SIGN CHANGES OF FOURIER COEFFICIENTS OF CUSP FORMS OF HALF-INTEGRAL WEIGHT OVER SPLIT AND INERT PRIMES IN QUADRATIC NUMBER FIELDS

ZILONG HE AND BEN KANE

Abstract. In this paper, we investigate sign changes of Fourier coefficients of half-integral weight cusp forms. In a fixed square class \( t\mathbb{Z}^2 \), we investigate the sign changes in the \( tp^2 \)-th coefficient as \( p \) runs through the split or inert primes over the ring of integers in a quadratic extension of the rationals. We show that sign changes occur in both sets of primes when there exists a prime dividing the discriminant of the field which does not divide the level of the cusp form and find an explicit condition that determines whether sign changes occur when every prime dividing the discriminant also divides the level.

1. Introduction

Throughout this paper, we let \( k \geq 1 \) and \( N \geq 4 \) be integers and \( 4 \mid N \). We denote by \( S_{k+1/2}(N, \psi) \) the space of cusp forms of weight \( k + 1/2 \) for the group \( \Gamma_0(N) \) with a Dirichlet character \( \psi \) modulo \( N \) and \( S_{3/2}^*(N, \psi) \) the orthogonal complement with respect to the Petersson scalar product of the subspace \( U(N, \psi) \) generated by unary theta functions. We also set \( S_{k+1/2}^*(N, \psi) = S_{k+1/2}(N, \psi) \) for \( k \geq 2 \).

Each \( f \in S_{k+1/2}^*(N, \psi) \) has a Fourier expansion given by

\[
 f(z) = \sum_{n=1}^{\infty} a_f(n) q^n,
\]

where \( q := e^{2\pi iz} \) with \( z \in \mathbb{H} \), the complex upper half-plane. Under the assumption that \( a_f(n) \in \mathbb{R} \), many authors have studied the change of signs \( \text{sgn}(a_f(n)) \) as \( n \) runs through natural sequences (for example, see [3, 7, 10, 11, 14]. For example, in [13, Theorem 1] it was shown that if \( a_f(t) \neq 0 \) for some \( f \in S_{k+1/2}^*(N, \chi) \), then there are infinitely many sign changes in the sequence

\[
 (a_f(tp^2))_{n=1}^{\infty},
\]

where \( p_n \) is the \( n \)-th prime. Letting \( K := \mathbb{Q}(\sqrt{D}) \) be a quadratic extension of \( \mathbb{Q} \), where \( D \in \mathbb{Z} \) is a fundamental discriminant, it is natural to ask whether there are infinitely many sign changes when the sequence (1.1) is restricted to the subsequence \( a_f(tp^2) \) with \( p \) running...
over all split (resp. inert) primes in the ring of integers $\mathcal{O}_K$, or even more generally in arithmetic progressions $p \equiv m \pmod{M}$ for some fixed $m$ and $M$. Since $p$ is split (resp. inert) in $\mathcal{O}_K$ if and only if $\chi_D(p) = 1$ (resp. $\chi_D(p) = -1$), where $\chi_D(n) := \left( \frac{D}{n} \right)$ denotes the Kronecker–Jacobi–Legendre symbol, we let $p_{D,n,+}$ denote the $n$-th prime which is split in $\mathcal{O}_K$ and $p_{D,n,-}$ denote the $n$-th prime which is inert in $\mathcal{O}_K$. For each $\varepsilon \in \{\pm\}$, we investigate sign changes across the sequences

\[(1.2) \quad (a_\ell \left( t p_{D,n,\varepsilon}^2 \right))_{n=1}^\infty.\]

**Theorem 1.1.** Let $k \geq 1$ be an integer, $N \geq 4$ an integer divisible by 4, and $\psi$ be a Dirichlet character modulo $N$. Suppose that $f \in S^*_k(N,\chi)$ has real Fourier coefficients and $t \geq 1$ is a squarefree integer such that $a_f(t) \neq 0$. If $D$ is a fundamental discriminant for which there exists an odd prime $\ell | D$ with $\ell \nmid N$, then there are infinitely many sign changes in both of the sequences

\[(a_\ell \left( t p_{D,n,+}^2 \right))_{n=1}^\infty \quad \text{and} \quad (a_\ell \left( t p_{D,n,-}^2 \right))_{n=1}^\infty.\]

More specifically, there exists a small constant $\delta = \delta_{f,t,D} > 0$ such that for sufficiently large $x$, there is a sign change with $p_{D,n,\varepsilon}$ in the interval $[x^\delta, x]$.

Since $f \in S^*_k(N,\chi)$ exhibits sign changes in the sequence (1.2) by [13, Theorem 1], one trivially obtains that $f \in S^*_k(N,\chi)$ exhibits sign changes across one of the sequences $(a_\ell \left( t p_{D,n,\varepsilon}^2 \right))_{n=1}^\infty$, but not both.

In order to describe the existence or non-existence of sign changes when every odd prime dividing $D$ also divides the level of the cusp form $f \in S^*_k(N,\chi)$, we require the Shimura lift [16]. For a squarefree positive integer $t$, $f$ can be lifted to a cusp form $f_t \in S_{2k}(N/2,\psi^2)$

\[f_t(z) = \sum_{n=1}^\infty a_{f_t}(n)q^n\]

by the $t$-th Shimura correspondence. Here the $n$-th coefficient $a_{f_t}(n)$ of $f_t$ is given by

\[(1.3) \quad a_{f_t}(n) = \sum_{d | n} \psi_{t,N}(d)d^{k-1}a_t(t \frac{n^2}{d^2}),\]

where $\psi_{t,N}$ denotes the character

\[\psi_{t,N}(d) := \psi(d) \left( \frac{(-1)^{kt}}{d} \right).\]

Thus in particular

\[(1.4) \quad a_{f_t}(p) = \psi_{t,N}(p)p^{k-1}a_t(t) + a_t \left( tp^2 \right).\]

We also require some properties of quadratic twists of modular forms. For $f \in S_k(M,\psi)$ with Fourier expansion ($q := e^{2\pi i z}$)

\[f(z) = \sum_{n \geq 1} a_f(n)q^n\]
and a character $\chi$, define
\[(f \otimes \chi)(z) := \sum_{n \geq 1} \chi(n) a_f(n) q^n.\]

If $f$ is a primitive form, then $f \otimes \chi$ is a Hecke eigenform, but not necessarily primitive. We write $f_{\chi}^*$ for the primitive form associated to the quadratic twist $f \otimes \chi$.

**Theorem 1.2.** Let $k \geq 1$ be an integer, $N \geq 4$ an integer divisible by 4, and $\psi$ be a Dirichlet character modulo $N$. Suppose that $f \in S_{k+1/2}^*(N,\chi)$ has real Fourier coefficients and $t \geq 1$ is a squarefree integer such that $a_f(t) \neq 0$. Let $D$ be a fundamental discriminant for which every odd prime dividing $D$ also divides $N$. Then the following hold.

1. The sequence (1.2) (with $\varepsilon = \pm$ fixed) restricted to the primes $p \nmid N$ does not exhibit sign changes if and only if $\chi_D = \psi t,N$ and the $t$-th Shimura correspondence satisfies
\[(1.6) \quad f_t \otimes \chi_{N^2} = \sum_i c_i \left( f_i - \varepsilon (f_i)_{\chi_D}^* \right) \otimes \chi_{N^2}\]
for some $c_i \in \mathbb{C}$ and where $f_i$ runs through a full set of primitive forms of level $M \mid N$. Moreover, if (1.2) does not exhibit sign changes for $\varepsilon$, then it does exhibit sign changes for $-\varepsilon$. There is at most one squarefree $t$ for which no sign changes occur and, if it exists, $t \mid N$.

2. There exists a choice of $N$, $t \geq 1$ squarefree, a Dirichlet character $\psi$ modulo $N$, a fundamental discriminant $D$, and $g \in S_{3/2}^*(N,\chi)$ such that $a_g(tp_D^2,\varepsilon)$ exhibits sign changes for precisely one of $\varepsilon = \pm$.

**Remark 1.3.** The twist by $\chi_{N^2}$ in Theorem 1.2 (1) precisely annihilates the coefficients that are not relatively prime to the level, which do not affect the sign changes except for possibly finitely many primes dividing $N$ and not dividing $D$.

If $f_i$ and $f_i^* = f_j$ are both newforms of level dividing $N$, then the term $f_i - \varepsilon f_i^*$ occurs twice (once for $i$ and once for $j$) unless $f_i = f_j^*$, in which case we say that $f_i$ has CM by $\chi_D$ (see (2.1)).

Note that although Theorem 1.2 (1) gives an if and only if statement, it is not immediately clear that the conditions are consistent with the assumption that $a_f(t) \neq 0$. The existence of such a form is the content of Theorem 1.2 (2). The counterexample from Theorem 1.2 (2) is constructed via the theory of quadratic forms, and in particular spinor genus theory. Although we only construct one explicit example, many can be constructed in an analogous way.

We determine a precise criteria which implies alternation of the coefficients in both cases and obtain Theorem 1.1 by showing that the criteria cannot be satisfied in this case; to obtain Theorem 1.2 (1) we need to determine precisely when this criteria holds. Arguing via orthogonality of characters, one should be able to generalize the results in this paper to show that there are sign changes in arithmetic progressions $p \equiv m \pmod{M}$ as long as $\gcd(M, N) = 1$, but the counterexample in Theorem 1.2 (2) implies that the gcd condition is necessary.

It might be interesting to investigate sign changes of Fourier coefficients of integral weight cusp forms across split or inert primes or arithmetic progressions as well.
The paper is organized as follows. In Section 2, we give some preliminaries and necessary information about quadratic twists. In Section 3, we give some useful information about the growth of convolution $L$-functions. In Section 4, we investigate the case when not every prime dividing $D$ divides the level of the modular form, proving Theorem 1.1. In Section 5, we investigate the case when all of the divisors of $D$ divide the level, proving Theorem 1.2.

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2. Preliminaries

2.1. Modular forms and quadratic twists. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ and $z \in \mathbb{H}$. A fractional linear transformation is defined by $\gamma z := \frac{az + b}{cz + d}$. Write $j(\gamma, z) := cz + d$. For a subgroup $\Gamma \subseteq \text{SL}_2(\mathbb{Z})$ and weight $w \in \mathbb{R}$, a multiplier system is defined as a function $\nu : \Gamma \to \mathbb{C}$ such that

$$\nu(\gamma A) j(\gamma A, z)^w = \nu(A) j(A, \gamma z)^w \nu(\gamma) j(\gamma, z)^w$$

for all $\gamma, A \in \Gamma$. Also, the slash operator $|_{w,\nu}$ of weight $w$ and multiplier system $\nu$ is defined as

$$f|_{w,\nu} := \nu(\gamma)^{-1} j(\gamma, z)^{-w} f(\gamma z).$$

Now we give a general definition of modular forms including integral and half-integral weight. A holomorphic modular form of weight $w \in \mathbb{R}$ and multiplier system $\nu$ for the subgroup $\Gamma$ is a function $f : \mathbb{H} \to \mathbb{C}$ satisfying the followings,

(1) $f$ is holomorphic on $\mathbb{H}$;
(2) $f|_{w,\nu} \gamma = f$;
(3) $f$ grows at most polynomially towards every cusp.

If moreover $f$ vanishes at every cusp, then $f$ is called a cusp form.

We are particularly interested in the case when $w$ is a half-integer and $\Gamma = \Gamma_0(L)$, where

$$\Gamma_0(L) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : L | c \right\}.$$

The theta multiplier is defined by

$$\nu_\Theta := \frac{\Theta(\gamma z)}{j(\gamma; z)^{\frac{w}{2}} \Theta(z)},$$

where $\Theta$ is the usual weight $1/2$ unary theta function

$$\Theta(z) := \sum_{n \in \mathbb{Z}} q^{n^2}.$$

For a character $\psi : \mathbb{Z} \to \mathbb{C}$ and $w \in \frac{1}{2} \mathbb{Z}$, we define the multiplier system

$$\nu_{\psi, w}(\gamma) := \psi(d) \nu_\Theta^{2w}$$

and call any modular form of weight $w$ and multiplier system $\nu_{\psi, w}$ on $\Gamma = \Gamma_0(L)$ a modular form of weight $w$ with level $L$ and Nebentypus character (or just character) $\psi$. Let $S_w(L, \psi)$ be
the subspace of holomorphic cusp forms of integral weight $\kappa \geq 2$ and level $L$ and Nebentypus $\psi$. We denote by $L^*$ the conductor of the character $\psi$. If $f \in S_\kappa^{new}(L, \psi)$ is a common eigenfunction for all Hecke operators and its first coefficient equals one, then $f$ is called a primitive form. We denote by $H_k^*(L, \psi)$ the set of all primitive forms of weight $k$, level $L$ and Nebentypus $\psi$.

We require some properties of quadratic twists of modular forms. For a real character $\chi = \chi_D$, we say that $f$ has CM by $\chi$ (or CM by the field $\mathbb{Q}(\sqrt{D})$) if $a_f(p) = 0$ whenever $p$ is a prime for which $\chi(p) = -1$, or in other words if

$$f \otimes \chi = f.$$  

The following is well-known, but we supply a proof for the convenience of the reader.

**Proposition 2.1.** Let $\chi$ be a Dirichlet character modulo $q$. If $f \in S_k(M, \psi)$ is a Hecke eigenform, then $f \otimes \chi \in S_k(Mq^2, \psi\chi^2)$ is also a Hecke eigenform.

**Proof.** Write $a_f(n)$ for the $n$-th Fourier coefficient of $f$. If $f \in S_k(M, \psi)$, then $f \otimes \chi \in S_k(Mq^2, \psi\chi^2)$ by [12, Proposition 17 (b), p. 127]. Since $f$ is a Hecke eigenform, we have

$$a_f(p^{n+1}) = a_f(p)a_f(p^n) - \psi(p)p^{k-1}a_f(p^{n-1})$$

for all primes $p$ and $n \geq 1$. Also, $a_f(d_1d_2) = a_f(d_1)a_f(d_2)$ for any $d_1, d_2 \in \mathbb{N}$ with $\gcd(d_1, d_2) = 1$, i.e., $a_f(n)$ is multiplicative. Combining this with the fact that $\psi(d)d^{k-1}$ is completely multiplicative, we furthermore have

$$\sum_{d|\gcd(n, m)} \psi(d)d^{k-1}a_f\left(\frac{nm}{d^2}\right) = a_f(m)a_f(n)$$

by [1, Exercises 30, 31, p. 49–50]. Hence one can check that

$$a_{T_m(f \otimes \chi)}(n) = \sum_{d|\gcd(n, m)} (\psi\chi^2)(d)d^{k-1}a_f\left(\frac{nm}{d^2}\right)\chi\left(\frac{nm}{d^2}\right)$$

$$= \chi(mn) \sum_{d|\gcd(n, m)} \psi(d)d^{k-1}a_f\left(\frac{nm}{d^2}\right)$$

$$= \chi(m)\chi(n)a_f(m)a_f(n)$$

$$= \chi(m)a_f(m)a_{f \otimes \chi}(n)$$

for all $m \in \mathbb{N}$. \hfill \Box

**Remark 2.2.** Given a Dirichlet character $\chi$ modulo $q$, if $f \in S_k(M, \psi)$ is a primitive form, then $f \otimes \chi \in S_k(Mq^2, \psi\chi^2)$ is a Hecke eigenform by Proposition [2,1]. In general, $f \otimes \chi$ may not be primitive, but there exists a unique primitive form $f^* \in S_k(M', \psi^*)$ with $M' | Mq^2$ such that $\psi\chi^2(n) = \psi^*(n)$ and $a_{f \otimes \chi}(n) = a_{f^*}(n)$ for any $n$ prime to $Mq^2$. [8, REMARK (2), p. 133 and EXERCISE 5, p. 376].

Given a Dirichlet character $\chi$ and a cusp form $f$, we denote by $f^*_\chi$ (or simply $f^*$, when the context is clear) the primitive form induced by a cusp form $f \otimes \chi$ as in Remark [2,2]. In particular, if $f \otimes \chi$ is primitive, then $f^*_\chi = f \otimes \chi$. When $\chi$ is real, $(f^*)^* = f$.

Although $f \otimes \chi$ need not be primitive, it is primitive under certain conditions. Specifically, from [8, Proposition 14.19 and 14.20], if $f \in H_k^*(L, \psi)$ and $\chi$ is a primitive character modulo $q$ with $\gcd(q, M) = 1$, then $f \otimes \chi \in H_k^*(Mq^2, \psi\chi^2)$.
Lemma 2.3. Let $k, M \in \mathbb{N}$ and $\psi$ and $\chi$ be characters such that the conductor of $\psi$ divides $M$ and there exists a prime $\ell$ such that $\ell$ divides the conductor of $\chi$ but $\ell$ does not divide $M$. Then for any $f \in H^*_k(M, \psi)$, we have that $f^*_\chi \in H_k(M'\ell^2, \psi\chi^2)$ for some $M' \in \mathbb{N}$.

In particular, if $\chi$ is real, $f$ does not have CM by $\chi$ and if $g \in H^*_k(N, \psi')$ with $\ell^2 \nmid N$ or $\psi' \neq \psi$, then $g \neq f^*$.

Proof. Note that if $\chi = \chi_1\chi_2$, then
\[(2.2) \quad f^*_\chi = f^*_\chi_1\chi_2 = (f^*_\chi_1)^*_{\chi_2}.
\]
Let $\ell \mid q$ with $\ell \nmid M$ be given and split $\chi = \chi_1\chi_2$ so that the conductor $q'$ of $\chi_1$ is relatively prime to $\ell$ and the conductor of $\chi_2$ is an $\ell$-power $\ell^r$. Note that since $h := f^*_\chi_1$ is primitive and its level $M_h$ (dividing $Mq^2$) is relatively prime to $\ell$,
\[f^*_\chi_1 \otimes \chi_2 \] is primitive of level $M_h\ell^{2r}$. Since $r \geq 1$, we see by (2.2) that
\[f^*_\chi = (f^*_\chi_1)^*_{\chi_2} = f^*_\chi_1 \otimes \chi_2
\]
has $\ell^2$ dividing its level. This is the first claim.

Since $\ell^2 \nmid M$, we immediately obtain that $f \neq f^*$, so $f$ does not have CM by $\chi$. Finally, if $g \in H^*_k(N, \psi')$ and $\ell^2 \nmid N$, then we immediately obtain that $g \neq f^*$, as they have different levels. If $\psi' \neq \psi$, then they are not equal because they have different Nebentypus. \(\square\)

2.2. Quadratic forms. Let $V$ be a quadratic space over $\mathbb{Q}$ associated with symmetric bilinear map $B : V \times V \to \mathbb{Q}$ and write $Q(x) = B(x, x)$, $x \in V$. We denote by $O(V)$ the orthogonal group of $V$ and $O'(V)$ the kernel of the homomorphism $\theta : O(V) \to \mathbb{Q}^\times/\mathbb{Q}^\times$ as usual. Let $O_h(V)$ and $O'_h(V)$ be the adelic groups of $O(V)$ and $O'(V)$, respectively. Let $L$ be a $\mathbb{Z}$-lattice on $V$. We define the class $\text{cls}(L)$, spinor genus $\text{spn}(L)$ and genus $\text{gen}(L)$ of $L$ by the orbits of $L$ under the actions of $O(V)$, $O(V)O'_h(V)$ and $O_h(V)$ respectively [see [9] for more details].

For $n \in \mathbb{N}$, if there exists some $x_0 \in L$ such that $Q(x_0) = n$, then we say that $n$ is represented by $L$ and denote by $r(n, L)$ the number of representation of $n$ by $L$. Also, we define the number of representations of $n$ by the genus (resp. spinor genus) of $L$ by the Siegel–Weil average
\[r(n, \text{gen}(L)) := \left( \sum_{K \in \text{gen}(L)} \frac{1}{|O(K)|} \right)^{-1} \sum_{K \in \text{gen}(L)} r(n, K) \frac{1}{|O(K)|},
\]
and
\[r(n, \text{spn}(L)) := \left( \sum_{K \in \text{spn}(L)} \frac{1}{|O(K)|} \right)^{-1} \sum_{K \in \text{spn}(L)} r(n, K) \frac{1}{|O(K)|},
\]
where the summation is over a complete set of representatives of the classes in the genus (resp. spinor genus) of $L$. A quadratic form $Q$ can be always associated with a lattice $L_Q$ and hence we abuse the notations $r(n, Q)$, $r(n, \text{gen}(Q))$ and $r(n, \text{spn}(Q))$ standing for $r(n, L_Q)$, $r(n, \text{gen}(L_Q))$ and $r(n, \text{spn}(L_Q))$. 
For a ternary quadratic form $Q$ of discriminant $D$ and level $N$, the theta series $\theta_Q$ associated with $Q$ is given by
\[ \theta_Q(z) := \sum_{n=0}^{\infty} r(n, Q) q^n, \]
and $\theta_Q \in M_{3/2}(N, \psi)$ for an appropriate $\psi$ (see for example [16, Proposition 2.1]). It is well known that the theta series can be expressed as
\[ (2.3) \quad \theta_Q(z) = E(z) + H(z) + f(z), \]
where $E(z)$ is in the space spanned by Eisenstein series, $H(z) = \sum_{n=0}^{\infty} a_H(n, Q) q^n \in U(N, \psi)$ and $f(z) = \sum_{n=0}^{\infty} a_f(n, Q) q^n \in S^*_3(N, \psi)$ ([3, Lemma 4]). In the theory of quadratic forms, the coefficients $a_E(n, Q)$ and $a_H(n, Q)$ can be interpreted as (see for example [5, Theorem 2]),
\[ (2.4) \quad a_E(n, Q) = r(n, \text{gen}(Q)) \quad \text{and} \quad a_H(n, Q) = r(n, \text{spn}(Q)) - r(n, \text{gen}(Q)). \]

3. Convolution $L$-series

The following lemma follows immediately by replacing $f$ by $f \otimes \chi$ in [13, Lemma 2.1] and it agrees with their results when $\chi$ is a trivial character.

Lemma 3.1. Let $f \in H^*_k(M_f, \psi_f)$ and $g \in H^*_k(M_g, \psi_g)$ whose $n$-th coefficient are $\lambda_f(n)n^{(k-1)/2}$ and $\lambda_g(n)n^{(k-1)/2}$. Let $\chi$ be a primitive character modulo $q$ and $\gcd(q, M_f) = 1$. Then as $x \to \infty$,
\[ (3.1) \quad \sum_{\substack{p \leq x \\ p \nmid M_fq}} \frac{\chi(p)\lambda_f(p)}{p} = O(1) \]
and
\[ (3.2) \quad \sum_{\substack{p \leq x \\ p \nmid M_fq}} \frac{|\lambda_f(p)|^2}{p} = \log \log x + O(1). \]

If $g \neq f \otimes \chi$, then
\[ (3.3) \quad \sum_{\substack{p \leq x \\ p \nmid M_fq}} \frac{\chi(p)\lambda_f(p)\lambda_g(p)}{p} = O(1). \]

The implied constants in (3.1) and (3.2) depend on the form $f$ and the character $\chi$ and that in (3.3) depends on the forms $f, g$ and the character $\chi$.

Although a primitive form $f$ twisted with the character $\chi$ may not be primitive in general, we are still able to make use of [13, Lemma 2.1] by taking $f^*$ instead of $f \otimes \chi$ from Remark 2.2.
Lemma 3.2. Let \( f \in H_k^*(M_f, \psi_f) \) and \( g \in H_k^*(M_g, \psi_g) \) whose \( n \)-th coefficient are \( \lambda_f(n)n^{(k-1)/2} \) and \( \lambda_g(n)n^{(k-1)/2} \). Let \( \chi \) be a primitive real character modulo \( q \) and \( f^* \) the primitive form induced by \( f \otimes \chi \).

1. We have

\[
\sum_{p \leq x \atop p \nmid M_f q \chi(p) = \pm 1} \frac{\lambda_f(p)}{p} = O(1).
\]

2. If \( f \) does not have CM by \( \chi \), then as \( x \to \infty \) we have

\[
\sum_{p \leq x \atop p \nmid M_f q \chi(p) = \pm 1} \frac{\lambda_f(p)\overline{\lambda_f(p)}}{p} = \frac{1}{2} \log \log x + O(1),
\]

\[
\sum_{p \leq x \atop p \nmid M_f q \chi(p) = \pm 1} \frac{\lambda_f(p)\overline{\lambda_f(p)}}{p} = \pm \frac{1}{2} \log \log x + O(1).
\]

If \( f \) has CM by \( \chi \), then

\[
\sum_{p \leq x \atop p \nmid M_f q \chi(p) = \pm 1} \frac{\lambda_f(p)\overline{\lambda_f(p)}}{p} = \frac{1}{2} \log \log x + O(1).
\]

If \( g \neq f \) and \( g \neq f^* \), then

\[
\sum_{p \leq x \atop p \nmid M_f q \chi(p) = \pm 1} \frac{\lambda_f(p)\overline{\lambda_g(p)}}{p} = O(1).
\]

3. Suppose that there exists an odd prime \( \ell \mid q \) such that \( \ell \nmid M_f \). Then as \( x \to \infty \) we have

\[
\sum_{p \leq x \atop p \nmid M_f q \chi(p) = \pm 1} \frac{\lambda_f(p)\overline{\lambda_f(p)}}{p} = \frac{1}{2} \log \log x + O(1).
\]

Moreover, if \( g \neq f \) and \( \ell^2 \nmid M_g \), then

\[
\sum_{p \leq x \atop p \nmid M_f q \chi(p) = \pm 1} \frac{\lambda_f(p)\overline{\lambda_g(p)}}{p} = O(1).
\]

The implied constants in (3.4), (3.9) and (3.6) depend on the form \( f \) and the character \( \chi \), and that in (3.8) depends on the forms \( f, g \) and the character \( \chi \).
Lemma 3.1, we have \( \chi(p)\lambda_f(p) = \lambda_f(p) \) for any prime \( p \nmid M_fq \). For (3.1), by Lemma 3.1 (3.1), we have

\[
2 \sum_{p \leq x \atop \chi(p) = \pm 1} \frac{\lambda_f(p)}{p} = \sum_{p \leq x \atop \chi(p) = \pm 1} \frac{\lambda_f(p)}{p} \pm \sum_{p \leq x \atop \chi(p) = \pm 1} \frac{\chi(p)\lambda_f(p)}{p} = \sum_{p \leq x \atop \chi(p) = \pm 1} \frac{\lambda_f(p)}{p} \pm \sum_{p \leq x \atop \chi(p) = \pm 1} \frac{\lambda_f(p)}{p} = O(1).
\]

(2) It is not difficult to see the relation

\[
2 \sum_{p \leq x \atop \chi(p) = \pm 1} \frac{\lambda_f(p)\overline{\lambda_g(p)}}{p} = \sum_{p \leq x \atop \chi(p) = \pm 1} \frac{\lambda_f(p)\overline{\lambda_g(p)}}{p} \pm \sum_{p \leq x \atop \chi(p) = \pm 1} \frac{\chi(p)\lambda_f(p)\overline{\lambda_g(p)}}{p}
\]

(3.11)

and the forms \( f, f^* = f_{\chi}^* \) and \( g \) are primitive. By Lemma 3.1, the first term in (3.11) contributes \( \log x + O(1) \) if \( g = f \) and \( O(1) \) otherwise, while the second term contributes \( \pm \log x + O(1) \) if \( g = f^* \) and \( O(1) \) otherwise.

For \( g = f \) with \( f \) not CM by \( \chi \), the first term in (3.11) thus contributes \( \log x + O(1) \) and the second contributes \( O(1) \), giving (3.5) (dividing (3.11) by 2).

For \( g = f^* \), the second term in (3.11) always contributes \( \pm \log x + O(1) \) and the first term contributes \( \log x + O(1) \) if and only if \( f \) has CM by \( \chi \) (and \( O(1) \) otherwise), giving (3.6) and (3.7).

If \( g \neq f \) and \( g \neq f^* \), then both terms in (3.11) contribute \( O(1) \), and we hence obtain (3.8).

(3) By Lemma 2.3, \( \ell \nmid M_f \) implies that \( f \neq f_{\chi}^* \). Since \( f \) does not have CM by \( \chi \), (3.5) implies (3.9).

Moreover, since \( \ell^2 \nmid M_f \) by Lemma 2.3 and \( \ell^2 \nmid M_g \) in (3.10), we have \( g \neq f^* \), and hence (3.10) follows immediately from (3.8).

4. Coefficients of arbitrary cusp forms and the proof of Theorem 1.1

Suppose that \( k, L \in \mathbb{N} \) and \( \psi \) is a character with conductor \( L_\psi \mid L \). Writing

\[
f| V_\ell(z) := f(\ell z),
\]

in \( \mathbb{Z} \), A. Atkin and J. Lehner obtain the well-known decomposition

\[
S_k(L, \psi) = \bigoplus_{M \mid L} \bigoplus_{f \in H^*_k(M, \psi)} \text{Span}_\mathbb{C} \{f| V_\ell : \ell \mid (L/M)\}.
\]

Lemma 4.1. Let \( k \geq 1 \) be an integer, \( N \geq 4 \) an integer divisible by 4 and \( \psi \) a Dirichlet character modulo \( N \). Let \( \chi \) be a primitive real character modulo \( q \) and \( \varepsilon \in \{ \pm 1 \} \). Suppose that \( \ell \in S^*_{k+1/2}(N, \psi) \) and \( t \geq 1 \) is a squarefree integer such that \( a(t) \neq 0 \). Assume that the sequence \( \{a(tn^2) \}_{n \in \mathbb{N}} \) is real.
(1) We have

\[ \sum_{p \leq x \atop p \nmid q} \frac{a_f(tp^2)}{p^{k+1/2}} = O_{f,t,\chi}(1). \]

(2) If there exists an odd prime \( r \mid q \) such that \( r \nmid N \), then

\[ \sum_{p \leq x \atop p \nmid q} \frac{a_f(tp^2)^2}{p^{2k}} = C \log \log x + O(1) \]

holds for some \( C > 0 \) for both \( \varepsilon = 1 \) and \( \varepsilon = -1 \).

(3) Suppose that every prime divisor of \( q \) divides \( N \). The equality (4.3) holds with \( C > 0 \) unless

\[ f_t \otimes \chi_{N^2} = \sum_i c_i (f_i - \varepsilon f_i^*) \otimes \chi_{N^2}, \]

where \( f_i \) run through all of the primitive forms of level dividing \( N \) and \( c_i \in \mathbb{C} \). If (4.4) holds, then \( C = 0 \) and moreover \( a_{f_t}(p) = 0 \) for every prime \( p \nmid N \) with \( \chi(p) = \varepsilon \).

The implied constants \( C \) and those occurring in the \( O \)-symbols depends on \( f, t, \chi \) and \( \varepsilon \).

Proof. Applying the M"obius inversion formula to (1.3), we have

\[ a_f(tn^2) = \sum_{d \mid n} \mu(d) \psi_{t,N}(d)d^{k-1}a_f \left( \frac{n}{d} \right), \]

where \( a_{f_t}(n) \) is the \( n \)-th coefficient of \( f_t \). Write \( a_{f_t}(n) = \lambda_{f_t}(n)n^{k-1/2} \) and recall that by Deligne’s bound \[4\],

\[ \lambda_{f_t}(n) \leq \sigma_0(n) \ll \log n, \]

where \( \sigma_s(n) := \sum_{d \mid n} d^s \) denotes the sum of divisors function. Then we may rewrite the above formula as

\[ \frac{a_f(tn^2)}{n^{k-1/2}} = \sum_{d \mid n} \frac{\mu(d)\psi_{t,N}(d)}{\sqrt{d}} \lambda_{f_t} \left( \frac{n}{d} \right). \]

Considering the special case that \( n = p \) is a prime and noting that \( \lambda_{f_t}(1) = a_f(t) \) yields

\[ \frac{a_f(tp^2)}{p^{k-1/2}} = \lambda_{f_t}(p) - \frac{\psi_{t,N}(p)}{\sqrt{p}}a_f(t). \]

Applying the decomposition (1.1) to \( S_{2k}(N/2, \psi^2) \), we obtain a basis

\[ \bigcup_{M \mid (N/2)} \bigcup_{L \in \mathcal{L}_{\psi^2}[M]} \left\{ f | V_{\ell} : \ell \mid \frac{N/2}{M}, f \in H^*_2(M, \psi^2) \right\}. \]

Hence \( f_t \in S_{2k}(N/2, \psi^2) \) can be written as

\[ f_t(z) = \sum_i \sum_{\ell \mid (N/(2M_i))} c_i, f_i(\ell z), \]
where \( f_i \in H^*_{2k}(M_i, (\psi^2)_{M_i}) \) is primitive of level \( M_i \) and \( c'_{i,s} \) are scalars depending on \( f \).

For any prime \( p \nmid Nq \), the terms with \( \ell \neq 1 \) do not contribute anything to the \( p \)-th Fourier coefficient, so, comparing coefficients of each side of (4.7), we see that

\[
\lambda_{f_i}(p) = \sum_i c_i \lambda_{f_i}(p),
\]

where \( c_i := c_{i,1} \). Moreover, since

\[
\sum_i c_i = \lambda_{f_i}(1) = a_i(t) \neq 0
\]

by assumption, not all \( c'_{i,s} \) are zero. Expressing \( \lambda_{f_i} \) by the linear combination of \( \lambda_{f_i} \) in (4.6), we have

\[
\frac{a_i(tp^2)}{p^{k-1/2}} = \sum_{i} c_i \lambda_{f_i}(p) - \frac{\psi_{t,N}(p)}{\sqrt{p}} a_i(t). \tag{4.8}
\]

(1) Dividing (4.8) by \( p \) and summing over \( p \leq x \) with \( \chi(p) = \pm 1 \) but \( p \nmid Nq \) on (4.8) and then applying Lemma 3.2 (3.4) to \( f_i \) in (4.8), we deduce that

\[
\sum_{p \leq x \atop p \nmid Nq} \frac{a_i(tp^2)}{p^{k+1/2}} = \sum_{i} c_i \sum_{p \leq x \atop \chi(p) = \pm 1} \frac{\lambda_{f_i}(p)}{p} - a_i(t) \sum_{p \leq x \atop p \nmid Nq} \frac{\psi_{t,N}(p)}{p^{3/2}} = O_{t,\lambda}(1),
\]

thereby showing (4.2).

(2) Since \( a_i(tn^2) \in \mathbb{R} \), multiplying (4.8) by its complex conjugate yields

\[
\frac{a_i(tp^2)}{p^{2k-1}} = \sum_{i} |c_i|^2 |\lambda_{f_i}(p)|^2 + \sum_{i \neq j} c_i c_j \overline{\lambda_{f_i}(p)} \overline{\lambda_{f_j}(p)}
\]

\[
+ |a_i(t)|^2 |\psi_{t,N}(p)|^2 \frac{1}{p} - 2 \text{ Re} \left( \sum_{i} c_i \overline{\lambda_{f_i}(p)} \overline{\psi_{t,N}(p)} a_i(t) \right). \tag{4.9}
\]

Dividing (4.9) by \( p \) and summing over \( p \leq x \) with \( \chi(p) = \pm 1 \) yields

\[
\sum_{p \leq x \atop p \nmid Nq} \frac{a_i(tp^2)}{p^{2k}} = \sum_{i} |c_i|^2 \sum_{p \leq x \atop \chi(p) = \pm 1} \frac{\lambda_{f_i}(p)}{p} + \sum_{i \neq j} c_i c_j \sum_{p \leq x \atop \chi(p) = \pm 1} \frac{\lambda_{f_i}(p)}{p} \overline{\lambda_{f_j}(p)}
\]

\[
+ |a_i(t)|^2 \sum_{p \leq x \atop \chi(p) = \pm 1} \frac{|\psi_{t,N}(p)|^2}{p^2} - 2 \sum_{p \leq x \atop \chi(p) = \pm 1} \text{ Re} \left( \sum_i c_i \overline{\lambda_{f_i}(p)} \overline{\psi_{t,N}(p)} a_i(t) \right). \tag{4.10}
\]

Using (4.5), the last two terms are clearly \( O_{t,\lambda}(1) \). Now consider the sum of the first and second terms.
Define the index sets $I^* := \{ i : f_i \text{ has CM by } \chi \}$ and $J := \{ (i,j) : i \neq j \text{ and } f_j = f_i^* \}$.

Write

$$S_1 := \sum_{i \notin I^*} |c_i|^2 \quad S_1^* := \sum_{i \notin I^*} |c_i|^2 \quad S_2 := \sum_{(i,j) \in J} c_i c_j.$$ 

Clearly, not both $S_1$ and $S_1^*$ are zero. For the first term in (4.10), we use (3.5) and (3.7) from Lemma 3.2 to obtain

$$\sum_i |c_i|^2 \sum_{p \leq x \atop p \nmid Nq \atop \chi(p) = \varepsilon} \frac{|\lambda_{f_i}(p)|^2}{p} = \frac{S_1 + (\varepsilon + 1)S_1^*}{2} \log \log x + O_{t,t,\chi}(1).$$

(4.11)

For the second term in (4.10), since $f_i \neq f_j$ for $i \neq j$, (3.6) implies

$$\sum_{(i,j) \in J} c_i c_j \sum_{p \leq x \atop p \nmid Nq \atop \chi(p) = \varepsilon} \frac{\lambda_{f_i}(p)\lambda_{f_j}(p)}{p} = \frac{\varepsilon S_2}{2} \log \log x + O_{t,t,\chi}(1).$$

(4.12)

If $i \neq j$ and $(i,j) \notin J$, then $f_j \neq f_i$ and $f_j \neq f_i^*$ (if $f_i$ has CM by $\chi$, then $f_i^* = f_i$, so $f_j \neq f_i$ implies that $f_j \neq f_i^*$). Hence for the remaining terms in (4.10) we may use (3.8) to obtain

$$\sum_{i \neq j \atop (i,j) \notin J} c_i c_j \sum_{p \leq x \atop p \nmid Nq \atop \chi(p) = \varepsilon} \frac{\lambda_{f_i}(p)\lambda_{f_j}(p)}{p} = O_{t,t,\chi}(1).$$

(4.13)

Combining (4.11), (4.12), and (4.13), the sum of the first and second terms in (4.10) is given by

$$\frac{S_1 + \varepsilon S_2 + (\varepsilon + 1)S_1^*}{2} \log \log x + O_{t,t,\chi}(1).$$

(4.14)

If there exists an odd prime $r \mid q$ such that $r \nmid N$, then Lemma 2.3 implies that $r^2 \mid M_{f_r}$. Since $r^2 \mid N$ and $M_j \mid N$ for every $j$ (including $j = i$), $f_i^* \neq f_j$ for every $j$ and hence $I^* = J = \emptyset$ and $S_2 = S_1^* = 0$ in this case. Since $S_1 + S_1^* > 0$, we furthermore obtain that from which we conclude that $S_1 > 0$. Hence (4.14) becomes

$$\frac{S_1}{2} \log \log x + O_{t,t,\chi}(1)$$

with $S_1 > 0$. This yields the claim for the case that such an odd prime $\ell$ exists.

(3) We write

$$S_1 + \varepsilon S_2 + (\varepsilon + 1)S_1^* =: C e^{i\theta}$$

with $C \geq 0$ and $-\pi < \theta \leq \pi$ and note that since the left-hand side of (4.3) is a sum of nonnegative real numbers, if $C \neq 0$ then we must have $\theta = 0$ (otherwise the limit in (4.14) would diverge to $+e^{i\theta} \infty$ as $x \to \infty$ while for each $x$ it equals a nonnegative real number, a contradiction).

Writing $a_i := \Re(c_i)$ and $b_i := \Im(c_i)$, we conclude that

$$S_1 + \varepsilon S_2 + (\varepsilon + 1)S_1^* = \Re(S_1 + \varepsilon S_2 + (\varepsilon + 1)S_1^*)$$

$$S_1 + \varepsilon S_2 + (\varepsilon + 1)S_1^* = : C e^{i\theta}$$

with $C \geq 0$ and $-\pi < \theta \leq \pi$ and note that since the left-hand side of (4.3) is a sum of nonnegative real numbers, if $C \neq 0$ then we must have $\theta = 0$ (otherwise the limit in (4.14) would diverge to $+e^{i\theta} \infty$ as $x \to \infty$ while for each $x$ it equals a nonnegative real number, a contradiction).
and hence
\[(4.15) \quad C = S_1 + \varepsilon S_2 + (\varepsilon + 1) S_1^* = \sum_{i \in I_*} (a_i^2 + b_i^2) + \varepsilon \sum_{(i,j) \in J} (a_i a_j + b_i b_j) + (\varepsilon + 1) \sum_{i \in I_*} (a_i^2 + b_i^2).\]

Consider the set
\[I_J := \{ i : \exists j \text{ s.t. } (i,j) \in J \}\]
and note that if \((i,j) \in J\) then \((j,i) \in J\), but since \((f_i^*)^* = f_i\), the tuples in \(J\) appear in pairs, and hence there does not exist \(j' \neq j\) such that \((i,j') \in J\). Thus \((4.15)\) becomes
\[(4.16) \quad C = \sum_{i \in (I^* \cup I_J)} (a_i^2 + b_i^2) + \sum_{(i,j) \in J} \frac{1}{2} \left( a_i^2 + b_i^2 + a_j^2 + b_j^2 \right) + \varepsilon \left( a_i a_j + b_i b_j \right) + (\varepsilon + 1) \sum_{i \in I_*} (a_i^2 + b_i^2).\]

Hence we conclude that \(C > 0\) unless all of the following hold:
- If \(i \notin (I^* \cup I_J)\), then \(c_i = 0\).
- If \((i,j) \in J\), then \(c_i = -\varepsilon c_j\).
- If \(\varepsilon = 1\), then \(c_i = 0\) for every \(i \in I_*\).

Noting that, since \(\chi_{N^2}\) annihilates \(f_i|V_\ell\) for every \(\ell > 1\),
\[f_i \otimes \chi_{N^2} = \sum_{i} c_i f_i \otimes \chi_{N^2}\]
and writing \(f_i = \frac{1}{2} (f_i + f_i^*)\) for \(i \in I^*\), these three conditions are equivalent to \((4.4)\).

Finally note that if \((4.4)\) holds, then for \(p \nmid N\) with \(\chi(p) = \varepsilon\) we have
\[a_{f_i}(p) = \sum_{i} c_i \left( a_{f_i}(p) - \varepsilon a_{f_i^*}(p) \right) = c_i (a_{f_i}(p) - \varepsilon \chi(p) a_{f_i}(p)) = c_i (a_{f_i}(p) - a_{f_i}(p)) = 0,\]
where we used the fact that
\[a_{f_i^*}(p) = a_{f_i} \otimes \chi(p) = \chi(p) a_{f_i}(p).\]

\[\Box\]

\textbf{Proof of Theorem 1.1.} We claim that if
\[(4.17) \quad \sum_{p \leq x} \frac{a_{f_i}(tp^2)^2}{p^{2k}} = C_{f,t,\chi,\varepsilon} \log \log x + O_{f,t,\chi,\varepsilon}(1),\]
for some \(C_{f,t,\chi,\varepsilon} > 0\), then the assertion is true for \(\chi(p) = \varepsilon\). By Deligne’s bound, \(|\lambda_{f_i}(p)| \leq 2\) (as \(|a_{f_i}(p)| \leq 2p^{-(k-1)/2}\)). Then \((4.8)\) is given by
\[|a_{f_i}(tp^2)p^{-(k-1/2)}| \leq 2 \sum_{i} |c_i| + |a_{f_i}(t)| =: C_{f,t}.\]
Suppose that \( a_i(tp^2) \) are of the same sign for \( y \leq p \leq x \) with \( p \nmid Nq \) and \( \chi(p) = \varepsilon \). Without loss of generality, assume that \( a_i(tp^2) > 0 \). Then

\[
(4.18) \sum_{y \leq p \leq x} \frac{a_i(tp^2)^2}{p^{2k}} \leq C_{t,t} \sum_{y \leq p \leq x} \frac{a_i(tp^2)}{p^{k+1/2}} = C_{t,t} \sum_{y \leq p \leq x} \frac{a_i(tp^2)}{p^{k+1/2}}.
\]

The left hand side of (4.18) given by

\[
C_{t,t,\chi,\varepsilon} \log \left( \frac{\log x}{\log y} \right) + O_{t,t,\chi,\varepsilon}(1)
\]

from the assumption (4.17). However, the right hand side is \( O_{t,t,\chi}(1) \) from Lemma 4.1 (4.2) for all \( x \geq y \geq 2 \). It is impossible if \( y = x^k \) with a small constant \( \delta = \delta(f, t, \chi, \varepsilon) > 0 \). The claim is proved. Combining the claim with (4.3) from Lemma 4.1, we are done.

5. Spinor genera and the proof of Theorem 1.2

In this section, we investigate the case when every odd prime dividing the conductor of \( \chi \) also divides the level of the modular form. We begin by showing Theorem 1.2 (1).

Proof of Theorem 1.2 (1). As in the proof of Theorem 1.1, the form exhibits sign changes whenever (4.17) holds. Hence for \( \ell \) such that (1.6) does not hold, Lemma 4.1 implies that (4.2) exhibits sign changes.

In the case that (1.6) holds, Lemma 4.1 implies that \( a_{f_1}(p) = 0 \) for every \( p \nmid N \) with \( \chi_D(p) = \varepsilon \). Then (1.4) implies

\[
0 = a_{f_1}(p) = \psi_{t,N}(p)p^{k-1}a_i\left(tp^2\right) + a_i(t).
\]

Thus

\[
a_i\left(tp^2\right) = -\psi_{t,N}(p)p^{1-k}a_i(t).
\]

If \( \psi_{t,N} = \chi_D \), then \( -\psi_{t,N}(p) = -\varepsilon \), so for all \( p \) with \( \chi(p) = \varepsilon \) we have

\[
\text{sgn} \left(a_i\left(tp^2\right)\right) = -\varepsilon \text{sgn} \left(a_i(t)\right).
\]

Thus there are no sign changes in this case.

On the other hand, if \( \psi_{t,N} \neq \chi_D \), then there exist infinitely many \( p \) for which \( \psi_{t,N}(p) = \chi_D(p) = \varepsilon \) and infinitely many \( p \) for which \( \psi_{t,N}(p) = -\chi_D(p) = -\varepsilon \). Hence (1.2) exhibits infinitely many sign changes in this case.

Finally note that (1.6) cannot hold for both \( \varepsilon \) and \( -\varepsilon \) because that would contradict the assumption that \( a_{f_1}(1) = \lambda_{f_1}(1) = a_i(t) \neq 0 \). Moreover, the condition \( \psi_{t,N} = \chi_D \) can occur for at most one choice of \( t \) and since \( D \mid N \) and \( \psi \) is a character modulo \( N \), we have \( t \mid N \).

While Theorem 1.2 (1) should yield many examples where sign changes are not exhibited, it is not entirely obvious that the conditions \( \psi_{t,N} = \chi_D, a_i(t) \neq 0 \), and (1.6) are simultaneously satisfied. We hence construct an explicit example where we can verify all three conditions. The construction goes through the theory of spinor genera of ternary quadratic forms.
Proof of Theorem 1.2 (2). Consider the quadratic forms
\[ Q_1(x, y, z) := x^2 + 48y^2 + 144z^2, \]
\[ Q_2(x, y, z) := 4x^2 + 48y^2 + 49z^2 + 4xz + 48yz, \]
discussed for example in [15, (4.18), p. 9] and [6, §7.3 An example].

The forms \( Q_1 \) and \( Q_2 \) are in the same spinor genus and are moreover representatives for the only two classes in their spinor genus.

The level \( N_{Q_i} \) and the determinant \( D_{Q_i} \) of \( Q_i \) (\( i = 1, 2 \)) is \( 2^63^2 \) and \( 2^83^3 \), respectively. Schulze-Pillot found that \( t = 1 \) is a primitive spinor exception and is represented by \( Q_1 \), but not represented by \( Q_2 \). By the theory of spinor genera, the spinor genus does not primitively represent the integers \( tp^2 \), or in other words, since all representations of \( tp^2 \) come from representations of \( t \),

\[ r(tp^2, Q_j) = r(t, Q_j). \]

Thus in particular

\[ r(tp^2, Q_2) = r(t, Q_2) = 0 \]

for any odd prime \( p \equiv -1 \pmod{3} \), i.e. \( (-3/p) = -1 \).

Plugging in the expansion (2.3) and noting that \( r(t, Q) \) is the \( t \)-th coefficient of \( \theta_Q \), (2.4) implies that

\[ 0 = r(t, Q_2) = a_E(t, Q_2) + a_H(t, Q_2) + a_f(t, Q_2) = r(t, \text{spn}(Q_2)) + a_f(t, Q_2). \]

Therefore, we have

\[ a_f(t, Q_2) = -r(t, \text{spn}(Q_2)) \neq 0. \]

For any inert prime \( p \) in \( \mathbb{Q}(\sqrt{-3}) \), i.e., \( (-3/p) = -1 \), replacing \( t \) by \( tp^2 \), we have \( a_f(tp^2, Q_2) = -r(tp^2, \text{spn}(Q_2)) \) analogously from \( r(tp^2, Q_2) = 0 \). Since \( t \) is represented by \( \text{spn}(Q_2) \), \( tp^2 \) is not primitively represented by \( \text{spn}(Q_2) \). Therefore,

\[ r(tp^2, \text{spn}(Q_2)) - r(t, \text{spn}(Q_2)) = r^*(tp^2, \text{spn}(Q_2)) = 0, \]

where \( r^*(n, \text{spn}(Q)) \) denotes the number of primitive representation of \( n \) by \( Q \). Namely, \( r(tp^2, \text{spn}(Q_2)) = r(t, \text{spn}(Q_2)) \). It follows that

\[ a_f(t, Q_2) = -r(t, \text{spn}(Q_2)) = -r(tp^2, \text{spn}(Q_2)) = a_f(tp^2, Q_2). \]

Hence we see that \( a_f(tp^2, Q_2) \) has the same sign for \( t = 1 \) and any prime \( p \equiv -1 \pmod{3} \). \( \square \)

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Department of Mathematics, University of Hong Kong, Pokfulam, Hong Kong

E-mail address: zilonghe@hku.hk

Department of Mathematics, University of Hong Kong, Pokfulam, Hong Kong

E-mail address: bkane@hku.hk