Application of thomas fermi model on the study of the transition phase from the inner crust to the core of neutron star

I Lathifa and E T Sulistyani
Department of Physics, Gadjah Mada University, Yogyakarta, Indonesia
E-mail: Isma.lathifa@mail.ugm.ac.id

Abstract. The study of the Thomas Fermi model and its application on the study of the inner crust of neutron star has been done in order to explain pasta phase (the geometric structure changing of atomic nuclei from spherical ones to homogenous liquid) on the transition area from the inner crust to the core of neutron star. The study was carried out by the Wigner Seitz approach on the inner crust that is obtaining the total energy equation as a function of the density of the system, as a Thomas Fermi's model. In this study Thomas Fermi's model explains atomic nuclei that consist of Fermi gas in a system of functional total energy to density by taking into account the contribution of kinetic energy, potential energy and the contribution of interactions between particles. The energy state equation obtained was used to explain the effect of increasing density on the inner crust energy and explain the pasta phase through the interpretation of the graph of Thomas Fermi energy per baryon as a function of baryon density for every geometric structure of atomic nuclei.

1. Introduction

The neutron star is a star that has a high density and contains an abundance of neutrons, especially in the core part of a star. The structure of a neutron star is distinguished by a star's core and a sheath consisting of the atmosphere, the ocean layer, the outer crust, the inner crust, and the mantle. The crust in a neutron star consists of electrons, neutron gas and neutron-rich nuclei. In the transition area between the inner crust and the core there is a mantle layer consisting of non-spherical nuclei which mean the shape of the atomic nucleus is not spherical in symmetry but resembles a pasta. This is caused by pressure and density in the mantle area which is increasingly close to the normal core density so that the nucleus changes shape towards the homogeneous phase, the transition stage changes form from the spherical nucleus to the homogeneous phase described as resembling pasta.

The existence of the pasta phase and neutron gas makes the state of the transition area from the inner crust to the neutron star core become so complex that it cannot be investigated in the laboratory. Because of the difficulty of describing the state of the transition region through the calculation of interactions between nucleons, the crust in a neutron star is studied through a fenomological model through semicyclical calculations such as the Thomas Fermi model. In 1993 the Thomas Fermi model approach was carried out on the inner crust of neutron star to explain the pasta phase by K. Oyamatsu [1] B. K. Sharma et all in 2015 [2], in addition, in 2015 X Vinas et al [3] reviewed the pasta phase using the CLDM and Thomas Fermi model approaches. In this study we will re-examine how Thomas Fermi's modeling describes the transition phase (pasta phase) from the inner crust to the core of neutron star by explaining the contribution of each energy and the effect of increasing neutron star density to each term of the energy density with a complete explanation.
2. Theory

2.1 Neutron stars and pasta phase

Some neutron stars, the most compact stars in the Universe, were given this name because their interior is largely composed of neutrons. A neutron star of the typical mass $M \sim 1 - 2 M_\odot$, where $M_\odot = 2 \times 10^{33}$ g is the solar mass, has the radius $R \approx 10 - 14$ km. The mass density $\rho$ in such star is $\sim 10^{15}$ g cm$^{-3}$, or roughly 3 times normal nuclear density (the typical density of a heavy atomic nucleus) $\rho_0 = 2.8 \times 10^{14}$ g cm$^{-3}$.

Two main qualitatively different regions, the core and the envelope, are distinguished in a neutron star. The core is in turn subdivided into the outer and inner core, and the envelope into the solid crust and the liquid ocean [4]. The crust of a neutron star comes from compacting the layers of a neutron star when cooling temperatures occur. The crust of neutron stars only covers a small portion of the total mass of neutron stars but has an important role to play in the development of neutron stars such as magnetic fields, cooling, explosions, and deformation. The crust of a neutron star consists of an inner crust and an outer crust.

The outer crust of a neutron star is hundreds of meters thick and consists of an electron-ion plasma that is completely ionized, that is, consists of ions in the form of atomic nuclei and strongly degenerate free electrons. Then the total pressure is determined by the pressure of degenerate electrons. The outer crust has a density about $10^7$ g cm$^{-3}$ [4]. The electron Fermi energy in deep-lying layers of the outer crust increases so as to enrich nuclei with neutrons by virtue of beta-captures. Finally, the inner-outer crust interface forms at $\rho = \text{neutron drip density}$ $\rho_{ND} = 4 \times 10^{11}$ g cm$^{-3}$, where free neutrons appear. This causes the core to become unstable. The instability of the nucleus makes neutrons start coming out of the nucleus of the atom and produce free neutron gas. This process is called neutron drops (drip neutrons). The presence of neutron gas marks the transition area from the outer crust to the crust in neutron stars. The inner crust is composed of free electrons, neutron gas, and a nucleus that is rich in neutrons. The pressure on the inner crust is produced by degenerated neutrons. The structure of the state of neutron star crust is presented in the following figure:

![Figure 1](https://example.com/figure1.png)

**Figure 1.** Schematic picture of the ground state structure of neutron stars along the density axis [5].

The crust of the neutron star is composed of Coulomb crystal while the core of the neutron star is believed to consist of homogeneous core material with density above the core saturation density, so it is predicted that changes in composition from the Coulomb crystal become homogeneous core fluid in the transition region from the inner crust to the neutron star core. At low densities, according to Bohr-Wheeler's fission conditions, the shape of the spherical nucleus on the crust in a neutron star will be stable if Coulomb energy is less than surface energy or it can be said that surface energy is more dominant. At higher density about $0.2 \rho_0 - 0.5 \rho_0$ ($\rho_0 = 2.8 \times 10^{14}$ g cm$^{-3}$) the bulk nuclear matter inside the nuclei, and the pure neutron gas outside the nuclei become more and more alike, the presence of the
neutron gas reduces the nuclear surface energy and the Coulomb interaction between nuclei, which keeps the nuclei in a lattice, becomes significant as the spacing between nuclei becomes comparable to the nuclear radius [6]. Under compression, the nuclei begin to touch and fuse forming complex shapes in order to minimize their energy. The competition between the nuclear attraction of protons and neutrons and the Coulomb repulsion between protons creates a variety of nonspherical nuclei. This transition is now believed to involve several pasta phases [7]. As the density approaches \( \rho_o \) the complex shapes transition to uniform nuclear matter. Phase changes that occur at the atomic nucleus aim to stabilize the nucleus. The bottom of the inner crust is reached when a homogeneous phase, consisting of neutrons, protons and electrons filling uniformly the whole Wigner Seitz cell, becomes energetically more stable than the possible phases of the inner crust [3].

The phase of the transition/pasta phase is a change in the geometric shape of the atomic nucleus from what originally resembled a ball to a plasma/homogeneous core liquid consisting mostly of neutrons and a little fraction of protons and electrons. The nucleus can change shape to resemble a rod (spaghetti phase), plates (lasagna phase), tubes, and bubbles. It is called the pasta phase due to changes in the atomic nuclei geometric shape that resembles the type of pasta such as spaghetti and lasagna. The change in the shape of the nuclei occurs as the density increases in the crust in a neutron star. Changing the atomic nuclei phase in the crust in a neutron star to the neutron star core in sequence can be seen in the following figure:

![Figure 2. Schematic depiction of the structure of the neutron star inner crust from the outer crust-inner crust boundary (left-hand side) to the inner crust-core boundary (right-hand side) [9].](image)

The red ones indicate the area occupied by free neutron gas, while the blue color indicates the area occupied by neutrons and protons. Free neutrons that are in the inner crust are in a superfluid state. Free neutron gas is the only gas that completely occupies the inner crustal region to reach a depth of 250 meters seen from the initial position of neutron drops. Whereas for a depth of more than 250 meters (precisely when the atomic nucleus is shaped like a plate), not only the neutron gas fills the pasta region but there are additional protons. For regions with a depth greater than 250 meters (seen from the initial position of neutron drops), the atomic nucleus consists of only neutrons and protons released from the nucleus combine with neutron gas outside the nucleus to form a mixture of neutrons. At the bottom of the inner crust, there is competition between Coulomb inter-core energy and core surface energy, which results in a variety of nuclei such as cylinders, slabs, and bubbles, which is then known as the "pasta phase" [9].

2.2 Thomas fermi model

Thomas Fermi is one model of the Theory of Functional Density (DFT) or Density Functional Theory (DFT), a theory for calculating the energy of an atom based on an approach through electron density and electron energy acting on a ground state. This method is a method that connects the properties of a
limited quantum system to a homogeneous system. This model assumes that a quantum system is equivalent to fermi gas free of functional total energy to density by taking into account the contribution of kinetic energy, external potential and the contribution of interactions between particles. This method is also a method that uses physical quantity density to determine the energy profile of an atom, without having to use a complicated Schrodinger wave equation. Another thing that makes Thomas Fermi's method interesting to study is its wide application, Thomas Fermi's method can be used to study ions, atoms, molecules, nuclei, classical statistics, semiconductors, solids, chemicals, and others. In general, the energy formulation of a system reviewed in the Thomas Fermi model in the density function can be written as follows:

\[ H = T(n) + V(n) + V_{pp}(n) \]  

With each parameter symbolizing a different energy contribution among others \( T \) is the contribution of system kinetic energy, \( V \) is the contribution of the external potential of the system which is energy caused by the interaction of the nuclei with electrons with negative values due to interactions between the nuclei and electrons, where the nuclei is charged positive and negatively charged electrons so that if the distance between the two gets closer, the energy between the two becomes smaller. \( V_{pp} \) is the contribution of internal interaction which is a rejecting potential between electrons, which is nothing but a Coulomb interaction between two electrons located at position \( r \) dan \( r' \). The three energy contributions each have the following equation:

\[ T(n) = \frac{3}{5} \int d^3r E_F(n(r)) n(r) \]  
\[ V(n) = -Z \int \frac{n(r)}{r} dr \]  
\[ V_{pp}(\rho) = \frac{1}{2} \iint \frac{n(r)n(r')}{|r-r'|} dr dr' \]

With \( Z \) is the core charge and \( E_F \) is Fermi energy for free fermi gas as a function of density \( n(r) \) which can be formulated as follows:

\[ E_F(n(r)) = \frac{\hbar^2}{2m} \left( \frac{3\pi^2}{2} \right) ^{2/3} n(r)^{2/3} \]

Based on the use function of the Thomas Fermi model, the Thomas Fermi model can be used to form the energy equation of the crust in a neutron star as a density function to determine the crustal conditions in neutron stars and explain the transition phase of the atomic nucleus (paste phase). In this study, the Thomas Fermi model was used to review core energy density. Thomas Fermi's approach to the nucleus has different physical conditions, there is no central interaction, the binding energy comes from interactions between nucleons, the Coulomb potential has a short-range and will even produce a zero range value.

3. Application of thomas fermi model
The neutron star studied in this study is a neutron star which has a temperature of \( T = 0 \) K, this is because neutron stars consist of strongly degenerated fermions so that the fermi energy of neutrons and protons is much greater than the energy due to the influence of neutron star temperature so the influence of temperature can be ignored. In addition, neutron stars are studied in a ground.state, do not rotation (no rotating) and also do not accretion/addition of regions (not-accretion). The electrons in the neutron star studied are assumed to be uniformly distributed. Through the Thomas Fermi model, the energy of the inner crust of neutron stars will be reviewed. The energy of the inner crust of neutron star is influenced by the contribution of internal energy which is the energy of particles and external energy due to pressure
on the system. In the crust in neutron stars pressure is generated by electrons and strong degenerates. This second combination of energy is called total free energy or Gibbs free energy. Gibbs free energy in the system can be formulated as follows:

\[ G = E + PV_c \]  

\( E \) is particle energy, \( P \) is the pressure on the system and \( V_c \) is the volume of the system, so that in order to determine the total energy on the crust in neutron stars a study is needed for each contribution, it is necessary to know how the formulation of energy produced by particles and pressure on the system.

### 3.1 The energy density of particles on the inner crusts of neutron stars with the wigner seitz approach

The Wigner Seitz (WS) approach to the inner crust of neutron star is carried out to obtain the equation of the inner crust energy. In this approach, cells in the bcc lattice are replaced by WS cells. The total energy for each WS cell is as follows [3]:

\[ E = \epsilon_{\text{inti}} + \epsilon_{\text{elec}} + \epsilon_{\text{Coul}} + \epsilon_{\text{ex}} \]  

The total energy of a neutron, proton and electron ensemble system is sought through the WS cell approach with volume \( V_c = \frac{4\pi(R_c)^3}{3} \) for the spherical geometric shapes that resemble spheres as follows:

\[ E = \int_{V_c} \left[ H(n_n, n_p) + \epsilon_{\text{elec}} + \epsilon_{\text{Coul}} + \epsilon_{\text{ex}} + m_n n_n c^2 + m_p n_p c^2 \right] dV \]  

\( \epsilon_{\text{elec}} \) is the energy density related to the movement of electrons. \( \epsilon_{\text{Coul}} \) is the density of coulomb energy which is affected by interactions between protons and neutrons, protons with neutrons, and neutrons with neutrons. \( \epsilon_{\text{ex}} \) is an energy density related to the interaction of the exchange of protons with protons and electrons with electrons. \( H \) is the contribution of core energy density which was approached by the Thomas Fermi model [2].

#### 3.1.1 The core energy density with the application of the thomas fermi model

In this study, Thomas Fermi’s approach was carried out to fulfill the hamiltonian value \( H(n_n, n_p) \) in equation (7) which is the contribution of nuclei energy density. The magnitude of the contribution of the nuclei energy density to the crust in a neutron star is given by the following equation [3]:

\[ H(n_n, n_p) = T(n_n, n_p) + V(n_n, n_p) \]  

\( T(n_n, n_p) \) is the nuclei kinetic energy density consisting of neutrons and protons given by the equation:

\[ T(n_n, n_p) = \frac{3}{5}(3\pi^2)^{2/3} \left[ \frac{\hbar^2}{2m_n n_n^3} + \frac{\hbar^2}{2m_p n_p^3} \right] \]  

On the inner crust of neutron star, it is assumed that there are no protons in the neutron drop region so that \( n_p(R_c) = 0 \), thus contributing to the nuclei kinetic energy density only as a function of neutron density \( T(n_n, 0) \). Below, a graph of the kinetic energy of spherical atomic nuclei is presented as a function of the baryon density:
Figure 3. Graph of the core kinetic energy density as a function of the baryon density of spherical atomic nuclei.

The increase of the nuclei kinetic energy is caused by increasing density, the number of particles per volume in the system is also increasing, along with the increasing number of particles per volume in the system, the more friction or collision between particles causes particles to move to result in kinetic energy in the system.

The nuclei potential energy density $V_{\text{inti}}(n_n, n_p)$ obtained from the functional BCPM (Barcelona-Catania-Paris-Madrid). A more detailed explanation of the BPCM model has been presented in Ref. [2]. The nuclei potential energy density in this model is divided into two terms, namely the contribution of the potential energy density of the surface core $V_{\text{surf}}$ to determine the core potential energy density value on the surface, and the bulk potential energy density of bulk $V_{\text{bulk}}$ is given by the following equation [2]:

$$V_{\text{bulk}}(n_n, n_p) = \int \left[ P_c(n)(1 - \beta^2) + P_p(n)\beta^2 \right] n \, d^3r$$  \hspace{1cm} (11)$$

$$V_{\text{surf}} = \frac{\pi}{2} \sum_{q,q'} a^2 V_{q,q'} n_q(r) \left[ \frac{1}{r} \int_0^\infty \left( n_{q'}(r') \left( e^{-\frac{(r-r')^2}{a^2}} - e^{-\frac{(r+r')^2}{a^2}} \right) r' \right) dr' \right] - \frac{1}{2} \pi^2 a n_{q'}(r) \right]$$  \hspace{1cm} (12)$$

On the inner crust of neutron star, it is assumed that there are no protons in the neutron drip region so $n_p(R_c) = 0$, then the potential energy density of the nuclei of the inner crust of neutron star contributes as a function of neutron density and proton density changes to the nuclei potential energy density as a function of neutron density only or $V(n_n, n_p) = V(n_n, 0)$. The contribution of the bulk potential density depends on the polynomials for symmetric nuclear matter (SNM) and pure neutron matter (PNM) material because of the value of $n_p = 0$ then $V_{\text{bulk}}$ is only influenced by polynomials for PNM material. Whereas the potential energy density of the surfaces $V_{\text{surf}}$ is influenced by the geometric shape of the atomic nuclei due to the existence of the volume integral term at, so the formulation of the $V_{\text{surf}}$ for each atomic geometry has a different formulation. In this study, equation (12) has not been derived so that a graph of the density of the surface potential of atomic nuclei cannot be carried out as a function of the density of the baryon. Below will be presented a graph of the bulk potential energy of spherical and rod-shaped atomic nuclei as a function of baryon density:
Based on the graph of the bulk potential energy density for the spherical atomic nuclei (figure 4), it can be seen that at a density of 0.06-0.08 fm⁻³ there is a fluctuation of bulk potential energy which shows instability in the geometry of atomic nuclei. However, in the graph of the bulk potential energy density for the rod-shaped atomic nuclei (figure 5) at the density of 0.06-0.08 fm⁻³ shows the energy density of the bulk potential which is relatively more stable compared to the spherical atomic geometry, this shows in the density range the geometric shape of the rod atomic nuclei is more stable than the spherical ones. Below will be presented a graph of the bulk potential energy density of atomic nuclei in the form of plates as a function of baryon density:
Figure 6. Graph of the bulk potential energy density as a function of the baryon density of plate atomic nuclei.

In the graph (figure 5) at a density of around $0.07 \, \text{fm}^3$, there is a slight fluctuation in the bulk potential energy density which shows the instability of the geometry of the stem nucleus at this density. However, in the density graph of bulk potential energy for the geometry of plate atomic nuclei (figure 6), it can be seen that at a density of $0.07-0.08 \, \text{fm}^3$ the density value of bulk potential energy tends to be more stable. This shows that the geometric shape of the atomic nucleus of the plate is more stable than the shape of the rod.

3.1.2 The electron energy density. The inner crust of neutron star has a very high density, this results in higher Fermi energy than Coulomb energy, so that electrons are degenerated and modeled as relatively free Fermi gas with a density energy $\varepsilon_{elec}$ distributed uniformly on WS cells with a constant density which can be formulated as follows:

$$\varepsilon_{elec} = \left(\frac{k_{Fe}}{8\pi^2} \left(2k_{Fe}^2 + m_e^2\right) \sqrt{k_{Fe}^2 + m_e^2}\right) - \left(\frac{m_e^2}{8\pi^2} \ln \left[\frac{k_{Fe} + \sqrt{k_{Fe}^2 + m_e^2}}{m_e}\right]\right) \tag{13}$$

$k_{Fe}$ is an electron Fermi constant, which is:

$$k_{Fe} = (3\pi^2 n_e)^{\frac{1}{3}} \tag{14}$$

Based on equation (13) above, electron energy density depends on the mass of the electron whose value is constant and the fermi constant of the electron whose value is affected by electron density. The electron energy density is directly proportional to the fermi constant of the electron, so that the increasing density of the baryon, the higher the electron energy density will be. Below will be presented a graph of the relationship between electron energy density $\varepsilon_{elec}$ as a function of the baryon density $n_b$: 

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\[ Ig\]
3.1.3 The Coulomb energy density. The amount of Coulomb energy density ($\varepsilon_{\text{coul}}$) is influenced by the interaction between protons and protons, protons with neutrons, and neutrons with neutrons. Assuming that electrons are evenly distributed, the amount of Coulomb energy is given by the following equation:

$$
\varepsilon_{\text{coul}}(n_p, n_e) = \frac{1}{2}(n_p - n_e)(V_p(r) - V_e(r))
$$

(15)

On the inner crust of neutron star, the number of protons is equal to an electron, because the assumption of a system consisting of protons and electrons is uniformly distributed, the density of the number of electrons and protons is equal, which means $V_p(R_c) = V_e(R_c)$, so that $\varepsilon_{\text{coul}}(R_c)$ Coulomb energy density reviewed in the system is 0.

3.1.4 The energy density of the exchange nucleon interaction. Electromagnetic interaction between two electrons is described as a transition-changes particle called a photon. According to quantum theory, the force between charges is considered from the displacement of virtual photon particles. When two electrons enter the interaction area (Coulomb repulsive force), a photon is displaced, then two electrons appear with the resultant trajectory (speed and direction of motion) made by the electromagnetic force communicated. Changes in the momentum of charged particles due to emissions and absorption power produce force. According to what is needed the tribe for the electron energy meeting consists of a meeting of electromagnetic energy in the system. The intermediate energy density between protons and neutrons with electrons with electrons can be given by the following equation:

$$
\varepsilon_{\text{ex}}(n_p, n_n) = -\frac{3}{4} \left(\frac{2}{\pi}\right)^{1/3} e^2 \left(n_p^{4/3} + n_n^{4/3}\right)
$$

(16)

The energy density data graph of nucleon exchange interactions as a function of baryon density is presented in the following figure:
3.2 Determine the equation of pressure on the crust in a neutron star

To find out the inner crust’s equation of state (EoS), it must be considered how the pressure of the system contributes to EoS, because the inner crust energy is also affected by the pressure generated from neutrons and electrons that are strongly degenerated. Based on the laws of thermodynamics, the magnitude of the pressure is given by the following equation:

\[ P = -\left(\frac{\partial E}{\partial V}\right) = -\frac{1}{4\pi R_c^2} \left(\frac{\partial E}{\partial R_c}\right) \]  

(17)

Based on the derived in the equation, it is known that the pressure in the inner crust is affected by the neutron gas pressure \( P_{ng} \), the free electron pressure \( P_{el}^{free} \) and also the electron exchange \( P_{el}^{ex} \) which can be formulated as follows:

\[ P = P_{ng} + P_{el}^{free} + P_{el}^{ex} \]  

(18)

\[ P_{ng} = \mu_n n_n - H (n_n (R_c), 0) - m_n n_n \]  

(19)

\[ P_{el}^{free} = n_e \sqrt{k_F^2 + \frac{m_e^2}{2}} - \varepsilon_{el} n_e \]  

(20)

\[ P_{el}^{ex} = \varepsilon_{el}^{ex} = -\frac{1}{3}\left(\frac{3}{4\pi}\right) a^2 n_e^{\frac{4}{3}} \]  

(21)

3.3 Determine the minimum energy equation per baryon on the crust in a neutron star

To be able to explain changes in the geometry of atomic nuclei on the inner crust of neutron star, it is necessary to know how the minimum energy per baryon for each atomic geometry. Minimum energy per baryon can be obtained by minimizing the density of particle energy on the inner crust of neutron stars and determining the energy formulation per baryon based on the Gibbs free energy equation on the system. The particle energy density in a ground state of a neutron star is obtained by minimizing the total particle energy density to the baryon density, assuming that the system is in beta equilibrium and has a neutral charge. To obtain the minimum energy value in the spherical shape of the nuclei geometry, a derived in the value of density of neutrons, protons, and electrons is carried out, so that the following results are obtained [3]:

![Graph of the exchange nucleon interaction energy density as a function of the baryon density of spherical atomic nuclei.](image)
\[
\frac{\delta E}{\delta n_n} = \frac{\delta H}{\delta n_n} + m_n - \mu_n = 0
\]  \hspace{1cm} (22)

\[
\frac{\delta E}{\delta n_p} = \frac{\delta H}{\delta n_p} + V_p(r) - V_e(r) - \left(\frac{3}{\pi}\right)^{\frac{1}{3}} e^2 n_p^{\frac{1}{3}}(r) + m_p - \mu_p = 0
\]  \hspace{1cm} (23)

\[
\frac{\delta E}{\delta n_e} = \sqrt{k_F e} + m_e + V_e(r) - V_p(r) - \left(\frac{3}{\pi}\right)^{\frac{1}{3}} e^2 n_e^{\frac{1}{3}}(r) - \mu_e = 0
\]  \hspace{1cm} (24)

Assuming that the system is in beta equilibrium, where \(\mu_e = \mu_n - \mu_p\). While the formulation of Gibbs energy per particle \(G/A\) is done by minimizing energy per unit volume to \(R_c/WS\) cell radius \([2]\) which obtained:

\[
G = \mu_n A
\]  \hspace{1cm} (25)

Free Gibbs energy is equivalent to the chemical potential of neutrons multiplied by the number of baryons. So the energy per baryon of the system is given by the following equation:

\[
\frac{E}{A} = \mu_n - \frac{PV}{A}
\]  \hspace{1cm} (26)

With \(\mu_n\) is the neutron chemical potential given by equation (22), \(P\) is the pressure on the system, \(V\) is the cell volume of Wigner Seitz and \(A\) is the number of baryons.

The Thomas Fermi model used for spherical WS cells can be expanded to be used in other geometric shapes such as rods, plates, tubes, and bubbles. In this review, it is assumed that the length of the rod, tube and plate area is unlimited. This aims to simplify Coulomb interactions and provide limited energy per baryon. The forms of atomic nuclei that are approached by the Wigner-Seitz cell approach are considered to have a certain volume. Assuming that each core form has a cell radius \(R_c\), the volume equation for the nonspherical nuclei form was explained in Ref \([8]\).

To be able to explain the paste phase on the crust in a neutron star, it is necessary to examine how the configuration of WS cells and also the minimum energy per baryon as a function of baryon density for each atomic geometry. A plot of the graph of the minimum energy per baryon \(E/A\) relative to energy per baryon for uniform \(npe\) material as a function of the baryon density \(n_b\). This is done because of the division of energy (energy separation) between different geometries of atomic nuclei. Through the graph plot, it can be explained how the geometry of atomic nuclei in the neutron star changes with increasing density in the system. Based on equation (26), it is necessary to know how the pressure equation and also the neutron chemical potential on the crust in neutron stars, while the two equations require the equation of surface potential energy density of atomic nuclei that were not found in this study so that the graph plot of the minimum energy per baryon cannot be done for various geometric shapes of atomic nuclei. Therefore, a study of changes in the geometry of atomic nuclei can be done through a literature review. The following will be presented graphically the relationship between minimum energy per baryon \(E/A\) relative to energy per baryon \(npe\) as a function of baryon density \(n_b\) for each geometry of atomic nuclei in the crust in neutron stars that should be obtained based on literature review \([2]\):
Figure 9. Energy per baryon of different shapes relative to uniform npe matter as a function of baryon density in the inner crust for (a) low density and (b) higher density [2].

The graph plot (figure 9a) is carried out for baryon density $n_b$ 0 to 0.08 fm$^{-3}$. On the graph (figure 9a), it can be seen that the shape of a spherical atomic nucleus apparently dominates the crust in a neutron star at density $\leq 0.005$ fm$^{-3}$. And the atomic nuclei geometry that resembles a rod and plate has begun to be seen at a density that tends to below, which is around 0.005 fm$^{-3}$. When the crust density reaches 0.065 fm$^{-3}$ (10$^{14}$ gr cm$^{-3}$), the ratio of the size of the atomic nucleus to the volume of WS cells increases significantly and it is found that the non-spherical nuclei have higher energy than the shape atomic geometry that resembles a sphere. When the density of 0.067 fm$^{-3}$, it was seen that the plot points of the data for the nuclei shape resembled a sphere and rod coincide, this indicates that the minimum energy per baryon for the nuclei geometry that resembles a sphere and rod has the same value. So that at this density the shape of the spherical atomic nuclei changes to the rod-shaped atomic nuclei. If observed again, on the graph (figure 9a), the energy value per baryon at the time of the change in shape for the nuclei with the tube and bubble geometry cannot be observed, so that more clearly displayed on the graph (figure 9b) for higher densities [2].

In the graph (figure 9b) the data plot is carried out for the density of baryon $n_b$ 0.07 fm$^{-3}$ to 0.0825 fm$^{-3}$. When the density of baryon 0.076 fm$^{-3}$, it can be seen that the data plot points for the nuclei in the form of rods and plates coincide, this states that the minimum energy per baryon for the core geometry that resembles rods and plates has the same value. So that changes in the shape of the atomic nucleus from the rod to the shape of the plate (slab) occur at a baryon density of 0.076 fm$^{-3}$. As the density increases, the energy per baryon in the form of rods and bubbles is closer to the shape of the plate. The change in the geometric shape of the atomic nucleus from the plate to the tube occurs when the density of baryon is 0.082 fm$^{-3}$ and changes in the geometric shape of the atomic nucleus from the tube to bubble occur when the density of baryon is 0.0825 fm$^{-3}$. The most depth part of the inner crust of a neutron star is achieved when the atomic nucleus has become a homogeneous phase (liquid uniform). Changes in the geometric shape of the atomic nucleus from bubbles to uniform occur at the density of $n_b = 0.0825$ fm$^{-3}$, so it is estimated that at that density, there is a transition from the crust in the neutron star to the neutron star core. For more details, the baryon density value in the event of a change in core geometry is presented in the following table:

| Drop/ | Rod/ | Slab/ | Tube/ | Bubble/ |
|-----------------|------|-------|-------|--------|
| 0.065 (10$^{14}$ gr cm$^{-3}$) | 0.067 | 0.076 | 0.082 | 0.0825 |

Table 1. Baryon density of the successive changes of energetically favorable topology in the inner crust. Units are fm$^{-3}$ [2].
Based on the table above it can be seen that the geometry of atomic nuclei changes with increasing density of the inner crust, make a changes of the geometry of the nuclei from the spherical, turns into a rod and becomes a plate, turns into a tube and then becomes a bubble and eventually turns into liquid uniform. The liquid uniform is a homogeneous phase where electrons, protons, and neutrons are distributed uniformly within WS cells. There is no limit to the atomic nucleus in it. The form of change in the atomic nuclei is obtained in accordance with the theory presented by Hashimoto that the nuclei geometry will change with increasing density as shown in Figure 2. This shows that the application of the Thomas Fermi model on the inner crust of neutron stars can be applied to spherical atomic nuclei and nonspherical atomic nuclei so well, besides that, Thomas Fermi’s model is also suitable to explain changes in the geometry of atomic nuclei (paste phase) from the inner crust of neutron stars to the core of neutron star.

4. Conclusion
The formulation of the total energy of the inner crust of neutron stars with a review of the Thomas Fermi model is carried out with the TF model approach on the energy density of atomic nuclei where the nuclei material in the inner crust can be broken down into kinetic energy and potential as a function of density. The total energy equation obtained shows that the increase in density in the crust in a neutron star will result in an increase in kinetic energy density in the system, a decrease in bulk potential energy density in the system, an increase in free-electron energy density and a decrease in nucleon exchange energy density. In this study, there was no result in a decrease in the potential energy density of the atomic core surface so that the review of the paste phase can only be done in a literature review. Based on the literature review carried out, Thomas Fermi’s model can explain the changes of the geometric shape of the atomic nuclei (paste phase) at the transition region from the inner crust of neutron stars to the core of neutron star well according to the theory of existing geometric changes in atomic nuclei.

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