FINITE SUBGROUPS OF HAM AND SYMP

IGNASI MUNDET I RIERA

Abstract. Let \((X, \omega)\) be a compact symplectic manifold of dimension \(2n\) and let \(\text{Ham}(X, \omega)\) be its group of Hamiltonian diffeomorphisms. We prove the existence of a constant \(C\), depending on \(X\) but not on \(\omega\), such that any finite subgroup \(G \subset \text{Ham}(X, \omega)\) has an abelian subgroup \(A \subseteq G\) satisfying \([G : A] \leq C\), and \(A\) can be generated by \(n\) elements or fewer. If \(b_1(X) = 0\) we prove an analogous statement for the entire group of symplectomorphisms of \((X, \omega)\). If \(b_1(X) \neq 0\) we prove the existence of a constant \(C'\) depending only on \(X\) such that any finite subgroup \(G \subset \text{Symp}(X, \omega)\) has a subgroup \(N \subseteq G\) which is either abelian or 2-step nilpotent and which satisfies \([G : N] \leq C'\).

These results are deduced from the classification of the finite simple groups, the topological rigidity of hamiltonian loops, and the following theorem, which we prove in this paper. Let \(E\) be a complex vector bundle over a compact, connected, smooth and oriented manifold \(M\); suppose that the real rank of \(E\) is equal to the dimension of \(M\), and that \(\langle e(E), [M] \rangle \neq 0\), where \(e(E)\) is the Euler class of \(E\); then there exists a constant \(C''\) such that, for any prime \(p\) and any finite \(p\)-group \(G\) acting on \(E\) by vector bundle automorphisms preserving an almost complex structure on \(M\), there is a subgroup \(G_0 \subseteq G\) satisfying \(M^{G_0} \neq \emptyset\) and \([G : G_0] \leq C''\).

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1. Introduction

1.1. For any symplectic manifold \((X, \omega)\) denote by \(\text{Ham}(X, \omega)\) (resp. \(\text{Symp}(X, \omega)\)) the group of Hamiltonian (resp. symplectic) diffeomorphisms of \((X, \omega)\). In this paper we are interested on the finite subgroups of \(\text{Ham}(X, \omega)\) or \(\text{Symp}(X, \omega)\) for arbitrary compact \(X\),
that is, the Hamiltonian or symplectic finite transformation groups of compact symplectic manifolds.

Let $\text{Symp}_0(X,\omega) \subseteq \text{Symp}(X,\omega)$ and $\text{Diff}_0(X) \subseteq \text{Diff}(X)$ denote the identity components. Let

$$H(X,\omega), \ S(X,\omega), \ D(X)$$

denote the set of isomorphism classes of finite subgroups of $\text{Ham}(X,\omega)$, $\text{Symp}_0(X,\omega)$, $\text{Diff}_0(X)$ respectively. There are obvious inclusions $H(X,\omega) \subseteq S(X,\omega) \subseteq D(X)$ for any $X$. A consequence of our results combined with those in [27] is the existence of infinitely many compact symplectic manifolds $(X,\omega)$ for which both $H(X,\omega)$ and $D(X) \setminus H(X,\omega)$ are infinite. We also deduce the existence of infinitely many compact symplectic manifolds $(X,\omega)$ for which both $H(X,\omega)$ and $S(X,\omega) \setminus H(X,\omega)$ are infinite.

One can easily construct plenty of examples of Hamiltonian finite transformation groups of compact manifolds by taking any compact symplectic manifold $X$ endowed with a Hamiltonian action of a connected Lie group $K$ and restricting the action to the finite subgroups of $K$. In particular for any finite group $G$ there is some compact symplectic manifold $X$ on which $G$ acts effectively by Hamiltonian diffeomorphisms (embed $G$ in $U(n)$ for some $n$ and take $X$ to be any Hamiltonian $U(n)$-manifold, e.g. $\mathbb{C}P^n$).

Of course, this construction is far from capturing all Hamiltonian finite transformation groups: one should not expect an arbitrary finite subgroup of $\text{Ham}(X,\omega)$ to be included in a compact connected subgroup of $\text{Ham}(X,\omega)$. On the negative side, there are examples of compact symplectic manifolds whose group of Hamiltonian diffeomorphisms is known to contain no nontrivial finite subgroup; this is the case, for example, if $\pi_2 = 0$, as proved by Polterovich [30, Theorem 1.2].

Roughly speaking, our main result says that for any compact symplectic manifold $X$ there is an upper bound (depending only on the topology of $X$) on how much nonabelian the finite subgroups of $\text{Ham}(X,\omega)$ are. If $X$ is simply connected we prove the same property for finite subgroups of $\text{Symp}(X,\omega)$. In contrast, there are many examples of compact symplectic manifolds $(X,\omega)$ admitting no such bound for finite subgroups of $\text{Diff}(X)$, so the properties we prove on $\text{Ham}(X,\omega)$ are genuinely symplectic. Similarly, if $X$ is a non simply connected symplectic manifold it is known [28] that there is in general no upper bound depending only on $X$ on how much nonabelian the finite subgroups of $\text{Symp}(X,\omega)$ are (although an upper bound depending on $\omega$ might exist in all situations). Hence our result on $\text{Ham}(X,\omega)$ is sharp.

As a measure of how much nonabelian a finite group is, we take the index of an abelian subgroup of maximal size. So the question we are interested in is the so-called Jordan property for Hamiltonian and symplectomorphism groups, which we now recall.

Let $C$ be a natural number. A group $\mathcal{G}$ is $C$-Jordan if each finite subgroup $G \subseteq \mathcal{G}$ has an abelian subgroup $A \subseteq G$ satisfying $[G : A] \leq C$. A group $\mathcal{G}$ is Jordan if it is $C$-Jordan for some $C$. This terminology, which was introduced by V. Popov in [31], is inspired by a classical theorem of C. Jordan [17] which states that $GL(n,\mathbb{C})$ is Jordan for any $n$.

We will consider the following more refined notion. Let $C, d$ be natural numbers. A group $\mathcal{G}$ is $(C, d)$-Jordan if each finite subgroup $G \subseteq \mathcal{G}$ has an abelian subgroup $A \subseteq G$ satisfying $[G : A] \leq C$ and such that $A$ can be generated by $d$ (or fewer) elements.

Some twenty years ago É. Ghys raised the question of whether the diffeomorphism group of any smooth compact manifold is Jordan. A number of papers have recently
been devoted to Ghys’s question (see e.g. [26] and the references therein), and it is known that the answer is affirmative in many cases and negative in many other ones [12, 27], although a description of which compact smooth manifolds have Jordan diffeomorphism group seems to be at present a widely open problem. As we will see, this is in sharp contrast with the analogous question for Hamiltonian diffeomorphism groups of compact symplectic manifolds, which are always Jordan. Regarding our more refined notion of Jordanness, we remark that a theorem of Mann and Su [20] (see Theorem 5.4 below) implies that if $X$ is a compact smooth manifold, $\mathcal{G}$ is any subgroup of $\text{Diff}(X)$, and $\mathcal{G}$ is $C$-Jordan for some $C$, then $\mathcal{G}$ is also $(C,d)$-Jordan for some $d$ depending only on $X$.

Analogous questions have been studied in algebraic geometry. J.–P. Serre [41] proved that the classical Cremona group is Jordan, and asked whether its higher dimensional analogues are also Jordan. More generally, one may ask whether the automorphism or birational transformation groups of general affine or projective varieties are Jordan (see e.g. the survey [33] and, more recently, [2, 23, 34, 35]).

The question has also been studied previously in symplectic geometry. The symplectomorphism group of $(T^2 \times S^2, \omega)$ has been proved to be Jordan for any choice of symplectic structure $\omega$ in [28]. An interesting consequence of this is that it gives an example of a symplectic manifold with Jordan symplectomorphism group but non Jordan diffeomorphism group: in fact, $\text{Diff}(T^2 \times S^2)$ was the first given example of a non Jordan diffeomorphism group of a compact manifold, see [12].

In this paper we only consider manifolds without boundary, so for us compact manifold is synonym of closed manifold.

1.2. Finite subgroups of $\text{Ham}(X, \omega)$. This is our first main result.

**Theorem 1.1.** Let $(X, \omega)$ be a $2n$-dimensional compact and connected symplectic manifold. Then $\text{Ham}(X, \omega)$ is $(C,n)$-Jordan for some $C$ depending only on $H^*(X)$.

Here $H^*(X)$ denotes the integral cohomology of $X$. In particular, $C$ is independent of $\omega$. As mentioned earlier, Theorem 1.1 is not a theorem on smooth finite transformation groups in symplectic disguise. Indeed, there are plenty of examples of compact symplectic manifolds whose diffeomorphism group is not Jordan, and in many cases even the identity component of the diffeomorphism group fails to be Jordan. So we can not replace $\text{Ham}(X, \omega)$ by $\text{Diff}_0(X)$. In fact, we will see below that one can neither replace $\text{Ham}(X, \omega)$ by $\text{Symp}_0(X, \omega)$.

For example, if $Y$ is any compact smooth manifold supporting an effective action of $\text{SU}(2)$ or $\text{SO}(3, \mathbb{R})$, then $\text{Diff}_0(T^2 \times Y)$ is not Jordan [12, 27]; more concretely, for any prime $p$ there is a subgroup of $\text{Diff}_0(T^2 \times Y)$ which is isomorphic to the Heisenberg group over $\mathbb{F}_p[1]$. This implies that if $Y$ is a compact symplectic manifold supporting an effective smooth action of $\text{SU}(2)$ or $\text{SO}(3, \mathbb{R})$ then, denoting by $\Omega$ the set of all symplectic forms on $T^2 \times Y$,

$$\mathcal{D}(T^2 \times Y) \setminus \bigcup_{\omega \in \Omega} \mathcal{H}(T^2 \times Y, \omega)$$

contains infinitely many elements.

For some symplectic manifolds $(X, \omega)$ Theorem 1.1 implies that $\text{Symp}_0(X, \omega)$ is Jordan:

\[\text{Actually the results in [27] refer to the full diffeomorphism group, but it is easy to check that all group actions that are defined there give rise to finite subgroups of the identity component of Diff.}\]
Corollary 1.2. Let \(\dim X = 2n\). If the flux homomorphism 
\[
\pi_1(\text{Symp}_0(X,\omega)) \to H^1(X;\mathbb{R})
\]
vanishes, then \(\text{Symp}_0(M,\omega)\) is \((C,n)\)-Jordan for some \(C\) depending only on \(H^*(X;\mathbb{Z})\).

See [22, §10.2] for the definition of the flux homomorphism.

Proof. If \(\Gamma\) denotes the image of the flux map \(\pi_1(\text{Symp}_0(X,\omega)) \to H^1(X;\mathbb{R})\) then there is an exact sequence
\[
\text{Ham}(X,\omega) \to \text{Symp}_0(X,\omega) \overset{F}{\to} H^1(X;\mathbb{R})/\Gamma,
\]
where \(F\) denotes the flux (see [22, Corollary 10.18]). If \(\Gamma = 0\) then any finite subgroup \(G \subset \text{Symp}_0(X,\omega)\) satisfies \(F(G) = 0\), so \(G \subset \text{Ham}(X,\omega)\).

There exist many examples of manifolds \((X,\omega)\) with nonzero first Betti number for which the flux homomorphism \(\pi_1(\text{Symp}_0(X,\omega)) \to H^1(X;\mathbb{R})\) vanishes, see e.g. [18].

1.3. Finite subgroups of \(\text{Symp}(X,\omega)\). It is natural to wonder to what extent Theorem 1.1 is true if one replaces \(\text{Ham}(X,\omega)\) by the entire group of symplectomorphisms of \(X\). The answer turns out to be yes in some cases.

Theorem 1.3. Let \((X,\omega)\) be a \(2n\)-dimensional compact and connected symplectic manifold satisfying \(b_1(X) = 0\). Then \(\text{Symp}(X,\omega)\) is \((C,n)\)-Jordan for some \(C\) which only depends on \(H^*(X)\).

The situation for manifolds \((X,\omega)\) with \(b_1(X) \neq 0\) is more involved. The case \(X = T^2 \times S^2\) has been studied in detail in [28], and it turns out that for any symplectic form \(\omega\) on \(T^2 \times S^2\) the group of symplectomorphisms \(\text{Symp}(T^2 \times S^2,\omega)\) is \(C\)-Jordan for some \(C = C(\omega)\) depending on \(\omega\), but there is no upper bound for \(C(\omega)\) as \(\omega\) moves along the set of symplectic forms on \(T^2 \times S^2\). This is related to the fact that \(\text{Diff}(T^2 \times S^2)\) is not Jordan, and a similar phenomenon occurs for any product \(T^2 \times Y\) with \(Y\) symplectic and supporting a Hamiltonian action of SU(2): namely, for any prime \(p\) there is a symplectic form \(\omega\) on \(T^2 \times Y\) such that \(\text{Symp}(T^2 \times Y,\omega)\) contains a subgroup isomorphic to the Heisenberg group over \(\mathbb{F}_p\) (this is a straightforward generalization of [28, §3]).

Hence, the hypothesis \(b_1(X) = 0\) cannot be removed in Theorem 1.3 and from this perspective Theorem 1.1 is optimal. On the other hand, the author does not know any example of compact symplectic manifold whose symplectomorphism group fails to be Jordan (compactness is crucial here, since clearly the symplectomorphism group of the cotangent bundle of \(T^2 \times S^2\) is not Jordan). It is also apparently unknown at present whether there exists a simply connected compact smooth manifold whose diffeomorphism group is not Jordan.

We actually prove a stronger result than Theorem 1.3. To state it, the following notation will be useful: if \(X\) is a smooth manifold and \(J\) is an almost complex structure on \(X\) then we denote by
\[
\text{Diff}(X,J) \subset \text{Diff}(X)
\]
the group of all diffeomorphisms of \(X\) preserving \(J\).
Theorem 1.4. Let \((X, J)\) be a \(2n\)-dimensional almost complex, compact and connected smooth manifold satisfying \(b_1(X) = 0\), and assume that there exists \(\omega \in H^2(X; \mathbb{R})\) such that \(\omega^n \neq 0\). Then \(\text{Diff}(X, J)\) is \((C, n)\)-Jordan for a constant \(C\) depending only on \(H^*(X)\).

In particular, the constant \(C\) is independent of \(J\) and \(\omega\) (and \(J\) and \(\omega\) need not be related in any way). Theorem 1.4 implies a particular case of a recent result of Meng and Zhang [23] stating that automorphism groups of (not necessarily smooth) projective varieties over any algebraically closed field of characteristic zero are Jordan.

Remark 1.5. Theorem 1.3 follows from Theorem 1.4 because for any symplectic manifold \((X, \omega)\) and any symplectic action of a compact Lie group \(K\) on \(X\) there exists a \(K\)-invariant almost complex structure on \(X\) (see e.g. [22, Lemma 5.49] and the remark before it).

Although we do not know at present whether the symplectomorphism group of every compact symplectic manifold is Jordan, we can prove the following weaker statement.

Theorem 1.6. Let \((X, \omega)\) be a compact symplectic manifold. There exists a constant \(C\) depending only on \(H^*(X)\) with the property that any finite subgroup \(\Gamma \subset \text{Symp}(X, \omega)\) has a subgroup \(N \subseteq \Gamma\) satisfying \([\Gamma : N] \leq C\) and \(N\) is either abelian or 2-step nilpotent.

Recall that a group \(N\) is 2-step nilpotent (equivalently, \(N\) has nilpotency class 2) if \(N\) is not abelian and every three elements \(a, b, c \in N\) satisfy \([[[a, b], c] = 1\).

1.4. Fixed points of finite symplectic \(p\)-group actions. To prove Theorems 1.1 and 1.4 we will use the general criterion of Jordanness given in [20]. This will allow us to restrict attention to fixed point properties of finite \(p\)-group actions. The theorems will then be a consequence of the following results.

Theorem 1.7. Let \((X, \omega)\) be a compact symplectic manifold. There exists a constant \(C\), depending only on \(H^*(X)\), such that for any prime \(p\) and any finite \(p\)-subgroup \(G \subset \text{Ham}(X, \omega)\) there is a subgroup \(G_0 \subseteq G\) satisfying \([G : G_0] \leq C\) and \(X^{G_0} \neq \emptyset\).

In particular, if \(p > C\) then any finite \(p\)-subgroup \(G \subset \text{Ham}(X, \omega)\) satisfies \(X^G \neq \emptyset\), and since the differential of the action at any \(x \in X^G\) embeds \(G\) inside \(\text{GL}(T_xM)\) (see e.g. the proof of Corollary 5.3) we may combine this with Jordan’s theorem (see Corollary 5.3) to obtain the following result:

Corollary 1.8. Let \((X, \omega)\) be a \(2n\)-dimensional compact symplectic manifold. There exists a constant \(C'\), depending only on \(H^*(X)\), such that for any prime \(p > C'\) and any finite \(p\)-group \(G \subset \text{Ham}(X, \omega)\) we have: \(X^G \neq \emptyset\), \(G\) is abelian, and \(G\) can be generated by \(n\) (or fewer) elements.

This has the interesting consequence that for many compact symplectic manifolds \((X, \omega)\) such that \(\pi_2(X) \neq 0\) (hence, not satisfying the hypothesis of the theorem of Polterovich which was mentioned earlier) the difference

\[\mathcal{S}(X, \omega) \setminus \mathcal{H}(X, \omega)\]

contains infinitely many elements. For example, if \(X = T^{2n} \times S^2\) is endowed with a product symplectic form \(\omega\) and the restriction of \(\omega\) to \(T^{2n}\) is translation invariant then
for any prime $p$ there is a subgroup of $\text{Symp}_0(X,\omega)$ isomorphic to $(\mathbb{Z}/p)^{2n+1}$, whereas if $p$ is big enough then any subgroup of $\text{Ham}(X,\omega)$ isomorphic to $(\mathbb{Z}/p)^r$ satisfies $r \leq n+1$.

**Theorem 1.9.** Let $(X, J)$ be a 2$n$-dimensional almost complex, compact and connected smooth manifold satisfying $b_1(X) = 0$, and assume that there exists $\omega \in H^2(X; \mathbb{R})$ such that $\omega^n \neq 0$. There exists a constant $C$, depending only on $H^*(X)$, with the following property. Let $p$ be any prime, and let $G \subset \text{Diff}(X, J)$ be a finite $p$-subgroup. Then there is a subgroup $G_0 \subseteq G$ satisfying $[G : G_0] \leq C$ and $X^{G_0} \neq \emptyset$.

Theorem 1.9 solves a weaker version of a classical problem in transformation groups, namely that of proving existence of fixed points of finite $p$-group actions on projective varieties. There exist many partial results on this question, assuming different restrictions on the group, the manifold, or the action, see e.g. [7, Theorem (1.5)], [8, Theorem (4.10)], or more recently [35, Theorem 4.2] combined with [2].

Combining the previous theorem with Jordan’s theorem (see Corollary 5.3) we obtain.

**Corollary 1.10.** Let $(X, J)$ be a 2$n$-dimensional almost complex, compact and connected smooth manifold satisfying $b_1(X) = 0$, and assume that there exists $\omega \in H^2(X; \mathbb{R})$ such that $\omega^n \neq 0$. There exists a constant $C$, depending only on $H^*(X)$, such that for any prime $p > C$, any finite $p$-subgroup $G \subset \text{Diff}(X, J)$ satisfies $X^G \neq \emptyset$, $G$ is abelian, and $G$ can be generated by $n$ (or fewer) elements.

1.5. Actions of finite $p$-groups on vector bundles. The main technical tool developed in this paper, from which we will deduce Theorems 1.7 and 1.9 is a fixed point theorem for actions of finite $p$-groups on complex vector bundles.

Before explaining the theorem, let us recall some standard terminology to avoid potential confusions. Let $G$ be a group acting smoothly on (the total space of) a vector bundle $E \to X$. We say that the action is by vector bundle automorphisms if it sends fibers to fibers and the action is compatible with the vector space structure on the fibers. In particular, there is an induced action of $G$ on $X$ such that for any $g \in G$, $x \in X$ and $e \in E_x$ we have $g \cdot e \in E_{g \cdot x}$. We say that the action of $G$ on $E$ lifts the action on $X$.

If $E \to X$ is a complex vector bundle, with $X$ a smooth manifold, and $J$ is an almost complex structure on $X$, we denote by

$$\text{Aut}(E, X, J)$$

the group of all vector bundle automorphisms of $E \to X$ lifting elements of $\text{Diff}(X, J)$.

**Theorem 1.11.** Let $(X, J)$ be an almost complex, compact and connected smooth manifold and let $E \to X$ be a complex vector bundle satisfying $\text{rk}_\mathbb{R}(E) = \dim_{\mathbb{R}} X$. Suppose that $\langle [X], e(E) \rangle \neq 0$, where $e(E)$ is the Euler class of $E$ and $X$ is oriented by $J$. Then there exists a constant $C$, depending only on $\langle [X], e(E) \rangle$ and $H^*(X)$, with the following property. Let $p$ be any prime, and let $G$ be a finite $p$-subgroup of $\text{Aut}(E, X, J)$. Then there exists an abelian subgroup $A \subseteq G$ satisfying $[G : A] \leq C$ and $X^A \neq \emptyset$.

The proof of Theorem 1.11 proceeds in two steps. First we assume that $G$ has an abelian normal subgroup $B$ such that $G/B$ is cyclic, and then the result is extended to arbitrary finite $p$-groups by combining that particular case with Jordan’s theorem. Of these two steps, the first one is substantially more involved than the second. To briefly
If $p$ does not divide $\langle [X], e(E) \rangle$ then the proof is similar to that of the main theorem in [6]. We proceed by induction on $|G|$. Let $H \subseteq G$ be a subgroup isomorphic to $\mathbb{Z}/p$. Since $\langle [X], e(E) \rangle$ is not divisible by $p$, $X^H$ is nonempty. If $H = G$ then we are done, otherwise we consider the induced action of $G/H$ on $X^H$. To be able to apply induction it is necessary to generalize the statement to include several vector bundles instead of only one and to replace the Euler class by a product of Chern classes of the vector bundles (and then the numerical hypothesis is that such product can be chosen so that its pairing with $[X]$ is not divisible by $p$). In the induction process we add the normal bundle $N$ of $X^H \hookrightarrow X$ to the collection of vector bundles, and all vector bundles (the old ones and $N$, the new one) have to be decomposed and twisted by characters of $H$ so that the action of $G/H$ on $X$ lifts to an action on each of them. Here we need all involved vector bundles (including $N$) to carry an invariant complex structure. This is the only moment where we use our assumption that the action of $G$ preserves an almost complex structure $J$ on $X$: the complex structure on $N$ is taken to be the restriction of $J$. Finally, to guarantee that the numerical hypothesis is inherited by the action of $G/H$ on $X^H$ and the new collection of vector bundles we use the standard fact that the equivariant Euler class of $N$ with $\mathbb{Z}/p$ coefficients is invertible up to inverting a generator of $H^*(BG)$.

If $p$ divides $\langle [X], e(E) \rangle$ things are more involved. Suppose that $p^k$ is the smallest power of $p$ that does not divide $\langle [X], e(E) \rangle$ and that $k \geq 2$. Let $H \subseteq G$ be a subgroup isomorphic to $\mathbb{Z}/p^k$ (if such $H$ does not exist, then $|G|$ is smaller than a constant depending on $X$ and $k$, so we are done). Now it is not necessarily true that $X^H$ is nonempty, but certainly $X^{H'} \neq \emptyset$, where $H' = p^{k-1}H \cong \mathbb{Z}/p$. Let $N$ be the normal bundle of $X^{H'} \subseteq X$. Then $N$ is a $G$-equivariant vector bundle, and the main step in the proof of Theorem 1.11 is the proof that its equivariant Euler class with $\mathbb{Z}/p^k$ coefficients is invertible up to inverting a generator of $H^*(BG)$, see Theorem 2.7. We use for that a well known nilpotence argument in equivariant cohomology due to Quillen (see [30] §3 — we only need the compact case, whose proof is elementary). The remaining details are the same as in the case where $p$ does not divide $\langle [X], e(E) \rangle$.

Remark 1.12. We mentioned above that the only place in the proof of Theorem 1.11 where we need that the action of $G$ preserves an almost complex structure on $X$ is to guarantee the existence of a $G_1$-invariant complex structure on the normal bundles of inclusions of the type $X^{G_1} \subseteq X^{G_0}$, where $G_1$ is abelian, $G_0 \subseteq G_1$ and $G_1/G_0 \cong \mathbb{Z}/p$. If $p$ is odd such complex structures always exist (see e.g. [11] Theorem (38.3)), so if we only consider odd primes Theorem 1.11 is true without assuming the existence of an almost complex structure (see [6] for similar considerations).

To deduce Theorems 1.6, 1.9 and 1.7 from Theorem 1.11 we apply the following result.

Theorem 1.13. Let $X$ be compact smooth manifold, and let $G \subseteq \text{Diff}(X)$ be a finite subgroup. Let $L \rightarrow X$ be a complex line bundle satisfying $g^*L \cong L$ for every $g \in G$. Suppose that one of the following conditions holds true.

1. $b_1(X) = 0$, or
2. there exists a symplectic form $\omega$ on $X$ such that $G \subseteq \text{Ham}(X, \omega)$, or
(3) there exists a finite group $\Gamma \subset \text{Diff}(X)$, whose action on $H^1(X)$ is trivial, which satisfies $G \subseteq [\Gamma, \Gamma]$ and $\gamma^*L \simeq L$ for every $\gamma \in \Gamma$.

Then there exists a finite group $G'$ sitting in a short exact sequence
\[1 \to H \to G' \xrightarrow{\pi} G \to 1,\]
where $H$ is finite cyclic and $|H|$ divides $|G|$, and a smooth action of $G'$ on $L$ by bundle automorphisms lifting the action of $G$ on $X$. The latter means that for any $\gamma \in G'$, any $x \in X$, and any $\lambda \in L_x$ we have $\gamma \cdot \lambda \in L_{\pi(\gamma)(x)}$.

The proof of Theorem 1.13 under hypothesis (2) (see Theorem 6.4) uses a result of Lalonde, McDuff and Polterovich [19, 21] based on the Seidel representation and implying the topological rigidity of Hamiltonian loops. In contrast, the proof under hypothesis (1) or (3) (see Theorems 6.1 and 6.5) only uses elementary cohomological arguments.

Remark 1.14. Since Theorem 1.13 does not involve any almost complex structure, it follows from Remark 1.12 that if we only consider odd primes then Theorem 1.9 is true for finite $p$-subgroups of $\text{Diff}(X)$ (so there is no need to consider almost complex structures).

1.6. Contents. Section 2 reviews some material on equivariant cohomology, introduces the notion of generic cohomology class of a cyclic $p$-group, and proves (Theorem 2.7) the invertibility of the Euler class of normal bundles of partially fixed point submanifolds of smooth $\mathbb{Z}/p^k$ actions (see the comments after the statement of Theorem 1.11). In Section 3 we prove a localization theorem for smooth actions of $\mathbb{Z}/p^k$ on manifolds. This will be the main building block of the proof of Theorem 1.11. Other ingredients will be a few technical results on finite $p$-groups proved in Section 4. The proof of Theorem 1.11 is given in Section 5. Section 6 contains the proof of Theorem 1.13. Finally, all theorems stated in this introduction except for Theorems 1.11 and 1.13 are proved in Section 7.

1.7. Conventions. By a natural number we understand a strictly positive integer.

Whenever we say that a group $G$ can be generated by $d$ elements we mean that there is a generating set $\{g_1, \ldots, g_d\}$ for $G$, where the $g_j$’s need not be distinct. If $G$ is a group and $S_1, \ldots, S_r$ are subsets of $G$, we denote by $\langle S_1, \ldots, S_r \rangle$ the subgroup of $G$ generated by the elements in the union of the $S_j$’s. Abusing notation, if a set has a unique element $s$, we will sometimes denote the set using the same symbol $s$ instead of $\{s\}$. If $A$ is an abelian group, $\hat{A} := \text{Hom}(A, \mathbb{C}^*)$ denotes the group of characters of $A$.

As mentioned previously, we only consider manifolds without boundary. Manifolds are not necessarily supposed to be connected, and vector bundles over disconnected manifolds need not have constant rank. Group actions on smooth manifolds will always be assumed to be smooth.

All cohomology groups will be, unless otherwise specified, with integer coefficients.

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2. Equivariant cohomology

This section contains the results from equivariant cohomology that will be used later in the localization arguments. Let us first briefly recall the basic definitions (see e.g. [12, Chap. III, §1] for details). If $G$ is a topological group and $X$ is a topological space acted on continuously by $G$ the $G$-equivariant cohomology of $X$ is by definition

$$H^*_G(X) = H^*(EG \times_G X),$$

where $EG \to BG$ is the universal $G$-principal bundle. The space $X_G := EG \times_G X$ is called the Borel construction, and its natural projection $X_G \to BG$ endows $H^*_G(X)$ with the structure of a module over $H^*(BG)$.

The two extreme cases are $G$ acting freely on $X$ (in this case $X_G \simeq X/G$, at least if the action admits slices) and $G$ acting trivially on $X$ (in this case $X_G \simeq BG \times X$). In particular, if $X$ is a $G$-space consisting of a unique point then $H^*_G(X) \simeq H^*(BG)$. For a general $G$-space the equivariant cohomology can be used to obtain information on isotropy groups. In our case $G$ will always be a finite group, and for this reason it will be essential to work with integer coefficients: rational and real coefficients are of no use in this situation, since if $G$ is finite then $H^*_G(X; \mathbb{Q}) \simeq H^*(X/G; \mathbb{Q})$.

If $V \to X$ is a vector bundle and the action of $G$ on $X$ lifts to an action on $V$ by vector bundle automorphisms, then $V_G$ is in a natural way a vector bundle over $X_G$. The equivariant characteristic classes of $V$ are the ordinary characteristic classes of $V_G$. So if $V$ is a complex (resp. real oriented) vector bundle then the equivariant Chern classes (resp. Euler class) of $V$ are $c_j^G(V) := c_j(V_G)$ (resp. $e^G(V) := e(V_G)$).

2.1. The pushforward map. Of fundamental importance in the localization arguments that we are later going to use is the so-called pushforward\footnote{There is no consensus in the literature on how to name this notion. Pushforward map seems to be the most usual name in the recent literature on equivariant cohomology in symplectic geometry, see e.g. [14, 15]. Umkehr/Umkehrung (the German word for "reversal") was the name used by Hopf [16] in the first paper on the subject (in the non equivariant context), and it is still used in homotopy theory [10]. Atiyah and Bott use it in their classical paper [11] on equivariant cohomology in symplectic geometry. For ordinary non-equivariant cohomology, one also uses shriek or transfer map [4, Chap. VI, Def. 11.2]. However, in equivariant cohomology transfer map usually means something different, see e.g. [3, 5, 42].} map in equivariant cohomology. Let $M$ and $N$ be compact smooth oriented manifolds endowed with orientation preserving actions of a compact Lie group $G$, and let $f : M \to N$ be a $G$-equivariant smooth map. The pushforward is a map

$$f^*_G : H^*_G(M) \to H^*_{G^{\dim N - \dim M}}(N),$$

which enjoys the following properties:

(P1) (composition) if $g : M \to R$ is another $G$-equivariant map, with $R$ a compact smooth oriented $G$-manifold, then $(g \circ f)^*_G = g^*_G \circ f^*_G$;

(P2) (product formula) for any $\alpha \in H^*_G(N)$ and any $\beta \in H^*_G(M)$ we have

$$f^*_G((f^*\alpha)\beta) = \alpha(f^*_G\beta);$$

combining (P1) and (P2) it follows that $f^*_G$ is a morphism of $H^*(BG)$-modules;
(P3) (embeddings) if \( f \) is an embedding, then \( f_*^G \) factors through the Thom isomorphism of the normal bundle \( \nu \to M \); more precisely, \( f_*^G \) is equal to the composition

\[
H_G^*(M) \xrightarrow{T} H_G^*(\nu, \nu \setminus \nu_0) \xrightarrow{\varepsilon} H_G^*(D(\nu), D(\nu) \setminus \nu_0) \xrightarrow{(\eta)^{-1}} H^*_G(exp(D(\nu)), exp(D(\nu)) \setminus M) \xrightarrow{\varepsilon} H_G^*(N, N \setminus M) \to H_G^*(N),
\]

where \( \nu_0 \subset \nu \) is the zero section, \( T \) is Thom isomorphism, \( D(\nu) \subset \nu \) is the open unit disk bundle defined with respect to a \( G \)-invariant Riemannian metric \( h \) on \( N, \eta : D(\nu) \to N \) is the exponential map w.r.t. \( h \) (which we assume to have injectivity radius at least 1, so that \( D(\nu) \) is a tubular neighborhood of \( M \) — otherwise we rescale \( D(\nu) \) in the fiberwise directions), \( \varepsilon \) denotes excision, and the last map is the natural map in cohomology; since all maps except the last one are isomorphisms, the image of \( f_*^G \) can be identified with the kernel of the restriction map \( H_G^*(N) \to H_G^*(N \setminus M) \); another consequence is the formula

\[
f_*^G(\alpha) = e^G(\nu),
\]

where \( e^G \) is the equivariant Euler class;

(P4) (functoriality) let \( \rho : K \to G \) be a morphism of groups; using \( \rho \) and the \( G \)-action, we may define an action of \( K \) on \( M \); let \( \pi : M \to \{ \ast \} \) be the map to a point; let \( \rho^*_M : H^*_G(M) \to H^*_K(M) \) and \( \rho^* : H^*(BG) \to H^*(BK) \) be the natural maps induced by \( \rho \); then we have

\[
\rho^* \pi_*^G = \pi_*^K \rho^*_M.
\]

In the context of symplectic geometry these properties and their application to localization arguments are well known, but they are typically applied to compact connected Lie group actions, where real coefficients are enough (see e.g. [1][13][15]). The proof of (P1)—(P4) for real coefficients is easily obtained using the Cartan–Weil model for equivariant cohomology [13] §10.7.

Since for us the use of integral coefficients is crucial, we briefly sketch for the reader’s convenience the definition of \( f_*^G \) over \( \mathbb{Z} \). Consider first the case \( G = \{1\} \). Then \( f_*^G \) can be defined as \( D_N^{-1} \circ f_* \circ D_M \), where \( D_M : H^*(M) \to H_{\dim M - \ast}(M) \) and \( D_N : H^*(N) \to H_{\dim N - \ast}(N) \) are the Poincaré duality maps and \( f_* : H_\ast(M) \to H_\ast(N) \) is the map induced in homology by \( f \) (see [4] Chap. VI, Def. 11.2]). To define \( f_*^G \) for arbitrary finite \( G \) one can use finite dimensional approximations of the classifying space \( BG \). More precisely, in order to define \( f_*^G : H^*_G(M) \to H^{r + \dim N - \dim M}(N) \) one considers a principal \( G \)-bundle \( P \to B \) with \( B \) a smooth compact, connected and oriented manifold, such that \( P \) is \( k \)-connected for \( k \) big enough so that the classifying map \( B \to BG \) for \( P \) induces isomorphisms

\[
H^r(P \times_G M) \simeq H_G^*(M), \quad H^{r + \dim N - \dim M}(P \times_G N) \simeq H_G^{r + \dim N - \dim M}(N)
\]

(one can take \( P \) to be a Stiefel manifold, see e.g. [14] Example C.1]). Combining these isomorphisms with the map

\[
D^{-1} \circ f_* \circ D : H^r(P \times_G M) \to H^{r + \dim N - \dim M}(P \times_G N)
\]

one obtains \( f_*^G \) (here \( D \) is Poincaré duality and \( f_* : H_\ast(P \times_G M) \to H_\ast(P \times_G N) \) is the map induced by \( f \)). The following lemma, whose proof is left to the reader (see [4]
Lemma 2.1. Consider a commutative diagram
\[
\begin{array}{ccc}
X & \xrightarrow{i} & Y \\
f \downarrow & & \downarrow g \\
U & \xrightarrow{h} & V
\end{array}
\]
where \(U, V, X, Y\) are compact, connected and oriented smooth manifolds and \(f, g, h, i\) are smooth maps. Suppose that \(i\) and \(h\) are embeddings and the normal bundles \(\nu_h \rightarrow U\) and \(\nu_i \rightarrow X\) satisfy \(\nu_i \simeq f^*\nu_h\). Then we have
\[
D_U f_* D_X i^* = h^* D_V g_* D_Y
\]
as maps from \(H^*(Y)\) to \(H^*(U)\), where \(D_M\) denotes Poincaré duality on the manifold \(M\).

To apply the previous lemma, note that for any two principal \(G\)-bundles \(P_0 \rightarrow B_0\) and \(P_1 \rightarrow B_1\) one can find another principal \(G\)-bundle \(P' \rightarrow B'\) admitting \(G\)-equivariant embeddings \(P_i \rightarrow P'\) \((i = 0, 1)\) in such a way that the resulting diagrams (with \(f_i\) and \(f'\) the maps induced by \(f\))
\[
\begin{array}{ccc}
P_i \times_G M & \xrightarrow{i} & P' \times_G M \\
f_i \downarrow & & \downarrow f' \\
P_i \times_G N & \xrightarrow{h} & P' \times_G N
\end{array}
\]
satisfy the hypothesis of Lemma 2.1. This immediately implies that \(f_G^*\) is well defined.

The proof that the pushforward map satisfies properties (P1)–(P3) is straightforward using standard results as in [4, Chap. VI]. To prove (P4) one may use a variant of Lemma 2.1 in which \(f\) and \(g\) are assumed to be submersions, the square (1) is Cartesian, and the maps \(i\) and \(f\) can be identified with the projections \(U \times_Y Y \rightarrow Y\) and \(U \times_Y Y \rightarrow U\).

2.2. Cohomology of cyclic \(p\)-groups. Let us identify the circle \(S^1\) with the complex numbers of modulus 1, and let
\[
\mu_{p^k} \subset S^1
\]
be the group of \(p^k\)-th roots of unity, where \(p\) is a prime and \(k\) a natural number. The following lemma is standard (see e.g. [12, Chap. III, §2]).

Lemma 2.2. Let \(t = c_1(ES^1) \in H^2(BS^1)\). We have \(H^*(BS^1) = \mathbb{Z}[t]\). The map
\[
\iota_k^* : H^*(BS^1) = \mathbb{Z}[t] \rightarrow H^*(B\mu_{p^k})
\]
induced by the inclusion \(\iota_k : \mu_{p^k} \hookrightarrow S^1\) is surjective and its kernel is the ideal of \(H^*(BS^1)\) generated by \(p^k t\). Furthermore, \(\iota_k^* t = c_1(L)\), where \(L = E\mu_{p^k} \times_{\iota_k} \mathbb{C}\).
Lemma 2.3. Let $j : \mu_{p^{-1}} \hookrightarrow \mu_p$ be the inclusion. For any nonnegative integer $r$ there is a commutative diagram with exact rows

$$
\begin{array}{cccccc}
p^k \mathbb{Z}(t^r) & \longrightarrow & \mathbb{Z}(t^r) & \xrightarrow{i_k} & H^{2r}(BS^1) & \longrightarrow & 0 \\
p^{k-1} \mathbb{Z}(t^r) & \longrightarrow & \mathbb{Z}(t^r) & \xrightarrow{i^*_{k-1}} & H^{2r}(B\mu_{p^{k-1}}) & \longrightarrow & 0.
\end{array}
$$

Proof. The exactness of the rows follows from Lemma 2.2. The commutativity of the square is a consequence of the identity $i_k \circ j = i_{k-1}$. \hfill \Box

2.3. Generic cohomology classes of cyclic $p$-groups. Let $k$ be a natural number and let

$$
\tau_k = i_k^* t \in H^2(B\mu_{p^k}),
$$

where $i_k : \mu_{p^k} \hookrightarrow S^1$ is the inclusion. We say that a cohomology class $\beta \in H^*(B\mu_{p^k})$ is generic if for any natural $r$ the class $\tau_k^r \beta$ is nonzero.

It follows from Lemma 2.2 that if $\beta \in H^d(B\mu_{p^k})$ then

$$
\beta \text{ is generic } \iff \begin{cases} 
\beta \text{ is not divisible by } p^k & \text{if } d = 0, \\
\beta \neq 0 & \text{if } d > 0.
\end{cases}
$$

The following two properties are obvious and will be used repeatedly in our arguments:

1. if $\alpha_1, \ldots, \alpha_n \in H^*(B\mu_{p^k})$ and $\alpha_1 + \cdots + \alpha_n$ is generic, then at least one of the $\alpha_j$’s is generic;
2. if $\alpha \in H^*(B\mu_{p^k})$ and $r$ is a nonnegative integer, then $\alpha$ is generic if and only if $\tau_k^r \alpha$ is generic.

Lemma 2.4. Let $X$ be a smooth oriented compact manifold endowed with an orientation preserving action of $\mu_{p^k}$. Let $\pi : X \to \{\ast\}$ be the map to a point. If for some $\alpha \in H^*_\mu(B\mu_{p^k})$ the class $\pi^\mu_{p^k} \alpha \in H^*(B\mu_{p^k})$ is generic, then the action of $\mu_{p^k}$ on $X$ is not free.

Proof. Denote for convenience $G = \mu_{p^k}$. By the product formula ((P2) in §2.1) we have $\pi^G_\pi((\pi^* \tau_k^s)\alpha) = \tau_k^r \pi^G_\pi \alpha$ for any natural $s$ and any $\alpha \in H^*_\mu(X)$. Suppose that $G$ acts freely on $X$. Then the Borel construction $X_G$ is homotopy equivalent to the smooth manifold $X/G$. If $s + r$ is bigger than the dimension of each connected component of $X/G$ then for any $\alpha$ we have

$$
\tau_k^s \alpha \in H^{s+r}(X) = H^{s+r}(X/G) = 0,
$$

so $\tau_k^s \pi^G_\pi \alpha = \pi^G_\pi (\tau_k^s \alpha) = 0$. Hence $\pi^G_\pi \alpha$ is not generic. \hfill \Box

The definition of generic cohomology classes can be naturally extended to abstract cyclic $p$-groups. Namely, if $G$ is one such group we may take any isomorphism $\theta : G \xrightarrow{\sim} \mu_{p^k}$ for some $k$ and declare that a cohomology class $\beta_G \in H^*(BG)$ is generic if it is equal to $\theta^* \beta$ for some generic class $\beta \in H^*(B\mu_{p^k})$. By [3] this definition is independent of $\theta$. 
2.4. **A nilpotence lemma.** Let $G = \mu_p k$ and $G_0 = \mu_{p^{-1}} k \subset G$ for some integer $k \geq 2$. Suppose that $G$ acts smoothly on a compact smooth manifold $X$. Denote by

$$\lambda : H^*_G(X) \to H^*_G(X)$$

the restriction map.

**Lemma 2.5.** If $\alpha, \beta \in H^*_G(X)$ satisfy $\lambda(\alpha) = \lambda(\beta)$ then $\alpha^{p^r} = \beta^{p^r}$ for some $r$.

**Proof.** We first prove that for any $\delta \in \text{Ker} \lambda$ we have $p^r \delta = 0$. Fixing some model $EG \to BG$ for the universal bundle of $G$, we can identify $X_{G_0}$ with $EG \times_{G_0} X$, so that the map

$$q : X_{G_0} \to X_G = EG \times_G X$$

inducing $\lambda$ can be chosen to be a degree $p$ covering (namely, the quotient by the residual action of $G/G_0$). This immediately implies that any nontrivial element in the kernel of $\lambda$ has order $p$. To prove this, let $S_\ast$ be the group of singular chains, define a “trace” map $\tau_\ast : S_\ast (X_G) \to S_\ast (X_{G_0})$ by sending a singular simplex $s : \Delta^r \to X_G$ to the sum of its $p$ different lifts $\Delta^r \to X_{G_0}$, and extend the definition linearly to singular chains; then $\tau_\ast$ is a morphism of complexes, and $q_\ast \tau_\ast : S_\ast (X_G) \to S_\ast (X_{G_0})$ is multiplication by $p$; dualising, the map $\tau^\ast : S^\ast (X_{G_0}) \to S^\ast (X_G)$ satisfies $\tau^\ast q^\ast (\alpha) = p \alpha$ for every $\alpha \in S^\ast (X_G)$.

Let $\pi : X \to X/G$ be the quotient map. Consider the sheaf $\mathcal{H}^*_G$ (resp. $\mathcal{H}^*_{G_0}$) of graded rings on $X/G$ defined by $\mathcal{H}^*_G(U) = H^*_G(\pi^{-1}(U))$ (resp. $\mathcal{H}^*_{G_0}(U) = H^*_{G_0}(\pi^{-1}(U))$) for every open subset $U \subseteq X/G$ (see [36 §3]). We have a commutative diagram of rings

$$(4) \quad \begin{array}{ccc}
H^*_G(X) & \xrightarrow{\lambda} & H^*_G(X) \\
\sigma \downarrow & & \downarrow \sigma_0 \\
H^0(X/G, \mathcal{H}^*_G) & \xrightarrow{\mu} & H^0(X/G, \mathcal{H}^*_{G_0})
\end{array}$$

where $\mu$ is defined by restriction similarly to $\lambda$ and $\sigma, \sigma_0$ are defined (also by restriction) in [36 §3]. We next prove that for any $\beta \in H^0(X/G, \mathcal{H}^*_G)$ such that $\mu(\beta) = 0$ we have $\beta^2 = 0$. We need for that to describe the stalks of $\mathcal{H}^*_G$ and $\mathcal{H}^*_{G_0}$ and the action of the restriction map $\mathcal{H}^*_G \to \mathcal{H}^*_{G_0}$ at the level of stalks. Let $y \in X/G$ and let $x \in \pi^{-1}(y)$ be any preimage. Let $G_1 = G_x$ be the stabiliser of $x$. Clearly, either $G_1 \subseteq G_0$ or $G_1 = G$. If $G_1 \subseteq G_0$ then $(\mathcal{H}^*_G)_y \simeq H^*(BG_1)$, $(\mathcal{H}^*_{G_0})_y \simeq (H^*(BG_1))^{\oplus p}$, and the natural restriction map $\rho_y : (\mathcal{H}^*_G)_y \to (\mathcal{H}^*_{G_0})_y$ can be identified with the diagonal inclusion. In particular, $\rho_y$ is injective in this case. If $G_1 = G$ then $(\mathcal{H}^*_G)_y \simeq H^*(BG)$, $(\mathcal{H}^*_{G_0})_y \simeq H^*(BG_0)$, and $\rho_y : (\mathcal{H}^*_G)_y \to (\mathcal{H}^*_{G_0})_y$ can be identified with the map $j^*$ in Lemma 2.3. The latter has the property that for any $a \in \text{Ker} j^*$ we have $a^2 = 0$. With these observations, the claim is clear.

We next claim that if $\delta \in \text{Ker} \lambda$ then $\delta^{p^r} = 0$ for sufficiently big $r$. By the commutativity of (4) if $\delta \in \text{Ker} \lambda$ then $\mu(\sigma(\delta)) = 0$, which by the previous claim implies that $\sigma(\delta^2) = \sigma(\delta)^2 = 0$. Now the claim follows from [36 Prop. 3.2 and Remark 3.4].
We are now ready to prove the lemma. Suppose that \( \alpha, \beta \in H_G^*(X) \) satisfy \( \lambda(\alpha) = \lambda(\beta) \), so that \( \delta = \beta - \alpha \in \text{Ker} \lambda \). Let \( r \) be a natural number such that \( \lambda^p = 0 \). Then

\[
\beta^p = (\alpha + \delta)^p = \alpha^p + \sum_{j=1}^{p-1} \binom{p}{j} \alpha^j \delta^{p-j} + \delta^p = \alpha^p + \delta^p = \alpha^p,
\]

because \( \binom{p}{j} \) is divisible by \( p \) for any \( 1 \leq j \leq p - 1 \). \( \square \)

**Lemma 2.6.** Let \( G' \subseteq G = \mu_{p^k} \) be any nontrivial subgroup. Let \( \nu : H_G^*(X) \rightarrow H_G^*(X) \) be the restriction map. If \( \alpha, \beta \in H_G^*(X) \) satisfy \( \nu(\alpha) = \nu(\beta) \) then \( \alpha^{p^s} = \beta^{p^s} \) for some \( s \).

**Proof.** Combine the previous lemma with induction on \( |G/G'| \). \( \square \)

### 2.5. Decomposing vector bundles according to characters

Let \( A \) be a finite abelian group acting smoothly on a manifold \( X \) and let \( E \rightarrow X \) be an \( A \)-equivariant complex vector bundle. Recall that \( \hat{A} \) denotes the group \( \text{Hom}(A, \mathbb{C}^*) \) of characters of \( A \). For any \( \xi \in \hat{A} \) we define \( E^{A, \xi} \) to be the subbundle of \( E|_{\hat{A}} \) consisting of those vectors on which the action of \( A \) is given by multiplication by the character \( \xi \). Namely,

\[
E^{A, \xi} = \{ v \in E_x \mid x \in X^A, \alpha \cdot v = \xi(\alpha)v \text{ for every } \alpha \in A \}.
\]

Since \( A \) is abelian, the irreducible representations of \( A \) are one-dimensional. This implies

\[
E|_{X^A} = \bigoplus_{\xi \in \hat{A}} E^{A, \xi}.
\]

### 2.6. Rings generated by Chern classes

Let \( X \) be a smooth manifold endowed with a smooth action of a finite group \( \Gamma \). Let \( W_1, \ldots, W_s \) be \( \Gamma \)-equivariant complex vector bundles over \( X \). Let \( A \subseteq \Gamma \) be a normal abelian subgroup. We denote

\[
C^A(A_1, \ldots, W_s) \subseteq H^*(X^A)
\]

the subring generated by \( 1 \) and the Chern classes of the vector bundles \( \{W_j^{A, \xi}\} \), where \( j \) and \( \xi \) run over the sets \( \{1, \ldots, s\} \) and \( \hat{A} \) respectively. When \( A \) is trivial we usually suppress it from the notation, so we write \( C(A_1, \ldots, W_s) \subseteq H^*(X) \).

Let \( G \subseteq \Gamma \) be any subgroup contained in the centralizer of \( A \). Then \( X^A \) is \( G \)-invariant and the \( G \)-equivariant Chern classes of the bundles \( W_j^{A, \xi} \) are naturally defined. We denote by \( C^G(A_1, \ldots, W_s) \subseteq H_G^*(X^A) \) the \( G \)-equivariant counterpart of \( C^A(A_1, \ldots, W_s) \). Namely, \( C^G(A_1, \ldots, W_s) \) is the subring generated by \( 1 \) and the \( G \)-equivariant Chern classes of the vector bundles \( \{W_j^{A, \xi}\} \), where \( j \) and \( \xi \) run over the sets \( \{1, \ldots, s\} \) and \( \hat{A} \) respectively. Again, if \( A \) is trivial we denote \( C^G(A_1, \ldots, W_s) \) by \( C^G(A_1, \ldots, W_s) \).

These definitions make sense more generally if each \( W_j \) is a \( \Gamma \)-equivariant complex vector bundle over a \( \Gamma \)-invariant submanifold \( Y_j \subseteq X \) containing \( X^A \), where the submanifolds \( Y_1, \ldots, Y_s \) need not be equal to \( X \) and may even be different among themselves.
2.7. Inverting the equivariant Euler class up to $\tau_k$. Let $k$ be any natural number, let $G = \mu_p^k$ and let $H = \mu_p \subseteq G$. Let $W$ be a complex vector bundle over a compact smooth manifold $X$, and suppose that $G$ acts complex linearly on $W$ lifting a smooth action on $X$. Suppose that $W^H = X$, so that:

(1) the action of $H$ on $X$ is trivial, and
(2) for any $x \in X$, any $h \in H \setminus \{1\}$ and any $v \in W_x \setminus \{0\}$ we have $h \cdot v \neq v$.

Define $\tau_k$ as in Subsection 2.2. We have:

**Theorem 2.7.** There exists some $f \in \mathcal{E}_H^G(W)[\tau_k] \subseteq H^*_G(X)$ and some positive integer $U$ such that $e^G(W)f = \tau_k^U$.

**Proof.** We identify the group of characters of $H$ with $\mathbb{Z}/p$, by assigning to $\chi \in \mathbb{Z}/p$ the character $\rho_\chi : H \to \mathbb{C}^*$, $\rho_\chi(\theta) = \theta^\chi$. Define $W_\chi := W^{H,\rho_\chi}$. We have a decomposition as a Whitney sum of complex vector bundles

$$W = \bigoplus_{\chi \in \mathbb{Z}/p} W_\chi.$$

Assumption (2) above implies that $W_0 = 0$. If $k$ is an integer we denote for convenience $W_k = W_{\overline{k}}$, where $\overline{k}$ is the class of $k$ in $\mathbb{Z}/p$. Then

$$\mathcal{E}_H^G(W) = \mathcal{E}_H^G(W_1, \ldots, W_{p-1}).$$

Since $H$ acts trivially on $X$, we have $X_H = X \times BH$, so by Künneth there is a natural inclusion

$$H^*(X) \otimes H^*(BH) = H^*(X)[\tau]/\{p\tau = 0\} \to H^*_H(X)$$

where $\tau$ corresponds to $\tau_1 \in H^2(BH)$. Let $r_\chi$ denote the (complex) rank of $W_\chi$. For any $\chi \in \mathbb{Z}/p$ let $\mathbb{C}_\chi = X \times \mathbb{C}$ denote the trivial line bundle over $X$ with the action of $H$ given by $h \cdot (x, z) = (x, \rho_\chi(h)z)$. By Lemma 2.2 the first $H$-equivariant Chern class of $\mathbb{C}_\chi$ is

$$c_1^H(\mathbb{C}_\chi) = \chi \tau.$$

Let $W_{\chi,0} = W_\chi \otimes \mathbb{C}_{-\chi}$. Then $H$ acts trivially on $W_{\chi,0}$, so the $H$-equivariant Chern class $c_j^H(W_{\chi,0})$ can be identified with the usual (non equivariant) Chern class $c_j(W_{\chi,0})$ (which of course coincides with $c_j(W_\chi)$) via the composition of maps

$$H^*(X) \ni \alpha \mapsto \alpha \otimes 1 \in H^*_H(X) \otimes H^0(BH) \subseteq H^*_H(X) \otimes H^*(BH) \to H^*_H(X).$$

Indeed, the Borel construction applied to $W_{\chi,0}$ gives a vector bundle on $X_H$ which can be identified with the pullback of $W_\chi$ via the projection map $X_H \to X$. We may thus write, somewhat abusively, $c_j^H(W_{\chi,0}) = c_j(W_\chi)$, where here (and below) we implicitly regard $c_j(W_\chi)$ as a class in $H^*_H(X)$. Since $W_\chi = W_{\chi,0} \otimes \mathbb{C}_\chi$, we have

$$c_j^H(W_\chi) = c_j(W_\chi) + \sum_{k=1}^{j} c_{j-k}(W_\chi)(\chi \tau)^k \binom{r_\chi - j + k}{k}.$$

Hence $c_j^H(W_\chi) \in \mathcal{E}(W_\chi)[\tau]$. These formulas also imply, using induction on $j$, that $c_j(W_\chi) \in \mathcal{E}^H(W_\chi)[\tau]$. Hence we have:

(5) $$\mathcal{E}(W_1, \ldots, W_{p-1})[\tau] = \mathcal{E}^H(W_1, \ldots, W_{p-1})[\tau]$$
Let $r = \sum r_{\chi}$ be the rank of $W$. We have

$$e^H(W) = \prod_{\chi \in (\mathbb{Z}/p)^{\times}} e^H(W_{\chi}) = \prod_{\chi \in (\mathbb{Z}/p)^{\times}} c^H_{r_{\chi}}(W_{\chi}) = \prod_{\chi \in (\mathbb{Z}/p)^{\times}} \left( \sum_{j=0}^{r_{\chi}} (\chi \tau)^j c_{r_{\chi}-j}(W_{\chi}) \right),$$

so we may write

$$e^H(W) = ar^s + P$$

where $a \in (\mathbb{Z}/p)^{\times}$ and $P$ is a polynomial in $\tau$ and the non-equivariant Chern classes $\{c_j(W_{\chi})\}_{j \geq 0, \chi \in (\mathbb{Z}/p)^{\times}}$, all of whose monomials have at least one factor of the form $c_j(W_{\chi})$ with $j \geq 1$. In particular, there exists some natural number $s$ such that $P^{s+1} = 0$ (because $X$ is compact), so

$$\psi_H = a^{-(s+1)} \left( a^s \tau^s + \sum_{j=1}^{s} (-1)^j a^{s-j} \tau^{r(s-j)} P^j \right)$$

satisfies $e^H(W) \psi_H = \tau^R$, where $R = r(s+1)$. Furthermore,

$$\psi_H \in C^H(W_1, \ldots, W_{p-1})[\tau]$$

by (3), so we may write

$$\psi_H = Q(\tau, c^H_1(W_1), \ldots, c^H_{r_1}(W_1), \ldots, c^H_1(W_{p-1}), \ldots, c^H_{r_{p-1}}(W_{p-1})),$$

for some polynomial $Q \in \mathbb{Z}[x_0, x_1, \ldots, x_r]$.

Now let

$$\nu : H^*_{G}(X) \rightarrow H^*_{H}(X)$$

be the restriction map and define

$$\psi_G := Q(\tau_k, c^G_1(W_1), \ldots, c^G_{r_1}(W_1), \ldots, c^G_1(W_{p-1}), \ldots, c^G_{r_{p-1}}(W_{p-1})) \in H^*_{G}(X).$$

Since $\nu(\tau_k) = \tau$, $\nu(e^G(W)) = e^H(W)$ and $\nu(c^G_j(W_{\chi})) = c^H_j(W_{\chi})$, we have

$$\nu(e^G(W) \psi_G) = e^H(W) \psi_H = \tau^R = \nu(\tau_k^R).$$

By Lemma 2.6 there exists some $S$ such that

$$(e^G(W) \psi_G)^S = \tau_k^{RS}.$$ \hspace{1cm} \Box$$

Writing $U = RS$ and $f = e^G(W)^{S-1} \psi_G^S$, it follows that $e^G(W)f = \tau_k^U$, as desired.

### 3. Localization

Let $p$ be a prime and $k$ a natural number. A $p^k$-admissible tuple is a tuple of the form

$$(X, G, A, V_1, \ldots, V_r),$$

where:

- $X$ is a closed smooth manifold, endowed with an almost complex structure $J$,
- $G$ is a finite $p$-group acting smoothly (but non necessarily effectively) on $X$ and preserving $J$,
- $A$ is an abelian normal subgroup of $G$ such that $X^A \neq \emptyset$,
- $V_1, \ldots, V_r$ are $G$-equivariant complex vector bundles over $X$, 

...
subject to the following condition. Let $N \to X^A$ be the normal bundle of the inclusion in $X$, with the complex structure induced by $J$. Denote by $Z_G(A)$ the centralizer of $A$ in $G$. Then there exist a $Z_G(A)$-invariant open and closed submanifold $M \subseteq X^A$ and a cohomology class
\[
\alpha \in \mathcal{C}_A(V_1, \ldots, V_r, N)
\]
such that $\langle [M], \alpha \rangle$ is not divisible by $p^k$.

The following is the main result of the present section.

**Theorem 3.1.** Let $(X, G, A, V_1, \ldots, V_r)$ be a $p^k$-admissible tuple. Suppose that there exists some $h \in Z_G(A)$ whose class $[h]$ in $G/A$ is $G$-invariant and has order $p^k$. Then there exists some $0 \leq i < k$ such that setting $A' = \langle A, h^p \rangle$ the tuple $(X, G, A', V_1, \ldots, V_r)$ is $p^k$-admissible.

**Proof.** Let $M$ be an open and closed submanifold of $X^A$ such that $M$ is preserved by the action of $Z_G(A)$ and such that there exists some $\alpha \in \mathcal{C}_A(V_1, \ldots, V_r, N)$ whose pairing $\langle [M], \alpha \rangle$ is not divisible by $p^k$. Suppose that $h \in Z_G(A)$ has the property that $[h]$ in $G/A$ is $G$-invariant and $[h]$ has order $p^k$ in $G/A$, so $h^p \in A$. Define these subgroups of $G$:
\[
\Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_k \subseteq G, \quad \Gamma_j := \langle h^{p^j} \rangle \subseteq G.
\]
The action of $\Gamma := \Gamma_k$ on $X$ preserves $M$. Since $\Gamma_0 \subseteq A$, the action of $\Gamma_0$ on $M$ is trivial, so the action of $\Gamma$ on $M$ induces an action of $\Gamma/\Gamma_0 \simeq \mathbb{Z}/p^k\mathbb{Z}$ (which need not be effective).

To simplify the notation, denote by
\[
E_1, \ldots, E_u
\]
the collection of all vector bundles on $M$ of the form $V_i^{A, \xi}|_M$ for some $i$ and $\xi \in \hat{A}$, or $N^{A, \xi}|_M$ for some $\xi \in \hat{A}$, where $N$ is the normal bundle of the inclusion $X^A \subseteq X$. Then we have
\[
\alpha \in \mathcal{C}(E_1, \ldots, E_u).
\]

For any $x \in M^{\Gamma_j}$ define
\[
\text{md}_j(x) = (\dim R T_x M^{\Gamma_0}, \dim R T_x M^{\Gamma_1}, \ldots, \dim R T_x M^{\Gamma_j}) \in \mathbb{Z}^{j+1}
\]
(“md” stands for multi-dimension), and for any $\mu \in \mathbb{Z}^{j+1}$ let
\[
M^\mu = \{x \in M^{\Gamma_j} \mid \text{md}_j(x) = \mu\}.
\]
Define also for any $i \geq 1$ the vector bundle
\[
N_i = \frac{TM^{\Gamma_{i-1}}|_{M^{\Gamma_i}}}{TM^{\Gamma_i}}.
\]
The bundle $N_i$ inherits from $J$ a structure of complex vector bundle. The normal bundle $N_{M^{\Gamma_j}|_{M^{\Gamma_j}}}$ of the inclusion $M^{\Gamma_j} \subseteq M$ satisfies
\[
N_{M^{\Gamma_j}|_{M^{\Gamma_j}}} \simeq \bigoplus_{i=1}^j N_i|_{M^{\Gamma_j}}.
\]

In terms of this notation, we have the following.
Lemma 3.2. There exists some $1 \leq l \leq k$, some $\mu \in \mathbb{Z}^{l+1}$, and some cohomology class $\beta \in \mathcal{C}_{\Gamma_l}(E_1, \ldots, E_u, N_1, \ldots, N_l)$ such that $\langle [M^\mu], \beta \rangle$ is not divisible by $p^k$. In particular $M^\mu$, and hence $M^{\Gamma_l}$, is nonempty.

The proof of Lemma 3.2 is given in Subsection 3.1 below. We now prove how the lemma implies Theorem 3.1. Let $l, \mu, \beta$ be the output of Lemma 3.2. We are going to prove that if $A' = \langle A, \Gamma_l \rangle = \langle A, h^p \rangle$ then $(X, G, A', V_1, \ldots, V_r)$ is $p^k$-admissible.

Since $A$ is normal in $G$ and the class of $h$ in $G/A$ is $G$-invariant, $A'$ is a normal subgroup of $G$. Since $M^{\Gamma_l} \neq \emptyset$ and $M \subseteq X^A$, we have $X^{A'} \neq \emptyset$. We next observe that $M^\mu$ is open and closed in $X^{A'}$. To prove this it suffices to check that $M^\mu$ is open and closed in $M^{\Gamma_l}$ (since $M$ on its turn is open and closed in $X^A$), and this follows from the fact that the map $\text{md}_l : M^{\Gamma_l} \to \mathbb{Z}^{l+1}$ is locally constant.

The submanifold $M^\mu \subseteq X$ is preserved by the action of $Z_G(A')$. This follows from the next two observations. First, $M \subseteq X$ is preserved by $Z_G(A)$ and $Z_G(A') \subseteq Z_G(A)$, hence $M^{\Gamma_l}$ is preserved by $Z_G(A')$. Second, the fact that $[h] \in G/A$ is $G$-invariant implies that the map $\text{md}_l : M^{\Gamma_l} \to \mathbb{Z}^{l+1}$ is $Z_G(A')$-invariant.

Now let $N$ be the normal bundle of the inclusion $M^\mu \subseteq X$. We claim that

$$
\beta \in \mathcal{C}_{A'}(V_1, \ldots, V_r, N).
$$

Since by Lemma 3.2 we have $\langle [M^\mu], \beta \rangle \notin p^k \mathbb{Z}$, (6) implies that $(X, G, A', V_1, \ldots, V_r)$ is $p^k$-admissible. The inclusion (6) is a consequence of

$$
\mathcal{C}_{\Gamma_l}(E_1, \ldots, E_u, N_1, \ldots, N_l) \subseteq \mathcal{C}_{A'}(V_1, \ldots, V_r, N),
$$

which follows from the product formula for the Chern class of a direct sum of vector bundles and these two facts:

- for any $j$ and $\xi \in \tilde{\Gamma}_l$ the vector bundle $E_j^{\Gamma_l, \xi}$ is the direct sum of vector bundles of the form $V_i^{A', \eta}$, for any $i$ and any $\eta \in \tilde{A}'$, and of the form $N^{A', \eta}$ for any $\eta \in \tilde{A}'$; this follows from the definition of the bundles $E_1, \ldots, E_u$, and the fact that $E_j^{\Gamma_l, \xi} \subseteq E_j$ is preserved by the action of $A'$ (which on its turn follows from the fact that $A$ acts on $E_j$ by homotheties);

- for any $j \leq l$ and $\xi \in \tilde{\Gamma}_l$ the vector bundle $N_j^{\Gamma_l, \xi}$ is the direct sum of vector bundles of the form $N^{A', \eta}$ for any $\eta \in \tilde{A}'$; this follows from the fact that $N_j^{\Gamma_l, \xi} \subseteq N$ is $A'$-invariant (which is a consequence of the assumption that $[h] \in G/A$ is $G$-invariant).

The proof of Theorem 3.1 is now complete.

3.1. Proof of Lemma 3.2 We first introduce some notation.

Denote $K_i = \Gamma / \Gamma_i$.

For any group $G$ acting on a space $X$ and every character $\chi \in \hat{G}$, $\mathbb{C}_\chi$ is the trivial line bundle $\mathbb{C}_\chi = X \times \mathbb{C}$ endowed with the action of $G$ defined by the formula $\gamma \cdot (x, z) = (\gamma \cdot x, \chi(\gamma)z)$.

Since $\Gamma$ is abelian, for every $j$ the inclusion $\Gamma_j \subseteq \Gamma$ induces by restriction a surjective morphism of character groups $r_j : \hat{\Gamma} \to \hat{\Gamma}_j$. Choose, for each $j$, a set theoretical section
\[ \psi_j \colon \hat{\Gamma}_j \to \hat{\Gamma} \text{ of } r_j. \] We emphasize that we do not require any compatibility relation between the morphisms \( \psi_j \) for different values of \( j \). Define, for every \( i, j \) and any \( \xi \in \hat{\Gamma}_j \), the following \( \Gamma \)-bundle over \( \hat{\Gamma} \):

\[ F_i(\Gamma_j, \xi) = E_i \Gamma_j, \xi \otimes C_{\psi_j(\xi)^{-1}}, \]

where we are using multiplicative notation in \( \hat{\Gamma} \). The effect of tensoring \( E_i \Gamma_j, \xi \) by \( C_{\psi_j(\xi)^{-1}} \) is to trivialize the action of \( \Gamma_j \). Hence, the natural action of \( \Gamma \) on \( F_i(\Gamma_j, \xi) \) factors through an action of \( K_j \), which lifts the action of \( K_j \) on \( M^\Gamma \). Similarly, we define for every \( i \leq j \)

\[ R_i(\Gamma_j, \xi) = N_i \Gamma_j, \xi \otimes C_{\psi_j(\xi)^{-1}}, \]

which again can be seen as a \( \Gamma \)-bundle or a \( K_j \)-bundle over \( M^\Gamma \).

Let \( \rho_j : \hat{\Gamma}_j \to \hat{\Gamma}_{j-1} \) be the restriction map. Take an arbitrary element \( \eta \in \hat{\Gamma}_j \) and let \( \xi = \rho_j(\eta) \). Since the characters \( \psi_j(\eta) \in \hat{\Gamma}_j \) and \( \psi_{j-1}(\xi) \in \hat{\Gamma}_{j-1} \) extend \( \eta \) and \( \xi \) respectively and the restriction of \( \eta \) to \( \Gamma_{j-1} \) coincides with \( \xi \), the character \( \psi_j(\eta) \psi_{j-1}(\xi)^{-1} \) is trivial on \( \Gamma_{j-1} \). Hence, it induces a character of \( K_{j-1} \), which we denote by \( \rho(\eta, \xi) \). For any \( j \) and any \( \xi \in \hat{\Gamma}_{j-1} \) we have the following equalities between \( K_{j-1} \)-bundles:

\[ F_i(\Gamma_{j-1}, \xi)|_{M^{\rho_j}} = \bigoplus_{\rho_j(\eta) = \xi} F_i(\Gamma_j, \eta) \otimes C_{\rho(\eta, \xi)}, \]

and

\[ R_i(\Gamma_{j-1}, \xi)|_{M^{\rho_j}} = \bigoplus_{\rho_j(\eta) = \xi} R_i(\Gamma_j, \eta) \otimes C_{\rho(\eta, \xi)}. \]

After these preliminaries, we proceed to prove Lemma 3.2. The main ingredient of the proof will be an inductive argument, which is the content of Lemma 3.3 below.

We now prepare the setting to run the inductive process.

Since \( \alpha \in \mathfrak{e}(E_1, \ldots, E_u) \), we may write \( \alpha \) as a sum of monomials on the Chern classes of the bundles \( E_i \), and the hypothesis of the lemma implies that the pairing of \([M]\) with at least one of these monomials yields an integer that is not divisible by \( p^k \). We then replace \( \alpha \) by this monomial, so we have

\[ \alpha = \prod_{l=1}^v c_{j_l}(E_{i_l}) \]

and the pairing \( \langle [M], \alpha \rangle \) is not divisible by \( p^k \). Let \( \mu_0 = \sum_{l=1}^v 2j_l \). Since the degree of \( \alpha \) is \( \mu_0 \), we have \( \langle [M], \alpha \rangle = \langle [M^{\mu_0}], \alpha \rangle \), so \( \langle [M^{\mu_0}], \alpha \rangle \) is not divisible by \( p^k \). Using the splitting \( E_i \simeq \bigoplus_{\xi \in F_0} E_i \Gamma_{0, \xi} \) and the multiplicativity of the total Chern class with respect to Whitney sum, and expanding, it follows that there exists some cohomology class

\[ \alpha_0 = \prod_{l=1}^{v_0} c_{j_{0,l}}(E_{i_{0,l}}, \xi_{0,l}) \]

such that

\[ \langle [M^{\mu_0}], \alpha_0 \rangle \]

is not divisible by \( p^k \).
Let 
\[ \gamma_0 := \prod_l c_{j_{0,l}}^K (F_{i_{0,l}} (\Gamma_0, \xi_{0,l})) \in H_0^K (M^{\mu_0}). \]

Let \( \pi_0 : M^{\mu_0} \to \{ \ast \} \) the map to a point. Then 
\[ (\pi_0)_K^0 (\gamma_0) \in H^0 (BK_0) \simeq \mathbb{Z}, \]
and it follows from (10) and property (P4) of Lemma 3.3 applied to the inclusion of groups \( \{1\} \to K_0 \) that \( \pi_0^K (\gamma_0) \) is generic in the sense of Section 2. Now we apply repeatedly the following lemma, beginning at \( s = 0 \) and stopping as soon as we fall in case (I), in which case the proof of Lemma 3.2 is completed.

**Lemma 3.3.** Let \( s \geq 0 \). Suppose there is some \( \mu = (\mu_0, \ldots, \mu_s) \in \mathbb{Z}^{s+1} \) and a class 
\[ \gamma_s = \prod_l c_{j_{s,l}}^K (F_{i_{s,l}} (\Gamma_s, \xi_{s,l})) \prod_l c_{j'_{s,l}}^K (R_{i'_{s,l}} (\Gamma_s, \xi'_{s,l})) \in H_0^K (M^{\mu}), \]
where \( i'_{s,l} \leq s \) for every \( l \) (it should be understood that when \( s = 0 \) no bundle of the type \( R_j \) appears in the formula) such that, letting \( \pi_s : M^{\mu} \to \{ \ast \} \) denote the map to a point, the cohomology class \( (\pi_s)_s^K (\gamma_s) \in H^s (BK_s) \) is generic. Then there exists an integer \( \mu_{s+1} \) such that, letting \( \nu = (\mu_0, \ldots, \mu_s, \mu_{s+1}) \), at least one of the following two properties holds true:

(I) there exists some \( \beta \in \mathcal{C}_{\Gamma_{s+1}} (E_1, \ldots, E_u, N_1, \ldots, N_{s+1}) \) such that \( [M^{\nu}], \beta \) is not divisible by \( p^k \), or

(II) denoting by \( \pi_{s+1} : M^{\nu} \to \{ \ast \} \) the map to a point, there exists a class 
\[ \gamma_{s+1} = \prod_l c_{j_{s+1,l}}^{K_{s+1}} (F_{i_{s+1,l}} (\Gamma_{s+1}, \xi_{s+1,l})) \prod_l c_{j'_{s+1,l}}^{K_{s+1}} (R_{i'_{s+1,l}} (\Gamma_{s+1}, \xi'_{s+1,l})) \]

in \( H_{s+1}^s (M^{\mu}) \) such that \( (\pi_{s+1})_{s+1}^K (\gamma_{s+1}) \) is generic and \( i'_{s+1,l} \leq s + 1 \) for every \( l \).

**Proof.** Since \( (\pi_s)_s^s (\gamma_s) \in H^s (BK_s) \) is generic, Lemma 2.4 implies that the action of \( K_s \) on \( M^{\mu} \) is not free, so there is some subgroup \( K' \subseteq K_s \) such that \( K' \neq \{1\} \) and \( (M^{\mu})_{K'} \neq \emptyset \). Since any subgroup \( K' \subseteq K_s \) different from \( \{1\} \) contains the projection of \( \Gamma_{s+1} \) to \( K_s = \Gamma/\Gamma_s \), it follows that \( (M^{\mu})_{\Gamma_{s+1}} \neq \emptyset \). For any \( n \in \mathbb{Z} \geq 1 \) let 
\[ M^n_{\mu} = \bigcup_{X \subseteq M^{\mu} \text{ connected component}} X, \quad \dim_{X=n, X \cap (M^{\mu})_{\Gamma_{s+1}} \neq \emptyset} \]

and let 
\[ M^n_{\mu-1} = M^{\mu} \setminus \bigcup_{n \geq 0} M^n_{\mu}. \]

Note that for each \( k \geq -1 \) the subset \( M^k_\mu \subseteq M^{\mu} \) is \( K_s \)-invariant. Since the action of \( K_s \) on \( M^n_{\mu-1} \) is free, \( (\pi_s)_s^K (\gamma_s|_{M^n_{\mu-1}}) \) is not generic. Since 
\[ \gamma_s|_{M^{\mu}} = \sum_{k \geq -1} \gamma_s|_{M^k_\mu} \]
(where \( \gamma_s|_{M^k_\mu} \in H^s_0 (M^{\mu}) \) is meant to be zero on \( M^{\mu} \setminus M^k_\mu \)), it follows that there is some \( \mu_{s+1} \in \mathbb{Z} \geq 0 \) such that \( (\pi_s)_s^K (\gamma_s|_{M^n_{\mu+1}}) \) is generic. Let \( \nu = (\mu_0, \ldots, \mu_s, \mu_{s+1}) \), let
\( \iota_s : M^\nu \to M^\mu \) be the inclusion and let \( \pi_{s+1} : M^\nu \to \{\ast\} \) be the projection to a point. Since \( M^\nu = M^\mu_{\mu+1} \cap M^{\Gamma+1} \), the action of \( K_s \) on 
\[
(M_{\mu+1}^\mu)^* = M^\mu_{\mu+1} \setminus M^\nu
\]
is free. Let \( \theta : K_s \to \mathbb{C}^* \) be an injective morphism and let 
\[
\tau = c^{K_s}(E K_s \times_\theta \mathbb{C}) \in H^2_{K_s}(B K_s).
\]
We also denote by \( \tau \) the pullback of this cohomology class to \( H^*_{K_s}(M^\mu) \) via the morphism induced by the projection \( \pi_s \). Since the action of \( K_s \) on \((M_{\mu+1}^\mu)^*\) is free, \( H^*_{K_s}((M_{\mu+1}^\mu)^*\) vanishes for big enough \( v \). Hence there exists some \( r \) such that 
\[
\tau^r |_{(M_{\mu+1}^\mu)^*} = 0.
\]
The long exact sequence in equivariant cohomology for the pair \((M_{\mu+1}^\mu, (M_{\mu+1}^\mu)^*)\) implies that \( \tau^r \gamma_s \) belongs to the image of the map 
\[
H^*_{K_s}(M_{\mu+1}^\mu \cap M^{\Gamma+1}) \to H^*_{K_s}((M_{\mu+1}^\mu)^*).
\]
By (P3) in \([2, 1]\) there exists some \( \lambda_s \in H^2_{K_s}(M^\nu) \) such that \( \tau^r \gamma_s = (\iota_{s+1})^* \lambda_s \) so 
\[
\iota_s^* (\tau^r \gamma_s) = \lambda_s e^{K_s}(N_{s+1}|_{M^\nu}).
\]
According to Theorem 2.7 there exists some \( \mu \) \( f \in e^{K_s}(\{ N_{s+1}^{r+1} \})_{\chi \in \Gamma_{s+1}}[\tau] \subseteq H^*_{K_s}(M^\nu) \) and some nonnegative integer \( U \) such that 
\[
\lambda_s e^{K_s}(N_{s+1}|_{M^\nu}) f = \tau^U \lambda_s.
\]
Then 
\[
(\pi_{s+1})^{K_s} (\iota_s^* (\tau^r \gamma_s) f) = (\pi_{s+1})^{K_s} (\tau^U \lambda_s)
\]
\[
= \tau^U (\pi_{s+1})^{K_s} (\lambda_s) \quad \text{by the product formula, (P2) in \([2, 1]\)}
\]
\[
= \tau^U (\pi_s \circ \iota_s)^{K_s} (\lambda_s) \quad \text{since } \pi_{s+1} = \pi_s \circ \iota_s
\]
\[
= \tau^U (\pi_s)^{K_s} (\tau^r \gamma_s)
\]
\[
= \tau^{U+r} (\pi_s)^{K_s} (\gamma_s),
\]
so \((\pi_{s+1})^{K_s} (\iota_s^* (\tau^r \gamma_s))\) is generic. By \([1]\), 
\[
e^{K_s}(\{ N_{s+1}^{r+1} \})_{\chi \in \Gamma_{s+1}}[\tau] = e^{K_s}(\{ R_{s+1}(\Gamma_{s+1}, \chi) \})_{\chi \in \Gamma_{s+1}}[\tau],
\]
so we may write 
\[
f = \tau^{e_1} f_1 + \tau^{e_2} f_2 + \ldots,
\]
where each \( f_i \) is a monomial on the \( K_s \)-equivariant Chern classes of the bundles 
\[
\{ R_{s+1}(\Gamma_{s+1}, \chi) \}_{\chi \in \Gamma_{s+1}}.
\]
Since \((\pi_{s+1})^{K_s} (\iota_s^* (\tau^r \gamma_s) f)\) is generic, by the product formula and the basic properties of generic elements in \( H^*(B K_s) \), there exists some \( w \) such that \((\pi_{s+1})^{K_s} (\iota_s^* (\gamma_s) f_w)\) is generic. We may write 
\[
\iota_s^* (\gamma_s) f_w = \prod_l c^{K_s}_{j_s,l} (F_{j_s,l}(\Gamma_s, \xi_{s,l})) \prod_l c^{K_s}_{j'_s,l} (R_{j'_s,l}(\Gamma_s, \xi'_{s,l})) \prod_l c^{K_s}_{j''_s,l} (R_{s+1}(\Gamma_s, \xi''_{s,l})),
\]
where the factors in the first two products are implicitly restricted to \( M^\nu \).
By formulas (8) and (9) we have
\[ c^K_j(F_i(\Gamma_s, \xi))|_{M^\nu} \in \mathcal{E}^{K_j}(\{F_i(\Gamma_{s+1}, \chi)\}_{\chi \in \Gamma_{s+1}})[\tau] \]
for every \( j, i, \xi \) and similarly
\[ c^K_j(R_i(\Gamma_s, \xi))|_{M^\nu} \in \mathcal{E}^{K_j}(\{R_i(\Gamma_{s+1}, \chi)\}_{\chi \in \Gamma_{s+1}})[\tau] \]
for every \( j, i \leq s + 1, \xi \). Consider the following collection of vector bundles over \( M^\nu \):
\[ \mathcal{B} = \{F_i(\Gamma_{s+1}, \chi)\}_{i, \chi \in \Gamma_{s+1}} \cup \{R_i(\Gamma_{s+1}, \chi)\}_{i \leq s+1, \chi \in \Gamma_{s+1}}. \]
We have
\[ t^*\gamma_f w \in \mathcal{E}^{K_j}(\mathcal{B})[\tau], \]
so we may write
\[ t^*\gamma_f w = \tau^{d_1} g_1 + \tau^{d_2} g_2 + \ldots \]
where each \( g_i \) is a monomial on the \( K_s \)-equivariant Chern classes of the bundles in \( \mathcal{B} \). For at least one \( z \), the class \( (\pi_{s+1})^K \mathcal{B}(g_z) \) is generic. Write
\[ g_z = \prod_{s} c^K_{j_{s+1}, l_1} (F_{s+1,l_1}(\Gamma_{s+1}, \xi_{s+1,l_1})) \prod_{s} c^K_{j_{s+1}, l_2} (R_{s+1,l_2}(\Gamma_{s+1}, \xi_{s+1,l_2})) \in H^*_K(M^\nu). \]
Now we distinguish two cases.
Suppose first that \( \text{deg } g_z = \text{dim } M^\nu \). In that case we define
\[ \beta := \prod_{s} c^K_{j_{s+1}, l_1} (F_{s+1,l_1}(\Gamma_{s+1}, \xi_{s+1,l_1})) \prod_{s} c^K_{j_{s+1}, l_2} (R_{s+1,l_2}(\Gamma_{s+1}, \xi_{s+1,l_2})) \]
and by property (P4) of \( \mathcal{G} \) we may identify
\[ (\pi_{s+1})^K \mathcal{B}(g_z) \in H^0(\bar{BK}_s) \cong \mathbb{Z} \]
with the paring \( \langle [M^\nu], \beta \rangle \). Since the former is generic, it follows that \( \langle [M^\nu], \beta \rangle \) is not divisible by \( p^{k-s} = |K_s| \), so a fortiori it is not divisible by \( p^k \). Hence we fall in case (I) of the statement of the lemma.
Now suppose that \( \text{deg } g_z > \text{dim } M^\nu \) and define
\[ \gamma_{s+1} = \prod_{s} c^K_{j_{s+1}, l_1} (F_{s+1,l_1}(\Gamma_{s+1}, \xi_{s+1,l_1})) \prod_{s} c^K_{j_{s+1}, l_2} (R_{s+1,l_2}(\Gamma_{s+1}, \xi_{s+1,l_2})) \in H^*_K(M^\nu). \]
Let \( \rho : K_s \rightarrow K_{s+1} \) be the natural projection map, and let \( \rho^*_s : H^*_K(M^\nu) \rightarrow H^*_K(M^\nu), \rho^* : H^*(\bar{BK}_{s+1}) \rightarrow H^*(\bar{BK}_s) \) be the maps induced by \( \rho \) (for the first map, recall that the action of \( K_s \) on \( M^\nu \) factors through \( \rho \)). Then we have \( \rho^*_s \gamma_{s+1} = g_z \). By (P4) in \( \mathcal{G} \),
\[ \rho^*((\pi_{s+1})^K \mathcal{B}(g_z)) \in (\pi_{s+1})^K \mathcal{B}(g_z). \]
Since \( (\pi_{s+1})^K \mathcal{B}(g_z) \in H^*(\bar{BK}_s) \) is generic, in particular it is nonzero, which implies that \( (\pi_{s+1})^K \mathcal{B}(\gamma_{s+1}) \) is nonzero. On the other hand,
\[ (\pi_{s+1})^K \mathcal{B}(\gamma_{s+1}) \in H^>0(\bar{BK}_{s+1}) \]
because \( \text{deg } \gamma_{s+1} = \text{deg } g_z > M^\nu \). Since a class in \( H^>0(\bar{BK}_{s+1}) \) is generic if and only if it is nonzero, it follows that \( (\pi_{s+1})^K \mathcal{B}(\gamma_{s+1}) \) is generic. So we fall in case (II) of the lemma, and the proof is complete. \( \square \)
4. Finite $p$-groups

This section contains several results on finite $p$-groups that will be used in the proof of Theorem 1.11. Before going to the results, we introduce some notation.

For any finite group $G$ we define

$$\alpha(G) := \min\{[G : A] \mid A \subseteq G \text{ abelian subgroup}\}.$$ 

We denote by $e(G)$ the exponent of $G$ (the least common multiple of the orders of its elements). For a finite abelian group $A$ we denote by $\text{rk}(A)$ the minimal number of elements needed to generate $A$. The rank of a finite group $G$, denoted $\text{rk}(G)$, is by definition the maximum of $\text{rk}(A)$ as $A$ runs over the set of all abelian subgroups of $G$. If $p$ is a prime number and $G$ is a finite $p$-group then $\text{rk}(G)$ coincides with the biggest $r$ such that $G$ contains a subgroup isomorphic to $(\mathbb{Z}/p)^r$.

Throughout this section we use additive notation for finite abelian groups, and multiplicative notation for not necessarily abelian groups (such as automorphism groups of abelian groups).

4.1. Abelian subgroups of finite $p$-groups of bounded rank. The main result of this subsection is the following theorem.

**Theorem 4.1.** Let $a, r$ be natural numbers and let $p$ be a prime. Suppose that $G$ is a finite $p$-group satisfying $\text{rk}(G) \leq r$ and $\alpha(G) > a^{18r^5}$. Then there exist subgroups $B \unlhd \Gamma \subseteq G$ such that $B$ is abelian, $\Gamma/B$ is cyclic, and $\alpha(\Gamma) > a$.

In the proof of theorem we will need the following four lemmas, the first three of which are well known. The proof of Theorem 4.1 is given after Lemma 4.5.

In what follows, $p$ denotes an arbitrary prime. The first lemma is well known (see e.g. [37, §5.2.3]) and will be used several times.

**Lemma 4.2.** Let $B$ be a maximal abelian normal subgroup of a finite $p$-group $G$. Then the action by conjugation on $B$ identifies $G/B$ with a subgroup of $\text{Aut}(B)$.

The following lemma gives an upper bound on the size of a finite $p$-group in terms of the rank and the exponent. Such bounds have been much studied in the literature. We give a simple argument for completeness, but much better bounds can be obtained using more elaborated arguments (see for example [39]).

**Lemma 4.3.** If $G$ is a finite $p$-group satisfying $\text{rk}(G) = r$ and $e(G) = p^d$ then $|G| \leq p^{2dr^2}$.

**Proof.** Let $B \subseteq G$ be a maximal abelian normal subgroup. Since $\text{rk}(B) \leq r$ and $e(B) \leq p^d$, we have $|B| \leq p^{rd}$. A theorem of Hall (see e.g. [37, §5.3.3]) implies that the order of $\text{Aut}(B)$ divides $p^{r^2(d-1)}|\text{GL}_r(F_p)| = p^{r^2(d-1)}(p^r - 1)(p^r - p)\ldots(p^r - p^{r-1})$. Hence, any Sylow $p$-subgroup $S \subseteq \text{Aut}(B)$ satisfies

$$|S| \leq p^{r^2(d-1)}p^{1+2+\ldots+(r-1)} = p^{r^2(d-1)}p^{\frac{r(r-1)}{2}} = p^{\frac{r^2d-r+1}{2}}.$$ 

By Lemma 4.2 we can identify $G/B$ with a $p$-subgroup of $\text{Aut}(B)$. Consequently,

$$|G| = |B| \cdot |G/B| \leq p^{rd}p^{\frac{r^2d-r+1}{2}} \leq p^{2dr^2}.$$ 

$\square$
The following is Gorchakov–Hall–Merzlyakov–Roseblade’s lemma (see [38, Lemma 5]).

**Lemma 4.4.** Let $B$ be a finite abelian $p$-group and let $r = \text{rk}(B)$. We have

$$\text{rk}(\text{Aut}(B)) \leq r(5r - 1)/2.$$ 

The proof of the following lemma was kindly supplied to us by A. Jaikin.

**Lemma 4.5.** Suppose that $B \trianglelefteq \Gamma$ are finite $p$-groups, with $B$ abelian, and that the morphism $\Gamma/B \to \text{Aut}(B)$ given by conjugation is injective. Let $r = \text{rk}(B)$. Then

$$\alpha(\Gamma) \geq |\Gamma/B|^\frac{1}{r}.$$ 

**Proof.** Let $A$ be an arbitrary abelian subgroup of $\Gamma$. Since $B$ has rank $r$, we can choose characters $\lambda_1, \ldots, \lambda_r \in \hat{B} = \text{Hom}(B, \mathbb{C}^\times)$ such that $\bigcap_{i=1}^r \text{Ker} \lambda_i = \{1\}$. Let $C = B/A \cap B$. To each $\bar{a} \in A/A \cap B$ we associate the $r$-tuple $(\lambda_1(\bar{a}), \ldots, \lambda_r(\bar{a})) \in (\hat{C})^r$. Since the centraliser of $B$ in $A$ is $A \cap B$, different elements of $A/A \cap B$ correspond to different $r$-tuples. Thus

$$|A/A \cap B| \leq |(\hat{C})^r| = |C|^r.$$ 

Let $G$ be the subgroup of $\Gamma$ generated by $A$ and $B$. Since $B$ is normal in $G$, $G = AB$. Thus $[G : A] = [B : A \cap B]$ and $[G : B] = [A : A \cap B]$. Hence we obtain

$$[\Gamma : A] = [\Gamma : G][G : A] = [\Gamma : G][B : A \cap B] \geq [\Gamma : G][A/A \cap B]^{1/r} \geq ([\Gamma : G][G : B])^{1/r} = [\Gamma : B]^{1/r},$$

which is what we wanted to prove. \hfill $\square$

We are now ready to prove Theorem 4.1. Let $a, r$ be natural numbers, and let $p$ be a prime. Let $G$ be a finite $p$-group satisfying $\text{rk}(G) \leq r$ and $\alpha(G) > a^{18r^5}$. Let $B \trianglelefteq G$ be a maximal normal abelian subgroup. By Lemma 4.2, the action of $G$ on $B$ by conjugation induces an injective morphism $c : G/B \to \text{Aut}(B)$. We have

$$|G/B| \geq \alpha(G) > a^{18r^5}.$$ 

By Lemma 4.4, $\text{rk}(G/B) \leq 3r^2$. Lemma 4.3 implies that

$$e(G/B) \geq |G/B|^{\frac{1}{3r^2}} > (a^{18r^5})^{\frac{1}{18r^5}} = a^r.$$ 

Since $G/B$ is a $p$-group, the exponent of $G/B$ coincides with the maximum of the orders of its elements. So there exists some element $\gamma \in G/B$ whose order satisfies $\text{ord}(\gamma) > a^r$. Let $Z = \langle \gamma \rangle \subseteq G/B$ and let $\pi : G \to G/B$ be the quotient map. Define $\Gamma := \pi^{-1}(Z)$. Then the groups $B \trianglelefteq \Gamma$ satisfy the hypothesis of Lemma 4.5, so

$$\alpha(\Gamma) \geq |\Gamma/B|^\frac{1}{r} = |Z|^\frac{1}{r} = \text{ord}(\gamma)^\frac{1}{r} > a.$$ 

This completes the proof of the theorem.
4.2. Some technical results. We begin recalling a well known and elementary lemma.

**Lemma 4.6.** Let $B$ be a finite abelian $p$-group, let $\Phi \subseteq \text{Aut}(B)$ be a $p$-subgroup, and let $A \subset B$ be a $\Phi$-invariant proper subgroup. There exists some $h \in B$ whose class in $B/A$ is $\Phi$-invariant and has order $p$.

**Proof.** Consider the natural action of $\Phi$ on $B/A$. The element $[0] \in B/A$ is fixed by $\Phi$, so its orbit has one element. Since $|B/A|$ is divisible by $p$, there must be some nontrivial $[g] \in B/A$ whose orbit consists also of a unique element. Let $p^r$ be the order of $[g]$. Then $h := p^{r-1}g$ has the desired properties. $\square$

Our aim in this section is to generalize the previous lemma so as to obtain elements whose class in $B/A$ is $\Phi$-invariant and has order $p^k$, where $k$ is an arbitrary natural number. Actually, we will generalize a weaker form of the lemma, in which $\Phi$ is required to act trivially on $B/pB$. To state the result we introduce some notation.

Suppose that $B$ is a finite abelian $p$-group, and $k$ is a natural number. Define

$$\text{Aut}_k(B) = \{ \phi \in \text{Aut}(B) \mid \phi(g) \in g + p^kB \text{ for all } g \in B \}.$$ 

The binomial formula implies that for any $\phi \in \text{Aut}_k(B)$ we have $\phi^p \in \text{Aut}_{k+1}(B)$ (write $\phi = \text{Id} + \psi$, where $\psi(g) \in p^kB$ for every $g$; then $\phi^p = \text{Id} + \sum_{k \geq 1} \binom{p}{k} \psi^k$). Since $\text{Aut}_k(B) = \{1\}$ for big enough $k$, it follows that $\text{Aut}_k(B)$ is a $p$-group.

We are now ready for our generalization of Lemma 4.6.

**Theorem 4.7.** Let $B$ be a finite abelian $p$-group, let $r = \text{rk}(B)$, and let $k$ be a natural number. Let $A \subseteq B$ be an $\text{Aut}_{k}(B)$-invariant subgroup. There exists some $h \in B$ whose class in $B/A$ is $\text{Aut}(B)$-invariant and has order $p^k$, unless $|B/A| \leq p^{r(k-1)}$.

We remark that the condition that $hA \in B/A$ is $\text{Aut}_{k}(B)$-invariant implies that $\langle A, nh \rangle \subseteq B$ is $\text{Aut}_{k}(B)$-invariant for every $n \in \mathbb{Z}$.

**Proof.** Let $\{b_1, \ldots, b_r\}$ be a generating set of $B$ and define, for every $i$,

$$c_i = \min\{j \in \mathbb{Z}_{\geq 0} \mid p^jb_i \in A\}.$$ 

Let $c = \max\{c_1, \ldots, c_r\}$.

Suppose that $c \geq k$. We prove that in this case there exists some $h \in B$ with the desired properties. Suppose that $c = c_i$ and define

$$h = p^{c_i-k}b_i = p^{c-k}b_i.$$ 

Let us check that $[h] \in B/A$ is $\text{Aut}_{k}(B)$-invariant. Equivalently, we have to prove that $\phi(h) \in h + A$ for all $\phi \in \text{Aut}_{k}(B)$. Let $\phi \in \text{Aut}_{k}(B)$. We have

$$\phi(b_i) = b_i + p^k(m_1b_1 + \cdots + m_rb_r)$$ 

for some $m_1, \ldots, m_r \in \mathbb{Z}$, so

$$\phi(h) = \phi(p^{c-k}b_i) = p^{c-k}b_i + p^k(m_1b_1 + \cdots + m_rb_r) \in h + A$$ 

because $p^j b_j \in A$ for every $j$. The order of $[h] \in B/A$ is a power of $p$, so it follows from the definitions of $c$ and $h$ that $[h]$ has order $p^k$.

Hence if $h$ does not exist then $c \leq k - 1$. But in that case the map

$$(\mathbb{Z}/p^{k-1})^r \ni (l_1, \ldots, l_r) \mapsto \sum l_ib_i \in B/A$$ 

is a group homomorphism.
Lemma 4.8. Let $B$ be a finite abelian $p$-group and let $k$ be a natural number. Then $\text{Aut}(B)$ is a normal $p$-subgroup of $\text{Aut}(B)$, so it is contained in any Sylow $p$-subgroup of $\text{Aut}(B)$. Let $r = \text{rk}(B)$ and let $\text{Aut}(B)_p \subseteq \text{Aut}(B)$ be a Sylow $p$-subgroup. We have

$$[\text{Aut}(B)_p : \text{Aut}(\langle k \rangle)(B)] \leq p^{kr^2}.$$ 

Proof. The subgroup $p^k B \subseteq B$ is characteristic, so $\text{Aut}(\langle k \rangle)(B)$ is a normal subgroup of $\text{Aut}(B)$. Moreover, the quotient $\text{Aut}(B)/\text{Aut}(\langle k \rangle)(B)$ can be identified with a subgroup of $\text{Aut}(B/p^k B)$. By the arguments in the proof of Lemma 1.3 if $S \subseteq \text{Aut}(B/p^k B)$ is a $p$-subgroup then $|S| \leq p^{kr^2}$, so $[\text{Aut}(B)_p : \text{Aut}(\langle k \rangle)(B)] = |\text{Aut}(B)/\text{Aut}(\langle k \rangle)(B)| \leq p^{kr^2}$. 

5. Vector bundles with nonzero Euler number

The main purpose of this section is to prove Theorem 1.11.

5.1. Preliminaries.

Lemma 5.1. Let $X$ be a manifold, let $p$ be a prime, and let $G$ be a finite $p$-group acting continuously on $X$. We have

$$\sum_j b_j(X^G, \mathbb{F}_p) \leq \sum_j b_j(X; \mathbb{F}_p).$$

In particular, the number of connected components of $X^G$ is at most $\sum_j b_j(X; \mathbb{F}_p)$.

Proof. If $|G| = p$ then the statement follows from [3, Theorem III.4.3]. For general $G$ use ascending induction on $|G|$. In the induction step, choose a central subgroup $G_0 \subseteq G$ of order $p$ and apply the inductive hypothesis to the action of $G/G_0$ on $X^{G_0}$. 

The following classical theorem of Camille Jordan was already mentioned in the introduction (see [17], and [13, 25] for modern proofs).

Theorem 5.2 (Jordan). For any natural number $n$ there is some constant $\text{Jor}_n$ such that any finite subgroup $G \subseteq \text{GL}(n, \mathbb{R})$ has an abelian subgroup $A$ satisfying $[G : A] \leq \text{Jor}_n$.

Corollary 5.3. Let $X$ be an $m$-dimensional connected smooth manifold and let $G$ be a finite group acting smoothly and effectively on $X$. If $X^G \neq \emptyset$ then there exists an abelian subgroup $A \subseteq G$ such that $[G : A] \leq \text{Jor}_m$. If furthermore $m = 2n$ and there is a $G$-invariant almost complex structure on $X$ then $A$ can be generated by $n$ elements.

Proof. Take a $G$-invariant Riemannian metric $\nu$ on $X$. For any subgroup $H \subseteq G$ and any $x \in X^H$ the exponential map $\exp_x^\nu : T_x X \to X$ is $H$-equivariant and induces a diffeomorphism between a neighborhood of $0$ in $T_x X$ and a neighborhood of $x$ in $X$. Hence, $X^H$ is a smooth submanifold of $X$, and $X^H \neq X$ unless $H = \{1\}$, because $G$ acts effectively. Since $X$ is connected, we deduce that $X^H$ has empty interior unless $H = \{1\}$. So for any $x \in X^G$ the derivative of the action of $G$ at $x$ induces an injective morphism $G \to \text{GL}(T_x X)$; indeed, if the action of some nontrivial element $g \in G$ on $T_x X$ were trivial, then by the $G$-equivariance of $\exp_x^\nu : T_x X \to X$ there would exist some neighbourhood
Theorem 5.4. For any compact manifold \( U \) an abelian subgroup of \( \mathbb{Z}/p\mathbb{Z} \) for some \( p \) depending only on \( H^*(U) \) with the property that for any prime \( p \) and any elementary \( p \)-group \( (\mathbb{Z}/p)^s \) admitting an effective action on \( U \) we have \( s \leq r \).

5.2. Proof of Theorem 1.11. Let \((X, J)\) be an almost complex, compact, connected and orientable manifold, let \( E \to X \) be a complex vector bundle satisfying \( \text{rk}_\mathbb{C}(E) = \dim_\mathbb{R} X \) and assume that \( \langle [X], e(E) \rangle \neq 0 \).

Let \( E^* \subset E \) be the complementary of the zero section, and let \( S(E) \) be the sphere bundle of \( E \), defined as the quotient of \( E^* \) by the action of \( \mathbb{R}_{>0} \) given by multiplication. Any finite group acting effectively on \( E \) by vector bundle automorphisms acts also effectively on \( S(E) \). Let \( r \) be the number given by Mann and Su’s Theorem 5.4 applied to \( S(E) \). The cohomology of \( H^*(S(E)) \) can be computed in terms of \( H^*(X) \) and the Euler class \( e(E) \) by Gysin, and \( e(E) \) is uniquely determined by \( \langle [X], e(E) \rangle \), so \( r \) can be chosen to depend only on \( H^*(X) \) and \( \langle [X], e(E) \rangle \).

Fix some prime \( p \). Define the natural number \( k \) by the condition that \( p^k \) is the smallest power of \( p \) not dividing \( \langle [X], e(E) \rangle \). Consider a finite \( p \)-group \( G \) acting smoothly and effectively on \( E \) by vector bundle automorphisms, and lifting an action on \( X \) which preserves \( J \). It follows from the definition of \( r \) in the previous paragraph that
\[
\text{rk}(G) \leq r.
\]

We will prove the theorem in two steps.

**Step 1.** Suppose that \( G \) sits in an exact sequence
\[
1 \to B \to G \xrightarrow{\pi} H \to 1,
\]
where \( B \) is an abelian normal subgroup of \( G \) and \( H \) is cyclic. The action of \( G \) on \( B \) by conjugation induces a morphism \( c : H \to \text{Aut}(B) \). Define
\[
H_0 := c^{-1}(\text{Aut}(B))
\]
if \( k > 1 \) (see Section 4.2) and \( H_0 := H \) if \( k = 1 \). By Lemma 4.8,
\[
[H : H_0] \leq p^{kr^2} \quad \text{if } k > 1, \quad [H : H_0] = 1 \quad \text{if } k = 1.
\]

Let \( G_0 = \pi^{-1}(H_0) \). The tuple \((X, G_0, \{1\}, E)\) is \( p^k \)-admissible in the sense of Section 3.

**Lemma 5.5.** There exists an abelian subgroup \( A_0 \subseteq B \), such that:

1. \( A_0 \) is normal in \( G_0 \);
2. \( X^{A_0} \neq \emptyset \), and there exists some cohomology class \( \alpha \in \text{C}_A(E, N) \), where \( N \) is the normal bundle of the inclusion \( X^{A_0} \subseteq X \), and an open and closed submanifold \( M \subseteq X^{A_0} \) such that \( \langle [M], \alpha \rangle \) is not divisible by \( p^k \);
(3) \([B : A_0] \leq p^{r(k-1)}.\)

Proof. Let \(A_0\) be the set of subgroups of \(B\) satisfying (1) and (2). We have \(\{1\} \in A_0.\) We claim that any maximal element of \(A_0\) satisfies (3). To prove this we consider two cases. If \(k > 1\) then this follows from Theorems 4.7 and 3.1. If \(k = 1\) it follows from Lemma 4.6 (with \(\Phi\) given by the adjoint action of \(G_0\)) and Theorem 3.1. \(\square\)

Let \(A_0 \subseteq B\) be one of the subgroups given by Lemma 5.5. Since \(A_0\) is normal in \(G_0\), the action of \(G_0\) on \(X\) preserves \(X^{A_0}\). Let \(G_1 \subseteq G_0\) be the subgroup consisting of those elements that preserve each connected component of \(X^{A_0}\) (in particular, \(G_1\) preserves \(M\)). By Lemma 5.1 we have

\[|\pi_0(X^{A_0})| \leq b(X; \mathbb{F}_p) := \sum_j b_j(X; \mathbb{F}_p),\]

so

(13) \([G_0 : G_1] \leq b(X; \mathbb{F}_p)! .\)

Define the following sets of characters of \(A_0:\)

\[\Xi_E = \{\xi \in \hat{A}_0 \mid E^{A_0,\xi}|_M \neq 0\}, \quad \Xi_N = \{\xi \in \hat{A}_0 \mid N^{A_0,\xi}|_M \neq 0\} .\]

Let

\[m = \dim_{\mathbb{R}} X = \text{rk}_\mathbb{R} E .\]

Since the decomposition of the fibers of \(E|_M\) and \(N|_M\) in irreducible representations of \(A_0\) is locally constant (because \(A_0\) is finite), we may bound

\[|\Xi_E| \leq \frac{\text{rk}_\mathbb{R}(E)}{2} |\pi_0(M)| \leq \frac{m}{2} \cdot b(X; \mathbb{F}_p),\]

and \(|\Xi_N|\) can be bounded above by the same quantity. Since \(A_0\) is a normal subgroup of \(G_1\), conjugation in \(G_1\) induces an action of \(G_1\) on \(\hat{A}_0\) which preserves the set \(\Xi_E \cup \Xi_N\) because the action of \(A_0\) on \(E\) and \(N\) extends to an action of \(G_1\). Let \(G_2 \subseteq G_1\) be the subgroup fixing each element of \(\Xi_E \cup \Xi_N\). We have

(14) \([G_1 : G_2] \leq (m \cdot b(X; \mathbb{F}_p))! .\)

It follows from the definition that for any \(\xi \in \Xi_E\) the action of \(G_2\) on \(E|_M\) preserves the subbundle \(E^{A_0,\xi}|_M \subseteq E|_M\), and the same holds replacing \(E\) by \(N\).

Now let

\[\{V_1, \ldots, V_r\} = \{E^{A_0,\xi}|_M\}_{\xi \in \Xi_E} \cup \{N^{A_0,\xi}\}_{\xi \in \Xi_N} .\]

For each \(j\) the action of \(G_2\) on \(M\) lifts to an action on \(V_j\), and we may identify \(\alpha\) with an element of \(C(V_1, \ldots, V_r)\). Let \(\gamma\) be a generator of the cyclic subgroup \(H_2 := \pi(G_2) \subseteq H_0\), let \(g \in \pi^{-1}(\gamma)\) be any element, and let \(K = \langle g \rangle \subseteq G_2\), so that \(\pi(K) = H_2\). Then \((M, K, \{1\}, V_1, \ldots, V_r)\) is a \(p^k\)-admissible tuple.

Arguing similarly as in the proof of Lemma 5.5 using Theorems 3.1 and 4.7 if \(k > 1\), and Theorem 3.1 and Lemma 4.6 (with \(\Phi = \{1\}\)) if \(k = 1\), we deduce the existence of an abelian subgroup \(A_1 \subseteq K\) satisfying \([K : A_1] \leq p^{k-1}\) and \(M^{A_1} \neq \emptyset\).

Let \(\Gamma = \langle A_0, A_1 \rangle\). We have \([G : \Gamma] = [B : B \cap \Gamma][H : \pi(\Gamma)]\). On the other hand,

\[[B : B \cap \Gamma] \leq [B : A_0] \leq p^{r(k-1)},\]
because \( A_0 \subseteq \Gamma \) and (3) in Lemma 5.3. We have
\[
[H_2 : \pi(\Gamma) \cap H_2] \leq [K : A_1]
\]
because \( \pi : K \to H_2 \) is a surjection and \( \pi(A_1) \subseteq \pi(\Gamma) \cap H_2 \). Consequently if \( k > 1 \) we may bound, using in particular the first part of (12),
\[
[H : \pi(\Gamma)] \leq [H : \pi(\Gamma) \cap H_2]
= [H : H_2][H_2 : \pi(\Gamma) \cap H_2]
\leq [H : H_0][H_0 : H_2][K : A_1]
\leq p^{kr_2}[G_0 : G_2]p^{k-1}
\leq p^{kr_2}b(X; \mathbb{F}_p)!(m \cdot b(X; \mathbb{F}_p))!p^{k-1},
\]
and hence
\[
[G : \Gamma] \leq p^{r(k-1)}p^{kr_2}b(X; \mathbb{F}_p)!(m \cdot b(X; \mathbb{F}_p))!p^{k-1}.
\]
If \( k = 0 \) similar arguments using the second part of (12) yield
\[
[G : \Gamma] \leq b(X; \mathbb{F}_p)!(m \cdot b(X; \mathbb{F}_p))!.
\]
Define \( C_0' = 1 \) if \( \langle [X], e(E) \rangle = 1 \), and if \( \langle [X], e(E) \rangle \neq 1 \) define
\[
C_0' = \sup_p \{p^{r(k-1)}p^{kr_2}b(X; \mathbb{F}_p)!(m \cdot b(X; \mathbb{F}_p))!p^{k-1} \mid p^{k-1} \text{ divides } \langle [X], e(E) \rangle, k > 1 \},
\]
where the supremum runs over the set of prime numbers. This number is finite because the set of primes \( p \) dividing \( \langle [X], e(E) \rangle \) is finite. Define also
\[
C_0'' = \sup_p \{b(X; \mathbb{F}_p)!(m \cdot b(X; \mathbb{F}_p))! \mid p \text{ does not divide } \langle [X], e(E) \rangle \}.
\]
This is also clearly finite, because \( b(X; \mathbb{F}_p) \) can be bounded above independently of \( p \). Both \( C_0' \) and \( C_0'' \) are bounded above by constants depending only on \( H^*(X) \) and \( \langle [X], e(E) \rangle \) (recall that \( m = \dim X \), so it can be recovered from \( H^*(X) \)). Let
\[
C_0 = \max\{C_0', C_0''\}.
\]
We have proved that any prime \( p \) and any finite \( p \)-group \( G \) acting on \( E \) sitting in an exact sequence as in (11) there is a subgroup \( \Gamma \subset G \) such that \( X^\Gamma \neq \emptyset \) and \( [G : \Gamma] \leq C_0 \). By Corollary 5.3, \( \Gamma \) has an abelian subgroup \( A \subseteq \Gamma \) satisfying \( [\Gamma : A] \leq \text{Jor}_{2m} \). So, setting
\[
C_1 = \text{Jor}_{2m} C_0,
\]
we have proved that \( G \) has an abelian subgroup \( A \) of index \( [G : A] \leq C_1 \) satisfying \( X^A \neq \emptyset \).

**Step 2.** Now suppose \( G \) is arbitrary. We claim that there exists some abelian subgroup \( A_0 \subseteq G \) satisfying \( [G : A_0] \leq C_1^{18r^3} \). Otherwise, Theorem 4.1 would imply the existence of subgroups \( B \leq \Gamma \subseteq G \) with \( B \) abelian, \( \Gamma/B \) cyclic, and \( \alpha(\Gamma) > C_1 \). However, the group \( \Gamma \) has the form considered in Step 1, so by the previous results we must have \( \alpha(\Gamma) \leq C_1 \), a contradiction.

Now, if \( A_0 \subseteq G \) is an abelian subgroup, then by the result of Step 1 there is a subgroup \( A \subseteq A_0 \) satisfying \( X^A \neq \emptyset \) and \( [A_0 : A] \leq C_1 \), so \( [G : A] \leq C := C_1^{1+18r^3} \).
6. Lifting finite group actions to line bundles

In this section we prove Theorem 1.13. Actually we divide it in in three different statements: Theorems 6.1, 6.4 and 6.5.

6.1. The case of vanishing first Betti number.

Theorem 6.1. Let $X$ be a connected, smooth manifold satisfying $b_1(X) = 0$ and let $L → X$ be a complex line bundle. Suppose that $G ⊂ \text{Diff}(X)$ is a finite subgroup satisfying $g^*L ≃ L$ for every $g ∈ G$. Then there exists a finite group $G'$ sitting in a short exact sequence

$$1 → H → G' → G → 1,$$

where $H$ is finite cyclic and $|H| = |G|$, and a smooth action of $G'$ on $L$ by bundle automorphisms lifting the action of $G$ on $X$.

Proof. The following proof was kindly provided by the referee. Consider the complex vector bundle

$$V = \bigoplus_{g ∈ G} (g^{-1})^*L.$$

We can identify the fiber of $V$ over $x$ with $V_x = \{v : G → L | v(g) ∈ L_{g^{-1}x}\}$. Define an action of $G$ on $V$ lifting the action on $X$ as follows: if $v ∈ V_x$ and $γ ∈ G$ then $γv : G → L$ is the map given by $γv(g) := v(γ^{-1}g)$, so that $γv ∈ V_{γx}$. The action of $G$ on $V$ induces an action on $\det V = \bigotimes_{g ∈ G} (g^{-1})^*L$. Choosing for every $g ∈ G$ an isomorphism $(g^{-1})^*L ≃ L$ we obtain an isomorphism $\det V ≃ L^\otimes d$, where $d = |G|$. We have thus defined an action of $G$ on $L^\otimes d$ lifting the action on $X$. This action is encoded in the assignment to every $(g, x) ∈ G × X$ of an isomorphism $φ_{g, x} : L_x^\otimes d → L_{gx}^\otimes d$ varying continuously with $x$. Let $I_{g, x}$ denote the set of the $d$ different isomorphisms $ψ : L_x → L_{gx}$ satisfying $ψ^\otimes d = φ_{g, x}$. Then

$$I_g = \bigcup_{x ∈ X} I_{g, x}$$

is a subset of the (total space of the) line bundle $L^\vee \otimes g^*L$. Endowing $I_g$ with the subspace topology, the projection $I_g → X$ becomes a principal $μ_d$-bundle, where $μ_d ⊂ S^1$ is the group of $d$-th roots of unity. Its topology is therefore encoded in a monodromy morphism $π_1(X) → μ_d$. Since by assumption $b_1(X) = 0$ the monodromy is trivial, so $I_g → X$ can be trivialized. Now let

$$G' = \{(g, s) | g ∈ G, s : X → I_g \text{ a section of } I_g → X\}$$

and consider the action of $G'$ on $L$ defined as follows: if $g' = (g, s) ∈ G'$ and $x ∈ X$, then the action of $g'$ sends $L_x$ to $L_{gx}$ via the map $s(x)$. This defines an injective map $G' → \text{Diff}(L)$ with identifies $G'$ with a subgroup of $\text{Diff}(L)$ and hence endows $G'$ with a group structure. Clearly, the projection $G' ⊃ (g, s) → g ∈ G$ is a surjective morphism of groups with respect to which the action of $G'$ on $L$ lifts the action of $G$ on $X$, and the kernel of $G' → G$ can be identified with $μ_d$ because $X$ is connected. □
6.2. A criterion for existence of actions of central extensions on line bundles.

Recall (see e.g. [9 §III.1]) that for a group $G$ and a left $G$-module $M$ the bar complex of $G$ with coefficients in $M$ has $n$-th term $C^n(G, M) = \text{Map}(G^n, M)$ and the coboundary map $\delta : C^{n-1}(G, M) \to C^n(G, M)$ sends $f : G^{n-1} \to M$ to the map $\delta f : G^n \to M$ defined as

$$\delta f(g_1, \ldots, g_n) = g_1 f(g_2, \ldots, g_n) - f(g_1 g_2, \ldots, g_n) + \cdots + (-1)^n f(g_1, \ldots, g_{n-1}).$$

Then $H^*(G, M)$ is the cohomology of the complex $(C^*(G, M), \delta)$. If $N$ is another $G$-module then any morphism $\phi : M \to N$ of $G$-modules induces a morphism in cohomology $\phi_* : H^*(G, M) \to H^*(G, N)$.

Assume that $X$ is a connected smooth manifold, $G \subset \text{Diff}(X)$ is a finite subgroup, and $L \to X$ is a complex line bundle satisfying $L \simeq g^*L$ for every $g \in G$. Take a Hermitian metric $| \cdot |$ on $L$.

Choose for every $g \in G$ a bundle automorphism $\beta_g : L \to L$ lifting $g^{-1}$ and satisfying $|\beta_g(\lambda)| = |\lambda|$ for every $\lambda \in L$ (that $\beta_g$ exists is a consequence of the assumption $L \simeq (g^{-1})^*L$). We call $\beta = (\beta_g)_{g \in G}$ a set of lifts.

For every $g_1, g_2 \in G$ the composition $\beta_{g_1}^{-1} \beta_{g_2} \beta_{g_1}$ is a bundle automorphism of $L$ lifting the identity and preserving the Hermitian metric, and hence can be identified with pointwise multiplication by a function $f_\beta(g_1, g_2) \in \mathcal{C}^\infty(X, S^1)$. To simplify the notation we use this convention: if $c \in \mathcal{C}^\infty(X, S^1)$ is any map, we denote by the same symbol $c$ the map $L \to L$ given by pointwise multiplication by $c$. For example, we write $\beta_{g_1}^{-1} \beta_{g_2} \beta_{g_1} = f_\beta(g_1, g_2)$. Consider the $G$-action on $\mathcal{C}^\infty(X, S^1)$ defined as

$$(g \cdot \phi)(x) = \phi(g^{-1} \cdot x)$$

for any $g \in G$, $\phi \in \mathcal{C}^\infty(X, S^1)$ and $x \in X$. For any $c \in \mathcal{C}^\infty(X, S^1)$ we have

$$\beta_g^{-1} c \beta_g = g \cdot c.$$

Clearly, the set of lifts $\beta$ defines a lift of the action of $G$ to $L$ (for which the action of $g$ is given by the map $\beta_{g^{-1}}$) if and only if $f_\beta(g_1, g_2) \equiv 1$ for every $g_1, g_2$. The following identity shows that $f_\beta$ is cocycle in $C^2(G, \mathcal{C}^\infty(X, S^1))$ (we use multiplicative notation for cochains with values in $\mathcal{C}^\infty(X, S^1)$, instead of additive notation as in (15)):

$$\frac{f_\beta(g_1 g_2, g_3)}{f_\beta(g_1, g_2 g_3)} = f_\beta(g_1 g_2, g_3) \frac{f_\beta(g_1, g_2)}{f_\beta(g_1, g_2 g_3)} = f_\beta(g_1, g_2) \frac{f_\beta(g_1 g_2, g_3)}{f_\beta(g_1, g_2)} = f_\beta(g_1, g_2) \frac{f_\beta(g_1 g_2, g_3)}{f_\beta(g_1, g_2 g_3)} = f_\beta(g_1, g_2) \frac{f_\beta(g_1 g_2, g_3)}{f_\beta(g_1, g_2 g_3)}.$$

If $\beta' = (\beta_g' = \beta_g c_g)_{g \in G}$ is another set of lifts (with $c_g \in \mathcal{C}^\infty(X, S^1)$) then

$$f_{\beta'}(g_1, g_2) = \beta_{g_1}^{-1} \beta_{g_2} \beta_{g_1}' = c_{g_1}^{-1} \beta_{g_1}^{-1} \beta_{g_2} \beta_{g_1} \beta_{g_2} c_{g_2} \beta_{g_1} c_{g_1}$$

$$= \beta_{g_1}^{-1} \beta_{g_2} \beta_{g_1} (g_1 \cdot c_{g_2}) c_{g_1},$$

by (16),

$$= \beta_{g_1}^{-1} \beta_{g_2} \beta_{g_1} c_{g_1}^{-1} (g_1 \cdot c_{g_2}) c_{g_1},$$

because $\beta_{g_1}^{-1} \beta_{g_2} \beta_{g_1}$ lifts the identity on $X$,

$$= f_\beta(g_1, g_2) c_{g_1}^{-1} (g_1 \cdot c_{g_2}) c_{g_1}.$$

So if $c \in C^1(G, \mathcal{C}^\infty(X, S^1))$ is the cochain defined as $c(g) = c_g$, then $f_{\beta'} = f_\beta \delta c$ (multiplicative notation!), so $f_\beta$ and $f_{\beta'}$ are cohomologous.

It follows that the cohomology class $[f_\beta] \in H^2(G, \mathcal{C}^\infty(X, S^1))$ is independent of the choice of $\beta$, and that the action of $G$ on $X$ lifts to an action on $L$ if and only if $[f_\beta] = 0$. 

\textit{Finite Subgroups of Ham and Symp}
Now define
\[ b(L) \in H^2(G, H^1(X; \mathbb{Z})) \]
to be the image of \([f_\beta]\) under the map in cohomology induced by passing to homotopy classes \(C^\infty(X, S^1) \to [X, S^1] = H^1(X; \mathbb{Z})\), where \(H^1(X; \mathbb{Z})\) is endowed with the action of \(G\) inherited from that on \(C^\infty(X, S^1)\).

**Theorem 6.2.** Then there exists a finite group \(G'\) sitting in a short exact sequence
\[ 1 \to H \to G' \to G \to 1, \]
where \(H\) is finite cyclic, and a smooth action of \(G'\) on \(L\) by bundle automorphisms lifting the action of \(G\) on \(X\), if and only if \(b(L) = 0\). Furthermore, if \(G'\) exists, then it can be chosen so that \(|H| = |G|\).

Note that Theorem 6.2 includes Theorem 6.1 as a particular case.

Although we will not use this fact here, it is worth pointing out that \(b(L)\) may be interpreted as the image of \(c_1(L)\) via the differential
\[ d^0_2 : H^2(X; \mathbb{Z})^G \to H^2(G, H^1(X; \mathbb{Z})) = H^2(BG, H^1(X; \mathbb{Z})) \]
in the Serre spectral sequence for the fibration \(X_G \to BG\). Hence the vanishing of \(b(L)\) is the first obstruction to lifting \(c_1(L)\) to an equivariant cohomology class. The second and last obstruction is the vanishing of the map \(d^0_3 : \operatorname{Ker} b \to H^3(G, \mathbb{Z})\), which also admits a geometric interpretation: if \(b(L) = 0\), then \(d^0_3(c_1(L)) \in H^3(G, \mathbb{Z}) \simeq H^2(G, S^1)\) defines a central extension \(1 \to S^1 \to G' \to G \to 1\), and the group \(G'\) in Theorem 6.2 can be identified with a subgroup of \(G\).

**Proof.** Suppose that there is an extension \(1 \to H \to G' \to G \to 1\) with the properties in the statement of the theorem, and that \(G'\) acts on \(L\) lifting the action of \(G\) on \(X\). Then the action of \(H\) on \(L\) is necessarily given by constant maps \(X \to S^1\). Choose for every \(g \in G\) a lift \(g' \in G'\) and let \(\beta_g : L \to L\) be the map given by the action of \((g')^{-1}\). Then \(\beta^{-1}_g \beta_{g_1} \beta_{g_2} \beta_{g_1} \beta_{g_2} \beta_{g_1}\) is given by the action of some element of \(H\), and hence is a constant map. It follows that \(b(L) = 0\).

Now assume, conversely, that \(b(L) = 0\). It follows from the discussion before the theorem that we may choose a set of lifts \(\beta = (\beta_g)\) such that \(\beta^{-1}_g \beta_{g_1} \beta_{g_2} \beta_{g_1} \beta_{g_2} \beta_{g_1}\) is homotopic to a constant map for every \(g_1, g_2\). This is equivalent to the existence of a function \(\phi_\beta(g_1, g_2) \in C^\infty(X, \mathbb{R})\) such that
\[ f_\beta(g_1, g_2) = \exp(2\pi i \phi_\beta(g_1, g_2)). \]

Since \(X\) is connected, the cocycle condition \(\delta f_\beta = 0\) implies that
\[ g_1 \cdot \phi_\beta(g_2, g_3) - \phi_\beta(g_1 g_2, g_3) + \phi_\beta(g_1, g_2 g_3) - \phi_\beta(g_1, g_2) \]
is a constant integer.

Since this integer need not be zero, in general the cochain \(\phi_\beta\) defined by the functions \(\phi_\beta(g_1, g_2)\) is not a cocycle in \(C^2(G, C^\infty(X, \mathbb{R}))\).

Let \(x_0 \in X\) be any point and define
\[ C_0^\infty(X, \mathbb{R}) = \{ f \in C^\infty(X, \mathbb{R}) | \sum_{g \in G} f(g \cdot x_0) = 0 \}. \]
Consider the linear projection $\Pi : \mathcal{C}_0^\infty(X, \mathbb{R}) \to \mathcal{C}_0^\infty(X, \mathbb{R})$ defined as

$$\Pi(f) = f - \frac{1}{|G|} \sum_{g \in G} f(g \cdot x_0).$$

The subspace $\mathcal{C}_0^\infty(X, \mathbb{R}) \subset \mathcal{C}^\infty(X, \mathbb{R})$ is $G$-invariant, the map $\Pi$ is $G$-equivariant, and the kernel of $\Pi$ consists of the constant functions. So if we define $\phi_0^0(g_1, g_2) := \Pi(\phi_0(g_1, g_2))$ then (17) implies that

$$g_1 \cdot \phi_0^0(g_2, g_3) - \phi_0^0(g_1g_2, g_3) + \phi_0^0(g_1, g_2g_3) - \phi_0^0(g_1, g_2) = 0,$$

i.e., $\phi_0^0$ is a cocycle in $C^2(G, \mathcal{C}_0^\infty(X, \mathbb{R}))$. Since $\mathcal{C}_0^\infty(X, \mathbb{R})$ is an $\mathbb{R}$-vector space and $G$ is finite, $H^*(G, \mathcal{C}_0^\infty(X, \mathbb{R})) = 0$ (see e.g. [9, Ch. III, Cor. 10.2]). It follows that there exists some $k \in C^1(G, \mathcal{C}_0^\infty(X, \mathbb{R}))$ such that $\phi_0^0 = \delta k$. So if we replace each $\beta_g$ by $\beta_g \exp(-2\pi i \kappa(g))$ then the new set of lifts, which we still denote by $\beta_1, \beta_2, \ldots$, has the property that $f_\beta(g_1, g_2)$ is a constant function $X \to S^1$ for every $g_1, g_2 \in G$. In other words, we may now regard $f_\beta$ as defining a cocycle in $C^2(G, S^1)$.

**Lemma 6.3.** Let $d = |G|$ and let $\mu_d \subset S^1$ be the subgroup of $d$-th roots of unity. We regard $\mu_d$ as a $G$-module with trivial $G$-action. The inclusion $\mu_d \subset S^1$ induces for every $k > 0$ a surjective map $H^k(G, \mu_d) \twoheadrightarrow H^k(G, S^1)$.

**Proof.** Let $e : \mathbb{R} \to S^1$ be the map $e(t) = \exp(2\pi it)$. Consider the following commutative diagram of trivial $G$-modules with exact rows

$$\begin{array}{ccc}
0 & \longrightarrow & \mathbb{Z}^G \\
| & | & | \\
0 & \longrightarrow & \mathbb{R} \\
| & | & | \\
e & \longrightarrow & S^1 \\
| & | & | \\
\mu_d & \longrightarrow & 1 \\
& & \\
0 & \longrightarrow & \mathbb{Z}^G \\
| & | & | \\
0 & \longrightarrow & \mathbb{R} \\
| & | & | \\
e & \longrightarrow & S^1 \\
| & | & | \\
\mu_d & \longrightarrow & 1.
\end{array}$$

Let $k > 0$ be an integer. Taking cohomology we obtain the following commutative diagram with exact rows (see e.g. [9, Ch. III, Prop. 6.1]):

$$\begin{array}{cccccc}
H^k(G, \mathbb{Z}) & \longrightarrow & H^k(G, \mu_d) & \longrightarrow & H^{k+1}(G, \mathbb{Z}) & \longrightarrow & H^{k+1}(G, \frac{1}{d} \mathbb{Z}) \\
| & | & | & | & | & | \\
H^k(G, \mathbb{R}) & \longrightarrow & H^k(G, S^1) & \longrightarrow & H^{k+1}(G, \mathbb{Z}) & \longrightarrow & H^{k+1}(G, \mathbb{R}).
\end{array}$$

As before, [9, Ch. III, Cor. 10.2] implies that $H^k(G, \mathbb{R}) = H^{k+1}(G, \mathbb{R}) = 0$, so $\partial_S$ is an isomorphism. Hence we need to prove that $\partial_\mu$ is surjective, and this will follow if we prove that $\iota_d = 0$. The vanishing of $\iota_d$ follows again from [9, Ch. III, Cor. 10.2]. Indeed, if $a \in C^{k+1}(G, \mathbb{Z})$ is a cocycle, then $\frac{a}{d} \in C^{k+1}(G, \frac{1}{d} \mathbb{Z})$ is also a cocycle, and we have $\iota_d[a] = d[a/d]$. Since $d = |G|$, [9, Ch. III, Cor. 10.2] implies that $d[a/d] = 0$. \hfill \Box

We have so far constructed a set of lifts $\beta$ with the property that for every $g_1, g_2$ the function $f_\beta(g_1, g_2) : X \to S^1$ is constant. By Lemma 6.3 there exists some $\lambda \in C^1(G, S^1)$ such that $(f_\beta \delta \lambda)(g_1, g_2) : X \to S^1$ is constant and its value is a $d$-th root of unity. We now replace each $\beta_g$ by $\beta_g \lambda(g)$, and denote again by $\beta_g$ the resulting map. Then we have $f_\beta(g_1, g_2) \in \mu_d$ for every $g_1, g_2$.

We claim that for every $g_1, \ldots, g_k \in G$ there is some $\theta \in \mu_d$ such that

$$\beta_{g_1 \cdot \ldots \cdot g_k}^{-1} \beta_{g_1} \beta_{g_2} \cdots \beta_{g_k} = \theta.$$
We prove the claim using induction on \( k \). The case \( k = 1 \) is trivial, and the case \( k = 2 \) is the statement that \( f_\beta(g_1, g_2) \in \mu_d \) for every \( g_1, g_2 \). If \( k > 2 \) and the claim is true for lower values of \( k \), then we may write
\[
\beta_{g_1} \ldots \beta_{g_{k-1}} = \beta_{g_1 \cdots g_{k-1}} \theta'
\]
for some \( \theta' \in \mu_d \). Then
\[
\beta_{g_1}^{-1} \beta_{g_{k-1}} \beta_{g_1} \ldots \beta_{g_k} = \beta_{g_1}^{-1} \beta_{g_1 \cdots g_{k-1}} \theta' \beta_{g_k} = \beta_{g_1}^{-1} \beta_{g_2}^{-1} \ldots \beta_{g_{k-1}}^{-1} \beta_{g_k} \theta' = f_\beta(g_1 \cdots g_{k-1}, g_k) \theta' \in \mu_d,
\]
which proves the induction step and hence the claim.

At this point we define \( G' \subset \text{Diff}(L) \) to be the subgroup generated by \( \{ \beta_g \mid g \in G \} \). By construction \( G' \) acts on \( L \) by bundle automorphisms. Consider the surjective morphism
\[
\pi : G' \to G
\]
satisfying \( \pi(\beta_g) = g \). To conclude the proof of the theorem we prove that any element in \( \text{Ker} \pi \) can be identified with multiplication by an element of \( \mu_d \). Now, if \( g_1, \ldots, g_k \in G \) satisfy \( g_1 \ldots g_k = \text{Id} \) then by the previous claim there exists some \( \theta \in \mu_d \) such that
\[
\beta_{g_1} \ldots \beta_{g_k} = \beta_{g_1 \cdots g_k} \theta = \beta_{\text{Id}} \theta.
\]
Since \( f_\beta(\text{Id}, \text{Id}) = \beta_{\text{Id}}^{-1} \beta_{\text{Id}} \beta_{\text{Id}} = \beta_{\text{Id}} \in \mu_d \), it follows that \( \beta_{g_1} \ldots \beta_{g_k} \in \mu_d \).

**6.3. Lifting Hamiltonian finite group actions.**

**Theorem 6.4.** Let \((X, \omega)\) be a compact and connected symplectic manifold and let \( L \to X \) be a complex line bundle. For any finite subgroup \( G \subset \text{Ham}(X, \omega) \) there is a finite group \( G' \) sitting in a short exact sequence
\[
1 \to H \to G' \to G \to 1,
\]
where \( H \) is finite and cyclic and \(|H|\) divides \(|G|\), and a smooth action of \( G' \) on \( L \) by bundle automorphisms lifting the action of \( G \) on \( X \).

**Proof.** Take a Hermitian structure on \( L \) and let \( \nabla \) be a unitary connection on \( L \).

Choose for each \( g \in G \) a time dependent Hamiltonian \( H_g : X \times [0, 1] \to \mathbb{R} \) whose flow at time 1 is equal to \( g \). Reparametrizing \( H_g \) in the time coordinate (and rescaling accordingly) if necessary, we may assume that the support of each \( H_g \) is contained in \( X \times (a, b) \) for some open subinterval \((a, b) \subset [0, 1]\).

Denote by \( \beta_g : L \to L \) the bundle automorphism obtained using parallel transport with respect to \( \nabla \) along the paths given by the Hamiltonian flow of \( H_{g^{-1}} \). More precisely, let \( \{ \phi_{g,t} \}_{t \in [0, 1]} \) denote the path in \( \text{Ham}(X, \omega) \) defined by the conditions \( \phi_{g,0} = \text{Id}_X \) and, for each \( t \in [0, 1] \) and \( x \in X \),
\[
\frac{\partial}{\partial t} \phi_{g,t}(x) \bigg|_{t=\tau} = X_{g,\tau}(\phi_{g,\tau}(x)),
\]
where \( X_{g,\tau} \) is the Hamiltonian vector field associated to \( H_g(\cdot, \cdot) \), which means that
\[
d(H_g(\cdot, \cdot)) = i X_{g,\cdot},
\]
where \( i \) denotes the standard contraction map. By our choice of \( H_g \) we have \( g = \phi_{g,1} \).
Let $\Pi : L \rightarrow X$ be the projection map. For each $g \in G$ and each $\lambda \in L$ there is a unique smooth map $\psi^\lambda_g : [0, 1] \rightarrow L$ satisfying $\psi^\lambda_g(0) = \lambda$, $\Pi(\psi^\lambda_g(t)) = \phi_{g,t}(x)$, with $x = \Pi(\lambda)$, and $(\psi^\lambda_g)'(t)$ belongs to the horizontal distribution of $\nabla$ for each $t$. Define

$$\beta_g(\lambda) := \psi^\lambda_g(1).$$

Then $\beta_g$ lifts $g^{-1}$ and preserves the Hermitian structure on $L$. Consider, as in the previous subsection, the cocycle $f_\beta \in C^2(G, C^\infty(X, S^1))$ defined by

$$f_\beta(g_1, g_2) = \beta_{g_2}^{-1} \beta_{g_1},$$

for every $g_1, g_2 \in G$. We claim that for every $g_1, g_2 \in G$ the map $f_\beta(g_1, g_2) : X \rightarrow S^1$ is null-homotopic (i.e. homotopic to a constant map). By Theorem 6.2 the claim implies the present theorem. On the other hand, the claim is equivalent to the condition that for any smooth map $\gamma : S^1 \rightarrow X$ the composition $f_\beta(g_1, g_2) \circ \gamma : S^1 \rightarrow S^1$ is null-homotopic, or equivalently that the degree of $f_\beta(g_1, g_2) \circ \gamma$ is 0.

Fix elements $g_1, g_2 \in G$ and a smooth map $\gamma : S^1 \rightarrow X$. To compute $\deg(f_\beta(g_1, g_2) \circ \gamma)$ we consider the concatenation of the flows defining $\alpha_{g_1^{-1}}$, $\alpha_{g_2^{-1}}$ and $\alpha_{g_2^{-1}g_1^{-1}}$. Namely let $Z : X \times [0, 1] \rightarrow \mathbb{R}$ be defined as follows:

$$Z(x, t) = \begin{cases} 3H_{g_1^{-1}}(x, 3t) & \text{if } t \in [0, \frac{1}{3}], \\ 3H_{g_2^{-1}}(x, 3t - 1) & \text{if } t \in [\frac{1}{3}, \frac{2}{3}], \\ 3H_{g_2^{-1}g_1^{-1}}(x, 3t - 2) & \text{if } t \in [\frac{2}{3}, 1]. \end{cases}$$

Note that $Z$ is smooth because of our assumption on the support of each $H_g$. Let $\{\Phi_t\}_{t \in [0, 1]}$ be the path in $\text{Ham}(X, \omega)$ defined by $Z$, which we view as a time depending Hamiltonian. Then $\Phi_1(x) = x$ for every $x \in X$, so we may define a map

$$\Psi : S^1 \times S^1 \rightarrow X, \quad \Psi(e^{2\pi i u}, e^{2\pi i v}) = \Phi_u(\gamma(e^{2\pi i v})).$$

We can identify $f_\beta(g_1, g_2)(\gamma(\theta))$ with the holonomy of $(\Psi^*L, \Psi^*\nabla)$ along $S^1 \times \{\theta\}$ with the orientation given by the path $[0, 1] \ni u \mapsto (e^{2\pi i u}, \theta) \in S^1 \times S^1$. Define $\eta : [0, 1] \rightarrow S^1$ by $\eta(v) = f_\beta(g_1, g_2)(\gamma(e^{2\pi i v}))$. Let $[S^1 \times S^1] \in H_2(S^1 \times S^1)$ denote the fundamental class, defined taking the counterclockwise orientation on each factor. We have:

$$\deg f_\beta(g_1, g_2) \circ \gamma = \frac{1}{2\pi i} \int_0^1 \frac{\eta'(v)}{\eta(v)} dv = \frac{1}{2\pi} \int_{S^1 \times S^1} \Psi^*F_\nabla \quad \text{by Stokes theorem}$$

$$= (c_1(\Psi^*L), [S^1 \times S^1]) \quad \text{by Chern–Weil theory}$$

$$= (c_1(L), \Psi_*[S^1 \times S^1]).$$

Now, Theorem 1.A in [19] implies that $\Psi_*[S^1 \times S^1] \in H_2(X; \mathbb{Q})$ is zero; indeed, $\Phi$ defines a loop in $\text{Ham}(X, \omega)$, and $\Psi_*[S^1 \times S^1]$ is equal to $\partial_\Phi(\gamma_*[S^1])$ in the notation of [19] (the proof of Theorem 1.A in [19] requires $X$ to be spherically monotone; the proof for a general compact symplectic manifold is given in [21]). It follows that $\deg(f_\beta(g_1, g_2) \circ \gamma) = 0$, so the proof of the claim is complete. \qed
6.4. Lifting actions of commutators of finite groups.

**Theorem 6.5.** Let $X$ be a compact and connected smooth manifold and let $L \to X$ be a complex line bundle. Suppose that $\Gamma \subset \text{Diff}(X)$ is a finite subgroup satisfying $\gamma^* L \simeq L$ for every $\gamma \in \Gamma$, and suppose that the action of $\Gamma$ on $H^1(X)$ is trivial. There is a finite group $G'$ sitting in a short exact sequence

$$1 \to H \to G' \to [\Gamma, \Gamma] \to 1,$$

where $H$ is finite and cyclic and $|H|$ divides $|[\Gamma, \Gamma]|$, and a smooth action of $G'$ on $L$ by bundle automorphisms lifting the action of $G$ on $X$.

**Proof.** Let $G = [\Gamma, \Gamma]$. By Theorem 6.2 it suffices to construct a set of lifts $(\beta_g)_{g \in G}$ such that $\beta^{-1} g_2 \beta g_1$ is homotopic to a constant map for every $g_1, g_2 \in G$.

Let $\eta_0 \in \Omega^2(X)$ be de Rham representative of $c_1(L)$. For every $\gamma \in \Gamma$ we have $\gamma^* L \simeq L$, and this implies that $\gamma^* \eta_0$ is cohomologous to $\eta_0$. Consequently

$$\eta = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma^* \eta_0$$

is $\Gamma$-invariant and also represents $c_1(L)$. Take a Hermitian metric on $L$, and let $A_\eta$ denote the space of all unitary connections on $L$ whose curvature is equal to $-i2\pi \eta$. By de Rham’s theorem $A_\eta$ is nonempty. Addition of closed one forms gives $A_\eta$ the structure of torsor over $\text{Ker}(d : \Omega^1(X, i\mathbb{R}) \to \Omega^2(X, i\mathbb{R}))$.

Let $M_\eta = A_\eta / \text{Map}(X, S^1)$, where elements of $\text{Map}(X, S^1)$ act on $A_\eta$ as gauge transformations of $L$. The torsor structure on $A_\eta$ induces on $M_\eta$ a struture of torsor over

$$T := \frac{\text{Ker}(d : \Omega^1(X, i\mathbb{R}) \to \Omega^2(X, i\mathbb{R}))}{d \log(\text{Map}(X, S^1))} \simeq H^1(X; \mathbb{Z}) \simeq (S^1)^{b_1(X)},$$

where $d \log$ sends $\sigma : X \to S^1$ to $\sigma^{-1} d\sigma \in \text{Ker}(d : \Omega^1(X, i\mathbb{R}) \to \Omega^2(X, i\mathbb{R}))$.

The group $\Gamma$ acts on $M_\eta$ by pullback: if $[A] \in M_\eta$ is the gauge equivalence class of a connection $A$ and $\gamma \in \Gamma$, then $\gamma^* A$ is a connection on $\gamma^* L$, so if $\iota : L \to \gamma^* L$ is an isomorphism of Hermitian line bundles (which exists, since $L$ and $\gamma^* L$ are isomorphic line bundles) then $\iota^* \gamma^* A$ is a connection on $L$, whose gauge equivalence class is independent of $\iota$. Hence we may define

$$\gamma^*[A] := [\iota^* \gamma^* A].$$

This action is compatible with the torsor structure, in the sense that if $\tau \in T$ then $\gamma^*(\tau \cdot [A]) = \tau \cdot \gamma^*[A]$ for every $\gamma$ and $[A]$. Indeed, one may choose as a representative of $\tau$ a closed 1-form $\alpha$, so that $\tau \cdot [A] = [A + \alpha]$, and compute

$$\gamma^*(\tau \cdot [A]) = \gamma^*[A + \alpha] = [\gamma^* A + \gamma^* \alpha] = [\gamma^* A + (\gamma^* \alpha - \alpha) + \alpha]$$

$$= \tau \cdot [\gamma^* A + (\gamma^* \alpha - \alpha)] = \tau \cdot [\gamma^* A] = \tau \cdot \gamma^*[A].$$

Here we have used the fact that $\gamma^* \alpha - \alpha$ is exact, which follows from the assumption that $\Gamma$ acts trivially on $H^1(X)$.

It follows that for every $\gamma \in \Gamma$ there exists some $\tau_\gamma \in T$ such that $\gamma^*[A] = \tau_\gamma \cdot [A]$ for every $[A] \in M_\eta$. The map $\Gamma \to T$ sending $\gamma$ to $\tau_\gamma$ is necessarily a morphism of groups, and since $T$ is abelian its restriction to $G$ is trivial. Hence $G$ acts trivially on $M_\eta$. 

Fix a connection $A \in \mathcal{A}_q$. For every $g \in G$ let $\beta_g : L \to L$ be an isomorphism lifting $g^{-1}$ and satisfying $\beta_g^*A = A$. Such $\beta_g$ exists because $G$ acts trivially on $\mathcal{M}_q$. Then, for every $g_1, g_2 \in G$, $\beta_{g_1}^{-1}\beta_{g_2}\beta_{g_1}$ is a gauge transformation of $L$ which fixes the connection $A$. This implies that $\beta_{g_1}^{-1}\beta_{g_2}\beta_{g_1}$ is constant as a map from $X$ to $S^1$, and so $(\beta_g)_{g \in C}$ is the required set of lifts. \hfill \Box

7. Proofs of the main theorems

7.1. Proof of Theorem 1.9. Take a class $w' \in H^*(X; \mathbb{Q})$ sufficiently close to $w$ so that $(w')^n \neq 0$. Multiplying $w'$ by a suitable integer we obtain an integral class $v \in H^2(X)$ satisfying $v^n \neq 0$. Let $L \to X$ be a complex line bundle such that $c_1(L) = v$. Let $E = L^\oplus n$. Then $e(E) = c_1(L)^n = v^n$, so $\langle [X], e(E) \rangle \neq 0$.

Let $p$ be a prime and let $G$ be a finite $p$-group acting on $X$ preserving an almost complex structure. Since $b_1(X) = 0$, by Theorem 6.1 there exists a short exact sequence

$$1 \to H \to G_L \xrightarrow{\pi} G \to 1$$

such that $G_L$ is a finite $p$-group and the action of $G$ on $X$ lifts to an action of $G_L$ on $L$. Let $G'$ be the fiber product of $n$ copies of $G_L$ over $G$, i.e.,

$$G' = \{(g_1, \ldots, g_n) \in G_L^n \mid \pi(g_1) = \cdots = \pi(g_n)\},$$

and let $\pi' : G' \to G$ be the map $\pi'(g_1, \ldots, g_n) = \pi(g_1)$. The action of $G_L$ on $L$ induces an action of $G'$ on $E$ lifting the action of $G$ on $X$. Since the action of $G$ on $X$ preserves $J$, the action of $G'$ on $E$ is given by a morphism $G' \to \text{Aut}(E, X, J)$.

Applying Theorem 1.11 to the image of the morphism $G' \to \text{Aut}(E, X, J)$ we deduce the existence of an abelian subgroup $A' \subseteq G'$ such that $X^{A'} \neq \emptyset$ and $[G' : A'] \leq C$. It follows that $A := \pi'(A')$ satisfies $X^A \neq \emptyset$ and $[G : A] \leq C$.

7.2. Proof of Theorem 1.7. The hypothesis that $G$ is a finite subgroup of $\text{Ham}(X, \omega)$ implies that the action of $G$ on $X$ preserves an almost complex structure on $X$ (see Remark 1.5). Then the proof is almost the same as that of Theorem 1.9 which we just gave, replacing Theorem 6.1 by Theorem 6.3.

7.3. A criterion to test Jordan property. Suppose that $\mathcal{C}$ is a set of finite groups and let $C, d$ be natural numbers. We say that $\mathcal{C}$ satisfies the property $\mathcal{J}(C, d)$ if each $G \in \mathcal{C}$ has an abelian subgroup $A$ such that $[G : A] \leq C$ and $A$ can be generated by $d$ elements. Let $\mathcal{T}(\mathcal{C})$ the set of all $T \in \mathcal{C}$ such that there exist distinct primes $p$ and $q$, a Sylow $p$-subgroup $P$ of $T$ (which might be trivial), and a normal Sylow $q$-subgroup $Q$ of $T$, such that $T = PQ$. The following is the main result in [29]. Note that the proof uses the classification of finite simple groups.

Theorem 7.1. If $\mathcal{C}$ is closed under taking subgroups and if $\mathcal{T}(\mathcal{C})$ satisfies $\mathcal{J}(C, d)$ then $\mathcal{C}$ satisfies $\mathcal{J}(C', d)$ for some natural number $C'$ depending on $C$ and $d$.

Theorem 7.1 is false if one replaces $\mathcal{T}(\mathcal{C})$ by the collection of all $p$-groups in $\mathcal{C}$ for varying prime $p$. However, for certain collections of finite groups it does suffice to consider only the $p$-groups in order to obtain property $\mathcal{J}(C, d)$, as the following theorem shows.

Theorem 7.2. Let $X$ be a $2n$-dimensional smooth, compact and connected manifold. Let $\mathcal{C}$ be a collection of finite subgroups of $\text{Diff}(X)$. Suppose that:
(1) $\mathcal{C}$ is closed under taking subgroups,
(2) for any $G \in \mathcal{C}$ there is a $G$-invariant almost complex structure on $X$,
(3) there exists a constant $C_0$ such that, for any prime $q$, any $G \in \mathcal{C}$ which is a $q$-group has a subgroup $G_0 \subseteq G$ satisfying $X^{G_0} \neq \emptyset$ and $[G : G_0] \leq C_0$.

Then $\mathcal{C}$ satisfies $\mathcal{J}(C, n)$, where $C$ only depends on $C_0$ and $H^*(X)$.

Proof. This statement is implicit in the proof of Theorem 5.2 in [26], so we just sketch the main ideas and refer to [26] for more details.

By Theorem 7.4 we only need to prove that $\mathcal{J}(\mathcal{C})$ satisfies $\mathcal{J}(C, n)$ for some $C$ depending only on $C_0$ and $H^*(X)$.

Let us say that some quantity is $H^*(X)$-bounded if it admits an upper bound which depends only on $H^*(X)$. Of course, $2n$ is $H^*(X)$-bounded.

Let $T \in \mathcal{J}(\mathcal{C})$ and suppose that $p, q$ are distinct prime numbers, $P \subseteq T$ (resp. $Q \subseteq T$) is a Sylow $p$-subgroup (resp. normal Sylow $q$-subgroup) and $T = PQ$. By assumption, there is a subgroup $Q_0 \subseteq Q$ satisfying $X^{Q_0} \neq \emptyset$ and $[Q : Q_0] \leq C_0$. By Corollary 5.3, there is an abelian subgroup $Q_a \subseteq Q_0$ satisfying $[Q_0 : Q_a] \leq \text{Jor}_2n$ and $Q_a$ can be generated by $n$ elements. Hence we have

\begin{equation}
[Q : Q_a] \leq C_0 \text{Jor}_2n.
\end{equation}

Applying the same argument to $P$ we conclude the existence of an abelian subgroup $P_a \subseteq P$ satisfying $[P : P_a] \leq C_0 \text{Jor}_2n$ and such that $P_a$ can be generated by $n$ elements.

The bound (19) implies that for any $g \in P$ we have

$$[Q_a : Q_b \cap gQ_a g^{-1}] \leq C_0 \text{Jor}_2n.$$  

Since $Q_a$ can be generated by $d$ elements, the group $Q_b = \bigcap_{g \in P} gQ_a g^{-1}$ satisfies

$$[Q_a : Q_b] \leq ((C_0 \text{Jor}_2n)!)^d$$

(see e.g. [26] Corollary 3.2). Since $Q_b \subseteq Q_0$ and $X^{Q_b} \neq \emptyset$, we have $X^{Q_b} \neq \emptyset$. By definition $Q_b$ is a normal subgroup of $T$, so the action of $P_a$ on $X$ preserves $X^{Q_b}$. By Lemma 5.1 the number of connected components of $X^{Q_b}$ is $H^*(X)$-bounded, so there exists a subgroup $P_b \subseteq P_a$ such that $[P_a : P_b]$ is $H^*(X)$-bounded and $P_b$ preserves each connected component of $X^{Q_b}$.

Let $Y \subseteq X^{Q_b}$ be any connected component, and let $N \to Y$ be the normal bundle of its inclusion in $X$. The actions of $Q_b$ and $P_b$ on $Y$ lift to actions on $N$. Consider the splitting

\begin{equation}
N \otimes \mathbb{C} = \bigoplus_{\xi \in Q_b} N^{Q_b, \xi}
\end{equation}

given by the action of $Q_b$ on $N$ (see Subsection 2.5). Let $\Xi \subseteq \hat{Q}_b$ be the set of characters $\xi$ such that $N^{Q_b, \xi} \neq 0$. We have $|\Xi| \leq \dim X$, so $|\Xi|$ is $H^*(X)$-bounded. The action by conjugation of $P_b$ on $Q_b$ induces an action on $\hat{Q}_b$ which necessarily preserves $\Xi$, so there is a subgroup $P_c \subseteq P_b$ which fixes each $\xi \in \Xi$ and such that $[P_b : P_c]$ is $H^*(X)$-bounded. Since $P_c$ fixes each $\xi \in \Xi$, the action of $P_c$ on $N \otimes \mathbb{C}$ preserves the splitting (20). Since the action of $Q_b$ on each $N^{Q_b, \xi}$ is given by scalar multiplication, it follows that the actions of $P_c$ and $Q_b$ on $N \to Y$ commute. By the arguments in the proof of Corollary 5.3 this implies that the actions of $P_c$ and $Q_b$ on $X$ commute. Consequently, $A := P_cQ_b$ is an
abelian group. Since 
$[T : A] \leq [P : P_c][Q : Q_b]$ and both $[P : P_c]$ and $[Q : Q_b]$ are $H^*(X)$-bounded, it follows that $[T : A]$ is $H^*(X)$-bounded.

Both $P_c$ and $Q_b$ can be generated by $n$ elements because they are subgroups of abelian subgroups that can be generated by $n$ elements. Since the orders of $P_c$ and $Q_b$ are coprime, this implies that $T$ can be generated by $n$ elements.

\[ \Box \]

7.4. **Proof of Theorems 1.1 and 1.3.** Theorem 1.2 follows immediately from combining Theorem 1.9 with Theorem 7.2 applied to the collection of all finite subgroups of $\text{Symp}(X, \omega)$. Similarly, Theorem 1.1 follows from Theorem 1.7 and Theorem 7.2 applied to the collection of all finite subgroups of $\text{Ham}(X, \omega)$.

7.5. **Proof of Theorem 1.6.** We begin with some preliminary results.

We claim that there is a constant $C_0$ such that for any finite subgroup $\Gamma \subset \text{Symp}(X, \omega)$ acting trivially on $H^1(X)$ and for any prime $p$, any $p$-subgroup $G \subseteq [\Gamma, \Gamma]$ has a subgroup $G_0 \subseteq G$ such that $X^{G_0} \neq \emptyset$ and $[G : G_0] \leq C$. Furthermore, $C_0$ only depends on $H^*(X)$. This claim follows from Theorem 7.5 and Theorem 1.11 using the same arguments as in Subsection 7.1.

Let $\mathcal{E}$ be the collection of all finite subgroups $G \subset \text{Symp}(X, \omega)$ for which there exist a finite subgroup $\Gamma \subset \text{Symp}(X, \omega)$ acting trivially on $H^1(X)$ and satisfying $G \subseteq [\Gamma, \Gamma]$. Combining the previous claim with Theorem 7.2 we conclude that $\mathcal{E}$ satisfies $j(C_1, n)$ for some constant $C_1$ depending on $H^*(X)$.

To conclude the preliminaries, we recall that a well known lemma of Minkowski states that for any natural number $k$ and any finite group $H \subseteq \text{GL}(k, \mathbb{Z})$ the restriction to $H$ of the natural map $\text{GL}(k, \mathbb{Z}) \to \text{GL}(k, \mathbb{Z}/3)$ is injective (see [24], and [40, Lemma 1] for a modern exposition). Hence no finite subgroup of $\text{Aut}(H^1(X))$ has more than $3^{b_1(X)}$ elements.

We are now ready to prove Theorem 1.6.

Let $\Gamma \subset \text{Symp}(X, \omega)$ be a finite subgroup. Let $h^1 : \Gamma \to \text{Aut}(H^1(X))$ be the natural map. Its image $h^1(\Gamma)$ is a finite subgroup of $\text{Aut}(H^1(X))$ so by Minkowski’s lemma it has at most $3^{b_1(X)}$ elements. Hence $\Gamma_0 = \ker h^1 \subset \Gamma$ satisfies $[\Gamma : \Gamma_0] \leq 3^{b_1(X)}$. The commutator $G := [\Gamma_0, \Gamma_0]$ is an element of $\mathcal{E}$, so there is an abelian subgroup $A \subseteq G$ satisfying $[G : A] \leq C_1$ and $A$ can be generated by $n$ elements, where $\dim X = 2n$.

For any prime $p$ let $A_p$ be the $p$-part of $A$. Since $A_p \subseteq [\Gamma_0, \Gamma_0]$, there is a subgroup $A_{p,0} \subseteq A_p$ satisfying $X^{A_{p,0}} \neq \emptyset$ and $[A_p : A_{p,0}] \leq C_0$. In particular, if $p > C_0$ then $A_{p,0} = A_p$. Hence setting $A_0 := \prod_p A_{p,0}$ we have the following rough estimate

$[A : A_0] \leq C_2 := C_0^{\alpha(C_0)}$,

where $\pi(C_0)$ denotes the number of primes not bigger than $C_0$.

Define $A_1 = \bigcap_{\phi \in \text{Aut}(G)} \phi(A_0)$. We have $[G : A_1] \leq C_3$ for some $C_3$ depending only on $C_1C_2$ and $n$ (this is standard, see e.g. [26, Corollary 3.2]), and $A_1$ is a characteristic subgroup of $G$. Since $G$ is a normal subgroup of $\Gamma_0$, it follows that $A_1$ is normal in $\Gamma_0$.

Let $\Gamma_1 = \Gamma_0/A_1$. Since $A_1$ is a normal subgroup of $G = [\Gamma_0, \Gamma_0]$, we have $[\Gamma_1, \Gamma_1] = [\Gamma_0, \Gamma_0]/A_1 = G/A_1$ and $\Gamma_1/[\Gamma_1, \Gamma_1] = \Gamma_0/[\Gamma_0, \Gamma_0]$. Hence the exact sequence

$1 \to [\Gamma_1, \Gamma_1] \to \Gamma_1 \to \Gamma_1/[\Gamma_1, \Gamma_1] \to 1$
can be rewritten as

\[ 1 \to G/A_1 \to \Gamma_1 \to \Gamma_0/[\Gamma_0, \Gamma_0] \to 1. \]

By Mann and Su’s Theorem \([5,4]\) there exists some constant \(r\) depending on \(H^*(X)\) such that for any monomorphism \((\mathbb{Z}/p)^s \to \Gamma\) we have \(s \leq r\). This implies that the Sylow \(p\)-subgroups of \(\Gamma\) can be generated by at most \(r(5r + 1)/2\) elements (such bound follows easily from Lemmas \([4.2]\) and \([4.4]\)). Consequently, the abelian group \(\Gamma_0/[\Gamma_0, \Gamma_0]\) can be generated by \(r(5r + 1)/2\) elements. By Lemma \(2.2\) in \([25]\) there is an abelian subgroup

\[ A_2 \subseteq \Gamma_1 \]

satisfying \([\Gamma_1 : A_2] \leq C_4\), where \(C_4\) depends only on \(|G/A_1| \leq C_3\) and \(r(5r + 1)/2\); consequently, \(C_4\) can be bounded above by a constant depending only on \(H^*(X)\).

Let \(S\) be the preimage of \(A_2\) under the projection map \(\Gamma_0 \to \Gamma_1\). Clearly \(S\) is solvable, since it fits in an exact sequence

\[ 0 \to A_1 \to S \to A_2 \to 0, \]

and \(A_1, A_2\) are abelian. In general we cannot expect \(S\) to be abelian or 2-step nilpotent, since the action of \(A_2\) by conjugation on \(A_1\) might be nontrivial. We want to prove that, nevertheless, \(S\) contains an abelian or 2-step nilpotent subgroup of bounded index.

For any prime \(p\) we have \(X^{A_1,p} \neq \emptyset\), where \(A_1,p\) is the \(p\)-part of \(A_1 \subseteq A_0\). Let \(N_p\) be the normal bundle of the inclusion \(X^{A_1,p} \hookrightarrow X\) and let \(N_p = \bigoplus_{\xi \in A_1,p} N_{p,\xi}\) be its decomposition according to the characters of \(A_1,p\) (see Subsection \([2.5]\)). Let \(\Xi_p = \{\xi \in \widehat{A_1,p} \mid N_{p,\xi} \neq 0\}\). We may bound, as in Subsection \([5.2]\)

\[ |\Xi_p| \leq 2n \sum_j b_j(X; \mathbb{F}_p). \]

The right hand side can be bounded by a constant \(B\) depending only on \(H^*(X)\), not on \(p\). The action of \(A_2\) on \(A_1\) by conjugation preserves \(A_1,p\) and hence permutes the elements of \(\Xi_p\); this way we get a morphism \(\sigma_p : A_2 \to \text{Perm}(\Xi_p)\). By Corollary \([5.3]\) any \(\alpha \in \text{Ker } \sigma_p\) acts trivially on \(A_1,p\) (this is the same idea that is used at the end of the proof of Theorem \([7.2]\)). We have \([A_2 : \text{Ker } \sigma_p] \leq B!\), so the intersection

\[ A_3 := \bigcap_{\{A' \leq A_2 \mid A' \leq B!\}} A', \]

acts trivially on \(A_1,p\) for each prime \(p\), and hence acts trivially on \(A_1\). Since the rank of \(A_2\) is at most \(r(5r + 1)/2\), we can bound \([A_2 : A_3] \leq C_5\) where \(C_5\) depends only on \(r\) and \(B\), hence only on \(H^*(X)\). Let \(N \subseteq S\) be the preimage of \(A_3\) under the projection map \(S \to A_2\). Then \(N\) is abelian or 2-step nilpotent and

\[ [\Gamma : N] = [\Gamma : \Gamma_0][\Gamma_0 : S][S : N] = [\Gamma : \Gamma_0][\Gamma_1 : A_2][A_2 : A_3] \leq 3^{b_1(X)}C_4C_5 =: C, \]

where \(C\) depends only on \(H^*(X)\). So the proof of Theorem \([7.6]\) is complete.
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