A PIE Representation of Coupled Linear ODE-PDE Systems with Constant Delay and Stability Analysis using LPIs

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Abstract—Partial Integral Equations (PIEs) have been used to represent both systems with delay and systems of Partial Differential Equations (PDEs) in one or two spatial dimensions. In this paper, we show that these results can be combined to obtain a PIE representation of any suitably well-posed 1D PDE model with delay. In particular, we represent these PDE systems with delay as coupled systems of 1D and 2D PDEs. We then show that a PIE representation of both the 1D and 2D subsystems can be derived, and combine these representations to obtain a 2D PIE representation for PDEs with delay. We show that this PIE representation can also be coupled to that of Ordinary Differential Equations (ODEs) with delay, deriving a PIE representation of general linear ODE-PDE systems with delay. Next, based on the PIE representation, we formulate the problem of stability analysis as a Linear Operator Inequality (LPI) optimization problem which can be solved using the PIETOOLS software suite. We apply the result to several examples from the existing literature involving delay in the dynamics as well as the boundary conditions of the PDE.

I. INTRODUCTION

We consider the problem of analysis of coupled systems of Ordinary Differential Equations (ODEs) and Partial Differential Equations (PDEs). Such ODE-PDE systems are frequently used to model physical processes, modeling the dynamics on the interior of the domain with the PDE subsystem, and the dynamics at the boundaries using ODE subsystems.

In both modeling and control of ODE-PDE systems, the dynamics of the system often exhibit a delayed response to external disturbances or changes in the internal state, giving rise to delays in different components of the model. For example, these delays may be inherent to the dynamics of the system itself, appearing within the PDE (sub)system, as in the following wave equation,

\[
\ddot{x}(t, s) = x_{ss}(t - \tau, s),
\]

\[
\dot{x}(t, 1) = -x(t, s), \quad x(t, 0) = 0.
\]

Alternatively, delay may occur in the interaction between coupled systems, explicitly appearing in the boundary conditions of the PDE as in

\[
\dot{x}(t) = -x(t - \tau) + x_{s}(t, 1)
\]

\[
\dot{x}(t, s) = x_{ss}(t, s)
\]

\[
\dot{x}(t, 1) = -x(t), \quad x(t, 0) = 0
\]

or defining the dynamics at the boundary of the domain as

\[
\dot{x}(t) = -x(t - \tau) + x_{s}(t, 1)
\]

\[
\dot{x}(t, s) = x_{ss}(t, s)
\]

\[
\dot{x}(t, 1) = -x(t), \quad x(t, 0) = 0
\]

In each case, the presence of delays naturally complicates analysis of solution properties such as stability of the system, as at any time \( t \geq 0 \), the state of the system involves not only the current value of the state \( x(t) \), but also the value of the ODE and PDE states \( x(s) \) and \( x(s) \) at all \( s \in [t - \tau, t] \).

To test stability of ODE-PDE systems with delay, one common approach involves searching for a Lyapunov-Krasovskii functional (LKFs) [1]. For a system with state \( u(t) \) and delayed state \( v(t, s) = u(t - s) \), stability can be verified by testing for existence of such an LKF \( V(u, v) \), satisfying \( \dot{V}(u, v) \geq 0 \) with equality if and only if \( u = v = 0 \), and \( \dot{V}(u(t), v(t)) \leq 0 \) along any solution \( (u(t), v(t)) \) to the system. Over the last few decades, LKFs for ODEs have been extensively studied, proving conditions on the structure of the LKF that are both necessary and sufficient for stability (see e.g. [2]), and that allow stability to be tested as a Linear Matrix Inequality (LMI). Unfortunately, the more complicated structure of PDEs, involving differential operators and Boundary Conditions (BCs), makes the task of finding a LKF \( V \) and verifying negativity of the derivative \( \dot{V}(u(t), v(t)) \) along solutions for PDEs with delay substantially more challenging.

Because of the complicated nature of testing stability of PDEs with delay, most prior work in this field considers either specific delayed ODE-PDEs or restricted classes of such systems. The analysis then becomes ad hoc, exploiting the structure of the PDE (parabolic, hyperbolic, elliptic), the type of BCs (Dirichlet, Neumann, Robin), and the location of the delay (in the PDE or a connected ODE) to prove stability of the system. For example, stability conditions for wave equations were proven in [3], [4], using Green’s function to show that the energy \( \dot{V}(u, v) \) of the solutions decreases. In [5]–[7], stability tests for linear and semi-linear diffusive PDEs with delay were derived, using, e.g. the Hellinger, Wirtinger, Jensen or Poincaré inequalities to prove LMI constraints for negativity of the derivative \( \dot{V}(u, v) \). Using similar approaches, LMIs for stability of reaction-diffusion PDEs with delayed boundary inputs [8] were also derived, as well as for ODE-PDE systems with delay in both the ODE and the PDE [9]. Finally, stability conditions have been found for systems with delayed feedback in the PDE [10] and in a coupled ODE [11] system.

Unfortunately, in each of these cases, the stability tests consider only particular systems or restricted classes, assume

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the LKF $V$ to be of a particular form, and rely on specific inequalities to verify negativity of the derivative $\dot{V}(u, v)$. The problem with such approaches, of course, is that extending the results for even slightly different models may require significant expertise from the user.

In this paper, we propose an alternative, LMI-based method for testing stability of a general class of linear ODE-PDE systems with delay, by representing them as Partial Integral Equations (PIEs). A PIE is an alternative representation of linear ODE-PDE systems, taking the form

$$T\dot{u}(t) = A\dot{u}(t),$$

where the operators \{\mathcal{T}, \ldots, D\} are Partial Integral operators. In [12] and [13], it was shown that the sets of 1D and 2D PI operators form *-algebras, meaning that the sum, composition, and adjoint of such PI operators is a PI operator as well. As such, Lyapunov functions for a PIE can be readily parameterized by positive PI operators $\mathcal{P}$. Moreover, since the PIE representation does not impose any BCs or continuity constraints of the fundamental state $\dot{u} \in L_2$, stability can be verified by testing feasibility of a Linear PI Inequality (LPI)

$$T^\dagger \mathcal{P} A + A^* T \leq 0,$$  \hspace{1cm} (1)

without having to account for BCs. Parameterizing positive PI operators by positive matrices, these operator inequalities can be enforced as LMI s, allowing problems of stability analysis, optimal control, and optimal estimation to be solved using semidefinite programming.

In [14], it was shown that a general class of linear Delay Differential Equations (DDEs) can be equivalently represented as PIEs. Similarly, in [15], it was shown that any suitably well-posed ODE-PDE system without delay can also be equivalently represented as a PIE. However, constructing a PIE representation for 1D ODE-PDE systems with delay is complicated by the fact that the delayed state $v(t, s, x) = u(t - s, x)$ varies in two spatial variables. To address this problem, we decompose the delayed ODE-PDE into the feedback interconnection of delayed and PDE subsystems where the interconnection signals are infinite-dimensional. In Section V, we prove that each of these subsystems can be equivalently represented as an associated PIE with infinite-dimensional inputs and outputs, extending prior work on PIE input-output systems to the case of infinite dimensional inputs and outputs. In Section III we consider the feedback interconnection of PIEs with infinite-dimensional inputs and outputs and derive formulae for the resulting closed-loop PIE. Finally, in Section V, we apply the results to a general class of delayed ODE-PDEs to obtain stability conditions expressed as Linear Partial Integral Operator Inequalities (LPIs). These LPIs are then converted to Semidefinite Programming problems using the PIETOOLS software package and tested on several examples of delayed ODE-PDEs in Section VI.

II. PRELIMINARIES

A. Notation

For a given domain $\Omega \subset \mathbb{R}^d$, let $L_2^n(\Omega)$ denote the set of $\mathbb{R}^n$-valued square-integrable functions on $\Omega$, where we omit the domain when clear from context. Define intervals $\Omega^b_k := [a, b]$, and let $\Omega^{bd}_{ac} := \Omega^b_a \times \Omega^d_c$ be the corresponding 2D domain. For $n = (n_0, n_1) \in \mathbb{N}^2$, define $L_2^{n_0}(\Omega^b_{a_1})$ and $L_2^{n_1}(\Omega^d_{c_1})$, and for $n = (n_1, n_2) \in \mathbb{N}^2$, define $Z_{n_1}(\Omega_{ac}^{bd}) := Z_{n_1}^a(\Omega_{ac}^{bd}) \times Z_{n_2}^b(\Omega_{ac}^{bd})$, where we also omit the domain when clear from context. For given $n \in \mathbb{N}^2$ and any $u = (u^a, u^b) \in Z_{n_2}^a$, define the inner product

$$\langle u, v \rangle_{Z_{n_2}^a} = u^a \cdot \dot{w} + \langle q, x \rangle_{L_2} + \langle r, \pi \rangle_{L_2} + \langle s, z \rangle_{L_2},$$

where $(\langle \cdot, \cdot \rangle_{L_2})$ denotes the standard inner product on $L_2$. For $k = (k_1, k_2) \in \mathbb{N}^2$, define Sobolev subspaces $H^k_{\Omega_{ac}^{bd}}$ and $H_0^{k_{bd}}$ of $L_2$ as

$$H^k_{\Omega_{ac}^{bd}} := \left\{ v \mid \partial_x^\alpha \partial_y^\beta v \in L_2^2(\Omega_{ac}^{bd}) \right\}, \forall \alpha, \beta \in \mathbb{N}^2 : \alpha_1 \leq k_1, \beta_1 \leq k_2.$$

For $v \in H^k_{\Omega_{ac}^{bd}}$, we denote the Dirac delta operators

$$\left[ \Delta^\alpha \delta \right](y) := (v, y) \quad \text{and} \quad \left[ \Delta^\beta \delta \right](x) := (v, x).$$

For a function $N \in L_2^{n \times m}(\Omega_{ac}^{bd})$, and any $v \in L_2^{m}(\Omega_{ac}^{bd})$, we define the multiplier operator $M$ and integral operator $f$ as

$$(M[N]v)(x, y) := N(x,y)v(x, y),$$

$$\left( f[N] \right)(x) := \int_a^b N(x, y)v(x, y)dx.$$
We denote the set of 4-PI operators as $\Pi_4^{n \times m}$, so that $\mathcal{P} \in \Pi_4^{n \times m}$ if and only if $\mathcal{P} = \mathcal{P}[B]$ for some $B \in \mathcal{N}_4^{n \times m}$.

Although a general class of PI operators in 2D has been defined in [13], to reduce notation, we will not introduce this class in this paper. Instead, through some abuse of notation, we allow 4-PI operators to act on $Z_2 := \left[ L_2 \left[ \mathcal{L}_1 \right] \right]$ as well, acting as multipliers along $y < \Omega^d$. Then, we define a subset of the PI operators on 2D as follows.

**Definition 3 (PI Operators on 2D ($\Pi_{12}$)):** For given $m := (m_1, m_2)$, $n := (n_1, n_2) \in \mathbb{N}^4$, define

$$N^{m \times n}_{12} \subset \Omega_{ac}^{bd} : \begin{bmatrix} N_{m_1 n_1}^{m_2 n_2} (\mathcal{P}_n) & N_{m_1 n_2}^{m_2 n_1} (\mathcal{P}_n) \\ N_{m_2 n_1}^{m_1 n_2} (\mathcal{P}_n) & N_{m_2 n_2}^{m_1 n_1} (\mathcal{P}_n) \end{bmatrix}.$$ 

Then, for given parameters $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \in N^{m \times n}_{12} \subset \Omega_{ac}^{bd}$, we define the associated PI operator for $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in Z_{21}^{m \times n}$ as well, and $(x, y) \in \Omega_{ac}^{bd}$ as

$$(\mathcal{P}[B]u)(x, y) = \begin{bmatrix} (P[B_{11}]u_1)(x) + \int^y_0 (P[B_{12}]u_2)(x, y) dy \\ (P[B_{21}]u_1)(x) + \int^y_0 (P[N]u_2)(x, y) dy \end{bmatrix},$$

where for $N = \{N_0, N_1, N_2\} \in \mathcal{N}^{m \times n}_{12}$, we define

$$(P[N]u_2)(x, y) = \begin{bmatrix} (P[N_0]u_1)(x, y) + \int^y_0 (P[N_1]u_2)(x, y, \nu) d\nu \\ \int^y_0 (P[N_2]u_2)(x, \nu, d\nu) \end{bmatrix}.$$ 

We denote the set of 2D PI operators as $\Pi_{12}^{n \times m}$, so that $\mathcal{P} \in \Pi_{12}^{n \times m}$ if and only if $\mathcal{P} = \mathcal{P}[B]$ for some $B \in \mathcal{N}_4^{n \times m}$.

We will denote the general set of PI operators as $\Pi$, so that $\mathcal{P} \in \Pi$ if and only if $\mathcal{P}$ belongs to one of the classes of PI operators presented in [15] (1D) and [13] (2D). In this paper, we make extensive use of the following properties of PI operators $\mathcal{P} \in \Pi$.

1. The sum of two PI operators is a PI operator. We denote the associated parameter map using the standard symbol $+$, so that for any $Q, R \in \mathcal{N}^{n \times m}_{12}$,

$$\mathcal{P}[Q] \mathcal{P}[R] = \mathcal{P}[Q + R],$$

if and only if $P = Q + R$.

2. The product of two PI operators is a PI operator. We denote the associated parameter map as $\mathcal{L}$, so that, for any $Q \in \mathcal{N}^{m \times p}_{12}$ and $R \in \mathcal{N}^{p \times m}_{12}$,

$$\mathcal{P}[Q] \mathcal{P}[R] = \mathcal{P}[Q \mathcal{P}[R]],$$

if and only if $P = \mathcal{L}(Q, R)$.

3. The adjoint $\mathcal{P}^*$ of a PI operator $\mathcal{P}$ is a PI operator.

### III. Feedback Interconnection of Partial Integral Equations

In order to prove that delayed ODE-PDE systems in Section [IV] can be equivalently represented as PIES, we will represent these systems as the feedback interconnection of delayed ODE and delayed PDE subsystems. After separately deriving a PI representation of the delayed ODE and delayed PDE subsystems, we can then construct the interconnection of the resulting PI representations to arrive at a PI representation of the original ODE-PDE with delay. In order to do this, in this section, we prove that the feedback interconnection of PIES can indeed be represented as a PIE. This result was already proven in [15], for the case of finite-dimensional interconnection signals, and will be extended here to include infinite-dimensional interconnection signals. In particular we consider PIES of the form

$$\begin{bmatrix} T_u w(t) + T \hat{u}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \hat{w}(t) \end{bmatrix},$$

where the operators $T$ through $D_{xy}$ are all PI operators, and where at each time $t \geq 0$, $\hat{u}(t) \in Z_2^{n_2}$, $w(t) \in Z_2^{n_2}$, and $z(t) \in Z_2^{n_2}$, for some $n_w, n_z, n_r, n_\theta \in \mathbb{N}^4$. We denote the parameters defining the PI operators in this PIE as

$$\mathcal{G}_{pie} := \begin{bmatrix} \hat{T} & \hat{T}_w \\ \hat{A} & \hat{B} \end{bmatrix} \begin{bmatrix} \hat{C} & \hat{D} \end{bmatrix} \subset \mathcal{N}^{n_2 \times n_2}_{12} \times \mathcal{N}^{n_2 \times n_2}_{12} \times \mathcal{N}^{n_2 \times n_2}_{12}.$$ 

and write

$$\mathcal{P}[\mathcal{G}_{pie}] := \begin{bmatrix} \hat{T} & \hat{T}_w \\ \hat{A} & \hat{B} \end{bmatrix} \begin{bmatrix} \hat{C} & \hat{D} \end{bmatrix}.$$ 

**Definition 4 (Solution to the PIE):** For a given input signal $w$ and given initial conditions $\hat{u}_i \in \mathcal{N}_2$, we say that $(\hat{u}, z)$ is a solution to the PIE defined by $\mathcal{G}_{pie}$ if $\hat{u}$ is Frechet differentiable, $\hat{u}(0) = \hat{u}_i$, and for all $t \geq 0$, $\hat{u}(t, z(t), w(t))$ satisfies Eqn. (2) with operators defined as in (3).

Using the composition and addition rules of PI operators, it is easy to show that the interconnection of two suitable PIES can also be represented as a PIE.

**Proposition 5 (Interconnection of PIES):** Let $\mathcal{G}_{pie,1}$ and $\mathcal{G}_{pie,2}$ define PIES with $\hat{D}_{zw} = 0$ in $\mathcal{G}_{pie,1}$. Define the associated PIE interconnection $\mathcal{G}_{pie} := \mathcal{L}_{pie \times pie}(\mathcal{G}_{pie,1}, \mathcal{G}_{pie,2})$ where the linear parameter map $\mathcal{L}_{pie \times pie}$ is as defined in Block [III]. Then, $[\hat{u}, z]$ solves the (autonomous) PIE defined by $\mathcal{G}_{pie}$ with initial conditions $[\hat{u}_0, z_0]$ if and only if $(\hat{u}, z)$ and $(\hat{v}, w)$ solve the PIES defined by $\mathcal{G}_{pie,1}$ and $\mathcal{G}_{pie,2}$, respectively, with initial conditions $\hat{u}_0$ and $\hat{v}_0$ and inputs $w$ and $z$ as defined in (4).

**Proof:** A proof is given in Block [IV].

### IV. A PIE Representation of 1D ODE-PDE Systems with Delay

In this section, we provide the main technical contribution of this paper, showing that for suitably well-posed linear 1D ODE-PDE systems, with delay in either the ODE or in the PDE, there exists an equivalent PIE representation. To reduce notational complexity, we will not explicitly derive the parameters defining this PI representation in full detail here, instead leveraging earlier results from e.g. [15] and [14]. In particular, in Subsection [IV-A], we repeat the result from [14], showing that an equivalent 1D PIE representation exists for any linear ODE with constant delays. In Subsection [IV-B] we then use the results from [15] to prove that, for any well-posed, linear, 1D PDE with constant delays, there exists an equivalent 2D PIE representation. Finally, in Subsection [IV-C] we combine these results, proving that any well-posed, linear, ODE-PDE system with constant delay can be equivalently represented as a PIE as well.
A. PIE Representation of ODEs with Delay

In [14], it was shown that a general class of linear Delay Differential Equations (DDEs) can be equivalently represented as PIEs. In this paper, we consider only a particular subset of such DDEs, taking the form

\[
\begin{bmatrix}
\dot{u}(t) \\
z(t)
\end{bmatrix} = \begin{bmatrix}
A & B_w \\
C_z & 0
\end{bmatrix} \begin{bmatrix}
u(t) \\
w(t)
\end{bmatrix} + \sum_{j=1}^{K} \begin{bmatrix}
A_j \\
C_{z,j}
\end{bmatrix} u(t - \tau_j),
\]

where \(0 < \tau_1 < \ldots < \tau_K\) are the delays, and where \(u(t)\) and \(z(t)\) are solutions to the PIEs defined by \(G_{\text{pie},1}\) and \(G_{\text{pie},2}\), respectively, with inputs \(w\) and \(u\), respectively, at any time \(t \geq 0\).

We collect the signals into the PDE state vector \(v = (v_1, \ldots, v_K) \in (Y_u, \ldots, Y_u) =: Y_u^K\).

Definition 6 (Solution to the DDE): For a given input signals \(w\) and given initial conditions \((u_0, v_0) \in \mathbb{R}^{n_u} \times \mathbb{R}^{n_v}\), we say that \((u, v, z)\) is a solution to the DDE defined by \(G_{\text{ode}, G_{\text{dde}}} = I\) if \((u, v)\) is Frechét differentiable, \((u(0), v(0)) = (u_0, v_0)\), and for all \(t \geq 0\), \((u(t), v(t), z(t), w(t))\) satisfies Eqn. (7).

To derive a PIE representation associated to the DDE (7), we first have to define a fundamental state \(\hat{v}(t)\). We say that \(\hat{v}(t)\) solves the PIE [5] defined by \(G_{\text{pie}}\) if and only if \((\hat{u}, \hat{z})\) and \((\hat{v}, \hat{w})\) solve the PIEs defined by \(G_{\text{pie},1}\) and \(G_{\text{pie},2}\), respectively, with inputs \(w\) and \(z\) as defined in [4].
$\mathbb{R}^{n_0} \times L^2_{2n}[\Omega_0^1]$ if and only if $(u, v, z)$ with

$$\begin{bmatrix} u \\ v \\ z \end{bmatrix} = P[T] \begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{z} \end{bmatrix} \in \mathbb{R}^{n_0} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n}$$

is a solution to the DDE defined by $\{G_{ode}, G_{dde}\}$ with and initial conditions $\{\tilde{u}_0, \tilde{v}_0, \tilde{z}_0\} = P[T] \begin{bmatrix} u_0 \\ v_0 \\ z_0 \end{bmatrix}$.

**Proof:** A proof is given in [14], Lemma 4.

### B. PIE Representation of 1D PDEs with Delay

Having shown that an equivalent 1D PIE representation exists for suitable ODEs with delay, in this subsection, we show that there exist an equivalent 2D PIE representation exists for any well-posed, linear 1D PDE with delay. In particular, we consider a system of the form

$$\begin{align*}
\dot{u}(t) &= \left[P[A_p] \quad A_b\right] \begin{bmatrix} (\partial_{int} u)(t) \\ (\partial_{int} v)(t) \end{bmatrix} + \sum_{j=1}^{K} \left[P[A_{p,j} \quad A_{b,j}\right] \begin{bmatrix} (\partial_{int} u)(t - \tau_j) \\ (\partial_{int} v)(t - \tau_j) \end{bmatrix},
\end{align*}
$$

for $n_p = (n_1, n_2, n_3) \in \mathbb{N}^3$, defining an associated differential operator $\partial_{int} : U^{n_p} \rightarrow L^2_{2n}$ and boundary Dirac operator $\Lambda_{bf} : U^{n_p} \rightarrow L^2_{2n}$ for $n_{int} = n + 1 + 2 + n_2 + 3n_3$ and $n_{bf} = 3n_{bc}$ with $n_{bc} = n_2 + 2n_3$ as

$$\partial_{int} := \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Lambda_{bf} := \begin{bmatrix} \Delta_0 & \Delta_{1X} \\ \Delta_{1X} & \Delta_{1X} \end{bmatrix},$$

where

$$D_{int} := \begin{bmatrix} D_{int} \\ D_{bf} \end{bmatrix} := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$
fundamental state \( \hat{u} \) using PI operators. Using this relation, we can define a PIE representation of the PDE Subcomponent \( \| \) with arbitrary input \( r \).

**Lemma 9 (PIE Representation of 1D PDE):** Let \( G_{bc} \) define a well-posed set of boundary conditions on the PDE state \( u \in U^\text{bc} \) as in \( \| \). Let the linear parameter map \( L_{\text{pie}-\text{pde}} \) be as defined in Thm. 12 of [15], and let

\[
\{ \hat{T}, \hat{A} \} = G_{\text{pie}-0} = L_{\text{pie}-\text{pde}}(G_{\text{pde}}, G_{bc}).
\]

Let further \( G_{\text{pde}} \) be defined, and give definite parameters \( B_r \) such that

\[
\mathcal{P}[B_r] = [\mathcal{P}[B_r,1] \ldots \mathcal{P}[B_r,K]], \quad \mathcal{P}[B_r,j] := [\mathcal{P}[A_{\hat{B},j}]].
\]

Define

\[
G_{\text{pie}} := \begin{bmatrix} \hat{T} & 0 \\ \hat{A} & B_r \end{bmatrix} C_q = P \times (C_{qp}, \hat{T}_{\text{int}}) + P \times (C_{qb}, \hat{T}_{\text{bd}}),
\]

where \( \hat{T}_{\text{int}} \in \mathcal{N}_3^{[n_t] \times [n_{u}]} \), and \( \hat{T}_{\text{bd}} \in \mathcal{N}_3^{[n_t] \times [n_{u}]} \) such that

\[
\mathcal{P}[\hat{T}_{\text{int}}] = D_{\text{int}} \circ \hat{T}, \quad \mathcal{P}[\hat{T}_{\text{bd}}] = \Lambda_{\text{bd}} \circ \hat{T}.
\]

Then, \( (\hat{u}, q) \) is a solution to the PDE defined by \( G_{\text{pie},0} \) with input \( r \) and initial conditions \( u_0 \in L_2^{[n_t]}(\Omega_0^d) \) if and only if \((u, q) \) with \( u = \hat{T} \hat{u} \) is a solution to the PDE defined by \( \{ G_{\text{pde}}, G_{bc} \} \) with initial conditions \( u_0 = \hat{T} u_0 \). Since the BCs in the PDE defined by \( \{ G_{\text{pde}}, G_{bc} \} \) do not depend on the input signal \( r \), it follows that for any \( \hat{u} \in L_2^{[n_t]}(\Omega_0^d), \hat{u} = \hat{T} \hat{u} \in X_{\text{bc}}^0 \) also satisfies the BCs of the PDE defined by \( \{ G_{\text{pde}}, G_{bc} \} \). Moreover, \( \hat{u} \) satisfies the initial conditions defined by \( u_0 \) if and only if \( u \) satisfies the initial conditions defined by \( u_0 \).

Proof: In [15] it was proven that, if \( G_{\text{pie},0} = L_{\text{pie}-\text{pde}}(G_{\text{pde}}, G_{bc}) \), then \( \hat{u} \) is a solution to the autonomous PDE defined by \( G_{\text{pie}} \) with initial conditions \( u_0 \), if and only if \( u = \hat{T} \hat{u} \) is a solution to the PDE defined by \( \{ G_{\text{pde}}, G_{bc} \} \) with initial conditions \( u_0 = \hat{T} u_0 \). Since the BCs in the PDE defined by \( \{ G_{\text{pde}}, G_{bc} \} \) do not depend on the input signal \( r \), it follows that for any \( \hat{u} \in L_2^{[n_t]}(\Omega_0^d), u = \hat{T} \hat{u} \in X_{\text{bc}}^0 \) also satisfies the BCs of the PDE defined by \( \{ G_{\text{pde}}, G_{bc} \} \). Moreover, \( \hat{u} \) satisfies the initial conditions defined by \( u_0 \) if and only if \( u \) satisfies the initial conditions defined by \( u_0 \).

Now, let \( \hat{u} \in L_2^{[n_t]}(\Omega_0^d) \) be arbitrary, and let \( u = \hat{T} \hat{u} \in X_{\text{bc}}^0 \). Since \( \hat{u} \) is a solution to the PDE defined by \( G_{\text{pie},0} \) if and only if \( u \) is a solution to the PDE defined by \( G_{\text{pde}}, G_{bc} \), it follows that, for any \( t \geq 0 \),

\[
\hat{u}(t) - [\mathcal{P}[A] B_0] \begin{bmatrix} D_{\text{int}}(u) \\ A_{\text{bd}}(u) \end{bmatrix} - \mathcal{P}[B_r] r(t)
= \hat{T} \hat{u}(t) - \hat{A} \hat{u}(t) - \hat{B}_r r(t).
\]

Similarly, applying the definition of the parameters \( \hat{C}_q \),

\[
q(t) - C_{qp}(D_{\text{int}} u) - C_{qb}(A_{\text{bd}} u)
= q(t) - C_{qp}(D_{\text{int}} \hat{T} \hat{u}) - C_{qb}(A_{\text{bd}} \hat{T} \hat{u}) - D_{\text{pie}} w(t)
= q(t) - C_q \hat{u}(t).
\]

It follows that, \( (\hat{u}, q) \) satisfies the PIE defined by \( G_{\text{pie}} \) if and only if \( (\hat{T} \hat{u}, q) \) satisfies the PDE defined by \( \| \).

Having shown that the PDE \( \| \) can be equivalently represented as a PIE, we now show that the PDE \( \| \) can also be equivalently represented as a PIE. For this, we define the fundamental state associated to the delayed states \( v_j \) as

\[
\hat{v}_j = \partial_{\hat{u}} \circ \hat{D}_u v_j = \hat{D}_u v_j,
\]

where \( \hat{D} \) is such that \( \hat{u} = \hat{D} u \) is the fundamental state associated to the PDE state \( u_0 \). Since the BCs imposed upon the states \( v_j \) are defined by the same \( G_{bc} \) as for the PDE state \( u \), the same PI operator \( \hat{T} \) can also be used to derive the PIE representation of the PDE \( \| \), as we prove in the following Lemma.

**Lemma 10:** Let \( G_{bc} \) define a well-posed set of boundary conditions, and let \( \hat{T} = L_{\text{pie}-bc}(G_{bc}) \) be as defined in Thm. 10 of [15]. Define the 2D PIEs

\[
E_0 q(t) + \hat{T}_{\text{pie}} \hat{v}_j(t) = \begin{bmatrix} \hat{A}_j & 0 \\ C_j & 1 \end{bmatrix} \begin{bmatrix} \hat{v}_j(t) \\ q(t) \end{bmatrix},
\]

where, for \( s \in \Omega_0^d \) and \( \hat{v}_j \in Z_2^{(\|n_t\| \times [n_{u}])}(\Omega_0^d) \),

\[
(\hat{v}_j(t), s) = \int_0^s \hat{T} \hat{v}_j(t,\theta) d\theta, \quad (\hat{A}_j, \hat{v}_j(t), s) = \frac{1}{\tau_j} \hat{T} \hat{v}_j(t, s),
\]

where \( T = \mathcal{P}[\hat{T}], \hat{T}_{\text{int}} := D_{\text{int}} \circ \hat{T}, \) and \( \hat{T}_{\text{bd}} := \Lambda_{\text{bd}} \circ \hat{T} \). Then, for an input of the form \( q = (\hat{q}_0, \hat{q}_0)(t) \), \( (\hat{v}_j, r_j) \) is a solution to the \( j \)th PIE \( \| \) with initial conditions \( v_0, j \in Y_{q(0)} \) if and only if \( (v_j, r_j) \) is a solution to the \( j \)th PDE \( \| \) with initial conditions \( v_0, j = \hat{T} v_0, j + E_0 q(t) \).

Proof: Since the BCs are the same as in Lemma 9 the states \( v_j(t, s) \) must satisfy

\[
v_{p,j}(s) = \hat{T} \hat{D} v_{p,j}(s), \quad s \in \Omega_0^d.
\]

By the fundamental theorem of calculus, it follows that

\[
v_j(s) = v_j(0) + \int_0^s (\partial_s \hat{T} \hat{D} v_j)(\theta) d\theta = E_0 q + \int_0^s (\hat{T} \hat{v}_j)(\theta) d\theta.
\]

Substituting this expression into the PDE \( \| \), we obtain the PIE \( \| \).

Having shown that both the PDE \( \| \) and the PDE \( \| \) can be equivalently represented as PIES, it follows that their interconnection can also be represented as a PIE.
follows that \( \hat{u} \) is a solution to the PIE defined by \( G_{\text{pie}} = L_{\text{pie}} \times \text{pie} \left( G_{\text{pie},1}, G_{\text{pie},2} \right) \) with initial conditions \( \hat{u}_0 = v_0 \) if and only if \( (u, q) \) and \( (v, r) \) are solutions to the PDEs defined by \( \{G_{\text{pde}}, G_{\text{pde},2}, G_{\text{bc}}\} \), with initial conditions \( u_0 \) and \( v_0 \) and inputs \( r \) and \( q \), where \( \hat{u} = \mathcal{P}[\hat{T}] \hat{u} \) and \( \hat{u}_0 = \mathcal{P}[\hat{T}] \hat{u}_0 \).

C. A PIE Representation of ODE-PDEs with Delays

Having shown that both ODEs and PDEs with delay can be represented as PIs, in this Subsection we combine these results to prove that suitable ODE-PDE with delays can also be equivalently represented as PIs. In particular, we consider an ODE with delay

\[
\begin{bmatrix}
    \dot{z} \\
    z(t)
\end{bmatrix} = \begin{bmatrix}
    A & B_w \\
    C_z & 0
\end{bmatrix} \begin{bmatrix}
    z(t) \\
    u(t) + \sum_{j=1}^{K} A_j C_{z,j} u(t - \tau_j)
\end{bmatrix}, \tag{14}
\]

coupled to a PDE with delay as

\[
\begin{bmatrix}
    \dot{u}_p(t) \\
    w(t)
\end{bmatrix} = \begin{bmatrix}
    \mathcal{P}[A_p] & B_p \\
    C_{wp} & D_{wz}
\end{bmatrix} \begin{bmatrix}
    u(t) + \sum_{j=1}^{K} A_{pj} B_{wp} u(t - \tau_j) \\
    \mathcal{P}[D_{wp}] u(t) + \sum_{j=1}^{K} A_{pj} C_{wp} u(t - \tau_j)
\end{bmatrix}, \tag{15}
\]

with BCs

\[
0 = \begin{bmatrix}
    a_p \\
    b_p
\end{bmatrix} \begin{bmatrix}
    u_p(t) \\
    \mathcal{P}[D_{wp}] u(t)
\end{bmatrix}.
\]

We note that, in this representation, the delayed PDE subsystem now includes input and output signals. We collect the PDE parameters associated to these signals into

\[
G_{w} = \{C_{w}, C_{wb}, D_{wz}\} \in L^{w \times n_{\text{int}}} \times R^{n_w \times n_{\text{bc}}} \times R^{n_w \times n_z},
\]

\[
G_{wud,.} = \{C_{w}, C_{wb}, D_{wz}\} \in L^{w \times n_{\text{int}}} \times R^{n_w \times n_{\text{bc}}} \times R^{n_w \times n_z},
\]

which we combine into \( G_r = \{G_{r,q}, G_{r,d,1}, \ldots, G_{r,d,K}\} \).

Then, the full delayed ODE-PDE system is defined by parameters \( G_{\text{dde}-\text{pde}} = \{G_{\text{dde}}, G_{\text{pde}}, G_{\text{pde},2}, G_{\text{pde},3}\} \), where we define \( G_{\text{dde}} = \{G_{\text{dde}}, G_{\text{pde}}\} \) and \( G_{\text{pde}} = \{G_{\text{pde}}, G_{\text{pde},2}, G_{\text{bc}}\} \).

Definition 12 (Solution to the PDE): For a given input signal \( w \) and given initial conditions \( (u_0, v_0) \in R^{n_0} \times Y_{k_0} \)

\[
(\hat{u}_0, v_0, 0, v_p, 0) \in X^{n_0} \times Y_{k_0},
\]

where \( q_0 = \mathcal{B}_{\text{int}}(0, 0, 0, 0) \), we say that \( (u, v, u_p, v_p) \) is a solution to the coupled ODE-PDE system defined by \( \{G_{\text{dde}}, G_{\text{pde}}, G_{\text{pde},2}, G_{\text{pde},3}, G_{\text{bc}}\} \) if

\[
\begin{bmatrix}
    u \\
    v \\
    u_p \\
    v_p
\end{bmatrix} = \mathcal{P}[\hat{T}] \begin{bmatrix}
    \hat{u} \\
    \hat{v} \\
    \hat{u}_p \\
    \hat{v}_p
\end{bmatrix}
\]

is a solution to the ODE-PDE with delay defined by \( \{G_{\text{dde}-\text{pde}}, G_{\text{bc}}\} \) with initial conditions

\[
\begin{bmatrix}
    u_0 \\
    v_0 \\
    u_{p,0} \\
    v_{p,0}
\end{bmatrix} = \mathcal{P}[\hat{T}] \begin{bmatrix}
    \hat{u}_0 \\
    \hat{v}_0 \\
    \hat{u}_{p,0} \\
    \hat{v}_{p,0}
\end{bmatrix}.
\]

Proof: The result follows immediately by combining Cor. 7 Prop. 13 and Prop. 5.

V. STABILITY AS AN LPI

Having established a bijective map between the solution of a suitable ODE-PDE system with delay and the solution of
an associated PIE, we now show that the PIE representation can be used to formulate a convex optimization problem used to verify stability of the delayed ODE-PDE. In particular, the following theorem shows that existence of a quadratic Lyapunov function for a delayed ODE-PDE can be posed as a convex Linear PI Inequality (LPI) optimization problem.

**Theorem 15:** Let $G_{dpe}$ define an ODE-PDE system with delay, and let $G_{pie} = \{ T, \hat{A} \} = L_{dpe} - dpe(G_{dpe})$ define the parameters of the associated PIE. Let $\mathcal{T} = \{ T \} \in \Pi_{12}^{n \times n}$ and $\hat{A} = \{ \hat{A} \} \in \Pi_{12}^{n \times n}$. Suppose that there exist $\epsilon, \delta > 0$ and $P \in \Pi_{12}^{n \times n}$ such that the PI operator $\mathcal{P} := \{ P \mathcal{P} \}$ satisfies $\mathcal{P} = \mathcal{P}^*, \ P \geq \epsilon I$, and

$$A^* \mathcal{P} T + T^* \mathcal{P} A \leq -\delta T^* T.$$

Finally, let $\zeta = \| \mathcal{P} \|_{\mathcal{L}_2}$. Then, for any solution $u(t) \in \mathcal{Z}_{12}$ to the PIE defined by $G_{pie}$, the solution $u(t)$ to the ODE-PDE defined by $G_{dpe}$ satisfies

$$\left\| u(t) \right\|_{L_2}^2 \leq \frac{\zeta}{\epsilon} \left\| u(0) \right\|_{L_2}^2 e^{-\frac{\delta}{\epsilon} t}.$$

**Proof:** Let $\hat{u}(t) \in \mathcal{Z}_{12}$ be an arbitrary solution to the PIE defined by $G_{pie}$. Consider the candidate Lyapunov function $V : \mathcal{Z}_{12} \to \mathbb{R}$ defined for $\hat{v} \in \mathcal{Z}_{12}$ as

$$V(\hat{v}) = \left\langle \hat{T} \hat{v}, \mathcal{P} T \hat{v} \right\rangle_{Z} \geq \epsilon \left\| \hat{T} \hat{v} \right\|_{Z}^2.$$ 

Since $\| \mathcal{P} \|_{\mathcal{L}_2} = \zeta$, this function is bounded from above as

$$V(\hat{v}) = \left\langle \hat{T} \hat{v}, \mathcal{P} T \hat{v} \right\rangle_{\mathcal{L}_2} \leq \zeta \left\| \hat{T} \hat{v} \right\|_{\mathcal{L}_2}^2.$$

In addition, since $\hat{u}$ is a solution to the PIE, the temporal derivative of $V$ along $\hat{u}$ satisfies

$$\dot{V}(\hat{u}(t)) = \left\langle \dot{\hat{T}} \hat{u}, \mathcal{P} T \hat{u} \right\rangle_{Z} + \left\langle \hat{T} \dot{\hat{u}}, \mathcal{P} T \hat{u} \right\rangle_{Z}$$

$$= \langle \hat{A} \hat{u}, \mathcal{P} T \hat{u} \rangle_{Z} + \langle \dot{\hat{T}} \hat{u}, \mathcal{P} T \hat{u} \rangle_{Z}$$

$$= \langle \dot{\hat{u}}, \left( A^* \mathcal{P} T + T^* \mathcal{P} A \right) \hat{u} \rangle_{Z} \leq -\delta \left\| \hat{T} \hat{u} \right\|_{Z}^2 \leq -\frac{\delta}{\epsilon} V(\hat{u}).$$

Applying the Grönwall-Bellman inequality, it immediately follows that

$$V(\hat{u}(t)) \leq V(\hat{u}(0)) e^{-\frac{\delta}{\epsilon} t}.$$

This implies that

$$\left\| \hat{T} \hat{u}(t) \right\|_{Z}^2 \leq \frac{\zeta}{\epsilon} \left\| \hat{T} \hat{u}(0) \right\|_{Z}^2 e^{-\frac{\delta}{\epsilon} t},$$

proving that

$$\left\| u(t) \right\|_{L_2}^2 \leq \frac{\zeta}{\epsilon} \left\| u(0) \right\|_{L_2}^2 e^{-\frac{\delta}{\epsilon} t}.$$ 

In the MATLAB software package PIETOOLS, a cone of positive semidefinite PI operators $\mathcal{P} \geq 0$ is parameterized by positive semidefinite matrices $P \geq 0$ as

$$\Pi_{+} := \{ P \in \Pi \mid \mathcal{P} = Z^* M[P] Z, \ Z \in \Pi, \ P \geq 0 \}.$$ 

Then, if $\mathcal{P} \in \Pi_{+}$, there exists some $P = [P^{1/2}]^T P \geq 0$ and $Z \in \Pi$ such that

$$\langle u, \mathcal{P} u \rangle = \langle M[P^{1/2}] Z u, M[P^{1/2}] Z u \rangle \geq 0,$$

for any $u$ in the domain of $\mathcal{P}$, guaranteeing that $\mathcal{P} \geq 0$. In this manner, LPIs such as the one in Thm. [15] can be posed as semidefinite programs, and can be numerically solved. In the next section, we apply this approach to test stability of several delayed ODE-PDE systems.

**VI. NUMERICAL EXAMPLES**

In this section, we provide several numerical examples, illustrating how stability of different ODE-PDE systems with delay can be numerically tested by verifying feasibility of the LPI from Thm. [15]. In each case, the PIETOOLS software package [16] is used to declare the delayed system as a coupled systems of PDEs and if applicable ODEs, convert the system to equivalent PEs, and subsequently declare and solve the stability LPI.

**A. Heat Equation with Delay in PDE**

We first consider a heat with a delayed reaction term,

$$\dot{u}(t, x) = \partial_x^2 u(t, x) + ru(t, x) - u(t - \tau, x),$$

$$u(t, 0) = u(t, \pi) = 0.$$ 

We expand this system as

$$\dot{u}(t, x) = \partial_x^2 u(t, x) + ru(t, x) - v(t, 1, x), \quad x \in [0, \pi],$$

$$v(t, s, x) = -\frac{1}{\tau} \partial_s v(t, s, x), \quad s \in [0, 1],$$

$$u(t, 0) = u(t, \pi) = 0, \quad v(t, s, 0) = v(t, s, \pi) = 0,$$

$$v(t, 0, x) = u(t, x).$$

In [17], it was shown that for $r = 1.9$, this system is stable for any $\tau < 1.0347$. Using PIETOOLS, stability of the system can be proven for any $\tau < 1.03$.

**B. Wave Equation with Delay in Boundary**

We now consider a wave equation with delay in the boundary,

$$u(t, x) = \partial_x^2 u(t, x), \quad x \in [0, 1],$$

$$u(t, 0) = 0, \quad \partial_x u(t, 1) = (1 - \mu) u(t, 1) + \mu u(t - \tau, 1),$$

where $\mu \in (0, 1)$. Introducing

$$u_0(t) = (1 - \mu) u(t, 1) + \mu u(t - \tau, 1),$$

$$u_1(t, x) = u(t, x), \quad u_2(t, x) = \dot{u}(t, x),$$

$$v_1(t, x) = \partial_x v_1(t, x), \quad v_2(t, x) = \partial_x v_2(t, x),$$

$$u_1(t, 0) = 0, \quad u_2(t, 0) = 0,$$

$$v_1(t, 0) = u_1(t, 1), \quad v_2(t, 0) = u_2(t, 1),$$

this system can be equivalently represented as

$$\dot{u}_0(t) = \partial_x u_1(t, 1),$$

$$\dot{u}_1(t, x) = u_2(t, x), \quad \dot{u}_2(t, x) = \partial_x^2 u_1(t, x),$$

$$\dot{v}_1(t, x) = -\frac{1}{\tau} \partial_x v_1(t, x), \quad \dot{v}_2(t, x) = -\frac{1}{\tau} \partial_x v_2(t, x),$$

$$u_1(t, 0) = 0, \quad u_2(t, 0) = 0,$$

$$v_1(t, 0) = u_1(t, 1), \quad v_2(t, 0) = u_2(t, 1),$$

$$u_0(t) = (1 - \mu) u_1(t, 1) + \mu v_1(t, 1),$$

$$\partial_x u_1(t, 1) = (1 - \mu) u_2(t, 1) + \mu v_2(t, 1).$$

The system can be proven to be stable for any $\mu < \frac{1}{2}$ [18]. Letting $\tau = 1$, using PIETOOLS, stability can be verified for any $\mu \leq 0.5 - 10^{-5}$. 

VII. Conclusion

In this paper, an LMI-based method for verifying stability of coupled, linear, delayed, ODE-PDE systems in a single spatial dimension was presented. In particular, it was shown that for any suitably well-posed ODE-PDE with delay, there is an associated (1D or 2D) PIE with a corresponding bijective map from solution of the ODE-PDE to that of the PIE. The PIE representation was then used to propose a stability test for the delayed ODE-PDE. This stability test was posed as a linear operator inequality expressed using PI operator variables (an LPI). Finally, the PIETOOLS software package was used to convert the LPI to a semidefinite programming problem and the resulting stability conditions were applied to several common examples of delayed PDEs.

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