PFAFFIAN SUM FORMULA FOR THE SYMPLECTIC GRASSMANNIANS

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Abstract. We study the torus equivariant Schubert classes of the Grassmannian of non-maximal isotropic subspaces in a symplectic vector space. We prove a formula that expresses an equivariant Schubert class as a sum of multi Schur-Pfaffians, whose entries are appropriate quadratic polynomials in the equivariantly modified special Schubert classes. Furthermore our result gives a proof to Wilson’s formula, which generalizes the Giambelli formula for the ordinary cohomology proved by Buch-Kresch-Tamvakis, given in terms of Young’s raising operators.

1. Introduction

The classical Giambelli formula [8] expresses a general Schubert class of the Grassmannian as the determinant of a matrix whose entries are the so-called special Schubert classes. Those special classes are defined by the locus of meeting a fixed subspace nontrivially and are also the Chern classes of the universal quotient bundle over the Grassmannian. Various extensions of the formula has been obtained (see for example [7], [22] and the references therein). The Giambelli problem is to find a “closed formula” of a Schubert class in terms of those special classes, and in torus equivariant setting, the problem is closely related to the theory of degeneracy loci of vector bundles (cf. [1], [7], [22]).

This paper concerns the equivariant Giambelli problem for the Grassmannians obtained as the homogeneous spaces of symplectic groups. For the symplectic or orthogonal groups, there is a natural notion of special Schubert classes, taking account of the isotropic conditions for symplectic or orthogonal form. For the Grassmannian of maximal isotropic subspaces, the Giambelli formula proved by Pragacz [20] expresses a general Schubert class as a Pfaffian whose entries are appropriate quadratic polynomials of the special Schubert classes. Equivariant versions for this case were obtained by Kresch-Tamvakis [16] and Kazarian [15] in the context of degeneracy loci independently. Kazarian’s formula is more direct generalization of Pragacz’s formula in a sense that it is written as a single Pfaffian and later also proved in [10] and [12] by using more algebraic methods. Buch, Kresch, and Tamvakis solved the Giambelli problem for non-maximal isotropic Grassmannians [5], [6]. Their formula expresses an arbitrary Schubert class in terms of the special Schubert classes as a polynomial defined by Young’s raising operators. These polynomial expressions can be thought of as certain “combinatorial interpolations” between the Jacobi-Trudi determinant and the Schur Pfaffian. Wilson [24] conjectured an equivariant version of their formula in the symplectic case. Our main result provides a formula expressing an arbitrary equivariant Schubert class as a sum of Pfaffians with the entries of equivariantly modified

2000 Mathematics Subject Classification. Primary 14M15, Secondary 05E05.
special Schubert classes. By showing the equivalence between our formula and Wilson’s, we obtain a proof to the conjecture.

1.1. Symplectic Grassmannian and its Schubert varieties. Throughout the paper, we fix a non-negative integer $k$. For any positive integer $n \geq k$, let IG$(n-k, 2n)$ denote the Grassmannian of $(n-k)$-dimensional isotropic subspaces in $\mathbb{C}^{2n}$ equipped with a symplectic form. There is a maximal parabolic subgroup $P_k$ of the symplectic group $G := \text{Sp}_{2n}(\mathbb{C})$ such that IG$(n-k, 2n)$ can be realised as the homogeneous space $G/P_k$.

A partitions $\lambda$ is $k$-strict if no part greater than $k$ is repeated. The Schubert varieties of IG$(n-k, 2n)$ are indexed by $k$-strict partitions whose Young diagrams fit in the $(n-k) \times (n+k)$ rectangle. We denote the set of such partitions by $\mathcal{P}_n^{(k)}$. Given $\lambda \in \mathcal{P}_n^{(k)}$ and a complete flag of subspaces $0 = F_0 \subset F_1 \subset \cdots \subset F_{2n} = \mathbb{C}^{2n}$ such that $F_{n+i} = (F_{n-i})^\perp$ for $0 \leq i \leq n$, the corresponding Schubert variety is defined as

$$\Omega_\lambda = \{ \Sigma \in \text{IG}(n-k, 2n) \mid \dim(\Sigma \cap F_{\ell(p_j(\lambda))}) \geq j, \quad 1 \leq \forall j \leq \ell(\lambda) \}, \quad (1.1)$$

where $\ell(\lambda)$ denotes the number of non-zero parts of $\lambda$ and

$$p_j(\lambda) = n + k + j - \lambda_j - \#\{i \mid i < j, \lambda_i + \lambda_j > 2k + j - i\}.$$ 

The codimension of $\Omega_\lambda$ is $|\lambda| = \sum_i \lambda_i$. The corresponding class $[\Omega_\lambda] \in H^*(\text{IG}(n-k, 2n))$ is the Schubert class. The special Schubert varieties are given by

$$\Omega_r = \{ \Sigma \in \text{IG}(n-k, 2n) \mid \dim(\Sigma \cap F_{n+k+1-r}) \geq 1 \},$$

for $1 \leq r \leq n+k$, and their classes $[\Omega_r]$ are called the special Schubert classes. They are equal to the $r$-th Chern classes $c_r(Q)$ of the universal quotient bundle $Q$ over IG$(n-k, 2n)$.

1.2. Double Schubert polynomials of type $C$. We recall the results in [11]. Let $B$ be a Borel subgroup of $G = \text{Sp}_{2n}(\mathbb{C})$, and $T$ the maximal torus contained in $B$. The Weyl group $N_G(T)/T$ is denoted by $W_n$ and identified with the group of the signed permutations of $\{\pm1, \ldots, \pm n\}$. We often denote $-i$ by $\overline{i}$. The flag variety $\mathcal{F}l_n$ is defined as the quotient space $G/B$. For each $w \in W_n$, the Schubert variety $X_w$ is defined as the Zariski closure of $B^-$-orbit of the corresponding point $e_w \in \mathcal{F}l_n$, where $B^-$ is the Borel subgroup such that $B \cap B^- = T$. The codimension of $X_w$ is precisely the length $\ell(w)$ of $w$ as a Weyl group element of $W_n$. We denote by $[X_w]_T$ the corresponding $T$-equivariant Schubert class in $H^*_T(\mathcal{F}l_n)$.

Let $\Gamma$ be the ring generated by the Schur $Q$-functions $Q_r(x)$ ($r \geq 1$), where $x$ is an infinite sequence of variables $x_1, x_2, \ldots$. Let $\mathcal{R}_\infty$ be the polynomial ring $\Gamma[z, t]$ in the variables $z_i, t_i$ ($i \geq 1$) with coefficients in $\Gamma$. Let $W_\infty$ be the Weyl group of type $C_\infty$, where we can regard it as the union of $W_n$ ($n \geq 1$). There are two commuting actions of $W_\infty$ on the ring $\mathcal{R}_\infty$ (see [2]). The double Schubert polynomials are defined as the elements of a distinguished $\mathbb{Z}[t]$-basis $\{ \mathcal{C}_w(z, t; x) \mid w \in W_\infty \}$ of $\mathcal{R}_\infty$, characterized by the two series of divided difference equations involving the left and the right divided difference operators $\{ \delta_i \mid i \geq 0 \}$ and $\{ \partial_i \mid i \geq 0 \}$.

The integral $T$-equivariant cohomology ring $H^*_T(\mathcal{F}l_n)$ of $\mathcal{F}l_n$ has an $H^*_T(pt)$-algebra structure given by the pullback of $\mathcal{F}l_n \rightarrow pt$. Together with an appropriate identification
there is a canonical homomorphism
\[ \pi_n : \mathcal{R}_\infty \rightarrow H^*_T(\mathcal{F}l_n), \]
of \( H^*_T(pt) \)-algebras such that \( \pi_n \) sends \( \mathcal{C}_w(z, t; x) \) to \([X_w]_T\) if \( w \in W_n \) and to zero if \( w \not\in W_n \).

**1.3. Equivariant Schubert classes of \( H^*_T(\text{IG}(n-k, 2n)) \).** Let \( s_i \) (\( i \geq 0 \)) be the standard (Coxeter) generators of \( W_\infty \) called the simple reflections (see \[2\]). Let \( W_{(k)} \) be the subgroup of \( W_\infty \) generated by \( s_i \) (\( i \geq 0 \), \( i \neq k \)). Let \( \mathcal{R}^{(k)}_\infty \) denote the invariant subring of \( \mathcal{R}_\infty \) with respect to the 2nd (“right”) action of \( W_{(k)} \). There is the following commutative diagram

\[
\begin{array}{ccc}
\pi^{(k)}_n & \longrightarrow & \pi_n \\
\downarrow & & \downarrow \\
H^*_T(\text{IG}(n-k, 2n)) & \overset{\text{pr}^*}{\longrightarrow} & H^*_T(\mathcal{F}l_n)
\end{array}
\]

where the horizontal arrow \( \text{pr}^* \) in the second row is the inclusion defined by the pullback of the natural projection \( \text{pr} : \mathcal{F}l_n \rightarrow \text{IG}(n-k, 2n) \) and \( \pi^{(k)}_n \) is obtained by restricting \( \pi_n \) to \( \mathcal{R}^{(k)}_\infty \).

Let \( W^{(k)}_\infty \) be the set of minimum length coset representatives of \( W_\infty / W_{(k)} \) and \( \mathcal{P}^{(k)}_\infty \) = \( \bigcup_{n \geq k} \mathcal{P}^{(k)}_n \) the set of all \( k \)-strict partitions, then there is a natural bijection

\[ \mathcal{P}^{(k)}_\infty \rightarrow W^{(k)}_\infty, \quad \lambda \mapsto w_\lambda, \]
such that \( |\lambda| = \ell(w_\lambda) \) and the image of \( \mathcal{P}^{(k)}_n \) is \( W^{(k)}_n := W_n \cap W^{(k)}_\infty \). If \( \lambda \in \mathcal{P}^{(k)}_n \), we have \( \text{pr}^*[\Omega_\lambda]_T = [X_{w_\lambda}]_T \) and

\[ \pi^{(k)}_n(\mathcal{C}_{w_\lambda}(z, t; x)) = [\Omega_\lambda]_T. \]

In particular, the special Schubert class \([\Omega_r]_T\) of degree \( r \) is the image of \( \mathcal{C}_{w_r}(z, t; x) \) where \( w_r \) is the element of \( W^{(k)}_\infty \) corresponding to the partition with \( r \) boxes in one row. The set of functions \( \mathcal{C}_w(z, t; x), \ w \in W^{(k)}_\infty \) forms a \( \mathbb{Z}[t] \)-basis of \( \mathcal{R}^{(k)}_\infty \).

**1.4. Main results.** Our goal is to give an explicit closed formula to describe \( \mathcal{C}_w(z, t; x) \) (\( w \in W^{(k)}_\infty \)) as a polynomial in terms of the double Schubert polynomials \( \mathcal{C}_{w_r}(z, t; x) \) associated to the special classes \([\Omega_r]_T\).

There is an equivariant (or double) version of the theta polynomials that can be found in Wilson’s thesis \[24\] (also mentioned in \[22\]). Although the following definition is slightly different from it, those appearing in our main theorem and the conjecture in \[24\] are identical after applying appropriate changes of indices. See Remark \[7.5\].

**Definition 1.1.** Define \( k\vartheta_{r}^{(l)}(x, z|t) \) for \( l, r \geq 0 \) by

\[
\sum_{r=0}^{\infty} k\vartheta_{r}^{(l)}(x, z|t) \cdot u^r = \sum_{r=0}^{\infty} k\vartheta_{r}^{(-l)}(x, z|t) \cdot u^r = \prod_{i=1}^{k} \frac{1 + x_i u}{1 - x_i u} \prod_{i=1}^{k} \frac{1 + z_i u}{1 - z_i u} \prod_{i=1}^{l} \frac{1}{1 + t_i u}.
\]

In the case when \( r < 0 \), we regard \( k\vartheta_{r}^{(l)}(x, z|t) = 0 \). We omit \( k \) when it is clear.
We show that $\mathcal{R}_\infty^{(k)}$ contains $k\vartheta^{(l)}_r(x,z|t)$ (see §1.3) and has the description (Remark 5.2):

$$\mathcal{R}_\infty^{(k)} = \mathbb{Z}[t][\vartheta_1, \vartheta_2, \ldots], \quad \vartheta_r := k\vartheta_r^{(0)}. \quad (1.2)$$

Wilson [24, Prop. 6] proved that

$$c_{w_r}(z; t; x) = k\vartheta_r^{(r-k-1)}(x, z|t). \quad (1.3)$$

Let $\lambda$ be a $k$-strict partition in $\mathcal{P}_n^{(k)}$. In the one-line notation of signed permutations (see [2]), we can write the corresponding element $w_\lambda$ of $W_n^{(k)}$ as

$$w_\lambda = v_1v_2 \ldots v_k|\zeta_1 \cdot \cdot \cdot \zeta_s u_1 \cdot \cdot \cdot u_{n-k-s}, \quad v_1 < \cdot \cdot \cdot < v_k, \quad \zeta_1 > \cdot \cdot \cdot > \zeta_s, \quad u_1 < \cdot \cdot \cdot < u_{n-k-s}, \quad (1.4)$$

where $s$ is a non-negative integer. Let $\chi_\lambda = (\chi_1, \ldots, \chi_{n-k})$ be the following sequence

$$\chi_\lambda := (\zeta_1 - 1, \ldots, \zeta_s - 1, -u_1, \ldots, -u_{n-k-s}) \in \mathbb{Z}^{n-k}. \quad (1.5)$$

We call $\chi_\lambda$ the characteristic index of $\lambda$. Let $\Delta_m := \{(i, j) \mid 1 \leq i < j \leq m\}$. Define a subset $D(\lambda)$ of $\Delta_{n-k}$ by

$$D(\lambda) := \{(i, j) \in \Delta_{n-k} \mid \chi_i + \chi_j < 0\}. \quad (1.6)$$

We use the multi Schur-Pfaffian due to Kazarian [15], which is a natural variation of the Schur Pfaffian [21]. Let $c_1^{(1)}, c_2^{(2)}, \ldots (r \in \mathbb{Z})$ be an ordered set of infinite formal variables. The multi Schur-Pfaffian $\text{Pf}[c_1^{(1)} c_2^{(2)} \cdots c_m^{(r)}]$ is defined in [31]. This is a finite $\mathbb{Z}$-linear combination of $c_3^{(1)} \cdots c_m^{(r)}$, $(s_1, \ldots, s_m) \in \mathbb{Z}^m$. For each $l \in \mathbb{Z}^m$, the substitution of $\vartheta_{c_l}^{(i)}$ to $c_{s_l}^{(i)}$ in this linear combination is denoted by $

\text{Pf}[\vartheta_{c_1}^{(l_1)} \vartheta_{c_2}^{(l_2)} \cdots \vartheta_{c_m}^{(l_m)}].$

The main result of this paper is the following.

**Theorem 1.2** (Pfaffian sum formula). Let $\lambda$ be a $k$-strict partition in $\mathcal{P}_n^{(k)}$. We have

$$c_{w_\lambda} = \sum_{I \subseteq D(\lambda)} \text{Pf} \left[ \vartheta_{\chi_1}^{(1)} \vartheta_{\chi_2}^{(2)} \cdots \vartheta_{\chi_{n-k}}^{(n-k)} \right], \quad (1.7)$$

where $I$ runs over all subsets of $D(\lambda)$ and $a_s^I = \#\{j \mid (s, j) \in I\} - \#\{i \mid (i, s) \in I\}$.

Note that essentially the right hand side does not depend on $n$, i.e., it depends only on $\lambda \in \mathcal{P}_\infty^{(k)}$. See Remark 6.3 for a more precise statement.

**Example 1.3.** Let $k = 1, n = 5$. Let $\lambda = (5, 3, 2, 1)$ be a $k$-strict partition. Then $w_\lambda = 5\vartheta_5(3)\vartheta_3(1)\vartheta_2(0)\vartheta_1(-3)$ and $D(\lambda) = \{(2, 4), (3, 4)\}$. We have

$$c_{w_\lambda} = \text{Pf}[\vartheta_5(3)\vartheta_3(1)\vartheta_2(0)\vartheta_1(-3)] + \text{Pf}[\vartheta_5(3)\vartheta_4(1)\vartheta_2(0)\vartheta_0(-3)] + \text{Pf}[\vartheta_5(3)\vartheta_3(1)\vartheta_3(0)\vartheta_0(-3)].$$

In [24], Wilson employed the raising operators to define double theta polynomials $\Theta_\lambda$, and proved that the polynomials satisfy the equivariant Chevalley formula. In her thesis, it was further conjectured that $\Theta_\lambda$ is equal to $c_{w_\lambda}$ and, in [22], it was announced that Tamvakis and Wilson completed the proof of the conjecture. As a corollary to Theorem 1.2, we can also give a proof that Wilson’s conjecture is true (see [7]).

Our result is more explicit than Wilson’s formula, in the sense that it only involves the subsets of the explicitly defined set $D(\lambda)$, while the formula in [21] involves polynomials
defined in terms of the raising operators, which we can determine only after some calculations. By a formal computation, we will derive Wilson’s formula from the Pfaffian sum formula. Once we read the formula in terms of raising operators, the following corollary is immediate.

**Corollary 1.4.** If \( D(\lambda) = \Delta_{n-k} \), in particular, if \( \lambda \) is contained in the \((n-k) \times k \) rectangle, then \( C_{w_{\lambda}} \) is a single determinant \((22, \S 1)\)

\[
\text{Det}[^{\vartheta}_{\lambda_1} \cdots ^{\vartheta}_{\lambda_{n-k}}] := \det(^{\vartheta}_{\lambda_1+j-i})_{1 \leq i,j \leq n-k}.
\]

If \( D(\lambda) = \emptyset \), in particular, if \( \lambda \) is a strict partition containing the \((n-k) \times k \) rectangle, then \( C_{w_{\lambda}} \) is a single Pfaffian

\[
Pf[^{\vartheta}_{\lambda_1} \cdots ^{\vartheta}_{\lambda_{n-k}}].
\]

The case when \( \lambda \) is contained in the \((n-k) \times k \) rectangle, the result was proved by Wilson \[24\]. Note that, even if \( D(\lambda) \neq \emptyset \), it is possible that the double Schubert polynomial is a single Pfaffian in the formula. For example,

\[
C_{13|\bar{5}\bar{4}2} = Pf[^{\vartheta}_{7(4)}^{\vartheta}_{5(2)}^{\vartheta}_{0(-4)}] + Pf[^{\vartheta}_{7(4)}^{\vartheta}_{4(2)}^{\vartheta}_{-1(-4)}] = Pf[^{\vartheta}_{7(4)}^{\vartheta}_{5(2)}^{\vartheta}_{0(-4)}].
\]

Applications of our result to the problem of degeneracy loci formulas of vector bundles are straightforward (cf. \[1\], \[2\], \[22\]). We only give the Chern class interpretations of \( k_{\vartheta_r}^{(l)}(z,x|t) \) in \( \S 5.2 \). It is worth mentioning that those special cases in Corollary 1.4 look precisely the same as the classical Kempf-Laksov determinantal formula for type A degeneracy loci \[14\] and the Pfaffian formula for Lagrangian degeneracy loci \[10, 15\] (see also Remark 4.2), although the functions are associated to the isotropic Grassmannians.

The main tool of the proof is the left divided difference operators \( \delta_i \). By explicit calculations, we show that the right hand side of (1.7) satisfies the part of the defining properties of the double Schubert polynomials corresponding to the left divided difference operators. We can use a uniqueness theorem available for the parabolic case to finish the proof. It is worth noting that our method is totally different from the method of the raising operators developed in \[5\], \[22\].

1.5. **Related results.** Anderson and Fulton \[2\] defined a notion of vexillary signed permutation in type B,C, and D. They showed the double Schubert polynomials associated to vexillary signed permutations are given by explicit Pfaffian formulas. Naruse \[19\] also independently proved a formula that express the corresponding double Schubert polynomials as a specialization of the factorial \( Q \)- and \( P \)-functions. Since our formula also express the some Schubert classes as single Pfaffians, therefore there is an overlap between our results and the results of \[2\], \[19\]. However, not all \( k \)-Grassmannian permutations are vexillary and there are non-vexillary \( k \)-Grassmannian permutations whose corresponding classes are written as single Pfaffians, e.g. \( 13|\bar{5}\bar{4}2 \) is not vexillary but \( C_{13|\bar{5}\bar{4}2} \) is a single Pfaffian as above.

Tamvakis \[23\] proved a combinatorial formula which expresses an arbitrary (equivariant) Schubert class of any classical \( G/P \) space as a polynomials in the special Schubert classes (see also \[22\]). The formula involves a combinatorial data related to the reduced decompositions of Weyl group element, and also the theta polynomials and Schur \( S \)-functions.
A natural question is the possibility of extending our results to type D. Also it is natural to ask if our formula can be derived by using Kazarian’s pushforward formula. If it is possible, there will arise a new perspective hopefully applicable to K-theory case. We hope to address these problems elsewhere.

1.6. Organization. This paper is organized as follows. In Section 2 we review the double Schubert polynomials (DSP) following [11]. In Section 3 we give some preliminary discussions on the symplectic Grassmannian. In Section 4 we introduce the multi Schur-Pfaffian used by Kazarian in a slightly generalized form. In Section 5 we introduce the double theta polynomials and establish some basic properties of them. Section 6 is devoted to the proof of Theorem 1.2. In Section 7 we introduce the raising operators and their action on formal power series to prove the equivalence of our main theorem and the conjecture in Wilson’s thesis [24]. In Section 8 we list the computation when $(n, k) = (5, 2), (5, 3)$.

Acknowledgements. We are specially grateful to Hiroshi Naruse for explaining his results. We thank Dave Anderson, Changzheng Li, Leonardo Mihalcea, Masaki Nakagawa, and Harry Tamvakis for the helpful conversations and their comments. The most part of the paper is written during T.I. stay at KAIST. The hospitality and perfect working conditions there are gratefully acknowledged.

2. Double Schubert polynomials of type $C$

In this section, we review the construction of the double Schubert polynomials, following [11]. The expository article [22] by Tamvakis will be also helpful to grasp more geometric backgrounds of this construction.

Let $W_\infty$ be the group defined by the generators \{s_i \mid i = 0, 1, \ldots\} and the relations

\[ s_i^2 = e \ (i \geq 0), \quad s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \ (i \geq 1), \]

\[ s_i s_j = s_j s_i \ (|i - j| \geq 2). \]

The corresponding Dynkin diagram is depicted by

```
C_\infty
  s_0 --- s_1 --- s_2 --- \cdots --- s_k --- \cdots --- s_{n-1}
```

The group $W_\infty$ is identified with the set of all permutations $w$ of the set \{1, 2, \ldots\}∪\{\bar{1}, \bar{2}, \ldots\} such that $w(i) \neq i$ for only finitely many $i$, and $w(\bar{i}) = w(\bar{i})$ for all $i$. The simple reflections are identified with the transpositions $s_0 = (1, \bar{1})$ and $s_i = (i + 1, \bar{i})(\bar{i}, i + 1)$ for $i \geq 1$. The Weyl group $W_n$ is identified with the subgroup of $W_\infty$ consisting of $w$ such that $w(i) = i$ for all $i > n$. In one-line notation, we often denote an element $w \in W_n$ by the finite sequence $(w(1), \ldots, w(n))$.

The function $Q_r(x)$ is defined by the following generating function:

\[
\sum_{r=0}^{\infty} Q_r(x) u^r = \prod_{i=1}^{\infty} \frac{1 + x_i u}{1 - x_i u}. \tag{2.1}
\]
Let $\Gamma$ be the ring generated by $Q_r(x)$ ($r \geq 1$). Let $R_\infty$ be the polynomial ring $\Gamma[t, z]$ in the variables $t = (t_1, t_2, \ldots)$, and $z = (z_1, z_2, \ldots)$ with coefficients in $\Gamma$. There are two actions of $W_\infty$ on the ring $R_\infty$ defined below. We denote the corresponding operators on $R_\infty$ by $s^+_i$ (right action) and $s^-_i$ (left action).

For $i \geq 1$, let $s^+_i(z_i) = z_{i+1}$, $s^+_i(z_{i+1}) = z_i$, $s^+_i(z_j) = z_j$ ($j \neq i, i + 1$), and $s^+_i(Q_r(x)) = Q_r(x)$. There is an automorphism $s_0^+$ of $Z[t]$-algebra on $R_\infty$ characterized by the following:

$$s_0^+(z_1) = -z_1, \quad s_0^+(z_i) = z_i \quad (i \geq 1), \quad \sum_{r=0}^{\infty} s_0^+(Q_r(x))u^r = \frac{1 + z_1u}{1 - z_1u} \prod_{i=1}^{\infty} \frac{1 + x_iu}{1 - x_iu}.$$ 

The last equation is equivalently written as

$$s_0^+Q_r(x_1, x_2, \ldots) = Q_r(z_1, x_1, x_2, \ldots).$$

Clearly we can extend $s^+_i$ to $R_\infty$ as an automorphism of $Z[t]$-algebra and show that $s_i \mapsto s^+_i$ ($i \geq 0$) gives a right action of $W_\infty$ on $R_\infty$. Similarly, there are operators $s^-_i$ ($i \geq 0$) on $R_\infty$ such that $s_i \mapsto s^-_i$ ($i \geq 0$) gives a left action of $W_\infty$ on $R_\infty$ as $Z[z]$-algebra automorphisms. In order to define this action, we can use the following ring automorphism $\omega$:

$$\omega(t_i) = -z_i, \quad \omega(z_i) = -t_i, \quad \omega(Q_r(x)) = Q_r(x).$$

Then $s^-_i = \omega s^+_i \omega$ ($i \geq 0$). In particular, we have

$$s^-_i Q_r(x_1, x_2, \ldots) = Q_r(-t_1, x_1, x_2, \ldots).$$

Define the simple roots by

$$\alpha_0 = 2t_1, \quad \alpha_i = t_{i+1} - t_i \quad (i \geq 1).$$

The right and left divided difference operators are defined by

$$\partial_i f = \frac{f - s^+_i f}{\omega(\alpha_i)}, \quad \delta_i f = \frac{f - s^-_i f}{\alpha_i} \quad (i \geq 0, \ f \in R_\infty).$$

**Theorem 2.1** ([11]). There exists a unique $Z[t]$-free basis \{ $\mathcal{E}_w(z, t; x) \mid w \in W_\infty$ \} of $R_\infty$ satisfying the equations

$$\partial_i \mathcal{E}_w = \begin{cases} \mathcal{E}_{ws_i} & \text{if } \ell(ws_i) < \ell(w) \\ 0 & \text{otherwise} \end{cases}, \quad \delta_i \mathcal{E}_w = \begin{cases} \mathcal{E}_{s_iw} & \text{if } \ell(s_iw) < \ell(w) \\ 0 & \text{otherwise} \end{cases}, \quad (2.2)$$

for all $i \geq 0$, and such that $\mathcal{E}_w$ has no constant term except for $\mathcal{E}_e = 1$.

\footnote{As an abstract ring, $\Gamma$ can be defined as the quotient of the polynomial ring $Z[Q_1, Q_2, Q_3, \ldots]$ of the variables $Q_1, Q_2, \ldots$ by the ideal generated by the following elements

$$Q_r^2 + 2 \sum_{i=1}^{r} (-1)^i Q_{r+i}Q_{r-i} \quad (r \geq 1),$$

with $Q_0 = 1$. See [17] for more detailed properties of the ring $\Gamma$.}
3. Preliminaries on the symplectic Grassmannian

3.1. $k$-strict partitions. We develop some combinatorics related to the Schubert classes of $\mathrm{IG}(n-k, 2n)$. The set of the minimum length coset representatives for $W_{\infty}/W(k)$ is given by

$$W^{(k)}_{\infty} = \{ w \in W_{\infty} | \ell(w) > \ell(ws_i) (\forall i \geq 1, i \neq k) \}.$$  

We will review the bijection $P^{(k)}_{\infty} \rightarrow W^{(k)}_{\infty}$ ($\lambda \mapsto w_{\lambda}$) such that $|\lambda| = \ell(w_{\lambda})$, which is due to [5]. Each $w \in W^{(k)}_{\infty}$ is called a $k$-Grassmannian permutation and if $w \in W_n$, we can write in the one-line notation (1.4)

$$w = v_1 \cdots v_k | \zeta_1 \cdots \zeta_s u_1 \cdots u_{n-k-s}.$$  

For each $i$ with $1 \leq i \leq k$, let $\mu_i$ be the number of the elements of $\{u_1, \ldots, u_{n-k-s}\}$ less than $v_i$. Then $\mu = (\mu_k, \ldots, \mu_1)$ is a partition whose Young diagram fits inside the $k \times (n-s-k)$ rectangle. Let $\nu$ be the conjugate of $\mu$ (the transpose of the Young diagram). Consider the strict partition $\zeta := (\zeta_1, \ldots, \zeta_s)$ defined by the entries with bars in the one-line notation. The $k$-strict partition $\lambda$ corresponding to the $k$-Grassmannian permutation $w$ is given by

$$\lambda_i = \begin{cases} 
\zeta_i + k & (1 \leq i \leq s) \\
\nu_{i-s} & (s+1 \leq i \leq n-k) 
\end{cases}.$$  

Note that the correspondence is independent of $n$ as far as $w \in W_n$.

**Example 3.1.** The 2-Grassmannian permutation $w = 58|431267$ corresponds to the 2-strict partition $\lambda = (6, 5, 3, 2, 1, 1)$.

$$\begin{array}{c}
\begin{array}{c}
\zeta \\
\nu \\
s \\
k
\end{array}
\end{array}$$

**Remark 3.2.** If $k > 0$, the poset structure on $P^{(k)}_{\infty}$ induced from the one on $W^{(k)}_{\infty}$ defined by the Bruhat order is different from the order on $P^{(k)}_{\infty}$ given by the inclusion of the Young diagrams. For example, let $k = 2$, $\lambda = (3, 2), \mu = (5, 1)$. Then $w_{\lambda} = 3412 \cdots, w_{\mu} = 1432 \cdots$. We have $w_{\lambda} \leq w_{\mu}$ in the Bruhat order, but $\lambda \not\leq \mu$. It would be an interesting problem to give a good combinatorial model for $W^{(k)}_{\infty}$ which enable us to see the Bruhat order manifestly. One candidate is the Maya diagrams introduced below.

**Remark 3.3.** We can depict the permutation $w$ as the following “Maya diagrams”:

$$\begin{array}{cccccccc}
\bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc
\end{array}$$
The set \{v_1, \ldots, v_k\} is indicated by the boxes with \circ, while \{\zeta_1, \ldots, \zeta_s\} are the positions of the boxes with \bullet. Then \mu_i is the number of vacant boxes to the left of the \(i\)th box with \circ. We have \(\zeta = (4,3,1)\) and \(\mu = (3,1)\), so \(\nu = (2,1,1)\).

We record the following lemmas without proofs and will use them later in the proof of the main theorem (cf. [12]).

**Lemma 3.4.** Let \(w = v_1 \cdots v_k \bar{\zeta}_1 \cdots \bar{\zeta}_s u_1 \cdots u_{n-k-s} \in W_n^{(k)}\). Suppose \(i \geq 1\). \(\ell(s_i w) < \ell(w)\) if and only if one of the following holds:

\((L1)\) \(w = ( \cdots | i+1 | \cdots i \cdots )\), i.e., \(\zeta_p = i+1\) and \(u_q = i\) for some \(p\) and \(q\);
\((L2)\) \(w = ( i \cdots | i+1 \cdots)\), i.e., \(\zeta_p = i+1\) and \(v_q = i\) for some \(p\) and \(q\);
\((L3)\) \(w = (i+1 \cdots | i \cdots)\), i.e., \(u_p = i\) and \(v_q = i+1\) for some \(p\) and \(q\).

Note that, in this case, \(s_i w \in W_n^{(k)}\).

**Lemma 3.5.** Let \(w = v_1 \cdots v_k \bar{\zeta}_1 \cdots \bar{\zeta}_s u_1 \cdots u_{n-k-s} \in W_n^{(k)}\).

\((L0)\) \(\ell(s_0 w) < \ell(w)\) if and only if \(w = ( \cdots | \bar{\cdot} \cdots )\), i.e., \(\zeta_s = 1\).

3.2. **Remarks on the Schubert conditions.** In this section, we review the definition of Schubert classes of \(IG(n-k, 2n)\) for the sake of the precise comparison of our conventions and those in [11]. It is worth noting that the characteristic index \(\chi\) appears in the Schubert conditions in an apparent manner.

Let \(e_1, \ldots, e_n, e_1^*, \ldots, e_n^*\) be a standard symplectic basis of \(\mathbb{C}^{2n}\). Define a symplectic form by

\[\langle e_i, e_j \rangle = \langle e_i^*, e_j^* \rangle = 0, \quad \langle e_i, e_j^* \rangle = \delta_{ij}.\]

For \(1 \leq i \leq n\), define a complete flag \(F^\bullet: F^n \subset \cdots \subset F^1 \subset F^\bar{1} \subset \cdots \subset F^n\) by

\[F^i = \langle e_i, \ldots, e_n \rangle, \quad F^\bar{i} = \langle e_i^*, \ldots, e_1^* \rangle + F^1.\]

Let \(\lambda \in \mathcal{P}_n^{(k)}\). Then the Schuber variety \(\Omega_\lambda\) with respect to \(F^\bullet\) can be also defined as

\[\Omega_\lambda := \{\Sigma \in IG(n-k, 2n) \mid \dim(\Sigma \cap F^{w_\lambda(k+j)}) \geq j \ (1 \leq j \leq n-k)\}.\]

Indeed, if we re-label the above flag by \(F_1 \subset \cdots \subset F_{2n}\), i.e., \(F^i = F_{n+1-i}\) and \(F^\bar{i} = F_{n+i}\) for \(1 \leq j \leq n\), then

\[F^{w_\lambda(k+j)} = \begin{cases} F^{\zeta_j} = F_{n+1-\zeta_j} = F_{n-\chi_j} & 1 \leq j \leq s \\ F^{w_{n-j}} = F_{n+u_j-s} = F_{n-\chi_j} & s+1 \leq j \leq n-k, \end{cases}\]

where \(v_1 \cdots v_k | \bar{\zeta}_1 \cdots \bar{\zeta}_s u_1 \cdots u_{n-k-s}\) is the one-line notation of \(w_\lambda\). Therefore the equivalence of the definitions of \(\Omega_\lambda\) at [11] and here follows from

\[p_j(\lambda) = n - \chi_j \ (1 \leq j \leq \ell(\lambda))\]  \hspace{1cm} (3.1)

and the fact that the condition is redundant for \(j > \ell(\lambda)\). We can prove the equation (3.1) as follows. If \(1 \leq j \leq s\), then the RHS is \(n - p_j(\lambda) = \zeta_j - 1 = \chi_j\). Suppose that
\(1 \leq j \leq \ell(\lambda)\). Then from the correspondence in \[3.1\] it is clear that

\[
-\chi_j = u_{j-s} = k - \lambda_j + 1 + j - s - 1 + \# \{ \zeta_i \mid \zeta_i \leq u_{j-s} \} = k - \lambda_j + 1 + \# \{ i \mid i < j, \chi_i + \chi_j \leq -1 \} = k - \lambda_j + 1 + j - 1 - \# \{ i \mid i < j, \chi_i + \chi_j \geq 0 \} = k - \lambda_j + 1 + j - 1 - \# \{ i \mid i < j, \lambda_i + \lambda_j > 2k + j - i \} = p_j(\lambda) - n.
\]

The second last equality follows from the following lemma.

**Lemma 3.6.** Let \(\chi\) be the characteristic index of \(\lambda\). Suppose \(1 \leq i < j \leq n - k\). Then \(\chi_i + \chi_j \geq 0\) if and only if \(\lambda_i + \lambda_j > 2k + j - i\).

**Proof.** Let \(v_1 \cdots v_k \bar{u}_1 \cdots u_{n-k-s}\) be the one-line notation for \(w_\lambda\). The only non-trivial case is when \(i \leq s\) and \(j \geq s + 1\). In this case, observe that \(\chi_j = k - \# \{ p \mid v_p < u_{j-s} \}\) and that \(\zeta_i > u_{j-s}\) if and only if \(\chi_i \geq j - i + \# \{ p \mid v_p < u_{j-s} \}\). Thus \(\chi_i + \chi_j = \zeta_i - 1 - u_{j-s} \geq 0\) if and only if

\[
\lambda_i + \lambda_j = \zeta_i + k + \lambda_j = 2k + \zeta_i + \lambda_j - k = 2k + \zeta_i - \# \{ p \mid v_p < u_{j-s} \} > 2k + j - i.
\]

\(\square\)

The \(T\)-fixed point of \(IG(n - k, 2n)\) corresponding to \(\lambda\) is \(\langle e^{*}_{w_\lambda(k+1)}, \ldots, e^{*}_{w_\lambda(n)} \rangle\), which is the image of \(e_w \in \mathcal{F}_n\) under the projection \(pr\) onto \(IG(n - k, 2n)\).

### 3.3. Invariant subring \(\mathcal{R}_\infty^{(k)}\)

Let \(\mathcal{R}_\infty^{(k)}\) be the sub-ring of elements in \(\mathcal{R}_\infty\) which are fixed by the right action of \(W_\infty^{(k)}\):

\[
\mathcal{R}_\infty^{(k)} := \{ f \in \mathcal{R}_\infty \mid s_i^\ast(f) = f \quad (\forall i \neq k) \}.
\]

Since the right action of \(W_\infty\) is \(\mathbb{Z}[t]\)-linear, \(\mathcal{R}_\infty^{(k)}\) is a \(\mathbb{Z}[t]\)-subalgebra of \(\mathcal{R}_\infty\).

**Proposition 3.7.** We have

\[
\mathcal{R}_\infty^{(k)} = \bigoplus_{w \in W_\infty^{(k)}} \mathbb{Z}[t]e_w.
\]

**Proof.** In order to prove the inclusion “\(\supseteq\)”, it is enough to show \(e_w \in \mathcal{R}_\infty^{(k)}\) for \(w \in W_\infty^{(k)}\). Let \(w \in W_\infty^{(k)}\). Then for any \(j \neq k\) we have \(\ell(ws_j) = \ell(w) + 1\), and hence \(\partial_j e_w = 0\). This is equivalent to \(s^\ast_j e_w = e_w\) for \(j \neq k\). Thus we have \(e_w \in \mathcal{R}_\infty^{(k)}\). To prove the reverse inclusion “\(\subseteq\)”, we write an arbitrary element \(f\) of \(\mathcal{R}_\infty^{(k)}\) as \(f = \sum_{w \in W_\infty} c_w e_w\) (\(c_w \in \mathbb{Z}[t]\)). If \(v \notin W_\infty^{(k)}\), then there is \(i\) such that \(i \neq k\) and \(\ell(ws_i) < \ell(v)\). We have \(0 = \partial_i f = \sum_{w \in W_\infty, \ell(ws_i) = \ell(v) - 1} c_w e_{ws_i}\). It follows that \(c_v = 0\). \(\square\)

**Lemma 3.8.** Let \(f \in \mathcal{R}_\infty^{(k)}\). If \(\partial_i f = 0\) for all \(i \geq 0\), then \(f = 0\).

**Proof.** By Proposition 3.7, we can choose an \(n\) large enough so that \(f = \sum_{w \in W_\infty^{(k)}} c_w e_w\). Observe that

\[
\delta_i f = \sum_{w \in W_\infty^{(k)}} (\delta_i c_w) e_w + \sum_{w \in W_\infty^{(k)}, \ell(ws) = \ell(w) - 1} s_i c_w e_{ws}.
\]
The coefficient of $C_w$ on RHS is
\[
\delta_i c_w + s_i c_{s_i w} \quad \text{if } s_i w \in W_n^{(k)} \quad \text{and} \quad \ell(s_i w) = \ell(w) + 1
\]
\[
\delta_i c_w \quad \text{otherwise}.
\]
Therefore if $w_{\text{max}}$ is the longest element in $W_n^{(k)}$, then $\delta_i c_{w_{\text{max}}} = 0$ for all $i$. Thus $c_{w_{\text{max}}} = 0$. By induction on $\ell(w_{\text{max}}) - \ell(w)$, we have $c_w = 0$ for all $w \in W_n^{(k)}$ and hence for all $w \in W_\infty^{(k)}$. \hfill \square

3.4. Duality of $\text{IG}(n - k, 2n)$. There is a unique longest element in $W_n^{(k)}$, which we denote by $w_{\text{max}}$. In the one-line notation, it is given by $12 \cdots k \overline{n} \cdots n - k$. For $w \in W_n^{(k)}$, define $w^\vee = w w_{\text{max}}$. We have $w^\vee \in W_n^{(k)}$ and $\ell(w^\vee) = \ell(w_{\text{max}}) - \ell(w)$. Moreover, we have $w_{\text{max}}^2 = e$, and so the operation $w \mapsto w^\vee$ is an involution on the set $W_n^{(k)}$. Note that this involution does depend on $n$.

Remark 3.9. If $w = v_1 \cdots v_k | \zeta_1 \cdots \zeta_s u_1 \cdots u_{n-k-s}$, then
\[
w^\vee = v_1 \cdots v_k | \overline{u_{n-k-s}} \cdots \overline{u_1} \zeta_s \cdots \zeta_1.
\]
In other words, the involution in terms of Maya diagram is given by exchanging the vacant boxes and the boxes occupied by “•”.

Let $w, v \in W_n^{(k)}$ and $i \in \mathcal{I} := \{0, 1, \ldots, n - 1\}$. We write $w \overset{i}{\rightarrow} v$ if $s_i w = v$ and $\ell(v) = \ell(w) + 1$. The relation is called the covering relation for weak left Bruhat order. The weak Bruhat graph is the graph such that the set vertices as $W_n^{(k)}$ and the (oriented) arrows are the covering relation for weak left Bruhat order.

Example 3.10. Let $n = 4, k = 2$. We can draw the weak Bruhat graph as follows. The involution is given by reflection with respect to the dashed horizontal line.
Let \( w \in W_n^{(k)} \). Define the following sets:

\[
\mathcal{I}_-(w) := \{ i \in \mathcal{I} \mid \ell(s_iw) = \ell(w) - 1 \},
\]

\[
\mathcal{I}_+(w) := \{ i \in \mathcal{I} \mid \ell(s_iw) = \ell(w) + 1 \text{ and } s_iw \in W_n^{(k)} \},
\]

\[
\mathcal{I}_0(w) := \{ i \in \mathcal{I} \mid \ell(s_iw) = \ell(w) + 1 \text{ and } s_iw \not\in W_n^{(k)} \}.
\]

Note that if \( i \in \mathcal{I}_-(w) \) then \( s_iw \in W_n^{(k)} \).

**Example 3.11.** Let \( n = 4, k = 2 \). If \( w = 23|1 \). Then \( \mathcal{I}_-(w) = \{1, 3\} \), \( \mathcal{I}_+(w) = \{0\} \), and \( \mathcal{I}_0(w) = \{2\} \).

**Lemma 3.12.** Let \( w \in W_n^{(k)} \). Then the following holds.

1. \( \mathcal{I}_-(w) = \mathcal{I}_+(w^\vee), \mathcal{I}_+(w) = \mathcal{I}_-(w^\vee), \)
2. \( \mathcal{I}_0(w) = \mathcal{I}_0(w^\vee). \)

**Proof.** (1) We will show \( \mathcal{I}_-(w) \subset \mathcal{I}_+(w) \). Let \( i \in \mathcal{I}_-(w) \). Then \( s_iw \in W_n^{(k)} \) as noted above. Since \( (s_iw)^\vee = s_iww_{\max} = s_iw^\vee \in W_n^{(k)} \), we have

\[
\ell(s_iw^\vee) = \ell(s_iw)^\vee = \ell(w_{\max}) - \ell(s_iw) = \ell(w_{\max}) - (\ell(w) - 1) = \ell(w^\vee) + 1.
\]

Thus \( i \in \mathcal{I}_+(w) \). The proof of the opposite inclusion relation is similar. The second statement follows from the fact \( (w^\vee)^\vee = w \).

(2) \( \mathcal{I}_0(w) \) is the complement of \( \mathcal{I}_-(w) \cup \mathcal{I}_+(w) \) in \( \mathcal{I} \). Hence the result follows from (1).

**Lemma 3.13.** Let \( w \in W_n^{(k)} \). If \( i \in \mathcal{I}_0(w) \), then \( s_iw = ws_j \) for some \( j \neq k \).

**Proof.** Because \( s_iw \not\in W_n^{(k)} \), there exists \( j(\neq k) \) such that \( \ell(s_iss_j) = \ell(s_iw) - 1 \). This means that \( s_iw \) has a reduced expression of a form \( s_{i_1} \cdots s_j w \) with \( l = \ell(w) \) (cf. [3, Cor. 1.4.6.]). Since \( \ell(s_iw) = \ell(w) + 1 \), \( w \) is obtained from the reduced expression of \( s_iw \) by deleting one unique simple reflection ("exchange condition" cf. [3, p.117]). Now the right end \( s_j \) is the unique one to be deleted (if otherwise, it contradicts with the assumption \( w \in W_n^{(k)} \)), and hence \( w = s_{i_1} \cdots s_{i_l} \), and the lemma follows.

3.5. **Lemma on left divided difference operators.** For \( w \in W_\infty \), we choose a reduced expression \( s_{i_1} \cdots s_{i_l} \) for \( w \). Then \( \delta_w := \delta_{i_1} \cdots \delta_{i_l} \) does not depend on the reduced expression. The following fact is well-known (cf. [5, §2]).

**Lemma 3.14.** Let \( u, v \in W \). Then

\[
\delta_u \delta_v = \begin{cases} 
\delta_{uv} & \text{if } \ell(u) + \ell(v) = \ell(uv) \\
0 & \text{if } \ell(u) + \ell(v) > \ell(uv) 
\end{cases}
\]

The following proposition will be used in the proof of the main theorem.

**Proposition 3.15.** Let \( w \in W_n^{(k)} \). We have the following.

1. If \( i \in \mathcal{I}_-(w) \) then \( \delta_i \delta_{w^\vee} = \delta_{(s_iw)^\vee} \).
2. If \( i \in \mathcal{I}_+(w) \) then \( \delta_i \delta_{w^\vee} = 0 \).
3. If \( i \in \mathcal{I}_0(w) \) then there exists \( j \neq k \) such that \( \delta_i \delta_{w^\vee} = \delta_{w^\vee} \delta_j \).
Proof. (1) If \( i \in \mathcal{I}_-(w), \) then from Lemma 3.12 we have \( i \in \mathcal{I}_+(w^\vee), \) i.e., \( s_i w^\vee \in W_n^{(k)} \) and \( \ell(s_i w^\vee) = \ell(w^\vee) + 1. \) Recall that \( s_i w^\vee = (s_i w)^\vee. \) Then the result follows from Lemma 3.14.

(2) If \( i \in \mathcal{I}_+(w), \) then from Lemma 3.12 we have \( i \in \mathcal{I}_-(w^\vee). \) This means that \( \ell(s_i w^\vee) = \ell(w^\vee) - 1. \) Hence \( \delta_{s_i w^\vee} = 0 \) by Lemma 3.14.

(3) If \( i \in \mathcal{I}_0(w), \) then from Lemma 3.12 we have \( i \in \mathcal{I}_0(w^\vee). \) Then from Lemma 3.13 there exists some \( j \neq k \) such that \( s_i w^\vee = w^\vee s_j, \) where the products in both hand sides are length-additive. Then the result follows from Lemma 3.14. \( \square \)

4. Multi Schur-Pfaffian

In this section, we recall the multi Schur-Pfaffian due to Kazarian, but in a slightly more general form.

Let \( c_r^{(1)}, c_r^{(2)}, \ldots (r \in \mathbb{Z}) \) be infinite indeterminates. The multi Schur-Pfaffian

\[
\text{Pf}[c_r^{(1)} \ldots c_r^{(m)}] \in \mathbb{Z}[c_r^{(1)}, c_r^{(2)}, \ldots (r \in \mathbb{Z})]
\]

is defined as follows:

- for \( m = 1 \) we set \( \text{Pf}[c_r^{(1)}] = c_r^{(1)}. \)
- for \( m = 2 \), we set \( \text{Pf}[c_r^{(1)} c_r^{(2)}] = c_r^{(1)} c_r^{(2)} + 2 \sum_{s=1}^{r_2} (-1)^s c_r^{(1)} c_r^{(2)}. \)
- for any even \( m \geq 4 \), we set

\[
\text{Pf}[c_r^{(1)} \ldots c_r^{(m)}] = \sum_{s=2}^{m} (-1)^s \text{Pf}[c_r^{(1)} c_r^{(s)}] \cdot \text{Pf}[c_r^{(2)} \ldots \widehat{c_r^{(s)}} \ldots c_r^{(m)}].
\]

- for any odd \( m \geq 3 \), we set

\[
\text{Pf}[c_r^{(1)} \ldots c_r^{(m)}] = \sum_{s=1}^{m} (-1)^{s-1} c_r^{(s)} \cdot \text{Pf}[c_r^{(1)} \ldots \widehat{c_r^{(s)}} \ldots c_r^{(m)}].
\]

Remark 4.1 (Pfaffian in classical literature). If we assume \( r_i \geq 0, c_r^{(i)} = 1 \) and \( \text{Pf}[c_r^{(i)}, c_r^{(j)}] + \text{Pf}[c_r^{(j)}, c_r^{(i)}] = 0 \) hold for all \( 1 \leq i, j \leq m, \) then \( \text{Pf}[c_r^{(1)} \ldots c_r^{(m)}] \) coincides with Kazarian's. In particular, in the case when \( m \) is even, then \( \text{Pf}[c_r^{(1)} \ldots c_r^{(m)}] \) is the classical Pfaffian of the skew symmetric matrix whose \( i, j \) entry is given by \( \text{Pf}[c_r^{(i)}, c_r^{(j)}]. \) If we further assume that \( c_r^{(i)} = c_r^{(j)} \) for all \( i, j, \) then it is due to Schur.

Remark 4.2. Let \( q_r^{(i)} := q_{\lambda_r^{(i)}}. \) Let \( \lambda \) be a strict partition of length \( \ell(\lambda), \) then

\[
\text{Pf}[q_{\lambda_1^{(1)}}^{(1)} \ldots q_{\lambda_\ell(\lambda)}^{(\ell(\lambda))}] := \text{Pf}[c_r^{(1)} \ldots c_r^{(\ell(\lambda))}]_{c=r} (4.1)
\]

is equal to the factorial Q-function \( Q_\lambda(x|t) \) defined by Ivanov [13]. This expression is obtained in [11], §11], which is also equivalent to Kazarian’s Lagrangian degeneracy loci formula [15]. Note that \( (4.1) \) is a variant of Ivanov’s original Pfaffian formula [13] Thm 9.1. In particular, \( \text{Pf}[q^{(0)}_{\lambda_1^{(1)}} \ldots q^{(0)}_{\lambda_\ell(\lambda)}] \) is the classical Schur Q-function [21]. The function \( q_r^{(0)} \) is also denoted by \( Q_r(x) \) at (2.1).

Remark 4.3. The multi Schur-Pfaffian can be defined in terms of the raising operators (cf. [3]). This aspect will be postponed until Section 7 since we will not use it in the proof of our main theorem.

We record the following properties of Pf which can be proved easily by induction on \( m. \)
Since it is a symmetric polynomial in $z$. These facts in the proof of the main theorem.

Although we will not use difference operators in Section 6. In Section 5.2, we give the geometric interpretation of Proposition 5.8 is essential for computing the double Schubert polynomials via the divided difference operators.

We have

We check the invariance of $s$.

Clearly this is equal to

The following are clear from the definition of Pfaffian

Remark 4.5. The following are clear from the definition of Pfaffian

where $|_{≥0}$ denotes the substitution $c_0^(i) = 1$ and $c_r^(i) = 0$ ($r < 0$).

5. Double theta polynomials

In this section, first we list basic formulas for the double theta polynomials. In particular, Proposition 5.8 is essential for computing the double Schubert polynomials via the divided difference operators in Section 6. In Section 5.2 we give the geometric interpretation of those polynomials in terms of the Chern classes of vector bundles, although we will not use these facts in the proof of the main theorem.

5.1. $kθ_r^(l)(x, z | t)$. Recall Definition 1.1 of the double theta polynomial $kθ_r^(l)(x, z | t)$. The generating function is denoted by

$$f_l(u) = \sum_{r=0}^{∞} kθ_r^(l)(x, z | t) \cdot u^r. \quad (5.1)$$

Proposition 5.1. We have $kθ_r^(l)(x, z | t) ∈ R_∞^(k)$.

Proof. We check the invariance of $kθ_r^(l)(x, z | t)$ under the action of $s_i^±$ ($i ≥ 0, i ≠ k$). Since it is a symmetric polynomial in $z_1, . . . , z_k$, it is obvious when $i ≥ 1, i ≠ k$. To show $s_i^0(kθ_r^(l)(x, z | t)) = kθ_r^(l)(x, z | t)$, it suffices to consider the case $l = 0$, since $s_i^0$ is $Z[t]$-linear. The action of $s_i^0$ is given by substitutions $(x_1, x_2, . . .) ↦ (z_1, x_1, x_2, . . .)$ and $z_1 = −z_1$. Thus $s_i^0f_0(u)$ is

$$\frac{1 + z_1 u}{1 - z_1 u} \prod_{i=1}^{∞} \frac{1 + x_i u}{1 - x_i u} \cdot (1 - z_1 u) \prod_{i=2}^{k} (1 + z_i u).$$

Clearly this is equal to $f_0(u)$.

Remark 5.2. $R_∞^(k)$ is generated by $θ_i := kθ_i^(0)$ ($i ≥ 1$) as an algebra over $Z[t]$ since

$$θ_i^2 + 2 \sum_{j=1}^{i} (-1)^j θ_{i+j} θ_{i-j} = \begin{cases} e_2(z_1^2, . . . , z_k^2) \\ 0. \end{cases}$$

We simply denote $kθ_r^(l)(x, z | t)$ by $θ_r^(l)$ if there is no ambiguity.

Lemma 5.3. For all $l > 1$, we have

$$θ_r^(l) = θ_r^(l-1) - t_l θ_r^(l-1).$$
For $l \geq 0$, we have
\[ \varrho_r^{(-l)} = \varrho_r^{(-l-1)} + t_{l+1} \varrho_{r-1}^{(-l-1)} = \sum_{i=0}^{r} (-t_i)^i \varrho_{r-i}^{(-l+1)} . \] (5.2)

**Proof.** The first equation is obtained by extracting the coefficient of $u^r$ in the equation
\[ f_i(u) = f_{l-1}(u) \cdot (1 - t_i u) \]
which is obvious from the definition of $\varrho_r^{(l)}$. The second identities are the consequence of the following equations
\[ f_{-l}(u) = f_{-l-1}(u) \cdot (1 + t_{l+1} u) = f_{-l+1}(u)(1 + t_{l+1} u)^{-1} . \]

\[ \square \]

**Lemma 5.4.** We have
\[
\begin{align*}
\varrho_r^{(l)} & = \varrho_r^{(l)} \quad (l \neq \pm i), \quad (5.3) \\
\varrho_r^{(i)} & = \varrho_r^{(i-1)} - t_{i+1} \varrho_{r-1}^{(i-1)} \quad (i \geq 0), \quad (5.4) \\
\varrho_r^{(-i)} & = \varrho_r^{(-i-1)} + t_i \varrho_{r-1}^{(-i-1)} \quad (i > 0). \quad (5.5)
\end{align*}
\]

**Proof.** Since $\varrho_r^{(l)}$ is a polynomial symmetric in $t_1, \ldots, t_{|l|}$, the identity (5.3) for $i \geq 1$ is obvious. The case $i = 1$, i.e., the invariance of $\varrho_r^{(l)} (l \neq 0)$ under $s_0^l$, can be shown in the same manner as in the proof of Proposition 5.1.

Let $g_k(u) := \prod_{j=1}^{\infty} \frac{1 + z_j u}{1 - z_j u} \prod_{j=1}^{k} (1 + z_j u)$. For $i \geq 1$, we have
\[ s_i f_i(u) = g_k(u)(1 - t_{i-1} u) \cdots (1 - t_i u)(1 - t_i u) = f_{i-1}(u)(1 - t_{i+1} u). \]
Thus the equation (5.4) for $i \geq 1$ is obtained by comparing the coefficients of $u^r$. The case $i = 0$ is derived from the following equation
\[ s_i f_0(u) = \frac{1 - t_{1} u}{1 + t_{1} u} g_k(u) = (1 - t_{1} u)f_{-1}(u). \]
The equation (5.5) follows from
\[
\begin{align*}
s_i f_{-i}(u) & = g_k(u)(1 + t_{1} u)^{-1} \cdots (1 + t_{i-1} u)^{-1}(1 + t_i u)^{-1} \\
& = g_k(u)(1 + t_{1} u)^{-1} \cdots (1 + t_{i-1} u)^{-1}(1 + t_i u)^{-1}(1 + t_{i+1} u)^{-1}(1 + t_i u) \\
& = f_{-i-1}(u)(1 + t_{i} u).
\end{align*}
\]

\[ \square \]

**Lemma 5.5.** For all $i \geq 0$, we have
\[
\delta_i \varrho_r^{(l)} = \begin{cases} 
0 & l \neq \pm i \\
\varrho_r^{(l-1)} & l = \pm i
\end{cases}.
\]
Proof. The case \( l \neq \pm i \) is obvious from the invariance result \((5.3)\). Let \( i \geq 1 \). The cases \( l = \pm i \) follow from the following equations:

\[
\begin{align*}
    f_i(u) - s_i^tf_i(u) &= f_{i-1}(u)((1-t_iu) - (1-t_{i+1}u)) \\
    &= f_{i-1}(u)(t_{i+1} - t_i)u, \\
    f_{-i}(u) - s_i^tf_{-i}(u) &= f_{-i+1}(u)((1+t_iu)^{-1} - (1+t_{i+1}u)^{-1}) \\
    &= f_{-i+1}(u)(1+t_iu)^{-1}(1+t_{i+1}u)^{-1}(t_{i+1} - t_i) \\
    &= f_{-i-1}(u)(t_{i+1} - t_i)u.
\end{align*}
\]

Finally we show \( \delta_0 \varphi_r^{(0)} = \varphi_r^{(-1)} \). This is a consequence of the following equation:

\[
f_0(u) - s_0^tf_0(u) = f_0(u)\left(1 - \frac{1-t_1u}{1+t_1u}\right) = f_0(u)\frac{2t_1u}{1+t_1u} = f_{-1}(u)2t_1u.
\]

\(\square\)

Lemma 5.6. For \( i > 0 \), we have

\[
\delta_i(\varphi_r^{(i)}\varphi_s^{(-i)}) = \varphi_r^{(i-1)}\varphi_s^{(-i-1)} + \varphi_r^{(i-1)}\varphi_s^{(-i-1)}.
\]

Proof. By the Leipnitz rule, Lemma \((5.5)\), Equation \((5.4)\), and Equation \((5.2)\), we have

\[
\begin{align*}
    \delta_i(\varphi_r^{(i)}\varphi_s^{(-i)}) &= \delta_i(\varphi_r^{(i)}\varphi_s^{(-i)}) + s_i(\varphi_r^{(i)})\delta_i(\varphi_s^{(-i)}) \\
    &= \varphi_r^{(i-1)}\varphi_s^{(-i)} + \varphi_r^{(i-1)}\varphi_s^{(-i-1)} + s_i(\varphi_s^{(-i)})\delta_i(\varphi_r^{(i)}) \\
    &= \varphi_r^{(i-1)}(\varphi_s^{(-i)} - t_{i+1}\varphi_s^{(-i-1)}) + \varphi_r^{(i-1)}\varphi_s^{(-i-1)} + s_i(\varphi_s^{(-i)})\delta_i(\varphi_r^{(i)}) \\
    &= \varphi_r^{(i-1)}\varphi_s^{(-i-1)} + \varphi_r^{(i-1)}\varphi_s^{(-i-1)} + s_i(\varphi_s^{(-i)})\delta_i(\varphi_r^{(i)}) \\
    &= \varphi_r^{(i-1)}\varphi_s^{(-i-1)} + \varphi_r^{(i-1)}\varphi_s^{(-i-1)}. \quad \square
\end{align*}
\]

We use the following notation in the rest of the paper.

Definition 5.7. For all \((r_1, \ldots, r_m), (l_1, \ldots, l_m) \in \mathbb{Z}^m\), let

\[
Pf[\varphi_r^{(l_1)} \cdots \varphi_r^{(l_m)}] := Pf[c_r^{(1)} \cdots c_r^{(m)}]|_{c=\varphi(r)}.
\]

where \( |_{c=\varphi(r)} \) means that we substitute \( \varphi_s^{(l_i)} \) to \( c_s^{(i)} \) for all \( i \) and \( s \in \mathbb{Z} \). We emphasize that we substitute theta polynomials after we write the Pfaffian as polynomials in the formal variable \( c_s^{(i)} \)’s. For example,

\[
Pf[\varphi_r^{(l_1)} \varphi_s^{(l_2)}] = Pf[c_s^{(1)} c_s^{(2)}]|_{c=\varphi(r)} = \left(c_s^{(1)} c_s^{(2)} - 2c_r^{(1)} c_s^{(2)}\right)|_{c=\varphi(r)} = -2\varphi_s^{(l_1)} \varphi_s^{(l_2)} = -2.
\]

By induction on \( m \) in the axioms of Pfaffian, the above two lemmas give the following proposition.

Proposition 5.8.

(a) Let \( i \geq 0 \). If \( l_p \neq \pm i \) for all \( p \), then \( \delta_i Pf[\varphi_r^{(l_1)} \cdots \varphi_r^{(l_m)}] = 0 \).

(b) Let \( i \geq 0 \). Suppose that \( l_p = i \) or \( l_p = -i \) for some \( p \) and that \( l_q \neq \pm i \) for all \( q \neq p \).

Then we have

\[
\delta_i Pf[\varphi_r^{(l_1)} \cdots \varphi_r^{(l_{p-1})} \varphi_r^{(\pm i)} \varphi_r^{(l_{p+1})} \cdots \varphi_r^{(l_m)}] = Pf[\varphi_r^{(l_1)} \cdots \varphi_r^{(l_{p-1})} \varphi_r^{(\pm i-1)} \varphi_r^{(l_{p+1})} \cdots \varphi_r^{(l_m)}].
\]
(c) Let $i > 0$. Suppose that $l_p = i$ and $l_q = -i$ for some $p < q$ and that $l_s \neq \pm i$ for all $s \neq p, q$. Then we have
\[
\delta_i \text{Pf}[\vartheta^{(i)}_{r_1} \cdots \vartheta^{(i)}_{r_p} \cdots \vartheta^{(-i)}_{q} \cdots \vartheta^{(-i)}_{l_m}]
\]
\[
= \text{Pf}[\vartheta^{(i)}_{r_1} \cdots \vartheta^{(i)}_{r_p-1} \cdots \vartheta^{(-i)}_{q} \cdots \vartheta^{(-i)}_{l_m}] + \text{Pf}[\vartheta^{(i)}_{r_1} \cdots \vartheta^{(i)}_{r_p-1} \cdots \vartheta^{(-i)}_{q-1} \cdots \vartheta^{(-i)}_{l_m}].
\]

5.2. **Double theta polynomials as equivariant Chern classes.** In this section, we show that the double theta polynomials $\vartheta_r^{(i)}$ correspond to the Chern classes of vector bundles. The result is not used in the proof of the main theorem.

Let $\mathcal{E}$ be the trivial vector bundle over $\mathcal{F}l_n$, and $\mathcal{L}_i, \mathcal{L}^*_i \subset \mathcal{E}$ the subbundles whose fibers are $\text{Span}(e_i), \text{Span}(e^*_i)$ respectively. Let $T = (\mathbb{C}^\times)^n$ be the $n$-dimensional torus and let $t_1, \cdots, t_n$ be the standard basis of $t^*_n$. Then $T$ acts on $\mathcal{L}_i$ with the weight $-t_i$ and $\mathcal{L}^*_i$ with the weight $t_i$. Note that $t_i = -c^T_1(\mathcal{L}_i)$. Let
\[
\mathcal{L} = \bigoplus_{i=1}^n \mathcal{L}_i, \quad \mathcal{L}^* = \bigoplus_{i=1}^n \mathcal{L}^*_i
\]
and hence $\mathcal{E} = \mathcal{L} \oplus \mathcal{L}^*$. Let $\mathcal{V}_n \subset \cdots \subset \mathcal{V}_1 = \mathcal{V} \subset \mathcal{E}$ be the tautological flag of vector bundles over the complete flag variety $\mathcal{F}l_n$ of isotropic subspaces of $V$ where rank $\mathcal{V}_i = n - i + 1$.

Let $z_i = c^T_r(\mathcal{V}_i/\mathcal{V}_{i+1})$. Note that the tautological bundle of $\text{IG}(n-k, 2n)$ is $pr^*(\mathcal{V}_{k+1})$, where $pr : \mathcal{F}l_n \to \text{IG}(n-k, 2n)$ is the natural projection. Note that $Q = \mathcal{E}/\mathcal{V}_{k+1}$ is the universal quotient bundle of $\text{IG}(n-k, 2n)$.

From the geometric construction of $\pi_n : R_\infty \to H^*_T(\mathcal{F}l_n)$ ([11, §10]) we have $\pi_n(Q_r(x)) = c_r(\mathcal{L}^* - \mathcal{V}_1) = c_r(\mathcal{V}^*_1 - \mathcal{L})$. In other words,
\[
\pi_n : \prod_{i=0}^\infty \frac{1 + x_i u}{1 - x_i u} \mapsto \prod_{i=1}^n \frac{1 + t_i u}{1 + z_i u} = \prod_{i=1}^n \frac{1 - z_i u}{1 - t_i u}. \tag{5.6}
\]

Define
\[
\mathcal{U}_l := \bigoplus_{i=1}^n \mathcal{L}_i \quad \text{if } l > 0, \quad \mathcal{U}_{-l} := \mathcal{L} \bigoplus \bigoplus_{i=1}^{l+1} \mathcal{L}^*_i \quad \text{if } l \geq 0.
\]

**Proposition 5.9.** For all $-n \leq l \leq n - 1$, we have
\[
\pi_n(\vartheta_r^{(l)}(x, z \mid t)) = c^T_r(\mathcal{E} - \mathcal{V}_{k+1} - \mathcal{U}_{l+1}).
\]

In particular, we have
\[
c^T_r(\mathcal{E} - \mathcal{V}_{k+1} - \mathcal{U}_{r-k}) = [\Omega_r]_T \quad (1 \leq r \leq n + k).
\]

**Proof.** In view of the relation (5.6), the proposition can be shown by the following formal calculations. For $l \geq 0$, we have
\[
c^T(\mathcal{E} - \mathcal{V}_{k+1} - \mathcal{U}_{l+1}) = \frac{\prod_{i=1}^n (1 + t_i^2 u^2)}{\prod_{i=k+1}^n (1 + z_i u) \prod_{i=l+1}^n (1 - t_i u)} = \prod_{i=1}^n \frac{1 + t_i u}{1 + z_i u} \prod_{i=1}^k (1 + z_i u) \prod_{i=1}^l (1 - t_i u).
\]
and for \( l > 0 \), we have
\[
c^T(\mathcal{E} - \mathcal{V}_{k+1} - U_{l+1}) = \frac{\prod_{i=1}^{n}(1 - t_i^2 u^2)}{\prod_{i=k+1}^{n}(1 + z_i u) \prod_{i=1}^{n}(1 - t_i u) \prod_{i=1}^{l}(1 + t_i u)}
\]
\[
= \prod_{i=1}^{n} \frac{1 + t_i u}{1 + z_i u} \prod_{i=1}^{k} (1 + z_i u) \prod_{i=1}^{l} \frac{1}{1 + t_i u}.
\]

The second statement follows from the result \((1.3)\) due to Wilson.

6. Proof of the main theorem

Recall that for \( w \in W_n^{(k)} \) we defined \( w' \in W_n^{(k)} \) in \((3.4)\).

**Definition 6.1.** Let
\[
\Theta_{\max}(x, z \mid t) := \text{Pf}[\theta_{n+k}^{(n-1)} \theta_{n+k-1}^{(n-2)} \cdots \theta_{2k+1}^{(k)}].
\]

For each \( \lambda \in \mathcal{P}_n^{(k)} \), define
\[
\Theta_{\lambda}^{(n,k)}(x, z \mid t) := \delta_{w'} \Theta_{\max}(x, z \mid t).
\]

**Theorem 6.2 (Pfaffian sum formula for \( \Theta_{\lambda} \)).** Let \( \lambda \in \mathcal{P}_n^{(k)} \). We have
\[
\Theta_{\lambda}^{(n,k)}(x, z \mid t) = \sum_{I \subset D(\lambda)} \text{Pf} \left[ \theta_{\lambda_1+a'}^{(x_1)} \cdots \theta_{\lambda_{n-k}+a_{n-k}}^{(x_{n-k})} \right],
\]
where \( I \) runs over all subsets of \( D(\lambda) \) and \( a'_i = \# \{ j \mid (s, j) \in I \} - \# \{ i \mid (i, s) \in I \} \).

**Proof.** We proceed by induction on \( \ell(w) \). If \( \ell(w') = 0 \), then \( w_\lambda = w_{\max} \), and the result is obvious from the definition. Suppose that \( \ell(w') > 0 \). There is a strict \( k \)-partition \( \lambda' \) and \( i \in \{0, 1, \ldots, n-1\} \) such that \( w_{\lambda'} \in W_n^{(k)} \), \( s_i w_{\lambda'} = w_\lambda \). This implies \( \ell(w_\lambda) = \ell(w_{\lambda'}) - 1 \). By Lemma \((3.4)\) and \((3.5)\), \( w_{\lambda'} \) is in one of the cases \( L1, L2, L3 \) and \( L0 \). Let \( \lambda_\lambda = (\lambda_1, \ldots, \lambda_{n-k}) \) and \( \lambda_{\lambda'} = (\lambda_1', \ldots, \lambda_{n-k}') \) be the characteristic index of \( \lambda \) and \( \lambda' \) respectively. By the hypothesis of induction we have
\[
\delta_{w'} \text{Pf}[\theta_{n+k}^{(n-1)} \theta_{n+k-1}^{(n-2)} \cdots \theta_{2k+1}^{(k)}] = \sum_{I \subset D(\lambda')} \text{Pf} \left[ \theta_{\lambda_1'+a_1'}^{(x_1)} \cdots \theta_{\lambda_{n-k}'+a_{n-k}'}^{(x_{n-k})} \right],
\]

In the cases \( L2, L3 \), or \( L0 \), we have \( D(\lambda') = D(\lambda) \). Furthermore, for some \( p \), \( \lambda'_p = \pm i = \lambda_p + 1 \) and \( \lambda'_q = \lambda_q + 1 \); \( \lambda'_q = \lambda_q \) and \( \lambda_q = \lambda'_q \) for all \( q \neq p \). Thus by Proposition \((5.8)\) we can compute
\[
\delta_{w'} \text{Pf}[\theta_{n+k}^{(n-1)} \theta_{n+k-1}^{(n-2)} \cdots \theta_{2k+1}^{(k)}] = \sum_{I \subset D(\lambda')} \text{Pf} \left[ \theta_{\lambda_1'+a_1'}^{(x_1)} \cdots \theta_{\lambda_{n-k}'+a_{n-k}'}^{(x_{n-k})} \right].
\]
where the first equality follows from Proposition 3.15, the second is the induction hypothesis, and the third follows by Proposition 5.8.

In the case $L1$, we have $D(\lambda) = D(\lambda') \cup \{(p, q)\}$. Furthermore $\chi'_p = i = \chi_p + 1$, $\chi'_q = -i = \chi_q + 1$, $\lambda'_p = \lambda'_p - 1$ and $\lambda'_q = \lambda_q$ for some $p$ and $q$; $\chi'_r = \chi_r$ and $\lambda_r = \lambda'_r$ for all $r \neq p, q$. Here note that $\lambda'_p \neq 0$ so that $\lambda_p \geq 0$. The claim now follows from the computation:

$$
\begin{align*}
\delta_{w_{\chi'}} \text{Pf} & \left[ \vartheta_{n+k}^{(n-1)} \vartheta_{n+k-1}^{(n-2)} \cdots \vartheta_{2k+1}^{(k)} \right] \\
= & \delta_1 \delta_{w_{\chi'}} \text{Pf} \left[ \vartheta_{n+k}^{(n-1)} \vartheta_{n+k-1}^{(n-2)} \cdots \vartheta_{2k+1}^{(k)} \right] \\
= & \sum_{I \subset D(\lambda')} \delta_I \text{Pf} \left[ \vartheta_{\lambda'_1+a_1}^{(i)} \vartheta_{\lambda'_p+a'_p}^{(i)} \cdots \vartheta_{\lambda'_q+a'_q}^{(i)} \cdots \vartheta_{\lambda'_n+a'_n}^{(i)} \right] \\
= & \sum_{I \subset D(\lambda')} \left( \text{Pf} \left[ \vartheta_{\lambda'_1+a_1}^{(i)} \vartheta_{\lambda'_p+a'_p}^{(i)} \cdots \vartheta_{\lambda'_q+a'_q}^{(i)} \cdots \vartheta_{\lambda'_n+a'_n}^{(i)} \right] \\
& \quad \quad + \text{Pf} \left[ \vartheta_{\lambda'_1+a_1}^{(i)} \vartheta_{\lambda'_q+a'_q}^{(i)} \cdots \vartheta_{\lambda'_n+a'_n}^{(i)} \right] \right) \\
= & \sum_{I \subset D(\lambda')} \text{Pf} \left[ \vartheta_{\lambda_1+a_1}^{(1)} \cdots \vartheta_{\lambda_p+a_p}^{(p)} \cdots \vartheta_{\lambda_q+a_q}^{(q)} \cdots \vartheta_{\lambda_n+a_n}^{(n)} \right] \\
& \quad \quad + \sum_{(p, q) \in I \subset D(\lambda')} \text{Pf} \left[ \vartheta_{\lambda_1+a_1}^{(1)} \cdots \vartheta_{\lambda_p+a_p}^{(p)} \cdots \vartheta_{\lambda_q+a_q}^{(q)} \cdots \vartheta_{\lambda_n+a_n}^{(n)} \right] \\
= & \sum_{I \subset D(\lambda')} \text{Pf} \left[ \vartheta_{\lambda_1+a_1}^{(1)} \cdots \vartheta_{\lambda_p+a_p}^{(p)} \cdots \vartheta_{\lambda_q+a_q}^{(q)} \cdots \vartheta_{\lambda_n+a_n}^{(n)} \right] \\
& \quad \quad + \sum_{(p, q) \in I \subset D(\lambda')} \text{Pf} \left[ \vartheta_{\lambda_1+a_1}^{(1)} \cdots \vartheta_{\lambda_p+a_p}^{(p)} \cdots \vartheta_{\lambda_q+a_q}^{(q)} \cdots \vartheta_{\lambda_n+a_n}^{(n)} \right]
\end{align*}
$$

where the first equality follows from Proposition 3.15, the second is the induction hypothesis, the third follows by Proposition 5.8, and the second last equality holds, since, for each $I \subset D(\lambda')$ and $J := I \cup \{(p, q)\} \in D(\lambda)$, $a_i'$ and $a_j'$ are related by

$$
a'_p + 1 = a'_p', \quad a'_q - 1 = a'_q', \quad \text{and} \quad a'_r = a'_r' \forall r \neq p, q.
$$

\[\square\]

Remark 6.3. We owe to Naruse that we learned from him, in the early stage of this work, that $v_{\omega_{max}}$ has a Pfaffian expression.

Remark 6.4. The expression of the right hand side of Theorem 5.2 is essentially independent of $n$. More precisely, if $\lambda \in P_n^{(k)}$, then we have obviously $\lambda \in P_{n+1}^{(k)}$ and all nonzero Pfaffians appearing in the formulas for $\Theta^{(n,k)}$ and $\Theta^{(n+1,k)}$ naturally coincide. We have $\Theta^{(n,k)} = \Theta^{(n+1,k)}$; in particular. This fact can be checked by using Remark 4.5. In fact, one can check that the lower indexes (degree) of the right end $\vartheta$ in the Pfaffians appearing in the formula of $\Theta^{(n+1,k)}$ are less than or equal to zero.
Proposition 6.5 (Stability of $\Theta_\lambda$). Let $\lambda \in \mathcal{P}_n^{(k)}$. For all $m \geq n$, we have
$$\Theta^{(m,k)}_\lambda(x, z \mid t) = \Theta^{(n,k)}_\lambda(x, z \mid t).$$

Proof. This is a consequence of the Pfaffian sum formula in Theorem 6.2 (see Remark 6.4).

By the above proposition, for each $\lambda \in \mathcal{P}_n^{(k)}$, we can define $\Theta^{k}_\lambda(x, z \mid t)$ to be $\Theta^{(n,k)}_\lambda(x, z \mid t)$ for any $n$ such that $\lambda \in \mathcal{P}_n^{(k)}$.

Theorem 6.6. Let $\lambda \in \mathcal{P}_\infty^{(k)}$. We have
$$\mathcal{C}_{\omega_\lambda}(z, t; x) = \Theta^{k}_\lambda(x, z \mid t).$$

Remark 6.7. Note that the dependence of $w_\lambda$ on $k$ is implicit.

Lemma 6.8. We have $\delta_j \Theta_{\text{max}} = 0$ for $j \neq k$.

Proof. Use Proposition 5.8 and Proposition 4.4.

Lemma 6.9. Let $w \in W_\infty^{(k)}$. We have
$$\delta_i \Theta_{w\lambda} = \begin{cases} \Theta_{s_i w\lambda} & \text{if } \ell(s_i w\lambda) = \ell(w\lambda) - 1 \\ 0 & \text{otherwise} \end{cases}.$$

Proof. From the definition of $\Theta_{w\lambda}$ and the fact $\delta_i \Theta_{\text{max}} = 0$ for $i \neq k$ (Lemma 6.8), the result follows from Proposition 3.11 immediately.

Proof of Theorem 6.6. Consider the collection of differences $\{\mathcal{C}_w - \Theta_w \mid w \in W_\infty^{(k)}\}$. Since $\Theta_w$ and $\mathcal{C}_w$ both satisfy the properties in Lemma 6.9 by the standard argument using induction on $\ell(w)$ (II), we can conclude that the difference $\mathcal{C}_w - \Theta_w$ is annihilated by all $\delta_i$. Therefore by Lemma 3.8 $\mathcal{C}_w = \Theta_w$.

7. Raising operators and Wilson’s conjecture

7.1. Basics on raising operators. Let $R_{ij}, 1 \leq i < j \leq m$ be the operator that acts on $\mathbb{Z}^m$ by
$$R_{ij} : (a_1, \ldots, a_i, \ldots, a_j, \ldots, a_m) \mapsto (a_1, \ldots, a_i + 1, \ldots, a_j - 1, \ldots, a_m)$$

Let $c^{(1)}_r, \ldots, c^{(m)}_r, r \in \mathbb{Z}$ be an ordered set of $m$ sequences of infinite variables. The action of $R_{ij}$ on a degree $m$ monomial $c^{(1)}_{r_1} \cdots c^{(m)}_{r_m}$ is defined by
$$R_{ij}(c^{(1)}_{r_1} \cdots c^{(i)}_{r_i} \cdots c^{(j)}_{r_j} \cdots c^{(m)}_{r_m}) = c^{(1)}_{r_1} \cdots c^{(i)}_{r_i+1} \cdots c^{(j)}_{r_j-1} \cdots c^{(m)}_{r_m}.$$

The action of any polynomial in $R_{ij}$ on the set $\mathcal{A}$ of all $\mathbb{Z}$-linear combinations of monomials $c^{(1)}_{r_1} \cdots c^{(m)}_{r_m}, (r_1, \ldots, r_m) \in \mathbb{Z}$ is naturally defined. For example,
$$(1 + R_{ij})c^{(1)}_{r_1} \cdots c^{(m)}_{r_m} = c^{(1)}_{r_1} \cdots c^{(i)}_{r_i} \cdots c^{(j)}_{r_j} \cdots c^{(m)}_{r_m} + c^{(1)}_{r_1} \cdots c^{(i)}_{r_i+1} \cdots c^{(j)}_{r_j-1} \cdots c^{(m)}_{r_m}.$$

Since the actions of the operators $R_{ij}, i < j$ commute, they are extended to the action of the polynomial ring $\mathbb{Z}[R_{ij}, i < j]$. For example,
$$(1 + R_{ij})(1 - R_{ij^*}) = (1 - R_{ij^*})(1 + R_{ij})c^{(1)}_{r_1} \cdots c^{(m)}_{r_m} = (1 - R_{ij^*} + R_{ij} - R_{ij^*} R_{ij})c^{(1)}_{r_1} \cdots c^{(m)}_{r_m}.$$
Consider a formal power series $F = \sum_{s=0}^{\infty} F_s$ where each $F_s$ is a homogeneous polynomial in $R_{ij}$ of degree $s$, regarding $R_{ij}$’s as formal variables. Each $F_r$ acts on $A$ and so $F_s(c_r^{(1)} \cdots c_r^{(m)})$ is in $A$. Thus we obtain the following formal series of $c_r^{(1)} \cdots c_r^{(m)}, (r_1, \ldots, r_m) \in \mathbb{Z}$

$$F(c_r^{(1)} \cdots c_r^{(m)}) := \sum_{s=0}^{\infty} F_s(c_r^{(1)} \cdots c_r^{(m)}).$$

It is well-defined since the coefficient of each $c_r^{(1)} \cdots c_r^{(m)}$ in the sum is finite. Indeed, the degree $s$ of the operator $F_s$ that creates a particular monomial $c_r^{(1)} \cdots c_r^{(m)}$ is bounded. It also has the property that the only appearing terms are such that $s_1 + \cdots + s_m = r_1 + \cdots + r_m$. Considering those properties, we can conclude that

$$F(c_r^{(1)} \cdots c_r^{(m)}) \big|_{c_r^{(i)} = 0}$$

is a polynomial where $|_{c_r^{(i)} = 0}$ denotes the substitution $c_r^{(i)} = 0$ for all $r < 0$ and all $i$. If $F_1$ and $F_2$ are two formal power series as above, then the product $F_1 F_2$ is also such a formal power series and therefore we have

$$(F_1 F_2)(c_r^{(1)} \cdots c_r^{(m)}) = F_1(F_2(c_r^{(1)} \cdots c_r^{(m)})).$$

The RHS of this identity is well-defined, i.e. it is a formal power series such that the coefficient of each $c_r^{(1)} \cdots c_r^{(m)}, (s_1, \ldots, s_m) \in \mathbb{Z}^m$ is finite.

**Example 7.1.** The following formal power series is important for our purpose:

$$F := \frac{1 - R_{12}}{1 + R_{12}} = 1 - 2R_{12} + 2R_{12}^2 - \cdots = 1 + \sum_{s=1}^{\infty} (-1)^s 2R_{12}^s.$$

For example, we have $F(c_{-2}^{(1)} c_1^{(2)}) = c_{-2}^{(1)} c_1^{(2)} - 2c_{-1}^{(1)} c_0^{(2)} + 2c_0^{(1)} c_1^{(2)} - 2c_0^{(1)} c_2^{(2)} + \cdots$, and hence we have $F(c_{-2}^{(1)} c_1^{(2)}) \big|_{c_r^{(i)} = 0} = 0$, while $F(c_{-1}^{(1)} c_3^{(2)}) \big|_{c_r^{(i)} = 0} = -2c_0^{(1)} c_2^{(2)} + 2c_1^{(1)} c_1^{(2)} - 2c_2^{(1)} c_0^{(2)}$.

The following lemma is obvious from the definition.

**Lemma 7.2.** Let $I(F)$ be the set of $i$’s such that $R_{ij}$ or $R_{ji}$ appear in $F$. If $I(F_1) \cap I(F_2) = \emptyset$, then we have the following well-defined identity of formal power series

$$(F_1 F_2)(c_r^{(1)} \cdots c_r^{(m)}) = \prod_{i \in I(F_1) \cup I(F_2)} c_i^{(i)} \cdot F_1 \left( \prod_{i \in I(F_1)} c_i^{(i)} \right) \cdot F_2 \left( \prod_{i \in I(F_2)} c_i^{(i)} \right).$$

**7.2. Pfaffians in terms of raising operators.** We have the following description of Pfaffians. Let $\Delta_m := \{(i, j) \in \mathbb{Z}^2 \mid 1 \leq i < j \leq m\}$.

**Proposition 7.3.** We have

$$\operatorname{Pf}[c_r^{(1)} \cdots c_r^{(m)}] \big|_{c_r^{(i)} = 0} = \left( \prod_{(i, j) \in \Delta_m} \frac{1 - R_{ij}}{1 + R_{ij}} c_r^{(1)} \cdots c_r^{(m)} \right) \big|_{c_r^{(i)} = 0}.$$

**Proof.** We proceed by induction on $m$. The cases $m = 1$ is obvious. For $m = 2$, the identity

$$\operatorname{Pf}[c_r^{(1)} c_r^{(2)}] \big|_{c_r^{(i)} = 0} = \left( \frac{1 - R_{12}}{1 + R_{12}} (c_r^{(1)} c_r^{(2)}) \right) \big|_{c_r^{(i)} = 0}$$
follows clearly from the definition (cf. Example 7.1). The general case can be deduced from the following identity of formal series: for \( m \) even,

\[
\prod_{(i,j)\in \Delta_m} \frac{1 - R_{ij}}{1 + R_{ij}} = \sum_{s=2}^{m} (-1)^{s-1} \prod_{i,j\neq 1,s} \frac{1 - R_{ij}}{1 + R_{ij}}
\]

and, for \( m \) odd,

\[
\prod_{(i,j)\in \Delta_m} \frac{1 - R_{ij}}{1 + R_{ij}} = \sum_{s=1}^{m} (-1)^{s-1} \prod_{i,j\neq s \in \Delta_m} \frac{1 - R_{ij}}{1 + R_{ij}}
\]

Since \( R_{ij} R_{js} = R_{is} \), the proof of these equations can be reduced to show equations for rational functions obtained from replacing \( R_{ij} \) with \( y_i/y_j \). Such equations go back to Schur (21 p.226).

\[\square\]

### 7.3. Pfaffian sum formula and Wilson’s conjecture.

**Definition 7.4.** Let \( \lambda \) be a \( k \)-strict partition in the \((n-k) \times (n+k)\) rectangle and \( \chi \in \mathbb{Z}^{n-k} \) the corresponding characteristic index. Let \((\cdot)_{c=\vartheta(\chi)}\) be the substitution of \( \vartheta_{r_i}^{(\chi_i)} \) to \( c_{r_i}^{(i)} \) for each \( r_i \in \mathbb{Z} \) and \( i = 1, \ldots, n-k \). Define

\[
R_{\lambda}[\vartheta_{\lambda_1}^{(\chi_1)} \cdots \vartheta_{\lambda_{n-k}}^{(\chi_{n-k})}] := \left( \prod_{(i,j)\in \Delta_{n-k}} \frac{1 - R_{ij}}{1 + R_{ij}} \prod_{i,j\in \Delta_{n-k}} \frac{1 - R_{ij}}{1 + R_{ij}} \right)_{c=\vartheta(\chi)}.
\]

By Lemma 3.6 and the remark below, this function coincides with the one defined in the Wilson’s thesis [24, Definition 10]. Note that if \( D(\lambda) = \Delta_{n-k} \), it is a single determinant. In fact, the argument in [22, §1] shows

\[
\left( \prod_{(i,j)\in \Delta_{n-k}} (1 - R_{ij})(c_{\chi_1}^{(i)} \cdots c_{\chi_{n-k}}^{(n-k)}) \right)_{c=\vartheta(\chi)} = \text{Det}[\vartheta_{\lambda_1}^{(\chi_1)} \cdots \vartheta_{\lambda_{n-k}}^{(\chi_{n-k})}] .
\]

If \( D(\lambda) = \emptyset \), then the definition gives a single Pfaffian \( \text{Pf}[\vartheta_{\lambda_1}^{(\chi_1)} \cdots \vartheta_{\lambda_{n-k}}^{(\chi_{n-k})}] \) by Proposition 7.3

**Remark 7.5.** Let \( \vartheta_{p}^{r}[j] \) be the function defined at Definition 7 in [24], then

\[
\vartheta_{p}^{r}[j] = \begin{cases} 
k \vartheta_{p+j}^{r} & k < p \text{ and } r \leq p - k - 1 \\
k \vartheta_{p+j}^{r} & k \geq p \end{cases}.
\]

Finally the following proposition shows that Theorem 1.2 is equivalent to Conjecture 1 in [24], and therefore Corollary 1.4 follows.

**Proposition 7.6.** Let \( \lambda \) be a \( k \)-strict partition in \( \mathcal{P}^{(k)}_n \) and \( \chi \) the corresponding characteristic index. We have

\[
\sum_{I \subseteq D(\lambda)} \text{Pf} \left[ \vartheta_{\lambda_1+\alpha_1}^{(\chi_1)} \cdots \vartheta_{\lambda_{n-k}+\alpha_{n-k}}^{(\chi_{n-k})} \right] = R_{\lambda}[\vartheta_{\lambda_1}^{(\chi_1)} \cdots \vartheta_{\lambda_{n-k}}^{(\chi_{n-k})}] .
\]
Proof. We have

\[
\sum_{I \subseteq D(\lambda)} \text{Pf}\left[ \varphi^{(\chi_1)}_{\lambda_1+a_1^I} \cdots \varphi^{(\chi_{n-k})}_{\lambda_{n-k}+a_{n-k}^I} \right] = \left( \prod_{(i,j) \in \Delta_{n-k}} \frac{1 - R_{ij}}{1 + R_{ij}} \sum_{I \subseteq D(\lambda)} c_{\lambda_1+a_1^I}^{(1)} \cdots c_{\lambda_{n-k}+a_{n-k}^I}^{(n-k)} \right)_{c=\varphi^{(\chi)}} = \left( \prod_{(i,j) \in \Delta_{n-k}} \frac{1 - R_{ij}}{1 + R_{ij}} \prod_{(i,j) \in D(\lambda)} (1 + R_{ij})(c_{\lambda_1}^{(1)} \cdots c_{\lambda_{n-k}}^{(n-k)}) \right)_{c=\varphi^{(\chi)}} \right)_{c=\varphi^{(\chi)}} = R_{\lambda}[\varphi^{(\chi_1)}_{\lambda_1} \cdots \varphi^{(\chi_{n-k})}_{\lambda_{n-k}}]
\]

where the first equality follows from the linearity of \((\cdot)\big|_{c=\varphi^{(\chi)}}\) and the operators, the second follows from the definition of \(a^I\), the third follows from Lemma 7.2, and the last is the definition of \(R_{\lambda}\). \(\square\)
8. **Example:** 

\((n, k) = (5, 2), (5, 3)\)

**\((5, 3)\) Computation**

\[
\begin{align*}
123|45 & \quad (00) \ \text{Det}[\vartheta_0^{-4}\vartheta_0^{-5}] \\
124|35 & \quad (10) \ \text{Det}[\vartheta_1^{-3}\vartheta_0^{-5}] \\
134|25 & \quad (20) \ \text{Det}[\vartheta_2^{-2}\vartheta_0^{-5}] \\
135|34 & \quad (11) \ \text{Det}[\vartheta_1^{-4}] \\
234|15 & \quad (30) \ \text{Det}[\vartheta_3^{-1}\vartheta_0^{-5}] \\
135|24 & \quad (21) \ \text{Det}[\vartheta_2^{-4}] \\
135|24 & \quad (40) \ \text{Det}[\vartheta_0^{-5}] \\
235|14 & \quad (31) \ \text{Det}[\vartheta_1^{-4}] \\
235|14 & \quad (50) \ \text{Det}[\vartheta_0^{-5}] \\
245|13 & \quad (41) \ \text{Det}[\vartheta_1^{-4}] \\
245|13 & \quad (60) \ \text{Det}[\vartheta_0^{-5}] \\
345|12 & \quad (51) \ \text{Det}[\vartheta_1^{-4}] \\
345|12 & \quad (61) \ \text{Det}[\vartheta_0^{-5}] \\
145|23 & \quad (70) \ \text{Det}[\vartheta_2^{-2}] \\
145|23 & \quad (80) \ \text{Pf}[\vartheta_2^{-4}] \\
135|34 & \quad (62) \ \text{Det}[\vartheta_1^{-4}] \\
135|34 & \quad (71) \ \text{Pf}[\vartheta_2^{-3}] \\
145|32 & \quad (81) \ \text{Det}[\vartheta_0^{-5}] \\
145|32 & \quad (72) \ \text{Pf}[\vartheta_2^{-3}] \\
245|31 & \quad (82) \ \text{Pf}[\vartheta_2^{-4}] \\
345|21 & \quad (73) \ \text{Pf}[\vartheta_0^{-5}] \\
145|32 & \quad (83) \ \text{Det}[\vartheta_3^{-1}] \\
145|32 & \quad (74) \ \text{Pf}[\vartheta_2^{-3}] \\
245|31 & \quad (84) \ \text{Pf}[\vartheta_2^{-4}] \\
345|21 & \quad (75) \ \text{Pf}[\vartheta_0^{-5}] \\
145|32 & \quad (85) \ \text{Det}[\vartheta_3^{-1}] \\
345|21 & \quad (76) \ \text{Pf}[\vartheta_2^{-3}] \\
245|31 & \quad (86) \ \text{Pf}[\vartheta_2^{-4}] \\
235|41 & \quad (87) \ \text{Pf}[\vartheta_0^{-5}] \\
145|31 & \quad (88) \ \text{Pf}[\vartheta_2^{-4}] \\
234|51 & \quad (90) \ \text{Det}[\vartheta_1^{-4}] \\
234|51 & \quad (91) \ \text{Pf}[\vartheta_2^{-3}] \\
125|43 & \quad (92) \ \text{Det}[\vartheta_3^{-1}] \\
125|43 & \quad (93) \ \text{Pf}[\vartheta_2^{-3}] \\
124|53 & \quad (94) \ \text{Det}[\vartheta_1^{-4}] \\
124|53 & \quad (95) \ \text{Pf}[\vartheta_2^{-3}] \\
235|51 & \quad (96) \ \text{Det}[\vartheta_1^{-4}] \\
235|51 & \quad (97) \ \text{Pf}[\vartheta_2^{-3}] \\
125|43 & \quad (98) \ \text{Det}[\vartheta_3^{-1}] \\
125|43 & \quad (99) \ \text{Pf}[\vartheta_2^{-3}] \\
124|53 & \quad (100) \ \text{Det}[\vartheta_1^{-4}] \\
124|53 & \quad (101) \ \text{Pf}[\vartheta_2^{-3}] \\
\end{align*}
\]
(5, 2) computation

\[
\begin{align*}
12|345 &\quad (000) &\quad \text{Det}[\theta_0^{-3}\theta_0^{-4}\theta_0^{-5}] \\
13|245 &\quad (100) &\quad \text{Det}[\theta_1^{-2}\theta_0^{-4}\theta_0^{-5}] \\
23|145 &\quad 14|235 &\quad \text{Det}[\theta_2^{-1}\theta_0^{-4}\theta_0^{-5}] &\quad \text{Det}[\theta_1^{-2}\theta_1^{-3}\theta_0^{-5}] \\
(200) &\quad (110) \\
13|245 &\quad 24|135 &\quad 25|134 &\quad 15|234 &\quad (300) &\quad (210) &\quad (220) &\quad (111) \\
\text{Det}[\theta_2^{-3}\theta_0^{-4}\theta_0^{-5}] &\quad \text{Pf}[\theta_2^{-1}\theta_1^{-1}\theta_0^{-5}] &\quad \text{Det}[\theta_1^{-2}\theta_1^{-3}\theta_1^{-4}] \\
(200) &\quad (110) \\
13|245 &\quad 14|325 &\quad 34|215 &\quad 25|134 &\quad 35|124 &\quad (400) &\quad (310) &\quad (320) &\quad (311) &\quad (221) \\
\text{Det}[\theta_2^{-3}\theta_0^{-4}\theta_0^{-5}] &\quad \text{Det}[\theta_1^{-2}\theta_0^{-5}] &\quad \text{Det}[\theta_0^{-2}\theta_0^{-5}] &\quad \text{Pf}[\theta_0^{-3}\theta_1^{-1}\theta_0^{-4}] &\quad \text{Pf}[\theta_0^{-4}\theta_0^{-4}] &\quad \text{Pf}[\theta_2^{-2}\theta_1^{-2}\theta_0^{-4}] &\quad \text{Det}[\theta_2^{-2}\theta_2^{-2}\theta_0^{-4}] &\quad \text{Det}[\theta_2^{-2}\theta_2^{-2}\theta_0^{-3}] \\
(500) &\quad (410) &\quad (320) &\quad (311) &\quad (221) \\
\text{Det}[\theta_2^{-3}\theta_0^{-3}\theta_0^{-5}] &\quad \text{Pf}[\theta_0^{-2}\theta_1^{-2}\theta_0^{-5}] &\quad \text{Pf}[\theta_1^{-2}\theta_0^{-5}] &\quad \text{Pf}[\theta_1^{-2}\theta_1^{-3}\theta_0^{-1}] &\quad \text{Pf}[\theta_1^{-2}\theta_1^{-2}\theta_0^{-1}] &\quad \text{Det}[\theta_0^{-2}\theta_0^{-2}\theta_0^{-4}] &\quad \text{Det}[\theta_2^{-2}\theta_0^{-2}\theta_0^{-4}] &\quad \text{Det}[\theta_2^{-2}\theta_0^{-2}\theta_0^{-3}] \\
(600) &\quad (510) &\quad (420) &\quad (411) &\quad (321) &\quad (322) \\
\text{Det}[\theta_2^{-3}\theta_0^{-3}\theta_0^{-4}] &\quad \text{Pf}[\theta_0^{-2}\theta_1^{-2}\theta_0^{-5}] &\quad \text{Pf}[\theta_1^{-2}\theta_0^{-5}] &\quad \text{Det}[\theta_0^{-2}\theta_1^{-2}\theta_0^{-1}] &\quad \text{Pf}[\theta_1^{-2}\theta_1^{-1}\theta_0^{-1}] &\quad \text{Det}[\theta_2^{-2}\theta_0^{-2}\theta_0^{-4}] &\quad \text{Det}[\theta_2^{-2}\theta_0^{-2}\theta_0^{-3}] \\
(700) &\quad (610) &\quad (520) &\quad (430) &\quad (511) &\quad (421) &\quad (322) \\
\text{Det}[\theta_2^{-3}\theta_0^{-3}\theta_0^{-4}] &\quad \text{Pf}[\theta_0^{-2}\theta_1^{-2}\theta_0^{-5}] &\quad \text{Pf}[\theta_1^{-2}\theta_0^{-5}] &\quad \text{Det}[\theta_0^{-2}\theta_1^{-2}\theta_0^{-1}] &\quad \text{Pf}[\theta_1^{-2}\theta_1^{-1}\theta_0^{-1}] &\quad \text{Det}[\theta_2^{-2}\theta_0^{-2}\theta_0^{-4}] &\quad \text{Det}[\theta_2^{-2}\theta_0^{-2}\theta_0^{-3}] \\
(710) &\quad (620) &\quad (530) &\quad (611) &\quad (521) &\quad (421) &\quad (422) \\
\text{Pf}[\theta_2^{-2}\theta_0^{-4}] &\quad \text{Pf}[\theta_0^{-2}\theta_2^{-1}\theta_0^{-5}] &\quad \text{Pf}[\theta_2^{-2}\theta_0^{-5}] &\quad \text{Pf}[\theta_0^{-2}\theta_2^{-1}\theta_0^{-1}] &\quad \text{Pf}[\theta_2^{-2}\theta_0^{-1}] &\quad \text{Pf}[\theta_2^{-2}\theta_0^{-1}] &\quad \text{Pf}[\theta_0^{-2}\theta_2^{-1}\theta_0^{-3}] &\quad \text{Pf}[\theta_0^{-2}\theta_0^{-2}\theta_0^{-1}] &\quad \text{Pf}[\theta_0^{-2}\theta_0^{-2}\theta_0^{-1}] \\
(720) &\quad (630) &\quad (540) &\quad (711) &\quad (621) &\quad (531) &\quad (522) &\quad (423) \\
\text{Pf}[\theta_2^{-2}\theta_2^{-1}\theta_0^{-4}] &\quad \text{Pf}[\theta_0^{-2}\theta_2^{-1}\theta_0^{-5}] &\quad \text{Pf}[\theta_2^{-2}\theta_0^{-5}] &\quad \text{Pf}[\theta_0^{-2}\theta_2^{-1}\theta_0^{-3}] &\quad \text{Pf}[\theta_0^{-2}\theta_0^{-1}\theta_0^{-4}] &\quad \text{Pf}[\theta_0^{-2}\theta_2^{-1}\theta_0^{-2}] &\quad \text{Pf}[\theta_2^{-2}\theta_2^{-1}\theta_0^{-2}] \\
23|514 &\quad 23|415 &\quad 14|325 &\quad 14|523 &\quad 25|134 &\quad 25|314 &\quad 35|314 &\quad 45|312 &\quad 45|213 &\quad (720) &\quad (630) &\quad (540) &\quad (711) &\quad (621) &\quad (531) &\quad (522) &\quad (423) \\
\text{Pf}[\theta_2^{-2}\theta_2^{-1}\theta_0^{-4}] &\quad \text{Pf}[\theta_0^{-2}\theta_2^{-1}\theta_0^{-5}] &\quad \text{Pf}[\theta_2^{-2}\theta_0^{-5}] &\quad \text{Pf}[\theta_0^{-2}\theta_2^{-1}\theta_0^{-3}] &\quad \text{Pf}[\theta_0^{-2}\theta_0^{-1}\theta_0^{-4}] &\quad \text{Pf}[\theta_0^{-2}\theta_2^{-1}\theta_0^{-2}] &\quad \text{Pf}[\theta_2^{-2}\theta_2^{-1}\theta_0^{-2}] \\
(720) &\quad (630) &\quad (540) &\quad (711) &\quad (621) &\quad (531) &\quad (522) &\quad (423) \\
\end{align*}
\]
References

[1] D. Anderson, Introduction to equivariant cohomology in algebraic geometry. In Contributions to algebraic geometry, EMS Ser. Congr. Rep. Eur. Math. Soc., Zürich, 2012, 71–92.

[2] D. Anderson and W. Fulton, Degeneracy loci, Pfaffians and vexillary permutations in types B,C, and D, available on arXiv:1210:2066v1.

[3] A. Björner and F. Brenti, Combinatorics of Coxeter groups, Graduate Texts in Math. 231, Springer 2005.

[4] S. Billey and M. Haiman, Schubert polynomials for the classical groups, J. Amer. Math. Soc. 8 (1995), No. 2, 443–482.

[5] A. S. Buch, A. Kresch, and H. Tamvakis, A Giambelli formula for isotropic Grassmannians, available on arXiv: 0811.2781.

[6] A. S. Buch, A. Kresch, and H. Tamvakis, A Giambelli formula for even orthogonal Grassmannians, to appear in J. reine angew. Math.

[7] W. Fulton and P. Pragacz, Schubert varieties and degeneracy loci, Lecture Notes in Math. 1689, Springer-Verlag, Berlin, 1998.

[8] G. Z. Giambelli, Risoluzione del problema degli spazi secanti, Mem. R. Accad. Sci. Torino (2) 52 (1902), 171–211.

[9] J. E. Humphreys, Reflection Groups and Coxeter Groups (Cambridge Studies in Advanced Mathematics, No. 29).

[10] T. Ikeda, Schubert classes in the equivariant cohomology of the Lagrangian Grassmannian, Adv. Math. 215 (2007), 1–23.

[11] T. Ikeda, L. Mihalcea, and H. Naruse, Double Schubert polynomials for the classical groups Adv. Math. 226 (2011), 840–886.

[12] T. Ikeda and H. Naruse, Excited Young diagrams and equivariant Schubert calculus, Trans. Amer. Math. Soc. 361 (2009), 5193–5221.

[13] V. N. Ivanov, Interpolation analogue of Schur Q-functions, Zap. Nauc. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. 307 (2004), 99–119.

[14] G. Kempf and D. Laksov, The determinantal formula of Schubert calculus. Acta Math. 132 (1974), 153–162.

[15] M. Kazarian, On Lagrange and symmetric degeneracy loci, preprint. available at: http://www.newton.cam.ac.uk/preprints2000.html

[16] A. Kresch and H. Tamvakis, Double Schubert polynomials and degeneracy loci for the classical groups, Annales de l’Institut Fourier, 52 no. 6 (2002),1681-1727.

[17] I. G. Macdonald, Symmetric functions and Hall polynomials, 2nd edition, Oxford Univ. Press, Oxford 1995.

[18] L. Manivel, Symmetric Functions, Schubert Polynomials and Degeneracy Loci, SMF/AMS TEXT and MONOGRAPHS, vol. 6, 2001.

[19] H. Naruse, private communication.

[20] P. Pragacz, Algebraico-geometric applications of Schur S- and Q-polynomials, in Séminaire d’Algèbre Dubreil-Malliavin 1989-1990, Springer Lecture Notes in Math. 1478 (1991), 130–191.

[21] I. Schur, Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen. J. reine angew. Math. 139 (1911), 155–250.

[22] H. Tamvakis, Giambelli and degeneracy locus formulas for classical G/P spaces, available on arXiv: 1305.3543

[23] H. Tamvakis, A Giambelli formula for classical G/P spaces, to appear in J. of Algebraic Geometry.

[24] E. Wilson, Equivariant Giambelli Formulae for Grassmannians, Ph.D. Thesis, University of Maryland (2010).

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