Aging dynamics of quantum spin glasses of rotors

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We study the long time dynamics of quantum spin glasses of rotors using the non-equilibrium Schwinger-Keldysh formalism. These models are known to have a quantum phase transition from a paramagnetic to a spin glass phase, which we approach by looking at the divergence of the spin relaxation rate at the transition point. In the aging regime, we determine the dynamical equations governing the time evolution of the spin response and correlation functions, and show that all terms in the equations that arise solely from quantum effects are irrelevant at long times under time reparametrization group (RpG) transformations. At long times, quantum effects enter only through the renormalization of the parameters in the dynamical equations for the classical counterpart of the rotor model. Consequently, quantum effects only modify the out of equilibrium fluctuation dissipation relation (OEFD), i.e. the ratio $X$ between the temperature and the effective temperature, but not the form of the classical OEFDR.

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I. INTRODUCTION

In recent years the focus of studies of glassy systems has moved from equilibrium to non-equilibrium properties, reflecting the understanding that glassiness is an intrinsically non-equilibrium phenomenon. The effect of quantum mechanics on spin glasses has also been a topic of interest. On the experimental side, there have been several studies of spin systems which are in the vicinity of a quantum phase transition from a spin-glass to a spin-fluid\textsuperscript{15,16} including the dipolar, transverse-field Ising magnet Li Ho\textsubscript{2} Y\textsubscript{1−x} F\textsubscript{x} \textsuperscript{17}. On the theoretical side, studies have mainly focussed on the Ising model in a transverse field or quantum rotors\textsuperscript{18,19}.

There has also been much recent work on the quantum-classical connection in spin glasses\textsuperscript{18,20,21} such as the derivation of quantum TAP equations\textsuperscript{19} and the large $N$ solution of SU($N$) models of spin glasses\textsuperscript{22,23}. Quantum fluctuations have been shown to be responsible for redefining the boundary between the spin glass and the quantum disordered phases in systems such as the rotors or the transverse field Ising model. In systems such as the quantum extension of the spherical $p$-spin-glass, quantum fluctuations are responsible for a crossover from a second to a first order phase transition\textsuperscript{24}.

Another system in which quantum effects and glassiness coexist is in the stripe glasses found in doped Mott insulators (e.g. Ref. \textsuperscript{25}).

In principle, quantum effects could also alter the non-equilibrium dynamics in the glassy regime at very low temperatures. Whilst the properties near the $T = 0$ quantum critical point of these systems can be understood within a static formulation\textsuperscript{26} studying the non-equilibrium effects in the glassy system requires a quantum dynamical approach.

Aging effects in classical spin glasses and other glassy systems have been investigated considerably in recent years. The out of equilibrium nature of the system displays itself through a persistent dependence of susceptibilities on the waiting time. The time since the system entered the glassy phase\textsuperscript{27} The correlation functions in such systems often display quite general scaling features\textsuperscript{28} and may also have an invariance under time reparametrizations\textsuperscript{29}. It is a natural question to ask whether quantum spin glasses have a similar aging behaviour to their classical counterparts.

Understanding aging dynamics in quantum systems requires formulating the problem within a non-equilibrium closed time path (CTP) formalism, such as the Schwinger-Keldysh approach. This dynamical approach is particularly attractive for studying disordered systems, since it eliminates the need to use replicas in carrying out the average over quenched disorder. This is because the generating functional in the CTP formalism is independent of the disorder realization. Several different disordered quantum systems have been studied recently using this approach, e.g. disordered and interacting electronic systems\textsuperscript{30,31}, the infinite-range quantum $p$-spin glass in the spherical limit\textsuperscript{32} and a semionic representation for quantum spin systems\textsuperscript{33}.

In this paper, using the Schwinger-Keldysh dynamical approach, we investigate the aging behaviour of a spin glass of $M$ component quantum rotors that have infinite range interactions and on site self interactions with a coefficient $u$. The model of quantum rotors is considerably simpler than that of true quantum Heisenberg spins present in any isotropic antiferromagnet or systems like the doped cuprates; the different components of the rotor variables all commute with each other, unlike the quantum spins. As a consequence, the path-integral written in the rotor variables has an action which contains no Berry phases and is purely real.

The quantum rotors are particularly interesting in elucidating the role of quantum mechanics in the long time
dynamics of the glassy phase for a number of reasons. They provide an example of a quantum system with continuous replica symmetry breaking (RSB), in contrast to the one-step RSB quantum $p$-spin model with a spherical constraint, which was recently studied by Cugliandolo and Lozano.  

Another feature of the rotor system is that it contains an expansion parameter, $u$, which we can use to organize a perturbative expansion. When we consider the model to $O(u)$, we get equations that look very similar to those obtained in the case of the $p = 2$ spherical spin model, which has no RSB. However, when we expand to $O(u^2)$ we end up with terms that are similar to those from the $p = 4$ spherical model and also some terms that do not arise in $p$-spin models. Therefore, we can investigate very generally how each term contributes to the long time dynamics of the model, and how the quantum effects enter in the problem through each of these terms.  

We show that the terms in the dynamical equations that appear as a consequence of quantum mechanics are irrelevant at long times. The precise sense in which the terms are irrelevant was defined in Ref. 10, using the language of time reparametrization invariance and the reparametrization group (RpG) of time transformations. Hence, at very long times, the dynamics of the quantum rotors is completely determined by a renormalized classical version of the model. In this time regime, quantum mechanics enters the problem only through a renormalization of the coefficients in the dynamical equations, and there is a complete correspondence between the classical and quantum versions of the model in the aging regime. In particular, this implies that the response and correlation functions of the quantum rotors are related by a modified fluctuation-dissipation relation, with an effective temperature $T_{\text{eff}} > T$, just as found in the classical spin glass models. Our results within the RpG framework extend the connection between the aging regimes of quantum and classical systems, as found in the quantum $p$-spin model\textsuperscript{14} with one-step RSB, to a wider and more general class of quantum glassy models.  

The paper is organized as follows. In section II we describe the model that we study and the CTP formalism that we use. In section III we obtain the saddle point dynamical equations for the model and develop the perturbation theory in $u$ that we need to obtain these equations. In section IV we study the solutions to these equations in the paramagnetic phase. In section V we investigate the solutions to the dynamical equations in the spin glass phase, and show that a selected number of terms can be made invariant under reparametrisations of the time coordinate in the aging regime. We analyze the terms stemming from the quantum dynamics imposed on the system that are not present in the classical model, and show that they become irrelevant at long times under the time reparametrization transformations. Finally, in section VI we discuss differences between classical and quantum spin glasses.

II. THE MODEL

We consider the model of a glass of quantum rotors introduced in Refs. 12, 13 with long range interactions on a $d$-dimensional lattice with $N$ sites. (We let $d$ and $N$ go to infinity later). An important point to note is that the $M$ components of angular momentum of the rotors commute on the same site, unlike Heisenberg spins, for which there are non-trivial commutation relations between the components. This simplification allows us to write a path integral in which there are no Berry phase terms, and makes the rotors much easier to treat analytically.

To derive the dynamical equations we use a CTP (Schwinger-Keldysh) formalism\textsuperscript{13, 28}. The CTP approach provides a way to study the non-equilibrium response of a system. The price that has to be paid for this is the introduction of a second component to the system, with time flowing in the opposite direction. The non-equilibrium formulation has the advantage that it leads to a generating functional that is automatically normalized to unity. The property of normalization allows averages over disorder realizations in a way that bypasses using replicas.

There are many possibilities for the choice of integration contour in the complex time plane.\textsuperscript{13, 14} The contour $C$ that we work with is illustrated in Fig. 1. It starts at $t = 0$, runs to $t = \infty$, and then returns in the negative $t$ direction to $t = 0$. The contours more usually used, for example in problems involving the calculation of nonlinear response, run from $t = -\infty$ to $t = \infty$ and back again\textsuperscript{30, 31}; however after an infinite time, the system will have equilibrated, so it is no longer possible to study the non-equilibrium dynamics that we are interested in.

![FIG. 1. The contour $C$ used in the formalism.](image)

For a system at equilibrium, the usual way to introduce finite temperature is through the density matrix with the form of the Gibbs-Boltzmann distribution. In a non-equilibrium situation, the system cannot be described by the Gibbs-Boltzmann distribution, so an alternative approach is required. The solution is to couple the system to a heat bath\textsuperscript{25} (chosen here to be a set of independent harmonic oscillators) and then allow the system to reach a constant temperature. A detailed account of integrating the bath variables to obtain an effective action in the $p$-spin model is given in Ref. 14, and we follow...
their approach in our study of the quantum rotors.

A. Effective action for the rotors and bath system

The action for the system of rotors interacting with the bath takes the form

$$S = S_{\text{free}} + S_{\text{int}} + S_{\text{dis}} + S_T,$$

(1)

where $S_{\text{free}}$ is the free action, $S_{\text{int}}$ describes the self-interaction of the rotors, $S_{\text{dis}}$ contains the spin exchange interactions which introduce disorder and frustration to the model, and $S_T$ describes the interaction with an external heat bath. The Lagrangian for the free rotors is

$$\mathcal{L}_{\text{free}} = \sum_i \frac{1}{2g} \left( \partial_t S^a_{\mu i} \right) \sigma^a_{3b} \left( \partial_t S_{\mu i}^b \right) + \frac{1}{2} m^2 S_{\mu i}^a \sigma_{3a} S_{\nu i}^b.$$

(2)

The indices on the spin variable $S_{\mu i}^a$ refer to the Keldysh contour branch ($\mu$), site ($i$), and spin component ($\nu$); sums over repeated Keldysh and spin component indices are always implied unless explicitly noted otherwise. The $\sigma_{123}$ are the standard Pauli sigma matrices. Note that the $S^2$ terms enter with an opposite sign to the $S^1$ terms because the direction of time integration is reversed on the second part of the contour. The first term in the Lagrangian $\mathcal{L}_{\text{free}}$ is a kinetic energy term ($1/g$ is the moment of inertia), and the second is a potential energy term ($m^2$ also acts as the parameter that tunes between harmonic oscillators on a lattice, each at site $i$.

The self-interactions are described by

$$\mathcal{L}_{\text{int}} = \sum_i \frac{u}{2} \left\{ \left[ \left( S^1_{\mu i} \right)^2 \right]^2 - \left[ \left( S^2_{\mu i} \right)^2 \right]^2 \right\}.$$

(3)

The term in the Lagrangian that contains quenched disorder is

$$\mathcal{L}_{\text{dis}} = \sum_{i \neq j} J_{ij} S^a_{\mu i} \sigma_{3}^{ab} S^b_{\nu j}.$$

(4)

Thermal effects are introduced by placing the system in contact with a heat bath using the method introduced by Feynman and Vernon. The contribution of this coupling to the action after integrating out heat bath variables is

$$S_T = -\int dt_1 dt_2 \sum_i \eta(t_1 - t_2) S^a_{\mu i}(t_1) \left[ \sigma^a_{11} + i \sigma^a_{22} \right] S^b_{\nu i}(t_2) + \nu(t_1 - t_2) S^a_{\mu i}(t_1) \left[ \delta^{ab} - \sigma^a_{11} \right] S^b_{\nu i}(t_2),$$

(5)

where the noise and dissipative kernels, $\nu$ and $\eta$ respectively, are given by

$$\nu(t) = \int_0^\infty d\omega \, I(\omega) \coth \left( \frac{\beta \hbar \omega}{2} \right) \cos(\omega t),$$

(6)

$$\eta(t) = -\theta(t) \int_0^\infty d\omega \, I(\omega) \sin(\omega t),$$

(7)

and $I(\omega)$ is the spectral density of the bath

$$I(\omega) = \frac{N_b}{\pi} \frac{|C_n|^2}{2Mn_\omega^2},$$

(8)

where $N_b$ is the number of oscillators in the bath, $\omega_n$ is the natural frequency of the $n^{th}$ oscillator, $M_n$ is its mass, and $C_n$ is its coupling to the system. Here we only consider the case of ohmic dissipation

$$I(\omega) = \frac{\gamma}{\pi} \omega e^{-\omega/\Lambda}, \quad \text{for } \omega < \Lambda,$$

where $\gamma$ plays the role of a friction coefficient (to relate to the notation in previous works, $\gamma = M\gamma_0$).

Properties to note are that both kernels are purely real, $\nu(t) = \nu(-t)$, and both kernels decay rapidly at large time differences. Note that the quantum fluctuation-dissipation theorem (QFDT) holds for the bath variables when there is ohmic dissipation. The QFDT relating the heat bath variables is

$$\eta(\omega) = \frac{1}{\hbar} \lim_{\epsilon \to 0} \int \frac{d\omega'}{2\pi} \frac{1}{\omega' - \omega + i\epsilon} \tanh \left( \frac{\beta \hbar \omega'}{2} \right) \hbar \nu(\omega').$$

(9)

B. Disorder average and definition of correlation functions

Having described all of the terms in the action, we now move on to performing the average over disorder realizations and defining the correlation and response in terms of the CTP two-point correlators.

The closed time path generating functional is

$$Z = \int [DS^a_{\mu i}] \, e^{\bar{S}}.$$

(10)

We perform an average over the quenched disorder

$$Z = \int [DJ] \, \mathcal{P}(J_{ij}) \int [DS^a_{\mu i}] \, e^{\bar{S}},$$

(11)

where

$$\mathcal{P}(J_{ij}) = \left( \frac{N}{2\pi J^2} \right)^{\frac{N}{2}} \exp \left[ -\frac{N J_{ij}^2}{2J^2} \right],$$

(12)
for a long range interaction. The form Eq. (12) for the
disorder probability distribution comes from the assump-
tion that the disorder is Gaussian distributed and that
\( \overline{J_{ij}^2} = J^2/N \). We can perform the integration over disor-
der without replicas because of the normalization prop-
erty of the generating functional in the CTP formulation,
since the disorder and the initial conditions are uncorre-
lated:

\[
\int \left[ DJ \right] P(J_{ij}) e^{i S(t)} \int dt \sum_{i \neq j} J_{ij} S_{ij}(t) \sigma_i^{ab} S_{j\mu}(t) = \left( \frac{N}{2 \pi J^2} \right)^{\frac{1}{4}} e^{-\frac{\beta^2}{2N}\int dt \sum_{i \neq j} \left[ S_{ij}^a(t) \sigma_i^{ab} S_{j\mu}(t) S_{j\nu}(t) \sigma_j^{ac} S_{j\nu}(t) \right]}
\]

The dynamical equations are written in terms of the cor-
relation and response, which are defined below. We use
an overbar \( \overline{\cdot} \) to indicate an average over realizations
of disorder and angular brackets \( \langle \ldots \rangle \) to indicate an aver-
age with respect to the action. The correlation, \( C(t_1, t_2) \),
and response, \( R(t_1, t_2) \), are

\[
C_{ij,\mu\nu}(t_1, t_2) = \frac{1}{2} \left\langle \overline{S_{ij}^a(t_1) S_{ij}^a(t_2) + S_{ij}^b(t_1) S_{ij}^b(t_2)} \right\rangle,
\]

and \( R_{ij,\mu\nu}(t_1, t_2) = \frac{\delta \langle S_{ij}^a(t_1) \rangle}{\delta h_{j\mu}(t_2)} \), which in linear response theory may be written as

\[
R_{ij,\mu\nu}(t_1, t_2) = \frac{i}{\hbar} \left\langle \overline{S_{ij}^a(t_1)[S_{ij}^a(t_2) - S_{ij}^b(t_2)]} \right\rangle.
\]

An alternative approach to the dynamics is to intro-
duce a matrix propagator as in the Keldysh formalism,
which takes the form for bosonic fields (e.g. Ref. [24]),

\[
G_{ij,\mu\nu}^{11}(t_1, t_2) = -i \langle T \left[ S_{ij}^a(t_1) S_{ij}^a(t_2) \right] \rangle,
\]

\[
G_{ij,\mu\nu}^{12}(t_1, t_2) = -i \langle S_{ij}^b(t_2) S_{ij}^a(t_1) \rangle,
\]

\[
G_{ij,\mu\nu}^{21}(t_1, t_2) = -i \langle S_{ij}^a(t_1) S_{ij}^b(t_2) \rangle,
\]

\[
G_{ij,\mu\nu}^{22}(t_1, t_2) = -i \langle \overline{\tilde{T} \left[ S_{ij}^b(t_1) S_{ij}^a(t_2) \right]} \rangle,
\]

where \( T \) and \( \tilde{T} \) are the operators for time ordering
and anti-time ordering respectively. Under the transforma-
tion in Keldysh space \( G \to \tilde{G} = LGL^\dagger \), where

\[
L = \frac{1}{\sqrt{2}} (1 - i \sigma_2),
\]

the propagator takes the form

\[
\tilde{G} = LGL^\dagger = \begin{pmatrix} 0 & G^A \\ G^R & G^K \end{pmatrix},
\]

where

\[
G^R = G^{11} - G^{12} = G^{21} - G^{22},
\]

\[
G^A = G^{11} + G^{22} = G^{21} + G^{12},
\]

\[
G^K = G^{11} - G^{21} = G^{12} - G^{22},
\]

and the results follow from the definitions of the propa-
gators in terms of time ordered products. The definitions
above imply the following properties for the propagators

\[
G_{ij,\nu \mu}^R(t_2, t_1) = G_{ij,\mu \nu}^A(t_1, t_2),
\]

\[
G_{ij,\nu \mu}^K(t_2, t_1) = G_{ij,\mu \nu}^K(t_1, t_2).
\]

The retarded and advanced propagators are purely real
and the Keldysh propagator is purely imaginary. Under
the transformation, \( L \), the spins are also rotated, \( \tilde{S} = LS \), so for

\[
S = \begin{pmatrix} S^1 \\ S^2 \end{pmatrix},
\]

we have

\[
\tilde{S} = \frac{1}{\sqrt{2}} \begin{pmatrix} S^1 - S^2 \\ S^1 + S^2 \end{pmatrix} = \begin{pmatrix} \tilde{S} \\ S \end{pmatrix}.
\]

The \( \tilde{S} \) and \( S \) variables in the quantum CTP formulation
naturally become the usual variables within the Martin-
Siggia-Rose (MSR) formalism \[24\] when the classical vari-
ables is taken. The relation between the Keldysh and retarded
propagators and the correlation and response is

\[
G^K(t_1, t_2) = -2iC(t_1, t_2),
\]

\[
G^R(t_1, t_2) = -\hbar R(t_1, t_2).
\]

We will work with the correlation and response rather
than the propagators in the dynamical equations. How-
ever, it is easier to use the matrix propagator in the Feyn-
man rules for including interactions, rather than using a
diagrammatic technique where response and correlation
are treated differently.\[24\]

In anticipation of the saddle-point evaluation for the
disorder term, we perform a Hubbard-Stratonovich trans-
formation to decouple the four spin term generated by the
disorder average. Introducing the Hubbard-Stratonovich field \( Q_{ij,\mu\nu}^{ab}(t_1, t_2) \), the effective action can be rewritten as

\[
Z = \int [DQ] e^{-\frac{\beta}{2N} \int dt_1 dt_2 \sum_i Q_{i,\mu\nu}(t_1, t_2) Q_{i,\mu\nu}^{ab}(t_1, t_2) Z[Q],
\]

\[
Z[Q] = \int [DS] e^{\sum T_{\mu\nu} + Q_{\mu\nu} + \sum \sigma_i^{ab} \tilde{S}_{ij}(t)}
\]

where

\[
S_{\text{tree}} = \int dt \sum_i \tilde{S}_{ij}^a(t) \Gamma_{\mu\nu}^{ab} \tilde{S}_{ij}^b(t),
\]

\[
S_Q = \int dt_1 dt_2 \sum_i Q_{i,\mu\nu}^{ab}(t_1, t_2) \tilde{S}_{ij}^a(t_1) \sigma_i^{ab} \tilde{S}_{ij}^b(t_2).
\]

Next we introduce a modified version of the Hubbard-
Stratonovich field, \( \tilde{Q}_{i,\mu\nu} = Q_{i,\mu\nu}^{ac} \sigma_i^{cb} \), thus

\[
Z = \int [DQ] e^{-\frac{\beta}{2N} \int dt_1 dt_2 \sum_i \sigma_i^{ab} \tilde{Q}_{i,\mu\nu}^{ab}(t_1, t_2) Q_{i,\mu\nu}^{ab}(t_1, t_2) \tilde{S}_{ij}(t_1, t_2) \tilde{S}_{ij}(t_2)} \times \int [D\tilde{S}] e^{\sum T_{\mu\nu} + Q_{\mu\nu} + \sum \sigma_i^{ab} \tilde{S}_{ij}(t)},
\]
Rearranging gives

\[ S_Q = \int dt_1 dt_2 \sum_i \hat{\dot{Q}}_{i,\mu\nu}^a(t_1, t_2) \tilde{S}_{i\mu}^a(t_1) \tilde{S}_{i\nu}^b(t_2). \]  

(36)

The \( \Gamma_{\mu\nu} \) term has the structure

\[ \hat{\Gamma}_{\mu\nu} = \delta_{\mu\nu} \left( \frac{\Gamma_K}{\Gamma_A} \right), \]

(37)

where

\[ \Gamma_R = \Gamma_A = -\frac{1}{2g} \frac{\partial^2}{\partial t^2} + \frac{1}{2} m^2, \quad \Gamma_K = 0. \]

(38)

### III. DYNAMICAL EQUATIONS

Using propagators, the self-consistent mean field equations can be represented in the form

\[ G^{-1} = G_0^{-1} - \Sigma_J - \Sigma_T - \Sigma_u, \]

(39)

where the \( \Sigma \) terms are the self energies from the disorder, the interaction with the thermal bath and from self-interaction respectively. The general strategy that we will adopt in obtaining the dynamical equations for this model is to perform a saddle point evaluation of \( \Sigma_J \) and then a perturbation expansion in \( u \). We obtain a solution to \( O(u) \) in the paramagnetic phase, in analogy with the replica symmetric solution in the equilibrium problem,[13] whilst in the spin glass phase we need to consider terms to \( O(u^2) \) in the interaction, which are the terms found to contribute to RSB.

### A. Saddle point evaluation of \( \Sigma_J \)

At the saddle point the functional derivative with respect to the Hubbard-Stratonovich field is zero

\[ \frac{\delta Z}{\delta Q} = 0. \]

The variation leads to the following equations

\[ 0 = \frac{Nhi}{2J^2} \left\langle \sigma_1^{bb'} \hat{Q}_{\mu\nu}^{b'c'}(t_2, t_1) \sigma_1^{c'a} \right\rangle + \sum_i \left\langle \tilde{S}_{i\mu}^a(t_1) \tilde{S}_{i\nu}^b(t_2) \right\rangle. \]

(40)

Rearranging gives

\[ \left\langle \hat{Q}_{\mu\nu}^{ab}(t_1, t_2) \right\rangle = -\frac{2J^2}{\hbar} \sigma_1^{bb'} \left( G_{\mu\nu}^{c'a'}(t_2, t_1) \sigma_1^{c'a} \right). \]

(41)

By substituting the saddle solution into Eq. (36), one obtains the self-energy contribution coming from the disorder:

\[ \Sigma_J(t_1, t_2) = \frac{2J^2}{\hbar} \sigma_1 \left( G(t_1, t_2) \sigma_1 \right) \]

\[ \frac{2J^2}{\hbar} \left( G^A G^R \right) (t_1, t_2) . \]

(42)

The notation of a matrix with \((t_1, t_2)\) following it is used to indicate that all of the propagators in the matrix have that as their argument. At the mean field level the solution is homogeneous in the site index, so we drop it.

In the absence of a magnetic field, we also drop the spin indices, i.e. \( G_{\mu\nu} = G \delta_{\mu\nu} \).

### B. Diagrammatic perturbation in \( u \)

The interaction terms in the action may be treated in a diagrammatic way by noting that in the \( \hat{S} \) basis, the interaction term is written as

\[ S_{int} = \frac{u}{2} \int_0^\infty dt \sum_i \tilde{S}_{i\mu}^a(t) \tilde{S}_{i\nu}^b(t) \sigma_1^{c'd'} \tilde{S}_{i\mu}^c(t) \sigma_1^{d'a}. \]

(43)

The Feynman rules (see Fig. 2) for calculating \( \Sigma_u \) term by term in \( u \) are that the propagator (solid line) is \( iG_{\mu\nu}(t, t') \) and the interaction propagator (dashed line with a dot at one end) is \( i\frac{u}{2} \delta_{\mu\nu} \). The \( \sigma_1 \) is inserted at the end with the dot.

\[ \int \frac{a}{b} \cdots \frac{c}{d} = i \frac{u}{2} \sigma_1^{ab} \delta_{cd} \delta_{\mu\nu}. \]

FIG. 2. The Feynman rules for the perturbation theory in \( u \).

This leads to four contributions to the self energy \( \Sigma_u \) to first order in \( u \), as shown in Fig. 3. Note that the bottom two diagrams in Fig. 3 are proportional to \( M \), the number of spin components of the rotors, since there is a closed loop around a closed loop, and that the trace includes spin indicies. Note also that \( tr(G\sigma_1^2) = 0 \), since \( G^R(t, t) = 0 \). The sum of these contributions is

\[ \Sigma_u^{(1)}(t, t') = \frac{iu}{2} (1 + M) \sigma_1 G^K(t, t) \delta(t - t'), \]

(44)

where the superscript (1) indicates that the self energy is to first order in \( u \).

To second order in \( u \) there are six different one particle irreducible diagram topologies to consider which contribute to the self energy and four different diagrams for each topology (the different topologies are shown in Fig. 4). We treat each topology separately, starting with the propagator with two self interactions, which contributes

\[ \Sigma_u^{(2a)}(t, t') = -\frac{u^2}{8\hbar} \times \left( \frac{3(G^A)^2 + 3(G^R)^2 + (G^K)^2}{[3(G^K)^2 + (G^A)^2]} \right)(t, t'). \]

(45)
The next diagrams are the vertex correction to the closed loop that contributes to $\Sigma_u^{(2)}$ and the propagator with one self interaction, which contribute

$$\Sigma_u^{(2b)}(t,t') = -\frac{u^2}{4\hbar} \delta(t-t') \times \left[ \int_0^\infty d\tilde{t} G^K(i,\tilde{t}) G^R(t,\tilde{t}) G^K(t,\tilde{t}) \right]. \quad (46)$$

The remainder of the self energies are expressible in terms of the expressions found in equations (45) and (47): $\Sigma^{(2c)} = M\Sigma^{(2a)}$, $\Sigma^{(2d)} = M\Sigma^{(2b)}$, $\Sigma^{(2c)} = \Sigma^{(2b)}$ and $\Sigma^{(2f)} = \Sigma^{(2d)}$. The diagram that plays the most important role in the physics is (2a) because it depends on two times rather than only one.

![Diagrams](image)

**FIG. 3.** Diagrams contributing to the self energy to first order in $u$.

**FIG. 4.** Diagrams contributing to the self energy to second order in $u$.

### C. The dynamical equations

From the self consistent Eq. (30) we obtain the dynamical equation

$$1 = \left( G_0^{-1} - \Sigma_J - \Sigma_T - \Sigma_u \right) G. \quad (47)$$

By substituting the self-energies $\Sigma_J$, $\Sigma_T$ and $\Sigma_u$ that we determined above to order $u^2$, and shifting notation to response and correlation, we obtain the dynamical equations for the quantum rotor model

$$\delta(t_1 - t_2) = \left\{ \frac{1}{2g} \frac{\partial^2}{\partial t_1^2} - \frac{1}{2} m^2 - u(1 + M)C(t_1,t_1) \right. \left. -2(1 + M)u^2 C(t_1,t_1) \times \int_0^\infty dt R(t_1,t)C(t_1,t) \right\} R(t_1,t_2)$$

$$+ \int_0^\infty dt \eta(t_1 - t)R(t,t_2)$$

$$-2J^2 \int_0^\infty dt R(t_1,t)R(t,t_2)$$

$$\left. - \frac{3}{2}(1 + M)u^2 \int_0^\infty dt C(t_1,t)^2 R(t_1,t)R(t,t_2) \right.$$  

$$+ \frac{1}{2}(1 + M)u^2 \hbar^2 \int_0^\infty dt R(t_1,t)^3 R(t,t_2). \quad (48)$$

and

$$0 = \left\{ \frac{1}{2g} \frac{\partial^2}{\partial t_1^2} - \frac{1}{2} m^2 - u(1 + M)C(t_1,t_1) \right. \left. -2(1 + M)u^2 C(t_1,t_1) \int_0^\infty dt R(t_1,t)C(t_1,t) \right\} C(t_1,t_2)$$

$$+ \int_0^\infty dt \eta(t_1 - t)C(t,t_2) - \frac{\hbar}{2} \int_0^\infty dt \nu(t_1 - t)R(t_2,t)$$

$$-2J^2 \int_0^\infty dt [C(t_1,t)R(t_2,t) + R(t_1,t)C(t_2)]$$

$$- \frac{1}{2}(1 + M)u^2 \int_0^\infty dt C(t_1,t)^3 R(t_2,t)$$

$$- \frac{3}{2}(1 + M)u^2 \int_0^\infty dt C(t_1,t)^2 R(t_1,t)C(t,t_2)$$

$$+ \frac{1}{2}(1 + M)u^2 \hbar^2 \int_0^\infty dt [R(t_1,t)^3 C(t,t_2)$$

$$+ 3R(t_1,t)^2 R(t_2,t)C(t_1,t)]. \quad (49)$$

The terms that arise in this expansion look partly like a $p = 4$ spherical spin glass (the $C^3 R$ term in the correlation equation), but there are also terms (multiplied by $\hbar^2$) which do not appear in the classical spherical model.

### IV. Paramagnetic phase

In the paramagnetic phase the correlation and response are time translation invariant (TTI) after the initial transients have died out. We find a solution to the dynamical equations (45) and (47) to $O(u)$ in this phase. With the assumption of TTI correlators, let $\tau = t_1 - t_2$, and then let $t_1 \to \infty$. Define $C_\infty = \lim_{t_1 \to \infty} C(t_1,t_1)$, which is the equilibrium limit of the equal time correlation (we also use $C_\infty$ in the spin glass phase; as a one time quantity
the equal time correlation has a limit as \( t_1 \to \infty \). TTI allows us to solve the problem in the paramagnetic phase by performing a Fourier transformation:

\[
-1 = \left\{ \frac{\omega^2}{2g} + \frac{1}{2}m^2 + u(1 + M)C_\infty \right\} R(\omega) + i \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{\eta(\omega') R(\omega')}{\omega' - \omega + i0} - 2J^2i \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{R(\omega') R(\omega')}{\omega' - \omega + i0},
\]

(50)

and

\[
0 = \left\{ \frac{\omega^2}{2g} + \frac{1}{2}m^2 + u(1 + M)C_\infty \right\} C(\omega) + i \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{\eta(\omega') C(\omega')}{\omega' - \omega + i0} - \frac{\hbar}{2} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{\nu(\omega') R(\omega')}{\omega' - \omega + i0} - 2J^2i \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{C(\omega') R(\omega') + R(\omega') C(\omega')}{\omega' - \omega + i0}.
\]

(51)

The following properties hold for the correlation and response and their Fourier transforms: \( C(t) \) is invariant under time reversal and is real, hence \( C(\omega) = C^*(-\omega) \), \( C^*(\omega) = C(\omega) \) and \( C(\omega) \) is real. The response is real, hence \( R(\omega) + R(-\omega) = 2Re[R(\omega)] \) and \( R(\omega) - R(-\omega) = 2iIm[R(\omega)] \). In addition, the noise kernel \( \nu(\omega) \) is real and has the property \( \nu(\omega) = \nu(-\omega) \). The real part of the equation (51) combined with the QFDT for the heat bath variables (52) gives

\[
\text{Im}[R(\omega)] = \frac{1}{\hbar} \tanh \left( \frac{\beta h \omega}{2} \right) C(\omega).
\]

(52)

Hence the correlation and response are related by the QFDT if the same is true for the heat bath variables. In this regime, we can solve explicitly for the response, and hence the correlation through the QFDT (52). We use these solutions to investigate the self-consistency condition in the paramagnetic regime and investigate critical slowing down as discussed by Sompolinsky.

A. Response, correlation, and phase boundary

Assuming that the integrands in the resonant integrals in equations (52) and (51) fall off sufficiently quickly at large \( \omega \), then they may be rewritten and solved for \( R(\omega) \) to give

\[
R(\omega) = -\frac{x(\omega)}{4J^2} - \frac{1}{4J^2} \sqrt{x(\omega)^2 - 8J^2},
\]

(53)

where

\[
x(\omega) = \frac{\omega^2}{2g} + \frac{1}{2}m^2 + u(1 + M)C_\infty - \eta(\omega).
\]

(note that \( \eta(\omega) \) has both real and imaginary parts). In the limit that \( \omega \to 0 \), the real part of the kernel \( \eta \) is proportional to the frequency cutoff, \( \Lambda \), so to remove this, define \( m_1^2 = m^2 - 2\eta(0) \). Then at zero frequency at the critical point,

\[
\frac{1}{2}m_1^2 + u(1 + M)C_\infty = 2\sqrt{2} J,
\]

(54)

and the imaginary part of the response is

\[
\text{Im}[R(\omega)] = -\frac{\text{Im}[\eta(\omega)]}{4J^2} - \frac{1}{4J^2} \sqrt{2} |\text{Im}[\eta(\omega)]| \sqrt{|\text{Im}[\eta(\omega)]|^2 + 8J^2}.
\]

(55)

To get the self-consistency condition for \( C_\infty \) at the critical point, we use equation (54) and the QFDT

\[
C_\infty = \hbar \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \coth \left( \frac{\beta h \omega}{2} \right) \text{Im}[R(\omega)].
\]

(56)

Using

\[
\text{Im}[\eta(\omega)] = -\frac{\gamma}{2} e^{-|\omega|/\Lambda} \theta(\Lambda - |\omega|),
\]

(57)

and noting that the second integral vanishes, since the integrand is odd, we get the result

\[
C_\infty = \frac{\pi \gamma}{48h^2\beta^2 J^2} + O(\Lambda^2).
\]

(58)

The quantity \( C_\infty \) fixes the equal-time correlation, and provides a constraint for the rotor size in this model, since we do not impose a spherical constraint. In the spin glass phase \( C(t, \omega) \) should relax to an equilibrium value \( q \), the Edwards-Anderson order parameter. This is because it is a one time quantity – the same is not true for two time quantities. The quantity \( m_1^2 \) acts to “tune” the system through the spin glass transition. Using equation (51) and separating \( m_1^2 \) into a piece that depends on the cutoff and a temperature dependent piece, we have an expression analogous to the one found for \( r_C(T) \) in the equilibrium study of quantum rotors

\[
m_1^2(\beta) = m_1^2 - \frac{u(1 + M)\gamma}{24h^2\beta^2 J^2}.
\]

(59)

B. Critical slowing down

The dynamic transition from the paramagnetic to the spin glass phase was investigated by Sompolinsky and Zippelius. However, in their treatment they took the infinite time limit before, rather than after the \( N \to \infty \) limit, the opposite order of limits than that considered here. This leads to finite energy barriers between traps and hence an infinite hierarchy of time scales and an ergodic solution. In the situation we consider the system
is confined to a single ergodic component and we are studying the relaxation within a trap.

To connect with the work of Sompolinsky and Zippelius, define

\[ \Gamma^{-1}(\omega) = \frac{\partial}{\partial \omega} R^{-1}(\omega), \]  

(60)

which leads to \( \Gamma(\omega) \simeq 1/\omega \) at small frequencies and

\[ \Gamma^{-1}(\omega) = \frac{\partial n}{\partial \omega} - \frac{\omega}{g} \left(1 - 2J^2 R(\omega)^2\right), \]  

(61)

which is analogous to their result.

C. Alternative approach – Schwinger-Keldysh Landau Theory

In previous work on quantum rotor systems[3], the approach taken has been to obtain a replicated field theory in the Hubbard-Stratonovich \( Q \) fields. In this and related work[4] we have obtained directly dynamical equations for spin correlation functions. In the following we will briefly show how to set up a Landau theory for the fields \( Q \) within a completely dynamical approach, and connect to both the replica results of Ref. [2,3] and the results obtained in the preceding sections.

In the dynamical Schwinger-Keldysh approach, if the system does reach equilibrium with the bath (as is the case in the paramagnetic phase), then the temperature of the heat bath can be introduced through the use of a density matrix rather than with the heat bath kernels referred to in the previous section.

Just as in section[1] integrating over disorder and introducing a Hermitian Hubbard-Stratonovich field leads to the following action

\[ Z = \int [\mathcal{D}\tilde{Q}] e^{-\frac{1}{2J} \int dt_1 dt_2 \sum_{ij} \tilde{Q}_i \sigma_1 K_{ij} \sigma_1 \tilde{Q}_j - \log Z_0[\tilde{Q}]}, \]  

(62)

where \( Z_0 \) is the single site generating functional

\[ Z_0[\tilde{Q}] = \int \mathcal{D} S^0 \exp[iS_0], \]

\[ S_0 = S_{\text{free}} + S_{\text{nl}} \]

\[ - \int dt dt' \sum_i \tilde{Q}_{i\mu}(t,t') S_{i\mu}^0(t') S_{i\mu}^0(t), \]  

(63)

where \( S_{\text{nl}} \) includes the self-interactions and interaction with the bath. We can expand \( Z_S[\tilde{Q}] \) by obtaining the perturbative vertices in powers of the \( \tilde{Q} \) fields, noting that the mean-field solution is isotropic in space. Using standard diagrammatic methods and in simplified notation, the effective action up to the cubic order is

\[ S = \int \frac{d\omega}{2\pi} Tr(QG_0^3) + \int \frac{d\omega_1}{2\pi} \frac{d\omega_2}{2\pi} \frac{d\omega_3}{2\pi} Tr(QG_0)^2 \]

\[ -i u \int \frac{d\omega_1}{2\pi} \frac{d\omega_2}{2\pi} \frac{d\omega_3}{2\pi} Tr[(G_0 Q G_0^3) \sigma_1 (G_0 Q G_0^3)] \]

\[ -i u \int \frac{d\omega_1}{2\pi} \frac{d\omega_2}{2\pi} \frac{d\omega_3}{2\pi} Tr(G_0 Q G_0^3) \]

\[ Tr(\sigma_1 G_0 Q G_0^3) + \ldots, \]  

(64)

where \( \ldots \) means higher order terms. It is important to point out that terms with coefficient \( u \) are important to get stable saddle point solutions. We study the saddle point solutions in the absence of spatial fluctuations.

Consider the following saddle point solution which is \( O(M) \) and TTI:

\[ Q_{\mu\nu}(\omega_1,\omega_2) = 2\pi \delta(\omega_1 + \omega_2) \delta_{\mu\nu} Q^{ab}(\omega_1), \]

\[ Q = \begin{pmatrix} Q_K & Q_R \\ Q_A & 0 \end{pmatrix}. \]

(65)

(66)

This Ansatz has the same structure in frequency and \( O(M) \) indices as in the replica approach[3]. However, here we are dealing with a \( 2 \times 2 \) matrix in Keldysh space in contrast to the \( n \times n \) \( (n \to 0 \) limit ) replica matrix. This leads to the saddle point action

\[ S_{\text{sp}} = \int \frac{d\omega}{2\pi} (Q_K R^0_H + Q_A G^0_A) \]

\[ + \int \frac{d\omega}{2\pi} [(Q_K R^0_H)^2 + (Q_A G^0_A)^2] \]

\[ + \int \frac{d\omega}{2\pi} [(Q_K R^0_H)^3 + (Q_A G^0_A)^3] \]

\[ - i u(1 + M) \int \frac{d\omega}{2\pi} [Q_R (G^0_H)^2 + Q_A (G^0_A)^2] \]

\[ \times \int \frac{d\omega}{2\pi} [(Q_K R^0_H + Q_A G^0_A) K_K + Q_K R^0_H G^0_H + \ldots]. \]  

(67)

By using the analytic properties of retarded and advanced Green functions, we can remove the quadratic term by a shift \( Q(\omega) \to Q(\omega) - CT(\omega) \). In the low energy limit, using free propagators: \( G^0_H \sim G^0_A \sim \omega^2 - r, G^0_K \sim 0 \), we get:

\[ S_{\text{sp}} = \int \frac{d\omega}{2\pi} (\omega^2 - r)(Q_R + Q_A) + \frac{\kappa}{3} \int \frac{d\omega}{2\pi} [Q_R^3 + Q_A^3] \]

\[ - i u(1 + M) \int \frac{d\omega}{2\pi} [Q_R + Q_A] \int \frac{d\omega}{2\pi} Q_K + \ldots. \]  

(68)

Solution in the Paramagnetic phase

Minimizing Equation (68) with respect to \( Q_R \) and \( Q_A \), gives

\[ \omega^2 - r + Q_R^2 - i u(1 + M) \int \frac{d\omega}{2\pi} Q_K = 0, \]

(69)

\[ \omega^2 - r + Q_A^2 - i u(1 + M) \int \frac{d\omega}{2\pi} Q_K = 0. \]  

(70)
The two equations are complex conjugates of each other, therefore there is only one independent equation. Solving for $Q_R$ gives

$$Q_R^2 = - (\omega^2 - \tilde{r}) , \quad \tilde{r} = r - u (1 + M) \int \frac{d\omega}{2\pi} \coth \frac{\beta \omega}{2} \chi''(\omega) , \quad \chi''(\omega) = \text{sgn}(\omega)(\omega^2 - \tilde{r})^{1/2} \theta(\omega - \sqrt{\tilde{r}}) .$$

The critical point is located at $\tilde{r} = 0$ where $r = r_c(T) = u (1 + M) \int \frac{d\omega}{2\pi} \omega \coth \frac{\beta \omega}{2}$. These results are the same as those obtained by the replica approach in Ref. [3].

V. GLASSY PHASE

In the spin glass phase the correlation and response are no longer time translation invariant. Instead, there is an aging piece that remains a function of two times. Below we investigate the dynamical behavior of the correlation and response functions for the quantum rotors in the glassy phase.

A. Weak ergodicity breaking and weak long term memory

Weak ergodicity breaking (WEB) and weak long term memory (WLTM) are phenomena that have been observed in the solutions to classical mean field spin glass models. The numerical results for the quantum version of the $p$-spin model are in agreement with those predicted from these scenarios, and we also assume WEB and WLTM in our solution of the quantum rotor model. WEB can be summarized as follows: the correlation and response functions behaves as the sum of a TTI piece and a piece which depends on two times

$$C(t_1, t_2) = C_{ST}(t_1 - t_2) + C_{AG}(t_1, t_2) .$$

The stationary (TTI) piece decays to zero as $t_1 - t_2 \to \infty$, and the aging piece satisfies $C_{AG}(t, t) = q$, and

$$\lim_{t_1 \to \infty} C_{AG}(t_1, t_2) = 0 ,$$

so that both times are important in the decay of the aging piece of the correlation. We separate the response in a similar way to the correlation

$$R(t_1, t_2) = R_{ST}(t_1 - t_2) + R_{AG}(t_1, t_2) .$$

The stationary piece decays to zero as $t_1 - t_2 \to \infty$, whilst the decay of the aging piece depends on both times. The assumption of WLTM applies to the integral of the response function: the integral of the response over any finite time interval vanishes, i.e.

$$\lim_{t_1 \to \infty} \int_0^{t_1} dt_2 R(t_1, t_2) = 0 .$$

for fixed $t$, however the integral over an interval that grows with time is finite

$$\lim_{t_1 \to \infty} \int_0^{t_1} dt_2 R(t_1, t_2) \neq 0 .$$

B. Modified QFDT

Treating the spin glass regime in a similar way to the paramagnetic phase, and looking in the limit $t_1 \to \infty$, it is possible to get an equation that includes the TTI pieces, and also some static pieces due to a non-zero Edwards-Anderson order parameter.

In the spin glass phase, we need to take into account the aging that has occurred up until the waiting time.

To do this we split $R$ and $C$ into stationary and aging pieces before letting $t_1 \to \infty$ and then Fourier transforming. The relation we find between the stationary parts of correlation and response is

$$\text{Im}[R_{ST}(\omega)] = \frac{1}{\hbar} \tanh \left( \frac{\beta \hbar \omega}{2} \right) C_{ST}(\omega) + \frac{2\pi \beta J^2}{\gamma} q (\chi_{\infty} + (RC)_{\infty}) \delta(\omega) ,$$

where

$$\chi_{\infty} = \lim_{t_1 \to \infty} \int_0^{t_1} dt R_{ST}(t_1 - t) ,$$

and $(RC)_{\infty}$ is (noting that $t_1 \sim t_2 \gg |t_1 - t_2|$ in the evaluation of the integral)

$$(RC)_{\infty} = \lim_{t_1 \to \infty} \int_0^{t_1} dt R_{AG}(t_1, t) (C_{AG}(t_1, t_1) + C_{AG}(t_1, t)) .$$

Equation (29) shows how the QFDT is modified in the spin glass phase for the rotor model. There is the usual piece that is present in the paramagnetic phase, and then a delta function at zero frequency that grows with decreasing temperature.

Modified QFDT within the dynamical Landau theory

As in the treatment above, in the spin glass state, the Keldysh component $Q_K$ acquires a non-trivial $\delta$ function part due to a nonzero Edwards-Anderson order parameter

$$iQ_K = iQ_K^{\text{F}} + 2\pi q \delta(\omega) .$$

Substituting the ansatz into Equation (79) gives
\[
\omega^2 - r + Q_R^2 - iu(1 + M) \int \frac{d\omega}{2\pi} Q_K^{reg} - (u + Mv)q = 0,
\]
and solving for \(Q_R\) and \(q\) leads to
\[
Q_R = -i\omega, \quad (84)
\]
\[
q = \frac{1}{u(1 + M)}(r_c(T) - r). \quad (85)
\]
Again, these results are the same as those obtained by the replica approach in Ref. 43.

C. Aging regime

Aging behaviour has been shown to be a feature in the long time dynamics of many glassy systems, both in experiments and theoretical models (see Ref. 22 for a review). In the case of spin glasses, the major models that have been studied in the aging regime are the Sherrington-Kirkpatrick (SK) model \(^2\), the spherical \(p\)-spin model \(^2\), and the quantum version of the spherical \(p\)-spin model \(^2\). In our work we consider the quantum version of the rotor model, which, in the absence of a magnetic field, is similar to the soft spin SK model with quantum dynamics rather than Langevin dynamics, having \(M\) components instead of only one.

One of the features of equilibrium solutions of mean field spin glass models is replica symmetry breaking, introduced by Parisi \(^4\) in the context of the SK model. Cugliandolo and Kurchan \(^4\) started from a set of equations for the long time dynamics of the SK model and showed that the correlations have an ultrametric structure analogous to that found in the Parisi RSB solution of the SK model. This ultrametric structure in the long time dynamics can be obtained in the model here only by including terms to \(O(u^2)\) in the dynamical equations – there is a direct analogy with the equilibrium solution of quantum rotors \(^4\), where RSB occurs when \(O(u^2)\) terms are included in the equilibrium solutions. (The approach of Read, Sachdev, and Ye was slightly different from ours, since they expressed all terms in the action in terms of the Hubbard-Stratonovich field \(Q^{ab}\), rather than working directly with the spin correlations, as we do here).

An important feature in the aging regime is the occurrence of triangle relations between correlations at different time scales. These triangle relations have an ultrametric structure analogous to that found in RSB. At sufficiently large times \(t_1\) and \(t_2\), it is possible to express the correlation at intermediate times, \(t_2 < t < t_1\), in terms of a function \(f\), which depends only on the correlations and has no explicit time dependence. Mathematically,
\[
C(t_1, t_2) = f[C(t_1, t), C(t, t_2)], \quad (86)
\]
and there is an inverse function \(\tilde{f}\)
\[
C(t_1, t) = \tilde{f}[C(t_1, t_2), C(t_2, t)]. \quad (87)
\]
Cugliandolo and Kurchan \(^4\) give a complete account of these triangle relations and the properties of the functions \(f\) and \(\tilde{f}\). The function \(f\) can have fixed points \(a\), such that \(a = f(a, a)\). Each of these fixed points constitutes a correlation scale. We make use of this approach to calculate the FDT violation factor \(X\), in the long time dynamics of the quantum rotor model.

The saddle point equations are written down to \(O(u^2)\) in Section 11. To obtain the appropriate equations in the long time regime, the response must be split into a stationary part and an aging piece, as in Equations (74) and (76). At long times, the contributions to the dynamical equations from the bath kernels should be negligible, since the system has been in contact with the bath sufficiently long for one time quantities to have reached a limit. What is studied here is how the two time quantities evolve after the interaction with the bath has ceased. We assume WEB and WTM, and treat integrals in the manner explained by Cugliandolo and Lozano \(^4\). Hence the equations we wish to solve are (for \(t_1 \neq t_2\))
\[
0 = -\frac{1}{2\lambda} \frac{\partial^2}{\partial t_1^2} R(t_1, t_2) + \lambda_1 R(t_1, t_2)
+ \lambda_2 R(t_1, t_2)C(t_1, t_2)^3 - \frac{1}{3} \lambda_3 R(t_1, t_2)^3
+ 2J^2 \int_0^\infty dt R(t_1, t)R(t, t_2)
+ \lambda_4 \int_0^\infty dt C(t_1, t)^2 R(t_1, t)R(t, t_2)
- \frac{1}{3} \lambda_5 \int_0^\infty dt R(t_1, t)^3 R(t, t_2), \quad (88)
\]
and
\[
0 = -\frac{1}{2\lambda} \frac{\partial^2}{\partial t_2^2} C(t_1, t_2) + \lambda_3 C(t_1, t_2) + \lambda_2 R(t_1, t_2)
+ \frac{1}{3} \lambda_3 C(t_1, t_2)^3 - \frac{1}{9} \lambda_2 R(t_1, t_2)^3
+ \lambda_4 \int_0^\infty dt C(t_1, t)^2 R(t_1, t)R(t, t_2)
+ \int_0^\infty dt C(t_1, t)^3 R(t, t_2)
+ \lambda_4 \int_0^\infty dt C(t_1, t)^2 R(t_1, t)C(t, t_2)
- \frac{1}{3} \lambda_4 \int_0^\infty dt [3R(t_1, t)^2 R(t_2, t)C(t_1, t)
+ R(t_1, t)^3 C(t_1, t)], \quad (89)
\]
where we drop the aging subscript (since the only parts of the correlation and response we work with here are the aging pieces). The values for the coefficients are displayed in Appendix B.

The equations derived here are very similar to those that have been written down for the soft spin SK model.
If we assume that at sufficiently long times only terms that are invariant under time reparametrizations remain (more explanation is given in Section 6), then the dynamical equations take the form

\[
0 = \lambda_1 R(t_1, t_2) + \lambda_2 C(t_1, t_2)^2 R(t_1, t_2) \\
+ 2J^2 \int_0^\infty dt \, R(t_1, t) R(t, t_2) \\
+ \lambda_4 \int_0^\infty dt \, C(t_1, t)^2 R(t_1, t) R(t, t_2),
\]

(90)

\[
0 = \lambda_1 C(t_1, t_2) + \frac{1}{3} \lambda_2 C(t_1, t_2)^3 \\
+ 2J^2 \int_0^\infty dt \, [C(t_1, t) R(t_2, t) + R(t_1, t) C(t, t_2)] \\
+ \frac{1}{3} \lambda_4 \int_0^\infty dt \, C(t_1, t)^3 R(t, t_2) \\
+ \lambda_4 \int_0^\infty dt \, C(t_1, t)^2 R(t_1, t) C(t, t_2).
\]

(91)

None of the terms with coefficients proportional to \( h^2 \) enter into these asymptotic equations, hence the dependence on quantum effects is only through the coefficients \( \lambda_1, \lambda_2 \), and \( \lambda_4 \). There are also classical terms (in the sense that they do not have a coefficient proportional to \( h^2 \)) that are not reparameterization invariant.

The equations above have extra terms relative to the terms considered by Cugliandolo and Kurchan, which are those with coefficient \( \lambda_4 \) – we can recover their results when these terms are unimportant. The reason that we have these terms is that we include all terms of \( O(u^3) \) before looking at the long time limit. Following Cugliandolo and Kurchan we can convert the equations into a manifestly time reparameterization invariant form by introducing functionals \( F(C) \) and \( H(C) \)

\[
F[C] = - \int_C dC' X[C']^a, \\
H[C] = - \int_C dC' C' A X[C'],
\]

(92)

and postulating a modified version of the FDT to relate the correlation and response

\[
R(t_1, t_2) = \beta X[C(t_1, t_2)] \frac{\partial}{\partial t_2} C(t_1, t_2) \theta(t_1 - t_2), \\
= \beta \frac{\partial}{\partial t_2} F[C(t_1, t_2)] \theta(t_1 - t_2).
\]

(94)

The relation (94) is also invariant under time reparameterization. We follow a similar reasoning to Ref. 40 to show that within an ultrametric scale

\[
X[a^*_1] = \frac{\lambda_2 a^*_1}{\beta} \frac{(2J^2 + \lambda_4 q^2)^\Delta_1}{(2J^2 + \lambda_4 q^2)^\Delta_2}.
\]

(95)

We also find that within an ultrametric scale \( X[C] = X[a^*_1] \), and that the scales have vanishing measure and there is thus a continuous ultrametric solution at correlation scale \( a^*_1 \), provided the following condition is satisfied

\[
\frac{2x_1 F[a^*_1] - x_2 - \lambda_2 a^*_1}{\lambda_2 a^*_1 + \beta \lambda_4 F[a^*_1]}(2a^*_1 X[a^*_1] - F[a^*_1]) > 0,
\]

(96)

where

\[
x_1 = \beta(2J^2 + \lambda_4 q^2), \\
x_2 = \lambda_1 + 2 \lambda_4 \beta \int_0^t d\tau \, C' F[C'],
\]

(97)

\[
F[a^*_1] = \frac{\lambda_2}{\beta \lambda_4} \left[ 1 - \sqrt{\frac{2J^2 + \lambda_4 q^2}{2J^2 + \lambda_4 q^2}} \right].
\]

(99)

The results here reduce to those found earlier for the classical SK model in the limit that \( \lambda_4 \to 0 \). Hence the solution for the FDT violation factor \( X[C] \) is

\[
X[C] = \frac{\lambda_2 C}{\beta} \frac{(2J^2 + \lambda_4 q^2)^\Delta_1}{(2J^2 + \lambda_4 q^2)^\Delta_2}.
\]

(100)

Note that if we define \( q \) to be the value of the correlation for which \( X[C] = 1 \), then this implies \( q = \beta/\lambda_2 = \frac{\lambda_2}{\lambda_2} \beta/(u^2 \lambda_\infty) \).

### D. Reparameterization symmetry

In deriving Equations (90) and (91) we assumed that the equations at long times are reparameterization invariant. As shown recently this assumption is justified, since at long times the dynamical equations governing the system flow to a “fixed point” of the reparameterization group at which the equations are invariant under time reparameterization transformations. The time reparameterizations are transformations of the form

\[
t \to \tilde{t} = h(t),
\]

(101)

where \( h(t) \) is a differentiable function with \( dh/dt \geq 1 \) (so as to stretch time). Transforming time also leads to transformations of the correlation functions – for a two-time correlation function \( G(t_1, t_2) \), we define a transformation \( G \to \tilde{G} \) such that

\[
\tilde{G}(t_1, t_2) = \left( \frac{dh}{dt_1} \right)^\Delta_1^G \left( \frac{dh}{dt_2} \right)^\Delta_2^G G[h(t_1), h(t_2)],
\]

(102)

where \( \Delta_1^G \) and \( \Delta_2^G \) the advanced and retarded scaling dimensions respectively. The “fixed point” dynamical equations are a set of equations which are invariant under an Rpq transformation, i.e. if they are satisfied by \( G \), then they are also satisfied by \( \tilde{G} \). The idea of irrelevancy under Rpq transformations, introduced in Ref. 40 where certain terms in the long time limit scale to progressively negligible perturbations about an Rpq fixed point, is the means used to obtain the dynamical equations at long times.
We have studied spin glasses of quantum rotors from a purely dynamical perspective. In the dynamical approach, the phase transition between the paramagnetic and glassy phase can be identified by looking at the critical slowing down of the dynamics. Alternatively, we also identify the transition at the breakpoint where it is no longer possible to satisfy the QFDT relation between correlators and response functions in the solution of the equations of motion. We find that, if one insists on a TTI solution, it is necessary to include a singular ($\delta$ function) piece in the correlation to balance the fluctuation dissipation relation. The amplitude of this singular term is proportional to $q$, and this simple minded TTI solution is the analog of the usual replica symmetric equilibrium solution.

For the glassy phase, we study the non-equilibrium dynamics in the aging regime, where TTI is broken. We find that the relation between correlation and response for the quantum rotor model has the same character as in the classical case — specifically the OEFDR has the same form, but different coefficients from the classical case. We show that all of the terms in the dynamical equations that are only present in the quantum version of the model (with an explicit $\hbar$ dependence) are not invariant under time reparametrizations, but instead they are RpG irrelevant and do not contribute in the long-time limit. The quantum terms, however, alter the short-time dynamics and consequently renormalize the coefficients of the classical (RpG invariant) terms in the long-time dynamical equations.

These findings help us understand rather more precisely how the behaviour of a spin glass system is modified by the introduction of quantum dynamics. Bhatt has argued that the scalings at the spin glass transition should be the same as in classical models, due to the different timescales involved — $\beta T$ is the timescale for the quantum case (this corresponds to a frequency of $\omega_q = 8 \times 10^{11} \text{Hz}$, where $T$ is the temperature in Kelvin), which at finite temperature will be much shorter than other relevant time scales in the problem. Another way to argue this is to look at the QFDT, which in the low frequency (long time) limit corresponds to the classical FDT. However, the glassy systems we consider here are never at equilibrium, and hence the long time limit in the non-equilibrium system is less obvious. One suggestion is that the QFDT should be generalized in a manner similar to the generalization to an OEFDR that differs from the classical equations of motion appears to be a generic feature of classical spin glass models. The invariance of the SK model was noticed by several authors, and we naturally encounter it in the rotor model as well. The form of the OEDFR seems to be a generic consequence of the retarded and advanced scaling dimensions of $C, R$ which in turn may be fixed by the form of the FDT. In this sense, it would be interesting to discover whether it is possible to construct different RpG fixed points, with different scaling dimensions.

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APPENDIX A: TTI SOLUTIONS FOR THE RESPONSE

One can attempt a solution for the response of the form $R(\omega) = P(\omega) + iQ(\omega)$. It is then possible to solve for $P$ and $Q$ and determine the quantum critical point, by writing $x(\omega) = a(\omega) + ib(\omega)$. The solution is

$$R(\omega) = -\frac{1}{4J^2}(a + ib) \pm \frac{1}{4J^2} \sqrt{(a^2 - b^2 - 8J^2)^2 + 4a^2b^2},$$

(A1)

which can be separated into real and imaginary parts:

$$P(\omega) = -\frac{a}{4J^2} \pm \frac{1}{4J^2} \left( (a^2 - b^2 - 8J^2)^2 + 4a^2b^2 \right)^{1/2} \times \cos \left( \tan^{-1} \left( \frac{2ab}{a^2 - b^2 - 8J^2} \right) \right),$$

(A2)
\[ Q(\omega) = -\frac{b}{4J^2} + \frac{1}{4J^2} \left( (a^2 - b^2 - 8J^2)^2 + 4a^2b^2 \right)^{1/2} \times \sin \left( \tan^{-1} \left( \frac{2ab}{\sqrt{a^2 - b^2 - 8J^2}} \right) \right). \]  

(A3)

We obtain the critical point when the square roots in the arguments of the sine and cosine vanish.

APPENDIX B: COEFFICIENTS FOR THE AGING EQUATIONS

The coefficients that appear for the dynamical equations in the aging regime are summarized below:

\[ \lambda_1 = \frac{1}{2} m^2 + u(1 + M)q + 2(1 + M)u^2B_\infty + 4J^2\chi_\infty \]
\[ + \frac{3}{2}(1 + M)u^2\gamma_\infty^{21} - \frac{1}{2}(1 + M)u^2h^2\gamma_\infty^{03}, \quad (B1) \]
\[ \lambda_2 = \frac{3}{2}(1 + M)u^2\chi_\infty, \quad (B2) \]
\[ \kappa_2 = 2J^2\gamma_\infty^{10}, \quad (B3) \]
\[ \lambda_4 = \frac{3}{2}(1 + M)u^2, \quad (B4) \]
\[ \kappa_4 = \frac{3}{2}(1 + M)u^2\gamma_\infty^{10}, \quad (B5) \]

and

\[ \gamma_\infty^{mn} = \lim_{t_1 \to +\infty} \int_0^{t_1} dt C_{ST}(t_1 - t)^m R_{ST}(t_1 - t)^n, \]
\[ B_\infty = \gamma_\infty^{11} + q\chi_\infty + \lim_{t_1 \to -\infty} \int_0^{t_1} dt R_{AG}(t_1, t)C_{AG}(t_1, t). \quad (B6) \]
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