EQUIDISTRIBUTION OF RANDOM WAVES ON SMALL BALLS

XIAOLONG HAN AND MELISSA TACY

Abstract. In this paper, we investigate of the equidistribution, at small scale, of randomised sums of Laplacian eigenfunctions on a compact manifold $M$. We prove small scale expectation and variance results for random combinations of eigenfunctions on all compact manifolds; the scale we prove here for which equidistribution is valid approaches the Planck scale. For balls at larger scales, we also obtain estimates showing that the probability that a randomised sum of Laplacian eigenfunctions fails to equidistribute decays exponentially with the eigenvalue.

1. Introduction

Studying the behaviour of random combinations of either plane waves or eigenfunctions has lately proved to be an exciting research area. It is conjectured, by Berry [B] in the 1970s, that eigenfunctions of chaotic systems such as billiards behave like random waves. That is, their behaviour is modelled by functions of the form

$$\sum_j c_j e^{i\lambda(x,\xi_j)}$$

where the $\xi_j$ are chosen as a set of equidistributed (on scale $\lambda^{-1}$) directions in $S^{n-1}$ and the coefficients $c_j$ are chosen according to Gaussian distribution. On a compact manifold $M$, the natural class of objects that replace plane waves are eigenfunctions. That is, we consider sums

$$\sum_{\lambda_j \in \Lambda} a_j e_j(x),$$

where $e_j$ is an eigenfunction of the Laplacian on $M$, $\Lambda \subset \mathbb{R}$, and the coefficients $a_j$ are prescribed in a random fashion. The obvious first question is how to pick the set $\Lambda$. Initially it may seem natural to fix an eigenspace $E_\lambda$ and randomise only over the eigenfunctions with eigenvalue exactly $\lambda$, as is done in [Ha2]. However, the multiplicity of this eigenvalue may be low (and in fact in chaotic cases such as when $M$ has negative curvature it is conjectured that the eigenvalues have very low multiplicity). Therefore, to capture the random behaviour, we allow ourselves to randomise over eigenfunctions whose eigenvalues sit in a spectral window. (Such randomisations were introduced in Zelditch [Z1].) That is, we consider functions

$$u = \sum_{\lambda_j \in [\lambda-W,\lambda]} a_j e_j(x)$$

for spectral window widths $1 \leq W \leq \lambda$. Such functions are commonly referred to as “random waves”. We adopt this terminology and reserve the term “random eigenfunctions” for those combinations taken over a single eigenspace. Particular attention should be given to the case where $W = 1$. In this case from Weyl law asymptotics we know that there are of order $\lambda^{n-1}$ eigenfunctions in the spectral window. The corresponds with the $\lambda^{n-1}$ directions needed to cover $S^{n-1}$ by $\xi_j$ spaced at $\lambda^{-1}$ distance from each other. We then ask about the expected behaviour of such functions as well as the variance in behaviour. In particular, we focus on...
small scale behaviour in this paper. We want to understand where random combinations of eigenfunctions equidistribute in small balls.

There are two parts to understanding this equidistribution. The first is to ascertain when

$$\mathbb{E} \left( \int_{B(x,r)} |u|^2 d\text{Vol} \right) \rightarrow \frac{\text{Vol}(B(x,r))}{\text{Vol}(\mathbb{M})} \quad \text{as} \quad \lambda \rightarrow \infty. \quad (1.1)$$

However, while the expectation value might equidistribute, it is still possible that the probability of non-equidistribution is quite high. To that end we also determine, for given \( r \), whether

$$\text{Var} \left( \int_{B(x,r)} |u|^2 d\text{Vol} \right) = o \left( \text{Vol}(B,r)^2 \right) \quad \text{as} \quad \lambda \rightarrow \infty. \quad (1.2)$$

The variance estimate tells us whether we may expect that a typical eigenfunction equidistributes. We are able to obtain these estimate down to the Planck scale \( \lambda^{-1} \).

Finally we consider the question of uniformity across balls of radius \( r \) covering \( \mathbb{M} \). For a fixed \( x \in \mathbb{M} \), (1.1) and (1.2) tell us that the measure (in \( \mathbb{S}^{n-1} \)) of the set of \( a \) such that the random wave \( u \) does not equidistribute on \( B(x,r) \) decays as \( \lambda \rightarrow \infty \). However it is possible that the sets of decaying measure associated with different points \( x \) on the manifold do not significantly overlap. Therefore it may still be the case that the probability \( u \) fails to equidistribute on some ball is large. We address this problem in Section 4 and show that for larger balls (for example above the square root Planck scale \( \lambda^{-1/2} \) for \( W = 1 \)) the probability that \( u \) fails to equidistribute on any ball decays exponentially as \( \lambda \rightarrow \infty \).

For the background of small scale equidistribution of eigenfunctions and its relations with quantum chaos, randomization, and other estimates of eigenfunctions, we refer to the recent works of Han [Ha1, Ha2], Hezari [He1, He2, He3], Hezari-Rivière [HR1, HR2], Lester-Rudnick [LR], Sogge [So1, So2], Zelditch [Z2], etc.

Let \( (\mathbb{M}, g) \) be an \( n \)-dim compact and smooth Riemannian manifold without boundary. Denote \( \Delta = \Delta_g \) the (positive) Laplace-Beltrami operator. Let \( \{e_j\}_{j=0}^\infty \) be an orthonormal basis of eigenfunctions (i.e. eigenbasis) of \( \Delta \) with eigenvalues \( \lambda_j^2 \) (counting multiplicities), i.e. \( \Delta e_j = \lambda_j^2 e_j \), where \( \lambda_j \) is called the eigenfrequency. Denote \( \text{Inj} \mathbb{M} \) the injectivity radius of \( \mathbb{M} \). We assume, without loss of generality, that \( \text{Inj} \mathbb{M} \geq 1 \).

We formally define our probability space in a similar fashion to Zelditch [Z1].

**Definition 1.1.** Let \( N_W(\lambda) \) be the number of eigenfunctions (counted with multiplicity) in \( [\lambda - W, \lambda] \). Then define

$$\mathcal{H}_W(\lambda) = \text{span}_{\lambda_j \in [\lambda-W,\lambda]} \{e_{\lambda_j}\}. \quad (1.3)$$

We introduce the following Gaussian probability measure on the space \( \mathcal{H}_W(\lambda) \):

$$\gamma_W(\lambda) := \frac{N_W(\lambda)}{\pi} e^{-\frac{N_W(\lambda)}{2}} da, \quad u_\lambda = \sum_{\lambda_j \in [\lambda-W,\lambda]} a_j e_j.$$  

Here, \( da \) is the Lebesgue measure in \( \mathbb{R}^{N_W(\lambda)} \).

Hence, the above Gaussian ensemble is equivalent to choosing \( u_\lambda \in \mathcal{H}_W(\lambda) \) at random from the unit sphere in \( \mathcal{H}_W(\lambda) \) with respect to the \( L^2 \) inner product, that is, the expected value of \( \|u_\lambda\|^2_{L^2(\mathbb{M})} \) with respect to \( \gamma_W(\lambda) \) is 1. We will leverage this dual interpretation of randomisation a number of times. Our main theorem states

**Theorem 1.2.** On a compact manifold \( \mathbb{M} \), let \( 1 \leq W \leq \lambda \). Then

- for \( r > 0 \), the expected value with respect to the probability measure \( \gamma_W(\lambda) \)

$$\mathbb{E} \left( \int_{B(x,r)} |u_\lambda|^2 d\text{Vol} \right) = \frac{\text{Vol}(B(x,r))}{\text{Vol}(\mathbb{M})} \left[ 1 + O \left( W^{-1} \right) \right] \quad (1.4)$$

uniformly for all \( x \in \mathbb{M} \);
Corollary 1.3. Suppose that $r^{-1} = o(\lambda)$, we have that the variance with respect to the probability measure $\gamma_W(\lambda)$

$$\text{Var} \left( \int_{B(x,r)} |u_\lambda|^2 \, d\text{Vol} \right) = \text{Vol}(B(x,r))^2 \left[ o(1) + O(W^{-2}) \right] \quad \text{as } \lambda \to \infty$$

uniformly for all $x \in \mathbb{M}$.

In the cases where the window width is growing $\lambda$ we may therefore conclude that random spectral clusters become equidistributed up to the Planck scale $\lambda^{-1}$ as $\lambda \to \infty$.

**Corollary 1.3.** Suppose that $W = W(\lambda)$ and $W^{-1} = o(1)$ as $\lambda \to \infty$. Then

$$\mathbb{E} \left( \int_{B(x,r)} |u_\lambda|^2 \, d\text{Vol} \right) = \frac{\text{Vol}(B(x,r))}{\text{Vol}(\mathbb{M})} + o(\text{Vol}(B(x,r))) \quad \text{as } \lambda \to \infty$$

uniformly for all $x \in \mathbb{M}$. If in addition $r^{-1} = o(\lambda)$, then

$$\text{Var} \left( \int_{B(x,r)} |u_\lambda|^2 \, d\text{Vol} \right) = o(\text{Vol}(B(x,r))^2) \quad \text{as } \lambda \to \infty$$

uniformly for all $x \in \mathbb{M}$.

**Remark.** In particular, if we choose $W = \lambda$, then $\mathcal{H}_W(\lambda)$ is the cut-off frequency ensemble considered in Zelditch [Z1]. In this case, Corollary 1.3 states that the random waves with cut-off frequency are equidistributed up to the Planck scale $\lambda^{-1}$.

If we look at fixed (but large) window widths, i.e. $W$ independent of $\lambda$, we may only conclude that the $L^2$ mass of $u$ concentrated in the ball is proportional to the normalised volume of the ball, according to (1.4) and (1.5) in Theorem 1.2. However, with a geometric condition on the manifold $\mathbb{M}$, we may recover the result of Corollary 1.3. The relevant condition is the set of the geodesic loop directions

$$\mathcal{L}_x := \{ \xi \in S^* \mathbb{M} : G_t(x, \xi) = (x, \eta) \text{ for some } t > 0 \text{ and } \eta \in S^* \mathbb{M} \}$$

is of measure zero in $S^* \mathbb{M}$ for all $x \in \mathbb{M}$. Here, $S^* \mathbb{M}$ is the cosphere space of $\mathbb{M}$ at $x$ and $S^* \mathbb{M}$ is the cosphere bundle of $\mathbb{M}$. Such pointwise aperiodic condition is called the “non self-focal” condition. Examples of manifolds satisfying the non-focal condition include the negatively curved manifolds (i.e. all sectional curvatures are negative everywhere.) Since manifolds with negative curvature are a key class of manifolds that we wish to understand using randomisation making such an assumption is not as restrictive as may first appear.

The above non self-focal condition is a natural dynamical condition to study the precise behavior of eigenfunctions restricted to a fixed-length spectrum window. See Section 2.1 for the background.

Concerning the small scale equidistribution of random eigenfunctions at asymptotically fixed frequency, we prove that

**Theorem 1.4.** Let $\mathbb{M}$ be a compact manifold with the property that the set of loop directions $\mathcal{L}_x$ is of measure zero in $S^* \mathbb{M}$ for all $x \in \mathbb{M}$. Then we have that for $r > 0$, the expected value with respect to the probability measure $\gamma_W(\lambda)$

$$\mathbb{E} \left( \int_{B(x,r)} |u_\lambda|^2 \, d\text{Vol} \right) = \frac{\text{Vol}(B(x,r))}{\text{Vol}(\mathbb{M})} + o(\text{Vol}(B(x,r))) \quad \text{as } \lambda \to \infty$$

uniformly for all $x \in \mathbb{M}$. If, in addition, $r^{-1} = o(\lambda)$, then

$$\text{Var} \left( \int_{B(x,r)} |u_\lambda|^2 \, d\text{Vol} \right) = o(\text{Vol}(B(x,r))^2) \quad \text{as } \lambda \to \infty$$

uniformly for all $x \in \mathbb{M}$.
Remark. In particular, if we choose \( W = 1 \), then \( \mathcal{H}_W(\lambda) \) is the asymptotically fixed frequency ensemble considered in Zelditch \([Z]\). In this case, Theorem 1.4 states that the random waves with asymptotically fixed frequency are equidistributed up to the Planck scale \( \lambda^{-1} \).

Throughout this paper, \( A \lesssim B \) (\( A \gtrsim B \)) means \( A \leq cB \) (\( A \geq cB \)) for some constant \( c \) depending only on the manifold; \( A \approx B \) means \( A \lesssim B \) and \( B \lesssim A \); the constants \( c \) and \( C \) may vary from line to line.

2. Preliminaries

A key technique in the study of randomisations of eigenfunctions on a manifold is to reduce questions about the expectation or variance of a random variable to problems involving sums of eigenfunctions over a spectral window. This approach is profitable because there is a considerable pool of literature focused on understanding the average behaviour of eigenfunctions. In particular, the highest order asymptotics of the kernel of the spectral projector

\[
E_{[0,\lambda]}(x, y) = \sum_{\lambda_j < \lambda} e_j(x)e_j(y)
\]

are well understood and there are a number of estimates linking the geometry of \( \mathbb{M} \) to the behaviour of lower order terms. In this section, we recall the spectral estimates of Laplacian and their connection to underlying geometry. We also discuss the key ideas from probability theory that allow us to link expected behaviour to spectral estimates.

2.1. Spectral estimates. On a compact manifold \( \mathbb{M} \), let \( \{e_j\}_{j=0}^\infty \) be an eigenbasis of \( \Delta \) with eigenvalues \( \lambda_j^2 \). The following estimates of the spectral projections are from Hörmander \([Ho]\).

Let \( T^*\mathbb{M} = \{(x, \xi) : x \in \mathbb{M}, \xi \in T_x^*\mathbb{M}\} \) be the cotangent bundle of \( \mathbb{M} \) and \( |\cdot|_x \) be the induced metric on the cotangent space \( T_x^*\mathbb{M} \). We denote \( \exp_y \) the exponential map at \( y \). Since \( \text{Inj} \mathbb{M} \geq 1 \), \( \exp_y(x) \) is diffeomorphic if the distance of \( x \) and \( y \) is small enough.

**Theorem 2.1** (Spectral projections). On a compact manifold \( \mathbb{M} \), the spectral projection operator onto the space

\[
\text{span}_{\lambda_j \in [0,\lambda]} \{e_j\}
\]

has the kernel

\[
E_{[0,\lambda]}(x, y) = \sum_{\lambda_j \in [0,\lambda]} e_j(x)e_j(y).
\]

Then for \( x, y \) close enough on \( \mathbb{M} \), we have that

\[
E_{[0,\lambda]}(x, y) = \frac{1}{(2\pi)^n} \int_{|\xi|_y < \lambda} e^{i(\exp_y^{-1}(x),\xi)} \frac{d\xi}{\sqrt{|g_y|}} + R(x, y, \lambda),
\]

where \( R(x, y, \lambda) \) is \( O(\lambda^{n-1}) \). In particular, letting \( x = y \), the pointwise Weyl asymptotic asserts that

\[
\sum_{\lambda_j \leq \lambda} |e_j(x)|^2 = c_n \lambda^n + R(\lambda, x), \quad \text{where } R(\lambda, x) = O(\lambda^{n-1}) \text{ as } \lambda \to \infty,
\]

uniformly for all \( x \in \mathbb{M} \). Here, \( c_n \) is a constant depending only on \( n \) (more precisely, \( c_n \) is the volume of the unit ball in \( \mathbb{R}^n \).) Integrating the above equation on \( \mathbb{M} \) with respect to \( x \), we have the Weyl asymptotic of the distribution of eigenvalues. Let \( N(\lambda) := \#\{j : \lambda_j \leq \lambda\} \).

Then

\[
N(\lambda) = c_n \text{Vol}(\mathbb{M}) \lambda^n + R(\lambda), \quad \text{where } R(\lambda) = O(\lambda^{n-1}) \text{ as } \lambda \to \infty. \quad (2.2)
\]

The remainder term estimate \( R(\lambda, x) = O(\lambda^{n-1}) \) in (2.1) is sharp on the sphere \( \mathbb{S}^n \). The \( \lambda^{n-1} \) growth rate is achieved at the poles of zonal harmonics on \( \mathbb{S}^n \). (See Hörmander \([Ho]\) Section 6.)
However, on some other manifolds than the spheres, the above estimates of $R(\lambda, x)$ and $R(\lambda)$ may be improved. Such improvements are related to the dynamical properties of the geodesic flow on $\mathbb{M}$. The geodesic flow $G_t$ is the Hamiltonian flow with Hamiltonian defined on $T^*\mathbb{M}$ as $H(x, \xi) = |\xi|^2_x$. The geodesic flow $G_t$ preserves the Liouville measure on $T^*\mathbb{M}$. Write the cosphere bundle $S^*\mathbb{M} = \{(x, \xi) \in T^*\mathbb{M} : |\xi|^2_x = 1\}$. Then $G_t$ acts on $S^*\mathbb{M}$ by homogeneity and leaves the induced Liouville measure on $S^*\mathbb{M}$ invariant.

Denote the set of periodic geodesics on $S^*\mathbb{M}$ as

$$\Pi = \{(x, \xi) \in S^*\mathbb{M} : G_t(x, \xi) = (x, \xi) \text{ for some } t > 0\}.$$ 

Duistermaat-Guillemin [DG] proved that

**Theorem 2.2** (Improved Weyl asymptotics). Assume that the set of periodic geodesics $\Pi$ is of Liouville measure zero in $S^*\mathbb{M}$. Then

$$N(\lambda) = c_n \text{Vol}(\mathbb{M})\lambda^n + R(\lambda), \quad \text{where } R(\lambda) = o(\lambda^{n-1}) \text{ as } \lambda \to \infty.$$  

To get the improvement of pointwise Weyl law, we need a pointwise dynamical condition on the geodesics that is similar to the one in Theorem 2.2. A geodesic loop through $x$ is a geodesic $L(t)$ parametrized by arclength so that for some $t_0 > 0$ such that $L(0) = L(t_0) = x$. Define the loop directions at $x$ as

$$L_x := \{\xi \in S^*\mathbb{M} : G_t(x, \xi) = (x, \eta) \text{ for some } t > 0 \text{ and } \eta \in S^*_x\mathbb{M}\}.$$ 

Canzani-Hanin [CH] proved that

**Theorem 2.3** (Improved spectral projection estimate). Assume that $L_x$ is of measure zero in $S^*_x\mathbb{M} \forall x \in \mathbb{M}$. Then

$$E_{[0,\lambda]}(x, y) = \frac{1}{(2\pi)^n} \int_{|\xi|_{g_y} < \lambda} e^{i\langle \exp_{y}^{-1}(x), \xi \rangle} \frac{d\xi}{\sqrt{|g_y|}} + R(x, y, \lambda),$$

where $R(x, y, \lambda) = o(\lambda^{n-1})$ uniformly for all $x, y \in \mathbb{M}$. In particular, the pointwise Weyl asymptotic asserts that

$$\sum_{\lambda_j \leq \lambda} |e_j(x)|^2 = c_n \lambda^n + R(\lambda, x), \quad \text{where } R(\lambda, x) = o(\lambda^{n-1}) \text{ as } \lambda \to \infty$$

uniformly for all $x \in \mathbb{M}$.

**Remark.**

1. If $L_x$ is of measure zero on $S^*_x\mathbb{M}$ for all $x \in \mathbb{M}$, then the set of periodic geodesics $\Pi$ is of Liouville measure zero on $S^*\mathbb{M}$. Hence, one has that $R(\lambda) = o(\lambda^{n-1})$ as $\lambda \to \infty$ as a corollary of Theorem 2.3. (One can also instead integrate (2.4) on $\mathbb{M}$ directly.)

2. Safarov [Sa] and Sogge-Zelditch [SZ] proved (2.4) under the loopset condition. In Sogge-Toth-Zelditch [STZ], they weakened the condition in Theorem 2.3. That is, suppose that the set of “recurrent loop directions” is of measure zero in $S^*_x\mathbb{M}$ for all $x \in \mathbb{M}$, then (2.4) holds uniformly for all $x \in \mathbb{M}$. Under the non-focal condition, the “off-diagonal” estimate of the spectral projection that $R(x, y, \lambda) = o(\lambda^{n-1})$ in Theorem 2.3 is investigated by Canzani-Hanin [CH].

3. There is a long history to investigate relation between the geometric condition of the manifold and the improved pointwise Weyl asymptotic (2.4) over (2.2). Sogge-Zelditch [SZ] and Sogge-Toth-Zelditch [STZ] addressed the problem: Determine the conditions of $\mathbb{M}$ that ensure the maximal growth rate of eigenfunctions $||e_j||_{L^\infty(\mathbb{M})} = \Omega(\lambda_j^{(n-1)/2})$ holds (or does not hold.) We refer to their work for more details on the relations between this problem and the estimates of the remainder $R(\lambda, x)$. 


2.2. Probabilistic estimates. Let \( S^d \subset \mathbb{R}^{d+1} \) be the \( d \)-dim unit sphere endowed with the uniform probability measure \( \mu_d \). Let

\[
u = \sum_{j=1}^{d+1} a_j s_j, \quad \text{where } a = (a_1, ..., a_{d+1}) \in S^d \text{ and } s = (s_1, ..., s_{d+1}) \in \mathbb{R}^{d+1}.
\]

Notice that

\[|\nu| > t \quad \text{if and only if} \quad |\langle (a_1, ..., a_{d+1}), (s_1(x), ..., s_{d+1}(x)) \rangle_{\mathbb{R}^{d+1}}| > t.\]

We therefore have the following fact. See e.g. [BuLe, Appendix A] for an elementary proof.

**Lemma 2.4.**

\[
\mu_d(|\nu| > t) = \begin{cases} 
1 - \frac{t^2}{|s|^2} & \text{if } 0 \leq t < |s|,
0 & \text{if } t \geq |s|,
\end{cases}
\]

where \(|s|\) is the length of \( s = (s_1, ..., s_{d+1}) \in \mathbb{R}^{d+1}\).

To estimate the probability that a function \( u_c \) deviates from the expectation we use the principle of concentration of measure. It is here that the high dimensionality of the probability spaces we consider comes into play. Concentration of measure requires that a random variable \( F(a) \) cannot take values away from its median too often. Exactly how close to the median depends on regularity properties of \( F \).

**Theorem 2.5** (Levy concentration of measures). Consider a Lipschitz function \( F \) on \( S^d \).

Then for any \( t > 0 \), we have

\[
\mu_d(|F - \mathcal{M}(F)| > t) \leq \exp \left(- \frac{(d-1)t^2}{2\|\mathcal{F}\|_{\text{Lip}}^2}\right).
\]

3. Proof of Theorems 1.2 and 1.4

In this section, we prove the small scale equidistribution results of in Theorems 1.2 and 1.4. For \( a \in S^{N(\lambda)-1} \), let \( u_{\lambda,a} = \sum_{\lambda_j \in (\lambda-W,\lambda)} a_j e_j \in H_W(\lambda) \). Write

\[
F_{x_0,r}(a) = \int_{B(x_0,r)} |u_{\lambda,a}(x)|^2 \, dx \quad \text{for } x_0 \in \mathbb{M}.
\]

**Proposition 3.1.** Suppose that \( F_{x_0,r}(a) \) is given by (3.1). Then

\[
\mathbb{E}(F_{x_0,r}) = \frac{\text{Vol}(B(x_0,r))}{\text{Vol}(M)} \left[1 + O(W^{-1})\right],
\]

where the expectation is taken over \( \gamma_W(\lambda) \) in Definition 1.7. If in addition \( \mathbb{M} \) satisfies the non-focal property as in Theorem 2.3 then

\[
\mathbb{E}(F_{x_0,r}) = \frac{\text{Vol}(B(x_0,r))}{\text{Vol}(M)} [1 + o(1)].
\]
Remark. If $W$ grows with $\lambda$ then (3.2) gives equidistribution in expectation on any manifold. If however $W$ is fixed we need to make the non-focal assumption to insure that $E(F_{x_0,r})$ is equidistributed in the limit $\lambda \to \infty$.

Proof. Denote
\[ e_{W,\lambda}(x) = |(e_1(x), \ldots, e_{N_W(\lambda)}(x))|, \]
that is, the length of the vector $(e_1(x), \ldots, e_{N_W(\lambda)}(x)) \in \mathbb{R}^{N_W(\lambda)}$. Then by the pointwise Weyl asymptotic (2.1) in Theorems 2.1 and 2.3 we have that
\[ e_{W,\lambda}^2(x) = c_n(\lambda^n - (\lambda - W)^n) + R(x, \lambda), \]
where
\[ |R(x, \lambda)| = \begin{cases} O(\lambda^{n-1}) & \text{on any manifold;} \\ o(\lambda^{n-1}) & \text{on manifolds with non-focal condition.} \end{cases} \]

Therefore, we have the local asymptotic
\[ e_{W,\lambda}^2(x) = \begin{cases} c_n n W \lambda^{n-1} + O(\lambda^{n-1}) & \text{on any manifold;} \\ c_n n W \lambda^{n-1} + o(\lambda^{n-1}) & \text{on manifolds with non-focal condition.} \end{cases} \]

Integration over $M$ gives
\[ N_W(\lambda) = \begin{cases} c_n n W \lambda^{n-1} \text{Vol}(M) + O(\lambda^{n-1}) & \text{on any manifold;} \\ c_n n W \lambda^{n-1} \text{Vol}(M) + o(\lambda^{n-1}) & \text{on manifolds with non-focal condition.} \end{cases} \]

We compute the expected value of $F_{x_0,r}$ with respect to $\gamma_W(\lambda)$, equivalently, the expected value of the square of the $L^2$ mass of a random wave $u_\lambda$ from the unit sphere in $H_W(\lambda)$. That is,
\[ E(F_{x_0,r}) = \int_{S^{N_W(\lambda)-1}} \int_{B(x_0,r)} \sum_{i,j=1}^{N_W(\lambda)} a_i a_j e_i(x) e_j(x) \, dx \, d\mu_{N_W(\lambda)-1}. \]

Recall the $\mu_d$ is the uniform probability measure on the sphere $S^d$.

Since each of the $a_i$ have mean zero and $E(a_i^2) = 1/N_W(\lambda)$, we have that
\[ E(F_{x_0,r}) = \frac{1}{N_W(\lambda)} \int_{B(x_0,r)} \sum_{i=1}^{N_W(\lambda)} e_i^2(x) \, dx. \]

Inserting our asymptotics for $N_W(\lambda)$ and $e_{W,\lambda}(x)$ we arrive at
\[ E(F_{x_0,r}) = \begin{cases} \text{Vol}(B(x_0,r)) \text{Vol}(M) \left[ 1 + O(W^{-1}) \right] & \text{on any manifold;} \\ \text{Vol}(B(x_0,r)) \text{Vol}(M) \left[ 1 + o(1) \right] & \text{on manifolds with non-focal condition.} \end{cases} \]

We now in a position to control the variance of $F_{x_0,r}$.

Proposition 3.2. Suppose that $F_{x_0,r}$ is given by (3.1) and $r^{-1} = o(1)$.

Then
\[ \text{Var}(F_{x_0,r}) = \text{Var} \left( \int_{B(x_0,r)} |u_\lambda|^2 \, d\text{Vol} \right) = \text{Vol}(B(x_0,r))^2 \left[ o(1) + O(W^{-2}) \right] \text{ as } \lambda \to \infty. \]

Further if $\mathcal{M}$ satisfies the non-focal assumption as in Theorem 1.4, then
\[ \text{Var}(F_{x_0,r}) = \text{Var} \left( \int_{B(x_0,r)} |u_\lambda|^2 \, d\text{Vol} \right) = o \left( \text{Vol}(B(x_0,r))^2 \right) \text{ as } \lambda \to \infty. \]
Proof. Recall our notation that for $a \in S_{NW}^{W(\lambda)-1}$,
\[ u_{\lambda,a} = \sum_{\lambda_j \in [\lambda-W,\lambda]} a_j e_j \in \mathcal{H}_W(\lambda). \]

We directly compute the variance
\[
\text{Var}(F_{x_{0},r}) = \int_{S_{NW}^{W(\lambda)-1}} \left| F_{x_{0},r}(a) - \mathbb{E}(F_{x_{0},r}) \right|^2 d\mu_{S_{NW}^{W(\lambda)-1}}
\]
\[
= \int_{S_{NW}^{W(\lambda)-1}} \left| \int_{B(x_{0},r)} \sum_{i,j=1}^{NW(\lambda)} a_i a_j e_i(x)e_j(x) dx - \mathbb{E}(F_{x_{0},r}) \right|^2 d\mu_{NW(\lambda)-1}
\]
\[
= \int_{S_{NW}^{W(\lambda)-1}} \int_{B(x_{0},r)} \int_{B(x_{0},r)} \sum_{i,j,k,l=1}^{NW(\lambda)} a_i a_j a_k a_l e_i(x)e_j(x)e_k(y)e_l(y) dx dy d\mu_{NW(\lambda)-1}
- 2\mathbb{E}(F_{x_{0},r}) \int_{S_{NW}^{W(\lambda)-1}} \int_{B(x_{0},r)} \sum_{i,j=1}^{NW(\lambda)} a_i a_j e_i(x)e_j(x) dx d\mu_{NW(\lambda)-1} + \mathbb{E}(F_{x_{0},r})^2.
\]

Note that since each $a_i$ has mean zero on $S_{NW}^{W(\lambda)-1}$, any term containing odd powers of the $a_i$ is zero in expectation. So we are left with terms with even powers only. Among the even-powered terms in the above equality, the terms $a_i^2$ have expectation $\mathbb{E}(a_i^2) = 1/N_W$, while the terms $a_i^2 a_j^2$ have expectation $\mathbb{E}(a_i^2 a_j^2) = 1/N_W^2$. Hence,

\[
\text{Var}(F_{x_{0},r}) = \frac{1}{N_W^2} \int_{B(x_{0},r)} \int_{B(x_{0},r)} \sum_{i,j=1}^{NW(\lambda)} e_i^2(x)e_j^2(y) dx dy
+ \frac{2}{N_W} \int_{B(x_{0},r)} \int_{B(x_{0},r)} \sum_{i,j=1}^{NW(\lambda)} e_i(x)e_i(y)e_j(x)e_j(y) dx dy
- \frac{2\mathbb{E}(F_{x_{0},r})}{NW(\lambda)} \int_{B(x_{0},r)} \sum_{i=1}^{NW(\lambda)} e_i^2(x) dx + \mathbb{E}(F_{x_{0},r})^2
= \frac{2}{N_W^2} \int_{B(x_{0},r)} \int_{B(x_{0},r)} \sum_{i,j=1}^{NW(\lambda)} e_i(x)e_i(y)e_j(x)e_j(y) dx dy
= \frac{2}{N_W^2} \int_{B(x_{0},r)} \int_{B(x_{0},r)} \left( \sum_{i=1}^{NW(\lambda)} e_i(x)e_i(y) \right) \left( \sum_{j=1}^{NW(\lambda)} e_j(x)e_j(y) \right) dx dy.
\]

Here, we use the fact that
\[
\mathbb{E}(F_{x_{0},r}) = \frac{1}{NW(\lambda)} \int_{B(x_{0},r)} \sum_{i=1}^{NW(\lambda)} e_i^2(x) dx,
\]
see Proposition 3.1.

Adopting the notation of Theorem 2.1, we write
\[
\sum_{i=1}^{NW(\lambda)} e_i(x)e_i(y) = E_{[\lambda-W,\lambda]}(x,y).
\]

where $E_{[\lambda-W,\lambda]}$ is the kernel of the spectral projection onto the space $\mathcal{H}_W(\lambda)$. So $\text{Var}(F_{x_{0},r})$ simplifies to
\[
\text{Var}(F_{x_{0},r}) = \frac{2}{N_W^2} \int_{B(x_{0},r)} \int_{B(x_{0},r)} E_{[\lambda-W,\lambda]}^2(x,y) dx dy.
\]
By Theorem 2.1,
\[ E_{[\lambda-W]}^2(x, y) = \frac{1}{(2\pi)^n} \int_{\lambda-W<|\xi|<\lambda} e^{i\langle \exp^{-1}(x, \xi) \rangle} \frac{d\xi}{\sqrt{|g_y|}} + R(x, y, \lambda, W), \tag{3.4} \]
where
\[ R(x, y, \lambda, W) = R(x, y, \lambda) + R(x, y, \lambda - W). \]
Further we have the asymptotic bounds
\[ \|R(x, y, \lambda, W)\| = \begin{cases} O(\lambda^{n-1}) & \text{on any manifold;} \\ o(\lambda^{n-1}) & \text{on manifolds with non-focal condition.} \end{cases} \]
Since \( x \) and \( y \) are close, say that they are in the same coordinate patch and indeed we may assume that \( y \) is the centre of that patch and \( g_y = \text{Id} \). Therefore, the integral becomes
\[ \frac{1}{(2\pi)^n} \int_{\lambda-W<|\xi|<\lambda} e^{i\langle \exp^{-1}(x, \xi) \rangle} d\xi. \]
Now
\[ \left| \int_{|\xi|=\rho} e^{i\langle \exp^{-1}(x, \xi) \rangle} d\xi \right| \leq \rho^{n-1} (1 + \rho |\exp^{-1}(x)|)^{-\frac{n-1}{2}} \]
by classical Fourier restriction calculations. Therefore,
\[ \frac{1}{(2\pi)^n} \int_{\lambda-W<|\xi|<\lambda} e^{i\langle \exp^{-1}(x, \xi) \rangle} d\xi \lesssim \begin{cases} W\lambda^{n-1}, & \text{if } 0 \leq |x-y| \leq \lambda^{-1}; \\ W\lambda^{-\frac{n-1}{2}} |x-y|^{-\frac{n-1}{2}}, & \text{if } \lambda^{-1} \leq |x-y| \leq 1. \end{cases} \]
Setting
\[ R(\lambda, W) = \sup_{x,y} |R(x, y, \lambda, W)|, \]
we have
\[ \int_{B(x, r)} \int_{B(x, r)} E_{[\lambda-W]}^2(x, y) \, dxdy \]
\[ \lesssim \int_{B(x, r)} \int_{B(x, r)\cap B(x, \lambda^{-1})} W^2\lambda^{2(n-1)} \, dxdy + R(\lambda, W)^2 \text{Vol}(B(x, r))^2 \]
\[ + \int_{B(x, r)} \int_{B(x, r)\setminus B(x, \lambda^{-1})} \left( W^2\lambda^{n-1}|x-y|^{-(n-1)} + R(\lambda, W)W^{-1} \lambda^{-1} |x-y|^{-\frac{n-1}{2}} \right) \, dxdy \]
\[ \lesssim W^2\lambda^{-2} \text{Vol}(B(x, r)) + W^2\lambda^{-1} \text{Vol}(B(x, r)) + R(\lambda, W)^2 \text{Vol}(B(x, r))^2. \]
By Theorems 2.1 and 2.3 again, we have that
\[ N_W(\lambda) = \begin{cases} c_n W \lambda^{n-1} \text{Vol}(M) + O(\lambda^{n-1}) & \text{on any manifold;} \\ c_n W \lambda^{n-1} \text{Vol}(M) + o(\lambda^{n-1}) & \text{on manifolds with non-focal condition.} \end{cases} \]
So
\[ \text{Var}(F_{x, r}) = \frac{2}{N_W^2} \int_{B(x, r)} \int_{B(x, r)} E_{[\lambda-W]}^2(x, y) \, dxdy \]
\[ \lesssim \lambda^{-n} \text{Vol}(B(x, r)) + \lambda^{-(n-1)} r \text{Vol}(B(x, r)) \]
\[ + W^2 \lambda^{-2(n-1)} R(\lambda, W)^2 \text{Vol}(B(x, r))^2. \tag{3.5} \]
Hence, if \( r^{-1} = o(\lambda) \), then
\[ \lambda^{-n} \text{Vol}(B(x, r)) = o(\text{Vol}(B(x, r))^2) \quad \text{and} \quad \lambda^{-(n-1)} r \text{Vol}(B(x, r)) = o((\text{Vol}(B(x, r))^2), \]
and we arrive at
\[ \text{Var}(F_{x, r}) \leq \text{Vol}(B(x, r))^2 \left[ o(1) + O(W^2 \lambda^{-2(n-1)} R(\lambda, W)^2) \right]. \]
On any manifold $R(\lambda, W) = O(\lambda^{n-1})$ which yields

$$\text{Var}(F_{x_0, r}) = \text{Vol}(B(x_0, r))^2 \left[ o(1) + O(W^{-2}) \right].$$

If in addition $M$ satisfies the non-focal assumption, then $R(\lambda, W) = o(\lambda^{n-1})$, and

$$\text{Var}(F_{x_0, r}) = o \left( \text{Vol}(B(x, r))^2 \right).$$

□

4. Uniform equidistribution

Note that for a fixed point $x \in M$, the expectation and variance results in Theorems 1.2 and 1.4 tell us that for $r \gg \lambda^{-1}$ the probability that the random wave $u$ (as in Definition 3.1) does not equidistribute is decaying in $\lambda$. However, there can be a set of small measure (on $S^{N_W(\lambda)-1}$) for which $u$ displays concentration near $x$. In this section, we look at the problem of determining whether given a radius $r(\lambda) \to 0$ as $\lambda \to \infty$ we can say that there is some uniformity in the set of measure zero. That is given a set of balls $B(x_k, r)$ that cover $M$ can we say that the probability that $u$ does not equidistribute on any one of the balls decays to zero as $\lambda \to \infty$?

We address this question by using concentration of measure to estimate the probability that on any individual ball $u$ fails to equidistribute. The key point is that the estimates provided by Levy concentration of measure provides exponential decay while only of polynomial number of balls are required to cover $M$. We therefore need control on the worst decay rate of $\|u\|_{L^2(B(x, r))}$ in a small ball $B(x, r)$ when $r \to 0$ as $\lambda \to \infty$. This estimate will directly allow us to control Lipschitz norm of $F_{x_0, r}$, which in turn controls the level of concentration. Sogge [So2, (4.1)] proved the following estimate for smoothed spectral clusters on small balls (see Sogge-Toth-Zelditch [STZ] and Sogge-Zelditch [SZ2] for the argument that shows that estimates on smoothed spectral clusters imply estimates on quasimodes).

**Lemma 4.1.** On a compact manifold $M$, let $u = \sum_{\lambda_j \in [\lambda - 1, \lambda]} a_j e_j$. Then for all $x \in M$ and $\lambda^{-1} \leq r \leq \text{Inj} M$, we have that

$$\int_{B(x, r)} |u|^2 \, d\text{Vol} \leq cr \|u\|_{L^2(M)}^2,$$

where $c$ depending only on $M$.

Lemma 4.1 is of course an improvement on the trivial estimate $\int_{B(x, r)} |u|^2 \, d\text{Vol} \leq 1$. In fact, it is already sharp on $S^n$, as the equation is achieved by the zonal harmonics on balls centered at its poles on $S^n$. See Sogge [So2, Section 4] for more discussion.

Since we are considering window widths that may be much larger than the one in Lemma 4.1 we need an equivalent statement for

$$u = \sum_{\lambda_j \in [\lambda - W, \lambda]} a_j e_j$$

Fortunately we are able to use Lemma 4.1 to very easily obtain the necessary estimates.

**Lemma 4.2.** Suppose that

$$u = \sum_{\lambda_j \in [\lambda - W, \lambda]} a_j e_j.$$

Then

$$\int_{B(x, r)} |u|^2 \, d\text{Vol} \leq \begin{cases} cr \|u\|_{L^2(M)}^2, & \text{if } \lambda^{-1} \leq r \leq W^{-1}; \\ |a_j|^2_{L^2(M)}^2, & \text{if } W^{-1} \leq r \leq \text{Inj} M. \end{cases}$$

(4.1)
where.

So by applying Lemma 4.1 and the Cauchy-Schwartz inequality, we have that

Given $u, v$ so we may apply Lemma 4.1 to each of the $u_k$ in which

Note that each $u_k$ is a fixed window spectral cluster at frequency $\mu_k = \lambda - W + k + 1 > \lambda/2$ so we may apply Lemma 4.1 to each of the $u_k$ separately. Now

$$
\int_{B(x_0,r)} |u|^2 \, d\text{Vol} = \sum_{m=0}^{W-1} \sum_{k=0}^{W-1} \int_{B(x_0,r)} u_k(x) \bar{u}_{(k+m)\lambda}(x) \, d\text{Vol},
$$

where

$$(k + m)\lambda = k + m \mod (W - 1).$$

So by applying Lemma 4.1 and the Cauchy-Schwartz inequality, we have that

$$
\int_{B(x_0,r)} |u|^2 \, d\text{Vol} \lesssim r \sum_{m=0}^{W-1} \sum_{k=0}^{W-1} \left\| u_k \right\|_{L^2(M)} \left\| u_{(k+m)\lambda} \right\|_{L^2(M)}
$$

$$
\lesssim r \left( \sum_{m=0}^{W-1} \sum_{k=0}^{W-1} \left\| u_k \right\|_{L^2(M)}^2 \right)^{1/2} \left( \sum_{k=0}^{W-1} \left\| u_{(k+m)\lambda} \right\|_{L^2(M)}^2 \right)^{1/2}
$$

$$
\lesssim r W \left\| u \right\|_{L^2(M)}^2.
$$

Remark. It turns out that these simple estimates are sharp. Spectral clusters $u$ of window width $W$ in Lemma 4.2 can be thought of as approximate eigenfunctions with $L^2$ error no greater than $W\lambda$. That is

$$
\left\| (\Delta - \lambda^2) u \right\|_{L^2(M)} \lesssim W\lambda \left\| u \right\|_{L^2(M)}.
$$

Such functions can be localised so that all their $L^2$ is located in one $W^{-1}$ size ball. See e.g. Tacy [1].

**Proposition 4.3.** Suppose that $F_{x_0,r}$ is given by (3.1). Then

$$
\left\| F_{x_0,r} \right\|_{Lip} \leq \begin{cases} crW, & \text{if } \lambda^{-1} \leq r \leq W^{-1}; \\ 1, & \text{if } W^{-1} \leq r \leq \text{Inj}(M). \end{cases}
$$

*Proof.* Given $u, v \in H_M(\lambda)$, let

$$
u = \sum_{j=1}^{N_M(\lambda)} a_j e_j \quad \text{and} \quad v = \sum_{j=1}^{N_M(\lambda)} b_j e_j,$$

where $a = (a_1, ..., a_{N_M(\lambda)})$ and $b = (b_1, ..., b_{N_M(\lambda)})$ are in $S^{N_M(\lambda)-1}$. Thinking of $F_{x_0,a}$ as the square of the $L^2$ mass of the function $u$ (or $v$) we have that

$$
|F_{x_0,a} - F_{x_0,b}| \leq \int_{B(x_0,r)} \left| u(x) \right|^2 - \left| v(x) \right|^2 \, dx
$$

$$
= \int_{B(x_0,r)} \left| \left| u(x) \right| - \left| v(x) \right| \right| \left( \left| u(x) \right| + \left| v(x) \right| \right) \, dx
$$

$$
\leq \left( \int_{B(x_0,r)} \left| u(x) - v(x) \right|^2 \, dx \right)^{1/2} \left( \int_{B(x_0,r)} \left( \left| u(x) \right| + \left| v(x) \right| \right)^2 \, dx \right)^{1/2}.
$$
Now if $r \leq W^{-1}$ by applying Lemma 4.2 we obtain that
\[ |u - v|_{L^2(B(x, r))} \leq C r^{1/2} W^{1/2} \|u - v\|_{L^2(M)}, \]
and
\[ |u + v|_{L^2(B(x, r))} \leq C r^{1/2} W^{1/2} \|u - v\|_{L^2(M)}; \]
and if $r > W^{-1}$, we obtain that
\[ |u - v|_{L^2(B(x, r))} \leq C \|u - v\|_{L^2(M)}; \]
and
\[ |u + v|_{L^2(B(x, r))} \leq C \|u - v\|_{L^2(M)}. \]
Since
\[ u(x) - v(x) = \sum_{j=1}^{N_W(\lambda)} (a_j - b_j)e_j(x) \]
and $e_j$ are orthonormal, we have that
\[ \|u - v\|_{L^2(M)} \approx \text{dist}(a, b). \]
So
\[ \|F_{x,r}\|_{\text{Lip}} \leq \begin{cases} \epsilon r W & \text{if } \lambda^{-1} \leq r \leq W^{-1}, \\ 1 & \text{if } W^{-1} \leq r \leq \text{Inj}(M). \end{cases} \]
\[ \square \]

We can now control the probability that for any $x$, $F_{x,r}$ deviates from its median, we will use this to show that the probability that the random wave is not equidistributed on any ball decays exponentially as $\lambda \to \infty$.

**Theorem 4.4.** Let $x_k$ such that the set of $B(x_k, r)$ cover $M$. Set
\[ C_{\delta,r} = \left\{ c \in S^{N_W(\lambda)-1} \mid \exists k \left| F_{x_k,r} - \frac{\text{Vol}(B(x_k, r))}{\text{Vol}(M)} \right| \geq r^n \lambda^{-\delta} \right\}. \]
Suppose $\lambda^{-1} \leq r \leq W^{-1}$ and $r = \lambda^{-\frac{1}{2} + \frac{1}{2} W^{-1}}$, then there exists a constant $C$ so that if $\epsilon > 2\delta/(n - 1)$, then
\[ \mu_{N_W(\lambda)-1}(C_{\delta,r}) \lesssim e^{-C \lambda^{(n-1)-2\delta}}. \] (4.2)
On the other hand if $W^{-1} \leq r \leq \text{Inj}(M)$ and $r = \lambda^{-\frac{n-1}{2} + \frac{1}{2} W^{-1}}$, then there exists a constant $C$ so that if $\epsilon > 2\delta/n$,
\[ \mu_{N_W(\lambda)-1}(C_{\delta,r}) \lesssim e^{-C \lambda^{n-2\delta}}. \] (4.3)

**Proof.** Clearly
\[ \mu_{N_W(\lambda)-1}(C_{\delta,r}) \leq \sum_k \mu_{S^{N_W(\lambda)-1}} \left( \left\{ c \in S^{N_W(\lambda)-1} \mid \left| F_{x_k,r} - \frac{\text{Vol}(B(x_k, r))}{\text{Vol}(M)} \right| > r^n \lambda^{-\delta} \right\} \right), \]
so since $W$ can at worst grow as a power of $\lambda$ it is enough so show that in the case $\lambda^{-1} \leq r \leq W^{-1}$ we have for any $k$,
\[ \mu_{N_W(\lambda)-1} \left( \left\{ c \in S^{N_W(\lambda)-1} \mid \left| F_{x_k,r} - \frac{\text{Vol}(B(x_k, r))}{\text{Vol}(M)} \right| > r^n \lambda^{-\delta} \right\} \right) \leq e^{-C \lambda^{(n-1)-2\delta}}, \] (4.4)
and when $W^{-1} \leq r \leq \text{Inj}(M)$,
\[ \mu_{N_W(\lambda)-1} \left( \left\{ c \in S^{N_W(\lambda)-1} \mid \left| F_{x_k,r} - \frac{\text{Vol}(B(x_k, r))}{\text{Vol}(M)} \right| > r^n \lambda^{-\delta} \right\} \right) \leq e^{-C \lambda^{(n-1)-2\delta}}. \] (4.5)
We use the Levy concentration of measures in Theorem 2.5. First we need to relate the median \( \mathcal{M}(F_{x,k,r}) \) to the expectation \( \mathbb{E}(F_{x,k,r}) \). This is a standard calculation we have
\[
\mathcal{M}(F_{x,k,r}) = \mathbb{E}(F_{x,k,r}) - \int_{\mathbb{S}^n_{W}(\lambda)^{-1}} (F_{x,k,r} - \mathcal{M}(F_{x,k,r})) \, d\mu_{NW}(\lambda)^{-1}.
\]
Using the concentration of measure, we have that
\[
\left| \int_{\mathbb{S}^n_{W}(\lambda)^{-1}} (F_{x,k,r} - \mathcal{M}(F_{x,k,r})) \, d\mu_{NW}(\lambda)^{-1} \right| \leq r^n \lambda^{-\delta} + \exp \left( -\frac{N_W(\lambda)r^{2n}\lambda^{-2\delta}}{2\|F\|_{Lip}^2} \right).
\]
So when \( \lambda^{-1} \leq r \leq W^{-1} \), we have
\[
|\mathcal{M}(F_{x,k,r}) - \mathbb{E}(F_{x,k,r})| \leq r^n \left( \lambda^{-\delta} + r^{-n}e^{-\lambda^{(n-1)-2\delta}} \right).
\]
If \( |F_{x,k,r} - \mathbb{E}(F_{x,k,r})| > r^n \lambda^{-\delta} \), we must have (for large \( \lambda \)) \( |F_{x,k,r} - \mathcal{M}(F_{x,k,r})| > r^n \lambda^{-\delta}/2 \). Similarly in the case where \( W^{-1} \leq r \leq \text{Inj}(\mathcal{M}) \), we have
\[
|\mathcal{M}(F_{x,k,r}) - \mathbb{E}(F_{x,k,r})| \leq r^n \left( \lambda^{-\delta} + r^{-n}e^{-\lambda^{n-2\delta}} \right).
\]
So likewise, \( |F_{x,k,r} - \mathbb{E}(F_{x,k,r})| > r^n \lambda^{-\delta} \) implies \( |F_{x,k,r} - \mathcal{M}(F_{x,k,r})| > r^n \lambda^{-\delta}/2 \). Then by applying the Levy concentration of measure we obtain (4.3) and (4.5).

\[\square\]

Remark.

(1) Estimates (4.2) and (4.3) imply that as \( \lambda \to \infty \) the probability that a random wave does not equidistribute on any ball not only decays but does so at an exponential rate.

(2) If \( W = 1 \) (that is we study a fixed frequency window) we see that the threshold radius is \( r = \lambda^{-1/2} \), the square root Planck scale \( \lambda^{-1} \). Therefore any small scale fluctuations must occur between Planck scale \( \lambda^{-1} \) and square root Planck scale \( \lambda^{-1/2} \).

(3) In some cases, e.g. the spheres, where explicit formula are known for \( E_{[0,\lambda]}(x,y) \) it is likely that more can be determined for the scales between \( \lambda^{-1} \) and \( \lambda^{-1/2} \). See Courcy-Ireland [CI].

References

[B] M. Berry, Regular and irregular semiclassical wavefunctions. J. Phys. A 10 (1977), no. 12, 2083–2091.
[BuLe] N. Burq and G. Lebeau, Injections de Sobolev probabilistes et applications. Ann. Sci. Éc. Norm. Supér. 46 (2013), no. 6, 917–962.
[CH] Y. Canzani and B. Hanin, Scaling limit for the kernel of the spectral projector and remainder estimates in the pointwise Weyl law. Anal. PDE 8 (2015), no. 7, 1707–1731.
[CI] Matthew de Courcy-Ireland, Small-scale equidistribution for random spherical harmonics. arXiv:1711.01317.
[DG] J. J. Duistermaat and V. W. Guillemin, The spectrum of positive elliptic operators and periodic bicharacteristics. Invent. Math. 29 (1975), 39–79.
[Ha1] X. Han, Small scale quantum ergodicity in negatively curved manifolds. Nonlinearity 28 (2015), no. 9, 3263–3288.
[Ha2] X. Han, Small scale equidistribution of random eigenbases. Comm. Math. Phys. 349 (2017), no. 1, 425–440.
[He1] H. Hezari, Applications of small scale quantum ergodicity in nodal sets. arXiv:1606.02057.
[He2] H. Hezari, Inner radius of nodal domains of quantum ergodic eigenfunctions. arXiv:1606.03499.
[He3] H. Hezari, Quantum ergodicity and \( L^p \) norms of restrictions of eigenfunctions. arXiv:1606.08066.
[HR1] H. Hezari and G. Rivièrè, \( L^p \) norms, nodal sets, and quantum ergodicity. Adv. Math. 290 (2016), 938–966.
[HR2] H. Hezari and G. Rivièrè, Quantitative equidistribution properties of toral eigenfunctions. arXiv:1503.02794. To appear in the Journal of Spectral Theory.
[Ho] L. Hörmander, The spectral function of an elliptic operator. Acta Math. 121 (1968), 193–218.
[Le] M. Ledoux, *The concentration of measure phenomenon*, American Mathematical Society, Providence, RI, 2001.

[LR] S. Lester and Z. Ruchnick, *Small scale equidistribution of eigenfunctions on the torus*. arXiv:1508.01074

[Sa] Y. Safarov, *Asymptotics of a spectral function of a positive elliptic operator without a nontrapping condition*, Funktsional. Anal. i Prilozhen. 22:3 (1988), 53–65, 96. In Russian: translated in Funct. Anal. Appl. 22:3 (1988), 213–223.

[So1] C. Sogge, *Localized $L^p$-estimates of eigenfunctions: A note on an article of Hezari and Rivière*. Adv. Math. 289 (2016), 384–396.

[So2] C. Sogge, *Problems related to the concentration of eigenfunctions*. arXiv:1510.07723

[STZ] C. Sogge, J. Toth, and S. Zelditch, *About the blowup of quasimodes on Riemannian manifolds*. J. Geom. Anal. 21 (2011), no. 1, 150–173.

[SZ] C. Sogge and S. Zelditch, *Riemannian manifolds with maximal eigenfunction growth*. Duke Math. J. 114 (2002), no. 3, 387–437.

[SZ2] C. Sogge and S. Zelditch, *A note on $L^p$-norms of quasi-modes*. Some Topics in Harmonic Analysis and Applications. ed. / Junfeng Li; Xiaochun Li; Guozhen Lu. , (Advanced Lectures in Mathematics; Vol. 34), International Press of Boston, Inc. 2016. p. 385-397.

[T] M. Tacy, *A note on constructing sharp examples for $L^p$ norms of eigenfunctions and quasimodes near submanifolds*. arXiv:1605.03698

[Z1] S. Zelditch, *Real and complex zeros of Riemannian random waves*. Spectral analysis in geometry and number theory, 321–342, Contemp. Math., 484, Amer. Math. Soc., Providence, RI, 2009.

[Z2] S. Zelditch, *Logarithmic lower bound on the number of nodal domains*. arXiv:1510.05315

Department of Mathematics, California State University, Northridge, CA 91330, USA

E-mail address: Xiaolong.Han@csun.edu

Department of Mathematics, The Australian National University, Canberra, ACT 2601, Australia

E-mail address: Melissa.Tacy@anu.edu.au