A CARLITZ–VON STAUDT TYPE THEOREM FOR FINITE RINGS

APOORVA KHARE AND AKAKI TIKARADZE

Abstract. We compute the $k$th power-sum polynomials (for $k \geq 0$) over an arbitrary finite ring $R$, obtained by summing the $k$th powers of $(T + r)$ for $r \in R$. For $R$ non-commutative, this extends the work of Brawley–Carlitz–Levine [Duke Math. J. 41], and resolves a conjecture by Fortuny, Grau, Oller-Marcén, and Rúa (2015). For $R$ commutative, our results bring together two classical programs in the literature: von Staudt–Clausen type results on computing zeta values in finite rings [J. reine angew. Math. 21]; and computing power-sum polynomials over finite fields, which arises out of the work of Carlitz on zeta functions [Duke Math. J. 5, 7]. Our proof in this case crucially uses symmetric function theory. Along the way, we also classify the translation-invariant polynomials over a wide class of finite commutative rings.

1. Introduction and power-sum formulas

Given a finite unital ring $R$, in this work we are interested in computing the “power-sum polynomials"

$$P^R_k(T) := \sum_{r \in R} (T + r)^k.$$ 

When $T = 0$ and $R = \mathbb{Z}/n\mathbb{Z}$, this involves computing the sum of the first $n$ $k$th powers modulo $n$, i.e., the zeta value $\zeta_R(-k)$. Interest in this problem goes back at least as far as the work of Clausen [Cl] and von Staudt [vSt]; clearly, the problem is also connected with Bernoulli numbers and Bernoulli polynomials.

In a series of papers, Carlitz worked on more general variants of the above problem; we include [Ca1, Ca2] as well as the survey article [Ca3]. Carlitz primarily worked in the case where $R$ is a finite field, with the above sum replaced by the sum over all monic polynomials of specified degree $d$, i.e.,

$$P_{F_q, d}^k(T) := \sum_{f \in F_q[T], \deg(f) < d} (T^d + f(T))^k, \quad k \in \mathbb{Z}.$$ 

Summing over $d \geq 0$ yields the zeta function of $F_q[T]$, which has important connections to (rank one) Drinfeld modules and to $q$-expansions of Eisenstein series. The congruence properties of these sums are also linked to class number formulas, leading to Kummer-type criteria for abelian extensions of $F_q(T)$; see e.g. [Gos2]. See also [Ge, Gos1] and the recent survey article [Th] for more on these connections, related topics, and references.

As described in [Th, Theorem 1] and subsequent remarks, if $k \leq 0$ then $P_{F_q, d}^k(T)$ vanishes for $d$ sufficiently large, while the non-vanishing is more subtle. There are also formulae shown in the literature (e.g. in [Ge]) which compute the sum $P_{F_q, d}^k(T)$ for various values of the parameters $(d, k)$. Our goal is to contribute to this area in a more general setting.

Date: December 26, 2017.

2010 Mathematics Subject Classification. Primary: 16P10; Secondary: 11T06, 16K20, 05E05, 13F35, 11T55.

Key words and phrases. Staudt–Clausen, translation-invariant polynomial, power sums, symmetric function, finite ring.

A.T. was partially supported by the University of Toledo Summer Research Awards and Fellowships Program.
Since the time of Carlitz, there has been much work on extending the von Staudt–Clausen–Carlitz theorem to other rings $R$, i.e., computing the zeta value $\zeta_R(-k)$. See e.g. [BCL, FGO, FGOR, GOS, Mor]. For completeness we also remind that zeta values over finite fields, $\sum_{r \in \mathbb{F}_q} r^k = \begin{cases} -1, & \text{if } (q - 1)|k, \\ 0, & \text{otherwise}, \end{cases}$ have modern applications to error-correcting codes such as the (dual) Reed–Solomon code, and more generally the BCH (Bose–RayChaudhuri–Hocquenghem) code, via the discrete Fourier transform. The question of computing $\zeta_R(-k)$ was resolved very recently in [GO] for an arbitrary finite commutative ring.

Over non-commutative rings $R$, few results are known. The study of zeta values in this case originated again in the work of Carlitz and his coauthors. See e.g. [BCL], where the authors showed the following result.

**Theorem 1.1** (Brawley, Carlitz, Levine, [BCL Section 3]). If $R = M_{n \times n}(\mathbb{F}_q)$ for $n \geq 2$, and $k \geq 0$, then

$$
\sum_{r \in R} r^k = \begin{cases} \text{Id}_{2 \times 2}, & \text{if } n = q = 2 \text{ and } 1 < k \equiv -1, 0, 1 \text{ mod } 6; \\ 0, & \text{otherwise}. \end{cases}
$$

Since the aforementioned work, to our knowledge the only rings for which power-sums/zeta-values have been computed, are matrix rings over $\mathbb{Z}/n\mathbb{Z}$ or $\mathbb{F}_q$ for various $n, q$.

In the present work, we begin to combine both of the aforementioned variants of the problem. Our main contribution in this paper is to compute all power sums

$$
P_k^R(T) := \sum_{r \in R} (T + r)^k
$$

over an arbitrary finite ring $R$. In particular, we extend the aforementioned works, and bring closure to the problem of computing zeta values over arbitrary finite rings.

When $R$ is commutative, we approach the problem from the viewpoint of symmetric polynomials. Observe that the polynomials in (1.2) are translation-invariant, as well as power-sum polynomials in the variables $\{(T + r) : r \in R\}$. This suggests connections to the rich and beautiful theory of symmetric functions, which we leverage in order to prove the following

**Theorem A.** Suppose a field $\mathbb{F}_q$ has size $q$ and characteristic $p$. Then for $k \in \mathbb{N}$,

$$
P_k^{\mathbb{F}_q}(T) = -\sum_{\gamma = \lfloor k/(q-1) \rfloor + 1}^{\lfloor k/(q-1) \rfloor} \binom{\gamma - 1}{k - (q - 1)\gamma} (T^q - T)^{k-(q-1)\gamma} \in \mathbb{F}_p[T^q - T].
$$

Notice, the right-hand side is a polynomial in $T^q - T$, which is itself translation-invariant on $\mathbb{F}_q$, and which generates all translation-invariant polynomials in $\mathbb{F}_q[T]$, as shown below in greater generality. Moreover, the coefficients of the polynomial $\sum_{r \in \mathbb{F}_q} (T + r)^k$ lie in $\mathbb{F}_p$ because they are necessarily fixed by every algebra automorphism of $\mathbb{F}_q$, in particular, the Frobenius.

Our next goal is to extend Theorem A from $\mathbb{F}_q$ to an arbitrary finite unital ring $R$. To do so, notice that if $R$ has characteristic $\prod_{i=1}^f p_i^{t_i}$ for pairwise distinct primes $p_i$, with $t_i > 0 \forall i$, then $R = R_1 \times \cdots \times R_f$, where each ring $R_i$ has characteristic $p_i^{t_i}$. Equipped with this notation, it is possible to state the main result of the paper.
Theorem B. Suppose as above that $R = \times_{i=1}^l R_i$ is an arbitrary finite unital ring. Then
\[
P_k^R(T) = \sum_{i=1}^l \frac{|R|}{|R_i|} P_k^{R_i}(T),
\]
where $k \in \mathbb{N}$, and the $i$th summand lives in $R_i[T]$.

Now consider the problem for each factor $R = R_i$, with characteristic equal to a power of a prime $p > 0$. Let $T_{n \times n}(\mathbb{F}_p)$ denote the upper triangular $n \times n$ matrices over $\mathbb{F}_p$, and $x := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Then,
\[
P_k^R(T) = \begin{cases}
P_k^R(T), & \text{if } R = \mathbb{F}_q, \text{ where } q = p^e; \\
p^{e-1}P_k^R(T), & \text{if } R = \mathbb{Z}/p^e\mathbb{Z}, \text{ and } p > 2 \text{ or } p = 2\neq k; \\
2^{m-1}P_k^R(T) + 2^{m-1}P_{k-1}^R(T), & \text{if } R = \mathbb{Z}/2^m\mathbb{Z}, \text{ and } 2 \nmid k; \\
xP_{k-1}^R(T) = x \sum_{1 \leq j \text{ odd }, \frac{k}{j}} T^{k-j}, & \text{if } R = \mathbb{F}_2[x]/(x^2) \text{ and } 2 \nmid k; \\
xP_{k-1}^R(T) = x \sum_{1 \leq j \text{ odd }, \frac{k}{j}} T^{k-j}, & \text{if } R = T_{2 \times 2}(\mathbb{F}_2) \text{ and } 2 \nmid k; \\
\text{Id}_{2 \times 2} \cdot \sum_{1 \leq j \equiv 0, \pm 1 \text{ mod } 6} \binom{k}{j} T^{k-j}, & \text{if } R = M_{2 \times 2}(\mathbb{F}_2); \\
0, & \text{otherwise.}
\end{cases}
\]

Here, $\mathbb{F}_2[x]/(x^2)$ is identified with the span of unipotent matrices in $T_{2 \times 2}(\mathbb{F}_q)$, with $1 \mapsto \text{Id}_{2 \times 2}$.

When $R = \mathbb{Z}/p^m\mathbb{Z}$ in Theorem [B] by $P_k^R(T)$ we mean any lift of this polynomial to $\mathbb{Z}/p^m\mathbb{Z}$; multiplying by $p^{e-1}$ gives independence from this choice of lift. Also note, the penultimate case of upper triangular $2 \times 2$ matrices can be replaced by the appropriate assertion for lower triangular $2 \times 2$ matrices, or for any isomorphic ring $R$.

As an application of our main result, we settle a conjecture in recent [FGOR].

Conjecture 1.6 (Fortuny, Grau, Oller-Marcén, and Rúa, 2015). Let $d > 1$ and $R$ be a finite (commutative) ring. Then all power sums $\sum_{A \in M_{d \times d}(R)} A^k$ vanish unless the following conditions hold:

1. $d = 2$,
2. $|R| \equiv 2 \mod 4$ and $1 < k \equiv 0, \pm 1 \mod 6$,
3. The unique element $e \in R \setminus \{0\}$ such that $2e = 0$ satisfies: $e^2 = e$.

Moreover, in this case the aforementioned power sum equals $e \cdot \text{Id}_{2 \times 2}$.

We recall that the authors stated two conjectures in [FGOR], first for the commutative case and then for the non-commutative case. We prove the latter, stronger version, for unital rings.

Proof. Write the finite unital ring $R = \times_{i=1}^l R_i$ as a product of factor rings of distinct prime power characteristics $p_i^{l_i}$. Then,
\[
M_{d \times d}(R) = \times_{i=1}^l M_{d \times d}(R_i)
\]
is a product of non-commutative rings. By Theorem [B] it follows that some $p_i = 2$ if there is a nonzero power sum. Let $p_1 = 2$ without loss of generality; then proving the conjecture for $R_1$ shows it for $R$, via Equation (1.4). Now note that $R_1$ and $M_{d \times d}(R_1)$ have the same characteristic. Applying Theorem [B] and dimension considerations, $R_1 = \mathbb{F}_2$, and the conditions in the conjecture are easily seen to hold.

Remark 1.7. Before proving the above theorems, we note a small computational error in [BCL, Theorem 5.4] by Carlitz et al. Let $T_{n \times n}(\mathbb{F}_q)$ denote the set of upper triangular matrices in $M_{n \times n}(\mathbb{F}_q)$; then the authors claimed that every power sum over $T_{n \times n}(\mathbb{F}_q)$ is zero, for all $n \geq 2$.
and all $q \geq 2$. In particular, they remarked that for $n = q = 2$, “a direct computation verifies the result” (see the last line of their proof argument in [BCL]). As we state in our main theorem and verify in Proposition 4.7 below, this computation actually shows that all odd-power sums over $T_{2 \times 2}(\mathbb{F}_2)$ are nonzero, except for the sum of elements of $T_{2 \times 2}(\mathbb{F}_2)$. For all other (triangular) matrix rings over finite fields, $R = M_{n \times n}(\mathbb{F}_q)$ or $T_{n \times n}(\mathbb{F}_q)$, the power sums were computed by Carlitz et al in [BCL, Section 3 and Theorem 5.4].

We now turn to the proofs of the above results. Observe that in all commutative cases, the formulas for $\mathcal{P}_k^R(T)$ can be expressed in terms of the “translation-invariant” polynomials $\mathcal{P}_k^R(T)$ from Theorem A. In fact, the proofs of the above theorems require an understanding of translation-invariant polynomials over prime power characteristic rings. This is addressed in the next section, where we classify such polynomials over a large family of finite rings.

2. Translation-invariant polynomials over prime power characteristic rings

The proofs of Theorems A and B given below use several ingredients. One of these is symmetric function theory and is discussed in the next section. Another essential ingredient involves translation-invariant polynomials. Given a commutative ring $R$, recall a polynomial $f \in R[T]$ is said to be translation-invariant on $R$ if $f(T + r) = f(T)$ for all $r \in R$. Now the power-sum polynomials $\mathcal{P}_k^R(T)$ are all translation-invariant over the finite ring $R$. More generally, every symmetric polynomial in the set of variables $\{(T + r) : r \in R\}$ is translation-invariant, a fact crucially used below.

Thus, the goal of the present section is to classify all such polynomials over the three finite commutative rings that feature in Theorem B: $F_2$, $\mathbb{Z}/p^m\mathbb{Z}$, and $\mathbb{F}_2[x]/(x^2)$. In our next result, we combine the three rings into a common framework using Witt vectors [Wi], and classify the translation-invariant polynomials over all such rings, finite or not.

**Theorem 2.1.** Fix integers $e, m > 0$ and a prime power $q = p^e$ with $p > 0$. Denote by $\mathbb{W}_m(\mathbb{F}_q)$ the corresponding Witt ring of vectors of length $m$. Now suppose $R = m \oplus \mathbb{W}_m(\mathbb{F}_q)$ is a commutative unital local ring, where $m$ is a free $\mathbb{W}_m(\mathbb{F}_q)$-module, as well as an ideal of nilpotence class $k$ for some $1 \leq k \leq p$. Then the set of translation-invariant polynomials in $R[T]$ is given by

$$\bigoplus_{n \geq 0} R \cdot (T^n - T)^{np^m} \oplus \bigoplus_{i=0}^{m-1} \bigoplus_{p^m} p^{m-1-i} \Ann_R(p^{m-1}m) \cdot (T^q - T)^{np^i}. \tag{2.2}$$

Observe that the formula in (2.2) does not depend on the nilpotence class $k$ as long as $k \leq p$.

**Theorem 2.1** fits into a broader setting, studied in [SV]: consider a commutative unital ring $R_m$ that is flat over $\mathbb{Z}/p^m\mathbb{Z}$ for some $m > 0$ and a fixed prime integer $p > 0$, and suppose $A_m$ is a flat $R_m$-algebra. (In our setting, $R_m = \mathbb{W}_m(\mathbb{F}_q)$, or more generally $\mathbb{W}_m(\mathbb{F}_q)[x]/(x^k)$ for $k > 0$, and $A_m = R_m[T]$.) The notion of translation-invariance is now replaced by the more general situation of invariance under a set $S$ of $R_m$-algebra automorphisms of $A_m$. Then the treatment in [SV] suggests the following recipe to produce $S$-invariants in $A_m$.

**Lemma 2.3.** Under the above setup, define for $1 \leq j \leq m$ the algebra

$$R_j := R_m \otimes_{\mathbb{Z}/p^m \mathbb{Z}} \mathbb{Z}/p^j \mathbb{Z},$$

and $A_j$ to be the corresponding algebra to $A_m$ but defined over $R_j$. Now if $\overline{S}$ denotes the group of automorphisms induced on the $R_1$-algebra $A_m/pA_m = A_1$, and $a_1 \in A_1^{\overline{S}}$ is fixed by $\overline{S}$, then the elements

$$p^{m-1-i}a_1^i \in A_m, \quad 0 \leq i < m$$

are all fixed by $S$, where $\tilde{a}_1 \in A_m$ denotes any lift of $a_1 \in A_1$. 


Lemma 2.3 provides a recipe to produce invariant elements in $A_m$ using $S$-invariants of $A_t$. A more challenging question in this general setting and in specific examples involves understanding if this recipe generates all $S$-invariants in $A_m$. Therefore, Theorem 2.4 shows that this is indeed the case for the family $R_m = \mathbb{W}_m(\mathbb{F}_q)$, including the finite rings $\mathbb{F}_q$ and $\mathbb{Z}/p^m\mathbb{Z}$ that feature in Theorem 2.3 above.

We now turn to Theorem 2.1. Before showing the result, we present an alternate formulation in terms of the coefficients $c_n \in R$ for which $c_n(T^q - T)^n$ is translation-invariant. First define for each $n > 0$ the integer $v_p(n)$ to be the $p$-adic valuation, and also,

$$n_\downarrow := \lfloor p^{m-1-v_p(n)} \rfloor p^{v_p(n)} = \begin{cases} 
0, & \text{if } p^n \mid n, \\
 p^{m-1}, & \text{otherwise}.
\end{cases}$$

Now the set of translation-invariant polynomials is (claimed to be) precisely the space

$$R \oplus \bigoplus_{n > 0} [p^{m-1-v_p(n)}] \cdot \text{Ann}_R(n_\downarrow \cdot m) \cdot (T^q - T)^n. \quad (2.4)$$

Proof of Theorem 2.1. We begin with the following observation:

Suppose $R$ is a commutative unital ring with prime ideal generated by a prime integer $p > 0$. Now suppose the following equation holds in $R$: $f_1 = f_0 + p^jX$, with $j > 0$. Fix $n > 0$ such that $p^2 \nmid n$. Then there exists $X' \in R$ such that

$$f_1^n = \begin{cases} 
f_0^n + p^jX', & \text{if } p \nmid n, \\
f_0^n + p^{j+1}X', & \text{if } p \mid n, \ p^2 \nmid n.
\end{cases} \quad (2.5)$$

If moreover $p \nmid X$ and $p > 2$, then we can choose $p \nmid X'$.

To see why (2.5) holds, first suppose $p \nmid n$. Then,

$$f_1^n = (f_0 + p^jX)^n = f_0^n + p^j(nX) + p^{2j}R = f_0^n + p^j(nX + p^jR).$$

Since $n \not\in (p)$, the assertion follows, including if $X \not\in (p)$ and $p > 2$. Next if $p \mid n$ but $p^2 \nmid n$, then write $n = pn'$ with $p \mid n'$. Now compute:

$$f_1^{pn'} = (f_0 + p^jX)^{pn'} = f_0^{pn'} + p^{j+1}(n'X) + p^{2j+1}(n'(pn' - 1)X^2/2) + p^{3j}R$$

$$\subset f_0^{pn'} + p^{j+1}(n'X + (p^j/2)R),$$

and a similar reasoning as in the previous case concludes the proof of (2.5).

Now we continue with the proof, breaking it up into steps for ease of exposition.

Step 1: We first show that the claimed polynomials in (2.2) are indeed translation-invariant. This is itself shown in a series of sub-steps. First we consider the case $m = 1$, i.e., $R = \mathbb{W}_m(\mathbb{F}_q) = \mathbb{F}_q$. Here the polynomial $T^q - T$ is translation-invariant (by repeatedly using the Frobenius map $Frob_q : r \mapsto r^q$), whence the assertion follows for $R = \mathbb{F}_q$. Now since $(T^q - T)^n$ is translation-invariant over $\mathbb{F}_q$ for $n \geq 0$, Lemma 2.3 shows the polynomials in (2.2) are translation-invariant over $R_m = \mathbb{W}_m(\mathbb{F}_q)$, where $m = 0$ and $J = R_m$.

We next show the assertion for $R = \mathbb{F}_m \oplus \mathbb{W}_m(\mathbb{F}_q)$. Begin by defining and studying

$$f_{i,n}(T) := (T^q - T)^{np^i}, \quad i, n \geq 0. \quad (2.6)$$

Repeatedly using Frobenius shows $(T + r)^n \equiv T^q + r^q \mod (p)$ in $R[T]$, for any $r \in R$. Thus,

$$f_{0,1}(T + r) - f_{0,1}(T) \equiv (T^q + r^q) - (T + r) - (T^q - T) \equiv r^q - r \mod pT. \quad (2.7)$$
Now to show the assertion, it suffices to show the invariance of \((2.2)\) under translation by any \(r \in m\). Start with \((2.7)\), noting by assumption on \(m\) that \(r^q = 0\). Thus \(f_{0,1}(T + r) = (T + r)^q - (T + r) \equiv T^q - T - r \mod p\). Applying \((2.5)\) for \(p \nmid n\),
\[
f_{0,n}(T + r) = f_{0,1}(T + r)^n \equiv (T^q - T - r)^n \mod p,
\]
whence repeated applications of \((2.5)\) now yield:
\[
f_{i,n}(T + r) \equiv (T^q - T - r)^{np^i} \mod p^{i+1}. \tag{2.8}
\]

There are now two sub-cases. If \(i \geq m\) and \(n > 0\), then by the binomial theorem,
\[
f_{m,n}(T + r) = f_{m-1,n}(T + r)^p = ((T^q - T - r)^{np^{m-1}})^p
\]
\[
= (T^q - T)^{np^m} + \sum_{l=1}^{p-1} \binom{np^m}{l} (T^q - T)^{np^m-l} (-r)^l
\]
\[
\in f_{m,n}(T) + p^m \cdot R[T] = f_{m,n}(T),
\]
where the summation stops at most by \(l = p - 1\) since \(r^p \in m^p = 0\). This calculation shows that \((T^q - T)^{np^m}\) is translation-invariant. Next, suppose \(0 < i < m - 1\), and \(r' \in \text{Ann}_R(p^{m-1}m)\). Given \(r \in m\), compute as above, working modulo \(p^{m-1-i} \cdot p^{i+1} = p^m = 0\):
\[
p^{m-1-i} r' f_{i,n}(T + r) = p^{m-1-i} r' (T^q - T)^{np^i}
\]
\[
= p^{m-1-i} r' (T^q - T)^{np^i} + p^{m-1-i} r' \sum_{l=1}^{p-1} \binom{np^i}{l} (T^q - T)^{np^i-l} (-r)^l
\]
\[
\in p^{m-1-i} r' f_{i,n}(T) + r' p^{m-1-i} \cdot p^i \cdot m[T].
\]
By choice of \(r'\), it follows \(r' p^{m-1-i} f_{i,n}(T)\) is indeed translation-invariant on \(R\), as desired.

**Step 2:** The remaining steps will prove the reverse inclusion, which is more involved. In this step, we show that for all \(m > 0\) and prime powers \(q\), every translation-invariant polynomial \(f(T)\) on \(R = m \oplus \mathbb{W}_m(F_q)\) is a polynomial in \(T^q - T\).

To see why, first use the \(\mathbb{W}_m(F_q)\)-freeness of \(m\) to write \(f(T) = \sum_j r_j g_j(T)\), where the \(r_j\) comprise a \(\mathbb{W}_m(F_q)\)-basis of \(R\), and \(g_j \in W_m(F_q)[T] \forall j\). Then each \(g_j\) is translation-invariant on \(\mathbb{W}_m(F_q)\), and it suffices to show that \(g_j \in \mathbb{W}_m(F_q)[T^q - T]\), for each \(j\). We may replace \(g_j\) by \(g_j(T) - g_j(0)\), whence \(T g_j(T)\).

Let \(\omega_m : F_q \to R = \mathbb{W}_m(F_q)\) denote the Teichmuller character, which restricts to an injective group morphism : \(F_q^\times \to R^\times = R \setminus \{0\}\). From this it follows that \(\prod_{a \in F_q^\times} (T - \omega_m(a)) = T^q - 1\). Moreover, the linear polynomials \(T - \omega_m(a) : a \in F_q\) are pairwise coprime.

Returning to the proof, since \(T g_j(T)\), it follows that \(T - \omega_m(a)\) also divides \(g_j(T)\) for \(a \in F_q\), whence \(T^q - T = \prod_{a \in F_q} (T - \omega_m(a))\) divides \(g_j(T)\). From this it follows by induction on \(\text{deg}(g_j)\) that each \(g_j\), whence \(f\), is a polynomial in \(T^q - T\).

**Step 3:** By the two previous steps, any \(f \in R[T]\) which is translation-invariant on \(R\), is a polynomial in \(T^q - T\); moreover, we may subtract all terms of the form \(c(T^q - T)^{np^m}\) as all such terms are translation-invariant. Thus, assume henceforth that
\[
f(T) \in \bigoplus_{i=0}^{m-1} \bigoplus_{p \nmid n} R \cdot (T^q - T)^{np^i}. \tag{2.9}
\]

In this step we complete the proof for \(m = 1\) and all \(p, q\). Notice that it suffices to show translation-invariance by \(m\). Let \(f(T) = \sum_{j=0}^n r_j T^j\), with \(n = \text{deg}(f)\). As \(p \nmid n\), use the
Frobenius repeatedly to compute for \( r \in m \):

\[
f(T + r) - f(T) = \sum_{j=0}^{n} r_j \left[ ((T + r)^q - (T + r)^j) - (T^q - T)^j \right]
\]

so the highest degree term in \( T \) comes from the leading term:

\[-r_n \cdot n \cdot (T^q - T)^{n-1} \cdot r, \quad r \in m.
\]

This expression must vanish; as \( n \) is invertible, \( r_n \in \text{Ann}_R(m) \). Subtracting the translation-invariant polynomial \( r_n(T^q - T)^n \) from \( f(T) \), we are done by induction on \( \deg(f) = n \).

**Step 4:** We next claim that if a translation-invariant polynomial \( f \) is as in \((2.9)\), then \( f(T) \) in fact lies in the second direct sum in \((2.2)\). We will show the claim first for \( R = \mathbb{W}_m(\mathbb{F}_p) = \mathbb{Z}/p^m\mathbb{Z} \), then for \( R = \mathbb{W}_m(\mathbb{F}_q) \), and finally (in the next step) for \( R = m \oplus \mathbb{W}_m(\mathbb{F}_q) \), to conclude the proof of the theorem.

Begin by assuming \( R = \mathbb{Z}/p^m\mathbb{Z} = \mathbb{W}_m(\mathbb{F}_p) \). Suppose \( f(T) = aT^n + \cdots \), where \( n = \deg(f) \) is positive and divisible by \( q = p \); and \( p^n \mid a \). Since \( f(T + 1) \equiv f(T) \), it follows that \( an = 0 \mod p^m \). Suppose \( n = n'p^{i+1} \) and \( a = xp^{m-1-i} \) for some \( 0 \leq i < m \) and \( n' > 0 \), and \( 0 \neq x \in \mathbb{Z}/p^m\mathbb{Z} \). Then \( f(T) - xp^{m-1-i}f_{i,n'}(T) \) is a translation-invariant polynomial of strictly smaller degree. Therefore the claim for \( \mathbb{W}_m(\mathbb{F}_p) \) follows by induction on \( \deg(f) \).

Next, suppose \( R = \mathbb{W}_m(\mathbb{F}_q) \), and \( f \) as in \((2.9)\) is translation-invariant on \( R \). Use the \( \mathbb{Z}/p^m\mathbb{Z} \)-freeness of \( \mathbb{W}_m(\mathbb{F}_q) \) to write \( f(T) = \sum_j r_j g_j(T) \), where the \( r_j \) comprise a \((\mathbb{Z}/p^m\mathbb{Z})\)-basis of \( R \), and \( g_j \in (\mathbb{Z}/p^m\mathbb{Z})[T] \forall j \). Then \( g_j \) is translation-invariant on \( \mathbb{Z}/p^m\mathbb{Z} \) for all \( j \). Now use the above analysis in this step for \( R = \mathbb{Z}/p^m\mathbb{Z} \) to write:

\[
f(T) = \sum_{i=0}^{m-2} \sum_{j=0}^{l_i} c_{ij}p^{m-1-i}(T^p - T)^{n_{ij}p^i} + \sum_{j=1}^{l_{m-1}} c_{m-1,j}(T^p - T)^{n_{m-1,j}p^{m-1}},
\]

where \( p \nmid n_{ij} \) if \( i < m - 1 \). We now show that \( f \) is of the desired form \((2.9)\) by induction on \( m \). The base case of \( m = 1 \) was proved in Step 2 above; moreover if \( p \) divides \( c_{m-1,j} \) for all \( j \) then \( p^{-1}f(T) \) makes sense and is translation-invariant in \( \mathbb{W}_{m-1}(\mathbb{F}_q) \). But then the result follows by the induction hypothesis.

Thus, assume without loss of generality that \( m > 1 \) and \( p \nmid f(T) \). Now consider \( f(T) \mod p \), which is a polynomial over \( \mathbb{F}_q = \mathbb{W}_1(\mathbb{F}_q) \) that is translation-invariant. Note that

\[
f(T) \mod p \equiv \sum_{j=1}^{l_{m-1}} c_{m-1,j}(T^p - T)^{n_{m-1,j}p^{m-1}}
\]

\[
\equiv \text{Frob}^{-1}(f_{1}(T)), \quad f_{1}(T) := \sum_{j=1}^{l_{m-1}} \text{Frob}^{-1}(c_{m-1,j})(T^p - T)^{n_{m-1,j}}.
\]

It follows that \( f_{1}(T) \) is translation-invariant on \( \mathbb{F}_q \). Now since \( m = 1 \) for \( f_1 \), it follows by Step 2 that \( 0 \neq f_{1}(T) \in \mathbb{F}_q[T^q - T] \), say \( f_{1}(T) = \sum_j a_j(T^q - T)^j \). Then,

\[
f(T) \mod p = \text{Frob}^{-1}(f_{1}(T)) = \sum_{j} a_j^{p^{m-1}}(T^q - T)^{jp^{m-1}}.
\]
Fixing lifts $b_j \in \mathbb{W}_m(\mathbb{F}_q)$ of $a_j^{m^{-1}} \in \mathbb{F}_q$ for all $j$, define

$$h(T) := \sum_j b_j(T^q - T)^{jp^{m^{-1}}} \in \mathbb{W}_m(\mathbb{F}_q)[T].$$

By Step 1, $h(T)$ is translation-invariant on $\mathbb{W}_m(\mathbb{F}_q)[T]$, and $f - h$ is a translation-invariant polynomial divisible by $p$. As mentioned above, now divide by $p$ and work in $\mathbb{W}_{m-1}(\mathbb{F}_q)[T]$ by induction on $m$, to conclude the proof in the case $R = \mathbb{W}_m(\mathbb{F}_q)$.

**Step 5:** The final step is to show the result for $R = \mathfrak{m} \oplus \mathbb{W}_m(\mathbb{F}_q)$, with $f(T)$ as in (2.3) being translation-invariant over $R$. Use a $\mathbb{W}_m(\mathbb{F}_q)$-basis of $R$, say $\{r_j\}$, to write $f(T) = \sum_j r_jg_j(T)$; then each $g_j$ is translation-invariant over $\mathbb{W}_m(\mathbb{F}_q)$. Thus by Step 4,

$$f(T) = \sum_{n>0} c_{nm}(T^q - T)^{np^m} + \sum_{i=0}^{m-1} \sum_{p|m} c_{ni}p^{m-1-i}(T^q - T)^{np^i} + c_0,$$

with all $c_{ni} \in R$. As above, assume that $c_{nm} = 0 \forall n > 0$, using Step 1. We show that $c_{ni} \in \text{Ann}_R(p^{m-1}\mathfrak{m})$ by induction on $\deg f$. The base case of $f(T) = c_0$ is obvious. Now given $f(T) = g(T^q - T)$ as above, notice by the uniqueness of the representation of $f$ in the preceding equation, the leading degree term of $f$ comes from a unique summand, say corresponding to $(n, i)$ with $i < m$. Expanding in powers of $T^q - T$ via computations as above, the identically zero polynomial

$$f(T + r) - f(T) = g((T + r)^q - (T + r)) - g(T^q - T) \equiv g(T^q - T - r) - g(T^q - T) \mod p$$

has leading term

$$p^{m-1-i}c_{ni}\left(\frac{np^i}{1}\right)(T^q - T)^{np^i-1}(-r) = p^{m-1}\cdot (-n)rc_{ni}(T^q - T)^{np^i-1} \mod p^m.$$

This term must vanish for all $r \in \mathfrak{m}$. As $n \in (\mathbb{Z}/p^m\mathbb{Z})^\times$, this shows $c_{ni} \in \text{Ann}_R(p^{m-1}\mathfrak{m})$. Subtracting $c_{ni}p^{m-1-i}(T^q - T)^{np^i}$ from $f(T)$, the result holds by induction on $\deg(f)$. □

We conclude our discussion of translation-invariant polynomials by observing that Theorem 2.11 helps classify all such polynomials over a large class of finite commutative rings. Indeed, given a finite commutative ring $R$, write $R = \times_j R_j$ as a product of local rings $R_j$, each of which has prime power characteristic $p_j^{m_j}$, as discussed prior to Theorem 2.12. Now if each $R_j$ is of the form in Theorem 2.11 then the translation-invariant polynomials in $R[T]$ can be classified using the following result.

**Lemma 2.10.** Given a commutative ring $R = \times_j R_j$, write $f \in R[T]$ as $\sum_j f_j$, with $f_j \in R_j[T]$. Then $f$ is translation-invariant over $R$ if and only if each $f_j$ is translation-invariant over $R_j$.

**Proof.** Writing $f = \sum_j f_j$ uniquely as above, it follows for $i \neq j$ and $r_j \in R_j$ that

$$f_i(T + r_j) - f_i(T) \in r_jR_i[T] = 0.$$

This implies the result, since

$$f\left(T + \sum_j r_j\right) - f(T) = \sum_j (f_j(T + r_j) - f_j(T)).$$

□
3. Power-sum polynomials over finite fields

With the above classification in Theorem [2.1] in hand, we now prove our first main result, Theorem [A] using the theory of symmetric functions. Suppose $R$ is any commutative unital ring. Fix $n \in \mathbb{N}$, and variables $x_1, \ldots, x_n$. A polynomial $p \in R[x_1, \ldots, x_n]$ is symmetric if $p$ is invariant under any permutation of the variables $(x_1, \ldots, x_n)$. Distinguished families of symmetric polynomials include the power sum symmetric polynomials

$$p_k := \sum_{1 \leq i \leq n} x_i^k, \quad k \geq 0$$

as well as the elementary symmetric polynomials

$$\sigma_k := \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k} \quad (1 \leq k \leq n).$$

The polynomials $\sigma_k$ appear when expanding a linear factorization of a monic polynomial:

$$\prod_{i=1}^n (T - x_i) = T^n - \sigma_1 T^{n-1} + \sigma_2 T^{n-2} + \cdots + (-1)^n \sigma_n. \quad (3.1)$$

The following well-known result relates the families of symmetric polynomials $p_k$ and $\sigma_k$, and can be found in numerous sources.

**Theorem 3.2** (Waring’s Formula; see e.g. [Gou]). Given $R[x_1, \ldots, x_n]$, for all $k > 0$,

$$p_k = \sum (-1)^{i_2 + i_3 + \cdots + i_n - 1} k! \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_n} \in \mathbb{Z}[\sigma_1, \sigma_2, \ldots, \sigma_n],$$

where the sum is over all $(i_1, \ldots, i_n) \in \mathbb{Z}^n$ such that all $i_j \geq 0$ and $\sum_j j i_j = k$.

Now we use the classification in Theorem [2.1] to compute a class of elementary symmetric polynomials over $\mathbb{F}_q$, which will be crucial in proving Theorem [A].

**Proposition 3.3.** Given $1 \leq k \leq q$, define

$$\mathcal{F}_k := \{ S \subseteq \mathbb{F}_q : |S| = k \}, \quad \Sigma_k(T) := \sum_{s \in \mathcal{F}_k} \prod_{r \in s} (T + r), \quad \Sigma_{-k}(T) := \sum_{s \in \mathcal{F}_k} \prod_{r \in s} (T + r)^{-1}.$$

Then for $1 \leq |k| \leq q$,

$$\Sigma_k(T) = \begin{cases} 1, & \text{if } k = q - 1; \\ T^q - T, & \text{if } k = q; \\ -1/(T^q - T) = S_{-1}(T), & \text{if } k = -1; \\ 1/(T^q - T), & \text{if } k = -q; \\ 0, & \text{otherwise.} \end{cases} \quad (3.4)$$

**Proof.** Since each $\Sigma_k(T)$ is also translation-invariant, by Theorem [2.1] it is a polynomial in $T^q - T$, with $T$-degree at most $k$. Thus for $0 \leq k < q$, $\Sigma_k(T)$ is a constant, hence equals $\Sigma_k(0)$. Now compute using Equation (3.1):

$$T^q - T = \prod_{r \in \mathbb{F}_q} (T - r) = T^q - T^{q-1} \Sigma_1(0) + T^{q-2} \Sigma_2(0) - \cdots + (-1)^q \Sigma_q(0).$$

More precisely, replace $n$ by $q$ and apply the homomorphism $:\mathbb{F}_q[\{x_r : r \in \mathbb{F}_q\}] \to \mathbb{F}_q$, sending $x_r \mapsto r$, to Equation (3.1). Thus, $\Sigma_1(0) = \cdots = \Sigma_{q-2}(0) = 0$, and

$$(-1)^{q-1} \Sigma_{q-1}(0) = -1.$$
If $q$ is even, then $p = 2$, and $(-1)^{q-1} = \pm 1 = 1$ in $\mathbb{F}_q$. Otherwise, $q$ is odd, and $(-1)^{q-1} = 1$. In either case, we get: $\Sigma_{q-1}(0) = -1$.

Next if $k = q$, then $\mathcal{F}_k = \{\mathbb{F}_q\}$, so $\Sigma_q(T) = \prod_{r \in \mathbb{F}_q}(T - r) = T^q - T$. This also shows that $\Sigma_1(T) = 1/(T^q - T)$. Finally, if $0 < k < q$, then taking common denominators yields: $\Sigma_{-k}(T) = \Sigma_{q-k}(T)/\Sigma_q(T)$. This concludes the proof. \hfill \Box

Using the above results, we prove the first main theorem in the paper.

**Proof of Theorem A** We begin by showing the following equation:

$$
P_k^{\mathbb{F}_q}(T) = \sum_{0 \leq \alpha, \beta \in \mathbb{Z}_{(q-1)\alpha + q\beta = k}} \frac{(\alpha + \beta - 1)!k}{\alpha!\beta!} (T^q - T)^{\beta} \in \mathbb{F}_p[T]. \quad (3.5)
$$

To see (3.5), apply Waring’s Formula (from Theorem 3.2) and Proposition 3.3. Set $R = \mathbb{F}_q$ and consider the $\mathbb{F}_p$-algebra homomorphism $\mathbb{F}_p[x_r : r \in \mathbb{F}_q] \to \mathbb{F}_q[T]$, sending $x_r \mapsto T + r$. This sends $\sigma_k \mapsto \Sigma_k(T)$ for all $1 \leq k \leq q$, and $p_k \mapsto \mathcal{P}_k^{\mathbb{F}_q}(T) \forall k > 0$. Hence $\mathcal{P}_k^{\mathbb{F}_q}(T)$ is computable using Waring’s Formula. Notice that every summand containing a positive power of $\sigma_1, \sigma_2, \ldots$, or $\sigma_{q-2}$ vanishes by Proposition 3.3. Now changing indices from $i_{q-1}, i_q$ to $\alpha, \beta$ respectively,

$$
P_k^{\mathbb{F}_q}(T) = \sum_{0 \leq \alpha, \beta \in \mathbb{Z}_{\alpha \gamma + \beta \alpha = 0}} (-1)^{\epsilon} \frac{(\alpha + \beta - 1)!k}{\alpha!\beta!} \Sigma_{-\alpha}(T) \Sigma_\alpha(T)^{\beta},
$$

with two cases. If $q$ is odd, then $\epsilon = \alpha$, and the summand equals

$$
\frac{(\alpha + \beta - 1)!k}{\alpha!\beta!} \cdot (\alpha - 1)^{\alpha} \Sigma_{-\alpha}(T)^{\alpha} = \frac{(\alpha + \beta - 1)!k}{\alpha!\beta!} \cdot (\alpha - 1)^{\alpha} = \frac{(\alpha + \beta - 1)!k}{\alpha!\beta!},
$$

which proves (3.5). If instead $q$ is even, then $\epsilon = \beta$, whence (3.5) again follows:

$$
\frac{(\alpha + \beta - 1)!k}{\alpha!\beta!} \cdot (\beta + 1)^{\beta} \Sigma_{-\beta}(T)^{\beta} = \frac{(\alpha + \beta - 1)!k}{\alpha!\beta!} \cdot (\beta + 1)^{\beta} = \pm 1 = \frac{(\alpha + \beta - 1)!k}{\alpha!\beta!}.
$$

Having shown (3.5), we now derive (1.3) from it. If $\alpha = 0$ then the corresponding coefficient of $(T^q - T)^{\beta}$ is $\frac{(\beta - 1)!^q}{\beta!} = q = 0$ in $\mathbb{F}_q$. Thus we may assume $\alpha > 0$. Now change variables from $\beta$ to $\gamma = \alpha + \beta$. Then Equation (3.5) yields:

$$
P_k^{\mathbb{F}_q}(T) = \sum_{0 < \alpha \leq \gamma : q\gamma - \alpha = k} \frac{(\gamma - 1)!k}{\alpha!(\gamma - \alpha)!} (T^q - T)^{\gamma - \alpha}.
$$

Since $\alpha > 0$, and $\beta = \gamma - \alpha = k - (q - 1)\gamma$, it follows that

$$
\frac{(\gamma - 1)!k}{\alpha!(\gamma - \alpha)!} = \frac{(\gamma - 1)!q\gamma}{\alpha!(\gamma - \alpha)!} - \frac{(\gamma - 1)!k}{\alpha!(\gamma - \alpha)!} = -\left(\frac{\gamma - 1}{k - (q - 1)\gamma}\right).
$$

Moreover, the summation over $0 < \alpha \leq \gamma$, where $\alpha = q\gamma - k$, translates to

$$
[k/q] < \gamma \leq [k/(q - 1)].
$$

Now the result (1.3) follows by putting together the above arguments. Finally, note that the coefficients are in $\mathbb{Z}/p\mathbb{Z}$, because Waring’s Formula has integer coefficients. \hfill \Box
Remark 3.6. Using Proposition 3.3 and Waring’s formula, one shows similarly that
\[ P^R_k(T) = \sum_{0 \leq \alpha, \beta \in \mathbb{Z} : \alpha + q\beta = k} \frac{(\alpha + \beta - 1)!k}{\alpha!\beta!} \frac{(-1)^\alpha}{(T^q - T)^{\alpha + \beta}} \]
\[
= \sum_{\beta = 0}^{\lceil k/q \rceil - 1} \left( k - (q - 1)\beta - 1 \right) \frac{(-1)^{k - q\beta}}{(T^q - T)^{k - (q - 1)\beta}},
\]
where \( \alpha = k - q\beta \), and \( k > 0 \). Note, the translation-invariant polynomial (in the variables \( 1/(T + r) \) for \( r \in \mathbb{F}_q \)) lies in \( \mathbb{F}_p[(T^q - T)^{-1}] = \mathbb{F}_p[\Sigma_q(T)] \).

4. Power sums over arbitrary finite rings

The entirety of this final section is devoted to proving Theorem B. The first reduction (1.4) to rings of prime power characteristic is obtained via the following straightforward computation, which is also used below. In it, we consider the more general case of a direct product of rings \( R = \times_i R_i \). Writing an element \( r \in R \) uniquely as \( r_1 + \cdots + r_l \), with \( r_i \in R_i \), we obtain:

\[ P^R_k(T) = \sum_{r \in R} (T + r_1 + \cdots + r_l)^k = \sum_{r \in R} \left( T^k + \sum_{j=1}^k \binom{k}{j} T^{k-j} \sum_{i=1}^l r_i^j \right) \]
\[ = (1 - l)|R|T^k + \sum_{i=1}^l \sum_{r_i \in R_i} \sum_{j=0}^k \binom{k}{j} T^{k-j} r_i^j \]
\[ = \sum_{i=1}^l \frac{|R|}{|R_i|} P^R_{k_i}(T). \]  

(4.1)

Now Theorem B is proved in a series of steps. The first few steps isolate the rings for which there exist non-vanishing power sums, and for ease of exposition we present these steps as individual propositions. First if \( R = M_{2 \times 2}(\mathbb{F}_2) \), then the result follows from Theorem 1.1 via the equation:

\[ P^R_k(T) = \sum_{r \in R} (T + r)^k = |R|T^k + \sum_{j=1}^k \binom{k}{j} T^{k-j} \sum_{r \in R} r^j. \]  

(4.2)

In particular, if \( \sum_{r \in R} r^k = 0 \) for all \( k > 0 \), then \( P^R_k(T) = |R|T^k = 0 \).

We begin by addressing the case of finite rings with odd prime power characteristic.

Proposition 4.3. Suppose \( R \) is a finite ring of characteristic \( p^m \) for some \( m \geq 1 \), with \( p \) odd as above. Also suppose \( R \) is neither a cyclic group (as an additive group) nor a field. Then all power sums over \( R \) vanish.

The following technical lemma will be useful in the proof (and beyond).

Lemma 4.4. Suppose \( R \) has prime characteristic \( p > 2 \), and a nonzero two-sided ideal \( I \) with \( I^2 = 0 \). Then \( P^R_k(T) = 0 \), i.e., the power sums \( \sum_{r \in R} r^k \) vanish for all \( k \geq 0 \). The same is true if \( p = 2 \) and \( \dim_{\mathbb{F}_2}(I) \geq 2 \).

In proving Proposition 4.3 at least when the characteristic is \( p \), the idea is to apply the lemma for \( I \) a suitable power of the Jacobson radical of \( R \).
Proof. Choose and fix coset representatives \( \{ c \in C \} \) of the subspace \( I \subset R \). Writing every element \( r \in R \) uniquely as \( c + y \), for \( c \in C \) and \( y \in I \), and rearranging sums, we have

\[
\sum_{r \in R} r^k = \sum_{c \in C} \sum_{y \in I} (c + y)^k = \sum_{c \in C} |I| c^k + \sum_{c \in C} \sum_{y \in I} \sum_{j=0}^{k-1} c^j y c^{k-1-j} = \sum_{c \in C} \sum_{j=0}^{k-1} \left( \sum_{y \in I} y \right) c^{k-1-j},
\]

and the innermost sum vanishes if \( p \) is odd, or if \( p = 2 \) and \( \dim_{\mathbb{F}_2}(I) \geq 2 \). \( \square \)

Proof of Proposition 4.3. The proof is by induction on \( m \); we will consider the case \( m = 1, 2 \) separately from higher \( m \). Also assume without loss of generality that \( R \) does not have characteristic \( p^{m-1} \).

First suppose \( m = 1 \). If \( R \) has nonzero Jacobson radical \( J(R) \), say of nilpotence degree \( d > 1 \), then we are done by Lemma 4.4 for \( I := J(R)^{d-1} \). Otherwise \( J(R) = 0 \), whence \( R \) is isomorphic to a direct product of matrix rings:

\[
R = \times_{j=1}^l M_{d_1 \times d_1}(D_j),
\]

for finite division rings \( D_j \) of characteristic \( p \). By Wedderburn’s Little Theorem, each \( D_j \) is a field \( \mathbb{F}_{p^j} \), say. Now use Equation (4.1) to show every power sum is zero if there is more than one factor. Finally, if \( R = M_{d \times d}(\mathbb{F}_{p^1}) \) with \( d > 1 \), then we are done by Theorem 1.1.

Now suppose \( R \) has characteristic \( p^m \), with \( m > 1 \), and the result holds for \( m - 1 \). Without loss of generality, we may assume \( p^{m-1} R \neq 0 \). Define \( I := p^{m-1} R \); then \( I^2 = 0 \) since \( m > 1 \). Since \( p I = 0 \), note as in Lemma 4.4 that \( \sum_{y \in I} y = 0 \). Also choose a set \( C \) of coset representatives of the subgroup \( I \) in \( R \); then as in the proof of Lemma 4.4

\[
\sum_{r \in R} r^k = \sum_{c \in C} \sum_{y \in I} (c + y)^k = |I| \sum_{c \in C} c^k + \sum_{c \in C} \sum_{y \in I} \sum_{j=0}^{k-1} c^j \left( \sum_{y \in I} y \right) c^{k-1-j} = \sum_{c \in C} \sum_{j=0}^{k-1} \left( \sum_{y \in I} y \right) c^{k-1-j}.
\]

For future use, note the first two equalities also hold in rings of characteristic \( 2^m \) for \( m > 1 \).

There are now two cases. First if \( m = 2 \), then \( R/(p) \) is a module over \( \mathbb{F}_p \). By the above analysis in the \( m = 1 \) case, \( R/(p) \) can have nonzero power sums only when \( R/(p) = \mathbb{F}_q \). If \( q = p \) then \( R/(p) \) is cyclic, whence so is \( R \) by Nakayama’s Lemma for \( \mathbb{Z}/p^2 \mathbb{Z} \), but this is impossible by assumption. Hence \( p^2 | q \). Now \( I = (p) \) is a module over \( R/(p) = \mathbb{F}_q \), so \( p^2 \) divides \( |I| \), and therefore the power sum in (4.5) vanishes as desired.

Finally, if \( m > 2 \), then it suffices to show, via (lifting using) the quotient map \( R \to R/(p^{m-1}) \), that the power sum \( \sum_{c \in R/(p^{m-1})} c^k \) vanishes. But since \( R/(p^{m-1}) \) has characteristic \( p^{m-1} \), by the induction hypothesis it can only have nonzero power sums if it is cyclic. In this case, Nakayama’s Lemma implies \( R \) is also cyclic, which is impossible.

The next case to consider is if \( R \) has characteristic \( p = 2 \). In what follows, define a ring to be indecomposable if it is not the direct product of two factor rings.

Proposition 4.6. Suppose \( R \) is a finite ring of characteristic \( 2 \). Then \( R \) has nonzero power sums only if \( R \) is one of \( \mathbb{F}_2, \mathbb{M}_{2 \times 2}(\mathbb{F}_2), \mathbb{F}_2[x]/(x^2), \) or upper triangular \( 2 \times 2 \) matrices over \( \mathbb{F}_2 \).

Proof. At the outset we note that \( R \) is indecomposable, for if \( R = R_1 \times R_2 \), both of even order, then \( P^k_R(R) = 0 \) by (1.1), whence all power sums are zero.

First suppose \( J(R) = 0 \). Then \( R = \times_{j=1}^l M_{d_1 \times d_2}(\mathbb{F}_{2^j}) \), as in the proof of Proposition 4.3. Again by Equation (1.1), every power sum is zero if there is more than one factor. Now by Theorem 1.1 the only cases when \( R \) has nonzero power sums is if \( R \) is a field or \( M_{2 \times 2}(\mathbb{F}_2) \).

Next, suppose \( J(R) \neq 0 \). We propose an argument similar to Lemma 4.4. First suppose \( \dim_{\mathbb{F}_2} J(R) \geq 2 \). If \( J(R) \) has nilpotence degree \( d \geq 2 \), define \( I := J(R)^{d-1} \); then either
dim_{\mathbb{F}_2} \mathcal{I} \geq 2$, in which case we are done by Lemma 4.4 or $\mathcal{I}$ is one-dimensional and $\mathcal{I} \cdot J(R) = J(R) \cdot \mathcal{I} = 0$. We show in this latter case that all power sums vanish on $R$.

Let $\mathcal{C}$ (respectively $\mathcal{B}$) denote any fixed choice of coset representatives of $J(R)$ in $R$ (respectively of $\mathcal{I}$ in $J(R)$). Now compute as in (14.5), using that $\mathcal{I}^2 = 0$ and $b\mathcal{I} = \mathcal{I}b = 0$ for $b \in \mathcal{B}$:

$$\sum_{r \in R} r^k = \sum_{c \in \mathcal{C}} \sum_{b \in \mathcal{B}} \sum_{y \in \mathcal{I}} ((c + b) + y)^k = |\mathcal{I}| \sum_{c,b} (c + b)^k + \sum_{c,b} \sum_{j=0}^{k-1} (c + b)^j \left( \sum_{y \in \mathcal{I}} y \right) (c + b)^{k-1-j}.$$ 

Since $\mathcal{I} \neq 0$, $|\mathcal{I}| = 0$ in $R$. Moreover, since $(c + b)^j yc^{k-1-j} = c^j ye^{k-1-j}$, we have

$$\sum_{r \in R} r^k = 0 + \sum_{c \in \mathcal{C}} \sum_{j=0}^{k-1} c^j \left( \sum_{y \in \mathcal{I}} y \right) c^{k-1-j} = \sum_{c \in \mathcal{C}} \sum_{j=0}^{k-1} \frac{|J(R)|}{|\mathcal{I}|} c^j \left( \sum_{y \in \mathcal{I}} y \right) c^{k-1-j} = 0.$$

This shows the result if dim_{\mathbb{F}_2} J(R) \geq 2 or $J(R) = 0$. The final case is if dim_{\mathbb{F}_2} J(R) = 1, whence $J(R)^2 = 0$. By Wedderburn’s theorem, $R$ is a semidirect product of $J(R)$ and the product ring $R/J(R) \cong \times_{j=1}^{n} M_{d_j \times d_j}(\mathbb{F}_{2^j})$ for some integers $d_j, e_j > 0$, with $J(R)$ a bimodule over $R/J(R)$. Now if $d_j > 1$ or $e_j > 1$ for any $j$, then the simple algebra $M_{d_j \times d_j}(\mathbb{F}_{2^j})$ cannot have a one-dimensional representation $J(R)$; thus it must act trivially on $J(R)$. It follows by the indecomposability of $R$ that $d_j = e_j = 1 \forall j$. Denote the idempotent in $\mathbb{F}_{2^e_j}$ by $1_j$. Then there are unique $i, j$ such that $1_i J(R) = J(R) = J(R)1_j$.

There are now two sub-cases. If $i = j$ then the product of finite fields is a single field $\mathbb{F}_2$, whence $1_i$ is the unit and hence the semidirect product $R$ is abelian. But then $R \cong \mathbb{F}_2[x]/(x^2)$, which shows the result. The other case is if $i \neq j$, in which case the algebra $R$ is explicitly isomorphic to upper triangular $2 \times 2$ matrices over $\mathbb{F}_2$. \hfill \Box

As an intermediate step, we verify our main result in the last two cases of Proposition 4.6

**Proposition 4.7.** Theorem 13 holds for $R = \mathbb{F}_2[x]/(x^2)$ and for the upper triangular matrices $T_{2 \times 2}(\mathbb{F}_2)$.

**Proof.** First suppose $R = \mathbb{F}_2[x]/(x^2)$. If $k$ is even, then compute using that $k = 0 = x^2$ in $R$:

$$\mathcal{P}_k^R(T) = \left(T^k + (T + x)^k\right) + \left((T + 1)^k + (T + 1 + x)^k\right) = \left(2T^k + kT^{k-1}x\right) + \left(2(T + 1)^k + k(T + 1)^{k-1}x\right) = 0.$$ 

Otherwise if $k$ is odd, then

$$\mathcal{P}_k^R(T) = \left(T^k + (T + 1)^k\right) + \left((x + T)^k + (x + (T + 1))^k\right) = \sum_{j=0}^{k-1} \binom{k}{j} x^{k-j}(T^j + (T + 1)^j) = kx\mathcal{P}_{k-1}^R(T) = x\mathcal{P}_{k-1}^R(T),$$

as stated in the theorem. The final equality in this case in the theorem, follows from Equation (1.2).

Next set $R = T_{2 \times 2}(\mathbb{F}_2)$, and note the powers of individual matrices in $R$ are as follows:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},$$

the nilpotent cone is $\mathbb{F}_2x = \mathbb{F}_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and the remaining five matrices are idempotent. Now a straightforward computation shows that $\sum_{r \in R} r^k$ vanishes if $k$ is even, and equals $x$ for $k > 1$ odd.
Finally, it follows using Equation (4.2) that
\[ P^R_k(T) = x \sum_{1 < j \text{ odd}} \binom{k}{j} T^{k-j}. \]

By the above analysis for \( \mathbb{F}_2[x]/(x^2) \), the result follows for \( k \) odd. If instead \( k \) is even, it suffices to show that \( \binom{k}{j} = 0 \) for \( j \) odd. To see why, consider the coefficient of \( y^j \) in expanding \((1 + y)^k\) over \( \mathbb{Z}/2\mathbb{Z} \). If \( k = 2l \), then we compute using the Frobenius:
\[(1 + y)^k = ((1 + y)^2)^l = (1 + y^2)^l,\]
from which it follows that the coefficient of \( y^j \) vanishes modulo 2, as desired. \( \square \)

Our next result proves Theorem B when \( R \) is not a cyclic group and has characteristic \( 2^m \) for some \( m > 1 \).

**Proposition 4.8.** Suppose \( R \) has characteristic \( 2^m \) for some \( m > 1 \). If \( R \) is not a cyclic group, then all power sums vanish.

**Proof.** Without loss of generality, assume \( 2^{m-1} R \neq 0 \). We show the result by induction on \( m \geq 2 \). For the base case of \( m = 2 \), there are two sub-cases. Set \( \mathcal{I} := 2^{m-1} R \), and first suppose \( \dim_{\mathbb{F}_2} \mathcal{I} \geq 2 \). Then the result follows from the calculation in Equation (4.5), since \( |\mathcal{I}| = 0 \) in \( R \) by assumption, and as explained in the proof of Lemma 4.4, \( \sum_{y \in \mathcal{I}} y = 0 \) for the \( \mathbb{F}_2 \)-vector space \( \mathcal{I} \).

In the other sub-case \( \dim_{\mathbb{F}_2}(2^{m-1}) = 1 \), we have \( \mathcal{I} = \{0, 2^{m-1}\} \), so \( \mathcal{I} \) is central. Now Equation (4.5) shows:
\[ \sum_{r \in R} r^k = 2 \sum_{c \in C} c^k + 2^{m-1} k \sum_{c \in C} c^{k-1}. \] (4.9)

If \( R \) has nonzero power sums, then by (4.9), so does the \( \mathbb{F}_2 \)-algebra \( R/2R \). It follows by Proposition 4.6 that \( R/2R \) is the algebra of upper triangular matrices \( T_{2 \times 2}(\mathbb{F}_2) \), or else \( \mathbb{F}_2[x]/(x^2), \mathbb{F}_{2e}, \) or \( M_{2 \times 2}(\mathbb{F}_2) \). Since \( R/2R \) has a one-dimensional simple module \( 2R \) of size 2, we have \( R/2R \neq M_{2 \times 2}(\mathbb{F}_2), \mathbb{F}_{2e} \) for \( e > 1 \). If \( R/2R = \mathbb{F}_2 \), then by Nakayama’s Lemma \( R \) is also cyclic, which is impossible.

Thus, \( R/2R \cong T_{2 \times 2}(\mathbb{F}_2) \) or \( \mathbb{F}_2[x]/(x^2) \). In both cases, recall by Proposition 4.7 that every power sum is nilpotent, say \( \sum_{c \in R/2R} c^k =: s_k \), with \( s_k^2 = 0 \). Choose any lift \( \overline{s}_k \in R \) of \( s_k \); then (4.9) implies:
\[ \sum_{r \in R} r^k = 2\overline{s}_k + 2k\overline{s}_{k-1}. \]

We now claim that if \( s^2 = 0 \) in \( R/2R \), then \( 2\overline{s} = 0 \) for any lift \( \overline{s} \); note this claim proves the \( m = 2 \) case. Indeed, if \( 2\overline{s} \in 2R \setminus \{0\} \) then \( 2\overline{s} = 2 \), so \( s^2 = 2 \). But \( s^2 \in 2R \), so we get \( 2 = 2\overline{s}^2 = 0 \), which is impossible.

Having proved the result for \( m = 2 \), the inductive step uses Equation (4.5), by adapting for \( p = 2 \) the proof of Proposition 4.3—specifically, the \( m > 2 \) case. \( \square \)

Combining all of the above analysis, we now show the main theorem.

**Proof of Theorem B.** The first assertion in (1.1) was shown above in (4.1). Thus, assume henceforth that \( R \) has prime power characteristic. Now the result is shown for \( R = \mathbb{F}_q \) in Theorem A; for \( R = M_{2 \times 2}(\mathbb{F}_2) \) via Equation (4.2) and Theorem 1.1; and for \( R = \mathbb{F}_2[x]/(x^2), T_{2 \times 2}(\mathbb{F}_2) \) in Proposition 4.4.

Next, suppose \( R \) is indecomposable and of prime power characteristic. By the above analysis in Propositions 4.3, 4.6, 4.7 and (4.8), it remains to compute the power-sum polynomials \( P^R_k(T) \) for \( R = \mathbb{Z}/p^n \mathbb{Z} \) for \( p > 2 \) a prime.
First assume $p$ is odd or $k$ is even. We show that for all $e, k \in \mathbb{N}$,
\[ P_k^{\mathbb{Z}/p^{e+1}\mathbb{Z}}(T) = pP_k^{\mathbb{Z}/p^e\mathbb{Z}}(T) \mod p^{e+1}. \] (4.10)
Indeed, compute for $k \in \mathbb{N}$ and any indeterminate $x$:
\[ x^k + (x+p^e)^k + \cdots + (x+(p-1)p^e)^k \equiv px^k + \binom{k}{1} x^{k-1} p^e \sum_{j=0}^{p-1} j \mod p^{2e}, \]
whence the same equality holds modulo $p^{e+1}$. Now there are two cases: if $p$ is odd, then the sum on the right-hand side equals \( \binom{p}{e} = 0 \mod p \). Otherwise if $p, k$ are even then the second term on the right-hand side is divisible by $2^{e+1}$. In either case, it follows that
\[ P_k^{\mathbb{Z}/2^{e+1}\mathbb{Z}}(T) = \sum_{i=0}^{2^{e+1}-1} (T+i)^k = \sum_{i=0}^{2^e-1} [(T+i)^k + (T+2^e+i)^k] = \sum_{i=0}^{2^e-1} [2(T+i)^k + 2^e k(T+i)^{k-1}] \]
upon applying the above analysis to $x \rightsquigarrow T+i$. This shows (4.10), and hence, the result.

The other case is when $p = 2$ and $k$ is odd. We first show the “induction step”, relating $P_k^{\mathbb{Z}/2^{e+1}Z}$ to $P_k^{\mathbb{Z}/2^eZ}$,
\[ P_k^{\mathbb{Z}/2^{e+1}\mathbb{Z}}(T) = 2P_k^{\mathbb{Z}/2^{e}\mathbb{Z}}(T) + 2P_k^{\mathbb{Z}/2^{e-1}\mathbb{Z}}(T) \mod 2^{e+1}. \]
By the previous case, the last term equals
\[ 2^e P_k^{\mathbb{Z}/2^{e}\mathbb{Z}}(T) \equiv 2^e \cdot 2^{e-1} P_k^{\mathbb{Z}/2^{e-1}\mathbb{Z}}(T) \mod 2^{e+1}, \]
and this vanishes if $e \geq 2 = p$. Repeating this computation inductively,
\[ P_{k-1}^{\mathbb{Z}/2^{m}\mathbb{Z}}(T) = 2P_{k-1}^{\mathbb{Z}/2^{m-1}\mathbb{Z}}(T) + 2^{m-1} P_{k-1}^{\mathbb{Z}/2^{m-1}\mathbb{Z}}(T) \equiv 4P_{k-1}^{\mathbb{Z}/2^{m-2}\mathbb{Z}}(T) + 2^{m-1} P_{k-1}^{\mathbb{Z}/2^{m-2}\mathbb{Z}}(T) + 0 \equiv \ldots \equiv 2^{m-2} P_{k-1}^{\mathbb{Z}/2^2\mathbb{Z}}(T) + 2^{m-1} P_{k-1}^{\mathbb{Z}/2^2\mathbb{Z}}(T) + 0 \equiv 2^{m-1} P_{k-1}^{\mathbb{Z}/2^2\mathbb{Z}}(T) + 2^{m-1} P_{k-1}^{\mathbb{Z}/2^2\mathbb{Z}}(T) + 0 \mod 2^m. \]
This shows the result when $p = 2$ and $k$ is odd, and hence, the theorem holds for every indecomposable finite unital ring $R$.

Finally, suppose $R$ is a general finite unital ring of prime power characteristic, say $p^m$. Write $R = \times_j R_j$ as a product of indecomposable factors; then by the above analysis, the prime $p$ kills the power-sum polynomial $P_{R_j}(T)$ for all $j$. Now if there are at least two factors $R_j$, then it follows by Equation (4.11) that $P_k^R(T) = 0$. This concludes the proof of the theorem. \(\square\)

References

[BCL] J.V. Brayley, L. Carlitz, and J. Levine, Power sums of matrices over a finite field, Duke Mathematical Journal 41(1):9–24, 1974.

[Ca1] L. Carlitz, Some sums involving polynomials in a Galois field, Duke Mathematical Journal 5(4):941–947, 1939.

[Ca2] L. Carlitz, An analogue of the Staudt–Clausen theorem, Duke Mathematical Journal 7(1):62–67, 1940.

[Ca3] L. Carlitz, The Staudt–Clausen theorem, Mathematics Magazine 34:131–146, 1961.
[Cl] T. Clausen, *Lehrsatz aus einer Abhandlung über die Bernoullischen Zahlen*, Astronomische Nachrichten 17(22):351–352, 1840.

[FGO] P. Fortuny Ayuso, J.M. Grau, and A.M. Oller-Marcén, *A von Staudt-type result for* $\sum_{z \in \mathbb{Z}_n}\zeta^k$, Monatshefte für Mathematik 178(3):345–359, 2015.

[FGOR] P. Fortuny Ayuso, J.M. Grau, A.M. Oller-Marcén, and I.F. Rúa, *On power sums of matrices over a finite commutative ring*, preprint, http://arxiv.org/abs/1505.08132.

[Ge] E.-U. Gekeler, *On power sums of polynomials over finite fields*, Journal of Number Theory 30(1):11–26, 1988.

[Gos1] D. Goss, *Von Staudt for* $\mathbf{F}_q[T]$, Duke Mathematical Journal 45(4):885–910, 1978.

[Gos2] D. Goss, *Kummer and Herbrand criterion in the theory of function fields*, Duke Mathematical Journal 49(2):377–384, 1982.

[Gou] H.W. Gould, *The Girard–Waring Power Sum Formulas for Symmetric Functions, and Fibonacci Sequences*, The Fibonacci Quarterly 37(2):135–140, 1999.

[GO] J.M. Grau and A.M. Oller-Marcén, *Power sums over finite commutative unital rings*, preprint, http://arxiv.org/abs/1603.05787.

[GOS] J.M. Grau, A.M. Oller-Marcén, and J. Sondow, *On the congruence $1^m + 2^m + \cdots + m^m \equiv n(\text{mod } m)$ with $n/m$*, Monatshefte für Mathematik 177(3):421–436, 2015.

[Mor] P. Moree, *On a theorem of Carlitz–von Staudt*, C. R. Math. Rep. Acad. Sci. Canada 16(4):166–170, 1994.

[SV] A. Stewart and V. Vologodsky, *On the center of the ring of differential operators on a smooth variety over $\mathbf{Z}/p^n\mathbf{Z}$*, Compositio Mathematica 149(1):63–80, 2013.

[Th] D.S. Thakur, *Power sums of polynomials over finite fields and applications: A survey*, Finite Fields and Their Applications 32:171–191, 2015.

[vSt] K.G.C. von Staudt, *Beweis eines Lehrratzes, die Bernoullischen Zahlen betreffen*, Journal für die reine und angewandte Mathematik 21:372–374, 1840.

[Wi] E. Witt, *Zyklische Körper und Algebren der Charakteristik $p$ vom Grad $p^n$. Struktur diskret bewerteter perfekter Körper mit vollkommenem Restklassenkörper der Charakteristik $p$*, Journal für die reine und angewandte Mathematik 176:126–140, 1937.

(A. Khare) Department of Mathematics, Stanford University
E-mail address: khare@stanford.edu

(A. Tikaradze) Department of Mathematics, University of Toledo
E-mail address: tikar06@gmail.com