Quantum measurements and Kolmogorovian probability theory

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Abstract

We establish connections between the requirement of measurability of a probability space and the principle of complementarity in quantum mechanics. It is shown that measurability of a probability space implies the dependence of results of quantum measurement not only on the properties of a quantum object under consideration, but also on the classical characteristics of the measuring device which is used. We show that if one takes into account the requirement of measurability in a quantum case, the Bell inequality does not follow from the hypothesis about the existence of an objective reality.

1 Introduction

In the review [1] by Home and Whitaker, devoted to interpretation of quantum mechanics one can find the statement: "The fundamental difficulty in interpretation of quantum theory is that it provides in general only probabilities of obtaining given results. Thus much of any discussion of quantum theory must depend on what one means by probability — how one defines or interprets the term".

In this review a lot of place is occupied by the interpretation of concept "probability" but Kolmogorovian approach [2] is mentioned only casually. At the same time namely Kolmogorovian probability theory is the most consecutive and mathematically strict. Besides, it most fully corresponds to that particular situation which takes place in quantum mechanics in the author’s opinion. In present paper the application of Kolmogorovian probability theory to the problem of quantum measurements will be described.

2 Probability space

Let us recollect original positions of Kolmogorovian probability theory (see, for example [3]). The so-called probability space $(\Omega, \mathcal{F}, P)$ lays in the foundation of probability-theoretic scheme. Here $\Omega$ is a set (space) of the elementary events $\omega$. The elementary event is understood as a possible result of a single experiment. Besides the elementary event the concept of "event" is introduced. Each event $F$ is identified with some subset
of set $\Omega$. It is assumed that in the experiment under consideration an event $F$ is carried out if the result of experiment belongs to $F$ ($\omega \in F$).

Collections of subsets of the set $\Omega$ (including the set $\Omega$ and the empty set $\emptyset$) are allotted with the structure of Boolean algebras. Algebraic operations are: intersection of subsets, joining of them, and complement with respect to $\Omega$.

The second ingredient of a probability space is the so-called $\sigma$-algebra $\mathcal{F}$. It is some Boolean algebra, closed in respect of denumerable number of operations of joining and intersection. The set $\Omega$ in which the particular $\sigma$-algebra $\mathcal{F}$ is chosen, refers to as measurable space. Further on the measurability will play a key role.

Finally, the third ingredient of probability space is a probabilistic measure $P$. This is a mapping of algebra $\mathcal{F}$ into the set of real numbers satisfying conditions: a) $0 \leq P(F) \leq 1$ for all $F \in \mathcal{F}$; b) $P(\sum_j F_j) = \sum_j P(F_j)$ for any denumerable collection of nonintersecting subsets $F_j \in \mathcal{F}$. Let us pay attention that the probabilistic measure is defined only for the events which are included in the algebra $\mathcal{F}$. For the elementary events the probability, generally speaking, is not defined.

The mapping $X$ of the set $\Omega$ in the expanded real straight line $\bar{R} = [-\infty, +\infty]$ refers to as the real random quantity on $\Omega$.

$$X(\omega) = X \in \bar{R}$$

It is supposed that the set $\bar{R}$ is allotted by the property of measurability. In the set $\bar{R}$ as a $\sigma$-algebra we can take Boolean algebra $\mathcal{F}_R$, generated by semiopen intervals $(x_i, x_j]$. This is a $\sigma$-algebra which is obtained by an application of algebraic operations to various intervals. We designate by $\{\omega \in \mathcal{F}_R\}$, $F_R \in \mathcal{F}_R$ the subset of elementary events $\omega$ for which $X(\omega) \in F_R$. The subsets $F = \{\omega \in \mathcal{F}_R\}$ form the $\sigma$-algebra $\mathcal{F}$ in the space $\Omega$.

### 3 Quantum measurements

We consider now the application of formulated main principles of probability theory to a problem of quantum measurements. We carry out our consideration starting from positions of “the objective local theory”, traced back to works of Bell [4, 5], and also to the paper by Einstein, Podolsky, Rosen [6].

It is usually considered that “the objective local theory” contradicts corollaries of the standard quantum mechanics. In present paper we try to show that such contradiction does not arise when we use Kolmogorovian probability theory.

Let us suppose that there is a certain objective reality, which determines possible result of any individual measurement. We name this objective reality by a physical state of a quantum object. One should not mix this notion with what is usually denominated by the term ”state” in standard quantum mechanics. We shall use the term ”quantum state” in latter case.

It is possible to read about a correlation between physical and quantum states, and the mathematical concept corresponding to a physical state in the papers [7, 8]. Here we only note that the quantum state is a certain equivalence class of physical states. This class has a potency of continual set. In the probability theory we associate an elementary event with a physical state. Correspondingly, we associate the set of physical states of
a quantum object with the space $\Omega$. Further, we need to make this space measurable, i.e. to choose certain $\sigma$-algebra $F$. Here, a peculiarity of quantum measurement, which has the name ”principle of complimentarity” in standard quantum mechanics, has crucial importance.

As opposed to measurements in classical physics, in a quantum case the measurements can be compatible and incompatible. Correspondingly, in quantum mechanics the observable quantities are subdivided into compatible (simultaneously measurable) and incompatible (additional). In principle, measurements of compatible observables can be carried out so that measurements of one observable do not disturb the measurement of other observable. We name the corresponding system of measuring devices in coordination with collection of these compatible observables. As a matter of principle it cannot be made for incompatible observables. As it is told in the paper by Zeilinger [9]: ”Quantum complementarity then is simply an expression of the fact that in order to measure quantities, we would have to use apparatuses which mutually exclude each other”.

Thus, we can organize each individual experiment only in such a way that compatible observables are measured in it. The results of measurement can be random. That is, observables correspond to the real random quantities in probability theory.

The main purpose of typical quantum experiment is the finding of probabilistic distributions of those or other observable quantities. We can obtain such distribution for certain collection of compatible observables on the application of certain measuring device. From the point of view of probability theory we choose certain $\sigma$-algebra $F$, choosing the certain measuring device. For example, let us use the device intended for measurement of momentum of a particle. Let us suppose that we can ascertain by means of this device that the momentum of particle hits an interval $(p_i, p_j]$. For definiteness we have taken a semiopen interval though it is not necessary. Hit of momentum of the particle in this or that interval is the event for the measuring device, which we use. These events are elements of certain $\sigma$-algebra. Thus, the probability space $(\Omega, F, P)$ is determined not only by the explored quantum object (by collection of its physical states) but also by the measuring device which we use.

Let us assume that we carry out some typical quantum experiment. We have an ensemble of the quantum systems, belonging to a certain quantum state. For example, the particles have spin $1/2$ and the spin projection on the $x$ axis equals $1/2$. Let us investigate the distribution of two incompatible observables (for example, the spin projections on the directions forming angles $\theta_1$ and $\theta_2$ with regard to the $x$ axis). We cannot measure both observables in one experiment. Therefore, we should carry out two groups of experiments which use different measuring devices. ”Different” is classically distinct. In our concrete case the devices should be oriented by various manners in the space.

We can describe one group of experiments with the help of a probability space $(\Omega, F_1, P_1)$, another group with the help of $(\Omega, F_2, P_2)$. Although in both cases the space of elementary events $\Omega$ is the same, the probability spaces are different. Certain $\sigma$-algebras $F_1$ and $F_2$ are introduced in these spaces to give them the property of measurability.

Formally, only mathematically, in the space $\Omega$ we can construct $\sigma$-algebra $F_{12}$ which incorporates algebras $F_1$ and $F_2$. Such algebra refers to as generated by the algebras $F_1$ and $F_2$. It contains besides the subsets $F_i^{(1)} \in F_1$ and $F_j^{(2)} \in F_1$ also any possible intersections and joins of the subsets $F_i^{(1)} \in F_1$ and $F_j^{(2)} \in F_2$. 

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However, physically such $\sigma$-algebra is unacceptable. Really, the event $F_{ij} = F_i^{(1)} \cap F_j^{(2)}$ is that the values of two incompatible observables belong to strictly defined regions for one quantum object. It is impossible to carry out experiment which could pick out such event for a quantum system in principle. Therefore, for such event there is no concept of "probability". That is, there is no any probabilistic measure corresponding to the subset $F_{ij}$, and the $\sigma$-algebra $\mathcal{F}_{12}$ is not good for construction of a probability space.

Here, basic characteristic of the application of probability theory to quantum systems become apparent: not every mathematically possible $\sigma$-algebra is physically allowable.

How does the probability space materialize in quantum experiment? The definition of probability implies repeated realization of tests. We should carry out these tests in identical conditions. It concerns both the object measured, and the measuring device. Evidently, we can supervise completely a microstate for neither one nor other. Therefore, the term "identical conditions" should be understood as some equivalence classes of states of the quantum object and the measuring device.

We can supervise only classical characteristics of the physical system. Therefore, fixing some classical parameters of the physical system, we fix a corresponding equivalence class. For the object measured such fixing usually is the choice of the certain quantum state. For example, for particles with spin — it is selection of particles with the certain orientation of spin. For the measuring device we also should choose the certain classical characteristic which fixes some equivalence class. For example, in the measuring device the initial united beam of particles should be split in several beams sufficiently well separated from each other, corresponding to different values of the spin projection onto a chosen direction.

Thus, in experiment the element of measurable space $(\Omega, \mathcal{F})$ corresponds to the pair — a quantum object (for example, belonging to the certain quantum state) plus the certain type of measuring device, which allows to fix events of the certain form. Such device can pick out the events corresponding to some set of compatible observable quantities. Therefore, in a quantum case the measuring devices should be divided into various types. Each type is coordinated with the separate collection of compatible observables.

Existence of various types of the measuring device leads to one more peculiarity of quantum measurements. The same observable can belong to two (or more) various collections of compatible observables. Such observable is compatible with all observables, which are included in various collections. However, other observables of different collections are not compatible among themselves. For the measurement of the chosen observable we can use various types of measuring device. It means that we can carry out experiment in various conditions. There is no guarantee that the result of measurement will not depend on these conditions.

Thus, the result of individual quantum measurement can depend not only on the internal properties of a measured object (a physical state), but also on the type of measuring device. In terms of probability theory it is expressed as follows. For quantum system the random quantity $X$ can be multiple-valued function of the elementary event $\omega$.

In the classical case all observables are compatible. Correspondingly, all measuring devices belong to one type. Therefore, the classical random quantity $X$ is the single-valued function of $\omega$. Let us notice that if in the quantum case we consider magnitude $X$ as a function not on space $\Omega$ but on measurable space $(\Omega, \mathcal{F})$, this function is single-valued.

This statement allows to look in a new fashion on the result obtained in the paper by
Kochen and Specker [10], where the no-go theorem is proved. The sense of this theorem can be expressed by the statement that for a particle with spin 1 there is no such internal characteristic which uniquely predetermines the values of squares of the spin projections on three mutually orthogonal directions.

In the standard quantum mechanics such three squares are described by mutually commuting operators. Therefore, the corresponding observables \( \hat{S}^2_x, \hat{S}^2_y, \hat{S}^2_z \) are compatible. The observables \( \hat{S}^2_{y'}, \hat{S}^2_{z'} \) are also compatible. Here, the \( x, y', z' \) directions are orthogonal among themselves, but the \( y, z \) directions are not parallel to the \( y', z' \) directions. The observables \( \hat{S}^2_y, \hat{S}^2_z \) are not compatible with the observables \( \hat{S}^2_{y'}, \hat{S}^2_{z'} \). The devices coordinated with the observables \( \hat{S}^2_{z}, \hat{S}^2_{z'} \) and \( \hat{S}^2_{y}, \hat{S}^2_{y'} \) belong to different types. Therefore, these devices not necessarily should give the same result for square of spin projection on the \( x \) direction.

We cannot use simultaneously two types of measuring devices in one experiment. Therefore, it is impossible to carry out direct experiment for check of this statement. However, it is possible to try to carry out indirect measurement.

For example, it is possible to organize such experiment as follows. To take a physical system in singlet state which decays in two massive particles with spin 1. For one of particles to measure \( \hat{S}^2_x \) with the help of the device coordinated with the observables \( \hat{S}^2_{y}, \hat{S}^2_{z} \), and for the other to measure \( \hat{S}^2_x \) with the help of the device coordinated with the observables \( \hat{S}^2_{y'}, \hat{S}^2_{z'} \). After that it is necessary to compare the results which appear for particles in each individual experiment. It is necessary to observe that the \( x \) direction is not special for decay of the singlet state. Otherwise, additional correlation between spin projections on the \( x \) direction for both particles can take place. It disturbs cleanliness of the experiment.

If the measurement result for squares of spin projection on a picked out axis really depends on the type of device used, then the basic condition of the no-go theorem of Kochen and Specker appears outstanding. In that case this theorem cannot be used as an argument in dispute about the existence of an objective reality.

4 The Bell inequality

We now look how the condition of measurability of a probability space proves oneself in such important case as deriving the Bell inequality [4]. There are many variants of this inequality. We follow the variant proposed in the work [11]. This variant is usually designated by the abbreviation CHSH.

Let a particle whose spin is equal to 0 disintegrate into two particles \( A \) and \( B \) whose spins are equal to 1/2. These particles fly apart to a large distance and are registered by the respective device \( D_a \) and \( D_b \). The measurements in the devices are independent. For the particle \( A \) the device \( D_a \) measures the spin projection on the \( a \) direction, and for the particle \( B \) the device \( D_b \) measures the spin projection on the \( b \) direction. We let \( \hat{A}_a \) and \( \hat{B}_b \) denote the corresponding observables and let \( A_a \) and \( B_b \) denote the measurement results.

Let us assume that the initial particle possesses the certain physical reality which can be marked by the parameter \( \lambda \). We shall use the same parameter for the description of
physical realities of products of disintegration. Correspondingly, it is possible to consider measurement results of the observables $\hat{A}_a, \hat{B}_b$ as the functions $A_a(\lambda), B_b(\lambda)$ of the parameter $\lambda$. Let the distribution of the events with respect to the parameter $\lambda$ be characterized by the probabilistic measure $P(\lambda)$:

$$\int dP(\lambda) = 1, \quad 0 \leq P(\lambda) \leq 1.$$ 

Let us introduce the correlation function $E(a, b)$:

$$E(a, b) = \int dP(\lambda) A_a(\lambda) B_b(\lambda).$$ (1)

Also we consider the combination

$$I = |E(a, b) - E(a, b')| + |E(a', b) + E(a', b')|$$

$$= \left| \int dP(\lambda) A_a(\lambda) [B_b(\lambda) - B_{b'}(\lambda)] \right| + \left| \int dP(\lambda) A_{a'}(\lambda) [B_b(\lambda) + B_{b'}(\lambda)] \right|.$$ (2)

For any directions $a$ and $b$

$$A_a(\lambda) = \pm 1/2, \quad B_b(\lambda) = \pm 1/2.$$ (3)

Therefore,

$$I \leq \int dP(\lambda) \left[ |A_a(\lambda)| |B_b(\lambda) - B_{b'}(\lambda)| + |A_{a'}(\lambda)| |B_b(\lambda) + B_{b'}(\lambda)| \right]$$

$$= \frac{1}{2} \int dP(\lambda) \left[ |B_b(\lambda) - B_{b'}(\lambda)| + |B_b(\lambda) + B_{b'}(\lambda)| \right].$$ (4)

Due to the equality (3) for each $\lambda$ one of the expressions

$$|B_b(\lambda)B_{b'}(\lambda)|, \quad |B_b(\lambda) + B_{b'}(\lambda)|$$

is equal to zero, and the other is equal to one. Here it is crucial that the same value of the parameter $\lambda$ appears in both expressions. Hence, the Bell inequality (CHSH) is obtained

$$I \leq \frac{1}{2} \int dP(\lambda) = 1/2.$$ (6)

The correlation function can be easily calculated within standard quantum mechanics. We obtain

$$E(a, b) = -1/4 \cos \theta_{ab},$$

where $\theta_{ab}$ is the angle between the directions $a$ and $b$. For the directions $a = 0, b = \pi/8, a' = \pi/4, b' = 3\pi/8$ we have

$$I = 1/\sqrt{2}.$$ 

It contradicts the inequality (6).

The experiments carried out correspond to quantum-mechanical calculations and do not confirm the Bell inequality. These results are considered as deciding certificates against the hypothesis about existence of an objective local reality in quantum physics.
It is easy to make sure that, if to take into account peculiarity of the application of probability theory to quantum systems, it is impossible to carry out such a derivation of the Bell inequality.

Because in a quantum case the $\sigma$-algebra and, correspondingly, the probabilistic measure depend on the measuring device used, it is necessary to make replacement $dP(\lambda) \rightarrow dP_{ab}(\omega)$ in the equation (1). Besides, the subset $\Omega_{ab}$ is necessary to take as the regions of integration instead of set $\Omega$. Here, $\Omega_{ab}$ is the set of physical states which appear in the experiments using the device for measurement of the spin projections on the axes $a$ and $b$. In real experiment (collection of experiments) the sample $\Omega_{ab}$ is finite, in ideal experiment (collection of experiments) this sample is denumerable. Therefore, we can always consider that $\Omega_{ab}$ is random denumerable sample of the space $\Omega$.

Thus the equation (1) should be rewritten in the form

$$E(a, b) = \int_{\Omega_{ab}} dP_{ab}(\omega) A_a(\omega) B_b(\omega)$$

Now the equation (2) looks

$$I = \left| \int_{\Omega_{ab}} dP_{ab}(\omega) A_a(\omega) B_b(\omega) - \int_{\Omega_{ab}'} dP_{ab'}(\omega) A_a(\omega) B_{b'}(\omega) \right| +$$

$$+ \left| \int_{\Omega_{a'b}} dP_{a'b}(\omega) A_{a'}(\omega) B_b(\omega) + \int_{\Omega_{a'b'}} dP_{a'b'}(\omega) A_{a'}(\omega) B_{b'}(\omega) \right| .$$

If the directions $a$ and $a'$ ($b$ and $b'$) are not parallel to each other, then the observables $\hat{A}_a \hat{B}_b$, $\hat{A}_a \hat{B}_{b'}$, $\hat{A}_{a'} \hat{B}_b$, $\hat{A}_{a'} \hat{B}_{b'}$ are mutually incompatible. There is no united physically allowable $\sigma$-algebra which corresponds all these observables. This implies that there is no united probabilistic measure for these observables. Correspondingly, four integrals in the equation (7) cannot be united in one.

The sets $\Omega_{ab}$, $\Omega_{a'b'}$, $\Omega_{ab'}$, $\Omega_{a'b}$ are different random samples of the space $\Omega$. The probability of their intersection is equal to zero because the space $\Omega$ has a potency of the continuum. Therefore, with the probability one there is no physical state which could appear in expressions of type (5). Correspondingly, transition to an inequality of type (6) is impossible.

Thus, in the quantum case the hypothesis about existence of a local objective reality does not drive to the Bell inequalities. Therefore, the numerous experimental verifications of the Bell inequalities, which were carried out earlier and are carried out now, loose solid theoretical base. Of course, experimental confirmation of the Bell inequalities would be of outstanding significance. However, the negative result proves nothing.

5 Conclusions

As a rule the experimental investigation in connection with a question of the existence of local objective reality in quantum physics is reduced to examination of an interference pattern. It is possible to get acquainted with some examples in the paper by Zeilinger [9] quoted above.
Evidently the interference pattern is determined by the probabilistic distribution describing physical process under consideration. At the same time, as shown above, type of measuring device, which is used in present observation, determines to a large degree the probabilistic distribution.

In such context, results of Dopfer [12] described in paper [9] look quite natural. He has practically carried out the experiment similar to gedanken experiment with a so-called Heisenberg microscope. In this experiment the presence or absence of an interference pattern depends on what distance from the lens the detector is placed on.

It then follows that the concept of a local objective reality and, traced back to Bohr, "the situational (contextual) approach" are not in such antagonistic contradiction as it is considered to be. It is quite allowable that there is a physical reality which is inherent to the quantum object under consideration and which predetermines the result of any experiment. However, this result can depend on conditions in which this experiment is carried out. One of these conditions is the classical characteristic (type) of the measuring device, which is used in a concrete case.

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