Abstract. We compute q-holonomic formulas for the HOMFLY polynomials of 2-bridge links colored with one-column (or one-row) Young diagrams.

1. Introduction and statement of results

The colored HOMFLY polynomial is an invariant of framed, oriented links $L$ in $S^3$ whose components are colored with Young diagrams (or alternatively partitions of integers). It takes values in the ring $\mathbb{Z}[a^{\pm 1}][q]$. In this note we mostly consider colorings by one-column Young diagrams, i.e. partitions of the form $1^r := (1, \ldots, 1)$ and refer to them as $r$-colorings.

Let $L$ be a framed, oriented link with components numbered $1, \ldots, k$. Let $P_{c_1, \ldots, c_k}(L)$ be the colored HOMFLY polynomial of $L$ with coloring $c_i$ on the $i$th component. As expected, this infinite set $\{P_{c_1, \ldots, c_k} \mid c_i \in \mathbb{N}\}$ of invariants carries only a finite amount of information. More precisely, Garoufalidis [Gar] proved that $P_{c_1, \ldots, c_k}(L)$ is q-holonomic in the $c_i$.

**Definition 1.** A one-parameter sequence $(f_n \mid n \in \mathbb{N})$ of polynomials in $\mathbb{Z}[a^{\pm 1}][q]$ is q-holonomic if there exist $d \in \mathbb{N}$ and polynomials $a_l \in \mathbb{Z}[u, v]$ for $0 \leq l \leq d$ such that:

$$\sum_{l=0}^{d} a_l(q, q^n)f_{n+l} = 0 \quad \text{for all } n \geq 0$$

This recurrence relation can be encoded as a polynomial $A \in \mathbb{Z}[a^{\pm 1}, q^{\pm 1}][M](L)$, with $LM = qML$. Here $L$ and $M$ are considered as operators that act on the sequence $f_n$ by $(Mf_n)(q) = q^n f_n(q)$ and $(Lf_n)(q) = f_{n+1}(q)$. The recurrence relation then takes the compact form $Af_n = 0$. We will not need the multivariable generalization of q-holonomic sequences [Zei], [GL] in this note, see part (1) of Theorem 4.

As intermediate step for computing these recurrence relations, it suffices to find a formula for the colored HOMFLY polynomial that is a multi-dimensional sum of q-proper hypergeometric summands. Then the recursion relation can be computed algorithmically [WZe], see [Gar]. Explicit q-holonomic formulas are known for torus links [BMS] and finitely many twist knots [Kaw]. The aim of this note is to compute explicit q-holonomic formulas for the colored HOMFLY polynomials of 2-bridge links.

**Definition 2.** A 2-bridge link is a link $L$ such that the pair $(S^3, L)$ can be split by an embedded $(S^2, \{4 \text{ points}\})$ into two pairs that are both homeomorphic to $(D^2 \times [0, 1], \{2 \text{ points}\} \times [0, 1])$.

A 2-bridge link has one or two components and in the latter case they are unknots. Every 2-bridge link has a special link diagram that can be constructed as follows:
(1) Start with the trivial tangle $\bigcirc$.
(2) Glue a finite number of crossings $\bigotimes$ (or $\bigotimes$) to the top endpoints.
(3) Glue a finite number of crossings $\bigotimes$ (or $\bigotimes$) to the right endpoints.
(4) Repeat (2) and (3) finitely many times. This produces a positive (or negative) rational tangle.
(5) Close the rational tangle up by connecting the four endpoints by two arcs, without introducing new crossings and without making crossings nugatory.

The sequence of natural numbers of crossings added alternately in steps (2) and (3) can be interpreted as the continued fraction expansion of a rational number $p/q \in \mathbb{Q}$ (with a minus sign in the case of negative rational tangles) and it is a well known fact that this is a complete invariant of 2-bridge links. Without loss of generality we only consider 2-bridge links that are closures of positive rational tangles.

**Example 3.**

The link associated to $\frac{7}{2} = [3, 2]$ is

Our main result is:

**Theorem 4.** Let $L$ be a 2-bridge link that is the closure of a positive rational tangle with continued fraction expansion $[a_r, \ldots, a_1]$ with a total of $n$ crossings. Then the following hold:

1. There exists a sequence $\{\tilde{P}_j(L) \in \mathbb{Z}[a^{\pm 1}, s^{\pm 1}](q) \mid j \in \mathbb{N}\}$, which only depends on the oriented link $L$ such that:
   
   $$P_j(L) = \tilde{P}_j(L) \big|_{s=1} P_j(\bigcirc) \quad \text{if } L \text{ is a knot.}$$
   $$P_{i,j}(L) = \tilde{P}_j(L) \big|_{s=q^{i-j}} P_i(\bigcirc) \quad \text{if } L \text{ has two components.}$$

   In particular, the colored HOMFLY polynomial of a two-component 2-bridge link, reduced with respect to color $i$, becomes independent of $i$ up to shifts in $q$-degree.

2. $\tilde{P}_j(L)$ is given (up to multiplication by a monomial in $a$, $q$ and $s$) by the $q$-holonomic formula:

   $$\tilde{P}_j(L) = \sum_{i_1=0}^{j} \cdots \sum_{i_{a_1}=i_{a_1-1}}^{j} \sum_{i_{a_1+1}=0}^{i_{a_1+a_2-1}} \cdots \sum_{i_{a_1+a_2}=0}^{i_{a_1=0}} \cdots \prod_{l=1}^{n} (-1)^{i_l} b_l(j, i_l, i_{l-1}) \left[ \frac{n_l(j, i_k)}{m_l(i_l, i_{l-1})} \right] Cl[\partial L](i_n)$$

   This is a multi-dimensional sum with one index per crossing. Each index $i_l$ runs between $i_{l-1}$ and $j$ in the case of a top crossing and between 0 and $i_{l-1}$ in the case of a right crossing. $b_l(j, i_l, i_{l-1})$ is a monic monomial in $q, a, s$ which depends only on $j, i_l, i_{l-1}$ and the boundary data of the tangle before the $l$th step of the inductive construction process. $\left[ \frac{n_l(j, i_l)}{m_l(i_l, i_{l-1})} \right]$ is a $q$-binomial coefficient with $n_l(j, i_l)$ and $m_l(i_l, i_{l-1})$ depending at most linearly on $j, i_l$ and $i_{l-1}$. Finally $Cl[\partial L](i_n)$ is a quotient of products of $q$-Pochhammer symbols, which depends only on $j, i_n$ and the boundary data of the rational tangle before closing up.

Theorem 4 provides an explicit $q$-holonomic formula for the $r$-colored HOMFLY polynomials of 2-bridge links. An implementation in Wolfram Mathematica can be found on the author’s website [https://www.dpmms.cam.ac.uk/~pw360/](https://www.dpmms.cam.ac.uk/~pw360/).
Remark 5. The HOMFLY polynomials $P_{rt}(L)$ with respect to colorings with one-row partitions with $r$ boxes are related to $P_r(L)$ as follows:

$$P_{rt}(L)(a, q) = (-1)^r P_r(L)(a, q^{-1})$$

This is proved, for example, in [LP] Lemma 4.2.

Acknowledgements. This note is a follow-up to [Wed], where we compute categorified $sl(N)$ invariants of positive rational tangles and verify conjectures of Gukov and Stosić in the setting of tangles, see also [GS], [GNSS]. I would like to thank Jacob Rasmussen for many interesting and fruitful discussions and Stavros Garoufalidis for his encouragement to write this note.

2. Proof of Theorem 4

(1) is well known for knots; in the more interesting case of links the statement holds for arbitrary colored links with an unknot component and is proved in section 5 of [Wed]. We prove (2) in two steps. First we use the replacement rules of [MOY] to evaluate the rational tangle associated to $[a_r, \ldots, a_1]$ in an appropriate skein module and expand in a distinguished basis. This can be done by induction on the number of crossings and in the $l$th step we get a new one-parameter sum over $i_l$, a monomial $b_l(j, i_l, i_{l-1})$ and a q-binomial coefficient $\left[\frac{n_l(j, i_l)}{m_k(i_l, i_{l-1})}\right]$. Second, we replace basis elements in the expansion by the $i$-reduced MOY evaluations of their closures, this accounts for the factor $Cl[\partial L](i_n)$.

Step 1: evaluating a rational tangle. Via the crossing replacement rules of [MOY] a rational tangle can be interpreted as an element of the HOMFLY skein of a disc with four boundary points whose colors and orientations are determined by the tangle. Such a skein is a module over the base ring $\mathbb{Z}[a^{\pm 1}, s^{\pm 1}](q)$ spanned by planar, oriented, trivalent graphs with a flow on the edges, modulo local relations and boundary preserving isotopy, see e.g. [Wed] section 2 for details. In the case of colors $i, j$ with $i \geq j$ each of these skeins is a free module of rank $j + 1$ over the base ring with basis elements (indexed by $0 \leq k \leq j$) as shown in Figure 1. The Figure also displays the four different closure operations and the linear operators $T$ and $R$ given by composing a skein element with a crossing on top or on the right. Throughout we assume $i \geq j$ and that the lower left strand is incoming and colored by $i$.

The skein element of the rational tangle associated to $[a_r, \ldots, a_1]$ is thus the image of the trivial tangle $UP[i, j, 0]$ or $OP[i, j, 0]$ (depending on the chosen orientation) under the linear operator $\cdots T^{a_3} R^{a_2} T^{a_1}$. The following Proposition describes the action of the operators $T$ and $R$ on the basis elements shown in Figure 1 up to overall multiplication by a monomial in $a$, $q$, and $s$. We write the first equation graphically and use more compact notation for the other eleven equations.

Proposition 6. (1) $TUP[i, j, k] = \sum_{h=k}^{j} (-1)^h s^k q^{h(k+1)} \left[\frac{h}{k}\right]$
Proof. Elementary MOY skein theory, or decategorification \((t = -1)\) of computations in section 3.3. of [Wen].

Note that the \(b_i(j, i_l = h, i_{l-1} = k)\), \(n_i(j, i_l = h)\) and \(m_i(i_l = h, i_{l-1} = k)\) can be read off from the formulas in Proposition 6. Furthermore, the summation indices agree with the statement of Theorem 4; top twists produce summations from \(k\) up to \(j\) and right twists from \(0\) to \(k\).

**Step 2: closing off.** Suppose \(L\) is a two-component 2-bridge link, then the invariant of the corresponding rational tangle is a linear combination of terms of one of four forms: \(UP[i, j, k]\), \(OP[i, j, k]\), \(RI[i, j, k]\), \(OPS[i, j, k]\). The first two can be closed off by connecting lower and upper endpoints (without introducing new crossings); the latter two are closed off by connecting left to right endpoints. The resulting graphs evaluate to the ground ring \(\mathbb{Z}[a^{\pm 1}](q)\) as follows:

**Proposition 7.** We introduce the notation \(\left[ \begin{array}{c} m \\ n \end{array} \right]_a : = \frac{a^n(-m)}{a^n} \frac{((aq^n)^2 q^{-2})_n}{(q^n q^n)_n} = \prod_{l=0}^{n-1} \frac{a q^{m-l} a^{-1} q^{l-m}}{q^{l+1} q^{q-l+1}}\) so that, in particular, \(P_n(\bigcirc) = \left[ \begin{array}{c} 0 \\ n \end{array} \right]_a\). Then the closures of \(UP[i, j, k]\), \(OP[i, j, k]\), \(RI[i, j, k]\)
and \( OPs[i, j, k] \) evaluate as follows:

\[
\begin{align*}
ClUP[i, j, k] &= \left[ \begin{array}{c} -k \\ j - k \end{array} \right] a \left[ \begin{array}{c} -i \\ k \end{array} \right] P_i(\bigcirc) = \left[ \begin{array}{c} -k \\ j - k \end{array} \right] a \left( \prod_{l=0}^{k-1} \frac{as^{-j-l} - a^{-1}s^{l+j}}{q^{l+1} - q^{-l-1}} \right) P_i(\bigcirc) |_{s=q^{-j}} \\
ClOP[i, j, k] &= \left[ \begin{array}{c} -k \\ j - k \end{array} \right] a \left[ \begin{array}{c} i \\ k \end{array} \right] P_i(\bigcirc) = \left[ \begin{array}{c} -k \\ j - k \end{array} \right] a \left( \prod_{l=0}^{k-1} \frac{sq^{j-l} - s^{-1}q^{-l-j}}{q^{l+1} - q^{-l-1}} \right) P_i(\bigcirc) |_{s=q^{j}} \\
ClRI[i, j, k] &= \left[ \begin{array}{c} k - j \\ k \end{array} \right] a \left[ \begin{array}{c} -i \\ j - k \end{array} \right] P_i(\bigcirc) = \left[ \begin{array}{c} k - j \\ k \end{array} \right] a \left( \prod_{l=0}^{j-k-1} \frac{as^{-j-l} - a^{-1}s^{l+j}}{q^{l+1} - q^{-l-1}} \right) P_i(\bigcirc) |_{s=q^{-j}} \\
ClOPS[i, j, k] &= \left[ \begin{array}{c} k - j \\ k \end{array} \right] a \left[ \begin{array}{c} i \\ j - k \end{array} \right] P_i(\bigcirc) = \left[ \begin{array}{c} k - j \\ k \end{array} \right] a \left( \prod_{l=0}^{j-k-1} \frac{sq^{j-l} - s^{-1}q^{-l-j}}{q^{l+1} - q^{-l-1}} \right) P_i(\bigcirc) |_{s=q^{j}}
\end{align*}
\]

Proof. Elementary \[ MOY \] skein theory. \( \square \)

Note that \( Cl[\partial L](i_n) := \frac{ClX[i,j]_{i_n}}{P_i(\bigcirc)} \in \mathbb{Z}[a^{\pm 1}, s^{\pm 1}](q) \) can easily be read off from the right hand sides of the above equations (here we have written \( X \) for the appropriate type of basis element), and as a function of \( s \) \( Cl[\partial L](i_n) \) is independent of \( i \). This completes the proof of Theorem 4 in the case of two-component 2-bridge links.

Let \( L \) be a 2-bridge knot. Then we can still use the complex for the rational tangle as above, but have to set \( i = j \) and hence \( s = 1 \). In this case \( UPS[j, j, k] = UP[j, j, k] \) and \( RIS[j, j, k] = RI[j, j, k] \) which allows to compute the evaluation of the closure of these graphs via the formulas in Proposition 7.

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