THE COMPLETION THEOREM IN TWISTED EQUIVARIANT K-THEORY FOR PROPER ACTIONS.

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Abstract. We compare different algebraic structures in twisted equivariant K-Theory for proper actions of discrete groups. After the construction of a module structure over untwisted equivariant K-Theory, we prove a completion Theorem of Atiyah-Segal type for twisted equivariant K-Theory. Using a Universal coefficient Theorem, we prove a cocompletion Theorem for Twisted Borel K-Homology for discrete Groups.

The Completion Theorem in equivariant K-theory by Atiyah and Segal [6] had a remarkable influence on the development of topological K-theory and computational methods related to it.

Twisted equivariant K-theory for proper actions of discrete groups was defined in [9] and further computational tools, notably a version of Segal’s spectral sequence have been developed by the authors and collaborators in [10], and [11].

In this work, we examine Twisted equivariant K-theory with the above mentioned methods as a module over its untwisted version and prove a generalization of the completion theorem by Atiyah and Segal.

It turns out that in the case of groups which admit a finite model for the classifying space for proper actions $EG$, the ring defined as the zeroth (Untwisted ) $G$-equivariant K-theory ring $K^0_G(EG)$ is Noetherian. Hence, usual commutative algebraic methods can be applied to deal with completion problems on noetherian modules over it, as it has been done in other contexts in the literature, [6], [21], [14], [18].

Using a universal coefficient theorem developed in the analytical setting [23], we prove a version of the co-completion theorem in twisted Borel Equivariant K-homology, thus extending results in [17] to the twisted case.

This work is organized as follows:

In section 1 we collect results on the multiplicative (twist-mixing) structures on twisted equivariant K-theory following its definition in [9]. We also recall in this section the spectral sequence of [10] and the needed notions of Bredon-type cohomology and $G$-CW complexes.

In section 2 we examine the ring Structure over the ring $K^0_G(EG)$, and establish the noetherian condition for certain relevant modules over it given by twisted equivariant K-theory groups.

The main theorem, 3.6 is proved in section 3.

Theorem. Let $G$ be a group which admits a finite model for $EG$, the classifying space for proper actions. Let $X$ be a finite, proper $G$-CW complex. Then, the pro-homomorphism

$$
\varphi_{n,p} : \left\{ K^*_G(X,P)/I_G,EG^n K^*_G(X,P) \right\} \rightarrow \left\{ K^*_G(X \times EG^{n-1}, P^*(P)) \right\}
$$

is a pro-isomorphism. In particular, the system $\left\{ K^*_G(X \times EG^{n-1}, P^*(P)) \right\}$ satisfies the Mittag-Leffler condition and the $\lim^1$ term is zero.

Finally, section 4 deals with the proof of the cocompletion theorem 4.6 involving Twisted Borel K-homology.
Theorem. Let $G$ be a discrete group. Assume that $G$ admits a finite model for $EG$. Let $X$ be a finite $G$-CW complex and $P \in H^3(X \times G, EG, \mathbb{Z})$. Let $I_{G,EG}$ be the augmentation ideal. Then, there exists a short exact sequence

$$
\text{colim}_{n \geq 1} \text{Ext}^1_{\mathbb{Z}}(K^*_G(X, P) / I_{G,EG}, \mathbb{Z}) \rightarrow K^*_*(X \times G, EG, p^*(P)) \rightarrow \text{colim}_{n \geq 1} K^*_n(X, P) / I_{G,EG}^n
$$

Contents

Aknowledgments 2

1. Preliminaries on (twisted) Equivariant K-theory for Proper and Discrete actions

Twisted equivariant $K$-Theory.

1.1. Topologies on the space of Fredholm Operators

1.2. Additive structure

1.3. Multiplicative structure

Bredon Cohomology and its Čech Version

2. Module Structure for twisted Equivariant K-theory 10

3. The completion Theorem 11

4. The cocompletion Theorem 13

References 15

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1. Preliminaries on (twisted) Equivariant K-theory for Proper and Discrete actions

**Definition 1.1.** Recall that a $G$-CW complex structure on the pair $(X, A)$ consists of a filtration of the $G$-space $X = \cup_{-1 \leq n} X_n$ with $X_1 = \emptyset$, $X_0 = A$ where every space is inductively obtained from the previous one by attaching cells in pushout diagrams

\[
\begin{array}{c}
\prod_i S^{n-1} \times G/H_i \longrightarrow X_{n-1} \\
\prod_i D^n \times G/H_i \longrightarrow X_n
\end{array}
\]

We say that a proper $G$-CW complex is finite if it is constructed out of a finite number of cells $G/H \times D^n$.

We recall the notion of the classifying space for proper actions:

**Definition 1.2.** Let $G$ be a discrete group. A model for the classifying space for proper actions is a $G$-CW complex $EG$ with the following properties:

- All isotropy groups are finite.
- For any proper $G$-CW complex $X$ there exists up to $G$-homotopy a unique $G$-map $X \to EG$.

The classifying space for proper actions always exists, it is unique up to $G$-homotopy and admits several models. The following list contains some examples. We remit to [19] for further discussion.

- If $G$ is a compact group, then the singleton space is a model for $EG$.
- Let $G$ be a group acting properly and cocompactly on a Cat(0) space $X$. Then $X$ is a model for $EG$.
- Let $G$ be a Coxeter group. The Davis complex is a model for $EG$.
- Let $G$ be a mapping class group of a surface. The Teichmüller space is a model for $EG$.

Let $G$ be a discrete group. A model for the classifying space for free actions $EG$ is a free contractible $G$-CW complex. Given a model $EG$ for the classifying space for proper actions, the space $BG$ is the $CW$-complex $EG/G$.

The following result is proved in [17], lemma 26 in page 6.

**Lemma 1.3.** Let $X$ be a finite proper $G$-CW complex. Then $X \times_G EG$ is homotopy equivalent to a $CW$ complex of finite type.

**Twisted equivariant K-Theory.** Twisted Equivariant K-Theory for proper actions of discrete groups was introduced in [9]. In what follows we will recall its definition using Fredholm bundles and its properties following the above mentioned article. The crucial difference to [9] is the use of graded Fredholm bundles, which are needed for the definition of the multiplicative structure.

Let $\mathcal{H}$ be a separable Hilbert space and

\[
\mathcal{U}(\mathcal{H}) := \{ U : \mathcal{H} \to \mathcal{H} \mid U \circ U^* = U^* \circ U = \text{Id} \}
\]

the group of unitary operators acting on $\mathcal{H}$. Let $\text{End}(\mathcal{H})$ denote the space of endomorphisms of the Hilbert space and endow $\text{End}(\mathcal{H})_{c.o.}$ with the compact open topology. Consider the inclusion

\[
\mathcal{U}(\mathcal{H}) \to \text{End}(\mathcal{H})_{c.o.} \times \text{End}(\mathcal{H})_{c.o.}
\]

\[
U \mapsto (U, U^{-1})
\]

and induce on $\mathcal{U}(\mathcal{H})$ the subspace topology. Denote the space of unitary operators with this induced topology by $\mathcal{U}(\mathcal{H})_{c.o.}$ and note that this is different from the usual
compact open topology on $\mathcal{U}(\mathcal{H})$. Let $\mathcal{U}(\mathcal{H})_{c.g}$ be the compactly generated topology associated to the compact open topology, and topologize the group $\mathcal{P}U(\mathcal{H})$ from the exact sequence

$$1 \to S^1 \to \mathcal{U}(\mathcal{H})_{c.g} \to \mathcal{P}U(\mathcal{H}) \to 1.$$ 

Let $\mathcal{H}$ be a Hilbert space. A continuous homomorphism $a$ defined on a Lie group $G$, $a : G \to \mathcal{P}U(\mathcal{H})$ is called stable if the unitary representation $\tilde{\mathcal{H}}$ induced by the homomorphism $\tilde{a} : \tilde{G} = a^* \mathcal{U}(\mathcal{H}) \to \mathcal{U}(\mathcal{H})$ contains each of the irreducible representations of $\tilde{G}$

**Definition 1.4.** Let $X$ be a proper $G$-CW complex. A projective unitary $G$-equivariant stable bundle over $X$ is a principal $\mathcal{P}U(\mathcal{H})$-bundle

$$\mathcal{P}U(\mathcal{H}) \to P \to X$$

where $\mathcal{P}U(\mathcal{H})$ acts on the right, endowed with a left $G$ action lifting the action on $X$ such that:

- the left $G$-action commutes with the right $\mathcal{P}U(\mathcal{H})$ action, and
- for all $x \in X$ there exists a $G$-neighborhood $V$ of $x$ and a $G_x$-contractible slice $U$ of $x$ with $V$ equivariantly homeomorphic to $U \times_{G_x} G$ with the action

$$G_x \times (U \times G) \to U \times G, \quad k \cdot (u, g) = (ku, gk^{-1}),$$

together with a local trivialization

$$P|_V \cong (\mathcal{P}U(\mathcal{H}) \times U) \times_{G_x} G$$

where the action of the isotropy group is:

$$G_x \times ((\mathcal{P}U(\mathcal{H}) \times U) \times G) \to (\mathcal{P}U(\mathcal{H}) \times U) \times G$$

$$(k, ((F, y), g)) \mapsto ((f_x(k)F, ky), gk^{-1})$$

with $f_x : G_x \to \mathcal{P}U(\mathcal{H})$ a fixed stable homomorphism.

**Definition 1.5.** Let $X$ be a proper $G$-CW complex. A $G$-Hilbert bundle is a locally trivial bundle $E \to X$ with fiber on a Hilbert space $\mathcal{H}$ and structural group the group of unitary operators $\mathcal{U}(\mathcal{H})$ with the strong* operator topology. Note that in $\mathcal{U}(\mathcal{H})$ the strong* operator topology and the compact open topology are the same \[25\]. The_bundle of Hilbert-Schmidt operators with the strong topology between Hilbert Bundles $E$ and $F$ will be denoted by $L_{HS}(E, F)$.

The following result resumes some facts concerning projective unitary stable $G$-equivariant bundles.

**Lemma 1.6.**

(i) Given a projective unitary, stable $G$-equivariant Bundle $P$, there exists a $G$-Hilbert bundle $E \to X$ such that the bundle $\text{End}_{L_{HS}}(E, E)$ has an associated $\mathcal{P}U(\mathcal{H})$ principal, stable $G$-equivariant bundle isomorphic to $P$, where $\mathcal{P}U(\mathcal{H})$ carries the *-strong topology.

(ii) Given projective unitary stable $G$-equivariant bundles $P_1$ and $P_2$, the isomorphism class of the $\mathcal{P}U(\mathcal{H})$ bundle associated to $L_{HS}(E_1^*, E_2)$ does not depend on the choice of the Hilbert bundles $E_1$.

**Proof.**

(i) Given a central extension $1 \to S^1 \to \tilde{G} \to G \to 1$ of $G$, consider the Hilbert space $L^2_{\tilde{G}}(\tilde{G}) \subset L^2(\tilde{G})$ defined as the closure of the direct sum of all $V$-isotypical subspaces, where $V$ is a $\tilde{G}$-representation where $S^1$ acts by multiplication. Form the completed sum $\mathcal{H}$ indexed by isomorphism classes of $S^1$-central extensions $\tilde{G}$ of $G$. In symbols:

$$\mathcal{H} = \bigoplus_{\tilde{G} \in \text{Ext}(G, S^1)} L^2_{\tilde{G}}(\tilde{G}) \otimes l^2(\mathbb{N}),$$
and consider the trivial bundle $E = X \times \mathcal{H} \to X$. Form the Bundle of Hilbert endomorphisms $\text{End}_{\mathcal{H}}(E, E)$ in the $*$-strong topology \[ \text{[25]} \].

The stability of the projective unitary bundle $P$ gives a group homeomorphism between $P(\mathcal{U}(\mathcal{H}))$ and the structural group of the bundle $\text{End}_{\mathcal{H}}(E, E^*)$, which is $P(\mathcal{U}(\mathcal{H}))$.

(ii) Follows from the reduction of the structural group $\mathcal{U}(\mathcal{H})$ in the $*$-strong topology to $\mathcal{PU}(\mathcal{H})$ (in the $*$-strong topology, since the central $S^1$ acts trivially on $L_{\mathcal{H}}(E_1, E_2)$). The equivalence of principal bundles and associated bundles, as well as the classification of projective unitary, stable $G$-equivariant bundles from \[ \text{[3]} \] finish the argument.

\[ \square \]

**Definition 1.7.** Define $P_1 \otimes P_2$ as the principal $\mathcal{PU}(\mathcal{H})$-bundle associated to $L_{\mathcal{H}}(E_1, E_2)$.

In \[ \text{[3]} \], Theorem 3.8, the set of isomorphism classes of projective unitary stable $G$-equivariant bundles, denoted by $\text{Bun}_{\mathcal{PU}}^G(X, \mathcal{PU}(\mathcal{H}))$ was seen to be in bijection with the third Borel cohomology groups with integer coefficients $H^3(X \times_G EG, \mathbb{Z})$.

**Proposition 1.8.** The map $\text{Bun}_{\mathcal{PU}}^G(X, \mathcal{PU}(\mathcal{H})) \to H^3(X \times_G EG, \mathbb{Z})$ is an abelian group isomorphism if the left hand side is furnished with the tensor product as additive structure.

**Proof.** In \[ \text{[3]} \], a classifying $G$-space $\mathcal{B}$, a universal projective unitary stable $G$-equivariant bundle $\mathcal{E} \to \mathcal{B}$, as well as a homotopy equivalence $f : \text{Maps}(X, \mathcal{B})^G \to \text{Maps}(X \times_G EG, \mathcal{PU}(\mathcal{H}))$ were constructed in Theorem 3.8. (This was only stated for $\pi_0$ there, but the argument goes over to higher homotopy groups). On the other hand, Theorem 3.8 in \[ \text{[3]} \] gives an isomorphism of sets to the equivalence classes of projective unitary stable $G$-equivariant bundles $\text{Bun}_{\mathcal{PU}}^G(X, \mathcal{PU}(\mathcal{H}))$. On the isomorphic sets $\pi_0(\text{Maps}(X, \mathcal{B})^G) \cong \pi_0(\text{Maps}(X \times_G EG, \mathcal{PU}(\mathcal{H})))$ define the operations

- The operation $*$, given by the unique $H$-space structure in $\mathcal{PU}(\mathcal{H}) = K(\mathbb{Z}, 3)$, and
- The operation $*$, defined in $\pi_0(\text{Maps}(X, \mathcal{B})^G)$ as follows. Given maps $f_0$ and $f_1$ consider the projective unitary stable $G$-equivariant bundles $f_i^*(\mathcal{E})$, where $\mathcal{E}$ is the universal bundle and form the classifying map $\psi$ of the projective unitary stable, $G$-equivariant bundle $f_1^*(\mathcal{E}) \otimes f_2^*(\mathcal{E})$. Define $f_1 * f_2 = \psi$.

The classification of bundles yields that these operations are mutually distributive and associative, and have a common neutral element given by the constant map. The two operations agree then because of the standard Lemma, see for example Lemma 2.10.10, page 56 in \[ \text{[1]} \].

\[ \square \]

**Definition 1.9.** Let $X$ be a proper $G$-CW complex and let $\mathcal{H}$ be a separable Hilbert space. The space $\text{Fred}(\mathcal{H})$ consists of pairs $(A, B)$ of bounded operators on $\mathcal{H}$ such that $AB - 1$ and $BA - 1$ are compact operators. Endow $\text{Fred}(\mathcal{H})$ with the topology induced by the embedding

\[
\text{Fred}(\mathcal{H}) \to B(\mathcal{H}) \times B(\mathcal{H}) \times K(\mathcal{H}) \times K(\mathcal{H})
\]

\[
(A, B) \mapsto (A, B, AB - 1, BA - 1)
\]

where $B(\mathcal{H})$ denotes the bounded operators on $\mathcal{H}$ with the compact open topology and $K(\mathcal{H})$ denotes the compact operators with the norm topology.
We denote by $\tilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$ a $\mathbb{Z}_2$-graded, infinite dimensional Hilbert space.

**Definition 1.10.** Let $U(\tilde{\mathcal{H}})_{c.g.}$ be the group of even, unitary operators on the Hilbert space $\tilde{\mathcal{H}}$ which are of the form

$$
\begin{pmatrix}
u_1 & 0 \\ 0 & \nu_2
\end{pmatrix},
$$

where $\nu_i$ denotes a unitary operator in the compactly generated topology defined as before.

We denote by $PU(\tilde{\mathcal{H}})$ the group $U(\tilde{\mathcal{H}})_{c.g.} / S^1$ and recall the central extension

$$
1 \to S^1 \to U(\tilde{\mathcal{H}}) \to PU(\tilde{\mathcal{H}}) \to 1
$$

**Definition 1.11.** Let $X$ be a proper $G$-CW complex. The space $\text{Fred}''(\tilde{\mathcal{H}})$ is the space of pairs $(\tilde{A}, \tilde{B})$ of self-adjoint, bounded operators of degree 1 defined on $\tilde{\mathcal{H}}$ such that $\tilde{A}\tilde{B} - I$ and $\tilde{B}\tilde{A} - I$ are compact.

Given a $\mathbb{Z}/2$-graded, stable Hilbert space $\hat{\mathcal{H}}$, the space $\text{Fred}''(\hat{\mathcal{H}})$ is homeomorphic to $\text{Fred}'(\mathcal{H})$.

**Definition 1.12.** We denote by $\text{Fred}^{(0)}(\tilde{\mathcal{H}})$ the space of self-adjoint degree 1 Fredholm operators $A$ in $\tilde{\mathcal{H}}$ such that $A^2$ differs from the identity by a compact operator, with the topology coming from the embedding $A \mapsto (A, A^2 - I)$ in $\mathcal{B}(\mathcal{H}) \times K(\mathcal{H})$.

The following result was proved in [3], Proposition 3.1:

**Proposition 1.13.** The space $\text{Fred}^{(0)}(\tilde{\mathcal{H}})$ is a deformation retract of $\text{Fred}''(\tilde{\mathcal{H}})$.

The above discussion can be concluded telling that $\text{Fred}^{(0)}(\tilde{\mathcal{H}})$ is a representing space for $K$-theory. The group $U(\tilde{\mathcal{H}})_{c.g.}$ of degree 0 unitary operators on $\tilde{\mathcal{H}}$ with the compactly generated topology acts continuously by conjugation on $\text{Fred}^{(0)}(\tilde{\mathcal{H}})$, therefore the group $PU(\tilde{\mathcal{H}})$ acts continuously on $\text{Fred}^{(0)}(\tilde{\mathcal{H}})$ by conjugation. In [3] twisted $K$-theory for proper actions of discrete groups was defined using the representing space $\text{Fred}'(\mathcal{H})$, but in order to have multiplicative structure we proceed using $\text{Fred}^{(0)}(\tilde{\mathcal{H}})$.

Let us choose the operator

$$
\hat{I} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}
$$

as the base point in $\text{Fred}^{(0)}(\tilde{\mathcal{H}})$.

Choosing the identity as a base point on the space $\text{Fred}'(\mathcal{H})$, gives a diagram of pointed maps

$$
\begin{array}{ccc}
\text{Fred}^{(0)}(\tilde{\mathcal{H}}) & \xrightarrow{i} & \text{Fred}''(\tilde{\mathcal{H}}) & \xrightarrow{f} & \text{Fred}'(\mathcal{H}) \\
& & r & & \\
& & \text{Fred}^{(0)}(\tilde{\mathcal{H}})
\end{array}
$$

where $i$ denotes the inclusion, $r$ is a strong deformation retract and $f$ is a homeomorphism. Moreover, the maps are compatible with the conjugation actions of the groups $U(\tilde{\mathcal{H}})_{c.g.}$, $U(\mathcal{H})_{c.g.}$ and the map $U(\tilde{\mathcal{H}})_{c.g.} \to U(\mathcal{H})_{c.g.}$.

Let $X$ be a proper compact $G$-ANR and let $P \to X$ be a projective unitary stable $G$-equivariant bundle over $X$. Denote by $\tilde{P}$ the projective unitary stable bundle obtained by performing the tensor product with the trivial bundle $\mathbb{P}(\tilde{\mathcal{H}})$, $\tilde{P} = P \otimes \mathbb{P}(\tilde{\mathcal{H}})$.
The space of Fredholm operators is endowed with a continuous right action of the group $PU(\hat{H})$ by conjugation, therefore we can take the associated bundle over $X$

$$\text{Fred}^{(0)}(\hat{P}) := \hat{P} \times_{PU(\hat{H})} \text{Fred}^{(0)}(\hat{H}),$$

and with the induced $G$ action given by

$$g \cdot ([\lambda, A]) := ([g\lambda, A])$$

for $g$ in $G$, $\lambda$ in $\hat{P}$ and $A$ in $\text{Fred}^{(0)}(\hat{H})$. Denote by

$$\Gamma(X; \text{Fred}^{(0)}(\hat{P}))$$

the space of sections of the bundle $\text{Fred}^{(0)}(\hat{P}) \to X$ and choose as base point in this space the section which chooses the base point $\hat{I}$ on the fibers. This section exists because the $PU(\hat{H})$ action on $\hat{I}$ is trivial, and therefore

$$X \cong \hat{P}/PU(\hat{H}) \cong \hat{P} \times_{PU(\hat{H})} \{\hat{I}\} \subset \text{Fred}^{(0)}(\hat{P});$$

let us denote this section by $s$.

**Definition 1.14.** Let $X$ be a connected $G$-space and $P$ a projective unitary stable $G$-equivariant bundle over $X$. The **Twisted $G$-equivariant $K$-theory** groups of $X$ twisted by $P$ are defined as the homotopy groups of the $G$-equivariant sections

$$K^{P}_{G}(X; P) := \pi_{P} \left( \Gamma(X; \text{Fred}^{(0)}(\hat{P}))^{G}, s \right).$$

where the base point $s = \hat{I}$ is the section previously constructed.

1.1. **Topologies on the space of Fredholm Operators.** In [24] a Fredholm picture of twisted $K$-theory is introduced, using the strong-* operator topology on the space of Fredholm Operators. For the sake of completeness, we establish here the isomorphism of these twisted equivariant $K$-theory groups with the ones described here.

Denote by $\text{Fred}'(\mathcal{H})_{ss}$ the space whose elements are the same as $\text{Fred}'(\mathcal{H})$ but with the strong *-topology on $B(\mathcal{H})$.

**Definition 1.15.** [24] Thm. 3.15] Let $X$ be a connected $G$-space and $P$ a projective unitary stable $G$-equivariant bundle over $X$. The **Twisted $G$-equivariant $K$-theory** groups of $X$ (in the sense of Tu-Xu-Laurent) twisted by $P$ are defined as the homotopy groups of the $G$-equivariant strong*-continuous sections

$$K^{P}_{G}(X; P) := \pi_{P} \left( \Gamma(X; \text{Fred}'(P))^{G}, s \right).$$

The bundle $\text{Fred}'(P)_{ss}$ is defined in a similar way as $\text{Fred}'(P)$.

We will prove that the functors $K^{P}_{G}(-, P)$ and $K^{*}_{G}(-, P)$ are naturally equivalent.

**Lemma 1.16.** The spaces $\text{Fred}'(\mathcal{H})$ and $\text{Fred}'(\mathcal{H})_{ss}$ are $PU(\mathcal{H})$-weakly homotopy equivalent.

*Proof.* The strategy is to prove that $\text{Fred}'(\mathcal{H})_{ss}$ is a representing of equivariant $K$-theory. The same proof for $\text{Fred}'(\mathcal{H})$ in [3] Prop. A.22 applies. In particular $GL(\mathcal{H})_{ss}$ is $G$-contractible because the homotopy $h_{t}$ constructed in [3] Prop. A.21] is continuous in the strong*-topology and then the proof applies. \qed

Using the above lemma one can prove that the identity map defines an equivalence between (twisted) cohomology theories $K^{P}_{G}(-, P)$ and $K^{*}_{G}(-, P)$. Then we have that the both definitions of twisted $K$-theory are equivalents. Summarizing
Theorem 1.17. For every proper $G$-CW-complex $X$ and every projective unitary stable $G$-equivariant bundle over $X$. We have an isomorphism 

$$K^*_G(X; P) \cong K^*_G(X; P).$$

Remark 1.18. In order to simplify the notation from now on we denote by $\mathcal{H}$ a $\mathbb{Z}_2$-graded separable Hilbert space and we denote by $\text{Fred}^{(0)}(P)$ the bundle $\text{Fred}^{(0)}(\hat{P})$.

1.2. Additive structure. There exists a natural map 

$$\Gamma(X; \text{Fred}^{(0)}(\hat{P})^G) \times \Gamma(X; \text{Fred}^{(0)}(\hat{P}))^G \to \Gamma(X; \text{Fred}^{(0)}(\hat{P})^G),$$

inducing an abelian group structure on the twisted equivariant $K$-theory groups, which we will define below. Consider for this the following commutative diagram.

$$
\begin{array}{ccc}
\text{Fred}^{(0)}(\hat{H}) \times \text{Fred}^{(0)}(\hat{H}) & \xrightarrow{i} & \text{Fred}^{(0)}(\hat{H}) \\
\downarrow & & \downarrow \\
\text{Fred}^{(0)}(\hat{H}) & \xleftarrow{f^{-1} \sigma} & \text{Fred}^{(0)}(\hat{H})
\end{array}
$$

where the vertical map denotes composition. As the maps involved in the diagram are compatible with the conjugation actions of the groups $U(\hat{H})_{\cdot, g}$, respectively $U(\hat{H})_{\cdot, g}$ and $G$, for any projective unitary, stable $G$-equivariant bundle $P$, this induces a pointed map 

$$\Gamma(X; \text{Fred}^{(0)}(\hat{P})^G, s) \times (\Gamma(X; \text{Fred}^{(0)}(\hat{P}))^G, s) \to (\Gamma(X; \text{Fred}^{(0)}(\hat{P})^G, s).$$

Which defines an additive structure in $K^*_G(X; P)$.

1.3. Multiplicative structure. We define an associative product on twisted $K$-theory.

$$K^*_G(X; P) \times K^*_G(X; P') \to K^*_G(X; P \otimes P')$$

Induced by the map

$$\langle A, A' \rangle \mapsto A \hat{\otimes} I + I \hat{\otimes} A'$$

defined in $\text{Fred}^{(0)}(\hat{H})$, and $\hat{\otimes}$ denotes the graded tensor product, see [7] in pages 24-25 for more details. We denote this product by $\bullet$.

Let $0$ be the projective unitary, stable $G$-equivariant bundle associated to the neutral element in $H^4(X \times_G EG, \mathbb{Z})$. The groups $\pi_2(\text{Fred}(0))$ define un twisted, equivariant, representable $K$-Theoy in negative degree for proper actions. The extended version via Bott periodicity agrees with the usual definitions of untwisted, equivariant $K$-theory groups for compact $G$-CW complexes [22], [21] as a consequence of Theorem 3.8, pages 8-9 in [10].

Bredon Cohomology and its Čech Version. (Untwisted) Bredon cohomology has been an useful tool to approximate equivariant cohomology theories with the use of spectral sequences of Atiyah-Hirzebruch type [15], [10].

We will recall a version of Bredon cohomology with local coefficients which was introduced in [10] and compared there to other approaches. These approaches fit all into the general approach of spaces over a category [15], [8].

Let $\mathcal{U} = \{ U_\sigma \mid \sigma \in I \}$ be an open cover of the proper $G$-CW complex $X$ which is closed under intersections and has the property that each open set $U_\sigma$ is $G$-equivariantly homotopic to an orbit $G/H_\sigma \subset U_\sigma$ for a finite subgroup $H_\sigma$. The existence of such a cover, sometimes known as contractible slice cover, is guaranteed for proper $G$-ANR’s by an appropriate version of the slice Theorem (see [2]).
Definition 1.19. Denote by $\mathcal{N}_G\mathcal{U}$ the category with objects $\mathcal{U}$ and where a morphism is given by an inclusion $U_\sigma \rightarrow U_\tau$. A twisted coefficient system with values on $R$-Modules is a contravariant functor $\mathcal{N}_G\mathcal{U} \rightarrow R\text{-Mod}$.

Definition 1.20. Let $X$ be a proper $G$-space with a contractible slice cover $\mathcal{U}$, and let $M$ be a twisted coefficient system. Define the Bredon equivariant homology groups with respect to $U$ as the homology groups of the category $\mathcal{N}_G\mathcal{U}$ with coefficients in $M$,

$$H^n_{\mathcal{G}}(X,\mathcal{U}; M) := H^n(\mathcal{N}_G\mathcal{U}, M).$$

These are the homology groups of the chain complex defined as the $R$-module

$$C^\mathcal{Z}_\ast(\mathcal{N}_G\mathcal{U}) \otimes_{\mathcal{N}_G\mathcal{U}} M,$$

given as the balanced tensor product of the contravariant, free $\mathcal{Z}_\ast\mathcal{N}_G\mathcal{U}$-chain complex $C^\mathcal{Z}_\ast(\mathcal{N}_G\mathcal{U})$ and $M$. This is the $R$-module

$$\bigoplus_{U_\sigma \in \mathcal{N}_G\mathcal{U}} R \otimes_R M(U_\sigma)/K$$

where $K$ is the $R$-module generated by elements

$$r \otimes x - r \otimes i^\ast(x),$$

for an inclusion $i : U_\sigma \rightarrow U_\tau$.

Remark 1.21 (Coefficients of twisted equivariant $K$-Theory on contractible Covers). Let $i_\sigma : G/H_\sigma \rightarrow U_\sigma \rightarrow X$ be the inclusion of a $G$-orbit into $X$ and consider the Borel cohomology group $H^3(EG \times_G G/H_\sigma, \mathbb{Z})$. Given a class $P \in H^3(EG \times_G X, \mathbb{Z})$, we will denote by $H^3_{P_\sigma}$ the central extension $1 \rightarrow S^1 \rightarrow H^3_{P_\sigma} \rightarrow H_\sigma \rightarrow 1$ associated to the class given by the image of $P$ under the maps

$$\omega_\sigma : H^3(EG \times X, \mathbb{Z}) \overset{i_\ast}{\rightarrow} H^3(EG \times_G G/H_\sigma, \mathbb{Z}) \overset{\sim}{\rightarrow} H^3(BH_\sigma, \mathbb{Z}) \overset{\sim}{\rightarrow} H^2(BH_\sigma, S^1).$$

Restricting the functors $K^p_G(X, P)$ and $K^1_G(X, P)$ to the subsets $U_\sigma$ gives contravariant functors defined on the category $\mathcal{N}_G\mathcal{U}$.

As abelian groups, the functors $K^p_G(X, P)$ satisfy:

$$K^p_G(U_\sigma, P) = \begin{cases} R_{S^1}(H^3_{P_\sigma}) & \text{if } j = 0 \\ 0 & \text{if } j = 1 \end{cases}$$

The Symbol $R_{S^1}(H^3_{P_\sigma})$ denotes the subgroup of the abelian group of isomorphisms classes of complex $H^3_{P_\sigma}$-representations, where $S^1$ acts by complex multiplication.

We recall the key result from [10], proposition 4.2

Proposition 1.22. spectral sequence associated to the locally finite and equivariantly contractible cover $\mathcal{U}$ and converging to $K^p_G(X, P)$, has for second page $E_2^{p,q}$ the cohomology of $\mathcal{N}_G\mathcal{U}$ with coefficients in the functor $K^q_G(?, P|?)$ whenever $q$ is even, i.e.

$$E_2^{p,q} := H^p(\mathcal{N}_G\mathcal{U}; K^q_G(?)),$$

and is trivial if $q$ is odd. Its higher differentials

$$d_r : E_r^{p,q} \rightarrow E_r^{p+r,q-r+1}$$

vanish for $r$ even.
2. Module Structure for Twisted Equivariant K-theory

Let \( X \) be a proper \( G \)-CW complex, and let \( P \) be a stable projective unitary \( G \)-equivariant bundle over \( X \). Recall that up to \( G \)-equivariant homotopy, there exists a unique map \( \lambda : X \rightarrow EG \). The map \( \lambda \) together with the multiplicative structure give an abelian group homomorphism

\[
K^0_G(EG) \rightarrow K^0_G(X, P),
\]

which gives \( K^0_G(X, P) \) the structure of a module over the ring \( K^0_G(EG) \).

We will analyze the structure of \( K^0_G(EG) \) as a ring. The results in the following lemma are proved inside the proofs of Theorem 4.3, page 610 in [21], and Theorem 6.5, page 21 in [20].

**Proposition 2.1.** Let \( G \) be a group which admits a finite model for the classifying space for proper actions \( EG \). Then,

- \( K^0_G(EG) \) is isomorphic to the Grothendieck Group of \( G \)-equivariant, finite dimensional complex vector bundles.
- The ring \( K^0_G(EG) \) is noetherian.
- Let \( \mathcal{O}_{FZN}(G) \) be the orbit category consisting of homogeneous spaces \( G/H \) with \( H \) finite and \( G \)-equivariant maps. Denote by \( R(?) \) the contravariant \( \mathcal{O}_{FZN}(G) \)-module given by assigning to an object \( G/H \) the complex representation ring \( R(H) \) and to a morphism \( G/H \rightarrow G/K \) the restriction \( R(K) \rightarrow R(H) \). Then, there exists a ring homomorphism

\[
K^0_G(EG) \rightarrow \lim_{\mathcal{O}_{FZN}(G)} R(?)
\]

which has nilpotent kernel and cokernel.

- Given a prime number \( p \), there exists a vector bundle \( E \) of dimension prime to \( p \), such that for every point \( x \in EG \), the character of the \( G_x \) representation \( E \mid_x \) evaluated on an element of order not a power of \( p \) is 0.

**Proof.**

- This is proved in [21], [22], [16], 3.8 in pages 8-9.
- Given a finite proper \( G \)-CW complex \( X \), there exists an equivariant Atiyah-Hirzebruch spectral sequence abutting to \( K^*_G(X) \) with \( E_2 \) term given by \( E_2^{p,q} = H^p_{Z\mathcal{O}_{FZN}(G)}(X, K^q(G/\Sigma^?)) \), where the right hand side denotes untwisted Bredon cohomology, defined over the Orbit Category \( \mathcal{O}_{FZN}(G) \) rather than over the category \( \mathcal{N}\mathcal{O}H \).

The group \( E_2^{p,q} \) can be identified with Bredon cohomology with coefficients on the representation ring if \( q \) is even and is zero otherwise. Since the Bredon cohomology groups of the spectral sequence are finitely generated if \( EG \) is a finite \( G \)-CW complex, this proves the first assumption.

- The edge homomorphism of the Atiyah-Hirzebruch spectral sequence of [15] gives a ring homomorphism \( K^*_G(X) \rightarrow H^*_\mathcal{O}_{FZN}(G)(X, R) \). The right hand side can be identified with the ring \( \lim_{\mathcal{O}_{FZN}(G)} R(?) \). The rational collapse of the equivariant Atiyah-Hirzebruch spectral sequence gives the second part.

- Let \( m \) be the least common multiple of the orders of isotropy groups \( H \) in \( EG \). For any finite subgroup \( H \), pick up a homomorphism \( \alpha_H : H \rightarrow \Sigma_m \) corresponding to a free action of \( H \) on \( \{1, \ldots, m\} \). Let \( n \) be the order of the group \( \Sigma_m/Symp(\Sigma_m) \) and let \( \rho : \Sigma_m \rightarrow U(n) \) be the permutation representation. Consider the element \( \{V_H\} = \{C^n[\rho \circ \alpha_H]\} \) in the inverse limit \( \lim_{\mathcal{O}_{FZN}(G)} R(?) \). According to the second part, there exists a vector bundle \( E \) which is mapped to some power \( \{V_H \otimes \} \). The Vector bundle satisfies the required properties.
Lemma 2.2. Let $G$ be a discrete group admitting a finite model for $EG$ and $P$ be a stable projective unitary $G$-bundle over a finite $G$-CW complex $X$. Then, the $K^0_G(EG)$-modules $K^i_G(X, P)$ are noetherian for $i = 0, 1$.

Proof. There exists ([10], Theorem 4.9 in page 14), a spectral sequence abutting to $K^*_G(X, P)$. Its $E_2$ term consists of groups $E_2^{q,p}$, which can be identified with a version of Bredon cohomology associated to a compact, open, $G$-invariant cover $U$ consisting of open sets which are $G$-homotopy equivalent to proper orbits.

These groups are denoted by $H^q_{\text{NCG}}(X, K^0_G(U))$ and are zero if $q$ is odd. Since $X$ is a proper, compact $G$-CW complex, the cover can be assumed to be finite. Given an element of the cover $U$, the group $K^0_G(U)$ is a finitely generated, free abelian group, as it is seen from A.3.4, page 40 in [9], where the groups $K^0_G(U)$ are identified with groups of projective complex representations. Compare also remark ([12]).

In particular the groups $H^q_{\text{NCG}}(X, K^0_G(U))$ in the spectral sequence abutting to $K^*_G(X, P)$ are finitely generated. By induction, the groups $E_2^{q,p}$ are finitely generated for all $r$ and hence the term $E_\infty$. Hence $K^*_G(X, P)$ is it for $i = 0, 1$. Since $K^*_G(E_G)$ is a noetherian ring, the result follows. □

3. The Completion Theorem

Definition 3.1 (Augmentation ideal). Let $G$ be a discrete group. Given a proper $G$-CW complex, the augmentation ideal $I_{G,X} \subset K^0_G(X)$ is defined to be the kernel of the homomorphism

$$K^0_G(X) \to K^0_G(X_0) \to K^0_{(e)}(X_0)$$

defined by restricting to the zeroth skeleton and restricting the acting group to the trivial group.

Proposition 3.2. Let $X$ be an $n$-dimensional proper $G$-CW complex. Then, any product of $n+1$ elements in $I_{G,X}$ is zero.

Proof. This is proved in [21], lemma 4.2 in page 609. □

We fix now our notations concerning pro-modules and pro-homomorphisms.

Let $R$ be a ring. A pro-module indexed by the integers is an inverse system of $R$-modules:

$$M_0 \xleftarrow{\alpha_0} M_1\xleftarrow{\alpha_1} M_2 \xleftarrow{\alpha_2} M_3, \ldots$$

We write $\alpha_n = \alpha_{n+1} \circ \ldots \circ \alpha_n$.

A strict pro-homomorphism $\{M_n, \alpha_n\} \to \{N_n, \beta_n\}$ consists of a collection of homomorphisms $\{f_n : M_n \to N_n\}$ such that $\beta_n \circ f_n = f_{n+1} \circ \alpha_n$ holds for each $n \geq 2$. A pro $R$-module $\{M_n, \alpha_n\}$ is called pro-trivial if for each $m \geq 1$ there is some $n \geq m$ such that $\alpha_n = 0$. A strict homomorphism $f$ as above is called a pro isomorphism if $\ker(f)$ and $	ext{coker}(f)$ are both pro-trivial. A sequence of strict homomorphisms

$$\{M_n, \alpha_n\} \xrightarrow{f_n} \{M'_n, \alpha'_n\} \xrightarrow{g_n} \{M''_n, \alpha''_n\}$$

is called pro-exact if $g_n \circ f_n = 0$ holds for $n \geq 1$ and the pro-$R$-module $\ker(g_n)/\text{im}(f_n)$ is pro-trivial. The following lemmas are proved in [5], Chapter 10, section 2, see also [21]:

Lemma 3.3. Let $0 \to \{M', \alpha'_n\} \to \{M_n, \alpha_n\} \to \{M''_n, \alpha''_n\} \to 0$ be a pro-exact sequence of pro-$R$-modules. Then there is a natural exact sequence
for any exact sequence

Fix any commutative noetherian ring

Lemma 3.4. Let

Theorem 3.6. Let

\[ \{M\}' \rightarrow M \rightarrow M'' \] of pro-R-modules is pro-exact.

Definition 3.5 (Completion Map). Let \( X \) be a proper \( G \)-CW complex. Let \( p : X \times EG \rightarrow X \) be the projection to the first coordinate. The up to \( G \)-homotopy unique map \( \lambda : X \rightarrow EG \), combined with Proposition 3.2 defines a pro-homomorphism

\[ \varphi_{\lambda,p} : \left\{ K_G^*(X,P)/I_G EG^nK_G^*(X,P) \right\} \rightarrow \left\{ K_G^*(X \times EG^{n-1},p^*(P)) \right\} \]

Theorem 3.6. Let \( G \) be a group which admits a finite model for \( EG \), the classifying space for proper actions. Let \( X \) be a finite, proper \( G \)-CW complex. Then, the pro-homomorphism

\[ \varphi_{\lambda,p} : \left\{ K_G^*(X,P)/I_G EG^nK_G^*(X,P) \right\} \rightarrow \left\{ K_G^*(X \times EG^{n-1},p^*(P)) \right\} \]

is a pro-isomorphism. In particular, the system \( \{K_G^*(X \times EG^{n-1},p^*(P))\} \) satisfies the Mittag-Leffler condition and the \( \text{lim}^1 \) term is zero.

Proof. Due to propositions 2.1 and 2.2 we are dealing with a noetherian ring

and the noetherian modules \( K_G^*(X,P) \) over it. Hence, we can use lemmas 3.3 and 3.4 and the 5-lemma for pro-modules and pro-homomorphisms to prove the result by induction on the dimension of \( X \) and the number of cells in each dimension.

Assume that \( X = G/H \) for a finite group \( H \). Then, the completion map fits in the following diagram

\[ \begin{array}{ccc}
K_G^*(G/H,P)/I_G EG & \rightarrow & K_G^*(G/H \times EG^{n-1},p^*(P)) \\
\text{ind}_{H \rightarrow G} \cong & & \cong \text{ind}_{H \rightarrow G} \\
\left\{ K_H^*(\{\bullet\},P|_{eH})/J^n \right\} & \rightarrow & \left\{ K_H^*(EH^{n-1},p^*(P)) \right\} \\
\downarrow & & \downarrow = \\
\left\{ K_H^*(\{\bullet\},P|_{eH})/I_{H,\{\bullet\}} \right\} & \rightarrow & \left\{ K_H^*(EH^{n-1},p^*(P)) \right\}
\end{array} \]

The higher vertical maps are induction isomorphisms, and the ideal \( J \) is generated by the image of \( I_G EG \) under the map \( \text{ind}_{H \rightarrow G} \circ \lambda \). The lower horizontal map
is a pro-isomorphism as a consequence of the Atiyah-Segal Completion Theorem for Twisted Equivariant K-theory of finite groups, Theorem 1, page 1925 in [18], where it is proved even for compact Lie groups. We will analyze now the lower vertical map and verify that it is a pro-isomorphism of pro-modules. This amounts to prove that \( I_H, (\star) / J \) is nilpotent. Since the representation ring of \( H \), \( R(H) \) is noetherian, this holds if every prime ideal which contains \( J \) also contains \( I_H, (\star) \).

For an element \( v \in H \), denote by \( \chi_v \) the characteristic function of the conjugacy class of \( v \). Let \( H \) be a finite group. Let \( \zeta \) be the primitive \(|H|\)-root of unity given by \( e^{2\pi i/|H|} \). Put \( A = \mathbb{Z}[\zeta] \).

Recall [3], lemma 6.4 in page 63, that given a finite group \( H \), and a prime ideal of the representation ring \( \mathcal{P} \), there exists a prime ideal \( p \subset A \) an an element in \( H \), \( v \) such that \( \mathcal{P} = \chi_v^{-1}(p) \).

Let \( \mathcal{P} \) be a prime ideal containing \( J \). We can assume that there exist \( s, t \in H \) with \( \chi_v^{-1}(t) \in p \) and such that if \( p \) is the characteristic of the field \( A/p \), then the order of \( s \) is prime to \( p \).

According to part 3 of proposition 2.1, there exists a complex vector bundle \( E \) over \( EG \) such that \( p \) is prime to \( \dim_C E \), and the character \( \chi_E|_v \) is zero after evaluation at the conjugacy class of \( s \). Let \( k = \dim_E \). Then, \( C^k \mid \chi|_{(G/H)} \) is in \( I_H, (\star) \). It follows that \( \mathcal{P} \) contains \( I_H, (\star) \).

This proves that the lower horizontal arrow is a pro-isomorphism, the \( \lim^1 \) term is zero, and the theorem holds for 0-dimensional G-CW complexes \( X \). Assume that the theorem holds for all \( n-1 \)-dimensional, finite proper G-CW complexes. Given a \( k \)-dimensional, finite, proper G-CW complex, \( X \) there exists a pushout

\[
\begin{array}{ccc}
\coprod S^{k-1} \times G/H & \longrightarrow & \coprod D^k \times G/H \\
\downarrow & & \downarrow \\
Y & \longrightarrow & X
\end{array}
\]

where \( Y \) is a \( k \)-dimensional, finite proper G-CW complex. The Mayer-Vietoris sequence for twisted equivariant K-theory gives pro-homomorphisms

\[
\cdots \left\{ K_G^*(X, P)/I_{G, EG^n} \right\} \longrightarrow \\
\left\{ K_G^*(Y, P)/I_{G, EG^n} \right\} \bigoplus \bigoplus \left\{ K_G^*(D^k \times G/H, P)/I_{G, EG^n} \right\} \longrightarrow \\
\bigoplus \left\{ K_G^*(S^{k-1} \times G/H, P)/I_{G, EG^n} \right\} \longrightarrow \left\{ K_G^{n+1}(X, P)/I_{G, EG^n} \right\} \cdots
\]

By induction, the completion maps for the \( n-1 \)-dimensional G-CW complexes are isomorphisms. By the 5-lemma for pro-groups, the completion map for \( X \) is an isomorphism.

\[\square\]

**Corollary 3.7.** Let \( G \) be a discrete group with a finite model for \( EG \). Let \( P \in H^3(BG, \mathbb{Z}) \cong H^3(EG \times_G EG, \mathbb{Z}) \) be a discrete torsion twisting. Consider \( I = I_G(EG) \) Then,

\[ K^*(BG, p^*(P)) \cong K_G^*(EG, P)_I \]

4. The cocompletion Theorem

Given a CW complex \( X \), and a class \( P \in H^3(X, \mathbb{Z}) \), the twisted K-homology groups are defined in terms of Kasparov bivariant groups involving continuous trace
algebras. We remit the reader for preliminaries on Kasparov KK-Theory and its relation to $K$-homology and Brown-Douglas-Fillmore Theory of extensions to [12], Chapter VII.

Let $H$ be a separable Hilbert space. Let $K$ be the $C^*$-algebra of compact operators in $H$. Recall that the automorphism group of the $C^*$-algebra $K$ consists of the unitary operators with the norm topology $U(H)$ and the inner automorphisms can be identified with the central $S^1$. Hence, there is an action of the group $PU(H) = U(H)$ on the algebra $K$.

**Remark 4.1.** The norm topology and the compactly generated topology agree on compact operators, hence, there is also a conjugation action of the group $U(H)_{c.g}$ of unitary operators in the compactly generated topology, as well as a group homomorphism $PU(H) \rightarrow \text{out}(K)$ to the outer automorphism group of the $C^*$-algebra $K$.

**Definition 4.2** (Continuous trace Algebras). Let $X$ be a CW complex. Given a cohomology class in the third cohomology group, $H^3(X, \mathbb{Z})$, represented by a principal projective unitary bundle $P : E \rightarrow X$, the continuous trace algebra associated to $P$ is the algebra $AP$ of continuous sections of the bundle $K \times_{PU(H)} E \rightarrow X$.

**Definition 4.3** (KK-picture of twisted $K$-homology). Let $X$ be a locally compact space and $P$ be a $P(U(H))$-principal bundle. The twisted equivariant $K$-homology groups associated to the projective unitary principal bundle $P$ are defined as the $KK$-groups

$$K_*(X, P) = KK_*(AP, \mathbb{C})$$

Continuous trace algebras, used in the operator theoretical definition of twisted $K$-theory and $K$-homology belong to the Bootstrap class [13] Proposition IV.1.4.16, in page 334. Hence, the following form of the Universal Coefficient Theorem for $KK$-Groups holds. It was proved in [23], page 439, Theorem 1.17:

**Theorem 4.4** (Universal coefficient Theorem for Kasparov KK-Theory). Let $A$ be a $C^*$-algebra belonging to the smallest full subcategory of separable nuclear $C^*$-algebras and which is closed under strong Morita equivalence, inductive limits, extensions, ideals, and crossed products by $\mathbb{R}$ and $\mathbb{Z}$. Then, there is an exact sequence

$$0 \rightarrow \text{Ext}_{\mathbb{Z}}(K^*(A), K^*(B)) \rightarrow KK_*(A, B) \rightarrow \text{Hom}_{\mathbb{Z}}(K^*(A), K^*(B)) \rightarrow 0$$

Where $K^*$ denotes the topological $K$-theory groups for $C^*$-algebras.

Specializing to the algebras $AP$ one has:

**Theorem 4.5.** Let $X$ be a locally compact space and $P$ be a $P(U(H))$-principal bundle. Then, there is an exact sequence

$$0 \rightarrow \text{Ext}_{\mathbb{Z}}(K^{*-1}(X, P), \mathbb{Z}) \rightarrow K_*(X, P) \rightarrow \text{Hom}_{\mathbb{Z}}(K^*(X, P), \mathbb{Z}) \rightarrow 0$$

We will prove the following cocompletion Theorem, inspired by the methods and results of [17].

**Theorem 4.6.** Let $G$ be a discrete group. Assume that $G$ admits a finite model for $EG$. Let $X$ be a finite $G$-CW complex and $P \in H^3(X \times_G EG, \mathbb{Z})$. Let $I_{G,EG}$ be the augmentation ideal. Then, there exists a short exact sequence

$$\text{colim}_{n \geq 1} \text{Ext}_{\mathbb{Z}}^n(K_G^*(X, P)/I_G^n EG, \mathbb{Z}) \rightarrow K_*(X \times_G EG, p^*(P)) \rightarrow \text{colim}_{n \geq 1} K_G^*(X, P)/I_G^n EG$$
Proof. Choose a CW complex $Y$ of finite type and a cellular homotopy equivalence $f : Y → X \times_G EG$. Let $f^n : Y^n → X \times_G EG^n$ be the map restricted to the skeletons. The pro-homomorphisms

\[ \left\{ K^*(X \times_G EG^n, p^*(P)) \right\} \longrightarrow \left\{ K^*(Y^n, p^*(P) \mid Y_n) \right\} \]

are a pro-isomorphism of abelian pro-groups. On the other hand, due to the completion theorem, there is a pro-isomorphism

\[ \varphi_{\lambda,P} : \left\{ K^*_G(X,P)/I_G,EG^nK^*_G(X,P) \right\} \longrightarrow \left\{ K^*_G(X \times_G EG^{n-1}, p^*(P)) \right\} \]

Using [13] one gets the exact sequence

\[ 0 → \text{Ext}_G(K_{*+1}(Y, p^*(P)), \mathbb{Z}) → K^*(Y, p^*(P)) → \text{Hom}_{\mathbb{Z}}(K_*(Y, p^*(P)), \mathbb{Z}) → 0. \]

Combining this exact sequence with the pro-isomorphisms given previously, one has the exact sequence

\[ \text{colim}_{n≥1} \text{Ext}^1_G(K^*_G(X,P)/I_G,EG^n,\mathbb{Z}) → K^*_G(X \times_G EG, p^*(P)) → \text{colim}_{n≥1} K^*_G(X,P)/I_G,EG \]

□

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