CONTINUOUS QUANTITATIVE HELLY-TYPE RESULTS

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Abstract. Brazitikos’ results on quantitative Helly-type theorems (for the volume and for the diameter) rely on the work of Srivastava on approximate John’s decompositions with few vectors. We change this technique by a stronger recent result due to Friedland and Youssef that allow us to obtain Helly-type versions which are sensitive to the number of convex sets involved.

1. Introduction

Helly’s classical theorem states that if \( \mathcal{C} = \{ C_i : i \in I \} \) is a finite family of at least \( n + 1 \) convex sets in \( \mathbb{R}^n \) and if any \( n + 1 \) members of \( \mathcal{C} \) have non-empty intersection then \( \bigcap_{i \in I} C_i \) is non-empty. In general, a Helly-type property is a property \( \Pi \) for which there exists a number \( s \in \mathbb{N} \) such that if \( \{ C_i : i \in I \} \) is a finite family of certain objects and every subfamily of \( s \) elements fulfills \( \Pi \), then the whole family fulfills \( \Pi \).

In the eighties, Bárány, Katchalski and Pach proved the following quantitative “volume version” of Helly’s theorem [BKP82, BKP84]:

\[
\text{Let } \mathcal{C} = \{ C_i : i \in I \} \text{ be a finite family of convex sets in } \mathbb{R}^n. \text{ If the intersection of any } 2n \text{ or fewer members of } \mathcal{H} \text{ has volume greater than or equal to } 1, \text{ then } \text{vol}(\bigcap_{i \in I} C_i) \geq c(n), \text{ where } c(n) > 0 \text{ is a constant depending only on } n.
\]

Thus, the previous result express the fact that “the intersection has large volume” is a Helly-type property for the family of convex sets.

Since every (closed) convex set is the intersection of a family of closed half-spaces; a simple compactness argument (see [BKP82]) shows that one can remove the restriction that \( \mathcal{C} \) is finite and also assume that each convex set is a closed half-space i.e.,

\[
\{ x \in \mathbb{R}^n : \langle x, v_i \rangle \leq 1 \},
\]

for some vector \( v_i \in \mathbb{R}^n \). Therefore, the theorem of Bárány et. al. is equivalent to the following statement:

Let \( \mathcal{H} = \{ H_i : i \in I \} \) be a family of closed half-spaces in \( \mathbb{R}^n \) such that \( \text{vol}(\bigcap_{i \in I} H_i) = 1 \). There exist \( s \leq 2n \) and \( i_1, \ldots, i_s \in I \) such that

\[
\text{vol}(H_{i_1} \cap \cdots \cap H_{i_s})^{1/n} \leq c(n),
\]

where \( c(n) > 0 \) is a constant depending only on \( n \).

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Figure 1. A convex body defined as the intersection of half-spaces which is enclosed by a convex set given by the intersection of a few of them.

Of course one cannot replace $2n$ by $2n - 1$ in the statement above. Indeed, the cube $[-1/2, 1/2]^n$ in $\mathbb{R}^n$ can be written as the intersection of the $2n$ closed half-spaces

$$H_\pm^j := \left\{ x : \langle x, \pm \frac{1}{2} e_j \rangle \leq 1 \right\}$$

and that the intersection of any $2n - 1$ of these half-spaces has infinite volume.

The authors of [BKP82] gave the bound $c(n) \leq n^{2n}$ for the constant $c(n)$ and conjectured that one might actually have polynomial growth i.e., $c(n) \leq n^d$ for an absolute constant $d > 0$. Naszódi [Nas16] has verified this conjecture; namely, he proved that $c(n) \leq cn^2$, where $c > 0$ is an absolute constant. A clever but slight refinement of Naszódi’s argument, due to Brazitikos [Bra17a, Theorem 3.1.], leads to the exponent $\frac{3}{2}$ instead of $2$.

Moreover, Brazitikos showed in [Bra17a, Theorem 1.4.] that if we relax the condition on the number $s$ of half-spaces that we use (but still require that it is proportional to the dimension $n$) one can improve significantly the estimate, giving a bound of order $n$.

**Theorem 1.1.** [Bra17a, Theorem 1.4.] There exists an absolute constant $\alpha > 0$ with the following property: for every family $\mathcal{H} = \{H_i : i \in I\}$ of closed half-spaces in $\mathbb{R}^n$,

$$H_i = \{ x \in \mathbb{R}^n : \langle x, v_i \rangle \leq 1 \},$$

with $\text{vol}(\bigcap_{i \in I} H_i) = 1$, there exist $s \leq \alpha n$ and $i_1, \ldots, i_s \in I$ such that

$$\text{vol}(H_{i_1} \cap \cdots \cap H_{i_s})^{1/n} \leq cn,$$

where $c > 0$ is an absolute constant.

Bárány, Katchalski and Pach also studied the question whether “the intersection has large diameter” is a sort of Helly-type property for convex sets. They provided the following quantitative answer to this question:
Let \( \{C_i : i \in I\} \) be a family of closed convex sets in \( \mathbb{R}^n \) such that \( \text{diam} (\bigcap_{i \in I} C_i) = 1 \). There exist \( s \leq 2n \) and \( i_1, \ldots, i_s \in I \) such that
\[
\text{diam} (C_{i_1} \cap \cdots \cap C_{i_s}) \leq (cn)^{n/2},
\]
where \( c > 0 \) is an absolute constant.

In the same work the authors conjectured that the bound \( (cn)^{n/2} \) should be polynomial in \( n \). Leaving aside the requirement that \( s \leq 2n \), Brazitikos in [Bra17b] provided the following relaxed positive answer:

**Theorem 1.2.** There exists an absolute constant \( \alpha > 1 \) with the following property: if \( \{C_i : i \in I\} \) is a finite family of convex bodies in \( \mathbb{R}^n \) with \( \text{diam} (\bigcap_{i \in I} C_i) = 1 \), there exist \( s \leq \alpha n \) and \( i_1, \ldots, i_s \in I \) such that
\[
\text{diam}(C_{i_1} \cap \cdots \cap C_{i_s}) \leq cn^{3/2},
\]
where \( c > 0 \) is an absolute constant.

It should be mentioned that when symmetry is assumed better bounds in both problems can be obtained.

Brazitikos’ proofs of Theorem 1.1 and Theorem 1.2 rely on the work of Batson, Spielman and Srivastava on approximate John’s decompositions with few vectors [BSST12]. For Theorem 1.1 this is successfully combined with a new and very useful estimate for corresponding ‘approximate’ Brascamp-Lieb-type inequality while, for Theorem 1.2 the argument is based on a clever lemma of Barvinok from [Bar14]. This lemma in turn, exploits again the theorem of Batson et. al. or to be precise, a more delicate version of Srivastava from [Sri12].

Of course if one is willing to further relax the number of convex sets involved in the statements of Theorems 1.1 and 1.2 then one should expect to obtain better bounds/estimates. The aim of this note is to present the following continuous quantitative Helly-type results (i.e., Helly-type results which are sensitive to the number of sets considered).

**Theorem 1.3.** (Continuous Helly-type theorem for the volume) Let \( 1 \leq \delta \leq 2 \), there is an absolute constant \( \alpha > 1 \) with the following property: for every \( n \in \mathbb{N} \) and every family \( \mathcal{H} = \{H_i : i \in I\} \) of closed half-spaces in \( \mathbb{R}^n \),
\[
H_i = \{x \in \mathbb{R}^n : \langle x, v_i \rangle \leq 1\},
\]
with \( \text{vol}(\bigcap_{i \in I} H_i) = 1 \), there exists \( s \leq \alpha n^\delta \) and \( i_1, \ldots, i_s \in I \) such that
\[
\text{vol}(H_{i_1} \cap \cdots \cap H_{i_s})^{1/n} \leq d_n n^{\frac{\delta}{2} - \frac{\delta}{2}},
\]
where \( d_n \to 1 \) as \( n \to \infty \).

**Theorem 1.4.** (Continuous Helly-type theorem for the diameter) Let \( 1 \leq \delta \leq 2 \), there is an absolute constant \( \alpha > 1 \) with the following property: for every \( n \in \mathbb{N} \) and every finite family \( \{C_i : i \in I\} \) of convex bodies in \( \mathbb{R}^n \) with \( \text{diam} (\bigcap_{i \in I} C_i) = 1 \), there exist \( s \leq \alpha n^\delta \) and \( i_1, \ldots, i_s \in I \) such that
\[
\text{diam}(C_{i_1} \cap \cdots \cap C_{i_s}) \leq cn^{3/2 + \delta/4},
\]
where \( c > 0 \) are absolute constant.
Note that in both theorems we recover the previous mentioned results when the number of sets is linear in \( n \) (i.e., when \( \delta = 1 \)). If the number of sets is \( n^2 \) then the bounds are the known ones which, of course, follow by directly applying John’s classical theorem. Therefore, the dependencies in the exponent of both results obtained seem to be accurate. Moreover, for a linear number of spaces (i.e., \( \delta = 1 \)) the constant that appears in Theorem 1.3 is better than the one in [Bra17a, Theorem 1.4.], since \( d_n \to 1 \) as \( n \) goes to infinity.

To obtain Theorems 1.3 and 1.4 we carefully follow Brazitikos’s proofs of Theorems 1.1 and 1.2 but instead of using Batson et. al. or Srivastava’s statement on the approximate John’s decomposition we replace it with the following stronger result due to Friedland and Youssef (who exploited the recent solution of the Kadison-Singer problem [MSS15], by showing that any \( n \times m \) matrix \( A \) can be approximated in operator norm by a submatrix with a number of columns of order the stable rank of \( A \)).

**Theorem 1.5.** [FY19, Theorem 4.1] Let \( \{u_j, a_j\}_{1 \leq j \leq m} \) be a John’s decomposition of the identity i.e, the identity operator \( I_n \) is decomposed in the form \( I_n = \sum_{j=1}^{m} a_j u_j \otimes u_j \). Then for any \( \varepsilon > 0 \) there exists a multi-set \( \sigma \subset [m] \) (i.e., it allows repetitions of the elements) with \( |\sigma| \leq n/c \varepsilon^2 \) so that

\[
(1 - \varepsilon)I_n \preceq \frac{n}{|\sigma|} \sum_{j \in \sigma} (u_j - u) \otimes (u_j - u) \preceq (1 + \varepsilon)I_n
\]

where \( u = \frac{1}{|\sigma|} \sum_{i \in \sigma} u_j \) satisfies \( \|u\|_2 \leq \frac{2 \varepsilon}{3 \sqrt{n}} \), and \( c > 0 \) is an absolute constant.

In the words of Friedland and Youssef, Thorem 1.5 improves Srivastava theorem [Sri12, Theorem 5] in three different ways. First the approximation ratio \( (1 + \varepsilon)/(1 - \varepsilon) \) can be made arbitrary close to 1 (while in Srivastava result one could only get a \((4 + \varepsilon)-approximation\)). Secondly, it gives an explicit expression of the weights appearing in the approximation. Finally, there is a big difference in the dependence on \( \varepsilon \) in the estimate of the norm of \( u \): Srivastava obtains a similar bound but with \( \varepsilon \) replaced by \( \sqrt{\varepsilon} \). This behaviour on \( \varepsilon \) will be crucial for our purposes allowing us to obtain the bounds on our main results. With this at hand, we take the \( \varepsilon \) parameter small but depending explicitly on \( n \).

## 2. Notation and background

We refer to the book of Artstein-Avidan, Giannopoulos and V. Milman [AAGM15] for basic facts from convexity and asymptotic geometry.

Recall that a convex body in \( \mathbb{R}^n \) is a compact convex subset \( K \) of \( \mathbb{R}^n \) with non-empty interior. We say that the body \( K \) is symmetric if \( x \in K \) implies that \( -x \in K \). For any set \( X \) we write \( \text{conv}(X) \) for its convex hull. For convex body \( K \) we write \( p_K \) for the Minkowski’s functional of \( K \), that is

\[
p_K(x) := \inf\{\lambda > 0 : x \in \lambda K\}.
\]

If \( 0 \in \text{int}(K) \) then the polar body \( K^0 \) of \( K \) is given by

\[
K^0 := \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in K\}.
\]
Volume is denoted by \( \text{vol}(\cdot) \) and diameter by \( \text{diam}(\cdot) \). We consider in \( \mathbb{R}^n \) the Euclidean structure \( \langle \cdot, \cdot \rangle \) and denote by \( \| \cdot \|_2 \) the corresponding Euclidean norm. We write \( B^n_2 \) and \( S^{n-1} \) for the corresponding Euclidean unit ball and unit sphere respectively.

We say that a convex body \( K \) is in John’s position if the ellipsoid of maximal volume inscribed in \( K \) is the Euclidean unit ball \( B^n_2 \). John’s classical theorem states that \( K \) is in John’s position if and only if \( B^n_2 \subseteq K \) and there exist \( u_1, \ldots, u_m \in \text{bd}(K) \cap S^{n-1} \) (contact points of \( K \) and \( B^n_2 \)) and positive real numbers \( a_1, \ldots, a_m \) such that

\[
\sum_{j=1}^{m} a_j u_j = 0
\]

and the identity operator \( I_n \) is decomposed in the form

\[
I_n = \sum_{j=1}^{m} a_j u_j \otimes u_j,
\]

where the rank-one operator \( u_j \otimes u_j \) is simply \( (u_j \otimes u_j)(y) = (u_j, y)u_j \).

If \( u_1, \ldots, u_m \) are unit vectors that satisfy John’s decomposition (1) with some positive weights \( a_j \). Then, one has the useful equalities

\[
\sum_{j=1}^{m} a_j = \text{tr}(I_n) = n \quad \text{and} \quad \sum_{j=1}^{m} a_j \langle u_j, z \rangle^2 = 1
\]

for all \( z \in S^{n-1} \). Moreover,

\[
\operatorname{conv}\{v_1, \ldots, v_m\} \supseteq \frac{1}{n} B^n_2.
\]

The body \( K \) is in Lwner position if the minimal volume ellipsoid that contains it is the Euclidean ball \( B^n_2 \). In that case, we also have a decomposition of the identity as before.

Given two matrices \( A, B \in \mathbb{R}^{n \times n} \) we write \( A \preceq B \) whenever \( B - A \) is positive semidefinite.

The letters \( c, c', C, C' \) etc. will always denote absolute positive constants which may change from line to line.

3. **Continuous Helly-type result for the volume: Theorem 3.1.**

As mentioned above we follow the proof of \cite[Theorem 1.4.]{Bra17a}. We include all the steps for completeness.

The following Brascamp-Lieb type inequality for approximate John’s decomposition of the identity will be crucial.

**Theorem 3.1.** \cite[Theorem 5.4]{Bra17a} Let \( \gamma > 1 \). Let \( u_1, \ldots, u_s \in S^{n-1} \) and \( a_1, \ldots, a_s > 0 \) satisfy

\[
I_{d_n} \preceq A := \sum_{j=1}^{s} a_j u_j \otimes u_j \preceq \gamma I_{d_n}
\]
and let $k_j = a_j \langle A^{-1} u_j, u_j \rangle > 0$, $1 \leq j \leq s$. If $f_1, \cdots, f_s : \mathbb{R} \rightarrow \mathbb{R}^+$ integrable functions then
\[
\int_{\mathbb{R}^n} \prod_{j=1}^s f_j^{k_j} (\langle x, u_j \rangle) \, dx \leq \gamma^n \prod_{j=1}^s \left( \int_{\mathbb{R}} f_j(t) \, dt \right)^{k_j}.
\]

We now prove Theorem 1.3.

Proof. (of Theorem 1.3)

Without loss of generality we assume that $P := \bigcap_{i \in J} H_i$ is in John’s position. Therefore there exist $J \subseteq I$ and vectors $(u_j)_{j \in J}$ which are contact points between $P$ and $S^{n-1}$ and $(a_j)_{j \in J}$ positive numbers, such that
\[
Id_n = \sum_{j \in J} a_j u_j \otimes u_j\quad \text{and} \quad \sum_{j \in J} a_j u_j = 0.
\]

Using Friedland and Youssef’s approximate decomposition, Theorem 1.5, we can find a multi-set $\sigma \subseteq J$ with $|\sigma| \leq \frac{1}{\sqrt{n}}$ and a vector $u = \frac{1}{|\sigma|} \sum_{j \in \sigma} u_j$ such that
\[
(1 - \varepsilon) Id_n \preceq \frac{n}{|\sigma|} \sum_{j \in \sigma} (u_j + u) \otimes (u_j + u) \preceq (1 + \varepsilon) Id_n,
\]
also satisfying that $\frac{2}{|\sigma|} \sum_{j \in \sigma} u_j + u = 0$ and $|u| \leq \frac{2n}{\sqrt{n}}$.

We consider the vector $w := \frac{3n}{2\sqrt{n}}$. Recall that $\frac{1}{n} B^2_2 \subseteq \text{conv}\{u_j, j \in J\}$, thus $\|w\|_2 \leq \frac{1}{n}$ and hence $w \in \text{conv}\{u_j, j \in J\}$. By Carathodory’s Theorem, we know that there is $\tau \subseteq J$, with $|\tau| \leq n + 1$ and $\rho_i > 0$, $i \in \tau$ such that
\[
w = \sum_{i \in \tau} \rho_i u_i\quad \text{and} \quad \sum_{i \in \tau} \rho_i = 1.
\]

Also notice that, since $u = \frac{1}{|\sigma|} \sum_{j \in \sigma} u_j$ and $\sum_{j \in \sigma} \frac{1}{|\sigma|} = 1$, $-u \in \text{conv}\{u_j, j \in \sigma\}$. Therefore, we have that the segment $[-u, \frac{3n}{2\sqrt{n}}]$ is contained in $\text{conv}\{u_j, j \in \sigma \cup \tau\}$. For $j \in \sigma$ we define
\[
v_j := \sqrt{\frac{n}{n+1}} \left( -u_j, \frac{1}{\sqrt{n}} \right)\quad \text{and} \quad b_j = \frac{n+1}{|\sigma|}.
\]

Set $v := -\sqrt{\frac{n}{n+1}(u, 0)}$. So, we have
\[
\sum_{j \in \sigma} b_j (v_j + v) \otimes (v_j + v) = \sum_{j \in \sigma} \frac{n}{|\sigma|} \left( -(u_j + u), \frac{1}{\sqrt{n}} \right) \otimes \left( -(u_j + u), \frac{1}{\sqrt{n}} \right)
\]
\[
= \left( \sum_{j \in \sigma} \frac{n}{|\sigma|} (u_j + u) \otimes (u_j + u) \right) \frac{n}{|\sigma|} \sum_{j \in \sigma} (u_j + u)
\]
\[
= \left( \sum_{j \in \sigma} \frac{n}{|\sigma|} (u_j + u) \otimes (u_j + u) \right) 0
\]
\[
= \left( \sum_{j \in \sigma} \frac{1}{|\sigma|} (u_j + u) \otimes (u_j + u) \right) 1,
\]

which implies
\[
(1 - \varepsilon) Id_{n+1} \preceq \sum_{j \in \sigma} b_j (v_j + v) \otimes (v_j + v) \preceq (1 + \varepsilon) Id_{n+1}.
\]
The sum $\sum_{j \in \sigma} b_j(v_j + v) \otimes (v_j + v)$ can be written as

$$\sum_{j \in \sigma} b_j v_j \otimes v_j + v \otimes \left( \sum_{j \in \sigma} b_j v_j \right) + \left( \sum_{j \in \sigma} b_j v_j \right) \otimes v + (n + 1)v \otimes v,$$

and notice that since

$$\sum_{j \in \sigma} b_j v_j = \sum_{j \in \sigma} \frac{n + 1}{|\sigma|} \sqrt{\frac{n}{n + 1}} \left( -u_j, \frac{1}{\sqrt{n}} \right)$$

$$= \sqrt{\frac{n + 1}{n}} \left( -\sum_{j \in \sigma} \frac{n}{|\sigma|} u_j, \frac{1}{|\sigma|} \sum_{j \in \sigma} \sqrt{n} \right)$$

$$= \sqrt{\frac{n + 1}{n}} (nu, \sqrt{n}),$$

we obtain that

$$\left( \sum_{j \in \sigma} b_j v_j \right) \otimes v = \sqrt{\frac{n + 1}{n}} (nu, \sqrt{n}) \otimes \sqrt{\frac{n}{n + 1}} (-u, 0) = \left( -nu \otimes u \ 0 \ 0 \right),$$

$$v \otimes \left( \sum_{j \in \sigma} b_j v_j \right) = \left( -nu \otimes u \ -\sqrt{nu} \ 0 \right),$$

and $(n + 1)v \otimes v = \left( nu \otimes u \ 0 \ 0 \right)$.

Hence, we can write Equation (3) as

$$(1 - \varepsilon)Id_{n+1} - T \preceq \sum_{j \in \sigma} b_j v_j \otimes v_j \preceq (1 + \varepsilon)Id_{n+1},$$

where $T = v \otimes \left( \sum_{j \in \sigma} b_j v_j \right) + \left( \sum_{j \in \sigma} b_j v_j \right) \otimes v + (n + 1)v \otimes v = \left( V \ z \ 0 \right)$, with $V = -nu \otimes u \ y \ z = -\sqrt{nu}$. Now, for $(x, t) \in S^n$ we have that

$$\langle T(x, t), (x, t) \rangle = \langle (Vx + zt, \langle z, x \rangle), (x, t) \rangle$$

$$= \langle (Vx, 0), (x, t) \rangle + \langle (zt, \langle z, x \rangle), (x, t) \rangle$$

$$\leq \langle Vx, x \rangle + \|zt, \langle z, x \rangle\| |t, x| = \langle Vx, x \rangle + |zt|^2 + |\langle z, x \rangle|^2$$

$$\leq \|V\| |x|^2 + |z|^2 + |\langle z, x \rangle|^2 \leq \|V\| + |z|^2 + |x|^2$$

$$= \|V\| + |z| = \|V\| + |z| = n|u|^2 + \sqrt{n}|u|$$

$$\leq n \frac{4\varepsilon^2}{9n} + \frac{2\varepsilon}{3} \frac{2\varepsilon}{9} + \frac{2\varepsilon}{3}$$

$$\leq \varepsilon,$$

for $\varepsilon$ small enough (say $\varepsilon \leq \frac{4}{9}$). So, $\|T\| \leq \varepsilon$, and hence Equation (3) implies that

$$(1 - 2\varepsilon)Id_{n+1} \preceq A := \sum_{j \in \sigma} b_j v_j \otimes v_j \preceq (1 + 2\varepsilon)Id_{n+1},$$
or equivalently
\[ \text{Id}_{n+1} \preceq \sum_{j \in \sigma} \frac{b_j}{1 - 2\varepsilon} v_j \otimes v_j \preceq \gamma \text{Id}_{n+1}, \]
with \( \gamma = \frac{1 + 2\varepsilon}{1 - 2\varepsilon} \). Applying Theorem 3.1 if \( f_j : \mathbb{R} \rightarrow \mathbb{R}^+ \) are measurable functions, then
\[
\int_{\mathbb{R}^{n+1}} \prod_{j \in \sigma} f_j^k_j \, (\langle x, v_j \rangle) \, dx \leq \gamma \frac{n+1}{1 - 2\varepsilon} \left( \int_{\mathbb{R}} f_j(t) \, dt \right)^{k_j},
\]
where
\[
k_j = \frac{b_j}{1 - 2\varepsilon} \left( \left( \frac{1}{1 - 2\varepsilon} A \right)^{-1} v_j, v_j \right) = b_j \langle A^{-1} v_j, v_j \rangle.
\]
Since \( A^{-1} \preceq \frac{1}{1 - 2\varepsilon} \text{Id}_{n+1} \), we have that \( k_j b_j \leq \frac{1}{1 - 2\varepsilon} \). Now for \( j \in \sigma \) we consider \( f_j(t) := e^{-\frac{b_j}{b_j} t} \chi_{[0,\infty)}(t) \). So,
\[
\int_{\mathbb{R}^{n+1}} \prod_{j \in \sigma} f_j^k_j \, (\langle x, v_j \rangle) \, dx \leq \gamma \frac{n+1}{1 - 2\varepsilon} \left( \int_{\mathbb{R}} f_j(t) \, dt \right)^{k_j}
= \frac{\gamma \frac{n+1}{1 - 2\varepsilon}}{b_j} \prod_{j \in \sigma} k_j
\leq \gamma \frac{n+1}{1 - 2\varepsilon} \frac{1}{\sum_{j \in \sigma} k_j} = \gamma \frac{n+1}{1 - 2\varepsilon} \frac{1}{n+1}
= \left( \frac{1 + 2\varepsilon}{1 - 2\varepsilon} \right)^{n+1}.
\]
Set
\[ Q = \bigcap_{i \in \sigma \cup \tau} H_i = \{ x \in \mathbb{R}^n : \langle x, u_j \rangle < 1, \ j \in \sigma \cup \tau \}, \]
and let \( y = (x, r) \in \mathbb{R}^{n+1} \). Assume that \( r > 0 \) and \( x \in \frac{r}{\sqrt{n}} Q \). Then we have that \( \langle x, u_j \rangle < \frac{1}{\sqrt{n}} \) for every \( j \in \sigma \), which implies that \( \langle y, v_j \rangle > 0 \) for every \( j \in \sigma \), and then \( \prod_{j \in \sigma} f_j^k_j (\langle y, v_j \rangle) > 0 \). We also have that
\[
\left\langle \frac{1}{|\sigma|} \sum_{j \in \sigma} u_j, x \right\rangle = \langle -u, x \rangle = \frac{2\sqrt{n}\varepsilon}{3} \langle -w, x \rangle
= \frac{2\sqrt{n}\varepsilon}{3} \left( \sum_{i \in \tau} \rho_i u_i, x \right) \geq \frac{-2\sqrt{n}\varepsilon}{3} \left( \sum_{i \in \tau} \rho_i \right) \frac{r}{\sqrt{n}}
= \frac{-2\varepsilon r}{3}.
\]
Thus, if \( y = (x, r) \in \frac{2}{\sqrt{n}}Q \times (0, \infty) \), then

\[
\prod_{j \in \sigma} f^k_j((y, v_j)) = \exp \left( - \sum_{j \in \sigma} b_j \left( \frac{r}{\sqrt{n}+1} - \sqrt{\frac{n}{n+1}} \langle x, u_j \rangle \right) \right) \\
= \exp \left( \frac{-r}{\sqrt{n}+1} \sum_{j \in \sigma} b_j \right) \exp \left( \sqrt{n} \sqrt{n+1} \left( x, \frac{1}{|\sigma|} \sum_{j \in \sigma} u_j \right) \right) \\
\geq e^{-r \sqrt{n+1}} e^{-r \sqrt{n+1} \epsilon r} = e^{-r \sqrt{n+1} (1+\frac{2}{3} \sqrt{n})}.
\]

Now, by Theorem 3.1,

\[
\frac{\text{vol}(Q)}{n^{\frac{m}{2}}} \int_0^\infty r^n e^{-r \sqrt{n+1}} (1+\frac{2}{3} \sqrt{n}) \, dr = \int_0^\infty \int_{\frac{2}{\sqrt{n}}Q} e^{-r \sqrt{n+1}} (1+\frac{2}{3} \sqrt{n}) \, dx \, dr \\
\leq \int_0^\infty \prod_{j \in \sigma} f^k_j((y, v_j)) \, dy \\
\leq \left( \frac{1+2\epsilon}{1-2\epsilon} \right)^{\frac{n+1}{n}}.
\]

Using that \( B_2^m \subseteq P \), and the fact that

\[
\int_0^\infty r^n e^{-r \sqrt{n+1}} (1+\frac{2}{3} \sqrt{n}) \, dr = \frac{n!}{(n+1)^{\frac{n+1}{2}} (1+\frac{2}{3} \sqrt{n})^{n+1}},
\]

we obtain, by taking \( 1+\epsilon' = \frac{1+2\epsilon}{(1-2\epsilon)} \),

\[
\text{vol}(\bigcap_{i \in \sigma \cup \tau} H_i) = \text{vol}(Q) \leq \frac{(1+\epsilon')^{\frac{n+1}{2}} n^{\frac{m}{2}} (n+1)^{\frac{m+1}{2}} (1+\frac{2}{3} \sqrt{n})^{n+1}}{n!} \text{vol}(P) \\
= \frac{(1+\epsilon')^{\frac{n+1}{2}} n^{\frac{m}{2}} (n+1)^{\frac{m+1}{2}} (1+\frac{2}{3} \sqrt{n})^{n+1}}{n!} \frac{\Gamma \left( \frac{m}{2} + 1 \right) \text{vol}(P)}{\pi^{\frac{m}{2}}}.
\]

By Stirling's formula we get, for a constant \( C > 0 \), the inequality

\[
\text{vol}(\bigcap_{i \in \sigma \cup \tau} H_i) \leq C \frac{(1+\epsilon')^{\frac{n+1}{2}} n^{\frac{m}{2}} (n+1)^{\frac{m+1}{2}} (1+\frac{2}{3} \sqrt{n})^{n+1}}{n!} \frac{n^{n(n+1)}}{2 \sqrt{2\pi n (\frac{2}{3} \sqrt{n})^n}} \frac{4\pi n}{n!} \left( \frac{n}{2\epsilon} \right)^{\frac{h}{2}} \text{vol}(P) \\
= C \frac{(1+\epsilon')^{\frac{n+1}{2}} (1+\frac{2}{3} \sqrt{n})^{n+1}}{n^{n(n+1)}} \frac{n^n n(n+1) \epsilon}{(n+1)^{\frac{m+1}{2}}} \left( \frac{\epsilon}{2\pi} \right)^{\frac{1}{2}} \sqrt{\frac{1}{2}} \text{vol}(P) \\
= C \frac{(1+\epsilon')^{\frac{n+1}{2}} (1+\frac{2}{3} \sqrt{n})^{n+1}}{n^{n(n+1)}} \frac{n^{n(n+1)}}{(n+1)^{\frac{m+1}{2}}} \left( \frac{\epsilon}{2\pi} \right)^{\frac{1}{2}} \sqrt{\frac{1}{2}} \text{vol}(P).
\]

Fix \( \epsilon := \frac{1}{2} n^{1-\delta}/2 \), using that \( 1+\epsilon' = \frac{1+2\epsilon}{(1-2\epsilon)} \), we have

\[
(1+\epsilon') \left( 1+\frac{2}{3} \sqrt{n} \right)^{2} \frac{e}{2\pi} = (1+\epsilon') \left( 1+\frac{1}{6} n^{(2-\delta)/2} \right)^{2} \frac{e}{2\pi} < cn^{2-\delta}.
\]
Therefore,
\[
\text{vol}\left( \bigcap_{i \in \sigma \cup \tau} H_i \right) \leq C_n \left( 1 + \frac{1}{n} \right)^{\frac{2}{\delta}} \sqrt{n + 1} \frac{n^{(2-\delta)/2}}{n^{\sqrt{2 \pi \sqrt{2 \pi}}} \text{vol}(P)}
\]
\[
\leq C_1 \sqrt{\frac{\varepsilon(n + 1) \pi}{\varepsilon}} n^{\frac{\delta}{2}} n^{(2-\delta)/2} \text{vol}(P)
\]
\[
= \left( C_1 \sqrt{n + 1} n^{(2-\delta)/2} \sqrt{\pi} \right) n^{\frac{(3-\delta)}{2}} \text{vol}(P).
\]

We conclude that
\[
\text{vol}\left( \bigcap_{i \in \sigma \cup \tau} H_i \right) \leq C_n n^{\frac{(3-\delta)}{2}} \text{vol}(P),
\]
where the intersection is taken over at most \(|\sigma \cup \tau| \leq \frac{2n}{\varepsilon} + n + 1 = \frac{n^\delta}{c} + n + 1 \leq \alpha n^\delta\) half-spaces. Since the constant \(C_n\) is of order \(n^{(3-\delta)/2}\), we have that \(d_n := C_1/n \to 1\) as \(n \to \infty\).

It should be mentioned that the case \(\delta = 2\) is of course easier (we just use John’s decomposition of the identity and the classical Brascamp-Lieb inequality directly). \(\square\)

4. Continuous Helly-type theorem for the diameter

To obtain Theorem 1.4 we prove the following proposition, which is a continuous version of [Bra17b, Proposition 4.2.]. We feel it is interesting in its own right. Again we include all the steps for completeness.

**Proposition 4.1.** Let \(1 \leq \delta \leq 2\). If \(K\) is a convex body whose minimal volume ellipsoid is the Euclidean unit ball, then there is a subset \(X \subseteq K \cap S^{n-1}\) of cardinality \(\text{card}(X) \leq \alpha n^\delta\) and
\[
K \subseteq B_2^n \subseteq Cn^{2-\frac{\delta}{2}} \text{conv}(X),
\]
where \(\alpha, C > 0\) are absolute constant.

**Proof.** By John’s theorem there exist \(v_j \in K \cap S^{n-1}\) and \(a_j > 0, j \in J\), such that
\[
I_n = \sum_{j \in J} a_j v_j \otimes v_j \text{ and } \sum_{j \in J} a_j v_j = 0.
\]
Let \(\varepsilon > 0\) small enough to be fixed later. By Theorem 1.3 we can find a multiset \(\sigma \subseteq J\) of cardinal \(|\sigma| \leq \frac{n}{\varepsilon^2}\) such that
\[
(1 - \varepsilon)I_n \leq \sum_{j \in \sigma} (v_j + v) \otimes (v_j + v) \leq (1 + \varepsilon)I_n,
\]
where \(v = \frac{1}{|\sigma|} \sum_{j \in \sigma} v_j\) satisfies \(\|v\|_2 \leq \frac{2}{\varepsilon \sqrt{n}}\).

Then, the vector \(w = \frac{3}{2\sqrt{\varepsilon n}}\) satisfies \(\|w\|_2 \leq \frac{1}{n}\) and therefore by Equation 2, it belongs to \(\text{conv}\{v_j : j \in J\}\). By Carathéodory’s theorem there exist \(\tau \subseteq J\) with \(|\tau| \leq n + 1\) and \(\rho_i > 0, i \in \tau\) such that
\[
w = \sum_{i \in \tau} \rho_i v_i, \text{ and } \sum_{i \in \tau} \rho_i = 1.
\]
Indeed, let 

\[ x \]

Note that 

\[ |v - \sum_{j \in \sigma} v_j| = |v - \sum_{j \in \sigma} \sum_{k \in \sigma} v_j| \leq \sum_{j \in \sigma} |v_j - \sum_{k \in \sigma} v_j| \leq \sum_{j \in \sigma} |v_j| \]

Observe also that 

\[ (C/E) \sum_{j \in \sigma} v_j \leq \sum_{j \in \sigma} \sum_{j \in \sigma} v_j \leq |v - \sum_{j \in \sigma} v_j| \leq |v - \sum_{j \in \sigma} v_j| \]

Therefore we have 

\[ (1 - 2\varepsilon)I_n \leq (1 - \varepsilon)I_n - T \leq \frac{n}{|\sigma|} \sum_{j \in \sigma} v_j - 2 \|\sum_{j \in \sigma} v_j\| \leq (1 + \varepsilon)I_n - T \leq (1 + 2\varepsilon)I_n. \]

Define \( X := \{v_j : j \in \sigma \cup \tau\} \) and \( E := \text{conv}(X) \). Let us show that \( B^n_2 \subseteq c \varepsilon n^{3/2} E \). Indeed, let \( x \in S^{n-1} \); set \( A := \frac{n}{|\sigma|} \sum_{j \in \sigma} v_j \) and \( \rho := \min\{\langle x, v_j \rangle : j \in \sigma\} \). Note that \( |\rho| \leq 1 \) and \( \langle x, v_j \rangle - \rho \leq 2 \) for all \( j \in \sigma \).

If \( \rho < 0 \) we have 

\[ p_E(Ax) \leq p_E \left( Ax - \rho \frac{n}{|\sigma|} \sum_{j \in \sigma} v_j \right) + p_E \left( \rho \frac{n}{|\sigma|} \sum_{j \in \sigma} v_j \right) \]

\[ = p_E \left( \sum_{j \in \sigma} \frac{n}{|\sigma|} \langle x, v_j \rangle v_j - \rho v \right) + p_E(n\rho(-v)) \]

\[ \leq \sum_{j \in \sigma} \frac{n}{|\sigma|} \langle x, v_j \rangle v_j - n\rho p_E(v) \]

\[ \leq n \left( 2 + \frac{2\sqrt{n\varepsilon}}{3} p_E(w) \right) \]

\[ \leq c_1 \varepsilon n^{3/2}, \]

where we are using that \( w \in K \) and therefore \( p_E(w) \leq 1 \).

On the other hand, if \( \rho \geq 0 \), then \( \langle x, v_j \rangle \geq 0 \) for all \( j \in \sigma \), therefore 

\[ p_E(Ax) = p_E \left( \frac{n}{|\sigma|} \sum_{j \in \sigma} \langle x, v_j \rangle v_j \right) \leq \frac{n}{|\sigma|} \sum_{j \in \sigma} \langle x, v_j \rangle p_E(v_j) \leq n. \]

This say that 

\[ p_{A^{-1}(E)}(x) \leq c_2 \varepsilon n^{3/2} \]

for all \( x \in S^{n-1} \), where \( c_2 > 0 \) is an absolute constant.

Therefore we have 

\[ (1 - 2\varepsilon)B^n_2 \subseteq A(B^n_1) \subseteq c_2(1 + 2\varepsilon)\varepsilon n^{3/2} E. \]

Finally, fix \( \varepsilon := \frac{1}{4} n^{1/2 - \delta} \). Since \( K \) is in Lwnner’s position

\[ K \subseteq B^n_2 \subseteq C_{2\varepsilon} \frac{1 + 2\varepsilon}{1 - 2\varepsilon} \varepsilon n^{3/2} \subseteq Cn^{2 - \delta} \text{conv}(X), \]

with \( |X| = |\sigma \cup \tau| \leq cn^\delta + n + 1 \leq cn^\delta \).

Let us now see the proof of the Theorem 1.3.
Proof. Consider $P := \bigcap_{i \in I} C_i$. Without loss of generality we can assume that $0 \in \text{int}(P)$ and that the polar body

$$P^o = \text{conv}(\bigcap_{i \in I} C_i^o)$$

is in Lwner’s position. Using Proposition 4.1 for the body $K = P^o$, we know there exists a set $X = \{v_1, \ldots, v_s\} \subseteq P^o \cap S^{n-1}$ such that $|X| \leq an^\delta$ and

$$P^o \subseteq Cn^{2-\frac{\delta}{2}} \text{conv}(X),$$

where $C > 0$ is an absolute constant. Since $v_1, \ldots, v_s$ are contact points between $P^o$ and $B_2^n$, then we have that $v_j \in \bigcap_{i \in I} C_i^o$ for all $j = 1, \ldots, s$. This implies that there exist $s \leq an^\delta$ and bodies $\{C_i\}$, such that $v_j \in C_i$ for all $j = 1, \ldots, s$. Then

$$\text{conv}(X) \subseteq \text{conv}(C_1 \cup \cdots \cup C_s),$$

and hence

$$P^o \subseteq Cn^{2-\frac{\delta}{2}} \text{conv}(C_1 \cup \cdots \cup C_s).$$

This shows that

$$C_{i_1} \cap \cdots \cap C_{i_s} \subseteq cn^{2-\frac{\delta}{2}} P,$$

and therefore we have the following estimate for the diameter

$$\text{diam}(C_{i_1} \cap \cdots \cap C_{i_s}) \leq cn^{2-\frac{\delta}{2}}.$$

This concludes the proof.

5. Final comments: symmetry assumption

It is well-known that if all the bodies are symmetric the bounds for these kind of results are better (see, for example, [Bra17a, Theorem 1.2] and [Bra17b, Theorem 1.2]). In that case, for a linear number of convex sets, the bounds are of order $n^{1/2}$. One should be tempted to think that relaxing the number of sets in these statements provides again stronger estimates but, unfortunately, we cannot have these type of continuous versions as before. Indeed, the exponent in $n$ in cannot be improved by allowing more sets: for example we can find $w_1, \ldots, w_N \in S^{n-1}$ (assuming that $N$ is exponential in the dimension $n$) such that

$$B_2^n \subseteq \bigcap_{j=1}^N H_j \subseteq 2B_2^n,$$

where $H_j$ is defined as the strip

$$H_j = \{x \in \mathbb{R}^n : |\langle x, w_j \rangle| \leq 1\}.$$

Thus, if $s = n^\delta$ with $\delta > 1$, for any choice of $j_1, \ldots, j_s \in \{1, \ldots, N\}$ we can use the classical lower bound for the volume due to Carl-Pajor [CP88] and Gluskin [Glu89], which shows that

$$|H_{j_1} \cap \cdots \cap H_{j_s}|^{1/n} \geq \frac{C}{\sqrt{\log(n)}}.$$

Therefore, if $H_{j_1} \cap \cdots \cap H_{j_s} \subseteq \beta \bigcap_{j=1}^N H_j$ for some $\beta > 0$, by comparing its volumes we obtain that

$$\beta \geq \frac{|H_{j_1} \cap \cdots \cap H_{j_s}|^{1/n}}{|2B_2^n|^{1/n}} \geq \frac{\sqrt{n}}{\sqrt{\log n}},$$

assuming that $N$ is exponential in the dimension $n$. 

where $c > 0$ is an absolute constant.

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