( \text{Pk} \), even for dup-free and \( ^*\)-free programs (Theorem 7.1). Surprisingly, we show that two programs are equivalent iff they agree on the set \( \{ \pi \pi \mid \pi \in \text{Pk} \} \), a set of packet histories of length two (Theorem 7.2). This is quite remarkable, as it shows that the equivalence of two programs reduces to their equivalence on a single input. The proof involves an alternative correlated semantics that is possibly of independent technical interest.

We first show that probabilistic programs, even dup-free and \( ^*\)-free ones, are not determined by their actions on sets of packets (histories of length one).

**Theorem 7.1.** There exist dup-free, \( ^*\)-free probabilistic programs \( p, q \) such that \( \|p\| \) and \( \|q\| \) agree on \( 2\text{Pk} \) but not on \( 2^\text{H} \).

**Proof.** Suppose \( \text{Pk} = \{\pi, \rho\} \). Write \( \pi! \) for the program that rewrites the input packet to \( \pi \), and \( \pi? \) for the test that succeeds precisely on \( \pi \) (and similar for \( \rho \)). Consider the two programs

\[
p \triangleq (\pi?; \text{drop} \& \rho?; \text{swap}) \oplus (\pi?; \text{swap} \& \rho?; \text{drop}) \oplus \text{skip}
\]

\[
q \triangleq (\pi?; \text{skip} \& \rho?; \text{drop}) \oplus (\pi?; \text{drop} \& \rho?; \text{skip}) \oplus \text{swap},
\]

where \( \text{swap} = (\pi?; \rho!) \& (\rho?; \pi!) \). It is easily checked that both programs give

\[
\{\pi\}, \{\pi\} \mapsto \frac{1}{3} \delta_{(\pi)} + \frac{1}{3} \delta_{(\rho)} + \frac{1}{3} \delta_{(\varnothing)} \quad \{\pi, \rho\} \mapsto \frac{1}{3} \delta_{(\pi)} + \frac{1}{3} \delta_{(\rho)} + \frac{1}{3} \delta_{(\pi, \rho)},
\]

and of course \( \varnothing \mapsto \delta_{\varnothing} \), whereas on \( \{\pi \pi, \rho \rho\} \) the two programs give

\[
\frac{1}{3} \delta_{(\pi \rho)} + \frac{1}{3} \delta_{(\rho \pi)} + \frac{1}{3} \delta_{(\pi, \rho \pi, \rho \rho)} \quad \frac{1}{3} \delta_{(\pi \pi)} + \frac{1}{3} \delta_{(\rho \rho)} + \frac{1}{3} \delta_{(\rho \pi, \pi \rho)},
\]

respectively. \( \square \)
On the other hand, the following theorem shows that the equivalence of programs reduces to their equivalence on a single input set of histories of length two.

**Theorem 7.2.** \( \|p\| = \|q\| \) iff \( \|p\|(a) = \|q\|(a) \), where \( a = \{\pi \pi \ | \ \pi \in P_k\} \).

The proof of this theorem depends on the development of an alternative correlated semantics of type \( \langle p \rangle : (2^H)^I \rightarrow D((2^H)^I) \), where \( I \) is any index set. Intuitively, \( \langle p \rangle \) takes an \( I \)-tuple of input sets \( (a_n \ | \ n \in I) \) and performs the same actions on all the sets simultaneously. This allows us to express a set \( a \in 2^H \) as a union of subsets in an arbitrary way, run the program on these subsets simultaneously, then take the union of the result, and the outcome is the same as running the program on the whole set \( a \). This works because the same sequences of deterministic actions are performed on all the subsets, maintaining correlation.

The correlated semantics is defined by induction. For example,

\[
\langle p \land q \rangle(a_1, \ldots, a_n) = \text{let } (b_1, \ldots, b_n) = \text{sample } \langle p \rangle(a_1, \ldots, a_n) \text{ in } (c_1, \ldots, c_n) = \text{sample } \langle q \rangle(a_1, \ldots, a_n) \text{ in } (b_1 \cup c_1, \ldots, b_n \cup c_n)
\]

\[
\langle p \oplus_r q \rangle(a_1, \ldots, a_n) = r\langle p \rangle(a_1, \ldots, a_n) + (1 - r)\langle q \rangle(a_1, \ldots, a_n)
\]

\[
\langle p^* \rangle(a_1, \ldots, a_n) = \sup_m \langle p^{(m)} \rangle(a_1, \ldots, a_n),
\]

etc. (Here we are using let \( c = \text{sample } \langle p \rangle(a) \text{ in } \langle q \rangle(c) \text{ as an abbreviation for } \langle p ; q \rangle(a) \). Of course, the important clause is (14), which says that the outcome of each random choice is applied in the same way to each element of the tuple.

**Lemma 7.3.** \( \langle p \rangle(\{\pi\}a) = c = \text{sample } \langle p \rangle(\{\pi\}) \) in \( ca = \{hh \ | \ h \in c, \ h \in a\} \).

**Proof.** See Appendix A.

**Theorem 7.4.** For all \( a_1, \ldots, a_n \),

\[
\langle p \rangle(\bigcup_{i=1}^n a_i) = \text{let } (c_1, \ldots, c_n) = \text{sample } \langle p \rangle(a_1, \ldots, a_n) \text{ in } \bigcup_{i=1}^n c_i.
\]

**Proof.** This is proved by a straightforward induction on the structure of \( p \). See Appendix A.

**Proof of Theorem 7.2.** Let \( d_{\pi}(a) = \{h \ | \ \pi h \in a\} \) be the Brzozowski derivative of \( a \) with respect to \( \pi \). We have

\[
\|p\|(a) = \|p\|(\bigcup_{\pi \in P_k} \{\pi\} d_{\pi}(a))
\]

\[
= \text{let } (c_\pi \ | \ \pi \in P_k) = \text{sample } \langle p \rangle(\{\pi\} d_{\pi}(a) \ | \ \pi \in P_k) \text{ in } \bigcup_{\pi \in P_k} c_\pi
\]

\[
= \text{let } (c_\pi \ | \ \pi \in P_k) = \text{sample } \langle p \rangle(\{\pi\} \ | \ \pi \in P_k) \text{ in } \bigcup_{\pi \in P_k} c_\pi \cdot d_{\pi}(a),
\]

where in (15) we have used Lemma 7.3. This says that \( \|p\| \) is uniquely determined by \( \langle p \rangle(\{\pi\} \ | \ \pi \in P_k) \). If \( a = \{\pi \pi \ | \ \pi \in P_k\} \), then \( d_{\pi}(a) = \{\pi\} \), so

\[
\|p\|(a) = \text{let } (c_\pi \ | \ \pi \in P_k) = \text{sample } \langle p \rangle(\{\pi\} \ | \ \pi \in P_k) \text{ in } \bigcup_{\pi \in P_k} c_\pi \{\pi\}.
\]

But in this case \( \bigcup_{\pi \in P_k} c_\pi \{\pi\} \) determines \( (c_\pi \ | \ \pi \in P_k) \) uniquely, since the elements of \( c_\pi \) are exactly the elements of \( \bigcup_{\pi \in P_k} c_\pi \{\pi\} \) ending in \( \pi \).
Deciding Probabilistic Program Equivalence in NetKAT

Corollary 7.5. $\llbracket p \rrbracket = \llbracket q \rrbracket$ iff $\llbracket \text{dup} : p \rrbracket (\text{Pk}) = \llbracket \text{dup} : q \rrbracket (\text{Pk})$.

Proof. By Theorem 7.2, $\llbracket p \rrbracket$ is uniquely determined by $\llbracket p \rrbracket (a)$, where $a = \{\pi \pi \mid \pi \in \text{Pk}\}$. But $\llbracket \text{dup} \rrbracket (\text{Pk}) = \delta_a$, therefore $\llbracket p \rrbracket$ is uniquely determined by

$$\llbracket p \rrbracket (a) = \llbracket p \rrbracket^\dagger (\delta_a) = \llbracket p \rrbracket^\dagger (\llbracket \text{dup} \rrbracket (\text{Pk})) = \llbracket \text{dup} : p \rrbracket (\text{Pk})$$

The counterexample of Theorem 7.1 may have appeared to fall out of thin air, but in light of Theorem 7.4, it is actually not difficult to produce such counterexamples. For two packets $\pi$, $\rho$,

$$\llbracket p \rrbracket (\{\pi, \rho\}) = \text{let } (c, d) = \text{sample } \llbracket p \rrbracket (\{\pi\}, \{\rho\}) \text{ in } c \cup d.$$

We can create any joint distribution whatsoever on $\{\emptyset, \{\pi\}, \{\rho\}, \{\pi, \rho\}\}^2$ with some $\llbracket p \rrbracket (\{\pi\}, \{\rho\})$. There are 16 outcomes $(a, b)$. We create a tree of depth four with probabilistic branches $\emptyset$, so that the leaf corresponding to $(a, b)$ has the desired probability. At that leaf, we make $\{\pi\}$ generate $a$ and $\{\rho\}$ generate $b$ with the program $(\pi? ; \&_{\sigma \in a} \sigma!) \& (\rho? ; \&_{\sigma \in b} \sigma!)$.

Given such a joint distribution $\llbracket p \rrbracket (\{\pi\}, \{\rho\})$, note that $\llbracket p \rrbracket (\{\pi\})$ and $\llbracket p \rrbracket (\{\rho\})$ are the two marginal distributions, $\llbracket p \rrbracket (\{\pi, \rho\})$ is given by (16), and of course $\llbracket p \rrbracket (\emptyset) = \emptyset$. For $\llbracket p \rrbracket$ and $\llbracket q \rrbracket$ to agree on $\text{O}^{\text{Pk}}$, the two joint distributions $\llbracket p \rrbracket$ and $\llbracket q \rrbracket$ must have the same marginals and the same probabilities (16) of unions and the sum of all the elements must be 1. This gives 13 equations in 16 unknowns, so it is an underconstrained linear system. With $n$ packets, we would get $3 \cdot 2^n + 1$ linear equations in $(2^n)^2$ unknowns.

8 RELATED WORK

One of the key ingredients in this paper is representing the iteration of dup-free ProbNetKAT programs with absorbing Markov chains and exploiting the ability to directly compute limiting distributions on them.

Markov chains have been used by several authors to represent and to analyze probabilistic programs. One of the early references is a paper by Sharir, Pnueli, and Hart [24], presenting a general method for proving properties of probabilistic programs. In their work, a probabilistic program is modeled by a Markov chain and an assertion on the output distribution is extended into an invariant assertion on all intermediate distributions (providing a probabilistic generalization of Floyd’s inductive assertion method). Their approach can assign semantics to infinite Markov chains for infinite processes, using stationary distributions of absorbing Markov chains in a similar way to the one used in this paper. One difference with our work is that their state-space includes the program location, while we can work directly on the space of packets sets.

More recently, Di Pierro, Hankin, and Wiklicky have used probabilistic abstract interpretation (PAl) to statically analyze probabilistic $\lambda$-calculus [3]. They introduce a linear operator semantics (LOS) and demonstrate a strictness analysis, which can be used in deterministic settings to replace lazy with eager evaluation without loss. Their work is later extended to a language called pWhile, using a store plus program location state-space similar to [24]. The language pWhile is, like our restricted version of ProbNetKAT (§ 6), a basic imperative language comprising while-do and if-then-else constructs, but augmented with random choice between program blocks with a rational probability, and limited to a finite number of finitely-ranged variables (in our case, packet fields).

Olejnik, Wicklicky, and Cheraghchi provided a probabilistic compiler pwc for a variation of pWhile [20], implemented in OCaml, together with a testing framework. The pwc compiler has optimizations involving, for instance, the Kronecker product to help control matrix size, and a
Julia backend. Their optimizations based on the Kronecker product might also be applied in, for instance, the generation of $S\parallel p$ from $B\parallel p$, but have not pursued this further yet.

There is a lot of other work on finding explicit distributions of probabilistic programs. Gordon, Henzinger, Nori, and Rajaman surveyed the state of the art with regard to probabilistic inference [8]. They show how stationary distributions on Markov chains can be used for the semantics of infinite probabilistic processes, and how they converge under certain conditions. Similarly to our paper, they use absorbing strongly-connected-components to represent termination.

Markov chains are used in many probabilistic model checkers, of which PRISM [17] is a prime example. PRISM supports analysis of discrete-time Markov chains, continuous-time Markov chains, and Markov decision processes. The models are checked against specifications written in temporal logics like PCTL and CSL. PRISM is written in Java and C++ and provides three model checking engines: a symbolic one with (multi-terminal) binary decision diagrams ((MT)BDDs), a sparse matrix one, and a hybrid. The use of PRISM to analyse ProbNetKAT programs is an interesting research avenue and we intend to explore it in particular when we design the full decision procedure.

9 CONCLUSION

This paper settles an important open question about probabilistic network programs: the decidability of program equivalence for history-free programs. The key technical challenge is overcome by modeling the iteration operator as an absorbing Markov chain, which makes it possible to compute a closed-form solution for its semantics. In future work we are interested in investigating equivalence for full ProbNetKAT, further optimizing our implementation, and exploring applications to additional problems in networks and beyond.

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We now give an alternative proof that relies only on material from results: $A$ is a Cantor-clopen set by [26] (i.e., both $A$ and $\overline{A}$ are Cantor-open), so its indicator function $1_A$ is Cantor-continuous. But $\mu_n \triangleq \|p^{(n)}\|(a)$ converges weakly to $\mu \triangleq \|p^*\|(a)$ in the Cantor topology (Theorem 4 in [4]), so

$$\lim_{n \to \infty} \|p^{(n)}\|(A) = \lim_{n \to \infty} \int 1_A d\mu_n = \int 1_A d\mu = \|p^*\|(A)$$

(To see why $A$ and $\overline{A}$ are open in the Cantor topology, note that they can be written in disjunctive normal form over atoms $B_{(h)}$.)

We now give an alternative proof that relies only on material from Section 3 and gives some more insight into the structure of $(2^H, B)$, at the cost of reiterating some basic results from [26]. First observe that $A$ is a finite boolean combination of sets of the form $B_h \triangleq B_{(h)} = \{a \in 2^H \mid h \in a\}$, because any basic open set $B_a = \bigcap_{h \in a} B_h$ is a finite boolean combination of these simple sets.

Now let $b \in 2^H$ be such that $h \in b$ iff $B_h$ is part of the finite boolean combination $A$; note that $b$ is finite. Consider the boolean algebra $B_b$ generated by $\{B_h\}_{h \in b}$. Its atoms are of the form

$$A_{ab} \triangleq \bigcap_{h \in a} B_h \cap \bigcap_{h \in b-a} \overline{B_h} = B_a - \bigcup_{h \in b-a} B_h = B_a - \bigcup_{a \subseteq b \setminus c} B_c$$

(17)
for \(a \subseteq b\), and, as first observed in [26], by the inclusion-exclusion principle any probability measure \(\mu\) satisfies

\[
\mu(A_{ab}) = \mu(B_a) - \mu\left(\bigcup_{a \subseteq c \subseteq b} B_c\right) = \sum_{a \subseteq c \subseteq b} (-1)^{|c-a|} \cdot \mu(B_c)
\]

(18)

Thus, convergence of \(\mu_n\) to \(\mu\) on the sets \(B_c\) (recall Equation (7)) implies \(\mu_n \to \mu\) on the atoms \(A_{ab}\) as well. But any element of the boolean algebra \(\mathcal{B}_b\), in particular \(A\), is a finite disjoint union of these atoms, so the claim follows by additivity. \(\square\)

**Proof of Lemma 4.2.** By structural induction on \(p\).

For all predicates, the claim follows directly from Lemma 3.1. The claim is obvious for \(f \leftarrow n\).

For \(p \& q\) and \(p \oplus r\), let \(\mu \triangleq \|p\|(a)\) and \(\nu \triangleq \|q\|(a)\). Then

\[
\|p \& q\|(a)(2^{Pk}) = (\mu \times \nu)(\{(b, c) \mid b \cup c \in 2^{Pk}\}) = (\mu \times \nu)(2^{Pk} \times 2^{Pk}) = \mu(2^{Pk}) \cdot \nu(2^{Pk}) = 1
\]

\[
\|p \oplus r\ q\|(a)(2^{Pk}) = r \cdot \mu(2^{Pk}) + (1 - r) \cdot \nu(2^{Pk}) = 1
\]

For \(p ; q\), let \(\mu \triangleq \|p\|(a)\) and \(\nu_b \triangleq \|q\|(b)\). Then

\[
\|p ; q\|(a)(2^{Pk}) = \int_{b \in 2^H} \nu_b(2^{Pk}) \cdot \mu(db) = \sum_{b \in 2^{Pk}} \nu_b(2^{Pk}) \cdot \mu(\{b\}) = \sum_{b \in 2^{Pk}} \mu(\{b\}) = 1
\]

For \(p^*\), let \(\mu_n \triangleq \|p^{(n)}\|(a)\) and \(\mu \triangleq \|p^*\|(a) = \lim_{n \to 0} \mu_n\). Then

\[
\|p^*\|(a)(2^H - 2^{Pk}) = \mu(A_{\emptyset, Pk}) = \sup_{n \to \infty} \mu_n(A_{\emptyset, Pk}) = 0
\]

(see (17))

(by Lemma 3.2)

(by IH)

\(\square\)

**Lemma A.1.** The matrix \(X = I - Q\) in Equation (10) of §5.1 is invertible.

**Proof.** Let \(S\) be a finite set of states, \(|S| = n, M\) an \(S \times S\) substochastic matrix \((M_{st} \geq 0, M1 \leq 1)\). A state \(s\) is **defective** if \((M1)_s < 1\). We say \(M\) is **stochastic** if \(M1 = 1\), **irreducible** if \((\sum_{t=0}^{n-1} M^t)_{st} > 0\) (that is, the support graph of \(M\) is strongly connected), and **aperiodic** if all entries of some power of \(M\) are strictly positive.
We show that if $M$ is substochastic such that every state can reach a defective state via a path in the support graph, then the spectral radius of $M$ is strictly less than 1. Intuitively, all weight in the system eventually drains out at the defective states.

Let $e_s, s \in S$, be the standard basis vectors. As a distribution, $e_s^T$ is the unit point mass on $s$. For $A \subseteq S$, let $e_A = \sum_{s \in A} e_s$. The $L_1$-norm of a substochastic vector is its total weight as a distribution. Multiplying on the right by $M$ never increases total weight, but will strictly decrease it if there is nonzero weight on a defective state. Since every state can reach a defective state, this must happen after $n$ steps, thus $\|e_s^T M^n\|_1 < 1$. Let $c = \max_s \|e_s^T M^n\|_1 < 1$. For any $y = \sum_s a_s e_s$,

$$\|y^T M^n\|_1 = \|\sum_s a_s e_s\|^T M^n \|_1 \leq \sum_s |a_s| \cdot \|e_s^T M^n\|_1 \leq \sum_s |a_s| \cdot c = c \cdot \|y^T\|_1.$$ 

Then $M^n$ is contractive in the $L_1$ norm, so $|\lambda| < 1$ for all eigenvalues $\lambda$. Thus $I - M$ is invertible because 1 is not an eigenvalue of $M$. \hfill $\Box$

**Proof of Lemma 7.3.** It should be clear that

$$[p](\{\pi h\}) = let \ c = sample \ [p](\{\pi\}) in \ c(\h), \quad (19)$$ 

since $[p](\{\pi h\})$ produces the same set of histories that $[p](\{\pi\})$ would produce with $h$ appended, as programs only look at head packets. Then

$$[p](\{\pi\} a) = [p](\bigcup_{h \in a} \{\pi h\})$$ 

$$= let \ (d_h \ | \ h \in a) = sample \ \{p\}(\{\pi h\} \ | \ h \in a) in \ \bigcup_{h \in a} d_h \quad (20)$$ 

$$= let \ (c_h \ | \ h \in a) = sample \ \{p\}(\{\pi\} \ | \ h \in a) in \$$ 

$$let \ (d_h \ | \ h \in a) = (c_h \{h\} \ | \ h \in a) in \ \bigcup_{h \in a} d_h \quad (21)$$ 

$$= let \ c = sample \ \{p\}(\{\pi\}) in$$ 

$$let \ (d_h \ | \ h \in a) = (c_h \{h\} \ | \ h \in a) in \ \bigcup_{h \in a} d_h \quad (22)$$ 

$$= let \ c = sample \ \{p\}(\{\pi\}) in \ \bigcup_{h \in a} c\{h\}$$ 

$$= let \ c = sample \ \{p\}(\{\pi\}) in ca.$$ 

In (20), we have used Theorem 7.4 with a possibly infinite product. Step (21) holds by (19). Finally, (22) holds because all actions are correlated. If the input is a tuple of identical sets, then the outcome will be a tuple with identical sets, that is,

$$[p](b, b, b, \ldots) = let \ c = sample \ [p](b) in \ (c, c, c, \ldots),$$ 

since exactly the same sequence of deterministic actions are applied to every component. \hfill $\Box$
Proof of Theorem 7.4. For \( n \geq 1 \) or \( n = \omega \) and all \( a = (a_i)_{i \leq n} \in (2^H)^n \), we define
\[
\langle p \rangle : (2^H)^n \rightarrow D((2^H)^n)
\]
\[
\langle p \rangle(a) \triangleq \prod_{0 \leq i < n} \llbracket p \rrbracket(a_i) \quad \text{for} \ p = t, f \leftarrow n, \ \text{dup}
\]
\[
\langle p \land q \rangle(a) \triangleq D(\langle b \cdot c \rangle \prod_{0 \leq i < n} (b_i \cup c_i))\langle p \rangle(a) \times \langle q \rangle(a)
\]
\[
\langle p ; q \rangle(a) \triangleq \langle q \rangle(\langle p \rangle(a))
\]
\[
\langle p \lor q \rangle(a) \triangleq r \cdot \langle p \rangle(a) + (1 - r) \cdot \langle q \rangle(a)
\]
\[
\langle p^* \rangle(a) \triangleq \bigsqcup_{n \in \mathbb{N}} \llbracket p^{(n)} \rrbracket(a)
\]
The supremum is taken in the \( n \)-ary product \( CPO \) of \((2^H, \sqsubseteq)\). We use that the probability (or Giry) monad \( D \) can be defined in the category of inductively complete partial orders, following Jones and Plotkin [10]. The fact that \( D \) is a monad is almost all that is needed in the following proof.

Letting \( \cup : (2^H)^n \rightarrow 2^H\) denote \( n \)-ary union, we must show
\[
\llbracket p \rrbracket \circ \cup = D(\cup) \circ \langle p \rangle.
\]
Let \( a \in (2^H)^n = (a_i)_{i \leq n} \) be arbitrary. We proceed by structural induction on \( p \).

For \( p = t, f \leftarrow n, \ \text{dup} \) we know by [26, Lemma 3] that \( \llbracket p \rrbracket(b) = \{f_p(h) \mid h \in b\} \) for some partial function \( H \rightarrow H \). Thus
\[
(D(\cup) \circ \langle p \rangle)(a) = D(\cup)(\prod_{0 \leq i < n} \llbracket p \rrbracket(a_i))
\]
\[
= D(\cup)(\prod_{0 \leq i < n} \delta_{\{f_p(h) \mid h \in a_i\}})
\]
\[
= D(\cup)(\prod_{0 \leq i < n} \{f_p(h) \mid h \in a_i\})
\]
\[
= \delta_{\cup(\{f_p(h) \mid h \in a\})}
\]
\[
= \llbracket p \rrbracket(\cup(a)).
\]

For \( p \land q \), let \( \mu \triangleq \llbracket p \rrbracket(a) \) and \( \nu \triangleq \llbracket q \rrbracket(a) \) and \( f(b, c) = \prod_{0 \leq i < n} (b_i \cup c_i) \) and \( f(b, c) = b \cup c \). Then
\[
(D(\cup) \circ \langle p \land q \rangle)(a) = D(\cup)(D(f)(\mu \times \nu))
\]
\[
= D(\cup \circ f)(\mu \times \nu)
\]
\[
= D(f \circ (\cup \times (\cup))(\mu \times \nu))
\]
\[
= D(f)(D(\cup)(\mu) \times D((\cup)(\nu))
\]
\[
= D(f)(\llbracket p \rrbracket(\cup(a)) \times \llbracket q \rrbracket(\cup(a))
\]
\[
= \llbracket p \land q \rrbracket(\cup(a)).
\]
For \( p \oplus q \), let \( \mu \triangleq (p)(a) \) and \( \nu \triangleq (q)(a) \). Then
\[
(\mathcal{D}(\bigcup) \circ (p \oplus q))(a) = \mathcal{D}(\bigcup)(r \cdot \mu + (1 - r) \cdot \nu)
= r \cdot (\mathcal{D}(\bigcup)(\mu)) + (1 - r) \cdot (\mathcal{D}(\bigcup)(\nu))
= r \cdot \|p\|\bigcup a + (1 - r) \cdot \|q\|\bigcup a
= \|p \oplus q\|\bigcup a.
\]

For \( p : q \),
\[
(\mathcal{D}(\bigcup) \circ (p : q))(a) = \mathcal{D}(\bigcup)((q)^\dagger ((p)(a)))
= (\mathcal{D}(\bigcup) \circ (q))^\dagger ((p)(a))
= \|q\| \bigcup (\mathcal{D}(\bigcup)((p)(a)))
= \|q\| \bigcup (\|p\| \bigcup a)
= \|p : q\| \bigcup a.
\]

Finally, for \( p^* \), observe that \( \bigcup \) is a Scott-continuous function on the product CPO \( \prod_{0 \leq i < n}(2^H, \subseteq) \). Since \( \mathcal{D} \) is a monad in the category of inductively complete partial orders [10], this implies that \( \mathcal{D}(\bigcup) \) is a Scott-continuous function on the CPO \( \mathcal{D}(\prod_{0 \leq i < n}(2^H, \subseteq)) \). Letting \( \mu_n = (p^{(n)})(a) \), we have
\[
(\mathcal{D}(\bigcup) \circ (p^*))(a) = \mathcal{D}(\bigcup)(\bigcup_{n \geq 0} \mu_n)
= \bigcup_{n \geq 0} \mathcal{D}(\bigcup)(\mu_n)
= \bigcup_{n \geq 0} \|p^{(n)}\| \bigcup a
= \|p^*\| \bigcup a.
\]

### B Complexity Analysis

The general decision procedure for equality of dup-free ProbNetKAT programs runs in time linear with the length of the program and exponential with the number of distinct packets. In our restricted language implementation, the latter relation becomes polynomial owing to being able to work on singletons rather than general sets. These complexity bounds can be derived as shown below.

Use \( b \) to denote the dimension of the big-step matrices \( B\|p\| \) used for the program \( p \). Naive matrix operations for parallel composition (\&), sequential composition (;), and random choice (\oplus) take time quartic, cubic, and quadratic with \( b \), the dimension of the matrix, respectively. Each big-step matrix for tests, assignments, drops, and skips can be naively determined element by element in \( O(b^2) \) time. However, these complexities are all dwarfed by that of the asterate (\*), so improvements here do not impact the complexity upper bound.

As shown in Section 5, the matrix for \( B\|p^*\| \) can be determined by creating the matrices \( S\|p\| = S \) and \( U \), computing their product, picking out transient transition submatrices \( Q \) and \( R \) from that
product, and computing \((I - Q)^{-1}R\). This process yields the \(B[p^*]\) entries for transient states; all other absorbing states are already known to transition to themselves with probability 1.

Use \(s = b^2\) to denote the dimension of the small-step matrices \(S[p]\). Naive construction of \(S[p]\) from \(B[p]\) takes \(O(s^2)\) time, as there are \(s^2\) elements. \(U\) can be found by picking out \(S\)'s strongly connected components in \(O(s^2)\) time, topologically sorting them in \(O(s^2)\) time, and determining saturation of the components in order from the terminal end. A component's saturation can be determined solely from the components it directly transitions to: a component is unsaturated if and only if it transitions to a component with more elements in its accumulator set, or to another unsaturated component. By starting at the terminal end of the topological sort, the saturation of all components a given component could transition to must be known when it comes time to determine that component's saturation, so a single pass from the terminal end suffices. The saturation of a component can then be applied directly to its constituent states. So \(U\) can be found in \(O(s^2)\) time total.

Multiplication of \(S\) and \(U\) takes \(O(s^\alpha)\) time, where \(2 \leq \alpha < 3\) will denote the complexity exponent of matrix multiplication. The indices for \(Q\) and \(R\) can be picked out in time linear with \(s\) merely by checking along the diagonal for ones (corresponding to an absorbing transition), and the remaining matrix algebra is limited by multiplication complexity again, \(O(s^\alpha)\).

The complexity upper bound for computing the asterate is therefore dependent on matrix multiplication and is given by \(O(s^\alpha) = O(b^{2\alpha})\). A program of length \(L\) could have \(O(L)\) many asterates, so the big-step matrix for the entire program can be found in \(O(L \cdot b^{2\alpha})\) time. Checking equality on the matrices is as simple as checking element-by-element, so this does not increase the complexity from finding the matrix. In the general case, \(b\) could count every set of packets from \(p\) such that \(b = 2^p\), so this is equivalent to \(O(L \cdot 4^{p\alpha})\), whereas \(b = p + 1\) in our language restricted implementation, for \(O(L \cdot p^{2\alpha})\). We can further count the number of packets \(p\) relevant to the program based on the number of fields used \(f\) in that program, where we can assume without loss of generality that each field is a Boolean, such that the fields refer to individual bits in the packet (and thus packet length is in \(O(f)\)). This yields a complexity of \(O(L \cdot 4^{f \cdot 2\alpha})\) in general case and \(O(L \cdot 4^{f \alpha})\) in the our restricted language implementation, where \(L\) is the program length and \(f\) is the number of fields used.