Abstract

It is argued that Majorana zero modes in a system of quantum fermions can mediate a teleportation-like process with the actual transfer of electronic material between well-separated points. The problem is formulated in the context of a quasi-realistic and exactly solvable model of a quantum wire embedded in a bulk p-wave superconductor. An explicit computation of the tunneling amplitude is given.

Teleportation by quantum tunneling in one form or another has been the physicist’s dream since the invention of the quantum theory. The simplest idea makes use of the fact that the quantum wave-function can have support in classically forbidden regions and can thus reach across apparent barriers. Wherever the wave-function has support, the object whose probability amplitude it describes can in principle be found.

Of course, the typical profile of a wave-function inside a forbidden region decays exponentially with distance, so its amplitude on the other side of that region should be vanishingly small, particularly if any appreciable distance is involved. A slightly
more sophisticated idea would be to consider a system where the wave-function could have support that is peaked in two spatially separated regions. In that case, one could imagine populating the quantum state corresponding to that particular wave-function with an object by interacting with the system in the region of one of the peaks. Then, once the state is populated, the object has non-zero probability of occupying the second peak and, with some efficiency, which is governed by the statistical rules of quantum mechanics, could be extracted from the perhaps far away region of the second peak. It would have been teleported.

This idea could work, however, only if the state described by the wave-function is well separated from other quantum states that the object could also take up and which would interfere with this process. Separation is defined, for example, by considering stationary states which are eigenstates of a Hamiltonian. Then it means that the energy eigenvalue of the state is well separated from the energy eigenvalues of all of the other states. In fact, it is this isolation of a two-peaked wave-function that is difficult to achieve.

To understand the problem, consider a mechanical system of a quantum particle moving in a potential energy landscape where the potential has two degenerate minima separated by an energy barrier. Then, the ground state of the system will have a wave-function with two peaks, one near each of the minima, and at least approximately symmetric under exchange of the positions of the minima. So, what happens when we attempt to populate the ground state by interacting with the system in the region of one of the minima? Can we place an object into that state and thereby teleport it to the location of the other wave-function peak at the other minimum?

The answer seems to be “No.” In such a system, there should always be a second quantum state, nearly degenerate in energy with the ground state, which is at least approximately antisymmetric in the positions of the minima. The larger the energy barrier to regular tunnelling between the minima, the closer to degenerate are the antisymmetric and symmetric states. When we attempt to create a particle in the ground state by interacting with the system at the location of one of the minima, instead of populating the ground state, we place the particle into a superposition of the symmetric and anti-symmetric states, the wave-function of which is localized at the position where we interacted with the system, and having practically zero amplitude at the second location. It then proceeds to tunnel to symmetrize its state at the regular rate for quantum tunnelling, a very slow process in any system where our semi-classical reasoning is valid and we are no further ahead.

What we need to find is a quantum system where a quantum state which is well isolated from other states in the spectrum can have peaks at different locations. From the argument above, it will be difficult to find states of this kind which obey the regular Schrödinger equation. It seems to have a built-in protection against the sort of non-locality that we are looking for.

Single-particle states that are in some sense isolated are well known to occur for Dirac equations, particularly when interacting with various topologically non-trivial background fields such as solitons, monopoles and instantons. The consequences of fermion zero modes such as chiral anomalies [2] and fractional fermion number [3], [4] are well known. Consider, for example, the simple one-dimensional model with Dirac equation \[ i\gamma^\mu \partial_\mu + \phi(x) \psi(x, t) = 0. \] This describes a fermion moving in one
dimension and interacting with a scalar field $\phi(x)$ which we shall take to have a soliton-antisoliton profile,

$$
\phi(x) = \begin{cases} 
\phi_0 & x < 0, \ x > L \\
-\phi_0 & 0 < x < L 
\end{cases}
$$

(1)

It is easy to see that the equation for energy eigenvalues, which, choosing a basis for Dirac matrices, we can write as

$$
i \left( \begin{array}{cc} 0 & \frac{d}{dx} + \phi(x) \\ \frac{d}{dx} - \phi(x) & 0 \end{array} \right) \left( \begin{array}{c} u_E(x) \\ v_E(x) \end{array} \right) = E \left( \begin{array}{c} u_E(x) \\ v_E(x) \end{array} \right)
$$

(2)

has exactly two bound states with energies and wave-functions

$$
E_+ \approx \phi_0 e^{-\phi_0 L} \quad \psi_+ \approx \sqrt{\phi_0} \left( \frac{e^{-\phi_0 |x|}}{i e^{-\phi_0 |L-x|}} \right) + O(e^{-\phi_0 L})
$$

(3)

$$
E_- \approx -\phi_0 e^{-\phi_0 L} \quad \psi_- \approx \sqrt{\phi_0} \left( \frac{e^{-\phi_0 |x|}}{i e^{-\phi_0 |L-x|}} \right) + O(e^{-\phi_0 L})
$$

(4)

where, sufficient for our purposes, we give only the large $L$ asymptotics – corrections to all quantities are of higher orders in $e^{-\phi_0 L}$. These states have energy well separated from the rest of the spectrum, which is continuous and begins at $E = \pm \phi_0$. $E_\pm$ are exponentially close to zero as $L$ is large. Each wave-function has two peaks, one near $x = 0$ and one near $x = L$.

The ground state of the many fermion system has the negative energy states filled and the positive energy states empty. A fermion or anti-fermion are then excited by populating a positive energy state or de-populating a negative energy state, respectively. If we create a fermion by populating the positive energy bound state, it has the wave-function $\psi_+$ given in eqn. (3) which indeed has peaks at both locations, $x = 0$ and $x = L$. Similarly, the anti-fermion has wave-function $\psi_-$ in eqn. (4). Note, however, that the positive and negative energy states are not isolated from each other. They are practically degenerate. This degeneracy will prove fatal to our attempt to use these states for teleportation, for interesting reasons which we shall now relate. When $\phi_0 L$ is large, the energy of the electron and positron states are almost zero. In this case, one can consider a second set of almost stationary states which are the superpositions

$$
\psi_0 = \frac{1}{\sqrt{2}} \left( e^{i E_0 t} \psi_+ + e^{-i E_0 t} \psi_- \right) \approx \sqrt{2 \phi_0} \left( \frac{e^{-\phi_0 |x|} \cos E_0 t}{e^{-\phi_0 |L-x|}} \sin E_0 t \right) + O(e^{-\phi_0 L})
$$

(5)

which, for $t < \frac{1}{E_0}$, has most of its support near $x = 0$ and

$$
\psi_L = \frac{1}{\sqrt{2i}} \left( e^{i E_0 t} \psi_+ - e^{-i E_0 t} \psi_- \right) \approx \sqrt{2 \phi_0} \left( \frac{e^{-\phi_0 |x|} \sin E_0 t}{e^{-\phi_0 |L-x|}} \cos E_0 t \right) + O(e^{-\phi_0 L})
$$

(6)

which has most of its support near $x = L$. By interacting with the system at $x = 0$, we could as well be dropping the fermion into the state $\psi_0$, which is localized there.
and which has exponentially vanishing probability of occurring at \(x = L\) (until \(\sin E_0 t\) becomes appreciable, which is just the usual estimate of tunnelling time through a barrier of height \(\phi_0\) and width \(L\)).

It might seem bizarre that the relevant state would be anything but the ground state that has \(\psi_-(x)\) populated and \(\psi_+(x)\) empty. It has been shown that as \(L \to \infty\), this 'ground state' is an entangled state of (appropriately defined) fermion number \([5, 6]\). If the system is prepared in this state, subsequent measurement of the fermion number of one of the solitons will collapse the wave-function to one where the fermion, rather than occupying the negative energy state \(\psi_-,\) occupies either the state \(\psi_0\) or the state \(\psi_L\), which are localized at \(x = 0\) and \(x = L\), respectively. In these states, the "fractional fermion number" of the solitons is a sharp quantum observable. As seen from the vicinity of each soliton, they are identical to the Jackiw-Rebbi states \([3]\) of the fermion in a single soliton background field, which have fermion number \(\pm \frac{1}{2}\). This issue has recently been re-examined \([7]\) in conjunction with some ideas about entangled electron states in Helium bubbles \([8]\).

We would now imagine that our dumping the fermion into the bound state, if performed near \(x = 0\) would populate the state \(\psi_0\), rather than \(\psi_+\), as this is the state with sharp local fermion number and it would have appreciable probability of appearing at \(x = L\) only after a time over order \(\frac{1}{\phi_0} e^{\phi_0 L}\). Our quest for a teleportation device has been foiled again by the existence of degenerate states, this time the slightly more subtle case of a fermion and anti-fermion state.

The situation is somewhat better if we consider a different type of fermion, called a Majorana fermion. The Hamiltonian of a Majorana fermion has a symmetry which maps positive energy states onto negative energy states. In the case of \((2)\), we have \(\psi_{E}(x) = \psi_{E}^*(x)\). Then, a fermion and an anti-fermion have the same spectrum, and we simply identify them as the same particle, a Majorana fermion. The fermion no longer has a conserved total fermion number. However, it still has fermion parity: fermion number conservation mod 2. In any process, Majorana fermions must be created or annihilated in pairs. This means that we should be able to classify all quantum states by eigenvalues of an operator \((-1)^F\), such that a state with an even number of fermions has \((-1)^F = 1\) whereas a state with an odd number has \((-1)^F = -1\). This classification of states has been argued to be of fundamental importance in three-dimensional quantum physics \([9, 10]\). Even though our present example is one-dimensional, we can imagine that it is an effective theory which is embedded in the three dimensional world. The example we will discuss later is of this sort.

Now, for the Majorana fermion, there is only one bound state, the wave-function of which is \(\psi_+\) (the complex conjugate of which is \(\psi_-\)), and that bound state can be either occupied or empty. We can assign \((-1)^F = 1\) for the empty state and \((-1)^F = -1\) for the occupied state. The states corresponding to \(\psi_0\) and \(\psi_L\) do not have a definite fermion parity. If we begin with the system where the quantum state is an eigenstate of fermion parity and we by some process dump a fermion into the bound state near \(x = 0\), its wave-function automatically has a second peak at \(x = L\) and it could in principle be extracted there. This is what we mean by "teleportation". We will argue later that, locally, if we live near \(x = 0\) and are unaware of the region near \(x = L\), a vestige of this phenomenon will appear as either violation of conservation of fermion parity or the existence of a hidden variable in the local theory. Similar
peculiar phenomena involving Majorana zero modes have previously been discussed in the context of supersymmetric field theories with solitons [11, 12].

Unfortunately, Majorana fermions are not easy to come by in nature. The electron is a complex fermion. Ordinarily, if we decompose it into its real and imaginary parts, which would be Majorana fermions, they are rapidly re-mixed by electromagnetic interactions. One place where this decomposition is done for us more efficiently is in a superconductor where, because electric charges are efficiently screened, the Bogoliubov quasi-fermions behave as if they are neutral excitations. These quasi-particles can be Majorana fermions when the super-conducting condensate is parity odd, P-wave being the simplest example [13]. In these materials, another common occurrence are mid-gap bound states, the analog of our fermion zero modes, called Andreev states [14], which typically live at surface of the superconductor. Majorana zero modes of the type that we are discussing are also known to be bound to vortices in p-wave superconductors where they have the remarkable effect of giving vortices non-Abelian fractional statistics [15]-[18]. For concreteness we will consider a slightly simpler system that was originally discussed by Kitaev [19]. We consider a quantum wire embedded in a bulk P-wave superconductor, depicted in Fig.1. We assume that

![Figure 1: A quantum wire on a bulk P-wave superconductor.](image)

the wire has a single channel and that the electrons are adequately described by a tight-binding model. We also assume a weak coupling to the superconductor whereby the electrons can enter and leave the wire as Cooper pairs. The Hamiltonian is

$$H = \sum_{n=1}^{L} \left( \frac{t}{2} a_{n+1}^\dagger a_n + \frac{t^*}{2} a_n^\dagger a_{n+1} + \frac{\Delta}{2} a_{n+1}^\dagger a_n^\dagger + \frac{\Delta^*}{2} a_n a_{n+1} + \mu a_n^\dagger a_n \right)$$  (7)

The first terms in the Hamiltonian are the contribution of hopping of electrons between sites, labelled $n = 1, 2, ..., L$, on the quantum wire, where $t$ is the hopping amplitude and \{ $a_m, a_n^\dagger$ \} = $\delta_{mn}$ are annihilation and creation operators for electrons. The second pair of terms arise from the presence of the super-conducting environment. The last term is the energy of an electron sitting on a site of the wire. We shall assume that $|\mu| < |t|$ and $|\Delta| < |t|$.

For simplicity, we shall consider spinless fermions. This is possible when the electron spectrum is strongly polarized so that the energy spectra of spin up and spin down electrons are well separated. This is a common occurrence in anisotropic superconductors [13]. In particular, the Cooper pair can be a bound state of spin up electrons only: in that case the terms with $\Delta$ in (7) which take into account that electron on the quantum wire can form Cooper pairs would only couple to the spin up electron. The other spin state would behave as a spectator and to the extent that it would be coupled to the spin up state, it could be taken into account by an effective Hamiltonian. We have not confirmed that this would not destroy the
unpaired Majorana zero mode that we shall find, but it is quite plausible that, at least for some range of parameters, the zero mode and teleportation will persist.

If \( t = |t|e^{i\phi} \) and \( \Delta = |\Delta|e^{i2\theta} \), by redefining \( a_n \to e^{i(\phi+\theta)}a_n \) for \( n \) odd and \( a_n \to e^{i(\phi-\theta)}a_n \) for \( n \) even, we remove the complex phases of \( t \) and \( \Delta \), which we henceforth assume to be positive real numbers. The equation of motion for the fermion is

\[
i \frac{d}{dt}a_n = \frac{t}{2}(a_{n+1} + a_{n-1}) - \frac{\Delta}{2}(a_{n+1} - a_{n-1}) + \mu a_n
\]

We decompose the fermion into real and imaginary parts, \( a_n = b_n + ic_n \), and assemble them into a spinor \( \psi_n = \left( \begin{array}{c} b_n \\ c_n \end{array} \right) \) for which we make the ansatz \( \psi_n(t) = e^{i\omega t}\psi_n \). It obeys the difference equation

\[
\psi_{n+2} = -N\psi_{n+1} - M\psi_n
\]

where

\[
M = \begin{bmatrix}
t + \Delta & t - \Delta \\
0 & t + \Delta
\end{bmatrix}
\quad N = \begin{bmatrix}
\frac{2\mu}{t^2 - \Delta^2} & \frac{2i\omega}{t + \Delta} \\
\frac{2i\omega}{t^2 - \Delta^2} & \frac{2\mu}{t^2 - \Delta^2}
\end{bmatrix}
\]

To solve (8) we define a generating function from which we can recover the wave-function by doing a contour integral

\[
\psi(\zeta) = \sum_{n=1}^{L} \zeta^n \psi_n \quad \psi_n = \oint_C \frac{d\zeta}{2\pi i} \frac{1}{\zeta^{n+1}} \psi(\zeta)
\]

where \( C \) is a contour of infinitesimal radius encircling the origin. We will consider a semi-infinite wire by putting \( L \to \infty \). In this case, the problem is exactly solvable. The generating function is easily obtained from (8) as

\[
\psi(\zeta) = \frac{\zeta}{1 + N\zeta + M\zeta^2} \psi_1 = \frac{\Omega(\theta)}{[t\cosh \theta + \mu]^2 - [\Delta \sinh \theta]^2 - \omega^2} \tilde{\psi}_1
\]

where \( \zeta = e^{\theta} \), \( \Omega(\theta) = \begin{bmatrix}
t \cosh \theta + \mu - \Delta \sinh \theta & -i\omega \\
in\omega & t \cosh \theta + \mu + \Delta \sinh \theta
\end{bmatrix} \) and \( \tilde{\psi}_1 = \begin{bmatrix} t + \Delta \\ 0 \\ \frac{t - \Delta}{2} \end{bmatrix} \psi_1 \). We will do the integral in (9) by inverting the contour, using the fact that the integrand vanishes at infinity so that the contour can be closed on the poles of \( \psi(\zeta) \). For generic real values of the frequency \( \omega \), \( \psi(\zeta) \) has four poles which solve a quartic equation \([t\cosh \theta_i + \mu]^2 - [\Delta \sinh \theta_i]^2 = \omega^2\), which is also the dispersion relation. The integral is then given by the sum of four residues

\[
\psi_n = -\sum_{i=1}^{4} e^{-n\theta_i} \left\{ \frac{4e^{\theta_i}}{t^2 - \Delta^2} \prod_{j \neq i} \left( e^{\theta_i} - e^{\theta_j} \right) \right\} \Omega(\theta_i) \tilde{\psi}_1
\]

For allowed values of \( \omega \), these are the (un-normalized) wave-function of the quasiparticles. The physically allowed values of \( \omega \) are determined by requiring normalizability of the wave-function.

There is a region of continuum spectrum where \( \theta_i \) are imaginary and

\[
\omega(k) = \pm \sqrt{[t \cos k + \mu]^2 + [\Delta \sin k]^2} \quad , \quad k \in (-\pi, \pi)
\]
For $\omega$ in this range, the four roots are $\theta_i = (ik_+, -ik_+, ik_-, -ik_-)$ which satisfy

$$k_\pm = \arccos \left[ -\frac{\mu t}{t^2 - \Delta^2} \pm \sqrt{\left( \frac{\mu t}{t^2 - \Delta^2} \right)^2 - \left( \frac{\mu^2 + \Delta^2 - \omega^2}{t^2 - \Delta^2} \right)} \right]$$

They must be used in (11) to get the continuum wave-function with energy $\omega$.

When $\omega$ is not in the range of continuum spectrum, the roots obey

$$\cosh \theta = -\frac{\mu t \pm i \sqrt{\Delta^2 (t^2 - \Delta^2 - \mu^2) - \omega^2 (t^2 - \Delta^2)}}{t^2 - \Delta^2}$$

and two of them, say $e^\theta$ and $e^{\theta^*}$, must have moduli less than one, leading to a potential diverging part in the wave-function. The divergence can only be avoided if the spinor $\tilde{\psi}_1$ is chosen as a simultaneous zero eigenvector of the two matrices $\Omega(\theta)$ and $\Omega(\theta^*)$. Recall that the dispersion relation ensures that these matrices have vanishing determinant, so each has a vanishing eigenvalue. They can have a simultaneous vanishing eigenvalue only if they commute,

$$[\Omega(\theta), \Omega(\theta^*)] = i\omega (\sinh \theta - \sinh \theta^*) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Since in the range of frequencies of interest, $\sinh \theta$ cannot be real, the only possible bound states are zero modes, $\omega = 0$. By appropriate choice of the spinor, $\tilde{\psi}_1$ we find the zero mode wave-function is $b_0^n = 0$

$$c_0^n = \frac{i \left[ (-\mu + i\sqrt{t^2 - \Delta^2 - \mu^2})^n - (-\mu - i\sqrt{t^2 - \Delta^2 - \mu^2})^n \right]}{\sqrt{2(t^2 - \Delta^2 - \mu^2)(t + \Delta)^n}}$$

(12)

There is exactly one bound state of one of the real components of the electron living at the edge of the quantum wire. Its energy is separated from the rest of the quasiparticles which are in the continuum spectrum.

Note that the existence of a mid-gap state for a Majorana fermion is quite robust. Since, in order to have a Majorana fermion in the first place, there must exist a one-to-one mapping between positive and negative energy states of the single-fermion Hamiltonian, smooth perturbations of it can only change the number of zero modes by an even integer. Generically, such perturbations would lift the zero modes, leaving the one that we have found.

Let us assume that the wire is effectively semi-infinite in the sense that we are only aware of the edge at $n = 1$. We would second quantize the fermions as

$$b_n(t) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \left( b_n(k)e^{i\omega t} \alpha_k + b_n^*(k)e^{-i\omega t} \alpha_k^\dagger \right)$$

$$c_n(t) = c_0^n \beta_0 + \int_{-\pi}^{\pi} \frac{dk}{2\pi} \left( c_n^0(k)e^{i\omega t} \beta_k + c_n^0(k)e^{-i\omega t} \beta_k^\dagger \right)$$

(13)

(14)

where we now assume that all wave-functions are normalized and the creation and annihilation operators for quasiparticles obey the usual anti-commutator algebra

$$\{\alpha_k, \alpha_{k'}^\dagger\} = \delta(k - k') \quad , \quad \{\beta_k, \beta_{k'}^\dagger\} = \delta(k - k') \quad , \quad \{\beta_k, \alpha_{k'}\} = 0$$
The zero mode operator obeys \((\beta^0)^2 = 1\) and anti-commutes with all other operators. A representation of the anticommutator algebra begins with an eigenstate of \(\beta^0\),

\[
\beta^0|+\rangle = |+\rangle
\]  

which is annihilated by all other operators \(\alpha_k|+\rangle = 0 = \beta_k|+\rangle\). Quasi-particle states are then given by creation operators acting on this ground state, \(\alpha^\dagger_k \beta^\dagger_{k_1} \ldots |+\rangle\).

This is an irreducible representation of the creation and annihilation operator algebra and is what we would use if we were unaware of the other end of the wire. The ground state \(|+\rangle\) is not an eigenstate of fermion parity, so fermion number mod 2 is not conserved in this system. We could then imagine a process where we probe the system with a source which couples to electric charge density, for example

\[
H_{\text{int}} = \sum_n V(n, t) a_n^\dagger(t) a_n(t).
\]

Recalling that \(a_n(t) = \frac{1}{\sqrt{2}} (b_n + i c_n)\) and using (13) and (14), see that there is a finite amplitude for the annihilation of a single fermionic quasi-particle.

\[
< + | H_{\text{int}} \alpha^\dagger_k |+\rangle = i e^{i \omega_k t} \sum_n \left( V(n, t) b_n(k) c_n^0 \right)
\]

It literally looks like the fermion vanished while traversing the region near the edge of the wire. The inverse process of making a quasi-fermion appear has an amplitude which is the complex conjugate of the above.

Of course, the total system that we are considering does have fermion parity symmetry. The Hamiltonian is quadratic in fermion operators, as is the probe that we are using. The symmetry is being broken by our insistence on the choice of an irreducible representation of the anti-commutator algebra. It can be restored by using a reducible representation instead. The minimal such restoration is equivalent to introducing a single hidden variable.

If the system has another boundary at large but finite \(L\), the most significant effect is that the bound state is not longer an exact zero mode. Now there are a pair of states with positive and negative energies of order \(\pm \Delta e^{-\Delta L}\). We shall assume that \(L\) is large enough that these energies are negligibly small. Similar to the relativistic case that we studied earlier \([3]\) and \([4]\), the wave-functions of each of these bound states has support at both ends of the wire and is vanishingly small in between. When \(L\) is large, the rest of the spectrum resembles the continuum spectrum of the semi-infinite wire. Now, second quantization introduces two operators corresponding to the (almost) zero modes, rather than the one which was found for the semi-infinite wire: we can think of it as producing the hidden variable which we discussed above.

The zero mode localized near \(x = L\) will be a mode of \(b_n(t)\) with wave-function \(b_n^0 = c_n^0\). To second quantize the Majorana fermions, we modify (13) as

\[
b_n(t) = c_{L+1-n}^0 \alpha^0 + \int_{-\pi}^{\pi} \frac{dk}{2\pi} \left( b_n(k) e^{i\omega_k t} \alpha_k + b_n^*(k) e^{-i\omega_k t} \alpha^\dagger_k \right)
\]

\[
c_n(t) = c_n^0 \beta^0 + \int_{-\pi}^{\pi} \frac{dk}{2\pi} \left( c_n(k) e^{i\omega_k t} \beta_k + c_n^*(k) e^{-i\omega_k t} \beta^\dagger_k \right)
\]
To represent the anti-commutator algebra we now need the second eigenstate of $\beta^0$, $\beta^0|\rightarrow > = -|\rightarrow >$. Since $\alpha^0$ must obey $\{ \alpha^0, \beta^0 \} = 0$ and $(\alpha^0)^2 = 1$, its action on these states must be (up to choice of phase) 

$$\alpha^0|+ >= |\rightarrow > , \quad \alpha^0|- >= |\leftarrow >$$

Now, we have two degenerate ground states $|\rightarrow >$ and $|\leftarrow >$. We can in fact find states which are eigenstates of fermion parity by taking superpositions

$$|\uparrow > = \frac{1}{\sqrt{2}} (|\rightarrow > + i|\leftarrow >) , \quad |\downarrow > = \frac{1}{\sqrt{2}} (|\rightarrow > - i|\leftarrow >)$$

In this space $\beta^0|\uparrow > = |\downarrow >$, and $\beta^0|\downarrow > = |\uparrow >$, $\alpha^0|\uparrow > = i|\rightarrow >$ and $\alpha^0|\downarrow > = -i|\leftarrow >$. We can choose $|\uparrow >$ and $|\downarrow >$ to be eigenstates of fermion parity, $(-1)^F|\uparrow > = |\uparrow >$, $(-1)^F|\downarrow > = -|\downarrow >$.

The process analogous to the one in (16) has the non-zero matrix element

$$<\uparrow | H_{\text{int}} \alpha_k \downarrow > = i e^{i \omega_k t} \sum_n (V(n, t) b_n(k) c_n^0)$$

and the destruction of a single quasi-fermion is accompanied by a flip in the vacuum state, the total process conserving $(-1)^F$.

What about teleportation? Let us imagine that we begin with the system in one of its ground states and inject an electron at site #1, to create the state $a_1^\dagger |\uparrow >$. We then ask what is the quantum transition amplitude for the transition, after a time $T$, of this state to one with the electron located at position #L, $a_L^\dagger |\uparrow >$. The amplitude is given by

$$A_{1L} = \frac{<\uparrow | a_L e^{iHT} a_1^\dagger |\uparrow >}{|a_1^\dagger |\uparrow > |a_L^\dagger |\uparrow >} = 2|c_1^0|^2 + (T \text{ and } L\text{-dependent})$$

The right-hand-side of this equation has two kinds of terms. The $T$- and $L$-dependent parts of this matrix element represent the usual propagation via excited quasi-particles which must travel across the wire. The first term, on the other hand, is unusual in that it is $T$ and $L$-independent. By ‘teleportation’, we are referring to this part of the amplitude. The probability that the electron, after being injected at site #1 is teleported to any site #n within the support of the zero mode at $n = L$ is

$$P = \sum_n |A_{1n}|^2 = 4 \frac{\Delta}{t} \frac{t^2 - \mu^2}{(t + \Delta)^2}$$

which can be significant.

Note that this tunneling process does not violate causality. There is a finite probability for an electron to appear at site #L spontaneously accompanied by a flip between the ground states,

$$<\uparrow | a_L e^{iHT} |\downarrow > \neq 0$$

In order for an observer at site #L to know that the electron arriving there was not produced by this spontaneous flip, the information about the state in which
the system was initially prepared would have to be sent to her/him by independent means.

We note that, if the fermion parity superselection rule is respected, the degenerate ground states behave like a classical switch. The switch can be in either of two positions $|\uparrow\rangle$ or $|\downarrow\rangle$ but a quantum superposition of the two states is not allowed. The quantum coherence of this system leads to this surprising classical behavior.

This has a further interesting effect when one considers two such systems. This consideration is in fact natural if there is spin degeneracy. For simplicity, we consider two spin states that are sufficiently weakly coupled to each other that they satisfy independent single-fermion Schrödinger equations and the quantum states are direct products of the states of the two systems. Then, the quantum states that are allowed by the fermion parity superselection rule cover two Bloch spheres, for the separate cases where fermion parity is odd or even, respectively

$$|\theta\phi\rangle = \cos\frac{\theta}{2} |\uparrow\downarrow\rangle + e^{i\phi} \sin\frac{\theta}{2} |\downarrow\uparrow\rangle, \quad |\theta'\phi'\rangle = \cos\frac{\theta'}{2} |\uparrow\uparrow\rangle + e^{i\phi'} \sin\frac{\theta'}{2} |\downarrow\downarrow\rangle$$

The expectation value of the spin vanishes in both of these states, independently of the angles. This means that the ground state degeneracy is not split by a sufficiently weak Zeeman interaction, in line with the fact that the existence of the Majorana zero modes is independent of the interactions – the Hamiltonians for the two spin species need not be identical, they only need to each have the conditions for existence of the Majorana zero mode in the first place. In the present case, the energy splitting of the spin states due to a Zeeman interaction with an external magnetic field would simply be carried by a relative shift of chemical potential $\mu$ for each spin which, if small enough, leaves the zero mode sector intact. The amplitude that we have computed for tunneling into zero modes in this case is just a sum of the amplitudes for each spin state. The states of a given fermion parity form a qubit whose properties are very stable and which could be of use for quantum computation.

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