Quadratic invariants of the elasticity tensor

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Abstract

We study the quadratic invariants of the elasticity tensor in the framework of its unique irreducible decomposition. The key point is that this decomposition generates the direct sum reduction of the elasticity tensor space. The corresponding subspaces are completely independent and even orthogonal relative to the Euclidean (Frobenius) scalar product. We construct a basis set of seven quadratic invariants that emerge in a natural and systematic way. Moreover, the completeness of this basis and the independence of the basis tensors follow immediately from the direct sum representation of the elasticity tensor space. We define the Cauchy factor of an anisotropic material as a dimensionless measure of a closeness to a pure Cauchy material and a similar isotropic factor is as a measure for a closeness of an anisotropic material to its isotropic prototype. For cubic crystals, these factors are explicitly displayed and cubic crystal average of an arbitrary elastic material is derived.

Key index words: anisotropic elasticity tensor, irreducible decomposition, quadratic invariants

1 Introduction

In the linear elasticity theory of anisotropic materials, the relation between the strain tensor \( u_{ij} \) and the stress tensor \( \sigma^{ij} \), the generalized Hooke’s law, is expressed by the use of the elasticity (stiffness) tensor \( C^{ijkl} \)

\[ \sigma^{ij} = C^{ijkl} u_{kl}. \] (1)

In 3-dimensional space, a generic 4th order tensor has 81 independent components. However, due to the standard symmetry assumptions for the stress and strain tensors,

\[ C^{ijkl} = C^{ijjk} = C^{klij}, \] (2)

the elasticity tensor is left with 21 independent components only. These components are not really the intrinsic characteristics of the material because they depend on the choice of the coordinate system. Thus, in order to deal with the proper material parameters, one must look for the invariants of the elasticity tensor.
There are only two linearly independent invariants of the first order of $C_{ijkl}$. They are usually taken as follows, see [20],
\[ A_1 = C_{ij}^{ij} = C_{ijij} \quad \text{and} \quad A_2 = C_{ij}^{ij} = C_{iijj}. \] (3)
Here and subsequently, we use the standard tensor conventions that strictly distinguish between covariant and contravariant indices.

Quadratic invariants of the elasticity tensor were studied by Ting [25]. He presented two such invariants,
\[ B_1 = C_{ijkl}^{ijkl} = C_{ijkl}^{ijkl} \quad \text{and} \quad B_2 = C_{ijkl}^{ijkl} = C_{ijkl}^{ijkl}. \] (4)

Ahmad [1] has contributed the two additional quadratic invariants,
\[ B_3 = C_{ijkl}^{ijkl} = C_{ijkl}^{ijkl} \quad \text{and} \quad B_4 = C_{ijkl}^{ijkl} = C_{ijkl}^{ijkl}. \] (5)
He also proved that the set of seven quadratic invariants
\[ \{ A_1^2, A_2^2, A_1 A_2, B_1, B_2, B_3, B_4 \} \] (6)
is linearly independent. Norris [20] studied the problem of the quadratic invariants and proved that the set (6) is complete. It means that every quadratic invariant of the elasticity tensor is a linear combination of the seven invariants listed in (6).

In order to prove this fact, Norris presented a generic quadratic invariant in the form
\[ I = f_{ijklpqrs} C_{ijkl}^{ijkl} C_{pqrs}, \] (7)
where $f_{ijklpqrs}$ is a numerical tensor and provided a detailed analysis of this tensor. It is proven that due to the symmetries (2), the most components of $f_{ijklpqrs}$ vanish and a lot of the remaining components are linearly related. This way, exactly seven invariants are left over. In particular, Norris demonstrated that a rather natural additional invariant
\[ B_5 = C_{ijkl}^{ijkl} = C_{ijkl}^{ijkl} \] (8)
can be in fact represented as a linear combination of the invariants listed in (6), namely
\[ B_5 = \frac{1}{2} A_1^2 + \frac{1}{2} A_2^2 - A_1 A_2 + B_1 - 2 B_2 + 4 B_3 - 2 B_4. \] (9)

In this situation, some principal questions arise:

- Is there some preferable basis of quadratic invariants?
- Is there a systematic way to construct such a basis?
• Which quadratic invariants can be used as characteristic parameters for different elastic materials?

• Which physical interpretation can be given to various quadratic invariants of the elasticity tensor?

In the current paper, we analyze the quadratic invariants problem in the framework of the unique irreducible decomposition of the elasticity tensor. In Section 2, we present the principle algebraic facts of this decomposition. The key point that such resolution of the elasticity tensor in the simple pieces generates the direct sum reduction of the elasticity tensor space. The corresponding subspaces are completely independent and even orthogonal relative to the Euclidean (Frobenius) scalar product. In this framework, we construct in Section 3 the basis set of seven quadratic invariants that emerge in a natural and systematic way. Moreover the completeness of this basis and its independence of basis tensors follow immediately from the direct sum representation of the elasticity tensor space. We compare this basis to the basis given in (6). In section 4, we provide some applications of quadratic invariants to physics motivated problems. We define Cauchy factor for an arbitrary anisotropic material. It can be used as a measure of deviation of a material from an analogical Cauchy material. We also prove Fedorov’s relation for an isotropic material closest to an anisotropic one. This result follows immediately from the irreducible orthogonal decomposition. Correspondingly, we define the isotropic factor that measures the closeness of an anisotropic material to its isotropic prototype. In section 5, we study a simplest non-trivial example of cubic crystal. It is naturally represented by three independent quadratic invariants. This low-dimensional case allows the visualization of the Cauchy and isotropic factors. Corresponding graphs are presented. In Conclusion section we present our main results and propose some possible direction for future investigations.

2 Irreducible decomposition

To describe the irreducible decomposition of the elasticity tensor we first observe two groups acting on it simultaneously, see [15]. For an arbitrary tensor of the range \( p \) defined on \( \mathbb{R}^n \), these are the permutation (symmetry) group \( S_p \) and the group of rotations \( SO(n, \mathbb{R}) \).

2.1 Irreducible decomposition under the permutation group

The symmetry group \( S_4 \) provides permutations of the elasticity tensor indices. The decomposition of \( C^{ijkl} \) under this group is described by two Young diagrams:

\[
\begin{array}{c}
\bigotimes \\
\bigotimes \\
\bigotimes \\
\bigotimes
\end{array} \oplus \begin{array}{c}
\bigotimes \\
\bigotimes \\
\bigotimes \\
\bigotimes
\end{array} = \begin{array}{c}
\bigotimes \\
\bigotimes \\
\bigotimes \\
\bigotimes
\end{array} \oplus \begin{array}{c}
\bigotimes \\
\bigotimes \\
\bigotimes \\
\bigotimes
\end{array}.
\]

(10)

All other 4-the order Young’s diagrams are non-relevant in our case due to the original symmetries \( [2] \) of \( C^{ijkl} \). The left-hand side of (10) represents a generic 4th rank tensor. On the right-hand side, two diagrams describe two tensors of different symmetries. The explicit expression of these tensors can be computed by the use of the corresponding symmetrization and antisymmetrization operations. For the elasticity tensor, the result is obvious. The first (row) diagram represents the totally symmetric tensor. The second (square) diagram represents an additional tensor that can be considered merely as a remainder.
As a result, we arrive at the unique and irreducible decomposition of the elasticity tensor under the action of the group $S_4$:

$$C^{ijkl} = S^{ijkl} + A^{ijkl},$$

with

$$S^{ijkl} := C^{(ijkl)} = \frac{1}{3} \left( C^{ijkl} + C^{iklj} + C^{iljk} \right)$$

and

$$A^{ijkl} := C^{ijkl} - C^{(ijkl)} = \frac{1}{3} \left( 2C^{ijkl} - C^{ilkj} - C^{iklj} \right).$$

We use the standard normalized Bach parentheses for symmetrization and antisymmetrization of indices. We observe that every antisymmetrization of $S^{ijkl}$ gives zero and every symmetrization preserves this tensor. As for the second part $A^{ijkl}$, its total symmetrization $A^{(ijkl)}$ vanishes. Consequently, we have a useful symmetry relation

$$A^{ijkl} + A^{iklj} + A^{iljk} = 0.$$  

As it is shown in [13] and [14], the equation $A^{ijkl} = 0$ describes the well-known Cauchy relation. Thus, we call $S^{ijkl}$ the Cauchy part and $A^{ijkl}$ the non-Cauchy part of the elasticity tensor.

Observe the main algebraic properties of this decomposition:

- The partial tensors $S^{ijkl}$ and $A^{ijkl}$ satisfy the minor symmetries,

$$S^{[ij][kl]} = S^{ijkl} = 0 \quad \text{and} \quad A^{[ij][kl]} = A^{ijkl} = 0,$$

and the major symmetry,

$$S^{ijkl} = S^{klij} \quad \text{and} \quad A^{ijkl} = A^{klij}.$$  

- The partial tensors can themselves serve as elasticities of some hypothetic material.

- Moreover, any additional symmetrization or antisymmetrization preserves the tensors $S^{ijkl}$ and $A^{ijkl}$ or nullifies them.

- The decomposition $(11)$ is preserved under arbitrary linear transformations. Thus, it can be referred to as irreducible $GL(3, \mathbb{R})$-decomposition.

- The irreducible decomposition of the tensor $C$ provides the decomposition of the corresponding tensor space $\mathcal{C}$ into a direct sum of two subspaces $S \subset \mathcal{C}$ (for the tensor $S$) and $A \subset \mathcal{C}$ (for the tensor $A$),

$$\mathcal{C} = S \oplus A.$$  

In particular, we have

$$\dim \mathcal{C} = 21, \quad \dim S = 15, \quad \dim A = 6.$$  

- The irreducible pieces $S^{ijkl}$ and $A^{ijkl}$ are orthogonal to one another in the following sense:

$$S^{ijkl}A^{ijkl} = 0.$$  

Indeed,

$$S^{ijkl}A^{ijkl} = S^{(ijkl)}A^{ijkl} = S^{(ijkl)}A^{(ijkl)} = 0.$$
• “Pythagorean theorem:” The Euclidean (Frobenius) squares of the tensors
\[ \tilde{C}^2 = C^{ijkl} C_{ijkl}, \quad \tilde{S}^2 = S^{ijkl} S_{ijkl}, \quad \tilde{A}^2 = A^{ijkl} A_{ijkl} \] (21)
satisfy the relation
\[ \tilde{C}^2 = \tilde{S}^2 + \tilde{A}^2. \] (22)

2.2 Irreducible decomposition under the rotation group

We are looking now for the decomposition of the elasticity tensor under the action of the group of rotations. The following fact is due to the classical theory of invariants: Relative to the subgroup \( SO(3) \) of \( GL(3) \), a basis of an arbitrary system of tensors coincides with a basis of the same system with the metric tensor added \[27\]. We use the Euclidean metric tensor \( g_{ij} \). In rectangular coordinates, \( g_{ij} = \text{diag}(1, 1, 1) \).

We start with the totally symmetric Cauchy part \( S^{ijkl} \). From the contraction of \( S^{ijkl} \) with the metric tensor, we construct a unique symmetric second-rank tensor
\[ S^{ij} := g_{kl} S^{ijkl} = S_{ij} = \frac{1}{3} (C_{ijkl} + 2C_{iklj}) \] (23)
and a unique scalar
\[ S := g_{ij} S^{ij} = S_{ijkl} = \frac{1}{3} (C_{ijkl} + 2C_{ikkl}). \] (24)

We denote the traceless part of the tensor \( S^{ij} \) as
\[ P^{ij} := S^{ij} - \frac{1}{3} S g^{ij}, \quad \text{with} \quad g_{ij} P^{ij} = 0. \] (25)

Now we turn to the decomposition of the tensor \( S^{ijkl} \). We denote the two subtensors
\[ (1) S^{ijkl} := \alpha S g^{ijkl}, \quad (2) S^{ijkl} := \beta P^{ijkl}. \] (26)

It can be checked now by the straightforward calculations that the remainder
\[ R^{ijkl} := S^{ijkl} - (1) S^{ijkl} - (2) S^{ijkl} \] (27)
is totally traceless if and only if
\[ \alpha = \frac{1}{5}, \quad \beta = \frac{6}{7}. \] (28)

Hence, we obtain the decomposition of the totally symmetric tensor \( S^{ijkl} \) into the sum of three independent pieces:
\[ S^{ijkl} = (1) S^{ijkl} + (2) S^{ijkl} + (3) S^{ijkl}, \] (29)
where
\[ (1) S^{ijkl} = \frac{1}{5} S g^{ijkl} = \frac{1}{15} S \left( g^{ij} g^{kl} + g^{ik} g^{jl} + g^{il} g^{jk} \right). \] (30)
\[ (2) S^{ijkl} = \frac{6}{7} P^{ijkl} = \frac{1}{7} \left( P^{ij} g^{kl} + P^{ik} g^{jl} + P^{il} g^{jk} + P^{jk} g^{il} + P^{jl} g^{ik} + P^{kl} g^{ij} \right). \] (31)
and
\[ (3) S^{ijkl} = R^{ijkl}. \] (32)
These pieces are unique and invariant under the action of the group $SO(3)$. Moreover, the corresponding subspaces $(1)\mathcal{S}$, $(2)\mathcal{S}$ and $(3)\mathcal{S}$ are mutually orthogonal. Indeed, for $A, B = 1, 2, 3$ with $A \neq B$,

$$^{(A)}S_{ijkl}^{(B)}S_{ijkl} = 0 .$$  \hspace{1cm} (33)

This fact follows immediately from the tracelessness of the tensors $P^{ij}$ and $R^{ijkl}$.

Consequently, the vector space $\mathcal{S}$ of the totally symmetric tensor $S^{ijkl}$ is decomposed into the direct sum of three subspaces

$$\mathcal{S} = (1)\mathcal{S} \oplus (2)\mathcal{S} \oplus (3)\mathcal{S}$$

with the corresponding dimensions

$$15 = 1 \oplus 5 \oplus 9 .$$  \hspace{1cm} (34)

We turn now to the second part of the elasticity tensor. The irreducible piece $A^{ijkl}$ is a fourth rank tensor with 6 independent components. It is quite naturally that it can be represented as a symmetric second-rank tensor, see [3], [12], [13], and [14] for detailed discussions. We define

$$\Delta_{mn} := \frac{1}{3} \epsilon_{mil} \epsilon_{njk} A^{ijkl} ,$$

where $\epsilon_{ijk} = 0, \pm 1$ denotes the 3-dimensional Levi-Civita permutation pseudo-tensor. Consequently, $\Delta_{mn}$ is a symmetric tensor, $\Delta_{mn} = \Delta_{nm}$, and we have

$$A^{ijkl} = \epsilon^{im(k} \epsilon^{l)jn} \Delta_{mn} .$$  \hspace{1cm} (37)

For a proof of this proposition it is enough to substitute (36) into (37) and to apply the standard relations for the Levi-Civita pseudo-tensor.

In order to decompose the non-Cauchy part $A^{ijkl}$, it is convenient to use its representation by the tensor density $\Delta_{ij}$. We denote

$$A := g^{mn} \Delta_{mn} .$$  \hspace{1cm} (38)

By using the relation

$$g^{mn} \epsilon_{mil} \epsilon_{njk} = g_{ijkl} g_{lk} - g_{ik} g_{jl}$$  \hspace{1cm} (39)

and Eq. (37), we derive

$$A = \frac{1}{3} (g_{ijkl} g_{lk} - g_{ik} g_{jl}) A^{ijkl} \equiv \frac{1}{3} (A_{ikkk} - A_{ikik}) = \frac{1}{3} (C_{ikkk} - C_{ikik}) .$$  \hspace{1cm} (40)

The tensor density $\Delta_{ij}$ can be decomposed into the scalar and traceless pieces:

$$\Delta_{ij} = Q_{ij} + \frac{1}{3} A g_{ij} ,$$

where the traceless piece is given by

$$Q_{ij} := \Delta_{ij} - \frac{1}{3} A g_{ij} .$$  \hspace{1cm} (41)

Substituting (41) into (37), we obtain

$$A^{ijkl} = (1)A^{ijkl} + (2)A^{ijkl} ,$$  \hspace{1cm} (43)
where the scalar part is given by

\[
(1) A_{ijkl} := \frac{1}{6} A \left( 2g^{ij} g^{kl} - g^{il} g^{jk} - g^{ik} g^{jl} \right)
\] (44)

and the remainder reads

\[
(2) A_{ijkl} := \frac{1}{2} \left( \epsilon^{ikm} \epsilon^{jln} + \epsilon^{ilm} \epsilon^{jkn} \right) Q_{mn}.
\] (45)

We recall that the tensor \( Q_{mn} \) is symmetric and traceless. Since the product of two Levi-Civita pseudo-tensors is represented by the determinant of the metric tensor \( g_{ij} \), the latter equation can be rewritten as

\[
(2) A_{ijkl} := g^{ik} Q_{jl} + g^{jk} Q_{il} + g^{il} Q_{jk} + g^{jl} Q_{ik} - 2g^{kl} Q_{ij} - 2g^{ij} Q_{kl}.
\] (46)

The decomposition given in Eq. (43) is unique, invariant, and irreducible under the action of the rotation group \( SO(3, \mathbb{R}) \) and of the permutation group \( S_4 \).

Correspondingly, the vector space \( \mathcal{A} \) of the tensor \( A_{ijkl} \) is irreducibly decomposed into the direct sum of two subspaces

\[
\mathcal{A} = (1) \mathcal{A} \oplus (2) \mathcal{A},
\] (47)

with the corresponding dimensions

\[
6 = 1 \oplus 5.
\] (48)

Since the trace of \( (2) A_{ijkl} \) equals zero, these subspaces are orthogonal to one another,

\[
(1) A_{ijkl} (2) A_{ijkl} = 0.
\] (49)

Collecting our results, we formulate the following

**Theorem 1.** Under the simultaneous action of the groups \( S_4 \) and \( SO(3, \mathbb{R}) \), the elasticity tensor is uniquely irreducibly decomposed into the sum of five parts

\[
C_{ijkl} = \sum_{A=1}^{5} (A) C_{ijkl} = (1) S_{ijkl} + (2) S_{ijkl} + (3) S_{ijkl} + (1) A_{ijkl} + (2) A_{ijkl}.
\] (50)

This decomposition corresponds to the direct sum decomposition of the vector space of the elasticity tensor into five subspaces

\[
\mathcal{C} = (1) \mathcal{C} \oplus (2) \mathcal{C} \oplus (3) \mathcal{C} \oplus (4) \mathcal{C} \oplus (5) \mathcal{C},
\] (51)

with the dimensions

\[
21 = (1 \oplus 5 \oplus 9) \oplus (1 \oplus 5),
\] (52)

The irreducible pieces are orthogonal to one another: For \( A \neq B \)

\[
(\text{A}) C_{ijkl} (\text{B}) C_{ijkl} = 0.
\] (53)

The Euclidean squares, \( C^2 = C_{ijkl} C^{ijkl} \) and \( (\text{A}) C^2 = (\text{A}) C_{ijkl} (\text{A}) C^{ijkl} \) with \( A = 1, \cdots, 5 \), fulfill the “Pythagorean theorem:”

\[
C^2 = (1) C^2 + (2) C^2 + (3) C^2 + (4) C^2 + (5) C^2.
\] (54)
2.3 Irreducible decompositions

The decomposition \([51]\) involves two scalars \(S\) and \(A\), two second order traceless tensors \(P_{ij}\) and \(Q_{ij}\), and a fourth order totally traceless tensor \(R_{ijkl}\). Exactly the same types of tensors emerge in the harmonic decomposition that is widely used in elasticity theory. Such decomposition is generated by expressing the partial tensors in term of the harmonic polynomials, i.e., the polynomial solutions of the Laplace equation. The corresponding tensors are required to be completely symmetric and totally traceless. As it was demonstrated by Backus \([3]\), such a harmonic decomposition is not applicable in general in a space of the dimension greater than three.

The most compact expression of this type was proposed by Cowin \([8]\),

\[
C_{ijkl} = a g^{ij} g^{kl} + b \left( g^{ik} g^{jl} + g^{il} g^{jk} \right) + \left( g^{ij} \hat{A}^{kl} + g^{kl} \hat{A}^{ij} \right) + \\
\left( g^{ik} \hat{B}^{jl} + g^{il} \hat{B}^{jk} + g^{jk} \hat{B}^{il} + g^{lj} \hat{B}^{ik} \right) + Z^{ijkl}.
\tag{55}
\]

An alternative expression was proposed by Backus \([3]\). It reads, see \([4]\),

\[
C_{ijkl} = H^{ijkl} + \left( H^{ij} g^{kl} + H^{ik} g^{jl} + H^{il} g^{jk} + H^{jk} g^{il} + H^{jl} g^{ik} + H^{kl} g^{ij} \right) + \\
H \left( g^{ij} g^{kl} + g^{ik} g^{jl} + g^{il} g^{jk} \right) + \\
\left( h^{ij} g^{kl} + h^{ik} g^{jl} + h^{il} g^{jk} - \frac{1}{2} h^{il} g^{jk} - \frac{1}{2} h^{ik} g^{jl} - \frac{1}{2} h^{jk} g^{il} - \frac{1}{2} h^{jl} g^{ik} \right) + \\
h \left( g^{ij} g^{kl} - \frac{1}{2} g^{il} g^{jk} - \frac{1}{2} g^{ik} g^{jl} \right) \tag{56}
\]

Let us compare these two expressions. First we observe that two totally traceless tensors must be equal to one another, \(H^{ijkl} = Z^{ijkl}\). As for the scalar terms in Eq.(55), they are merely linear combinations of the corresponding terms in Eq.(56). Indeed, it is enough to take

\[
a = H + h \quad \text{and} \quad b = H - (1/2)h.
\tag{57}
\]

Quite similarly, the traceless second order tensors of Eq.(55) are linear combinations of the corresponding terms in Eq.(56) with the identities

\[
\hat{A}^{kl} = H^{kl} + h^{kl} \quad \text{and} \quad \hat{B}^{jl} = H^{jl} - (1/2)h^{jl}.
\tag{58}
\]

Both decompositions are irreducible under the action of the rotation group. The key difference between the two is that that the decomposition of Backus is also irreducible under the action of permutation group. Cowin’s decomposition is reducible in this sense.

Let us compare now the harmonic decomposition Eq.(56) to our decomposition as it is given in Eq.(50). We immediately identify

\[
(1)S^{ijkl} = H \left( g^{ij} g^{kl} + g^{ik} g^{jl} + g^{il} g^{jk} \right),
\tag{59}
\]

\[
(2)S^{ijkl} = H^{ij} g^{kl} + H^{ik} g^{jl} + H^{il} g^{jk} + H^{jk} g^{il} + H^{jl} g^{ik} + H^{kl} g^{ij},
\tag{60}
\]

\[
(3)S^{ijkl} = H^{ijkl},
\tag{61}
\]

\[
(1)A^{ijkl} = h \left( g^{ij} g^{kl} - \frac{1}{2} g^{il} g^{jk} - \frac{1}{2} g^{ik} g^{jl} \right),
\tag{62}
\]

\[
(2)A^{ijkl} = h^{ij} g^{kl} + h^{ik} g^{jl} - \frac{1}{2} h^{il} g^{jk} - \frac{1}{2} h^{ij} g^{il} - \frac{1}{2} h^{jk} g^{il} - \frac{1}{2} h^{jl} g^{ik}.
\tag{63}
\]
We can straightforwardly derive the relations

\[ H = \frac{1}{15} S, \quad H^{ij} = \frac{1}{7} P^{ij}, \quad H^{ijkl} = R^{ijkl}, \quad (64) \]

and

\[ h = \frac{1}{3} A, \quad h^{ij} = -2Q^{ij}. \quad (65) \]

Thus the two decompositions are equivalent. The difference is that in Eq.(50) the partial tensors are identified as elasticities themselves. These partial tensors generate the minimal direct sum decomposition of the elasticity tensor space. Moreover, the corresponding subspaces are mutually orthogonal and the squares of the tensors satisfy the “Pythagorean theorem”. We will see in the following section how those properties can be applied to the problem of quadratic invariants.

3 Linear and quadratic invariants

3.1 Linear invariants

Each linear invariant of the elasticity tensor can be represented as

\[ I = h_{ijkl} C^{ijkl}, \quad (66) \]

where \( h_{ijkl} \) is a tensor. Using the decomposition (51), it can be rewritten as a sum of irreducible parts with different leading coefficients

\[ I = \sum_{l=1}^{5} (I) h_{ijkl} (I) C^{ijkl}. \quad (67) \]

We observe that the tensor \( h_{ijkl} \) can be constructed only as a product of two components of the metric tensor, \( h_{ijkl} \sim (g \cdot g)_{ijkl} \). Since the tensors \( P_{ij}, Q_{ij}, \) and \( R_{ijkl} \) are totally traceless they do not contribute to the sum in Eq.(67). Consequently we are left with

\[ I = \alpha S + \beta A. \quad (68) \]

Hence, every linear invariant is represented as a linear combination of the two basic linear invariants

\[ L_1 = S = \frac{1}{3} (C_{iikk} + 2C_{ikik}) , \quad L_2 = A = \frac{1}{3} (C_{iikk} - C_{ikik}) . \quad (69) \]

Their independence can be seen from their explicit expressions. It follows also from the fact that \( S \) and \( A \) are related to two different subspaces \( (1)S \) and \( (1)A \). We readily obtain the expression of the linear invariants given in Eq.(2),

\[ C_{ikik} = S - A , \quad C_{iikk} = S + 2A . \quad (70) \]
3.2 Quadratic invariants

Norris [20] presented a generic quadratic invariant of the elasticity tensor in the form

\[ f_{ijklpqrs} C^{ijkl} C^{pqrs}. \]  

(71)

Here \( f_{ijklpqrs} \) is a numerical tensor. Using the irreducible decomposition (51) it can be rewritten as

\[ \sum_{I,J=1}^{5} (I,J) f_{ijklpqrs} (I) C^{ijkl} (J) C^{pqrs}. \]  

(72)

The tensors with the components \((I,J) f_{ijklpqrs}\) can be constructed only as a product of four components of the metric tensor, \((I,J) f_{ijklpqrs} \sim (g \cdot g \cdot g \cdot g)_{ijklpqrs}\). Using the traceless property of the tensors \(P_{ij}, Q_{ij}\), and \(R_{ijkl}\), we can show that the quadratic invariants can be chosen uniquely as

\[ Z_1 = S^2, \quad Z_2 = AS, \quad Z_3 = A^2, \]  

(73)

\[ Z_4 = P_{ij} P^{ij}, \quad Z_5 = P_{ij} Q^{ij}, \quad Z_6 = Q_{ij} Q^{ij}, \quad \text{and} \quad Z_7 = R_{ijkl} R^{ijkl}. \]  

(74)

It is clear that this set of invariants is complete. Indeed, when two tensors in (72) are irreducibly decomposed in the form (51) with an arbitrary tensor \(f_{ijklpqrs}\) constructed from the metric tensor and numbers, only the terms (73, 74) can appear. Moreover, these invariants are independent. It is due to the fact that they are taken from independent and even orthogonal subspaces of the elasticity tensor space.

3.3 Relations between two sets of quadratic invariants

Let us display the quadratic invariants of the set (6) in terms of \(Z_I\). We present the details of calculations in the Appendix. For the quadratic invariants constructed from the linear ones, we have straightforwardly

\[ A_1^2 = (S - A)^2 = Z_1 - 2Z_2 + Z_3, \]  

(75)

\[ A_1 A_2 = (S - A)(S + 2A) = Z_1 + Z_2 - 2Z_3, \]  

(76)

\[ A_2^2 = (S + 2A)^2 = Z_1 + 4Z_2 + 4Z_3. \]  

(77)

For the first invariant of Ting, we write

\[ B_1 = C^{ijkl} C_{ijkl} = \sum_{I=1}^{5} \sum_{J=1}^{5} (I) C^{ijkl} (J) C_{ijkl}. \]  

(78)

Due to the orthogonality of the set, we are left here with

\[ B_1 = (1) S^2 + (2) S^2 + (3) S^2 + (4) A^2 + (5) A^2 \]  

(79)

Let us list the expressions of these invariants

\[ (1) S^2 = \frac{1}{5} S^2, \quad (2) S^2 = \frac{6}{7} P_{ij} P^{ij}, \quad (3) S^2 = R^{ijkl} R_{ijkl}, \]  

(80)
and
\[(1) A^2 = A^2, \quad (2) A^2 = 12Q^{mn}Q_{mn}.\] (81)
Consequently,
\[B_1 = \frac{1}{5} Z_1 + Z_3 + \frac{6}{7} Z_4 + 12Z_6 + Z_7.\] (82)

The second invariant of Ting takes the form
\[B_2 = \frac{1}{3} Z_1 + \frac{4}{3} Z_2 + \frac{4}{5} Z_3 + Z_4 - 4Z_5 + 4Z_6.\] (83)

The first and the second invariants of Ahmad read
\[B_3 = \frac{1}{3} Z_1 + \frac{1}{3} Z_2 - \frac{2}{3} Z_3 + Z_4 - Z_5 - 2Z_6\] (84) and
\[B_4 = \frac{1}{3} Z_1 - \frac{2}{3} Z_2 + \frac{1}{3} Z_3 + Z_4 + 2Z_5 + Z_6,\] (85)
respectively. From these expressions we see that the set of invariants (6) is complete and the invariants are independent. The same is true for our set \(Z_i\). We calculate also the additional invariant of Norris:
\[B_5 = C^{ijkl}C_{ikjl} = \frac{1}{5} Z_1 - \frac{1}{2} Z_3 + \frac{6}{7} Z_4 - 6Z_6 + Z_7.\] (86)
Substituting the expressions (73–74) we obtain the formula (9) of Norris.

4 Applications: Invariants as characteristics of materials

Since invariants of the elasticity tensor are independent of the coordinate system used in specific measurements, they can be used as intrinsic characteristics of the materials. It is clear that linear independence is not enough for this goal. Indeed, although the invariants \(Z_1, Z_2, \text{ and } Z_3\) are linear independent, they are related by a quadratic relation \(Z_2^2 = Z_1Z_3\). We will show that an intrinsic meaning can be assigned to the five invariants that correspond to different direct subspaces of the elasticity tensor space, namely
\[\{Z_1, Z_3, Z_4, Z_6, Z_7\}.\] (87)
All these invariants are positive.

4.1 Cauchy relations and Cauchy factor

In the early days of the elasticity theory, Cauchy formulated a molecular model for elastic bodies, based on 15 independent elasticity constants. In this way 6 constraints, called Cauchy relations were assumed. A lattice-theoretical analysis shows, see [12], [16], that the Cauchy relations are valid provided the following conditions hold:

- The interaction forces between the molecules of a crystal are central forces;
- each molecule is a center of symmetry;
- the interaction forces between the building blocks of a crystal can be well approximated by a harmonic potential.
More recent discussions of the Cauchy relations can be found, e.g., in [1], [3], [4], or [7]. Different compact expressions of the Cauchy relations can be found in literature. For instance in [12], they are presented as

\[ C^{ijk} - C^{ijk} = 0. \]  

(88)

An alternative form is widely used, see [22], [26], [10], [7],

\[ C^{ijkl} - C^{ikjl} = 0. \]  

(89)

The irreducible decomposition technique [13] yields

\[ A^{ijkl} = 0, \quad \text{or} \quad Q_{ij} = 0. \]  

(90)

As it was demonstrated experimentally already by Voigt, the Cauchy relations do not hold even approximately. Thus, elastic properties of the generic anisotropic material is described by the whole set of 21 independent component. In fact, the situation with the Cauchy relations is much more interesting, see [12]. One can look for the deviation of the elasticity tensor from its Cauchy part. As it was pointed out by Haussühl [12], this deviation, even being a macroscopic characteristic, can provide some important information about microscopic structure of the material. To have such deviation term we must have a unique proper decomposition of the elasticity tensor into two independent parts that can be referred to as Cauchy and non-Cauchy parts.

In [12], the deviation from the Cauchy part was presented by the value of the corresponding combination given in the left hand side of Eq.(88). Such way of expression is valid only in the case when the non-Cauchy part is presented by only one component. Moreover this expression is dimension-full and depends on the choice of the coordinate system.

With the use of the quadratic invariants we can introduce an invariant characteristic of deviation of a material from its Cauchy prototype. Due to the “Pythagorean theorem” (51), we define the dimensionless quantity, which we will call the Cauchy factor

\[ \mathcal{F}_{\text{Cauchy}} = \sqrt{\frac{S_{ijkl}S_{ijkl}}{C^{ijkl}C^{ijkl}}}. \]  

(91)

Evidently, 0 ≤ \( \mathcal{F}_{\text{Cauchy}} \) ≤ 1. A pure Cauchy material is determined by \( \mathcal{F}_{\text{Cauchy}} = 1 \). For \( \mathcal{F}_{\text{Cauchy}} = 0 \), we have a hypothetic material without Cauchy part at all. Comparing two materials, we must conclude that a material with higher Cauchy factor has a microscopic structure closer to spherical symmetry.

4.2 Fedorov’s problem

In linear elasticity for anisotropic materials one must deal with a big set of elasticity constants. But in some problems, the elastic body can only be slightly different from an isotropic one. Fedorov [11], in a classical book on the propagation of elastic waves in anisotropic crystals, has demonstrated how the anisotropic elastic tensor can be averaged over the 3-dimensional spatial directions in order to find some kind of isotropic approximation.

Recently Norris [21] took up this program and defined an Euclidean distance function for solving the Fedorov problem in a novel way. He succeeded in doing so and even extended the formalism for averaging the given set of elastic parameters relative to less symmetric classes. The corresponding procedure can be outlined as follows:
For a given 4th order elasticity tensor, one constructs the corresponding 2nd order Christoffel tensor, which is quadratic in the wave vector.

One consider the $C^2$ norm of the difference between the given anisotropic Christoffel tensor and a generic isotropic one. The isotropic Christoffel tensor is taken with two unknown parameters.

Since Christoffel tensor is quadratic in the wave vector $n$, the mentioned norm is quartic likewise. Its average is computed in space directions.

The resulting expression is left to be a function of two isotropic parameters. Its minimization is applied and the resulting pair of isotropic parameters is derived.

The result of this consideration is given in \[21\] as

$$\kappa^* = \frac{1}{9} C_{iijj}, \quad \mu^* = \frac{1}{10} C_{ijij} - \frac{1}{30} C_{iijj}.$$  \hspace{1cm} (92)

In \[20\], \[21\] and \[18\], Norris explained that minimizing a Euclidean distance function is equivalent to projecting the tensor of elastic stiffness onto the appropriate symmetry.

Let us consider an elasticity tensor of 21 independent components. It is irreducibly decomposed to the sum of five independent pieces. Two scalar pieces, namely $(1)S_{ijkl}$ and $(1)A_{ijkl}$ has a special property: They are invariant under arbitrary $SO(3)$ transformation. Using the direct sum of the corresponding subspaces, we can construct a subspace

$$3\mathbb{O} = (1)S \otimes (1)A.$$  \hspace{1cm} (93)

This 2-dimensional subspace is $SO(3)$ invariant and orthogonal to all other subspaces of an arbitrary elasticity tensor. Evidently it must be identified as an isotropic part of an elasticity tensor. Consequently we constructed an isotropic part of the generic elasticity tensor in the form

$$(\text{iso})C_{ijkl} = (1)S_{ijkl} + (1)A_{ijkl},$$  \hspace{1cm} (94)

or, explicitly,

$$(\text{iso})C_{ijkl} = \frac{1}{15} S\left(g_{ij}g_{kl} + g_{ik}g_{jl} + g_{il}g_{jk}\right) + \frac{1}{6} A\left(2g_{ij}g_{kl} - g_{il}g_{jk} - g_{ik}g_{jl}\right).$$  \hspace{1cm} (95)

Let us compare this expression to the standard representation of an isotropic material in terms of the Lamé moduli $\lambda$ and $\mu$

$$(\text{iso})C_{ijkl} = \lambda g_{ij}g_{kl} + \mu\left(g_{ik}g_{jl} + g_{il}g_{jk}\right).$$  \hspace{1cm} (96)

We derive the effective Lamé moduli for an anisotropic material

$$\lambda = \frac{1}{15} S + \frac{3}{4} A, \quad \mu = \frac{1}{15} S - \frac{1}{6} A.$$  \hspace{1cm} (97)

Recall that

$$S^* = \frac{1}{3} (C_{iikk} + 2C_{ikik}), \quad A^* = \frac{1}{3} (C_{iikk} - C_{ikik}).$$  \hspace{1cm} (98)
Substituting these expressions into equations (97), we derive

$$\lambda^* = \frac{2}{15} C_{ijij} - \frac{1}{15} C_{ijij}, \quad \mu^* = \frac{1}{10} C_{ijij} - \frac{1}{30} C_{ijij}. \quad (99)$$

With the bulk constant \( \kappa = \lambda + \frac{2}{3} \mu \) we recover both expressions given in Eq. (92).

In order to express the deviation of the given anisotropic material from its effective isotropic prototype, one uses the distance between two tensors. This quantity is dimensionful and depends on the average magnitude of the elasticity tensor. Instead, we define the \textit{isotropy factor} of an anisotropic material in the form

$$F_{\text{iso}} = \sqrt{\frac{(\text{iso}) C_{ijkl} C_{ijkl}}{C_{ijkl} C_{ijkl}}} = \sqrt{\frac{(1) S_{ijkl} (1) S_{ijkl} + (1) A_{ijkl} (1) A_{ijkl}}{C_{ijkl} C_{ijkl}}} \quad (100)$$

In terms of the constants \( S \) and \( A \) it reads

$$F_{\text{iso}} = \sqrt{\frac{S^2 + 5 A^2}{5 C_{ijkl} C_{ijkl}}} = \sqrt{\frac{Z_1 + 5 Z_3}{Z_1 + 5 Z_3 + (30/7) Z_4 + 60 Z_6 + 5 Z_7}} \quad (101)$$

We observe that \( 0 \leq F_{\text{iso}} \leq 1 \). It is equal to one for pure isotropic materials and equal to zero for some hypothetic material without isotropic part, i.e., in the case when the effective Lamé moduli vanish.

### 4.3 Irreducibility factors

As a natural extension of the Cauchy and the isotropy factors described above, we introduce dimensionless numerical factors that describe the contribution of the irreducible pieces to the elasticity tensor. For the 5 irreducible parts \((I) C_{ijkl}\) with \( I = 1, \cdots, 5 \), we define the \textit{irreducibility factors}

$$\mathcal{F}_{\text{irr}} = \sqrt{\frac{(I) C_{ijkl} (1) C_{ijkl}}{C_{ijkl} C_{ijkl}}} \quad (102)$$

In particular, the Cauchy factor is expressed as

$$\mathcal{F}_{\text{Cauchy}} = \sqrt{\frac{(1) A_{ijkl} (1) A_{ijkl} + (2) A_{ijkl} (2) A_{ijkl}}{C_{ijkl} C_{ijkl}}} = \sqrt{(4) F_{\text{irr}}^2 + (5) F_{\text{irr}}^2} \quad (103)$$

and the isotropy factor as

$$\mathcal{F}_{\text{iso}} = \sqrt{\frac{(1) S_{ijkl} (1) S_{ijkl} + (1) A_{ijkl} (1) A_{ijkl}}{C_{ijkl} C_{ijkl}}} = \sqrt{(1) F_{\text{irr}}^2 + (4) F_{\text{irr}}^2} \quad (104)$$
5 Cubic crystals

5.1 Definition

Cubic crystals are described by three independent elasticity constants. In a properly chosen coordinate system, they can be put, see Nayfeh [19], into the following Voigt matrix:

\[
\begin{bmatrix}
C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1131} & C_{1112} \\
* & C_{2222} & C_{2233} & C_{2223} & C_{2231} & C_{2212} \\
* & * & C_{3333} & C_{3323} & C_{3331} & C_{3312} \\
* & * & * & C_{2323} & C_{2331} & C_{2312} \\
* & * & * & * & C_{3131} & C_{3112} \\
* & * & * & * & * & C_{1212}
\end{bmatrix}
= \begin{bmatrix}
C_{11} & C_{12} & C_{112} & 0 & 0 & 0 \\
* & C_{11} & C_{12} & 0 & 0 & 0 \\
* & * & C_{11} & 0 & 0 & 0 \\
* & * & * & C_{66} & 0 & 0 \\
* & * & * & * & C_{66} & 0 \\
* & * & * & * & * & C_{66}
\end{bmatrix}.
\]  

(105)

Taking into account the multiplicities of the elements exhibited in (105), we calculate

\[
C^2 = C_{ijkl}C^{ijkl} = 3(C_{11})^2 + 6(C_{12})^2 + 12(C_{66})^2.
\]  

(106)

5.2 \(S_4\)-decomposition

We decompose (105) irreducibly and find the Cauchy part

\[
S_{ijkl} = \begin{bmatrix}
S_{1111} & S_{1122} & S_{1133} & S_{1123} & S_{1131} & S_{1112} \\
* & S_{2222} & S_{2233} & S_{2223} & S_{2231} & S_{2212} \\
* & * & S_{3333} & S_{3323} & S_{3331} & S_{3312} \\
* & * & * & S_{2323} & S_{2331} & S_{2312} \\
* & * & * & * & S_{3131} & S_{3112} \\
* & * & * & * & * & S_{1212}
\end{bmatrix}
= \begin{bmatrix}
\alpha & \beta & \beta & 0 & 0 & 0 \\
* & \alpha & \beta & 0 & 0 & 0 \\
* & * & \alpha & 0 & 0 & 0 \\
* & * & * & \beta & 0 & 0 \\
* & * & * & * & \beta & 0 \\
* & * & * & * & * & \beta
\end{bmatrix}.
\]  

(107)

where

\[
\alpha = C_{11}, \quad \beta = \frac{1}{3}(C_{12} + 2C_{66})
\]  

(108)

The square of this tensor takes the value

\[
\tilde{S}^2 = S_{ijkl}S^{ijkl} = 3\alpha^2 + 18\beta^2 = 3(C_{11})^2 + 2(C_{12} + 2C_{66})^2.
\]  

(109)

The non-Cauchy part is represented by

\[
A_{ijkl} = \begin{bmatrix}
A_{1111} & A_{1122} & A_{1133} & A_{1123} & A_{1131} & A_{1112} \\
* & A_{2222} & A_{2233} & A_{2223} & A_{2231} & A_{2212} \\
* & * & A_{3333} & A_{3323} & A_{3331} & A_{3312} \\
* & * & * & A_{2323} & A_{2331} & A_{2312} \\
* & * & * & * & A_{3131} & A_{3112} \\
* & * & * & * & * & A_{1212}
\end{bmatrix}
= \begin{bmatrix}
0 & 2\gamma & 2\gamma & 0 & 0 & 0 \\
* & 0 & 2\gamma & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 & 0 \\
* & * & * & -\gamma & 0 & 0 \\
* & * & * & * & -\gamma & 0 \\
* & * & * & * & * & -\gamma
\end{bmatrix}.
\]  

(110)

where

\[
\gamma = \frac{1}{3}(C_{12} - C_{66}).
\]  

(111)

Its square reads

\[
\tilde{A}^2 = A_{ijkl}A^{ijkl} = 36\gamma^2 = 4(C_{12} - C_{66})^2.
\]  

(112)
We can straightforwardly check the identities
\[ \tilde{A} \cdot \tilde{S} = 0, \]  
and
\[ C^2 = \tilde{S}^2 + \tilde{A}^2 = 3\alpha^2 + 18\beta^2 + 36\gamma^2. \]  
The Cauchy factor is expressed as
\[ F_{\text{Cauchy}} = \sqrt{\alpha^2 + 6\beta^2 + 12\gamma^2}, \]  
or
\[ F_{\text{Cauchy}} = \sqrt{\frac{3 (C_{11})^2 + 2 (C_{12} + 2C_{66})^2}{3 (C_{11})^2 + 6 (C_{12})^2 + 12 (C_{66})^2}}. \]  
For the Cauchy relation we have
\[ \gamma = 0, \quad \text{or} \quad C^{12} = C^{66}. \]  
In this case, Eq.\((115)\) yields \(F_{\text{Cauchy}} = 1.\)

For cubic crystals, the Cauchy factor in Eqs.\((115,116)\) depends on two independent parameters and thus allows 3-dimensional visualization, see Fig. 1 and Fig 2.

Figure 1: Functional dependence of the Cauchy factor \(F_{\text{Cauchy}}\) with respect to the parameters \(\beta/\alpha\) and \(\gamma/\alpha\). The pure Cauchy materials are depicted by the straight line lying on the axis \(\gamma = 0.\)

5.3 \(SO(3)\)-decomposition

Let us determine the \(SO(3)\) components of the totally symmetric tensor. Using Eq.\((23)\) we derive from \((107)\)
\[ S^{ij} = (\alpha + 2\beta)g^{ij} = \left( C^{11} + \frac{2}{3}C^{12} + \frac{4}{3}C^{66} \right) g^{ij}. \]  
Consequently,
\[ S = 3(\alpha + 2\beta) = 3C^{11} + 2C^{12} + 4C^{66}, \]  
(119)
Figure 2: Functional dependence of the Cauchy factor $F_{\text{Cauchy}}$ with respect to the parameters $C^{12}/C^{11}$ and $C^{66}/C^{11}$. The pure Cauchy materials are depicted by the straight line $C^{12} = C^{66}$.

and

$$P^{ij} = 0.$$  \hspace{1cm} (120)

The total traceless remainder can be expressed as

$$R^{ijkl} = \begin{bmatrix} R^{1111} & R^{1122} & R^{1133} & R^{1123} \\ R^{2222} & R^{2233} & R^{2223} & R^{2231} \\ R^{3333} & R^{3323} & R^{3331} & R^{3312} \\ R^{2323} & R^{2331} & R^{2312} & R^{1212} \end{bmatrix} = \frac{\alpha - 3\beta}{5} \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ * & 2 & -1 & 0 & 0 & 0 \\ * & * & 2 & 0 & 0 & 0 \\ * & * & * & -1 & 0 & 0 \\ * & * & * & * & -1 & 0 \\ * & * & * & * & * & -1 \end{bmatrix},$$  \hspace{1cm} (121)

The two non-zero quadratic invariants read

$$S^2 = 9(\alpha + 2\beta)^2 = (3C^{11} + 2C^{12} + 4C^{66})^2$$  \hspace{1cm} (122)

and

$$R^2 = \frac{6}{5}(\alpha - 3\beta)^2 = \frac{6}{5}(C^{11} - C^{12} - C^{66})^2.$$  \hspace{1cm} (123)

We check the formula

$$\bar{S}^2 = \frac{1}{5} S^2 + R^2,$$  \hspace{1cm} (124)

that must hold with the correspondence with the expressions (80,81) of the first Ting’s invariant.

Let us turn now to the asymmetric part $A^{ijkl}$. Eq. (110) yields

$$\Delta^{ij} = 2\gamma g^{ij}.$$  \hspace{1cm} (125)

Consequently,

$$A = 6\gamma, \quad Q^{ij} = 0.$$  \hspace{1cm} (126)

Thus,

$$\bar{A}^2 = A^2 = 36\gamma^2 = 4(C^{12} - C^{66})^2.$$  \hspace{1cm} (127)
Accordingly, the isotropic factor takes the form

\[ F_{\text{iso}} = \sqrt{\frac{(1/5)S^2 + A^2}{C_{ijkl}C_{ijkl}}} = \sqrt{\frac{(3/5)(\alpha + 2\beta)^2 + 12\gamma^2}{\alpha^2 + 6\beta^2 + 12\gamma^2}}, \]  

(128)

or

\[ F_{\text{iso}} = \sqrt{\frac{(3C_{11} + 2C_{12} + 4C_{66})^2 + 20 (C_{12} - C_{66})^2}{15 ((C_{11})^2 + 2 (C_{12})^2 + 4 (C_{66})^2)}}. \]  

(129)

It is well known that an isotropic medium can be described as a cubic crystal with \( C_{66} = (1/2)(C_{11} - C_{12}) \). It is equivalent to \( \alpha = 3\beta \). When this relation is substituted into (128), we obtain \( F_{\text{iso}} = 1 \). In Fig. 3 and Fig 4, we present the functional dependence of the isotropy factor for a cubic crystal with respect to the homogeneous fractions of its parameters.

Figure 3: Functional dependence of the isotropy factor \( F_{\text{iso}} \) is depicted with respect to the variables \( \beta/\alpha \) and \( \gamma/\alpha \). The straight line with \( \beta/\alpha = 1/3 \) corresponds to the pure isotropic materials.

Figure 4: Functional dependence of the isotropy factor \( F_{\text{iso}} \) with respect to the variables \( C_{12}/C_{11} \) and \( C_{66}/C_{11} \). The straight line \( C_{66} = (1/2)(C_{11} - C_{12}) \) correspond to the isotropic medium.
5.4 Cubic averaging

We consider now a problem of averaging of an arbitrary elastic material by a cubic symmetry prototype. Let in some coordinate system a material be described by a full set of 21 elastic parameters $C_{ijkl}$. We are looking for a cubic crystal that is mostly close to our material with respect to the Euclidean metric. We assume that in the same coordinate system the cubic crystal is represented in the canonical form (105). Since for cubic crystal $P_{ij} = Q_{ij} = 0$, its elasticity tensor is left

$$C_{ijkl}^{\text{cub}} = (1)S_{ijkl}^{\text{cub}} + (3)S_{ijkl}^{\text{cub}} + (1)A_{ijkl}^{\text{cub}},$$

where

$$\begin{align*}
(1)S_{ijkl}^{\text{cub}} &= \frac{1}{15}m \left( g_{ij}g_{kl} + g_{ik}g_{jl} + g_{il}g_{jk} \right), \\
(1)A_{ijkl}^{\text{cub}} &= \frac{1}{6}n \left( 2g_{ij}g_{kl} - g_{il}g_{jk} - g_{ik}g_{jl} \right), \\
(3)S_{ijkl}^{\text{cub}} &= k\sigma_{ijkl}.
\end{align*}$$

Here $\sigma_{ijkl}$ is a set of parameters (not a tensor) that is represented in the $6 \times 6$ notations by the constant matrix in the right hand side of Eq.(121). Thus $m, k, n$ are our unknown variables. The square distance between two elasticity tensors is given by

$$\left(C_{ijkl} - C_{ijkl}^{\text{cub}}\right)^2 = K_1(S - m)^2 + K_2(A - n)^2 + (R_{ijkl} - k\sigma_{ijkl})^2 + K_3,$$

where $K_1, K_2, K_3$ are positive numerical constants. This expression reaches its minimal value for

$$m = S, \quad n = A,$$

and

$$(R_{ijkl} - k\sigma_{ijkl})\sigma_{ijkl} = 0,$$

We have

$$k = \frac{R_{ijkl}\sigma_{ijkl}}{\sigma_{ijkl}\sigma_{ijkl}},$$

Using the expression of $\sigma_{ijkl}$ listed in Eq.(121), we derive

$$k = \frac{R_{1111}^{1111} + R_{2222}^{2222} + R_{3333}^{3333}}{6}.$$ 

Consequently the parameters of the cubic crystal prototype take the values

$$\alpha = \frac{1}{5}S + 2k = \frac{1}{3} \left( C_{1111}^{1111} + C_{2222}^{2222} + C_{3333}^{3333} \right) - \frac{2}{15}S,$$

$$\beta = \frac{1}{15}S - k = -\frac{1}{6} \left( C_{1111}^{1111} + C_{2222}^{2222} + C_{3333}^{3333} \right) - \frac{7}{30}S,$$

$$\gamma = \frac{1}{6}A.$$

6 Conclusion

In the framework of the irreducible decomposition of elasticity tensor, we studied the problem of its quadratic invariants. Since this decomposition is orthogonal, the invariants emerge in a natural and systematic way. Their independence and completeness follow straightforwardly from the direct sum decomposition of the tensor space. For arbitrary anisotropic materials, we defined the Cauchy factor as a dimensionless measure of a closeness to a pure Cauchy material. Quite similarly, we defined isotropy factor as a measure for a closeness to an isotropic prototype of a given material. The irreducible factors are defined in order to characterize the contributions of different irreducible parts of an anisotropic elasticity tensor. This formalism can be useful for various elasticity problems:

- Elasticity wave propagation [2];
- complete set of anisotropy invariants, see [28];
- material symmetries and wavefront symmetries [5], [6];
- averaging of anisotropic material by a higher symmetric prototype [18], [21].

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A Calculating the relations between invariants

A.1 The first Ting invariant

For the first invariant of Ting, we write

\[ B_1 = C^{ijkl}C_{ijkl} = \sum_{i=1}^{5} \sum_{j=1}^{5} (i)C^{ijkl} = \sum_{j=1}^{5} (j)C_{ijkl}. \]  

Due to the orthogonality of the decomposition, we are left with

\[ B_1 = (1)S^2 + (2)S^2 + (3)S^2 + (1)A^2 + (2)A^2 \]  

We calculate step by step

\[ (1)S^2 = \frac{1}{225} S^2 \left( g^{ij} g^{kl} + g^{ik} g^{jl} + g^{il} g^{jk} \right)^2 = \frac{1}{5} S^2 = \frac{1}{5} Z_1, \]  

\[ (2)S^2 = \frac{1}{49} \left( p^{ij} g^{kl} + p^{ik} g^{jl} + p^{il} g^{jk} + p^{jk} g^{il} + p^{jl} g^{ik} + p^{kl} g^{ij} \right)^2 = \frac{6}{7} P^{ij} P_{ij} = \frac{6}{7} Z_4, \]  

\[ (3)S^2 = R^{ijkl} R_{ijkl} = Z_7, \]  

and

\[ (1)A^2 = \frac{1}{36} A^2 \left( 2g^{ij} g^{kl} - g^{il} g^{jk} - g^{ik} g^{jl} \right)^2 = A^2 = Z_3 \]  

\[ (2)A^2 = \frac{1}{4} \left( e^{ikm} e^{jln} + e^{ilm} e^{jkn} \right) \left( \epsilon_{ikp} \epsilon_{jql} + \epsilon_{ilp} \epsilon_{jkl} \right) Q_{mn} Q^{pq} = 12Q^{mn} Q_{mn} = 12Z_6. \]
Consequently, the first invariant of Ting reads

\[ B_1 = \frac{1}{5} Z_1 + Z_3 + \frac{6}{7} Z_4 + 12Z_6 + Z_7. \] (150)

**A.2 The second Ting invariant**

For the second invariant of Ting, \( B_2 \), we first calculate the trace

\[ g^{ij} S_{ijkl} = P_{kl} + \frac{1}{3} S_{g_{kl}}, \] (152)

\[ g^{ij} A_{ijkl} = g^{ij} (g_{ik} g_{jl} - g_{il} g_{jk} - g_{ij} g_{kl}) = \frac{2}{3} A_{g_{kl}}. \] (154)

Hence,

\[ C_{ijkl} = P_{kl} - 2Q_{kl} + \frac{1}{3} (S + 2A) g_{kl}. \] (155)

Consequently, the second invariant of Ting reads

\[ B_2 = C_{i k l} C_{j k l} = P_{kl} P^{kl} - 4P_{kl} Q^{kl} + 4Q_{kl} Q^{kl} + \frac{1}{3} (S + 2A)^2, \] (156)

or

\[ B_2 = \frac{1}{3} Z_1 + \frac{4}{3} Z_2 + \frac{4}{3} Z_3 + Z_4 - 4Z_5 + 4Z_6. \] (157)

**A.3 The first Ahmad invariant**

For the first invariant of Ahmad, \( B_3 \), we need the trace

\[ g^{ij} S_{ijkl} = P_{kl} + \frac{1}{3} S_{g_{kl}}, \] (159)

\[ g^{ij} A_{ijkl} = g^{ij} (g_{ij} g_{kl} - g_{il} g_{jk} - g_{ik} g_{jl}) = -\frac{1}{3} A_{g_{kl}}. \] (160)

Consequently, the first invariant of Ahmad reads

\[ B_3 = C^i_{ikl} + \frac{1}{3} g_{kl} (S - A). \] (161)
Hence using (155) and (162) we get

\[
B_3 = \left( P_{kl} + Q_{kl} + \frac{1}{3} g_{kl} (S - A) \right) \left( P_{kl} - 2Q_{kl} + \frac{1}{3} (S + 2A) g_{kl} \right) 
= P_{kl} P_{kl} - P_{kl} Q_{kl} - 2Q_{kl} Q_{kl} + \frac{1}{3} (S + 2A) (S - A),
\]

or

\[
B_3 = \frac{1}{3} Z_1 + \frac{1}{3} Z_2 - \frac{2}{3} Z_3 + Z_4 - Z_5 - 2Z_6.
\]

### A.4 The second Ahmad invariant

This invariant is obtained by the use of the formula (162),

\[
B_3 = \left( P_{kl} + Q_{kl} + \frac{1}{3} g_{kl} (S - A) \right) \left( P_{kl} + Q_{kl} + \frac{1}{3} g_{kl} (S - A) \right) 
= P_{kl} P_{kl} + 2P_{kl} Q_{kl} + Q_{kl} Q_{kl} + \frac{1}{3} (S - A)^2,
\]

or

\[
B_4 = \frac{1}{3} Z_1 - \frac{2}{3} Z_2 + \frac{1}{3} Z_3 + Z_4 + 2Z_5 + Z_6.
\]

### A.5 The Norris invariant

We put the invariant of Norris first in the form

\[
B_5 = C_{ijkl}^i C_{ikjl} = \left( S_{ijkl}^i + A_{ijkl}^i \right) \left( S_{ikjl}^i + A_{ikjl}^i \right) = S_{ijkl}^i S_{ijkl} + A_{ijkl}^i A_{ikjl}.
\]

Using (144,145,146), we have

\[
S_{ijkl}^i S_{ikjl} = S_{ijkl}^i S_{ijkl} = \frac{1}{5} Z_1 + \frac{6}{7} Z_4 + Z_7.
\]

We observe

\[
A_{ijkl}^i A_{ikjl} = \left( (1) A_{ijkl}^i + (2) A_{ijkl}^i \right) \left( (1) A_{ikjl}^i + (2) A_{ikjl}^i \right) = (1) A_{ijkl}^i (1) A_{ikjl}^i + (2) A_{ijkl}^i (2) A_{ikjl}^i.
\]

Thus we find

\[
(1) A_{ijkl}^i A_{ikjl}^i = \frac{1}{36} A^2 \left( 2g_{ij} g_{kl} - g_{il} g_{jk} - g_{ik} g_{jl} \right) \left( 2g_{jk} g_{ij}^i - g_{ij}^i g_{ij}^k - g_{ij}^k g_{ij}^l \right) = -\frac{1}{2} A^2,
\]

\[
(2) A_{ijkl}^i A_{ikjl}^i = \left( g_{ik} Q_{jl} + g_{jk} Q_{il} + g_{jl} Q_{ik} + g_{lj} Q_{ij} - 2g_{kl} Q_{ij} - 2g_{lj} Q_{ik} \right) \times \left( g_{ij}^k Q^{kl} + g_{ij}^l Q^{jl} + g_{ij}^l Q^{ij} + g_{ij}^k Q^{kl} - 2g_{il} Q^{ij} - 2g_{ik} Q^{jl} \right) 
= -6Q_{ij} Q^{ij}.
\]

Consequently,

\[
B_5 = C_{ijkl}^i C_{ikjl} = \frac{1}{5} Z_1 - \frac{1}{2} Z_3 + \frac{6}{7} Z_4 - 6Z_6 + Z_7.
\]
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