STABILITY FOR STOCHASTIC MCKEAN-VLASOV EQUATIONS
WITH NON-LIPSCHITZ COEFFICIENTS*

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Abstract. In this paper we consider the stability of a type of stochastic McKean-Vlasov equations with non-Lipschitz coefficients. Firstly, sufficient conditions are given for the exponential stability of the second moments for their solutions in terms of a Lyapunov function. Then we weaken the conditions and furthermore obtain exponentially 2-ultimate boundedness of their solutions. Finally, almost surely asymptotic stability of their solutions is proved.

1. Introduction

Given a complete filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, \infty)}, \mathbb{P})\). Consider the following stochastic McKean-Vlasov equation on \(\mathbb{R}^d\):

\[
\begin{aligned}
X_t &= \xi + \int_0^t b(X_s, \mu_s)ds + \int_0^t \sigma(X_s, \mu_s)dW_s, \\
\mu_s &= \text{the probability distribution of } X_s,
\end{aligned}
\]

(1)

where \(\xi\) is a \(\mathcal{F}_0\)-measurable random variable with \(\mathbb{E}|\xi|^{2p} < \infty\) for any \(p > 1\), \(W_t = (W^1_t, W^2_t, \cdots, W^d_t)\) is a \(\mathcal{F}_t\)-adapted standard \(d\)-dimensional Brownian motion and the coefficients \(b : \mathbb{R}^d \times \mathcal{M}_2(\mathbb{R}^d) \to \mathbb{R}^d\), \(\sigma : \mathbb{R}^d \times \mathcal{M}_2(\mathbb{R}^d) \to \mathbb{R}^d \times \mathbb{R}^d\) are Borel measurable. \((\mathcal{M}_2(\mathbb{R}^d))\) is defined in Section 2.1.

If \(b, \sigma\) are independent of \(\mu_s\), Eq. (1) becomes a usual stochastic differential equation (SDE). Moreover, in recent years, the stability for solutions of SDEs has been studied extensively in the literature. Most of these papers are concerned with exponential stability of \(p\)-th moments for their solutions, exponential stability of sample paths for their solutions and exponential stability, exponentially 2-ultimate boundedness, or almost surely asymptotic stability of their solutions. Let us mention some works. For linear SDEs, Arnold collected a number of results on exponential stability of their solutions in his monograph [1]. For SDEs in infinite dimensional Hilbert spaces, Ichikawa [6] proved the stability of moments and exponential stability of sample paths for their solutions under Lipschitz and linear growth conditions. For SDEs with jumps, Deng-Krstić-Williams [5] studied the almost surely asymptotic stability by using a strong Markov property. Later,
the second named author and Duan [9] offered some general conditions of exponentially 2-ultimate boundedness of solutions for SDEs with jumps. For SDEs with jumps in infinite dimensional Hilbert spaces, Bao, Truman and Yuan [2] discussed the almost surely asymptotic stability for their solutions under local Lipschitz condition but without a linear growth condition.

If \( b, \sigma \) depend on \( \mu_s \), Eq. (1) is called as a stochastic McKean-Vlasov equation (SMVE). And there is few results about the stability of its solution due to its specialty including distributions. Recently, for a semilinear stochastic McKean-Vlasov evolution equation, Govindan and Ahmed [4] investigated the exponential stability for its solution under Lipschitz and linear growth conditions.

In this paper, we study the stability of SMVEs under non-Lipschitz conditions. In [3], we have proved that Eq. (1) has a unique strong solution. Here we continue and consider three types of stability of the strong solution for Eq. (1). Firstly, we give sufficient conditions to prove exponential stability of the second moment in terms of Lyapunov functions. Then by a similar way, it is shown that exponentially 2-ultimate boundedness of its solution hold. Finally, motivated by [2], we take a nonrandom initial condition and prove the almost surely asymptotic stability under more restricted conditions of Lyapunov functions.

The rest of the paper is organized as follows. In Section 2, we recall some basic notations and notions, and give some necessary assumptions. And then we prove exponential stability of the second moment for the strong solution to Eq. (1) in Section 3. In Section 4, the exponentially 2-ultimate boundedness of the strong solution for Eq. (1) is investigated. Finally, the almost surely asymptotic stability of the strong solution for Eq. (1) is proved in Section 5.

The following convention will be used throughout the paper: \( C \) with or without indices will denote different positive constants (depending on the indices) whose values may change from one place to another.

2. THE FRAMEWORK

In the section, we recall some basic notations and notions, and give some necessary concepts and assumptions.

2.1. Notations and notions. In the subsection, we introduce notations and notions used in the sequel.

Let \( C^2_+(\mathbb{R}^d) \) denote the class of twice continuously differentiable nonnegative functions defined on \( \mathbb{R}^d \). Let \( \Gamma \) denote the family of functions \( \gamma : \mathbb{R}_+ \to \mathbb{R}_+ \), which are continuous, strictly increasing, and \( \gamma(0) = 0 \). And \( \Gamma_\infty \) means the family of functions \( \gamma \in \Gamma \) with \( \gamma(x) \to \infty \) as \( x \to \infty \). Let \( \partial_{ij} \) denote the differentiation with respect to the coordinates with corresponding numbers (e.g. \( \partial_{ij}(f) := \frac{\partial^2 f(x)}{\partial x^i \partial x^j} \)). Let \( \mathcal{B}(\mathbb{R}^d) \) be the Borel \( \sigma \)-algebra on \( \mathbb{R}^d \) and \( \mathcal{M}(\mathbb{R}^d) \) be the space of all probability measures defined on \( \mathcal{B}(\mathbb{R}^d) \) carrying the usual topology of weak convergence.

For convenience, we shall use \( | \cdot | \) and \( \| \cdot \| \) for norms of vectors and matrices, respectively. Furthermore, let \( \langle \cdot, \cdot \rangle \) denote the scalar product in \( \mathbb{R}^d \). Let \( A^* \) denote the transpose of the matrix \( A \).
Define the Banach space
\[ C_p(\mathbb{R}^d) := \left\{ \varphi \in C(\mathbb{R}^d), \| \varphi \|_{C_p(\mathbb{R}^d)} = \sup_{x \in \mathbb{R}^d} \frac{|\varphi(x)|}{(1 + |x|)^2} + \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x - y|} < \infty \right\}. \]

Let \( M^*_{\lambda^2}(\mathbb{R}^d) \) be the Banach space of signed measures \( m \) on \( \mathcal{B}(\mathbb{R}^d) \) satisfying
\[ \|m\|_{\lambda^2}^2 := \int_{\mathbb{R}^d} (1 + |x|)^2 |m|(dx) < \infty, \]
where \( |m| = m^+ + m^- \) and \( m = m^+ - m^- \) is the Jordan decomposition of \( m \). Let \( M_{\lambda^2}(\mathbb{R}^d) = M^*_{\lambda^2}(\mathbb{R}^d) \cap M(\mathbb{R}^d) \) be the set of probability measures on \( \mathcal{B}(\mathbb{R}^d) \). We put on \( M_{\lambda^2}(\mathbb{R}^d) \) a topology induced by the following metric:
\[ \rho(\mu, \nu) := \sup_{\|\varphi\|_{C_p(\mathbb{R}^d)} \leq 1} \left| \int_{\mathbb{R}^d} \varphi(x) \mu(dx) - \int_{\mathbb{R}^d} \varphi(x) \nu(dx) \right|. \]

Then \( (M_{\lambda^2}(\mathbb{R}^d), \rho) \) is a complete metric space.

2.2. Some assumptions. In the subsection, we give out some assumptions:

(H.1.1) The functions \( b, \sigma \) are continuous in \( (x, \mu) \) and satisfy for \( (x, \mu) \in \mathbb{R}^d \times M_{\lambda^2}(\mathbb{R}^d) \)
\[ |b(x, \mu)|^2 + \|\sigma(x, \mu)\|^2 \leq L_1 (1 + |x|^2 + \|\mu\|_{\lambda^2}^2), \]
where \( L_1 > 0 \) is a constant.

(H.1.2) The functions \( b, \sigma \) satisfy for \( (x_1, \mu_1), (x_2, \mu_2) \in \mathbb{R}^d \times M_{\lambda^2}(\mathbb{R}^d) \)
\[ 2\langle x_1 - x_2, b(x_1, \mu_1) - b(x_2, \mu_2) \rangle + \|\sigma(x_1, \mu_1) - \sigma(x_2, \mu_2)\|^2 \leq L_2 \left( \kappa(|x_1 - x_2|^2) + \rho^2(\mu_1, \mu_2) \right), \]
where \( L_2 > 0 \) is a constant, and \( \kappa(x) \) is a positive, strictly increasing, continuous concave function and satisfies \( \kappa(0) = 0, \int_0^1 \frac{1}{\kappa(x)+x}dx = \infty \).

By [3, Theorem 3.1], we know that Eq. (11) has a unique strong solution denoted as \( X_t \) under (H.1.1)-(H.1.2). And then we assume some other conditions to prove exponential stability of the second moment for \( X_t \).

(H.2.1) There exists a function \( v : \mathbb{R}^d \to \mathbb{R} \) satisfying
(i) \( v \in C^2_+(\mathbb{R}^d) \),
(ii) \( \mathcal{L}v(x) + \alpha v(x) \leq 0 \), where \( \alpha > 0 \) is a constant and
\[ \mathcal{L}v(x) = b^i(x, \mu) \partial_i v(x) + \frac{1}{2} (\sigma^i(x, \mu) \sigma(x, \mu))^{ij} \partial_j^2 v(x), \]
(iii) \( a_1 |x|^2 \leq v(x) \leq a_2 |x|^2 \), where \( a_1, a_2 > 0 \) are two constants.

Here and hereafter we use the convention that the repeated indices stand for the summation. In the following, we weaken (H.2.1) to show the exponentially 2-ultimate boundedness of \( X_t \).

(H.2.2) There exists a function \( v : \mathbb{R}^d \to \mathbb{R} \) satisfying
(i) \( v \in C^2(\mathbb{R}^d) \),
(ii) \( \mathcal{L}v(x) + \alpha v(x) \leq M_1 \),
(iii) \( a_1 |x|^2 - M_2 \leq v(x) \leq a_2 |x|^2 + L_3 \),
where \( M_1, M_2, M_3 \geq 0 \) are constants.

To obtain the almost surely asymptotic stability of \( X_t \), we strengthen (H.2.1).
(H2.3) There exists a function \(v : \mathbb{R}^d \to \mathbb{R}\) satisfying

(i) \(v \in C^2_+ (\mathbb{R}^d)\),
(ii) \(\mathcal{L}v(x) + \alpha v(x) \leq 0\),
(iii) \(\gamma_1(|x|) \leq v(x) \leq \gamma_2(|x|)\), where \(\gamma_i \in \Gamma_\infty (i = 1, 2)\).

3. Exponential stability of the second moment

In the section, we study exponential stability of the second moment for the strong solution to Eq.(1).

Theorem 3.1. Assume that \((H_{1.1})-(H_{1.2})\) and \((H_{2.1})\) hold. Then \(X_t\) satisfies

\[
\mathbb{E} |X_t|^2 \leq \frac{a_2}{a_1} e^{-\alpha t} \mathbb{E} |\xi|^2, \quad t \geq 0.
\]

Proof. Applying the Itô formula to \(e^{\alpha t} v(X_t)\), we have

\[
e^{\alpha t} v(X_t) - v(\xi) = \int_0^t \alpha e^{\alpha s} v(X_s)ds + \int_0^t e^{\alpha s} b^i(X_s, \mu_s) \partial_i v(X_s)ds
\]

\[
+ \int_0^t e^{\alpha s} \partial_i v(X_s) \sigma^{ij}(X_s, \mu_s)dW^j_s
\]

\[
+ \frac{1}{2} \int_0^t e^{\alpha s} (\sigma^*(X_s, \mu_s)\sigma(X_s, \mu_s))^{ij} \partial_{2ij}^2 v(X_s)ds
\]

\[
= \int_0^t \alpha v(X_s) + b^i(X_s, \mu_s) \partial_i v(X_s)
\]

\[
+ \frac{1}{2} (\sigma^*(X_s, \mu_s)\sigma(X_s, \mu_s))^{ij} \partial_{2ij}^2 v(X_s)ds
\]

\[
+ \int_0^t e^{\alpha s} \partial_i v(X_s) \sigma^{ij} (X_s, \mu_s)dW^j_s. \tag{3}
\]

Then it follows from (ii) in \((H_{2.1})\) that

\[
e^{\alpha t} v(X_t) - v(\xi) \leq \int_0^t e^{\alpha s} \partial_i v(X_s) \sigma^{ij} (X_s, \mu_s)dW^j_s.
\]

By taking the expectation on two sides, one can get that

\[
e^{\alpha t} \mathbb{E} v(X_t) - \mathbb{E} v(\xi) \leq 0,
\]

which yields

\[
\mathbb{E} v(X_t) \leq e^{-\alpha t} \mathbb{E} v(\xi).
\]

Moreover, by (iii) in \((H_{2.1})\) it holds that

\[
 a_1 \mathbb{E} |X_t|^2 \leq \mathbb{E} v(X_t) \leq e^{-\alpha t} \mathbb{E} v(\xi) \leq a_2 e^{-\alpha t} \mathbb{E} |\xi|^2.
\]

Thus, we obtain

\[
\mathbb{E} |X_t|^2 \leq \frac{a_2}{a_1} e^{-\alpha t} \mathbb{E} |\xi|^2.
\]

The proof is completed. \(\square\)
4. Exponentially 2-ultimate boundedness

In the section, we study exponentially 2-ultimate boundedness of the solution of Eq. (1).

First of all, we introduce the concept of exponentially 2-ultimate boundedness.

**Definition 4.1.** If there exist positive constants $K$, $\beta$, $M$ such that

$$
\mathbb{E} |X_t|^2 \leq K e^{-\beta t} \mathbb{E} |\xi|^2 + M, \quad t \geq 0,
$$

then the solution $X_t$ for Eq. (1) is called exponentially 2-ultimately bounded.

**Theorem 4.2.** Suppose that $(H_{1.1})-(H_{1.2})$ and $(H_{2.2})$ hold. Then $X_t$ is exponentially 2-ultimately bounded, i.e.

$$
\mathbb{E} |X_t|^2 \leq \frac{a_2}{a_1} e^{-\alpha t} \mathbb{E} |\xi|^2 + \frac{\alpha (M_2 + M_3) + M_1}{a a_1}, \quad t \geq 0.
$$

Since the proof of the above theorem is similar to that in Theorem 3.1, we omit it.

5. Almost surely asymptotic stability

In the section, we require that $\xi = x_0$ is non-random and study almost surely asymptotic stability of the strong solution for Eq. (1).

First of all, we introduce the concept of almost surely asymptotic stability.

**Definition 5.1.** The solution of Eq. (1) is said to be almost surely asymptotically stable if for all $x_0 \in \mathbb{R}^d$, it holds that

$$
\mathbb{P} \left\{ \lim_{t \to \infty} |X_t| = 0 \right\} = 1.
$$

**Theorem 5.2.** Assume that $(H_{1.1})-(H_{1.2})$ and $(H_{2.3})$ hold. Then $X_t$ is almost surely asymptotically stable.

**Proof.** To prove that for all $x_0 \in \mathbb{R}^d$,

$$
\mathbb{P} \left\{ \lim_{t \to \infty} |X_t| = 0 \right\} = 1,
$$

by (iii) of $(H_{2.3})$ we only need to show that for all $x_0 \in \mathbb{R}^d$,

$$
\mathbb{P} \left\{ \lim_{t \to \infty} v(X_t) = 0 \right\} = 1. \quad (4)
$$

Next, we study $v(X_t)$ for $t \geq 0$. Applying the Itô formula to $v(X_t)$, one can obtain

$$
v(X_t) = v(X_s) + \int_s^t b^i(X_u, \mu_u) \partial_i v(X_u) du + \int_s^t \frac{1}{2} (\sigma^{ij} (X_u, \mu_u) \sigma(X_u, \mu_u))^{ij} \partial^2_{ij} v(X_u) du
$$

$$
+ \int_s^t \partial_i v(X_u) \sigma^{ij}(X_u, \mu_u) dW^j_u, \quad 0 \leq s < t. \quad (5)
$$

Thus by (ii) in $(H_{2.3})$, it holds that

$$
v(X_t) \leq v(X_s) - \alpha \int_s^t v(X_u) du + \int_s^t \partial_i v(X_u) \sigma^{ij}(X_u, \mu_u) dW^j_u
$$

$$
\leq v(X_s) + \int_s^t \partial_i v(X_u) \sigma^{ij}(X_u, \mu_u) dW^j_u, \quad 0 \leq s < t.
$$

5
Set $\mathcal{G}_t := \sigma(\mathcal{F}_t^W \cup \mathcal{N})$ for $t \geq 0$, where $\{\mathcal{F}_t^W\}_{t \geq 0}$ is the $\sigma$-algebra generated by $W$ and $\mathcal{N}$ is the collection of $\mathbb{P}$ null sets. And then we have that $v(X)$ is adapted to $\{\mathcal{G}_t\}_{t \geq 0}$ when $X$ is the strong solution of Eq. ($\mathbb{H}_23$), and $\int_0^t \partial_t v(X_u) \sigma^{ij}(X_u, \mu_u) dW_u^j$ is a martingale with respect to $\{\mathcal{G}_t\}_{t \geq 0}$. Therefore, we obtain

$$E(v(X_t) \mid \mathcal{G}_s) \leq E(v(X_s) \mid \mathcal{G}_s) + E\left(\int_s^t \partial_t v(X_u) \sigma^{ij}(X_u, \mu_u) dW_u^j \mid \mathcal{G}_s\right) \leq v(X_s),$$

which implies that $v(X)$ is a supermartingale with respect to $\{\mathcal{G}_t\}_{t \geq 0}$. If $s = 0$ and $X_0 = x_0 = 0$, by (iii) of ($\mathbb{H}_23$) and the supermartingale property of $v(X)$, it holds that $v(X_t) = 0$ for $t \geq 0$. Thus, ($4$) is right.

In the following, we assume $x_0 \neq 0$ and prove that ($4$) is right. So, set

$$A_1 := \{\omega : \liminf_{t \to \infty} v(X_t) > 0\},$$

$$A_2 := \{\omega : \limsup_{t \to \infty} v(X_t) > 0\},$$

and then we just need to prove $\mathbb{P}(A_1) = \mathbb{P}(A_2) = 0$. Taking the expectation on two sides of ($3$), one can have that

$$E(v(X_t)) = v(x_0) + E\left(\int_0^t b^i(X_s, \mu_s) \partial_i v(X_s) ds + \int_0^t \frac{1}{2} (\sigma^* (X_s, \mu_s) \sigma(X_s, \mu_s))^{ij} \partial^2_{ij} v(X_s) ds\right).$$

Thus, (ii) and (iii) in ($\mathbb{H}_23$) admits us to obtain that

$$0 \leq E\gamma_1(\{X_t\}) \leq E(v(X_t)) \leq v(x_0) - \alpha \int_0^t E v(X_s) ds, \quad t \geq 0,$$

and furthermore

$$\int_0^t E v(X_s) ds \leq \frac{v(x_0)}{\alpha}, \quad t \geq 0.$$

Let $t \to \infty$, we obtain that

$$\int_0^\infty E v(X_s) ds \leq \frac{v(x_0)}{\alpha} < \infty. \quad (6)$$

Hence, by the Fatou lemma it holds that

$$\liminf_{t \to \infty} v(X_t) = 0, \quad a.s.$$  

that is, $\mathbb{P}(A_1) = 0$.

Next, we prove that $\mathbb{P}(A_2) = 0$. It follows from the simple calculation that

$$\mathbb{P}(A_2) = \mathbb{P}(A_2 A^c_1) + \mathbb{P}(A_2 A_1) = \mathbb{P}(A_2 A^c_1),$$

where $A^c_1$ stands for the complementary set of $A_1$. Therefore, we only need to prove $\mathbb{P}(A_2 A^c_1) = 0$. Note that $A_2 A^c_1 = \left\{\omega : \liminf_{t \to \infty} v(X_t) = 0, \limsup_{t \to \infty} v(X_t) > 0\right\}$. Suppose that $\mathbb{P}(A_2 A^c_1) \neq 0$, there exists $\varepsilon_1, \varepsilon_2 > 0$ such that

$$\mathbb{P} \{v(X) \text{ crosses from below } \varepsilon_1 \text{ to above } 2\varepsilon_1 \text{ and back infinitely many times}\} \geq \varepsilon_2. \quad (7)$$
Set \( \tau_n := \inf\{ t \geq 0, |X_t| \geq n \} \). Now applying the Itô formula to \( |X_{s \wedge \tau_n} - x_0|^2 \) for \( s \geq 0 \), one can get that

\[
|X_{s \wedge \tau_n} - x_0|^2 = 2 \int_0^{s \wedge \tau_n} (X^i_u - x^i_0)b^i(X_u, \mu_u)du + \int_0^{s \wedge \tau_n} (\sigma^*(X_u, \mu_u)\sigma(X_u, \mu_u))^{ij} du
+ 2 \int_0^{s \wedge \tau_n} (X^i_u - x^i_0)\sigma^{ij}(X_u, \mu_u)dW^j_u
\leq 2 \int_0^{s \wedge \tau_n} |X_u - x_0| |b(X_u, \mu_u)| du + \int_0^{s \wedge \tau_n} \|\sigma(X_u, \mu_u)\|^2 du
+ 2 \int_0^{s \wedge \tau_n} (X^i_u - x^i_0)\sigma^{ij}(X_u, \mu_u)dW^j_u.
\]

By (2) and the BDG inequality, we can derive that

\[
\mathbb{E}\left( \sup_{0 \leq s \leq t} |X_{s \wedge \tau_n} - x_0|^2 \right) \leq \mathbb{E} \int_0^{t \wedge \tau_n} |X_u - x_0|^2 du + L_1 \mathbb{E} \int_0^{t \wedge \tau_n} (1 + |X_u|^2 + \|\mu_u\|^2_{L^2}) du
+ C \mathbb{E} \left( \int_0^{t \wedge \tau_n} |X_u - x_0|^2 \|\sigma(X_u, \mu_u)\|^2 du \right)^{1/2}
\leq \mathbb{E} \int_0^{t \wedge \tau_n} |X_u - x_0|^2 du + L_1 \mathbb{E} \int_0^{t \wedge \tau_n} (1 + |X_u|^2 + \|\mu_u\|^2_{L^2}) du
+ \frac{1}{4} \mathbb{E}\left( \sup_{0 \leq s \leq t} |X_{s \wedge \tau_n} - x_0|^2 \right) + C \mathbb{E} \int_0^{t \wedge \tau_n} \|\sigma(X_u, \mu_u)\|^2 du,
\]

which yields

\[
\mathbb{E}\left( \sup_{0 \leq s \leq t} |X_{s \wedge \tau_n} - x_0|^2 \right) \leq C \mathbb{E} \int_0^{t \wedge \tau_n} |X_u - x_0|^2 du + C \mathbb{E} \int_0^{t \wedge \tau_n} (1 + |X_u|^2 + \|\mu_u\|^2_{L^2}) du
\leq C \mathbb{E} \int_0^{t \wedge \tau_n} |X_u - x_0|^2 du + C \mathbb{E} \int_0^{t \wedge \tau_n} \left( 1 + |X_u|^2 + \mathbb{E}(1 + |X_u|^2) \right) du.
\]

It follows from the boundedness of \( X_t \) that

\[
\mathbb{E}\left( \sup_{0 \leq s \leq t} |X_{s \wedge \tau_n} - x_0|^2 \right) \leq C \mathbb{E}(t \wedge \tau_n) \leq Ct,
\]

where \( C > 0 \) is depending on \( L_1, x_0 \) and \( n \). Then for any \( \lambda > 0 \), Chebyshev’s inequality gives that

\[
P\left\{ \sup_{0 \leq s \leq t} |X_{s \wedge \tau_n} - x_0| > \lambda \right\} \leq \frac{Ct}{\lambda^2}.
\]

(8)
Besides, for the supermartingale \( v(X) \), by [3, Theorem 3.6, P.13], it holds that for any \( \delta(\cdot) \in \Gamma_\infty \)
\[
\mathbb{P} \left\{ \sup_{0 \leq s \leq t} v(X_s) \geq \delta \left( v(x_0) \right) \right\} \leq \frac{v(x_0)}{\delta(v(x_0))}.
\]

On one hand, by (iii) of \((H_{2.3})\), we know that \( \sup_{0 \leq s \leq t} |X_s| \geq \eta(|x_0|) \) implies \( \sup_{0 \leq s \leq t} v(X_s) \geq \delta(v(x_0)) \), where \( \eta(\cdot) = \gamma_1^{-1} \circ \delta \circ \gamma_2(\cdot) \) and \( \gamma_1^{-1} \) is the inverse function of \( \gamma_1 \). On the other hand, for any \( \epsilon_1 > 0 \), we can choose \( \delta(\cdot) \in \Gamma_\infty \) such that \( \frac{v(x_0)}{\delta(v(x_0))} \leq \epsilon_1 \). Thus, we obtain
\[
\mathbb{P} \left\{ \sup_{0 \leq s \leq t} |X_s| \geq \eta(|x_0|) \right\} \leq \epsilon_1, \quad \forall t \geq 0.
\]  

Next, since \( v(x) \) is continuous in \( \mathbb{R}^d \), it must be uniformly continuous in \( B := \{ x \in \mathbb{R}^d : |x| < \eta(|x_0|) \} \). Therefore, for any \( \epsilon_2 > 0 \), we can choose a function \( \gamma \in \Gamma \) such that for any \( |x - y| < \gamma(\epsilon_2) \),
\[
|v(x) - v(y)| \leq \epsilon_2,
\]
where \( x, y \in B \). Thus, combining [3] with (10), one can obtain
\[
\mathbb{P} \left\{ \sup_{0 \leq s \leq t} |v(X_s) - v(x_0)| > \epsilon_2 \right\} = \mathbb{P} \left\{ \sup_{0 \leq s \leq t} |v(X_s) - v(x_0)| > \epsilon_2, \sup_{0 \leq s \leq t} |X_s| < \eta(|x_0|) \right\} + \mathbb{P} \left\{ \sup_{0 \leq s \leq t} |v(X_s) - v(x_0)| > \epsilon_2, \sup_{0 \leq s \leq t} |X_s| \geq \eta(|x_0|) \right\} \leq \mathbb{P} \left\{ \sup_{0 \leq s \leq t} |X_s - x_0| > \gamma(\epsilon_2), \sup_{0 \leq s \leq t} |X_s| < \eta(|x_0|) \right\} + \mathbb{P} \left\{ \sup_{0 \leq s \leq t} |X_s| \geq \eta(|x_0|) \right\} \leq \mathbb{P} \left\{ \sup_{0 \leq s \leq t} |X_s \wedge \tau_{\eta(|x_0|)} - x_0| > \gamma(\epsilon_2) \right\} + \mathbb{P} \left\{ \sup_{0 \leq s \leq t} |X_s| \geq \eta(|x_0|) \right\} \leq \frac{Ct}{\gamma^2(\epsilon_2)} + \epsilon_1.
\]

Taking \( t^* \) such that \( \frac{Ct^*}{\gamma^2(\epsilon_2)} \leq \frac{1}{4} \) and \( \epsilon_1 \leq \frac{1}{2} \), we get that
\[
\mathbb{P} \left\{ \sup_{0 \leq s \leq t^*} |v(X_s) - v(x_0)| > \epsilon_2 \right\} \leq \frac{3}{4},
\]
that is,
\[
\mathbb{P} \left\{ \sup_{0 \leq s \leq t^*} |v(X_s) - v(x_0)| \leq \epsilon_2 \right\} \geq \frac{1}{4}.
\]

Now, set
\[
S_1 := \inf\{t \geq 0 : v(X_t) \leq \epsilon_1\},
\]
\[
S_2 := \inf\{t \geq \tau_1 : v(X_t) \geq 2\epsilon_1\},
\]
Thus,

\[ S_3 : = \inf\{t \geq \tau_2 : v(X_t) < \varepsilon_1\}, \]

\[ \vdots \]

\[ S_{2k} : = \inf\{t \geq \tau_{2k-1} : v(X_t) > 2\varepsilon_1\}, \]

\[ S_{2k+1} : = \inf\{t \geq \tau_{2k} : v(X_t) < \varepsilon_1\}, \quad k \in \mathbb{N}^+, \]

and then \( \{S_i\}_{i=1}^\infty \) is a sequence of stopping times. Thus, by (6) it holds that

\[ \infty > \mathbb{E}\left(\int_0^\infty v(X_s)ds\right) \geq \sum_{k=1}^\infty \mathbb{E}\left[I_{\{S_{2k} < \tau_n\}}\int_{S_{2k}}^{S_{2k+1}} v(X_s)ds\right] \]

\[ \geq \varepsilon_1 \sum_{k=1}^\infty \mathbb{E}\left[I_{\{S_{2k} < \tau_n\}}(S_{2k+1} - S_{2k})\right] \]

\[ = \varepsilon_1 \sum_{k=1}^\infty \mathbb{E}\left[I_{\{S_{2k} < \tau_n\}}\mathbb{E}(S_{2k+1} - S_{2k} | \mathcal{G}_{S_{2k}})\right]. \quad (12) \]

We estimate \( \mathbb{E}(S_{2k+1} - S_{2k} | \mathcal{G}_{S_{2k}}) \) on \( \{S_{2k} < \tau_n\} \). First of all, by the similar deduction to that of [7] Theorem 4.20, P.322, one can get that the strong solution \( X_t \) of Eq. (11) have the strong Markov property for any \( \{\mathcal{G}_t\}_{t \geq 0} \)-stopping time. Setting \( \varepsilon_2 = \frac{\varepsilon_1}{2} \) and following the argument of Deng and Williams in [5, P.1241], we obtain that on \( \{S_{2k} < \tau_n\} \)

\[ \mathbb{E}(S_{2k+1} - S_{2k} | \mathcal{G}_{S_{2k}}) \geq \mathbb{E}\left((S_{2k+1} - S_{2k}) I_{\{\sup_{0 \leq s \leq t^*} |v(X_s) - v(X_0)| \leq \varepsilon_1\}} | \mathcal{G}_{S_{2k}}\right) \]

\[ \geq t^* \mathbb{P}\left\{\sup_{0 \leq s \leq t^*} |v(X_s) - v(X_0)| \leq \varepsilon_1 \right\} \]

\[ = t^* \mathbb{P}\left\{\sup_{0 \leq s \leq t^*} |v(X_s) - v(x_0)| \leq \varepsilon_2 \right\} \]

\[ \geq \frac{t^*}{4}, \]

where \( \tilde{X} := X_{\cdot+S_{2k}} \) and (11) is used in the last inequality. Therefore, (12) gives that

\[ \frac{t^*}{4} \varepsilon_1 \sum_{k=1}^\infty \mathbb{P}\{S_{2k} < \tau_n\} < \infty. \]

It follows from the Borel-Cantelli lemma that

\[ \mathbb{P}\{S_{2k} < \tau_n \text{ for infinitely many } k\} = 0. \]

Note that

\[ \{S_{2k} < \tau_n \text{ for infinitely many } k\} = \{S_{2k} < \tau_n \text{ for infinitely many } k \text{ and } \tau_n = \infty\} \]

\[ \cup \{S_{2k} < \tau_n \text{ for infinitely many } k \text{ and } \tau_n < \infty\}. \]

Thus,

\[ \mathbb{P}\{S_{2k} < \tau_n \text{ for infinitely many } k \text{ and } \tau_n = \infty\} = 0. \]

Note that \( \sup_{t \geq 0} \mathbb{E}(-v(X_t))^+ = 0 \), by [7] Theorem 3.15, P.17 we get that \( v(X_\infty) = \lim_{t \to \infty} v(X_t) \) exists and \( \mathbb{E}v(X_\infty) < \infty \). Therefore it follows from the supermartingale
inequality that
\[ P \left\{ \sup_{t \geq 0} v(X_t) \geq \gamma_2(n) \right\} \leq \mathbb{E} \sup_{t \geq 0} v(X_t) \leq \frac{v(x_0)}{\gamma_2(n)}, \]
which gives
\[ P\{\tau_n = \infty\} = P\left\{ \sup_{t \geq 0} |X_t| < n \right\} \geq P\left\{ \sup_{t \geq 0} v(X_t) < \gamma_2(n) \right\} \geq 1 - \frac{v(x_0)}{\gamma_2(n)}. \]
So, we have \( P\{\tau_n = \infty\} \to 1 \) as \( n \to \infty \), and furthermore
\[ P\{S_{2k} < \infty \text{ for infinitely many } k\} = 0, \]
which is a contradiction of (7). This completes the proof. \( \square \)

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