A note on the entropy of rotating BPS AdS$_7 \times S^4$ black holes

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Abstract: In this note we show that the entropy of BPS, rotating, electrically charged AdS$_7 \times S^4$ black holes can be obtained by an extremization principle involving a particular combination of anomaly coefficients of the six-dimensional $\mathcal{N} = (2,0)$ theory. This result extends our previous finding for BPS, rotating AdS$_5 \times S^5$ black holes.
1 Introduction

There has been some recent progress in deriving the entropy of BPS static, asymptotically AdS$_4$ magnetically charged black holes that can be embedded in string/M-theory [1–6]. The method uses a dual field theory computation based on localization. There are also many examples of BPS, electrically charged, rotating black holes in various dimensions whose entropy cannot be yet explained in this way. The most famous ones are asymptotic to AdS$_5 \times S^5$ [7–11]. They depend on three electric charges $Q_I$ ($I = 1, 2, 3$), associated with rotations in $S^5$, and two angular momenta $J_{\phi}, J_\psi$ in AdS$_5$. Supersymmetry actually requires a constraint among the charges and only four of them are independent. The derivation of their entropy in terms of states of the dual $\mathcal{N} = 4$ SU($N$) super Yang-Mills (SYM) theory is still an open problem [12–14]. The natural place where to look for such derivation is the superconformal index [12, 15]

$$I_{S^3 \times S^1}(\Delta_I, \omega_i) = \text{Tr}_\mathcal{H}(-1)^F e^{-2\pi i \left(\sum_{i=1}^{3} \Delta_i r_i - \sum_{i=1}^{2} \omega_i h_i\right)},$$  

(1.1)

where $h_i$ are the generators of angular momentum, $r_i$ are the Cartan generators of the SO(6) R-symmetry and $\omega_i, \Delta_I$ are the conjugate chemical potentials, respectively. (1.1) is defined for $\Delta_1 + \Delta_2 + \Delta_3 + \omega_1 + \omega_2 \in \mathbb{Z}$ since the exponent should commute with the relevant supercharge and chemical potentials are only defined modulo one. The index counts states preserving the same supersymmetries of the black holes and it depends on a number of
fugacities equal to the number of conserved charges of the black holes. However, due to a large cancellation between bosonic and fermionic states, the superconformal index is a quantity of order one for generic values of the fugacities while the entropy scales like $N^2$ [12]. We recently observed [16] that the Bekenstein-Hawking entropy of these black holes can be obtained as the Legendre transform with respect to $\omega_i$ and $\Delta_I$ of the quantity

$$E^{SU(N)} = -i\pi N^2 \frac{\Delta_1 \Delta_2 \Delta_3}{\omega_1 \omega_2},$$

with the determination

$$\sum_{I=1}^{3} \Delta_I + \sum_{i=1}^{2} \omega_i = 1.$$  \hspace{1cm} (1.3)

This constraint among chemical potentials reflects the constraint among charges of the black holes and is compatible with the constraint in the index (1.1). The quantity (1.2) can be expressed in terms of the flavored cubic t’Hooft anomaly coefficients of $\mathcal{N} = 4$ SYM. Indeed it can be obtained by an equivariant integral of the anomaly polynomial of the theory, as shown in [17].

In this short note we extend our observation to BPS, electrically charged, rotating black holes in $\text{AdS}_7 \times S^4$. We expect a family of black holes depending on two electric charges $Q_I$ ($I = 1, 2$), associated with rotations in $S^4$, and three angular momenta $J_i$ ($i = 1, 2, 3$) in $\text{AdS}_5$. Supersymmetry again requires a constraint among the charges and only four of them are independent. The dual field theory is the $A_{N-1} \mathcal{N} = (2, 0)$ theory in six dimensions. Inspired by the $\text{AdS}_5$ result, we consider the expression for the equivariant integral of the anomaly polynomial of the theory, which, at large $N$, is given by [17]

$$E^{(A_{N-1})}(\Delta_i, \omega_i) = i\pi N^3 \frac{(\Delta_1 \Delta_2)^2}{12 \omega_1 \omega_2 \omega_3}.$$ \hspace{1cm} (1.4)

We will show that the Bekenstein-Hawking entropy of seven-dimensional BPS black holes can be obtained by extremizing the quantity

$$- E^{(A_{N-1})}(\Delta_I, \omega_i) - 2\pi i \sum_{I=1}^{2} \Delta_I Q_I - 2\pi i \sum_{i=1}^{3} \omega_i J_i,$$ \hspace{1cm} (1.5)

with respect to $\Delta_I, \omega_i$ with the constraint

$$\sum_{I=1}^{2} \Delta_I + \sum_{i=1}^{3} \omega_i = 1.$$ \hspace{1cm} (1.6)

\footnote{With a different choice of angular determination for the chemical potentials $\Delta_I$ and $\omega_i$, the quantity (1.2) becomes the supersymmetric Casimir energy of the theory (see for example Eq. (4.50) in [17]). The supersymmetric Casimir energy can be interpreted as the energy of the vacuum [18] and it arises both as a prefactor in the relation that connects the supersymmetric partition function on $S^3 \times S^1$ to the index [19, 20] and also as a term in the high-temperature expansion of the index [21, 22]. All these results seem to have been obtained assuming a particular angular determination of fugacities, which implies, in particular, $\Delta_1 + \Delta_2 + \Delta_3 + \omega_1 + \omega_2 = 0$, instead of (1.3).}
The general black hole solution depending on all four conserved quantities is not available but we will check that (1.5) correctly reproduces the entropy of the existing solutions. A two-parameter black hole, with two electric charges and one angular momentum, was found in [23] as the BPS limit of a non-extremal solution [24]. The solution was later extended to a three-parameter one, with three rotations and one electric charge, in [25]. We have explicitly checked the validity of (1.5) in both cases and are thus confident that the result holds in general.

It is remarkable that the equivariant integral of the anomaly polynomial accounts for the entropy of both the AdS$_5 \times S^5$ and AdS$_7 \times S^4$ supersymmetric black holes. Moreover, it is noteworthy that, in both cases, the solutions to the extremization equations associated with (1.2) and (1.4) are complex but the value of the Legendre transform at the critical points, the entropy, is a real function of the black hole charges. This result still needs a proper field theory interpretation. Here we just make few observations.

According to the standard holographic dictionary, the black hole entropy should account for the 1/16–BPS states in the $\mathcal{N} = (2, 0)$ theory with given electric charge and angular momentum. As in four dimensions, the partition function for such BPS states is too difficult to compute due to the small amount of preserved supersymmetry. The superconformal index of the $\mathcal{N} = (2, 0)$ theory, on the other hand, counts states weighted with signs:

$$I_{S^5 \times S^1}(\Delta_I, \omega_i) = \text{Tr}_H(-1)^F e^{-2\pi i \left(\sum_{I=1}^2 \Delta_I r_I - \sum_{i=1}^3 \omega_i h_i\right)},$$

(1.7)

where $h_i$ are the generators of angular momentum, $r_I$ are the Cartan generators of the SO(5) R-symmetry, and $\Delta_1 + \Delta_2 + \omega_1 + \omega_2 + \omega_3 \in \mathbb{Z}$. The index is explicitly computable but, as in four dimensions, is a quantity of order one for generic values of the fugacities. The $S^5 \times S^1$ partition function, on the other hand, is related to the index in the large $N$ limit by [26]

$$\log Z_{S^5 \times S^1}(\Delta_I, \omega_i) = -E_{\text{susy}}(\Delta_I, \omega_i) + \log I_{S^5 \times S^1}(\Delta_I, \omega_i),$$

(1.8)

where $E_{\text{susy}}$ is the supersymmetric Casimir energy and scales like $\mathcal{O}(N^3)$. It would be interesting to see if the quantity (1.4) dominates the index or the partition function in some particular regime for the chemical potentials and a choice of angular determination compatible with the constraint (1.6). An expression for $E_{\text{susy}}$ has been conjectured in [17] by integrating the anomaly polynomial of the $\mathcal{N} = (2, 0)$ theory, and formally coincides with (1.5). However, the conjecture seems to assume a different angular determination for the chemical potentials, compatible with $\sum_{I=1}^2 \Delta_I + \sum_{i=1}^3 \omega_i = 0$ rather than (1.6).

The AdS$_5 \times S^5$ and AdS$_7 \times S^4$ black holes behave quite differently from their magnetically charged relatives in AdS$_4 \times S^7$ whose entropy has been recently derived [1, 2]. The main difference comes from the magnetic charges that have a dual interpretation as a topological twist [27, 28]. The topologically twisted index [29, 30] has been shown to scale like $N^{3/2}$ in the large $N$ limit [1, 31, 32], suggesting that there is no cancellation between

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2In our notations, $Q_I$ are eigenstates of $r_I$ and $J_i$ of $-h_i$. 
bosonic and fermionic ground states, while the superconformal index is a quantity of order one. It should also be noticed that the derivation of the entropy of AdS$_4 \times S^7$ black holes is a purely microscopic counting with no reference to the ubiquitous Cardy formula [33]. On the other hand, whatever its field theory interpretation is, the extremization principles in five and seven dimensions suggests that some sort of Cardy mechanism is at work. First of all, as already said, (1.2) and (1.4) can be written in terms of anomaly coefficients for the R and flavor symmetries of the dual theory. This is simple to see for (1.2). Indeed, by an obvious redefinition of the chemical potentials (see Appendix A), (1.2) can be written as the large $N$ limit of

$$E^{\text{SU}(N)} = \frac{4\pi i}{27} \frac{(\omega_1 + \omega_2 - 1)^3}{\omega_1 \omega_2} a(\hat{\Delta}_I).$$

(1.9)

Here

$$a(\hat{\Delta}_I) = \frac{9}{32} \text{Tr} R^3(\hat{\Delta}_I) = \frac{27}{32}(N^2 - 1)\hat{\Delta}_1\hat{\Delta}_2\hat{\Delta}_3,$$

(1.10)

together with $\hat{\Delta}_1 + \hat{\Delta}_2 + \hat{\Delta}_3 = 2$ is the trial central charge of $\mathcal{N} = 4$ SYM. (1.4) can be written similarly since it arises from an equivariant integration of the eight-form anomaly polynomial of the 6D $\mathcal{N} = (2,0)$ theory [17]. Moreover, for an extremal BTZ black hole in AdS$_3$ the relevant quantity to consider is the elliptic genus, whose logarithm in the large $N$ limit goes as $c_l/\omega$ where $\omega$ is the chemical potential associated with rotation and $c_l$ is the left-moving central charge. The entropy of a black hole of angular momentum $j$ is then correctly reproduced by the Legendre transform with respect to $\omega$, i.e. $S \propto \sqrt{j c_l}$. Obviously, this is nothing else than Cardy formula.

The paper is organized as follows. In section 2 we first review the basic features of the relevant truncation of seven-dimensional maximal gauged supergravity and we later discuss the BPS, rotating black holes of interest. In section 3, we show that the Bekenstein-Hawking entropy of the black holes can be obtained as the Legendre transform of the quantity (1.4). We conclude in section 4 with discussions and future directions. In the appendices we briefly discuss a conjecture to extend our previous result for black holes in AdS$_5 \times S^5$ to more general compactifications and the dimensional reduction of the seven-dimensional black holes to six dimensions.

2 Supersymmetric rotating AdS$_7$ black holes

The supersymmetric rotating AdS$_7$ black holes of interest are solutions of the SO(5) maximal ($\mathcal{N} = 4$) gauged supergravity in seven dimensions [34], obtained by reducing eleven-dimensional supergravity on $S^4$ [35, 36]. We will work with a $U(1)^2$ consistent truncation [37] of the theory, which consists of the metric, a three-form potential $A_{(3)}$, two Abelian

$^3$Comparing to expressions for the supersymmetric Casimir energy that can be found in [18–20] we have an extra minus one in the numerator which is due to the constraint (1.6).
gauge fields $A^I_1$ ($I = 1, 2$) in the Cartan of SO(5) and two real scalars $\varphi_1$ and $\varphi_2$. The bosonic Lagrangian is given by [34]

$$\mathcal{L}_7 = R \ast 1 - \frac{1}{2} \sum_{i=1}^{2} \ast d\varphi_i \wedge d\varphi_i - \frac{1}{2} \sum_{I=1}^{2} L_I^{-2} \ast F^I_2 \wedge F^I_2 - \frac{1}{2} (L_1 L_2)^2 \ast F_4 \wedge F_4$$

$$- 2g^2 \left[ (L_1 L_2)^{-4} - 8L_1 L_2 - 4L_1^{-1}L_2^{-2} - 4L_1^{-2}L_2^{-1} \right] \ast 1$$

$$- gF_4 \wedge A_3 + F^1_2 \wedge F^2_2 \wedge A_3,$$

where

$$F^I_2 = dA^I_1,$$

$$F_4 = dA_3,$$

$$L_1 = e^{-\frac{1}{\sqrt{2}}\varphi_1 - \frac{1}{\sqrt{10}}\varphi_2},$$

$$L_2 = e^{\frac{1}{\sqrt{2}}\varphi_1 - \frac{1}{\sqrt{10}}\varphi_2},$$

and $g$ is the gauge coupling constant. There is a “first-order self-duality” condition for the four-form field strength that has to be imposed after the variation of the Lagrangian and that can be conveniently written by including a two-form potential $A_2$, and defining

$$F_3 = dA_2 - \frac{1}{2} A^1_1 \wedge dA^2_1 - \frac{1}{2} A^2_1 \wedge dA^1_1.$$

The self-duality equation then reads

$$(L_1 L_2)^2 \ast F_4 = -2gA_3 - F_3.$$

We are interested in supersymmetric black holes with electric charges $Q_I$ ($I = 1, 2$) under the U(1)$^2$ and angular momenta $J_i$ ($i = 1, 2, 3$) in AdS$_7$. We expect supersymmetry to impose a constraint among the five charges, leaving four independent ones. The most general family of such black holes has not been written yet. A two-parameter black hole, with two electric charges and one angular momentum with a constraint among them, was found in [23]. A three-parameter family of black holes, with three rotations and one electric charge with a constraint, was later found in [25]. Note that in both cases the near-horizon geometry is a warped product of AdS$_3$ and a squashed $S^5$. We now write explicitly these solutions and their thermodynamic quantities. We have corrected few misprints in [23].

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Footnote: Here we use the conventions of [23].
\[2.1\] Single-rotation two-charge black holes

The solution can be written as [23]

\[ ds^2 = (H_1 H_2)^{1/5} \left( - \frac{V}{H_1 H_2 B} r^2 dt^2 + B(\sigma + f dt)^2 + \frac{dr^2}{V} + r^2 ds_{CP}^2 \right), \]

\[ A_{(1)} = \frac{2ms_1}{\rho^4 \Xi H_1} (\alpha_1 \Xi dt + \beta_1 \sigma), \]

\[ A_{(2)} = \frac{mas_1 s_2}{\rho^4 \Xi^-} \left( \frac{1}{H_1} + \frac{1}{H_2} \right) dt \wedge \sigma, \quad A_{(3)} = \frac{2mas_1 s_2}{\rho^4 \Xi^-} \sigma \wedge J, \]

\[ L_I = (H_1 H_2)^{3/5} H^{-1}_I, \quad H_I = 1 + \frac{2ms_I}{\rho^4}, \quad \rho = \sqrt{\Xi r}, \]

\[ \alpha_1 = c_1 - \frac{1}{2} (1 - \Xi^2) (c_1 - c_2), \quad \alpha_2 = c_2 + \frac{1}{2} (1 - \Xi^2) (c_1 - c_2), \]

\[ \beta_1 = -a \alpha_2, \quad \beta_2 = -a \alpha_1, \quad \Xi^\pm = 1 \pm a g, \quad \Xi = 1 - a^2 g^2, \]

\[ s_I \equiv \sinh \delta_I, \quad c_I \equiv \cosh \delta_I. \]

The metric functions \( V, B \) and \( f \) depends on the radial coordinate \( r \) and are given by

\[ V = \frac{Y}{\Xi^6 \rho^5}, \quad B = \frac{f_1}{H_1 H_2 \Xi^2 \rho^4}, \quad f = -\frac{2f_2}{f_1}, \]

where

\[ f_1 = \Xi^6 H_1 H_2 - \frac{4 \Xi^2 a^2 s_1^2 s_2^2}{\rho^4} \left[ 4 \Xi^2 - 4c_1 c_2 (\Xi^4 - 1) + (c_1^2 + c_2^2)(\Xi^2 - 1)^2 \right], \]

\[ f_2 = -\frac{1}{2} g \Xi^6 H_1 H_2 + \frac{1}{4} m a \left[ 2c_1 c_2 (\Xi^4 + 1) - (c_1^2 + c_2^2)(\Xi^2 - 1) \right], \]

\[ Y = g^2 \rho^8 H_1 H_2 + \Xi^6 + \frac{1}{2} m a^2 \left[ 4 \Xi^2 - 2c_1 c_2 (\Xi^4 - 1) + (c_1^2 + c_2^2)(\Xi^2 - 1)^2 \right] \]

\[ - \frac{1}{2} m \rho^2 \left[ 4 \Xi + 2c_1 c_2 a^2 g^2 (3a^2 g^2 + 8ag + 6) - (c_1^2 + c_2^2)a^2 g^2 (ag + 2)(3ag + 2) \right]. \]

Only two parameters are independent due to the constraints\(^5\)

\[ e^{\delta_1 + \delta_2} = 1 - \frac{2}{3ag}, \]

\[ m = \frac{128 e^{3\delta_1} (3e^{\delta_1 + \delta_2} - 1)^3}{729g^4 (e^{2\delta_2} - 1)(e^{2\delta_2} - 1) (e^{\delta_1 + \delta_2} + 1)^2 (e^{\delta_1 + \delta_2} - 1)^4}. \]

The former comes from the BPS condition and the latter is required in order to avoid naked closed timelike curves (CTCs). With these choices, the function \( V \) becomes

\[ V = \frac{g^2 (r^2 - r_0^2)^2}{r^2} \left( 1 + \frac{9 e^{2(\delta_1 + \delta_2)} - 6 e^{\delta_1 + \delta_2} + 17}{3(e^{\delta_1 + \delta_2} + 1)(3e^{\delta_1 + \delta_2} - 5)g^2r^2} + \frac{h}{g^4r^4} \right), \]

\(^5\)We correct a misprint in [23] here, i.e. \( m_{\text{here}} = (3e^{\delta_1 + \delta_2} - 1)m_{\text{there}}. \)
where
\[ h = \left[ 32( - 2d_1^2 - 2d_2^2 + 9d_1d_2 + 9d_1^2d_2^2 - 3d_1^2d_2(d_1 + d_2)^2 + 2d_1^2d_2^2(2d_1^2 - 3d_1d_2 + 2d_2^2) \\
- d_1d_2(3d_1^2 - 2d_1d_2 + 3d_2^2)) \right] \sqrt{[9d_1d_2(d_1^2 - 1)(d_2^2 - 1)(d_1d_2 + 1)(3d_1d_2 - 5)^2]}, \] (2.10)
and we defined \( d_I \equiv e^{\delta t} \) (\( I = 1, 2 \)). The black hole has an event horizon at \( V(r_0) = 0 \) which reads
\[ r_0^2 = \frac{16}{3g^2(e^{\delta t} + 1)(3e^{\delta t} - 5)} . \] (2.11)

The line element \( ds^2_{\mathbb{CP}^2} \) in (2.5) is the standard Fubini-Study metric on \( \mathbb{CP}^2 \):
\[ ds^2_{\mathbb{CP}^2} = d\xi^2 + \frac{1}{4} \sin^2 \xi \left( \sigma_1^2 + \sigma_2^2 + \cos^2 \xi \sigma_3^2 \right) , \] (2.12)
where \( \sigma_i \) (\( i = 1, 2, 3 \)) are left-invariant one-forms on \( SU(2) \), satisfying \( d\sigma_i = -\frac{1}{2} \epsilon_{ijk} \sigma_j \wedge \sigma_k \).

Note that, the Kähler form on \( \mathbb{CP}^2 \) is \( J = \frac{i}{2} dB \) with \( B = \frac{1}{2} \sin^2 \xi \sigma_3 \) being the connection of the \( U(1) \) bundle over \( \mathbb{CP}^2 \) whose total space is the unit \( S^5 \). We also have \( \sigma = d\psi + B \) and \( 0 \leq \psi \leq 2\pi \) is the coordinate along the \( U(1) \) fiber of \( S^5 \). The thermodynamic quantities are given by\(^6\)
\[ E = \frac{m\pi^2}{32G_{\text{N}}\Xi^4} \left[ 12\Xi^2(\Xi_+^2 - 2) - 2c_1c_2a^2g^2(21\Xi_+^4 - 20\Xi_+^3 - 15\Xi_+^2 - 10\Xi_+ - 6) \\
+ (c_1^2 + c_2^2)(21\Xi_+^6 - 62\Xi_+^5 + 40\Xi_+^4 + 13\Xi_+^2 - 2\Xi_+ + 6) \right], \]
\[ S = \frac{\pi^3}{4G_{\text{N}}} [B(r_0)H_1(r_0)H_2(r_0)]^{1/2} r_0^4, \]
\[ J = \frac{m\pi^2}{16G_{\text{N}}\Xi^4} \left[ 4ag\Xi_+^2 - 2c_1c_2(2\Xi_+^3 - 3\Xi_+^2 - 1) + ag(c_1^2 + c_2^2)(\Xi_+ + 1)(2\Xi_+^3 - 3\Xi_+^2 - 1) \right], \]
\[ Q_I = \frac{m\pi^2 s_I}{4G_{\text{N}}\Xi^3} \left[ a^2g^2 c_1c_2 \frac{c_I}{2}(2\Xi_+ + 1) - c_I(2\Xi_+^3 - 3\Xi_+^2 - 1) \right], \]
\[ T = 0 , \quad \Omega = -g , \quad \Phi_I = -1. \] (2.13)

The charges satisfy the BPS condition \( E + 3gJ - \sum_{I=1}^2 Q_I = 0 \).

\(^6\) We correct a misprint in [23] here, \( i.e. S_{\text{here}} = \frac{1}{4} S_{\text{there}} \).
2.2 Three-rotation single-charge black holes

The solution reads

$$ds^2 = H^{2/5} \left\{ \frac{(r^2 + y^2)(r^2 + z^2)}{r} dr^2 + \frac{(r^2 + y^2)(y^2 - z^2)}{Y} dy^2 + \frac{(r^2 + y^2)(z^2 - y^2)}{Z} dz^2 \right. $$

$$\left. - \frac{H}{H^2(r^2 + y^2)(r^2 + z^2)} A^2 \right. $$

$$+ \frac{Y}{(r^2 + y^2)(y^2 - z^2)} \left[ dt + (z^2 - r^2) d\psi_1 - r^2 z^2 d\psi_2 - \frac{q}{H(r^2 + y^2)(r^2 + z^2)} A \right]^2 $$

$$+ \frac{Z}{(r^2 + y^2)(z^2 - y^2)} \left[ dt + (y^2 - r^2) d\psi_1 - r^2 y^2 d\psi_2 - \frac{q}{H(r^2 + y^2)(r^2 + z^2)} A \right]^2 $$

$$+ \frac{a_1^2 a_2 a_3}{r^2 y^2 z^2} \left[ dt + (y^2 + z^2) d\psi_1 + (y^2 z^2 - r^2 y^2 - r^2 z^2) d\psi_2 - r^2 y^2 z^2 d\psi_3 \right. $$

$$- \frac{q}{H(r^2 + y^2)(r^2 + z^2)} \left( 1 + \frac{g y^2 z^2}{a_1 a_2 a_3} \right) A \right]^2 \right\}, \tag{2.14}$$

$$L = H^{-1/5}, \quad A_{(1)} = -\frac{q(1 - a_1 g - a_2 g - a_3 g)}{H(r^2 + y^2)(r^2 + z^2)} A,$$

$$A_{(2)} = \frac{q}{H(r^2 + y^2)(r^2 + z^2)} A^\wedge \left\{ dt + \sum_{i=1}^3 a_i^2 (g^2 dt + d\psi_1) + \sum_{i<j} a_i a_j^2 (g^2 d\psi_1 + d\psi_2) + a_1^2 a_2 a_3^2 (g^2 d\psi_2 + d\psi_3) \right.$$

$$- g^2 (y^2 + z^2) dt - g^2 y^2 z^2 d\psi_1 + a_1 a_2 a_3 g \left[ d\psi_1 + (y^2 + z^2) d\psi_2 + y^2 z^2 d\psi_3 \right] \right\},$$

$$A_{(3)} = qa_1 a_2 a_3 \left[ d\psi_1 + (y^2 + z^2) d\psi_2 + y^2 z^2 d\psi_3 \right]$$

$$\wedge \left[ \frac{1}{(r^2 + y^2) z} dz \wedge (d\psi_1 + y^2 d\psi_2) + \frac{1}{(r^2 + z^2) y} dy \wedge (d\psi_1 + z^2 d\psi_2) \right]$$

$$- g g A \wedge \left[ \frac{z}{r^2 + y^2} dz \wedge (d\psi_1 + y^2 d\psi_2) + \frac{y}{r^2 + z^2} dy \wedge (d\psi_1 + z^2 d\psi_2) \right],$$

where

$$R = \left( \frac{r^2 - y^2}{r^2} \right)^2 \left\{ g^2 r^4 + [1 + (a_1^2 + a_2^2 + a_3^2) g^2 + 2 g^2 r^2 \psi] r^2 + \frac{(a_1 a_2 a_3 - q g)^2}{r^4} \right\},$$

$$Y = 1 - \frac{g^2 y^2}{y^2} \prod_{i=1}^3 (a_i^2 - y^2), \quad Z = 1 - \frac{g^2 z^2}{z^2} \prod_{i=1}^3 (a_i^2 - z^2), \tag{2.15}$$

$$\mathcal{A} = dt + (y^2 + z^2) d\psi_1 + y^2 z^2 d\psi_2,$$

$$H = 1 + \frac{q}{(r^2 + y^2)(r^2 + z^2)}.$$

The black hole has an event horizon at $r = r_0$:

$$r_0^2 = \frac{a_1 a_2 + a_2 a_3 + a_3 a_1 - a_1 a_2 a_3 g}{1 - a_1 g - a_2 g - a_3 g}. \tag{2.16}$$
We also denote \( \Xi_i = 1 - a_i^2 g^2 \) and \( \Xi_{i \pm} = 1 \pm a_i g \), which are positive to have a correct signature. The parameters \( q \) and \( a_i \) have to satisfy

\[
q = \frac{\Xi_1 - \Xi_2 - \Xi_3 - (a_1 + a_2)(a_2 + a_3)(a_1 + a_3)}{(1 - a_1 g - a_2 g - a_3 g)^2 g},
\]

in order for the solution to be free from naked CTCs. The thermodynamic quantities are given by

\[
E = -\pi^2 \frac{\prod_{k<l} (a_k + a_l)(\sum_i \Xi_i + \sum_{i<j} \Xi_{i<j} (1 + a_1 a_2 a_3 g^3) \left(2 + \sum_i a_i g + \sum_{i<j} a_i a_j g^2\right))}{8G_N \Xi_1 + \Xi_2 + \Xi_3 + (1 - a_1 g - a_2 g - a_3 g)^2 g r_0},
\]

\[
S = -\frac{\pi^2}{4G_N} \frac{(a_1 + a_2)(a_2 + a_3)(a_1 + a_3) (a_1 a_2 + a_2 a_3 + a_1 a_3 - a_1 a_2 a_3 g)}{\Xi_1 + \Xi_2 + \Xi_3 + (1 - a_1 g - a_2 g - a_3 g)^2 g r_0},
\]

\[
J_i = -\frac{\pi^2}{8G_N} \frac{(a_1 + a_2)(a_2 + a_3)(a_1 + a_3)[a_i - (a_i^2 + 2a_i \sum_{j \neq i} a_j + \sum_{j \neq i} a_j g + a_1 a_2 a_3 g^2)]}{\Xi_1 + \Xi_2 + \Xi_3 + (1 - a_1 g - a_2 g - a_3 g)^2 g},
\]

\[
Q = -\frac{\pi^2}{4G_N} \frac{(a_1 + a_2)(a_2 + a_3)(a_1 + a_3)}{\Xi_1 + \Xi_2 + \Xi_3 + (1 - a_1 g - a_2 g - a_3 g)^2 g},
\]

\[
T = 0, \quad \Omega_i = -g, \quad \Phi = -1.
\]

Finally, the charges satisfy the BPS condition

\[
E + g \sum_{i=1}^{3} J_i - 2Q = 0.
\]

### 3 An extremization principle for the entropy

In this section we will show that the Bekenstein-Hawking entropy of the BPS black holes (2.5) and (2.14) can be obtained as a Legendre transform of a combination of anomaly coefficients of the dual \( \mathcal{N} = (2, 0) \) theory in six dimensions. The result is a natural generalization of the analogous one for \( \text{AdS}_5 \times S^5 \) black holes [16].

#### 3.1 Anomaly polynomials for 6D \( \mathcal{N} = (2, 0) \) field theories

Our analysis involves a quantity formally equal to the supersymmetric Casimir energy of the theory, which we now briefly review.

The supersymmetric Casimir energy, \( E^{(g)} \), for an \( \mathcal{N} = (2, 0) \) theory with algebra \( \mathfrak{g} \) arises in the regularization of the \( S^5 \times S^1 \) partition function [26, 38–43] and is related to the superconformal index defined in (1.7) by

\[
\log Z_{S^5 \times S^1}(\Delta_l, \omega_l) = -E^{(g)}(\Delta_l, \omega_l) + \log Z_{S^5 \times S^1}(\Delta_l, \omega_l).
\]

It is the leading contribution to \( \log Z_{S^5 \times S^1} \) for \( \beta \to \infty \), where \( \beta \) is the radius of \( S^1 \) when the chemical potentials are rescaled as \( \Delta_l = \beta \Delta_l \) and \( \omega_l = \beta \omega_l \).\footnote{We have reabsorbed a standard factor of \( \beta \) in the definition of \( E^{(g)} \) for convenience. (3.1) is usually written as \( \log Z_{S^5 \times S^1} = -\beta E^{(g)} + \log Z_{S^5 \times S^1} \).} Since the superconformal
index is a quantity of order one for generic values of the fugacities, the supersymmetric Casimir energy, which scales as $N^3$, is also the leading contribution to the $S^5 \times S^1$ partition function in the large $N$ limit. The supersymmetric Casimir energy of the $\mathcal{N} = (2, 0)$ theory has been conjectured to be equal to an equivariant integral of the eight-form anomaly polynomial and it reads \[17\]

$$E^\theta(\Delta_I, \omega_i) = r_g E^{(1)}(\Delta_I, \omega_i) + \frac{i\pi}{12} d_g h^\vee_g \frac{(\Delta_1 \Delta_2)^2}{\omega_1 \omega_2 \omega_3},$$

(3.2)

where $r_g, d_g$ and $h^\vee_g$ are the rank, dimension and dual Coxeter number of the simply laced Lie algebra $g$, respectively; $E^{(1)}$ is the supersymmetric Casimir energy of the Abelian tensor multiplet theory:

$$E^{(1)}(\Delta_I, \omega_i) = \frac{i\pi}{24 \omega_1 \omega_2 \omega_3} \left[ (\Delta_1 \Delta_2)^2 - \sum_{i<j} (\omega_i \omega_j)^2 + \frac{1}{4} \left( \sum_{i=1}^3 \omega_i^2 - \Delta_1^2 - \Delta_2^2 \right) \right].$$

(3.3)

Here $\Delta_I$ ($I = 1, 2$) are the chemical potentials conjugate to the R-symmetry generators $r_I$ and $\omega_i$ ($i = 1, 2, 3$) are the chemical potentials conjugate to the Cartan generators of rotations $h_i$ in three orthogonal planes in $\mathbb{R}^6$.

Superconformal indices are defined in general as

$$I(\mu_a) = \text{Tr}_H(-1)^F e^{-\beta \{Q, Q^\dagger\}} e^{-\sum_a \mu_a R_a},$$

(3.4)

for a choice of supercharge $Q$, and $R_a$ is the set of all R and flavor symmetries that commute with $Q$. For the superconformal index (1.7), the linear combination $\sum_{i=1}^3 h_i + \sum_{I=1}^2 r_I$ does not commute with $Q$. This translates into a linear constraint among the chemical potentials

$$2 \sum_{I=1}^2 \Delta_I + 3 \sum_{i=1}^3 \omega_i = n, \quad n \in \mathbb{Z}.$$  

(3.5)

Notice that, since $r_I$ and $J_i$ have integer eigenvalues, the chemical potentials are only defined modulo one. For this reason the right-hand side of (3.5) is not required to vanish but it must be an integer. In four dimensions, where an analogous constraint appears in the definition of the four-dimensional superconformal index, the statement (3.1) has been derived under the assumption $n = 0$. In six dimensions, things are less clear.

In this paper we shall consider the quantity (3.2), arising from the equivariant integral of the eight-form anomaly polynomial, for a general choice of angular ambiguities in (3.5). For $g = A_{N-1}$, the equivariant integral at large $N$ reads

$$E^{(A_{N-1})}(\Delta_I, \omega_i) = i\pi N^3 \frac{(\Delta_1 \Delta_2)^2}{12 \omega_1 \omega_2 \omega_3}.$$  

(3.6)
3.2 Reproducing the entropy

We now show that the Bekenstein-Hawking entropy of the BPS black holes (2.5) and (2.14) can be obtained by extremizing the quantity

\[ S(\Delta_I, \omega_i) \equiv -E^{(A_{N-1})}(\Delta_I, \omega_i) - 2\pi i \sum_{I=1}^{2} \Delta_I Q_I - 2\pi i \sum_{i=1}^{3} \omega_i J_i, \]

where \( E^{(A_{N-1})} \) is given in (3.6), with respect to \( \Delta_I, \omega_i \) and subject to the constraint

\[ \sum_{I=1}^{2} \Delta_I + \sum_{i=1}^{3} \omega_i = 1. \]

In order to check it, it is convenient to work with the following parameterization of the chemical potentials:

\[ \omega_1 = \frac{1}{1 + z_1 + z_2 + z_3 + z_4}, \quad \Delta_1 = \frac{z_1}{1 + z_1 + z_2 + z_3 + z_4}, \]
\[ \Delta_2 = \frac{z_2}{1 + z_1 + z_2 + z_3 + z_4}, \quad \omega_2 = \frac{z_3}{1 + z_1 + z_2 + z_3 + z_4}, \]
\[ \omega_3 = \frac{z_4}{1 + z_1 + z_2 + z_3 + z_4}. \]

Note that \( \sum_{I=1}^{2} \Delta_I + \sum_{i=1}^{3} \omega_i = 1 \). Then the extremization equations become

\[ \frac{2^7 g^6 G_N}{\pi^2} (q_a - q_0) = -\frac{z_1^2 z_2^2}{z_3 z_4} \left(1 + 2/z_a\right), \quad \text{for } a = 1, 2, \]
\[ \frac{2^7 g^6 G_N}{\pi^2} (q_b - q_0) = -\frac{z_1^2 z_2^2}{z_3 z_4} \left(1 - 1/z_b\right), \quad \text{for } b = 3, 4, \]

where we have relabeled the black hole charges as

\[ J_1 = q_0, \quad Q_1 = q_1, \quad Q_2 = q_2, \quad J_2 = q_3, \quad J_3 = q_4. \]

The value of \( S \) at the critical point \( \tilde{z}_i \), as a function of the charges, is given by

\[ S(Q_I, J_i) = \frac{i\pi^3}{2^6 g^6 G_N} \tilde{z}_1 \tilde{z}_2 \tilde{z}_3 \tilde{z}_4 - 2\pi i J_1. \]

By an explicit computation one can check that the solution to the extremization equations is complex; however, quite remarkably, \( S \) at the critical point is a real function of the black hole charges. Moreover, by equating two electric charges or three angular momenta, one can check that it precisely coincides with the entropy of the black holes (2.5) and (2.14)

\[ S|_{\text{crit}}(Q_I, J_i) = S_{BH}(Q_I, J_i). \]
In order to compare the field theory inspired result with the gravity ones in (3.13) we made use of the relation between field theory and gravitational parameters in the large $N$ limit, which is given by

$$N^3 = \frac{3\pi^2}{16g^5G_N}. \quad (3.14)$$

It is quite remarkable that the entropy of the black holes is reproduced as a Legendre transform of the integrated anomaly polynomial with the correct field theory normalization.

4 Discussion and conclusions

In this note we have extended our previous observation [16] that the entropy of BPS, rotating $\text{AdS}_5 \times S^5$ black holes can be written as the Legendre transform of a combination of anomaly coefficients for $R$ and flavor symmetries of the dual theory to the case of $\text{AdS}_7 \times S^4$ black holes. It would be interesting to see if the same results hold only for maximally supersymmetric dual theories or it can be also extended to rotating black holes asymptotic to $\text{AdS}_5 \times Y_5$, where $Y_5$ is a five-dimensional Sasaki-Einstein manifold. For such black holes there is a natural conjecture that we discuss in Appendix A.

An important rôle in our analysis is played by the angular ambiguities (3.5) in the definition of chemical potentials, which affect both the partition function on $S^5 \times S^1$ and the index. A choice of determination for the chemical potentials should be made when performing limits, for example low- and high-temperature, or modular transformations of the integrand of the corresponding matrix models, since these operations typically involve multi-valued functions. Examples in the analogous four-dimensional case can be found in [19–22, 44]. It is then interesting to ask whether there exists a limit in the fugacities, subject to the constraint (1.6), where the quantity (1.4) dominates the partition function or the index. We notice that, also for static magnetically charged black holes in $\text{AdS}_4$, the ambiguities played a crucial rôle. It was shown indeed in [1, 31, 32] that, if we assume that all the real parts of chemical potentials live in the interval $[0, 2\pi]$, one can find a consistent saddle point for the topologically twisted index and reproduce the entropy of the black holes, only if the sum of all chemical potentials is a very specific multiple of $2\pi$.\(^8\)

It would be also very interesting to compute, using supersymmetric holographic renormalization, the on-shell action of the black holes in $\text{AdS}_5 \times S^5$ and $\text{AdS}_7 \times S^4$. It has been shown in [45–47] that, for a class of BPS static $\text{AdS}_4$ black holes, the on-shell action indeed reproduces the entropy of the black holes and, in the grand canonical picture, the large $N$ limit of the twisted index. It would be interesting to see if we can reproduce (1.4) via a holographic computation. Notice also that anomalies seem to affect the field theory and

\(^8\)More precisely, with the determination $\text{Re} \Delta_I \in [0, 2\pi]$, one finds a saddle point, up to discrete symmetries, only if the sum of all $\Delta_I$ appearing in each superpotential term is $2\pi$. The interval $[0, 2\pi]$ in [1, 31, 32] is analogous to the interval $[0, 1]$ in this paper.
holographic computation [48, 49]. These anomalies could also be responsible for the choice of determination (1.6).

Finally, we noticed in [16] that the extremization for $\text{AdS}_5 \times S^5$ black holes with equal rotations has a nice interpretation in terms of an attractor mechanism for static black holes in four-dimensional gauged supergravity upon dimensional reduction of the squashed $S^3$ horizon geometry along the Hopf fiber. It would be interesting to show that a similar mechanism is at work here, using $\mathcal{N} = (1, 1)$ six-dimensional gauged supergravity for the solution that is obtained by dimensional reduction of the squashed $S^5$ horizon geometry of the black hole with equal angular momenta along the Hopf fiber. We briefly discuss the physical interpretation of this reduction in Appendix B.

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A AdS$_5$ black holes entropy and anomalies in four dimensions

In this appendix we make some remarks on a possible generalization of the relation between the entropy of BPS, rotating AdS$_5$ black holes and the anomaly polynomials of their field theory duals originally presented in [16]. In particular, we consider supersymmetric black hole solutions asymptotic to $\text{AdS}_5 \times Y_5$, where the internal space $Y_5$ is a Sasaki-Einstein manifold.

Consider five-dimensional gauged supergravity with $n_V$ massless vector multiplets and Fayet-Iliopoulos (FI) gauging. The Lagrangian is completely determined by the symmetric coefficients $C_{IJK}, I, J, K \in \{1, ..., n_V\}$ which can be read off from the Chern-Simons terms in the Lagrangian,

$$L_5 = eR_5 + \ldots - \frac{1}{6} C_{IJK} F^I \wedge F^J \wedge A^K + \ldots ,$$

(A.1)

and the FI parameters $\xi_I$, that specify the linear combination $\xi_I A^I$ used for electrically gauging the R-symmetry. Here, $A^I$ are the U(1) gauge fields and $F^I$ their corresponding field strengths. See for example [50] and references therein for a comprehensive description of five-dimensional gauged supergravity.
The general supersymmetric rotating black holes in the above class of five-dimensional gauged supergravity were written down in [11] after the seminal paper [7] and further developments. The black hole solutions, apart from explicitly depending on the numbers \( C_{IJK} \) and \( \xi_I \), depend on the set of asymptotic charges given by \( n_V \) electric charges \( Q_I \) and two angular momenta \( J_\pm \). Due to the requirement of supersymmetry and the existence of a smooth black hole horizon, there is one additional constraint among the set of asymptotic charges. It is particularly useful to consider a Scherk-Schwarz dimensional reduction down to four dimensions as it was done in [16], where one can explicitly write down the black hole attractor mechanism. The resulting four-dimensional supergravity has \((n_V + 1)\) U(1) vector fields (the new Kaluza-Klein gauge field is labeled by \( A^0 \)) and is uniquely specified by a the holomorphic prepotential,

\[
\mathcal{F}(X^\Lambda) = -\frac{1}{6} C_{IJK} X^I X^J X^K X^0.
\] (A.2)

The prepotential uniquely determines the scalar manifold given by the holomorphic sections \( X^\Lambda, \Lambda \in \{0, I\} \), in turn defining all kinetic terms in the four-dimensional Lagrangian. The R-symmetry in four dimensions is again gauged by the linear combination \( \xi_\Lambda A^\Lambda \) where the new gauge field \( A^0 \) is included with a weight \( \xi_0 = 1 \). From a four-dimensional perspective the same black holes can be described by \((n_V + 1)\) electric charges \((q_0, q_I) = G_N^{(5)} (J_+/2, -Q_I)/\pi\), an angular momentum \( j = G_N^{(5)} J_-/2\pi\), and the KK magnetic charge \( p^0 = 1 \). Note that \( G_N^{(5)} = 4\pi G_N^{(4)} \) and the black hole entropy remains the same upon reduction to four dimensions. The static limit \((J_+ = 0)\) is particularly useful since we can write down the black hole entropy, in terms of four-dimensional variables, in a compact form [51, 52]\(^9\)

\[
\mathcal{S}(X^\Lambda) = -\frac{i\pi}{2G_N^{(4)}} \left( q_\Lambda X^\Lambda - p^\Lambda \frac{\partial \mathcal{F}(X^\Lambda)}{\partial X^\Lambda} \right) = -\frac{i\pi}{2G_N^{(4)}} \left( q_0 X^0 + q_I X^I - \frac{1}{6} C_{IJK} X^I X^J X^K \right),
\] (A.3)

under the constraint

\[
\xi_\Lambda X^\Lambda = X^0 + \xi_I X^I = 1.
\] (A.4)

Upon extremizing \( \mathcal{S}(X^\Lambda) \) as given above, one fixes the scalar fields in terms of the conserved charges and recovers the correct Bekenstein-Hawking entropy at the extremum.

Introducing an extra parameter \( X^- \) (being conjugate to \( J_- \)), in [16] we showed that the function \( \mathcal{S} \) can be extended to include also the last remaining charge \( J_- \) for the \( \textit{stu} \) model. In this case the only nonvanishing triple intersection numbers are \( C_{123} = 1 \) (and cyclic permutation) and \( \xi_I = 1 \). It would be interesting to similarly generalize also the \((J_- \propto j \neq 0)\) case to arbitrary parameters \( C_{IJK} \), but at the moment we are lacking proper understanding of the four-dimensional rotating attractor mechanism.

However, we can try to apply these arguments to the case of BPS, rotating black holes in \( \text{AdS}_5 \times Y_5 \). The five-dimensional effective theory contains \( n_V \) massless vector multiplets,

\(^9\)Here we correct a sign mistake in (4.23), (4.24) and (4.30) in [16].
corresponding to the R- and global symmetries of the dual field theory. Generically the reduction on $Y_5$ leads to other matter multiplets in five-dimensional supergravity, such as hypermultiplets and massive vector multiplets. These, however, do not carry additional U(1) gauge symmetries and we will work under the assumptions that they decouple in the description of the the black hole near-horizon geometry. With this working assumption, we could expect that the entropy is given by the minimum of (A.3) in the case of equal angular momenta and by its natural extension for $J_{\pm} \neq 0$. This is particularly intriguing because the coefficients $C_{IJK}$ in a compactification on AdS$_5 \times Y_5$ are proportional to the anomaly coefficients $\text{Tr} \, Q_I Q_J Q_K$ for the $n_V$ symmetries $Q_I$ associated with the gauge fields $A^I$ in the bulk five-dimensional theory [53, 54]. As a consequence, it is tempting to speculate that the entropy of a black hole with electric charges $Q_I$ and angular momenta $J_i$ should be obtained as a Legendre transform of

$$E(\Delta_I, \omega_i) = -i\pi N^2 \sum_{I,J,K=1}^{n_V} \frac{C_{IJK} \Delta_I \Delta_J \Delta_K}{6 \omega_1 \omega_2},$$  

(A.5)

with respect to $\Delta_I$ and $\omega_i$ with the constraint

$$\omega_1 + \omega_2 + \sum_{I=1}^{n_V} \Delta_I = 1.$$  

(A.6)

The expression (A.5) is fully determined by anomalies. By setting $\Delta_I = (1 - \omega_1 - \omega_2) \hat{\Delta}_I/2$, it can be written as

$$E(\Delta_I, \omega_i) = \frac{4\pi i (\omega_1 + \omega_2 - 1)^3}{27 \omega_1 \omega_2} a(\hat{\Delta}_I),$$  

(A.7)

where

$$a(\hat{\Delta}_I) = \frac{9N^2}{64} \sum_{I,J,K=1}^{n_V} C_{IJK} \hat{\Delta}_I \hat{\Delta}_J \hat{\Delta}_K,$$  

(A.8)

subject to $\sum_{I=1}^{n_V} \hat{\Delta}_I = 2$ is the trial R-charge of the conformal field theory in the large $N$ limit [53, 54]. The expression (A.7) has a strong resemblance with the refined supersymmetric Casimir energy for the Hopf surface $H_{p,q} \simeq S^3 \times S^1$ in the large $N$ limit [18–20]. Indeed it differs from it only by the $-1$ in the numerator.\(^{11}\) (A.5) reduces exactly to the supersymmetric Casimir energy if we impose $\omega_1 + \omega_2 + \sum_{I=1}^{n_V} \Delta_I = 0$ instead of (A.6), corresponding to a different choice of angular determinations for the chemical potentials.

### B Dimensional reduction and topological twist on $\mathbb{C}P^2$

For the case of asymptotically AdS$_5$ BPS rotating black holes that we considered previously in [16], the reduction of the solutions from five to four dimensions gave us an additional

\(^{10}\)To compare with (A.3), we set $X^I = \Delta_I$, $X^0 = \omega_1 + \omega_2$, $X^- = \omega_1 - \omega_2$ and $J^\pm = J_1 \pm J_2$.

\(^{11}\)See, for example, Eq. (C.3) in [16] with $\omega_i = -i|b_i|$ where $p = e^{-2\pi |b_1|}$, $q = e^{-2\pi |b_2|}$. Recall also that in the large $N$ limit $a = c$. 

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physical understanding. The rotating black holes with equal angular momenta reduce to static domain-wall solutions in four dimensions with near horizon geometry $\text{AdS}_2 \times S^2$. We showed that from four-dimensional perspective supersymmetry is preserved by the $U(1)_R$ gauge field canceling the spin connection on the internal $S^2$ manifold via a topological twist [27, 28]. Without going into so much details, now we would like to argue that a similar dimensional reduction gives analogous understanding of the asymptotically $\text{AdS}_7$ black holes from six-dimensional point of view.

We focus on the single-rotation class of solutions described in section 2.1. We can dimensionally reduce the metric and all other fields along the $U(1)$ fiber of $S^5$, with the remaining $\mathbb{CP}^2$ retaining all its symmetries and obtaining a static solution. We consider the usual Kaluza-Klein (KK) ansatz for the metric,

$$ds^2_7 = e^{\phi^{KK}} ds^2_6 + e^{-4\phi^{KK}} (d\psi + A^{KK})^2 ,$$

where $ds^2_6$ is the resulting 6D line element, $\phi^{KK}$ is the KK scalar field, and $A^{KK}$ the $U(1)$ KK vector field. We see that the resulting six-dimensional line-element has the usual time and radial directions, as well as internal space $\mathbb{CP}^2$, whose metric we repeat again here,

$$ds^2_{\mathbb{CP}^2} = d\xi^2 + \frac{1}{4} \sin^2 \xi \left( \sigma_1^2 + \sigma_2^2 + \cos^2 \xi \sigma_3^2 \right) ,$$

where $\sigma_i$ ($i = 1, 2, 3$) are left-invariant one-forms on $SU(2)$, satisfying $d\sigma_i = -\frac{1}{2} \epsilon_{ijk} \sigma_j \wedge \sigma_k$. The Kähler form on $\mathbb{CP}^2$ is $J = \frac{1}{2} dB$ with $B = \frac{1}{2} \sin^2 \xi \sigma_3$. Comparing to the explicit solution in (2.5), we see that the KK vector field has a leg along the time direction carrying an electric charge in 6D (which corresponds to the angular momentum in 7D) but also has a leg along the internal $\mathbb{CP}^2$ manifold,

$$A^{KK}_{\mathbb{CP}^2} = B = \frac{1}{2} \sin^2 \xi \sigma_3 .$$

The reduction of the remaining fields give rise to additional electric charges for the other six-dimensional vector fields.

The only vector field along the internal manifold is the KK vector (B.3), therefore we are interested in seeing how the Killing spinor covariant derivative depends on it. Via a general Scherk-Schwarz ansatz for the reduction of fermions along a $U(1)$ isometry, see e.g. [55], the Killing spinor covariant derivative looks like

$$D_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{4} \left( \omega^a_{\mu} \gamma_{ab} + 2 g A^{KK}_\mu \Gamma_R \right) \epsilon + \ldots ,$$

where the ellipsis denotes additional connections that will not be important below. The coupling constant $g$ is left arbitrary, the $\gamma_{ab}$ are spatial gamma matrices, and the matrix $\Gamma_R$ allows for some internal structure of the spinors as typically the $U(1)$ KK gauge field becomes part of a bigger R-symmetry mixing the fermions.

An arbitrary four-manifold has an $SO(4) = SU(2)_l \times SU(2)_r$ holonomy, but for Kähler manifolds such as $\mathbb{CP}^2$ we have a further simplification and one of the two $SU(2)$ factors,
say SU(2), becomes U(1). Explicitly, in the coordinates we already introduced, the nonvanishing components of the spin connection read

\[\omega_{\sigma_1 14} = \omega_{\sigma_1 32} = \frac{1}{2} \cos \xi, \quad \omega_{\sigma_2 13} = \omega_{\sigma_2 24} = \frac{1}{2} \cos \xi,\]

\[\omega_{\sigma_3 12} = \frac{1}{2} \cos^2 \xi - 1, \quad \omega_{\sigma_3 34} = \frac{1}{2} (\cos^2 \xi - \sin^2 \xi).\] (B.5)

It is easy to see that splitting the spin connection into a self-dual and antiself-dual part \(\omega^\pm\) is equivalent to splitting it into a U(1) and an SU(2) factor, respectively. In order for the supersymmetric twist to be performed, we need to cancel completely both \(\omega^+\) and \(\omega^-\) in the Killing spinor covariant derivative, (B.4). The SU(2) part drops out automatically if we impose the projection

\[\gamma^{1234} \epsilon = -\epsilon \quad \Rightarrow \quad \omega^{-ab}_{\mu} \gamma^{-ab}_\epsilon = 0,\] (B.6)

where \(2\gamma^{-ab} \equiv \gamma_{ab} - \varepsilon_{abcd} \gamma^{cd}\). The U(1) part of the spin connection is then

\[\omega^{+12} = -\frac{3}{2} \sin^2 \xi \sigma_3 = -3B = -3A_{\mathbb{CP}^2}^{KK}.\] (B.7)

We therefore see that the spin connection is precisely canceled and the supersymmetric twist is completed upon imposing

\[\gamma^{12} \epsilon = \Gamma_R \epsilon, \quad g = \frac{3}{2},\] (B.8)

in (B.4).

Note that supersymmetric flows in six dimensions with a U(1) twist on \(\mathbb{CP}^2\) have already been explicitly found in [56] and further studied in [57], in the absence of electric charges, two-form field and additional scalars. Here we have shown that the 7D rotating black holes we consider, upon reduction to six dimensions, fit in the same category of solutions in [56, 57] with additional conserved charges.

References

[1] F. Benini, K. Hristov, and A. Zaffaroni, “Black hole microstates in AdS4 from supersymmetric localization,” JHEP 05 (2016) 054, arXiv:1511.04085 [hep-th].
[2] F. Benini, K. Hristov, and A. Zaffaroni, “Exact microstate counting for dyonic black holes in AdS4,” Phys. Lett. B771 (2017) 462–466, arXiv:1608.07294 [hep-th].
[3] F. Azzurli, N. Bobev, P. M. Crichigno, V. S. Min, and A. Zaffaroni, “A universal counting of black hole microstates in AdS4,” JHEP 02 (2018) 054, arXiv:1707.04257 [hep-th].
[4] S. M. Hosseini, K. Hristov, and A. Passias, “Holographic microstate counting for AdS4 black holes in massive IIA supergravity,” JHEP 10 (2017) 190, arXiv:1707.06884 [hep-th].

– 17 –
[5] F. Benini, H. Khachatryan, and P. Milan, “Black hole entropy in massive Type IIA,” 
Class. Quant. Grav. 35 no. 3, (2018) 035004, arXiv:1707.06886 [hep-th].

[6] N. Bobev, V. S. Min, and K. Pilch, “Mass-deformed ABJM and Black Holes in AdS$_4$,” 
arXiv:1801.03135 [hep-th].

[7] J. B. Gutowski and H. S. Reall, “Supersymmetric AdS(5) black holes,” JHEP 02 (2004) 006, arXiv:hep-th/0401042 [hep-th].

[8] J. B. Gutowski and H. S. Reall, “General supersymmetric AdS(5) black holes,” JHEP 04 (2004) 048, arXiv:hep-th/0401129 [hep-th].

[9] Z. W. Chong, M. Cvetic, H. Lu, and C. N. Pope, “Five-dimensional gauged supergravity black holes with independent rotation parameters,” Phys. Rev. D72 (2005) 041901, arXiv:hep-th/0505112 [hep-th].

[10] Z. W. Chong, M. Cvetic, H. Lu, and C. N. Pope, “General non-extremal rotating black holes in minimal five-dimensional gauged supergravity,” Phys. Rev. Lett. 95 (2005) 161301, arXiv:hep-th/0506029 [hep-th].

[11] H. K. Kunduri, J. Lucietti, and H. S. Reall, “Supersymmetric multi-charge AdS(5) black holes,” JHEP 04 (2006) 036, arXiv:hep-th/0601156 [hep-th].

[12] J. Kinney, J. M. Maldacena, S. Minwalla, and S. Raju, “An Index for 4 dimensional super conformal theories,” Commun. Math. Phys. 275 (2007) 209–254, arXiv:hep-th/0510251 [hep-th].

[13] L. Grant, P. A. Grassi, S. Kim, and S. Minwalla, “Comments on 1/16 BPS Quantum States and Classical Configurations,” JHEP 05 (2008) 049, arXiv:0803.4183 [hep-th].

[14] C.-M. Chang and X. Yin, “1/16 BPS states in $\mathcal{N} = 4$ super-Yang-Mills theory,” Phys. Rev. D88 no. 10, (2013) 106005, arXiv:1305.6314 [hep-th].

[15] C. Romelsberger, “Counting chiral primaries in N = 1, d=4 superconformal field theories,” Nucl. Phys. B747 (2006) 329–353, arXiv:hep-th/0510060 [hep-th].

[16] S. M. Hosseini, K. Hristov, and A. Zaffaroni, “An extremization principle for the entropy of rotating BPS black holes in AdS$_5$,” JHEP 07 (2017) 106, arXiv:1705.05383 [hep-th].

[17] N. Bobev, M. Bullimore, and H.-C. Kim, “Supersymmetric Casimir Energy and the Anomaly Polynomial,” JHEP 09 (2015) 142, arXiv:1507.08553 [hep-th].

[18] B. Assel, D. Cassani, L. Di Pietro, Z. Komargodski, J. Lorenzen, and D. Martelli, “The Casimir Energy in Curved Space and its Supersymmetric Counterpart,” JHEP 07 (2015) 043, arXiv:1503.05537 [hep-th].

[19] B. Assel, D. Cassani, and D. Martelli, “Localization on Hopf surfaces,” JHEP 08 (2014) 123, arXiv:1405.5144 [hep-th].

[20] J. Lorenzen and D. Martelli, “Comments on the Casimir energy in supersymmetric field theories,” JHEP 07 (2015) 001, arXiv:1412.7463 [hep-th].

[21] A. Arabi Ardehali, J. T. Liu, and P. Szepietowski, “High-Temperature Expansion of
Supersymmetric Partition Functions,” *JHEP* **07** (2015) 113, arXiv:1502.07737 [hep-th].

[22] A. Arabi Ardehali, “High-temperature asymptotics of supersymmetric partition functions,” *JHEP* **07** (2016) 025, arXiv:1512.03376 [hep-th].

[23] M. Cvetic, G. W. Gibbons, H. Lu, and C. N. Pope, “Rotating black holes in gauged supergravities: Thermodynamics, supersymmetric limits, topological solitons and time machines,” arXiv:hep-th/0504080 [hep-th].

[24] Z. W. Chong, M. Cvetic, H. Lu, and C. N. Pope, “Non-extremal charged rotating black holes in seven-dimensional gauged supergravity,” *Phys. Lett.* **B626** (2005) 215–222, arXiv:hep-th/0412094 [hep-th].

[25] D. D. K. Chow, “Equal charge black holes and seven dimensional gauged supergravity,” *Class. Quant. Grav.* **25** (2008) 175010, arXiv:0711.1975 [hep-th].

[26] H.-C. Kim and S. Kim, “M5-branes from gauge theories on the 5-sphere,” *JHEP* **05** (2013) 144, arXiv:1206.6339 [hep-th].

[27] E. Witten, “Topological Quantum Field Theory,” *Commun. Math. Phys.* **117** (1988) 353.

[28] E. Witten, “Mirror manifolds and topological field theory,” arXiv:hep-th/9112056 [hep-th].

[29] F. Benini and A. Zaffaroni, “A topologically twisted index for three-dimensional supersymmetric theories,” *JHEP* **07** (2015) 127, arXiv:1504.03698 [hep-th].

[30] N. A. Nekrasov and S. L. Shatashvili, “Bethe/Gauge correspondence on curved spaces,” *JHEP* **01** (2015) 100, arXiv:1405.6046 [hep-th].

[31] S. M. Hosseini and A. Zaffaroni, “Large N matrix models for 3d \( \mathcal{N} = 2 \) theories: twisted index, free energy and black holes,” *JHEP* **08** (2016) 064, arXiv:1604.03122 [hep-th].

[32] S. M. Hosseini and N. Mekareeya, “Large N topologically twisted index: necklace quivers, dualities, and Sasaki-Einstein spaces,” *JHEP* **08** (2016) 089, arXiv:1604.03397 [hep-th].

[33] J. L. Cardy, “Operator Content of Two-Dimensional Conformally Invariant Theories,” *Nucl. Phys.* **B270** (1986) 186–204.

[34] M. Pernici, K. Pilch, and P. van Nieuwenhuizen, “Gauged Maximally Extended Supergravity in Seven-dimensions,” *Phys. Lett.* **143B** (1984) 103–107.

[35] H. Nastase, D. Vaman, and P. van Nieuwenhuizen, “Consistent nonlinear KK reduction of 11-d supergravity on AdS(7) x S(4) and selfduality in odd dimensions,” *Phys. Lett.* **B469** (1999) 96–102, arXiv:hep-th/9905075 [hep-th].

[36] H. Nastase, D. Vaman, and P. van Nieuwenhuizen, “Consistency of the AdS(7) x S(4) reduction and the origin of selfduality in odd dimensions,” *Nucl. Phys.* **B581** (2000) 179–239, arXiv:hep-th/9911238 [hep-th].

[37] J. T. Liu and R. Minasian, “Black holes and membranes in AdS(7),” *Phys. Lett.* **B457** (1999) 39–46, arXiv:hep-th/9903269 [hep-th].
[38] J. Kallen, J. A. Minahan, A. Nedelin, and M. Zabzine, “N^3-behavior from 5D Yang-Mills theory,”  
*JHEP* **10** (2012) 184, arXiv:1207.3763 [hep-th].

[39] H.-C. Kim and K. Lee, “Supersymmetric M5 Brane Theories on R x CP2,”  
*JHEP* **07** (2013) 072, arXiv:1210.0853 [hep-th].

[40] G. Lockhart and C. Vafa, “Superconformal Partition Functions and Non-perturbative Topological Strings,”  
arXiv:1210.5909 [hep-th].

[41] H.-C. Kim, J. Kim, and S. Kim, “Instantons on the 5-sphere and M5-branes,”  
arXiv:1211.0144 [hep-th].

[42] J. A. Minahan, A. Nedelin, and M. Zabzine, “5D super Yang-Mills theory and the correspondence to AdS_7/CFT_6,”  
*J. Phys.* **A46** (2013) 355401, arXiv:1304.1016 [hep-th].

[43] H.-C. Kim, S. Kim, S.-S. Kim, and K. Lee, “The general M5-brane superconformal index,”  
arXiv:1307.7660 [hep-th].

[44] F. Brunner, D. Regalado, and V. P. Spiridonov, “Supersymmetric Casimir energy and SL(3,Z) transformations,”  
*JHEP* **07** (2017) 041, arXiv:1611.03831 [hep-th].

[45] N. Halmagyi and S. Lal, “On the On-Shell: The Action of AdS_4 Black Holes,”  
arXiv:1710.09580 [hep-th].

[46] A. Cabo-Bizet, U. Kol, L. A. Pando Zayas, I. Papadimitriou, and V. Rathee, “Entropy functional and the holographic attractor mechanism,”  
arXiv:1712.01849 [hep-th].

[47] K. Hristov, I. Lodato, and V. Reys, “On the quantum entropy function in 4d gauged supergravity,”  
arXiv:1803.05920 [hep-th].

[48] I. Papadimitriou, “Supercurrent anomalies in 4d SCFTs,”  
*JHEP* **07** (2017) 038, arXiv:1703.04299 [hep-th].

[49] O. S. An, “Anomaly-corrected supersymmetry algebra and supersymmetric holographic renormalization,”  
*JHEP* **12** (2017) 107, arXiv:1703.09607 [hep-th].

[50] D. Klemm and W. A. Sabra, “General (anti-)de Sitter black holes in five-dimensions,”  
*JHEP* **02** (2001) 031, arXiv:hep-th/0011016 [hep-th].

[51] S. L. Cacciatori and D. Klemm, “Supersymmetric AdS(4) black holes and attractors,”  
*JHEP* **01** (2010) 085, arXiv:0911.4926 [hep-th].

[52] G. Dall’Agata and A. Gnecchi, “Flow equations and attractors for black holes in N = 2 U(1) gauged supergravity,”  
*JHEP* **03** (2011) 037, arXiv:1012.3756 [hep-th].

[53] Y. Tachikawa, “Five-dimensional supergravity dual of a-maximization,”  
*Nucl. Phys.* **B733** (2006) 188–203, arXiv:hep-th/0507057 [hep-th].

[54] S. Benvenuti, L. A. Pando Zayas, and Y. Tachikawa, “Triangle anomalies from Einstein manifolds,”  
*Adv. Theor. Math. Phys.* **10** no. 3, (2006) 395–432, arXiv:hep-th/0601054 [hep-th].

[55] K. Hristov and A. Rota, “6d-5d-4d reduction of BPS attractors in flat gauged supergravities,”  
*Nucl. Phys.* **B897** (2015) 213–228, arXiv:1410.5386 [hep-th].
[56] M. Naka, “Various wrapped branes from gauged supergravities,” arXiv:hep-th/0206141 [hep-th].

[57] N. Bobev and P. M. Crichigno, “Universal RG Flows Across Dimensions and Holography,” JHEP 12 (2017) 065, arXiv:1708.05052 [hep-th].