Majorana spin liquid and dimensional reduction in Cs$_2$CuCl$_4$

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The low-temperature behavior of the magnetic insulator Cs$_2$CuCl$_4$ can be modeled by an anisotropic triangular lattice spin-1/2 Heisenberg antiferromagnet with two different exchange couplings $J$ and $J' \approx J/3$. We show that in a wide range of magnetic fields the experimentally observed field dependence of the crossover temperature $T_c$ for spin-liquid behavior can be explained within a mean-field theory based on the representation of spin operators in terms of Majorana fermions. We also show that for small magnetic fields the specific heat and the spin susceptibility both exhibit a maximum as a function of temperature at $T_c = J/2$. In the spin-liquid regime, the Majorana fermions can only propagate along the direction of the strongest bond, in agreement with the dimensional reduction scenario advanced by Balents [Nature (London) 464, 199 (2010)].

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I. INTRODUCTION

In the past decade, the physical properties of the magnetic insulator Cs$_2$CuCl$_4$ have been explored experimentally using a variety of different experimental techniques, such as inelastic neutron scattering, ultrasound measurements, and nuclear magnetic resonance. It is now generally accepted that at low temperatures the magnetic properties of Cs$_2$CuCl$_4$ can be modeled by a quasi-two-dimensional spin-1/2 antiferromagnetic Heisenberg model where the spins within the layers form an anisotropic triangular lattice with two different nearest neighbor exchange couplings $J = 4.34$ K and $J' = 1.49$ K, as shown in Fig. 1. In addition, the spins are coupled by a weak inter-plane exchange coupling $J'' \approx 0.20$ K and a slightly larger in-plane Dzyaloshinskii-Moriya interaction $D = 0.23$ K; these couplings are responsible for the emergence of long-range magnetic order at sufficiently low temperatures. The phase diagram of Cs$_2$CuCl$_4$ as a function of the temperature $T$ and an external magnetic field $H \hat{z}$ perpendicular to the layers is shown schematically in Fig. 2. In this work, we shall focus on the finite-temperature spin-liquid phase of Cs$_2$CuCl$_4$ in the regime where the external magnetic field is not too close to the critical field $H_c = 8.5$ T. Because in this part of the phase diagram the temperature is large compared with the inter-plane interaction $J''$ and the Dzyaloshinskii-Moriya interaction $D$, these interactions can be neglected for our purpose. It is therefore reasonable to describe the spin-liquid phase of Cs$_2$CuCl$_4$ within a purely two-dimensional triangular lattice antiferromagnetic Heisenberg model,

$$\mathcal{H} = \frac{1}{2} \sum_{ij} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j - h \sum_i S_i^z,$$

where the spin $S = 1/2$ operators $\mathbf{S}_i \equiv \mathbf{S}_{R_i}$ are localized on the sites $R_i$ of a distorted triangular lattice and the exchange couplings $J_{ij} \equiv J(R_i - R_j)$ are only finite if $R_i - R_j$ connect nearest neighbor sites on the lattice. At this point, we assume different exchange couplings $J_{ij} = J(\pm \delta_{ij})$ in each of the three directions $\delta_1$, $\delta_2$, and $\delta_3$ shown in Fig. 3 where $\mu = 1, 2, 3$ labels the directions. Later, we shall set $J_1 = J = 4.34$ K and $J_2 = J_3 = J' = 1.49$ K to describe Cs$_2$CuCl$_4$. The energy $h = g\mu_B H$ in Eq. (1) is the Zeeman energy in the external magnetic field of magnitude $H$, where $g \approx 2.19$ is the effective $g$-factor associated with the Cu spins and $\mu_B$ is the Bohr magneton.

Given the fact that in Cs$_2$CuCl$_4$ the ratio $J'/J \approx 1/3$ is not really small, it is at first sight reasonable to expect that the nature of the spin-fluid phase is such that the elementary excitations of the spin liquid can propagate coherently in all directions on the two-dimensional lattice. However, a theory where the elementary excitations of the spin liquid resemble the one-dimensional fermionic

![Figure 1](image1.png)

**FIG. 1.** (Color online) (a) Anisotropic triangular lattice with nearest neighbor exchange coupling $J_1$ (thick lines), $J_2$ (thin lines) and $J_3$ (thin dashed lines); the corresponding link vectors are $\delta_1 = b \hat{x}$, $\delta_2 = -x b \hat{y} + y \hat{z}$ and $\delta_3 = -x \hat{y} - y b \hat{z}$. To describe Cs$_2$CuCl$_4$, we should set $J_1 = J = 4.34$ K and $J_2 = J_3 = J' = 1.49$ K; the lattice structure is orthorhombic with in-plane lattice parameters $b = 7.48$ Å and $c = 12.26$ Å; the crystallographic $a$ axis is perpendicular to the plane of the paper. (b) Topologically equivalent square lattice with diagonal bonds.

![Figure 2](image2.png)

**FIG. 2.** Schematic of the anisotropic triangular lattice with nearest neighbor exchange couplings $J_1$ (thick lines), $J_2$ (thin lines) and $J_3$ (thin dashed lines). The corresponding link vectors are $\delta_1 = b \hat{x}$, $\delta_2 = -x b \hat{y} + y \hat{z}$ and $\delta_3 = -x \hat{y} - y b \hat{z}$. To describe Cs$_2$CuCl$_4$, we should set $J_1 = J = 4.34$ K and $J_2 = J_3 = J' = 1.49$ K; the lattice structure is orthorhombic with in-plane lattice parameters $b = 7.48$ Å and $c = 12.26$ Å; the crystallographic $a$ axis is perpendicular to the plane of the paper. (b) Topologically equivalent square lattice with diagonal bonds.
spinon excitations of a Heisenberg chain has been highly successfu,

suggesting that the spin-liquid phase in Cs₂CuCl₄ supports elementary excitations which can only propagate coherently along the direction δ₁ of the strongest bond. In a simple picture, this dimensional reduction in Cs₂CuCl₄ arises from a strong frustration-induced reduction of the effective coupling J′ associated with the weaker bonds. However, a quantitative microscopic conformation of this scenario using many-body methods is rather involved. In this work, we shall show that a straightforward mean-field theory based on the well-known representation of the spin operators in terms of Majorana fermions naturally explains the dimensional reduction in Cs₂CuCl₄. Specifically, we find that an anisotropic spin-liquid state where the fractionalized fermionic excitations can only propagate coherently along the direction of the strongest bond minimizes the free energy already at the mean-field level.

If the external magnetic field has a component parallel to the layers, the phase diagram of quasi-two-dimensional frustrated antiferromagnets is more complex, as discussed in a series of recent theoretical works by Starykh and co-authors. Here we consider only the case where the magnetic field points along the crystallographic a axis.

II. ROTATIONALLY INVARIANT MAJORANA MEAN-FIELD THEORY

A recent NMR study of Cs₂CuCl₄ found evidence that the spin-liquid phase in this material exhibits gapless fermionic excitations. To describe this phase theoretically, one should therefore express the spin operators of the underlying Heisenberg model in terms of fermionic degrees of freedom. One possibility is to use Abrikosov pseudofermions, where the spin-operator at lattice site \( \mathbf{R}_i \) is expressed in terms of a pair of canonical fermion operators \( c_{i\uparrow} \) and \( c_{i\downarrow} \) as

\[
S_i = \frac{1}{2} (c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger + c_{i\downarrow} c_{i\uparrow}) \sigma^\alpha \left( \begin{array}{c} c_{i\uparrow} \\ c_{i\downarrow} \end{array} \right).
\]

Here, the components of the vector \( \sigma \) are the usual Pauli matrices. Substituting this representation into Eq. (1), the exchange part of the Hamiltonian is quartic in the pseudofermions, which has to be replaced by a quadratic form in order to obtain a mean-field description. Of course, there is no unique way of doing this and for Cs₂CuCl₄ the different possibilities have been classified by Zhou and Wen using the projective symmetry group associated with the mean-field decouplings. One disadvantage of the representation (2) is that the fermionic Hilbert space contains two unphysical states per lattice site, corresponding to empty and doubly occupied sites. In order to describe the physical spin system, these unphysical states must be projected out. According to Popov and Fedotov, this can be done by formally imposing on the system a fictitious imaginary chemical potential \( \mu_f = i\pi T/2 \). In frequency space, this is equivalent to replacing the fermionic Matsubara frequencies \( 2\pi T(n+1/2) \) by semionic ones, \( 2\pi T(n+1/4) \). If no further approximations are made, the semionic Matsubara frequencies automatically eliminate the unphysical states from the fermionic Hilbert space. Recently, this procedure has been used to study the triangular lattice antiferromagnet by means of a diagrammatic Monte Carlo method. To avoid the complications associated with an imaginary chemical potential, it is sometimes sufficient to implement the projection only on average, which formally amounts to setting \( \mu_f = 0 \). Unfortunately, at finite temperature this approximation can introduce uncontrollable error.

In this work, we shall use a different fermionic representation based on Majorana fermions; introducing for each lattice site \( \mathbf{R}_i \) three Majorana fermions \( \eta_i^x, \eta_i^y, \eta_i^z \), and \( \eta_i^\dagger \) satisfying the anticommutation relations

\[
\eta_i^\alpha \eta_j^\beta + \eta_j^\beta \eta_i^\alpha = \delta_{ij} \delta^{\alpha\beta},
\]

the spin algebra can be reproduced by setting

\[
S_i^x = -i \eta_i^y \eta_i^z, \quad S_i^y = -i \eta_i^z \eta_i^x, \quad S_i^z = -i \eta_i^x \eta_i^y.
\]

Note that with our normalization \( (\eta_i^\alpha)^2 = 1/2 \). The above Majorana representation has been used previously by several authors to study quantum spin systems. Moreover, a coherent state path integral for the Majorana fermions can be constructed so that the usual field theoretical methods can be used to study the underlying spin model. An advantage of the above Majorana representation is that it does not introduce any unphysical
of the Majorana fermions, it is then useful to introduce the lattice Fourier transform:

\[ C_{ij}^{\alpha} = \eta_i^{\alpha} \eta_j^{\dagger \alpha}, \]

and use the operator identity

\[ S_i \cdot S_j = \frac{1}{2} \sum_{\alpha \neq \beta} \eta_i^{\alpha} \eta_j^{\bar{\beta}} \eta_j^{\beta} \eta_i^{\bar{\beta}} = \frac{1}{2} \sum_{\alpha \neq \beta} C_{ij}^{\alpha \beta} C_{ij}^{\beta \alpha}, \]

to write the Heisenberg Hamiltonian [1] for vanishing magnetic field as

\[ H = \frac{1}{4} \sum_{ij} \sum_{\alpha \neq \beta} J_{ij} C_{ij}^{\alpha \beta} C_{ij}^{\beta \alpha}. \]

Performing now a simple mean-field decoupling [25]

\[ C_{ij}^{\alpha \beta} \rightarrow C_{ij}^{\alpha} \langle C_{ij}^{\beta} \rangle + \langle C_{ij}^{\alpha} \rangle C_{ij}^{\beta} - \langle C_{ij}^{\alpha} \rangle \langle C_{ij}^{\beta} \rangle, \]

and assuming spin-rotational invariance so that the expectation values

\[ \langle C_{ij}^{\alpha} \rangle = \langle \eta_i^{\alpha} \eta_j^{\dagger \alpha} \rangle \equiv iZ_{ij} \]

are independent of the flavor index \( \alpha \), we obtain the mean-field Hamiltonian

\[ H_{\text{MF}} = i \sum_{ij\alpha} t_{ij} \eta_i^{\alpha} \eta_j^{\dagger \alpha} + U_0, \]

with hopping energies

\[ t_{ij} = J_{ij} Z_{ij} = -t_{ji}, \]

and the interaction energy

\[ U_0 = \frac{3}{2} \sum_{ij} J_{ij} Z_{ij}^2. \]

Note that by definition \( Z_{ij} = -Z_{ji} \). Assuming that the mean-field state is translationally invariant, we may set

\[ Z_{Rk, Rk + \delta_{\mu}} = \pm Z_{\mu}. \]

It is then useful to introduce the lattice Fourier transform of the Majorana fermions,

\[ \eta_R^{\alpha} = \frac{1}{\sqrt{N}} \sum_k e^{i k \cdot R} \eta_k^{\alpha}, \]

where the \( k \) sum is over a full unit cell in the reciprocal space of the underlying Bravais lattice. Our mean-field Hamiltonian [10] can then be written as

\[ H_{\text{MF}} = \frac{1}{2} \sum_{k, \alpha} \epsilon_k \eta_k^{\alpha} \eta_k^{\dagger \alpha} + U_0, \]

with mean-field energy dispersion

\[ \epsilon_k = -4 \sum_{\mu=1}^{3} J_{\mu} Z_{\mu} \sin(k \cdot \delta_{\mu}), \]

and interaction energy

\[ U_0 = 3N \sum_{\mu=1}^{3} J_{\mu} Z_{\mu}^2. \]

At finite temperature, the mean-field free energy is

\[ F = -\frac{3}{2\beta} \sum_k \ln(1 + e^{-\beta \epsilon_k}) + U_0, \]

leading to the three self-consistency equations

\[ Z_{\mu} = \frac{1}{N} \sum_k f(\epsilon_k) \sin(k \cdot \delta_{\mu}), \quad \mu = 1, 2, 3. \]

Here, \( f(\epsilon_k) = 1/(e^{\beta \epsilon_k} + 1) \) is the Fermi function.

Let us now analyze the possible solutions of the above mean-field equations. At sufficiently high temperatures Eq. (19) has only the trivial solution \( Z_{\mu} = 0 \) for all directions \( \mu \), but there is a critical temperature below which at least one of the order parameters \( Z_{\mu} \) is finite. In the vicinity of the critical temperature, the order parameters are small and we may expand the free energy in powers of the \( Z_{\mu} \). We obtain

\[ \frac{\beta F}{N} = -\frac{3 \ln 2}{2} + \frac{\beta U_0}{N} - \frac{3\beta^2}{16N} \sum_k \epsilon_k^2 \]

\[ + \frac{\beta^4}{128N} \sum_k \epsilon_k^4 + O(Z_{\mu}^6). \]

To carry out the momentum integrations over the first Brillouin zone, it is convenient to map the unit cell in reciprocal space onto a rectangle using the volume-preserving transformation

\[ k_x = k_1, \quad k_y = k_2 + \frac{2b}{c} k_1. \]

Note that with the definitions of the lattice constants shown in Fig. 1(a) the volume of the Brillouin zone is \( V_{BZ} = (2\pi/b)(4\pi/c) \). In the thermodynamic limit \( N \rightarrow \infty \), the Brillouin zone integration of any function \( f(k_x, k_y) \) can then be written as

\[ \frac{1}{N} \sum_k f(k_x, k_y) = \frac{1}{V_{BZ}} \int_0^{2\pi} dk_1 \int_0^{4\pi} dk_2 f(k_1, k_2 + \frac{2b}{c} k_1) \]

\[ = \int_0^{2\pi} dq_1 \int_0^{2\pi} dq_2 \frac{d\epsilon}{2\pi} f \left( \frac{q_1}{b}, \frac{q_1 + q_2}{c} \right), \]
where in the last line we have set $q_1 = bk_1$ and $q_2 = \frac{1}{2}k_2$. This transformation maps the original anisotropic triangular lattice onto a square lattice, as shown in Fig. 1(b).

Using the fact that with these definitions $k \cdot \delta_1 = q_1$, $k \cdot \delta_2 = q_2$, and $k \cdot \delta_3 = -q_1 - q_2$, the mean-field energy dispersion can be written as

$$\epsilon_k = -4\{t_1 \sin q_1 + t_2 \sin q_2 - t_3 \sin(q_1 + q_2)\},$$

where we have defined the hopping energies

$$t_\mu = J_\mu Z_\mu.$$ (24)

The Brillouin zone integrations in Eq. (20) can now easily be carried out,

$$\frac{1}{N} \sum_k \epsilon_k^2 = 8 \sum_\mu t_\mu^2,$$ (25)

$$\frac{1}{N} \sum_k \epsilon_k^4 = 96 \left[t_1^4 + t_2^4 + t_3^4 + 4(t_1^2 t_2^2 + t_1^2 t_3^2 + t_2^2 t_3^2)\right].$$ (26)

Defining $K_\mu = \beta J_\mu$, we obtain for the dimensionless free energy per site,

$$\frac{\beta F}{N} = -\frac{3}{2} \ln 2 + \frac{3}{2} \sum_\mu K_\mu (2 - K_\mu) Z_\mu^2 + \frac{3}{4} \sum_\mu K_\mu^2 Z_\mu^4 + 3 \left[(K_1 K_2 Z_1 Z_2)^2 + (K_2 K_3 Z_2 Z_3)^2 + (K_3 K_1 Z_3 Z_1)^2\right] + O(Z_\mu^6).$$ (27)

Minimization gives the following three conditions

$$\frac{K_1 - 2}{K_1} = K_1^2 Z_1^2 + 2(K_2^2 Z_2^2 + K_3^2 Z_3^2), \quad \text{if } Z_1 \neq 0,$$ (28a)

$$\frac{K_2 - 2}{K_2} = K_2^2 Z_2^2 + 2(K_3^2 Z_3^2 + K_1^2 Z_1^2), \quad \text{if } Z_2 \neq 0,$$ (28b)

$$\frac{K_3 - 2}{K_3} = K_3^2 Z_3^2 + 2(K_1^2 Z_1^2 + K_2^2 Z_2^2), \quad \text{if } Z_3 \neq 0.$$ (28c)

Let us first consider the isotropic case $K_1 = K_2 = K_3 = K = \beta J$. Naively, one might then look for a solution $Z_1 = Z_2 = Z_3 = Z$ in the low-temperature regime. In this case, the three self-consistency equations (28a–28c) reduce to the single equation

$$Z^2 = \frac{K - 2}{3K^3}$$ (29)

which has only a solution if $K \geq 2$, i.e. $T \leq J/2$. The corresponding free energy is

$$\frac{\beta F}{N} = -\frac{3}{2} \ln 2 - \frac{(K - 2)^2}{4K^2}.$$ (30)

It turns out, however, that even for the isotropic triangular lattice antiferromagnet a one-dimensional Majorana state has lower energy. To see this, let us assume that only $Z_1$ is non-zero while $Z_2 = Z_3 = 0$. Then we obtain from Eq. (28a)

$$Z_1^2 = \frac{K - 2}{K^3}$$ (31)

and for the corresponding free energy,

$$\frac{\beta F}{N} = -\frac{3}{2} \ln 2 - \frac{3(K - 2)^2}{4K^2}.$$ (32)

The energy gain for $K > 2$ is three times as large as in the isotropic Majorana state. Hence, our Majorana mean-field theory predicts that in the isotropic triangular lattice the discrete three-fold rotational symmetry of the lattice is spontaneously broken for temperatures below $T_c = J/2$. The emergent Majorana fermions can then propagate coherently only in one direction. Note that finite temperature phase transitions with spontaneous breaking of the discrete rotational symmetry of the underlying lattice have also been found in other frustrated continuous spin models.\(^{29,30}\)

Consider now the anisotropic triangular lattice relevant to Cs$_2$CuCl$_4$ with couplings $J_1 = J$ and $J_2 = J_3 = J' \approx J/3$. By repeating the above analysis, it is easy to see that also in this case the free energy is minimized by a one-dimensional Majorana state, where the Majorana fermions can only propagate along the direction of the largest exchange coupling associated with the crystallographic $b$ axis (the $x$ axis in our notation) in Cs$_2$CuCl$_4$. With $J = 4.34$ K, we predict that for vanishing magnetic field the transition to the spin-liquid phase in Cs$_2$CuCl$_4$ should occur at $T_c = J/2 = 2.17$ K. A simple calculation shows that at this temperature the specific heat $C$ should exhibit a maximum, as shown in Fig. 3. Indeed, in the experimental work by Radu et al.\(^{19}\) the temperature of the spin-liquid transition was identified with the temperature where the specific heat exhibits a maximum, which yields $T_c \approx 2.1$ K for vanishing magnetic field, in excellent agreement with our prediction. An alternative estimate of $T_c$ due to Coldea et al.\(^{23}\) identified the transition temperature to the spin-liquid phase with the temperature where the spin susceptibility exhibits a maximum, leading to the estimate $T_c \approx 2.65$ K for vanishing magnetic field, based on measurements by Carlin et al.\(^{31}\), while a more recent study of magnetic susceptibilities by Tokiwa et al.\(^{32}\) finds $T_c \approx 2.8$ K. Another alternative estimate of the transition temperature by Vachon et al.\(^{24}\) which is based on NMR measurements, leads to $T_c \approx 2.5$ K.

Although in Cs$_2$CuCl$_4$ the weak interplane exchange and the Dzyaloshinskii-Moriya interaction stabilize a magnetically ordered state for temperatures below $T_N \approx 0.62$ K, let us briefly discuss the mean-field results for the two-dimensional anisotropic triangular lattice antiferromagnet (1) in the limit of vanishing temperature. The free energy (18) then reduces to the ground state energy

$$E_0 = \lim_{\beta \to \infty} F = \frac{3}{2} \sum_k \Theta(-\epsilon_k)\epsilon_k + U_0,$$ (33)

and the self-consistency equations (19) can be written as

$$Z_\mu = \frac{1}{N} \sum_k \Theta(-\epsilon_k) \sin(k \cdot \delta_\mu), \quad \mu = 1, 2, 3.$$ (34)
III. MAJORANA MEAN-FIELD THEORY IN A MAGNETIC FIELD

If Cs$_3$CuCl$_4$ is exposed to a magnetic field along the crystallographic $a$ axis, the critical temperature for spin-liquid behavior is reduced, as shown by the dashed line in Fig. 2. In this section, we shall calculate the magnetic field dependence of the transition temperature $T_c(H)$ using Majorana mean-field theory. Of course, in the vicinity of the quantum critical point at $H_c = 8.5$ T spin fluctuations play an important role so that mean-field theory is not reliable. However, for $H \lesssim 0.8H_c$ our Majorana mean-field theory describes the experimental data for $T_c(H)$ quite well.

Using again the representation (1) of the spin operators in terms of Majorana fermions, our spin Hamiltonian can be written as

$$\mathcal{H} = \frac{1}{4} \sum_{ij} C_{ij} \gamma_i \gamma_j + ih \sum_{k} \eta_k^x \eta_k^y,$$

where $C_{ij} = \eta_i^x \eta_j^x$. As a first try, let us follow Ref. [20] and decouple the exchange term in exactly the same way as in zero field, see Eq. (38). Using the same notations as in Sec. II we then obtain the mean-field Hamiltonian

$$\mathcal{H}_{MF} = i \sum_{ij\alpha} t_{ij} \eta_i^\alpha \eta_j^\alpha + ih \sum_{i} \eta_i^x \eta_i^y + U_0. \quad (37)$$

In momentum space, this assumes the form

$$\mathcal{H}_{MF} = \frac{1}{2} \sum_{k} (\eta_k^x, \eta_k^y, \eta_k^z) \begin{pmatrix} \epsilon_k & ih & 0 \\ -ih & \epsilon_k & 0 \\ 0 & 0 & \epsilon_k \end{pmatrix} \begin{pmatrix} \eta_k^x \\ \eta_k^y \\ \eta_k^z \end{pmatrix} + U_0. \quad (38)$$

For a given $k$ the above $3 \times 3$ matrix has the eigenvalues $\epsilon_k + sh$, where $s$ assumes the values $-1, 0, 1$. The free energy is therefore

$$F = -\frac{1}{2\beta} \sum_{k,s} \ln \left[ 1 + e^{-\beta(\epsilon_k + sh)} \right] + U_0. \quad (39)$$

The self-consistency equations for the variational parameters $Z_\mu$ are

$$Z_\mu = \frac{1}{3N} \sum_{k,s} f(\epsilon_k + sh) \sin(k \cdot \delta_\mu), \quad \mu = 1, 2, 3. \quad (40)$$

Expanding the free energy to fourth order in the variational parameters $Z_\mu$, we obtain

$$\frac{\beta F}{N} = -\frac{1}{2} [\ln 2 + \ln(1 + e^{\beta h}) + \ln(1 + e^{-\beta h})] + \frac{3}{2} \sum_\mu K_\mu [2 - K_\mu f_2(\beta h)] Z_\mu^2 + \frac{3}{4} f_4(\beta h) \left[ \sum_\mu K_\mu^2 Z_\mu^4 + 4(K_1 K_2 Z_1 Z_2)^2 + 4(K_2 K_3 Z_2 Z_3)^2 + 4(K_3 K_1 Z_3 Z_1)^2 \right], \quad (41)$$
FIG. 4. (Color online) Numerical evaluation of the mean-field result \( (33) \) of the ground state energy per site as a function of \( Z = Z_1 \) and \( Z' = Z_2 = Z_3 \) for different values of \( J'/J \) as indicated.

\[
f_2(x) = \frac{1}{3} + \frac{2}{3 \cosh^2(x/2)}, \quad f_4(x) = \frac{8}{3} \sum_s e^{sx}(4e^{sx} - 1 - e^{2sx}) / (1 + e^{sx})^4. \tag{42, 43}
\]

We have normalized the above functions such that \( f_2(0) = f_4(0) = 1 \). The magnetic field dependence of the critical temperature is obtained from the condition that the coefficient of the quadratic term in the expansion \( (41) \) of the free energy vanishes, leading to the self-consistency equation

\[
T_c/J = \frac{1}{6} + \frac{1}{3 \cosh^2[h/(2T_c)]}. \tag{44}
\]

A numerical solution of this equation for the parameters relevant for Cs\(_2\)CuCl\(_4\) gives the dashed line in Fig. 6. Obviously, the shape of this curve does not agree with the experimentally observed \( T_c(H) \) shown in Fig. 2 so that at first sight it seems that the magnetic field dependence of the transition temperature to the spin-liquid phase in Cs\(_2\)CuCl\(_4\) is not well described by Majorana mean-field theory. However, the mean-field decoupling used to derive Eq. \( (44) \) is not self-consistent, because in the presence of a magnetic field the expectation values \( \langle \eta_x \eta_y \rangle \) are finite and should be taken into account in our mean-field decoupling. In the presence of a magnetic field, we should therefore replace the decoupling \( (8) \) by

\[
S_i \cdot S_j = \frac{1}{2} \sum_{\alpha \neq \beta} \eta_x^\alpha \eta_y^\alpha \eta_x^\beta \eta_y^\beta
\]

\[
- \frac{1}{2} \sum_{\alpha \neq \beta} [C_{ij}^{\alpha} (C_{ij}^{\beta}) + (C_{ij}^{\alpha}) (C_{ij}^{\beta}) - (C_{ij}^{\alpha}) (C_{ij}^{\beta}) - (C_{ij}^{\alpha}) (C_{ij}^{\beta})]
-
[\eta_x^\alpha \eta_y^\alpha \eta_x^\beta \eta_y^\beta + \eta_x^\alpha \eta_y^\alpha \eta_x^\beta \eta_y^\beta - \eta_x^\alpha \eta_y^\alpha \eta_x^\beta \eta_y^\beta] \tag{45}
\]
The self-consistency equations (48) and (49) can be obtained by setting the derivative of the free energy with respect to the magnetic moment \( m \) to zero. The energy dispersion (16) but with variational parameters \( Z_\mu \) determined by

\[
Z_\mu = \frac{1}{3N} \sum_{k,s} f(\epsilon_k + sb) \sin(k \cdot \delta_\mu), \quad \mu = 1, 2, 3.
\]

The self-consistency equations (48) and (49) can be obtained by calculating the extremum of the free energy

\[
F = -\frac{1}{2\beta} \sum_{k,s} \ln \left[ 1 + e^{-\beta(\epsilon_k + sb)} \right] + U_0,
\]

where the potential \( U_0 \) is now

\[
U_0 = N \sum_\mu J_\mu \left[ 3Z_\mu^2 - m^2 \right].
\]

We find that the critical temperature satisfies

\[
\frac{T_c}{J} = \frac{1}{6} + \frac{1}{3 \cosh^2 ((h - \tilde{J}_0 m_c)/(2T_c))},
\]

where the effective magnetic moment \( m_c \) at the critical temperature is determined by

\[
m_c = \frac{1}{2} \tanh \left( (h - \tilde{J}_0 m_c)/(2T_c) \right).
\]

For a given value of the magnetic field, the coupled equations (52) and (53) should be solved simultaneously to obtain \( T_c \) and \( m_c \) as a function of \( h \). Substituting the parameters relevant for \( \text{Cs}_2\text{CuCl}_4 \) (\( h_c/J = 2.85 \) and \( \tilde{J}_0/h_c = 1.18 \)), the resulting critical temperature is shown as a solid line in Fig. 6, which agrees quite well with the experimentally determined crossover temperature up to fields \( H \lesssim 0.8H_c \). For completeness, we show in Fig. 7 the self-consistent magnetic moment \( m_c \) and the effective magnetic field \( h - \tilde{J}_0 m_c \) at the critical temperature. Note that the antiferromagnetic coupling tends to screen the external magnetic field, so that a stronger external field is needed to generate a given effective field. As a
FIG. 8. (Color online) Magnetic susceptibility $\chi$ as a function of $T/J$ for $J'/J = 1/3$ and $h/J = 0.01$, corresponding to an external magnetic field $H \approx 0.03T$.

IV. SUMMARY AND CONCLUSIONS

In summary, we have developed a simple mean-field description of the finite temperature spin-liquid phase in Cs$_2$CuCl$_4$ based on the representation of the spin operators in terms of Majorana fermions. We have argued that the experimentally observed crossover temperature for spin-liquid behavior in Cs$_2$CuCl$_4$ can be identified with the critical temperature $T_c(H)$ below which the mean-field equations for the dispersion of the Majorana fermions have a finite solution. For small external fields, the emergence of the spin-liquid state gives rise to a maximum in the specific heat and the spin susceptibility as a function of temperature at $T_c = J/2$. We have found that a coherent motion of the Majorana fermions is only possible along the direction of the strongest bond, in agreement with the dimensional reduction scenario discussed by Balents. The emergent one-dimensional Fermi surface of the Majorana fermions is associated with a nematic instability where the discrete rotational symmetry of the lattice is broken.

Given the values of the exchange couplings, our mean-field theory yields an expression for $T_c(H)$ without further adjustable parameters, which agrees quantitatively with the experimentally observed crossover temperature for spin-liquid behavior in Cs$_2$CuCl$_4$ up to fields $H \lesssim 0.8H_c$. For larger fields our Majorana mean-field theory is not reliable any more because other types of excitations such as spin fluctuations become important.

Our Majorana mean-field theory is complementary to the approach developed by Starykh and co-authors, where Cs$_2$CuCl$_4$ is regarded as an array of weakly coupled Heisenberg chains which can be analyzed using bosonization techniques. Given the rather large value of $J'/J \approx 1/3$ in Cs$_2$CuCl$_4$, the validity of this approach is not obvious. In contrast, our Majorana approach treats the system $a priori$ as two dimensional; the one-dimensional nature of the Majorana fermions in the spin-liquid phase appears simply as the result of our mean-field calculation. In both methods, the dimensional reduction in the presence of a substantial value of $J'$ is somewhat surprising. The agreement of our Majorana mean-field theory with experiments probing the spin-liquid phase suggests that the Majorana fermions which are formally introduced via the representation have a significant overlap with the dominant physical excitations in the finite temperature spin-liquid phase of Cs$_2$CuCl$_4$.

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