To question about moment at elasticplastic bending

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Abstract. Sheet metal is one of the main forms used in the metal-working, it can be cut or bent into various shapes. Bending of sheet metal on the vertical presses is the most common technological operation in metallurgy and mechanical engineering at the production of various products. Below the mathematical model of elastic-plastic deformation at bending of sheet metal on the vertical presses is proposed.

1. Introduction
The plastic deformation of steels in the hardening zone is widely used in the production of metalwares from the steel rods, sheets and pipes [1–25].

Let the long rod with the initial perpendicular plane $A_0$ and the initial length $l_0$ be stretched by the longitudinal force $F$. Let $l$ and $\Delta l$ be the length and absolute lengthening of the rod under tension, then the relative elongation of the rod under tension $\varepsilon = \Delta l/l_0$. Let $\varepsilon_y$ and $\varepsilon_u$ be the yield strength and tensile strength of steel, and $\varepsilon_y$ and $\varepsilon_u$ be the relative elongation’s values at which the yield strength and tensile strength are achieved.

The dependence of the stress on the relative elongation of the rod in the elastic zone obeys the classical linear Hooke’s law (Hooke R.): $\sigma(\varepsilon) = E\varepsilon$ ($0 \leq |\varepsilon| \leq \varepsilon_y = \sigma_y/E$), where $\sigma = F/A_0$ is a normal stress in the perpendicular plane of the steel rod, $\varepsilon = \Delta l/l_0$ is the relative elongation of the rod under tension, $E$ is the young's modulus (the modulus of elasticity) of steel, $\sigma_y$ is the yield strength of steel.

The direct non-linear Shinkin’s approximation (Shinkin V. N.) for the hardening zone has the form

$$
\sigma(\varepsilon) = \begin{cases} 
E\varepsilon, & 0 \leq \varepsilon < \varepsilon_y = \sigma_y/E; \\
\sigma_y + \sum_n P_n \left[ \left( \frac{\varepsilon - \varepsilon_y}{E} + \varepsilon_n \right)^n - \varepsilon_n^n \right], & \varepsilon \geq \varepsilon_y, \quad \varepsilon_n \geq 0; \\
\sigma(\varepsilon) = -\sigma(-\varepsilon), & \varepsilon \leq 0; \\
\sigma(\varepsilon_y) = \sigma_y, \\
\frac{d\sigma(\varepsilon_y)}{d\varepsilon} = \sum_n nP_n \varepsilon_n^{n-1} = P_y;
\end{cases}
$$

where $P_y$ is the hardening module of steel at $\varepsilon = \varepsilon_y$, $P_n$ are the coefficients of the series (may have both
positive and negative values), \( \varepsilon_n \) are the displacements of the relative elongation \( \varepsilon \) relatively \( \varepsilon_y, n \) are the real numbers (can have both positive and negative values).

Under \( P_n = 0 \) we obtain the Prandtl’s approximation. Under one term of the series, \( \varepsilon_n = 0 \) and \( n = 1 \), we obtain the linear approximation.

The peculiarities of the Shinkin’s approximation are the introduction of the displacements \( \varepsilon_n \) (which allow us to have a finite value of the derivative of the hardening curve at the point corresponding to the yield strength) and the approximation of the hardening curve in the form of a finite or infinite series (which allows us to approximate the hardening curve from the yield strength to the ultimate strength with any precision).

Remark. Under the approximation of the whole hardening zone of steel (from the yield strength to the ultimate strength) by three members of the series and under the boundary conditions (the four boundary conditions)

\[ \sigma(\varepsilon) = \sigma_y, \quad \sigma(\varepsilon_u) = \sigma_u, \quad \frac{d\sigma(\varepsilon)}{d\varepsilon} = P_y, \quad \frac{d\sigma(\varepsilon_u)}{d\varepsilon} = 0; \]

the direct non-linear Shinkin’s approximation is a cubic multinomial (polynomial) by \( \varepsilon \) and has the form

\[ \sigma(\varepsilon) = \begin{cases} E\varepsilon, & 0 \leq \varepsilon < \varepsilon_y = \frac{\sigma_y}{E}, \\ \sigma_y + P_y(\varepsilon - \varepsilon_y) - \frac{2P_y(\varepsilon_y - \varepsilon_y)}{(\varepsilon_y - \varepsilon_y)^2}(\varepsilon - \varepsilon_y)^2 + \frac{P_y(\varepsilon_y - \varepsilon_y)}{(\varepsilon_y - \varepsilon_y)^3}(\varepsilon - \varepsilon_y)^3, & \varepsilon_y \leq \varepsilon \leq \varepsilon_u; \\ \sigma(\varepsilon) = -\sigma(-\varepsilon), & \varepsilon \leq 0; \end{cases} \]

where \( \sigma_u \) is the ultimate strength of steel, \( \varepsilon_u \) is the relative elongation of the rod corresponding to the ultimate strength.

2. Bend moment

Let \( h \) and \( b \) be, respectively, the thickness and width of the rectangular rod, and \( \rho \) be the radius of curvature of the non-stresses axis (the non-stresses surface) of the rod.

The plastic deformation on the surface of a rectangular bar occurs when the curvature of the non-stresses axis of the bar \( [3, 4, 11, 12] \) \( \rho < \rho_\text{res} = Eh/(2\sigma_y) \). Under the bend of the rod, the normal stresses in the height of the perpendicular plane of the rod are equal to

\[ \sigma(y) = \begin{cases} \sigma_y, & 0 \leq y < y_y = \frac{\sigma_y\rho}{E}, \\ \sigma_y + \sum \frac{P_n}{\rho_n} \left[ \left( y - \frac{\sigma_y\rho}{E} + \varepsilon_n\rho \right)^n - \varepsilon_n\rho^n \right], & y_y \leq y \leq \frac{h}{2}, \\ \sigma(y) = -\sigma(-y), & y \leq 0; \end{cases} \]

\[ \sigma(y_y) = \sigma_y, \quad \frac{d\sigma(y_y)}{dy} = \sum \frac{nP_n}{\rho^n} (\varepsilon_n\rho)^{n-1} = \frac{1}{\rho} \sum nP_n \varepsilon_n^{n-1} = \frac{P_y}{\rho}, \]
The bend moment in the rod’s perpendicular plane has the form

\[
M = \frac{1}{4}bh^2\sigma_y \left[ 1 - \frac{4}{3} \left( \frac{\sigma_0}{E} \right)^2 \right] \left[ 1 - 4 \left( \frac{\sigma_0}{E} \right)^2 - \left( \frac{P_n}{\sigma_y} \right) \sum_n P_n \varepsilon_n^n + \right]
\]

For the linear hardening \((n = 1, \varepsilon_n = 0)\)

\[
M = \frac{1}{4}bh^2\sigma_y \left[ 1 - \frac{4}{3} \left( \frac{\sigma_0}{E} \right)^2 \right] + \frac{1}{3} \left( \frac{P}{E} \right) \left( \frac{Eh}{\sigma_y} \right) \left[ 1 - 2 \left( \frac{\sigma_0}{E} \right) \left( 1 + \frac{\sigma_0}{E} \right) \right].
\]

For the Prandtl’s approximation \((n = 1, \varepsilon_n = 0, P_n = 0)\)

\[
M = \frac{1}{4}bh^2\sigma_y \left[ 1 - \frac{4}{3} \left( \frac{\sigma_0}{E} \right)^2 \right].
\]

### 3. Spring-back

After a removal of the external loads (forces), the rod straightens, but not to the end. The residual curvature of the non-stresses line of the rod is determined by the Hencky’s theorem, according to which the difference between the true stresses and the imagined perfectly elastic stresses in the rod is equal to the final stresses. In this case [11, 12, 19, 20]

\[
\frac{1}{\rho_{\text{re}}} = \frac{1}{\rho} - \frac{12M}{Eh^2b}, \quad \beta = \frac{\rho_{\text{re}}}{\rho} = \left( 1 - \frac{12M\rho}{Eh^2b} \right)^{-1},
\]

where \(\rho_{\text{re}}\) is the final radius of curvature of the longitudinal non-stresses axis of the rod after its straightening (spring back), \(\beta\) is the spring-back factor.

Substituting in this formula the value of the bend moment, we obtain

\[
\frac{1}{\beta} = 1 - \frac{3\sigma_0}{Eh} \left[ 1 - \frac{4}{3} \left( \frac{\sigma_0}{E} \right)^2 \right] \left[ 1 - 4 \left( \frac{\sigma_0}{E} \right)^2 - \left( \frac{P_n}{\sigma_y} \right) \sum_n P_n \varepsilon_n^n + \right]
\]

For the linear hardening \((n = 1, \varepsilon_n = 0)\)

\[
\beta = \left( 1 - \frac{P}{E} \right) \left[ 1 - 2 \left( \frac{\sigma_0}{E} \right)^2 \left( 1 + \frac{\sigma_0}{E} \right) \right]^{-1}.
\]

For the Prandtl’s approximation \((n = 1, \varepsilon_n = 0, P_n = 0)\)
4. Final stresses

According to the Hencky’s theorem, the spring back of the rod takes place according to the linear law [19, 20] \( \sigma_{unl}(y) = \alpha_{unl} y \) (\( \alpha_{unl} = \text{const} \)). The final stresses are equal to the difference between the true stresses and the imagined perfectly elastic stresses: \( \sigma_{res} = \sigma - \sigma_{unl} \). The bend moments at loading and unloading are equal:

\[
M = M_{unl} = \frac{1}{12} \alpha_{unl} bh^3, \quad \alpha_{unl} = \frac{12M}{bh^3},
\]

\[
\alpha_{unl} = 3 \frac{\sigma_y}{h} - \frac{4}{3} \frac{\sigma_y}{h} \left( \frac{\sigma_s \rho}{Eh} \right)^2 - \frac{1}{2} \frac{P_n}{\rho} \left( 1 - 2 \frac{\sigma_s \rho}{Eh} \right)^2 \left( 1 + \frac{\sigma_s \rho}{Eh} \right) - \left( \frac{\sigma_y}{h} \right)^2 \left( \frac{\sigma_s \rho}{Eh} \right).
\]

For the linear hardening \( n = 1, \varepsilon_n = 0 \)

\[
\alpha_{unl} = 3 \frac{\sigma_y}{h} - 4 \frac{\sigma_y}{h} \left( \frac{\sigma_s \rho}{Eh} \right)^2 + \frac{P_n}{\rho} \left( 1 - 2 \frac{\sigma_s \rho}{Eh} \right)^2 \left( 1 + \frac{\sigma_s \rho}{Eh} \right).
\]

For the Prandtl’s approximation \( n = 1, \varepsilon_n = 0, P_n = 0 \)

\[
\alpha_{unl} = 3 \frac{\sigma_y}{h} - 4 \frac{\sigma_y}{h} \left( \frac{\sigma_s \rho}{Eh} \right)^2.
\]

The minimum final stress is observed on the surface of the rod, is less than zero and equal to

\[
\sigma_{res} = \sigma_y + \sum_n \frac{P_n}{\rho} \left( h + \frac{\sigma_s \rho}{E} + \varepsilon_n \rho \right)^n - \varepsilon_n \rho^n.
\]

\[
\beta = \left( 1 - \frac{2 \sigma_s \rho}{Eh} \right)^2 \left( 1 + \frac{\sigma_s \rho}{Eh} \right)^{-1}.
\]
\[
\sigma_{res}^1 = -\frac{1}{2} \sigma_y \left( 1 - \frac{P}{E} \right) \left( 1 - 4 \left( \frac{\sigma_y \rho}{Eh} \right)^2 \right) < 0.
\]

For the Prandtl’s approximation \((n = 1, \varepsilon_n = 0, P_n = 0)\)
\[
\sigma_{res}^1 = -\frac{1}{2} \sigma_y \left( 1 - 4 \left( \frac{\sigma_y \rho}{Eh} \right)^2 \right).
\]

The maximum final stress is observed at \(y = y_{res}\), is greater than zero and equal to
\[
\sigma_{res}^2 = \sigma_y - 
\left[
1 - \frac{4}{3} \left( \frac{\sigma_y \rho}{Eh} \right)^2
\right]
- \left[
1 - 4 \left( \frac{\sigma_y \rho}{Eh} \right)^2
\right] \sum_n \frac{P_n}{\sigma_y} \varepsilon_n^n + 
\sum_n \frac{4}{(n+1)(n+2)} \frac{P_n}{\sigma_y} \left( \frac{h}{2\rho} \right)^n \left( \left( 1 - 2 \frac{\sigma_y \rho}{Eh} + 2 \varepsilon_n \frac{\rho}{h} \right)^{n+1} \left( 1 + \frac{\sigma_y \rho}{Eh} - \varepsilon_n \frac{\rho}{h} \right)^{-n} - \left( 2 \varepsilon_n \frac{\rho}{h} \right)^{n+1} \left( (n+2) \frac{\sigma_y \rho}{Eh} - \varepsilon_n \frac{\rho}{h} \right) \right].
\]

For the linear hardening \((n = 1, \varepsilon_n = 0)\)
\[
\sigma_{res}^2 = \sigma_y \left( 1 - \frac{P}{E} \right) \left( 1 - 2 \frac{\sigma_y \rho}{Eh} \right)^2 \left( 1 + \frac{\sigma_y \rho}{Eh} \right) > 0.
\]

For the Prandtl’s approximation \((n = 1, \varepsilon_n = 0, P_n = 0)\)
\[
\sigma_{res}^2 = \sigma_y \left( 1 - 2 \frac{\sigma_y \rho}{Eh} \right)^2 \left( 1 + \frac{\sigma_y \rho}{Eh} \right) > 0.
\]

5. Numerical calculations

The results of mechanical tests (the stretching diagram) of two flat specimens from the pipe steel of the strength class 485 are shown in Figure 1. The length and width of the specimens, respectively, is 180 mm and 40.2 mm, the test temperature is +20 °C, the test procedure is ISO 6892. The test results of specimens: the conventional yield strength \(\sigma_{0.5}\) \((0.5 \%)\) for specimens, respectively, equals 509 MPa and 517 MPa, the ultimate strength equals 609 MPa and 615 MPa, the relative elongation of specimens at the time of rupture equals 24 % and 25 %, the ratio of the yield strength to the ultimate strength equals 0.836 and 0.841.

From Figure 1 it is clearly seen that the hardening curve in the stretching diagram is clearly nonlinear. The derivative (the tangent of an inclination angle) of the stress curve at the point, corresponding to the yield strength of the steel, has a bend and some finite (not infinite) value. In the hardening zone, the stress curve is a monotonically increasing convex curve, having a maximum (the zero derivative) at a point corresponding to the ultimate strength \(\sigma_u\).

Consider the approximation of the hardening zone of two steel rods, the stretching diagram of which is shown in Figure 1. In this case, the hardening modulus at \(\varepsilon = \varepsilon_y\) is equal to \(P_y = 2155\) MPa, the average yield strength is \(\sigma_y = (509 + 517)/2 = 513\) MPa, the average ultimate strength is \(\sigma_u = (609 + 615)/2 = 612\) MPa, the average value of the relative elongation (corresponding to the
ultimate strength $\sigma_u$ is $\varepsilon_u = 0.125$. The experimental hardening curve (Figure 1) and the curve of the direct approximation by Shinkin for the hardening zone are shown in Figure 1.

Consider the bend of the rectangular rod, the mechanical characteristics of the steel which correspond to Figure 1. Let the young’s modulus of steel $E = 2 \cdot 10^5$ MPa, the rod’s thickness $h = 0.02$ m, the rod’s width $b = 1.8$ m, and the radius of curvature of the non-stresses surface of the rod $\rho = 0.25$ m. Then the bend moment of the perpendicular plane of the rectangular rod, the spring-back factor and the final stresses are respectively equal to $M = 99.89$ kPa-m, $\beta = 1.116$, $\sigma_{res}^1 = -257.9$ MPa, $\sigma_{res}^2 = 459.6$ MPa.

![Experimental curve and Shinkin's approximation](image.png)

Figure 1. The experimental dotted curve (1) and the curve of the direct approximations by Shinkin (2) in the steel’s hardening zone; $\varepsilon_y \leq \varepsilon \leq \varepsilon_u$.

6. Conclusions

The mathematical model of elastic-plastic deformation at bending of sheet metal (with taking into account the nonlinearity of the hardening curve) on the vertical presses is proposed. The analytical expressions for the bend moment, the spring-back factor and the final stresses are obtained.

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