A REPLACEMENT LEMMA FOR OBTAINING POINTWISE ESTIMATES IN PHASE TRANSITION MODELS

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Abstract. We establish a replacement lemma for a variational problem, which is not based on a local argument. We then apply it to a phase transition problem and obtain pointwise estimates.

1. Introduction

We consider the elliptic system

$$\Delta u - W_u(u) = 0, \text{ for } u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m,$$

(1)

where $W : \mathbb{R}^m \to \mathbb{R}$ a nonnegative $C^1$ potential possessing several minima and $W_u(u) := (\partial W/\partial u_1, \ldots, \partial W/\partial u_n)^\top$. The system (1) is variational with associated functional

$$J_\Omega(u) = \int_\Omega \left( \frac{1}{2} |\nabla u|^2 + W(u) \right) dx.$$

(2)

In what follows, we take $\Omega$ to be a bounded, open, and connected set in $\mathbb{R}^n$, with Lipschitz boundary. We introduce the hypothesis

$$(H) \text{ Let } \lambda \to W(a + \lambda w), \text{ with } |w| = 1, \text{ be a strictly increasing function on } [0, r_0). \text{ The vector } a \text{ is a global minimum of } W \text{ and } r_0 \text{ is positive and fixed.}$$

Note that (H) is a very weak nondegeneracy hypothesis that was introduced in [3].

The main purpose of this note is to establish the following

Lemma. Let $\Omega$ be as above and let $A \subset \Omega$ be an open, Lipschitz set with $\partial A \neq \emptyset$. Moreover, suppose that

(1) $u(\cdot) \in W^{1,2}(\Omega) \cap C^1(\Omega),$

(2) $|u(x) - a| \leq r \text{ on } \partial A \cap \Omega, \text{ for some } r \text{ with } 2r \in (0, r_0),$

(3) there is an $x_0 \in A$ such that $|u(x_0) - a| > r.$

Then, there exists $\tilde{u}(\cdot) \in W^{1,2}(\Omega)$ such that

$$\tilde{u}(x) = u(x), \quad \text{in } \Omega \setminus A,$$

$$|\tilde{u}(x) - a| \leq r, \quad \text{in } A,$$

$$J_\Omega(\tilde{u}) < J_\Omega(u).$$

We note that in the lemma, no a priori bound is imposed on the max$_A |u(x) - a|$ and, thus, the lemma is not of local nature. Its meaning is that from the point of view of minimizing $J$ for a function that is in part close to the minimum value of $W$, independently of the structure of $W$, it is more efficient to remain close to the minimum throughout (see Figure 1).
Corollary. Let \( n = m = 2 \) and let \( W \) have exactly one global minimum at \( a = (\alpha, 0) \) on the right half-plane \( \mathbb{R}_+^2 = \{ u_1, u_2 \mid u_1 \geq 0 \} \), while \( W > 0 \) in \( \mathbb{R}_+^2 \setminus \{a\} \). Consider the family of variational problems

\[
\min_{\Omega_R^\mu} J_{\Omega_R^\mu}, \quad \text{where} \quad \Omega_R^\mu = \{ (x_1, x_2) \mid 0 < x_1 < \mu R \text{ and } |x_2| \leq R \},
\]

with corresponding global minimizers \( \{u_{R,\mu}\} \) and suppose that

(i) \( u_{R,\mu} \) maps \( \Omega_R^\mu \) in \( \{(u_1, u_2) \mid u_1 \geq 0\} \), (positivity)

(ii) \( J_{\Omega_R^\mu}(u_{R,\mu}) \leq CR \), where \( C \) a universal constant,

(iii) \( |u_{R,\mu}(x) - a| \leq r \) on \( \{(\mu R, x_2) \mid |x_2| \leq R\} \) and \( \partial u_{R,\mu}/\partial n = 0 \) on the remaining three sides of \( \partial \Omega_R^\mu \).

Then, there exist \( R_0 > 0 \), \( \mu_0 > 0 \), and \( \eta_0 > 0 \) such that

\[
|u_{R,\mu}(x) - a| \leq \frac{r}{2} \quad \text{in} \quad \{(x_1, x_2) \in \Omega_R^\mu \mid \eta_0 R \leq x_1 \leq \mu R\},
\]

for all \( R \geq R_0 \) and \( \mu \geq \mu_0 \).

The proof of the corollary is a two-dimensional measure-theoretic argument, where the kinetic and potential terms in the energy are estimated independently. It would be very interesting to extend this to higher dimensions. The one-dimensional version of the lemma above appeared in [3], and subsequently in [5], where an extension from balls to convex sets was given. For hypotheses (i) and (ii) see [1], [4].

2. Proofs

Proof of the Lemma. We utilize the polar representation

\[
u(x) = a + |u(x) - a| \frac{u(x) - a}{|u(x) - a|} =: a + \rho(x)n(x)
\]

and note that

\[
|\nabla u(x)|^2 = |\nabla \rho(x)|^2 + \rho^2(x)|\nabla u(x)|^2.
\]
**Step 1.** We begin by settling the lemma under the additional hypothesis
\[ \rho(x) \leq 2r < r_0, \quad \text{in } A. \]  

(4)

We choose \( \varepsilon > 0 \) so that \( \rho(x) > r + \varepsilon \), where \( r + \frac{\varepsilon}{2} \) is not a critical value of \( \rho \) in \( A \).

(5)

Therefore, the set
\[ \Gamma_\varepsilon = \partial C_\varepsilon \cap A, \quad \text{where } C_\varepsilon = \{ x \in A \mid \rho(x) > r + \frac{\varepsilon}{2} \}, \]

is a \( C^1 \) manifold in \( A \).

Now, define \( \tilde{u}_\varepsilon \) as follows.

\[
\begin{cases}
\tilde{u}_\varepsilon(x) = u(x), & \text{in } A \setminus C_\varepsilon, \\
\tilde{u}_\varepsilon(x) = a + \left( r + \frac{\varepsilon}{2} \right) n(x), & \text{in } C_\varepsilon, \\
\tilde{u}_\varepsilon(x) = u(x), & \text{in } \Omega \setminus A.
\end{cases}
\]

(6)

Notice that \( \tilde{u}_\varepsilon \) is continuous on \( \Gamma_\varepsilon \). There also holds
\[ |\nabla \tilde{u}_\varepsilon(x)|^2 = \left( r + \frac{\varepsilon}{2} \right)^2 |\nabla n(x)|^2 \leq \rho^2(x) |\nabla n(x)|^2 \leq |\nabla u(x)|^2 \]
in \( C_\varepsilon \). It follows that \( \tilde{u}_\varepsilon \in W^{1,2}(\Omega) \) and, moreover,
\[ \int_\Omega |\nabla u|^2 \, dx \geq \int_\Omega |\nabla \tilde{u}_\varepsilon|^2 \, dx. \]

(7)

Hence, \( \tilde{u}_\varepsilon \rightharpoonup \tilde{u} \) in \( W^{1,2} \) as \( \varepsilon \to 0 \), and by weak lower semi-continuity,
\[ \int_\Omega |\nabla u|^2 \, dx \geq \int_\Omega |\nabla \tilde{u}|^2 \, dx. \]

(8)

Clearly
\[
\begin{cases}
\tilde{u}(x) = a + rn(x), & \text{in } C_0 = \{ x \in A \mid \rho(x) > r \}, \\
\tilde{u} = u(x), & \text{in } \Omega \setminus C_0.
\end{cases}
\]

Finally,
\[ \int_A W(u(x)) \, dx = \int_{A \setminus C_0} W(a + \rho(x)n(x)) \, dx + \int_{C_0} W(a + \rho(x)n(x)) \, dx. \]

By (H), (iii), and the hypothesis \( A^+ = \{ x \in A \mid \rho(x) > 2r \} = \emptyset \),
\[ \int_{C_0} W(a + \rho(x)n(x)) \, dx > \int_{C_0} W(a + rn(x)) \, dx. \]

Therefore,
\[ \int_\Omega W(u) \, dx > \int_\Omega W(\tilde{u}) \, dx, \]

(9)

and so, \( J_\Omega(u) > J_\Omega(\tilde{u}) \).

Also by (4),
\[
\begin{cases}
\tilde{u}(x) = u(x), & \text{in } A \setminus C_0, \\
\tilde{u}(x) = a + rn(x), & \text{in } C, \\
\tilde{u}(x) = u(x), & \text{in } \Omega \setminus A,
\end{cases}
\]

thus, the lemma is established under hypothesis (4).
Step 2. We may therefore assume that
\[ |A^+| > 0. \]  
(10)

We first assume that \( r \) is not a critical value of \( \rho \) in \( A \) and later we remove this assumption.

Define the Lipschitz function
\[
\alpha(\tau) = \begin{cases} 
1, & \text{for } \tau \leq r, \\
\frac{2r - \tau}{r}, & \text{for } r \leq \tau \leq 2r, \\
0, & \text{for } \tau \geq 2r,
\end{cases}
\]  
(11)

and recall that compositions of Lipschitz functions with \( W^{1,2} \) functions render \( W^{1,2} \) functions.

Set
\[
\begin{aligned}
w(x) &= u(x), & \text{in } A \setminus C_0 \\
w(x) &= a + r\alpha(\rho(x))n(x), & \text{in } C \\
w(x) &= u(x), & \text{in } \Omega \setminus A.
\end{aligned}
\]  
(12)

Note that \( W \) is continuous on \( \partial C \) (\( C^1 \) manifold) and so \( w \) is in \( W^{1,2}(\Omega) \).

In \( \{ x \in A \mid r \leq \rho(x) \leq 2r \} \) there holds
\[
|\nabla w(x)|^2 = |\nabla \rho(x)|^2 + r^2\alpha^2|\nabla n(x)|^2
\leq |\nabla \rho(x)|^2 + r^2|\nabla n(x)|^2 \quad (\text{since } \alpha \leq 1)
\leq |\nabla \rho(x)|^2 + \rho^2|\nabla u(x)|^2
= |\nabla u(x)|^2.
\]  
(13)

Also \( \nabla w = 0 \) in \( A^+ \) and \( \nabla w = \nabla u \) in the rest of \( A \). It follows that
\[
\int_{\Omega} |\nabla u|^2 \, dx \geq \int_{\Omega} |\nabla w|^2 \, dx.
\]  
(14)

In \( \{ x \in A \mid r \leq \rho(x) \leq 2r \} \) there holds
\[
W(w(x)) = W(a + r\alpha(\rho(x))n(x))
\leq W(a + rn(x))
\leq W(a + \rho(x)n(x))
= W(u(x)),
\]  
(15)

while
\[
W(w(x)) = 0 < W(u(x)), \text{ in } A^+,
\]  
since \( a \) is a global minimum.

Now, since \( |A^+| > 0 \), we obtain
\[
\int_{\Omega} W(u(x)) \, dx > \int_{\Omega} W(w(x)) \, dx.
\]  
(16)

We also note that
\[ |w(x) - a| \leq r, \text{ in } A. \]

Thus, the lemma is established in this case as well.

Step 3. Finally, suppose that \( r \) is a critical value of \( \rho \) in \( A \). We can choose a decreasing and noncritical sequence \( r_n \to r \). Then, the hypotheses (i), (ii), (iii) of
the lemma are satisfied with \( r = r_n \) and, thus, we obtain a sequence \( \{ \tilde{u}_n \} \) with the following properties:

\[
\begin{cases}
\tilde{u}_n(x) = u(x), & \text{in } \Omega \setminus A, \\
|\tilde{u}_n(x) - a| \leq r_n, & \text{in } A, \\
J_\Omega(\tilde{u}_n) < J_\Omega(u).
\end{cases}
\]

Moreover, by construction,

\[
\int_\Omega |\nabla u|^2 \, dx \geq \int_\Omega |\nabla \tilde{u}_n|^2 \, dx.
\]

Hence, by taking possibly a subsequence, there holds \( \tilde{u}_n \rightharpoonup \tilde{u} \) in \( W^{1,2} \) as \( n \to \infty \) and thus,

\[
\int_\Omega |\nabla u|^2 \, dx \geq \int_\Omega |\nabla \tilde{u}|^2 \, dx.
\]

By the compactness of the embedding \( W^{1,2}_{loc} \hookrightarrow L^2_{loc} \) and from

\[
W(\tilde{u}_n(x)) \leq W(u(x)), \quad \text{in } \Omega,
\]

we obtain

\[
W(\tilde{u}(x)) \leq W(u(x)), \quad \text{a.e. in } \Omega.
\]

However,

\[
\int_{A^+} W(u) \, dx > \int_{A^+} W(\tilde{u}) \, dx,
\]

thus, it follows that

\[
\begin{cases}
\tilde{u}(x) = u(x), & \text{in } \Omega \setminus A, \\
|\tilde{u}(x) - a| \leq r, & \text{in } A, \\
J_\Omega(\tilde{u}) < J_\Omega(u).
\end{cases}
\]

The proof of the lemma is complete. \( \square \)

We continue with the

**Proof of the Corollary.** In what follows, we write \( u \) for \( u_{R,\mu} \), \( \rho \) for \( \rho_{R,\mu} \) etc. Consider the sets \( j_R \subset i_R \subset \mathbb{R} \), with

\[
i_R := \{ x_1 \in (0, \eta R) \mid \text{there exists } x_2 \in (0, R) \text{ with } \rho(x_1, x_2) \geq \frac{r}{2} \}
\]

and

\[
j_R := \{ x_1 \in i_R \mid \text{there exists } x_2 \in (0, R) \text{ with } \rho(x_1, x_2) \geq \frac{r}{4} \}
\]

Then, the positivity property (i) implies the lower bound

\[
R w_0 |i_R \setminus j_R| \leq \int_0^R \int_{i_R \setminus j_R} W(u) \, dx_1 \, dx_2,
\]

where \( w_0 := \min_{|u - a| > r/4} W(u) > 0 \).

From the definition of \( j_R \), we conclude that for \( x_1 \in j_R \) there is an interval \( L_{x_1} = (a_{x_1}, b_{x_2}) \) of \( x_2 \) values such that

\[
\frac{r}{4} = \rho(x_1, a_{x_1}) \leq \rho(x_1, x_2) \leq \rho(x_1, b_{x_2}) = \frac{r}{2}, \quad \text{for all } x_2 \in L_{x_1}.
\]

It follows that

\[
\int_{L_{x_1}} W(u(x_1, \tau)) \, d\tau \geq w_0 |L_{x_1}|, \quad \text{for all } x_1 \in j_R.
\]
Moreover, we have
\[
\frac{r}{4} \leq \int_{L_{x_1}} \left| \frac{\partial \rho}{\partial x_2}(x_1, \tau) \right| \ d\tau \leq \left( |L_{x_1}| \int_{L_{x_1}} \left| \frac{\partial \rho}{\partial x_2}(x_1, \tau) \right|^2 \ d\tau \right)^{1/2}
\leq \left( |L_{x_1}| \int_{L_{x_1}} |\nabla u(x_1, \tau)|^2 \ d\tau \right)^{1/2},
\]
(19)

From (18) and (19) we have
\[
\rho \text{ we conclude that }
\]

Remark.

Moreover, we have
\[
\frac{1}{32} \frac{1}{|L_{x_1}|} r^2 + w_0 |L_{x_1}| \leq \int_{L_{x_1}} \frac{1}{2} |\nabla u(x_1, \tau)|^2 \ d\tau + \int_{L_{x_1}} W(u(x_1, \tau)) \ d\tau,
\]
thus,
\[
\frac{r \sqrt{w_0}}{2 \sqrt{2}} \leq \int_{L_{x_1}} \frac{1}{2} |\nabla u(x_1, \tau)|^2 \ d\tau + \int_{L_{x_1}} W(u(x_1, \tau)) \ d\tau.
\]
(20)

Concluding,
\[
CR \geq 2 \int_{\Omega_{R,\mu}} \left( \frac{1}{2} |\nabla u|^2 + W(u) \right) \ dx \geq \int_0^R \int_{j_R} \left( \frac{1}{2} |\nabla u|^2 + W(u) \right) \ dx_1 dx_2
\]
\[
= \int_0^R \int_{j_R} \left( \frac{1}{2} |\nabla u|^2 + W(u) \right) \ dx_1 dx_2 + \int_0^R \int_{j_R} \left( \frac{1}{2} |\nabla u|^2 + W(u) \right) \ dx_1 dx_2
\]
\[
\geq Rw_0 |j_R \setminus j_R| + \frac{r \sqrt{w_0}}{2 \sqrt{2}} |j_R|,
\]
(21)

where the last inequality follows from (17), (20). Hence,
\[
CR \geq A |j_R| + B (|i_R| - |i_R|) R, \text{ for } A := r \sqrt{w_0}/2 \sqrt{2}, \ B := w_0,
\]
\[
\geq \min\{A, BR\} |i_R|
\]
\[
\geq A |j_R|, \text{ if } R \geq r/2 \sqrt{2w_0} =: R_0.
\]
(22)

Consequently, if we take \( R \) large, we obtain that
\[
|i_R| \leq \frac{2 \sqrt{2} CR}{r \sqrt{w_0}} =: \eta_0 R.
\]

If we take \( \eta > \eta_0 \) and fix it, then \( |i_R| < \eta R \) and therefore there is an \( \bar{x}_1 \in (0, \eta R) \), which does not belong to \( i_R \), and such that
\[
\rho(\bar{x}_1, x_2) < \frac{r}{2}, \text{ for all } x_2 \in (0, R).
\]
(23)

Applying now the lemma for the choice \( A = \{ (x_1, x_2) \mid \bar{x}_1 \leq x_1 \leq \mu R, |x_2| < R \} \), we conclude that \( \rho \leq r/2 \) in \( A \), thus, \( \rho < r \) on the line \( x_1 = \eta R \). \( \square \)

Remark. The intuition behind hypothesis (ii) is that if \( u_{R,\mu} \) is bounded away from \( a \) on a large set, then
\[
\int_{\Omega_R} W(u_{R,\mu}(x)) \ dx \geq CR^2,
\]
therefore, by (ii) this cannot happen.

The \textit{a priori} bound (ii) is related to the fact that (2) is linked to a perimeter functional (see (2)). In general dimensions, the appropriate \textit{a priori} estimate is \( J_{\Omega_R}(u) \leq CR^{n-1} \).
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