GENERALIZATIONS OF REID INEQUALITY

SOUHEYB DEHIMI AND MOHAMMED HICHEM MORTAD

Abstract. In this paper, we improve the famous Reid Inequality related to linear operators. Some monotony results for positive operators are also established with a different approach from what is known in the existing literature. Lastly, Reid and Halmos-Reid inequalities are extended to unbounded operators.

1. Introduction

First, assume that readers are familiar with notions and result on $B(H)$. We do recall a few definitions and results though:

1. Let $A \in B(H)$. We say that $A$ is positive (we then write $A \geq 0$) if $< Ax, x > \geq 0$, $\forall x \in H$.

2. For every positive operator $A \in B(H)$, there is a unique positive $B \in B(H)$ such that $B^2 = A$. We call $B$ the positive square root of $A$.

3. The absolute value of $A \in B(H)$ is defined to be the (unique) positive square root of the positive operator $A^*A$. We denote it by $|A|$.

4. If $A \geq B \geq 0$, then $\sqrt{A} \geq \sqrt{B}$ (a particular case of the so-called Heinz Inequality).

5. We say that $A \in B(H)$ is hyponormal if $AA^* \leq A^*A$. Equivalently, $\|A^*x\| \leq \|Ax\|$ for all $x \in H$.

6. We also need the following lemma:

Lemma 1.1. (see e.g. [10]) Let $H$ be a complex Hilbert space. If $A, B \in B(H)$, then $\forall x \in H$:

$$\|Ax\| \leq \|Bx\| \iff \exists K \in B(H) \text{ contraction: } A = KB.$$ 

The inequality of Reid which first appeared in [8] is recalled next:

Theorem 1.2. Let $A, K \in B(H)$ be such that $A$ is positive and $AK$ is self-adjoint. Then

$$|< AKx, x >| \leq \|K\||Ax, x >$$

for all $x \in H$.

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* Corresponding author.
Remark. As shown in e.g. [3], Reid Inequality is equivalent to the operator monotony of the positive square root on the set of positive operators.

Halmos in [2] improved the inequality by replacing $\|K\|$ by $r(K)$ where $r(K)$ is the usual spectral radius. We shall call this the Halmos-Reid Inequality. Other generalizations of Theorem 1.2 are known in the literature from which we only cite [4] and [5].

In an earlier version of this paper (see [7]), the corresponding author showed the following:

**Theorem 1.3.** Let $A, K \in B(H)$ be such that $A$ is positive and $AK$ is normal. Then

$$| < AKx, x > | \leq ||K|| < Ax, x >$$

for all $x \in H$.

Can we go up to hyponormal $AK$? In fact, the result is not true even when $AK$ is quasinormal (and so we cannot go up to subnormal either). The counterexample is given next:

**Example 1.4.** Let $S$ be the shift operator on $\ell^2$. Setting $A = SS^*$, we see that $A \geq 0$. Now, take $K = S$ (and so $||K|| = 1$). It is clear that $AK = SS^*S = S$ is quasinormal. If Reid Inequality held, then we would have

$$| < Sx, x > | \leq SS^*x, x > = ||S^*x||^2$$

for each $x \in \ell^2$. This inequality clearly fails to hold for all $x$. Indeed, taking $x = (2,1,0,0,\cdots)$, we see that

$$| < Sx, x > | = 2 \leq ||S^*x||^2 = 1$$

which is absurd.

The good news is that Reid Inequality can yet be improved as it holds if $AK$ is co-hyponormal, that is, if $(AK)^*$ is hyponormal. The proof, however, relies on the following result:

**Lemma 1.5.** ([4], cf. Theorem 3.2) Let $A \in B(H)$ be hyponormal. Then

$$| < Ax, x > | \leq |A||x, x > .$$

This and some interesting consequences may be found in Section 2.

In Section 3 we treat Reid (and Halmos-Reid) Inequality for unbounded operators. Fortunately, the latter inequality does hold for co-hyponormal unbounded operators as well. The proof is a little more technical and so it seems appropriate to recall a couple of definitions here (other notions on unbounded operators are assumed, cf. [9]):

**Definition.** Let $T$ and $S$ be unbounded positive self-adjoint operators. We say that $S \geq T$ if $D(S^{\frac{1}{2}}) \subseteq D(T^{\frac{1}{2}})$ and $\|S^{\frac{1}{2}}x\| \geq \|T^{\frac{1}{2}}x\|$ for all $x \in D(S^{\frac{1}{2}})$.

**Remark.** Heinz Inequality is valid for positive unbounded operators as well. See e.g. [9].
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Now, we recall the definition of an unbounded hyponormal operator.

**Definition.** A densely defined operator $T$ with domain $D(T)$ is called hyponormal if

$$D(T) \subset D(T^*) \text{ and } \|T^*x\| \leq \|Tx\|, \forall x \in D(T).$$

2. MAIN RESULTS: THE BOUNDED CASE

**Theorem 2.1.** Let $A, K \in B(H)$ be such that $A$ is positive and $(AK)^*$ is hyponormal. Then

$$| < AKx, x > | \leq \|K\| < Ax, x >$$

for all $x \in H$.

**Proof.** The inequality is evident when $K = 0$. So, assume that $K \neq 0$. It is then clear that $\frac{K}{\|K\|}$ satisfies

$$KK^* \leq \|K\|^2 I.$$

Hence

$$|(AK)^*|^2 = AKK^*A \leq \|K\|^2 A^2$$

or simply $|(AK)^*| \leq \|K\|A$ after passing to square roots.

Now, for all $x \in H$

$$| < AKx, x > | = | < x, (AK)^*x > | = |(AK)^*x, x > | = | < (AK)^*x, x > |.$$

Since $(AK)^*$ is hyponormal, Lemma 1.5 combined with $|(AK)^*| \leq \|K\|A$ give

$$| < AKx, x > | = | < (AK)^*x, x > | \leq | < (AK)^*x, x > | \leq \|K\| < Ax, x >$$

and this marks the end of the proof. \(\square\)

**Theorem 2.2.** Theorem 2.1 can be deduced from the operator monotony of the positive square root on the set of positive operators, and vice versa.

**Proof.** Let $A, B \in B(H)$.

(1) Assume that $0 \leq A \leq B$. Let $x \in H$. Since $A \leq B$, we easily see that:

$$\|\sqrt{A}x\|^2 \leq \|\sqrt{B}x\|^2.$$

So, by Lemma 1.1 we know that $\sqrt{A} = K\sqrt{B}$ for some contraction $K \in B(H)$. Since $\sqrt{A}$ is self-adjoint, it follows that $K\sqrt{B}$ too is self-adjoint (hence co-hyponormal!). As $\sqrt{B} \geq 0$, then by Reid Inequality (Theorem 2.1) we obtain:

$$< \sqrt{A}x, x > \leq < \sqrt{B}K^*x, x > \leq < \sqrt{B}x, x >$$

or

$$\sqrt{A} \leq \sqrt{B},$$

as required.

(2) The other implication, which uses the fact that the square root is increasing, has already been presented in the proof of Theorem 2.1.
The idea of the first part of the proof of the preceding result may be used to produce new proofs of other known results on monotony.

**Theorem 2.3.** Let \( A, B \in B(H) \). If \( 0 \leq A \leq B \) and if \( A \) is invertible, then \( B \) is invertible and \( B^{-1} \leq A^{-1} \).

**Proof.** As in the proof of Theorem 2.2, we know that \( \sqrt{A} = K \sqrt{B} \) for some contraction \( K \in B(H) \). Since \( \sqrt{A} \) is invertible (as \( A \) is), it follows that \( I = (\sqrt{A})^{-1} K \sqrt{B} \), i.e. the self-adjoint \( \sqrt{B} \) is left invertible and so \( B \) or simply \( B \) is invertible (cf. [1]) and

\[
(\sqrt{B})^{-1} = (\sqrt{A})^{-1} K = K^* (\sqrt{A})^{-1}
\]

by the self-adjointness of both \( (\sqrt{B})^{-1} \) and \( (\sqrt{A})^{-1} \).

Let \( x \in H \). Then (since \( K^* \) too is a contraction)

\[
< B^{-1} x, x > = \|(\sqrt{B})^{-1} x\|^2 = \|K^* (\sqrt{A})^{-1} x\|^2 \leq \|(\sqrt{A})^{-1} x\|^2 = < A^{-1} x, x >,
\]

as needed. \(\square\)

The following improvement of Lemma 1.1 makes proofs in case of commutativity very simple.

**Lemma 2.4.** Let \( H \) be a complex Hilbert space. If \( A, B \in B(H) \) are self-adjoint and \( BA \geq 0 \), then

\[
\forall x \in H : \|Ax\| \leq \|Bx\| \iff \exists K \in B(H) \text{ positive contraction : } A = KB.
\]

**Proof.**

(1) "\(\Rightarrow\)" : Let \( x \in H \). Then

\[
0 \leq < KBx, Bx > = < Ax, Bx > = < BAx, x >,
\]

that is, \( BA \geq 0 \).

(2) "\(\Leftarrow\)" : Since \( BA \geq 0 \), it follows that \( BA \) is self-adjoint, i.e. \( AB = BA \). As a consequence, \( \ker A \) reduces \( A \) and \( B \), and the restriction of \( A \) to \( \ker A \) is the zero operator on \( \ker A \). Hence, we can assume that \( A \) is injective. Therefore, because \( \ker B \subset \ker A = \{0\} \), we see that \( B^{-1} \) is self-adjoint and densely defined. Set \( K_0 = AB^{-1} \). Then \( K_0 \) is densely defined and

\[
\|K_0(Bx)\| = \|AB^{-1}Bx\| = \|Ax\| \leq \|Bx\|, \forall x \in H,
\]

signifying that \( K_0 \) is a contraction with a unique contractive extension \( K \) to the whole \( H \). Since

\[
< K_0(Bx), Bx > = < Ax, Bx > = < BAx, x > \geq 0
\]

for all \( x \in H \), we see that \( K \) is positive as well. Clearly

\[
KBx = K_0(Bx) = Ax
\]

for all \( x \in H \), and this completes the proof. \(\square\)
Remark. By scrutinizing the previous proof, we see that we can replace each word "positive" by "self-adjoint" in the statement and in the proof. Hence we also have:

**Proposition 2.5.** Let $A, B \in B(H)$ be self-adjoint such that $AB$ is self-adjoint (that is, iff $AB = BA$). Then

$$\forall x \in H : \|Ax\| \leq \|Bx\| \iff \exists K \in B(H) \text{ self-adjoint contraction : } A = KB.$$ 

The next result is known. Its proof is a simple application of Lemma 2.4.

**Corollary 2.6.** Let $A, B \in B(H)$ be positive and commuting. Then

$$0 \leq A \leq B \implies A^2 \leq B^2.$$ 

**Proof.** Since $AB \geq 0$, we know by Lemma 2.4 that $\sqrt{A} = K\sqrt{B}$ for some positive contraction $K \in B(H)$ and $K\sqrt{B} = \sqrt{BK}$. Hence

$$A = K\sqrt{B}K\sqrt{B} = K^2 B.$$ 

So for all $x \in H$:

$$\|Ax\|^2 = \|K^2 Bx\|^2 \leq \|Bx\|^2$$

or merely

$$\langle A^2 x, x \rangle = \langle Ax, Ax \rangle = \|Ax\|^2 \leq \|Bx\|^2 = \langle B^2 x, x \rangle,$$ 

as required. 

**Remark.** The previous proof could even be shortened by directly using Reid Inequality. As is presented, it can be given in courses which do not cover Reid Inequality.

By invoking the functional calculus of positive operators, we know that we can define $A^\alpha$ for any (real) $\alpha > 0$ (we may allow $\alpha = 0$) whenever $A \geq 0$. We also know that if $B$ is also positive and it commutes with $A$, then (using the spectral theorem or else)

$$(AB)^\alpha = A^\alpha B^\alpha.$$ 

As a generalization of Corollary 2.6 we have

**Proposition 2.7.** Let $A, B \in B(H)$ be positive and commuting. Then

$$0 \leq A \leq B \implies A^\alpha \leq B^\alpha$$

for any $\alpha \in (0, \infty)$.

To prove it, we need the following perhaps known result:

**Lemma 2.8.** Let $A \in B(H)$ be such that $A \geq 0$. Then

$$\|A^\alpha\| = \|A\|^\alpha$$

for any $\alpha \in (0, \infty)$.

**Proof.** One of the ways of seeing this is to use the spectral radius theorem and the fact that $A^\alpha$ is positive. 

\[ \Box \]
Now, we give the proof of Proposition 2.7.

**Proof.** Since $AB \geq 0$, we know by Lemma 2.4 that $\sqrt{A} = K \sqrt{B}$ for some positive contraction $K \in B(H)$ and $K \sqrt{B} = \sqrt{BK}$. Hence

$$A = K \sqrt{BK} \sqrt{B} = K^2 B = BK^2.$$  

Ergo

$$A^\alpha = (BK^2)^\alpha = B^\alpha K^{2\alpha} \quad \text{(because } K^2 B = BK^2).$$

Finally, for any $x \in H$, Reid Inequality (and a glance at Lemma 2.8) allows us to write:

$$< A^\alpha x, x > = < B^\alpha K^{2\alpha} x, x > \leq \|K\|^{2\alpha} < B^\alpha x, x > \leq < B^\alpha x, x >,$$

as desired. □

3. **Main Results: The Unbounded Case**

We start with the next practical result (probably known):

**Proposition 3.1.** Let $T$ be a closed hyponormal operator. Then

$$T^* T \geq TT^*.$$

**Proof.** First, since $T$ is closed, both $T^* T$ and $TT^*$ are self-adjoint and positive (cf. [3]). As in the bounded case, write $|T| = (T^* T)^{\frac{1}{2}}$. Then it is known that

$$D(|T|) = D(T) \subseteq D(T^*) = D(|T^*|).$$

Finally, for all $x \in D(T)$, we have

$$||T^* x|| = ||T^* x|| \leq ||Tx|| = ||T|x||.$$

Therefore, according to Definition 1, we have $T^* T \geq TT^*$, as required. □

**Remark.** By Definition 1 above and some trivial observations, $T^* T \geq TT^*$ clearly implies that $T$ is hyponormal.

Now, we prove the analogue of Lemma 1.5 for unbounded operators. The proof uses the Generalized Cauchy-Schwarz Inequality for unbounded self-adjoint positive operators.

**Theorem 3.2.** Let $T$ be a closed hyponormal operator. Then

$$\left| <Tx, x> \right| \leq |T|x, x > \quad \text{for all } x \in D(T).$$

**Proof.** Let $T = U|T|$ be the polar decomposition of $T$ where $U$ is partial isometry. Remember that (see e.g. [3])

$$|T^*| = U|T|U^*$$

and

$$U^*U|T| = |T|.$$  

By Proposition 3.1 $TT^* \leq T^* T$. Hence by Heinz Inequality ("unbounded" version), we infer that $|T^*| \leq |T|$. 

Now, to show the required inequality, let \( x \in D(T) \). Then, we have
\[
| \langle Tx, x \rangle |^2 = | \langle U |T| x, x \rangle |^2 \\
= | \langle |T| x, U^* x \rangle |^2 \\
\leq | \langle |T| x, x \rangle \rangle < |T| U^* x, x \rangle \\
= | \langle |T| x, x \rangle \rangle |T^*| x, x \rangle \\
\leq | \langle |T| x, x \rangle \rangle |T| x, x \rangle \) (as \( T \) is hyponormal).
\]
Accordingly,
\[
| \langle Tx, x \rangle | \leq | \langle T |x, x \rangle |,
\]
as required. \( \square \)

We are ready to give the "unbounded" analogue of Theorem 2.1.

**Theorem 3.3.** Let \( K \) be a bounded operator and let \( A \) be a non-necessarily bounded self-adjoint positive operator such that \((AK)^*\) is hyponormal. Then
\[
| \langle AKx, x \rangle | \leq \|K\| \langle Ax, x \rangle
\]
for all \( x \in D(A) \).

The proof of the preceding theorem is based on the following simple lemma.

**Lemma 3.4.** Let \( A \) be a non-necessarily bounded self-adjoint operator. Let \( K \in B(H) \) be such that \( KK^* \leq \alpha I \) for some \( \alpha > 0 \). Then
\[
AK(\alpha A)^* \leq \alpha A^2
\]
whenever \( AK \) is densely defined.

**Proof.** Since \( AK \) is closed as \( A \) is and \( K \) is bounded, we clearly have
\[
|((\alpha A)^*)|^2 = AK(\alpha A)^* \leq \alpha A^2.
\]
To show the required inequality, notice first that
\[
D(\alpha A) = D(A) = D(K^* A) \subseteq D((\alpha A)^*) = D((\alpha A)^*).
\]
According to Definition 3.1, we need only check that \( \|((\alpha A)^*)x\| \leq \sqrt{\alpha} \|Ax\| \) for all \( x \in D(A) \). So, let \( x \in D(A) \). Then
\[
K^* Ax = (\alpha A)^* x.
\]
Hence for all \( x \in D(A) \):
\[
\|((\alpha A)^*)x\|^2 = \|(\alpha A)^* x\|^2 \\
= \|K^* Ax\|^2 \\
\leq \|K\|^2 \|Ax\|^2
\]
or simply \( |(\alpha A)^*| \leq \|K\| A \) (where obviously \( \alpha = \|K\|^2 \)). \( \square \)

We are ready to give a proof of Theorem 3.3.
Proof. First, since $(AK)^*$ is hyponormal, we clearly have

$$D(A) \subseteq D((AK)^*) \subseteq D((AK)^{**}) = D(AK)$$

because $AK$ is also closed.

Now, the inequality is evident when $K = 0$. So, assume that $K \neq 0$. It is then clear that $\frac{K}{\|K\|} \neq 0$ satisfies

$$KK^* \leq \|K\|^2 I.$$  

Lemma 3.4 then yields

$$|(AK)^*|^2 = AK (AK)^* \leq \|K\|^2 A^2.$$  

Therefore, for all $x \in D(A)$

$$| < AKx, x > | = | < x, (AK)^* x > |$$

$$= | < (AK)^* x, x > |$$

$$= | < (AK)^* x, x > |$$

$$\leq | < (AK)^* x, x > |$$

$$\leq \|K\| | < Ax, x > |$$

and this marks the end of the proof. \qquad \square

In the end, we give the generalization of Halmos-Reid Inequality. The proof, which uses a standard argument (cf. [2] or [6]), is more technical in the unbounded case.

**Theorem 3.5.** Let $K$ be a bounded operator, and let $A$ be an unbounded self-adjoint positive operator such that $K^* A \subseteq AK$, then

$$| < AKx, x > | \leq r(K) < Ax, x > \text{ for all } x \in D(A).$$

where $r(K)$ denotes the spectral radius of $K$.

**Proof.** Since $K^* A \subseteq AK$, we can get by induction that

$$K^n A \subseteq AK^n \text{ or merely } (AK^n)^* \subseteq AK^n$$

for all $n$. Hence $(AK^n)^*$ is hyponormal. We also observe that as $A^{\frac{1}{2}} A^{\frac{1}{2}} = A$, then $D(A) \subseteq D(A^{\frac{1}{2}})$.

Now, let us prove the following key result:

$$| < AKx, x > | \leq | < AK^{2n}x, x > |^{\frac{1}{2n}} < Ax, x >^{\frac{1}{2n}}$$

$$\leq \|K\|^{\frac{1}{2n}} | < Ax, x >^{\frac{1}{2n}}$$

$$\leq \|K\|^{\frac{1}{2n}} | < Ax, x >^{\frac{1}{2n}}$$

This completes the proof. \qquad \square
We use a proof by induction. For $n = 1$, we have for all $x \in D(A)$:

\[ |<AKx, x>| \leq |<A_{1/2}^nKx, x>| \]

\[ = |<A_{1/2}^nKx, A_{1/2}^nKx>| \text{ because } A_{1/2}^nKx \in D(A_{1/2}^n). \]

\[ \leq \|A_{1/2}^nKx\| \|A_{1/2}^nKx\| \]

\[ = <A_{1/2}^nKx, A_{1/2}^nKx>^{1/2} <Ax, x>^{1/2} \]

\[ = <AKx, Kx>^{1/2} <Ax, x>^{1/2} \]

\[ = <K^*AKx, x>^{1/2} <Ax, x>^{1/2} \]

\[ = <AK^2x, x>^{1/2} <Ax, x>^{1/2} \]

Using a similar argument we can prove it for $n + 1$ if it holds for $n$.

Therefore, for all $n$ (and all $x \in D(A)$)

\[ |<AKx, x>| \leq |<AK^{2n}x, x>|^{1/2n} <Ax, x>^{2n-1}. \]

Since $(AK^{2n})^*$ is hyponormal, we might apply Theorem 3.3 to get

\[ |<AKx, x>| \leq |<AK^{2n}x, x>|^{1/2n} <Ax, x>^{2n-1} \]

\[ \leq ||K^{2n}||^{1/2n} <Ax, x>^{1/2n} <Ax, x>^{2n-1} \]

\[ = ||K^{2n}||^{1/2n} <Ax, x> \]

whichever $n$. Passing to the limit finally yields

\[ |<AKx, x>| \leq r(K) <Ax, x> \quad \text{for all } x \in D(A), \]

as needed.

\[ \square \]

**Corollary 3.6.** Let $K$ be a bounded operator, and let $A$ be an unbounded self-adjoint positive operator such that $AK$ is self-adjoint, then

\[ |<AKx, x>| \leq r(K) <Ax, x> \quad \text{for all } x \in D(A). \]

*Proof.* The proof simply follows from

\[ K^*A \subset (AK)^* = AK. \]

\[ \square \]

4. **Conclusion**

The new bounded version of Reid Inequality is equivalent to a bunch of other properties. Indeed, since we have shown that it is equivalent to the fact that the square root is increasing on the set of positive operators, we may call on [5] to list other equivalent conditions.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ORAN 1, AHMED BEN BELLA, B.P. 1524, EL MENOUAR, ORAN 31000, ALGERIA.

Mailing address:
Pr Mohammed Hichem Mortad
BP 7085 SEDDIKIA ORAN
31013
ALGERIA
E-mail address: sohayb20091@gmail.com
E-mail address: mhmortad@gmail.com, mortad@univ-oran.dz.