A proof of Catalan’s Convolution formula

Alon Regev
Department of Mathematical Sciences
Northern Illinois Univeristy
DeKalb, IL
regev@math.niu.edu

Abstract
We give a new proof of the $k$-fold convolution of the Catalan numbers. This is done by enumerating a certain class of polygonal dissections called $k$-in-$n$ dissections. Furthermore, we give a formula for the average number of cycles in a triangulation.

1 Introduction

The Catalan numbers are defined as follows.

**Definition 1.** For any $n \geq 0$,

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$ 

For $n < 0$, $C_n = 0$.

The Catalan $k$-fold convolution formula is due to Catalan.

**Theorem 2.** [2] Let $1 \leq k \leq n$. Then

$$\sum_{i_1+\ldots+i_k=n} C_{i_1-1} \cdots C_{i_k-1} = \frac{k}{2n-k} \binom{2n-k}{n}. \quad (1)$$

Catalan’s original proof [2, 3, 4, 5] uses Lagrange inversion. Gessel and Lacrombe [4] give two proofs which use hypergeometric identities. Tedford [6] exhibits several interpretations of the left-hand side of (1). In this note we use another such interpretation, in terms of dissections of polygons, to give a new proof of Theorem 2. We arrive at this proof using Theorem 5, which enumerates a class of polygonal dissections called $k$-in-$n$ dissections. As another consequence of this enumeration, in Corollary 7 we give a formula for the average number of cycles in a triangulation.

2 The $k$-in-$n$ dissections

**Definition 3.** Let $n \geq 3$ and let $0 \leq k \leq n - 3$. 

1. A $k$-dissection of an $n$-gon is a partition of the $n$-gon into $k + 1$ parts by $k$ noncrossing diagonals.

2. A triangulation of an $n$-gon is an $(n - 3)$-dissection.

3. For $k \geq 4$, an $k$-in-$n$ dissection is an $(n - k)$-dissection of an $n$-gon into one $k$-gon and $n - k + 1$ triangles (see Figure 1). A 3-in-$n$ dissection is a triangulation with one of its $n - 3$ triangles marked.

4. Let $f_k(n)$ be the number of $k$-in-$n$ dissections.

It is well known that for $n \geq 3$ the number of triangulations of an $n$-gon is $C_{n-2}$.

**Lemma 4.** Let $3 \leq k \leq n$. Then

\[(n - k)f_k(n) = n \sum_{i=2}^{n-k+1} C_{i-1} f_k(n - i + 1). \tag{2}\]

**Proof.** The left-hand side of (3) is the number of $k$-in-$n$ dissections, with one of the $n - k$ diagonals marked. These can also be chosen as follows. Choose one vertex $v$ out of the $n$ vertices, then choose $2 \leq i \leq n - k + 1$. Form the diagonal from $v$ to a vertex which is a distance $i$ from $v$ (proceeding, say, counterclockwise along the edges of the $n$-gon). Mark this diagonal. Now choose a triangulation of the resulting $(i + 1)$-gon and a $k$-in-$((n - i) + 1)$ dissection of the resulting $((n - i) + 1)$-gon. Each such choice results in a unique $k$-in-$n$ dissection with one of the diagonals marked.

Lemma 4 can be used to enumerate the $k$-in-$n$ dissections.

**Theorem 5.** Let $3 \leq k \leq n$. The number of $k$-in-$n$ dissections is

\[f_k(n) = \binom{2n - k - 1}{n - 1}. \tag{3}\]
Note 6. There is a bijection between \( k \)-in-\( n \) dissections and \( k \)-crossing partitions of \( \{1, \ldots, n\} \), as defined in [1]. Thus Theorem 5 is equivalent to [1, Theorem 1].

Theorem 5 implies the following corollary:

**Corollary 7.** Let \( 3 \leq k < n \). The average number of cycles of length \( k \) in a triangulated \( n \)-gon is

\[
\frac{\binom{2n-k-1}{n-1}}{n} \frac{C_{k-2}}{C_{n-2}}.
\]

**Proof.** Each cycle of length \( k \) in a triangulation of an \( n \)-gon uniquely corresponds to a \( k \)-in-\( n \) dissection together with a triangulation of a \( k \)-gon. The result then follows from (3). \( \square \)

The following lemmas will be used in the proof of Theorem 5. It is well known that for any \( n \geq 0 \),

\[
\sum_{i \geq 0} C_i C_{n-i} = C_{n+1}. \tag{4}
\]

**Lemma 8.** For any \( n \geq 1 \),

\[
\sum_{i \geq 0} i C_i C_{n-i} = \binom{2n+1}{n-1}. \tag{5}
\]

**Proof.** Note that

\[
\sum_{i \geq 0} i C_i C_{n-i} = \sum_{i \geq 0} (n-i) C_i C_{n-i}.
\]

Therefore by (4),

\[
\sum_{i \geq 0} i C_i C_{n-i} = \frac{1}{2} \sum_{i \geq 0} n C_i C_{n-i} = \frac{n}{2} C_{n+1} = \binom{2n+1}{n-1}.
\]

\( \square \)

**Lemma 9.** Let \( 1 \leq q \leq p \leq 2q-1 \). Then

\[
\sum_{i \geq 0} C_i \binom{p-1-2i}{q-1-i} = \frac{p}{q}. \tag{6}
\]

**Proof.** We use induction on \( q \). If \( q = 1 \) then \( p = 1 \) and both sides of (6) are equal to 1. Now suppose \( q \geq 2 \). If \( p = q \) then both sides are equal to 1. If \( p = 2q-1 \) then (6) follows from (4) and (5), since

\[
\sum_{i \geq 0} C_i \binom{2q-2-2i}{q-1-i} = \sum_{i \geq 0} C_i (q-i) C_{q-1-i} = q \sum_{i \geq 0} C_i C_{q-1-i} - \sum_{i \geq 0} i C_i C_{q-1-i} = q C_q - \binom{2q-1}{q-2} = \binom{2q-1}{q}.
\]

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Now suppose \( q + 1 \leq p \leq 2q - 2 \). Note that \( q - 1 \leq p - 1 \leq 2q - 2 - 1 = 2(q-1) - 1 \). Therefore by the induction hypothesis, (6) holds for \( p - 1 \) and \( q - 1 \). Also \( q \leq p - 1 \) and \( p - 1 \leq 2q - 3 < 2q - 1 \), so that (6) holds for \( p - 1 \) and \( q \). Thus

\[
\binom{p}{q} = \binom{p-1}{q-1} + \binom{p-1}{q} = \sum_{i \geq 0} \frac{1}{i+1} \binom{2i}{i} \binom{p-2-2i}{q-2-i} + \sum_{i \geq 0} \frac{1}{i+1} \binom{2i}{i} \binom{p-2-2i}{q-1-i}
\]

\[
= \sum_{i \geq 0} \frac{1}{i+1} \binom{2i}{i} \binom{p-1-2i}{q-1-i}.
\]

### 2.1 Proof of Theorem 3

**Proof.** Fix \( k \geq 3 \) and proceed by induction on \( n \). If \( n = k \) then both sides are equal to 1. Now let \( n \geq k + 1 \). By Lemma 4 and by the induction hypothesis,

\[
f_k(n) = \frac{n}{n-k} \sum_{i=2}^{n-k-1} C_{i-1} f_k(n-i+1)
\]

\[
= \frac{n}{n-k} \sum_{i=2}^{n-k-1} C_{i-1} \left( \frac{2(n-i+1)-k-1}{n-i} \right)
\]

\[
= \frac{n}{n-k} \left( \sum_{i \geq 1} C_{i-1} \left( \frac{2(n-i+1)-k-1}{n-i} \right) - f_k(n) \right).
\]

Solving for \( f_k(n) \) and applying Lemma 9, with \( q = n \) and \( p = 2n - k \),

\[
f_k(n) = \frac{n}{2n-k} \sum_{i \geq 0} C_i \left( \frac{2n-k-2i-1}{n-i-1} \right) = \frac{n}{2n-k} \left( \frac{2n-k}{n} \right) = \left( \frac{2n-k-1}{n-1} \right).
\]

### 3 Proof of the Catalan convolution formula

The next Lemma gives the relation between the number of \( k \)-in-\( n \) dissections and the Catalan convolution.

**Lemma 10.** Let \( 3 \leq k < n \). Then

\[
k f_k(n) = n \sum_{i_1 + \ldots + i_k = n} C_{i_1-1} \cdots C_{i_k-1}.
\]
Proof. The left-hand side of (7) is the number of \( k \)-in-\( n \) dissections, with one of the vertices of the \( k \)-gon marked. These can also be chosen as follows. Choose any vertex \( v \) of the \( n \)-gon. For each vertex \( v \), choose \( i_1, \ldots, i_k \) such that \( i_1 + \ldots + i_k = n \). This determines the lengths of the sides of a \( k \)-gon by starting at \( v \) and proceeding, say, counterclockwise. For example, in Figure 1, if \( v \) is the bottom vertex then the lengths are 1, 4, 2, 2, 3. For each \( 1 \leq r \leq k \), there is a resulting \((i_r + 1)\)-gon sharing one edge of the \( k \)-gon. Each of these \((i_r + 1)\)-gon can be triangulated in \( C_{i_r-1} \) ways, forming a uniquely determined \( k \)-in-\( n \) dissection with one of the of the \( k \)-gon marked.

The proof of Theorem 2 now follows from Lemma 10, since

\[
\sum_{i_1+\ldots+i_k=n} C_{i_1-1} \cdots C_{i_k-1} = \frac{k}{n} f_k(n) = \frac{k}{n} \binom{2n-k-1}{n-1} = \frac{k}{2n-k} \binom{2n-k}{n}.
\]

References

[1] M. Bergerson and A. Miller and A. Pliml and V. Reiner and P. Shearer and D. Stanton and N. Switala, A note on 1-crossing partitions, available at http://www.math.umn.edu/~reiner/Papers/onecrossings.pdf.

[2] E. Catalan, Sur les nombres de Segner, Rend. Circ. Mat. Palermo, 1 (1887) 190–201.

[3] D. R. French and P. J. Larcombe, The Catalan number \( k \)-fold self-convolution identity: the original formulation, J. Combin. Math. Combin. Comput., 46 (2003) 191–204.

[4] I. Gessel and P. J. Lacrombe, A forgotten convolution type identity of Catalan: two hypergeometric proofs, Util. Math., 59 (2001), 97-109.

[5] P. J. Lacrombe, A forgotten convolution type identity of Catalan, Util. Math., 57 (2000), 65-72.

[6] S. J. Tedford, Combinatorial interpretations of convolutions of the Catalan numbers, Integers 11 (2011).