Perfect random number generator is unnecessary for secure quantum key distribution

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Abstract

Quantum key distribution(QKD) makes it possible for two remotely separated parties do unconditionally secure communications. In principle, the security is guaranteed by the uncertainty principle in quantum mechanics: if any third party watches the key, she must disturbs the quantum bits therefore she has a risk to be detected. However, the security in practice is quite different, since many of the assumptions of the ideal case do not exist. Our presently existing secure proof of QKD protocols require the perfect random number generators. Actually, we can never have perfect generators in the real world. Here we show that the imperfect random numbers can also be used for secure QKD, if they satisfy certain explicit condition.

Quantum key distribution(QKD) has abstracted strong interests of scientists since it makes it possible to set up unconditional secure key between two remote parties by principles of quantum mechanics. However, the unconditional security in principle does not necessarily give rise to the unconditional security in practice, where many non-ideal factors occur. "The most important question in quantum cryptography is to determine how secure it really is" [1]. Different from the assumed ideal situation, there are many imperfections in realizing

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anything in the real world. Consider the case of QKD. These imperfections may include the channel noise, small errors in source and devices, biased random number generators and so on. So far, the security proof with channel noise have been given by a number of authors [2,3,5–12]. The security was then extended to the case including small errors in source and devices [13,14]. However, the effect of imperfection of random numbers is still unknown. This causes problems in practice, though it is many people’s belief that sufficiently good random numbers must also work very well. However, if the security is based on such a belief, then it is still conditional security instead of the unconditional one and we don’t know why this belief must be stronger than the beliefs on the assumed complexity of certain mathematical problems which are the base of classical key distribution. Therefore a strict proof is needed here for unconditional security of QKD in the real world. Without an explicit analysis on the effect of imperfect random numbers, we don’t know how good is sufficient therefore we have no choice but to blindly increase the quality of random number generators. This can in principle raise the total cost unlimitedly. No matter how much we have done in improving the quality of our random numbers, we still worry the security a little bit, e.g., we don’t know whether a bias of $10^{-10}$ or $10^{-30}$ undermines the security severely. The best way to solve the issue is to give an explicit study on the effect of the imperfections with certain operational criterion. We show that, a QKD with a good imperfect random number generator (IRNG) with certain explicit condition is secure if the same QKD protocol with a perfect random number generator (PRNG) is secure.

We start from the definition of a PRNG and quantifying of an IRNG. Consider the case that the generator produces an $\omega$-bit string, $s$. There are $2^\omega$ possible different strings in all, we define all $\omega$ string as set $\{s_i, i = 1, 2 \cdots 2^\omega\}$. An $\omega$-bit PRNG is defined as a generator which generates every string with equal probability over $\{s_i\}$, i.e., every string in $\{s_i\}$ have the same probability, $2^{-\omega}$ to be generated. If a generator generates $\omega$-bit strings with a non-uniform probability distribution over $\{s_i\}$, it is an imperfect random number generator, IRNG. Specifically, we quantify the quality of an $\omega$-bit random number generator by the value of entropy: Suppose it generates string $s_i$ with probability $p_i$, the entropy
\[ H(s) = -\sum_{i=1}^{2^\omega} p_i \log_2 p_i. \]  

In particular, we denote \( R_{\omega, \theta} \) for an IRNG which generates \( \omega \)-bit binary strings with entropy \( -\sum p_i \log_2 p_i = \theta \). In case of PRNG, the probability distribution is uniform and the above function reaches its maximum value \( \omega \). Given an IRNG, the entropy is always less than \( \omega \). Here we shall consider the case of a good IRNG, of which the entropy value is \( \omega - \epsilon \).

In practice, it is the case that Alice and Bob assume that they are using a perfect random string but actually they are using an imperfect random string which is a little bit different from the perfect one. It can also be the case that they only know the lower bound of the quality of their generator, say \( H(s) \geq \omega - \epsilon \), but they don’t know the explicit pattern of the strings and they have no way to change the generator to a perfect one. However, we should assume the worst case that Eavesdropper (Eve) knows the pattern of their string though they themselves don’t know it. For example, a very smart Eve could find out the pattern from the history of the data. Also, in QKD, we assume Eve knows the protocol itself. This means Eve knows the specific status of all the devices involved, including the random number generators. Therefore, given the existing security proofs with PRNGs, we still worry a little bit that Eve could take advantage of her knowledge about the IRNG being used in the protocol and obtain a larger amount of information than the theoretical upper bound in the case PRNGs are used. Our purpose is to see whether Eve can obtain significantly large information to the final key generated by certain QKD protocol with good IRNG, if Eve’s information to the final key by the same protocol with PRNG is in principle bounded by a very small value. In all existing QKD protocols, both Alice and Bob needs some random numbers for the task. For example, in BB84 protocol [4], Alice needs the random numbers to prepare the initial quantum state \( |0\rangle \) or \( |1\rangle \) for each qubits; he needs to choose a subset of the qubits for the error test and he also needs random numbers to make error correction, privacy amplification finally. For all these issues, Alice only needs to prepare an \( \omega_a \)-bit binary random string \( s \) in the beginning, if the protocol needs \( \omega_a \) random bits in all at Alice’s side. In carrying out the protocol, Alice just reads \( s \) from left to right, whenever a
random bit is needed. Bob also needs random numbers to determine his measurement bases \((\{|0\rangle, |1\rangle\})\) or \(\{± = \frac{1}{\sqrt{2}}(|0\rangle ± |1\rangle)\}\). We at this moment assume Bob has perfect random numbers while Alice does not. After we complete the proof of our **Theorem**, we extend it to the case that Bob does not have perfect random numbers either. We shall first show the following theorem:

**Theorem:** Given any QKD protocol \(P\), suppose Bob always uses a PRNG and Eve knows what random generators are used by Alice and Bob, if Eve’s information is bounded by \(\varepsilon_0\) in the case that Alice uses a PRNG, then Eve’s information to the \(k\)-bit final key is bounded by

\[
\varepsilon_0 + (4k + 1)\sqrt{\varepsilon_A/2 + O(\varepsilon_A^{3/2})} + O(\varepsilon_A)
\]

to the final key in the case Alice uses an IRNG, \(R_{\omega_A, \varepsilon_A}\).

For clarity let us first recall the theorem of Holevo bound \([15,16]\).

Clare announces the following facts: He will Alice an \(\omega_a\)-bit state which can be either \(\rho_0\) or \(\rho_1\), with equal probability. He sets his bit value \(X = 0\) if he passes \(\rho_0\) to Alice, and \(X = 1\) if he passes \(\rho_1\) to Bob. It’s known to all parties that \(\rho_0 = (2^{−\omega_a}) \sum_{i=1}^{2^{\omega_a}} |s_i\rangle \langle s_i|\) and \(\rho_1 = \sum_{i=1}^{2^{\omega_a}} p_i |s_i\rangle \langle s_i|\). State \(|s_i\rangle\) is a product state of \(\omega_a\) qubits with each of them being prepared in \(\{|0\rangle, |1\rangle\}\) basis. String \(s_i\) gives the full information of state of each qubits, e.g., if \(s_j = 01010011 \cdots 10\), then \(|s_j\rangle = |01010011 \cdots 10\rangle\). In such a case, using Holevo’s theorem \([15]\) we find that Alice’s information to bit \(X\) is bounded by

\[
h = H(\bar{p}) - \frac{\omega_a}{2} - \frac{1}{2}H(p)
\]

and \(p = p_i\). In fact, in this case, if Alice directly observes each qubits in \(\{|0\rangle, |1\rangle\}\) basis she can reach the upper bound of information to \(X\). With a little bit calculation, one immediately obtain the fact that

\[
h \leq \varepsilon_a/2.
\]

In what follows we shall show that, with the restriction by Holevo’s theorem, Eve’s information to the final key must be negligible in a QKD with good IRNG \(R_{\omega_A, \varepsilon_A}\), if her information
is in principle negligible in the same QKD protocol with perfect random numbers. We now consider the following game

**Game G:** Clare announces that he will pass Alice an $\omega_a$-bit state which can be either $\rho_0$ or $\rho_1$, with equal probability, and he sets his bit value $X = 0$ if he passes $\rho_0 = (2^{-\omega_a}) \sum_{i=1}^{2^{\omega_a}} |s_i\rangle\langle s_i|$ to Alice, and $X = 1$ if he passes state $\rho_1 = \sum_{i=1}^{2^{\omega_a}} p_i |s_i\rangle\langle s_i|$ to Bob. Alice measures each qubits of the state from Clare in $\{|0\rangle, |1\rangle\}$ basis and obtain a classical string $s$. Using $s$ as the random string Alice runs QKD protocol $P$ with Bob. At this moment Eve attacks the protocol just as if she were a real eavesdropper. If the protocol does not pass the error test, Alice gives it up and uses string $s$ to obtain the information about $X$, in such a case she can reach the Holevo bound. If the protocol passes the error test, they continue and set up a $k$-bit final key $Y$ and then Alice announces it. In such a case, Eve can obtain information about $X$ by reading the final key, and Eve reports her information about $X$ to Alice and Alice uses this as her own information about $X$. Obviously, Eve’s information about $X$ must also be bounded by $h$, otherwise the result of our game violates Holevo’s theorem.

Suppose scheme $T$ is the optimal attack to QKD protocol $P$ with imperfect random string whose Shannon entropy is $\epsilon_A$ (but $T$ is not necessarily optimal to the same protocol with perfect random string). Suppose Eve attacks $Y$ by scheme $T$. Without any loss of generality, $T$ has the following property: if string $s$ used by Alice is perfectly random, Eve acquires information $\epsilon'$ about $Y$. If $s$ is from generator $R_{\omega_a, \epsilon_A}$, Eve’s information about the final key is optimized, we denote it by $\eta$ in such a case. In our game $R_{\omega_a, \epsilon_A}$ corresponds to the case $X = 1$. Intuitively, $\eta$ should not be too large given $\epsilon'$ being very small, since otherwise after Alice announces $Y$, Eve may easily see whether her actual information about $Y$ prior to the announcement is $\eta$ or $\epsilon'$ therefore she can access an unreasonably large amount of information about Clare’s bit $X$. After attack $T$, Eve has 2 sets of probability distribution $P = \{P_i\}, Q = \{Q_i\}$ about the $k$-bit final key $Y$, conditional on $X = 0, 1$, respectively. Before reading final key $Y$, these two sets of distribution about $Y$ have equal probability. More specifically, after reading the final key $Y$, the two distributions $P$ and $Q$ can be
different. Therefore probability of \( X = 0 \) and \( X = 1 \) can also be different after Eve reads \( Y \). That is to say, in reading \( Y \), Eve may obtain different probabilities for the probability distribution \( P \) and \( Q \).

For simplicity, the two probabilities for a specific possible final key, \( Y_j \). Consider one possible way for Alice to violate the Holevo’s theorem: If the final key is not \( Y_j \), she disregard the QKD result just uses string \( s \) itself and obtain information about \( X \) in the amount of Holevo bound. If the final key is \( Y_j \), she announces \( Y_j \) and uses Eve’s information about \( X \) as her own information. Therefore, Eve’s two probabilities \( (P_j, Q_j) \) about any \( Y_j \) cannot be too different, otherwise Alice has non-zero chance to violate Holevo’s theorem. Specifically, we have the following restriction

\[
I_E(X : Y_j) = 1 - \frac{1}{2} H(P'_j) - \frac{1}{2} H(Q'_j) \leq h. \tag{4}
\]

Here \( P'_j = \frac{P_j}{P_j + Q_j}, \quad Q'_j = \frac{Q_j}{P_j + Q_j} \), and \( H(t) = -t \log_2 t - (1-t) \log_2 t \). For simplicity we shall use \( \log \) instead of \( \log_2 \) hereafter. The above formula is equivalent to

\[
I_E(X : Y_j) = 1 + \frac{1}{2 + \delta} \log \frac{1}{2 + \delta} + \frac{1 + \delta}{2 + \delta} \log \frac{1 + \delta}{2 + \delta} \leq h. \tag{5}
\]

Here \( \delta_j \) is defined by \( \delta_j = Q_j/P_j - 1 = Q'_j/P'_j - 1 \). After a further reduction we obtain

\[
I_E(X : Y_j) = -\frac{1}{(2 + \delta_j)} \log(1 + \delta_j/2) + \frac{1 + \delta_j}{2 + \delta_j} \log \left(1 + \frac{\delta_j}{2 + \delta_j}\right) \tag{6}
\]

If \( \delta_j \geq 0 \), we have

\[
I_E(X : Y_j) \geq -\frac{\delta_j}{2(2 + \delta_j)} + \frac{1 + \delta_j}{(2 + \delta_j)} \left[ \frac{\delta_j}{2 + \delta_j} - \frac{\delta_j^2}{2(2 + \delta_j)^2} \right] \geq \frac{\delta_j^2}{2(2 + \delta_j)^2}; \tag{7}
\]

if \( \delta_j < 0 \), we have

\[
I_E(X : Y_j) \geq -\frac{1}{(2 + \delta_j)}(\delta_j/2 + \delta_j^2/8) + \frac{\delta_j(1 + \delta_j)}{(2 + \delta_j)^2} \geq \frac{2\delta_j^2 - \delta_j^3}{8(2 + \delta_j)^2} \geq \frac{2\delta_j^2}{8(2 + \delta_j)^2}. \tag{8}
\]

In any case, we have

\[
|\delta_j| \leq |\Delta = 4[1 + O(\sqrt{h})]\sqrt{h} \tag{9}
\]
for any $j$, given the restriction of formula (4). With this formula, we can now calculate the lower bound of $H(Q)$, the entropy to the whole $k$-bit final key, given distribution $Q = \{Q = i\}$.

$$H(Q) = H(\{P_i(1 + \delta_i)\}) = - \sum_i P_i(1 + \delta_i) \log[P_i(1 + \delta_i)]$$

$$\geq (1 - |\Delta|)H(P) - (1 + |\Delta|) \log(1 + |\Delta|)$$

$$\geq k - \epsilon' - (4k + 1)\left[1 + O(\sqrt{h})\right]\sqrt{h} - O(h).$$  (10)

Since $h \leq \epsilon_A/2$, we have

$$\eta \leq \epsilon' + (4k + 1)\left[1 + O(\sqrt{h})\right]\sqrt{h} - O(h).$$  (11)

Note that we always have $\epsilon' \leq \epsilon_0$, since $\epsilon_0$ is the upper bound of any attacks to a QKD protocol with PRNG, $\epsilon'$ is the information through $T$, which is an optimized attack to QKD with IRNG, but not necessarily also an optimized attack to the same QKD protocol with PRNG. Therefore we complete the proof of our theorem by replacing $\epsilon'$ with $\epsilon_0$. Now we consider the case that Bob’s random string is also imperfect, say, he uses an IRNG $R_{\omega_b, \epsilon_B}$. Since we have already known that Eve’s information is bounded by $\epsilon_0 + (4k + 1)\left[1 + O(\sqrt{h})\right]\sqrt{h} - O(h)$ we the case that Alice uses IRNG and Bob uses PRNG, we now just consider game $G'$ where David passes Bob an $\omega_b$-qubit state $|\rho_0\rangle = \sum_{i=0}^{2^\omega_b} |s_i\rangle\langle s_i|$ if he sets $X = 0$ and passes Bob an $\omega_b$-qubit state $|\rho_1\rangle = \sum_{i=0}^{2^\omega_b} |s_i\rangle\langle s_i|$ if he sets $X = 0$ and passes Bob an $\omega_b$-qubit state $|\rho_1\rangle = \sum_{i=0}^{2^\omega_b} p_i'|s_i\rangle\langle s_i|$ if he sets $X = 1$. Similarly to the proof for our theorem, we have the following corollary: Suppose Eve always knows what random number generators are used by Alice and Bob in a certain QKD protocol $P$. If Eve’s information is upper bounded by $\epsilon_0$ in the case PRNGs are used, then Eve’s information is upper bounded by $\eta \leq \epsilon_0 + (4k + 1)\sqrt{\epsilon_A/2} + (4k + 1)\sqrt{\epsilon_B/2} + O(\epsilon_A^{3/2}) + O(\epsilon_A)$ to the $k$-bit final key in the case Alice uses IRNG $R_{\omega_a, \epsilon_A}$ and Bob uses IRNG $R_{\omega_b, \epsilon_B}$.

We conclude that, if a QKD protocol is secure with perfect random numbers being used, it must be also secure with exponentially small imperfections in the random numbers. Therefore perfect random number generators which never exist in the real world are not necessary for
secure QKD.

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