ABSTRACT HYPERNORMALISATION, AND NORMALISATION-BY-TRACE-EVALUATION FOR GENERATIVE SYSTEMS

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Abstract. Jacobs’ hypernormalisation is a construction on finitely supported discrete probability distributions, obtained by generalising certain patterns occurring in quantitative information theory. In this paper, we generalise Jacobs’ notion in turn, by describing a notion of hypernormalisation in the abstract setting of a symmetric monoidal category endowed with a linear exponential monad—a structure arising in the categorical semantics of linear logic.

We show that Jacobs’ hypernormalisation arises in this fashion from the finitely supported probability measure monad on the category of sets, which can be seen as a linear exponential monad with respect to a non-standard monoidal structure on sets which we term the convex monoidal structure. We give the construction of this monoidal structure in terms of a quantum-algebraic notion known as a tricycloid.

Besides the motivating example, and its natural generalisations to the continuous context, we give a range of other instances of our abstract hypernormalisation, which swap out the side-effect of probabilistic choice for other important side-effects such as non-deterministic choice, logical choice via tests in a Boolean algebra, and input from a stream of values.

Finally, we exploit our framework to describe a normalisation-by-trace-evaluation process for behaviours of various kinds of coalgebraic generative systems, including labelled transition systems, probabilistic generative systems [56], and stream processors [21].

1. Introduction

Hypernormalisation was introduced by Jacobs in [25] in order to provide, among other things, a smooth category-theoretic formulation of certain concepts [36] of quantitative information flow. The main theoretical contribution of this paper is to analyse, in turn, Jacobs’ notion of hypernormalisation, showing how it arises naturally out of well-studied category-theoretic concepts, and how it generalises to other settings. The main application of this paper uses our framework in order to relate bisimilarity and trace equivalence for a range of coalgebraic generative systems. As is well-known, for such systems, bisimulation equivalence is a finer relation than trace equivalence, so that multiple different behaviours (i.e., states

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up to bisimilarity) may have the same underlying trace. We will describe a process of “normalisation-by-trace-evaluation” which normalises any given behaviour to a maximally-efficient one with the same trace.

To motivate our development, it will be useful to take a step back, and first describe the ideas of quantitative information flow which motivated [25]. These ideas are concerned with the following question: given a probabilistic process $P$ which, when run on a private input returns a public output, how can we measure the leakage inherent in $P$, that is, the extent to which knowledge of the output allows an adversary to infer knowledge about the input?

The natural way of answering this question is information-theoretic: we model the input and output types as finite sets $A$ and $B$, and $P$ as a discrete channel, that is, a function assigning to each input in $A$ a discrete probability distribution over possible outputs in $B$. Using this, any prior distribution $\pi$ on $A$ yields a joint distribution on $A \times B$ with associated marginals on $A$ (viz. $\pi$) and $B$; we can now define the leakage of $P$ as the “mutual information”, i.e., the difference between the sum of the marginals’ entropies and the entropy of the joint distribution; said another way, the leakage of $P$ is the difference between the entropy of the prior distribution on $A$, and the expected entropy (over all possible observed outputs) of the posterior distribution on $A$.

Now, if $Q$ is another process of input type $A$—but possibly different output type—then we can compare their drops in entropy to ascertain whether $P$ or $Q$ leaks more information about $A$. **Prima facie**, the answer to this depends not only on a choice of prior $\pi$ for $A$, but also on the flavour of entropy chosen for the calculations—which has to do with questions like: must a successful attack guess the input precisely, or would partial knowledge suffice? A key contribution of [36] is to exhibit a leakage-ordering which is **robust**, in the sense of being largely independent of these choices—and moreover, depends not on the channels $P$ and $Q$ themselves, but only on their so-called **abstract channel** denotations.

According to [36], an **abstract channel** of type $A$ is a function $\mathcal{D}A \to \mathcal{D}(\mathcal{D}A)$ from the set $\mathcal{D}A$ of probability distributions on $A$ to the set of finitely supported probability distributions on $\mathcal{D}A$; said another way, it is a discrete time-homogeneous Markov chain with set of states $\mathcal{D}A$. In the case of the abstract channel $P_r$ associated to a channel $P$ from $A$ to $B$, this Markov chain encodes the probabilities that a given prior distribution will update to a given posterior distribution on account of an observation in $B$. Crucially, however, the identity of these observations is suppressed, which is what makes the abstract channel “abstract”.

Now in [36], the status of probabilistic channels as a kind of monadic computation [40] is clearly acknowledged; indeed, as is well known, the operation $A \mapsto \mathcal{D}A$ underlies the finitely supported discrete distribution monad $\mathcal{D}$ on the category of sets. However, the construction in [36] of an abstract channel $P_r$ from a channel $P$ does not exploit this fact. This is where Jacobs’ [25] enters the picture; one of its objectives is to explain the construction $P \mapsto P_r$ via the calculus of monadic computation, so providing a framework for generalisation beyond the finite discrete case.
In Jacobs’ analysis, the abstract channel \( P \) associated to a discrete channel \( P : A \to DB \) is found as a composite

\[
(1.1) \quad DA \xrightarrow{P} D(A \times B) \xrightarrow{N} D(DA \times B) \xrightarrow{D(\pi_1)} D(DA) ,
\]

whose three terms we now explain. The last map is the action of the monad \( D \) on arrows by pushforward—which in this case takes a joint distribution on \( DA \times B \) to the associated marginal distribution on \( DA \). The first map, by contrast, takes a prior distribution \( \pi \) on \( A \) to the associated joint distribution on \( A \times B \) given by \( a, b \mapsto \pi(a) \cdot (Pa)(b) \). In categorical terms, this \( \tilde{P} \) arises as the composite

\[
(1.2) \quad DA \xrightarrow{D(1,P)} D(A \times DB) \xrightarrow{\text{str}} DD(A \times B) \xrightarrow{\mu} D(A \times B) ,
\]

where \( \mu \) is the monad multiplication, and \( \text{str} \) is its cartesian strength.

The remaining part of (1.1) is the map \( N \), which is Jacobs’ hypernormalisation. It can be described as follows. Given a joint distribution \( \omega \in D(A \times B) \), we have for each \( b \in B \) the marginal probability \( \omega(b) = \sum_a \omega(a,b) \); and when this is non-zero, we have also the conditional distribution \( \omega_{A\mid b} \) on \( A \) with \( \omega_{A\mid b}(a) = \omega(a,b)/\omega(b) \). Now \( N(\omega) \in D(DA \times B) \) is the distribution which takes the value \( (\omega_{A\mid b}, b) \) with probability \( \omega(b) \), for all \( b \in B \) with \( \omega(b) > 0 \) (note that it is important for this that \( B \) is a finite set). In particular, \( N \) encodes the process of normalising a non-zero sub-probability distribution \( \omega(\cdot,b) \) on \( A \) to the probability distribution \( \omega_{A\mid b} \)—while avoiding the impossibility of normalising \( \omega(\cdot,b) \) when it is everywhere-zero. This explains the name “hypernormalisation” chosen for this map.

In [25], Jacobs introduces hypernormalisation by an element-based definition, and verifies by hand a number of desirable equational properties—the implication being that, to generalise away from the finite discrete setting, it would suffice to define a corresponding map \( N \), and to verify the corresponding properties. Our objective here is to replace this axiomatic approach with a synthetic one: rather than defining hypernormalisation on a case-by-case basis, we will show how it arises naturally from a certain well-known categorical framework. This will, in particular, allow us in a principled way to generalise hypernormalisation (and so also channel-abstraction) to diverse other settings.

The framework in question is that of a symmetric monoidal category endowed with a linear exponential monad. A linear exponential monad \( T \) is one for which the symmetric monoidal structure of the base category lifts to the category of \( T \)-algebras and there becomes finite coproduct. Linear exponential monads originate in the categorical semantics of linear logic [4], but also have applications in studying abstract differentiation in mathematics and computer science [10]. A key observation of this paper is that linear exponential monads always have an associated notion of hypernormalisation, satisfying all the equational axioms one may hope for.

The motivating example fits into this framework via the discrete distribution monad \( D \) on the category of sets. This turns out to be a linear exponential monad, but with respect to a non-standard monoidal structure on \( Set \) which we term the convex monoidal structure. The convex monoidal structure has the empty set as unit and binary tensor given by

\[
(1.3) \quad A \star B = A + ((0,1) \times A \times B) + B
\]
where \((0, 1)\) denotes the open interval; while its associativity constraints are controlled by a map \(v\): \((0, 1) \times (0, 1) \rightarrow (0, 1) \times (0, 1)\) which encodes a particular change of coordinates for points of the topological 2-simplex.

The unfamiliar aspect here is the monoidal structure (1.3); but it turns out that this can, in turn, be understood via another established piece of category theory. A tricocycloid [49] in a symmetric monoidal category is an object \(H\) endowed with an invertible map \(v\): \(H \otimes H \rightarrow H \otimes H\) satisfying suitable axioms; and by a general construction of loc. cit., any tricocycloid \(H\) in \(\mathcal{C}\) gives a new monoidal structure on \(\mathcal{C}\) defined by \(A \star B = A + H \otimes A \otimes B + B\).

Typical examples of tricocycloids arise in the \(k\)-linear context from Hopf algebras [50] and multiplier Hopf algebras [55]; but their relevance here stems from the fact that \((0, 1)\) is a tricocycloid in the cartesian monoidal category of sets, so that we can derive the convex monoidal structure via the general construction described above. We refer to \((0, 1)\) endowed with the map \(v\) as the convex tricocycloid; it is a basic combinatorial object which lies at the heart of probability theory.

Beyond recapturing our motivating example, we also give a range of other examples of hypernormalisation arising from other linear exponential monads. One obvious direction of generalisation we pursue exhibits various continuous probability monads as linear exponential monads. However, and more interestingly, we also give a range of non-probabilistic examples. From the well-known perspective of [40, 42], monads encode computational effects, and the discrete distribution monad \(D\) in particular encodes (finite) probabilistic choice. We will see that the monads encoding other computational effects, including non-deterministic choice, logical choice over tests valued in a Boolean algebra, and input from a stream of \(B\)-values, also admit hypernormalisation.

These non-probabilistic examples will play a role in the main application of this paper, which uses our abstract hypernormalisation to relate bisimilarity and trace equivalence for automata of a certain kind. Given a monad \(T\) on a category \(\mathcal{C}\) with finite coproducts, and a finite set \(A\), we define a generative \(T\)-system with alphabet \(A\) to be a set of states \(S\) with a transition map \(S \rightarrow T(\sum_{a \in A} S)\). For suitable choices of \(T\) this yields labelled transition systems, probabilistic generative systems [56], a “logical” analogue of probabilistic generative systems, and the stream processors of [21].

In these examples, there are various notions of equivalence on states available [57], the finest being bisimilarity, and the coarsest trace equivalence. On the one hand, it is well-known that bisimilarity can be captured category-theoretically in a number of ways [48], the cleanest of which is that two states are bisimilar precisely when they become equal under the unique map to the final generative \(T\)-system. This final object exists under very mild conditions, and may be thought of as an object of behaviours \(\text{Beh}\) for generative \(T\)-systems.

On the other hand, there is a category-theoretic characterisation of trace equivalence, originally due to [45] and developed further in inter alia [22]; this involves viewing a generative \(T\)-system as a coalgebra \(S \rightarrow \sum_{a \in A} S\) in the Kleisli category of \(T\), and seeking a final such coalgebra in \(\mathcal{K}(T)\). When this exists, it provides an object of traces, and so a notion of trace equivalence on states, given as before by equality under the unique map to the object of traces. However,
since $\mathcal{X}l(T)$ is a badly-behaved category, existence of a final coalgebra is not guaranteed—and in particular, one does not exist in our examples.

Now, the badly-behaved $\mathcal{X}l(T)$ can be identified with the full subcategory of free algebras within the much better-behaved category $\mathcal{C}^T$ of Eilenberg–Moore $T$-algebras. So if we define an object of traces for generative $T$-systems $Tr$ to be a final object among coalgebras $S \to \sum_{a \in A} S$ not in $\mathcal{X}l(T)$ but rather in $\mathcal{C}^T$, then existence will hold under very mild conditions. In particular, this is the case in each of our examples—for example, the object of traces for probabilistic generative systems is precisely the set of probability distributions on $A^N$—and the induced notion of trace equivalence is precisely the expected one.

Now, when the objects of behaviours and traces for generative $T$-systems exist, there is a canonical map $\text{reflect}: \text{Beh} \to \text{Tr}$ which captures the fact that bisimilar states are also trace equivalent. But as we will see, when the monad $T$ admits hypernormalisation, more is true: the reflection map admits a canonical section $\text{reify}: \text{Tr} \to \text{Beh}$ which produces a “minimal” behaviour realising each trace—where, informally, the minimality of a behaviour expresses that it carries out the least possible amount of $T$-computation in order to determine the next output token from $A$. In particular, the idempotent $\text{reify} \circ \text{reflect}: \text{Beh} \to \text{Beh}$ implements “normalisation-by-trace-evaluation” for behaviours of generative $T$-systems.

We conclude this introduction with a brief overview of the contents of the paper. We begin Section 2 by recalling Jacobs’ notion of hypernormalisation; as in (1.1), this is a certain map associated to the monad $D$ for finitely supported probability distributions on the category of sets. The algebras for this monad are abstract convex spaces—the variety of algebras generated by the quasivariety of convex subsets of affine spaces (cf. [41]), and our first contribution is to explain how hypernormalisation can be understood in terms of finite coproducts in the category of abstract convex spaces. The key is the (well-known) observation that the binary coproduct of abstract convex spaces $A$ and $B$ is given by $A \star B$ as in (1.3), endowed with a suitable convex structure.

In fact, the results just described do not quite recapture hypernormalisation. In Section 3, we rectify this, and in doing so arrive at the key idea of this paper: that an appropriate general setting for hypernormalisation is a symmetric monoidal category endowed with a linear exponential monad. In this setting, we define a notion of hypernormalisation, and show that it inherits almost all of the good equational properties of hypernormalisation noted in [25]; we also show that the qualifier “almost” can be removed so long as the symmetric monoidal structure on the base category is co-affine—meaning that the unit object is initial—and the linear exponential monad $T$ is affine—meaning that $T1 \cong 1$. We also show that, when $\mathcal{C}$ is a category with finite products and distributive finite coproducts, and $T$ has a cartesian strength, we have a perfect analogue of the channel-to-abstract-channel construction following Jacobs’ pattern (1.1).

In Section 4, we exhibit the motivating example of hypernormalisation as an instance of our general setting by showing that the convex monoidal structure (1.3) is indeed a symmetric monoidal structure on $\text{Set}$ for which the discrete distribution monad $D$ is a linear exponential monad. As discussed, we do this by first constructing the convex tricocycloid, and applying the general construction
of [49]. Along the way, we fill out some aspects of the theory of tricycloids, in particular relating them to operads in the sense of [35].

In Sections 5 and 6, we turn to examples of hypernormalisations beyond the motivating one. In Section 5, we exhibit the structure required to obtain hypernormalisation maps for three non-discrete probability monads: the expectation monad on sets [26]; the monad of Radon probability measures on compact Hausdorff spaces [38]; and the Kantorovich monad [54] on 1-bounded metric spaces. Then in Section 6, we turn to combinatorial examples, including the monad for total finite non-determinism \( P^f \); a monad for “logical distributions” valued in a Boolean algebra \( B \); and the monad for \( B \)-ary branching trees.

Finally, in Section 7, we develop our application of abstract hypernormalisation to normalisation-by-trace-evaluation for generative \( T \)-systems. We begin by introducing these systems, and characterise the associated objects of behaviours and traces, along with the behaviour map and trace map associated to any generative \( T \)-system. We then explain our framework for normalisation-by-trace-evaluation in the context of a monad \( T \) which admits hypernormalisation. Finally, we describe this process explicitly for a range of examples of monads admitting hypernormalisation drawn from elsewhere in the paper.

2. Hypernormalisation and convex coproducts

2.1. Hypernormalisation. In this section, we first recall from [25] the notion of hypernormalisation for finitely supported discrete probability distributions, and then explain its relation to coproducts in the category of abstract convex spaces—the category of algebras for the discrete distribution monad.

Definition 1. A finitely supported sub-probability distribution on a set \( A \) is a function \( \omega : A \to [0, 1] \) such that \( \text{supp}(\omega) \) is finite and \( \omega(A) \leq 1 \). We call \( \omega \) a probability distribution if \( \omega(A) = 1 \).

Here, we write \( \text{supp}(\omega) \) for the set \( \{ a \in A : \omega(a) > 0 \} \) and, for any \( B \subseteq A \), write \( \omega(B) \) for \( \sum_{b \in B} \omega(b) \). It will often be convenient to write a sub-probability distribution \( \omega \) on \( A \) as a formal convex combination

\[
\sum_{a \in \text{supp}(\omega)} \omega(a) \cdot a
\]

of elements of \( A \); so, for example, \( \omega : \{ a, b, c, d \} \to [0, 1] \) with \( \omega(a) = \omega(c) = \frac{1}{3} \), \( \omega(d) = \frac{1}{6} \) and \( \omega(b) = 0 \) could also be written as \( \frac{1}{3} \cdot a + \frac{1}{3} \cdot c + \frac{1}{6} \cdot d \).

Definition 2. If \( \omega \) is a sub-probability distribution on a set \( A \) such that \( \omega(A) > 0 \), then its normalisation is the probability distribution \( \nabla \) with \( \nabla(a) = \omega(a)/\omega(A) \).

Of course, if \( \omega : A \to [0, 1] \) is everywhere-zero, then we cannot normalise it. One way of understanding Jacobs’ hypernormalisation [25] is as a principled way of avoiding this singularity. In the definition, and henceforth, we write \( \mathcal{D}A \) for the set of probability distributions on a set \( A \).

Definition 3. Let \( A \) be a set and \( n \in \mathbb{N} \). The \( n \)-ary hypernormalisation function

\[
N : \mathcal{D}(A + \cdots + A) \longrightarrow \mathcal{D}(n \cdot \mathcal{D}A)
\]
is given as follows. For $1 \leq i \leq n$ and $a \in A$, we write $(a, i)$ for the image of $a$ under the $i$th coproduct injection $A \to A + \cdots + A$. Each $\omega \in \mathcal{D}(A + \cdots + A)$ yields $n$ sub-probability distributions $\omega_i$ on $A$ with $\omega_i(a) = \omega(a, i)$, and we take

$$N(\omega) = \sum_{1 \leq i \leq n} \omega_i(A) \cdot (\overline{w_i}, i).$$

In other words $N(\omega)$ “normalises the non-zero sub-probability distributions among $\omega_1, \ldots, \omega_n$ and records the total weights”. Note that this agrees with the description of hypernormalisation given in the introduction: the only difference is that there, we wrote $B$ for the finite set $\{1, \ldots, n\}$, and wrote $A \times B$ for $A + \cdots + A$, so that hypernormalisation became a map $\mathcal{D}(A \times B) \to \mathcal{D}(\mathcal{D}A \times \mathcal{D}B)$.

As suggested in Section 8 of [25], one may generalise the hypernormalisation maps by replacing the $n$ copies of $A$ with $n$ possibly distinct sets, yielding maps

$$N: \mathcal{D}(A_1 + \cdots + A_n) \to \mathcal{D}(\mathcal{D}A_1 + \cdots + \mathcal{D}A_n)$$

defined in an entirely analogous manner to before. In this paper it will be this asymmetric version of hypernormalisation that we use. In fact, the key features of hypernormalisation are fully alive in the $n = 2$ case, and so in large part we will concentrate on the binary hypernormalisation maps

$$N: \mathcal{D}(A + B) \to \mathcal{D}(\mathcal{D}A + \mathcal{D}B).$$

Of particular note is the case of (2.4) where $B$ is a singleton set $1 = \{\ast\}$, so that we have a map $N: \mathcal{D}(A + 1) \to (\mathcal{D}A + \mathcal{D}1) \cong \mathcal{D}(\mathcal{D}A + 1)$. An element of $\mathcal{D}(A + 1)$ can be identified with a sub-probability distribution $\omega$ on $A$, with the one additional point $\ast$ necessarily being given the weight $1 - \omega(A)$; likewise, an element of $\mathcal{D}(\mathcal{D}A + 1)$ can be identified with a sub-probability distribution on $\mathcal{D}A$. Under these identifications, the action of $N$ can be described as follows:

- If $\omega$ is the zero sub-probability distribution on $A$, then $N(\omega)$ is the zero sub-probability distribution on $\mathcal{D}A$;
- Otherwise, $N(\omega)$ is the sub-probability distribution on $\mathcal{D}A$ which assigns the weight $\omega(A)$ to the single point $\overline{a}$.

2.2. Convex coproducts. As explained in the introduction, hypernormalisation is closely bound up with the discrete distribution monad $\mathcal{D}$ on the category of sets. We now recall this monad, and explain how hypernormalisation is related to coproducts in the category of $\mathcal{D}$-algebras.

**Definition 4.** The functor $\mathcal{D}: \text{Set} \to \text{Set}$ takes $A$ to $\mathcal{D}A$ on objects; while on maps, $\mathcal{D}f: \mathcal{D}A \to \mathcal{D}B$ sends $\omega \in \mathcal{D}A$ to the pushforward $f_*(\omega) \in \mathcal{D}B$ given by

$$f_*(\omega)(b) = \omega(f^{-1}(b)).$$

The unit $\eta: 1_{\text{Set}} \Rightarrow \mathcal{D}$ and multiplication $\mu: \mathcal{D}\mathcal{D} \Rightarrow \mathcal{D}$ of the discrete distribution monad $\mathcal{D}$ have respective components at a set $A$ given by

$$\eta_A: A \to \mathcal{D}A \quad \eta_A(a) = 1 \cdot a$$

and

$$\mu_A: \mathcal{D}\mathcal{D}A \to \mathcal{D}A \quad \mu_A(a) = \sum_{1 \leq i \leq n} \lambda_i \omega_i \mapsto (a \mapsto \sum_{1 \leq i \leq n} \lambda_i \omega_i(a)).$$
(Note that, in giving $\mu_A$, we have to the left a formal convex combination of elements of $\mathcal{D}A$, and to the right, an actual convex combination in $[0,1]$.)

In [25], the discrete distribution monad is discussed in terms of its Kleisli category; our interest here is in the algebras of this monad, which are sometimes known as abstract convex spaces.

**Definition 5.** An abstract convex space is a set $A$ endowed with an operation

$$\tag{2.6} (0,1) \times A \times A \to A$$

$$ (r,a,b) \mapsto r(a,b)$$

satisfying the following axioms for all $a,b,c \in A$ and $r,s \in (0,1)$:

(i) $r(a,a) = a$;

(ii) $r(a,b) = r^* (b,a)$ (recall we write $r^*$ for $1-r$);

(iii) $r(s(a,b),c) = (rs)(a,r\frac{s^*}{rs}(b,c))$.

A map of convex spaces from $A$ to $B$ is a function $f : A \to B$ such that $f(r(a,b)) = r(fa,fb)$ for all $a,b \in A$ and $r \in (0,1)$.

If we view the operation (2.6) as an “abstract convex combination” $r(a,b) = r \cdot a + r^* \cdot b$, then the axioms are just what is needed to ensure that this behaves as expected. The two main classes of examples of abstract convex spaces are:

- Convex subsets of vector spaces (so convex spaces in the usual sense) under the usual convex combination operation; and

- Meet-semilattices under the operation $r(a,b) = a \wedge b$ for all $r \in (0,1)$.

If we extend the operation of an abstract convex space to one $[0,1] \times A \times A \to A$ by defining $1(a,b) = a$ and $0(a,b) = b$, then axioms (i)–(iii) are still validated for the new edge cases in $\{0,1\}$ wherever this makes sense (i.e., so long as $rs \neq 1$ in (iii)). This yields the axiomatisation of abstract convex spaces found in [41, §2, Axioms B1–B3]. In particular, these axioms ensure that each of the valid ways of interpreting a formal convex combination

$$\tag{2.7} \sum_{1 \leq i \leq n} r_i \cdot a_i \in \mathcal{D}A$$

as an element of $A$ via repeated application of the operation (2.6) will give the same result, so that we have a well-defined function $\mathcal{D}A \to A$. This function endows $A$ with $\mathcal{D}$-algebra structure, which is the key step in proving:

**Lemma 6.** [23, Theorem 4]. The category $\mathcal{C}\text{onv}$ of abstract convex spaces and convex maps is isomorphic over Set to the category $\text{Set}^D$ of $\mathcal{D}$-algebras.

This result justifies us in using expressions of the form (2.7) to denote an element of an abstract convex space $A$, and we do so without further comment.

The relation between abstract convex spaces and hypernormalisation lies in the construction of finite coproducts in $\mathcal{C}\text{onv}$. While coproducts in algebraic categories are usually messy and syntactic, for abstract convex spaces they are quite intuitive. Given $A, B \in \mathcal{C}\text{onv}$, their coproduct must certainly contain copies of $A$ and $B$; and must also contain a formal convex combination $r \cdot a + r^* \cdot b$ for each $a \in A$, $b \in B$ and $r \in (0,1)$. For a general algebraic theory, this process of
convex combination operator whose most involved case is coproduct $A \star B$ in Conv is the set $A + ((0, 1) \times A \times B) + B$, endowed with the convex combination operation whose most involved case is

$$r_{A \star B}((s, a, b), (t, a', b')) = (rs + rt^*, (\frac{rs}{(rs + rt^*)})A(a, a'), (\frac{rt^*}{(rs + rt^*)})B(b, b')) .$$

This formula was obtained by expanding out the formal convex combination $r \cdot (s \cdot a + s^* \cdot b) + r^* \cdot (t \cdot a' + t^* \cdot b')$, rearranging, and partially evaluating the terms from $A$ and from $B$. The reader should have no difficulty giving the remaining, simpler, cases (where one or both arguments of $r_{A \star B}$ come from $A$ or $B$), and in then proving that the resulting object is an abstract convex space.

If we write elements $a \in A$, $(r, a, b) \in (0, 1) \times A \times B$ and $b \in B$ in the three summands of $A \star B$ as, respectively,

$$\iota_1(a) , \quad r \cdot a + r^* \cdot b \quad \text{and} \quad \iota_2(b) ,$$

then the two coproduct injections are given by $\iota_1: A \to A \star B \leftarrow B: \iota_2$; and as for the universal property of coproduct, if $f: A \to C$ and $g: A \to C$ are convex maps, then the unique induced map $\langle f, g \rangle: A \star B \to C$ sends $\iota_1(a)$ or $\iota_2(b)$ to $f(a)$ or $g(b)$ respectively, and sends $r \cdot a + r^* \cdot b = r_{A \star B}(a, b)$ to $r \cdot f(a) + r^* \cdot g(b) = r_C(fa, gb)$.

To draw the link with hypernormalisation, consider the free-forgetful adjunction

\begin{equation}
\begin{array}{c}
\text{Conv} \\
\xleftarrow{F^D} \\
\xrightarrow{\eta^D} \\
\text{Set}
\end{array}
\end{equation}

associated to the monad $D$. The left adjoint $F^D$ sends the set $A$ to the set $\mathcal{D}A$, seen as an abstract convex space under the convex combination operation given pointwise by the usual one on $[0, 1]$. Being a left adjoint, $F^D$ preserves coproducts, and so we have for any set $A$ and any $n \in \mathbb{N}$ a bijection of abstract convex spaces

\begin{equation}
\varphi: \mathcal{D}(A + B) \to \mathcal{D}A \star \mathcal{D}B ,
\end{equation}

which, if we spell it out, we see is really just hypernormalisation:

**Proposition 8.** The isomorphism (2.9) is given by

$$\varphi(\omega) = \begin{cases} 
\iota_1(\omega_1) & \text{if } \omega_1(A) = 1; \\
\iota_2(\omega_2) & \text{if } \omega_2(B) = 1; \\
\omega_1(A) \cdot \overline{\omega_1} + \omega_2(B) \cdot \overline{\omega_2} & \text{otherwise},
\end{cases}$$

where $\omega_1$ and $\omega_2$ are the sub-probability distributions obtained by restricting $\omega$ to $A$ and $B$.

**Proof.** $\varphi$ is the extension of the composite function

\begin{equation}
A + B \xrightarrow{\eta + \eta} \mathcal{D}A + \mathcal{D}B \leftarrow \mathcal{D}A \star \mathcal{D}B
\end{equation}

to a convex map $\mathcal{D}(A+B) \to \mathcal{D}A \star \mathcal{D}B$. Precomposing (2.10) with $\iota_1: A \to A+B$ yields

$$A \xrightarrow{\eta} \mathcal{D}A \xrightarrow{\iota_1} \mathcal{D}A \star \mathcal{D}B$$
whence \( \varphi \) identifies \( DA \leadsto DA \star DB \). This proves the first case of the desired formula; the second is similar.

Finally, consider \( \omega \in DA \star DB \) which does not factor through either \( DA \) or \( DB \). Since both sub-probability distributions \( \omega_1 \) and \( \omega_2 \) are non-zero, we can form both \( \omega_1 \in DA \) and \( \omega_2 \in DB \), and in these terms we now have

\[
\omega = \omega_1(A) \cdot (t_1)_*(\overline{\omega_1}) + \omega_2(B) \cdot (t_2)_*(\overline{\omega_2})
\]

in \( DA \star DB \). As \( \varphi \) is a convex map, it follows that \( \varphi(\omega) = \omega_1(A) \cdot \overline{\omega_1} + \omega_2(B) \cdot \overline{\omega_2} \)

from the two cases already proved.

\[ \square \]

3. Abstract Hypernormalisation

The map (2.9) of Proposition 8 is related to the hypernormalisation map (2.4), but is not quite the same. In this section, we explain how to derive the latter map from the former one, and isolate the general structure required for this derivation: that of a linear exponential monad. Guided by this, we define an abstract notion of hypernormalisation with respect to a linear exponential monad, show that it has the desired equational properties, and explain how it may give rise to a version of the channel-to-abstract channel construction from the introduction.

3.1. Recapturing hypernormalisation. Towards bridging the gap between (2.4) and (2.9), we observe that (2.4) is not a map of abstract convex spaces, so that to recapture it, we must necessarily leave the category \( \text{Conv} \). We do so in an apparently simple-minded fashion, by considering the category \( \text{Conv}_{arb} \) whose objects are abstract convex spaces and whose maps are arbitrary functions.

Now, the binary coproduct \( \star \) on \( \text{Conv} \) is part of a symmetric monoidal structure, whose unit is the empty convex space, and whose coherence isomorphisms are induced from the universal properties of finite coproducts. This symmetric monoidal structure extends to \( \text{Conv}_{arb} \); by this we mean simply that \( \text{Conv}_{arb} \) has a symmetric monoidal structure with respect to which the inclusion \( \text{Conv} \to \text{Conv}_{arb} \) becomes symmetric strict monoidal. This monoidal structure is (necessarily) given on objects as before, while on maps \( f : A \to A' \) and \( g : B \to B' \) of \( \text{Conv}_{arb} \), the tensor \( f \star g : A \star B \to A' \star B' \) is given by

\[
f \star g = f + ((0,1) \times f \times g) + g,
\]

i.e., exactly the same formula as the definition of \( \star \) on maps in \( \text{Conv} \).

Now suppose we are given abstract convex spaces \( A \) and \( B \). Using the extended monoidal structure on \( \text{Conv}_{arb} \), we obtain a function

\[
A \star B \xrightarrow{\eta_A \star \eta_B} DA \star DB \xrightarrow{\varphi^{-1}} DA \star DB
\]

whose second part is the inverse of (2.9) and whose first part is the tensor (3.1) of the (non-convex) functions \( \eta_A \) and \( \eta_B \). Working through the definitions, we see that this sends elements \( t_1(a) \) and \( t_2(b) \) of \( A \star B \) to the distributions \( 1 \cdot a \) and \( 1 \cdot b \) on \( A \star B \) concentrated at a single point; while an element \( r \cdot a + r^* \cdot b \in A \star B \) is sent to the two-point distribution \( r \cdot a + r^* \cdot b \) on \( A \star B \). Combining this description of (3.2) with Proposition 8, we immediately obtain:
**Proposition 9.** The hypernormalisation map (2.4) is the composite

\[
\mathcal{D}(A + B) \xrightarrow{\varphi} \mathcal{D}A \star \mathcal{D}B \xrightarrow{\eta D A \star D B} \mathcal{D}D A \star D D B \xrightarrow{\psi^{-1}} \mathcal{D}(D A + D B).
\]

Thus, the hypernormalisation map (2.4) arises inevitably from the isomorphism (2.9) together with the fact that the coproduct monoidal structure on \(\text{Conv}_{\text{arb}}\) extends to \(\text{Conv}_{\text{arb}}\). We now give an explanation of why this extension of monoidal structure should exist.

To motivate this explanation, observe that the formula (3.1) for the extended tensor product on \(\text{Conv}_{\text{arb}}\) works because the underlying set of \(A \star B\) depends only on the underlying sets of \(A\) and \(B\), and not on their convex structure. So could the symmetric monoidal structure \(\star\) on \(\text{Conv}\) be a lifting of a symmetric monoidal structure on \(\text{Set}\)? In other words, is there a symmetric monoidal structure \((\star, 0)\) on \(\text{Set}\)—which as in the introduction we term the *convex monoidal structure*—such that \(U^D: (\text{Conv}, \star, 0) \to (\text{Set}, \star, 0)\) is strict symmetric monoidal?

In Section 4 below, we will see that this is indeed the case; for the moment, let us see how, assuming this fact, we can recover the symmetric monoidal structure of \(\text{Conv}_{\text{arb}}\). To do this, we consider the evident factorisation \(\text{Conv} \to \text{Conv}_{\text{arb}} \to \text{Set}\) of \(U^D\) through \(\text{Conv}_{\text{arb}}\), and apply the following result:

**Lemma 10.** [43] Let \(F: \mathcal{E} \to \mathcal{C}\) be a strict symmetric monoidal functor between symmetric monoidal categories, and let

\[
F = \mathcal{E} \xrightarrow{G} \mathcal{D} \xrightarrow{H} \mathcal{C}
\]

be a factorisation of the underlying functor \(F\) wherein \(G\) is bijective on objects and \(H\) is fully faithful. There is a unique symmetric monoidal structure on \(\mathcal{D}\) making both \(G\) and \(H\) strict symmetric monoidal.

**Proof.** Define the unit and the tensor on objects in \(\mathcal{D}\) to be those of \(\mathcal{E}\), and define the tensor on maps and the coherence morphisms to be those of \(\mathcal{C}\). □

At this point, if we still take for granted the existence of the convex monoidal structure \((\star, 0)\) on \(\text{Set}\), then one final category-theoretic transformation will allow us to derive the hypernormalisation maps (2.4) purely in terms of the structure of the discrete distribution monad. We begin by recalling:

**Definition 11.** A monad \(T\) on a symmetric monoidal category \((\mathcal{C}, \otimes, I)\) is *symmetric opmonoidal* if it comes endowed with a map \(\psi_I: TI \to I\) and maps \(\psi_{XY}: T(X \otimes Y) \to TX \otimes TY\) for \(X, Y \in \mathcal{C}\), subject to seven coherence axioms; see, for example, [39, Section 7].

The relevance of this definition for us is captured by:

**Lemma 12.** [39, Theorem 7.1]. For any monad \(T\) on a symmetric monoidal category \((\mathcal{C}, \otimes, I)\), symmetric opmonoidal monad structures on \(T\) correspond bijectively to liftings of the symmetric monoidal structure of \(\mathcal{C}\) to \(\mathcal{C}^T\).

**Proof.** Given symmetric opmonoidal structure on \(T\), we define the lifted tensor product of \(T\)-algebras by

\[
(TX \xrightarrow{x} X) \otimes (TY \xrightarrow{y} Y) = (T(X \otimes Y) \xrightarrow{\psi_{XY}} TX \otimes TY) \xrightarrow{x \otimes y} X \otimes Y)
\]
with as unit the $T$-algebra $\nu_I : TI \to I$. Conversely, given a lifted tensor product on $T$-algebras, we obtain the opmonoidal structure map $\nu_{XY}$ as the composite
\[
T(X \otimes Y) \xrightarrow{T(\eta_X \otimes \eta_Y)} T(TX \otimes TY) \xrightarrow{\theta} TX \otimes TY
\]
where $\theta$ is the $T$-algebra structure of $(\mu_X : TTX \to TX) \otimes (\mu_Y : TTY \to TY)$, and obtain $\nu_I : TI \to I$ as the $T$-algebra structure of the lifted unit. □

Thus, the fact that the coproduct monoidal structure on Conv lifts the (assumed) convex monoidal structure $(\ast, 0)$ on Set can be re-expressed by saying that the discrete distribution monad $D$ on Set is symmetric opmonoidal with respect to $(\ast, 0)$. Actually, more is true: $D$ is a linear exponential monad.

**Definition 13.** A linear exponential monad on a symmetric monoidal category $(C, \otimes, I)$ is a symmetric opmonoidal monad $T$ on $C$ such that the lifted symmetric monoidal structure on the category of algebras $C^T$ is given by finite coproducts. More precisely, we means by this that the lifted unit object $(I, \nu_I)$ should be initial in $C^T$; and that, for any pair of algebras $(X, x)$ and $(Y, y)$, the cospan
\[
(X, x) \xrightarrow{\rho_X} (X, x) \otimes (I, \nu_I) \xrightarrow{1 \otimes !} (X, x) \otimes (Y, y) \xleftarrow{\lambda_Y} (I, \nu_I) \otimes (Y, y)
\]
should define a binary coproduct in $C^T$, where we use $!$ to denote the unique maps out of the initial object $(I, \nu_I)$.

Linear exponential $\text{co}$monads originate in the categorical semantics of linear logic [4, Definition 3] where they interpret the exponential modality which allows a resource to be copied freely. Importantly, the co-Kleisli category of a linear exponential comonad on a symmetric monoidal closed category is cartesian closed; this is a categorical formulation of the translation of intuitionistic logic into linear logic [19, §5]. Linear exponential comonads also arise in connection with [10]'s differential categories, which are categories endowed with an abstract notion of differentiation, encoded by a comonad which in many cases is linear exponential (note that in this context, the term “monoidal coalgebra modality” is often used rather than “linear exponential comonad”).

The dual notion of linear exponential $\text{monad}$ appears both in linear logic, where it models the de Morgan dual connective $\otimes$ of $!$, and in the study of codifferential categories, of which there are many natural examples; see, for example, [8]. Furthermore, as we will show in the next section, a linear exponential monad is exactly the structure one needs for an good abstract notion of hypernormalisation.

The salience of this last observation is not so much that it establishes a deep connection between probabilistic structures and linear logic, but rather that it makes available the well-understood calculus of reasoning for linear exponential (co)monads, as discussed in, for example, [37, Section 7] or [9]. As we will see, this allows to show that our abstract notion of hypernormalisation verifies all the equational axioms one could wish for.

### 3.2. Abstract Hypernormalisation.

Given a symmetric monoidal category $(C, \otimes, I)$ with finite coproducts and a linear exponential monad $T$ on $C$, we continue to write $\varphi : T(A + B) \to TA \otimes TB$ for the map underlying the $T$-algebra
isomorphism $F^T(A + B) \to F^T(A) \otimes F^T(B)$; more generally, we write
\begin{equation}
\varphi : T(A_1 + \cdots + A_n) \to TA_1 \otimes \cdots \otimes T A_n
\end{equation}
for the corresponding $n$-ary isomorphism. Note that these isomorphisms are natural in maps of $\mathcal{C}$, which is to say that all diagrams of the following form commute:
\begin{equation}
\begin{array}{ccc}
T(A_1 + \cdots + A_n) & \xrightarrow{\varphi} & TA_1 \otimes \cdots \otimes T A_n \\
\downarrow T(f_1 + \cdots + f_n) & & \downarrow T f_1 \otimes \cdots \otimes T f_n \\
T(B_1 + \cdots + B_n) & \xrightarrow{\varphi} & TB_1 \otimes \cdots \otimes T B_n
\end{array}
\end{equation}

**Definition 14.** Let $(\mathcal{C}, \otimes, I)$ be a symmetric monoidal category with finite coproducts, and let $T$ be a linear exponential monad on $\mathcal{C}$. The binary hypernormalisation map $N: T(A + B) \to T(TA + TB)$ is the composite
\[ T(A + B) \xrightarrow{\varphi} TA \otimes TB \xrightarrow{T\eta \otimes T\eta} TTA \otimes TTB \xrightarrow{\varphi^{-1}} T(TA + TB). \]
More generally, given objects $A_1, \ldots, A_n$, the $n$-ary hypernormalisation map $N: T(\sum_i A_i) \to T(\sum_i TA_i)$ is the composite
\begin{equation}
T\left(\sum_i A_i\right) \xrightarrow{\varphi} \bigotimes_i TA_i \xrightarrow{\otimes \eta \otimes \eta} \bigotimes_i TTA_i \xrightarrow{\varphi^{-1}} T\left(\sum_i T A_i\right)
\end{equation}
The leading example, as we confirm in Section 4, is Jacobs' original hypernormalisation, which arises on taking $(\mathcal{C}, \otimes, I)$ to be $(\text{Set}, \star, 0)$ and $T = D$. However, as we will see in Section 5, there are many other interesting examples of abstract hypernormalisation, including continuous probability monads on suitable categories, and examples related to continuous functions on streams.

Before turning to these examples, we investigate the degree to which our abstract hypernormalisation inherits the good equational properties of Jacobs' original definition. In the statement and proof of the following result, we write $\langle f_i \rangle_{i \in I} : \sum_i A_i \to B$ to denote a copairing of maps $f_i : A_i \to B$ out of a coproduct.

**Proposition 15.** Let $T$ be a linear exponential monad on the symmetric monoidal category $(\mathcal{C}, \otimes, I)$. The hypernormalisation maps (3.7) satisfy the conditions expressed by the commutativity of the following diagrams:
\begin{enumerate}
\item Hypernormalisation has a left inverse:
\[ T(\sum_i A_i) \xrightarrow{N} T(\sum_i TA_i) \]
\item Hypernormalisation is idempotent:
\[ T(\sum_i A_i) \xrightarrow{N} T(\sum_i TA_i) \]
\end{enumerate}
(3) Hypernormalisation is natural in maps $f_i: A_i \rightarrow B_i$:

\[
\begin{align*}
T(\Sigma_i A_i) & \xrightarrow{N} T(\Sigma_i T A_i) \\
T(\Sigma_i f_i) & \downarrow \\
T(\Sigma_i B_i) & \xrightarrow{N} T(\Sigma_i T B_i)
\end{align*}
\]

and in Kleisli maps $f_i: A_i \rightarrow \mathbb{T}B_i$:

\[
\begin{align*}
T(\Sigma_i A_i) & \xrightarrow{N} T(\Sigma_i T A_i) \xrightarrow{T(\Sigma_i f_i)} T(\Sigma_i T B_i) \\
T(\Sigma_i f_i) & \downarrow \\
T(\Sigma_i B_i) & \xrightarrow{T(\Sigma_i \mu_{B_i})} T(\Sigma_i T B_i).
\end{align*}
\]

**Proof.** To prove (1), we first claim that each diagram as to the left below commutes. This is a general fact about linear exponential monads—see, for example [37, Section 7]—but we include the proof for the sake of self-containedness.

\[
\begin{align*}
T(\Sigma_i T A_i) & \xrightarrow{\varphi} \otimes_i T T A_i \\
T(\Sigma_i f_i) & \downarrow \\
T(\Sigma_i B_i) & \xrightarrow{\otimes_i \mu_{A_i}} T(\Sigma_i T B_i)
\end{align*}
\]

Note that both paths are $T$-algebra maps $F^T(\Sigma_i T A_i) \rightarrow \otimes_i F^T A_i$ with as domain a coproduct of the $T$-algebras $F^T A_i$. So it suffices to show commutativity on precomposing by a coproduct coprojection $T\iota_i: T T A_i \rightarrow T(\Sigma_i T A_i)$. This means showing the outside of the diagram to the right above commutes, wherein we write $j_i$ for a coproduct coprojection $X_i \rightarrow \otimes_i X_i$ in the category of $T$-algebras. But the bottom triangle commutes by definition of $\varphi$, the left region by naturality of $\mu$ and the right region by naturality of the coproduct coprojections $j_i$.

Now commutativity in (3.8) yields commutativity in the right part of:

\[
\begin{align*}
T(\Sigma_i A_i) & \xrightarrow{N} T(\Sigma_i T A_i) \xrightarrow{T(\Sigma_i f_i)} T(\Sigma_i T B_i) \\
\varphi & \downarrow \quad \varphi^{-1} \\
\otimes_i T A_i & \xrightarrow{\otimes_i \mu_{T A_i}} \otimes_i T T A_i \xrightarrow{\otimes_i \mu_{A_i}} \otimes_i T A_i
\end{align*}
\]

whose left part commutes by definition of $N$. So the outside commutes; now by the monad axioms for $T$, the lower composite is the identity, whence also the upper one as required for (1).
Turning to (2), we observe that pre-composing by $\varphi^{-1}$ and post-composing by $\varphi$ yields the square

$$
\begin{array}{ccc}
\otimes_i TA_i & \xrightarrow{\otimes_i \eta T A_i} & \otimes_i T^2 A_i \\
\downarrow & & \downarrow \\
\otimes_i T^2 A_i & \xrightarrow{\otimes_i \eta T^2 A_i} & \otimes_i T^3 A_i
\end{array}
$$

which commutes by functoriality of $\otimes$ and naturality of $\eta$. Finally, for (3), commutativity of the first diagram is clear from the naturality (3.6) of the maps $\varphi: T(\Sigma_i A_i) \to \otimes_i TA_i$ in the $A_i$, the functoriality of $\otimes$, and the naturality of the unit $\eta: 1 \Rightarrow T$. Commutativity of the second diagram follows trivially from the first after postcomposing by $T(\Sigma_i \mu_{B_i})$. \hfill $\square$

The preceding conditions generalise ones appearing in [25, Lemma 5]. Our (2) and (3) correspond exactly to its (3) and (5), while our (1) corresponds either to the right diagram of its (2) or to its (4). We have no correlate of the right diagram of part (1) of [25, Lemma 5], since it uses the canonical strength of the discrete distribution monad $D$ with respect to the cartesian monoidal structure of $\text{Set}$, and it is not clear what this should be replaced with in general. This leaves only the left diagrams appearing in (1) and (2) of [25, Lemma 5]. Interestingly, while these make sense in our setting, they do not hold without additional assumptions. For the left diagram of (2), this condition is:

**Definition 16.** A monad $T$ on a category $\mathcal{C}$ with a terminal object $1$ is **affine** if the unique map $T1 \to 1$ is invertible (necessarily with inverse $\eta_1: 1 \to T1$).

**Proposition 17.** Let $T$ be an affine linear exponential monad on the symmetric monoidal category $(\mathcal{C}, \otimes, I)$ with terminal object $1$. The hypernormalisation maps (3.7) satisfy the additional condition that:

(4) Destroying the output structure destroys hypernormalisation:

$$
\begin{array}{ccc}
T(\Sigma_i A_i) & \xrightarrow{N} & T(\Sigma_i TA_i) \\
\downarrow & & \downarrow \\
T(\Sigma_i 1) & \xrightarrow{T(\Sigma_i !)} & T(\Sigma_i !)
\end{array}
$$

**Proof.** We may precompose (4) by $\varphi^{-1}: \otimes_i TA_i \to T(\Sigma_i A_i)$, postcompose by $\varphi: T(\Sigma_i 1) \to \otimes_i T1$, and rewrite using the definition of $N$ and the naturality (3.6) to obtain the triangle to the left in:

$$
\begin{array}{ccc}
\otimes_i TA_i & \xrightarrow{\otimes_i \eta T A_i} & \otimes_i TTA_i \\
\downarrow & & \downarrow \\
\otimes_i T1 & \xrightarrow{\otimes_i T!} & \otimes_i T1
\end{array}
\quad
\begin{array}{ccc}
\otimes_i TA_i & \xrightarrow{\otimes_i !} & \otimes_i 1 \\
\downarrow & & \downarrow \\
\otimes_i T1 & \xrightarrow{\otimes_i \eta_1} & \otimes_i T1
\end{array}
$$
whose commutativity is equivalent to that of (4). But by naturality of $\eta$, this triangle is equally the triangle on the right above, which commutes since post-composing by the invertible map $\otimes \eta!$: $\otimes \eta T1 \to \otimes \eta 1$ yields along both sides the map $\otimes \eta!$: $\otimes \eta TA_i \to \otimes \eta 1$. □

Finally, we consider what is necessary for the left diagram of part (1) of [25, Lemma 5] to commute in our setting.

Definition 18. A symmetric monoidal category $(\mathcal{C}, \otimes, I)$ is said to be co-affine if its unit object is initial.

So, for example, the convex monoidal structure and the cocartesian monoidal structure on $\mathbf{Set}$ are co-affine, while the cartesian monoidal structure is not so. The point of this extra condition is that it allows us to prove:

Lemma 19. Let $T$ be a linear exponential monad on the symmetric monoidal co-affine category $(\mathcal{C}, \otimes, I)$. Finite coproduct coprojections $\eta_i: X_i \to \otimes_i X_i$ in the category of $T$-algebras are natural with respect to arbitrary maps of $\mathcal{C}$.

Proof. Since non-empty finite coproducts can be constructed from binary ones, it suffices to prove the binary case. Given $T$-algebras $(X, x)$ and $(Y, y)$, we know from Definition 13 that the coproduct coprojection $(X, x) \to (X, x) \otimes (Y, y)$ is given by the composite

$$(X, x) \xrightarrow{\rho_X} (X, x) \otimes (I, \nu_I) \xrightarrow{1 \otimes !} (X, x) \otimes (Y, y),$$

where $!: (I, \nu_I) \to (Y, y)$ is the unique map of $T$-algebras induced by the initiality of $(I, \nu_I)$ in $\mathcal{T}$. By co-affineness, the underlying map in $\mathcal{C}$ of this composite is

$$X \xrightarrow{\rho_X} X \otimes I \xrightarrow{1 \otimes !} X \otimes Y,$$

where $!$ is the unique map out of the initial object $I \in \mathcal{C}$. Given this description, the desired naturality with respect to arbitrary maps of $\mathcal{C}$ is now immediate. □

Proposition 20. Let $T$ be a linear exponential monad on the symmetric monoidal co-affine category $(\mathcal{C}, \otimes, I)$. The hypernormalisation maps (3.7) satisfy the additional condition that:

(5) Normalising trivial input gives trivial output:

$$\begin{align*}
\Sigma_i TA_i \xrightarrow{\eta_i} T(\Sigma_i A_i) \\
\downarrow^{\lambda_i} \quad \quad \downarrow^{N} \\
\Sigma_i T A_i \xrightarrow{\eta_{\Sigma_i TA_i}} T(\Sigma_i TA_i).
\end{align*}$$

Proof. By definition of $N$ and naturality of $\eta$, this is equally to show that the diagram below left commutes. Since $\varphi$ is the underlying map of the unique comparison between the coproducts $F^T(\Sigma_i A_i)$ and $\otimes_i F^T A_i$ in $\mathcal{C}^T$, it in particular commutes with the coproduct coprojections, so that this diagram is equally the
3.3. **Channel abstraction.** In this short section, we explain how, in our abstract setting, hypernormalisation can be used to build an analogue of the channel-to-abstract-channel construction of [36] described in the introduction. We are motivated in doing this by the examples of hypernormalisation for continuous probability monads described in Sections 5.1–5.3 below; thus, in what follows, the reader should keep in mind the interpretation that $\mathcal{C}$ is some category of “spaces”, and that $\mathcal{T}$ is a monad of “distributions” on $\mathcal{C}$.

Let us recap what the construction should do. The input data, a channel, is simply a map $P : A \to TB$, which we think of as giving probabilities that a private input in $A$ will give rise to a public output in $B$. The output data, the associated abstract channel, is a map $P^r : TA \to TTA$, thought of as giving the probabilities that a given prior distribution on $A$ should update to a given posterior distribution on $A$ via conditioning on an observed output in $B$.

Following Jacobs’ lead, we will build $P^r$ from $P$ via a composite (1.1). The main difficulty comes in finding an analogue of the first map $\tilde{P}$ therein. This map, we recall, was itself a composite (1.2), one of whose terms is the canonical cartesian strength of the finite discrete distribution monad $D$. We already remarked in the previous section that it was not clear what to replace this with in general, so here we adopt the most simple-minded approach that is compatible with our examples: we simply assume that, again, our monad $\mathcal{T}$ has a cartesian strength. This is enough to generalise the first map in (1.1); however, there is still a small problem with the second map, which should be a map $T(A \times B) \to T(TA \times B)$ given by hypernormalisation. In the motivating example, this was unproblematic: we could use the fact that $B$ was a finite set $\{1, \ldots, n\}$ to express $A \times B$ as an $n$-fold coproduct $A + \cdots + A$, and then apply $n$-ary hypernormalisation. In our general context, we can do something similar so long as finite coproducts distribute over finite products in $\mathcal{C}$, and we assume that $B$ is an $n$-ary coproduct $1 + \cdots + 1$; then by distributivity we have $A \times B \cong A + \cdots + A$ and can proceed as before. The above discussion thus justifies giving:

**Definition 21.** Let $(\mathcal{C}, \otimes, I)$ be a symmetric monoidal category with finite products and distributive finite coproducts. Let $\mathcal{T}$ be a linear exponential monad on $\mathcal{C}$ endowed with a cartesian strength. Given a map $P : A \to TB$, where $B = 1 + \cdots + 1$ is an $n$-fold coproduct of the terminal object, we define the associated abstract channel to be the map $P^r : TA \to TTA$ given by

\[
TA \xrightarrow{P} T(A \times B) \xrightarrow{\eta} T(TA \times B) \xrightarrow{T(\pi_1)} TTA
\]

wherein the first map is the composite

\[
TA \xrightarrow{T(1,P)} T(A \times TB) \xrightarrow{T(\text{str})} TT(A \times B) \xrightarrow{\mu} T(A \times B),
\]
and the second map is the composite of the \( n \)-ary hypernormalisation map 
\[
T(A+\cdots+A) \to T(TA+\cdots+TA)
\]
with isomorphisms 
\[
T(A \times B) \cong T(A+\cdots+A)
\]
and 
\[
T(TA + \cdots + TA) \cong T(TA \times B).
\]

The restriction we impose on the form of \( B \) above is a real one; in our examples, it means that our channel \( P: A \to TB \) involves a continuous space of hidden inputs but only a finite discrete space of observable outputs. While this is already progress, allowing an arbitrary observation space \( B \) would require something, more general than hypernormalisation, which gave a smooth categorical treatment of disintegration for probability measures on a product space.

4. Tricocycloids and the convex monoidal structure

In this section, we complete our description of the convex monoidal structure \((\ast, 0)\) on \( \text{Set} \) with respect to which the discrete distribution monad is linear exponential, and by doing so exhibit Jacobs’ hypernormalisation as a particular instance of our abstract hypernormalisation.

The aspects of the convex monoidal structure we have not yet discussed are its unit, associativity and symmetry constraints. Given that it should lift to the coproduct monoidal structure on \( \text{Conv} \), we can read off these constraints from the corresponding ones for coproducts in \( \text{Conv} \). However, there is still work to do: we must show the maps involved can be defined in a way that does not depend on any convex structure, but only on the underlying sets.

We clarify the combinatorics involved in this by using a notion from quantum algebra known as a tricocycloid. We begin this section by explaining how tricocycloids give rise to monoidal structures, and how they relate to operads in the sense of [35]; we then exhibit a tricocycloid in \( \text{Set} \) which will allow us to construct the desired convex monoidal structure.

4.1. Tricocycloids. Although our applications will primarily be in the category of sets, the construction we are about to give naturally exists in a more general setting. Rather than just the cartesian monoidal category of sets, it starts from a symmetric monoidal category \((\mathcal{C}, \otimes, I)\) with finite distributive coproducts—i.e., finite coproducts that are preserved by tensor in each variable. For simplicity, we write \( \otimes \) as if it were strictly associative, and for brevity, we may denote tensor by mere juxtaposition. We can now ask: given an object \( H \in \mathcal{C} \)—which in the motivating case will be the set \((0, 1)\)—under what circumstances is there a symmetric monoidal structure \((\ast, 0)\) on \( \mathcal{C} \) with unit the initial object, and tensor

\[
A \ast B := A + H \otimes A \otimes B + B
\]

First let us consider what is necessary to get a monoidal structure. The unit constraints \( A \ast 0 \to A \) and \( 0 \ast A \to A \) are easy; we have canonical isomorphisms

\[
A + HA0 + 0 \to A + 0 + 0 \to A + 0 + H0B + B \to 0 + 0 + B \to B
\]

using the preservation of the initial object by tensor on each side. The associativity constraint \((A \ast B) \ast C \to A \ast (B \ast C)\) is more interesting; it involves a map

\[
(A+HAB+B)+H(A+HAB+B)C+C \to A+HAB+B+HBC+C+B+HBC+C
\]
which, since tensor preserves binary coproducts in each variable, is equally a map
\[ A + HAB + B + HAC + HHABC + HBC + C \rightarrow A + HAB + HAHBC + HAC + B + HBC + C. \]

Now the coherence axioms relating the associativity and the unit constraints force this map to take the summands \( A, B, C, HAC, HAC, HBC \) of the domain to the corresponding summands of the codomain via identity maps, so leaving only the \( HHABC \)-summand of the domain unaccounted for. Though we are not forced to, it would be most natural to map this summand to the \( HAHBC \)-summand of the codomain via a composite
\[
HHABC \xrightarrow{v111} HHABC \xrightarrow{1\sigma11} HAHBC,
\]
where here \( v : HH \to HH \) is some fixed invertible map, and \( \sigma \) is the symmetry.

At this point, we have all the data of a monoidal structure, satisfying all the axioms except perhaps for the Mac Lane pentagon axiom, which equates the two arrows \( ((A \otimes B) \otimes C) \otimes D \Rightarrow A \otimes (B \otimes (C \otimes D)) \) constructible from the associativity constraint cells. If we expand out the definitions, we find that this equality is automatic on most summands of the domain; the only non-trivial case to be checked is the equality of the two morphisms \( HHHABC \Rightarrow HAHBCD \) given by the respective string diagrams (read from top-to-bottom):

\[
(v \otimes 1)(1 \otimes \sigma)(v \otimes 1) = (1 \otimes v)(v \otimes 1)(1 \otimes v) : H \otimes H \otimes H \rightarrow H \otimes H \otimes H.
\]

The preceding argument shows:

**Proposition 23.** Let \( (\mathcal{C}, \otimes, I) \) be a symmetric monoidal category with finite distributive coproducts, let \( H \in \mathcal{C} \) and let \( v : H \otimes H \to H \otimes H \). The pair \((H, v)\) is
a tricocycloid if and only if there is a monoidal structure $(\ast, 0)$ with $\ast$ as in (4.1), and with unit and associativity constraints as in (4.2) and (4.3).

We can think of the object $H$ underlying a tricocycloid as parametrising “ways of non-trivially combining two things”; the map $v$ then compares two ways in which $H \otimes H$ could parametrise “ways of non-trivially combining three things”. This intuition may be clarified in terms of the notion of operad. Operads were introduced by May in [35] as a tool for describing certain kinds of topological-algebraic theory arising in homotopy theory, and involve objects of “$n$-ary operations” for each $n$, with suitable composition laws. The following notion of pseudo-operad, due to Markl, is concerned with the case where the objects of nullary and unary operations are trivial, and so can be omitted.

**Definition 24.** A pseudo-operad [34] in a symmetric monoidal category $\mathcal{C}$ is a sequence $(H_n)_{n \geq 2}$ of objects endowed with maps

\begin{equation}
\circ_i : H_n \otimes H_m \to H_{n+m-1}
\end{equation}

for $n, m \geq 1$ and $1 \leq i \leq n$

rendering commutative the following diagrams for $n, m, k \geq 1$ and $1 \leq i < j \leq n$:

\[
\begin{array}{ccc}
H_n \otimes H_m \otimes H_k & \xrightarrow{1 \otimes \sigma} & H_n \otimes H_k \otimes H_m \\
\circ_i \otimes 1 & \downarrow & \circ_i \otimes 1 \\
H_{n+m-1} \otimes H_k & \xrightarrow{\circ_{j+m-1}} & H_{n+m+k-2}
\end{array}
\]

and the following diagrams for $n, m, k \geq 1$ and $1 \leq i \leq n$ and $1 \leq j \leq m$:

\[
\begin{array}{ccc}
H_n \otimes H_m \otimes H_k & \xrightarrow{1 \otimes \circ_j} & H_n \otimes H_{m+k-1} \\
\circ_i \otimes 1 & \downarrow & \circ_i \\
H_{n+m-1} \otimes H_k & \xrightarrow{\circ_{j+i-1}} & H_{n+m+k-2}
\end{array}
\]

We think of the objects $H_n$ involved in a pseudo-operad as parametrising “ways of non-trivially combining $n$ things”; the maps $\circ_i$ then describe the way of combining $n + m - 1$ things induced by a way of combining $n$ things and a way of combining $m$ things, according to the following schema:

\[
\begin{array}{ccc}
\begin{array}{c}
\alpha \\
\otimes \\
\beta
\end{array} & \mapsto & \\
\begin{array}{c}
\alpha \\
\otimes \\
\beta
\end{array}
\end{array}
\]

In general, there is no reason to expect the objects $H_n$ parametrising $n$-ary combinations for $n \geq 3$ to be determined by the object $H_2$ of binary combinations; but when this is the case, we get a tricocycloid. This idea dates back to Day [11], is explained in detail in the introduction of [12], and is made precise by:

**Lemma 25.** To give a tricocycloid in a symmetric monoidal category $\mathcal{C}$ is equally to give a pseudo-operad for which the maps (4.7) are all invertible.
Proof (sketch). From a pseudo-operad $H$ we obtain a tricocycloid with underlying object $H_2$ and with

\[(4.8) \quad v = H_2 \otimes H_2 \xrightarrow{\sigma_1} H_3 \xrightarrow{(\alpha_2)^{-1}} H_2 \otimes H_2.\]

The tricocycloid axiom follows by constructing a suitable commutative diagram relating the various composition operations $H_2 \otimes H_2 \otimes H_2 \to H_3$, and using invertibility of the maps $\sigma_i$. Conversely, from a tricocycloid $(H,v)$ we construct a pseudo-operad with $H_n = H^\otimes(n-1)$, and with the maps $\sigma_i$ given by suitable composites of $v$ which we will not spell out in general; but let us at least say that, in the lowest dimension, we have $\sigma_1, \sigma_2 : H_2 \otimes H_2 \to H_3$ given by $v, \text{id} : H \otimes H \to H \otimes H$.

We now describe, following [49, Section 4], the additional structure on a tricocycloid needed to induce a symmetry on the associated monoidal structure. Such a symmetry is given by coherent isomorphisms $\sigma_{AB} : A \star B \to B \star A$, i.e., maps $A + HAB + B \to B + HBA + A$, and the coherence axiom relating $\sigma$ with the unit constraints force the $A$- and $B$-summands of the domain to be mapped to the corresponding summands of the codomain. Like before, it is now natural to map the remaining $HAB$-summand to the $HBA$-summand via a composite

\[(4.9) \quad HAB \xrightarrow{\gamma^{11}} HAB \xrightarrow{\sigma} HBA\]

for some fixed map $\gamma : H \to H$. Since a symmetry must satisfy $\sigma_{BA} \circ \sigma_{AB} = 1$, it follows that $\gamma$ must be an involution (i.e., $\gamma^2 = 1$). As for the hexagon axiom relating the symmetry to the associativity, its only non-trivial case expresses the equality of the maps $HHABC \Rightarrow HAHBC$ given by the respective diagrams:

\[(4.10) \quad \begin{array}{c}
\begin{array}{ccc}
H & H & A \\
\gamma & \gamma & \gamma
\end{array}
\end{array} \quad \text{and} \quad \begin{array}{c}
\begin{array}{ccc}
H & H & A \\
\gamma & \gamma & \gamma
\end{array}
\end{array} .\]

Like before, it is necessary and sufficient for this that we should have equality of the diagrams obtained from (4.10) by deleting the $A$-, $B$- and $C$-strings; we encapsulate this requirement in:

**Definition 26.** Let $(H,v)$ be a tricocycloid in the symmetric monoidal category $(\mathcal{C}, \otimes, I)$. A symmetry for $H$ is an involution $\gamma : H \to H$ satisfying the equality

\[(4.11) \quad (1 \otimes \gamma)v(1 \otimes \gamma) = v(\gamma \otimes 1)v : H \otimes H \to H \otimes H.\]

The preceding argument thus shows:

**Proposition 27.** Let $(\mathcal{C}, \otimes, I)$ be a symmetric monoidal category with finite distributive coproducts, and let $(H,v)$ be a tricocycloid in $\mathcal{C}$. An involution $\gamma : H \to H$ is a symmetry for $(H,v)$ just when the maps $\sigma_{AB} : A \star B \to B \star A$ determined by (4.9) endow the associated monoidal structure $(\star, 0)$ on $\mathcal{C}$ with a symmetry.
A symmetry on a tricocycloid can also be described via the corresponding pseudo-operad. We call a pseudo-operad $H$ in $\mathcal{C}$ symmetric if each $H_n$ carries a symmetric group action $\alpha : S_n \to \mathcal{C}(H_n, H_n)$, with respect to which composition is equivariant. It is now straightforward to show that giving a symmetric tricocycloid is the same as giving a symmetric pseudo-operad with all maps (4.7) invertible.

4.2. The convex monoidal structure. Using the preceding theory, we can obtain the associativity and symmetry constraints of the desired convex monoidal structure $A, B \mapsto A + (0, 1) \times A \times B + B$ on $\text{Set}$ by endowing the set $(0, 1)$ with the structure of a symmetric tricocycloid.

This tricycloid is most easily understood by deriving it from a symmetric pseudo-operad. Indeed, for each $n \geq 2$ we may consider the set

$$H_n = \{(r_1, \ldots, r_n) \in (0, 1)^n : r_1 + \cdots + r_n = 1\}.$$

We now have maps $\circ_i : H_n \times H_m \to H_{n+m-1}$ defined by

$$\circ_i((r_1, \ldots, r_n), (s_1, \ldots, s_m)) \mapsto (r_1, \ldots, r_{i-1}, r_is_1, \ldots, r_is_m, r_{i+1}, \ldots, r_n)$$

and maps $\sigma : S_n \times H_n \to H_n$ defined by

$$(g, (r_1, \ldots, r_n)) \mapsto (r_{g(1)}(1), \ldots, r_{g(n)})$$

which easily satisfy the axioms for a symmetric pseudo-operad. Moreover, each of the maps $\circ_i$ is invertible, with inverse

$$(t_1, \ldots, t_{n+m-1}) \mapsto \left((t_1, \ldots, t_{i-1}, u, t_{i+m}, \ldots, t_{n+m-1}), \left(\frac{t_1}{u}, \ldots, \frac{t_{i+m-1}}{u}\right)\right)$$

where here $u := \sum_{j=i}^{i+m-1} t_j$. So by Lemma 25, $H_2$ is a symmetric tricocycloid; transporting this structure along the isomorphism $H_2 \cong (0, 1)$ given by $(r, s) \mapsto r$, we conclude that $(0, 1)$ is a symmetric tricocycloid. The following result spells the structure out, and gives a direct proof of the symmetric tricocycloid axioms.

**Proposition 28.** In the cartesian monoidal category of sets, $(0, 1)$ is a symmetric tricocycloid, under the operations

$$v : (0, 1)^2 \mapsto (0, 1)^2, \quad \gamma : (0, 1) \mapsto (0, 1),$$

$$(r, s) \mapsto (rs, \frac{r \cdot s^*}{(rs)^*}), \quad r \mapsto r^*,$$

where, as before, we write $r^* = 1 - r$ for any $r \in (0, 1)$.

**Proof.** We begin by checking that $((0, 1), v)$ is a tricocycloid. It is easy arithmetic to see that $rs$ and $r \cdot s^*/(rs)^*$ are in $(0, 1)$ whenever $r$ and $s$ are, so that $v$ is well-defined. For the tricocycloid axiom, the function $(v \times 1)(1 \times \sigma)(v \times 1) : (0, 1)^3 \to (0, 1)^3$ is given by

$$(r, s, t) \mapsto (rs, \frac{r \cdot s^*}{(rs)^*}, t) \mapsto (rs, t, \frac{r \cdot s^*}{(rs)^*}) \mapsto (rst, \frac{rs \cdot t}{(rst)^*}, \frac{r \cdot s^*}{(rs)^*})$$

while the map $(1 \times v)(v \times 1)(1 \times v) : (0, 1)^3 \to (0, 1)^3$ is given by

$$(r, s, t) \mapsto (r, st, \frac{s \cdot r^*}{(sr)^*}) \mapsto (rst, \frac{r \cdot s^*}{(rst)^*}, \frac{s \cdot r^*}{(sr)^*}) \mapsto (rst, \frac{r \cdot s^*}{(rst)^*}, \frac{s \cdot r^*}{(sr)^*})^* / \left(\frac{r \cdot s^*}{(rst)^*}\right)^*.$$
so that the desired equality follows from the calculation
\[
\left(\frac{r(s)^*}{(r \cdot s)^*}\right) \left(\frac{s^*}{(s^*)^*}\right) = \left(\frac{r(s)^*}{(r \cdot s)^*}\right) \left(\frac{s^*}{(s^*)^*}\right) = \frac{r \cdot s^*}{(r \cdot s)^*}.
\]

We note also that \(v\) is invertible, with inverse \(v^{-1}(t, u) = \left(\frac{(t^* \cdot u)^*}{(t \cdot u)^*}\right)\); that \(v \circ v^{-1}\) and \(v^{-1} \circ v\) are identities follows by a short calculation using (4.15).

We now show that \(\gamma\) provides a symmetry for the tricocycloid \(((0, 1), v)\). Clearly \(\gamma\) is an involution, so it remains to check the coherence axiom. The map 
\[(1 \times \gamma) v (1 \times \gamma) : (0, 1)^2 \to (0, 1)^2\]
is given by 
\[(r, s) \mapsto (r, s^*) \mapsto (r \cdot s^*, \frac{r \cdot s^*}{(r \cdot s^*)^*}) \mapsto \left(r \cdot s^*, \left(\frac{r \cdot s^*}{(r \cdot s^*)^*}\right)^*\right),\]
while \(v(\gamma \times 1) v: (0, 1)^2 \to (0, 1)^2\) is given by 
\[(r, s) \mapsto (rs, \frac{r \cdot s^*}{(r \cdot s^*)^*}) \mapsto (rs^*, \frac{r \cdot s^*}{(r \cdot s^*)^*}) \mapsto \left(r \cdot s^*, \left(\frac{r \cdot s^*}{(r \cdot s^*)^*}\right)^*\right).\]

To check the equality of the second terms, we calculate using (4.15) twice that:
\[(rs)^* \left(\frac{r \cdot s^*}{(r \cdot s^*)^*}\right)^* = (rs)^* \frac{r \cdot s^*}{(r \cdot s^*)^*} = \frac{r \cdot s^*}{(r \cdot s^*)^*} = \left(\frac{r \cdot s^*}{(r \cdot s^*)^*}\right)^* .\]

Since the definition of \(v\) derives from the coproduct of abstract convex spaces, it is not unreasonable that the same coefficients should appear here as in the convex space axioms. We will see the deeper reason for this in Section 6.

**Definition 29.** The **convex monoidal structure** is the symmetric monoidal structure \((\ast, 0)\) on the category of sets associated to the convex tricocycloid \(((0, 1), v, \gamma)\).

Working through the details of Lemma 7, the reader should have no difficulty in verifying that the forgetful functor \(\text{Conv} \to \text{Set}\) is strict symmetric monoidal with respect to the coproduct monoidal structure on \(\text{Conv}\) and the convex monoidal structure on \(\text{Set}\). In light of Lemma 6 and Lemma 12, we have thus verified:

**Proposition 30.** With respect to the convex monoidal structure on \(\text{Set}\), the discrete distribution monad \(\mathbb{D}\) is a linear exponential monad.

It is now easy to check that the abstract hypernormalisation maps (3.7) reduce in this case to Jacobs’ hypernormalisation maps (2.3), as desired.

## 5. Probabilistic examples

In the following two sections, we describe instances of abstract hypernormalisation which go beyond the motivating case. In this section, we explore examples involving continuous probability monads; the hypernormalisation arising here is suitable for the channel-to-abstract-channel construction of Section 3.3 above, and so for non-discrete generalisations of the theory of [36].

### 5.1. The expectation monad

The next simplest probabilistic monad beyond the finite discrete case is the so-called **expectation monad** \(\mathcal{E}\) on the category of sets. This was named and investigated in [26], but dates back to [51], where it was described, implicitly, as the monad generated by the chain of adjunctions:

\[(\mathcal{K}\text{Conv} \xrightarrow{\text{id}} \mathcal{K}\mathcal{H} \xrightarrow{\text{id}} \text{Set}).\]

Here, \(\mathcal{K}\mathcal{H}\) is the category of compact Hausdorff spaces; while \(\mathcal{K}\text{Conv}\) is the category whose objects are compact convex subsets \(A\) of locally convex vector
spaces, and whose morphisms are continuous affine maps (where affineness is the condition \( f(ra + r'a') = rf(a) + r^*f(a') \)). The two right adjoints in (5.1) are the obvious forgetful functors, while the two left adjoints send, respectively, a set \( X \) to its space of ultrafilters \( \beta X \) with the Stone topology, and a compact Hausdorff space \( Y \) to its space of Radon probability measures, identified via the Riesz representation theorem with the positive elements of norm 1 in the ordered Banach space of continuous linear functionals \( C(Y, \mathbb{R}) \to \mathbb{R} \).

As explained in [26], the monad \( E \) induced by the composite adjunction (5.1) can be described in various ways; the most direct is as follows. We write \( E X \) for the set of normalised, finitely additive functions \( \omega : P X \to [0,1] \), i.e., functions such that \( \omega(X) = 1 \) and \( \omega(A \cup B) = \omega(A) + \omega(B) \) whenever \( A, B \subseteq X \) are disjoint. The action of \( E \) on morphisms is given by pushforward, \( (E f)(\omega)(B) = \omega(f^{-1}(B)) \); the monad unit \( \eta_X : X \to E X \) takes \( x \in X \) to the Dirac distribution with \( \eta_X(x)(A) = 1 \) if \( x \in A \) and \( \eta_X(x)(A) = 0 \) otherwise; while the monad multiplication is given by a suitable notion of integration against a valuation:

\[
\mu_X(\omega)(A) = \int_{\tau \in E X} \tau(A) \, d\omega.
\]

The details may be found in [26]; however, we will not need them to describe hypernormalisation for \( E \). Rather, we need only make \( E \) into a linear exponential monad, which we can do from the perspective of the category of algebras using:

**Proposition 31.** ([51, Theorem 4]) The composite adjunction in (5.1) is monadic.

Thus \( E \)-algebras can be identified with compact convex subsets of locally convex vector spaces; and so if we can understand finite coproducts of these, we can obtain the desired linear exponential structure on \( E \).

**Proposition 32.** If \( A \subseteq V \) and \( B \subseteq W \) are objects of \( \mathsf{KC}_{\text{Conv}} \), then their coproduct may be given as

\[
\{ (ra, r^*b, r) : r \in [0,1], a \in A, b \in B \} \subseteq V \oplus W \oplus \mathbb{R}.
\]

**Proof.** This is [46, Proposition 7]. \( \square \)

At the level of underlying sets, the coproduct of \( A \) and \( B \) in (5.2) is given by

\[
\{(a,0,1) : a \in A\} + \{(ra, r^*b, r) : r \in (0,1), a \in A, b \in B\} + \{(0, b, 0) : b \in B\}
\]

which is clearly isomorphic to \( A + (0,1) \times A \times B + B = A \star B \). In a similar way, the coherence constraints for the coproduct in \( \mathsf{KC}_{\text{Conv}} \) lift the coherence constraints for the convex monoidal structure on \( \mathsf{Set} \), and so we have:

**Proposition 33.** The expectation monad on \( \mathsf{Set} \) is a linear exponential monad with respect to the convex monoidal structure.

It follows that the expectation monad admits a notion of hypernormalisation. To calculate this, we first define, like before, the normalisation of a valuation \( \omega : P A \to [0,1] \) with \( \omega(A) > 0 \) to be the normalised valuation \( \overline{\omega} \) with \( \overline{\omega}(U) = \omega(U)/\omega(A) \). Noting that each set \( E A \) admits a structure of abstract
convex space where \( r(\omega_1, \omega_2)(U) = r\omega_1(U) + r^*\omega_2(U) \), we may now describe the hypernormalisation map \( N: \mathcal{E}(\Sigma_i A_i) \to \mathcal{E}(\Sigma_i \mathcal{E}A_i) \) as in (2.2) by

\[
N(\omega) = \sum_{1 \leq i \leq n} \omega_i(A_i) \cdot t_i(\mathcal{E}I).
\]

5.2. The Radon monad. The expectation monad \( \mathbb{E} \) on \( \text{Set} \) arose from the composite adjunction in (5.1); on the other hand, the left-hand adjunction in (5.1) induces the Radon monad \( \mathcal{R} \) on \( \mathbb{KH} \). Again, this was introduced implicitly in [51], with the details now being provided by [38].

**Definition 34.** Let \( X \) be a compact Hausdorff space. A Radon probability measure on \( X \) is a probability measure \( \omega: \Sigma_X \to [0, 1] \) on the Borel \( \sigma \)-algebra of \( X \) such that \( \omega(M) = \sup\{\omega(K): K \subseteq M, K \text{ compact}\} \) for all \( M \in \Sigma_X \). We write \( \mathcal{R}(X) \) for the space of Radon probability measures on \( X \) with the weak topology: the coarsest topology such that, for each continuous \( f: X \to \mathbb{R} \), the integration map \( \omega \mapsto \int f \, d\omega \) is continuous as a function \( \mathcal{R}(X) \to \mathbb{R} \).

The remaining aspects of the Radon monad \( \mathcal{R} \) on \( \mathbb{KH} \) are much as before: the action on morphisms is by pushforward, the monad unit selects the Dirac valuations, and the multiplication is given by integration against a measure. To obtain our notion of hypernormalisation, we will again exploit monadicity, using:

**Proposition 35.** ([51, Theorem 4]) The left-hand adjunction in (5.1) is monadic.

So, identifying the category of \( \mathcal{R} \)-algebras with the category \( \mathbb{KH}^{\text{conv}} \) on \( \mathbb{KH} \), it only remains to relate the coproduct (5.2) of compact convex spaces with a suitable monoidal structure on \( \mathbb{KH} \). This will be the well-known topological join:

**Definition 36.** The join of two topological spaces \( X \) and \( Y \) is the quotient space of the product space \( [0, 1] \times X \times Y \) under the smallest equivalence relation \( \sim \) for which \( (0, x, y) \sim (0, x', y) \) and \( (1, x, y) \sim (1, x, y') \) for all \( x, x' \in X \) and \( y, y' \in Y \).

We can realise the topological join \( X \ast Y \) as the set \( X + (0, 1) \times X \times Y + Y \), with a basis for the topology generated by sets of three forms

\[
U + (0, a) \times U \times Y + \emptyset \quad \emptyset + (a, b) \times U \times V + \emptyset \quad \emptyset + (b, 1) \times X \times V + V
\]

for all rationals \( 0 < a < b < 1 \) and all \( U \subseteq X, V \subseteq Y \) open. Presented in this way, it is easy to see that topological join is part of a monoidal structure \( (\ast, 0) \) on \( \mathbb{KH} \) which lifts the convex monoidal structure on \( \mathbb{Set} \). On the other hand, comparing with the formula (5.2), we conclude that the forgetful functor \( \mathbb{KH}^{\text{conv}} \to \mathbb{KH} \) sends coproduct to topological join, and so we obtain:

**Proposition 37.** The Radon monad on \( \mathbb{KH} \) is a linear exponential monad with respect to the join monoidal structure \( (\ast, 0) \) on \( \mathbb{KH} \).

We thus obtain hypernormalisation maps \( N: \mathcal{R}(\Sigma_i A_i) \to \mathcal{R}(\Sigma_i \mathcal{R}A_i) \) whose action on a distribution \( \omega \) is obtained in much the same way as previously. The key point is that we get continuity of \( N \) for free: something which otherwise would have required some fairly messy calculation.
5.3. The Kantorovich monad. For our final example of probabilistic hypernor-
malisation, we consider the Kantorovich monad [54] on the category \( \mathbb{C}\text{Met}_1 \) whose
objects are complete metric spaces which are 1-bounded (i.e., \( d(x, y) \leq 1 \) for all
\( x, y \)) and whose morphisms are 1-Lipschitz mappings (i.e., \( d(fx, fy) \leq d(x, y) \)
for all \( x, y \)).

**Definition 38.** Let \( X \) be a complete 1-bounded metric space. \( \mathcal{K}(X) \) is the
complete metric space whose elements are Radon probability measures on \( X \),
under the Kantorovich (or “earth-mover’s”) metric:
\[
\inf \left\{ \int dX(x, y) \, d\mu(x, y) : \mu \in \mathcal{K}(X \times X); (\pi_1)_*(\mu) = \omega, (\pi_2)_*(\mu) = \pi \right\}
\]
where the infimum is over joint distributions on \( X \times X \) with marginals \( \omega \) and \( \pi \).

This operation underlies a monad on \( \mathbb{C}\text{Met}_1 \) following the established pattern;
and to exhibit a notion of hypernormalisation, we also follow the established
pattern, by investigating coproducts of \( K \)-algebras. We begin with the characteris-
tisation of these algebras.

**Definition 39.** A **convex metric space** is a metric space \( X \) which is also a convex
space in the sense of Definition 5, subject to the compatibility condition
\[
(5.3) \quad d(r(x, z), r(y, z)) \leq rd(x, y) \quad (\text{or equally, } d(r(x, y), r(x, z)) \leq r^*d(y, z)) \, .
\]
We write \( \mathbb{C}\text{ConvMet}_1 \) for the category of 1-bounded complete convex metric
spaces, and convex 1-Lipschitz maps.

**Proposition 40.** The category of \( K \)-algebras is isomorphic to \( \mathbb{C}\text{ConvMet}_1 \) over
\( \mathbb{C}\text{Met}_1 \).

**Proof.** This is [16, Theorem 5.2.1] (though see also [33, Theorem 10.9]). \( \square \)

And now we characterise finite coproducts in this category.

**Proposition 41.** The coproduct of \( X, Y \in \mathbb{C}\text{ConvMet}_1 \) is the coproduct
\( X \star Y \) of underlying convex spaces, endowed with the metric:
\[
(5.4) \quad d_{X \star Y}(r \cdot x + r^* \cdot y, s \cdot w + s^* \cdot z) = rd(x, w) + (s - r) + (1 - s)d(y, z)
\]
for any \( 0 \leq r, s \leq 1 \); here, by convention, we allow \( 1 \cdot x + 0 \cdot y \) to denote
\( x \in X \subseteq X \star Y \), and correspondingly for \( 0 \cdot x + 1 \cdot y \).

**Proof.** It is a straightforward calculation to show that this is indeed a complete,
1-bounded metric satisfying \( (5.3) \); indeed, it is easy to see that \( d_{X \star Y} \) is really just
the Kantorovich metric restricted to distributions concentrated at two points.

To exhibit the universal property of coproduct, let \( f : X \to Z \) and \( g : Y \to Z \)
be maps in \( \mathbb{C}\text{ConvMet}_1 \), and let \( (f, g) : X \star Y \to Z \) be the induced unique map
of convex spaces as in Section 2.2. It suffices to show that \( (f, g) \) is 1-Lipschitz.
For the most involved case, consider elements \( r \cdot x + r^* \cdot y \) and \( s \cdot w + s^* \cdot z \) in
\( X \star Y \) with \( 0 < r, s < 1 \). We must show that
\[
d(r(fx, gy), s(fw, gz)) \leq rd(x, w) + (s - r) + (1 - s)d(y, z) \, .
\]
Now, the left-hand side is by the triangle inequality smaller than
\[
d(r(fx, gy), r(fw, gy)) + d(r(fw, gy), s(fw, gy)) + d(s(fw, gy), s(fw, gz))
\]
and we calculate that $d(r(fx,gy), r(fw,gy)) \leq rd(fx,fw) \leq rd(x,w)$ and $d(s(fw,gy), s(fw,gz)) \leq s^*d(gy,gz) \leq s^*d(y,z)$ using (5.3) and contractivity of $f$ and $g$. So it suffices to show that $d(r(fw,gy), s(fw,gy)) \leq s - r$ in $Z$. Writing $u = fw$ and $v = gy$, this follows by the calculation

$$d(r(u,v), s(u,v)) = d(s\left(\frac{r}{s}(u,v), v\right), s(u,v)) \leq sd\left(\frac{r}{s}(u,v), u\right) = sd\left(\frac{r}{s} - \frac{r}{s}(u,u)\right) \leq (s-r)d(v,u) \leq s - r.$$ 

We are thus in the familiar situation that the $\mathbb{C} \text{Met}_1$-object underlying the coproduct of $X, Y \in \mathbb{C} \text{ConvMet}_1$ depends only on the underlying $\mathbb{C} \text{Met}_1$-objects of $X$ and $Y$. Accordingly, we make:

**Definition 42.** The *join* of two 1-bounded complete metric spaces $X$ and $Y$ is the set $X \star Y = X + (0,1) \times X \times Y + Y$ endowed with the metric (5.4). This provides the binary tensor of the convex monoidal structure $(\star, 0)$ on $\mathbb{C} \text{Met}_1$, whose remaining data is all lifted from the convex monoidal structure on $\text{Set}$.

And so obtain:

**Proposition 43.** The Kantorovich monad on $\mathbb{C} \text{Met}_1$ is a linear exponential monad with respect to the join monoidal structure $(\star, 0)$ on $\mathbb{C} \text{Met}_1$.

As such, we have a good notion of hypernormalisation for the Kantorovich monad. Once again, the maps $N: \mathcal{K}(\Sigma_i A_i) \to \mathcal{K}(\Sigma_i \mathcal{K}A_i)$ are defined in the expected way—but we obtain with no extra work the fact that this is a contractive mapping, as required.

### 5.4. Other probability monads.

There are many other probability monads in existence; for example, the Giry monads [20] on the category of measurable spaces, and on the category of Polish spaces; the probabilistic powerdomain on the category of dcpos [28], and on the category of continuous dcpos; and so on.

In each case it would be reasonable to try and derive a notion of hypernormalisation following the pattern set out above. However, in many of these other cases, we are hampered by a lack of a concrete description of the monad algebras: while they are always some kind of barycentric algebra, the precise structure involved is hard to pin down. Without a concrete description of the algebras, we cannot give a concrete description of their finite coproducts; and so cannot in this way obtain the required linear exponential structure.

However, another approach is possible. As explained in [6], to obtain a linear exponential structure on a monad $T$ with respect to a monoidal structure $(\otimes, I)$, it suffices to exhibit suitably coherent isomorphisms $T(A + B) \cong TA \otimes TB$ and $T(0) \cong I$. For other probability monads, we can perfectly well do just this—but careful analysis will be necessary. For example, in the case of the Giry monad on measurable spaces, we would need to exhibit the space $\mathcal{G}(X + Y)$ of measures on a coproduct of measurable spaces as obtained by combining in some manner the measurable spaces $\mathcal{G}X$ and $\mathcal{G}Y$. At the level of underlying sets, this is easy: we have as usual $\mathcal{G}(X + Y) \cong \mathcal{G}X + (0,1) \times \mathcal{G}X \times \mathcal{G}Y + \mathcal{G}Y$. But, describing the $\sigma$-algebra of $\mathcal{G}(X + Y)$ in terms of those for $\mathcal{G}X$ and $\mathcal{G}Y$ seems much harder—and so we leave this to future work.
6. Combinatorial examples

In this section, we first consider further examples of our framework that arise by replacing \((0, 1)\) by a different symmetric tricocycloid in the category of sets. One important situation that is not quite captured by this is that where finite probability distributions on a set are replaced by “logical distributions”—convex combinations whose coefficients are drawn not from \([0, 1]\) but from a given Boolean algebra \(B\); however, we will see that we can capture this example by instantiating our framework in a category other than the category of sets. Finally in this section, we consider an example in which hypernormalisation implements an extensional collapse for programs of type \(\text{Stream}(A) \to B\).

6.1. Other tricocylcoids. As noted above, our next examples of abstract hypernormalisation will arise from other tricocylcoids in the category of sets. To motivate the manner in which this will happen, we first explain how we can derive the finite discrete distribution monad \(D\) from the convex tricocylcoid.

First of all, the convex tricocylcoid induces the convex monoidal structure \((\star, 0)\), and so, as with any symmetric monoidal structure, we can consider the commutative \(\star\)-monoids. It is easy to see that these are almost abstract convex spaces: they are sets \(A\) endowed with an operation \((0, 1) \times A \to A \to A\) satisfying axioms (ii) and (iii), but not necessarily (i) from Definition 5. The missing axiom (i) is the idempotency condition \(r(a, a) = a\), which we can capture abstractly as follows.

**Definition 44.** A monoidal diagonal for a symmetric monoidal category \((\mathcal{C}, \otimes, I)\) is a monoidal natural transformation \(\delta : \text{id} \Rightarrow \otimes \circ \Delta : \mathcal{C} \to \mathcal{C}\); this comprises a natural family of maps \(\delta_A : A \to A \otimes A\) rendering commutative each diagram:

\[
\begin{array}{c}
\delta_A \otimes \delta_B \\
\downarrow \quad \downarrow \\
A \otimes A \otimes B \otimes B \\
\delta_A \otimes B \\
\end{array}
\quad \quad \quad \quad
\begin{array}{c}
1 \otimes \delta_B \\
\downarrow \\
A \otimes B \otimes A \otimes B \\
\end{array}
\]

A commutative \(\otimes\)-monoid \((A, m, e)\) in \(\mathcal{C}\) is idempotent if, whenever \(\delta\) is a monoidal diagonal for \((\mathcal{C}, \otimes, I)\), we have that \(m \circ \delta_A = \text{id}_A : A \to A\).

When \((\mathcal{C}, \otimes, I)\) is \((\text{Set}, \times, 1)\), the only monoidal diagonal is the usual diagonal \((\text{id}, \text{id}) : A \to A \times A\), and so idempotency in the above sense coincides with idempotency in the usual sense. On the other hand:

**Lemma 45.** A commutative monoid in \((\text{Set}, \star, 0)\) is idempotent just when it satisfies axiom (i) as well as axioms (ii) and (iii) in Definition 5—in other words, just when it is an abstract convex space.

**Proof.** By considering naturality with respect to maps \(1 \to A\), we see that the possible monoidal diagonals \(\delta_r : \text{id} \Rightarrow \star \circ \Delta\) are indexed by \(r \in [0, 1]\); we have that \(\delta_0, \delta_1\) are the left and right coproduct coprojections \(A \to A + ((0, 1) \times A \times A) + A\), and that \(\delta_r(a) = (r, a, a)\) for \(0 < r < 1\). It follows that a commutative \(\star\)-monoid is idempotent just when it satisfies the additional axiom \(r(a, a) = a\) for all \(0 < r < 1\)—that is, just when it is an abstract convex space. \(\square\)
So from the convex tricocycloid, we can obtain abstract convex spaces as the idempotent commutative monoids for the associated monoidal structure \((\star, 0)\), and obtain the monad \(D\) as the free idempotent commutative \(\star\)-monoid monad. More generally, for any symmetric tricocycloid \(H\) with structure maps \(v: H^2 \mapsto H^2\), \((r, s) \mapsto (rs, r \circ s)\), \(\gamma: H \mapsto H\), \(r \mapsto r^*\),

we can consider idempotent commutative monoids for the associated monoidal structure \(\star_H\); these will be sets \(A\) equipped with an operation \(H \times A \times A \to A\), written like before as \(r, a, b \mapsto r(a, b)\), that satisfies

\[
\begin{align*}
  r(a, a) &= a \\
  r(a, b) &= r^*(b, a) \\
  r(s(a, b), c) &= (rs)(a, (r \circ s)(b, c)).
\end{align*}
\]

Since this is clearly algebraic structure, we obtain a monad \(D_H\) on \(\text{Set}\) whose algebras are these “\(H\)-convex sets”. In fact, this monad is automatically linear exponential for the \(\star_H\)-monoidal structure, and so admits a notion of hypernormalisation.

We now illustrate this with some other examples of symmetric tricocycloids in \(\text{Set}\). In giving these examples, we can exploit the fact that \(D_H\) is linear exponential to calculate the action of the monads \(D_H\). Indeed, it is clear that \(D_H(1) \cong 1\), whence for any finite set \(n \cong 1 + \cdots + 1\) we have \(D_H(n) \cong 1 \star_H \cdots \star_H 1\); and to extend to infinite sets, we note that the theory of \(H\)-convex sets is finitary, so that \(D_H(A)\) is the directed colimit of the sets \(D_H(n)\) for all finite \(n \subseteq A\).

**Example 46.** Consider the one-element symmetric tricocycloid \(1\). In this case, the induced monoidal structure is given by \(A \star 1 B = A + A \times B + B\), which it is also fruitful to think of as

\[
(6.2) \quad A \star 1 B = (\{A + \{\bot\}\} \times (B + \{\bot\})) \setminus (\{\bot, \bot\}).
\]

In this case, the idempotent commutative \(\star_1\)-monoids are *join-semilattices* (possibly without bottom element), the monad \(D_1\) is the non-empty finite powerset monad, and hypernormalisation \(N: D_1(A + B) \to D_1(D_1A + D_1B)\) is given by

\[
\begin{align*}
  \{a_1, \ldots, a_n\} &\mapsto \{\{a_1, \ldots, a_n\}\} \\
  \{b_1, \ldots, b_m\} &\mapsto \{\{b_1, \ldots, b_m\}\} \\
  \{a_1, \ldots, a_n, b_1, \ldots, b_m\} &\mapsto \{\{a_1, \ldots, a_n\}, \{b_1, \ldots, b_m\}\}.
\end{align*}
\]

Computationally, going from \(D\) to \(D_1\) amounts to stepping back from probabilistic to non-deterministic (terminating) computation; we are interested in *possibility* rather than *probability*. 
Example 47. Let $X$ be a topological space, and let $H$ be the set of continuous functions $X \to (0, 1)$ with the tricocycloid structure given pointwise as in $(0, 1)$. In this case, a typical example of a $H$-convex set is the set of global sections of a sheaf of vector spaces on $X$; while the action of the monad $D_H$ is given by

$$D_H(A) = \{ S \subseteq A \text{ finite, } \omega : X \to \Delta_S \text{ continuous} \}$$

where $\Delta_S$ is the interior of the standard topological $|S|$-simplex, given by a singleton when $|S| = 1$, and by $\{ r \in (0, 1)^S : \sum_{s \in S} r(s) = 1 \}$ otherwise. In other words, the elements of $D_H(A)$ are those of $A$, together with all non-trivial “finite convex combinations”

$$\sum_{1 \leq i \leq n} f_i \cdot a_i$$

where each $a_i \in A$, and where the $f_i$’s are continuous maps $X \to (0, 1)$ satisfying $\sum_i f_i = 1$ (i.e., constituting a partition of unity). In this case, the notion of hypernormalisation carries over mutatis mutandis from the motivating case, where arithmetic on the coefficients $f_i$ is done pointwise in $(0, 1)$.

Before continuing, we take a slight detour in order to deepen our understanding of tricocycloids. As we have just seen, we can obtain the discrete distribution monad from the tricocycloid $(0, 1)$. However, as shown in [24], we may also obtain it in an apparently different way: from the effect monoid structure on $[0, 1]$.

Definition 48. (cf. [14]) A partial commutative monoid is a set $M$ with a constant 0 and partial binary operation $\cdot : M \times M \rightharpoonup M$ satisfying the axioms:

$$r \cdot 0 \simeq r \simeq 0 \cdot r \quad (r \cdot s) \cdot t \simeq r \cdot (s \cdot t) \quad r \cdot s \simeq s \cdot r$$

where $\simeq$ denotes Kleene equality of partially defined functions, i.e., one side is defined just when the other is, and they are then equal. $M$ is an effect algebra if it is equipped with a constant 1 and (total) unary operation $(\cdot) \perp$ such that:

(i) For all $r \in M$, the element $r \perp$ is unique such that $r \cdot r \perp \simeq 1$;

(ii) If $r \cdot 1$ is defined, then $r = 0$.

An effect algebra is an effect monoid if it comes equipped with a (total) binary operation $r, s \mapsto r \cdot s$ which is associative and has unit 1, and which distributes over $\cdot$; i.e., we have equalities

$$r \cdot 0 = 0 = 0 \cdot r \quad r \cdot (s \cdot t) \simeq' (r \cdot s) \cdot (r \cdot t) \quad (r \cdot s) \cdot t \simeq' (r \cdot s) \cdot (r \cdot t)$$

where $\simeq'$ means “if the left-hand side is defined, then so is the right-hand side, and they are equal”.

Examples 49. (i) $[0, 1]$ is an effect monoid where 0, 1 have their usual meanings; $\cdot$ is ordinary multiplication; $r \cdot s$ is given by $r + s$ if this sum lies in $[0, 1]$, and is undefined otherwise; and where $r \perp = 1 - r$.

(ii) Any Boolean algebra is an effect monoid, where 0, 1 are $\bot, \top$, where $\cdot$ is intersection $\land$, where $r \cdot s$ is given by $r \lor s$ if $r \land s = \bot$, and is undefined otherwise; and where $r \perp$ is the complement of $r$. 
As shown in [24], any effect monoid $M$ induces a monad $D_M$ on Set whose action on objects is given by

$$D_M(A) = \{ \omega: A \to M \mid \text{supp}(\omega) \text{ finite and } \bigotimes_{a \in A} \omega(a) \simeq 1 \}$$

and whose remaining structure is defined by analogy with the discrete distribution monad; in particular, if $M = [0,1]$, we get the discrete distribution monad itself.

**Remark 50.** It is natural to ask how these two constructions of $D$—from the tricocycloid $(0,1)$ and the effect monoid $[0,1]$—relate to each other. More generally, we may ask whether the monad $D_M$ associated to an effect monoid $M$ also arises as the monad $D_H$ associated to some tricocycloid, or vice versa. This question was investigated in detail by Kaddar [30]. One of his main results is that the assignments

$$M \mapsto M \setminus \{0,1\} \quad \text{and} \quad H \mapsto H \amalg \{0,1\}$$

give a bijection between tricocycloids satisfying certain cancellativity properties, and effect monoids with *normalisation*, i.e., effect monoids such that for all $a,b$ with $a \otimes b$ defined and $a \neq 1$, there is a unique $c$ with $b = a \ltimes c$. In particular, this construction relates the tricocycloid $(0,1)$ and the effect monoid $[0,1]$.

An intuitive way of understanding this correspondence is by way of the notion of pseudo-operad from Definition 24. Any effect monoid $M$ gives a symmetric pseudo-operad with underlying sets

$$H_n = \{(m_1, \ldots, m_n) \in (M \setminus \{0,1\})^n : m_1 \otimes \cdots \otimes m_n \simeq 1\},$$

and structure maps $\circ_i$ and $\sigma$ defined as in (4.12) and (4.13). By Lemma 25, this pseudo-operad yields a tricocycloid just when each map $\circ_i: H_n \times H_m \to H_{n+m-1}$ is invertible: which is exactly the condition that $M$ admit normalisation.

Although this is not proven in detail in [30], it flows from the above understanding that the monad $D_M$ of an effect monoid with normalisation coincides with the monad $D_H$ of the corresponding tricocycloid: the point is that the $n$-ary “$M$-convex combination operations” for $D_M$ can via normalisation be decomposed into composites of binary convex combinations.

6.2. **Logical hypernormalisation.** So far we have said nothing about the second example of an effect monoid from Examples 49, that of a Boolean algebra. One might hope this example to be associated to some kind of “logical hypernormalisation”. This does turn out to be the case, but there are some subtleties, as we now explain.

**Example 51.** Let $B$ be a non-trivial Boolean algebra; we may view $B$ as an effect monoid as in Examples 49, and so may form the associated monad $D_B$ as in (6.4), with action on objects given by:

$$D_B(X) = \{ \omega: X \to B \mid \omega \text{ has finite support and } \text{im}(\omega) \text{ is a partition of } B \}.$$

The Eilenberg–Moore algebras for $D_B$ admit a characterisation via binary operations due to Bergman [5, Theorem 14]: they are sets $A$ endowed with an operation $B \times A \times A \to A$, written $r, a, b \mapsto r(a, b)$ as usual, satisfying the following axioms:
(i) \( r(a, a) = a \);
(ii) \( r(a, b) = r^\perp(b, a) \);
(iii) \( 0(a, b) = b \);
(iv) \( r(r(a, b), c) = r(a, c) \);
(v) \( r(s(a, b), b) = (rs)(a, b) \).

But in fact, in the presence of (i)–(iii), axioms (iv)–(v) may be replaced by:
(iv)' \( r(s(a, b), c) = (sr)(a, r(b, c)) \).

The easier direction is that taking \( b = c \) in (iv)' yields (v), while taking \( s = r^\perp \) and using (iii) yields (iv). Conversely, if we assume (iv) and (v), then we obtain (iv)' by the calculation

\[
(sr)(a, r(b, c)) = s(r(a, r(b, c)), r(b, c)) = s(r(a, c), r(b, c)) = r(s(a, b), s(c, c)) = r(s(a, b), c)
\]

using, in turn: (v); the equality \( r(a, r(b, c)) = r(a, c) \) obtained from (iv) and (ii); the non-trivial equality \( r(s(w, x), s(y, z)) = s(r(w, y), r(x, z)) \) of [5, Proposition 11]; and (i).

Thus, if we write \( r(a, b) \) as \( a +_r b \), then this alternate axiomatisation becomes:

\[
a +_r a = a \quad a +_r b = b +_r a \quad (a +_s b) +_r c = a +_sr (b +_r c) \quad a +_0 b = b
\]

which are the axioms \( U1–U3, U7 \) for the operation of \textit{guarded union} in the theory of guarded Kleene algebra with tests [47].

**Remark 52.** The nice description of the \( D_B \)-algebras in this example does not follow from the arguments of Remark 50. For indeed, seen as an effect monoid, \( B \) does not admit normalisation, since in any effect monoid with normalisation, \( r \neq 0 \) and \( rs = rt \) implies \( s = t \), which is clearly not true in a Boolean algebra.

However, we can give a more careful explanation as to why \( D_B \) is generated by binary operations, by considering, again, the associated pseudo-operad of \( B \) with underlying sets (6.5). Since \( B \) does not admit normalisation, the \( \circ_i \)-maps of this pseudo-operad as in (4.12) are not invertible. However, they do admit well-behaved sections \( (\circ_i)^* \) given by

\[
(t_1, \ldots, t_{n+m-1}) \mapsto ((t_1, \ldots, t_{i-1}, u, t_{i+m}, \ldots, t_{n+m-1}), (u \Rightarrow t_i, t_{i+1}, \ldots, t_{i+m-1}))
\]

where we write \( u \) for \( \bigvee_{j=i}^{j=m-1} t_i \) and \( u \Rightarrow t_i \) for \( \neg u \lor t_i \). Using these sections, we can obtain a map \( v: H_2 \times H_2 \to H_2 \times H_2 \) much like in (4.8) as the composite

\[
H_2 \otimes H_2 \overset{\circ_1}{\longrightarrow} H_3 \overset{(\circ_2)^*}{\longrightarrow} H_2 \otimes H_2;
\]

and while the \( v \) so obtained is not invertible, it does satisfy the tricocycloid axiom (4.6). This means that applying the construction of Section 4.1 \textit{mutatis mutandis} yields not a monoidal structure, but a \textit{skew monoidal} structure in the sense of [52], with a symmetry induced by the (invertible) \( (\cdot)^*: H_2 \to H_2 \). In this circumstance, there is no problem in still considering idempotent commutative monoids—and on doing, we recover Bergman’s description of the \( D_B \)-algebras via binary operations given above.

The fact that a Boolean algebra \( B \) \textit{qua} effect monoid lacks normalisation prevents us from carrying out “logical hypernormalisation” for the monad \( D_B \).
Indeed, consider an element \( \omega \in D_B(X + Y) \). We have the (complementary) elements \( \omega(X) = \bigvee_{x \in X} \omega(x) \) and \( \omega(Y) = \bigvee_{y \in Y} \omega(y) \) of \( B \), and from this may attempt to produce an element \( N(\omega) \in D_B(D_B X + D_B Y) \). We focus on the interesting case where neither \( \omega(X) \) nor \( \omega(Y) \) are \( \bot \); here, writing \( \omega_X \) and \( \omega_Y \) for the restriction of \( \omega \) to \( X \) and \( Y \), we would like to take

\[
N(\omega) = \omega(X) \cdot \omega_X \omega + \omega(Y) \cdot \omega_Y .
\]

The issue is that there is no obvious meaning to be assigned to \( \omega_X \) or \( \omega_Y \). Indeed, \( \omega_X \) represents a sum \( \sum r_i \cdot x_i \) of elements of \( X \) weighted by disjoint elements \( r_i \in B \) of total weight \( \omega(X) \), and there is no sensible way of distributing the missing weight \( \omega(Y) \) among the \( r_i \)'s to obtain a normalised distribution \( \omega_X \). (More precisely: there is no way of doing so which would make \( N \) into a natural transformation.)

However, it turns out we can describe a kind of logical hypernormalisation by changing our perspective. Observe that elements of \( D_B(X) \) can be seen as total computations which switch on an element of the Boolean algebra \( B \) before returning an element in \( X \). However, it also make sense to consider partial computations (like \( \omega_X \) and \( \omega_Y \) above) where one branch of the switch is left undefined. In this situation, we can still obtain a monad, but to do so we must allow not only computations but also values to be defined only on some part of \( B \). Thus, we must change both the monad and the base category, as follows.

**Definition 53.** Let \( B \) be a non-trivial Boolean algebra. The category \( \text{Set}_B \) of \( B \)-labelled sets has:

- **Objects** being pairs \( \mathbf{X} = (X, |-|_X) \) where \( X \) is a set and \( |-|_X : X \to B \setminus \{\bot\} \);
- **Morphisms** \( \mathbf{X} \to \mathbf{Y} \) being functions \( f : X \to Y \) such that \( |f(x)|_Y = |x|_X \) for all \( x \in X \).

As suggested above, we think of a \( B \)-labelled set \( \mathbf{X} \) as a set of partially defined values, where the value \( x \) is defined only when the test \( |x|_X \in B \) —which we term the **domain** of \( x \)—evaluates to true. We choose to disallow elements of domain \( \bot \); the real reason for doing this is that, as we shall see shortly, it allows the formulae we will write to resemble more closely those for \( \mathcal{D} \). However, for the moment, we may justify the choice on the grounds that there should always be a unique totally undefined element—which, as such, need not be recorded explicitly.

**Definition 54.** Let \( \mathbf{X} \) be a \( B \)-labelled set. A **finitely supported logical subdistribution** on \( \mathbf{X} \) is a function \( \omega : X \to B \) such that \( \text{supp}(\omega) \) is non-empty and finite, the image of \( \omega \) is pairwise-disjoint, and \( \omega(x) \leq |x|_X \) for all \( x \in X \). We may also write \( \omega \) as a formal convex sum as in (2.1).

The **domain** of a subdistribution \( \omega \) is the value \( \omega(X) \), where like before we write \( \omega(A) = \bigvee_{a \in A} \omega(a) \) for any \( A \subseteq X \). Of course, \( \omega \) is a logical distribution if it has domain \( \top \). We write \( s\mathcal{D}_B(\mathbf{X}) \) for the \( B \)-set of logical subdistributions on \( \mathbf{X} \).

Note that we exclude the totally undefined logical subdistribution in this definition. On the one hand, this is forced by the fact that our labelled \( B \)-sets may not have elements of domain \( \bot \); on the other, we can justify this as capturing the fact that the totally undefined subdistribution is “unnormalizable”, in the sense that it can’t be expressed as a distribution on a non-trivial Boolean algebra.
The assignment \( X \mapsto s\mathcal{D}_B(X) \) underlies a monad on \( \mathsf{Set}_B \), whose action on objects is given by pushforward as in (2.5), and whose unit and multiplication are given by

\[
\eta_X : X \to s\mathcal{D}_B(X) \quad \mu_X : s\mathcal{D}_Bs\mathcal{D}_B(X) \to s\mathcal{D}_B(X)
\]

\[
x \mapsto |x| \cdot x \quad \sum_{1 \leq i \leq n} r_i \cdot \omega_i \mapsto \left( x \mapsto \bigvee_{1 \leq i \leq n} r_i \land \omega_i(x) \right).
\]

As with the finite distribution monad on \( \mathsf{Set} \), the algebras for the logical distribution monad on \( \mathsf{Set}_B \) are well known. In what follows, if \( X \) is a \( B \)-labelled set, then we write \( X(b) \subseteq X \) for the set of elements of domain \( b \).

**Definition 55.** Let \( B \) be a Boolean algebra. A presheaf on \( B \) is a \( B \)-labelled set \( X \) endowed with restriction maps \( (-)_{\mid c} : X(b) \to X(c) \) for all \( c \leq b \) in \( B \setminus \{ \top \} \), satisfying the evident functoriality axioms. A presheaf is a sheaf if for any disjoint \( b, c \in B \setminus \{ \top \} \), the following diagram is a product:

\[
\begin{array}{ccc}
X(b) & \xrightarrow{(-)_{\mid b}} & X(b \lor c) \xrightarrow{(-)_{\mid c}} X(c).
\end{array}
\]

We write \( \mathcal{P}\mathsf{sh}(B) \) for the category of presheaves on \( B \), whose maps are \( B \)-labelled set maps commuting with the restriction operations, and \( \mathsf{Sh}(B) \leq \mathcal{P}\mathsf{sh}(B) \) for the full subcategory of sheaves.

**Proposition 56.** The category of Eilenberg–Moore algebras of the logical subdistribution monad \( s\mathcal{D}_B \) is isomorphic over \( \mathsf{Set}_B \) to the category of sheaves \( \mathsf{Sh}(B) \).

This result is a simple exercise in the theory of sheaves; we include a sketch proof for completeness.

**Proof (sketch).** Given a sheaf \( X \), we endow its underlying \( B \)-labelled set with \( s\mathcal{D}_B \)-algebra structure \( \theta : s\mathcal{D}_B(X) \to X \) by taking \( \theta(\sum r_i \cdot x_i) \) to be the unique element \( y \in X(\bigvee_i r_i) \) such that \( y_{\mid r_i} = x_{\mid r_i} \) for each \( i \in I \). Conversely, if \( \theta : s\mathcal{D}_B(X) \to X \) is a \( \mathcal{D} \)-algebra, then we make its underlying \( B \)-set into a presheaf on \( B \) by taking

\[
(-)_{\mid c} : X(b) \to X(c)
\]

\[
x \mapsto \theta(c \cdot x).
\]

For the sheaf axiom, if \( b, c \in B \setminus \{ \top \} \) are disjoint, \( x \in X(b) \) and \( y \in X(c) \), then the unique \( z \in X(b \lor c) \) with \( z_{\mid b} = x \) and \( z_{\mid c} = y \) is given by \( z = \theta(b \cdot x + c \cdot y) \). \( \square \)

Given this result, we can exploit the well-known characterisation of coproducts in categories of sheaves to give a concrete description of finite coproducts in the category of \( s\mathcal{D}_B \)-algebras; alternatively, we can give a direct proof paralleling Lemma 7. Either way, we have:

**Lemma 57.** The initial sheaf on \( B \) is the empty \( B \)-labelled set \( \emptyset \); while if \( X \) and \( Y \) are sheaves on \( B \), then their coproduct \( X \star Y \) in \( \mathsf{Sh}(B) \) is the \( B \)-labelled set with elements of domain \( b \) determined by:

\[
(X \star Y)(b) = X(b) + \left( \sum_{b_1, b_2 \in B \setminus \{ \top \}, b = b_1 \lor b_2} X(b_1) \times Y(b_2) \right) + Y(b),
\]
and whose sheaf restriction operation \((X \star Y)(b) \to (X \star Y)(c)\) is inherited from \(X\) and \(Y\) on the outer summands, and on the middle summand is given by

\[
\sum_{b_1, b_2 \in B' \perp} \sum_{b = b_1 \oplus b_2} X(b_1) \times Y(b_2) \to X(c) + \left( \sum_{c_1, c_2 \in B' \perp} X(r) \times Y(s) \right) + Y(c)
\]

\[(x, y) \mapsto \begin{cases} \text{in}_1(x|_c) & \text{if } c \leq b_1; \\ \text{in}_3(y|_c) & \text{if } c \leq b_2; \\ \text{in}_2(x|_{b_1 \cap c}, y|_{b_2 \cap c}) & \text{otherwise.} \end{cases}
\]

Since the underlying \(B\)-labelled set of the sheaf coproduct \(X \star Y\) relies only on the underlying \(B\)-labelled sets of \(X\) and \(Y\), and not on their sheaf structure, we may expect that \((\star, 0)\) extends to a monoidal structure on \(\text{Set}_B\) with respect to which \(s\mathcal{D}_B\) is linear exponential. This is in fact the case. The only non-trivial point is obtaining the unitality, associativity and symmetry constraints for \(\star\); and like before, these are determined by the corresponding constraints for coproducts of sheaves—so long as these descend to \(\text{Set}_B\).

We concentrate here on associativity. Note first that, if we define \(X(\perp)\) to be a singleton set for any \(B\)-labelled set \(X\), then the binary tensor can be written as

\[(X \star Y)(b) = \sum_{b = b_1 \oplus b_2} X(b_1) \times X(b_2).
\]

With this convention, if \(X, Y\) and \(Z\) are \(B\)-labelled sets, then the three-fold tensors \((X \star Y) \star Z\) and \(X \star (Y \star Z)\) satisfy

\[(X \star (Y \star Z))(b) = \sum_{b = q \oplus t} (X \star Y)(q) \times Z(t) = \sum_{b = q \oplus t} \left( \sum_{q = r \oplus s} X(r) \times Y(s) \right) \times Z(t)
\]

\[(X \star (Y \star Z))(b) = \sum_{b = r \oplus u} X(r) \times (Y \star Z)(u) = \sum_{b = r \oplus u} X(r) \times \left( \sum_{u = s \oplus t} Y(s) \times Z(t) \right)
\]

which are isomorphic to each other by way of the set

\[
\sum_{b = r \oplus s \perp t} X(r) \times Y(s) \times Z(t).
\]

Assembling these isomorphisms as \(b \in B\) varies gives the desired natural isomorphisms \((X \star Y) \star Z \to X \star (Y \star Z)\). It is now straightforward to verify the Mac Lane coherence axioms making this into a a symmetric monoidal structure on \(\text{Set}_B\), and to see that this structure lifts to the coproduct monoidal structure on the category of \(s\mathcal{D}_B\)-algebras (i.e., sheaves on \(B\)). Thus we have shown:

**Proposition 58.** The logical distribution monad \(s\mathcal{D}_B\) is linear exponential with respect to the monoidal structure \((\star, 0)\) on \(\text{Set}_B\).

The induced hypernormalisation maps \(N: s\mathcal{D}_B(\Sigma_i X_i) \to s\mathcal{D}_B(\Sigma_i s\mathcal{D}_B(X_i))\) can be described as follows. An element of \(s\mathcal{D}_B(\Sigma_i X_i)\) of domain \(b\) is a function \(\omega: \Sigma_i X_i \to B\) of finite support whose image is a partition of \(b \in B\), such that \(\omega(x) \leq |x|_{X_i}\) for all \(x \in X_i\). For each \(i\), we have the elements \(\omega(X_i) \in B\), which themselves constitute an \(I\)-fold partition of \(b\); and we also have for each \(i\) the
restricted function $\omega_i = \omega|_{X_i} : X_i \to B$, which, so long as $\omega(X_i) \neq \perp$, is an element of $\mathcal{D}X_i$ of domain $\omega(X_i)$. As such, we can define $\mathcal{N}(\omega)$ to be the element

$$\sum_{i \in I} \omega(X_i) \cdot \epsilon_i(\omega_i) \in \mathcal{D}(\Sigma_i \mathcal{D}X_i).$$

In this way, we have sidestepped the problem of normalising the subdistributions $\omega_i$ to total ones, as we would have needed to do for the monad $\mathcal{D}B$ on $\mathbf{Set}$; instead, we can allow them to remain as subdistributions, and so obtain a good notion of hypernormalisation.

One way of seeing this is that we have decoupled the two aspects of hypernormalisation of probability distributions from each other: on the one hand, collecting like terms to obtain an outer distribution; and on the other, normalising the inner sub-distributions to genuine distributions. Our logical hypernormalisation does the former, but not the latter; and while this may seem to limit its value, we will see in Section 7 that this is not the case.

6.3. Normalisation of continuous functions on streams. For our final example, we describe an instance of hypernormalisation of a different flavour; in it, the hypernormalisation map will allow us to normalise programs encoding continuous functions defined on a type of streams.

By a stream over a finite alphabet $B$, we mean simply an $\mathbb{N}$-indexed family $\vec{b} = (b_0, b_1, \ldots) \in B^\mathbb{N}$. The set $\mathbf{Stream}(B)$ of streams can be topologised, as usual, as a product of discrete spaces, and then a function $f : \mathbf{Stream}(B) \to A$ to a finite discrete space $A$ is continuous precisely when it is locally constant:

$$\forall \vec{b} \in \mathbf{Stream}(B). \exists n \in \mathbb{N}. \forall \vec{c} \in \mathbf{Stream}(B). b_{\leq n} = c_{\leq n} \implies f(\vec{b}) = f(\vec{c});$$

where we define $b_{\leq n} = (b_0, \ldots, b_n)$. In other words, a continuous $f$ uses only finitely many tokens of its input to compute its output.

It is an idea which goes back to Brouwer (though see [21] for a modern treatment) that such functions can be represented, non-uniquely, by well-founded $B$-ary branching trees with leaves labelled in $A$, i.e., elements of the initial algebra $T_B(A) = \mu X. X^B + A$. For example, the binary tree to the left in:

```
    a
   / \     /
  0 * 1   b 0 * 1
 /   /     /   \
0 * 1   0 * 1 0 * 1
```

is a decision tree encoding the function $f : \mathbf{Stream}(\{0, 1\}) \to \{a, b, c\}$ with

$$f(00\ldots) = a, \quad f(010\ldots) = f(1\ldots) = b, \quad \text{and} \quad f(011\ldots) = c.$$

However, this tree is an inefficient encoding, since, on receiving an first input token of 1, we request needless additional input tokens before returning the output value $b$. There is an obvious algorithm which normalises such decision trees to maximally efficient ones, in this case the tree right above: starting from
the leaves, we recursively contract all subtrees as left below to leaves as to the right:

\[ x \xrightarrow{\ast} \begin{array}{c} x \\ \end{array} \xrightarrow{\sim} x. \]

(6.6)

While this normalisation algorithm is clearly motivated from a computational perspective, it is less clear how to understand it structurally. We will show that, in fact, it arises as an instance of hypernormalisation.

To see it in this way, we consider the monad \( T_B \), whose value at \( A \) is the set \( T_B(A) \) of \( B \)-ary branching trees with leaves in \( A \). This is the free monad on the endofunctor \( X \mapsto X^B \) of Set, so that \( T_B \)-algebras are exactly \( B \)-ary magmas: sets \( X \) endowed with a \( B \)-ary operation \( X^B \to X \). We will show that \( T_B \) is a linear exponential monad on Set for a suitable monoidal structure, and so admits hypernormalisation. From this, we obtain for any finite set \( A \) a function

(6.7) \[ T_B(A) \cong T_B(\Sigma_{a \in A} 1) \xrightarrow{N} T_B(\Sigma_{a \in A} T_B 1) \xrightarrow{T_B(\Sigma_{a \in A})} T_B(\Sigma_{a \in A} 1) \cong T_B(A) \]

which we will see implements the normalisation algorithm described above.

To find the monoidal structure on Set for which \( T_B \) is linear exponential, the first step, as usual, is to characterise finite coproducts of \( T_B \)-algebras—so equalling, finite coproducts of \( B \)-ary magmas. Clearly, the empty set underlies the initial \( B \)-ary magma. The following result characterises binary coproducts of \( B \)-ary magmas; in its statement, and henceforth, if \( \tau \in T_B(A) \) and \( A' \subseteq A \), then we say that \( \tau \) is \( A' \)-labelled if it lies in \( T_B(A') \subseteq T_B(A) \).

**Lemma 59.** Let \( (X, \xi : X^B \to X) \) and \( (Y, \gamma : Y^B \to Y) \) be \( B \)-ary magmas. Their coproduct in the category of \( B \)-ary magmas is given by

\[ X \otimes Y := \{ \tau \in T_B(X + Y) : \tau \text{ has no non-trivial } X- \text{ or } Y-\text{labelled subtree} \}. \]

The magma operation \( \theta : (X \otimes Y)^B \to X \otimes Y \) takes a family of trees \( \langle \tau_b : b \in B \rangle \) to the disjoint union of the \( \tau_b \)'s, joined together at a fresh root vertex,

\[ \theta(\tau_b : b \in B) = \tau_b \cdots \tau_\psi, \]

except in the cases where this would create a non-trivial \( X \)- or \( Y \)-labelled subtree.

These exceptional cases are where:

- For each \( b \in B \), the tree \( \tau_b \) is a one-vertex tree \( \bullet_{x_b} \), labelled by some \( x_b \in X \); in this case, we take \( \theta(\tau_b : b \in B) = \xi(\langle x_b : b \in B \rangle) \).
- For each \( b \in B \), the tree \( \tau_a \) is a one-vertex tree \( \bullet_{y_b} \), labelled by some \( y_b \in Y \); in this case, we take \( \theta(\tau_a : b \in B) = \gamma(\langle y_b : b \in B \rangle) \).

**Proof.** By its construction, the magma operation on \( X \otimes Y \) is well-defined, and the two maps \( \iota_1 : X \to X \otimes Y \leftarrow Y : \iota_2 \) sending an element of \( X \) or \( Y \) to the corresponding one-vertex labelled tree are magma homomorphisms. Moreover, given \( B \)-ary magma morphisms \( f : X \to Z \) and \( g : Y \to Z \), the unique magma morphism \( \langle f, g \rangle : X \otimes Y \to Z \) with \( \langle f, g \rangle \iota_1 = f \) and \( \langle f, g \rangle \iota_2 = g \) is given as the
composite of the inclusion \( X \odot Y \hookrightarrow T_B(X + Y) \) with the unique \( B \)-ary magma morphism \( T_B(X + Y) \to Z \) extending the function \( (f, g): X + Y \to Z \). \( \square \)

Note that the set \( X \odot Y \) underlying the coproduct of \((X, \xi)\) and \((Y, \gamma)\) does not rely on \( \xi \) and \( \gamma \), but only on the underlying sets \( X \) and \( Y \). So, like before, we may posit the existence of a symmetric monoidal structure \((\odot, 0)\) on \( \text{Set} \) for which \( T_B \) is linear exponential. Once again, the only point requiring work is defining the associativity, unitality and symmetry isomorphisms, and, like before, we concentrate on the case of associativity. For this, we follow the idea of Lemma 25 by interposing a ternary tensor product \( X \odot Y \odot Z \subseteq T_B(X + Y + Z) \) composed of those trees without any non-trivial \( X \)-, \( Y \)- or \( Z \)-labelled subtree.

**Lemma 60.** For any sets \( X, Y, Z \) we have isomorphisms
\[
(X \odot Y) \odot Z \xrightarrow{\ell} X \odot Y \odot Z \xleftarrow{r} X \odot (Y \odot Z)
\]

**Proof (sketch).** An element \( \tau \in (X \odot Y) \odot Z \) is an \((X \odot Y) + Z\)-labelled tree, and each \( X \odot Y \)-leaf is itself an \( X + Y \)-labelled tree. These data are equally encapsulated by an \( X + Y + Z \)-labelled tree with a collection of vertices marked: namely, the roots of the \( X \odot Y \)-trees at the leaves of the original \( \tau \). Forgetting this vertex marking yields the element \( \ell(\tau) \in X \odot Y \odot Z \); and to show invertibility of the \( \ell \) so defined, we need to reconstruct \( \tau \)'s marking uniquely from \( \ell(\tau) \). But this is easy; it is uniquely characterised by the following properties:

(i) The subtree above every marked vertex is \( X + Y \)-labelled;
(ii) Every leaf vertex is above some marked vertex;
(iii) No non-leaf vertex has all of its \( B \) children marked,
and we can obtain it via the following algorithm: first mark each \( X \)- or \( Y \)-leaf; then recursively move markings towards the root until (ii) is satisfied. This must terminate by well-foundedness. This defines the invertible \( \ell \); now \( r \) is dual. \( \square \)

We can thus take \( r^{-1} \ell: (X \odot Y) \odot Z \to X \odot (Y \odot Z) \) as the desired associativity constraint. The pentagon axiom equating the two maps \(((X \odot Y) \odot Z) \odot W \to X \odot (Y \odot (Z \odot W))\) now follows in the spirit of Lemma 25 from the observation that each edge in this pentagon is simply a way of redistributing markings on a particular element of the quaternary tensor \( X \odot Y \odot Z \odot W \). Proceeding similarly for the unit and symmetry constraints, we may complete the construction of the symmetric monoidal structure \((\odot, 0)\) on \( \text{Set} \) and see that it lifts to the coproduct monoidal structure on the category of \( T \)-algebras. In other words, we have:

**Proposition 61.** The \( B \)-ary magma monad \( T_B \) is linear exponential with respect to the monoidal structure \((\odot, 0)\) on \( \text{Set} \).

The associated hypernormalisation maps \( N: T_B(\Sigma_i X_i) \to T_B(\Sigma_i T_B(X_i)) \) may be described as follows. An element \( \tau \in T_B(\Sigma_i X_i) \) is a \( \Sigma_i X_i \)-labelled \( B \)-ary tree. There is a unique way of marking vertices in \( \tau \) such that:

(i) The subtree above any marked vertex is \( X_i \)-labelled for some \( i \);
(ii) No vertex has all \( B \) of its children marked.

On constructing this marking, the subtree above each marked vertex is an element of \( \Sigma_i T_B(X_i) \); so viewing each such subtree as a leaf labelled in \( \Sigma_i T_B(X_i) \), we have obtained the element \( N(\tau) \in T_B(\Sigma_i T_B(X_i)) \).
More intuitively, if we think of $\tau \in T_B(\Sigma_i X_i)$ as a decision tree computing a continuous function $f: \text{Stream}(B) \rightarrow \Sigma_i X_i$, then $N(\tau) \in T_B(\Sigma_i T_B(X_i))$ computes a function $f': \text{Stream}(B) \rightarrow \Sigma_i T_B(X_i)$ as follows: given $S \in \text{Stream}(B)$, we run the computation of $f(S)$ using $\tau$, and halt at the precise moment that the summand $X_i \subseteq \sum_{i} X_i$ in which $f(S)$ lies has been determined. We then return as $f'(S)$ the $X_i$-labelled subtree lying above the halting vertex, i.e., the continuation of the computation of $f(S)$ as an element of the set $X_i$.

We now use our understanding of hypernormalisation to describe the map $T_B(A) \rightarrow T_B(A)$ of (6.7). This first applies $N: T_B(\Sigma_{a \in A} 1) \rightarrow T_B(\Sigma_{a \in A} T_B 1)$, whose effect on $\tau \in T_B(A)$ is to mark the roots of the largest subtrees whose leaves are all labelled with a single element $a \in A$. It then applies the function $T_B(\Sigma_a!): T_B(\Sigma_{a \in A} T_B 1) \rightarrow T_B(\Sigma_{a \in A} 1)$, which has the effect of collapsing the marked vertex at the root of each $\{a\}$-labelled subtree to the bare leaf $a$. The endofunction of $T_B(A)$ so resulting acts on a tree $\tau$ precisely by carrying out the contractions in (6.6)—in other words, it normalises $\tau$ to its most efficient representative, as desired.

7. Application: relating behavioural and trace equivalence

In this final section, we use our framework for abstract hypernormalisation to relate behavioural equivalence and trace equivalence for certain kinds of automata; more precisely, we will use it to describe a normalisation-by-evaluation process which normalises behaviours—states modulo bisimilarity—to maximally efficient representatives of the corresponding traces—states modulo trace equivalence.

7.1. Generative systems. We begin by introducing the kinds of automata that we will be concerned with.

**Definition 62.** Let $T$ be a monad on a category $C$ with finite coproducts, and $A$ a finite set. A generative $T$-system with alphabet $A$ is an object $S \in C$ endowed with a map $\sigma: S \rightarrow T(\Sigma_{a \in A} S)$. We write $\text{Gen}_A(T)$ for the category of generative $T$-systems over $A$, with as maps the obvious homomorphisms.

**Example 63.** When $T$ is the identity monad on $\text{Set}$, this definition yields deterministic generative systems $\sigma: S \rightarrow A \times S$. We see $S$ as a set of states, and $\sigma$ as associating to each state in $S$ an output token from $A$, and a next state in $S$; so we have a degenerate kind of Mealy machine.

**Example 64.** On the other hand, for the non-empty finite powerset monad on $\text{Set}$, a generative $P_f^+$-system $\sigma: S \rightarrow P_f^+(A \times S)$ is a non-terminating, finitely branching labelled transition system.

We will use $T = \text{id}_{\text{Set}}$ and $T = P_f^+$ as our running examples in the next few sections; in Sections 7.4 and 7.5 below, we will consider other monads capturing stream processors [21, 18], probabilistic generative systems [56], and a “logical” version of generative systems in the spirit of Section 6.2 above.

7.2. Behaviours and traces. For our general notion of generative system, there are two natural notions of equivalence between states. On the one hand, we have bisimilarity; for example, in the case of a labelled transition $\sigma: S \rightarrow P_f^+(A \times S)$,
states \( s, t \in S \) are bisimilar if they are related by a bisimulation \( R \) on \( S \), i.e., an equivalence relation with

\[
(7.1) \quad x \ R \ y \text{ and } (a, x') \in \sigma(x) \implies \exists y'. x' \ R \ y' \text{ and } (a, y') \in \sigma(y).
\]

On the other hand, we have trace equivalence; for labelled transition systems, the trace of a state \( s_0 \in S \) is the subset \( \text{tr}(s_0) \subseteq \text{Stream}(A) \) satisfying

\[
(7.2) \quad \text{tr}(s_0) = \{(a_0, a_1, \ldots) \mid \exists (s_1, s_2, \ldots) \text{ s.t. } (a_i, s_{i+1}) \in \sigma(s_i) \forall i\}
\]

and two states are deemed trace equivalent if they have the same trace.

We now provide the general definitions of bisimilarity and trace equivalence for generative \( T \)-systems. We begin with the better-known case of bisimilarity:

**Definition 65.** Let \( T \) be a monad on a category \( \mathcal{C} \) with finite coproducts and \( A \) a finite set. An object of behaviours for generative \( T \)-systems over \( A \) is a final object \( \beta : \text{Beh} \to T(\Sigma_{a \in A} \text{Beh}) \) in the category \( \text{Gen}_A(T) \). The behaviour map for a generative \( T \)-system \( \sigma : S \to T(\Sigma_{a \in A} S) \) is the unique homomorphism \( \text{beh} : (S, \sigma) \to (\text{Beh}, \beta) \). Two states are bisimilar if they have the same image under the behaviour map.

**Example 66.** When \( T = \text{id}_{\text{Set}} \), the object of behaviours is \( \text{Stream}(A) \) with the structure map \( (\text{hd}, \text{tl}) : \text{Stream}(A) \to A \times \text{Stream}(A) \) given by

\[
(a_0, a_1, \ldots) \mapsto (a_0, (a_1, a_2, \ldots)).
\]

Given a generative system \( \sigma = (\sigma_0, \sigma_1) : \text{Beh} \to T(\Sigma_{a \in A} \text{Beh}) \) in the category \( \text{Gen}_A(T) \), we have

\[
\text{beh}(s) = (\sigma_0(s), \sigma_0(\pi_1(s)), \sigma_0(\pi_2(s)), \ldots).
\]

**Example 67.** When \( T = \text{Pow}_+ \), the object of behaviours can be described via the techniques of [1]. Consider the set \( \text{Beh}' \) of rooted trees which are finitely branching, purely infinite (i.e., with no leaf vertices), and have edges labelled by elements of \( A \). This becomes a labelled transition system \( \text{Beh}' \to \text{Pow}_+(A \times \text{Beh}') \) via

\[
\begin{array}{cccc}
\tau_1 & \cdots & \tau_n, \\
\tau_1 & \cdots & \tau_n, \\
\end{array}
\]

and the object of behaviours is the quotient of \( \text{Beh}_w \) by the largest bisimulation as in (7.1). Alternatively, following [58, 2], we can describe it as the set of strongly extensional trees of \( \text{Beh}_w \) (i.e., those admitting no tree bisimulation).

Given a labelled transition system \( \sigma : S \to \text{Pow}_+(A \times S) \) and \( s \in S \), the behaviour \( \text{beh}(s) \) is given coinductively by

\[
\text{beh}(s) = \begin{array}{c}
\tau_1 \cdots \tau_n, \\
\tau_1 \cdots \tau_n, \\
\tau_1 \cdots \tau_n, \\
\end{array}
\]

where \( \sigma(s) = \{(a_1, t_1), \ldots, (a_n, t_n)\} \).

We now turn to trace equivalence. As for bisimulation, this will be characterised in terms of equality under a map, this time to a suitably-defined object of traces.

**Definition 68.** Let \( \mathcal{C} \) be a category with finite coproducts, and \( A \) a finite set. An \( A \)-ary comagma in \( \mathcal{C} \) is an object \( X \) endowed with a map \( X \to \sum_{a \in A} X \); we write \( \text{Comag}_A(\mathcal{C}) \) for the category of \( A \)-ary commagmas in \( \mathcal{C} \) and their homomorphisms.
Given a monad $T$ on $\mathcal{C}$, an object of traces for generative $T$-systems over $A$ is defined to be a final object $\tau: Tr \to \Sigma_{a \in A} Tr$ in the category $\mathbf{Comag}_{A}(\mathcal{C}^T)$.

To motivate this definition, note that a generative $T$-system $\sigma: S \to T(\Sigma_{a \in A} S)$ is equally well an $A$-ary comagma in the Kleisli category $\mathcal{K}(T)$; and it is an idea going back to $[45]$ that a suitable notion of object of traces in this context should be given by a final object in $\mathbf{Comag}_{A}(\mathcal{K}(T))$. Note this is different from a final object in $\mathbf{Gen}_{A}(T)$, since morphisms $(S, \sigma) \to (S, \sigma')$ in the latter category involve maps $S \to T(S')$.

However, the category $\mathcal{K}(T)$ is not well-behaved, and a final $A$-ary comagma is not guaranteed to exist. Often, this problem can be resolved by expanding the category $\mathcal{C}$ and the monad $T$—see, for example, $[22, 31]$—but this must be done in an ad hoc manner which requires thought. However, the category $\mathcal{K}(T)$ can always be embedded into the much better-behaved category $\mathcal{C}^T$ of Eilenberg–Moore algebras—as the free algebras therein—and this better behaviour now guarantees, under very mild side-conditions, the existence of a final $A$-ary comagma, which we may thus declare to be our object of traces.

**Example 69.** When $T = id_{\mathbf{Set}}$, the category of $A$-ary comagmas in $\mathbf{Set}^{id}$ is precisely the category of deterministic generative systems $S \to A \times S$. So the object of traces is simply the object of behaviours $(\mathbf{Stream}(A), (hd, tl))$.

**Example 70.** When $T = P^+_\triangledown$, an $A$-ary comagma in $\mathbf{Set}^{P^+_\triangledown}$ is a join-semilattice $S$ endowed with a $\vee$-preserving map $S \to S \triangledown \cdots \triangledown 1$, where $1$ is as in Example 46. In light of the alternative presentation $(6.2)$ of this monoidal product, such a map can equally be described as a $\vee$-preserving map $\sigma: S \to (S_\perp)^A$—where $S_\perp$ is $S$ with a new bottom element adjoined—such that no $\sigma(s)$ is constant at $\perp$.

In this context, the following proposition describes the object of traces.

**Proposition 71.** The final $A$-ary comagma in the category $\mathbf{Set}^{P^+_\triangledown}$ of join-semilattices is the set $Tr$ of non-empty closed subsets of $\mathbf{Stream}(A)$, with join-semilattice structure given by union, and with comagma structure map

$$\tau: Tr \to (Tr \cup \{\emptyset\})^A$$

$$\emptyset \mapsto a_0 \mapsto \{(a_1, a_2, \ldots) \mid (a_0, a_1, \ldots) \in \emptyset\}.$$

The closed sets here are those which are closed in the product topology. To make this explicit, recall that for $\bar{a} \in \mathbf{Stream}(A)$, we write $a_{\leq n}$ for the initial segment $(a_0, \ldots, a_n)$. If, for $V \subseteq \mathbf{Stream}(A)$, we also write $V_{\leq n}$ for $\{a_{\leq n} : \bar{a} \in V\}$, then a set $V \subseteq \mathbf{Stream}(A)$ is closed if $\bar{a} \in V$ whenever $a_{\leq n} \in V_{\leq n}$ for all $n \in \mathbb{N}$.

**Proof.** Consider an $A$-ary comagma $\sigma: S \to (S_\perp)^A$ in $\mathbf{Set}^{P^+_\triangledown}$. We may extend $\sigma$ to a $\vee$- and $\perp$-preserving map $S_\perp \to (S_\perp)^A$ by defining $\sigma(\perp)(a) = \perp$; and now, given $s_0 \in S$ and $\bar{a} \in \mathbf{Stream}(A)$, we may define a stream $\sigma^*(s_0, \bar{a}) = (s_0, s_1, \ldots) \in \mathbf{Stream}(S_\perp)$ by taking $s_{i+1} = \sigma(s_i)(a_i)$ for each $i$. We will say that $s_0$ generates $\bar{a}$ if no element of $\sigma^*(s_0, \bar{a})$ is $\perp$.

With these conventions, we can now define the map $u: S \to Tr$ by

$$u(s) = \{\bar{a} \in \mathbf{Stream}(A) \mid s \text{ generates } \bar{a}\}.$$
To see that \( u(s) \) is closed, consider \( \vec{a} \in \text{Stream}(A) \) with \( a \leq n \in \text{Tr}_{\leq n} \) for each \( n \). Then for each \( n \), there exists \( \vec{b} = (a_0, \ldots, a_n, b_{n+1}, \ldots) \in \text{Stream}(A) \) such that \( s \) generates \( \vec{b} \); in particular, \( \sigma^*(s, \vec{b}) \leq n \) contains no \( \perp \) for each \( n \), whence also \( \sigma^*(s, \vec{a}) \leq n = \sigma^*(s, \vec{b}) \leq n \) contains no \( \perp \) for each \( n \). Thus \( s \) generates \( \vec{a} \) so that \( \vec{a} \in u(s) \) as required.

To see that \( u \) preserves \( \lor \), consider \( s, t \in S \) and \( \vec{a} \in \text{Stream}(A) \). Since \( \sigma \) preserves \( \lor \)'s, we see that \( \sigma^*(s \lor t, \vec{a}) \) is the pointwise join of \( \sigma^*(s, \vec{a}) \) and \( \sigma^*(t, \vec{a}) \); in particular, this means that \( s \lor t \) generates \( \vec{a} \) if and only if either \( s \) generates \( \vec{a} \) or \( t \) generates \( \vec{a} \); which is to say that \( u(s \lor t) = u(s) \lor u(t) \), as desired.

Next, to show that \( u \) is a comagma homomorphism, we must verify that, for any \( s_0 \in S \) and \( a_0 \in A \), we have

\[
\{(a_1, a_2, \ldots) : (a_0, a_1, \ldots) \in u(s_0)\} = \begin{cases} u(\sigma(s_0)(a_0)) & \text{if } \sigma(s_0)(a_0) \neq \perp; \\
\emptyset & \text{if } \sigma(s_0)(a_0) = \perp. \end{cases}
\]

We can rephrase this as saying

\[
(a_0, a_1, \ldots) \in u(s_0) \iff \sigma(s_0)(a_0) = s_1 \neq \perp \text{ and } (a_1, a_2, \ldots) \in u(s_1),
\]

which is clear on observing that if \( \sigma(s_0)(a_0) = s_1 \neq \perp \), then \( \sigma^*(s_0, (a_0, a_1, \ldots)) = \sigma^*(s_1, (a_1, a_2, \ldots)) \). This proves that \( u \) is a comagma homomorphism \( S \to \text{Tr} \), and it remains to prove it is unique. So let \( f \) be another such homomorphism; then for each \( s_0 \in S \) we have

\[
(7.4) \quad (a_0, a_1, \ldots) \in f(s_0) \iff \sigma(s_0)(a_0) = s_1 \neq \perp \text{ and } (a_1, a_2, \ldots) \in f(s_1),
\]

and we must show \( f(s_0) = u(s_0) \).

In one direction, if \( \vec{a} \in f(s_0) \), then \( \sigma^*(s_0, \vec{a}) \) is clearly never \( \perp \), and so \( \vec{a} \in u(s_0) \). Conversely, suppose that \( \vec{a} \in u(s_0) \). If the sequence \( \sigma^*(s_0, \vec{a}) \) is \((s_0, s_1, \ldots)\), then for each \( n \), the set \( f(s_{n+1}) \) is non-empty; so letting \((b_{n+1}, b_{n+2}, \ldots)\) be any element of it, we may apply the leftward implication in (7.4) \( n + 1 \) times to see that \((a_0, a_1, \ldots, a_n, b_{n+1}, \ldots) \in f(s_0) \). But this means that \( a \leq n \in f(s_0) \leq n \) for each \( n \); since \( f(s_0) \) is closed, we conclude that \( \vec{a} \in f(s_0) \) as required. \( \square \)

Returning to the general situation, we now explain how every generative \( T \)-system has a trace map valued in the object of traces. As we noted above, any such system \( \sigma : S \to T(\Sigma_{a \in A} S) \) is equally an \( A \)-ary comagma in the Kleisli category \( \mathcal{K}l(T) \) which may in turn be seen as an \( A \)-ary comagma on the free algebra \( F^T(S) \) in \( \mathcal{C}^T \). Now using the finality of the object of traces yields the desired trace map, as the following definition makes precise:

**Definition 72.** Let \( \sigma : S \to T(\Sigma_{a \in A} S) \) be a generative \( T \)-system over \( A \). The **associated free \( A \)-ary comagma** in \( \mathcal{C}^T \) is \((F^T S, \sigma^z)\), where

\[
\sigma^z := F^T(S) \xrightarrow{T_\sigma} F^T(T(\Sigma_{a \in A} S)) \xrightarrow{\mu_{\Sigma_{a \in A} S}} F^T(\Sigma_{a \in A} F^T(S)) \cong \Sigma_{a \in A} F^T(S).
\]

This assignment is the action on objects of a functor

\[
(7.5) \quad (-)^z : \text{Gen}_A(T) \to \text{Comag}_A(\mathcal{C}^T)
\]

which on maps sends \( f : (S, \sigma) \to (U, v) \) to \( F^T f : (F^T S, \sigma^z) \to (F^T U, v^z) \).
If $T$ has an object of traces $(\text{Tr}, \tau)$, then the trace map of a generative $T$-system $(S, \sigma)$ is the map $\text{tr}: S \to \text{Tr}$ obtained as the restriction along $\eta_S: S \to U^\uplus T S$ of the unique comagma map $(F^\uplus T S, \sigma^\#) \to (\text{Tr}, \tau)$.

**Example 73.** When $T = \text{id}_{\text{Set}}$, the object of traces is just the object of behaviours, and the trace map is just the behaviour map.

**Example 74.** When $T = P_f^+$, a generative system is a labelled transition system $\sigma: S \to P_f^+(A \times S)$. The associated $A$-comagma in the category of join-semilattices is given by

$$\sigma^\#: P_f^+(S) \to (P_f^+(S) \cup \{\emptyset\})^A$$

$$V \mapsto a \mapsto \{s_1 \in S: (a, s_1) \in \sigma(s_0) \text{ for some } s_0 \in V\}.$$  

It follows from this description that $\{s_0\}$ generates $\vec{a}$ if and only if there exist $s_1, s_2, \ldots \in S$ such that $(a, s_i+1) \in \sigma(s_i)$ for each $i$; which is to say that the non-empty subset $\text{tr}(s_0) \subseteq \text{Stream}(A)$ associated to $s_0$ is the trace of $(7.2)$.

### 7.3. Normalisation by trace evaluation for behaviours.

As is well known, bisimilarity is a finer equivalence on states of labelled transition systems than trace equivalence; for example, we may consider the labelled transition systems

$$\sigma: \{s, t, u\} \to P_f^+([0,1] \times \{s, t, u\}) \quad \sigma': \{s', t', u'\} \to P_f^+([0,1] \times \{s', t', u'\})$$

$$s \mapsto \{(0, t), (1, t), (1, u)\} \quad s' \mapsto \{(0, t'), (1, u')\}$$

$$t \mapsto \{(0, t)\} \quad t' \mapsto \{(0, t')\}$$

$$u \mapsto \{(1, t)\} \quad u' \mapsto \{(0, t'), (1, t')\}.$$  

The states $s$ and $s'$ both have trace $\{00\ldots, 100\ldots, 1100\ldots\}$; however, their respective behaviours are the trees

\begin{align*}
\begin{array}{c|c|c|c|c}
& & & & \\
0 & 0 & 0 & &
\end{array} & \text{and} & \begin{array}{c|c|c|c|c}
& & & & \\
0 & 0 & 0 & &
\end{array}
\end{align*}

which are not bisimilar. The difference between $s$ and $s'$ which this captures is that an execution from state $s$ must immediately decide whether it will output zero, one or two 1’s, while an execution from state $s'$ need only decide the first bit now, and can defer the decision on bit 2 until the next computation step.

In fact, $s'$ above is “minimal” in the sense that, at each step, it only requires a decision on the very next output bit. We might hope that this minimality can be captured by somehow “normalising” the behaviour of $s$ to that of $s'$. What we will now see is that this is in fact possible: we have an embedding-retraction pair

\[ (7.6) \]
wherein \texttt{reflect} takes the trace of a behaviour, while its section \texttt{reify} takes the minimal realisation of each trace as a behaviour. In particular, the “normalisation function” \texttt{reify} \circ \texttt{reflect} sends does the right thing for our example above: \texttt{beh}(s′) is mapped to \texttt{beh}(s).

We now explain how to construct the maps in (7.6) for generative T-systems, where T is any monad which admits hypernormalisation, i.e., any linear exponential monad. We first define the maps in each direction; note that the easier direction, \texttt{reflect}, does not rely on hypernormalisation.

\textbf{Definition 75.} If the object of behaviours \texttt{Beh} and the object of traces \texttt{Tr} exist for generative T-systems over A, then the \textit{reflection} map \texttt{reflect}: \texttt{Beh} \rightarrow \texttt{Tr} is the trace map of \texttt{Beh} qua generative T-system.

\textbf{Definition 76.} Let T be a linear exponential monad on a symmetric monoidal category \((\mathcal{C}, \otimes, I)\) with finite coproducts. If \(\sigma: S \rightarrow S \otimes \cdots \otimes S\) is an A-ary comagma in \(\mathcal{C}^T\), then the \textit{associated generative} T-system is \((S, \sigma^\flat)\), where

\[
\sigma^\flat = S \xrightarrow{\sigma} S \otimes \cdots \otimes S \xrightarrow{n_S \otimes \cdots \otimes n_S} T(S) \otimes \cdots \otimes T(S) \xrightarrow{\varphi^{-1}} T(S + \cdots + S).
\]

This assignment is the action on objects of a functor

\(\text{(7.7)} \quad (-)^\flat: \mathcal{Comag}_A(T) \rightarrow \mathcal{Gen}_A(T)\)

which on maps sends \(f: (S, \sigma) \rightarrow (U, \upsilon)\) to \(f: (S, \sigma^\flat) \rightarrow (U, \upsilon^\flat)\).

If the object of behaviours \texttt{Beh} and the object of traces \texttt{Tr} exist for generative T-systems over A, then the \textit{reification} map \texttt{reify}: \texttt{Tr} \rightarrow \texttt{Beh} is the behaviour map of the associated generative T-system of \texttt{Tr}.

\textbf{Example 77.} Recall that when \(T = P_f^+\), the set \texttt{Beh} is the set of purely infinite, finitely branching A-labelled trees modulo bisimilarity, while \texttt{Tr} is the set of closed subsets of \text{Stream}(A). In this case, the reflection map sends a tree \(\tau\) to the set of A-streams that label the infinite paths from the root of \(\tau\). On the other hand, the reflection map sends a closed subset \(V \subseteq \text{Stream}(A)\) to the tree

\[
\tau = \begin{array}{ccc}
\tau_1 & \cdots & \tau_n \\
a_1 & \cdots & a_n
\end{array}
\]

where \(a_1, \ldots, a_n\) are the distinct elements of the set \(V_{\leq 0}\), and where \(\tau_i\) is (coinductively) the tree associated to the closed subset \(\{(b_0, b_1, \ldots) : (a_i, b_0, b_1, \ldots) \in V\}\).

In the preceding example, it is easy to see that reification is a section of reflection; and in fact, this is true in general:

\textbf{Proposition 78.} \textit{In the situation of Definition 76, we have that} \texttt{reflect} \circ \texttt{reify} = \texttt{id}.

\textit{Proof.} Let the coalgebra structures of \texttt{Tr} and \texttt{Beh} be \(\tau: \texttt{Tr} \rightarrow \texttt{Tr} \otimes \cdots \otimes \texttt{Tr}\) and \(\beta: \text{Beh} \rightarrow T(\text{Beh} + \cdots + \text{Beh})\) respectively. By definition, \texttt{reflect} is the unique homomorphism

\[
u: (\text{Tr}, \tau^\flat) \rightarrow (\text{Beh}, \beta) \quad \text{in} \quad \mathcal{Gen}_A(T)
\]

while \texttt{reflect} is the unique homomorphism

\[
u: (F^T(\text{Beh}), \beta^\flat) \rightarrow (\text{Tr}, \tau) \quad \text{in} \quad \mathcal{Comag}_A(\mathcal{C}^T)
\]
precomposed by $\eta_{\text{Beh}}$. So $\reflect \circ \reify = v \circ \eta_{\text{Beh}} \circ u = v \circ T u \circ \eta_{\text{Tr}}$, which is equally the precomposition by $\eta_{\text{Tr}}$ of the composite homomorphism

$$\tag{7.8} \left( (F^T(\text{Tr}), (\tau^\flat)^\sharp) \xrightarrow{\tau^\flat = Tu} (F^T(\text{Beh}), \beta^\sharp) \xrightarrow{\nu} (\text{Tr}, \tau) \right).$$

We claim that (7.8) is in fact equal to the algebra structure map $\alpha : T(\text{Tr}) \to \text{Tr}$ of the Eilenberg–Moore $T$-algebra $\text{Tr}$; since this structure map satisfies $\alpha \circ \eta_{\text{Tr}} = \text{id}$, this will complete the proof. To prove the claim, note that, since (7.8) is the unique map into a terminal object, it suffices to show that $\alpha$ is itself a map $(F^T(\text{Tr}), (\tau^\flat)^\sharp) \to (\text{Tr}, \tau)$. Now, $(\tau^\flat)^\sharp$ is, by Definitions 72 and 76, the following composite

$$T(\text{Tr}) \xrightarrow{T\tau^{-1}} T^2(\Sigma_a \text{Tr}) \xrightarrow{\mu} T(\Sigma_a \text{Tr}) \xrightarrow{\nu} \otimes_a T(\text{Tr}) ;$$

however, because $T$ is a linear exponential monad, the composite along the bottom row is, by [7, Theorem 3.1.6], exactly the opmonoidal structure map $\nu : (\otimes_a \text{Tr}) \to \otimes_a T(\text{Tr})$, and so to say that $\alpha : (F^T(\text{Tr}), (\tau^\flat)^\sharp) \to (\text{Tr}, \tau)$ is to say that the following diagram commutes:

As the bottom row is, by (3.4), the $T$-algebra structure of $\otimes_a (\text{Tr}, \tau)$, commutativity of this diagram is precisely the fact that $\tau : (\text{Tr}, \tau) \to \otimes_a (\text{Tr}, \tau)$ in $\mathcal{C}^T$. □

As indicated above, the composite $\reify \circ \reflect$ is an idempotent endofunction of the set of behaviours, which implements “normalisation by trace evaluation”. In the case of $P^+_f$, it normalises each purely infinitely, finitely branching $A$-labelled tree to a trace-equivalent one in which the children of any given node have distinct $A$-labels: thus, behaviours which, at each step, only decide the next output token.

7.4. Stream processors. In the final sections of this paper, we investigate the reification–reflection pair for other computationally meaningful linear exponential monads $T$. We begin with the free $B$-ary magma monad $T_B$ from Section 6.3.

Generative $T_B$-systems. A generative $T_B$-system $\sigma : S \to T_B(A \times S)$ is a stream processor [21]. The function $\sigma$ assigns to each state $s$ a decision tree which, by consuming an initial segment of a stream of $B$-values, decides an output $A$-value and a next state in $S$. When continued indefinitely, this process turns a stream of $B$-values into a stream of $A$-values; whence the name.

Behaviours. In this case, the object of behaviours $\text{Beh}$ is the final coalgebra $\nu_X. T_B(A \times X)$; since it is a final coalgebra of a polynomial endofunctor, it has an entirely standard description, which can be stated as follows: it is the set of (possibly infinite) $B$-ary branching trees wherein each non-leaf vertex is labelled
by a finite list of $A$-elements; each leaf is labelled by a stream of $A$-elements; and there is no infinite simple path of nodes labelled by the empty string.

We introduce the following notation: given a tree $t \in \text{Beh}$ and some $a \in A$, we write $a \cdot t$ for the tree $t$ with the element $a$ adjoined to the front of the string labelling the root. Using this, we may describe the coalgebra structure of $\text{Beh}$ as follows. For a tree $t \in \text{Beh}$, a cut vertex is one labelled by a non-empty string, but whose proper ancestors are all unlabelled. For a cut vertex $v$, we define $\text{hd}(v) \in A$ and $\text{tl}(v) \in \text{Beh}$ to be such that the subtree of descendents of $v$ is $\text{hd}(v) \cdot \text{tl}(v)$. Now $\beta : \text{Beh} \to T_B(A \times \text{Beh})$ takes $t$ to the well-founded $B$-ary tree obtained by replacing each cut vertex $v \in t$ with the leaf labelled by $(\text{hd}(v), \text{tl}(v))$.

**Behaviour map.** Given a generative $T_B$-system $\sigma : S \to T_B(A \times S)$, the behaviour $\text{beh}(s) \in S$ is the tree obtained by coinductively replacing each $(a, t)$-labelled leaf of $\sigma(s)$ by the tree $a \cdot \text{beh}(t)$.

**Traces.** An $A$-ary comagma in $\mathsf{Set}^{T_B}$ is a $B$-ary magma $(X, \xi)$ endowed with a $B$-ary magma homomorphism

$$
(X, \xi) \to (X, \xi) \circ \cdots \circ (X, \xi)
$$

where $\circ$ is as in Lemma 59. In this case, [18, Theorem 37] proves that the object of traces $\text{Tr}$ is the set of continuous functions $\text{Stream}(B) \to \text{Stream}(A)$, endowed with the $B$-ary magma structure $\zeta$ given by

$$
(\zeta(f_b : b \in B))(c_0, c_1, \ldots) = f_{c_0}(c_1, c_2, \ldots)
$$

and the $A$-ary comagma structure $\tau$ found as follows. Given a continuous function $f : \text{Stream}(B) \to \text{Stream}(A)$, we call a string $b_0 \cdots b_k \in B^*$ determining if the composite $f(-)_0 : \text{Stream}(B) \to A$ is constant on the clopen set

$$(7.9) \quad [b_0 \cdots b_k] = \{ \tilde{b} \in \text{Stream}(B) : b_{\leq k} = (b_0, \ldots, b_k) \}.
$$

We call $\seq{b}$ minimal if it is determining, but $\seq{b_{\leq k-1}}$ is not so. Now by continuity of $f$, we can find minimal strings $m_1, \ldots, m_k \in B^*$, elements $a_1, \ldots, a_k \in A$ and continuous functions $f_1, \ldots, f_k \in \text{Tr}$ such that the following clauses totally define $f$:

$$
f(m_1 \cdot \tilde{b}) = a_1 \cdot f_1(\tilde{b}) \quad \cdots \quad f(m_k \cdot \tilde{b}) = a_k \cdot f_k(\tilde{b})
$$

where $\cdot$ denotes concatenation. The finite strings $m_1, \ldots, m_k$ can be seen as vertex addresses in the purely infinite $B$-ary tree: and now replacing the subtree rooted at $m_i$ by a leaf vertex labelled by $(a_i, f_i) \in A \times \text{Tr}$ yields an element of $T_B(A \times \text{Tr})$; and the minimality of each $m_i$ ensures that this tree in fact lies in $\text{Tr} \circ \cdots \circ \text{Tr} \subseteq T_B(A \times \text{Tr})$, so that we may define it to be $\tau(f)$.

**Trace map.** For a given stream processor $\sigma : S \to T_B(A \times S)$, the trace of a state $s \in S$ is determined as in [21, §3]. First, for each set $V$, we write $\theta_V$ for the function $\text{Stream}(B) \times T_B(V) \to \text{Stream}(B) \times V$ determined by

$$
\theta_V(\tilde{b}, \nu_v) = (\tilde{b}, v)
$$

and

$$
\theta_V \left( \seq{b_0, b_1, \ldots}, \begin{array}{c}
\tau_b \\
\cdots \\
\tau_V
\end{array} \right) = \theta((b_1, b_2, \ldots), \tau_{b_0}).
$$
Now the trace $\text{tr}(s)$ of $s \in S$ is the homomorphism $\text{Stream}(B) \to \text{Stream}(A)$ determined by

$$\text{tr}(s)(\vec{b})_0 = a \quad \text{and} \quad \text{tr}(s)(\vec{b})_{i+1} = \text{tr}(s')(\vec{c}),$$

where $\theta(\vec{b}, \sigma(s)) = (\vec{c}, a, s')$.

The fact that this is a correct description of the trace map is [18, Proposition 42].

**Reflection and reification.** The reflection map $\text{Beh} \to \text{Tr}$ is simply the instantiation of the above trace map at $(\text{Beh}, \beta)$. On the other hand, the generative $T_B$-system structure on $\text{Tr}$ is the composite

$$\text{Tr} \xrightarrow{\tau} \text{Tr} \circ \cdots \circ \text{Tr} \xrightarrow{\subseteq} T_B(A \times \text{Tr})$$

and the reflection map $\text{Tr} \to \text{Beh}$ is the associated behaviour map of this $T_B$-system. We illustrate this with an example drawn from [18, Example 22]. Consider the following two stream processors:

$$\sigma: \{s\} \to T_B(A \times \{s\}) \quad \text{and} \quad \tau: \{t\} \to T_B(A \times \{t\})$$

$$s \mapsto (a, \bullet_s) \quad \quad \quad t \mapsto \begin{cases} (a, t) \quad \text{...} \quad (a, t). \end{cases}$$

The traces $\text{tr}(s)$ and $\text{tr}(t)$ are both the continuous function $\text{Stream}(B) \to \text{Stream}(A)$ sending every stream to $(a, a, a, \ldots)$. On the other hand, as shown in loc. cit., their behaviours are distinct: indeed, $\text{beh}(s)$ is the one-vertex tree whose root is labelled by $(a, a, a, \ldots)$, while $\text{beh}(t)$ is the purely infinite $B$-ary tree whose every vertex is labelled with a single $a$. The difference which this captures is that $s$ produces its stream of $a$’s while completely ignoring its input, and on the other, $t$ inefficiently consumes a single input token before outputting each $a$. In fact, $s$ is the most efficient realisation of this function, and the normalisation-by-trace-evaluation map $\text{reify} \circ \text{reflect}$ sends $\text{beh}(t)$ to $\text{beh}(s)$. In general, the image of this normalisation function comprises those trees $t$ in which no vertex has child subtrees of the form $a \cdot t_1, \ldots, a \cdot t_k$ for the same $a$.

### 7.5. Probabilistic and logical generative systems.

In this section, we now consider the case where $T$ is either the discrete distribution monad $D$ on $\text{Set}$, or else the logical subdistribution monad $sD_B$ on $\text{Set}_B$.

**Generative $D$- and $sD_B$-systems.** When $T = D$, a generative $D$-system is a (non-terminating) *generative probabilistic system* in the sense of [56]; its transition map $\sigma: S \to \mathcal{D}(A \times S)$ associates to each state a finite probability distribution over output tokens and next states. On the other hand, when $T = sD_B$, a generative $sD_B$-system $\sigma: S \to s\mathcal{D}_B(\Sigma_{a \in A} S)$ assigns to each state $s \in S$ of domain $b$ a finitely supported $B$-valued logical subdistribution $\omega$ of domain $b$ on the $B$-set $A \times S$ (with domains given by $|(a, s)|_{A \times S} = |s|_S$).

**Behaviours.** The objects of behaviours can calculated, as in Example 67, by following the approach of [1]; we first exhibit a *weakly final* coalgebra, and then quotient it by bisimulation. For generative probabilistic systems, we consider the set $\text{Beh}'$ of finitely branching, purely infinite trees, whose edges are labelled by elements of $(0, 1] \times A$, and where the $(0, 1]$-weights on the children of any vertex sum to 1. There is an evident structure map $\beta': \text{Beh}' \to \mathcal{D}(A \times \text{Beh}')$ and
the object of behaviours is its quotient by the largest bisimulation in the sense of [32], i.e., the largest equivalence relation \( R \) satisfying

\[(7.10) \quad x \ R \ y \implies \beta'(x)(\{a\} \times C) = \beta'(y)(\{a\} \times C) \text{ for all } a \in A, \ C \in \text{Beh}'/R.\]

For generative s\(D_B\)-systems, we take \( \text{Beh}' \) to be the \( B \)-labelled set of finitely branching, purely infinite trees whose vertices are labelled by elements of \( B \setminus \perp \), whose edges are labelled in \((B \setminus \perp) \times A\), and where the \( B \)-weights on the children of a vertex labelled by \( b \) are a finite partition of \( b \); the domain of such a tree is the label of its root. For the analogous notion of bisimulation to (7.10), we again obtain \( \text{Beh} \) as the quotient of \( \text{Beh}' \) by the largest bisimulation.

**Behaviour map.** Given a generative probabilistic system \( \sigma: S \to \mathcal{D}(A \times S) \) and \( s \in S \), the behaviour \( \text{beh}(s) \) is given coinductively by

\[
\text{beh}(s) = \frac{\text{beh}(t_1) \ldots \text{beh}(t_n)}{(p_1,a_1) \ldots (p_n,a_n)} \quad \text{where } \sigma(s) = p_1 \cdot (a_1,t_1) + \cdots + p_n \cdot (a_n,t_n).
\]

The behaviour map for a generative s\(D_B\)-system is entirely analogous.

**Traces.** An \( A \)-ary comagma in \( \text{Set}^D \) is an abstract convex space \( X \) endowed with a convex map \( X \to X \ast \cdots \ast X \). In this case, we have the following characterisation of the final such comagma. In the statement and proof, we make use of the basic clopen sets of \( \text{Stream}(A) \) defined in (7.9).

**Proposition 79.** The final \( A \)-ary comagma in \( \text{Set}^D \) is the set \( \mathcal{R}(\text{Stream}(A)) \) of (necessarily) Radon probability distributions on \( \text{Stream}(A) \), endowed with the usual convex space structure and with the comagma structure

\[
\mathcal{R}(\text{Stream}(A)) \xrightarrow{(\text{hd},t)} \mathcal{R}(A \times \text{Stream}(A)) \xrightarrow{\rightsquigarrow} \mathcal{R}(\text{Stream}(A)) \ast \cdots \ast \mathcal{R}(\text{Stream}(A));
\]

more explicitly, this is the comagma structure given by

\[
\omega \mapsto \sum_{a \in A} \omega([a]) \cdot t_a(\overline{\omega_a})
\]

where \( \overline{\omega_a} \) is the distribution with \( \overline{\omega_a}([a_1a_2\cdots]) = \omega([a_1a_2\cdots])/\omega([a]). \)

The reader should compare this result to [31, Theorem 3.33] which, among other things, proves that \( \text{Stream}(A) \) is terminal in the category \( \text{Comag}_A(\mathcal{K}(\Pi(\mathcal{P}))) \), where \( \mathcal{P} \) is the probability monad on the category of measurable spaces. Neither result implies the other, but one might expect, as a common generalisation, that \( \mathcal{P}(\text{Stream}(A)) \) is a final \( A \)-ary comagma in \( \text{Meas}^\mathcal{P} \).

**Proof.** Let \( (S,\sigma): S \to S \ast \cdots \ast S \) be an \( A \)-ary comagma in the category of convex spaces. We define a map \( u: S \to \mathcal{R}(\text{Stream}(A)) \) by

\[
u(s)([])=1 \quad \text{and} \quad u(s)([a_0 \ldots a_n]) = \begin{cases} p_{a_0} \cdot u(s_{a_0})([a_1 \ldots a_n]) & \text{if } a_0 \in A'; \\ 0 & \text{otherwise}, \end{cases}
\]

where here we assume that \( \sigma(s) = \sum_{a \in A'} p_a \cdot t_a(s_a) \) for some \( A' \subseteq A \). It is easy to see that \( u(s) \) extends to a finitely additive map on the Boolean algebra of all clopen sets of \( \text{Stream}(A) \), and so by [13, Lemma IV.9.11] extends uniquely to a probability measure on \( \text{Stream}(A) \). So \( u \) is well-defined; it is now straightforward
to verify, following the pattern of Proposition 71, that it is a convex map and a map of $A$-ary comagmas.

Suppose now that $h : S → \mathcal{R}(\text{Stream}(A))$ is a convex map and a map of $A$-ary comagmas; we must show that $u = h$. For this it suffices to show that

$$u(s)([a_0 \cdots a_n]) = h(s)([a_0 \cdots a_n])$$

for all states $s$ and finite strings $a_0 \cdots a_n \in A^*$. We do so by induction on $n$. For the base case $n = 0$, suppose that $σ(s) = \sum_{a \in A} p_a · t_a(s_a)$ as above; now as $h$ is a comagma homomorphism, we must have that

$$\sum_{a \in A} h(s)([a]) · t_a(h(s)_a) = \sum_{a \in A'} p_a · t_a(h(s_a)) ;$$

so in particular, $u(s)([a]) \neq 0$ iff $a \in A'$ iff $h(s)([a]) \neq 0$, and when these equivalent conditions hold, we have $u(s)([a]) = p_a = h(s)([a])$, as required.

For the inductive step, we suppose we have verified the equality for $n - 1$, and will now verify it for $n$ as in (7.11). If $a_0 \notin A'$ then $u(s)([a_0]) = 0 = h(s)([a_0])$ and both sides of (7.11) are zero. If $a_0 \in A'$, then from (7.12) we must have $h(s)_a = h(s_0)$. But since $h(s)_a = h(s)([a_1 \cdots a_n]) = h(s)([a_0 \cdots a_n])/h(s)([a_0])$ we conclude using the inductive hypothesis that

$$h(s)([a_0 \cdots a_n]) = h(s)([a_0]) · h(s_0)([a_1 \cdots a_n]) = p_a · u(s_0)([a_1 \cdots a_n]) = u(s)([a_0 \cdots a_n]) .$$

In the logical case, an $A$-ary comagma in $(\text{Set}_B)^{\text{D}^0}$ is a sheaf $X$ endowed with a sheaf map $X → X * \cdots * X$. An entirely similar proof to the above shows that:

**Proposition 80.** The final $A$-ary comagma in $\text{Set}_B^{\text{D}^0}$ is the $B$-sheaf of continuous functions valued in $\text{Stream}(A)$. Via Stone duality, this is equally well the $B$-sheaf $\mathbf{Tr}$ whose elements of domain $b \in B$ are homomorphisms from the Boolean algebra of clopen sets of $\text{Stream}(A)$ to the Boolean algebra $B/b$ of elements below $b$ in $B$, with comagma structure $\mathbf{Tr} → \mathbf{Tr} * \cdots * \mathbf{Tr}$ given by

$$\omega \mapsto \sum_{a \in A} \omega([a]) · t_a(\omega_n)$$

where $\omega_n$ is the homomorphism $\text{Clopen}(\text{Stream}(A)) → B/\omega([a])$ defined by $\omega_n([a_1 a_2 \cdots]) = \omega([a a_1 a_2 \cdots])$.

**Trace map.** The following result characterises traces for probabilistic generative systems. The construction of the probabilities in (7.13) goes back at least to [29].

**Proposition 81.** Let $σ : S → \mathcal{D}(A × S)$ be a probabilistic generative system. The probability distribution $\text{tr}(s)$ associated to each state $s$ is characterised by

$$\text{tr}(s)([]) = 1 \quad \text{tr}(s)([a_0 a_1 \cdots a_n]) = \sum_{t \in S} σ(s)(a_0, t) · \text{tr}(t)([a_1 \cdots a_n]) .$$

**Proof.** Let $\overline{\text{tr}} : \mathcal{D}(S) → \mathcal{R}(\text{Stream}(A))$ be the unique extension of $\text{tr}$ to a convex map satisfying $\overline{\text{tr}} ◦ η_S = \text{tr}$. It suffices to check that $\overline{\text{tr}}$ is a map of $A$-ary comagmas $(F^\mathcal{D}(S), σ^\mathcal{D}) → (\text{Beh}, β)$; for then $\overline{\text{tr}}$ is the unique such homomorphism, and its
precomposition with $\eta_S$, which is $\text{tr}$, is the desired trace map. Now, to check that $\text{tr}$ is a map of $A$-ary comagmas, it suffices to do so on Dirac distributions $\eta(s) \in \mathcal{D}(S)$: in other words, we need only check commutativity in:

$$\begin{array}{c}
S \xrightarrow{\sigma} \mathcal{D}(A \times S) \cong \mathcal{D}(S) \ast \cdots \ast \mathcal{D}(S) \\
\text{tr} \downarrow \quad \downarrow \in \mathcal{D}(\text{Stream}(A)) \cong \mathcal{R}(\text{Stream}(A)) \ast \cdots \ast \mathcal{R}(\text{Stream}(A)).
\end{array}$$

Around the top of this diagram we have

\[ s \mapsto \sum_{\begin{array}{c} a \in A \\ \sigma(s)(a,S) \neq 0 \end{array}} \sigma(s)(a,S) \cdot \text{tr}(\sigma(s)(a,-)) \]

\[ \sum_{\begin{array}{c} a \in A \\ \sigma(s)(a,S) \neq 0 \end{array}} \sigma(s)(a,S) \cdot \text{tr}(\sum_{t \in S} \sigma(s)(a,t) \cdot \text{tr}(t)) \]

where we write $\sigma(s)(a,S)$ for $\sum_{t \in S} \sigma(s)(a,t)$; while around the bottom we have

\[ s \mapsto \sum_{\begin{array}{c} a \in A \\ \text{tr}(s)([a]) \neq 0 \end{array}} \text{tr}(s)([a]) \cdot \text{tr}(\sigma(s)(a,-)) \]

Thus, we need only verify that for all $a \in A$ and $s \in S$ we have $\sigma(s)(a,S) = \text{tr}(s)([a])$ for all $a, s$—which is clear from (7.13)—and that, if $\sigma(s)(a,S) \neq 0$, we have

\[ \text{tr}(s)([a]) = \sum_{t \in S} \frac{\sigma(s)(a,t)}{\sigma(s)(a,S)} \cdot \text{tr}(t). \]

But evaluating at $[a_1 \cdots a_n]$, this is the condition that

\[ \frac{\text{tr}(s)([a_1 \cdots a_n])}{\text{tr}(s)([a])} = \sum_{t \in S} \frac{\sigma(s)(a,t)}{\sigma(s)(a,S)} \cdot \text{tr}(t)[a_1 \cdots a_n] \]

which since $\text{tr}(s)([a]) = \sigma(s)(a,S)$, follows from (7.13). \hfill \square

By a corresponding argument, we can show:

**Proposition 82.** For a generative $s\mathcal{D}_B$-system $\sigma : S \rightarrow s\mathcal{D}_B(\sum_{a \in A} S)$, the trace $\text{tr}(s)$ of some $s \in S$ of domain $b$ is the homomorphism $\text{Clopen}(\text{Stream}(A)) \rightarrow B/b$ specified on the generating basic clopen sets by

$$\text{tr}(s)([a]) = b \quad \text{tr}(s)([a_0 a_1 \cdots a_n]) = \bigvee_{t \in S} \sigma(s)(a_0, t) \land \text{tr}(t)([a_1 \cdots a_n]).$$

**Reflection and reification.** In both the examples of this section, the reflection map $\text{Beh} \rightarrow \text{Tr}$ is the trace map of the object of behaviours. As for the reification maps, we concentrate on the probabilistic case; the logical case is entirely
analogous. The probabilistic generative system structure $\text{Tr} \rightarrow \mathcal{D}(A \times \text{Tr})$ on the object of traces is given by

$$\omega \mapsto \sum_{\substack{a \in A \\omega([a]) \neq 0}} \omega([a]) \cdot (a, \omega_a) ,$$

and as such, the reflection map $\text{Tr} \rightarrow \text{Beh}$ realises each trace as the behaviour which at each step makes the minimal specialisation of the sample space required to resolve the next output token. For example, we may consider the following refinement of the example from Section 7.3:

$\sigma : \{s, t, u\} \rightarrow \mathcal{D}(\{0, 1\} \times \{s, t, u\})$

$s \mapsto \frac{1}{3} \cdot (0, t) + \frac{1}{3} \cdot (1, t) + \frac{1}{3} \cdot (1, u)$

$t \mapsto 1 \cdot (0, t)$

$u \mapsto 1 \cdot (1, t)$

and

$\tau : \{s', t', u'\} \rightarrow \mathcal{D}(\{0, 1\} \times \{s', t', u'\})$

$s' \mapsto \frac{1}{3} \cdot (0, t') + \frac{2}{3} \cdot (1, u')$

$t' \mapsto 1 \cdot (0, t')$

$u' \mapsto \frac{1}{2} \cdot (0, t') + \frac{1}{2} \cdot (1, t') .$

The traces of $s$ and $s'$ are both the probability distribution which picks uniformly between the infinite binary strings 00\ldots, 100\ldots and 1100\ldots; however, their respective behaviours are the trees

\[
\begin{align*}
\bullet & \quad \bullet & \quad \bullet & \quad \bullet & \quad \bullet \\
(1, 0) & \quad (1, 0) & \quad (1, 0) & \quad (1, 0) & \quad (1, 0) \\
(1, 0) & \quad (1, 0) & \quad (1, 1) & \quad (1, 0) & \quad (1, 0) \\
(\frac{1}{3}, 0) & \quad (\frac{1}{3}, 1) & \quad (\frac{1}{3}, 1) & \quad (\frac{1}{2}, 0) & \quad (\frac{1}{2}, 1) \\
\end{align*}
\]

which are not bisimilar; but nonetheless, applying the normalisation function $\text{reify} \circ \text{reflect}$ sends $\text{beh}(s)$ to $\text{beh}(s')$.

7.6. Concluding remarks. The application developed in this section can be extended in two directions. On the one hand, we can apply it as-is to other examples of monads which admit hypernormalisation, for instance the continuous probability monads of Section 5.

More interestingly, we can change the kind of automaton we consider. We considered generative $\text{T}$-systems, which produce a stream of output values with as only external stimulus the computational effects encoded by the monad $\text{T}$. However, we can just as easily consider automata that both consume and produce tokens: indeed, if we define a Mealy $\text{T}$-machine with input alphabet $I$ and output alphabet $O$ to be a map of the form $\Sigma_{i \in I} S \rightarrow T(\Sigma_{o \in O} S)$, then the notions of behaviour, trace, reflection and reification given above adapt unproblematically.

However, there are other directions in which we might wish to generalise our automata. Most obviously, we might wish to introduce the possibility of termination, by looking at coalgebras of the form $S \rightarrow T(A \times S + 1)$ or $S \rightarrow T(A \times S) + 1$. To capture the notion of behaviour and trace in such situations, we must look not at $A$-ary conmagmas but comodels of more general
algebraic theories, which involve not only “cooperations” $S \to \sum_{a \in A} S$ but also equations between derived cooperations; see, for example [44, 53, 3, 17]. *Prima facie*, it is not at all clear that our normalisation-by-trace-evaluation extends to this context, since coequations between derived operations do not play well with the hypernormalisation structure. However, this is not to say that a more nuanced approach might not be possible: but this is to be left for further work.

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