EXISTENCE AND REGULARITY RESULTS FOR STOCHASTIC FRACTIONAL PSEUDO-PARABOLIC EQUATIONS DRIVEN BY WHITE NOISE

TRAN NGOC THACH
Applied Analysis Research Group, Faculty of Mathematics and Statistics
Ton Duc Thang University
Ho Chi Minh City, Vietnam

DEVENDRA KUMAR*
Department of Mathematics
University of Rajasthan
Jaipur 302004, Rajasthan, India

NGUYEN HOANG LUC
Division of Applied Mathematics
Thu Dau Mot University
Binh Duong Province, Vietnam

NGUYEN HUY TUAN1,2,*

1 Division of Applied Mathematics, Science and Technology Advanced Institute
Van Lang University
Ho Chi Minh City, Vietnam

2 Faculty of Technology, Van Lang University
Ho Chi Minh City, Vietnam

Abstract. Solutions of a direct problem for a stochastic pseudo-parabolic equation with fractional Caputo derivative are investigated, in which the non-linear space-time-noise is assumed to satisfy distinct Lipschitz conditions including globally and locally assumptions. The main aim of this work is to establish some existence, uniqueness, regularity, and continuity results for mild solutions.

1. Introduction. Let $X \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be a bounded domain with the boundary is smooth enough. We investigate a direct problem for a stochastic fractional pseudo-parabolic equation

\[
\begin{cases}
C D_t^\beta (u - \alpha \Delta u) + (-\Delta)^s u = I_t^{1-\beta}[|\varphi(t,u(t))|W(t)], & (t, x) \in (0, T] \times X, \\
u(0, x) = u^{ini}(x), & x \in X, \\
u(t, x) = 0, & (t, x) \in (0, T] \times \partial X.
\end{cases}
\]

(1)

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*Corresponding authors: Nguyen Huy Tuan (nguyenhuytuan@vlu.edu.vn) and Devendra Kumar (devendra.maths@gmail.com).
Here, $\Delta$ is the Laplacian operator, $\lambda, s > 0$, $(-\Delta)^s$ is defined in Section 2, $\alpha > 0$ is the diffusion coefficient (or called the diffusivity),

$$C_D^\beta u(t) = \frac{1}{\Gamma(1 - \beta)} \int_0^t (t - r)^{-\beta} \frac{\partial u}{\partial r}(r)dr, \quad 0 < \beta < 1,$$

is the mean square random Caputo fractional derivative [11, 12, 13, 42] of the stochastic process $u(t)$,

$$I_t^{1-\beta} u(t) = \frac{1}{\Gamma(1 - \beta)} \int_0^t (t - r)^{-\beta} u(r)dr,$$

is the mean square random Riemann–Liouville fractional integral [13] of the stochastic process $u(t)$, $W(t)$ is an $\mathcal{F}_t$-adapted Wiener process defined on a completed probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, and $\dot{W}(t) = \frac{\partial W(t)}{\partial t}$ describes a white noise.

Recent decades, stochastic fractional differential equations (SFDEs) have received significant attentions with remarkable applications in distinct fields of science [2, 26, 27, 30, 37, 48, 52]. It is undeniable that SFDEs are of paramount importance in describing distinct physical phenomena [29, 33, 35, 36, 41, 43].

This work aims to investigate a direct problem for a SFDE that is pseudo-parabolic equation with fractional Caputo derivative (1). Noting that if $C_D^\beta$ is replaced by the classical derivative $\partial_t$ and the order $s$ of the negative Laplacian operator is changed by one, then the equation in (1) turns to be the classic pseudo-parabolic equation, which has successful applications in modeling homogeneous fluids through a fissured rock, long waves in nonlinear dispersive systems, and the aggregation of populations [7, 40, 50]. For some impressive studies on the such integer order equation, the reader can refer to [3, 14, 15, 19, 20] [23, 24, 28, 31, 38, 51].

It is the fact that dealing with several problems in media and systems with fractal properties leads to problems of solving fractional differential equations [16, 33, 39, 42, 49, 54], which contain fractional derivatives, instead of traditional partial differential equations. A similar situation occurs when considering pseudo-parabolic equations, which have an important role to play in modeling mass transfer processes in various media and systems [6, 25, 44, 46]. Since the intense of mass transfer processes in media with fractal and non-fractal structures is different, pseudo-parabolic equations with classical derivative are no longer appropriate to be used in media and systems with fractal properties. Motivated by this reason, in this paper, we include the fractional Caputo derivative in the pseudo-parabolic equations to describe better mass transfer processes in fractal structure. Let us now introduce some few works on pseudo-parabolic equations equations containing fractional derivative. In two recent years (2018 and 2019), Beshtokov [8, 9, 10] studied boundary value problems for this equation, investigated the uniqueness and the stability of the solution. In 2019, Sousa [47] presented the stability of the solution in the sense of the Ulam-Hyers and generalized Ulam–Hyers–Rassias separately. We would like to emphasize that, as we know, regularity properties of the solutions have not been handled until now.

It should be noted that all of above results are considered in the deterministic case; in contrast, as we know, the equation in the stochastic case has not been studied until now. Motivated of this reason, our study here aims at studying a direct problem for a stochastic fractional pseudo-parabolic equation and establishing the existence, uniqueness, as well as the regularity of the solutions, which has been of
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paramount importance in the scientific community, especially in fractional calculus and stochastic analysis.

The challenge of the present paper lies on the fact that analysis techniques used to deal with the deterministic case cannot be applied here. Furthermore, due to the presence of the new quantity that is fractional Caputo derivative, the solution operators become more complicated and it is required to construct sharp estimates for them, which helps us obtain various existence and regularity results. Without flexible estimates for such operators, hardly could we guarantee the continuity of the solution. We emphasize that in this study, our problem is investigated under distinct Lipschitz conditions for the space-time-noise \( \phi \) in the form

\[
\| \phi(t, v_1) - \phi(t, v_2) \|_{L^p(\Omega; L^2_0(H, \dot{H}^\nu))} \leq K(t, v_1, v_2) \| v_1 - v_2 \|_{L^p(\Omega, \dot{H}^\sigma)},
\]

where the function \( K \) is considered in three cases that are

1. \( K \) is a constant (globally assumption),
2. \( K \) only depend on \( t \), and
3. \( K \) only depend on \( v_1, v_2 \) (locally assumption).

The main contribution of the present article is to construct the existence, uniqueness, and regularity of the solutions to Problem (1) under globally and locally Lipschitz conditions for the non-linear space-time-noise \( \phi \). In Section 2, some material including stochastic background, fractional calculus, definition of mild solutions, and pivotal properties of solution operators are given. In Section 3, the main results for the considered problem with two distinct cases of non-linear space-times-noise are shown and proved.

2. Preliminaries.

2.1. Some functional spaces. Initially, the Hilbert scale space \( \dot{H}^\sigma \) and the fractional operator \((-\Delta)^\sigma\) are introduced. Let \((\cdot, \cdot)\) and \(\| \cdot \|\) be the inner product and the norm in \( H := L^2(X) \). Let \( \lambda_k \) and \( e_k \) be Dirichlet eigenvalues and corresponding eigenfunctions of the negative Laplacian operator \(-\Delta\) respectively, which forms an increasing positive sequence tending to infinity. Let \( \dot{H}^\sigma \), with \( \sigma > 0 \), be the Hilbert scale space of \( H \)-valued functions \( f \) satisfying

\[
\| f \|_{\dot{H}^\sigma} := \left( \sum_{k=1}^{\infty} |(f, e_k)|^2 \lambda_k^{2\sigma} \right)^{\frac{1}{2}} < \infty.
\]

Identifying the dual space \( H^* = H \), we can set \( \dot{H}^{-\sigma} = (\dot{H}^\sigma)^* \). It is known from [1] that \( \dot{H}^{-\sigma} \) is a Hilbert space endowed with the norm

\[
\| f \|_{\dot{H}^{-\sigma}} := \left( \sum_{k=1}^{\infty} |(f, e_k)|^2 \lambda_k^{-2\sigma} \right)^{\frac{1}{2}},
\]

where \((\cdot, \cdot)\) stands for the duality bracket between \( \dot{H}^{-\sigma} \) and \( \dot{H}^\sigma \). The fractional operator \((-\Delta)^\sigma\) from \( \dot{H}^{-\frac{\sigma}{2}} \) to \( \dot{H}^{\frac{\sigma}{2}} \) can be defined by (see [22], [32])

\[
(-\Delta)^\sigma := \sum_{k=1}^{\infty} (\cdot, e_k) \lambda_k^\sigma e_k.
\]

It is observed that if \( \sigma = 0 \) then \( \dot{H}^\sigma \) and \((-\Delta)^\sigma\) turn to be \( H \) and \(-\Delta\) respectively.

Now, some stochastic spaces and a representation of Wiener process \( \{W(t)\} \) are introduced. Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\) be a completed probability space with a normal...
filtration \( \{ \mathcal{F}_t \} \) satisfying that it is right continuous and all \( \mathbb{P} \)-null sets belong to \( \mathcal{F}_0 \). Let \( L^p(\Omega, \mathcal{H}^\sigma) \), \( p \geq 2 \), be the space of all random variable \( \xi : \Omega \to \mathcal{H}^\sigma \) such that
\[
\|\xi\|_{L^p(\Omega, \mathcal{H}^\sigma)} := \mathbb{E}\|\xi\|_{\mathcal{H}^\sigma}^p < \infty.
\]
Let \( C([0, T]; L^p(\Omega, \mathcal{H}^\sigma)) \) be the space of continuous functions \( w : [0, T] \to L^p(\Omega, \mathcal{H}^\sigma) \) satisfying
\[
\|w\|_{C([0, T]; L^p(\Omega, \mathcal{H}^\sigma))} := \sup_{t \in [0, T]} \|w(t)\|_{L^p(\Omega, \mathcal{H}^\sigma)} < \infty.
\]
For \( \kappa \geq 0 \), we define by \( L^{\infty, \kappa}(0, T; L^p(\Omega, \mathcal{H}^\sigma)) \), the space of functions \( w : (0, T) \to L^p(\Omega, \mathcal{H}^\sigma) \) satisfying
\[
\|w\|_{L^{\infty, \kappa}(0, T; L^p(\Omega, \mathcal{H}^\sigma))} := \text{ess sup}_{t \in (0, T)} \|w(t)\|_{L^p(\Omega, \mathcal{H}^\sigma)} < \infty.
\]
If \( \kappa = 0 \), it turns to be the well-known space \( L^\infty(0, T; L^p(\Omega, \mathcal{H}^\sigma)) \).

Denote by \( \mathcal{L}(\mathcal{H}^\sigma, \mathcal{H}^{\sigma'}) \) the space of all bounded linear operators from \( \mathcal{H}^\sigma \) to \( \mathcal{H}^{\sigma'} \) and \( \mathcal{L}(\mathcal{H}^\sigma) := \mathcal{L}(\mathcal{H}^\sigma, \mathcal{H}^\sigma) \). In what follows, we briefly elaborate the noise term \( W(t) \). We begin with some well-known results from the spectral theorem [33, 53].

**Proposition 2.1.** Let \( Q \in \mathcal{L}(H) \) be a non-negative and self-adjoint operator and the trace of \( Q \) is finite. Then, \( Q \) satisfies \( Qe_k = \gamma_k e_k \), where \( \{ e_k \} \) is an orthonormal basis of \( H \). Furthermore, this operator possesses the following expression
\[
Qf = \sum_{k=1}^{\infty} \gamma_k (f, e_k) e_k, \quad f \in H.
\]

**Proposition 2.2** (see [34]). For each \( m \in H \) and \( Q \in \mathcal{L}(H) \) satisfies Proposition 2.1, there exists a Gaussian measure \( \mu = \mathcal{N}(m, Q) \).

Now, fixed \( Q \) satisfying Proposition 2.1, we can give the definition of a standard \( Q \)-Wiener process as follows.

**Definition 2.1.** A \( H \)-valued stochastic process \( W(t) \) is said to be a \( Q \)-Wiener process if it satisfies the four conditions

i) \( W(0) = 0 \);

ii) \( W \) has \( \mathbb{P} \)-a.s. continuous trajectories.

iii) the increments of \( W \) are independent, i.e. \( W(t_1), W(t_2), \ldots W(t_n) - W(t_{n-1}) \) independent for all \( 0 \leq t_1 < \cdots < t_n \leq T \);

iv) the increments have Gaussian laws
\[
\mathbb{P} (W(t) - W(s)) = \mathcal{N}(0, (t-s)Q), \quad 0 \leq s \leq t \leq T.
\]

It is known from [4, 5, 17, 18, 34], for \( Q \) satisfying Proposition 2.1, there always exist a \( Q \)-Wiener process \( W(t) \); moreover, it has the following representation
\[
W(t) = \sum_{k \geq 1} Q^{\frac{1}{2}} e_k b_k(t) = \sum_{k \geq 1} \gamma_k^\frac{1}{2} e_k b_k(t),
\]
where \( b_k(t) \) are one-dimensional Brownian motions. A good example to think about is of course cylindrical Wiener process
\[
W(t) = \sum_{k \geq 1} e_k b_k(t),
\]
which is a \( Q \)-Wiener process when \( Q \equiv I \). Another example is \( N \)-dimensional Brownian Motion, which is obtained when \( Q \equiv I \) and \( H \) is replaced by \( \mathbb{R}^N \).
Let $L_0^2(H, \dot{H}^\sigma) = L^2(Q^{\frac{1}{2}}(H), \dot{H}^\sigma)$ be the space of Hillbert-Schmidt operators $P$ from $Q^{\frac{1}{2}}(H)$ to $\dot{H}^\sigma$ such that

$$\|P\|_{L_0^2(H, \dot{H}^\sigma)} := \left( \sum_{k=1}^\infty \|P Q^{\frac{1}{2}} \epsilon_k\|^2_{H^\sigma} \right)^{\frac{1}{2}} < \infty.$$  

Let $L^r(\Omega, L_0^2(H, \dot{H}^\sigma))$, $r \geq 2$, be the space of functions $\Phi : \Omega \rightarrow L_0^2(H, \dot{H}^\sigma)$ satisfying

$$\|\Phi\|_{L^r(\Omega, L_0^2(H, \dot{H}^\sigma))} := \mathbb{E} \|\Phi\|^r_{L_0^2(H, \dot{H}^\sigma)} < \infty.$$  

In the case $\sigma = 0$, we set $L_0^2 := L_0^2(H, H)$ and $L^r(\Omega, L_0^2) := L^r(\Omega, L_0^2(H, H))$ for short.

2.2. Mittag-Leffler function and some properties. Next, let us introduce the following function defined by series expansion, which is called the Mittag-Leffler type \[42\] function

$$E_{\beta, \beta'}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\beta k + \beta')}, \quad \beta, \beta' > 0.$$  

For convenience, let us set $E_\beta(z) := E_{\beta, 1}(z)$ and $E_{\beta, \beta}(z) := E_{\beta, \beta}(z)$. It is known from [57] that the two functions have the following integral representation

$$E_\beta(z) = \int_0^\infty M_\beta(\zeta) \exp(z \zeta) d\zeta, \quad E_{\beta, \beta}(z) = \int_0^\infty \beta \zeta M_\beta(\zeta) \exp(z \zeta) d\zeta, \quad (2)$$  

where $M_\beta$ is the Wright function defined by

$$M_\beta(z) := \sum_{k \geq 1} \frac{(-z)^k}{k! \Gamma(-\beta(k + 1) + 1)}.$$  

Some pivotal properties of the aforementioned functions are presented in the following proposition. Throughout this paper, $a \lesssim b$ is used to describe that there exists $m > 0$ such that $a \leq mb$.

Proposition 2.3. Let $\beta \in (0, 1)$, $\beta' > 0$, $\delta \in [0, 1]$, and $h > 0$. Setting

$$\mathcal{E}_\beta^h(z, t) := E_\beta(-z(t + h)^\beta) - E_\beta(-zt^\beta), \quad \overline{\mathcal{E}_\beta^h}(z, t) := E_{\beta, \beta}(-z(t + h)^\beta) - E_{\beta, \beta}(-zt^\beta). \quad (3)$$  

Then, there holds

(E1) $|E_{\beta, \beta'}(-z)| \lesssim (1 + z)^{-1}$,

(E2) $\int_0^\infty \zeta^\delta M_\beta(\zeta) d\zeta = \frac{\Gamma(1+\eta)}{\Gamma(1+\beta\eta)}$, for $\eta \in (-1, \infty)$,

(E3) $|\mathcal{E}_\beta^h(z, t)| \lesssim z^\delta h^\beta$, $|\overline{\mathcal{E}_\beta^h}(z, t)| \lesssim z^\delta h^\beta$.

Proof. The proof of two first properties (E1), (E2) can be found in [42, 45, 57]. The last $\lesssim$-inequality can be proved by using the representations (2) and the identity (E2) as follows

$$\mathcal{E}_\beta^h(z, t) = E_\beta(-z(t + h)^\beta) - E_\beta(-zt^\beta) = \int_0^\infty M_\beta(\zeta)(\exp(-\zeta(t + h)^\beta) - \exp(-\zeta t^\beta)) d\zeta,$$

$$\overline{\mathcal{E}_\beta^h}(z, t) = E_{\beta, \beta}(-z(t + h)^\beta) - E_{\beta, \beta}(-zt^\beta) = \int_0^\infty \beta \zeta M_\beta(\zeta)(\exp(-\zeta(t + h)^\beta) - \exp(-\zeta t^\beta)) d\zeta.$$  

By using the inequality $|\exp(-x)-\exp(-y)| \leq C_\delta |x-y|^\delta$, for $\delta \in [0,1]$, one directly obtain

$$|E_\beta^h(z,t)| \leq \int_0^\infty M_\beta(\zeta) |\exp(-\zeta z(t+h)^\beta) - \exp(-\zeta z t^\beta)|d\zeta$$

$$\lesssim \int_0^\infty M_\beta(\zeta) \zeta^\beta z^\beta (t+h)^\beta - t^\beta \zeta d\zeta \lesssim z^\delta h^\delta \int_0^\infty \zeta^\delta M_\beta(\zeta) d\zeta,$$

where we note that $(t+h)^\beta - t^\beta \lesssim h^\beta$ due to $\beta \in (0,1)$. By exactly the same technique as in above, one can check that

$$|\overline{E}_\beta^h(z,t)| \lesssim z^\delta h^\delta \int_0^\infty \zeta^\delta M_\beta(\zeta) d\zeta$$

Now, by combining two latter estimates and using the identity (E2), it is clear to see that (E3) holds.

\[\square\]

### 2.3. Mild formulation and estimates for solution operators.

In this subsection, a notation of mild solution of Problem (1) will be given and some properties of the solution operators will be investigated.

Initially, we aim to find an expression for $u(t)$ in the series expansion $\sum_{k=1}^\infty (u(t), e_k)c_k$. Taking the inner product of two sides of the equation in (1), one arrives at

$$C D_\beta^t (u(t), e_k) + \lambda_k^\beta (1 + \alpha \lambda_k)^{-1} (u(t), e_k) = (1 + \alpha \lambda_k)^{-1} I_1^{1-\beta} [\lambda \phi(t, u(t)) \gamma_\beta^2 b_k(t)].$$

In order to solve the above equation, we refer to the Laplace transform method (see Subsection 5.3 of [33]). In this way, an explicit form for the $k$-th coefficients is obtained as

$$(u(t), e_k) = E_\beta \left( \frac{-t^\beta \lambda_k^\beta}{1 + \alpha \lambda_k} \right) (u^{ini}, e_k)$$

$$+ \lambda \int_0^t (1 + \alpha \lambda_k)^{-1} \gamma_\beta^\frac{1}{2} E_\beta \left( \frac{-\gamma_\beta\lambda_k^\frac{1}{2}}{1 + \alpha \lambda_k} \right) \phi(r, u(r))db_k(r).$$

By defining two operators $S_\beta(t), \overline{S}_\beta(t)$ as follows

$$S_\beta(t) := \sum_{k=1}^\infty (\cdot, e_k)E_\beta \left( \frac{-t^\beta \lambda_k^\beta}{1 + \alpha \lambda_k} \right) e_k,$$

$$\overline{S}_\beta(t) := \sum_{k=1}^\infty (\cdot, e_k)(1 + \alpha \lambda_k)^{-1} E_\beta \left( \frac{-t^\beta \lambda_k^\beta}{1 + \alpha \lambda_k} \right) e_k,$$

one obtain the following expression for $u(t)$ which is called mild formulation.

**Definition 2.2.** $u(t)$ is called a mild solution of (1) if $u$ belongs to $C([0,T]; L^p(\Omega, H^\sigma))$, with $p \geq 2, \sigma \geq 0$, and satisfies

$$u(t) = S_\beta(t) u^{ini} + \lambda \int_0^t \overline{S}_\beta(t-r) \phi(r, u(r))dW(r), \quad \mathbb{P} - a.e.,$$

where the solution operators $S_\beta(t), \overline{S}_\beta(t)$ are defined in (4).

**Definition 2.3.** $u(t)$ is called a local mild solution of (1) in $C([0,T_*]; L^p(\Omega, \dot{H}^\sigma))$ with stopping time $T_* > 0$ if it satisfies Definition 2.2 on $[0,T_*]$. Since two solution operators $S_\beta(t)$ and $\overline{S}_\beta(t)$ play an pivotal role in our results, it is necessary to construct some properties for them. The following result gives some information of such solution operators.
Proposition 2.4. Let $s \in (0, 1]$, $0 \leq \sigma - \nu \leq 1$, $\delta \in (0, 1]$, $h > 0$. Then, for $t \in [0, T]$, there holds
\[
\|S_\beta(t)\|_{L(\mathcal{H}^\nu_x, \mathcal{H}^\nu)} \lesssim 1, \quad \|\mathcal{S}_\beta(t)\|_{L(\mathcal{H}^\nu, \mathcal{H}^\nu)} \lesssim 1, \tag{6}
\]
\[
\|S_\beta(t + h) - S_\beta(t)\|_{L(\mathcal{H}^\nu_x, \mathcal{H}^\nu)} \lesssim h^{\delta}, \quad \|\mathcal{S}_\beta(t + h) - \mathcal{S}_\beta(t)\|_{L(\mathcal{H}^\nu, \mathcal{H}^\nu)} \lesssim h^{\delta}. \tag{7}
\]

Proof. As a consequence, it can be seen from the property (E1) of Proposition 2.3 that $E_{\beta, \nu}(-z) \lesssim 1$. By this and noting that $(1 + \alpha \lambda_k)^{-1} \lesssim \lambda_k^{-(\sigma - \nu)}$, one can bound the $k$-coefficients in the series expansion (4) as
\[
E_\beta \left( \frac{-t^\beta \lambda_k^s}{1 + \alpha \lambda_k} \right) \lesssim 1, \quad \text{and} \quad (1 + \alpha \lambda_k)^{-1} E_\beta \left( \frac{-t^\beta \lambda_k^s}{1 + \alpha \lambda_k} \right) \lesssim \lambda_k^{-(\sigma - \nu)}, \quad \text{for } t \in [0, T].
\]

From the explicit expansions (4) of $S_\beta(t)$, $\mathcal{S}_\beta(t)$, and the above properties, one obtains
\[
\|S_\beta(t)\|_{\mathcal{H}^\nu} = \|(-\Delta)^{\sigma} S_\beta(t)\| \lesssim \|\varphi\|_{\mathcal{H}^\nu}, \quad \|\mathcal{S}_\beta(t)\|_{\mathcal{H}^\nu} = \|(-\Delta)^{\nu}(-\Delta)^{\sigma-\nu} \mathcal{S}_\beta(t)\| \lesssim \|\varphi\|_{\mathcal{H}^\nu},
\]
which implies that (6) for $t \in [0, T]$.

We now continue to verify the results (7) by a similar way as in above. From (3), (4), one can observe that
\[
S_\beta(t + h) - S_\beta(t) := \sum_{k=1}^\infty \langle \cdot, e_k \rangle \mathcal{E}_k^h \left( \frac{\beta \lambda_k^s}{1 + \alpha \lambda_k}, t \right) e_k,
\]
\[
\mathcal{S}_\beta(t + h) - \mathcal{S}_\beta(t) := \sum_{k=1}^\infty \langle \cdot, e_k \rangle (1 + \alpha \lambda_k)^{-1} \mathcal{E}_k^h \left( \frac{\beta \lambda_k^s}{1 + \alpha \lambda_k}, t \right) e_k.
\]

By applying the property (E3) of Proposition 2.3, the $k$-coefficients in the first series expansion can be bounded as
\[
\mathcal{E}_k^h \left( \frac{\beta \lambda_k^s}{1 + \alpha \lambda_k}, t \right) \lesssim \left( \frac{\beta \lambda_k^s}{1 + \alpha \lambda_k} \right)^{\delta} h^{\delta} \lesssim \lambda_k^{-\delta(1-s)} h^{\delta} \lesssim h^{\delta \delta}, \tag{8}
\]
where we have used the fact that $\lambda_k^{-\delta(1-s)} \leq \lambda_1^{-\delta(1-s)}$. It follows that
\[
\|(S_\beta(t + h) - S_\beta(t))\|_{\mathcal{H}^\nu} = \|(-\Delta)^{\sigma} (S_\beta(t + h) - S_\beta(t))\| \lesssim h^{\delta \delta}\|\varphi\|_{\mathcal{H}^\nu}.
\]
By a similar technique as in above, one can check at
\[
\|(\mathcal{S}_\beta(t + h) - \mathcal{S}_\beta(t))\|_{\mathcal{H}^\nu} = \|(-\Delta)^{\nu}(-\Delta)^{\sigma-\nu} (\mathcal{S}_\beta(t + h) - \mathcal{S}_\beta(t))\| \lesssim h^{\delta \delta}\|\varphi\|_{\mathcal{H}^\nu},
\]
which implies that (7) holds.

Proposition 2.5. Let $s > 1$, $0 \leq \sigma - \nu < 1$, $\delta > 0$ be small enough such that $\sigma - \nu + \delta(1-s) < 1$, $\mu > \sigma, \epsilon > 0$ be small enough such that $\sigma + \epsilon(s-1) < \mu$. Then, for $t \in [0, T]$, there holds
\[
\|S_\beta(t)\|_{L(\mathcal{H}^\nu_x, \mathcal{H}^\nu)} \lesssim 1, \quad \|\mathcal{S}_\beta(t)\|_{L(\mathcal{H}^\nu, \mathcal{H}^\nu)} \lesssim 1, \tag{9}
\]
\[
\|S_\beta(t + h) - S_\beta(t)\|_{L(\mathcal{H}^\nu_x, \mathcal{H}^\nu)} \lesssim h^{\delta \epsilon}, \quad \|\mathcal{S}_\beta(t + h) - \mathcal{S}_\beta(t)\|_{L(\mathcal{H}^\nu, \mathcal{H}^\nu)} \lesssim h^{\delta \epsilon}. \tag{10}
\]

Proof. Since $\mu - \sigma > 0$ and $\sigma - \nu \in [0, 1)$, it can be seen that
\[
E_\beta \left( \frac{-t^\beta \lambda_k^s}{1 + \alpha \lambda_k} \right) \lesssim 1 \lesssim \lambda_k^{\nu-s}, \quad (1 + \alpha \lambda_k)^{-1} E_\beta \left( \frac{-t^\beta \lambda_k^s}{1 + \alpha \lambda_k} \right) \lesssim \lambda_k^{\nu-\sigma}, \quad \text{for } t \in [0, T].
\]
By two above estimates, one can check that the properties (9) hold. Next, by a similar technique as in (8), one can see

\[
E_{\beta} \left( \frac{\beta \lambda_k^s}{1 + \alpha \lambda_k}, t \right) \lesssim \left( \frac{\beta \lambda_k^s}{1 + \alpha \lambda_k} \right)^{\frac{s}{s-1}} \lesssim \lambda_k^{s-1} \lesssim \lambda_k^{\sigma-1}, \quad (1 + \alpha \lambda_k)^{-1} E_{\beta} \left( \frac{\beta \lambda_s^k}{1 + \alpha \lambda_k}, t \right) \lesssim \lambda_k^{-s} \lesssim \lambda_k^{\sigma-1} \lesssim \lambda_k^{\sigma-1},
\]

which helps us to obtain (10).

\[\Box\]

**Proposition 2.6.** Let \(1 < \sigma - \nu < s\). Then, for \(t \in (0, T]\), there holds

\[
\|S_\beta(t)\|_{L(H^{\nu+1}, H^\sigma)} \lesssim t^{-\frac{\beta(s-\nu)}{s-1}}, \quad \|S_\beta(t)\|_{L(H^\sigma, H^\nu)} \lesssim t^{-\frac{\beta(s-\nu)}{s-1}},
\]

(13)

**Proof.** As a consequence, it can be seen from the property (E1) of Proposition 2.3 that

\[
E_{\beta, \sigma}(-z) \lesssim (1 + z)^{-1} \lesssim z^{-\xi}, \quad \text{for any } \xi \in (0,1).
\]

(14)

Applying (14) for \(\xi = \frac{s-\nu-1}{s-1} \in (0,1)\), for \(0 < t \leq T\), it is obvious that

\[
E_{\beta} \left( \frac{-t^\beta \lambda_k^s}{1 + \alpha \lambda_k} \right) \lesssim \left( \frac{t^\beta \lambda_k^s}{1 + \alpha \lambda_k} \right)^{\frac{s-\nu-1}{s-1}} \lesssim t^{-\frac{\beta(s-\nu-1)}{s-1}} \lesssim \lambda_k^{-\nu},
\]

\[
(1 + \alpha \lambda_k)^{-1} E_{\beta} \left( \frac{-t^\beta \lambda_k^s}{1 + \alpha \lambda_k} \right) \lesssim \lambda_k^{-1} \left( \frac{t^\beta \lambda_k^s}{1 + \alpha \lambda_k} \right)^{-\xi} \lesssim t^{-\frac{\beta(s-\nu-1)}{s-1}} \lesssim \lambda_k^{-\nu}.
\]

By the above properties, one can easily obtain

\[
\|S_\beta(t)\|_{H^\sigma} \lesssim t^{-\frac{\beta(s-\nu-1)}{s-1}} \|\varphi\|_{H^{\nu+1}}, \quad \|S_\beta(t)\|_{H^\sigma} \lesssim t^{-\frac{\beta(s-\nu-1)}{s-1}} \|\varphi\|_{H^\nu},
\]

which implies that (6) for \(t \in (0, T]\). This completes the proof. \(\Box\)

In the following proposition, we introduce a stochastic tool, which is called the Burkholder-Davis-Gundy-type inequality.

**Proposition 2.7 (see [21]).** Let \(p \geq 2\) and \(\Psi\) be a mapping from \([0, T]\) to \(L_0^2\) satisfying

\[
\mathbb{E} \left( \int_0^t \|\Psi(r)\|_{L_0^2}^2 dr \right)^{\frac{p}{2}} < \infty.
\]

Then, it holds that

\[
\mathbb{E} \left\| \int_0^t \Psi(r) dW(r) \right\|^p \lesssim \mathbb{E} \left[ \left( \int_0^t \|\Psi(r)\|_{L_0^2}^2 dr \right)^{p/2} \right],
\]

where the hidden constant only depend on \(p\).

3. **Existence, uniqueness, and regularity results.** Primary purpose now is to establish the existence, uniqueness and regularity of the solution of Problem (1). Let \(\sigma, \mu, \nu \geq 0\) and \(p \geq 2\). To do this end, the following assumptions are given:

(H1) The initial condition \(u_{ini} \in L^p(\Omega, \dot{H}^\sigma)\),

(H2) The initial condition \(u_{ini} \in L^p(\Omega, \dot{H}^\nu)\),

(7) By the above properties, one can easily obtain

\[
E_{\beta, \sigma}(-z) \lesssim (1 + z)^{-1} \lesssim z^{-\xi}, \quad \text{for any } \xi \in (0,1).
\]

(14)
(H3) The time-spatial-noise \( \phi \) satisfies \( \phi(t, 0) = 0 \) and the globally Lipschitz condition
\[
\|\phi(t, v_1) - \phi(t, v_2)\|_{L^p(\Omega; L^2(H, \dot{H}^s))} \leq K_{\text{glo}} \|v_1 - v_2\|_{L^p(\Omega, \dot{H}^s)},
\]
for \( v_1, v_2 \in L^p(\Omega, \dot{H}^s) \), \( K_{\text{glo}} > 0 \) is a real number.

(H4) The time-spatial-noise \( \phi \) satisfies \( \phi(t, 0) = 0 \) and the locally Lipschitz condition
\[
\|\phi(t, v_1) - \phi(t, v_2)\|_{L^p(\Omega; L^2(H, \dot{H}^s))} \leq K_{\text{loc}} \left( \|v_1\|_{L^p(\Omega, \dot{H}^s)} + \|v_2\|_{L^p(\Omega, \dot{H}^s)} \right) \|v_1 - v_2\|_{L^p(\Omega, \dot{H}^s)},
\]
for \( v_1, v_2 \in L^p(\Omega, \dot{H}^s) \), \( K_{\text{loc}} > 0 \) is a real number.

(H5) The time-spatial-noise \( \phi \) satisfies \( \phi(t, 0) = 0 \) and the locally Lipschitz condition
\[
\|\phi(t, v_1) - \phi(t, v_2)\|_{L^p(\Omega; L^2(H, \dot{H}^s))} \leq K(t) \|v_1 - v_2\|_{L^p(\Omega, \dot{H}^s)},
\]
for \( v_1, v_2 \in L^p(\Omega, \dot{H}^s) \), \( K : [0, T] \to \mathbb{R}^+ \) does not depend on \( v_1, v_2 \).

3.1. Problem (1) under globally Lipschitz time-space-noise. In this subsection, Problem (1) in the case \( \phi \) satisfies globally Lipschitz condition (H3) is considered. In the following theorems, the results are established in two distinct cases including \( s \leq 1 \) and \( s > 1 \) separately.

**Theorem 3.1.** Let \( s \in (0, 1] \), \( u^{\text{ini}} \) and \( \phi \) fulfill (H1),(H3) for \( p \geq 2 \) and \( \sigma, \nu \) satisfying \( 0 \leq \sigma - \nu \leq 1 \). Then, there holds

i) Problem (1) has a unique mild solution \( u \) on \([0, T]\).

ii) the solution \( u \) satisfies
\[
\|u\|_{C([0, T]; L^p(\Omega, \dot{H}^s))} \lesssim \|u^{\text{ini}}\|_{L^p(\Omega, \dot{H}^s)},
\]
where the hidden constant depends on the constant \( K_{\text{glo}} \).

**Theorem 3.2.** Let \( s > 1 \), \( u^{\text{ini}} \) and \( \phi \) fulfill (H2),(H3) for \( p \geq 2 \) and \( \sigma, \mu, \nu \) satisfying \( 0 \leq \sigma - \nu < 1 \), \( \mu > \sigma \). Then, Problem (1) has a unique mild solution on \([0, T]\) satisfying
\[
\|u\|_{C([0, T]; L^p(\Omega, \dot{H}^s))} \lesssim \|u^{\text{ini}}\|_{L^p(\Omega, \dot{H}^s)},
\]
where the hidden constant depends on the constant \( K_{\text{glo}} \).

**Theorem 3.3.** Let \( s > 1 \), \( u^{\text{ini}} \) and \( \phi \) fulfill (H2),(H3) for \( \sigma, \mu, \nu \) satisfying \( 1 < \sigma - \nu < s \), \( \mu = \nu + 1 \), and \( p > 2/(1 - \kappa) \), with \( \kappa := \frac{\beta(\sigma - \nu - 1)}{s - 1} \). Then, Problem (1) has a unique solution \( u \in L^\infty, \kappa(0, T; L^p(\Omega, \dot{H}^s)) \) satisfying
\[
\|u\|_{L^\infty, \kappa(0, T; L^p(\Omega, \dot{H}^s))} \lesssim \|u^{\text{ini}}\|_{L^p(\Omega, \dot{H}^{\nu + 1})},
\]
where the hidden constant depends on the constant \( K_{\text{glo}} \).

**Remark 3.1.** It should be noted that in Theorem 3.3, the initial condition \( u^{\text{ini}} \in L^p(\Omega, \dot{H}^\mu) \), where \( \mu = \nu + 1 \) is less than \( \sigma \). This assumption is less strict than (H1).

**Proof of part i) in Theorem 3.1.** The proof here is split into two steps. In Step 1, we verify that \( \exists : C([0, T]; L^p(\Omega, \dot{H}^s)) \to C([0, T]; L^p(\Omega, \dot{H}^s)) \) defined by
\[
\exists(u)(t) = S_\beta(t)u^{\text{ini}} + \lambda \int_0^t \overline{S}_\beta(t - r)\phi(r, u(r))dW(r).
\]
is well-defined, i.e. \( \mathcal{S}(C([0,T]; L^p(\Omega, H^\nu))) \subset C([0,T]; L^p(\Omega, H^\nu)) \). In Step 2, a standard method that is Banach fixed point theorem is applied to show that this map is a contraction from \( C([0,T]; L^p(\Omega, H^\nu)) \) to itself.

**Step 1 (Verifying 3 is well-defined).** For \( u \in C([0,T]; L^p(\Omega, H^\nu)) \), we aim at showing that

\[
\sup_{t \in [0,T]} \| \mathcal{S}(u)(t) \|_{L^p(\Omega, H^\nu)} \lesssim \| u \|_{L^p(\Omega, H^\nu)} + \| u \|_{C([0,T]; L^p(\Omega, H^\nu))},
\]

(19)

and for \( h \) small enough there holds

\[
\| \mathcal{S}(u)(t + h) - \mathcal{S}(u)(t) \|_{L^p(\Omega, H^\nu)} \lesssim h^{\min(k, \frac{1}{2})} (\| u \|_{L^p(\Omega, H^\nu)} + \| u \|_{C([0,T]; L^p(\Omega, H^\nu))}).
\]

(20)

We begin by checking the first result (19). Indeed, it can be seen from (18) that

\[
\| \mathcal{S}(u)(t) \|_{L^p(\Omega, H^\nu)} \lesssim \| S_\beta(t) u^{ini} \|_{L^p(\Omega, H^\nu)} + \left\| \int_0^t \mathcal{S}_\beta(t - r) \phi(r, u(r)) dW(r) \right\|_{L^p(\Omega, H^\nu)}
\]

\[
= \left( E \| S_\beta(t) u^{ini} \|_{H^\nu}^p \right)^{\frac{1}{p}} + \left( E \left\| \int_0^t \mathcal{S}_\beta(t - r) \phi(r, u(r)) dW(r) \right\|_{H^\nu}^p \right)^{\frac{1}{p}}.
\]

(21)

An upper bound for the first term can be found by using the property (6) of \( S_\beta(t) \) in Proposition 2.4 as

\[
\left( E \| S_\beta(t) u^{ini} \|_{H^\nu}^p \right)^{\frac{1}{p}} \lesssim \left( E \| u^{ini} \|_{H^\nu}^p \right)^{\frac{1}{p}} = \| u^{ini} \|_{L^p(\Omega, H^\nu)}.
\]

(22)

For the second term in (21), by applying Proposition 2.7 and the property (6) of \( \mathcal{S}_\beta(t) \) in Proposition 2.4, one obtains

\[
\left( E \left\| \int_0^t \mathcal{S}_\beta(t - r) \phi(r, u(r)) dW(r) \right\|_{H^\nu}^p \right)^{\frac{1}{p}} = \left( E \left\| \int_0^t (-\Delta)^{\nu} \mathcal{S}_\beta(t - r) \phi(r, u(r)) dW(r) \right\|_{H^\nu}^p \right)^{\frac{1}{p}}
\]

\[
\lesssim \left( E \left( \int_0^t \| (-\Delta)^{\nu} \mathcal{S}_\beta(t - r) \phi(r, u(r)) \|_{L^2_\beta}^2 dr \right)^{\frac{1}{p}} \right)^{\frac{1}{p}}
\]

\[
\lesssim \left( E \left( \int_0^t \| (-\Delta)^{\nu} \phi(r, u(r)) \|_{L^2_\beta}^2 dr \right)^{\frac{1}{p}} \right)^{\frac{1}{p}},
\]

(23)

where we note that \( \| (-\Delta)^{\nu} \mathcal{S}_\beta(t) \Phi \|_{L^2_\beta} \lesssim \| \mathcal{S}_\beta(t) \|_{L^p(H^\nu, H^\nu)} \| (-\Delta)^{\nu} \Phi \|_{L^2_\beta} \lesssim \| (-\Delta)^{\nu} \Phi \|_{L^2_\beta} \) for \( \Phi \in L^2_\beta(H, H^\nu) \). On the other hand, the Hölder inequality allows that

\[
E \left( \int_0^t \| (-\Delta)^{\nu} \phi(r, u(r)) \|_{L^2_\beta}^2 dr \right)^{\frac{1}{2}} \lesssim E \left[ \left( \int_0^t \| (-\Delta)^{\nu} \phi(r, u(r)) \|_{L^2_\beta}^2 dr \right)^{\frac{1}{2}} \right]^{\frac{1}{2}}
\]

\[
= t^{\frac{\nu}{2} - 1} \int_0^t E \| \phi(r, u(r)) \|_{L^2_\beta(H, H^\nu)}^p dr,
\]

(24)

and the hypothesis (H3) leads to \( \| \phi(t, u(t)) \|_{L^p(H, L^2(H, H^\nu))} \leq K_{glo} \| u(t) \|_{L^p(\Omega, H^\nu)} \). From three latter observations, it is clear that

\[
\left( E \left\| \int_0^t \mathcal{S}_\beta(t - r) \phi(r, u(r)) dW(r) \right\|_{H^\nu}^p \right)^{\frac{1}{p}} \lesssim t^{\frac{\nu}{2} - \frac{1}{2}} \left( \int_0^t E \| \phi(r, u(r)) \|_{L^2_\beta(H, H^\nu)}^p dr \right)^{\frac{1}{2}}
\]

\[
\lesssim K_{glo} t^{\frac{\nu}{2} - \frac{1}{2}} \left( \int_0^t \| u(r) \|_{L^p(\Omega, H^\nu)}^p dr \right)^{\frac{1}{2}}.
\]

(25)
Combining (21)-(25) and noting that \( t^{\frac{1}{2} - \frac{3}{p}} \leq T^{\frac{1}{2} - \frac{3}{p}} \) due to \( p \geq 2 \), one can see
\[
\| \Theta(u)(t) \|_{L^p(\Omega, H^s)} \lesssim \| u^{ini} \|_{L^p(\Omega, H^s)} + \left( \int_0^t \| u(r) \|_{L^p(\Omega, H^s)}^p \, dr \right)^{\frac{1}{p}} \leq \| u^{ini} \|_{L^p(\Omega, H^s)} + K_{glob} t^{\frac{1}{2}} \| u \|_{C([0,T];L^p(\Omega, H^s))}. \quad (26)
\]

By the observation \( t^{\frac{1}{2}} \leq T^{\frac{1}{2}} \), the above estimate leads to (19) as desired.

Now, we continue to show that the continuity result (20) holds. It can be seen from the integral equation (18) that
\[
\Theta(u)(t+h) - \Theta(u)(t) = (S_\beta(t+h) - S_\beta(t)) u^{ini} + \lambda \int_t^{t+h} S_\beta(t+h-r) \phi(r, u(r)) \, dW(r) + \lambda \int_0^t \left( S_\beta(t+h-r) - S_\beta(t-r) \right) \phi(r, u(r)) \, dW(r). \quad (28)
\]
The first term can be estimated by using the property (7) of Proposition 2.4 as
\[
\mathbb{E} \| (S_\beta(t+h) - S_\beta(t)) u^{ini} \|_{H^s}^p \lesssim h^{p\beta} \mathbb{E} \| u^{ini} \|_{H^s}^p = h^{p\beta} \| u^{ini} \|_{L^p(\Omega, H^s)}. \quad (29)
\]
For the second term, one can find an upper bound a similar technique as in (23)
\[
\mathbb{E} \left\| \lambda \int_t^{t+h} S_\beta(t+h-r) \phi(r, u(r)) \, dW(r) \right\|_{L^p(\Omega, H^s)}^p \lesssim \mathbb{E} \left( \int_t^{t+h} \| (-\Delta)^\nu \phi(r, u(r)) \|_{L^2(\Omega)}^2 \, dr \right)^{\frac{p}{2}}. \quad (30)
\]
On the other hand, the Hölder inequality allows that
\[
\mathbb{E} \left( \int_t^{t+h} \| (-\Delta)^\nu \phi(r, u(r)) \|_{L^2(\Omega)}^2 \, dr \right)^{\frac{p}{2}} \lesssim \left( \int_t^{t+h} \| (-\Delta)^\nu \phi(r, u(r)) \|_{L^2(\Omega)}^p \, dr \right)^{\frac{1}{p}} \lesssim h^{\frac{p}{2} - 1} \int_t^{t+h} \mathbb{E} \| \phi(r, u(r)) \|_{L^2(\Omega)}^p \, dr,
\]
and the hypothesis (H3) leads to \( \| \phi(t, u(t)) \|_{L^p(\Omega, L^2(\Omega, H^s))} \leq K_{glo} \| u(t) \|_{L^p(\Omega, H^s)}. \)

From three latter observations, it is clear that
\[
\mathbb{E} \left\| \lambda \int_t^{t+h} S_\beta(t+h-r) \phi(r, u(r)) \, dW(r) \right\|_{L^p(\Omega, H^s)}^p \lesssim |K_{glo}|^p h^{\frac{p}{2} - 1} \int_t^{t+h} \mathbb{E} \| u(r) \|_{L^p(\Omega, H^s)}^p \, dr \lesssim |K_{glo}|^p h^{\frac{p}{2}} \| u \|_{C([0,T];L^p(\Omega, H^s))}. \quad (31)
\]

By a similar way as in (23)-(25) and using the property (7), one can check that
\[
\mathbb{E} \left\| \lambda \int_0^t (S_\beta(t+h-r) - S_\beta(t-r)) \phi(r, u(r)) \, dW(r) \right\|_{L^p(\Omega, H^s)}^p \lesssim |K_{glo}|^p h^{\beta (\frac{3}{2} - \frac{1}{p}) - 1} \int_0^t \mathbb{E} \| u(r) \|_{L^p(\Omega, H^s)}^p \, dr \lesssim |K_{glo}|^p h^{\beta \frac{3}{2}} \| u \|_{C([0,T];L^p(\Omega, H^s))}. \quad (32)
\]

Now, combining (28)-(31), one deduces that
\[
\| \Theta(u)(t+h) - \Theta(u)(t) \|_{L^p(\Omega, H^s)} \lesssim h^{\beta \delta} \| u^{ini} \|_{L^p(\Omega, H^s)} + K_{glob} (h^{\frac{1}{2}} + h^{\beta \delta} t^{\frac{1}{2}}) \| u \|_{C([0,T];L^p(\Omega, H^s))}. \quad (33)
\]

Since \( t^{\frac{1}{2}} \leq T^{\frac{1}{2}} \), \( h^{\beta \delta} \leq h^{\min(\beta \delta, \frac{1}{2})} \), and \( h^{\frac{1}{2}} \leq h^{\min(\beta \delta, \frac{1}{2})} \) for \( h \) is small enough, we conclude that (20) holds and finish Step 1 here.

Step 2 (Verifying \( \Theta \) is a contraction). Let \( v, w \in C([0,T];L^p(\Omega, H^s)) \), it is clear that
\[
\| \Theta(v)(t) - \Theta(w)(t) \|_{L^p(\Omega, H^s)} \lesssim \left\| \int_0^t \Phi(t-r) \left( \phi(v(r)) - \phi(r, w(r)) \right) \, dW(r) \right\|_{L^p(\Omega, H^s)}.
\]
On the other hand, in a similar way as in (23)-(24) an then using the Lipschitz condition (H3), one arrives at
\[ \| \int_0^t \mathcal{S}_\beta(t-r)(\phi(r,v(r)) - \phi(r,w(r)))dW(r) \|_{L^p(\Omega, H^s)} \]
\[ \lesssim t^{2-\frac{1}{p}} \left( \int_0^t \| \phi(r,v(r)) - \phi(r,w(r)) \|_{L^2(\Omega, H^s)}^p dr \right)^{\frac{1}{p}} \]
\[ \lesssim K_{\|v\|_p} T^{\frac{3}{2}-\frac{1}{p}} \left( \int_0^t \| v(r) - w(r) \|_{L^p(\Omega, H^s)}^p dr \right)^{\frac{1}{p}}. \]  
(32)

It follows from the above estimates that there exists \( M_1 > 0 \) such that
\[ \| \Imag(v)(t) - \Imag(w)(t) \|_{L^p(\Omega, H^s)}^p \leq M_1 \int_0^t \| v(r) - w(r) \|_{L^p(\Omega, H^s)}^p \, dr. \]  
(33)

Using the fact that \( \| v(r) - w(r) \|_{L^p(\Omega, H^s)} \leq \| v - w \|_{C([0,T];L^p(\Omega, H^s))} \), the above inequality immediately lead to
\[ \| \Imag(v)(t) - \Imag(w)(t) \|_{L^p(\Omega, H^s)}^p \leq M_1 t \| v - w \|_{C([0,T];L^p(\Omega, H^s))} \cdot \]

which implies that the following result holds for \( n = 1 \)
\[ \| \Imag^N(v)(t) - \Imag^N(w)(t) \|_{L^p(\Omega, H^s)}^p \leq M_1^N t^n N!^{-1} \| v - w \|_{C([0,T];L^p(\Omega, H^s))} \].

Now, we will show that (34) holds for any \( n \geq 1 \). The strategy here is to assume that (34) holds for \( N \) and then prove it still hold for \( N + 1 \). Indeed, applying a similar argument employed to obtain (33), one gets
\[ \| \Imag^{N+1}(v)(t) - \Imag^{N+1}(w)(t) \|_{L^p(\Omega, H^s)}^p \leq M_1 \int_0^t \| \Imag^{N}(v)(r) - \Imag^{N}(w)(r) \|_{L^p(\Omega, H^s)}^p \, dr \]
\[ \leq M_1 \int_0^t M_1^N r^N N!^{-1} \| v - w \|_{C([0,T];L^p(\Omega, H^s))}^p \, dr \]
\[ = M_1^{N+1} r^{N+1} ((N+1)!)^{-1} \| v - w \|_{C([0,T];L^p(\Omega, H^s))}^p. \]

Hence, we conclude that (34) holds for any \( n \geq 1 \), which directly leads to
\[ \| \Imag^n(v)(t) - \Imag^n(w)(t) \|_{C([0,T];L^p(\Omega, H^s))}^p \leq M^n T^n (n!)^{-1} \| v - w \|_{C([0,T];L^p(\Omega, H^s))}^p. \]

By the fact that \( M^n T^n (n!)^{-1} \) tends to zero as \( n \) tends to infinity, there exists \( m \geq 1 \) such that \( \Imag^m \) is a contraction, which leads to \( \Imag^m(u) = u \) has a unique solution \( u \in C([0,T];L^p(\Omega, H^s)) \); therefore, \( \Imag^m(\Imag(u)) = \Imag(\Imag^m(u)) = \Imag(u) \). Hence, \( \Imag(u) = u \) has a unique solution \( u \) in \( C([0,T];L^p(\Omega, H^s)) \).

**Proof of part ii) in Theorem 3.1.** By using a similar technique employed to obtain (26), one arrives at
\[ \| u(t) \|_{L^p(\Omega, H^s)}^p \lesssim \| u^{ini} \|_{L^p(\Omega, H^s)}^p + |K_{\|v\|_p}| \int_0^t \| u(r) \|_{L^p(\Omega, H^s)}^p \, dr. \]  
(35)

The Grönwall inequality is now applied to obtain
\[ \| u(t) \|_{L^p(\Omega, H^s)}^p \lesssim \| u^{ini} \|_{L^p(\Omega, H^s)}^p. \]

Therefore, the regularity result (15) holds.

**Proof of Theorem 3.2.** The results in Theorem 3.2 can be verified by a similar technique as in the proof of Theorem 3.1, where Proposition 2.5 is used instead of Proposition 2.4.
Proof of Theorem 3.3. We begin by checking that \( J : L^{\infty,\kappa}(0, T; L^p(\Omega, \dot{H}^\sigma)) \to L^{\infty,\kappa}(0, T; L^p(\Omega, \dot{H}^\sigma)) \) defined by as follows is a contraction.

\[
J(u)(t) = S_\beta(t)u^{ini} + \lambda \int_0^t \overline{S}_\beta(t-r)\phi(r, u(r))dW(r). \tag{36}
\]

An upper bound for the first term can be found by using the property (13) of \( S_\beta(t) \) in Proposition 2.6. In this way, one arrives at

\[
t^\kappa \|S_\beta(t)u^{ini}\|_{L^p(\Omega, \dot{H}^\sigma)} = t^{\frac{\beta(\sigma - \nu - 1)}{\nu - 1}} \left( E \|S_\beta(t)u^{ini}\|_{L^p(\Omega, \dot{H}^\sigma)}^p \right)^{\frac{1}{p}} \lesssim \|u^{ini}\|_{L^p(\Omega, \dot{H}^{\nu+1})}. \tag{37}
\]

By applying Proposition 2.7 and the property of \( \overline{S}_\beta(t) \) in Proposition 2.6, one obtains

\[
\left( E \left\| \int_0^t \overline{S}_\beta(t-r)\phi(r, u(r))dW(r) \right\|_{L^p(\Omega, \dot{H}^\sigma)}^p \right)^{\frac{1}{p}} = \left( E \left\| \int_0^t (-\Delta)^{\beta} \overline{S}_\beta(t-r)\phi(r, u(r))dW(r) \right\|_{L^p(\Omega, \dot{H}^{\nu+1})}^p \right)^{\frac{1}{p}} \lesssim \left( E \left( \int_0^t \|(-\Delta)^{\beta} \overline{S}_\beta(t-r)\phi(r, u(r))\|_{L^p(\Omega, \dot{H}^{\nu+1})}^2 dr \right)^{\frac{1}{2}} \right)^{\frac{1}{p}} \lesssim \left( E \left( \int_0^t r^{\frac{\beta(\sigma - \nu - 1)}{\nu - 1}} \|(-\Delta)^{\nu} \phi(r, u(r))\|_{L^p(\Omega, \dot{H}^{\nu+1})}^2 dr \right)^{\frac{1}{2}} \right)^{\frac{1}{p}}, \tag{38}
\]

where it should be noted that

\[
\|(-\Delta)^{\beta} \overline{S}_\beta(t)\Phi\|_{L^p_0} \lesssim \|\overline{S}_\beta(t)\|_{L(\dot{H}^\nu, \dot{H}^{\nu+1})} \|(-\Delta)^{\nu} \Phi\|_{L^p_0} \lesssim t^{-\frac{\beta(\sigma - \nu - 1)}{\nu - 1}} \|(-\Delta)^{\nu} \Phi\|_{L^p_0}.
\]

On the other hand, \( L^p \)-Hölder inequality allows that

\[
E \left( \int_0^t \|(-\Delta)^{\nu} \phi(r, u(r))\|_{L^p(\Omega, \dot{H}^{\nu+1})}^2 dr \right)^{\frac{1}{2}} \lesssim \left\{ \left( \int_0^t r^{-\frac{\nu}{\nu - 1}} dr \right)^{\frac{\nu - 2}{2}} \left( \int_0^t \|\phi(r, u(r))\|_{L^p(\Omega, \dot{H}^{\nu+1})}^p dr \right)^{\frac{p}{2}} \right\}^{\frac{1}{2}} = \left( \int_0^t r^{-\frac{\nu}{\nu - 1}} dr \right)^{\frac{\nu - 2}{2}} \left( \int_0^t \|\phi(r, u(r))\|_{L^p(\Omega, \dot{H}^{\nu+1})}^p dr \right)^{\frac{p}{2}} \lesssim |K_{\beta\phi}| t^{\frac{\nu - 1}{2} - \frac{\nu}{p}} \int_0^t \|u(r)\|_{L^p(\Omega, \dot{H}^{\nu+1})} dr, \tag{39}
\]

where we have used the fact that \( \|\phi(t, u(t))\|_{L^p(\Omega, L^2(\dot{H}^{\nu+1}))} \leq K_{\beta\phi} \|u(t)\|_{L^p(\Omega, \dot{H}^{\nu+1})} \), which follows from the condition (H3), and \( \int_0^t r^{-\frac{\nu}{\nu - 1}} dr \lesssim t^{1 - \frac{\nu}{2}} \) due to \( p > \frac{2}{1 - \kappa} \).

Combining (36)-(39), and noting that \( t^{\frac{\nu - 1}{2} - \frac{\nu}{p}} \leq T^{\frac{\nu - 1}{2} - \frac{\nu}{p}} \), it is obvious that

\[
t^\kappa \|J(u)(t)\|_{L^p(\Omega, \dot{H}^{\nu+1})} \lesssim \|u^{ini}\|_{L^p(\Omega, \dot{H}^{\nu+1})} + t^\kappa \left( \int_0^t \|u(r)\|_{L^p(\Omega, \dot{H}^{\nu+1})}^p dr \right)^{\frac{1}{p}} \tag{40}
\]

\[
\lesssim \|u^{ini}\|_{L^p(\Omega, \dot{H}^{\nu+1})} + \|u\|_{L^{\infty,\kappa}(0, T; L^p(\Omega, \dot{H}^{\nu+1}))}, \tag{41}
\]

which helps us to obtain an upper bound for \( J(u) \) as

\[
\|J(u)\|_{L^{\infty,\kappa}(0, T; L^p(\Omega, \dot{H}^{\nu+1}))} \lesssim \|u^{ini}\|_{L^p(\Omega, \dot{H}^{\nu+1})} + \|u\|_{L^{\infty,\kappa}(0, T; L^p(\Omega, L^2(\dot{H}^{\nu+1})))}
\]

In order words, \( J \) is well-defined. The step of verifying \( J \) is a contraction is easy to handle, one can check it by a similar technique as in the proof of Theorem 3.1.

Now, in a similar way employed to obtain (40), one arrives at

\[
t^{\nu \kappa} \|u(t)\|_{L^p(\Omega, \dot{H}^{\nu+1})} \lesssim \|u^{ini}\|_{L^p(\Omega, \dot{H}^{\nu+1})} + t^{\nu \kappa} \int_0^t \|u(r)\|_{L^p(\Omega, \dot{H}^{\nu+1})}^p dr \lesssim \|u^{ini}\|_{L^p(\Omega, \dot{H}^{\nu+1})} + \int_0^t r^{-\kappa} \|u(r)\|_{L^p(\Omega, \dot{H}^{\nu+1})}^p dr \tag{42}
\]
By applying the generalized Grönwall inequality (see [55]), one obtains
\[ p^\nu \|u(t)\|_{L^p(\Omega, H^\sigma)} \leq \vartheta \|u^{\text{ini}}\|_{L^p(\Omega, H^\sigma)}^{\nu} E_{1-\kappa} \left( \vartheta (1 - \kappa) t^{1-\kappa} \right), \]
where \( \vartheta \) is the hidden constant in (42). In addition
\[ E_{1-\kappa} \left( \vartheta (1 - \kappa) t^{1-\kappa} \right) \leq \exp \left( \vartheta (1 - \kappa) t^{1-\kappa} \right) \leq \exp \left( \vartheta (1 - \kappa) T^{1-\kappa} \right). \]
From two latter observations, it is clear that
\[ \|u(t)\|_{L^p(\Omega, H^\sigma)} \leq \|u^{\text{ini}}\|_{L^p(\Omega, H^\sigma)}, \]
which implies that the regularity property (17) holds. \( \Box \)

### 3.2. Problem (1) under locally Lipschitz time-space-noise

In this subsection, Problem (1) in the case \( \phi \) satisfies locally Lipschitz condition (H4) (or (H5)) is considered. The results here are established in two distinct cases including \( s \leq 1 \) and \( s > 1 \) separately.

In the following couple of theorems, we study Problem (1) in the first case when \( s \leq 1 \).

**Theorem 3.4.** Let \( s \in (0, 1), u^{\text{ini}} \) and \( \phi \) fulfill (H1), (H4) for \( p \geq 2 \) and \( \sigma, \nu \) satisfying \( 0 \leq \sigma - \nu \leq 1 \). Then, Problem (1) has a unique local mild solution on \([0, T_*]\), for \( T_* > 0 \) small enough.

**Remark 3.2.** Let \( s \in (0, 1), u^{\text{ini}}, \phi \) fulfill Theorem 3.4. Let \( u \) be the mild solution Problem (1) on \([0, T_{\text{max}}]\), where \( T_{\text{max}} \) is the maximal time of existence of \( u \). Then,
\[ T_{\text{max}} = \infty, \quad \text{or} \quad \lim_{t \to T_{\text{max}}} \sup_{t} \|u(t)\|_{L^p(\Omega, H^\sigma)} = \infty. \]

**Theorem 3.5.** Let \( s \in (0, 1), u^{\text{ini}} \) and \( \phi \) fulfill (H1), (H5) for \( p \geq 2 \) and \( \sigma, \nu \) satisfying \( 0 \leq \sigma - \nu \leq 1 \). Assume further that \( K \in L^p(0, T) \). Then, Problem (1) has a unique mild solution on \([0, T]\) satisfying
\[ \|u\|_{C([0, T]; L^p(\Omega, H^\sigma))} \leq \|u^{\text{ini}}\|_{L^p(\Omega, H^\sigma)}, \]
where the hidden constant depends on the function \( K \).

Next, two following theorems state similar results in the case when \( s > 1 \).

**Theorem 3.6.** Let \( s > 1, u^{\text{ini}} \) and \( \phi \) fulfill (H2), (H4) for \( p \geq 2 \) and \( \sigma, \mu, \nu \) satisfying \( 0 \leq \sigma - \nu < 1, \mu > \sigma \). Then, Problem (1) has a unique local mild solution on \([0, T_*]\), for \( T_* > 0 \) small enough.

**Remark 3.3.** Let \( s > 1, u^{\text{ini}}, \phi \) fulfill Theorem 3.6. Let \( u \) be the mild solution Problem (1) on \([0, T_{\text{max}}]\), where \( T_{\text{max}} \) is the maximal time of existence of \( u \). Then,
\[ T_{\text{max}} = \infty, \quad \text{or} \quad \lim_{t \to T_{\text{max}}} \sup_{t} \|u(t)\|_{L^p(\Omega, H^\sigma)} = \infty. \]

**Theorem 3.7.** Let \( s > 1, u^{\text{ini}} \) and \( \phi \) fulfill (H2), (H5) for \( p \geq 2 \) and \( \sigma, \mu, \nu \) satisfying \( 0 \leq \sigma - \nu < 1, \mu > \sigma \). Assume further that \( K \in L^p(0, T) \). Then, Problem (1) has a unique mild solution on \([0, T]\) satisfying
\[ \|u\|_{C([0, T]; L^p(\Omega, H^\sigma))} \leq \|u^{\text{ini}}\|_{L^p(\Omega, H^\sigma)}, \]
where the hidden constant depends on the function \( K \).

**Proof of Theorem 3.4.** Defining
\[ X_* := \left\{ u \in C([0, T_*], L^p(\Omega, H^\sigma)) : \sup_{0 \leq t \leq T_*} \|u(t)\|_{L^p(\Omega, H^\sigma)} \leq \varphi \right\}. \]
The proof here also aim at verifying that $\mathcal{S} : X_* \to X_*$ is a contraction. By a similar technique as in Step 1 of the proof of part i) in Theorem 3.1, one can check that $\mathcal{S} : X_* \to X_*$ is well-defined under the assumption (H4). For $0 \leq t \leq T_*$, it can be seen that

$$
\|\mathcal{S}(v)(t) - \mathcal{S}(w)(t)\|_{L^p(\Omega, H^\sigma)} \leq \left\| \int_0^t \bar{S}_\beta(t-r)(\phi(r, v(r)) - \phi(r, w(r)))dW(r) \right\|_{L^p(\Omega, H^\sigma)}.
$$

On the other hand, in a similar way as in (23)-(24) and using the Lipschitz condition (H4), one arrives at

$$
\left\| \int_0^t \bar{S}_\beta(t-r)(\phi(r, v(r)) - \phi(r, w(r)))dW(r) \right\|_{L^p(\Omega, H^\sigma)} \\
\leq t^{\frac{1}{2} - \frac{1}{p}} \left( \int \mathbb{E}\|\phi(r, v(r)) - \phi(r, w(r))\|^p_{L^p(\Omega, H^\sigma)}dr \right)^{\frac{1}{p}} \\
\leq K_{loc} t^{\frac{1}{2} - \frac{1}{p}} \left( \int \|v(r)\|_{L^p(\Omega, H^\sigma)} + \|w(r)\|_{L^p(\Omega, H^\sigma)} \right)^p \|v(r) - w(r)\|_{L^p(\Omega, H^\sigma)}^{\frac{p}{2}} \\
\leq cK_{loc} t^{\frac{1}{2} - \frac{1}{p}} \left( \int \|v(r) - w(r)\|_{L^p(\Omega, H^\sigma)}^p \right)^{\frac{1}{2}}.
$$

From two latter observations, one can see there exists $\mathcal{M}_2 > 0$ such that

$$
\|\mathcal{S}(v)(t) - \mathcal{S}(w)(t)\|_{L^p(\Omega, H^\sigma)} \leq \mathcal{M}_2 T_*^{\frac{1}{2}} \|v - w\|_{X_*}.
$$

By choosing $T_* > 0$ such that $\mathcal{M}_2 T_*^{\frac{1}{2}} < 1$, one can see clearly that $\mathcal{S}$ is a contraction. 

\textbf{Proof of Remark 3.2.} Assume that $T_{\text{max}} \leq \infty$. Let us pick a positive sequence $\{t_n\}$ satisfying $\lim_{n \to \infty} t_n = T_{\text{max}}$. For $t_m, t_n \in [0, T_{\text{max}}]$, it can be seen that

$$
\|u(t_n) - u(t_m)\|_{L^p(\Omega, H^\sigma)} \\
\leq \|S_\beta(t_n) - S_\beta(t_m)\|_{L^p(\Omega, H^\sigma)} + \lambda \left\| \int_{t_m}^{t_n} \bar{S}_\beta(r)\phi(r, u(r))dW(r) \right\|_{L^p(\Omega, H^\sigma)} \\
+ \lambda \left\| \int_{t_m}^{t_n} (S_\beta(t_n) - r)\bar{S}_\beta(r)\phi(r, u(r))dW(r) \right\|_{L^p(\Omega, H^\sigma)}.
$$

By a similar technique as in Step 1 of the proof of Theorem 3.1, one can arrive at

$$
\|u(t_n) - u(t_m)\|_{L^p(\Omega, H^\sigma)} \\
\leq |t_n - t_m|^{\frac{1}{2}} \|u^{ini}\|_{L^p(\Omega, H^\sigma)} + \left( K_{loc} |t_n - t_m|^\frac{3}{2} + K_{loc} T_*^{\frac{3}{2}} |t_n - t_m|^\frac{3}{2} \right) \|u^{ini}\|_{L^p(\Omega, H^\sigma)}.
$$

Let $\epsilon$ be an arbitrary positive number. We now find $N(\epsilon)$ such that for any $n \geq m > N(\epsilon)$ there holds

$$
\|u(t_n) - u(t_m)\|_{L^p(\Omega, H^\sigma)} \leq \epsilon.
$$

Due to the fact that $\lim_{n \to \infty} t_n = T_{\text{max}}$, there exists $N_1(\epsilon)$ such that $t_n \geq T_{\text{max}}$, for any $n > N_1(\epsilon)$. Denote by $c$ the hidden constant in (47). Since $\{t_n\}$ is a Cauchy sequence in $\mathbb{R}$, there exists $N_2(\epsilon), N_3(\epsilon), N_4(\epsilon)$ such that

$$
c|t_n - t_m|^{\frac{1}{2}} \|u^{ini}\|_{L^p(\Omega, H^\sigma)} \leq \frac{\epsilon}{3}, \quad \text{for any } n > m > N_3(\epsilon),$$

$$
cK_{loc} |t_n - t_m|^\frac{3}{2} \|u^{ini}\|_{L^p(\Omega, H^\sigma)} \leq \frac{\epsilon}{3}, \quad \text{for any } n > m > N_4(\epsilon),$$

$$
cK_{loc} T^{\frac{3}{2}} |t_n - t_m|^\frac{3}{2} \|u^{ini}\|_{L^p(\Omega, H^\sigma)} \leq \frac{\epsilon}{3}, \quad \text{for any } n > m > N_5(\epsilon).$$
Choosing \( N(\epsilon) = \max \{ N_3(\epsilon), N_4(\epsilon), N_5(\epsilon) \} \), we have (49) holds, which implies that \( \{u(t_n)\} \) is a Cauchy sequence in \( L^p(\Omega, \dot{H}^\sigma) \). Assume that it tends to \( u_* \in L^p(\Omega, \dot{H}^\sigma) \) as \( n \to \infty \), we deduce that
\[
\lim_{t \to T_{\text{max}}} \|u(t)\|_{L^p(\Omega, \dot{H}^\sigma)} = \|u_*\|_{L^p(\Omega, \dot{H}^\sigma)},
\]
where we note that \( t_n \) is arbitrary. We may extend \( u \) over \([0,T_{\text{max}}]\) and obtain a contradiction with the maximality of \( T_{\text{max}} \).

\[\square\]

**Proof of Theorem 3.5.** Reconsider the mapping \( \mathcal{Z} : C([0,T]; L^p(\Omega, \dot{H}^\sigma)) \to C([0,T]; L^p(\Omega, \dot{H}^\sigma)) \). We first show that \( \mathcal{Z} \) is well-defined and it is a contraction. By the same way employed to obtain (21)-(24) and then using the locally Lipschitz condition (H5), one obtains
\[
\|\mathcal{Z}(u)\|_{L^p(\Omega, \dot{H}^\sigma)} \lesssim \|u^{ini}\|_{L^p(\Omega, \dot{H}^\sigma)} + t^{\frac{1}{2}-\frac{1}{p}} \left( \int_0^t E(\|\phi(r, u(r))\|_{L^p_0(\dot{H}, \dot{H}^\sigma)}^p) dr \right)^{\frac{1}{p}}\]
\[\lesssim \|u^{ini}\|_{L^p(\Omega, \dot{H}^\sigma)} + t^{\frac{1}{2}-\frac{1}{p}} \left( \int_0^t |\mathcal{K}(r)|^p E(\|u(r)\|_{L^p(\Omega, \dot{H}^\sigma)}^p) dr \right)^{\frac{1}{p}}.\]  

(50)

Since \( \mathcal{K} \in L^p(0,T) \), we know \( \int_0^1 |\mathcal{K}(r)|^p dr < \infty \); therefore, the above estimate leads to
\[
\sup_{t \in [0,T]} \|\mathcal{Z}(u)(t)\|_{L^p(\Omega, \dot{H}^\sigma)} \lesssim \|u^{ini}\|_{L^p(\Omega, \dot{H}^\sigma)} + T^{\frac{1}{2}-\frac{1}{p}} \|\mathcal{K}\|_{L^p(0,T)} \|u\|_{C([0,T]; L^p(\Omega, \dot{H}^\sigma))}.\]  

(51)

By a similar way as in (28)-(31) and the locally Lipschitz condition (H5), one obtains
\[
\|\mathcal{Z}(u)(t + h) - \mathcal{Z}(u)(t)\|_{L^p(\Omega, \dot{H}^\sigma)} \lesssim h^{\delta\beta} \|u^{ini}\|_{L^p(\Omega, \dot{H}^\sigma)} + h^{\frac{1}{2}} \left( \int_{t}^{t+h} |\mathcal{K}(r)|^p E(\|u(r)\|_{L^p(\Omega, \dot{H}^\sigma)}^p) dr \right)^{\frac{1}{p}} + \\
y + h^{\delta\beta} \left( \int_{t}^{t+h} |\mathcal{K}(r)|^p E(\|u(r)\|_{L^p(\Omega, \dot{H}^\sigma)}^p) dr \right)^{\frac{1}{p}} \\
\lesssim h^{\delta\beta} \|u^{ini}\|_{L^p(\Omega, \dot{H}^\sigma)} + h^{\min \{\beta\delta, \frac{p}{2} - 1\} \left( \int_{t}^{t+h} |\mathcal{K}(r)|^p E(\|u(r)\|_{L^p(\Omega, \dot{H}^\sigma)}^p) dr \right)^{\frac{1}{p}},
\]

Hence, for \( 0 \leq t \leq T \), there holds
\[
\|\mathcal{Z}(u)(t + h) - \mathcal{Z}(u)(t)\|_{L^p(\Omega, \dot{H}^\sigma)} \lesssim h^{\delta\beta} \|u^{ini}\|_{L^p(\Omega, \dot{H}^\sigma)} + h^{\min \{\beta\delta, \frac{p}{2} - 1\} \left( \int_{t}^{t+h} |\mathcal{K}(r)|^p E(\|u(r)\|_{L^p(\Omega, \dot{H}^\sigma)}^p) dr \right)^{\frac{1}{p}}.\]  

(52)

From (51) and (52), we conclude that \( \mathcal{Z} \) is also well-defined under the conditions (H1) and (H5). The step of showing \( \mathcal{Z} \) is a contraction is similar to Step 2 in the proof of Theorem 3.1, therefore; it is skipped here.

By using a similar argument employed to obtain (50), one arrives at
\[
\|u(t)\|_{L^p(\Omega, \dot{H}^\sigma)}^p \lesssim \|u^{ini}\|_{L^p(\Omega, \dot{H}^\sigma)}^p + \int_0^t |\mathcal{K}(r)|^p E(\|u(r)\|_{L^p(\Omega, \dot{H}^\sigma)}^p) dr.\]  

(53)
Using the Grönwall inequality, one obtains

\[
\|u(t)\|_{L^p(\Omega, \dot{H}^\sigma)} \lesssim \|u_{ini}\|_{L^p(\Omega, \dot{H}^\sigma)} \exp \left( \int_0^t |\mathcal{K}(r)|^p \, dr \right) = \|u_{ini}\|_{L^p(\Omega, \dot{H}^\sigma)} \exp(\|\mathcal{K}\|_{L^p(0,T)}^p),
\]

which implies that the regularity result (43) holds. \(\square\)

**Proof of Theorem 3.6 and Theorem 3.7.** The results in Theorem 3.6 (Theorem 3.7) can be verified by a similar technique as in the proof of Theorem 3.4 (Theorem 3.5), where Proposition 2.5 is used instead of Proposition 2.4. \(\square\)

**Proof of Remark 3.3.** The results in Remark 3.3 be obtained by a similar technique as in Remark 3.2; therefore, we skip here. \(\square\)

4. **Conclusion.** In this work, a direct problem for a fractional psuedo-parabolic equation perturbed by white noise is considered. Under distinct Lipschitz assumptions for the non-linear space-time-noise, some existence, uniqueness, regularity, and continuity results are established. In order to deal with the stochastic problem with the presence of fractional Caputo derivative, we use stochastic analysis and fractional calculus throughout this study flexibly.

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E-mail address: traangocthach@tdtu.edu.vn
E-mail address: devendra.maths@gmail.com
E-mail address: nguyenhoangluc@tdmu.edu.vn
E-mail address: nguyenhuytuan@vlu.edu.vn