Abstract

In this paper, we derive a stability result for $L_1$ and $L_\infty$ perturbations of diffusions under weak regularity conditions on the coefficients. In particular, in contrast to [17, 18], the drift terms we consider can be unbounded with at most linear growth, and we do not require uniform convergence of perturbed diffusions. Instead, we require a weaker convergence condition in a special metric introduced in this paper, related to the Holder norm of the diffusion matrix differences. Our approach is based on a special version of the McKean-Singer parametrix expansion.

Contents

1 Introduction 1
1.1 Setting . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 1
1.2 Assumptions and Main Result . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 3

2 McKean-Singer parametrix method 5
2.1 Parametrix Representation of transition Densities . . . . . . . . . . . . . . . . . . . . . . . . 5
2.2 Flow associated with drift . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 7
2.3 Convergence of parametrix series for diffusions with unbounded drift . . . . . . . . . . . . . 7

3 Stability of parametrix series 10
3.1 $L_1$ - perturbation case . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 10
3.2 Derivation of the main result in $L_\infty$ case . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 14
3.3 Simulation study . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 16

4 Appendix 18
4.1 Proof of Lemma 3.1 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 18
4.2 Proof of Lemma 3.2 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 20

References 21

1 Introduction

1.1 Setting

For a fixed deterministic horizon $T > 0$, let us consider the following $d$ - dimensional, non-degenerate Ito's diffusion

$$dX_t = b(t, X_t) \, dt + \sigma(t, X_t) \, dW_t, \quad t \in [0, T].$$  (1.1)
In this framework, some explicit convergence results can be found in [18] and for measurable bounded drift in the one of the model. Another important application includes the case of mollification by spatial convolution. Parameters \((b, \sigma)\) in space, with potentially unbonded drift. Also \(a(t, x) := \sigma^*(t, x)\) is assumed to be uniformly elliptic. The infinitesimal generator of (1.1) at time \(u\) and for all \(\varphi \in C^2_{b}(\mathbb{R}^d, \mathbb{R})\), \(z \in \mathbb{R}^d\), is defined as follows:

\[
L_u \varphi(z) = \frac{1}{2} \text{Tr} \left( \sigma \sigma^*(u, z) D_z^2 \varphi(z) \right) + \langle b(u, z), D_z \varphi(z) \rangle = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(u, z) \frac{\partial^2 \varphi(u, z)}{\partial z_i \partial z_j} + \sum_{i=1}^{d} b_i(u, z) \frac{\partial \varphi(u, z)}{\partial z_i}.
\]

We now introduce, for a given parameter \(\varepsilon > 0\), a perturbed version of (1.1) with dynamics

\[
dX^\varepsilon_T = b_\varepsilon(t, X^\varepsilon_t) dt + \sigma_\varepsilon(t, X^\varepsilon_t) dW_t, \quad t \in [0, T],
\]

where the coefficients \(b_\varepsilon, \sigma_\varepsilon\) satisfy at least the same assumptions as \(b, \sigma\) and are in some sense meant to be close to \(b, \sigma\) when \(\varepsilon\) is small. When both coefficients \(b, \sigma\) are bounded and Hölder continuous, it is well known that there exists a unique weak solution to (1.1), which has a density \([14, 15, 17]\). Moreover, combining the parametrix method to get the upper bound \([17, 11]\), and the chaining method to obtain the lower bound \([4]\), it can be shown \([2, 3]\) that the transition density \(p(0, x, t, y)\) satisfies the two sided Gaussian bounds

\[
\mathbb{E} \left[ \left| \frac{p(x, t, y) - p(0, x, t, y)}{\varepsilon} \right| \right] \leq C \varepsilon \leq \mathbb{E} \left[ \left| \frac{p(x, t, y) - p(0, x, t, y)}{\varepsilon} \right| \right],
\]

where \(C > 0\) is the same for both \(n\) and \(m\) sides.

Such methods were successfully developed for more general cases, namely, for operators satisfying strong Hörmander conditions and Kolmogorov operators with linear drifts \([16, 21, 13]\). When the drift is unbounded but with at most linear growth, namely,

\[
|b(x, t)| \leq C (|x|^\beta + 1), \quad \beta \in (0, 1],
\]

the generalization of the parametrix method was obtained in \([11]\), but the method developed there fails for the drifts with linear growth. It seems quite plausible that a linearly growing drift is exactly the boundary case, starting from which, it is necessary to introduce a forward flow corresponding to the transport of the initial condition or, equivalently, a backward flow corresponding to the transport of the terminal condition \([20, 12, 16]\). For unbounded at most linearly growing drift, the upper bound for the transition density may be obtained by using the truncation method introduced in \([12]\), see also \([20]\). This method consists in viewing a Fleming logarithmic transformation (see \([12, 22]\)) of a transition density as a value function of a certain stochastic control problem \([10]\), the desired density estimates are obtained by choosing appropriate controls. In the case of unbounded drift, the truncation method allows one to obtain an upper bound for a transition density, but, in order to study the sensitivity of transition densities w.r.t. the perturbations of coefficients one needs to work with a complete parametrix series without truncation.

Note also that the upper bounds for the total variation, entropy and Kantorovich distances between two solutions to Fokker-Planck-Kolmogorov equations with different diffusion matrices and drifts on \(\mathbb{R}^d \times [0, T]\) with fixed \(T > 0\) were obtained in \([7]\), and upper bounds for the distances between stationary solutions to Fokker-Planck-Kolmogorov equations with different diffusion matrices and drifts were obtained in \([8]\).

The aim of the present paper is to extend the stability results \([17]\) to the case of nonlinear unbounded drift with at most linear growth. Moreover, we replace the condition of \(L_\infty\) - closeness by the weaker one, namely, by \(L_1\) - closeness. To this end, the complete parametrix series will be constructed, which, in contrast to \([11]\), will be convergent also for drifts with linear growth.

Diffusions with the dynamics (1.1) and unbounded drifts appear in many applied problems. Important applications can, for instance, be found in mathematical finance. It is often very useful to know how a perturbation of the volatility \(\sigma\) impacts the density, and therefore the associated option prices \([9, 6]\). In the framework of parameter estimation, it can be useful, having at the hand estimators \((b_\varepsilon, \sigma_\varepsilon)\) of the true parameters \((b, \sigma)\) and some controls for the differences \(|b - b_\varepsilon|, |\sigma - \sigma_\varepsilon|\) in a suitable sense, to quantify the difference \(p - p_{\varepsilon}\) of the densities corresponding, respectively, to the dynamics with the estimated parameters and the one of the model. Another important application includes the case of mollification by spatial convolution. This specific kind of perturbation is useful for investigating the error between the densities of a non-degenerate diffusion of type (1.1) with Hölder coefficients (or with piecewise smooth bounded drift) and its Euler scheme. In this framework, some explicit convergence results can be found in \([18]\) and for measurable bounded drift in...
In summary, stability results can be useful in every applied field where the diffusion coefficients might be misspecified.

This paper is organized as follows. In the next subsection, we introduce our assumptions and state the main result. Section 2 contains an introduction to some crucial facts about parametrix expansion and deterministic flow associated with a drift of corresponding diffusion. In Section 3, we derive our main results in $L_1$ and $L_\infty$ cases, respectively. Additional proofs of technical lemmas are given in Appendix. In what follows, the letter $C$ stands for a positive constant, only depending on the quantities in the assumptions; in the context of the proofs, it's value may vary from line to line.

1.2 Assumptions and Main Result

In this paper, we study the difference between transition densities of the process (1.1) and its perturbed version (1.3). A similar problem for processes with bounded drift was discussed in [17]. In contrast, we consider a more general class of processes with growing drift. In our setting, we prove that the difference between transition densities admits an upper bound in terms of $L_1 - L_1$ norm: for probability measure $\mu$, we denote

$$||f(x, y)||_{L_1^p(\mathbb{R}^d, \mathcal{A}, \mu)} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x, y)| dy \mu(dx).$$

For any Borel sets $A \in \mathcal{B}([t, s])$, $B \in \mathcal{B}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, define the probability measures

$$\beta(A) = \int_A \mathfrak{B} \left( \frac{\gamma}{2} \right) du, \quad \lambda(B) = \int_B \bar{p}(t, s, x, y) dy,$$

and $L_1$-space of functions defined on $([t, s] \times \mathbb{R}^d \times \mathbb{R}^d, \beta \times \lambda \times \mu)$

$$||f(u, x, y)||_{L_1} = \left[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{[t, s]} \mathfrak{B} \left( \frac{\gamma}{2} \right) |f(u, x, y)| \bar{p}(t, s, x, y) du dy \mu(dx) \right].$$

Here $\mathfrak{B}(u; 1, \frac{\gamma}{2})$ is a beta-density on the interval $[t, s]$, $0 \leq t < s \leq T$, and $\bar{p}(t, s, x, y)$ is the transition density of some an auxiliary diffusion process that will be introduced in (2.8). Recall that

$$\mathfrak{B}(u; \alpha, \beta) = \frac{1}{B(\alpha, \beta)} (u - t)^{\alpha - 1} (s - u)^{\beta - 1} (s - t)^{1 - \alpha - \beta}, \quad u \in [t, s],$$

where $B(\alpha, \beta)$ is the Beta-function. It is natural in this case to consider the perturbation in $L_1$ sense and to use the beta-density as a weight function in the time variable. This choice is well-adapted to the beta function appearing in the upper bounds for the parametrix series terms.

Let us introduce the following assumptions. Below, the parameter $\varepsilon > 0$ is fixed and the constants appearing in the assumptions do not depend on it. In what follows, it is important that we consider the problem globally, namely on the product space $[0, T] \times \mathbb{R}^d$.

(A1) (Uniform Ellipticity.) Matrices $a = \sigma^* \sigma$ and $a_\varepsilon = \sigma_\varepsilon \sigma^*_\varepsilon$ are uniformly elliptic, i.e., there exists a positive constant $\Lambda \geq 1$, such that $\forall (t, x) \in [0, T] \times \mathbb{R}^d$,

$$\Lambda^{-1} |w|^2 \leq \langle a(t, x)w, w \rangle \leq \Lambda |w|^2, \quad \forall w \in \mathbb{R}^d,$$

$$\Lambda^{-1} |w|^2 \leq \langle a_\varepsilon(t, x)w, w \rangle \leq \Lambda |w|^2, \quad \forall w \in \mathbb{R}^d.$$ 

This condition also implies the uniform ellipticity of matrices $a^{-1}$ and $a_\varepsilon^{-1}$.

(A2) (Regularity.) The diffusion coefficients and the drifts of (1.1) and (1.3) are H"older and Lipschitz continuous in space, respectively: for $\gamma \in (0, 1], K > 0$ and $\forall (t, x), (s, y) \in [0, T] \times \mathbb{R}^d$,

$$|\sigma(t, x) - \sigma(t, y) + |\sigma_\varepsilon(t, x) - \sigma_\varepsilon(t, y)| \leq K |x - y|^\gamma,$$

$$|b(t, x) - b(t, y)| + |b_\varepsilon(t, x) - b_\varepsilon(t, y)| \leq K |x - y|,$$

$$|b(t, 0)| + |b_\varepsilon(t, 0)| \leq K, \quad 0 \in \mathbb{R}^d.$$
Clearly, if $\sigma(t, \cdot)$ is bounded and Hölder continuous so is $a(t, \cdot)$. Also, the regularity conditions imply at most linear growth of the drift coefficients:

$$|b(t, x)| + |b_t(t, x)| \leq K (1 + |x|) \quad \forall (t, x) \in [0, T] \times \mathbb{R}^d.$$

Thanks to the Lipschitz continuity and at most linear growth of the drift term in (1.1), the following Cauchy problem has a unique solution for fixed $(s, y) \in [0, T] \times \mathbb{R}^d$, defined globally on $[0, s]$:

$$\dot{\theta}_{w,s}(y) = b(w, \theta_{w,s}(y)), \quad \theta_{w,s}(y) = y. \quad (1.4)$$

Throughout the paper, we will call the deterministic flow solving ODE (1.4) the flow associated with drift $b$. In the same way, $\theta^\varepsilon_{w,s}(y)$ is the flow associated with $b^\varepsilon$.

**A3** (Almost equivalence of the flows.)

There exists a positive constant $C_0$ such that $\forall 0 \leq t < s \leq T$, uniformly in space,

$$\int_t^s |b_x(u, x) - b(u, x)| \, du \leq C_0 \sqrt{s - t}.$$ 

As a consequence, the flows $\theta_{w,s}(y)$ and $\theta^\varepsilon_{w,s}(y)$ associated with the drifts of (1.1) and (1.3) are almost equivalent:

$$|\theta^\varepsilon_{w,s}(y) - \theta_{w,s}(y)| \leq K \int_t^s |b_x(u, \theta^\varepsilon_{u,s}(y)) - b(u, \theta_{u,s}(y))| \, du \leq K \int_t^s |\theta^\varepsilon_{u,s}(y) - \theta_{u,s}(y)| \, du + \int_t^s |b_x(u, \theta^\varepsilon_{u,s}(y)) - b(u, \theta_{u,s}(y))| \, du \leq C_0 \sqrt{s - t}$$

by the Grönwall inequality.

For a given $\varepsilon > 0$, we say that the assumption (A) holds when conditions (A1) - (A3) are in force. Let us now introduce, under (A), the quantities that will bound the difference of the densities $p$ and $p^\varepsilon$ in our main result below. Set for $\varepsilon > 0$:

$$\Delta_{\varepsilon,b}(t, s) = \| [(b - b^\varepsilon)(u, \theta_{u,s}(y))] \|_{L_1}, \quad (1.5)$$

$$\Delta_{\varepsilon,\sigma}(t, s) = \| [\sigma - \sigma^\varepsilon(u, \theta_{u,s}(y))] \|_{L_1},$$

$$\Delta_{\varepsilon} := \Delta_{\varepsilon,b} + \Delta_{\varepsilon,\sigma}.$$ 

Here $\| \cdot \|_\beta$ stands for the expression consisting of two terms:

$$|f(t, x)|_\beta := |f(t, x)| + \sup_{y \in \mathbb{R}^d, y \neq x} \frac{|f(t, x) - f(t, y)|}{|x - y|^\beta}.$$

Now, for an arbitrary positive constant $\delta \in (\gamma/2, \gamma)$ and parameter $\alpha > 0$, we consider diagonal and off-diagonal maxima:

$$M_{\varepsilon,\alpha,\delta} = \max_{(s-t) \leq \alpha} (s-t)^{\delta-\gamma/2} (\Delta_{\varepsilon}(t, s))^{\gamma-\delta},$$

$$M_{\varepsilon,\alpha,\delta}^C = \max_{(s-t) > \alpha} (\Delta_{\varepsilon}(t, s))^{\gamma-\delta},$$

$$M_{\varepsilon,\alpha,\delta}^C = \max_{(s-t) \geq \alpha} (\Delta_{\varepsilon}(t, s))^{\gamma-\delta},$$

Let us state the main result of the paper.
Theorem 1.1. Let \( p(t, s, x, y) \) and \( \tilde{p}_\varepsilon(t, s, x, y) \) be the transition densities of (1.1) and (1.3) under (A). Then there exists a constant \( C > 0 \) depending only on \( T \) and parameters from the assumptions such that \( \forall(t, x), (s, y) \in [0, T] \times \mathbb{R}^d \),

\[
\| p - p_\varepsilon \|_{L^1([R^d, L_1([R^d)])}(t, s, x, y) \leq \frac{C}{\delta - \gamma/2} \cdot \left( M_{\varepsilon, \alpha, \delta} + M_{\varepsilon, \alpha, \delta}^C \right).
\]

Remark. Suppose that \( \Delta_\varepsilon \rightarrow 0 \). Then the right-hand side of the last inequality in (1.6) tends to zero for a suitable choice of \( \alpha(\varepsilon) \rightarrow 0 \).

In the following we will denote by \( \langle \cdot, \cdot \rangle \) and \( |\cdot| \) the Euclidean scalar product and the norm on \( \mathbb{R}^d \). Also we keep the notation \( D_\varepsilon^2 = \prod_{i=1}^d \gamma_i^2 \) to indicate the differentiation with respect to the multi-index \( \gamma = (\gamma_1, \cdots, \gamma_d) \in \mathbb{N}^d \) and for which we denote \( |\gamma| = \sum_{i=1}^d \gamma_i \).

2 \ McKean-Singer parametrix method

2.1 Parametrix Representation of transition Densities

Assume that (A1) and (A2) are in force. These assumptions imply the existence of transition densities of (1.1) and (1.3) and, therefore, will allow us to apply the PDE technique, namely, find the transition densities of processes solving (1.1) and (1.3) as fundamental solutions of the corresponding Kolmogorov equations.

Now let \( \tilde{X}_{u,t}^{(\tau,z)}(x) \) denote the process starting at \( x \) at time \( t \) with dynamics

\[
d\tilde{X}_{u}^{(\tau,z)} = b(u, \theta_{u,\tau}(z))du + \sigma(u, \theta_{u,\tau}(z))dW_u, \quad u \in [t, T].
\]

(2.1)

Usually there are two different choices for parameters \( (\tau, z) \). The first possibility is to identify \( (\tau, z) \) with the "frozen" initial point \( (t, x) \) of the process (2.1). However, we will choose the second possibility and freeze the terminal point \( y \) at time \( s \), letting \( (\tau, z) = (s, y) \). This approach is called backward parametrix.

In fact, after freezing the parameters \( (\tau, z) = (s, y) \), (2.1) is inhomogeneous Gaussian process with mean and covariance matrix given, respectively, by

\[
\tilde{\theta}_{s,t}(x) = x + \int_t^s b(u, \theta_{u,s}(y))du = x + y - \theta_{t,s}(y), \\
\tilde{C}_{s,t}(x) = \int_t^s \sigma^*(u, \theta_{u,s}(y))du.
\]

The idea is to use the density of the Gaussian process in (2.1) to derive an estimate on the transition density \( \tilde{p}(t, s, x, y) \) of the process with mollified versions \( \tilde{b}_\varepsilon(t, x) \) and \( \tilde{\sigma}_\varepsilon(t, x) \) of \( b(t, x) \) and \( \sigma(t, x) \). After that, the parametrix expansion for the transition density of the process (1.1) can be obtained by taking \( \varepsilon \rightarrow 0 \) [12, 20]. In this case, \( \tilde{p}(t, s, x, y) \) is called a proxy.

An important property is that the Gaussian transition density \( \tilde{p}(t, s, x, y) \) of (2.1) satisfies the backward Kolmogorov equation:

\[
\begin{cases}
\partial_t \tilde{p}(u, s, x, y) + \tilde{L}_u \tilde{p}(u, s, x, y) = 0, & t \leq u < s, \quad x, y \in \mathbb{R}^d \\
\tilde{p}(u, s, x, y) \xrightarrow{u \uparrow s} \delta_y(\cdot),
\end{cases}
\]

(2.2)

where

\[
\tilde{L}_u \varphi(z, y) = \frac{1}{2} \text{Tr} \left( \sigma \sigma^*(u, \theta_{u,s}(y)) D_z^2 \varphi(z, y) \right) + \langle b(u, \theta_{u,s}(y)), D_z \varphi(z, y) \rangle
\]
denotes the infinitesimal generator of (2.1) with frozen \((s, y) \in [0, T] \times \mathbb{R}^d\).

We now introduce a mollification procedure for the coefficients in (1.1), i.e., setting \(b_\zeta = b \ast \zeta\), \(\sigma_\zeta = \sigma \ast \zeta\), where \(\zeta(t) = \zeta^{-(d+1)}(t/\zeta)\), \(\zeta \in C_0^\infty(\mathbb{R}^{d+1}, \mathbb{R}_+), \int_{\mathbb{R}^{d+1}} \zeta(z)dz = 1\), is a time-space mollifier on \(\mathbb{R}^{d+1}\) and \(\ast\) denotes the time-space convolution. It then follows from the Hörmander theorem that the equation with dynamics

\[
dX^\zeta_t = b_\zeta(t, X^\zeta_t)dt + \sigma_\zeta(t, X^\zeta_t)dW_t, \quad t \in [0, T],
\]

admits a smooth transition density \(p^\zeta(t, s, x, y)\) for \(t < s\). Therefore, \(p^\zeta(t, s, x, y)\) must satisfy the forward Kolmogorov equation:

\[
\begin{cases}
\partial_t p^\zeta(t, u, x, z) - L^*_\zeta \ast p^\zeta(t, u, x, z) = 0, & t < u \leq s, z \in \mathbb{R}^d \\
p^\zeta(t, u, x, .) \xrightarrow{\text{u \to t}} \delta_x(\cdot),
\end{cases}
\]

where \(L^*_\zeta\) is an adjoint operator for the infinitesimal generator (1.2).

Our aim in this step is to estimate the transition density \(p^\zeta(t, s, x, y)\) at every point using the Gaussian proxy (2.1). To this end, we introduce a convolution, which will play a crucial role in our analysis:

\[
f \otimes g(t, s, x, y) = \int_t^s du \int_{\mathbb{R}^d} dz f(t, u, x, z)g(u, s, z, y).
\]

Equations (2.2), (2.3) yield the formal expansion, which is initially due to McKean and Singer ([19]):

\[
(p^\zeta - \overline{p}) (t, s, x, y) = \int_t^s du \partial_u \int_{\mathbb{R}^d} dz p^\zeta(t, u, x, z)\overline{p}(u, s, z, y) = \int_t^s du \int_{\mathbb{R}^d} dz \left(\partial_u p^\zeta(t, u, x, z)\overline{p}(u, s, z, y) + p^\zeta(t, u, x, z)\overline{p}(u, s, z, y)\right)
\]

\[
= \int_t^s du \int_{\mathbb{R}^d} dz \left(L^*_\zeta \ast p^\zeta(t, u, x, z)\overline{p}(u, s, z, y) - p^\zeta(t, u, x, z)\overline{L}_u \overline{p}(u, s, z, y)\right) = \int_t^s du \int_{\mathbb{R}^d} dz \left[p^\zeta(t, u, x, z)\left(L^*_\zeta - \overline{L}_u\right)\overline{p}(u, s, z, y)\right] = p^\zeta(t, s, x, y),
\]

where we used the Newton-Leibniz formula and the Dirac convergence for the first equality, equations (2.2) and (2.3) for the third one. We finally take the adjoint for the fourth equality. The function \(H^\zeta(t, s, x, y) = (L^*_\zeta - \overline{L}_u)\overline{p}(t, s, x, y)\) is called a parametrix kernel. With these notations the equality above rewrites as

\[
p^\zeta(t, s, x, y) = \overline{p}(t, s, x, y) + p^\zeta \otimes H^\zeta(t, s, x, y) = \overline{p}(t, s, x, y) + \overline{p} \otimes H^\zeta(t, s, x, y) + p^\zeta \otimes \overline{H}^\zeta(t, s, x, y) \Longrightarrow p^\zeta(t, s, x, y) = \sum_{i=0}^{+\infty} \overline{p} \otimes H^{\zeta,i}(t, s, x, y),
\]

where \(\overline{p} \otimes H^{\zeta,i}(t, s, x, y) = [\overline{p} \otimes H^{\zeta,i-1}] \otimes H^\zeta(t, s, x, y)\) and \(\overline{p} \otimes H^{\zeta,0}(t, s, x, y) := \overline{p}(t, s, x, y)\).

Now it can be shown that \(H^\zeta \xrightarrow{\zeta \to 0} H := (L - \overline{L}_u)\overline{p}(t, s, x, y)\) pointwise. Therefore, by the bounded convergence theorem, \(p^\zeta(t, v, x, y) \xrightarrow{\zeta \to 0} \overline{p}(t, s, x, y) + \sum_{i=1}^{+\infty} \overline{p} \otimes H^i(t, s, x, y)\). Moreover, Theorem 11.1.4 in [23] implies that for any bounded continuous function \(f : \mathbb{R}^d \to \mathbb{R}\),

\[
E \left[f(\overline{X}^\zeta_u)\right] \xrightarrow{\zeta \to 0} E \left[f(X_u)\right].
\]

This eventually gives (see the details in [17])

\[
p(t, s, x, y) = \overline{p}(t, s, x, y) + \sum_{i=1}^{+\infty} \overline{p} \otimes H^i(t, s, x, y).
\]

Recall that the decomposition (2.4) is formal until we have proved the uniform convergence of the series on the right-hand side.
2.2 Flow associated with drift

In this subsection, we derive the properties of the flow solving the ODE (1.4). These properties will be important for further analysis.

Proposition 2.1 (Lipschitz continuity of the flow). \( \theta_{t,s}(\cdot) \) is Lipschitz continuous in the spatial variable, i.e., there exists \( C > 0 \) such that \( \forall x, y \in \mathbb{R}^d \)

\[
|\theta_{t,s}(x) - \theta_{t,s}(y)| \leq C |x - y|.
\]

Proof. Let’s fix the initial and terminal times \( t \) and \( s \), respectively. Let \( x, y \in \mathbb{R}^d \). Then

\[
|\theta_{t,s}(x) - \theta_{t,s}(y)| = |x - y + \int_t^s (b(u, \theta_{u,s}(x)) - b(u, \theta_{u,s}(y))) \, du| \\
\leq |x - y| + K \cdot \int_t^s |\theta_{u,s}(y) - \theta_{u,s}(x)| \, du.
\]

By the Grönwall inequality,

\[
|\theta_{t,s}(x) - \theta_{t,s}(y)| \leq |x - y| \cdot \exp(C(s - t)) \leq |x - y| \cdot \exp(CT).
\]

\( \square \)

It is well-known, that under assumption (A2), the flow \( \theta_{t,s}(\cdot) \) has Lipschitz inverse and enjoys the semigroup property, i.e.,

\[
\begin{align*}
\theta_{t,s}(\theta_{s,u}(y)) &= \theta_{t,u}(y), \\
\theta_{s,t}(\theta_{t,u}(y)) &= \theta_{s,u}(y).
\end{align*}
\]

When \( t = u \) or \( s = u \), respectively, the equalities above take the form

\[
\theta_{t,s}(\theta_{s,u}(y)) = y, \theta_{s,t}(\theta_{t,u}(y)) = y. \tag{2.5}
\]

The direct consequence of the semigroup property is the next proposition (see [12]):

Proposition 2.2 (Bi-Lipschitz property of flow). The deterministic flow \( \theta_{u,s}(\cdot) \) solving (1.4) enjoys the bi-Lipschitz property, i.e., there exists a constant \( C \geq 1 \) such that for all \( x, y \in \mathbb{R}^d \)

\[
C^{-1} |y - \theta_{s,u}(x)| \leq |x - \theta_{u,s}(y)| \leq C |y - \theta_{s,u}(x)|.
\]

2.3 Convergence of parametrix series for diffusions with unbounded drift

Recall that the transition density of the process (1.1) can be expanded into the formal parametrix series (2.4). We recall for the sake of completeness the key steps in the proof of its uniform convergence. In this subsection, we suppose that the coefficients of (1.1) satisfy (A1), (A2).

From direct computations, for all \( 0 \leq t < s \leq T \), \( x, y \in \mathbb{R}^d \) and any multiindex \( \nu \) with \( |\nu| \leq 4 \), there exists \( C > 0 \) such that

\[
|D_{\nu}^x \tilde{p}(t, s, x, y)| \leq \frac{C}{(s - t)^{|\nu| + d}/2} \cdot \exp \left( - \frac{|\theta_{t,s}(y) - x|^2}{C(s - t)} \right). \tag{2.6}
\]

Next, applying Hölder continuity of the coefficients and (2.6), we readily get that there is a constant \( C > 0 \) such that

\[
|H(t, s, x, y)| \leq \frac{C}{(s - t)^{d/2 + 1 - \gamma/2}} \cdot \exp \left( - \frac{|\theta_{t,s}(y) - x|^2}{C(s - t)} \right). \tag{2.7}
\]

The next step is estimating the convolutions of higher order. Direct iteration of (2.1) leads to unbounded growth of constant \( C \) and, as a consequence, (2.4) will diverge. The following proposition allows one to manage the iteration procedure without the constants deteriorating at every step. This estimate was told us by S. Menozzi (personal communication). For the readers convenience we give a sketch of the proof.
Proposition 2.3. For any $C$, $0 < C < \infty$ the expression

$$\frac{C}{(s-t)^{d/2}} \cdot \exp \left( -\frac{|\theta_{t,s}(y) - x|^2}{C(s-t)} \right)$$

is a lower bound for the transition density $\bar{p}$ of the auxiliary diffusion process $\bar{X}_t$ with the dynamics

$$d\bar{X}_t = b(t, \bar{X}_t) \, dt + \lambda \, dW_t, \quad t \in [0,T],$$

(2.8)

where $\lambda = \lambda(C)$ is a positive large parameter depending on the quantities in the assumptions. In other words, the transition density $\bar{p}$ of the process $\bar{X}_t$ satisfies

$$\bar{C}\bar{p}(t, s, x, y) \geq C(s-t)^{-\frac{d}{2}} \exp \left( -\frac{|\theta_{s,t}(x) - y|^2}{C(s-t)} \right)$$

for a constant $\bar{C}$.

Importantly, $\bar{p}$ satisfies the Chapman-Kolmogorov identity. Namely,

$$\int_{\mathbb{R}^d} \bar{p}(t, v, x, z)\bar{p}(v, s, z, y) \, dz = \bar{p}(t, s, x, y).$$

It also holds that

$$\bar{p}(t, s, x, y) \leq \frac{C}{(s-t)^{\frac{d}{2}}} \exp \left( -\frac{|\theta_{s,t}(y) - x|^2}{C(s-t)} \right),$$

for the flow $\theta_{s,t}(y)$ chosen above that solves (1.4), up to a possible modification of $C$.

Proof. Let us consider

$$C_T(s-t)^{-\frac{d}{2}} \exp \left( -\frac{|\theta_{s,t}(x) - y|^2}{C_T(s-t)} \right)$$

(2.9)

with $C_T$ depending only on (A). We have to show that this expression can serve as a lower bound for an auxiliary diffusion $\bar{X}_t$ with sufficiently large $\lambda(C_T)$. For the linear drift $b(t, x) = G_t x$, it follows directly from (3.2), (3.4) and (3.7) in Proposition 3.1 in [12]. Indeed, in this case the Gaussian density $\bar{p}(t, s, x, y)$ can be written explicitly

$$\bar{p}(t, s, x, y) = (2\pi)^{-\frac{d}{2}} \det^{-\frac{1}{2}} (K_s) \exp \left( -\frac{1}{2} (K_s^{-1} R(s,t)x - y, R(s,t)x - y) \right),$$

where

$$K_s = \lambda^2 \int_0^s R(s,u)R^*(s,u) \, du := \lambda^2 \bar{R}(s,t), \quad K_s^{-1} = \lambda^{-2} \bar{R}^{-1}(s,t).$$

$R(s,t)_{0 \leq t, s \leq T}$ stands for the resolvent associated with $(G_t)_{0 \leq t \leq T}$, $R(s,t)x = \theta_{s,t}(x)$. The density $\bar{p}(t, s, x, y)$ can be rewritten as

$$\bar{p}(t, s, x, y) = (2\pi)^{-\frac{d}{2}} \lambda^{-d} \det^{-1/2} \bar{R}(s,t) \exp \left( -\frac{1}{2\lambda^2} (\bar{R}^{-1}(s,t) (\theta_{s,t}(x) - y), \theta_{s,t}(x) - y) \right) \geq K_T(s-t)^{-\frac{d}{2}} \lambda^{-d} \exp \left( -\frac{|\theta_{s,t}(x) - y|^2}{\lambda^2 K_T(s-t)} \right).$$

(2.10)

Now we take $\lambda$ such that $\lambda^2 K_T > C_T$ and then with this fixed $\lambda$ take $\bar{C}$ such that $\bar{C} K_T \lambda^{-d} > C_T$. Then we obtain from (2.9) and (2.10)

$$\bar{C}\bar{p}(t, s, x, y) \geq \bar{C} K_T(s-t)^{-\frac{d}{2}} \lambda^{-d} \exp \left( -\frac{|\theta_{s,t}(x) - y|^2}{\lambda^2 K_T(s-t)} \right) \geq C_T(s-t)^{-\frac{d}{2}} \exp \left( -\frac{|\theta_{s,t}(x) - y|^2}{C_T(s-t)} \right).$$

8
The general case uses linearization and follows the strategy developed in [12], pages 1599 – 1613. Many objects become simple or trivial because of a constant diffusion matrix $\lambda I_d$. We only indicate the key points resulting to a desire majorization. Inequality of Proposition 4.2 becomes

$$\sup_{0 \leq s \leq t} \frac{1}{C} \left| \Theta_\mu \right| \leq C \lambda(t)^{-\frac{1}{2} + \frac{\eta}{2}}$$

with a constant $C$ depending on $\mu$ and $T$ only. The control of Proposition 4.3 has the following form

$$\mathbb{P} \left\{ \forall t \in [0, T - \varepsilon], \quad \left| \Theta_\mu - \Gamma_\mu \right| \leq C \left( T - t \right)^{-\frac{1}{2} + \frac{\eta}{2}} \right\} \geq 1 - \mu,$$

and (4.27) and (4.29) become

$$\mathbb{E} \left( \sum_{i=1}^{6} R_{T-\varepsilon}^{i} \right) \leq C_1(\lambda) + \frac{C_2 \left| \theta_T(x_0) - y_0 \right|^2}{T}$$

for sufficiently large $\lambda$ and then sufficiently large $C$ such that

$$\bar{C} \bar{p}(t, s, x, y) \geq \bar{C} \exp \left( -C_1(\lambda) \right) (s - t)^{-\frac{3}{2}} \exp \left( -\frac{C_2 \left| \theta_{s,t}(x) - y \right|^2}{s - t} \right) \geq \frac{C_T(s - t)^{-\frac{3}{2}} \exp \left( -\frac{\left| \theta_{s,t}(x) - y \right|^2}{C_T(s - t)} \right)}{C_T(s - t)}.$$

\[ \square \]

**Corollary 2.4.** By Theorem 1.1 in [12]

$$p(t, s, x, y) \leq C_T(s - t)^{-\frac{3}{2}} \exp \left( -\frac{\left| \theta_{s,t}(x) - y \right|^2}{C_T(s - t)} \right)$$

with $C_T$ depending only on $\mu$. Proposition 2.3 implies

$$\bar{C} \bar{p}(t, s, x, y) \geq p(t, s, x, y)$$

for sufficiently large constant $\bar{C}$.

**Remarks.**

1. In fact, these modifications of the proofs allow us to construct a majorizing diffusion for a broader class of degenerate Kolmogorov-type systems considered in [12].

2. It should also be noted that in [1], when studying Geometric Average Asian options, the authors considered Kolmogorov-type operators and obtained two-sided estimates for the fundamental solution, these estimates are also given in terms of the fundamental solutions of the corresponding equations with constant diffusions.

3. Sometimes, in our paper, we will automatically renew the majorizing density $\bar{p}(t, s, x, y)$, meaning the following sequence of inequalities:

$$|\theta_{s,t}(y) - x|^\beta \bar{p}(t, s, x, y) \leq |\theta_{s,t}(y) - x|^\beta \frac{C_1}{(s - t)^{d/2}} \exp \left( -\frac{\left| \theta_{s,t}(y) - x \right|^2}{C_1(s - t)} \right) \leq \frac{C_2}{(s - t)^{d/2}} \exp \left( -\frac{\left| \theta_{s,t}(y) - x \right|^2}{C_2(s - t)} \right) \leq \bar{p}(t, s, x, y).$$
We can now rewrite the estimates (2.6) and (2.7) in the appropriate form using the majorizing density 
\( \tilde{p}(t, s, x, y) \):

\[
|\tilde{p}(t, s, x, y)| \leq C \cdot \tilde{p}(t, s, x, y),
\]

(2.11)

\[
|H(t, s, x, y)| \leq \frac{C}{(s-t)^{1-\gamma/2}} \cdot \tilde{p}(t, s, x, y).
\]

(2.12)

The following proposition shows that the same upper bounds can be derived for the transition density 
\( \tilde{p}_\varepsilon(t, s, x, y) \) if we additionally assume that (A3) holds.

**Proposition 2.5.** The transition density \( \tilde{p}_\varepsilon(t, s, x, y) \) of the perturbed process (1.3) admits an upper bound (2.11) with the same majorizing density \( \tilde{p} \):

\[
|\tilde{p}_\varepsilon(t, s, x, y)| \leq C \cdot \tilde{p}(t, s, x, y).
\]

**Proof.** Similarly to (2.6), \( \tilde{p}_\varepsilon(t, s, x, y) \) satisfies the following estimate:

\[
|\tilde{p}_\varepsilon(t, s, x, y)| \leq \frac{C}{(s-t)^{\gamma/2}} \cdot \exp \left( \frac{-|\theta_{t,s}(y) - x|^2}{C(s-t)} \right) = \frac{C}{(s-t)^{\gamma/2}} \cdot \exp \left( \frac{-|\theta_{t,s}(y) - \theta_{t,s}(y) + \theta_{t,s}(y) - x|^2}{C(s-t)} \right) \leq \frac{2C}{(s-t)^{\gamma/2}} \cdot \exp \left( \frac{-|\theta_{t,s}(y)|^2}{2C(s-t)} \right) \cdot \exp \left( \frac{|\theta_{t,s}(y) - \theta_{t,s}(y)|^2}{C(s-t)} \right).
\]

The factor \( \exp \left( \frac{|\theta_{t,s}(y) - \theta_{t,s}(y)|^2}{C(s-t)} \right) \) is bounded due to the consequence of the assumption (A3), while the remaining factor can be viewed as a lower bound for majorizing density. \( \square \)

Recall that the proposition above immediately implies the control of the perturbed parametrix kernel \( H_\varepsilon \):

\[
|H_\varepsilon(t, s, x, y)| \leq \frac{C}{(s-t)^{1-\gamma/2}} \tilde{p}(t, s, x, y).
\]

Using the Chapman-Kolmogorov equation, we get that there is an upper bound for the n-order convolution of the parametrix kernels with the transition densities \( \tilde{p}(t, s, x, y) \) and \( \tilde{p}_\varepsilon(t, s, x, y) \):

\[
|\tilde{p} \otimes H^n|(t, s, x, y) \leq C^{n+1} \frac{\Gamma \left( \frac{\gamma}{2} \right)^n}{\Gamma(1 + \frac{n\gamma}{2})} (s-t)^{n\gamma/2} \cdot \tilde{p}(t, s, x, y),
\]

\[
|\tilde{p}_\varepsilon \otimes H^n|(t, s, x, y) \leq C^{n+1} \frac{\Gamma \left( \frac{\gamma}{2} \right)^n}{\Gamma(1 + \frac{n\gamma}{2})} (s-t)^{n\gamma/2} \cdot \tilde{p}(t, s, x, y).
\]

(2.13)

The above estimate (2.13) means that parametrix series (2.4) converges absolutely and uniformly on \( (0, T] \times \mathbb{R}^d \):

\[
\sum_{n=0}^{\infty} |\tilde{p} \otimes H^n|(t, s, x, y) \leq \frac{C}{(s-t)^{1-\gamma/2}} \cdot \tilde{p}(t, s, x, y)
\]

for a constant \( C > 0 \). The same argument is valid for the parametrix series of \( p_\varepsilon \).

3 Stability of parametrix series

3.1 \( L_1 \) - perturbation case

In this subsection, we will investigate the stability of (1.1) under \( L_1 \)-type perturbation (1.5) through the difference of the respective series. For a given fixed parameter \( \varepsilon \), under (A1)–(A2), the densities \( p(t, s, x, y), p_\varepsilon(t, s, x, y) \) of the processes in (1.1), (1.3) both admit a parametrix expansion of the type (2.4). This decomposition allows us to get an upper bound for the difference of the transition densities using "term-by-term" estimation.
To achieve this, we split the convolution into two parts: (Difference of the main terms and their derivatives)

\[
\text{Lemma 3.1}
\]

\[
H(p(t, s, x, y) - p_c(t, s, x, y)) = \sum_{r=0}^{\infty} \tilde{p} \otimes H^r(t, s, x, y) - \tilde{p}_c \otimes H^r_c(t, s, x, y)
\]

\[
\leq |\tilde{p}(t, s, x, y) - \tilde{p}_c(t, s, x, y)| + \sum_{r=1}^{\infty} |\tilde{p} \otimes H^r(t, s, x, y) - \tilde{p}_c \otimes H^r_c(t, s, x, y)|.
\]

The strategy is to study the above difference by using some well-known properties of the Gaussian kernels and their derivatives. First, we study the difference of the \textit{main terms}. From now on, we fix a positive parameter $\delta \in (\gamma/2, \gamma)$, which will be necessary to avoid singularities when estimating the differences of the higher order convolutions. Let us introduce some notations:

\[
\varphi(t, s; y) = \left(\int_t^s \left| (b - b_c)(u, \theta_{u,s}(y)) \right| \, du \right)^{2 - \delta},
\]

\[
\psi(t, s; y) = \left| (b - b_c)(t, \theta_{t,s}(y)) \right| + \left| (\sigma - \sigma_c)(t, \theta_{t,s}(y)) \right|,
\]

\[
\|f(t, s, x, y)\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |f(t, s, x, y)| dy.
\]

\textbf{Lemma 3.1 (Difference of the main terms and their derivatives).} \textit{There exists $C > 0$ such that for all $0 \leq t < s \leq T$, $x, y \in \mathbb{R}^d$, $\delta \in (\gamma/2, \gamma)$, and $\nu, |\nu| \leq 4$:

\[
\left| \frac{D_{\nu}^s \tilde{p}(t, s, x, y) - D_{\nu}^s \tilde{p}_c(t, s, x, y)}{(s-t)^{\nu/2}} \right| \leq C p(t, s, x, y) \left( \int_t^s \left| (b - b_c)(u, \theta_{u,s}(y)) \right| \, du \right)^{\gamma - \delta} + \left( \int_t^s \left| (\sigma - \sigma_c)(u, \theta_{u,s}(y)) \right| \, du \right)^{\gamma - \delta}.
\]

Consequently, in terms of $L_1$ norm, the difference of the main terms can be rewritten as

\[
\left| (\tilde{p}_c - \tilde{p})(t, s, x, y) \right|_{L^1(\mathbb{R}^d)} \leq C(s - t)^{(\gamma - \delta)/2} \left[ \left( \Delta_{\epsilon,b}^{(\gamma - \delta)} \right)^{\gamma - \delta} + \left( \Delta_{\epsilon,a}^{(\gamma - \delta)} \right)^{\gamma - \delta} \right].
\]

The previous lemma estimates the difference of the main terms of the expansions. The next lemma quantifies the difference between the parametrix kernels $H(t, s, x, y)$ and $H_c(t, s, x, y)$.

\textbf{Lemma 3.2 (Difference of parametrix kernels).} \textit{For all $0 \leq t < s \leq T$ and $x, y \in \mathbb{R}^d$, there exists $C > 0$ such that

\[
|H - H_c|(t, s, x, y) \leq C p(t, s, x, y) \left( \frac{\psi(t, s; y)}{(s-t)^{1-\gamma/2}} + \frac{\varphi(t, s; y)}{(s-t)^{1+\gamma/2-\delta}} \right).
\]

For technical reasons, we postpone the proofs of Lemmas 3.1 and 3.2 till the Appendix.

To complete the proof of the main theorem, we need to handle the difference of the $n$-th order convolutions. To achieve this, we split the convolution into two parts:

\[
|\tilde{p} \otimes H^{n+1} - \tilde{p}_c \otimes H_c^{n+1}|(t, s, x, y) \leq |(\tilde{p}_c \otimes H_c^n) \otimes (H - H_c)|(t, s, x, y) + |(\tilde{p} \otimes H^n - \tilde{p}_c \otimes H_c^n) \otimes H|(t, s, x, y).
\]

The next lemma allows one to control the first term of (3.1).
Lemma 3.3. For all $0 \leq t < s \leq T$ and $x, y \in \mathbb{R}^d$, there exists $C > 0$ such that

$$
|\langle \tilde{p}_e \otimes H_{\epsilon}^n \rangle \otimes (H - H_e) \rangle (t, s, x, y) \leq C^{n+2} (s-t)^{\gamma/2} \int_t^s \frac{\psi(u; s, y)}{(s-u)^{1-\gamma/2}} du + 
$$

$$
+ C^{n+2} (s-t)^{(n-1)/2} \int_t^s \frac{\psi(u; s, y)}{(s-u)^{1-\gamma/2}} du + C^{n+2} (s-t)^{(n+1)/2} \int_t^s \frac{(u-t)^{(n+1)/2}}{(s-u)^{1+\gamma/2-\delta} du} \varphi(t, s, y).
$$

Proof. Using the control of $n$-th order convolution (2.13) and Lemma 3.2, we obtain:

$$
||\langle \tilde{p}_e \otimes H_{\epsilon}^n \rangle \otimes (H - H_e) \rangle (t, s, x, y) \leq 
\leq C^{n+2} \frac{\Gamma(\gamma/2)}{\Gamma(1+n\gamma/2)} \tilde{p}(t, s, x, y) \left( (s-t)^{\gamma/2} \int_t^s \frac{\psi(u; s, y)}{(s-u)^{1-\gamma/2}} du + \varphi(t, s, y) \int_t^s \frac{(u-t)^{(n+1)/2}}{(s-u)^{1+\gamma/2-\delta} du} \right) =
$$

$$
= C^{n+2} (s-t)^{\gamma/2} \frac{\Gamma(\gamma/2)}{\Gamma(1+n\gamma/2)} \tilde{p}(t, s, x, y) \int_t^s \frac{\psi(u; s, y)}{(s-u)^{1-\gamma/2}} du + 
$$

$$
+ C^{n+2} (s-t)^{(n+1)/2} \frac{\Gamma(\gamma/2)}{\Gamma(1+n\gamma/2)} \tilde{p}(t, s, x, y) \varphi(t, s, y).
$$

To get an upper bound for the second term of (3.1), we will iterate it. We estimate the difference between convolutions of the first order.

Lemma 3.4. For all $0 \leq t < s \leq T$ and $x, y \in \mathbb{R}^d$, there exists $C > 0$ such that

$$
|\tilde{p} \otimes H - \tilde{p}_e \otimes H_e \rangle (t, s, x, y) \leq \frac{C^2}{\gamma/2} \tilde{p}(t, s, x, y) \left( B(1+\delta-\gamma; \gamma/2) \left( \int_t^s \frac{\psi(u; s, y)}{(s-u)^{1-\gamma/2}} du \right) \right).
$$

Proof. Using the strategy described above, we write

$$
||\langle \tilde{p} \otimes H - \tilde{p}_e \otimes H_e \rangle \otimes (H - H_e) \rangle (t, s, x, y) \leq ||\langle \tilde{p} - \tilde{p}_e \otimes H \rangle (t, s, x, y) + ||\tilde{p}_e \otimes (H - H_e) \rangle (t, s, x, y).
$$

To estimate the first term in the expression above, using the semigroup property, we write

$$
|b - b_e\rangle (t, \theta_{t,u}(z)) \tilde{p}(u; s, z, y) \leq ||(b-b_e\rangle (t, \theta_{t,s}(\theta_{s,u}(z))) - (b-b_e\rangle (t, \theta_{t,s}(y))) + 
$$

$$
+ ||(b-b_e\rangle (t, \theta_{s,s}(y))) \tilde{p}(u, s, z, y) \leq C ||(\theta_{s,u}(z) - y) + 1 \rangle \langle b - b_e\rangle (t, \theta_{t,s}(y)) \tilde{p}(u, s, z, y) \leq C ||(b-b_e\rangle (t, \theta_{t,s}(y)) \tilde{p}(u, s, z, y).
$$

Recall that the same argument is valid for the diffusion coefficients $\sigma, \sigma_e$:

$$
|\sigma - \sigma_e\rangle (t, \theta_{t,u}(z)) \tilde{p}(u, s, z, y) \leq C ||(\sigma - \sigma_e\rangle (t, \theta_{t,s}(y)) \tilde{p}(u, s, z, y).
$$

Now, from Lemma 3.1 and (2.12),

$$
||\langle \tilde{p} - \tilde{p}_e \rangle \otimes H \rangle (t, s, x, y) \leq 
C^2 \tilde{p}(t, s, x, y) \left( \int_t^s \frac{|(b-b_e\rangle (u, \theta_{u,s}(y))|}{\psi(u; s, y)} du \right)^{\gamma-\delta} \int_t^s \frac{(u-t)^{\delta/2-\gamma/2} (s-u)^{\gamma/2-1} du}. \right.
$$
Proof. Continuing the descent until the difference of the first order convolutions, we get

Now we are ready to prove the main theorem 1.1 of this paper.

\[ \varepsilon \]

Let us consider the perturbation argument \( \Delta \)

\[ \int_t^s \left| (\sigma - \sigma_\varepsilon) (u, \theta_{u,s}(y)) \right| \gamma \, du \leq \]

\[ \leq C^2 B \frac{(1 + \delta - \gamma; \gamma/2)}{(s - t)^{\gamma/2 - \delta}} \hat{p}(t, s, x, y) \left[ \left( \int_t^s \left| (b - b_\varepsilon) (u, \theta_{u,s}(y)) \right| \gamma \, du \right)^{\gamma - \delta} + \right] \]

\[ \left( \int_t^s \left| (\sigma - \sigma_\varepsilon) (u, \theta_{u,s}(y)) \right| \gamma \, du \right)^{\gamma - \delta} = C^2 B \frac{(1 + \delta - \gamma; \gamma/2)}{(s - t)^{\gamma/2 - \delta}} \hat{p}(t, s, x, y) \varphi(t, s, y). \]

Combining the upper bound derived above with Lemma 3.2 completes the proof.

Now we are ready to prove the main theorem 1.1 of this paper.

**Proof.** From Lemmas 3.4, 3.3 and Jensen's inequality, we have the following \( L_1 - L_1 \) upper bounds:

\[ \| (\tilde{p} \otimes H - \tilde{p}_\varepsilon \otimes H_{\varepsilon}) (t, s, x, y) \|_{L_t^p(\mathbb{R}^d, L_1(\mathbb{R}^d))} \leq \]

\[ \leq C^2 \frac{B}{\delta - \gamma/2} (1 + \delta - \gamma; \gamma/2) (s - t)^{\gamma + \delta/2} \left[ (\Delta_{\varepsilon,b})^{\gamma - \delta} + (\Delta_{\varepsilon,\sigma})^{\gamma - \delta} \right], \]

\[ \| (\tilde{p}_\varepsilon \otimes H_\varepsilon^p) \otimes (H - H_{\varepsilon}) (t, s, x, y) \|_{L_t^p(\mathbb{R}^d, L_1(\mathbb{R}^d))} \leq \]

\[ \leq C^{n+2} (s - t)^{\delta + (n-1)\gamma/2} \frac{\Gamma_n(\gamma/2)}{\Gamma(1 + n\gamma/2)} \left[ (\Delta_{\varepsilon,b})^{\gamma - \delta} + (\Delta_{\varepsilon,\sigma})^{\gamma - \delta} \right]. \]

The next step is the iteration of (3.1) using the above estimates:

\[ \| (\tilde{p} \otimes H^{n+1} - \tilde{p}_\varepsilon \otimes H_{\varepsilon}^{n+1}) (t, s, x, y) \|_{L_t^p(\mathbb{R}^d, L_1(\mathbb{R}^d))} \leq \]

\[ \leq C^{n+2} (s - t)^{\delta + (n-1)\gamma/2} \frac{\Gamma_n(\gamma/2)}{\Gamma(1 + n\gamma/2)} \left[ (\Delta_{\varepsilon,b})^{\gamma - \delta} + (\Delta_{\varepsilon,\sigma})^{\gamma - \delta} \right] + \]

\[ + C \int_{\mathbb{R}^d} \int_t^s \left| (\tilde{p} \otimes H^n - \tilde{p}_\varepsilon \otimes H_{\varepsilon}^n) \right| (u, u, z, y) \frac{\hat{p}(u, s_1, z, y)}{(s - u)^{1-\gamma/2}} dz dy \mu(dx) = \]

\[ = C^{n+2} (s - t)^{\delta + (n-1)\gamma/2} \frac{\Gamma_n(\gamma/2)}{\Gamma(1 + n\gamma/2)} \left[ (\Delta_{\varepsilon,b})^{\gamma - \delta} + (\Delta_{\varepsilon,\sigma})^{\gamma - \delta} \right] + \]

\[ + C \int_t^s \frac{1}{(s - u)^{1-\gamma/2}} \left\| (\tilde{p} \otimes H^n - \tilde{p}_\varepsilon \otimes H_{\varepsilon}^n) \right\|_{L_t^p(\mathbb{R}^d, L_1(\mathbb{R}^d))} du. \]

Continuing the descent until the difference of the first order convolutions, we get

\[ \| (\tilde{p} \otimes H^{n+1} - \tilde{p}_\varepsilon \otimes H_{\varepsilon}^{n+1}) (t, s, x, y) \|_{L_t^p(\mathbb{R}^d, L_1(\mathbb{R}^d))} \leq \]

\[ \leq C^{n+2} (s - t)^{\delta + (n-1)\gamma/2} \frac{\Gamma_n(\gamma/2)}{\Gamma(1 + n\gamma/2)} \left[ (\Delta_{\varepsilon,b})^{\gamma - \delta} + (\Delta_{\varepsilon,\sigma})^{\gamma - \delta} \right] + \]

\[ + C^n \int_t^s \frac{1}{(s - s_1)^{1-\gamma/2}} \int_{s_1}^s \frac{1}{(s_1 - s_2)^{1-\gamma/2}} \cdots \int_{s_{n-1}}^s \left\| (\tilde{p} \otimes H - \tilde{p}_\varepsilon \otimes H_{\varepsilon}) \right\|_{L_t^p(\mathbb{R}^d, L_1(\mathbb{R}^d))} ds_n \cdots ds_1+ \]

\[ + \sum_{i=1}^{n-1} \frac{1}{\Gamma \left( 1 + (n-i)\gamma/2 \right)} \int_t^s \frac{1}{(s - s_1)^{1-\gamma/2}} \int_t^{s_1} \frac{1}{(s_1 - s_2)^{1-\gamma/2}} \cdots \]

\[ \cdots \int_t^{s_{i-2}} \frac{1}{(s_{i-2} - s_{i-1})^{1-\gamma/2}} \int_t^{s_{i-1}} \frac{1}{(s_{i-1} - s_i)^{1-\gamma/2}} \left[ (\Delta_{\varepsilon,b})^{\gamma - \delta} + (\Delta_{\varepsilon,\sigma})^{\gamma - \delta} \right] \left( t, s_i, x \right) ds_i \cdots ds_1. \]

Let us consider the perturbation argument \( \Delta_{\varepsilon}(t, s) \). On the diagonal we have

\[ \Delta_{\varepsilon,b}(t, x) + \Delta_{\varepsilon,\sigma}(t, x) = | b - b_\varepsilon | (t, x) + | \sigma - \sigma_\varepsilon | (t, x). \]
Clearly,

\[ \Delta \varepsilon(t, s) \leq M_{\varepsilon, \alpha, \delta} + M_{\varepsilon, \alpha, \delta}^C \leq \alpha^{\delta - \gamma/2} M_{\varepsilon, \alpha, \delta} + T^{\delta - \gamma/2} M_{\varepsilon, \alpha, \delta}^C. \]  

(3.3)

So, substituting the right-hand side of the estimate (3.3) instead of the perturbation argument in (3.2) and integrating, we get

\[ \| (\tilde{\rho} \otimes H^{n+1} - \tilde{\rho} \otimes H^{n+1}) (t, s, x, y)\|_{L^p_t(L^q_x(L^\gamma(R^d)))} \leq \]

\[ \leq C^{n+2} \left( \alpha^{\delta - \gamma/2} M_{\varepsilon, \alpha, \delta} + T^{\delta - \gamma/2} M_{\varepsilon, \alpha, \delta}^C \right) (s - t)^{\delta + (n-1)\gamma/2} \frac{1}{\Gamma(1 + n\gamma/2)} \sum_{i=1}^{\infty} \Gamma(\gamma/2). \]

The claim of Theorem 1.1 follows from the asymptotics of the Gamma function.

\[ \square \]

3.2 Derivation of the main result in \( L_\infty \) case

It turns out that in the case of uniform perturbations, under assumptions (A1), (A2), the difference of transition densities \( p(t, s, x, y) \) and \( p_\varepsilon(t, s, x, y) \) admits an estimate in terms of \( L_\infty^p - L_\infty^p \) norm. Recall that for arbitrary function \( f(x, y) \), where \( x, y \in \mathbb{R}^d \), its \( L_\infty^p - L_\infty^p \) norm with Borel measures \( \mu, \lambda \) and \( p, q \in [1, \infty] \) has the following form:

\[ \| f(x, y) \|_{L_q^p(\lambda(R^d), \mu)} = \left[ \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |f(x, y)|^p \lambda(dy) \right)^{q/p} \mu(dx) \right]^{1/q}. \]

For a given parameter \( \varepsilon > 0 \), let us introduce the perturbation arguments on the fixed time interval \( 0 \leq t < s \leq T \):

\[ \Delta^{\infty}_{\varepsilon, b} = \sup_{(u, x) \in [t, s] \times \mathbb{R}^d} |b(u, x) - b_\varepsilon(u, x)|, \]

\[ \Delta^{\infty}_{\varepsilon, \sigma} := \sup_{u \in [t, s]} |\sigma(u, .) - \sigma_\varepsilon(u, .)|_\gamma, \]

(3.4)

where \( | \cdot |_\alpha \) stands for the usual Hölder norm with exponent \( \alpha \) in the space of Hölder continuous bounded functions, i.e.,

\[ |f|_\alpha := \sup_{x \in \mathbb{R}^d} |f(t, x)| + \sup_{x \neq y, x, y \in \mathbb{R}^d} \frac{|f(t, x) - f(t, y)|}{|x - y|^\alpha}. \]

**Theorem 3.5.** Let \( p(t, s, x, y) \) and \( p_\varepsilon(t, s, x, y) \) be the transition densities of diffusions (1.1) and (1.3) under (A1), (A2). Then there exists a constant \( C > 0 \) depending only on \( T \) and parameters from the assumptions (A) such that \( \forall (t, x), (s, y) \in [0, T] \times \mathbb{R}^d \),

\[ |p - p_\varepsilon| (t, s, x, y) \leq C \cdot (\Delta^{\infty}_\varepsilon)^\gamma \cdot \tilde{\rho}(t, s, x, y). \]

As a consequence,

\[ \| (p - p_\varepsilon)(t, s, x, y) \|_{L_q^p(\lambda(R^d), \mu)} \leq C (\Delta^{\infty}_\varepsilon)^\gamma \cdot \| \tilde{\rho}(t, s, x, y) \|_{L_q^p(\lambda(R^d), \mu)}. \]

**Remark.** Despite the fact that we defined the uniform perturbations (3.4) of the coefficients taking supremum in spatial variable over the whole \( \mathbb{R}^d \) can be reformulated through the flow (1.4). Indeed, using the semigroup property (2.5) of the flow \( \theta_{t, s}(\cdot) \), we get

\[ \sup_{(u, y) \in [t, s] \times \mathbb{R}^d} |(b - b_\varepsilon) (u, \theta_{t,u}(y))| \leq \sup_{(u, x) \in [t, s] \times \mathbb{R}^d} |b(u, x) - b_\varepsilon(u, x)| = \]

\[ = \sup_{(u, x) \in [t, s] \times \mathbb{R}^d} |b(u, \theta_{t,u}(\theta_{s,u}(x))) - b_\varepsilon(u, \theta_{t,u}(\theta_{s,u}(x))))| \leq \sup_{(u, y) \in [t, s] \times \mathbb{R}^d} |(b - b_\varepsilon) (u, \theta_{t,u}(y))|. \]

**Proposition 3.6.** For all \( 0 \leq t < s \leq T \) and \( y \in \mathbb{R}^d \), there exists a constant \( C > 0 \) such that

\[ |\theta_{t,s}(y) - \theta^\varepsilon_{t,s}(y)| \leq C \Delta^{\infty}_{\varepsilon,b} \cdot (s - t). \]
Lemma 3.7. There exists $C > 0$ such that for all $0 \leq t < s \leq T$, $x, y \in \mathbb{R}^d$, and multiindex $\nu$, $|\nu| \leq 2$: 
\[
|D_x^\nu \tilde{p}(t, s, x, y) - D_x^\nu \tilde{p}_c(t, s, x, y)| \leq \frac{C\hat{p}(t, s, x, y)}{(s - t)^{|
u|/2}} \left(\Delta_x^\infty\right)^{|\nu|/2} .
\]
Consequently, in terms of $L^1_t - L^2_t$ norm, the difference of the main terms can be rewritten as 
\[
||\tilde{p}_c - \tilde{p}||_{L^1_t(L^\infty_x), L^2_t}(t, s, x, y) \leq \frac{C}{(s - t)^{|
u|/2}} ||\hat{p}(t, s, x, y)||_{L^1_t(L^\infty_x), L^2_t} \left(\Delta_x^\infty\right)^{|\nu|/2} .
\]

Lemma 3.8. For all $0 \leq t < s \leq T$ and $x, y \in \mathbb{R}^d$, there exists $C > 0$ such that 
\[
|H - H_c|(t, s, x, y) \leq \frac{C\hat{p}(t, s, x, y)}{(s - t)^{1-\gamma/2}} \left(\Delta_x^\infty\right)^{|\nu|/2} .
\]

To complete the proof of Theorem 3.5, we consider the difference between $n$-th order convolutions. We will derive the respective upper bound by induction.

Lemma 3.9. For all $0 \leq t < s \leq T$ and $x, y \in \mathbb{R}^d$, there exists $C > 0$ such that 
\[
|\tilde{p} \odot H^n - \tilde{p}_c \odot H^n_c|(t, s, x, y) \leq (n + 1)C^{n+2} \cdot \left(\Delta_x^\infty\right)^{|\nu|/2} \cdot (s - t)^{1-\gamma/2} \frac{\Gamma^n(\gamma/2)}{\Gamma(1+n\gamma/2)} \hat{p}(t, s, x, y).
\]

Proof. Let us write decomposition (3.1): 
\[
|\tilde{p} \odot H^{n+1} - \tilde{p}_c \odot H_c^{n+1}|(t, s, x, y) \leq \left(|\tilde{p} \odot H^n - \tilde{p}_c \odot H^n_c| \odot H\right)(t, s, x, y) + |(\tilde{p}_c \odot H^n_c) \odot (H - H_c)|(t, s, x, y) := i + ii).
\]
From (2.13) and Lemma 3.8, 
\[
ii) \leq C^{n+2} \left(\Delta_x^\infty\right)^{|\nu|/2} \frac{\Gamma^n(\gamma/2)}{\Gamma(1+n\gamma/2)} \int_t^s du \int_{\mathbb{R}^d} dz \frac{(u - t)^{1-\gamma/2}}{(s - u)^{1-\gamma/2}} \hat{p}(t, u, x, z)\hat{p}(u, s, z, y) =
\]
The aim of this section is to give an example of perturbations with $\Delta_{\sigma}^3$. Simulation study

Stirling’s formula.

Following form:

$$\int_{t}^{s} (s-u)^{\gamma-1} du \quad \text{for}\quad |\varepsilon| \text{small for}$$

First, here:

$$\int_{b}^{t,s} \leq C^2 \hat{p}(t,s,x,y) (\Delta_{\sigma}^\infty) \Gamma \left( \frac{\gamma}{2} \right) \Gamma \left( \frac{1}{2} \right) (s-t)^{\gamma/2}.$$

Hence,

$$|\hat{p} \otimes H - \hat{p}_s \otimes H|_{(t,s,x,y)} \leq 2C^2 \hat{p}(t,s,x,y) (\Delta_{\sigma}^\infty) \Gamma \left( \frac{\gamma}{2} \right) \Gamma \left( \frac{1}{2} \right) (s-t)^{\gamma/2}.$$

The induction on $n$ completes the proof.

Theorem 3.5 now readily follows from Lemmas 3.7, 3.9 combined with parametrix expansion (3.1) and Stirling’s formula.

3.3 Simulation study

The aim of this section is to give an example of perturbations with $\Delta_{\sigma}(t,s) \to 0$, $t < s$ when $\varepsilon \to 0$, but $|b(t,x) - b_{\varepsilon}(t,x)| \neq 0$ uniformly. The idea is to consider the case of highly oscillating perturbations when the integral is small for $\varepsilon \to 0$, but the difference $|b(t,x) - b_{\varepsilon}(t,x)|$ remains of a constant order in the neighborhood of the point $t = \sqrt{\varepsilon}$. We consider the simplest case $d = \gamma = 1$, $\Delta_{\varepsilon,\sigma}(t,s) \equiv 0$, and the drift perturbation of the following form:

$$b(t,x) = x,$$

$$b_{\varepsilon}(t,x) = b(t,x) + e^{\frac{t}{\sqrt{\varepsilon}}} \sin^2 \left( \frac{t}{\sqrt{\varepsilon}} \right) \cos(x).$$

Here, $q = \frac{p}{p-1}$, $2 < q < \infty$. Clearly, in this case, (A) holds true. Let us estimate $\Delta_{\varepsilon,b}(t,s)$ when $t < s$.

$$|b(t,x) - b_{\varepsilon}(t,x)|_1 \leq e^{-\frac{t}{\sqrt{\varepsilon}}} \sin^2 \left( \frac{t}{\sqrt{\varepsilon}} \right),$$

$$\Delta_{\varepsilon,b}(t,s) \leq \int_{t}^{s} \mathbb{B} \left( u; 1, \frac{1}{2} \right) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |b(u, \theta_{u,s}(y)) - b_{\varepsilon}(u, \theta_{u,s}(y))|_1 \times$$

$$\times \hat{p}(t,s,x,y) dy \, du \leq M \int_{t}^{s} \mathbb{B} \left( u; 1, \frac{1}{2} \right) e^{-\frac{u^2}{\sqrt{\varepsilon}}} \sin^2 \left( \frac{u}{\sqrt{\varepsilon}} \right) du \leq$$

$$\leq M \left( \int_{t}^{s} \mathbb{B} \left( u; 1, \frac{1}{2} \right) du \right)^{\frac{1}{p}} \left( \int_{t}^{s} e^{-\frac{u^2}{\sqrt{\varepsilon}}} \sin^2 \left( \frac{u}{\sqrt{\varepsilon}} \right) du \right)^{\frac{1}{q}} \frac{1}{p} + \frac{1}{q} = 1.$$

First,

$$\left( \int_{t}^{s} \mathbb{B} \left( u; 1, \frac{1}{2} \right) du \right)^{1/p} = \frac{1}{B \left( 1, \frac{1}{2} \right) (s-t)^{\frac{1}{2}}} \left( \int_{t}^{s} (s-u)^{\frac{1}{2}} du \right)^{1/p} =$$

$$= \frac{(s-t)^{1/2-1}}{B \left( 1, \frac{1}{2} \right) (1-\frac{1}{2})^{\frac{1}{2}}}.$$

Next, using the representation $\sin(x) = \frac{\exp(ix) - \exp(-ix)}{2i}$, we get:

$$\int_{0}^{\infty} e^{-x^2} \sin^2(x) dx = \int_{0}^{\infty} \left( e^{-x^2} - \frac{1}{4} e^{-(x-i)^2-1} - \frac{1}{4} e^{-(x+i)^2-1} \right) dx = \frac{\sqrt{\pi}}{4} \left( 1 - \frac{1}{e} \right).$$

As a consequence,

$$\left( \int_{t}^{s} e^{-\frac{u^2}{\sqrt{\varepsilon}}} \sin^2 \left( \frac{u}{\sqrt{\varepsilon}} \right) du \right)^{\frac{1}{q}} \leq \left( \int_{0}^{\infty} e^{-\frac{u^2}{\sqrt{\varepsilon}}} \sin^2 \left( \frac{u}{\sqrt{\varepsilon}} \right) du \right)^{\frac{1}{4}} =$$
By Theorem 1.1,
\[
\| (p - p_\varepsilon) (t, s, x, y) \|_{L^\mu_1(\mathbb{R}, L^1_1(\mathbb{R}))} \leq C \left(1 - \frac{\varepsilon}{2}\right)^{q \left(\delta - \frac{1}{2}\right)} \left(\varepsilon^{k\left(\delta - \frac{1}{2}\right)} + \varepsilon^{\left(\delta - \kappa\right)(1 - \frac{1}{p})(1 - \delta)}\right),
\]
where \(0 < \kappa < \frac{1}{2}, \frac{1}{2} < \delta < 1, 1 < p < 2\).

Equating the powers, we get for \(\delta = \frac{3}{4}, q = \frac{p - 1}{p} > 2\)
\[
\| (p - p_\varepsilon) (t, s, x, y) \|_{L^\mu_1(\mathbb{R}, L^1_1(\mathbb{R}))} \leq \frac{C}{\left(1 - \frac{\varepsilon}{2}\right)^{7/2} \left(\delta - \frac{1}{2}\right)} \left[\varepsilon^{(1 - \delta)\left(\delta - \frac{1}{2}\right)} \right]^{1/(1 + q)} = \frac{C}{\left(1 - \frac{\varepsilon}{2}\right)^{7/2}} \varepsilon^{1/8(1 + q)} = \frac{C}{\left(1 - \frac{\varepsilon}{2}\right)^{7/2}} \varepsilon^{\frac{1}{2^{4\rho}}} \varepsilon^{\frac{1}{2^{7/2}}}
\]
for any \(\rho > 0\) and an appropriate choice of \(q \to 2^+\).

Importantly, the perturbation does not tend to zero uniformly. Indeed, for \(\varepsilon \geq \frac{1}{2}\),
\[
|b_\varepsilon(t, x) - b(t, x)| \geq 2^{-\frac{7}{2}} \varepsilon^{-\frac{3}{2}} |\cos(x)|.
\]

The following numerical experiments demonstrate the behavior of the perturbation argument \(\Delta_{\varepsilon,b}(t, s)\) in the neighborhood of the point \(t = \sqrt{\varepsilon}\) for different \(\varepsilon\). We consider the time step \(\Delta t\), the constant \(q = 2.01\) and the probability measure \(\mu(\cdot) = \delta_1(\cdot)\), where \(\delta_\varepsilon(\cdot)\) is a Dirac delta measure centred on some fixed point \(x \in \mathbb{R}\).

Figure 1: \(\Delta_{\varepsilon,b}(t, s), 0 \leq t < s \leq 1\), where \(\Delta t = 0.1, \varepsilon = 1\).

Figure 2: \(\Delta_{\varepsilon,b}(t, s), 0 \leq t < s \leq 1\), where \(\Delta t = 0.1, \varepsilon = 0.5\).
4 Appendix

4.1 Proof of Lemma 3.1

Let us consider the case when $|\nu|=0$. Let us now identify the transition densities $\tilde{p}(t, s, x, y)$ and $\tilde{p}_\varepsilon(t, s, x, y)$ with $(d+1) \times d$ matrices $\Omega$ and $\Omega^\varepsilon$ consisting of the first rows that are the components of the respective mean vectors $\theta_{t,s}(y) = (\theta^1_{t,s}(y), \ldots, \theta^d_{t,s}(y))$ and $\theta^\varepsilon_{t,s}(y) = (\theta^1_{t,s}^\varepsilon(y), \ldots, \theta^d_{t,s}^\varepsilon(y))$, and the $d \times d$ covariance matrices, namely, $\Omega_{ij} = \int_s^t a_{ij}(u, \theta_{u,s}(y))du$, $\Omega_{ij}^\varepsilon = \int_s^t a_{ij}^\varepsilon(u, \theta_{u,s}^\varepsilon(y))du$, where $i > 1$.

For $(d+1) \times d$ matrix $A$, we will denote by $A_1$ the first row and by $A_{2:d+1}$ the square matrix composed of
rows from 2 to \( d + 1 \). We can rewrite \( \tilde{p}(t, s, x, y) \) and \( \tilde{p}_\varepsilon(t, s, x, y) \) in terms of \( \Omega \) and \( \Omega^\varepsilon \):

\[
\tilde{p}(t, s, x, y) = f(\Omega), \\
\tilde{p}_\varepsilon(t, s, x, y) = f(\Omega^\varepsilon),
\]

where

\[
f : \mathbb{R}^{(d+1)\times d} \to \mathbb{R},
A \mapsto f(A) = \frac{1}{(2\pi)^{d/2} \det(A_{2:d+1})^{1/2}} \exp \left( -\frac{1}{2} ((A_{2:d+1})^{-1}(A_1 - x), A_1 - x) \right).
\]

So, applying the Taylor expansion and (2.6) combined with (A4) gives:

\[
|\tilde{p}_\varepsilon - \tilde{p}(t, s, x, y)| = |f(\Omega) - f(\Omega_\varepsilon)| =
\]

\[
= \left| \sum_{|\nu|=1} (\Omega_\varepsilon - \Omega)^\nu \cdot \int_0^1 (1 - \lambda) \mathcal{D}^\nu f \{ \Omega + \lambda(\Omega_\varepsilon - \Omega) \} d\lambda \right|
\]

\[
= \left| \sum_{i=1}^d \left( \theta_{\varepsilon,i}(y) - \theta_{i,0}(y) \right) \cdot \int_0^1 (1 - \lambda) \mathcal{D}^\nu f \{ \Omega + \lambda(\Omega_\varepsilon - \Omega) \} d\lambda +
\]

\[
+ \sum_{i,j=1}^d \left( \int_t^s a_{ij}(u, \theta_{\varepsilon,i,s}(u)) du - \int_t^s a_{ij}(u, \theta_{i,0,s}(u)) du \right) \cdot \int_0^1 (1 - \lambda) \mathcal{D}^\nu f \{ \Omega + \lambda(\Omega_\varepsilon - \Omega) \} d\lambda \right| \leq
\]

\[
\leq \frac{C}{(s-t)^{(1+d)/2}} \exp \left( -\frac{|\theta_{\varepsilon,s}(y) + c(\theta_{\varepsilon,i,s}(y) - \theta_{i,0,s}(y)) - x|^2}{C(s-t)} \right) \| \theta_{\varepsilon,s}(y) - \theta_{i,0,s}(y) \| +
\]

\[
+ \frac{C}{(s-t)^{(1+d)/2}} \exp \left( -\frac{|\theta_{i,0,s}(y) + c(\theta_{\varepsilon,i,s}(y) - \theta_{i,0,s}(y)) - x|^2}{C(s-t)} \right) \left( \int_t^s |a(u, \theta_{\varepsilon,i,s}(u)) - a(u, \theta_{i,0,s}(u))| du \right) \leq
\]

\[
\leq \frac{C}{(s-t)} \tilde{p}(t, s, x, y) \exp \left( \frac{c |\theta_{\varepsilon,i,s}(y) - \theta_{i,0,s}(y)|^2}{C(s-t)} \right) \left( \int_t^s |(b - b_\varepsilon)(u, \theta_{\varepsilon,i,s}(u))| du \right) +
\]

\[
+ \frac{C}{(s-t)} \tilde{p}(t, s, x, y) \exp \left( \frac{c |\theta_{\varepsilon,i,s}(y) - \theta_{i,0,s}(y)|^2}{C(s-t)} \right) \left( s - t \right) \left( \int_t^s |(b - b_\varepsilon)(u, \theta_{\varepsilon,i,s}(u))| du \right) \gamma +
\]

\[
+ \int_t^s |(\sigma - \sigma_\varepsilon)(u, \theta_{\varepsilon,i,s}(u))| du \right| \leq
\]

\[
\leq C \tilde{p}(t, s, x, y) \left( \frac{\int_t^s |(b - b_\varepsilon)(u, \theta_{\varepsilon,i,s}(u))| du}{(s-t)^{\gamma/2}} \right) \left( \int_t^s |(\sigma - \sigma_\varepsilon)(u, \theta_{\varepsilon,i,s}(u))| du \right) \right) \frac{\gamma}{(s-t)^{\gamma}}.
\]

We will use a slightly different bound:

\[
|\tilde{p}_\varepsilon - \tilde{p}(t, s, x, y)| \leq
\]

\[
\leq C \tilde{p}(t, s, x, y) \left( \frac{\int_t^s |(b - b_\varepsilon)(u, \theta_{\varepsilon,i,s}(u))| du}{(s-t)^{\gamma/2}} \right) \left( \int_t^s |(\sigma - \sigma_\varepsilon)(u, \theta_{\varepsilon,i,s}(u))| du \right) \right) \frac{\gamma}{(s-t)^{\gamma}}.
\]

(4.1)

Now the upper bound on the difference in \( L_1 \) norm easily follows from (4.1) and Jensen’s inequality:

\[
\| (\tilde{p}_\varepsilon - \tilde{p})(t, s, x, y) \|_{L_1(A_t^\varepsilon)} \leq
\]

\[
\leq \frac{C}{(s-t)^{\gamma-\delta}} \int \tilde{p}(t, s, x, y) \cdot \varphi(t, s; y) dy dx \mu(dx) \leq
\]

19
\[ C \left[ \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^2 \mathfrak{M}(u; 1, \gamma/2) \left( (b - b_c)(u, \theta_{u,s}(y)) \right) \, dudy \, d\mu(dx) \right)^{\gamma - \delta} + \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^2 \mathfrak{M}(u; 1, \gamma/2) \left( (\sigma - \sigma_c)(u, \theta_{u,s}(y)) \right) \, dudy \, d\mu(dx) \right)^{\gamma - \delta} = C \left[ (\Delta_{\varepsilon,b})^{\gamma - \delta} + (\Delta_{\varepsilon,\sigma})^{\gamma - \delta} \right]. \]

The bounds for \(|\nu| \geq 1\) follow from differentiation of the Taylor expansion and similar bounds (2.6) for the derivatives of the Gaussian densities \(\bar{p}\) and \(\bar{p}_c\).

### 4.2 Proof of Lemma 3.2

In order to prove the Lemma 3.2, we decompose the difference \(|H - H_c|(t, s, x, y)\) into 6 parts in the following way:

\[ |H - H_c|(t, s, x, y) \leq \frac{1}{2} \sum_{i,j=1}^d |a_{ij}(t, x) - \alpha^*_j(t, x) - a_{ij}(t, \theta_{t,s}(y)) + \alpha^*_j(t, \theta_{t,s}(y))| \left| \frac{\partial^2 \tilde{p}(t, s, x, y)}{\partial x_i \partial x_j} \right| + \frac{1}{2} \sum_{i,j=1}^d |a_{ij}^*(t, \theta_{t,s}(y)) - a_{ij}^*(t, \theta_{t,s}(y))| \left| \frac{\partial^2 \tilde{p}(t, s, x, y)}{\partial x_i \partial x_j} \right| + \frac{1}{2} \sum_{i,j=1}^d |a_{ij}(t, x) - a_{ij}(t, \theta_{t,s}(y))| \left| \frac{\partial^2 (\tilde{p}(t, s, x, y) - \bar{p}_c(t, s, x, y))}{\partial x_i \partial x_j} \right| + \sum_{i=1}^d |b_i(t, x) - b_i(t, \theta_{t,s}(y))| \left| \frac{\partial \tilde{p}(t, s, x, y)}{\partial x_i} \right| + \sum_{i=1}^d |b_i^*(t, \theta_{t,s}(y)) - b_i^*(t, \theta_{t,s}(y))| \left| \frac{\partial \tilde{p}(t, s, x, y)}{\partial x_i} \right| + \sum_{i=1}^d |b_i^*(t, \theta_{t,s}(y)) - b_i^*(t, \theta_{t,s}(y))| \left| \frac{\partial \tilde{p}(t, s, x, y)}{\partial x_i} \right| := i) + ii) + iii) + iv) + v) + vi). \]

For ii) and v), by regularity assumptions and (2.6), we have

\[ ii) \leq \frac{C}{(s-t)^{1/2}} \bar{p}(t, s, x, y) \left| \theta_{t,s}(y) - \theta_{t,s}(y) \right| \right\} \leq \frac{C}{(s-t)^{1/2}} \bar{p}(t, s, x, y) \left( \int_t^s \left| (b - b^*(u, \theta_{u,s}(y))) \right| \, du \right) \right\} \leq \frac{C}{(s-t)^{1/2}} \bar{p}(t, s, x, y) \left( \int_t^s \left| (b - b^*(u, \theta_{u,s}(y))) \right| \, du \right). \]

To obtain an upper bound for i), we first estimate the first part of the corresponding expression:

\[ |a(t, x) - \alpha^*(t, x) - a(t, \theta_{t,s}(y)) + a^*(t, \theta_{t,s}(y))| = \]

\[ = |\sigma \sigma^*(t, x) - \sigma^*(\sigma^*)^*(t, x) - \sigma \sigma^*(t, \theta_{t,s}(y)) + \sigma^*(\sigma^*)^*(t, \theta_{t,s}(y))| \leq \left| |\sigma(t, x) - \sigma^*(t, x) - (\sigma(t, \theta_{t,s}(y)) - \sigma^*(t, \theta_{t,s}(y)))| \sigma^*(t, x) + \right| \sigma(t, x) - \sigma^*(t, x) - (\theta(t, \theta_{t,s}(y)) - \sigma^*(t, \theta_{t,s}(y)))| (\sigma^*)^*(t, x) + \right| \sigma^*(t, x) - (\sigma^*)^*(t, x) - (\sigma^*(t, \theta_{t,s}(y)) + (\sigma^*)^*(t, \theta_{t,s}(y)))| (\sigma^*)^*(t, x) + + |\sigma(t, \theta_{t,s}(y)) - \sigma^*(t, \theta_{t,s}(y))| (\sigma^*)^*(t, x) + \]
\[ + |\sigma^z(t, x) - \sigma^z(t, \theta_{t,s}(y))|\, |\sigma^z(t, x) - (\sigma^z)^* (t, x)| \leq \]
\[ \leq C \left( |x - \theta_{t,s}(y)|^\gamma + |x - \theta_{t,s}(y)|^{2\gamma} \right) |(\sigma - \sigma^z) (t, \theta_{t,s}(y))|_1. \]

Combining the estimate above and Lemma 2.6 with the semigroup property of the flows, we have for \( i \) and \( iv \)
\[ i) \leq \frac{C}{s-t} \bar{p}(t, s, x, y) |a(t, x) - a^*(t, x) - a(t, \theta_{t,s}(y)) + a^*(t, \theta_{t,s}(y))| \leq \]
\[ \leq \frac{C}{s-t} \bar{p}(t, s, x, y) \left( |x - \theta_{t,s}(y)|^\gamma + |x - \theta_{t,s}(y)|^{2\gamma} \right) |(\sigma - \sigma^z) (t, \theta_{t,s}(y))|_1 \leq \frac{C}{(s-t)^{1-\gamma/2}} |(\sigma - \sigma^z) (t, \theta_{t,s}(y))|_1. \]

Finally, using the control Lemma 3.1 for \( iii \) and \( vi \), we get
\[ iii) \leq C \bar{p}(t, s, x, y) \left( \frac{\int_t^s |(b-b^z) (u, \theta_{u,s}(y))| \, du}{(s-t)^{1-\gamma/2}} \right)^{\gamma/2} + \left( \frac{\int_t^s |(\sigma - \sigma^z) (u, \theta_{u,s}(y))| \, du}{(s-t)^{1+\gamma/2}} \right)^{\gamma/2}. \]
\[ vi) \leq \left( \frac{\int_t^s |(b-b^z) (u, \theta_{u,s}(y))| \, du}{(s-t)^{\gamma/2}} \right)^{\gamma/2} + \left( \frac{\int_t^s |(\sigma - \sigma^z) (u, \theta_{u,s}(y))| \, du}{(s-t)^{\gamma/2}} \right)^{\gamma/2}. \]

Summing up the bounds derived above, we complete the proof.

References

[1] Anceschi F., Muzzioli S., Polidoro S. Existence of a Fundamental Solution of Partial Differential Equations associated to Asian Options. arXiv:2007.09037v1, 2020.

[2] Aronson, D. G. The fundamental solution of a linear parabolic equation containing a small parameter. Ill. Journ. Math. 3 (1959), 580 – 619.

[3] Aronson, D. G. Bounds for the fundamental solution of a parabolic equation. Bull. Amer. Math. Soc. 73 (1967), 890–896.

[4] Bass, R. F. Diffusions and Elliptic Operators. Springer, 1997.

[5] O. Bencheikh and B. Jourdain. Convergence in total variation of the Euler-Maruyama scheme applied to diffusion processes with measurable drift coefficient and additive noise. arXiv:2005.09354v1, 2020.

[6] E. Benhamou, E. Gobet and M. Miri, Expansion formulas for European options in a local volatility model. Inter. J. Theor. Appl. Finance 13 (2010) 602–634.

[7] V. Bogachev, M. Röckner, S. Shaposhnikov. Distances between transition probabilities of diffusions and applications to nonlinear Fokker–Planck–Kolmogorov equations. J. Funct. Anal. 271, (2016), 1262-1300.

[8] V. Bogachev, M. Röckner, S. Shaposhnikov. The Poisson equation and estimates for distances between stationary distributions of diffusions. J. of Math. Sciences, v. 232, 3, 2018, 254-282.

[9] F. Corielli, P. Foschi and A. Pascucci, Parametrix approximation of diffusion transition densities. SIAM J. Financial Math. 1 (2010) 833–867.

[10] J.-M. Coron. Control and nonlinearity. Mathematical Surveys and Monographs, 136, AMS, 2007.

[11] T. Deck, S. Kruse, Parabolic differential equations with Holder continuous and unbounded coefficients. Acta Applicandae Mathematicae, vol. 74, 1 (2002), 71-91.
[12] Delarue, F., and Menozzi, S. Density estimates for a random noise propagating through a chain of differential equations. J. Funct. Anal. 259, 6 (2010), 1577–1630.

[13] Di Francesco, M., and Pascucci, A. On a class of degenerate parabolic equations of Kolmogorov type. AMRX Appl. Math. Res. Express 3 (2005), 77–116.

[14] A. Friedman, Partial Differential Equations of Parabolic Type. Prentice-Hall (1964).

[15] A.M. Il’ in, A.S. Kalashnikov and O.A. Oleinik, Second-order linear equations of parabolic type. Uspehi Mat. Nauk 17 (1962) 8 3–146.

[16] V. Konakov, S. Menozzi and S. Molchanov, Explicit parametrix and local limit theorems for some degenerate diffusion processes. Annales de l’Institut Henri Poincare, Serie B 46 (2010) 908–923.

[17] V. Konakov, A. Kozhina and S. Menozzi, Stability of densities for perturbed diffusions and Markov chains. ESAIM: Probability and Statistics, v.21, (2017), 88-112.

[18] V. Konakov and S. Menozzi, “Weak error for the Euler scheme approximation of diffusions with non-smooth coefficients,” Electron. J. Probab. 22 Paper No. 46 (2017).

[19] H.P. McKean and I.M. Singer, Curvature and the eigenvalues of the Laplacian. J. Differ. Geometry 1 (1967) 43–69.

[20] S. Menozzi, A. Pesce, X. Zhang, Density and gradient estimates for non-degenerate Brownian SDEs with unbounded measurable drift, arXiv:2006.07158v1, 2020.

[21] Polidoro, S. On a class of ultra-parabolic operators of Kolmogorov-Fokker-Planck type. Matematicae (Catania) 49, 1 (1994), 53–105.

[22] Sheu, S. J. Some estimates of the transition density of a nondegenerate diffusion Markov process. Ann. Probab. 19, 2 (1991), 538–561.

[23] D.W. Stroock and S.R.S. Varadhan, Multidimensional diffusion processes. Springer-Verlag Berlin Heidelberg New York (1979).