On the Quantum Query Complexity of Detecting Triangles in Graphs

Mario Szegedy*
Rutgers University

Abstract

We show that in the quantum query model the complexity of detecting a triangle in an undirected graph on \( n \) nodes can be done using \( O(n^{1+\frac{3}{7}} \log^2 n) \) quantum queries. The same complexity bound applies for outputting the triangle if there is any. This improves upon the earlier bound of \( O(n^{1+\frac{1}{2}}) \).

1 Introduction

The classical black box complexity of graph properties has made its fame through the notoriously hard evasiveness conjecture of AAnderaa, Karp and Rosenberg (see e.g. [LY]) which states that every non-trivial Boolean function on graphs whose value remains invariant under the permutation of the nodes has deterministic query complexity exactly \( \binom{n}{2} \), where \( n \) is the number of nodes of the input graph.

The question of quantum query complexity of graph properties was first raised in [BCWZ]. Here the authors show that the quantum model behaves differently from the deterministic model in the zero error case, although an \( \Omega(n^2) \) lower bound still holds.

The bounded error quantum query model is more analogous to the classical randomized model. About the latter we know much less, in particular the general lower bounds are far from the conjectured \( \Omega(n^2) \). For a long time Peter Hajnal’s \( \Omega(n^{4/3}) \) bound [H] was the best, until it was slightly improved in [CK]. The best general lower bound, \( \Omega(n^{2/3}) \), in the bounded error quantum case is due to Yao and Santha (not yet published).

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The quantum query complexity of the Triangle Detection Problem was first treated in H. Buhrman, C. Dürr, M. Heiligman, P. Høyer, F. Magniez, M. Santha and R. de Wolf [BDHH], where the authors show that in the case of sparse graphs the folklore $n^{1.5}$ upper bound can be improved. Their method breaks down when the graph has $\theta(n^2)$ edges. The folklore bound is based on searching the space of all $n^3$ triangles with Grover’s algorithm [G]. Since the above search idea is quite natural, one might conjecture that the upper bound is the best possible.

Below we show however, using combinatorial ideas, that a quantum query machine can solve the Triangle Detection Problem in time $O(n^{1+\frac{4}{7}} \log^2 n)$. The lower bound remains $\Omega(n)$.

2 Notations

We denote the set $\{1, 2, \ldots, n\}$ by $[n]$. A simple undirected graph is a set of edges $G \subseteq \{(a, b) \mid a, b \in [n]; a \neq b\}$ with the understanding that $(a, b) \overset{\text{def}}{=} (b, a)$. The complete graph on a set $\nu \subseteq [n]$ is denoted by $\nu^2$. The neighborhood of a $v \in [n]$ in $G$ is denoted by $\nu_G(v)$. Its definition:

$$\nu_G(v) = \{b \mid (v, b) \in G\}.$$  

We denote $|\nu_G(v)|$ by $\deg_G v$. For sets $A, B \subseteq [n]$ let

$$G(A, B) = \{(a, b) \mid a \in A; b \in B; (a, b) \in G\}.$$  

The following function will play a major role in our proof: We denote the number of paths of length two from $a \in [n]$ to $b \in [n]$ in $G$ with $t(G, a, b)$. In formula:

$$t(G, a, b) = |\{x \mid (a, x) \in G; (b, x) \in G\}|.$$  

t($G$) is the number of triangles in $G$. We define an operation on graphs. For a graph $G \subseteq [n]^2$ define

$$G^{(t)} = \{(a, b) \in [n]^2 \mid t(G, a, b) \leq t\}.$$  

3 The Triangle Detection Problem (TDP)

Input: An undirected graph $G$ with vertex set $[n]$, given by its incidence matrix, $G(a, b)$. ($G(a, b)$ is a symmetric zero-one valued matrix with all zeros in the diagonal.)
Output: “No,” if $G$ is triangle free. Else, a set $\{a, b, c\}$ such that $(a, b), (b, c), (c, a) \in G$.

Any classical machine that solves the Triangle Detection Problem (TDP) has to query all edges of the graph, and even a classical randomized machine has to query a constant fraction of all edges. In contrast, it has been known that a quantum query machine can successfully output a triangle of $G$, if exists, making only $O(n^{1.5})$ quantum queries to $G$, with high probability. In this section we give an algorithm that improves on the exponent 1.5.

The basis states of a quantum query machine for the TDP are of the form $|a, b\rangle|c\rangle$, where $1 \leq a \leq b \leq n$ and $c \in S$ is specific to the the machine. The starting state is $|1, 1\rangle|c_0\rangle$ for some distinguished $c_0 \in S$. A query step is a unitary operator $O = O_G$ acting on the basis elements as:

$$O_G|a, b\rangle|c\rangle = (-1)^{G(a,b)}|a, b\rangle|c\rangle.$$ 

The computation is a sequence $U_0O_G\ldots U_{t-1}O_GU_t$ of query steps and non-query steps. Each non-query step is an arbitrary unitary operator on the state space with the only restriction that it does not depend on $G$. The output is read from a measurement of the final state, and needs to be correct with probability at least $1 - \epsilon$. We will describe our algorithm as a one that runs on a classical machine that calls quantum subroutines, but makes classical queries as well. Although this type of the machine seems more general than the one we explained above, the two models can be shown to have the same power.

The quantum query complexity of the Triangle Detection Problem as well as of many of its kins with small one-sided certificate size are notoriously hard to analyze, because one of the main lower bounding methods breaks down near the square root of the instance size:

**Lemma 1** If the 1-certificate size of a Boolean function on $n$ Boolean variables is $k$ then the general weighted version of the quantum adversary method (also known as the method of Ambainis) in the sense of [BSS] can prove only a lower bound of $(1 - 2\sqrt{\epsilon(1 - \epsilon)})\sqrt{nk}$.

Problems with small certificate complexity include various collision type problems such as the 2-1 collision problem and the element distinctness problem. Using the degree bound of R. Beals, H. Buhrman, R. Cleve, M. Mosca and Ronald de Wolf [BBCM] for the 2-1 collision problem the first polynomial lower bound was shown by Aaronson [Aa]. Shi [S] showed tight $\Omega(n^{1/3})$ lower bound and Kutin [K] removed the "small range assumption". For element distinctness, an $\Omega(n^{1/2})$ lower bound follows by an easy reduction from Grover’s search. Shi got $\Omega(n^{2/3})$ and Ambainis [A] removed the "small range assumption." With a recent ingenious algorithm of Ambainis [A2] the complexity of the element distinctness problem was finally determined to be in $\theta(n^{2/3})$. In a sequel of this paper using Ambainis’s new
technique with Santha and Magniez [MSS] we could improve the present upper bound for the TDP to $n^{1.3}$. The algorithm presented here is based on three combinatorial observations:

The first observation is due to H. Buhrman, C. Dürr, M. Heiligman, P. Høyer, F. Magniez, M. Santha and R. de Wolf [BDHH]:

**Lemma 2** If graph $G_1$ is known then detecting a triangle with at least one edge $G_1$ can be determined with $O(\sqrt{n|G \cap G'|})$ queries.

**Proof:** The proof of the above statement is exactly the same as that of Theorem 8 in [BDHH], and is based on the Amplitude Amplification technique of G. Brassard, P. Hoyer, M. Mosca, A. Tapp [BHMT]. □

Note that the lemma can also be proven using Grover search, and we lose only a factor of $\log n$ in the analysis. Throughout the paper we do not try to optimize $\log n$ factors, since we conjecture that the exponent we present here is not tight. Perhaps the most crucial observation to the algorithm is the following trivial-looking one:

**Lemma 3** For every $v \in [n]$, using $O(n \log n)$ queries, we either find a triangle in $G$ or verify that $G \subseteq [n]^2 \setminus \nu_G(v)^2$ with probability $1 - \frac{1}{n^3}$.

**Proof:** We query all edges incident to $v$ classically using $n - 1$ queries. This determines $\nu_G(v)$. With Grover’s search [G] we find an edge of $G$ in $\nu_G(v)^2$, if there is any. In order to achieve $1 - \frac{1}{n^3}$ success probability we use the following safe variant of the search:

**Definition 1 (Safe Grover Search)** Let $c \geq 1$ be an arbitrary constant. The Safe Grover Search on a database of $N$ items is the usual Grover search iterated $O(\log N)$ times independently. The safe Grover search uses $O_c(N \log N)$ queries and it finds the (type of) item we are looking for, if it is the database, with probability at least $1 - \frac{1}{n^c}$.

□

This lemma with the observation that hard instances have to be dense, already enable us to show that the quantum query complexity of the TDP is $o(n^{1.5})$, using the lemma of Szemerédi. However another fairly simple observation can help us to decrease the exponent:

**Lemma 4** Let $k = \lceil 4n^4 \log n \rceil$, and let $v_1, v_2, \ldots, v_k$ randomly chosen from $[n]$ (with no repetitions). Let $G' = [n]^2 \setminus \cup_{i=1}^k \nu_G(v_i)^2$. Then

$$\text{Prob}_{v_1, v_2, \ldots, v_k} \left( G' \subseteq G^{(n^{1-\epsilon})} \right) > 1 - \frac{1}{n^3}.$$
Proof: Let us first remind the reader about the following lemma that is useful in many applications:

**Lemma 5** Let \( X \) be a fixed subset of \([n]\) of size \(pn\) and \(Y\) be a random subset of \([n]\) of size \(qn\), where \(p+q < 1\). Then the probability that \(X \cap Y\) is empty is \((1 - pq)^n(1 + O(p^3 + q^3 + 1/n))\).

Proof: The probability we are looking for is estimated using the Stirling formula as

\[
\frac{(n(1-p))}{(nq)} = \frac{[n(1-p)]!}{[nq]!} \frac{[n(1-q)]!}{[n-1-p-q]!} = \frac{(1-p)(1-q)^n}{(1-p)(1-q)^{(1-p)(1-q)^n}} (1 \pm o(1)) = (1 - pq)^n(1 + O(p^3 + q^3 + 1/n)).
\]

Consider now a fixed edge \((a, b)\) such that \(t(G, a, b) \geq n^{1-\epsilon}\). The probability that \((a, b) \in G'\) is the same as the probability that the set \(X = \{x \in [n] \mid (x, a) \in G \land (x, b) \in G\}\) is disjoint from the random set \(\{v_1, v_2, \ldots, v_k\}\). Notice that \(|X| = t(G, a, b)\). By Lemma 5 we can estimate now this probability as

\[
\left(1 - \frac{4n^\epsilon \log n}{n} \times \frac{n^{1-\epsilon}}{n}\right)^{n(1+o(1))} = \left(1 - \frac{4 \log n}{n}\right)^{n(1+o(1))} < e^{-3 \log n} = n^{-3}.
\]

Then the lemma follows from the union bound, since the number of possible edges \((a, b)\) is at most \(n^2\).

Our algorithm will have three parameters \(\epsilon = \frac{3}{7}, \epsilon' = \delta = \frac{1}{7}\), and it either returns a triangle during its run or it:

1. Randomly creates a graph \(G'\) such that \(G \subseteq G' \subseteq G^{(n^{1-\epsilon})}\) with high probability;
2. In a number of steps classifies the edges of \(G'\) into \(T\) and \(E\) such that \(T\) contains only a small number of triangles and \(E \cap G\) is small;
3. Searches for a triangle in \(G\) among all triangles inside \(T\);
4. Searches for a triangle of \(G\) intersecting with \(E\);
5. In case steps 3 and 4 run without success the algorithm returns “No.”

See the algorithm in details in Figure 1.
| Command | Complexity |
|---------|------------|
| 1. Set \( k = \lceil 4n^\epsilon \log n \rceil \) Select \( \{v_1, v_2, \ldots, v_k\} \) randomly from \([n]\) and query all edges of the form \((v_i, v)\) for \(1 \leq i \leq k\) and \(v \in [n]\). | \( O(n^{1+\epsilon} \log n) \) |
| 2. For each \( v_i \) check with the Safe Grover search if \( G \cap \nu_G(v_i)^2 \) is empty. If some \(1 \leq i \leq k\) it is not, after verification return \((v, w), (v_i, v), (v_i, w)\), where all three edges are in \(G\). Otherwise define \( G' = [n]^2 \cup \bigcup_{i=1}^{k} \nu_G(v_i)^2 \). | \( O(n^{1+\epsilon}(\log n)^2) \) |
| 3. Set \( T = \emptyset, E = \emptyset \). | 0 |
| 4. Find an edge \((v, w) \in G'\) for which \( t(G', v, w) < \lceil n^{1-\epsilon} \rceil \) if exists, and put this edge to \(T\), delete it from \(G'\). Keep repeating this until no such edge is found. | 0 |
| 5. Pick a vertex \( v \) of \( G' \) with non-zero degree and choose one of the 1. low degree hypothesis: \( |\nu_G(v)| \leq 10\lceil n^{1-\delta} \rceil \); 2. high degree hypothesis: \( |\nu_G(v)| \geq 0.1\lceil n^{1-\delta} \rceil \); such that the chosen hypothesis holds with probability at least \( 1 - \frac{1}{n^3} \). | \( O(n^{\delta} \log n) \) |
| 6. If in 5. we chose the low degree hypothesis, add all edges of \( G' \) with end point \( v \) into \( E \) and delete them from \( G' \). | 0 |
| 7. If in 5. we chose the high degree hypothesis, then find all vertices in \( A \equiv \nu_G(v) \). First run the Safe Grover to determine if there is an edge in \( \nu_G(v)^2 \). If one is found, after verification, output the triangle of \( G \) induced by \( v \) and this edge. If no triangle is found add all edges in \( G'(A, A') \) to \( E \), where \( A' = \nu_{G'}(v) \). Also, delete these edges from \( G' \). | \( O(n \log n) \) |
| 8. Repeat 4.-7. until \( G' \) becomes empty. | \( \text{See analysis} \) |
| 9. Try to find a triangle inside \( T \) using Grover search, verify and output if found. If not, go to 10. | \( \sqrt{n^{3-\epsilon} \log n} \) |
| 10. Search for a triangle in \( G \) with non-empty intersection with \( E \), verify and output if found. | \( \sqrt{n|G \cap E|} \) |

Figure 1: Quantum query algorithm for the Triangle Detection Problem
4 Analysis

Lemma 6 The algorithm for the TDP described in the previous section runs using

\[ O(n^{1+\epsilon}(\log n)^2 + n^{1+\delta+\epsilon'} \log n + \sqrt{n^{3-\epsilon'}} \log n + \sqrt{n^{3-\min(\delta,\delta'-\epsilon')}}). \]

quantum queries and returns “No” with probability one if there is no triangle in \( G \), otherwise returns a triangle of \( G \) with some constant positive probability.

Proof:

We prove the correctness and the bound on the running time together. Clearly, if there is no triangle in the graph, the algorithm outputs “No,” since the algorithm places a triplet into the output register only after checking that it is a triangle in \( G \). Therefore the correctness proof requires only to calculate the probability with which the algorithm outputs a triangle if there is any.

Step 1:

For each \( v_i \) we use \( n-1 \) deterministic queries to find \( v_G(v_i) \). Since \( 1 \in [k] \), where \( k = O(n^\epsilon \log n) \), Step 1 costs \( O(n^{1+\epsilon} \log n) \) queries.

Step 2:

For each \( v_i \) we use Safe Grover to find out if \( v_G(v_i)^2 \) is empty. A single run of Safe Grover takes \( O(n \log n) \) queries, thus the total number of queries made is \( O(n^{1+\epsilon}(\log n)^2) \). We now claim that with probability \( 1 - O(\frac{1}{n^\epsilon}) \) we have that \( G \subseteq G' \subseteq G^{(n^{1-\epsilon})} \), or else a triangle of \( G \) is found.

Indeed, if no triangle is found then the Safe Grover for each \( v_i \) with probability \( 1 - \frac{1}{n^\epsilon} \) verifies that \( G \cap v_G(v_i)^2 = \emptyset \). So \( G' \supseteq G \) holds. That \( G' \subseteq G^{(n^{1-\epsilon})} \) holds follows from Lemma 4.

Step 4:

Realize that after steps 1. and 2. we know \( G' \), so Step 4 costs us no queries. One of the important, but simple observations is that even repeated applications of step 4 creates a graph \( T \) with a small number of triangles. This is shown in the following lemma:

Lemma 7 Let \( H \) be a graph on \([n]\). Assume that a graph \( T \) is built by a process that starts with an empty set, and at every step either discards some edges from \( H \) or adds an edge \((a, b)\) of \( H \) to \( T \) for which \( t(H, a, b) \leq \tau \) holds. For the \( T \) created by the end of the process we have \( t(T) \leq \left(\frac{n}{2}\right)\tau \).
Proof: Let us denote by $T[i]$ the edge of $T$ that $T$ acquired when it was incremented for the $i$th time, and let us use the notation $H^i$ for the current version of $H$ before the very moment when $T[i] = (a_i, b_i)$ was copied into $T$. Since $\{T[i], T[i+1], \ldots \} \subseteq T^i \subseteq H^i$, we have

$$t(T^i, a_i, b_i) \leq t(H^i, a_i, b_i) \leq \tau.$$ 

Now the lemma follows from

$$t(T) = \sum_i t(T^i, a_i, b_i) \leq \left(\frac{n}{2}\right) \tau,$$

since $i$ can go up to at most $\left(\frac{n}{2}\right)$. \Box

Step 5:

In this step we use an obvious sampling strategy: Set a counter $C$ to 0. Query $\lceil n^\delta \rceil$ random edge candidates from $v \times [n]$ (the constants are important). If there is an edge of $G$ among them, add one to $C$. Repeat this process $K = c_0 \log n$ times, where $c_0$ is a sufficiently large constant. Accept the low degree hypothesis if by the end $C < K/2$, otherwise accept the large degree hypothesis. Then

1. The probability that $\deg_G(v) > 10n^{1-\delta}$ and the low degree hypothesis is accepted is at most $1/n^3$.

2. The probability that $\deg_G(v) < 0.1n^{1-\delta}$ and the high degree hypothesis is accepted is at most $1/n^3$.

Indeed, using Lemma 5, considering a single round of sampling the probability that our sample set does not contain an edge from $G$ even though $\deg_G(v) > 10n^{1-\delta}$ is

$$\left(1 - \frac{10n^{1-\delta}}{n} \times \frac{n^\delta}{n}\right)^{n(1+o(1))} = \left(1 - \frac{10}{n}\right)^{n(1+o(1))} < 0.1$$

Similarly, the probability that our sample set contains an edge from $G$ even though $\deg_G(v) < 0.1n^{1-\delta}$ is

$$1 - \left(1 - \frac{0.1n^{1-\delta}}{n} \times \frac{n^\delta}{n}\right)^{n(1+o(1))} = 1 - \left(1 - \frac{1}{10n}\right)^{n(1+o(1))} < 0.2.$$ 

Now for $K = c_0 \log n$ rounds, where $c_0$ is large enough the Chernoff bound gives the claim.
Step 6:
Since we know $G'$ the operation costs us no queries.

Step 7:
Finding out $\nu_G(v)$ costs us $n - 1$ classical queries. Finding out if $\nu_G(v)^2 \cap G = \emptyset$ costs us $O(n \log n)$ queries, using the safe Grover.

Step 8:
The key to estimating the complexity of this step is an upper estimate on the number of executions of Step 7. In turn, this is done by lower bounding $|G'(A, A')|$. For each $x \in A$ we have $t(G', v, x) \geq n^\epsilon$, otherwise in Step 4 we would have classified $(v, x)$ into $T$. A triangle $(v, x, y)$ contributing to $t(G', v, x)$ contributes with the edge $(x, y)$ to $G'(A, A')$. Two triangles $(v, x, y)$ and $(v, x', y')$ can give the same edge in $G'(A, A')$ only if $x = y'$ and $y = x'$. Thus:

$$|G'(A, A')| \geq \frac{1}{2} \sum_{x \in \nu_G(v)} t(G', v, x) \geq |A|n^{1-\epsilon}/2. \quad (1)$$

Since we executed Step 7 only under the large degree hypothesis on $v$, if the hypothesis is correct, the right hand side of Equation 1 is at least $0.1n^{1-\delta}n^{1-\epsilon}/2 = \Omega(n^{2-\delta-\epsilon'})$. Since $G'$ has at most $\binom{n}{2}$ edges, it can execute Step 7 at most $O(n^{\delta+\epsilon'})$ times.

How about the number of executions of Step 5? Every execution of Step 5 leads to either the execution of Step 6 or that of Step 7. We have already seen that the number of executions of Step 7 is in $n^{\delta+\epsilon}$, which is easily in $O(n)$ (and it would not help to choose the parameters otherwise). We claim that each vertex is processed in Step 6 at most once. Indeed, if a vertex $v$ gets into Step 6, its incident edges are all removed, and its degree in $G'$ becomes 0 making it ineligible for being processed in Step 5 again. Thus the total complexity of Step 8 is:

$$n^\delta O(n \log n) + O(n)O(n^\delta \log n).$$

Step 9:
By Lemma 7 we have that there are at most $n^2n^{1-\epsilon'}$ triangles are in $T$. $T$ is a graph that is known to us, and so we can find out if one of these triangles belong to $G$ in time $O(\sqrt{n^3-\epsilon'^3} \log n)$, using Safe Grover.

Step 10:
In order to estimate $G \cap E$ observe that we added edges to $E$ only in Steps 6 and 7. Assume the Safe Grover worked throughout the whole algorithm. In each execution of Step 6 we added at most $10n^{1-\delta}$ edges to $E$, and we had $O(n)$ such executions that give a total
of $O(n^{2-\delta})$ edges. The number of executions of Step 7 is $O(n^{\delta+\epsilon'})$. Our task is now to bound now the the number of edges of $G$ each such execution adds to $E$.

We estimate $|G \cap G'(A, A')|$ from the $A'$ side. This is the only place where we use the fact that $G' \subseteq G^{(n^{1-\epsilon})}$. For every $x \in A'$ we have $t(G, v, x) \leq n^{1-\epsilon}$. On the other hand every edge $(y, x)$, $y \in A$, $x \in A'$, $(y, x) \in G'$ creates a $(v, x)$-based triangle. Thus

$$|G \cap G'(A, A')| \leq |A'|n^{1-\epsilon} \leq n^{2-\epsilon}.$$ 

Therefore the total number of edges of $G$ Step 7. contributes to $E$ is $n^{2-\epsilon+\delta+\epsilon'}$. In conclusion,

$$|G \cap E| \leq O(n^{2-\delta} + n^{2-\epsilon+\delta+\epsilon'}).$$

By Lemma 2 the complexity of finding a triangle in $G$ that contains an edge from $E$ is $O \left( \sqrt{n^{3-\min(\delta, \epsilon-\delta-\epsilon')}} \right)$.

### 4.1 Conclusions of the Analysis

The probability that not all the Safe Grovers run correctly can be upper bounded by $\frac{1}{n}$, for large $n$, using the union bound. Thus in the sequel we assume that none of the Safe Grovers fails. From the analysis we conclude that the total number of queries is upper bounded by:

$$O(n^{1+\epsilon} \log n + n^{1+\epsilon}(\log n)^2 + (n^{1+\delta+\epsilon'} \log n + n^{1+\delta} \log n) + \sqrt{n^{3-\epsilon'} \log n} + \sqrt{n^{3-\min(\delta, \epsilon-\delta-\epsilon')}}).$$

With $\epsilon = \frac{3}{7}$, $\epsilon' = \delta = \frac{1}{7}$, this gives $O(n^{1+\delta} \log^2 n)$ for the total number of queries. When the Safe Grover does not fail, the structural assumptions we made based on it hold. $G'$ eventually lends all its edges to $T$ and $E$. Since $G \subseteq G'$, every triangle in $G$ either has to be contained totally in $T$ or it has to have a non-empty intersection with $E$. Finally, we mention that in case one of the Safe Grovers goes wrong, the algorithm cannot run beyond the given time bound: quantum query machines by definition execute a prescribed number of steps.

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5 Appendix

Proof of Lemma 1:

For a matrix $M$, let $\lambda(M)$ denote the greatest eigenvalue of $M$. Recall the general form of Ambainis from [BSS]:

**Theorem 1** Let $f : S \rightarrow T$ be a partial boolean function with $S \subseteq \{0, 1\}^n$. Let $\Gamma$ be an arbitrary $S \times S$ nonnegative symmetric matrix that satisfies $\Gamma[x, y] = 0$ whenever $f(x) = f(y)$.

For $i \in \{1, \ldots, n\}$ let $\Gamma_i$ be the matrix:

$$\Gamma_i[x, y] = \begin{cases} 0 & \text{if } x_i = y_i, \\ \Gamma[x, y] & \text{if } x_i \neq y_i. \end{cases}$$

Then:

$$QQC_\epsilon(f) \geq \frac{(1 - 2\sqrt{\epsilon(1-\epsilon)})\lambda(\Gamma)}{2 \max_{1 \leq i \leq n} \lambda(\Gamma_i)}.$$

Above $QQC_\epsilon(f)$ denotes the $\epsilon$ error quantum query complexity of $f$. We need to work in the case when $S \subseteq \Sigma^n$ and $T = \{0, 1\}$. Let $A = f^{-1}(1)$. Let $k$ be the 1-certificate size of $f$. For an input $x \in A$ let $A_x$ be its 1-certificate. Let

$$\Gamma'[x, y] = \begin{cases} \Gamma[x, y] & \text{if } f(x) = 1 \text{ and } f(y) = 0, \\ 0 & \text{otherwise}. \end{cases}$$

Let $v \in \mathbb{R}^D$ be the unit vector with $\lambda(\Gamma) = v\Gamma v^*$. Notice that $v$ is positive. Because $\Gamma[x, y]$ is 0 whenever $f(x) = f(y)$ and $\Gamma$ is symmetrical, it holds that $v\Gamma'v = \lambda/2$.

For $1 \leq i \leq n$ let $v_i$ be the vector which equals to $v$ on those coordinates $x$ where $i \in A_x$ and 0 otherwise. We now have:

$$\langle v_i, v \rangle = \langle v_i, v_i \rangle \quad \text{for } 1 \leq i \leq n. \quad (2)$$

$$v_1 + \ldots + v_n \leq kv; \quad (3)$$

$$v_1\Gamma_1v + v_2\Gamma_2v + \ldots + v_n\Gamma_nv \geq v\Gamma'v = \lambda/2; \quad (4)$$

$$\frac{\lambda(\Gamma)}{\max_{1 \leq i \leq n} \lambda(\Gamma_i)} \leq 2(|v_1| + \ldots + |v_n|). \quad (5)$$

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Here the first inequality is meant entry-wise, and Inequality (5) follows from Inequality (4) and from $v_i \Gamma_i v = |v_i| |v_i| \Gamma_i v \leq |v_i| \lambda(\Gamma_i)$. Now from (2) and (3) it follows that

$$\sum_{i=1}^{n} |v_i| \leq \sqrt{n \sum_{i=1}^{n} |v_i|^2} = \sqrt{n \sum_{i=1}^{n} \langle v_i | v \rangle} \leq \sqrt{n \langle k | v \rangle} = \sqrt{nk}.$$

The above together with (5) gives the lemma. □