Appendix to STOPS: Short-term Volatility-controlled Policy Search and its Global Convergence

Liangliang Xu, Daoming Lyu, Yangchen Pan, Aiwen Jiang, Bo Liu

1 Notation Systems

- \((S, A, P, r, \gamma)\) with state space \(S\), action space \(A\), the transition kernel \(P\), the reward function \(r\), the initial state \(S_0\) and its distribution \(\mu_0\), and the discounted factor \(\gamma\).
- \(r_{\text{max}} > 0\) is a constant as the upper bound of the reward.
- State value function \(V_\pi(s)\) and state-action value function \(Q_\pi(s, a)\).
- The normalized state and state action occupancy measure of policy \(\pi\) is denoted by \(\nu_\pi(s)\) and \(\sigma_\pi(s, a)\).
- \(T\) is the length of a trajectory.
- The return is defined as \(G\). \(J(\pi)\) is the expectation of \(G\).
- Policy \(\pi_\theta\) is parameterized by the parameter \(\theta\).
- \(\tau\) is the temperature parameter in the softmax parameterization of the policy.
- \(F(\theta)\) is the Fisher information matrix.
- \(\eta_T D\) is the learning rate of TD update. Similarly, \(\eta_N PG\) is the learning rate of NPG update. \(\eta_P PO\) is the learning rate of PPO update.
- \(\beta\) is the penalty factor of KL difference in PPO update.
- \(f((s, a); \theta)\) is the two-layer over-parameterized neural network, with \(m\) as its width.
- \(\phi_\theta\) is the feature mapping of the neural network.
• $D$ is the parameter space for $\theta$, with $Y$ as its radius.
• $M > 0$ is a constant as the initialization upper bound on $\theta$.
• $J_G^\lambda(\pi)$ is the mean-variance objective function.
• $J_\lambda(\pi)$ is the reward-volatility objective function, with $\lambda$ as the penalty factor.
• $J_y^\lambda(\pi)$ is the transformed reward-volatility objective function, with $y$ as the auxiliary variable.
• $\tilde{r}$ is the reward for the augmented MDP. Similarly, $\tilde{V}_\pi(s)$ and $\tilde{Q}_\pi(s,a)$ are state value function and state-action value function of the augmented MDP, respectively. $\tilde{J}(\pi)$ is the risk-neural objective of the augmented MDP.
• $\hat{y}_k$ is an estimator of $y$ at $k$-th iteration.
• $\omega$ is the parameter of critic network.
• $\delta_k = \arg\min_{\delta \in D} \| \tilde{F}(\theta_k)\delta - \tau_k \nabla_\theta J(\pi_{\theta_k}) \|_2$.
• $\xi(\delta) = \tilde{F}(\theta_k)\delta - \tau_k \nabla_\theta J(\pi_{\theta_k}) - E[\tilde{F}(\theta_k)\delta - \tau_k \nabla_\theta J(\pi_{\theta_k})]$.
• $\sigma_{\xi}$ is a constant associated with the upper bound of the gradient variance.
• $\varphi_k, \psi_k, \varphi'_k, \psi'_k$ are the concentrability coefficients, upper bounded by a constant $c_0 > 0$.
• $\varphi_k^* = \mathbb{E}_{(s,a)\sim \sigma_\pi} \left( \frac{d\pi^*_\theta}{d\pi_0} - \frac{d\pi_{\theta_k}}{d\pi_0} \right)^2 \right)^{1/2}$.
• $\psi_k^* = \mathbb{E}_{(s,a)\sim \sigma_\pi} \left( \frac{d\sigma^*_\pi}{d\sigma_\pi} - \frac{d\sigma_{\theta_k}}{d\sigma_\pi} \right)^2 \right)^{1/2}$.
• $K$ is the total number of iterations. Similarly, $K_{TD}$ is the total number of TD iterations.
• $c_3 > 0$ is a constant as to quantify the difference in risk-neutral objective between optimal policy and any policy.

2 Algorithm Details

We provide a comparison between MVPI and STOPS. Note that neither NPG nor PPO solve $\theta_k := \arg\max_\theta J(\pi_{\theta_k})$ directly, but instead solve an approximation optimization problem at each iteration. We provide pseudo-code for the implementation of MVPI and VARAC in Algorithm 3 and 4.
Algorithm 2: A comparison between STOPS and MVPI

1. for \( k = 1, \ldots, K \) do
2.   \[ y_k := (1 - \gamma)J(\pi_k); \]
3.   \[ \tilde{J}(\pi_{\theta_k}) := \mathbb{E}_{(s,a) \sim \pi_{\theta_k}}(r_{s,a} - \lambda r_{s,a}^2 + 2\lambda r_{s,a} y_k); \]
4.   \[ \text{if MVPI: then} \]
5.     \[ \theta_k := \text{arg max}_\theta(\tilde{J}(\pi_{\theta_k})); \]
6.     \[ \text{// This is achieved by line 9 to 15 in Algorithm 3} \]
7.   \[ \text{else if STOPS: then} \]
8.     \[ \text{if select NPG update then} \]
9.     \[ \text{update } \theta_k \text{ according to Eq. (11)}; \]
10.    \[ \text{else if select PPO update then} \]
11.   \[ \text{update } \theta_k \text{ according to Eq. (12)}; \]
12. \end
13. \text{Output: } \pi_{\theta_K};

3 Theoretical Analysis Details

In this section, we discuss the theoretical analysis in detail. We first present the overview in Section 3.1. Then we provide additional assumptions in Section 3.2. In the rest of the section, we present all the supporting lemmas and the proof for Theorem 1 and 2.

3.1 Overview

We provide Figure 9 to illustrate the structure of the theoretical analysis. First, under Assumption 3 and 4 as well as Lemma 13, we can obtain Lemma 14, 15 and 16. These are the building blocks of lemma 2 which is a shared component in the analysis of both NPG and PPO. The shared components also include Lemma 3 as well as Lemma 1 obtained under Assumption 5. For PPO analysis, under Assumption 2 and 4 we obtain Lemma 7 and 8 from Lemma 2 and 6. Then combined with Lemma 3, 4 and 9, we obtain Theorem 1, the major result of PPO analysis. Likely for NPG analysis, we first obtain Lemma 11 and 12 under Assumption 1, 2 and 4. Then together with Lemma 2, 3, 4 and 10, we obtain Theorem 2, the major result of NPG analysis.

3.2 Additional Assumptions

Assumption 3. (Action-value function class) We define

\[ \mathcal{F}_{\Upsilon, \infty} := \left\{ f(s,a; \theta) = f_0(s,a) + \int \mathbb{1}_{\{\theta^T(s,a) > 0\}}(s,a)^T c(\theta) \, dw : \|c(\theta)\|_{\infty} \leq \Upsilon / \sqrt{d} \right\} \]
Algorithm 3: MVPI with over-parameterized networks

1. **Input**: number of iteration \( K \), learning rate for natural policy gradient (resp. PPO) TD \( \eta_{\text{NPG}} \) (resp. \( \eta_{\text{PPO}} \)), temperature parameters \( \{\tau_k\}_{k=1}^K \).

2. **Initialization**: Initialize policy network \( f((s,a); \theta,b) \) as defined in Eq. (5). Set \( \tau_1 = 1 \). Initialize Q-network with \((b, \omega_1)\) similarly;

3. **for** \( k = 1, \cdots, K \)** do
   4. Sample a batch of transitions \( \{s_t,a_t,r_t,s'_t\}_t^T \) following current policy with size of \( T \);
   5. \( y = \frac{1}{T} \sum_{t=1}^T r_t \);
   6. **for** \( t = 1, \cdots, T \)** do
      7. \( \tilde{r}_t = r_t - \lambda r_t^2 + 2\lambda r_t y, a'_t \sim \pi(a|s'_t) \);
   8. **end**
   9. **repeat**
   10. **Q-value update**: update \( \omega_k \) according to Eq. (13);
   11. **if** select NPG update **then**
      12. update \( \theta_k \) according to Eq. (11);
   13. **else if** select PPO update **then**
      14. update \( \theta_k \) according to Eq. (12);
   15. **until** CONVERGE;
   16. **end**
17. **Output**: \( \pi_{\theta_K} \).

Where \( \mu : \mathbb{R}^d \to [0, 1] \) is a probability density function of \( \mathcal{N}(0, I_d/d) \). \( f_0(s,a) \) is the two-layer neural network corresponding to the initial parameter \( \Theta_{\text{init}} \), and \( \iota : \mathbb{R}^d \to \mathbb{R}^d \) is a weighted function. We assume that \( Q_\pi \in \mathcal{F}_{\Upsilon, \infty} \) for all \( \pi \).

**Assumption 4.** (Regularity of stationary distribution) For any policy \( \pi \), and \( \forall x \in \mathbb{R}^d, \forall \|x\|_2 = 1, \forall u > 0 \), we assume that there exists a constant \( c > 0 \) such that \( \mathbb{E}_{(s,a) \sim \pi}(1 \{\|x^\top(s,a)\| \leq u\}) \leq cu \).

Assumption 3 is a mild regularity condition on \( Q_\pi \), as \( \mathcal{F}_{\Upsilon, \infty} \) is a sufficiently rich function class and approximates a subset of the reproducing kernel Hilbert space (RKHS) \([71]\). Similar assumptions are widely imposed \([21, 44, 47]\). Assumption 4 is a regularity condition on the transition kernel \( \mathcal{P} \). Such regularity holds so long as \( \sigma_\pi \) has an upper bound density, satisfying most Markov chains.

In \([22]\) Lemma 4.15, they make a mistake in the proof. They accidentally flip a sign in \( y^* - \bar{y} \) when transitioning from the first equation in the proof to Eq.(4.15). This invalidates the conclusion in Eq.(4.17), an essential part of the proof. We tackle this issue by proposing the next assumption.

**Assumption 5.** (Convergence Rate of \( J(\pi) \)) We assume \( \pi^* \) (the optimal policy to the risk-averse objective function \( J_\lambda(\pi) \)) converges to the risk-neutral objective \( J(\pi) \) for both NPG and PPO with the over-parameterized neural network to be...
Algorithm 4: VARAC

1. **Input**: number of iteration $K$, learning rate for natural policy gradient (resp. PPO) TD $\eta_{NPG}$ (resp. $\eta_{PPO}$), temperature parameters $\{\tau_k\}_{k=1}^K$.

2. **Initialization**: Initialize policy network $f((s,a);\theta,b)$ as defined in Eq. (6). Set $\tau_1 = 1$. Initialize Q-network with $(b,\omega_1)$ similarly.

3. for $k = 1, \ldots, K$ do

4. Sample a batch of transitions $\{(s_t,a_t,r_t,s'_t)\}_{t=1}^T$ following current policy with size of $T$;

5. $y = \frac{1}{T} \sum_{t=1}^T r_t$;

6. **Q-value update**: update both networks’ $\omega_k$ according to Eq. (13);

7. Output $Q_k$ and $W_k$;

8. update $\theta_k$ with NPG or PPO;

9. end

10. **Output**: $\pi_{\theta_K}$;

$O(1/\sqrt{K})$. Specifically, there exists a constant $c_3 > 0$ such that,

$$J(\pi^*) - J(\pi_k) \leq \frac{c_3}{\sqrt{K}}$$

It was proved [20,21] that the optimal policy w.r.t the risk-neutral objective $J(\pi)$ obtained by NPG and PPO method with the over-parameterized two-layer neural network converges to the globally optimal policy at a rate of $O(1/\sqrt{K})$, where $K$ is the number of iteration. Since our method uses similar settings, we assume the convergence rates of risk-neutral objective $J(\pi)$ in our paper follow their results.

In the following subsections, we study STOPS’s convergence of global optimality and provide a proof sketch.

### 3.3 Proof of Theorem 1

We first present the analysis of policy evaluation error, which is induced by TD update in Line 9 of Algorithm 1. We characterize the policy evaluation error in the following lemma:

**Lemma 2.** (Policy Evaluation Error) We set learning rate of TD $\eta_{TD} = \min\{(1 - \gamma)/3(1 + \gamma)^2, 1/\sqrt{K_{TD}}\}$. Under Assumption 3 and 4, it holds that, with probability of $1 - \delta$,

$$\|\hat{Q}_{\omega_k} - \hat{Q}_{\pi_k}\|_{\mathbb{P}_{\pi_k}}^2 = \mathcal{O}(\gamma^3 m^{-1/2} \log(1/\delta) + \gamma^{5/2} m^{-1/4} \sqrt{\log(1/\delta)}$$

$$+ \gamma r_{max}^2 m^{-1/4} + \mathcal{T}^2 K_{TD}^{-1/2} + \mathcal{Y}),$$

(14)

where $\hat{Q}_{\pi_k}$ is the Q-value function of the augmented MDP, and $\hat{Q}_{\omega_k}$ is its estimator at the $k$-th iteration. We provide the proof and its supporting lemmas...
In the following, we establish the error induced by the policy update. Eq. (8) can be re-expressed as

$$J^y_\lambda(\pi) = \sum_{s,a} \sigma_{r_{s,a}} - \lambda r_{s,a}^2 + 2\lambda r_{s,a}y_{k+1} - \lambda y_{k+1}^2$$  \hspace{1cm} (15)$$

It can be shown that for $\pi$, $\max_y J^y_\lambda(\pi) = J_\lambda(\pi) [23,32]$. We denote the optimal policy to the augmented MDP associated with $y^*$ by $\pi^*(y^*)$. By definition, it
is obvious that $\pi^*$ and $\pi^*(y^*)$ are equivalent. For simplicity, we will use the unified term $\pi^*$ in the rest of the paper. We present Lemma 3 and 4.

**Lemma 3.** (Policy’s Performance Difference) For reward-volatility objective w.r.t. auxiliary variable $y$ as $J^y_\lambda(\pi)$ defined in Eq. (15). For any policy $\pi$ and $\pi'$, we have the following,

$$J^y_\lambda(\pi') - J^y_\lambda(\pi) = (1 - \gamma)^{-1} \mathbb{E}_{s \sim \nu_{\pi'}} \left[ \mathbb{E}_{a \sim \pi}[\tilde{Q}_{\pi,y}] - \mathbb{E}_{a \sim \pi}[\tilde{Q}_{\pi,y}] \right],$$

where $\tilde{Q}_{\pi,y}$ is the state-action value function of the augmented MDP, and its rewards are associated with $y$.

**Proof.** When $y$ is fixed,

$$J^y_\lambda(\pi') - J^y_\lambda(\pi) = \sum_{s,a} \sigma_{\pi'} \tilde{r}_{s,a} - \sum_{s,a} \sigma_{\pi} \tilde{r}_{s,a} = J(\pi') - J(\pi) \tag{16}$$

We then follow Lemma 6.1 in [72]:

$$J(\pi') - J(\pi) = (1 - \gamma)^{-1} \mathbb{E}_{(s,a) \sim \pi'} \left[ \tilde{A}_{\pi} \right] \tag{17}$$

where $\tilde{A}_{\pi} = \tilde{Q}_{\pi} - \tilde{V}_{\pi}$ is the advantage function of policy $\pi$. Meanwhile,

$$\mathbb{E}_{a \sim \pi'}[\tilde{A}_{\pi}] = \mathbb{E}_{a \sim \pi'}[\tilde{Q}_{\pi}] - \mathbb{V}_{\pi} = \mathbb{E}_{a \sim \pi'}[\tilde{Q}_{\pi}] - \mathbb{E}_{a \sim \pi}[\tilde{Q}_{\pi}] \tag{18}$$

From Eq. (16), Eq. (17) and Eq. (18), we complete the proof. $\square$

Lemma 3 is inspired by [72] and adopted by most work on global convergence [17, 20, 25]. Next, we derive an upper bound for the error of the critic update in Line 5 of Algorithm 1.

**Lemma 4.** (y Update Error) We characterize the error induced by the estimation of auxiliary variable $y$ w.r.t. the optimal value $y^*$ at $k$-th iteration as,

$$J^y_\lambda(\pi^*) - J^y_{\hat{\lambda}}(\pi^*) = \frac{c_3 r_{\text{max}}(1 - \gamma)^{\lambda}}{\sqrt{k}},$$

where $r_{\text{max}}$ is the bound of the original reward, and $c_3$ is a constant error term.

**Proof.** We start from the subproblem objective defined in Eq. (15) with $y^*$ and
\[
\hat{y}_k:
J^y_\lambda (\pi^*) - J^\hat{y}_\lambda (\pi^*)
= \left( \sum_{s,a} \sigma_{\pi^*} (r_{s,a} - \lambda r^2_{s,a} + 2\lambda r_{s,a}y^*) - \lambda y^* \right) - \\
\left( \sum_{s,a} \sigma_{\pi^*} (r_{s,a} - \lambda r^2_{s,a} + 2\lambda r_{s,a}\hat{y}_k) - \lambda \hat{y}_k \right)
= 2\lambda \left( \sum_{s,a} \sigma_{\pi^*} r_{s,a} \right) (y^* - \hat{y}_k) - \lambda (y^* - \hat{y}_k)
= (1 - \gamma) \lambda (y^* - \hat{y}_k, J(\pi^*) - J(\pi^* - \hat{J}(\pi))]
\]

where we obtain the final two equalities by the definition of \( J_\pi \) and \( y \).

Because \( r_{s,a} \) is upper-bounded by a constant \( r_{\text{max}} \), we have
\[
|y^* - \hat{y}_k| \leq 2r_{\text{max}}.
\]

Under Assumption 5, we have,
\[
J^y_\lambda (\pi^*) - J^\hat{y}_\lambda (\pi^*)
= \frac{2c_3r_{\text{max}}(1-\gamma)\lambda}{\sqrt{k}}.
\]

Thus we finish the proof.

From Lemma 3 and 4, we can also obtain the following Lemma.

**Lemma 5.** (Performance Difference on \( \pi \) and \( y \)) For reward-volatility objective w.r.t. auxiliary variable \( y \) as \( J^y_\lambda(\pi) \) defined in Eq. 15. For any \( \pi, y \) and the optimal \( \pi^*, y^* \), we have the following,
\[
J^y_\lambda (\pi^*) - J^\hat{y}_\lambda (\pi^*) = (1 - \gamma)^{-1} \mathbb{E}_{s \sim \nu_\pi} \mathbb{E}_{a \sim \pi^*} [\tilde{Q}_{\pi,y}] - \mathbb{E}_{a \sim \pi} [\tilde{Q}_{\pi,y}] + \frac{2c_3r_{\text{max}}(1-\gamma)\lambda}{\sqrt{k}}.
\]

where \( \tilde{Q}_{\pi,y} \) is the state-action value function of the augmented MDP, and its rewards are associated with \( y \).

**Proof.** It is easy to see that \( J^y_\lambda (\pi^*) - J^y_\lambda (\pi) = J^y_\lambda (\pi^*) - J^\hat{y}_\lambda (\pi^*) + J^\hat{y}_\lambda (\pi^*) - J^\hat{y}_\lambda (\pi) \).

Then replace \( J^y_\lambda (\pi^*) - J^\hat{y}_\lambda (\pi) \) with Lemma 3 and \( J^y_\lambda (\pi^*) - J^\hat{y}_\lambda (\pi^*) \) with Lemma 4, we finish the proof.

**Lemma 5** quantifies the performance difference of \( J^y_\lambda(\pi) \) between any pair \( \pi, y \) and the optimal \( \pi^*, y^* \), while **Lemma 3** only quantifies the performance difference of \( J^\hat{y}_\lambda(\pi) \) between \( \pi \) and \( \pi' \) when \( y \) is fixed.

We now study the global convergence of STOPS with neural PPO as the policy update component. First, we define the neural PPO update rule.
Lemma 6. \cite{20}. Let $\pi_{\theta_k} \propto \exp\{\tau_{k-1} f_{\theta_k}\}$ be an energy-based policy. We define the update

$$\hat{\pi}_{k+1} = \arg \max_{\pi} \mathbb{E}_{s \sim \nu_k} [\mathbb{E}_{s}[Q_{\omega_k}] - \beta_k KL(\pi_{\theta_k})]$$

, where $Q_{\omega_k}$ is the estimator of the exact action-value function $Q^{\pi_{\theta_k}}$. We have

$$\hat{\pi}_{k+1} \propto \exp\{\beta_{k}^{-1} Q_{\omega_k} + \tau_{k-1}^{-1} f_{\theta_k}\}$$

And to represent $\hat{\pi}_{k+1}$ with $\pi_{\theta_{k+1}} \propto \exp\{\tau_{k-1}^{-1} f_{\theta_{k+1}}\}$, we solve the following subproblem,

$$\theta_{k+1} = \arg \min_{\theta \in \mathbb{R}} \mathbb{E}_{(s,a) \sim \sigma_{\pi}} [(f_{\theta}(s,a) - \tau_{k+1}(\beta_{k}^{-1} Q_{\omega_k}(s,a)) + \tau_{k-1}^{-1} f_{\theta}(s,a))^2]$$

We analyze the policy improvement error in Line 13 of Algorithm 1 \cite{20}. Let $\varepsilon$ derivatives \cite{73}. We denote RHS in Eq. (20) by $\mathcal{PPO}$ improves error as follows: Under Assumptions 3 and 4, we set the learning rate of $\mathcal{PPO}$

$$\|f_{\theta} - \tau_{k+1}(\beta_{k}^{-1} Q_{\omega_k} + \tau_{k}^{-1} f_{\theta_k})\|^2$$

$$= \mathcal{O}(\mathcal{Y}^{3} m^{-1/2} \log(1/\delta) + \mathcal{Y}^{5/2} m^{-1/4} \sqrt{\log(1/\delta)}$$

$$+ \mathcal{Y}^{2} \max m^{-1/4} + \mathcal{Y}^{2} K_{\text{TD}}^{-1/2} + \mathcal{Y})$$

(19)

We quantify how the errors propagate in neural $\mathcal{PPO}$ \cite{20} in the following.

Lemma 7. \cite{20}. (Error Propagation) We have,

$$\mathbb{E}_{\sim \nu_{\pi}} \left[ \mathbb{E}_{a \sim \pi} \left[ \log(\pi_{\theta_{k+1}}/\pi_{k+1}) \right] - \mathbb{E}_{a \sim \pi_{\theta_k}} \left[ \log(\pi_{\theta_{k+1}}/\pi_{k+1}) \right] \right] \leq \tau_{k+1}^{-1} \varepsilon'' \varphi_{k+1} + \beta_{k}^{-1} \varepsilon'' \psi_{k}$$

(20)

$\varepsilon''$ are defined in Eq. (14) as well as Eq. (19). $\varphi_{k} = \mathbb{E}_{(s,a) \sim \sigma_{\pi}} \left[ \left( \frac{d\pi_{\theta_k}}{d\pi_{\theta_0}} - \frac{d\pi_{\theta_0}}{d\pi_{\theta_0}} \right)^2 \right]^{1/2}$, $\psi_{k} = \mathbb{E}_{(s,a) \sim \sigma_{\pi}} \left[ \left( \frac{d\pi_{\theta_k}}{d\pi_{\theta_0}} - \frac{d\pi_{\theta_0}}{d\pi_{\theta_0}} \right)^2 \right]^{1/2}$ are the Radon-Nikodym derivatives \cite{73}. We denote RHS in Eq. (20) by $\varepsilon_k = \tau_{k+1}^{-1} \varepsilon'' \varphi_{k+1} + \beta_{k}^{-1} \varepsilon'' \psi_{k}$.

Lemma 7 essentially quantifies the error from which we use the two-layer neural network to approximate the action-value function and policy instead of having access to the exact ones. Please refer to \cite{20} for complete proofs of Lemma 6 and 7.
We then characterize the difference between energy functions in each step \[20\]. Under the optimal policy \(\pi^*\),

**Lemma 8.** \[20\]. *(Stepwise Energy Function difference)* Under the same condition of Lemma 7, we have

\[
E_{s \sim \nu^*}\left[\|\tau_{k+1}^{-1}f_{\theta_{k+1}} - \tau_k^{-1}f_{\theta_k}\|_\infty^2\right] \leq 2\varepsilon_k' + 2\beta^{-2}U,
\]

where \(\varepsilon_k' = |A|\tau_k^{-2}\epsilon_k^2\)

and \(U = 2E_{s \sim \nu^*}\left[\max_{a \in A}(\tilde{Q}_\omega)^2\right] + 2\Upsilon^2\).

**Proof.** By the triangle inequality, we get the following,

\[
\|\tau_{k+1}^{-1}f_{\theta_{k+1}} - \tau_k^{-1}f_{\theta_k}\|_\infty^2 \leq 2\left(\|\tau_{k+1}^{-1}f_{\theta_{k+1}} - \tau_k^{-1}f_{\theta_k} - \beta^{-1}\tilde{Q}_\omega\|_\infty^2 + \|\beta^{-1}\tilde{Q}_\omega\|_\infty^2\right)
\]

(22)

We take the expectation of both sides of Eq. (22) with respect to \(s \sim \nu^*\). With the 1-Lipshitz continuity of \(\tilde{Q}_\omega\) in \(\omega\) and \(\|\omega - \Theta_{\text{init}}\|_2 \leq \Upsilon\), we have,

\[
E_{\nu^*}\left[\|\tau_{k+1}^{-1}f_{\theta_{k+1}} - \tau_k^{-1}f_{\theta_k}\|_\infty^2\right] \leq 2(|A|\tau_k^{-2}\epsilon_k^2 + E_{s \sim \nu^*}\left[\max_{a \in A}(\tilde{Q}_\omega)^2\right] + \Upsilon^2)
\]

Thus complete the proof. \(\square\)

We then derive a difference term associated with \(\pi_{k+1}\) and \(\pi_{\theta_k}\), where at the \(k\)-th iteration \(\pi_{k+1}\) is the solution for the following subproblem,

\[
\pi_{k+1} = \arg\max_{\pi} \left\{ E_{s \sim \nu^*}[E_{a \sim \pi}[\tilde{Q}_{\tau_k,\hat{y}_k} - \beta\text{KL}(\pi||\pi_{\theta_k})]] \right\}
\]

and \(\pi_{\theta_k}\) is the policy parameterized by the two-layered over-parameterized neural network. The following lemma establishes the one-step descent of the KL-divergence in the policy space:

**Lemma 9.** *(One-step difference of \(\pi\)) For \(\pi_{k+1}\) and \(\pi_{\theta_k}\), we have

\[
\begin{align*}
\text{KL}(\pi^*||\pi_{\theta_k}) - \text{KL}(\pi^*||\pi_{\theta_{k+1}}) \\
\geq \left( E_{a \sim \pi^*}[\log(\pi_{\theta_{k+1}})] - E_{a \sim \pi_{\theta_k}}[\log(\pi_{\theta_k})]\right) \\
+ \beta^{-1}\left( E_{a \sim \pi^*}[\tilde{Q}_{\pi_k,\hat{y}_k}] - E_{a \sim \pi_{\theta_k}}[\tilde{Q}_{\pi_k,\hat{y}_k}]\right) \\
+ \frac{1}{2}\|\pi_{\theta_{k+1}} - \pi_{\theta_k}\|_2^2 + \left( E_{a \sim \pi_{\theta_k}}[\tau_{k+1}^{-1}f_{\theta_{k+1}} - \tau_k^{-1}f_{\theta_k}] \\
- E_{a \sim \pi_{\theta_k}}[\tau_{k+1}^{-1}f_{\theta_{k+1}} - \tau_k^{-1}f_{\theta_k}]\right)
\end{align*}
\]

(23)
We then have,

\[ \text{KL}(\pi^*||\pi_k) - \text{KL}(\pi^*||\pi_{k+1}) = E_{a\sim\pi^*}[\log(\frac{\pi_{k+1}}{\pi_k})] \]

(By definition, \(\text{KL}(\pi_{k+1}||\pi_k) = E_{a\sim\pi_{k+1}}[\log(\frac{\pi_{k+1}}{\pi_k})] \))

\[ = (E_{a\sim\pi^*}[\log(\frac{\pi_{k+1}}{\pi_k})] - E_{a\sim\pi_{k+1}}[\log(\frac{\pi_{k+1}}{\pi_k})]) + \text{KL}(\pi_{k+1}||\pi_k) \]

We then add and subtract terms.

\[ = E_{a\sim\pi^*}[\log(\frac{\pi_{k+1}}{\pi_k})] - E_{a\sim\pi_{k+1}}[\log(\frac{\pi_{k+1}}{\pi_k})] + \text{KL}(\pi_{k+1}||\pi_k) \]

We then add and subtract terms,

\[ = E_{a\sim\pi^*}[\log(\frac{\pi_{k+1}}{\pi_k})] - E_{a\sim\pi_{k+1}}[\log(\frac{\pi_{k+1}}{\pi_k})] + \text{KL}(\pi_{k+1}||\pi_k) \]

\[ = (E_{a\sim\pi^*}[\log(\frac{\pi_{k+1}}{\pi_k})] - E_{a\sim\pi_{k+1}}[\log(\frac{\pi_{k+1}}{\pi_k})]) + \text{KL}(\pi_{k+1}||\pi_k) \]

Rearrange the terms and we get,

\[ = (E_{a\sim\pi^*}[\log(\frac{\pi_{k+1}}{\pi_k})] - E_{a\sim\pi_{k+1}}[\log(\frac{\pi_{k+1}}{\pi_k})]) + \text{KL}(\pi_{k+1}||\pi_k) \]

Recall that \(\pi_{k+1} \propto \exp\{T_k^{-1}f_{\theta_k} + \beta^{-1}\hat{Q}_{\pi_k} \}.\) We define the two normalization factors associated with ideal improved policy \(\pi_{k+1}\) and the current parameterized policy \(\pi_k\) as,

\[ Z_{k+1}(s) := \sum_{a'\in A} \exp(T_k^{-1}f_{\theta_k}(s, a') + \beta^{-1}\hat{Q}_{\pi_k}(s, a')) \]

\[ Z_{\theta_k+1}(s) := \sum_{a'\in A} \exp(T_k^{-1}f_{\theta_k+1}(s, a')) \]

We then have,

\[ \pi_{k+1}(a|s) = \frac{\exp(T_k^{-1}f_{\theta_k}(s, a) + \beta^{-1}\hat{Q}_{\pi_k}(s, a))}{Z_{k+1}(s)}, \]

\[ \pi_{\theta_k+1}(a|s) = \frac{\exp(T_k^{-1}f_{\theta_k+1}(s, a))}{Z_{\theta_k+1}(s)} \]

For any \(\pi, \pi'\) and \(k\), we have,

\[ E_{a\sim\pi}[\log Z_{\theta_k+1}] - E_{a\sim\pi'}[\log Z_{\theta_k+1}] = 0 \]

\[ E_{a\sim\pi}[\log Z_{k+1}] - E_{a\sim\pi'}[\log Z_{k+1}] = 0 \]
Now we look back at a few terms on RHS from Eq. (24):

\[
\mathbb{E}_{a \sim \pi^*} \left[ \log(\pi_{\theta_{k+1}}) + \beta^{-1} \tilde{Q}_{\pi_{k+1}, \hat{y}_k} \right] - \mathbb{E}_{a \sim \pi_{\theta_k}} \left[ \log(\pi_{\theta_k}) + \beta^{-1} \tilde{Q}_{\pi_{\theta_k}, \hat{y}_k} \right] \\
= \left( \mathbb{E}_{a \sim \pi^*} [\tau_{k+1}^{-1} f_{\theta_k} + \beta^{-1} \tilde{Q}_{\pi_{k+1}, \hat{y}_k} - \log Z_{\theta_{k+1}}] - \mathbb{E}_{a \sim \pi_{\theta_k}} [\tau_{k}^{-1} f_{\theta_k} + \beta^{-1} \tilde{Q}_{\pi_{\theta_k}, \hat{y}_k} - \log Z_{\theta_k}] \right) \\
= \mathbb{E}_{a \sim \pi^*} \left[ \log \frac{\exp\{\tau_{k+1}^{-1} f_{\theta_k} + \beta^{-1} \tilde{Q}_{\pi_{k+1}, \hat{y}_k}\}}{Z_{k+1}} \right] - \mathbb{E}_{a \sim \pi_{\theta_k}} \left[ \log \frac{\exp\{\tau_{k}^{-1} f_{\theta_k} + \beta^{-1} \tilde{Q}_{\pi_{\theta_k}, \hat{y}_k}\}}{Z_{k+1}} \right] \\
= \mathbb{E}_{a \sim \pi^*} [\log \pi_{\theta_{k+1}}] - \mathbb{E}_{a \sim \pi_{\theta_k}} [\log \pi_{\theta_k}] \\
\text{(29)}
\]

For Eq. (29), we obtain the first equality by Eq. (26). Then, by swapping Eq. (27) with Eq. (28), we obtain the second equality. We achieve the concluding step with the definition in Eq. (25). Following a similar logic, we have,

\[
\mathbb{E}_{a \sim \pi_{\theta_k}} [\log(\pi_{\theta_{k+1}})] - \mathbb{E}_{a \sim \pi_{\theta_k}} [\log(\pi_{\theta_k})] \\
= \mathbb{E}_{a \sim \pi_{\theta_k}} [\tau_{k+1}^{-1} f_{\theta_{k+1}} - \log Z_{\theta_{k+1}} - \tau_{k}^{-1} f_{\theta_k} + \log Z_{\theta_k}] - \mathbb{E}_{a \sim \pi_{\theta_k}} [\tau_{k+1}^{-1} f_{\theta_{k+1}} - \tau_{k}^{-1} f_{\theta_k}] \\
= \mathbb{E}_{a \sim \pi_{\theta_k}} [\tau_{k+1}^{-1} f_{\theta_{k+1}} - \tau_{k}^{-1} f_{\theta_k} - \mathbb{E}_{a \sim \pi_{\theta_{k+1}}} [\tau_{k+1}^{-1} f_{\theta_{k+1}} - \tau_{k}^{-1} f_{\theta_k}]] \\
\text{(30)}
\]

Finally, by using the Pinsker’s inequality [74], we have,

\[
\text{KL}(\pi_{\theta_{k+1}} \| \pi_{\theta_k}) \geq 1/2 \| \pi_{\theta_{k+1}} - \pi_{\theta_k} \|^2 \\
\text{(31)}
\]

Plugging Eqs. (29), (30), and (31) into Eq. (24), we have

\[
\text{KL}(\pi^* \| \pi_{\theta_k}) - \text{KL}(\pi^* \| \pi_{\theta_{k+1}}) \\
\geq \left( \mathbb{E}_{a \sim \pi^*} [\log(\pi_{\theta_{k+1}})] - \mathbb{E}_{a \sim \pi_{\theta_k}} [\log(\pi_{\theta_{k+1}})] \right) - \left( \mathbb{E}_{a \sim \pi_{\theta_k}} [\log(\pi_{\theta_k})] - \mathbb{E}_{a \sim \pi_{\theta_{k+1}}} [\log(\pi_{\theta_{k+1}})] \right) - \log(\pi_{\theta_k}) + \beta^{-1} (\mathbb{E}_{a \sim \pi^*} [\tilde{Q}_{\pi_{k+1}, \hat{y}_k}] - \mathbb{E}_{a \sim \pi_{\theta_k}} [\tilde{Q}_{\pi_{k+1}, \hat{y}_k}]) \\
\geq 1/2 \| \pi_{\theta_{k+1}} - \pi_{\theta_k} \|^2 + \left( \mathbb{E}_{a \sim \pi_{\theta_k}} [\tau_{k+1}^{-1} f_{\theta_{k+1}} - \tau_{k}^{-1} f_{\theta_k}] - \mathbb{E}_{a \sim \pi_{\theta_{k+1}}} [\tau_{k+1}^{-1} f_{\theta_{k+1}} - \tau_{k}^{-1} f_{\theta_k}] \right)
\]

Rearranging the terms, we obtain Lemma 9.

Lemma 9 serves as an intermediate-term for the major result’s proof. We obtain upper bounds by telescoping this term in Theorem 1. Now we are ready to present the proof for Theorem 1.
Proof. First we take expectation of both sides of Eq. \((23)\) with respect to \(s \sim \nu_{\pi}\) from Lemma \([3]\) and insert Eq. \((20)\) to obtain,

\[
\begin{align*}
\mathbb{E}_{s \sim \nu_{\pi}}[\text{KL}(\pi^* || \pi_{\theta_{k+1}})] & - \mathbb{E}_{s \sim \nu_{\pi}}[\text{KL}(\pi^* || \pi_{\theta_k})] \\
\leq & \varepsilon_k - \beta^{-1} \mathbb{E}_{s \sim \nu_{\pi}}[\mathbb{E}_{a \sim \pi^*}[\hat{Q}_{\pi_k, \hat{\gamma}_k}]] - \mathbb{E}_{a \sim \pi_{\theta_k}}[\hat{Q}_{\pi_k, \hat{\gamma}_k}] \\
& - 1/2 \mathbb{E}_{s \sim \nu_{\pi}}[||\pi_{\theta_{k+1}} - \pi_{\theta_k}||_2^2] - \mathbb{E}_{s \sim \nu_{\pi}}[\mathbb{E}_{a \sim \pi_{\theta_k}}[\tau_{k+1}^{-1} f_{\theta_{k+1}} - \tau_k^{-1} f_{\theta_k}]] \\
& \left[\tau_{k+1}^{-1} f_{\theta_{k+1}} - \tau_k^{-1} f_{\theta_k}\right]
\end{align*}
\]

Then, by Lemma \([3]\) we have,

\[
\begin{align*}
\beta^{-1} \mathbb{E}_{s \sim \nu_{\pi}}[\mathbb{E}_{a \sim \pi^*}[\hat{Q}_{\pi_k, \hat{\gamma}_k}]] - \mathbb{E}_{a \sim \pi_{\theta_k}}[\hat{Q}_{\pi_k, \hat{\gamma}_k}] \\
= & \beta^{-1} (1 - \gamma) (J^k_{\pi^*} (\pi^*) - J^\theta_k (\pi))
\end{align*}
\]

And with Hölder’s inequality, we have,

\[
\begin{align*}
\mathbb{E}_{s \sim \nu_{\pi}}[\mathbb{E}_{a \sim \pi_{\theta_k}}[\tau_{k+1}^{-1} f_{\theta_{k+1}} - \tau_k^{-1} f_{\theta_k}]] - \mathbb{E}_{a \sim \pi_{\theta_k}}[\tau_{k+1}^{-1} f_{\theta_{k+1}} - \tau_k^{-1} f_{\theta_k}]
\end{align*}
\]

Insert Eqs. \((33)\) and \((34)\) into Eq. \((32)\), we have,

\[
\begin{align*}
\mathbb{E}_{s \sim \nu_{\pi}}[\text{KL}(\pi^* || \pi_{\theta_{k+1}})] & - \mathbb{E}_{s \sim \nu_{\pi}}[\text{KL}(\pi^* || \pi_{\theta_k})] \\
\leq & \varepsilon_k - (1 - \gamma) \beta^{-1} (J^k_{\pi^*} (\pi^*) - J^\theta_k (\pi)) - 1/2 \mathbb{E}_{s \sim \nu_{\pi}}[||\pi_{\theta_{k+1}} - \pi_{\theta_k}||_2^2] + \mathbb{E}_{s \sim \nu_{\pi}}[||\tau_{k+1}^{-1} f_{\theta_{k+1}} - \tau_k^{-1} f_{\theta_k}||_2] \\
& + (1 - \gamma) \beta^{-1} \left[\left(1 - \gamma\right) J^\theta_k (\pi) - J^\theta_k (\pi) + J^\theta_k (\pi)\right] + 1/2 \mathbb{E}_{s \sim \nu_{\pi}}[||\tau_{k+1}^{-1} f_{\theta_{k+1}} - \tau_k^{-1} f_{\theta_k}||_2^2]
\end{align*}
\]

The second inequality holds by using the inequality \(2AB - B^2 \leq A^2\), with a minor abuse of notations. Here, \(A := ||\tau_{k+1}^{-1} f_{\theta_{k+1}} - \tau_k^{-1} f_{\theta_k}||_2\) and \(B := ||\pi_{\theta_k} - \pi_{\theta_{k+1}}||_1\). Then, by plugging in Lemma \([3]\) and Eq. \((21)\) we end up with,

\[
\begin{align*}
\mathbb{E}_{s \sim \nu_{\pi}}[\text{KL}(\pi^* || \pi_{\theta_{k+1}})] & - \mathbb{E}_{s \sim \nu_{\pi}}[\text{KL}(\pi^* || \pi_{\theta_k})] \\
\leq & \varepsilon_k - (1 - \gamma) \beta^{-1} (J^k_{\pi^*} (\pi^*) - J^\theta_k (\pi)) + (1 - \gamma) \beta^{-1} \left(2c^3 M (1 - \gamma) \lambda \sqrt{k}\right) + (\varepsilon'_k + \beta_k^{-2} U)
\end{align*}
\]
Rearrange Eq. (35), we have

\[(1 - \gamma)\beta^{-1}(J_{\Lambda}^*(\pi^*) - J_{\Lambda}^{\theta_k}(\pi_k))\]
\[\leq \mathbb{E}_{s \sim \nu_{\pi^*}}[\text{KL}(\pi^* || \pi_{\theta_k})] - \mathbb{E}_{s \sim \nu_{\pi^*}}[\text{KL}(\pi^* || \pi_{\theta_k+1})]\]
\[+ \left(\frac{2c_3M(1 - \gamma)^2}{\beta^2K}\right) + \varepsilon_k + \varepsilon'_k + \beta^{-2}K\]

And then telescoping Eq. (36) results in,

\[(1 - \gamma)\sum_{k=1}^{K} \beta^{-1} \min_{k \in [K]} (J_{\Lambda}^*(\pi^*) - J_{\Lambda}^{\theta_k}(\pi_k))\]
\[\leq (1 - \gamma)\sum_{k=1}^{K} \beta^{-1}(J_{\Lambda}^*(\pi^*) - J_{\Lambda}^{\theta_k}(\pi_k))\]
\[\leq \mathbb{E}_{s \sim \nu_{\pi^*}}[\text{KL}(\pi^* || \pi_0)] - \mathbb{E}_{s \sim \nu_{\pi^*}}[\text{KL}(\pi^* || \pi_K)]\]
\[+ \lambda r_{\max}(1 - \gamma)\sum_{k=1}^{K} \beta^{-1} \left(\frac{2c_3}{\sqrt{K}}\right) + U \sum_{k=1}^{K} \beta^{-2}K\]
\[+ \sum_{k=1}^{K}(\varepsilon_k + \varepsilon'_k)\] (37)

We complete the final step in Eq. (37) by plugging in Lemma 4 and Eq. (20). For the observation we make in the proof of Theorem 2,

1. \(\mathbb{E}_{s \sim \nu_{\pi^*}}[\text{KL}(\pi^* || \pi_0)] \leq \log |\mathcal{A}|\) due to the uniform initialization of policy.
2. \(\text{KL}(\pi^* || \pi_K)\) is a non-negative term.

We now have,

\[\min_{k \in [K]} J_{\Lambda}^*(\pi^*) - J_{\Lambda}^{\theta_k}(\pi_k)\]
\[\leq \frac{\log |\mathcal{A}| + UK\beta^{-2} + \sum_{k=1}^{K}(\varepsilon_k + \varepsilon'_k)}{(1 - \gamma)K\beta^{-1}}\]
\[+ \lambda r_{\max}(1 - \gamma)\left(\frac{2c_3}{\sqrt{K}}\right)\]

Replacing \(\beta\) with \(\beta_0\sqrt{K}\) finishes the proof.

3.4 Proof of Theorem 2

In the following part, we focus the convergence of neural NPG. We first define the following terms under neural NPG update rule.
Lemma 10. \[21\] For energy-based policy \(\pi_\theta\), we have policy gradient and Fisher information matrix,

\[
\nabla_\theta J(\pi_\theta) = \tau E_{d_{\pi_\theta}(s,a)}[Q_{\pi_\theta}(s,a) - E_{\pi_\theta}[\phi_\theta(s,a)]] \\
F(\theta) = \tau^2 E_{d_{\pi_\theta}(s,a)}[(\phi_\theta(s,a) - E_{\pi_\theta}[\phi_\theta(s,a)])]^T 
\]

We then derive an upper bound for \(J^*_\lambda(\pi^*) - J^*_\lambda(\pi_k)\) for the neural NPG method in the following lemma:

Lemma 11. (One-step difference of \(\pi\)) It holds that, with probability of \(1 - \delta\),

\[
(1 - \gamma)(J^*_\lambda(\pi^*) - J^*_\lambda(\pi_k)) \leq \eta_{NPG} E_{s \sim \nu^*}[KL(\pi^*||\pi_k) - KL(\pi^*||\pi_{k+1})] + \eta_{NPG} (9\Upsilon^2 + \tau_{\text{max}}^2) + 2c_0'\epsilon'_k + \eta_{NPG} \epsilon''_k,
\]

where

\[
\epsilon'_k = O(\Upsilon^3 m^{-1/2} \log(1/\delta) + T^{5/2} m^{-1/4} \sqrt{\log(1/\delta)}) \\
+ \Upsilon^2 T_{\text{TD}} + \Upsilon - 1/4, \\
\epsilon''_k = 8\eta_{NPG} \Upsilon^{1/2} c_0' \sigma_\xi^{1/2} T^{-1/4} + O((\tau_{k+1} + \eta_{NPG}) T^{3/2} m^{-1/4}) + \eta_{NPG} \Upsilon^{5/4} m^{-1/8},
\]

\(c_0\) is defined in Assumption 2 and \(\sigma_\xi\) is defined in Assumption 1. Meanwhile, \(\Upsilon\) is the radius of the parameter space, \(m\) is the width of the neural network, and \(T\) is the sample batch size.

Proof. We start from the following,

\[
KL(\pi^*||\pi_k) - KL(\pi^*||\pi_{k+1}) - KL(\pi_{k+1}||\pi_k) \\
= E_{a \sim \pi^*}[\log(\frac{\pi_{k+1}}{\pi_k})] - E_{a \sim \pi_{k+1}}[\log(\frac{\pi_{k+1}}{\pi_k})] \\
(\text{by KL's definition})
\]

We now show the building blocks of the proof. First, we add and subtract a few terms to RHS of Eq. (38), then take the expectation of both sides with respect to \(s \sim \nu^*\). Rearrange these terms, we get,

\[
E_{s \sim \nu^*}[KL(\pi^*||\pi_k) - KL(\pi^*||\pi_{k+1}) - KL(\pi_{k+1}||\pi_k)] \\
= \eta_{NPG} E_{s \sim \nu^*}[E_{a \sim \pi^*}[\tilde{Q}_{\pi_k, \hat{g}_k}] - E_{a \sim \pi_{k+1}}[\tilde{Q}_{\pi_{k+1}, \hat{g}_k}]] + H_k
\]

15
where $H_k$ is denoted by,

$$H_k := E_{s \sim \nu_{s*}} \left[ E_{a \sim \pi^*} \left[ \log \left( \frac{\pi_k + 1}{\pi_k} \right) \right] - \eta_{\text{NPG}} \tilde{Q}_{\omega_k} \right]$$

$$- E_{a \sim \pi_k} \left[ \log \left( \frac{\pi_k + 1}{\pi_k} \right) \right] - \eta_{\text{NPG}} E_{s \sim \nu_{s*}} \left[ E_{a \sim \pi^*} \left( \tilde{Q}_{\omega_k} - \tilde{Q}_{\pi_k \omega_k} \right) \right]$$

$$+ \eta_{\text{NPG}} E_{s \sim \nu_{s*}} \left[ E_{a \sim \pi_k} \left( \tilde{Q}_{\omega_k} - \tilde{Q}_{\pi_k \omega_k} \right) \right]$$

$$+ E_{s \sim \nu_{s*}} \left[ \log \left( \frac{\pi_k + 1}{\pi_k} \right) \right]$$

$$- E_{a \sim \pi_{k+1}} \left[ \log \left( \frac{\pi_k + 1}{\pi_k} \right) \right]$$

(40)

By Lemma 3, we have

$$\eta_{\text{NPG}} E_{s \sim \nu_{s*}} \left[ E_{a \sim \pi^*} \left[ \tilde{Q}_{\pi_k \omega_k} \right] - E_{a \sim \pi_k} \left[ \tilde{Q}_{\pi_k \omega_k} \right] \right]$$

$$= \eta_{\text{NPG}} \left( \gamma \right) \left( J_{\lambda}^{\hat{y}_k} (\pi^*) - J_{\lambda}^{\hat{y}_k} (\pi_k) \right)$$

(41)

Insert Eqs. (41) back to Eq. (39), we have,

$$\eta_{\text{NPG}} \left( \gamma \right) \left( J_{\lambda}^{\hat{y}_k} (\pi^*) - J_{\lambda}^{\hat{y}_k} (\pi_k) \right)$$

$$= E_{s \sim \nu_{s*}} \left[ KL(\pi^* || \pi_k) - KL(\pi^* || \pi_{k+1}) - KL(\pi_{k+1} || \pi_k) \right]$$

$$- H_k$$

$$\leq E_{s \sim \nu_{s*}} \left[ KL(\pi^* || \pi_k) - KL(\pi^* || \pi_{k+1}) - KL(\pi_{k+1} || \pi_k) \right]$$

$$+ |H_k|$$

(42)

We reach the final inequality of Eq. (42) by algebraic manipulation. Second, we follow Lemma 5.5 of [21] and obtain an upper bound for Eq. (40). Specifically, with probability of $1 - \delta$,

$$E_{a \sim \text{init}} \left[ |H_k| - E_{s \sim \nu_{s*}} \left[ KL(\pi_{k+1} || \pi_k) \right] \right]$$

$$\leq \eta_{\text{NPG}}^2 (9 \Upsilon^2 + r_{\max}^2) + 2 \eta_{\text{NPG}} c_0 \epsilon'_k + \epsilon''_k$$

(43)

The expectation is taken over randomness. With these building blocks of Eqs. (42) and (43), we are now ready to reach the concluding inequality. Plugging Eqs. (43) back into Eq. (42), we end up with, with probability of $1 - \delta$,

$$\eta_{\text{NPG}} \left( \gamma \right) \left( J_{\lambda}^{\hat{y}_k} (\pi^*) - J_{\lambda}^{\hat{y}_k} (\pi_k) \right)$$

$$\leq E_{s \sim \nu_{s*}} \left[ KL(\pi^* || \pi_k) - KL(\pi^* || \pi_{k+1}) \right]$$

$$+ \eta_{\text{NPG}}^2 (9 \Upsilon^2 + r_{\max}^2) + 2 \eta_{\text{NPG}} c_0 \epsilon'_k + \epsilon''_k$$

(44)

Dividing both sides of Eq. (44) by $\eta_{\text{NPG}}$ completes the proof. The details are included in the Appendix.

We have the following Lemma to bound the error terms $H_k$ defined in Eq. (40) of Lemma 11.
Lemma 12. [21] Under Assumptions [4] we have

\[
E_{a \sim \text{init}} \left[ |H_k| - E_{a \sim \nu_k} [ \text{KL}(\pi_k \| \pi_{k+1})] \right] \
\leq \eta_{\text{NPG}}^2 (9Y^2 + r_{\text{max}}^2) + \eta_{\text{NPG}} (\varphi_k + \psi_k) \epsilon_k' + \epsilon_k''
\]

Here the expectation is taken over all the randomness. We have \( \epsilon_k' := \|Q_{\omega_k} - Q_{\pi_k}\|_{\nu_k}^2 \) and

\[
\epsilon_k'' = \sqrt{2} Y^{1/2} \eta_{\text{NPG}} (\varphi_k + \psi_k) \tau_k^{-1} \left\{ E_{(s,a) \sim \sigma_{\pi_k}} [\|\xi_k(\omega_k)\|_2^2] \
+ E_{(s,a) \sim \sigma_{\pi_k}} [\|\xi_k(\omega_k)\|_2^2] \right\}^{1/2} \
+ O((\tau_k + \eta_{\text{NPG}}) Y^{3/2} m^{-1/4} + \eta_{\text{NPG}} Y^{3/4} m^{-1/8}).
\]

Recall \( \xi_k(\omega_k) \) and \( \xi_k(\omega_k) \) are defined in Assumption [2] while \( \varphi_k, \psi_k, \varphi_k', \) and \( \psi_k \) are defined in Assumption [3].

Please refer to [21] for complete proof. Finally, we are ready to show the proof for Theorem 2.

Proof. First, we combine Lemma 4 and 11 to get the following:

\[
(1 - \gamma) (J^{\star}_{\lambda}(\pi^{\star}) - J^{\hat{\pi}}_{\lambda}(\pi^{\star}) + J^{\hat{\pi}}_{\lambda}(\pi^{\star}) - J^{\hat{\pi}}_{\lambda}(\pi_k)) \
\leq \eta_{\text{NPG}}^{-1} E_{s \sim \nu_k} [\text{KL}(\pi^{\star} \| \pi_k) - \text{KL}(\pi^{\star} \| \pi_{k+1})] \
+ \eta_{\text{NPG}} (9Y^2 + r_{\text{max}}^2) + 2c_0 \epsilon_k' + \eta_{\text{NPG}}^{-1} \epsilon_k'' \
+ 2c_3 M (1 - \gamma)^2 \lambda 
\]

We can then see this:

1. \( E_{s \sim \nu_k} [\text{KL}(\pi^{\star} \| \pi_1)] \leq \log |A| \) due to the uniform initialization of policy.

2. \( \text{KL}(\pi^* \| \pi_{K+1}) \) is a non-negative term.

And by setting \( \eta_{\text{NPG}} = 1/\sqrt{K} \) and telescoping Eq. (45), we obtain,

\[
(1 - \gamma) \min_{k \in [K]} (J^{\star}_{\lambda}(\pi^{\star}) - J^{\hat{\pi}}_{\lambda}(\pi_k)) \
\leq (1 - \gamma) \frac{1}{K} \sum_{k=1}^{K} E (J^{\star}_{\lambda}(\pi^{\star}) - J^{\hat{\pi}}_{\lambda}(\pi_k)) \
\leq \frac{1}{\sqrt{K}} (E_{s \sim \nu_k} [\text{KL}(\pi^{\star} \| \pi_1)] + 9Y^2 + r_{\text{max}}^2) + \frac{1}{K} \sum_{k=1}^{K} (2\sqrt{K} c_0 \epsilon_k' + \eta_{\text{NPG}}^{-1} \epsilon_k'' + 2c_3 M (1 - \gamma)^2 \lambda) \]

(46)
plug $\epsilon_k'$ and $\epsilon_k''$ defined in Lemma 11 into Eq. (46), and set $\epsilon_k$ as,

$$
\epsilon_k = \sqrt{8c_0}Y^{1/2}T^{-1/4} \\
+ O\left(\tau_{k+1}Y^{3/2}m^{-1/4} + Y^{5/4}m^{-1/8}\right) \\
+ c_0O(Y^3m^{-1/2}log(1/\delta) + Y^{5/2}m^{-1/4}\sqrt{log(1/\delta)}) \\
+ Y_{\tau_{\max}}^2m^{-1/4} + T^2K_{TD}^{-1} + \mathcal{Y}
$$

we complete the proof.

3.5 Proof of Lemma 1

Proof. First, we have $E[G] = 1 \gamma E[R]$, i.e., the per-step reward $R$ is an unbiased estimator of the cumulative reward $G$. Second, it is proved that $V(G) \leq V(R)$ \cite{11}. Given $\lambda \geq 0$, summing up the above equality and inequality, we have

$$
\frac{1}{1 - \gamma} J^G(\pi) = \frac{1}{1 - \gamma} \left( E[R] - \frac{\lambda}{1 - \gamma} V(R) \right) \\
\leq E[G] - \lambda V(G) = J^G(\pi).
$$

It completes the proof.

3.6 Proof of Lemma 2

We first provide the supporting lemmas for Lemma 2. We define the local linearization of $f((s,a); \theta)$ defined in Eq. (4) at the initial point $\Theta_{init}$ as,

$$
\hat{f}((s,a); \theta) = \frac{1}{\sqrt{m}} \sum_{v=1}^{\infty} b_v \mathbb{I}\{[\Theta_{init}]_v^T (s,a) > 0\} [\theta]_v^T (s,a)
$$

We then define the following function spaces,

$$
\mathcal{F}_{Y,m} := \left\{ \frac{1}{\sqrt{m}} \sum_{v=1}^{\infty} b_v \mathbb{I}\{[\Theta_{init}]_v^T (s,a) > 0\} [\theta]_v^T (s,a) : \|\theta - \Theta_{init}\|_2 \leq Y \right\},
$$

and

$$
\mathcal{F}_{Y,m} := \left\{ \frac{1}{\sqrt{m}} \sum_{v=1}^{\infty} b_v \mathbb{I}\{[\Theta_{init}]_v^T (s,a) > 0\} [\theta]_v^T (s,a) : \|[\theta]_v - [\Theta_{init}]_v\|_\infty \leq Y/\sqrt{md} \right\}.
$$
\([\Theta_{\text{init}}], \sim \mathcal{N}(0, I_d/d)\) and \(b_v \sim \text{Unif}\{\{-1, 1\}\}\) are the initial parameters. By the definition, \(\mathcal{F}_{Y,m}\) is a subset of \(\mathcal{F}_{Y,\infty}\). The following lemma characterizes the deviation of \(\mathcal{F}_{Y,m}\) from \(\mathcal{F}_{Y,\infty}\).

**Lemma 13. (Projection Error)** [71]. Let \(f \in \mathcal{F}_{Y,\infty}\), where \(\mathcal{F}_{Y,\infty}\) is defined in Assumption 3. For any \(\delta > 0\), it holds with probability at least \(1 - \delta\) that

\[
\|\Pi_{\mathcal{F}_{Y,m}} f - f\|_2 \leq \Upsilon m^{-1/2}[1 + \sqrt{2\log(1/\delta)}]
\]

where \(\varsigma\) is any distribution over \(S \times A\).

Please refer to [71] for a detail proof.

**Lemma 14. (Linearization Error) Under Assumption 4**, for all \(\theta \in \mathcal{D}\), where \(\mathcal{D} = \{\xi \in \mathbb{R}^{md} : \|\xi - \Theta_{\text{init}}\|_2 \leq \Upsilon\}\), it holds that,

\[
\mathbb{E}_{\nu_v} \left[ \left( f((s,a);\theta) - \hat{f}((s,a);\theta) \right)^2 \right] \leq \frac{4c_1 \Upsilon^3}{\sqrt{m}}
\]

where \(c_1 = c \sqrt{\mathbb{E}_{\xi}(0, I_d/d)[1/\|\xi(s,a)\|_2^2]}\), and \(c\) is defined in Assumption 4.

**Proof.** We start from the definitions in Eq. (4) and Eq. (47),

\[
\mathbb{E}_{\nu_v} \left[ \left( f((s,a);\theta) - \hat{f}((s,a);\theta) \right)^2 \right] = \mathbb{E}_{\nu_v} \left[ \left( \frac{1}{\sqrt{m}} \sum_{v=1}^{m} \left( \mathbb{I}\{[\theta]_v(s,a) > 0\} - \mathbb{I}\{[\Theta_{\text{init}}]_v(s,a) > 0\} \right) \right)^2 \right]
\]

\[
\leq \frac{1}{m} \mathbb{E}_{\nu_v} \left[ \left( \sum_{v=1}^{m} \left( \mathbb{I}\{[\theta]_v(s,a) > 0\} - \mathbb{I}\{[\Theta_{\text{init}}]_v(s,a) > 0\} \right) \right)^2 \right] \]

(48)

The above inequality holds because the fact that \(|\sum W| \leq \sum |W|\), where \(W = (\mathbb{I}\{[\theta]_v(s,a) > 0\} - \mathbb{I}\{[\Theta_{\text{init}}]_v(s,a) > 0\})b_v[\theta]_v(s,a)\). \(\Theta_{\text{init}}\) is defined in Eq. 6. Next, since \(\mathbb{I}\{[\Theta_{\text{init}}]_v(s,a) > 0\} \neq \mathbb{I}\{[\theta]_v(s,a) > 0\}\), we have,

\[
\|\Theta_{\text{init}}\|_v(s,a) \leq |[\theta]_v(s,a) - \Theta_{\text{init}}\|_v(s,a)| \leq \|\theta\|_v - \|\Theta_{\text{init}}\|_v, \quad (49)
\]

where we obtain the last inequality from the Cauchy-Schwartz inequality. We also assume that \(|(s,a)|_2 \leq 1\) without loss of generality. [20][21]. Eq. (49) further implies that,

\[
\mathbb{I}\{[\theta]_v(s,a) > 0\} - \mathbb{I}\{[\Theta_{\text{init}}]_v(s,a) > 0\} \leq \mathbb{I}\{[\Theta_{\text{init}}]_v(s,a) > 0\} \leq \|\theta\|_v - \|\Theta_{\text{init}}\|_v \leq 1
\]

(50)
Then plug Eq. (50) and the fact that $|b_v| \leq 1$ back to Eq. (48), we have the following,

\[
E_{\nu^v} \left[ \left( f((s,a);\theta) - \hat{f}((s,a);\theta) \right)^2 \right] \\
\leq \frac{1}{m} E_{\nu^v} \left[ \left( \sum_{v=1}^{m} \mathbb{1} \left\{ |[\Theta_{init}]_v^\top (s,a)| \leq \| [\theta]_v - [\Theta_{init}]_v \|_2 \right\} \right)^2 \right] \\
\leq \frac{1}{m} E_{\nu^v} \left[ \left( \sum_{v=1}^{m} \mathbb{1} \left\{ |[\Theta_{init}]_v^\top (s,a)| \leq \| [\theta]_v - [\Theta_{init}]_v \|_2 \right\} \left( \| [\theta]_v - [\Theta_{init}]_v \|_2 + \| [\Theta_{init}]_v^\top (s,a) \|_2 \right) \right)^2 \right] \\
\leq \frac{1}{m} E_{\nu^v} \left[ \left( \sum_{v=1}^{m} \mathbb{1} \left\{ |[\Theta_{init}]_v^\top (s,a)| \leq \| [\theta]_v - [\Theta_{init}]_v \|_2 \right\} \left( \| [\theta]_v - [\Theta_{init}]_v \|_2 + \| [\Theta_{init}]_v^\top (s,a) \|_2 \right) \right)^2 \right] \\
\leq \frac{1}{m} E_{\nu^v} \left[ \left( \sum_{v=1}^{m} \mathbb{1} \left\{ |[\Theta_{init}]_v^\top (s,a)| \leq \| [\theta]_v - [\Theta_{init}]_v \|_2 \right\} \right)^2 \right] \\
\leq \frac{2}{m} \left\| [\theta]_v - [\Theta_{init}]_v \right\|_2^2 \tag{51}
\]

We obtain the second inequality by the fact that $|A| \leq |A - B| + |B|$. Then follow the Cauchy-Schwartz inequality and $\|(s,a)\|_2 \leq 1$ we have the third equality. By inserting Eq. (49) we achieve the fourth inequality. We continue Eq. (51) by following the Cauchy-Schwartz inequality and plugging $\| [\theta] - [\Theta_{init}] \|_2 \leq 1$, we get

\[
E_{\nu^v} \left[ \left( f((s,a);\theta) - \hat{f}((s,a);\theta) \right)^2 \right] \\
\leq \frac{4\gamma^2}{m} E_{\nu^v} \left[ \left( \sum_{v=1}^{m} \mathbb{1} \left\{ |[\Theta_{init}]_v^\top (s,a)| \leq \| [\theta]_v - [\Theta_{init}]_v \|_2 \right\} \right)^2 \right] \\
\leq \frac{4\gamma^2}{m} \sum_{v=1}^{m} \mathbb{1} \left\{ |[\Theta_{init}]_v^\top (s,a)| \leq \| [\theta]_v - [\Theta_{init}]_v \|_2 \right\} \\
\leq \frac{4\gamma^2}{m} \sum_{v=1}^{m} \frac{\| [\theta]_v - [\Theta_{init}]_v \|_2}{\| [\Theta_{init}]_v \|_2} \\
\leq \frac{4\gamma^2}{m} \left( \sum_{v=1}^{m} \frac{\| [\theta]_v - [\Theta_{init}]_v \|_2}{\| [\Theta_{init}]_v \|_2} \right)^{-1/2} \left( \sum_{v=1}^{m} \frac{1}{\| [\Theta_{init}]_v \|_2^2} \right)^{-1/2} \\
\leq \frac{4\gamma^2}{m^{1/2}} \tag{52}
\]
We obtain the second inequality by imposing Assumption A 4 and the third by following the Cauchy-Schwartz inequality. Finally, we set $c_1 := c \sqrt{E_N(\theta,t^2/\delta)}[1/(\|s,a\|_2^2)]$. Thus, we complete the proof.

In the $t$-th iterations of TD iteration, we denote the temporal difference terms w.r.t $\hat{f}(s,a); \theta_t)$ and $f((s,a); \theta_t)$ as

$$\delta^0_t((s,a),(s,a)'); \theta_t) = \hat{f}(s,a); \theta_t) - \gamma \hat{f}(s,a); \theta_t) \leftarrow r_{s,a},$$

$$\delta^0_t((s,a),(s,a)'; \theta_t) = f((s,a); \theta_t) - \gamma f((s,a); \theta_t) \leftarrow r_{s,a}.$$ 

For notation simplicity, in the sequel we write $\delta^0_t((s,a),(s,a)'; \theta_t)$ and $\delta^0_t((s,a),(s,a)'; \theta_t)$ as $\delta^0_t$ and $\delta^0_t$. We further define the stochastic semi-gradient $g_t(\theta_t) := \delta^0_t \nabla_\theta f((s,a); \theta_t)$, its population mean $\bar{g}_t(\theta_t) := E_{\nu_t}[g_t(\theta_t)]$. The local linearization of $\bar{g}_t(\theta_t)$ is $\hat{g}_t(\theta_t) := E_{\nu_t} [\delta^0_t \nabla_\theta \hat{f}((s,a); \theta_t)]$. We denote them as $g_t, \bar{g}_t, \hat{g}_t$, respectively.

**Lemma 15.** Under Assumption A 4, for all $\theta_t \in D$, where $D = \{\xi \in \mathbb{R}^{md} : \|\xi - \Theta_{init}\|_2 \leq \gamma\}$, it holds with probability of $1 - \delta$ that,

$$\|\bar{g}_t - \hat{g}_t\|_2 = O\left(\gamma^{1/2}m^{-1/4}(1 + (m \log \frac{1}{\delta})^{-1/2}) + \gamma^{1/2}r_{max}m^{-1/4}\right)$$

**Proof.** By the definition of $\bar{g}_t$ and $\hat{g}_t$, we have

$$\|\bar{g}_t - \hat{g}_t\|_2$$

$$= E_{\nu_t} [\delta^0_t \nabla_\theta f((s,a); \theta_t) - \delta^0_t \nabla_\theta \hat{f}((s,a); \theta_t)]_2^2$$

$$= E_{\nu_t} [\delta^0_t - \delta^0_t]_2^2 \nabla_\theta f((s,a); \theta_t) + \delta^0_t \nabla_\theta (\nabla_\theta f((s,a); \theta_t) - \nabla_\theta \hat{f}((s,a); \theta_t))_2^2$$

$$\leq 2E_{\nu_t} [\delta^0_t - \delta^0_t]_2^2 \nabla_\theta f((s,a); \theta_t)_2^2 + 2E_{\nu_t} [\delta^0_t \nabla_\theta f((s,a); \theta_t) - \nabla_\theta \hat{f}((s,a); \theta_t))_2^2]$$

(53)

We obtain the inequality because $(A + B)^2 \leq 2A^2 + 2B^2$. We first upper bound $E_{\nu_t} [\delta^0_t - \delta^0_t]_2^2 \nabla_\theta f((s,a); \theta_t)_2^2$ in Eq. (53). Since $(\|s,a\|_2 \leq 1$, we have
||∇θf((s, a); θt)||₂ ≤ 1. Then by definition, we have the following first inequality,

\[ E_νπ[ (δ^0_t − δ^0_0) \left\| ∇θf((s, a); θ_t) - ∇θ\hat{f}((s, a); θ_t) \right\|_2^2 ] \]

\[ \leq E_νπ \left[ (f((s, a); θ_t) - \hat{f}((s, a); θ_t) - γ( (f((s', a'); θ_t) - \hat{f}((s', a'); θ_t))) \right] \]

\[ \leq E_νπ \left[ |f((s, a); θ_t) - \hat{f}((s, a); θ_t)| + |f((s', a'); θ_t) - \hat{f}((s', a'); θ_t)| \right] \]

\[ \leq 2E_νπ\left[ (f((s, a); θ_t) - \hat{f}((s, a); θ_t))^2 \right] + 2E_νπ \left[ ( (f((s', a'); θ_t) - \hat{f}((s', a'); θ_t)))^2 \right] \]

\[ \leq 4E_νπ\left[ (f((s, a); θ_t) - \hat{f}((s, a); θ_t))^2 \right] \leq \frac{16c_1 Y^3}{\sqrt{m}} \quad (54) \]

We obtain the second inequality by |γ| ≤ 1, then obtain the third inequality by the fact that \((A + B)^2 \leq 2A^2 + 2B^2\). We reach the final step by inserting Lemma 14. We then proceed to upper bound \(E_νπ[ |δ^0_t||∇θf((s, a); θ_t) - ∇θ\hat{f}((s, a); θ_t)||_2| ]\). From Hölder’s inequality, we have,

\[ E_νπ \left[ (|δ^0_t||∇θf((s, a); θ_t) - ∇θ\hat{f}((s, a); θ_t)||_2) \right] \]

\[ \leq E_νπ \left[ (δ^0_t)^2 \right] E_νπ \left[ ||∇θf((s, a); θ_t) - ∇θ\hat{f}((s, a); θ_t)||_2^2 \right] \]

\[ (55) \]

We first derive an upper bound for first term in Eq. (55), starting from its defi-
We obtain the first and the third inequality by the fact that \((A + B + C)^2 \leq 3A^2 + 3B^2 + 3C^2\). Recall \(r_{\text{max}}\) is the boundary for reward function \(r\), which leads to the second inequality. We obtain the last inequality in Eq. (56) following the fact that \(|\hat{f}(s, a; \theta_i) - \hat{f}(s, a; \theta_{\pi^*})| \leq \|\theta_i - \theta_{\pi^*}\| \leq 2\Upsilon\) and \(Q_{\pi} \leq (1 - \gamma)^{-1}r_{\text{max}}\).

Since \(\overline{F}_{\Upsilon, m} \subset F_{\Upsilon, m}\), by Lemma 13, we have,

\[
E_{\nu_{\pi}} \left[ (\hat{f}(s, a; \theta_{\pi^*}) - Q_{\pi})^2 \right] \leq \frac{\Upsilon^2(1 + \sqrt{2\log(1/\delta)}/m)^2}{m} \tag{57}
\]

Combine Eq. (56) and Eq. (57), we have with probability of \(1 - \delta\),

\[
E_{\nu_{\pi}} [(\delta_t^0)^2] \leq 72\Upsilon^2(1 + \frac{\log(1/\delta)}{m}) + 21(1 - \gamma)^{-2}r_{\text{max}}^2 \tag{58}
\]
Lastly we have
\[
E_{\nu}[\|\nabla f((s,a);\theta_t) - \nabla \hat{f}((s,a);\theta_t)\|^2]
\]
\[
= E_{\nu}\left[ \left( \frac{1}{m} \sum_{v=1}^{m} \left( \mathbb{1}\{[\theta_v]^{T}(s,a) > 0\} - \mathbb{1}\{[\Theta_{\text{init}}]^{T}(s,a) > 0\} \right) \right)^2 ||(s,a)||^2 \right]
\]
\[
\leq E_{\nu}\left[ \frac{1}{m} \sum_{v=1}^{m} \left( \mathbb{1}\{|[\Theta_{\text{init}}]^{T}(s,a)| \leq ||[\theta_v] - [\Theta_{\text{init}}]||_2\} \right) \right]
\]
\[
\leq c_1 \sqrt{\frac{\Upsilon}{m}}
\]  
(59)
We obtain the first inequality by following Eq. (50) and the fact that \(|b_v| \leq 1\) and \(||(s,a)||_2 \leq 1\). Then for the rest, we follow the similar argument in Eq. (52).
To finish the proof, we plug Eq. (54), Eq. (58) and Eq. (59) back to Eq. (53),
\[
||\bar{g}_t - \hat{g}_t||^2_2
\]
\[
\leq 2\left( \frac{16c_1 \Upsilon^3}{\sqrt{m}} + \left( 72\Upsilon^2 (1 + \frac{\log(1/\delta)}{m}) + 21(1 - \gamma)^{-2}r_{\text{max}}^2 \right) \right)
\]
\[
\leq \frac{176c_1 \Upsilon^3}{\sqrt{m}} + \frac{144c_1 \Upsilon^3 \log(1/\delta)}{m^{3/2}} + \frac{42c_1 \Upsilon r_{\text{max}}^2}{(1 - \gamma)^{-2} \sqrt{m}}
\]
Then we have,
\[
||\bar{g}_t - \hat{g}_t||_2
\]
\[
\leq \sqrt{\frac{176c_1 \Upsilon^3}{\sqrt{m}} + \frac{144c_1 \Upsilon^3 \log(1/\delta)}{m^{3/2}} + \frac{42c_1 \Upsilon r_{\text{max}}^2}{(1 - \gamma)^{-2} \sqrt{m}}}
\]
\[
= \mathcal{O}\left( \Upsilon^{3/2} m^{-1/4} (1 + (m \log \frac{1}{\delta})^{-1/2}) + \Upsilon^{1/2} r_{\text{max}} m^{-1/4} \right)
\]
\[
\square
\]
Next, we provide the following lemma to characterize the variance of \(g_t\).

**Lemma 16.** (Variance of the Stochastic Update Vector) [20]. There exists a constant \(\xi_g^2 = \mathcal{O}(\Upsilon^2)\) independent of \(t\) such that for any \(t \leq T\), it holds that
\[
E_{\nu}[||g_t(\theta_t) - \hat{g}_t(\theta_t)||^2_2] \leq \xi_g^2
\]
A detailed proof can be found in [20]. Now we provide the proof for Lemma 2.
Proof.

\[ ||\theta_{t+1} - \theta^\star||^2_2 \]
\[ = ||\Pi_D(\theta_t - \eta g_t(\theta_t)) - \Pi_D(\theta^\star - \eta \hat{g}_t(\theta^\star))||^2_2 \]
\[ \leq ||(\theta_t - \theta^\star) - \eta (g_t(\theta_t) - \hat{g}_t(\theta^\star))||^2_2 \]
\[ = ||\theta_t - \theta^\star||^2_2 - 2\eta (g_t(\theta_t) - \hat{g}_t(\theta^\star))^T (\theta_t - \theta^\star) \]
\[ + \eta^2 ||g_t(\theta_t) - \hat{g}_t(\theta^\star)||^2_2 \] (60)

The inequality holds due to the definition of \( \Pi_D \). We first upper bound \( ||g_t(\theta_t) - \hat{g}_t(\theta^\star)||^2_2 \) in Eq. (60),

\[ \leq 3 \left( ||g_t(\theta_t) - \hat{g}_t(\theta_t)||^2_2 + ||\hat{g}_t(\theta_t) - \hat{g}_t(\theta^\star)||^2_2 \right) \] (61)

The inequality holds due to fact that \( (A + B + C)^2 \leq 3A^2 + 3B^2 + 3C^2 \). Two of the terms on the right hand side of Eq. (61) are characterized in Lemma 15 and Lemma 16. We therefore characterize the remaining term,

\[ \leq \text{E}_{\nu^*} \left[ \left( \dot{f}(\theta_t; \theta^\star) - \dot{f}(s'; \theta^\star) \right) \left( \dot{\theta}_t - \dot{\theta}(s'; \theta^\star) \right) \right] \]
\[ \leq \text{E}_{\nu^*} \left[ \dot{f}(\theta_t; \theta^\star) - \dot{f}(s'; \theta^\star) \right] \left( \dot{\theta}_t - \dot{\theta}(s'; \theta^\star) \right) \]
\[ + 2\gamma \text{E}_{\nu^*} \left[ \left( \dot{f}(\theta_t; \theta^\star) - \dot{f}(s'; \theta^\star) \right) \left( \dot{\theta}_t - \dot{\theta}(s'; \theta^\star) \right) \right] \] (62)

We obtain the first inequality by the fact that \( ||\nabla_\theta \hat{f}((s, a); \theta_t)|| \leq 1 \). Then we use the fact that \((s, a)\) and \((s', a')\) have the same marginal distribution as well as \( \gamma < 1 \) for the second inequality. Follow the Cauchy-Schwarz inequality and
the fact that \((s, a)\) and \((s', a')\) have the same marginal distribution, we have
\[
\mathbb{E}_{\nu_\pi} \left[ \left( \hat{f}(s', a'; \theta_t) - \hat{f}(s, a; \theta_t) \right) \left( \hat{f}(s, a; \theta_t) - \hat{f}(s, a'; \theta_{\pi^*}) \right) \right] \\
\leq \mathbb{E}_{\nu_\pi} \left[ \left( \hat{f}(s', a'; \theta_t) - \hat{f}(s', a'; \theta_{\pi^*}) \right) \right] \mathbb{E}_{\nu_\pi} \left[ \left( \hat{f}(s, a; \theta_t) - \hat{f}(s, a; \theta_{\pi^*}) \right) \right] \\
= \mathbb{E}_{\nu_\pi} \left[ \left( \hat{f}(s', a'; \theta_t) - \hat{f}(s', a'; \theta_{\pi^*}) \right)^2 \right] 
\]
We plug Eq. (63) back to Eq. (62),
\[
\| \hat{g}_t(\theta_t) - \hat{g}_t(\theta_{\pi^*}) \|^2 \\
\leq (1 + \gamma)^2 \mathbb{E}_{\nu_\pi} \left[ \left( \hat{f}(s, a; \theta_t) - \hat{f}(s, a'; \theta_{\pi^*}) \right)^2 \right]. 
\]
Next, we upper bound \((g_t(\theta_t) - \hat{g}_t(\theta_{\pi^*}))^T (\theta_t - \theta_{\pi^*})\). We have,
\[
(g_t(\theta_t) - \hat{g}_t(\theta_{\pi^*}))^T (\theta_t - \theta_{\pi^*}) \\
= (g_t(\theta_t) - \hat{g}_t(\theta_t))^T (\theta_t - \theta_{\pi^*}) + (\hat{g}_t(\theta_t) - \hat{g}_t(\theta_{\pi^*}))^T (\theta_t - \theta_{\pi^*}) \\
\geq - \| \hat{g}_t(\theta_t) - \hat{g}_t(\theta_t) \| \| \theta_t - \theta_{\pi^*} \|_2 \\
\geq - 2\Upsilon \| \hat{g}_t(\theta_t) - \hat{g}_t(\theta_t) \|_2 
\]
We obtain the second inequality since \(\| \theta_t - \theta_{\pi^*} \|_2 \leq 2\Upsilon\) by definition. For the
We then bound the error terms by rearrange Eq. (68). First, we have, with Eq. (66) and (67), we have,

\[
\begin{align*}
    &\geq \mathbb{E}_\nu \left[ \left( \hat{f}((s, a); \theta_t) - \hat{f}((s, a); \theta_{\pi^*}) \right)^2 \right] - \\
    &\geq \mathbb{E}_\nu \left[ \left( \hat{f}((s, a); \theta_t) - \hat{f}((s, a); \theta_{\pi^*}) \right)^2 \right] - \\
    &\geq (1 - \gamma) \mathbb{E}_\nu \left[ \left( \hat{f}((s, a); \theta_t) - \hat{f}((s, a); \theta_{\pi^*}) \right)^2 \right].
\end{align*}
\]  

where the inequality follows from Eq. (63). Combine Eqs. (60), (61), (64), (65), (66) and (67), we have,

\[
\begin{align*}
    &\geq \mathbb{E}_\nu \left[ \left( \hat{f}((s, a); \theta_t) - \hat{f}((s, a); \theta_{\pi^*}) \right)^2 \right] - \\
    &\geq \mathbb{E}_\nu \left[ \left( \hat{f}((s, a); \theta_t) - \hat{f}((s, a); \theta_{\pi^*}) \right)^2 \right] - \\
    &\geq (1 - \gamma) \mathbb{E}_\nu \left[ \left( \hat{f}((s, a); \theta_t) - \hat{f}((s, a); \theta_{\pi^*}) \right)^2 \right].
\end{align*}
\]  

We then bound the error terms by rearrange Eq. (68). First, we have, with probability of 1 − δ,

\[
\begin{align*}
    &\mathbb{E}_\nu \left[ \left( f((s, a); \theta_t) - \hat{f}((s, a); \theta_{\pi^*}) \right)^2 \right] \\
    &= \mathbb{E}_\nu \left[ \left( f((s, a); \theta_t) - \hat{f}((s, a); \theta_{\pi^*}) \right)^2 \right] - \\
    &\leq 2 \mathbb{E}_\nu \left[ \left( f((s, a); \theta_t) - \hat{f}((s, a); \theta_{\pi^*}) \right)^2 \right] - \\
    &\leq \left( \eta (1 - \gamma) - 1.5 \eta^2 (1 + \gamma)^2 \right)^{-1} \left( \| \theta_t - \theta_{\pi^*} \|_2^2 \\
    &- \| \theta_{t+1} - \theta_{\pi^*} \|_2^2 + 4 \eta \| \xi_g \|_2 + 3 \eta^2 \xi_g^2 \right) + \epsilon_g
\end{align*}
\]  

\[27\]
where
\[
\epsilon_g = \mathcal{O}(\gamma^3 m^{-1/2} \log(1/\delta) + \gamma^5/2 m^{-1/4} \sqrt{\log(1/\delta)} + \gamma r^2_{\text{max}} m^{-1/4})
\]
We obtain the first inequality by the fact that \((A+B)^2 \leq 2A^2 + 2B^2\). Then by Eq. (68), Lemma 14 and Lemma 15, we reach the final inequality. By telescoping Eq. (3.6) for \(t = T\), we have, with probability of \(1 - \delta\),
\[
\|f((s, a); \theta_T) - \hat{f}((s, a); \theta_{\pi^*})\|^2 \\
\leq \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_{\nu_{\pi^*}} \left[ (f((s, a); \theta_t) - \hat{f}((s, a); \theta_{\pi^*})^2 \right] \\
\leq T^{-1}(2\eta(1 - \gamma) - 3\eta^2(1 + \gamma)^2)^{-1}(\|\Theta_{\text{init}} - \theta_{\pi^*}\| + 4\gamma T \eta |\xi_g| + 3T \eta^2 \xi_g^2) + \epsilon_g
\]
Set \(\eta = \min\{1/\sqrt{T}, (1 - \gamma)/3(1 + \gamma)^2\}\), which implies that \(T^{-1/2}(2\eta(1 - \gamma) - 3\eta^2(1 + \gamma)^2)^{-1} \leq 1/(1 - \gamma)^2\), then we have, with probability of \(1 - \delta\),
\[
\|f((s, a); \theta_T) - \hat{f}((s, a); \theta_{\pi^*})\| \\
\leq \frac{1}{T} \sqrt{T}(\|\Theta_{\text{init}} - \theta_{\pi^*}\|^2 + 4\gamma \sqrt{T}|\xi_g| + 3\xi_g^2) + \epsilon_g \\
\leq \frac{\gamma^2 + 4\gamma \sqrt{T}|\xi_g| + 3\xi_g^2}{(1 - \gamma)^2 \sqrt{T}} + \epsilon_g \\
= \mathcal{O}(\gamma^3 m^{-1/2} \log(1/\delta) + \gamma^5/2 m^{-1/4} \sqrt{\log(1/\delta)} + \gamma r^2_{\text{max}} m^{-1/4} + \gamma^2 T^{-1/2} + \gamma)
\]
We obtain the second inequality by the fact that \(\|\Theta_{\text{init}} - \theta_{\pi^*}\|_2 \leq \gamma\). Then by definition we replace \(\hat{Q}_{\omega_k}\) and \(\hat{Q}_{\pi_k}\). □

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