Research Article

Complexity Analysis of a Modified Predator-Prey System with Beddington–DeAngelis Functional Response and Allee-Like Effect on Predator

Shuangte Wang and Hengguo Yu

College of Mathematics and Physics, Wenzhou University, Wenzhou, China

Correspondence should be addressed to Shuangte Wang; wanshuangte@126.com

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In this paper, complex dynamical behaviors of a predator-prey system with the Beddington–DeAngelis functional response and the Allee-like effect on predator were studied by qualitative analysis and numerical simulations. Theoretical derivations have given some sufficient and threshold conditions to guarantee the occurrence of transcritical, saddle-node, pitchfork, and nondegenerate Hopf bifurcations. Computer simulations have verified the feasibility and effectiveness of the theoretical results. In short, we hope that these works could provide a theoretical basis for future research of complexity in more predator-prey ecosystems.

1. Introduction

In reference [1], the authors simply considered a predator-prey system with Holling type II functional response and Allee-like effect on predator, which is described by the following nonlinear ordinary differential equations (ODEs):

\[ \dot{x} = r_1 x \left( 1 - \frac{x}{K_1} \right) - \frac{q_1 x y}{a + x} - m_1 x - d x^2, \quad (1a) \]

\[ \dot{y} = r_2 y \left( 1 - \frac{y}{K_2} \right) + \frac{e_1 q_1 x y}{a + y} - m_2 y, \quad (1b) \]

subject to initial conditions \( x(0), y(0) \geq 0 \). Here we replace the Holling type II functional response \((q x)/(a + x)\) with a functional response \((q x)/(a + bx + cy)\) and denote the parameter \( q \) as \( q_1 \) for later use, and thus above system (1a) and system (1b) have a modified version:

\[ \dot{x} = r_1 x \left( 1 - \frac{x}{K_1} \right) - \frac{q_1 x y}{a + bx + cy} - m_1 x - d x^2, \quad (2a) \]

\[ \dot{y} = r_2 y \left( 1 - \frac{y}{K_2} \right) + \frac{e_1 q_1 x y}{a + bx + cy} - m_2 y, \quad (2b) \]

where functions \( x = x(t) \) and \( y = y(t) \) are the densities of prey and predator at time \( t \), respectively. In terms of biology, all above positive constants have practical considerations. Parameters \( r_1 \) and \( r_2 \) denote the intrinsic growth rate of the prey and predator, respectively; \( K_1 \) and \( K_2 \) represent the carrying capacity of the environment; \( a \) is the half-saturation constant; \( q_1 \) is the search efficiency of predator for prey; \( m_1 \) and \( m_2 \) are the mortality rate of the prey and predator species, respectively; \( e_1 \) is the biomass conversion and we denote \( e_1 q_1 \) as \( q_2 \) for convenience; \( d \) is the intraspecific competition coefficient of the prey; \( e \) is the Allee effect.
constant. The specific growth term \( r_1x(1 - (x/K_1)) \) governs the increase of prey in the lack of predator, while the specific growth term \( r_2y(1 - (y/K_2)) \) governs the increase of predator. The square term \( dx^2 \) is an intrinsic decrease term on prey. The term \( y/(y + e) \) on the specific growth term of predator with multiply form is called the Allee-like effect [2, 3] and is different from predator-prey systems in [4–8] with Allee effect on prey. The coupled term \( q_1x(a + bx + cy) \) is named Beddington–DeAngelis (B-D) functional response (named after Beddington and DeAngelis et al.) [9, 10]. It is similar to the Holling type II functional response incorporating an extra term \( cy \) in denominator, which describes mutual interference among predators [11, 12]. This functional response has some of the same qualitative features as the ratio-dependent form but avoids some of singular behaviors of ratio-dependent models at low densities [11]. Obviously, when \( b = 1 \) and \( c = 0 \), system (2a) and system (2b) reduce to original system (1a) and system (1b).

When parameters \( d = m_1 = e = 0 \) (without Allee effect) and \( r_2 - m_2 < 0 \), system (2a) and system (2b) reduce to System (2.1) in [13] and the author particularly conducted stability (local and global) and bifurcation (saddle-node, transcritical, Hopf–Andronov, and Bogdanov–Takens) analysis with a detailed mathematical analysis. When \( d = m_1 = r_2 = 0 \), system (2a) and system (2b) become model system (3a) and model system (3b) in [14], which was also independently and originally proposed in [9–11], while in [14], the authors discussed local and global asymptotic stability behavior of various equilibria and Hopf bifurcation occurs when parameter \( m \) corresponding to reserved region crosses some critical values. To mimic the real-world scenario, they solved the inverse problem of estimation of system parameter \( m \) by using the sampled data. System (1.3) in [15] is similar to above system in [14] except the constant rate harvesting term. In this reference, the authors showed that it undergoes several kinds of bifurcations, such as the saddle-node bifurcation, the subcritical and supercritical Hopf bifurcation, and Bogdanov–Takens bifurcation by choosing the death rate of the predator and the harvesting rate of the prey as bifurcation parameters.

Motivated by previous progress of predator-prey systems with B-D functional response or Allee-like effect, this paper mainly concentrates on dynamical analysis of system (2a) and system (2b). The rest of this paper is structured as follows. Preliminaries, such as boundedness, permanence, and existence of trivial equilibria, are given in Section 2. The existence of interior equilibrium is presented in Section 3 by virtues of the cobweb model and polynomial equations, respectively. In Section 4, we give stability analysis of equilibria and nonexistence of limit cycles. In Section 5, local codimension one bifurcations are analyzed, especially the Hopf bifurcation incorporating numerical simulations and Hopf bifurcation curves. In Section 6, we carry out short conclusions for our system.

## 2. Preliminaries

In this section, we devote to give some priori foundations. Before presenting the main results, we denote the first quadrant as \( \mathbb{R}^2_+ \), and its closure is \( \mathbb{R}^2_+ = \mathbb{R}^2 \). For biological consideration, system (2a) and system (2b) are defined on the domain \( \mathbb{R}^2_+ \) and all the solutions are non-negative with initial conditions \( x(0), y(0) \geq 0 \), i.e., \( \mathbb{R}^2_+ \) is an invariant set and any orbits starting from it cannot cross the coordinate axes. Furthermore, all solutions are uniformly bounded. But now we only need to prove following theorems.

### 2.1. Boundedness and Permanence

**Theorem 1** (uniform boundedness). Suppose that a non-negative function \( \varphi(x, y) \) and its partial derivatives \( \varphi_x \) and \( \varphi_y \) are continuous in \( \mathbb{R}^2_+ \), then the system

\[
\begin{align*}
\dot{x} &= r_1x \left(1 - \frac{x}{K_1}\right) - \varphi(x, y) - m_1x - dx^2, \\
\dot{y} &= r_2y \left(1 - \frac{y}{K_2}\right) \frac{y}{y+e} + e \varphi(x, y) - m_2y,
\end{align*}
\]

subject to \( x(0), y(0) \geq 0 \) is uniformly bounded.

This Theorem 1 holds obviously after introducing an auxiliary function \( z = e_x x + y \) [1, 16]. The following theorem with the help of comparison principle in ODEs is about permanence of system (2a) and system (2b).

**Theorem 2** (permanence). If parameters satisfy

\[
\begin{align*}
\omega_1 &= \frac{r_1 - m_1 - ((q_1M_2)/(a + cM_2))}{(r_1/K_1)_x + d} > 0, \\
\omega_0 &= \frac{q_2\omega_1 (1 - \lambda)}{a + bw_1 (1 - \lambda)} - m_2 > 0, \\
a_1 &= \frac{r_2}{e} - \frac{cq_2\omega_1 (1 - \lambda)}{[a + bw_1 (1 - \lambda)]^2} < 0, \\
cp &= 1, e \geq \rho [a + bw_1 (1 - \lambda)],
\end{align*}
\]
where $M_2$ is a positive constant, $\lambda \in (0, 1)$, and $\rho = \sqrt[3]{((r_2e(K_2 + e))/K_2)^2q_2\omega_1(1 - \lambda))}$, then system (2a) and system (2b) are permanent.

Proof. From above Theorem 1, we have a positive upper bound $\xi_1$ such that

$$\max\left\{\limsup_{t \to -\infty} x(t), \limsup_{t \to -\infty} y(t)\right\} \leq \xi_1.$$  \hfill (5)

Then, we obtain $M_2 > 0$ and sufficiently large $T_1$, such that

$$\dot{x} \geq x \left[ r_1 - m_1 - \frac{q_1M_2}{a + cM_2} \left( \frac{r_1}{K_1} + d \right) \right], \quad \forall t \geq T_1.$$ \hfill (6)

By using the lemmas in [17], we admit $\liminf_{t \to -\infty} x(t) \geq \omega_1$. That is to say, there exist sufficiently large $T_2$, such that $x(\infty) \geq \omega_1(1 - \lambda), \forall t \geq T_2$. Then, we consider equation (2b) again. It yields $\dot{y} \geq f(y)y$; here the function $f(y)$ incorporating its Taylor expression is

$$f(y) = \frac{r_2y}{y + e} \left( 1 - \frac{y}{K_2} \right) - m_2 + \frac{q_2\omega_1(1 - \lambda)}{a + b\omega_1(1 - \lambda) + cy}$$

$$= a_0 + a_1y + \frac{1}{2} f''(\xi_y) y^2, \quad \xi_y \in [0, y],$$ \hfill (7)

in which the second order derivative of $f(y)$ is

$$f''(y) = \frac{2r_2e(K_2 + e)}{K_2(y + e)^3} + \frac{2q_2\omega_1(1 - \lambda)}{[a + b\omega_1(1 - \lambda) + cy]^2}.$$ \hfill (8)

From the conditions in this theorem, we know that $f''(y) \geq 0$ when $y \geq 0$. Thus, $f(y) \geq a_0 + a_1y, \forall y \geq 0$. By using the lemmas in [17] again, the proof is completed. \hfill $\Box$

2.2. Existence of Trivial Equilibria. In this section, we will discuss the existence conditions of trivial equilibria of system (2a) and system (2b). Firstly, from [1], it is obvious that system (2a) and system (2b) have trivial equilibria: $E_0 = (0, 0)$, $E_1 = (x_1, 0)$, and $E_2^{(i)} = (y_{1,2}, 0), k = 1, 2$, where

$$x_1 = \frac{(r_1 - m_1)(r_1/K_1 + d)}{(r_2/m_2) + (e/K_2)}$$

$$y_{1,2} = \frac{r_2 - m_2 \pm \sqrt{\Delta}}{2r_2/K_2}, \quad \Delta = (r_2 - m_2)^2 - 4r_2m_2e/K_2.$$ \hfill (9)

The point $E_1$ exists when $r_1 > m_1$, while the points $E_2^{(i)}$ all exist if $r_2 > m_2$, and $\Delta > 0$. If $r_2 > m_2$ but $\Delta = 0$, then the two equilibria $E_2^{(1)}$ and $E_2^{(2)}$ collide with each other and we denote this equilibrium as $E_2 = (0, y_2)$.

3. Existence of Interior Equilibrium

Here and below, we denote the interior equilibrium $E_k$ of system (2a) and system (2b) as $(x_k, y_k)$ or $(s_1, s_2)$. This equilibrium must satisfy the following coupled algebraic equations:

$$r_1 \left( 1 - \frac{x}{K_1} \right) - \frac{q_1y}{a + bx + cy} - m_1 - dx = 0,$$ \hfill (10a)

$$r_2 \left( 1 - \frac{y}{K_2} \right) - \frac{q_2y}{a + bx + cy} - m_2 = 0.$$ \hfill (10b)

From equation (10a), we have

$$y \longrightarrow ( ((r_1 - m_1)a)/(q_1 - (r_1 - m_1)c) ) \text{ or } \infty \quad \text{if} \quad q_1 = (r_1 - m_1)c \text{ when } x \longrightarrow 0; \quad \text{or} \quad ( (am_2)/(q_2 - m_2b) ) \text{ or } \infty \quad \text{if} \quad q_2 = m_2b.$$ The implicit derivative $y'(x)$ from equation (10b) is

$$y'(x) = \frac{q_2K_2(a + cy)(y + e)^2}{r_2(y^2 + 2ey - K_2e)(a + bx + cy)^2 + q_2cK_2x(y + e)^2},$$ \hfill (11)

and we denote the positive root of a quadratic equation $y^2 + 2ey - K_2e = 0$ as $y_{1,2} = -e + \sqrt{e^2 + K_2e}$ for later use.

3.1. Cobweb Model. Based on above approximate analysis and the cobweb model, some cases about the existence of the interior equilibrium $E_k$ will be illustrated when isoclines from equations (10a) and (10b) all fall in $R_+^2$.

Case 1. If parameters satisfy

$$\frac{(r_1 - m_1)a}{q_1 - c(r_1 - m_1)} > y_k > y_{1,2}, \quad x_1 > 0,$$ \hfill (12)

then an interior equilibrium exists. Here, we take $r_1 = 1$, $r_2 = 1$, $K_1 = 20$, $K_2 = 8$, $q_1 = 0.12$, $q_2 = 0.06$, $a = 4$, $m_1 = 0.3$, $m_2 = 0.5$, $e = 0.3$, $b = 1$, $c = 0.1$, and $d = 5$; the first interior equilibrium is $E_1 \approx (0.119085, 3.688431)$ and the second equilibrium is $E_2 = (0.136769, 0.323758)$.

Case 2. If parameters satisfy

$$q_1 = c(r_1 - m_1), \quad y_k > y_{1,2}, \quad x_1 > 0,$$ \hfill (13)

then an interior equilibrium exists. Here, we take $r_1 = 1$, $r_2 = 1$, $K_1 = 20$, $K_2 = 8$, $q_1 = 0.6$, $a = 4$, $m_1 = 0.3$, $m_2 = 0.5$, $e = 0.3$, $b = 1$, $c = 0.1$, and $d = 5$; the first interior equilibrium is $E_1 = (0.126835, 3.83244)$, and the second equilibrium is $E_2 = (0.137621, 0.298566)$.

Case 3. If parameters satisfy

$$0 < x_1 < \frac{am_2}{q_2 - m_2b}$$

$$0 < y_k < \frac{\min\{y_{1,2}, (r_1 - m_1)a/(q_1 - c(r_1 - m_1))\}}{q_1 - c(r_1 - m_1)},$$ \hfill (14)

then an interior equilibrium exists. Here, we take $r_1 = 1$, $r_2 = 1.5$, $K_1 = 20$, $K_2 = 50$, $q_1 = 5$, $q_2 = 1.1$, $a = 4$, $b = 1$, $c = 0.1$, $m_1 = 0.1$, $m_2 = 0.5$, $e = 1$, and $d = 3.565$; an interior equilibrium is $E_2 = (0.0916118, 0.470837)$.

Case 4. If parameters satisfy...
0 < x_1,
q_2 = m_2 b,
\begin{align}
0 < y_k &< \min \left\{ \frac{(r_1 - m_1)a}{q_1 - c(r_1 - m_1)}, y_k \right\}, \\
0 < y_k &< y_1,
q_1 = c(r_1 - m_1),
\end{align}
then an interior equilibrium exists. Here, we take
r_1 = 1, r_2 = 1, K_1 = 20, K_2 = 8, q_1 = 0.12, a = 4,
m_1 = 0.3, m_2 = 0.5, e = 0.3, b = 1, c = 0.1, and d = 5; an
interior equilibrium is E_k \approx \{0.136885, 0.303164\}. Notice
that another interior equilibrium E_k \approx \{0.118552, 3.79777\} verifies Case 1.

Case 5. If parameters satisfy
\begin{align}
0 < x_1 &< \frac{am_2}{q_2 - m_2 b}, \\
0 < y_k &< y_1,
q_1 > c(r_1 - m_1),
\end{align}
then an interior equilibrium exists (see the second
equilibrium E_2 in Case 2).

Case 6. If y_k does not exist and parameters satisfy
\begin{align}
0 < \frac{am_2}{q_2 - m_2 b} &< x_1, \\
q_1 &> c(r_1 - m_1),
\end{align}
then an interior equilibrium exists. Here, we take
r_1 = 1, r_2 = 1.5, K_1 = 10, K_2 = 2, q_1 = 5, q_2 = 1.1,
a = 4, b = 1, c = 0.1, m_1 = 0.1, m_2 = 0.5, e = 1, and d = 0.05389805885 \approx d_{16}^{[10]}, an
interior equilibrium is E_k \approx \{0.399541, 0.750397\}.

Case 7. If y_k does not exist and parameters satisfy
\begin{align}
0 < \frac{am_2}{q_2 - m_2 b} &< x_1, \\
q_1 &> c(r_1 - m_1),
\end{align}
then an interior equilibrium exists. Here, we take
r_1 = 1, r_2 = 1.5, K_1 = 10, K_2 = 2, q_1 = 5, q_2 = 1.1,
a = 4, b = 1, c = 0.1, m_1 = 0.1, m_2 = 0.5, e = 1, and d = 0.11; an
interior equilibrium is E_k \approx \{4.178700, 2.09452\}.

3.2. Polynomial Equations of x_*, and y_*. From equation
(10a), an expression of y is
\[
y = \frac{(x_1 - x)(r_1/K_1) + d(a + bx)}{q_1 - c(r_1 - m_1) + c[r_1/K_1 + d]} x.
\]
Substituting it into the equation (10b), a quintic algebraic
equation of x_1 can be derived in the form of
\[
p(x_1) = \frac{K_x e q_1}{K_1 d + r_1} \left[ a m_2 (K_1 d + r_1) + K_1 (r_1 - m_1) (b m_2 - q_1) > 0, \right.
\]
where x_1 > 0 and a_0 < 0, then there is a positive root x_*, such
that x_* < x_1.

Case 1. If parameters satisfy
\begin{align}
p(x_1) &> 0, \\
x_1 &> 0, \\
a_0 &< 0,
q_1 &> c(r_1 - m_1),
\end{align}
then an interior equilibrium exists. Here, we take
r_1 = 1, r_2 = 1, K_1 = 10, K_2 = 8, q_1 = 0.12, a = 4,
m_1 = 0.3, m_2 = 0.5, e = 0.3, b = 1, c = 0.1, and d = 5; an
interior equilibrium is E_k \approx \{3.491956, 0.016748\}, and the following lemma is verified.

Lemma 1. Suppose that f(x) = \sum_{k=0}^{n} a_k x^n is a polynomial
with real coefficients, a_n \neq 0, n > 1. If a_n a_0 < 0, then the
equation f(x) = 0 has a positive root.

Proof
We only consider the special case a_n > 0, and thus a_0 < 0. It is
clear that the polynomial f(x) has a decomposition
\[
f(x) = \sum_{k=0}^{n-1} \left( a_n x^{n-k} + a_k \right) x^k.
\]
If we take a sufficiently large positive number X such that
X > \max_{0 \leq k \leq n-1} \{ n \lvert a_k \rvert / \lvert a_n \rvert \}^{(n-k)}}, then f(X) > 0,
and thus we complete the proof. \hfill \square

Lemma 2. Suppose that f(x) = \sum_{k=0}^{n} a_k x^n is a real poly-
nomial, a_n \neq 0, n > 1. If there is a positive number x_0 > 0 such
that a_n f(x_0) < 0, then the equation f(x) = 0 has a positive
root.

On the other hand, from equation (10b), we obtain an
expression
\[
x = \frac{(a + cy)[m_2 (y + e) - r_2 y (1 - y/K_2)]}{q_2 - b m_2} (y + e) + b_2 y (1 - y/K_2)\right).
\]
And a quintic algebraic equation of y_2 can be written in the
form of q(y) = \sum_{k=0}^{5} b_k y^k = 0 (see coefficients b_k in Appendix A.2). Suppose that the denominator
and numerator of expression (23) are all positive for some
y_2 > 0; then, x_2 > 0. Notice that if b_3 > 0 and b_0 < 0,
there must exist a positive root for the quintic equation
q(y) = 0 (see Lemma 1).

Case 2. If parameters satisfy b_3 < 0, m_2 \geq r_2, q_2 > b m_2,
y_2 > 0, and q(y_2) > 0, where
Here we rewrite equations (10a) and (10b) as
\[ y_u = \frac{q_2 - bm_2 + br_2 + \sqrt{(q_2 - bm_2 + br_2)^2 + (4br_2e/K_2)(q_2 - bm_2)}}{(2br_2/K_2)}, \] (24)

is a root of the quadratic equation in the denominator of (23), then an interior equilibrium exists. Here, we take \( r_1 = 1, r_2 = 0.5, K_1 = 10, K_2 = 8, m_1 = 0.3, m_2 = 0.6, d = 0.1, q_1 = 0.8, q_2 = 0.6, e = 0.3, a = 0.2, b = 0.2, \) and \( c = 0.1, \) and an interior equilibrium is \( E_{11} \approx (0.169319, 0.212419). \)

Case 3. If parameters satisfy \( b_0 < 0, y_2 > 0, q(y_2) > 0, \) \( y_u > 0, \) and \( q(y_u) > 0, \) then an interior equilibrium exists. Here, we take \( r_1 = 1, r_2 = 1, K_1 = 20, K_2 = 80, q_1 = 3, q_2 = 10, a = 2, b = 1, c = 1, m_1 = 0.1, m_2 = 0.5, \) \( e = 2, \) and \( d = 2.092753894 \approx d_1^{[12]}, \) and an interior equilibrium is \( E_{12} \approx (0.067460, 0.695847). \)

**Remark 1.** Here we rewrite equations (10a) and (10b) as
\[ f(x, y) = \left[ r_1 \left(1 - \frac{x}{K_1}\right) - m_1 - dx \right] (a + bx + cy) - q_1 y = 0, \] (25a)
\[ g(x, y) = r_2 y \left(1 - \frac{y}{K_2}\right)(a + bx + cy) + q_2 x (y + e) - m_2 (a + bx + cy) (y + e) = 0. \] (25b)

Furthermore, if we sort them as \( f(x, y) = a_0(x)y^2 + a_1(x)y, \)
\( g(x, y) = b_0(x)y^3 + b_1(x)y^2 + b_2(x)y + b_3(x), \) the first eliminant
\[ R_x(f, g) = \begin{bmatrix} a_0(x) & a_1(x) & 0 & 0 \\ 0 & a_0(x) & a_1(x) & 0 \\ 0 & 0 & a_0(x) & a_1(x) \\ b_0(x) & b_1(x) & b_2(x) & b_3(x) \end{bmatrix}, \] (26)

also yields above quintic equation \( p(x) = 0. \) Similarly, if we sort them as \( f(x, y) = a_0(y)x^2 + a_1(y)x + a_2(y), \)
\( g(x, y) = b_0(y)x + b_1(y), \) the second eliminant
\[ R_y(f, g) = \begin{bmatrix} a_0(y) & a_1(y) & a_2(y) \\ 0 & b_0(y) & b_1(y) \\ 0 & 0 & b_1(y) \end{bmatrix}, \] (27)
yields the quintic equation \( q(y) = 0. \)

**4. Stability Analysis of System (2a) and System (2b)**

In this section, we use the Routh–Hurwitz criterion and Perron’s theorems to analyze local stability of above equilibria in their existence interval, respectively. A theorem about global asymptotic stability and a theorem about nonexistence of limit cycles are also considered.

4.1. Local Stability Analysis. The Jacobian matrix of system (2a) and system (2b) takes the following form \( J = (I_{ij})_{2 \times 2}, \) where four components are
\[ J_{11} = r_1 - m_1 - 2x \left(\frac{r_1}{K_1} + d\right) - \frac{q_1(a + cy)y}{(a + bx + cy)^2}, \]
\[ J_{12} = -\frac{q_1x(a + bx)}{(a + bx + cy)^2}, \]
\[ J_{21} = \frac{q_2y(a + cy)}{(a + bx + cy)^2}, \]
\[ J_{22} = \frac{r_2y}{K_2(y + e)} (-2y^2 - 3ey + K_2y + 2K_2e) + \frac{q_2x(a + bx)}{(a + bx + cy)^2} - m_2. \] (28)

For the trivial equilibrium \( E_0 \) with \( r_1 \neq m_1 \) and the axial equilibrium \( E_1 \) with \( (q_2x_1/(bx_1 + a)) \neq m_2, \) we omit their stability [1]. When \( r_1 = m_1, \) the transformation \( \tau = m_2t \) and equations (2a) and (2b) yield \( \bar{x} = ((r_1/K_1) + d)/m_2 x^2 \) (we still use symbol \( t \), and thus \( E_2 \) is a saddle node and the parabolic sector is on the right half plane.

In the case that two axial equilibria \( E_2^{(k)} (k = 1, 2) \) exist, the Jacobian matrices are
\[ J(E_2^{(k)}) = \begin{bmatrix} r_1 - m_1 - \frac{q_1 y_k}{a + cy_k} & 0 \\ \frac{q_2 y_k}{a + cy_k} & \frac{(m_2 - r_2)y_k + 2m_2e}{y_k + e} \end{bmatrix}. \] (29)

Since \( J_{22}(E_2^{(1)}) < 0, \) we have the following: (a) \( E_2^{(1)} \) is a saddle point if \( r_1 - m_1 > ((q_1 y_1)/(a + cy_1)); \) (b) \( E_2^{(1)} \) is an asymptotically stable node if \( r_1 - m_1 < ((q_1 y_1)/(a + cy_1)). \) For the equilibrium \( E_2^{(2)}, \) it is obvious that \( J_{22}(E_2^{(2)}) > 0, \) and we have the following: (a) \( E_2^{(2)} \) is an unstable node if \( r_1 - m_1 > ((q_1 y_2)/(a + cy_2)); \) (b) \( E_2^{(2)} \) is a saddle point if \( r_1 - m_1 < ((q_1 y_2)/(a + cy_2)); \) (c) \( E_2^{(2)} \) is an unstable higher-order singular point if \( r_1 - m_1 = ((q_1 y_2)/(a + cy_2)). \) In the special case that two axial equilibria \( E_2^{(k)} (k = 1, 2) \) collide with each other, the Jacobian matrix is
\[ J(E_2) = \begin{bmatrix} r_1 - m_1 - \frac{q_1 y_2}{a + cy_2} & 0 \\ \frac{q_2 y_2}{a + cy_2} & 0 \end{bmatrix}, \] (30)
and thus \( E_2 \) is a higher-order singular point. If \( r_1 - m_1 > ((q_1 y_2)/(a + cy_2)), \) \( E_2 \) is unstable.
For the interior equilibrium $E_*$, its Jacobian matrix is
\[
J(E_*) = \begin{bmatrix}
J_{11}(E_*) & -\frac{q_1 x_* (a + b x_*)}{(a + b x_* + c y_*)^2} \\
\frac{q_2 y_* (a + c y_*)}{(a + b x_* + c y_*)^2} & J_{22}(E_*)
\end{bmatrix},
\]
where
\[
J_{11}(E_*) = \frac{r_1}{K_1} + d x_* + \frac{q_1 b x_* y_*}{(a + b x_* + c y_*)},
\]
\[
J_{22}(E_*) = \frac{r_2 y_*}{K_2(y_* + e)} (-y_*^2 - 2e y_* + K_2 e) - \frac{q_2 c x_* y_*}{(a + b x_* + c y_*)}. \tag{32}
\]

Denoting a new discriminant $\Delta_* = A_1^2 - 4A_2$ with the trace $A_1 := \text{tr } J(E_*)$ and the determinant $A_2 := \det J(E_*)$, we have
\begin{itemize}
  \item[(a)] If $A_1 < 0$ and $A_2 > 0$, $\Delta_* \geq 0$, then $E_*$ is an asymptotically stable node; (a2) $A_2 > 0$, $\Delta_* < 0$, then $E_*$ is an asymptotically stable focus; (a3) $A_2 < 0$, then $E_*$ is a saddle point.
  \item[(b)] If $A_1 = 0$ and (b1) $A_2 > 0$, then $E_*$ is a center or a focus; (b2) $A_2 < 0$, then $E_*$ is a saddle point.
  \item[(c)] If $A_1 > 0$, then $E_*$ is unstable and (c1) $\Delta_* = 0$, then $E_*$ is a node; (c2) $\Delta_* < 0$, then $E_*$ is a focus; (c3) $\Delta_* > 0$ and $A_2 > 0$, then $E_*$ is a saddle point.
\end{itemize}

When $A_2 = 0$ but $A_1 \neq 0$, $E_*$ is a stable (unstable) node if $A_1 < 0 \, (A_1 > 0)$ (see Theorem 7.1 in Zhifen Zhang’s book [18] for more details). It is probable that $E_*$ has a cusp case of codimension at least 2 which ensures potential Bogdanov–Takens bifurcation when $A_1 = A_2 = 0$.

### 4.2. Global Asymptotic Stability

Combining the stability conclusions of the point $E_0$ in above section, the positive definite Lyapunov function $V = e_1 x + y$ ensures that $E_0$ is globally asymptotically stable if one of the following conditions holds:
\[
\begin{align*}
  \text{(a)} & \quad r_1 \leq m_1, r_2 \leq m_2; \\
  \text{(b)} & \quad r_1 \leq m_1, \Delta < 0.
\end{align*}
\]

Furthermore, conditions $r_1 < m_1$, $r_2 + (q_2/b) - m_2 \leq 0$ and Theorem 1 deduce global asymptotical stability of $E_0$ pronto. If $r_1 > m_1$, $r_2 + (q_2/b) - m_2 \leq 0$, and $(q_1 x_*)/(a + b x_*) \leq m_1 \leq 0$, equilibria $E_1^{(k)}$, $E_2$, and $E_*$ do not exist, and $E_0$ is unstable, then Theorem 1 ensures that $E_1$ is globally asymptotically stable. For the further consideration of point $E_*$, the following theorem explains its global asymptotic stability.

**Theorem 3** (global asymptotic stability of $E_*$). If a unique interior equilibrium $E_*$ exists and parameters satisfy
\[
r_1 \frac{K_1}{K_2} + d > \frac{q_1 b y_*}{a(a + b x_* + c y_*)}, \quad y_* > K_2, \tag{34}
\]
then $E_*$ is globally asymptotically stable.

**Proof.** Here we take an unbounded positive definite Lyapunov function
\[
V = V(x, y) = \left( x - x_* - x_* \ln \frac{x}{x_*} \right) + A \left( y - y_* - y_* \ln \frac{y}{y_*} \right), \tag{35}
\]
with $A = ((q_1 (a + b x_*)/(q_2 (a + c y_*))))$. Introducing new variables $\tilde{x} = x - x_*$ and $\tilde{y} = y - y_*$, computing derivative along orbits of system (2a) and system (2b), we have
\[
\begin{align*}
  \frac{dV}{dt}(2) &= \left[ \frac{r_1}{K_1} - d + \frac{q_1 b y_*}{(a + b x + c y_*)} \right] \tilde{x}^2 \\
  &+ A \left[ \frac{r_2}{(y + e)(y_* + e)} \left( e - y y_* + e(y + y_*) \right) \right] \\
  &- \frac{q_2 c x_*}{(a + b x + c y_*)} \tilde{y}^2. \tag{36}
\end{align*}
\]

It is obvious that $(dV/dt)(2)$ is negative definite. Consequently, the Lyapunov function $V$ satisfies the asymptotic stability theorem in [19]. Thus, we complete the proof.

If conditions $((q_1 b y_*)/(a(a + b x_* + c y_*))) \leq (r_1/K_1) + d$ and $y_* \geq K_2$ hold, then $A_1 < 0$, $A_2 > 0$ and $E_*$ is asymptotically stable node or focus. Additionally, we could solve a potential interior equilibrium $E_*$ with two control variables $\lambda \in (0, 1)$ and $\mu \geq 1$ from such conditions, where
\[
x_* = -a(K_1 d + r_1)(K_2 e \mu + a) \lambda + q_1 b y_* K_2, \quad y_* = \mu K_2, \tag{37}
\]
and parameters $m_1$ and $m_2$ are constrained by
where coefficients are
\[a_3 = -K_2^2 b_1 q_1 r_2, \]
\[a_2 = \theta (a c d \lambda - b q_1) q_2 - r_2 b_2 q_1 K_1 + a \lambda c q_2 r_1 K_2, \]
\[a_1 = \frac{a (K_1 d + r_1) (c e + a) \lambda - b e K_1 q_1}{K_2 b_2 q_2}, \]
\[a_0 = -a^2 q_2 a e (K_1 d + r_1). \]

Here we set some values of parameters as \( r_1 = 1.21, \)
\( r_2 = 1.5, \)
\( K_1 = 2, \)
\( K_2 = 5, \)
\( a = 1, \)
\( b = 1.5, \)
\( c = 1.7, \)
\( d = 0.3, \)
\( e = 0.3, \)
\( q_1 = 1.2, \)
\( q_2 = 0.21, \)
and the unique globally asymptotically stable node \( E_0 = (0.555936, 5.04) \) with \( \lambda = 0.95 \) and \( \mu = 1.008 \) is depicted in Figure 1 with characteristic direction \( \theta = \theta_2 = 0.101084 \) which is solved from the characteristic function
\[\Theta (\theta) = 0.076926 \cos (\theta)^2 - 0.898114 \sin (\theta) \cos (\theta) + 0.014025, \]
\[= C_2 (\theta - \theta_2) + O((\theta - \theta_2)^2), \]
after we make the polar-coordinate-transformation \( x = r \cos (\theta) + x_0, \)
\( y = r \sin (\theta) + y_0. \) The residual real simple root of equation \( \Theta (\theta) = 0 \) in interval \([0, 2\pi] \) is \( \theta_1 = 1.555157. \) The power series of above characteristic function \( \Theta (\theta) \) up to order two at the point \( \theta_2 \) admits indexes \( C_2 (\theta - \theta_2) < 0 \) and \( R (\theta_2) < 0 \) with same negative sign, i.e., \( p = 1 \text{(odd number)} \) and \( k = 2, \)
where function
\[R (\theta) = 0.898114 \cos (\theta)^2 + 0.076926 \sin (\theta) \cos (\theta) - 1.438292. \]

Thus, \( \theta = \theta_2 \) actually shows a fact that trajectories enter the stable node along this direction. \( \square \)

4.3. Nonexistence of Limit Cycles. In this section, we consider nonexistence of closed orbits and limit cycles of system (2a) and system (2b). Firstly, taking a diffeomorphism \( \varphi: u = x, v = y, \)
\( \tau = f((a + bx + cy)(y + e)) \) which preserves the orientation of time, system (2a) and system (2b) are topologically equivalent to following system:
\[\dot{x} = P(x, y) = [r_1 x \left(1 - \frac{x}{K_1}\right) - m_1 x - \frac{dx^2}{x}] \]
\[\cdot (a + bx + cy)(y + e) - q_1 x y (y + e), \]
\[\dot{y} = Q(x, y) = r_2 y^2 \left(1 - \frac{y}{K_2}\right) (a + bx + cy) + q_2 x y (y + e) - m_2 y (a + bx + cy)(y + e), \]
\[\frac{\partial (BP)}{\partial x} + \frac{\partial (BQ)}{\partial y} = \frac{1}{K_1 K_2 x y} \sum_{1 \leq i, j \leq 3} a_{ij} x^i y^j, \]
where coefficients
\[a_{02} = -2K_1 \left(c (m_2 - r_2) K_2 + a r_2\right), \]
\[a_{11} = -K_2 \left[(-r_1 - r_2 + m_1 + m_2) b + c d e + a d - q_2\right] \cdot K_1 + r_1 (c e + a)], \]
\[a_{10} = -e [(a d + bm_1 - br_1) K_1 + a r_1] K_2, \]
\[a_{01} = -K_2 [(ce + a) m_2 - ar_2] K_1, \]
and other unlisted coefficients are all nonpositive. Thus, we complete the proof.

Here we take over values of parameters from Section 4.2 but \( m_1 = m_2 = 2, \) and system (2a) and system (2b) do not have meaningful equilibria except a globally asymptotically stable node \( E_0 \) (the origin \( O \)) with the characteristic direction \( \theta = 0 \) (the positive \( \sigma \)-axis) since the characteristic equation \( \Theta (\theta) = (m_1 - m_2 - r_2) \cos (\theta) \), and conditions in Theorem 4 all hold. Hence, there are no closed orbits and limit cycles in this numerical case.

In addition, system (2a) and system (2b) merely own a saddle point \( E_0 \) and a globally asymptotically stable node \( E_1 \).
when we use parameters from Section 4.2 but set $a = 2$, $d = 3$, $m_1 = 0.4$, and $m_2 = 1.4$. Conditions in Theorem 4 are also satisfied.

5. Local Bifurcations of System (2a) and System (2b)

In this section, we will give sufficient conditions to show the existence of saddle-node bifurcation, transcritical bifurcation, and Hopf bifurcation of system (2a) and system (2b). Firstly, we denote this system in the following form:

$$
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = F(x, y),
$$

for simplicity and convenience.

5.1. Saddle-Node Bifurcation. The two trivial equilibria $E_2^{(k)}$ $(k = 1, 2)$ collide with each other and system (47) has a unique boundary equilibrium $E_1$ when $y > 0$, if $r_2 > m_2$ and $y_1 = y_2$, or

$$m_2 = m_2^{[SN]} = r_2 + \frac{2r_2 e}{K_2} (1 - \sqrt{1 + \frac{K_2}{e}}).$$

Then, there is a chance of bifurcation around this higher-order singular point. Here we choose $m_2$ as a bifurcation parameter and select eigenvector $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ corresponding to the zero eigenvalue for matrix (30). The eigenvector corresponding to the zero eigenvalue for the transpose of matrix (30) is $w = \begin{pmatrix} 1 \\ w_2 \end{pmatrix}$, where

$$w_2 = -\frac{r_1 - m_1 - ((q_1 y_2)/(a + cy_2))}{(d_2 y_2)/(a + cy_2)}.$$ 

Suppose $r_1 - m_1 - ((q_1 y_2)/(a + cy_2)) < 0$, then the following transversality conditions hold:

$$w^T F_{m_2}(E_2, m_2^{[SN]}) = -w_2 y_2 \neq 0,$$

$$w^T [D^2 F(E_2, m_2^{[SN]})(v, v)] = \frac{2r_2 w_2}{K_2} \left( \frac{e}{K_2 + e} - 1 \right) \neq 0.$$

Thus, we have the following theorem by using Sotomayor’s theorem [22, 23].

**Theorem 5** (saddle-node bifurcation). Suppose that the point $E_1$ exists; if $r_1 - m_1 - ((q_1 y_2)/(a + cy_2)) < 0$, then system (47) undergoes a saddle-node bifurcation around point $E_2$ with respect to the bifurcation parameter $m_2$.

5.2. Transcritical and Pitchfork Bifurcation. The equilibrium $E_2^{(1)}$ changes its stability when $r_1$ crosses the threshold $r_1^{[TC]} = m_1 + ((q_1 y_1)/(a + cy_1))$; in other words, $E_2^{(1)}$ is a higher-order equilibrium when $r_1 = r_1^{[TC]}$. Let $J_{22}(E_2^{(1)}) < 0$. Thus, we choose the parameter $r_1$ as a bifurcation parameter and an eigenvector $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ corresponding to the zero eigenvalue for the Jacobian matrix $J(E_2^{(1)})$ when $r_1 = r_1^{[TC]}$, where $v_2 = -((q_2 y_2)/(J_{22}(E_2^{(1)})(a + cy_1))) > 0$. The eigenvector corresponding to the zero eigenvalue for the transpose of matrix $J(E_2^{(1)})$ is $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; then, the transversality conditions are

$$w^T F_{r_1}(E_2^{(1)}, r_1^{[TC]}) = 0,$$

$$w^T D F_{r_1}(E_2^{(1)}, r_1^{[TC]})(v, v) = 1,$$

$$w^T D^2 F(E_2^{(1)}, r_1^{[TC]})(v, v) = \frac{-2r_1^{[TC]} K_1}{K_1} + \frac{2q_1 by_1}{(a + cy_1)^2} - 2 d$$

$$+ \frac{2q_1 q_2 a y_1}{(a + cy_1)^3} J_{22}(E_2^{(1)}).$$

Thus, we have the following theorem by using Sotomayor’s theorem [22, 23].

**Theorem 6** (transcritical bifurcation). Suppose that the two axial equilibria $E_2^{(k)}$ $(k = 1, 2)$ coexist and $J_{22}(E_2^{(1)}) < 0$; if $a + cy_1 \geq bK_1$ or $J_{22}(E_2^{(1)}) \geq -((q_2 a)/(b(a + cy_1)))$, then system (47) undergoes a transcritical bifurcation around the point $E_2^{(1)}$ with respect to the bifurcation parameter $r_1$.

For the special case

$$w^T D^2 F(E_2^{(1)}, r_1^{[TC]})(v, v) = 0,$$

there is a chance the system (47) undergoes a pitchfork bifurcation. We still use the bifurcation parameter $r_1$ and eigenvectors $v$ and $w$; then, the fourth transversality condition is

$$\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = F(x, y),
$$

for simplicity and convenience.
\[ w^T D^3 F(E_2^{(1)}, r_1^{[PF]}) (v, v, v) = \frac{6 q_1 \left[ b c f_{22} y_1 + a (f_{22} b + q_2) \right] \left[ c (f_{22} b - q_2) y_1 + a b f_{22} \right]}{(c y_1 + a)^3} \tag{53} \]

where \( f_{22} = f_{22}(E_2^{(1)}) \), \( r_1^{[PF]} = m_1 + ((q_1 y_1)/(a + c y_1)) \).

Thus, we have another theorem by using Sotomayor’s theorem [22, 23].

**Theorem 7** (pitchfork bifurcation). Suppose that the two axial equilibria \( E_2^{(k)} (k = 1, 2) \) coexist and \( f_{22}(E_2^{(1)}) < 0 \); if condition (52) and \( f_{22}(E_2^{(1)}) \neq -((aq_2)/(b(c y_1 + a))) \) hold, then system (47) undergoes a pitchfork bifurcation around the point \( E_2^{(1)} \) with respect to the bifurcation parameter \( r_1 \).

5.3. *Hopf Bifurcation*. In this section, we consider the Hopf bifurcation of system (2a) and system (2b). Here the point \( E_* \) exists and we choose \( d \) as bifurcation parameter. Suppose that \( \lambda(d) = a(d) \pm i \omega(d) \) are a pair of conjugate eigenvalues of matrix \( f(E_*) \), where \( a(d) = (1/2) A_1(d) \). The critical value \( d^{[H]} \) satisfies

\[
\begin{align*}
& a(d^{[H]}) = 0, \\
& a'(d^{[H]}) \neq 0, \\
& A_2(d^{[H]}) > 0.
\end{align*}
\tag{54}
\]

Then, system (47) undergoes a Hopf bifurcation around the point \( E_* \), with respect to the bifurcation parameter \( d \).

We will calculate the first Lyapunov number \( \sigma \) at the point \( E_* \), which is used to determine the stability of limit cycles and Hopf bifurcation direction. The method and calculations of the first and second Lyapunov coefficients can be found in [22, 24, 25]. Therefore, translating the point \( E_* \) to the origin \( O = (0, 0) \) by a linear transformation (I): \( X = x - x_*, Y = y - y_* \), system (47) in power series around the origin is

\[
\begin{align*}
\dot{X} &= f_1(X, Y) = \sum_{1 \leq i+j \leq 3} a_{ij} X^i Y^j + O(|X, Y|^4), \\
\dot{Y} &= g_1(X, Y) = \sum_{1 \leq i+j \leq 3} b_{ij} X^i Y^j + O(|X, Y|^4),
\end{align*}
\tag{55}
\]

where coefficients are

\[
\begin{align*}
a_{10} &= \frac{r_1 s_1 - d s_1 + q_1 b s_1 s_2}{K_1 (b s_1 + c s_2 + a)^2}, \\
a_{01} &= \frac{-q_1 s_1 (b s_1 + a)}{(b s_1 + c s_2 + a)^2}, \\
a_{20} &= \frac{r_1}{K_1} + \frac{q_1 s_2 b (cs_2 + a)}{(b s_1 + c s_2 + a)^3} - d, \\
a_{02} &= \frac{q_1 s_2 c (b s_1 + a)}{(b s_1 + c s_2 + a)^3}, \\
a_{11} &= \frac{a^2 + (b s_1 + c s_2) a + 2 s_1 s_2 b c q_1}{(b s_1 + c s_2 + a)^4}, \\
a_{30} &= \frac{q_1 s_2 b^2 (c s_2 + a)}{(b s_1 + c s_2 + a)^6}, \\
a_{03} &= \frac{q_1 s_2 c^2 (b s_1 + a)}{(b s_1 + c s_2 + a)^3}, \\
a_{21} &= \frac{b (s_1 (2 c s_2 + a) b - c^2 s_2^2 + a^2) q_1}{(b s_1 + c s_2 + a)^4}, \\
a_{12} &= \frac{c (s_2 (2 b s_1 + a) c - b^2 s_1^2 + a^2) q_1}{(b s_1 + c s_2 + a)^4}, \\
b_{10} &= \frac{q_2 s_2 (c s_2 + a)}{(b s_1 + c s_2 + a)^2}, \\
b_{01} &= \frac{r_2 s_2 (-K_2 c e + 2 e s_2 + s_2^2)}{K_2 (s_2 + e)^2} - \frac{q_2 c s_2}{(b s_1 + c s_2 + a)^3}, \\
b_{20} &= \frac{-q_2 s_2 b (c s_2 + a)}{(b s_1 + c s_2 + a)^3}, \\
b_{02} &= \frac{r_2 ((K_2 - 3 s_2) e^2 - 3 s_2^2 c - s_2)}{K_2 (s_2 + e)^3} - \frac{q_2 s_2 c (b s_1 + a)}{(b s_1 + c s_2 + a)^3}, \\
b_{11} &= \frac{(a^2 + (b s_1 + c s_2) a + 2 s_1 s_2 b c) q_2}{(b s_1 + c s_2 + a)^3},
\end{align*}
\]
where

\( b_{30} = \frac{q_2 s_2 b^2 (cs_2 + a)}{(bs_1 + cs_2 + a)^4}, \)

\( b_{03} = -\frac{r_2 c^2 (K_2 + c) + q_2 s_1 c^2 (bs_1 + a)}{(bs_1 + cs_2 + a)^4}, \)

\( b_{21} = -\frac{b (s_1 (2cs_2 + a) - c^2 s^2 + a^2) q_2}{(bs_1 + cs_2 + a)^4}, \)

and \( O(|X, Y|^4) \) stands for some smooth functions. After that, by using a transformation (II): \( u = Y, v = (-Xb_{10} + Ya_{10})/\beta \) with \( \beta = \sqrt{A_1(E_c)} > 0 \), the above system becomes a standard form:

\[
\dot{u} = f_2(u, v) = -\beta v + \sum_{2 \leq i + j \leq 3} A_{ij}u^i v^j + O(|u, v|^4),
\]

\[
\dot{v} = g_2(u, v) = \beta u + \sum_{2 \leq i + j \leq 3} B_{ij}u^i v^j + O(|u, v|^4),
\]

where

\[
A_{20} = \frac{(a_{10}b_{20} + a_{10}b_{11}b_{11} + b_{02}b_{10}^2)}{b_{10}^2},
\]

\[
A_{11} = -\frac{(2a_{10}b_{20} + b_{10}b_{11})\beta}{b_{10}^2},
\]

\[
A_{02} = -\frac{b_{20}(a_{01}b_{10} + a_{10}^2)}{b_{10}^2},
\]

\[
A_{30} = a_{10}b_{30} + a_{10}^2 b_{10}b_{21} + a_{10}b_{12}b_{12} + b_{03}b_{10}^4,
\]

\[
A_{21} = -\frac{(3a_{10}b_{30} + 2a_{10}b_{10}b_{21} + b_{10}b_{12})\beta}{b_{10}^2},
\]

\[
A_{12} = -\frac{(3a_{10}b_{30} + b_{10}b_{21})(a_{01}b_{10} + a_{10}^3)}{b_{10}^2},
\]

\[
A_{03} = -\frac{b_{03}\beta^3}{b_{10}^2},
\]

\[
B_{20} = -a_{02}b_{10}^2 - a_{10}b_{20} + a_{10}^2 a_{20}b_{10} - a_{10}^2 b_{10}b_{11} + a_{10}a_{11}b_{10}^2 - a_{10}b_{02}b_{10}^2,
\]

\[
B_{11} = \frac{2a_{10}^2 b_{20} - 2a_{10}a_{20}b_{10} + a_{12}b_{10}b_{11} - a_{11}b_{10}^2}{b_{10}^2},
\]

\[
B_{02} = \frac{(a_{10}b_{20} - a_{20}b_{10})\beta}{b_{10}^2},
\]

\[
B_{30} = -a_{03}b_{10}^4 - a_{10}^4 b_{30} + a_{10}^3 a_{30}b_{10} - a_{10}^3 b_{10}b_{11} + a_{10}a_{21}b_{10}^2 - a_{10}^2 b_{12}b_{12} + a_{10}a_{11}b_{10}^2 - a_{10}b_{03}b_{10}^2,
\]

\[
B_{21} = -\frac{3a_{10}^3 b_{30} - 3a_{10}a_{30}b_{10} + 2a_{10}^2 b_{10}b_{21} - 2a_{10}a_{21}b_{10}^2 + a_{10}b_{10}b_{12}^2 - a_{12}b_{10}^4}{b_{10}^2},
\]

\[
B_{12} = \frac{\beta(3a_{10}^3 b_{30} - 3a_{10}a_{30}b_{10} + a_{10}b_{10}b_{21} - a_{21}b_{10}^2)}{b_{10}^2},
\]

\[
B_{03} = \frac{(a_{10}b_{30} - a_{30}b_{10})(a_{01}b_{10} + a_{10}^3)}{b_{10}^3}
\]
Let \( p = q = (1/\sqrt{2}) \begin{pmatrix} i \\ 1 \end{pmatrix} \) be two corresponding eigenvectors of a matrix \( A \) such that \( Aq = i\beta q, \quad A^\dagger p = -i \beta p \) and \( < p, q > = 1 \); the operation \( < x, y > = x^T y \ (x, y \in C) \) with the Hermitian transpose (upper symbol) \( \dagger \) represents the usual inner product, and the Jacobian matrix is
\[
A = \begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix}.
\] (59)

We should rewrite the functions in the right hand side of system (57) in the form of power series
\[
F_2(x) = \begin{pmatrix} f_2(x) \\ g_2(x) \end{pmatrix} = Ax + \frac{1}{2!} B(x, x) + \frac{1}{3!} C(x, x, x) + O(\|x\|^4),
\] (60)

where the components of linear functions \( B \) and \( C \) are
\[
B_i(x, y, z) = \sum_{j,k=1}^2 \frac{\partial^2 (F_2)_k(0)}{\partial y_i \partial z_j} x_j y_k,
\]
\[
C_i(x, y, z) = \sum_{j,k,l=1}^2 \frac{\partial^3 (F_2)_j(0)}{\partial x_i \partial y_k \partial z_l} x_j y_k z_l, \quad i = 1, 2,
\] (61)

while \( \|x\| \) is the two-dimensional Euclidean norm of \( x \). Define \( T^\ast \) be the largest subspace invariant by the matrix \( A \) and the generalized eigen subspace corresponding to the pair of purely imaginary eigenvalues \( \pm i\beta \), i.e., \( T^\ast = \text{span} [q, q] \). That is to say, for any element \( z \in T^\ast \), there must exist a linear expansion \( z = wq + \bar{w} \bar{q} \). Now we can construct a two-dimensional parameterized center manifold
\[
H = H(w, \bar{w}) = wq + \bar{w}q + \sum_{j+k \geq 2} \frac{1}{j! k!} h_{j,k} w^j \bar{w}^k,
\] (62)

in which \( h_{j,k} \in C^2 \) and its complex conjugate is \( \bar{h} {j,k} = h_{j,k} \). Combining equations (57) and \( H(w, \bar{w}) \), we arrive at a complex equation without consideration:
\[
\frac{\partial H}{\partial w} \dot{w} + \frac{\partial H}{\partial \bar{w}} \dot{\bar{w}} = F_2(H).
\] (63)

The "chart" \( w \) for the central manifold \( H \) should be extracted from following differential equation:
\[
\dot{w} = i\beta w + \frac{1}{2} G_2 w |w|^2 + \cdots.
\] (64)

So, substituting it into equation (63) and comparing coefficients of these \( w^j \bar{w}^k \), we recursively derive
\[
h_{11} = -A^{-1} B(q, \bar{q}), \quad h_{20} = (2i\beta I - A)^{-1} B(q, \bar{q}),
\] (65)

and an equality from coefficient of the cubic term \( w|w|^2 \):
\[
(i\beta I - A)h_{21} + G_{21} q = B(q, \bar{q}, h_{20}) + 2B(q, h_{11}) + C(q, q, \bar{q}).
\] (66)

Here letter \( I \) represents the unit matrix with rank 2. It is quite apparent that equality (66) admits a solution:
\[
G_{21} = < p, B(q, h_{20}) + 2B(q, h_{11}) + C(q, q, \bar{q}) >.
\] (67)

At last, the first Lyapunov coefficient is calculated as
\[
l_1 = \frac{1}{2} \text{Re}(G_{21})
\]
\[
= \frac{1}{4\beta} \left[ (B_{21} + A_{12} + 3A_{30} + 3B_{03})\beta + (A_{11} + 2B_{02})A_{02} + A_{11}A_{20} - 2A_{20}B_{20} - B_{11}(B_{02} + B_{20}) \right].
\] (68)

Thus, the first Lyapunov number \( \sigma_1 \) for the focus of planar system (57) is given by the formula \( \sigma = (6\pi/\beta)l_1 \). Since above expression is much too complicated, we need to present some numerical simulations and figures around the point \( E \), with computer simulations.

**Theorem 8** (nondegenerate Hopf bifurcation). Assume that the equilibrium \( E_* \) exists and parameters satisfy condition (54) and \( \sigma \neq 0 \); then, system (47) undergoes a nondegenerate Hopf bifurcation around this equilibrium as parameter \( \delta = d \) passes through the critical value \( d^{[1]} \). The Hopf bifurcation is supercritical (subcritical), the interior equilibrium \( E_* \) is a stable focus (unstable focus) with multiplicity one, and limit cycles are stable (unstable) if \( \sigma < 0 \) (\( \sigma > 0 \)).

**Remark 2.** The first Lyapunov coefficient \( l_1 \) can be defined by \( (1/2\beta)\text{Re}(G_{21}) \) at times. On occasion, there are times when our system (55) exists some values of parameters such that \( \sigma = 0 \) or the system may undergoes a degenerate Hopf bifurcation. Accompanied by a proper transformation, for planar system (55), the first Lyapunov number is given by the following formula [22]:

\[
\sigma_1 = \frac{3\pi}{2a_{00}A_2^2} \left\{ \begin{array}{l}
[a_{10}b_{10}(a_{11}^2 + a_{11}b_{02} + a_{02}b_{11}) + a_{10}a_{01}(b_{11}^2 + a_{20}b_{11} + a_{11}b_{20})] \\
+ b_{10}^2 (a_{11}a_{02} + 2a_{02}b_{02}) - 2a_{10}b_{10}(b_{11}^2 - a_{20}a_{02}) - 2a_{10}a_{01}(a_{20}^2 - b_{20}b_{02}) \\
- a_{11}^2 (2a_{20}b_{20} + b_{11}b_{20}) + (a_{01}b_{10} - 2a_{11}^2)(b_{11}b_{02} - a_{11}a_{20}) - (a_{11}^2 + a_{01}b_{10})[3(b_{10}b_{03} - a_{01}a_{20})] \\
+ 2a_{10}(a_{21} + b_{12}) + (b_{10}a_{12} - a_{01}b_{21}) \end{array} \right\}.
\] (69)
which is an advanced version of \( \sigma \) or \( l_1 \).

Finally, for the Hopf bifurcation, we numerically give following examples to simulate how the parameter \( d \) controls dynamical behavior of system (2a) and system (2b).

**Example 1.** Consider Case 6 in Section 3.1 and Theorem 4 again. Figure 2(a) reveals the existence of this interior equilibrium \( E_8 \) in Case 6 by using the cobweb method. To investigate how the control parameter \( d \) affects dynamical behavior of our system (2a) and system (2b), Figures 2(c) and 3 depict phase diagrams corresponding to values \( d = 0.0595 \) and \( d = 0.053 \), respectively. When \( d = 0.0595 \), this interior equilibrium is an asymptotically stable focus since \( A_1 \approx -0.0011352489, \) \( A_2 \approx 0.07340334276, \) and \( \Delta_* \approx -0.2936121311 \). When \( d = 0.053 \), it gets \( A_1 \approx 0.00017832135, \) \( A_2 \approx 0.07359407961, \) and \( \Delta_* \approx -0.2943762866 \), and there exists a limit cycle (by using the Poincare–Bendixson theorem) around this unstable focus. Here we notice that

\[
\alpha(d_1) - \alpha(d_2) = \alpha'(d^{[HI]}_1)(d_1 - d_2) + o(10^{-6}). \tag{70}
\]

This implies that the Hopf bifurcation occurs in system (2a) and system (2b) when \( d = d^{[HI]}_8 \). The first Lyapunov coefficient \( l_1 \approx -0.122574 \), and thus the Hopf bifurcation is supercritical and a limit cycle generated by the critical point...
is stable. This interior equilibrium is a multiple stable focus with multiplicity one.

Besides, for a perturbed system with sufficiently small parameter vector $\delta = (\delta_1, \delta_2)$ in a neighborhood of the origin $O$ in the parameter plane

$$x = r_1 x \left(1 - \frac{x}{K_1} - \frac{q_1 xy}{a + bx + cy} - m_1 x - (d + \delta_1)x^2, \quad (71a)\right)$$

$$y = r_2 y \left(1 - \frac{y}{K_2} \right) \frac{y}{y + e} + \frac{q_2 xy}{a + bx + cy} -(m_2 + \delta_2)y, \quad (71b)$$

and Hopf bifurcation analysis with two bifurcation parameters $d$ and $m_2$, we let $\delta \neq 0$ and suppose that $E_* = (x_*, y_*)$ is an interior equilibrium of above perturbed system, where $y_* = y_0 + w$, $|w|$ is sufficiently small, and

$$x_* = -\frac{(cy_* + a)[((\delta_2 + m_2 - r_2)y_* + e(m_2 + \delta_2)][K_2 + r_1y_*^2]}{[((\delta_2 + m_2 - r_2)b - q_1)y_* + e[(m_2 + \delta_2)b - q_1]]K_2 + br_1y_*^2} \quad (72)$$

Substituting it into $A_1$ and $A_2$, the solutions $\delta_1 = \delta_1(w)$ and $\delta_2 = \delta_2(w)$ can be directly solved, which are written as the form up to third order:

$$\delta_1 = -1.503836w + 24.893383w^2 + O(w^3),$$

$$\delta_2 = 3.309615w - 56.757836w^2 + O(w^3), \quad (73)$$

and hence the slope $k = \lim_{w \to 0} ((\delta_2(w))/(\delta_1(w)))$ of approximation straight line of the Hopf bifurcation curve $H_p$ (see Figure 2(b)) in a small neighborhood of the origin in parameter plane is approximately about $-2.200781$. The supercritical Hopf bifurcation curve $H_p$ of system (71a) and system (71b) at the point $E_*$ is numerically defined by solution (73), i.e., $H_p = \{\delta | \delta satisifies (73)\}$, while the variables $\delta$ and $w$ both ensure the existence of $E_*$ and

$$A_2(w) = 0.073568 + 2.087448w - 47.606989w^2 + O(w^3) > 0. \quad (74)$$

As a matter of fact, we perceive the phenomenon that Theorem 8 describes the special case once $\delta$ lies on the horizontal line $\delta_2 = 0$.

**Example 2.** Here we set parameters as well as Case 3 in Section 3.2. Figures 4 and 5 depict phase diagrams
corresponding to values $d_1 = 2.109$ and $d_2 = 2.09$, respectively. One should notice that

$$a(d_1) - a(d_2) \approx a'(d_{12}) (d_1 - d_2) + o(10^{-6}). \tag{75}$$

Similar to above example, system (2a) and system (2b) undergo a Hopf bifurcation when $d$ passes through $d_{12}^{[H]}$. The first Lyapunov coefficient $l_1 = -0.002729$ is also found to be negative. Thus, the Hopf bifurcation is supercritical and a limit cycle generated by the critical point is stable. The interior equilibrium is also a multiple stable focus with multiplicity one. Figure 6 is the Hopf bifurcation curve corresponding to system (69) with slope $k = -0.094044$ and

$$Hp = \{ \delta | \delta_1 = -40.550983w + 639.229323w^2 + O(w^3), \delta_2 \approx 3.813575w - 4.617377w^2 + O(w^3) \}. \tag{76}$$

6. Conclusions

In summary, we firstly considered stability analysis and bifurcations of system (2a) and system (2b) with B-D functional response and Allee-like effect, which is a modified version of a predator-prey system in [1]. The polynomial's method, derived from eliminants $R_1(f, g)$ and $R_2(f, g)$, can be extended to more complicated polynomial systems. Some conclusions are the same as reference [1], such as the uniform boundedness (Theorem 1), the existence of equilibria $E_*$, the nonexistence (Theorem 4) of limit cycles, and so on. It is supposed that some methods and conclusions can be available in original system (1a) and system (1b), such as pitchfork bifurcation. Lemmas 1 and 2 are available in more complicated predator-prey systems. Some critical cases, such as $r_1 - m_1 = ((q_1y_1)/(a + cy_1))$ and $A_2 = 0$, need to be researched further with the help of topologically equivalent systems or the "blow-up" method (horizontal and vertical blow-ups). Other parameters can also be considered as a bifurcation parameter $\delta$ in Hopf bifurcation, although it is described by Hopf bifurcation curve $Hp$, for instance $\delta = x_*$. Notice that Theorem 2 (permanence) can be extended to its reaction-diffusive version [26, 27]. Under the conditions or the discussion of Theorem 3, the Turing instability of its corresponding reaction-diffusion system subject to the homogeneous Neumann boundary conditions [28]:

$$u_t - D_1 u_{xx} = r_1u\left(1 - \frac{u}{K_1}\right) - \frac{q_1uv}{a + bu + cv} - m_1u - du^2, \tag{77a}$$

$$v_t - D_2 v_{xx} = r_2v\left(1 - \frac{v}{K_2}\right) + \frac{q_2uv}{a + bu + cv} - m_2v, \tag{77b}$$

$$\partial_x u = \partial_x v = 0, \quad t \geq 0, x \in \partial(\Omega), \tag{77c}$$
will not occur, and thus it is available to find potential Hopf bifurcation points and consider transversality conditions if we choose $u_*$ as Hopf bifurcation parameter. Here $D_1$ and $D_2$ are two positive diffusive constants; $\Delta$ is the Laplacian operator; $u = u(t, x)$ and $v = v(t, x)$ are the densities of prey and predator, respectively; $\Omega$ is an one-dimensional bounded domain with smooth boundary $\partial (\Omega)$; the symbol $\partial_+$ is the outer flux, and no flux boundary condition is imposed; thus, the system is closed \([29]\); the admissible initial functions $u_0(x)$ and $v_0(x)$ are all continuous functions on $\overline{\Omega}$ to describe an environment surrounded by dispersal barriers, we take zero flux at $\partial (\Omega)$ \([28]\). The method of calculating the first Lyapunov coefficient $l_1$ in Section 5.3 is a reference for deducing of Hopf bifurcation direction in reaction-diffusion systems \([29, 30]\).

Though the dynamical behavior of predator-prey systems in single species or multispecies has been researched by many previous literatures, we still need further study in biomathematics, especially the phytoplankton and zooplankton systems. Meanwhile, how to keep ecosystems in balance or coexistence states and avoid harmful effect is our next direction in this area.

### Appendix

#### A. Polynomials $p(x)$ and $q(y)$

The polynomial $p(x)$ in Section 3.2 is $p(x) = \sum_{k=0}^{5} a_k x^k$, where coefficients are
\[ + d^2 (m_2 - r_2) a^3 + 4 ((m_2 - r_2) b - q_2) m_1 d a + ((m_2 - r_2) b - 2 q_2) m_1 b q_1 \]
\[ + 3 \left( - de c + a d - \frac{1}{3} r_1 b + \frac{1}{3} b m_1 \right) c^2 q_2 (r_1 - m_1)^2 K_2 + a r_1 q_1 \left( 3 r_1 b^2 + (-6 a b d - 6 b^2 r_1) r_1 \right) \]
\[ + a^2 d^2 + 6 a b d m_1 + 3 b^2 m_1^2 \right) K_2^3 - 2 \left( \left( - \left( m_2 b - \frac{3}{2} q_2 \right) \right) c + a \left( (m_2 - r_2) b \right) \]
\[ - \frac{1}{2} q_1 c + c \left( - c ((-b m_2 + 3 q_2) r_1 + a m_2 d + m_1 (b m_2 - 3 q_2)) c + a \left( (2 m_2 + 2 r_2) b \right) \]
\[ + 2 q_2 r_1 + d (m_2 - r_2) a + 2 ((m_2 - r_2) b - q_2) m_1) q_1 \left( - \frac{3}{2} q_2 c^2 (r_1 - m_1)^2 (-c e + a) K_2 \right) \]
\[ - a^2 r_1 (a d + 3 b m_1 - 3 b r_1) r_1 K_2^2 + a \left( (m_2 - r_2) K_2 + a^2 r_2 q_1 r_1 K_1 \right), \]
\[ a_1 = \left( e (b m_2 - q_2) q_1^3 + (2 e \left( (m_2 b + \frac{3}{2} q_2) r_1 + a m_2 d + m_1 \left( m_2 b - \frac{3}{2} q_2 \right) \right) c \right) \]
\[ - a \left( ((-2 m_2 + 2 r_2) b + q_2) r_1 + d (m_2 - r_2) a + 2 \left( (m_2 - r_2) b - \frac{1}{2} q_2 \right) m_1 \right) q_1^2 \]
\[ + 2 \left( e \left( - \frac{1}{2} m_2 b + \frac{3}{2} q_2 \right) r_1 + a m_2 d + \frac{1}{2} m_1 (b m_2 - 3 q_2) \right) e c \]
\[ + a \left( (m_2 + r_2) \right) \left( ((-m_2 + r_2) b + q_2) r_1 + d (m_2 - r_2) a + ((m_2 - r_2) b - q_2) m_1) c (r_1 - m_1) \right) q_1 \]
\[ - q_2 c^2 (r_1 - m_1)^3 (-c e + a) K_2 - 2 a^2 r_1 (r_1 - m_1) \left( a d - \frac{3}{2} r_1 b + \frac{3}{2} b m_1 \right) q_1 K_2^3 \]
\[ - 2 a \left( (m_2 - m c + \frac{1}{2} a (m_2 - r_2)) \right) q_2 + (- (m_2 - m c + a) (m_2 - r_2) (r_1 - m_1) K_2 + a^2 r_1 (r_1 - m_1) K_2 + a^2 r_1 (r_1 - m_1) K_2 + a^2 r_2 (r_1 - m_1)^2) q_1 K_3^2. \]

The polynomial \( q(y) \) in Section 3.2 is \( q(y) = \sum_{k=0}^{5} b_k y^k \), where coefficients are

\[ b_5 = q_2 c^2 \left( K_1 d + r_1 \right) r_2 K_2 + K_1 b^2 q_1 r_2^2, \]
\[ b_4 = q_2 c^2 \left( K_1 d + r_1 \right) (m_2 - r_2) K_2^2 + \left( \left( c^2 d e + 2 a c d - 2 \left( c \left( - \frac{1}{2} r_1 + \frac{1}{2} m_1 \right) c + q_1 \right) b \right) K_1 \right) \]
\[ + 2 \left( \frac{1}{2} c e + a \right) c r_1 \right) q_2 + 2 q_2 b^2 K_1 (m_2 - r_2) r_2 K_2, \]
\[ b_3 = \left( ((-r_1 + m_1) c + q_1) K_1 q_2^2 + \left( 2 \left( - \frac{1}{2} m_2 + \frac{1}{2} r_2 \right) c^2 d e + 2 (m_2 - r_2) \right) \left( a c d - \left( - \frac{1}{2} r_1 \right) \right) \]
\[ + \frac{1}{2} m_1 c + q_1 b \right) b \right) K_1 + 2 \left( c \left( - \frac{1}{2} r_1 - \frac{1}{2} m_1 \right) c + q_1 \right) e \]
\[ + q_2 b^2 K_1 (m_2 - r_2)^2 K_2^2 + \left( \left( 2 a c d - 2 \left( - \frac{1}{2} r_1 - \frac{1}{2} m_1 \right) c + q_1 \right) b \right) e \]
\[ + (a d + b (r_1 - m_1)) a \right) K_1 + a r_1 (2 c e + a) q_2 + 2 m_2 q_2 b^2 e K_1 r_2 K_2, \]
\[ b_2 = \left( - (2 r_1 - 2 m_1) c - 2 q_1) e + a (r_1 - m_1) K_1 q_2^2 + \left( m_2 c^2 d e^2 + 4 \left( - \frac{1}{2} r_1 \right) a c d - \left( - \frac{1}{2} r_1 \right) \right) \]
\[ b_1 = 2e \left( - \left( \left( \frac{1}{2} \right)^2 - \frac{1}{2} m_1 \right) e + a(r_1 - m_1) \right) + \frac{1}{2} m_1 c + q_1 b c + (ad + b(r_1 - m_1)) a(m_2 - \frac{1}{2} r_2) \]
\[ + a^2 (m_2 - r_2) K_1 + 2m_2q_1 b^2 e K_1 (m_2 - r_2) K^2 + aq_2 e((a d + b(r_1 - m_1)) K_1 + ar_1) r_2 K_2, \]
\[ b_0 = ( - K_1 (r_1 - m_1) q_2 + m_2 (a d + b(r_1 - m_1)) K_1 + ar_1) aq_2 e^2 K_2. \]

(A.2)

Data Availability
The data used to support the findings of this study are included within the article.

Conflicts of Interest
The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References
[1] S. T. Wang, H. G. Yu, and C. J. Dai, “The dynamical behavior of a predator-prey system with holling type II functional response and allee effect,” Applied Mathematics, vol. 11, no. 5, pp. 407–425, 2020.
[2] G. Wang, X. G. Liang, and F. Z. Wang, “The competitive dynamics of populations subject to an Allee effect,” Ecological Modelling, vol. 124, no. 2-3, pp. 183–192, 1999.
[3] S. R. Zhou and G. Wang, “Allee-like effects in metapopulation dynamics,” Mathematical Biosciences, vol. 189, no. 1, pp. 103–113, 2004.
[4] P. Aguirre, J. D. Flores, and E. González-Olivares, “Bifurcations and global dynamics in a predator-prey model with a strong Allee effect on the prey, and a ratio-dependent functional response,” Nonlinear Analysis: Real World Applications, vol. 16, pp. 235–249, 2014.
[5] E. González-Olivares and A. Rojas-Palma, “Limit cycles in a Gause-type predator-prey model with sigmoid functional response and weak Allee effect on prey,” Mathematical Methods in Applied Sciences, vol. 35, pp. 963–975, 2012.
[6] J. Zu, “Global qualitative analysis of a predator-prey system with Allee effect on the prey species,” Mathematics and Computers in Simulation, vol. 94, pp. 33–54, 2013.
[7] J. Zu and M. Mimura, “The impact of Allee effect on a predator-prey system with Holling type II functional response,” Applied Mathematics and Computation, vol. 217, no. 7, pp. 3542–3556, 2010.
[8] Y. Cai, C. Zhao, W. Wang, and J. Wang, “Dynamics of a Leslie-Gower predator-prey model with additive Allee effect,” Applied Mathematical Modelling, vol. 39, no. 7, pp. 2092–2106, 2015.
[9] D. L. DeAngelis, R. A. Goldstein, and R. V. O’Neill, “A model for trophic interaction,” Ecology, vol. 56, no. 4, pp. 881–892, 1975.
[10] J. R. Beddington, “Mutual interference between parasites or predators and its effect on searching efficiency,” The Journal of Animal Ecology, vol. 44, no. 1, pp. 331–340, 1975.
[11] M. Fan and Y. Kuang, “Dynamics of a nonautonomous predator-prey system with the Beddington-DeAngelis functional response,” Journal of Mathematical Analysis and Applications, vol. 295, no. 1, pp. 15–39, 2004.
[12] P. J. Pal and P. K. Mandal, “Bifurcation analysis of a modified Leslie-Gower predator-prey model with Beddington-DeAngelis functional response and strong Allee effect,” Mathematics and Computers in Simulation, vol. 97, pp. 123–146, 2014.
[13] H. Mainul, “A detailed study of the Beddington-DeAngelis predator-prey model,” Mathematical Biosciences, vol. 234, pp. 1–16, 2011.
[14] J. P. Tripathi, S. Abbas, and M. Thakur, “Dynamical analysis of a prey-predator model with Beddington-DeAngelis type function response incorporating a prey refuge,” Nonlinear Dynamics, vol. 80, no. 1-2, pp. 177–196, 2015.
[15] J. Lee and H. Baek, “Dynamics of a Beddington-DeAngelis type predator-prey system with constant rate harvesting,” Electronic Journal of Qualitative Theory of Differential Equations, vol. 1, no. 1, pp. 1–20, 2017.
[16] G. Birkhoff and G. C. Rota, Ordinary Differential Equations Introductions to Higher Mathematics, Ginn and Company, Boston, MA, USA, 1962.
[17] F. D. Chen, “On a nonlinear nonautonomous predator-prey model with diffusion and distributed delay,” Journal of Computational and Applied Mathematics, vol. 180, no. 1, pp. 33–49, 2005.
[18] Z. F. Zhang, T. R. Ding, W. Z. Huang, and Z. X. Dong, Qualitative Theory Of Differential Equations (Translations Of Mathematical Monographs), Vol. 101, American Mathematical Society, Providence, RI, USA, 1992.
[19] D. R. Merkin, F. F. Afagh, and A. L. Smirnov, Introduction to the Theory of Stability, Springer, Berlin, Germany, 1997.
[20] C. Chicone, Ordinary Differential Equations with Applications, Springer, Berlin, Germany, 2006.
[21] A. A. Andronov, E. A. Leontovich, I. I. Gordon, and A. G. Maier, *Qualitative Theory of Second-Order Dynamic Systems*, Halsted Press, New York, NY, USA, 1973.

[22] L. Perko, *Differential Equations and Dynamical Systems*, Springer, Berlin, Germany, 2001.

[23] B. Pirayesh, A. Pazirandeh, and M. Akbari, “Local bifurcation analysis in nuclear reactor dynamics by Sotomayor’s theorem,” *Annals of Nuclear Energy*, vol. 94, pp. 716–731, 2016.

[24] Y. A. Kuznetsov, *Elements of Applied Bifurcation Theory*, Springer, Berlin, Germany, 1995.

[25] Y. A. Kuznetsov, “Numerical normalization techniques for all codim 2 bifurcations of equilibria in ODE’s,” *SIAM Journal on Numerical Analysis*, vol. 36, no. 4, pp. 1104–1124, 1999.

[26] Q. X. Ye, Z. Y. Li, M. X. Wang, and Y. P. Wu, *Introduction to Reaction-Diffusion Equations*, Science Press, Beijing, China, 2011.

[27] M. X. Wang and P. Y. H. Pang, “Qualitative analysis of a diffusive variable-territory prey-predator model,” *Discrete & Continuous Dynamical Systems-A*, vol. 23, no. 3, pp. 1061–1072, 2009.

[28] M. Pascual, “Diffusion-induced chaos in a spatial predator-prey system,” *Proceedings of the Royal Society of London B*, vol. 251, pp. 1–7, 1993.

[29] F. Q. Yi, J. J. Wei, and J. P. Shi, “Bifurcation and spatio-temporal patterns in a homogeneous diffusive predator-prey system,” *Journal of Differential Equations*, vol. 246, no. 5, pp. 1944–1977, 2009.

[30] A. Y. Wan, Z. Q. Song, and L. F. Zheng, “Patterned solutions of a homogenous diffusive predator-prey system of holling type-III,” *Acta Mathematicae Applicatae Sinica (English Series)*, vol. 32, no. 1, pp. 1073–1086, 2016.