Reduced bispectrum seeded by helical primordial magnetic fields

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Received January 28, 2017
Revised May 27, 2017
Accepted May 30, 2017
Published June 9, 2017

Abstract. In this paper, we investigate the effects of helical primordial magnetic fields (PMFs) on the cosmic microwave background (CMB) reduced bispectrum. We derive the full three-point statistics of helical magnetic fields and numerically calculate the even contribution in the collinear configuration. We then numerically compute the CMB reduced bispectrum induced by passive and compensated PMF modes on large angular scales. There is a negative signal on the bispectrum due to the helical terms of the fields and we also observe that the biggest contribution to the bispectrum comes from the non-zero IR cut-off for causal fields, unlike the two-point correlation case. For negative spectral indices, the reduced bispectrum is enhanced by the passive modes. This gives a lower value of the upper limit for the mean amplitude of the magnetic field on a given characteristic scale. However, high values of IR cut-off in the bispectrum, and the helical terms of the magnetic field relaxes this bound. This demonstrates the importance of the IR cut-off and helicity in the study of the nature of PMFs from CMB observations.

Keywords: primordial magnetic fields, non-gaussianity

ArXiv ePrint: 1511.02991
1 Introduction

Recent observational evidence of intergalactic magnetic fields from γ-ray observations of blazars and constraints imposed by CMB power spectrum suggest the existence of magnetic fields created in the early Universe [1–5]. Interesting theoretical models have been proposed to explain generation processes which gave rise to this likely primordial field. Some of them are originated by causal requirements via cosmological phase transitions or by non-linear evolution of primordial density perturbations [6–8]; and the other ones could be generated during inflation, with them being the most appealing models due to the production of large-scale magnetic fields beyond the horizon scale [9–29]. Moreover, several additional attempts from string theory or extra dimensions have been done to generate the seed magnetic fields needed to be coherent on cosmological scales [30–33]. One way to find out the process which gave rise to this field and determine its main features is by making theoretical predictions about the signatures in the CMB from primordial magnetic fields (PMFs) [34–37, 43]. Indeed, some authors have shown that vector modes dominate all the temperature and polarization anisotropies for higher multipolar numbers while the scalar mode contribution is larger for lower ℓ [38, 44, 45]. Other effect of PMFs on the CMB comes from the non-Gaussian (NG) signals because its contribution to energy-momentum tensor is quadratic in the fields. The relevant NG signal from PMFs with an amplitude similar to the curvature turns out to be
a feature that is important for constraining mean-field amplitude depending on what signal is induced by passive or compensated modes. Studies of NG signals via bispectrum on CMB have found upper limits of PMFs around $2 - 22$ nG derived from scalar magnetic modes, and $3.2 - 10$ nG derived from vector-tensor magnetic modes smoothed on a scale of $1$ Mpc [47–55]. The Planck Collaboration also reported limits on the amplitude of $B_{1Mpc} < 3nG$ for $n_B = -2.9$; $B_{1Mpc} < 0.07nG$ for $n_B = -2$; and $B_{1Mpc} < 0.04nG$ for $n_B = 2$ from compensated modes; and $B_{1Mpc} < 4.5nG$ for $n_B = -2.9$ from the passive-scalar mode [56]. PMFs have also been constrained by the POLARBEAR experiment, where they reported that the PMF amplitude from the two-point correlation functions is less than $3.9$ nG at the 95% confidence level [57]. On the other hand, distinct signatures on the parity-odd CMB cross correlations would carry valuable clues about a primordial magnetic helicity. In fact, helical contribution in the field (in the perfect conductivity limit) has been widely studied because it produces efficient transference of power from smaller to larger scales, and thus be able to explain the actual observed magnetic fields [6, 58]. Further, observational evidence of helical primordial magnetic fields would offer a window for probing physics beyond the standard models of particle physics, particularly processes of parity violation in the early Universe [59–61].

The study of helical fields via NG will give a deeper understanding of the magnetic field generation model and help us to strengthen the constraints of PMF amplitude. Thus, the main goal of this paper is to investigate the effects of helical PMFs on the CMB bispectrum. Following previous formulation for calculating the bispectrum, we have found signals that arise by considering a minimal cut-off and we observed that local-type shape contains the biggest contribution to the bispectrum. This paper is organized as follows. Section 2 describes the statistical properties of PMFs. In section 3 we define the magnetic bispectrum and in section 4 through the numerical computation, we solve exactly the bispectrum for a collinear configuration and observe some important signals. Section 5 is defined the reduced bispectrum and in section 6 we present numerical results of the primary bispectrum sourced by helical PMFs. Section 7 is devoted to further discussion and conclusions.

2 Statistical aspects of primordial magnetic fields

We consider a stochastic primordial magnetic field (PMF) generated in the very early Universe which could have been produced during inflation (non-causal field) or after inflation (causal field). This field acts like a source of fluctuations on the CMB anisotropies under a FLRW background Universe described by the metric

$$ds^2 = a^2(t)(-dt^2 + \delta_{ij}dx^i dx^j)$$

with $t$ being the conformal time. The PMF power spectrum which is defined as the Fourier transform of the two point correlation can be written as

$$\langle B_i(k)B_j(k') \rangle = (2\pi)^3\delta^3(k - k') \left( P_{ij}(k)P_B(k) + i\epsilon_{ijk}\hat{k}^i P_H(k) \right),$$

where $P_{ij}(k) = \delta_{ij} - \hat{k}_i\hat{k}_j$ is a projector onto the transverse plane, $\epsilon_{ijk}$ is the 3D Levi-Civita tensor and, $P_B(k)$, $P_H(k)$ are the symmetric/anti-symmetric parts of the power spectrum

$^1$This projector has the property $P_{ij}\hat{k}^i = 0$ with $\hat{k}^i = \frac{k^i}{r}$ and we use the Fourier transform notation $B_i(k) = \int d^3x \exp^{ikx} B_i(x)$. 

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which represent the magnetic field energy density and absolute value of the kinetic helicity respectively [38]

\[ \langle B_i(k)B^*_i(k') \rangle = 2(2\pi)^3\delta^3(k-k')P_B(k), \] (2.3)
\[ -i(\epsilon_{ijk}k^IB_i(k)B^*_j(k')) = 2(2\pi)^3\delta^3(k-k')P_H(k). \] (2.4)

We assume that power spectrum scales as a simple power law [37]

\[ P_B(k) = A_Bk^{n_B}, \quad \text{with} \quad A_B = \frac{B^2_\lambda 2\pi^2\lambda^{n_B+3}}{\Gamma(\frac{n_B+3}{2})}, \] (2.5)
\[ P_H(k) = A_Hk^{n_H}, \quad \text{with} \quad A_H = \frac{H^2_\lambda 2\pi^2\lambda^{n_H+3}}{\Gamma(\frac{n_H+4}{2})}, \] (2.6)

with \( \Gamma \) being the Gamma function. In order to avoid infrared divergences (when we do not consider an infrared cutoff), \( n_B > -3, n_H > -4 \). Also, \( B_\lambda, H_\lambda \) are the comoving PMF strength and magnetic helicity smoothing over a Gaussian sphere of comoving radius \( \lambda \) [62, 63]. The more general case of the power spectrum for magnetic fields can be studied if we assume that it is defined for \( k_m \leq k \leq k_D \), where \( k_D \) is an ultraviolet cut-off corresponding to damping scale where the field is suppressed on small scales [3] as \( k_D \sim \mathcal{O}(10)\text{Mpc}^{-1} \) and we also consider a possible dependence on an infrared cut-off, \( k_m \). Given the Schwarz inequality [62],

\[ \lim_{k'\to k} \langle B(k) \cdot B^*(k') \rangle \geq | \lim_{k'\to k} \langle (\hat{k} \times B(k)) \cdot B^*(k') \rangle |, \] (2.7)

an additional constraint is found for these fields

\[ |A_H| \leq A_Bk^{n_B-n_H}. \] (2.8)

In the case where \( A_H = A_B \) and \( n_B = n_H \) we define the maximal helicity condition. We will also use the procedure in [37] to parametrize the infrared cut-off by a single constant parameter \( \alpha \),

\[ k_m = \alpha k_D, \quad 0 \leq \alpha < 1 \] (2.9)

which in the case of inflationary scenarios would correspond to the wave mode that exits the horizon at inflation epoch and for causal modes would be important when this scale is larger than the wavenumber of interest (as claimed by Kim et al. [42]). Thus, this infrared cut-off would be important in order to constrain PMF parameters and magnetogenesis models [37, 39–42]. Equation (2.9) gives only an useful mathematical representation to constrain these cut-off values via cosmological datasets (for this case, the parameter space would be given by \((\alpha, k_D, B_\lambda, H_\lambda, n_H, n_B))\), and therefore we want to point out that latter expression does not state any physical relation between both wave numbers. In [39, 40], they showed constraints on the maximum wave number \( k_D \) as a function of \( n_B \) via big bang nucleosynthesis (BBN), and they considered the maximum and minimum wave numbers as independent parameters. In fact, we have found out that the integration scheme used for calculating the spectrum and bispectrum of PMFs is exactly the same if we parametrize \( k_m \) as seen in (2.9), or if we consider \((k_m, k_D, B_\lambda, H_\lambda, n_H, n_B))\) as independent parameters. Thus the inclusion of \( k_m \) is done only for studying at a phenomenological level its effects on the CMB bispectrum. On the other hand, the equations for the adimensional energy density of magnetic field and
spatial part of the electromagnetic energy momentum tensor respectively written in Fourier space are given as

\[ \rho_B(k1) = \frac{1}{8\pi\rho_{\gamma,0}} \int \frac{d^3p}{(2\pi)^3} B_l(p)B^l(k1 - p), \]

\[ \Pi_{ij}(k1) = \frac{1}{4\pi\rho_{\gamma,0}} \int \frac{d^3p}{(2\pi)^3} \left[ \frac{\delta_{ij}}{2} B_l(p)B^l(k1 - p) - B_i(p)B_j(k1 - p) \right], \tag{2.10} \]

where in the last expressions we are considering high conductivity so, the electric field is suppressed; the magnetic field evolves as \( B^2 \sim a^{-4}(t) \), and therefore we can express each component of the energy momentum tensor in terms of photon energy density \( \rho_\gamma = \rho_{\gamma,0}a^{-4} \), with \( \rho_{\gamma,0} \) being its present value.\(^2\)

Given that spatial electromagnetic energy momentum tensor is symmetric, we can decompose this tensor into the two scalar \( (\rho_B, \Pi^{(S)}) \), one vector \( (\Pi^{(V)}_i) \) and one tensor \( (\Pi^{(T)}_{ij}) \) components as

\[ \Pi_{ij} = \frac{1}{3} \delta_{ij} \rho_B + \left( \hat{k}_i \hat{k}_j - \frac{1}{3} \delta_{ij} \right) \Pi^{(S)} + (\hat{k}_i \Pi^{(V)}_j + \hat{k}_j \Pi^{(V)}_i) + \Pi^{(T)}_{ij} \tag{2.11} \]

which obey to \( \hat{k}^i \Pi^{(V)}_i = \hat{k}^i \Pi^{(T)}_i = \Pi^{(T)}_{ii} = 0 \) \([37, 64, 65]\). The components of this tensor are recovered by applying projector operators defined as

\[ \rho_B = \delta^{ij} \Pi_{ij} \]

\[ \Pi^{(S)} = \left( \delta^{ij} - \frac{3}{2} \Pi^{ij} \right) \Pi_{ij} = \mathcal{P}^{ij} \Pi_{ij} \]

\[ \Pi^{(V)}_i = \hat{k}^j \Pi^{(V)}_{ij} = \mathcal{Q}^{ij}_i \Pi_{ij} \]

\[ \Pi^{(T)}_{ij} = \left( P_i^a P^b_j - \frac{1}{2} \Pi^{ab} \Pi_{ij} \right) \Pi_{ab} = \mathcal{P}^{ab} \Pi_{ab}, \tag{2.12} \]

where \( (\ldots) \) in the indices denotes symmetrization \([47]\).

\section{The magnetic bispectrum}

Since the magnetic field is assumed as a Gaussianly-distributed stochastic helical field and the electromagnetic energy momentum tensor is quadratic in the fields, the statistics must be non-Gaussian and the bispectrum is non-zero as was claimed by \([47]\). Using eq. (2.10) we have that three-point correlation function is expressed as

\[ \langle \Pi_{ij}(k1)\Pi_{kl}(k2)\Pi_{mn}(k3) \rangle = \frac{-1}{(4\pi\rho_{\gamma,0})^3} \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3s}{(2\pi)^3} \times \]

\[ \times \langle B_l(p)B_j(k1 - p)B_l(k2 - q)B_n(k3 - s)B_l(q)B_m(s) \rangle \]

\[ \times \left( \frac{\delta_{ij}}{2} \ldots - \frac{\delta_{jl}}{2} \ldots - \frac{\delta_{mn}}{2} \ldots + \frac{\delta_{ij} \delta_{mn}}{4} \ldots + \frac{\delta_{ij} \delta_{tl}}{4} \ldots \right) + \frac{\delta_{ij} \delta_{mn}}{4} \ldots - \frac{\delta_{ij} \delta_{mn} \delta_{tl}}{8} \ldots. \tag{3.1} \]

\(^2\)The adimensional energy density of magnetic field showed here is written with different notation in \([54]\): \( \Omega_B \equiv \frac{B^2}{8\pi\rho_{\gamma}} \), and in \([48, 53]\): \( \Delta_B \equiv \frac{B^2}{8\pi\rho_{\gamma}} \).
Figure 1. Geometrical configuration for bispectrum. The wavevectors $k_1$, $k_2$ and $k_3$ are free, while $p$ is the integration mode.

Where $\langle \ldots \rangle$ describes an ensemble average over six stochastic fields. We can use Wicks theorem to decompose the six point correlation function into products of the magnetic field power spectrum expressed in eq. (2.3). Eight of fifteen terms contribute to the bispectrum and they are proportional to $\delta(k_1 + k_2 + k_3)$ due to the homogeneity condition. In [65] they point out that expression (3.1) can be reduced to just one contribution if the projection tensor used for extracting each one of the contribution is symmetric in $(ij)$, $(tl)$ and $(mn)$. Therefore one can write the bispectrum as follows

$$\langle \Pi_{ij}(k_1)\Pi_{tl}(k_2)\Pi_{mn}(k_3) \rangle = \delta(k_1 + k_2 + k_3) \times$$

$$\times \frac{8}{(4\pi \rho_{\gamma,0})^3} \int \frac{d^3p}{(2\pi)^3} F_{il}(p) F_{jm}(k_1 - p) F_{ln}(k_2 + p),$$

being $F_{ij}(k) = P_{ij}(k)P_B(k) + i\epsilon_{ijm}k^mP_H(k)$. Wavevectors that appear in eq. (3.2) generate a tetrahedron configuration (see figure 1) such that fifteen angles must be included for calculating the bispectrum. So, in order to make comparison with previous works, we use non only the geometry configuration for bispectrum but as well the notation of these angles defined in [65] given as

$$\begin{align*}
\beta &= \hat{p} \cdot \hat{k_1} - \hat{p}, & \gamma &= \hat{p} \cdot \hat{k_2} + \hat{p}, & \mu &= \hat{k_1} - \hat{p} \cdot \hat{k_2} + \hat{p}, & \theta_{kp} &= \hat{k_1} \cdot \hat{k_2} \\
\theta_{kq} &= \hat{k_1} \cdot \hat{k_3}, & \theta_{pq} &= \hat{k_2} \cdot \hat{k_3}, & \alpha_k &= \hat{k_1} \cdot \hat{p}, & \alpha_p &= \hat{k_2} \cdot \hat{p}, & \alpha_q &= \hat{k_3} \cdot \hat{p} \\
\beta_k &= \hat{k_1} \cdot \hat{k_1} - \hat{p}, & \beta_p &= \hat{k_2} \cdot \hat{k_1} - \hat{p}, & \beta_q &= \hat{k_3} \cdot \hat{k_1} - \hat{p}, & \gamma_k &= \hat{k_1} \cdot \hat{k_2} + \hat{p} \\
\gamma_p &= \hat{k_2} \cdot \hat{k_2} + \hat{p}, & \gamma_q &= \hat{k_3} \cdot \hat{k_2} + \hat{p}.
\end{align*}$$

As we will see, the bispectrum has two main contributions, the first contribution contains terms proportional to $A_B^3$ or $A_B A_H^2$ and this is called the even contribution denoted here with $B^{(S)}$. The second contribution is proportional to terms like $A_B^3 H$ or $A_B^2 H A_H$ and it is called the
odd contribution denoted as $B^{(A)}$. Hence, we can define the three-point correlation for the scalar modes as

$$\langle Z_1(k_1)Z_2(k_2)Z_3(k_3) \rangle = \delta \left( \sum_{i=1}^{3} k_i \right) \left( B^{(S)}_{Z_1Z_2Z_3} - i\epsilon_{ijk}B^{(A)}_{Z_1Z_2Z_3} \right), \quad (3.4)$$

where $Z_{1,2,3} = \{\rho_B, \Pi_B^{(S)}\}$. We begin calculating the bispectrum of the magnetic energy density. To do so, we shall apply the projector defined in eq. (2.12) three times $\delta_i\delta_i\delta_{mn}$ on eq. (3.1) to obtain the following

$$\langle \Pi_{ij}(k_1)\Pi_{il}(k_2)\Pi_{mn}(k_3) \rangle \delta_i\delta_l\delta_{mn} = \langle \rho_B(k_1)\rho_B(k_2)\rho_B(k_3) \rangle, \quad (3.5)$$

using eq. (3.4), a straightforward calculation gives the following expression

$$B^{(S)}_{\rho_B\rho_B\rho_B} = \frac{8}{(2\pi)^3(4\pi\rho_\gamma)^3} \int d^3p \left( P_B(\rho_B)P_B(|k_1 - p|)P_B(|p + k_2|)F_{\rho\rho\rho}^1 \right. \left. - P_B(\rho_B)P_B(|k_1 - p|)P_B(|p + k_2|)F_{\rho\rho\rho}^0 \right. \left. + P_H(|p + k_2|)P_B(|k_1 - p|)F_{\rho\rho\rho}^2 \right. \left. - P_B(|p + k_2|)P_H(|k_1 - p|)F_{\rho\rho\rho}^3 \right. \left. + P_B(|p + k_2|)P_B(|k_1 - p|)F_{\rho\rho\rho}^4 \right), \quad (3.6)$$

for the even contribution. The values of $F_{\rho\rho\rho}$ are shown along with the odd contribution in appendix A. In order to find the three-point cross-correlation between scalar anisotropic stress and magnetic energy density, we will apply the three projections $\delta_i\delta_l\mathcal{P}_{mn}$ defined in eq. (2.12) on eq. (3.1) which gives us

$$\langle \Pi_{ij}(k_1)\Pi_{il}(k_2)\Pi_{mn}(k_3) \rangle \delta_i\delta_l\mathcal{P}_{mn} = \langle \rho_B(k_1)\rho_B(k_2)\Pi_B^{(S)}(k_3) \rangle, \quad (3.7)$$

and using the three-point correlation eq. (3.4), the even contribution yields

$$B^{(S)}_{\rho_B\rho_B\Pi_B^{(S)}} = \frac{8}{(2\pi)^3(4\pi\rho_\gamma)^3} \int d^3p \left( -P_B(\rho_B)P_B(|k_1 - p|)P_B(|p + k_2|)F_{\rho\rho\Pi}^1 \right. \left. - P_B(\rho_B)P_H(|k_1 - p|)P_B(|p + k_2|)F_{\rho\rho\Pi}^0 \right. \left. + P_H(|p + k_2|)P_B(|k_1 - p|)F_{\rho\rho\Pi}^2 \right. \left. + P_B(|p + k_2|)P_B(|k_1 - p|)F_{\rho\rho\Pi}^3 \right. \left. + P_B(|p + k_2|)P_B(|k_1 - p|)F_{\rho\rho\Pi}^4 \right), \quad (3.8)$$

Other cross-bispectrum is obtained by applying $\delta_i\mathcal{P}_{il}\mathcal{P}_{mn}$ on eq. (3.1) and this gives

$$\langle \Pi_{ij}(k_1)\Pi_{il}(k_2)\Pi_{mn}(k_3) \rangle \delta_i\mathcal{P}_{il}\mathcal{P}_{mn} = \langle \rho_B(k_1)\Pi_B^{(S)}(k_2)\Pi_B^{(S)}(k_3) \rangle, \quad (3.9)$$

as result we found the following expression

$$B^{(S)}_{\rho_B\Pi_B^{(S)}\Pi_B^{(S)}} = \frac{8}{4(2\pi)^3(4\pi\rho_\gamma)^3} \int d^3p \left( P_B(\rho_B)P_B(|k_1 - p|)P_B(|p + k_2|)F_{\rho\Pi\Pi}^1 \right. \left. + P_B(\rho_B)P_H(|p + k_2|)P_B(|k_1 - p|)F_{\rho\Pi\Pi}^0 \right. \left. - P_H(|p + k_2|)P_B(|k_1 - p|)F_{\rho\Pi\Pi}^2 \right. \left. - P_B(|p + k_2|)P_B(|k_1 - p|)F_{\rho\Pi\Pi}^3 \right. \left. + P_B(|p + k_2|)P_B(|k_1 - p|)F_{\rho\Pi\Pi}^4 \right), \quad (3.10)$$

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Finally the three-point correlation of scalar anisotropic stress is obtained by applying \( P_{ij} P_{il} P_{mn} \) on eq. \((3.1)\) finding that
\[
\langle \Pi_{ij}(k_1) \Pi_{il}(k_2) \Pi_{mn}(k_3) \rangle P_{ij} P_{il} P_{mn} = \langle \Pi_B^{(S)}(k_1) \Pi_B^{(S)}(k_2) \Pi_B^{(S)}(k_3) \rangle,
\]
thus, the expression for the bispectrum for that contribution is
\[
B^{(S)}_{\Pi_B^{(S)} \Pi_B^{(S)} \Pi_B^{(S)}} = \frac{1}{(2\pi)^3(4\pi \rho_s)^3} \int d^3 p \left( -P_B(p)P_B(|k_1 - p|)P_B(|p + k_2|)F^1_{\Pi_B^{(S)} \Pi_B^{(S)} \Pi_B^{(S)}} \right.
\]
\[-P_B(p)P_H(|p + k_2|)P_H(|k_1 - p|)F^2_{\Pi_B^{(S)} \Pi_B^{(S)} \Pi_B^{(S)}}
\]
\[+P_H(p)P_B(|p + k_2|)P_H(|k_1 - p|)F^3_{\Pi_B^{(S)} \Pi_B^{(S)} \Pi_B^{(S)}}
\]
\[+ P_H(p)P_B(|k_1 - p|)P_H(|p + k_2|)F^4_{\Pi_B^{(S)} \Pi_B^{(S)} \Pi_B^{(S)}} \right).
\]
Again, the \( F_{\Pi_B^{(S)} \Pi_B^{(S)} \Pi_B^{(S)}} \)’s values can be checked along with the odd contribution in appendix A. In the case where the helicity of the field is not considered (\( A_H = 0 \)), the only contribution that remains is sourced from \( A_B^3 \). The results of this contribution were reported in [65] and we have found the same expressions. Therefore, we have generalized those previous results to even and odd contributions of the PMFs bispectrum and thus these findings are the first results of the paper.

4 Full evaluation

With the derivation of the angular part of the three-point correlation for each component of the magnetic tensor, we proceed to make the evaluation of the above integrals. Due to the complexity of the calculation, we are going to follow the exact methodology proposed in [48–50]. They consider two cases for finding solution of the correlator. In the first case, the terms dependent on the integration vector \( p \) are not considered in the evaluation. For the second case, the squeezed collinear configuration is used to make predictions. We will use five representative shapes of the bispectrum which are shown in the figure 2. The odd signal arising from \( B^{(A)} \) will not be considered here but will be deferred for later work.

4.1 \( p \)-independent

For this case, the only terms which appear in the evaluation are those angles given in eq. \((3.3)\) independent of \( p \); they are \( (\theta_{kp}, \theta_{kq}, \theta_{pq}) \). The values of these angles for each configuration are shown in figure 2. The \( F \)'s functions defined above take the following values under this approximation
\[
F^1_{\rho \rho \Pi} = \mu^2, \quad F^2_{\rho \rho \Pi} = \mu, \quad F^3_{\rho \rho \Pi} = F^4_{\rho \rho \Pi} = 0, \quad (4.1)
\]
\[
F^1_{\rho \rho \Pi} = \mu^2 - 3, \quad F^2_{\rho \rho \Pi} = -\mu, \quad F^3_{\rho \rho \Pi} = F^4_{\rho \rho \Pi} = 0, \quad (4.2)
\]
\[
F^1_{\rho \Pi \Pi} = \mu^2 - 6 + 9\theta^2_{pq}, \quad F^2_{\rho \Pi \Pi} = (-7 + 9\theta^2_{pq})\mu, \quad F^3_{\rho \Pi \Pi} = F^4_{\rho \Pi \Pi} = 0, \quad (4.3)
\]
\[
F^1_{\Pi \Pi \Pi} = -9 + 9\theta^2_{kp} - 27\theta_{kp}\theta_{kq}\theta_{pq} + 9\theta^2_{pq} + \mu^2 + 9\theta^2_{kq}, \quad F^3_{\Pi \Pi \Pi} = 0,
\]
\[
F^2_{\Pi \Pi \Pi} = (-13 + 18\theta^2_{kp} + 9\theta^2_{kq} - 27\theta_{kp}\theta_{kq}\theta_{pq} + 9\theta^2_{pq})\mu, \quad F^4_{\Pi \Pi \Pi} = 0, \quad (4.4)
\]
where the result given for \( F_{\Pi \Pi \Pi} \) is in agreement with the reported in [48].\(^3\) The values of \( F \) for each geometrical representation of the bispectrum are shown in table 1 and \( \mu = 0 \) for all configuration except to squeezed configuration where it takes \( \mu \sim -1 \).

\(^3\)There is a difference with a minus sign because we are taking a different signature in the metric.
Figure 2. Geometrical representations for the bispectrum. The figure shows a visual representation of the triangles and the collinear configuration of the bispectrum shape. The table in the bottom-right panel describes the values of the $p$-independent terms for each configuration.

Table 1. Values of $F$ for different geometrical configurations in the $p$-independent case.

4.2 Squeezed collinear configuration

In this approximation, the magnitude of one wave vector ($k_3$) is small while the others have equal magnitudes but have opposing directions ($k_1 = -k_2$) as shown in figure 2. With this assumption the angles can be reduced to

$$\beta = \hat{p} \cdot \hat{k} - p \sim -\hat{p} \cdot \hat{k} - p \sim -\gamma, \quad \mu \sim -\hat{k} \cdot p \sim -\gamma,$$

$$\alpha_k \sim -\alpha_p \sim \alpha_q,$$

$$\beta_k \sim -\beta_p \sim \beta_q \sim -\gamma_k \sim \gamma_p \sim -\gamma_q.$$  \hfill (4.5)

By using this approximation the values of the $F$’s are simplified to

$$F_{\rho\rho\rho}^1 = 1 + \beta^2, \quad F_{\rho\rho\rho}^2 = -(1 + \beta^2), \quad F_{\rho\rho\rho}^3 = -2\beta, \quad F_{\rho\rho\rho}^4 = 2\beta,$$  \hfill (4.6)

$$F_{\rho\rho\Pi}^1 = -2 + 3\alpha_k^2 + \beta^2 - 6\alpha_k\beta_k + 3\beta_k^2 + 3\beta^2_k, \quad F_{\rho\rho\Pi}^2 = 1 - 3\alpha_k^2 - 2\beta^2 + 6\alpha_k\beta_k, \quad F_{\rho\rho\Pi}^3 = -\beta(-1 + 3\beta_k^2), \quad F_{\rho\rho\Pi}^4 = \beta(-1 + 3\beta_k^2).$$  \hfill (4.7)
\begin{align*}
F_{\rho \eta}^{1} &= -9\alpha_{k}\beta_{k}^{3} + (2 - 3\beta_{k}^{2})^{2} + \beta^{2}(1 + 3\beta_{k}^{2}) + \alpha_{k}^{2}(-3 + 9\beta_{k}^{2}), \quad F_{\rho \eta}^{3} = 3\alpha_{k}\beta_{k} - \beta(4 - 3\beta_{k}^{2}), \\
F_{\rho \pi}^{1} &= -2 - 2\beta^{2} + 3\alpha_{k}\beta_{k} + 3\beta_{k}^{2}, \quad F_{\rho \pi}^{4} = -6\alpha_{k}\beta_{k} + 9\alpha_{k}\beta_{k}^{3} - \beta(-5 + 6\beta_{k}^{2}), \quad (4.8) \\
F_{\eta \pi}^{1} &= -8 + \beta^{2} + 18\beta_{k}^{2} + 3\beta_{k}^{2} - 9\beta_{k}^{2} + 6\alpha_{k}\beta_{k}(1 - 3\beta_{k}^{2}) + 9\alpha_{k}^{2}(1 - 3\beta_{k}^{2} + 3\beta_{k}^{4}), \\
F_{\eta \pi}^{2} &= 4 - 2\beta^{2} - 3\beta_{k}^{2} + \alpha_{k}^{2}(-6 + 9\beta_{k}^{2}), \quad F_{\eta \pi}^{3} = F_{\eta \pi}^{4} = (2\beta - 3\alpha_{k}\beta_{k})(1 - 3\beta_{k}^{2}). \quad (4.9)
\end{align*}

Same results have been obtained in [48] for \( F_{\eta \pi}^{1} \) (case where there is not helicity). The angular part of the integrals must be written in spherical coordinates \( d^{3}p = p^{2}d\rho d\alpha d\theta \), where \( \theta \) is the azimuthal angle. Since we consider an upper cut-off \( k_{D} \), we must introduce the \( (k1, k2) \)-dependence on the angular integration domain; this implies that we should split the integral domain in different regions such as \( 0 < k1, k2 < 2k_{D} \). The integration domains we use for calculating the integrals are shown in appendix B. By using the power spectrum expression for the magnetic fields eqs. (2.5)–(2.6) and the \( F \)’s values for the \( p \)-independent case given above, we get the causal magnetic bispectrum \( (n_{B} = n_{H} = 2) \) which is shown in the figure 3. We see that the most contribution for the bispectrum occurs when \( k1 \sim k2 \) and \( (\Pi_{B}\Pi_{B}\Pi_{B}) \) generates the largest value for scalar mode. Hence, we conclude that the shape of the non-Gaussian associated with the PMF can be classified into the local-type configuration as was previously reported in [51] for a scale invariant shape. We also observe that effects from \( A_{B}A_{H}^{3} \) contribution are smaller with respect to \( A_{B}^{3} \). The figures 3d, 3e show the cross-correlation of the bispectrum obtaining the same behavior and a large contribution (with respect to the energy density bispectrum).

On the other hand, in figure 4 we present the results under the squeezed collinear configuration driven also by causal fields. Here, we see that the bispectrum for this configuration is less than the \( p \)-independent case. We also see that magnetic anisotropic stress change the sign under this configuration for wavenumbers larger than \( k_{D}/2 \), while the \( A_{B}A_{H}^{2} \) contribution is practically negative. Note that all the palettes shown here are sequential colour palettes, so they are very well suited to show the amplitude of the bispectrum. Finally, for negative spectral indices, an approximate solution for the bispectrum can be found using the formula (5.13) in [54] (see also equation (B.6) in appendix B). Assuming \( (n_{B} = n_{H} = n) \) along with \( k2 < k1 < k_{D} \), the expression for bispectrum becomes

\begin{align*}
B_{\rho_{B}\rho_{B}\rho_{B}}^{(S)} \sim & \frac{16}{(2\pi)^{3}(4\pi\rho_{\gamma},a)^{3}} \left( \frac{nk^{1n}k2^{2n+3}}{(n + 3)(2n + 3)} + \frac{nk^{13n+3}}{(2n + 3)(3n + 3)} + \frac{k^{3n+3}}{3n + 3} \right) \\
& \times (A_{s}^{3}F_{\rho p\rho}^{1} + A_{H}^{2}A_{s}(F_{\rho p\rho}^{3} - F_{\rho p\rho}^{4} - F_{\rho p\rho}^{2})) , \quad (4.10) \\
B_{\rho_{B}n_{B}\rho_{B}n_{B}}^{(S)} \sim & \frac{2}{(2\pi)^{3}(4\pi\rho_{\gamma},a)^{3}} \left( \frac{nk^{1n}k2^{2n+3}}{(n + 3)(2n + 3)} + \frac{nk^{13n+3}}{(2n + 3)(3n + 3)} + \frac{k^{3n+3}}{3n + 3} \right) \\
& \times (A_{s}^{3}F_{\Pi\Pi\Pi}^{1} + A_{H}^{2}A_{s}(F_{\Pi\Pi\Pi}^{3} + F_{\Pi\Pi\Pi}^{4} - F_{\Pi\Pi\Pi}^{2})) . \quad (4.11)
\end{align*}

4.3 Infrared cut-off

We now analyze the effect of an infrared cut-off parametrized by \( \alpha \) on the magnetic bispectrum (see equation (2.9)). We saw that the non-Gaussianity peaks at \( k1 \sim k2 \) under a squeezed configuration, so we compute the magnetic bispectrum using the strategy adopted in [37] and in appendix C. In figure 5, the effect of this IR cut-off for causal fields is illustrated. The top figures 5a and 5b show the effect of \( \alpha \) on \( k1^{3}\langle \rho_{B}\rho_{B}\rho_{B} \rangle \) where the lines refer to different values of the cut-off, while the bottom figures 5c and 5d show the effect of \( \alpha \) on \( k1^{3}\langle \Pi_{B}\Pi_{B}\Pi_{B} \rangle \). What we read off these figures is how the peak of the bispectrum moves to
Figure 3. Total contribution of three-point correlation of all scalar modes described in the text using the $p$-independent approximation. The figures (a), (b) (c) show the three-point correlation of the energy density of the magnetic field without, with $A_B A_H^2$ and full contribution respectively. The figures (d), (e) and (f) show the cross three-point correlation of the field. Finally, the figures (g), (h) (i) show the cross three-point correlation field in the equilateral configuration, where the $\langle \rho_B \Pi_B \Pi_B \rangle$ has a total negative contribution. We can see that largest contribution to the bispectrum is obtained when $k_1 \sim k_2$. We can also see that biggest contributions to the scalar modes is given by $\langle \Pi_B \Pi_B \Pi_B \rangle$ when we consider a squeezed configuration.
Figure 4. Total contribution of three-point correlation of non-crossing scalar modes described in the text using the squeezed collinear configuration. The figures (a), (b) (c) show the three-point correlation of the energy density of the magnetic field without, with $A_B A_H^2$ and full contribution respectively; while figures (d), (e) and (f) show the three-point correlation of the anisotropic stress of the magnetic field without, with $A_B A_H^2$ and full contribution respectively. On the other hand, the figures (g), (h) and (i) show the cross three-point correlation of the field in this configuration. We can see that $k_1 \sim k_2$ has the biggest contribution to the bispectrum. One important feature of the anisotropic stress mode is the negative contribution to the total bispectrum for wavevectors larger than $K_D/2$. 

(a) Three-point correlation of $\langle \rho_\mathcal{B} \rho_\mathcal{B} \rho_\mathcal{B} \rangle$ in units of $(2\pi)^3 (4\pi \rho_{\gamma,0})^3$ without $A_B A_H^2$.

(b) Three-point correlation of $\langle \rho_\mathcal{B} \rho_\mathcal{B} \rho_\mathcal{B} \rangle$ in units of $(2\pi)^3 (4\pi \rho_{\gamma,0})^3$ only with $A_B A_H^2$.

(c) Even contribution of three-point correlation of $\langle \rho_\mathcal{B} \rho_\mathcal{B} \rho_\mathcal{B} \rangle$ in units of $(2\pi)^3 (4\pi \rho_{\gamma,0})^3$.

(d) Three-point correlation of $\langle \Pi_\mathcal{B} \Pi_\mathcal{B} \Pi_\mathcal{B} \rangle$ in units of $(2\pi)^3 (4\pi \rho_{\gamma,0})^3$ without $A_B A_H^2$.

(e) Three-point correlation of $\langle \Pi_\mathcal{B} \Pi_\mathcal{B} \Pi_\mathcal{B} \rangle$ in units of $(2\pi)^3 (4\pi \rho_{\gamma,0})^3$ only with $A_B A_H^2$.

(f) Even contribution of three-point correlation of $\langle \Pi_\mathcal{B} \Pi_\mathcal{B} \Pi_\mathcal{B} \rangle$ in units of $(2\pi)^3 (4\pi \rho_{\gamma,0})^3$.

(g) Three-point correlation of $\langle \rho_\mathcal{B} \Pi_\mathcal{B} \Pi_\mathcal{B} \rangle$ in units of $(2\pi)^3 (4\pi \rho_{\gamma,0})^3$.

(h) Three-point correlation of $\langle \rho_\mathcal{B} \Pi_\mathcal{B} \Pi_\mathcal{B} \rangle$ in units of $(2\pi)^3 (4\pi \rho_{\gamma,0})^3$ only with $A_B A_H^2$.

(i) Even contribution of three-point correlation of $\langle \rho_\mathcal{B} \Pi_\mathcal{B} \Pi_\mathcal{B} \rangle$ in units of $(2\pi)^3 (4\pi \rho_{\gamma,0})^3$. 

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(a) Change of $k^3 \langle \rho_B \rho_B \rho_B \rangle$ respect to the infrared cut-off without $A_B A_H^2$.

(b) Change of $k^3 \langle \rho_B \rho_B \rho_B \rangle$ respect to the infrared cut-off only with $A_B A_H^2$.

(c) Change of $k^3 \langle \Pi_B \Pi_B \Pi_B \rangle$ respect to the infrared cut-off without $A_B A_H^2$.

(d) Change of $k^3 \langle \Pi_B \Pi_B \Pi_B \rangle$ respect to the infrared cut-off only with $A_B A_H^2$.

Figure 5. Effects of a lower cut-off at the three-point correlation of non-crossing scalar modes described in the text using the squeezed collinear configuration. The figures (a), (b) show the three-point correlation of the energy density of the magnetic field without and only with $A_B A_H^2$ respectively, while figures (c) and (d) show the three-point correlation of the anisotropic stress of the magnetic field without and only with $A_B A_H^2$ respectively. The black, red, blue, violet and green lines refer to lower cut-off for $\alpha = 0.01, \alpha = 0.4, \alpha = 0.6, \alpha = 0.7, \alpha = 0.9$ respectively. Here the units are normalized respect to the values in figure (a) and we use $n_B = n_H = 2$.

High wavenumbers when we increase the value of $k_m$ in the same way that magnetic power spectrum (and its amplitude decreases also due to reduction of the wavenumber space) and how the effects of the $A_B A_H^2$ contributions PMF are tiny compared with the non-helical case. On the other hand, the figure 6 shows the effect of the infrared cut-off when we are considering non-causal fields. The top panel shows the three-point correlation of the energy density of the magnetic field for $n_H = n_B = -5/2$ while the bottom panel shows the three-point correlation for $n_H = n_B = -1.9$. Here we can see that the contribution driven by $A_B A_H^2$ is bigger than $A_B^3$, meaning that for negative spectral indices the effect of helicity becomes relevant for our studies.
5 Reduced bispectrum from PMF

In this section, we estimate the reduced bispectrum and give a careful review the results of \[48, 66, 67\]. The CMB temperature perturbation at a direction of photon momentum $\hat{n}$ can be expanded into spherical harmonics

$$
\frac{\Delta T^{(Z)}(\hat{n})}{T}(\hat{n}) = \sum_{lm} a_{lm}^{(Z)} Y_{lm}(\hat{n}),
$$

(5.1)
where $Z = S, V, T$ refers to the contribution given by scalar, vector or tensor perturbations. The coefficient $a^{(Z)}_{l_m}$ is written as [53]

$$a^{(Z)}_{l_m} = 4\pi (-i)^l \int \frac{d^3 k}{(2\pi)^3} \Delta^Z_l(k) \sum_\lambda [\text{sgn}(\lambda)]^3 \xi^{(\lambda)}_{l_m}(k),$$

$$\xi^{(\lambda)}_{l_m}(k) = \int d^2 k \xi^{(\lambda)}(k) \lambda Y^*_{l_m}(\hat{k}),$$

(5.2)

where $\lambda = 0, \pm 1, \pm 2$ describes the helicity of the scalar, vector, tensor mode; $\lambda Y^*_{l_m}$ is the spin-weight spherical harmonics; $\xi^{(\lambda)}(k)$ is the primordial perturbation and $\Delta^Z_l(k)$ is the transfer function. Let us define the CMB angular bispectrum as

$$B^{m_1 m_2 m_3}_{l_1 l_2 l_3} = \left\langle \prod_{n=1}^3 a^{(Z)}_{l_n m_n} \right\rangle,$$

(5.3)

where only scalar perturbations ($Z=S$) will be considered in the paper. Now, by substituting the equation (5.2) into equation (5.3) we can find

$$\left\langle \prod_{n=1}^3 \xi^{(\lambda)}(k) \right\rangle = A_P \delta(k_1 + k_2 + k_3) B^{(S)}_{\Pi_B \Pi_B \Pi_B}.$$

(5.4)

We also consider a rough approximation for the transfer function that works quite well at large angular scales and for primordial adiabatic perturbations given by $\Delta_l(k) = \frac{3}{2} \Xi_l(k(\eta_0 - \eta_*))$, where $\Xi_l(x)$ the spherical Bessel function, $\eta_0 = 14.38$ Gpc the conformal time at present and $\eta_* = 284.85$ Mpc the conformal time at the recombination epoch [67]. This is the large scale Sachs Wolfe effect. Since we want to evaluate the contribution of the bispectrum by PMFs, the three-point correlator for the primordial perturbation must satisfy the relation

$$\left\langle \prod_{n=1}^3 \xi^{(\lambda)}(k) \right\rangle = A_P \delta(k_1 + k_2 + k_3) B^{(S)}_{\Pi_B \Pi_B \Pi_B}.$$

(5.5)

Here, $A_P$ is a constant that depends on the type of perturbation (passive or compensated magnetic mode) and $B^{(S)}_{\Pi_B \Pi_B \Pi_B}$ is the magnetic bispectrum computed above. Using the following relations [66]

$$\delta(k_1 + k_2 + k_3) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \exp(i(k_1 + k_2 + k_3) \cdot x) d^3 x,$$

(5.6)

$$\exp(ik \cdot x) = 4\pi \sum_l i^l j_l(kx) \sum_m Y_{lm}(\hat{k}) Y^*_{lm}(\hat{x}),$$

(5.7)

with the Gaunt integral $G^{m_1 m_2 m_3}_{l_1 l_2 l_3}$ defined by

$$G^{m_1 m_2 m_3}_{l_1 l_2 l_3} = \int d^2 \hat{n} Y_{l_1 m_1}(\hat{n}) Y_{l_2 m_2}(\hat{n}) Y_{l_3 m_3}(\hat{n})$$

$$= \sqrt{\frac{(2l_1 + 1)(2l_2 + 1)(2l_3 + 1)}{4\pi}} \left( \begin{array}{ccc} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{array} \right),$$

(5.8)
and along with the equation (5.5), the equation (5.4) takes the form

\begin{equation}
\langle \prod_{n=1}^{3} a_{l_{n} m_{n}} \rangle = A_P \sqrt{\frac{2(l_{1} + 1)(2l_{2} + 1)(2l_{3} + 1)}{4\pi}} \left( \begin{array}{ccc}
l_{1} & l_{2} & l_{3} \\
0 & 0 & 0
\end{array} \right) \left( \begin{array}{ccc}
l_{1} & l_{2} & l_{3} \\
m_{1} & m_{2} & m_{3}
\end{array} \right) \times \sum_{n=1}^{3} \frac{1}{3\pi^{2}} \int kn^{2} \int j_{l_{n}}(knx)j_{l_{n}}(kn(\eta_{0} - \eta_{s}))dkn \right] B^{(S)}_{\Pi \Pi \Pi}(x^{2}dx).
\end{equation}

Here the matrix is the Wigner-3j symbol and it vanishes unless the selection rules are satisfied.\(^4\) Given the rotational invariance of the Universe, Komatsu-Spergel [66] defined a real symmetric function of \(l\) called the reduced bispectrum \(b_{l_{1}l_{2}l_{3}}\)

\begin{equation}
\langle \prod_{n=1}^{3} a_{l_{n} m_{n}} \rangle \equiv \mathcal{g}_{l_{1}l_{2}l_{3}}^{m_{1}m_{2}m_{3}} b_{l_{1}l_{2}l_{3}}.
\end{equation}

Checking the last two equations, the properties of the bispectrum generated by PMFs can be expressed via the reduced bispectrum as

\begin{equation}
b_{l_{1}l_{2}l_{3}} = A_P \left[ \prod_{n=1}^{3} \frac{1}{3\pi^{2}} \int kn^{2} \int j_{l_{n}}(knx)j_{l_{n}}(kn(\eta_{0} - \eta_{s}))dkn \right] B^{(S)}_{\Pi \Pi \Pi}(x^{2}dx).
\end{equation}

In order to calculate \(A_P\), we must clarify the sources of primordial perturbations. Prior to neutrino decoupling (\(\tau_{\nu} = 1\text{MeV}^{-1}\)), the Universe is dominated by radiation and it is tightly coupled to baryons such that they cannot have any anisotropic stress contribution. Since we are also considering magnetic fields, they would be the only ones that develop anisotropic stress and therefore at superhorizon scales the curvature perturbation depends on the primordial magnetic source [53]. But after neutrino decoupling, neutrinos generated anisotropic stress which compensates the one coming from PMF finishing the growth of the perturbations. Shaw-Lewis [68] showed that curvature perturbation is given by

\begin{equation}
\xi(k) \sim -\frac{1}{3} R_{\gamma} \ln \left( \frac{\tau_{\nu}}{\tau_{B}} \right) \Pi^{(S)}_{\Pi \Pi \Pi}(k),
\end{equation}

commonly known as passive mode, where \(R_{\gamma} = \frac{\rho_{s}}{\rho} \sim 0.6\) and \(\tau_{B}\) is the epoch of magnetic field generation. Another contribution comes from the density-sourced mode with unperturbed anisotropic stresses, the magnetic compensated scalar mode, this is proportional to the amplitude of the perturbed magnetic density just as the magnetic Sachs Wolfe effect [68]. So, if the primordial perturbation is associated with the initial gravitational potential, in the limit on large-angular scales the compensated modes is expressed as [48, 69]

\begin{equation}
\xi(k) \sim \frac{1}{4} R_{\gamma} \rho_{B}(k).
\end{equation}

Therefore if we use the passive mode contribution, \(B^{(S)}_{\Pi \Pi \Pi}\) is given by \(B^{(S)}_{\Pi \Pi \Pi} = \left( \frac{1}{3} R_{\gamma} \ln \left( \frac{\tau_{\nu}}{\tau_{B}} \right) \right)^{3}\), whilst compensated mode the primordial

\(^4\)The Wigner-3j symbols satisfy that: \(|m_{1}| \leq l_{1}, |m_{2}| \leq l_{2}, |m_{3}| \leq l_{3}, m_{1} + m_{2} + m_{3} = 0, l_{1} + l_{2} + l_{3} = Z\) and \(|l_{1} - l_{2}| \leq l_{3} \leq l_{1} + l_{2}\) [53].
three-point correlation is described by \( B_{\rho B\rho B}^{(S)} \) (see equation (3.6)) with \( A_P = \left( \frac{1}{4} R_\gamma \right)^3 \). Since the magnetic bispectrum only depends on \( (k_1; k_2) \), the \( k_3 \) integral in the equation (5.11) gives \( \frac{\pi}{2} \delta (x - (\eta_0 - \eta_\ast)) \) due to the closure relation [48], and integrating out the delta function one finally obtains

\[
b_{l_1l_2l_3} = A_P \frac{\pi}{2} \left[ \prod_{n=1}^{2} \frac{1}{3\pi^2} \int k_n^2 j_{l_n}(kn(\eta_0 - \eta_\ast))^2 dkn \right] B_{\Pi_B\Pi_B\Pi_B}^{(S)}.
\]

(5.14)

This is the master formula that we shall use in the following section in order to calculate the CMB reduced bispectrum.

6 Analysis

In this section we show the numerical results of the CMB reduced bispectrum produced by helical PMFs. In order to numerically solve eq. (5.14), we use the adaptive strategy implemented in Mathematica called Levin-type rule which estimates the integral of an oscillatory function with a good accuracy [73].

6.1 Causal fields

Figure 7 presents the CMB reduced bispectrum generated by compensated PMFs modes under collinear configuration. In this figure we observe the signal produced by only the \( A_B^3 \) contribution 7a, as well as the signal by the whole 7b. We found that \( A_B^3 A_B^2 \) contribution (helical) is smaller than non-helical part \( A_B^3 \). Here we plot the change of the reduced bispectrum with respect to \( l_1 \), finding a large contribution for large values of \( l_1 \). We also see that helical contribution reaches a maximum around \( l_2 \sim 400 \) whilst non-helical contribution tends to increase at least until \( l_2 \sim 500 \). In figures 7c and 7d, the effect of an IR cut-off on the reduced bispectrum are shown. Each of these plots show the signal for different values of \( l_1 \). We see that signal is biggest for small \( \alpha \) values (being the biggest contribution for spectrum without IR cut-off) similar to the found with the power spectrum case [37].

In figure 8 we show the CMB reduced bispectrum generated by PMFs passive modes under collinear configuration. Meanwhile, figures 8a and 8b describe the signal for the non-helical and total contribution respectively. An interesting feature is that reduced bispectrum generated by helical contribution is totally negative, then, by using this unusual behavior we would have direct evidence of a helical component in the field.

Other important result of this paper is reported in figures 8c and 8d. Here we show again the effect of an IR cut-off on the reduced bispectrum and we have found out that the biggest contribution of the bispectrum comes from an IR cut-off near \( \alpha \sim 0.5 \) instead of \( \alpha = 0 \). This peak might correspond to a type of dynamics in large scales and help us to determine the nature of PMFs (Since we are trying with causal fields, this infrared cut-off would correspond to the maximum scale in which magnetic fields may be generated at later times). Therefore the evidence of this cut-off in the bispectrum would reveal an interesting signal from passive magnetic scalar mode. In addition, the change of the reduced bispectrum for helical PMFs in presence of an IR cut-off is showed in figure 9. Here we observe how the signal decreases when the IR cut-off increases for the compensated mode and how change the behavior for the passive case. We want to remark on some approximations used so far. Since we assume that effects of PMFs are important for small multipolar numbers, we write the transfer functions in terms of spherical Bessel function. Previous papers have worked
Figure 7. Reduced bispectrum seeded by compensated PMFs with $n = 2$ using the squeezed collinear configuration. The figures (a) shows the reduced bispectrum of the magnetic field with only $A^3_B$, while figure (b) shows the total contribution of the compensated mode; here the lines refers to different values of $l_1$, violet($l_1 = 11$), black($l_1 = 61$), red($l_1 = 110$), green($l_1 = 161$) and blue line($l_1 = 210$). The figures (c), (d) show the effects of an infrared cut-off on the reduced bispectrum for different values of multipolar numbers $l_1$. Black; red; green; blue; and violet lines refer to lower cut-off of $\alpha = 0.001$, $\alpha = 0.4$, $\alpha = 0.5$, $\alpha = 0.7$, $\alpha = 0.8$ respectively. The reduced bispectrum is in units of $4\pi 10^{-8} A_P / (8\pi^2 \rho_{\gamma,0})^3$.

without this approximation. For instance, [55] computes the full radiation transfer function taking into account the effect of PMFs. The full numerical integration of the bispectrum can be done via second-order Einstein-Boltzmann codes like SONG [74] improving the estimation of the amplitude of PMFs. Moreover, we must note that [70] found a WMAP bound on non-helical passive mode for tensor temperature bispectrum of $B_{1 Mpc} < 3.1 nG$ and the Planck paper [56] reported $B_{1 Mpc} < 2.8 nG$ all of them for scale-invariant fields. Actually, the tensor mode is dominant in the passive mode, and can give quite tighter constraint of the PMF amplitude than the scalar mode. Thus, tensor mode contribution and a full transfer function determined by the presence of these fields will improve our results and therefore they will be interesting subjects of our future work.
6.2 Non-causal fields

Now let us consider a red magnetic spectrum. Figure 10 shows the reduced bispectrum for compensated mode with $n_H = n_B = n = -5/2$. Again, we plot the non-helical 10a and total 10b contribution of the bispectrum while plots 10c and 10d correspond to the change of the signal due to an IR cut-off. Since the PMFs bispectrum is almost determined by the poles in each $k$, the value of it peaked for $l_1 = l_2 = l_3$ as we can observe in figures 10, 11, 12. Additionally, the figure 11 shows signals for $n_H = n_B = n = -3/2$ and $n_H = n_B = n = -1.9$.

As a matter of fact, some mechanisms of inflationary magnetogenesis with parity violating terms which lead to helical magnetic field ($n = -1.9$), stand for the lower bound for the which the field can satisfy the intensity of magnetic fields in the intergalactic medium; thereby the
(a) Effects of infrared cut-off on the reduced bispectrum with $l_1 = 161$ seeded by total contribution of compensated helical PMFs.

(b) Effects of infrared cut-off on the reduced bispectrum with $l_1 = 161$ seeded by total contribution of passive helical PMFs.

Figure 9. Reduced bispectrum seeded by compensated (a) and passive (b) helical PMFs with only $A_B A_H^2$ contribution using the squeezed collinear configuration. Black; red; green; blue; and violet lines refer to lower cut-off of $\alpha = 0.001$, $\alpha = 0.4$, $\alpha = 0.5$, $\alpha = 0.7$, $\alpha = 0.8$ respectively. The reduced bispectrum is in units of $4\pi 10^{-8} A_P / 2(8\pi^2 \rho_0)$. The signal described in figure 11 constraints models for providing the seed for galactic magnetic fields [29]. On the other hand, figure 12 shows the reduced bispectrum taking into account the p-independent approximation implemented in section 4.1. Due to complexity of the angular structure on the PMF bispectrum for passive mode, the numerical computation for reduced bispectrum requires a great deal of time. To avoid this problem we can use the p-independent approximation by reducing the PMF bispectrum to an independent angular form as we studied above. From figures 12a and 12b we observe an increase in the signal as we expected due to the lack of angular terms in the bispectrum. Finally, figures 12c and 12d show the total contribution of passive modes. Note that the amplitude is larger than the compensated mode. This behavior have also been reported in [48]. This result is interesting because an estimation of $B_\lambda$ through a local-type primordial NG in curvature perturbation generates constraints stronger than the compensated ones. Finally, if we compare the results reported in this section with the ones shown in above section for causal fields, we can observe that effect of $k_m$ is more significant in negative spectral indices, specially for nearly scale invariant scale fields. We then conclude that $k_m$ plays an important role in the study of these non-causal fields and this generates the possibility of determining some important clues in the mechanisms of magnetogenesis.

6.3 Estimation of the magnetic field amplitude

In fact, it is possible to obtain a rough estimate of $B_\lambda$ using the formula for the primary reduced bispectrum found in [66]

$$b_{lll} \sim l^{-4} \times 2 \times 10^{-17} f_{NL},$$

where the non-Gaussianity (local) is fully specified by a single constant parameter $f_{NL}$. As we mentioned above, the k-dependence on the magnetic bispectrum is similar to the CMB bispectrum arising from the local type NG of curvature perturbations, therefore, by
(a) Reduced bispectrum given by $A_3^3$ contribution of compensated PMFs.

(b) Reduced bispectrum given by total contribution of compensated PMFs.

(c) Effects of infrared cut-off on the reduced bispectrum with $l_1 = 111$.

(d) Effects of infrared cut-off on the reduced bispectrum with $l_1 = 210$.

Figure 10. Absolute value of reduced bispectrum seeded by compensated PMFs with $n = -5/2$ using the squeezed collinear configuration. The figure (a) shows reduced bispectrum of the magnetic field with only $A_3^3$ contribution, while figure (b) shows the total contribution of the compensated mode; here the lines refers to different values of $l_1$, violet($l_1 = 11$), black($l_1 = 61$), red($l_1 = 110$), green($l_1 = 161$) and blue line($l_1 = 210$). The figures (c) and (d) show the effects of an infrared cut-off on the reduced bispectrum for difference values of multipolar numbers $l_1$. Black; red; green; and violet lines refer to lower cut-off of $\alpha = 0.001$, $\alpha = 0.4$, $\alpha = 0.6$, $\alpha = 0.8$ respectively. The reduced bispectrum is in units of $4\pi 10^{16} A_P/(8\pi^2 \rho_\gamma G)^3$.

Comparing the last equation with eq. (5.14), allow us to express a simple relation between $F_{NL}$ and $B_\lambda$ given by

$$f_{NL} \propto \left( \frac{B_\lambda}{10^{-9} G} \right)^6.$$  \hspace{2cm} (6.2)

In table 2 we present the constant of proportionality of the last expression. Here we use $\frac{\tau_B}{\tau_B} = 10^{17}$ which corresponds to the PMF generated at the grand unification energy scale(GUT) scale, $\left( \frac{B_\lambda^2}{8\pi G \rho_\gamma} \right) \sim 10^{-7} \left( \frac{B_\lambda}{10^{-9} G} \right)^2$, and $R_\gamma \sim 0.6$. In order to constrain the smoothed amplitude of the magnetic field on a scale of 1Mpc ($B_1$), we will use the $f_{NL}$ value reported by Planck Collaboration [71] of $f_{NL} < 5.8$ at 68% CL. The results of $B_1$ are shown in table 3. Our results for compensated modes lead to upper bounds on the PMF smoothed
amplitude which are consistent with the 2015 Planck analysis [56], but for passive modes our results are slightly tighter because these were based on a rough estimation and may involve some uncertainties (except for the causal case where the bound coincides with Planck analysis), however notice that for passive modes the limits are almost 10 times more stringent than the compensated ones as was reported in [48] for no-helical and scale invariant case, hence CMB-observation are sensitive to the magnetic induced modes. Since our results were obtained under the Sachs-Wolfe approximation, we expected a lower value of $B_1$ for causal fields with respect to the non causal fields, and therefore the blue spectra generated by these fields is strongly disfavoured by the CMB bispectrum. On the other hand, we constrain $B_1$ through the helical contribution and we observed an enhance of its amplitude. We see this same effect in the two point correlation as was reported by Planck analysis $B_1 < 5.6nG$ at 95% CL [56] for that contribution. In the tables also show the bounds when a high value of

Figure 11. Absolute values of reduced bispectrum seeded by compensated PMFs using the squeezed collinear configuration. The figures (a), (b) show the total contribution of the compensated mode of the magnetic field for $n = -3/2$ and $n = -1.9$ respectively; here the lines refers to different values of $l_1$, violet($l_1 = 11$), black($l_1 = 61$), red($l_1 = 110$), green($l_1 = 161$) and blue line($l_1 = 210$). The last figures (c),(d) show the effects of an infrared cut-off on the reduced bispectrum for difference values of multipolar numbers $l_1$. Blue; green; red; and black lines refer to lower cut-off of $\alpha = 0.001$, $\alpha = 0.4$, $\alpha = 0.6$, $\alpha = 0.8$, respectively. The bispectrum is in units of $4\pi10^{16}A_P/(8\pi^2\rho_0)^3$. 

(a) Reduced bispectrum of compensated PMFs for $n = -3/2$. 

(b) Reduced bispectrum of compensated PMFs for $n = -1.9$. 

(c) Effects of infrared cut-off on the reduced bispectrum with $l_1 = 161$ for $n = -3/2$. 

(d) Effects of infrared cut-off on the reduced bispectrum with $l_1 = 161$ for $n = -1.9$. 

...
(a) Reduced bispectrum of compensated PMFs for $n = -5/2$.

(b) Reduced bispectrum of compensated PMFs for $n = -3/2$.

(c) Reduced bispectrum of passive PMFs for $n = -5/2$.

(d) Reduced bispectrum of passive PMFs for $n = -3/2$.

Figure 12. Absolute values of reduced bispectrum seeded by passive and compensated PMFs under p-independent approximation. The figures (a), (b) show the total contribution of the compensated mode of the magnetic field for $n = -5/2$ and $n = -3/2$ respectively. The figures (c), (d) show the total contribution of the passive mode of the magnetic field for $n = -5/2$ and $n = -3/2$ respectively. Here the lines refer to different values of $l_1$, violet$(l_1 = 11)$, black$(l_1 = 61)$, red$(l_1 = 110)$, green$(l_1 = 161)$ and blue line$(l_1 = 210)$.

the IR cut-off ($\alpha \sim 0.8$) is used. Since cut-off reduces the amplitude magnetic bispectrum signal, the upper limit of $B_\lambda$ becomes somewhat relaxed and therefore, we are able to illustrate the impact $k_m$ can have on constraints on the PMF amplitude. Although this effect becomes very small for a tiny value of $k_m$, the presence of this scale in the analysis of NG are complementary to the ones found by the two point correlation case and will provide new insight into the nature of primordial magnetic fields.

7 Discussion and conclusions

In this paper we investigate the effects of helical PMFs in the CMB reduced bispectrum. One of the main motivations to introduce the helicity comes from the fact that these fields are good observables to probe parity-violation in the early stages of the Universe. Furthermore, since magnetic fields depend quadratically on the field, it must induce non-Gaussian signals
on CMB anisotropies at lower order instead of the standard inflationary mechanisms where this signal appears only at high orders [50]. We started our work deriving the full even and odd parts of the bispectrum which comes from the helical magnetic fields, thus extending the previous results reported in [65]. We obtained the full expression for the PMF bispectrum but, we did not consider modes that arise from odd intensity-intensity-intensity bispectrum. Although these signals are smaller than the even ones, the evidence of the odd signals would be a decisive observable to probe parity-violating processes in the early Universe [51]. We will provide more details of the parity-odd signals in a future paper. Then, through the methodology used in [48, 49], we found that PMFs bispectrum peaks at $k_1 \sim k_2$ under a squeezed configuration implying that statistical properties of the PMFs are similar to those of the local-type NG of curvature perturbations. By calculating the bispectrum given by PMFs anisotropic stress, we observed that its amplitude is larger that the density one and also has a negative contribution for values less than $k_D$. Through numerical calculations of the intensity-intensity-intensity reduced bispectra of the scalar modes, we also studied the total contributions of the helical PMF bispectrum and the presence of an IR cut-off in the convolution integrals. Here we observe the same behavior seen in the power spectral case and the presence of negative contribution due to helical terms $A_B A_B^2 H$. Nevertheless, in the computation in the passive modes for causal fields we observed an unusual behavior in the bispectrum. Indeed, in figure 8 we have found out that biggest contribution of the bispectrum comes from an IR cut-off near to $\alpha \sim 0.5$ instead of $\alpha = 0$. Since $k_m$ is dependent on PMF generation model, this behavior might set strong limits on PMF amplitude. Finally, we investigated the effects of $k_m$ on the reduced bispectrum for $n < 0$. Due to the fact that the magnetic field intensity can be enhanced when we use passive modes, it is expected that those modes determine a very strong constraints on the amplitude of the magnetic field on a given characteristic scale $\lambda$. We verify this statement by using the primary reduced bispectrum found in [66] and calculating $B_\lambda$. Our results showed in tables 2, 3 reflect the fact that the corresponding bound

| $n$ | $\frac{\alpha}{k_D}$ | $n = -\frac{5}{2}$ | Passive | $n = -1.9$ | Passive | $n = -\frac{3}{2}$ | Passive | $n = 2$ | Passive |
|-----|----------------------|------------------|---------|-----------|---------|-----------|---------|---------|---------|
| Comp. | 0.86 (1.92) | 0.12 | 0.47 (0.80) | 0.35 (0.49) | 0.008 | 0.021 | 0.0069 |
| Helical | 1.04 (2.43) | 0.13 | 0.38 | 0.33 | 0.067 | 0.018 | 0.0050 |
| Total | 0.92 (1.85) | 0.14 | 0.40 (0.73) | 0.38 (0.44) | 0.064 | 0.017 (0.021) | 0.0051 (0.006) |

| $n$ | $\frac{\alpha}{k_D}$ | $n = -\frac{5}{2}$ | Passive | $n = -1.9$ | Passive | $n = -\frac{3}{2}$ | Passive | $n = 2$ | Passive |
|-----|----------------------|------------------|---------|-----------|---------|-----------|---------|---------|---------|
| Comp. | 1.15 (2.58) | 0.16 | 0.64 (1.07) | 0.47 (0.66) | 0.098 | 0.029 | 0.0092 |
| Helical | 1.39 (3.25) | 0.17 | 0.52 | 0.44 | 0.089 | 0.024 | 0.0068 |
| Total | 1.24 (2.48) | 0.19 | 0.54 (0.98) | 0.52 (0.59) | 0.085 | 0.023 (0.029) | 0.0068 (0.0083) |

Table 2. Constant of proportionality of the eq. (6.2) for different spectral indices and modes(compensated(comp) or passive) of the PMF without considering an IR cut-off. Parentheses are used to represent this value for $\frac{\alpha}{k_D} \sim 0.8$.

Table 3. Bound on smoothed amplitude of the magnetic field on a scale of 1Mpc ($B_1$ in units of $nG$) for different spectral indices and modes(compensated(comp) or passive) of the PMF without considering an IR cut-off. Parentheses are used to represent $B_1$ for $\frac{\alpha}{k_D} \sim 0.8$. Here we use $f_{NL} < 5.8$ reported by Planck Collaboration, 2016 [56].
on the mean amplitude of the field is dependent on strong values of the minimal cut-off and the helical contribution relaxing the constraints of $B_\lambda$. We also found that for passive modes the limits are almost 10 times more stringent than the compensated ones for both helical and non helical contribution, this result was also reported in [50] for non helical fields. However, we can observe that effect of $k_m$ is more significant in the magnetic bispectrum driven by negative spectral indices. Hence, the presence of $k_m$ plays an important role in the analysis of the signatures that these non-causal fields may leave in cosmological observations.

In conclusion, we have studied the effects of helicity and a minimal cut-off on the constraints of the PMFs amplitude by computing the CMB reduced bispectrum induced on large angular scales by those fields. Even though $k_m$ for causal modes would be important when this scale is larger than the wavenumber of interest, for non-causal modes it is related with the horizon scale of the beginning of inflation [37, 72], and therefore the study of this cut-off on the bispectrum give us information about the PMF generation mechanisms.

A Scalar bispectra of the magnetic field

Here we present the results of the section 2 for the even and odd contributions of the magnetic field bispectrum. The expressions for the magnetic bispectrum reported in this paper were performed with the help of the tensor computer algebra xAct package in Mathematica [75].

Scalar correlation ($<\rho\rho\rho>$). For the energy density of PMF bispectrum ($<\rho\rho\rho>$) we have

\[
F_{\rho\rho\rho}^1 = \beta^2 + \gamma^2 + \mu^2 - \beta\gamma\mu, \tag{A.1}
\]
\[
F_{\rho\rho\rho}^2 = \beta\gamma + \mu, \tag{A.2}
\]
\[
F_{\rho\rho\rho}^3 = \beta\mu + \gamma, \tag{A.3}
\]
\[
F_{\rho\rho\rho}^4 = \beta + \gamma\mu, \tag{A.4}
\]

for the even part, and using definition of eq. (3.4) we have for the odd part

\[
B_{\rho\rho\rho\rho\rho\rho}^{(A)}_{ijk} = B_{\rho\rho\rho\rho\rho\rho}^{(A)} \rho\rho\rho\rho\rho\rho \rho\rho\rho\rho\rho\rho, \tag{A.5}
\]

with

\[
B_{\rho\rho\rho\rho\rho\rho}^{(A)} = \frac{8}{(2\pi)^3(4\pi)^3} \int d^3p \left( P_H(|p + k_2|)P_H(p|p - k_1| - P_B(p)|k_1 - p|)\beta \right.
\]
\[
+ P_H(|p + k_2|)P_B(p|k_1 - p|)\gamma - P_H(p)|k_1 - p|\mu). \tag{A.6}
\]

Scalar cross-correlation($<\rho\rho\Pi>$). For the three-point cross-correlation ($<\rho\rho\Pi>$) we obtain

\[
F_{\rho\rho\Pi}^{(s)} = 3(-1 + \alpha^2_q + \beta^2_q + \gamma^2_q) + \gamma^2 + \beta^2 + \mu^2 - \beta\gamma\mu + 3\beta\gamma\gamma\gamma_q - 3\alpha_q(\beta\gamma + \gamma\gamma_q) - 3\mu\beta\gamma\gamma_q, \tag{A.7}
\]
\[
F_{\rho\rho\Pi}^{(s)} = 2\beta\gamma - 3\alpha_q\beta\gamma\gamma - 3\alpha_q\beta\gamma\gamma_q - \mu + 3\alpha^2_q\mu, \tag{A.8}
\]
\[
F_{\rho\rho\Pi}^{(s)} = \gamma(-2 + 3\beta^2_q) + 3\alpha_q\gamma\gamma_q + \beta\mu - 3\alpha_q\beta\gamma\gamma_q, \tag{A.9}
\]
\[
F_{\rho\rho\Pi}^{(s)} = \beta(-2 + 3\gamma^2_q) + 3\alpha_q\beta\gamma + \gamma\mu - 3\alpha_q\gamma\gamma_q\mu, \tag{A.10}
\]
for the even contribution, and with the definition of the three point correlation eq. (3.4), the odd part is given by

$$B^{(A)ijk}_{\rho B B H_B^{(s)}} = B^{(A1)}_{\rho B B H_B^{(s)}} k_2 \left( p k_3 + p \right)^k + B^{(A2)}_{\rho B B H_B^{(s)}} k_1 \left( p k_3 + p \right)^k$$

$$+ B^{(A3)}_{\rho B B H_B^{(s)}} k_1 - p k_2 + p j \left( p k_3 + p \right)^k + B^{(A4)}_{\rho B B H_B^{(s)}} k_1 - p k_2 + p j \left( p k_3 + p \right)^k,$$  \hspace{0.5cm} (A.11)

where

$$B^{(A1)}_{\rho B B H_B^{(s)}} = -\frac{12 \left( 2 \pi \right)^3 \left( 4 \pi \right)^3}{(4 \pi)^3} \int \left( d^3 p \left( \rho B B H_B^{(s)} \right) \right) p B \left( \rho B B H_B^{(s)} \right) k_1 - p | \gamma q \rangle q$$

$$B^{(A2)}_{\rho B B H_B^{(s)}} = -\frac{12 \left( 2 \pi \right)^3 \left( 4 \pi \right)^3}{(4 \pi)^3} \int \left( d^3 p \left( \rho B B H_B^{(s)} \right) \right) p B \left( \rho B B H_B^{(s)} \right) k_1 - p | \gamma q \rangle q$$

$$B^{(A3)}_{\rho B B H_B^{(s)}} = -\frac{12 \left( 2 \pi \right)^3 \left( 4 \pi \right)^3}{(4 \pi)^3} \int \left( d^3 p \left( \rho B B H_B^{(s)} \right) \right) p B \left( \rho B B H_B^{(s)} \right) k_1 - p | \gamma q \rangle q$$

$$B^{(A4)}_{\rho B B H_B^{(s)}} = -\frac{12 \left( 2 \pi \right)^3 \left( 4 \pi \right)^3}{(4 \pi)^3} \int \left( d^3 p \left( \rho B B H_B^{(s)} \right) \right) p B \left( \rho B B H_B^{(s)} \right) k_1 - p | \gamma q \rangle q.$$

Scalar cross-correlation \(< \rho \Pi \Pi \Pi \rangle \). The result for the even contribution of the three-point cross-correlation \(< \rho \Pi \Pi \Pi \rangle \) is given by

$$F_{\rho \Pi \Pi (s)}^{(1)} = -6 + 3 \left( \alpha^2 + \beta^2 + \gamma^2 \right) + 3 \left( \alpha^2 + \beta^2 + \gamma^2 \right) + 3 \left( \alpha^2 + \beta^2 + \gamma^2 \right) + 3 \left( \alpha^2 + \beta^2 + \gamma^2 \right) + 3 \left( \alpha^2 + \beta^2 + \gamma^2 \right)$$

$$- 3 \beta_\rho \gamma_\rho \mu - 9 \beta_\rho (\beta_\rho + \beta_\gamma - \alpha_\gamma q)$$

$$F_{\rho \Pi \Pi (s)}^{(2)} = -3 \left( \alpha^2 + \beta^2 + \gamma^2 \right) + 9 \left( \alpha^2 + \beta^2 + \gamma^2 \right) + 3 \left( \alpha^2 + \beta^2 + \gamma^2 \right) + 3 \left( \alpha^2 + \beta^2 + \gamma^2 \right)$$

$$- 3 \beta_\rho \gamma_\rho \mu - 9 \beta_\rho (\beta_\rho + \beta_\gamma - \alpha_\gamma q)$$

$$F_{\rho \Pi \Pi (s)}^{(3)} = 9 \left( \alpha^2 + \beta^2 + \gamma^2 \right) + 3 \left( \alpha^2 + \beta^2 + \gamma^2 \right) + 3 \left( \alpha^2 + \beta^2 + \gamma^2 \right)$$

$$- 3 \beta_\rho \gamma_\rho \mu - 9 \beta_\rho (\beta_\rho + \beta_\gamma - \alpha_\gamma q)$$

$$F_{\rho \Pi \Pi (s)}^{(4)} = 3 \left( \alpha^2 + \beta_\rho \gamma_\rho \mu + 9 \beta_\rho (\beta_\rho + \beta_\gamma - \alpha_\gamma q)$$

$$+ \beta (\mu - 3 \beta_\rho \gamma_\rho \mu + 9 \beta_\rho (\beta_\rho + \beta_\gamma - \alpha_\gamma q)$$

$$+ \beta (\mu - 3 \beta_\rho \gamma_\rho \mu + 9 \beta_\rho (\beta_\rho + \beta_\gamma - \alpha_\gamma q)).$$

(A.15)
Using eq. (3.4), the odd contribution can be written as

\[
B^{(A)\, \ell j k}_{\rho_B \Pi_B^{(S)}} = B^{(A1)}_{\rho_B \Pi_B^{(S)}} \Phi^{(\ell)}_{j k} + B^{(A2)}_{\rho_B \Pi_B^{(S)}} \Phi^{(\ell)}_{j k} + B^{(A3)}_{\rho_B \Pi_B^{(S)}} \Phi^{(\ell)}_{j k} + B^{(A4)}_{\rho_B \Pi_B^{(S)}} \Phi^{(\ell)}_{j k} + B^{(A5)}_{\rho_B \Pi_B^{(S)}} \Phi^{(\ell)}_{j k} + B^{(A6)}_{\rho_B \Pi_B^{(S)}} \Phi^{(\ell)}_{j k} + B^{(A7)}_{\rho_B \Pi_B^{(S)}} \Phi^{(\ell)}_{j k} + B^{(A8)}_{\rho_B \Pi_B^{(S)}} \Phi^{(\ell)}_{j k} + B^{(A9)}_{\rho_B \Pi_B^{(S)}} \Phi^{(\ell)}_{j k} + B^{(A10)}_{\rho_B \Pi_B^{(S)}} \Phi^{(\ell)}_{j k},
\]

with

\[
B^{(A1)}_{\rho_B \Pi_B^{(S)}} = \frac{6}{(2\pi)^3(4\pi)^3} \int d^3p P_B(|k1 - p|) (P_B(p) P_H(|p + k2|)(\alpha_q - \beta_q) + P_H(p) P_B(|p + k2|)\gamma_q),
\]

\[
B^{(A2)}_{\rho_B \Pi_B^{(S)}} = \frac{18}{(2\pi)^3(4\pi)^3} \int d^3p P_B(|k1 - p|) P_H(p) P_B(|p + k2|)(-\gamma_p \gamma_q + \theta_{pq}),
\]

\[
B^{(A3)}_{\rho_B \Pi_B^{(S)}} = \frac{18}{(2\pi)^3(4\pi)^3} \int d^3p P_H(|p + k2|) (P_B(p) P_H(|k1 - p|)(\beta_p \alpha_q - \beta_\theta pq) + P_B(|k1 - p|) P_B(p)(-\alpha_p \alpha_q + \alpha_1 \beta_\gamma q - \beta_\beta \gamma q + \theta_{pq})),
\]

\[
B^{(A4)}_{\rho_B \Pi_B^{(S)}} = \frac{2}{(2\pi)^3(4\pi)^3} \int d^3p P_B(|p + k2|) P_B(|k1 - p|)(\gamma - 3\alpha_p \gamma p) - P_B(|k1 - p|) P_B(|p + k2|) \beta + P_H(p) (P_H(|k1 - p|) P_B(|p + k2|) + P_B(|k1 - p|) P_B(|p + k2|)(3\beta_3 \gamma q - \mu)),
\]

\[
B^{(A5)}_{\rho_B \Pi_B^{(S)}} = \frac{6}{(2\pi)^3(4\pi)^3} \int d^3p P_B(|p + k2|) (-P_H(p) P_B(|k1 - p|)\beta_q + P_H(|k1 - p|) P_B(p)(\alpha_q - \gamma_q + 3\alpha_1 \gamma p + 3\alpha_p \gamma q - 3\alpha_p \theta_{pq})),
\]

\[
B^{(A6)}_{\rho_B \Pi_B^{(S)}} = \frac{6}{(2\pi)^3(4\pi)^3} \int d^3p P_B(|p + k2|) P_B(|k1 - p|) P_H(|k1 - p|) \gamma_p + P_B(|k1 - p|) P_B(|p + k2|)(\alpha_p \beta - \beta_p)),
\]

\[
B^{(A7)}_{\rho_B \Pi_B^{(S)}} = \frac{6}{(2\pi)^3(4\pi)^3} \int d^3p P_B(|p + k2|) P_B(|k1 - p|) (-\gamma_p \gamma_q + \theta_{pq}),
\]

\[
B^{(A8)}_{\rho_B \Pi_B^{(S)}} = \frac{6}{(2\pi)^3(4\pi)^3} \int d^3p P_B(|p + k2|) P_B(|k1 - p|) \alpha_p + P_B(|k1 - p|) P_H(p)(-\beta_p - 3\beta_3 \gamma p \gamma q + 3\beta_3 \theta_{pq} + \gamma_p \mu)),
\]

\[
B^{(A9)}_{\rho_B \Pi_B^{(S)}} = \frac{6}{(2\pi)^3(4\pi)^3} \int d^3p H(|p + k2|) P_B(p) P_B(|k1 - p|) \alpha_p + P_H(p) (P_B(|k1 - p|) P_B(|k2 + p|) \beta_p - P_H(|k1 - p|) P_H(|k2 + p|) \gamma_p).}
\]
Scalar cross-correlation (< ΠΠΠ >). Finally, the even contribution for the three-point cross-correlation of scalar anisotropic stress is written as

\[
F_{\Pi(3)\Pi(3)\Pi(3)}^{1} = -9 + 3(\alpha_{p}^{2} + \alpha_{q}^{2} + \alpha_{p}^{2}) - 9\theta_{pq}(\beta_{p}\beta_{q} + \gamma_{p}\gamma_{q} - 3\beta_{k}\beta_{q}\theta_{kp} + 3\theta_{kp}\theta_{kq})
+ 3(\beta_{q}^{2} + \beta_{p}^{2} + \beta_{k}^{2}) + \beta_{p}^{2} + \mu^{2} - \beta_{p}\mu + 3(\beta_{k}^{2} + \beta_{q}^{2} + \gamma_{q}^{2}) - 3\alpha_{q}(\beta_{p} + \gamma_{q})
+ 3\beta_{q}(\gamma_{p}\theta_{pq} - 3\beta_{p}\theta_{kp} + \gamma_{p}\gamma_{q} - 9\theta_{kp}(\beta_{p}\beta_{q} + \gamma_{p}\gamma_{q} + 3\beta_{k}\beta_{q}\gamma_{kp} - \theta_{kp})
- 3\alpha_{q}(\gamma_{p}\gamma_{q} + 3\theta_{pq}(\beta_{p} + 3\beta_{q}\theta_{pq} - 3\beta_{q}\theta_{pq} - \gamma_{p}\mu)) + 9\theta_{pq}^{2}
+ 9\theta_{pq}(\beta_{p}\beta_{q} - 3\alpha_{q}\beta_{q}\gamma_{kp} - \beta_{k}\gamma_{k} + 3\beta_{k}\beta_{q}\gamma_{kp} + 3\alpha_{q}\theta_{kp} - 3\gamma_{q}\theta_{kq} - \beta_{k}\mu
- 3\alpha_{p}(\gamma_{p}\gamma_{k} - \theta_{kp} - 3\gamma_{p}\gamma_{k}\theta_{kq} + 3\beta_{k}\gamma_{k}\theta_{pq} + \beta_{k}(\beta_{p} + 3\beta_{q}\gamma_{kp} - 3\beta_{k}\theta_{pq} - \gamma_{p}\mu))
- 3\mu(\beta_{k}\gamma_{k} + \gamma_{p}\gamma_{p} + \beta_{q}\gamma_{q} - 3\beta_{k}\gamma_{kp} - 9\theta_{kp}(\beta_{k}\beta_{q} + \gamma_{k}\gamma_{q} - 3\beta_{k}\theta_{kp}\gamma_{pq})), \tag{A.31}
\]

\[
F_{\Pi(3)\Pi(3)\Pi(3)}^{2} = 3\beta_{p}(3\alpha_{p}\alpha_{q}\gamma_{k} - \alpha_{p}\gamma_{p} + 6\gamma_{q} - 3\alpha_{q}^{2}\gamma_{q} - 3\gamma_{k}\theta_{kq}) + \alpha_{q}^{2}(9\beta_{q} + 6\mu)
+ 6\beta_{q}(\beta_{p} - 3\gamma_{p}\theta_{pq}) - 9\gamma_{q}(3\beta_{p}\theta_{kq} + \beta_{k}\theta_{kq} - 3\beta_{q}\theta_{kp} + \beta_{p}\theta_{pq})
+ \beta(2\gamma - 6\gamma_{k}\gamma_{k} - 3\alpha_{q}\gamma_{k}\theta_{kq} + 9\theta_{p}(3\theta_{kq}\beta_{k} - \gamma_{q}\theta_{kq})) + 6\beta_{k}\gamma_{k}
+ \mu(-13 + 6\alpha_{p}^{2} + 3\alpha_{q}^{2} + 18\theta_{k}\gamma_{k} + 18\theta_{k}\gamma_{k} + 3\alpha_{q}\theta_{kq} + 9\theta_{k} + 3\theta_{k}\gamma_{k} + 27\theta_{k}\theta_{k}\theta_{pq} + 9\theta_{pq}^{2})
+ 3\alpha_{p}(\alpha_{q}\gamma_{k}\theta_{kp} + \beta_{q}(2\gamma + 6\alpha_{q}\gamma_{k} + 3\alpha_{q}\gamma_{k}\theta_{kq} + 3\gamma_{k}\gamma_{k}\theta_{pq}
- 9\alpha_{k}\beta_{q}\theta_{pq} + 6\gamma_{q}\theta_{pq} + 9\alpha_{k}\theta_{pq}\mu + 3\alpha_{q}\theta_{pq}\mu)), \tag{A.32}
\]

\[
F_{\Pi(3)\Pi(3)\Pi(3)}^{3} = 6\beta_{q}(\gamma_{p} - \alpha_{p}) - 6\alpha_{q}(\beta_{p} + 3\beta_{p}\gamma_{pq} - 27\theta_{k}\theta_{k}\theta_{pq} - 2\mu(\gamma - 3\alpha_{q} \gamma_{q})
+ 3\beta_{q}(\gamma_{p} - \alpha_{p}\gamma_{k}\gamma_{k} + 3\alpha_{q}\theta_{kp} - 3\gamma_{p}\gamma_{k}\theta_{pq} + 9\alpha_{q}\gamma_{q}\theta_{pq} + 3\alpha_{q}\theta_{pq}) + 18\alpha_{q}\beta_{p}\theta_{pq}
- \beta(-13 + 3\beta_{k}^{2} + 2\gamma_{p} + 3\alpha_{q}\gamma_{k}\gamma_{k} + 2\theta_{k}\gamma_{k} + 2\theta_{pq}^{2} - 9\gamma_{k}(\gamma_{p}\theta_{kp} + \gamma_{q}\theta_{kq}))
+ 9\gamma_{p}(3\gamma_{p}\theta_{kq} - 2\theta_{pq} - 27\theta_{k}\theta_{k}\theta_{pq}) + 3\alpha_{k}(3\beta_{k}\gamma_{k} + \beta_{p}(-3\gamma_{p}\gamma_{k} + 3\theta_{k}\gamma_{k} - 9\theta_{k} + 3\theta_{k}\gamma_{k} + 3\theta_{k}\gamma_{k} + 9\gamma_{k}(\gamma_{p}\theta_{kp} + \gamma_{q}\theta_{kq})))
+ 9\gamma_{p}(\gamma_{p} + 3\beta_{k}^{2} + \gamma_{p} - 3\gamma_{p}\gamma_{k}\theta_{pq} + 3\theta_{pq}^{2}) + \mu(\gamma - 3\gamma_{p}\gamma_{k} - 3\gamma_{p}\gamma_{k}\theta_{pq})), \tag{A.33}
\]

\[
F_{\Pi(3)\Pi(3)\Pi(3)}^{4} = -27\alpha_{q}\beta_{k}\beta_{q}\gamma_{p}\theta_{kp} - 3\beta_{p}\gamma_{p} + \mu(2\beta - 3\alpha_{p}\beta_{p} - 6\alpha_{q}\beta_{q} + 9\alpha_{q}\beta_{q}\theta_{pq})
+ 6\gamma_{k}(\beta_{p} - 3\alpha_{p}\beta_{p} + 3\alpha_{q}\beta_{q}) + 9\alpha_{p}\beta_{p}\gamma_{q} + 3\alpha_{q}\beta_{q}(2\alpha_{p} - 3\alpha_{p}\beta_{p} + 3\alpha_{q}\beta_{q}\theta_{pq}
+ 9\gamma_{q}(9\alpha_{p}\beta_{p}\gamma_{q} - \beta_{k}\gamma_{k}\gamma_{k} + 2\alpha_{q}\theta_{kp}\theta_{kq} - 3\alpha_{q}\gamma_{q}\theta_{pq} + 3\alpha_{q}\gamma_{q}\theta_{pq})
+ \alpha_{q}(3\gamma_{p}\theta_{kp} + 3\beta_{q}\beta_{q}\theta_{kp} + \gamma_{k}(\gamma_{p} + 3\beta_{k}\gamma_{k} - 3\gamma_{p}\gamma_{k}\theta_{pq})))
+ \gamma(-13 + 3(2\beta_{p}^{2} + \beta_{p}^{2} + \beta_{q}^{2}) + 9(\theta_{k} + 2\theta_{k}\theta_{pq}^{2} - 9\theta_{pq}(\beta_{p}\beta_{q} + 3\theta_{kp}\theta_{kq})), \tag{A.34}
\]

for the odd contribution we found

\[
B_{\Pi(3)\Pi(3)\Pi(3)}^{(A1)} = \frac{3}{\langle 2\pi \rangle^{3}} \int d^{3} p (P_{H}(p)P_{H}(\langle k1 - p \rangle)P_{H}(\langle p + k2 \rangle)(-2\alpha_{q} + 3\alpha_{k}\theta_{kq})
- P_{B}(p)P_{B}(\langle k1 - p \rangle)P_{B}(\langle p + k2 \rangle)(-2\alpha_{q} + 3\alpha_{k}\theta_{kq})) \tag{A.35}
\]

\[
B_{\Pi(3)\Pi(3)\Pi(3)}^{(A2)} = \frac{1}{\langle 2\pi \rangle^{3}} \int d^{3} p (P_{B}(p)P_{B}(\langle k1 - p \rangle)P_{B}(\langle p + k2 \rangle)(-3\alpha_{p}\beta_{k})
- P_{B}(p)P_{B}(\langle k1 - p \rangle)P_{B}(\langle p + k2 \rangle)(-3\alpha_{p}\beta_{k})) \tag{A.36}
\]

\[
B_{\Pi(3)\Pi(3)\Pi(3)}^{(A3)} = \frac{3}{\langle 2\pi \rangle^{3}} \int d^{3} p (P_{B}(p)P_{B}(\langle k1 - p \rangle)P_{B}(\langle p + k2 \rangle)(-\alpha_{q} + \gamma_{q} - 3\alpha_{p}\gamma_{p}\gamma_{q} + 3\alpha_{p}\theta_{pq})), \tag{A.37}
\]
\begin{align}
B^{(A4)}_{\gamma^2 p} & \equiv \frac{3}{(2\pi)^3(4\pi)^3} \int d^3 p P_B(p) \left( P_B(|p+k|^2) P_H(|k-l-p|) \gamma_p ight. \\
& + P_B(|k-l-p|) P_H(|p+k|^2) (\alpha_P - 3\alpha_k \alpha_P \beta_k - 3\beta_k \theta_k p), \\
B^{(A5)}_{\gamma^2 p} & \equiv \frac{9}{(2\pi)^3(4\pi)^3} \int d^3 p P_B(p) P_B(|p+k|^2) P_H(|k-l-p|)(-\gamma_p \gamma_q + \theta_q), \\
B^{(A6)}_{\gamma^2 p} & \equiv \frac{3}{(2\pi)^3(4\pi)^3} \int d^3 p P_B(|p+k|^2) (P_B(p) P_H(|k-l-p|) \alpha_p \\
& + P_B(|k-l-p|) P_H(p)(-\beta_p - 3\beta_q \gamma_p \gamma_q + 3\beta_q \theta_q p + \gamma_q \mu)), \\
B^{(A7)}_{\gamma^2 p} & \equiv \frac{3}{(2\pi)^3(4\pi)^3} \int d^3 p P_B(|k-l-p|) (P_B(|p+k|^2) P_H(p) \\
& + P_B(p) P_H(|p+k|^2) (\alpha_q - \beta_q + 3\alpha_k \beta_q - 3\alpha_k \theta_k q)), \\
B^{(A8)}_{\gamma^2 p} & \equiv \frac{9}{(2\pi)^3(4\pi)^3} \int d^3 p P_B(|p+k|^2) P_B(|k-l-p|)(-\gamma_p \gamma_q + \theta_q), \\
B^{(A9)}_{\gamma^2 p} & \equiv \frac{3}{(2\pi)^3(4\pi)^3} \int d^3 p P_B(|p+k|^2) (P_B(p) P_B(|k-l-p|) \\
& - 3\theta_k p \theta_k q + \alpha_k (-\alpha_q - \beta_k - 3\alpha_k \beta_k - 3\beta_k \theta_k q) + P_H(|k-l-p|) P_H(p)(-2\alpha_k \beta_k p \\
& + 3\alpha_q \beta_k \theta_k p + 3\alpha_k \beta_k \theta_k q - 3\beta_k \theta_k p + 3\theta_k p - 3\alpha_k \beta_k \theta_k q)), \\
B^{(A10)}_{\gamma^2 p} & \equiv \frac{3}{(2\pi)^3(4\pi)^3} \int d^3 p (P_B(|p+k|^2) P_H(|k-l-p|) \alpha_k \\
& - P_B(p)(P_B(|k-l-p|) P_H(|p+k|^2) \beta_k + P_H(|k-l-p|) P_B(|p+k|^2)(\alpha_k \gamma \\
& - \gamma_k - 3\alpha_k \alpha_P \gamma_P + 3\gamma_P \theta_k p)), \\
B^{(A11)}_{\gamma^2 p} & \equiv \frac{9}{(2\pi)^3(4\pi)^3} \int d^3 p P_B(p) P_B(|p+k|^2) P_H(|k-l-p|)(-\alpha_k \alpha_P + \theta_k p), \\
B^{(A12)}_{\gamma^2 p} & \equiv \frac{9}{(2\pi)^3(4\pi)^3} \int d^3 p P_B(p) P_B(|p+k|^2) P_H(|k-l-p|)(-\gamma_k \gamma_q + 3\gamma_P \gamma_q \theta_k p \\
& + \theta_k q - 3\theta_k p \theta_q + \alpha_k (-\alpha_q + \gamma_q + 3\alpha_q \gamma_q + 3\alpha_q \theta_q)), \\
B^{(A13)}_{\gamma^2 p} & \equiv \frac{9}{(2\pi)^3(4\pi)^3} \int d^3 p P_B(|p+k|^2) (P_B(p) P_B(|k-l-p|)(-\beta_k \beta_q + \theta_k q) \\
& + P_H(|k-l-p|) P_H(p)(-\alpha_q \beta_k + \beta \theta_k q)), \\
B^{(A14)}_{\gamma^2 p} & \equiv \frac{3}{(2\pi)^3(4\pi)^3} \int d^3 p (-P_H(|p+k|^2) P_B(p) P_B(|k-l-p|) \alpha_k \\
& + P_H(p)(P_H(|k-l-p|) P_H(|p+k|^2) \beta_k + P_B(|k-l-p|) P_B(|p+k|^2)(-\gamma_k \\
& - 3\beta_k \beta_q \gamma_q + 3\gamma_q \theta_k q + \beta_k \mu))), \\
B^{(A15)}_{\gamma^2 p} & \equiv \frac{9}{(2\pi)^3(4\pi)^3} \int d^3 p P_H(p) P_B(|p+k|^2) P_B(|k-l-p|)(-\beta_k \beta_q + \theta_k q), \\
B^{(A16)}_{\gamma^2 p} & \equiv \frac{9}{(2\pi)^3(4\pi)^3} \int d^3 p P_H(|p+k|^2) (P_B(p) P_B(|k-l-p|)(\alpha_k \alpha_P - \theta_k p) \\
& + P_H(|k-l-p|) P_H(p)(\alpha_k \beta_p - \beta \theta_k p)), \\
B^{(A17)}_{\gamma^2 p} & \equiv \frac{3}{(2\pi)^3(4\pi)^3} \int d^3 p P_H(p) P_B(|p+k|^2) P_B(|k-l-p|)(-\gamma_k \gamma_p + \theta_k p \\
& + 3\gamma_p \gamma_q \theta_k = 3\theta_k \theta_q \theta_p + \beta_k (-\beta_p - 3\beta_q \gamma_q \gamma_q + 3\beta_q \theta_q \theta_p + \gamma_q \mu)), \\
\end{align}
\[ B^{(A_{18})}_{\Pi_B^{(S)}\Pi_B^{(S)}\Pi_B^{(S)}} = \frac{-3}{(2\pi)^3} \int d^3 p P_B(p) P_H(|p + k2|) P_B(|k1 - p|) \alpha_p \]
\[ + P_H(p) (P_B(|p + k2|) P_B(|k1 - p|)) \gamma_p + P_H(|p + k2|) P_H(|k1 - p|) (2\beta_p - 3\beta_\theta \theta_{kp}) , \quad (A.52) \]
\[ B^{(A_{19})}_{\Pi_B^{(S)}\Pi_B^{(S)}\Pi_B^{(S)}} = \frac{-3}{(2\pi)^3} \int d^3 p P_B(|p + k2|) (P_H(|k1 - p|) P_B(p) \alpha_k \]
\[ - P_H(p) P_B(|k1 - p|) \beta_k) , \quad (A.53) \]

where
\[ B^{(A_{\ell j k})}_{\Pi_B^{(S)}\Pi_B^{(S)}\Pi_B^{(S)}} = B^{(A_{1})}_{\Pi_B^{(S)}\Pi_B^{(S)}\Pi_B^{(S)}} k1 - p k2 + p p k3 + B^{(A_{2})}_{\Pi_B^{(S)}\Pi_B^{(S)}\Pi_B^{(S)}} k1 - p k2 + p \]
\[ + B^{(A_{3})}_{\Pi_B^{(S)}\Pi_B^{(S)}\Pi_B^{(S)}} k1 - p k3 + p p k \]
\[ + B^{(A_{4})}_{\Pi_B^{(S)}\Pi_B^{(S)}\Pi_B^{(S)}} k1 - p k2 + p k3 + p \]
\[ + B^{(A_{5})}_{\Pi_B^{(S)}\Pi_B^{(S)}\Pi_B^{(S)}} k1 - p \]
\[ + B^{(A_{6})}_{\Pi_B^{(S)}\Pi_B^{(S)}\Pi_B^{(S)}} k1 - p k2 + p k \]
\[ + B^{(A_{7})}_{\Pi_B^{(S)}\Pi_B^{(S)}\Pi_B^{(S)}} k1 - p \]
\[ + B^{(A_{8})}_{\Pi_B^{(S)}\Pi_B^{(S)}\Pi_B^{(S)}} k1 - p k2 + p k \]
\[ + B^{(A_{9})}_{\Pi_B^{(S)}\Pi_B^{(S)}\Pi_B^{(S)}} k1 - p \]
\[ + B^{(A_{10})}_{\Pi_B^{(S)}\Pi_B^{(S)}\Pi_B^{(S)}} k1 - p k2 + p k \]
\[ + B^{(A_{11})}_{\Pi_B^{(S)}\Pi_B^{(S)}\Pi_B^{(S)}} k1 - p \]
\[ + B^{(A_{12})}_{\Pi_B^{(S)}\Pi_B^{(S)}\Pi_B^{(S)}} k1 - p k2 + p k \]
\[ + B^{(A_{13})}_{\Pi_B^{(S)}\Pi_B^{(S)}\Pi_B^{(S)}} k1 - p \]
\[ + B^{(A_{14})}_{\Pi_B^{(S)}\Pi_B^{(S)}\Pi_B^{(S)}} k1 - p k2 + p k \]
\[ + B^{(A_{15})}_{\Pi_B^{(S)}\Pi_B^{(S)}\Pi_B^{(S)}} k1 - p \]
\[ + B^{(A_{16})}_{\Pi_B^{(S)}\Pi_B^{(S)}\Pi_B^{(S)}} k1 - p k2 + p k \]
\[ + B^{(A_{17})}_{\Pi_B^{(S)}\Pi_B^{(S)}\Pi_B^{(S)}} k1 - p \]
\[ + B^{(A_{18})}_{\Pi_B^{(S)}\Pi_B^{(S)}\Pi_B^{(S)}} k1 - p k2 + p k \]
\[ + B^{(A_{19})}_{\Pi_B^{(S)}\Pi_B^{(S)}\Pi_B^{(S)}} k1 - p \]

(A.54)

Without helical contributions of the field(A_H = 0), our results are in agreement with the ones found in [65], however there is an additional factor of 3 in the eq. (A.18) in three terms.

B Integration domain

The angular part of the integrals must be written in spherical coordinates \( d^3 p = 2\pi p^2 dp d\alpha_k \), where \( 2\pi \) comes from of the integration of \( \theta \). Since we consider an upper cut-off \( k_D \) that corresponds to the damping scale at the spectrum, we must introduce the \((k1, k2)\)-dependence on the angular integration domain. This implies that we should split the integral domain in different regions such that

\[ |k1 - p| \leq k_D, \quad |k2 + p| \leq k_D, \quad (B.1) \]

obtaining that region of the wave vectors where \( 0 < k1, k2 < 2k_D \). Since we expect that most important contribution comes from \( k1 \to -k2 \) and using the above constraints we get the following integration domain in a squeezed configuration

\[ k_D > k2 > 0 \]
\[ k2 > k1 > 0 \]
\[ \int_0^{k_D - k2} dp \int_{-1}^{1} d\alpha_k + \int_{k_D - k2}^{k_D} dp \int_{\frac{1}{2\alpha_D}}^{1} \frac{d\alpha_k}{\alpha_D} \]

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Thus, the expression (3.2) can be written as

\[ k_D > k_1 > k_2 \quad \int_0^{k_D - k_1} dp \int_{-1}^1 d\alpha_k + \int_{k_D - k_1}^{k_D} dp \int_{\frac{k_1^2 + p^2 - k_2^2}{2k_1p}}^1 d\alpha_k \]

\[ 2k_D > k_1 > k_D \quad \int_{k_1 - k_D}^{k_D} dp \int_{\frac{k_1^2 + p^2 - k_2^2}{2k_1p}}^1 d\alpha_k \]

\[ 2k_D > k_2 > k_D \]

\[ k_2 > k_1 > 0 \quad \int_{k_2 - k_D}^{k_D} dp \int_{\frac{k_2^2 + p^2 - k_3^2}{2k_2p}}^1 d\alpha_k \]

\[ 2k_D > k_1 > k_2 \quad \int_{k_1 - k_D}^{k_D} dp \int_{\frac{k_1^2 + p^2 - k_2^2}{2k_1p}}^1 d\alpha_k. \]  

(B.2)

The above integration domain was used to calculate the bispectrum for causal fields shown in figures 3 and 4. However, for the case of non-causal primordial magnetic fields (negative spectral indices) we can approximate the above result by selecting only regions where we can get the biggest contribution to the bispectrum (in fact, in [46] they claimed that the biggest contribution comes from the poles of the integral). Then, we can work with the approximation made in [48, 50, 54] where \( k_2 < k_1 < k_D \) and the angular part is neglected, finding that scheme of integration is reduced to

\[ k_D > k_2 > 0 \]

\[ k_D > k_1 > k_2 \quad \int_0^{k_D} dp, \]  

(B.3)

and therefore the bispectrum can be approximated in following way: the wave vector can be expressed in the basis defined in figure 1 as follows

\[ \mathbf{k}_1 = \hat{e}_z, \quad \mathbf{p} = \sin \theta \cos \phi \hat{e}_x + \sin \theta \sin \phi \hat{e}_y + \cos \theta \hat{e}_z, \]

\[ \mathbf{k}_2 = -\sin \theta' \hat{e}_x - \cos \theta' \hat{e}_z, \quad \mathbf{k}_3 = \sin \theta'' \hat{e}_x + \cos \theta'' \hat{e}_z, \]

(B.4)

being \( \theta, \theta', \theta'' \) the polar angle of \( \mathbf{p}, \mathbf{k}_2 \) and \( \mathbf{k}_3 \) respectively. With these formulas we can find the inner product between different wave vectors

\[ \mathbf{p} \cdot \mathbf{k}_2 = -\sin \theta \cos \phi \sin \theta' - \cos \theta \cos \theta' \]

\[ \mathbf{p} \cdot \mathbf{k}_3 = \sin \theta'' \sin \theta \cos \phi + \cos \theta \cos \theta'' \]

\[ \mathbf{k}_2 \cdot \mathbf{k}_3 = -\sin \theta'' \sin \theta' - \cos \theta' \cos \theta''. \]  

(B.5)

Thus, the expression (3.2) can be written as

\[ \int p^n |\mathbf{k}_1 - \mathbf{p}|^n |\mathbf{k}_2 + \mathbf{p}|^n d^3p \sim 2\pi \int dp p^{n+2} \left( |k_1 - p|^n (p^2 + k_2^2 - 2pk_2 \cos \theta')^{n/2} + |k_2 - p|^n (p^2 + k_1^2 - 2pk_1 \cos \theta')^{n/2} \right) \]

\[ \sim \int dp p^{n+2} \left( k_1^n \left| 1 - \frac{p}{k_1} \right|^n k_2^n (1 + \left( \frac{p}{k_2} \right)^2 - 2 \frac{p}{k_2} \cos \theta')^{n/2} + k_2^n \left| 1 - \frac{p}{k_2} \right|^n k_1^n (1 + \left( \frac{p}{k_1} \right)^2 - 2 \frac{p}{k_1} \cos \theta')^{n/2} \right) \]

\[ \sim 2 \left( \frac{nk_1 nk_2 2^{n+3}}{(n+3)(2n+3)} + \frac{nk_1 3^{n+3}}{(2n+3)(3n+3)} + \frac{k_D^{3n+3}}{(3n+3)} \right), \]  

(B.6)
where in the last equality we have accounted eq. (B.3) and split into sub-ranges: $0 < q < k_2$, $k_2 < q < k_1$ and $k_1 < q < k_D$. This result was derived analytically in [54].

C Integration domain for $\alpha \neq 0$

We use the convolutions for the PMFs spectra with the parametrization for the magnetic field given in eqs. (2.2), (2.5) and (2.6). Since $P_B \neq 0$ and $P_H \neq 0$ for $k_m < k_1,k_2 < k_D$, some conditions need to be taken into account: $k_m < p < k_D$, $k_m < |k_1 - p| < k_D$ and $k_m < |k_2 + p| < k_D$. The latter conditions introduce a $k$-dependence on the angular integration domain and using the squeezed configuration $(k_1 = -k_2 \equiv k, k_3 \approx 0)$, the bispectrum is non zero only for $0 < k < 2k_D$. Such constraints split the integrals in different parts as you can see in the appendix in [37]. However, as claimed in [38], the $p$-integrals need a further splitting for odd $n_H,n_B$. Here we will show only the result for $k_D > 5k_m$ and $2k_m > k_D > k_m$.

For $k_D > 5k_m$, we have:

\[
\begin{align*}
&k_m > k > 0 \\
&\int_{k_m}^{k+k_m} d^3p(p>k) \int_{-1}^{\frac{k^2+p^2-k_m^2}{2kp}} d\gamma + \int_{k_m+k}^{k_D} d^3p(p>k) \int_{-1}^{\frac{k^2+p^2-k_m^2}{2kp}} d\gamma + \int_{k_m}^{k} d^3p(p>k) \int_{-1}^{\frac{k^2+p^2-k_m^2}{2kp}} d\gamma \quad \text{(C.1)}\\
&2k_m > k > k_m \\
&\int_{k_m}^{k} d^3p(p>k) \int_{-1}^{\frac{k^2+p^2-k_m^2}{2kp}} d\gamma + \int_{k_m+k}^{k_D} d^3p(p>k) \int_{-1}^{\frac{k^2+p^2-k_m^2}{2kp}} d\gamma \\
&+ \int_{k_m}^{k_D-k} d^3p(p>k) \int_{-1}^{\frac{k^2+p^2-k_m^2}{2kp}} d\gamma + \int_{k_m+k}^{k_D} d^3p(p>k) \int_{-1}^{\frac{k^2+p^2-k_m^2}{2kp}} d\gamma \quad \text{(C.2)}\\
&\frac{k_D - k_m}{2} > k > 2k_m \\
&\int_{k_m}^{k_D-k} d^3p(p>k) \int_{-1}^{\frac{k^2+p^2-k_m^2}{2kp}} d\gamma + \int_{k_m+k}^{k_D} d^3p(p>k) \int_{-1}^{\frac{k^2+p^2-k_m^2}{2kp}} d\gamma \\
&+ \int_{k_m}^{k_D-k} d^3p(p>k) \int_{-1}^{\frac{k^2+p^2-k_m^2}{2kp}} d\gamma + \int_{k_m+k}^{k_D} d^3p(p>k) \int_{-1}^{\frac{k^2+p^2-k_m^2}{2kp}} d\gamma \quad \text{(C.3)}\\
&\frac{k_D - k_m}{2} > k > \frac{k_D - k_m}{2} \\
&\int_{k_m}^{k_D-k} d^3p(p>k) \int_{-1}^{\frac{k^2+p^2-k_m^2}{2kp}} d\gamma + \int_{k_m+k}^{k_D} d^3p(p>k) \int_{-1}^{\frac{k^2+p^2-k_m^2}{2kp}} d\gamma \\
&+ \int_{k_m}^{k_D-k} d^3p(p>k) \int_{-1}^{\frac{k^2+p^2-k_m^2}{2kp}} d\gamma + \int_{k_m+k}^{k_D} d^3p(p>k) \int_{-1}^{\frac{k^2+p^2-k_m^2}{2kp}} d\gamma \quad \text{(C.4)}\\
&\frac{k_D - k_m}{2} > k < \frac{k_D + k_m}{2} \\
&\int_{k_m}^{k_D-k} d^3p(p>k) \int_{-1}^{\frac{k^2+p^2-k_m^2}{2kp}} d\gamma + \int_{k_m+k}^{k_D} d^3p(p>k) \int_{-1}^{\frac{k^2+p^2-k_m^2}{2kp}} d\gamma \\
&+ \int_{k_m}^{k_D-k} d^3p(p>k) \int_{-1}^{\frac{k^2+p^2-k_m^2}{2kp}} d\gamma + \int_{k_m+k}^{k_D} d^3p(p>k) \int_{-1}^{\frac{k^2+p^2-k_m^2}{2kp}} d\gamma 
\end{align*}
\]
\[ + \int_{k_{D-k}}^{k} \, d^3 p(p>k) \int_{k_{2}+p_{2}+k_{2}^{2}}^{\frac{k^2+p^2+k^2}{2kp}} \, d\gamma + \int_{k_{k+}}^{k_{D+k}} \, d^3 p(p>k) \int_{k_{2}+p_{2}+k_{2}^{2}}^{\frac{k^2+p^2+k^2}{2kp}} \, d\gamma \]

\[ \text{C.5} \]

\[ k_D - k_m > k > \frac{k_D + k_m}{2} \]

\[ \int_{k_m}^{k_{D-k}} \, d^3 p(p>k) \int_{-1}^{1} \, d\gamma + \int_{k_{D-k}}^{k_{k-m}} \, d^3 p(p>k) \int_{k_{2}+p_{2}+k_{2}^{2}}^{\frac{k^2+p^2+k^2}{2kp}} \, d\gamma + \int_{k_{k+}}^{k_{D+k}} \, d^3 p(p>k) \int_{k_{2}+p_{2}+k_{2}^{2}}^{\frac{k^2+p^2+k^2}{2kp}} \, d\gamma \]

\[ \text{C.6} \]

\[ k_D > k > k_D - k_m \]

\[ \int_{k_{k-m}}^{k_{k-m}} \, d^3 p(p>k) \int_{k_{2}+p_{2}+k_{2}^{2}}^{\frac{k^2+p^2+k^2}{2kp}} \, d\gamma + \int_{k_{D}}^{k_{k-m}} \, d^3 p(p>k) \int_{k_{2}+p_{2}+k_{2}^{2}}^{\frac{k^2+p^2+k^2}{2kp}} \, d\gamma \]

\[ \text{C.7} \]

\[ 2k_D > k > k_D + k_m \]

\[ \int_{k_{k-m}}^{k_{k-m}} \, d^3 p(p>k) \int_{k_{2}+p_{2}+k_{2}^{2}}^{\frac{k^2+p^2+k^2}{2kp}} \, d\gamma \]

\[ \text{C.8} \]

\[ k_D + k_m > k > k_D \]

\[ \int_{k_m}^{k_{D-k}} \, d^3 p(p>k) \int_{-1}^{1} \, d\gamma + \int_{k_{k-m}}^{k_{D-k}} \, d^3 p(p>k) \int_{k_{2}+p_{2}+k_{2}^{2}}^{\frac{k^2+p^2+k^2}{2kp}} \, d\gamma \]

\[ \text{C.9} \]

For the case where \(2k_m > k_D > k_m\), we have

\[ \frac{k_D - k_m}{2} > k > 0 \]

\[ \int_{k_m}^{k_{k-m}} \, d^3 p(p>k) \int_{-1}^{1} \, d\gamma + \int_{k_{k-m}}^{k_{D-k}} \, d^3 p(p>k) \int_{k_{2}+p_{2}+k_{2}^{2}}^{\frac{k^2+p^2+k^2}{2kp}} \, d\gamma \]

\[ \text{C.10} \]

\[ k_D - k_m > k > \frac{k_D - k_m}{2} \]

\[ \int_{k_m}^{k_{D-k}} \, d^3 p(p>k) \int_{-1}^{1} \, d\gamma + \int_{k_{k-m}}^{k_{D-k}} \, d^3 p(p>k) \int_{k_{2}+p_{2}+k_{2}^{2}}^{\frac{k^2+p^2+k^2}{2kp}} \, d\gamma \]

\[ \text{C.11} \]

\[ k_m > k > k_D - k_m \]

\[ \int_{k_m}^{k_{D-k}} \, d^3 p(p>k) \int_{k_{2}+p_{2}+k_{2}^{2}}^{\frac{k^2+p^2+k^2}{2kp}} \, d\gamma \]

\[ \text{C.12} \]
\[
\int_{k_m}^{k} d^3p_{(k>p)} \left( \frac{k^2 + p^2 - k_m^2}{2kp} \right) d\gamma + \int_{k}^{k_D} d^3p_{(p>k)} \left( \frac{k^2 + p^2 - k_D^2}{2kp} \right) d\gamma 
\]
\[\text{C.13}\]

\[
2k_m > k > k_D
\]
\[\int_{k_m}^{k_D} d^3p_{(k>p)} \left( \frac{k^2 + p^2 - k_m^2}{2kp} \right) d\gamma
\]
\[\text{C.14}\]

\[
k_m + k > k_m + k_D
\]
\[\int_{k_m}^{k_D} d^3p_{(k>p)} \left( \frac{k^2 + p^2 - k_m^2}{2kp} \right) d\gamma + \int_{k_m}^{k} d^3p_{(p>k)} \left( \frac{1}{k^2 + p^2 - k_D^2} \right) d\gamma
\]
\[\text{C.15}\]

\[
2k_D > k > k_m + k_D
\]
\[\int_{k_m}^{k_D} d^3p_{(k>p)} \left( \frac{k^2 + p^2 - k_D^2}{2kp} \right) d\gamma.
\]
\[\text{C.16}\]

The integration domain above generalizes the results obtained in [37]. With this we can calculate the spectrum and bispectrum (under certain configurations) of PMFs for any value of the magnetic spectral index.

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