A NOTE ON KASPAROV PRODUCT AND DUALITY

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ABSTRACT. Using Paschke-Higson Duality [Hig, Pa], we get a natural index pairing 

\[ K_i(A) \times K_{i+1}(D_\Phi) \to \mathbb{Z} \]

where \( i = 0, 1 \text{ (mod 2)} \)
and \( A \) is a separable \( C^* \)-algebra, \( \Phi \) is a representation of \( A \) on a separable Hilbert space \( \mathbb{H} \). We prove this is a special case of Kasparov Product [Kas].

1. INTRODUCTION

The purpose of this paper is to give a clear exposition of a connection between KK-theory [Kas] and index pairing

\[ K_i(A) \times K_{i+1}(D_\Phi) \to \mathbb{Z} \]

where \( i = 0, 1 \text{ (mod 2)} \)

which is defined in Section 2. In fact, we are going to show that each index pairing is a Kasparov product only using elementary ingredients of K-theory and KK-theory. (see proposition 3.1 below and lemma 2.9, 2.12 below.) As an application of this approach, we show an alternate proof of Bott periodicity in KK-theory [Kas]; cf. Theorem 18.10.2 in [Bl]. In this proof, we do not use geometric argument (for example, use of Clifford algebra [Kas]) but operator theory and pure algebra.

2. PASCHKE-HIGSON DUALITY AND INDEX PAIRING

In this section, we review Paschke-Higson duality theory [Hig]. Throughout this article, \( \mathbb{H} \) is a separable infinite dimensional Hilbert space. \( \mathcal{B}(\mathbb{H}) \) is the set of linear bounded operators on \( \mathbb{H} \). \( \mathcal{K}(\mathbb{H}) \) (shortly \( \mathcal{K} \)) is an ideal of compact operators on \( \mathbb{H} \), and \( \mathcal{Q}(\mathbb{H}) \) (shortly \( \mathcal{Q} \)) is the Calkin algebra.

We use the following notation: if \( X \) and \( Y \) are operators in \( \mathcal{B}(\mathbb{H}) \) we shall write

\[ X \sim Y \]

if \( X \) and \( Y \) differ by a compact operator.

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Note that every \( \* \)-representation \( \Phi : A \to B(\mathbb{H}) \) determines a \( \* \)-homomorphism \( \hat{\Phi} \) of \( A \) into the Calkin algebra.

**Definition 2.1.** Let \( A \) be \( C^* \)-algebra. A \( \* \)-representation \( \Phi : A \to B(\mathbb{H}) \) is called admissible if it is non-degenerate and \( \ker(\hat{\Phi}) = 0 \).

**Remark 2.2.** If a \( \* \)-representation is admissible, then it is necessarily faithful and its image contains no non-zero compact operator.

**Definition 2.3.** Let \( \Phi \) be a \( \* \)-representation of \( A \) on \( \mathbb{H} \). Denote by \( D_\Phi(A) \) the essential commutant of \( \Phi(A) \) in \( B(\mathbb{H}) \). Thus

\[
D_\Phi(A) = \{ T \in B(\mathbb{H}) \mid \forall a \in A, [\Phi(a), T] \sim 0 \}
\]

Given two representations \( \Phi_0, \Phi_1 \) on \( \mathbb{H}_0, \mathbb{H}_1 \) respectively, we say they are approximately unitarily equivalent if there exists a sequence \( \{ U_n \} \) consisting of unitaries in \( B(\mathbb{H}_0, \mathbb{H}_1) \) such that

\[
\text{Ad}(U_n)\Phi_0(a) \sim \Phi_1(a) \quad \text{for all} \quad a \in A
\]

\[
\| \text{Ad}(U_n)\Phi_0(a) - \Phi_1(a) \| \to 0 \quad \text{as} \quad n \to \infty
\]

We write \( \Phi_0 \sim_u \Phi_1 \) in this case.

**Theorem 2.4.** (Voiculescu) Let \( A \) be a separable \( C^* \)-algebra and \( \Phi_i : A \to B(\mathbb{H}_i) \), \( i = 0, 1 \) be non-degenerate \( \* \)-representations. Then if \( \ker(\hat{\Phi}_0) \subset \ker(\hat{\Phi}_1) \), then \( \Phi_0 \oplus \Phi_1 \sim_u \Phi_0 \) where \( \hat{\Phi} \) is the natural \( \* \)-homomorphism into the Calkin algebra induced by a \( \* \)-representation \( \Phi \).

**Proof.** See Corollary 1 in p343 of [Ar]. \( \square \)

**Corollary 2.5.** Assume \( \Phi_i : A \to B(\mathbb{H}_i) \) are admissible representations for \( i = 0, 1 \). Then \( \Phi_0 \sim_u \Phi_1 \).

**Proof.** Admissibility implies \( \ker(\hat{\Phi}) = 0 \). From this, the result is straightforward. \( \square \)

Recall that an extension of a unital separable \( C^* \)-algebra \( A \) is a unital \( \* \)-monomorphism \( \tau \) of \( A \) into the Calkin algebra. We say \( \tau \) is split if there is a non-degenerate \( \* \)-representation \( \rho \) such that \( \hat{\rho} = \tau \) and semi-split if there is a completely positive map \( \rho \) such that \( \hat{\rho} = \tau \).

**Corollary 2.6.** Let \( A \) be a separable unital \( C^* \)-algebra. If \( \tau \) is a unital injective extension of \( A \) and if \( \sigma \) is a split unital extension of \( A \), then \( \tau \oplus \sigma \) is unitarily equivalent \( \tau \).

**Proof.** See p352-353 in [Ar]. \( \square \)

Now we will prove the existence of admissible representation of \( A \).
Proposition 2.7. There is a non-degenerate $^*$-representation $\Phi$ of for a separable $C^*$-algebra $A$ such that $\text{Ker}(\Phi) = 0$.

Proof. Let $\pi$ be a faithful representation of $A$ on $H_\pi$. Take $\Phi$ as $\pi^\infty = \pi \oplus \pi \oplus \cdots$ and $H = H_\pi \oplus H_\pi \oplus \cdots$ $\square$

Definition 2.8. When $\Phi$, $\Psi$ are admissible representations of $A$ on $H$, $D_\Phi(A)$ is isomorphic to $D_\Psi(A)$ by Corollary 2.5. Therefore we define $D(A) = D_\Phi(A)$ as the dual algebra of $A$ up to unitary equivalence.

If $p$ is a projection in $D_\Phi(A)$, we call it ample and can define an extension

$$\tau = \tau_{\Phi, p} : A \xrightarrow{\Phi(p) \mapsto \Phi_p} B(pH) \xrightarrow{\pi} Q(pH)$$

In general, this extension is not injective. However, if the extension $\tau_{\Phi, p}$ is injective, we call that an abstract Toeplitz extension.

To define the map from $K_0(D(A))(= K_0(D_\Phi(A)))$ to $\text{Ext}^{-1}(A)$, we need the following two technical lemmas.

Lemma 2.9. Let $A$ be a unital $C^*$-algebra. For any $\alpha \in K_0(D(A))$, there exists a ample projection $p \in D(A)$ such that $\alpha = [p]_0$.

Proof. Step1: By Corollary 2.6, there is a unitary $u \in B(pH \oplus H, H)$ such that $\text{Ad}(u)(\Phi_p \oplus \Phi)(a) \sim \Phi(a)$ for any $a \in A$ if $p$ is ample. Let $U = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$. We can easily check that $U \in M_2(D_\Phi(A))$ and $UU^* = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$, $U^*U = \begin{pmatrix} p & 0 \\ 0 & I \end{pmatrix}$. Therefore we have $[p \oplus I]_0 = [p \oplus 0]_0$. This implies $[p]_0 + [I]_0 = [P]_0$. In particular, $[I]_0 = 0$. Also you can conclude every two ample projections are Murray-von Neumann equivalent.

Step2: Note that $p \oplus I$ is always ample whether $p$ is ample or not because $(\Phi \oplus \Phi)_{p \oplus I}(a)$ is never compact unless $a \in A$ is zero.

Step3: Any element in $K_0(D_\Phi(A))$ is written by $[q]_0 - [I]_0$ for some $q \in M_n(D_\Phi(A))$. As we observed in Step1, this is just $[q]_0$. By Step2, we may assume $q$ is ample for $\Phi \oplus \Phi \oplus \cdots \oplus \Phi$. Now if we can show $[q]_0 = [p]_0$ for some ample $p \in D_\Phi(A)$, we are done.

Since $\Phi \oplus \Phi \oplus \cdots \oplus \Phi \sim_u \Phi$, there exists $v : \mathbb{H}^n \mapsto \mathbb{H}$ s.t.

1. $v^*v = 1_{B(\mathbb{H}^n)}$, $vv^* = 1_{B(\mathbb{H})}$

2. $\text{Ad}(v) \Phi \oplus \Phi \oplus \cdots \oplus \Phi \in K \quad \forall a \in A$. 

Let $V = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $v \in \mathbb{M}_n(D_\Phi(A))$. Then $[q]_0 = [VqV^*]_0 = [vqv^*]_0$. It’s left to the reader to check that $vqv^*$ is also ample. \hfill \Box

Lemma 2.10. Let $\phi : A \to \mathcal{B}(\mathbb{H}_1 \oplus \mathbb{H}_2)$ be a $*$-representation. Write $\phi(a) = \begin{pmatrix} \phi_{11}(a) & \phi_{12}(a) \\ \phi_{21}(a) & \phi_{22}(a) \end{pmatrix}$. Suppose $\phi_{11}$ is $*$-homomorphism modulo $K(\mathbb{H}_1)$. i.e., $\phi_{11}$ is $*$-homomorphism. Then $\phi_{12}(a)$, $\phi_{21}(a)$ are compacts for any $a \in A$ and $\phi_{22}$ is $*$-homomorphism.

Proof. Using $\phi(aa^*) = \phi(a)\phi(a^*)$ with decomposition of $\phi$ on $\mathbb{H}_1 \oplus \mathbb{H}_2$ and the fact $\phi_{11}$ is $*$-homomorphism modulo $K(\mathbb{H}_1)$, we have $\phi_{12}(a)\phi_{12}(a^*)$ is compact therefore $\phi(a)$ is compact for $a \in A$. Similarly, using $\phi(a^*a) = \phi(a)^*\phi(a)$, we have $\phi_{21}(a)$ is compact for $a \in A$. It follows that $\phi_{22}$ is $*$-homomorphism modulo $K(\mathbb{H}_2)$. \hfill \Box

Proposition 2.11. $K_0(D_\Phi(A)) \cong \text{Ext}^{-1}(A)$ where $\Phi$ is a admissible representation of $A$ on a separable Hilbert space $\mathbb{H}$.

Proof. With the Lemma 2.9 in hand, we define the map from $K_0(D_\Phi(A))$ to $\text{Ext}^{-1}(A)$ as follows $

[p]_0 \mapsto [\tau_{\Phi,p}]

$ When $[p] = [q]$, as we have seen in the proof of Lemma 2.9, $p$ and $q$ are Murray- von-Neumann equivalent in $D_\Phi(A)$ so that partial isometry implementing this equivalence induces the equivalence between $\tau_{\Phi,p}$ and $\tau_{\Phi,q}$. Conversely, unitary equivalence between $\tau_{\Phi,p}$ and $\tau_{\Phi,q}$ induces Murray- von-Neumann equivalence between $p$ and $q$ evidently. From $\Phi \oplus \Phi \sim_u \Phi$, we get a unitary $u \in \mathcal{B}(\mathbb{H} \oplus \mathbb{H}, \mathbb{H})$ which induces a natural isomorphism $Ad(u) : \mathbb{M}_2(D_\Phi(A)) \to D_\Phi(A)$. Note $\pi \circ Ad(u) = Ad(u) \circ (\pi \otimes id_2)$. Since $[p] + [q] = [p \oplus q]$ and $p \oplus q$ is in $D_{\Phi \oplus \Phi}(A)$, $[p] + [q]$ is mapped to $[\pi \circ Ad(u) \circ (\Phi \oplus \Phi)p \oplus q] = [Ad(u) \circ ((\pi \otimes id_2) \circ (\Phi \oplus \Phi)p \oplus q]$ which is indeed $[\tau_{\Phi,p}] + [\tau_{\Phi,q}]$. So far we have shown the map is a monomorphism. It is remained to show the map is onto.

Suppose $\rho$ is semi-split extension of $A$ with a completely positive lifting $\psi : A \to \mathcal{B}(\mathbb{H})$. By Steinspring’s dilation theorem, there is a non-degenerate $*$-representation $\phi : A \to \mathcal{B}(\mathbb{H}_0)$ and an isometry $V : \mathbb{H} \to \mathbb{H}_0$ such that $\psi(a) = V^*\phi(a)V$ for all $a \in A$. If we set $P_1 = VV^*$ and $P_2 = 1 - P_1$, then $\mathbb{H}_0 = P_1(\mathbb{H}_0) \oplus P_2(\mathbb{H}_0) = \mathbb{H}_1 \oplus \mathbb{H}_2$. If we decompose $\phi$ on $\mathbb{H}_0 = \mathbb{H}_1 \oplus \mathbb{H}_2$, we have $V\psi(a)V^* = VV^*\phi(a)V^* = P_1\phi(a)P_1 = \phi_{11}(a)$. Since $\psi = \rho$ is (injective) $*$-homomorphism, we
can conclude \( \phi_{11} \) is (injective)*-homomorphism modulo compact. By the Lemma 2.10, \( \phi_{12}(a), \phi_{21}(a) \) are compacts for \( a \in A \) and \( \phi_{22} \) is *-homomorphism modulo compact. This implies that \([P_1, \phi] \in \mathcal{K}\). Thus \( \dot{\phi}_{11} \) is an abstract Toeplitz extension. Viewing \( V : \mathbb{H} \rightarrow \mathbb{H}_1 \) as an unitary, we can also see that \( \rho \) is unitarily equivalent to \( \dot{\phi}_{11} \). Hence we finish the proof. \[ \square \]

Similarly, we are going to define the map from \( K_1(D(A)) \) to \( KK(A, \mathbb{C}) \). We begin with the following lemma which is expected as we have Lemma 2.9.

**Lemma 2.12.** Let \( A \) be as above. For any \( \alpha \in K_1(D(A)) \), there exists an unitary \( u \in D(A) \) such that \( \alpha = [u]_1 \).

**Proof.** Assume \( T \in M_n(D_\Phi(A)) \approx D_\Phi \oplus \Phi \oplus \ldots \oplus \Phi(A) \) is a unitary which represents \( \alpha \in K_1(D_\Phi(A)) \). Let \( V \) be as above and \( S = \begin{pmatrix} V & 1 - VV^* \\ 0 & V^* \end{pmatrix} \in M_{2n}(D_\Phi(A)) \). It is easy to check that \( VTV^* + 1 - VV^* = \begin{pmatrix} vTv^* & 0 \\ 0 & 1 \end{pmatrix} \) and \( S \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix} S^* = \begin{pmatrix} vTv^* + 1 - VV^* & 0 \\ 0 & 1 \end{pmatrix} \).

Therefore \([T]_1 = [vTv^*]_1\). It is left to the reader to check that \( vTv^* \in D(A) \) is also unitary. \[ \square \]

**Proposition 2.13.** \( KK(A, \mathbb{C}) \cong K_1(D_\Phi) \) where \( \Phi \) is an admissible representation of unital separable \( C^* \)-algebra \( A \) on a separable Hilbert space \( \mathbb{H} \).

**Proof.** With the Lemma 2.12 we define the map from \( K_1(D_\Phi) \) to \( KK(A, \mathbb{C}) \) as follows.

\[
[u]_1 \mapsto \left[ \left( \hat{\mathbb{H}}, \begin{pmatrix} \Phi & 0 \\ 0 & \Phi \end{pmatrix} \right), \left( \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix} \right) \right]
\]

where \( \hat{\mathbb{H}} \) is a graded Hilbert \( \mathbb{C} \)-module \( \mathbb{H} \oplus \mathbb{H} \) with the standard even grading. (See Chapter 14.2 in [Bi]). Indeed, this construction gives rise to well-defined group homomorphism. If \([u] = [v]\), then \( u \oplus 1 \) is homotopic to \( v \oplus 1 \) up to unitary.

Therefore \( \left( \hat{\mathbb{H}} \oplus \hat{\mathbb{H}}, \begin{pmatrix} \Phi & 0 \\ 0 & \Phi \end{pmatrix} \right), \left( \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix} \right) \) is operator homotopic to \( \left( \hat{\mathbb{H}}, \begin{pmatrix} \Phi & 0 \\ 0 & \Phi \end{pmatrix} \right), \left( \begin{pmatrix} 0 & v^* \\ v & 0 \end{pmatrix} \right) \).
Let $\alpha \in KK(A, \mathbb{C})$ be represented by \((H_0 \oplus H_1, (\phi_0 \ 0, 0 \ u^* \ \phi_1), (0 \ u \ 0))\) where $u$ is a unitary in $\mathcal{B}(H_0, H_1)$. Let $\Psi = \cdots \phi_0 \oplus \phi_0 \oplus \phi_1 \oplus \phi_1 \cdots$ and $H = \cdots H_0 \oplus H_0 \oplus H_1 \oplus H_1 \cdots$. We consider a degenerate cycle \[
(H \oplus H, (\Psi \ 0 \ 0), (0 \ I \ 0))\] Then \[
(H_0 \oplus H_1, (\phi_0 \ 0, 0 \ u^* \ \phi_1), (0 \ u \ 0)) \oplus (H \oplus H, (\Psi \ 0 \ 0), (0 \ I \ 0))\] is unitarily equivalent to \[
(H \oplus H, (\Psi \ 0 \ 0), (0 \ F^* \ 0))\] where $F = \begin{pmatrix} I & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & I \end{pmatrix}$ or shifting to the right i.e. $F$ sends $(\cdots, \eta_1, \eta_0, \xi_0, \xi_1, \cdots)$ to $(\cdots, \eta_2, \eta_1, u\eta_0, \xi_0, \cdots)$.

Again by adding a degenerate cycle \[
(\hat{H}, (\Phi \ 0 \ 0), (0 \ I \ 0))\], we get \[
\alpha = \left[\begin{pmatrix} H \oplus H, (\Psi \oplus \Phi \ 0 \ 0), (0 \ F^* \oplus I) \end{pmatrix}\right] \]
Since $\Phi$ is admissible, we obtain a unitary $U \in \mathcal{B}(H \oplus H, H)$ such that $Ad(U) \circ \Psi \oplus \Phi \sim \Phi$,
\[
= \left[\begin{pmatrix} \hat{H}, (Ad(U) \circ \Psi \oplus \Phi) \oplus (Ad(U) \circ \Psi \oplus \Phi), (0 \ Ad(U)(F^* \oplus I) \ Ad(U)(F^* \oplus I)) \end{pmatrix}\right]
\]
By the lemma 4.1.10. in [JenThom]
\[
= \left[\begin{pmatrix} \hat{H}, \Phi \oplus \Phi, (0 \ Ad(U)(F^* \oplus I) \ Ad(U)(F^* \oplus I)) \end{pmatrix}\right]
\]
It is not hard to check that $Ad(U)(F \oplus I) \in D_\Phi(A)$ and $\alpha$ is the image of it. So we finish the proof. \(\square\)

Remark 2.14. A unital $C^*$-algebra $A$ is said to have $K_1$-surjectivity if the natural map from $U(A)/U_0(A)$ to $K_1(A)$ is surjective and is said to have (strong)$K_0$-surjectivity if the group $K_0(A)$ is generated by $\{[p] \ | \ p$ is a projection in $A\}$. Therefore Lemma 2.9 and Lemma 2.12 show $D_\Phi(A)$ has (strong)$K_0$-surjectivity and $K_1$-surjectivity.
Now we are ready to define the index pairing between $K_i(A)$ and $K_{i+1}(D\Phi(A))$ for all $i = 0, 1$. For the following two definitions, we mean Index as the classical Fredholm index.

Given a projection $p \in M_k(A)$ and $u \in K_1(D\Phi(A))$, when $\Phi_k$ is k-th amplification of $\Phi$, the operator

$$\Phi_k(p)u^{(k)}\Phi_k(p) : \Phi_k(p)(\mathbb{H}^k) \to \Phi_k(p)(\mathbb{H}^k)$$

is essentially unitary, and therefore Fredholm.

**Definition 2.15.** The (even)index pairing $K_0(A) \times K_1(D\Phi(A)) \to \mathbb{Z}$ is defined as follows.

$$([p], [u]) \longrightarrow \text{Index}\left(\Phi_k(p)u^{(k)}\Phi_k(p)\right)$$

where $p \in M_k(A)$ and $\Phi_k$ is k-th amplification of $\Phi$.

Similarly, given $v \in M_k(A)$ and $p \in K_0(D\Phi(A))$, the operator

$$p^{(k)}\Phi_k(v)p^{(k)} - (1 - p^{(k)}) : \mathbb{H}^k \to \mathbb{H}^k$$

is essentially unitary, and therefore Fredholm.

**Definition 2.16.** The (odd)index pairing $K_1(A) \times K_0(D\Phi(A)) \to \mathbb{Z}$ is defined as follows.

$$([v], [p]) \longrightarrow \text{Index}\left(p^{(k)}\Phi_k(v)p^{(k)} - (1 - p^{(k)})\right)$$

if $v \in M_k(A)$ and $\Phi_k$ is k-th amplification of $\Phi$.

3. **Kasparov Product and Duality**

In this section, we prove main results: Each index pairing is a special case of Kasparov product. Before doing this, we need some elementary computations of Kasparov groups.

**Proposition 3.1.** $KK(S, B) = K_1(B)$ where $S = \{ f \in C(\mathbb{T}) \mid f(1) = 0 \}$ and $B$ is a $C^*$-algebra.

**Proof.** Most of proof can be found in [Lee]. We just note that any unitary in $K_1(B)$ can be liftable to $\phi \in KK(S, B)$ here $\phi : S \to B$ which is determined by sending $z - 1$ to $u - 1$.

**Theorem 3.2.** (Even case) the mapping $K_0(A) \times K_1(D\Phi) \to \mathbb{Z}$ is the Kasparov product $KK(S, SA) \times KK(SA, S) \to \mathbb{Z}$.

**Proof.** Without loss of generality, we may assume $p$ is the element of $A$. (If necessary, consider $\mathbb{C}^k \otimes A$) Using the K-theory Bott map, $p$ is mapped to $f_p(z) \in K_1(SA)$. Then as we have noted in Proposition 3.1 $f_p(z)$ is lifted to $\Psi$ as the element of $KK(S, SA)$ where $\Psi$ is the
*-homomorphism from $S$ to $SA$ which is determined by sending $z - 1$ to $(z - 1)p$.

On the other hand, $[u] \in K_1(D\Phi(A))$ is mapped to $[\mathcal{E}]$ in $KK(A, \mathbb{C})$ where $\mathcal{E} = \left( \mathbb{H}, (\Phi \circ \Phi), (0 \ 0) \right)$ is a Kasparov A-$\mathbb{C}$ module by Proposition 2.13. Using natural isomorphism $\tau_S : KK(A, \mathbb{C}) \mapsto KK(SA, S)$, we can think of a Kasparov product $\Psi$ by $[\tau_S(\mathcal{E})]$. Using elementary functorial properties, we can check $\Psi$ to $\tau$ to $\Psi$. Using natural isomorphism $2.13$. where $E$ is determined by sending $\rho$ to $(\hat{H}, \rho \oplus \rho, G)$.

Note that $\rho : \mathbb{C} \mapsto \mathcal{B}(\mathbb{H})$. Similarly, we have

Corollary 3.4. Let $x \in KK(C_1, S) \cong Ext(\mathbb{C}, S)$ be represented by the extension $0 \mapsto S \mapsto C \mapsto \mathbb{C} \mapsto 0$ and $y \in KK(S, C_1) \cong Ext(S, \mathbb{C})$ be
represented by the extension $0 \to K \to C^*(v - 1) \to S \to 0$ where $v$ is a coisometry of Fredholm index 1 (e.g. the adjoint of the unilateral shift). Then $\mathbf{x} \cdot \mathbf{y} = 1_{C_1}$. 

Proof. Note that $\mathbf{x}$ corresponds to the unitary $t \mapsto e^{2\pi it}$ in $K_1(S)$ by the Brown’s Universal Coefficient Theorem [Br]. Also, the Busby invariant of $0 \to K \to C^*(v - 1) \to S \to 0$ is the homomorphism $\tau : S \to \mathbb{Q}$ sends $e^{2\pi it} - 1$ to $\pi(v) - 1$. Since $KK(C_1, C_1) \cong \mathbb{Z}$, using Theorem 3.3 we can conclude $\mathbf{x} \cdot \mathbf{y} = 1_{C_1}$. □

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