MAPPING CLASS GROUPS OF ONCE-STABILIZED
HEEGAARD SPLITTINGS

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Abstract. We show that if a Heegaard splitting is the result of stabilizing a high distance Heegaard splitting exactly once then its mapping class group is finitely generated.

A Heegaard splitting for a compact, connected, closed, orientable 3-manifold $M$ is a triple $(\Sigma, H^-, H^+)$ where $\Sigma$ is a compact, separating surface in $M$ and $H^-, H^+$ are handlebodies in $M$ such that $M = H^- \cup H^+$ and $\partial H^- = \Sigma = H^- \cap H^+ = \partial H^+$. The mapping class group $\text{Mod}(M, \Sigma)$ of the Heegaard splitting is the group of homeomorphisms $f : M \to M$ that take $\Sigma$ onto itself, modulo isotopies that fix $\Sigma$ setwise.

Heegaard splittings of distance greater than 3 are known to have finite mapping class groups [8] and certain distance two Heegaard splittings are known to have virtually cyclic mapping class groups [6]. By distance, we mean the distance $d(\Sigma)$ defined by Hempel [5], which we will review below. However, for stabilized (distance zero) Heegaard splittings, the problem of understanding their mapping class groups is much harder. Genus-two Heegaard splittings of the 3-sphere have finitely presented mapping class groups [1][4][12], but determining whether the mapping class groups of higher genus splittings of $S^3$ are finitely generated has proved to be a very difficult problem. We will study Heegaard splittings that results from stabilizing a high distance Heegaard splitting exactly once:

1. Theorem. Let $(\Sigma', H^-_{\Sigma'}, H^+_{\Sigma'})$ be a Heegaard surface with genus $g$ and distance $d(\Sigma') > 2g + 2$. If $(\Sigma, H^-_{\Sigma}, H^+_{\Sigma})$ is the Heegaard splitting that results from stabilizing $\Sigma'$ exactly once then $\text{Mod}(M, \Sigma)$ is finitely generated.

In fact, we will describe an explicit generating set in Section 1. Because every automorphism of $(M, \Sigma)$ is an automorphism of $M$, there is a canonical map $i : \text{Mod}(M, \Sigma) \to \text{Mod}(M)$. We will write $\text{Isot}(M, \Sigma)$ for the kernel of the map $i$, and will call this the isotopy subgroup.

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When $M$ is hyperbolic, $Isot(M, \Sigma)$ will be a finite index subgroup of $Mod(M, \Sigma)$.

To prove Theorem 1, we show that $Isot(M, \Sigma)$ is finitely generated in a very specific way. This will imply that $Mod(M, \Sigma)$ is finitely generated because $d(\Sigma') > 2g + 2 > 3$, so $M$ must be hyperbolic, and $Isot(M, \Sigma)$ is finite index in $Mod(M, \Sigma)$. We will see that $Isot(M, \Sigma)$ is generated by two subgroups that have been recently classified by Scharlemann [13], with a well understood intersection. We conjecture that $Isot(M, \Sigma)$ is in fact a free product with amalgamation, but we are unable to prove this.

Note that every element in $Isot(M, \Sigma)$ is determined by an isotopy of $\Sigma$ in $M$, i.e. a continuous family of embedded surfaces $\{\Sigma_r\}$ such that $\Sigma_0 = \Sigma_1 = \Sigma$. The two subgroups that generate $Isot(M, \Sigma)$ will be defined as follows:

The stabilized Heegaard surface $\Sigma$ can be isotoped into either of the handlebodies $H_{\Sigma}^\pm$ bounded by the original Heegaard surface $\Sigma'$. After such an isotopy it forms a Heegaard surface for the handlebody. Define $St^-(\Sigma)$ as the subgroup of $Isot(M, \Sigma)$ corresponding to isotopies entirely in $H_{\Sigma}^-$ and let $St^+(\Sigma)$ be the subgroup corresponding to isotopies in $H_{\Sigma}^+$.

These two subgroups are the isotopy subgroups of the mapping class groups for $\Sigma$, thought of as a Heegaard splitting for $H_{\Sigma}^-$ or $H_{\Sigma}^+$. Scharlemann [13] shows that such groups are finitely generated. We will show that $Isot(M, \Sigma)$ is generated by these two subgroups.

We review this generating set in Section 1, then determine a condition on isotopies of $\Sigma$ in $M$ that will guarantee that the corresponding element of $Isot(M, \Sigma)$ is generated by elements of $St^-(\Sigma) \cup St^+(\Sigma)$. The proof is based on the double sweep-out machinery developed by Cerf [2], Rubinstein-Scharlemann [11] and the author [7], which we review Sections 2 and 3. The proof of this Lemma and Theorem 1 are completed in Section 5.

1. Mapping class groups in handlebodies

A genus $g$ handlebody $H$ has, up to isotopy, a unique Heegaard splitting for each genus $h \geq g$ [14]. For $h = g$, this Heegaard splitting is a boundary parallel surface, which cuts $H$ into a genus $g$ handle-body and a trivial compression body $\partial H \times [0, 1]$. The higher genus Heegaard splittings come from adding unknotted handles to the genus $g$ Heegaard splitting, as in Figure 1. This construction is called stabilization.

An equivalent way to construct a genus $g + k$ Heegaard splitting of a handlebody $H$ is to take $k$ boundary parallel, properly embedded arcs
\( \alpha_1, \ldots, \alpha_k \) in \( H \) and let \( \Sigma \) be the boundary of a regular neighborhood of \( \partial H \cup \alpha_1 \cup \cdots \cup \alpha_k \).

Let \( \Sigma \) be the genus \( g + 1 \) Heegaard splitting for \( H \). Scharlemann \cite{Scharlemann1993} has determined a very simple generating set for the isotopy subgroup \( \text{Isot}(H, \Sigma) \) in terms of the boundary parallel arc \( \alpha \subset H \) that defines \( \Sigma \). The surface \( \Sigma \) bounds a genus \( g + 1 \) handlebody on one side and a compression body with a single one-handle on the other side. There is exactly one non-separating compressing disk in the compression body, dual to the arc \( \alpha \), so the isotopy subgroup of \( \Sigma \) can be described entirely in terms of isotopies of \( \alpha \).

Let \( D \subset H \) be a disk whose boundary consists of the arc \( \alpha \) and an arc in \( \partial H \) (since \( \alpha \) is unknotted) and let \( E_1, \ldots, E_g \) be a collection of compressing disks for \( \partial H \) that are disjoint from \( D \) and cut \( H \) into a single ball. By Theorem 1.1 in \cite{Scharlemann1993}, \( \text{Isot}(H, \Sigma) \) is generated by the following two subgroups. (We use slightly different notation here.)

1. Let \( \mathcal{F}(H, \Sigma) \) be the subgroup generated by isotoping the disk \( D \) around \( H \), and dragging \( \alpha \) with it. Because \( D \cap \partial H \) is an arc, each element is defined by a path in \( \partial H \), modulo spinning around a regular neighborhood of \( D \cap \partial H \). Thus this subgroup is isomorphic to an extension of \( \pi_1(\partial H) \) by the integers.

2. Let \( \mathcal{A}(H, \Sigma) \) be the subgroup generated by fixing one endpoint of \( \alpha \) and dragging the other around in the complement of a collection of properly embedded disks \( D_1, \ldots, D_g \) whose complement in \( H \) is a single ball. Because the complement of the disks is the ball, any path of one endpoint can be extended to an isotopy of \( \alpha \) that ends back where it started. Any such mapping
class is determined by an element of the fundamental group of the planar surface \( \partial H \setminus (\bigcup D_i) \), so this group is isomorphic to the fundamental group of the planar surface.

**Figure 2.** The isotopies that generate \( \mathcal{G}(H, \Sigma) \) and \( \mathfrak{A}(H, \Sigma) \).

In particular, each of these subgroups is finitely generated, so \( Isot(H, \Sigma) \) is finitely generated. Each of the subgroups \( St^\pm(\Sigma) \subset Isot(M, \Sigma) \) is isomorphic to \( Isot(H, \Sigma) \). Their intersection contains the subgroups \( \mathcal{G}(H^\pm \Sigma, \Sigma) \) in each handlebody because the isotopies defining these subgroups can be carried out within a regular neighborhood of the boundary.

We would like to show that if \( \Sigma' \) is a high distance Heegaard splitting then every element of \( Isot(M, \Sigma) \) is a product of elements of \( St^-(\Sigma) \) and \( St^+(\Sigma) \). Fix spines \( K^-, K^+ \) of the handlebodies \( H^\pm \Sigma' \). The key will be the following Lemma:

2. **Lemma.** Let \( \{\Sigma_r\} \) be an isotopy of the surface \( \Sigma \) and assume there is a sequence of values \( 0 = r_0 < r_1 < \cdots < r_k = 1 \) with the property that for \( r \in [r_i, r_{i+1}] \), the surface \( \Sigma_r \) is disjoint from \( K^- \) when \( i \) is even and disjoint from \( K^+ \) when \( i \) is odd. Moreover, assume that each \( \Sigma_{r_i} \) is a Heegaard surface for the complement of the two spines. Then the element of \( Isot(M, \Sigma) \) defined by \( \{\Sigma_r\} \) is generated by the elements of \( St^-(\Sigma) \cup St^+(\Sigma) \).

*Proof.* If every \( \Sigma_r \) in the isotopy is disjoint from both spines \( K^\pm \) then there is a value \( \epsilon \) such that each \( \Sigma_r \) is contained in \( f^{-1}([\epsilon, 1 - \epsilon]) \). This isotopy is conjugate to an isotopy in \( H_{\Sigma}^- \) as well as to an isotopy in \( H_{\Sigma}^+ \), so it determines an element in the subgroup \( St^-(\Sigma) \cap St^+(\Sigma) \).

If an isotopy \( \{\Sigma_r\} \) ends with \( \Sigma_1 \) disjoint from both spines and isotopic to but not equal to \( \Sigma \), then we can extend the isotopy so that \( \Sigma_{1+\delta} = \Sigma \). This extension is not unique, but it is well defined up to multiplication...
by elements in \( St^-(\Sigma) \cap St^+(\Sigma) \). Thus such an isotopy ending disjoint from \( K^- \cup K^+ \) determines a coset of the intersection.

If an isotopy is disjoint from \( K^+ \) and ends with the image of \( \Sigma \) isotopic to a Heegaard surface for the complement of \( K^+ \) and \( K^- \) then it can be extended to an isotopy that takes \( \Sigma \) onto itself. Moreover, because the isotopy is disjoint from \( K^+ \), it is disjoint from the closure of some regular neighborhood \( N \) of \( K^+ \) and \( N \) is isotopic to \( H^+ (\Sigma') \).

Thus by conjugating the isotopy of \( \Sigma \) with an ambient isotopy that takes \( N \) onto \( H^+ (\Sigma') \), we can turn the original isotopy into an isotopy of \( \Sigma \) in \( H^- (\Sigma') \). Such an isotopy determines a coset of the intersection subgroup inside \( St^-(\Sigma) \). Similarly, if the isotopy is disjoint from \( K^- \), it determines a coset inside \( St^+ (\Sigma) \).

We have assumed that our isotopy \( \{ \Sigma_r \} \) can be cut into finitely many sub-intervals such that in each interval, \( \Sigma_r \) is always disjoint from \( K^- \) or always disjoint from \( K^+ \). The restriction of the isotopy \( \{ \Sigma_r \} \) to each interval determines a coset of \( St^-(\Sigma) \cap St^+(\Sigma) \) in either \( St^-(\Sigma) \) or \( St^+(\Sigma) \). The element of \( Isot(M, \Sigma) \) defined by the entire isotopy is a product of representatives for these cosets, and is thus in the subgroup generated by \( St^-(\Sigma) \cup St^+(\Sigma) \). \( \square \)

To prove Theorem 1 we must show that we can always find an isotopy of this form. The rest of the paper will be devoted to proving this:

3. Lemma. If \( d(\Sigma') \) is greater than \( 2g + 2 \) then every element of \( Isot(M, \Sigma) \) is represented by an isotopy satisfying the conditions of Lemma 2.

2. Sweep-outs and graphics

A sweep-out is a smooth function \( f : M \to [-1, 1] \) such that for each \( t \in (-1, 1) \), the level set \( f^{-1}(t) \) is a closed surface. Moreover, \( f^{-1}(-1) \) and \( f^{-1}(1) \) is each a graph, called a spine of the sweep-out. The preimages \( f^{-1}([-1, t]) \) and \( f^{-1}([t, 1]) \) are handlebodies for each \( t \in (-1, 1) \) so each level surface \( f^{-1}(t) \) is a Heegaard surface for \( M \) and the spines of the sweep-outs are spines of the two handlebodies in this Heegaard splitting. See [7] for a more detailed description of the methods described in this section.

We will say that a sweep-out represents a Heegaard splitting \( (\Sigma', H^-_{\Sigma'}, H^+_{\Sigma'}) \) if \( f^{-1}(-1) \) is isotopic to a spine for \( H^-_{\Sigma'} \) and \( f^{-1}(1) \) is isotopic to a spine for \( H^+_{\Sigma'} \). The level surfaces \( f^{-1}(t) \) of such a sweep-out will be isotopic to \( \Sigma' \). Because the complement of the spines of a Heegaard splitting is a surface cross an interval, we can construct a sweep-out for any Heegaard splitting, i.e. we have the following:
4. **Lemma.** Every Heegaard splitting of a compact, connected, closed orientable, smooth 3-manifold is represented by a sweep-out.

Given two sweep-outs, \( f \) and \( h \), their product is a smooth function \( f \times h : M \to [-1, 1] \times [-1, 1] \). (That is, we define \( (f \times h)(x) = (f(x), h(x)) \).) The discriminant set for \( f \times h \) is the set of points where the level sets of the two functions are tangent.

Generically, the discriminant set will be a one dimensional smooth submanifold in the complement in \( M \) of the spines \([9, 10]\). The function \( f \times h \) defines a piecewise smooth map from this collection of arcs and loops into the square \([-1, 1] \times [-1, 1]\). At the finitely many non-smooth points, the image is a cusp, which we will think of as a valence-two vertex. At the finitely many points where the restriction of \( f \) is two-to-one, we see a crossing which we will think of as a valence-four vertex. There are also valence-one and -two vertices in the boundary of the square. The resulting graph is called the Rubinstein-Scharlemann graphic (or just the graphic for short). Kobayashi-Saeki’s approach \([9]\) uses singularity theory to recover the machinery originally constructed by Rubinstein and Scharlemann in \([11]\) using Cerf theory \([3]\).

5. **Definition.** The function \( f \times h \) is *generic* if the discriminant set is a smooth one-dimensional manifold and each arc \( \{t\} \times [-1, 1] \) or \([-1, 1] \times \{s\} \) contains at most one vertex of the graphic.

Kobayashi and Saeki \([9]\) have shown that after an isotopy of \( f \) and \( h \), we can assume that \( f \times h \) is generic. The author has generalized this to an isotopy of sweep-outs as follows:

6. **Lemma** (Lemma 34 in \([7]\)). Every isotopy of the sweep-out \( h \) is conjugate to an isotopy \( \{h_r\} \) so that the graphic defined by \( f \) and \( h_r \) is generic for all but finitely many values of \( r \). At the finitely many non-generic points, one of six changes can occur to the graphic, indicated in Figure 3:

1. A pair of cusps forming a bigon may cancel with each other or be created.
2. A pair of cusps adjacent to a common crossing may cancel or be created (similar to a Reidemeister one-move).
3. Three crossing may perform a Reidemeister three-move.
4. Two parallel edges may pinch together to form a pair of cusps, or vice versa.
5. Two edges may perform a Reidemeister two-move.
6. A cusp may pass across an edge.

For a more details description of these three moves, see \([7]\).
3. Spanning and Splitting

Let $f$ and $h$ be sweep-outs. For each $s \in (-1, 1)$, define $\Sigma_s = h^{-1}(s)$, $H_s^- = h^{-1}([-1, s])$ and $H_s^+ = h^{-1}([s, 1])$. Similarly, for $t \in (-1, 1)$, define $\Sigma_t = f^{-1}(t)$. Following [7], we will say that $\Sigma_t$ is mostly above $\Sigma_s$ if each component of $\Sigma_t \cap H_s^-$ is contained in a disk subset of $\Sigma_t$. Similarly, $\Sigma_t$ is mostly below $\Sigma_s$ if each component of $\Sigma_t \cap H_s^+$ is contained in a disk in $\Sigma_t$.

Figure 4 shows the three possible positions for $\Sigma_t$ (shown in blue) with respect to $\Sigma_s$ for three different values of $s$ (outlined in red). For the highest value of $s$, $\Sigma_t$ is mostly below $\Sigma_s$. For the lowest value, it’s mostly above, and for the middle value (in which $\Sigma_s$ looks like a quadrilateral, $\Sigma_t$ is neither mostly above nor mostly below.

Let $R_a \subset (-1, 1) \times (-1, 1)$ be the set of all values $(t, s)$ such that $\Sigma_t$ is mostly above $\Sigma_s$. Let $R_b \subset (-1, 1) \times (-1, 1)$ be the set of all values $(t, s)$ such that $\Sigma_t$ is mostly below $\Sigma_s$. For any fixed $t \in (-1, 1)$, there will be values $a, b$ such that $\Sigma_t$ will be mostly above $\Sigma_s$ if and only if $s \in [-1, a)$ and mostly above $\Sigma_s$ if and only if $s \in (b, 1]$. In particular, both regions will be vertically convex.

As noted in [7], the closure of $R_a$ in $(-1, 1) \times (-1, 1)$ is bounded by arcs of the Rubinstein-Scharlemann graphic, as is the closure of $R_b$. The closures of $R_a$ and $R_b$ are disjoint (as long as the level surfaces of $f$ have genus at least two.)

7. Definition. Given a generic pair $f, h$, we will say $h$ spans $f$ if there is a horizontal arc $[-1, 1] \times \{s\}$ that intersects the interiors of both
regions $R_a$ and $R_b$, as on the left side of Figure 5. Otherwise, we will say that $h$ splits $f$, as on the right of the figure.

Let $f$ be a sweep-out representing $(\Sigma', H_{\Sigma'}, H_{\Sigma'}^+)$. By Lemma 16 in [7] we can choose a sweep-out $h$ for $(\Sigma, H_{\Sigma}, H_{\Sigma}^+)$ that spans $f$.

We can identify $\Sigma$ with any level surface of $h$ by an isotopy and we will choose a specific surface in a moment. For now, note that once we have chosen a level surface of $h$ to represent $\Sigma$, every element of $Isot(M, \Sigma)$ is represented by an isotopy of $\Sigma$, which can be extended to an ambient isotopy of $h$. By Lemma 6 we can choose the induced family of sweep-outs $\{h_r\}$ to be generic for all but finitely many values of $r$. At all these values, $h_r$ either spans or splits $f$. We will rule out splitting using the following Lemma:

8. Lemma (Lemma 27 in [7]). If $h$ splits $f$ then $d(\Sigma')$ is at most twice the genus of $\Sigma$.

Note that the notation here is slightly different than that used in [7]. In particular, the surfaces $\Sigma$ and $\Sigma'$ play the opposite roles in [7] as
they do here. In the statement of Theorem 1, we assume that $d(\Sigma')$ is strictly greater than $2g + 2$, so Lemma 8 implies that $h_r$ spans $f$ for every generic value of $r$. This will be the key to proving Lemma 9.

4. Bicompressible surfaces in handlebodies

Recall that a two-sided surface $S \subset M$ is bicompressible if there are compressing disks for $S$ on both sides of the surface. We will say that $S$ is reducible if there is a sphere $P \subset M$ such that $P \cap S$ is a single loop that is essential in $S$. This term is usually used only for Heegaard surfaces, but we will apply it here to general bicompressible surfaces.

9. **Lemma.** Let $F$ be a closed, orientable, genus $g$ surface and $S \subset (F \times [0,1])$ a closed, embedded, bicompressible surface of genus $g + 1$ that separates $F \times \{0\}$ from $F \times \{1\}$. Then $S$ is reducible.

**Proof.** Let $C^-$ be the closure of the component of $F \times [0,1] \setminus S$ adjacent to $F \times \{0\}$ and let $C^+$ be closure of the other component. Because $S$ is bicompressible, there are compressing disks $D^- \subset C^-$ and $D^+ \subset C^+$ for $S$.

If $\partial D^-$ is non-separating in $S$ then compressing $S$ across $D^-$ produces a genus $g$ surface $S^-$ that separates $F \times \{0\}$ from $F \times \{1\}$. Because $F$ is also genus $g$, the surface $S^-$ must be isotopic to $F \times \{\frac{1}{2}\}$. In other words, $S^-$ separates $F \times [0,1]$ into two pieces homeomorphic to $F \times [0,1]$. If we reattach the tube to produce $S$ from $S^-$, we see that $C^-$ is a compression body that results from attaching a single one-handle to $F \times [0,1]$.

The same argument applies to $D^+$ and $C^+$. Thus if we can choose $D^-$ and $D^+$ to be non-separating, $S$ will be a genus $g + 1$ Heegaard surface for $F \times [0,1]$.

Every genus $g + 1$ Heegaard surface for $F \times [0,1]$ is reducible by Scharlemann-Thompson’s classification of Heegaard splittings for surface-cross-intervals [15], so in this case we conclude that $S$ is reducible.

Otherwise, assume without loss of generality that every compressing disk for $S$ in $C^-$ is separating. If we compress $S$ along such a disk $D^-$ then the resulting surface $S^-$ consists of a genus $g$ component and a torus. As above, the genus $g$ component must be isotopic to $F \times \{\frac{1}{2}\}$ so that the torus component of $S^-$ is contained in $F \times [\frac{1}{2}, 1]$.

Because $F \times [\frac{1}{2}, 1]$ is atoroidal, the torus component $T S^-$ can be compressed to form a sphere $T'$. Because $F \times [\frac{1}{2}, 1]$ is irreducible, $T'$ bounds a ball and we can recover $T$ by attaching a tube to $T'$. Thus either bounds a solid torus or is contained in a ball (and bounds a knot complement), depending on which side of $T'$ the tube us attached. In
either case, $T$ has a (non-separating) compressing disk $D^+$. There is an arc $\alpha$ dual to the disk $D^-$ from the genus $g$ component of $S^-$ to $T$. Because $D^-$ and $D^+$ are on opposite sides of $S$, the arc must be adjacent to $T$ on the same side of the surface as the disk $D^+$. This is impossible if $T$ bounds a solid torus on the side containing $D^+$. Thus $T$ must be contained in a ball that intersects the arc $\alpha$ in a single point. This sphere will intersect $S$ in a single essential loop, so $S$ is reducible (though not necessarily a Heegaard surface).

\section{The Proof of Theorem \ref{main} \label{main Proof}}

\textit{Proof of Lemma \ref{main}.} By assumption, $d(\Sigma') > 2g + 2$ where $g$ is the genus of $\Sigma'$. Because $\Sigma$ is a stabilization of $\Sigma'$, its genus is $g + 1$, so by Lemma \ref{main} the graphic $f \times h_r$ can never be spanning. Thus there is a value $s_r$ for each sweep-out $h_r$ such that the horizontal line $[0,1] \times \{s_r\}$ passes through both regions $R_a$, $R_b$ of the graphic. Moreover, we can choose the values $s_r$ so that they vary continuously with $r$ and $s_0 = s_1$. We will further choose values $a_r$, $b_r$ that vary continuously, except for finitely many jumps, such that $(a_r, s_r) \in R_a$ and $(b_r, s_r) \in R_b$. (A jump occurs when $a_r$ or $b_r$ is in a “tooth” of $R_a$ or $R_b$ that is moved away from the horizontal arc, as in Figure \ref{jump} and we choose a new point in a different tooth that still intersects the arc.)

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{jump.png}
\caption{A jump in the value of $b_r$.}
\end{figure}

For each $r$, define $\Sigma_r = h^{-1}(s_r)$. This family of surfaces defines an isotopy of $\Sigma$ corresponding to some element of $\text{Isot}(M, \Sigma)$. We will modify this isotopy so that at every time $r$, the surface is disjoint from one of the spines $K^\pm$ as follows:

The restriction of $f$ to $\Sigma_r$ is a Morse function on the surface, and the level sets at $a_r$ and $b_r$ form loops in this surface. In the level surface of $f$, these loops are trivial since $\Sigma$ is mostly above or mostly below $\Sigma'$ at these points. Thus we can compress $\Sigma$ along these collections of loops to a surface that separates $K^-$ from $K^+$. Because $\Sigma$ has genus...
exactly one more than $\Sigma'$, at most one of these loops can be an actual compression. For each $r_i$, this compression is either at $a_r$ or $b_r$.

At each of the finitely many values of $r$ where the values $a_r$ are discontinuous, let $a_r$ and $a'_r$ be the left and right limits. If the level set at level $b_r$ contains an essential loop then the level sets at both levels $a_r$ and $a'_r$ must be trivial, so neither defines a compression. If either of the level sets at $a_r$ or $a'_r$ does contain a compression then $b_r$ does not.

The same argument holds for a jump in $b_r$.

Thus we can cut the interval $[0, 1]$ at points $0 = r_0 < r_1 < \cdots < r_k = 1$ so that for $r \in [r_i, r_{i+1}]$, the level sets at level $b_r$ are trivial when $i$ is even and the loops at level $a_r$ are trivial when $i$ is odd. Moreover, we can assume that for each even $i$, there is an $r \in [r_i, r_{i+1}]$ such that the level loops of $a_r$ are non-trivial and vice-versa for odd $i$. In other words, we want to cut the interval into as few subintervals as possible.

For $r \in [r_0, r_1]$, the level loops in $\Sigma_r$ at level $b_r$ are trivial in $\Sigma_r$. Each of these loops bounds a disk in each of the surfaces $\Sigma_r, \Sigma'_b$, and there is, up to isotopy, a unique way to project the disk in $\Sigma_r$ onto the disk $\Sigma'_a$, as in Figure 7. Let $S_r$ be the result of projecting all the disks in this way and assume we have chosen the projections to vary continuously with $r$ along the interval.

After the projection, the surface $S_r$ is entirely below level $b_r$ and is thus disjoint from $K^+$. For the intervals in which the level loops at level $a_r$ are trivial, $S_r$ will be disjoint from $K^-$. Repeat this construction for each $i$. The resulting isotopy defines the same element of $\Mod(M, \Sigma)$ as the original isotopy, so all that remains is to show that $\Sigma_{r_i}$ is a Heegaard surface for the complement of both spines for each $i$.

We will start with $r_1$ and then repeat the argument for each $i$. By assumption, there is an $r \in [r_0, r_1]$ such that the level set of $\Sigma_r$ at level $b_r$ is non-trivial. If we compress along all these loops, the resulting

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure7.png}
\caption{Removing trivial intersections between $\Sigma$ and a spine.}
\end{figure}
surface will contain a component isotopic to the boundary of a regular neighborhood of $K^-$, which is on the negative side of $\Sigma_r$. Thus at least one of the compressions must have been on the negative side of $\Sigma_r$.

Similarly, if we choose $r' \in [r_1, r_2]$, we can find a compressing disk on the positive side of $\Sigma_{r'}$. If we choose $r$ to be the last such value and $r'$ to be the first such value, then these compressing disks will determine compressing disks on both sides of $\Sigma_{r_1}$. Since $\Sigma_{r_1}$ is bicompressible, it is reducible by Lemma 9, as in Figure 8.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure8.png}
\caption{Two possible reducible surfaces in $M \setminus K^+$.}
\end{figure}

For each $r \in [r_0, r_1]$, the surface $\Sigma_r$ is disjoint from $K^-$. The surfaces thus determine an isotopy of the surface inside the handlebody $M \setminus K^+$, so each $\Sigma_r$ is a Heegaard surface for this handlebody. Any reducing sphere for a Heegaard surface in an irreducible three-manifold determines a Heegaard surface for the ball bounded by the reducing sphere. In this case, we get a genus-one Heegaard surface, which is a standard unknotted torus [16]. Thus the reducing sphere for $\Sigma_{r_1}$, which is disjoint from both spines, bounds an unknotted handle, as on the left in Figure 8, and $\Sigma_{r_1}$ is a Heegaard surface for the complement of the two spines. By repeating this argument for each successive $i$, we complete the proof. □

**Proof of Theorem 4.** Let $\gamma \in Isot(M, \Sigma)$ be an element of the isotopy subgroup of $\Sigma$, which is the result of stabilizing a Heegaard surface $\Sigma'$ exactly once. Assume the distance $d(\Sigma')$ is strictly greater than $2g + 2$. Then by Lemma 3, we can represent $\gamma$ by an isotopy satisfying the conditions of Lemma 2. Then by Lemma 2, $\gamma$ is in the subgroup generated by $St^- (\Sigma) \cup St^+ (\Sigma) \subset Isot(M, \Sigma)$.

Since $\gamma$ was an arbitrary element, $St^- (\Sigma) \cup St^+ (\Sigma)$ must generate the entire group $Isot(M, \Sigma)$. Because $d(\Sigma) > 2g + 2 > 3$, $M$ is hyperbolic by Hempel’s Theorem [5] (and geometrization). Thus $Mod(M)$ is finite, so $Isot(M, \Sigma)$ is a finite index subgroup of $Mod(M, \Sigma)$. Since a finite
index subgroup of $\text{Mod}(M, \Sigma)$ is finitely generated, the entire group is finitely generated. 

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