Determinants and conformal anomalies of GJMS operators on spheres

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The conformal anomalies and functional determinants of the Branson–GJMS operators, $P_{2k}$, on the $d$–dimensional sphere are evaluated in explicit terms for any $d$ and $k$ such that $k \leq d/2$ (if $d$ is even). The determinants are given in terms of multiple gamma functions and a rational multiplicative anomaly, which vanishes for odd $d$. Taking the mode system on the sphere as the union of Neumann and Dirichlet ones on the hemisphere is a basic part of the method and leads to a heuristic explanation of the non–existence of ‘super–critical’ operators, $2k > d$ for even $d$. Significant use is made of the Barnes zeta function. The results are given in terms of ratios of determinants of operators on a $(d + 1)$–dimensional bulk dual sphere. For odd dimensions, the log determinant is written in terms of multiple sine functions and agreement is found with holographic computations, yielding an integral over a Plancherel measure. The N–D determinant ratio is also found explicitly for even dimensions. Ehrhart polynomials are encountered.

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1. Introduction.

The conformally covariant higher derivative GJMS operators, $P_{2k}$, [1], have been the subject of a certain amount of activity in the general area of conformal geometry e.g. Juhl, [2], Gover, [3]. The paper of Diaz, [4], contains some physical applications and a useful survey.

Branson has computed the ratio of the (log) determinants of such operators in conformally related spaces, and has defined the notion of $Q$–curvature to be of use in this endeavour. The method is the standard one, well known to physicists since the late 1970s, of integrating the conformal anomaly, which is the constant term in the heat–kernel expansion, or the value of the $\zeta$–function at zero, and is a local quantity easily computed.

If one wishes to know the actual value of the determinant using this Jacobian, or cocycle function, it is necessary to evaluate it in some fiducial metric where the eigenvalue problem is sufficiently tractable, in one way or another. Traditionally, spheres and generalised cylinders have been used for this purpose, in other situations. This is a more difficult problem as it leads to non–local quantities, in particular the derivative of the $\zeta$–function at zero, in $\zeta$–function regularisation.

Branson, [5], constructed the GJMS conformally covariant higher derivative operators in the special case of the (round) sphere as an explicit product, and I intend here to use direct, simple–minded spectral techniques to work out the value of the $\zeta$–function of $P_{2k}$ at argument 0 (the conformal anomaly) and of its derivative at 0, which, as mentioned, determines the determinant.

For comparison, Guillarmou, [6], has already computed the determinant of $P_{2k}$, holographically, on the conformal infinity of a factored hyperbolic space. Trivial factoring gives the case of a bounding sphere. See also Diaz, [4]. Diaz and Dorn, [7], gave an independent conformal field theory computation of an equivalent quantity on this boundary using dimensional regularisation, again in a holographic setting.

2. The Branson GJMS operators

I denote by $Y_d$ the (non–negative) conformally invariant Laplacian,

$$-Y_d = \Delta_2 + \frac{d - 2}{4(d - 1)} R,$$

often referred to, in mathematical works, as the Yamabe operator.
Branson’s construction of $P_{2k}$, [5], in the special case of the (round) $d$–sphere is a simple product $^2$, $k \in \mathbb{Z}^+$,

$$P_{2k} = \prod_{j=0}^{k-1} (Y_d - j(j + 1))$$

$$= \prod_{j=0}^{k-1} \left( \sqrt{Y_d + 1/4} - j - 1/2 \right) \left( \sqrt{Y_d + 1/4} + j + 1/2 \right). \quad (1)$$

There are not many derivations of this factorisation. Branson uses Lie group theory to construct an object which has the required properties and then invokes a uniqueness argument. Amongst the few other derivations, Graham’s, [8], is the most direct, using stereographic projection from flat space. Morpurgo gives a similar discussion earlier, [9], and makes contact with Branson’s method.

I assume, in a physicist’s non–rigorous way, that the eigenvalue problem for $P_{2k}$ is solved by that for $Y_d$, and this is classically known. It has been shown, for example, with a certain amount of effort, that the $P_{2k}$ are self–adjoint. Some function analytic discussion can be found in [9].

The reason for writing $P_{2k}$ in the above way is because the eigenvalues of $Y_d + 1/4$ are perfect squares. Branson, [5], also gives this form. His notation is $B = \sqrt{Y_d + 1/4}$.

Furthermore, I am going to obtain the sphere expressions by adding the hemispherical Dirichlet and Neumann ones, as in previous work, [10–12]. The eigenvalues of $\sqrt{Y_d + 1/4}$ then take the general linear form $\mathbf{m} \cdot \mathbf{\omega} + a$ where $\mathbf{m}$ and $\mathbf{\omega}$ are $d$–dimensional vectors with $m_i \in \mathbb{Z}^+$ and (here) $\omega_i = 1$, $\forall i$. $^3$ This is the way the eigenvalues immediately appear in the classic, and ancient, separation of variables, the degeneracies arising from coincidences. It is better not to do the combinatorics which give them explicitly in terms of factorials.

The constant, $a$, takes the values $a = a_N = (d - 1)/2$ and $a = a_D = (d + 1)/2$ for, respectively, Neumann (N) and Dirichlet (D) conditions on the rim of the hemisphere. I write out the eigenvalues of $P_{2k}$,

$$\lambda^k(\mathbf{m}) = \prod_{j=0}^{k-1} \left( \mathbf{m} \cdot \mathbf{\omega} + a - j - 1/2 \right) \left( \mathbf{m} \cdot \mathbf{\omega} + a + j + 1/2 \right), \quad (2)$$

$^2$ I make a different sign choice to others.

$^3$ I retain $\omega$ for a while because the expressions have validity when the sphere is replaced by a quotient.
and remark that the first factor has a zero in the Neumann case at the upper limit first, as \( k \) increases, when \( k = d/2 \). (I am assuming initially that \( d \) is even). Increasing \( k \) to \( d/2+1 \) and over, gives zeros for \( N \), but also, more vitally, for Dirichlet conditions, which is impossible \(^4\), and so we must restrict \( k \) to \( k \leq d/2 \). The available general proofs of this depend upon the existence of appropriate tensors.

In the calculations, the critical case, \( k = d/2 \), must be treated separately. It corresponds, roughly, to the minimally coupled scalar. I now turn to the construction of some spectral invariants.

### 3. The zeta function and its value at zero

In the usual way, I define the relevant \( \zeta \)-function as the continuation of the sum,

\[
Z_d(s, a, k) = \sum_{m=0}^{\infty} \frac{1}{(\lambda^k(m))^s}
\]

\[
= \sum_{m=0}^{\infty} \prod_{j=0}^{k-1} \frac{1}{((m \cdot \omega + a)^2 - (j + 1/2)^2)^{s}},
\]

and am required to find \( Z_d(0, a, k) \) and \( Z_d'(0, a, k) \). \(^5\) The evaluation at \( s = 0 \) is the condition that allows an explicit solution along the lines followed in [11]. This reference deals with the lowest case, \( k = 1 \). In order to make comparison simpler I rewrite (3)

\[
Z_d(s, a, k) = \sum_{m=0}^{\infty} \frac{1}{((m \cdot \omega + a)^2 - \alpha_j^2)^s \cdots ((m \cdot \omega + a)^2 - \alpha_{k-1}^2)^s},
\]

where \( \alpha_j = j + 1/2 \). In [11] it was shown that, in order to deduce \( Z_d(0, a, k) \) and \( Z_d'(0, a, k) \), it was sufficient to expand in powers of \( \alpha \) and I will employ the same tactic here, for every factor. The procedure is adequately illustrated by the \( k = 2 \),

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\(^4\) The argument is an extension of the one familiar in potential theory and depends on the alternating signs of the powers of the Laplacian in \( P_{2k} \) to give a sum of squares on application of Green’s theorem. If \( 2k > d \), we are led to operators that have a Dirichlet mode on the hemisphere. This is impossible and so, while one can write the operators down, they have to be multiplied by zero, which is always an option.

\(^5\) Another task would be to determine the singularity structure.
Paneitz operator case,

\[ Z_d(s, a, 2) = \sum_{m=0}^{\infty} \frac{1}{((m \cdot \omega + a)^2 - \alpha^2)^s ((m \cdot \omega + a)^2 - \alpha'^2)^s} \]

\[ = \sum_{r, r'=0}^{\infty} \alpha^{2r} \alpha'^{2r'} \frac{s(s+1) \ldots (s+r-1)}{r!} \frac{s(s+1) \ldots (s+r'-1)}{r'!} \]

\[ \times \zeta_d(4s + 2r + 2r', a | \omega), \]

with \( \alpha = 1/2, \alpha' = 3/2 \) and where the mode sum has now been performed to give a Barnes \( \zeta \)-function as in [11], and elsewhere. This expression does not constitute a complete continuation. It does not give the poles, for example, but the values when \( s \) is a non-positive integer can be obtained fairly easily.

The general definition of the Barnes \( \zeta \)-function is,

\[ \zeta_d(s, a | \omega) = \frac{i \Gamma(1 - s)}{2\pi} \int_L d\tau \frac{\exp(-a\tau)(-\tau)^s-1}{\prod_{i=1}^{d} (1 - \exp(-\omega_i\tau))} \]

\[ = \sum_{m=0}^{\infty} \frac{1}{(a + m \cdot \omega)^s}, \quad \text{Re } s > d, \quad (6) \]

where I refer to the components, \( \omega_i \), of the \( d \)-vector, \( \omega \), as the degrees or parameters. For simplicity, I assume that the \( \omega_i \) are real and positive. For the hemisphere they are all equal to 1, as given above. If \( a \) is zero, the origin, \( \mathbf{m} = \mathbf{0} \), is to be excluded. The contour, \( L \), is the standard Hankel one.

If \( s \) is set to zero in (3) one has to use the fact that the Barnes \( \zeta \)-function has poles,

\[ \zeta_d(s + r, a | \omega) \to \frac{N_r(d, a)}{s} + R_r(d, a) \quad \text{as } s \to 0, \quad (7) \]

where \( 1 \leq r \leq d \). Then very easily,

\[ Z_d(0, a, 2) = \zeta_d(0, a | \omega) + \frac{1}{4} \sum_{r=1}^{u} \frac{\alpha^{2r} + \alpha'^{2r}}{r} N_{2r}(d, a), \]

where \( u = d/2 \) if \( d \) is even and \( u = (d - 1)/2 \) if \( d \) is odd.

The residues are known in terms of generalised Bernoulli polynomials but they are not needed because it was shown in [11] that there is an identity,

\[ \zeta_d(0, a + \alpha | \omega) + \zeta_d(0, a - \alpha | \omega) - 2\zeta_d(0, a | \omega) = \sum_{r=1}^{u} \frac{\alpha^{2r}}{r} N_{2r}(d, a), \quad (8) \]
and then,

\[ Z_d(0, a, 2) = \frac{1}{4} (\zeta_d(0, a + \alpha | \omega) + \zeta_d(0, a - \alpha | \omega) + \zeta_d(0, a + \alpha' | \omega) + \zeta_d(0, a - \alpha' | \omega)) , \]

which is the average of the regularised dimensions of the linear operators occurring in (2).

It is clear that the same structure will hold for all levels, and so, for the sub–critical cases,

\[ Z_d(0, a, k) = \frac{1}{2k} \sum_{j=0}^{k-1} (\zeta_d(0, a + \alpha_j) + \zeta_d(0, a - \alpha_j)) , \quad k < d/2 . \] (9)

To obtain the hemisphere expression, the appropriate \( a \) coefficient has to be used. For the Dirichlet case, (9) holds for the critical \( k = d/2 \) case also, while for Neumann conditions, the existence of a zero mode means that 1 has to be subtracted from the right–hand side.

Barnes, [13] gives the required values of the \( \zeta \)–function, in particular,

\[ \zeta_d(0, a | \omega) = \frac{(-1)^d}{d!} B_d^{(d)}(a | \omega) \]

so that from (9), if \( d \) is even,

\[ Z_d(0, a_D, k) = \frac{1}{2k d!} \sum_{j=0}^{k-1} \left( B_d^{(d)}(d/2 + j + 1) + B_d^{(d)}(d/2 - j) \right) \]

\[ = \frac{1}{2k d!} \sum_{j=0}^{k-1} \left( B_d^{(d)}(d/2 - j - 1) + B_d^{(d)}(d/2 + j) \right) \]

\[ = Z_d(0, a_N, k) , \] (10)

where now I have dropped the label \( \omega \) and used a symmetry of the Bernoulli polynomials, which shows that the N and D quantities are equal, for \( k < d/2 \). Adding these two then just doubles the expression and so, on the full \( i.e. \) periodic sphere,

\[ Z_d(0, k) \bigg|_{\text{sphere}} = \frac{1}{k d!} \sum_{j=0}^{k-1} \left( B_d^{(d)}(d/2 + j + 1) + B_d^{(d)}(d/2 + j) \right) - \delta_{k,2d} , \] (11)

which is the expression quoted in an earlier work, [14], where some further comments can be found.
The expression (11) can be re-expressed in various ways. Use of the integral relation
\[ B_\nu^{(n-1)}(x) = \int_0^1 dt \, B_\nu^{(n)}(x + t), \]
and the product form, (e.g. [15] p.186, [16], §8),
\[ B_d^{(d+1)}(x | 1) = (x - 1)(x - 2) \ldots (x - d), \]
yields the compact form,
\[ Z_d(0, k) \bigg|_{\text{sphere}} + \delta_{k, 2d} \frac{2(-1)^{d/2}}{k \, d!} \int_0^k dt \, \prod_{i=1}^{d/2-1} (i^2 - t^2), \]
which is identical to a formula deduced by Diaz, [4], on a holographic argument.

If \( d \) is odd, the (anti–) symmetry of the Bernoulli polynomials shows that
\[ Z_d(0, a, k) = -Z_d(0, a, k) \]
and so the conformal anomaly on odd spheres is zero, as it must be.

4. The determinants in subcritical cases

It is necessary to find the derivative \( Z_d'(0, a, k) \). I will again use the \( k = 2 \) expression (5) to exemplify the calculation.

The sums over \( r \) and \( r' \) are split up to give
\[
Z_d(s, a, 2) = \zeta_d(4s, a) + \sum_{r=1}^{\infty} (\alpha^{2r} + \alpha'^{2r}) \frac{s(s+1) \ldots (s+r-1)}{r!} \zeta_d(4s+2r, a)
+ \sum_{r, r'=1}^{\infty} \alpha^{2r} \alpha'^{2r'} \frac{s(s+1) \ldots (s+r-1)}{r!} \frac{s(s+1) \ldots (s+r'-1)}{r'!} \times \zeta_d(4s+2r+2r', a).
\]

The calculation of the derivative of the first two terms proceeds as in [11]. The difference is the sum \((\alpha^{2r} + \alpha'^{2r})\) in place of \(\alpha^{2r}\) and the argument of 4s instead of 2s. Carrying through the algebra is straightforward and yields for the derivative at zero
\[
Z_d'(0, a, 2) = \zeta_d'(0, a + \alpha) + \zeta_d'(0, a - \alpha) + \zeta_d'(0, a + \alpha') + \zeta_d'(0, a - \alpha')
- \sum_{r=1}^{u} \frac{\alpha^{2r} + \alpha'^{2r}}{r} H_2(r) N_2r(d, a) + X,
\]
where \( H_k \) is defined by,

\[
H_k(r) = H(2r - 1) - \frac{1}{2k}H(r - 1)
\]

in terms of the harmonic series

\[
H(r) = \sum_{n=1}^{r} \frac{1}{n}.
\]

\( X \) is the contribution of the third term in (12), now to be calculated. This term is the novelty.

The pole structure, (7), and the existence of a factor of \( s^2 \) in the summand quickly shows that the remainder term, \( R \), can be ignored and also that the sum over the region where the Barnes \( \zeta \)–function has no poles can be dropped. From the argument of the \( \zeta \)–function this region is \( r + r' > u \). The sum is therefore restricted to a finite set of terms, which is a big calculational simplification.

Taking the elementary derivative, and setting \( s \) to 0 gives, combined with (13), the complete derivative at zero,

\[
Z'_d(0, a, 2) = \zeta'_d(0, a + \alpha) + \zeta'_d(0, a - \alpha) + \zeta'_d(0, a + \alpha') + \zeta'_d(0, a - \alpha')
\]

\[
- \sum_{r=1}^{u} \frac{\alpha^{2r} + \alpha'^{2r}}{r} H_2(r) N_{2r}(d, a)
\]

\[
+ \frac{1}{4} \sum_{r, r'} \alpha^{2r} \alpha'^{2r'} \left( \frac{1}{r} H(r - 1) + \frac{1}{r'} H(r' - 1) \right) N_{2r+2r'}(d, a). \tag{14}
\]

The first four terms are to be expected from the complete linear factorisation of the eigenvalues, (2). The remaining ones are evidence of multiplicative anomalies \(^6\). The fifth term is the anomaly associated with the factorisation of each \( \alpha^2 \) bracket, of which there are two. I term this a first order anomaly. The last term is a mixing, or cross term, corresponding to an interaction between the different \( \alpha^2 \) factors. I call this a second order anomaly.

If one raises the level to \( k \) by adding further factors to (5) one sees, by splitting the sums, that there are \( k \) contributions of first order and \( k(k-1)/2 \) of second. Further, the increasing powers of \( s \) in the summand show that there are no third order contributions. The calculation of the determinants for the \( P_{2k} \) is then straightforward and a matter of bookkeeping.

\(^6\) In odd dimensions there is no anomaly or, rather, the N and D hemisphere anomalies cancel.
The derivatives of the Barnes function are formally expressed in terms of multiple gamma functions, by definition of the latter, and I write the general log determinant \((\rho_d)\) is a normalising modulus, [13], and is independent of \(a\),

\[
Z_d'(0, a, k) = \log \left( \frac{1}{\rho_d} \prod_{j=0}^{k-1} \Gamma_d(a + \alpha_j) \Gamma_d(a - \alpha_j) \right) + M(d, a, k),
\]

where \(M(d, a, k)\) reflects the multiplicative anomaly and is composed of first and second order terms, \(M(d, a, k) = M_1(d, a, k) + M_2(d, a, k)\), given by,

\[
M_1(d, a, k) = -\sum_{r=1}^{u} \left( \sum_{j=0}^{k-1} \alpha_j^{2r} \right) \frac{1}{r} H_k(r) N_{2r}(d, a)
\]

\[
M_2(d, a, k) = \frac{1}{2k} \sum_{r_i=1}^{u} \sum_{r_j=1}^{u-r_i} \sum_{i<j=0}^{k-1} \alpha_i^{2r_i} \alpha_j^{2r_j} \left( \frac{1}{r_i} H(r_i - 1) + \frac{1}{r_j} H(r_j - 1) \right) N_{2r_i+2r_j}(d, a),
\]

respectively.

5. Computations

Expression (15), with (16), gives the derivative of the \(\zeta\)-function for \(P_{2k}\) on the hemisphere for N and D conditions after substitution of the appropriate values for the \(a\) and the \(\alpha\). Adding these gives the full sphere value which is the one most frequently requested.

The first part of (15) is difficult to find (numerically), being transcendental. The second part is easy, being rational. I only need to give the residues,

\[
N_r(d, a) = \frac{(-1)^{d-r}}{(r-1)!(d-r)!} B_{d-r}(d, a)
\]

\[
= \frac{1}{(r-1)!(d-r)!} B_{d-r}(d, a),
\]

again in terms of generalised Bernoulli polynomials, [15,13]. These are easily calculated from recursion and the symmetry shown in (17) shows that the N and D residues are equal, \(N_r(d, a_N) = N_r(d, a_D)\). \(M\) is well adapted for evaluation by machine algebra.

It might appear that the transcendental part of (15) becomes more involved for increasing \(k\). I will now show this is not so, at least formally, by using the relation, (13),

\[
\frac{\Gamma_d(a)}{\rho_d} = \frac{\Gamma_{d+1}(a)}{\Gamma_{d+1}(a+1)},
\]
to effect cancellations which also eliminate the moduli $\rho_d$. Adding the N and D contributions, and substituting the explicit $\alpha_j$, the relation (18) allows staggered numerator–denominator cancellations which result in the compact formula,

$$Z_d'(0, k) \bigg|_{\text{sphere}} = \log \frac{\Gamma_{d+1}(d/2 - k) \Gamma_{d+1}(d/2 - k + 1)}{\Gamma_{d+1}(d/2 + k) \Gamma_{d+1}(d/2 + k + 1)} + 2M(d, k),$$

dropping the $a_N$ argument by defining $M(d, k) = M(d, a_N, k)$.

I do not intend to evaluate the first term numerically nor to take it much further formally for even $d$ (see section 10). An expression for $\log \Gamma_n$ as a sum of derivatives of the Hurwitz $\zeta$–function can be found, e.g. by using the expansion in [16] §9, and elsewhere.

However, I do present some numbers for the multiplicative anomaly part obtained from (16) in the form of polynomials in $k$, for a few $d$. There is no anomaly for $k = 1/2$, i.e. for just one linear factor, and so I list the polynomials $\overline{M}(d, k) \equiv M(d, k)/(1 - 2k)$,

$$\overline{M}(4, k) = \frac{1}{17280}(1 + 2k)(132k^3 - 36k^2 - 137k + 31),$$

$$\overline{M}(6, k) = \frac{1}{29030400}(13152k^6 + 2256k^5 - 79344k^4 - 22032k^3 + 76330k^2 + 26195k - 5805),$$

and give the values for the Paneitz operator for different dimensions, $M(4, 2) = -659/1152 \approx -0.57205$, $M(6, 2) = 5141/276480 \approx 0.01839$ and $M(8, 2) = -15666659/6502809600 \approx -0.00241$.

6. The determinant in the critical case

When $k = d/2$ there is a zero mode in the Neumann problem and it is necessary to remove this from the $\zeta$–function, i.e. the entire $m = 0$ term has to be omitted in (3), and hence in all the arising Barnes $\zeta$–functions.

The first thing to say is that the multiplicative anomaly contribution is unaltered because removing one term does not affect the analytic structure of the $\zeta$–functions and, although it does affect the value at zero, this cancels in the relevant identity, (8).

Referring again to the $k = 2$ example, it is the last $\zeta'(0)$ term, with $a_N = \alpha'$, in (14) that gives the trouble, solved by replacing all the Barnes $\zeta$–functions by subtracted ones, called $\tilde{\zeta}_d$ in [11]. This can be easily allowed for in the other $\zeta'(0)$.
parts by adding and subtracting their $m = 0$ terms, producing a correction. For the offending term, however, I have to use the limit, \[ \lim_{\epsilon \to 0} \zeta'_d(0, \epsilon) = -\log \epsilon - \log \rho_d. \]

These adjustments mean that (19) is replaced by
\[
Z'_d(0, d/2) \bigg|_{\text{sphere}} = -\log \frac{(d - 1)! \Gamma_{d+1}(d) \Gamma_{d+1}(d + 1)}{\Gamma_{d+1}(1)} + 2M(d, d/2). \tag{20}
\]

The cancellation pattern has altered, slightly, and the correction has been added. I also note that $\rho_d = \Gamma_{d+1}(1)$.

7. Explicit evaluation in the odd case

As has been mentioned, if $d$ is odd, the anti–symmetry of the appropriate Bernoulli polynomials means that the multiplicative anomalies for N and D conditions have opposite signs, and so cancel for the complete sphere. Furthermore, $P_{2k}$ has no zero modes\(^7\), in either case, and so the result for the log determinant is as in (19) except there is no final $M$ term, and all $k$ are allowed. The only difference is that the existence of negative eigenvalues for $k > d/2$ introduces an overall sign factor, $(-1)^{\sharp(\lambda < 0)}$ (see section 8).

It is well known that the odd and even $d$–spheres form two, somewhat distinct, families. In the former, for conformal coupling in $d + 1$ dimensions, the heat–kernel expansion terminates\(^8\). The heat–kernels can be related by a derivative which increases $d$ by 2 and the simpler odd dimension properties can be traced to the facts that $S^1$ and $S^3$ are group manifolds.

Another consequence is that $\zeta'(0)$ (equivalently, the effective action) can be computed in terms of ‘elementary’ functions and I will perform the calculation that illustrates this using (19), which I repeat,
\[
Z'_d(0, k) \bigg|_{\text{sphere}} = \log \left( \frac{\Gamma_{d+1}(d/2 - k + 1)}{\Gamma_{d+1}(d/2 + k)} + \log \frac{\Gamma_{d+1}(d/2 - k)}{\Gamma_{d+1}(d/2 + k + 1)} \right) \tag{21}
\]
\[
= \log \text{Sin}_{d+1}(d/2 + k) - \log \text{Sin}_{d+1}(d/2 - k), \quad d \text{ odd},
\]

\(^7\) There is no analogue of minimal coupling in the class of $P_{2k}$ operators for odd $d$.

\(^8\) In the dual hyperbolic space this expansion is exact.
where I have employed Kurokawa’s multiple sine function \( S_d(x) = \text{Sin}_d(z) \) (see the appendix). For computation purposes, it is best to relate this to the multiple cotangent function (see the appendix) and write (21) as an integral. This will also enable me to compare with other formulae. Thus,

\[
Z'_d(0,k) \bigg|_{\text{sphere}} = \int_{d/2-k}^{d/2+k} dz \cot_{d+1}(z) \\
= \frac{(-1)^d}{d!} \int_{d/2-k}^{d/2+k} dz B^{(d+1)}_d(z) \pi \cot(\pi z) \\
= \int_0^k dz \frac{(-1)^{d+1}}{d!} (B^{(d+1)}_d(d/2 + z) - B^{(d+1)}_d(d/2 - z)) \pi \tan(\pi z) \\
= \int_0^k dz \frac{2(-1)^{d+1}}{d!} \pi \tan(\pi z) \prod_{j=1}^{(d-1)/2} (z^2 - (j - 1/2)^2) \\
\equiv \int_0^k dz P(d,z) \pi \tan(\pi z),
\]

which is expressed in terms of elementary functions, (cf Das and Dunne, [17]. Kamela and Burgess, [18]). The even \( d \) result cannot be expressed in terms of multiple trig functions. (But see the section 9.)

8. Negative modes and lattice considerations

Expression (22) is not suitable for numerical computation. It is better to use a form involving the \( \psi \) function. However, it is instructive when considering the behaviour as \( k \) increases, because for \( k > d/2 \) (allowed for odd \( d \)) the integrand diverges at \( z = d/2 + n, \quad (n = 0, 1, \ldots) \) and there appears to be a problem in the value of the integral. This is solved by taking the integral to be a contour one. Further, since the residues at the poles are integers, as the contour is changed, the left–hand side is uncertain only up to an integer multiple of \( 2\pi i \) so that the determinant is well defined. This is the same argument used by Gangolli, [19], in connection with the Selberg \( \zeta \)–function.

Furthermore, splitting up the integral into principle part and residue contributions, the latter are integer multiples of \( \pi i \) and count the number of negative modes that are encountered as \( k \) increases. (I am thinking of \( k \) as a continuous variable.) The principle part corresponds to the \( \zeta \)–function constructed from the moduli of the eigenvalues and the residues give the sign factor, \((-1)^Z\lambda<0\).
Looking back to the eigenvalues, (2), the condition for a negative eigenvalue is
the union of the N and D conditions,
\[
\sum_{i=1}^{d} m_i < j + 1 - d/2, \quad N \\
\leq j + 1 - (d + 1)/2
\]
\[
\sum_{i=1}^{d} m_i < j - d/2, \quad D \\
\leq j - (d + 1)/2,
\]
with 0 ≤ j ≤ k − 1, for a given k, (j ∈ ℤ).

The problem posed by (23) to find all \( m_i \geq 0 \) is the same as that underlying
the notion of the Ehrhart polynomial, [20], which is to find the number of non–negative
integer lattice points, \( m_i \in ℤ^d \), in the integer dilation of the rational polytope,
\[
\sum_{i=1}^{d} m_i \omega_i \leq 1,
\]
where the \( \omega_i \) are positive integers. In (23), we are dealing with problems in
a standard integer simplex in \( d \)-dimensions: all the \( \omega_i \) are one. This is an easy and
much studied example. Here, the integer dilations, \( t \), are given by the right–hand
sides of (23) and the answers are Ehrhart polynomials, in \( t \). They are the binomial
coefficients, \( \binom{d+t}{d} \), [20]. This agrees with the residues in (22), as mentioned, if we
note that,
\[
B_{d}^{(d+1)}(d + t + 1) = \binom{d+t}{d}.
\]

9. The N-D determinant ratio in the even, subcritical case

For even \( d \), as said, subtracting the N and D hemisphere log determinants
removes the anomaly. This difference is easily found from (15) to be,
\[
Z_d'(0, k) \bigg|_{N-D} = \log \frac{\Gamma_d(d/2 - k)}{\Gamma_d(d/2 + k)} \\
= \log \frac{\Gamma_{d+1}(d/2 - k) \Gamma_{d+1}(d/2 + k + 1)}{\Gamma_{d+1}(d/2 + k) \Gamma_{d+1}(d/2 - k + 1)} \\
= \log \frac{\sin_{d+1}(d/2 + k)}{\sin_{d+1}(d/2 - k)}, \quad d \text{ even},
\]
i.e. exactly the same form as the odd $d$–sphere log determinant.

The hemisphere approach puts a geometrical slant on Kurokawa’s construction of the multiple sine function.

The explicit calculation proceeds almost identically to the odd case,

$$Z'_d(0, k) \bigg|_{N-D} = \int_{d/2-k}^{d/2+k} dz \cot_{d+1}(z)$$

$$= \frac{(-1)^d}{d!} \int_{d/2-k}^{d/2+k} dz \, B^{(d+1)}_{d}(z) \, \pi \cot(\pi z)$$

$$= \int_0^k dz \frac{(-1)^{d+1}}{d!} \left( B^{(d+1)}_{d}(d/2 + z) - B^{(d+1)}_{d}(d/2 - z) \right) \pi \cot(\pi z)$$

$$= \int_0^k dz \frac{(-1)^d}{(d-1)!} \pi z \cot \pi z \prod_{j=1}^{d-2} \left( z^2 - j^2 \right), \quad k < d/2.$$

(24)

Extending the range of $k$ beyond its limit, $d/2$, by taking the integral as a contour one, still makes sense. This time, the pole residues count the difference in number of Neumann and Dirichlet negative modes which equals the number of Neumann negative modes in one dimension less, corresponding to the Ehrhart polynomial, \( \binom{d-1+t}{d-1} = \binom{d+t}{d} - \binom{d+t-1}{d} \).

10. Holographic aspect

The integrand in the odd $d$ expression, (22), is recognised as (proportional to) the continuation of the hyperbolic Plancherel measure on $\mathbb{H}^{d+1}$, e.g. [21]. This equation is related to, but not the same as, a theorem of Kurokawa’s, [22,23], in the special case of the trivial factoring of the $(d+1)$–dimensional hyperbolic bulk, $\mathbb{H}^{d+1}$. The integral over the Plancherel measure came from the original functional relation of the Selberg $\zeta$–function which was then connected with the determinant of a ‘linear’ pseudo–differential operator constructed from the Laplacian on the dual space, $S^{d+1}$. (Sarnak gave the lowest dimension treatment in [24].) My result, on the other hand, has a holographic aspect as it relates a determinant on an $S^d$ boundary to quantities on the hyperbolic bulk.

This can be made more explicit by recognising in (19) and (21) the ratio of two determinants of the $(d+1)$–dimensional spherical operators, \( Y_{d+1}^{1/2} \pm k \), computed again by considering the union of N and D $(d+1)$–hemisphere spectra, \((m.\omega + a_D \pm k) \cup (m.\omega + a_N \pm k)\), where $a_N = d/2$ and $a_D = d/2 + 1$. This quickly
gives,

\[- \log \det P_{2k} \equiv Z'_d(0, k) \bigg|_{d-\text{sphere}} = \log \frac{\det [(Y_{d+1} + 1/4)^{1/2} + k]}{\det [(Y_{d+1} + 1/4)^{1/2} - k]} + 2M(d, k), \quad (25)\]

as a holographic–type statement, the top line referring to \(d\)–dimensional quantities and the bottom line to \((d + 1)\)–dimensional, apart from the polynomial, \(M(d, k)\). If \(d\) is odd, there is no \(M\) term in (25).

For odd \(d\), the \((d + 1)\)–dimensional, purely bulk, formula that follows on combining (25) and (22), \(i.e.

\[\log \frac{\det [(Y_{d+1} + 1/4)^{1/2} + k]}{\det [(Y_{d+1} + 1/4)^{1/2} - k]} = \int_0^k dz P(d, z) \pi \tan \pi z \quad (26)\]

is effectively given in [22], and written out in [25], Corollary 4.6. For Riemann surfaces, Cartier and Voros, [26], give equivalent equations.

It is clear from (26), say by differentiating with respect to \(k\) to give a resolvent, that the residues at the poles of the integrand at \(n + d/2\) are the degeneracies of the eigenvalues, \(n + d/2\), of \((Y_{d+1} + 1/4)^{1/2}\). Since it is a general theorem (see \(e.g\). Camporesi, [21] §10, Helgason, [27]) that the residues of the Plancherel measure are (proportional to) the degeneracies on the dual compact space, here \(S^{d+1}\), this proves that the polynomial \(P(d, z)\) is the Plancherel polynomial up to a factor, as it has here been computed to be, \(cf\) [23]. Also, the \(N \cup D\) hemisphere mode structure is equivalent to Theorem 3 in [23], for \(SO(d + 2)\).

Different bulks give different Plancherel measures (\(e.g\). [23] and [28]). This is clear for the Yamabe case, \(k = 1\), when the calculation for the other symmetric spaces can be followed through. Minakshisundaram, [29], gave an early analysis of the \(\zeta\)–function for the unitary sphere and the other cases can be easily worked out. Cahn and Wolf [30], construct the heat–kernels and Ikeda [31] goes through the corresponding details of the \(\zeta\)–functions. See also [23]. General and specific information can be found in the rather detailed Camporesi and Higuchi, [32], and [21].

For \(k \neq 1\), the analysis remains to be done. The equation equivalent to (26) obviously holds in terms of the Plancherel measure on the symmetric space and the Laplace–Beltrami operator on its compact dual. The question is deriving the identity (25) for a corresponding boundary GJMS operator, \(P_{2k}\), given, possibly, as
a group representation intertwinor, cf Fontana, Branson and Morpurgo, [33]. One could conjecture a product structure similar to (1).

The lattice considerations of section 8 again show the relation to the dimensions of the spherical representations of $SO(d + 2)$. I also conjecture that this discussion extends to the other Plancherel measures and leads to Ehrhart polynomials for the various Cartan–Weyl weight polytopes, cf Welleda Beldoni, Beck and Cochet, [34]. Indeed the results of Panyushev, [35], Conway and Sloane, [36], and of Bachet, de la Harpe and Venkov, [37], can be used to obtain Theorem 4 of Kurokawa, [23], or, perhaps, vice versa.

It is also possible to finitely factor the bulk. Apparently, some factorings correspond to spectral problems on the boundary that cannot be resolved classically, [6].

11. A generalisation

It is clear that the formal developments do not depend on the particular values of the $\alpha_j$ constants in (3) giving the GJMS operator. One can therefore define a more general $\zeta$–function, $\Upsilon_d$, from $Z_d$, with any $\alpha_j$, by,

$$Z'_d(0, a, k, \alpha) = \log \frac{\Upsilon_d(a, k, \alpha)}{\varrho_d(k)}$$

maintaining Barnesian normalisation with $\varrho_\nu$ the necessary modulus.

The N and D anomalies have opposite signs for odd $d$ and the same signs for even $d$. To eliminate the anomaly, therefore, one can add the $Z'(0)$ for odd dimensions and subtract them for even, cf [12]. This motivates the combination,

$$\left[ \frac{\Upsilon_d(a, k, \alpha)}{\varrho_d(k)} \right]^{-1} \left[ \frac{\Upsilon_d(a^*, k, -\alpha)}{\varrho_d(k)} \right]^{(-1)^d}$$

$$= \prod_{j=0}^{k-1} \left[ \frac{\Gamma_d(a + \alpha_j) \Gamma_d(a - \alpha_j)}{\rho_d^2} \right]^{-1} \left[ \frac{\Gamma_d(a^* + \alpha_j) \Gamma_d(a^* - \alpha_j)}{\rho_d^2} \right]^{(-1)^d}$$

(27)

which equality follows from (15). Here $a$ and $a^* = d - a$ are dual boundary condition constants, such as $a_N$ and $a_D$.

The definition of the Barnes multiple sine function (see the appendix) suggests that the left–hand side of (27) be defined as the multiple sine function, $\mathcal{Sin}_d$, for the $Z_d$ $\zeta$–function, and then this equation is more compactly written,

$$\mathcal{Sin}_d(a, k, \alpha) = \prod_{j=0}^{k-1} \sin_d(a + \alpha_j) \sin_d(a - \alpha_j).$$
12. Comments and conclusion

The conformal anomaly and functional determinant of the GJMS operators, $P_{2k}$, on the sphere have been evaluated in explicit formal and analytical forms. As such, they should be useful as test cases for more general situations and also as fiducial expressions.

Equation (22) agrees with the form extracted from the results of Guillarmou, [6], Theorems 1.3 and 4.2, derived on the basis that the $P_{2k}$ operators are the residues at a pole of the scattering operator on a hyperbolic bulk, of which the sphere is the conformal infinity. I have not needed to employ the Selberg $\zeta$–function.

Equation (22) is also consonant with the expression derived by Diaz and Dorn, [7] in a boundary conformal field theory evaluation using dimensional regularisation, with $k$ now playing the role of conformal field dimension. These authors also produce an expression when $d$ is even which should agree with the one here, up to a polynomial in $k$.

Some history of ordinary sphere determinants was attempted in [38] and an improved technique, over that in [11], was given, but I have not used it. To the references in this paper, I should add those to Kurokawa mentioned above.

The $\zeta$–function, (3), is an example of one defined by Friedman and Ruijsenaas, [39], as a generalisation of Shintani’s $\zeta$–function, [40]. The methods in the present work can be thought of as elementary and explicit derivations of some of the results given in this reference, applied to the GJMS operators. My approach is different and, in its domain of validity, simpler.\(^9\)

Acknowledgments

I thank Colin Guillarmou for explanation and information.

Appendix

In this appendix I place some, more or less standard, technical information on the Barnes functions that is best separated from the main text. I use Barnes’ original definitions.

\(^9\) The method of expanding in $\alpha$ can be applied to a general product of linear factors and will be described at another time.
The multiple $\psi$–functions are defined in terms Barnes’ multiple $\Gamma$–function,

$$
\psi_d^{(p)}(a) = \frac{\partial^p}{\partial a^p} \log \Gamma_d(a).
$$

(28)

I set $\psi_d(a) \equiv \psi_d^{(1)}(a)$ for the first $\psi$–function. Barnes, [13] §52 gives the recursion,

$$
\psi_{d+1}(a + 1) = -\frac{a}{d} \psi_d(a) - \frac{1}{d} \zeta_d(0, a)
$$

(29)

with

$$
\zeta_d(0, a) = \frac{(-1)^d}{d!} B_d^{(d)}(a).
$$

(30)

I have put all parameters equal to 1 and rearranged the formula a little.

Also, the recursion formula for the Barnes $\zeta$–function leads to the recursion for the multiple gamma function, [13],

$$
\frac{\Gamma_d(a)}{\rho_d} = \frac{\Gamma_{d+1}(a)}{\Gamma_{d+1}(a + 1)},
$$

which gives

$$
\psi_{d+1}(a + 1) = \psi_{d+1}(a) - \psi_d(a).
$$

Hence

$$
\psi_{d+1}(a) = \frac{d - a}{d} \psi_d(a) - \frac{1}{d} \zeta_d(0, a),
$$

(31)

which can be iterated down to $\psi_1$, the standard function, $\psi$. The result is,

$$
\psi_d(a) = \frac{(-1)^{d-1}}{(d-1)!} B_{d-1}^{(d)}(a) \psi(a) - \frac{1}{(d-1)!} \sum_{k=1}^{d-1} \frac{(-1)^k}{k} B_{d-k-1}(d-a) B_k^{(k)}(a),
$$

(32)

where some products have been rewritten as Bernoulli polynomials. Pochhammer symbols could be used instead.

The first three cases are,

$$
\psi_2(z) = -(z - 1) \psi(z) + z - \frac{1}{2},
$$

$$
\psi_3(z) = \frac{z(z - 1)}{2} \psi(z) - \frac{18z^2 + 42z - 17}{24},
$$

$$
\psi_4(z) = -\frac{z(z - 1)(z - 2)}{6} \psi(z) + \frac{22z^3 - 114z^2 + 167z - 60}{72}.
$$

The general expression is easily programmed.
Similar expressions can be found in Kanemitsu et al, [41] and Onodera, [42]. Because of different normalisations and definitions, the formulae in some of these references are complicated by the appearance of derivatives of the Hurwitz $\zeta$–function.

The formula (31) is useful for the evaluation of $\log \Gamma_d(z)$ needed in the construction of the spherical log determinants. Thus

$$\log \frac{\Gamma_d(z_2)}{\Gamma_d(z_1)} = \int_{z_1}^{z_2} dz \psi_d(z)$$

$$= \frac{(-1)^{d-1}}{(d-1)!} \int_{z_1}^{z_2} dz \left( B_{d-1}^{(d)}(z) \psi(z) + Q_d(z) \right)$$

where the polynomial $Q$ is given by

$$Q_d(z) = -(-1)^{d-1} \sum_{k=1}^{d-1} \frac{(-1)^k}{k} B_{d-k-1}^{(d-k)}(d-z) B_k^{(k)}(z)$$

and there are no poles of $\psi(z)$ in the integration range.

Combinations of the $\Gamma$ or $\psi$ functions often occur, in particular those leading to multiple trigonometric functions. These are usually defined using a differently normalised $\Gamma$–function. I have been using Barnes’ original definition which is,

$$\log \frac{\Gamma_d(z)}{\rho_d} = \zeta_d'(0, z),$$

where $\rho_d$ is a normalising modulus. In [22], the $\Gamma$–function is defined without this factor,

$$G_d(z) = \frac{\Gamma_d(z)}{\rho_d},$$

which is Vardi’s notation, [43]. In terms of $G_d$, the multiple sine function is defined by,

$$\text{Sin}_d(z) = \frac{(G_d(d-z))^{(-1)^d}}{G_d(z)},$$

and I note that for even $d$ this is independent of $\rho_d$.

The $\psi$–function, (28), also does not depend on $\rho_d$ and one can define a multiple cotangent function by,

$$\text{Cot}_d(z) = \frac{d}{dz} \log \text{Sin}_d(z)$$

$$= -\psi_d(z) - (-1)^d \psi_d(d-z),$$

\[\text{10} \text{ This topic is beset by conflicting definitions and notations. The } G_d \text{ should not be confused with the quantity having the same symbol in Vigneras, [44]. Vardi denotes this quantity by } \Gamma_d.\]
which has the nice property that it satisfies the homogeneous recursion,

\[ \text{Cot}_{d+1}(a) = \frac{d-a}{d} \text{Cot}_d(a), \]

easily proved from the \( \psi \) recursions using the symmetry,

\[ B_d^{(d)}(d-a) = (-1)^d B_d^{(d)}(a). \]

Hence one obtains the explicit connection with the ordinary cotangent,

\[ \text{Cot}_d(a) = \frac{(-1)^{d-1}}{(d-1)!} B_{d-1}^{(d)}(a) \pi \cot(\pi a). \quad (35) \]

These and other relations can be found in [25], and elsewhere, derived in various ways. For example, from the definition (34) one has the duality (or complementarity) symmetry,

\[ \text{Cot}_d(d - z) = (-1)^d \text{Cot}_d(z). \]

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