Form factors of the $SU(2)$ invariant massive Thirring model with boundary reflection

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Abstract

The $SU(2)$ invariant massive Thirring model with a boundary is considered on the basis of the vertex operator approach. The bosonic formulae are presented for the vacuum vector and its dual in the presence of the boundary. The integral representations are also given for form factors of the present model.

1 Introduction

Integrable two dimensional field theory possesses an infinite set of mutually commuting integrals of motion. For bulk (i.e., without boundary) massive theories, the integrability is ensured by the factorized scattering condition or the Yang–Baxter equation, in addition to the unitarity and crossing symmetry condition [1]. Cherednik [2] showed that the integrability in the presence of reflecting boundary is ensured by the boundary Yang–Baxter equation (the reflection equation) and the Yang–Baxter equation for bulk theory. A systematic treatment of determining the spectrum of integrable models with boundary reflection was initiated by Sklyanin [3] in the framework of the algebraic Bethe ansatz.

The earliest studies on off-shell quantities such as correlation functions of integrable models with non-trivial boundary condition involved the Ising model [4, 5]. Correlation functions for an impenetrable Bose gas with Neumann/Dirichlet boundary conditions were obtained in [6]. However, the success in the Ising model and the impenetrable Bose gas model is rather special because they are equivalent to the free fermion theory.

In 1994 the method to extract off-shell quantities was found for other integrable models with a boundary. The boundary crossing symmetry condition was proposed in [7] on the basis of the boundary bootstrap approach, in order to obtain the boundary vacuum vectors. Jimbo et al. [8] developed this idea to obtain correlation functions of the XXZ spin chain with a boundary magnetic field, using the vertex operator approach [9, 10, 11]. The $U_q(\hat{sl}_n)$-generalization of [8] was given in [12]. The generalization
to the face model was given in [13]. Throughout the studies on the integral formulae of form factors for bulk sine-Gordon models Smirnov found three axioms that form factors should satisfy [14]. The boundary analogue of Smirnov’s axioms were formulated in [15].

Let us briefly remind you of the boundary crossing symmetry condition. This condition is the equation involving the boundary $S$-matrix ($K$-matrix) and the generators of the Zamolodchikov–Faddev (ZF) algebra [1, 16], which determines the boundary state, the vacuum of integrable models with boundary reflection. In [3] the ZF generators in the condition are the asymptotic operators, corresponding to the type II vertex operators in the terminology of [1]. When the transfer matrix of boundary integrable model is expressed in terms of the type I vertex operators and $K$-matrix, the asymptotic operators in the condition should be replaced by the type I vertex operators in order to get integral formulae for correlations. Concerning the XXZ chain, its $U_q(\hat{sl}_n)$-generalization and the face-model, see [8, 12, 13].

In this paper we study the $SU(2)$ invariant massive Thirring model with a boundary. In [17] Lukaynov constructed two kinds of generators of the ZF algebra: the asymptotic operators and the local operators, the analogue of the type II and I vertex operators, respectively. He also obtained the integral formulae of form factors for the $SU(2)$ invariant massive Thirring model and the sine-Gordon model in the bulk case [17], which satisfy Smirnov’s axioms [14]. In [18] the boundary crossing symmetry condition involving the local operators (the analogue of the type I vertex operators) was solved for the $SU(2)$ invariant massive Thirring model to construct the boundary states in terms of free bosons. The physical meaning of their boundary state is thus unclear. In [19] form factors satisfying the boundary analogue of Smirnov’s axioms [15] were obtained for the sine-Gordon model with a boundary.

The rest of this paper is organized as follows. In section 2 we review the ZF algebra for the present case, and give the asymptotic and the local operators in terms of free boson. In section 3 solving the boundary crossing symmetry condition involving the asymptotic operators (the analogue of the type II vertex operators), we construct boundary state and its dual of the $SU(2)$ invariant massive Thirring model with a boundary. In section 4 we present explicit integral formulae for form factors of the model.

2 Formulation of the problem

The purpose of this section is to set up the problem, thereby fixing the notation.

2.1 The model

The $SU(2)$ invariant massive Thirring model with boundary reflection is defined by describing its boundary $S$-matrix ($K$-matrix) in addition to the bulk $S$-matrix. The bulk $S$-matrix $S(\beta)$ of the model is given by

$$S(\beta) = \frac{\Gamma\left(\frac{1}{2} + \frac{\beta}{2\pi i}\right) \Gamma\left(-\frac{\beta}{2\pi i}\right)}{\Gamma\left(\frac{1}{2} - \frac{\beta}{2\pi i}\right) \Gamma\left(\frac{\beta}{2\pi i}\right)} \begin{pmatrix} 1 & \frac{\beta}{\beta - \pi i} & -\frac{\pi i}{\beta - \pi i} \\ \frac{\beta}{\beta - \pi i} & 1 & \frac{\pi i}{\beta - \pi i} \\ -\frac{\pi i}{\beta - \pi i} & \frac{\pi i}{\beta - \pi i} & 1 \end{pmatrix}.$$ (2.1)
The $S$-matrix satisfies the Yang–Baxter equation:

$$S_{12}(\beta_1 - \beta_2)S_{13}(\beta_1 - \beta_3)S_{23}(\beta_2 - \beta_3) = S_{23}(\beta_2 - \beta_3)S_{13}(\beta_1 - \beta_3)S_{12}(\beta_1 - \beta_2); \quad (2.2)$$

the unitarity symmetry:

$$S_{12}(\beta)S_{21}(-\beta) = 1; \quad (2.3)$$

and the crossing symmetry:

$$S_{12}(\pi i - \beta) = C_1 S_{12}(\beta) C_1, \quad C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (2.4)$$

The Zamolodchikov–Faddeev (ZF) operators $Z^a_\pm(\beta)$, $Z^a(\beta)$ ($a = \pm$) generate the following algebra:

$$Z^a_\pm(\beta_1)Z^b_\pm(\beta_2) = \sum_{cd} S^c_{ab}(\beta_1 - \beta_2)Z^c_\pm(\beta_2)Z^c_\pm(\beta_1), \quad Z^a(\beta_1)Z^b(\beta_2) = \sum_{cd} Z^d(\beta_2)Z^c(\beta_1)S^a_{cd}(\beta_1 - \beta_2), \quad Z^a(\beta_1)Z^b_\pm(\beta_2) = \sum_{cd} Z^d_\pm(\beta_2)S^c_{ab}(\beta_1 - \beta_2)Z^c(\beta_1) + g\delta^a_\pm\delta(\beta_1 - \beta_2), \quad (2.5)$$

where $g$ is a constant. The boundary $S$-matrix ($K$-matrix) $K(\beta)$ is given by

$$K(\beta) = \frac{h(-\beta)}{h(\beta)} \begin{pmatrix} 1 & \mu - \beta \\ \frac{\mu - \beta}{\mu + \beta} & \frac{1 - \beta}{1 + \beta} \end{pmatrix}, \quad h(\beta) = \frac{\Gamma\left(\frac{-\mu}{2\pi i} - \frac{\beta}{2\pi i}\right)\Gamma\left(\frac{1}{2} - \frac{\beta}{2\pi i}\right)}{\Gamma\left(\frac{\mu}{2\pi i} - \frac{\beta}{2\pi i}\right)\Gamma\left(-\frac{\beta}{2\pi i}\right)}. \quad (2.6)$$

The $K$-matrix satisfies the boundary Yang–Baxter equation (reflection equation) [2]:

$$K_2(\beta_2)S_{21}(\beta_1 + \beta_2)K_1(\beta_1)S_{12}(\beta_1 - \beta_2) = S_{21}(\beta_1 - \beta_2)K_1(\beta_1)S_{12}(\beta_1 + \beta_2)K_2(\beta_2), \quad (2.7)$$

the boundary unitarity symmetry:

$$K(\beta)K(-\beta) = 1, \quad (2.8)$$

and the boundary crossing symmetry [3]:

$$C_{a - a}K_{- a}^{\pm a}\left(\frac{\pi i}{2} - \beta\right) = \sum_{c = \pm} S^a_{c - c}(2\beta)C_{- c c}K^c_\pm\left(\frac{\pi i}{2} + \beta\right). \quad (2.9)$$

From the same argument in [3, 4] the boundary state $|B\rangle$ and its dual state $\langle B|$ should satisfy

$$Z^a_\pm(\beta)|B\rangle = K^a_\pm(\beta)Z^a_\pm(-\beta)|B\rangle, \quad \langle B|Z^a(-\beta) = \langle B|Z^a(\beta)K^a_\pm(\beta), \quad (2.10)$$

where $Z^a_\pm(\beta)$ and $Z^a(\beta)$ is connected by

$$Z^a(\beta) = \sum_{b = \pm} C^{ab}Z^b_\pm(\pi i + \beta). \quad (2.11)$$
2.2 Ultraviolet regularization

Following Lukyanov [17] we consider the ultraviolet regularization of the original model such that the bosonic field has the oscillator decomposition.

Let us fix the parameter $\epsilon > 0$. In what follows we use the abbreviations, $x = e^{-\frac{\pi}{\epsilon}}$, $\zeta = e^{i\epsilon \beta}$, $w = e^{i\epsilon \gamma}$ and $r = e^{i\epsilon \mu}$. We also use the symbol

$$ (z; p_1, \cdots, p_m)_\infty := \prod_{i_1, \cdots, i_m \geq 0} (1 - z p_1^{i_1} \cdots p_m^{i_m}), \quad |p_i| < 1 \ (1 \leq i \leq m). $$

The $q$-analog of $\Gamma$ function is defined by

$$ \Gamma_q(z) = (1 - q)^{1-z} \frac{(q; q)_\infty}{(q^z; q)_\infty}. \quad (2.12) $$

You can easily see that $\Gamma_q(z) \rightarrow \Gamma(z)$ as $q \rightarrow 1$.

The regularized bulk $S$-matrix $S_\epsilon(\beta)$ is given by

$$ S_\epsilon(\beta) = \frac{1}{\sqrt{\zeta}} \frac{g_\epsilon(-\beta)}{g_\epsilon(\beta)} \left( \begin{array}{ccc} 1 & -\frac{\sinh \frac{\pi \epsilon}{2}}{\sinh \frac{\pi \beta}{2}} & -\frac{\sinh \frac{\pi \gamma}{2}}{\sinh \frac{\pi \beta}{2}} \\ \sinh \frac{\epsilon (\pi - \beta)}{2} & \sinh \frac{\epsilon (\pi - \beta)}{2} & \sinh \frac{\epsilon (\pi - \beta)}{2} \\ \sinh \frac{\epsilon (\pi - \beta)}{2} & \sinh \frac{\epsilon (\pi - \beta)}{2} & 1 \end{array} \right), \quad g_\epsilon(\beta) = \frac{(\zeta^{-1}; x^4)_\infty}{(x^2 \zeta^{-1}; x^4)_\infty}. \quad (2.13) $$

The regularized boundary $S$-matrix ($K$-matrix) is given by

$$ K_\epsilon(\beta) = \frac{h_\epsilon(-\beta)}{h_\epsilon(\beta)} \left( \begin{array}{cc} 1 & \zeta - r \\ 1 - r \zeta & 1 \end{array} \right), \quad h_\epsilon(\beta) = \frac{(x^2 r^{-1} \zeta^{-1}; x^4)_\infty (\zeta^{-2}; x^8)_\infty}{(x^{-1} \zeta^{-1}; x^4)_\infty (x^2 \zeta^{-2}; x^8)_\infty}. \quad (2.14) $$

In the limit $\epsilon \rightarrow 0$, these regularized matrices behave like

$$ S_{\epsilon \rightarrow 0}^{cd}(\beta) \rightarrow (-1)^{1+c+d} S_{ab}^{cd}(\beta), \quad K_\epsilon(\beta) \rightarrow K(\beta). \quad (2.15) $$

2.3 Bosonizations of the ZF algebra

Let us consider the following free boson [17]

$$ \phi_\epsilon(\beta) = \frac{1}{\sqrt{2}} (Q - \epsilon \beta P) + \sum_{m \neq 0} \frac{2 \alpha_m \zeta^m}{i(x^{-2m} - x^{2m})}, \quad (2.16) $$

where the oscillator modes $a_m$ and zero modes $P$, $Q$ satisfy the commutation relations

$$ [a_m, a_n] = \frac{(x^m - x^{-m})(x^{2m} - x^{-2m})x^{-|m|}}{4m} \delta_{m+n,0}, \quad [P, Q] = \frac{1}{l}. \quad (2.17) $$

Let $\phi_\epsilon^+(\beta)$ and $\phi_\epsilon^-(\beta)$ denote the positive and negative frequency part of $\phi_\epsilon(\beta)$, respectively. Then we have

$$ [\phi_\epsilon^+(\beta_1), \phi_\epsilon^-(\beta_2)] = -\log g_\epsilon(\beta_2 - \beta_1). \quad (2.18) $$
We now introduce the elementary vertex operators

\[ V(\beta) = \zeta^{\beta} : e^{i\phi_{\lambda}(\beta)} :, \quad \bar{V}(\gamma) = w^{-1} : e^{-i\bar{\phi}_{\bar{\lambda}}(\gamma)} :, \quad (2.19) \]

where

\[ \bar{\phi}_{\lambda}(\gamma) = \phi_{\lambda}(\gamma + \frac{\pi}{2}i) + \phi_{\lambda}(\gamma - \frac{\pi}{2}i) = \sqrt{2}(Q - \epsilon \gamma P) + \sum_{m \neq 0} \frac{2a_m w^m}{i(x^{-m} - x^m)}. \quad (2.20) \]

The positive and negative frequency part of \( \bar{\phi}_{\lambda}(\gamma) \), \( \bar{\phi}_{\lambda}^+(\gamma) \) and \( \bar{\phi}_{\lambda}^-(\gamma) \) are also defined. Lukyanov [17] prove that the bosonizations of the asymptotic operators of the ZF algebra (the analogue of the type II vertex operators) are given as follows:

\[ Z^e_{\pm}(\beta) = \zeta^{\beta} V(\beta), \]
\[ Z^e_-(\beta) = c_e \zeta^{\beta} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\gamma}{2\pi} \left( \bar{V}(\gamma)V(\beta) + xV(\beta)\bar{V}(\gamma) \right) \]
\[ = c_e(x - x^{-1}) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\gamma}{2\pi} \left( 1 - \frac{w/(x\zeta)}{1 - \zeta/(xw)} \right) : \bar{V}(\gamma)V(\beta) :, \quad (2.21) \]

where \( c_e \) is an irrelevant constant.

In order to construct the local operators of the ZF algebra (the analogue of the type I vertex operators), let us also consider the fields

\[ \phi'_e(\alpha) = -\frac{1}{\sqrt{2}}(Q - \epsilon \alpha P) - \sum_{m \neq 0} \frac{2x^{\alpha} |a_m|}{i(x^{-2m} - x^{2m})} \xi^m, \]
\[ \bar{\phi}'(\delta) = \phi'_e(\delta + \frac{\pi}{2}i) + \phi'_e(\delta - \frac{\pi}{2}i) = -\sqrt{2}(Q - \epsilon \delta P) - \sum_{m > 0} \frac{2x^m (a_m v^m - a_{-m} v^{-m})}{i(x^{-m} - x^m)}, \quad (2.22) \]

where \( \xi = e^{i\alpha} \) and \( v = e^{i\delta} \). We now define the local operators \( Z'_e(\alpha) (a = \pm) \)

\[ Z^e_{\pm}(\alpha) = \xi V'(\alpha), \]
\[ Z^e_-(\alpha) = i c'_e \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\delta}{2\pi} V'(\delta)V'(\alpha) + V'(\alpha)V'(\delta), \quad (2.23) \]

where \( c'_e \) is another irrelevant constant, and

\[ V'(\alpha) = :e^{i\phi'_e(\alpha)} :, \quad \bar{V}'(\delta) = v^{-1} : e^{-i\bar{\phi}'(\delta)} :, \quad (2.24) \]

Those local operators satisfy the following commutation relations [17]:

\[ Z_{\alpha}(\alpha_1)Z_{\alpha}(\alpha_2) = -\sum_{cd} c_{\alpha \alpha_2 \alpha_1} Z_{\alpha}(\alpha_2)Z_{\alpha}(\alpha_1) \]
\[ Z_{\alpha}(\alpha)Z_{\alpha}(\beta) = ab \tan \left( \frac{\pi}{4} + i \frac{\alpha - \beta}{2} \right) Z_{\alpha}(\beta)Z_{\alpha}(\alpha). \quad (2.25) \]

The generators \( (2.21) \) and \( (2.23) \) act on the regularized Fock space \( F_e \), whose vacuum vector \( |0\rangle \) is characterized as follows:

\[ P|0\rangle = 0, \quad a_m|0\rangle = 0, \quad (m > 0). \]
2.4 Observable local fields of the model

In this paper we are interested in the soliton form factors of the observable local field \( O \) given by

\[
G^{O}_{\alpha_{1}, \ldots, \alpha_{n}}(\beta_{1}, \ldots, \beta_{n}) = \frac{\langle B | O Z_{\alpha_{1}}^{\ast} (\beta_{1}) \cdots Z_{\alpha_{n}}^{\ast} (\beta_{n}) | B \rangle}{\langle B | B \rangle}.
\]

(2.26)

The space of observable local fields are identified with the set of fields commuting with the type II vertex operators [10], or the asymptotic operators of the ZF algebra [17] up to a phase factor. In [17] the set of observable local fields are introduced in terms of the local operators of the ZF algebra (the analogue of the type I vertex operators) as follows:

\[
\Lambda_{m}(\alpha) = \frac{i}{\eta'} \left[ \begin{array}{c} \frac{1}{2} \\ \frac{a}{2} \\ \frac{b}{2} \\ 1 \\ m \end{array} \right] Z_{a_{m}}^{\prime}(\alpha + i \frac{\pi}{2}) Z_{b_{m}}^{\prime}(\alpha - i \frac{\pi}{2}),
\]

(2.27)

where the Clebsch-Gordan coefficients are given by

\[
\begin{align*}
\left[ \begin{array}{c}
\frac{1}{2} \\
\frac{a}{2} \\
\frac{b}{2} \\
1 \\
m
\end{array} \right]_{-1} &= \delta_{a+b, \pm 1}, \\
\left[ \begin{array}{c}
\frac{1}{2} \\
\frac{a}{2} \\
\frac{b}{2} \\
1 \\
0
\end{array} \right]_{-1} &= (-1)^{\frac{i}{2}-a+b} \delta_{a+b, 0} \sqrt{2}.
\end{align*}
\]

(2.28)

The fields \( \Lambda_{m}(\alpha) \)'s are expressed in terms of bosons [17]

\[
\begin{align*}
\Lambda_{1}(\alpha) &= \xi \tilde{V}^{\prime}(\alpha), \\
\Lambda_{0}(\alpha) &= \frac{i}{\sqrt{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\delta}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dv_{1} v_{2}}{2\pi} \xi \left( \tilde{V}^{\prime}(\delta_{1}) \tilde{V}^{\prime}(\delta_{2}) \right), \\
\Lambda_{-1}(\alpha) &= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\delta_{1}}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\delta_{2}}{2\pi} v_{1} v_{2} \xi \left( \tilde{V}^{\prime}(\delta_{1}) \tilde{V}^{\prime}(\delta_{2}) \right) - 2 \tilde{V}^{\prime}(\delta_{1}) \tilde{V}^{\prime}(\alpha) \tilde{V}^{\prime}(\delta_{2}) + \tilde{V}^{\prime}(\alpha) \tilde{V}^{\prime}(\delta_{1}) \tilde{V}^{\prime}(\delta_{2})
\end{align*}
\]

(2.30)

where

\[
\tilde{V}^{\prime}(\alpha) = \xi^{-1} : e^{i \tilde{\phi}^{\prime}(\alpha)} :.
\]

(2.31)

The fields \( \Lambda_{m}(\alpha) \)'s (anti-)commute with the asymptotic operators:

\[
\Lambda_{m}(\alpha) Z^{\ast}_{\epsilon a}(\beta) = (-1)^{m} Z^{\ast}_{\epsilon a}(\beta) \Lambda_{m}(\alpha).
\]

(2.32)

3 Boundary states

In this section we wish to determine the boundary state satisfying the relation [7, 8]

\[
Z^{\ast}_{\epsilon a}(\beta) B_{\epsilon} = K^{\ast}_{\epsilon a}(\beta) Z^{\ast}_{\epsilon a}(-\beta) B_{\epsilon}, \quad (a = \pm).
\]

(3.1)

In order to solve (3.1) we make the ansatz that the boundary state has the following form

\[
| B_{\epsilon} \rangle = e^{B_{\epsilon}} | 0 \rangle, \quad B_{\epsilon} = \frac{1}{2} \sum_{m=1}^{\infty} \alpha_{m} a_{-m}^{2} + \sum_{m=1}^{\infty} \beta_{m} a_{-m}^{2}.
\]

(3.2)
Note that \([a_m, a_{-m}]\) appearing in the denominator is a c-number because of (2.17). From the boundary Yang–Baxter equation, the coefficients \(\alpha_m, \beta_m\) do not depend on the spectral parameter \(\beta\). The presence of \(e^{B_x}\) has the effect of a Bogoliubov transformation

\[
e^{-B_x} a_n e^{B_x} = a_n + \alpha_n a_{-n} + \beta_n, \quad e^{-B_x} a_{-n} e^{B_x} = a_{-n},
\]

for \(n > 0\). Using the bosonization formulae, eq. (3.1) for \(a = +\) reduces to

\[
e^{i\phi_+ (\gamma)} |B_k\rangle = h_\epsilon (-\beta) e^{i\phi_- (\gamma)} |B_k\rangle.
\]

By straightforward calculation, the coefficients \(\alpha_m, \beta_m\) are to be found

\[
\alpha_m = -1, \quad \beta_m = \frac{1}{2m} (1 - x^{2m}) x^{-2m} e^{-m} - \frac{1}{2m} \theta_m (1 - x^m) (1 - x^{2m}) x^{-2m},
\]

where

\[
\theta_m = \begin{cases} 1, & m : \text{even}, \\ 0, & m : \text{odd}. \end{cases}
\]

Let us prove (3.1) for \(a = -\). From (2.20) and (1.2) we have

\[
e^{-i\bar{\phi}_+ (\gamma)} |B_k\rangle = I_\epsilon (\gamma) e^{-i\bar{\phi}_- (\gamma)} |B_k\rangle,
\]

where

\[
I_\epsilon (\gamma) = (1 - w^2) (1 - w/(xr)).
\]

Thus eq. (3.1) for \(a = -\) reduces to

\[
\int_{-\pi}^{\pi} d\gamma \frac{I_\epsilon (\gamma)(1 - r \zeta)}{w (1 - w/(x \zeta))(1 - 1/(xw \zeta))(1 - w/(x \zeta))} e^{-i\bar{\phi}_- (\gamma) - i\phi_- (\gamma)} |B_k\rangle = (\zeta \leftrightarrow \zeta^{-1}).
\]

Here we change the integral variable \(\gamma \rightarrow -\gamma\) in both sides and add it to the original one. Then (3.8) holds if the following the relation concerning the integrand

\[
\frac{I_\epsilon (\gamma)}{I_\epsilon (-\gamma)} = -w^2 \frac{(1 - w/(xr))}{(1 - 1/(xr w))}.
\]

Since (3.10) is compatible with (3.8), we have (3.9), which implies (3.1) for \(a = -\).

The dual boundary state \(\langle B_k|\) is determined by

\[
\langle B_k| Z_\epsilon^0 (-\beta) = \langle B_k| Z_\epsilon^0 (\beta) K_\epsilon^0 (\beta), \quad (a = \pm).
\]

In order to solve (3.11) we make the ansatz that the dual boundary state has the following form

\[
\langle B_k| = \langle 0| e^{G_x}, \quad G_x = \frac{1}{2} \sum_{m=1}^{\infty} \frac{\gamma_m}{[a_m, a_{-m}]} a_m^2 + \sum_{m=1}^{\infty} \frac{\delta_m}{[a_m, a_{-m}]} a_m.
\]

From the same arguments as before, the coefficients \(\gamma_m, \delta_m\) are to be found

\[
\gamma_m = -x^{4m}, \quad \delta_m = \frac{1}{2m} (1 - x^{2m}) x^m r^{-m} + \frac{1}{2m} \theta_m (1 - x^m) (1 - x^{2m}).
\]
The action of the elementary operators to the dual boundary state are given by

\begin{equation}
\langle B_\epsilon | e^{i\phi_-(\pi) - \beta} \rangle = (1 - r^{-1}) \langle B_\epsilon | e^{i\phi_+(\pi+\beta)} \rangle, \quad (3.14)
\end{equation}

\begin{equation}
\langle B_\epsilon | e^{-i\phi_-(\pi+\gamma)} \rangle = I_\epsilon^\ast (\gamma) \langle B_\epsilon | e^{-i\phi_+(\pi-\gamma)} \rangle, \quad (3.15)
\end{equation}

where

\begin{equation}
I_\epsilon^\ast (\gamma) = \frac{1 - 1/(u^2)}{1 - x/(rw)}.
\end{equation}

Note that our boundary states are different from the ones constructed by Chao et al. \[18\], because they solved the condition

\begin{equation}
Z_{+a}^\ast (\beta) | B_\epsilon \rangle = K_{+a}^\ast (\beta) Z_{-a}^\ast (\beta) | B_\epsilon \rangle, \quad (3.16)
\end{equation}

instead of \[2.10\]. In order to compute form factors one should identify \( |B_\epsilon \rangle \) and \( \langle B_\epsilon | \) satisfying \[2.10\] with the vacuum and its dual of the present model.

## 4 Form Factors

Let us consider the form factor of the form

\begin{equation}
G_{e,a_1,\ldots,a_n}^{m_1,\ldots,m_k} (\alpha_1, \ldots, \alpha_k | \beta_1, \ldots, \beta_n) = \frac{\langle B_\epsilon | \Lambda_{m_1} (\alpha_1) \cdots \Lambda_{m_k} (\alpha_k) Z_{+a_1}^\ast (\beta_1) \cdots Z_{+a_n}^\ast (\beta_n) | B_\epsilon \rangle}{\langle B_\epsilon | B_\epsilon \rangle}, \quad (4.1)
\end{equation}

where \( \Lambda_m (\alpha) \) \((m = 0, \pm 1)\) are the observable local fields of the present model introduced in section 2.4.

In what follows we use the following abbreviations:

\begin{equation}
\xi_j = e^{i\epsilon \alpha_j}, \quad \nu_j = e^{i\epsilon \beta_j}, \quad \zeta_j = e^{i\epsilon \gamma_j}, \quad \omega_j = e^{i\epsilon \gamma_j}.
\end{equation}

Let \( A_\pm \) and \( M_m \) \((m = 0, \pm 1)\) signify the sets such that

\begin{equation}
A_\pm := \{ a_j | a_j = \pm \}, \quad M_m := \{ m_j | m_j = m \},
\end{equation}

and \( r = \#(A_-), \quad p = 2\#(M_-) + \#(M_0) \). Then we find the integral formula for the form factor \[4.1\] as follows:

\begin{equation}
G_{e,a_1,\ldots,a_n}^{m_1,\ldots,m_k} (\alpha_1, \ldots, \alpha_k | \beta_1, \ldots, \beta_n)
= \prod_{a \in A_\pm} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\epsilon_a}{2\pi} \zeta_a^{-\frac{1}{2}} \left( P_\epsilon (\zeta_a, v_a) + x \right) \prod_{j = a + 1}^{n} P_\epsilon (\zeta_j, v_a)
\times \prod_{m \in M_{-1}} \left[ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\epsilon_m}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\delta_m}{2\pi} \xi_m v_m^2 \left( P_\epsilon (\xi_m, v_m^1) P_\epsilon (\xi_m, v_m^2) - 2P_\epsilon (\xi_m, v_m^1) + 1 \right) \right]
\times \prod_{j = m + 1}^{k} P_\epsilon (\xi_j, v_m^1)
\times R_\epsilon (\alpha_1, \ldots, \alpha_k | \delta_m, \delta_m^2) \prod_{m \in M_{0} \setminus M_{-1}} \left[ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\delta_m}{2\pi} \xi_m v_m^1 \left( P_\epsilon (\xi_m, v_m^1) - 1 \right) \prod_{j = m + 1}^{k} P_\epsilon (\xi_j, v_m^1) \right)
\end{equation}

where

\begin{equation}
P_\epsilon (\xi, w) := \frac{\zeta - x w}{w - x \zeta}, \quad P_\epsilon^\ast (\xi, v) := \frac{x \xi - x^{-1} v}{x^{-1} \xi - x^2 v}, \quad (4.4)
\end{equation}
and the auxiliary function
\[
R_e(\alpha_1, \ldots, \alpha_k|\delta_1, \ldots, \delta_p|\beta_1, \ldots, \beta_n|\gamma_1, \ldots, \gamma_r) = \frac{\langle B_e| V'(\alpha_k) \cdots V'(\alpha_1)V'(\delta_p) \cdots V'(\delta_1)V(\beta_n) \cdots V(\beta_1)V(\gamma_r) \cdots V(\gamma_1)|B_e \rangle}{\langle B_e| B_e \rangle}. \tag{4.5}
\]

Taking the normal ordering, we have
\[
R_e(\alpha_1, \ldots, \alpha_k|\delta_1, \ldots, \delta_p|\beta_1, \ldots, \beta_n|\gamma_1, \ldots, \gamma_r) = \\
\delta_{2k-2p+n-2r,0} \prod_{1 \leq j < l \leq n} g_e(\beta_j - \beta_l) \prod_{j=1}^k \xi_j^{2p+n-2r-1} \prod_{j=1}^p w_j^{-n+2r-1} \prod_{j=1}^r w_j^{-1} \prod_{j=1}^n \xi_j^{r-\frac{1}{2}} \\
\times \prod_{1 \leq j < l \leq k} (1 - x^2 \xi_j / \xi_j) \prod_{1 \leq j < l \leq p} (1 - x^2 v_j / v_j) \prod_{1 \leq j < l \leq r} (1 - w_l / w_j)(1 - w_l / (x^2 w_j)) \\
\times \prod_{j=1}^p \prod_{l=1}^k (1 - \xi_j / \xi_j) \prod_{j=1}^r \prod_{l=1}^p (1 - x^2 v_j / w_j)(1 - v_l / (x^2 w_j)) \prod_{j=1}^r \prod_{l=1}^n (1 - \xi_j / (x w_j)) \\
\times \ I_e(\alpha_1, \ldots, \alpha_k|\delta_1, \ldots, \delta_p|\beta_1, \ldots, \beta_n|\gamma_1, \ldots, \gamma_r). \tag{4.6}
\]

Here \(I_e\) is defined as
\[
I_e(\alpha_1, \ldots, \alpha_k|\delta_1, \ldots, \delta_p|\beta_1, \ldots, \beta_n|\gamma_1, \ldots, \gamma_r) = \\
\frac{\langle 0| e^{G^* \exp \left( \sum_{m=1}^{\infty} a_m X_m \right) \exp \left( - \sum_{m=1}^{\infty} a_m Y_m \right)} e^{B^*} | 0 \rangle}{\langle 0| e^{G^*} e^{B^*} | 0 \rangle}, \tag{4.7}
\]

where
\[
X_m = \frac{2}{1 - x^{2m}} \left( x^{2m} \sum_{j=1}^k \xi_j^{-m} - x^{2m} \sum_{j=1}^p v_j^{-m} + x^m \sum_{j=1}^r w_j^{-m} \right) + \frac{2}{1 - x^{4m}} \left( -x^{2m} \sum_{j=1}^n \xi_j^{-m} \right),
\]
\[
Y_m = \frac{2}{1 - x^{2m}} \left( x^{2m} \sum_{j=1}^k \xi_j^{-m} - x^{2m} \sum_{j=1}^p v_j^{-m} + x^m \sum_{j=1}^r w_j^{-m} \right) + \frac{2}{1 - x^{4m}} \left( -x^{2m} \sum_{j=1}^n \xi_j^{-m} \right). \tag{4.8}
\]

The vacuum expectation value \([4.7]\) becomes
\[
I_e(\alpha_1, \ldots, \alpha_k|\delta_1, \ldots, \delta_p|\beta_1, \ldots, \beta_n|\gamma_1, \ldots, \gamma_r) = \\
\exp \left( \sum_{m=1}^{\infty} [a_m, a_m] \right) \frac{1}{1 - a_m \gamma_m} \left\{ \frac{1}{2} \gamma_m X_m^2 + \frac{1}{2} a_m Y_m^2 - a_m \gamma_m X_m Y_m \right\} \\
+ \frac{1}{1 - a_m \gamma_m} \left\{ (\delta_m + \gamma_m \beta_m) X_m - (\beta_m + a_m \delta_m) Y_m \right\}. \tag{4.9}
\]

In order to derive the formulae of the vacuum expectation value \([4.9]\) we prepare the coherent states,
\[
|\eta\rangle = \exp \left( \sum_{m=1}^{\infty} \frac{\eta_m a_m}{[a_m, a_m]} \right) |0\rangle, \quad \langle \bar{\eta}| = \langle 0| \exp \left( \sum_{m=1}^{\infty} \frac{\bar{\eta}_m a_m}{[a_m, a_m]} \right), \tag{4.10}
\]

which enjoy the following properties:
\[
a_n |\eta\rangle = \eta_n |\eta\rangle, \quad \langle \bar{\eta}| a_n = \bar{\eta}_n \langle \bar{\eta}. \tag{4.11}
\]
The completeness relation

\[ id = \int \prod_{m>0} \frac{d\eta_m d\bar{\eta}_m}{[a_m, a_{-m}]} \exp \left( - \sum_{m=1}^{\infty} \frac{\eta_m \bar{\eta}_m}{[a_m, a_{-m}]} \right) |\eta\rangle \langle \bar{\eta}|, \]  

(4.12)
can be easily proved. Here the integral \( \int d\eta d\bar{\eta} \) implies \( \frac{1}{2i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy \).

By using the Bogoliubov transformation, we obtain

\[ \langle 0 | e^{G_1} e^{B_2} \exp \left( \sum_{m=1}^{\infty} a_{-m} X_m \right) \exp \left( - \sum_{m=1}^{\infty} a_m Y_m \right) e^{B_1} |0 \rangle \]

\[ = \langle 0 | e^{G_1} e^{B_2} \exp \left( \sum_{m=1}^{\infty} a_{-m} (X_m - \alpha_m Y_m) \right) |0 \rangle \]

\times \exp \left( - \sum_{m=1}^{\infty} \beta_m Y_m \right) \exp \left( \frac{1}{2} \sum_{m=1}^{\infty} [a_m, a_{-m}] \alpha_m Y_m^2 \right). \]

(4.13)

Inserting the completeness relation between \( e^{G_1} \) and \( e^{B_2} \) and calculating the Gaussian-type integral, we have (4.9), and

\[ \langle B_2 | B_1 \rangle = \prod_{m=1}^{\infty} \frac{1}{\sqrt{1 - \alpha_m \gamma_m}} \exp \left( \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{1 - \alpha_m \gamma_m} \frac{1}{[a_m, a_{-m}]} (\alpha_m \delta_m^2 + \gamma_m \beta_m^2 + 2 \beta_m \delta_m) \right) \]  

(4.14)

The vacuum expectation value (4.9) is evaluated by

\[ \exp \left( - \sum_{m=1}^{\infty} \frac{1}{m} \frac{z^m}{(1 - p_1^m) \cdots (1 - p_n^m)} \right) = (z; p_1, \cdots, p_n)_{\infty}, \quad |p_i| < 1 \quad (1 \leq i \leq n). \]  

(4.15)

The norm of the vacuums is given as

\[ \langle B_2 | B_1 \rangle = \frac{(x^2 r^2; x^4)_{\infty} (x^{10} r^2; x^8)_{\infty}}{(x^6; x^8)_{\infty} (x^6 r^2; x^8)_{\infty}}. \]  

(4.16)
The integral \( I_\epsilon(\alpha_1, \ldots, \alpha_k|\beta_1, \ldots, \beta_n|\gamma_1, \ldots, \gamma_r) \) is given by

\[
I_\epsilon(\alpha_1, \ldots, \alpha_k|\beta_1, \ldots, \beta_n|\gamma_1, \ldots, \gamma_r) = (x^4+x^2)^{k+p}(x^2;x^r)\left(\frac{(x^4,x^4)_{\infty}}{(x^4;x^4)_{\infty}}\right)^n \times \prod_{j=1}^{k}(x^2\xi_j;x^4)_{\infty}(x^6\xi_j^{-2};x^4_{\infty}) \prod_{j=1}^{p}(v_j;x^4_{\infty})(x^4v_j^{-2};x^4_{\infty}) \prod_{j=1}^{r}(w_j;x^4_{\infty})(x^4w_j^{-2};x^4_{\infty}) \times \prod_{j=1}^{n}\left(\frac{(\zeta_j^2;x^4)_{\infty}(x^4\zeta_j^{-2};x^4_{\infty})(\zeta_j^2;x^8)_{\infty}(x^4\zeta_j^{-2};x^8)_{\infty}}{(x^2\zeta_j;x^4)_{\infty}(x^6\zeta_j^{-2};x^4_{\infty})(x^2\zeta_j^2;x^4_{\infty})(x^6\zeta_j^{-2};x^4_{\infty})}\right) \times \prod_{j=1}^{n}(1-r^{-1}\xi_j) \prod_{j=1}^{p}\frac{1}{1-r^{-1}v_j} \prod_{j=1}^{r}(1-r^{-1}w_j) \prod_{j=1}^{n}(x^{2r-1}\zeta_j;x^4)_{\infty}^{\frac{1}{1-r^{-1}\zeta_j;x^4_{\infty}}}
\]

(4.17)

Here we have obtained the integral representation of the form factor in the ultraviolet regularization scheme. Next we consider the original problem by taking the limit \( \epsilon \to 0 \) of (4.3).

Let us set the function

\[
Q_\epsilon(\beta) := \frac{\gamma(\zeta^2)}{(x^4;x^4)_{\infty}},
\]

(4.18)

where

\[
\gamma(\zeta^2) = \frac{(\zeta^2;x^4)_{\infty}(x^4\zeta^{-2};x^4_{\infty})}{(x^2\zeta^2;x^4)_{\infty}(x^6\zeta^{-2};x^4_{\infty})}.
\]

Then as \( \epsilon \to 0 \) the function \( Q_\epsilon(\beta) \) behave like

\[
Q_\epsilon(\beta) \sim \exp\left(-\int_0^{\infty} dy \frac{e^{\pi y/2}}{y} \left(\frac{sh^2\left(\frac{\pi y}{2}\right) - sh^2\frac{\pi y}{4}}{sh\pi y ch\frac{\pi y}{2}}\right)\right) = Q(\beta).
\]

(4.19)
In the limit \( \epsilon \to 0 \), the integral (4.17) thus behaves like

\[
I_\epsilon \sim \prod_{j=1}^{k} \frac{1}{\Gamma\left(\frac{1}{2} + \frac{\alpha_j}{\pi i}\right)\Gamma\left(\frac{3}{2} - \frac{\alpha_j}{\pi i}\right)} \prod_{j=1}^{p} \frac{1}{\Gamma\left(\delta_i + \frac{\beta_j}{\pi i}\right)\Gamma\left(1 - \delta_i - \frac{\beta_j}{\pi i}\right)} \\
\times \prod_{j=1}^{r} \frac{1}{\Gamma\left(\frac{1}{2} \pm \frac{\beta_j + \delta_i}{2\pi i}\right)\Gamma\left(\frac{3}{2} \pm \frac{\beta_j - \delta_i}{2\pi i}\right)} Q(\beta_j) \\
\times \prod_{1 \leq i < j \leq k} \frac{1}{\Gamma\left(-\frac{1}{2} + \frac{\alpha_i + \alpha_j}{\pi i}\right)\Gamma\left(\frac{3}{2} - \frac{\alpha_i + \alpha_j}{\pi i}\right)} Q\left(\frac{\beta_i + \beta_j - \delta_i - \delta_j}{2}\right) \\
\times \prod_{1 \leq i < j < k} \frac{1}{\Gamma\left(\frac{1}{2} + \frac{\beta_i - \beta_j}{2\pi i}\right)\Gamma\left(\frac{3}{2} + \frac{\beta_i - \beta_j}{2\pi i}\right)} \\
\times \prod_{i=1}^{k} \prod_{j=1}^{r} \frac{1}{\Gamma\left(-\frac{1}{2} + \frac{\delta_i + \gamma_j}{\pi i}\right)\Gamma\left(\frac{3}{2} - \frac{\delta_i + \gamma_j}{\pi i}\right)} \\
\times \prod_{i=1}^{k} \prod_{j=1}^{n} \frac{1}{\Gamma\left(-\frac{1}{2} + \frac{\alpha_i + \beta_j}{2\pi i}\right)\Gamma\left(\frac{3}{2} - \frac{\alpha_i + \beta_j}{2\pi i}\right)} \\
\times \prod_{i=1}^{k} \prod_{j=1}^{n} \frac{1}{\Gamma\left(-\frac{1}{2} + \frac{\beta_i + \beta_j}{2\pi i}\right)\Gamma\left(\frac{3}{2} - \frac{\beta_i + \beta_j}{2\pi i}\right)} \\
=: I(\alpha_1, \ldots, \alpha_k, \delta_1, \ldots, \delta_p, \beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_r).
\]

Note that as \( \epsilon \to 0 \),

\[
P_\epsilon(\zeta, w) \longrightarrow P(\beta - \gamma), \quad P'_\epsilon(\xi, v) \longrightarrow P'(\alpha - \delta),
\]

where

\[
P(\beta) = \frac{i\beta + \frac{\pi}{4}}{i\beta - \frac{\pi}{4}}, \quad P'(\alpha) = \frac{i\alpha - \pi}{i\alpha + \pi}.
\]

After all, we obtain the following integral representation of the form factor for the \( \text{SU}(2) \) invariant
massive Thirring model with boundary reflection:

\[
\begin{align*}
& \quad G^{m_1 \cdots m_k}_{a_1 \cdots a_n} (\alpha_1, \ldots, \alpha_k|\beta_1, \ldots, \beta_n) \\
& = \delta_{2k-2p+n-2r,0} \prod_{\alpha \in A} \int_{-\infty}^{\infty} \frac{d\gamma}{2\pi} (P(\beta_a - \gamma_a) + x) \prod_{j=n+1}^{n} P(\beta_j - \gamma_a) \\
& \quad \times \prod_{m \in M_{-1}} \int_{-\infty}^{\infty} \frac{d\delta_m}{2\pi} \xi_m v_m^2 \left( P'(\alpha_m - \delta_m^1) P'(\alpha_m - \delta_m^2) - 2P'(\alpha_m - \delta_m^3) + 1 \right) \\
& \quad \times \prod_{1 \leq j < l \leq n} \left( \pi - i(\delta_l - \delta_j) \right) \left( \pi + i(\delta_l - \delta_j) \right) \\
& \quad \times \prod_{1 \leq j < l \leq p} \left( \pi - i(\delta_l - \delta_j) \right) \left( \pi + i(\delta_l - \delta_j) \right) \\
& \quad \times I(\alpha_1, \ldots, \alpha_k|\beta_1, \ldots, \beta_n|\gamma_1, \ldots, \gamma_r),
\end{align*}
\]

where \( I(\alpha_1, \ldots, \alpha_k|\gamma_1, \ldots, \delta_p|\beta_1, \ldots, \beta_n|\gamma_1, \ldots, \gamma_r) \) is defined in (4.20).

In this paper we have obtained the boundary state and its dual of the boundary \( SU(2) \) invariant massive Thirring model, by solving the boundary crossing symmetry condition. We have also presented the integral formulae of the soliton form factor of the model. For an application of our integral formulae, the asymptotic behaviors of the soliton form factors may be evaluated along the line of Smirnov’s pioneering work [14].

It is known that the \( SU(2) \) invariant massive Thirring model can be obtained from the sine-Gordon model by taking the limit such that an appropriate parameter of the latter model goes to infinity [14]. Nevertheless, it is not easy to see that our expression (4.21) is reduced from the corresponding form factor obtained in [14]. A part of technical reasons of the difficulty is as follows. For the boundary sine-Gordon model the integral contour for the asymptotic generators of the ZF algebra is more complicated than that for the local generators. On the other hand, for the present case, the integral contours for both the asymptotic and local generators of the ZF algebra can be taken on the real line.

In this paper we used the diagonal solution of the boundary Yang–Baxter equation. It may be interesting to study a nondiagonal boundary condition case.

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