Local Conformal Instability and Local Non-Collapsing in the Ricci flow of Quantum Spacetime

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It is known that the conformal instability or bottomless problem rises in the path integral method in quantizing the general relativity. Does quantum spacetime itself really suffer from such conformal instability? If so, does the conformal instability cause the collapse of local spacetime region or even collapse the whole spacetime? The problems are studied in the framework of the Quantum Spacetime Reference Frame (QSRF) and induced spacetime Ricci flow. We find that if the lowest eigenvalue of an operator, associated with the F-functional in a local compact (closed and bounded) region, is positive, the local region is conformally unstable and will tend to volume-shrinking and curvature-pinching along the Ricci flow-time \( t \); if the eigenvalue is negative or zero, the local region is conformally stable up to a trivial rescaling. However, the local non-collapsing theorem in the Ricci flow proved by Perelman ensures that the instability will not cause the local compact spacetime region collapse into nothing. The total effective action is also proved positive defined and bounded from below keeping the whole spacetime conformally stable, which can be considered as a generalization of the classical positive mass theorem of gravitation to the quantum level.

I. INTRODUCTION

Through a standard Wick rotation, the resulting Euclidean Einstein-Hilbert action

\[
S_E = -\frac{1}{16\pi G} \int d^4x \sqrt{g} R = -\frac{1}{16\pi G} \int d^4x \sqrt{\tilde{g}} \left[ \Omega^2 \tilde{R} + 6(\nabla \Omega)^2 \right]
\]

of General Relativity suffers a well known “bottomless” problem [1, 2], in which \( \Omega \) is a conformal factor

\[
g_{\mu\nu}(x) = \Omega^2(x) \tilde{g}_{\mu\nu}(x).
\]

The problem is associated with the conformal instability of the theory, a naive observation is that the kinetic term \((\nabla \Omega)^2\) of the conformal factor of gravity has a “wrong sign”, making the Euclidean action potentially become arbitrarily negative or unbounded from below. For this reason, the functional integral quantization of the corresponding theory

\[
Z = \int [\mathcal{D}g_{\mu\nu}] e^{-S_E}
\]

is divergent and ill defined. This divergence is not related to the ultraviolet divergences of quantum gravity, it must be taken care of first before one can renormalize the theory.

Various possible solutions for dealing with the conformal instability problem have been proposed. For example, it was suggested that the integration of the conformal factor should be performed by distorting the integration contour in the complex plane to avoid the unboundness action [1, 2]. However, the contour deformations in functional integral would lead to complex non-perturbative contributions to the calculations [4, 5]. Another approach has been proposed that the theory should be formulated in terms of dynamics and physical (transverse-traceless) degrees of freedom by taking account of certain Jacobian factors in the functional integral measure [6–8], leading to a non-standard Wick rotation of the conformal factor. In the approach, the “wrong sign” of the conformal factor can flip to be “right” depending on specific range of value of an undetermined constant in the supermetric. Furthermore, the approach works only at the linearized level, and beyond the linearized level the Euclidean action is rather complicated, and it is not obvious whether the “wrong sign” will also flip beyond the linearized level.

An alternative possibility is that the unboundedness of the Euclidean action may be not necessarily a real problem in defining its quantum theory. On the one hand, there are a number of bottomless theories at naive classical level which are well defined and having good properties at the quantum level. For example, electron in an attractive Coulomb well (e.g. the hydrogen atom) has a similar instability before the quantum mechanics is discovered, since

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the action is also unbounded from below which should cause the extranuclear electron falling into the infinitely deep Coulomb well and hence the atom finally collapses. But now it is known that the quantum mechanical treatment ultimately evades such instability problem: the energy eigenvalue is bounded from below and the electron has almost vanishing wavefunction and probability amplitude to fall into the infinitely deep well. On the other hand, similar type of conformal instability in gravitational system may be necessary and even crucial in understanding an inflationary universe at early epoch. For example, the conformon inflation [11] is caused by such instability of the conformal factor: the conformon fields.

Without a consistent quantum theory of spacetime and gravity, the conformal instability problem may not be cleared up and resolved. Recently, a possible framework of quantum spacetime is proposed, i.e. Quantum Spacetime Reference Frame (QSRF) [12–18] based on a non-linear sigma model (NLSM). The fields of the NLSM map the coordinates of $d = 4 - \epsilon$ fiducial lab spacetime to a $D = 4$ relativistic quantum scalar fields playing the role of frames of reference, and it is found that the RG flow of QSRF formally generalizes the Ricci flow in mathematics. The aim of the paper is to review the framework and consider the conformal instability problem on the basis of the QSRF and its induced spacetime Ricci flow.

The Ricci flow was initially introduced in 1980s by Friedan in $d = 2 + \epsilon$ NLSM [14, 20] and independently by Hamilton in mathematics [21, 22]. The main motivation of the Ricci flow from the mathematical point of view is to classify 3-manifolds, a specific goal is to prove the Poincare conjecture. Hamilton used it as a useful tool to gradually deform a manifold into a more and more “simple and good” manifold whose topology can be readily recognized for some simple cases. A general realization of the program is achieved by Perelman at around 2003 [23–25], who introduced several monotonic functionals (the F-functional and the W-functional) to prove the local non-collapsing theorem for finite flow-time local singularities which may be developed in general initial manifolds. By using the local non-collapsing theorem, he ruled out the cigar-type of singularity during the Ricci flow, which removes a main stumbling block of Hamilton’s program to the Poincare conjecture. Some basic theorems can be generalized to the $D = 4$ Riemannian manifolds (compact case [22, 24], and noncompact [27, 29]).

The Ricci flow is a very powerful tool to study geometry. Perelman’s breakthrough is his discovery of the variational structure that the Ricci flow can be seen as a gradient flow of some monotonic functionals, it is such variational structure paves a way to study the stability problem of a geometry. The main goal of the paper is to study the conformal instability problem by using the monotonic functionals of the Ricci flow of spacetime.

Although the standard mathematical results of the Ricci flow were oriented to prove the conjecture of 3-space, they were not generalized directly for Lorentzian spacetime. The framework of QSRF provides us a possible physical foundation and approach to generalize the Ricci flow to a Riemannian or Lorentzian 4-spacetime, when the relativistic quantum frame fields are considered measuring rods and clocks of the spacetime. The Ricci flow of the spacetime is then considered as a smearing and coarse-graining process induced by 2nd order central moment quantum fluctuations of the spacetime frame fields. Furthermore, monotonic functionals of Perelman’s type are also deduced from the diffeomorphism anomaly in the functional integral quantization of QSRF. So we could consider QSRF may lay a possible physical foundation to the spacetime Ricci flow and related monotonic functional of Perelman for physical spacetime. In this sense, at the physical rigorous level, the Ricci flow and Perelman’s monotonic functionals can be formally generalized to the Lorentzian spacetime. Particularly, the physical important spacetime configurations are those Ricci solitons as the flow limit. The Ricci soliton equation weakly cares about the spacetime signature, and the signature of its solution depends on the boundary condition. The Ricci flow of a Lorentzian manifolds is also an important direction in geometry (see e.g. [30, 31]). We have noticed that even without a clear and generally accepted physical foundation, there already have formal applications of the Ricci flow to the physical spacetime (see e.g. [33–41]), although they may lead to some conceptual confusions in interpreting their results. We also notice that various relativistic generalization and non-Riemannian modifications of Perelman’s works have been considered in gravity and information theories (see e.g. [42, 44] and references therein), which is important for a consistent theory of the spacetime Ricci flow.

The structure of the paper is as follows. In the section II, we briefly review the framework of QSRF and its induced Ricci flow of spacetime, in which monotonic functionals of Perelman’s type are deduced from the diffeomorphism anomaly of QSRF, and the effective gravity theory of QSRF is also discussed. In the section III, by using the monotonic F-functional of the theory which generalizes Perelman’s F-functional, the conformal instability problem can be reconsidered within the framework. In the section IV, by using the non-local-collapsing theorem and related W-functional developed by Perelman, the final destiny of the local conformal instability of quantum spacetime can be studied. Finally, we discuss and conclude the paper in section V.
II. A THEORY OF QUANTUM SPACETIME

To make the paper self-contained, in this section, we briefly review the framework of Quantum Spacetime Reference Frame (QSRF) as a possible theory of quantum spacetime for a preliminary, which have certain overlaps with the previous works [12, 18]. The section provides a framework and methodology to the conformal instability problem. We hope this section provides some general basis and background of the Ricci flow and related effective gravity theory to the readers. If the readers have been familiar with related literature, they could skip the section and directly go to the section III and IV.

A. Quantum Spacetime Reference Frame (QSRF)

The quantum spacetime reference frame starts from the idea that spacetime is nothing but an ideal and standard reference that a moving body is relative to. In fact “ideal” can not be realized in rigor at the quantum level, all things are quantum fluctuating. A quantum standard reference can be measured (relative to a fiducial lab) by quantum rods and quantum clocks which are also subject to quantum fluctuations. Surprisingly, gravitation is nothing but emergent phenomenon coming from relative measurement of the under-studied moving body being reference to the quantum standard reference, i.e. quantum reference frame. If the classical equivalence principle is generalized to the quantum level, then some universal (classical or quantum) properties (e.g. the universal acceleration of a falling body which is independent to its mass, the Hubble redshift (universal recession velocity) and even the broadening (universal acceleration) of spectral lines) are independent to theirs energies) are not merely the (classical or quantum) properties of matter, they are actually the (classical or quantum) properties of the spacetime itself. Thus in this framework, the most fundamental thing is not individual quantum state but the relations between two quantum states, i.e. the under-studied quantum body $|\psi\rangle$ and the quantum reference frame system $|X\rangle$, described by an entangled state $|\Psi\rangle = \sum_{ij} \psi_{i,j} \otimes |X\rangle_j$ in the whole Hilbert space $\mathcal{H}_{\phi} \otimes \mathcal{H}_X$.

If the quantum spacetime reference frame $|X_\mu\rangle$ ($\mu = 0, 1, 2, \ldots, D-1$) itself is considered as the under-studied quantum body, then we have a quantum theory of the quantum spacetime. In this case the reference system could be the fiducial lab spacetime $|x_a\rangle$, ($a = 0, 1, 2, \ldots, d-1$). The entangled state is constructed by a one-to-one correspondence between two states, i.e. $|x_a\rangle \rightarrow |X_\mu\rangle$, which is nothing but a general non-linear differentiable mapping $X : \mathbb{R}^d \rightarrow M^D$. From the geometric point of view, the entangled state $\sum_{ij} |X\rangle_i \otimes |x\rangle_j$ or $X(x)$ is a mapping from a local coordinate flat patch $x \in \mathbb{R}^d$ to a Riemannian or Lorentzian manifolds $X \in M^D$ (the signature depends on the boundary condition). From the physics point of view, the general non-linear mapping can be realized by a kind of field theory, the non-linear sigma model (NLSM)

$$S[X] = \frac{1}{2} \lambda \int d^d x g_{\mu\nu} \frac{\partial X^\mu}{\partial x_a} \frac{\partial X^\nu}{\partial x_a},$$

where $\lambda$ is a constant with dimension of energy density $[L^{-d}]$ taking the value of the critical density $\rho_c$ of the universe. $x_a$ having dimension of length $L$ is the base space of NLSM. It will be interpreted as the lab wall and clock frame as the starting reference, which is fiducial and classical (infinite precise and free from quantum fluctuations). Frame field $X_\mu$ also having dimension of length is the target space of NLSM interpreted as physical rods and clocks, i.e. the reference frame fields measuring the quantum spacetime coordinate. $X_\mu$ are the coordinates of a general Riemannian or Lorentzian manifolds $M^D$ with curved metric $g_{\mu\nu}$, called the target space in NLSM's terminology. We will work with the real-defined coordinates for the target spacetime, and the Wick rotated case has been generally included into the general coordinates transformation of the time component. The frame fields will be promoted to quantum fields, in the language of quantum fields theory, $X_\mu(x)$ or $X^\mu(x) = g^{\mu\nu}X_\nu(x)$ are the real scalar frame fields. $D = 4$ is the least number of the frame fields capable to measure the coordinates of a spacetime event. $d$ is the dimensions of the fiducial lab spacetime, it could take $d = 4 - \epsilon$, where $0 < \epsilon \ll 1$ is for topological reason that the homotopic group $\pi_d(M^D)$ should be trivial so that the mapping $X(x) : \mathbb{R}^d \rightarrow M^D$ should be free from topological obstacles and singularities. Since a quantum fields theory is well-formulated in an inertial frame, and the local patch is also flat, so the fiducial lab spacetime $x_a$ is considered flat and rigid. Note that $d^d x = d^d x \det e$ (det $e$ is the Jacobian with $|\det e| = 1$) is a classical invariant volume element no matter in Minkovskian or Euclidean, since if one replaces the Euclidean time to the Minkowskian time $d\tau(E) \rightarrow id\tau(M)$, the Jacobian changes correspondingly $\det e(E) \rightarrow i \det e(M)$, so the Euclidean theory will give the same result as the Minkowskian one at least classically, and the action is always real no matter it is Minkowskian or Euclidean one. Without loss of generality, we consider the base space as the Euclidean one which is better defined when one uses the functional integral quantization method. It is well-known that the NLSM in $d = 2$ is perturbative renormalizable, and some numerical calculations also support $d = 3$ and $d = 4 - \epsilon < 4$ are non-perturbative renormalizable, so it is a well-defined relativistic quantum fields theory.
The quantum frame fields model of spacetime has practical physical interpretation, for instance, in the lab scale, it can be considered as a multi-wire proportional chamber system constructed relative to the wall and clock of a lab that are used to measure coordinates of events in the lab, where the frame fields can be interpreted as the spin-less electron signal. When the spacetime measuring scale is beyond the lab’s scale, for instance, to the cosmic scale, the quantum fluctuation or broadening of the frame fields (e.g. by using spin-less light signal) become unignorable, and curved metric $g_{\mu\nu}$ measured by the comparison between the frame fields and fiducial coordinates become non-trivial and important, since the quantum fluctuation $X_{\mu} = (e^a_{\mu})x_a + \delta X_{\mu}$ has nontrivial physical consequence. The lowest 2nd order central moment (variance) modifies the quadratic form distance of Riemannian or Lorentzian geometry and hence gives correction to the quantum expectation of the metric

$$\langle g_{\mu\nu} \rangle = \left\langle \frac{\partial X_{\mu}}{\partial x_a} \frac{\partial X_{\nu}}{\partial x_a} \right\rangle = \left\langle \frac{\partial X_{\mu}}{\partial x_a} \right\rangle \left\langle \frac{\partial X_{\nu}}{\partial x_a} \right\rangle + \frac{\partial^2}{\partial x_a^2} \langle \delta X_{\mu} \delta X_{\nu} \rangle = g^{(1)}_{\mu\nu}(X) + \delta g^{(2)}_{\mu\nu}(X).$$

(5)

The 2nd order moment quantum fluctuation

$$\delta g^{(2)}_{\mu\nu}(X) = \frac{R^{(1)}_{\mu\nu}(X)}{32\pi^2\lambda} \delta k^{d-2}$$

(6)
deforms the metric, where $R^{(1)}_{\mu\nu}$ is the Ricci curvature given by $g^{(1)}_{\mu\nu}$ and $k^{d-2}$ is the cutoff energy scale of the Fourier component of the frame fields. For $d = 4 - \epsilon$, the validity of the perturbation calculation $R^{(1)} \delta k^2 \ll \lambda$ is the validity of the Gaussian approximation. It will be shown later that $\lambda$ is nothing but the critical density $\rho_c$ of the universe, $\lambda \sim O(H_0^2/G)$, $H_0$ the current Hubble’s parameter, $G$ the Newton’s constant. Thus for our concern of pure gravity in which matter is ignored, the condition $R^{(1)} \delta k^2 \ll \lambda$ is equivalent to $\delta k^2 \ll 1/G$ which is reliable except for some local singularities may develop when the Gaussian approximation is failed.

**B. The RG flow of QSRF as the Ricci flow of Spacetime**

The deformation of the metric at the Gaussian approximation is driven by the 2nd moment quantum fluctuation and hence the Ricci curvature, which gives rise to the Ricci flow equation

$$\frac{\partial g_{\mu\nu}}{\partial t} = -2R_{\mu\nu},$$

(7)

where the flow-time interval

$$\delta t = -\frac{1}{64\pi^2\lambda} \delta k^{d-2}$$

(8)

has the dimension of distance squared [$L^2$] for any dimension of the base space $d$, in our relativistic quantum frame fields setting, we have taken $d = 4 - \epsilon$.

For the Ricci curvature is non-linear for the metric, the Ricci flow equation is a non-linear version of the heat equation for the metric, the metric flowing along $t$ introduces an averaging or coarse-graining process to the intrinsic non-linear quantum spacetime which is highly non-trivial. In general, if the flow is free from local singularities there exists long flow-time solution in $t \in (-\infty, 0)$, which is often called ancient solution in mathematical literature. This range of the $t$-parameter corresponds to $k \in (0, \infty)$, that is from $t = -\infty$, i.e. the short distance (high energy) UV scale $k = \infty$ forwardly to $t = 0$ i.e. the long distance (low energy) IR scale $k = 0$. The metric at certain scale $t$ is given by being averaged out the shorter distance details which produces an effective correction to the metric at that scale. So along $t$, the manifolds loss its information in shorter distance, thus the flow is irreversible, i.e. generally having no backwards solution, which is the underlying reason for the entropy of a spacetime.

As it is shown in $[5]$, the 2nd order moment fluctuation modifies the local (quadratic) distance of the spacetime, so the flow is non-isometry. The non-isometry is not important for its topology, so along $t$, the flow preserves the topology of the spacetime but its local metric, shape and size (volume) change. There also exists a very special solution of the Ricci flow called Ricci soliton, which only changes the local volume while keeps its local shape. The Ricci soliton, and its generalized version, the Gradient Ricci Soliton, as the flow limits (of the generalized Ricci-DeTurck flow), are the generalization of the notion of fixed point in the sense of RG flow. The solution of the Ricci soliton as the fixed point solution of the Ricci flow can be either Riemannian or Lorentzian manifolds, whose signature is determined by the boundary condition of the solution.
Although the Ricci flow is not strongly parabolic, DeTurck provides a simple way to prove the short-flow-time existence of the Ricci flow by slightly modifying it, named Ricci-DeTurck flow \[47\]

$$\frac{\partial g_{\mu\nu}}{\partial t} = -2 \left( R_{\mu\nu} + \nabla_\mu \nabla_\nu f \right), \tag{9}$$

which is equivalent to the standard Ricci flow equation up to a diffeomorphism given by \( u = e^{-f} \). From the geometric point of view, \( u = e^{-f} \) introduces a positive defined density bundle over the local geometric, so it is also called a density of manifolds. From the physics point of view, \( u \) can also be interpreted as a positive defined density matrix \[18\] of the frame fields system, \( u \) is usually normalized by the condition

$$\lambda \int u \, d^D x = 1, \tag{10}$$

which also defines an invariant and fiducial volume (the rigid lab frame) from the language of density manifolds. The condition, together with the Ricci-DeTurck flow, gives the flow equation of \( u \),

$$\frac{\partial u}{\partial t} = (-\Delta + R) u. \tag{11}$$

Note the minus sign in front of the Laplacian, it is a backwards heat-like equation. Naively speaking, the solution of the backwards heat flow will not exist. But we could also note that if one let the Ricci flow flows to certain singular IR scale \( t_* \), (if the flow is free from global singularity from the mapping \( X(x) : \mathbb{R}^d \to M^D \), the solution is ancient, and hence \( t_* = 0 \), and at \( t_* \) one may then choose an appropriate \( u(t_*) = u_0 \) arbitrarily (up to a diffeomorphism gauge) and flows it backwards in \( \tau = t_* - t \) to obtain a solution \( u(\tau) \) of the backwards equation. In this case, \( u \) satisfies the heat-like equation

$$\frac{\partial u}{\partial \tau} = (\Delta - R) u, \tag{12}$$

which does admit a solution along \( \tau \), often called the conjugate heat equation in mathematical literature.

So far \[12\] together with \[9\] the mathematical problem of the Ricci flow of a Riemannian/Lorentzian manifolds is transformed to coupled equations

$$\begin{cases}
\frac{\partial g_{\mu\nu}}{\partial t} = -2 \left( R_{\mu\nu} - \nabla_\mu \nabla_\nu f \right) \\
\frac{\partial u}{\partial t} = (-\Delta + R) u \\
\frac{\partial \lambda}{\partial t} = -1
\end{cases} \tag{13}$$

and the manifolds \((M^D, g)\) are generalized to density manifolds \((M^D, g, u) \) \[48–50\] with the constraint \[10\].

### C. Anomaly Induced Action of QSRF

The quantum fluctuation and hence the Ricci flow does not preserve the quadratic distance of a Riemannian or Lorentzian geometry. The non-isometry of the quantum fluctuation induces a breakdown of diffeomorphism or general coordinate transformation at the quantum level, namely the anomaly. It will be surprise to see that the monotonic functionals of Perelman’s type and an effective action of gravity can be deduced from the anomaly of QSRF.

Here we consider functional quantization of the pure frame fields, the partition function is

$$Z(M^D) = \int [D X] \exp \left( -S[X] \right) = \int [D X] \exp \left( -\frac{1}{2} \lambda \int d^4 x g^{\mu\nu} \partial_\alpha X_\mu \partial_\alpha X_\nu \right), \tag{14}$$

where \( M^D \) is the target spacetime, and the base space is Euclidean. Note that a general coordinate transformation

$$X_\mu \to \tilde{X}_\mu = \frac{\partial \tilde{X}_\mu}{\partial X_\nu} X_\nu = e^\nu_\mu X_\nu \tag{15}$$
where the Jacobian of the diffeomorphism. The Jacobian is nothing but a local relative volume element $Z$ then the partition function volume comparison. formalism of the framework formally the same with the Perelman’s standard form even in the Lorentzian signature. into (19) to keep the condition (10). In fact the definition of the volume form and the manifolds density ensure the in the Lorentz signature. Otherwise, for the Lorentz signature, there should introduce some extra imaginary factor $i$ into (19) to keep the condition (10). In fact the definition of the volume form and the manifolds density ensure the formalism of the framework formally the same with the Perelman’s standard form even in the Lorentzian signature. The manifolds density encodes a most important information of a Riemannian or Lorentzian geometry, i.e. the local volume comparison.

In this situation, if we parameterize a dimensionless solution $u$ of the conjugate heat equation (12) as

$$u(\hat{X}) = \frac{dV(X_\mu)}{dV(\hat{X}_\mu)} = |\text{det } e^a_\mu| = \frac{1}{|\text{det } e^a_\mu|}. \quad (18)$$

Until now, the previous derivations are straightforward, the above $u$ density followed by (11) is the first explicit generalization from the standard 3-space to a Riemannian or Lorentzian 4-spacetime. It is worth stressing that here the absolute symbol of the determinant is because the density $u$ and the volume form are kept positive defined even in the Lorentz signature. Otherwise, for the Lorentz signature, there should introduce some extra imaginary factor $i$ into (19) to keep the condition (10). In fact the definition of the volume form and the manifolds density ensure the formalism of the framework formally the same with the Perelman’s standard form even in the Lorentzian signature. The manifolds density encodes a most important information of a Riemannian or Lorentzian geometry, i.e. the local volume comparison.

In this situation, if we parameterize a dimensionless solution $u$ of the conjugate heat equation (12) as

$$u(\hat{X}) = \frac{1}{\lambda(4\pi \tau)^{D/2}} e^{-f(\hat{X})}, \quad (19)$$

then the partition function $Z(M^D)$ is transformed to

$$Z(\hat{M}^D) = \int [D\hat{X}] \exp \left( -S[\hat{X}] \right) = \int \left( \prod_x |\text{det } e| \right) [DX] \exp \left( -S[X] \right)$$

$$= \int \left( \prod_x e^{f + \frac{D}{2} \log(4\pi \tau)} \right) [DX] \exp \left( -S[X] \right)$$

$$= \int [DX] \exp \left\{ -S[X] + \lambda \int d^4x \left[ f + \frac{D}{2} \log(4\pi \tau) \right] \right\}$$

$$= \int [DX] \exp \left\{ -S[X] + \lambda \int_{\hat{M}^D} d^D X u \left[ f + \frac{D}{2} \log(4\pi \tau) \right] \right\}. \quad (20)$$

Note that the change of the partition function

$$Z(\hat{M}^D) = e^{\lambda N(\hat{M}^D)} Z(M^D) \quad (21)$$

is nothing but a pure real Shannon entropy in terms of the density $u$

$$N(\hat{M}^D) = \int_{\hat{M}^D} d^D X u \left[ f + \frac{D}{2} \log(4\pi \tau) \right] = - \int_{\hat{M}^D} d^D X u \log u. \quad (22)$$

The classical action $S[X]$ is invariant under the general coordinates transformation or diffeomorphism, but the quantum partition function is no longer invariant under the general coordinates transformation or diffeomorphism, which is called diffeomorphism anomaly, meaning a breaking down of the diffeomorphism at the quantum level. The diffeomorphism anomaly is purely due to the quantum fluctuation and Ricci flow of the frame fields which do not
preserve the functional integral measure and change the spacetime volume at the quantum level. The diffeomorphism anomaly has many profound consequences to the theory of quantum reference frame, e.g. non-unitarity, the trace anomaly, the notion of entropy, reversibility, and the cosmological constant \[16, 18\].

Without loss of generality, if we simply consider the under-transformed coordinates $X_\mu$ as the coordinates of the fiducial lab $x_a$ which can be treated as a classical parameter coordinates, in this situation the classical action of NLSM is just a topological invariant, i.e. half the dimension of the target spacetime

$$\exp \left( -\frac{1}{2} \lambda \int d^4x g_{\mu\nu}^{(1)} \partial_\mu x_\alpha \partial_\nu x_\beta \right) = \exp \left( -\frac{1}{2} \lambda \int d^4x g_{\mu\nu}^{(1)}g_{\mu\nu}^{(1)} \right) = e^{-\frac{D}{2}}.$$  \tag{23}$$

Thus the total partition function of the frame fields takes a simple form

$$Z(M^D) = e^{\lambda N(M^D) - \frac{D}{2}}. \tag{24}$$

A relative density

$$u_r(X) = \frac{u}{u_*} \tag{25}$$

can be defined by a density $u(X)$ being relative to

$$u_*(X) = \frac{1}{\lambda (4\pi \tau)^{D/2}} \exp \left( -\frac{1}{4\pi \tau} |g_{\mu\nu}X^\mu X^\nu| \right), \tag{26}$$

in which the absolute symbol in the exponential is used to keep the quadratic distance and hence the volume form induced from the Gaussian integral positive even in the Lorentzian signature. $u_*$ corresponds to the maximum Shannon entropy $N_*$

$$N_* = -\int d^DX u_* \log u_* = \int d^DX u_* \frac{D}{2} \left[ 1 + \log(4\pi \tau) \right] = \frac{D}{2\lambda} \left[ 1 + \log(4\pi \tau) \right], \tag{27}$$

when the spacetime becomes a gradient shrinking Ricci soliton (GSRS) satisfying

$$R_{\mu\nu} + \nabla_\mu \nabla_\nu \tilde{f} = \frac{1}{2\tau} g_{\mu\nu}. \tag{28}$$

By using the relative density, a relative Shannon entropy $\tilde{N}$ can be defined by

$$\tilde{N}(M^D) = -\int d^DX u \log u_r = -\int d^DX u \log u + \int d^DX u_* \log u_* = N - N_* = -\log Z_P \leq 0, \tag{29}$$

where $Z_P$ is nothing but the Perelman’s partition function \[23\]

$$\log Z_P = \int_{M^D} d^DX u \left( \frac{D}{2} - f \right) \geq 0. \tag{30}$$

In terms of the relative Shannon entropy, the total partition function \[24\] of the frame fields is normalized by the extreme value

$$Z(M^D) = \frac{e^{\lambda N - \frac{D}{2}}}{e^{\lambda N_*}} = e^{\lambda \tilde{N} - \frac{D}{2}}. \tag{31}$$

### D. Effective Theory of Gravity

Followed by different route to generalize the Ricci flow, a Ricci flow of a Lorentzian 4-spacetime is constructed on the physical foundation of the QSRF, which may provide us an alternative way to generalize the 3-space Ricci flow to the 4-spacetime version, we take $D = 4$ in this subsection. Together with a quantum generalization of Equivalence Principle, an effective gravity is emerged from the QSRF.

The relative Shannon entropy $\tilde{N}$ as the anomaly vanishes at gradient shrinking Ricci soliton (GSRS) or IR scale, however, it is non-zero at ordinary lab scale up to UV where the fiducial volume of the lab is considered rigid and fixed $\lambda \int d^4x = 1$. The cancellation of the anomaly at the lab scale up to UV is physically required, which leads to the
counter term $\nu(M^D_{\tau=\infty})$ or cosmological constant. Since the $\tilde{N}$ is monotonic non-decreasing along $t$ or non-increasing along $\tau$

$$\frac{\partial \tilde{N}(M^D)}{\partial \tau} = \frac{\partial N(M^D)}{\partial \tau} - \frac{\partial N_*}{\partial \tau} = F(M^D) - \frac{D}{2\lambda}\tau \equiv \tilde{F}(M^D) \leq 0. \quad (32)$$

The monotonicity of $\tilde{N}$ and the W-functional implies a lower bound at UV $\nu$.

$$\nu(M^D_{\tau=\infty}) = \lim_{\tau \to \infty} \lambda \tilde{N}(M^D, u, \tau) = \lim_{\tau \to \infty} \lambda \tilde{W}(M^D, u, \tau) = \inf_{\tilde{\tau}} \lambda \tilde{W}(M^D, u, \tau) < 0, \quad (33)$$

where $\tilde{W}$, the W-functional, is the Legendre transformation of $\tilde{N}$ w.r.t. $\tau^{-1}$,

$$\tilde{W} = \frac{\partial \tilde{N}}{\partial \tau} + \tilde{N} = \tau \tilde{F} + \tilde{N}. \quad (34)$$

The Ricci-DeTurck flow turns out to be a gradient flow of the monotonic functionals. The functionals are formally identified with Perelman’s, with the monotonicities $\{23\}$.

$$\frac{\partial F}{\partial t} \geq \frac{2}{D} \int d^D X u (R + \Delta f)^2 \geq \frac{2}{D} \left[ \int d^D X u (R + \Delta f) \right]^2 = \frac{2}{D} \tilde{F}^2 \geq 0, \quad (35)$$

$$\frac{\partial \tilde{W}}{\partial t} = 2\tau \int d^D X u \left| R_{\mu\nu} + \nabla_\mu \nabla_\nu f - \frac{1}{2\tau} g_{\mu\nu} \right|^2 \geq 0. \quad (36)$$

In this sense, QSRF provides us a framework to calculate and generalize Perelman’s W-functional to a Lorentzian spacetime by using the anomaly or Shannon entropy in terms of the density $u$ given by the local volume form of a Lorentzian geometry $\{13\}$.

In other words, the difference between the effective actions (relative Shannon entropies) at UV and IR is finite

$$\nu = \lambda (\tilde{N}_{\tau=\infty} - \tilde{N}_{\tau=0}) < 0. \quad (37)$$

In fact $e^\nu < 1$ (usually called the Gaussian density $\{52, 53\}$) is a relative volume or the reduced volume $\tilde{V}(M^D_{\tau=\infty})$ of the backwards limit manifolds introduced by Perelman, or the inverse of the initial condition of the manifolds density $u_{\tau=0}$. A finite value of it makes an initial spacetime with unit volume from UV flow and converge to a finite $u_{\tau=0}$, and hence the manifolds finally converge to a finite relative volume/reduced volume instead of shrinking to a singularity point at $\tau = 0$.

As an example, for a homogeneous and isotropic universe for which the sizes of space and time (with a “ball” radius $a_\tau$) are on an equal footing, i.e. $ds^2 = a_\tau^2 (-dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2)$, which is a Lorentzian shrinking soliton configuration. Note that the shrinking soliton equation $R_{\mu\nu} = \frac{2}{\tau} g_{\mu\nu}$ it satisfies and its volume form $\{18\}$ are independent to the signature, so it can be approximately given by a 4-ball value $\nu(B^4_0) \approx -0.8 \quad \{13, 16\}$.

So the partition function, which is anomaly canceled at UV and having a fixed-volume fiducial lab, is

$$Z(M^D) = e^{\lambda N - \frac{D}{2} - \nu}. \quad (38)$$

Since $\lim_{\tau \to 0} \tilde{N}(M^D) = 0$, so at small $\tau$, $\tilde{N}(M^D)$ can be expanded by powers of $\tau$

$$\tilde{N}(M^D) = \int_{M^D} d^D X u \left[ R_{\tau=0} + |\nabla f_{\tau=0}|^2 - \frac{D}{2\tau} \right] \tau + O(\tau^2), \quad (39)$$

in which $\lambda \int d^D X u \tau |\nabla f_{\tau=0}|^2 \approx \frac{D}{2\tau}$ (at GSRS) has been used.

For $D = 4$ and small $\tau$, the effective action of $Z(M^4)$ can be given by

$$-\log Z(M^4) = S_{eff} \approx \int_{M^4} d^4 X u (2\lambda - \lambda R_0 + \lambda \nu) \quad \text{(small $\tau$).} \quad (40)$$
Considering \( ud^4X \) as the invariant and fiducial volume element \( d^4X \sqrt{|g_k|} \) at certain scale \( k \) and \( \tau = -t = \frac{1}{64\pi^2}k^2 \) (when \( t = 0 \) i.e. the global geometry is free from singularity), we have

\[
S_{\text{eff}} = \int_{M^4} d^4X \sqrt{|g_k|} \left( 2\lambda - \frac{R_0}{64\pi^2}k^2 + \lambda\nu \right) \quad \text{(small \( k \)).} \tag{41}
\]

The effective action can be interpreted as a low energy effective action of pure gravity. As the cutoff scale \( k \) ranges from the lab scale to the solar system scale (\( k > 0 \)), the action must recover the well-tested Einstein-Hilbert (EH) action. But at the cosmic scale (\( k \to 0 \)), we know that the EH action deviates from observations and the cosmological constant becomes important. In this picture, as \( k \to 0 \), the action leaving \( 2\lambda + \lambda\nu \) should play the role of the standard EH action with a limit constant background scalar curvature \( R_0 \) plus the cosmological constant, so

\[
2\lambda + \lambda\nu = \frac{R_0 - 2\Lambda}{16\pi G}. \tag{42}
\]

While at UV \( k \to \infty \), \( \lambda \bar{N} \to \nu \), the action leaves only the fiducial Lagrangian \( \frac{D}{2}\lambda = 2\lambda \) which should be interpreted as a constant EH action without the cosmological constant

\[
2\lambda = \frac{R_0}{16\pi G}. \tag{43}
\]

Thus we have the cosmological term

\[
\lambda\nu = -\frac{2\Lambda}{16\pi G} = -\rho_\Lambda. \tag{44}
\]

The action can be rewritten as an effective EH action plus a cosmological term

\[
S_{\text{eff}} = \int_{M^4} d^4X \sqrt{|g_k|} \left( \frac{R_k}{16\pi G} + \lambda\nu \right) \quad \text{(small \( k \)),} \tag{45}
\]

where

\[
\frac{R_k}{16\pi G} = 2\lambda - \frac{R_0}{64\pi^2}k^2, \tag{46}
\]

which is nothing but the flow equation of the scalar curvature

\[
R_k = \frac{R_0}{1 + \frac{1}{4\pi G}k^2}, \quad \text{or} \quad R_\tau = \frac{R_0}{1 + \frac{1}{D}\frac{R_0}{\tau}}. \tag{47}
\]

Since at the cosmic scale \( k \to 0 \), the effective scalar curvature is bounded by \( R_0 \) which can be measured by the “Hubble’s constant” \( H_0 \) at the cosmic scale,

\[
R_0 = D(D - 1)H_0^2 = 12H_0^2, \tag{48}
\]

so \( \lambda \) is nothing but the critical density of the 4-spacetime Universe

\[
\lambda = \frac{3H_0^2}{8\pi G} = \rho_c, \tag{49}
\]

so the cosmological constant is always of order of the critical density with a “dark energy” fraction

\[
\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c} = -\nu \approx 0.8, \tag{50}
\]

which is not far from observations. The detailed discussions about the cosmological constant problem and the observational effect in the cosmology, especially the modification of the Distance-Redshift relation leading to the acceleration parameter \( q_0 \approx -0.68 \) can be found in [13–16].

If we include matter into the gravity theory, consider the entangled system in \( \mathcal{H}_\psi \otimes \mathcal{H}_X \) between the under-studied quantum body and the quantum reference frame fields system. Without loss of generality, we could take a scalar field \( \psi \) as the under-studied system, which shares the base space with the frame fields, the total action of the two entangled systems is a direct sum of each system

\[
S[\psi, X] = \int d^dX \left[ \frac{\lambda}{2} \frac{\partial \psi}{\partial x_a} \frac{\partial \psi}{\partial x_a} - V(\psi) + \frac{1}{2} \lambda g_{\mu\nu} \frac{\partial X^\mu}{\partial x_a} \frac{\partial X^\nu}{\partial x_a} \right], \tag{51}
\]
where $V(\psi)$ is some potential of the $\psi$ fields. Since both $\psi$ field and the frame fields $X$ share the same base space (fiducial lab), here they are formulated on the usual lab spacetime $x$. If we interpret the frame fields as the physical spacetime coordinates, the coordinates or reference frames of $\psi$ field must be transformed from $x$ to $X$. At the semi-classical level, or 1st moment approximation when the fluctuation of $X$ can be ignored, it is simply a coordinates transformation

$$S[\psi, X] \approx S[\psi(X)] = \int d^4X \sqrt{|\det g^{(1)}|} \left[ \frac{1}{4} \left( g^{(1)\mu\nu} \frac{\partial X^\mu}{\partial x^a} \frac{\partial X^\nu}{\partial x^a} \right) \left( \frac{1}{2} g^{(1)\mu\nu} \frac{\delta \psi}{\delta X^\mu} \frac{\delta \psi}{\delta X^\nu} + 2\lambda \right) - V(\psi) \right]$$

$$\approx \int d^4X \sqrt{|\det g^{(1)}|} \left[ \frac{1}{2} g^{(1)\mu\nu} \frac{\delta \psi}{\delta X^\mu} \frac{\delta \psi}{\delta X^\nu} - V(\psi) + 2\lambda \right],$$

in which $^{(1)}\approx$ stands for the 1st moment approximation, and $\frac{1}{4} \left( g^{(1)\mu\nu} \frac{\partial X^\mu}{\partial x^a} \frac{\partial X^\nu}{\partial x^a} \right) = \frac{1}{4} \left( g^{(1)\mu\nu} g^{(1)\mu\nu} \right) = \frac{1}{4} D = 1$ has been used. It is easy to see, at the semi-classical level, i.e. only consider the 1st moment of $X$ while 2nd moment fluctuations are ignored, the (classical) coordinates transformation reproduces the scalar field action in general coordinates $X$ up to a constant $2\lambda$, and the derivative $\frac{\partial}{\partial x^a}$ is replaced by the functional derivative $\frac{\delta}{\delta X^a}$. $\sqrt{|\det g^{(1)}|}$ is the Jacobian determinant of the coordinate transformation, note that the determinant requires the coordinates transformation matrix a square matrix, so at semi-classical level $d$ must be very close to $D = 4$, which is not necessarily true beyond the semi-classical level, when the 2nd moment quantum fluctuations are important. For instance, since $d$ is a parameter but an observable in the theory, it could even not necessary be an integer but effectively fractal at the quantum level, and we have chosen $d = 4 - \epsilon$.

When the gravity and quantum spacetime frame fields are normalized by the Ricci flow, $2\lambda$ term in eq.(52) is normalized by 2nd moment fluctuation, by using eq.(41) and eq.(40), a matter-coupled-gravity is emerged from the Ricci flow

$$S[\psi, X] \approx \int d^4X \sqrt{|g_k|} \left[ \frac{1}{2} g^{\mu\nu} \frac{\delta \psi}{\delta X^\mu} \frac{\delta \psi}{\delta X^\nu} - V(\psi) + 2\lambda - \frac{R_0}{64\pi} k^2 + \lambda \nu \right]$$

$$= \int d^4X \sqrt{|g_k|} \left[ \frac{1}{2} g^{\mu\nu} \frac{\delta \psi}{\delta X^\mu} \frac{\delta \psi}{\delta X^\nu} - V(\psi) + \frac{R_k}{16\pi G} + \lambda \nu \right]$$

### III. CONFORMAL STABILITY OF LOCAL QUANTUM SPACETIME AND THE F-FUNCTIONAL

Note that, defined by the variation of the effective action of gravity (31) or the Shannon entropy (22), there is also a “wrong sign” in the F-functional for the spacetime Ricci flow with $D = 4$

$$F(g, u) = \frac{\partial N}{\partial \tau} = \int d^D X \left( R + |\nabla f|^2 \right).$$

It is formally similar with the Euclidean Einstein-Hilbert action (11) in which $d^D X$ plays the role of the scale $k$ dependent measure $d^D \sqrt{|g_k|}$ in the action. The functional is a generalization of Perelman’s F-functional of a 3-space (23), so it can be considered for the 4-spacetime with Euclidean signature.

From the point of view of the F-functional, the conformal stability of a spacetime configuration is determined by the sign of the lowest eigenvalue of an operator $-4\Delta + R (\Delta$ the Laplace-Beltrami operator and $R$ the scalar curvature for the spacetime) associated with the F-functional $F(g, \phi)$ [52, 54],

$$\Lambda(g) = \inf \left\{ F(g, \phi) = \int d^4X \left( R\phi^2 + 4|\nabla \phi|^2 \right), \text{with } \lambda \int \phi^2 d^4X = 1 \right\}$$

rather than naively depends on the sign of the local kinetic term. Just as the stability of quantum hydrogen atom, the competition between the kinetic term $|\nabla \phi|^2$ and the potential term $R\phi^2$ is important for the conformal stability.

More precisely, to probe the stability of a local compact (closed and bounded) spacetime region centered at spacetime point $X$, one could choose $\phi^2(X) = u(X) = e^{-f(X)}$, an approximate delta function centered at $X$ to calculate $\Lambda(g(X))$. From the backwards heat equation (11) and (19), it is clear that if the eigenvalue $\Lambda(g(X)) > 0$, the local compact 4-volume $d^4X \sim u^{-1}$ of the region will decay and shrink by the increasing of $u$ along the flow-time $t$ (but the physical-time) into a shrinking soliton and tend to disappear (not completely collapse, see later) or develop local neckpinch and tend to thorns in finite flow-time $t$ (finite scale) during the Ricci flow, so the local compact region around $X$ is linearly unstable. The positive eigenvalue $\Lambda$ is similar with the negative unbounded classical action $S_E$, leading to a conformally unstable region, and the function $\phi$ plays a similar role of the conformal factor $\Omega$, not only they both
have similar “wrong sign” in their kinetic terms, but also $\phi = \sqrt{u}$ represents the longitudinal and trace part of the degrees of freedom of metric. Actually $\phi = \sqrt{u}$ is indeed a conformal factor of the gravity up to a constant multiple, a positive sign of $\Lambda$ will indeed induce a conformal instability of gravity at least in a local compact region. Furthermore, it is also found [17] that such local unstable shrinking region modeled by a shrinking Ricci soliton reproduces a spatial inflationary region at a local physical-time. If $\Lambda(g(X)) \leq 0$, the spacetime compact region expands or keeps its volume, and hence it is stable up to a rescaling, so it is equivalent to the positive mass theorem in that region. The flat spacetime with $R = 0$ is an example of $\Lambda = 0$ which is conformally stable.

Certainly, the similarity between the F-functional and the Einstein-Hilbert action is formal but exact. From the previous effective action of gravity [25] with long distance approximation [29], we can see that (up to a normalization factor) the effective action given in terms of the Shannon entropy $N$ differs from the F-functional by a $\tau$ parameter. $F = \frac{\partial F}{\partial \tau} = 0$ plays the role of a fixed point action (corresponding to the steady soliton configuration), and $\tilde{F} = \frac{\partial \tilde{F}}{\partial \tau} = 0$ the fixed point action of the shrinking soliton, while the relative Shannon entropy $\tilde{N} = N - N_{c}$ plays the role of the scale dependent effective action. In this sense, to investigate the final destiny of the unstable region (where $\Lambda > 0$) and the stability of the whole spacetime we need to study the exact action $\tilde{N}$ and the related entropy functional.

IV. LOCAL NON-COLLAPSING OF QUANTUM SPACETIME AND THE W-FUNCTIONAL

What is the final destiny of the unstable region, does the local unstable compact region completely collapse into nothing? Or even worse, does the instability finally leads to a disastrous collapse of the whole spacetime? A no-go answer to the second question seems guaranteed by the positive mass theorem [53, 54] at the classical level, but what is the case at the quantum level?

The monotonicity of the F-functional $\frac{dF}{dt} \geq 0$ (see eq. (35)) claims that $\Lambda(g)$ is nondecreasing along the flow-time $t$ during any Ricci flow without any curvature condition, and bounded above by $F \leq \frac{D}{2\Lambda(t_{*} - t)} = \frac{D}{2\tau}$ (followed from (32)), where $D$ is the dimension of spacetime, $t$ the flow-time, $t_{*}$ is certain finite singular flow-time when the local curvature diverges. The equal sign is saturated if and only if the local spacetime region flows and finally becomes to a gradient shrinking Ricci soliton configuration eq. (28).

However, the right hand side of $F \leq \frac{D}{2\Lambda(t_{*} - t)}$ also diverges when $t \to t_{*}$, the bound still does not answer the question whether the local shrinking Ricci soliton region finally local collapse as $t \to t_{*}$. More precisely, to define the notion “local collapse” of a compact local spacetime region, we consider if there exists a sequence $(i = 1, 2, 3,...)$ of flow-time $t_{i} \to t_{*}$ and radii $r_{i} \in (0, \infty)$ of the local compact space and time region with $r_{i}^{2}/t_{i}$ uniformly bounded from above, and the Riemannian curvature of the local region is bounded comparable to the radius $|Rm(g(t_{i}))| \leq r_{i}^{-2}$ in the compact spacetime “ball” $B_{g(t_{i})}(r_{i})$, here the volume of the spacetime “ball” in the local compact region

$$V(B_{g(t_{i})}(r_{i})) \equiv \int_{B(r_{i})} d^{4}X_{t_{i}}$$

shrinks to zero in the limit of the sequence $\lim_{i \to \infty} \frac{V(B_{u(t_{i})}(r_{i}))}{r_{i}^{4}} = 0$. If the situation actually happens, the local compact spacetime region is said to be local collapse.

To probe whether the local compact region is local collapse, we need a dimensionless and scale invariant version of the F-functional. To achieve the scale invariance, Perelman includes an explicit insertion of the scale parameter $\sqrt{\tau}$, telling us at what distance scale to localize (i.e. $\sqrt{\tau}$). The W-functional will be essential for understanding the critical structure near a shrinking spacetime region. First, it is invariant under simultaneous rescaling of $\tau$ and $g$. Second, it is non-decreasing along the flow-time during any Ricci flow $\frac{\partial W}{\partial t} \geq 0$ (see eq. (36)) without any assumption on curvature, for this reason, it is also often called the W-entropy.

One can also use the $u$ density to probe the local compact region of $g(t)$ where one concerns. For example, the collapse or non-collapse of the region near a point $X$ can be detected from the value of the W-functional. If one chooses $u(X,t)$ an approximate delta function centered at $X$ at flow-time $t$, then the more collapse of the region, the more negative the value of $W[g(X),u(X),t]$. However, by the monotonicity eq. (36) of the W-functional during the
Ricci flow, since the W-functional of a certain initial metric of the region is bounded from below at certain initial scales (bounded from above and below), after a certain amount of flow-time, the W-functional must also be bounded from below at all bounded scales, so the local collapsing of the compact spacetime region corresponding to the arbitrary negative value of the W-functional must be ruled out by the monotonicity of it.

This leads to the (4-spacetime generalization of) local non-collapsing theorem of Perelman, which states that if a local compact spacetime region around point \( X \) is unstable and hence tends to shrink, as one approaches the singular finite flow-time \( t \), when the local curvature diverges, collapsing of the spacetime region cannot actually occur at the scales \( r \sim O(\sqrt{t}^{-1}) \), the volume of a unit ball in the compact space and time region or essentially the volume ratio is bounded from below

\[
\frac{V(B_{g(t)}(X, r))}{r^D} \geq \kappa(g, D) > 0. \tag{58}
\]

The local non-collapsing theorem has important physical consequences. Since the inequality \( \frac{\partial W}{\partial t} \geq 0 \) saturates when the spacetime region is a gradient shrinking Ricci soliton eq. (28). Although the volume of the shrinking soliton is shrinking due to the conformal instability, the scale invariant \( W \) entropy has already maximized to be a finite constant value, the local shape of the shrinking soliton does not change, the information of its shape or topology is preserved rather than lost (when \( W \) entropy is arbitrarily negative and hence collapse, then the information is completely lost). The information of its size or volume is not preserved (relative to an observer and depends on the definition of ruler and clock of the observer), but its structure encoded by its shape and topology can always be “zoomed in” by the observer if it does not completely collapse into nothing. More physical speaking, the conformal instability may shrink a local compact spacetime region but will not collapse it into nothing, otherwise the scale-invariant residue information of its local topology will be lost during the Ricci flow. As a consequence, local conformal instability may happen in some places where \( \Lambda(g(X)) > 0 \) for a given general initial spacetime manifolds, and the local Lagrangian is not necessarily positive defined, but the total effective action \( S_{eff}(M^D) = -\log Z = \frac{D}{2} - \lambda \bar{N}(M^D) \) should be positive defined and bounded from below, where the effective action \( \frac{D}{2} - \lambda \bar{N}(M^D) \) is given by QSRF corresponding to Perelman’s partition function \( \tilde{N}(M^D) \) in the Ricci flow. It is easy to verify that it is indeed the case, because the Relative Shannon entropy \( \bar{N}(M^D) \) is bounded above \( \bar{N}(M^D) \leq 0 \) from eq. (29) as is shown in Section-II.

The positive and boundedness of the total effective action \( S_{eff}(M^D) \) ensures the stability of the whole spacetime, which can be seen as a quantum generalization of the classical positive mass theorem in certain sense. In the geometric point of view, starting from an arbitrary initial spacetime manifolds, the Ricci flow as a RG flow gradually deforms and smooths out local irregularities on it, after certain proper treatment of the local neckpinch (“surgeries” by hand or by internal mechanism of the full RG flow) the spacetime will finally flow to a stable and non-collapse spacetime up to a rescaling.

V. DISCUSSIONS AND CONCLUSIONS

In this paper, the framework quantum spacetime reference frame (QSRF) is reviewed. Differing from other routes, the framework provides us an alternative way to generalize a 3-space Ricci flow to a Riemannian or Lorentzian 4-spacetime version and has clear physical meaning. Under the physical interpretation of QSRF, the volume form and density \( u \) defined by eq. (33) in the framework ensure the monotonic functionals induced from the anomaly of QSRF formally the same with Perelman’s standard functionals in a Riemannian or Lorentzian spacetime. The Ricci flow is a gradient flow of the monotonic functionals, the variational structure makes the functionals suitable for studying the stability problem of spacetime.

The conformal instability problem of quantum spacetime is studied in the framework of QSRF and induced spacetime Ricci flow. Instead of naively observing a “wrong sign” in front of the kinetic term of the conformal factor in the path integral of general relativity, the conformal stability of quantum spacetime and gravitation actually depend on the sign of the lowest eigenvalue \( \Lambda(g(X)) \) of the operator \(-\Delta + R\), associated with the F-functional around a local compact (closed and bounded) spacetime region. If \( \Lambda(g(X)) > 0 \) the local compact region is conformally unstable and will tend to shrink its volume (along the flow-time \( t \) but the physical-time), if \( \Lambda(g(X)) \leq 0 \) the local compact region is conformally stable up to a trivial rescaling. Thus a given general spacetime will possibly develop local volume shrinking and local curvature pinching in some places. Although conformal instability may happen in some places of a given general spacetime, the instability will not cause the structure of the local region collapse into nothing, a finite residue information taken by the scale-invariant W-functional of that region is preserved. The residue information of the shrinking local region will not be lost, otherwise the W-entropy will become infinitely negative which is ruled out by the monotonicity of it.

Different from the possible approaches to the conformal instability of the Euclidean general relativity in literature, which try to locally flip the “wrong sign” into a right one in the Lagrangian, the framework of QSRF and induced
Ricci flow allow the local Lagrangian in the functional integral being negative, leading to conformally unstable in certain places of the spacetime, but the total effective action is proved positive defined and bounded below, so the whole spacetime is stable, which generalizes the classical positive mass theorem of spacetime to the quantum level.

The presence of the conformal instability of a quantum spacetime may have important physical consequence, for instance, an inflationary universe at early epoch may be driven by the mechanism of the conformal instability [17], in other words, the spatial inflationary region near the local physical-time origin may be modeled by a gradient shrinking Ricci soliton configuration, leading to a possible quantum theory of early universe different from the textbook inflation models driven by inflaton. The instability may also lead to the formation of black hole as a temporal static shrinking Ricci soliton [18]. The conformal instability tends to shrink the local spacetime volume but keep it non-collapsing, it implies a possible way to test the work is to probe (directly or indirectly) the characteristic spectrum of quantum fluctuations produced from the shrinking soliton-like configurations such as the early universe and black hole, which are worth further studying.

We also notice limitations of the framework. The basic limitation is that it is not clear if the monotonic functionals of Perelman’s type are mathematically well-defined for the Lorentzian 4-spacetime, a rigorous mathematical foundation is still under development. In this paper, based on the proposed physical foundation, i.e. the QSRF approach, the generalization of Ricci flow and its monotonic functionals to 4-spacetime by using the relativistic frame fields theory is formal and direct. In general, ghost states and/or horizon effects may arise in a Lorentzian or indefinite-metric quantum theories, it is not clear if it is well-defined or has richer consequences from the quantum Lorentzian spacetime than the standard Riemannian version, or just as is shown in the paper that the two versions are formally the same due to the fact that they share common definitions of the positive-defined volume forms $\sqrt{|g|}$ and the manifolds density $u$.

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