CALABI-YAU VARIETIES WITH SEMI-STABLE FIBRE STRUCTURES

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Abstract. Motivated by the Strominger-Yau-Zaslow conjecture, we study Calabi-Yau varieties with semi-stable fibre structures. We use Hodge theory to study the higher direct images of wedge products of relative cotangent sheaves of certain semi-stable families over higher dimensional quasi-projective bases, and obtain some results on positivity. We then apply these results to study nonisotrivial Calabi-Yau varieties fibred by semi-stable Abelian varieties (or hyperkähler varieties).

The mirror symmetry conjecture is based on the suggestion that two sigma-models in superstring theory are equivalent. The targets of the models are Calabi-Yau 3-folds, so mirror symmetry predicts that these should come in pairs, $M$ and $\tilde{M}$, satisfying $h^{p,q}(M) = h^{p,3−q}(\tilde{M})$. Recently Strominger-Yau-Zaslow noticed in [17] that String Theory suggests that if $M$ and $\tilde{M}$ are mirror pairs of $n$-dimensional CY manifolds, then on $M$ should exist a special Lagrangian $n$-tori fibration $f : M \rightarrow B$, (with some singular fibres) such that $\tilde{M}$ is obtained by finding some suitable compactification of the dual fibration.

Motivated by the Strominger-Yau-Zaslow conjecture, we study the fibration $f : X \rightarrow Y$, where $X$ is a Calabi-Yau type projective manifold.

Definition 0.1. (Calabi-Yau manifolds).

1. A projective complex manifold $Y$ is called Calabi-Yau if the following conditions are satisfied:
   a) the canonical line bundle $\omega_Y$ of $Y$ is trivial;
   b) $H^p(Y, \Omega^p_Y) = 0$ for $p$ with $0 < p < \dim Y$.

2. A compact Kähler manifold $Y$ is called hyperkähler if it is of dimension $2n \geq 4$ and the following conditions are satisfied:
   a) there is a non-zero holomorphic two form $\beta_Y$ unique up to scalar such that $\det(\beta_Y)$ is nowhere zero;
   b) $H^1(Y, \mathcal{O}_Y) = 0$.

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A compact Kähler manifold $Y$ is called \textit{Calabi-Yau type} if its canonical line bundle $\omega_Y$ is trivial. (Thus, Abelian varieties and hyperkähler manifolds are Calabi-Yau-type.)

\textbf{Remark.} If $Y$ is hyperkähler, then $\dim H^2(Y, \mathcal{O}_Y) = 1$, and $\omega_X$ is trivial.

Studying projective manifolds with semi-stable fiber structures by Deligne’s Hodge theory and the semi-positivity package, we then have:

\textbf{Theorem 0.2.} Let $f : X \to Y$ be a semi-stable family between two non-singular projective varieties with

$$f : X_0 = f^{-1}(Y_0) \to Y_0$$

smooth, $S = Y \setminus Y_0$ a reduced normal crossing divisor and $\Delta = f^*S$ a relative reduced normal crossing divisor in $X$. Assume that $f$ satisfies that

a) the polarized VHS $R^k f_* \mathbb{Q}_{X_0}$ is strictly of weight $k$;

b) $H^k(X, \mathcal{O}_X) = 0$;

c) $Y$ is simply connected.

Then $f_* \Omega^k_{X/Y}(\log \Delta)$ is locally free on $Y$ without flat quotient, and $S \neq \emptyset$. Moreover,

$$\deg_C(f_* \Omega^k_{X/Y}(\log \Delta)) > 0$$

for any sufficiently general curve $C \subset \overline{M}$.

Based on the above theorem, we consider Calabi-Yau varieties fibred by semi-stable Abelian varieties (or by hyperkähler varieties) over $\mathbb{C}\mathbb{P}^1$, and then we obtain the following one of our main results:

\textbf{Theorem 0.3.} Let $f : X \to \mathbb{P}^1$ be a semi-stable family fibred by Abelian varieties such that

$$f : X_0 = f^{-1}(C_0) \to C_0$$

is smooth with finite singular values $S = \mathbb{P}^1 \setminus C_0$ and a normal crossing $\Delta = f^{-1}(S)$. Assume that $X$ is a projective manifold with trivial canonical sheaf $\omega_X$ and $H^1(X, \mathcal{O}_X) = 0$. Then, $f$ is nonisotrivial and $\dim X \leq 3$.

In particular, if $X$ is a Calabi-Yau manifold then $X$ is one of the following:

a) $K3$ with $\#S \geq 6$. Moreover, if $\#S = 6$ then $X \to \mathbb{P}^1$ is modular, i.e., $C_0$ is the quotient of the upper half plane $\mathcal{H}$ by a subgroup of $\text{SL}_2(\mathbb{Z})$ of finite index.

b) Calabi-Yau threefold with $\#S \geq 4$. Moreover, if $\#S = 4$ then this family is rigid and there exists an étale covering $\pi : Y' \to \mathbb{P}^1$ such that $f' : X' = X \times_{\mathbb{P}^1} Y' \to Y'$ is isogenous over $Y'$ to a product $E \times_Y E$, where $h : E \to Y$ is a family of semi-stable elliptic curves and modular.

Furthermore, we consider Calabi-Yau varieties fibred by semi-stable Abelian varieties with higher dimension base $Y$, the result in [25] shows that the base $Y$ must be rational connected (cf. Fact 2.4 in section 2).
Consider a rationally connected manifold $Z$. Let $D_\infty$ be a reduced normal crossing divisor in $Z$. One can choose a very freely rational curve (not necessarily smooth) $C$ intersecting each component of $D_\infty$ transversely, then

$$\pi_1(C \cap (Z - D_\infty)) \to \pi_1(Z - D_\infty) \to 0$$

is surjective.

Actually, Kollár’s results on fundamental groups show the following fact:

**Proposition 0.4** (cf. [13]). Let $X$ be a smooth projective variety and $U \subset Z$ be an open dense subset such that $Z \setminus U$ is a normal crossing divisor. Assume that $Z$ is rationally connected. Then, there exists a very free rational curve $C \subset Z$ such that it intersects each irreducible component of $Z \setminus U$ transversally and the map of topological fundamental groups $\pi_1(C \cap U) \to \pi_1(U)$ is surjective; moreover

a) if $\dim Z \geq 3$ then $C$ is a smooth rational curve in $Z$, and

b) if $\dim Z = 2$ then there is an immersion $h : \mathbb{P}^1 \to C \subset Z$.

With the above proposition 0.4, we apply Theorem 0.3 and immediately obtain the following result:

**Theorem 0.5.** Let $f : X \to Y$ be a semi-stable family of Abelian varieties between two non-singular projective varieties with $f : X_0 = f^{-1}(Y_0) \to Y_0$ smooth, a reduced normal crossing divisor $S = Y \setminus Y_0$ and a relative reduced normal crossing divisor $\Delta = f^* (S)$ in $X$. Assume that

a) the period map of the VHS $R^1 f_* (\mathbb{Q}^X)$ is injective at one point in $Y_0$;

b) the canonical bundle $\omega_X$ is trivial and $H^0(X, \Omega^1_X) = 0$.

Then, the dimension of a general fibre is bounded above by a constant dependent on $Y$.

With similar methods, we consider any Calabi-Yau variety fibred by semi-stable hyperkähler varieties, and show the base manifold is rationally connected such that the dimension of a general fibre is bounded above by a constant depending on the base, moreover if the base is $\mathbb{CP}^1$ then the dimension of a general fibre is four (see Theorem 3.1 and Theorem 3.2 in section 3).

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1. Preliminary

Let $M$ be a smooth $m$-dimensional complex quasi-projective variety, $M \subset \overline{M}$ a smooth projective compactification such that $\overline{M} - M = D_\infty$ is a reduced normal crossing divisor. Let $j : M \hookrightarrow \overline{M}$ be the open embedding.

Let $H$ be an arbitrary polarized variation of $\mathbb{R}$-Hodge structures (VHS for short) over $M$ with unipotent monodromies around $D$. Let $H = H \otimes O_M$, a holomorphic vector bundle, $\mathcal{H} = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \cdots \supset \mathcal{F}^w \supset 0$ the Hodge filtration with vector bundles (Hodge bundles). $w$ will be called the weight of the VHS. $\mathcal{F}^w$ will be denoted $\mathcal{F}^b$, here the sign $b$ is for bottom, i.e., $\mathcal{F}^b$ is the lowest piece of the Hodge filtration (cf. [15]).

Deligne’s canonical extension of a VHS. We consider a special coordinate around $o \in D$, i.e., it is a coordinate neighborhood $\Delta_o \subset \overline{M}$ of $o$ which is isomorphic to $\Delta^n$, and

\[ M \cap \Delta_o = \{ z = (z_1, \cdots, z_n) \mid z_1 \neq 0, \cdots, z_l \neq 0 \text{ for some } 1 \leq l \leq n \} \cong (\Delta^*)^l \times \Delta^{n-l}. \]

In particular,

\[ \Delta_o \cap D_\infty = \{ z = (z_1, \cdots, z_n) \mid z_1 \cdots z_l = 0 \}. \]

Let $\gamma_\alpha$ be a local monodromy around $z_\alpha = 0$ in $\Delta_o$ for $\alpha = 1, \cdots, l$. $N_\alpha = \log \gamma_\alpha$ is thus nilpotent.

Choose a flat multivalued basis $(v.)$ of $\mathcal{H}$ over $\Delta_o \cap M$. The formula

\[ (\tilde{v})(z) := \exp \left( -\frac{1}{2 \pi \sqrt{-1}} \sum_{\alpha=1}^{l} \log z_\alpha N_\alpha \right)(v.)(z) \]

gives a single-valued basis basis of $H$ over $\Delta_o \cap M$. The formula

\[ (\tilde{v})(z) := \exp \left( -\frac{1}{2 \pi \sqrt{-1}} \sum_{\alpha=1}^{l} \log z_\alpha N_\alpha \right)(v.)(z) \]

gives a single-valued basis basis of $H$. Deligne’s canonical extension $\mathcal{H}$ of $\mathcal{H}$ to $\Delta_o$ is generated by $(\tilde{v})$ (cf. [3], [15]). The construction of $\mathcal{H}$ is independent of the choice of $\tilde{z}$’s and $(v.)$. Obviously, $\mathcal{H}$ is locally free.

Denote by $\mathcal{F}^p := \mathcal{H} \cap j_* \mathcal{F}^p$. Thanks to the nilpotent orbit theorem (cf. [2], [15]),

\[ \mathcal{H} = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \cdots \supset \mathcal{F}^w \supset 0 \]

is a filtration of locally free sheaves.

The Higgs bundle associated to a VHS. Let $\nabla$ be the Gauss-Manin connection on the Hodge bundle $\mathcal{H}$. The Deligne canonical extension $\mathcal{H}$ of $\mathcal{H}$ and the nilpotent theorem allow the Gauss-Manin connection extend to be an regular connection

\[ \nabla : \mathcal{H} \to \mathcal{H} \otimes \Omega^1_{\overline{M}}(\log D_\infty) \]

with the Griffiths transversality:

\[ \nabla : \mathcal{F}^p \to \mathcal{F}^{p-1} \otimes \Omega^1_{\overline{M}}(\log D_\infty) \quad p = 1, \cdots, w. \]
Denote by
\[ E^{p, w-p} = \mathcal{F}^p / \mathcal{F}^{p+1} \quad p = 1, \cdots, w. \]
the Griffiths transversality allows \( \mathcal{O}_M \)-linear morphisms
\[ \theta^{p, q} : E^{p, q} \rightarrow E^{p-1, q+1} \otimes \Omega^1_M(\log D) \]
for \( p = 1, \cdots, w \) and \( p + q = w \). We then have the Higgs bundle
\[ (E := \bigoplus E^{p, q}, \theta := \bigoplus \theta^{p, q}), \]
i.e., \( \theta : E \rightarrow E \otimes \Omega^1_M(\log D) \) is an \( \mathcal{O}_M \)-linear morphism with \( \theta \wedge \theta = 0 \) (cf. [16]).

Kollár’s decomposition. We recite the results of Proposition 4.10 and Remark 4.11 in [10].

Definition 1.1. A smooth projective curve \( C \subset \overline{M} \) is sufficiently general if it satisfies that
\begin{enumerate}
    \item \( C \) intersects \( D_\infty \) transversely;
    \item \( \pi_1(C_0) \rightarrow \pi_1(M) \rightarrow 0 \) is surjective where \( C_0 = C \cap M \).
\end{enumerate}

Remark. There are many sufficiently general curves: Let \( C \) be a complete intersection of very ample divisors such that it is a smooth projective curve in \( \overline{M} \) intersecting \( D_\infty \) transversally. The quasi-projective version of the Lefschetz hyperplane theorem (cf. [6]) guarantees the subjectivity of \( \pi_1(C_0) \rightarrow \pi_1(M) \). But we note that a curve \( C \) is sufficiently general does not mean that it must be a complete intersection of hyperplanes.

Theorem 1.2 (Kollar). Let \( M \) be a quasi-projective \( n \)-fold with a smooth projective completion \( \overline{M} \) such that \( D_\infty = \overline{M} - M \) is a reduced normal crossing divisor. Consider the lowest piece \( \mathcal{F}^b \) of the Hodge filtration of a polarized VHS \( H \) on \( M \). Assume that all local monodromies of \( V \) are unipotent. There is a unique decomposition for the canonical extension \( \mathcal{F}^b \) of \( \mathcal{F}^b \)
\[ \mathcal{F}^b = \mathcal{A} \oplus \mathcal{U}, \]
such that \( \mathcal{A} \) has no flat quotient and \( \mathcal{U} \) is a unitary flat bundle on \( \overline{M} \). Moreover, if \( C \) is a sufficiently general curve in \( \overline{M} \) then \( \mathcal{A}|_C \) is an ample vector bundle on \( C \).

The Fujita-Kawamata’s Semi-positive package. We recite the Semi-positive package in [7] and [8].

Definition 1.3. Let \( \pi : X \rightarrow Y \) be an algebraic fibration with \( d = \dim X - \dim Y \). We say \( \pi \) satisfies the unipotent reduction condition (URC) if the following are held:
\begin{enumerate}
    \item \( U \) is a unitary flat bundle on \( \overline{M} \).
    \item \( \mathcal{A}|_C \) is an ample vector bundle on \( C \).
\end{enumerate}
(1) there is a Zariski open dense subset $Y_0$ of $Y$ such that $D = Y \setminus Y_0$ is a divisor of normal crossing on $Y$, i.e., $D$ is a reduced effective divisor and if $D = \sum_{i=1}^{N} D_i$ is the decomposition to irreducible components, then all $D_i$ are non-singular and cross normally;

(2) $\pi: X_0 \to Y_0$ is smooth where $X_0 = \pi^{-1}(Y_0)$;

(3) all local monodromies of $R^d\pi_*\mathbb{Q}_{X_0}$ around $D$ are unipotent.

The URC holds automatically for any semi-stable family.

**Theorem 1.4** (Kawamata). Let $\pi: X \to Y$ be a proper algebraic family with connected fibre and $\omega_{X/Y} := \omega_X \otimes \pi^*\omega_{-1}^Y$ be the relative dualizing sheaf. Let $\mathcal{F}$ be the bottom filtration of the VHS $R^d\pi_*\mathbb{Q}_{X_0}$ where $X_0 = f^{-1}(Y_0)$ and $d = \dim X - \dim Y$. Assume that $\pi$ satisfies URC, then we have:

1. $\pi_*\omega_{X/Y} = \overline{\mathcal{F}}$ is locally free, where $\overline{\mathcal{F}}$ is the Deligne canonical extension of $\mathcal{F}$ over $\overline{M}$.

2. $\pi_*\omega_{X/Y}$ is semi-positive, i.e., for every projective curve $T$ and morphism $g: T \to Y$ every quotient line bundle of $g^*(\pi_*\omega_{X/Y})$ has non-negative degree.

In this paper, we also need the following result in [9]:

**Corollary 1.5** (Kawamata). Let $Y$ be projective manifold and let $Y_0$ be a dense open set of $Y$ such that $S = Y \setminus Y_0$ is a reduced normal crossing divisor. Let $H$ be a polarized VHS of strict weight $w$ over $Y_0$ such that all local monodromies are unipotent.

Assume that

$$T_{Y,p} \to \text{Hom}(\mathcal{F}_p^w, \mathcal{F}_p^{w-1}/\mathcal{F}_p^w)$$

is injective at $p \in Y_0$ where $\mathcal{F}_p^w =: \mathcal{F}^b$ is the lowest piece of the Hodge filtration of the VHS $H$. Then, $\det \mathcal{F}^b$ is a big line bundle, where $\mathcal{F}^b$ is the canonical extension of $\mathcal{F}^b$.

**Proof.** The form $c_1(\mathcal{F}^b, h)$ on $Y_0$ is a current on $Y$ and represents the Chern class $C_1(\mathcal{F}^b)$ due to Cattani-Kaplan-Schmid’s theorem (cf. [2]). We can calculate $c_1(\mathcal{F}^b, h)$ as follows:

Let $\mathcal{N} = (\mathcal{F}^b)^\vee$ and $h$ is the Hodge metric on $\mathcal{H} = H \otimes \mathcal{O}_{Y_0}$. Then, we have the curvature form

$$\Theta(\mathcal{N}, h) = -\theta \wedge \bar{\theta}|_{\mathcal{N}} + \bar{A}_h \wedge A$$

on $Y_0$ where $A$ is the second fundamental form of $\mathcal{N}$ (cf. [15]), and $\theta(\mathcal{N}) = 0$ on $Y_0$ by the Griffiths transversality.

On the other hand, the lemma 2.6 in [20] or Theorem 0.1 in [27] shows that

$$\int_{Y_0} c_1(\mathcal{F}^b, h)^{\dim Y} > 0,$$
by the injectivity of the morphism $T_{g, p} \rightarrow \text{Hom}(\mathcal{F}_p, \mathcal{F}_{p-1})$. Since $\det(\mathcal{F})$ is a nef line bundle by Theorem 1.3, The Riemann-Roch Theorem says that $\det(\mathcal{F})$ is big, or we can obtain the bigness by Sommese-Kawamata-Siu’s numerical criterion:

If $L$ is a hermitian semi-positive line bundle on a compact complex manifold $X$ such that $\int_X \wedge^{\text{dim} X} c_1(L) > 0$, then $L$ is a big line bundle on $X$. In particular, if the line bundle $L$ is nef with $(L)^{\text{dim} X} > 0$, then $L$ is big.

\[ \square \]

Families of Calabi-Yau-type manifolds. Let $(Y, L)$ be a polarized manifold. Let

\[ \pi : (\mathcal{Y}, (Y, L)) \rightarrow (\mathcal{M}_{c_1(L)}, 0) \]

be the polarized Kuranishi family and $D$ the classifying space of the polarized $\mathbb{Q}$-Hodge structure of

\[ H^n_{\text{prim}}(Y, \mathbb{C}) := \text{Ker}(H^n(Y, \mathbb{C}) \overset{\wedge c_1(L)}{\longrightarrow} H^{n+2}(Y, \mathbb{C})). \]

Theorem 1.6 (Bogomolov-Tian-Todorov \[18, 19\]). Assume that $Y$ has trivial canonical line bundle $\omega_Y$. Then, the Kuranishi family of $(Y, L)$ is universal and the base manifold is a smooth open set in the Euclidean space of dimension

\[ \dim_{\mathbb{C}} H^1(Y, \Theta_Y)_{c_1(L)}, \]

where $H^1(Y, \Theta)_{c_1(L)} := \text{Ker}(H^1(Y, \Theta_Y) \overset{\wedge c_1(L)}{\longrightarrow} H^2(Y, \mathcal{O}_Y)).$

Let

\[ f : (\mathcal{Y}, (Y, L)) \rightarrow (Z, 0) \]

be a family of polarized manifolds. Locally, near $0 \in Z$, one has a commutative diagram of period maps

\[ \begin{array}{ccc}
\mathcal{M}_{c_1(L)} & \xrightarrow{\Phi} & Z \\
\Phi \downarrow & & \Phi_Z \\
D & & \\
\end{array} \]

Let $\mu_0 = (d\Phi)_0$ and $\lambda_0 = (d\Phi_Z)_0$. Also we have the Kodaira-Spencer map $\rho = (d\iota)_0$.

Theorem 1.7 (Griffiths’s Infinitesimal Torelli Theorem, \[4\]). We have a natural morphism

\[ \mu_0 : T_{\mathcal{M}_{c_1(L)}, 0} = H^1(Y, \Theta_Y)_{c_1(L)} \rightarrow \bigoplus_p \text{Hom}(H^{n-p, p}_{\text{prim}}, H^{n-p-1, p+1}_{\text{prim}}) \]
and $\lambda_0 = \mu_0 \circ \rho$, where $T_{\mathfrak{m}}(L,0)$ is the Zariski tangent space. Write $\mu_0 = (\mu_0^0, \cdots, \mu_0^n)$ where

$$
\mu_0^i : T_{\mathfrak{m}}(L,0) = H^1(Y, \Theta_Y)_{c_1} \to \text{Hom}(H_{prim}^{n-i,i}, H_{prim}^{n-i-1,i+1}).
$$

If $Y$ has trivial $\omega_Y$, then $\mu_0^0$ is an isomorphism and $\mu_0$ is an injective map.

In this paper, we will use the following fact frequently: For a non-isotrivial family of manifolds with trivial canonical line bundle, the period map is not degenerate at a general point of the base. The Infinitesimal Torelli Theorem shows that the condition that the period map $\Phi$ is not degenerate at $0$ is equivalent to the condition that the Kodaira-Spencer map is injective at $0$.

2. Calabi-Yau type manifolds with semi-stable fibrations.

From now on, we always assume that $X$ is a Kähler manifold with trivial canonical line bundle $\omega_X \cong \mathcal{O}_X$.

Some Facts. Let $f : X \to C$ be a semi-stable family from a projective manifold $X$ with trivial $\omega_X$ to a smooth projective curve $C$. Since

$$
\omega_C^{-1} = \mathcal{O}_C(\sum t_i - \sum t_j) \text{ with } \#\{i\} - \#\{j\} = 2 - 2g(C),
$$

we then have

$$
\omega_{X/C} = \omega_X \otimes f^*\omega_C^{-1} = f^*\omega_C^{-1} = \mathcal{O}_X(\sum X_{t_i} - \sum X_{t_j}).
$$

Thus, $f_*\omega_{X/C} = \omega_C^{-1}$ and $\mathcal{O}_X(X_{t_i})|_{X_t} = \mathcal{O}_{X_t}(X_{t_i} \cdot X_t) = \mathcal{O}_{X_t} \forall t \in C$ where $X_t = f^{-1}(t)$. Hence,

$$
\omega_{X_t} = \omega_{X/C}|_{X_t} = \mathcal{O}_{X_t} \forall t \in C.
$$

i.e., the canonical sheaf of each closed fibre is trivial.

On the other hand, if all closed fibres have trivial canonical bundle then $\omega_X$ is trivial if and only if $f_*\omega_{X/C} = \omega_C^{-1}$. In particular, if the total space is a Calabi-Yau manifold, then the fibres can be Abelian varieties, lower dimensional Calabi-Yau varieties or hyperkähler varieties.

Fact 2.1. Let $f : X \to Y$ be a semi-stable family of Calabi-Yau varieties over a higher dimension base such that $f$ is smooth over $Y_0$ and $Y \setminus Y_0$ is a reduced normal crossing divisor. We have:

1. If the induced moduli map is a generically finite morphism, then the line bundle $f_*\omega_{X/Y}$ is big and nef.
2. If $f$ is a smooth family and the induced period map has no degenerated point, then the line bundle $f_*\omega_{X/Y}$ is ample.
In particular, we have:

**Corollary 2.2.** Let $f : X \to C$ be a semi-stable non-isotrivial family over a smooth projective curve $C$, where $X$ has trivial $\omega_X$. Then, $f_*\omega_{X/C}$ is a big line bundle and $C$ is $\mathbb{P}^1$.

Let $f : X \to Y$ be a semi-stable proper family smooth over a Zariski open dense set $Y_0$ such that $S = Y - Y_0$ is a reduced normal crossing divisor. Suppose that $X$ is a projective manifold. Then, a general fibre is a smooth projective manifold with trivial canonical bundle and it has certain geometric type ‘K’. We know that the coarse quasi-projective moduli scheme $\mathcal{M}_K$ exists for the set of all polarized projective manifolds with trivial canonical sheaf and with given type ‘K’ (cf.[21]).

By the infinitesimal Torelli theorem, that the family $f$ satisfies the condition in the corollary 1.5 if and only if the unique moduli morphism $\eta_f : Y_0 \to \mathcal{M}_K$ for $f$ is a generically finite morphism. Moreover, the condition is equivalent to that $f$ contains no isotrivial subfamily whose base is a subvariety passing through a general point of $Y$. If $Y$ is a curve, that $f$ satisfies the condition in the corollary 1.5 if and only if the family $f$ is non-isotrivial. Actually, we have a result from Kollár’s decomposition and the semi-positive package:

**Corollary 2.3.** Let $f : X \to Y$ be a surjective morphism between two non-singular projective varieties such that every fibre is irreducible and $f : X_0 = X \setminus \Delta \to Y \setminus S$ be the maximal smooth subfamily where $S = Y \setminus Y_0$ is a reduced normal crossing divisor. Let $\mathcal{F}$ be the lowest piece $\mathcal{F}^b$ of the Hodge filtration of a polarized VHS $R^{n-1}f_*(\mathbb{Q}_{X_0})$ on $Y_0$, i.e.,

$$\mathcal{F} = F^{n-1}(R^{n-1}f_*(\mathbb{Q}_{X_0}) \otimes O_{Y_0}).$$

Assume that $X$ is a projective $n$-fold with trivial canonical sheaf $\omega_X$ and $f$ is semi-stable, i.e., $\Delta = f^*S$ is a relative reduced normal crossing divisor in $X$.

If the moduli morphism of $f$ is generically finite, then we have:
1. $f_*\Omega^{n-1}_{X/Y}(\log \Delta) = f_*\omega_{X/Y} = \mathcal{F}$ is a big and nef line bundle on $Y$, where $\mathcal{F}$ is the canonical extension of $\mathcal{F}$.
2. Moreover, for any sufficient general curve $C \subset Y \mathcal{F}|_C$ is ample on $C$.

Recently, Zhang(cf.[23]) proves that log $Q$-Fano varieties are rationally connected, and it implies a result which was obtained by Kollár-Miyaoka-Mori in case of threefold (cf.[12]): Any higher dimensional variety with a big and nef anticanonical bundle must be rationally connected.

Therefore, we have:

**Fact 2.4.** Let $f : X \to Y$ be a semi-stable proper family between two non-singular projective varieties. Assume that $X$ has trivial canonical sheaf $\omega_X$ and
the induced moduli morphism $\eta_f$ is generically finite. Then, the anti-canonical line bundle $\omega_Y^{-1}$ of $Y$ is big and nef, and so $Y$ is rationally connected.

**Vanishing of unitary flat subbundles.** Now we consider the cohomology geometry in case of Calabi-Yau manifolds. Before proving Theorem 0.2, we recite Deligne’s Theory as follows:

Let $f : X \rightarrow Y$ be a semi-stable family between two non-singular projective varieties with $f : X_0 = f^{-1}(Y_0) \rightarrow Y_0$ smooth, $S = Y \setminus Y_0$ a reduced normal crossing divisor and $\Delta = f^*S$ a relative reduced normal crossing divisor in $X$. Let $y \in Y_0$ be a fixed point. Consider the following commutative diagram:

$$
\begin{array}{ccc}
H^m(X, \mathbb{Q}) & \xrightarrow{i^*} & H^m(X_0, \mathbb{Q}) \\
\downarrow \iota_y & & \downarrow \iota_{y_0} \\
H^m(X_y, \mathbb{Q}) & &
\end{array}
$$

where $i_y : X_y \hookrightarrow X_0$, $\iota_y : X_y \hookrightarrow X$ are natural embeddings. $H^m(X_0, \mathbb{Q})$ can be equipped with a functorial mixed Hodge structure $(W_m, F^p, H^m(X_0, \mathbb{Q}))$ of pure weight $m$ with

$$W_m(H^m(X_0, \mathbb{Q})) = \text{Image}(H^m(X, \mathbb{Q}) \xrightarrow{i^*} H^m(X_0, \mathbb{Q})).$$

In addition, $H^m(X, \mathbb{Q}) H^m(X_y, \mathbb{Q})$ also have a pure Hodge structure of weight $m$, and all of $i^*, \iota_{y*}, \iota_{y_0}$ are morphisms of mixed Hodge structures. It follows that

$$\text{Image}(\iota_{y*}) = \text{Image}(\iota_{y_0}).$$

(cf. Section 3, in particular Corollary 3.2.18 of [3]).

$$i_y^* : H^m(X, \mathbb{Q}) \rightarrow H^m(X_y, \mathbb{Q})$$

is a morphism of mixed Hodge structures.

We note that $\pi_1(Y_0, y)$ acts on the $H^n(X_y, \mathbb{Q})$. In particular, there is a $\pi_1$-invariant subspace

$$H^m(X_y, \mathbb{Q})^{\pi_1} \hookrightarrow H^m(X_y, \mathbb{Q}).$$

We glue these invariant subspaces together into a constant sheaf $\mathcal{I}$ on $Y_0$. $\mathcal{I}$ coincides with the constant sheaf on $Y_0$ with the fiber $H^0(Y_0, R^m f_*\mathbb{C}_{X_0})$, which is a subsheaf of $R^m f_*\mathbb{Q}_{X_0}$ generated by the global sections of $R^m f_*\mathbb{Q}_{X_0}$, i.e.,

$$\mathcal{I} = (R^m f_*\mathbb{Q})^{\pi_1}.$$

There is a natural isomorphism of $\mathbb{Q}$-vector spaces

$$\kappa_y : H^0(Y_0, R^m f_*\mathbb{Q}_{X_0}) \xrightarrow{\sim} H^m(X_y, \mathbb{Q})^{\pi_1}.$$
Deligne shows that Leray’s spectral sequence for \( f : X_0 \to Y_0 \)
\[
E_2^{p,q} = H^p(Y_0, R^q f_\ast \mathbb{Q}_{X_0}) \Rightarrow H^{p+q}(X_0, \mathbb{Q})
\]
degenerates at \( E_2 \) (cf. Theorem 4.1.1 (i) of [3]), and so it follows that the canonical mapping
\[
j : H^m(X_0, \mathbb{Q}) \to H^0(Y_0, R^m f_\ast \mathbb{Q}_{X_0}) \to 0
\]
is surjective. Moreover, \( i_y^\ast \) and \( i_y^\ast \) can be decomposed as follows:
\[
i_y^\ast : H^m(X_0, \mathbb{Q}) \to H^0(Y_0, R^m f_\ast \mathbb{Q}_{X_0}) \to H^m(X_y, \mathbb{Q}) \to 0
\]
\[
i_y^\ast : H^m(X, \mathbb{Q}) \to H^0(Y_0, R^m f_\ast \mathbb{Q}_{X_0}) \to H^m(X, \mathbb{Q}) \to 0
\]
Since \( i_y^\ast \) and \( i_y^\ast \) are morphisms of mixed Hodge structure and
\[
\text{Image}(i_y^\ast) = \text{Image}(i_y^\ast) = H^m(X_y, \mathbb{Q}) \cap i_x^\ast \mathbb{Q}_{X_0},
\]
\( H^m(X_y, \mathbb{Q}) \) as an image of a Hodge structure, is a Hodge substructure in
\( H^m(X_y, \mathbb{Q}) \) (cf. the proof of Theorem 4.11 (ii) of [3]).

We can introduce a Hodge structure on \( H^0(Y_0, R^m f_\ast \mathbb{Q}_{X_0}) \), as a quotient structure on \( H^m(X_0, \mathbb{Q}) \). This quotient structure is independent of \( y \in Y_0 \) (cf. Corollary 4.1.2 of [3]).

Proof of Theorem 0.2. At first, we note that \( f_\ast \Omega^k_{X/Y}(log \Delta) \) is just the canonical extension of \( F^k(R^k f_\ast (\mathbb{Q}_{X_0}) \otimes \mathcal{O}_{Y_0}) \) which is the lowest piece of the Hodge filtration of the VHS \( R^k f_\ast \mathbb{Q}_{X_0} \), thus \( f_\ast \Omega^k_{X/Y}(log \Delta) \) is locally free (cf. Lemma 1 in [8] for detail). That the polarized VHS \( R^k f_\ast \mathbb{Q}_{X_0} \) is of weight \( k \) guarantees that \( f_\ast \Omega^k_{X/Y}(log \Delta) \neq 0 \).

By Kollar’s decomposition [12] we have:
\[
f_\ast \Omega^k_{X/Y}(log \Delta) = \mathcal{A} \oplus \mathcal{U}
\]
such that \( \mathcal{A} \) has no flat quotient and \( \mathcal{U} \) is flat (so \( \mathcal{U} \) is trivial here since \( Y \) is simply connected). We show that \( f_\ast \Omega^k_{X/Y}(log \Delta) \) has no flat direct summand: Otherwise, there is a nonzero global section \( s \in H^0(Y_0, R^k f_\ast (\mathbb{C})) \) of \((k, 0)\)-type.

Consider Deligne’s Hodge theory, we have the following commutative diagram:
\[
\begin{array}{ccc}
H^m(X, \mathbb{C}) & \xrightarrow{i_y^\ast} & H^m(X_0, \mathbb{C}) \\
\downarrow{i_y^\ast} & & \downarrow{i_y^\ast} \\
H^m(X_y, \mathbb{C}) & &
\end{array}
\]
where \( i_y : X_y \hookrightarrow X_0, i_y : X_y \hookrightarrow X \) are natural embeddings. For each pair \((p, q)\) with \( p + q = m \), the following restriction map induced by \( X_y \subset X \) is a Hodge
morphism:
\[ r_{y}^{p,q} : H^q(X, \Omega_X^{p}) \cong H^m(X, \mathbb{C}) \rightarrow H^q(X_y, \mathbb{C}) \rightarrow H^q(X_y, \Omega_{X_y}^{p}) \]
where \( y \in Y_0 \) is a fixed point.

Since
\[ \text{image}(i_{*}y) = \text{image}(i_{*}y) = H^m(X_y, Q^\pi_1(Y_0, y)) \]
we then have:

The component of type \((p, q)\) of the group \( H^m(X_y, C) \pi_1(Y_0, y) \) is just the image of \( H^q(X, \Omega_X^{p}) \) under \( r_{y}^{p,q} \).

Let \( m = k \). We then have a nonzero lifting \( \tilde{s} \in H^0(X, \Omega_X^{k}) \) of \( s \), it is a contradiction. Similarly, we have \( S \neq \emptyset \). \( \square \)

**Corollary 2.5.** Let \( f : X \rightarrow Y \) be a semi-stable family between two non-singular projective varieties with \( f : X_0 = f^{-1}(Y_0) \rightarrow Y_0 \) smooth, \( S = Y \setminus Y_0 \) a reduced normal crossing divisor and \( \Delta = f^{-1}(S) \) a relative reduced normal crossing divisor in \( X \). Assume that \( X \) is a Calabi-Yau \( n \)-fold and \( Y \) is simply connected. Then,

a) \( f \) is a nonisotrivial family with \( S \neq \emptyset \); 
b) \( f_*\omega_{X/Y} \) is an ample line bundle over any sufficiently general curve.

**Proof.** Consider the polarized VHS \( R^{n-1}f_*(Q_{X_0}) \). Suppose that \( f \) is isotrivial then the holomorphic period map for the VHS \( R^{n-1}f_*(Q_{X_0}) \) is constant over \( Y_0 \) by the infinitesimal Torelli theorem. Then the line bundle \( f_*\omega_{X_0/Y_0} \) is unitary flat over \( Y_0 \). Since \( f \) is semi-stable, all local monodromies around the VHS are unipotent, and so \( f_*\omega_{X/Y} \) is unitary flat. Since \( Y \) is simply-connected, \( f_*\omega_{X/Y} \) then is a trivial line bundle on \( Y \), it is a contradiction to the theorem 0.2. By similar arguments we then obtain \( S \neq \emptyset \). \( \square \)

**Remark.** Without assumption that \( X \) is Calabi-Yau manifold and \( Y \) is simply connected. When only \( X \) has trivial \( \omega_X \), we still have the following results:

1. By infinitesimal Torelli theorem, if \( f \) is isotrivial then the line bundle \( f_*\omega_{X/Y} \) is unitary flat.
2. Conversely, if \( f_*\omega_{X/Y} \) is unitary flat and the global Torelli theorem hold for a general fiber (e.g. a general fiber is K3 or Abelian variety) then \( f \) is isotrivial.

**Corollary 2.6.** Let \( f : X \rightarrow \mathbb{P}^1 \) be a semi-stable family with \( f : X_0 = f^{-1}(C_0) \rightarrow C_0 \) smooth, \( S = \mathbb{P}^1 \setminus C_0 \), and \( \Delta = f^*S \). Assume that \( X \) is a projective manifold with \( H^k(X, \mathcal{O}_X) = 0 \) and the polarized VHS \( R^k f_*\mathcal{Q}_{X_0} \) is strictly of weight \( k \). Then, \( S \neq \emptyset \) and \( f_*\mathcal{Q}_{X/\mathbb{P}^1}^k (\log \Delta) \) is an ample bundle on \( \mathbb{P}^1 \).
Proposition 2.7. Let \( f : X \to C \) be a semi-stable family over a smooth projective curve with \( f : X_0 = f^{-1}(C_0) \to C_0 \) smooth, \( S = C \setminus C_0 \), and \( \Delta = f^{-1}(S) \). If \( X \) is a projective \( n \)-fold with trivial \( \omega_X \), then the following conditions are equivalent:

1. \( f \) is a non-isotrivial family.
2. \( f_\ast \omega_{X/C} \) is an ample line bundle on \( C \).
3. \( C = \mathbb{P}^1 \) and \( S \geq 3 \).

Proof. At first, we note that each smooth closed fibre of \( f \) has trivial canonical sheaf, and

\[
2 - 2g(C) = \deg \omega_C^{-1} = \deg f_\ast \omega_{X/C} \geq 0.
\]

a) \( (1) \iff (2) \) is proven as follows:
- The infinitesimal Torelli theorem holds for the VHS \( R^{n-1} f_\ast (\mathbb{Q}_{X_0}) \) because there is an isomorphism for any \( t \in C_0 \):
  \[
  H^1(X_t, T_{X_t}) \to \text{Hom}(H^0(X_t, \Omega^{n-1}_{X_t}), H^1(X_t, \Omega^{n-2}_{X_t})).
  \]
- \( f \) is isotrivial \iff \( f_\ast \omega_{X_0/C_0} \) is a unitary flat line bundle on \( C_0 \).
- Since \( f \) is semi-stable, we have
  \[
  f_\ast \omega_{X_0/C_0} \text{ is unitary flat on } C_0 \iff f_\ast \omega_{X/C} = \omega_C^{-1} \text{ is unitary flat on } C.
  \]
- \( \omega_C^{-1} \) is unitary flat on \( C \iff 0 = \deg \omega_C^{-1} \iff C \) is elliptic.

b) \( (1) \implies (3) \) : Otherwise, \( C = \mathbb{P}^1 \) since \( C \) is elliptic will induce that \( f \) is isotrivial by the above arguments. If \( S \leq 2 \) then the global monodromy is generated by the locally monodromies around points in \( S \). Since Deligne’s complete reducible theorem(cf.\[3\]) says that the global monodromy is semi-simple, it implies that all local monodromies are identity, and so the global monodromy is trivial. Then the VHS \( R^{n-1} f_\ast (\mathbb{Q}_{X_0}) \) is trivial, which contradicts to that \( f \) is non-isotrivial by the infinitesimal Torelli theorem.

c) \( (3) \implies (1) \) is obvious since that \( f \) is isotrivial is equivalent to that \( C \) is elliptic.

\[ \square \]

Remark. The proof actually implies the following equivalent conditions:

i. \( f \) is an isotrivial family;
ii. \( f_\ast \omega_{X/C} \) is an unitary flat line bundle on \( C \);
iii. \( C \) is an elliptic curve.

3. Dimension counting for fibered Calabi-Yau manifolds

Calabi-Yau manifolds fibred by Abelian varieties. We now use Theorem 0.2 and its corollaries to deal with Calabi-Yau manifolds over \( \mathbb{C} \mathbb{P}^1 \) fibred by semi-stable Abelian varieties. We then have the dimension of such Calabi-Yau manifold \( \leq 3 \), see Theorem 0.3.
**Proof of Theorem 0.3.** Consider the Higgs bundle \((E, \theta)\) corresponding to \(R^1f_*Q_{X_0}\). The semi-stability of \(f\) shows that all local monodromies of \(R^1f_*Q_{X_0}\) are unipotent, thus there is the Deligne canonical extension
\[
\overline{E} = f_*\Omega^1_{X/P^1}(\log \Delta) \bigoplus R^1f_*(\mathcal{O}_X)
\]
where \(\Delta = f^*(S)\), and both pieces are locally free.

1. Let \(n = \dim f^{-1}(t)\). Then, \(n = \text{rk} f_*\Omega^1_{X/P^1}(\log \Delta)\) as a general fibre is an Abelian variety. It has been shown in 2.6 and 2.5 that \(f\) is a non-isotrivial family and the bundle \(f_*\Omega^1_{X/P^1}(\log \Delta)\) is ample on \(P^1\). The Grothendieck splitting theorem says that there is a decomposition:
\[
f_*\Omega^1_{X/P^1}(\log \Delta) = \bigoplus_{i=1}^n \mathcal{O}_{P^1}(d_i),
\]
and all integers \(d_i\) are positive by the ampleness of the \(f_*\Omega^1_{X/P^1}(\log \Delta)\).

On the other hand, the commutative diagram of morphisms
\[
\begin{array}{ccc}
\wedge^n f_*\Omega^1_{X/P^1}(\log \Delta) & \xrightarrow{\neq 0} & f_*(\wedge^n\Omega^1_X(\log \Delta)) \\
\downarrow & & \downarrow \\
\mathcal{O}_{P^1}(\sum_{i=1}^n d_i) & \xrightarrow{\neq 0} & f_*(\omega_{X/P^1})
\end{array}
\]
induces that
\[
n \leq \sum_{i=1}^n d_i \leq \deg f_*(\omega_{X/P^1}).
\]

By Zariski main theorem, \(f\) only having connected fibres is same as \(f_*\mathcal{O}_X = \mathcal{O}_{P^1}\). Hence \(\deg f_*(\omega_{X/P^1}) = 2\), and so the dimension of a general fibre is less than 3.

2. If \(X\) is a Calabi-Yau threefold then
\[
f_*\Omega^1_{X/P^1}(\log \Delta) = \mathcal{O}_{P^1}(1) \oplus \mathcal{O}_{P^1}(1).
\]

There is so called Arakelov-Yau inequality (cf. [4][24]).
\[
\deg f_*\Omega^1_{X/P^1}(\log \Delta) \leq \text{rk} f_*\Omega^1_{X/P^1}(\log \Delta) \leq \frac{\deg(\Omega^1_{P^1}(\log S))}{2} = 2g(P^1) - 2 + \#S.
\]

Thus \(\#S \geq 4\), and if \(\#S = 4\) then there is an étale covering \(\pi : Y' \to \mathbb{P}^1\) such that \(f' : X' = X \times_{P^1} Y' \to Y'\) is isogenous over \(Y'\) to a product \(E \times_{Y'} \cdots \times_{Y'} E\), where \(h : E \to Y'\) is a family of semi-stable elliptic curves reaching the Arakelov bound.
3. If $X$ is a K3 surface then
\[ f_* \Omega^1_{X/P^1}(\log \Delta) = f_* \omega_{X/P^1} = \mathcal{O}_{P^1}(2). \]

We deduce $#S \geq 6$ from the Arakelov-Yau inequality for weight one VHS:
\[ \deg f_* \Omega^1_{X/P^1}(\log \Delta) \leq \frac{\text{rk} f_* \Omega^1_{X/P^1}(\log \Delta)}{2} \deg(\Omega^1_{P^1}(\log S)) = \frac{#S}{2} - 1. \]

4. The modularity is due to recent results by Viehweg-Zuo (cf. [23]).

Now we deal with any semi-stable fibration $f : X \to Y$ with higher dimension base $Y$. We obtain the dimension of a general fibre is bounded above by a constant dependent on $Y$.

Proof of Theorem 0.5. The proof follows from the next steps.
1. By 1.2, $f_* \Omega^1_{X/Y}(\log \Delta)$ has no flat quotient. Moreover, $f_* \Omega^1_{X/Y}(\log \Delta)$ is ample on any sufficiently general curve in $Y$. We always have:
\[ \wedge^n f_* \Omega^1_{X/Y}(\log \Delta) = \det f_* \Omega^1_{X/Y}(\log \Delta), \]
where $n$ is the dimension of a general fibre and also is the rank of the locally free sheaf $f_* \Omega^1_{X/Y}(\log \Delta)$. On the other hand, the non-zero map
\[ \wedge^n f_* \Omega^1_{X/Y}(\log \Delta) \not\rightarrow f_*(\wedge^n \Omega^1_{X/Y}(\log \Delta)) = f_* \omega_{X/Y}, \]
induces that the line bundle $\omega^{-1} = f_* \omega_{X/Y}$ is big and nef. Thus $Y$ is a rationally connected projective manifold and so $U = 0$ by [0.2] 2.

2. By Proposition 0.4 we have a very free morphism $g : \mathbb{P}^1 \to \overline{M}$ which is sufficiently general, i.e., the image curve $C := g(\mathbb{P}^1)$ satisfies:
\begin{enumerate}
\item $C$ intersects $S$ transversely;
\item $\pi_1(C_0) \to \pi_1(Y_0) \to 0$ is surjective where $C_0 = C \cap Y_0$.
\end{enumerate}

3. If $\dim Y \geq 3$, $g$ is an embedding and we have a very free smooth rational curve $C \subset Y$. Since $f_* \Omega^1_{X/Y}(\log \Delta)$ is ample over $C$,
\[ f_* \Omega^1_{X/Y}(\log \Delta)|_C \cong \bigoplus_{i=1}^n \mathcal{O}_{P^1}(d_i) \text{ with } \forall d_i > 0. \]

Denote by $l = -\omega_Y \cdot C$. The commutative diagram of morphisms
\[ \begin{array}{ccc}
\wedge^n f_* \Omega^1_{X/Y}(\log \Delta)|_C \not\rightarrow & f_*(\wedge^n \Omega^1_{X/Y}(\log \Delta))|_C \\
\downarrow = & \downarrow = \\
\mathcal{O}_{P^1}(\sum_{i=1}^n d_i) \not\rightarrow & f_* \omega_{X/Y}|_C.
\end{array} \]
induces that
\[ n \leq \sum_{i=1}^{n} d_i \leq \deg_C f_\omega_X = -\deg_C(\omega_Y) = l. \]

4. If \( \dim Y = 1 \), \( Y \) is then \( \mathbb{P}^1 \) and \( l = 2 \). If \( \dim Y = 2 \), then \( Y \) is a smooth Del Pezzo surface and \( g \) is an immersion by the theorem 0.4.

a) Let \( \Gamma \) be the graph of the morphism \( g \), i.e.,
\[ \Gamma = \{(x, y) \in \mathbb{P}^1 \times X \mid y = g(x)\} \subset \mathbb{P}^1 \times X. \]
\( \Gamma \) is a smooth curve, actually it is isomorphic to \( \mathbb{P}^1 \). We have
\[ \mathbb{P}^1 \times X \xrightarrow{pr_1} \mathbb{P}^1 \xrightarrow{pr_2} X \]
and the projections \( pr_1 \) and \( pr_2 \) both are proper morphisms. Hence \( pr_1 : \Gamma \to g(\mathbb{P}^1) \) is a finite morphism. Denote by \( C := g(\mathbb{P}^1) \). We then have a finite set \( B \subset C \) such that \( g^{-1}(B) \) is a finite set in \( \mathbb{P}^1 \) and
\[ g : \mathbb{P}^1 \setminus g^{-1}(B) \to C \setminus B \]
is an étale covering.

b) Since the real-codimension of \( B \) in \( Y \) is four, we have a natural isomorphism of topological fundamental groups \( \pi_1(Y_0 \setminus B) \xrightarrow{\cong} \pi_1(Y_0) \), and so a surjective homomorphism \( \pi_1(C_0 \setminus B) \twoheadrightarrow \pi_1(Y_0) \to 0 \). Denote by
\[ T_0 := \mathbb{P}^1 - g^{-1}(B \cup (C - C_0)) \]
and \( \phi := g|_{T_0} \).

We have an étale covering \( \phi : T_0 \to C_0 \setminus B \), and an injective
\[ 0 \to \mathcal{O}_{C_0 \setminus B} \to \phi_* \mathcal{O}_{T_0}. \]
Without lost generality we may assume that \( \phi \) is a Galois covering, then the above short sequence has a split and so
\[ \phi_* \mathcal{O}_{T_0} = \mathcal{O}_{C_0 \setminus B} \oplus \text{Galois conjugates}. \]

c) Let \( \mathcal{F} = f_* \Omega_X^{1}/(\log \Delta) \) and \( \mathcal{L} := g^*(\mathcal{F}|_C) \). Then,
\[ \mathcal{L} = \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(d_i) \text{ with } d_i \geq 0 \ \forall i. \]

We claim that all \( d_i > 0 \).
Suppose that one \( d_i = 0 \), then \( \phi_* (\mathcal{L}|_{T_0}) \) has a nonzero flat quotient
\[ \phi_* (\mathcal{L}|_{T_0}) \twoheadrightarrow \mathcal{O}_{C_0 \setminus B} \to 0. \]
Thus, $\mathcal{F}|_{C_0 \setminus B}$ has a nonzero flat quotient since we have
\[ 0 \longrightarrow \mathcal{F}|_{C_0 \setminus B} \longrightarrow \phi_* \phi^*(\mathcal{F}|_{C_0 \setminus B}) = \phi_*(\mathcal{L}|_{T_0}). \]
and the projection formula
\[ \phi_* \phi^*(\mathcal{F}|_{C_0 \setminus B}) = \mathcal{F}|_{C_0 \setminus B} \otimes \phi_*(\mathcal{O}_{T_0}). \]
The subjectivity of $\pi_1(C_0 \setminus B) \to \pi_1(M) \to 0$ implies that $\mathcal{F}|_M$ has a flat quotient, and so $\phi_* \Omega^2_{X/Y} (\log \Delta)$ itself has a unitary flat quotient. It is a contradiction. Hence, all integers $d_i$ are positive and
\[ n \leq -\deg_{\mathbb{P}^1} g^* \omega_Y = -g_* [\mathbb{P}^1] \cdot \omega_Y = l. \]

\[ \square \]

**Calabi-Yau manifolds fibred by hyperkähler varieties.** Using similar methods as the case of Calabi-Yau manifolds fibred by Abelian varieties, we have the following result:

**Theorem 3.1.** Let $f : X \to \mathbb{P}^1$ be a semi-stable family fibred by hyperkähler varieties with
\[ f : X_0 = f^{-1}(C_0) \to C_0 \]
smooth, $S = \mathbb{P}^1 \setminus C_0$, and $\Delta = f^{-1}(S)$. Assume that $X$ is a projective manifold with trivial canonical sheaf $\omega_X$ and $H^2(X, \mathcal{O}_X) = 0$. Then, $f$ is nonisotrivial with $\# S \geq 3$ and the dimension of a general fibre is four.

**Proof.** Consider the Higgs bundle $(E, \theta)$ corresponding to $R^2 f_* \mathcal{Q}_{X_0}$. The semi-stability of $f$ shows that there is the Deligne canonical extension:
\[ \overline{E} = f_* \Omega^2_{X/\mathbb{P}^1} (\log \Delta) \bigoplus R^1 f_* \Omega^1_{X/\mathbb{P}^1} (\log \Delta) \bigoplus R^2 f_* (\mathcal{O}_X), \]
such that $f_* \Omega^2_{X/\mathbb{P}^1} (\log \Delta), R^2 f_* (\mathcal{O}_X)$ are line bundles. By 2.6,
\[ f_* \Omega^2_{X/\mathbb{P}^1} (\log \Delta) = \mathcal{O}_{\mathbb{P}^1}(d) \]
and $d$ is a positive integer.

Let $F$ be a general fibre and let $n = \dim F/2$. The commutative diagram
\[ \text{Sym}^n f_* \Omega^2_{X/\mathbb{P}^1} (\log \Delta) \xrightarrow{\neq 0} f_* (\wedge^n \Omega^1_{X/\mathbb{P}^1} (\log \Delta)) \]
\[ \downarrow \hspace{1 cm} \downarrow \]
\[ \mathcal{O}_{\mathbb{P}^1}(nd) \xrightarrow{\neq 0} f_* \omega_{X/\mathbb{P}^1}, \]
induces $n \leq \deg f_* \omega_{X/\mathbb{P}^1} = 2$. Thus $\dim F = 4$ by the definition, and so
\[ f_* \Omega^2_{X/\mathbb{P}^1} (\log \Delta) = \mathcal{O}_{\mathbb{P}^1}(1). \]

$\# S \geq 3$ is a well-known result since $f$ is nonisotrivial. $\square$
Theorem 3.2. Let $f : X \to Y$ be a semi-stable family of hyperkähler varieties between two non-singular projective varieties with $f : X_0 = f^{-1}(Y_0) \to Y_0$ smooth, $S = Y \setminus Y_0$ a reduced normal crossing divisor and $\Delta = f^*(S)$ a relative reduced normal crossing divisor in $X$. Assume that the period map for the VHS $R^2f_*\mathbb{Q}_{X_0}$ is injective at one point in $Y_0$ and $X$ has trivial canonical sheaf $\omega_X$ with $H^0(X, \Omega^2_X) = 0$. Then, the dimension of a general fibre is bounded above by a constant depending on $Y$.

Remark 3.3. If a fibration $f : X \to Y$ has $\dim X = 2n$ and $X$ is a projective irreducible sympletic manifold, then $\dim Y = n$(cf. [14]). Here we obtain some necessary conditions for a question asked by Naichung Conan Leung in CUHK algebraic geometry working seminar: Does there exist a Calabi-Yau manifold fibred by Abelian varieties (resp. hyperkähler varieties)?

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