On Characterization of Poisson Integrals of Schrödinger Operators with Morrey Traces

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Abstract  Let \( L \) be a Schrödinger operator of the form \( L = -\Delta + V \) acting on \( L^2(\mathbb{R}^n) \) where the nonnegative potential \( V \) belongs to the reverse Hölder class \( B_q \) for some \( q \geq n \). In this article we will show that a function \( f \in L^2,\lambda(\mathbb{R}^n), 0 < \lambda < n \), is the trace of the solution of \( Lu = -u_{tt} + Lu = 0, u(x,0) = f(x) \), where \( u \) satisfies a Carleson type condition
\[
\sup_{x_B, r_B} \int_0^r \int_{B(x_B, r_B)} t|\nabla u(x,t)|^2 \, dx \, dt \leq C < \infty.
\]

Its proof heavily relies on investigate the intrinsic relationship between the classical Morrey spaces and the new Campanato spaces \( L^{2,\lambda}(\mathbb{R}^n) \) associated to the operator \( L \), i.e.
\[
L^{2,\lambda}(\mathbb{R}^n) = L^{2,\lambda}(\mathbb{R}^n).
\]

Conversely, this Carleson type condition characterizes all the \( L \)-harmonic functions whose traces belong to the space \( L^{2,\lambda}(\mathbb{R}^n) \) for all \( 0 < \lambda < n \).

Keywords Schrödinger operators, Dirichlet problem, Morrey spaces, Campanato spaces, Poisson semigroup

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1 Introduction

Consider Schrödinger operators
\[
L = -\Delta + V(x) \quad \text{on} \quad \mathbb{R}^n, \quad n \geq 3,
\]
where \( V \in L^1_{\text{loc}}(\mathbb{R}^n) \) is nonnegative and not identically zero. It follows that the operator \( L \) is a self-adjoint positive definite operator. From the Feynman–Kac formula, the kernel \( p_t(x,y) \) of the semigroup \( e^{-tL} \) satisfies the estimate
\[
0 \leq p_t(x,y) \leq \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}}.
\]

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In addition, suppose that $V(x)$ also belong to the reverse Hölder class $B_q$ for some $q \geq n$, which by definition means that $V \in L^q_{\text{loc}}(\mathbb{R}^n), V \geq 0$, and there exists a constant $C > 0$ such that the reverse Hölder inequality
$$\left( \frac{1}{|B|} \int_B V(x)^q \, dx \right)^{1/q} \leq C \frac{1}{|B|} \int_B V(x) \, dx$$
holds for each ball $B$ in $\mathbb{R}^n$. We refer to [7, 14, 19] for the properties of these Schrödinger operators.

For $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, it is well known that the Poisson extension $u(x, t) = e^{-t\sqrt{L}}f(x)$, $t > 0, x \in \mathbb{R}^n$, is a solution to the equation
$$Lu = -u_{tt} + Lu = 0 \quad \text{in } \mathbb{R}^{n+1}_+$$
with the boundary data $f$ on $\mathbb{R}^n$. The equation $Lu = 0$ is understood in the weak sense, that is, $u \in W^{1,2}_{\text{loc}}(\mathbb{R}^{n+1})$ is a weak solution of $Lu = 0$ if it satisfies
$$\int_{\mathbb{R}^{n+1}_+} \nabla u \cdot \nabla \psi \, dY + \int_{\mathbb{R}^n} Vu \psi \, dx = 0, \quad \forall \psi \in C_0^1(\mathbb{R}^{n+1}).$$
At the end-point space $L^\infty(\mathbb{R}^n)$, the study of singular integrals has a natural substitution, the BMO space, i.e. the space of functions of bounded mean oscillation. It was shown in [7] that a BMO$_L(\mathbb{R}^n)$ function is the trace of the solution of $-u_{tt} + Lu = 0, u(x, 0) = f(x)$, whenever $u$ satisfies
$$\sup_{x \in \mathbb{R}^n} \frac{1}{r_B^n} \int_{r_B} \int_{B(x, r)} t |\nabla u(x, t)|^2 \, dx \, dt \leq C < \infty,$$
where $\nabla = (\nabla_x, \partial_t)$. Conversely, this Carleson condition characterizes all the $L$-harmonic functions whose traces belong to the space BMO$_L(\mathbb{R}^n)$ associated to an operator $L$, which extends the analogous characterization founded by Fabes et al. in [9] for the classical BMO space of John–Nirenberg. See also Chapter 2 of the standard textbook [21].

The main goal of this paper is to continue this line of research [7, 9, 21] to study the cases of $f$ in Morrey spaces. Recall that Morrey spaces were introduced in 1938 by Morrey [15] in relation to regularity problems of solutions to partial differential equations. Recall that for every $1 \leq p < \infty$ and $\lambda \geq 0$, the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ is defined as
$$L^{p,\lambda}(\mathbb{R}^n) = \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} \int_{B(x, r)} |f(y)|^p \, dy < \infty \right\}.$$ (1.5)
This is a Banach space with respect to the norm
$$\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} = \left( \sup_{x \in \mathbb{R}^n} \int_{B(x, r)} |f(y)|^p \, dy \right)^{1/p}$$ (1.6)
It is known that $L^{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $L^{p,n}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. When $\lambda > n$, the space $L^{p,\lambda}(\mathbb{R}^n)$ is trivial ($L^{p,\lambda}(\mathbb{R}^n) = \{0\}$). In the case $\lambda \in (0, n]$, the space $L^{p,\lambda}(\mathbb{R}^n)$ is non-separable. We refer to [1, 2, 11, 16–18, 22] for more properties of the Morrey spaces.

The following theorem is our main result of this paper.

**Theorem 1.1** Suppose $V \in B_q$ for some $q \geq n$ and $0 < \lambda < n$. We denote by $H^{2,\lambda}_{L}(\mathbb{R}^{n+1}_+)$ the class of all $C^1$-functions $u(x, t)$ of the solution of $Lu = 0$ in $\mathbb{R}^{n+1}_+$ such that
$$\|u\|_{H^{2,\lambda}_{L}(\mathbb{R}^{n+1}_+)}^2 = \sup_{x \in \mathbb{R}^n} \int_{B(x, r)} t |\nabla u(x, t)|^2 \, dx \, dt < \infty,$$ (1.7)
where $\nabla = (\nabla_x, \partial_t)$. Then we have

1. If $f \in L^{2,\lambda}(\mathbb{R}^n)$, then the function $u = e^{-t\sqrt{\Delta}}f$ is in $H^ {2,\lambda}_{L}(\mathbb{R}^{n+1}_+)$, and $\|u\|_{H^{2,\lambda}_{L}(\mathbb{R}^{n+1}_+)} \leq C\|f\|_{L^{2,\lambda}(\mathbb{R}^{n})}$.

2. If $u \in H^ {2,\lambda}_{L}(\mathbb{R}^{n+1}_+)$, then there exists some $f \in L^{2,\lambda}(\mathbb{R}^{n})$ such that $u(x,t) = e^{-t\sqrt{\Delta}}f(x)$, and $\|f\|_{L^{2,\lambda}(\mathbb{R}^{n})} \leq C\|u\|_{H^{2,\lambda}_{L}(\mathbb{R}^{n+1}_+)}$.

We would like to mention that in the case of $L = -\Delta$, Theorem 1.1 was obtained by Jiang et al. [13]. When $L$ is the Schrödinger operators as in (1.1) above, the proof of (1) of our Theorem 1.1 follows by the standard argument to use the definition of Morrey spaces and the full gradient estimates on the kernel of the Poisson semigroup in the $(x,t)$ variables.

We will prove that for every $1 \leq p < \infty$ and $\lambda > 0$, provided

$$\|f\|_{L^{p,\lambda}(\mathbb{R}^{n})} = \left( \sup_{x \in \mathbb{R}^{n}, r > 0} r^{-\lambda} \int_{B(x,r)} |f(y) - e^{-r^2L}f(y)|^p \, dy \right)^{1/p} < \infty. \quad (1.8)$$

We will prove that for every $1 \leq p < \infty$ and $0 < \lambda < n$, the Campanato spaces $L^{p,\lambda}(\mathbb{R}^{n})$ (modulo the kernel spaces) coincides with the Morrey spaces $L^{p,\lambda}(\mathbb{R}^{n})$, i.e.,

$$L^{p,\lambda}(\mathbb{R}^{n})/K_{L,p} = L^{p,\lambda}(\mathbb{R}^{n}) \quad \text{and} \quad \|f\|_{L^{p,\lambda}(\mathbb{R}^{n})} \approx \left\| f - \lim_{t \to +\infty} e^{-tL}f \right\|_{L^{p,\lambda}}. \quad (1.9)$$

See Theorem 2.2 below. Observe that for the Schrödinger operator $L$ as in (1.1) under the additional assumption $V \in B_{q}$ for some $q \geq n$, we have that for $0 < \lambda < n, 1 \leq p < \infty$ there holds: $K_{L,p} = K_{\sqrt{L},p} = \{0\}$, and hence

$$L^{p,\lambda}(\mathbb{R}^{n}) = L^{p,\lambda}(\mathbb{R}^{n}) = L^{p,\lambda}(\mathbb{R}^{n}). \quad (1.10)$$

From the equivalence (1.10) between the Morrey spaces and the Campanato spaces $L^{p,\lambda}(\mathbb{R}^{n})$, we follow the line of the proof as in [7, 9, 13] to obtain the proof of (2) of Theorem 1.1.

Throughout, the letters $C$ and $c$ will denote (possibly different) constants that are independent of the essential variables.

2 Properties of Campanato Spaces Associated to Operators

In this section, we assume that $L$ is a linear operator on $L^{2}(\mathbb{R}^{n})$ which generates an analytic semigroup $e^{-tL}$ with a kernel $p_t(x,y)$ satisfying

$$|p_t(x,y)| \leq Ct^{-n/m} \left(1 + \frac{|x-y|}{t^{1/m}}\right)^{-(n+\epsilon)}, \quad \forall x, y \in \mathbb{R}^{n}, \quad (2.1)$$

where $C, m$ and $\epsilon$ are positive constants.

We now define the class of functions that the operators $e^{-tL}$ act upon. Fix $1 \leq p < \infty$. For any $\beta > 0$, a complex-valued function $f \in L^{p}_{loc}(\mathbb{R}^{n})$ is said to be a function of type $(p;\beta)$ if $f$...
It follows from (2.1) that the operators
\[ p \] and write
\[ \|f\|_{\mathcal{M}(p;\beta)} = \inf\{C > 0 : (2.2) \text{ holds}\}. \]

It is not hard to see that \( \mathcal{M}(p;\beta) \) is a complex Banach space under \( \|f\|_{\mathcal{M}(p;\beta)} < \infty \). For any given operator \( L \), let
\[ \Theta(L) = \sup\{\epsilon > 0 : (2.1) \text{ holds}\} \]
and write
\[ \mathcal{M}_p = \begin{cases} \mathcal{M}(p;\Theta(L)) & \text{if } \Theta(L) < \infty; \\ \{\mathcal{M}(p;\beta) : 0 < \beta < \infty\} & \text{if } \Theta(L) = \infty. \end{cases} \]

Note that if \( L = -\Delta \) or \( L = \sqrt{-\Delta} \) on \( \mathbb{R}^n \), then \( \Theta(-\Delta) = +\infty \) or \( \Theta(\sqrt{-\Delta}) = 1 \), respectively. For any \((x,t) \in \mathbb{R}^{n+1}\) and \( f \in \mathcal{M}_p \), define
\[ e^{-tL}f(x) = \int_{\mathbb{R}^n} p_t(x,y)f(y)dy. \]

It follows from (2.1) that the operators \( p_tf \) is well defined. Following [4], we can define Campanato spaces \( \mathcal{L}^{p,\lambda}_{L}(\mathbb{R}^n) \) associated to an operator \( L \) as follows.

**Definition 2.1** Let \( 1 \leq p < \infty \) and \( 0 < \lambda < n \). We say that a function \( f \in \mathcal{M}_p \) belongs to the space \( \mathcal{L}^{p,\lambda}_{L}(\mathbb{R}^n) \) associated to an operator \( L \), if
\[ \|f\|_{\mathcal{L}^{p,\lambda}_{L}(\mathbb{R}^n)} = \left( \sup_{B \subset \mathbb{R}^n} \int_B f(x) - e^{-r_B^\lambda L}f(x) dx \right)^{1/p} < \infty, \]
where \( m \) is a fixed positive constant in (2.1).

It can be verified that \( \mathcal{L}^{p,\lambda}_{L}(\mathbb{R}^n) / \mathcal{K}_{L,p} \) is a Banach space, where \( \mathcal{K}_{L,p} \) is the kernel space and defined by
\[ \mathcal{K}_{L,p} = \{ f \in \mathcal{M}_p : e^{-tL}f(x) = f(x) \text{ for almost all } x \in \mathbb{R}^n \text{ and all } t > 0 \}. \]

It is shown in [4] that the space \( \mathcal{L}^{p,\lambda}_{\Delta}(\mathbb{R}^n) \) coincides with \( \mathcal{L}^{p,\lambda}(\mathbb{R}^n) \) for \( 0 < \lambda < n \), and a necessary and sufficient condition for the classical space \( \mathcal{L}^{p,\lambda}(\mathbb{R}^n) \subseteq \mathcal{L}^{p,\lambda}_{L}(\mathbb{R}^n) \) with \( \|f\|_{\mathcal{L}^{p,\lambda}_{L}(\mathbb{R}^n)} \leq C\|f\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} \) is that for every \( t > 0 \), \( e^{-tL}(1) = 1 \) almost everywhere.

Recall that the classical Campanato spaces were introduced in 1963 by Campanato [2], which are a generalization of the BMO spaces of functions of bounded mean oscillation introduced by John and Nirenberg [12]. Recall that for every \( 1 \leq p < \infty \) and \( \lambda > 0 \), the Campanato space \( \mathcal{L}^{p,\lambda}(\mathbb{R}^n) \) is defined as
\[ \mathcal{L}^{p,\lambda}(\mathbb{R}^n) = \{ f \in L^p_{\text{loc}}(\mathbb{R}^n) : \|f\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} < \infty \} \]
with the Campanato seminorm being given by
\[ \|f\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} = \left( \sup_{x \in \mathbb{R}^n, r > 0} r^{-\lambda} \int_{B(x,r)} |f(y) - f_{B(x,r)}|^p dy \right)^{1/p} < \infty. \]
The relationship between Morrey spaces and Campanato spaces is the following important result (see [11, 17]):
(i) \( \mathcal{L}^{p,\lambda}(\mathbb{R}^n)/C = L^{p,\lambda}(\mathbb{R}^n) \), when \( \lambda \in [0,n) \) and \( C \) denotes the space of all constant functions;
(ii) \( \mathcal{L}^{p,n}(\mathbb{R}^n) = \text{BMO}(\mathbb{R}^n) \), when \( \lambda = n \);
(iii) \( \mathcal{L}^{p,\lambda}(\mathbb{R}^n)/C = \text{Lip}_{\alpha}(\mathbb{R}^n) \) with \( \alpha = (\lambda-n)/p \), when \( \lambda \in (n,n+p) \) and \( \text{Lip}_{\alpha}(\mathbb{R}^n) \) denotes the homogeneous Lipschitz space in \( \mathbb{R}^n \).

2.1 The Intrinsic Relationship Between Campanato Spaces Associated to Operators and Classical Morrey Spaces

The goal of this subsection is to prove the following theorem.

**Theorem 2.2** For every \( 1 \leq p < \infty \) and \( 0 < \lambda < n \), Campanato spaces \( \mathcal{L}^{p,\lambda}(\mathbb{R}^n) \) (modulo the kernel spaces) coincide with the classical Morrey spaces \( L^{p,\lambda}(\mathbb{R}^n) \), i.e.,

\[
\mathcal{L}^{p,\lambda}(\mathbb{R}^n)/\mathcal{K}_{L,p} = L^{p,\lambda}(\mathbb{R}^n) \quad \text{and} \quad \|f\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} \approx \left\| f - \lim_{t \to +\infty} e^{-tL}f \right\|_{L^{p,\lambda}(\mathbb{R}^n)},
\]

where \( \mathcal{K}_{L,p} \) is given in (2.7). More precisely, for every \( f \in \mathcal{L}^{p,\lambda}(\mathbb{R}^n) \), the mapping \( f \mapsto f - \lim_{t \to +\infty} e^{-tL}f \) is bijective and bicontinuous from \( \mathcal{L}^{p,\lambda}(\mathbb{R}^n)/\mathcal{K}_{L,p} \) to \( L^{p,\lambda}(\mathbb{R}^n) \).

The proof of Theorem 2.2 is based on the following lemmas.

**Lemma 2.3** Let \( 1 \leq p < \infty \) and \( 0 < \lambda < n \). Suppose \( f \in L^{p,\lambda}(\mathbb{R}^n) \). Then for every \( t > 0 \) and \( x \in \mathbb{R}^n \), \( |e^{-tL}f(x)| \leq C t^{-n/m} \|f\|_{L^{p,\lambda}(\mathbb{R}^n)} \). As a consequence, we have

\[
f \in \mathcal{L}^{p,\lambda}(\mathbb{R}^n) \quad \text{and} \quad \|f\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} \leq C \|f\|_{L^{p,\lambda}(\mathbb{R}^n)}.
\]

**Proof** Let \( f \in L^{p,\lambda}(\mathbb{R}^n) \). From kernel estimate (2.1) of the semigroup \( e^{-tL} \), we have

\[
|e^{-tL}f(x)| \leq C t^{-n/m} \int_{\mathbb{R}^n} \left( 1 + \frac{|x-y|}{t^{1/m}} \right)^{-(n+\epsilon)} |f(y)|dy
\]

\[
\leq C t^{-n/m} \int_{B(x,2t^{1/m})} |f(y)|dy + C t^{-n/m} \sum_{k=2}^{\infty} \int_{B(x,2kt^{1/m}) \setminus B(x,2(k-1)t^{1/m})} \left( 1 + \frac{|x-y|}{t^{1/m}} \right)^{-(n+\epsilon)} |f(y)|dy
\]

\[
\leq C \sum_{k=1}^{\infty} 2^{-k(n+\epsilon)} t^{-n/m} \int_{B(x,2kt^{1/m})} |f(y)| dy.
\]

By Hölder’s inequality, we obtain that for every \( k \in \mathbb{N} \),

\[
t^{-n/m} \int_{B(x,2kt^{1/m})} |f(y)| dy \leq C t^{-n/m} (2k^{1/m})^{(1-\frac{1}{p})n+\frac{n}{p}} \int_{B(x,2kt^{1/m})} |f(y)|^p dy \frac{1}{(2k^{1/m})^\lambda} \left( \int_{B(x,2kt^{1/m})} |f(y)|^p dy \right)^{1/p}
\]

\[
\leq C 2^{(n+\frac{n-\lambda}{p})} t^{\frac{n-\lambda}{p}} \|f\|_{L^{p,\lambda}(\mathbb{R}^n)},
\]

which gives

\[
|e^{-tL}f(x)| \leq C \sum_{k=1}^{\infty} 2^{-k(n+\epsilon)} 2^{(n+\frac{n-\lambda}{p})} t^{\frac{n-\lambda}{p}} \|f\|_{L^{p,\lambda}(\mathbb{R}^n)} \leq C t^{\frac{n-\lambda}{p}} \|f\|_{L^{p,\lambda}(\mathbb{R}^n)}.
\]

(2.10)

It follows from \( f \in L^{p,\lambda}(\mathbb{R}^n) \) that \( \int_{\mathbb{R}^n} |f(x)|^p (1 + |x|)^{-(n+\beta)} dx \leq C \|f\|^p_{L^{p,\lambda}(\mathbb{R}^n)} \) for any \( \beta > 0 \), and so \( f \in \mathcal{M}_p \). Note that for any ball \( B = B(x_B,r_B) \subset \mathbb{R}^n \), we apply estimate (2.10)
Using this notation, we have that
\[ \left( \frac{1}{r_B} \int_B |f - e^{-r_B^p L} f(x)|^p \, dx \right)^{1/p} \leq \|f\|_{L^{p,\lambda}(\mathbb{R}^n)} + \left( \frac{1}{r_B} \int_B |e^{-r_B^p L} f(x)|^p \, dx \right)^{1/p} \]
\[ \leq C \|f\|_{L^{p,\lambda}(\mathbb{R}^n)}. \]

Taking the supremum over all the balls \( B \) in \( \mathbb{R}^n \), we have that \( \|f\|_{L^{p,\lambda}(\mathbb{R}^n)} \leq C \|f\|_{L^{p,\lambda}(\mathbb{R}^n)}. \)

The proof of Lemma 2.3 is complete.

**Lemma 2.4** Let \( 1 \leq p < \infty \) and \( 0 < \lambda < n \). Suppose \( f \in L^{p,\lambda}_L(\mathbb{R}^n) \). Then

(i) For any \( t > 0 \) and \( K > 1 \), there exists \( C > 0 \) independent of \( t \) and \( K \) such that
\[ \|e^{-tL} f(x) - e^{-KL} f(x)\|_{L^\infty(\mathbb{R}^n)} \leq C t^{\frac{\lambda}{n-\lambda}} \|f\|_{L^{p,\lambda}(\mathbb{R}^n)}, \]

(ii) For any \( \delta > 0 \), there exists \( C(\delta) > 0 \) such that
\[ \int_{\mathbb{R}^n} \frac{|e^{-tL} f(x) - f(x)|^p}{(t^{1/m} + |x|)^{n+\delta}} \, dx \leq C(\delta) t^{-(n-\lambda+\delta)/m} \|f\|_{L^{p,\lambda}(\mathbb{R}^n)}^p. \]

**Proof** For the proof of (i), we refer to (i) of [4, Lemma 3]. The proof of (ii) can be obtained by making minor modifications with that of (ii) of [4, Lemma 3], and we skip it.

**Lemma 2.5** Let \( 1 \leq p < \infty \) and \( 0 < \lambda < n \). For every \( f \in L^{p,\lambda}_L(\mathbb{R}^n) \), there exists a function \( \sigma_L(f) \in M_p \) such that \( \lim_{t \to +\infty} \|e^{-tL} f - \sigma_L(f)\|_{L^\infty(\mathbb{R}^n)} = 0 \). Moreover, \( \sigma_L(f) \in K_{L,p} \).

**Proof** Let \( f \in L^{p,\lambda}_L(\mathbb{R}^n) \). For any \( 0 < t < s \), we apply (i) of Lemma 2.4 to obtain
\[ |e^{-tL} f(x) - e^{-sL} f(x)| \leq C t^{\frac{\lambda}{n-\lambda}} \|f\|_{L^{p,\lambda}(\mathbb{R}^n)}, \]
for some constant \( C \) independent of \( t \) and \( x \), and thus \( |e^{-tL} f(x) - e^{-sL} f(x)| \to 0 \) uniformly as \( t \to +\infty \). By the completeness of \( \mathbb{R}^n \), there exists a function \( \sigma_L(f)(x) = \lim_{t \to +\infty} e^{-tL} f(x) \) such that
\[ \lim_{t \to +\infty} \|e^{-tL} f - \sigma_L(f)\|_{L^\infty(\mathbb{R}^n)} = 0. \tag{2.11} \]

Let us prove that \( \sigma_L(f) \in M_p \). Set
\[ \beta_0 := \begin{cases} \Theta(L), & \text{if } \Theta(L) < \infty; \\ \text{any positive number}, & \text{if } \Theta(L) = \infty. \end{cases} \]

Using this notation, we have that \( f \in M_{p,\beta} \) for any \( \beta \in (0, \beta_0) \). It suffices to show that
\[ \int_{\mathbb{R}^n} \frac{|\sigma_L f(x)|^p}{(1 + |x|)^{n+\beta}} \, dx < \infty. \tag{2.12} \]

To prove (2.12), we choose some \( t_0 > 1 \) in (2.11) large enough such that \( \|e^{-t_0L} f - \sigma_L(f)\|_{L^\infty(\mathbb{R}^n)} \leq 1 \). One writes
\[ |\sigma_L f(x)| \leq |\sigma_L f(x) - e^{-t_0L} f(x)| + |e^{-t_0L} f(x) - e^{-t_0L} f(x)| + |f(x) - e^{-t_0L} f(x)| + |f(x)|. \]

From (i) of Lemma 2.4, \( \|e^{-tL} f(x) - e^{-t_0L} f(x)\|_{L^\infty(\mathbb{R}^n)} \leq C \|f\|_{L^{p,\lambda}(\mathbb{R}^n)} \). This, in combination with (ii) of Lemma 2.4 and the fact that \( f \in M_{p,\beta} \), shows
\[ \int_{\mathbb{R}^n} \frac{|\sigma_L f(x)|^p}{(1 + |x|)^{n+\beta}} \, dx \leq C + C \|f\|_{L^{p,\lambda}(\mathbb{R}^n)}^p + C \int_{\mathbb{R}^n} \frac{|f(x)|^p}{(1 + |x|)^{n+\beta}} \, dx < \infty, \tag{2.13} \]
and hence \( \sigma_L f \in \mathcal{M}_p \).

Finally, we apply (2.11) to obtain that for any \( s > 0 \),
\[
e^{-sL} \sigma_L f(x) = e^{-sL} \lim_{t \to +\infty} e^{-tL} f(x) = \lim_{t \to +\infty} e^{-sL} e^{-tL} f(x) = \lim_{t \to +\infty} e^{-(s+t)L} f(x) = \sigma_L f(x)
\]
for almost all \( x \in \mathbb{R}^n \), which yields that \( \sigma_L(f) \in \mathcal{K}_{L,p} \). This completes proof of Lemma 2.5. \( \square \)

**Proof of Theorem 2.2** Suppose that \( f \in L^{p,\lambda}(\mathbb{R}^n) \). From Lemma 2.3, we have that \( f \in \mathcal{L}^p_{p,\lambda}(\mathbb{R}^n) \) and \( \|f\|_{\mathcal{L}^p_{p,\lambda}(\mathbb{R}^n)} \leq C\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} \). Next we prove that
\[
f - \sigma_L(f) \in \mathcal{L}^p_{p,\lambda}(\mathbb{R}^n) \quad \text{and} \quad \|f - \sigma_L(f)\|_{\mathcal{L}^p_{p,\lambda}(\mathbb{R}^n)} \leq C\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}, \quad (2.14)
\]
and it reduces to show that \( \sigma_L(f)(x) = 0 \), a.e. \( x \in \mathbb{R}^n \). For any \( t > 0 \) and almost everywhere \( x \in \mathbb{R}^n \), we use Lemma 2.3 to obtain
\[
|e^{-t^nL}(f)(x)| \leq Ct^{(\lambda-\nu)/p}\|f\|_{L^{p,\lambda}}.
\]
Since \( \lambda \in (0,n) \), we have
\[
|\sigma_L(f)(x)| \leq \lim_{t \to +\infty} |e^{-tL}(f)(x)| = 0, \quad \text{for a.e. } x \in \mathbb{R}^n
\]
and (2.14) holds.

For the converse part of Theorem 2.2, we will show that for any \( f \in \mathcal{L}^p_{p,\lambda}(\mathbb{R}^n) \), there exists constant \( C > 0 \) such that for any ball \( B \subset \mathbb{R}^n \),
\[
\left( \frac{1}{r^\lambda_B} \int_B |f(x) - \sigma_L(f)(x)|^p \, dx \right)^{1/p} \leq C\|f\|_{\mathcal{L}^p_{p,\lambda}(\mathbb{R}^n)}. \quad (2.15)
\]
Indeed, one can write
\[
\left( \frac{1}{r^\lambda_B} \int_B |f(x) - \sigma_L(f)(x)|^p \, dx \right)^{1/p} \leq \|f\|_{\mathcal{L}^p_{p,\lambda}(\mathbb{R}^n)} + \left( \frac{1}{r^\lambda_B} \int_B |e^{-t^nL}(f(x) - \sigma_L(f)(x)|^p \, dx \right)^{1/p}. \quad (2.16)
\]
By Lemmas 2.4 and 2.5,
\[
\|e^{-t^nL}(f(x) - \sigma_L(f)(x)\|_{L^\infty(\mathbb{R}^n)} \leq \lim_{k \to +\infty} \|e^{-t^nL}f(x) - e^{-kr^nL}f(x)\|_{L^\infty(\mathbb{R}^n)} + \lim_{k \to +\infty} \|e^{-kr^nL}f(x) - \sigma_L(f)(x)\|_{L^\infty(\mathbb{R}^n)} \leq Cr_B^{\frac{\lambda-n}{p}}\|f\|_{\mathcal{L}^p_{p,\lambda}(\mathbb{R}^n)}.
\]
This, in combination with (2.16), shows the desired estimate (2.15). Hence, the proof of Theorem 2.2 is complete.

2.2 Campanato Spaces Associated to Schrödinger Operators and Classical Morrey Spaces Are Equivalent

In this subsection, let us consider the Schrödinger operators as (1.1)
\[
L = -\Delta + V(x) \quad \text{on } \mathbb{R}^n, \quad n \geq 3,
\]
where \( V \in L^1_{\text{loc}}(\mathbb{R}^n) \) and \( V \not\equiv 0 \) is a non-negative function in \( B_q \) for some \( q \geq n \). For such Schrödinger operator \( L \), the semigroup kernels \( \mathcal{P}_t(x,y) \) of the operators \( e^{-t\sqrt{L}} \) satisfy the following properties:
Lemma 2.6 For every \( N > 0 \), there exists a constant \( C = C_N \) such that for \( x, y \in \mathbb{R} \) and \( t > 0 \),
\[
|\mathcal{P}_t(x, y)| \leq C \frac{t}{(t^2 + |x - y|^2)^{\frac{n+1}{2}}} \left( 1 + \frac{(t^2 + |x - y|^2)^{1/2}}{\rho(x)} + \frac{(t^2 + |x - y|^2)^{1/2}}{\rho(y)} \right)^{-N}; \tag{2.17}
\]
\[
|t\nabla \mathcal{P}_t(x, y)| \leq C \frac{t}{(t^2 + |x - y|^2)^{\frac{n+1}{2}}} \left( 1 + \frac{(t^2 + |x - y|^2)^{1/2}}{\rho(x)} + \frac{(t^2 + |x - y|^2)^{1/2}}{\rho(y)} \right)^{-N}. \tag{2.18}
\]
Here,
\[
\rho(x) = \sup \left\{ r > 0 : \frac{1}{r^{n-2}} \int_{B(x, r)} V(y) dy \leq 1 \right\}.
\]

Proof For the proof, see Proposition 3.6, [14]. \( \square \)

As an application of Theorem 2.2, we have the following result.

Theorem 2.7 For \( 0 < \lambda < n \), \( 1 \leq p < \infty \), there holds
\[
\mathcal{L}^p\mathcal{L}^\lambda(\mathbb{R}^n) = \mathcal{L}^{p,\lambda}(\mathbb{R}^n) = L^{p,\lambda}(\mathbb{R}^n).
\]

Proof By Theorem 2.2, it suffices to show that for every \( 1 \leq p < \infty \),
\[
\mathcal{K}_{L,p} = \mathcal{K}_{\mathcal{L}^p} = \{0\}. \tag{2.19}
\]
The proof of (2.19) can be obtained by making modifications with [6, Proposition 6.5]. See also [20]. We give a brief argument of this proof for completeness and convenience for the reader.

Let us prove that \( \mathcal{L}^{p,\lambda}(\mathbb{R}^n) = L^{p,\lambda}(\mathbb{R}^n) \). For any \( d \geq 0 \), one writes
\[
\mathcal{H}_d(L) = \{ f \in W^{1,2}_{\text{loc}}(\mathbb{R}^n) : Lf = 0 \text{ and } |f(x)| = O(|x|^d) \text{ as } |x| \to \infty \}
\]
and
\[
\mathcal{H}_L = \bigcup_{d: 0 \leq d < \infty} \mathcal{H}_d(L).
\]
By Lemma 2.7 of [7], it follows that for any \( d \geq 0 \),
\[
\mathcal{H}_L = \mathcal{H}_d(L) = \{0\}.
\]
Assume that \( f \in \mathcal{K}_{L,p} \cap \mathcal{M}_p \). From estimates (2.17) and (2.18), we see that \( f = e^{-t\sqrt{T}} f \in W^{1,2}_{\text{loc}}(\mathbb{R}^n) \) and \( |f(x)| = O(|x|^\epsilon) \) for some \( \epsilon > 0 \). Because of the growth of \( f \), we use a standard approximation argument through a sequence \( f_k \) as follows. For any \( k \in \mathbb{N} \), we denote by \( \eta_k \) a standard \( C^\infty \) cut-off function which is 1 inside the ball \( B(0, k) \), zero outside \( B(0, k + 1) \), and let \( f_k = f \eta_k \in W^{1,2}(\mathbb{R}^n) \). Since \( f = e^{-t\sqrt{T}} f \), we have that for any \( \varphi \in C^1_0(\mathbb{R}^n) \),
\[
\langle Lf, \varphi \rangle = \langle Le^{-t\sqrt{T}} f, \varphi \rangle = \lim_{k \to \infty} \langle Le^{-t\sqrt{T}} f_k, \varphi \rangle
\]
\[
= \lim_{k \to \infty} \left\langle \frac{d}{dt} e^{-t\sqrt{T}} f_k, \varphi \right\rangle = \left\langle \frac{d}{dt} e^{-t\sqrt{T}} f, \varphi \right\rangle
\]
\[
= \left\langle \frac{d^2}{dt^2} f, \varphi \right\rangle = 0,
\]
which proves \( f \in \mathcal{H}_L = \{0\} \). Hence, \( \mathcal{K}_{L,p} = \{0\} \). Similar argument above shows that \( \mathcal{K}_{L,p} = \{0\} \). This completes the proof of Theorem 2.7. \( \square \)
Corollary 2.8  Let \( L = -\Delta + V \), where \( V \neq 0 \) is a non-negative potential in \( B_q \) for some \( q \geq n \). From Theorem 2.7 and the previous results proved in \([6, 8, 14]\), we have the following results:

(i) \( L_{p,\lambda}^p(\mathbb{R}^n) = L^p(\mathbb{R}^n) \), when \( \lambda \in (0, n) \);
(ii) \( L_{p,\lambda}^p(\mathbb{R}^n) \subseteq \text{BMO}(\mathbb{R}^n) \), when \( \lambda = n \) (see \([6, 8]\));
(iii) \( L_{p,\lambda}^p(\mathbb{R}^n) \subseteq \text{Lip}_\alpha(\mathbb{R}^n) \) with \( \alpha = (\lambda - n)/p \), when \( \lambda \in (n, n + p) \) (see \([14]\)).

3  Proof of Theorem 1.1

Proof of (1) of Theorem 1.1  Suppose \( f \in L^{2,\lambda}(\mathbb{R}^n) \). From Lemmas 2.4 and 3.9 of \([7]\), it follows that \( u(x, t) = e^{-t\sqrt{t}} f(x) \in C^1(\mathbb{R}^n_+) \). It will be enough if we have proved

\[
\|u\|_{H^{1,\lambda}_{L^2}(\mathbb{R}^n_+)} \leq C \|f\|_{L^{2,\lambda}(\mathbb{R}^n)}.
\]

In fact,

\[
\left( \frac{1}{r^B_0} \int_0^{r^B_0} \int_B |t \nabla e^{-t\sqrt{t}} f(x)|^2 \, dx \, dt \right)^{1/2} \leq \sum_{k=0}^{\infty} \frac{1}{r^B_0} \left( \int_0^{r^B_0} \int_B |t \nabla e^{-t\sqrt{t}} f_k(x)|^2 \, dx \, dt \right)^{1/2} =: \sum_{k=0}^{\infty} J_k,
\]

where \( f_0 = f \chi_{2B} \) and \( f_k = f \chi_{2^{k+1}B \setminus 2^k B} \) for \( k = 1, 2, \ldots \). Obviously, \( |J_0| \leq C \|f\|_{L^{2,\lambda}(\mathbb{R}^n)} \). For any \( x \in B \) and \( k \in \mathbb{N} \), we apply estimates (2.17) and (2.18) to obtain

\[
|t \nabla e^{-t\sqrt{t}} f_k(x)| \leq C \int_{2^{k+1}B} (t + |x - y|)^{n+1} |f(y)| \, dy
\]

\[
\leq C \left( \frac{t}{(2^k r_B)^{n+1}} \int_{2^{k+1}B} |f(y)| \, dy \right)
\]

\[
\leq C \left( \frac{t}{(2^k r_B)^{n+1}} \right)^{1 + \frac{n}{2}} \|f\|_{L^{2,\lambda}(\mathbb{R}^n)},
\]

which yields

\[
|J_k| \leq C 2^{-k(1 + \frac{n}{2})} \|f\|_{L^{2,\lambda}(\mathbb{R}^n)}.
\]

Hence, \( \sum_{k=1}^{\infty} |J_k| \leq C \|f\|_{L^{2,\lambda}(\mathbb{R}^n)} \), and then \( \|u\|_{H^{1,\lambda}_{L^2}(\mathbb{R}^n_+)} \leq C \|f\|_{L^{2,\lambda}(\mathbb{R}^n)} \).

Before we give the proof of (2) of Theorem 1.1, we need some auxiliary lemmas.

Lemma 3.1  Suppose \( 0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n) \) for some \( q > n/2 \). Let \( u \) be a weak solution of \( \mathbb{L} u = 0 \) in the ball \( B(Y_0, 2r) \subset \mathbb{R}^{n+1} \). Then for any \( p \geq 1 \), there exists a constant \( C = C(n, p) > 0 \) such that

\[
\sup_{B(Y_0, r)} |u(Y)| \leq C \left( \int_{B(Y_0, 2r)} |u(Y)|^p \, dY \right)^{1/p}.
\]

Proof  For the proof, we refer to Lemma 2.6 of \([7]\). \(\square\)

Lemma 3.2  Suppose \( V \in B_q \) for some \( q \geq n/2 \). Assume that \( u \in W^{1,2}_{\text{loc}}(\mathbb{R}^n) \) is a weak solution of \( Lu = (-\Delta + V)u = 0 \) in \( \mathbb{R}^n \). Also assume that there is a \( d > 0 \) such that

\[
\int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+d}} \, dx \leq C_d < \infty.
\]

Then \( u = 0 \) in \( \mathbb{R}^n \).
Lemma 3.3 Let 0 < λ < n. For every \( u \in \text{HL}^{2,\lambda}_{L}({\mathbb{R}^{n+1}_{+}}) \) and for every \( k \in \mathbb{N} \), there exists a constant \( C(k,n,\lambda) > 0 \) such that
\[
\int_{\mathbb{R}^{n}} \frac{|u(x,1/k)|^2}{(1+|x|)^{2n}} dx \leq C(k,n,\lambda) < \infty,
\]
hence \( u(x,1/k) \in L^{2}((1+|x|)^{-2n}dx) \). Therefore for all \( k \in \mathbb{N} \), \( e^{-t\sqrt{2}}(u(\cdot,1/k))(x) \) exists everywhere in \( \mathbb{R}^{n+1}_{+} \).

Proof Since \( u \in C^{1}(\mathbb{R}^{n+1}_{+}) \), it reduces to show that for every \( k \in \mathbb{N} \),
\[
\int_{|x| \geq 1} \frac{|u(x,1/k) - u(x/|x|,1/k)|^2}{(1+|x|)^{2n}} dx \leq C(k,n,\lambda)\|u\|^{2}_{\text{HL}^{2,\lambda}_{L}({\mathbb{R}^{n+1}_{+}})} < \infty. \tag{3.1}
\]
To do this, we write
\[
u(x,1/k) - u(x/|x|,1/k) = [u(x,1/k) - u(x,|x|)] + [u(x,|x|) - u(x/|x|,|x|)] + [u(x/|x|,|x|) - u(x/|x|,1/k)].
\]
Let
\[
I = \int_{|x| \geq 1} \frac{|u(x,1/k) - u(x,|x|)|^2}{(1+|x|)^{2n}} dx,
\]
\[
II = \int_{|x| \geq 1} \frac{|u(x,|x|) - u(x/|x|,|x|)|^2}{(1+|x|)^{2n}} dx,
\]
and
\[
III = \int_{|x| \geq 1} \frac{|u(x/|x|,|x|) - u(x/|x|,1/k)|^2}{(1+|x|)^{2n}} dx.
\]
Set \( Y_0 = (x,t) \) and \( r = t/4 \). We use Lemma 3.1 for \( \partial_t u \) and Schwarz’s inequality to obtain
\[
|\partial_t u(x,t)| \leq C \left( \frac{1}{r + 1} \int_{B(Y_0,2r)} |\partial_y u(y,s)|^2 \, dy \right)^{1/2},
\]
\[
\leq C \left( \frac{1}{r + 1} \int_{B(x,t/2)} \int_{t/2}^{3t/2} |\partial_s u(y,s)|^2 \, ds \, dy \right)^{1/2},
\]
\[
\leq Ct^{-1} \left( \frac{1}{|B(x,2t)|} \int_{0}^{2t} \int_{B(x,2t)} s|\partial_y u(y,t)|^2 \, dy \, ds \right)^{1/2},
\]
\[
\leq Ct^{-1 + \frac{2 - \lambda}{2}} \|u\|^{2}_{\text{HL}^{2,\lambda}_{L}({\mathbb{R}^{n+1}_{+}})}. \tag{3.2}
\]
Thus,
\[
|u(x,|x|) - u(x,1/k)| = \int_{1/k}^{1} |\partial_t u(x,t)| \, dt \leq \frac{2}{n - \lambda} \left( k^{n-\lambda}/2 \right) \|u\|^{2}_{\text{HL}^{2,\lambda}_{L}({\mathbb{R}^{n+1}_{+}})}. \tag{3.3}
\]
It follows that
\[
I + III \leq C(k,n,\lambda)\|u\|^{2}_{\text{HL}^{2,\lambda}_{L}({\mathbb{R}^{n+1}_{+}})} \int_{|x| \geq 1} \frac{1}{(1+|x|)^{2n}} dx
\]
\[
\leq C(k,n,\lambda)\|u\|^{2}_{\text{HL}^{2,\lambda}_{L}({\mathbb{R}^{n+1}_{+}})}.
\]
For the term II, we have that for any $x \in \mathbb{R}$,

$$u(x, |x|) - u(x/|x|, |x|) = \int_1^{|x|} D_r u(r\omega, |x|) dr, \quad x = |x|\omega.$$  

Let $B = B(0, 1)$ and $2^m B = B(0, 2^m)$. Note that for every $m \in \mathbb{N}$, we have

$$\int_{2^m B \setminus 2^{m-1} B} \left| \int_1^{|x|} D_r u(r\omega, |x|) |dr|^2 dx \right| = \int_1^{2m} \left| D_r u(r\omega) \right|^2 \rho^{n-1} dr dp \leq 2^m \int_1^{2m} \left| D_r u(r\omega) \right|^2 dr dp \leq 2^m \int_1^{2m} \left| \nabla u(y, t) \right|^2 |y|^{1-n} dy dt \leq 2^m \int_1^{2m} \int_{2^m B} \left| \nabla u(y, t) \right|^2 dy dt,$$

which gives

$$\int_{2^m B \setminus 2^{m-1} B} \left| u(x, |x|) - u(x/|x|, |x|) \right|^2 dx \leq C 2^{m(2n-1)} \left( \frac{1}{|2^m B|} \int_0^2 \left| \nabla u(y, t) \right|^2 dy dt \right)^{1/2} \leq C 2^{m(n+\lambda-1)} \| u \|_{\text{HL}_{2,\lambda}(\mathbb{R}^{n+1})}^2.$$  

Therefore,

$$\text{II} \leq C \sum_{m=1}^{\infty} \frac{1}{2^m} \int_{2^m B \setminus 2^{m-1} B} \left| u(x, |x|) - u(x/|x|, |x|) \right|^2 dx \leq C(n, \lambda) \| u \|_{\text{HL}_{2,\lambda}(\mathbb{R}^{n+1})}^2.$$  

Combining estimates of I, II and III, we have obtained (3.1).

Note that by Lemma 2.6, the semigroup kernels $P_t(x, y)$, associated to $e^{-t\sqrt{\gamma}}$, decay faster than any power of $1/|x - y|$. Hence, for all $k \in \mathbb{N}$, $e^{-t\sqrt{\gamma}}(u(\cdot, 1/k))(x)$ exists everywhere in $\mathbb{R}^{n+1}$. This completes the proof.

\[\square\]

**Proof of (2) of Theorem 1.1** To prove it, we will adapt the argument as in [7, 9, 13] and apply the key Theorem 2.7. Suppose $u \in \text{HL}_{2,\lambda}(\mathbb{R}^{n+1})$. Our aim is to look for a function $f \in L^{2,\lambda}(\mathbb{R}^n)$ such that

$$u(x, t) = e^{-t\sqrt{\gamma}} f(x), \quad \text{for each } (x, t) \in \mathbb{R}^{n+1}.$$  

To do this, for every $k \in \mathbb{N}$, we write $f_k(x) = u(x, k^{-1})$. We will first show that

$$\sup_{k \in \mathbb{N}} \| f_k \|_{L^{2,\lambda}(\mathbb{R}^n)} \leq C \| u \|_{\text{HL}_{2,\lambda}(\mathbb{R}^{n+1})}. \quad (3.4)$$  

In fact, Lemmas 3.1, 3.2, 3.3 are at our disposal, we can follow the arguments of Lemmas 3.2 and 3.4 of [7] to obtain the following two facts of function $u$ in the space $\text{HL}_{2,\lambda}(\mathbb{R}^{n+1})$ with $0 < \lambda < n$:

(i) For every $k \in \mathbb{N}$, $u(x, t + 1/k) = e^{-t\sqrt{\gamma}}(u(\cdot, 1/k))(x)$ for all $x \in \mathbb{R}^n$ and $t > 0$.

(ii) There exists a constant $C > 0$ (depending only on $n$ and $\lambda$) such that for all $k \in \mathbb{N}$,

$$\sup_{x \in B} \int_{(x, r_B)} \left| \partial_t e^{-t\sqrt{\gamma}}(u(\cdot, 1/k))(x) \right|^2 dx dt \leq C \| u \|_{\text{HL}_{2,\lambda}(\mathbb{R}^{n+1})}^2 < \infty.$$
By Lemma 3.5 of [7], we have
\[ \|f_k\|_{\mathcal{L}^{2,\lambda}(\mathbb{R}^n)} \leq C \sup_B \left( \frac{1}{r^n} \int_0^r \int_B |t \partial_t e^{-t\sqrt{T}} f_k(x)|^2 \frac{dxdt}{t} \right)^{1/2} \leq C \|u\|_{\mathcal{HL}^{2,\lambda}_{L^2}(\mathbb{R}^n)} < \infty, \]
which implies (3.4).

Then by combining (3.4) and Theorem 2.7: \( L^{2,\lambda}(\mathbb{R}^n) = \mathcal{L}^{2,\lambda}(\mathbb{R}^n) \), we have that
\[ \|f_k\|_{L^{2,\lambda}(\mathbb{R}^n)} = \|f_k\|_{\mathcal{L}^{2,\lambda}(\mathbb{R}^n)} \leq C \|u\|_{\mathcal{HL}^{2,\lambda}_{L^2}(\mathbb{R}^n)} . \quad (3.5) \]

Next, we will look for a function \( f \in L^{2,\lambda} \) through \( L^2(B(0,2^j)) \)-boundedness of \( \{f_k\} \) for each \( j \in \mathbb{N} \). Indeed, for every \( j \in \mathbb{N} \) we use (3.5) to obtain
\[ \int_{B(0,2^j)} |f_k(x)|^2 \leq C 2^{j\lambda} \|u\|_{\mathcal{HL}^{2,\lambda}_{L^2}(\mathbb{R}^n+1)}^2 . \]
This tells us that the sequence \( \{f_k\}_{k=1}^\infty \) is bounded in \( L^2(B(0,2^j)) \). So, after passing to a subsequence, the sequence converges weakly to a function \( g_j \in L^2(B(0,2^j)) \). Now we define a function \( f(x) \) by
\[ f(x) = g_j(x), \quad \text{if } x \in B(0,2^j), \quad j = 1, 2, \ldots. \]
It is easy to see that \( f \) is well defined on \( \mathbb{R}^n = \bigcup_{j=1}^\infty B(0,2^j) \). Also we can check that for any open ball \( B \subset \mathbb{R}^n \),
\[ \int_B |f(x)|^2 dx \leq C r^\lambda \|u\|_{\mathcal{HL}^{2,\lambda}_{L^2}(\mathbb{R}^n+1)}^2, \]
which implies
\[ \|f\|_{L^{2,\lambda}(\mathbb{R}^n)} \leq C \|u\|_{\mathcal{HL}^{2,\lambda}_{L^2}(\mathbb{R}^n+1)}. \]

Finally, we will show that \( u(x,t) = e^{-t\sqrt{T}} f(x) \). Since \( u(x,.) \) is continuous on \( \mathbb{R}_+ \), we have \( u(x,t) = \lim_{k \to +\infty} u(x,t+k^{-1}) \), and by (i), \( u(x,t) = \lim_{k \to +\infty} e^{-t\sqrt{T}} (u(\cdot,k^{-1}))(x) \). It reduces to show
\[ \lim_{k \to +\infty} e^{-t\sqrt{T}} (u(\cdot,k^{-1}))(x) = e^{-t\sqrt{T}} f(x). \quad (3.6) \]
Indeed, we recall that \( \mathcal{P}_t(x,y) \) is the kernel of \( e^{-t\sqrt{T}} \), and for any \( \ell \in \mathbb{N} \), we write
\[ e^{-t\sqrt{T}} (u(\cdot,k^{-1}))(x) = \int_{B(x,2^\ell t)} \mathcal{P}_t(x,y) f_k(y) dy + \int_{\mathbb{R}^n \setminus B(x,2^\ell t)} \mathcal{P}_t(x,y) f_k(y) dy. \]
Using the Hölder inequality, we obtain
\[ \left| \int_{\mathbb{R}^n \setminus B(x,2^\ell t)} \mathcal{P}_t(x,y) f_k(y) dy \right| \leq C \sum_{i=\ell}^\infty 2^{-i(2^\ell t)^{-\lambda}} \int_{B(x,2^{i+\ell} t)} |f_k(y)| dy \leq C \sum_{i=\ell}^\infty 2^{-i(2^\ell t)^{(\lambda-n)/2}} \|f_k\|_{L^{2,\lambda}(\mathbb{R}^n)} \leq C 2^{-\ell(1+\frac{\alpha}{2})}(2^\ell t)^{(\lambda-n)/2} \|f_k\|_{L^{2,\lambda}(\mathbb{R}^n)} . \]
From (3.5), we have that \( \|f_k\|_{L^{2,\lambda}(\mathbb{R}^n)} \leq C \|u\|_{\mathcal{HL}^{2,\lambda}_{L^2}(\mathbb{R}^n+1)} \) for some constant \( C > 0 \) independent of \( k \). Hence,
\[ \lim_{\ell \to +\infty} \lim_{k \to +\infty} \int_{\mathbb{R}^n \setminus B(x,2^\ell t)} \mathcal{P}_t(x,y) f_k(y) dy = \lim_{\ell \to +\infty} \left( C 2^{-\ell(1+\frac{\alpha}{2})}(2^\ell t)^{(\lambda-n)/2} \|u\|_{\mathcal{HL}^{2,\lambda}_{L^2}(\mathbb{R}^n+1)} \right) = 0. \]
since $\lambda \in (0, n)$. Therefore,

$$
\lim_{k \to +\infty} e^{-t\sqrt{\mathcal{L}}}(u(\cdot, k^{-1}))(x) = \lim_{t \to +\infty} \lim_{k \to +\infty} \int_{B(x, 2t)} \mathcal{P}_t(x, y)f_k(y)dy = e^{-t\sqrt{\mathcal{L}}f(x)},
$$

and (3.6) follows readily. Then we have showed that

$$u(x, t) = e^{-t\sqrt{\mathcal{L}}f(x)}.$$  

The proof of Theorem 1.1 is complete.

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