Note on Coleman’s formula for the absolute Frobenius on Fermat curves

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Abstract
Coleman calculated the absolute Frobenius on Fermat curves explicitly. In this paper we show that a kind of $p$-adic continuity implies a large part of his formula. To do this, we study a relation between functional equations of the ($p$-adic) gamma function and monomial relations on ($p$-adic) CM-periods.

1 Introduction
We modify Euler’s gamma function $\Gamma(z)$ into

$$\Gamma_{\infty}(z) := \frac{\Gamma(z)}{\sqrt{2\pi}} = \exp(\zeta'(0, z)) \quad (z > 0)$$

and focus on its special values at rational numbers. Here we put $\zeta(s, z) := \sum_{k=0}^{\infty} (z + k)^{-s}$ to be the Hurwitz zeta function. The last equation is due to Lerch. One has a “simple proof” in [Yo, p17]. The gamma function enjoys some functional equations:

Euler’s Reflection formula:

$$\Gamma_{\infty}(z)\Gamma_{\infty}(1-z) = \frac{1}{2\sin\pi z}, \quad (1)$$

Gauss’ Multiplication formula:

$$\prod_{k=0}^{d-1} \Gamma_{\infty}(z + \frac{k}{d}) = d^{\frac{1}{2}-dz} \Gamma_{\infty}(dz) \quad (d \in \mathbb{N}). \quad (2)$$

For proofs, see [Ar, §3, 4]. The main topic of this paper is a relation between such functional equations and monomial relations of CM-periods, and its $p$-adic analogue. We introduce some notations.

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Definition 1.1. Let $K$ be a CM-field. We denote by $I_K$ the $\mathbb{Q}$-vector space formally generated by all complex embeddings of $K$:

$$I_K := \bigoplus_{\sigma \in \text{Hom}(K, \mathbb{C})} \mathbb{Q} \cdot \sigma.$$ 

We identify a subset $S \subset \text{Hom}(K, \mathbb{C})$ as an element $\sum_{\sigma \in S} \sigma \in I_K$. Shimura’s period symbol is the bilinear map

$$p_K : I_K \times I_K \to \mathbb{C}^\times / \mathbb{Q}^\times$$

characterized by the following properties (P1), (P2).

(P1) Let $A$ be an abelian variety defined over $\overline{\mathbb{Q}}$, having CM of type $(K, \Xi)$. Namely, for each $\sigma \in \text{Hom}(K, \mathbb{C})$, there exists a non-zero “$K$-eigen” differential form $\omega_\sigma$ of the second kind satisfying

$$k^*(\omega_\sigma) = \sigma(k) \omega_\sigma \quad (k \in K),$$

where $k^*$ denotes the action of $k \in K$ via $K \cong \text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ on the de Rham cohomology $H^1_{\text{dR}}(A, \mathbb{C})$. Then we have

$$\Xi = \{ \sigma \in \text{Hom}(K, \mathbb{C}) \mid \omega_\sigma \text{ is holomorphic} \},$$

and

$$p_K(\sigma, \Xi) \equiv \begin{cases} \pi^{-1} \int_\gamma \omega_\sigma & (\sigma \in \Xi) \\ \int_\gamma \omega_\sigma & (\sigma \in \text{Hom}(K, \mathbb{C}) - \Xi) \end{cases} \mod \overline{\mathbb{Q}}^\times$$

for an arbitrary closed path $\gamma \subset A(\mathbb{C})$ satisfying $\int_\gamma \omega_\sigma \neq 0$.

(P2) Let $\rho$ be the complex conjugation. Then we have

$$p_K(\sigma, \tau) p_K(\rho \circ \sigma, \tau) \equiv p_K(\sigma, \tau) p_K(\sigma, \rho \circ \tau) \equiv 1 \mod \overline{\mathbb{Q}}^\times \quad (\sigma, \tau \in \text{Hom}(K, \mathbb{C})).$$

Strictly speaking, Shimura’s $p_K$ in [Sh, §32] is a bilinear map on $\bigoplus_{\sigma \in \text{Hom}(K, \mathbb{C})} \mathbb{Z} \cdot \sigma$. The period symbol also enjoys the following relations:

(P3) Let $\iota : K' \cong K$ be an isomorphism of CM-fields. Then we have

$$p_K(\sigma, \tau) \equiv p_{K'}(\sigma \circ \iota, \tau \circ \iota) \mod \overline{\mathbb{Q}}^\times \quad (\sigma, \tau \in \text{Hom}(K, \mathbb{C})).$$

(P4) Let $K \subset L$ be a field extension of CM-fields. We define two linear maps defined as

$$\text{Res} : I_L \to I_K, \ \tilde{\sigma} \mapsto \tilde{\sigma}|_K \quad (\tilde{\sigma} \in \text{Hom}(L, \mathbb{C})),
\text{Inf} : I_K \to I_L, \ \sigma \mapsto \sum_{\tilde{\sigma} \in \text{Hom}(L, \mathbb{C}), \tilde{\sigma}|_K = \sigma} \tilde{\sigma} \quad (\sigma \in \text{Hom}(K, \mathbb{C})).$$

Then we have

$$p_K(\text{Res}(X), Y) \equiv p_L(X, \text{Inf}(Y)) \mod \overline{\mathbb{Q}}^\times \quad (X \in I_L, \ Y \in I_K).$$
Theorem 1.2 ([Gr, Theorem in Appendix]). Let \( F_N : x^N + y^N = 1 \) be the \( N \)th Fermat curve, \( \eta_{r,s} := x^{r-1}y^s dx \) its differential forms of the second kind \( (0 < r, s < N, r + s \neq N) \).
Then we have for any closed path \( \gamma \) on \( F_N(\mathbb{C}) \) with \( \int_{\gamma} \eta_{r,s} \neq 0 \)
\[
\int_{\gamma} \eta_{r,s} \equiv \frac{\Gamma\left(\frac{r}{N}\right)\Gamma\left(\frac{s}{N}\right)}{\Gamma\left(\frac{r+s}{N}\right)} \mod \mathbb{Q}(\zeta_N)^\times. \tag{3}
\]

Theorem 1.3 ([Gr, §2], [Yo, §2, Chap. III]). The CM-type corresponding to \( \eta_{r,s} \) is
\[
\Xi_{r,s} := \{ \sigma_b \mid 1 \leq b \leq N, (b, N) = 1, \langle \frac{N}{N} \rangle + \langle \frac{r}{N} \rangle + \langle \frac{s}{N} \rangle = 1 \}. \tag{4}
\]
That is, we have
\[
p_{\mathbb{Q}(\zeta_N)}(\text{id}, \Xi_{r,s}) = \begin{cases} \pi^{-1} \int_{\gamma} \eta_{r,s} & \text{if } r + s < N, \\ \int_{\gamma} \eta_{r,s} & \text{if } r + s > N \end{cases} \mod \mathbb{Q}^\times.
\]

Corollary 1.4 ([Ka2, Theorem 3]). We have for any \( \frac{a}{N} \in \mathbb{Q} - \mathbb{Z} \)
\[
\Gamma_\infty\left(\frac{a}{N}\right) \equiv \pi^{-\frac{1}{2}}(\frac{a}{N})^2 p_{\mathbb{Q}(\zeta_N)}(\text{id}, \sum_{(b,N)=1} \left(\frac{1}{2} - \langle \frac{ab}{N} \rangle\right) \cdot \sigma_b) \mod \mathbb{Q}^\times. \tag{5}
\]
Here the sum runs over all \( b \) satisfying \( 1 \leq b \leq N, (b, N) = 1 \).

Note that (5) holds true even if \( (a, N) > 1 \), essentially due to (P4). Although the following is just a toy problem, we provide its proof by using the period symbol, in order to explain the theme of this paper: we may say that some functional equations of the gamma function “correspond” to some monomial relations of CM-periods.

Proposition 1.5 (A toy problem). The explicit formula (5) implies the following “functional equations mod \( \mathbb{Q}^\times \)” on \( \Gamma\left(\frac{a}{N}\right) \):

“Reflection formula”:
\[
\Gamma_\infty\left(\frac{a}{N}\right) \Gamma_\infty\left(\frac{N-a}{N}\right) \equiv 1 \mod \mathbb{Q}^\times,
\]

“Multiplication formula”:
\[
\prod_{k=0}^{d-1} \Gamma_\infty\left(\frac{a}{N} + \frac{k}{d}\right) \equiv \Gamma_\infty\left(\frac{da}{N}\right) \mod \mathbb{Q}^\times.
\]

Proof. “Reflection formula” follows from (P2) immediately. Concerning “Multiplication formula”, we may assume that \( d \mid N \). Under the expression (5), “Multiplication formula” is equivalent to
\[
\sum_{k=0}^{d-1} \pi^{-\frac{1}{2}}(\frac{a}{N} + \frac{k}{d}) \cdot \sigma_b \equiv \pi^{-\frac{1}{2}}(\frac{a}{N})^2 p_{\mathbb{Q}(\zeta_N)}(\text{id}, \sum_{(b,N)=1} \left(\frac{1}{2} - \langle \frac{ab}{N} \rangle\right) \cdot \sigma_b).
\]
This follows from the multiplication formula
\[
\sum_{k=0}^{d-1} B_1(x + \frac{k}{d}) = B_1(dx)
\]
for the 1st Bernoulli polynomial \( B_1(x) = x - \frac{1}{2}. \)

The aim of this paper is to study a \( p \)-adic analogue of such “correspondence”. More precisely, we shall characterize the \( p \)-adic gamma function by its functional equations and some special values. Then we show that the period symbol and its \( p \)-adic analogue satisfy the corresponding properties to such functional equations. As an application, we provide an alternative proof of a large part of Coleman’s formula (Theorem 2.4-(i)): originally, Coleman’s formula was proved by calculating the absolute Frobenius on all Fermat curves. We shall see that it suffices to calculate it on only one curve (Remark 3.7).

**Remark 1.6.** Yoshida and the author formulated conjectures in [KY1, KY2, Ka2] which are generalizations of Coleman’s formula, from cyclotomic fields to arbitrary CM-fields: Coleman’s formula implies “the reciprocity law on cyclotomic units” [Ka1] and “the Gross-Koblitz formula on Gauss sums” [GK, Co1] simultaneously. The author conjectured a generalization [Ka2, Conjecture 4] of Coleman’s formula which implies a part of Stark’s conjecture and a generalization of (the rank 1 abelian) Gross-Stark conjecture simultaneously. The results in this paper (in particular Remark 3.7) are very important toward this generalization, since we know only a finite number of algebraic curves (e.g., [BS]) whose Jacobian varieties have CM by CM-fields which are not abelian over \( \mathbb{Q} \).

The outline of this paper is as follows. First we introduce Coleman’s formula [Co2] for the absolute Frobenius on Fermat curves in §2. The author rewrote it in the form of Theorem 2.4: roughly speaking, we write Morita’s \( p \)-adic gamma function \( \Gamma_p \) in terms of Shimura’s period symbol \( p_K \), its \( p \)-adic analogue \( p_{K,p} \), and modified Euler’s gamma function \( \Gamma_\infty \). In §3, we show that some functional equations almost characterize \( \Gamma_p \) (Corollary 3.3), and the corresponding properties ((13), Theorem 3.5) hold for \( p_K, p_{K,p}, \Gamma_\infty \). Then we see that a large part (Corollary 3.6) of Coleman’s formula follows automatically, without explicit computation, under assuming certain \( p \)-adic continuity properties. Unfortunately, our results have a root of unity ambiguity although the original formula is a complete equation, since some definitions are well-defined only up to roots of unity. In §4, we confirm that we can show (at least, a part of) needed \( p \)-adic continuity properties relatively easily.

## 2 Coleman’s formula in terms of period symbols

Coleman explicitly calculated the absolute Frobenius on Fermat curves [Co2]. The author rewrote his formula in [Ka1, Ka2] as follows.

### 2.1 \( p \)-adic period symbol

Let \( p \) be a rational prime, \( \mathbb{C}_p \) the \( p \)-adic completion of the algebraic closure \( \overline{\mathbb{Q}}_p \) of \( \mathbb{Q}_p \), and \( \mu_\infty \) the group of all roots of unity. For simplicity, we fix embeddings \( \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}, \mathbb{C}_p \) and
consider any number field as a subfield of each of them. Let \( B_{\text{cris}} \subset B_{dR} \) be Fontaine’s \( p \)-adic period rings. We consider the composite ring \( B_{\text{cris}} \otimes_p B_{dR} \). Let \( A \) be an abelian variety with CM defined over \( \mathbb{Q} \), \( \gamma \) a closed path on \( \subset A(\mathbb{C}) \), and \( \omega \) a differential form of the second kind of \( A \). Then the \( p \)-adic period integral

\[
\int_p : H^1_B(A(\mathbb{C}), \mathbb{Q}) \times H^1_{dR}(A, \mathbb{Q}) \to B_{\text{cris}} \otimes_p (\gamma, \omega) \mapsto \int_{\gamma, p} \omega
\]

is defined by the comparison isomorphisms of \( p \)-adic Hodge theory, instead of the de Rham isomorphism (e.g., [Ka1, §6], [Ka2, §5.1]). Here \( H^B \) denotes the singular (Betti) homology. Then, in a similar manner to \( p_K \), we can define the \( p \)-adic period symbol

\[
p_{K,p} : I_K \times I_K \to (B_{\text{cris}} \otimes_p - \{0\})^Q/Q^	imes
\]

satisfying \( p \)-adic analogues of (P1), (P2), (P3), (P4). Here we put \((B_{\text{cris}} \otimes_p - \{0\})^Q := \{x \in B_{dR} | \exists n \in \mathbb{N} \text{ s.t. } x^n \in B_{\text{cris}} \otimes_p - \{0\}\}. Moreover the “ratio”

\[
\left[ \int_\gamma \omega_\sigma : \int_{\gamma, p} \omega_\sigma \right] \in (\mathbb{C}^\times \times (B_{\text{cris}} \otimes_p - \{0\}))/\mathbb{Q}^\times
\]

depends only on \( \sigma \in \text{Hom}(K, \mathbb{C}) \) and the CM-type \( \Xi \). That is, if we replace \( A, \omega_\sigma, \gamma \) with \( A', \omega'_\sigma, \gamma' \) for the same \( \Xi, \sigma \), then we have

\[
\frac{\int_{\gamma'} \omega'_\sigma}{\int_{\gamma} \omega_\sigma} = \frac{\int_{\gamma'} \omega'_\sigma}{\int_{\gamma} \omega_\sigma} \in \mathbb{Q}^\times.
\]

Therefore we may consider the following ratio of the symbols \([p_K : p_{K,p}]\), which is well-defined up to \( \mu_\infty \).

**Proposition 2.1** ([Ka2, Proposition 4]). There exists a bilinear map

\[
[p_K : p_{K,p}] : I_K \times I_K \to (\mathbb{C}^\times \times (B_{\text{cris}} \otimes_p - \{0\}))/Q^\times/Q^\times
\]

satisfying the following.

(i) Let \( A, \Xi, \sigma, \omega_\sigma, \gamma \) be as in (P1). Then

\[
[p_K : p_{K,p}](\sigma, \Xi) \equiv \begin{cases}
[(2\pi i)^{-1} \int_\gamma \omega_\sigma : (2\pi i)^{-1} \int_{\gamma, p} \omega_\sigma] & (\sigma \in \Xi) \\
[\int_\gamma \omega_\sigma : \int_{\gamma, p} \omega_\sigma] & (\sigma \in \text{Hom}(K, \mathbb{C}) - \Xi)
\end{cases} \mod (\mu_\infty \times \mu_\infty)\mathbb{Q}^\times.
\]

Here \( (2\pi i)^{-1} \in B_{\text{cris}} \) is the \( p \)-adic counterpart of \( 2\pi i \) defined in, e.g., [Ka2, §5.1].

(ii) We have for \( \sigma, \tau \in \text{Hom}(K, \mathbb{C}) \) and for the complex conjugation \( \rho \)

\[
[p_K : p_{K,p}](\sigma, \tau) \cdot [p_K : p_{K,p}](\rho \circ \sigma, \tau) \equiv 1 \mod (\mu_\infty \times \mu_\infty)\mathbb{Q}^\times,
\]

\[
[p_K : p_{K,p}](\sigma, \tau) \cdot [p_K : p_{K,p}](\sigma, \rho \circ \tau) \equiv 1 \mod (\mu_\infty \times \mu_\infty)\mathbb{Q}^\times.
\]

(iii) Let \( \iota : K' \cong K \) be an isomorphism of CM-fields. Then we have for \( \sigma, \tau \in \text{Hom}(K, \mathbb{C}) \)

\[
[p_K : p_{K,p}](\sigma, \tau) \equiv [p_{K',p} : p_{K',p}](\sigma \circ \iota, \tau \circ \iota) \mod (\mu_\infty \times \mu_\infty)\mathbb{Q}^\times.
\]

(iv) Let \( K \subset L \) be a field extension of CM-fields. Then we have for \( X \in I_L, Y \in I_K \)

\[
[p_K : p_{K,p}](\text{Res}(X), Y) \equiv [p_L : p_{L,p}](X, \text{Inf}(Y)) \mod (\mu_\infty \times \mu_\infty)\mathbb{Q}^\times.
\]
2.2 Coleman’s formula

Theorem 2.4 below is essentially due to Coleman [Co2, Theorems 1.7, 3.13]. Note that the original formula does not have a root of unity ambiguity. First we prepare some notations. We assume that $p$ is an odd prime.

**Definition 2.2.**

(i) Let $C_p^1 := \{ z \in C_p^\times \mid |z|_p = 1 \}$. We fix a group homomorphism

$$\exp_p : C_p \to C_p^1$$

which coincides with the usual power series $\exp_p(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!}$ on the convergence region. For $\alpha \in C_p^\times$, $\beta \in C_p$, we put

$$\alpha^\beta := \exp_p(\beta \log_p \alpha)$$

with $\log_p$ Iwasawa’s $p$-adic log function.

(ii) For $z \in C_p^\times$, we put

$$z^* := \exp_p(\log_p(z)), \quad z^b := p^{\ord_p z} z^*.$$

Here we define $\ord_p z \in \mathbb{Q}$ by $|z|_p = |p|_p^{\ord_p z}$. Note that $z \equiv z^* \mod \mu_\infty (z \in C_p^\times)$.

(iii) We define the $p$-adic gamma function on $\mathbb{Q}_p$ as follows.

(a) On $\mathbb{Z}_p$, $\Gamma_p(z)$ denotes Morita’s $p$-adic gamma function which is the unique continuous function $\Gamma_p : \mathbb{Z}_p \to \mathbb{Z}_p^\times$ satisfying

$$\Gamma_p(n) := (-1)^n \prod_{1 \leq k \leq n-1, \ p \nmid k} k \quad (n \in \mathbb{N}).$$

(b) On $\mathbb{Q}_p - \mathbb{Z}_p$, we use $\Gamma_p : \mathbb{Q}_p - \mathbb{Z}_p \to \mathcal{O}_{\mathbb{Q}_p}^\times$ defined in [Ka1, Lemma 4.2], which is a continuous function satisfying

$$\Gamma_p(z + 1) = z^* \Gamma_p(z), \quad \Gamma_p(2z) = 2^{2z-\frac{1}{2}} \Gamma_p(z) \Gamma_p(z + \frac{1}{2}).$$

Such a continuous function on $\mathbb{Q}_p - \mathbb{Z}_p$ is unique up to multiplication by $\mu_\infty$.

(iv) For $z \in \mathbb{Z}_p$, we define $z_0 \in \{1, 2, \ldots, p\}$, $z_1 \in \mathbb{Z}_p$ by

$$z = z_0 + p z_1.$$

Note that when $p \mid z$, we put $z_0 = 0$, instead of $0$.

(v) Let $W_p$ be the Weil group defined as

$$W_p := \{ \tau \in \Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \mid \tau|_{\mathbb{Q}_p^{ur}} = \sigma_p^{\deg \tau} \text{ with } \deg \tau \in \mathbb{Z} \}.$$

Here $\mathbb{Q}_p^{ur}$ denotes the maximal unramified extension of $\mathbb{Q}_p$, $\sigma_p$ the Frobenius automorphism on $\mathbb{Q}_p^{ur}$.
(vi) We define the action of \( W_p \) on \( \mathbb{Q} \cap [0, 1) \) by identifying \( \mathbb{Q} \cap [0, 1) = \mu_\infty \). Namely
\[
\tau(\frac{a}{N}) := \frac{b}{N} \quad \text{if} \quad \tau(\zeta_N^b) = \zeta_N^b \quad (\tau \in W_p).
\]

(vii) Let \( \Phi_{\text{cris}} \) be the absolute Frobenius automorphism on \( B_{\text{cris}} \). We consider the following action of \( W_p \) on \( B_{\text{cris}} \mathbf{Q}_p \cong B_{\text{cris}} \otimes_{\text{Qur}} \mathbf{Q}_p \):
\[
\Phi_\tau := \Phi_{\text{cris}}^{\deg \tau} \otimes \tau \quad (\tau \in W_p).
\]

(viii) For \( \frac{a}{N} \in \mathbb{Q} \cap (0, 1) \) we put
\[
P(\frac{a}{N}) := \frac{\Gamma_\infty(\frac{a}{N}) \cdot (2\pi i)^{\frac{1}{2} - (\frac{a}{N})} p_{\text{Q}(\zeta_N)p} \left( \text{id}, \sum_{(b,N)=1} \left( \frac{1}{2} - (\frac{ab}{N}) \right) \sigma_b \right)}{(2\pi i)^{\frac{1}{2} - (\frac{a}{N})} p_{\text{Q}(\zeta_N)} \left( \text{id}, \sum_{(b,N)=1} \left( \frac{1}{2} - (\frac{ab}{N}) \right) \sigma_b \right)} \in (B_{\text{cris}} \mathbf{Q}_p - \{0\})^Q / \mu_\infty.
\]
This definition makes sense since
\[
\frac{\Gamma_\infty(\frac{a}{N})}{(2\pi i)^{\frac{1}{2} - (\frac{a}{N})} p_{\text{Q}(\zeta_N)} \left( \text{id}, \sum_{(b,N)=1} \left( \frac{1}{2} - (\frac{ab}{N}) \right) \sigma_b \right)} \in \mathbf{Q} \subset B_{\text{cris}} \mathbf{Q}_p
\]
by (5) and the ratio \([p_K : p_{K,p}]\) is well-defined up to \( \mu_\infty \) by Proposition 2.1.

Remark 2.3. (i) Let \( p_{p-1} \) be the group of all \((p-1)\)st roots of unity, \( p^\mathbb{Z} := \{p^n \mid n \in \mathbb{Z}\} \), \( 1 + p\mathbb{Z}_p := \{1 + pz \mid z \in \mathbb{Z}_p\} \). Then we have the canonical decomposition
\[
\mathbb{Q}_p^x \rightarrow \langle \mu_{p-1} \rangle \times p^\mathbb{Z} \times 1 + p\mathbb{Z}_p,
\]
where \( \omega \) denotes the Teichmüller character. The maps \( z \mapsto z^*, z^b \) provide a similar (but non-canonical) decomposition of \( \mathbb{C}_p^x \). Moreover, we note that the maps \( z \mapsto \exp_p(z), z^*, z^b \) are continuous homomorphisms.

(ii) We easily see that
\[
\tau(z) = \langle pz \rangle, \quad \tau^{-1}(z) = z + 1 \quad (z \in \mathbb{Z}_{(p)} \cap (0, 1), \quad \tau \in W_p, \quad \text{deg} \tau = 1).
\]

Theorem 2.4 ([Ka2, Theorem 3]). Let \( p \) be an odd prime.

(i) Assume that \( z \in \mathbb{Z}_{(p)} \cap (0, 1) \). Then we have
\[
\Gamma_p (\tau(z)) \equiv p_{\frac{1}{2} - \tau^{-1} (z)} \frac{P(z)}{\Phi_\tau (P(\tau^{-1}(z)))} \mod \mu_\infty \quad (\tau \in W_p, \quad \text{deg} \tau = 1).
\]

(ii) Assume that \( z \in (\mathbb{Q} - \mathbb{Z}_{(p)}) \cap (0, 1) \). Then we have
\[
\frac{\Gamma_p (\tau(z))}{\Gamma_p(z)} \equiv p_{(z-\tau(z)) \text{ord}_p z} \frac{P(\tau(z))}{\Phi_\tau (P(z))} \mod \mu_\infty \quad (\tau \in W_p).
\]

Remark 2.5. As a result, we see that the right-hand sides of Theorem 2.4-(i), (ii) are \( p \)-adic continuous on \( z, (z, \tau(z)) \) respectively, since the left-hand sides are so. We use only the \( p \)-adic continuity in the next section, in order to recover Theorem 2.4-(i).
3 Main results

Morita’s $p$-adic gamma function $\Gamma_p : \mathbb{Z}_p \to \mathbb{Z}_p^\times$ is the unique continuous function satisfying

$$
\Gamma_p(0) = 1, \quad \frac{\Gamma_p(z + 1)}{\Gamma_p(z)} = \begin{cases} 
-\frac{z}{z} (z \in \mathbb{Z}_p^\times), \\
-1 (z \in p\mathbb{Z}_p).
\end{cases}
$$

(7)

In this section, we study other functional equations characterizing $\Gamma_p$ and provide an alternative proof of Coleman’s formula in the case $z \in \mathbb{Z}(p)$. Strictly speaking, we only “assume” that the right-hand sides of Theorem 2.4-(i), (ii) are continuous on $z$, $(z, \tau(z))$ respectively (of course, this is correct). Then we can recover a “large part” (Corollary 3.6) of Theorem 2.4-(i). We assume that $p$ is an odd prime.

3.1 A characterization of Morita’s $p$-adic gamma function

$\Gamma_p(z)$ satisfies the following $p$-adic analogues of multiplication formulas, which we consider only up to roots of unity in this paper. For the detailed formulation and its proof, see [Ko, “Basic properties of $\Gamma_p$,” in §2 of Chap. IV].

Proposition 3.1. Let $d \in \mathbb{N}$ with $p \nmid d$. Then we have for $z \in \mathbb{Z}_p$

$$
\prod_{k=0}^{d-1} \Gamma_p(z + \frac{k}{d}) \equiv d^{1-dz+(dz)} \Gamma_p(dz) \mod \mu_\infty.
$$

(8)

Note that if $p \mid d$, then $z + \frac{k}{d}$ is not in the domain of definition of Morita’s $\Gamma_p$. In the rest of this subsection, we show that multiplication formulas (8) and some conditions characterize Morita’s $p$-adic gamma function (at least up to $\mu_\infty$).

Proposition 3.2. Assume a continuous function $f(z) : \mathbb{Z}_p \to \mathbb{C}_p^\times$ satisfies

$$
\prod_{k=0}^{d-1} f(z + \frac{k}{d}) \equiv f(dz) \mod \mu_\infty \quad (p \nmid d).
$$

(9)

Then the following holds.

(i) $\frac{f(z+1)}{f(z)} \mod \mu_\infty$ depends only on $\text{ord}_p z$.

(ii) The values

$$
c_k := \left( \frac{f(p^k + 1)}{f(p^k)} \right)^b
$$

characterize the function $f(z)$ up to $\mu_\infty$. More precisely, for $z \in \mathbb{Z}_p$, we write the $p$-adic expansion of $z - 1$ as

$$
z - 1 = \sum_{k=0}^{\infty} x_k p^k \quad (x_k \in \{0, 1, \ldots, p - 1\}).$$
Then we have
\[ f(z) \equiv \prod_{k=0}^{\infty} \alpha_k^{x_k - \frac{k-1}{2}} \mod \mu_\infty \text{ with } \alpha_k := c_k \prod_{i=0}^{k-1} c_i^{p_i^{k-i}(p-1)}. \]

Conversely, assume that
\[ f \left( 1 + \sum_{k=0}^{\infty} x_k p^k \right) \equiv \prod_{k=0}^{\infty} \alpha_k^{x_k - \frac{k-1}{2}} \mod \mu_\infty \quad (x_k \in \{0, 1, \ldots, p-1\}) \tag{10} \]
for constants \( \alpha_k \in \mathbb{C}_p^\times \) satisfying \( \alpha_k \to 1 \) (\( k \to \infty \)). Then \( f(z) \) satisfies the functional equations (9).

Proof. We suppress \( \mod \mu_\infty \). Assume (9). Replacing \( z \) with \( z + \frac{1}{d} \), we obtain \( \prod_{k=1}^{d} f(z + \frac{k}{d}) \equiv f(dz + 1) \). It follows that \( \frac{f(z + 1)}{f(z)} \equiv \frac{f(dz + 1)}{f(dz)} \). That is,
\[ g(z) := \frac{f(z + 1)}{f(z)} \equiv g(dz) \quad (p \nmid d \in \mathbb{N}). \]

Then the assertion (i) is clear. Let \( c_k := (g(p^k))^b, a_n := x_0 + x_1 p + \cdots + x_n p^n \) (\( 0 \leq x_i \leq p - 1 \)). We easily see that
\[ \#\{y = 1, 2, \ldots, a_n \mid \text{ord}_p y = k\} = x_k + \sum_{i=k+1}^{n} x_i p^{i-k-1}(p-1) \quad (0 \leq k \leq n). \]

Then we can write
\[ f(a_n + 1)^b = (f(1) g(1) g(2) \cdots g(a_n))^b = f(1)^b \alpha_0^{x_0} \alpha_1^{x_1} \cdots \alpha_n^{x_n} \]
with \( \alpha_k = c_k \prod_{i=0}^{k-1} c_i^{p^{k-i}(p-1)} \). Since \( \lim_{n \to \infty} f(a_n + 1) \) converges, so do \( \lim_{n \to \infty} f(a_n + 1)^b \) and \( \prod_{k=0}^{\infty} \alpha_k^{x_k} \). Moreover we can write
\[ f(z) \equiv f(1) \prod_{k=0}^{\infty} \alpha_k^{x_k}. \]

Consider the case of \( d = 2, z = \frac{1}{2} \) of (9): \( f(\frac{1}{2}) f(1) \equiv f(1) \). Therefore, noting that
\[ -\frac{1}{2} = \sum_{k=0}^{\infty} \frac{p-1}{2} p^k, \]
we obtain
\[ 1 \equiv f(\frac{1}{2}) \equiv f(1) \prod_{k=0}^{\infty} \alpha_k^{\frac{p-1}{2}}, \text{ that is, } f(1) \equiv \prod_{k=0}^{\infty} \alpha_k^{\frac{p-1}{2}}. \]

Then the assertion (ii) is also clear.

Next, assume (10). When \( \text{ord}_p z = k \), we see that \( \frac{f(z+1)}{f(z)} \equiv \frac{\alpha_k}{\alpha_{k-1}} \) (resp. \( \alpha_0 \)) if \( k > 0 \) (resp. \( k = 0 \)). In particular, \( g(z) := \frac{f(z+1)}{f(z)} \mod \mu_\infty \) depends only on \( \text{ord}_p z \). When \( z + z' = 1 \),
the $p$-adic expansions $z - 1 = \sum_{k=0}^{\infty} x_k p^k$, $z' - 1 = \sum_{k=0}^{\infty} x'_k p^k$ satisfy $x_k + x'_k = p - 1$ for any $k$. Then we have

$$f(z)f(z') = \prod_{k=0}^{\infty} \alpha_k^0 = 1.$$  

Therefore the case $z = 0$ of (9) holds true since $\left(\prod_{k=1}^{d-1} f(\frac{k}{d})\right)^2 = \prod_{k=1}^{d-1} f(\frac{k}{d})f(1 - \frac{k}{d}) \equiv 1$. Then (9) for $z \in \mathbb{N}$ follows by mathematical induction on $z$ noting that

$$\prod_{k=0}^{d-1} f(z + \frac{k}{d}) \equiv \prod_{k=0}^{d-1} f(z + \frac{k}{d})g(z + \frac{k}{d}),$$

$$f(dz + d) \equiv f(dz)g(dz) \cdots g(dz + d - 1),$$

$$\text{ord}_p(dz + k) = \text{ord}_p(z + \frac{k}{d}).$$

Since $\mathbb{N}$ is dense in $\mathbb{Z}_p$, we see that (9) holds for any $z \in \mathbb{Z}_p$. □

The following corollary provides a nice characterization of $\Gamma_p(z) \mod \mu_\infty$ in terms of functional equations and one or two special values.

**Corollary 3.3.** Assume a continuous function $f(z): \mathbb{Z}_p \rightarrow \mathbb{C}_p^\times$ satisfies

$$\prod_{k=0}^{d-1} f(z + \frac{k}{d}) \equiv f(dz) \mod \mu_\infty \ (p \nmid d)$$

and put

$$c_n := \left(\frac{f(p^n + 1)}{f(p^n)}\right)^b.$$  

Then the following equivalences hold:

(i) $c_0 = c_1 = \cdots \Leftrightarrow f(z) \equiv c_0^{z-\frac{1}{2}} \mod \mu_\infty$.

(ii) $c_1 = c_2 = \cdots \Leftrightarrow f(z) \equiv c_0^{z-\frac{1}{2}}(c_1/c_0)^{z_1+\frac{1}{2}} \mod \mu_\infty$.

**Proof.** We suppress mod$\mu_\infty$. For (i), assume that $c_0 = c_1 = \cdots$. Then

$$\alpha_k := c_k \prod_{i=0}^{k-1} c_i^{p^{k-1-i}(p-1)} = c_0^{p^k}.$$  

Hence we have by Proposition 3.2

$$f \left(1 + \sum_{k=0}^{\infty} x_k p^k\right) \equiv \prod_{k=0}^{\infty} x_k^{p^k-\frac{1}{2}} = c_0^{\sum_{k=0}^{\infty} x_k p^k-\frac{1}{2}} = c_0^{z-1+\frac{1}{2}} = c_0^{z-\frac{1}{2}}.$$  

The opposite direction is trivial by definition $c_n := \left(\frac{f(p^n + 1)}{f(p^n)}\right)^b$. For (ii), the assumption $c_1 = c_2 = \cdots$ implies $\alpha_0 = c_0$, $\alpha_k = c_0^{p^k}(c_1/c_0)^{p^{k-1}}$ ($k \geq 1$). In this case we have

$$f \left(1 + \sum_{k=0}^{\infty} x_k p^k\right) \equiv c_0^{\sum_{k=0}^{\infty} x_k p^k-\frac{1}{2}}(c_1/c_0)^{\sum_{k=1}^{\infty} x_k p^{k-1}-\frac{1}{2}} = c_0^{z_1+\frac{1}{2}}(c_1/c_0)^{z_1+\frac{1}{2}}$$

since $\sum_{k=1}^{\infty} x_k p^{k-1} = \frac{z_1 - x_0}{p} = z_1$. □
3.2 Alternative proof of a part of Coleman’s formula

We fix \( \tau \in W_p \) with \( \deg \tau = 1 \) and put

\[
G_1(z) := \left( p^{\frac{1}{2} - \tau^{-1}(z)} \frac{P(z)}{\Phi_{\tau}(P(\tau^{-1}(z)))} \right)^b \quad (z \in Z_p(\cap (0, 1)),
\]

\[
G_2(z) := \left( p^{(\tau^{-1}(z) - z)\text{ord}_p z} \frac{P(z)}{\Phi_{\tau}(P(\tau^{-1}(z)))} \right)^b \quad (z \in (Q - Z_p) \cap (0, 1)).
\]

Here we added \((\ ))^b\) to the right-hand sides of Coleman’s formulas (Theorem 2.4), in order to resolve a root of unity ambiguity, only superficially. Note that \(G_2\) corresponds to Theorem 2.4-(ii) replaced \(z\) with \(\tau^{-1}(z)\).

By Theorem 2.4-(i), we see that \(G_1\) is continuous for the \(p\)-adic topology. \(G_2\) is not \(p\)-adically continuous in the usual sense, on the whole of \((Q - Z_p) \cap (0, 1)\) (for details, see Remark 3.8). Theorem 2.4-(i) only implies the following “continuity”:

\[
G_1(z) \text{ is continuous for the relative topology induced by } z \in (Q - Z_p) \cap (0, 1) \hookrightarrow \mathbb{Q}_p \times \mathbb{Q}_p, \ z \mapsto (z, \tau^{-1}(z)).
\]

In Corollary 3.6, oppositely, we show that the \(p\)-adic continuity of \(G_1, G_2\) implies a “large part”

\[
G_1(z) \equiv a z^\frac{1}{2} b z^{1 + \frac{1}{2} \Gamma_p(z)} \mod \mu_\infty \quad (a, b \in \mathbb{C}_p^\times)
\]

of Theorem 2.4-(i):

\[
G_1(z) \equiv \Gamma_p(z) \mod \mu_\infty.
\]

Besides we shall show the continuity of \(G_1(z)\) in §4, independently of Theorem 2.4.

Hereinafter in this section, we forget Theorem 2.4. We assume the following Assumption instead.

**Assumption 3.4.** \(G_1(z)\) is \(p\)-adically continuous and \(G_2(z)\) is continuous in the sense of (12). In particular, we regard \(G_1\) as a \(p\)-adic continuous function:

\[
G_1(z) : Z_p \to \mathbb{C}_p.
\]

First we derive “multiplication formula”:

\[
\prod_{k=0}^{d-1} G_1(z + \frac{k}{d}) \equiv d^{1 - d z + (dz)^1} G_1(dz) \mod \mu_\infty \quad (p \nmid d \in \mathbb{N})
\]

independently of Theorem 2.4.

**Proof of (13).** We suppress \(\mod \mu_\infty\). Let \(z \in Z_p(\cap (0, \frac{1}{d}))\). By Definition 2.2-(viii) and (11) we can write

\[
\prod_{k=0}^{d-1} G_1(z + \frac{k}{d}) G_1(dz) \equiv \prod_{k=0}^{d-1} \frac{\Gamma_\infty(z + \frac{k}{d})}{\Gamma_\infty(dz)} \Phi_{\tau} \left( \frac{\Gamma_\infty(\tau^{-1}(dz))}{\prod_{k=0}^{d-1} \Gamma_\infty(\tau^{-1}(z + \frac{k}{d}))} \right) \prod_{k=0}^{d-1} p^{\frac{1}{2} - \tau^{-1}(z + \frac{k}{d})} \frac{P(z)}{p^{\frac{1}{2} - \tau^{-1}(dz)}}
\]

\times “products of classical or \(p\)-adic periods”,

\[
\equiv d^{1 - d z + (dz)^1} G_1(dz) \mod \mu_\infty.
\]
where the “products of classical or $p$-adic periods” become trivial by (6), as we saw in the
proof of Proposition 1.5. Besides we see that
\[
\{ \tau^{-1}(z + \frac{k}{d}) \mid k = 0, \ldots, d - 1 \} = \{ \frac{\tau^{-1}(dz)}{d} + \frac{k}{d} \mid k = 0, \ldots, d - 1 \}.
\]
To see this, it suffices to show that \( \{ \tau^{-1}(\zeta_N^a \zeta_d^k) \mid k = 0, \ldots, d - 1 \} \) and \( \{ \tau^{-1}(\zeta_N^d) \zeta_d^k \mid k = 0, \ldots, d - 1 \} \) coincide with each other. We easily see that both of them are the inverse
image of \( \tau^{-1}(\zeta_N^d) \) under the \( d \)th power map \( \mu_\infty \to \mu_\infty, x \mapsto x^d \). Hence we obtain
\[
\frac{\prod_{k=0}^{d-1} G_1(z + \frac{k}{d})}{G_1(dz)} \equiv \prod_{k=0}^{d-1} \Gamma_\infty(z + \frac{k}{d}) \cdot \Phi_\tau \left( \frac{\Gamma_\infty(\tau^{-1}(dz))}{\prod_{k=0}^{d-1} \Gamma_\infty(\tau^{-1}(dz) + \frac{k}{d})} \right) \cdot \prod_{k=0}^{d-1} \frac{p^\frac{1}{2} \cdot \tau^{-1}(dz)}{p^\frac{1}{2} \cdot \tau^{-1}(dz)} = d^{\frac{1}{2} - dz} \cdot \Phi_\tau(d^{-\tau^{-1}(dz)} - 1) \equiv d^{\frac{1}{2} - dz} \cdot d^{\tau^{-1}(dz)} - 1
\]
by (2), (6). For the last “\( \equiv \)”, we note that \( \Phi_\tau \) acts on \( \mathbb{Q}_p \supset d^{\tau^{-1}(dz)} - \frac{1}{2} \) as \( \tau \). By Remark 2.3-(ii), we have \( \tau^{-1}(dz) = (dz)_1 + 1 \). Then the assertion is clear. \( \square \)

Furthermore we can show that \( c_n = \left( \frac{f(p^n+1)}{f(p^n)} \right)^b \) for \( f(z) := \frac{G_1(z)}{\Gamma_p(z)} \) is constant, at least for \( n \geq 1 \).

**Theorem 3.5.** We assume Assumption 3.4 and put \( f(z) := \frac{G_1(z)}{\Gamma_p(z)} \).

(i) The following functional equations hold.
\[
\prod_{k=0}^{d-1} f(z + \frac{k}{d}) \equiv f(dz) \mod \mu_\infty (p \nmid d).
\]

(ii) We have \( c_1 = c_2 = \cdots \) for \( c_n := \left( \frac{f(p^n+1)}{f(p^n)} \right)^b \).

**Proof.** We suppress \( \mod \mu_\infty \). (i) follows from (8), (13). For (ii), we need for \( z \in p\mathbb{Z}_p \)
\[
\frac{G_1(pz)G_1(z+1)}{G_1(pz+1)G_1(z)} = \frac{\Gamma_p(pz)\Gamma_p(z+1)}{\Gamma_p(pz+1)\Gamma_p(z)}.
\]

Since the right-hand side is equal to \( \begin{cases} 1 & (p \nmid z) \\ z & (p \mid z) \end{cases} \) by (7), it suffices to show that
\[
\frac{G_1(pz)G_1(z+1)}{G_1(pz+1)G_1(z)} \equiv 1 \quad (z \in p\mathbb{Z}_p).
\]

Note that we can not use the definition (11) directly since \( z, z + 1, pz, pz + 1 \) are not contained in \( (0, 1) \) simultaneously. Therefore a little complicated argument is needed as follows. Let \( z \in \mathbb{Z}_p \cap (0, \frac{1}{p}) \). By Remark 2.3-(ii), we have
\[
\tau(z) = \langle pz \rangle = pz, \text{ hence } \tau^{-1}(pz) = z.
\]

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We can write
\[ H_1(z) := \frac{G_1(z)G_2(z + \frac{1}{p}) \cdots G_2(z + \frac{p-1}{p})}{G_1(pz)} \]
\[ \equiv p^{z+(z+\frac{1}{p})+\cdots+(z+\frac{p-1}{p})-\tau^{-1}(z)-\tau^{-1}(z+\frac{1}{p})-\cdots-\tau^{-1}(z+\frac{p-1}{p})} P(z) P(z + \frac{1}{p}) \cdots P(z + \frac{p-1}{p}) \]
\[ \times \Phi_{\tau} \left( \frac{P(z)}{P(\tau^{-1}(z))P(\tau^{-1}(z + \frac{1}{p})) \cdots P(\tau^{-1}(z + \frac{p-1}{p}))} \right). \]

Here we note that \( \text{ord}_p(z + \frac{k}{p}) = -1 \) for \( k = 1, \ldots, p - 1 \). We have
\[ \{ \tau^{-1}(z + \frac{k}{p}) | k = 0, \ldots, p - 1 \} = \{ \frac{z+k}{p} | k = 0, \ldots, p - 1 \} \] (14)

since both of \( \{ \tau^{-1}(\zeta_N^a)^k \} | k = 0, \ldots, p - 1 \} \), \( \{ \zeta_{pN}^{a+Nk} \} | k = 0, \ldots, p - 1 \} \) are the set of the \( p \)th roots of \( \zeta_N^a \) when \( z = \frac{a}{N} \). Therefore the \( p \)-power parts of \( H_1 \) become
\[ p^{z+(z+\frac{1}{p})+\cdots+(z+\frac{p-1}{p})-\frac{z+k}{p}-\frac{z+k}{p}-\cdots-\frac{z+k}{p}} = p^{p(z-1)}. \]

Moreover the “period parts” of \( H_1 \) become trivial by (6), (14). Namely we can write
\[ H_1(z) \equiv p^{p(z-1)} \frac{\Gamma_{\infty}(z) \Gamma_{\infty}(z + \frac{1}{p}) \cdots \Gamma_{\infty}(z + \frac{p-1}{p})}{\Gamma_{\infty}(pz)} \Phi_{\tau} \left( \frac{\Gamma_{\infty}(z)}{\Gamma_{\infty}(\frac{z+k}{p}) \Gamma_{\infty}(\frac{z+k}{p}) \cdots \Gamma_{\infty}(\frac{z+k}{p})} \right). \]

By using the original Multiplication formula (2) for \( \Gamma_{\infty} \), we obtain
\[ H_1(z) \equiv p^{p(z-1)} p^{\frac{1}{2} - pz} p^{\frac{1}{p}} = 1. \]

Next, let \( z = \frac{a}{N} \in \mathbb{Z}_p \cap (-\frac{1}{p}, 0) \). Then we have
- \( \tau(z+1) = pz+1 \). Hence \( \tau^{-1}(pz+1) = z+1 \).
- \( \{ \tau^{-1}(\zeta_N^a)^k \} | k = 1, \ldots, p \} = \{ \zeta \} \), \( \zeta = \zeta_N^a \) \( \{ \zeta_{pN}^{a+Nk} \} | k = 1, \ldots, p \}. \) Hence
\[ \{ \tau^{-1}(z + \frac{k}{p}) | k = 1, \ldots, p \} = \{ \frac{z+k}{p} | k = 1, \ldots, p \}. \]

Then we can prove similarly that
\[ H_2(z) := G_2(z + \frac{1}{p}) \cdots G_2(z + \frac{p-1}{p}) G_1(z+1) \]
\[ \equiv p^{(z+\frac{1}{p})+\cdots+(z+\frac{p-1}{p})+(z+1)-\tau^{-1}(z)+\cdots-\tau^{-1}(z+\frac{p-1}{p})-\tau^{-1}(z+1)} P(z + \frac{1}{p}) \cdots P(z + \frac{p-1}{p}) \]
\[ \times \Phi_{\tau} \left( \frac{P(z + 1)}{P(\tau^{-1}(z + \frac{1}{p})) \cdots P(\tau^{-1}(z + \frac{p-1}{p}))} \right) \]
\[ \equiv p^{(z+\frac{1}{p})+\cdots+(z+\frac{p-1}{p})+(z+1)-\frac{1}{p}-\cdots-\frac{1}{p}-\frac{1}{p}-\frac{1}{p}} p^{\frac{1}{2} - (pz+1)} p^{\frac{1}{2}} = 1. \]
Here $H_i(z) \equiv 1 \mod \mu_\infty$ implies $H_i(z) = 1$ ($i = 1, 2$) since we have $x^b = \exp_p(\log_p x) = \exp_p(0) = 1$ for $x \in \mu_\infty$. $(G_1(z), G_2(z)$ are in the image under $(\cdot)^2$ by definition, so are $H_i(z)$.) In particular, we have

$$
\frac{G_1(pz)}{G_1(z)} = G_2(z + \frac{1}{p}) \cdots G_2(z + \frac{p-1}{p}) \quad (z \in \mathbb{Z}_p(\mu) \cap (0, \frac{1}{p})),
$$

$$
\frac{G_1(pz + 1)}{G_1(z + 1)} = G_2(z + \frac{1}{p}) \cdots G_2(z + \frac{p-1}{p}) \quad (z \in \mathbb{Z}_p(\mu) \cap (-\frac{1}{p}, 0)).
$$

Let $z \in p\mathbb{Z}(\mu)$. Then there exist $z^+_n \in p\mathbb{Z}(\mu) \cap (0, \frac{1}{p})$, $z^+_n \in p\mathbb{Z}(\mu) \cap (-\frac{1}{p}, 0)$ which converge to $z$ when $n \to \infty$ respectively. Then we can write

$$
\frac{G_1(pz)}{G_1(z)} = \lim_{n \to \infty} \frac{G_1(pz^+_n)}{G_1(z^+_n)} = \lim_{n \to \infty} G_2(z^+_n + \frac{1}{p}) \cdots G_2(z_n^+ + \frac{p-1}{p}),
$$

$$
\frac{G_1(pz + 1)}{G_1(z + 1)} = \lim_{n \to \infty} \frac{G_1(pz^-_n + 1)}{G_1(z^-_n + 1)} = \lim_{n \to \infty} G_2(z^-_n + \frac{1}{p}) \cdots G_2(z^-_n + \frac{p-1}{p}).
$$

Recall that $G_2(z)$ is continuous in the sense of (12). Clearly we have for $k = 1, \ldots, p-1$

$$
z^+_n + \frac{k}{p} \to z + \frac{k}{p} \quad (n \to \infty).
$$

Additionally we see that

$$
\tau^{-1}(z^+_n + \frac{k}{p}) = \frac{z^+_n}{p} + \tau^{-1}(\frac{k}{p}) \to \tau^{-1}(\frac{k}{p}) \quad (n \to \infty)
$$

by noting that $\tau^{-1}(z + z') \equiv \tau^{-1}(z) + \tau^{-1}(z') \mod \mathbb{Z}$ ($\forall z, z'$), $\tau^{-1}(z) \equiv \frac{z}{p} \mod \mathbb{Z}$ if $p \mid z$, $\frac{z}{p} \in (-\frac{1}{p}, \frac{1}{p})$, $\tau^{-1}(\frac{k}{p}) \in [\frac{1}{p}, \frac{p-1}{p}]$. It follows that

$$
\lim_{n \to \infty} G_2(z^+_n + \frac{k}{p}) = \lim_{n \to \infty} G_2(z^-_n + \frac{k}{p}).
$$

Then the assertion is clear. \hfill \Box

By Corollary 3.3, we obtain the following.

**Corollary 3.6.** Assume Assumption 3.4. Then there exist constants $a, b$ satisfying

$$
G_1(z) \equiv a^{z-\frac{1}{2}}b^{\frac{1}{2}}\Gamma_p(z) \mod \mu_\infty.
$$

**Remark 3.7.** In addition to the above results, by computing the absolute Frobenius on only one Fermat curve, we obtain Coleman’s formula $G_1(z) \equiv \Gamma_p(z) \mod \mu_\infty$. For example, when $p = 3$, we obtain it for $z = 1, 2$ by the computation on $F_5$. It follows that $a^{\frac{1}{6}}b^{\frac{1}{6}} \equiv a^{\frac{1}{6}}b^{\frac{1}{6}} \equiv 1$, hence $a \equiv b \equiv 1$.

**Remark 3.8.** We used the assumption $p \mid z$ only in the last paragraph of the proof for Theorem 3.5 because $G_2$ is not $p$-adically continuous on the whole of $(\mathbb{Q} - \mathbb{Z}_p(\mu)) \cap (0, 1)$. For example, we put

$$
z_n := \frac{1}{p^2} + \frac{p^{n+1}}{p^{n+2} + (1-p)^n} \in (\mathbb{Q} - \mathbb{Z}_p(\mu)) \cap (0, 1) \quad (n \in \mathbb{N})
$$
and take \( \tau \in W_p \) with \( \deg \tau = 1 \) so that
\[
\tau_0 \tau = \zeta_p^{-1}.
\]
In particular we see that
\[
z_n \to \frac{1}{p^2} \text{ for the } p\text{-adic topology.}
\]
On the other hand we see that
\[
\tau_0^{-1}(z_n) \equiv \tau_0^{-1}\left(\frac{1}{p^2}\right) + \tau_0^{-1}\left(\frac{p^{n+1}}{p^{n+2}+1} \frac{2^2}{p} + \frac{p^n}{p^{n+2}+1}\right) = 1 - \frac{(1-p)^n}{p^2(p^{n+2}+1)} \mod \mathbb{Z},
\]
\[
1 - \frac{(1-p)^n}{p^2(p^{n+2}+1)} \in \begin{cases} (1,2) & \text{if } n \text{ is odd,} \\ (0,1) & \text{if } n \text{ is even.} \end{cases}
\]
Hence we have
\[
\tau_0^{-1}(z_n) = \begin{cases} -\frac{(1-p)^n}{p^2(p^{n+2}+1)} \to -\frac{1}{p^2} & \text{if } n = 2k+1, k \to \infty, \\ 1 - \frac{(1-p)^n}{p^2(p^{n+2}+1)} \to 1 - \frac{1}{p^2} & \text{if } n = 2k, k \to \infty. \end{cases}
\]
Then, by Theorem 2.4-(ii), we see that \( G_2(z_n) = (\Gamma_p(z_n)/\Gamma_p(\tau_0^{-1}(z_n)))^b \) does not converge \( p\)-adically although \( z_n \) does.

4 On the \( p\)-adic continuity

In the previous section, we showed that the \( p\)-adic continuity of the right-hand sides of Theorem 2.4-(i), (ii) implies a large part of Theorem 2.4-(i) itself. In this section, we see that it is relatively easy to show such \( p\)-adic continuity properties, without explicit computation. For simplicity, we consider only the case \( z \in \mathbb{Z}_p \). Assume that \( p \nmid N \).

Lemma 4.1 ([Co1, §VI]). Let \( 1 \leq r, s < N \) with \( r + s \neq N \). We consider the formal expansion of the differential form \( \eta_{r,s} = x^r y^s - N \frac{dx}{x} \) on \( F_N : x^N + y^N = 1 \) at \( (x,y) = (0,1) \):
\[
\eta_{r,s} = \sum_{n=0}^{\infty} b_{r,s}(n)x^n \frac{dx}{x},
\]
\[
b_{r,s}(n) := \begin{cases} (-1)^{\frac{n}{N}} \left( \frac{s}{N} - 1 \right) & (n \equiv r \mod N), \\ 0 & (n \not\equiv r \mod N). \end{cases}
\]
Let \( \Phi \) be the absolute Frobenius on \( H^1_{dR}(F_N, \mathbb{Q}_p) \). Then there exists \( \alpha_{r',s'} \in \mathbb{Q}_p \) satisfying
\[
\Phi(\eta_{r,s}) = \alpha_{r',s'} \eta_{r',s'} \text{ for } 1 \leq r', s' < N, \ ps \equiv r' \mod N, \ ps \equiv s' \mod N.
\]
Then we have
\[
\alpha_{r',s'} = \lim_{n \to N \mod N} \frac{pb_{r,s}(n)}{b_{r',s'}(pn)} = \lim_{n \to N \mod N} \frac{(p-1)^k}{p} \frac{p\left(\frac{s}{N} - 1\right)}{p\left(\frac{s'}{N} - 1\right)} \left(\frac{p-1}{pk + \frac{ps-r'}{N}}\right).
\]
We note that \( \alpha_{r',s'} \) depends only on \((\frac{r'}{N}, \frac{s'}{N})\). That is \( \alpha_{r',s'} \) with \( N = N_1 \) is equal to \( \alpha_{r',s'} \) with \( N = tN_1 \).

**Proposition 4.2.** \( \alpha_{r',s'} \) is p-adically continuous on \((\frac{r'}{N}, \frac{s'}{N}) \in (\mathbb{Z}_p \cap (0, 1))^2 \).

**Proof.** It suffices to show that \( \alpha_{r'_1,s'_1} \) with \( N = N_1 \) is close to \( \alpha_{r'_2,s'_2} \) with \( N = N_2 \) when \( \frac{r'_1}{N_1} \) is close to \( \frac{r'_2}{N_2} \) and \( \frac{s'_1}{N_1} \) is close to \( \frac{s'_2}{N_2} \). We may assume \( N := N_1 = N_2 \) by considering \( N = N_1N_2 \). First we fix \( r' := r'_1 \) and assume that \( s'_1 \) is close to \( s'_2 \). Then we can take the same \( k \) for the limit expressions (15) of \( \alpha_{r',s'_1}, \alpha_{r',s'_2} \). We easily see that if \( p^k | (s'_1 - s'_2) \), then \( p^k - 1 \mid (s_1 - s_2) \). In fact, we can write \( s'_1 = ps_i - l_iN \) with \( l_i = 0, 1, \ldots, p - 1 \) since \( 0 < s_i, s'_i < N \) for \( i = 1, 2 \). If \( p \mid (s'_1 - s'_2) \), then we have \( p \mid (l_1 - l_2) \), so \( l_1 = l_2 \). Therefore we obtain \( s_1 - s_2 = \frac{s'_1 - s'_2}{p} \). It follows that \( s_1 \) also is close to \( s_2 \). Hence the continuity on \( \frac{r'}{N} \) is clear since the numerator (resp. the denominator) of the expression (15) is a polynomial on \( \frac{r}{N} \) (resp. \( \frac{s}{N} \))

For the variable \( \frac{s'}{N} \), we replace \( x \) with \( y \). In other words, replace the point \((x, y) = (0, 1)\) for the expansion with \((0, 1)\). Then the continuity on \( \frac{s'}{N} \) also follows from the same argument. \( \square \)

**Corollary 4.3.** \( G_1(z) \) defined in (11) is p-adically continuous on \( z \in \mathbb{Z}_p \cap (0, 1) \). In particular, we may regard \( G_1(z) \) as a continuous function on \( \mathbb{Z}_p \).

**Proof.** CM-types \( \Xi_{r,s} \) of (4), corresponding to \( \eta_{r,s} \), generate the \( \mathbb{Q} \)-vector space \( \{ \sum_{\sigma} c_{\sigma} \cdot \sigma \mid c_{\sigma} + c_{\rho_{\sigma}} \) is a constant \} \). More explicitly, we claim that

\[
\sum_{(b,N)=1} \left( \frac{1}{2} - \left\langle \frac{ab}{N} \right\rangle \right) \sigma_b = \frac{1}{N} \sum_{1 \leq s < N, \ a+s \neq N} \Xi_{a,s} - \frac{N-2}{2N} \sum_{(b,N)=1} \sigma_b,
\]

where \( s \) runs over \( 1 \leq s < N \) with \( a+s \neq N \) in the first sum of the right-hand side. By the definition (4), \( \sigma_b \in \Xi_{a,s} \) if and only if \( \left\langle \frac{ab}{N} \right\rangle + \left\langle \frac{ab}{N} \right\rangle < 1 \). Namely \( \left\langle \frac{ab}{N} \right\rangle = \frac{1}{N}, \frac{2}{N}, \ldots, 1 - \frac{1}{N} - \left\langle \frac{ab}{N} \right\rangle \). The number of such \( b \) is congruent to \(-1 - ab \mod N \). Hence we have

\[
\frac{1}{N} \sum_{1 \leq s < N, \ a+s \neq N} \Xi_{a,s} = \sum_{(b,N)=1} \left( \frac{1}{N} - \left\langle \frac{ab}{N} \right\rangle \right) \sigma_b = \sum_{(b,N)=1} \left( 1 - \frac{1}{N} - \left\langle \frac{ab}{N} \right\rangle \right) \sigma_b.
\]

Here we note that \( ab \neq 0 \mod N \) since \( (b,N) = 1 \), \( a \neq 0 \mod N \). Then the above claim follows. By substituting this into Definition 2.2-(viii), we can write

\[
P\left( \frac{a}{N} \right) \equiv \frac{\Gamma_{\infty}\left( \frac{a}{N} \right)(2\pi i)^{\frac{1}{2} - \frac{a}{N}} \prod_{1 \leq s < N, \ a+s \neq N} \left( (2\pi i)^{e_s} \int_{\gamma_p} \eta_{a,s} \right)^{\frac{1}{N}}}{(2\pi i)^{\frac{1}{2} - \frac{a}{N}} \prod_{1 \leq s < N, \ a+s \neq N} \left( (2\pi i)^{e_s} \int_{\gamma_p} \eta_{a,s} \right)^{\frac{1}{N}}} \mod \mu_{\infty},
\]

where \( e_s := \begin{cases} -1 & (a+s < N) \\ 0 & (a+s > N) \end{cases} \)

since the part \( \sum_{(b,N)=1} \sigma_b \) becomes trivial by Proposition 2.1-(ii). We can strengthen the congruence relation \( \equiv \) of the formula (3) into an equality =, by selecting a specific closed
path \( \gamma_0 \) (e.g., \( \gamma_0 = N \gamma_N \) with \( \gamma_N \) in [Ot, Proposition 4.9]). Then we have

\[
P\left( \frac{a}{N} \right) \equiv c \cdot (2\pi i)_p^{-\frac{1}{2} + \frac{1}{N}} \prod_{1 \leq s < N, \ a + s \not\equiv N} \left( \int_{\gamma_0 \cdot p} \eta_{a,s} \right)^{\frac{1}{N}} \mod \mu_\infty,
\]

where we put

\[
c := \frac{\Gamma\left( \frac{a}{N} \right)}{(2\pi)^{\frac{1}{2}}} \left( \prod_{1 \leq s < N, \ a + s \not\equiv N} \frac{\Gamma\left( \frac{a+s}{N} \right)}{\Gamma\left( \frac{a}{N}\Gamma\left( \frac{a}{N} \right) \right)} \right)^{\frac{1}{N}}.
\]

Since (2) implies that

\[
\prod_{1 \leq s \leq N} \frac{\Gamma\left( \frac{a+s}{N} \right)}{\Gamma\left( \frac{a}{N}\Gamma\left( \frac{a}{N} \right) \right)} = \frac{N^{-a}a!}{\Gamma\left( \frac{a}{N}\right)^{N}},
\]

we obtain

\[
c = \frac{\Gamma\left( \frac{a}{N} \right)}{(2\pi)^{\frac{1}{2}}} \left( \frac{\Gamma\left( \frac{a}{N}\Gamma\left( \frac{N-a}{N} \right) \right)}{\Gamma(1)} \frac{\Gamma\left( \frac{a}{N}\Gamma\left( \frac{N-a}{N} \right) \right)}{\Gamma\left( \frac{a}{N}\Gamma\left( \frac{a}{N} \right) \right)} \right)^{\frac{1}{N}} = \left( \frac{N^{1-a}a!}{2 \sin\left( \frac{a}{N}\pi \right)} \right)^{\frac{1}{N}}.
\]

For the last equality we used (1) and the difference equation \( \Gamma(z+1) = z\Gamma(z) \). Take \( \tau \in W_p \) with \( \deg \tau = 1 \). Then we have

\[
G_1\left( \frac{a'}{N} \right) \equiv p^{\frac{1}{2} - \frac{1}{N}} \frac{P\left( \frac{a'}{N} \right)}{\Phi\left( P\left( \frac{a}{N} \right) \right)} = \left( \frac{N^{a-a'}(a'-1)!}{p^{a-N-1}(a-1)!} \prod_{1 \leq s < N, \ a + s \not\equiv N} \alpha_{a',s}^{-1} \right)^{\frac{1}{N}} \mod \mu_\infty,
\]

by noting that \( \Phi(2\pi i p) = p(2\pi i)_p \) and \( \Phi(\sin\left( \frac{a}{N}\pi \right)) = \tau(\sin\left( \frac{a}{N}\pi \right)) = \pm \sin\left( \frac{a'}{N}\pi \right) \). Here \( a',s' \) denote integers satisfying \( 1 \leq a',s' < N \), \( pa \equiv a' \mod N \), \( ps \equiv s' \mod N \) as above. By Proposition 4.2, \( \alpha_{a',s'} \) are continuous for \( a' \). When \( a \) is in a small open ball, as we saw in the proof of Proposition 4.2, we may write \( a' = pa - M \) for a fixed \( M \) (\( M \) is \( lN \) in the proof of Proposition 4.2). Then the remaining part becomes

\[
\frac{N^{a-a'}(a'-1)!}{p^{a-N-1}(a-1)!} = \pm \Gamma_p(a' + M + 1) \frac{p^N N^{(a-p)a'+M} (a' + M)}{a'(a'+1)(a'+2) \cdots (a'+M)},
\]

which is also continuous as desired. \( \square \)

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