Sketch-Guided Equality Saturation

Scaling Equality Saturation to Complex Optimizations of Functional Programs

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Generating high-performance code for diverse hardware and application domains like image processing, physics simulation, and machine learning is challenging. Functional array programming languages with patterns like map and reduce have been successfully combined with semantics-preserving term rewriting to define and explore optimization spaces. However, deciding what sequence of rewrites to apply is hard and has a huge impact on the performance of the rewritten program. Equality saturation avoids the issue by automatically exploring many possible ways to apply rewrites. It is made feasible by an efficient representation of many equivalent programs in an e-graph data structure.

Equality saturation has some limitations for compiler optimizations that rewrite functional language terms. Currently, there are only naive encodings of the lambda calculus for equality saturation. We present new techniques for encoding polymorphically typed lambda calculi, and show that the efficient encoding reduces the runtime and memory consumption of equality saturation by orders of magnitude.

Moreover, equality saturation does not yet scale to complex compiler optimizations. These emerge from long rewrite sequences of thousands of rewrite steps, and may use pathological combinations of rewrite rules that cause the e-graph to quickly grow too large. This paper introduces sketch-guided equality saturation, a semi-automatic technique that allows programmers to provide program sketches to guide rewriting. Sketch-guided equality saturation is evaluated for seven complex matrix multiplication optimizations, including loop blocking, vectorization, and multi-threading. Even with efficient lambda calculus encoding, unguided equality saturation can locate only the two simplest of these optimizations, the remaining five are undiscovered even with an hour of compilation time and 60GB of RAM. By specifying three or fewer sketch guides all seven optimizations are found in seconds of compilation time, using under 1GB of RAM, and generating high performance code.

1 INTRODUCTION

Term rewriting has been effective in optimizing compilers for decades [Dershowitz 1993]. More recently, functional array languages like LIFT [Steuwer et al. 2015] and Rise [Hagedorn et al. 2020] produce high-performance code for diverse hardware by using rewrite rules to define and explore optimization spaces. However, deciding when to apply each rewrite rule, the so-called phase ordering problem, is hard and has a huge impact on the performance of the rewritten program. The challenge is that the global benefit of applying a rewrite rule depends on future rewrites. Maximizing local benefit in a greedy fashion is not sufficient in the absence of a convergence property, i.e. confluence and termination, as local optima may be far away from global optima.

Rewriting strategies [Visser et al. 1998] allow programmers to control when to apply each rewrite rule, step-by-step. Prior work on ELEVATE [Hagedorn et al. 2020] has shown that rewriting strategies can achieve complex optimizations of Rise programs in less than a second of compilation time. However, these complex optimizations emerge from thousands of rewrite steps making the rewriting strategies challenging to write, as the phase ordering problem is passed on to the programmer.

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Equality saturation [Tate et al. 2009; Willsey et al. 2021] mitigates the phase ordering problem by exploring many possible ways to apply rewrite rules. Starting from an input program, equality saturation grows an equality graph (e-graph) by applying all possible rewrites iteratively until reaching a fixed point (saturation), achieving a performance goal, or timing out. An e-graph efficiently represents a large set of equivalent programs, and is grown by repeatedly applying rewrite rules in a purely additive way. Instead of replacing the matched left-hand side of a rewrite rule by its right-hand side, the equality between the left-hand side and the right-hand side is recorded in the e-graph. After growing the e-graph, the best program found is extracted from it using a cost model, e.g. one that selects the fastest program.

Unfortunately, equality saturation does not scale to complex optimizations such as the ones applied to RISE with ELEVATE, producing huge e-graphs that exceed the memory available in most machines. To scale equality saturation to complex optimizations of functional programs, this paper makes advances in two directions, as shown by the two axes in fig. 1.

**Rewriting Guidance.** On each equality saturation iteration, the e-graph tends to grow larger as every possible rewrite rule is applied in a purely additive way. The growth rate is extremely rapid for some combinations of useful rewrite rules, like associativity and commutativity. This makes exploring long rewrite sequences requiring many iterations unfeasible. One way to address this issue is to limit the number of rules applied [Wang et al. 2020; Willsey et al. 2021], but this risks not finding optimizations that require an omitted rule. An alternative is to use an external solver to speculatively add equivalences [Nandi et al. 2020], but this requires the identification of sub-tasks that can benefit from being delegated.

This paper proposes sketch-guided equality saturation that factors complex optimizations into a sequence of smaller optimizations, each sufficiently simple to be found by equality saturation. The programmer guides rewriting by describing how a program should evolve during optimization, through a sequence of sketches: program patterns that leave details unspecified. While sketches have previously been used as a starting point for program synthesis [Lezama 2008], our work uses sketches in a novel way as checkpoints to guide program optimization.

**Lambda Calculus Encoding.** As almost all programming languages use variables, and hence name binding, this paper explores practical ways of efficiently encoding languages with name bindings for the purposes of equality saturation. We focus on encoding the lambda calculus as it is the standard formalism for functional languages such as RISE. Previously, equality saturation has used a naive lambda calculus encoding that is simple but inefficient [Willsey et al. 2021]: the size of the e-graph quickly blows up. The inefficiency can be avoided by explicitly avoid name

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Fig. 1. This paper is about scaling equality saturation to complex optimizations of functional programs by combining an efficient λ calculus encoding with sketch-guided equality saturation.
bindings in the rewritten language [Smith et al. 2021]. We show that lambda calculus name bindings can be managed by using De Bruijn indices to avoid overloading the e-graph with $\alpha$-equivalent terms, and that an approximate implementation of substitution avoids overloading the e-graph with intermediate substitution steps.

Our evaluation demonstrates that combining sketch-guided equality saturation with an efficient lambda calculus encoding enables complex optimizations of Rise functional programs, that require tens of thousands of rewrite steps in term rewriting. We first evaluate the effectiveness of our lambda calculus encoding techniques for three rewrite goals. Thereafter, we show that sketch guiding enables seven complex optimizations of matrix multiplication to be applied. Elevate rewriting strategies apply these complex optimizations in less than a second with low memory consumption, but require ordering thousands of rewrite rules. Unguided equality saturation abstracts over rewrite ordering, but can only locate the simplest of the seven optimizations before the e-graph exceeds the available memory. Sketch-guided equality saturation applies all seven optimizations in seconds with low memory consumption, and only requires ordering up to three sketch guides.

To summarize, the main contributions of this paper are:

- Proposing sketch-guided equality saturation, a semi-automated process offering a novel, practical trade-off between rewriting strategies and equality saturation. The programmer guides multiple equality saturations by specifying a sequence of sketches describing how the program should evolve during optimization (section 3).
- Exploring new techniques to efficiently encode a polymorphically typed lambda calculus such as Rise for the purpose of equality saturation. In particular, De Bruijn indices are used to avoid overloading the e-graph with $\alpha$-equivalent terms, and an approximate (extraction-based) substitution is used to avoid overloading the e-graph with intermediate substitution steps (section 4).
- Two systematic evaluations of Rise applications. The effectiveness of our lambda calculus encoding is demonstrated by optimizing a binomial filter application (section 5.1). Sketch-guided equality saturation is evaluated by exploring seven complex optimizations of a matrix multiplication application, including loop blocking, vectorization, and multi-threading. These optimizations are infeasible with unguided equality saturation due to excessive runtime and memory consumption, but sketch-guided equality saturation performs the optimizations in seconds and with low memory consumption. No more than three sketch guides need to be written for any of the seven optimizations (section 5.2).

2 MOTIVATION AND BACKGROUND

This section introduces Rise programs and their optimization, as well as prior work on Elevate rewriting strategies and equality saturation. We highlight limitations and motivate our work.

2.1 The Rise functional language

Rise [Hagedorn et al. 2020] is a functional array programming language. It is a spiritual successor of Lift [Steuwer et al. 2015, 2017] that demonstrated performance portability across hardware by automatically applying semantics-preserving rewrite rules to optimize programs from various domains, including scientific code [Hagedorn et al. 2018] and convolutions [Mogers et al. 2020].

As a typed lambda calculus, Rise provides standard lambda abstraction ($\lambda x. b$), function application ($f x$), identifiers and literals. Rise expresses data-parallel computations as compositions of high-level computational patterns over multi-dimensional arrays, such as map which applies a function to each element of an array. reduce combines all elements of an array to a single value given a binary reduction operator. split, join, transpose, zip, unzip and slide reshape arrays in various ways.
Fig. 2. Applying a blocking optimization to matrix multiplication via rewriting in Rise. In the initial program (top-left), a `dot` product is computed between each row of a (aRow) and column of b (bCol). To define the `dot` product, the `zip` pattern combines two arrays a and b whose elements are multiplied pairwise using `map` before they are summed using `reduce`. In the final program (right), a blocking optimization has been applied. Intuitively, each red circle identifies a loop characteristic of the optimization (bottom-left). Understanding the remaining program details is not required, as they are only shown to visualize program complexity.

Listing 1. An Elevate strategy that applies the blocking optimization to matrix multiplication [Hagedorn et al. 2020]

High-level programs, such as a matrix multiplication in the top left of fig. 2, express their computations without committing to a particular implementation strategy.

Implementation choices are explicitly encoded in Rise programs by applying rewrite rules that introduce low-level patterns which directly correspond to a particular implementation strategy. For example, `reduceSeq` is a sequential implementation of `reduce`. For `map`, there exist multiple low-level patterns corresponding to different sequential and parallel implementations. After rewriting, Rise programs are translated to low-level imperative code such as C or OpenCL for execution.

The Rise program on the right of fig. 2 shows an optimized version of matrix multiplication. A common loop blocking optimization that improves data locality, and therefore memory usage, has been introduced by rewriting. Its impact is visualized in the bottom-left of fig. 2.

2.2 Elevate rewriting strategies

To specify optimizations Rise is complemented by a second language: Elevate [Hagedorn et al. 2020] that describes complex optimizations as compositions of rewrite rules, called rewriting strategies. The performance of the code generated by Rise and Elevate is comparable with state-of-the-art compilers, e.g. with the TVM deep learning compiler [Chen et al. 2018] for matrix multiplication [Hagedorn et al. 2020]; and with the Halide image processing compiler [Ragan-Kelley et al. 2012] for the Harris corner detection [Koehler and Steuwer 2021].
For example, the Elevate matrix multiplication blocking strategy is shown in listing 1. This strategy describes the sequence of rewrite steps that are required to rewrite the Rise program on the left of Fig. 2 to the one on the right. 32×32 blocks (or tiles) are created in line 2 and another block of 4 is created in line 4. Finally, loops are reordered in line 5 to create 4×32×32 blocks and hence produce the loop nest in Fig. 2. All abstractions like tile, split, reorder, outermost and innermost are not built-in but programmer-defined.

Limitations of manual rewriting with strategies. Although Elevate enables the development of abstractions that help write concise strategies, strategies remain challenging to write. The authors of [Hagedorn et al. 2020] and [Koehler and Steuwer 2021] estimate that they spent between two and five person-weeks developing the Elevate strategies for their matrix multiplication and image processing case studies. These case studies implement complex optimizations by applying tens of thousands of rewrite steps.

A particular limitation is that strategies are often program-specific and complex to implement. For example, the implementation of the reorder strategy is 43 lines long, involves the definition of 8 internal strategies, and carefully and recursively composes them together with some generic strategies. Despite its name, this reorder strategy is not capable of reordering arbitrary nestings of map and reduce patterns, but only works for the matrix multiplication example. Developing generic strategies is difficult because small program differences require adjustments to the rewrite sequence. The fundamental problem is that while Elevate empowers programmers to manually control the rewrite process, this requires the programmer to order all rewrite steps deterministically, i.e. to deal with fine-grained phase ordering.

To reduce the programmer effort required to optimize Rise programs, we look at equality saturation as an automation technique that promises to mitigate the phase ordering problem.

2.3 Equality saturation

Equality saturation [Tate et al. 2009; Willsey et al. 2021] is a technique for efficiently implementing rewrite-driven compiler optimizations without committing to a single rewrite choice. We demonstrate how equality saturation mitigates the phase ordering problem with a rewriting example where greedily reducing a cost function is not sufficient to find the optimal program.

Rewriting is often used to fuse operators and avoid writing results to memory, for example:

\[(\text{map} (\text{map} f)) \circ (\text{transpose} \circ (\text{map} (\text{map} g))) \quad (A)\]

\[(\text{map} (\text{map} (f \circ g))) \circ \text{transpose} \quad (B)\]

The initial term (A) applies function g to each element of a two-dimensional matrix (using two nested maps), transposes the result, and then applies function f to each element. The optimized term (B) avoids storing an intermediate matrix in memory and transposes the input before applying g and f to each element. Applying the following rewrite rules in the correct order is sufficient to perform this optimization:

\[\text{transpose} \circ \text{map} (\text{map} a) \leftrightarrow \text{map} (\text{map} a) \circ \text{transpose} \quad (1)\]

\[a \circ (b \circ c) \leftrightarrow (a \circ b) \circ c \quad (2)\]

\[\text{map} a \circ \text{map} b \leftrightarrow \text{map} (a \circ b) \quad (3)\]
Fig. 3. Growing an e-graph for the term \((\text{map} (\text{map} f)) \circ (\text{transpose} \circ (\text{map} (\text{map} g)))\). An e-graph is a set of e-classes themselves containing equivalent e-nodes. The dashed boxes are e-classes, and the solid boxes are e-nodes. New e-nodes and e-classes are shown in red.

Rule (1) states that transposing a two-dimensional array before or after applying a function to the elements is equivalent. Rule (2) states that function composition is associative. Finally, rule (3) is the rewrite rule for map fusion. In this example, minimizing the term size results in maximizing fused maps and is, therefore, a good cost model.

If we greedily apply rewrite rules that lower term size, we will only apply rule (3) as this is the only rule that reduces term size. However, rule (3) cannot be directly applied to term \((A)\): we are in a local optimum. The only way to reduce term size further is to first apply the other rewrite rules, which may or may not pay off depending on future rewrites.

Equality saturation minimizes term size by exploring many rewrites while avoiding local minima. First, an equality graph (e-graph) representing the initial term is constructed (fig. 3a). An e-graph is a set of equivalence classes (e-classes). An e-class is a set of equivalent nodes (e-nodes). An e-node \(F(e_1, .., e_n)\) is an \(n\)-ary function symbol \((F)\) from the term language, associated with \(n\) child e-classes \((e_i)\). Examples of symbols are \(\text{map}\), \(\text{transpose}\), and \(\circ\). The e-graph data structure is used during equality saturation to efficiently represent and rewrite a set of equivalent programs.

The initial e-graph is iteratively grown by applying rules non-destructively (figs. 3b to 3d). While standard term rewriting picks a single possible rewrite in a depth-first manner, equality saturation explores all possible rewrites in a breadth-first manner. In an equality saturation iteration, rewrites are applied independently: they only depend on rewrites from previous iterations. For the sake of simplicity, we only apply a handful of rewrite rules in fig. 3. When applying a rewrite rule, the equality between its left-hand side and its right-hand side is recorded in the e-graph. Rewrite rules stop being applied when a fixed point is reached (saturation), or when another stopping criteria is reached (e.g. timeout). If saturation is reached, it means that all possible rewrites have been explored.

An e-graph represents many equivalent terms and is far more compact than some set of terms as equivalent sub-terms are shared. E-graphs can represent exponentially many terms in polynomial space, and even infinitely many terms in the presence of cycles [Willsey et al. 2021]. To maximize sharing, a congruence invariant is maintained: intuitively identical e-nodes should not be in different e-classes (fig. 4). Later we will see that even extensive sharing does not necessarily prevent e-graph sizes from exploding.

After rewriting completes, an extraction procedure selects the best term from the e-graph using a cost function. Extraction can use a relatively simple bottom-up e-graph traversal if a local cost function \(c\) can be defined [Panchekha et al. 2015]. With costs of type \(K\), \(c\) has signature \(\text{cost}(\text{e-graph}) = K\).
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Fig. 4. The congruence invariant simplifies the e-graph on the left by merging two identical e-nodes for \( f \) into a single e-node as shown on the right.

Fig. 5. Smallest term size computed using an e-class analysis, shown for each e-class in the top-right corner in blue. Where it differs from its e-class value, the smallest term size of e-nodes is also shown. The e-class and e-nodes of the smallest term (B) are shown in green.

c\((F(k_1 : K, .., k_n : K)) : K\). More complex cost functions require more complex extraction procedures [Wang et al. 2020; Wu et al. 2019].

An e-class analysis [Willsey et al. 2021] propagates analysis data of type \( D \) in a bottom-up fashion, and can be used for extraction when the cost function is local. An e-class analysis is defined by providing two functions: one to make the analysis data from an \( n \)-ary symbol \( F \) combined with the data \( d_i \) of its child e-classes; and one to merge the analysis data of e-nodes in the same e-class.

\[
\text{make}(F(d_1 : D, .., d_n : D)) : D
\]

\[
\text{merge}(d_1 : D, d_2 : D) : D
\]

To demonstrate e-class analysis, we compute the smallest term in each e-class of our example using an e-class analysis with analysis data \( D = (\text{term size}, \text{term}) \) as well as make and merge functions below. The term sizes from the analysis are shown in fig. 5. The analysis reveals that there is a smaller term (B) of size 7 in the same e-class as the original term (A) which has a size of 9 (top left in fig. 5).

\[
\text{make}(F(d_1, .., d_n)) = (1 + \sum_i f_{\text{st}}(d_i), F(s_{\text{nd}}(d_1), .., s_{\text{nd}}(d_n)))
\]

\[
\text{merge}(d_1, d_2) = \text{if } f_{\text{st}}(d_1) \leq f_{\text{st}}(d_2) \text{ then } d_1 \text{ else } d_2
\]

Limitations of automatic rewriting with equality saturation. In practice, the applicability of equality saturation is limited by scaling issues. As the e-graph grows, iterations become slower and require more memory. The growth rate is aggravated by some combinations of rewrite rules, such as associativity and commutativity, that generate an exponential number of equivalent permutations [Nandi et al. 2020; Wang et al. 2020; Willsey et al. 2021]. This makes exploring long rewrite sequences inherently hard, as the breadth-first exploration of all possible rewrites leads to an exponential increase of the search space. We encounter these issues when attempting to optimize matrix multiplication in Rise using equality saturation, as we discuss in Section 5.

Our contributions ameliorate these limitations by minimizing the size of the e-graphs required. Sketch-guiding factors unfeasible equality saturations into a sequence of smaller, and hence feasible, equality saturations (section 3). Efficient encoding of the \( \lambda \) calculus reduces the sizes of the e-graphs produced when optimizing functional programs, like Rise programs (section 4).
3 SKETCH-GUIDED EQUALITY SATURATION

This section introduces sketch-guided equality saturation, a novel semi-automated process offering a trade-off between manually written rewriting strategies and fully automated equality saturation. The programmer guides multiple equality saturations by specifying a sequence of sketches that describe how a program should evolve during optimization. While rewriting strategies require fine-grained phase ordering, sketch-guided equality saturation enables coarse-grained phase ordering. For example, instead of specifying how to sequence thousands of rewrite rules to achieve matrix multiplication blocking, specifying just two sketches suffices.

3.1 The Intuition for Sketches

When designing optimizations, programmers often visualize the desired shape of the optimized program. An optimization, e.g. with rewriting strategies or schedules, is often illustrated with program snippets that explain its effect [Adams et al. 2019; Anderson et al. 2021; Chen et al. 2018; Ikarashi et al. 2021; Koehler and Steuwer 2021; Ragan-Kelley et al. 2013; Sioutas et al. 2020]. Indeed, this is exactly how we have explained the loop nest blocking optimization in fig. 2.

Our key new insight is that explanatory program snippets can be formalized as sketches and used to guide an optimization search, instead of manually writing strategies or schedules that perform the optimization directly. Sketches are program patterns that capture intuitions about the program shape while leaving details unspecified. The guided optimization search still performs rewriting with semantic preserving rules, allowing sketches to leave out details without sacrificing correctness. We define sketches formally in section 3.2.

We illustrate by presenting the sketches for matrix multiplication blocking. Listing 2 shows the sketch for the unoptimized baseline goal, specifying the desired program structure as two nested map patterns and a nested reduce, with innermost addition and multiplication operations. The formal definitions of containsMap, containsReduceSeq and containsAddMul are in section 3.2. The comments on the right show the equivalent nested for loops, using the same intuition as in fig. 2.
A sequence of sketches (e.g., listing 4 followed by listing 3) may be used to achieve a desired pattern of elements of domain type \( n \times 4 \times R \), each processed by the three innermost for loops.

Search for the *blocking* goal can be made more tractable by specifying intermediate *sketch guides*, and listing 4 is an example. This sketch guide describes a program shape where the *map* and *reduce* patterns have been split but not yet reordered.

Elevate rewriting strategies, as seen before in listing 1, are detailed imperative specifications of how to rewrite the program. In contrast, a sketch is a declarative specification of the optimization goal, and equality saturation is used to search for a sequence of rewrites to achieve that goal. A sequence of sketches (e.g., listing 4 followed by listing 3) may be used to achieve a desired optimization when the equality saturation search with a single sketch as a goal does not succeed.

### 3.2 Sketch Definition

Sketches are specified in a SketchBasic language with just four constructors. The syntax of SketchBasic and the set of terms that the constructors represent are defined in fig. 6. A sketch \( s \) represents a set of terms \( R(s) \), such that \( R(s) \subset T \) where \( T \) denotes all terms in the language we rewrite. We say that any \( t \in R(s) \) satisfies sketch \( s \).

The \( ? \) sketch is the least precise, representing all terms in the language. The \( F(s_1, ..., s_n) \) sketch represents all terms that match a specific \( n \)-ary function symbol \( F \) from the term language, and whose \( n \) children satisfy sketches \( s_i \). The \( \text{contains}(s) \) sketch represents all terms containing a term that satisfies sketch \( s \). Finally, the \( s_1 \lor s_2 \) sketch represents terms satisfying either \( s_1 \) or \( s_2 \).

When rewriting terms in a typed language, sketches may be annotated with a type sketch \( s :: pt \) constraining the type of terms. If \( R(pt) \) denotes the set of terms satisfying the type sketch \( pt \), then \( R(s :: pt) = R(s) \cap R(pt) \). The grammar of type sketches depends on the language we rewrite. We elide type sketches from our definition of SketchBasic for simplicity, but use them for RISE.

Writing a useful sketch to guide an optimization search requires striking a balance between being too precise and too vague. An overly precise sketch may exclude valid optimized programs with a slightly different structure. An overly vague sketch may lead to finding undesirable programs. This balance also interacts with the rewrite rules involved, since programs that may be found by the search are \( R(s) \cap EQ(t, \text{rules}) \) where \( EQ(t, \text{rules}) \) represents the set of terms that can be discovered to be equivalent to the initial term \( t \) according to the given rules. This means that using a more restricted set of rules generally enables specifying less precise sketches. How to best select rules and write effective sketches are topics for future work.

**Sketch Abstractions.** Sketch abstractions are defined by combining generic constructs from SketchBasic with type annotations from the term language. To illustrate, listing 5 shows the sketch abstractions for our RISE matrix multiplication case study. Here \( \rightarrow \) is a function type, and \( n, dt \) an array type of \( n \) elements of domain type \( dt \). The type annotations restrict the iteration domains of patterns like *map* and *reduceSeq*. We use similar definitions for other language constructs.


3.3 Sketch-Guided Equality Saturation

The process of guiding equality saturation with a sequence of sketches is illustrated in fig. 7. The programmer provides a sequence of $N$ intermediate sketch guides and a final goal sketch: $\text{sketch}_1, \ldots, \text{sketch}_{N+1}$. Successive equality saturation searches are performed to find equivalent terms that satisfy each sketch in the sequence. As each sketch may be satisfied by many terms, the programmer must also provide a sequence of cost models $\text{cost}_1, \ldots, \text{cost}_{N+1}$ to select the term to be used as the starting point for the next search. Sets of rewrite rules ($\text{rules}_1, \ldots, \text{rules}_{N+1}$) are provided to grow the e-graph in each search. The cost model and set of rules may be identical for many or all of the searches, but we show examples in section 5 where selectively restricting the set of rules reduces search runtime. Figure 8 shows how each search is performed, and how these searches differ from standard equality saturation.

The pseudo-code for the sketch-guided equality saturation algorithm is shown in listing 6. The entry point is the $\text{SGES}$ function (line 1) that takes a term and a sequence of sketches, cost models and rewrite rules ($\text{params}$). It repeatedly searches (line 4) for each sketch using the associated cost model and rewrite rules, and outputs a term if found, otherwise nothing.

![Fig. 7. Sketch-Guided Equality Saturation](image)

![Fig. 8. Sketch-Satisfying Equality Saturation implementing each search](image)
At the beginning of each search, we may normalize the input term (line 10) to apply destructive rewrites that are always desired before starting a purely additive equality saturation. For our matrix multiplication running example we use $\beta\eta$ normal form. The `extract` function (line 15) is used to extract a term from the e-graph that satisfies the specified sketch while minimizing the specified cost model, and we describe it in the next subsection. In this paper, we terminate equality saturation as soon as a program satisfying the current sketch is found, whether or not the cost could be further improved by a longer search. This is because we give more value to satisfying the sketch than to minimizing the cost. Other applications of sketch-guided equality saturation could use different stopping criteria. The `extract` function (line 14) is used to stop growing the e-graph by checking whether `extract` would succeed.

### 3.4 Sketch-Satisfying Extraction

To `extract` the best program that satisfies a `SketchBasic` sketch $s$ from an e-graph $g$ we define a helper function $E(c,s,g)$, where $c$ is a cost function that must be monotonic and local. While `extract` returns a program from an e-class $e$, the helper $E$ returns a map from e-classes to optional tuples of costs and terms. After invoking $E$ we simply look up the e-class $e$ in the map and extract the term from the optional tuple. For efficiency, we memoize previously computed results of $E$. The `extract` function is recursively defined over the four `SketchBasic` cases as follows.

**Case 1:** $s = \emptyset$. This case is equivalent to extracting the programs minimizing $c$ from the e-graph. We implement this extraction as an e-class analysis (section 2.3) with data type $D = \text{Option}[(k,t)]$ and functions `make` that constructs analysis data and `merge` that combines analysis data from e-nodes in the same e-class:

$$
\text{make}(F(d_1, \ldots, d_n)) = \begin{cases}
\text{Some} & \left( \begin{array}{l} c(F(k_1, \ldots, k_n)), \\ F(t_1, \ldots, t_n) \end{array} \right) \\
\text{None} & \text{otherwise}
\end{cases}
$$

$$
\text{merge}(d_1, d_2) = \begin{cases}
d_1 & \text{if } k_1 \leq k_2 \text{ then } d_1 \text{ else } d_2 \\
d_2 & \text{if } k_1 \neq k_2 \text{ then } (k_j, \_) \in d_1 \\
\text{None} & \text{otherwise}
\end{cases}
$$

**Case 2:** $s = F(s_1, \ldots, s_n)$. We consider each e-class $e$ containing $F(e_1, \ldots, e_n)$ e-nodes and the terms that should be extracted for each child e-class $e_i$. We write $E(c,s,g)[e]$ for indexing into the map returned by $E$:

$$
E(c,F(s_1, \ldots, s_n), g)[e] = \begin{cases}
\text{Some}(c(F(k_1, \ldots, k_n)), F(t_1, \ldots, t_n)) & (k_i, t_i) \in E(c, s_i, g)[e_i] \\
\text{None} & \text{otherwise}
\end{cases}
$$

**Case 3:** $s = \text{contains}(s_2)$. We use another e-class analysis and initialize the analysis data to $E(c,s_2,g)$ corresponding to the base case where $R(s_2) \subset R(\text{contains}(s_2))$. To `make` the analysis data we consider all terms that would contain terms from $s_2$ and keep the best by folding them using `merge`:

$$
\text{make}(F(d_1, \ldots, d_n)) = \text{foldl merge} \text{ None}
\{ \text{Some}(c(F(k_1, \ldots, k_j, \ldots, k_n)), F(t_1, \ldots, t_j, \ldots, t_n)) \mid i \neq j, (k_i, t_i) \in E(c, s, g)[e_i], (k_j, t_j) \in d_j \}
$$

To `merge` the analysis data, we do the same as for $s = \emptyset$.

**Case 4:** $s = s_1 \lor s_2$. We `merge` the results from $s_1$ and $s_2$:

$$
E(c,s_1 \lor s_2,g)(e) = \text{merge}(E(c,s_1,g)(e), E(c,s_2,g)(e))
$$
**4 EFFICIENT EQUALITY SATURATION FOR THE LAMBDA CALCULUS**

We now investigate the second direction shown in fig. 1 for scaling equality saturation to complex optimizations of functional programs, by exploring the engineering design choices required to efficiently encode a polymorphically typed lambda calculus within equality saturation. A set of design choices are realized for the **Rise** language in the new **Risegg** implementation that is heavily inspired by the egg library [Willsey et al. 2021].

Lambda calculus terms can be encoded as terms of shape \( F(t_1, \ldots, t_n) \), as shown in table 1. Variable names are not modeled directly as terms, but as operator metadata: 'lam x', 'lam y', 'var x' and 'var y' are all distinct operators.

Applying equality saturation to lambda calculus terms requires the efficient support of standard operations and rewrites. Figure 9 shows the standard rules of **\( \beta \)**-reduction and **\( \eta \)**-reduction that require dealing with substitution, name bindings and freshness predicates. The other two rules encode map-fusion and map-fission, and are interesting because they introduce new name bindings on their right-hand-side.

### 4.1 Substitution

**\( \beta \)**-reduction requires substituting \( b[e/x] \), but standard term substitution cannot be used during equality saturation as \( b \) and \( e \) are not terms but e-classes. A simple way to address this is to use *explicit substitution* as in egg’s lambda calculus example [Willsey et al. 2021]. A syntactic constructor is added to represent substitution, with rewrite rules to encode its small-step behavior:

\[
(\lambda x. b) e \mapsto b[e/x] \\
\lambda x. f x \mapsto f \quad \text{if } x \text{ not free in } f \\
\text{map } f (\text{map } g \text{ arg}) \mapsto \text{map } (\lambda x. f (g x)) \text{ arg} \\
\text{map } (\lambda x. g x) \mapsto \lambda y. \text{map } f (\text{map } (\lambda x. g x) y) \quad \text{if } x \text{ not free in } f
\]

Unfortunately explicit substitution adds all intermediate substitution steps to the e-graph, quickly exploding its size. Section 5.1 shows that this is a major problem in practice, making relatively simple rewrites unfeasible. To avoid adding intermediate substitution steps to the e-graph, we propose *extraction-based substitution* that works as follows.

1. extract a term for each e-class involved in the substitution (i.e \( b \) and \( e \));
2. perform standard term substitution;
3. add the resulting term to the e-graph.

| \( \lambda \) calculus | \( F \) | \( t_1, \ldots, t_n \) |
|------------------------|-------|---------------------|
| \( \lambda x. e \)     | lam x| \( e \)              |
| \( e_1, e_2 \)         | app  | \( e_1, e_2 \)       |
| \( x \)                | var x|                     |

Table 1. Encoding of \( \lambda \) calculus terms as \( F(t_1, \ldots, t_n) \) terms for the purposes of equality saturation.
Section 5.1 demonstrates that extraction-based substitution is far more efficient than explicit substitution. Extraction-based substitution is, however, an approximation as it computes the substitution for a subset of the terms represented by \( b \) and \( e \), and ignores the rest. Figure 10 shows an example where the initial e-graph is in the middle, and the e-graph after extraction-based substitution with \( b = x \) and \( e = y \) on the right. This particular choice results in an e-graph lacking the \( id \ x \) program that is included in the e-graph without approximation (left in fig. 10).

In practice, we have not observed the approximation to be an issue when optimizing \( \text{Rise} \) programs (section 5), and believe that two main reasons account for this. Firstly, the substitution is computed on each equality saturation iteration, where different terms may be extracted, increasing coverage of the set of terms represented by \( b \) and \( e \). Secondly, many of the ignored equivalences are recovered either by e-graph congruence, or by applying further rewrite rules. Future work may investigate alternative substitution implementations to balance efficiency with non-approximation.

### 4.2 Name Bindings

In equality saturation inappropriate handling of name bindings easily leads to serious efficiency issues. Consider rewrite rules like map-fusion that create a new lambda abstraction on their right-hand side. What name should be introduced when they are applied? In standard term rewriting, generating a fresh name using a global counter (aka. gensym) is a common solution. But if a new name is generated each time the rewrite rule is applied, the e-graph is quickly burdened with many \( \alpha \)-equivalent terms\(^3\).

Fewer \( \alpha \)-equivalent terms are introduced if fresh names are generated as a function of the matched e-class identifiers. However as the e-graph grows and e-classes are merged, e-class identifiers change, and \( \alpha \)-equivalent terms are still generated and duplicated in the e-graph.

De Bruijn indices [de Bruijn 1972] are a standard technique for representing lambda calculus terms without naming the bound variables, and avoid the need for \( \alpha \) conversions. If De Bruijn indices enable two \( \alpha \)-equivalent terms to be structurally equivalent, the standard e-graph congruence invariant prevents their duplication, by ensuring that equivalent e-nodes are not allocated to different e-classes. Hence we translate our terms and rewrite rules to use De Bruijn indices instead of names, and achieve significant efficiency gains (section 5.1).

**True equality modulo \( \alpha \)-renaming.** While De Bruijn indices give a significant performance improvement, they do not provide equality modulo \( \alpha \)-renaming for sub-terms. Consider \( f \ (\lambda x. \ f) = %0 \ (\lambda. \ %1) \), where \( %i \) are De Bruijn indices. Although \( %0 \) and \( %1 \) are structurally different, they both correspond to the same variable \( f \). In practice, we have not observed this to be a significant issue when optimizing \( \text{Rise} \) programs, but it does require care when comparing sub-terms that

\(^3\)Two \( \lambda \) terms are \( \alpha \)-equivalent if one can be made equivalent to the other simply by renaming variables.
have a different number of surrounding lambdas. Future work may investigate alternatives to De Bruijn indices, for example through hashing modulo $\alpha$-renaming [Maziarz et al. 2021], or through nominal rewriting techniques [Fernández and Gabbay 2007].

*Translating name-based rules into index-based rules.* Using De Bruijn indices means that rewrite rules must manipulate terms with De Bruijn indices. Thankfully, more user-friendly name-based rewrite rules can be automatically translated to the index-based rules used internally [Bonelli et al. 2000]. An example demonstrating this is given in section 4.5.

*Explicit or extraction-based substitution.* Both explicit substitution and extraction-based substitution are compatible with De Bruijn indices, and for explicit substitution we use the $\lambda s$ calculus [Kamareddine and Ríos 1995].

*Shifting De Bruijn indices.* De Bruijn indices must be shifted when a term is used with different surrounding lambdas (example in section 4.5). As for substitution, shifting can be implemented with explicit rewrite rules, or with *extraction-based index shifting*:

- extract a term from the e-class whose indices need shifting;
- perform index shifting on the term;
- add the resulting term to the e-graph.

In RISEGG we use extraction-based index shifting whenever extraction-based substitution is used.

*Avoiding Name Bindings using Combinators.* It is also possible to avoid name bindings entirely [Smith et al. 2021]. For example, it is possible to introduce a function composition combinator '$\circ$' as in section 2.3, greatly simplifying the map-fusion and map-fission rules:

\[
\begin{align*}
  f \, (g \, x) & \longmapsto (f \circ g) \, x \quad \text{(\circ-intro)} \\
  \text{map } f \circ \text{map } g & \longmapsto \text{map } (f \circ g) \quad \text{(map-fusion$_2$)} \\
  \text{map } (f \circ g) & \longmapsto \text{map } f \circ \text{map } g \quad \text{(map-fission$_2$)}
\end{align*}
\]

However, this approach has its own downsides. Associativity rules are required, which we know increases the growth rate of the e-graph [Wilsley et al. 2021]. Only using a left-/right-most associativity rule avoids generating too many equivalent ways to parenthesize terms. But other rewrite rules now have to take this associativity convention into account, making their definition more difficult and their matching more expensive. In general, matching modulo associativity or commutativity are algorithmically hard problems [Benanav et al. 1987].

The function composition $\circ$ combinator on its own is also not sufficient to remove the need for name bindings. At one extreme, combinatorial logic could be used as any lambda calculus term can be represented, replacing function abstraction by a limited set of combinators. However, translating a lambda calculus term into combinatory logic results in a term of size $O(n^3)$ in the worst case, where $n$ is the size of the initial term [Lachowski et al. 2018]. Translating existing rewrite systems to combinatory logic would be challenging in itself.

### 4.3 Freshness Predicates

Handling predicates is not trivial in equality saturation. The $\eta$-reduction has the side condition that "if $x$ not free in $f$", but in an e-graph $f$ is an e-class and not a term.

The predicate could be precisely handled by filtering $f$ into $f'' = \{ t \mid t \in f, x$ not free in $t \}$, and using $f''$ on the right-hand-side of the rule. However, this requires splitting an e-class in two: one that satisfies the predicate, and one that does not. We do not attempt such a split as it would reduce sharing, increase e-graph size and be incompatible with the congruence invariant unless the very notion of equality is changed.
The design of \textsc{Risegg} makes the engineering trade-off to only apply the $\eta$-reduction rewrite rule if $\forall t \in f. x$ not free in $t$, following egg’s lambda calculus example \cite{Willsey2021}. Advantages are that this predicate is efficient to compute using an e-class analysis, and that there is no need to split the e-class. The disadvantage is that it is an approximation that ignores some valid terms.

Figure 11 shows an example where $\eta$-reduction is not applied. In practice, we have not observed the approximation to be an issue, e.g. for the results presented in section 5.

### 4.4 Adding Polymorphic Types

Typed lambda calculi are pervasive, e.g. as the foundation for almost all functional languages, and a key consideration is how to add types to the e-graph. More specifically, we look at how polymorphic types interact with equivalence classes. If types can be computed in a bottom-up fashion, an e-class analysis can be used, similar to how the size and shape of tensors is computed in \cite{Wang2020}. However, if polymorphic types are monomorphized, their instantiation is context-dependent and cannot be computed in a bottom-up fashion. For example, consider the terms $(\lambda x. x) (0 : \text{int})$ and $(\lambda x. x) (0.0 : \text{float})$. In \textsc{Rise}, the identify function is monomorphized and has two different type instantiations that should live in different e-classes: $\lambda x. x : \text{int} \rightarrow \text{int}$ and $\lambda x. x : \text{float} \rightarrow \text{float}$. Hence, in \textsc{Risegg} instantiated types are embedded in the e-graph instead of computed by an analysis. Each e-class is associated with a type that all of its e-nodes must satisfy.

In an e-graph there is a tension between sharing and the availability of contextual information in a given e-class. Type instantiation prevents sharing, as the same polymorphic expression produces a different e-class at each type. However, instantiation provides additional contextual information, as each e-class is associated with a precise monotype.

\textit{Hash-consing types.} Since types are duplicated many times in the e-graph, and since structural type equality is often required, we hash-cons types for efficiency \cite{Filliâtre2006}. Alternatively, types could be stored in the e-graph itself to provide equational reasoning at the type level, but this is not necessary for this paper.

### 4.5 Compiling user-defined rewrite rules

To avoid explicit typing or explicit use of De Bruijn indices in user-defined rewrite rules, user-friendly name-based and partially typed rewrite rules are compiled internally as required.

Types are inferred with \textsc{Rise} type inference: both sides of rewrite rules can be seen as terms where free variables correspond to pattern variables. After inferring the types on the left-hand-side, we check that the right-hand-side is well-typed for any well-typed left-hand-side match. When applied, typed rewrite rules match (deconstruct) types with their left-hand-side, and construct types on their right-hand-side. Type annotations can be used to constrain the inferred types.
Bound variables are replaced with their corresponding De Bruijn index, and indices are shifted as required for terms with differing numbers of surrounding lambdas. We illustrate with examples.

Example 1: η-reduction.

\[ \lambda x. f \, x \rightarrow f \quad \text{if } x \text{ not free in } f \]

RISEGG translates this rule into:

\[ (\lambda. f (\%0 : t_0)) : t_0 \rightarrow t_1 \rightarrow (\varphi^{-1}_1 f) : t_0 \rightarrow t_1 \quad \text{if } \%0 \text{ not free in } f \]

The transformed rule uses a De Bruijn index \%0 for the bound variable, and pattern variables otherwise: \( f \), \( t_0 \) and \( t_1 \). It provides index shifting through \( \varphi^{-1}_1 f \) that shifts all indices \( \geq 1 \) by \( -1 \) as a surrounding \( \lambda \) has been removed. Some types are matched on the left-hand-side (\( t_0 \) and \( t_1 \)), and used to construct types on the right-hand-side (\( t_0 \rightarrow t_1 \)). Some types, like the type of \( f \), are not matched on the left-hand-side as RISEGG avoids matching on redundant type information to some extent, assuming that the terms matched are well-typed.

Example 2: η-abstraction.

\[ f : t_0 \rightarrow t_1 \rightarrow \lambda x. f \, x \]

η-abstraction illustrates how type annotations may be used, and are sometimes required. RISEGG translates this rule into:

\[ f : t_0 \rightarrow t_1 \rightarrow (\lambda. (((\varphi^1_0 f) : t_0 \rightarrow t_1) (\%0 : t_0)) : t_1) : t_0 \rightarrow t_1 \]

A type error occurs if the type of \( f \) is not annotated as in \( f : t_0 \rightarrow t_1 \) on the left-hand-side, since the right-hand-side requires it to be a function type.

5 EVALUATION

We now evaluate our two proposed techniques for scaling equality saturation to complex optimizations of functional programs. Figure 12 illustrates our evaluation process. Starting from the naive lambda calculus encoding used in egg’s example [Willsey et al. 2021], we first investigate the effectiveness of the new encoding from section 4, implemented in RISEGG, for unguided equality saturation (section 5.1). Thereafter, we adopt the new lambda calculus encoding in RISEGG, and compare unguided and sketch-guided equality saturation to reproduce seven complex matrix multiplication optimizations previously applied with ELEVATE rewriting strategies (section 5.2).

![Fig. 12. We first evaluate our efficient λ calculus encoding before evaluating sketch-guided equality saturation using this encoding.](image_url)
5.1 Impact of Lambda Calculus Encoding

The efficiency of equality saturation for the lambda calculus is evaluated by attempting to discover three rewrite goals using four combinations of the substitution and name binding techniques outlined in section 4. The naive lambda calculus encoding uses explicit substitution and variable names. The efficient encoding uses extraction-based substitution to avoid intermediate substitution steps, and De Bruijn indices to avoid duplicating $\alpha$-equivalent terms. Discovering a rewrite goal means that it is feasible to grow an e-graph starting from the initial program until the goal program is represented in the e-class of the initial program.

Experimental Setup. The alternate encodings are realized for an untyped subset of the Rise language in an early prototype of Risegg. To measure search runtime, we use egg’s built-in mechanisms, falling back to the GNU time utility in case of an out-of-memory exception. Maximum memory residency is measured with the GNU time utility. The experiments are run on a laptop with an AMD Ryzen 5 PRO 2500U processor, and we limit the available RAM to 2 GB.

Rewrite Goals. We use three rewrite goals with increasing complexity.

The reduction rewrite goal in fig. 13 is based on a unit test from egg’s lambda calculus example, and only uses $\eta$-reduction and $\beta$-reduction rules. The lambda calculus examples from egg are relatively simple, as the rewrite rules involved do not introduce new names on their right-hand side and in most cases do not increase term size: the e-graph size does not grow explosively.

The fission rewrite goal in fig. 14 adds the use of map-fusion and map-fission rewrite rules that introduce new name bindings on their right-hand-side, and interact with each other as well as $\beta$-reduction to create many possibilities: the e-graph size starts to explode.

The binomial rewrite goal from fig. 15 is a real optimization that requires 6 more rewrite rules, increasing interactions between rewrite rules and aggravating the e-graph growth rate. A binomial filter is an essential component of many image processing pipelines, where it reduces noise or

Fig. 13. reduction rewrite goal. The initial program creates a $\text{comp}$ combinator for function composition and uses it to compose the $\text{add1}$ function with itself 7 times. All uses of $\text{comp}$ and $\text{add1}$ are $\beta$-reduced in the final program, which simply applies $+1$ to its input value 7 times.

Fig. 14. fission rewrite goal. The initial program successively applies $f_1$ to $f_5$ inside a map pattern. The final program first applies $f_1$ and $f_2$ inside one map pattern, before applying $f_3$ to $f_5$ inside another map pattern.
Fig. 15. **binomial rewrite goal.** The initial Rise program iterates over 2D neighborhoods (nbh). A dot product is computed between the weights (weights2d shown in section 5.1) and each neighborhood. The final program iterates over two 1D neighborhoods (vertical and horizontal) instead.

detail. It is a 2D convolution, and in Rise uses a range of patterns including zip, transpose and slide that reshape arrays in various ways. The purpose of the rewrite is to separate the 2D convolution into two 1D convolutions according to the well-known convolution kernel equation:

$$
\begin{bmatrix}
1 & 2 & 1 \\
2 & 4 & 2 \\
1 & 2 & 1
\end{bmatrix}
= \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}
$$

This separation optimization reduces both memory accesses and arithmetic complexity. Although more complex than the previous two rewrite goals, this rewrite goal is still relatively simple and we consider that unguided equality saturation should at least scale to this goal to be useful in practice for Rise. Prior work achieved this optimization by orchestrating 30 rewrite rules – including 17 \( \eta/\beta \)-reductions – with Elevate rewriting strategies [Koehler and Steuwer 2021].

**Results.** Table 2 compares the performance of equality saturation using different combinations of substitution (explicit/extraction-based) and name binding (named/De Bruijn) techniques. It reports whether the goal is found, the search runtime, memory consumption, number of applied rewrite rules, and e-graph size. The simple reduction goal is found by all combinations, although explicit substitution with De Bruijn indices is less efficient with 25K rewrite rules, 25K e-classes,
and occupying 35 MB. The \textit{fission} goal is not found if explicit substitution is used with named variables, exhausting the 2 GB memory and showing that this encoding is particularly inefficient.

The \textit{binomial} goal is only found by combining extraction-based substitution and De Bruijn indices. With this encoding, all three rewrite goals are found by applying less than 5K rewrite rules, producing e-graphs with fewer than 3K e-nodes and 1K e-classes, and occupying less than 8 MB of memory. For all three rewrite goals, this combination provides the fastest searches and most compact e-graphs, often by orders of magnitude. We conclude that \textit{combining extraction-based substitution and De Bruijn indices gives an efficient encoding of lambda calculus for equality saturation.}

5.2 Impact of Sketch Guidance

This section compares unguided equality saturation to the new sketch-guided equality saturation to achieve complex optimization goals. In the evaluation, \textit{both} equality saturation techniques use the efficient lambda calculus encoding from section 5.1. Matrix multiplication is selected as the case study as it allows us to compare against published \textsc{Elevate} strategies that specify optimizations equivalent to TVM schedules [Hagedorn et al. 2020]. TVM is a state-of-the-art deep learning compiler [Chen et al. 2018], and Hagedorn et al. [2020] demonstrate that expressing optimizations performed by TVM as compositions of rewrites is possible and achieves the same high performance as TVM. We evaluate the seven matrix multiplication optimizations described in the TVM manual and presented in [Hagedorn et al. 2020]. The optimizations are typical compiler optimizations, including loop blocking, loop permutation, vectorization, and multi-threading.

In this evaluation, we compare how much runtime and memory are required for unguided equality saturation and our sketch-guided equality saturation. For both guided and unguided equality saturation the optimization goal is specified as a sketch, that acts as the stopping criteria. This is less restrictive than the searches for a goal program in the previous subsection as the sketch goal may be satisfied by many programs.

We validate that the result of each complex optimizations is high performance code as follows. When a program satisfying the optimization goal sketch is found, we check that the generated C code is equivalent, modulo variable names, to the manually optimized versions that already demonstrated performance competitive with TVM.

\textit{Experimental Setup.} The full version of \textsc{Risegg}\footnote{https://github.com/rise-lang/shine/tree/sges/src/main/scala/rise/eqsat} is implemented in Scala, and we use standard Java utilities for measurements: System.nanoTime() to measure search runtime, and the Runtime api to approximate maximum heap memory residency with regular sampling.

The experiments are performed on two platforms. For \textsc{Elevate} strategies and our sketch-guided equality saturation, we use a less powerful AMD Ryzen 5 PRO 2500U with 4 GB of RAM available to the JVM. For unguided equality saturation, we use a more powerful Intel Xeon E5-2640 v2 with 60 GB of RAM available to the JVM.

\textit{Unguided Equality Saturation.} Table 3 shows the runtime and memory consumption required to find the optimization goals with unguided equality saturation. The search terminates when the sketch describing the optimization goal is found in the e-graph. Most optimization goals are not found before exhausting the 60 GB of available memory. Only the \textit{baseline} and \textit{blocking} goals are found, and the search for \textit{blocking} requires more than 1h and about 35 GB of RAM. Millions of rewrite rules are applied, and the e-graph contains millions of e-nodes and e-classes. More complex optimizations involve more rewrite rules, creating a richer space of equivalent programs but exhausting memory faster. As examples, \textit{vectorization} and \textit{loop-perm} use vectorization rules, while \textit{array-packing}, \textit{cache-blocks}, and \textit{parallel} use rules for optimizing memory storage.
Table 3. Runtime and memory consumption for unguided equality saturation with efficient lambda calculus encoding. Only the baseline and blocking optimization goals are found, with other optimizations exceeding 60 GB.

| goal          | found? | runtime | RAM  | rules | e-nodes | e-classes |
|---------------|--------|---------|------|-------|---------|-----------|
| baseline      | ✓      | 0.5s    | 0.02 GB | 2     | 51      | 49        |
| blocking      | ✓      | >1h     | 35 GB | 5M    | 4M      | 2M        |
| vectorization | ✗      | >1h     | >60 GB|       |         |           |
| loop-perm     | ✗      | >1h     | >60 GB|       |         |           |
| array-packing | ✗      | 35mn    | >60 GB|       |         |           |
| cache-blocks  | ✗      | 35mn    | >60 GB|       |         |           |
| parallel      | ✗      | 35mn    | >60 GB|       |         |           |

Table 4. Runtime and memory consumption for sketch-guided equality saturation with efficient lambda calculus encoding. All optimizations are found in seconds using less than 0.5 GB of memory, and requiring at most 3 sketch guides.

| goal          | sketch guides | found? | runtime | RAM  | rules | e-nodes | e-classes |
|---------------|---------------|--------|---------|------|-------|---------|-----------|
| baseline      | 0             | ✓      | 0.5s    | 0.02 GB | 2     | 51      | 49        |
| blocking      | 1             | ✓      | 7s      | 0.3 GB | 11K   | 11K     | 7K        |
| vectorization | 2             | ✓      | 7s      | 0.4 GB | 11K   | 11K     | 7K        |
| loop-perm     | 2             | ✓      | 4s      | 0.3 GB | 6K    | 10K     | 7K        |
| array-packing | 3             | ✓      | 5s      | 0.4 GB | 9K    | 10K     | 7K        |
| cache-blocks  | 3             | ✓      | 5s      | 0.5 GB | 9K    | 10K     | 7K        |
| parallel      | 3             | ✓      | 5s      | 0.4 GB | 9K    | 10K     | 7K        |

Sketch-Guided Equality Saturation. Table 4 shows the runtime and memory consumption for sketch-guided equality saturation, where sketches guide the optimization process and break a single equality saturation search into multiple. All optimizations are found in less than 10s, using less than 0.5 GB of RAM. Interestingly, the number of rewrite rules applied by sketch-guided equality saturation is in the same order of magnitude as for the manual ELEVATE strategies reported in [Hagedorn et al. 2020]. On one hand, equality saturation applies more rules than necessary because of its explorative nature. On the other hand, ELEVATE strategies apply more rules than necessary because they re-apply the same rule to the same sub-expression and do not necessarily orchestrate the shortest possible rewrite path. The e-graphs contain no more than $10^4$ e-nodes and e-classes, two orders of magnitude less than the $10^6$ required for blocking without sketch-guidance.

E-Graph Evolution in Guided and Unguided Search. Figure 16 plots the growth of the e-graphs during unguided and sketch-guided equality saturation searches for the blocking and parallel optimization goals from Tables 3 and 4. The e-graphs produced by unguided equality saturation grow exponentially with each search iteration. The e-graph contains millions of e-nodes and e-classes, and millions of rules have been applied, within a small number of iterations (less than 10). Such rapid growth limits the scalability of unguided search, for example in the 7th iteration of the parallel search the e-graph exhausts 60GB memory.

While the e-graphs produced by sketch-guided equality saturation typically also grow exponentially with each iteration, sketches are satisfied within a small number of iterations thanks to an appropriate selection of sketch guides. In consequence the number of rewrites and crucially the maximum size of the e-graphs is three orders of magnitudes smaller than for unguided search: no more than 11K in our example searches. Once a program satisfying a sketch guide is found,
Fig. 16. The evolution of the e-graph, and the number of rewrite rules applied, during searches for two optimization goals. Sketch guides are depicted with purple vertical lines. Note that the scale of the y-axes for unguided graphs (a) and (b) is millions, while for guided graphs (c) and (d) it is thousands.

A new search is started for the next sketch using that program, growing a fresh e-graph that remains small. Hence sketch-guided equality saturation can scale to more complex optimizations of functional programs, such as parallel. Conceptually factoring optimizations into a long sequence of sketch-guided searches means that there is no limit on the complexity of the optimizations, i.e. the number of rewrites, that may be searched for.

The search for the final parallel sketch goal shows linear rather than exponential growth. This is likely because the set of rewrite rules used by the search have little interaction. We will discuss the impact of choosing suitable sets of rewrite rules shortly.

**Sketches Guiding the Search.** Table 5 shows how each optimization goal is described by logical optimization steps, each corresponding to a sketch describing the program after the step is applied. It transpires that the split sketch in listing 4 is a useful first guide for all optimization goals. While the sketch sizes range from 7 to 12, programs are of size 90 to 124, showing that a sketch elides around 90% of the program. Even when 4 sketches must be written, the total size of the sketches is still small: the largest total being 38. Intricate aspects of the optimized program never need to be specified in the sketches, for example array reshaping patterns such as split, join and transpose.

**Choice of Rules and Cost Model.** Besides the sketches, programmers also specify the rules used in each search and a cost model. For the split sketch, 8 rules are required explaining how to split map and reduce. The reorder sketches require 9 rules that swap various nestings of map and reduce. The store sketch requires 4 rules and the lower sketches 10 rules including map-fusion, 6 rules for vectorization, 1 rule for loop unrolling and 1 rule for loop parallelization. If we naively use all rules for the blocking search, the search runtime increases by about 25×, still finding the optimizations in minutes but showing the importance of selecting a small set of rules.
Table 5. Decomposition of each optimization goal into logical steps. A sketch is defined for each logical step. In this table, sketch size counts operators such as `containsMap`, program size counts operators such as `map`, lambdas, variables and constants: λ applications are not counted.

| goal          | sketch guides       | sketch goal       | sketch sizes | program size |
|---------------|---------------------|-------------------|--------------|--------------|
| blocking      | split               | reorder1          | 7            | 90           |
| vectorization | split + reorder1    | lower1            | 7            | 124          |
| loop-perm     | split + reorder2    | lower2            | 7            | 104          |
| array-packing | split + reorder2 + store | lower3          | 7-12         | 121          |
| cache-blocks  | split + reorder2 + store | lower4          | 7-12         | 121          |
| parallel      | split + reorder2 + store | lower5          | 7-12         | 121          |

We use a simple cost model that minimizes weighted term size. Rules and cost models may be reused and packaged into libraries for recurring logical steps.

6 RELATED WORK

Controlling optimizations. Historically, programmers had either to explicitly write optimized code, e.g., explicit vectorization or loop ordering, or to entrust optimization to a black box compiler.

Automating optimization. Some compiler optimizations are fully automated via equality saturation [Tate et al. 2009; Yang et al. 2021] or heuristic searches [Mullapudi et al. 2015; Steuwer et al. 2015]. Although this approach can automatically yield high performance, it is not always feasible or even desirable, as it may result in poor performance or may be too time-consuming [Maleki et al. 2011; Parello et al. 2004]. When automatic optimization is unsatisfactory, programmers often fall back to manual optimization in order to achieve their performance goals [Lacassagne et al. 2014; Lemaitre et al. 2017; Niittylahti et al. 2002].

Rewriting strategies and schedules. More recently, compiler optimizations can be precisely controlled by the programmer specifying rewriting strategies [Hagedorn et al. 2020; Koehler and Steuwer 2021; Visser et al. 1998] or schedules [Chen et al. 2018; Ragan-Kelley et al. 2012]. However, these are challenging to write as the phase ordering problem is passed on to the programmer [Ikarashi et al. 2021], as discussed in section 2.2.

Guidance used outside of optimizations. Although equality saturation exploits e-graphs for program optimization [Tate et al. 2009], e-graphs were originally designed for efficient congruence closure in theorem provers [De Moura and Bjørner 2008; Nelson 1980], and are useful in other settings such as program synthesis, or semantic code search [Premtoon et al. 2020].

Guidance in theorem proving. In the realm of theorem proving, guidance is typically required because automation is both theoretically undecidable and difficult in practice. Just as rewriting strategies specify how to transform a program step-by-step, proof tactics [Gordon et al. 1979] specify how to transform a proof state step-by-step in procedural proof languages. In declarative proof languages, proof sketches specify partial proofs and can guide theorem proving [Corbineau 2007; Wiedijk 2003]. Proof sketches are analogous to program sketches that specify partial programs to guide optimization.

Sketching for synthesizing programs. The idea of sketching has been used for program synthesis [Lezama 2008], along with counterexample guided inductive synthesis that combines a synthesizer with a validation procedure. Our approach differs as we use sketches for optimizations rather than program synthesis. We use sketches as program patterns to filter a set of equivalent programs generated via equality saturation, and as a result do not require a validation procedure.
Other techniques for scaling equality saturation. There exists a number of prior techniques for scaling equality saturation in practice.

Languages with binding. We are not the first to attempt applying equality saturation to languages with binding. Willsey et al. [2021] implements a partial evaluator for the lambda calculus using explicit substitution. Although conceptually simple, this is too inefficient for complex optimizations, as demonstrated in section 5.1. Smith et al. [2021] proposes access patterns to avoid the need for binding structures when representing tensor programs. In section 4, we instead present lambda calculus encoding techniques that make equality saturation significantly more efficient.

Rewrite rule scheduling. To reduce e-graph growth, previous work proposes rewrite rule schedulers as a way to control which rewrite rules should be applied on a given equality saturation iteration [Willsey et al. 2021]. By default, the egg library uses a BackoffScheduler preventing specific rules from being applied too often, and reducing e-graph growth in the presence of “explosive” rules such as associativity and commutativity. Our experience with Rise optimization is that using the BackoffScheduler is often counterproductive as the desired optimization depends on some explosive rules. Future work may explore better ways to schedule rewrite rules, but at present Riseegg does not use a rewrite rule scheduler.

External solvers. External solvers may add equivalences to the e-graph [Nandi et al. 2020], but this requires the identification of sub-tasks that can benefit from being delegated.

7 CONCLUSION

This paper broadens the applicability of equality saturation by making it scale to complex optimizations in a functional language in two ways.

Firstly, sketch-guided equality saturation is proposed as a semi-automated technique for scaling to more complex optimizations by factoring a single equality saturation search into a sequence of smaller equality saturation searches (section 3). Programmers guide the process by describing how a program should evolve using a sequence of sketches. We demonstrate that sketch-guided equality saturation enables seven complex optimizations of matrix multiplication to be applied within seconds in the Rise functional language, using under 1 GB of RAM, using no more than three sketch guides (table 4). By contrast, traditional unguided equality saturation cannot discover the five more complex optimizations even with an hour of runtime and 60 GB of RAM (table 3). For each optimization, the generated code is identical to the high-performance code generated by manually ordering thousands of rewrites via rewriting strategies. The performance of this code is comparable with code produced by the state-of-the-art TVM compiler [Hagedorn et al. 2020].

Secondly, to effectively apply equality saturation to functional languages, we propose new techniques to efficiently encode lambda calculi for equality saturation. The key innovations are extraction-based substitution and representing identifiers as De Bruijn indices (section 4). Specifically for our Rise use case, we apply the techniques to a polymorphically typed lambda calculus. Combining the techniques reduces the runtime and memory consumption of equality saturation over lambda terms by orders of magnitude, enabling equality saturation to scale to more complex optimization goals (table 2).

Future work may investigate how to identify appropriate sets of rules to be applied in each search. We are interested in exploring how to design effective sketch guides for more diverse applications, and to explore ways to possibly even synthesize sketch guides automatically. We believe that combining imperative, step-by-step approaches (e.g. rewriting strategies) with more declarative approaches (e.g. sketch guidance) deserves further research and can lead to the creation of practical, interactive optimization assistants. Finally, we hope that the community will be inspired to apply and extend sketch-guided equality saturation in new domains.
REFERENCES

Andrew Adams, Karima Ma, Luke Anderson, Riyadh Baghdadi, Tzu-Mao Li, Michael Gharbi, Benoit Steiner, Steven Johnson, Kayvon Fatahalian, Frédo Durand, et al. 2019. Learning to optimize halide with tree search and random programs. TOG (2019).

Luke Anderson, Andrew Adams, Karima Ma, Tzu-Mao Li, Tian Jin, and Jonathan Ragan-Kelley. 2021. Efficient automatic scheduling of imaging and vision pipelines for the GPU. OOPSLA (2021).

Dan Benanav, Deepak Kapur, and Paliath Narendran. 1987. Complexity of matching problems. *Journal of symbolic computation* 3, 1-2 (1987), 203–216.

Eduardo Bonelli, Delia Kesner, and Alejandro Rios. 2000. A de Bruijn notation for higher-order rewriting. In *International Conference on Rewriting Techniques and Applications*. Springer, 62–79.

Tianqi Chen, Thierry Moreau, Ziheng Jiang, Lianmin Zheng, Eddie Yan, Haichen Shen, Meghan Cowan, Leyuan Wang, Yuwei Hu, Luis Ceze, et al. 2018. {TVM}: An automated end-to-end optimizing compiler for deep learning. In *13th {USENIX} Symposium on Operating Systems Design and Implementation ( {OSDI} 18)*. 578–594.

Pierre Corbineau. 2007. A declarative language for the Coq proof assistant. In *International Workshop on Types for Proofs and Programs*. Springer, 69–84.

N.G de Bruijn. 1972. Lambda calculus notation with nameless dummies, a tool for automatic formula manipulation, with application to the Church-Rosser theorem. *Indagationes Mathematicae (Proceedings)* 75, 5 (1972), 381–392.

Leonardo De Moura and Nikolaj Bjørner. 2008. Z3: An efficient SMT solver. In *International conference on Tools and Algorithms for the Construction and Analysis of Systems*. Springer, 337–340.

Nachum Dershowitz. 1993. A taste of rewrite systems. In *Functional Programming, Concurrently, Simulation and Automated Reasoning*. Springer, 199–228.

Maribel Fernández and Murdoch J Gabbay. 2005. Nominal rewriting. *Information and Computation* 205, 6 (2007), 917–965.

Jean-Christophe Filliâtre and Sylvain Conchon. 2006. Type-safe modular hash-consing. In *Proceedings of the 2006 Workshop on ML*. 12–19.

Michael J Gordon, Arthur J Milner, and Christopher P Wadsworth. 1979. *Edinburgh LCF: a mechanised logic of computation*. Springer.

Bastian Hagedorn, Johannes Lenfers, Thomas Koehler, Xueying Qin, Sergei Gorlatch, and Michel Steuwer. 2020. Achieving high-performance the functional way: a functional pearl on expressing high-performance optimizations as rewrite strategies. *Proceedings of the ACM on Programming Languages* 4, ICFP (2020), 1–29.

Bastian Hagedorn, Larisa Stoltzfus, Michel Steuwer, Sergei Gorlatch, and Christophe Dubach. 2018. High performance stencil code generation with lift. In *Proceedings of the 2018 International Symposium on Code Generation and Optimization, CCG 2018*, Vösendorf / Vienna, Austria, February 24-28, 2018. ACM, 100–112.

Yuka Ikarashi, Jonathan Ragan-Kelley, Tsukasa Fukusato, Jun Kato, and Takeo Igarashi. 2021. Guided Optimization for Image Processing Pipelines. In *VL/HCC*.

Bastian Hagedorn and Michel Steuwer. 2021. Towards a Domain-Extensible Compiler: Optimizing an Image Processing Pipeline on Mobile CPUs. In *2021 IEEE/ACM International Symposium on Code Generation and Optimization (CGO)*. IEEE, 27–38.

Lionel Lacassagne, Daniel Etiemble, Ali Hassan Zahraee, Alain Dominguez, and Pascal Vezolle. 2014. High level transforms for SIMD and low-level computer vision algorithms. In *Proceedings of the 2014 Workshop on Programming Models for SIMD/Vector processing*. 49–56.

Łukasz Lachowski et al. 2018. On the complexity of the standard translation of lambda calculus into combinatory logic. *Reports on Mathematical Logic* 53 (2018), 19–42.

Florian Lemaitre, Benjamin Couturier, and Lionel Lacassagne. 2017. Cholesky factorization on SIMD multi-core architectures. *Journal of Systems Architecture* 79 (2017), 1–15.

A Solar Lezama. 2008. *Program synthesis by sketching*. Ph.D. Dissertation. PhD thesis, EECS Department, University of California, Berkeley.

Saeed Maleki, Yaoqing Gao, Maria J Garzar, Tommy Wong, David A Padua, et al. 2011. An evaluation of vectorizing compilers. In *2011 International Conference on Parallel Architectures and Compilation Techniques*. IEEE, 372–382.

Krzysztof Maziarz, Tom Ellis, Alan Lawrence, Andrew Fitzgibbon, and Simon Peyton Jones. 2021. Hashing modulo alpha-equivalence. In *Proceedings of the 42nd ACM SIGPLAN International Conference on Programming Language Design and Implementation*. 960–973.

Naums Mogers, Valentin Radu, Lu Li, Jack Turner, Michael O’Boyle, and Christophe Dubach. 2020. Automatic generation of specialized direct convolutions for mobile GPUs. In *Proceedings of the 13th Annual Workshop on General Purpose Processing using Graphics Processing Unit*. 41–50.
Ravi Teja Mullapudi, Vinay Vasista, and Uday Bondhugula. 2015. PolyMage: Automatic Optimization for Image Processing Pipelines. *SIGARCH Comput. Archit. News* 43, 1 (March 2015), 429–443.

Chandrakana Nandi, Max Willsey, Adam Anderson, James R Wilcox, Eva Darulova, Dan Grossman, and Zachary Tatlock. 2020. Synthesizing structured CAD models with equality saturation and inverse transformations. In *Proceedings of the 41st ACM SIGPLAN Conference on Programming Language Design and Implementation*. 31–44.

Charles Gregory Nelson. 1980. *Techniques for program verification*. Stanford University.

Jarkko Niittylahti, Juha Lemmetti, and Juhana Helovuo. 2002. High-performance implementation of wavelet algorithms on a standard PC. *Microprocessors and Microsystems* 26, 4 (2002), 173–179.

Pavel Panchekha, Alex Sanchez-Stern, James R Wilcox, and Zachary Tatlock. 2015. Automatically improving accuracy for floating point expressions. *ACM SIGPLAN Notices* 50, 6 (2015), 1–11.

D. Parello, O. Temam, A. Cohen, and J.-M. Verdun. 2004. Towards a Systematic, Pragmatic and Architecture-Aware Program Optimization Process for Complex Processors. In *SC ’04: Proceedings of the 2004 ACM/IEEE Conference on Supercomputing*, 15–15. https://doi.org/10.1109/SC.2004.61

Varot Premtoon, James Koppel, and Armando Solar-Lezama. 2020. Semantic code search via equational reasoning. In *Proceedings of the 41st ACM SIGPLAN Conference on Programming Language Design and Implementation*. 1066–1082.

Jonathan Ragan-Kelley, Andrew Adams, Sylvain Paris, Marc Levy, Saman P. Amarasinghe, and Frédéric Durand. 2012. Decoupling algorithms from schedules for easy optimization of image processing pipelines. *ACM Trans. Graph.* 31, 4 (2012), 32:1–32:12.

Jonathan Ragan-Kelley, Connelly Barnes, Andrew Adams, Sylvain Paris, Frédéric Durand, and Saman Amarasinghe. 2013. Halide: a language and compiler for optimizing parallelism, locality, and recomputation in image processing pipelines. *ACM Sigplan Notices* (2013).

Savvas Sioutas, Sander Stuijk, Twan Basten, Henk Corporaal, and Lou Somers. 2020. Schedule synthesis for halide pipelines on gpus. *TACO* (2020).

Gus Henry Smith, Andrew Liu, Steven Lyubomirsky, Scott Davidson, Joseph McMahan, Michael Taylor, Luis Ceze, and Zachary Tatlock. 2021. Pure tensor program rewriting via access patterns (representation pearl). *arXiv preprint arXiv:2105.09377* (2021).

Michel Steuwer, Christian Fensch, Sam Lindley, and Christophe Dubach. 2015. Generating performance portable code using rewrite rules: from high-level functional expressions to high-performance OpenCL code. In *Proceedings of the 20th ACM SIGPLAN International Conference on Functional Programming, ICFP 2015, Vancouver, BC, Canada, September 1-3, 2015*. ACM, 205–217.

Michel Steuwer, Toomas Remmelg, and Christophe Dubach. 2017. Lift: a functional data-parallel IR for high-performance GPU code generation. In *CGO*. ACM, 74–85.

Ross Tate, Michael Stepp, Zachary Tatlock, and Sorin Lerner. 2009. Equality saturation: a new approach to optimization. In *Proceedings of the 36th annual ACM SIGPLAN-SIGACT symposium on Principles of programming languages*. 264–276.

Eelco Visser, Zine-el-Abidine Benaissa, and Andrew Tolmach. 1998. Building program optimizers with rewriting strategies. *ACM Sigplan Notices* 34, 1 (1998), 13–26.

Yisu Remy Wang, Shana Hutchison, Jonathan Leang, Bill Howe, and Dan Suciu. 2020. SPORES: sum-product optimization via relational equality saturation for large scale linear algebra. *arXiv preprint arXiv:2002.07951* (2020).

Freek Wiedijk. 2003. Formal proof sketches. In *International Workshop on Types for Proofs and Programs*. Springer, 378–393.

Max Willsey, Chandrakana Nandi, Yisu Remy Wang, Oliver Flatt, Zachary Tatlock, and Pavel Panchekha. 2021. Egg: Fast and extensible equality saturation. *Proceedings of the ACM on Programming Languages* 5, POPL (2021), 1–29.

Chenming Wu, Haisen Zhao, Chandrakana Nandi, Jeffrey I Lipton, Zachary Tatlock, and Adriana Schulz. 2019. Carpentry compiler. *ACM Transactions on Graphics (TOG)* 38, 6 (2019), 1–14.

Yichen Yang, Phitchaya Phothilimthana, Yisu Wang, Max Willsey, Sudip Roy, and Jacques Pienaar. 2021. Equality saturation for tensor graph superoptimization. *Proceedings of Machine Learning and Systems* 3 (2021).