On the Solution of a Optimal Control Problem for a Hyperbolic System

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Abstract: In this study, the problem of determining the control function that is at the right hand side of a hyperbolic system from the final observation is investigated. Using the Fourier-Galerkin method, the weak solution of this hyperbolic system is obtained. The necessary conditions for the existence and uniqueness of the optimal solution are proved. We also find the approximate solutions of the test problems in numerical examples by a MAPLE® program. Finally, the numerical results are presented in the form of tables.

Keywords: Optimal Control, Partial Differential Equation, Numerical Approximation

1. Introduction

The problem of determining the control function that is at the right hand side of the hyperbolic system has been studied by different authors. Lions [3] examined the problems in detail when the control function is at the right hand side of the hyperbolic problem by using different cost function. Periago [4] has investigated the problem of optimizing the shape and position of the support of the internal exact control of minimal $L_2(0,T)$–norm for the 1-D wave equation.

Yamamoto [5] has studied the inverse problem of determining $^\dagger f(x)$ from $\frac{\partial u(f)}{\partial n}$ subject to the hyperbolic problem

$$u''(x, t) = Du(x, t) + \sigma(t)f(x), x \in \Omega, t > 0$$
$$u(x, 0) = 0, u'(x, 0) = 0, x \in \Omega$$
$$u(x, t) = 0, x \in \partial \Omega, t > 0$$

where $\sigma \in C^4(0,T)$.

Benamou [6] has used the domain decomposition method to solve the optimal control problem in the hyperbolic system and has taken the set of admissible control as a convex subset of $L^2((0,T) \times \Omega)$.

Kim and Pavol [7] have minimized the cost functional

$$J(v) = \int_0^T \int_0^\pi \left( q(u(x,t)) + h(v(x,t)) \right) dx dt$$

governed by periodic nonlinear 1-D wave equation. The necessary and sufficient conditions for an admissible pair $(u^*, v^*) \in L^\infty(\Omega) \times L^\infty(\Omega), \Omega = (0,\pi) \times (0,T)$ to be an optimal pair have given by authors.

Lopez at all. [8] have considered problem of controlling the function $f(x,t)$ related to the hyperbolic problem

$$\varepsilon u_{tt} - \Delta u + u_t = f_1 w(x,t) \in \Omega \times (0,T)$$
$$u(x,0) = u^0(x), u_t(x,0) = u^1(x), x \in \Omega$$
$$u(x,t) = 0, (x,t) \in \partial \Omega \times (0,T).$$

Privat et al. [9] have minimized the norm of the control for given initial data in the wave equation defined on $(0,\pi)$ with Homogeneous Dirichlet boundary condition when the control is in at the right hand side of the equation.

Subaşı and Saraç [10] have obtained a minimizer function for the optimal control problem of the initial velocity in a wave equation.

Saraç and Şener [11] have determined the transverse distributed load in Euler-Bernoulli beam problem from of admissible control. The set of admissible controls has been taken as a subspace of the space $L_2[a,b]$.

Saraç [12] has obtained symbolic and numeric solutions by
using the initial velocity as a control function in hyperbolic problem.

Şener et al. [14] have explained applications of the Galerkin method to wave equation.

The problem of determining of unknown spatial load distributions in a vibrating Euler–Bernoulli beam from limited measured data has been solved in [16].

The space \( L_2(0, l) \) consist of the functional which are square integrable, inner product and norm in \( L_2(0, l) \) are defined respectively as;

\[
(u, v)_{L_2} = \int_0^l u(x)v(x)dx \quad \text{and} \quad \|u\|_{L_2} = \sqrt{(u, u)_{L_2}}.
\]

Let \( U_{ad} \) be closed, convex subset of \( L_2(0, l) \).

In this study, we consider an optimal control problem for a wave equation with homogeneous Dirichlet boundary conditions, the control being the one from the functions that

\[
\begin{align*}
    u_{tt} - a^2u_{xx} &= f(t)v(x), \\
    u(x, 0) &= \varphi(x), u_t(x, 0) = \psi(x), \\
    u(0, t) &= 0, u(l, t) = 0,
\end{align*}
\]

where \( y \) is given target function and \( \varphi, \psi \) and \( f \) are known functions.

With the choice of the functional in (1), we mentioned the observation of \( u(x,T;v) \) in \( L_2(0, l) \) for the control \( v \) in \( L_2(0, l) \). Our aim is to obtain suitable function \( v^* \) which approaches the solution of the problem (2) to desired target \( y(x) \in L_2(0, l) \) at the final time \( t = T \). Another word, we want to find the function \( v^* \in U_{ad} \) such that

\[
\inf J_a(v) = J_a(v^*).
\]

Here \( \alpha > 0 \) is a regularization parameter which ensures both the uniqueness of the solution and a balance between the norms \( \|u(x,T;v) - y(x)\|_{L_2(0,l)}^2 \) and \( \|v(x)\|_{L_2(0,l)}^2 \). Detailed information as regards the regularization parameter can be found in [2]. The term \( \|v\|_{L_2}^2 \) is called penalization term; its role is to avoid using too large controls in the minimization of \( J_a(v) \).

In system (2), the term \( f(t)v(x) \) is considered to be an external force. External forces in this form of separation of variables are important in modelling vibrations. In [5] Yamamoto point out that the system (2) is regarded as

\[
\int_0^T \int_0^l (-u_t\eta_t + a^2 u_x\eta_x)dxt = \int_0^T \int_0^l f v\eta dxt + \int_0^l \psi(x,0) \eta dx \tag{3}
\]

for all \( \eta \in H_0^1(\Omega) \) with \( \eta(x,T) = 0 \). To have this solution the followings are needed;

\[
f \in L_2(0,T), v \in L_2(0,l), \varphi \in H_0^1(0,l), \psi \in L_2(0,l) \tag{4}
\]

Theorem 2.2. Suppose that the condition (4) holds, then the problem (2) has a unique generalized solution and the following estimate is valid for this solution;

\[
\|u\|_{H_0^1(\Omega)}^2 \leq c_0 (\|\varphi\|_{H_0^1(0,l)}^2 + \|\psi\|_{L_2(0,l)}^2 + \|f\|_{L_2(0,T)}^2 \|v\|_{L_2(0,l)}^2) \tag{5}
\]

Proof of this theorem can easily be obtained by Galerkin method used in [1].

Let's give the increment \( \Delta v \) to \( v \) such that \( v + \Delta v \in U_{ad} \) and show the solution of (2) corresponding \( v + \Delta v \) by \( u_{\Delta} = u(x,t; v + \Delta v) \). Then the function \( \Delta u = u_{\Delta} - u \) will be the solution of the following difference problem:

2. Existence of Unique Optimal Solution

In this section, we give the solvability of the optimal control problem (1)-(2). First we state the generalized solution of the hyperbolic problem (2) in view of [1].

Definition 2.1. The generalized (weak) solution of the hyperbolic problem (2) is the function \( \varphi \in H_0^1(0,l) \) with \( u(x,0) = \varphi(x), x \in (0, l) \) which satisfies the following integral identity:

\[
\int_0^T \int_0^l (-u_t\eta_t + a^2 u_x\eta_x)dxt = \int_0^T \int_0^l f v\eta dxt + \int_0^l \psi(x,0) \eta dx \tag{3}
\]

for all \( \eta \in H_0^1(\Omega) \) with \( \eta(x,T) = 0 \). To have this solution the followings are needed;

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\[ \Delta u_{tt} = a^2 \Delta u_{xx} + f(t) \Delta v(x) \]
\[ \Delta u(x, 0) = 0, \Delta u_t(x, 0) = 0 \]
\[ \Delta u(0, t) = 0, \Delta u(t, 0) = 0 \]

\[ \|\Delta u(x, T)\|_{L^2(0, l)} \leq c_1 \|\Delta v\|_{L^2(0, l)} \quad (7) \]

**Proof:** We can prove this lemma in view of [15]. We multiply both sides of the hyperbolic equation (6) by \( \Delta u_t \), then integrate it on \([0, l] \). After some transformations, we have

\[ 1 \frac{d}{dt} \left( \int_0^l (\Delta u_t)^2 + a^2 (\Delta u_v)^2 \right) dx = \int_0^l f(t) \Delta v(x) \Delta u_t(x, t; v) dx + a^2 (\Delta u_v \Delta u_t)|_{x=0}^{x=l}. \]

Using here the homogeneous boundary conditions of the system (6), we get

\[ \frac{d}{dt} \left( \int_0^l (\Delta u_t)^2 + a^2 (\Delta u_v)^2 \right) dx = \int_0^l f(t) \Delta v(x) \Delta u_t(x, t; v) dx. \]

Integrating both sides on \([0, t], t \in [0, T] \), we get

\[ I^2(t) = \int_0^t \int_0^l f(\tau) \Delta v(x) \Delta u_t(x, \tau; v) dx d\tau, \forall t \in [0, T] \]

where

\[ I^2(t) = \frac{1}{2} \int_0^l (\Delta u_t)^2 + a^2 (\Delta u_v)^2 \right) dx, t \in [0, T]. \]

We differentiate both the sides

\[ 2I(t)I'(t) = \int_0^l f(t) \Delta v(x) \Delta u_t(x, t; v) dx, \forall t \in [0, T]. \]

Applying to the right-hand side the Cauchy inequality, we obtain

\[ 2I(t)I'(t) \leq f(t) \|\Delta u\|_{L^2(0, l)} \|\Delta u_t\|_{L^2(0, l)}, \forall t \in [0, T]. \]

Since we have

\[ \|\Delta u_t\|_{L^2(0, l)}^2 \leq \int_0^l (\Delta u_t)^2 + a^2 (\Delta u_v)^2 \right) dx = 2I^2(t), \forall t \in [0, T] \]

we get

\[ I'(t) \leq \frac{1}{\sqrt{2}} f(t) \|\Delta v\|_{L^2(0, l)}, \forall t \in [0, T]. \]

Integrating both the sides on \([0, t], t \in [0, T] \) and taking into account \( I(0) = 0 \), we obtain

\[ I(t) \leq \frac{1}{\sqrt{2}} \|\Delta v\|_{L^2(0, l)} \int_0^t f(\tau) d\tau, \forall t \in [0, T]. \]

Substituting in last inequality \( t = T \), we write

\[ I(T) \leq \frac{C}{\sqrt{2}} \|\Delta v\|_{L^2(0, l)} \]

where \( \int_0^T f(t) dt \leq C \) (C is a constant).

We have
\[
\int_0^T [\Delta u_t(x,t)]^2 dx = \int_0^T \left( \int_0^T \Delta u_t(x,t) dt \right)^2 dx \\
\leq T \int_0^T \int_0^T [\Delta u_t(x,t)]^2 dt dx \\
\leq 2T \int_0^T I^2(t) dt.
\]

Combining last inequalities, we get
\[
\int_0^T [\Delta u(x,T)]^2 dx \leq 2T \int_0^T C^2 \|\Delta v\|_{L^2(0,T)}^2 dt \\
\leq (2C)^2 \|\Delta v\|_{L^2(0,T)}^2
\]
which implies the required estimate (7).

We can write the cost functional (1) in the following way;
\[
J_\alpha(v) = \int_0^T [u(x,T;v) - u(x,T;0) + u(x,T;0) - y(x)]^2 dx + \alpha \int_0^T v^2 dx
\]

So we rewrite \(J_\alpha(v)\) as
\[
J_\alpha(v) = \pi(v,v) - 2Lv + b
\] (8)
\[
\pi(v,v) = \int_0^T [u(x,T;v) - u(x,T;0)]^2 dx + \alpha \int_0^T v^2 dx
\] (9)
\[
Lv = \int_0^T [u(x,T;v) - u(x,T;0)][y(x) - u(x,T;0)] dx
\] (10)
and
\[
b = \int_0^T [y(x) - u(x,T;0)]^2 dx
\] (11)

Due to the linearity of the transform \(v \rightarrow u[v] - u[0]\), it can easily be seen that the functional \(\pi(v,v)\) is bilinear and symmetric. Further, we write the following;
\[
|\pi(v,v)| \geq \alpha \|v\|_{L^2(0,T)}^2
\] (12)
and this implies the coercivity of \(\pi(v,v)\). Since
\[
\pi(v,\eta) = \int_0^T [u(x,T;v) - u(x,T;0)] [u(x,T;\eta) - u(x,T;0)] dx + \alpha \int_0^T \eta dx
\]
applying Cauchy-Schwartz inequality and using (7), we get
\[
|\pi(v,\eta)| \leq c_2 \|v\|_{L^2(0,T)} \|\eta\|_{L^2(0,T)}
\] (13)
for \(c_2 = \max\{c_1, \alpha\}\). Then \(\pi(v,\eta)\) is continuous.

The functional \(Lv\) is linear. We can easily write that
\[
Lv \leq c_4 \|v\|_{L^2(0,T)}
\] (14)
using (7). Hence we see that the functional \(Lv\) is continuous.

Theorem 2.4. Let \(\pi(v,v)\) be a continuous symmetric bilinear coercive form and \(Lv\) be a continuous linear form. Then there exists a unique element \(v^* \in U_{ad}\) such that
\[
J_\alpha(v^*) = \inf_{v \in U_{ad}} J_\alpha(v).
\]

Proof of this theorem can easily be obtained by showing the weak lower semi-continuity of \(J_\alpha\) same as in [3].

3. Frechet Differential of the Cost Functional and Minimizing Sequence

Let us introduce the Lagrangian \(L(u, v, z)\) given by
\[
L(u, v, z) = \int_0^T [u(x,T;v) - y(x)]^2 dx + \alpha \int_0^T v^2 dx + \int_0^T \int_0^T [u_{tt} - u_{xx} - f(t)u(x)] dz dx dt
\] (15)
Using the $\delta L = 0$ stationarity condition, we get the following adjoint problem:

$$
\begin{align*}
    z_{t t} - a^2 z_{xx} &= 0 \\
    z(x, T) &= 0, z_t(x, T) = 2[u(x, T; v) - y(x)] \\
    z(0, t) = 0, z(l, t) &= 0
\end{align*}
$$

(16)

Now, we investigate the variation of the functional $J_a(v)$. The difference functional

$$
\Delta J_a(v) = J_a(v + \Delta v) - J_a(v)
$$

is such as

$$
\Delta J_a(v) = \int_0^l \int_0^T [u(x, T; v) - y(x)] \Delta u(x, t) \, dx \, dt 
$$

(17)

Here, the term

$$
2 \int_0^l [u(x, T; v) - y(x)] \Delta u(x, T) \, dx
$$

must be evaluated. Using the problems (6) and (16), we have

$$
2 \int_0^l [u(x, T; v) - y(x)] \Delta u(x, T) \, dx = -\int_0^T \int_0^l f(t) z(x, t) \Delta v(x) \, dx \, dt
$$

So the relation (17) can be written as

$$
\Delta J_a(v) = \int_0^l \left\{ -\int_0^T f(t) z(x, t) + 2av \right\} \Delta v \, dx + \int_0^l [\Delta u(x, T)]^2 \, dx + \alpha \int_0^l (\Delta v)^2 \, dx
$$

(18)

Using Lemma 2.3 in the (18), we can write the following equality:

$$
\Delta J_a(v) = \left( -\int_0^T f(t) z(x, t) \, dt + 2av, \Lambda \right)_{L_2(0, l)} + o(\|v\|^2_{L_2(0, l)})
$$

By the definition of Frechet differential at $v \in U_{ad}$ we get the gradient

$$
J'_a(v) = -\int_0^T f(t) z(x, t) \, dt + 2av.
$$

So, we can state the following theorem in view of [2].

**Theorem 3.1.** The control $v^*$ and the state $u^* = u(v^*)$ are optimal if there exists a multiplier $z^* \in U_{ad}$ such that $z^*$ and $v^*$ satisfy the following optimality conditions:

$$
\left( -\int_0^T f(t) z^*(x, t) \, dt + 2av^*, v - v^* \right)_{L_2(0, l)} \geq 0
$$

(19)

for $\forall v \in U_{ad}$.

Now, we can apply standard steepest descent iteration. We write an iterative procedure to compute a sequence of controls $\{v_k\}$ convergent to the optimal one.

1. Select an initial control $v_0$. If $v_k$ is known ($k \geq 0$) then $v_{k+1}$ is computed according to the following scheme.
2. Solve the state problem (2) in the sense (3) and get corresponding $u_k$.
3. Knowing $u_k$ solve the adjoint problem (16).
4. Using $z_k$ get the gradient ($J'_a(u_k)$)
5. Set

$$
v_{k+1} = v_k - \beta_k J'_a(u_k)
$$

(20)

and select the relaxation parameter $\beta_k$ in order to assure that

$$
J_a(v_{k+1}) - J_a(v_k) = \beta_k \left[ -\|J'_a(u_k)\|^2 + o(\beta_k) \beta_k \right] < 0
$$

(21)

for sufficiently small $\beta_k > 0$. The term $o(\beta_k)$ is infinite decreasing term with high order respect to $\beta_k$. Computations of the $\beta_k$ can be carried out by one of the methods shown in [13].

One of the following can be taken as a stopping criterion to the iteration process;
\[ \| p_{k+1} - p_k \| < \varepsilon_1, \| J_a(p_{k+1}) - J_a(p_k) \| < \varepsilon_2, \| J'_a(p_k) \| < \varepsilon_3. \]

Lemma 3.2. The cost functional (1) is strongly convex with the strong convexity constant \( \alpha \).
From the following strongly convex functional definition for \( \lambda \in [0,1] \):
\[ J_a(\lambda v_1 + (1-\lambda)v_2) \leq \lambda J_a(v_1) + (1-\lambda)J_a(v_2) - \chi \lambda (1-\lambda)\|v_1 - v_2\|_{L_2(0,1)}^2 \]
we can see that cost functional (1) is strongly convex the constant \( \chi = \alpha \).
So, we can give the following theorem which states the convergence of the minimizer to optimal solution.

Theorem 3.3. Let \( v^* \) be optimum solution of the problem (1)-(2). Then the minimizer given in (20) satisfies the following inequality;
\[ \|v_k - v^*\|^2 \leq \frac{2}{\alpha}(J_a(v_k) - J_a(v^*)), k = 0,1,2,\ldots \] (22)

Proof of this theorem is obtained by taking \( \lambda = \frac{1}{2} \) in the definition of the above strongly convex functional.

4. Numerical Example

In this section we test the method in a numerical example. The used trigonometric basis functions are chosen such as:
\[ \left\{ \frac{2}{\pi} \sin \left( \frac{n \pi}{l} x \right), \frac{2}{2 \pi} \sin \left( \frac{2n \pi}{l} x \right), \frac{2}{3 \pi} \sin \left( \frac{3n \pi}{l} x \right), \ldots, \frac{2}{N \pi} \sin \left( \frac{Nn \pi}{l} x \right) \right\} \]
for the generalized solution of the hyperbolic problem (2).

Example 4.1: Let us consider the following problem of minimizing the cost functional:
\[ J_a(v) = \int_0^2 [u(x,1; v)]^2 dx + \alpha \int_0^2 v^2 dx \]
under the following condition:
\[ u_{tt} - u_{xx} = (t-1)u(x), (x,t) \in (0,2) \times (0,1] \]
\[ u(x,0) = \begin{cases} x^3 + x^2 & 0 < x \leq 1 \\ -7x^2 + 19x - 10 & 1 \leq x \leq 2 \end{cases} \]
\[ u_t(x,0) = \begin{cases} x^3 + x^2 & 0 < x \leq 1 \\ -7x^2 + 19x - 10 & 1 \leq x \leq 2 \end{cases} \]
\[ u(0,t) = 0, u(2,t) = 0, t \in (0,1]. \]
The weak solution of this problem is
\[ u(x,t) = (t-1) \left\{ \begin{array}{ll} x^3 + x^2 & 0 < x \leq 1 \\ -7x^2 + 19x - 10 & 1 \leq x \leq 2 \end{array} \right\} \]
The function \( u \) and its partial derivatives \( u_x, u_t \) belong to \( C(\Omega) \). The function \( u(x,t) \) is not a classical solution since \( u_{xx} \notin C(\Omega) \). Here the force function is discontinuous.
Rewrite the functional as
\[ J_a(v) = J_a^2(v) + \alpha J_a^2(v) \]
where
\[ J_a^2(v) = \int_0^2 [u(x,1; v)]^2 dx \]
\[ J_a^2(v) = \int_0^2 v^2 dx \]
Choosing \( \alpha = 0.1 \), starting the initial element \( v_0 = \sin \pi x \) and the relaxation parameter \( \beta_k = 0.1 \) assures the inequality \( J_{a,1}(v_{k+1}) < J_{a,1}(v_k) \).
We get the following approximate minimizing function and the values of the \( J_{a,1}(v_{98}) \) and \( J_{a,1}^2(v_{98}) \), respectively;
\[ v_{98} = -0.088086149 \sin(3.14159265x) - 1.66789652 \sin(1.57079632x) - 0.011854776 \sin(4.7123898x) + 0.00127332 \sin(6.28318530x) - 0.000781035 \sin(7.85398163x) - 0.00106163 \sin(9.4247779x) - 0.000162151 \sin(10.9955742x) + 0.00003977 \sin(12.5663706x) - 0.000042586 \sin(14.1371669x) - 0.00008255 \sin(15.7079632x) \]
\[ J_{a,1}(v_{98}) = 5.303067954, \]
\[ J_{a,1}^2(v_{98}) = 2.789781915 \]
when the stopping criteria \( J_{a,1}(v_{k+1}) - J_{a,1}(v_k) > -0.2 \times 10^{-9} \) are chosen.
5. Conclusion

In this paper, we show that the external force in the wave equation be controlled by minimizing the distance between final situation distance and the desired target function. By using the adjoint approach in the mathematical analysis of the optimal control problem for wave equation, the gradient of the cost functional can be obtained. The minimizing sequence is constructed via this gradient.

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