USE DG-METHODS TO BUILD A MATRIX FACTORIZATION

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ABSTRACT. Let $P$ be a commutative Noetherian ring, $\mathfrak{R}$ be an ideal of $P$ which is generated by a regular sequence of length four, $f$ be a regular element of $P$, and $\overline{P}$ be the hypersurface ring $P/(f)$. Assume that $\mathfrak{R} : f$ is a grade four Gorenstein ideal of $P$. We give a resolution $N$ of $\overline{P}/\mathfrak{R}\overline{P}$ by free $\overline{P}$-modules. The resolution $N$ is built from a Differential Graded Algebra resolution of $\overline{P}/(\mathfrak{R} : f)$ by free $\overline{P}$-modules, together with one homotopy map. In particular, we give an explicit form for the matrix factorization which is the infinite tail of the resolution $N$.

0. INTRODUCTION.

Let $P$ be a commutative Noetherian ring, $\mathfrak{R}$ be an ideal of $P$, $f$ be a regular element of $P$, and $\overline{P}$ be the hypersurface ring $P/(f)$. This paper grew out of a desire to find an efficient method for resolving $\overline{P}/\mathfrak{R}\overline{P}$ by free $\overline{P}$-modules. We are particularly interested in this problem when $\mathfrak{R}$ is generated by a regular sequence.

The ultimate goal is to compare the resolution of the Frobenius powers $\overline{P}/\mathfrak{R}[q]\overline{P}$ to the resolution of $\overline{P}/\mathfrak{R}\overline{P}$, for $q = p^e$, where $P$ is a ring of prime characteristic $p$. The most interesting feature of the $\overline{P}$-resolution of $\overline{P}/\mathfrak{R}[q]\overline{P}$ is the infinite tail of the resolution, which is a matrix factorization of $f$. One goal is to determine the number of infinite tails that appear as $q = p^e$ varies and the least positive value of $e'$ for which the infinite tail of the resolution of $\overline{P}/\mathfrak{R}[q]e'\overline{P}$ is isomorphic to the infinite tail of the resolution of $\overline{P}/\mathfrak{R}[q]e'\overline{P}$, with $q' = p^{e'}$.

This ultimate goal has been accomplished when $P = k[x,y,z]$, $\mathfrak{R}$ is the maximal ideal $(x,y,z)$, and $k$ is a field of characteristic $p$. If $f = x^n + y^n + z^n$, then the Betti numbers of $\overline{P}/\mathfrak{R}[q]e'\overline{P}$ are calculated in [10] and the resolution of $\overline{P}/\mathfrak{R}[q]e'\overline{P}$ is given in [9]. If $f$ is a generic homogeneous form of $P$, then the graded Betti numbers of $\overline{P}/\mathfrak{R}[q]e'\overline{P}$ are calculated in [11].

The present paper gives a resolution $N$ of $\overline{P}/\mathfrak{R}\overline{P}$ by free $\overline{P}$-modules when $\mathfrak{R}$ is generated by a regular sequence of length four, $(\mathfrak{R} : f)$ is a Gorenstein ideal of grade four in $P$, and $P$ is an arbitrary commutative Noetherian ring. The resolution $N$ is built from a Differential Graded Algebra resolution

$$M : 0 \rightarrow M_4 \xrightarrow{m_4} M_3 \xrightarrow{m_3} M_2 \xrightarrow{m_2} M_1 \xrightarrow{m_1} M_0 = P$$

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of \( P/(\mathfrak{K} : f) \) together with a homotopy map \( X : M_1 \to M_2 \). The resolution \( N \) is given in Theorem 9.1. The matrix factorization which comprises the infinite tail of \( N \) is given in Theorem 2.4. The precise properties of the homotopy map \( X : M_1 \to M_2 \) are also given in Theorem 2.4. The most important part of the paper is the proof that \( X \) exists. This proof is given in Sections 4, 5, and 6.

The cleanest version of the matrix factorization of Theorem 2.4 occurs when \( f \) is the element \( \beta_0(1) \) of \( P \) which corresponds to the product

\[
\alpha_1(e_1) \cdot \alpha_1(e_2) \cdot \alpha_1(e_3) \cdot \alpha_1(e_4)
\]

in \( M_4 \), where \( K = \bigwedge^\ast (\bigoplus P e_i) \) is a Koszul complex which resolves \( P/\mathfrak{K} \), and

\[
\alpha : K \to M
\]

is a map of DG\( \Gamma \)-algebras. In this case, the matrix factorization of \( f \) is given by

\[
\begin{bmatrix}
X_{|M_{1,2}} & \alpha_2 & m_3|_{M_{3,2}}
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
\text{proj}_{M_{1,2}} \circ m_2 \\
\beta_2 \\
\text{proj}_{M_{3,2}} \circ X^\dagger
\end{bmatrix}.
\]

The decompositions \( M_1 = M_{1,1} \oplus M_{1,2} \) and \( M_3 = M_{3,1} \oplus M_{3,2} \) are explained in the text, \( \alpha_2 \) is the degree two component of \( \alpha \), and \( \beta_2 \) and \( X^\dagger \) essentially are maps adjoint to \( \alpha_2 \) and \( X \), respectively. An arbitrary \( f \) has the form \( r\beta_0(1) + \kappa \), where \( r \in P \) and \( \kappa \in \mathfrak{K} \). Once one has a matrix factorization for \( \beta_0(1) \), then there is no added difficulty in finding the matrix factorization for an arbitrary \( f \); but the formulas become more complicated. In particular, a streamlined version of the paper can be read if one takes \( r = 1 \) and \( \sigma, z_i, w_i, Y, \) and \( W \) all to be zero.

In Section 10 we describe two other interpretations of the map \( X : M_1 \to M_2 \). On the one hand, \( X \) is a higher order multiplication in the sense of [12, 6]. On the other hand, \( X \) and its adjoint give a homotopy from the complex \( M \) to itself.

**CONTENTS**

0. Introduction. 
1. Notation, conventions, and elementary results. 
2. Matrix factorization. 
3. Preliminary calculations. 
4. There exists a homomorphism \( X \) which satisfies 2.4.(b) and 2.4.(c). 
5. There exists a homomorphism \( X \) which satisfies 2.4.(a), 2.4.(b), and 2.4.(c). 
6. The map \( X \) of Theorem 2.4 exists. 
7. Further properties of \( X \). 
8. The proof of Theorem 2.4. 
9. The matrix factorization of Theorem 2.4 induces the infinite tail of the resolution of \( P/(f, \mathfrak{K}) \) by free \( P/(f) \) modules. 
10. Other interpretations of \( X \).
1. Notation, conventions, and elementary results.

1.1. The grade of a proper ideal $I$ in a commutative Noetherian ring $P$ is the length of the longest regular sequence on $P$ in $I$. The ideal $I$ of $P$ is called perfect if the grade of $I$ is equal to the projective dimension of the $P$-module $P/I$. The grade $g$ ideal $I$ is called Gorenstein if it is perfect and $\text{Ext}^g_P(P/I,P) \cong P/I$. It follows from Bass [1, Prop. 5.1] that if $I$ is a Gorenstein ideal in a Gorenstein ring $P$, then $P/I$ is also a Gorenstein ring.

1.2. A complex $\cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow 0$ is called acyclic if the only non-zero homology occurs in position zero.

1.3. Let $P$ be a commutative Noetherian ring, $X$ be a free $P$-module, and $Y$ be a $P$-module. The rules for a divided power algebra $D_\bullet X$ are recorded in [4, Def. 1.7.1] or [3, Appendix 2]. (In practice these rules say that $x^{(a)}$ behaves like $x^a/(a!)$ would behave if $a!$ were a unit in $P$.) Two rules that we use often are

\[(px)^{(n)} = p^n x^{(n)}, \quad \text{for } p \in P \text{ and } x \in X, \text{ and} \]

\[(x+y)^{(n)} = \sum_{i=0}^{n} x^{(i)} y^{(n-i)}, \quad \text{for } x, y \in X.\]

If $x$ and $x'$ are elements of $X$, then $x \cdot x' = x' \cdot x$ in $D_2(X)$. The co-multiplication homomorphism

\[\text{comult} : D_2 X \rightarrow X \otimes_P X\]

sends $x^{(2)}$ to $x \otimes x$ and sends $x \cdot x'$ to $x \otimes x' + x' \otimes x$, for $x, x' \in X$. Often we will describe a homomorphism $\phi : D_2 X \rightarrow Y$ by giving the value of $\phi(x^{(2)})$ for each $x \in X$. One then automatically knows the value of $\phi(x \cdot x')$, for $x, x' \in X$ because

\[(x + x')^{(2)} = x^{(2)} + x \cdot x' + x'\cdot x^{(2)}.\]

1.4. If $P$ is a ring and $A$, $B$, and $C$ are $P$-modules, then the $P$-module homomorphism $\phi : A \otimes_P B \rightarrow C$ is a perfect pairing if the induced $P$-module homomorphisms $A \rightarrow \text{Hom}_P(B,C)$ and $B \rightarrow \text{Hom}_P(A,C)$, given by $a \mapsto \phi(a \otimes \_)$ and $b \mapsto \phi(\_ \otimes b)$, respectively, are isomorphisms.

1.5. A Differential Graded algebra $F$ (written DG-algebra) over the commutative Noetherian ring $P$ is a complex of finitely generated free $P$-modules $(F, d)$:

$$\cdots \rightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 = P,$$

together with a unitary, associative multiplication $F \otimes_P F \rightarrow F$, which satisfies

(a) $F_i F_j \subseteq F_{i+j}$,

(b) $d_{i+j}(x_i x_j) = d_i(x_i) x_j + (-1)^i x_i d_j(x_j)$,
(c) $x_i x_j = (-1)^{ij} x_j x_i$, and
(d) $x_i^2 = 0$, when $i$ is odd,
for $x_i \in F_i$. The DG-algebra $F$ is called a $\text{DG}_\Gamma$-algebra (or a DG-algebra with divided powers) if, for each positive even index $i$ and each element $x_i$ of $F_i$, there is a family of elements $\{x_i^{(k)}\}$ which satisfy the divided power axioms of 1.3, and which also satisfy
\begin{equation}
(1.5.1) \quad d_{ik}(x_i^{(k)}) = d_i(x_i)x_i^{(k-1)}.
\end{equation}

The DG-algebra $F$ exhibits Poincaré duality if there is an integer $m$ such that $F_i = 0$ for $m < i$, $F_m$ is isomorphic to $P$, and for each integer $i$, the multiplication map
$$F_i \otimes P F_{m-i} \rightarrow F_m$$
is a perfect pairing of $P$-modules.

**Example 1.6.** The Koszul complex is the prototype of a $\text{DG}_\Gamma$-algebra which exhibits Poincaré duality.

Lemma 1.7 is used at a critical spot in the proof of Lemma 5.1. The assertion is obvious if $P$ is a local ring or if $P$ is a domain; however the assertion holds without any hypothesis imposed on $P$.

**Lemma 1.7.** Let $P$ be a commutative Noetherian ring and $\mathfrak{R}$ be an ideal in $P$ which is generated by a regular sequence, then there exists a regular sequence $a_1, \ldots, a_n$ in $\mathfrak{R}$ which generates $\mathfrak{R}$ with the property that each $a_i$ is a regular element of $P$.

**Proof.** Let $a_1, \ldots, a_n$ be a regular sequence which generates $\mathfrak{R}$. Observe that for any choice of $p_2, \ldots, p_n$ in $P$, the elements $a_1, a_2 + p_2 a_1, \ldots, a_n + p_n a_1$ also form a regular sequence which generates $\mathfrak{R}$. Fix an integer $i$ with $2 \leq i \leq n$. We prove there exists an element $p_i \in P$ with $a_i + p_i a_1$ a regular element of $P$. Let
$$S = \{p \in \text{Ass}(P) \mid p \text{ is not properly contained in } q \text{ for any } q \in \text{Ass}(P)\}.$$ The point is that the set of zero divisors of $P$ is $\bigcup_{p \in S} p$ and no prime of $S$ contains another prime of $S$. Decompose $S$ into two subsets:
$$S_1 = \{p \in S \mid a_i \in p\} \quad \text{and} \quad S_2 = \{p \in S \mid a_i \notin p\}.$$ If $p \in S_2$, then $p \not\subseteq q$ for any $q$ of $S_1$. Thus, the prime avoidance lemma ensures that $p \not\subseteq \bigcup_{q \in S_1} q$ and there exists an element $p_p \in p \setminus \bigcup_{q \in S_1} q$. Observe that
$$a_i + (\prod_{p \in S_2} p_p) a_1$$is a regular element on $P$. \qed
2. MATRIX FACTORIZATION.

**Data 2.1.** Let $P$ be a commutative Noetherian ring, $f$ be a regular element in $P$, $\mathfrak{R}$ be an ideal of $P$ which is generated by a regular sequence of length four, and

\begin{equation}
M : 0 \rightarrow M_4 \xrightarrow{m_4} M_3 \xrightarrow{m_3} M_2 \xrightarrow{m_2} M_1 \xrightarrow{m_1} M_0 = P
\end{equation}

be a complex of length four which is a resolution of $P/(\mathfrak{R} : f)$ by free $P$-modules. Assume that

(a) $M$ is a DG $\Gamma$-algebra which exhibits Poincaré duality, and

(b) the module $M_1$ is the direct sum of two free submodules

$$M_1 = M_{1,1} \oplus M_{1,2},$$

with rank $M_{1,1} = 4$ and $m_1(M_{1,1}) = \mathfrak{R}$.

**Remarks 2.2.**

(a) According to [7], every self-dual resolution

$$M : 0 \rightarrow M_4 \rightarrow M_3 \rightarrow M_2 \rightarrow M_1 \rightarrow M_0 = P$$

is a DG $\Gamma$-algebra which exhibits Poincaré duality. Earlier versions of this theorem [8, 5] proved the result when $P$ is Gorenstein and local and $M$ is a minimal resolution. It is shown in [7] that these three hypotheses are unnecessary.

(b) It is not important for our purposes that the resolution $M$ of $P/(\mathfrak{R} : f)$ be a minimal resolution (when this notion is defined). Indeed, hypothesis 2.1.(b) might rule out the possibility of $M$ being a minimal resolution. Nonetheless, the result of [7] may be applied in order to obtain a resolution $M$ which satisfies both hypotheses 2.1.(a) and 2.1.(b).

There are three results in this paper. Theorem 2.4 gives an explicit matrix factorization of $f$ provided there exists a map $X : M_1 \rightarrow M_2$ which satisfies five properties; Theorem 6.3 states that the map $X$ exists; and Theorem 9.1 states that the matrix factorization of Theorem 2.4 induces the infinite tail in the resolution of $P/(f, \mathfrak{R})$ by free $P/(f)$-modules.

**Definition and Conventions 2.3.** Adopt Data 2.1. Let

\begin{equation}
K : 0 \rightarrow K_4 \xrightarrow{k_4} K_3 \xrightarrow{k_3} K_2 \xrightarrow{k_2} K_1 \xrightarrow{k_1} K_0 = P
\end{equation}

be the Koszul complex which is a resolution of $P/\mathfrak{R}$. Notice that $K$ is automatically a DG $\Gamma$-algebra which exhibits Poincaré duality. The elements of $K_i$ are denoted by $\phi_i$, and

$$[-]_K : K_4 \rightarrow P$$

is a fixed orientation isomorphism. The elements of $M_i$ are denoted by $\theta_i$ and

$$[-]_M : M_4 \rightarrow P$$
is a fixed orientation isomorphism. Define
\[ \alpha_0 : K_0 = P \to M_0 = P \]
to be the identity map and define
\[ \alpha_1 : K_1 \to M_1 = M_{1,1} \oplus M_{1,2} \]
so that \( \text{proj}_{M_{1,1}} \circ \alpha_1 : K_1 \to M_{1,1} \) is the isomorphism for which the diagram
\[
\begin{array}{c}
K_1 \\
\downarrow \text{proj}_{M_{1,1}} \circ \alpha_1 \sim \downarrow m_1|_{M_{1,1}} \\
M_{1,1}
\end{array}
\]
commutes; and
\[
\text{proj}_{M_{1,2}} \circ \alpha_1 : K_1 \to M_{1,2} \quad \text{is the zero map.}
\]
(Recall the decomposition of \( M_1 \) which is described in 2.1.(b).) Define \( \alpha : K \to M \) to be the map of \( \text{DG} \Gamma \)-algebras which extends \( \alpha_0 \) and \( \alpha_1 \).
Define
\[ \beta_i : M_i \to K_i \]
by
\[
[\beta_i(\theta_i) \wedge \phi_{4-i}]_K = [\theta_i \cdot \alpha_{4-i}(\phi_{4-i})]_M,
\]
for all \( \theta_i \in M_i \) and \( \phi_{4-i} \in K_{4-i} \). (The fact that \( M \) and \( K \) are Poincaré duality algebras ensures that this definition is meaningful.)

2.3.5. Notice that linkage theory guarantees that
\[ \mathcal{R} : \text{im} m_1 = (\mathcal{R}, \beta_0(1)) \quad \text{and} \quad \mathcal{R} : \beta_0(1) = \text{im} m_1. \]
On the other hand, linkage theory also guarantees that
\[ \mathcal{R} : \text{im} m_1 = (\mathcal{R}, f) \quad \text{and} \quad \mathcal{R} : f = \text{im} m_1. \]
So
\[
(2.3.6) \quad f = r\beta_0(1) + k_1(\sigma)
\]
for some \( r \in P \) and \( \sigma \in K_1 \). Usually, \( r \) will be a unit in \( P \); indeed, for example, if \( P \) is a local ring, then \( r \) is a unit.

2.3.7. Define submodules
\[ M_{3,1} = \{ \theta_3 \in M_3 \mid \theta_3 M_{1,2} = 0 \} \quad \text{and} \quad M_{3,2} = \{ \theta_3 \in M_3 \mid \theta_3 M_{1,1} = 0 \} \]
and that the multiplication maps
\[ M_{1,1} \otimes_P M_{3,1} \to M_4 \quad \text{and} \quad M_{1,2} \otimes_P M_{3,2} \to M_4 \]
are both perfect pairings.

**2.3.8.** The maps $\beta_4 : M_4 \to K_4$ and $\beta_3|_{M_3,1} : M_{3,1} \to K_3$ are isomorphisms. Indeed, the definitions yield that

$$[\beta_4(\theta_4)]_K = [\beta_4(\theta_4) \wedge 1 ]_K = [\theta_4 \cdot \alpha_0 (1)]_M = [\theta_4]_M,$$

and that $[-]_K$ and $[-]_M$ are both isomorphisms. Similarly, the definitions yield that the map

$$M_{3,1} \to \text{Hom}_P(K_1, P),$$

given by $\theta_{3,1} \mapsto [\beta_3(\theta_{3,1}) \wedge -]_K,$

is an isomorphism. On the other hand, $K$ is a Poincaré Duality algebra; so,

$$K_3 \to \text{Hom}_P(K_1, P),$$

given by $\phi_3 \mapsto [\phi_3 \wedge -]_K,$

is also an isomorphism. It follows that $\beta_3|_{M_{3,1}} : M_{3,1} \to K_3$ is an isomorphism.

**2.3.9.** For each homomorphism $h : M_1 \to M_2$, let

$$h^ \dagger : M_2 \to M_3$$

be the homomorphism defined by

$$h^ \dagger (\theta_2) \cdot \theta_1 = \theta_2 \cdot h(\theta_1),$$

for $\theta_i \in M_i$. (The existence of $h^ \dagger$ is also guaranteed by the assumption that $M$ is a Poincaré duality algebra.)

**2.3.10.** The homomorphisms

$$z_i : K_i \to K_{i+1} \quad \text{and} \quad w_i : M_i \to M_{i+1}$$

are defined by

$$z_i(\phi_i) = \phi_i \wedge \sigma \quad \text{and} \quad w_i(\theta_i) = \theta_i \cdot \alpha_1 (\sigma),$$

for $\phi_i \in K_i$ and $\theta_i \in M_i$. The homomorphisms

$$Y : M_2 \to K_2 \quad \text{and} \quad W : K_2 \to M_2$$

are defined by

$$Y = z_1 \circ (\text{proj}_{M_{1,1}} \circ \alpha_1)^{-1} \circ \text{proj}_{M_{1,1}} \circ m_2 \quad \text{and} \quad W = m_3 \circ (\beta_3|_{M_{3,1}})^{-1} \circ z_2.$$

Recall from (2.3.2) and 2.3.8 that the indicated inverse maps exist.

We are now able to state the result about matrix factorization. This result gives an explicit matrix factorization of the $f$ of Data 2.1 in terms of the maps defined in Definition 2.3 and one other map $X : M_1 \to M_2$ provided the map $X$ exists and satisfies five properties. Theorem 6.3 states that the map $X$ exists; and Theorem 9.1 states that the matrix factorization of Theorem 2.4 induces the infinite tail in the resolution of $P/(f, \delta)$ by free $P/(f)$-modules. Recall from 2.3.5 that the parameter $r$ is usually a unit. In this case, there is no reason to consider the matrix factorization
M.F.1. Indeed, in this case, M.F.2 is obtained from M.F.1 by splitting off a trivial factorization. Furthermore, as was observed in the Introduction, a streamlined, but still meaningful, version of the paper can be read if one takes $r = 1$ and $\sigma, z_i, w_i, Y,$ and $W$ all to be zero.

**Theorem 2.4.** Adopt the language of Definition 2.3. Suppose that

$$X : M_1 \to M_2$$

is an $R$-module homomorphism which satisfies

(a) $X \circ \alpha_1 = 0,$
(b) $m_2 \circ X = \beta_0(1) \cdot \id_{M_1} - \alpha_1 \circ \beta_1,$
(c) $X \circ m_2 + m_3 \circ X^\dagger = \beta_0(1) \cdot \id_{M_2} - \alpha_2 \circ \beta_2,$
(d) $X^\dagger \circ X = 0,$ and
(e) $X^\dagger \circ \alpha_2 = 0.$

Then the following statements hold.

M.F. 1. Let $G_{\text{even}}$ and $G_{\text{odd}}$ be the free $P$-modules

$$G_{\text{even}} = M_{1,2} \oplus K_2 \oplus M_3 \oplus K_4 \quad \text{and} \quad G_{\text{odd}} = M_2 \oplus K_3 \oplus M_4$$

and $g_{\text{even}} : G_{\text{even}} \to G_{\text{odd}}$ and $g_{\text{odd}} : G_{\text{odd}} \to G_{\text{even}}$ be the $P$-module homomorphisms

$$g_{\text{even}} = \begin{bmatrix}
(rX - w_1)|_{M_{1,2}} & \alpha_2 & m_3 & 0 \\
0 & -z_2 & r\beta_3 & -k_4 \\
0 & 0 & -w_3 & \alpha_4
\end{bmatrix} \quad \text{and} \quad g_{\text{odd}} = \begin{bmatrix}
\text{proj}_{M_{1,2}} \circ m_2 & 0 & 0 \\
r\beta_2 - Y & -k_3 & 0 \\
x^\dagger + w_2 & \alpha_3 & m_4 \\
0 & z_3 & r\beta_4
\end{bmatrix}.$$

Then the equalities

$$g_{\text{odd}} \circ g_{\text{even}} = f \cdot \id_{G_{\text{even}}} \quad \text{and} \quad g_{\text{even}} \circ g_{\text{odd}} = f \cdot \id_{G_{\text{odd}}}$$

hold.

M.F. 2. Assume that $r$ is a unit. Let $\hat{G}_{\text{even}}$ and $\hat{G}_{\text{odd}}$ be the free $P$-modules

$$\hat{G}_{\text{even}} = M_{1,2} \oplus K_2 \oplus M_{3,2} \quad \text{and} \quad \hat{G}_{\text{odd}} = M_2$$

$\hat{g}_{\text{even}} : \hat{G}_{\text{even}} \to \hat{G}_{\text{odd}}$ and $\hat{g}_{\text{odd}} : \hat{G}_{\text{odd}} \to \hat{G}_{\text{even}}$ be the $P$-module homomorphisms

$$\hat{g}_{\text{even}} = \begin{bmatrix}
(rX - w_1)|_{M_{1,2}} & \alpha_2 + r^{-1}W & m_3 |_{M_{3,2}}
\end{bmatrix}$$

$$\hat{g}_{\text{odd}} = \begin{bmatrix}
\text{proj}_{M_{1,2}} \circ m_2 & 0 \\
r\beta_2 - Y & 0 \\
\text{proj}_{M_{3,2}} \circ (x^\dagger + w_2)
\end{bmatrix}.$$
Then the equalities
\[ \hat{g}_{\text{odd}} \circ g_{\text{even}} = f \cdot \text{id}_{\phi_{\text{even}}} \quad \text{and} \quad \hat{g}_{\text{even}} \circ g_{\text{odd}} = f \cdot \text{id}_{\phi_{\text{odd}}} \]
hold.

**Remarks 2.5.**

(a) The proof of Theorem 2.4 is given in 8.1. First we make numerous preliminary calculations involving the maps of Data 2.1, Definition 2.3, and \( X \) itself.

(b) It should be noted that \( G_{\text{even}} \) and \( G_{\text{odd}} \) have the same rank. Indeed, \( K_2, K_3, K_4, \) and \( M_4 \) have rank 6, 4, 1, and 1, respectively; \( M_1 \) and \( M_3 \) have the same rank; \( \text{rank} M_2 = 2 \text{rank} M_1 - 2; \text{and rank} M_{1,2} = \text{rank} M_1 - 4. \) Similarly,

\[ \text{rank } \hat{G}_{\text{even}} = 6 + 2(\text{rank} M_1 - 4) = 2 \text{rank} M_1 - 2 = \text{rank} M_2 \]

\[ = \text{rank } \hat{G}_{\text{odd}}. \]

3. Preliminary Calculations.

In this section we prove many formulas involving the data of 2.1 and 2.3. These formulas are used in the the proof of the existence of \( X \) and in the proof of Theorem 2.4. There are many of these formulas; but each proof is straightforward. The hard work is involved in the proof of Theorem 6.3, where we establish the existence of \( X \).

3.1. We often use the graded product rule on \( M \) and \( K \) in the following form. If \( \theta_j \in M_j \) and \( \phi_i \in K_j \) then \( 0 = m_5(\theta_i \cdot \phi_{5-i}), 0 = k_5(\phi_i \wedge \phi_{5-i}) \); and therefore,

\[ 0 = m_i(\theta_i) \cdot \phi_{5-i} + (-1)^i \theta_i \cdot m_{5-i}(\phi_{5-i}) \quad \text{and} \quad 0 = k_i(\phi_i) \wedge \phi_{5-i} + (-1)^i \phi_i \wedge k_{5-i}(\phi_{5-i}), \]

(3.1.1)

**Observation 3.2.** *In the language of 2.3, the maps \( \beta_i \) form a map of complexes.*

**Proof.** It suffices to show that

\[ \beta_i \circ m_{i+1} = k_{i+1} \circ \beta_{i+1}. \]

It suffices to show that

\[ [(\beta_i \circ m_{i+1})(\theta_{i+1}) \wedge \phi_{4-i})]_K = [(k_{i+1} \circ \beta_{i+1})(\theta_{i+1}) \wedge \phi_{4-i}]_K. \]

We compute

\[ [(\beta_i \circ m_{i+1})(\theta_{i+1}) \wedge \phi_{4-i}]_K \]
\[ = [m_{i+1}(\theta_{i+1}) \cdot \alpha_{4-i}(\phi_{4-i})]_M, \]
\[ = (-1)^i [\theta_{i+1} \cdot (m_{4-i} \circ \alpha_{4-i})(\phi_{4-i})]_M, \quad \text{by (2.3.4)}, \]
\[ = (-1)^i [\theta_{i+1} \cdot (\alpha_{3-i} \circ k_{4-i})(\phi_{4-i})]_M, \quad \text{by (2.3.4)}, \]
\[ = (-1)^i [(\beta_{i+1} \theta_{i+1}) \wedge k_{4-i}(\phi_{4-i})]_K, \]
\[ = [(k_{i+1} \circ \beta_{i+1})(\theta_{i+1}) \wedge \phi_{4-i}]_K, \quad \text{by (3.1.1)}, \]

\[ \square \]
It is convenient to combine the maps of complexes $\alpha$ and $\beta$ into the following commutative diagram:

\[
\begin{array}{c}
0 \rightarrow K_4 \xrightarrow{k_4} K_3 \xrightarrow{k_3} K_2 \xrightarrow{k_2} K_1 \xrightarrow{k_1} K_0 \\
| & | & | & |
0 \rightarrow M_4 \xrightarrow{m_4} M_3 \xrightarrow{m_3} M_2 \xrightarrow{m_2} M_1 \xrightarrow{m_1} M_0
\end{array}
\]

\[\begin{array}{c}
\alpha_4 \downarrow \\
\beta_4 \downarrow
\end{array}\]

\[\begin{array}{c}
\alpha_3 \downarrow \\
\beta_3 \downarrow
\end{array}\]

\[\begin{array}{c}
\alpha_2 \downarrow \\
\beta_2 \downarrow
\end{array}\]

\[\begin{array}{c}
\alpha_1 \downarrow \quad \alpha_1 = \alpha_0 \\
\beta_1 \downarrow
\end{array}\]

\[\begin{array}{c}
\alpha_0 \downarrow \\
\beta_0 \downarrow
\end{array}\]

**Observation 3.3.** Adopt the language of 2.3. The following formulas hold for $\theta_i$ in $M_\ell$ and $\phi_\ell$ in $K_i$:

(a) $\beta_i \circ \alpha_i = \beta_0(1) \cdot \text{id}_{K_i}$, for $0 \leq i \leq 4$,
(b) $\theta_i \cdot (\alpha_{4-i} \circ \beta_{4-i})(\theta_{4-i}) = (\alpha_i \circ \beta_i)(\theta_i) \cdot \theta_{4-i}$, for $0 \leq i \leq 4$,
(c) $\beta_i(\theta_j \cdot \alpha_{i-j}(\phi_{i-j})) = \beta_j(\theta_j) \wedge \phi_{i-j}$, for $0 \leq j \leq i \leq 4$,
(d) $\beta_3|_{M_{3,2}} = 0$,
(e) $(\beta_3|_{M_{3,1}})^{-1} \circ \beta_3 = \text{proj}_{M_{3,1}}$, and
(f) $w_3 \circ (\beta_3|_{M_{3,1}})^{-1} \circ z_2 = 0$.

**Proof.**

(a) It suffices to show that

\[[(\beta_i \circ \alpha_i)(\phi_i) \wedge \phi_{4-i}]_K = [\beta_0(1) \cdot \phi_i \wedge \phi_{4-i}]_K,\]

for all $\phi_i \in K_i$ and $\phi_{4-i} \in K_{4-i}$. Observe that

\[
[(\beta_i \circ \alpha_i)(\phi_i) \wedge \phi_{4-i}]_K \\
= [\alpha_i(\phi_i) \cdot \alpha_{4-i}(\phi_{4-i})]_M, \quad \text{by (2.3.4),}
\]

\[= [\alpha_4(\phi_i \wedge \phi_{4-i})]_M, \quad \text{because $\alpha$ is an algebra map,}
\]

\[= [1 \cdot \alpha_4(\phi_i \wedge \phi_{4-i})]_M
\]

\[= [\beta_0(1) \cdot \phi_i \wedge \phi_{4-i}]_K, \quad \text{by (2.3.4).}
\]

(b) Apply (2.3.4) and graded commutativity multiple times:

\[
[\theta_i \cdot (\alpha_{4-i} \circ \beta_{4-i})(\theta_{4-i})]_M = [\beta_i(\theta_i) \wedge \beta_{4-i}(\theta_{4-i})]_K
\]

\[= (-1)^i[\beta_{4-i}(\theta_{4-i}) \wedge \beta_i(\theta_i)]_K
\]

\[= (-1)^i[\theta_{4-i} \cdot (\alpha_i \circ \beta_i)(\theta_i)]_M
\]

\[= [(\alpha_i \circ \beta_i)(\theta_i) \cdot \theta_{4-i}]_M.
\]
(c) Observe that
\[ [\beta_i(\theta_j \cdot \alpha_{i-j}(\phi_{i-j})) \wedge \phi_{4-j}]_M = [\theta_j \cdot \alpha_{i-j}(\phi_{i-j}) \cdot \alpha_{4-j}(\phi_{4-j})]_M, \]
by (2.3.4),
\[ = [\theta_j \cdot \alpha_{4-j}(\phi_{i-j} \wedge \phi_{4-i})]_M, \]
because \( \alpha \) is an algebra map.
\[ = [\beta_j(\theta_j) \wedge \phi_{i-j} \wedge \phi_{4-i}]_K, \]
by (2.3.4).

Multiplication is associative in both \( K \) and \( M \).

(d) If \( \theta_{3,2} \) is an element of \( M_{3,2} \), then
\[ [\beta_3(\theta_{3,2}) \wedge \phi_1]_K = [\theta_{3,2} \cdot \alpha_1(\phi_1)]_M = 0, \]
for all \( \phi_1 \in K_1 \) by the definition of \( M_{3,2} \) (see 2.3.7); hence \( \beta_3(\theta_{3,2}) = 0 \).

(e) Observe that
\[ (\beta_3|_{M_{3,1}})^{-1} \circ \beta_3 = (\beta_3|_{M_{3,1}})^{-1} \circ \beta_3 \circ (\text{proj}_{M_{3,1}} + \text{proj}_{M_{3,2}}) \]
\[ = (\beta_3|_{M_{3,1}})^{-1} \circ (\beta_3|_{M_{3,1}} \circ \text{proj}_{M_{3,1}} + \beta_3|_{M_{3,2}} \circ \text{proj}_{M_{3,2}}). \]

Recall from (d) that \( \beta_3|_{M_{3,2}} = 0 \). Conclude that
\[ (\beta_3|_{M_{3,1}})^{-1} \circ \beta_3 = (\beta_3|_{M_{3,1}})^{-1} \circ (\beta_3|_{M_{3,1}} \circ \text{proj}_{M_{3,1}}) = \text{proj}_{M_{3,1}}. \]

(f) If \( \phi_2 \in K_2 \), then
\[ [(w_3 \circ (\beta_3|_{M_{3,1}})^{-1} \circ \sigma_2)(\phi_2)]_M = [(\beta_3|_{M_{3,1}})^{-1}(\phi_2 \wedge \sigma) \cdot \alpha_1(\sigma)]_M \]
by 2.3.10
\[ = [\beta_3((\beta_3|_{M_{3,1}})^{-1}(\phi_2 \wedge \sigma)) \wedge \sigma]_K \]
by 2.3.4
\[ = [\phi_2 \wedge \sigma \wedge \sigma]_K = 0. \]

\[ \square \]

**Lemma 3.4.** In the language of 2.3, \( \ker m_3 \cap \ker \beta_3 = 0 \).

**Proof.** Let \( \theta_3 \) be an element of \( \ker m_3 \cap \ker \beta_3 \). The complex \( M \) is a resolution; so \( \theta_3 = m_4(\theta_4) \) for some \( \theta_4 \in M_4 \). Apply (3.2.1) to see that
\[ 0 = \beta_3(\theta_3) = (\beta_3 \circ m_4)(\theta_4) = k_4 \circ \beta_4(\theta_4). \]
The map \( k_4 \) is an injection; consequently, \( \beta_4(\theta_4) = 0 \). On the other hand,
\[ 0 = [\beta_4(\theta_4)]_K = [\beta_4(\theta_4) \wedge 1]_K = [\theta_4 \cdot \alpha_0(1)]_M = [\theta_4]_M. \]
Thus, \( \theta_4 = 0 \) and \( \theta_3 = m_4(\theta_4) \) is also zero. \[ \square \]
Observation 3.5. The maps and modules form maps of complexes.

Proof. The maps $z_i$ and $w_i$ are defined in (2.3.10). Elements of degree 1 in a DG-algebra square to zero. It follows that $z_i \circ z_{i-1} = 0$ and $w_i \circ w_{i-1} = 0$. To see that $\alpha$ is a map of complexes, we observe that

$$
(\alpha_{i+1} \circ z_i)(\phi_i) = \alpha_{i+1}(\phi_i \wedge \sigma), \quad \text{by the definition of } z,
$$

$$
= \alpha_i(\phi_i) \cdot \alpha_1(\sigma), \quad \text{because } \alpha \text{ is an algebra map},
$$

$$
= (w_i \circ \alpha_i)(\phi_i), \quad \text{by the definition of } w.
$$

To see that $\beta$ is a map of complexes, we observe that

$$
[(z_i \circ \beta_i)(\theta_i) \wedge \phi_{3-i}]_K
$$

$$
= [\beta_i(\theta_i) \wedge \sigma \wedge \phi_{3-i}]_K, \quad \text{by the definition of } z,
$$

$$
= [\theta_i \cdot \alpha_{4-i}(\sigma \wedge \phi_{3-i})]_M, \quad \text{by (2.3.4)},
$$

$$
= [(\theta_i \cdot \alpha_1(\sigma)) \cdot \alpha_{3-i}(\phi_{3-i})]_M, \quad \text{because } \alpha \text{ is an algebra map},
$$

$$
= [w_i(\theta_i) \cdot \alpha_{3-i}(\phi_{3-i})]_M, \quad \text{by the definition of } w,
$$

$$
= [(\beta_{i+1} \circ w_i)(\theta_i) \wedge \phi_{3-i}]_K, \quad \text{by (2.3.4)}.
$$

Of course, we used that the multiplication on $M$ is associative. \hfill \Box

Observation 3.6. The maps $z_i$ and $w_i$ of Definition 2.3.10 satisfy the following formulas:

(a) $z_{i-1} \circ k_i - k_{i+1} \circ z_i = (-1)^{i+1}k_1(\sigma) \cdot id_{K_i}$, and

(b) $w_{i-1} \circ m_i - m_{i+1} \circ w_i = (-1)^{i+1}k_1(\sigma) \cdot id_{M_i}$.

Proof. One uses the definition of $z$ and $w$, the graded product rule, and the commutativity of (3.2.1). (a) If $\phi_i \in K_i$, then

$$
(z_{i-1} \circ k_i - k_{i+1} \circ z_i)(\phi_i) = k_i(\phi_i) \wedge \sigma - k_{i+1}(\phi_i \wedge \sigma) = (-1)^{i+1}k_1(\sigma) \cdot \phi_i.
$$

(b) If $\theta_i \in M_i$, then

$$
(w_{i-1} \circ m_i - m_{i+1} \circ w_i)(\theta_i) = m_i(\theta_i) \cdot \alpha_1(\sigma) - m_{i+1}(\theta_i \cdot \alpha_1(\sigma))
$$

$$
= (-1)^{i+1}k_1(\sigma) \cdot \theta_i.
$$

\hfill \Box
4. **There exists a homomorphism** $X$ **which satisfies** 2.4.(b) **and** 2.4.(c).

Retain the data of 2.1 and 2.3. In this section we produce a formal complex $B$ which automatically has a partial multiplicative structure. We also produce a null homotopic map of complexes $c : B \to K$. Our first approximation of the map $X : M_1 \to M_2$ is manufactured from this homotopy. This version of $X$ satisfies 2.4.(b) and 2.4.(c).

Our inspiration for using $B$ comes from the proof of [2, Prop. 1.1] and from [13, Sect. 2]. A complex similar to $B$ plays a crucial role in [7, Lem. 3.2]. An earlier version of the present paper was able to prove the existence of $X$ only for rings in which 2 is a unit. The present version of the paper relies on the divided powers in $B, M$ and $K$ to avoid that hypothesis.

**Definition 4.1.** Adopt the notation of 2.1 and 2.3.

(a) Define

$$ B : 0 \to B_4 \overset{b_4}{\to} B_3 \overset{b_3}{\to} B_2 \overset{b_2}{\to} B_1 \overset{b_1}{\to} B_0 $$

to be the modules

$$ B_4 = D_2 M_2, \quad B_3 = M_1 \otimes M_2, \quad B_2 = \bigwedge^2 M_1 \oplus M_2, $$

$$ B_1 = M_1, \quad B_0 = M_0, $$

and the maps

$$ b_4(\theta^{(2)}_2) = m_2(\theta_2) \otimes \theta_2, $$

$$ b_3(\theta_1 \otimes \theta_2) = \begin{bmatrix} -\theta_1 \land m_2(\theta_2) \\ m_1(\theta_1) \cdot \theta_2 \end{bmatrix}, $$

$$ b_2 \left( \begin{bmatrix} \theta_1 \land \theta'_1 \\ \theta_2 \end{bmatrix} \right) = m_1(\theta_1) \cdot \theta'_1 - m_1(\theta'_1) \cdot \theta_1 + m_2(\theta_2), \text{ and} $$

$$ b_1 = m_1. $$

(b) Define $c_i : B_i \to K_i$ by

$$ c_4(\theta^{(2)}_2) = \beta_0(1) \cdot \beta_4(\theta^{(2)}_2) - \left( \beta_2(\theta_2) \right)^{(2)}, $$

$$ c_3(\theta_1 \otimes \theta_2) = \beta_0(1) \cdot \beta_3(\theta_1 \cdot \theta_2) - \beta_1(\theta_1) \land \beta_2(\theta_2), $$

$$ c_2 \left( \begin{bmatrix} \theta_1 \land \theta'_1 \\ \theta_2 \end{bmatrix} \right) = \beta_0(1) \cdot \beta_2(\theta_1 \theta'_1) - \beta_1(\theta_1) \land \beta_1(\theta'_1). $$

The maps $c_1$ and $c_0$ are both identically zero.

Notice that the divided powers on the left side of (4.1.2) take place in the formal divided power algebra $D_2 M_2$; the first divided power on the right side takes place in the DG-Γ-algebra $M$; and the second divided power on the right side take place in the DG-Γ-algebra $K$.

**Observation 4.2.** Retain the data of Definition 4.1. The following statements hold.
(a) The maps and modules of $B$ form a complex.
(b) The maps $c : B \to K$ form a map of complexes.
(c) There are homotopy maps $h_i : B_i \to K_{i+1}$, for $0 \leq i \leq 4$, such that
   (i) $h_0$, $h_1$, and $h_4$ all are zero,
   (ii) the restriction of $h_2$ to the summand $M_2$ of $B_2$ is identically zero, and
   (iii) $c_i = h_{i-1} \circ b_i + k_{i+1} \circ h_i$, for $1 \leq i \leq 4$.

Proof. Assertion (a) is obvious. We first prove (b). Observe that

\[
(k_4 \circ c_4)(\theta_2^{(2)}) = k_4 \left( \beta_0(1) \cdot \beta_4(\theta_2^{(2)}) - (\beta_2(\theta_2))^{(2)} \right)
\]

\[
= \beta_0(1) \cdot (\beta_3 \circ m_4)(\theta_2^{(2)}) - (k_2 \circ \beta_2)(\theta_2) \wedge \beta_2(\theta_2),
\]

by (3.2.1) and (1.5.1),

\[
= \beta_0(1) \cdot (\beta_3(m_2(\theta_2) \cdot \theta_2) - (\beta_1 \circ m_2)(\theta_2) \wedge \beta_2(\theta_2),
\]

by (1.5.1) and (3.2.1),

\[
= c_3(m_2(\theta_2) \otimes \theta_2) = (c_3 \circ b_4)(\theta_2^{(2)}).
\]

Observe also that

\[
(k_3 \circ c_3)(\theta_1 \otimes \theta_2)
\]

\[
= k_3 \left( \beta_0(1) \cdot \beta_3(\theta_1 \cdot \theta_2) - \beta_1(\theta_1) \wedge \beta_2(\theta_2) \right)
\]

\[
= \left\{ \begin{array}{l}
\beta_0(1) \cdot (\beta_2 \circ m_3)(\theta_1 \cdot \theta_2) \\
-(k_1 \circ \beta_1)(\theta_1) \wedge \beta_2(\theta_2) + \beta_1(\theta_1) \wedge (k_2 \circ \beta_2)(\theta_2),
\end{array} \right.
\]

by (3.2.1) and 1.5.(b),

\[
= \left\{ \begin{array}{l}
\beta_0(1) \cdot (m_1(\theta_1) \cdot \beta_2(\theta_2) - \beta_2(\theta_1 \cdot m_2(\theta_2)) \\
-(k_1 \circ m_1(\theta_1) \cdot \beta_2(\theta_2) + \beta_1(\theta_1) \wedge (k_1 \circ m_2(\theta_2)),
\end{array} \right.
\]

by 1.5.(b) and (3.2.1),

\[
= \beta_0(1) \cdot (\theta_1 \cdot m_2(\theta_2)) + \beta_1(\theta_1) \wedge (\beta_1 \circ m_2)(\theta_2)
\]

\[
= (c_2 \circ b_3)(\theta_1 \otimes \theta_2).
\]

Finally, observe that

\[
(k_2 \circ c_2) \left( \left[ \frac{\theta_1 \wedge \theta'_1}{\theta_2} \right] \right)
\]

\[
= k_2 \left( \beta_0(1) \cdot \beta_2(\theta_1 \theta'_1) - \beta_1(\theta_1) \wedge \beta_1(\theta'_1) \right)
\]

\[
= \left\{ \begin{array}{l}
\beta_0(1) \cdot (\beta_1 \circ m_2)(\theta_1 \theta'_1) \\
-(k_1 \circ \beta_1)(\theta_1) \cdot \beta_1(\theta'_1) + \beta_1(\theta_1) \cdot (k_1 \circ \beta_1)(\theta'_1),
\end{array} \right.
\]

by (3.2.1) and 1.5.(b),

\[
= \left\{ \begin{array}{l}
\beta_0(1) \cdot (m_1(\theta_1) \cdot \beta_1(\theta'_1) - m_1(\theta'_1) \cdot \beta_1(\theta_1)) \\
-(k_1 \circ m_1(\theta_1) \cdot \beta_1(\theta'_1) + \beta_1(\theta_1) \cdot (k_1 \circ \theta_1 \cdot m_1(\theta'_1)),
\end{array} \right.
\]

by 1.5.(b) and (3.2.1),

\[
= 0 = c_1 \circ b_2 \left( \left[ \frac{\theta_1 \wedge \theta'_1}{\theta_2} \right] \right).
\]

This completes the proof of (b); now we prove (c). The map $c : B \to K$ is a map of complexes with $c_0$ and $c_1$ both identically zero; furthermore, $K$ is a resolution. It
follows that there is a homotopy
\[ \{ h_i : B_i \to K_{i+1} \mid 0 \leq i \leq 4 \} \]
which satisfies condition (ciii). It is clear that \( h_0 \) and \( h_1 \) may be chosen to be zero. The target for \( h_4 \) is zero; so this map also is zero. The restriction of \( h_2 \) to \( M_2 \) may be taken to be any homomorphism which completes the homotopy in the sense that
\[ c_2|_{M_2} = h_1 \circ m_2 + k_3 \circ h_2|_{M_2}. \]
The maps \( c_2|_{M_2} \) and \( h_1 \) are already identically zero. Consequently, one may choose \( h_2|_{M_2} \) to be identically zero.

**Lemma 4.3.** Adopt the notation of 2.1 and 2.3. Then there exists a homomorphism \( X : M_1 \to M_2 \) which satisfies 2.4.(b) and 2.4.(c).

**Proof.** Let \( \{ h_i : B_i \to K_{i+1} \} \) be the homotopy of Observation 4.2.(c). Define \( X : M_1 \to M_2 \) by
\[
(4.3.1) \quad X(\theta_1) \cdot \theta_2 = (\beta_4^{-1} \circ h_3)(\theta_1 \otimes \theta_2).
\]
(Recall from 2.3.8 that \( \beta_4 \) is an isomorphism.) We first prove that \( X \) satisfies 2.4.(c). Observe that
\[
(X \circ m_2 + m_3 \circ X^1)(\theta_2) \cdot \theta'_2
= (\beta_4^{-1} \circ h_3)(m_2(\theta_2) \otimes \theta'_2) + \theta_2 \cdot X(m_2(\theta'_2)), \quad \text{by (3.1.1) and 2.3.9},
= (\beta_4^{-1} \circ h_3)(m_2(\theta_2) \otimes \theta'_2) + (\beta_4^{-1} \circ h_3)(m_2(\theta'_2) \otimes \theta_2)
= (\beta_4^{-1} \circ h_3)(b_4(\theta_2 \theta'_2))
= (\beta_4^{-1} \circ c_4)(\theta_2 \theta'_2), \quad \text{by Obs. 4.2.(ciii)},
\]
\[
= \beta_0(1) \cdot \theta_2 \theta'_2 - \beta_4^{-1}\left(\beta_2(\theta_2) \wedge \beta_2(\theta'_2)\right)
= \beta_0(1) \cdot \theta_2 \theta'_2 - \beta_4^{-1}\left(\beta_2(\theta_2) \cdot (\alpha_2 \circ \beta_2)(\theta'_2)\right), \quad \text{by Obs. 3.3.(c)},
= \beta_0(1) \cdot \theta_2 \theta'_2 - \theta_2 \cdot (\alpha_2 \circ \beta_2)(\theta'_2)
= \beta_0(1) \cdot \theta_2 \theta'_2 - (\alpha_2 \circ \beta_2)(\theta_2) \cdot \theta'_2 \quad \text{by Obs. 3.3.(b)},
= \left(\beta_0(1) \cdot \text{id}_{M_2} - \alpha_2 \circ \beta_2\right)(\theta_2) \cdot \theta'_2.
\]
Now we prove that \( X \) satisfies 2.4.(b). Recall from (3.1.1) that
\[
(4.3.2) \quad (m_2 \circ X)(\theta_1) \cdot \theta_3 = -X(\theta_1) \cdot m_3(\theta_3) = -(\beta_4^{-1} \circ h_3)(\theta_1 \otimes m_3(\theta_3)).
\]
Apply Observation 4.2.(ciii) to see that
\[ k_4 \circ h_2 + h_2 \circ b_3 = c_3. \]

Observe that
\[ b_3(\theta_1 \otimes m_3(\theta_3)) = \begin{bmatrix} 0 \\ m_1(\theta_1) \cdot m_3(\theta_3) \end{bmatrix}, \]
which is in the summand \( M_2 \) of \( T_2 \). It follows from Obs. 4.2.(cii) that
\[ (h_2 \circ b_3)(\theta_1 \otimes m_3(\theta_3)) = 0; \]
and therefore,
\[ (k_4 \circ h_3)(\theta_1 \otimes m_3(\theta_3)) = c_3(\theta_1 \otimes m_3(\theta_3)) \]
\[ = \beta_0(1) \cdot \beta_3((\theta_1 \cdot m_3(\theta_3)) - \beta_1(\theta_1) \land (\beta_2 \circ m_3)(\theta_3)). \]

Use the commutative diagram (3.2.1) to write \( \beta_2 \circ m_3 \) as \( k_3 \circ \beta_3 \) and then use the product rule 1.5.(b) on each summand. It follows that \( (k_4 \circ h_3)(\theta_1 \otimes m_3(\theta_3)) \) is equal to
\[ = \begin{cases} \beta_0(1) \cdot (m_1(\theta_1) \cdot \beta_3(\theta_3) - (\beta_3 \circ m_4)(\theta_1 \cdot \theta_3)) \\ - \left( \beta_0(1) \cdot m_1(\theta_1) \cdot \beta_3(\theta_3) - k_4(\beta_1(\theta_1) \land \beta_3(\theta_3)) \right) \end{cases} \]
\[ = k_4 \left( - \beta_0(1) \cdot \beta_4(\theta_1 \cdot \theta_3) + \beta_1(\theta_1) \land \beta_3(\theta_3) \right), \text{ by (3.2.1).} \]

The map \( k_4 \) is injective; hence,
\[ h_3(\theta_1 \otimes m_3(\theta_3)) = - \beta_0(1) \cdot \beta_4(\theta_1 \cdot \theta_3) + \beta_1(\theta_1) \land \beta_3(\theta_3), \]
and (4.3.2) now becomes
\[ \left( m_2 \circ X \right)(\theta_1) \cdot \theta_3 = - (\beta_4^{-1} \circ h_3)(\theta_1 \otimes m_3(\theta_3)) \]
\[ = \beta_4^{-1} \left( \beta_0(1) \cdot \beta_4(\theta_1 \cdot \theta_3) - \beta_1(\theta_1) \land \beta_3(\theta_3) \right). \]

Recall from assertions (c) and (b) of Observation 3.3 that
\[ \beta_4^{-1} \left( \beta_1(\theta_1) \land \beta_3(\theta_3) \right) = \beta_4^{-1} \left( \beta_4(\theta_1 \cdot (\alpha_3 \circ \beta_3)(\theta_3)) \right) = \theta_1 \cdot (\alpha_3 \circ \beta_3)(\theta_3) \]
\[ = (\alpha_1 \circ \beta_1)(\theta_1) \cdot \theta_3. \]
Thus, \( \left( m_2 \circ X \right)(\theta_1) \cdot \theta_3 = \beta_0(1) \cdot \theta_1 \cdot \theta_3 - (\alpha_1 \circ \beta_1)(\theta_1) \cdot \theta_3 \) and \( X \) satisfies 2.4.(b).

5. There exists a homomorphism \( X \) which satisfies 2.4.(a), 2.4.(b), and 2.4.(c).

Lemma 5.1 is the main result in this section; its proof is given in 5.7.

**Lemma 5.1.** Adopt the notation of 2.1 and 2.3. Let \( X : M_1 \rightarrow M_2 \) be the homomorphism of Lemma 4.3. Then there is a homomorphism \( U : M_1 \rightarrow M_3 \) such that \( X' = X - m_3 \circ U \) satisfies 2.4.(a), 2.4.(b), and 2.4.(c).
The map $u$ of Observation 5.2 is a first approximation of the map $U$ which is promised in Lemma 5.1. The map $u$ will be modified in Lemma 5.4 and Definition 5.5.

**Observation 5.2.** Adopt the notation of 2.1 and 2.3. Let $X : M_1 \rightarrow M_2$ be the homomorphism of Lemma 4.3. Then there exists a homomorphism $u : K_1 \rightarrow M_3$ such that

$$X \circ \alpha_1 = m_3 \circ u : K_1 \rightarrow M_3.$$ 

**Proof.** Consider $\alpha_1$ followed by 2.4.(b):

$$m_2 \circ (X \circ \alpha_1) = (\beta_0(1) \cdot \text{id}_{M_1} - \alpha_1 \circ \beta_1) \circ \alpha_1.$$

Apply Observation 3.3.(a) to see that the right side of the previous equation is zero. It follows that $m_2 \circ (X \circ \alpha_1)$ is identically zero. The complex $\text{Hom}_p(K_1, M)$ is acyclic; hence there exists a homomorphism $u : K_1 \rightarrow M_3$ such that

$$X \circ \alpha_1 = m_3 \circ u : K_1 \rightarrow M_3. \quad \square$$

In order to modify $u$ (and $X$), we use the homotopy of Observation 4.2.(c). The homotopy map $h_3$ gave rise to the homomorphism $X : M_1 \rightarrow M_2$ of Lemma 4.3; but $h_3$ contains information about $X$ that we have not yet exploited.

**Observation 5.3.** The restriction of the map $c_4$ of Definition 4.1 to $D_2(\text{im}\alpha_2)$ is identically zero.

**Proof.** The map $k_4$ is an injection. It suffices to show that

$$(k_4 \circ c_4) \left((\alpha_2(\phi_2))^{(2)}\right) = 0,$$

for each $\phi_2 \in K_2$. The left side of (5.3.1) is

$$\beta_0(1) \cdot (k_4 \circ \beta_4) \left((\alpha_2(\phi_2))^{(2)}\right) - k_4 \left( ((\beta_2 \circ \alpha_2)(\phi_2))^{(2)} \right).$$

Apply (3.2.1), (1.5.1), assertions (c) and (a) of Observation 3.3, and the Divided Power axiom (1.3.1) to see that

$$\beta_0(1) \cdot (k_4 \circ \beta_4) \left((\alpha_2(\phi_2))^{(2)}\right) = \beta_0(1) \cdot (\beta_3 \circ m_4) \left((\alpha_2(\phi_2))^{(2)}\right)$$

$$= \beta_0(1) \cdot \beta_3 \left((m_2 \circ \alpha_2)(\phi_2) \cdot \alpha_2(\phi_2)\right) = \beta_0(1) \cdot \beta_3 \left((\alpha_1 \circ k_2)(\phi_2) \cdot \alpha_2(\phi_2)\right)$$

$$= \beta_0(1) \cdot (\beta_1 \circ \alpha_1 \circ k_2)(\phi_2) \wedge \phi_2 = \beta_0(1)^2 \cdot k_2(\phi_2) \wedge \phi_2$$

$$= \beta_0(1)^2 \cdot k_4(\phi_2^{(2)}) = k_4 \left( ((\beta_0(1)(\phi_2))^{(2)}\right)$$

$$= k_4 \left( ((\beta_2 \circ \alpha_2)(\phi_2))^{(2)}\right).$$

Thus, (5.3.1) is established and the proof is complete. \quad \square

The serious work in this section is done in the proof of Lemma 5.4.
Lemma 5.4. Adopt the notation of 2.1 and 2.3 and let \( u \) be the homomorphism of Observation 5.2. Then there exists a homomorphism \( v : K_1 \to M_4 \) such that the homomorphism

\[(5.4.1) \quad u' = (u + m_4 \circ v) : K_1 \to M_3,\]

satisfies

\[u'(\phi_1) \cdot \alpha_1(\phi_1') + u'(\phi_1') \cdot \alpha_1(\phi_1) = 0 \quad \text{and} \quad u'(\phi_1) \cdot \alpha_1(\phi_1) = 0,\]

for all \( \phi_1, \phi_1' \) in \( K_1 \).

Proof. Let \( \phi_1 \) and \( \phi_1' \) be elements of \( K_1 \). Consider the element \( (\alpha_2(\phi_1 \land \phi_1'))^{(2)} \) of \( D_2M_2 \). Observe that

\[
0 = (\beta_4^{-1} \circ c_4) \left( (\alpha_2(\phi_1 \land \phi_1'))^{(2)} \right),
\]

by Obs. 5.3,

\[
= (\beta_4^{-1} \circ h_3 \circ b_4) \left( (\alpha_2(\phi_1 \land \phi_1'))^{(2)} \right),
\]

by Obs. 4.2.(iii),

\[
= (\beta_4^{-1} \circ h_3) \left( (m_2 \circ \alpha_2)((\phi_1 \land \phi_1') \oslash \alpha_2(\phi_1 \land \phi_1')) \right),
\]

by (4.1.1),

\[
= X \left( (m_2 \circ \alpha_2)((\phi_1 \land \phi_1')) \right) \cdot \alpha_2(\phi_1 \land \phi_1'),
\]

by (4.3.1),

\[
= (X \circ \alpha_1)(\phi_1 \land \phi_1') \cdot \alpha_2(\phi_1 \land \phi_1'),
\]

by (3.2.1),

\[
= ((m_3 \circ u)(k_2(\phi_1 \land \phi_1')))(\alpha_2(\phi_1 \land \phi_1')),
\]

by Obs. 5.2,

\[
= (u(k_2(\phi_1 \land \phi_1')))(((m_2 \circ \alpha_2)(\phi_1 \land \phi_1'))),
\]

by (3.1.1),

\[
= (u(k_2(\phi_1 \land \phi_1')))(((\alpha_1 \circ k_2)(\phi_1 \land \phi_1'))),
\]

by (3.2.1).

The differential in the Koszul complex yields

\[
0 = (k_1(\phi_1) \cdot u(\phi_1') - k_1(\phi_1') \cdot u(\phi_1)) \cdot (k_1(\phi_1) \cdot \alpha_1(\phi_1') - k_1(\phi_1') \cdot \alpha_1(\phi_1));
\]

hence,

\[
0 = \begin{cases} 
  k_1(\phi_1) \cdot k_1(\phi_1) \cdot (u(\phi_1') \cdot \alpha_1(\phi_1')) \\
  -k_1(\phi_1') \cdot k_1(\phi_1) \cdot (u(\phi_1) \cdot \alpha_1(\phi_1') + u(\phi_1') \cdot \alpha_1(\phi_1)) \\
  +k_1(\phi_1') \cdot k_1(\phi_1') \cdot (u(\phi_1) \cdot \alpha_1(\phi_1)).
\end{cases}
\]

for all \( \phi_1 \) and \( \phi_1' \) in \( K_1 \).

Thus,

\[
(k_1(\phi_1))^2 \cdot (u(\phi_1') \cdot \alpha_1(\phi_1')) \subseteq (k_1(\phi_1'))M_4
\]

for all \( \phi_1 \) and \( \phi_1' \) in \( K_1 \). The image of \( k_1 \) is an ideal in \( P \) of grade 4 and \( M_4 \) is isomorphic to \( P \). Assume that \( \phi_1' \) is an element of \( K_1 \) with \( k_1(\phi_1') \) a regular element in \( P \). In this case every associated prime of \( P/(k_1(\phi_1')) \) has grade one; and therefore,

\[
u(\phi_1') \cdot \alpha_1(\phi_1') \in (k_1(\phi_1'))M_4.
\]
Lemma 1.7 guarantees that $K_1$ has a basis $\phi_{1,1}, \phi_{1,2}, \phi_{1,3}, \phi_{1,4}$ with the property that $k_1(\phi_{1,i})$ is a regular element of $P$ for each $i$. For each $i$, we identify an element $v(\phi_{1,i}) \in M_4$ with

$$u(\phi_{1,i}) \cdot \alpha_1(\phi_{1,i}) = k_1(\phi_{1,i}) \cdot v(\phi_{1,i}).$$

(5.4.3)

Extend $v$ to be a homomorphism $v : K_1 \to M_4$. Take $a$ and $b$ from the set

$$\{\phi_{1,1}, \phi_{1,2}, \phi_{1,3}, \phi_{1,4}\}$$

and rewrite (5.4.2) as

$$0 = \begin{cases} +k_1(a)k_1(a) \cdot (k_1(b) \cdot v(b)) \\ -k_1(b)k_1(a) \cdot (u(a) \cdot \alpha_1(b) + u(b) \cdot \alpha_1(a)) \\ +k_1(b)k_1(b) \cdot (k_1(a) \cdot v(a)) \end{cases}$$

Use the fact that $k_1(a)$ and $k_1(b)$ are regular elements of $P$ in order to see that

$$0 = k_1(a) \cdot v(b) - (u(a) \cdot \alpha_1(b) + u(b) \cdot \alpha_1(a)) + k_1(b) \cdot v(a).$$

In other words,

$$u(a) \cdot \alpha_1(b) + u(b) \cdot \alpha_1(a) = k_1(a) \cdot v(b) + k_1(b) \cdot v(a)$$

$$= -m_4(v(b)) \cdot \alpha_1(a) - m_4(v(a)) \cdot \alpha_1(b),$$

for all $a, b \in \{\phi_{1,1}, \phi_{1,2}, \phi_{1,3}, \phi_{1,4}\}$. (The last equality uses the product rule of (3.1.1) and the equality $m_1 \circ \alpha_1 = k_1$ of the Commutative Diagram (3.2.1).) Similarly, we deduce directly from (5.4.3) that

$$u(a) \cdot \alpha_1(a) = -m_4(v(a)) \cdot \alpha_1(a),$$

for $a \in \{\phi_{1,1}, \phi_{1,2}, \phi_{1,3}, \phi_{1,4}\}$. Thus,

$$u(\phi_{1,i}) \cdot \alpha_1(\phi_{1,i}) = 0 \quad \text{and} \quad (u + m_4 \circ v)(\phi_{1,i}) \cdot \alpha_1(\phi_{1,i}) = 0,$$

(5.4.4)

for all $a, b \in \{\phi_{1,1}, \phi_{1,2}, \phi_{1,3}, \phi_{1,4}\}$. It now follows that (5.4.4) holds for all $a$ and $b$ in $K_1$. \hfill \Box

**Definition 5.5.** Adopt the notation of 2.1 and 2.3 and let $u'$ be the homomorphism of (5.4.1). Define

$$U : M_1 \to M_3$$

by

$$\begin{cases} U(\alpha_1(\phi_{1,i})) = u'(\phi_{1,i}), & \text{if } \phi_{1,i} \in K_1, \\ U(\theta_{1,2}) \cdot \alpha_1(\phi_{1,i}) = -u'(\phi_{1,i}) \cdot \theta_{1,2}, & \text{if } \theta_{1,2} \in M_{1,2} \text{ and } \phi_{1,i} \in K_1, \text{ and} \\ U(\theta_{1,2}) \cdot M_{1,2} = 0 & \text{if } \theta_{1,2} \in M_{1,2}. \end{cases}$$

**Remarks 5.6.** (a) Notice that

$$U(\theta_{1}) \cdot \theta'_{1} + U(\theta'_{1}) \cdot \theta_{1} = 0,$$

(5.6.1)

for all $\theta_{1}, \theta'_{1} \in M_1$. 


(b) Recall from 2.3 that $M_1 = M_{1,1} \oplus M_{1,2}$ and $\alpha_1 : K_1 \to M_{1,1}$ is an isomorphism. It follows that $U$ is a well-defined homomorphism on all of $M_1$.

5.7. Proof of Lemma 5.1. Let $U$ be the homomorphism of Definition 5.5 and let

\[(5.7.1) \quad X' = X - m_3 \circ U.\]

We prove that the homomorphism $X'$ of (5.7.1) satisfies hypotheses 2.4.(a), 2.4.(b), and 2.4.(c). Hypothesis 2.4.(b) holds because

\[m_2 \circ X' = m_2 \circ (X - m_3 \circ U) = m_2 \circ X.\]

Hypothesis 2.4.(c) holds because

\[\left( m_3 \circ U \circ m_2 + m_3 \circ (m_3 \circ U) \right) (\theta_2) \cdot \theta_2' = U(m_2 \theta_2) \cdot m_2(\theta_2') + U(m_2 \theta_2') \cdot m_2(\theta_2) = 0.\]

(The first equality uses (3.1.1), (2.3.9), and the graded-commutativity of $M$; the second uses (5.6.1).) It follows that

\[X' \circ m_2 + m_3 \circ (X') \downarrow = (X - m_3 \circ U) \circ m_2 + m_3 \circ (X - m_3 \circ U) \downarrow = X \circ m_2 + m_3 \circ X \downarrow.\]

Hypothesis 2.4.(a) holds because

\[X' \circ \alpha_1 = X \circ \alpha_1 - m_3 \circ U \circ \alpha_1, \quad \text{by (5.7.1)}, \]

\[= X \circ \alpha_1 - m_3 u', \quad \text{by Def. 5.5}, \]

\[= X \circ \alpha_1 - m_3 (u + m_4 \circ v), \quad \text{by (5.4.1)}, \]

\[= X \circ \alpha_1 - m_3 u, \quad \text{because $M$ is a complex}, \]

\[= 0, \quad \text{by Obs. 5.2}.\]

This completes the proof of Lemma 5.1. \qed

6. The map $X$ of Theorem 2.4 exists.

In Lemma 5.1 we produced a map $X : M_1 \to M_2$ which satisfies hypotheses 2.4.(a), 2.4.(b), and 2.4.(c). In Lemma 6.2 we show the $X$ also satisfies 2.4.(d), and 2.4.(e). No further modification is needed.

Lemma 6.1. Adopt the notation of 2.1 and 2.3. If $X : M_1 \to M_2$ is a homomorphism which satisfies 2.4.(a), then

\[\text{ker} m_3 \cap \text{im} X \downarrow = 0.\]

Proof. The complex $M$ is acyclic; so it suffices to prove that

\[\text{im} m_4 \cap \text{im} X \downarrow = 0.\]

Suppose that $\theta_4 \in M_4$ and $\theta_2 \in M_2$ with

\[(6.1.1) \quad m_4(\theta_4) = X \uparrow(\theta_2).\]
Let $\phi_1$ be an arbitrary element of $K_1$. Observe that
\begin{align*}
-k_1(\phi_1) \cdot \theta_4 &= -m_1(\alpha_1(\phi_1)) \cdot \theta_4, & \text{by (3.2.1)}, \\
&= m_4(\theta_4) \cdot \alpha_1(\phi_1), & \text{by (3.1.1)}, \\
&= X^\dagger(\theta_2) \cdot \alpha_1(\phi_1), & \text{by (6.1.1)}, \\
&= \theta_2 \cdot (X \circ \alpha_1)(\phi_1), & \text{by (2.3.9)}, \\
&= 0, & \text{by 2.4.(a)}.
\end{align*}
Thus, the ideal im$k_1$, which has positive grade, annihilates the element of $\theta_4$ of $M_4$. Recall that the module $M_4$ is isomorphic to $P$. It follows that $\theta_4$ is zero. □

**Lemma 6.2.** If $X : M_1 \to M_2$ is a homomorphism which satisfies 2.4.(a), 2.4.(b), and 2.4.(c), then $X$ also satisfies 2.4.(d) and 2.4.(e).

**Proof.** We first prove 2.4.(e). Consider $\alpha_2$ followed by 2.4.(c):
\begin{equation}
X \circ m_2 \circ \alpha_2 + m_3 \circ X^\dagger \circ \alpha_2 = (\beta_0(1) \cdot \id_{M_2} - \alpha_2 \circ \beta_2) \circ \alpha_2.
\end{equation}
The right side of (6.2.1) is zero by Observation 3.3.(a); the composition $X \circ m_2 \circ \alpha_2$ is equal to
\begin{equation*}
X \circ \alpha_1 \circ k_2 = 0
\end{equation*}
by (3.2.1) and 2.4.(a). Thus, equation (6.2.1) yields
\begin{equation*}
m_3 \circ X^\dagger \circ \alpha_2 = 0;
\end{equation*}
and therefore,
\begin{equation*}
\im(X^\dagger \circ \alpha_2) \subseteq \ker m_3 \cap \im X^\dagger = 0
\end{equation*}
by Lemma 6.1. This establishes 2.4.(e).

Now we prove 2.4.(d). Consider $X$ followed by 2.4.(c):
\begin{equation}
X \circ m_2 \circ X + m_3 \circ X^\dagger \circ X = \beta_0(1) \cdot \id_{M_2} \circ X - \alpha_2 \circ \beta_2 \circ X.
\end{equation}
Observe that
\begin{equation*}
\alpha_2 \circ \beta_2 \circ X = 0.
\end{equation*}
Indeed,
\begin{align*}
[\theta_2 \cdot (\alpha_2 \circ \beta_2)(X(\theta_1))]_M &= [\beta_2(\theta_2) \wedge \beta_2(X(\theta_1))]_K, & \text{by (2.3.4)}, \\
&= [\beta_2(X(\theta_1)) \wedge \beta_2(\theta_2)]_K, & \text{because $K$ is graded-commutative}, \\
&= [X(\theta_1) \cdot (\alpha_2 \circ \beta_2)(\theta_2)]_M, & \text{by (2.3.4)}, \\
&= ([\alpha_2 \circ \beta_2](\theta_2) \cdot X(\theta_1))_M, & \text{because $M$ is graded-commutative}, \\
&= [(X^\dagger \circ \alpha_2 \circ \beta_2)(\theta_2) \cdot \theta_1]_M, & \text{by (2.3.9)}, \\
&= 0, & \text{by 2.4.(e)}.
\end{align*}
Apply 2.4.(b) and 2.4.(a) to see that
\begin{equation*}
X \circ m_2 \circ X = X \circ (\beta_0(1) \cdot \id_{M_1} - \alpha_1 \circ \beta_1) = \beta_0(1) \cdot X.
\end{equation*}
Thus, equation (6.2.2) is
\[ \beta_0(1) \cdot X + m_3 \circ X^\dagger \circ X = \beta_0(1) \cdot X \]
or \(m_3 \circ X^\dagger \circ X = 0\). It follows that
\[ \text{im}(X^\dagger \circ X) \subseteq \ker m_3 \cap \text{im} X^\dagger = 0 \]
by Lemma 6.1. This establishes 2.4.(d).

**Theorem 6.3.** Adopt the language of 2.3. Then there exists a map
\[ X : M_1 \rightarrow M_2 \]
such that the hypotheses of Theorem 2.4 hold.

**Proof.** Apply Lemma 5.1 followed by Lemma 6.2.

**7. Further Properties of X.**

We continue Section 3. Now that we have proven that the map \( X \) of Theorem 2.4 exists, we deduce further properties of \( X \). These formulas, together with those of Section 3, provide the proofs of Theorems 2.4 and 9.1. There are many of these formulas; but each proof is straightforward.

**Observation 7.1.** The map \( X \) of Theorem 2.4 satisfies the following identities:

(a) \( \beta_3 \circ X^\dagger = 0 \),
(b) \( \beta_2 \circ X = 0 \),
(c) \( w_3 \circ X^\dagger = 0 \),
(d) \( \text{im} X^\dagger \subseteq M_{3,2} \), and
(e) \( X^\dagger \circ m_3 + \alpha_3 \circ \beta_3 = \beta_0(1) \cdot \text{id}_{M_3} \).

**Proof.** (a) Use (2.3.4), (2.3.9), and Hypothesis 2.4.(a) to see that
\[ [\beta_3 \circ X^\dagger](\theta_2) \wedge \phi_1]_K = [X^\dagger(\theta_2) \cdot \alpha_1(\phi_1)]_M = [\theta_2 \cdot (X \circ \alpha_1)(\phi_1)]_M = 0. \]
(b) Use (2.3.4), the graded-commutativity of \( M \), (2.3.9), and Hypothesis 2.4.(e) to see that
\[ [\beta_2 \circ X](\theta_1) \wedge \phi_2]_K = [X(\theta_1) \cdot \alpha_2(\phi_2)]_M = [\alpha_2(\phi_2) \cdot X(\theta_1)]_M = 0. \]
(c) Apply Definition 2.3.10, (2.3.9), and Hypothesis 2.4.(a) to see that
\[ (w_3 \circ X^\dagger)(\theta_2) = (X^\dagger(\theta_2)) \cdot \alpha_1(\sigma) = \theta_2 \cdot X(\alpha_1(\sigma)) = 0. \]
(d) It suffices to show that \( X^\dagger(\theta_2) \cdot \alpha_1(\phi_1) = 0 \) and this is obvious from the definition of \( \dagger \) and 2.4.(a) as shown in the proof of (c).
(e) Observe that
\[(X^\dagger \circ m_3)(\theta_3) \cdot \theta_1 = m_3(\theta_3) \cdot X(\theta_1),\]
by (2.3.9),
\[= \theta_3 \cdot (m_2 \circ X)(\theta_1),\]
by (3.1.1),
\[= \theta_3 \cdot (\beta_0(1) \cdot \theta_1 - (\alpha_1 \circ \beta_1)(\theta_1)),\]
by 2.4.(b),
\[= (\beta_0(1) \cdot \text{id}_{M_3} - \alpha_3 \circ \beta_3)(\theta_3) \cdot \theta_1,\]
by 3.3.(b).

\[\square\]

**Lemma 7.2.** In the language of Definition 2.3 and Theorem 2.4, the following identities hold:

(a) \(w_2 \circ X = X^\dagger \circ w_1,\)
(b) \(Y \circ w_1|_{M_{1,2}} = 0,\) and
(c) \((\beta_2 \circ w_1 + Y \circ X)|_{M_{1,2}} = 0,\)
(d) \(w_1 \circ \text{proj}_{M_{1,2}} \circ m_2 + \alpha_2 \circ Y = w_1 \circ m_2,\)
(e) \(W \circ \beta_2 + m_3 \circ \text{proj}_{M_{3,2}} \circ w_2 = m_3 \circ w_2,\)
(f) \(\text{proj}_{M_{3,2}} \circ w_2 \circ W = 0,\)
(g) \(\text{proj}_{M_{3,2}} \circ (X^\dagger \circ W + w_2 \circ \alpha_2) = 0,\) and
(h) \(\beta_2 \circ W - Y \circ \alpha_2 = k_1(\sigma) \cdot \text{id}_{K_2}.\)

**Proof.** (a) We prove
\[
\text{im}(w_2 \circ X - X^\dagger \circ w_1) \subseteq (\ker m_3 \cap \ker \beta_3)
\]
and then apply Lemma 3.4. Observe that \(\beta_3 \circ X^\dagger = 0\) by Observation 7.1.(a) and
\[
\beta_3 \circ w_2 \circ X = z_2 \circ \beta_2 \circ X = 0
\]
by Observation 3.5 and Observation 7.1.(b). It follows that
\[
\text{im}(w_2 \circ X - X^\dagger \circ w_1) \subseteq \ker \beta_3.
\]
We complete the proof by showing that \(\text{im}(w_2 \circ X - X^\dagger \circ w_1) \subseteq \ker m_3.\) Observe that
\[
m_3 \circ (w_2 \circ X - X^\dagger \circ w_1)
\]
\[
= w_1 \circ m_2 \circ X + (k_1(\sigma)) \cdot X - m_3 \circ X^\dagger \circ w_1,\]
by 3.6.(b),
\[
= w_1 \circ (\beta_0(1) \cdot \text{id}_{M_3} - \alpha_1 \circ \beta_1) + (k_1(\sigma)) \cdot X - m_3 \circ X^\dagger \circ w_1,\]
by 2.4.(b),
(Use Observation 3.5 twice to see that \(w_1 \circ \alpha_1 \circ \beta_1 = \alpha_2 \circ z_1 \circ \beta_1 = \alpha_2 \circ \beta_2 \circ w_1.\))
\[
= \left(\beta_0(1) \cdot \text{id}_{M_2} - \alpha_2 \circ \beta_2 - m_3 \circ X^\dagger\right) \circ w_1 + (k_1(\sigma)) \cdot X
\]
\[
= X \circ m_2 \circ w_1 + (k_1(\sigma)) \cdot X,\]
by 2.4.(c),
(Use Observation 3.6.(b), again, to see that \(m_2 \circ w_1 = w_0 \circ m_1 - k_1(\sigma) \cdot \text{id}_{M_1}.\))
\[
= X \circ w_0 \circ m_1 = 0,\]
by 2.4.(a),
since \(w_0(1) = \alpha_1(\sigma).\)
(b) If \( \theta_{1,2} \in M_{1,2} \), then apply Definition 2.3.10 twice to see that
\[
(Y \circ w_1)(\theta_{1,2}) = (z_1 \circ (\text{proj}_{M_{1,1}} \circ \alpha_1)^{-1} \circ \text{proj}_{M_{1,1}} \circ m_2)(\theta_{1,2} \cdot \alpha_1(\sigma)).
\]
The product rule yields that
\[
m_2(\theta_{1,2} \cdot \alpha_1(\sigma)) = m_1(\theta_{1,2}) \cdot \alpha_1(\sigma) - m_1(\alpha_1(\sigma)) \cdot \theta_{1,2}.
\]
The projection map \( \text{proj}_{M_{1,1}} \) acts like the identity map on \( \alpha_1(\sigma) \) and like the zero map on \( \theta_{1,2} \). Thus
\[
(Y \circ w_1)(\theta_{1,2}) = m_1(\theta_{1,2}) \cdot z_1 \circ (\text{proj}_{M_{1,1}} \circ \alpha_1)^{-1}(\alpha_1(\sigma))
\]
\[
= m_1(\theta_{1,2}) \cdot \sigma \wedge \sigma = 0.
\]

(c) Let \( \theta_{1,2} \in M_{1,2} \). According to Definition 2.3.10,
\[
(Y \circ X)(\theta_{1,2}) = (z_1 \circ (\text{proj}_{M_{1,1}} \circ \alpha_1)^{-1} \circ \text{proj}_{M_{1,1}} \circ m_2 \circ X)(\theta_{1,2}).
\]
Apply Hypothesis 2.4.(b) to write
\[
m_2 \circ X(\theta_{1,2}) = \beta_0(1) \cdot \theta_{1,2} - (\alpha_1 \circ \beta_1)(\theta_{1,2}).
\]
Recall that \( \text{proj}_{M_{1,1}} \) sends \( \theta_{1,2} \) to zero and acts like the identity map on the image of \( \alpha_1 \). It follows that
\[
(Y \circ X)(\theta_{1,2}) = - (z_1 \circ (\text{proj}_{M_{1,1}} \circ \alpha_1)^{-1} \circ \alpha_1 \circ \beta_1)(\theta_{1,2})
\]
\[
= - (z_1 \circ \beta_1)(\theta_{1,2})
\]
\[
= - (\beta_2 \circ w_1)(\theta_{1,2}), \quad \text{by Obs. 3.5.}
\]

(d) Use the definition of \( Y \), given in 2.3.10, and the Commutative Diagram 3.5 to see that
\[
\alpha_2 \circ Y = \alpha_2 \circ z_1 \circ (\text{proj}_{M_{1,1}} \circ \alpha_1)^{-1} \circ \text{proj}_{M_{1,1}} \circ m_2
\]
\[
= w_1 \circ \alpha_1 \circ (\text{proj}_{M_{1,1}} \circ \alpha_1)^{-1} \circ \text{proj}_{M_{1,1}} \circ m_2.
\]
The map \( \alpha_1 \circ (\text{proj}_{M_{1,1}} \circ \alpha_1)^{-1} \) is the identity on \( M_{1,1} \). Thus,
\[
(7.2.1) \quad \alpha_2 \circ Y = w_1 \circ \text{proj}_{M_{1,1}} \circ m_2.
\]

(e) The definition of \( W \) is given in 2.3.10. Observe that
\[
W \circ \beta_2 + m_3 \circ \text{proj}_{M_{3,2}} \circ w_2
\]
\[
= (m_3 \circ (\beta_3|_{M_{3,1}})^{-1} \circ z_2) \circ \beta_2 + m_3 \circ \text{proj}_{M_{3,2}} \circ w_2
\]
\[
= m_3 \circ (\beta_3|_{M_{3,1}})^{-1} \circ \beta_3 \circ w_2 + m_3 \circ \text{proj}_{M_{3,2}} \circ w_2, \quad \text{by Observation 3.5,}
\]
\[
= m_3 \circ (\text{proj}_{M_{3,1}} + \text{proj}_{M_{3,2}}) \circ w_2 = m_3 \circ w_2, \quad \text{by Observation 3.3.(e).}
\]

(f) Recall the definition of \( W \) from 2.3.10. We calculate the value of
\[
\text{proj}_{M_{3,2}} \circ w_2 \circ W = \text{proj}_{M_{3,2}} \circ w_2 \circ m_3 \circ (\beta_3|_{M_{3,1}})^{-1} \circ z_2.
\]
Apply 3.6.(b) to write
\[ w_2 \circ m_3 = m_4 \circ w_3 + k_1(\sigma) \cdot \text{id}_{M_3}. \]

Recall from Observation 3.3.(f) that \( w_3 \circ (\beta_3|_{M_{3,1}})^{-1} \circ z_2 = 0 \). Observe also, that the image of \( \text{id}_{M_3} \circ (\beta_3|_{M_{3,1}})^{-1} \) is contained in \( M_{3,1} \); hence
\[ \text{proj}_{M_{3,2}} \circ \text{id}_{M_3} \circ (\beta_3|_{M_{3,1}})^{-1} \]
is the zero map.

(g) Observe that
\[
\text{proj}_{M_{3,2}} \circ (X^\dagger \circ W) \\
= \text{proj}_{M_{3,2}} \circ (X^\dagger \circ m_3 \circ (\beta_3|_{M_{3,1}})^{-1} \circ z_2), \quad \text{by 2.3.10,}
\]
\[
= \text{proj}_{M_{3,2}} \circ \left( -(\alpha_3 \circ \beta_3 + \beta_0(1) \cdot \text{id}_{M_3}) \circ (\beta_3|_{M_{3,1}})^{-1} \circ z_2 \right), \quad \text{by 7.1.(e),}
\]
\[
= - \text{proj}_{M_{3,2}} \circ \alpha_3 \circ \beta_3 \circ (\beta_3|_{M_{3,1}})^{-1} \circ z_2,
\]
because \( \text{proj}_{M_{3,2}}(M_{3,1}) = 0 \). Use the fact that \( \beta_3 \circ (\beta_3|_{M_{3,1}})^{-1} = \text{id}_{K_3} \), together with Commutative Diagram 3.5, to see that
\[ \text{proj}_{M_{3,2}} \circ \alpha_3 \circ \beta_3 \circ (\beta_3|_{M_{3,1}})^{-1} \circ z_2 = \text{proj}_{M_{3,2}} \circ \alpha_3 \circ z_2 = \text{proj}_{M_{3,2}} \circ w_2 \circ \alpha_2. \]

(h) Observe that
\[ \beta_2 \circ W = \beta_2 \circ m_3 \circ (\beta_3|_{M_{3,1}})^{-1} \circ z_2, \quad \text{by 2.3.10,}
\]
\[ = k_3 \circ \beta_3 \circ (\beta_3|_{M_{3,1}})^{-1} \circ z_2, \quad \text{by (3.1),}
\]
\[ = k_3 \circ z_2
\]
and
\[ Y \circ \alpha_2 = z_1 \circ (\text{proj}_{M_{1,1}} \circ \alpha_1)^{-1} \circ \text{proj}_{M_{1,1}} \circ m_2 \circ \alpha_2, \quad \text{by 2.3.10,}
\]
\[ = z_1 \circ (\text{proj}_{M_{1,1}} \circ \alpha_1)^{-1} \circ \text{proj}_{M_{1,1}} \circ \alpha_1 \circ k_2, \quad \text{by (3.2.1),}
\]
\[ (7.2.2) \]
\[ = z_1 \circ k_2. \]

Apply Observation 3.6.(a) to conclude that
\[ \beta_2 \circ W - Y \circ \alpha_2 = k_3 \circ z_2 - z_1 \circ k_2 = k_1(\sigma) \cdot \text{id}_{K_2}. \]

\[ \square \]

**Lemma 7.3.** In the language of Definition 2.3 and Theorem 2.4, the following identities hold:

(a) \( (r\beta_2 - Y) \circ \alpha_2 + k_3 \circ z_2 = f \cdot \text{id}_{K_2} \),
(b) \( \text{proj}_{M_{1,2}} \circ m_2 \circ (rX - w_1)|_{M_{1,2}} = f \cdot \text{id}_{M_{1,2}} \),
(c) \( -w_3 \circ m_4 + r\alpha_4 \circ \beta_4 = f \cdot \text{id}_{M_4} \), and
(d) \( (rX - w_1)|_{M_{1,2}} \circ \text{proj}_{M_{1,2}} \circ m_2 + \alpha_2 \circ (r\beta_2 - Y) + m_3 \circ (rX^\dagger + w_2) = f \cdot \text{id}_{M_2} \).
Proof. (a) Recall, from Observation 3.3.(a), that $\beta_2 \circ \alpha_2 = \beta_0(1) \cdot \id_M$. We calculated in (7.2.2) that $Y \circ \alpha_2 = z_1 \circ k_2$. Recall from Observation 3.6.(a) that

$$-z_1 \circ k_2 + k_3 \circ z_2 = k_1(\sigma) \cdot \id_M.$$  

Use (2.3.6).

(b) If $\theta_{1,2} \in M_{1,2}$, then

$$(m_2 \circ X)_{\theta_{1,2}} = \beta_0(1) \cdot \theta_{1,2} - (\alpha_1 \circ \beta_1)(\theta_{1,2})$$

by Hypothesis 2.4.(b), and

$$(m_2 \circ w_1)_{\theta_{1,2}} = m_1(\theta_{1,2}) \cdot \alpha_1(\sigma) - k_1(\sigma) \cdot \theta_{1,2}$$

by Observation 3.6.(b). The projection map $\text{proj}_{M_{1,2}}$ acts like the identity on $\theta_{1,2}$ but annihilates the image of $\alpha_1$. Thus,

$$\text{proj}_{M_{1,2}} m_2 \circ (rX - w_1)_{\mid M_{1,2}} = r\beta_0(1) \cdot \text{id}_{M_{1,2}} + k_1(\sigma) \cdot \text{id}_{M_{1,2}} = f \cdot \text{id}_{M_{1,2}}.$$  

(c) Apply Observation 3.6.(b) to see that $-w_3 \circ m_4 = k_1(\sigma) \cdot \text{id}_{M_{4}}$. Let $\theta_4$ be an element of $M_{4}$. Notice that

$$(\alpha_4 \circ \beta_4)(\theta_4) = 1 \cdot (\alpha_4 \circ \beta_4)(\theta_4) = (\alpha_0 \circ \beta_0)(1) \cdot \theta_4 = \beta_0(1) \cdot \theta_4,$$

by Observation 3.3.(b). Hence,

$$-w_3 \circ m_4 + r\alpha_4 \circ \beta_4 = (k_1(\sigma) + r\beta_0(1)) \cdot \text{id}_{M_{4}} = f \cdot \text{id}_{M_{4}}.$$  

(d) Hypothesis 2.4.(a) states that $X_{\mid M_{1,1}}$ is identically zero; consequently,

$$X_{\mid M_{1,2}} \circ \text{proj}_{M_{1,2}} = X.$$  

Thus,

$$(rX - w_1)_{\mid M_{1,2}} \circ \text{proj}_{M_{1,2}} m_2 + \alpha_2 \circ (r\beta_2 - Y) + m_3 \circ (rX^\dagger + w_2)$$

$$= r(X \circ m_2 + \alpha_2 \circ \beta_2 + m_3 \circ X^\dagger) - w_1 \circ \text{proj}_{M_{1,2}} m_2 - \alpha_2 \circ Y + m_3 \circ w_2$$

$$= r\beta_0(1) \cdot \text{id}_{M_{2}} - w_1 \circ \text{proj}_{M_{1,2}} m_2 - \alpha_2 \circ Y + m_3 \circ w_2.$$  

The most recent equality is due to Hypothesis 2.4.(c). Recall from (7.2.1) that

$$\alpha_2 \circ Y = w_1 \circ \text{proj}_{M_{1,1}} m_2.$$  

It follows that

$$(rX - w_1)_{\mid M_{1,2}} \circ \text{proj}_{M_{1,2}} m_2 + \alpha_2 \circ (r\beta_2 - Y) + m_3 \circ (rX^\dagger + w_2)$$

$$= r\beta_0(1) \cdot \text{id}_{M_{2}} - w_1 \circ \text{proj}_{M_{1,2}} m_2 - \alpha_2 \circ Y + m_3 \circ w_2$$

$$= r\beta_0(1) \cdot \text{id}_{M_{2}} - w_1 \circ \text{proj}_{M_{1,2}} m_2 - w_1 \circ \text{proj}_{M_{1,1}} m_2 + m_3 \circ w_2$$

$$= r\beta_0(1) \cdot \text{id}_{M_{2}} - w_1 \circ m_2 + m_3 \circ w_2 = r\beta_0(1) \cdot \text{id}_{M_{2}} + k_1(\sigma) \cdot \text{id}_{M_{2}}$$

$$= f \cdot \text{id}_{M_{2}}.$$  

The penultimate equality is established in Observation 3.6.(b).  

\[\square\]
8. The Proof of Theorem 2.4.

8.1. The proof of Theorem 2.4. The proof follows quickly from the calculations of Sections 3 and 7.

(M.F.1) Observe first that \((g_{\text{even}}g_{\text{odd}})_{(1,1)}\) is equal to

\[
(rX - w_1)_{M_{1,2} \circ \text{proj}_{M_{1,2}} \circ m_2 + \alpha_2 \circ (r\beta_2 - Y) + m_3 \circ (rX^\dagger + w_2)
= f \cdot \text{id}_{M_2},
\]

by Lemma 7.3.(d). Observe further that

\[
(g_{\text{even}}g_{\text{odd}})_{(1,2)} = -\alpha_2 \circ k_3 + m_3 \circ \alpha_3 = 0, \quad \text{by (3.2.1)},
\]

\[
(g_{\text{even}}g_{\text{odd}})_{(1,3)} = m_3 \circ m_4 = 0, \quad \text{by (2.1.1)},
\]

\[
(g_{\text{even}}g_{\text{odd}})_{(2,1)} = z_2 \circ Y + r(-z_2 \circ \beta_2 + \beta_3 \circ w_2) + r^2 \beta_3 \circ X^\dagger = 0,
\]

by Observation 3.5 and Observation 7.1.(a). The homomorphism \(Y\) is defined in (2.3.10). The composition \(z_2 \circ Y\) is zero because \(z_2 \circ z_1 = 0\). Observe also that

\[
(g_{\text{even}}g_{\text{odd}})_{(2,2)} = r\beta_3 \circ \alpha_3 + z_2 \circ k_3 - k_4 \circ z_3 = f \cdot \text{id}_{K_3},
\]

by Observation 3.3.(a), Observation 3.6.(a), and (2.3.6),

\[
(g_{\text{even}}g_{\text{odd}})_{(2,3)} = r(\beta_3 \circ m_4 - k_4 \circ \beta_4) = 0, \quad \text{by (3.2.1)},
\]

\[
(g_{\text{even}}g_{\text{odd}})_{(3,1)} = -rw_3 \circ X^\dagger - w_3 \circ w_4 = 0, \quad \text{by 7.1.(c) and 3.5},
\]

\[
(g_{\text{even}}g_{\text{odd}})_{(3,2)} = -w_3 \circ \alpha_3 + \alpha_4 \circ z_3 = 0, \quad \text{by 3.5},
\]

\[
(g_{\text{even}}g_{\text{odd}})_{(3,3)} = -w_3 \circ m_4 + r\alpha_4 \circ \beta_4 = f \cdot \text{id}_{M_3}, \quad \text{by 7.3.(c)},
\]

\[
(g_{\text{odd}}g_{\text{even}})_{(1,1)} = \text{proj}_{M_{1,2}} \circ m_2 \circ (rX - w_1)_{M_{1,2}} = f \cdot \text{id}_{M_{1,2}}, \quad \text{by 7.3.(b)},
\]

\[
(g_{\text{odd}}g_{\text{even}})_{(1,2)} = \text{proj}_{M_{1,2}} \circ m_2 \circ \alpha_2 = \text{proj}_{M_{1,2}} \circ \alpha_1 \circ k_2 = 0,
\]

by (3.2.1) and (2.3.3),

\[
(g_{\text{odd}}g_{\text{even}})_{(1,3)} = \text{proj}_{M_{1,2}} \circ m_2 \circ m_3 = 0, \quad \text{by (2.1.1)},
\]

\[
(g_{\text{odd}}g_{\text{even}})_{(1,4)} = 0
\]

\[
(g_{\text{odd}}g_{\text{even}})_{(2,1)} = (r^2\beta_2 \circ X - r(\beta_2 \circ w_1 + Y \circ X) + Y \circ w_1)_{M_{1,2}} = 0,
\]

by Observation 7.1.(b), and items (c) and (b) of Lemma 7.2,

\[
(g_{\text{odd}}g_{\text{even}})_{(2,2)} = (r\beta_2 - Y) \circ \alpha_2 + k_3 \circ z_2 = f \cdot \text{id}_{K_2}, \quad \text{by 7.3.(a)},
\]

\[
(g_{\text{odd}}g_{\text{even}})_{(2,3)} = r(\beta_2 \circ m_3 - k_3 \circ \beta_3) - Y \circ m_3 = 0,
\]

by (3.2.1), 2.3.10, and (2.1.1),

\[
(g_{\text{odd}}g_{\text{even}})_{(2,4)} = k_3 \circ k_4 = 0, \quad \text{by (2.3.1)}
\]

\[
(g_{\text{odd}}g_{\text{even}})_{(3,1)} = r^2X^\dagger \circ X + r(w_2 \circ X - X^\dagger \circ w_1) - (w_2 \circ w_1) = 0,
\]
by Hypothesis 2.4.(d), Lemma 7.2.(a), and Observation 3.5,
\[(g_{odd}g_{even})(3,2) = (rX^\dagger + w_2) \circ \alpha_2 - \alpha_3 \circ z_2 = 0,\]
by Hypothesis 2.4.(e) and Observation 3.5,
\[(g_{odd}g_{even})(3,3) = r(X^\dagger \circ m_3 + \alpha_3 \circ \beta_3) + (w_2 \circ m_3 - m_4 \circ w_3) = f \cdot \text{id}_{M_1},\]
by Observation 7.1.(e) and Observation 3.6.(b),
\[(g_{odd}g_{even})(3,4) = -\alpha_3 \circ k_4 + m_4 \circ \alpha_4 = 0, \quad \text{by \,(3.2.1),}\]
\[(g_{odd}g_{even})(4,1) = 0,\]
\[(g_{odd}g_{even})(4,2) = -z_3 \circ z_2 = 0, \quad \text{by 3.5}\]
\[(g_{odd}g_{even})(4,3) = r(z_3 \circ \beta_3 - \beta_4 \circ w_3) = 0, \quad \text{by 3.5, and}\]
\[(g_{odd}g_{even})(4,4) = -z_3 \circ k_4 + r\beta_4 \circ \alpha_4 = f \cdot \text{id}_{K_4},\]
by Observation 3.6.(a) and Observation 3.3.(a).

(M.F.2) The product $g_{even}g_{odd}$ is equal to $rA + B + r^{-1}C$, where
\[
A = X \circ \text{proj}_{M_{1,2}} \circ m_2 + \alpha_2 \circ \beta_2 + m_3 \circ \text{proj}_{M_{2,2}} \circ X^\dagger, \]
\[
B = -w_1 \circ \text{proj}_{M_{1,2}} \circ m_2 - \alpha_2 \circ Y + W \circ \beta_2 + m_3 \circ \text{proj}_{M_{2,2}} \circ w_2, \quad \text{and}\]
\[
C = -W \circ Y.\]

Recall from Hypothesis 2.4.(a) and Observation 7.1.(d) that
\[XM_{1,1} = 0 \quad \text{and} \quad \text{im}X^\dagger \subset M_{3,2}.\]
It follows that $X \circ \text{proj}_{M_{1,2}} = X$ and $\text{proj}_{M_{3,2}} \circ X^\dagger = X^\dagger$. Thus,
\[
A = X \circ m_2 + \alpha_2 \circ \beta_2 + m_3 \circ X^\dagger = \beta_0(1) \cdot \text{id}_{M_2}.\]
The final equality is due to Hypothesis 2.4.(c).

The first two terms of $B$ add to $-w_1 \circ m_2$ and the last two terms add to $m_3 \circ w_2$ by items (d) and (e), respectively, of Lemma 7.2. Apply Observation 3.6.(b) to conclude that $B = k_{1}(\sigma) \cdot \text{id}_{M_2}$.

The composition $W \circ Y$ factors through $z_2 \circ z_1 = 0$ (see Definition 2.3.10); hence, $C = 0$ and
\[
g_{even}g_{odd} = (r\beta_0(1) + k_{1}(\sigma)) \cdot \text{id}_{M_2} = f \cdot \text{id}_{M_2}.\]
We compute the composition $g_{odd}g_{even}$. Observe that
\[(g_{odd}g_{even})_{1,1} = \text{proj}_{M_{1,2}} \circ m_2 \circ (rX - w_1)|_{M_{1,2}}.\]
Apply Hypothesis 2.4.(b) and Observation 3.6.(b) to write
\[ m_2 \circ X = \beta_0(1) \cdot \text{id}_{M_1} - \alpha_1 \circ \beta_1 \quad \text{and} \]
\[ -m_2 \circ w_1 = -w_0 \circ m_1 + k_1(\sigma) \cdot \text{id}_{M_1}. \]

Recall, from (2.3.3), that \( \text{proj}_{M_{1,2}} \circ \alpha_1 = 0 \). Notice that the image of \( w_0 \circ m_1 \) is contained in \( M_{1,1} \); hence, \( \text{proj}_{M_{1,2}} \circ w_0 \circ m_1 = 0 \). Thus,
\[ (g_{\text{odd}} g_{\text{even}})_{1,1} = (r\beta_0(1) + k_1(\sigma)) \cdot \text{id}_{M_{1,2}} = f \cdot \text{id}_{M_{1,2}}. \]

The map \((g_{\text{odd}} g_{\text{even}})_{1,2}\) is equal to
\[ \text{proj}_{M_{1,2}} \circ m_2 \circ (\alpha_2 + r^{-1}W). \]

Apply (3.2.1) and (2.3.3) to see that
\[ \text{proj}_{M_{1,2}} \circ m_2 \circ \alpha_2 = \text{proj}_{M_{1,2}} \circ \alpha_1 \circ k_2 = 0 \]
by (3.2.1) and (2.3.3). The equality \( m_2 \circ W = 0 \) follows immediately from the definition of \( W \) in 2.3.10. It follows that \((g_{\text{odd}} g_{\text{even}})_{1,2} = 0\).

The map \((g_{\text{odd}} g_{\text{even}})_{1,3}\) is
\[ \text{proj}_{M_{1,2}} \circ m_2 \circ m_3 |_{M_{3,2}} = 0. \]

The map \((g_{\text{odd}} g_{\text{even}})_{2,1}\) is
\[ r^2 \beta_2 \circ X |_{M_{1,2}} - r(\beta_2 \circ w_1 + Y \circ X) |_{M_{1,2}} + Y \circ w_1 |_{M_{1,2}} = 0 \]
by 7.1.(b), and items (c), and (b) of Lemma 7.2. Observe that
\[ (g_{\text{odd}} g_{\text{even}})_{2,2} = r\beta_2 \circ \alpha_2 + (\beta_2 \circ W - Y \circ \alpha_2) + r^{-1}Y \circ W \]
\[ = r\beta_0(1) \cdot \text{id}_{K_2} + k_1(\sigma) \cdot \text{id}_{K_2} + r^{-1} \cdot 0 = f \cdot \text{id}_{K_2} \]
by Observation 3.3.(a), Lemma 7.2.(h), and the fact that \( Y \circ W \) factors through \( m_2 \circ m_3 = 0; \)
see Definition 2.3.10. The map \((g_{\text{odd}} g_{\text{even}})_{2,3}\) equals
\[ r\beta_2 \circ m_3 |_{M_{3,2}} - Y \circ m_3 |_{M_{3,2}} = 0. \]

Indeed, \( \beta_2 \circ m_3 = K_3 \circ \beta_3 \) by Commutative Diagram 3.2.1, \( \beta_3 |_{M_{3,2}} = 0 \) by Observation 3.3.(d), and \( Y \circ m_3 \) factors through \( m_2 \circ m_3 = 0 \) by the definition of \( Y \), see 2.3.10. Observe that
\[ (g_{\text{odd}} g_{\text{even}})_{3,1} = \text{proj}_{M_{3,2}} \circ \left( r^2 X^\dagger \circ X + r(w_2 \circ X - X^\dagger \circ w_1) - w_2 \circ w_1 \right) |_{M_{1,2}} \]
and this is zero by Hypothesis 2.4.(d), Lemma 7.2.(a), and the fact that the rows Commutative Diagram 3.5 are complexes. Apply Hypothesis 2.4.(e) and items (f) and (g) of Lemma 7.2 to see that
\[ (g_{\text{odd}} g_{\text{even}})_{3,2} = \text{proj}_{M_{3,2}} \circ (rX^\dagger \circ \alpha_2 + (X^\dagger \circ W + w_2 \circ \alpha_2) + r^{-1}w_2 \circ W) \]
is equal to zero. The map \((g_{\text{odd}}g_{\text{even}})_{3,3}\) is equal to
\[
\text{proj}_{M_{3,2}} \circ (rX + w_2) \circ m_3|_{M_{3,2}}
\]
\[
= \text{proj}_{M_{3,2}} \circ (r(\beta_0(1) \cdot \text{id}_{M_3} - \alpha_3 \cdot \beta_3) + m_4 \circ w_3 + k_1(\sigma) \cdot \text{id}_{M_3})|_{M_{3,2}}
\]
by 7.1.(e) and 3.6.(b). Recall from Observation 3.3.(d) that \(\beta_3(M_{3,2}) = 0\). Recall, also, from 2.3.10 and 2.3.7, that \(w_3(M_{3,2}) \subseteq M_{3,2} \cdot M_{1,1} = 0\). It follows that \((g_{\text{odd}}g_{\text{even}})_{3,3}\) is
\[
\text{proj}_{M_{3,2}} \circ (r\beta_0(1) \cdot \text{id}_{M_3} + k_1(\sigma) \cdot \text{id}_{M_3})|_{M_{3,2}} = f \cdot \text{id}_{M_{3,2}}.
\]

9. The matrix factorization of Theorem 2.4 induces the infinite tail of the resolution of \(P/(f, \mathcal{R})\) by free \(P/(f)\) modules.

Let \(\mathcal{P}\) represent \(P/(f)\) and \(-\) represent the functor \(- \otimes P\).

**Theorem 9.1.** Adopt the language of Theorem 2.4. Then the following statements hold.

1. The maps and modules

\[
N : \cdots \overset{n_3}{\to} N_2 \overset{n_2}{\to} N_1 \overset{n_1}{\to} N_0
\]
form a resolution of \(\mathcal{P}/\mathfrak{R}\mathcal{P}\) by free \(\mathcal{P}\)-modules, where the modules of \(N\) are

\[
N_i = \begin{cases} 
K_0, & \text{if } i = 0, \\
K_1, & \text{if } i = 1, \\
M_{1,2} \oplus K_2, & \text{if } i = 2, \\
M_2 \oplus K_3, & \text{if } i = 3, \\
G_{\text{even}}, & \text{if } 4 \leq i \text{ and } i \text{ is even, and} \\
G_{\text{odd}}, & \text{if } 5 \leq i \text{ and } i \text{ is odd,}
\end{cases}
\]

and the differentials \(n_i\) are given by
\[
n_1 = -k_1; \\
n_2 = \left[ (r\beta_1 + z_0 \circ m_1)|_{M_{1,2}} \quad -k_2 \right]; \\
n_3 = \left[ \text{proj}_{M_{1,2}} \circ \overline{m_2} \quad 0 \\
\overline{r} \overline{\beta}_2 \overline{\sigma}_2 \quad -k_3 \right]; \\
n_4 = \left[ (r\overline{X} - \overline{w}_1)|_{M_{1,2}} \overline{\alpha}_2 \quad \overline{m}_3 \quad 0 \\
0 \quad -\overline{\sigma}_2 \quad \overline{r} \overline{\beta}_3 \quad -k_4 \right]; \\
n_i = \overline{g}_{\text{odd}}, & \text{if } 5 \leq i \text{ and } i \text{ is odd; and} \\
n_i = \overline{g}_{\text{even}}, & \text{if } 6 \leq i \text{ and } i \text{ is even.}
\]

2. If \(r\) is a unit, then the maps and modules

\[
\hat{N} : \cdots \overset{\hat{n}_3}{\to} \hat{N}_2 \overset{\hat{n}_2}{\to} \hat{N}_1 \overset{\hat{n}_1}{\to} \hat{N}_0
\]
form a resolution of \( \overline{P}/\overline{\mathfrak{g}P} \) by free \( \overline{P} \)-modules, where the modules of \( \tilde{N} \) are

\[
\tilde{N}_i = \begin{cases} 
K_0, & \text{if } i = 0, \\
K_1, & \text{if } i = 1, \\
M_{1,2} \oplus K_2, & \text{if } i = 2, \\
G_{\text{odd}}, & \text{if } 3 \leq i \text{ and } i \text{ is odd}, \\
G_{\text{even}}, & \text{if } 4 \leq i \text{ and } i \text{ is even, and}
\end{cases}
\]

and the differentials \( \hat{n}_i \) are given by

\[
\begin{align*}
\hat{n}_1 &= -k_1; \\
\hat{n}_2 &= \left( r\beta_1 + z_0 \circ m_1 \right)_{|M_{1,2}} - k_2; \\
\hat{n}_3 &= \left[ \text{proj}_{M_{1,2}} m_2 ight]_{|R^2 - \mathfrak{g}}; \\
\hat{n}_i &= \tilde{g}_{\text{even}}, \quad \text{if } 4 \leq i \text{ and } i \text{ is even}; \text{ and} \\
\hat{n}_i &= \tilde{g}_{\text{odd}}, \quad \text{if } 5 \leq i \text{ and } i \text{ is odd.}
\end{align*}
\]

**Proof.** The idea for this proof is inspired by the proof of [9, Lem. 2.3]. Recall the map of complexes \( \beta : M \to K \) of Observation 3.2. Consider the perturbation \( \beta' : M \to K \) of \( \beta \), where

\[
(9.1.3) \quad \beta'_i = \begin{cases} 
rb_i, & \text{for } 2 \leq i \leq 4, \\
r\beta_1 + z_0 \circ m_1, & \text{for } i = 1, \\
r\beta_0 + m_1 \circ w_0, & \text{for } i = 0,
\end{cases}
\]

for \( r \) defined in (2.3.5) and \( z_1 \) and \( w_0 \) defined in (2.3.10). In particular,

\[
(9.1.4) \quad \beta'_0(1) = r\beta_0(1) + k_1(\sigma) = f.
\]

It is easy to see that \( \beta' : M \to K \) is also a map of complexes. Indeed, the only interesting calculation occurs in the right most square; and this square commutes because

\[
m_1 \circ w_0 \circ m_1 = k_1 \circ z_0 \circ m_1,
\]

since

\[
m_1 \circ (w_0 \circ m_1) = m_1 \circ (k_1(\sigma) \cdot \text{id}_{M_1} + m_2 \circ w_1) = k_1(\sigma) \cdot m_1 \quad \text{by 3.6.(b), and}
\]

\[
(k_1 \circ z_0) \circ m_1 = (k_1(\sigma) \cdot \text{id}_{K_0}) \circ m_1 = k_1(\sigma) \cdot m_1 \quad \text{by 3.6.(a)}.
\]

Consider the short exact sequence

\[
0 \to P/(\mathfrak{g}: f) \to P/\mathfrak{g} \to P/(\mathfrak{g}, f) \to 0.
\]
The complexes \( M \) and \( K \) are resolutions of \( P/(\mathfrak{R} : f) \) and \( P/\mathfrak{R} \), respectively, by free \( P \)-modules. It follows that the mapping cone \( L \) of

\[
\begin{array}{ccccccc}
0 & \rightarrow & M_4 & \xrightarrow{m_4} & M_3 & \xrightarrow{m_3} & M_2 & \xrightarrow{m_2} & M_1 & \xrightarrow{m_1} & M_0 \\
& \downarrow{\rho_4} & \downarrow{\rho_3} & \downarrow{\rho_2} & \downarrow{\rho_1} & & & \downarrow{\rho_0} \\
0 & \rightarrow & K_4 & \xrightarrow{k_4} & K_3 & \xrightarrow{k_3} & K_2 & \xrightarrow{k_2} & K_1 & \xrightarrow{k_1} & K_0 \\
\end{array}
\]

is a resolution of \( P/(\mathfrak{R},f) \) by free \( P \)-modules. This resolution has the form

\[
L : 0 \rightarrow L_5 \xrightarrow{\ell_5} L_4 \xrightarrow{\ell_4} L_3 \xrightarrow{\ell_3} L_2 \xrightarrow{\ell_2} L_1 \xrightarrow{\ell_1} L_0,
\]

where

\[
L_5 = M_4, \quad L_4 = \oplus_{K_4}, \quad L_3 = \oplus_{K_3}, \quad L_2 = \oplus_{K_2}, \quad L_1 = \oplus_{K_1}, \quad L_0 = K_0,
\]

\[
\ell_5 = \begin{bmatrix} m_4 \\ \beta_4^1 \end{bmatrix}, \quad \ell_4 = \begin{bmatrix} m_3 & 0 \\ \beta_3^1 & -k_4 \end{bmatrix}, \quad \ell_3 = \begin{bmatrix} m_2 & 0 \\ \beta_2^1 & -k_3 \end{bmatrix}, \quad \ell_2 = \begin{bmatrix} m_1 & 0 \\ \beta_1^1 & -k_2 \end{bmatrix},
\]

and \( \ell_1 = \begin{bmatrix} \beta_0^1 & -k_1 \end{bmatrix} \). The element \( f \) of \( P \) is regular by hypothesis; hence \( \overline{L} \) is a complex with homology:

\[
H_i(\overline{L}) = \text{Tor}_i^P (\overline{\mathcal{P}}/\mathfrak{R}\overline{\mathcal{P}}, \overline{\mathcal{P}}) = \begin{cases} \overline{\mathcal{P}}/\mathfrak{R}\overline{\mathcal{P}}, & \text{if } i \text{ is 0 or 1, and} \\ 0, & \text{otherwise.} \end{cases}
\]

Furthermore, the cycle

(9.1.5)

\[
\xi = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

in \( \overline{L}_i \) represents a generator of \( H_1(\overline{L}) \). We kill the homology in \( \overline{L} \). Define \( P \)-module homomorphisms \( \rho_i : L_i \rightarrow L_{i+1} \) by

\[
\rho_4 = \begin{bmatrix} -w_3 & \alpha_4 \end{bmatrix}, \quad \rho_3 = \begin{bmatrix} -rX^1-w_2 & -\alpha_3 \\ 0 & -z_3 \end{bmatrix}, \quad \rho_2 = \begin{bmatrix} rX-w_1 & \alpha_2 \\ 0 & -z_2 \end{bmatrix},
\]

\[
\rho_1 = \begin{bmatrix} 0 & -\alpha_1 \\ 0 & -z_1 \end{bmatrix}, \quad \text{and} \quad \rho_0 = \begin{bmatrix} \alpha_0 \\ 0 \end{bmatrix}.
\]

It is shown in Lemma 9.2.(a) that

(9.1.6)

\[
\begin{array}{ccccccc}
0 & \rightarrow & L_5 & \xrightarrow{\ell_5} & L_4 & \xrightarrow{\ell_4} & L_3 & \xrightarrow{\ell_3} & L_2 & \xrightarrow{\ell_2} & L_1 & \xrightarrow{\ell_1} & L_0 & \rightarrow & 0 \\
& \downarrow{\rho_4} & \downarrow{\rho_3} & \downarrow{\rho_2} & \downarrow{\rho_1} & \downarrow{\rho_0} & & & & & & & \downarrow{\rho_0} \\
0 & \rightarrow & L_5 & \xrightarrow{\ell_5} & L_4 & \xrightarrow{\ell_4} & L_3 & \xrightarrow{\ell_3} & L_2 & \xrightarrow{\ell_2} & L_1 & \xrightarrow{\ell_1} & L_0 & \rightarrow & 0 \\
\end{array}
\]

is a map of complexes. It is clear that \( \rho_0 \) induces an isomorphism from \( H_0 \) of the top line of (9.1.6) to \( H_1 \) of the bottom line of (9.1.6). Let \( \overline{M} \) be the total complex of (9.1.6). We have shown that the homology of \( \overline{M} \) is concentrated in positions 0 and 3 and the \( \xi \) from (9.1.5) of the summand \( \overline{L}_1 \) in \( \overline{M}_3 = \overline{L}_1 \oplus \overline{L}_3 \) represents the \( \overline{L}_1 \)-component of a generator of \( H_3(\overline{M}) \). It is shown in Lemma 9.2.(b) that \( \overline{\rho}_1 \circ \overline{\rho}_0 = 0; \)
so indeed, the element $\xi$ of $M_3$ is a cycle of $M$. We kill the homology of $M$. In theory we need to give a map of complexes from $L[-4]$ to all of $M$; however, in practice, because of Lemma 9.2(b), it suffices to give a map of complexes from $L[-4]$ to the top line $L[-2]$ of (9.1.6). Iterate this process to see that $P/KP$ is resolved by the total complex $T$ of the infinite double complex given in Table 1.

We emphasize that it is shown in Lemma 9.2(b) that each column of Table 1 is a complex. Observe that the modules of $T$ are

$$T_i = \begin{cases} 
L_0 & \text{if } i = 0 \\
L_1 & \text{if } i = 1 \\
L_0 \oplus L_2, & \text{if } i = 2, \\
L_1 \oplus L_3, & \text{if } i = 3, \\
L_0 \oplus L_2 \oplus L_4, & \text{if } 4 \leq i \text{ and } i \text{ is even,} \\
L_1 \oplus L_3 \oplus L_5, & \text{if } 5 \leq i \text{ and } i \text{ is odd,}
\end{cases}$$

and the differential of $T$ is

$$t_1 = \ell_1, \quad t_2 = [p_0 \ell_2], \quad t_3 = \begin{bmatrix} \ell_1 & 0 & 0 \\
-p_1 & \ell_3 & 0 \\
0 & -p_3 & \ell_5
\end{bmatrix}, \quad t_4 = \begin{bmatrix} p_0 & \ell_2 & 0 \\
0 & p_2 & \ell_4 \\
0 & 0 & p_3
\end{bmatrix},$$

$$t_i = \begin{bmatrix} \ell_1 & 0 & 0 \\
-p_1 & \ell_3 & 0 \\
0 & -p_3 & \ell_5
\end{bmatrix}, \quad \text{if } 5 \leq i \text{ and } i \text{ is odd, and}

$$t_i = \begin{bmatrix} p_0 & \ell_2 & 0 \\
0 & p_2 & \ell_4 \\
0 & 0 & p_3
\end{bmatrix}, \quad \text{if } 6 \leq i \text{ and } i \text{ is even.}$$

In order to remove the parts of $T$ that obviously split off, we record $T$ explicitly and we employ the decomposition

$$M_1 = M_{1,1} \oplus M_{1,2}.$$
Thus, $T_i$ is equal to

\[
\begin{align*}
K_0, & \quad \text{if } i = 0, \\
M_0 \oplus K_1, & \quad \text{if } i = 1, \\
K_0 \oplus M_{1,1} \oplus M_{1,2} \oplus K_2, & \quad \text{if } i = 2, \\
M_0 \oplus K_1 \oplus M_2 \oplus K_3, & \quad \text{if } i = 3, \\
K_0 \oplus M_{1,1} \oplus M_{1,2} \oplus K_2 \oplus M_3 \oplus K_4, & \quad \text{if } 4 \leq i \text{ and } i \text{ is even}, \\
M_0 \oplus K_1 \oplus M_2 \oplus K_3 \oplus M_4, & \quad \text{if } 5 \leq i \text{ and } i \text{ is odd},
\end{align*}
\]

and the differentials $t_i$ are given by

\[
t_1 = \begin{bmatrix} \beta'_0 & -k_1 \end{bmatrix};
\]
\[
t_2 = \begin{bmatrix} \alpha_0 & m_{1|M_{1,1}} & m_{1|M_{1,2}} & 0 \\ 0 & \beta'_{1|M_{1,1}} & \beta'_{1|M_{1,2}} & -k_2 \end{bmatrix};
\]
\[
t_3 = \begin{bmatrix} \beta'_0 & -k_1 & 0 & 0 \\ 0 & \text{proj}_{M_{1,1}} \circ \alpha_1 & \text{proj}_{M_{1,1}} \circ m_2 & 0 \\ 0 & 0 & \text{proj}_{M_{1,2}} \circ m_2 & 0 \\ 0 & \bar{z}_4 & r \bar{\beta}_2 & -k_3 \end{bmatrix};
\]
\[
t_4 = \begin{bmatrix} \alpha_0 & m_{1|M_{1,1}} & m_{1|M_{1,2}} & 0 & 0 & 0 \\ 0 & \beta'_{1|M_{1,1}} & \beta'_{1|M_{1,2}} & -k_2 & 0 & 0 \\ 0 & -w_{1|M_{1,1}} & (rX - w_1)_{|M_{1,2}} & \bar{\alpha}_2 & m_3 & 0 \\ 0 & 0 & 0 & -\bar{z}_2 & r \bar{\beta}_3 & -k_4 \end{bmatrix};
\]
\[
t_i = \begin{bmatrix} \beta'_0 & -k_1 & 0 & 0 & 0 \\ 0 & \text{proj}_{M_{1,1}} \circ \alpha_1 & \text{proj}_{M_{1,1}} \circ m_2 & 0 & 0 \\ 0 & 0 & \text{proj}_{M_{1,2}} \circ m_2 & 0 & 0 \\ 0 & \bar{z}_4 & r \bar{\beta}_2 & -k_3 & 0 \\ 0 & 0 & (rX' + w_2) & \bar{\alpha}_3 & m_4 & \bar{z}_3 & r \bar{\beta}_4 \end{bmatrix} ,
\]

if $5 \leq i \text{ and } i \text{ is odd};$ and

\[
t_i = \begin{bmatrix} \alpha_0 & m_{1|M_{1,1}} & m_{1|M_{1,2}} & 0 & 0 & 0 \\ 0 & \beta'_{1|M_{1,1}} & \beta'_{1|M_{1,2}} & -k_2 & 0 & 0 \\ 0 & -w_{1|M_{1,1}} & (rX - w_1)_{|M_{1,2}} & \bar{\alpha}_2 & m_3 & 0 \\ 0 & 0 & 0 & -\bar{z}_2 & r \bar{\beta}_3 & -k_4 \end{bmatrix} ,
\]
if $6 \leq i$ and $i$ is even. The maps $\alpha_0$ and $\text{proj}_{M_{i,1}} \circ \overline{\alpha}$ are isomorphisms. One applies
elementary row and column operations to see that the complex $(\mathbb{T}, t)$ is isomorphic to the complex $(\mathbb{T}, t')$ where the differentials $t'_i$ are given by

$$t'_1 = \begin{bmatrix} 0 & -\overline{k}_1 \end{bmatrix};$$

$$t'_2 = \begin{bmatrix} \overline{\alpha}_0 & 0 & 0 & 0 \\ 0 & 0 & \overline{\beta}'_{M_{1,2}} & -\overline{k}_2 \end{bmatrix};$$

$$t'_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \text{proj}_{M_{1,1}} \circ \overline{\alpha}_1 & 0 & 0 \\ 0 & 0 & \text{proj}_{M_{1,2}} \circ \overline{m}_2 & 0 \\ 0 & 0 & \overline{r_3}\overline{\beta}_2 - \overline{Y} & -\overline{k}_3 \end{bmatrix};$$

$$t'_4 = \begin{bmatrix} \overline{\alpha}_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (r\overline{X} - \overline{w}_1)|_{M_{1,2}} & \overline{\alpha}_2 & \overline{m}_3 & 0 \\ 0 & 0 & 0 & -\overline{z}_2 & \overline{r_3}\overline{\beta}_3 & -\overline{k}_4 \end{bmatrix};$$

$$t'_i = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \text{proj}_{M_{1,1}} \circ \overline{\alpha}_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \text{proj}_{M_{1,2}} \circ \overline{m}_2 & 0 & 0 & 0 \\ 0 & 0 & \overline{r_3}\overline{\beta}_2 - \overline{Y} & -\overline{k}_3 & 0 & 0 \\ 0 & 0 & \overline{r_3}\overline{X} + \overline{w}_2 & \overline{\alpha}_3 & \overline{m}_4 & 0 \\ 0 & 0 & 0 & 0 & -\overline{z}_3 & \overline{r_3}\overline{\beta}_4 \end{bmatrix},$$

if $5 \leq i$ and $i$ is odd; and

$$t'_i = \begin{bmatrix} \overline{\alpha}_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (r\overline{X} - \overline{w}_1)|_{M_{1,2}} & \overline{\alpha}_2 & \overline{m}_3 & 0 \\ 0 & 0 & 0 & -\overline{z}_2 & \overline{r_3}\overline{\beta}_3 & -\overline{k}_4 \end{bmatrix},$$

if $6 \leq i$ and $i$ is even.

It is clear that the complex $N$ of (9.1.1) is a subcomplex of the resolution $(\mathbb{T}, t')$ and the inclusion map is a quasi-isomorphism. Thus, $N$ is a resolution. The completes the proof of statement 1.

The proof of statement 2 begins with the resolution $N$ from 1. The module $M_3$ is now written as $M_{3,1} \oplus M_{3,2}$. The differentials $n_1$, $n_2$, and $n_3$ are unchanged, and the other differentials are now written as follows:

$$n_4 = \begin{bmatrix} (r\overline{X} - \overline{w}_1)|_{M_{1,2}} & \overline{\alpha}_2 & \overline{m}_3|_{M_{3,1}} & \overline{m}_3|_{M_{3,2}} & 0 \\ 0 & -\overline{z}_2 & \overline{r_3}\overline{\beta}_3 & -\overline{k}_4 \end{bmatrix};$$
The maps and modules of (9.1.6) form a map of complexes.

Lemma 9.2.

(a) The maps and modules of (9.1.6) form a map of complexes.

(b) The maps and modules

\[
\begin{bmatrix}
\text{proj}_{M_{1,2}} m_2 & 0 & 0 \\
\alpha_2 & m_3|_{M_{3,1}} & m_3|_{M_{3,2}} \\
0 & -z_2 & r_{\beta_3}|_{M_{3,1}} \\
0 & 0 & -w_3|_{M_{3,1}} - w_3|_{M_{3,2}} \alpha_4
\end{bmatrix},
\]

if \(5 \leq i\) and \(i\) is odd; and

\[
\begin{bmatrix}
\text{proj}_{M_{1,2}} m_2 & 0 & 0 \\
\alpha_2 & m_3|_{M_{3,1}} & m_3|_{M_{3,2}} \\
0 & -z_2 & r_{\beta_3}|_{M_{3,1}} \\
0 & 0 & -w_3|_{M_{3,1}} - w_3|_{M_{3,2}} \alpha_4
\end{bmatrix},
\]

if \(6 \leq i\) and \(i\) is even. The map \(r_{\beta_3}|_{M_{3,2}}\) should appear in row 2, column 4 of the map \(n_i\), for even \(i\) with \(4 \leq i\). This map is zero according to Observation 3.3.(d).

Recall from (2.3.8) that \(r_{\beta_4} : M_4 \to K_4\) and \(r_{\beta_3}|_{M_{3,1}} : M_{3,1} \to K_3\) are isomorphisms. One uses elementary row and column operations, as was done above, to obtain a complex isomorphic to \(N\), which is quasi-isomorphic to \(\hat{N}\).

The two calculations in the next result were used in the proof of Theorem 9.1.

Proof. We compute in \(P\). Keep in mind that the image of \(f\) in \(\bar{P}\) is zero. Observe that

\[
(\ell_1 \circ \rho_0)_{1,1} = \beta'_0 \circ \alpha_0 = f \cdot \text{id}_{K_0}, \quad \text{by (9.1.4)};
\]

\[
(\rho_0 \circ \ell_1 - \ell_2 \circ \rho_1)_{1,1} = \alpha_0 \circ \beta'_0 = f \cdot \text{id}_{M_0}, \quad \text{by (9.1.4)};
\]

\[
(\rho_0 \circ \ell_1 - \ell_2 \circ \rho_1)_{1,2} = -\alpha_0 \circ k_1 + m_1 \circ \alpha_1 = 0, \quad \text{by (3.2.1)};
\]

\[
(\rho_0 \circ \ell_1 - \ell_2 \circ \rho_1)_{2,1} = 0;
\]

\[
(\rho_0 \circ \ell_1 - \ell_2 \circ \rho_1)_{2,2} = \beta'_1 \circ \alpha_1 - k_2 \circ z_1
\]

\[
= r_{\beta_0}(1) \cdot \text{id}_{K_1} + z_0 \circ k_1 - k_2 \circ z_1
\]

\[
= (r_{\beta_0}(1) + k_1(\sigma)) \cdot \text{id}_{K_1} = f \cdot \text{id}_{K_1},
\]

by 3.3.(a), (3.2.1), and 3.6.(a);

\[
(\rho_1 \circ \ell_2 - \ell_3 \circ \rho_2)_{1,1} = -\alpha_1 \circ \beta'_1 - m_2 \circ (rX - w_1)
\]

\[
= -r(\alpha_1 \circ \beta_1 + m_2 \circ X) - (w_0 \circ \alpha_0 \circ m_1 - m_2 \circ w_1)
\]

\[
= - (r_{\beta_0}(1) + k_1(\sigma)) \cdot \text{id}_{M_1} = - f \cdot \text{id}_{M_1},
\]
by 3.5, 2.4.(b), and 3.6.(b);

\[(\rho_1 \circ \ell_2 - \ell_3 \circ \rho_2)_{1,2} = \alpha_1 \circ k_2 - m_2 \circ \alpha_2 = 0, \quad \text{by (3.2.1)};\]

\[(\rho_1 \circ \ell_2 - \ell_3 \circ \rho_2)_{2,1} = -z_1 \circ \beta'_1 - \beta'_2 \circ (rX - w_1)
= r^2 \beta_2 \circ X + r(-z_1 \circ \beta_1 + \beta_2 \circ w_1) - z_1 \circ z_0 \circ m_1 = 0, \]

by 7.1.(b) and 3.5;

\[(\rho_1 \circ \ell_2 - \ell_3 \circ \rho_2)_{2,2} = z_1 \circ k_2 - k_3 \circ z_2 - r \beta_2 \circ \alpha_2
= - (k_1(\sigma) + r\beta_0(1)) \cdot \text{id}_{K_2} = -f \cdot \text{id}_{K_2}, \]

by 3.6.(a) and 3.3.(a);

\[(\rho_2 \circ \ell_3 - \ell_4 \circ \rho_3)_{1,1}
= r(X \circ m_2 + \alpha_2 \circ \beta_2 + m_3 \circ X^\dagger) - w_1 \circ m_2 + m_3 \circ w_2
= (r\beta_0(1) + k_1(\sigma)) \cdot \text{id}_{M_2} = f \cdot \text{id}_{M_2}, \]

by 2.4.(c) and 3.6.(b);

\[(\rho_2 \circ \ell_3 - \ell_4 \circ \rho_3)_{1,2} = -\alpha_2 \circ k_3 + m_3 \circ \alpha_3 = 0, \quad \text{by (3.2.1)};\]

\[(\rho_2 \circ \ell_3 - \ell_4 \circ \rho_3)_{2,1}
= r^2 (\beta_3 \circ X^\dagger) + r(-z_2 \circ \beta_2 + \beta_3 \circ w_2) = 0, \]

by 7.1.(a) and 3.5;

\[(\rho_2 \circ \ell_3 - \ell_4 \circ \rho_3)_{2,2} = r \cdot \beta_3 \circ \alpha_3 + z_2 \circ k_3 - k_4 \circ z_3
= (r\beta_0(1) + k_1(\sigma)) \cdot \text{id}_{K_3} = f \cdot \text{id}_{K_3}, \]

by 3.3.(a) and 3.6.(a);

\[(\rho_3 \circ \ell_4 - \ell_5 \circ \rho_4)_{1,1}
= -r(X^\dagger \circ m_3 + \alpha_3 \circ \beta_3) - w_2 \circ m_3 + m_4 \circ w_3
= - (r\beta_0(1) + k_1(\sigma)) \cdot \text{id}_{M_3} = -f \cdot \text{id}_{M_3}, \]

by 7.1.(e) and 3.6.(b);

\[(\rho_3 \circ \ell_4 - \ell_5 \circ \rho_4)_{1,2} = \alpha_3 \circ k_4 - m_4 \circ \alpha_4 = 0, \quad \text{by (3.2.1)};\]

\[(\rho_3 \circ \ell_4 - \ell_5 \circ \rho_4)_{2,1} = r(-z_3 \circ \beta_3 + \beta_4 \circ w_3) = 0, \quad \text{by 3.5};\]

\[(\rho_3 \circ \ell_4 - \ell_5 \circ \rho_4)_{2,2} = z_3 \circ k_4 - r \beta_4 \circ \alpha_4
= - (k_1(\sigma) + r\beta_0(1)) \cdot \text{id}_{K_4} = -f \cdot \text{id}_{K_4}, \]

by 3.6.(a) and 3.3.(a);

\[(\rho_4 \circ \ell_5)_{1,1} = -w_3 \circ m_4 + r \alpha_4 \circ \beta_4 = f \cdot \text{id}_{M_4}, \quad \text{by 7.3.(c)};\]
\[(\rho_1 \circ \rho_0) = 0; \]
\[(\rho_2 \circ \rho_1)_{1,1} = 0; \]
\[(\rho_2 \circ \rho_1)_{1,2} = -rX \circ \alpha_1 + w_1 \circ \alpha_1 - \alpha_2 \circ z_1 = 0, \]

by 2.4.(a) and 3.5;
\[(\rho_2 \circ \rho_1)_{2,1} = 0; \]
\[(\rho_2 \circ \rho_1)_{2,2} = z_2 \circ z_1 = 0, \]
\[(\rho_3 \circ \rho_2)_{1,1} = (-rX^\dagger - w_2) \circ (rX - w_1) \]
\[= -r^2X^\dagger \circ X + r(X^\dagger \circ w_1 - w_2 \circ X) + w_2 \circ w_1 = 0, \]

by 2.4.(d), Lemma 7.2.(a), and 3.5;
\[(\rho_3 \circ \rho_2)_{1,2} = -rX^\dagger \circ \alpha_2 - w_2 \circ \alpha_2 + \alpha_3 \circ z_2 = 0, \]

by 2.4.(e) and 3.5;
\[(\rho_3 \circ \rho_2)_{2,1} = 0; \]
\[(\rho_3 \circ \rho_2)_{2,2} = z_3 \circ z_2 = 0, \]
\[(\rho_4 \circ \rho_3)_{1,1} = rw_3 \circ X^\dagger + w_3 \circ w_2 = 0, \]

by 7.1.(c) and 3.5; and
\[(\rho_4 \circ \rho_3)_{1,2} = w_3 \circ \alpha_3 - \alpha_4 \circ z_3 = 0, \]

by 3.5. \(\square\)

10. OTHER INTERPRETATIONS OF X.

10.1. Adopt the notation of 2.1 and 2.3. Fix elements \(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\) in \(K_1\) with
\[\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3 \wedge \varepsilon_4]_K = 1.\]

It is not difficult to see that the homomorphism \(X : M_1 \rightarrow M_2\) satisfies 2.4.(b) if and only if \((m_2 \circ X)(\theta_1)\) is equal to
\[
\begin{align*}
&\left[\alpha_1(\varepsilon_1)\alpha_1(\varepsilon_2)\alpha_1(\varepsilon_3)\alpha_1(\varepsilon_4)\right]_M \cdot \theta_1 - \left[\theta_1 \alpha_1(\varepsilon_2)\alpha_1(\varepsilon_3)\alpha_1(\varepsilon_4)\right]_M \cdot \alpha_1(\varepsilon_1) \\
&+ \left[\theta_1 \alpha_1(\varepsilon_2)\alpha_1(\varepsilon_3)\alpha_1(\varepsilon_4)\right]_M \cdot \alpha_1(\varepsilon_1)\alpha_1(\varepsilon_4) - \left[\theta_1 \alpha_1(\varepsilon_1)\alpha_1(\varepsilon_3)\alpha_1(\varepsilon_4)\right]_M \cdot \alpha_1(\varepsilon_1)\alpha_1(\varepsilon_3) \\
&+ \left[\theta_1 \alpha_1(\varepsilon_1)\alpha_1(\varepsilon_2)\alpha_1(\varepsilon_3)\right]_M \cdot \alpha_1(\varepsilon_1)\alpha_1(\varepsilon_4)
\end{align*}
\]

and \(X\) satisfies 2.4.(c) if and only if \((X \circ m_2)(\theta_2)\) \((\theta'_2)\) is equal to
\[
\begin{align*}
&\left[-\theta_2 \alpha_1(\varepsilon_1)\alpha_1(\varepsilon_2)\alpha_1(\varepsilon_3)\alpha_1(\varepsilon_4)\right]_M \cdot \alpha_1(\varepsilon_1)\alpha_1(\varepsilon_2)\alpha_1(\varepsilon_3)\theta'_2 + \left[\theta_2 \alpha_1(\varepsilon_2)\alpha_1(\varepsilon_3)\alpha_1(\varepsilon_4)\right]_M \cdot \alpha_1(\varepsilon_1)\alpha_1(\varepsilon_2)\alpha_1(\varepsilon_3)\theta'_2 \\
&- \left[\theta_2 \alpha_1(\varepsilon_1)\alpha_1(\varepsilon_2)\alpha_1(\varepsilon_4)\right]_M \cdot \alpha_1(\varepsilon_1)\alpha_1(\varepsilon_2)\alpha_1(\varepsilon_4)\theta'_2 + \left[\theta_2 \alpha_1(\varepsilon_1)\alpha_1(\varepsilon_3)\alpha_1(\varepsilon_4)\right]_M \cdot \alpha_1(\varepsilon_1)\alpha_1(\varepsilon_2)\alpha_1(\varepsilon_4)\theta'_2 \\
&+ \left[\theta_2 \alpha_1(\varepsilon_1)\alpha_1(\varepsilon_2)\alpha_1(\varepsilon_3)\right]_M \cdot \alpha_1(\varepsilon_1)\alpha_1(\varepsilon_2)\alpha_1(\varepsilon_4)\theta'_2 - \left[\theta_2 \alpha_1(\varepsilon_1)\alpha_1(\varepsilon_2)\alpha_1(\varepsilon_3)\right]_M \cdot \alpha_1(\varepsilon_1)\alpha_1(\varepsilon_2)\alpha_1(\varepsilon_3)\theta'_2 \\
&+ \left[\alpha_1(\varepsilon_1)\alpha_1(\varepsilon_2)\alpha_1(\varepsilon_3)\alpha_1(\varepsilon_4)\right]_M \cdot \theta_2 \cdot \theta'_2.
\end{align*}
\]
Maps $X$ with the above two properties are considered in [12, 6]. In particular, in the language of [6, Def. 1.3], the map $M_2 \otimes M_1 \to P$, which is given by

$$\theta_2 \otimes \theta_1 \mapsto [X(\theta_1) \cdot \theta_2]_M,$$

is called a “partial higher order multiplication” on $M$. (The higher order multiplication is called partial, rather than complete, because the element $\alpha_1(\epsilon_1) \wedge \alpha_1(\epsilon_2) \wedge \alpha_1(\epsilon_3) \wedge \alpha_1(\epsilon_4)$ of $\wedge^4 M_1$ is held fixed, rather than allowed to be arbitrary.) The papers [12, 6] use higher order multiplication to prove that if $P$ is a local ring in which two is a unit, then the minimal resolution of the almost complete intersection ring $P/(\mathfrak{a}, f)$, by free $P$-modules, is a DG-algebra. In particular, the paper [12] proves that if $P$ is a local ring in which two is a unit, then $M$ has a complete higher order multiplication. In the present paper, we are able to obtain higher order multiplication over any commutative Noetherian ring; we do not require that two be a unit or that the ring be local. The present paper makes significant use of divided powers; see, in particular, the complex $B$ of Definition 4.1. The concept of divided powers barely appears in [12, 6]. In the present paper we did not consider complete higher order multiplications.

10.2. The map $X$ of Theorem 2.4 gives the following null homotopy:

$$
\begin{array}{cccccc}
0 & \rightarrow & M_4 & \rightarrow & M_3 & \rightarrow & M_2 & \rightarrow & M_1 & \rightarrow & M_0 \\
\downarrow & & m_4 & \downarrow & m_3 & \downarrow & m_2 & \downarrow & m_1 & \\
0 & \rightarrow & M_4 & \rightarrow & M_3 & \rightarrow & M_2 & \rightarrow & M_1 & \rightarrow & M_0,
\end{array}
$$

where $w_i : M_i \rightarrow M_i$ is given by

$$w_i(\theta_i) = \beta_0(1)\theta_i - (\alpha_i \circ \beta_i)\theta_i;$$

$h_0$ and $h_3$ are both zero; $h_1 = X$; and $h_2 = X^\dagger$.

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