Embedding Gödel’s universe in five dimensions

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Abstract
According to the Campbell-Magaard theorem, any analytical space-time can be locally and analytically embedded into a five-dimensional pseudo-Riemannian Ricci-flat manifold. We find explicitly this embedding for Gödel’s universe. The embedding space is Ricci-flat and has a non-Lorentzian signature of type \((++-\cdot-\cdot)\). We also show that the embedding found is global.

1 Introduction

In 1949, Kurt Gödel\textsuperscript{1} found a solution of Einstein’s field equations which soon became very popular because it described a spacetime possessing very strange properties. Among these was the existence, predicted by the model, of timelike closed curves which violate global causality. Even though it is not viable as a physical model of our universe, Gödel’s solution has some historical importance as it certainly stimulated a great deal of research on questions of causality and global properties of relativistic spacetimes\textsuperscript{2,3}.

Due to its peculiarity, different aspects of the so-called Gödel’s universe have always been studied with interest. For example, Rosen\textsuperscript{4} in 1965, was able to characterize Gödel’s model as a four-dimensional hypersurface embedded in a pseudo-Euclidean space with ten dimensions. An interesting result concerning the embedding of the Gödel spacetime in higher dimensions has been published which shows that the Gödel metric may also be viewed as a squashed anti-de Sitter geometry, thereby allowing the construction of a global algebraic isometric embedding in seven-dimensional flat spaces\textsuperscript{5}. Motivated by the current interest in higher dimensional theories of gravity, particularly the five-dimensional Randall-Sundrum models with non-compact extra dimension\textsuperscript{6}, the embedding of Gödel and Gödel type solutions on a 3-brane has been studied recently\textsuperscript{7}. 

1
Further motivation to study the embedding of Gödel’s solution comes from the so-called induced-matter proposal \[8\]. According to this proposal any solution of Einstein’s equations may be obtained from five-dimensional vacuum field equations, with matter in four-dimensions being generated or "induced" by purely geometrical means. Following this scheme, Wesson and collaborators have shown how to obtain from five-dimensional vacuum (or Ricci-flat) spaces a number of known solutions of the Einstein equations (regarded as hypersurfaces in five dimensions) whose energy-momentum tensor is generated by the extra-dimension \[9\]. In fact, the energy-momentum thus generated corresponds to the extrinsic curvature of the four-manifold embedded in five-dimensional vacuum space \[10\]. It has been later realised \[11\] that any energy-momentum tensor can be generated in this way, provided that any solution of Einstein’s equations has an embedding into a five-dimensional Ricci-flat solution, and this is almost precisely the content of an embedding theorem of differential geometry \[12, 13\]. Therefore, according to the this theorem (proposed by Campbell (1926) and later proved by Magaard (1963)), it is possible, to locally embed Gödel’s solution in a five-dimensional Ricci-flat pseudo-Riemannian space \[12, 13\]. In the light of the induced-matter theory, that means it must be possible to geometrically generate a source of matter and energy which is the source of Gödel’s universe with all its peculiarities. Let us note that conjectural considerations of such embeddings in the context of the induced-matter theory were already put forward some years ago \[14\].

Let us recall then the content of the Campbell-Magaard theorem \[12, 13\]. It states that any \(n\)-dimensional pseudo-Riemannian manifold \((M^n, g)\) can be locally, analytically and isometrically embedded in a Ricci-flat \((n+1)\)-dimensional manifold \((\tilde{N}^{n+1}, \tilde{g})\). Since its "rediscovery" in the nineties \[11\] the theorem has found a number of applications and has been discussed in various contexts in the literature \[15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26\]. Therefore, in view of the Campbell-Magaard theorem one would like to look at Gödel’s solution as a hypersurface embedded in a five-dimensional Ricci-flat space.

2 The embedding of Gödel’s universe in ten dimensions

In a classical paper, in which he performs the embedding of various relativistic spacetimes, Rosen exhibits an embedding of Gödel’s universe in a ten-dimensional pseudo-Euclidean flat space \(E^{10}\) with signature \((++++++----)\) by explicitly giving the embedding transformations. Let us see how it is done.

Gödel’s metric may be expressed in the form \[1\]

\[
    ds^2 = a^2(dt^* + e^{x^*}dy^*)^2 - dx^*2 - \frac{1}{2}e^{2x^*}dy^*2 - dz^*2
\]

where \(a\) is a constant. Let us choose a coordinate system \{\(Z^1, ..., Z^{10}\)\} of \(E^{10}\),
in which the line element is given by

\[ dS^2 = (dZ^1)^2 + (dZ^2)^2 + (dZ^3)^2 + (dZ^4)^2 + (dZ^5)^2 - \\
- (dZ^6)^2 - (dZ^7)^2 - (dZ^8)^2 - (dZ^9)^2 - (dZ^{10})^2 \] (2)

Now let us consider the four-dimensional hypersurface \( \Sigma_4 \) defined by the parametric equations

\[
\begin{align*}
Z^1 &= at^* \\
Z^2 &= (a/\sqrt{2})e^{x^*} \cos y^* \\
Z^3 &= (a/\sqrt{2})e^{x^*} \sin y^* \\
Z^4 &= a\sqrt{2}e^{1/2 x^*} \cos \frac{1}{2}(t^* + y^*) \\
Z^5 &= a\sqrt{2}e^{1/2 x^*} \sin \frac{1}{2}(t^* + y^*) \\
Z^6 &= a x^* \\
Z^7 &= a z^* \\
Z^8 &= (a/\sqrt{2})e^{x^*} \\
Z^9 &= a\sqrt{2}e^{1/2 x^*} \cos \frac{1}{2}(t^* - y^*) \\
Z^{10} &= a\sqrt{2}e^{1/2 x^*} \sin \frac{1}{2}(t^* - y^*)
\end{align*}
\]

A simple calculation shows that the metric induced in \( \Sigma_4 \) by the pseudo-Euclidean metric of \( \mathrm{E}^{10} \) is equal to Gödel’s metric (1).

The above result may be viewed as an application of the well-known Janet-Cartan theorem [27, 28] which claims that any \( n \)-dimensional Riemannian space \( M^n \) can be locally and isometrically embedded in an Euclidean space \( \mathrm{E}^m \) with \( m \leq n(n + 1)/2 \) dimensions. In fact, as Gödel’s metric is pseudo-Riemannian, one should consider the extension of Janet-Cartan theorem to pseudo-Euclidean spaces, obtained by Friedman [29]. In this extension the following condition is required for the embedding to take place: if \( ds^2 = g_{\alpha \beta} dx^\alpha dx^\beta \) and \( dS^2 = \eta_{ab} dZ^a dZ^b \) denote the line elements of \( M^n(p, q) \) e \( \mathrm{E}^m(r, s) \), respectively, with \( p \) and \( q \) (\( n = p + q \)) being the number of positive and negative eigenvalues of \( g_{\alpha \beta} \), \( r \) and \( s \) (\( m = r + s \)) the number of positive and negative eigenvalues of \( \eta_{ab} \), then \( r \geq p \) and \( s \geq q \). Thus, if \( n = 4 \), the maximum number of dimensions required for the embedding space is \( m = 10 \), which means that for performing the embedding of Gödel’s spacetime Rosen needed this maximum number.

Before we proceed, let us note that, by defining the new coordinates \( t = at^*, x = ax^*, y = ay^*, z = az^* \), with \( a = 1/\sqrt{2}w \), the line element [1] can be put in the form [30]

\[ ds^2 = dt^2 - dx^2 + \frac{1}{2} \exp(2\sqrt{2}wx) dy^2 - dz^2 + 2 \exp(\sqrt{2}wx) dtdy \] (3)
The embedding of G"odel’s universe in five dimensions

In this section we shall show how to obtain the embedding of G"odel’s spacetime in a five-dimensional Ricci-flat space, the metric of which has signature (+ − − − +). We already know that when \( n \geq 3 \), the Campbell-Magaard allows us to lower the number of dimensions of the embedding space \( N_{n+1} \) from \( \frac{n(n+1)}{2} \) to \( n+1 \), as long as \( N^{n+1} \) be Ricci-flat. However, we shall not employ directly the Campbell-Magaard; instead we shall make use of the following theorem due to Magaard [13]:

**Theorem (Magaard).** Let \( (M^n, g) \) be a \( n \)-dimensional pseudo-Riemannian manifold, \( \{x^\mu\} \) a local coordinate system of a neighbourhood \( U \) of \( p \in M^n \), with coordinates \((x_1^p, ..., x_n^p)\) defined by the parametrization \( x : U \to M^n \). A sufficient and necessary condition for \((M^n, g)\), with line element \( ds^2 = g_{\alpha\beta}(x)dx^\alpha dx^\beta \), to be locally, isometrically and analytically embedded in a \((n+1)\)-dimensional manifold \((M^{n+1}, \tilde{g})\) is that there exist analytical functions

\[
\tilde{g}_{\alpha\beta} = \tilde{g}_{\alpha\beta}(x^1, ..., x^n, x^{n+1})
\]

\[
\phi = \phi(x^1, ..., x^n, x^{n+1})
\]
defined in an open set \( D \subset x(U) \times \mathbb{R}^n \) containing the point \((x_1^p, ..., x_n^p, 0)\), satisfying the following conditions:

\[
\tilde{g}_{\alpha\beta}(x^1, ..., x^n, 0) = g_{\alpha\beta}(x^1, ..., x^n)
\]
in an open set of \( x(U) \): \( \tilde{g}_{\alpha\beta} = \tilde{g}_{\beta\alpha}, \left| \tilde{g}_{\alpha\beta} \right| \neq 0 ; \phi \neq 0 \), and that

\[
ds^2 = \tilde{g}_{\alpha\beta} dx^\alpha dx^\beta + \varepsilon \phi dx^{n+1} dx^{n+1}
\]

with \( \varepsilon = 1 \), represents the line element of \( M^{n+1} \) in a coordinate neighbourhood \( V \) of \( M^{n+1} \) [13, 31].

In the light of the above theorem let us take \( n = 4 \), \( \varepsilon = 1 \), \( \phi = -k^2 \), where \( k \) is a constant, and the set of analytical functions \( \{\tilde{g}_{\alpha\beta}(t, x, y, z, \psi)\}^{-1} \), \((\alpha, \beta = 0, 1, 2, 3)\) the non-null elements of which are \( \tilde{g}_{00} = 1; \tilde{g}_{02} = \tilde{g}_{20} = \exp(\sqrt{2}w(x + k\psi)); \tilde{g}_{11} = -1; \tilde{g}_{22} = \frac{1}{2} \exp(2\sqrt{2}w(x + k\psi)); \tilde{g}_{13} = \tilde{g}_{31} ; \tilde{g}_{13} = -1 \). Clearly the conditions \( \tilde{g}_{0\alpha} = \tilde{g}_{\alpha0}, \phi \neq 0 \) are satisfied, and also \( |\tilde{g}_{0\alpha}| = -\frac{1}{2} \exp(2\sqrt{2}w(x + k\psi)) \neq 0 \). Moreover, \( \tilde{g}_{\alpha\beta}(t, x, y, z, 0) = g_{\alpha\beta}(t, x, y, z) \), hence the functions \( g_{\alpha\beta} \) may be identified with the components of G"odel’s metric written in the form [3]. We conclude, therefore, from the above theorem that the G"odel’s universe can be embedded in a five-dimensional space \( M^5 \) with metric given by

\[
dS^2 = dt^2 - dx^2 + \frac{1}{2} \exp(2\sqrt{2}w(x + k\psi)) dy^2 - dz^2 + 2 \exp(\sqrt{2}w(x + k\psi)) dtdy + k^2 dy^2
\]

\[\text{(7)}\]

\[^1\text{We can choose the range of the new coordinates to be given by} -\infty < t, x, y, z, \psi < -\infty.\]
the embedding taking place for $\psi = 0$, i.e. by choosing the embedding functions given by $t \rightarrow t$, $x \rightarrow x$, $y \rightarrow y$, $z \rightarrow z$, $\psi = 0$.

If we calculate the components $R_{ab}$ of the Ricci tensor directly from (7) we get $R_{ab} = 0$. We see then that the five-dimensional manifold $M^5$, in which Gödel’s universe appears as the hypersurface $\psi = 0$, is a Ricci-flat space, and that proves our claim.

Let us conclude this section by noting two points. The first is that the manifold on which Gödel’s metric (3) is defined is $R^4$, i.e. $-\infty < t, x, y, z < \infty$ [39], and it is clear that the present embedding takes the whole of $R^4$ into $M^5$, irrespective of the domain chosen for $\psi$. Moreover, we see that the embedding functions and the metric of the embedded spacetime are analytic in $R^4$ while the metric of embedding space is analytical in $M^5$. It turns out then that in spite of the local character of the theorem mentioned previously in this particular case the embedding found happens to be global (a global version of the Campbell-Magaard theorem has been discussed recently in [32]). It is interesting to have a look at the components of the extrinsic curvature tensor $\Omega_{\alpha\beta}$ of the hypersurface $\psi = \text{const}$ of $M^5$. In the coordinates of (7) it can easily be shown that $\Omega_{\alpha\beta}$ is given by $\Omega_{\alpha\beta} = \frac{1}{2} \frac{i \partial g_{\alpha\beta}}{i \partial \psi}$, so that the nonvanishing components of $\Omega_{\alpha\beta}$ are $\Omega_{02} = \Omega_{20} = -\frac{1}{2} \sqrt{2} w \exp(\sqrt{2} w (x + k \psi))$, $\Omega_{22} = -\sqrt{2} w \exp(2 \sqrt{2} w (x + k \psi))$. As we see, the extrinsic curvature is also well-behaved (analytical) everywhere for any hypersurface of the foliation $\psi = \text{const}$, in particular for $\psi = 0$. As a consequence of the global character of the embedding, all global properties so characteristics of Gödel’s universe, such as the existence of closed timelike curves, are preserved in $M^5$.

The second point is concerned with the existence of timelike curves which may appear to be closed in the four-dimensional spacetime, hence appearing to violate global causality, but are not closed when viewed from the perspective of $M^5$. Let us explain what we mean. In the context of the induced-matter theory, for example, a geodesic motion in five-dimensions may appear accelerated when viewed from five dimensions. This anomalous acceleration manifest itself when geodesics lying on $M^5$ are projected (essentially by hiding the extra dimension) onto the hypersurface which represents ordinary four-dimensional spacetime [33, 34]. As the projected curve need not be a geodesic with respect to the metric of the hypersurface, there appears acceleration. Thus in this approach the projection of a non-closed curve in $M^5$ may appear to be closed (when we drop the fifth dimension) in four-dimensional spacetime, much in the same way as in ordinary tridimensional space $R^3$ we can project an helix into a circle. This fact perhaps suggests that concept of causality when extra dimensions of the spacetime are present needs further discussion.

\footnote{This can be done quickly and efficiently by employing algebraic computation programs such as SHEEP or GRTensor.}
4 Final remarks

We would like to call attention for the fact that the space \((M^5, \tilde{g})\), which is a solution of the Einstein vacuum field equations in five dimensions, has the peculiarity of possessing a non-Lorentzian (ultra-hyperbolic) metric, with two timelike dimensions. Spaces of these kind have been studied recently, mainly in connection with the idea that massless particles in five dimensions may appear "massive" when viewed from four-dimensional spacetime \([35, 36, 37]\). There are also claims that two times theories may find some motivation in M-theory \([38]\). One must recognise, however, that from the point of view of physics, ultra-hyperbolic spaces may pose some problems with respect to causality. On the other hand, examples of embedding spaces with extra timelike dimensions are many \([4]\), and include, for instance, the embedding of the Schwarzschild spacetime in a six-dimensional flat manifold obtained for the first time by Kasner \([39, 40]\). Isometric embeddings in flat spaces with two times have also been investigated in the context of branes \([41]\). Finally, it is interesting to note that it is not possible to globally embed a spacetime which is not globally hyperbolic into a pseudo-euclidean space with only one timelike dimension \([42]\). We do not know whether a similar result holds in the case of Ricci-flat embedding spaces.

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