MODELS OF FLUID FLOWING IN NON-CONVENTIONAL MEDIA: NEW NUMERICAL ANALYSIS

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Abstract. The concept of differentiation with power law reset has not been investigated much in the literature, as some researchers believe the concept was wrongly introduced, it is unable to describe fractal sharps. It is important to note that, this concept of differentiation is not to describe or display fractal sharps but to describe a flow within a medium with self-similar properties. For instance, the description of flow within a non-conventional media which does not obey the classical Fick's laws of diffusion, Darcy's law and Fourier's law cannot be handle accurately with conventional mechanical law of rate of change. In this paper, we pointed out the use of the non-conventional differential operator with fractal dimension and it possible applicability in several field of sciences, technology and engineering dealing with non-conventional flow. Due to the wider applicability of this concept and the complexities of solving analytically those partial differential equations generated from this operator, we introduced in this paper a new numerical scheme that will be able to handle this class of differential equations. We presented in general the conditions of stability and convergence of the numerical scheme. We applied to some well-known diffusion and subsurface flow models and the stability analysis and numerical simulations for each cases are presented.

1. Introduction. In recent decade, the movement of fluid within a non-conventional media has attracted attention of many scholars and this has also emerged a separate field of investigation[1, 4, 2]. This study of more general behavior of porous media including deformation of solid frame is called poro-mechanics. One can also add to this, fractured aquifers, turbulence and other anomalous media that show irregular structures. One of the must anomalous structure, is perhaps the turbulent flow which is a regime in fluid dynamic that accounts for chaotic changes and flow velocity. We must recall that, a porous medium is mostly classified by its porosity, we can also include other properties including but not limited to permeability, tensile strength, tortuosity and electrical conductivity. We confirm based on the literature review that, several natural substances for instance rocks,

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and soil which comprise (aquifers and petroleum reservoirs); biological tissues like bones, wood and cork; zeolites and also other man-made materials like cements and ceramics can be viewed as porous media. This concept is of course used in several fields of applied science, technology and engineering, for instance, we have in mechanics and filtration: acoustics, geo-mechanics, rock, soil mechanics. We have in engineering, petroleum engineering, construction engineering and bio-remediation. In geo-sciences, we have hydro-geology, petroleum geology, and geophysics. We can also add to this list, biophysics, biology and material science. We can mention that, this theory, is well applied in ink-jet printing and nuclear waste disposal technology. Porous materials frequently have a fractal-like configuration, having a pore superficial area that appears to propagate indeterminately once observed through gradually growing determination. Mathematically, this is described by assigning the pore surface a Hausdorff dimension greater than 2. Thus, with above theory being understood, we shall thus recall that, the porous media, subsurface, turbulence and several other kind of media commonly display fractal properties. It is therefore important to point out that, the flow within these systems which do not have classical physical laws do not obey to the Fick’s laws of diffusion, Darcy’s law and Fourier’s law, since they are no longer based on the classical and simpler Euclidean geometry. Noting that, this geometry, which doesn’t apply to media of non-integer fractal dimensions. As results, following the same line of ideas suggested by several researchers in the field for instance Hausdorff, the basic physical concepts including, distance and velocity which are constance can also be represented in fractal media, thus a reformulation of these is required; the scales for space and time should be reformulated employing the well-known Hausdorff formula \((x^\beta, t^\alpha)\). In this case, one can calculate the fractal dimension as:

\[
\alpha = \log_2(X) \tag{1}
\]

Where \(X\) is the to total distance cover in fractal space. Due to this non conventional flow, a new concept of differentiation was introduced, one needs to confess that, the new differential operator does not describe fractal sharps but describes the flow within media with self-similarities, the definition is given below. one can find several numerical methods for solving partial differential equations and other for ordinary differential equations [11, 9, 10, 12, 13, 3].

**Definition 1.1.** A discontinuous media can be described by fractal dimensions. Chen et al. suggested a fractal derivative defined as [5]:

\[
\frac{\partial u(x,t)}{\partial t^\alpha} = \lim_{t \to t_1} \frac{u(x,t) - u(x,t_1)}{t^\alpha - t_1^\alpha}, \quad \alpha > 0 \tag{2}
\]

The more generalized version is given as:

\[
\frac{\partial u^\beta(x,t)}{\partial t^\alpha} = \lim_{t \to t_1} \frac{u^\beta(x,t) - u^\beta(x,t_1)}{t^\alpha - t_1^\alpha}, \quad \alpha > 0 \tag{3}
\]

**Definition 1.2.** [5] If \(u(t)\) is continuous is an closed interval \([a,b]\), then the fractal integral of \(u\) with order \(\alpha\) is defined as:

\[
\mathring{\int}_a^t \mathring{t}_\alpha^\alpha u(x,t) = \alpha \int_0^t \tau^{\alpha-1} u(x,\tau) d\tau \quad \alpha > 0 \tag{4}
\]
**Definition 1.3.** The well known Mellin Transform is defined by [6]:

\[ \mathcal{M} \{ f(x); p \} = \int_0^\infty x^{p-1} f(x) dx \]  \hspace{1cm} (5)

and the inversion formula is given by [6]:

\[ f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-p} \mathcal{M} \{ f(x); p \} ds \]  \hspace{1cm} (6)

provided that the above integral exists.

Since the numerical method that will be suggested here is linked to the Mellin transform operator, thus in order to accommodate readers that are not familiar with this operator, we present here the definition of the Mellin transform. We shall note that, the Mellin transform is mainly use for differential equations with singularities, as this operator has ability to remove some kind if singularities.

**Definition 1.4.** The Mellin transform of the derivative is given by [6]:

**Lemma 1.5.**

\[ \mathcal{M} [ f^n(x) ] = (-1)^n \frac{\Gamma(p)}{\Gamma(p-n)} \hat{f}(p-n) \]  \hspace{1cm} (7)

Provided that \( x^{n-1} f(r)(x) = 0 \) as \( x \to 0 \) \( \forall \ r = 0, 1, 2 \ldots (n-1) \). also

\[ \mathcal{M} [ x^n f^n(x) ] = (-1)^n \frac{\Gamma(p+n)}{\Gamma(p)} \hat{f}(p) \]  \hspace{1cm} (8)

**Definition 1.6.** The Mellin transform of differential operator [6]:

**Lemma 1.7.**

\[ \mathcal{M} \left[ (x \frac{d}{dx})^n f(x) \right] = (-1)^n p^n \hat{f}(p) \]  \hspace{1cm} (9)

**Definition 1.8.** The Mellin transform of integrals [6]:

\[ \mathcal{M} [ I_n f(x) ] = \mathcal{M} \int_0^x I_{n-1} f(t) dt \]  \hspace{1cm} (10)

\[ (-1)^n \frac{\Gamma(p)}{\Gamma(p+n)} \hat{f}(p+n) \]

**Definition 1.9.** The Mellin transform convolution type theorem is given by [6]

\[ \mathcal{M} f(x) = \hat{f}(p) \text{ and } \mathcal{M} g(x) = \hat{g}(p), \text{ then } \]

\[ \mathcal{M} [ f(x) * g(x) ] = \mathcal{M} \int_0^\infty f(\xi) g \left( \frac{x}{\xi} \right) \frac{d\xi}{\xi} = \hat{f}(p)\hat{g}(p) \]  \hspace{1cm} (11)

\[ \mathcal{M} [ f(x)Og(x) ] = \mathcal{M} \int_0^\infty f(x\xi) g\xi d\xi = \hat{f}(p)\hat{g}(1-p) \]  \hspace{1cm} (12)
Definition 1.10. The Parseval’s Type Property of Mellin transform is given by [6]:

\[ \mathcal{M} f(x) = \tilde{f}(p) \quad \text{and} \quad \mathcal{M} g(x) = \tilde{g}(p), \text{then} \]

\[ \mathcal{M}[f(x)g(x)] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{f}(s)\tilde{g}(p-s)ds. \] (13)

2. New numerical method for fractal order P.D.E. To illustrate the method we consider the general fractal partial differential equation.

\[ \frac{\partial u(x,t)}{\partial t^\alpha} = Lu(x,t) + Nu(x,t) \] (14)

Where \( L \) is a linear operator and \( N \) is a non linear operator.

In this paper we introduce a new numerical scheme for fractal order differential equation. Applying Mellin- Transform (5) on both sides of equation (14), with respect to the variable \( x \) to obtain

\[ \mathcal{M} \left( \frac{\partial u(x,t)}{\partial t^\alpha} ; p \right) = \mathcal{M} [ (Lu(x,t) + Nu(x,t)) ; p ] \] (15)

For the fractal order partial derivative this will be

\[ \frac{du(p,t)}{dt^\alpha} = F(u,t) \] (16)

This is

\[ \frac{du(t)}{dt^\alpha} = F(u,t) \] (17)

where \( u(t) = u(p,t) \) and \( F(u,t) = [(Lu(x,t) + Nu(x,t)) ; p] \).

The next step is to apply the fractal operator on equation (17). Doing so we obtain

\[ u(t) - u(t_0) = \alpha \int_0^t \tau^{\alpha-1} F(u,\tau)d\tau \] (18)

When \( t = t_{n+1} \) we have

\[ u_{n+1} = u(t_{n+1}) = u_0 + \alpha \int_0^{t_{n+1}} \tau^{\alpha-1} F(u,\tau)d\tau \] (19)

When \( t = t_n \)

\[ u_n = u(t_n) = u_0 + \alpha \int_0^{t_n} \tau^{\alpha-1} F(u,\tau)d\tau \]

\[ u_{n+1} - u_n = \alpha \left[ \int_0^{t_{n+1}} \tau^{\alpha-1} F(u,\tau)d\tau - \int_0^{t_n} \tau^{\alpha-1} F(u,\tau)d\tau \right] \] (20)

\[ u_{n+1} - u_n = \alpha \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \tau^{\alpha-1} F(u,\tau)d\tau \] (21)

If we approximate \( f(t,\tau) \) with the following Lagrange polynomial

\[ P(t) \approx f(u,\tau) = \left\{ \begin{array}{ll} \frac{(t-t_{n-1})}{(t_n-t_{n-1})} & F(u,t_n) + \frac{(t-t_n)}{(t_{n-1}-t_n)} F(u,t_{n-1}) \\
\end{array} \right. \]

\[ P(t) = \left\{ \begin{array}{ll} \frac{(t-t_{n-1})}{(t_n-t_{n-1})} & F_n + \frac{(t-t_n)}{(t_{n-1}-t_n)} F_{n-1} \\
\end{array} \right. \] (23)
The fractal integral in equation (22) can then be expressed as

\[ u_{n+1} - u_n = \alpha \sum_{j=0}^{n} \left[ \frac{F_n}{h} \int_{t_j}^{t_{j+1}} t^\alpha \, dt - \frac{F_{n-1}}{h} \int_{t_j}^{t_{j+1}} t^\alpha \, dt \right] \]

or

\[ u_{n+1} - u_n = \alpha \sum_{j=0}^{n} \left[ \frac{F_n}{h} \int_{t_j}^{t_{j+1}} \left( t^\alpha - t_{j+1}^\alpha \right) \, dt - \frac{F_{n-1}}{h} \int_{t_j}^{t_{j+1}} \left( t^\alpha - t_{j+1}^\alpha \right) \, dt \right] \]

Since \( t_{n-2}, t_{n-1}, t_n \) and \( t_{n+1} \) are equally spaced then

\[ t_{n+1} - t_{n-2} = t_{n} - t_{n-1} = (t_{n+1} - t_{n}) = h \]

\[ u_{n+1} - u_n = \alpha \sum_{j=0}^{n} \left[ \frac{F_n}{h} \int_{t_j}^{t_{j+1}} [t^\alpha - (t - t_{n-1})] \, dt - \frac{F_{n-1}}{h} \int_{t_j}^{t_{j+1}} [t^\alpha - (t - t_{n-1})] \, dt \right] \]

or

\[ u_{n+1} - u_n = \alpha \sum_{j=0}^{n} \left[ \frac{F_n}{h} \int_{t_j}^{t_{j+1}} \left( t^\alpha - t_{n-1}^\alpha \right) \, dt - \frac{F_{n-1}}{h} \int_{t_j}^{t_{j+1}} \left( t_{n-1}^\alpha - t_{j+1}^\alpha \right) \, dt \right] \]

so we get

\[ u_{n+1} = u_n + \alpha \sum_{j=0}^{n} \left[ \frac{F_n}{h} \left( \frac{1}{\alpha + 1} (t_{j+1}^\alpha - t_{j+1}^\alpha) - \frac{t_{n-1}^\alpha}{\alpha} (t_{j+1}^\alpha - t_{j+1}^\alpha) \right) \right] \]

or

\[ u_{n+1} = u_n + \alpha \sum_{j=0}^{n} \left[ \frac{F_n}{h} \left( \frac{h^{\alpha+1}}{\alpha + 1} [(j+1)^{\alpha+1} - j^{\alpha+1}] - \frac{h^{\alpha}(j+1)^\alpha}{\alpha} \right) \right] \]

or

\[ u_{n+1} = u_n + \alpha \sum_{j=0}^{n} \left[ \frac{F_n}{h} \left( \frac{(j+1)^{\alpha+1} - j^{\alpha+1}}{\alpha + 1} - \frac{(j-1)^\alpha}{\alpha} \right) \right] \]
To test the effectiveness of the numerical scheme, we take the limit of $\alpha = 1$. Thus taking $\alpha = 1$ we obviously recover the classic Adam-Bashforth Numerical scheme for Partial differential equation.

$$u_{n+1} - u_n = h \sum_{j=0}^{n} \left[ F_n \left\{ \frac{(j+1)^2 - j^2}{2} - (j-1) [(j+1) - j] \right\} - F_{n-1} \left\{ \frac{(j+1)^2 - j^2}{2} - j [(j+1) - j] \right\} \right]$$

or we can write this as

$$u_{n+1} - u_n = h \sum_{j=0}^{n} \left[ F_n \left\{ \frac{(j+1)^2 - j^2}{2} - (j-1) \right\} - F_{n-1} \left\{ \frac{(j+1)^2 - j^2}{2} - j \right\} \right]$$

(31)

or

$$u_{n+1} - u_n = h \left[ \frac{3}{2} F_n - \frac{1}{2} F_{n-1} \right]$$

(32)

or

(33)

3. Error analysis.

**Theorem 3.1.** Let $y^\alpha = f(y,t)$ be a fractal partial differential equation for $\alpha > 0$ such that $f$ is bounded, then the numerical solution of $y^\alpha$ fractal partial differential equation is stable if the following inequality is satisfied.

$$u_{n+1} = u_n + \alpha \sum_{j=0}^{n} \left[ F_n h^\alpha \left\{ \frac{(j+1)^{\alpha+1} - j^{\alpha+1}}{\alpha+1} - \frac{(j-1) [(j+1)^\alpha - j^\alpha]}{\alpha} \right\} - F_{n-1} h^\alpha \left\{ \frac{(j+1)^{\alpha+1} - j^{\alpha+1}}{\alpha+1} - \frac{j [(j+1)^\alpha - j^\alpha]}{\alpha} \right\} \right] + R_n^\alpha$$

(34)

where

$$R_n^\alpha = \frac{h^2}{8} \left\| F^{(2)}(u,\varsigma) \right\| \left( \sum_{j=0}^{n} \frac{t_{j+1}^\alpha - t_j^\alpha}{\alpha} \right), \quad \alpha > 0$$

**Proof.** Following the derivation presented earlier

$$u_{n+1} - u_n = \alpha \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \tau^{\alpha-1} F(u,\tau) d\tau$$

where

$$f(u,\tau) = \left\{ \frac{(t-t_{n-1})}{(t_{n-1}-t_n)} \right\} F_n + \left\{ \frac{(t-t_n)}{(t_{n-1}-t_n)} \right\} F_{n-1} + \frac{F^{(2)}(u,\varsigma)}{2!} \prod_{i=0}^{1}(t-t_i)$$

(35)

$$u_{n+1} - u_n = \alpha \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \tau^{\alpha-1} \left\{ \left\{ \frac{(t-t_{n-1})}{(t_{n-1}-t_n)} \right\} F_n + \left\{ \frac{(t-t_n)}{(t_{n-1}-t_n)} \right\} F_{n-1} \right\} dt$$

$$+ \alpha \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \frac{F^{(2)}(u,\varsigma)}{2!} \prod_{i=0}^{1}(t-t_i)\tau^{\alpha-1} dt$$

(36)
This is equal to
\[ u_{n+1} = u_n + \alpha \sum_{j=0}^{n} F_n h^\alpha \left\{ \frac{(j+1)\alpha + 1 - j^\alpha + 1}{\alpha + 1} - \frac{(j-1)(j+1)\alpha - j^\alpha}{\alpha} \right\} \]
\[ -F_{n-1} h^\alpha \left\{ \frac{(j+1)\alpha + 1 - j^\alpha + 1}{\alpha + 1} - \frac{j(j+1)\alpha - j^\alpha}{\alpha} \right\} + R_n^\alpha \]

Let \( \|f(t)\|_\infty = \sup_{t \in [a,b]} |f(t)| \), we assume that
\[ \|F^{(2)}(u,\varsigma)\|_\infty < M < \infty \]
Then one can easily deduce that, the error which is a function of \( n \) and the fractal dimension \( \alpha \)
\[ R_n^\alpha = \alpha \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \frac{F^{(2)}(u,\varsigma)}{2!} \prod_{i=0}^{j} (t - t_i) t_i^{\alpha - 1} dt \]
\[ \|R_n^\alpha\| \leq \alpha \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \left\| \frac{F^{(2)}(u,\varsigma)}{2!} \prod_{i=0}^{j} (t - t_i) t_i^{\alpha - 1} dt \right\| \]
\[ \leq \alpha \left\| \frac{F^{(2)}(u,\varsigma)}{2!} \right\| \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \left\| \prod_{i=0}^{j} (t - t_i) t_i^{\alpha - 1} dt \right\| \]
\[ \leq \alpha \left\| \frac{F^{(2)}(u,\varsigma)}{2!} \right\| \frac{h^2}{4} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} t_i^{\alpha - 1} dt \]

because
\[ \prod_{i=1}^{n} (t - t_i) < \frac{n!}{4} h^{n+1} \]
so now
\[ \|R_n^\alpha\| \leq \alpha \left\| \frac{F^{(2)}(u,\varsigma)}{2!} \right\| \frac{h^2}{4} \sum_{j=0}^{n} \frac{t_{j+1}^{\alpha} - t_j^{\alpha}}{\alpha} \]
\[ \leq \frac{h^2}{8} \left\| \frac{F^{(2)}(u,\varsigma)}{2!} \right\| \sum_{j=0}^{n} \frac{t_{j+1}^{\alpha} - t_j^{\alpha}}{\alpha}, \quad \alpha > 0 \]

\[ \Box \]

4. **Numerical analysis for fractal diffusion equation.** Let us consider the following fractal diffusion equation.
\[ \frac{\partial u(x,t)}{\partial t^\alpha} = \kappa \frac{\partial^2 u(x,t)}{\partial x^2} \]

Applying Mellin transform on both side we get the following
\[ \mathcal{M} \left( \frac{\partial u(x,t)}{\partial t^\alpha} \right) = \kappa \mathcal{M} \left( \frac{\partial^2 u(x,t)}{\partial x^2} \right) \]
\[ = \kappa [(p-1)(p-2) \mathcal{M} \{ f(x); p-1 \}] \]
Silencing the variable $\mathcal{P}$ writing $u(x, t) = u(t)$ and $F(x, t, u(x, t)) = \frac{\partial u(x, t)}{\partial \alpha}$ then we can rewrite the equation as

$$\frac{du(t)}{dt^{\alpha}} = F(x, t, u(x, t))$$

we proved earlier that the solution in the mellin transform to the previous equation is given by (30)

$$u_{n+1} - u_n = \alpha \sum_{j=0}^{n} \left[ F_n h^\alpha \left\{ \frac{(j+1)^{\alpha+1} - j^{\alpha+1}}{\alpha + 1} - \frac{(j-1) [(j+1)^\alpha - j^\alpha]}{\alpha} \right\} - F_{n-1} h^\alpha \left\{ \frac{(j+1)^{\alpha+1} - j^{\alpha+1}}{\alpha + 1} - j [(j+1)^\alpha - j^\alpha] \right\} \right]$$

Applying the inverse mellin transform of (43) we have

$$u(x, t_{n+1}) = u(x, t_n) + h^\alpha \delta_n^\alpha \frac{\partial^2 u(x, t_n)}{\partial x^2} - h^\alpha \delta_n^{\alpha+1} \frac{\partial^2 u(x, t_{n-1})}{\partial x^2}$$

where

$$\delta_n^\alpha = \alpha \sum_{j=0}^{n} \left\{ \frac{(j+1)^{\alpha+1} - j^{\alpha+1}}{\alpha + 1} - \frac{(j-1) [(j+1)^\alpha - j^\alpha]}{\alpha} \right\}$$

and

$$\delta_n^{\alpha+1} = \alpha \sum_{j=0}^{n} \left\{ \frac{(j+1)^{\alpha+1} - j^{\alpha+1}}{\alpha + 1} - j [(j+1)^\alpha - j^\alpha] \right\}$$

Discretising in the space variable we have

$$u(x_i, t_{n+1}) = u(x_i, t_n) + h^\alpha \delta_n^\alpha \frac{u(x_{i+1}, t_n) - 2u(x_i, t_n) + u(x_{i-1}, t_n)}{(\Delta x)^2} - h^\alpha \delta_n^{\alpha+1} \frac{u(x_{i+1}, t_{n-1}) - 2u(x_i, t_{n-1}) + u(x_{i-1}, t_{n-1})}{(\Delta x)^2}$$

(45)

Let $u(x_i, t_n) = u_i^n$ and $(\Delta x) = l$ we have

$$u_i^{n+1} = u_i^n + h^\alpha \delta_n^\alpha \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{l^2} - h^\alpha \delta_n^{\alpha+1} \frac{u_{i+1}^{n-1} - 2u_i^{n-1} + u_{i-1}^{n-1}}{l^2}$$

(46)

$$u_i^{n+1} = \left( 1 - \frac{2h^\alpha \delta_n^\alpha}{l^2} \right) u_i^n + \frac{h^\alpha \delta_n^\alpha}{l^2} u_{i+1}^n - \frac{h^\alpha \delta_n^{\alpha+1}}{l^2} u_{i-1}^n$$

(47)
5. **Stability analysis of fractal diffusion equation.** In this section, we present in details the stability analysis of the numerical method for solving the considered equation. To achieve this, we employ the Von Neumann stability analysis method approach.

Assume we have Fourier expansion in space of

\[ u(x, t) = \sum_f \hat{u}(t) \exp(j fx) \]

while letting

\[ u^n_j = \hat{u}_n \exp(j f i \Delta x) = \hat{u}_n \exp(j f i l) \]

Equation (47) becomes

\[ \hat{u}_{n+1} e^{j f i l} = \left( 1 - \frac{2h^\alpha \delta_n^1 K}{l^2} \right) \hat{u}_n e^{j f i l} + \frac{h^\alpha \delta_n^1 K}{l^2} \hat{u}_n e^{j f (i+l) l} + \frac{h^\alpha \delta_n^1 K}{l^2} \hat{u}_n e^{j f (i-l) l} - \frac{h^\alpha \delta_n^1 K}{l^2} \hat{u}_{n-1} e^{j f i l} + \frac{2h^\alpha \delta_n^1 K}{l^2} \hat{u}_{n-1} e^{j f i l} - \frac{h^\alpha \delta_n^1 K}{l^2} \hat{u}_{n-1} e^{-j f i l} \]

This is

\[ \hat{u}_{n+1} = \left( 1 - \frac{2h^\alpha \delta_n^1 K}{l^2} \right) \hat{u}_n + \frac{h^\alpha \delta_n^1 K}{l^2} \hat{u}_n e^{j f i l} + \frac{h^\alpha \delta_n^1 K}{l^2} \hat{u}_n e^{-j f i l} - \frac{h^\alpha \delta_n^1 K}{l^2} \hat{u}_{n-1} e^{j f i l} + \frac{2h^\alpha \delta_n^1 K}{l^2} \hat{u}_{n-1} e^{j f i l} - \frac{h^\alpha \delta_n^1 K}{l^2} \hat{u}_{n-1} e^{-j f i l} \]

which implies

\[ \hat{u}_{n+1} = \left( 1 - \frac{2h^\alpha \delta_n^1 K}{l^2} \right) \hat{u}_n + \frac{2h^\alpha \delta_n^1 K}{l^2} \hat{u}_{n-1} + \frac{h^\alpha \delta_n^1 K}{l^2} \hat{u}_n \left( e^{j f i l} + e^{-j f i l} \right) - \frac{h^\alpha \delta_n^1 K}{l^2} \hat{u}_{n-1} \left( e^{j f i l} + e^{-j f i l} \right) \]

Further we have

\[ \hat{u}_{n+1} = \left( 1 - \frac{2h^\alpha \delta_n^1 K}{l^2} \right) \hat{u}_n + \frac{2h^\alpha \delta_n^1 K}{l^2} \hat{u}_{n-1} + \frac{h^\alpha \delta_n^1 K}{l^2} \hat{u}_n \left( 2 \cos(f l) \right) - \frac{h^\alpha \delta_n^1 K}{l^2} \hat{u}_{n-1} \left( 2 \cos(f l) \right) \]

and

\[ \hat{u}_{n+1} = \left( 1 - \frac{2h^\alpha \delta_n^1 K}{l^2} + \frac{h^\alpha \delta_n^1 K}{l^2} \left( 2 \cos(f l) \right) \right) \hat{u}_n + \left( \frac{2h^\alpha \delta_n^1 K}{l^2} - \frac{h^\alpha \delta_n^1 K}{l^2} \left( 2 \cos(f l) \right) \right) \hat{u}_{n-1} \]

Thus we can write

\[ \hat{u}_{n+1} = \left( 1 - \frac{2h^\alpha \delta_n^1 K}{l^2} (1 - 2 \cos(f l)) \right) \hat{u}_n + \left( \frac{2h^\alpha \delta_n^1 K}{l^2} (1 - 2 \cos(f l)) \right) \hat{u}_{n-1} \]
We aim to prove that for all $n \geq 1$ the following inequality is satisfied, $\|u_n\| < \|u_0\|$. From equation (53) we have

\[
\frac{\dot{u}_{n+1}}{\dot{u}_n} = \left(1 - \frac{2\alpha \delta_n^\alpha \kappa}{l^2} (1 - 2 \cos(fl))\right) + \left(\frac{2h\alpha \delta_n^{\alpha - 1} \kappa}{i^2} (1 - 2 \cos(fl))\right) \frac{\dot{u}_{n-1}}{\dot{u}_n} = \left(1 - \frac{4h\alpha \delta_n^\alpha \kappa}{l^2} \sin^2\left(\frac{fl}{2}\right)\right) + \left(\frac{4h\alpha \delta_n^{\alpha - 1} \kappa}{i^2} \sin^2\left(\frac{fl}{2}\right)\right) \frac{\dot{u}_{n-1}}{\dot{u}_n}
\]


\[
(54)
\]

For $n = 0$

\[
\frac{\dot{u}_1}{\dot{u}_0} = 1 - \frac{4h\alpha \delta_n^\alpha \kappa}{l^2} \sin^2\left(\frac{fl}{2}\right)
\]

\[
(55)
\]

\[
\left|\frac{\dot{u}_1}{\dot{u}_0}\right| < 1 \iff -1 < 1 - \frac{4h\alpha \delta_n^\alpha \kappa}{l^2} \sin^2\left(\frac{fl}{2}\right) < 1 \iff 0 < \frac{4h\alpha \delta_n^\alpha \kappa}{l^2} \sin^2\left(\frac{fl}{2}\right) < 1
\]

This will certainly be achieved if

\[
0 < \frac{2h\alpha \delta_n^\alpha \kappa}{l^2} < 1
\]

as

\[
\frac{2h\alpha \delta_n^\alpha \kappa}{l^2} \sin^2\left(\frac{fl}{2}\right) \leq \frac{2h\alpha \delta_n^\alpha \kappa}{l^2}
\]

Therefore

\[
0 < \frac{2h\alpha \delta_n^\alpha \kappa}{l^2} < 1
\]

\[
\frac{h^2}{l^2} < \frac{1}{2\delta_n^\alpha}
\]

we will now prove that $\forall n|u_n| < |u_0|$, we proved already that $|u_1| < |u_0|$

let us assume that $|u_j| < |u_0|, \forall j \geq n$ and prove that $|u_{n+1}| < |u_0|$

\[
|\dot{u}_{n+1}| = \left|\left(1 - \frac{2h\alpha \delta_n^\alpha \kappa}{l^2} (1 - 2 \cos(fl))\right) \dot{u}_n + \left(\frac{2h\alpha \delta_n^{\alpha - 1} \kappa}{i^2} (1 - 2 \cos(fl))\right) \dot{u}_{n-1}\right|
\]

\[
(57)
\]

Let $A_1 = \left(1 - \frac{4h\alpha \delta_n^\alpha \kappa}{l^2} \sin^2\left(\frac{fl}{2}\right)\right)$, $A_2 = \left(\frac{4h\alpha \delta_n^{\alpha - 1} \kappa}{i^2} \sin^2\left(\frac{fl}{2}\right)\right)$

Because of the condition (56) we have both $A_1, A_2 > 0$.

\[
|\dot{u}_{n+1}| = \left|\left(1 - \frac{4h\alpha \delta_n^\alpha \kappa}{l^2} \sin^2\left(\frac{fl}{2}\right)\right) \dot{u}_n + \left(\frac{4h\alpha \delta_n^{\alpha - 1} \kappa}{i^2} \sin^2\left(\frac{fl}{2}\right)\right) \dot{u}_{n-1}\right|
\]

\[
(58)
\]

By induction hypothesis we have

\[
|\dot{u}_{n+1}| < |A_1||\dot{u}_n| + |A_2||\dot{u}_{n-1}|
\]

This means

\[
|\dot{u}_{n+1}| < (A_1 + A_2)|\dot{u}_0|
\]

But

\[
A_1 + A_2 = \left(1 - \frac{4h\alpha \delta_n^\alpha \kappa}{l^2} \sin^2\left(\frac{fl}{2}\right)\right) + \left(\frac{4h\alpha \delta_n^{\alpha - 1} \kappa}{i^2} \sin^2\left(\frac{fl}{2}\right)\right)
\]

Therefore

\[
|\dot{u}_{n+1}| < |\dot{u}_0|
\]
This prove that $\forall n|u_n| < |u_0|$ the numerical scheme solution to the fractal order diffusion equation is stable.

6. Solution of groundwater flow equation. Groundwater models describe the groundwater flow and transport processes using mathematical equations based on certain simplifying assumptions. These assumptions typically involve the direction of flow, geometry of the aquifer, the heterogeneity or anisotropy of sediments or bedrock within the aquifer, the contaminant transport mechanisms and chemical reactions. Because of the simplifying assumptions embedded in the mathematical equations and the many uncertainties in the values of data required by the model, a model must be viewed as an approximation and not an exact duplication of field conditions.

Groundwater models, however, even as approximations are a useful investigation tool that groundwater hydrologists may use for a number of applications. The equation under investigation here is given below as:

$$ S \frac{\partial u}{T \partial t} = \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} $$

(62)

The equation (62) is subjected to the following initial and boundary conditions

$$ u(r, 0) = 0, \quad \lim_{r \to 0} \frac{\partial u(r, t)}{\partial r} = -\frac{Q}{2\pi T} $$

(63)

where:

$S$: is the specific storativity

$T$: the transmissivity of the aquifer

$\partial t$: the time derivative

$Q$: is the constant discharge rate of the borehole

Now we apply fractal derivative with respect to time and for $\partial t$ derivative to have

$$ S \frac{\partial u}{T \partial^{\alpha} t} = \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} $$

(64)

Now we find numerical solution using of this fractal groundwater flow equation. Applying Mellin transform on both side of equation (64) we get the following

$$ \mathcal{M} \left( \frac{\partial u(r, t)}{\partial^{\alpha} t} \right) = \mathcal{M} \left( \frac{T}{S} \left( \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} \right) \right) $$

(65)

Now here if we are taking

$$ F(r, t, u(r, t)) = \mathcal{M} \left[ \frac{T}{S} \left( \frac{1}{\alpha r^{\alpha}} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} \right) \right] $$

(66)

So the equation can be written as

$$ \frac{du}{dt^{\alpha}} = F(r, t, u(r, t)) $$

(67)

we prove earlier the solution of (67) is given as

$$ u_{n+1} - u_n = \alpha \sum_{j=0}^{n} \left[ F_n h^\alpha \left( \frac{(j + 1)^{\alpha+1} - j^{\alpha+1}}{\alpha + 1} - \frac{(j - 1) [(j + 1)^{\alpha} - j^{\alpha}]}{\alpha} \right) \right. $n-1 h^\alpha \left. \left( \frac{(j + 1)^{\alpha+1} - j^{\alpha+1}}{\alpha + 1} - j [(j + 1)^{\alpha} - j^{\alpha}] \right) \right] $$

(68)
Applying the inverse mellin transform of (67) and putting the value of \( F_n \) and \( F_{n-1} \) we have

\[
\begin{align*}
\psi_{n+1} &= \psi_n + h^{\alpha} \delta_n^\alpha \frac{T}{S} \left( \frac{1}{\alpha r^\alpha} \frac{\partial u(r, t_n)}{\partial r} + \frac{\partial^2 u(r, t_n)}{\partial r^2} \right) \\
&\quad - h^{\alpha} \delta_n^{\alpha,1} \frac{T}{S} \left( \frac{1}{\alpha r^\alpha} \frac{\partial u(r, t_{n-1})}{\partial r} + \frac{\partial^2 u(r, t_{n-1})}{\partial r^2} \right) \\
&= h^{\alpha} \delta_n^\alpha \frac{T}{S} \left( \frac{1}{\alpha r^\alpha} \left( \frac{u(r_{i+1}, t_n) - u(r_i, t_n)}{(\Delta x)^2} + \frac{u(r_{i+1}, t_n) - 2u(r_i, t_n) + u(r_{i-1}, t_n)}{(\Delta x)^2} \right) \right) \\
&\quad - h^{\alpha} \delta_n^{\alpha,1} \frac{T}{S} \left( \frac{1}{\alpha r^\alpha} \left( \frac{u(r_{i+1}, t_{n-1}) - u(r_i, t_{n-1})}{(\Delta x)^2} \right) \right) \\
&\quad + \frac{u(r_{i+1}, t_{n-1}) - 2u(r_i, t_{n-1})u(r_{i-1}, t_{n-1})}{(\Delta x)^2} + u(r_i, t_n)
\end{align*}
\]

Discretising in the space variable we have

\[
u(r_i, t_{n+1})
\]

Let \( u(r_n, t_n) = u_i^n \) and \( (\Delta x) = l \) we have

\[
\begin{align*}
\psi_{i+1} &= \psi_i + h^{\alpha} \delta_n^\alpha \frac{T}{S} \left( \frac{1}{\alpha r^\alpha} \frac{u_{i+1}^n - u_i^n}{l^2} + \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{l^2} \right) \\
&\quad - h^{\alpha} \delta_n^{\alpha,1} \frac{T}{S} \left( \frac{1}{\alpha r^\alpha} \frac{u_{i+1}^{n-1} - u_{i-1}^{n-1}}{l^2} + \frac{u_{i+1}^{n-1} - 2u_{i}^{n-1} + u_{i-1}^{n-1}}{l^2} \right) \\
&= \left[ 1 - T \frac{h^{\alpha} \delta_n^\alpha}{l^2} \left( \frac{1}{\alpha r^\alpha} + 2 \right) \right] u_i^n + \left[ h^{\alpha} \delta_n^\alpha T \frac{l}{S} \left( 1 + \frac{1}{\alpha r^\alpha} \right) \right] u_{i+1}^n + \frac{h^{\alpha} \delta_n^{\alpha,1} T}{l^2} \frac{1}{S} \sum u_{i-1}^{n-1} \\
&\quad + \left[ h^{\alpha} \delta_n^{\alpha,1} T \frac{l}{S} \left( 2 + \frac{1}{\alpha r^\alpha} \right) \right] u_{i-1}^{n-1} - \left[ h^{\alpha} \delta_n^{\alpha,1} T \frac{l}{S} \left( 1 + \frac{1}{\alpha r^\alpha} \right) \right] u_{i+1}^{n-1} - \frac{h^{\alpha} \delta_n^{\alpha,1} T}{l^2} \frac{1}{S} \sum u_{i-1}^{n-1}
\end{align*}
\]

7. Stability analysis for fractal groundwater flow equation. Assume we have Fourier expansion in space of

\[
u(x, t) = \sum u_i^n \exp(j \pi x) = \sum \hat{u}_n \exp(j \pi x)
\]

while letting

\[
u_i^n = \hat{u}_n \exp(j \pi x) = \hat{u}_n \exp(j \pi l)
\]
Equation (72) becomes

\[
\hat{u}_{n+1} e^{jfl} = \left[ 1 - \frac{T}{S} \left( \frac{1}{\alpha r^\alpha} + 2 \right) \right] \hat{u}_n e^{jfl} + \left[ \frac{h^\alpha \delta_n^\alpha}{l^2} \left( \frac{1}{S} \right) \frac{T}{\alpha r^\alpha} \hat{u}_{n-1} e^{jfl} \right] + \left[ \frac{h^\alpha \delta_n^\alpha}{l^2} \left( \frac{T}{S} \right) \frac{1}{\alpha r^\alpha} \hat{u}_{n-1} e^{jfl} \right] \tag{73}
\]

which implies

\[
\hat{u}_{n+1} = \left[ 1 - \frac{T}{S} \left( \frac{1}{\alpha r^\alpha} + 2 \right) \right] \hat{u}_n + \left[ \frac{h^\alpha \delta_n^\alpha}{l^2} \left( \frac{1}{S} \right) \frac{T}{\alpha r^\alpha} \hat{u}_{n-1} \right] + \left[ \frac{h^\alpha \delta_n^\alpha}{l^2} \left( \frac{T}{S} \right) \frac{1}{\alpha r^\alpha} \hat{u}_{n-1} \right] \tag{74}
\]

or

\[
\hat{u}_{n+1} = \left[ 1 - \frac{T}{S} \left( \frac{1}{\alpha r^\alpha} + 2 \right) \right] \hat{u}_n + \left[ \frac{h^\alpha \delta_n^\alpha}{l^2} \left( \frac{T}{S} \right) \frac{2}{\alpha r^\alpha} \hat{u}_{n-1} \right] + \left[ \frac{h^\alpha \delta_n^\alpha}{l^2} \left( \frac{T}{S} \right) \frac{2}{\alpha r^\alpha} \hat{u}_{n-1} \right] \tag{75}
\]

Further we have

\[
\hat{u}_{n+1} = \left[ 1 - \frac{T}{S} \left( \frac{1}{\alpha r^\alpha} + 2 \right) \right] \hat{u}_n + \left[ \frac{h^\alpha \delta_n^\alpha}{l^2} \left( \frac{T}{S} \right) \frac{2}{\alpha r^\alpha} \hat{u}_{n-1} \right] \tag{76}
\]

and

\[
\hat{u}_{n+1} = \left[ 1 - \frac{T}{S} \left( \frac{1}{\alpha r^\alpha} + 2 \right) \right] \hat{u}_n + \left[ \frac{h^\alpha \delta_n^\alpha}{l^2} \left( \frac{T}{S} \right) \frac{2}{\alpha r^\alpha} \hat{u}_{n-1} \right] \tag{77}
\]

Thus we can write

\[
\hat{u}_{n+1} = \left[ 1 - \frac{T}{S} \left( \frac{1}{\alpha r^\alpha} + 2 \right) \right] \hat{u}_n + \left[ \frac{h^\alpha \delta_n^\alpha}{l^2} \left( \frac{T}{S} \right) \frac{2}{\alpha r^\alpha} \hat{u}_{n-1} \right] \tag{78}
\]
Let us assume that

\[ \left| \hat{u}_{n+1} \right| < 1 \leftrightarrow -1 < 1 - \frac{2T h^\alpha}{S} \frac{\delta_n}{l^2} \sin^2 \left( \frac{f l}{2} \right) < 1 \]

or

\[ 0 < \frac{2T h^\alpha}{S} \frac{\delta_n}{l^2} \left( \frac{1}{\alpha r^\alpha} + 2 \right) \sin^2 \left( \frac{f l}{2} \right) < 2 \]

\[ \Rightarrow 0 < \frac{T h^\alpha}{S} \frac{\delta_n}{l^2} \left( \frac{1}{\alpha r^\alpha} + 2 \right) \sin^2 \left( \frac{f l}{2} \right) < 1 \]

This will certainly be achieved if

\[ 0 < \frac{T h^\alpha}{S} \frac{1}{l^2} \left( \frac{1}{\alpha r^\alpha} + 2 \right) \delta_n < 1 \]

as

\[ \frac{T h^\alpha}{S} \frac{1}{l^2} \left( \frac{1}{\alpha r^\alpha} + 2 \right) \sin^2 \left( \frac{f l}{2} \right) \delta_n \leq \frac{T h^\alpha}{S} \frac{1}{l^2} \frac{1}{\alpha r^\alpha} + 2 \right) \delta_n \]

Therefore

\[ 0 < \frac{T h^\alpha}{S} \frac{1}{l^2} \left( \frac{1}{\alpha r^\alpha} + 2 \right) \delta_n < 1 \]

or

\[ h^\alpha < \frac{S}{T} \frac{1}{\frac{1}{\alpha r^\alpha} + 2} \delta_n \]

We aim to prove that for all \( n \geq 1 \) the following inequity is satisfied, \(|u_n| < |u_0|\)

Let us assume that \(|u_j| < |u_0|\), \( \forall j \leq n \) and prove that \(|u_{n+1}| < |u_0|\)

From equation (82) we have

\[ |\hat{u}_{n+1}| = \left| \left[ 1 - \frac{2T h^\alpha}{S} \frac{\delta_n}{l^2} \left( \frac{1}{\alpha r^\alpha} + 2 \right) \sin^2 \left( \frac{f l}{2} \right) \right] \hat{u}_n \right| + \left| \left[ h^\alpha \frac{\delta_n}{l^2} \frac{1}{2T} \left( \frac{2}{S} + \frac{1}{\alpha r^\alpha} \right) \sin^2 \left( \frac{f l}{2} \right) \right] \hat{u}_{n-1} \right| \]

Let

\[ A_1 = \left[ 1 - \frac{2T h^\alpha}{S} \frac{\delta_n}{l^2} \left( \frac{1}{\alpha r^\alpha} + 2 \right) \sin^2 \left( \frac{f l}{2} \right) \right] \]

\[ A_2 = \left[ h^\alpha \frac{\delta_n}{l^2} \frac{1}{2T} \left( \frac{2}{S} + \frac{1}{\alpha r^\alpha} \right) \sin^2 \left( \frac{f l}{2} \right) \right] \]
Because of the equation (82) we have both $A_1, A_2 > 0$.

\[
|\hat{u}_{n+1}| = A_1 |\hat{u}_n| + A_2 |\hat{u}_{n-1}|
\]

\[
< A_1 |\hat{u}_n| + A_2 |\hat{u}_{n-1}|
\]

(84)

By induction hypothesis we have

\[
|\hat{u}_{n+1}| < |A_1||\hat{u}_0| + |A_2||\hat{u}_0|
\]

This means

\[
|\hat{u}_{n+1}| < (A_1 + |A_2|)|\hat{u}_0|
\]

But

\[
A_1 + A_2 = \left[ 1 - \frac{2T}{S} \frac{h^\alpha \delta_n}{l^2} \left( \frac{1}{\alpha r^\alpha} + 2 \right) \sin^2 \left( \frac{f l}{2} \right) \right] + \left[ \frac{h^\alpha \delta_n^1}{l^2} \frac{2T}{S} \left( 2 + \frac{1}{\alpha r^\alpha} \right) \sin^2 \left( \frac{f l}{2} \right) \right]
\]

Therefore

\[
|\hat{u}_{n+1}| < |\hat{u}_0|
\]

This prove that $\forall n |u_n| < |u_0|$ the numerical scheme solution to the fractal order ground water flow equation is stable.

8. **Graphical simulation.** In this section, we present the numerical approximation of the heat equation with non-conventional for different values of fractal dimensional order alpha. In this simulation, we consider the initial condition to be $u(x,1)=10 \sin(x)$, the boundaries conditions to be $u[1, x] = 10$, $u[10, x] = 0$. The numerical solutions are depicted in Figures 1, 2, 3, 4 and 5 for different values of fractal.
9. **Conclusion.** We make use of Mellin-transform operator and the Lagrange interpolation formula to introduce a new numerical scheme, to solve numerical those partial differential equations generated by the non-conventional local derivative with fractal dimensional order. A detailed analysis of stability condition and convergence of the numerical solution to the exact solution are given. The non-conventional differential operator used here is not well spread but has a wider applicability in several fields of science, technology and engineering, as this operator is able to describe the flow within non-conventional media with self-similarities. An example of fractured aquifer was considered and the model of groundwater flowing within this confined system we suggested using the non-conventional differential operator with fractal dimensional order. The diffusion model was also considered, both models were solved numerically via the proposed method and some simulations are made for different values of fractal dimensional order.

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