Accuracy and stability analysis of the modified Lax numerical method in solving the sea waves refraction equation

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Abstract. We analyse the accuracy and stability of the modified Lax numerical method used in solving the two dimensional sea waves refraction equation. The refraction phenomenon on the beach was observed by the fact that waves hit the shore line in perpendicular way. Angles of the incoming waves from offshore keep changing due to the smaller depth of water, and therefore become important to be determined. The accuracy was evaluated by means of the Taylor series expansion and the stability was analysed by the Von Neumann method. The analysis showed that the method is stable for a certain interval although it is only first order accurate.

1. Introduction
In this paper we discuss a numerical method performance for the sea waves refraction equation. We investigate analytically the accuracy and the stability of the selected numerical scheme, in this case the modified Lax method. The modified Lax scheme was chosen due to its stability for the simple wave equation. It generally improves stability because of the presence of the factor in the method to manage the calculation, although does not improve accuracy [1]. Another advantage of this method is its simplicity which is characterized by explicit form and only one computation stage needed at a point. The discretization of this equation was done by using finite difference method which is suitable for the numerical scheme.

The accuracy was evaluated by means of the Taylor series expansion. The truncation error of the method can be shown by substituting Taylor expansion to the equation. By assuming the increment intervals for horizontal and vertical axes are in the same order, then the order of accuracy of the method is determined [1].

The stability was analysed by the von Neumann method by using the Fourier expansion applied to the equation. The stability of the method can be achieved when the value of amplification factor or ratio of two successive steps is not larger than one [1].

Waves propagate from offshore to the beach following a line which is called a wave ray. Wave refraction can be seen as a bend of a ray of waves at an oblique angle when it enters the less depth of water [2]. Therefore angles of the incoming waves become important to be determined as input for obtaining other coastal quantities such as long shore currents [3], and wave energy [4]. However, the angle distributions solution will not be discussed here.
2. The modified Lax method
Before implementing the modified Lax method to the refraction equation, we first consider the method in its general form. In a two dimensional horizontal \(xy\)-plane, \(x\) refers to on-offshore axis and \(y\) is axis parallel to the shoreline.

Consider a rectangular grid of resolution \(\Delta x\) and \(\Delta y\) where \(i\) and \(j\) denote the indices in \(x\)- and \(y\)-direction, respectively (\(0 \leq i \leq N_x, 0 \leq j \leq N_y\)). The parameter of the method \(\alpha\) should satisfy \(0 < \alpha < 1\). Let \(t\) represents the time and the asterisk (*) indicates evaluation at the next time step for an interval \(\Delta t\). The general formulation of the modified Lax method is defined for one dimensional case determining derivative \(\frac{dq}{dt}\) of a quantity \(q\), as

\[
\frac{q^*_i - \frac{1}{2}\alpha(q_{i+1} + q_{i-1}) - (1 - \alpha)q_i}{\Delta t} = \text{right hand side (1)}
\]

For \(\alpha = 1\), it returns back to the Lax method. For \(\alpha = 0\), it returns to forward Euler [1]. The scheme is expanded to the two dimensional in spatial form as it is applied to the refraction equations.

2.1. Wave refraction
For the purpose of modelling the refraction equation, let \(\theta\) represents incident wave angle with respect to the positive \(x\)-axis, \(\omega\) is the angular frequency, and \(k\) is wave number. Figure 1 shows the definition sketch for the equation.

A wave ray is defined as a line tangent to the wave number vector at every point [2]. When the wave direction makes an angle with the beach normal, the wave number oriented in the direction is a normal vector of the wave phase function. This fact represents irrotational condition of the wave number vector. This implies that wave length and wave angle are reduced when the waves enter the shallow water. This property is expressed according to the following consideration.

For a scalar phase function \(\Omega = kx - \omega t\), define the wave number vector \(\mathbf{k} = (k_x, k_y)\) as \(\mathbf{k} = \nabla \Omega\). Since

\[
\nabla \times \mathbf{k} = 0,
\]

it follows that [5]

\[
\begin{align*}
\partial_x k_y - \partial_y k_x &= 0, \\
\partial_x(k \sin \theta) - \partial_y(k \cos \theta) &= 0.
\end{align*}
\]
These equations lead to another shoaling related property on the coastline, where $y$ variations of the variables are zero, (3) turns to

$$\partial_x (k \sin \theta) = 0, \quad k \sin \theta = \text{constant}. \quad (4)$$

2.2. Numerical scheme

In the computational plane, the central difference applies for the scheme with the points $i, j$ are positions for $k$ and $\theta$. For particular discretization purpose, it is helpful to define the indices $c = (i, j), w = (i, j - 1), e = (i, j + 1), \text{ and } s = (i + 1, j)$.

Employing Lax method (1) in forward $x$-direction which is the ‘timelike’ direction, the discretized form for refraction equation (3) is

$$k_s \sin \theta_s - \frac{\alpha}{2} (k_e \sin \theta_e + k_w \sin \theta_w) - (1 - \alpha) k_c \sin \theta_c = \frac{\Delta x}{2 \Delta y} (k_e \cos \theta_e - k_w \cos \theta_w). \quad (5)$$

Given $k_s$, then $\theta_s$ is determined, as shown by the corresponding molecule depicted in Figure 2.

![Figure 2. Discretization molecule for wave refraction. At $\theta$ location, $k$ is also defined.](image)

3. Numerical analysis

3.1. Accuracy

In order to investigate the accuracy for the modified Lax method (1), consider Taylor expansion for the solution $q$ with respect to $x$ around a point $i$:

$$q_{i+1} = q_i + \Delta x \partial_x q_i + \frac{1}{2!} (\Delta x)^2 \partial_x^2 q_i + \frac{1}{3!} (\Delta x)^3 \partial_x^3 q_i + O(\Delta x)^4, \quad (6)$$

and let

$$\hat{q}_i = \frac{1}{2} \alpha (q_{i-1} + q_{i+1}) + (1 - \alpha) q_i. \quad (7)$$

By using the Taylor expansion in spatial and temporal dimensions, the following relations are needed to be used in the accuracy analysis [1].

$$\hat{q}_i = \alpha (q_i + \frac{1}{2} (\Delta x)^2 \partial_x^2 q_i) + (1 - \alpha) q_i + O(\Delta x)^4,$$
\[ q_{i+1} - q_{i-1} = 2\Delta x \partial_x q_i + \frac{2}{3!} (\Delta x)^3 \partial_x^3 q_i + O(\Delta x)^5, \]
\[ q_i^* - q_i = \Delta t \partial_t q_i + \frac{1}{2} (\Delta t)^2 \partial_t^2 q_i + O(\Delta t)^3. \]

The truncation error for the two dimensional equation is determined in the same way as the one in one dimensional problem. Term with respect to \( x \)-direction was added to replace the term with respect to \( t \). Here we derive the accuracy for the wave refraction equation.

\[ \frac{k_s \sin \theta_s - \frac{\alpha}{2} (k_c \sin \theta_c + k_w \sin \theta_w) - (1 - \alpha) k_c \sin \theta_c}{\Delta x} = k_c \cos \theta_c - k_w \cos \theta_w. \]

Apply Taylor expansion around \( k_c \sin \theta_c \), the left hand side becomes

\[ \frac{k_s \sin \theta_s - \frac{\alpha}{2} (k_c \sin \theta_c + k_w \sin \theta_w) - (1 - \alpha) k_c \sin \theta_c}{\Delta x} = \frac{\Delta x \partial_x (k_c \sin \theta_c)}{\Delta x} + \frac{1}{2} (\Delta x)^2 \partial_x^2 (k_c \sin \theta_c) + O(\Delta x)^3 \]
\[ + \frac{\Delta x}{\Delta x} \frac{\alpha}{2} (2k_c \sin \theta_c + (\Delta y)^2 \partial_y^2 (k_c \sin \theta_c) + O(\Delta y)^4) - \alpha k_c \sin \theta_c \]
\[ = \partial_x (k_c \sin \theta_c) + \frac{1}{2} \Delta x \partial_x^2 (k_c \sin \theta_c) - \frac{\alpha}{2} \frac{(\Delta y)^2}{\Delta x} \partial_y^2 (k_c \sin \theta_c) + O(\Delta x)^2, \] (8)

and the right hand side becomes

\[ \frac{k_c \cos \theta_c - k_w \cos \theta_w}{2\Delta y} = \frac{2\Delta y \partial_y (k_c \cos \theta_c) + \frac{\alpha}{2} (\Delta y)^2 \partial_y^2 (k_c \cos \theta_c) + O(\Delta y)^5}{2\Delta y} \]
\[ = \partial_y (k_c \cos \theta_c) + \frac{1}{3!} (\Delta y)^2 \partial_y^3 (k_c \cos \theta_c) + O(\Delta y)^4 \] (9)

The truncation error for refraction equation is

\[ \partial_x (k_c \sin \theta_c) - \partial_y (k_c \cos \theta_c) = -\frac{1}{2} (\Delta x)^2 \partial_x^2 (k_c \sin \theta_c) + \frac{\alpha}{2} \frac{(\Delta y)^2}{\Delta x} \partial_y^2 (k_c \sin \theta_c) \]
\[ + \frac{1}{3!} (\Delta y)^2 \partial_y^3 (k_c \cos \theta_c) + O(\Delta x)^2. \] (10)

Assume \( \Delta x \) and \( \Delta y \) are in the same order, the method is first order accurate.

3.2. Stability

The von Neumann stability analysis was used to evaluate the stability of the method after being used used for the equation. \( \theta \) was written in the Fourier series \[6\]: \( \theta_j = \theta e^{ikjy}. \)

The stability of the method will be achieved when the amplification factor \( \rho \) (modulus of the comparison of the two consecutive steps) is less than 1. Stability of refraction equation in the 'timelike' direction is determined.

\[ \partial_x (k \sin \theta) - \partial_y (k \cos \theta) = 0. \]

Let \( P = \sin \theta, Q = \cos \theta \), the equation and its discretization becomes

\[ \partial_x (kP) - \partial_y (kQ) = 0, \] (11)

such that

\[ k_{i+1,j}P_{i+1,j} - \frac{\alpha}{2} (k_{i,j+1}P_{i,j+1} + k_{i,j-1}P_{i,j-1}) - (1 - \alpha)k_{i,j}P_{i,j} \]
\[ = \frac{\Delta x}{2\Delta y} (k_{i,j+1}Q_{i,j+1} - k_{i,j-1}Q_{i,j-1}). \] (12)
Consider a Fourier mode, \( P_{i,j} = \tilde{P}_{i} e^{iny_j} \), and let \( \rho = \frac{k_{ii+1} \tilde{P}_{i+1}}{k_i \tilde{P}_i} \) to be applied to the discretized form,

\[
\rho - \frac{\alpha}{2} (e^{iny_y} + e^{-iny_y}) - (1 - \alpha) - \frac{\Delta x}{2\Delta y} (e^{iny_y} \tilde{Q} \tilde{P} - e^{-iny_y} \tilde{Q} \tilde{P}) = 0,
\]

\[
\rho - 1 + \alpha(1 - \cos n \Delta y) - i \frac{\Delta x}{\Delta y} \tilde{Q} \sin n \Delta y = 0. \tag{13}
\]

By replacing back \( \tilde{Q} = \cot \theta_0 = C_0 \) constant, and defining \( \sigma = \frac{\Delta x}{\Delta y} \), stability condition is determined by the following relations,

\[
\rho = 1 - \alpha(1 - \cos n \Delta y) + i C_0 \sigma \sin n \Delta y, \tag{14}
\]

\[
|\rho|^2 = (1 - \alpha(1 - \cos n \Delta y))^2 + (C_0 \sigma \sin n \Delta y)^2,
\]

\[
= 1 + \alpha^2(1 - \cos n \Delta y)^2 - 2\alpha(1 - \cos n \Delta y) + (C_0 \sigma \sin n \Delta y)^2 \leq 1.
\]

\[
\alpha^2(1 - \cos n \Delta y)^2 - 2\alpha(1 - \cos n \Delta y) + (C_0 \sigma)^2(1 - \cos^2 n \Delta y) \leq 0,
\]

\[
\alpha^2(1 - \cos n \Delta y) - 2\alpha + (C_0 \sigma)^2(1 + \cos n \Delta y) \leq 0. \tag{15}
\]

This is a linear function in \( \cos n \Delta y \), so all values of the left hand side are in the interval for \( \cos n \Delta y = -1 \) and \( \cos n \Delta y = 1 \). Thus the interval is

\[
C_0^2 \sigma^2 < \alpha < 1. \tag{16}
\]

The solution of the above inequation gives the limits to choose the factor for stability criteria of the method in refraction equation.

4. Conclusion

The contribution of one dimensional model leads to an easier way to investigate the performance of the two dimensional equation. The modified Lax method shows that it is an appropriate method for doing the integration of the refraction equation due to its stability and simple computations. Although the method has only first order accuracy, the stability property of the method allows the simulation to maintain the stable execution until it is accomplished.

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