About a Super-Ashtekar-Renteln Ansatz

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Abstract

The Ashtekar-Renteln Ansatz gives the self-dual solutions to the Einstein equation. A direct generalization of the Ashtekar-Renteln Ansatz to N=1 supergravity is given both in the canonical and in the covariant formulation and a geometrical property of the solutions is pointed out. 

1 Introduction

The Ashtekar-Renteln Ansatz is a simple solution to all the constraints of gravitation in the Hamiltonian formulation using Ashtekar’s variables for the case of non-vanishing cosmological constant [1]. This Ansatz gives the solutions of the Einstein equation corresponding to a vanishing Weyl tensor. The Hamiltonian formulation of gravity using Ashtekar’s variables uses the following constraint system: The Hamilton constraint that ”pushes” the spacelike
hypersurface forward in time:

\[ \mathcal{H} = \frac{i}{2} f^{ij} E_i^a E_j^b F_{ab} + \frac{\lambda}{6} f^{ijk}\epsilon_{abc} E_i^a E_j^b E_k^c \approx 0 \]  

(1)

where \( F_{ab} \) is the field strength of the configuration space variable \( A_a^i \); \( E_i^a \) is the canonically conjugated momentum and \( \lambda \) is the cosmological constant. 

\( a,b,c,... \) are the space indices; \( i,j,k,... \) are the internal indices. The vector constraints that generate the spacelike diffeomorphisms look like:

\[ \mathcal{H}_a = E_i^b F_{ab}^i \approx 0 \]  

(2)

Finally the constraints called Gauss’ law are:

\[ G_i = D_a E_i^a \approx 0 \]  

(3)

The Ashtekar-Renteln Ansatz gives a simple relation between the ‘electric’ field \( E_i^a \) and the ‘magnetic’ field \( B_i^a \) obtained from the field strength \( F_{ab}^i \); a relation that solves all the constraints:

\[ E_i^a = -\frac{3i}{\lambda} B_i^a \]  

(4)

As we see it is crucial to have a non-vanishing cosmological constant. The field configuration obtained by using the Ashtekar-Renteln Ansatz has self-dual field strength and it contains some geometrical information as it will become clear later.

There is a relation equivalent to the Ashtekar-Renteln Ansatz related to the CDJ-action [10]. This pure connection action is built up from tensors of the form:

\[ \Omega_{ij} = \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta i} F_{\gamma\delta j} \]  

(5)

\( \alpha, \beta... \) being space-time indices. The Ashtekar-Renteln Ansatz is equivalent to demanding that:

\[ \Omega_{ij} \propto \delta_{ij} \]  

(6)

The equivalence can be seen from the definition of the conjugated momenta for \( A_a^i \).

The goal of the present paper is to generalize these results to \( N = 1 \) supergravity which was formulated in terms of Ashtekar’s variables in [1]. For some more results see also [3], [5] and [6]. The structure of this paper is
the following: in Section 2 after introducing the notation a generalization of the Ashtekar-Renteln Ansatz is given. Then a generalization of the $\Omega$ matrix is obtained together with some conditions (generalizations of eq. (6)), which when applied to the generalized $\Omega$, lead to the generalized Ashtekar-Renteln Ansatz. This Ansatz will be called the super-Ashtekar-Renteln Ansatz. Afterwards a geometrical interpretation of these results is briefly described, and it is shown that the field strength obtained by the super-Ashtekar-Renteln Ansatz (or equivalently by applying the conditions generalized from eq. (6) on the generalized $\Omega$) is self-dual and for non-degenerate metric it forms a basis for all self-dual two-forms on the manifold. In Section 3 the covariant form of the generalized Ansatz is given together with the covariant form of the above mentioned conditions. Section 4 contains some general comments about the self-duality of the solutions and a wave function of the Chern-Simons type discussed in \[4,5\] and \[6\]. The similarity between the super and the non-super case is stressed all along the paper.

2 The phase space form of the “super-Ashtekar-Renteln Ansatz”

The phase space action of supergravity is:

\[
S = \int \dot{A}_i^a E_i^a - \dot{\Psi}_{iA} \Pi^{A} - \Lambda^i G_i - \mathcal{N} \mathcal{H} - \mathcal{N}^a \mathcal{H}_a - \nu_A S_A^\dagger - \mu_A S^A \tag{7}
\]

where the basic variables are the “old” $A_i^a$ and $E_i^a$ and the Grassmann odd vectorspinor $\Psi_{iA}$ with its conjugate momenta $\Pi^{A}$. One can of course also begin with a Hilbert-Palatini type Lagrangian using the variables $\psi_{iA}$ and their complex conjugate $\bar{\psi}_{iA}$. One can then arrive at the momenta $\Pi^{A}$ as in \[2\]. The point is that one can compute $\bar{\psi}_{iA}$ knowing $\Pi^{A}$ and $\nu_A$.

The conventions used for the Pauli matrices are the following:

\[
[\sigma_i, \sigma_j]_- = \sqrt{2}f_{ijk}\sigma^k \tag{8}
\]

\[
[\sigma_i, \sigma_j]^+_B = \epsilon^i_B \delta_{ij} \tag{9}
\]

In the case when the spinor indices are not explicitly written the following convention is used:

\[
\Psi_{iA} \equiv \Psi_{iA} \sigma_i^A \Pi^{AB}
\]
The constraints are these:
- Gauss’ law:
  \[ G_i = D_a E_i^a + \frac{i}{\sqrt{2}} \Psi_a \sigma_i \Pi^a \approx 0 \]  \hspace{1cm} (10)
- Another constraint is the one usually called the left supersymmetry generator:
  \[ S = D_a \Pi^a - i \alpha E_i^a \sigma_i \Psi_a \]  \hspace{1cm} (11)
- The constraint which in some representation is the complex conjugate of \( S^A \) is \( S^{\dagger A} \) and called the right supersymmetry generator:
  \[ S^{\dagger} = f_{ijk} E_i^a E_j^b \sigma_k [-2 \sqrt{2} D_a \psi_b - i \bar{\alpha} \epsilon_{abc} \Pi^c] \]  \hspace{1cm} (12)
- The Hamilton and the vector constraints have the same role of generators of diffeomorphisms as in the non-super case. They look like:
  \[ H_a = E_i^b F_{ab} - 2 \Pi^b D_{[a} \psi_{b]} - i \alpha \psi_a \sigma_i E_i^b \psi_b \approx 0 \]  \hspace{1cm} (13)
  and
  \[ H = \frac{i}{2} \epsilon_{abc} f^{ijk} E_i^a E_j^b \sigma_k - \frac{1}{2} \alpha \bar{\alpha} \epsilon_{abc} E_i^a E_j^b E_k^c + i 2 \sqrt{2} \Pi^b E_i^a \sigma^i D_{[a} \psi_{b]} \]
  \[ + \bar{\alpha} \epsilon_{abc} \Pi^a E_i^b \sigma^i \Pi^c - \frac{1}{2} \alpha \epsilon_{abc} f^{ijk} E_i^a E_j^b \psi_a \sigma_k \psi_b \approx 0 \]  \hspace{1cm} (14)

Two remarks about this set of constraints might be useful. The first one is that this form of the constraints is equivalent to the form in \([2]\), though the notation is different. This notation was used in \([7]\). The second remark concerns the constants \( \alpha \) and its complex conjugate \( \bar{\alpha} \) which give the cosmological constant: \( \lambda = -\alpha \bar{\alpha} \). The first generalization of super gravity to the case of non-vanishing cosmological constant was given in \([8]\). It turned out that the cosmological constant had to be negative. This means that it can be written as minus the square of a constant, see \([2]\). Since there is no evident reason why this constant (\( \alpha \)) should be real, it seems reasonable to use a more general complex constant. The reality condition then implies that the cosmological constant should have the above mentioned form. The non-super case can be recovered by setting all the Grassmann odd variables equal to zero.
The algebra between Gauss’ law and the left supersymmetry generator looks like:

\[ \{ G_i, G_j \} = if_{ij}^k G_k \]  \hspace{1cm} (15)

\[ \{ G_i, S^A \} = -i \sqrt{2} \sigma_i^A S^B \]  \hspace{1cm} (16)

\[ \{ S^A, S^B \} = -i \alpha \sigma_i^{AB} \mathcal{G}_i \]  \hspace{1cm} (17)

This is evidently a semisimple graded algebra (GSU(2)). For a treatment of these algebras see e.g. [1]. Therefore the configuration space is spanned by a graded-SU(2) connection and it is natural to introduce a notation that reflects this fact. The “super” configuration space variables can be defined as:

\[ A_{\bar{i}}^a = (A_i^a, \psi^A_a) \]  \hspace{1cm} (18)

\[ E_{\bar{i}}^a = (E_i^a, \Pi^a_A) \]  \hspace{1cm} (19)

The barred indices are the supersymmetry indices \( \bar{i} = (i, A) \). The field strength for \( A_{\bar{i}}^a \) is:

\[ F_{\bar{i}}^{\bar{a}} = (F_{\bar{i}}^{\bar{a}}, 2D_{\bar{a} \bar{b}} \psi^A_{\bar{b}}) \]  \hspace{1cm} (20)

where

\[ F_{\bar{i}}^{\bar{a}} = F_{\bar{i}}^{\bar{a}} + i \alpha \psi^A_a \sigma^{i} \psi^A_b \]  \hspace{1cm} (21)

This field strength \( F_{\bar{i}}^{\bar{a}} \) becomes the “usual” field strength when the cosmological constant vanishes. Using it one can write the constraints in a somewhat simpler form.

One can now introduce a “super”-covariant derivative and a “super”-Gauss’ law:

\[ D_{\bar{i}} E_{\bar{a}}^a = \partial_{\bar{i}} E_{\bar{a}}^a + \Lambda_{\bar{i} \bar{j}}^{\bar{k}} A_{\bar{k}}^a E_{\bar{a}}^a \approx 0 \]  \hspace{1cm} (22)

where:

\[ \Lambda_{\bar{i} \bar{j}}^{\bar{k}} = if_{\bar{i} \bar{j}}^{\bar{k}} \]

\[ \Lambda_{\bar{i} A}^B = \frac{i}{\sqrt{2}} \sigma_{\bar{i} A}^B \]

\[ \Lambda_{A \bar{i}}^B = -i \alpha \sigma_{A \bar{i}}^B \]

\[ \Lambda_{A i}^B = -\frac{i}{\sqrt{2}} \sigma_{A i}^B \]
This constraint is of course Gauss’ law for $\bar{i}, j, k$ and the left supersymmetry generator for $\bar{i} = A, B$. Another notation that will be used is:

$$B^a_{\bar{i}} = (B^a_{\bar{i}}, \beta^a_A) \quad (23)$$

where:

$$B^a_{\bar{i}} = \frac{1}{2} \epsilon^{abc} F^i_{bc} \quad (24)$$

$$\beta^a_A = c^{abc} D_{[b} \psi_{c]A} \quad (25)$$

(Note that $F$ used here is the field strength defined by eq. (21).) The Hamilton and the vector constraints now take the form:

$$\mathcal{H} = \frac{i}{2} \epsilon_{abc} f^{ijk} E^a_i E^b_j (B^c_k + i\alpha \bar{\alpha} E^c_k) + \epsilon_{abc} E^a_i \Pi^b \sigma^i (i\sqrt{2}/\beta^c - \bar{\alpha} \Pi^c) \approx 0 \quad (26)$$

$$\mathcal{H}_a = E^b_i F^i_{ab} - 2\Pi^b D_{[a} \psi_{b]} \approx 0 \quad (27)$$

In order to solve the constraints one would like to find the “super-electric” field as a function of the “super-magnetic” field. One can start by trying to find the solutions of the following form:

$$E^a_i = M^{ij} B^a_j \quad (28)$$

where $M^{ij}$ is a 5x5 matrix. Since $B^a_{\bar{i}}$ and $\beta^a_A$ are independent of each other the vector constraints impose some conditions on the matrix $M$. Namely that the 3x3 submatrix $M^{ij}$ is symmetric, the 2x2 submatrix is antisymmetric and the “mixed” 2x3 and 3x2 parts are just the transposed of each other. Writing now equation (28) in the original notation:

$$E^{ai} = M^{(ij)} B^a_j + M^{iA} \beta^a_A \quad (29)$$

$$\Pi^{aA} = M^{Ai} B^a_i + \frac{1}{2} \epsilon^{AB} M^D_B \beta^a_B \quad (30)$$

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A simple Ansatz one can make for this matrix is that the first $3 \times 3$ sub-matrix is diagonal and the “mixed” part vanishes. One obtains then:

$$E_i^a = \frac{i}{\alpha \bar{\alpha}} B_i^a$$

$$\Pi^{aA} = i \sqrt{2} \frac{\epsilon^{abc} D_{[b} \psi_{c]}^A}{\bar{\alpha}}$$

$B$ is defined here as in eq.(24). This is our generalization of the Ashtekar-Renteln Ansatz and it will be referred to as the “super-Ashtekar-Renteln” Ansatz. This Ansatz solves trivially all the constraints as it can be seen using eq.(20). One can express the time-space components of the field strength from the equations of motion by varying the phase space action by the ‘electric’ field:

$$F_{0ai} = N \{ A_{ai}, H \} + N^b \{ A_{ai}, H_b \} + \{ A_{ai}, \nu AS_A^A \}$$

This could be done in the non-super case as well. There one could insert the result obtained from the equation of motion for $F_{0ai}$ in the matrix $\Omega$ defined in equation (5). If one inserts here the form of the ‘electric’ field from the Ashtekar-Renteln Ansatz one obtains equation (6). Or vice-versa: demanding that $\Omega_{ij} \propto \delta_{ij}$ one can obtain a relation between $E$ and the field strength. This relation was exactly the Ashtekar-Renteln Ansatz.

A similar structure can be found in supergravity too. One can define a matrix $\Omega$ as:

$$\Omega_{ij} = \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta i} F_{\alpha\beta j}$$

Inserting the super-Ashtekar-Renteln Ansatz here one can compute $N$:

$$N = \frac{\alpha \bar{\alpha}}{24} |B| \Omega_{ii}$$

$N^a$ can be computed from another matrix. Define $W$ as:

$$W_{ij} = F_{0ai} B^a_j$$

$\Omega$ is in fact 8 times the symmetric part of $W$. $N^a$ from here:

$$N^a = \frac{1}{2} W_{[ij]} \epsilon^{abc} (B^{-1})^i_c (B^{-1})^j_b$$
To complete the solution one has to find $\nu_A$. This is most easily done from the “mixed” part of $\Omega$ ($\Omega_{iA}$) by changing its tangent space index $i$ into spinor indices. One can always find a two-spinor corresponding to a space-time vector:

$$ v_{AA'} = v_I \sigma^I_{AA'} $$

and the Lorentz symmetry of the space-time vector space transforms into the $\text{SL}(2,\mathbb{C})$ symmetry of the spinor space. In our case however one only has space indices and the rotation symmetry $\text{SO}(3)$. This means we only need $\text{SU}(2)$ spinors and the two-spinor corresponding to a space vector is:

$$ v_{AB} = v_i \sigma^i_{AB} $$

where the $\sigma^i$-s are the Pauli matrices. In exactly the same way one can define:

$$ \Omega_{DEA} = \Omega_{iA} \sigma^i_{DE} $$

This $\Omega_{DEA}$ can be decomposed in irreducible parts. Since it is symmetric in the last two indices the decomposition looks like:

$$ \Omega_{DEA} = \Omega_{(DEA)} + \frac{1}{2} \Omega_{D[EA]} + \frac{1}{2} \Omega_{E[DA]} $$

The totally symmetric part vanishes when using the super-Ashtekar-Renteln Ansatz and the decomposition becomes:

$$ \Omega_{DEA} = \frac{1}{3} \epsilon_{AD} \Omega_{EB}^B + \frac{1}{3} \epsilon_{AE} \Omega_{DB}^B $$

and one obtains:

$$ \nu_A = \frac{i \alpha^2 \bar{\alpha}}{4 |B|} \Omega_{AB}^B $$

So the super-Ashtekar-Renteln Ansatz gives us a set of solutions and some constraints on the $\mathbf{W}$ matrix: its symmetric tracefree part vanishes as well as the totally symmetric part of the $\Omega = \Omega_i \sigma^i$ tensor.
Similarly to the non-super case one could start in searching for a solution by demanding that:
\[ \hat{\Omega}_{ij} = 0, \]  
(44)

where \( \hat{\Omega} \) is the traceless part of \( \Omega \)

\[ \Omega_{AB} \propto \epsilon_{AB}, \]  
(45)

\[ \Omega_{i(A} \sigma_{BC)} = 0 \]  
(46)

and solving the equations of motion one obtains for the electric field and the spinor momentum the same form as given by the super-Ashtekar-Renteln Ansatz given above. These three relations can then be understood as a covariant formulation of the super-Ashtekar-Renteln Ansatz.

There is a geometrical interpretation of these results related to the interpretation of the results of the Ashtekar-Renteln Ansatz in the non-super case [16]. There exists a one to one correspondence between the way the duality operator acts on two-forms and the conformal structure on a four dimensional space. Knowing the metric up to a conformal factor is enough to compute how the duality operator acts and vice-versa. This fact is expressed in four dimensions by Urbantke’s formula [17], [18] which gives the conformal structure as a function of a basis of the self-dual vector fields. If the field strength is self-dual and it is non-degenerate it can be used as a basis to all self-dual fields, that is it gives the metric up to a conformal factor by Urbantke’s formula:

\[ g^{\mu \nu} \propto f_{ijk} \epsilon^{i \alpha \beta \gamma} \epsilon^{\delta \nu \rho \sigma} F_{i}^{\alpha} F_{j}^{\beta} F_{k}^{\gamma} \]  
(47)

The field configuration obtained using the Ashtekar-Renteln Ansatz has self-dual field strength. This can be seen by writing the self-dual part of the curvature tensor as a sum of its irreducible components and inserting the solution of the Einstein equation. The field strength is non-degenerate if the determinant of the “magnetic” field does not vanish, as is the case for generic solutions. This means we can use equation (47). The conformal factor can be determined from the action.

One can use similar arguments to understand the nature of solutions in supergravity. Since it is not clear (at least to the author) how to decompose the curvature tensor in this case, it seems now easier to go the other way around and try instead to use Urbantke’s formula and see what kind of a
result it gives. The main question is: does Urbantke’s formula give a tensor that has the structure of the metric in the ADM decomposition? The ADM decomposition expressed in Ashtekar’s variables looks like:

\[
(-g)^{\frac{1}{2}} g^{\alpha\beta} = \begin{pmatrix}
-N^{-1} & N^{-1} N^b \\
N^{-1} N^a & NE_i^a E_i^b - N^{-1} N^a N^b
\end{pmatrix}
\]  

(48)

Inserting the form of \( F_0 \) from the equations of motion (33) into Urbantke’s formula (47) the obtained tensor in general is not of the form eq.(48). The time-time and the time-space components of the matrix depend explicitly on the ‘electric’ field. Using the super-Ashtekar-Renteln Ansatz to express \( E \) in terms of the field strength \( F \) one can again compute the time-time and the time-space components and notice that they are of the same form as in equation (48). This means that Urbantke’s formula gives the conformal structure for those solutions of the constraints that obey the super-Ashtekar-Renteln Ansatz. From here one can conclude that the super-Ashtekar-Renteln Ansatz gives a set of self-dual solutions to the constraint system of N=1 supergravity and when the ‘magnetic’ field is non-degenerate, that is corresponding to a non-degenerate metric, this set of field strengths is a basis for all self-dual vector fields.

To summarize: in contrast to the non-super case the field strength \( F \) does not in general correspond to a self-dual solution, but the field strengths obtained through the super-Ashtekar Ansatz do correspond to self-dual solutions.

3 The covariant form

The Ashtekar-Renteln Ansatz (in the non-super case) can be formulated in a covariant form too [14], [15]. To derive this form one can start from an action:

\[
I = \int \frac{1}{2} \Sigma^{AB} \wedge F_{AB} - \lambda \Sigma^{AB} \wedge \Sigma_{AB} - \Psi_{(ABCD)} \Sigma^{AB} \wedge \Sigma^{CD}
\]

(49)

The fundamental variables are the field-strength two-forms and the two-forms \( \Sigma \) which are obtained as:

\[
\Sigma^{AB} = \epsilon^{[A} \epsilon_{|B|} \Sigma^B
\]

(50)
The last term in the action with the totally symmetric Lagrange multiplier \( \Psi_{(ABCD)} \) is to ensure that \( \Sigma \) is of the form of eq. (50). \( \lambda \) is the cosmological constant. The following Ansatz is the covariant formulation of the Ashtekar-Renteln Ansatz:

\[
\frac{1}{2} F_{\alpha \beta} = \lambda \Sigma_{\alpha \beta}^{AB} \tag{51}
\]

A very similar construction can be made in the super case and one can obtain a generalization of eq.(51). This generalization will be the covariant form of the super-Ashtekar-Renteln Ansatz.

The action one can start with is the one given in [2] and [15]:

\[
I = \int \left[ \frac{i}{2} \Sigma^{AB} \wedge F_{AB} - i \sqrt{2} \chi^A \wedge D \psi_A - \frac{1}{2} \Psi_{(ABCD)} \Sigma^{AB} \wedge \Sigma^{CD} - \Omega_{(ABC)} \Sigma^{AB} \wedge \chi^C \\
+ \frac{\alpha \bar{\alpha}}{2 \sqrt{2}} \Sigma^{AB} \wedge \Sigma_{AB} + \frac{\alpha}{2} \Sigma^{AB} \wedge \psi_A \wedge \psi_B + \frac{\bar{\alpha}}{2} \chi^A \wedge \chi_A \right] \tag{52}
\]

The new variable here is:

\[
\chi^A_{\alpha \beta} = e^{A}_{\alpha B} \bar{\psi}^B_{\beta} \tag{53}
\]

This variable is related to the momenta of the \( \psi_{\alpha A} \) in the canonical formulation. The term with the totally symmetric Lagrange multiplier \( \Omega_{(ABC)} \) is there to ensure that eq.(53) holds. Details about how \( \Psi_{(ABCD)} \) and \( \Omega_{(ABC)} \) work can be found in [15]. \( \alpha \) and \( \bar{\alpha} \) give the cosmological constant as before. It is straightforward to show that this action is invariant under the left super-symmetry transformation (eq. (11)) if one writes the variations in the following form:

\[
\delta_{\epsilon} \Sigma^{AB} = i \chi^{(A} \epsilon^{B)} \tag{54}
\]

\[
\delta_{\epsilon} \psi^A = D \epsilon^A \tag{55}
\]

\[
\delta_{\epsilon} A^{AB} = i \alpha \psi^{(A} \epsilon^{B)} \tag{56}
\]

\[
\delta_{\epsilon} \chi^A = \frac{i \alpha}{\sqrt{2}} \Sigma^{AB} \epsilon_B \tag{57}
\]

(The space-time indices are suppressed.) One can now obtain the equations of motion by varying the action with respect to the different variables. If one varies first with respect to \( A \) one obtains the equation corresponding to the Gauss’ law (eq. (10)). Variation with respect to \( \psi \) gives the left super-symmetry generator (eq. (11)). In one sentence: variation with respect to
the super configuration-space variable $A^\alpha_\alpha$ gives us the "super"-Gauss' law (eq. (22)). The variation with respect to $\Sigma$ gives the following equation:

$$F_{\alpha \beta AB} + 2i \Psi_{(ABCD)} \Sigma^{CD}_{\alpha \beta} + 2i \Omega_{(ABC)} \chi^C_{\alpha \beta} - i\sqrt{2} \alpha \bar{\alpha} \Sigma_{\alpha \beta AB} - i\alpha \psi_{[\alpha A} \psi_{\beta]B} = 0 \quad (58)$$

Varying by $\chi$ one obtains:

$$- i\sqrt{2} D_{[\alpha} \psi_{\beta]}A - \Omega_{(ABC)} \Sigma^{BC}_{\alpha \beta} + \bar{\alpha} \chi_{\alpha \beta A} = 0 \quad (59)$$

Introducing a new notation in the first equation of motion (eq. (58))

$$F^{AB}_{\alpha \beta} = F_{\alpha \beta}^{AB} - i\alpha \psi_{[\alpha A} \psi_{\beta]}$$

one can notice the following: the Ansatz

$$F^{AB}_{\alpha \beta} = i\sqrt{2} \alpha \bar{\alpha} \Sigma_{\alpha \beta AB} \quad (60)$$

and

$$\chi_{\alpha \beta A} = i\sqrt{2} D_{[\alpha} \psi_{\beta]}A \quad (61)$$

is equivalent to demanding that:

$$\Psi_{(ABCD)} = 0 \quad (62)$$

and

$$\Omega_{(ABC)} = 0 \quad (63)$$

Equations (60) and (61) are the covariant formulation of the super-Ashtekar-Renteln Ansatz. In a more elegant way:

$$F = i\sqrt{2} \alpha \bar{\alpha} \Sigma$$

and

$$\chi = i\sqrt{2} \bar{\alpha} D \wedge \psi$$

The other two equations (eq. (52) and (53)) are equivalent to equations (44), (45) and (46). The easiest way to see that this covariant formulation of the super-Ashtekar-Renteln Ansatz is equivalent to the phase-space formulation expressed in equations (34) and (32) is to use the gauge where $e^0_\alpha = 0$. 

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4 Final comments

When quantizing a theory one wants to obtain wave function(s) that satisfy the constraint operators corresponding to the classical constraints. In the connection representation of gravity one needs wave functions depending on the connection: $\Psi[A]$. The variables become operators acting on the wave function: $A^a_i \Psi[A] = A^a_i \Psi[A]$ and $E^i \Psi[A] = \frac{\lambda}{8\pi G} \Psi[A]$. There are two unsolved problems that one meets here: regularization and factor ordering. For a discussion about the possible choices of factor ordering and wave functions corresponding to them see [19]. One can choose the ordering so that the functional derivatives are at the left and the functions of the connection at the right. This is called factor ordering I in [19]. There is a relatively simple wave function that solves the Gauss’ law and the Hamilton constraint in the non-super case. It is the exponential of the Chern-Simons term divided by the cosmological constant. This means that here too it is crucial to have a non-vanishing cosmological constant.

In super gravity we have of course two more fundamental variables that appear as operators in the quantum theory: $\hat{\psi}_a A \Psi = \psi_a A \Psi$ and $\hat{\Pi} ^a A \Psi = \frac{\delta}{\delta \psi_a A} \Psi$. Now a wave function very similar to the one existing in the non-super theory solves quite trivially the Gauss’ law, the left and right super symmetry and the Hamilton constraints if the super-Ashtekar-Renteln Ansatz holds [4],[5],[6]. This is true if the factor ordering is chosen as “factor ordering 1” in [19]. The vector constraint is however not eliminated by this wave function. Since in this ordering the vector constraint operator does not generate space-time diffeomorphisms this does not affect the diffeomorphism invariance of the wave function. This wave function has the following explicit form:

$$S = e^{-\frac{i}{\hbar} \int e^{abc} \{ \partial \alpha \bar{A}_b A_c + \frac{\sqrt{2}}{4\pi G} \bar{A}_a A_b A_c + i \alpha A_a \psi_b \psi_c + \sqrt{2} \bar{\alpha} \psi_a \partial_b \psi_c \} }$$ (64)

As we see there are many aspects of supergravity related to the super-Ashtekar-Renteln Ansatz which are simple generalizations of the non-super case: the form of the Ansatz is quite similar to the non-super Ansatz both in the canonical and the covariant formulation; with a proper choice of the fundamental variables we just have a GSU(2) constraint algebra instead of SU(2); one can also obtain almost similar conditions put on tensors depending on the ‘magnetic’ field, conditions that are equivalent to the Ansatz and one can obtain wave functions that are in principle similar to each other.
(By similar it is meant that the super case is a super-generalization of the non-super one.) What is different however is the self-duality of the different solutions. In pure gravity the field strength was self-dual by construction while in super-gravity the super-Ashtekar-Renteln Ansatz had to be imposed in order to obtain self-dual solutions and this was also reflected in the geometrical interpretation of the Ansatzes. The presence of spinor variables in the super case leads also to a non-vanishing torsion. This can be computed but its form is quite complicated.

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References

[1] A.Ashtekar,P.Renteln, unpublished
[2] T.Jacobson, Class.Quantum Grav.5, 923(1988)
[3] H.Nicolai and H.-J.Matschull, J.of Geom.and Phys., vol.11, no 1-4, 15(1993)
[4] H. Kodama, Phys.Rev.D, vol42, No 8, 2548(1990)
[5] Takashi Sano, preprint UT-621,Tokyo, (1992)
[6] Takashi Sano and Jun’ichi Shiaishi, preprint UT-622, Tokyo (1992)
[7] I.Bengtsson, unpublished
[8] P.K.Townsend, Phys.Rev.D, vol15, No 10, 2803(1977)
[9] A.Pais and V.Rittenberg, J.Math.Phys. 16, 10(1975)
[10] R.Capovilla, T.Jacobson and J.Dell, Class.Quantum Grav.8, 59(1991)
[11] R.Capovilla, T.Jacobson and J.Dell, Class.Quantum Grav. 7,L1(1990)
[12] S.G. Gindikin, Funct.Anal.Appl. 16, 51(1982)
[13] S.G. Gindikin, Sov.J.Nucl.Phys. 36, 313(1982)
[14] J.Samuel, Class.Quantum Grav.5, L123(1988)
[15] R.Capovilla, J.Dell, T.Jacobson and L.Mason, Class.Quantum Grav. 8, 41 (1991)
[16] C.G.Torre, Phys.Rev.D41, 3620(1990)
    S.Koshti and N.Dadich, Class.Quantum Grav. 7, L5(1990)
[17] H.Urbantke, J.Math.Phys. 25, 2321(1984)
[18] G.Harnett, J.Math.Phys. 32, 84(1991)
[19] B.Brügman, R.Gambini and J.Pullin, Nuclear Physics B 385 587(1992)