Forcing Total Outer Independent Edge Geodetic Number of a Graph

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Abstract: In this paper we learn the new idea of forcing total outer independent edge geodetic number of a graph. Let G be a connected graph and R be a minimum total outer independent edge geodetic set of G. A subset L ⊆ R is known as a forcing subset for R if R is the unique minimum total outer independent edge geodetic set containing L. A forcing subset for R of minimum cardinality is a minimum forcing subset of R. The forcing total outer independent edge geodetic number of G denoted by \( f_{tot}^G (R) \) is \( f_{tot}^G (R) = \min \{ f_{tot}^G (L) \} \), where the minimum is taken over all minimum total outer independent edge geodetic set R in G. Some general properties satisfied by this concept are studied. It is shown that for any couple of integers \( l, m \) with \( 0 < l \leq m - 4 \), there exists a connected graph G such that \( f_{tot}^G (G) = l \) and \( g_{tot}^G (G) = m \).

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1 Introduction

By a graph \( G = (V, E) \), we denote a finite undirected graph in the absence of loops or multiple lines. The order and size of G are denoted by p and q consecutively. For primary graph theoretic expressions we allude to Harary [2, 3]. The distance \( d(a, b) \) between two points \( a \) and \( b \) in a connected graph G is the length of a shortest \( a - b \) path in G. An \( a - b \) path of length \( d(a, b) \) is known as \( a - b \) geodesic. A point \( c \) is said to lie on a \( a - b \) geodesic \( Q \) if \( c \) is a point of \( Q \) as well as the points \( a \) and \( b \). A geodesic set of G is a set \( R \subseteq V (G) \) such that every point of G is accommodate in a geodesic connecting some couple of points in R. The geodetic number \( g(G) \) of G is the least order of its geodetic sets and any geodetic set of order \( g(G) \) is a geodetic basis. The geodetic number of a graph was established in [4]. An edge geodetic set of G is a set \( R \subseteq V (G) \) such that every line of G is accommodate in a geodesic connecting some couple of points in R. The edge geodetic number \( g_1(G) \) of G is the least order of its edge geodetic sets and any edge geodetic set of order \( g_1(G) \) is an edge geodesic basis of G or a set of \( g_1(G) \) is known as the neighborhood of the point \( c \) in G. For any set \( N \) of points of G, the induced subgraph \( < N > \) is the maximal subgraph of G with point set \( N \). A point is an extreme point of a graph G if \( < N > \) is complete. A set \( R \) of points of G is an independent geodetic set if \( R \) is an independent set and \( |R| = V \). The least cardinality of an independent geodetic set is the independent geodetic number \( g_1(G) \), it was established by P. Kazemi and doost Ali mojde [5]. An edge geodetic set \( R \subseteq V \) is known as total edge geodetic set if the subgraph induced by R has no isolated points. The least cardinality of a total edge geodetic set of G is the total edge geodetic number and is indicated by \( g_1(G) \) [6]. An edge geodetic set \( R \subseteq V \) is said to be total outer independent edge geodetic set if \( < R > \) has no isolated points.
and $\langle V - R \rangle$ is an independent set. The least cardinality of a total outer independent edge geodetic set and it is indicated by $g_{tt}^{oi}(G)$ is known as total outer independent edge geodetic number of $G$ [7]. In this paper we define forcing total outer independent edge geodetic set.

The following theorem is utilize in the consequences.

**Theorem 1.1.** [7] Each extreme vertex of a graph $G$ is in every total outer independent edge geodetic set of $G$.

### 2 Forcing Total Outer Independent Edge Geodetic Number of a Graph

**Definition 2.1** Let $G$ be a connected graph and $R$ be a minimum total outer independent edge geodetic set of $G$. A subset $L \subseteq R$ is known as a forcing subset for $R$ if $R$ is the unique minimum total outer independent edge geodetic set containing $L$. A forcing subset for $R$ of minimum cardinality is a minimum forcing subset of $R$. The forcing total outer independent edge geodetic number of $G$, denoted by $f_{tt}^{oi}(G)$ is $f_{tt}^{oi}(G) = \min \{f_{ii}^{oi}(R)\}$, where the minimum is taken over all minimum total outer independent edge geodetic set $R$ in $G$.

**Example 2.2** For the graph $G$ given in Figure 2.1 there are nine minimum total outer independent edge geodetic sets, they are $R_1 = \{a_1, a_3, a_4, a_5, a_6\}$, $R_2 = \{a_1, a_2, a_3, a_4, a_6\}$, $R_3 = \{a_2, a_3, a_5, a_6, a_7\}$, $R_4 = \{a_2, a_3, a_4, a_5, a_6, a_7\}$, $R_5 = \{a_2, a_3, a_4, a_5, a_6\}$, $R_6 = \{a_1, a_4, a_6, a_7\}$, $R_7 = \{a_1, a_2, a_4, a_5, a_7\}$, $R_8 = \{a_1, a_2, a_4, a_5, a_6, a_7\}$, $R_9 = \{a_2, a_4, a_5, a_6, a_7\}$ so that $f_{tt}^{oi}(R_1) = 4, f_{tt}^{oi}(R_2) = 3, f_{tt}^{oi}(R_3) = 4, f_{tt}^{oi}(R_4) = 4, f_{tt}^{oi}(R_5) = 4, f_{tt}^{oi}(R_6) = 3, f_{tt}^{oi}(R_7) = 3, f_{tt}^{oi}(R_8) = 5, f_{tt}^{oi}(R_9) = 4$. Hence $f_{tt}^{oi}(G) = \min \{3, 4, 5\} = 3$.

![Figure 2.1 Graph G](image)

**Theorem 2.3** For the connected graph $G$

(i) $f_{tt}^{oi}(G) = 0$ if and only if $G$ has a unique total outer independent edge geodetic set.

(ii) $f_{tt}^{oi}(G) = 1$ if and only if $G$ has atleast two total outer independent edge geodetic set, one of which is a unique total outer independent edge geodetic set containing one of its elements, and

(iii) $f_{tt}^{oi}(G) = g_{tt}^{oi}(G)$ if and only if no total outer independent edge geodetic set of $G$ is the unique total outer independent edge geodetic set containing any of its proper subsets.

**Proof.**
(i). We assume that $f_{1t}^{oi}(G) = 0$. Therefore by definition of forcing $f_{1t}^{oi}(R) = 0$, for some total outer independent edge geodetic set $R$ of $G$. Clearly the empty set $\phi$ is the minimum forcing subset for $R$. Note that the empty set $\phi$ is a subset of every set. Therefore, $R$ is the unique total outer independent edge geodetic set of $G$. Conversely we assume that $R$ is the unique total outer independent edge geodetic set of $G$. Clearly $\phi$ is the only forcing subset for $R$. Therefore, $f_{1t}^{oi}(R) = 0$. Hence $f_{1t}^{oi}(G) = 0$.

(ii). We assume that $f_{1t}^{oi}(G) = 1$. Then by definition of forcing $f_{1t}^{oi}(R) = 1$, for some total outer independent edge geodetic set $R$ of $G$. That is there is a singleton subset $L$ of a total outer independent edge geodetic set $R$ of $G$ such that $L$ is not a subset of any other total outer independent edge geodetic set of $G$. Hence $R$ is the unique total outer independent edge geodetic set of $G$ containing one of its element. Conversely to prove $f_{1t}^{oi}(G) = 1$. It is clear from the definition of forcing subset.

(iii). Assume that $f_{1t}^{oi}(G) = g_{1t}^{oi}(G)$. To prove that no total outer independent edge geodetic set of $G$ is the unique total outer independent edge geodetic set containing any of its proper subsets. By theorem 1.2 we know that $g_{1t}^{oi}(G) \geq 2$, for any total outer independent edge geodetic set of $G$. Also by our assumption no proper subset of $R$ is a forcing subset of $R$. Hence any total outer independent edge geodetic set of $G$ is the unique total outer independent edge geodetic set containing any of its proper subsets. Conversely, $G$ contains more than one total outer independent edge geodetic set $R$ other than $R$ is a forcing subset for $R$.

Therefore, $f_{1t}^{oi}(G) = g_{1t}^{oi}(G)$.

Observation 2.4 If $G = K_{m,n}$ ($m, n \geq 2$) is a complete bipartite graph then $f_{1t}^{oi}(G) = 1$.

Definition 2.5 In a connected graph $G$, a point $w$ is called a total outer independent edge geodetic point of $G$ if $w$ belongs to every minimum total outer independent edge geodetic sets of $G$.

Example 2.6 For the graph $G$ given in Figure 2.2. there are four minimum total outer independent edge geodetic sets namely $R_1 = \{a_1, a_2, a_3, a_5, a_6\}$, $R_2 = \{a_1, a_2, a_3, a_4, a_5\}$, $R_3 = \{a_1, a_3, a_4, a_5, a_7\}$, $R_4 = \{a_1, a_3, a_5, a_6, a_7\}$. Here the points $a_1$, $a_2$ and $a_5$ are total outer independent edge geodetic points of $G$.

\[\text{Figure 2.2. Graph } G\]

Theorem 2.7 Let $G$ be a connected graph and $Z$ be the set of all total outer independent edge geodetic points of $G$. Then $f_{1t}^{oi}(G) \leq g_{1t}^{oi}(G) - |Z|$.

Proof Let $R$ be any minimum total outer independent edge geodetic set of $G$. Therefore, $g_{1t}^{oi}(G) = |R|$, also $Z$ be the set of all total outer independent edge geodetic points of $G$, so that $Z \subseteq R$ and $R$ is the unique
minimum total outer independent edge geodetic set containing $R-Z$. Hence, $f^{ol}_{1l}(G) \leq |R-Z| = |R| - |Z| = g^{ol}_{1l}(G) - |Z|$.

**Remark 2.8** The relation in theorem 2.7 is sharp. For the graph $G$ given in Figure 2.2 $f^{ol}_{1l}(G) = 2$, $g^{ol}_{1l}(G) = 5$ and $|Z| = 3$. For the graph $G$ given in Figure 2.1 $f^{ol}_{1l}(G) = 3$, $g^{ol}_{1l}(G) = 5$, and $|Z| = 0$.

### 3. Realisation Result

**Theorem 3.1** For any couple of integers $l, m$ with $0 < l \leq m - 4$, there exist a connected graph $G$ such that $f^{ol}_{1l}(G) = l$ and $g^{ol}_{1l}(G) = m$.

**Proof:** There are two cases.

**Case 1** Suppose $l = 1$ and $m \geq 5$. Consider the cycle $C_4 : a_1, a_2, a_3, a_4, a_1$ of order 4. Construct a graph $G$ by adding $m - 3$ new points $b_1, b_2, \ldots, b_{m-3}$ to $C_4$ and join each $b_k$ ($1 \leq k \leq m - 3$) to the point $a_1$ in $C_4$. The resulting graph $G$ is given in figure 3.1. Let $X = \{b_1, b_2, \ldots, b_{m-3}\}$ be the set of all extreme points of $G$. Then $G$ contains exactly two $g^{ol}_{1l}(G)$ sets they are $R_1 = X \cup \{a_1, a_2, a_3\}$ and $R_2 = X \cup \{a_1, a_3, a_4\}$. Hence $g^{ol}_{1l}(G) = |X| + 3 = m$. Since $R_1$ is the unique $g^{ol}_{1l}$ set containing $a_2$, so by theorem 2.3 (ii), $f^{ol}_{1l}(G) = 1$.

![Figure 3.1 Graph G](image)

**Case 2** Suppose $l \geq 2$. Take the path $P_4 : a_1, a_2, a_3, a_4$ of length 4. For each integer $k (1 \leq k \leq l)$, let $E_k : b_k, c_k$ be a path. Construct a graph $G$ by join the path $P_4$ and a path $E_k (1 \leq k \leq l)$ and adding $2l$ lines $a_2b_k$ and $a_3c_k$ for $(1 \leq k \leq l)$ and also add $m - l - 4$ pendant lines $a_2e_n$ for $1 \leq n \leq m - l - 4$. The resulting graph $G$ is given in Figure 3.2.

First we prove that $g^{ol}_{1l}(G) = m$. Here $Y = \{a_1, a_4, e_1, e_2, \ldots, e_{m-l-4}\}$ are the set of all extreme points of $G$. By theorem 1.1, $Y$ is contained in every total outer independent edge geodetic set of $G$. It is clear that $a_2$ and $a_3$ also belong to every total outer independent edge geodetic set of $G$. Also we observed that $Y \cup \{a_2, a_3\}$ is not a total outer independent edge geodetic set of $G$. Now we claim that every total outer independent edge geodetic set contains at least one element from each $E_k (1 \leq k \leq l)$. Suppose we assume that there is a total outer independent edge geodetic set, say $R_1$ such that no edge of $E_k$ belong to $R_1$ for some $k$. Then we observed that the edge $b_kc_k (1 \leq k \leq l)$ do not lie on any geodesic joining a couple of points of $R_1$ so that $R_1$ is not an edge geodetic set of $G$. So $R_1$ is not a total outer independent edge geodetic
set of $G$, which is a contradiction to our assumption. Thus every total outer independent edge geodetic set of $G$ contains $Y \cup \{a_2, a_3\}$ and at least one element from each $E_k (1 \leq k \leq l)$. Consider $R_2 = Y \cup \{a_2, a_3, c_1, c_2, ... , c_l\}$. It is clear that $R_2$ is a minimum total outer independent edge geodetic set of $G$ and so $g^{\text{ol}}_I(G) = |R_2| = m$. Hence it follows that every minimum total outer independent edge geodetic set of $G$ contains $Y \cup \{a_2, a_3\}$ and at least one element from each $E_k (1 \leq k \leq l)$. Next we show that $f^{\text{ol}}(G) = l$. Since every minimum total outer independent edge geodetic set contains $Y \cup \{a_2, a_3\}$. Hence by theorem 2.7, $f^{\text{ol}}(G) \leq g^{\text{ol}}_I(G) - |Y \cup \{a_2, a_3\}| = l$. Moreover, a set $R_2$ is a minimum total outer independent edge geodetic set of $G$ iff $R_2$ is of the form $Y \cup \{a_2, a_3, c_1, c_2, ... , c_l\}$ where $c_k$ is any point in $E_k$ for $(1 \leq k \leq l)$. If $f^{\text{ol}}(G) < l$, then $L_1$ is a subset of $R_2$ with $|L_1| < l$ such that there is a point $y_k (1 \leq k \leq l)$ and $y_k \notin L_1$. Let $f_k$ be a point of $E_k$ different from $y_k$. Then $L' = (R_2 - \{y_k\}) \cup \{f_k\}$ is a minimum total outer independent edge geodetic set of $G$ different from $R_2$ such that $L'$ contains $L_1$. It is true for all minimum total outer independent edge geodetic set of $G$. Hence $f^{\text{ol}}(G) = l$.

![Figure 3.2 Graph G](image)

**Figure 3.2 Graph G**

### 4 Conclusion

In this paper we define and learn the idea forcing total outer independent edge geodetic number of a graph. This work can be extended to find forcing upper total outer independent edge geodetic number of a graph, forcing connected total outer independent edge geodetic number of a graph. The findings united in this paper would support the readers to develop various useful applications to the world of science and technology.
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