NEW RESULTS ON THE HADRONIC CONTRIBUTIONS
TO \( \alpha(M_Z^2) \) AND TO \( (g - 2)_\mu \)

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Abstract

We reevaluate the dispersion integrals of the leading order hadronic contributions to the running of the QED fine structure constant \( \alpha(s) \) at \( s = M_Z^2 \), and to the anomalous magnetic moments of the muon and the electron. Finite-energy QCD sum rule techniques complete the data from \( e^+e^- \) annihilation and \( \tau \) decays at low energy and at the \( c\bar{c} \) threshold. Global quark-hadron duality is assumed in order to resolve the integrals using the Operator Product Expansion wherever it is applicable. We obtain \( \Delta \alpha_{\text{had}}(M_Z^2) = (276.3 \pm 1.6) \times 10^{-4} \) yielding \( \alpha^{-1}(M_Z^2) = 128.933 \pm 0.021 \), and \( a_{\mu}^{\text{had}} = (692.4 \pm 6.2) \times 10^{-10} \) with which we find for the complete Standard Model prediction \( a_{\mu}^{\text{SM}} = (11\,659,\,159.6 \pm 6.7) \times 10^{-10} \). For the electron, the hadronic contribution reads \( a_{e}^{\text{had}} = (187.5 \pm 1.8) \times 10^{-14} \). The following formulae express our results on the running of \( \alpha \) at \( M_Z^2 \) as a function of the input value for \( \alpha_s(M_Z^2) \) and its error:

\[
\Delta \alpha_{\text{had}}(M_Z^2) = 249.8 + 221 \alpha_s(M_Z^2) \pm 1.5 \pm 221 \Delta \alpha_s(M_Z^2) \ ,
\]

\[
\alpha^{-1}(M_Z^2) = 129.297 - 3.03 \alpha_s(M_Z^2) \pm 0.020 \pm 3 \Delta \alpha_s(M_Z^2) \ .
\]

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Introduction

The running of the QED fine structure constant $\alpha(s)$ and the anomalous magnetic moment of the muon are observables whose theoretical precisions are limited by second order loop effects from hadronic vacuum polarization. Both magnitudes are related via dispersion relations to the hadronic production rate $R$ in $e^+e^-$ annihilation. While far from quark thresholds and at sufficiently high center-of-mass energy $\sqrt{s}$, $R(s)$ can be predicted by perturbative QCD, theory may fail when resonances occur, i.e., local quark-hadron duality is broken. Fortunately, one can circumvent this drawback by using $e^+e^-$ annihilation data for $R(s)$ and, as proposed in Ref. [1], hadronic $\tau$ decays benefitting from the conserved vector current (CVC). With help from a moment analysis using weighted integrals over low-energy $e^+e^-$ cross sections we showed in Ref. [2] that the Operator Product Expansion (OPE) [3] (also called SVZ approach [4]) can safely be applied down to energies of $\sqrt{s} = 1.8$ GeV. It turned out that at this energy nonperturbative contributions to the dispersion integrals are very small.

In this letter, we improve our previous determinations by using finite-energy QCD sum rule techniques in order to access theoretically energy regions where perturbative QCD fails. This idea was recently advocated in Ref. [5]. In principle, the method uses no additional assumptions beyond those applied in Ref. [2]. However, parts of the dispersion integrals evaluated at low-energy and the $c\bar{c}$ threshold are obtained from values of the Adler $D$-function itself, where we assume local quark-hadron duality to hold. We therefore perform an evaluation at rather high energies (3 GeV for $u,d,s$ quarks and 15 GeV for the $c$ quark contribution) to suppress deviations from the local duality assumption by nonperturbative phenomena. We present a thorough evaluation of the associated theoretical and experimental uncertainties.

Running of the QED Fine Structure Constant

The running of the electromagnetic fine structure constant $\alpha(s)$ is governed by the renormalized vacuum polarization function, $\Pi_\gamma(s)$. For the spin 1 photon, $\Pi_\gamma(s)$ is given by the Fourier transform of the time-ordered product of the electromagnetic currents $j_{\mu mn}(x)$ in the vacuum $(q^\mu q^\nu - q^2 g^{\mu\nu}) \Pi_\gamma(q^2) = i \int d^4x e^{iqx} \langle 0 | T (j_{\mu mn}(x) j_{\nu mn}(0)) | 0 \rangle$. With $\Delta \alpha(s) = -4\pi \alpha \text{Re} [\Pi_\gamma(s) - \Pi_\gamma(0)]$ and $\Delta \alpha(s) = \Delta \alpha_{\text{lep}}(s) + \Delta \alpha_{\text{had}}(s)$, which subdivides the running contributions into a leptonic and a hadronic part, one has

$$\alpha(s) = \frac{\alpha(0)}{1 - \Delta \alpha_{\text{lep}}(s) - \Delta \alpha_{\text{had}}(s)},$$

(1)

where $4\pi \alpha(0)$ is the square of the electron charge in the long-wavelength Thomson limit.

For the case of interest, $s = M_Z^2$, the leptonic contribution at three-loop order has recently been calculated to be [3]

$$\Delta \alpha_{\text{lep}}(M_Z^2) = 314.97686 \times 10^{-4}.$$
Using analyticity and unitarity, the dispersion integral for the contribution from the hadronic vacuum polarization reads\textsuperscript{[7]}

\[ \Delta \alpha_{\text{had}}(M_Z^2) = -\frac{\alpha(0)M_Z^2}{3\pi} \text{Re} \int ds \frac{R(s)}{s(s-M_Z^2)-i\epsilon}, \]  

(3)

and, employing the identity \[1/(x' - x - i\epsilon)\epsilon \rightarrow 0 = \text{P}\{1/(x' - x)\} + i\pi\delta(x' - x),\] the above integral is evaluated using the principle value integration technique.

**Muon Magnetic Anomaly**

It is convenient to separate the Standard Model prediction for the anomalous magnetic moment of the muon, \(a^\text{SM}_\mu \equiv (g-2)_\mu/2\), into its different contributions,

\[ a^\text{SM}_\mu = a^\text{QED}_\mu + a^\text{had}_\mu + a^\text{weak}_\mu, \]  

(4)

where \(a^\text{QED}_\mu = (11\,658\,470.6 \pm 0.2) \times 10^{-10}\)\textsuperscript{[8]} is the pure electromagnetic contribution (see \[8\] and references therein), \(a^\text{had}_\mu\) is the contribution from hadronic vacuum polarization, and \(a^\text{weak}_\mu = (15.1 \pm 0.4) \times 10^{-10}\)\textsuperscript{[8, 9, 10]} accounts for corrections due to exchange of the weak interacting bosons up to two loops.

Equivalently to \(\Delta \alpha_{\text{had}}(M_Z^2)\), by virtue of the analyticity of the vacuum polarization correlator, the contribution of the hadronic vacuum polarization to \(a_\mu\) can be calculated via the dispersion integral\textsuperscript{[11]}

\[ a^\text{had}_\mu = \frac{\alpha^2(0)}{3\pi^2} \int ds \frac{K(s)}{s} R(s), \]  

(5)

where \(K(s)\) denotes the QED kernel\textsuperscript{[12]},

\[ K(s) = x^2 \left(1 - \frac{x^2}{2}\right) + (1 + x)^2 \left(1 + \frac{1}{x^2}\right) \left(\ln(1+x) - x + \frac{x^2}{2}\right) + \frac{(1+x)}{(1-x)} x^2 \ln x, \]  

(6)

with \(x = (1 - \beta_\mu)/(1 + \beta_\mu)\) and \(\beta_\mu = (1 - 4m_\mu^2/s)^{1/2}\). The function \(K(s)\) decreases monotonically with increasing \(s\). It gives a strong weight to the low energy part of the integral\textsuperscript{[5]}. About 92% of the total contribution to \(a^\text{had}_\mu\) is accumulated at c.m. energies \(\sqrt{s}\) below 1.8 GeV and 72% of \(a^\text{had}_\mu\) is covered by the two-pion final state which is dominated by the \(\rho(770)\) resonance. Data from vector hadronic \(\tau\) decays published by the ALEPH Collaboration\textsuperscript{[13]} provide a precise spectrum for the two-pion final state as well as new input for the lesser known four-pion final states. This new information improves significantly the precision of the \(a^\text{had}_\mu\) determination\textsuperscript{[1]}.  

**Theoretical Prediction of \(R(s)\)**

Using a method based on weighted integrals over the low-energy \(e^+e^-\) cross sections, we showed in Ref.\textsuperscript{[2]} that perturbative QCD is safely applicable for the evaluation of the
dispersion integrals (3) and (5) at energies above 1.8 GeV, leaving out the $c\bar{c}$ threshold region where resonances occur. We stress that this approach uses global quark-hadron duality only. Deviations from local duality are averaged to a negligible contribution when calculating the integral. However, a systematic uncertainty is introduced through the cut at explicitly 1.8 GeV, i.e., non-vanishing oscillations could give rise to a bias after integration. In order to estimate the associated systematic error, we fitted different oscillating curves to the (rather imprecise) data around the cut region and obtained the following estimates

$$
\Delta(\Delta \alpha_{\text{had}}(M_Z^2)) = 0.15 \times 10^{-4},
\Delta \alpha_{\mu}^{\text{had}} = 0.24 \times 10^{-10},
$$

from the comparison of the integral over the oscillating simulated data to perturbative QCD. These numbers are added as systematic uncertainties to the corresponding low-energy integrals.

In asymptotic energy regions we use the formulae of Ref. [14] which include mass corrections up to order $\alpha_s^2$ to evaluate the perturbative prediction of $R(s)$ entering into the integrals (3) and (5).

### Theoretical Approach Using Low-Energy Spectral Moments

Following the suggestion made in Ref. [14], we can write the following identity:

$$
F = \int_{4m_n^2}^{s_0} ds R(s) [f(s) - p_{n,m}(s)] + \frac{1}{2\pi i} \int \frac{ds}{s} [P_{n,m}(s_0) - P_{n,m}(s)] D_{uds}(s),
$$

with $P_{n,m}(s) = \int_0^s dt p_{n,m}(t)$ and $f(s) = \alpha(0)^2 K(s)/(3\pi^2 s)$ for $F \equiv a_{\mu}^{\text{had}}(2m_n, \sqrt{s})$, as well as $f(s) = \alpha(0)M_Z^2/(3\pi s(s - M_Z^2))$ for $F \equiv \Delta \alpha_{\text{had}}(M_Z^2)[2m_n, \sqrt{s}]$. The contour integral runs counter-clockwise around the circle from $s = s_0 - i\epsilon$ to $s = s_0 + i\epsilon$. The regular functions $p_{n,m}(s)$ approximate the kernel $f(s)$ in order to reduce the contribution of the first integral in Eq. (3) which has a singularity at $s = 0$ and is thus evaluated using experimental data. The second integral in Eq. (8) can be evaluated theoretically applying QCD. To suppress unphysical subtractions we use the Adler $D$-function, defined as $D_{f_i}(s) = -12\pi^2 s d\Pi_{f_i}(s)/ds$ and $R_{f_i}(s) = 12\pi \text{Im} \Pi_{f_i}(s + i\epsilon)$ for the set of quark flavours $f_i$, to calculate theoretically the second integral in Eq. (3), rather than the correlator $\Pi_{f_i}$. Using the OPE, the D-function for massive quarks is given by [13, 14, 17]

$$
D_{f_i}(-s) = N_C \sum f_i Q_f^2 \left\{ 1 + d_0 \frac{\alpha_s(s)}{\pi} + d_1 \left( \frac{\alpha_s(s)}{\pi} \right)^2 + d_2 \left( \frac{\alpha_s(s)}{\pi} \right)^3 \right. \\
- \frac{m_f^2(s)}{s} \left( 6 + 28 \frac{\alpha_s(s)}{\pi} + (295.1 - 12.3 n_f) \left( \frac{\alpha_s(s)}{\pi} \right)^2 \right) \\
+ \frac{2\pi^2}{3} \left( 1 - \frac{11\alpha_s(s)}{18} \right) \left\langle \frac{\alpha_s}{\pi} G G \right\rangle \frac{1}{s^2}
$$

\[3\]
+ 8\pi^2 \left( 1 - \frac{\alpha_s(s)}{\pi} \right) \frac{\langle m_f \bar{q}_f q_f \rangle}{s^2} + \frac{32\pi^2 \alpha_s(s)}{27} \frac{\sum_k \langle m_k \bar{q}_k q_k \rangle}{s^2} + 12\pi^2 \frac{\langle O_6 \rangle}{s^3} + 16\pi^2 \frac{\langle O_8 \rangle}{s^4} \right),

(9)

where additional logarithms occur when \( \mu^2 \neq s \) and \( \mu \) being the renormalization scale\(^3\). The coefficients of the perturbative part read \( d_0 = 1, d_1 = 1.9857 - 0.1153 n_f, d_2 = d_2 + \beta_0^2 \pi^2/48 \) with \( \beta_0 = 11 - 2n_f/3 \) and \( n_f \) the number of involved quark flavours, and \( d_2 = -6.6368 - 1.2001 n_f - 0.0052 n_f^2 - 1.2395 (\sum_f Q_f)^2/N_C \sum_f Q_f^2 \). The nonperturbative operators in Eq. (8) are the gluon condensate, \( \langle (\alpha_s/\pi)GG \rangle \), and the quark condensates, \( \langle m_f \bar{q}_f q_f \rangle \). The latter obey approximately the PCAC relations

\[
(m_u + m_d) \langle \bar{u}u + \bar{d}d \rangle \simeq -2f_\pi^2 m_\pi, \quad m_s \langle ss \rangle \simeq -f_\pi^2 (m_K^2 - m_\pi^2),
\]

(10)

with the pion decay constant \( f_\pi = (92.4 \pm 0.26) \) MeV \(^[13]\). In the chiral limit the relations \( f_\pi = f_K \) and \( \langle \bar{u}u \rangle = \langle \bar{d}d \rangle = \langle ss \rangle \) hold. The complete dimension \( D = 6 \) and \( D = 8 \) operators are parametrized phenomenologically using the vacuum expectation values \( \langle O_6 \rangle \) and \( \langle O_8 \rangle \), respectively.

The functions \( p_{n,m}(s) \) introduced in Eq. (8) are chosen in order to reduce the uncertainty of the data integral. This approximately coincides with a low residual value of the integral, \textit{i.e.}, a good approximation of the integration kernel \( f(s) \) by the \( p_{n,m}(s) \). We use

\[
p_{n,m}(s) \equiv \frac{1}{(s + s_0 + \epsilon)^m} \sum_{i=1}^n c_i \left( 1 - \left( \frac{s}{s_0} \right)^i \right),
\]

(11)

where the form \((1 - s/s_0)\) ensures a vanishing integrand at the crossing of the positive real axis where the validity of the OPE is questioned. Polynomials of order \( s^n \) involve leading order nonperturbative contributions of dimension \( D = 2(n + 1) \). We therefore restrict the analysis to \( n < 2 \). Including additional powers \( n \geq 2 \) would require to fit the expectation values of the corresponding dimension \( D \geq 6 \) operators as they are not known theoretically and a direct application of the results on \( \tau \) data \([20]\) is dangerous since deviations from the vacuum saturation hypothesis used could be \( s \)-dependent. The new information is then spoiled since the additional fitted parameters deteriorate the accuracy of \( \Delta \alpha_{\text{had}}(M_\pi^2) \) and \( a_\mu^\text{had} \). Figure 1 shows the integration kernel of \( a_\mu^\text{had} \) and two of its \( p_{n,m} \) approximations obtained from a \( \chi^2 \) minimization. At low energies \( p_{1,20} \) shows a clearly better agreement with the kernel than \( p_{1,0} \) so that the uncertainty from the data integral in Eq. (8) is reduced. However, setting \( m > 0 \) increases dramatically the theoretical uncertainties from \( \alpha_s, m_s \) and the nonperturbative operators which are then much larger than the errors of the pure data dispersion integrals. Therefore, we choose \( m = 0 \) in the following analysis.

Several polynomials are used as spectral moments in order to adjust \( a_\mu^\text{had} \) and

\(^3\) The negative energy-squared in \( D(-s) \) of Eq. (4) is introduced when continuing the Adler function from the spacelike Euclidean space, where it is originally defined, to the timelike Minkowski space by virtue of its analyticity property.
\[ K(s) / s \]

\[ p_{1,0}(s) \]

\[ p_{1,20}(s) \]

\[ \sim T_{15}(s, 9 \text{ GeV}^2) \]

\[ s \text{ (GeV}^2) \]

Figure 1: The integration kernel \( K(s) / s \) of Eq. (3) and approximations according to the definitions Eqs. (11) and (14).

\( \Delta \alpha_{\text{had}}(M_Z^2,[2m_\pi, \sqrt{s_0}]) \) as well as the gluon condensate simultaneously in a combined fit using Eq (5). The experimental and theoretical correlations between the moments are calculated analytically from the total experimental covariance matrices and the variation of the theoretical parameters within their uncertainties.

**Theoretical Approach Using Dispersion Relations**

Another closely related method is the approximation of the integrals (3) and (5) via the dispersion relation of the Adler \( D \)-function

\[ D_f(Q^2) = Q^2 \int_{4m_f^2}^{\infty} ds \frac{R_f(s)}{(s + Q^2)^2}, \]

for space-like \( Q^2 = -q^2 \) and the quark flavour \( f \). Using the above integrand as approximation of the integration kernels in Eqs. (3) and (4), Eq. (8) becomes

\[ F = \int_{4m_f^2}^{s_0} ds R_{\text{Data}}(s) \left[ f(s) - \frac{A_F Q^2}{(s + Q^2)^2} \right] + A_F \left( D_{uds}(Q^2) - Q^2 \int_{s_0}^{\infty} ds \frac{R_{QCD}^{uds}(s)}{(s + Q^2)^2} \right), \]

with a normalization constant \( A_F \) to be optimized for both \( F \equiv a_{\mu,[2m_\pi, \sqrt{s_0}]}^{\text{had}} \) and \( F \equiv \Delta \alpha_{\text{had}}(M_Z^2,[2m_\pi, \sqrt{s_0}]) \). The last integral in Eq (13) corrects the contribution from the \( D \)-function and is calculated theoretically. One notices that in contrast to Eq. (8), where only global quark-hadron duality is assumed, Eq. (13) requires local duality to hold at \( Q^2 \). One therefore needs to choose \( Q^2 \) large against the scale where nonperturbative
phenomena appear. On the other hand, $Q^2$ too large deteriorates the approximation of
the kernel $f(s)$ in Eq. (13) enhancing the experimental uncertainty on $F$.

It is interesting to test whether derivatives of the Adler function (12), defined as

$$M_n(Q^2) \equiv \frac{(-1)^n}{(n+1)!} (Q^2)^{n+1} \frac{\partial^n (D_f(Q^2)/Q^2)}{\partial Q^{2n}} = \int \frac{ds}{4m^2} R_f(s) T_n(s, Q^2), \quad (14)$$

with $T_n(s, Q^2) = (Q^2)^{n+1}/(s + Q^2)^{n+2}$, ameliorate the evaluation of the hadronic contributions (13). Higher derivatives $n > 15$ indeed improve the approximation of the kernel function $f(s)$ by $A_f T_n(Q^2)$ as can be seen from Fig. 1, and thus reduce the experimental uncertainty from the first integral on the r.h.s. of Eq. (13). Unfortunately, it increases the theoretical errors of the $D$-function in Eq. (13). In order to demonstrate the effect we use the theoretical prediction (3) which, setting $\alpha_s(M_Z^2) = 0.1200$ and $\sqrt{Q^2} = 3$ GeV, yields for $u, d, s$ quark flavours the moments

$$M_0(9 \text{ GeV}^2) = 2.190 + 0.28 \hat{m}_s + 0.15 \langle \alpha_s/\pi \rangle GG + 1.8 \langle m_q \bar{q} q \rangle + 0.3 \langle O_6 \rangle + 0.05 \langle O_8 \rangle, \quad M_1(9 \text{ GeV}^2) = 0.468 + 0.45 \hat{m}_s + 0.46 \langle \alpha_s/\pi \rangle GG + 5.3 \langle m_q \bar{q} q \rangle + 2.3 \langle O_6 \rangle + 0.7 \langle O_8 \rangle.$$

Taking as uncertainties: $\Delta \alpha_s(M_Z^2) = 0.002$, $\Delta \hat{m}_s = 0.07$ GeV, $\Delta \langle \alpha_s/\pi \rangle GG = 0.02$ GeV$^4$, $\Delta \langle m_q \bar{q} q \rangle = 0.3 \times 10^{-4}$ GeV$^4$, $\Delta \langle O_6 \rangle = 0.01$ GeV$^6$ and $\Delta \langle O_8 \rangle = 0.01$ GeV$^8$, one obtains the relative errors $\delta M_0(9 \text{ GeV}^2) = 0.8\%$ and $\delta M_1(9 \text{ GeV}^2) = 7.9\%$, where the latter is far beyond the accuracy needed to improve the pure data results on $\Delta \alpha_{\text{had}}(M_Z^2)$ and $\alpha_{\mu}^{\text{had}}$.

**Theoretical Uncertainties**

Looking at Eq. (9) it is instructive to subdivide the discussion of theoretical uncertainties into three classes:

(i) The perturbative prediction. The estimation of theoretical errors of the perturbative series is strongly linked to its truncation at finite order in $\alpha_s$. Due to the incomplete resummation of higher order terms, a non-vanishing dependence on the choice of the renormalization scheme (RS) and the renormalization scale is left. Furthermore, one has to worry whether the missing four-loop order contribution $d_5(\alpha_s/\pi)^4$ gives rise to large corrections to the perturbative series. On the other hand, these are problems to which any measurement of the strong coupling constant is confronted with, while their impact decreases with increasing energy scale. The error on the input parameter $\alpha_s$ therefore reflects the theoretical uncertainty of the perturbative expansion in powers of $\alpha_s$. A very robust $\alpha_s$ measurement is obtained from the global electroweak fit performed at the $Z$-boson mass where uncertainties from perturbative QCD are reduced. The value found is $\alpha_s(M_Z^2) = 0.120 \pm 0.003$ [19]. A second precise $\alpha_s$ measurement is obtained from the fit of the OPE to the hadronic width of the $\tau$ lepton and to spectral moments [20], where the nonperturbative contribution was found to be lower than 1%. Additional tests in which the mass scale was reduced down to 1 GeV proved the excellent stability of the $\alpha_s$ determination. The value
recently reported by the ALEPH Collaboration is $\alpha_s(M_Z^2) = 0.1202 \pm 0.0026$ [20]. The consistency of the above values using quite different approaches at various mass scales is remarkable and supports QCD as the theory of strong interactions. Both measurements are almost uncorrelated so that we obtain the weighted average $\alpha_s(M_Z^2) = 0.1201 \pm 0.0020$, used in the following analysis.

Although contained in the above uncertainty of $\alpha_s$, we add the total difference between the results obtained using contour-improved fixed-order perturbation theory (FOPT_{CI}) and FOPT (see comments in Ref. [20]) as systematic error.

Due to the truncation of the perturbative series, the arbitrary choice of the renormalization scale $\mu$ leaves an ambiguity. When setting $\mu^2 \neq s$, additional logarithms enter the $D$-function [3]. We evaluate the corresponding uncertainty by varying $s \leq \mu^2 \leq (3/2)s$.

(ii) The quark mass correction. Since a theoretical evaluation of the integrals [8] and (13) is only applied far from quark thresholds, quark mass corrections are small. We use the following settings:

$m_{u,d} = 0$,
$m_s(1 \text{ GeV}) = (0.20 \pm 0.07) \text{ GeV}$,
$m_c(m_c) = (1.3 \pm 0.2) \text{ GeV}$,
$m_b(m_b) = (4.2 \pm 0.2) \text{ GeV}$,
$m_t(m_t) = (176 \pm 6) \text{ GeV}$.

(iii) The nonperturbative contribution. Using $n = 1$ only, the functions (11) receive direct, i.e., non-suppressed contributions from the dimension $D = 4$ terms. Associated nonperturbative parameters are the vacuum expectation values of the gluon and the quark condensates. While the latter can be obtained from the PCAC relation [10], for which a 20% uncertainty is assumed, the gluon condensate cannot be fixed theoretically. There fortunately exist experimental determinations using similar finite-energy sum rule techniques which are almost independent from the data used in this analysis: a fit using the $\tau$ vector plus axial-vector hadronic width and spectral moments yields $\langle (\alpha_s/\pi)GG \rangle = (0.001 \pm 0.015) \text{ GeV}^4$ [20] while a moment analysis using $c\bar{c}$ resonances results in $\langle (\alpha_s/\pi)GG \rangle = (0.017 \pm 0.004) \text{ GeV}^4$ [21]. An adjustment on $e^+e^-$ data showed consistent results [2]. Following the above estimates, we use a gluon condensate of

$$\langle (\alpha_s/\pi)GG \rangle = (0.015 \pm 0.020) \text{ GeV}^4$$

(15)

in this analysis.

Another tiny source of uncertainty is the error on the $Z$-boson mass.

Low-Energy Results

Due to the suppression of nonperturbative contributions in powers of the energy scale $s$, the critical domain where nonperturbative effects may give residual contributions to $R(s)$
is the low-energy regime with three active quark flavours. However, we have shown in Ref. [2] that at the scale $\sqrt{s_0} = 1.8$ GeV global duality holds and the small nonperturbative effects are described by the OPE. Up to this energy, $R(s)$ is obtained from the sum of the hadronic cross sections exclusively measured in the occurring final states.

The data analysis follows exactly the line of Ref. [4]. In addition to the $e^+e^-$ annihilation data we use spectral functions from $\tau$ decays into two- and four final state pions measured by the ALEPH Collaboration [13]. As described in Ref. [4], corrections to the charged $\rho^\pm$ width have to be applied to account for small CVC-violating effects. The magnitude of the width difference, $(\Gamma_{\rho^\pm} - \Gamma_{\rho^0})/\Gamma_{\rho} = (2.8 \pm 3.9) \times 10^{-3}$, translated into $a^{\text{had}}_\mu$ and $\Delta \alpha^{(5)}_{\text{had}}(M_Z^2)$ is evaluated using a parametrization of the $\rho$ line shape based on vector resonances [13]. One thus obtains the additive corrections

$$\delta a^{\text{had}}_\mu = - (1.3 \pm 2.0) \times 10^{-10}$$
$$\delta \Delta \alpha^{(5)}_{\text{had}}(M_Z^2) = - (0.09 \pm 0.12) \times 10^{-4}$$

for the $\tau^- \to \pi^- \pi^0 \nu_\tau$ spectral function which is applied in the present (and former) analysis. Corrections for the higher mass resonances $\rho(1450), \rho(1700)$ are expected to be negligible. Extensive studies have been performed in Ref. [4] in order to bound unmeasured modes, such as $K\bar{K}$ with pions or the $\pi^+\pi^-4\pi^0$ final states, via isospin constraints. We bring attention to the straightforward and statistically well-defined averaging procedure and error propagation used in this paper as in the preceding ones, which takes into account full systematic correlations between the cross section measurements. All technical details concerning the data analysis and the integration method used are presented in Ref. [4].

The experimental determination of the spectral moments in the first integral of Eq. (8) is performed as the sum over the respective moments of all exclusively measured $e^+e^-$ final states completed by $\tau$ data.

We fit the following equations ($\sqrt{s_0} = 1.8$ GeV):

$$\frac{\alpha^2(0)}{3\pi^2} \int \frac{ds}{4m^2_Z} R^{\text{Data}}(s) \left[ \frac{K(s)}{s} - \frac{p^{(i)}_{1,0}(s)}{s} \right] = a^{\text{had}}_{\mu, [2m_s, \sqrt{s_0}]}$$
$$\quad - \frac{\alpha^2(0)}{6\pi^2} \int_{i=1}^{9} \frac{ds}{s} \left[ p^{(i)}_{1,0}(s_0) - p^{(i)}_{1,0}(s) \right] D_{uds}(s) , \quad (i = 1, \ldots, 9)$$

$$\quad - \frac{\alpha(0)M_Z^2}{3\pi} \int \frac{ds}{4m^2_Z} R^{\text{Data}}(s) \left[ \frac{1}{s(s - M^2_Z)} - \frac{p^{(i)}_{1,0}(s)}{s} \right] = \Delta \alpha^{(5)}_{\text{had}}(M_Z^2)_{[2m_s, \sqrt{s_0}]}$$
$$\quad + \frac{\alpha(0)M_Z^2}{6\pi^2} \int_{i=10}^{18} \frac{ds}{s} \left[ p^{(i)}_{1,0}(s_0) - p^{(i)}_{1,0}(s) \right] D_{uds}(s) , \quad (i = 10, \ldots, 18)$$

(19)
Table 1: Polynomial moments: The horizontal lines separate the moments into Eqs. (17) and (18). The first column numbers the moments and the second defines the coefficients corresponding to Eq. (11). The following columns give the l.h.s integrals of Eqs. (17) and (18), which are the experimental results, and the r.h.s integrals, i.e., the theoretical values after fitting. The sums of both experimental and theoretical integrals give the values for \( a_{\mu, [2m_\pi, 1.8 \text{ GeV}]}^{\text{had}} \times 10^{10} \) and \( \Delta \alpha_{\text{had}}(M_Z^2)_{[2m_\pi, 1.8 \text{ GeV}]} \times 10^4 \), corresponding to the respective choice of polynomials. The last column shows the difference between the pure data results (given in the first and the 10th line, respectively, where the polynomial vanishes) and the sum in units of one standard deviation.

|   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|
|   | \( c_i \) | Data \( \Delta \) | Theory \( \Delta \) | Sum \( \Delta \) |   |   |
| (01) | 0 | 635.05 | 7.36 | 0.00 | 0.00 | 635.05 | 7.36 | 0.00 |
| (02) | 0.0005 | 601.49 | 6.93 | 33.47 | 0.81 | 644.96 | 6.99 | 0.01 |
| (03) | 0.001 | 567.56 | 6.51 | 66.95 | 1.62 | 634.51 | 6.73 | 0.07 |
| (04) | 0.002 | 500.18 | 5.81 | 133.89 | 3.23 | 634.07 | 6.65 | 0.12 |
| (05) | 0.003 | 432.50 | 5.72 | 200.84 | 4.95 | 633.34 | 6.17 | 0.19 |
| (06) | 0.004 | 364.98 | 5.27 | 267.78 | 6.47 | 632.77 | 8.15 | 0.23 |
| (07) | 0.005 | 297.36 | 4.90 | 334.73 | 8.08 | 632.08 | 9.46 | 0.27 |
| (08) | 0.006 | 229.78 | 4.90 | 401.67 | 9.70 | 631.45 | 10.99 | 0.30 |
| (09) | 0.007 | 162.23 | 5.59 | 468.62 | 11.32 | 630.85 | 12.64 | 0.31 |
| (10) | 0 | 56.77 | 1.06 | 0.00 | 0.00 | 56.77 | 1.06 | 0.00 |
| (11) | 0.0005 | 44.66 | 0.86 | 11.98 | 0.29 | 56.64 | 0.90 | 0.12 |
| (12) | 0.0007 | 39.80 | 0.78 | 16.78 | 0.41 | 56.58 | 0.88 | 0.17 |
| (13) | 0.0009 | 34.93 | 0.70 | 21.57 | 0.52 | 56.49 | 0.87 | 0.24 |
| (14) | 0.0011 | 30.08 | 0.62 | 26.36 | 0.74 | 56.44 | 0.89 | 0.27 |
| (15) | 0.0013 | 25.24 | 0.55 | 31.15 | 0.75 | 56.39 | 0.93 | 0.28 |
| (16) | 0.0015 | 20.46 | 0.48 | 35.95 | 0.87 | 56.40 | 0.99 | 0.27 |
| (17) | 0.0017 | 15.57 | 0.41 | 40.74 | 0.99 | 56.31 | 1.07 | 0.31 |
| (18) | 0.0019 | 10.68 | 0.35 | 45.53 | 1.10 | 56.21 | 1.16 | 0.35 |

where the polynomials \( p^{(i)}_{1,0} \) are defined in Eq. (11). Table 4 shows the measured polynomial moments (l.h.s. of Eqs. (17) and (18)), together with the fitted theoretical integrals (on the r.h.s. of Eqs. (17) and (18)) and the sum of both integrals for all 18 polynomials. The correlation matrix including experimental and theoretical correlations of the moments defined in Table 1 is given in Table 2. These correlations have been inserted into the \( \chi^2 \) minimization used to fit the free varying parameters for which we obtain:

\[
\begin{align*}
  a_{\mu, [2m_\pi, 1.8 \text{ GeV}]}^{\text{had}} &= (634.3 \pm 5.6_{\text{exp}} \pm 2.1_{\text{theo}}) \times 10^{-10}, \\
  \Delta \alpha_{\text{had}}(M_Z^2)_{[2m_\pi, 1.8 \text{ GeV}]} &= (56.53 \pm 0.73_{\text{exp}} \pm 0.39_{\text{theo}}) \times 10^{-4},
\end{align*}
\]

yielding a minimum \( \chi^2 = 1.0 \) for 16 degrees of freedom. The correlation between the fitted parameters (20) amounts to 69%.

The improvement in precision provided by this method can be easily understood from Table 4. As the chosen polynomial approaches better the actual kernel, the uncertainty from the integral on data is reduced, while the corresponding uncertainty from theory increases. Some optimum is reached for some particular choice of polynomial, providing...
Table 2: Total experimental and theoretical correlations between the spectral moments used in the combined fit.

| Moments | (01) | (02) | (03) | (04) | (05) | (06) | (07) | (08) | (09) | (10) | (11) | (12) | (13) | (14) | (15) | (16) | (17) | (18) |
|---------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
| (1)     | 1    | .97  | .94  | .82  | .65  | .46  | .31  | .19  | .10  | .75  | .71  | .66  | .59  | .51  | .42  | .34  | .25  | .18  |
| (2)     |      | 1    | .97  | .88  | .73  | .57  | .43  | .31  | .23  | .71  | .71  | .68  | .63  | .57  | .50  | .42  | .35  | .29  |
| (3)     |      |      | 1    | .93  | .82  | .68  | .55  | .44  | .36  | .66  | .70  | .69  | .66  | .62  | .57  | .51  | .45  | .39  |
| (4)     |      |      |      | 1    | .94  | .68  | .76  | .61  | .50  | .63  | .66  | .69  | .69  | .68  | .65  | .62  | .59  |      |
| (5)     |      |      |      |      | 1    | .95  | .90  | .85  | .80  | .31  | .51  | .59  | .65  | .70  | .73  | .74  | .73  | .73  |
| (6)     |      |      |      |      |      | 1    | .97  | .94  | .91  | .13  | .38  | .49  | .58  | .66  | .72  | .76  | .79  | .80  |      |
| (7)     |      |      |      |      |      |      | 1    | .97  | .96  | .00  | .28  | .40  | .52  | .62  | .70  | .76  | .80  | .82  |      |
| (8)     |      |      |      |      |      |      |      | 1    | .98  | -.10 | .19  | .33  | .45  | .57  | .67  | .74  | .79  | .83  |
| (9)     |      |      |      |      |      |      |      |      | 1    | -.17 | .13  | .27  | .40  | .53  | .63  | .72  | .78  | .82  |
| (10)    |      |      |      |      |      |      |      |      |      | 1    | .93  | .86  | .78  | .67  | .56  | .44  | .34  | .24  |
| (11)    |      |      |      |      |      |      |      |      |      |      | 1    | .97  | .93  | .87  | .79  | .70  | .62  | .54  |
| (12)    |      |      |      |      |      |      |      |      |      |      |      | 1    | .97  | .93  | .87  | .80  | .73  | .66  |
| (13)    |      |      |      |      |      |      |      |      |      |      |      |      | 1    | .97  | .93  | .88  | .83  | .77  |
| (14)    |      |      |      |      |      |      |      |      |      |      |      |      |      | 1    | .97  | .94  | .90  | .86  |
| (15)    |      |      |      |      |      |      |      |      |      |      |      |      |      |      | 1    | .97  | .95  | .92  |
| (16)    |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      | 1    | .97  | .96  |
| (17)    |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      | 1    | .98  |
| (18)    |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      | 1    |

The best constraint on the $a_{\mu,\text{had}}^\text{had}$ and $\Delta\alpha_{\text{had}}(M_Z^2)$ determinations. It should be noted that the experimental precision is not uniform throughout the energy range considered: for $\sqrt{s} \approx 0.8$ GeV, $R(s)$ is dominated by the $\rho$ spectral function, well measured in both $\tau$ decays and $e^+e^-$ annihilation, while at larger $s$ values the uncertainties from some poorer measurements such as $e^+e^- \rightarrow K\bar{K}\pi\pi$ or $6\pi$ take over. We clearly disagree with the conclusions reached in Ref. [5], stating that the final uncertainty in the determination of $\Delta\alpha_{\text{had}}(M_Z^2)$ becomes insensitive to the experimental errors. The experimental uncertainty can hardly be reduced to nothing even if the integral on data becomes very small. A vanishing error is only achieved if the polynomial approximates well the kernel, but the theoretical uncertainty becomes prohibitive in this case. This trend is illustrated in Table 1.

As described before we may also use Eq. (13) to constrain theoretically the low-energy dispersion relations (3) and (5). Here different moments are defined by varying values for $Q^2$ and $A_F$. Unfortunately such moments are almost degenerate, i.e., they have very large correlations so that a combined fit as performed in the case of the polynomials does not make much sense. We therefore optimize the choice of $Q^2$ and $A_F$ in order to minimize the resulting errors on $a_{\mu,\text{had}}(2m_\pi,\sqrt{s_0})$ and $\Delta\alpha_{\text{had}}(M_Z^2)(2m_\pi,\sqrt{s_0})$ setting again $\sqrt{s_0} = 1.8$ GeV. The corresponding results are given in Table 3. An analysis of the theoretical uncertainties shows that at $\sqrt{Q^2} = 3$ GeV nonperturbative contributions to the $D$-function are negligible. The theoretical errors given in Table 3 originate mainly from uncertainties of the massless perturbative series in Eq. (9), i.e., the error on $\alpha_s$ and the variation of $\mu$. The weighted averages of pure data and the theory-improved results given take into account
the correlations between both values. These values agree with those from the combined fit of the polynomial moments. The most precise of the two evaluations are retained for the final results.

**Results on the $c\bar{c}$ Threshold**

The $c\bar{c}$ threshold region involves the narrow resonances, $J/\psi(1S)$, $\psi(2S)$ and $\psi(3770)$, parametrized using relativistic Breit-Wigner formulae [22, 1], as well as the broad resonance and continuum region starting with the opening of the $e^+e^- \rightarrow DD$ mode at about 3.7 GeV. A compilation of the data used can be found in Ref. [1]. In this energy region we do not use the polynomial approach for which the precision is limited by the uncertainty on the $c$-quark mass and the threshold behaviour of the correlator. The first problem is less severe with the dispersion relation approach while the second one is inexistent.

Equation (13) now reads:

$$F = \int_{s_1}^{s_2} ds \mathcal{R}(s) \left[ f(s) - \frac{A_F Q^2}{(s + Q^2)^2} \right] + \sum_{\psi=1S,2S,3770} \int_{s_1}^{s_2} ds \mathcal{R}_{\psi}(s) \left[ f(s) - \frac{A_F Q^2}{(s + Q^2)^2} \right]$$

$$+ A_F \left( D_c(Q^2) - Q^2 \int_{s_2}^{\infty} ds \frac{R_{QCD}(s)}{(s + Q^2)^2} \right) + Q^2 \int_{s_1}^{s_2} ds \frac{R_{uds}(s)}{(s + Q^2)^2},$$

where we choose for the continuum the integration ranges $\sqrt{s_1} = 3.7$ GeV and $\sqrt{s_2} = 5$ GeV, and set $F \equiv a_{\mu}^{had}(\sqrt{s_1}, \sqrt{s_2})$ and $F \equiv \Delta\alpha_{\mu}^{had}(M_Z^2)\psi(\sqrt{s_1}, \sqrt{s_2})$, while $f(s)$ denotes the corresponding integration kernels, respectively. Table 3 gives the results of the individual terms in Eq. (21) again after optimizing the choice of $Q^2$ and the normalization $A_F$. The quoted error on $D_c(Q^2)$ is dominated by the uncertainty on the $c$-quark mass.

**Results for $\Delta\alpha_{\mu}^{had}(M_Z^2)$ and $a_{\mu}^{had}$**

In the previous sections we reevaluated the dispersion integrals yielding the hadronic contributions to $a_{\mu}$ and to $\alpha(M_Z^2)$ for the low-energy and $c\bar{c}$ threshold regions. According to Ref. [2] (see also Ref. [24]), we employ perturbative QCD using the formulae of Ref. [14] for the cross section ratio $\mathcal{R}(s)$ in other regions. For the crossing of the $b\bar{b}$ threshold, we assume nonperturbative effects to be negligible and use perturbative QCD in the context of global duality (see, e.g., Ref. [25]) to evaluate the hadronic contributions. The theoretical error is then dominated by the uncertainty on the $b$-quark production threshold $2M_b$, with $M_b$ being the pole mass of the $b$-quark.

Table 4 shows the experimental and theoretical evaluations of $\Delta\alpha_{\mu}^{had}(M_Z^2)$, $a_{\mu}^{had}$ and $a_{\epsilon}^{had}$ for the respective energy regimes. Experimental errors between different lines are assumed to be uncorrelated, whereas theoretical errors but those from $c\bar{c}$ and $b\bar{b}$ thresholds which are quark mass dominated are added linearly.

\[\text{Table 4}\]

\[\text{Table 4}\]
The optimized choices of the parameters $Q^2$ and $A_F$, the solutions of the corresponding integrals (“I”) in Eq. (13) (upper table) and Eq. (21) (lower table) as well as the corresponding values for the Adler $D$-functions. Additionally given are the correlations between the theory-improved and the pure data results used to calculate the average of both.

### Table 3

| $\Delta \alpha_{\text{had}}(M_Z^2)[2m_\pi, 1.8 \text{ GeV}] \times 10^4$ | $\alpha_{\mu, \text{had}}^0[2m_\pi, 1.8 \text{ GeV}] \times 10^{10}$ |
|---|---|
| $\sqrt{Q^2}$ | 3 GeV | 3 GeV |
| $A_F$ | -0.0003 | 0.009 |
| $I_{\text{exp}}^0(2m_\pi - 1.8 \text{ GeV})$ | $44.67 \pm 0.70$ | $534.1 \pm 6.2$ |
| $D_{uds}^\text{theo}(Q^2)$ | $42.18 \pm 0.23$ | $353.5 \pm 2.0$ |
| $-\Delta I_{uds}^\text{theo}(1.8 \text{ GeV} - \infty)$ | $-30.68 \pm 0.21$ | $-257.1 \pm 1.8$ |
| Total | $56.17 \pm 0.70_{\text{exp}} \pm 0.22_{\text{theo}}$ | $630.5 \pm 6.2_{\text{exp}} \pm 1.9_{\text{theo}}$ |
| Data only | $56.8 \pm 1.1^{(1)}$ | $635.1 \pm 7.4^{(1)}$ |
| Correlation | 95% | 95% |
| Average | $56.36 \pm 0.70_{\text{exp}} \pm 0.18_{\text{theo}}$ | $632.5 \pm 6.2_{\text{exp}} \pm 1.6_{\text{theo}}$ |

| $\Delta \alpha_{\text{had}}(M_Z^2)[\psi[3.7, 5 \text{ GeV}] \times 10^4$ | $\alpha_{\mu, \psi}[3.7, 5 \text{ GeV}] \times 10^{10}$ |
|---|---|
| $\sqrt{Q^2}$ | 15 GeV | 15 GeV |
| $A_F$ | -0.0009 | 0.0015 |
| $I_{\text{exp}}(3.7 - 5 \text{ GeV})$ | $7.02 \pm 0.45$ | $2.94 \pm 0.19$ |
| $I_{\text{exp}}(\psi(1S, 2S, 3770))$ | $6.25 \pm 0.37$ | $6.12 \pm 0.35$ |
| $D_{\text{theo}}^\psi(Q^2)$ | $79.95 \pm 0.83$ | $37.22 \pm 0.38$ |
| $-\Delta I_{\psi}(5 \text{ GeV} - \infty)$ | $-74.01 \pm 0.31$ | $-34.55 \pm 0.15$ |
| $I_{uds}^\text{theo}(3.7 - 5 \text{ GeV})$ | $5.32 \pm 0.03$ | $2.47 \pm 0.02$ |
| Total | $24.53 \pm 0.58_{\text{exp}} \pm 0.88_{\text{theo}}$ | $14.20 \pm 0.40_{\text{exp}} \pm 0.40_{\text{theo}}$ |
| Data only (3.7 – 5 GeV) | $15.80 \pm 1.00^{(2)}$ | $6.93 \pm 0.44^{(2)}$ |
| Data only $\psi(1S, 2S, 3770)$ | $9.24 \pm 0.68^{(3)}$ | $7.51 \pm 0.44^{(3)}$ |
| Correlation | 55% | 71% |
| Average | $24.75 \pm 0.84_{\text{exp}} \pm 0.50_{\text{theo}}$ | $14.31 \pm 0.50_{\text{exp}} \pm 0.21_{\text{theo}}$ |

1. The slight modification compared to our previous values of $\Delta \alpha_{\text{had}}(M_Z^2)[2m_\pi, 1.8 \text{ GeV}] = 56.9 \times 10^{-4}$ and $\alpha_{\mu, \text{had}}^0[2m_\pi, 1.8 \text{ GeV}] = 636.5 \times 10^{-10}$ are due to a reevaluation of the contributions from charm hadron data.
2. A reevaluation of the treatment of the experimental errors on the contribution from charm threshold led to a significant reduction of the very conservative uncertainties given in our previous paper [2].
3. The changes in the resonance contributions compared to Ref. [3] is due to the multiplicative correction $(\alpha/\alpha(M_\psi^2))^2$, erroneously not applied before [2].
According to Table 4, the combination of the theoretical and experimental evaluations of the integrals (3) and (8) yields the final results

\[ \Delta \alpha_{\text{had}}(M_Z^2) = (276.3 \pm 1.1_{\exp} \pm 1.1_{\text{theo}}) \times 10^{-4} \]
\[ \alpha^{-1}(M_Z^2) = 128.933 \pm 0.015_{\exp} \pm 0.015_{\text{theo}} \]
\[ a_{\mu}^{\text{had}} = (692.4 \pm 5.6_{\exp} \pm 2.6_{\text{theo}}) \times 10^{-10} \]
\[ a_{\mu}^{\text{SM}} = (11659159.6 \pm 5.6_{\exp} \pm 3.7_{\text{theo}}) \times 10^{-10} \]

and \( a_e^{\text{had}} = (187.5 \pm 1.7_{\exp} \pm 0.7_{\text{theo}}) \times 10^{-14} \) for the leading order hadronic contribution to \( a_e \). The total \( a_{\mu}^{\text{SM}} \) value includes an additional contribution from non-leading order hadronic vacuum polarization summarized in Refs. [23] to be \( a_{\mu}^{\text{had}}[(\alpha/\pi)^3] = (-10.0 \pm 0.6) \times 10^{-10} \). Also the light-by-light scattering (LBLS) contribution has recently been reevaluated to be \( a_{\mu}^{\text{had}}[\text{LBLS}] = (-7.9 \pm 1.5) \times 10^{-10} \) [27]. Together with the value \( a_{\mu}^{\text{had}}[\text{LBLS}] = (-9.2 \pm 3.2) \times 10^{-10} \) [28], we use the average \( \langle a_{\mu}^{\text{had}}[\text{LBLS}] \rangle = (-8.5 \pm 2.5) \times 10^{-10} \) so that the total higher order hadronic correction amounts to \( a_{\mu}^{\text{had}}[(\alpha/\pi)^3 + \text{LBLS}] = (-18.5 \pm 2.6) \times 10^{-10} \). Figures 2 and 3 show a compilation of published results for the hadronic contributions to \( \alpha(M_Z^2) \) and \( a_{\mu} \). Some authors give the hadronic contribution for the five light quarks only and add the top quark part separately. This has been corrected for in Fig. 3.
Figure 2: Comparison of $\Delta \alpha_{\text{had}}(M_Z^2)$ evaluations. The values are taken from Refs. [29, 30, 31, 32, 1, 2, 24, 37] and from this work.

Figure 3: Comparison of $a_{\mu}^{\text{had}}$ evaluations. The values are taken from Refs. [33, 34, 35, 22, 36, 1, 2] and from this work.
Conclusions

We have reevaluated the hadronic vacuum polarization contribution to the running of the QED fine structure constant, $\alpha(s)$, at $s = M_Z^2$ and to the anomalous magnetic moment of the muon, $a_\mu$. We employed perturbative and nonperturbative QCD in the framework of the Operator Product Expansion in order to extend the energy regime where theoretical predictions are reliable. Based on analyticity, we constrained the low-energy and $c\bar{c}$ threshold region theoretically using finite energy sum rules and dispersion relations. The standard evaluation using data from $e^+e^-$ annihilation and $\tau$ decays at low energies and near quark production thresholds is therefore improved. Our results are $\Delta \alpha_{\text{had}}(M_Z^2) = (276.3 \pm 1.6) \times 10^{-4}$, propagating $\alpha^{-1}(0)$ to $\alpha^{-1}(M_Z^2) = (128.933 \pm 0.021)$, and $a_\mu^{\text{had}} = (692.4 \pm 6.2) \times 10^{10}$ which yields the Standard Model prediction $a_\mu^{\text{SM}} = (11659.159.5 \pm 6.7) \times 10^{-10}$. For the electron we found $a_e^{\text{had}} = (187.5 \pm 1.8) \times 10^{-14}$. These results have direct implications for on-going experimental programs. On one hand, the precision on $\alpha(M_Z^2)$ is now such that it does not limit anymore the adjustment of the Higgs mass from accurate experimental determinations of $\sin^2 \theta_W$ [2]. On the other hand, the gain in accuracy for $a_\mu^{\text{had}}$ is even more rewarding as it will permit to exploit the foreseen precision increase of the $a_\mu$ measurement [38] in order to achieve a significant determination of the contribution $a_\mu^{\text{weak}}$.

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