THE RELATIVE HYPERBOLICITY OF ONE-RELATOR
RELATIVE PRESENTATIONS

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ABSTRACT. We prove that if $G$ is a free-torsion group and $w(t)$ is a word in the
alphabet $G \cup \{t^{\pm 1}\}$ with exponent sum one, then the group $\langle G, t|w(t)^k = 1 \rangle$, where $k \geq 2$, is relatively hyperbolic with respect to $G$.

1. Introduction

Let us recall that a group $\tilde{G}$ is called relatively hyperbolic with respect to a
collection of its subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$ if $\tilde{G}$ admits a relatively finite presentation
$\langle X, \{H_\lambda\}_{\lambda \in \Lambda}|R = 1, R \in \mathcal{R} \rangle$,
where $X$ and $\mathcal{R}$ are finite, satisfying a linear relative isoperimetric inequality. That
is, there exists $C > 0$ such that for every word $w$ in the alphabet $X^{\pm 1} \cup \mathcal{H}$, where
$\mathcal{H} = \sqcup_{\lambda \in \Lambda}(H_\lambda \setminus \{1\})$, representing the identity in the group $\tilde{G}$, there exists an
expression

$$w = \prod_{i=1}^{k} f_i^{-1} R_i^{\pm 1} f_i,$$

with the equality in the group $F = (*_{\lambda \in \Lambda} H_\lambda) * F(X)$, where $F(X)$ is the free group
with basis $X, R_i \in \mathcal{R}, f_i \in F$ and $k \leq C \|w\|$, where $\|w\|$ is the length of the word $w$.

The following theorem belongs to B. Newman [10], see also [8].

**Theorem 1.1.** Any one-relator group

$$\langle a, b, \ldots | R^k \rangle,$$

where $k > 1$, is hyperbolic.

We obtain a similar fact for one-relator relative presentations.

Let $G$ be a torsion-free group and let $t$ be a letter, $t \notin G$.

**Definition 1.2.** The word $w = \prod_{i=1}^{n} g_i t^{\epsilon_i} \in G * \langle t \rangle_\infty$, where $\epsilon_i = \pm 1$, is called
unimodular if $\sum \epsilon_i = 1$.

**Main theorem.** A group

$$\tilde{G} = \langle G, t|w^k = 1 \rangle := G * \langle t \rangle_\infty / \ll w^k \gg,$$

where the word $w$ is unimodular, $k \geq 2$ and $G$ is a torsion-free group, is relatively
hyperbolic with respect to $G$.

The following example shows that if $w$ is not unimodular and $G$ is not torsion-free
group then the theorem is not true.
Example. Suppose that \( G = \langle a \rangle_3 \) and \( \hat{G} = \langle G, t | (a^{-1}t^{-1}at)^3 = 1 \rangle \). Then the elements \( u = t^{-1}ata, v = at^{-1}at \) generate an abelian free group of rank 2, so the group \( \hat{G} \) cannot be relatively hyperbolic with respect to \( G \) (see [11]).

Recall that a group \( H \) is called \( SQ\)-universal if every countable group can be embedded into a quotient of \( H \). In [11] it was proved the \( SQ\)-universality of non-elementary properly relatively hyperbolic groups.

**Corollary 1.3.** Under the conditions of theorem [7] the group \( \hat{G} \) is \( SQ\)-universal.

Theorem [11] implies the following corollary (see [11]).

**Corollary 1.4.** If the word problem is solvable in the group \( G \), then it is solvable in \( \hat{G} \).

One-relator groups have been well studied. In these groups the word problem is solvable [8], [9]. In [3] J. Howie studied the quotient of the free products of the type \( \hat{G} = (A \ast B)/r^n \), where \( n > 3 \). For this type of groups the word problem is also solvable. In [4] A. Juhasz generalized this result for some amalgamated free products. These results were obtained by using small cancellation theory and it is unknown whether these results can be extended to the cases \( n = 2, 3 \).

In this work we use Howie diagrams over a relative presentation and Klyachko’s method of multiple motion on diagrams.

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## 2. Algebraic Lemmata

**Lemma 2.1.** The natural mapping \( G \to \hat{G} \) is injective.

**Proof.** In [5] it was proved that, if a word \( w \) is unimodular, then the natural mapping \( G \to \hat{G} = \langle G, t | w = 1 \rangle = G \ast \langle t \rangle_\infty / \langle \langle w \rangle_\infty \rangle \) is injective. Now, consider the natural mappings \( G \to \hat{G} \to \hat{G} \). If \( G \) does not embed into \( \hat{G} \) then \( G \) does not embed into \( \hat{G} \). This contradiction proves Lemma 2.1.

The following lemma is similar to Lemma 2 from [6].

**Lemma 2.2.** [6] Let the word \( w \) be unimodular and nonconjugate to a word of the form \( gt \), where \( g \in G \). Then the group \( \hat{G} \) has a (relative) presentation of the form

\[
\hat{G} = \langle H, t | \{p^t = p^\varphi, p \in P \setminus \{1\}, (ct \prod_{i=0}^m (b_ia_i))^{k} = 1 \} \rangle,
\]

where \( a_i, b_i, c \in H, P \) and \( P^\varphi \) are isomorphic subgroups of \( H, \varphi : P \to P^\varphi \) is an isomorphism and the following conditions hold:

1. \( m \geq 0 \) (i.e., the product in (1) is nonempty);
2. \( a_i \notin P, b_i \notin P^\varphi \);
3. \( \langle P, a_i \rangle = P \ast \langle a_i \rangle_\infty, \langle P^\varphi, b_i \rangle = P^\varphi \ast \langle b_i \rangle_\infty \) in \( H \);
4. The groups \( H, P, P^\varphi \) are free products of finitely many isomorphic copies of \( G \): \( H = G^{(0)} \ast \ldots \ast G^{(s)}, P = G^{(0)} \ast \ldots \ast G^{(s-1)}, P^\varphi = G^{(1)} \ast \ldots \ast G^{(s)}, \) where \( s \geq 0 \) (for \( s = 0, P \) and \( P^\varphi \) are assumed to be trivial), and the isomorphism \( \varphi \) is the shift: \( (G^{(i)})^\varphi = G^{(i+1)} \).
3. Howie diagrams

Consider a map on an oriented two-dimensional sphere. The corners of this map are labelled by elements of some group $H$. The edges are directed and labelled by the letter $t$.

The label of a vertex is defined as the product of the labels of all corners near this vertex listed clockwise. The label of a vertex is an element of $H$ defined up to conjugation.

To obtain the label of a face, we should walk along its boundary anticlockwise and write down the labels of all its corners and edges; the label of an edge should be written as its inverse if we walk through it against the arrow. The label of a face is an element of the free group of $H$ and the cyclic group with basis $t$, defined up to a cyclic permutation.

Such a labelled map is called a Howie disk diagram over a relative presentation (2) $\langle H, t | w_1 = 1, w_2 = 1, \ldots \rangle$, if

- one face is separated out and called exterior; the remaining faces are called interior;
- the label of each interior face is a cyclic permutation of one of the words $w_i \pm 1$;
- the label of each vertex is the identity element of $H$.

The diagram is said to be reduced if it contains no such edge $e$ that both faces containing $e$ are interior, these faces are different and their labels written starting with the label of edge $e$ are mutually inverse (such a pair of faces with a common edge is called a reducible pair).

The following lemmata are the key ingredients of the proof in the rest of the paper.

**Lemma 3.1.** [2] If the natural mapping from the group $H$ to the group with relative presentation (2) is injective, then the image of an element $u \in H \ast \langle t \rangle_\infty \setminus \{1\}$ is identity in the group (2) if and only if there exists a disk diagram over this presentation whose exterior face is labelled by $u$. A minimal (with respect to the number of faces) such diagram is reduced.

Let $\varphi : P \to P^\varphi$ be an isomorphism between two subgroups of a group $H$. A relative presentation of the form (3) $\langle H, t | p^t = p^\varphi; p \in P \setminus \{1\}, w_1 = 1, w_2 = 1, \ldots \rangle$ is called a $\varphi$-presentation. A diagram over a $\varphi$-presentation (3) is called $\varphi$-reduced if it is reduced and different interior cells with labels of the form $p^t p^{-\varphi}, p \in P$, have no common edges.

**Lemma 3.2.** [6] The complete $\varphi$-analog of lemma 3.1 is valid, i.e any minimal diagram over presentation (3) whose exterior face is labelled by $u$ is $\varphi$-reduced.

A multiple motion of period $T \in \mathbb{R}$ on a diagram with faces $\{D_i, i \in I\}$ is called a set of mappings $\{\alpha_{i,j} : \mathbb{R} \to \partial D_i; i \in I, j = 1, \ldots, d_i\}$ (called cars), satisfying the following periodicity conditions:

1) $\alpha_{i,j}(t + T) = \alpha_{i,j+1}(t)$ for any $t \in \mathbb{R}$ and $j \in \{1, \ldots, d_i\}$ (subscripts are modulo $d_i$);
2) there exists such a partitioning of each circle $\partial D_i$ into $d_i$ arcs that during the time interval $[0, T]$ each car $\alpha_{i,j}$ moves along the $j$th arc.

The positive integers $d_i$ are called the \textit{multiplicities} of the multiple motion.

The multiple motion $\alpha_{i,j}$ is called a \textit{multiple motion with separated stops} if every car moves without U-turns and infinite decelerations and accelerations moving around the boundary of its face anticlockwise, possible stopping for a finite time at some corners; and there exists a set of corners called the stop corners such that:

1) the cars stop only at stop corners (possibly, at some stop corners, the cars do not stop);
2) at each vertex $v$ having stop corners at it, the stops are separated in the following sense: let $c_1, ..., c_k$ be all stop corners at $v$ enumerated anticlockwise; it is required that, for each $i$, at corners $c_i$ and $c_{i+1}$ (subscripts are modulo $k$) cars are never located simultaneously (in particular, this implies that $k \geq 2$).

If the number of cars being at a moment $t \in \mathbb{R}$ at a point $p$ of the sphere equals the multiplicity of this point (in other words, either two cars simultaneously pass the same internal point $p$ of an edge at the moment $t$ or there are $k$ cars at a vertex $p$ of degree $k$ at the moment $t$), then we say that a \textit{complete collision} occurs at the point $p$ at the moment $t$; the point $p$ is called a \textit{point of complete collision}.

\textbf{Lemma 3.3.} \[6\] \textit{The number of points of complete collision of a multiple motion with separated stops on a sphere cannot be smaller than}

$$2 + \sum_{i \in I} (d_i - 1),$$

\textit{where $d_i$ are the multiplicities of the multiple motion.}

\section{4. Standard multiple motion}

In this section we will define a multiple motion on Howie diagram over the presentation \[1\].

A Howie diagram can have corners of four kinds: $(++)$, $(-)$, $(+-)$ and $(-+)$ (fig. 2).

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig2.png}
\caption{fig. 2}
\end{figure}

A vertex of the kind

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{fig3.png}
\caption{fig. 3}
\end{figure}

is called a \textit{sink},

and a vertex of the kind
is called a source.

The following lemma is obvious.

**Lemma 4.1.** In the anticlockwise listing of the corners at a vertex $v$, the corners of type $(++)$ alternate with corners of type $(--)$. If at a vertex $v$ there are no corners of type $(++)$, or, equivalently, there are no corners of type $(--)$ then either all corners at $v$ are of type $(+-)$ (the vertex is a sink), or all corners at $v$ are of type $(--)$ (the vertex is a source).

Now we define a standard multiple motion on the interior cells of a disk diagram over presentation (1), which is similar to the standard motion in [6].

- the car going around a face with label $p^{-}p'$ moves anticlockwise uniformly with unit speed (one edge per unit time) visiting the corner of type $(+-)$ at the even moments of time (fig.5a);
- on the boundary of a face with label $(ct \prod_{i=0}^{m} b_{i}a_{i})^{\pm 1}$ there are $k$ moving cars. Each of them at moment zero is in a corner with label $b_{0}^{\pm 1}$ and moves anticlockwise during the time intervals $[0, 4m + 2]$ on its own arc with label $(ct \prod_{i=0}^{m} b_{i}a_{i})^{\pm 1}$;
- for $m > 0$ the car moving on an arc with label $ct \prod_{i=0}^{m} b_{i}a_{i}$ stays at the corner of type $(++)$ during the time intervals $[2m + 2, 4m + 1] + (4m + 2)\mathbb{Z}$ and moves uniformly with unit speed all the remaining time; for $m = 0$ such a car moves without stops with speed 2 on the positive edges and with speed 1 on the negative ones (fig.5b);
- for $m > 0$ the car moving on an arc with label $(ct \prod_{i=0}^{m} b_{i}a_{i})^{-1}$ stays at the corners of type $(--)$ during the time intervals $[1, 2m] + (4m + 2)\mathbb{Z}$ and moves uniformly with unit speed all the remaining time; for $m = 0$ such a car moves without stops with speed 2 on the negative edges and with speed 1 on the positive ones (fig.5c).
0, 2, 4, ... $p^{-\varphi}$ $p$ 1, 3, 5, ... fig.5a

$0, b_0 \rightarrow a_0, a_1, \ldots, a_{m-1}, b_m, a_m, c \rightarrow 4m + 2$

$m > 0$

$0, b_0 \rightarrow a_0 \rightarrow c \rightarrow 2$

$m = 0$

$[1, 2m] \Rightarrow 2m + 1 \Rightarrow 2m + 2 \Rightarrow 4m \Rightarrow 4m + 1$

$m > 0$

$0, b_0^{-1} \rightarrow c_0^{-1} \rightarrow 1/2 \Rightarrow 1$

$m = 0$
Let $u$ be the label of the exterior face. For $m > 0$ we can suppose that on the exterior face there are at least one corner of type $(- -)$ and one of type $(++)$ (it means that in the word $u$ there are subwords of form $t^{-1}g_1t^{-1}$ and $tg_2t$, where $g_1, g_2 \in G$, such subwords will always be found, if necessary, conjugating $u$ by $t^n$, $n > 0$). There is one car moving on the exterior face. During the time intervals $[1, 2m] + (4m + 2)\mathbb{Z}$ and $[2m + 2, 4m + 1] + (4m + 2)\mathbb{Z}$ the car stays at the corners of type $(- -)$ and $(++)$. This car moves uniformly anticlockwise with the same speed all the remaining time.

For $m = 0$ the car moves uniformly with the same speed of period 2.

**Lemma 4.2.** The standard multiple motion on a diagram over relative representation $\langle 1 \rangle$ is a motion with separated stops. In the interior cells the complete collisions can occur only at vertices being sources or sinks.

*Proof.* Let all corners of types $(++)$ and $(- -)$ be the stop corners. Then, the stops are separated by Lemma 4.1 and the fact that the schedule of the standard motion is such that cars are never located at the same time at corners of type $(++)$ and $(- -)$.

A collision on an edge at a moment $t$ means that at this moment the direction of the motion of one of the cars coincides with the direction of the edge, while the direction of the motion of the other colliding car is opposite to the direction of the edge (fig.7)

But the schedule of the standard multiple motion on interior cells is such that, at each moment $t$, either all cars being on edges move in the direction of the edge (this is so when the integer part of $t$ is odd), or all cars being on edges move in the direction opposite to the direction of the edge (this is so when the integer part of $t$ is even). Therefore, collisions can occur only at vertices; the separateness of stops implies that a vertex of complete collision can not have stop corners and, therefore, is a source or a sink. The lemma is proven.

5. **Proof of main theorem**

By Lemma 2.1, the natural mapping $G \to \hat{G}$ is injective. So for every word $u \in G * \langle t \rangle_\infty$, where $u = \hat{C} \ 1$, there is a reduced disk diagram over the presentation of $\hat{G}$ with a exterior face with label $u$ (Lemma 3.1). We denote by $|u|$ the number of edges of exterior face of this diagram.
It is well-known that in this case \( u \) has a presentation
\[
 u = \prod_{i=1}^{h} f_i^{-1} w^{k_i} f_i,
\]
where \( h \) is the number of faces of the disk diagram.

We will prove the following inequality
\[
 (4) \quad h < C|u|,
\]
where \( C \) is a constant. In fact, this inequality is equivalent to the inequality in the definition of hyperbolicity, because the number of edges of the exterior face of the disk diagram exactly equals to a half of the length of \( u \).

We consider two cases:

Case 1. \( |w| > 1 \). By Lemma 2.2, \( \hat{G} \) has a relative presentation (1). The construction of \( H \) implies that the natural mapping \( H \to \hat{G} \) is injective, so for all words \( u \) in the alphabet \( H \cup \{t^\pm 1\} \), where \( u = \hat{G} 1 \), there is a \( \varphi \)-reduced disk diagram \( M \) over the presentation (1) whose exterior face is labelled by \( u \) (Lemma 3.2).

We define a standard multiple motion on \( M \) as in section 4. Let us show that all complete collisions of the standard motion on this map occur only at the boundary of exterior face.

Indeed, according to Lemma 4.2, a complete collision on interior cells can occur only at vertices being sinks or sources.

Suppose that a vertex of complete collision is a sink. Then, all of corners at this vertex are of type \((+, -)\); the label of each of these corners is either \( p^\varphi \), where \( p \in P \), or \( b_i^{\pm 1} \). If at this vertex there are a corner labelled by \( b_i^{\pm 1} \) and a corner labelled by \( b_j^{\pm 1} \), where \( i \neq j \), then a complete collision does not occur at this vertex, because these two corners are never visited at the same time. If at this vertex there is a corner labelled by \( b_i^{\pm 1} \) and a corner labelled by \( p^\varphi \) then the label of a vertex being a sink at which a complete collision occurs is
\[
 \prod_j (b_i^{\epsilon_j} p_j^{\varphi}),
\]
where \( \epsilon_j \in \mathbb{Z} \), \( p_j \in P \). But the label of an vertex must be 1, thus we have a nontrivial (because the diagram is \( \varphi \)-reduced) relation of the specified form, which contradicts property 3 from Lemma 2.2.

For the case, when a vertex of complete collision is a source, we can prove in the same way.

Suppose that \( d \) is the number of interior faces with label of type \( p^\varphi p^{-\varphi} \) (fig.5a), \( e \) is the number of other faces (fig.5b, fig.5c), \( f \) is the number of edges of exterior face of the disk diagram \( M \).

By Lemma 3.3, the number of complete collisions is not smaller than \( 2 + \sum_{i=1}^{e} (k - 1) = 2 + e(k - 1) \). Besides complete collision can occur only at the boundary of the exterior face, therefore the number of complete collisions is not greater than the number of edges of the exterior face of \( M \) (because at the boundary of the exterior face the car almost is located at stop corners and in the remaining time moves very quickly, so at each edge of the exterior face there is at most one point of complete collision). Thus, we have \( 2 + e(k - 1) \leq f \), i.e., \( e \leq \frac{f - 2}{k - 1} \).
So over the presentation (1) the word \( u \) has a presentation

\[
\begin{align*}
u &= \prod_{i=1}^{e} f_{i}^{-1} v^{\pm k} f_{i}^{-1} p_{i}^{-\varphi} f_{i},
\end{align*}
\]

where \( f_{i} \in H \ast \langle t \rangle \), \( v = ct \prod_{m=0}^{\infty} (b_{i} a_{i} t) \).

Then in the group \( \tilde{G} \) with the starting presentation \( \tilde{G} = \langle G, t \mid w^{k} = 1 \rangle \), the word \( u = \tilde{G} 1, u \in G \ast \langle t \rangle \) can be represented as

\[
\begin{align*}u &= \prod_{i=1}^{e} f_{i}^{-1} v^{\pm k} f_{i},
\end{align*}
\]

indeed, in the starting presentation of \( \tilde{G} \) all words of the type \( f_{i}^{-1} p_{i}^{-\varphi} f_{i} \) are reduced to the identity element, and the word \( v \) is simply the word \( w \), which is rewritten in new presentation.

Existence of this presentation of \( u \) guarantees the existence of a disk diagram whose exterior face is labelled by \( u \) over the starting presentation of the group \( \tilde{G} \). Moreover, let \( f' \) be the number of edges of the exterior face of this diagram, obviously \( f' \geq f \), thus \( e < \frac{f'}{k-1} \).

Hence we obtain the inequality (1) with constant \( C = \frac{1}{2(k-1)} \).

**Case 2.** \( |w| = 1 \). Then the group \( \tilde{G} \) has a presentation:

\[
\begin{align*}
\tilde{G} &= \langle G, t \mid (gt)^{k} = 1 \rangle \simeq G \ast \langle x \rangle_{k},
\end{align*}
\]

Such group is relative hyperbolic with respect to subgroup \( G \).

6. **Proof of Corollary 1.3**

Let us show that \( G \) is a proper subgroup of the group \( \tilde{G} \) and \( \tilde{G} \) is non-elementary.

The group \( G \) is proper, because the group \( \tilde{G} \) has nature mapping on non-trivial group \( \langle t \rangle_{k} \).

Consider the nature mapping \( \tilde{G} \in \hat{G} = \langle G, t \mid w = 1 \rangle \). In [7] it was proved that the group \( \hat{G} \) contains a nonabelian free subgroup (hence it is non-elementary), except in the following two cases:

1) the group \( \hat{G} \) is isomorphic to the Baumslag-Solitar group \( G_{1,2} = \langle g, t \mid g^{-1} tg = t^{2} \rangle \);

2) the word \( w \) is conjugate to the word \( gt \), where \( g \in G \).

The group \( G_{1,2} \) is non-elementary. So non-elementary quality of \( \tilde{G} \) is followed from non-elementary quality of \( \hat{G} \). In case 2) the group \( \tilde{G} \) has a presentation \( G \ast \langle t \rangle_{k} \), which is non-elementary.

By the Corollary 1.2 from [1], we obtain that the group \( \tilde{G} \) is SQ-universal.

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