CERES for Propositional Proof Schemata

Mikheil Rukhaia

*joint work with* T. Dunchev, A. Leitsch and D. Weller

Institute of Computer Languages,
Vienna University of Technology.

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Introduction
Schemata are very useful in mathematical proofs (avoids explicit use of the induction).

Schemata are used on meta-level.

Many problems can be expressed in propositional schema language, like:
- Circuit verification,
- Graph coloring,
- Pigeonhole principle, etc.
Set of index variables is a set of variables over natural numbers.

Linear arithmetic expression is as usual built on the signature 0, s, +, − and on a set of index variables.

Indexed proposition is an expression of the form $p_a$, where $a$ is a linear arithmetic expression.

Propositional variable is an indexed proposition $p_a$, where $a \in \mathbb{N}$. 
Syntax

- **Formula schema** is defined inductively:
  - Indexed proposition is a formula schema.
  - If $\phi_1$ and $\phi_2$ are formula schemata, then so are $\phi_1 \lor \phi_2$, $\phi_1 \land \phi_2$ and $\neg \phi_1$.
  - If $\phi$ is a formula schema, $a, b$ are linear arithmetic expressions and $i$ is an index variable, then $\bigwedge_{i=a}^{b} \phi$ and $\bigvee_{i=a}^{b} \phi$ are formula schemata, called iterations.
Semantics

- **Interpretation** is a pair of functions, $I = (\mathcal{I}, \mathcal{I}_p)$, s.t. $\mathcal{I}$ maps index variables to natural numbers and $\mathcal{I}_p$ maps propositional variables to truth values.

- **Truth value** $[\phi]_I$ of a formula schema $\phi$ in an interpretation $I$ is defined inductively:
  
  - $[p_a]_I = \mathcal{I}_p(p_{\mathcal{I}(a)})$.
  - $[-\phi]_I = \mathbf{T}$ iff $[\phi]_I = \mathbf{F}$.
  - $[\phi_1 \land (\lor) \phi_2]_I = \mathbf{T}$ iff $[\phi_1]_I = \mathbf{T}$ and (or) $[\phi_2]_I = \mathbf{T}$.
  - $[\bigwedge_{i=a}^b (\bigvee_{i=a}^b \phi)]_I = \mathbf{T}$ iff for every (there is an) integer $\alpha$ s.t. $\mathcal{I}(a) \leq \alpha \leq \mathcal{I}(b)$, $[\phi]_{I[\alpha/i]} = \mathbf{T}$. 
Cut-Elimination on Proof Schemata

**Aim:** describe syntactically sequence of cut-free proofs \((\chi_n)_{n \in \mathbb{N}}\) obtained by cut-elimination on proof sequences \((\varphi_n)_{n \in \mathbb{N}}\).

- Cut-free proofs of schema typically are described in meta-language.
- Find object language to define sequence \((\chi_n)_{n \in \mathbb{N}}\).
Which cut-elimination method?

- Reductive cut-elimination.
- CERES.
  - Efficient.
  - Strong methods of redundancy-elimination.
  - Atomic cut-normal form is constructed via parts of the original proof.
The CERES Method

- **CERES** is a cut-elimination method by resolution.

- Method consists of the following steps:
  1. Skolemization of the proof (if it is not already skolemized).
  2. Computation of the characteristic clause set.
  3. Refutation of the characteristic clause set.
  4. Computation of the Projections and construction of the Atomic Cut Normal Form.
Schematic LK
Basic Notions

- **Sequent Schema** is an expression of the form \( \Gamma \vdash \Delta \), where \( \Gamma \) and \( \Delta \) are multisets of formula schemata.

- **Initial Sequent Schema** is an expression of the form \( A \vdash A \), where \( A \) is an indexed proposition.

- **Proof Link** is a tuple \((\varphi, t)\), where \( \varphi \) is a proof name and \( t \) is a linear arithmetic expression.
Calculus LKS

- **Axioms:** initial sequent schemata or proof links.

- **Rules:**
Calculus LKS

- **Axioms:** initial sequent schemata or proof links.

- **Rules:** \( \land \) introduction:

  \[
  \frac{A, \Gamma \vdash \Delta}{A \land B, \Gamma \vdash \Delta} \quad \Lambda: l1
  \]

  \[
  \frac{B, \Gamma \vdash \Delta}{A \land B, \Gamma \vdash \Delta} \quad \Lambda: l2
  \]

  \[
  \frac{\Gamma \vdash \Delta, A \quad \Pi \vdash \Lambda, B}{\Gamma, \Pi \vdash \Delta, \Lambda, A \land B} \quad \Lambda: r
  \]

- **Equivalences:** \( A_0 \equiv \bigwedge_{i=0}^{0} A_i \) and \((\bigwedge_{i=0}^{n} A_i) \land A_{n+1} \equiv \bigwedge_{i=0}^{n+1} A_i\)
Calculus LKS

- **Axioms**: initial sequent schemata or proof links.

- **Rules**: \( \lor \) introduction:

\[
\frac{A, \Gamma \vdash \Delta \quad B, \Pi \vdash \Lambda}{A \lor B, \Gamma, \Pi \vdash \Delta, \Lambda} \lor : l
\]

\[
\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \lor B} \lor : r1 \quad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \lor B} \lor : r2
\]

**Equivalences**: \( A_0 \equiv \bigvee_{i=0}^0 A_i \) and \( (\bigvee_{i=0}^n A_i) \lor A_{n+1} \equiv \bigvee_{i=0}^{n+1} A_i \)
Calculus LKS

- **Axioms:** initial sequent schemata or proof links.

- **Rules:** \( \neg \) introduction:

\[
\frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta} \quad \neg : l
\]

\[
\frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A} \quad \neg : r
\]
Calculus **LKS**

- **Axioms**: initial sequent schemata or proof links.

- **Rules**: Weakening rules:
  
  \[
  \frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \quad w: l \\
  \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A} \quad w: r
  \]
Calculus LKS

- **Axioms**: initial sequent schemata or proof links.

- **Rules**: Contraction rules:
  
  \[
  \frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \quad c: l \quad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A} \quad c: r
  \]
Calculus LKS

- **Axioms**: initial sequent schemata or proof links.

- **Rules**: Cut rule:

\[
\frac{\Gamma \vdash \Delta, A \quad A, \Pi \vdash \Lambda}{\Gamma, \Pi \vdash \Delta, \Lambda} \quad \text{cut}
\]
LKS-proof

- **Derivation** is a directed tree with nodes as sequences and edges as rules.

- **LKS-proof** of the sequence $S$ is a derivation of $S$ with axioms as leaf nodes.

- An **LKS-proof** is called *ground* if it does not contain free parameters, index variables, or proof links.
Proof Schemata

Proof schema \( \psi \) is a tuple of pairs \( \langle (\psi^1_{\text{base}}, \psi^1_{\text{step}}), \ldots, (\psi^m_{\text{base}}, \psi^m_{\text{step}}) \rangle \) such that:

- \( \psi^1 \prec \psi^2 \prec \cdots \prec \psi^m \),
- \( \psi^i_{\text{base}} \) is a ground LKS-proof of \( S^i \{ n \leftarrow 0 \} \), for \( i \in \{1, \ldots, m\} \),
- \( \psi^i_{\text{step}} \) is an LKS-proof of \( S^i \{ n \leftarrow k + 1 \} \), where \( k \) is an index variable, and \( \psi^i_{\text{step}} \) contains proof links of the form (for \( i \prec j \)):

\[
\frac{\langle \psi^i, k \rangle}{S^i \{ n \leftarrow k \}} \quad \text{or} \quad \frac{\langle \psi^j, k^j \rangle}{S^j \{ n \leftarrow k^j \}}
\]

From now on \( m = 1 \).
Proof Evaluation

- An **evaluation** of a proof schema $\psi$ is a ground LKS-proof $eval(\psi, k)$, defined inductively:

  - $eval(\psi, 0) = \psi_{\text{base}}$, and

  - $eval(\psi, i + 1)$ is defined as $\psi_{\text{step}}$ with end-sequent $S \{k \leftarrow i\}$ and every proof link to $(\psi, k)$ in $\psi_{\text{step}}$ are replaced by $eval(\psi, i)$. 
An Example

\[ \psi_{\text{base}}: \]
\[
\frac{A_0 \vdash A_0}{\neg A_0, A_0 \vdash \neg: l} \quad \frac{A_1 \vdash A_1}{A_0, \neg A_0 \lor A_1 \vdash A_1} \quad \lor: l
\]

\[ \psi_{\text{step}}: \]
\[
\frac{A_0, \bigwedge_{i=0}^{k} (\neg A_i \lor A_{i+1}) \vdash A_{k+1}}{\neg A_{k+1}, A_{k+1} \vdash \neg: l} \quad \frac{A_{k+2} \vdash A_{k+2}}{A_{k+1}, \neg A_{k+1} \lor A_{k+2} \vdash A_{k+2}} \quad \lor: l
\]
\[
\frac{A_0, \bigwedge_{i=0}^{k} (\neg A_i \lor A_{i+1}) \vdash A_{k+2}}{\neg A_{k+1}, A_{k+1} \lor A_{k+2} \vdash A_{k+2}} \quad \land: l
\]
An Example (ctd.)

g. \textit{eval}(\psi, 0):

\[
\begin{align*}
A_0 & \vdash A_0 \\
\neg A_0, A_0 & \vdash \neg: l \\
& \vdash A_1 \\
A_0, \neg A_0 \lor A_1 & \vdash A_1 \\
\lor: l
\end{align*}
\]

\[
A_1 \vdash A_1 \\
\neg A_1, A_1 & \vdash \neg: l \\
A_2 & \vdash A_2 \\
\lor: l
\]

\[
\begin{align*}
A_0, \bigwedge_{i=0}^0 (\neg A_i \lor A_{i+1}) & \vdash A_1 \\
A_1, \neg A_1 \lor A_2 & \vdash A_2 \\
\text{cut}
\end{align*}
\]

\[
A_0, \bigwedge_{i=0}^1 (\neg A_i \lor A_{i+1}) \vdash A_2 \\
\land: l
\]

\[
\begin{align*}
A_0, \bigwedge_{i=0}^0 (\neg A_i \lor A_{i+1}) & \vdash A_1 \\
A_1, \neg A_1 \lor A_2 & \vdash A_2 \\
\text{cut}
\end{align*}
\]

\[
A_0, \bigwedge_{i=0}^1 (\neg A_i \lor A_{i+1}) \vdash A_2 \\
\land: l
\]
Schematic Characteristic Clause Set
Basic Notions

- **Cut-configuration** $\Omega$ of $\psi$ is a set of formula occurrences from the end-sequent of $\psi$.

- $\text{cl}_k^{\Omega,\psi}$ is an unique indexed proposition symbol for all cut-configurations $\Omega$ of $\psi$.

- The intended semantics of $\text{cl}_k^{\Omega,\psi}$ will be “the characteristic clause set of $\text{eval}(\psi, k)$, with the cut-configuration $\Omega$”.
Characteristic Clause Set

$CL_\rho(\psi, \Omega)$ is defined inductively:

- if $\rho$ is an axiom of the form $\Gamma_\Omega, \Gamma_C, \Gamma \vdash \Delta_\Omega, \Delta_C, \Delta$, then
  
  $$CL_\rho(\psi, \Omega) = \{\Gamma_\Omega, \Gamma_C \vdash \Delta_\Omega, \Delta_C\}.$$  

- if $\rho$ is a proof link of the form $\Gamma_\Omega, \Gamma_C, \Gamma \vdash \Delta_\Omega, \Delta_C, \Delta$
  
  then
  
  $$CL_\rho(\psi, \Omega) = \{\vdash cl_t^{\Omega', \psi}\}.$$
Characteristic Clause Set (ctd.)

- if $\rho$ is an unary rule with immediate predecessor $\rho'$, then

$$CL_\rho(\psi, \Omega) = CL_{\rho'}(\psi, \Omega).$$

- if $\rho$ is a binary rule with immediate predecessors $\rho_1, \rho_2$, then either

$$CL_\rho(\psi, \Omega) = CL_{\rho_1}(\psi, \Omega) \cup CL_{\rho_2}(\psi, \Omega)$$

or

$$CL_\rho(\psi, \Omega) = CL_{\rho_1}(\psi, \Omega) \otimes CL_{\rho_2}(\psi, \Omega).$$
Characteristic Clause Set (ctd.)

- $\text{CL}(\psi, \Omega) = \text{CL}_\rho(\psi, \Omega)$, where $\rho$ is the last inference of $\psi$.

- $\text{CL}(\varphi) = \text{CL}(\varphi, \emptyset)$, where $\varphi$ is a ground LKS-proof.

- $\text{CL}_{\text{base}} = \bigcup_{\Omega} (\{ \text{cl}_0^{\Omega, \psi} \vdash \} \otimes \text{CL}(\psi_{\text{base}}, \Omega))$.

- $\text{CL}_{\text{step}} = \bigcup_{\Omega} (\{ \text{cl}_{k+1}^{\Omega, \psi} \vdash \} \otimes \text{CL}(\psi_{\text{step}}, \Omega))$, for $0 \leq k \leq n$.

- $\text{CL}_s(\psi) = \{ \vdash \text{cl}_n^{\emptyset, \psi} \} \cup \text{CL}_{\text{base}} \cup \text{CL}_{\text{step}}$. 
Lemma (2.1)

Let $C$ be a clause and $C$ be a clause set. Then an interpretation $I \models \{C\} \otimes C$ iff $I \models C$ or $I \models \neg C$.

Lemma (2.2)

Let $\psi$ be a proof schema and $\text{CL}(\psi, \Omega)$ be a characteristic clause set as defined above. Assume that for all cut-configurations $\Omega$, $I \models \text{cl}_i^{\Omega, \psi}$ implies $I \models \text{CL}(\text{eval}(\psi, i), \Omega)$. Then $I \models \text{CL}(\psi_{\text{step}} \{k \leftarrow i\}, \Omega)$ implies $I \models \text{CL}(\text{eval}(\psi, i + 1), \Omega)$. 
Unsatisfiability of $\text{CL}_s(\psi)$ (ctd.)

**Proposition (2.1)**

Let $\varphi$ be a ground LKS-proof. Then $\text{CL}(\varphi)$ is unsatisfiable.

**Proposition (2.2)**

If $I \models \text{CL}_s(\psi)$ then $I \models \text{CL}(\text{eval}(\psi, I(n)))$.

**Corollary (2.1)**

Let $\psi$ be a proof schema and $\text{CL}_s(\psi)$ its characteristic clause set. Then $\text{CL}_s(\psi)$ is unsatisfiable.
An Example

$\nabla \quad \psi_{\text{base}}:\$

$$
\frac{A_0 \vdash A_0}{\neg A_0, A_0 \vdash \neg \colon l} \quad \frac{A_1 \vdash A_1}{A_0, \neg A_0 \vee A_1 \vdash A_1} \quad \vee : l
$$

$\nabla \quad \psi_{\text{step}}:\$

$$
\frac{A_0 \wedge \bigwedge_{i=0}^{k} (\neg A_i \vee A_{i+1}) \vdash A_{k+1}}{A_{k+1} \vdash A_{k+1}} \quad \frac{\neg A_{k+1}, A_{k+1} \vdash \neg \colon l}{A_{k+1}, \neg A_{k+1} \vee A_{k+2} \vdash A_{k+2}} \quad \vee : l
\quad \frac{A_{k+2} \vdash A_{k+2}}{\text{cut}}
\frac{A_0, \bigwedge_{i=0}^{k} (\neg A_i \vee A_{i+1}), \neg A_{k+1} \vee A_{k+2} \vdash A_{k+2}}{A_0, \bigwedge_{i=0}^{k+1} (\neg A_i \vee A_{i+1}) \vdash A_{k+2}} \quad \wedge : l
$$
The characteristic clause set schema of $\psi$ is:

\[
\begin{align*}
(1) & \quad \vdash \text{cl}^{\emptyset,\psi}_n \\
(2) & \quad \text{cl}^{\emptyset,\psi}_0 \vdash \\
(3) & \quad \text{cl}^{\{A_{k' + 1}\},\psi}_0 \vdash A_1 \\
(4) & \quad \text{cl}^{\{A_{k' + 1}\},\psi}_{k+1} \vdash \text{cl}^{\{A_{k' + 1}\},\psi}_k \\
(5) & \quad \text{cl}^{\{A_{k' + 1}\},\psi}_{k+1}, A_{k+1} \vdash A_{k+2} \\
(6) & \quad \text{cl}^{\emptyset,\psi}_{k+1} \vdash \text{cl}^{\{A_{k' + 1}\},\psi}_k \\
(7) & \quad \text{cl}^{\emptyset,\psi}_{k+1}, A_{k+1} \vdash
\end{align*}
\]
An Example (ctd.)

The characteristic clause set schema of $\psi$ is:

1. $\vdash \text{cl}_n^{\emptyset,\psi}$
2. $\text{cl}_0^{\emptyset,\psi} \vdash$
3. $\text{cl}_0^{\{A_{k'},+1\},\psi} \vdash A_1$
4. $\text{cl}_{k+1}^{\{A_{k'},+1\},\psi} \vdash \text{cl}_k^{\{A_{k'},+1\},\psi}$
5. $\text{cl}_{k+1}^{\{A_{k'},+1\},\psi}, A_{k+1} \vdash A_{k+2}$
6. $\text{cl}_0^{\emptyset,\psi} \vdash \text{cl}_k^{\{A_{k'},+1\},\psi}$
7. $\text{cl}_{k+1}^{\emptyset,\psi}, A_{k+1} \vdash$
Schematic Projections
Let $\rho$ be an unary and $\sigma$ a binary rule. Let $\phi, \psi$ be $\text{LKS}$-proofs, then $\rho(\phi)$ is the $\text{LKS}$-proof obtained from the $\phi$ by applying $\rho$, and $\sigma(\phi, \psi)$ is the proof obtained from the proofs $\phi$ and $\psi$ by applying $\sigma$. 
Basic Notions

Let $\rho$ be an unary and $\sigma$ a binary rule. Let $\phi, \psi$ be LKS-proofs, then $\rho(\phi)$ is the LKS-proof obtained from the $\phi$ by applying $\rho$, and $\sigma(\phi, \psi)$ is the proof obtained from the proofs $\phi$ and $\psi$ by applying $\sigma$.

$$\phi = A_0 \vdash A_0$$
Let $\rho$ be an unary and $\sigma$ a binary rule. Let $\phi, \psi$ be LKS-proofs, then $\rho(\phi)$ is the LKS-proof obtained from the $\phi$ by applying $\rho$, and $\sigma(\phi, \psi)$ is the proof obtained from the proofs $\phi$ and $\psi$ by applying $\sigma$.

\[
\neg(\phi) = \frac{A_0 \vdash A_0}{\neg A_0, A_0 \vdash \neg : l}
\]
Let \( \rho \) be an unary and \( \sigma \) a binary rule. Let \( \phi, \psi \) be \( \text{LKS} \)-proofs, then \( \rho(\phi) \) is the \( \text{LKS} \)-proof obtained from the \( \phi \) by applying \( \rho \), and \( \sigma(\phi, \psi) \) is the proof obtained from the proofs \( \phi \) and \( \psi \) by applying \( \sigma \).

\[
\neg(\phi) = \frac{A_0 \vdash A_0}{\neg A_0, A_0 \vdash \neg: l} \quad \psi = A_1 \vdash A_1
\]
Let $\rho$ be an unary and $\sigma$ a binary rule. Let $\phi, \psi$ be LKS-proofs, then $\rho(\phi)$ is the LKS-proof obtained from the $\phi$ by applying $\rho$, and $\sigma(\phi, \psi)$ is the proof obtained from the proofs $\phi$ and $\psi$ by applying $\sigma$.

$\forall (\neg(\phi), \psi) = \frac{A_0 \vdash A_0}{\neg A_0, A_0 \vdash \neg: l} \frac{A_1 \vdash A_1}{A_0, \neg A_0 \lor A_1 \vdash A_1 \lor: l}$
\[
P^{\Gamma \vdash \Delta} = \{ \psi^{\Gamma \vdash \Delta} \mid \psi \in P \}, \text{ where } \psi^{\Gamma \vdash \Delta} \text{ is } \psi \text{ followed by weakenings adding } \Gamma \vdash \Delta.
\]
Basic Notions (ctd.)

$P_{\Gamma \vdash \Delta} = \{\psi_{\Gamma \vdash \Delta} \mid \psi \in P\}$, where $\psi_{\Gamma \vdash \Delta}$ is $\psi$ followed by weakenings adding $\Gamma \vdash \Delta$.

$$
\psi = \\
A_0 \vdash A_0 \\
\neg A_0, A_0 \vdash \neg: l \\
A_1 \vdash A_1 \\
A_0, \neg A_0 \lor A_1 \vdash A_1 \\
\lor: l
$$
Basic Notions (ctd.)

\[ P^{\Gamma \vdash \Delta} = \{ \psi^{\Gamma \vdash \Delta} \mid \psi \in P \}, \]  where \( \psi^{\Gamma \vdash \Delta} \) is \( \psi \) followed by weakenings adding \( \Gamma \vdash \Delta \).

\[
\psi^{\Gamma \vdash \Delta} = \frac{A_0 \vdash A_0}{\neg A_0, A_0 \vdash} \quad \frac{A_1 \vdash A_1}{\lor: l} \\
\frac{A_0, \neg A_0 \lor A_1 \vdash A_1}{A_0, \neg A_0 \lor A_1, \Gamma \vdash A_1} \quad \frac{\lor: l}{w: l^*} \\
\frac{A_0, \neg A_0 \lor A_1, \Gamma \vdash \Delta, A_1}{w: r^*}
\]
$P \times_{\sigma} Q = \{ \sigma(\phi, \psi) \mid \phi \in P, \psi \in Q \}.$
Basic Notions (ctd.)

\[ P \times_{\sigma} Q = \{ \sigma(\phi, \psi) \mid \phi \in P, \psi \in Q \}. \]

\[ P = \left\{ \frac{A_0 \vdash A_0}{\neg A_0, A_0 \vdash -: l}, \frac{B_0 \vdash B_0}{\neg A_0, B_0 \vdash B_0} \right\} \]

\[ Q = \left\{ \frac{A_1 \vdash A_1}{-}, \frac{B_1 \vdash B_1}{A_1, B_1 \vdash B_1} \right\} \]
Basic Notions (ctd.)

\[ P \times \lor Q = \begin{cases} 
\frac{A_0 \vdash A_0}{\neg A_0, A_0 \vdash \neg: l} & \frac{A_1 \vdash A_1}{A_0, \neg A_0 \lor A_1 \vdash A_1} & \lor: l \\
\frac{B_0 \vdash B_0}{\neg A_0, B_0 \vdash B_0} & \frac{\neg A_0, B_0 \vdash B_0}{w: l} & \frac{A_1 \vdash A_1}{A_0, \neg A_0 \lor A_1 \vdash B_0, A_1} & \lor: l \\
\frac{A_0 \vdash A_0}{\neg A_0, A_0 \vdash \neg: l} & \frac{B_1 \vdash B_1}{A_1, B_1 \vdash B_1} & \frac{w: l}{w: l} & \lor: l \\
\frac{B_0 \vdash B_0}{\neg A_0, B_0 \vdash B_0} & \frac{B_1 \vdash B_1}{A_1, B_1 \vdash B_1} & \frac{w: l}{w: l} & \lor: l \\
\end{cases} \]
$PR(\psi, \rho, \Omega)$ is defined inductively:

- if $\rho$ is an axiom $S$, then $PR(\psi, \rho, \Omega) = \{S\}$.

- if $\rho$ is a proof link of the form

$$\Gamma \vdash \Delta, \Delta_C, \Delta$$

then $PR(\psi, \rho, \Omega)$ is:

$$pr_{\Omega', \psi}^\Delta, t$$

$$\Gamma \vdash \Delta, cl_t^{\Omega', \psi}$$
If $\rho$ is an unary inference with immediate predecessor $\rho'$ and

$$PR(\psi, \rho', \Omega) = \{\phi_1, \ldots, \phi_n\},$$

then either

$$PR(\psi, \rho, \Omega) = PR(\psi, \rho', \Omega)$$

or

$$PR(\psi, \rho, \Omega) = \{\rho(\phi_1), \ldots, \rho(\phi_n)\}. $$
If $\rho$ is a binary inference with immediate predecessors $\rho_1$ and $\rho_2$, then either

$$PR(\psi, \rho, \Omega) = PR(\psi, \rho_1, \Omega)^{\Gamma_2 \vdash \Delta_2} \cup PR(\psi, \rho_2, \Omega)^{\Gamma_1 \vdash \Delta_1}$$

or

$$PR(\psi, \rho, \Omega) = PR(\psi, \rho_1, \Omega) \times_\rho PR(\psi, \rho_2, \Omega)$$
The set of projections of $\psi$ is defined as follows:

$$PR(\psi) = \bigcup_{\Omega} (PR(\psi_{base}, \rho_{base}, \Omega) \cup PR(\psi_{step}, \rho_{step}, \Omega)).$$
### An Example

**ψ\text{base}:**

\[
\begin{align*}
  A_0 & \vdash A_0 \\
  \neg A_0, A_0 & \vdash \neg: l \\
  A_1 & \vdash A_1 \\
  A_0, \neg A_0 \lor A_1 & \vdash A_1 \\
  \lor: l
\end{align*}
\]

**ψ\text{step}:**

\[
\begin{align*}
  & (\psi, k) \\
  & A_0, \bigwedge_{i=0}^{k} (\neg A_i \lor A_{i+1}) \vdash A_{k+1} \\
  & A_{k+1} \vdash A_{k+1} \\
  & \neg A_{k+1}, A_{k+1} \vdash \neg: l \\
  & A_{k+2} \vdash A_{k+2} \\
  & A_{k+1}, \neg A_{k+1} \lor A_{k+2} \vdash A_{k+2} \\
  & \lor: l \\
  & \bigwedge_{i=0}^{k+1} (\neg A_i \lor A_{i+1}) \vdash A_{k+2} \\
  & A_0, \bigwedge_{i=0}^{k+1} (\neg A_i \lor A_{i+1}) \vdash A_{k+2} \\
  & \land: l
\end{align*}
\]
An Example (ctd.)

\( \bigcup_{\Omega \in \{\emptyset, \{A_{k'}+1\}\}} PR(\psi_{\text{base}}, \rho_{\text{base}}, \Omega) \) is:

\[
\begin{align*}
A_0 & \vdash A_0 \\
\neg A_0, A_0 & \vdash \neg: l \\
A_1 & \vdash A_1 \\
A_0, \neg A_0 \lor A_1 & \vdash A_1 \\
\lor: l
\end{align*}
\]

\( \bigcup_{\Omega \in \{\emptyset, \{A_{k'}+1\}\}} PR(\psi_{\text{step}}, \rho_{\text{step}}, \Omega) \) is:

\[
\begin{align*}
A_{k+1} & \vdash A_{k+1} \\
\neg A_{k+1}, A_{k+1} & \vdash \neg: l \\
A_{k+2} & \vdash A_{k+2} \\
A_{k+1}, \neg A_{k+1} \lor A_{k+2} & \vdash A_{k+2} \\
\lor: l \\
A_{k+1}, A_0, \bigwedge_{i=0}^{k} (\neg A_i \lor A_{i+1}), \neg A_{k+1} \lor A_{k+2} & \vdash A_{k+2} \\
\wedge: l
\end{align*}
\]
An Example (ctd.)

\[(pr_{\{A_k'+1\},\psi}, k)\]

\[A_0, \bigwedge_{i=0}^{k} (-A_i \lor A_{i+1}) \vdash cl_k^{\{A_{k+1}\},\psi}\]

\[w: l\]

\[\land: l\]

\[A_0, \bigwedge_{i=0}^{k} (-A_i \lor A_{i+1}) \vdash cl_k^{\{A_{k+1}\},\psi}\]

\[A_0, \bigwedge_{i=0}^{k+1} (-A_i \lor A_{i+1}) \vdash cl_k^{\{A_{k+1}\},\psi}\]

and

\[(pr_{\{A_k'+1\},\psi}, k)\]

\[A_0, \bigwedge_{i=0}^{k} (-A_i \lor A_{i+1}) \vdash cl_k^{\{A_{k+1}\},\psi}\]

\[w: l, r\]

\[\land: l\]

\[A_0, \bigwedge_{i=0}^{k} (-A_i \lor A_{i+1}) \vdash cl_k^{\{A_{k+1}\},\psi}, A_{k+2}\]

\[A_0, \bigwedge_{i=0}^{k+1} (-A_i \lor A_{i+1}) \vdash cl_k^{\{A_{k+1}\},\psi}, A_{k+2}\]
Ongoing and Future Work
Correctness of the definition of $PR(\psi)$

- Let $\psi$ be a proof schema and $PR(\psi)$ the set of projections of $\psi$ as defined above. Then by $Proj(\psi, k)$ we denote the set $\{eval(\phi, k) \mid \phi \in PR(\psi)\}$.

- Let $PR(eval(\psi, k), \Omega)$ be a set of projections for a ground LKS-proof $eval(\psi, k)$ with the cut-configuration $\Omega$. 
Correctness of the definition of $PR(\psi)$ (ctd.)

Lemma (3.1)

Let $\psi$ be a proof schema and $(\psi, k)$ an arbitrary proof link of $\psi$, then for all cut-configurations $\Omega$, $(pr^\Omega, \psi, k)$ evaluates to the set $PR(eval(\psi, k), \Omega)$.

Proposition (3.1)

Let $\psi$ be a proof schema, then $PR(eval(\psi, k), \emptyset) \subseteq Proj(\psi, k)$. 

Future Work

- Given the schemata of refutations and projections construct the schema of ACNF.
- Extend these results for the first order proof schemata.
- Cut-elimination on proof schema for Fürstenberg’s prime proof.
Questions?