SYMMETRIC CHAIN DECOMPOSITION OF NECKLACE POSETS

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ABSTRACT. A finite ranked poset is called a symmetric chain order if it can be written as a disjoint union of rank-symmetric, saturated chains. If \( \mathcal{P} \) is any symmetric chain order, we prove that \( \mathcal{P}^n/\mathbb{Z}_n \) is also a symmetric chain order, where \( \mathbb{Z}_n \) acts on \( \mathcal{P}^n \) by cyclic permutation of the factors.

1. Introduction

Let \( (\mathcal{P},<) \) be a finite poset. A chain in \( \mathcal{P} \) is a sequence of the form \( x_1 < x_2 < \cdots < x_n \) where each \( x_i \in \mathcal{P} \). For \( x, y \in \mathcal{P} \), we say \( y \) covers \( x \) (denoted \( x \lessdot y \)) if \( x < y \) and there does not exist \( z \in \mathcal{P} \) such that \( x < z \) and \( z < y \). A saturated chain in \( \mathcal{P} \) is a chain where each element is covered by the next. We say \( \mathcal{P} \) is ranked if there exists a function \( \text{rk} : \mathcal{P} \to \mathbb{Z}_{\geq 0} \) such that \( x \lessdot y \) implies \( \text{rk}(y) = \text{rk}(x) + 1 \). The rank of \( \mathcal{P} \) is defined as \( \text{rk}(\mathcal{P}) = \max\{\text{rk}(x) \mid x \in \mathcal{P}\} + \min\{\text{rk}(x) \mid x \in \mathcal{P}\} \). A saturated chain \( \{x_1 \lessdot x_2 \lessdot \cdots \lessdot x_n\} \) in a ranked poset \( \mathcal{P} \) is said to be rank-symmetric if \( \text{rk}(x_1) + \text{rk}(x_n) = \text{rk}(\mathcal{P}) \).

We say that \( \mathcal{P} \) has a symmetric chain decomposition if it can be written as a disjoint union of saturated, rank-symmetric chains. A symmetric chain order is a finite ranked poset for which there exists a symmetric chain decomposition.

A finite product of symmetric chain orders is a symmetric chain order. This result can be proved by induction \([1]\) or by explicit constructions (e.g. \([3]\)). Naturally, this raises the question of whether the quotient of a symmetric chain order under a given group action has a symmetric chain decomposition. For example, if \( X \) is a set then \( \mathbb{Z}_n \) acts on the set \( \text{Map}(\mathbb{Z}_n, X) \simeq X^n \). The elements of \( X^n/\mathbb{Z}_n \) are called \( n \)-bead necklaces with labels in \( X \). A symmetric chain decomposition of the poset of binary necklaces was first constructed by K. Jordan \([6]\), building on the work of Griggs-Killian-Savage \([4]\). There have been recent independent proofs and generalizations of these results \([2, 5]\).

The main result of this paper is the following:

1.1. Theorem. If \( \mathcal{P} \) is a symmetric chain order, then \( \mathcal{P}^n/\mathbb{Z}_n \) is a symmetric chain order.

We give a brief outline of the proof. First, we show that the poset of \( n \)-bead binary necklaces is isomorphic to the poset of partition necklaces, i.e. \( n \)-bead necklaces labeled by positive integers which sum to \( n \). It turns out to be convenient to exclude the maximal and minimal binary necklaces, which correspond to those partitions of \( n \) having \( n \) parts and 0 parts, respectively. Let \( \mathcal{Q}(n) \) denote the poset of partition necklaces...
with these two elements removed. We decompose $Q(n)$ into rank-symmetric sub-posets $Q_\alpha$, running over partition necklaces $\alpha$ where 1 does not appear. This decomposition corresponds to the “block-code” decomposition of binary necklaces defined in [4].

We can also extend this idea to non-binary necklaces. In fact, the poset of $n$-bead $(m+1)$-ary necklaces embeds into the poset of $mn$-bead binary necklaces, and the image corresponds to the union of those $Q_\alpha \subset Q(mn)$ such that every part of $\alpha$ is divisible by $m$.

Next, we prove a “factorization property” for $Q_\alpha \subset Q(n)$. If $P$ and $Q$ are finite ranked posets, we say that $P$ covers $Q$ (or $Q$ is covered by $P$) if there is a morphism of ranked posets from $P$ to $Q$ which is a bijection on the underlying sets. We denote this relation as $P \hookrightarrow Q$. Note that any ranked poset covered by a symmetric chain order is also a symmetric chain order. If $\alpha$ is aperiodic, then $Q_\alpha$ is covered by a product of symmetric chains. If $\alpha$ is periodic of period $d$, then $Q_\alpha$ is covered by the poset of $(n/d)$-bead necklaces labeled by $Q_\beta$, for some aperiodic $d$-bead necklace $\beta$.

Finally, if $\mathcal{P}$ is a symmetric chain order, then $\mathcal{P}^n/\mathbb{Z}_n$ has a decomposition into posets which are either products of chains, or posets of $d$-bead necklaces with labels in a product of chains (where $d < n$), or posets of $n$-bead $(m+1)$-ary necklaces for some $m \geq 1$. In each case, we apply induction to finish the proof.

### 2. Generalities on necklaces

We begin by recalling some basic facts about $\mathbb{Z}_n$-actions on sets. We will use additive notation for the group operation of $\mathbb{Z}_n$. The subgroups of $\mathbb{Z}_n$ are of the form $\langle d \rangle$ where $d$ is a positive divisor of $n$, and $\mathbb{Z}_n/\langle d \rangle \simeq \mathbb{Z}_d$. If $X$ is a set with $\mathbb{Z}_n$-action, let $X^{\langle d \rangle}$ denote the set of $\langle d \rangle$-fixed points in $X$. Equivalently:

$$X^{\langle d \rangle} = \{ x \in X \mid \langle d \rangle \subset \text{Stab}_{\mathbb{Z}_n}(x) \}.$$ 

Note that $X^{\langle c \rangle} \subset X^{\langle d \rangle}$ if $c$ is a divisor of $d$. Next, we define:

$$X^{\langle d \rangle} = \{ x \in X \mid \langle d \rangle = \text{Stab}_{\mathbb{Z}_n}(x) \}.$$ 

Of course, we have:

$$X = \bigsqcup_{d | n} X^{\langle d \rangle}$$

and the $\mathbb{Z}_n$ action on $X^{\langle d \rangle}$ factors through $\mathbb{Z}_d$. In other words, we have a bijection:

$$X/\mathbb{Z}_n \simeq \bigsqcup_{d | n} X^{\langle d \rangle}/\mathbb{Z}_d.$$ 

Now consider the special case where $X = \text{Map}(\mathbb{Z}_n, Y)$ for some arbitrary set $Y$, where $\mathbb{Z}_n$ acts on the first factor. In other words,

$$(af)(b) = f(a + b)$$
for any \( a, b \in \mathbb{Z}_n \) and \( f : \mathbb{Z}_n \to Y \). Now the previous paragraph implies that:

\[
\text{Map}(\mathbb{Z}_n, Y) = \bigsqcup_{d|n} \text{Map}(\mathbb{Z}_n, Y)[d]
\]

and

\[
\text{Map}(\mathbb{Z}_n, Y) / \mathbb{Z}_n = \bigsqcup_{d|n} \text{Map}(\mathbb{Z}_n, Y)[d] / \mathbb{Z}_d.
\]

The elements of \( \text{Map}(\mathbb{Z}_n, Y) / \mathbb{Z}_n \) are called \( n \)-bead necklaces with labels in \( Y \).

An element of \( \text{Map}(\mathbb{Z}_n, Y)\{d\} / \mathbb{Z}_d \) is said to be periodic of period \( d \). An element of \( \text{Map}(\mathbb{Z}_n, Y)\langle n \rangle / \mathbb{Z}_n \) is said to be aperiodic. Given a map \( g : \mathbb{Z}_n \to Y \), let \([g]\) denote the corresponding necklace in \( \text{Map}(\mathbb{Z}_n, Y) / \mathbb{Z}_n \). A \( n \)-bead necklace with labels in \( Y \) can be visualized as a sequence of \( n \) elements of \( Y \) placed evenly around a circle, where we discount the effect of rotation by any multiple of \( \frac{2\pi}{n} \) radians. Given \((y_1, \ldots, y_n) \in Y^n\), let \([y_1, \ldots, y_n]\) denote the corresponding \( n \)-bead necklace.

Our first observation is that an \( n \)-bead necklace of period \( d \) is uniquely determined by any sequence of \( d \) consecutive elements around the circle. Moreover, as we rotate the circle, these \( d \) elements will behave exactly like an aperiodic \( d \)-bead necklace.

2.1. Proposition. There is a natural bijection between \( n \)-bead necklaces of period \( d \) and aperiodic \( d \)-bead necklaces.

Proof. Recall the following general fact: if \( G \) is a group, \( H \) is a normal subgroup of \( G \), and \( Y \) is an arbitrary set, then there is an isomorphism of \( G \)-sets:

\[
\text{Map}(G, Y)^H \cong \text{Map}(G/H, Y)
\]

\[
f \mapsto (gH \mapsto f(g)).
\]

Moreover, the action of \( G \) on each side factors through \( G/H \). In particular, there is an isomorphism of \( \mathbb{Z}_n \)-sets:

\[
\text{Map}(\mathbb{Z}_n, Y)^{\langle d \rangle} \cong \text{Map}(\mathbb{Z}_d, Y)
\]

where the \( \mathbb{Z}_n \)-action factors through \( Z_d \). Looking at elements of period \( d \), we get:

\[
\text{Map}(\mathbb{Z}_n, Y)^{\langle d \rangle} \cong \text{Map}(\mathbb{Z}_d, Y)^{\langle d \rangle}
\]

and so:

\[
\text{Map}(\mathbb{Z}_n, Y)^{\langle d \rangle} / \mathbb{Z}_d \cong \text{Map}(\mathbb{Z}_d, Y)^{\langle d \rangle} / \mathbb{Z}_d.
\]

□

Now suppose that \( Y \) is a disjoint union of non-empty subsets:

\[
Y = \bigsqcup_{i \in I} Y_i
\]

where \( I \) is a finite set. Equivalently, we have a surjective map \( \pi : Y \to I \), where \( Y_i = \pi^{-1}(i) \) for each \( i \in I \). It follows that there is a surjective map:

\[
\pi_* : \text{Map}(\mathbb{Z}_n, Y) \to \text{Map}(\mathbb{Z}_n, I)
\]
\[ \pi_*(f) = \pi \circ f. \]

Given a map \( g : \mathbb{Z}_n \to I \), we define:
\[
\text{Map}_g(\mathbb{Z}_n, Y) = \pi^{-1}(g) = \{ f : \mathbb{Z}_n \to Y \mid \pi \circ f = g \}.
\]

In other words, \( f \in \text{Map}_g(\mathbb{Z}_n, Y) \) if and only if \( f(a) \in Y_{g(a)} \) for all \( a \in \mathbb{Z}_n \). Since \( \pi_* \) is surjective, we have a decomposition:
\[
\text{Map}(\mathbb{Z}_n, Y) = \bigcup_{g \in \text{Map}(\mathbb{Z}_n, I)} \text{Map}_g(\mathbb{Z}_n, Y).
\]

Note that \( \text{Map}_g(\mathbb{Z}_n, Y) \) is not necessarily stable under the action of \( \mathbb{Z}_n \). If \( a, b \in \mathbb{Z}_n \) and \( f \in \text{Map}_g(\mathbb{Z}_n, Y) \), then:
\[
a(f)(b) = f(a + b) \in Y_{g(a + b)}
\]
so we have a bijection:
\[
\text{Map}_g(\mathbb{Z}_n, Y) \simeq \text{Map}_{ag}(\mathbb{Z}_n, Y)
\]
induced by the action of \( a \in \mathbb{Z}_n \). We define:
\[
\text{Map}_{\{g\}}(\mathbb{Z}_n, Y) = \bigcup_{a \in \mathbb{Z}_n} \text{Map}_{ag}(\mathbb{Z}_n, Y).
\]

Note that \( \mathbb{Z}_n \) acts on \( \text{Map}_{\{g\}}(\mathbb{Z}_n, Y) \).

2.2. Remark. We recall a basic observation which will make it easier to define maps on sets of necklaces. Suppose \( S \) and \( T \) are sets equipped with equivalence relations \( \sim \) and \( \approx \), respectively. Let \( U \) be a subset of \( S \) which has a non-trivial intersection with each equivalence class in \( S \). Then \( U \) inherits the equivalence relation \( \sim \) and the natural map from \( U/\sim \) to \( S/\sim \) is a bijection. Given a map \( f : U \to T \) such that \( u_1 \sim u_2 \implies f(u_1) \approx f(u_2) \) for all \( u_1, u_2 \in U \), we obtain a map \( (S/\sim) \simeq (U/\sim) \to (T/\approx) \).

2.3. Remark. If \( \alpha \) is a periodic \( n \)-bead necklace of period \( d \) with labels in \( I \), then:
\[
\alpha = \underbrace{[\beta, \ldots, \beta]}_{d \text{ times}}
\]
where \( \beta = (\beta_1, \ldots, \beta_d) \) is a \( d \)-tuple of elements in \( I \) such that \( [\beta] \) is aperiodic.

2.4. Lemma. Let \( \pi : Y \to I \) be a surjective map where \( I \) is finite.

(1) There is a natural decomposition:
\[
\text{Map}(\mathbb{Z}_n, Y)/\mathbb{Z}_n = \bigcup_{d|n} \left( \bigcup_{\alpha \in \text{Map}(\mathbb{Z}_n, I)^{d}/\mathbb{Z}_d} \text{Map}_\alpha(\mathbb{Z}_n, Y)/\mathbb{Z}_n \right).
\]

(2) If \( \alpha = [\beta, \ldots, \beta] \in \text{Map}(\mathbb{Z}_n, I)^{d}/\mathbb{Z}_d \), where \( \beta = (\beta_1, \ldots, \beta_d) \), then there is a bijection:
\[
\text{Map}_\alpha(\mathbb{Z}_n, Y)/\mathbb{Z}_n \simeq (Y_{\beta_1} \times \cdots \times Y_{\beta_d})^{\mathbb{Z}_n}/\mathbb{Z}_d.
\]
Proof. (1) Since

\[
\text{Map}(\mathbb{Z}_n, Y) = \bigcup_{g \in \text{Map}(\mathbb{Z}_n, I)} \text{Map}_g(\mathbb{Z}_n, Y)
\]

and

\[
\text{Map}(\mathbb{Z}_n, I) = \bigcup_{d | n} \text{Map}(\mathbb{Z}_n, I)^{(d)}
\]

we see that:

\[
\text{Map}(\mathbb{Z}_n, Y) = \bigcup_{d | n} \left( \bigcup_{g \in \text{Map}(\mathbb{Z}_n, I)^{(d)}} \text{Map}_g(\mathbb{Z}_n, Y) \right).
\]

As noted above, in order to make this an equality of \(\mathbb{Z}_n\)-sets we need to take the coarser decomposition:

\[
\text{Map}(\mathbb{Z}_n, Y) = \bigcup_{d | n} \left( \bigcup_{[g] \in \text{Map}(\mathbb{Z}_n, I)^{(d)}/\mathbb{Z}_d} \text{Map}_g[\mathbb{Z}_n, Y] \right).
\]

Now we simply take the quotient by \(\mathbb{Z}_n\) on both sides:

\[
\text{Map}(\mathbb{Z}_n, Y)/\mathbb{Z}_n = \bigcup_{d | n} \left( \bigcup_{[g] \in \text{Map}(\mathbb{Z}_n, I)^{(d)}/\mathbb{Z}_d} \text{Map}_g[\mathbb{Z}_n, Y]/\mathbb{Z}_n \right).
\]

Note that we are simply organizing the \(n\)-bead \(Y\)-labeled necklaces by looking at the periods of the underlying \(n\)-bead \(I\)-labeled necklaces.

(2) Let \(g \in \text{Map}(\mathbb{Z}_n, I)^{(d)}\) and let \(a \in \mathbb{Z}_n\). By definition, \(ag = (a + x)g\) if and only if \(x \in \langle d \rangle\). So:

\[
\text{Map}_ag(\mathbb{Z}_n, Y) = \text{Map}_{(a+x)g}(\mathbb{Z}_n, Y)
\]

if \(x \in \langle d \rangle\). On the other hand, if

\[
h \in \text{Map}_ag(\mathbb{Z}_n, Y) \cap \text{Map}_{(a+x)g}(\mathbb{Z}_n, Y)
\]

for some \(x \in \mathbb{Z}_n\), then \(\pi \circ h = ag = (a + x)g\), which implies that \(x \in \langle d \rangle\). The upshot is that we can actually write \(\text{Map}_{[g]}(\mathbb{Z}_n, Y)\) as a disjoint union over \(\mathbb{Z}_d\):

\[
\text{Map}_{[g]}(\mathbb{Z}_n, Y) = \bigcup_{a \in \mathbb{Z}_d} \text{Map}_ag(\mathbb{Z}_n, Y).
\]

Now consider the sequence of values \(g(a)\) for \(a \in \mathbb{Z}_n\). This sequence is of the form \((\beta, \ldots, \beta)\), where \(\beta = (\beta_1, \ldots, \beta_d)\). Therefore:

\[
\text{Map}_g(\mathbb{Z}_n, Y) \simeq (Y_{\beta_1} \times \cdots \times Y_{\beta_d})^{\frac{n}{d}}
\]

and so:

\[
\text{Map}_{[g]}(\mathbb{Z}_n, Y) \simeq \bigcup_{j=0}^{d-1} (Y_{\beta_{j+1}} \times \cdots \times Y_{\beta_d} \times Y_{\beta_1} \times \cdots \times Y_{\beta_j})^{\frac{n}{d}}.
\]
Let us apply Remark 2.2 to the following sets:

\[
S = \bigoplus_{j=0}^{d-1} (Y_{\beta_{j+1}} \times \cdots \times Y_{\beta_d} \times Y_{\beta_1} \times \cdots \times Y_{\beta_j})^{\frac{n}{d}} \quad \text{and} \quad T = (Y_{\beta_1} \times \cdots \times Y_{\beta_d})^{\frac{n}{d}}.
\]

The equivalence relations on \(S\) and \(T\) are defined by group actions: \(\mathbb{Z}_n\) acts on \(S \simeq \text{Map}_{[g]}(\mathbb{Z}_n, Y)\) and \(\mathbb{Z}_d^{\frac{n}{d}}\) acts on \(T\) by cyclic permutation of the factors. Let \(U\) be the subset of \(S\) corresponding to the \(j = 0\) component:

\[
U = (Y_{\beta_1} \times \cdots \times Y_{\beta_d})^{\frac{n}{d}}.
\]

Each element of \(S\) is equivalent to an element of \(U\), and the restricted equivalence relation on \(U\) is given by the action of the subgroup \(\langle d \rangle\) which is exactly the same as the action of \(\mathbb{Z}_d^{\frac{n}{d}}\) by cyclic permutation of the factors. Therefore:

\[
S/\mathbb{Z}_n \simeq U/\langle d \rangle \simeq T/\mathbb{Z}_d^{\frac{n}{d}}.
\]

\[\square\]

2.5. Remark. We can visualize the above result as follows: we choose a place to “cut” an \(n\)-bead \(Y\)-labeled necklace in order to get an \(n\)-tuple of elements of \(Y\). We can always rotate the original necklace so that the underlying \(I\)-labeled necklace has a given position with respect to the cut. Moreover, if the underlying \(I\)-labeled necklace has period \(d\), then we can break the \(n\)-tuple into segments of size \(d\) so that the corresponding \(I\)-labeled \(d\)-bead necklaces are aperiodic. As we rotate the original necklace by multiples of \(\frac{2\pi}{d}\) radians, we will permute these segments among each other.

3. Partition necklaces

Let \(n\) be a positive integer. Consider the set of ordered partitions of \(n\) into \(r\) positive parts:

\[
\mathcal{P}(n,r) = \{(a_1, \ldots, a_r) \in \mathbb{Z}_{>0}^r \mid \sum_{i=1}^r a_i = n\}
\]

Define:

\[
\mathcal{P}(n) = \bigsqcup_{r=1}^{n-1} \mathcal{P}(n,r)
\]

In other words, \(\mathcal{P}(n)\) is the set of non-empty ordered partitions of \(n\) into positive parts, where at least one part is greater than 1. Note that refinement of partitions defines a partial order on \(\mathcal{P}(n)\), and the rank of a partition is given by the number of parts.

Let \(\mathcal{Q}(n)\) denote the set of necklaces associated to \(\mathcal{P}(n)\):

\[
\mathcal{Q}(n) = \bigsqcup_{i=1}^{n-1} \mathcal{P}(n,r)/\mathbb{Z}_r
\]

In other words:

\[
\mathcal{Q}(n) = \{[a_1, \ldots, a_r] \in \mathbb{Z}_{>0}^r/\mathbb{Z}_r \mid 1 \leq r \leq n-1, \sum_{i=1}^r a_i = n\}
\]
where \([a_1, \ldots, a_r]\) denotes the \(\mathbb{Z}_r\)-orbit of \((a_1, \ldots, a_r)\).

The elements of \(Q(n)\) are called partition necklaces. Note that \(Q(n)\) inherits the structure of a ranked poset from \(P(n)\).

Let \(N(n, 1)\) denote the set of \(n\)-bead binary necklaces with the necklaces \([0, \ldots, 0]\) and \([1, \ldots, 1]\) removed.

### 3.1. Proposition

For any \(n \geq 1\), there is an isomorphism of ranked posets:

\[
\psi_n : N(n, 1) \cong Q(n).
\]

**Proof.** Given a non-empty \(n\)-bead binary necklace \(\beta\) of rank \(r\), let \(\psi_n(\beta)\) be the necklace whose entries are given by the number of steps between consecutive non-zero entries of \(\beta\). More precisely, \(\psi_n\) is given by:

\[
[1, 0^{c_1}, 1, 0^{c_2}, \ldots, 1, 0^{c_r}] \mapsto [c_1 + 1, \ldots, c_r + 1]
\]

Note that the right hand side is the necklace of a partition of \(n\) into \(r\) positive parts.

The inverse of \(\psi_n\) is given by:

\[
[a_1, \ldots, a_r] \mapsto [1, 0^{a_1-1}, 1, 0^{a_2-1}, \ldots, 1, 0^{a_r-1}].
\]

Moreover, changing a “zero” to a “one” in a binary necklace corresponds to a refinement of the corresponding partition necklace, so the above bijection is compatible with the partial orders and rank functions on each poset. \(\square\)

An ordered partition \((a_1, \ldots, a_r)\) and the corresponding partition necklace \([a_1, \ldots, a_r]\) are said to be fundamental if each \(a_i \geq 2\). Let \(\mathcal{F}(n)\) denote the set of fundamental partition necklaces in \(Q(n)\).

Now we apply Remark 2.2 to the case where \(S = P(n)\) and \(T\) is the subset of \(P(n)\) consisting of fundamental partitions. Equip each set with the necklace equivalence relation, so \((S/\sim) = Q(n)\) and \((T/\approx) = F(n)\). Define the subset:

\[
U = \{(1^{n_1}, m_1, 1^{n_2}, m_2, \ldots, 1^{n_k}, m_k) \in P(n) \mid n_i \geq 0, m_i \geq 2 \text{ for all } 1 \leq i \leq k\}
\]

Since we have excluded \((1, \ldots, 1)\) from \(P(n)\), we see that any element of \(P(n)\) is equivalent to some element in \(U\). Now define:

\[
f : U \rightarrow T
\]

\[
(1^{n_1}, m_1, 1^{n_2}, m_2, \ldots, 1^{n_k}, m_k) \mapsto (m_1 + n_1, \ldots, m_k + n_k).
\]

Since \(f\) is compatible with the respective equivalence relations, we obtain a map:

\[
\pi_n : Q(n) \rightarrow \mathcal{F}(n)
\]

\[
[1^{n_1}, m_1, 1^{n_2}, m_2, \ldots, 1^{n_k}, m_k] \mapsto [m_1 + n_1, m_2 + n_2, \ldots, m_k + n_k].
\]

Note that \(\pi_n\) restricts to the identity on \(\mathcal{F}(n)\). In particular, \(\pi_n\) is surjective. Therefore, we get a decomposition of \(Q(n)\):

\[
Q(n) = \bigsqcup_{\alpha \in \mathcal{F}(n)} Q_{\alpha}
\]
where $Q_n = \pi_n^{-1}(\alpha)$. This decomposition is the same as the decomposition for binary necklaces defined in [4]. Indeed, the map $\pi_n \circ \psi_n$ is essentially the necklace version of the “block-code” construction.

If $m \geq 1$, a fundamental partition necklace $[a_1, \ldots, a_r] \in \mathcal{F}(n)$ is said to be divisible by $m$ if each $a_i$ is divisible by $m$. Define the following sub-poset of $Q(n)$:

$$Q(n, m) = \{ \alpha \in Q(n) \mid \pi_n(\alpha) \text{ is divisible by } m \} = \bigsqcup_{m|\alpha} Q_\alpha.$$

Let $N(n, m)$ denote the set of $n$-bead $(m+1)$-ary necklaces with the necklaces $[0, \ldots, 0]$ and $[m, \ldots, m]$ removed. We have the following generalization of Proposition 3.1.

3.2. Lemma. For any $n, m \geq 1$, there is an isomorphism of ranked posets:

$$\psi_{n,m} : N(n, m) \simeq Q(mn, m).$$

Proof. Given an $n$-bead $(m+1)$-ary necklace, we construct an $mn$-bead binary necklace via the substitution: $j \mapsto 1^{j0^{m-j}}$, and then we apply the map $\psi_{mn}$ from Proposition 3.1. This composition is clearly a morphism of ranked posets. Here is an explicit formula for $\psi_{n,m}$:

$$[b_1, b_2, \ldots, b_r] \mapsto [1^{b_1-1}, m(c_1+1) - b_1 + 1, \ldots, 1^{b_r-1}, m(c_r+1) - b_r + 1]$$

where each $b_i \geq 1$ and $c_i \geq 0$. The sum of the terms in the partition necklace is:

$$\sum_{i=1}^{r} (b_i - 1 + m(c_i + 1) - b_i + 1) = m(r + \sum_{i=1}^{r} c_i) = mn$$

as desired. Let us check that $\pi_{mn} \circ \psi_{n,m}(\alpha)$ is divisible by $m$ for all $\alpha \in N(n, m)$. Consider the element:

$$\alpha = [b_1, b_2, \ldots, b_r] \mapsto [1^{b_1-1}, m(c_1+1) - b_1 + 1, \ldots, 1^{b_r-1}, m(c_r+1) - b_r + 1]$$

where $\pi_{n,m}(\psi_{n,m}(\alpha)) = [me_1, \ldots, me_s]$ and this result is indeed divisible by $m$.

By reversing the above process, we get a formula for the inverse of $\psi_{n,m}$. An arbitrary element of $Q(mn, m)$ is of the form:

$$[1^{m_1}, m_1, 1^{m_2}, m_2, \ldots, 1^{m_k}, m_k]$$

where each $m_i \geq 1$, each $m_i + n_i$ is divisible by $m_i$, and $\sum_{i=1}^{k} (m_i + n_i) = mn$. The corresponding $mn$-bead binary necklace is:

$$[1^{n_1+1}, 0^{m_1-1}, \ldots, 1^{n_k+1}, 0^{m_k-1}]$$

Now we need to apply the substitution $1^{j0^{m-j}} \mapsto j$. Since $m_i + n_i$ is divisible by $m$, we can apply this to each block $[1^{n_i+1}, 0^{m_i-1}]$ separately. Furthermore, we should
break each block into segments of size \( m \) and apply the substitution to each segment. Therefore, \((1^{n_i+1}, 0^{m_i-1})\) looks like:

\[
\underbrace{(1^m, 1^m, \ldots, 1^m)}_{q_i \text{ times}}, 1^n, 0^{m-r}, 0^{m-1-(m-r_i)}.
\]

where \( q_i \) is the quotient of the division of \( n_i + 1 \) by \( m \) and \( r_i \) is the remainder. Note that \( m_i - 1 - (m - r_i) = m_i - 1 - m + (n_i + 1 - m q_i) = m_i + n_i - m q_i - m \), which is divisible by \( m \). Therefore, the inverse of \( \psi_{n,m} \) is given by the following formula:

\[
[1^{n_1}, m_1, 1^{n_2}, m_2, \ldots, 1^{n_k}, m_k] \mapsto [m^{q_1}, r_1, 0^{t_1}, \ldots, m^{q_k}, r_k, 0^{t_k}]
\]

where:

\[ n_i + 1 = m q_i + r_i \text{ such that } 0 \leq r_i < m \]

Note that the number of beads in the above necklace is:

\[
\sum_{i=1}^{k} (q_i + 1 + \frac{m_i + n_i}{m} - q_i - 1) = \frac{1}{m} \sum_{i=1}^{k} (m_i + n_i) = \frac{m n}{m} = n
\]

as desired. \( \square \)

3.3. Lemma. Let \( \alpha = [a_1, \ldots, a_r] \in \mathcal{F}(n) \). If \( \alpha \) is aperiodic, then:

\[ Q_{[a_1]} \times \cdots \times Q_{[a_r]} \lelihook Q_{\alpha}. \]

If \( \alpha \) is periodic of period \( d \) and \( \alpha = [\beta, \ldots, \beta] \), then:

\[ Q_{\frac{\beta}{r}} / \mathbb{Z}_{r^d} \lelihook Q_{\alpha}. \]

Proof. If \( m \geq 2 \), note that \( Q_{[m]} \) is a chain with \( m - 1 \) vertices. We will apply Lemma 2.4 to the following set:

\[ Q = \bigsqcup_{m=2}^{n} Q_{[m]}. \]

Note that our indexing set is \( I = \{2, \ldots, n\} \). Let \( \alpha = [a_1, \ldots, a_r] \in \mathcal{F}(n) \). Since \( a_1 + \cdots + a_r = n \), we know that each \( a_i \leq n \), which implies that \( \alpha \) is labeled by elements of \( I \). If \( \alpha \) is aperiodic, it follows from part (2) of Lemma 2.4 that we have a rank-preserving bijection:

\[ \text{Map}_\alpha(\mathbb{Z}_r, Q) / \mathbb{Z}_r \cong Q_{[a_1]} \times \cdots \times Q_{[a_r]} \]

On the other hand, if \( \alpha = [\beta, \ldots, \beta] \in \text{Map}(\mathbb{Z}_r, I)^{(d)} / \mathbb{Z}_d \), where \( \beta = (\beta_1, \ldots, \beta_d) \), then we have rank-preserving bijections:

\[ \text{Map}_\alpha(\mathbb{Z}_r, Q) / \mathbb{Z}_r \cong (Q_{[\beta_1]} \times \cdots \times Q_{[\beta_d]}) / \mathbb{Z}_d \cong Q_{[\beta]} / \mathbb{Z}_d \]

where the second bijection exists due to the fact that \( [\beta] \) is aperiodic. It remains to check that the poset relations are preserved. Indeed, any covering relation among two
necklaces labeled by $Q[\beta_1] \times \cdots \times Q[\beta_d]$ will correspond to a covering relation within a chain $Q[\beta_i]$ for some $i$, which will also be a covering relation among the corresponding $Q$-labeled necklaces.

3.4. Remark. The above Lemma provides an explanation of why it is easier to find a symmetric chain decomposition of $n$-bead binary necklaces if $n$ in prime [4]. Indeed, in this case all non-trivial necklaces are aperiodic, so each $Q_\alpha$ is covered by a product of symmetric chains and we can apply the Greene-Kleitman rule.

4. Proof of the theorem

4.1. Theorem. If $\mathcal{P}$ is a symmetric chain order, then $\mathcal{P}^n/\mathbb{Z}_n$ is a symmetric chain order.

Proof. The statement is trivial for $n = 1$. Assume that the theorem is true for any $n' < n$. Let $C_1, \ldots, C_r$ denote the chains in a symmetric chain decomposition of $\mathcal{P}$. We may assume that:

$$\mathcal{P} = \bigsqcup_{i=1}^r C_i.$$ 

If we let $I = \{1, 2, \ldots, r\}$ and apply part (1) of Lemma 2.4 to $\mathcal{P}$, we obtain:

$$\text{Map}(\mathbb{Z}_n, \mathcal{P})/\mathbb{Z}_n = \bigsqcup_{d|n} \left( \bigsqcup_{\alpha \in \text{Map}(\mathbb{Z}_n, I)_{(d)}/\mathbb{Z}_d} \text{Map}_\alpha(\mathbb{Z}_n, \mathcal{P})/\mathbb{Z}_n \right).$$

Now we apply part (2) of Lemma 2.4. If $\alpha = [a_1, \ldots, a_n]$ is an aperiodic $n$-bead necklace with labels in $I$, then:

$$C_{a_1} \times \cdots \times C_{a_n} \hookrightarrow \text{Map}_\alpha(\mathbb{Z}_n, \mathcal{P}).$$

Since $C_{a_1} \times \cdots \times C_{a_n}$ is a symmetric chain order, it follows that $\text{Map}_\alpha(\mathbb{Z}_n, \mathcal{P})$ is a symmetric chain order. Also note that $C_{\beta_1} \times \cdots \times C_{\beta_d}$ is a centered subposet of $\text{Map}(\mathbb{Z}_n, \mathcal{P})/\mathbb{Z}_n$. On the other hand, if $\alpha = [\beta_1, \ldots, \beta_d]$ is a periodic $n$-bead necklace with labels in $I$, where $\beta = (\beta_1, \ldots, \beta_d)$, then:

$$(C_{\beta_1} \times \cdots \times C_{\beta_d})_{\mathbb{Z}_d} \hookrightarrow \text{Map}_\alpha(\mathbb{Z}_n, \mathcal{P})/\mathbb{Z}_n.$$ 

Again, note that this poset is a centered subposet of $\text{Map}(\mathbb{Z}_n, \mathcal{P})/\mathbb{Z}_n$ since it is a cyclic quotient of a centered subposet of $\mathcal{P}^n$.

If $d > 1$, then $\frac{n}{d} < n$ and $(C_{\beta_1} \times \cdots \times C_{\beta_d})$ is a symmetric chain order, so

$$(C_{\beta_1} \times \cdots \times C_{\beta_d})_{\mathbb{Z}_d} \hookrightarrow \text{Map}_\alpha(\mathbb{Z}_n, \mathcal{P})/\mathbb{Z}_n$$

is a symmetric chain order by induction.

If $d = 1$, then:

$$C^n/\mathbb{Z}_n \hookrightarrow \text{Map}_\alpha(\mathbb{Z}_n, \mathcal{P})/\mathbb{Z}_n.$$
where $C$ is a chain with $m + 1$ vertices, for some $m \geq 1$. It suffices to consider the centered subposet $N(n, m)$. By Lemma 3.2, we have:

$$N(n, m) \simeq Q(mn, m).$$

If $Q_\alpha \subset Q(mn, m)$, then $\alpha = [ma_1, \ldots, ma_s]$, where $a_1 + \cdots + a_s = n$. In particular, note that $s \leq n$. By Lemma 3.3, there are two possibilities for $Q_\alpha$. If $\alpha$ is aperiodic, $Q_\alpha$ is a product of chains, so it is a symmetric chain order. If $\alpha$ is periodic of period $d$, then:

$$Q_\alpha^{\frac{n}{d}} / \mathbb{Z}_{\frac{n}{d}} \hookrightarrow Q_\alpha$$

where $[\beta]$ is a $d$-bead aperiodic necklace. In particular, $Q_{[\beta]}$ is itself a product of chains (hence a symmetric chain order). We know that $\beta = (mc_1, \ldots, mc_d)$, where $c_1 + \cdots + c_d = \frac{dn}{d}$. There are three possible cases:

(i) If $d > 1$, then $\frac{n}{d} < n$. Since $Q_{[\beta]}$ is a symmetric chain order, by induction we conclude that

$$Q_{[\beta]}^{\frac{n}{d}} / \mathbb{Z}_{\frac{n}{d}}$$

is a symmetric chain order.

(ii) If $d = 1$ and $s < n$ then $Q_{[\beta]}$ is a single chain, so $Q_{[\beta]}^{\frac{n}{d}} / \mathbb{Z}_{\frac{n}{d}}$ is a symmetric chain order by induction.

(iii) If $d = 1$ and $s = n$, then $\beta = (m)$ and $\alpha = [m, \ldots, m]$. In this case:

$$Q_{[m]}^{n} / \mathbb{Z}_{n} \hookrightarrow Q_\alpha.$$

Since $Q_{[m]}$ is a chain with $m - 1$ vertices, we see that we have returned to the case of the $\mathbb{Z}_n$-quotient of the $n$-fold power of a single chain. However, note that the we have managed to decrease the length of the chain by two, i.e. from $m + 1$ vertices to $m - 1$ vertices. Now we can again apply Lemma 3.2 and Lemma 3.3 to the centered subposet $N(n, m - 2)$, etc.

Eventually, after we go through this argument enough times, we will eventually reach the case of:

$$C^n / \mathbb{Z}_n$$

where $C$ is a chain with one or two vertices. If $|C| = 1$, there is nothing to show. So we are left with the case where $C$ is a chain with two vertices, i.e. the poset of binary necklaces. It suffices to look at the centered subposet $N(n, 1)$. By Proposition 3.1,

$$N(n, 1) \simeq Q(n).$$

Again, we consider the subposets $Q_\alpha$. As usual, if $\alpha$ is aperiodic then $Q_\alpha$ is covered by a product of symmetric chains. If $\alpha = [\beta, \ldots, \beta]$ is periodic of period $d$ then

$$Q_{[\beta]}^{\frac{n}{d}} / \mathbb{Z}_{\frac{n}{d}} \hookrightarrow Q_\alpha$$

where $[\beta]$ is an aperiodic $d$-bead necklace and $Q_{[\beta]}$ is a product of chains. If $d > 1$, then $\frac{n}{d} < n$ so

$$Q_{[\beta]}^{\frac{n}{d}} / \mathbb{Z}_{\frac{n}{d}}$$
is a symmetric chain order by induction. Finally, if $\alpha$ is periodic of period $d = 1$ then $\alpha$ is an $n$-bead partition necklace of period 1 whose entries sum to $n$, so $\alpha = [1, 1, \ldots, 1]$, but this element was explicitly excluded from the set $\Omega(n)$.

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