Heat trace asymptotics for wedge-like singularity.

Asilya Suleymanova

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Abstract

In this note we consider a heat trace expansion on a manifold with wedge-like singularity. We show that there are two terms in the expansion that contain information about the presence of the singularity, namely the logarithmic term $ct^{-1/2}\log t$ and the half power term $+bt^{-1/2}$. We also give a geometric expression for $c$.

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1 Geometric setup

Consider a Riemannian manifold, $(M, g)$, of dimension $m$ such that $M = M_1 \cup U$, where $M_1$ is a compact manifold with boundary and $U = (0, 1) \times N$ has a fibration $(0, 1) \times F \to U \to (0, 1) \times \Sigma$, where $N, \Sigma$ are compact. This fibration induces a vertical bundle of tangents to the fibres $T_VN \subset TN$ and a horizontal cotangent subbundle $T_H^*N$ of cotangents annihilating $T_VN$. We take a complementary tangent subbundle $T_HN \subset TN$, inducing a vertical $T_V^*N$ that annihilates $T_HN$ such that $TN = T_VN \oplus T_HN$. 
The metric on $U$ is

$$g_U = dr^2 + g_H + r^2 g_V(s), \quad r \in (0, 1),$$

where $g_H$ is induced from a metric on $\Sigma$ and $g_V(s)$ is a family of metrics on the fibres, $s \in \Sigma$.

Assume now that $\dim \Sigma = 1$, i.e. it is a circle. Assume furthermore that metric on the fibres does not depend on $s$, i.e. $g_V(s) = g_V$.

$$g_{\text{wedge}} = dr^2 + d\theta^2 + r^2 g_V, \quad r \in (0, 1),$$

We call such singularity a wedge-like singularity.

## 2 Curvature tensors near the singularity

In this section we give explicit formulas for the curvature tensors in $(U, g_{\text{wedge}})$ in terms of the curvature tensors on fibers. For the latter we use the classical tensor notations and for the tensors on $(U, g_{\text{wedge}})$ we use classical notations but with tilde.

Let $(x^1, \ldots, x^{m-2})$ be local coordinates on a fiber $F$ and $p = (r, \theta, x^1, \ldots, x^{m-2}) \in U$. For $i, j \in \{r, \theta, 1, \ldots, m-2\}$ denote by $\tilde{g}_{ij}$ the components of the metric tensor on $g_{\text{wedge}}$, and by $g_{ij}$ for $i, j \in \{1, \ldots, m-2\}$ the components of the metric tensor $g_V$. Then

$$\tilde{g}_{rr} = \tilde{g}_{\theta\theta} = 1,$$

$$\tilde{g}_{ir} = \tilde{g}_{ri} = \tilde{g}_{i\theta} = \tilde{g}_{\theta i} = 0, \quad \text{for } i \in \{\theta, 1, \ldots, m-2\} \text{ and } j \in \{r, 1, \ldots, m-2\}$$

and

$$\tilde{g}_{ij} = r^2 g_{ij}, \quad \text{for } i, j \in \{1, \ldots, m-2\}.$$

Then

$$\tilde{g}_{ij,k}(r, \theta, x) = \begin{cases} 
2r g_{ij}(x), & \text{if } k = r, \\
0 & \text{if } k = \theta, \\
r^2 g_{ij,k}(x) & \text{otherwise}.
\end{cases}$$

The Christoffel symbols are of course

$$\tilde{\Gamma}^i_{jk} = \frac{1}{2} \tilde{g}^{im} \left( \tilde{g}_{mj,k} + \tilde{g}_{mk,j} - \tilde{g}_{jk,m} \right),$$

and now we express them in terms of the Christoffel symbols $\Gamma^i_{jk}$ and the metric tensor $g_{ij}$.
Let $i = r$, then
\[
\tilde{\Gamma}_{jk}^r = \begin{cases} 
0, & \text{if } j = r \text{ or } k = r \text{ or } j = \theta \text{ or } k = \theta, \\
-r g_{jk}, & \text{otherwise}.
\end{cases}
\]

Let $i \notin \{r, \theta\}$ and $j = r$, then
\[
\tilde{\Gamma}_{rk}^i = \tilde{\Gamma}_{kr}^i = \begin{cases} 
0, & \text{if } k = r \text{ or } i = \theta \text{ or } k = \theta, \\
r^{-1} \delta_k^i, & \text{otherwise}.
\end{cases}
\]

Assume that $i, j, k \notin \{r, \theta\}$, then $\tilde{\Gamma}_{jk}^i = \Gamma_{jk}^i$. Finally $\tilde{\Gamma}_{jk}^\theta = \tilde{\Gamma}_{kj}^\theta = 0$. The scalar curvature $\tilde{\text{Scal}}(p)$ can be expressed in the following way
\[
\tilde{\text{Scal}}(p) = g^{ij} \left( \tilde{\Gamma}_{ij,m}^m - \tilde{\Gamma}_{im,j}^m + \tilde{\Gamma}_{ij}^l \tilde{\Gamma}_{ml}^m - \tilde{\Gamma}_{im}^l \tilde{\Gamma}_{jl}^m \right) = r^{-2} \left( (2.1) \right)
\]
where $p = (r, \theta, x) \in U$, $x \in F$ and $\tilde{\text{Scal}}(p)$ is the scalar curvature on $(U, g_{\text{wedge}})$, and $\text{Scal}(x)$ is the scalar curvature on $(F, g_V)$.

If $i, j, k, l \notin \{r, \theta\}$, we obtain
\[
\tilde{R}_{ijkl}(p) = r^{-2} (R_{ijkl}(x) - g_{ip}(x)g_{jm}(x)(\delta_k^p \delta_l^m - \delta_k^p \delta_l^m)). \quad (2.2)
\]

Similarly, for the tensor Ricci
\[
\tilde{Ric}_{ij}(p) = r^{-2} (Ric_{ij}(x) - (m - 3)g_{ij}(x)). \quad (2.3)
\]

### 3 Resolvent Expansion

Let $\Delta$ be the Friedrichs extension of the Laplace-Beltrami operator on $U$
\[
\Delta = -\partial^2_r + \Delta_{\Sigma} + r^{-2} A(s),
\]
where $\Delta_{\Sigma}$ is the Laplace operator on the singular stratum $\Sigma$, $A(s), s \in \Sigma$ is a smooth family of elliptic operators on functions on fibers $F$. In the case of the wedge-like singularity, we have $\Delta_{\Sigma} = -\partial^2_\theta$, and $A(s) = \Delta_F + (\frac{m-2}{2} \left( \frac{m-2}{2} - 1 \right))$, where $\Delta_F$ is the Laplace-Beltrami operator on the fiber $F$.

Let $\varphi(r, \theta)$ be a cutoff function supported near $r = 0$, by [2.5, Theorem 5.2], for $d > m/2$ the resolvent $(\Delta + z^2)^{-d}$ is trace class and the following is true
\[
\text{tr}(\varphi(r, \theta)(\Delta + z^2)^{-d}) = \int_0^\infty \varphi(r, \theta) \sigma(r, rz) dr,
\]
where \( \sigma(r, rz) := \sigma(r, \theta, rz) := \text{tr}_{L^2(F)}(\Delta + z^2)^{-d} \).

From the local heat kernel expansion of heat kernel away from the singularity we know, see [S, (3.6), (3.17)], that

\[
\text{tr}_{L^2(F)}(\Delta + \zeta^2/r^2)^{-d} \sim \zeta \rightarrow \infty (4\pi)^{-\frac{m}{2}} r^{-d-m} \int_F r^{m-2} u_j(p) \, dvol_F.
\]

Above \( u_j(p) \) are polynomials in the curvature tensors at \( p \in M \) that come from the local heat kernel expansion away from the singularity, in particular \( u_0(p) \equiv 1, u_1(p) = \text{Scal}(p) \) and

\[
u_2(p) = \frac{1}{360} (12\Delta \text{Scal}(p) + 5 \text{Scal}(p)^2 - 2|\text{Ric}(p)|^2 + 2|R(p)|^2).
\]

Hence

\[
\sigma(r, \zeta) \sim \zeta \rightarrow \infty \sum_{j=0}^{\infty} \zeta^{-2d+m-2j} \sigma_j(r),
\]

where

\[
s_j(r) = (4\pi)^{-\frac{m}{2}} r^{2d-m+2j} \int_F r^{m-2} u_j(p) \, dvol_N \frac{\Gamma(-\frac{m}{2} + d + j)}{(d-1)!} \int_F r^{m-2} u_j(p) \, dvol_N \frac{\Gamma(-\frac{m}{2} + d + j)}{(d-1)!}.
\]

We can apply the Singular Asymptotics Lemma to obtain an asymptotic expansion of the resolvent trace, see [BS, p. 287], Proposition 3.1.

**Proposition 3.1.**

\[
\int_0^\infty \varphi(r, \theta) \sigma(r, \zeta) dr \sim \sum_{l=0}^{\infty} \frac{1}{l!} \int_0^\infty \int_\Sigma \zeta^l \partial_r^l \left( \sigma(r, \zeta) \varphi(r, \theta) \right) \big|_{r=0} \, d\theta \, d\zeta
\]

\[
+ \sum_{j=0}^{\infty} \int_0^\infty \int_\Sigma \sigma_j(r)(rz)^{-2d+m-2j} \varphi(r) \, d\theta \, dr
\]

\[
+ \sum_{l=\frac{d}{2}-d+1}^{\infty} z^{-2d+m-2l} \log z \int_\Sigma \frac{\partial_r^{2d-m+2l-1} \left( \sigma_l(r) \varphi(r, \theta) \right) \big|_{r=0}}{(2d - m + 2l - 1)!} \, d\theta.
\]
Assume $\varphi(r, \theta) \equiv 1$ near $r = 0$ and consider the last sum in the expansion

$$L := \sum_{l=\frac{m}{2} + d + 1}^{\infty} z^{-2d+m-2l} \log z \int_{\Sigma} \frac{\partial^{2d-m+2l-1} \sigma_1(r) \varphi(r, \theta)}{(2d-m+2l-1)!} d\theta$$

$$= \sum_{j=\frac{m}{2} - d + 1}^{\infty} z^{-2d+m-2j} \log z \frac{\Gamma(-\frac{m}{2} + d + j) \text{vol}(\Sigma)}{(d-1)!(2d-m+2j-1)!} \times$$

$$\times (4\pi)^{-\frac{m}{2}} \partial^{2d-m+2j-1}_r \left( r^{2d-2+2j} \int_{F} u_j(p) \, d\text{vol}_F \right) |_{r=0}.$$  

To continue the computation we note that for $j \geq m/2$ we have

$$\partial^{2d-m+2j-1}_r \left( r^{2d-2+2j} \int_{F} u_j(p) \, d\text{vol}_F \right) |_{r=0}$$

$$= \partial^{2j-m+1}_r \left( r^{2j} \int_{F} u_j(p) \, d\text{vol}_F \right) |_{r=0},$$

this makes sense only for $j \geq \frac{m-1}{2}$, therefore

$$L = \sum_{j=\frac{m}{2} - d + 1}^{\infty} z^{-2d+m-2j} \log z \frac{\Gamma(-\frac{m}{2} + d + j) \text{vol}(\Sigma)}{(d-1)!(2d-m+2j-1)!} \times$$

$$\times (4\pi)^{-\frac{m}{2}} \partial^{2j-m+1}_r \left( r^{2j} \int_{F} u_j(p) \, d\text{vol}_F \right) |_{r=0}$$

$$= \sum_{l=0}^{\infty} z^{-2d-2l+1} \log z \frac{\Gamma(d+l - \frac{1}{2}) \text{vol}(\Sigma)}{(d-1)!(2d+2l-2)!} \times$$

$$\times (4\pi)^{-\frac{m}{2}} \partial^{2l}_r \left( r^{m+2l-1} \int_{F} u_{\frac{m-1}{2} + l}(p) \, d\text{vol}_F \right) |_{r=0}$$

$$= z^{-2d+1} \log z \frac{\Gamma(d - \frac{1}{2}) \text{vol}(\Sigma)}{(d-1)!(2d-2)!} \times$$

$$\times (4\pi)^{-\frac{m}{2}} \left( r^{m-1} \int_{F} u_{\frac{m-1}{2}}(p) \, d\text{vol}_F \right) |_{r=0}.$$  

The last equality is due to the fact that $r^{m-1+2l} u_{\frac{m-1}{2} + l}(p)$ is a smooth function with respect to $r$.

By [S, p.47], the logarithmic part in the heat trace expansion coming from this term is
\[-(4\pi)^{-\frac{m}{2}}t^{-\frac{1}{2}}\log t \times \frac{1}{2} \text{vol}(\Sigma) \left( r^{m-1} \int_{F} u^{\frac{m-1}{2}}(p) \, d\text{vol}_F \right) \big|_{r=0}. \quad (3.6)\]

4 Heat trace expansion

From Proposition 3.1 and (3.6) we obtain

\[\text{tr} e^{-t\Delta} \sim_{t \to 0^+} (4\pi t)^{-\frac{m}{2}} \sum_{j=0}^{\infty} \tilde{a}_j t^j + bt^{-1/2} + ct^{-1/2} \log t, \quad (4.1)\]

where

\[\tilde{a}_j = \begin{cases} \int_M u_j \, d\text{vol}_M & \text{for } j \leq m/2 - 1, \\ f_M u_j \, d\text{vol}_M & \text{for } j > m/2 - 1. \end{cases}\]

and

\[c = \frac{1}{2} \text{vol}(\Sigma) \left( r^{m-1} \int_{F} u^{\frac{m-1}{2}}(p) \, d\text{vol}_F \right) \big|_{r=0},\]

\[b\] is to be computed.

Note that if \(m\) is even, the logarithmic term is zero.

4.1 Logarithmic term in low dimensional cases

Let \(\dim M = 3\), then \(c = \frac{1}{12} \text{vol}(\Sigma) \text{Scal}(x) = 0\), so the logarithmic term is always zero in this case.

Let \(\dim M = 5\), then

\[c = \frac{1}{2} \text{vol}(\Sigma) \int_{F} \left( \frac{1}{360} \left( 5 \text{Scal}(p)^2 - 2|\text{Ric}(p)|^2 + 2|R(p)|^2 \right) \right) \, d\text{vol}_F \]

\[= -\frac{\text{vol}(\Sigma)}{720} (4\pi)^{-2} \int_{F} \left( 3(\text{Scal}(x) - 6)^2 + 6(\text{Ric}_{ij}(x) - 2g_{ij})^2 \right) \, d\text{vol}_F. \]

The above expression is equal to zero if and only if \(\text{Scal}(x) \equiv 6\) and \(\text{Ric}_{ij}(x) = 2g_{ij}(x)\); equivalently if and only if the sectional curvature of the fiber \((F, g_V)\) is equal to one. Therefore the logarithmic term in this case is equal to zero if and only if the fiber \((F, g_V)\) is isometric to a spherical space form.
References

[BS] J. Brüning, R. Seeley, *The expansion of the resolvent near a singular stratum of conical type*, J. Funct. Anal. 95 (1991), 255--290.

[S] A. Suleymanova, *On the spectral geometry of manifolds with conic singularities*, PhD thesis, https://edoc.hu-berlin.de/handle/18452/19097 (2017).