BOUNDS ON THE NODAL STATUSES OF SOME TRANSFINITE GRAPHS

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Abstract — The bounds on the statuses of the nodes in a finite graph established by Entringer, Jackson, and Snyder are extended herein so that they apply to the statuses of the nodes in transfinite graphs of a certain kind.

Key Words: Statuses in graphs, transfinite generalization of status, distances in transfinite graphs.

1 Introduction

The purpose of this note is to extend the known bounds on the statuses of the nodes of a finite graph to nodes in transfinite graphs of a certain kind. That known result was established by Entringer, Jackson, and Snyder [1]. It states that the status \( s(x) \) of any node \( x \) in a finite connected graph \( G \) having \( p \) nodes and \( q \) branches satisfies the inequalities

\[
p - 1 \leq s(x) \leq \frac{(p - 1)(p + 2)}{2} - q
\]

and that these bounds can be achieved for each \( q \) such that \( (p - 1) \leq q \leq p(p - 1)/2 \). A modification of this result holds for transfinite graphs satisfying certain conditions.

2 Some Preliminary Definitions and Known Results

We shall use some definitions and symbolism appearing in [3]. Also, we restrict our attention to transfinite graphs \( G^\mu \) of rank \( \mu \), where for the sake of some simplicity we restrict \( \mu \) to the positive natural numbers.\(^2\) The rather complicated recursive definitions of such \( \mu \)-graphs

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\(^1\)See also [2, pages 43-44] for an exposition of that result.

\(^2\)When \( \mu = 0 \), \( G^\mu \) is a conventional graph.
appear in Section 2.4 of [3]. All our arguments extend readily to graphs of higher ranks, that is, the transfinite-ordinal ranks. Since the distance from any nonmaximal node $z$ is the same as the distance from any node of higher rank containing $z$, we can restrict our attention to the maximal nodes in $G^\mu$, that is, to the nodes that are not contained in any node of higher rank. This is understood henceforth.

Transfinite nodes are defined in terms of tips (i.e., graphical extremities), which in turn are equivalence classes of one-ended paths, as stated in [3, page 11]. The $\mu$-nodes in $G^\mu$ are the nodes of highest rank in $G^\mu$. A $\mu$-node is said to be pristine if it does not contain a node of lower rank; We will assume that all the $\mu$-nodes are pristine. Also, a node $x^\rho$ of any rank $\rho$ ($\rho \leq \mu$) is called a nonsingleton if it contains at least two elements (either two $(\mu - 1)$-tips of a $(\mu - 1)$-tip and a node of rank lower than $\rho$).

Furthermore, two branches are said to be $\rho$-connected if there is a path of rank $\rho$ or less that terminates at nodes of those branches. Actually, such path-connectedness need not exist between all pairs of branches.\(^3\) To insure that such path-connectedness does exist, we impose the following Condition A [3, page 25]. We say that two tips are nondisconnectable if their representative paths meet infinitely often [3, page 25]. Also, a node is said to embrace a tip if that tip is part of that node (see [3, page 12] for the precise definition).

**Condition A.** If two tips are nondisconnectable, then either they are contained in the same node or at least one of them is the sole member of a maximal node.

Under this condition, for any two nonsingleton nodes there will be a path that terminates at them [3, Lemma 4.3-2], and moreover such path connectedness is a transitive binary relation for the nonsingleton nodes of $G^\mu$; in fact, it is an equivalence relation [3, Theorem 3.1-4].

Throughout this work we assume that $G^\mu$ is $\mu$-connected in the sense that every pair of branches are $\rho$-connected for some rank $\rho \leq \mu$ depending on the choice of those branches. As a result, the set of branches in $G^\mu$ is partitioned into subsets according to $(\mu - 1)$-connectedness, and the subgraph of rank $\mu - 1$ induced by such a subset is called a $(\mu - 1)$-section [3, page 23]. Because we are assuming that all the $\mu$-nodes are pristine, it follows

\(^3\)A more general concept of connectedness is based on walks. Such walk-connectedness always exists between branches. Our results extend to this case [3, page 67], as is indicated at the end of this paper.
that every node of rank less than $\mu$ is contained within some $(\mu - 1)$-section. Thus, the $(\mu - 1)$-sections also partition the set of nodes of ranks less than $\mu$. We say the a $\mu$-node $x^\mu$ is incident to a $(\mu - 1)$-section $S^{\mu-1}$ if $x^\mu$ contains a $(\mu - 1)$-tip whose representative paths lie within $S^{\mu-1}$. Moreover, if a $\mu$-node is incident to two or more $(\mu - 1)$-sections, it serves as a connection between them.

Lengths of paths and the distances between nodes are defined in [3, Sections 4.2 and 4.4]. By virtue of Condition A, the $\mu$-connectedness of $G^\mu$, and the assumption that all the $\mu$-sections are pristine, we have the following results as a consequence of [3, Lemma 4.7-4]. The length of any path $P$ within a $(\mu - 1)$-section $S^{\mu-1}$ that is incident to a $\rho$-node $x^\rho$ ($\rho < \mu$) in $S^{\mu-1}$ and reaches a $\mu$-node $x^\mu$ incident to $S^{\mu-1}$ is $\omega^\mu$. This is because $P$ can reach $x^\mu$ only through a $(\mu - 1)$-tip. Consequently, we can define the $\mu$-length of any two-ended path $P$ that reaches or passes through at least one $\mu$-node as $\omega^\mu \cdot n$, where $n$ is the number of incidences that $P$ makes with $\mu$-nodes; that is, when $P$ terminates at a $\mu$-node, there is one such incidence, and, when $P$ passes through a $\mu$-node from one $(\mu - 1)$-section to another adjacent $(\mu - 1)$-section, there are two such incidences. Furthermore, we define the $\mu$-distance between any two nodes $x$ and $y$ as the minimum of the $\mu$-lengths of all the paths that meet $x$ and $y$; such a $\mu$-distance exists because those $\mu$-lengths are ordinals and any set of ordinals is well-ordered and therefore has a minimum. It is a fact that under our assumptions there will be a path terminating at $x$ and $y$ whose $\mu$-length is that $\mu$-distance; such a path is called an $x, y$ geodesic.

3 The $\mu$-Statuses of Nonsingleton Nodes

Even though $G^\mu$ is branchwise $\mu$-connected, it can happen that there is no path between two nodes if at least one of them is a singleton, in which case no (path-based) distance will exist between them. However, under Condition A, distances between nonsingleton nodes always exist.\footnote{See [3, Section 3.1] for a discussion of this matter.} For this reason, we shall restrict our definition of nodal statuses to the nonsingleton nodes.\footnote{One might motivate this restriction by noting that a singleton node is a “dead end” in the sense that no path can pass through it, and so it does not contribute to the connectivity of $G^\mu$.}
Another convention that we shall adopt in order to extend (1) transfinitely is that the
distance between any two nodes within a \((\mu - 1)\)-section is taken to be 0. Thus, it is only a
transition to or from a \(\mu\)-node that contributes to the length of a geodetic path and thereby
to a distance. Without this assumption, the status of a node could be infinite. Also, we shall
henceforth assume that the \(\mu\)-connected \(\mu\)-graph \(G^\mu\) has only finitely many nonsingleton
\(\mu\)-nodes and only finitely many \((\mu - 1)\)-sections.

To define the \(\mu\)-status of any nonsingleton node \(x^\rho\) \((\rho \leq \mu)\), we first choose a single
node \(y^\alpha_m\) \((\alpha_m < \mu)\) for each \((\mu - 1)\)-section \(S^\mu_{m-1}\), one such node for each \((\mu - 1)\)-section,
and designate it as the representative node for \(S^\mu_{m-1}\). Because we have taken the distances
between nodes in a single \((\mu - 1)\)-section to be 0, we can take the distance from any node
in \(S^\mu_{m-1}\) to be the same as the distance from the representative node \(y^\alpha_m\) for \(S^\mu_{m-1}\). Then,
we define the \(\mu\)-status \(s^\mu(x^\rho)\) as the sum of the distances from \(x^\rho\) to all the nonsingleton
\(\mu\)-nodes plus the sum of the distances to the representative nodes of all the \((\mu - 1)\)-sections.
In symbols,
\[
s^\mu(x^\rho) = \sum_{k=1}^{K} d(x^\rho, x^\mu_k) + \sum_{m=1}^{M} d(x^\rho, y^\alpha_m) \tag{2}
\]
Here, \(k\) numbers the nonsingleton \(\mu\)-nodes, there being \(K\) of them, and \(m\) numbers the
\((\mu - 1)\)-sections, there being \(M\) of those. By our assumptions, \(K\) and \(M\) are natural
numbers.

4 Conditions Imposed upon \(G^\mu\)

Let us now list all the conditions we have assumed for \(G^\mu\).

4.1. The rank \(\mu\) of the transfinite graph \(G^\mu\) is a positive natural number.

4.2. \(G^\mu\) is \(\mu\)-connected (i.e., between every two branches there is path connecting them).

4.3. Condition A holds.

4.4. All \(\mu\)-nodes are pristine (i.e., none of them contains a node of lower rank).

4.5. There are only finitely many nonsingleton \(\mu\)-nodes and only finitely many \((\mu - 1)\)-
sections.
4.6. The distance between any two nodes within the same \((\mu - 1)\)-section is taken to be 0.

Also, bear in mind the following considerations.

**Note 4.7.** We have restricted our attention to only the maximal nodes because the distance from any nonmaximal node is the same as the distance from the maximal node containing it.

**Note 4.8.** Similarly, we have considered in our analysis only the nonsingleton nodes because the singleton nodes do not contribute to the connectivity of \(G^\mu\). On the other hand, as a result of Assumptions 4.2 and 4.3, for any two nonsingleton nodes \(x\) and \(y\) there is a path terminating at \(x\) and \(y\) [3, Lemma 4.3-2]. Consequently, a distance is defined between \(x\) and \(y\).

In view of all this, the \(\mu\)-status \(s(x^\rho)\) for any maximal nonsingleton node \(x^\rho\) \((\rho \leq \mu)\) is well-defined by (2).

5 The Replacement 0-graph

In order to extend (1) transfinitely, we replace \(G^\mu\) by a 0-graph in which the distances in \(G^0\) are the same as the \(\mu\)-distances in \(G^\mu\) except for a multiplicative factor \(\omega^\mu\). To do this, we adapt the replacement procedure given in [3, page 60]. Remember that \(\mu \geq 1\), that \(k = 1, 2, \ldots, K\) numbers the nonsingleton \(\mu\)-nodes, and that \(m = 1, 2, \ldots, M\) numbers the \((\mu - 1)\)-sections. Now, we replace each nonsingleton \(\mu\)-node \(x_k^\mu\) by a 0-node \(x_k^0\). Furthermore, having chosen a nonsingleton \(\rho_m\)-node \(y_m^{\rho_m}\) \((0 \leq \rho_m \leq \mu - 1)\) within each \((\mu - 1)\)-section, we replace \(y_m^{\rho_m}\) by a 0-node \(y_m^0\). (If \(\rho_m = 0\), no replacement is needed.) Thus, for each \(m\) we insert a branch between \(y_m^0\) and each \(x_k^0\) corresponding to a \(\mu\)-node incident to the \((\mu - 1)\)-section \(S_{m}^{\mu - 1}\) that contains \(y_m^\rho\). In this way, \(G^\mu\) is replaced by a finite 0-graph consisting of the 0-nodes \(x_k^0\) and \(y_m^0\) \((k = 1, 2, \ldots, K; m = 1, 2, \ldots, M)\) and of the said branches. In particular, each \((\mu - 1)\)-section \(S_{m}^{\mu - 1}\) is replaced by a star 0-graph with \(y_m^0\) as its center 0-node and the \(x_k^0\) corresponding to the \(\mu\)-nodes incident to \(S_{m}^{\mu - 1}\) as its peripheral 0-nodes.

Now, to each path \(P\) in \(G^\mu\) there corresponds a unique path \(Q\) that passes through the 0-nodes \(x_k^0\) corresponding to the \(\mu\)-nodes \(x_k^\mu\). If one terminal node \(y_m^{\rho_m}\) \((\rho_m < \mu)\) of \(P\) lies within a \((\mu - 1)\)-section, the path \(Q\) terminates correspondingly at the 0-node \(y_m^0\). On the
other hand, if $P$ terminates at a $\mu$-node $x_k^\mu$, then $Q$ terminates at $x_k^0$. Furthermore, we define the $\mu$-length $|P|$ of $P$ as $\omega^\mu$ times the number $n$ of incidences that $P$ makes with the $\mu$-nodes, where we count an incidence once if $P$ terminates at a $\mu$-node and we count the incidence twice if $P$ passes through a $\mu$-node. On the other hand, the number of branches in $Q$ is simply $n$. Thus, the length $|Q|$ of $Q$ is $n$. As a result of all this, the lengths of $P$ and $Q$ are related as follows:

$$|P| = \omega^\mu \cdot |Q| \quad (3)$$

6 Bounds on the Statuses

In conformity with the way we have defined the $\mu$-lengths of paths in $G^\mu$, we can define the $\mu$-distance between any two nodes in $G^\mu$ as $\omega^\mu$ times the distance between the corresponding 0-nodes in $G^0$, where any node within a $(\mu - 1)$-section $S_{m-1}^{\mu-1}$ is represented by the node $y_{m}^{0}$ for $S_{m-1}^{\mu-1}$.

Thus, the $\mu$-status of any node $x^\rho (\rho \leq \mu)$ of $G^\mu$, as defined by (2), is simply $\omega^\mu$ times the status of the 0-node in $G^0$.

To lift the bounds (1) to the transfinite case, we need merely determine the number $p$ of 0-nodes and the number $q$ of branches in $G^0$. Specifically, $p$ is the number of $\mu$-nodes in $G^\mu$ plus the number of $(\mu - 1)$-sections in $G^\mu$. With regard to $q$, note again that each $(\mu - 1)$-section $S_{m-1}^{\mu-1}$ has been replaced by a star graph with center at $y_{m}^{0}$ and branches between $y_{m}^{0}$ and every one of the $x_{k}^{0}$ corresponding to the $\mu$-nodes $x_{k}^{\mu}$ incident to $S_{m-1}^{\mu-1}$. Let $\delta_{m}$ be the number of such $x_{k}^{\mu}$; $\delta_{m}$ is the degree of $y_{m}^{0}$. Then,

$$q = \sum_{m=1}^{M} \delta_{m}.$$ 

Altogether then, under the assumptions 4.1 to 4.6, we have the desired bounds on the status $s(x^\rho)$ of any node $x^\rho$ in $G^\mu$ as follows:

$$\omega^\mu \cdot (p - 1) \leq s(x^\rho) \leq \omega^\mu \cdot [(p - 1)(p + 2)/2 - q] \quad (4)$$

These bounds can be achieved for each $q$ such that $p - 1 \leq q \leq p(p - 1)/2$. 

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7 Two Final Comments

7.1. We have restricted our analysis to only the nonsingleton nodes in $G^\mu$. Thus, the $\mu$-status $s^\mu(x)$ of any such node $x$ is the sum of the distances from $x$ to all (and only) the other nonsingleton $\mu$-nodes and the other representative nonsingleton nodes in the $(\mu-1)$-sections (one representative node to each $(\mu-1)$-section. This restriction can be relaxed somewhat by allowing distances from $x$ to include some of the singleton $\mu$-nodes. Specifically, we can also allow a finite number of singleton $\mu$-nodes such that the $\mu$-distances to them from any nonsingleton node exists. Such singleton nodes can occur; see the set $\mathcal{M}$ defined in [3, page 44].

7.2. We can also relax the restriction imposed by assumption 4.2 by taking definitions of entities in the transfinite graph to be based on walks rather than on paths. As a result, two branches that are not connected by a transfinite path may be connected by a transfinite walk, as is explained in [3, Chapter 5]. Thus, upon assuming that the walk-based transfinite graph $G^\mu$ is walk-connected, we can define walk-based distances between any two nodes of $G^\mu$ [3, Section 5.4]. In this case, we can discard assumption 4.3; it is no longer needed. Then, $G^\mu$ can be related to a unique 0-graph $G^0$ whereby the $\mu$-distance between nonsingleton walk-based nodes in $G^\mu$ is $\omega^\mu$ times the distance between the corresponding 0-nodes in $G^0$. The procedure for doing this is much the same as that presented above. As a result, we again have (2) and (3), and also (1) replaced by (4), where $s^\mu(x^\rho)$ is now defined as the sum of the walk-based $\mu$-distances from $x^\rho$ to the walk-based nonsingleton nodes in $G^\mu$. Moreover, finitely many singleton $\mu$-nodes can also be allowed in this case, as is explained in the preceding paragraph 7.1.

References

[1] R.C. Entringer, D.C. Jackson, and D.E. Snyder, Distance in graphs, Czech. Math. Journal, 26 (1976), 283-296.

[2] F. Buckley and F. Harary, Distance in Graphs, Addison-Wesley Publishing Co., Redwood City, CA, 1990.
[3] A.H. Zemanian, *Graphs and Networks: Transfinite and Nonstandard*, Birkhauser-Boston, Cambridge, MA, 2004.