Integrable systems associated to the filtrations of Lie algebras

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Abstract. In 1983 Bogoyavlenski conjectured that if the Euler equations on a Lie algebra $g_0$ are integrable, then their certain extensions to semisimple lie algebras $g$ related to the filtrations of Lie algebras $g_0 \subset g_1 \subset g_2 \cdots \subset g_{n-1} \subset g_n = g$ are integrable as well. In particular, by taking $g_0 = \{0\}$ and natural filtrations of $\mathfrak{so}(n)$ and $\mathfrak{u}(n)$, we have Gel’fand-Cetlin integrable systems. We proved the conjecture for filtrations of compact Lie algebras $g$: the system are integrable in a noncommutative sense by means of polynomial integrals. Various constructions of complete commutative polynomial integrals for the system are also given.

1. Introduction

Let $G$ be a compact connected Lie group. Consider a chain of connected compact Lie subgroups

$$G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_{n-1} \subset G_n = G$$

and the corresponding filtration of the Lie algebra $g = \text{Lie}(G)$

(1)

$$g_0 \subset g_1 \subset g_2 \cdots \subset g_{n-1} \subset g_n = g.$$

We study integrable Euler equations related to the filtration (1). One can consider non compact Lie algebras as well. In fact, one of the first contributions is given by Trofimov, who constructed integrable systems on Borel subgroups of complex semisimple Lie algebras (see [36]). Later, Bogoyavlenski [2] considered filtration of semisimple Lie algebras, such that the restrictions of the Killing form to $g_i$, $i = 0, \ldots, n$ are non-degenerate. We restrict ourself to the compact case and a generalization of Gel’fand-Cetlin systems on Lie algebras $\mathfrak{so}(n)$ and $\mathfrak{u}(n)$ given by filtrations (9) and (10) below in order to insure compact invariant manifolds of the flows. Similar statements can be formulated for reductive groups as well.

Fix an invariant scalar product $\langle \cdot, \cdot \rangle$ on $g$ and denote the restrictions of $\langle \cdot, \cdot \rangle$ to $g_i$, $i = 0, \ldots, n$ by $\langle \cdot, \cdot \rangle_i$. By the use of $\langle \cdot, \cdot \rangle$, we identify $g \cong g^*$ and $g_i \cong g_i^*$, $i = 0, \ldots, n$. Let $p_i$ be the orthogonal complement of $g_{i-1}$ in $g_i$, $p_0 = g_0$ and $\text{pr}_{p_i}$ and $\text{pr}_{g_i}$ the orthogonal projections onto $p_i$ and $g_i$, respectively. For $x \in g$, we denote

$$y_i = \text{pr}_{p_i}(x), \quad x_i = y_0 + y_1 + \cdots + y_i = \text{pr}_{g_i}(x), \quad i = 0, \ldots, n.$$

The Euler equations

(2)

$$\dot{x} = [x, \omega], \quad \omega = A(x)$$

associated with a symmetric positive operator of the form

(3)

$$A(x) = A_0(g_0) + \sum_{i=1}^{n} s_i y_i, \quad A_0 : g_0 \to g_0, \quad s_i \in \mathbb{R}, \quad i = 1, \ldots, n$$
were studied by Bogoyavlenski [2]. The equations are Hamiltonian with respect to the Lie–Poisson bracket
\begin{equation}
\{f,g\}_x = -\langle x, [\nabla f(x), \nabla g(x)] \rangle
\end{equation}
and the Hamiltonian function \( H(x) = \frac{1}{2}\langle x, \omega \rangle = \frac{1}{2}\langle A(x), x \rangle \).

Due to the relations
\[ [\mathfrak{p}_i, \mathfrak{p}_j] \subset \mathfrak{p}_j, \quad 0 \leq i < j \leq n, \]
the Euler equations (2) can be rewritten into the form
\begin{equation}
\dot{y}_i = [y_0, A_0(y_0)],
\end{equation}
\begin{equation}
\dot{\tilde{y}}_i = [\tilde{y}_i, A_0(y_0) - s_i y_0 + (s_1 - s_i) y_1 + \cdots + (s_{i-1} - s_i) y_{i-1}], \quad i = 1, \ldots, n.
\end{equation}

Specially, if \( \mathfrak{g}_0 = \{0\} \) is a trivial Lie algebra, we have \( y_1 = \text{const} \) and the components of \( y_2 \) are elementary functions of the time \( t \).

The system (5), (6) has obvious family of polynomial first integrals
\begin{equation}
\mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2 + \cdots + \mathcal{I}_n,
\end{equation}
where \( \mathcal{I}_i \) are invariants \( \mathbb{R}[\mathfrak{g}_i]^{G_i} \) lifted to \( \mathfrak{g} \) along the projection to \( \mathfrak{g}_i \): \( \mathcal{I}_i = \text{pr}_{\mathfrak{g}}^{\ast} \mathbb{R}[\mathfrak{g}_i]^{G_i}, \quad i = 1, \ldots, n \). According to the following (quite simple, but important) lemma, it is clear that \( \mathcal{I} \) is a commutative algebra with respect to the Lie-Poisson bracket (4).

**Lemma 1 ([2, 35, 36]).** If \( f \) and \( g \) Lie–Poisson commute on \( \mathfrak{g}_i \), then their lifts \( \tilde{f} = \text{pr}_{\mathfrak{g}}^{\ast} f \) and \( \tilde{g} = \text{pr}_{\mathfrak{g}}^{\ast} g \) Lie–Poisson commute on \( \mathfrak{g} \).

Bogoyavlenski found also a large class of additional first integrals obtained by the translations of invariants of \( \mathfrak{g}_i \) along the subalgebra \( \mathfrak{g}_{i-1} \)
\begin{equation}
p_{\alpha, \beta}(x) = p(\alpha x_i + \beta y_i), \quad p \in \mathbb{R}[\mathfrak{g}_i]^{G_i}, \quad i = 1, \ldots, n, \quad \alpha, \beta \in \mathbb{R}
\end{equation}
and conjectured that the equations (5), (6) are completely integrable if this is true for the Euler equations (5) (see [2]). In [25], Mikityuk proved that Bogoyavlenski’s integrals (8) imply complete commutative integrability in the case when \( (\mathfrak{g}_i, \mathfrak{g}_{i-1}) \) are symmetric pairs, that is when
\[ [\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{g}_{i-1}, \quad i = 1, \ldots, n. \]

On the other hand, Thimm used chains of subalgebras (1) in studying the integrability of geodesic flows on homogeneous spaces (see [35]). He proved that integrals (7) form a complete commutative algebras on the Lie algebras \( \mathfrak{so}(n) \) and \( \mathfrak{u}(n) \), with respect to the natural filtrations
\begin{equation}
\mathfrak{so}(2) \subset \mathfrak{so}(3) \subset \cdots \subset \mathfrak{so}(n-1) \subset \mathfrak{so}(n)
\end{equation}
and
\begin{equation}
\mathfrak{u}(1) \subset \mathfrak{u}(2) \subset \cdots \subset \mathfrak{u}(n-1) \subset \mathfrak{u}(n),
\end{equation}
respectively.

After [15], the corresponding integrable systems are referred as Gelfand-Cetlin systems on \( \mathfrak{so}(n) \) and \( \mathfrak{u}(n) \). Namely, Gel’fand and Cetlin constructed canonical bases for a finite-dimensional representation of the orthogonal and unitary groups by the decomposition of the representation by a chain of subgroups [12, 13]. The corresponding integrable systems on the adjoint orbits with integrals \( \mathcal{I} \) can be seen as a symplectic geometric version of the Gelfand-Cetlin construction [15]. Also, Thimm’s examples motivated Guillemin and Sternberg to introduce an important notion of multiplicity free Hamiltonain actions [14, 16].

The construction is also used in the study of integrability of geodesic flows on homogeneous

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1The gradient is determined by an invariant metric: \( df(\xi) = \langle \nabla f(x), \xi \rangle \). Also, to simplify notation, the Lie brackets, the Lie-Poisson brackets and the gradients of the functions on \( \mathfrak{g}_i \) will be denoted by the same symbols as on \( \mathfrak{g}, i = 1, \ldots, n \).
spaces and bi-quotients of Lie groups (see [1, 5, 7, 35]). The nonholonomic systems on compact Lie groups G with left invariant metrics defined by the Hamiltonians of the form $H = \frac{1}{2} \langle A(x), x \rangle$, where $A$ is given by (3), and left-invariant constraints are studied in [19].

In this paper we prove complete integrability in a noncommutative sense of the system (5), (6) and calculate the dimension of invariant isotropic tori $\delta$ (Theorem 2, Section 3):

**Main result 1.** Assume that the Euler equations (5) are integrable by polynomial integrals. Then the Euler equations (5), (6) are completely integrable in a noncommutative sense by means of polynomial integrals as well.

Concerning dynamics, noncommutative (or superintegrability) is a stronger characteristic than the usual commutative (or Liouville) integrability. The system is solvable by quadratures, regular compact invariant manifolds are isotropic tori, and there exist an appropriate action–angle coordinates in which the dynamics is linearized [29, 30]. It implies the Liouville integrability, at least by means of smooth functions: invariant isotropic tori can be always organized into resonant Lagrangian tori [6].

According to the Mishchenko-Fomenko conjecture, a natural algebraic problem is a construction of a complete commutative set of polynomial integrals. The problem can be formulated in terms of pairs $(G_{i-1}, G_i)$: we need to construct

$$b(g_i, g_{i-1}) = \frac{1}{2} (\dim p_i + \text{rank } g_i - \text{rank } g_{i-1})$$

mutually independent commuting $Ad_{G_{i-1}}$-invariants on $g_i$ that are independent from the polynomials on $g_{i-1}$ (Corollary 4, Theorem 5, Section 4). The basic examples are pairs $(G_{i-1}, G_i)$ where $G_{i-1}$ is a multiplicity free or an almost multiplicity free subgroup of $G_i$ (see Examples 1 and 2, Section 4).

In Section 5 we adopt various constructions of commutative polynomials on $g_i$ to provide complete commutative polynomial integrals for the system (5), (6). When $G_{i-1}$ is an isotropy subgroup of $a_i \in g_i$, one can use the Mishchenko-Fomenko shifting of argument method in both cases: when $a_i$ is regular (see [3, 28]), but also a singular element of $g_i$ (see Theorem 8). Further, as an example of a variation of Mikityuk’s construction and a complement to (9) and (10), a complete commutative sets of polynomials for the filtrations

$$\mathfrak{sp}(k_0) \subset \mathfrak{sp}(k_1) \subset \cdots \subset \mathfrak{sp}(k_n), \quad k_0 < k_1 < \cdots < k_n$$

are given (Proposition 10, see also [17]).

Also, in subsection 5.3 we estimate the number of independent Bogoyavlenski’s integrals (8) (Theorem 12).

**Main result 2.** Under certain generic assumptions, among Bogoyavlenski’s integrals (8) there are at least $b(g_i, g_{i-1})$ mutually independent polynomials (see (11)) that are independent from the polynomials on $g_{i-1}$.

Although the polynomials (8) do not commute in general, we provide examples of complete commutative sets of integrals obtained by Bogoyavlenski’s method (Example 5). Finally, in Theorems 13, 15, 16 we recall the results obtained in [3, 9, 20, 33], which solve the problem in several interesting cases.

For the completeness of the exposition, we begin the presentation by briefly recalling on the concept of noncommutative integrability.

### 2. Complete algebras of functions on Poisson manifolds

Let $(M, \Lambda)$ be a Poisson manifold. The Poisson bracket of two smooth functions is defined by the use of the Poisson tensor $\Lambda$ as usual $\{f, g\} = \Lambda_x(df(x), dg(x))$, giving the Lie algebra structure to $C^\infty(M)$. Let $x = (x_1, \ldots, x_n)$ be local coordinates on $M$. Let $f$ and $g$ be the first integrals of the Hamiltonian equations with the Hamiltonian $H$:

$$\dot{x} = X_H = \sum_i \{x_i, H\} \frac{\partial}{\partial x_i} = \sum_{i,j} \Lambda_{ij} \frac{\partial H}{\partial x_j} \frac{\partial}{\partial x_i},$$

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that is \( \{ f, H \} = \{ g, H \} = 0 \). Then, due to the Jacobi identity, their Poisson bracket \( \{ f, g \} \) is also the first integral: \( \{ \{ f, g \}, H \} = 0 \).

Therefore we can consider the Lie algebra \( \mathcal{F} \) of first integrals. Let \( F_x = \{ df \mid f \in \mathcal{F} \} \) be a subspace of \( T^*_x M \) spanned by differentials of the functions from \( \mathcal{F} \) at \( x \in M \). Assume that the dimensions of \( F_x \) and \( \dim \ker \Lambda_x |_{F_x} \) are constant on an open dense set \( U \) of \( M \). The corresponding dimensions are denoted by \( \text{ddim} \mathcal{F} \) (differential dimension of \( \mathcal{F} \)) and \( \text{dind} \mathcal{F} \) (differential index of \( \mathcal{F} \)), respectively.

**Remark 1.** The numbers \( l = \text{ddim} \mathcal{F} \) and \( k = \text{dind} \mathcal{F} \) have a clear geometrical meaning. Let \( y_1, y_2, \ldots, y_l \in \mathcal{F} \) be independent functions within a domain \( V \subset U \), such that

\[
\mathbf{F} : V \to W \subset \mathbb{R}^l, \quad \mathbf{F}(x) = (y_1(x), \ldots, y_l(x))
\]

is a submersion and

\[
\{ y_i, y_j \} = a_{ij}(y_1, \ldots, y_l),
\]

where \( a_{ij} \) are smooth functions on \( W \). Then (13) defines a Poisson structure on \( W \) of a constant corank \( k \).

\( \mathcal{F} \) is a complete algebra (or a complete algebra at \( x, x \in U \)) if

\[
\text{ddim} \mathcal{F} + \text{dind} \mathcal{F} = \dim M + \text{corank} \Lambda,
\]

or, equivalently, if

\[
F^\Lambda_x = \{ \xi \in T^*_x M \mid \Lambda_x(F_x, \xi) = 0 \} \subset F_x, \quad x \in U.
\]

Similarly, if \( \mathcal{F} \) is an arbitrary Lie subalgebra of \( C^\infty(M) \) and \( \mathcal{F}_0 \subset \mathcal{F} \) its subalgebra, we say that \( \mathcal{F}_0 \) is a complete subalgebra of \( \mathcal{F} \) if

\[
\text{ddim} \mathcal{F}_0 + \text{dind} \mathcal{F}_0 = \text{ddim} \mathcal{F} + \text{dind} \mathcal{F}.
\]

In particular, \( \mathcal{F} \) is a complete algebra on \( M \) if and only if it is a complete subalgebra of \( C^\infty(M) \). If \( \mathcal{F} \) is a complete algebra on \( M \) then \( \mathcal{F}|_S \) (the restrictions of the functions to \( S \)) will be a complete algebra on a generic symplectic leaf \( \mathcal{S} \subset M \). Specially, we may be interested in the competentness of \( \mathcal{F} \) not on \( M \) but on a particular, regular or singular, symplectic leaf \( \mathcal{S}_0 \) (see Lemma 3 below). Then the condition (14) is slightly changed: \( \mathcal{F}|_{\mathcal{S}_0} \) is complete

\[
\text{ddim} \mathcal{F}|_{\mathcal{S}_0} + \text{dind} \mathcal{F}|_{\mathcal{S}_0} = \dim \mathcal{S}_0,
\]

if \( F^\Lambda_x = \{ \xi \in T^*_x M \mid \Lambda_x(F_x, \xi) = 0 \} \subset F_x + \ker \Lambda_x \), for a generic \( x \in \mathcal{S}_0 \).

The Hamiltonian system (12) is called completely integrable in a noncommutative sense (or superintegrable), if it has a complete algebra \( \mathcal{F} \) of first integrals. The integrable system is solvable by quadratures (at least locally), the regular compact connected components of the level sets determined by functions in \( \mathcal{F} \) are \( \delta \)-dimensional (isotropic considered on the symplectic leaves of \( \Lambda \)) tori (\( \delta = \dim M - \text{ddim} \mathcal{F} = \text{dind} \mathcal{F} - \text{corank} \Lambda \)), and there exist an appropriate action–angle coordinates in which the dynamics is linearized \([29, 30]\).

When \( \mathcal{F} \) is a complete commutative algebra,

\[
\text{ddim} \mathcal{F} = \text{dind} \mathcal{F} = a(M) = \frac{1}{2}(\dim M + \text{corank} \Lambda),
\]

we have the usual Liouville integrability -- regular compact invariant level sets of the functions from \( \mathcal{F} \) are \( \delta_0 = \dim M - a(M) \) dimensional (Lagrangian on the symplectic leaves) tori.

**Remark 2.** Note that \( a(M) \) is the maximal number of Poisson commuting independent functions on \( M \). For an arbitrary \( \mathcal{F} \) we have the inequality

\[
\text{ddim} \mathcal{F} + \text{dind} \mathcal{F} \leq 2a(M),
\]

(or, concerning Remark 1, \( a(W) \leq a(M) \)) and equality holds if and only if \( \mathcal{F} \) is a complete algebra.
Mishchenko and Fomenko stated the conjecture that noncommutative integrability implies the Liouville integrability by means of an algebra of integrals that belong to the same functional class as the original one [29]. Note that in the case of noncommutative integrability trajectories of (12) belong to the tori of dimension \( \delta < \delta_0 = \dim M - \alpha(M) \), that is, \( \delta_0 \)-dimensional Lagrangian invariant tori are resonant: they are filled with \( (\delta_0 - \delta) \)-parametric family of \( \delta \)-dimensional invariant tori. In a smooth category the problem is easy to solve: we can always semi–locally reorganize isotropic toric foliation into the Lagrangian toric foliation (see [6]2). The polynomial Mishchenko-Fomenko conjecture says that one can find independent commuting functions \( p_1, \ldots, p_n \), that are polynomials in functions from \( \mathcal{F} \). The conjecture is solved for finite dimensional Lie algebras \( \mathcal{F} \) (see [4, 34]). From a point of view of the dynamics, noncommutative integrability is stronger then the Liouville one since Lagrangian tori are resonant and not an intrinsic property of the system.

3. Polynomial noncommutative integrability

Suppose that the Euler equations (5) are completely integrable by means of a complete algebra \( \mathfrak{A}_0 \) of polynomial integrals,

\[
dim \mathfrak{A}_0 + \dim \mathfrak{A}_0 = \dim \mathfrak{g}_0 + \text{rank} \mathfrak{g}_0,
\]

and let \( \delta_0 = \dim \mathfrak{A}_0 - \text{rank} \mathfrak{g}_0 \) be the dimension of generic invariant tori.

Let \( \mathfrak{A}_i \) be the algebra \( \mathbb{R}[\mathfrak{g}_i]^{\mathfrak{g}_i} \) of \( \mathfrak{g}_i \)-invariant polynomials on \( \mathfrak{g}_i \). Consider the lift of algebras \( \mathfrak{A}_i \) to the Lie algebra \( \mathfrak{g} \):

\[
\mathfrak{A}_i = \text{pr}_{\mathfrak{g}}^* \mathfrak{A}_i, \quad i = 0, \ldots, n.
\]

In particular, since the invariants on \( \mathfrak{g}_i \) belong to \( \mathfrak{A}_i \), we have \( \mathcal{I}_i \subset \mathfrak{A}_i \).

**Theorem 2.** The system (5), (6) is completely integrable with a complete set of polynomial integrals

\[
\mathcal{A} = \mathfrak{A}_0 + \mathfrak{A}_1 + \cdots + \mathfrak{A}_n.
\]

A generic motion is a quasiperiodic winding over \( \delta = \delta_0 + \text{rank} \mathfrak{g}_0 - \text{rank} \mathfrak{g} + \sum_{i=1}^n \dim \text{pr}_{\mathfrak{g}_i}(\mathfrak{g}_i(x_i)) \)

dimensional invariant tori determined by the integrals \( \mathcal{A} \). Here we take generic elements \( x_i \in \mathfrak{g}_i, i = 1, \ldots, n \).

Recall that for a generic \( x_i \in \mathfrak{g}_i \), \( \mathfrak{g}_i(x_i) \) is a Cartan subalgebra in \( \mathfrak{g} \) that is spanned by the gradients of rank \( \mathfrak{g}_0 \) basic invariant polynomials in \( \mathbb{R}[\mathfrak{g}_i]^{\mathfrak{g}_i} \), which also coincides with the kernel of the Lie-Poisson structure of \( \mathfrak{g}_i \) at \( x_i \).

**Proof.** First, note that polynomials \( \mathfrak{A}_i \) \((i > 0)\) are indeed integrals of the equations. The polynomial \( p \) belongs to \( \mathfrak{A}_i, i = 1, \ldots, n \), if and only if

\[
\langle \nabla p(x_i), [\xi, x_i] \rangle = 0, \quad \text{for all } \xi \in \mathfrak{g}_{i-1}, x_i \in \mathfrak{g}_i,
\]

which is equivalent to the Poisson commuting of \( p \) with the lifting of all polynomials from \( \mathbb{R}[\mathfrak{g}_{i-1}] \) to \( \mathbb{R}[\mathfrak{g}_i] \).

Let \( \tilde{p} = \text{pr}_{\mathfrak{g}}^* p \). Then

\[
\frac{d}{dt} \tilde{p}(x) = \langle \nabla p(x_i), \dot{x}_i \rangle = \langle \nabla p(x_i), [x_i, A(x_i)] \rangle = \langle \nabla p(x_i), [x_i, A(x_{i-1}) - s_i x_{i-1}] \rangle = 0.
\]

In [6] the symplectic case is considered, but the proof can be easily modified to the Poisson case.

\( \mathfrak{g}_l(x_k) \) denotes the isotropy algebra of \( x_k \in \mathfrak{g}_k \) within \( \mathfrak{g}_l \):

\[
\mathfrak{g}_l(x_k) = \{ \xi \in \mathfrak{g}_l \mid [\xi, x_k] = 0 \}, \quad l \leq k.
\]

Generic means that the dimensions of the isotropy algebras \( \mathfrak{g}_i(x_i) \) and \( \mathfrak{g}_{i-1}(x_i) \) are minimal.
Let $O_i \subset g_i$ be a generic $G_i$-adjoint orbit. With a sign convention (4), the momentum mapping of $G_i$ adjoint action is simply the inclusion mapping $i : O_i \hookrightarrow g_i$ multiplied by $-1$, while the momentum mapping of the adjoint action of $G_{i-1}$ is 

$$\Phi_{i-1} : O_i \rightarrow g_{i-1}, \quad \Phi_{i-1} = - pr_{g_{i-1}} \circ i.$$ 

According to the Lemma 3 on integrability related to Hamiltonian actions (see below), the algebra $C^\infty_{G_{i-1}} (O_i) + \Phi_{i-1}^*(\mathbb{R}[g_{i-1}])$ is a complete algebra on $O_i$, where $C^\infty_{G_{i-1}} (O_i)$ is the algebra of smooth $G_{i-1}$-invariant functions. Since generic orbit of $Ad_{G_{i-1}}$-action on $g_i$ are separated by invariant polynomials, we have that $\mathfrak{A}_i | O_i + \Phi_{i-1}^*(\mathbb{R}[g_{i-1}])$ is a complete polynomial algebra on $O_i$. Therefore

(16)

$$\mathfrak{A}_i + pr_{i-1}^* (\mathbb{R}[g_{i-1}])$$

is a complete algebra on $g_i$ (recall that the invariants $\mathbb{R}[g_i]^{G_i}$ are contained in $\mathfrak{A}_i = \mathbb{R}[g_i]^{G_{i-1}}$). Next, by induction using the item (ii) of Lemma 3, we get that $\mathfrak{A}$ is a complete algebra of integrals on $g$.

In order to determine the dimension of invariant tori, note that the dimension of invariant tori determined by the functions $\mathfrak{A}_i | O_i + \Phi_{i-1}^*(\mathbb{R}[g_{i-1}])$ on $O_i$, equals (Lemma 3):

$$\delta_i = \text{dind} (\mathfrak{A}_i | O_i + \Phi_{i-1}^*(\mathbb{R}[g_{i-1}])) = \text{dind} (\mathfrak{A}_i | O_i) = \text{dind} (\Phi_{i-1}^*(\mathbb{R}[g_{i-1}]))$$

$$= \dim g_{i-1}(x_i) - \dim g_{i-1}(x_{i-1})$$

$$= \text{rank } g_{i-1} - \text{rank } g_i + \text{pr}_{p_i}(g_{i}(x_i)).$$

Here $x_i \in O_i$ is a generic element and $x_{i-1} = \text{pr}_{g_{i-1}}(x_i)$.

Again, by induction using the item (ii) of Lemma 3, we get the dimension of invariant tori:

$$\delta = \delta_0 + \delta_1 + \cdots + \delta_n$$

$$= \delta_0 + \text{rank } g_0 - \text{rank } g_1 + \text{pr}_{p_1}(g_1(x_1)) + \text{rank } g_2 + \text{pr}_{p_2}(g_2(x_2))$$

$$+ \cdots + \text{rank } g_{n-1} - \text{rank } g_n + \text{pr}_{p_n}(g_n(x_n))$$

$$= \delta_0 + \text{rank } g_0 - \text{rank } g_n + \text{pr}_{p_1}(g_1(x_1)) + \cdots + \text{pr}_{p_n}(g_n(x_n)),$$

for a generic $x_i \in g_i$, $i = 1, \ldots, n$. $\square$

Here we recall on the construction of collective integrable systems. Consider the Hamiltonian action of a compact connected Lie group $K$ on the symplectic manifold $M$ with the equivariant momentum mapping $\Phi : M \rightarrow \mathfrak{k}^* \cong \mathfrak{k}$.

Let $C^\infty_K (M)$ be the algebra of $K$-invariant functions. According to the Noether theorem, \{f, \bar{p}\}_M = 0 for all $f \in C^\infty_K (M)$, $\bar{p} = \Phi \circ p$, $p \in \mathbb{R}[\mathfrak{k}]$. Also, if $p \in \mathbb{R}[\mathfrak{p}]^K$ is an invariant polynomial, then $\bar{p}$ is $K$-invariant. In [6] we proved the following quite simple but important statement.

**Lemma 3** ([6]). (i) The algebra of functions $C^\infty_K (M) + \Phi^* (\mathbb{R}[\mathfrak{k}])$ is complete on $M$:

$$\text{dind} (C^\infty_K (M) + \Phi^* (\mathbb{R}[\mathfrak{k}])) + \text{dind} (C^\infty_K (M) + \Phi^* (\mathbb{R}[\mathfrak{k}]))) = \dim M$$

and the dimension of invariant regular isotropic tori is

$$\delta_0 = \text{dind} (C^\infty_K (M) + \Phi^* (\mathbb{R}[\mathfrak{k}])) = \dim C^\infty_K (M) = \dim \Phi^* (\mathbb{R}[\mathfrak{k}]) = \dim K_\mu - \dim K_x,$$

where $K_x$ and $K_\mu$ are isotropic subgroups of a generic $x \in M$ and $\mu = \Phi (x) \in \mathfrak{k}$.

(ii) Let $\mathfrak{f} \subset \mathbb{R}[\mathfrak{k}]$ be complete on a generic adjoint orbit in the image of $M$,

$$\text{dind} (\mathfrak{f} | O) + \text{dind} (\mathfrak{f} | O) = \dim O, \quad O \subset \Phi (M),$$

and let $\mathcal{F} = \Phi^* \mathfrak{f}$. Then $C^\infty_K (M) + \mathcal{F}$ is complete on $M$:

$$\text{dind} (C^\infty_K (M) + \mathcal{F}) + \text{dind} (C^\infty_K (M) + \mathcal{F}) = \dim M$$
and the dimension of invariant isotropic tori is
\[ \delta_0 = \text{dind} (C_K^{\infty} (M) + F) = \delta_0 + \text{dind} (\mathfrak{g}|_{\mathfrak{o}}). \]

Note that all adjoint orbits in \( \Phi(M) \) could be singular and then the completeness of \( \mathfrak{g} \)
on \( \mathfrak{f} \) does not imply directly the completeness of the restriction \( \mathfrak{g}|_{\mathfrak{o}} \) on \( \mathfrak{o} \subset \Phi(M) \).

4. The problem of a polynomial commutative integrability

As above, we suppose that the Euler equations (5) are completely integrable by means of a complete commutative algebra \( \mathfrak{A}_0 \) of polynomial integrals and set \( \mathfrak{A}_0 = \text{pr}^*_\mathfrak{g}_0 \mathfrak{A}_0 \).

According to Lemma 1 and Theorem 2, we have

**Corollary 4.** Suppose that for every \( i = 1, \ldots, n \) there exist a commutative subalgebra \( \mathfrak{B}_i \) of the algebra of \( \text{Ad}G_{i-1} \)-invariants \( \mathfrak{A}_i = \mathbb{R}[\mathfrak{g}_i]^{G_{i-1}} \), such that \( \mathfrak{B}_i + \text{pr}^*_\mathfrak{g}_i (\mathbb{R}[\mathfrak{g}_i-1]) \) is a complete algebra on \( \mathfrak{g}_i \). Then
\[
(17) \quad \mathcal{B} = \mathfrak{A}_0 + \mathcal{B}_1 + \cdots + \mathcal{B}_n, \quad \mathcal{B}_i = \text{pr}^*_\mathfrak{g}_i (\mathfrak{B}_i), \quad i = 1, \ldots, n
\]
is a complete commutative set on \( \mathfrak{g} \):

\[
\text{ddim } \mathcal{B} = a(\mathfrak{g}) = \frac{1}{2} \left( \dim \mathfrak{g} + \text{rank } \mathfrak{g} \right).
\]

Therefore, the polynomial commutative integrability of the system (5), (6), reduces to a construction of commutative subalgebras \( \mathfrak{B}_i \) of \( \mathfrak{A}_i, i = 1, \ldots, n. \)

Mikityuk proved that Bogoyavlenski’s integrals (8) solves the problem in the case when \( (\mathfrak{g}_i, \mathfrak{g}_i-1) \) are symmetric pairs [25] (see also examples in [17, 31]).

To simplify notation, let us denote \( K_0 = G_{i-1}, G = G_i, \mathfrak{k} = \mathfrak{g}_i-1, \mathfrak{g} = \mathfrak{g}_i, \mathfrak{p} = \mathfrak{p}_i, x = x_i, y_0 = x_{i-1}, y_1 = y_i, \mathfrak{A} = \mathbb{R}[\mathfrak{g}]^K \). The algebra of polynomials
\[
(18) \quad \mathfrak{A} + \text{pr}^*_\mathfrak{p} (\mathbb{R}[\mathfrak{t}])
\]
is complete with respect to the Lie Poisson bracket on \( \mathfrak{g} \) (see (16)).

Let \( \mathfrak{B} \) be a commutative subset of the algebra of \( \text{Ad}K \)-invariants \( \mathfrak{A} \). We always assume that \( \mathfrak{B} \) contains \( \text{Ad}G \)-invariant polynomials \( \mathbb{R}[\mathfrak{g}]^G \) and the lift of \( \text{Ad}K \)-invariants \( \text{pr}^*_\mathfrak{p} (\mathbb{R}[\mathfrak{t}]^K) \) that belong to the center of \( \mathfrak{A} \).

**Definition 1.** We shall say that \( \mathfrak{B} \) is a **complete commutative set of \( \text{Ad}K \)-invariants** (or \( \text{ad}_K \)-invariants) if the set of polynomials
\[
\mathfrak{B} + \text{pr}^*_\mathfrak{p} (\mathbb{R}[\mathfrak{t}])
\]
is complete on \( \mathfrak{g} \) with respect to the Lie-Poisson bracket, or, equivalently, if \( \mathfrak{B} \) is a complete commutative subset of \( \mathfrak{A} \):

\[
\text{ddim } \mathfrak{B} = \text{dind } \mathfrak{B} = \frac{1}{2} (\text{ddim } \mathfrak{A} + \text{dind } \mathfrak{A}).
\]

**Theorem 5.** A commutative set of \( \text{Ad}K \)-invariants \( \mathfrak{B} \) is complete if it has
\[
\mathfrak{b}(\mathfrak{g}, \mathfrak{t}) = \frac{1}{2} \left( \dim \mathfrak{p} + \text{rank } \mathfrak{g} - \text{rank } \mathfrak{t} \right)
\]
polynomials independent from polynomials \( \text{pr}^*_\mathfrak{p} (\mathbb{R}[\mathfrak{t}]) \). In other words, \( \mathfrak{B} \) is complete if
\[
\dim \text{pr}_\mathfrak{p} \text{span} \{ \nabla p(x) \mid p \in \mathfrak{B} \} = \frac{1}{2} (\dim \mathfrak{p} + \text{rank } \mathfrak{g} - \text{rank } \mathfrak{t}),
\]
for a generic \( x \in \mathfrak{g} \).

**Proof.** Assume that \( \mathfrak{B} \) is a commutative set of \( \text{Ad}K \)-invariants and let
\[
\kappa = \dim \text{pr}_\mathfrak{p} \text{span} \{ \nabla p(x) \mid p \in \mathfrak{B} \}.
\]

Then, due to inequality (15) and the completeness of (18), we have
\[
(\kappa + \dim \mathfrak{t}) + (\kappa + \text{rank } \mathfrak{t}) = \text{ddim } (\mathfrak{B} + \text{pr}^*_\mathfrak{p} (\mathbb{R}[\mathfrak{t}])) + \text{ddim } (\mathfrak{B} + \text{pr}^*_\mathfrak{t} (\mathbb{R}[\mathfrak{t}]))
\]
Therefore, \( \mathfrak{B} + \text{pr}_t^* (\mathcal{R}[f]) \) is complete if and only if \( \kappa = b(\mathfrak{g}, t) \). \( \square \)

Note that the construction of \( \mathfrak{B} \) is closely related to the construction of complete \( G \)-invariant algebras of functions on the cotangent bundle of the homogeneous space \( G/K \) (see \([5, 7, 9, 10, 20, 21, 23, 26, 27]\)).

The simplest situation is the case when the algebra of \( \text{Ad}_K \)-invariants \( \mathfrak{A} \) is already commutative — the center of \( \mathfrak{A} \) coincides with \( \mathfrak{A} \). Then we say that \( K \subset G \) is a multiplicity free subgroup of \( G \). For example, \( \text{SO}(n - 1) \) and \( \text{U}(n - 1) \) are multiplicity free subgroups of \( \text{SO}(n) \) and \( \text{U}(n) \), respectively. This is the reason that the invariants (7) are sufficient for the integrability of the Gel’fand-Cetlin systems on \( \mathfrak{so}(n) \) and \( u(n) \) \([14, 16]\).

The next natural step is to consider a subgroup \( K \subset G \) when apart from \( \text{Ad}_G \)-invariants, for a complete commutative set of \( \text{Ad}_K \)-invariants we can take arbitrary \( \text{Ad}_K \)-invariant polynomial, which is not in the center of \( \mathfrak{A} \). Then we say that \( K \) is an almost multiplicity free subgroup of \( G \).

The classification of multiplicity free subgroups \( K \) of compact Lie groups \( G \) is given by Krämer \([22]\) (see also Heckman \([18]\)). If \( G \) is a simple group, the pair of corresponding Lie algebras \( (\mathfrak{g}, \mathfrak{t}) \) is

\[
(B_n, D_n), \quad (D_n, B_{n-1}), \quad \text{or} \quad (A_n, A_{n-1} \oplus u(1)).
\]

**Example 1.** Multiplicity free pairs are:

\[
(SU(n), S(U(1) \times U(n - 1))), \quad (SU(n), U(n - 1)), \quad (SU(4), Sp(2)), \\
(SO(n), SO(n - 1)), \quad (SO(4), U(2)), \quad (SO(4), SU(2)), \\
(SO(6), U(3)), \quad (SO(8), Spin(7)), \quad (Spin(7), SU(4)).
\]

In the next statement we obtained the classifications of almost multiplicity free subgroups of compact simple Lie groups:

**Theorem 6.** The pair of Lie algebras \((\mathfrak{g}, \mathfrak{t})\) corresponding to the almost multiplicity free subgroups \( K \subset G \) belongs to the following list:

\[
(A_n, A_{n-1}), \quad (A_3, A_1 \oplus A_1 \oplus u(1)), \quad (B_2, u(2)), \\
(B_2, B_1 \oplus u(1)), \quad (B_3, g_2), \quad (g_2, A_2).
\]

The proof will be given in a separate paper.

**Example 2.** Almost multiplicity free pairs:

\[
(SU(n), SU(n - 1)), \quad (SU(4), S(U(2) \times U(2))), \quad (SU(3), SO(3)), \\
(SO(5), SO(3) \times SO(2)), \quad (Sp(2), Sp(1) \times U(1)), \quad (SO(5), U(2)), \\
(Sp(2), U(2)), \quad (SO(6), SO(4) \times SO(2)), \quad (SO(6), SU(3)), \\
(Spin(7), G_2), \quad (G_2, SU(3)), \quad (SO(3) \times SO(4), SO(3)).
\]

### 5. Commutative polynomial integrals

As in Section 4, let us denote \( K = G_{i-1}, G = G_i, \mathfrak{t} = \mathfrak{g}_{i-1}, \mathfrak{g} = \mathfrak{g}_i, \mathfrak{p} = \mathfrak{p}_i, x = x_i, y_0 = x_{i-1}, y_1 = y_i, \) that is \( x = y_0 + y_1 \). Further, in this section by \( p_1, \ldots, p_r \) we denote the base of homogeneous invariant polynomials on \( \mathfrak{g}, r = \text{rank} \mathfrak{g} \).
5.1. Isotropy subgroups and the Mishchenko-Fomenko shifting of argument method. Mishchenko and Fomenko showed that the set of polynomials induced from the invariants by shifting the argument

$$\mathfrak{B} = \{ p_{j,a,k}(x) | k = 0, \ldots, \deg p_j, j = 1, \ldots, r \},$$

with

$$p_{j,a,\lambda} = p_j(x + \lambda a) = \sum_{k=0}^{\deg p_j} p_{j,a,k}(x)\lambda^k$$

is a complete commutative set on $\mathfrak{g}$, for a generic $a \in \mathfrak{g}$ (see [3, 28]).

The following Bolsinov’s statement will be useful to us frequently, hence we are stating it for the sake of completeness. The proof can be found in [3, Proposition 2.5].

**Lemma 7** ([3]). Consider a pencil of skew-symmetric linear forms $\Pi = \{ \lambda_1 A_1 + \lambda_2 A_2 | \lambda_1, \lambda_2 \in \mathbb{R}, |\lambda_1| + |\lambda_2| \neq 0 \}$ in $\mathbb{R}^n$ and set $R_0 = \max_{\lambda \in \Pi} \text{rank} \Lambda$. Let $\Lambda^1, \ldots, \Lambda^q \in \Pi^C = \{ \lambda_1 A_1 + \lambda_2 A_2 | \lambda_1, \lambda_2 \in \mathbb{C}, |\lambda_1| + |\lambda_2| \neq 0 \}$ be linearly independent skew-symmetric forms in $\mathbb{C}^n$ with rank less then $R_0$. Let $L \subset \mathbb{C}^n$ be the union of kernels of all forms in $\Pi^C$, $L_0 \subset L^\Lambda$ be the union of kernels of forms with the maximal rank, and let

$$L^\Lambda_0 = \{ \xi \in \mathbb{C}^n | \Lambda(\xi, \eta) = 0, \eta \in L_0 \}$$

be the $\Lambda$-orthogonal space of $L_0$. Then,

(i) $\Lambda$-orthogonal space of kernels of forms with the maximal rank $L^\Lambda_0$ does not depend on $\Lambda \in \Pi^C$. It is an isotropic subspace and contains the kernels of all forms in $\Pi^C$: $L \subset L^\Lambda_0$, $L \subset L^\Lambda_0$.

(ii) The equality $L^\Lambda_0 = L$ holds if and only if

$$\dim_{\mathbb{C}} \ker(\Lambda|_{\ker \Lambda'}) = \dim \{ \xi \in \ker \Lambda' | \Lambda(\xi, \eta) = 0, \eta \in \ker \Lambda' \} = n - R_0, \quad i = 1, \ldots, q,$$

where $\Lambda \in \Pi^C$ is a skew-symmetric form of rank $R_0$ at $x$.

**Theorem 8.** Let $\mathfrak{g}$ be a Lie algebra equal to the isotropy algebra of a element $a \in \mathfrak{g}$

$$\mathfrak{g} = \mathfrak{g}(a) = \{ x \in \mathfrak{g} | [x, a] = 0 \}$$

and let $K$ be its corresponding Lie group. The set (19) is a complete commutative set of $\text{Ad}_K$-invariant polynomials on $\mathfrak{g}$.

**Proof.** It is clear that polynomials $\mathfrak{B}$ are $\text{Ad}_K$-invariant. Based on Lemma 7, Bolsinov proved that for a given $x^0 \in \mathfrak{g}$ (regular or singular), one can find a regular element $a \in \mathfrak{g}$, such that $\mathfrak{B}$ is a complete commutative algebra on the adjoint orbit through $x^0$ (see [8] and [37, Theorem 5, page 230]), the statement is firstly proved by Mishchenko and Fomenko by using a different approach ([28, 37]). One can reverse the roll of $x^0$ and $a$ and slightly change the given theorem to prove the required statement. We will present the detailed proof since some of the arguments will be used in the proof of Theorem 12 given below.

The set (19) is the union of Casimir polynomials of the Poisson brackets of maximal rank $R_0 = \dim \mathfrak{g} - \text{rank} \mathfrak{g}$ within the pencil $\Pi$ of compatible Poisson brackets given by Lie Poisson structure $\Lambda$ and the $a$-bracket $\Lambda_a$ given by

$$\{ f, g \}_{|x^0} = \Lambda_a(x)(\nabla f(x), \nabla g(x)) = -\langle a, [\nabla f(x), \nabla g(x)] \rangle.$$

It is sufficient to consider the case when $a$ is a singular element of $\mathfrak{g}$. Since $\mathfrak{g}$ is compact, there exist $x = y_0 + y_1 \in \mathfrak{g}$, such that the complex plane

$$\ell(x, a) = \{ \lambda_0 y_0 + \lambda_1 y_1 | \lambda_0, \lambda_1 \in \mathbb{C} \}$$

intersect the set of singular points in the complexified Lie algebra $\mathfrak{g}^C$ only at the line $\mathbb{C} \cdot a$ and that $y_0$ is regular in $\mathfrak{g}^C$.

Consider the complexified pencil of skew-symmetric forms

$$\Pi^C_x = \{ \lambda_0 \Lambda(x) + \lambda_1 \Lambda_a(x) | \lambda_0, \lambda_1 \in \mathbb{C}, |\lambda_0| + |\lambda_1| \neq 0 \}.$$
The kernel of $\lambda_0 \Lambda(x) + \lambda_1 \Lambda_0(x)$ in $\mathfrak{g}^\mathbb{C}$ is the isotropy algebra
\begin{equation}
\mathfrak{g}^\mathbb{C}(\lambda_0 x + \lambda_1 \mathfrak{a}) = \{ \xi \in \mathfrak{g}^\mathbb{C} \mid [\xi, \lambda_0 x + \lambda_1 \mathfrak{a}] = 0 \}.
\end{equation}
Thus, all forms $\Pi^\mathbb{C}_x$ are regular except $\Lambda_0(x)$ with the kernel equals to $\mathfrak{t}^\mathbb{C}$. According to Lemma 7, we have
\begin{equation}
L_0^\mathbb{C} = L_0 + \mathfrak{t}^\mathbb{C}
\end{equation}
if and only if
\[
\dim \mathbb{C}\{ \xi \in \mathfrak{t}^\mathbb{C} \mid \langle x, [\xi, \eta] \rangle = 0, \eta \in \mathfrak{t}^\mathbb{C} \} = \dim \mathbb{C}\{ \xi \in \mathfrak{t}^\mathbb{C} \mid [\xi, \mathfrak{y}_0] = 0 \} = \text{rank } \mathfrak{g}.
\]
Since $\text{rank } \mathfrak{t} = \text{rank } \mathfrak{g}$ and $\mathfrak{y}_0$ is regular, the above identity is satisfied. The real part of $L_0$ is spanned by the gradients of the polynomials in $\mathfrak{B}$ at $x$. Therefore, in the real domain, (22) implies that $\mathfrak{B} + \mathfrak{p}_x^\mathbb{C}(\mathbb{R}[\mathfrak{t}])$ is a complete set of functions at $x$, and therefore on an open dense subset on $\mathfrak{g}$.

5.2. Mikityuk’s construction with symmetric pairs and its variation. In the above notation, Bogoyavlenski’s integrals (8) can be defined as coefficients in $\lambda$ in the expansion of $p_j(y_0 + \lambda y_1)$:
\begin{equation}
\mathfrak{B} = \{ p_{j,k}(x) \mid k = 0, \ldots, \deg p_j, j = 1, \ldots, r \},
\end{equation}
\[p_{j,k}(x) = p_j(y_0 + \lambda y_1) = \sum_k \lambda^k p_{j,k}(x), \quad j = 1, \ldots, r.
\]
Mikityuk has proved the following completeness statement for polynomials (23) (see [25, Proposition 3, Theorems 1 and 2]).

Theorem 9 ([25]). Assume that $(\mathfrak{g}, \mathfrak{t})$ is a symmetric pair. Then the set of polynomials (23) is a complete commutative set of $\mathfrak{ad}_\mathfrak{t}$-invariant polynomials on $\mathfrak{g}$. Therefore, if all pairs $(\mathfrak{g}_i, \mathfrak{g}_{i-1})$, $i = 1, \ldots, n$ are symmetric and $\mathfrak{A}_0$ is a complete commutative set on $\mathfrak{g}_0$, then the associated set (17) is a complete commutative set on $\mathfrak{g}$.

There is a small variation of Mikityuk’s construction that allows us to significantly extend the class of examples if not all of the pairs $(\mathfrak{g}_i, \mathfrak{g}_{i-1})$, $i = 1, \ldots, n$ are symmetric. Simply, we can extend the original filtration and use different methods at every step. As an illustration, consider an arbitrary chain of compact symplectic groups with standard inclusions
\begin{equation}
\mathfrak{sp}(k_0) \subset \mathfrak{sp}(k_1) \subset \cdots \subset \mathfrak{sp}(k_n), \quad k_0 < k_1 < \cdots < k_n.
\end{equation}
Then, we extend (24) to the filtration (also using natural inclusions):
\[\mathfrak{sp}(k_0) \subset \mathfrak{sp}(k_0) \times \mathfrak{sp}(k_1 - k_0) \subset \mathfrak{sp}(k_1) \subset \cdots \subset \mathfrak{sp}(k_{n-1}) \times \mathfrak{sp}(k_n - k_{n-1}) \subset \mathfrak{sp}(k_n).
\]
Now, the construction of functions in involution is clear. If in the step $(\mathfrak{g}_i, \mathfrak{g}_{i-1})$ we have a symmetric pair $(\mathfrak{sp}(k_i), \mathfrak{sp}(k_{i-1}) \times \mathfrak{sp}(k_i - k_{i-1}))$, then $\mathcal{B}_i$ is given by (23). On the other hand, if $(\mathfrak{g}_i, \mathfrak{g}_{i-1})$ is $(\mathfrak{sp}(k_{i-1}) \times \mathfrak{sp}(k_i - k_{i-1}), \mathfrak{sp}(k_{i-1}))$, then for $\mathcal{B}_i$ we take an arbitrary complete commutative set on $\mathfrak{sp}(k_i - k_{i-1})$ (for example using the argument shift method [28]).

Proposition 10. Assume that the Euler equations (5) on $\mathfrak{sp}(k_0)$ are integrable by polynomial integrals. Then the Euler equations (5), (6) associated to the filtration (24) are completely integrable in a commutative sense by means of polynomial integrals as well.

Of course, the above variation can be applied for other constructions of commutative polynomials. If for a given Lie subalgebra $\mathfrak{t} \subset \mathfrak{g}$ one can find a Lie subalgebra $\mathfrak{h}$, $\mathfrak{t} \subset \mathfrak{h} \subset \mathfrak{g}$, having a complete commutative set of $\mathfrak{ad}_\mathfrak{t}$-invariant polynomials on $\mathfrak{h}$ and a complete commutative set of $\mathfrak{ad}_\mathfrak{h}$-invariant polynomials on $\mathfrak{g}$, then the problem is solved for a pair $(\mathfrak{g}, \mathfrak{t})$ as well.

4By $(\cdot, \cdot)$ we also denote an invariant quadratic form on $\mathfrak{g}^\mathbb{C}$, the extension of the invariant scalar product from $\mathfrak{g}$ to $\mathfrak{g}^\mathbb{C}$. 

5.3. Bogoyavlenski’s integrals. First, we note the following lemma.

**Lemma 11.** The polynomials (23) are $\text{Ad}_K$–invariant.

**Proof.** As we already mentioned, the polynomials (23) commute with the lifting of polynomials from $\mathbb{R}[\mathfrak{t}]$ to $\mathbb{R}[\mathfrak{g}]$ if $(\mathfrak{g}, \mathfrak{t})$ is a symmetric pair (Theorem 9). The proof given there can be slightly modified, and adopted to a general case.

Let $p \in \mathbb{R}[\mathfrak{g}]^G$ be an invariant polynomial and let $p_\lambda(x) = p(y_0 + \lambda y_1)$. The gradient of $p_\lambda$ at $x$ is given by

$$P^\lambda = \nabla p_\lambda(x) = P^\lambda_0 + \lambda P^\lambda_1,$$

where

$$P^\lambda_0 = \text{pr}_\mathfrak{t} \nabla p|_{y_0 + \lambda y_1}, \quad P^\lambda_1 = \text{pr}_p \nabla p|_{y_0 + \lambda y_1}.$$

Since $p$ is an invariant polynomial, we have

$$[P^\lambda_0 + P^\lambda_1, y_0 + \lambda y_1] = 0. \tag{25}$$

Let $\xi$ be an arbitrary element in $\mathfrak{t}$. Then

$$\langle \nabla p_\lambda(x), [\xi, x] \rangle = \langle P^\lambda_0 + \lambda P^\lambda_1, [\xi, y_0 + \lambda y_1] \rangle = \langle P^\lambda_0, [\xi, y_0] \rangle + \lambda \langle P^\lambda_1, [\xi, y_1] \rangle. \tag{26}$$

On the other hand, from (25) we get

$$0 = \langle [P^\lambda_0 + P^\lambda_1, y_0 + \lambda y_1], \xi \rangle = \langle [P^\lambda_0, y_0], \xi \rangle + \lambda \langle [P^\lambda_1, \lambda y_1], \xi \rangle. \tag{27}$$

Therefore, according to (26) and (27), $p_\lambda$ is an $\text{Ad}_K$–invariant polynomial.

We can see this also directly by using the identities: $p(\text{Ad}_y(x)) = p(x)$, $\text{Ad}_y(x) = \text{Ad}_y(y_0) + \text{Ad}_y(y_1)$, $g \in G$, and $\text{Ad}_y(y_0) \in \mathfrak{t}$, $\text{Ad}_y(y_1) \in \mathfrak{p}$, $g \in K$. \hfill $\square$

Now, we would like to use a relationship between the argument shift method and translations along the subalgebras presented above to estimate the number of independent Bogoyavlenski’s integrals.

**Theorem 12.** Let $\text{Sing}(\mathfrak{g}^C)$ be the set of singular point in $\mathfrak{g}^C$ and let

$$\ell(x) = \{\lambda_0 y_0 + \lambda_1 y_1 | \lambda_0, \lambda_1 \in \mathbb{C} \}.$$

Assume that there exist $x \in \mathfrak{g}$ such that

$$\ell(x) \cap \text{Sing}(\mathfrak{g}^C) = \{0\}, \quad \text{or,} \quad \ell(x) \cap \text{Sing}(\mathfrak{g}^C) = \mathbb{C} : y_1.$$

Then, for a given $x$, for polynomials (23) we have

$$\dim B(x) \geq \text{b}(\mathfrak{g}, \mathfrak{t}) = \frac{1}{2}(\dim \mathfrak{p} + \text{rank} \mathfrak{g} - \text{rank} \mathfrak{t}),$$

where $B(x) = \text{pr}_p \text{span} \{\nabla p_{\lambda, j}(x) | j = 1, \ldots, r, \lambda \in \mathbb{R} \}$.

**Proof. Step 1.** We can obtain polynomials (23) by the translation of invariants in the direction of $\mathfrak{t}$ instead of $\mathfrak{p}$: $p_\lambda(x) = p(\lambda y_0 + y_1), p \in \mathbb{R}[\mathfrak{g}]^G$. Since

$$\nabla p_\lambda(x) = \lambda \text{pr}_\mathfrak{t} \nabla p|_{\lambda y_0 + y_1} + \text{pr}_p \nabla p|_{\lambda y_0 + y_1} = \lambda \text{pr}_\mathfrak{t} \nabla p|_{\lambda y_0 + y_1} + \text{pr}_p \nabla p|_{x + \mu y_0},$$

$\lambda = \mu + 1$, we need to estimate the dimension of the linear space

$$B(x) = \text{span} \{\text{pr}_p \nabla p_{j, \mu, \lambda}(x) | j = 1, \ldots, r, \mu \in \mathbb{R} \}.$$

The space $B(x)$ is equal to the projection to $\mathfrak{p}$ of the space $C(x)$ spanned by gradients of the polynomials $p_{\alpha, \mu}(x) = p(x + \mu a)$ obtained by shifting of argument in the direction $\alpha = y_0 = \text{pr}_\mathfrak{t}(x)$:

$$C(x) = \text{span} \{\nabla p_{\alpha, \mu}(x) | p \in \mathbb{R}[\mathfrak{g}]^G \}.$$

Note that the sets of polynomials (19) and (23) are different, but the projections to $\mathfrak{p}$ of their gradients at the given point $x$ are the same.
Step 2. Let us assume that the complex plane $\ell(x)$ intersects the set of singular points in $\mathfrak{g}^C$ only in 0 and consider the pencil of compatible Poisson structures spanned by Lie-Poisson bracket (4) and the $a$-bracket (20), where $a = y_0$. The kernel of the skew-symmetric form $\lambda_0\Lambda(x) + \lambda_1\Lambda_a(x)$ in $\mathfrak{g}^C$ is the isotropy algebra (21). Thus all forms in $\Pi^C_1$ have the maximal rank, and, according to Lemma 7, $L_0^\Lambda = L_0$ where $L_0 \subset \mathfrak{g}^C$ is the union of all kernels in in $\Pi^C_1$. Since $C(x) = L_0 \cap \mathfrak{g}$, the set $\{p_{a,\mu} \mid \mu \in \mathbb{R}\}$ is complete at $x$:

$$\dim C(x) = a(\mathfrak{g}) = \frac{1}{2}(\dim \mathfrak{g} + \text{rank} \mathfrak{g}).$$

Therefore

$$\dim B(x) = \dim \text{pr}_p C(x) = \frac{1}{2}(\dim \mathfrak{g} + \text{rank} \mathfrak{g}) - \dim (C(x) \cap \mathfrak{k}).$$

It remains to note that

$$\dim (C(x) \cap \mathfrak{k}) \leq \frac{1}{2}(\dim \mathfrak{k} + \text{rank} \mathfrak{k}),$$

implying that $\dim B(x) \geq b(\mathfrak{g}, \mathfrak{k})$. Indeed, we have

$$\{p_{a,\mu_1}, f_{a,\mu_2}\}(x) = \langle [x, [\nabla p|_{x+\mu_1 a}, \nabla f|_{x+\mu_2 a}]] \rangle = 0, \quad p, f \in \mathbb{R}[\mathfrak{g}]^G,$$

and, if $\nabla p|_{x+\mu_1 a}, \nabla f|_{x+\mu_2 a} \in \mathfrak{k}$, then also

$$\{y_0, [\nabla p|_{x+\mu_1 a}, \nabla f|_{x+\mu_2 a}] \} = 0.$$

Thus, $C(x) \cap \mathfrak{k}$ is an isotropic subspace of $\mathfrak{k}$ at $y_0$ with respect to the Lie-Poisson bracket on $\mathfrak{k}$. On the other hand, the maximal isotropic subspace at $y_0$ (it is regular in $\mathfrak{k}$) has the dimension $\frac{1}{2}(\dim \mathfrak{k} + \text{rank} \mathfrak{k})$.

Step 3. Now assume that the complex plane $\ell(x)$ intersects the set of singular points in $\mathbb{C} \cdot y_1$ and in addition that $\text{pr}_{\mathfrak{g}^C(y_1)} x$ is regular in $\mathfrak{g}^C(y_1)$, where $\mathfrak{g}^C(y_1)$ is the isotropy Lie algebra of $y_1$ within $\mathfrak{g}^C$. Then all skew-symmetric forms in $\Pi^C_1$, except $\Lambda_1 = \Lambda(x) - \Lambda_a(x)$, have the maximal rank ($a = y_0$). Since rank $\mathfrak{g}^C(y_1) = \text{rank} \mathfrak{g}$, we get

$$\dim \mathbb{C} \ker(\Lambda|_{\ker \Lambda_1}) = \dim \mathbb{C} \{\xi \in \mathfrak{g}^C(y_1) \mid \langle x, [\xi, \eta] \rangle = 0, \eta \in \mathfrak{g}^C(y_1)\}$$

$$= \dim \mathbb{C} \{\xi \in \mathfrak{g}^C(y_1) \mid [\xi, \text{pr}_{\mathfrak{g}^C(y_1)} x] = 0\} = \text{rank} \mathfrak{g}.$$

Thus, from Lemma 7, we have $L_0^\Lambda = L_0 + \mathfrak{g}^C(y_1)$, that in the real domain implies

$$C(x)^\Lambda = C(x) + \mathfrak{g}(y_1).$$

For $y_1$ that belong an open dense set of $\mathfrak{p}$ with minimal dimensions of the isotropy algebras $\mathfrak{g}(y_1)$ and $\mathfrak{k}(y_1)$, we have that the semisimple part of $\mathfrak{g}(y_1)$ belongs to $\mathfrak{k}(y_1)$ (e.g., see [26]):

$$[\mathfrak{g}(y_1), \mathfrak{g}(y_1)] = [\mathfrak{k}(y_1), \mathfrak{k}(y_1)].$$

Then

$$\mathfrak{g}(y_1) = \mathfrak{k}(y_1) + \mathfrak{z}(\mathfrak{g}(y_1)),$$

where $\mathfrak{z}(\mathfrak{g}(y_1))$ is the center of $\mathfrak{g}(y_1)$. Further, we have (e.g., see [1, Lemma 4])

$$\mathfrak{z}(\mathfrak{g}(y_1)) \subset \text{span} \{\nabla p_j(y_1) \mid j = 1, \ldots, r\} \subset C(x).$$

By combining (28), (29), and (30) we obtain

$$C(x)^\Lambda = C(x) + \mathfrak{k}(y_1).$$

Thus, there exist a subspace $D \subset \mathfrak{k}(y_1)$ such that $C_1(x) = C(x) + D$ is a maximal isotropic subspace in $\mathfrak{g}$ with respect to $\Lambda(x)$ and

$$\dim C_1(x) = \frac{1}{2}(\dim \mathfrak{g} + \text{rank} \mathfrak{g}).$$
Using the identity $B(x) = pr_p C(x) = pr_p C_1(x)$, the rest of the proof is the same as in Step 2 with $C(x)$ replaced by $C_1(x)$.

Thus, if $B$ is commutative and the conditions of Theorem 12 are satisfied, according to Theorem 5 and Lemma 11, (23) is a complete commutative set of $\text{ad}_\ell$-invariant polynomials on $g$.

Let $f, p$ be invariant polynomials, $f_\ell(x) = f(y_0 + \lambda y_1)$, and $p_\mu(x) = p(y_0 + \mu y_1)$, $\mu \neq \lambda$, $\lambda^2 + \mu^2 \neq 0$. With the above notation we have

\begin{align*}
[F_0^\lambda + F_1^\lambda, y_0 + \lambda y_1] &= 0, \quad [P_0^\mu + P_1^\mu, y_0 + \mu y_1] = 0,
\end{align*}

which implies (see (27)):

\begin{align*}
\langle \gamma_0, [F_0^\lambda, \xi]\rangle &= \lambda \langle \gamma_1, [\xi, F_1^\lambda]\rangle, \quad \langle \gamma_0, [P_0^\mu, \xi]\rangle = \mu \langle \gamma_1, [\xi, P_1^\mu]\rangle, \quad \xi \in \mathfrak{r}.
\end{align*}

In particular,

\begin{align*}
\langle \gamma_0, [F_0^\lambda, P_0^\mu]\rangle &= \lambda \langle \gamma_1, [P_0^\mu, F_1^\lambda]\rangle, \quad \langle \gamma_0, [P_0^\mu, F_0^\lambda]\rangle = \mu \langle \gamma_1, [F_0^\lambda, P_1^\mu]\rangle.
\end{align*}

Since $x = y_0 + y_1$, $y_0$, $y_1$ can be expressed as linear combinations of $y_0 + \lambda y_1$ and $y_0 + \mu y_1$, from (31) and (32) we get

\begin{align*}
0 &= \langle \gamma_0, [P_0^\mu + P_1^\mu, F_0^\lambda + F_1^\lambda]\rangle = \langle \gamma_0, [F_0^\lambda, P_1^\mu]\rangle = \langle \gamma_0, [P_1^\mu, F_1^\lambda]\rangle, \\
0 &= \langle \gamma_1, [P_0^\mu + P_1^\mu, F_0^\lambda + F_1^\lambda]\rangle = \langle \gamma_1, [P_0^\mu, F_1^\lambda]\rangle = \langle \gamma_1, [P_1^\mu, F_0^\lambda]\rangle.
\end{align*}

Now, by the use of the above identities, the Poisson bracket $\{f_\ell, p_\mu\}$ reads

\begin{align*}
\{f_\ell, p_\mu\}_\mathfrak{r} &= \langle \gamma_0, y_0 + y_1, [P_0^\mu + \mu P_1^\mu, F_0^\lambda + \lambda F_1^\lambda]\rangle \\
&= \langle \gamma_0, [P_0^\mu, F_1^\lambda]\rangle + \lambda \mu \langle \gamma_0, [P_1^\mu, F_0^\lambda]\rangle \\
&\quad + \lambda \langle \gamma_1, [P_0^\mu, F_1^\lambda]\rangle + \mu \langle \gamma_1, [P_1^\mu, F_0^\lambda]\rangle + \lambda \mu \langle \gamma_1, [P_1^\mu, F_1^\lambda]\rangle \\
&= \langle \gamma_0, [P_0^\mu, F_1^\lambda]\rangle + \lambda \mu \langle \gamma_0, [P_1^\mu, F_0^\lambda]\rangle \\
&\quad + \langle \gamma_0, [P_0^\mu, F_1^\lambda]\rangle + \langle \gamma_0, [P_1^\mu, F_0^\lambda]\rangle - (\lambda + \mu) \langle \gamma_0, [P_0^\mu, F_0^\lambda]\rangle \\
&= (1 - \lambda)(1 - \mu) \langle \gamma_0, [P_0^\mu, F_0^\lambda]\rangle.
\end{align*}

**Example 3.** If $\mathfrak{r}$ is a Cartan subalgebra, then $\{f_\ell, p_\mu\}_\mathfrak{r} = (1 - \lambda)(1 - \mu) \langle \gamma_0, [F_0^\mu, P_0^\lambda]\rangle = 0$ and integrals (23) provides a complete commutative set on $g$. The pair $(\mathfrak{g}, \mathfrak{r})$ is already described in Theorem 8: $\mathfrak{r} = \mathfrak{g}(a)$, where $a \in \mathfrak{r}$ is regular.

It is also obvious that if $(\mathfrak{g}, \mathfrak{r})$ is a symmetric pair that $B$ is commutative. In Bogoyavlenski [2, Theorem 1] it is claimed that the set $B$ is always commutative, however the presented proof contains a small gap. Recently, Panyshev and Yakimova gave an example of the case where $B$ is not commutative [32, Example 2.3].

**Example 4.** For symmetric pairs $(\mathfrak{so}(n), \mathfrak{so}(p) \times \mathfrak{so}(n-p))$, Theorem 12 provides another proof of Theorem 9.

**Example 5.** In the following example we verified the commutativity of $B$ by direct computations. The examples are also convenient in the discussion of the conditions in Theorem 12. Consider the case $\mathfrak{g} = \mathfrak{so}(5)$. The Lie subalgebras

\begin{align*}
\mathfrak{so}(2) \oplus \mathfrak{so}(2), \quad \mathfrak{so}(2) \oplus \mathfrak{so}(3), \quad \mathfrak{so}(4)
\end{align*}

satisfy conditions of Theorem 12. Moreover, $\mathfrak{so}(4)$ and $\mathfrak{so}(2) \times \mathfrak{so}(3)$ are multiplicity free and almost multiplicity free subgroups of $\mathfrak{so}(5)$. On the other side, if $\mathfrak{r} = \mathfrak{so}(3)$ or $\mathfrak{r} = \mathfrak{so}(2)$, a generic $y_0 \in \mathfrak{r}$ is not regular in $\mathfrak{so}(5)$ and we can not apply Theorem 12.

We have

\begin{align*}
\mathbf{b}(\mathfrak{so}(5), \mathfrak{so}(3)) &= 4, & \mathbf{b}(\mathfrak{so}(5), \mathfrak{so}(2)) &= 5.
\end{align*}
Let \( x = y_0 + y_1 \in \mathfrak{so}(5) \), with
\[
y_0 = \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix}, \quad y_1 = \begin{pmatrix} P_2 \\ -P_1^T \\ 0 \end{pmatrix},
\]
where \( P_1 \in \mathbb{R}^{3 \times 2}, P_2 \in \mathfrak{so}(2), Q \in \mathfrak{so}(3) \) in the case \( \mathfrak{g} = \mathfrak{so}(3) \) and \( P_1 \in \mathbb{R}^{2 \times 3}, P_2 \in \mathfrak{so}(3), Q \in \mathfrak{so}(2) \) in the case \( \mathfrak{g} = \mathfrak{so}(2) \). Bogoyavlenski’s integrals are given by:
\[
p_{1,1}(x) = p_{1,1}(y_0 + \lambda y_1) = \text{tr}(y_0 + \lambda y_1)^2 = \sum_{k=0}^{2} \lambda^k p_{j,k}(x),
p_{1,0}(x) = \text{tr}(y_0^2),
p_{1,1}(x) = \text{tr}(y_0 y_1 + y_1 y_0) \equiv 0,
p_{2,1}(x) = \text{tr}(y_0 y_1)^2,\]
\[
p_{2,0}(x) = \text{tr}(y_0^2),
p_{2,1}(x) = \text{tr}(y_0 y_1 y_0 + y_1 y_0 y_0 + y_0^2 y_1 y_0 + y_0 y_0^2) \equiv 0,
p_{2,0}(x) = \text{tr}(y_0^3),
p_{2,1}(x) = \text{tr}(y_0 y_1 y_0^2 + y_1 y_0^2 y_0 + y_1 y_0^2 y_0 + y_1 y_0 y_1 + y_1 y_0 y_1 + y_1 y_0 y_1),\]
\[
p_{2,0}(x) = \text{tr}(y_0^4).
\]
The polynomials \( p_{j,k} \) commute and we need to estimate the number of independent gradients after projections onto \( \mathfrak{p} \), at a general point \( x \in \mathfrak{so}(5) \). For this reason, we consider the gradients:
\[
\nabla p_{1,2}(x) = 2y_1, \\
\nabla p_{2,2}(x) = 4(y_0^2 y_1 + y_0 y_1 y_0 + y_1 y_0^2 + y_0^2 y_0 + y_1 y_0 y_1 + y_0 y_1^2), \\
\nabla p_{2,3}(x) = 4(y_1^3 + y_0 y_1^2 + y_1 y_0 y_1 + y_1^2 y_0), \\
\nabla p_{2,4}(x) = 4y_1^4.
\]

Let \( \mathfrak{g} = \mathfrak{so}(3) \) and let \( e_{ij} \) be the standard basis of \( \mathfrak{so}(5) \). The equation
\[
\mu_1 \text{pr}_p \nabla p_{1,2}(x) + \mu_2 \text{pr}_p \nabla p_{2,2}(x) + \mu_3 \text{pr}_p \nabla p_{2,3}(x) + \mu_4 \text{pr}_p \nabla p_{2,4}(x) = 0
\]
at the point \( x = e_{45} + e_{15} + e_{23} + e_{24} + e_{15} + e_{25} \) reduces to the system of equations
\[
\mu_2 + \mu_3 = 0, \quad \mu_1 - 8\mu_3 - 8\mu_4 = 0, \quad \mu_2 = 0, \\
\mu_3 - \mu_4 = 0, \quad \mu_1 - 10\mu_3 - 10\mu_4 = 0,
\]
which has only the trivial solution. Therefore, \( \mathfrak{B} \) is a complete set of \( \text{Ad}_{SO(3)} \)-invariant polynomials.

Obviously, according to (33), Bogoyavlenski’s integrals are not sufficient for \( \mathfrak{g} = \mathfrak{so}(2) \). However, we can consider the variation of the method, by taking the chain of subalgebras
\[
\mathfrak{so}(2) \subset \mathfrak{so}(2) \oplus \mathfrak{so}(2) \subset \mathfrak{so}(5) \quad \text{or} \quad \mathfrak{so}(2) \subset \mathfrak{so}(2) \oplus \mathfrak{so}(3) \subset \mathfrak{so}(5).
\]

### 5.4. Diagonal subgroups.
Let \( G_0 \) be a compact connected Lie group and \( \mathfrak{g}_0 = \text{Lie}(G_0) \) its Lie algebra. Consider the case when the group \( G \) is the product \( G = G_0^m \) and the subgroup \( K \subset G \) is \( G_0 \) diagonally embedded into the product:
\[
K = \text{diag}(G_0) = \{ (g, \ldots, g) \mid g \in G_0 \} \subset G.
\]

For a purpose of a construction of integrable geodesic flows on \( m \)-symmetric spaces \( Q = G/K \), the following construction related to the filtration
\[
\mathfrak{g}_0 \subset \mathfrak{g}_1 = \mathfrak{g}_0 \oplus \mathfrak{g}_0 \subset \cdots \subset \mathfrak{g}_{m-1} = (\mathfrak{g}_0)^m = \mathfrak{g}
\]
is given in [20]. Let \( f_1, \ldots, f_{r_0} \) be the base of homogeneous invariant polynomials on \( \mathfrak{g}_0 \), \( r_0 = \text{rank} \mathfrak{g}_0 \), and let
\[
\mathfrak{B} = \mathfrak{B}_1 + \mathfrak{B}_2 + \cdots + \mathfrak{B}_{m-1},
\mathfrak{B}_1 = \{ f_{a,k}^j(x) \mid k = 0, \ldots, \deg f_j, \; j = 1, \ldots, r_0 \},
\]
where the polynomials \( f_{j,k}^j(x) \) are defined by:
\[
f_j(y_0 + y_1 + \cdots + y_{i-1} + \lambda y_i) = \sum_{k=0}^{\deg f_j} f_{j,k}^j(y_0, y_1, \ldots, y_{m-1}) \lambda^k.
\]

**Theorem 13** ([20]). (i) The set \( \mathfrak{B} \) is a commutative set of \( \text{Ad}_K \)-invariant polynomials on \( \mathfrak{g} \).

(ii) The set \( \mathfrak{B} + \mu^* (\mathbb{R}[\mathfrak{g}_0]) \) is a complete set of polynomials on \( \mathfrak{g} \), where
\[
\mu(y_0, y_1, \ldots, y_{m-1}) = y_0 + y_1 + \cdots + y_{m-1}.
\]

Therefore, the set \( \mathfrak{B} \) solves our problem for the pair \( (G, K) = (G_0^m, \text{diag}(G_0)) \).

### 5.5. Reyman’s construction: the shifting of argument and symmetric pairs.

Here we present Reyman’s construction of commutative polynomials related to symmetric pairs [33]. Suppose that \( \mathfrak{g} \) is a Lie subalgebra of a semisimple Lie algebra \( \mathfrak{h} \), such that \( (\mathfrak{h}, \mathfrak{g}) \) is a symmetric pair:
\[
\{ \mathfrak{g}, \mathfrak{m} \} \subset \mathfrak{m}, \quad \{ \mathfrak{m}, \mathfrak{m} \} \subset \mathfrak{g},
\]
where \( \mathfrak{m} \) is the orthogonal complement of \( \mathfrak{g} \) with respect to an invariant scalar product \( \langle \cdot, \cdot \rangle \).

Further, suppose there exist \( a \in \mathfrak{m} \), such that \( \mathfrak{t} \) equals to the isotropy algebra of \( a \) within \( \mathfrak{g} \),
\[
\mathfrak{t} = \mathfrak{g}(a) = \{ \xi \in \mathfrak{g} \mid [\xi, a] = 0 \}.
\]

Let \( h_1, \ldots, h_s \) be the basic homogeneous invariant polynomials on \( \mathfrak{h} \), \( s = \text{rank} \mathfrak{h} \) and let \( \mathcal{K} \) be the set of linear functions on \( \mathfrak{t} \), considered as linear functions on \( \mathfrak{h} \).

On \( \mathfrak{h} \) we have a pencil \( \Pi \) of compatible Poisson bivectors spanned by
\[
\Lambda_1(\xi_1 + \eta_1, \xi_2 + \eta_2) = -\langle z, [\xi_1, \xi_2] + [\xi_1, \eta_2] + [\eta_1, \xi_2] \rangle,
\Lambda_2(\xi_1 + \eta_1, \xi_2 + \eta_2) = -\langle z + a, [\xi_1, \xi_2 + \eta_2] \rangle,
\]
where \( z \in \mathfrak{h} \), \( \xi_1, \xi_2 \in \mathfrak{g} \), \( \eta_1, \eta_2 \in \mathfrak{m} \) (see Reyman [33]). The Poisson bivectors \( \Lambda_{\lambda_1, \lambda_2} \), for \( \lambda_1 + \lambda_2 \neq 0 \) and \( \lambda_2 \neq 0 \), are isomorphic to the canonical Lie-Poisson bivector on \( \mathfrak{h} \). Thus, the union of their Casimir functions
\[
\mathcal{B} = \{ h_{\lambda,k}(z) = h_k(\lambda x + t + \lambda^2 a) \mid k = 1, 2, \ldots, s, \lambda \in \mathbb{R} \},
\]
where \( z = x + t, x \in \mathfrak{g}, t \in \mathfrak{m} \), is a commutative set with respect to the all brackets from the pencil \( \Pi \) (see [4, 33]). Moreover, for a generic \( a \in \mathfrak{m} \), the set of functions \( \mathcal{B} + \mathcal{K} \) is a complete non-commutative set on \( \mathfrak{h} \) with respect to \( \Lambda_1 \) (see [4, Theorem 1.5], for the detail proofs of the above statements, given for an arbitrary semi-simple symmetric pair, see [37, pages 234-237]).

The symplectic leaf within \( \mathfrak{h} \) (and the corresponding symplectic structure) of the bracket \( \Lambda_1 \) at a point \( x \in \mathfrak{g} \) coincide with the symplectic leaf through \( x \) of the Lie-Poisson bracket (and the corresponding symplectic structure) on \( \mathfrak{g} \). Therefore, the following statement holds.

**Proposition 14** ([3, 28, 33]). The restrictions of the polynomial (34)
\[
\mathcal{B} = \{ h_{j,a,k}(x) \mid k = 1, \ldots, \deg f_j, \; j = 1, \ldots, s \},
\]

\footnote{Again, we use the same symbol for different objects. The restriction of \( \langle \cdot, \cdot \rangle \) to \( \mathfrak{g} \) is a positive definite invariant scalar product we are dealing with. Also, as above, we identify \( \mathfrak{h} \) and \( \mathfrak{h}^* \) by \( \langle \cdot, \cdot \rangle \).}
\[ h_j(x + \lambda a) = \sum_{k=0}^{\deg h_j} h_{j,a,k}(x) \lambda^k \]

is a commutative set of $K$-invariant polynomials on $\mathfrak{g}$.

Moreover, the following is true (see the discussion in [3, before Theorem 1.6]).

**Theorem 15 ([3]).** For a generic $a \in \mathfrak{m}$, the the set $\mathfrak{B} + pr^*_f(\mathbb{R}[\mathfrak{f}])$ is complete on $\mathfrak{g}$.

Here, $a \in \mathfrak{m}$ is generic if the dimension of the isotropy algebras $\mathfrak{g}(a)$ and $\mathfrak{h}(a)$ are minimal. It would be interesting to prove the above statement in the singular case as well. In particular, we have the following statement (see [37, pages 241–244] and [9, Theorem 1]).

**Theorem 16 ([9, 37]).** Consider the symmetric pair $(\mathfrak{h}, \mathfrak{g}) = (\mathfrak{sl}(n), \mathfrak{so}(n))$ and let

\[ a = \text{diag}(a_1, \ldots, a_k, \ldots, a_k), \quad \mathfrak{f} = \mathfrak{so}(n)(a) = \mathfrak{so}(n_1) \oplus \cdots \oplus \mathfrak{so}(n_k). \]

The set $\mathfrak{B}$ is a complete commutative set of $\text{ad}_\mathfrak{f}$-invariant polynomials on $\mathfrak{so}(n)$.

Integrals $\mathfrak{B}$ are referred as Manakov integrals [24]. Theorem 16 implies complete integrability of a motion of a symmetric rigid body about a fixed point in $\mathbb{R}^n$ (see [11, 24]), having the operator of inertia $I = A^{-1}$ of the form

\[ x = I(\omega) = J\omega + \omega J, \quad \omega \in \mathfrak{so}(n), \]

where a mass tensor is $J = \text{diag}(b_1, \ldots, b_n)$, $a_i = b_i^2$ (see [9, Subsection 1.6]).

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**References**

[1] Bazaikin, Ya. V., Double quotients of Lie groups with an integrable geodesic flow, *Siborsk. Mat. Zh.* 2000, vol. 41, pp. 513–530 (Russian); English translation: *Siberian Math. J.* 2000, vol. 41, pp. 419–432.

[2] Bogoyavlenskii, O. I., Integrable Euler equations associated with filtrations of Lie algebras, *Mat. Sb.* 1983, vol. 121(163), pp. 233–242 (Russian). English translation: *Sbornik: Mathematics* 1991, vol. 55, no. 1, pp. 68–92 (Russian); English translation: *Math. USSR-Izv.* 1992, vol. 38, no. 1, pp. 69–90.

[3] Bolsinov, A. V., Complete Poisson brackets on Lie algebras and the completeness of families of functions in involution, *Izv. Acad. Nauk SSSR, Ser. matem.* 1991, vol. 55, no. 1, pp. 68–92 (Russian); English translation: *Math. USSR-Izv.* 1992, vol. 38, no. 1, pp. 69–90.

[4] Bolsinov, A. V., Complete commutative subalgebras in polynomial Poisson algebras: a proof of the Mischenko–Fomenko conjecture, *Theoretical and Applied Mechanics*, 2016, vol. 43, pp. 145–168.

[5] Bolsinov, A. V. and Jovanović, B., Integrable geodesic flows on homogeneous spaces, *Matem. Sbornik* 2001, vol. 192, no. 7, pp. 21–40 (Russian). English translation: *Siberian Math. J.* 2001, vol. 42, no. 7-8, 951–969.

[6] Bolsinov, A. V. and Jovanović, B., Non-commutative integrability, moment map and geodesic flows, *Annals of Global Analysis and Geometry*, 2003, vol. 23, pp. 305–322, arXiv: math-ph/0109031.

[7] Bolsinov, A. V. and Jovanović, B., Complete involutive algebras of functions on cotangent bundles of homogeneous spaces, *Mathematische Zeitschrift*, 2004, vol. 246, no. 1-2, pp. 213–236.

[8] Briolov, A.V., Construction of complete integrable geodesic flows on compact symmetric spaces. *Izv. Acad. Nauk SSSR, Ser. matem.* 1986, vol. 50 no. 2, pp. 661–674 (Russian); English translation: *Izvestiya: Mathematics*, 1987, vol. 29 no. 1, 19–31.

[9] Dragović, V., Gajić B. and Jovanović, B., Singular Manakov Flows and Geodesic Flows of Homogeneous Spaces of $\text{SO}(n)$, *Transformation Groups*, 2009, vol. 14, no. 3, 513–530, arXiv:0901.2444.

[10] Dragović, V., Gajić B. and Jovanović, B., On the completeness of the Manakov integrals, *Fundam. Prikl. Mat.*, 2015, vol. 20, no. 2, pp 35–49 (Russian). English translation: *J. Math. Sci.*, 2017, vol. 223, no. 6, pp. 675–685, arXiv:1504.07221.

[11] Fedorov, Yu. N. and Kozlov, V. V., Various aspects of $n$-dimensional rigid body dynamics, *Amer. Math. Soc. Transl. Series 2* 2, 1995, vol. 168, pp. 141–171.

[12] Gel’fand, I and Tsetlin M., Finite-dimensional representation of the group of unimodular matrices, *Dokl. Akad. Nauk SSSR*, 1950, vol. 71, pp. 825–828.

[13] Gel’fand, I and Tsetlin M., Finite-dimensional representation of the group of orthogonal matrices, *Dokl. Akad. Nauk SSSR*, 1950, vol. 71, pp. 1017–1020.
[14] Guillemin, V and Sternberg, S., On collective complete integrability according to the method of Thimm, *Ergod. Th. & Dynam. Sys.* 1983, vol. 3, pp. 219–230.

[15] Guillemin, V and Sternberg, S., The Gel’fand-Cetlin system and quantization of the complex flag manifolds, *Journal of Function Analysis*, 1983, vol. 52, 106–128.

[16] Guillemin, V and Sternberg, S., Multiplicity-free spaces, *J. Diff. Geometry*, 1984, vol. 19, pp. 31–56.

[17] Harada, M., The symplectic geometry of the Gel’fand-Cetlin-Molev bases for representation of $Sp(2n, \mathbb{C})$, *J. Sympl. Geom.* 2006, vol. 4, pp. 1–41, arXiv:math/0404485.

[18] Heckman, G. J., Projections of orbits and asymptotic behavior of multiplicities for compact connected Lie groups, *Invent. Math.* 1982, vol. 67, no. 2, pp. 333–356.

[19] Jovanović, B., Geometry and integrability of Euler–Poincaré–Suslov equations, *Nonlinearity*, 2001, Vol. 14, 1555–1567, arXiv:math-ph/0107024.

[20] Jovanović, B., Integrability of Invariant Geodesic Flows on n-Symmetric Spaces, *Annals of Global Analysis and Geometry*, 2010, vol. 38, pp. 305–316, arXiv:1006.3693 [math.DG].

[21] Jovanović, B., Integrability of Invariant Geodesic Flows on n-Symmetric Spaces, *Annals of Global Analysis and Geometry*, 2010, vol. 38, pp. 305–316, arXiv:1006.3693 [math.DG].

[22] Krämer, M., Multiplicity free subgroups of compact connected Lie groups, *Arch. Math.*, 1976, vol. 27, pp. 28–36.

[23] Lompert, K. and Panasyuk, A., Invariant Nijenhuis Tensors and Integrable Geodesic Flows, *Symmetry, Integrability and Geometry: Methods and Applications (SIGMA)*, 2019, vol. 15, 056, 30 pages, arXiv:1812.04511.

[24] Manakov, S. V., Note on the integrability of the Euler equations of n-dimensional rigid body dynamics, *Funk. Anal. Pril.*, 1976, vol. 10, no. 4, pp. 93–94 (Russian).

[25] Mikityuk, I. V., Integrability of the Euler equations associated with filtrations of semisimple Lie algebras, *Matem. Sbornik* 1984, vol. 125(167), no. 4 (Russian); English translation: *Math. USSR Sbornik* 1986 vol. 53, no. 2, pp. 541–549.

[26] Mikityuk, I.V., Integrability of geodesic flows for metrics on suborbits of the adjoint orbits of compact groups, *Transform. Groups*, 2016, vol. 21, pp. 531–553.

[27] Mykytyuk, I. V. and Panasyuk A., Bi-Poisson structures and integrability of geodesic flows on homogeneous spaces. *Transformation Groups*, 2004, vol. 9, no. 3, pp. 289–308.

[28] Mishchenko, A. S. and Fomenko, A. T., Euler equations on finite-dimensional Lie groups, *Izv. Acad. Nauk SSSR, Ser. matem.*, 1978, vol. 42, no. 2, pp. 396–415 (Russian); English translation: *Math. USSR Izv.* 1978, vol. 12, no. 2, pp. 371–389.

[29] Mishchenko, A. S. and Fomenko, A. T., Generalized Liouville method of integration of Hamiltonian systems. *Funkt. Anal. Prilozh.* 1978, vol. 12, no. 2, pp. 46–56 (Russian); English translation: *Funct. Anal. Appl.*, 1978, vol. 12, pp. 113–121.

[30] Nekhoroshev, N. N., Action-angle variables and their generalization. *Tr. Mosk. Mat. O.-va* 1972, vol. 26, pp. 181–198 (Russian); English translation: *Trans. Moscow Math. Soc.*, 1972, vol. 26, 180–198.

[31] Pan’yanov, D. I. and Yakimova, O. S., Poisson-commutative subalgebras of $S(g)$ associated with involutions, *International Mathematics Research Notices*, 2021, vol. 2021, pp. 18367–18406, arXiv:1809.00350.

[32] Pan’yanov, D. I. and Yakimova, O. S., Reductive subalgebras of semisimple Lie algebras and Poisson commutativity, *Journal of Symplectic Geometry*, 20 (2022), pp. 911–926, arXiv:2012.04014.

[33] Reyman, A. G., Integrable Hamiltonian systems connected with graded Lie algebras, *Zap. Nauchn. Semin. LOMI AN SSSR*, 1980, vol. 95, pp. 3–54 (Russian); English translation: *J. Sov. Math.*, 1982, vol. 19, pp. 1507–1545.

[34] Sadetov, S. T., A proof of the Mishchenko-Fomenko conjecture (1981). *Dokl. Akad. Nauk* 2004, vol. 397, no. 6, 751–754 (Russian).

[35] Thimm A., Integrable geodesic flows on homogeneous spaces, *Ergod. Th. & Dynam. Sys.* 1981, vol. 1, pp. 495–517.

[36] Trofimov, V. V., Euler equations on Borel subalgebras of semisimple Lie groups, *Izv. Acad. Nauk SSSR, Ser. matem.*, 1979, vol. 43, no. 3, pp. 714–732 (Russian).

[37] Trofimov, V. V. and Fomenko, A. T., *Algebra and geometry of integrable Hamiltonian differential equations*, Moskva, Faktorial, 1995 (Russian).