The maximum number of intersections of two polygons

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Abstract
We determine the maximum number of intersections between two polygons with \( p \) and \( q \) vertices, respectively, in the plane. The cases where \( p \) or \( q \) is even or the polygons do not have to be simple are quite easy and already known, but when \( p \) and \( q \) are both odd and both polygons are simple, the problem is more difficult. We prove that the conjectured maximum \( (p - 1)(q - 1) + 2 \) is correct for all odd \( p \) and \( q \).

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1 Introduction

In computational geometry it is a basic and important problem to determine the intersections between line segments, especially polygons; see for example [2]. From the viewpoint of discrete geometry and combinatorics, it is interesting to determine the number of intersections rather than the intersections themselves. In this paper we calculate the maximum number of intersections between two polygons with a given number of vertices, considering separately the case where both polygons are simple. According to Thomas Banchoff, this was one of the favorite problems of Oliver Selfridge.

If the polygons are not required to be simple, there are easy upper bounds on the number of intersections and these are attained. To give a complete answer to Selfridge’s problem, we include the solution of this case, given in [1], as well.

If the polygons are required to be simple, the maximum number of intersections coincides with the general case if one of the polygons has an even number of edges. But if both polygons have an odd number of vertices, the maximum number of intersections \( f(p, q) \) is different unless one of the polygons is a triangle. The right answer \( f(p, q) = (p - 1)(q - 1) + 2 \) was already stated in [1], but their proof turns out to be wrong as indicateded in [3]. We will show how to modify their argument such that it works.

In [3] it was already noted that there is a mistake in the proof of [1]. Using a different approach, they showed \( f(p, q) \leq pq - \left\lfloor \frac{p}{2} \right\rfloor - q \) if \( p \leq q \) are both odd. In the case \( p = 5 \) they obtained the exact answer \( f(p, q) = (p - 1)(q - 1) + 2 \). They conjectured the formula \( f(p, q) = (p - 1)(q - 1) + 2 \) for odd \( p, q \) both greater than 5.

In Section 2 we start with some preliminaries and the formulation of the main theorem. We deal with the case where both polygons have an even number of vertices in Section 3 and with the case where

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one polygon has an even and the other an odd number of vertices in Section 4 and with the case where both polygons have an odd number of vertices in Section 5.

2 Preliminaries

By a polygon we mean a finite sequence of vertices in the plane with straight line segments connecting consecutive vertices as well as the final and the first vertex. The polygon is called simple, if any vertex is contained in exactly two edges and any other point is contained in exactly one edge.

In this paper, we consider the intersection between two polygons $P$ and $Q$. We denote the number of vertices of $P$ by $p$ and the number of vertices of $Q$ by $q$.

For simplicity, we will assume that not more than two edges are passing through a point of intersection. This is no restriction, since there is always a slight deformation of vertices deforming the polygons into the above situation without changing the total number of intersections (if the number of intersections is counted appropriately, see [1]).

Our aim is to show the following theorem of [1].

**Theorem 1.** Let $P$ and $Q$ be two polygons having $p$ and $q$ vertices, respectively.

1. Let $p$ and $q$ be both even. Then the maximum number of intersections is $pq$, even if $P$ and $Q$ are simple.

2. Let $p$ be even and $q$ odd. Then the maximum number of intersections is $p(q - 1)$, even if $P$ and $Q$ are simple.

3. Let $p$ and $q$ be both odd. Then the maximum number of intersections is $pq - \max(p, q)$. If $P$ and $Q$ are simple, the maximum number of intersections is $(p - 1)(q - 1) + 2$.

3 Both polygons have an even number of vertices

Since any of the $p$ edges of $P$ can intersect at most $q$ edges of $Q$, the number of intersections is bounded by $pq$. We will now construct two simple polygons $P$ and $Q$ intersecting in $pq$ points.

For $P$, we take a triangle $\triangle ABC$ and choose a line $g$ parallel to the edge $a = BC$ intersecting the triangle very close to the point $A$. Now choose $\frac{p}{2} - 1$ distinct points on $g$ in the interior of the triangle. Also, choose $\frac{p}{2}$ distinct points on $a$ including the vertices $B$ and $C$. Finally, we delete the edge $BC$ and connect the point on $a$ closest to $B$ (which is $B$ itself) with the point on $g$ closest to the edge $AB$, continue with the point on $a$ second closest to $B$, and so on, until we end up with $C$. $P$ is a simple polygon with $p$ vertices.

Now take a triangle $\triangle A'B'C'$ which intersects $\triangle ABC$ in six points and construct the simple polygon $Q$ in a similar way as $P$.

Any edge of $P$ intersects all edges of $Q$. Hence, $P$ and $Q$ intersect in $pq$ points. This shows the first part of Theorem 1. An example for $p = 8$ (red) and $q = 16$ (blue) is shown in Figure 1.
4 One polygon has an even and the other an odd number of vertices

Without loss of generality, \( p \) is even and \( q \) is odd. A line corresponding to a particular edge of \( P \) divides the plane into two half spaces. If we go along \( Q \), we change the half spaces at each intersection with \( P \). Since \( Q \) is a polygon, it follows that the number of intersections of \( Q \) with these particular edge of \( P \) is even. But it can be at most the odd number \( q \), hence, it is at most \( q - 1 \). Thus, the number of intersections is bounded by \( p(q - 1) \). In the following, we will construct two simple polygons \( P \) and \( Q \) intersecting in \( p(q - 1) \) points.

For \( Q \), we take a zig-zag polygonal curve on \( q \) vertices, which starts on a horizontal base line, goes then alternately up and down being always slightly above the horizontal line, and ends on the base line. Now connect the end points of the zig-zag line by the corresponding segment of the base line. \( Q \) is a simple polygon with \( q \) vertices.

Choosing a triangle \( \triangle ABC \) intersecting all spikes of the zig-zag polygonal curve, we construct \( P \) exactly as in Section 3. Any edge of \( P \) intersects all but one edge of \( Q \), the missing edge is exactly the one corresponding to the base line. Thus, we get \( p(q - 1) \) intersections, which shows the second part of Theorem 1.

An example for \( p = 8 \) (red) and \( q = 9 \) (blue) is shown in Figure 2.

5 Both polygons have an odd number of vertices

We first consider the case of general, not necessarily simple polygons. By the same argumentation as in Section 4, the number of intersections of \( P \) and \( Q \) is bounded by \( p(q - 1) \) as well as by \((p-1)q \). Therefore, it is bounded by \( pq - \max(p,q) \). We now construct two polygons \( P \) and \( Q \) having \( pq - \max(p,q) \) intersections.

Choose the vertices of a regular \( p \)-gon and a regular \( q \)-gon with all vertices distinct and both of them having the same circumcircle \( C \) of circumference 1. For \( P \), we connect any two points of the \( p \)-gon...
Figure 2: Two simple polygons with 8 and 9 vertices intersecting 64 times

having exactly $\frac{p-3}{2}$ points of the $p$-gon in between. $Q$ is defined in a similar way.

$P$ and $Q$ are polygons with $p$ and $q$ vertices, respectively. Without loss of generality, $p \leq q$. Now fix an arbitrary edge $e$ of $Q$. It divides the circumcircle $C$ into two arcs of length $(q \pm 1)/(2q)$. Using that $(p-1)/(2p) \geq (q-1)/(2q)$, the endpoints of an edge $e'$ of $P$ which does not intersect $e$ both lie in the arc of length $(q+1)/(2q)$. If there was another edge of $P$ which does not intersect $e$, this would yield three points of $P$ lying inside the arc of length $(q+1)/(2q)$, two of them separating $C$ into two arcs of length $(p \pm 1)/(2p)$ and the third one being inside the arc of length $(p+1)/(2p)$. Since any two distinct points of $P$ have distance at least $1/p$ on $C$, an arc of length $(p-1)/(2p) + 1/p = (p+1)/(2p) \geq (q+1)/(2q)$ would lie inside the arc of length $(q+1)/(2q)$, contradiction.

Therefore, any edge of $Q$ intersects at least $p - 1$ edges of $P$, such that we obtain at least $q(p-1) = pq - \max(p,q)$ intersections. Since this number is already an upper bound, the number of intersections is exactly $pq - \max(p,q)$. This shows the third part of Theorem 1 in the case of general polygons.

An example for $p = 5$ (red) and $q = 7$ (blue) is shown in Figure 3.

Figure 3: Two non-simple polygons with 5 and 7 vertices intersecting 28 times

From now on, we assume that the polygons $P$ and $Q$ are simple. As in [1], we use the notation $|\cdot|$ for the cardinality of a set and denote by $E_P(e)$ the set of edges of $P$ which intersect the edge $e$ of $Q$. The claim in [1] from which the proof of the third part of Theorem 1 in the case of simple polygons
would follow, was the existence of two edges $e, e'$ of $Q$ such that $|E_P(e) \cap E_P(e')| \leq 1$. But in Figure 4 we can see that any pair of edges of the triangle $Q$ (blue) is intersected by exactly two edges of the heptagon $P$ (red). In the following, we modify the proof of [1] by considering triples instead of pairs of edges.

Figure 4: Each pair of edges of the triangle is intersected by two edges of the heptagon

Lemma 2. Let $P$ and $Q$ be two simple polygons both with an odd number of vertices. Then there exists a triple of edges $(e, e', e'')$ of $Q$ such that

$$|E_P(e) \cap E_P(e') \cap E_P(e'')| \leq 1.$$  

Proof. Assume, for a contradiction, that every triple, and therefore every pair, of edges of $Q$ intersect at least two edges of $P$. Choose an orientation of the polygon $Q$, i.e., we direct all edges of $Q$ in such a way, that at every vertex one edge is oriented towards the vertex and the other one is oriented away from it.

We assign a plus/minus sign to each edge of $Q$ as follows. Edge $e_0$ is conventionally assigned a plus sign. Consider a pair of edges $(e, e')$ and let $a_1, a_2, \ldots, a_n$ be the sequence of edges of $P$ intersecting $e$ and $e'$, the edges being ordered according to the order of their points of intersection with the directed edge $e$. Since $P$ is simple, the order of intersections on the directed edge $e'$ is either $a_1, a_2, \ldots, a_n$ or $a_n, a_{n-1}, \ldots, a_1$. We assign the same sign to $e$ and $e'$ in the first case and opposite signs in the second case. For an illustration, see Figure 5.

The signs on the edges of $Q$ are well-defined. Indeed, let $(e, e', e'')$ be a triple of edges of $Q$ and let $a_1, a_2, \ldots, a_m$ be the sequence of edges of $P$ intersecting $e, e'$ and $e''$, the edges being ordered according to the order of their points of intersection with the directed edge $e$. By assumption, $m \geq 2$. $e'$ respectively $e''$ is assigned the same sign as $e$ has, if and only if the order of intersections on the directed edge $e'$ respectively $e''$ is $a_1, a_2, \ldots, a_m$, and the opposite sign as $e$ has, if and only if the order of intersections is $a_m, a_{m-1}, \ldots, a_1$. Hence, the same respectively opposite sign is assigned to $e'$ and $e''$ if and only if the edges $a_1, a_2, \ldots, a_m$ intersect the directed edges $e'$ and $e''$ in the same respectively opposite order. This is consistent to the choice of signs for the pair $(e', e'')$.

The reason why the proof of [1] fails is that when considering pairs only, it is not possible to define the signs consistently: In Figure 5 we would have to assign different signs to any pair of edges of the triangle, which is impossible. The remainder of our proof of the lemma is now the same as in [1].

Because $Q$ has an odd number of edges, there exists two adjacent edges $(e_1, e_2)$ with the same sign. Let $v$ be their common vertex. Without loss of generality, $e_1$ is oriented towards $v$. Then $e_2$ is oriented
away from \( v \). By construction, there are at least two edges of \( P \) intersecting both \( e_1, e_2 \) in the same order \( a_1, a_2, \ldots, a_n \). But then \( a_n \) is closest to \( v \) on \( e_1 \) but farthest to \( v \) on \( e_2 \), such that \( a_1 \) and \( a_n \) have to intersect, contradicting the hypothesis that \( P \) is simple.

By Lemma 2 there exists a triple of edges \((e, e', e'')\) of \( Q \) such that at most one edge of \( P \) is intersecting all these three edges. All other edges of \( P \) can intersect at most two of the three edges of this triple. Thus, the number of intersections of the three edges \( e, e', e'' \) with \( P \) is bounded by \( 3 + 2(p-1) = 2p + 1 \).

By the argumentation of Section 4, any of the other \( q - 3 \) edges of \( Q \) can intersect at most \( p - 1 \) edges of \( P \) because \( p \) is odd. Therefore, the number of intersections of \( P \) and \( Q \) is bounded by

\[
(p-1)(q-3) + (2p+1) = (p-1)(q-1) + 3.
\]

It is well known that any two polygons intersect transversely in an even number of points (one may use the Jordan curve theorem and an argumentation as in Section 4). Since \( p, q \) are odd, \( (p-1)(q-1) + 3 \) is odd as well, such that the number of intersections of \( P \) and \( Q \) is actually bounded by \( (p-1)(q-1) + 2 \).

To finally prove the third part of Theorem 1 we have to construct simple polygons \( P \) and \( Q \) both with an odd number of vertices intersecting \( (p-1)(q-1) + 2 \) times. To do so, we take two zig-zag polygons as in Section 4 and place them in such a way that all spikes are pairwise intersecting, the two base lines do intersect and each of the base lines intersects one of the boundary spikes. \( P \) and \( Q \) are simple polygons intersecting \( (p-1)(q-1) + 2 \) times.

For an illustration for \( p = 9 \) (red) and \( q = 5 \) (blue), see Figure 5.

![Figure 5: Two simple polygons with 5 and 9 vertices intersecting 34 times](image)

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