Slow-roll inflation without fine-tuning *

Tiago C. Charters¹,² †, José P. Mimoso¹,² † & Ana Nunes¹,³ §

¹Dep. Física, F. C. L., Ed C1, Campo Grande, 1700 Lisboa, Portugal
²CFNUL and ³CMAF, Av. Prof. Gama Pinto, 2, 1699 Lisboa Codex, Portugal

(March 24, 2022)

The slow-roll approximation is the usual starting point to study the constraints imposed on the inflaton potential parameters by the observational data. We show that, for a potential exhibiting at least two extrema and giving rise to a limited inflationary period, slow-roll does not have to be taken as an additional hypothesis and is in fact forced by the constraints on the number of e-foldings.

I. INTRODUCTION

Inflation avoids the horizon, flatness and monopole problems of the standard cosmological model by just requiring a brief period of accelerated expansion during the very early universe [1]. It also provides an explanation for the origin of the fluctuations associated with the observed cosmic microwave background anisotropies [2]. The latter is a most valued feature at present, since after the detection made by COBE of these anisotropies the prospects for probing the physics of the inflationary models became real. Moreover the coming MAP and Planck missions will push our ability to contrast the theoretical models with observations to the precision required to rule out many of them.

---

*preprint

†email: charters@alf1.cii.fc.ul.pt

‡email: jpmimoso@alf1.cii.fc.ul.pt

§email: anunes@lmc.fc.ul.pt
The underlying ideas of inflation are well-known, but a fully consistent scenario is still unavailable. In most cases the predictions of the models are based on a few features that seem consensual. In the simplest version of inflation one ordinarily considers a self-interacting scalar field $\varphi$ (the *inflaton*) being damped by expansion of the universe and eventually yielding the negative pressures required to promote the violation of the strong energy principle, and hence the positive acceleration. The details of the process depend on the form of the potential, but the popular approach has been to associate the bulk of inflation with a stage during which the scalar field slow rolls along a potential well. The *slow-roll parameters* are defined in this context and a perturbative expansion in terms of these parameters is then used to predict the anisotropies of the cosmic microwave background (CMB), and conversely, to reconstruct from the latter data the shape of the inflaton potential.

A central issue regarding this usual procedure is then to ascertain the legitimacy of the slow-roll approximation. Given an arbitrary potential the slow-roll regime is not a generic dynamical pattern. However the vast majority of inflationary models rely on the slow-roll assumption and then show that it is possible to constrain the parameters of the potential in such a way as to meet the requirements of enough inflation and limited fluctuations in the CMB.

The issues of the genericity of the slow-roll regime and of the possibility of meeting the observational requirements without the slow-roll assumptions have not until now been dealt with satisfactorily. In what concerns the former of these questions, there has been some effort to either avoid it or argue in favour of an affirmative answer in the case of particular models. In particular the results of Ref. have shown that for the pseudo-Nambu-Goldstone-boson of the natural inflation model fine tuning can be avoided if we consider non-zero initial velocities for the field. We recover below this conclusion in an independent context. As for the latter question there has not been to our knowledge any attempt to address it directly. Leaving this question unanswered encourages criticism of the relevance of the theoretical estimates based on the slow-roll approximation. Recently, the reliability of the reconstruction procedure based on the lowest-order slow-roll expansion
has been questioned by Wang, Mukhanov and Steinhardt [13] who show that its validity depends on the effective equation of state of the inflaton field being almost stationary during the relevant epoch of horizon exit of the fluctuations. In this reference the authors explicitly consider a potential for which this condition is not met and show that the implications for the CMB anisotropies are significantly different from those predicted by the slow-roll approximation. In Ref. [16] it is shown that the particular potential considered in [13] is such that several requirements, among which the end of inflation, cannot be met.

Given that slow-roll is an atypical dynamical regime for damped oscillations, in the absence of a general negative answer to the second of the questions, the validity of the slow-roll approximation will remain a natural target to criticism such as in [13].

In the present work our aim has been to determine the consequences of imposing the constraints on the number of e-foldings and on the amplitude of the CMB fluctuations to a general scalar field model giving rise to a limited inflationary period. Our study relies on a numerical analysis of the dynamics, which is both rigorous and general because it stems from the knowledge of the global dynamics of the non-linear dynamical system under consideration. Contrary to what may be thought the numerical approach is more adequate than an alternative analytic approach which is bound here to be either local or too restrictive.

The results of this analysis show that the constraints on the number of e-foldings force the free parameters that control the basic features of the potential to take the values for which the orbits exhibit slow roll dynamics for a relatively long time of their evolution. This is a consequence of the known fact that rapid oscillations in convex potentials do not contribute to inflation [17–19]. We also show that these values of the parameters correspond to a big region in parameter space. These two conclusions are of great importance on the one hand to avoid the criticism of [13] and, on the other hand, to establish that inflation is likely to happen.

We consider the flat isotropic Friedmann-Robertson-Walker (FRW) universe characterized by the metric
\[ ds^2 = -dt^2 + a(t)^2 \left[ dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right] , \]

and assume that the matter content of the universe is a homogeneous scalar field with a self-interacting potential. The field equations are

\[ H^2 = \frac{8\pi}{3m_{pl}^2} \left( \frac{1}{2} \dot{\varphi}^2 + V(\varphi) \right) , \]

\[ \ddot{\varphi} + 3H \dot{\varphi} + V'(\varphi) = 0 , \]

where the overdots stand for the derivatives with respect to time, \( m_{pl}^2 \) is the Planck mass, \( H = \dot{a}/a \) is the Hubble parameter, and \( V' = dV/d\varphi \). It is also possible to deduce from the latter equations that

\[ \frac{\ddot{a}}{a} = \frac{8\pi}{3m_{pl}^2} \left( V(\varphi) - \dot{\varphi}^2 \right) \]

holds and hence that inflation requires \( V(\varphi) > \dot{\varphi}^2 \).

In our numerical study, we shall consider potentials of the form

\[ V_1(A, B, \varphi) = \frac{A}{B^2} (\varphi^2 - B)^2 , \]

\[ V_2(A, B, \varphi) = A \left( 1 + \cos(\frac{\pi \varphi}{\sqrt{B}}) \right) , \]

where \( A \) and \( B \) are positive constants. However, our results extend to potentials exhibiting several extrema. For these potentials, given any initial conditions outside a potential trap, the damping term in Eq. (3) will make the system evolve to approach the potential well where it will be caught by the stable equilibrium. Provided that this is vanishing (no false vacuum term), the last stage of the evolution of the system corresponds to a potential with one maximum and one vanishing minimum, such as in Eqs. (5,6), together with initial conditions slightly above the energy of the maximum. Furthermore, the behaviour of the solutions of the systems associated with the potentials \( V_1 \) of Eq. (3) or \( V_2 \) of Eq. (4) is typical in this more general class: the qualitative behaviour does not depend on the detailed form of the potential and is described in some detail below. In this general setting our results
apply to any non-degenerate potential with the aforementioned qualitative properties, since the only essential requirement is the non-degeneracy of the consecutive extrema. From a physical viewpoint, our conclusions hold for a non-degenerate potential susceptible of giving rise to a finite inflationary period whose end is followed by the death of the oscillating field, and also to sufficient inflation without fine-tuning the parameters. In fact, the first condition excludes potentials without extrema, such as the exponential potential or the arctan potential of Ref. [13], as well as potentials with a false vacuum term [14,15], while the second discards single minimum potentials as the quadratic potential [3,4,19].

Consider the Eqs. (2,3) with the potentials (5) or (6). In both cases and indeed for any potential with a maximum and a vanishing minimum, the problem reduces to the study of the dynamics of a non-linear, non-linearly damped, oscillator whose damping term tends to zero as the stable equilibrium is approached. In spite of the high non-linearity, the equations are those of an autonomous system in the plane, and therefore the possibility of chaotic behavior is excluded. Indeed, the qualitative description of the behavior of the system in phase-space is quite straightforward. Consider for instance the case of the standard double well potential: there are three equilibrium points, one saddle at \((\varphi, \dot{\varphi}) = (0, 0)\) and two stable focus at \((\varphi, \dot{\varphi}) = (\pm \sqrt{B}, 0)\). The stable manifolds of the saddle point separate the plane into two regions, I and II. One of the stable focuses attracts all the points in I, and the other attracts all of the points in II. Moreover, there is no divergence of nearby orbits, and, apart from the boundary of the two basins of attraction, the future behaviour of the orbits will change smoothly with the initial conditions. Bearing this picture in mind, it is possible to perform a thorough numerical exploration of the system, in order to determine the set of parameters and initial conditions which satisfy the observational constraints.

It is usual to consider the parameters [3,1]

\[
\epsilon(t) = \frac{3}{2} \frac{\dot{\varphi}^2}{\dot{\varphi}} + V(A, B, \varphi) \tag{7}
\]

\[
\eta(t) = -\frac{\ddot{\varphi}}{H \dot{\varphi}} \tag{8}
\]
Inflation is equivalent to having $\epsilon < 1$. The slow roll regime is defined by the following conditions: $\epsilon(t) \ll 1$, which is the condition for neglecting the kinetic energy contribution to the damping term in Eq. (3), and $\eta(t) \ll 1$, which is the condition for neglecting the acceleration term in Eq. (3).

To successfully solve the cosmological puzzles of standard cosmology an inflationary model must satisfy the following constraints [1, 2].

(i) **Sufficient inflation** - We demand that the scale factor of the universe inflates by at least 60 e-foldings.

(ii) **Energy bounds** - To keep full generality we allow inflation to take place between the Planck energy scale of $m_{pl}$ and the electroweak phase transition at $10^{-17}m_{pl}$. This provides energy bounds in phase-space for the dynamics of the scalar field $10^{-34}m_{pl}^2 \leq H^2 \leq m_{pl}^2$.

(iii) **Density perturbations** - The observations of the anisotropies of the CMB constraint the amplitude of density fluctuations, $\delta_H$, to satisfy the bound $\delta_H \leq \delta_H^{obs} \simeq 1.9 \times 10^{-5}$ [20, 21].

We shall now impose these constraints on the dynamics of Eqs. (2, 3) under the potentials $V_i(A, B, \varphi)$, $i = 1, 2$. First note that both potentials are proportional to $A$. Changing the time scale through $t = \tau/\sqrt{A}$ and redefining the field’s velocity and acceleration accordingly, the equations of motion (2, 3) become independent of the parameter $A$. The total number of e-foldings is given by

$$N(A, B) = \int_0^{+\infty} \sqrt{\frac{8\pi}{3m_{pl}^2}} \left(\frac{\dot{\varphi}^2(t)}{2} + V(A, B, \varphi(t))\right) dt , \quad (9)$$

where $\varphi(t)$ is a solution of the Eqs. (2, 3). Performing the rescaling of the time variable in the latter integral, we have $N(A, B) = N(1, B)$, and, thus, the number of e-foldings is independent of $A$. Notice that this scaling property is also independent of the detailed form of $V$.

When $\epsilon \ll 1$ and $\dot{\epsilon} \simeq 0$ hold, it is known that $\delta_H = \mathcal{O}(1)H^2/\dot{\varphi}$, where the right-hand side is evaluated at horizon crossing [4, 13]. Thus, assuming that $\epsilon(t)$ satisfies the latter conditions, the amplitude of the density fluctuations scales as $\delta_H(A) = \sqrt{A}\delta_H(1)$ and so restriction (iii) can be dealt with separately. Setting $\varphi(0) = 0$, we are thus left with the two
parameters \((B, \varphi'(0))\), where the prime denotes the derivative with respect to the rescaled time, and we have to satisfy the former two constraints, (i) and (ii).

FIG. 1. In this figure we represent the boundary of the admissible region in parameter space from the viewpoint of the constraints on the number of e-foldings for the double-well potential (labelled d-w) and the pseudo-Nambu-Goldstone-boson potential of natural inflation (N-G). The initial energy bounds translate in terms of \(\varphi'(0)\) in an upper bound of the order of \(10^5\).

The initial energy bounds define an interval of admissible values for \(\varphi'(0)\). The usual approach is to study the ensemble of initial conditions with energy equal or lower than the symmetry breaking energy \(A\). Here we consider initial energies that are equal or larger than \(A\). This is important to avoid fine tuning, since these orbits will have an additional contribution to the number of e-foldings with respect to the lower energy orbits usually considered in the literature.

The question then reduces to find for each \(\varphi'(0)\) in the admissible energy range the interval of values of \(B\) such that \(N(1, B) \in [60, +\infty]\). The results are shown in Figure 1,
and it can be seen that the constraint on the number of e-foldings can be met in a big region of parameter space. More interestingly, it turns out that for all the parameter values that satisfy the constraint on the number of e-foldings, the slow-roll parameters $\epsilon$ and $\eta$, as well as the time derivatives of $\epsilon$, behave in such a way that the slow-roll assumptions are fulfilled during the whole inflationary period. Thus the slow-roll approximation is valid. In Figure 2 we represent a typical orbit in the admissible region in parameter space, as well as the behaviour of $\epsilon$ and $\eta$. In Figure 3 we show the typical values of $\epsilon$ and $\eta$ for those orbits during the inflationary period. This means that the slow-roll assumption to deal with the constraint on $\delta_H$ is fully justified. Moreover, the fact that slow-roll holds during the inflationary period agrees with the observational constraints on the spectral index and on the gravitational waves contribution to the CMB \cite{2,23,27}.

More significantly, our analysis provides a negative answer to the question of whether it is possible to meet the basic inflationary requirements without slow-roll in the class of potentials with several extrema yielding a finite inflationary period. In fact, our numerical results amount to a global study of the $(B, \dot{\varphi}(0))$ parameter plane and show that parameters that give rise to orbits off the slow-roll regime also fail to satisfy the constraint on the number of e-foldings. This finding agrees with the results of \cite{16} regarding the potential considered in \cite{13}, tailor-made to break up the slow-roll approximation, and which they show gives rise to a non-flat perturbation spectrum. The idea that the slow-roll regime is forced by the observational constraints seems to prevail in the general setting of arbitrary potentials exhibiting symmetry breaking, or just several extrema, as well as for other specific models.

It is also clear that the three main constraints, on the number of e-foldings, on the amplitude of the perturbation spectrum, and on the initial energy of the field, can be met in a potential belonging to the broad class considered in this paper without fine tuning of the parameters. The final conclusion is then that in this setting the slow-roll approximation is a general starting point to study further constraints imposed by new observational data.
FIG. 2. In figure (a) we plot a typical slow-roll orbit in phase-space with $B = 50$ and initial conditions $\varphi(0) = 0$ and $\varphi'(0) = 5.41533$ for the double-well potential. The PNGB is analogous. In (b) we show the behaviour of the slow-roll parameters along the orbit in (a).
ACKNOWLEDGEMENTS

The authors wish to acknowledge the financial support from Fundação de Ciência e Tecnologia under the grant PBIC/C/FIS/2215/95 and A.N. thanks the project PRAXIS 2/2.1/MAT/125/94. We are also grateful to Andrew R. Liddle for helpful comments on an preliminary draft of this work.

[1] E. W. Kolb and M. S. Turner, The Early Universe, Addison-Wesley, New York, 1990.

[2] A. R. Liddle and D. H. Lyth, Phys. Rep. 231 (1993) 1.

[3] J. E. Lidsey, A. R. Liddle, E. W. Kolb, E. J. Copeland, T. Barreiro and M. Abney, Rev. Mod. Phys 69 (1997) 373.
[4] A. R. Liddle, P. Parsons and J. D. Barrow, Phys. Rev. D 50 (1994) 7222.

[5] A. D. Linde, Phys. lett. B129 (1983) 177.

[6] M. S. Madsen and P. Coles, Nuc. Phys. B298 (1988) 701.

[7] J. H. Kung and R. Brandenberger, Phys. Rev. D 40 (1989) 2532.

[8] K. Freese, J. A. Frieman and A. V. Olinto, Phys. Rev. Lett. 65 (1990) 3233.

[9] F. Adams, K. Freese and A. H. Guth, Phys. Rev. D 43 (1991) 965.

[10] F. Adams, J. R. Bond, K. Freese, J. A. Frieman, A. V. Olinto, Phys. Rev. D47 (1993) 426.

[11] D. S. Goldwirth and T. Piran, Phys. Rep. 214 (1992) 223.

[12] L. Knox and A. Olinto, Phys. Rev. D 48 (1993) 946.

[13] L. Wang, V. F. Mukhanov and P. J. Steinhardt, Phys. Lett. B414 (1997) 18.

[14] J. Garcia-Bellido and D. Wands, Phys. Rev. D 54 (1996) 7181.

[15] W. H. Kinney, Phys. Rev. D 56 (1997) 2002.

[16] E. J. Copeland, I. J. Grivell, E. W. Kolb and A. R. Liddle, Phys. Rev. D 58 (1998) 043002 [astro-ph/9802203].

[17] M. S. Turner, Phys. Rev. D28 (1983) 1243.

[18] T. Damour and V.F. Mukhanov, Phys. Rev. Lett 80 (1998) 3440.

[19] A.R. Liddle and A. Mazumdar, Phys. Rev. D 58 (1998) 083508.

[20] K. M. Górsksi et al, Astrophys. J. 430 (1994) L89.

[21] E. F. Bunn, A. R. Liddle and M. White, Phys. Rev. D 49 (1996) 5917R.

[22] E. D. Stewart and D. H. Lyth, Phys. Lett. B 302 (1993) 171.

[23] F. Adams and K. Freese, Phys. Rev. D 51 (1995) 6722.
[24] I. J. Grivell and A. R. Liddle, Phys. Rev. D 54 (1996) 7191.

[25] I. J. Grivell and A. R. Liddle, unpublished astro-ph/9906327.

[26] J. P. Zibin, D. Scott and M. White, unpublished astro-ph/9901028.

[27] J. P. Zibin, D. Scott and M. White, unpublished astro-ph/9904228.