Blow-up for a weakly coupled system of semilinear damped wave equations in the scattering case with power nonlinearities

Alessandro Palmieri\textsuperscript{a}, Hiroyuki Takamura\textsuperscript{b}

\textsuperscript{a}Institute of Applied Analysis, Faculty for Mathematics and Computer Science, Technical University Bergakademie Freiberg, Prüferstraße 9, 09596, Freiberg, Germany

\textsuperscript{b}Mathematical Institute, Tohoku University, Aoba, Sendai 980-8578, Japan

Abstract

In this work we study the blow-up of solutions of a weakly coupled system of damped semilinear wave equations in the scattering case with power nonlinearities. We apply an iteration method to study both the subcritical case and the critical case. In the subcritical case our approach is based on lower bounds for the space averages of the components of local solutions. In the critical case we use the slicing method and a couple of auxiliary functions, recently introduced by Wakasa-Yordanov, to modify the definition of the functionals with the introduction of weight terms. In particular, we find as critical curve for the pair \((p,q)\) of the exponents in the nonlinear terms the same one as for the weakly coupled system of semilinear wave equations with power nonlinearities.

Keywords: Semilinear weakly coupled system; Blow-up; Scattering producing damping; Critical curve; Slicing method.

2010 MSC: Primary 35L71, 35B44; Secondary 35G50, 35G55

1. Introduction

In this paper we consider a weakly coupled system of semilinear wave equations with time-dependent, scattering producing damping terms and power nonlinearities, namely,

\[
\begin{aligned}
  &u_{tt} - \Delta u + b_1(t)u_t = |v|^p, & x \in \mathbb{R}^n, t > 0, \\
  &v_{tt} - \Delta v + b_2(t)v_t = |u|^q, & x \in \mathbb{R}^n, t > 0, \\
  & (u, u_t, v, v_t)(0, x) = (\varepsilon u_0, \varepsilon u_1, \varepsilon v_0, \varepsilon v_1)(x) & x \in \mathbb{R}^n,
\end{aligned}
\]  

(1)

where \(b_1, b_2 \in \mathcal{C}([0, \infty)) \cap L^1([0, \infty))\) are nonnegative functions, \(\varepsilon\) is a positive parameter describing the size of initial data and \(p, q > 1\). We will prove blow-up results for (1) both in the subcritical case and in the critical case.

Let us provide now an historical overview on some results, which are strongly related to our model and the motivations that lead us to consider the nonlinear model (1). Recently, the Cauchy problem for the semilinear wave equation with damping in the scattering case

\[
\begin{aligned}
  &u_{tt} - \Delta u + b(t)u_t = f(u, \partial_t u), & x \in \mathbb{R}^n, t > 0, \\
  & (u, u_t)(0, x) = (\varepsilon u_0, \varepsilon u_1)(x) & x \in \mathbb{R}^n,
\end{aligned}
\]  

(2)

has been studied in [18, 36], [19], [20] in the cases \(f(u, \partial_t u) = |u|^p, |\partial_t u|^p, |\partial_t u|^p + |u|^q\) with \(p, q > 1\), respectively, provided that \(b\) is a continuous, nonnegative and summable function. In particular, for the...
power nonlinearity $|u|^p$, combining the result in the subcritical case from [18] and the result in the critical case from [36], we see that the range of values of $p$, for which a blow-up result can be proved, is the same as in case of the classical semilinear wave equation with power nonlinearity. Furthermore, in the above cited papers the upper bounds for the lifespan of the solutions are shown to be the same one (that means also the sharp one) for the classical semilinear wave model. More precisely, the condition for the exponent $p$ of the semilinear term, that implies the validity of a blow-up result, is $1 < p \leq p_0(n)$ for $n \geq 2$, where $p_0(n)$ denotes the Strauss exponent, i.e., the positive root of the quadratic equation

$$(n-1)p^2 - (n+1)p - 2 = 0,$$

and $p > 1$ in the one dimensional case. This condition on $p$ is equivalent to require

$$\frac{1 + p^{-1}}{p - 1} \geq \frac{n - 1}{2}. \quad (3)$$

For the corresponding results in the case of semilinear wave equations we refer to the works [13, 31, 14, 8, 7, 30, 29, 23, 5, 33, 12, 37, 40] for the proof of Strauss’ conjecture and to [30, 22, 38, 39, 24, 32, 41] for the proof of the sharp estimates of the lifespan of local in time solutions.

On the other hand, it is known that for the weakly coupled system of classical wave equations

$$\begin{cases}
u_{tt} - \Delta v = |u|^q, & x \in \mathbb{R}^n, \ t > 0, \\
v_{tt} - \Delta v = |u|^q, & x \in \mathbb{R}^n, \ t > 0, \\
(u, u_t, v, v_t)(0, x) = (\varepsilon u_0, \varepsilon u_1, \varepsilon v_0, \varepsilon v_1)(x) & x \in \mathbb{R}^n,
\end{cases} \quad (4)$$

the critical curve for the pair $(p, q)$ of exponents is given by the cubic relation

$$\max \left\{ \frac{p + 2 + q^{-1}}{pq - 1}, \frac{q + 2 + p^{-1}}{pq - 1} \right\} = \frac{n - 1}{2}.$$

For further details on the results for (4) we refer to [4, 2, 3, 1, 16, 15, 6, 17]. So, we see that the study of the weakly coupled system is not just a simple generalization of the result for the single semilinear equation. Indeed, it holds

$$\max \left\{ \frac{p + 2 + q^{-1}}{pq - 1}, \frac{q + 2 + p^{-1}}{pq - 1} \right\} \geq \max \left\{ \frac{1 + p^{-1}}{p - 1}, \frac{1 + q^{-1}}{q - 1} \right\}, \quad (5)$$

where the equality is satisfied only in the case $p = q$. Therefore, according to (3) and (5), for $p \neq q$ it may happen that

$$\max \left\{ \frac{p + 2 + q^{-1}}{pq - 1}, \frac{q + 2 + p^{-1}}{pq - 1} \right\} \geq \frac{n - 1}{2}.$$

(that is, $(p, q)$ belongs to the blow-up region in the $p$ - $q$ plane) even though one among $p, q$ is greater than the Strauss exponent.

The goal of this paper is to prove for the weakly coupled system (1) blow-up results for the same range of pair $(p, q)$ as in the corresponding results for (4) and, furthermore, the same upper bound estimates for the lifespan of local solutions.

From a more technical point of view, in this paper we will generalize the approaches for the Cauchy problem (2) in the case of a power nonlinearity developed by [18] in the subcritical case and [36] in the critical case to the study of a weakly coupled system of semilinear wave equations with damping terms in the scattering case. In the subcritical case the multiplier introduced in [18] plays a fundamental role, in order to make the iteration frame for our model analogous to the one for the corresponding case without damping. In the critical case, however, a nontrivial generalization of the approach by Wakasa-Yordanov is necessary, in order to take into account of the asymmetric behavior of the model on the critical curve except
for the cusp point \( p = q \). This situation will be dealt with the aid of an asymmetric frame in the iteration scheme. On the other hand, in the special case \( p = q \) the situation is completely symmetric to what happens in the case of a single equation.

Finally, let us point out that, due to the general structure of the coefficients for the damping terms, we may not apply the revisited test function method recently developed by Ikeda-Sobajima-Wakasa for the classical wave equation in [11], whose approach is based on a family of self-similar solutions (see also [9] for the application of this method to the semilinear heat, damped wave and Schrödinger equations and [10, 28, 25] in the scale-invariant case).

Before stating the main results of this paper, let us introduce a suitable notion of energy solutions according to [21].

**Definition 1.1.** Let \( u_0, v_0 \in H^1(\mathbb{R}^n) \) and \( u_1, v_1 \in L^2(\mathbb{R}^n) \). We say that \( (u, v) \) is an energy solution of (1) on \([0, T)\) if

\[
\begin{align*}
\int_{\mathbb{R}^n} \partial_t u(t, x) \phi(t, x) \, dx - \int_{\mathbb{R}^n} \varepsilon u_1(x) \phi(0, x) \, dx - \int_0^t \int_{\mathbb{R}^n} \partial_t u(s, x) \phi_s(x, s) \, dx \, ds \\
+ \int_0^t \int_{\mathbb{R}^n} \nabla u(s, x) \cdot \nabla \phi(s, x) \, dx \, ds + \int_0^t \int_{\mathbb{R}^n} b_1(s) \partial_t u(s, x) \phi(s, x) \, dx \, ds \\
= \int_0^t \int_{\mathbb{R}^n} |v(s, x)|^p \phi(s, x) \, dx \, ds \\
\end{align*}
\]

(6)

and

\[
\begin{align*}
\int_{\mathbb{R}^n} \partial_t v(t, x) \psi(t, x) \, dx - \int_{\mathbb{R}^n} \varepsilon v_1(x) \psi(0, x) \, dx - \int_0^t \int_{\mathbb{R}^n} \partial_t v(s, x) \psi_s(x, s) \, dx \, ds \\
+ \int_0^t \int_{\mathbb{R}^n} \nabla v(s, x) \cdot \nabla \psi(s, x) \, dx \, ds + \int_0^t \int_{\mathbb{R}^n} b_2(s) \partial_t v(s, x) \psi(s, x) \, dx \, ds \\
= \int_0^t \int_{\mathbb{R}^n} |u(s, x)|^q \psi(s, x) \, dx \, ds \\
\end{align*}
\]

(7)

for any \( \phi, \psi \in \mathcal{C}_c^\infty([0, T) \times \mathbb{R}^n) \) and any \( t \in [0, T) \).

After a further step of integrations by parts, requiring further that the functions \( b_1, b_2 \) are continuously differentiable, (6) and (7) provide

\[
\begin{align*}
&\int_{\mathbb{R}^n} (\partial_t u(t, x) \phi(t, x) - u(t, x) \phi_s(t, x) + b_1(t) u(t, x) \phi(t, x)) \, dx \\
&- \int_{\mathbb{R}^n} (\varepsilon u_1(x) \phi(0, x) - \varepsilon u_0(x) \phi_s(0, x) + b_1(0) \varepsilon u_0(x) \phi(0, x)) \, dx \\
&+ \int_0^t \int_{\mathbb{R}^n} u(s, x) \left( \phi_s(x, s) - \Delta \phi(s, x) - \partial_s(b_1(s) \phi(s, x)) \right) \, dx \, ds \\
&= \int_0^t \int_{\mathbb{R}^n} |v(s, x)|^p \phi(s, x) \, dx \, ds \\
\end{align*}
\]

(8)
and
\[
\int_{\mathbb{R}^n} \left( \partial_t v(t,x)\psi(t,x) - v(t,x)\psi_s(t,x) + b_2(t) v(t,x)\psi(t,x) \right) dx \\
- \int_{\mathbb{R}^n} \left( \varepsilon v_1(x)\psi(0,x) - \varepsilon v_0(x)\psi_s(0,x) + b_2(0) \varepsilon v_0(x)\psi(0,x) \right) dx \\
+ \int_{0}^{t} \int_{\mathbb{R}^n} v(s,x) \left( \psi_s(s,x) - \Delta \psi(s,x) - \partial_s (b_2(s) \psi(s,x)) \right) dx ds \\
= \int_{0}^{t} \int_{\mathbb{R}^n} |u(s,x)|^q \psi(s,x) dx ds.
\] (9)

In particular, letting \( t \to T \), we find that \((u,v)\) fulfills the definition of weak solution to \((1)\).

Let us state the blow-up result for \((1)\) in the subcritical case.

**Theorem 1.2.** Let \( b_1, b_2 \in C([0, \infty)) \cap L^1([0, \infty)) \) be nonnegative functions. Let us consider \( p, q > 1 \) satisfying
\[
\max \left\{ \frac{p + 2 + q^{-1}}{pq - 1}, \frac{q + 2 + p^{-1}}{pq - 1} \right\} > \frac{n - 1}{2}.
\] (10)

Assume that \( u_0, v_0 \in H^1(\mathbb{R}^n) \) and \( u_1, v_1 \in L^2(\mathbb{R}^n) \) are nonnegative, pairwise nontrivial and compactly supported in \( B_R = \{ x \in \mathbb{R}^n : |x| \leq R \} \) functions.

Let \((u,v)\) be an energy solution of \((1)\) with lifespan \( T = T(\varepsilon) \) such that
\[
\text{supp} u, \text{supp} v \subset \{ (t,x) \in [0,T) \times \mathbb{R}^n : |x| < t + R \}. \tag{11}
\]

Then, there exists a positive constant \( \varepsilon_0 = \varepsilon_0(u_0, u_1, v_0, v_1, n, p, q, b_1, b_2, R) \) such that for any \( \varepsilon \in (0, \varepsilon_0) \) the solution \((u,v)\) blows up in finite time. Moreover, the upper bound estimate for the lifespan
\[
T(\varepsilon) \leq C \varepsilon^{-\max\{F(n,p,q),F(n,q,p)\}^{-1}} \tag{12}
\]
holds, where \( C \) is an independent of \( \varepsilon \), positive constant and
\[
F(n,p,q) = \frac{p + 2 + q^{-1}}{pq - 1} - \frac{n - 1}{2}. \tag{13}
\]

**Corollary 1.3.** Let \( n = 1 \) and \( p, q > 1 \), or \( n = 2 \) and \( 1 < p, q < 2 \). Furthermore, we assume that \((p,q)\) and \((u_0, u_1, v_0, v_1)\) satisfy the same assumptions as in Theorem 1.2. If
\[
\int_{\mathbb{R}^n} u_1(x) dx \neq 0 \quad \text{and} \quad \int_{\mathbb{R}^n} v_1(x) dx \neq 0,
\]
then, the lifespan estimate \((12)\) can be improved as follow
\[
T(\varepsilon) \leq C \varepsilon^{-\max\{G(n,p,q),G(n,q,p)\}^{-1}},
\]
where
\[
G(n,p,q) = \frac{2(1 + p^{-1})}{pq - 1} - \frac{n}{p} + n - 2. \tag{14}
\]

**Corollary 1.4.** Let \( n = 2 \) and \( 1 < p < 2, q \geq 2 \). Furthermore, we assume that \((p,q)\) and \((u_0, u_1, v_0, v_1)\) satisfy the same assumptions as in Theorem 1.2. If
\[
\int_{\mathbb{R}^2} u_1(x) dx \neq 0,
\]
then, the lifespan estimate (12) can be improved as follow
\[ T(\varepsilon) \leq Ce^{-\max(F(n,p,q),G(n,p,q))^{-1}}, \]
where \(F(n,p,q)\) and \(G(n,p,q)\) are defined by (13) and (14), respectively.

**Corollary 1.5.** Let \( n = 2 \) and \( 1 < q < 2, p \geq 2 \). Furthermore, we assume that \((p,q)\) and \((u_0,u_1,v_0,v_1)\) satisfy the same assumptions as in Theorem 1.2. If
\[ \int_{\mathbb{R}^n} v_1(x)dx \neq 0, \]
then, the lifespan estimate (12) can be improved as follow
\[ T(\varepsilon) \leq Ce^{-\max(F(n,q,p),G(n,q,p))^{-1}}, \]
where \(F(n,p,q)\) and \(G(n,p,q)\) are defined by (13) and (14), respectively.

In the critical case we have the following result.

**Theorem 1.6.** Let \( b_1, b_2 \in C^{1}([0,\infty)) \cap L^1([0,\infty)) \) be nonnegative functions and let \( n \geq 2 \). Let us consider \( p,q > 1 \) satisfying
\[ \max \left\{ \frac{p + 2 + q^{-1}}{pq - 1}, \frac{q + 2 + p^{-1}}{pq - 1} \right\} = \frac{n - 1}{2}. \] (15)
Assume that \( u_0,v_0 \in H^1(\mathbb{R}^n) \) and \( u_1,v_1 \in L^2(\mathbb{R}^n) \) are nonnegative, pairwise nontrivial and compactly supported in \( B_R \).

Let \((u,v)\) be an energy solution of (1) with lifespan \( T = T(\varepsilon) \) that satisfies (11). Then, there exists a positive constant \( \varepsilon_0 = \varepsilon_0(u_0,u_1,v_0,v_1,n,p,q,b_1,b_2,R) \) such that for any \( \varepsilon \in (0,\varepsilon_0] \) the solution \((u,v)\) blows up in finite time. Moreover, the upper bound estimates for the lifespan
\[ T(\varepsilon) \leq \begin{cases} \exp \left( C\varepsilon^{-\min(q(pq^{-1}),p(pq^{-1}))} \right) & \text{if } p \neq q, \\ \exp \left( C\varepsilon^{-p(p^{-1})} \right) & \text{if } p = q \end{cases} \] (16)
hold, where \( C \) is an independent of \( \varepsilon \), positive constant and \( F = F(n,p,q) \) is defined by (13).

**Remark 1.7.** The upper bound estimates (12) and (16) for the lifespan coincide with the sharp estimates for the lifespan of local solutions to the weakly coupled system of semilinear wave equations with power nonlinearities. However, as we do not deal with global in time existence results for (1) in the present work, we do not derive a lower bound estimate for \( T(\varepsilon) \).

**Remark 1.8.** Let us point out explicitly that in the critical case we need to require more regularity for the time-dependent coefficients \( b_1,b_2 \) in comparison to the subcritical case. Namely, \( b_1,b_2 \) are assumed of class \( C^1 \) rather than being merely continuous. The reason of this stronger assumption is that in the critical case we shall employ (8)-(9) in place of (6)-(7), in order to find the coupled system of ordinary integral inequalities for suitable functionals, whose dynamic is studied to prove the blow-up result.

In this paper we study the nonexistence of global in time solutions for a semilinear weakly coupled system of damped wave equations in the scattering producing case with power nonlinearities and the corresponding upper bound for the lifespan in the same range of powers \((p,q)\) as for the analogous system without damping terms. In two forthcoming papers [26, 27] we will consider as well the case with nonlinearities of derivative type and of mixed type for a semilinear weakly coupled system of damped wave equations in the scattering case.

The remaining part of this paper is organized as follows: in Section 2 we recall a multiplier, that has been introduced in [18] in order to study the corresponding single semilinear equation, and its properties and we derive some lower bounds for certain functionals related to a local solution; then, in Section 3 we prove Theorem 1.2 by using the preparatory results from Section 2 and an iterative method. Finally, in Section 4 we prove the result in the critical case adapting the approach from [35, 36] for a weakly coupled system. In particular, the slicing method is employed in order to deal with logarithmic factors in the iteration argument.
Notations
Throughout this paper we will use the following notations: $B_R$ denotes the ball around the origin with radius $R$; $f \lesssim g$ means that there exists a positive constant $C$ such that $f \leq Cg$ and, similarly, for $f \gtrsim g$; finally, as in the introduction, $p_0(n)$ denotes the Strauss exponent.

2. Definition of the multipliers and lower bounds of the functionals

The arguments used in this section are similar to some of those employed in [18, Section 3]. However, for the sake of self-containedness and readability of the paper, we will provide them.

Definition 2.1. Let $b_1, b_2 \in \mathcal{C}([0, \infty)) \cap L^1([0, \infty))$ be the nonnegative, time-dependent coefficients in (1). We define the corresponding multipliers

$$m_j(t) = \exp \left( - \int_t^\infty b_j(\tau) d\tau \right) \quad \text{for } t \geq 0 \text{ and } j = 1, 2.$$ 

Since $b_1, b_2$ are nonnegative functions, it follows that $m_1, m_2$ are increasing functions. Moreover, due to the fact that these coefficients are summable, we get also that these multipliers are bounded and

$$m_j(0) \leq m_j(t) \leq 1 \quad \text{for } t \geq 0 \text{ and } j = 1, 2. \quad (17)$$

A fundamental property of these multipliers is the relation with the corresponding derivatives. More precisely,

$$m'_j(t) = b_j(t) \, m(t) \quad \text{for } j = 1, 2. \quad (18)$$

Such a relation will play a fundamental role in the remaining part of this section, which is devoted to the determination of lower bounds for the spatial integral of the nonlinear terms and to the deduction of a pair of coupled integral inequalities for the spatial averages of the components of a local solution to (1).

Lemma 2.2. Let us assume that $u_0, u_1, v_0, v_1$ are nonnegative, pairwise nontrivial and compactly supported in $B_R$ for some $R > 0$. Let $(u, v)$ be a local (in time) energy solution to (1) satisfying (11). Then, there exist two constants $C_1 = C_1(u_0, u_1, b_1, q, R) > 0$ and $K_1 = K_1(v_0, v_1, b_2, p, R) > 0$, independent of $\varepsilon$ and $t$, such that for any $t \geq 0$ and $p, q > 1$, the following estimates hold:

$$\int_{\mathbb{R}^n} |u(t, x)|^q \, dx \geq C_1 \varepsilon^\theta (1 + t)^{n-1} \omega^{-q}, \quad (19)$$

$$\int_{\mathbb{R}^n} |v(t, x)|^p \, dx \geq K_1 \varepsilon^\theta (1 + t)^{n-1} \omega^{-p}. \quad (20)$$

Proof. Let us define the functionals

$$U_1(t) = \int_{\mathbb{R}^n} u(t, x) \Psi(t, x) \, dx \quad \text{and} \quad V_1(t) = \int_{\mathbb{R}^n} v(t, x) \Psi(t, x) \, dx$$

where $\Psi = \Psi(t, x) \equiv e^{-t} \Phi(x)$ and

$$\Phi = \Phi(x) = \begin{cases} e^x + e^{-x} & \text{for } n = 1, \\ \int_{\mathbb{S}^{n-1}} e^{\omega \cdot \xi} \, dS_{\omega} & \text{for } n \geq 2 \end{cases} \quad (21)$$

is an eigenfunction of the Laplace operator, as $\Delta \Phi = \Phi$. Then, by Hölder inequality, we have

$$\int_{\mathbb{R}^n} |u(t, x)|^q \, dx \geq |U_1(t)|^q \left( \int_{|x| \leq t+R} \Phi^q(t, x) \, dx \right)^{-1(q-1)}, \quad (22)$$

$$\int_{\mathbb{R}^n} |v(t, x)|^p \, dx \geq |V_1(t)|^p \left( \int_{|x| \leq t+R} \Phi^p(t, x) \, dx \right)^{-1(p-1)},$$
where $p', q'$ denote the conjugate exponents of $p, q$, respectively. We will prove now (19) by using (22), the proof of (20) being analogous. The next steps consist in determining a lower bound for $U_1(t)$ and an upper bound for the integral $\int_{|x| \leq t + R} \Psi^q(t, x) dx$, respectively.

Due to the support property for $u$, we can apply the definition of energy solution with test functions that are not compactly supported. Applying the definition of energy solution with $\Psi$ as test function and differentiating with respect to $t$ the obtained relation, we find for any $t \in (0, T)$$$
\frac{d}{dt} \int_{\mathbb{R}^n} u_\varepsilon(t, x) \Psi(t, x) dx + \int_{\mathbb{R}^n} \left( -u_\varepsilon(t, x) \Psi_t(t, x) + \nabla u(t, x) \cdot \nabla \Psi(t, x) + b_1(t) u_\varepsilon(t, x) \Psi(t, x) \right) dx
= \int_{\mathbb{R}^n} |v(t, x)|^p \Psi(t, x) dx.$$

Rearranging the previous relation, we get
$$
\int_{\mathbb{R}^n} |v(t, x)|^p \Psi(t, x) dx = \frac{d}{dt} \int_{\mathbb{R}^n} u_\varepsilon(t, x) \Psi(t, x) dx - \int_{\mathbb{R}^n} u_\varepsilon(t, x) \Psi_t(t, x) dx
- \int_{\mathbb{R}^n} u(t, x) \Delta \Psi(t, x) dx + b_1(t) \int_{\mathbb{R}^n} u_\varepsilon(t, x) \Psi(t, x) dx
= \frac{d}{dt} \int_{\mathbb{R}^n} u_\varepsilon(t, x) \Psi(t, x) dx + b_1(t) \int_{\mathbb{R}^n} u_\varepsilon(t, x) \Psi(t, x) dx
+ \int_{\mathbb{R}^n} (u(t, x) \Psi(t, x) - u(t, x) \Psi(t, x)) dx,
$$

where in the last step we used the properties $\Psi_t = -\Psi$ and $\Delta \Psi = \Psi$. Multiplying both sides of the previous relation by the multiplier $m_1$ and employing (18), we obtain
$$
m_1(t) \int_{\mathbb{R}^n} |v(t, x)|^p \Psi(t, x) dx = \frac{d}{dt} \left( m_1(t) \int_{\mathbb{R}^n} u_\varepsilon(t, x) \Psi(t, x) dx \right)
+ m_1(t) \int_{\mathbb{R}^n} (u_\varepsilon(t, x) \Psi(t, x) - u(t, x) \Psi(t, x)) dx.
$$

Integrating the last equality over $[0, t]$, we find
$$
\int_0^t m_1(s) \int_{\mathbb{R}^n} |v(s, x)|^p \Psi(s, x) dx ds
= m_1(t) \int_{\mathbb{R}^n} u_\varepsilon(t, x) \Psi(t, x) dx - \varepsilon m_1(0) \int_{\mathbb{R}^n} u_\varepsilon(0, x) \Phi(x) dx
+ \int_0^t m_1(s) \int_{\mathbb{R}^n} (u_\varepsilon(s, x) \Psi(s, x) - u(s, x) \Psi(s, x)) dx ds.
$$

Noticing that
$$
\int_0^t m_1(s) \int_{\mathbb{R}^n} u_\varepsilon(s, x) \Psi(s, x) dx ds
= m_1(t) \int_{\mathbb{R}^n} u(s, x) \Psi(s, x) dx - \varepsilon m_1(0) \int_{\mathbb{R}^n} u_\varepsilon(0, x) \Phi(x) dx
- \int_0^t \int_{\mathbb{R}^n} u(s, x) \left( m_1'(s) \Psi(s, x) + m_1(s) \Psi_s(s, x) \right) dx ds
= m_1(t) \int_{\mathbb{R}^n} u(s, x) \Psi(s, x) dx - \varepsilon m_1(0) \int_{\mathbb{R}^n} u_\varepsilon(0, x) \Phi(x) dx
- \int_0^t \int_{\mathbb{R}^n} u(s, x) b_1(s) m_1(s) \Psi(s, x) dx ds + \int_0^t m_1(s) \int_{\mathbb{R}^n} u(s, x) \Psi(s, x) dx ds,
$$

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it follows
\[
\int_0^t m_1(s) \int_{\mathbb{R}^n} |v(s,x)| \Psi(s,x) \, dx \, ds + \int_0^t b_1(s) m_1(s) \int_{\mathbb{R}^n} u(s,x) \Psi(s,x) \, dx \, ds \\
+ \varepsilon m_1(0) \int_{\mathbb{R}^n} (u_0(x) + u_1(x)) \Phi(x) \, dx
\]
\[
= m_1(t) \int_{\mathbb{R}^n} (u(t,x) \Psi(t,x) + u(t,x) \Psi(t,x)) \, dx
\]
\[
= m_1(t) \frac{d}{dt} \int_{\mathbb{R}^n} u(t,x) \Psi(t,x) \, dx + 2m_1(t) \int_{\mathbb{R}^n} u(t,x) \Psi(t,x) \, dx.
\]
Using the definition of the functional $U_1$, from the previous relation we derive the inequality
\[
m_1(t)(U_1'(t) + 2U_1(t)) \geq \varepsilon m_1(0) C(u_0, u_1) + \int_0^t b_1(s) m_1(s) U_1(s) \, ds,
\]
where $C(u_0, u_1) \triangleq \int_{\mathbb{R}^n} (u_0(x) + u_1(x)) \Phi(x) \, dx$. Using the boundedness of the multiplier $m_1$, we get
\[
U_1'(t) + 2U_1(t) \geq \varepsilon \frac{m_1(0)}{m_1(t)} C(u_0, u_1) + \frac{1}{m(t)} \int_0^t b_1(s) m_1(s) U_1(s) \, ds
\]
\[
\geq \varepsilon m_1(0) C(u_0, u_1) + \frac{1}{m(t)} \int_0^t b_1(s) m_1(s) U_1(s) \, ds.
\] (23)

A multiplication of both sides in the last estimate by $e^{2t}$ and an integration over $[0,t]$ yield
\[
e^{2t} U_1(t) \geq U_1(0) + \varepsilon \frac{m_1(0)}{2} C(u_0, u_1)(e^{2t} - 1) + \int_0^t e^{2s} \frac{1}{m(s)} \int_0^t b_1(\tau) m_1(\tau) U_1(\tau) \, d\tau \, ds.
\] (24)

A comparison argument proves the positiveness of the functional $U_1$. Due to the fact that initial data are pairwise nontrivial, at least one among $u_0, u_1$ is not identically 0. In the first case $u_0 \neq 0$, since $u_0 \geq 0$ implies $U_1(0) > 0$, by continuity it holds $U_1(t) > 0$ at least in a right neighborhood of $t = 0$. If $t_0 > 0$ was the smallest value such that $U_1(t_0) = 0$, then, evaluation of (24) in $t = t_0$ would provide a contradiction. In the second case $u_0 \equiv 0$ and $u_1 \neq 0$, we can employ (23) to get a contradiction. Indeed, in this case we have $U_1(0) = 0$ and $U_1'(0) = \varepsilon \int_{\mathbb{R}^n} u_1(x) \Phi(x) \, dx > 0$. By continuity, $U_1'(t) > 0$ for any $t \in [0,t_1]$ with $t_1 > 0$. Therefore, $U_1$ is strictly increasing, and then positive, in $(0,t_1)$. Let us assume by contradiction that $t_2 > t_1$ is the smallest value such that $U_1(t_2) = 0$. Consequently, $U_1'(t_2) \leq 0$ (if $U_1'(t_2)$ was positive, then, $U_1$ would be strictly increasing in a neighborhood of $t_2$, but this would contradict the definition of $t_2$, since there would be a smaller zero, $U_1$ being negative in a left neighborhood of $t_2$). If we plug $U_1(t_2) = 0, U_1'(t_2) \leq 0$ and $U_1(t) > 0$ for $t \in (0,t_2)$ in (23), we find the contradiction we were looking for.

In particular, due to the fact that $U_1$ is positive, (24) implies
\[
U_1(t) \geq e^{-2t} U_1(0) + \varepsilon \frac{m_1(0)}{2} C(u_0, u_1)(1 - e^{-2t}) \geq \varepsilon.
\] (25)

The integral involving $\Psi^q$ in the right-hand side of (22) can be estimated in a standard way (cf. estimate (2.5) in [37]), namely,
\[
\int_{|x| \leq t + R} \Psi^q(t,x) \, dx \leq e^{-\frac{q}{p}} \int_{|x| \leq t + R} \Phi^q(x) \, dx \leq C_{q,R} (1 + t)^{n-1-\frac{m}{p} - \frac{m}{q} + \frac{m}{r}},
\] (26)
where $C_{q,R}$ is a suitable positive constant. Combing the estimate (25), (26) and (22), we find (19). This concludes the proof.

**Remark 2.3.** As we have already mentioned the proof of Lemma 2.2 follows the approach from Section 3 in [18]. However, the same estimates can be proved by following the proof of Lemma 5.1 in [35], by working with a different functional in place of $U_1$. 8
3. Subcritical case: Proof of Theorem 1.2

Let us consider a local solution \((u,v)\) of \((1)\) on \([0,T)\) and define the following couple of time-dependent functionals related to this solution:

\[
U(t) = \int_{\mathbb{R}^n} u(t,x) \, dx, \quad V(t) = \int_{\mathbb{R}^n} v(t,x) \, dx.
\]

The proof of Theorem 1.2 consists of two parts. In the first part we determine a pair of coupled integral inequalities for \(U\) and \(V\), while in the second one an iteration argument is used so that the blow-up of \((U,V)\) in finite time can be shown.

3.1. Determination of the iteration frame

If we choose \(\phi = \phi(s,x)\) and \(\psi = \psi(s,x)\) in (6) and in (7), respectively, satisfying \(\phi \equiv 1 \equiv \psi\) on \(\{(x,s) \in [0,t] \times \mathbb{R}^n : |x| \leq s + R\}\), then, we find

\[
\begin{aligned}
\int_{\mathbb{R}^n} \partial_t u(t,x) \, dx - \int_{\mathbb{R}^n} \partial_t u(0,x) \, dx + \int_0^t \int_{\mathbb{R}^n} b_1(s) \partial_t u(s,x) \, dx \, ds &= \int_0^t \int_{\mathbb{R}^n} |v(s,x)|^p \, dx \, ds, \\
\int_{\mathbb{R}^n} \partial_t v(t,x) \, dx - \int_{\mathbb{R}^n} \partial_t v(0,x) \, dx + \int_0^t \int_{\mathbb{R}^n} b_2(s) \partial_t v(s,x) \, dx \, ds &= \int_0^t \int_{\mathbb{R}^n} |u(s,x)|^q \, dx \, ds
\end{aligned}
\]

or, equivalently,

\[
\begin{aligned}
U'(t) - U'(0) + \int_0^t b_1(s)U'(s) \, ds &= \int_0^t \int_{\mathbb{R}^n} |v(s,x)|^p \, dx \, ds, \\
V'(t) - V'(0) + \int_0^t b_2(s)V'(s) \, ds &= \int_0^t \int_{\mathbb{R}^n} |u(s,x)|^q \, dx \, ds
\end{aligned}
\]

Differentiating with respect to \(t\) the previous equalities, we arrive at

\[
\begin{aligned}
U''(t) + b_1(t)U'(t) &= \int_{\mathbb{R}^n} |v(t,x)|^p \, dx, \\
V''(t) + b_2(t)V'(t) &= \int_{\mathbb{R}^n} |u(t,x)|^q \, dx.
\end{aligned}
\]

(27)

(28)

Multiplying (27) by \(m_1(t)\), we get

\[
m_1(t)U''(t) + m_1(t)b_1(t)U'(t) = \frac{d}{dt}(m_1(t)U'(t)) = m_1(t) \int_{\mathbb{R}^n} |v(t,x)|^p \, dx.
\]

Hence, integrating over \([0,t]\) and using the assumption \(u_1 \geq 0\), we obtain

\[
m_1(t)U'(t) = m_1(0)U'(0) + \int_0^t m_1(s) \int_{\mathbb{R}^n} |v(s,x)|^p \, dx \, ds \geq \int_0^t m_1(s) \int_{\mathbb{R}^n} |v(s,x)|^p \, dx \, ds.
\]

Consequently, using the boundedness of the multiplier \(m_1\), from (17) we have

\[
U'(t) \geq \int_0^t \frac{m_1(s)}{m_1(t)} \int_{\mathbb{R}^n} |v(s,x)|^p \, dx \, ds \geq m_1(0) \int_0^t \int_{\mathbb{R}^n} |v(s,x)|^p \, dx \, ds.
\]

Since \(u_0\) is nonnegative a further integration on \([0,t]\) provides

\[
U(t) \geq m_1(0) \int_0^t \int_{\mathbb{R}^n} |v(s,x)|^p \, dx \, ds \, d\tau.
\]

(29)
Moreover, due to Hölder inequality and the compactness of the support of solution with respect to $x$, from (29) we derive

\[ U(t) \geq C_0 \int_0^t \int_0^\tau (1 + s)^{-n(p-1)}|V(s)|^p ds d\tau, \]  

(30)

where $C_0 = m_1(0)(\text{meas}(B_1))^{1-p} R^{-n(p-1)} > 0$.

In a similar way, using the assumptions $v_0, v_1 \geq 0$ and the properties of the multiplier $m_2$, from (28) we may derive

\[ V(t) \geq m_2(0) \int_0^t \int_0^\tau |u(s,x)|^q dx ds d\tau \]

(31)

\[ \geq K_0 \int_0^t \int_0^\tau (1 + s)^{-n(q-1)}|U(s)|^q ds d\tau, \]  

(32)

where $K_0 = m_2(0)(\text{meas}(B_1))^{1-q} R^{-n(q-1)} > 0$.

### 3.2. Iteration argument

Now we can proceed with the second part of the proof, where we use a standard iteration argument (see for example [18, 34] in the case of a single equation or [1, 25] in the case of a weakly coupled system). We will apply an iteration method based on lower bound estimates (19), (20), (29), (31) and the iteration frame (30), (32).

By using an induction argument, we prove that

\[ U(t) \geq D_j(1 + t)^{-\alpha_j t^{\beta_j}} \quad \text{for} \quad t \geq 0, \]  

(33)

\[ V(t) \geq \Delta_j(1 + t)^{-\alpha_j t^{\beta_j}} \quad \text{for} \quad t \geq 0, \]  

(34)

where \{a_j\}_{j\geq 1}, \{b_j\}_{j\geq 1}, \{D_j\}_{j\geq 1}, \{\alpha_j\}_{j\geq 1}, \{\beta_j\}_{j\geq 1}$ and \{\Delta_j\}_{j\geq 1}$ are suitable sequences of nonnegative real numbers to be determined afterwards.

We prove first the base case $j = 1$. Plugging the lower bound estimate for the nonlinear term $|v|^p$ given by (20) in (29), we obtain for $t \geq 0$

\[ U(t) \geq m_1(0)K_1 e^p \int_0^t \int_0^\tau (1 + s)^{-n-1-(n-1)\frac{p}{2}} ds d\tau \]

\[ \geq m_1(0)K_1 e^p (1 + t)^{-(n-1)\frac{p}{2}} \int_0^t \int_0^\tau s^{n-1} ds d\tau \geq \frac{m_1(0)K_1}{n(n+1)} e^p (1 + t)^{-(n-1)\frac{p}{2}} t^{n+1}, \]

which is the desired estimate, provided that we define

\[ D_1 = \frac{m_1(0)K_1}{n(n+1)} e^p, \quad a_1 = (n-1)\frac{p}{2}, \quad b_1 = n+1. \]

Analogously, we can prove (34) for $j = 1$ combining (31) and (19), provided that

\[ \Delta_1 = \frac{m_2(0)C_1}{n(n+1)} e^q, \quad \alpha_1 = (n-1)\frac{q}{2}, \quad \beta_1 = n+1. \]

Let us proceed with the inductive step: (33) and (34) are assumed to be true for $j \geq 1$, hence, we prove them for $j+1$. Let us combine (34) in (30). Then, since $\alpha_j$ and $\beta_j$ are positive numbers, we obtain

\[ U(t) \geq C_0 \Delta_j^p \int_0^t \int_0^\tau (1 + s)^{-n(p-1)-\alpha_j p^{\beta_j p}} ds d\tau \geq C_0 \Delta_j^p (1 + t)^{-n(p-1)-\alpha_j p^{\beta_j p}} \int_0^t \int_0^\tau s^{\beta_j p} ds d\tau = \frac{C_0 \Delta_j^p}{(\beta_j p + 1)(\beta_j p + 2)} (1 + t)^{-n(p-1)-\alpha_j p^{\beta_j p^{\beta_j p+2}}}, \]
that is, (33) for \( j + 1 \) provided that
\[
D_{j+1} = \frac{C_0 \Delta_j^g}{(\beta_j p + 1)(\beta_j p - 1)}, \quad a_{j+1} = n(p - 1) + \alpha_j p, \quad b_{j+1} = \beta_j p + 2.
\]

Similarly, we can prove (34) for \( j + 1 \) combining (32) and (33), in the case in which
\[
\Delta_{j+1} = \frac{K_0 D_j^g}{(b_j q + 1)(b_j q + 2)}, \quad \alpha_{j+1} = n(q - 1) + a_j q, \quad \beta_{j+1} = b_j q + 2.
\]

So, we proved the inductive step. In particular, the positiveness of the exponents \( a_j, b_j, \alpha_j, \beta_j \) follows immediately by the recursive relations we required throughout the inductive step and by the fact that the initial terms \( a_1, b_1, \alpha_1, \beta_1 \) are nonnegative.

Let us determine now explicitly the representations for \( a_j, b_j, \alpha_j, \beta_j \). Let us begin with the case in which \( j \) is an odd integer. We start with \( a_j \). Using the previous definitions and applying iteratively the obtained relation, we have
\[
a_j = n(p - 1) + a_{j-1} p = n(p - 1) + (n(q - 1) + a_{j-2} q) p = n(pq - 1) + b_{j-2} pq
\]
\[
= n(pq - 1) \sum_{k=0}^{(j-3)/2} (pq)^k + a_1(pq)^{\frac{j-1}{2}} = (n + a_1)(pq)^{\frac{j-1}{2}} - n
\]
\[
= (n + \frac{n-1}{2} p)(pq)^{\frac{j-1}{2}} - n. \tag{35}
\]

In a completely analogous way, for odd \( j \) we get
\[
\alpha_j = (n + a_1)(pq)^{\frac{j-1}{2}} - n \tag{37}
\]
\[
= (n + \frac{n-1}{2} q)(pq)^{\frac{j-1}{2}} - n. \tag{38}
\]

For the sake of brevity, we do not derive the representations of \( a_j \) and \( \alpha_j \) for even \( j \), as it is unnecessary for the proof of the theorem.

Similarly, combining the definitions of \( b_j \) and \( \beta_j \), for odd \( j \) we have
\[
b_j = b_1(pq)^{\frac{j-1}{2}} + 2(p+1) \sum_{k=0}^{(j-3)/2} (pq)^k = \left(b_1 + \frac{2(p+1)}{pq-1}\right)(pq)^{\frac{j-1}{2}} - \frac{2(p+1)}{pq-1}
\]
\[
= \left(n + 1 + \frac{2(p+1)}{pq-1}\right)(pq)^{\frac{j-1}{2}} - \frac{2(p+1)}{pq-1}, \tag{39}
\]
\[
\beta_j = \beta_1(pq)^{\frac{j}{2}} + 2(q+1) \sum_{k=0}^{(j-3)/2} (pq)^k = \left(\beta_1 + \frac{2(q+1)}{pq-1}\right)(pq)^{\frac{j}{2}} - \frac{2(q+1)}{pq-1}
\]
\[
= \left(n + 1 + \frac{2(q+1)}{pq-1}\right)(pq)^{\frac{j}{2}} - \frac{2(q+1)}{pq-1}. \tag{40}
\]

In the case in which \( j \) is even, from (42) and (40) we have, respectively,
\[
b_j = \beta_j p + 2 = p \left(n + 1 + \frac{2(q+1)}{pq-1}\right)(pq)^{\frac{j}{2}} - \frac{2(q+1)}{pq-1} + 2, \tag{43}
\]
\[
\beta_j = b_j q + 2 = q \left(n + 1 + \frac{2(p+1)}{pq-1}\right)(pq)^{\frac{j}{2}} - \frac{2(p+1)}{pq-1} + 2. \tag{44}
\]
Thus, from (40), (42), (43) and (44), we see that for any $j \geq 1$ the following holds:\n\begin{align*}
b_j < B_0(pq)^{\frac{j+1}{2}}, & \quad \beta_j < \tilde{B}_0(pq)^{\frac{j+1}{2}} \quad \text{for } j \text{ odd}, \\
b_j < B_0(pq)^{\frac{j}{2}}, & \quad \beta_j < \tilde{B}_0(pq)^{\frac{j}{2}} \quad \text{for } j \text{ even},
\end{align*}
(45)
where $B_0 = B_0(p,q,n)$ and $\tilde{B}_0 = \tilde{B}_0(p,q,n)$ are positive and independent of $j$ constants.

The next step is to derive lower bounds for $D_j$ and $\Delta_j$. From the definition of $D_j$ and $\Delta_j$ it follows immediately\n\[ D_j \geq \frac{C_0}{b_j^p} \Delta_{j-1}^p \quad \text{and} \quad \Delta_j \geq \frac{K_0}{b_j^p} D_{j-1}^p, \]
(46)
Hence, due to (45), coupling the inequalities in (46), it follows\n\[ D_j \geq \frac{C_0 \Delta_{j-1}^p}{B_0(pq)^{j-1}} \geq \frac{C_0 K_0^p}{B_0^2 (pq)^{j-1} \beta_0^p} D_{j-2}^p \frac{D_{j-2}^p}{B_0^2 B_0^p ((pq)^{p+1})^{j-1}} = \tilde{C} D_{j-2}^p, \]
(47)\n\[ \Delta_j \geq \frac{K_0 D_{j-1}^p}{B_0^2 (pq)^{j-1}} \geq \frac{K_0 C_0^p \Delta_{j-2}^p}{B_0^2 B_0^p ((pq)^{p+1})^{j-1}} = \tilde{K} \Delta_{j-2}^p, \]
(48)\nwhere $\tilde{C} = C_0 K_0^p / B_0^2 B_0^p$ and $\tilde{K} = K_0 C_0^p / B_0^2 B_0^p$. By (47), if $j$ is odd, then, we have
\[ \log D_j \geq pq \log D_{j-2} - (j-1)(p+1) \log pq + \log \tilde{C} \]
\[ \geq (pq)^2 \log D_{j-4} - (j-1) + (j-3) pq (p+1) \log pq + (1+pq) \log \tilde{C} \]
\[ \geq \cdots \geq (pq)^{(j-1)/2} \log D_1 - \left( \sum_{k=1}^{(j-1)/2} (j+1-2k) (pq)^{k-1} \right) (p+1) \log pq + \left( \sum_{k=0}^{(j-3)/2} (pq)^k \right) \log \tilde{C}. \]
Using an inductive argument, the following formula can be shown:
\[ \sum_{k=1}^{(j-1)/2} (j+1-2k) (pq)^{k-1} = \frac{1}{pq-1} \left( 2(pq)^{\frac{j-1}{2}} \frac{pq}{pq-1} - 1 \right) = j+1. \]
Also,
\[ \log D_j \geq (pq)^{\frac{j-1}{2}} \left[ \log D_1 - \frac{2(pq)(p+1)}{(pq-1)^2} \log pq + \log \tilde{C} \frac{pq}{pq-1} \right] + \frac{2(pq)(p+1)}{(pq-1)^2} \log pq \]
\[ + (j-1) \frac{(pq)^{\frac{j-1}{2}}}{pq-1} \log pq - \log \tilde{C} \frac{pq}{pq-1} \]
Consequently, for an odd $j$ such that $j > \frac{\log \tilde{C}}{(p+1) \log pq} - \frac{2(pq)(p+1)}{(pq-1)^2} \log pq + 1$, it holds\n\[ \log D_j \geq (pq)^{\frac{j-1}{2}} \left( \log D_1 - S_{p,q}(\infty) \right), \]
(49)\nwhere $S_{p,q}(\infty) = \frac{2(pq)(p+1)}{(pq-1)^2} \log pq - \frac{\log \tilde{C}}{pq-1}$.

Similarly, by using (48), it is possible to prove for an odd $j$ the following estimate:
\[ \log \Delta_j \geq (pq)^{\frac{j-1}{2}} \left[ \log \Delta_1 - \frac{2(pq)(q+1)}{(pq-1)^2} \log pq + \log \tilde{K} \frac{pq}{pq-1} \right] + \frac{2(pq)(q+1)}{(pq-1)^2} \log pq \]
\[ + (j-1) \frac{(q+1)}{pq-1} \log pq - \log \tilde{K} \frac{pq}{pq-1} . \]
Thus, for \( j > \frac{\log \tilde{K}}{(q+1) \log(pq)} - \frac{2(pq)}{pq - 1} + 1 \) the last inequality implies
\[
\log \Delta_j \geq (pq)^{\frac{1}{2}} (\log \Delta_1 - \tilde{S}_{p,q}(\infty)), \tag{50}
\]
where \( \tilde{S}_{p,q}(\infty) \equiv \frac{2(pq)(q+1)}{(pq-1)^2} \log(pq) - \frac{\log \tilde{K}}{pq - 1} \). Let us set \( j_0 = \left[ \frac{1}{\log(pq)} \max\{ \frac{\log \tilde{C}}{q+1}, \frac{\log \tilde{K}}{q+1} \} - \frac{2pq}{pq - 1} + 1 \right] \), for the sake of brevity. Combining the iterative inequality in (33) and the lower bound in (49), for an odd \( j > j_0 \) and \( t \geq 0 \), employing (36) and (40), we arrive at
\[
\begin{align*}
U(t) \geq & \exp \left((pq)^{\frac{1}{2}} (\log D_1 - S_{p,q}(\infty))\right) (1 + t)^{-a_j} t^{k_j} \\
= & \exp \left((pq)^{\frac{1}{2}} (\log D_1 - S_{p,q}(\infty))\right) (1 + t)^{-(n + \frac{n-1}{2}p)} \left( n + \frac{n-1}{2}p \right)^{\frac{1}{2}} \\
= & \exp \left((pq)^{\frac{1}{2}} (\log D_1 - \left(n + \frac{n-1}{2}p\right) \log(1 + t) + \left(n + 1 + \frac{2(p+1)}{pq-1}\right) \log t - S_{p,q}(\infty)\right)\right) (1 + t)^{n t^{-\frac{2(p+1)}{pq-1}}}.
\end{align*}
\]
Consequently, for \( t \geq 1 \), using the inequality \( \log 2t \geq \log(1 + t) \), from the previous estimate we find
\[
U(t) \geq \exp \left((pq)^{\frac{1}{2}} J(t)\right) (1 + t)^{n t^{-\frac{2(p+1)}{pq-1}}} \tag{51},
\]
where
\[
J(t) \equiv \log D_1 + \left((n + 1 + \frac{2(p+1)}{pq-1}) \left(n + \frac{n-1}{2}p\right) \log t - \left(n + \frac{n-1}{2}p\right) \log 2 - S_{p,q}(\infty)\right)
= \log \left(D_1 t^{\frac{pq+2p+1}{pq-1} - \frac{n-1}{2}}\right) - \left(n + \frac{n-1}{2}p\right) \log 2 - S_{p,q}(\infty). \tag{52}
\]
Let us point out that the power of \( t \) in the above definition is positive if and only if \( F(n, q, p) > 0 \).

In an analogous way, from (34), (50), (38) and (42) we obtain for \( t \geq 1 \) and for an odd \( j > j_0 \)
\[
V(t) \geq \exp \left((pq)^{\frac{1}{2}} \tilde{J}(t)\right) (1 + t)^{n t^{-\frac{2(p+1)}{pq-1}}} \tag{53},
\]
where
\[
\tilde{J}(t) \equiv \log \left(\Delta_1 t^{\frac{pq+2p+1}{pq-1} - \frac{n-1}{2}}\right) - \left(n + \frac{n-1}{2}q\right) \log 2 - \tilde{S}_{p,q}(\infty) \tag{54}
\]
and in this case the power of \( t \) is positive if and only if \( F(n, p, q) > 0 \).

If \( F(n, q, p) > 0 \), then, we can find \( \varepsilon_0 = \varepsilon_0(u_0, v_0, v_1, v_2, n, p, q, b_1, b_2, R) > 0 \) such that
\[
\tilde{C} \varepsilon_0^{-F(n,q,p)^{-1}} \geq 1,
\]
where \( \tilde{C} \equiv \left(\frac{n(n+1)}{m_{(0)(0)}} \varepsilon_0^{n+\frac{n-1}{2}p} \exp(S_{p,q}(\infty))\right)^{-1} \). Therefore, for \( \varepsilon \in (0, \varepsilon_0] \) and \( t > \tilde{C} \varepsilon^{-F(n,q,p)^{-1}} \) we have \( t \geq 1 \) and \( \tilde{J}(t) > 0 \). Letting \( j \to \infty \) in (51), the lower bound blows up and, consequently, \( U \) may be finite only for \( t \leq \tilde{C} \varepsilon^{-F(n,q,p)^{-1}} \).

Analogously, in the other case \( F(n, p, q) > 0 \), assuming that
\[
\tilde{K} \varepsilon_0^{-F(n,p,q)^{-1}} \geq 1,
\]
where the multiplicative constant in this case is given by \( \tilde{K} \equiv \left(\frac{n(n+1)}{m_{(0)(0)}} \varepsilon_0^{n+\frac{n-1}{2}q} \exp(\tilde{S}_{p,q}(\infty))\right)^{-1} \), then, for any \( \varepsilon \in (0, \varepsilon_0] \) and any \( t > \tilde{K} \varepsilon^{-F(n,p,q)^{-1}} \) we get \( t \geq 1 \) and \( \tilde{J}(t) > 0 \). Thus, as \( j \to \infty \) in (53) the lower bound for \( V(t) \) diverges. Also, \( V \) may be finite only for \( t \leq \tilde{K} \varepsilon^{-F(n,p,q)^{-1}} \). So, we proved Theorem 1.2 and the estimate for the lifespan of the local solution given in (12).
3.3. Proof of Corollaries 1.3, 1.4 and 1.5

In this section we sketch how it is possible to modify the proof of Theorem 1.2 in order to show the improvement of (12) as stated in Corollary 1.3, in Corollary 1.4 and in Corollary 1.5.

The first remark is that (19) and (20) can be improved in the case \( n = 1 \) and in the case \( n = 2 \) for exponents \( p,q \) such that \( 1 < p,q < 2 \), provided that the initial speeds for \( u \) and \( v \) are nontrivial (i.e., when the integrals of \( u_1,v_1 \) over \( \mathbb{R}^n \) do not vanish). Indeed, as \( U'(0) = \varepsilon \int_{\mathbb{R}^n} u_1(x)dx > 0 \) and \( V'(0) = \varepsilon \int_{\mathbb{R}^n} v_1(x)dx > 0 \), since \( U,V \) are convex functions, we get immediately \( U(t) \geq U'(0)t \) and \( V(t) \geq V'(0)t \). Consequently, using again the support condition for \( u \) and \( v \) and Hölder’s inequality, we have

\[
\int_{\mathbb{R}^n} |u(t,x)|^p dx \geq \tilde{C}_1 (1+t)^{-n(q-1)} (U(t))^q \geq \tilde{C}_1 \varepsilon^q (I[u_1])^q (1+t)^{-n(q-1)p},
\]

\[
\int_{\mathbb{R}^n} |v(t,x)|^p dx \geq \tilde{K}_1 (1+t)^{-n(p-1)} (V(t))^p \geq \tilde{K}_1 \varepsilon^p (I[v_1])^p (1+t)^{-n(p-1)p},
\]

where \( \tilde{C}_1, \tilde{K}_1 \) are suitable constants depending on \( n,p,q,R \) and \( I[f] = \int_{\mathbb{R}^n} f(x)dx \). For large times, these lower bounds are stronger than (19) and (20) in the above mentioned cases. Hence, for the proofs of Corollaries 1.3, 1.4 and 1.5 it is possible to follow faithfully the steps of the proof of Theorem 1.2 with few crucial modifications. If \( n = 1 \) or \( n = 2 \) and \( 1 < p < 2 \), then, (33) in the base case is true for

\[ D_1 \geq \tilde{K}_1 (I[v_1])^p \varepsilon^p, \quad a_1 = p, \quad b_1 = (n-1)p, \]

and, similarly, if \( n = 1 \) or \( n = 2 \) and \( 1 < q < 2 \), then, (34) in the base case is true for

\[ \Delta_1 \geq \tilde{C}_1 (I[u_1])^q \varepsilon^q, \quad a_1 = q, \quad b_1 = (n-1)q. \]

If \( n = 1 \) or \( n = 2 \) and \( 1 < p < 2 \), then, we can replace (52) by

\[ J(t) = \log \left( D_1 t^{b_1-n-a_1+\frac{2(p+1)}{pq-1}} \right) - (n+p) \log 2 - S_{p,q}(\infty) = \log \left( D_1 t^{pG(n,p,q)} \right) - (n+p) \log 2 - S_{p,q}(\infty), \]

substituting the new values of \( a_1,b_1 \) in (35) and (39) instead of the ones used in Section 3.2. Analogously, if \( n = 1 \) or \( n = 2 \) and \( 1 < q < 2 \), then, we can replace (54) by

\[ \tilde{J}(t) = \log \left( \Delta_1 t^{b_1-n-a_1+\frac{2(q+1)}{pq-1}} \right) - (n+q) \log 2 - \tilde{S}_{p,q}(\infty) = \log \left( \Delta_1 t^{qG(n,q,p)} \right) - (n+q) \log 2 - \tilde{S}_{p,q}(\infty), \]

substituting now the new values of \( a_1,\beta_1 \) in (37) and (41) in place of the ones used in Section 3.2. Having in mind these changes, the proof of each corollary is a straightforward modification of the arguments used in Section 3.2.

4. Critical case: Proof of Theorem 1.6

In this section we will prove Theorem 1.6. The structure of the proof is organized as follows: in Section 4.1 we recall the definition of certain auxiliary functions, which are necessary in order to introduce the functionals that we will estimate throughout the proof, and lower bound estimates for a fundamental system of solutions of the family of ODEs \( \mathcal{D}_y y = 0 \), where \( \mathcal{D}_b = \partial^2_t + b(t) \partial_t - \lambda^2 \) and \( \lambda \) is a real parameter; moreover, using these estimates, we derive a couple of crucial estimates for the averages of the components of a local in time solution multiplied by one of the above cited auxiliary functions (these averages are actually the functionals whose dynamic we shall use to prove the blow-up result); then, in Section 4.2 we derive two coupled integral inequalities and lower bounds containing logarithmic terms for the functionals; in Section 4.3 we combine the lower bounds and the integral inequalities from Section 4.2 in order to prove a family of lower bound estimates via the slicing method; finally, in Section 4.4 we use this sequence of lower bound estimates to proved the blow-up result and to derive the upper bound estimate for the lifespan of local in time solutions.
4.1. Definition of the auxiliary functions

In this section we recall the definition of a pair of auxiliary functions from [35], which are necessary in order to introduce the time-dependent functionals that will be considered for the iteration argument.

Let \( r > -1 \) be a parameter. Then, we introduce the functions

\[
\xi_r(t, x) \doteq \int_0^{\lambda_0} e^{-\lambda(t + R)} \cosh(\lambda t) \Phi(\lambda x) \lambda^r \, d\lambda,
\]

\[
\eta_r(t, s, x) \doteq \int_0^{\lambda_0} e^{-\lambda(t + R)} \frac{\sinh(\lambda(t - s))}{\lambda(t - s)} \Phi(\lambda x) \lambda^r \, d\lambda,
\]

where \( \lambda_0 \) is a fixed positive parameter and \( \Phi \) is defined by (21).

Some useful properties of \( \xi_r \) and \( \eta_r \) are stated in the following lemma, whose proof can be found in [35, Lemma 3.1].

**Lemma 4.1.** Let \( n \geq 2 \). There exist \( \lambda_0 > 0 \) such that the following properties hold:

(i) if \( r > -1 \), \( |x| \leq R \) and \( t \geq 0 \), then,

\[
\xi_r(t, x) \geq A_0, \quad \eta_r(t, 0, x) \geq B_0(t)^{-1};
\]

(ii) if \( r > -1 \), \( |x| \leq s + R \) and \( t > s \geq 0 \), then,

\[
\eta_r(t, s, x) \geq B_1(t)^{-1}(s)^{-r};
\]

(iii) if \( r > \frac{n-3}{2} \), \( |x| \leq t + R \) and \( t > 0 \), then,

\[
\eta_r(t, t, x) \leq B_2(t)^{-\frac{n-3}{2}} (t - |x|)^{-\frac{n-3}{2} - r}.
\]

Here \( A_0 \) and \( B_k \), with \( k = 0, 1, 2 \), are positive constants depending only on \( \lambda_0 \), \( r \) and \( R \) and we denote \( \langle y \rangle \doteq 3 + |y| \).

**Remark 4.2.** Even though in [35] the previous lemma is stated requiring \( r > 0 \) in (i) and (ii), the proof provided in that paper is valid for any \( r > -1 \) as well.

**Lemma 4.3.** Let \( \lambda \) be a positive real parameter and let \( b \in C^1([0, \infty)) \cap L^1([0, \infty)) \) be a nonnegative function. We introduce the differential operators

\[
\mathcal{L}_b \doteq \partial_t^2 + b(t) \partial_t - \lambda^2, \quad \mathcal{L}_b^* \doteq \partial_s^2 - \partial_s b(s) - \lambda^2
\]

and the fundamental system of solutions \( y_j = y_j(t, s; \lambda, b) \), with \( j = 1, 2 \), such that

\[
\mathcal{L}_b y_1(t, s; \lambda, b) = 0, \quad y_1(s, s; \lambda, b) = 1, \quad \partial_s y_1(s, s; \lambda, b) = 0;
\]

\[
\mathcal{L}_b y_2(t, s; \lambda, b) = 0, \quad y_2(s, s; \lambda, b) = 0, \quad \partial_s y_1(s, s; \lambda, b) = 1.
\]

Then, \( \{y_1, y_2\} \) depends continuously on \( \lambda \) and satisfies for \( t \geq s \geq 0 \) the following estimates:

(i) \( y_1(t, s; \lambda, b) \geq e^{-\|b\|_{L^1}} \cosh \lambda(t - s) \),

(ii) \( y_2(t, s; \lambda, b) \geq e^{-2\|b\|_{L^1}} \frac{\sinh \lambda(t - s)}{\lambda} \).

Moreover,

(iii) \( \mathcal{L}_b^* y_2(t, s; \lambda, b) = 0 \),

(iv) \( y_1(t, 0; \lambda, b) = b(0) y_2(t, 0; \lambda, b) - \partial_s y_2(t, 0; \lambda, b) \),

(v) \( \partial_s y_2(t, t; \lambda, b) = -1 \).
Proof. See Lemma 2.3 in [36]. In particular, (v) follows by the first condition in [36, relation (4.7)].

**Proposition 4.4.** Let \( b_1, b_2 \in \mathcal{C}^1([0, \infty)) \cap L^1([0, \infty)) \) and let \( u_0, v_0 \in H^1(\mathbb{R}^n) \) and \( u_1, v_1 \in L^2(\mathbb{R}^n) \) be nonnegative, pairwise nontrivial and compactly supported in \( B_R \). Let \((u,v)\) be an energy solution to (1) on \([0,T)\) according to Definition 1.1 satisfying (11). Then, the following estimates hold:

\[
\int_{\mathbb{R}^n} u(t,x) \eta_{r_1}(t,t,x) \, dx \geq e^{-||b_1||_{L^1}} \int_{\mathbb{R}^n} u_0(x) \xi_{r_1}(t,x) \, dx + e^{-2||b_1||_{L^1}} \varepsilon t \int_{\mathbb{R}^n} u_1(x) \eta_{r_1}(t,0,x) \, dx \\
+ e^{-2||b_1||_{L^1}} \int_0^t \int_{\mathbb{R}^n} v(s,x) \eta_{r_1}(t,s,x) \, dx \, ds, 
\]

(55)

\[
\int_{\mathbb{R}^n} v(t,x) \eta_{r_2}(t,t,x) \, dx \geq e^{-||b_2||_{L^1}} \xi_{r_2}(t,x) \, dx + e^{-2||b_2||_{L^1}} \varepsilon t \int_{\mathbb{R}^n} v_1(x) \eta_{r_2}(t,0,x) \, dx \\
+ e^{-2||b_2||_{L^1}} \int_0^t \int_{\mathbb{R}^n} u(s,x) \eta_{r_2}(t,s,x) \, dx \, ds,
\]

(56)

for \( r_1, r_2 > -1 \) and any \( t \in (0,T) \).

Proof. Thanks to (11) we have that \( u(t,\cdot), v(t,\cdot) \) have compact support in \( B_{R+t} \) for any \( t \geq 0 \). Therefore, we may employ (6) and (7) also for noncompactly supported test function. Moreover, by using a density argument we can weaken the regularity for the test functions in Definition 1.1. Consequently, we may choose as test functions

\[
\phi = \phi(s,x) = \Phi(\lambda x) y_2(t,s;\lambda,b_1), \quad \psi = \psi(s,x) = \Phi(\lambda x) y_2(t,s;\lambda,b_2),
\]

where \( \Phi \) is defined by (21). As \( \Phi \) is an eigenfunction of the Laplace operator and \( y_2(t,s;\lambda,b_1), y_2(t,s;\lambda,b_2) \) solve \( \mathcal{L} y = 0 \) and \( \mathcal{L}_y y = 0 \), respectively, we get that \( \phi \) and \( \psi \) satisfy

\[
\phi_{ss} - \Delta \phi - \partial_s(b_1(s)\phi) = 0 \quad b_1(0)\phi(0,x) - \phi_s(0,x) = \Phi(\lambda x) y_1(t,0;\lambda,b_1) \quad \text{and} \quad \phi_s(t,x) = -\Phi(\lambda x),
\]

\[
\psi_{ss} - \Delta \psi - \partial_s(b_2(s)\psi) = 0 \quad b_2(0)\psi(0,x) - \psi_s(0,x) = \Phi(\lambda x) y_1(t,0;\lambda,b_2) \quad \text{and} \quad \psi_s(t,x) = -\Phi(\lambda x),
\]

where we employed (iv) and (v) from Lemma 4.3 to get the relations for the values of \( \phi \) and \( \psi \) at \( s = 0, t \).

Let us prove (55). Using the above defined \( \phi \) in (8) and its properties, we get

\[
\int_{\mathbb{R}^n} u(t,x) \Phi(\lambda x) \, dx = \varepsilon y_1(t,0;\lambda,b_1) \int_{\mathbb{R}^n} u_0(x) \Phi(\lambda x) \, dx + \varepsilon y_2(t,0;\lambda,b_1) \int_{\mathbb{R}^n} u_1(x) \Phi(\lambda x) \, dx \\
+ \int_0^t \int_{\mathbb{R}^n} v(s,x) \Phi(\lambda x) \, dx, 
\]

where we used also the condition \( \phi(t,x) = 0 \), which follows immediately from the initial values of \( y_2(t,s;\lambda,b_1) \) prescribed in the statement of Lemma 4.3. Using the estimates from below (i) and (ii) in Lemma 4.3, we obtain from the previous relation

\[
\int_{\mathbb{R}^n} u(t,x) \Phi(\lambda x) \, dx \geq \varepsilon e^{-||b_1||_{L^1}} \cosh \lambda t \int_{\mathbb{R}^n} u_0(x) \Phi(\lambda x) \, dx + \varepsilon e^{-2||b_1||_{L^1}} \frac{\sinh \lambda t}{\lambda} \int_{\mathbb{R}^n} u_1(x) \Phi(\lambda x) \, dx \\
+ e^{-2||b_1||_{L^1}} \int_0^t \frac{\sinh \lambda (t-s)}{\lambda} \int_{\mathbb{R}^n} |v(s,x)|^p \Phi(\lambda x) \, dx.
\]

Multiplying both sides of the last inequality by \( e^{-\lambda(t+R)|x|^2} \), integrating with respect to \( \lambda \) over \([0,\lambda_0]\) and applying Tonelli’s theorem, we get finally (55). In order to prove (56), it is sufficient to repeat the above steps after plugging the prescribed \( \psi \) function in (9). Hence, the proof is complete. \(\Box\)
4.2. Lower bound estimates

Hereafter until the end of Section 4, we will assume that \( u_0, u_1, v_0, v_1 \) satisfy the assumptions from the statement of Theorem 1.6. Furthermore, without loss of generality we assume that \( (p,q) \) satisfies the critical condition \( F(n,p,q) = 0 \), because the case \( F(n,q,p) = 0 \) is completely symmetric, assumed the switch of \( p,q \) and \( u,v \), respectively. Let \((u,v)\) be an energy solution of (1) on \([0,T)\). We introduce the following time-dependent functionals

\[
\mathcal{U}(t) = \int_{\mathbb{R}^n} u(t,x) \eta_r(t,t,x) \, dx, \\
\mathcal{V}(t) = \int_{\mathbb{R}^n} v(t,x) \eta_r(t,t,x) \, dx.
\]  

(57)

Let us point out that we will prescribe in the next proposition the exact assumptions for the parameters \( r_1, r_2 \). From Proposition 4.4 it follows immediately the positiveness of the functionals \( \mathcal{U}, \mathcal{V} \).

The next step is to derive two integral inequalities involving \( \mathcal{U} \) and \( \mathcal{V} \) in a "coupled way", and, as we have just mentioned, this goal will somehow fix the range for \((r_1,r_2)\). Let us point out explicitly that the case \( p > q \) and the case \( p = q \) (see Remark 4.5 below) will be treated separately with a different choice of the pair \((r_1,r_2)\) (which will correspond to a different frame for the iteration scheme).

**Remark 4.5.** Since we assume that \( (p,q) \) satisfies \( F(n,p,q) = 0 \leq F(n,q,p) \) it may be either \( F(n,q,p) < 0 \) or \( F(n,q,p) = 0 \). Due to the monotonicity of the function \( f = f(p) = p - p^{-1} \) for \( p > 1 \), in the first case we are in the case \( p > q \), while in the latter case we have \( p = q \). Moreover, the condition \( F(n,p,q) = 0 \) it equivalent to require \( p = p_0(n) \), so that \( F(n,p,q) = F(n,q,p) = 0 \) corresponds to the limit case \( p = q = p_0(n) \).

**Proposition 4.6.** Let us assume that \( r_1, r_2 \) are given parameters satisfying \( r_1 = \frac{n-1}{2} - \frac{1}{q} \) and

- \( r_2 > \frac{n-1}{2} - \frac{1}{p} \) if \( p > q \);
- \( r_2 = \frac{n-1}{2} - \frac{1}{p} \) if \( p = q \).

Let \( \mathcal{U}, \mathcal{V} \) be the functionals defined by (57). Then, there exist positive constants \( C \) and \( K \) depending on \( n,p,q,R,b_1,b_2 \) such that for any \( t \geq 0 \) the following estimates hold:

\[
\mathcal{U}(t) \geq C(t)^{-1} \int_0^t (t-s) (s)^{\frac{n-1}{2} + 1 + (r_2+1-n)p} (\mathcal{V}(s))^p \, ds,
\]

(58)

\[
\mathcal{V}(t) \geq K(t)^{-1} \int_0^t (t-s) (s)^{-r_2 - \frac{n+1}{q} + n-1} (\log(s))^{-(q-1)} (\mathcal{U}(s))^q \, ds
\]

(59)

for \( p > q \) and

\[
\mathcal{U}(t) \geq C(t)^{-1} \int_0^t (t-s) (s)^{-1} (\log(s))^{-(p-1)} (\mathcal{V}(s))^p \, ds,
\]

(60)

\[
\mathcal{V}(t) \geq K(t)^{-1} \int_0^t (t-s) (s)^{-1} (\log(s))^{-(q-1)} (\mathcal{U}(s))^q \, ds
\]

(61)

for \( p = q \).

**Proof.** For the proof of this result we will follow the main ideas of Proposition 4.2 in [35]. Let us begin with the proof in the case \( p > q \). By Hölder’s inequality and the support property for \( v(s,\cdot) \), we obtain

\[
\mathcal{V}(s) \leq \left( \int_{\mathbb{R}^n} |v(s,x)|^p \eta_{r_1}(t,s,x) \, dx \right)^{\frac{1}{p}} \left( \int_{B_{s+R}} \eta_{r_1}(t,s,x) \eta_{r_2}(s,s,x) \, dx \right)^{\frac{1}{p'}}.
\]

(62)
We estimate the second factor on the right hand side in the last inequality. By (ii) and (iii) in Lemma 4.1 (note that, according to our choice in the statement of this proposition, both conditions \( r_1, r_2 > \frac{q-1}{2} \) and \( r_1, r_2 > -1 \) are always fulfilled), we obtain
\[
\int_{B_{r+s}} \frac{\eta_{r_2}(s, t, x)}{\eta_{r_1}(t, s, x)}^{\gamma'} \frac{dx}{r} \leq \langle t \rangle^{\frac{q-1}{2}} \left\langle s - |x| \right\rangle^\gamma \int_{B_{r+s}} \frac{\eta_{r_2}(s, t, x)}{\eta_{r_1}(t, s, x)}^{\gamma'} \frac{dx}{r} \leq \langle t \rangle^{\frac{q-1}{2}} \left\langle s - |x| \right\rangle^\gamma \left\langle s - |x| \right\rangle^{\frac{q-1}{2}},
\]
where in the last step we used the assumption on \( r_2 \) which is equivalent to require a power smaller than -1 for the term \( \langle s - |x| \rangle \) in the integral. Combining (55), (62) and the previous estimate, we find
\[
\mathcal{U}(t) \geq \int_0^t (t-s) \int_{\mathbb{R}^n} |v(s, x)|^p \eta_{r_1}(t, s, x) dx \, ds \geq \int_0^t (t-s) \left( \eta(s) \right)^p \left( t \right)^{-1} \left\langle s - |x| \right\rangle^{\frac{n-1}{2} - p - r_1 + n + \frac{1}{3} (r_2 + 1)} \, ds
\]
\[
\geq \langle t \rangle^{-1} \int_0^t (t-s) \left( \eta(s) \right)^p \left( t \right)^{-1} \left\langle s - |x| \right\rangle^{\frac{n-1}{2} + 1 + \frac{1}{3} (r_2 + 1 - n)} \, ds.
\]
Let us prove now (59). Analogously to (62), we get
\[
\mathcal{U}(s) \leq \left( \int_{\mathbb{R}^n} |u(s, x)|^q \eta_{r_2}(t, s, x) dx \right)^\frac{1}{q} \left( \int_{B_{r+s}} \eta_{r_2}(t, s, x) \frac{dx}{r} \right)^\frac{1}{q}.
\]
(63)
Employing again (ii) and (iii) in Lemma 4.1 and thanks the choice of the parameter \( r_1 \), we arrive at
\[
\int_{B_{r+s}} \frac{\eta_{r_2}(s, t, x)}{\eta_{r_2}(t, s, x)}^{\gamma'} \frac{dx}{r} \leq \langle t \rangle^{\frac{q-1}{2}} \left\langle s - |x| \right\rangle^\gamma \int_{B_{r+s}} \frac{\eta_{r_2}(s, t, x)}{\eta_{r_2}(t, s, x)}^{\gamma'} \frac{dx}{r} \leq \langle t \rangle^{\frac{q-1}{2}} \left\langle s - |x| \right\rangle^\gamma \left\langle s - |x| \right\rangle^{\frac{q-1}{2}} \leq \langle t \rangle^{\frac{q-1}{2}} \left\langle s - |x| \right\rangle^\gamma \left\langle s - |x| \right\rangle^{\frac{q-1}{2}},
\]
If we combine (56), (63) and the last estimate, we have
\[
\mathcal{U}(t) \geq \int_0^t (t-s) \int_{\mathbb{R}^n} |u(s, x)|^q \eta_{r_2}(t, s, x) dx \, ds
\]
\[
\geq \int_0^t (t-s) \left( \mathcal{U}(s) \right)^q \left( t \right)^{-1} \left\langle s - |x| \right\rangle^{\frac{n-1}{2} - p - r_1 + n + \frac{1}{3} (r_2 + 1) (\log(s))^{-1} \, ds
\]
\[
= \langle t \rangle^{-1} \int_0^t (t-s) \left( \mathcal{U}(s) \right)^q \left( t \right)^{-1} \left\langle s - |x| \right\rangle^{\frac{n-1}{2} - p - r_1 + n + \frac{1}{3} (r_2 + 1) (\log(s))^{-1} \, ds.
\]
(64)
In the case \( p = q = p_0(n) \), if we plug the value of \( r_2 \) in (64), then, thanks to \( -\frac{n-1}{2} (q-1) + \frac{1}{3} = -1 \), we get immediately (61). Due to symmetry reasons the proof of (60) is totally analogous. This completes the proof of the proposition.

The integral inequalities derived in the last proposition will play a fundamental role in the iteration argument. However, in order to start with this iteration argument we have to derive a lower bound for the functional \( \mathcal{U} \) containing a logarithmic term. For this purpose we will combine the lower bounds for the nonlinearities that we have shown in the subcritical case in Lemma 2.2 with the estimates from Lemma 4.1 and Proposition 4.6.

**Lemma 4.7.** Let \( p, q > 1 \) satisfy \( F(n, p, q) = 0 \). Then, for any \( t \geq \frac{t}{2} \) the following estimates hold:
\[
\mathcal{U}(t) \geq \tilde{C} \varepsilon^{pq} \log \left( \frac{\varepsilon}{t} \right) \quad \text{if} \quad p > q,
\]
\[
\mathcal{U}(t) \geq \tilde{C} \varepsilon^{pq} \log \left( \frac{\varepsilon}{t} \right) \quad \text{if} \quad p = q,
\]
where \( \tilde{C} \) is a positive constant depending on \( n, p, q, u_0, u_1, v_0, v_1, b_1, b_2, R \).
Proof. From (55), estimate (ii) in Lemma 4.1, (20) and the definition of \( r_1 \) we obtain for \( t \geq 1 \)
\[
\mathcal{U}(t) \gtrsim \int_0^t (t - s) \left| v(s, x) \right|^p \eta_1(t, s, x) \, dx \, ds \gtrsim (t)^{-1} \int_0^t (t - s) (s)^{-r_1} \int_{\mathbb{R}^n} \left| v(s, x) \right|^p \, dx \, ds
\]
\[
\gtrsim \varepsilon^p(t)^{-1} \int_0^t (t - s) (s)^{-r_1 + n - 1 - \frac{n - 1}{2}} \, ds = \varepsilon^p(t)^{-1} \int_0^t (t - s) (s)^{\frac{n - 1}{2} + \frac{1}{2} - \frac{n - 1}{2}} \, ds.
\]  
(65)

Similarly, by (56) and (19) we get for \( t \geq 1 \)
\[
\mathcal{V}(t) \gtrsim \varepsilon^q(t)^{-1} \int_0^t (t - s) (s)^{-r_2 + n - 1 - \frac{n - 1}{2}} \, ds.
\]  
(66)

In the special case \( p = q = p_0(n) \), the power of \( (s) \) in the integral in the right hand side of (65) is \(-1\). Hence, we may estimate for \( t \geq \frac{3}{2} \)
\[
\int_1^t \frac{t - s}{s} \, ds = \int_1^t \log s \, ds = \int_{\frac{2}{3}}^t \log s \, ds \gtrsim t \log \left( \frac{3}{2} \right) \gtrsim (t) \log \left( \frac{3}{2} \right).
\]  
(67)

Combining (65) and (67), we get the desired estimate in the case \( p = q \). In the case \( q > p \), keeping on the estimate in (66), we find for \( t \geq 1 \)
\[
\mathcal{V}(t) \gtrsim \varepsilon^q(t)^{-1} \int_0^t (t - s) (s)^{n - 1} \, ds \gtrsim \varepsilon^q(t)^{-1} \int_0^t (t - s) (s)^{-r_2 + n - 1} (s)^{n - 1} \int_{\frac{2}{3}}^t \log s \, ds \gtrsim \varepsilon^q(t)^{-r_2 + \frac{n - 1}{2}} \int_{\frac{2}{3}}^t (s)^{n - 1} \, ds.
\]

Now we plug the previous lower bound in (58), after shrinking the domain of integration, and we arrive for \( t \geq \frac{3}{2} \) at
\[
\mathcal{U}(t) \gtrsim \varepsilon^{pq}(t)^{-1} \int_1^t (t - s) (s)^{\frac{n - 1}{2} + \frac{1}{2} + (r_2 + 1 - n)p - r_2 p - \frac{n - 1}{2} np + np} \, ds
\]
\[
= \varepsilon^{pq}(t)^{-1} \int_1^t (t - s) (s)^{-\frac{n - 1}{2}(p - 1) + p + \frac{1}{2} + \frac{n - 1}{2}} \, ds = \varepsilon^{pq}(t)^{-1} \int_1^t (t - s) (s)^{-1} \, ds \gtrsim \varepsilon^{pq} \log \left( \frac{2t}{3} \right),
\]
where we used the condition \( F(n, p, q) = 0 \) in the second last step and again (67) in the last step. This concludes the proof. \( \square \)

4.3. Iteration argument via slicing method

In this section we apply the so-called slicing method, which has been introduced for the first time in [1], in order to prove a family of lower bound estimates for \( \mathcal{U} \). Let us introduce the sequence \( \{ \ell_j \}_{j \in \mathbb{N}} \), where \( \ell_j \doteq 2 - 2^{-j+1} \). The goal of this iteration method is to prove
\[
\mathcal{U}(t) \geq C_j (\log(t))^{-b_j} \left( \log \left( \frac{t}{\ell_{2j}} \right) \right)^{a_j} \quad \text{for} \quad t \geq \ell_{2j} \quad \text{and for any} \quad j \in \mathbb{N},
\]  
(68)

where \( \{ C_j \}_{j \in \mathbb{N}}, \{ a_j \}_{j \in \mathbb{N}} \) and \( \{ b_j \}_{j \in \mathbb{N}} \) are sequences of nonnegative real numbers that we shall determine afterwards. For \( j = 0 \) we know that (68) is true thanks to Lemma 4.7 with
\[
C_0 = \begin{cases} 
\tilde{C} e^{pq} & \text{if} \quad p > q, \\
\tilde{C} e^p & \text{if} \quad p = q, \\
\end{cases} \quad a_0 = 1 \quad \text{and} \quad b_0 = 0.
\]

We are going to prove the validity of (68) by using an inductive proof. As we have already remarked the validity of the base case, it remains to prove the inductive step. Let us assume that (68) holds for \( j \geq 1 \), we want to prove it now for \( j + 1 \). Because of the different frame in (58)-(59) and in (60)-(61), we shall consider separately the cases \( p > q \) and \( p = q \).
Case $p > q$

Combining (59) and (68) for $j$, for any $s \geq \ell_{2j+1}$ it follows

$$
\mathcal{V}(s) \geq K(s)^{-1} \int_{\ell_{2j}}^{s} (s - \tau)(\tau - \tau_2 - \frac{a_{j+1}q}{2} \log(\tau))^{-q} (\mathcal{U}(\tau))^q d\tau \\
\geq KC_j^q \int_{\ell_{2j}}^{s} (s - \tau)(\tau - \tau_2 - \frac{a_{j+1}q}{2} \log(\tau))^{-q} \left( \log \left( \frac{s}{\tau} \right) \right)^{a_j q} d\tau \\
\geq KC_j^q \int_{\ell_{2j}}^{s} (s - \tau)(\tau - \tau_2 - \frac{a_{j+1}q}{2} \log(\tau))^{-q} \left( \log \left( \frac{s}{\tau} \right) \right)^{a_j q} d\tau.
$$

We can estimate now the integral in the last line of the previous chain of inequalities as follows

$$
\int_{\ell_{2j}}^{s} (s - \tau)(\tau - \tau_2 - \frac{a_{j+1}q}{2} \log(\tau))^{-q} \left( \log \left( \frac{s}{\tau} \right) \right)^{a_j q} d\tau \geq \int_{\ell_{2j+1}}^{s} (s - \tau)(\tau - \tau_2 - \frac{a_{j+1}q}{2} \log(\tau))^{-q} \left( \log \left( \frac{s}{\tau} \right) \right)^{a_j q} d\tau \\
\geq \left( \frac{\ell_{2j+1}}{\ell_{2j}} \right)^{-q} \left( \log \left( \frac{s}{\tau} \right) \right)^{a_j q} \int_{\ell_{2j+1}}^{s} (s - \tau) d\tau \\
\geq \frac{1}{2} \left( \frac{\ell_{2j+1}}{\ell_{2j}} \right)^{-q} \left( \log \left( \frac{s}{\tau} \right) \right)^{a_j q} \cdot n_{j+1} \left( \log \left( \frac{s}{\tau} \right) \right)^{a_j q},
$$

where we used the relation $\langle y \rangle \geq y \geq \frac{1}{4} \langle y \rangle$ for $y \geq 1$ in the first and last inequality. Moreover, in the first line we might restrict the domain of integration since $\ell_{2j} < \ell_{2j+1}$, in the third one we used the inequality $2\ell_{2j} \geq \ell_{2j+1}$ and, finally, we employed the condition $1 - \frac{\ell_{2j}}{\ell_{2j+1}} \geq 2^{-\langle \log 2 \rangle}$. Hence, plugging (70) in (69), for any $s \geq \ell_{2j+1}$ we find

$$
\mathcal{V}(s) \geq 2^{-4j-3n-q} KC_j^q \left( \log(s) \right)^{-q} \left( s - \tau_2 - \frac{a_{j+1}q}{2} \log\left( \frac{s}{\tau} \right) \right)^{a_j q}.
$$

Next we use the previous lower bound for $\mathcal{V}(s)$ in (58), so that for $t \geq \ell_{2j+2}$ we have

$$
\mathcal{U}(t) \geq 2^{-4j+3n+8p} KC_j^p C_j^{pq} \langle t \rangle^{-1} \int_{\ell_{2j+1}}^{\ell_{2j+2}} (t - s) (s - \tau)(\tau - \tau_2 - \frac{a_{j+1}q}{2} \log(\tau))^{-q} \left( \log \left( \frac{s}{\tau} \right) \right)^{a_j pq} d\tau \\
\geq 2^{-4j+3n+8p} KC_j^p C_j^{pq} \langle t \rangle^{-1} \int_{\ell_{2j+1}}^{\ell_{2j+2}} (t - s) (s - \tau)(\tau - \tau_2 - \frac{a_{j+1}q}{2} \log(\tau))^{-q} \left( \log \left( \frac{s}{\tau} \right) \right)^{a_j pq} d\tau \\
\geq 2^{-4j+3n+8p} KC_j^p C_j^{pq} \langle t \rangle^{-1} \int_{\ell_{2j+1}}^{\ell_{2j+2}} \left( \log \left( \frac{s}{\tau} \right) \right)^{a_j pq} d\tau,
$$

where in the second inequality the condition $F(n, p, q) = 0$ implies that the exponent of the factor $\langle s \rangle$ is exactly $-1$. Using integration by parts, we may estimate the last integral in the following way:

$$
\int_{\ell_{2j+1}}^{t} \left( \log \left( \frac{s}{\tau} \right) \right)^{a_j pq} d\tau = (a_j pq + 1)^{-1} \int_{\ell_{2j+1}}^{t} \left( \log \left( \frac{s}{\tau} \right) \right)^{a_j pq + 1} d\tau \\
\geq (a_j pq + 1)^{-1} \int_{\ell_{2j+1}}^{t} \left( \log \left( \frac{s}{\tau} \right) \right)^{a_j pq + 1} d\tau \\
\geq (a_j pq + 1)^{-1} \left( 1 - \frac{\ell_{2j+1}}{\ell_{2j+2}} \right) t \left( \log \left( \frac{s}{\tau} \right) \right)^{a_j pq + 1} \\
\geq 2^{-2j-6} (a_j pq + 1)^{-1} t \left( \log \left( \frac{s}{\tau} \right) \right)^{a_j pq + 1},
$$

(72)
where in the second step it is possible to shrink the domain of integration due to \( t \geq \ell_{2j+2} \). Also, combining (71) and (72), we have
\[
\mathcal{U}(t) \geq 2^{-(2p+1)2j-(3n+8)p-8}CK^pC_j^q(a_jpq + 1)^{-1} \left( \log(t) \right)^{-p(q-1)-b_jpq} \left( \log \left( \frac{t}{\ell_{2j+2}} \right) \right)^{a_jpq+1}.
\]
Therefore, if we put
\[
C_{j+1} \doteq 2^{-(2p+1)2j-(3n+8)p-8}CK^pC_j^q(a_jpq + 1)^{-1}, \quad a_{j+1} \doteq a_jpq + 1 \quad \text{and} \quad b_{j+1} \doteq p(q-1) + b_jpq, \tag{73}
\]
then, we proved (68) for \( j+1 \) in the case \( p > q \).

Let us determine explicitly the expressions of \( a_j \) and \( b_j \). By using recursively the above relations, we find
\[
a_j = a_{j-1}pq + 1 = a_0(pq)^j + \sum_{k=0}^{j-1} (pq)^k = (pq)^j + \frac{(pq)^{j+1}-1}{pq-1}, \tag{74}
\]
\[
b_j = p(q-1) + b_{j-1}pq = p(q-1) \sum_{k=0}^{j-1} (pq)^k + b_0(pq)^j = \frac{p(q-1)}{pq-1}((pq)^j - 1). \tag{75}
\]
In particular,
\[
a_{j-1}pq + 1 = \frac{(pq)^{j+1}-1}{pq-1} \leq \frac{pq}{pq-1}(pq)^j,
\]
which implies in turn
\[
C_j \geq M\Theta^{-1}C_{j-1}^pq, \tag{76}
\]
where \( \Theta \doteq 2^{2(2p+1)}pq \) and \( M \doteq 2^{-(3n+4)p-6}CK^p\frac{(pq-1)}{(pq)} \).

**Case \( p = q \)**

We have to modify slightly the procedure seen in the case \( p > q \), by using (60)-(61) in place of (58)-(59). Using (61) and (68) for \( j \), for any \( s \geq \ell_{2j+1} \) we have
\[
\mathcal{V}(s) \geq K\langle s \rangle^{-1} \int_{\ell_{2j}}^s (s-\tau)\langle \tau \rangle^{-q-1} \left( \log(\tau) \right)^{-p(q-1)-b_jpq} \left( \log \left( \frac{\tau}{\ell_{2j+1}} \right) \right)^{a_jpq+1} d\tau.
\]
Then,
\[
\int_{\ell_{2j}}^{s} \frac{s-\tau}{\tau} \left( \log \left( \frac{\tau}{\ell_{2j}} \right) \right)^{a_jpq} d\tau = (a_jpq + 1)^{-1} \int_{\ell_{2j}}^{s} \left( \log \left( \frac{\tau}{\ell_{2j}} \right) \right)^{a_jpq+1} d\tau
\]
\[
\geq (a_jpq + 1)^{-1} \int_{\ell_{2j}a_jpq}^{s} \left( \log \left( \frac{\tau}{\ell_{2j+1}} \right) \right)^{a_jpq+1} d\tau
\]
\[
\geq (a_jpq + 1)^{-1} \left( 1 - \frac{\ell_{2j}}{\ell_{2j+1}} \right) s \left( \log \left( \frac{\ell_{2j+1}}{\ell_{2j}} \right) \right)^{a_jpq+1}
\]
\[
\geq 2^{-(2j+5)}(a_jpq + 1)^{-1}(s) \left( \log \left( \frac{\ell_{2j+1}}{\ell_{2j}} \right) \right)^{a_jpq+1}. \tag{78}
\]
A crucial difference with respect to the case $p > q$ is that we can increase the power for the logarithmic term, using integration by parts, even in this first stage of the inductive step. Plugging (78) in (77), we get

\[ \mathcal{T}(s) \geq 2^{-(2j+7)} KC_j^q (a_j q + 1)^{-1} (\log(s))^{-(q-1)-b_j q} \left( \log \left( \frac{s}{\ell_{2j+1}} \right) \right)^{a_j q + 1}. \]

Then, we combine the previous lower bound for $\mathcal{T}'(s)$ with (60), so that for $t \geq \ell_{2j+2}$ it follows

\[
\begin{align*}
\mathcal{U}(t) &\geq 2^{-(2j+7)p} CK^p C_j^p (a_j q + 1)^{-p} (t)^{-1} \int_{\ell_{2j+1}}^{t} (t-s) (s)^{-1} \left( \log(s) \right)^{-(pq-1)-b_j pq} \left( \log \left( \frac{s}{\ell_{2j+1}} \right) \right)^{a_j pq + p} ds \\
&\geq 2^{-(2j+7)p} CK^p C_j^p (a_j q + 1)^{-p} (\log(t))^{-(pq-1)-b_j pq} (t)^{-1} \int_{\ell_{2j+1}}^{t} (t-s) (s)^{-1} \left( \log \left( \frac{s}{\ell_{2j+1}} \right) \right)^{a_j pq + p} ds \\
&\geq 2^{-(2j+7)p-2} CK^p C_j^p (a_j q + 1)^{-p} (\log(t))^{-(pq-1)-b_j pq} (t)^{-1} \int_{\ell_{2j+1}}^{t} \frac{t-s}{s} \left( \log \left( \frac{s}{\ell_{2j+1}} \right) \right)^{a_j pq + p} ds. \quad (79)
\end{align*}
\]

We use again integration by parts. Thus,

\[
\begin{align*}
\int_{\ell_{2j+1}}^{t} \frac{t-s}{s} \left( \log \left( \frac{s}{\ell_{2j+1}} \right) \right)^{a_j pq + p} ds &= (a_j pq + p + 1)^{-1} \int_{\ell_{2j+1}}^{t} \left( \log \left( \frac{s}{\ell_{2j+1}} \right) \right)^{a_j pq + p + 1} ds \\
&\geq (a_j pq + p + 1)^{-1} \int_{\ell_{2j+1}}^{t} \left( \log \left( \frac{s}{\ell_{2j+1}} \right) \right)^{a_j pq + p + 1} ds \\
&\geq (a_j pq + p + 1)^{-1} \left( 1 - \frac{\ell_{2j+1}}{\ell_{2j+2}} \right) t \left( \log \left( \frac{t}{\ell_{2j+2}} \right) \right)^{a_j pq + p + 1} \\
&\geq 2^{-(2j+3)} (a_j pq + p + 1)^{-1} (t) \left( \log \left( \frac{t}{\ell_{2j+2}} \right) \right)^{a_j pq + p + 1}. \quad (80)
\end{align*}
\]

If we combine (79) and (80), then, we arrive at

\[ \mathcal{U}(t) \geq 2^{-(p+1)2j-7p-8} CK^p C_j^p (a_j q + 1)^{-p} (a_j pq + p + 1)^{-1} (\log(t))^{-(pq-1)-b_j pq} \left( \log \left( \frac{t}{\ell_{2j+2}} \right) \right)^{a_j pq + p + 1}. \]

Putting

\[
C_{j+1} = 2^{-(p+1)2j-7p-8} CK^p C_j^p (a_j q + 1)^{-p} (a_j pq + p + 1)^{-1},
\]

\[
a_{j+1} = a_j pq + p + 1 \quad \text{and} \quad b_{j+1} = (pq-1) + b_j pq,
\]

from the last inequality we get (68) for $j + 1$ when $p = q$.

Let us write the expressions of $a_j$ and $b_j$,

\[
a_j = a_{j-1} pq + p + 1 = a_0 (pq)^j + (p + 1) \sum_{k=0}^{j-1} (pq)^k = (1 + \frac{p+1}{pq-1})(pq)^j - \frac{p+1}{pq-1}, \quad (81)
\]

\[
b_j = (pq-1) + b_{j-1} pq = (pq-1) \sum_{k=0}^{j-1} (pq)^k + b_0 (pq)^j = (pq)^j - 1. \quad (82)
\]

Therefore,

\[
a_{j-1} pq + p + 1 = \frac{p(q+1)}{pq-1}(pq)^j + 1 - \frac{p+1}{pq-1} \leq \frac{p(q+1)}{pq-1}(pq)^j,
\]

\[
a_{j-1} = \frac{p(q+1)}{pq-1}(pq)^j - \frac{p+1}{pq-1} \leq \frac{p(q+1)}{pq-1}(pq)^j,
\]

so that we have again (76) but now with $\Theta \doteq 2^{(p+1)(pq)^{p+1}}$ and $M \doteq 2^{-5p-6} CK^p \frac{(pq-1)^{p+1}}{(pq-1)^{p+1} (pq)^{p+1}}$. 

22
Lower bound for $C_j$

Let us derive now a lower bound for $C_j$, in which the dependence on $j$ can be more easily handled than in (76). Applying the logarithmic function to both sides of (76) and iterating the obtained relation, we get

$$\log C_j \geq (pq) \log C_{j-1} - j \log \Theta + \log M$$

$$\geq (pq)^2 \log C_{j-2} - (j + (j - 1)(pq)) \log \Theta + (1 + pq) \log M$$

$$\geq \cdots \geq (pq)^j \log C_0 - \sum_{k=0}^{j-1} (j - k)(pq)^k \log \Theta + \sum_{k=0}^{j-1} (pq)^k \log M$$

$$= (pq)^j \log C_0 - pq \sum_{k=1}^{j} \frac{k}{(pq)^k} \log \Theta + \frac{(pq)^j - 1}{pq - 1} \log M$$

$$= (pq)^j \left( \log C_0 - S_j \log \Theta + \log M \right) + \frac{\log M}{pq - 1},$$

where $S_j \doteq \sum_{k=1}^{j} \frac{k}{(pq)^k}$. By the ratio test it follows immediately that $\{S_j\}_{j=1}^\infty$ is the sequence of partial sums of a convergent series. Therefore, if we denote by $S$ the limit of this sequence, then, since $S_j \uparrow S$ as $j \to \infty$ we may estimate

$$C_j \geq M^{-(pq-1)} \exp \left( (pq)^j \log \left( C_0 \Theta^{-S} M^{pq-1} \right) \right). \quad (83)$$

4.4. Conclusion of the proof of Theorem 1.6

In this section we complete the proof in the critical case $F(n, p, q) = 0$. Summing up the results of the last section, from (74), (75), (81), (82) it follows the validity of (68) with

$$a_j = A(pq)^j + 1 - A, \quad b_j = B(pq)^j - B, \quad (84)$$

where

$$A = \begin{cases} \frac{pq}{pq - 1} & \text{if } p > q, \\ 1 + \frac{p+1}{pq - 1} & \text{if } p = q \end{cases}, \quad B = \begin{cases} \frac{p(q+1)}{pq} & \text{if } p > q, \\ \frac{1}{pq} & \text{if } p = q. \end{cases}$$

Combining (68), (83) and (84), we arrive at

$$\mathcal{U}(t) \geq M^{-(pq-1)} \exp \left( (pq)^j \log \left( C_0 \Theta^{-S} M^{pq-1} \right) \right) (\log(t))^{-B(pq)^j} + B \log \left( \log \left( \frac{t}{\ell_j} \right) \right)^{A(pq)^j + 1 - A}$$

$$\geq M^{-(pq-1)} \exp \left( (pq)^j \log \left( C_0 \Theta^{-S} M^{pq-1} \right) \right) (\log(t))^{-B(pq)^j} + B \log \left( \log \left( \frac{t}{\ell_j} \right) \right)^{A(pq)^j + 1 - A}$$

$$\geq M^{-(pq-1)} \exp \left( (pq)^j \log \left( C_0 \Theta^{-S} M^{pq-1} \right) (\log(t))^{-B(pq)^j} + B \log \left( \log \left( \frac{t}{\ell_j} \right) \right)^{A(pq)^j + 1 - A} \right)$$

for any $t \geq 2$. Since $\log(3 + t) \leq \log(2t) \leq 2 \log t$ and $\log \left( \frac{t}{\ell_j} \right) \geq \frac{1}{2} \log t$ for any $t \geq 4$, from the last estimate we may derive the following estimate for $t \geq 4$:

$$\mathcal{U}(t) \geq M^{-(pq-1)} \exp \left( (pq)^j \log \left( 2^{-B-A} C_0 \Theta^{-S} M^{pq-1} (\log(t))^{A-B} \right) \right) (\log(t))^{B} \left( \log \left( \frac{t}{\ell_j} \right) \right)^{1 - A}. \quad (85)$$

Let us consider separately the cases $p > q$ and $p = q.$
Case $p > q$

In this case $C_0 = \tilde{C}_\varepsilon^{pq}$. Hence, (85) implies

$$U(t) \geq M^{-(pq-1)} \exp \left( (pq)^j \log \left( E \varepsilon^{pq} (\log t)^{\frac{pq}{pq-1}} \right) \right) (\log(t))^B \left( \log \left( \frac{t}{2} \right) \right)^{1-A},$$

where $E \equiv 2^{1-p} C \varepsilon^{pq-1}$. Let us denote $K(t) \equiv \log \left( E \varepsilon^{pq} (\log t)^{\frac{pq}{pq-1}} \right)$.

We can choose $\varepsilon_0 = \varepsilon_0(u_0, v_0, v_1, n, p, q, b_1, b_2, R) > 0$ so small that

$$\exp \left( E^{-\frac{pq-1}{pq}} \varepsilon_0^{-q(pq-1)} \right) \geq 4.$$

Therefore, for any $\varepsilon \in (0, \varepsilon_0]$ and any $t \geq \exp \left( E^{-\frac{pq-1}{pq}} \varepsilon^{-q(pq-1)} \right)$ we get $t \geq 4 K(t) > 0$ and, consequently, taking the limit in (86) as $j \to \infty$ we find that $U(t)$ is not finite. Also, we proved the upper bound estimate for the lifespan $T \leq \exp \left( E^{-\frac{pq-1}{pq}} \varepsilon^{-q(pq-1)} \right)$.

Case $p = q$

In this case $C_0 = \tilde{C}_\varepsilon^p$. Therefore, (85) yields

$$U(t) \geq M^{-(pq-1)} \exp \left( (pq)^j \log \left( E \varepsilon^p (\log t)^{\frac{p}{p-1}} \right) \right) (\log(t))^B \left( \log \left( \frac{t}{2} \right) \right)^{1-A}.$$

Repeating the same steps as in the first case, we get the upper bound estimate for the lifespan

$$T \leq \exp \left( E^{-\frac{p-1}{p-2}} \varepsilon^{-p(p-1)} \right).$$

This conclude the proof of Theorem 1.6.

Acknowledgments

The first author is member of the Gruppo Nazionale per L’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Instituto Nazionale di Alta Matematica (INdAM). This paper was written partially during the stay of the first author at Tohoku University in 2018. He would like to thank the Mathematical Department of Tohoku University for the hospitality and the excellent working conditions during this period. The second author is partially supported by the Grant-in-Aid for Scientific Research (B)(No.18H01132).

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