A Scale-Separated Dynamic Mode Decomposition From Observations of the Ionospheric Electron Density Profile

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Abstract

We present a method for modeling a time series of ionospheric electron density profiles using modal decompositions. Our method is based on the Dynamic Mode Decomposition (DMD), which provides a means of determining spatiotemporal modes from measurements alone. DMD-derived models can be easily updated as new data is recorded and do not require any physics to inform the dynamics. However, in the case of ionospheric profiles, we find a wide range of oscillations, including some far above the diurnal frequency. Therefore, we propose nontrivial extensions to DMD using multiresolution analysis (MRA) via wavelet decompositions. We call this method the Scale-Separated Dynamic Mode Decomposition (SSDMD) since the MRA isolates fluctuations at different scales within the time series into separated components. We show that this method provides a stable reconstruction of the mean plasma density and can be used to predict the state of the vertical profile at future time steps. We demonstrate the SSDMD method on data sets covering periods of high and low solar activity.

1 Introduction

The need for accurate modeling and forecasting of the prevailing space weather conditions continues to play a critical role in the development and operation of a variety of radio communications and radar applications. The Earth’s ionosphere is of particular interest as it provides a medium for the propagation of radio waves far beyond the horizon [1, 2, 3]. As a result, the ionosphere has been the subject of intense study for decades, and efforts to enhance our ability to model and predict the vertical plasma density profile continue to this day. In general, there are two modeling approaches: physics-based and empirical.

In physics-based models, the equations of fluid mechanics and magnetohydrodynamics are solved. However, the ionosphere is driven by many exogenous systems, including solar and geomagnetic activity, tidal forcing from the lower troposphere [4], and thermospheric general circulation [5]. This means that while the physics are relatively well-understood, careful specification of these drivers is required in order to produce accurate simulations and forecasts. Additionally, even when physics-based models such as the thermosphere-ionosphere-mesosphere-electrodynamics general circulation model (TIME-GCM) [6, 7, 8] and SAMI3 [9, 10] offer accurate modeling capability, they often underestimate the variance observed in the measurements of the ionospheric plasma density [11].

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On the other hand, empirical models, such as the International Reference Ionosphere (IRI), are generally less intensive to run but can require large quantities of data from many different sources in order to account for the complex interactions between the various space weather systems. These sources include measurements from Mass Spectrometer Incoherent Scatter Radar (MSIS) to provide neutral composition derived from years of ground and space-based observations \cite{12}, as well as vertical soundings for the bottomside, GPS-based observations of the total electron content (TEC), and in situ satellite measurements for the relevant ion species composition \cite{13}. Such an undertaking requires decades of dedicated service with international collaboration and has resulted in IRI becoming the official ISO standard for the ionosphere. Nevertheless, IRI provides only statistical estimates of the monthly average plasma density given a number of user-defined inputs such as solar activity via the monthly smoothed sunspot number rather than simulating the dynamics.

More recently, determining reduced-order models (ROM) from data has been explored. In \cite{14}, a quasi-physical dynamic ROM is obtained for the thermospheric mass density using the thermosphere-ionosphere-electrodynamics general circulation model (TIE-GCM) \cite{15}, a precursor to TIME-GCM, as the source of observations. This ROM is based on a modal decomposition technique known as Dynamic Mode Decomposition (DMD) in which a set of spatiotemporal modes are determined via a linear best fit to data snapshots of a dynamical system \cite{16, 17, 18}. DMD has also been shown to be especially useful in a number of physics and engineering contexts, such as in \cite{19} where it was used to help identify structure in weakly turbulent flows. Prior work on adapting DMD to data with dynamics at multiple scales can be found in \cite{20, 21}.

In this work, we present a proof of concept for a novel approach to ionospheric modeling that does not model the ionospheric density globally but instead attempts to model only the local electron density profile (EDP). This approach is motivated by the prevalence of vertical ionospheric sounder stations worldwide, which can generate data streams at regular cadences regarding the height-dependent profile of the ionospheric plasma density. The Global Ionospheric Radio Observatory (GIRO) provides a database of sounder data along with scaled EDP \cite{22}. However, plasma irregularities and traveling ionospheric disturbances manifest as fluctuations in the EDP and occur at multiple scales. In fact, these irregularities are shown to range from the atmospheric scale height, where fluctuations are driven by gravity, down to the ion gyroradius, where fluctuations are driven by Earth’s magnetic field in \cite{23}.

We therefore see that modal analysis and dimensional reduction techniques, which facilitate the identification of simpler features within relatively complex data, would be of great utility in the study and use of ionospheric data. Likewise, measurement driven modeling techniques which bypass the intricate physics modeling that has been necessary to date to develop predictive capabilities would be especially desirable. To this end, we propose nontrivial extensions of DMD by way of a multiresolution analysis (MRA) that separates timescales in a time series of EDPs. We call this method Scale-Separated DMD (SSDMD) and demonstrate its utility in obtaining a dynamic model of the local ionospheric profile from a relatively short recording of data.

This paper will provide the necessary background and algorithmic details to perform SSDMD on a time series of EDPs, and is organized as follows. In Section 2.1 we introduce the concept of Koopman modes and how we may determine dynamics from a nonlinear time series of observations. Section 2.2 then presents the DMD algorithm as a numerical approach to implementing Koopman mode analysis; more in-depth treatments of the DMD method and its many forms can be found in \cite{16, 21, 25}. Section 2.3 provides background on multiresolution analysis and wavelet decompositions and demonstrates how we generate a scale-separated expansion from a time series of data, and Section 2.4 then provides a means of using these expansions with DMD to produce a model for the average dynamics of an electron density profile as well as a means of characterizing the fluctuations about the mean model. Section 2.5 summarizes the complete SSDMD algorithm in pseudocode. Finally, Section 3 gives results from this analysis on measured data from the Lowell DIDBase Digisonde sounder located in Boulder, CO \cite{22}. 

2
2 Methods

2.1 Koopman Mode Analysis

In order to obtain an equation-free model of the ionosphere, we take advantage of the work by Koopman [26] which demonstrates how the equations for a generic nonlinear dynamical system may be rewritten as a linear infinite-dimensional operator acting on measurement functions of the system. This begins by considering a generic dynamical system,

\[ \frac{d}{dt} y(t) = f(y(t)), \quad y(0) = y_0 \in \mathcal{M} \subseteq \mathbb{R}^{N_s}, \]

where \( \mathcal{M} \) is some connected, compact subset of \( \mathbb{R}^{N_s} \) and define an observable, \( g(y(t)) \), such that \( g : \mathcal{M} \rightarrow \mathbb{C} \). Denoting the affiliated flow, \( y(t) = S(t; y) \), we may rewrite the system using the Koopman operator, \( \mathcal{K}^t \),

\[ \mathcal{K}^t g(y) = g(S(t; y)). \]

We see \( \mathcal{K}^t \) is linear since

\[ \mathcal{K}^t(\alpha g_1(y) + \beta g_2(y)) = \alpha g_1(S(t; y)) + \beta g_2(S(t; y)) = \alpha \mathcal{K}^t g_1(y) + \beta \mathcal{K}^t g_2(y). \]

Following [27], we see that with some basic assumptions, i.e. if we choose observables such that they are square-integrable and suppose \( \mathcal{M} \) is invariant with respect to the flow, we have simplified a problem of determining some unknown nonlinear function \( f(y(t)) \) to one of finding an eigendecomposition of the linear operator, \( \mathcal{K}^t \). Moreover, by finding the Koopman eigenfunctions \( \{\phi_j\}_{j=1}^\infty \) and affiliated eigenvalues \( \{\lambda_j\}_{j=1}^\infty \), where

\[ \mathcal{K}^t \phi_j = e^{t\lambda_j} \phi_j, \quad j \in \{1, 2, \ldots\}, \]

then we have a modal decomposition for any other observable, \( g \), so that

\[ g(y) = \sum_{j=1}^{\infty} c_j \phi_j(y), \]

and we can track the evolution of \( g(y) \) along the flow with the formula,

\[ \mathcal{K}^t g(y) = \sum_{j=1}^{\infty} c_j e^{t\lambda_j} \phi_j(y). \]

See [28] and [29] for more in-depth treatments of the Koopman operator and its properties. Now, the challenge of determining the modes and eigenvalues of the infinite-dimensional operator, \( \mathcal{K}^t \), remains. In general, this is impossible to obtain in an analytic way, however, DMD allows us to approximate a finite number of the Koopman modes and eigenvalues by fitting to some appropriately chosen measurements of the system.

2.2 Dynamic Mode Decomposition

A practical approach to applied Koopman analysis is to approximate the action of the Koopman operator via DMD. In its most basic form, DMD starts with a given time series of measurements of the system,

\[ Y = \{y_1, y_2, \ldots, y_{N_T}\}, \]

where \( y_k = y(t_k) \) are snapshots of the system with \( t_k = k\delta t \) for some time step \( \delta t \). From this, we create two matrices,

\[ Y_- = \{y_1, y_2, \ldots, y_{N_T-1}\} \quad \text{and} \quad Y_+ = \{y_2, y_3, \ldots, y_{N_T}\}, \]

and
and look to find a matrix, $K$, such that

$$KY_\perp = Y_+.$$  

(9)

This can be done simply via regression by finding the matrix, $K$, that solves the optimization problem

$$K_o = \text{argmin}_K \|Y_+ - KY_\perp\|_F^2 = Y_+Y_+^\dagger,$$  

(10)

where $\|\cdot\|_F$ denotes the Frobenius norm and $Y_+^\dagger$ denotes the Moore-Penrose inverse of $Y_-$. The DMD model is then given by the eigendecomposition of the matrix $K_o$, however, solving (10) directly can generate highly unstable results due to ill-conditioning in $Y_-$. To address this, it is common in the DMD literature to use the singular-value decomposition (SVD) of $Y_-$ and apply a threshold to keep only the most significant singular values. If the SVD of $Y_-$ is

$$Y_- = U\Sigma V^*,$$  

(11)

then introducing a threshold, $c_{\text{svd}} > 0$, we truncate the columns of $U$ and $V$ corresponding to the singular values, $\Sigma_{jj}$, such that

$$\log_{10}\left(\frac{\Sigma_{jj}}{\Sigma_{11}}\right) > -c_{\text{svd}},$$  

(12)

where $\Sigma_{jj}$ are entries along the diagonal of $\Sigma$ and are ordered such that

$$\Sigma_{11} \geq \Sigma_{22} \geq \cdots \geq \Sigma_{N_\Sigma N_\Sigma}.$$  

(13)

We label the truncated versions of $U$, $\Sigma$, and $V$ as $\tilde{U}$, $\tilde{\Sigma}$, and $\tilde{V}$ respectively. A straightforward approximation of Equation (10) can then be given by

$$K_o \approx Y_+\tilde{V}\tilde{\Sigma}^{-1}\tilde{U}^*.$$  

(14)

However, $K_o$ will be an $N_\Sigma \times N_\Sigma$ matrix, so when $N_\Sigma$ is large it may be computationally expensive to compute the eigendecomposition. When this is the case, we define a modified matrix, $\tilde{K}_o$, using the exact-DMD algorithm [24],

$$\tilde{K}_o \triangleq \tilde{U}^*Y_+\tilde{V}\tilde{\Sigma}^{-1}.$$  

(15)

This is now an $r \times r$ matrix, where $r$ is the number of singular values that satisfy (12), from which we may still compute the DMD modes and eigenvalues of $K_o$ through the diagonalization,

$$\tilde{K}_o = W\Lambda W^{-1}.$$  

(16)

Indeed, as is shown in [24], by defining the DMD modes,

$$\phi \triangleq \frac{1}{\lambda}Y_+\tilde{V}\tilde{\Sigma}^{-1}w,$$  

(17)

where $w$ is an eigenvector from $W$ and $\lambda \neq 0$ from the diagonal entries of $\Lambda$, we find that

$$\tilde{K}_o w = \lambda w \implies K_o \phi = \lambda \phi.$$  

(18)

That is to say, each eigenvalue/DMD mode pair, $(\lambda, \phi)$, corresponds to an eigenvalue/eigenvector pair of $K_o$. Additionally, it can be shown that $\Lambda$ will in fact contain all nonzero eigenvalues of $K_o$.

Finally, given the time scale from the snapshots of the system, $\delta t$, representing the amount of time which has passed from observation $y_k$ to $y_{k+1}$, we construct a continuous time model of the system,

$$y(t) \approx \Phi \Lambda^{t/\delta t} \Phi^\dagger y(0),$$  

(19)
where $\Phi$ is a matrix whose columns are the DMD modes, $\phi$, and $y(0)$ is an initial condition. This also provides a time stepping mechanism for reconstructing our time series as well as forecasting future states.

The connections between DMD and the Koopman operator are explored further in [17, 18]. However, we point out that the Koopman operator is most naturally formulated with respect to Lagrangian data, while in this work we focus on analyzing Eulerian data, that is to say, we assume the $y_j$ observations in our data stream are measurements of the EDP at fixed positions in altitude. Were one to develop effective Euler-to-Lagrangian maps for the data sets studied herein, this would open up a wider range of tools related to the DMD method. This is a subject for future research.

Thus, beyond just providing a modal decomposition, the DMD method has the potential to, using data alone, provide a time-evolving model for said data. While a generally successful approach, this straightforward implementation is known to struggle with multiscale data or any data that has both very small and very large gradients from snapshot to snapshot. This motivates the inclusion of an MRA.

2.3 Multiresolution Analysis

The primary contribution of this paper is to provide a method of adapting the DMD algorithm to work on data with fluctuations at multiple scales, as is the case when modeling EDP measurements. To do this, we make use of MRA by way of wavelet decompositions [30]. Discrete wavelet analysis begins with a scaling function $\phi$, such that relative to some time scale $\Delta t$, the set of functions

$$\{\tau_n \phi\}_{n=-\infty}^{\infty},$$

forms an orthonormal set, where

$$\tau_n \phi(t) = \phi(t - n \Delta t).$$

We define the subspace $V_0 \subset H$ so that

$$V_0 = \text{Span} \{\{\tau_n \phi\}_{n=-\infty}^{\infty}\}.$$  

By introducing the corresponding wavelet function $\psi$, one is able to separate the space $V_0$ such that

$$V_0 = V_1 \oplus W_1,$$

where

$$V_1 = \text{Span} \left\{ \sqrt{2} \{\tau_{2n} \phi\}_{n=-\infty}^{\infty} \right\}, \quad W_1 = \text{Span} \left\{ \sqrt{2} \{\tau_{2n} \psi\}_{n=-\infty}^{\infty} \right\},$$

so that $V_1$ represents the parts of functions in $V_0$ which vary on double the length scale, while $W_1$ represents the details of the functions in $V_0$. In turn, one can then look at longer scales by separating $V_1 = V_2 \oplus W_2$ and so forth. This collection of separated spaces represents an MRA; for more in-depth treatments, see [31, 30].

Thus, for a given $y(t)$, we can decompose it across timescales consisting of $N_{lvl}$ levels, such that

$$y(t) \approx \sum_{j=1}^{N_{lvl}+1} d_j(t),$$

where $d_j(t) \in \mathbb{R}^{N_s}$, such that

$$d_j(t) = \sum_{n=-M_f}^{M_f} d_{j,n} \psi_{j,n}(t), 1 \leq j \leq N_{lvl},$$
and
\[ d_{N_{lvl}+1}(t) = \sum_{n=-M_f}^{M_f} d_{N_{lvl}+1,n} \phi_{N_{lvl},n}(t), \] (27)
with
\[ \psi_{j,n}(t) = 2^{-j/2} \tau_n \psi(2^{-j} t), \phi_{N_{lvl},n}(t) = 2^{-N_{lvl}/2} \tau_n \psi(2^{-N_{lvl}} t). \] (28)
The vectors \( d_{j,n} \), \( 1 \leq j \leq N_{lvl} \), denote the detail coefficients at the \( j^{th} \) scale while \( d_{N_{lvl}+1,n} \) denotes the approximation coefficients at the terminal scale \( N_{lvl} \). Note that the wavelet decompositions are performed in the temporal dimension and occur independently at each height in the profile, thus the vector quantities, \( d_j(t) \), represent only parts of the signal at the \( j^{th} \) scale. Given our discrete time series from Equation (7), these vector quantities form the columns of a new set of data matrices,
\[ Y_j = \left\{ d_j^{(1)} d_j^{(2)} \cdots d_j^{(N_{lvl})} \right\}, \] (29)
which are reconstructions of the original data at each scale and sum coherently, so that \( Y = \sum_{j=1}^{N_{lvl}+1} Y_j \).

2.3.1 Correlations Across Scales

Taking our data set to be given by the temporally evolving vector quantities, \( y(t) \), we produce a corresponding set of functions \( d_j(t) \). To compare these different timescales to one another, and also to take into account the role that we want the matrix \( \tilde{K}_o \) to play in advancing the data forward in time, we compute the following correlation matrix \( C \) whose entries are given by
\[ C_{jl} = \left[ \tilde{d}_j^T(\cdot) \odot \tilde{d}_j^T(\cdot) + \tilde{d}_l^T(\cdot) \odot \tilde{d}_l^T(\cdot) \right], \] (30)
with, \( j, l \in 1, \ldots, N_{lvl} \), and
\[ \tilde{d}_j(t) = \frac{d_j(t) - \bar{d}_j}{||d_j(\cdot) - \bar{d}_j||_{2,t}}, \] (31)
where \( \tau \) and \([\cdot] \) are temporal and spatial averaging respectively, \( ||\cdot||_{2,t} \) is an \( L_2 \)-norm over time, and \( \odot \) represents the Hadamard product between two matrices. Finally, the + and − superscripts denote whether we are looking a time-step forward or not.

Thus, we now have a quantitative means for comparing our time series across different timescales. By setting a threshold value, say \( c_{corr} \), we can then generate a matrix \( A^{tr} \) where
\[ A_{jl}^{tr} = \begin{cases} 1, & |C_{jl}| \geq c_{corr} \\ 0, & |C_{jl}| < c_{corr} \end{cases} \] (32)
The matrix \( C \) is symmetric, and thus \( A^{tr} \) is as well. Note, in practice these correlations will typically be larger for the longer time scales since we are looking at one-step correlations, with higher frequency oscillations becoming increasingly less correlated. We then generate a graph, \( G^{tr} \), defined by the adjacency matrix, \( A^{tr} \). Based on the connectivity of the graph, we are able to regroup timescales which are most correlated to one another and perform DMD separately on each group.

Thus, for a given choice of threshold \( c_{corr} \), we will have \( N_C \leq N_{lvl}+1 \) connected components within \( G^{tr} \). We then form \( N_C \) time series by summing only the \( Y_j \) which belong to the same connectect so that
\[ Y_n^C = \sum_{j \in G^{tr}_n} Y_j, \] (33)
where \( j \in G^{tr}_n \) denotes the scales that are in the \( n^{th} \) connected component in \( G^{tr} \), and \( Y_n^C \) is the time series for the \( n^{th} \) connected component. Then, finding a corresponding \( \tilde{K}_{o,n} \) via DMD, we
generate an affiliated expansion for each group of correlated scales so that the total time series can be approximated by

$$y(t) \approx \sum_{n=1}^{N_C} \Phi_n \Lambda_n^{t/\Delta t} \Phi_n^\dagger y_n(0).$$

### 2.4 A Dynamical Model for the Average

Having separated the original time series across timescales via MRA with some appropriately chosen correlation threshold \(c_{\text{corr}}\), we now have a collection of time series,

$$\{Y_C^1, Y_C^2, \ldots, Y_C^{N_C}\},$$

that represent scales within the data set whose one-step correlations are relatively weak. We treat these as being essentially independent with respect to our DMD approximation to the Koopman operator.

For the ionospheric profile data we model in this paper, we note that, using observations that span only several weeks in time and are restricted to a single geographic location, the one-dimensional EDP is essentially memoryless after twenty-four hours have passed \[32\]. This strongly suggests that before naively applying the DMD method to time series of arbitrary length, instead we should first average the data across 24 hour cycles for the duration of our measurement period. Denoting the number of time steps in a full day as \(T_D\) and assuming that \(N_T + 1\) is divisible by \(T_D\), so that the data set represents the number of days \(N_D\) where

$$N_D = \frac{N_T + 1}{T_D},$$

we separate the mean signal over 24-hour cycles from the fluctuations about the mean for each time series, \(Y_C^n\). This creates two new affiliated time series for each connected component that have the properties,

$$\bar{Y}_C^n(t_k + T_D) = Y_C^n(t_k),$$

and

$$\sum_{k=1}^{T_D} \bar{y}_C^n(t_k + mT_D) = 0, \quad m = 0, \ldots, N_D - 1,$$

where \(\bar{\cdot}\) and \(\hat{\cdot}\) denote the 24-hour mean signal and fluctuations about the 24-hour mean, respectively.

So, we have further separated each connected component from our MRA into their mean signal, \(Y_C^n\), and affiliated fluctuations around said means, \(Y_C^n\). From the fluctuations, we find estimates of the standard deviations at each time of day for each group of correlated scales. These are given by the time series \(s_C^n(t_k)\), where

$$s_C^n(t_k) = \left( \frac{1}{N_D - 1} \sum_{m=0}^{N_D-1} \left( \hat{y}_C^n(t_k + mT_D) \odot \hat{y}_C^n(t_k + mT_D) \right) \right)^{1/2}.$$

However, we do not use these data in generating the SSDMD model, rather, they simply provide insight into the spatial and temporal distribution of the fluctuations at each scale over the measurement time period.

Thus, for each \(Y_C^n\), we have now generated two affiliated time series,

$$\bar{Y}_C^n = \{\bar{y}_C^n(t_k)\}_{t_k=1}^{T_D}, \quad s_C^n = \{s_C^n(t_k)\}_{t_k=1}^{T_D}.$$

Finally, using Equation (19) on the 24-hour averaged data, we generate a DMD model for each connected component so that

$$\bar{y}_C^n(t) \approx \Phi_n \Lambda_n^{t/\Delta t} \Phi_n^\dagger \bar{y}_C^n(0).$$
Since the MRA allows us to sum all $N_C$ components coherently, we then have the SSDMD model,

$$
\bar{y}(t) \approx \sum_{n=1}^{N_C} \Phi_n A_n^{t/\Delta t} \Phi_n^\dagger \bar{y}_n^C(0).
$$

Equation (42) is a model for the dynamics of the average that accounts for nonlinear oscillations at multiple scales while preserving strong couplings between scales. Furthermore, by separating the mean signal from the fluctuations within each correlated collection from the MRA, the model denoises the data without erroneously removing oscillations from the original signal that may initially appear as noise. We found that the fluctuations of the 24-hour mean themselves offered little in the way of modal decompositions or dynamics. As expected, such noisy signals reliably produce DMD eigenvalues that have such large complex magnitudes that they induce rapid growth of the modes and are highly unstable. This makes sense intuitively since the DMD method fits a linear model to the one-step transition from each state vector, or column in $Y$, to the next. Thus large amplitude noise from observation to observation can cause the DMD method to be highly unstable.

2.5 Algorithm

The complete SSDMD method is summarized in Algorithm 1. We assume familiarity with standard numerical methods for computing the reduced Singular Value Decomposition (SVD), eigenvalue decomposition, solving an initial value problem, and computing 1-dimensional wavelet decompositions. When computing the mean profiles over 24-cycles, use Equation (38). The algorithm returns the reconstructed time series of the input data along with the DMD eigenvalues, modes, and eigenfunctions.

**Algorithm 1: SSDMD**

**Data:** $Y \in \mathbb{R}^{N_S \times N_T}$ such that each column, $y_i \in \mathbb{R}^{N_S}$, is an observation of the system $\delta t$ time from $y_{i-1}$.

**Result:** $\hat{Y}, W, \Lambda, \Phi$

Initialize: set DMD threshold $c_{dmd} > 0$, and correlation threshold $c_{corr} > 0$.

```
begin
    $\bar{Y} \leftarrow \text{discreteWaveletDecomposition}(Y)$
    $\bar{Y}_C, N_C \leftarrow \text{correlatedConnectedComponents}(\bar{Y}, c_{tr})$
    for $n=1 \ldots N_C$ do
        $\bar{Y}_C \leftarrow \text{meanDailyCycles}(Y_C^n)$
        $\bar{Y}_{n, -} \leftarrow [\bar{y}_1^{C, n}, \bar{y}_2^{C, n}, \ldots, \bar{y}_{n-1}^{C, n}]$
        $\bar{Y}_{n, +} \leftarrow [\bar{y}_n^{C, n}, \bar{y}_{n+1}^{C, n}, \ldots, \bar{y}_{n+m}^{C, n}]$
        $U, \Sigma, V^\dagger \leftarrow \text{reducedSVD}(Y_{n, -}, C_{dmd})$
        $K \leftarrow Y_{n, +} V \Sigma^{-1} U^\dagger$
        $W_n, \Lambda_n \leftarrow \text{eigenvalueDecomposition}(K)$
        $\Phi_n \leftarrow \text{solveIVP}(W_n, Y_{n, -}^C)$
        $\hat{Y}_n \leftarrow W_n \Lambda_n \Phi_n$
    $\hat{Y} \leftarrow \sum_{n=1}^{N_C} \hat{Y}_n$
    $W \leftarrow [W_1, W_2, \ldots, W_n]$
    $\Lambda \leftarrow [\Lambda_1, \Lambda_2, \ldots, \Lambda_n]$
    $\Phi \leftarrow [\Phi_1, \Phi_2, \ldots, \Phi_n]$
end
```

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### Results

#### 3.1 Data Description

We demonstrate the SSDMD algorithm on two data sets. Both are time series of ionospheric EDPs taken from the Didbase digital ionogram database for the Digisonde sounding system located in Boulder, CO and operated by the USAF [22]. This sounder generates estimates of the vertical EDP created using the ARTIST 5 algorithm to invert the raw ionogram [33] and provides reasonably accurate measurements of the bottomside plasma density. All results presented are in terms of plasma frequency in MHz rather than the electron number density. Dataset 1 consists of a 25 day window covering October 15, 2013 to November 9, 2013. The sounder had a measurement cadence of 15 minutes for a total of 2,400 observations. Also, this year was a period of solar maximum, with a smoothed sunspot number of 107 during the measurement window. We use the first 20 days to fit the SSDMD model and the remaining 5 to test. In Dataset 2, we use a 13 day window covering February 01, 2018 to February 13, 2018. The sounder had a measurement cadence of 5 minutes for a total of 3,744 observations. This period was during solar minimum, with a smoothed sunspot number of 12.6. Here we fit using the first 10 days and save the remaining 3 days for testing. See Table 1 for a summary of the parameters used for the SSDMD model for each data set.

In Section 3.3 we compare SSDMD to IRI output for both data sets. We used the IRI 2016 Python model with up-to-date solar, ionospheric, and magnetic indices that cover both periods of measurement used in this study. All switches in the IRI model were set to true except for the following: \{4, 5, 6, 22, 23, 28, 29, 30, 33, 34, 35\}. We generated profile time series for each data set at the Boulder, CO location and interpolated to the same grid as the sounder data.

Figure 1 shows Dataset 1 and Dataset 2 as profilograms in panels (a) and (b), respectively. We preprocess the raw profiles by interpolating each profile to a regular 1km resolution height grid and then clipping the profiles at 350km. This is done primarily because the sounder only provides accurate measurements of the bottomside structure. We then fill missing profiles using standard imputation. In Dataset 1, 12 of the 2,400 observations were missing, and in Dataset 2, 86 of the 3,744 observations were missing. Because there were relatively few missing values for both data sets, they were kept in the final estimates of error as they had a negligible impact. However, a large number of missing profiles relative to the size of the data set will likely affect the learned dynamics in the model. Both data sets are then centered on their respective means prior to training the models; however, the means are added back for the final reconstructions.

Figure 1 illustrates the time series for Dataset 1 and Dataset 2. In both cases, we observe high-frequency features interspersed over the much longer time scale of the diurnal cycle. The need to account for these oscillations at multiple scales is further motivated by the Hilbert spectrum of a slice through Dataset 1 at a vertical height of 330km. At this altitude, we see there is a significant degree of instantaneous energy at frequencies much higher than diurnal variation (1 cycle/day); see Figure 2. These relatively high-frequency transient events complicate an
Figure 1: (a) Dataset 1, a profilogram from the Digisonde Boulder, CO station covering the days of October 15, 2013 to November 09, 2013; profiles were measured every 15 minutes. This period of observation occurred near the last solar maximum. (b) Dataset 2, a profilogram from the Digisonde Boulder, CO station covering the days of February 01, 2018 to February 13, 2018; profiles were measured every 5 minutes. This period of observation occurred near the last solar minimum.

attempt at a modal decomposition and especially deter any direct application of DMD. This is because the standard DMD method finds a linear best fit operator that takes each profile in the time series one step into the future and is the chief motivation for the SSDMD algorithm.

3.2 SSDMD Method

We now illustrate each step of the process in generating an SSDMD model using Dataset 1. Following Algorithm 1 we decompose the data into several time series, each representing a single scale from the wavelet decomposition. In Figure 3 we see that Dataset 1 is expanded into 11 individual scales. Fourth-order Coiflets are used for the discrete wavelet transforms in the MRA. The type of wavelet used here is a hyperparameter, however, and the optimal choice may vary for different data sets.

We then compute correlations across the scales using Equation (30) and generate the correlation matrix and graph. For Dataset 1, we use a correlation threshold $c_{tr} = 10^{-1.95}$ and find there are three groups formed among the scales, Figure 4. This correlation threshold is yet another hyperparameter that requires manual tuning to achieve the best results. Determining the optimal setting for this parameter will be a topic of future research, though its value here was chosen such that the average pointwise reconstruction error was minimized. The graph $G^{tr}$ identifies the correlated scales using the affiliated correlation matrix, which are grouped back together through summation to get Figure 5. Note that the first group consists of the bulk of
Figure 2: The affiliated Hilbert spectrum for Dataset 1 of a slice through a height of 330km. The Hilbert spectrum plot reveals the instantaneous energy at different frequencies as time progresses. The stable diurnal cycle can be seen near 1 cycle/day while various time localized, spurious oscillations occur throughout at frequencies that are an order of magnitude higher.

the large scale features in the time series while the last two scales remain on their own. However, this will not always be the case, and subgroups within the higher frequency components can arise depending on the data observed.

Finally, averages over 24-hour lags are computed and the corresponding fluctuations about each mean, see Figure 6. We now have time series for the average profiles for the three correlated scale groups at each time of day. A DMD model is fit for each of these, and thus, we are able to sum the output of each coherently to generate a forecast of the future state of the profile, see Equation 42. While the fluctuations about the averages are currently not used in fitting the DMD models, they do offer qualitative information on the time-space distribution and magnitude of the noise in the data set. In this sense, then, by averaging only once we have separated the scales, our method allows one to filter a signal in a manner that is most consistent with the one-step fit for the $\tilde{K}_o$ linear operator without potentially losing information through standard denoising or filtering methods.

### 3.3 SSDMD Model Performance

The SSDMD model provides eigenvalues and DMD modes from which we may reconstruct the training data and generate a forecast for the test data. The reconstruction of the training data and forecasts over the test period for both data sets are shown in Figure 7, while Figures 8 and 9 show horizontal slices through the measured and modeled data at several heights for Datasets 1 and 2, respectively. In these figures, we get a qualitative sense of SSDMD’s ability to model the oscillatory patterns at different heights in the profile. The areas where SSDMD struggles to capture the true plasma frequency dynamics are seen readily in Dataset 2 below 150km during the night hours. However, the plasma density in these regions provided by the ARTIST5 inversion algorithm is relatively low, below 0.5 MHz, and is therefore outside the raw measurement bandwidth of the Digisonde station at Boulder, CO [34]. Therefore, one should exercise caution when using the model output in these regions, and we see that SSDMD erroneously predicts negative values at certain times. However, this is to be somewhat expected because the plasma frequency does indeed become confined to a low and narrow frequency range in the E-region at night, see bottom three panels in Figure 9. However, this overshooting of the model gets damped away as the modes evolve in time. Rather interestingly, we do not see such
drastic oscillations in Dataset 1 at similar altitudes. This effect is largely due to Dataset 2 having been measured at a higher cadence, and therefore containing far more uncorrelated components at the fastest scales, see Figure 10, which happen to also have relatively large magnitudes. As the time-resolution of sounder measurements increases, a wider spectrum of geophysical noise will be observed and thus additional data-processing techniques may be required prior to applying DMD.

We quantify the performance of the SSDMD models in Figures 11 and 12, which give the mean absolute percentage error (MAPE) for each data set. The mean is taken across all profiles at a given time of day, estimating the model error as a function of both time of day and height. From these figures, we see immediately how the SSDMD method can provide a reasonably accurate forecast for certain heights of the ionosphere given comparatively short periods of measurement, with the error remaining within a few tens of percent throughout the test period. However, note that the scale for Dataset 2 is increased compared to Dataset 1, and the test forecast MAPE is higher throughout the daytime hours. There is also a larger discrepancy between the train and test periods. It seems the relatively low plasma frequency following the solar minimum and higher measurement cadence can complicate even SSDMD since finding stable oscillatory behavior at any scale within such noisy measurements becomes a challenge.

We find that both data sets’ forecast model error is highest during the day-night and night-day transition. To understand why this occurs, we look at the SSDMD plasma frequency forecast for Dataset 1 at the height of 150km and compare it with that of IRI, Figure 13. Here we see that because the day-night transition can cause a very rapid jump in plasma frequency
Figure 4: (a) The correlation coefficient matrix $C$ and (b) the corresponding graph $G^{tr}$, computed using the threshold $c_{tr} = 10^{-1.95}$ for Dataset 1.

Figure 5: Dataset 1 decomposed into 3 different scale groups. Each reconstruction captures features of the data with correlated components according to the one-step spatiotemporal correlation.

At certain altitudes, even a small phase error between the model and measurement can result in substantial errors in the forecast.

Despite the difficulties during the transition, Figure 13 also shows that SSDMD produces a model with less bias than what is seen with IRI. This result is even more pronounced in Figure 14, in which we compare the mean absolute percentage error over the entire profile as a time series. Furthermore, Figure 15 provides a histogram of the pointwise error of SSDMD and IRI for each of the data set train and test periods. These results indicate a modest improvement over IRI using comparatively small computational and training data overhead.

Finally, the time-averaged error of the model at each height step is presented in Figure 16. Here we include a simple daily-median model, in which the profiles at each time of day are averaged over all days in the training data and simply repeated for each day in the forecast. SSDMD performs better than both IRI and the daily-median model in the F-region for both Dataset 1 and 2. However, we see again that SSDMD may not be the ideal method for modeling the E-region dynamics using sounder data.
Figure 6: Dataset 1 decomposed into (a) averages over 24-hour cycles of the correlated wavelet groups, $\bar{Y}_C^n$, and (b) fluctuations around the average over 24-hour cycles of the correlated wavelet groups, $S_C^n$. The data matrices in panel (a) are used to generate the SSDMD model.

4 Conclusions and Future Directions

We present extensions to the standard DMD algorithm that account for oscillations at multiple scales within measured data. By including a wavelet decomposition along each spatial dimension, we separate the various scales within the data that may otherwise appear as noise and often preclude a standard DMD approach. Subsequent correlation analysis across the time scales shows how we may recombine specific scales to preserve strong couplings between them in their one-step correlation. We call these correlated groups the connected components. In the case of modeling vertical profiles of the ionospheric plasma frequency, we perform an additional noise removal step for each connected component by averaging across 24-hour cycles. Preconditioning the DMD step by isolating the connected components alleviates the problem of having large single-step gradients in the measurement data that often prevent DMD from finding any stable modes. Performing DMD separately on each connected component produces a set of eigenvalues, modes, and eigenfunctions that can be combined coherently to form the SSDMD model.

The SSDMD algorithm is computationally efficient compared to physics-based models such as TIME-GCM or SAMI3, generating a model and simulating a three- to five-day forecast on the order of seconds. Additionally, SSDMD requires far less data to generate and update than empirical models like IRI. SSDMD is therefore lightweight enough to be updated in near-real-
time as additional data is obtained. Even with limited observations, such as a single vertical ionosonde, SSDMD can produce reasonable forecasts of the average profile dynamics in the mid-latitudes and can be used during periods of solar maximum and solar minimum with minimal tuning of hyperparameters.

The method is not without its limitations, however, and we see that even with wavelet filtering, the sharp transitions in plasma density during the day-night and night-day transition in the E-region can cause significant forecast error. A direction for future research will be to address the fact that the model does not account for any driving forces such as solar activity, tidal forcing, or geomagnetic activity. As such, model forecast accuracy is highly dependent on the average density at each time of day during the measurement period being highly correlated with the forecast period. Extending SSDMD further to incorporate external forcing is the topic of future development and, combined with longer measurements series, should allow for a significant increase in forecast accuracy.

The method was developed for one-dimensional observations of the ionosphere at a mid-latitude location. In future work, two- and three-dimensional data may be used, however, such spatially varied data sets will likely have to include modeled or assimilated data to fill in gaps between sounder locations and therefore detract somewhat from the purely data-driven approach of SSDMD. Finally, data spanning longer time periods may even be used to extract seasonal and solar cycle dynamics. Thus, the method of SSDMD is not inherently limited to ionospheric prediction and should be adaptable to many space weather domain that involves multiscale phenomena.

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Figure 7: SSDMD reconstruction of (a) Dataset 1 and (b) Dataset 2. The vertical dotted black line denotes the transition from training data to test data.
Figure 8: SSDMD model of the average plasma frequency for Dataset 1 at 6 different height slices. The modeled result (blue dotted line) is generated via the DMD modes and eigenvalues computed using SSDMD. The vertical black dotted line represents the transition from training data to test data.
Figure 9: SSDMD model of the average plasma frequency for Dataset 2 at 6 different height slices. The modeled result (blue dotted line) is generated via the DMD modes and eigenvalues computed using SSDMD. The vertical black dotted line represents the transition from training data to test data.
Figure 10: (a) The correlation coefficient matrix $\mathbf{C}$ and (b) the corresponding graph $G^{tr}$, computed using the threshold $c_{tr} = 10^{-1.5}$ for Dataset 2.
Figure 11: The mean absolute percentage error at each time of day over the train (a) and test (b) periods for Dataset 1.
Figure 12: The mean absolute percentage error at each time of day over the train (a) and test (b) periods for Dataset 2.
Figure 13: Forecast of the plasma frequency at 150km from SSDMD (blue line) compared against measurement (red dots) and IRI (black line with triangles).

Figure 14: Mean absolute percentage error comparison with IRI for Dataset 1 (a) and Dataset 2 (b). The mean error over each profile is computed and shown for the entire time series, with the vertical black dotted line denoting the transition from training to test data.
Figure 15: Histograms of the pointwise errors over the training (a) and test (b) sets for Dataset 1 using IRI (red) and SSDMD (blue). Histograms of the pointwise errors over the training (c) and test (d) sets for Dataset 2 using IRI (red) and SSDMD (blue).
Figure 16: Comparison of SSDMD with the IRI model and the daily mean profile for each time of day. The error is given as the absolute error and averaged over time steps for each height. Panels (a) and (b) show results for training and test sets for Dataset 1 and panels (c) and (d) give respective results for Dataset 2.
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