Abstract

We introduce the isoclinism of crossed modules. We also give GAP implementations for constructing the isoclinism families of finite crossed modules and consequently give an enumeration about isoclinic crossed modules existing in the GAP library.

Key words: Isoclinism, Crossed Module, GAP
1. Introduction

The notion of isoclinism was first defined in Hall (1940), for a classification of finite groups whose orders are prime powers. This work was detailed in Hall and Senior (1964). After then, some new results were given in many papers, such as Jones and Wiegold (1974); Modabbernia (2012); Mohammadzadeh et al. (2013); Parvaneh et al. (2011); Salemkar et al. (2008); Tappe (1976). Also the relation between groups and their stem extensions with Schur multiplicators were given in Beyl and Tappe (1982) where they consider the isoclinism of central group extensions. In this work we consider the 2-dimensional group version, called “crossed module”, of isoclinism and determine some basic results. This construction gives a new equivalence relation on crossed modules weaker then isomorphism. So we have a new classification of crossed modules and the resulting equivalence classes, called isoclinism families, are convenient with some algebraic invariants such as nilpotency classes, rank and middle length. For determine such a comparison, we defined some new concepts for crossed modules such as; class preserving actor, rank and middle length of crossed modules etc.

In general, the isoclinism is used for classification of finite groups, and there are many works concerning the enumeration of groups with finite order related to isoclinism, Hall and Senior (1964); James et al. (1990). So, we first construct GAP implementation for isoclinic groups, and as an example we give a character table consisting of the isoclinism families of the group with order 192. Enumeration of crossed modules and related structures are determined from many view points. In Alp and Wensley (2013); Brown and Wensley (2003); Ellis (2004); Ellis and Luyen (2012); Ellis (2013), one can found many computations about these notions with GAP. By a similar way, we give a GAP implementation for classification of finite crossed modules up to isoclinism. For this, we added some new functions which do not exist in XMod package, such as; DerivedSubXMod(XM), FactorXMod(XM,NM), IsIsoclinicXMod(XM1,XM2).

In order to get our goals, we organize the paper as follows;

In Section 2, we recall some needed results about crossed modules and introduce some new notions which will be used in the sequel of the paper.

In Section 3, we introduce the notion of isoclinism for crossed modules and establish the basic theory. As expected, we give the compatibility of this definition with nilpotency, solvability and class preserving actors of crossed modules as it was the case for groups.

In Section 4, we give GAP implementations for computing the isoclinism families of crossed modules in low order.

In Section 5, by using these implementations, we give character tables consisting isoclinism families of certain crossed modules. These tables are particular examples which show that the definition of isoclinism is convenient with the nilpotency classes and other invariants in low order.

2. Preliminaries

In this section we recall some needed material about crossed modules. Crossed modules were defined by Whitehead (1948), as an algebraic model for homotopy 3-types. We refer to Brown et al. (2011); Norrie (1990, 1987); Porter (2011), for a compreherensive and detailed work.
Definition 1. A crossed module is a group homomorphism

\[ d : G_1 \rightarrow G_0 \]

with a left action from \( G_0 \) on \( G_1 \) written \((g_0, g_1) \mapsto g_0g_1\), for \( g_0 \in G_0, g_1 \in G_1 \) satisfying the following conditions:

1) \( d(g_0g_1) = gad(g_1)g_0^{-1} \),

2) \( d(g_1')g_1' = g_1g_1'g_1^{-1} \),

for all \( g_0 \in G_0, g_1, g'_1 \in G_1 \).

We will denote such a crossed module by \( G : G_1 \xrightarrow{d} G_0 \).

Examples 1.

1) Let \( N \) be a normal subgroup of \( M \). Then \( \underline{N} \xrightarrow{\text{inc}} M \) is a crossed module with conjugate action of \( M \) on \( N \). Consequently, every group \( M \) can be thought as a crossed module in the two obvious way: \( \underline{1} \xrightarrow{\text{inc}} M \) or \( M \xrightarrow{id} M \).

2) \( 1 \xrightarrow{\alpha} 1 \) is a crossed module and it is called the trivial crossed module.

3) \( K \xrightarrow{1} L \) is a crossed module, where \( K \) is a \( L \)-module and the boundary operator is the zero map.

4) \( M \xrightarrow{c} \text{Aut}(M) \) is a crossed module, where \( c \) assigns to each element \( x \in M \), the inner automorphism \( c_x \) of \( M \) defined by \( c_x(m) = xmx^{-1} \), for all \( m \in M \).

Definition 2. A crossed module \( G : G_1 \xrightarrow{d} G_0 \) is called aspherical if \( \ker d = 1 \), i.e \( d \) is injective, and simply connected if \( \text{coker} d = 1 \), i.e \( d \) is surjective.

A morphism between two crossed modules \( G : G_1 \xrightarrow{d} G_0 \) and \( G' : G_1' \xrightarrow{d'} G_0' \) is a pair \((\alpha, \beta)\) of group homomorphisms \( \alpha : G_1 \rightarrow G_1' \), \( \beta : G_0 \rightarrow G_0' \), such that \( \beta d = d' \alpha \) and \( \alpha(g_0g_1) = \beta(g_0)\alpha(g_1) \), for all \( g_0 \in G_0 \), \( g_1 \in G_1 \). Consequently, we have the category \( \text{XMod} \) whose objects are the crossed modules and its morphisms are the morphisms of crossed modules.

Remark 1. Since \( M \xrightarrow{id} M \) is a crossed module, for any group \( M \), the category of groups can be thought as a full subcategory of crossed modules.

We say that \((\alpha, \beta)\) is an automorphism of \( G \) if \( \alpha \) and \( \beta \) are both automorphisms. We denote the group of automorphisms of the crossed module \( G : G_1 \xrightarrow{d} G_0 \) by \( \text{Aut}(G) \) where the multiplication is defined by componentwise composition.

Existing of zero object and equalizers give rise to subobjects and normal subobjects. A crossed module \( H : H_1 \xrightarrow{d_H} H_0 \) is a subcrossed module of a crossed module \( G : G_1 \xrightarrow{d_G} G_0 \) if \( H_1 \), \( H_0 \) are subgroups of \( G_1 \), \( G_0 \), respectively, \( d_H = d_G|_{H_1} \) and the action of \( H_0 \) on \( H_1 \) is induced by the action of \( G_0 \) on \( G_1 \). Also, a subcrossed module \( H : H_1 \xrightarrow{d_H} H_0 \) of a crossed module \( G : G_1 \xrightarrow{d_G} G_0 \) is normal if \( H_0 \) is a normal subgroup of \( G_0 \), \( g_0h_1 \in H_1 \) and \( b_{g_0}g_1g_1^{-1} \in H_1 \), for all \( g_0 \in G_0 \), \( g_1 \in G_1 \), \( h_0 \in H_0 \), \( h_1 \in H_1 \). Consequently, we have the quotient crossed module \( G/H : G_1/H_1 \xrightarrow{d_G} G_0/H_0 \) with the induced boundary map and action.
Definition 3. Let \((\alpha, \beta) : (G : G_1 \xrightarrow{d} G_0) \rightarrow (G' : G'_1 \xrightarrow{d'} G'_0)\) be a crossed module morphism. The kernel of \((\alpha, \beta)\) is the normal subcrossed module \((\ker \alpha, \ker \beta, d)\) of \(G\), denoted by \(\ker(\alpha, \beta)\) and the image Im\((\alpha, \beta)\) is the subcrossed module \((\Im \alpha, \Im \beta, d'|)\) of \(G'\).

Analogous to group theory, we have the third isomorphism theorem for crossed modules given in [Norrie 1987].

Let \(H : H_1 \xrightarrow{d} H_0\) and \(K : K_1 \xrightarrow{d} K_0\) be a subcrossed modules of \(G : G_1 \xrightarrow{d} G_0\). Then the intersection of \(H\) and \(K\) is defined as the subcrossed module
\[
H \cap K : H_1 \cap K_1 \xrightarrow{d} H_0 \cap K_0,
\]
which is normal in \(G\). \(HK\) is also defined as the crossed submodule \(HK : H_1K_1 \xrightarrow{d} H_0K_0\) when \(K\) is normal. Consequently, we have
\[
\frac{H}{H \cap K} \cong \frac{HK}{K}.
\]

Definition 4. Let \(G : G_1 \xrightarrow{d} G_0\) be a crossed module. A derivation from \(G_0\) to \(G_1\) is the map \(\partial : G_0 \rightarrow G_1\) such that \(\partial(xy) = \partial(x)^\partial(y)\), for all \(x, y \in G_0\).

The set of all derivations is denoted by \(\text{Der}(G_0, G_1)\).

As defined in [Whitehead 1948], \(\text{Der}(G_0, G_1)\) is a semigroup with the multiplication \(\partial_1 \circ \partial_2\) defined by
\[
(\partial_1 \circ \partial_2)(g_0) = \partial_1(d\partial_2(g_0)g_0)\partial_2(g_0) = (\partial_1d(g_1)g_0)\partial_2(g_0)\partial_1(g_0),
\]
for all \(g_0 \in G_0\), \(\partial_1, \partial_2 \in \text{Der}(G_0, G_1)\) where the identity element is the zero map. The Whitehead group \(D(G_0, G_1)\) is defined to be the group of units of \(\text{Der}(G_0, G_1)\), and the elements of \(D(G_0, G_1)\) are called regular derivations.

Due to [Norrie 1990], for a given crossed module \(G : G_1 \xrightarrow{d} G_0\), we have the crossed module
\[
\Delta : D(G_0, G_1) \rightarrow \text{Aut}(G)
\]
\[
\partial \mapsto (\partial d, d\partial)
\]
with the action of \(\text{Aut}(G)\) on \(D(G_0, G_1)\) given by \((\alpha, \beta)\partial = \alpha\partial\beta^{-1}\), for all \((\alpha, \beta) \in \text{Aut}(G)\), \(\partial \in D(G_0, G_1)\). This crossed module is called the actor of \(G\), and denoted by \(\text{Act}(G)\).

This structure was introduced by [Lue 1979] and developed in [Norrie 1990]. The actor objects are defined for introduce the actions in the category of crossed modules from which the objects such as centralizers, commutators, Abelian objects, semi direct products are defined.

There is a canonical morphism of crossed modules
\[
(\eta, \gamma) : G \rightarrow \text{Act}(G)
\]
given by \(\eta : G_1 \rightarrow D(G_0, G_1)\), \(\eta_{g_1}(g_0) = g_1g_0^{-1}g_1^{-1}\) and \(\gamma : G_0 \rightarrow \text{Aut}(G)\), \(\gamma(g_0) = (\alpha_{g_0}, \phi_{g_0})\) such that \(\alpha_{g_0}(g_1) = g_0g_1\) and \(\phi_{g_0}(g'_0) = g_0g_0'g_0^{-1}\), for all \(g_0, g_0' \in G_0, g_1 \in G_1\).

The image of the morphism \((\eta, \gamma)\) is called the inner actor of the crossed module \(G : G_1 \xrightarrow{d} G_0\), denoted by \(\text{InnAct}(G)\).
Definition 5. The center of the crossed module $G : G_1 \xrightarrow{d} G_0$ is defined as the normal subcrossed module

$$Z(G) : G_1^{G_0} \xrightarrow{d} St_{G_0}(G_1) \cap Z(G_0)$$

where

$$G_1^{G_0} = \{ g_1 \in G_1 : g_0 \circ g_1 = g_1, \text{ for all } g_0 \in G_0 \},$$

$$St_{G_0}(G_1) = \{ g_0 \in G_0 : g_0 \circ g_1 = g_1, \text{ for all } g_1 \in G_1 \}$$

and $Z(G_0)$ is the center of $G_0$. This definition recovers the generalized definition of Huq given in [Huq 1968]. (Here $G_1^{G_0}$ is called fixed point subgroup of $G_1$ and $St_{G_0}(G_1)$ is called the stabilizer of $G_1$ in $G_0$.) So, $G : G_1 \xrightarrow{d} G_0$ is called Abelian when $G = Z(G)$.

Definition 6. Let $G : G_1 \xrightarrow{d} G_0$ be a crossed module. The commutator subcrossed module $[G, G]$ of $G$ is defined by

$$[G, G] : D_{G_0}(G_1) \xrightarrow{d} [G_0, G_0]$$

where $D_{G_0}(G_1)$ is the subgroup generated by $\{ g_0 \circ g_1^{-1} : g_1 \in G_1, g_0 \in G_0 \}$ and $[G_0, G_0]$ is the commutator subgroup of $G_0$.

Proposition 1. Let $G : G_1 \xrightarrow{d} G_0$ be a crossed module. Then we have the following:

(i) If $G$ is simply connected, then $G_1^{G_0} = Z(G_1)$ and $D_{G_0}(G_1) = [G_1, G_1]$.

(ii) If $G$ is aspherical, then $Z(G_0) = St_{G_0}(G_1) \cap Z(G_0)$.

Proof. Can be checked by a direct calculation. □

Finally, we have stem crossed modules defined as the crossed modules whose centers are in their commutators. This definition was introduced in [Vieites and Casas 2002].

Example 2. The crossed module $G : Kl_4 \xrightarrow{d} C_3$ is a stem crossed module. Since $Z(G) = \{ e \} \xrightarrow{d} \{ e \}$ and $[G, G] : Kl_4 \xrightarrow{d} \{ e \}$, we have $Z(G) \subseteq [G, G]$.

Definition 7. A crossed module $G : G_1 \xrightarrow{d} G_0$ is called finite if $G_1$ and $G_0$ are finite groups.

The order of a finite crossed module is defined as the pair $[m, n]$ where $m$, $n$ are the orders of $G_1$, $G_0$, respectively.

Let $H : H_1 \xrightarrow{d} H_0$ be a subcrossed module of the crossed module $G : G_1 \xrightarrow{d} G_0$. Suppose that there is a finite sequence $H^i : (H_1^i \xrightarrow{d} H_0^i)_{0 \leq i \leq n}$ of the subcrossed modules of $G$ such that

$$H = H^0 \leq H^1 \leq \ldots \leq H^{n-1} \leq H^n = G.$$

This will be called a series of length $n$ from $H$ to $G$. The subcrossed modules $H^0$, $H^1$, ..., $H^n$ are called the terms of the series and quotient crossed modules $H^i/H^{i-1}$, $i = 1, \ldots, n$, are called the factors of the series. A series 1 to $G$ is shortly called a series of $G$. A series is called central if all factors are central. $G$ is called solvable if it has a series all of whose factors are Abelian crossed modules and is called nilpotent if it has a series all of whose factors are central factors of $G$. 

5
Definition 8. Let \( G : G_1 \to G_0 \) be nilpotent. Then, for any central series
\[
1 = G^0 \trianglelefteq G^1 \trianglelefteq \cdots \trianglelefteq G^r = G_0
\]
of \( G \), we have
\[
\Gamma_{r-i+1}(G) \leq G^i \leq \xi_i(G),
\]
i = 0, 1, \ldots, r where \( \Gamma_1(G) = (G) \), \( \xi_0(G) = 1 \) and
\[
\Gamma_n(G) = [\Gamma_{n-1}(G), G], \quad n > 1
\]
\[
\xi_n(G_0)/\xi_{n-1}(G_0) = \xi(G_0/\xi_{n-1}(G_0)), \quad n > 0.
\]
Furthermore, the least integer \( c \) such that \( \Gamma_{c+1}(G) = 1 \) is equal to the least integer \( c \) such that \( \xi_c(G) = G \). The integer \( c \) is called the nilpotency class of the crossed module \( G \).

Definition 9. Let \( G : G_1 \to G_0 \) be solvable and let \( n \) be the least integer such that \( G^n = 1 \) where \( G^n \) is the subcrossed module of \( G \) such that \( G^0 = G \) and for each integer \( n > 0 \), \( G^n = [G^{n-1}, G^{n-1}] \). Then we call \( n \) the derived length of \( G \).

3. Isoclinic Crossed Modules

In this section we introduce the notion of isoclinic crossed modules to have a new equivalence relation on crossed modules weaker than isomorphism, which gives rise to a new classification. First, we recall the definition of isoclinic groups from Hall (1940).

Let \( M \) and \( N \) be two groups. \( M \) and \( N \) are isoclinic if there exist isomorphisms \( \eta : M/Z(M) \to N/Z(N) \) and \( \xi : [M, M] \to [N, N] \) between central quotients and derived subgroups, respectively, such that, the following diagram is commutative:

\[
\begin{array}{ccc}
M/Z(M) \times M/Z(M) & \xrightarrow{c_M} & [M, M] \\
\eta \times \eta \downarrow & & \downarrow \xi \\
N/Z(N) \times N/Z(N) & \xrightarrow{c_N} & [N, N]
\end{array}
\]

where \( c_M, c_N \) are commutator maps of groups. The pair \((\eta, \xi)\) is called an isoclinism from \( M \) to \( N \), and denoted by \( (\eta, \xi) : M \sim N \).

Remark 2. As expected, isoclinism is an equivalence relation.

Examples 3.

(1) All Abelian groups are isoclinic to each other.
(2) The dihedral, quasidihedral and quaternion groups of order \( 2^n \) are isoclinic, for \( n \geq 3 \).
(3) Every group is isoclinic to a stem group. (See Hall (1940), for details.)

Now we are going to define the notion of isoclinic crossed modules.

Notation In the sequel of the paper, for a given crossed module \( G : G_1 \to G_0 \), we denote \( G/Z(G) \) by \( \overline{G} \) where \( \overline{G}_1 = G_1/G_1^G \) and \( \overline{G}_0 = G_0/(St_{G_0}(G_1) \cap Z(G_0)) \), for shortness.
**Definition 10.** The crossed modules $G : G_1 \xrightarrow{d_G} G_0$ and $H : H_1 \xrightarrow{d_H} H_0$ are **isoclinic** if there exist isomorphisms

$$(\eta_1, \eta_0) : (G_1 \xrightarrow{d_G} G_0) \longrightarrow (H_1 \xrightarrow{d_H} H_0)$$

and

$$(\xi_1, \xi_0) : (D_{G_0}(G_1) \xrightarrow{d_G} [G_0, G_0]) \longrightarrow (D_{H_0}(H_1) \xrightarrow{d_H} [H_0, H_0])$$

such that the diagrams

$$\begin{array}{c}
\begin{array}{ccc}
G_1 \times G_0 & \xrightarrow{c_1} & D_{G_0}(G_1) \\
\eta_1 \times \eta_0 & & \downarrow \xi_1 \\
H_1 \times H_0 & \xrightarrow{c_1'} & D_{H_0}(H_1)
\end{array}
\end{array}$$

are commutative where $c_1, c_1'$ defined by $c_1(g_1G_1, g_0(St_{G_0}(G_1) \cap Z(G_0))) = g_0g_1g_1^{-1}$, $c_1'(h_1H_1, h_0(St_{H_0}(H_1) \cap Z(H_0))) = h_0h_1h_1^{-1}$, for all $g_1 \in G_1, g_0 \in G_0, h_0 \in H_0, h_1 \in H_1$ and $c_0, c_0'$ defined by $c_0(g_0(St_{G_0}(G_1) \cap Z(G_0)), g_0'(St_{G_0}(G_1) \cap Z(G_0))) = [g_0, g_0]', c_0'(h_0(St_{H_0}(H_1) \cap Z(H_0)), h_0'(St_{H_0}(G_1) \cap Z(G_0))) = [h_0, h_0]'$, for all $g_0, g_0' \in G_0$ and $h_0, h_0' \in H_0$. (The well definition of the maps $c_1, c_0, c_1'$ and $c_0'$ are given in Appendix A.)

The pair $((\eta_1, \eta_0), (\xi_1, \xi_0))$ is called an **isoclinism** from $G$ to $H$ and this situation is denoted by $((\eta_1, \eta_0), (\xi_1, \xi_0)) : G \sim H$.

**Remark 3.** If the crossed modules $G$ and $H$ are simply connected or finite, then the commutativity of diagrams (1) with (2) in Definition 10 are equivalent to the commutativity of following diagram:

$$\begin{array}{c}
\begin{array}{ccc}
G/Z(G) \times G/Z(G) & \xrightarrow{c_1} & G/G \\
(\eta_1, \eta_0) \times (\eta_1, \eta_0) & & \downarrow (\xi_1, \xi_0) \\
H/Z(H) \times H/Z(H) & \xrightarrow{c_1'} & H/H
\end{array}
\end{array}$$

**Examples 4.**

1. All Abelian crossed modules are isoclinic.
2. Let $M$ and $N$ be isoclinic groups. Then $M \xrightarrow{id} M$ is isoclinic to $N \xrightarrow{id} N$.
3. Let $M$ be a group and let $N$ be a normal subgroup of $M$ with $NZ(M) = M$. Then $N \xrightarrow{inc} M$ is isoclinic to $M \xrightarrow{id} M$.
4. Some particular examples can be found in Section 5.

**Proposition 2.** Isoclinism is an equivalence relation.

**Proof.** One can easily check by a direct calculation. □
In [Hall 1940], it is proved that every group is isoclinic to a stem group. This property depends on the construction of Schur multiplicator and stem extensions of a group as given in [Beyl and Tappe 1982]. The same definitions and constructions were given for crossed modules in [Vieites and Casas 2002].

**Proposition 3.** Every crossed module is isoclinic to a stem crossed module.

**Proof.** It can be proved by using the related constructions and definitions given in [Beyl and Tappe 1982; Vieites and Casas 2002]. □

**Proposition 4.** Let \( G : G_1 \xrightarrow{d} G_0 \) be a crossed module and \( H : H_1 \xrightarrow{d_i} H_0 \) be its subcrossed module. If \( G = HZ(G) \), i.e. \( G_1 = H_1G_1^{G_0} \) and \( G_0 = H_0(St_{G_0}(G_1) \cap Z(G_0)) \), then \( G \) is isoclinic to \( H \).

**Proof.** First, we will show that \( H_1^{H_0} = H_1 \cap G_1^{G_0} \) and \( St_{H_0}(H_1) \cap Z(H_0) = H_0 \cap (St_{G_0}(G_1) \cap Z(G_0)) \).

Let \( h_1 \in H_1^{H_0} \). For any \( g_0 \in G_0 \), since \( G_0 = H_0(St_{G_0}(G_1) \cap Z(G_0)) \) there exist \( a_0 \in St_{G_0}(G_1) \cap Z(G_0) \) and \( h'_0 \in H_0 \) such that \( g_0 = h'_0a_0 \). We have \( g_0h_1 = (h'_0a_0)h_1 = h'_0(h_0h_1) = h'_0h_1 = h_1 \), so \( h_1 \in H_1 \cap G_1^{G_0} \). Conversely, for any \( h_1 \in H_1 \cap G_1^{G_0} \), we have \( h_1 \in H_1^{H_0} \). So, \( H_1^{H_0} = H_1 \cap G_1^{G_0} \).

Let \( h_0 \in St_{H_0}(H_1) \cap Z(H_0) \). For any \( g_1 \in G_1 \), there exist \( k_1 \in H_1 \) and \( a_1 \in G_1^{G_0} \) such that \( g_1 = k_1a_1 \). Then

\[
h_0g_1 = h_0(k_1a_1) = h_0k_1h_0a_1 = k_1a_1,
\]

which means that \( h_0 \in St_{G_0}(G_1) \). On the other hand, it is clear that \( h_0 \in Z(G_0) \). Then, we obtain \( h_0 \in H_0 \cap (St_{G_0}(G_1) \cap Z(G_0)) \). By a direct calculation we get \( St_{H_0}(H_1) \cap Z(H_0) = H_0 \cap (St_{G_0}(G_1) \cap Z(G_0)) \).

By the third isomorphism theorem for crossed modules, we have

\[
\frac{H}{Z(H)} = \frac{(H_1, H_0, d)}{(H_1^{H_0}, St_{H_0}(H_1) \cap Z(H_0), d)} = \frac{(H_1 \cap G_1^{G_0}, H_0 \cap (St_{G_0}(G_1) \cap Z(G_0)), d)}{(H_1, H_0, d)} = \frac{(G_1^{G_0}, St_{G_0}(G_1) \cap Z(G_0), d) \cap (H_1, H_0, d)}{G} \approx \frac{(G_1^{G_0}, St_{G_0}(G_1) \cap Z(G_0), d)}{G} = \frac{HZ(G)}{G} = \frac{Z(G)}{G},
\]

as required.

Let \( g_0g_1g_1^{-1} \in D_{G_0}(G_1) \), then there exist \( h_1 \in H_1, a_1 \in G_1^{G_0}, h_0 \in H_0, a_0 \in G_0 \).
the crossed module isomorphisms

\[
\begin{align*}
\eta_1 g_1 a_1^{-1} &= (h_0 a_0)(h_1 a_1)(h_1 a_1)^{-1} \\
&= h_0 a_0 (h_1) h_0 a_0 (a_1) (h_1 a_1)^{-1} \\
&= h_0 (a_0 h_1) h_0 (a_0 a_1) a_1^{-1} h_1^{-1} \\
&= (h_0 h_1) (h_0 a_1) a_1^{-1} h_1^{-1} \\
&= (h_0 h_1) a_1^{-1} h_1^{-1} \\
&= h_0 h_1 h_1^{-1},
\end{align*}
\]

we have \( g_0 g_1 a_1^{-1} \in D_{H_0}(H_1) \). On the other hand, for any \( [g_0, g'_0] \in [G_0, G_0] \) there exist \( h_0, h'_0 \in H_0 \), \( a_0, a'_0 \in (St_{G_0}(G_1) \cap Z(G_0)) \) such that \( g_0 = h_0 a_0 \), \( g'_0 = h'_0 a'_0 \), from which we get \( [g_0, g'_0] = [h_0 a_0, h'_0 a'_0] = [h_0, h'_0] \).

Finally, we have that the crossed modules \( G \) and \( H \) are isoclinic where the isomorphisms \( (\eta_1, \eta_0) \) and \( (\xi_1, \xi_0) \) are defined by \( (inc., inc.) \), \( (id_{G_1}, id_{G_0}) \), respectively. \( \Box \)

**Remark 4.** When \( H : H_1 \to H_0 \) is finite crossed module then the converse of Proposition 4 is true.

**Proposition 5.** Let \( G : G_1 \to G_0 \) and \( H : H_1 \to H_0 \) be isoclinic crossed modules.

(i) If \( G \) and \( H \) are aspherical, then \( G_0 \) and \( H_0 \) are isoclinic groups.

(ii) If \( G \) and \( H \) are simply connected, then \( G_1 \) and \( H_1 \) are isoclinic groups.

**Proof.** Let \( G : G_1 \to G_0 \) and \( H : H_1 \to H_0 \) be isoclinic crossed modules. Then we have the crossed module isomorphisms

\[
(\eta_1, \eta_0) : (G_1 \to G_0) \to (H_1 \to H_0)
\]

which makes diagrams (1) and (2) commutative.

(i) From asphericallity of the crossed modules, we have \( Z(G_0) \subseteq St_{G_0}(G_1) \), \( Z(H_0) \subseteq St_{H_0}(H_1) \). Consequently, \( \eta_0 \) is an isomorphism between \( G_0/Z(G_0) \) and \( H_0/Z(H_0) \). So the isomorphisms \( \eta_0 \) and \( \xi_0 \) give rise to an isoclinism from \( G_0 \) to \( H_0 \).

(ii) Since \( G \) and \( H \) are simply connected crossed modules, we have \( G_1^{G_0} = Z(G_1) \), \( H_1^{H_0} = Z(H_1) \), \( D_{G_0}(G_1) = [G_1, G_1] \) and \( D_{H_0}(H_1) = [H_1, H_1] \). So we have the isomorphisms \( \eta_1 : G_1/Z(G_1) \to H_1/Z(H_1) \), \( \xi_1 : [G_1, G_1] \to [H_1, H_1] \) which make \( G_1 \) and \( H_1 \) isoclinic. \( \Box \)

**Proposition 6.** Let \( G \) and \( H \) be isoclinic finite crossed modules. Then \( G_1 \) and \( G_0 \) are isoclinic to \( H_1 \) and \( H_0 \), respectively.

**Proof.** Let \( G : G_1 \to G_0 \) and \( H : H_1 \to H_0 \) be isoclinic crossed module. Then we have the crossed module isomorphisms
\[(\eta_1, \eta_0) : (G_1 \xrightarrow{d_{G_1}} G_0) \longrightarrow (H_1 \xrightarrow{d_{H_1}} H_0)\]
\[(\xi_1, \xi_0) : (D_{G_0}(G_1) \xrightarrow{d_{D_0}} [G_0, G_0]) \longrightarrow (D_{H_0}(H_1) \xrightarrow{d_{D_0}} [H_0, H_0])\]

which makes diagrams (1) and (2) commutative. The isomorphism \(\xi_1 : D_{G_0}G_1 \longrightarrow D_{H_0}H_1\) gives rise to the restriction \(\xi_1| : [G_1, G_1] \longrightarrow [H_1, H_1]\) which is also an isomorphism by the finiteness of \(G_1\) and \(H_1\). Similarly, we have the isomorphisms \(\eta_1 : G_1/Z(G_1) \longrightarrow H_1/Z(H_1), \eta_1(g_1Z(G_1)) = h_1Z(H_1), \eta_0 : G_0/Z(G_0) \longrightarrow H_0/Z(H_0), \eta_0(g_0Z(G_0)) = h_0Z(H_0),\) and \(\xi_0\) which makes \(G_1, G_0\) isoclinic to \(H_1, H_0\), respectively. \(\square\)

**Remark 5.** In general, the finiteness of a crossed module \(G : G_1 \xrightarrow{d} G_0\) does not give the equation \([G_1, G_1] = D_{G_0}(G_1)\). For the crossed module \(C_8 \xrightarrow{d} C_2\), we have \([C_8, C_8] = \{e\}\) and \(D_{C_8}(C_8) = C_4\). So \([C_8, C_8] \neq D_{C_8}(C_8)\).

As indicated in [Hall and Senior (1964)](Hall1964), the isoclinism of two groups doesn’t give rise to the isomorphism of their automorphism groups.

**Example 5.** Despite the fact that \(C_{32}\) isoclinic to \(Kl(4), Aut(C_{32})\) isn’t isomorphic to \(Aut(Kl_4)\). Since, \(|Aut(C_{32})| = 16 \neq 6 = |Aut(Kl_4)|\). But, in [Yadav (2008)](Yadav2008), it is shown that, class preserving automorphism groups of isoclinic groups are isomorphic.

In the crossed module case, we obtain the same results. For this, first we introduce the class preserving actor of a crossed module as follows;

**Proposition 7.** Let \(G : G_1 \longrightarrow G_0\) be a crossed module, \(D_C(G_0, G_1) = \{\delta \in D(G_0, G_1) \mid \text{there exists } g_1 \in G_1 \text{ such that } \delta(g_0) = g_1 \alpha(g_1)^{-1}, \text{ for all } g_0 \in G_0\}\) and \(Aut_C(G) = \{ (\alpha, \beta) \in Aut(G) \mid \text{there exists } g_0 \in G_0 \text{ such that } \alpha(g_1) = \alpha(g_1)^{-1}, \beta(g_0) = g_0 \alpha(g_1)^{-1}, \text{ for all } g_1 \in G_1, g_0 \in G_0\}\). Then, we have the following:
(a) \(D_C(G_0, G_1)\) is a subgroup of \(D(G_0, G_1)\).
(b) \(Aut_C(G)\) is a subgroup of \(Aut(G)\).

**Proof.** The proof is given in Appendix A. \(\square\)

**Proposition 8.** \(Act_C(G) : D_C(G_0, G_1) \xrightarrow{\Delta} Aut_C(G)\) is a crossed module with the action induced from the action of \(Aut(G)\) over \(D(G_0, G_1)\) such that
\[Aut_C(G) \times D_C(G_0, G_1) \longrightarrow D_C(G_0, G_1)\]
\[((\alpha, \beta), \delta) \longmapsto (\alpha, \beta)\delta\]
\[(\alpha, \beta)\delta(h_0) = \alpha(g_1) h_0 \alpha(g_1)^{-1}, \text{ for all } h_0 \in G_0 \text{ and } g_1 \in G_1.\]

**Proof.** It can be shown by a direct calculation. \(\square\)

**Definition 11.** Let \(G : G_1 \xrightarrow{d_{G_1}} G_0\) be a crossed module. The crossed module
\[Act_C(G) : D_C(G_0, G_1) \xrightarrow{\Delta} Aut_C(G)\]
define in Proposition 8 will be called class preserving actor of \(G\) and will be denoted by \(Act_C(G)\).
Theorem 9. Let $G : G_1 \xrightarrow{d_G} G_0$ and $H : H_1 \xrightarrow{d_H} H_0$ be isoclinic crossed modules. Then, we have $D_C(G_0, G_1) \cong D_C(H_0, H_1)$.

Proof. Suppose that $G$ and $H$ are isoclinic crossed modules. Then we have the isomorphisms

$$(\eta_1, \eta_0) : (G_1 \xrightarrow{d_G} G_0) \longrightarrow (H_1 \xrightarrow{d_H} H_0)$$

$$(\xi_1, \xi_0) : (D_{G_0}(G_1) \xrightarrow{d_G} [G_0, G_0]) \longrightarrow (D_{H_0}(H_1) \xrightarrow{d_H} [H_0, H_0])$$

which makes the diagrams (1) and (2) commutative. Let $\delta_G \in D_C(G_0, G_1)$, $h_0 \in H_0 - (\text{St}_{H_0}(H_1) \cap Z(H_0))$ and $g_0 \in St_{G_0}(G_1) \cap Z(G_0) = \eta_0^{-1}(h_0) \in G_0$. Since $g_0 \in G_0$, there exists an element $a_1 \in G_1$ such that $\delta_G(g_0) = a_1 g_0 a_1^{-1}$.

Let $\eta_1(g_0) = a_1$. Now we define a map $\delta_H : H_0 \longrightarrow H_1$ by

$$\delta_H(h_0) = \begin{cases} a_1' h_0 (a_1')^{-1} & h_0 \in H_0 - (\text{St}_{H_0}(H_1) \cap Z(H_0)) \\ e_{H_1} & h_0 \in \text{St}_{H_0}(H_1) \cap Z(H_0). \end{cases}$$

We will give the proof step-by-step.

**Step 1** $\delta_H$ is well defined.

Proof: Let $h_0, h_0' \in H_0 - (\text{St}_{H_0}(H_1) \cap Z(H_0))$. Then $h_0, h_0' \in H_0$ and $g_0 = \eta_0^{-1}(h_0) \in G_0$. Let $g_0 = \eta_0^{-1}(h_0)$ and $g_0' = \eta_0^{-1}(h_0') \in G_0$. So $g_0, g_0' \in G_0$ and there exist elements $a_1, b_1 \in G_1$ such that $\delta_G(g_0) = a_1 g_0 a_1^{-1}$ and $\delta_G(g_0') = b_1 g_0' b_1^{-1}$. Now we will show that if $h_0 = h_0'$, then $a_1' h_0 (a_1')^{-1} = b_1' h_0' (b_1')^{-1}$ where $a_1' = \eta_1(g_0)$, $b_1' = \eta_1(g_0')$.

Let $h_0 = h_0'$. Then $\overline{\eta_0} = \overline{\eta_0'}$ which means $(g_0')^{-1} g_0 \in St_{G_0}(G_1) \cap Z(G_0)$. Say, $(g_0')^{-1} g_0 = g$ where $g \in St_{G_0}(G_1) \cap Z(G_0)$. We have

$$\delta_G(g_0) = \delta_G(g_0') = \delta_G(g_0')^{g_0} \delta_G(g)$$

Since $\delta_G(g_0) = \delta_G(g_0') = a_1 g_0 a_1^{-1}$, we get $a_1' h_0 (a_1')^{-1} = b_1' h_0' (b_1')^{-1}$ which shows the well definition of $\delta_H$.

**Step 2** $\delta_H \in D_C(H_0, H_1)$.

Proof: Let $h_0, h_0' \in H_0$. We must check that $\delta_H(h_0 h_0') = \delta_H(h_0)(h_0' \delta_H(h_0'))$.

(i) If $h_0, h_0' \in St_{H_0}(H_1) \cap Z(H_0)$, then $h_0 h_0' \in St_{H_0}(H_1) \cap Z(H_0)$. So, $\delta_H(h_0 h_0') = e_{H_1}$ and $\delta_H(h_0)(h_0' \delta_H(h_0')) = e_{H_1} = e_{H_1}$. That is, $\delta_H(h_0 h_0') = \delta_H(h_0)(h_0' \delta_H(h_0'))$.

(ii) If $h_0 \in H_0 - (\text{St}_{H_0}(H_1) \cap Z(H_0))$ and $h_0' \in St_{H_0}(H_1) \cap Z(H_0)$, then $h_0 h_0' \in H_0 - (\text{St}_{H_0}(H_1) \cap Z(H_0))$. Since $\delta_H(h_0) = a_1' h_0 (a_1')^{-1}$ and $\delta_H(h_0') = e_{H_1}$, we have

$$\delta_H(h_0) \cdot \delta_H(h_0') = a_1' h_0 (a_1')^{-1} \cdot e_{H_1} = e_{H_1}.$$
\[ \delta_H(h_0h'_0) = a'_1 h_{a_0}^1 (a'_1)^{-1} \]
\[ = a'_1 h_{a_0}^1 (h_0)^{-1} (a'_1)^{-1} \]
\[ = a'_1 (h_{a_0}^1 (h_0)^{-1}) (a'_1)^{-1} \]
\[ = a'_1 h_{a_0}^1 (a'_1)^{-1} e_{H_1} \]
\[ = (a'_1 h_{a_0}^1 (a'_1)^{-1}) h_0 e_{H_1} \]
\[ = \delta_H(h_0) (h_0) \delta_H(h_0'). \]

(iii) If \( h_0, h'_0 \in H_0 - (\text{St}_{H_0}(H_1) \cap Z(H_0)) \), then \( h_0h'_0 \in H_0 - (\text{St}_{H_0}(H_1) \cap Z(H_0)) \) and
\[ \eta_1^{-1}(h_0h'_0) = \eta_0^{-1}(h_0) \eta_0^{-1}(h'_0) = g_0g_0 = g_0g_0, \] where \( g_0, g'_0 \in G_0 \). Let \( \delta_G(g_0g'_0) = a_1 b_1 \)
\[ g_0^* g'_0 (a_1 b_1)^{-1}, \delta_G(g_0) = a_1 g_0 a_1^{-1} \] and \( \delta_G(g'_0) = b_1 g'_0 b_1^{-1} \). Since \( \delta_G(g_0g'_0) = \delta_G(g_0)^g_0 \delta_G(g'_0), \) we get
\[ a_1 b_1 \ g_0^* g'_0 (a_1 b_1)^{-1} = (a_1 g_0 a_1^{-1}) (a_1 g_0 b_1 g'_0 b_1^{-1}) \]
\[ = (a_1 g_0 a_1^{-1}) (g_0 b_1 g'_0 b_1^{-1}) \]
\[ = (a_1 g_0 a_1^{-1}) (g_0 b_1 g'_0 b_1^{-1}) (b_1 g_0^* g'_0 b_1^{-1}). \]

By applying \( \xi_1 \), we get
\[ a'_1 b'_1 \ h_{a_0}^1 (a'_1 b'_1)^{-1} = (a'_1 h_{a_0}^1 (a'_1)^{-1}) (h_{a_0}^1 b'_1 (a'_1 b'_1)^{-1}) \]
\[ = (a'_1 h_{a_0}^1 (a'_1)^{-1}) (h_{a_0}^1 b'_1 h_{a_0}^1 (a'_1 b'_1)^{-1}) \]
\[ = (a'_1 h_{a_0}^1 (a'_1)^{-1}) h_{a_0}^1 (a'_1 b'_1 h_{a_0}^1 (a'_1 b'_1)^{-1}). \]

That is, \( \delta_H(h_0h'_0) = \delta_H(h_0) h_0 \delta_H(h'_0). \)

From definition of \( \delta_H \), we obtain \( \delta_H \in D_C(H_0, H_1). \)

Step 3 The map
\[ \phi : D_C(G_0, G_1) \rightarrow D_C(H_0, H_1) \]
\[ \delta_G \rightarrow \delta_H \]
is an isomorphism.

Proof: Let \( \delta_G, \delta'_G \in D_C(G_0, G_1), h_0 \in H_0 \) and \( g_0 = \eta_0^{-1}(h_0) \). Since \( \delta_G, \delta'_G \in D_C(G_0, G_1), \) there exists \( a \in G_1 \) such that \( \delta_G a \ delta'_G \in D_C(G_0, G_1), \) there exists \( a \in G_1 \) such that \( \delta_G(a_0) = a_1 g_0 a_1^{-1}, \delta'_G(g_0) = b_1 g_0 b_1^{-1} \). Since \( \delta_G a \ delta'_G \) \( (g_0) = a_1 b_1 \ g_0 a_1^{-1}, \) we get
\[ a_1 g_0 a_1^{-1} = a_1 b_1 \ g_0 (a_1 b_1)^{-1}. \]

By applying \( \xi_1 \), we get
\[ a'_1 h_{a_0}^1 (a'_1)^{-1} = a'_1 h_{a_0}^1 (h_{a_0}^1 b'_1 (a'_1 b'_1)^{-1}), \]
\[ \text{since } \eta_1(a_1 b_1) = \eta_1(a_1 b_1) = a_1 b_1 = a_1 b_1. \]

Finally, since \( \phi_{\delta_G \delta'_G}(h_0) = a'_1 h_{a_0}^1 (a'_1)^{-1} \)
and \( \phi_{\delta_G \delta'_G}(h_0) = a'_1 h_{a_0}^1 (a'_1 b'_1)^{-1}, \) for all \( h_0 \in H_0, \)
we have \( \phi_{\delta_G \delta'_G} = \phi_{\delta_G \delta'_G}, \) i.e \( \phi \) is a homomorphism.
Similarly, for each $\delta_H \in D_C(H_0, H_1)$, we can define $\delta_G \in D_C(G_0, G_1)$ and the homomorphism $\varphi : D_C(H_0, H_1) \to D_C(G_0, G_1)$, $\varphi(\delta_H) = \delta_G$ as follows;

Let $\delta_H \in D_C(H_0, H_1)$ and $g_0 \in G_0 - (St_{G_0}(G_1) \cap Z(G_0))$. Define $\overline{h_0} = h_0(St_{H_0}(H_1) \cap Z(H_0)) = \eta_0(g_0) \in \overline{H_0}$. So there exists an element $a'_1 \in H_1$ such that $\delta_H(h_0) = a'_1 \overline{h_0(a'_1)^{-1}}$.

Let $\eta_1^{-1}(a'_1) = \overline{a_1}$. Now we define the map $\delta_G : G_0 \to G_1$ by

$$\delta_G(g_0) = \begin{cases} a_1 \overline{g_0(a_1)^{-1}} & g_0 \in G_0 - (St_{G_0}(G_1) \cap Z(H_0)) \\ e_{G_1} & g_0 \in St_{G_0}(G_1) \cap Z(G_0) \end{cases}$$

Clearly, $\phi \varphi = id_{D_C(H_0, H_1)}$ and $\varphi \phi = id_{D_C(G_0, G_1)}$. Thus the homomorphism $\phi$ is an isomorphism. \Box

**Proposition 10.** If $G$ and $H$ are finite crossed modules, then $Aut_C(G) \cong Aut_C(H)$.

*Proof.* It can be easily checked by a similar way of Theorem 4.1 in Yadav (2008). \Box

**Corollary 11.** Let $G : G_1 \xrightarrow{d_G} G_0$ and $H : H_1 \xrightarrow{d_H} H_0$ be two finite isoclinic crossed modules. Then $Act_C(G) \cong Act_C(H)$.

**Example 6.** Let $M = K_{14}$ and $N = C_{32}$. The crossed modules $M \xrightarrow{id} M$ and $N \xrightarrow{id} N$ are isoclinic but their actors $(Aut(M), Aut(M), \Delta) \not\cong (Aut(N), Aut(N), \Delta)$. On the other hand, their class preserving actors $(Aut_C(M), Aut_C(M), \Delta)$ and $(Aut_C(N), Aut_C(N), \Delta)$ are isomorphic. (See Example 5, for details.)

In group theory, if $M$ and $N$ are isoclinic groups, then $M$ is nilpotent (solvable) if and only if $N$ is nilpotent (solvable), and they have the same nilpotency class (derived length).

On the other hand, for crossed modules, if $G : G_1 \xrightarrow{d} G_0$ is nilpotent (solvable) then all subcrossed modules and all quotient crossed modules of $G$ are nilpotent (solvable). Also, if $G/Z(G)$ is nilpotent (solvable), then $G$ is nilpotent (solvable). So, we have the following result:

**Proposition 12.** Let $G : G_1 \xrightarrow{d_G} G_0$ and $H : H_1 \xrightarrow{d_H} H_0$ be two isoclinic crossed modules.

(i) $G$ is nilpotent (solvable) crossed module if and only if $H$ is nilpotent (solvable).

(ii) If $G$ and $H$ are nilpotent (solvable) and both nontrivial, then they have the same nilpotency class (derived length).

**Remark 6.** When we consider the groups as crossed modules, then we recover classical results for isoclinic groups. In fact, if $M \xrightarrow{id} M$ and $N \xrightarrow{id} N$ are isoclinic crossed modules then we find that $M$ and $N$ are isoclinic. On the other hand, let $N$, $M$ be finitely generated groups and $N' \leq N$, $M' \leq M$. Then the isoclinism of inclusion crossed modules $N' \xleftarrow{inc} N$ and $M' \xleftarrow{inc} M$ give rise to the isoclinism between the pair groups $(N', N)$ and $(M', M)$. Also, the converse is true, that is, if $(G_1, G_0)$ and $(H_1, H_0)$ are isoclinic pair groups, then the resulting inclusion crossed modules $G_1 \xleftarrow{inc} G_0$ and $H_1 \xleftarrow{inc} H_0$ are isoclinic. (See Salemkar et al. (2007), for the definition of isoclinic pair groups.)
4. Computer Implementations

GAP is an open-source system for discrete computational algebra. The system consists of a library of mathematical algorithm implementations, a database about some algebraic properties of small order groups, vector spaces, modules, algebras, graphs, codes, designs etc. and some character tables of these algebraic structures. The system has worldwide usage in the area of education and scientific researches. GAP is free software and user contributions to the system are supported. These contributions are organized in a form of GAP packages and are distributed together with the system. Contributors can submit additional packages for inclusion after a reviewing process.

Since, no standard GAP function yet exist for checking that two groups $M$ and $N$ are isoclinic or not, first we add the function $\text{IsIsoclinicGroup}(M,N)$. In the following GAP session it is seen that the dihedral group with order 8 and the quaternion group are isoclinic. Notice that two isoclinic groups may have different orders.

\begin{verbatim}
gap> Q8 := QuaternionGroup(8);
<pc group of size 8 with 3 generators>
gap> D8 := DihedralGroup(8);
<pc group of size 8 with 3 generators>
gap> IsIsoclinicGroup(Q8,D8);
true
\end{verbatim}

In James et al. (1990), the 115 isoclinism families induced from 2328 groups with order 128 and their basic properties were given. We add the function $\text{IsoclinismFamily}(M)$ to determine the isoclinism classes of the group $M$ with order $n$. We give a table consisting isoclinism classes of 1543 group with order 192 in Appendix B, by using this function.

The computer applications of crossed modules were given by Alp and Wensley, in Alp and Wensley (2013), by the shared package XMod. To add a function checks if any two crossed modules are isoclinic or not, first we need to define the functions; factor crossed modules, commutator crossed modules which haven’t implemented in the XMod package. Also we redefine a function to find center of a given crossed module, to make compatible with the other defined functions.

4.1. Implementations for centers of crossed modules

Up to the definition of isoclinic crossed modules, we first need to construct the center of a given crossed module by using Definition 5 which is called by $\text{CenterXMod}(XM)$. The step-by-step construction of this function is given as follows:

**Step 1**: We added the function $\text{G1G0(XM)}$, for computing the subgroup $G_1G_0$, the fixed point of $G_1$, induced from the crossed module $XM : G_1 \xrightarrow{d} G_0$.

\begin{verbatim}
gap> G1G0(XM);
<pc group of size 2 with 1 generators>
gap> IsSubgroup(Source(XM),last);
true
\end{verbatim}

**Step 2**: We added the function $\text{StG0G1(XM)}$, for computing the subgroup $\text{St}_{G_0}G_1$, the stabilizer of $G_1$ in $G_0$, induced from the crossed module $XM : G_1 \xrightarrow{d} G_0$.
Step 3: By the definition of center of a crossed module $XM : G_1 \longrightarrow G_0$ given in Definition 5, we added the function CenterXMod(XM), for computing the center.

4.2. Implementations for factor and commutator subcrossed modules

Since no standard GAP function yet exist for computing the commutator subcrossed module of a given crossed module, we added the function CommutatorSubXMod(XM). The step-by-step construction of this function is as follows:

Step 1: We added the function DG0G1(XM), for computing the subgroup $D_{G_0}(G_1)$ induced from the crossed module $XM : G_1 \longrightarrow G_0$ defined in Definition 6.

Step 2: We added the function DerivedSubXMod(XM) for computing the subcrossed module $[XM, XM]$ defined in Definition 6.

Additionally, we added the function FactorXMod(XM,NM), for computing the quotient crossed modules.
4.3. Implementations for isoclinic crossed modules

We have added a function `IsIsoclinicXMod(XM1,XM2)`, for checking two crossed modules are isoclinic or not. Step-by-step construction of this function is as follows:

**Step 1:** First of all, we needed a function for isomorphism of crossed modules and so we added the function `IsIsomorphicXMod(XM1,XM2)`. In the following GAP session, it is proved that the constructed crossed modules XM and XM2 are not isomorphic.

```
gap> C2 := Cat1(32,9,1);
[(C8 x C2) : C2=>Group( [ f2, f2 ] )]
gap> XM2 := XMod(C2);
[Group( [ f1*f2*f3, f3, f4, f5 ] )->Group( [ f2, f2 ] )]
gap> IsIsomorphicXMod(XM,XM2);
false
```

**Step 2:** We determined all isomorphisms between the factor crossed modules $XM1/Z(XM1)$ and $XM2/Z(XM2)$. If there is no such isomorphism, we arrange the function `IsIsoclinicXMod(XM1,XM2)` to make its output `false`.

```
gap> ZXM2 := CenterXMod(XM2);;
gap> IsIsomorphicXMod(FactorXMod(XM,ZXM),FactorXMod(XM2,ZXM2));
true
```

**Step 3:** We continued the same procedure given in Step 2 for the commutator sub-crossed modules.

```
gap> KM2 := DerivedSubXMod(XM2);;
gap> IsIsomorphicXMod(KM,KM2);
true
```

**Step 4:** Then, after determining the existence of two isomorphism given in Step 2 and Step 3, we arrange `IsIsoclinicXMod(XM1,XM2)` to give out put `true` if the isomorphisms make the diagrams (1) and (2) in Definition 10 commutative.

```
gap> IsIsoclinicXMod(XM,XM2);
true
gap> Size(XM2);
[ 16, 2 ]
```

Remark 7. This GAP session shows that two crossed modules whose orders are different can be isoclinic as it is the case for groups.

5. Character Tables

For determining isoclinism families of order $[n,m]$, we added a function `AllXMods(n,m)` to find all crossed modules of order $[n,m]$.

```
gap> list := AllXMods(4,4);;
gap> Length(list);
60
```
Then we added the function \texttt{AllXModsByIso(list)} to choose one representative from all isomorphism families and construct the new list of crossed modules. Naturally, in the new list, there is no isomorphic crossed modules.

\begin{verbatim}
gap> ilist := AllXModsByIso(list);;
gap> Length(ilist);
18
\end{verbatim}

We added the function \texttt{IsoclinismXModFamily(XM,ilist)}, to get the isoclinism families for a given order \([n,m]\). There are two isoclinism families of crossed modules of order \([4,4]\) which is given in the following GAP session.

\begin{verbatim}
gap> IsoclinismXModFamily(iso_list[3],ilist);
[ 1, 3, 4, 6, 8, 10, 12, 14, 16, 18 ]
gap> IsoclinismXModFamily(iso_list[2],ilist);
[ 2, 5, 7, 9, 11, 13, 15, 17 ]
\end{verbatim}

Now, we introduce the notion “rank” and “middle length” of a crossed module.

\textbf{Definition 12.} Let \(G : G_1 \xrightarrow{d} G_0\) be a finite crossed module. Then the pair

\[
\left( \log_2 \left| G_1^{G_0} \cap D_{G_0}(G_1) \right|, \ \log_2 \left| \left( St_{G_0}(G_1) \cap Z(G_0) \right) \cap [G_0,G_0] \right| \right)
\]

\[
+ \left( \log_2 \left| G_1 / G_1^{G_0} \right|, \ \log_2 \left| G_0 / \left( St_{G_0}(G_1) \cap Z(G_0) \right) \right| \right)
\]

is called the \textit{rank} of \(G\). Also the pair

\[
\left( \log_2 \left| D_{G_0}(G_1) / (G_1^{G_0} \cap D_{G_0}(G_1)) \right|, \ \log_2 \left| \left( St_{G_0}(G_1) \cap Z(G_0) \right) \cap [G_0,G_0] \right| \right)
\]

is called the \textit{middle length} of \(G\).

The rank, middle length, nilpotency class and lower central series of the crossed modules in the same isoclinism family are equal. For computing these gadgets (these can be thought as the gadgets for correcting our isoclinism definition) we added the functions \texttt{RankOfXMod(XM)}, \texttt{MiddleLengthOfXMod(XM)}, \texttt{LowerCentralSeriesOfXMod(XM)} and \texttt{NilpotencyClassOfXMod(XM)}.

\begin{verbatim}
gap> RankOfXMod(XM);
[ 3, 1 ]
gap> MiddleLengthOfXMod(XM);
[ 1, 0 ]
gap> LowerCentralSeriesOfXMod(XM);
[ [Group( [ f1*f2*f3, f3, f4 ] )->Group( [ f2, f2 ] )],
  [Group( [ f3 ] )->Group( <identity> of ... )],
  [Group( [ f4 ] )->Group( <identity> of ... )],
  [Group( <identity> of ... )->Group( <identity> of ... )] ]
gap> Length(last);
4
gap> NilpotencyClassOfXMod(XM);
17
\end{verbatim}
Example 7. The groups of order 8 has 5 isomorphism classes and 2 isoclinism families which are listed as follows;

| Family | Numbers | Represent. | Rank | Middle Length | Nilpotency Class | G/Z | γ2(G) |
|--------|---------|------------|------|---------------|-----------------|-----|-------|
| 1      | 3       | [8,1]      | 0    | 0             | 1               | [1,1]|       |
| 2      | 2       | [8,3]      | 3    | 0             | 2               | [2,1]|       |

By using these 5 isomorphism class, we get 9008 crossed modules with order [8,8], 294 isomorphism classes and 20 isoclinism families.

The following informations is listed in Table II for each isoclinism family;

1) the number of crossed modules in the family,

2) the rank of crossed modules in the family,

3) the middle length of crossed modules in the family,

4) the nilpotency class, c > 0, of the crossed modules in the family.

5) the size of the central quotient, XM/Z(XM), of a crossed modules XM in the family.

6) the size of the non-trivial or non-repeatedly terms, γ2(XM),...,γc(XM), of the lower central series of XM; here γ1(XM) = XM and γi+1(XM) = [γi(XM), XM], for 1 ≤ i ≤ c.
| Fam. | Num. | Rank | M. L. | Class | $|XM/ZXM|$ | $\gamma_2(XM)$ | $\gamma_3(XM)$ |
|------|------|------|------|-------|-----------------|-----------------|-----------------|
| 1    | 37   | [0,0]| [0,0]| 1     | [1,1]           |                 |                 |
| 2    | 79   | [2,1]| [0,0]| 2     | [2,2]           | [2,1]           |                 |
| 3    | 18   | [3,1]| [1,0]| 3     | [4,2]           | [4,1]           | [2,1]           |
| 4    | 8    | [3,2]| [1,0]| 3     | [4,4]           | [4,1]           | [2,1]           |
| 5    | 14   | [0,3]| [0,0]| 2     | [1,4]           | [1,2]           |                 |
| 6    | 42   | [2,3]| [0,0]| 2     | [2,4]           | [2,2]           |                 |
| 7    | 12   | [3,3]| [1,0]| 3     | [4,4]           | [4,2]           | [2,1]           |
| 8    | 8    | [3,2]| [1,0]| 3     | [4,4]           | [4,2]           | [2,1]           |
| 9    | 4    | [3,3]| [1,0]| 3     | [4,4]           | [4,2]           | [2,1]           |
| 10   | 4    | [3,2]| [1,0]| 3     | [4,4]           | [4,1]           | [2,1]           |
| 11   | 10   | [3,2]| [0,0]| 2     | [2,4]           | [4,1]           |                 |
| 12   | 15   | [3,2]| [0,0]| 2     | [4,4]           | [2,1]           |                 |
| 13   | 10   | [3,3]| [0,0]| 2     | [2,4]           | [4,2]           |                 |
| 14   | 2    | [3,3]| [1,1]| 3     | [4,8]           | [4,2]           | [2,1]           |
| 15   | 15   | [3,3]| [0,0]| 2     | [4,4]           | [2,2]           |                 |
| 16   | 6    | [2,3]| [0,0]| 2     | [2,4]           | [2,2]           |                 |
| 17   | 2    | [3,3]| [1,1]| 3     | [4,8]           | [4,2]           | [2,1]           |
| 18   | 2    | [3,3]| [0,0]| 2     | [4,4]           | [2,2]           |                 |
| 19   | 2    | [3,2]| [0,0]| 2     | [4,4]           | [2,1]           |                 |
| 20   | 4    | [3,2]| [1,0]| 3     | [4,4]           | [4,1]           | [2,1]           |
**Example 8.** The group of order 18 has 5 isomorphism classes and 4 isoclinism families.

**Table III**
Number of Groups in Each Isoclinism Family
and Some Family Invariants

| Fam. | Num. | Rep. | Rank | M. L. | Class | $G/Z\gamma_2$($G$) |
|------|------|------|------|------|-------|---------------------|
| 1    | 1    | [18,1]| 4.17 | 3.17 | 0     | [18,1] [9,1]       |
| 2    | 2    | [18,2]| 0.00 | 0.00 | 1     | [1,1]            |
| 3    | 1    | [18,3]| 2.58 | 1.58 | 0     | [6,1] [3,1]       |
| 4    | 1    | [18,4]| 4.17 | 3.17 | 0     | [18,4] [9,2]     |

By using these 5 isomorphism classes, we get 2222 crossed modules with order [18,18],
97 isomorphism classes, 46 isoclinism families.

**Table IV**
Number of Crossed Modules in Each Isoclinism Family
and Some Family Invariants

| Fam. | Num. | Rank | M. L. | Class | $|XM/Z(XM)|$ | $|\gamma_2(XM)|$ | $|\gamma_3(XM)|$ |
|------|------|------|------|-------|------------|----------------|----------------|
| 1    | 1    | [4.17,4.17]| [3.17,3.17]| 0     | [18,18]   | [9,9]        |
| 2    | 2    | [0.00,4.17]| [0.00,3.17]| 0     | [1,18]    | [1,9]        |
| 3    | 1    | [3.17,4.17]| [3.17,3.17]| 0     | [9,18]    | [9,9]        |
| 4    | 1    | [3.17,4.17]| [3.17,3.17]| 0     | [9,18]    | [9,9]        |
| 5    | 1    | [3.17,4.17]| [3.17,3.17]| 0     | [9,18]    | [9,9]        |
| 6    | 20   | [0.00,0.00]| [0.00,0.00]| 1     | [1,1]     | [1,1]        |
| 7    | 2    | [3.17,2.58]| [3.17,0.00]| 0     | [9,6]     | [9,1]        |
| 8    | 16   | [3.17,1.58]| [0.00,0.00]| 2     | [3,3]     | [3,1]        |
| 9    | 2    | [3.17,1.00]| [3.17,0.00]| 0     | [9,2]     | [9,1]        |
| 10   | 4    | [0.00,2.58]| [0.00,1.58]| 0     | [1,6]     | [1,3]        |
| 11   | 1    | [3.17,4.17]| [3.17,1.58]| 0     | [9,18]    | [9,3]        |
| 12   | 2    | [3.17,4.17]| [0.00,1.58]| 0     | [3,18]    | [3,3] [1,3]  |
| 13   | 1    | [3.17,2.58]| [3.17,1.58]| 0     | [9,6]     | [9,3]        |
| 14   | 1    | [3.17,4.17]| [3.17,1.58]| 0     | [9,18]    | [9,3]        |
| 15   | 1    | [3.17,2.58]| [3.17,1.58]| 0     | [9,6]     | [9,3]        |

*table continued*
| Fam. | Num. | Rank | M. L. | Class | $|XM/Z(XM)|$ | $|\gamma_2(XM)|$ | $|\gamma_3(XM)|$ |
|------|------|------|------|-------|-------------|--------------|--------------|
| 16   | 2    | [0.00,4.17] | [0.00,3.17] | 0     | [1,18]      | [1.9]        |              |
| 17   | 1    | [3.17,4.17] | [3.17,3.17] | 0     | [9,18]      | [9.9]        |              |
| 18   | 1    | [3.17,4.17] | [3.17,3.17] | 0     | [9,18]      | [9.9]        |              |
| 19   | 2    | [2.58,2.58] | [1.58,1.58] | 0     | [6,6]       | [3.3]        |              |
| 20   | 1    | [4.17,4.17] | [3.17,3.17] | 0     | [18,18]     | [9.9]        |              |
| 21   | 1    | [1.58,4.17] | [1.58,3.17] | 0     | [3,18]      | [3.9]        |              |
| 22   | 1    | [3.17,4.17] | [1.58,3.17] | 0     | [9,18]      | [9.9]        |              |
| 23   | 1    | [3.17,4.17] | [1.58,3.17] | 0     | [9,18]      | [9.9]        |              |
| 24   | 1    | [3.17,4.17] | [3.17,3.17] | 0     | [9,18]      | [9.9]        |              |
| 25   | 1    | [1.58,4.17] | [1.58,3.17] | 0     | [3,18]      | [3.9]        |              |
| 26   | 1    | [3.17,4.17] | [1.58,3.17] | 0     | [3,18]      | [9.9]        |              |
| 27   | 1    | [3.17,4.17] | [3.17,3.17] | 0     | [9,18]      | [9.9]        |              |
| 28   | 4    | [1.58,1.00] | [1.58,0.00] | 0     | [3,2]       | [3.1]        |              |
| 29   | 2    | [3.17,2.58] | [3.17,0.00] | 0     | [9,6]       | [9.1]        |              |
| 30   | 2    | [3.17,1.00] | [3.17,0.00] | 0     | [9,2]       | [9.1]        |              |
| 31   | 2    | [1.58,2.58] | [1.58,1.58] | 0     | [3,6]       | [3.3]        |              |
| 32   | 1    | [3.17,4.17] | [3.17,1.58] | 0     | [9,18]      | [9.3]        |              |
| 33   | 1    | [3.17,2.58] | [3.17,1.58] | 0     | [9,6]       | [9.3]        |              |
| 34   | 2    | [3.17,2.58] | [1.58,1.58] | 0     | [9,6]       | [3.3]        |              |
| 35   | 1    | [3.17,2.58] | [1.58,1.58] | 0     | [3,6]       | [9.3]        |              |
| 36   | 2    | [1.58,2.58] | [1.58,1.58] | 0     | [3,6]       | [3.3]        |              |
| 37   | 1    | [3.17,4.17] | [3.17,1.58] | 0     | [9,18]      | [9.3]        |              |
| 38   | 1    | [3.17,2.58] | [3.17,1.58] | 0     | [9,6]       | [9.3]        |              |
| 39   | 1    | [1.58,4.17] | [1.58,3.17] | 0     | [3,18]      | [3.9]        |              |
| 40   | 1    | [3.17,4.17] | [1.58,3.17] | 0     | [9,18]      | [3.9]        |              |
| 41   | 1    | [3.17,4.17] | [1.58,3.17] | 0     | [3,18]      | [9.9]        |              |
| 42   | 1    | [3.17,4.17] | [3.17,3.17] | 0     | [9,18]      | [9.9]        |              |
| 43   | 1    | [1.58,4.17] | [1.58,3.17] | 0     | [3,18]      | [3.9]        |              |
| 44   | 1    | [3.17,4.17] | [1.58,3.17] | 0     | [3,18]      | [9.9]        |              |
| 45   | 1    | [3.17,4.17] | [3.17,3.17] | 0     | [9,18]      | [9.9]        |              |
| 46   | 1    | [3.17,4.17] | [3.17,3.17] | 0     | [9,18]      | [9.9]        |              |
Appendix A

Well definition of the maps $c_1$ and $c_0$ used in Definition 10:

Let $(g_1 G_1^{G_0}, g_0 St_{G_0}(G_1) \cap Z(G_0)) = (a_1 G_1^{G_0}, a_0 St_{G_0}(G_1) \cap Z(G_0))$, then $g_1^{-1}(a_1) \in G_1^{G_0} \subseteq Z(G_1)$ and $g_0^{-1}(a_0) \in St_{G_0}(G_1) \cap Z(G_0) \subseteq Z(G_0)$. Then we have

$$g_0 g_1 g_1^{-1} = (g_0 g_1 g_1^{-1}(a_1)(a_1)^{-1}$$

$$= (g_0 g_1)^{g_0 (g_1^{-1} a_1)}(a_1)^{-1} \ (\because g_1^{-1} a_1 \in G_1^{G_0})$$

$$= g_0 (g_1 g_1^{-1} a_1)(a_1)^{-1} \ (\because (g_0 g_1) = (g_0 g_1))$$

$$= (g_0 a_1)(a_1)^{-1}$$

$$= (g_0^{-1} a_0) (g_0 a_1)(a_1)^{-1} \ (\because g_0^{-1} a_0 \in St_{G_0}(G_1))$$

$$= (g_0^{-1} a_0) g_0 (g_1 g_1^{-1} a_1)(a_1)^{-1} \ (\because g_0 g_1 = g_0 g_1)$$

$$= (g_0 (g_0^{-1} a_0) a_1)(a_1)^{-1} \ (\because g_0^{-1} a_0 \in Z(G_0))$$

$$= (g_0 a_1)(a_1)^{-1}$$

which gives the well-definition of $c_1$.

Similarly, let $\overrightarrow{g_0}, \overrightarrow{a_0}, \overrightarrow{g_0'}, \overrightarrow{a_0'} \in \overrightarrow{G_0}$. If $(\overrightarrow{g_0}, \overrightarrow{g_0'}) = (\overrightarrow{a_0}, \overrightarrow{a_0'})$, then $g_0^{-1} a_0 \in St_{G_0}(G_1) \cap Z(G_0) \subseteq Z(G_0)$ and $(g_0')^{-1} a_0' \in St_{G_0}(G_1) \cap Z(G_0) \subseteq Z(G_0)$. Then, we have

$$[g_0, g_0'] = g_0 g_0^{-1} g_0'(g_0')^{-1}$$

$$= g_0 g_0^{-1} a_0 a_0^{-1} (g_0')^{-1}$$

$$= g_0 g_0' g_0^{-1} a_0 a_0^{-1} (g_0')^{-1}$$

$$= g_0 (g_0^{-1} a_0) g_0 a_0^{-1} (g_0')^{-1} \ (\because g_0^{-1} a_0 \in Z(G_0))$$

$$= a_0 g_0 a_0^{-1} (g_0')^{-1}$$

$$= a_0 g_0 a_0^{-1} (g_0')^{-1} (a_0' a_0^{-1})$$

$$= a_0 g_0 a_0^{-1} (g_0')^{-1} a_0 a_0^{-1}$$

$$= a_0 g_0 a_0^{-1} (g_0')^{-1} a_0 a_0^{-1} \ (\because (g_0')^{-1} a_0' \in Z(G_0))$$

$$= a_0 a_0' a_0^{-1}$$

which gives the well-definition of $c_0$.

Well definition of the maps $c_1'$ and $c_0'$ can be shown by a similar way.

Proof of Proposition 8:

(a) i) Let $\delta, \delta' \in D_c(G_0, G_1)$. We first show that $\delta \delta' \in D_c(G_0, G_1)$. Since $\delta, \delta' \in D_c(G_0, G_1)$, there exist $g_1, g_1' \in G_1$ such that $\delta(g_0) = g_1^{-1} g_0 g_1$ and $\delta'(g_0) = (g_1')^{-1} g_0 g_1'$, for
all \( g_0 \in G_0, g_1, g'_1 \in G_1 \). Then,

\[
\delta \delta'(g_0) = \delta(\delta'(g_0)g_0)\delta'(g_0)
\]

\[
= \delta(d(\delta'(g_0))g_0)\delta'(g_0)
\]

\[
= \delta(d(\delta'(g_0))d(\delta'(g_0))\delta(g_0)\delta'(g_0) \quad (\because \delta \in D(G_0, G_1))
\]

\[
= \delta(d(\delta'(g_0))\delta'(g_0)(\delta'(g_0))^{-1}\delta'(g_0) \quad (\because d \sim \text{crossed module})
\]

\[
g_1 \delta'(g_0)g_1^{-1}(g'_1)^{\beta_0}(g'_1)^{-1}g_1 g_0 g_1^{-1}
\]

\[
g_1((g'_1)^{\beta_0}(g'_1)^{-1})g_1^{-1}((g'_1)^{\beta_0}(g'_1)^{-1})^{-1}(g'_1)^{\beta_0}(g'_1)^{-1}g_1 g_0 g_1^{-1}
\]

\[
g_1g'_1^{-\beta_0}(g'_1)^{-1}g_1^{-1}(g'_1)^{-1}g_1^{-1}(g'_1)^{-1}g_1^{-1}g_0 g_1^{-1}
\]

\[
g_1g'_1^{-\beta_0}(g'_1)^{-1}g_1^{-1}g_1 g_0 g_1^{-1}
\]

\[
= g_1g'_1^{-\beta_0}(g'_1)^{-1}g_1^{-1}g_1 g_0 g_1^{-1}
\]

\[
= (g_1g'_1)^{-\beta_0}(g_1g'_1)^{-1}
\]

i.e \( \delta \delta' \in D_c(G_0, G_1) \).

ii) Let \( \delta \in D_c(G_0, G_1) \). Since \( \delta \in D_c(G_0, G_1) \), there exists \( g_1 \in G_1 \) such that \( \delta(g_0) = g_1 \cdot g_0^{-1} \), for all \( g_0 \in G_0 \). Define \( \delta^{-1}(g_0) = (g_1^{-1})^{\beta_0}g_1 \), then we have

\[
\delta \delta^{-1}(g_0) = g_1g_1^{-1}g_0(g_1^{-1}g_1)^{-1}
\]

\[
= g_1g_1^{-1}g_0e_{G_1}
\]

\[
= e_{G_1}e_{G_1}
\]

\[
= e_{G_1}
\]

\[
= id_{D_c(G_0, G_1)}(g_0).
\]

So \( D_c(G_0, G_1) \leq D(G_0, G_1) \).

(b) Let \( (\alpha, \beta), (\alpha', \beta') \in Aut_c(G) \). Since \( (\alpha, \beta), (\alpha', \beta') \in Aut_c(G) \), there exist \( g_0, g'_0 \in G_0 \) such that \( \alpha(g_1) = g_0 g_1; \alpha'(g_1) = g'_0 g_1 \); \( \beta(h_0) = g_0 h_0 g_0^{-1} \); \( \beta'(h_0) = g'_0 h_0 g'_0^{-1} \), for all \( g_0, h_0 \in G_0, g_1 \in G_1 \). Then

\[
(\alpha \circ \alpha')(g_1) = \alpha(\alpha'(g_1))
\]

\[
= \alpha(g_0 g_1)
\]

\[
= g'_0 g_0 g_1
\]

\[
= (g'_0 g_0)g_1
\]

and
\[(\beta \circ \beta')(h_0) = \beta(\beta'(h_0)) \]
\[= \beta(g_0'h_0(g_0')^{-1}) \]
\[= g_0(g_0'h_0(g_0')^{-1})g_0^{-1} \]
\[= g_0g_0'h_0(g_0g_0')^{-1}. \]

So \((\alpha, \beta) \circ (\alpha', \beta') \in \text{Aut}_c(G)\), for all \((\alpha, \beta), (\alpha', \beta') \in \text{Aut}_c(G)\).

Let \((\alpha, \beta) \in \text{Aut}_c(G)\). If we define \((\alpha, \beta)^{-1} = (\alpha^{-1}, \beta^{-1})\) by \(\alpha^{-1}(g_1) = g_0^{-1}g_1\), \(\beta^{-1}(g_0) = g_0^{-1}g_0g_0\), for all \(g_0, g_0' \in G_0\), \(g_1 \in G_1\), then we have

\[(\alpha \circ \alpha^{-1})(g_1) = \alpha(\alpha^{-1}(g_1)) \]
\[= \alpha(g_0^{-1}g_1) \]
\[= g_0g_0^{-1}g_1 \]
\[= (g_0g_0')^{-1}g_1 \]
\[= e_{G_0} g_1 \]
\[= g_1 \]
\[= id_{G_1}(g_1) \]

and similarly \(\beta \circ \beta^{-1} = id_{G_0}\). So \(\text{Aut}_c(G) \leq \text{Aut}(G)\).

Appendix B

Table V

| Fam. | Num. | Rep.   | Rank | M. L. | Class | $G/Z_{\gamma}$ | $\gamma_2(G)$ | $\gamma_3(G)$ | $\gamma_4(G)$ | $\gamma_5(G)$ | $\gamma_6(G)$ |
|------|------|--------|------|-------|-------|---------------|---------------|---------------|---------------|---------------|---------------|
| 1    | 19   | [192,1]| 2.58 | 1.58  | 0     | [6,1]         | [3,1]         |               |               |               |               |
| 2    | 11   | [192,2]| 0.00 | 0.00  | 1     | [1,1]         |               |               |               |               |               |
| 3    | 1    | [192,3]| 7.58 | 6.00  | 0     | [192,3]      | [64,2]        |               |               |               |               |
| 4    | 1    | [192,4]| 7.58 | 4.00  | 0     | [48,3]       | [64,19]       |               |               |               |               |
| 5    | 53   | [192,6]| 4.58 | 1.58  | 0     | [12,4]       | [6,2]         | [3,1]         |               |               |               |
| 6    | 3    | [192,7]| 7.58 | 4.58  | 0     | [96,6]       | [48,2]        | [24,2]        | [12,2]        | [6,2]         | [3,1]         |
| 7    | 10   | [192,10]| 7.58 | 2.58  | 0     | [48,14]      | [24,9]        | [12,5]        | [6,2]         | [3,1]         |               |
| 8    | 25   | [192,15]| 5.58 | 2.58  | 0     | [24,6]       | [12,2]        | [6,2]         | [3,1]         |               |               |
| 9    | 5    | [192,25]| 7.58 | 1.58  | 0     | [48,11]      | [12,2]        | [6,2]         | [3,1]         |               |               |
| 10   | 12   | [192,27]| 6.58 | 2.58  | 0     | [48,14]      | [12,5]        | [6,2]         | [3,1]         |               |               |

*table continued*
| Fam. | Num. | Rep. | Rank | M. L. | Class | G/Z | \( \gamma_2(G) \) | \( \gamma_3(G) \) | \( \gamma_4(G) \) | \( \gamma_5(G) \) | \( \gamma_6(G) \) |
|------|------|------|------|-------|-------|-----|----------------|----------------|----------------|----------------|----------------|
| 11   | 4    | [192,30] | 7.58 | 3.58 | 0 | [96,13] | [24,15] | [12,5] | [6,2] | [3,1] |
| 12   | 4    | [192,32] | 7.58 | 3.58 | 0 | [96,13] | [24,9] | [12,5] | [6,2] | [3,1] |
| 13   | 35   | [192,38] | 5.58 | 2.58 | 0 | [24,8] | [12,2] | [6,2] | [3,1] |
| 14   | 2    | [192,46] | 7.58 | 3.58 | 0 | [96,15] | [24,2] | [12,2] | [6,2] | [3,1] |
| 15   | 3    | [192,47] | 7.58 | 3.58 | 0 | [96,16] | [24,2] | [12,2] | [6,2] | [3,1] |
| 16   | 14   | [192,48] | 6.58 | 3.58 | 0 | [48,15] | [24,2] | [12,2] | [6,2] | [3,1] |
| 17   | 10   | [192,62] | 6.58 | 3.58 | 0 | [48,7] | [24,2] | [12,2] | [6,2] | [3,1] |
| 18   | 1    | [192,72] | 7.58 | 3.58 | 0 | [96,24] | [24,2] | [12,2] | [6,2] | [3,1] |
| 19   | 3    | [192,75] | 7.58 | 3.58 | 0 | [96,28] | [24,2] | [12,2] | [6,2] | [3,1] |
| 20   | 4    | [192,78] | 7.58 | 4.58 | 0 | [96,33] | [48,2] | [24,2] | [12,2] | [6,2] | [3,1] |
| 21   | 11   | [192,84] | 6.58 | 2.58 | 0 | [48,19] | [12,5] | [6,2] | [3,1] |
| 22   | 2    | [192,95] | 7.58 | 3.58 | 0 | [96,41] | [24,15] | [12,5] | [6,2] | [3,1] |
| 23   | 7    | [192,96] | 7.58 | 2.58 | 0 | [48,19] | [24,9] | [12,5] | [3,1] |
| 24   | 4    | [192,100] | 7.58 | 3.58 | 0 | [96,41] | [24,9] | [12,5] | [6,2] | [3,1] |
| 25   | 3    | [192,122] | 7.58 | 3.58 | 0 | [96,39] | [24,2] | [12,2] | [6,2] | [3,1] |
| 26   | 31   | [192,128] | 3.00 | 0.00 | 2 | [4,2] | [2,1] |
| 27   | 11   | [192,129] | 5.00 | 1.00 | 3 | [16,3] | [4,2] | [2,1] |
| 28   | 25   | [192,131] | 4.00 | 1.00 | 3 | [8,3] | [4,1] | [2,1] |
| 29   | 7    | [192,133] | 6.00 | 1.00 | 3 | [16,3] | [8,2] | [4,2] |
| 30   | 3    | [192,143] | 6.00 | 0.00 | 2 | [16,2] | [4,1] |
| 31   | 2    | [192,157] | 6.00 | 2.00 | 4 | [32,6] | [8,5] | [4,2] | [2,1] |
| 32   | 4    | [192,159] | 6.00 | 2.00 | 4 | [32,6] | [8,2] | [4,2] | [2,1] |
| 33   | 10   | [192,163] | 5.00 | 2.00 | 4 | [16,7] | [8,1] | [4,1] | [2,1] |
| 34   | 3    | [192,166] | 6.00 | 2.00 | 4 | [32,9] | [8,1] | [4,1] | [2,1] |
| 35   | 1    | [192,171] | 6.00 | 2.00 | 4 | [32,13] | [8,1] | [4,1] | [2,1] |
| 36   | 3    | [192,177] | 6.00 | 3.00 | 5 | [32,18] | [16,1] | [8,1] | [4,1] | [2,1] |
| 37   | 2    | [192,180] | 7.58 | 5.58 | 0 | [96,64] | [96,3] |
| 38   | 2    | [192,182] | 6.58 | 5.58 | 0 | [96,64] | [48,3] |
| 39   | 11   | [192,183] | 5.58 | 3.58 | 0 | [24,12] | [24,3] |
| 40   | 1    | [192,184] | 7.58 | 5.58 | 0 | [192,184] | [48,50] |
| Fam. | Num. | Rep.   | Rank | M. L. | Class | $G/Z$ | $\gamma_2(G)$ | $\gamma_3(G)$ | $\gamma_4(G)$ | $\gamma_5(G)$ | $\gamma_6(G)$ |
|------|------|--------|------|-------|-------|-------|---------------|---------------|---------------|---------------|---------------|
| 41   | 1    | [192,185] | 7.58 | 5.58  | 0     | [192,185] | [48,3]        |               |               |               |               |
| 42   | 7    | [192,186] | 4.58 | 3.58  | 0     | [24,12]  | [12,3]        |               |               |               |               |
| 43   | 2    | [192,188] | 5.58 | 4.00  | 0     | [48,3]   | [16,2]        |               |               |               |               |
| 44   | 2    | [192,189] | 6.58 | 4.00  | 0     | [48,3]   | [32,2]        |               |               |               |               |
| 45   | 2    | [192,191] | 6.58 | 4.00  | 0     | [96,70]  | [16,14]       |               |               |               |               |
| 46   | 2    | [192,192] | 6.58 | 4.00  | 0     | [96,71]  | [16,2]        |               |               |               |               |
| 47   | 2    | [192,193] | 6.58 | 4.00  | 0     | [96,72]  | [16,2]        |               |               |               |               |
| 48   | 2    | [192,194] | 7.58 | 4.00  | 0     | [96,72]  | [32,2]        |               |               |               |               |
| 49   | 2    | [192,195] | 7.58 | 4.00  | 0     | [96,70]  | [32,24]       |               |               |               |               |
| 50   | 2    | [192,197] | 7.58 | 4.00  | 0     | [96,71]  | [32,2]        |               |               |               |               |
| 51   | 7    | [192,200] | 4.58 | 2.00  | 0     | [12,3]   | [8,4]         |               |               |               |               |
| 52   | 2    | [192,201] | 7.58 | 4.00  | 0     | [96,70]  | [32,49]       |               |               |               |               |
| 53   | 5    | [192,203] | 3.58 | 2.00  | 0     | [12,3]   | [4,2]         |               |               |               |               |
| 54   | 106  | [192,205] | 6.58 | 1.58  | 0     | [24,14]  | [12,5]        | [3,1]         |               |               |               |
| 55   | 49   | [192,207] | 5.58 | 1.58  | 0     | [24,14]  | [6,2]         | [3,1]         |               |               |               |
| 56   | 36   | [192,215] | 7.58 | 1.58  | 0     | [24,14]  | [24,15]       | [3,1]         |               |               |               |
| 57   | 35   | [192,238] | 6.58 | 1.58  | 0     | [24,14]  | [12,5]        | [3,1]         |               |               |               |
| 58   | 13   | [192,239] | 7.58 | 2.58  | 0     | [48,36]  | [24,9]        | [6,2]         | [3,1]         |               |               |
| 59   | 9    | [192,261] | 7.58 | 2.58  | 0     | [48,36]  | [24,9]        | [6,2]         | [3,1]         |               |               |
| 60   | 13   | [192,269] | 6.58 | 2.58  | 0     | [48,36]  | [12,1]        | [6,2]         | [3,1]         |               |               |
| 61   | 24   | [192,280] | 7.58 | 2.58  | 0     | [48,36]  | [24,9]        | [6,2]         | [3,1]         |               |               |
| 62   | 6    | [192,299] | 7.58 | 2.58  | 0     | [96,87]  | [12,5]        | [6,2]         | [3,1]         |               |               |
| 63   | 2    | [192,300] | 7.58 | 3.58  | 0     | [96,89]  | [24,15]       | [6,2]         | [3,1]         |               |               |
| 64   | 4    | [192,305] | 7.58 | 3.58  | 0     | [96,89]  | [24,9]        | [6,2]         | [3,1]         |               |               |
| 65   | 4    | [192,307] | 7.58 | 3.58  | 0     | [96,91]  | [24,9]        | [6,2]         | [3,1]         |               |               |
| 66   | 30   | [192,315] | 6.58 | 2.58  | 0     | [48,38]  | [12,2]        | [6,2]         | [3,1]         |               |               |
| 67   | 26   | [192,316] | 6.58 | 2.58  | 0     | [48,38]  | [12,2]        | [6,2]         | [3,1]         |               |               |
| 68   | 64   | [192,318] | 7.58 | 2.58  | 0     | [48,38]  | [24,9]        | [6,2]         | [3,1]         |               |               |
| 69   | 24   | [192,319] | 7.58 | 2.58  | 0     | [48,38]  | [24,9]        | [6,2]         | [3,1]         |               |               |
| 70   | 20   | [192,323] | 7.58 | 2.58  | 0     | [48,38]  | [24,9]        | [6,2]         | [3,1]         |               |               |

*table continued*
| Fam. | Num. | Rep. | Rank | M. L. | Class | $G/Z$ | $\gamma_2(G)$ | $\gamma_3(G)$ | $\gamma_4(G)$ | $\gamma_5(G)$ | $\gamma_6(G)$ |
|------|------|------|------|------|-------|-------|---------------|---------------|---------------|---------------|---------------|
| 71   | 4    | [192,381] | 7.58 | 3.58 | 0     | [96,89]| [24,9]       | [6,2]         | [3,1]         |
| 72   | 3    | [192,455] | 7.58 | 3.58 | 0     | [96,102]| [24,9]      | [6,2]         | [3,1]         |
| 73   | 2    | [192,467] | 7.58 | 3.58 | 0     | [96,110]| [24,2]      | [12,2]        | [6,2]         | [3,1]         |
| 74   | 6    | [192,469] | 7.58 | 3.58 | 0     | [96,117]| [24,2]      | [12,2]        | [6,2]         | [3,1]         |
| 75   | 4    | [192,470] | 7.58 | 3.58 | 0     | [96,117]| [24,2]      | [12,2]        | [6,2]         | [3,1]         |
| 76   | 21   | [192,523] | 6.58 | 2.58 | 0     | [48,43]  | [12,2]      | [6,2]         | [3,1]         |
| 77   | 24   | [192,526] | 7.58 | 2.58 | 0     | [48,43]  | [24,9]      | [6,2]         | [3,1]         |
| 78   | 2    | [192,591] | 7.58 | 3.58 | 0     | [96,144]| [24,15]    | [6,2]         | [3,1]         |
| 79   | 24   | [192,592] | 7.58 | 2.58 | 0     | [48,43]  | [24,9]      | [6,2]         | [3,1]         |
| 80   | 18   | [192,597] | 7.58 | 2.58 | 0     | [48,43]  | [24,9]      | [6,2]         | [3,1]         |
| 81   | 16   | [192,598] | 7.58 | 2.58 | 0     | [48,43]  | [24,9]      | [6,2]         | [3,1]         |
| 82   | 4    | [192,620] | 7.58 | 3.58 | 0     | [96,144]| [24,9]      | [6,2]         | [3,1]         |
| 83   | 3    | [192,700] | 7.58 | 3.58 | 0     | [96,137]| [24,9]      | [6,2]         | [3,1]         |
| 84   | 4    | [192,706] | 7.58 | 3.58 | 0     | [96,138]| [24,2]      | [12,2]        | [6,2]         | [3,1]         |
| 85   | 3    | [192,719] | 7.58 | 3.58 | 0     | [96,145]| [24,9]      | [6,2]         | [3,1]         |
| 86   | 4    | [192,757] | 7.58 | 3.58 | 0     | [96,144]| [24,9]      | [6,2]         | [3,1]         |
| 87   | 4    | [192,758] | 7.58 | 3.58 | 0     | [96,146]| [24,9]      | [6,2]         | [3,1]         |
| 88   | 4    | [192,800] | 7.58 | 3.58 | 0     | [96,160]| [24,9]      | [6,2]         | [3,1]         |
| 89   | 2    | [192,802] | 7.58 | 3.58 | 0     | [96,160]| [24,15]    | [6,2]         | [3,1]         |
| 90   | 35   | [192,812] | 5.00 | 0.00 | 2     | [8,5]   | [4,2]       |               |               |
| 91   | 10   | [192,825] | 6.00 | 0.00 | 2     | [8,5]   | [8,5]       |               |               |
| 92   | 13   | [192,850] | 5.00 | 1.00 | 3     | [16,11] | [4,1]      | [2,1]         |               |
| 93   | 24   | [192,880] | 6.00 | 1.00 | 3     | [16,11] | [8,2]      | [2,1]         |               |
| 94   | 4    | [192,886] | 6.00 | 2.00 | 3     | [32,27] | [8,2]      | [2,1]         |               |
| 95   | 2    | [192,890] | 6.00 | 2.00 | 3     | [32,27] | [8,5]      | [2,1]         |               |
| 96   | 13   | [192,898] | 6.00 | 2.00 | 3     | [32,27] | [8,5]      | [2,1]         |               |
| 97   | 9    | [192,901] | 6.00 | 1.00 | 3     | [16,11] | [8,2]      | [2,1]         |               |
| 98   | 3    | [192,904] | 6.00 | 2.00 | 3     | [32,28] | [8,2]      | [2,1]         |               |
| 99   | 2    | [192,942] | 6.00 | 2.00 | 4     | [32,39] | [8,1]      | [4,1]         | [2,1]         |
| 100  | 10   | [192,945] | 7.58 | 3.58 | 0     | [48,48] | [48,32]    | [24,3]        |               |

Table continued
| Fam. | Num. | Rep. | Rank | M. L. | Class | $G/Z$ | $\gamma_2(G)$ | $\gamma_3(G)$ | $\gamma_4(G)$ | $\gamma_5(G)$ | $\gamma_6(G)$ |
|------|------|------|------|------|-------|-------|---------------|---------------|---------------|---------------|---------------|
| 101  | 10   | [192,947] | 6.58 | 3.58 | 0     | [48,48] | [24,3]        |               |               |               |               |
| 102  | 1    | [192,955] | 7.58 | 5.58 | 0     | [192,955] | [48,50]      |               |               |               |               |
| 103  | 1    | [192,956] | 7.58 | 5.58 | 0     | [192,956] | [48,3]       |               |               |               |               |
| 104  | 3    | [192,957] | 7.58 | 4.58 | 0     | [96,187] | [48,31]      | [24,13]       | [12,3]        |               |               |
| 105  | 10   | [192,959] | 6.58 | 3.58 | 0     | [48,48] | [24,13]      | [12,3]        |               |               |               |
| 106  | 3    | [192,962] | 7.58 | 4.58 | 0     | [96,187] | [48,33]      | [24,3]        |               |               |               |
| 107  | 4    | [192,973] | 7.58 | 4.58 | 0     | [96,195] | [48,31]      | [24,13]       | [12,3]        |               |               |
| 108  | 4    | [192,987] | 7.58 | 4.58 | 0     | [96,195] | [48,33]      | [24,3]        |               |               |               |
| 109  | 6    | [192,994] | 6.58 | 2.00 | 0     | [48,49] | [8,5]        | [4,2]         |               |               |               |
| 110  | 6    | [192,998] | 6.58 | 2.00 | 0     | [48,49] | [8,4]        |               |               |               |               |
| 111  | 5    | [192,1003] | 7.58 | 2.00 | 0     | [48,49] | [16,12]     | [8,4]         |               |               |               |
| 112  | 1    | [192,1008] | 7.58 | 4.00 | 0     | [192,1008] | [16,2]      |               |               |               |               |
| 113  | 1    | [192,1009] | 7.58 | 4.00 | 0     | [192,1009] | [16,14]     |               |               |               |               |
| 114  | 3    | [192,1014] | 7.58 | 3.00 | 0     | [96,197] | [16,10]     | [8,5]         | [4,2]         |               |               |
| 115  | 3    | [192,1017] | 7.58 | 3.00 | 0     | [96,197] | [16,13]     | [8,4]         |               |               |               |
| 116  | 1    | [192,1020] | 7.58 | 6.00 | 0     | [192,1020] | [64,192]    |               |               |               |               |
| 117  | 1    | [192,1021] | 7.58 | 4.00 | 0     | [48,50] | [64,224]    |               |               |               |               |
| 118  | 1    | [192,1022] | 7.58 | 4.00 | 0     | [48,50] | [64,239]    |               |               |               |               |
| 119  | 1    | [192,1023] | 7.58 | 6.00 | 0     | [1092,1023] | [64,242]   |               |               |               |               |
| 120  | 1    | [192,1024] | 7.58 | 4.00 | 0     | [48,50] | [64,242]    |               |               |               |               |
| 121  | 1    | [192,1025] | 7.58 | 6.00 | 0     | [192,1025] | [64,245]    |               |               |               |               |
| 122  | 24   | [192,1042] | 7.58 | 1.58 | 0     | [48,51] | [12,5]      | [3,1]         |               |               |               |
| 123  | 20   | [192,1045] | 6.58 | 1.58 | 0     | [48,51] | [6,2]       | [3,1]         |               |               |               |
| 124  | 50   | [192,1049] | 7.58 | 1.58 | 0     | [48,51] | [12,5]      | [3,1]         |               |               |               |
| 125  | 26   | [192,1145] | 7.58 | 1.58 | 0     | [48,51] | [12,5]      | [3,1]         |               |               |               |
| 126  | 55   | [192,1146] | 7.58 | 1.58 | 0     | [48,51] | [12,5]      | [3,1]         |               |               |               |
| 127  | 50   | [192,1148] | 7.58 | 1.58 | 0     | [48,51] | [12,5]      | [3,1]         |               |               |               |
| 128  | 19   | [192,1153] | 7.58 | 1.58 | 0     | [48,51] | [12,5]      | [3,1]         |               |               |               |
| 129  | 3    | [192,1310] | 7.58 | 2.58 | 0     | [96,207] | [12,2]      | [6,2]         | [3,1]         |               |               |
| 130  | 6    | [192,1316] | 7.58 | 2.58 | 0     | [96,209] | [12,2]      | [6,2]         | [3,1]         |               |               |

*table continued*
| Fam. | Num. | Rep. | Rank | M. L. | Class | $G/Z$ | $\gamma_2(G)$ | $\gamma_3(G)$ | $\gamma_4(G)$ | $\gamma_5(G)$ | $\gamma_6(G)$ |
|------|------|------|------|-------|-------|-------|--------------|--------------|--------------|--------------|--------------|
| 131  | 4    | [192,1331] | 7.58 | 2.58  | 0     | [96,209] | [12,2]       | [6,2]        | [3,1]        |               |              |
| 132  | 4    | [192,1333] | 7.58 | 2.58  | 0     | [96,209] | [12,2]       | [6,2]        | [3,1]        |               |              |
| 133  | 4    | [192,1394] | 7.58 | 2.58  | 0     | [96,219] | [12,2]       | [6,2]        | [3,1]        |               |              |
| 134  | 7    | [192,1407] | 5.00 | 0.00  | 2     | [16,14]  | [2,1]        |              |              |              |              |
| 135  | 11   | [192,1423] | 6.00 | 0.00  | 2     | [16,14]  | [4,2]        |              |              |              |              |
| 136  | 15   | [192,1434] | 6.00 | 0.00  | 2     | [16,14]  | [4,2]        |              |              |              |              |
| 137  | 5    | [192,1449] | 6.00 | 0.00  | 2     | [16,14]  | [4,2]        |              |              |              |              |
| 138  | 3    | [192,1465] | 6.00 | 1.00  | 3     | [32,46]  | [4,1]        | [2,1]        |              |              |              |
| 139  | 4    | [192,1472] | 7.58 | 3.58  | 0     | [96,226] | [24,13]      | [12,3]       |              |              |              |
| 140  | 4    | [192,1483] | 7.58 | 3.58  | 0     | [96,226] | [24,3]       |              |              |              |              |
| 141  | 2    | [192,1489] | 7.58 | 5.58  | 0     | [96,227] | [96,203]     |              |              |              |              |
| 142  | 2    | [192,1491] | 7.58 | 5.58  | 0     | [96,227] | [96,204]     |              |              |              |              |
| 143  | 2    | [192,1492] | 7.58 | 5.58  | 0     | [96,227] | [96,204]     |              |              |              |              |
| 144  | 2    | [192,1495] | 6.58 | 5.58  | 0     | [96,227] | [48,50]      |              |              |              |              |
| 145  | 2    | [192,1505] | 5.58 | 4.00  | 0     | [48,50]  | [16,14]      |              |              |              |              |
| 146  | 2    | [192,1506] | 6.58 | 4.00  | 0     | [48,50]  | [32,47]      |              |              |              |              |
| 147  | 2    | [192,1508] | 6.58 | 4.00  | 0     | [48,50]  | [32,49]      |              |              |              |              |
| 148  | 2    | [192,1524] | 7.58 | 1.58  | 0     | [96,230] | [6,2]        | [3,1]        |              |              |              |
| 149  | 2    | [192,1525] | 7.58 | 1.58  | 0     | [96,230] | [6,2]        | [3,1]        |              |              |              |
| 150  | 1    | [192,1541] | 7.58 | 6.00  | 0     | [192,1541] | [64,267]    |              |              |              |              |

The following informations is listed in Table V for each isoclinism family:
(1) the number of groups in the family;
(2) the rank; the rank of $G$ is $\log_2 |Z(G) \cap G'| + \log_2 |G/Z(G)|$.
(3) the middle length; the middle length of $G$ is $\log_2 |G'/Z(G) \cap G'|$.
(4) the nilpotency class, $c > 0$, of the groups in the family.
(5) the group id of $G/Z(G)$, of a group $G$ in the family.
(6) the group id of the non-trivial or non-repeatedly terms, $\gamma_2(G), ..., \gamma_c(G)$, of the lower central series of $G$; here $\gamma_1(G) = G$ and $\gamma_{i+1}(G) = [\gamma_i(G), G]$ for $1 \leq i \leq c$. 

29
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