DOUBLE QUANTIZATION ON THE COADJOINT REPRESENTATION OF SL(n) [1]

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Abstract

For \( g = sl(n) \) we construct a two parametric \( U_h(g) \)-invariant family of algebras, \( (Sg)_{t,h} \), which defines a quantization of the function algebra \( Sg \) on the coadjoint representation and in the parameter \( t \) gives a quantization of the Lie bracket. The family induces a two parametric deformation of the function algebra of any maximal orbit which is a quantization of the Kirillov-Kostant-Souriau bracket in the parameter \( t \). In addition we construct a quantum de Rham complex on \( g^* \).

1 Introduction

Let \( G \) be a simple Lie group over the field of complex numbers \( \mathbb{C} \) with the Lie algebra \( g \). Let \( M \) be a homogeneous space and \( A \) the algebra of algebraic functions on \( M \). The algebra \( A \) is commutative and has a \( G \)-invariant multiplication.

The first problem we consider in this note is to construct a quantized algebra \( A_h \) in which the deformed multiplication \( m_h \) is invariant under the action of the Drinfeld-Jimbo quantum group \( U_h(g) \) defined over \( \mathbb{C}[[h]] \), i.e.

\[
xm_h(a \otimes b) = m_h(x)(a \otimes b),
\]

where \( a, b \in A_h \), \( x \in U_h(g) \), and \( \Delta_h \) denote the comultiplication in \( U_h(g) \).

Let \( A_t \) be a \( G \)-invariant quantization of \( A \) by an invariant Poisson bracket on \( M \). The second problem is to construct a two parametrized \( U_h(g) \)-invariant family of algebras, \( A_{t,h} \), such that \( A_t = A_{t,0} \).

More explicitly, by a quantization (or deformation) of \( A \) we mean an algebra \( A_h \) over the algebra of formal power series \( \mathbb{C}[[h]] \) in a variable \( h \) that is isomorphic to \( A[[h]] = A \otimes \mathbb{C}[[h]] \) as a \( \mathbb{C}[[h]] \)-module and \( A_0 = A_h/hA_h = A \) as an algebra (note that when we consider modules over \( \mathbb{C}[[h]] \) the symbol \( \otimes \) denotes the tensor product completed in the \( h \)-adic topology). We say in this case that \( A_h \) is a flat deformation of \( A \).

Similarly, the flatness of two parametric family, \( A_{t,h} \), is defined.

In [4], [5], [6] the first problem is solved for flag varieties, for orbits of highest weight vector in irreducible representations of \( G \), and for semisimple orbits in \( g^* \). The second problem is solved in [7] for hermitian symmetric spaces.

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In this note we consider the coadjoint representation $g^*$ of $U(g)$ for $g = sl(n)$ and show that the second problem can be solved when $M$ is a maximal orbit in $g^*$ and $\mathcal{A}_t$ is a quantization of the Kirillov-Kostant-Souriau bracket.

Moreover, we define a $U_h(g)$-invariant algebra $(Sg)_{t,h}$, quantization of the function algebra on $g^*$, and show that the quantized algebra $\mathcal{A}_{t,h}$ for a maximal orbit can be presented as a quotient algebra of $(Sg)_{t,h}$.

To obtain $(Sg)_{t,h}$ we use an idea of the paper of Lyubashenko and Sudbery concerning the construction of a quantum analog of Lie algebra for $sl(n)$, but we deal with the quantum group $U_h(g)$ over $C[[h]]$ instead of $U_q(g)$ and use for our construction the R-matrix, that allows us to simplify the proofs. Using the fact that the quantum Casimir $C_V$ is invertible in $U_h(g)$ (see Section 2), we define a quantum Lie algebra as an embedding of the deformed adjoint representation, $g_h$, in $U_h(g)$ such that the kernel of the extension to an algebra homomorphism $T(g_h) \to U_h(g)$ is defined by quadratic-linear relations, as in the classical case. This definition differs from the definition given in [3]. For $U_q(g)$ our results are valid as well (see Remark 3.2).

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2 Quantum Lie algebra for $U_h(sl(n))$

Let $R = R_i^r \otimes R_j^r \in U_h(g) \otimes U_h(g)$ be the R-matrix (summation by $i$ is assumed). It satisfies the properties

$$\Delta'(x) = R\Delta(x)R^{-1}, \quad x \in U_h(g),$$

(2.1)

where $\Delta$ is the comultiplication in $U_h(g)$ and $\Delta'$ the opposite one,

$$(\Delta \otimes 1)R = R_i^{13} R_j^{23} = R_i^r \otimes R_j^r \otimes R_j^r R_i^r$$

and

$$(1 \otimes \Delta)R = R_i^{13} R_j^{12} = R_i^r R_j^r \otimes R_j^r \otimes R_i^r,$$

(2.2)

and

$$(1 \otimes \varepsilon)R = (\varepsilon \otimes 1)R = 1 \otimes 1,$$

(2.3)

where $\varepsilon$ is the counit in $U_h(g)$.

Consider the element $Q = Q_i^r \otimes Q_i^r = R^{21} R$. It follows from (2.1) that $Q$ commutes with elements from $U_h(g) \otimes U_h(g)$ of the form $\Delta(x)$. This is equivalent to $Q$ being invariant under the adjoint action of $U_h(g)$ on $U_h(g) \otimes U_h(g)$.

Let $V$ be an irreducible finite dimensional representation of $U_h(g)$ and $\rho : U_h(g) \to \text{End}(V)$ the corresponding map of algebras. Consider the dual space $\text{End}(V)^*$ as a left $U_h(g)$-module setting

$$(x\varphi)(a) = \varphi(\gamma(x(1))ax(2)),$$

where $\varphi \in \text{End}(V)^*$, $a \in \text{End}(V)$, $\Delta_h(x) = x(1) \otimes x(2)$ in Sweedler notions, and $\gamma$ denotes the antipode in $U_h(g)$.
Consider the map $f : \text{End}(V)^* \rightarrow U_h(\mathfrak{g})$ defined as $\varphi \mapsto \varphi(\rho(Q))Q''$. From the invariance of $Q$ it follows that $f$ is a $U_h(\mathfrak{g})$-equivariant map, so $\mathcal{T} = \text{Im}(f)$ is a $U_h(\mathfrak{g})$-submodule.

It follows from (2.2) that $\mathcal{T}$ is a left coideal in $U_h(\mathfrak{g})$, i.e. $\Delta(x) \in U_h(\mathfrak{g}) \otimes \mathcal{T}$ for any $x \in \mathcal{T}$. Indeed, $Q = R''R'_j \otimes R'_jR''$. Applying (2.2) we obtain

$$(1 \otimes \Delta_h)R^{21} = R''R'_j \otimes R'_jR''$$

Let $\varphi \in \text{End}(V)^*$. Define $\psi_\varphi \in \text{End}(V)^*$ setting $\psi_\varphi(a) = \varphi(R'_jaR'_j)$ for $a \in \text{End}(V)$. Then $\Delta\varphi(R'_jR'_j)R'_jR'' = R'_jR'' \otimes \psi_\varphi(R'_jR'_jR'j_R''_kR''_k)$, which obviously belongs to $U_h(\mathfrak{g}) \otimes \mathcal{T}$.

Recall that $R = F e^{\frac{b}{2} t} F^{-1}$. Here $t = \sum t_i \otimes t_i$ is the split Casimir, where $t_i$ form an orthonormal basis in $\mathfrak{g}$ with respect to the Killing form, $F = 1 \otimes 1 + \frac{h}{2}(r + o(h))$, and $r$ is the classical Drinfeld-Jimbo R-matrix. Therefore,

$$Q = R^{21}R = Fe^{ht}R^{-1} = 1 \otimes 1 + ht + \frac{h^2}{2}(t^2 + [r,t]) + o(h^2). \quad (2.4)$$

Denote by $\text{Tr}$ the unique (up to a factor) invariant element in $\text{End}(V)^*$. Let $Z_0 = \rho_0(\mathfrak{g})$ and denote by $Z_h$ some $U_h(\mathfrak{g})$-invariant deformation of $Z_0$ in $\text{End}(V)$. Then we have a decomposition $\text{End}(V) = I \oplus Z_h \oplus W$, where $I$ is one dimensional subspace of invariant elements, $W$ is a subspace complement to $I \oplus Z_h$. This gives a decomposition $\text{End}(V)^* = I^* \oplus Z_h^* \oplus W^*$ where $W^*$ consists of all the elements which are equal to zero on $I \oplus Z_h$. The space $I^*$ is generated by $\text{Tr}$, and after normalizing in such a way that $\text{Tr}(id) = 1$, we obtain that $C_V = f(\text{Tr})$ is of the form

$$C_V = \text{Tr}(\rho(Q))Q_2 = 1 + h^2c + o(h^2), \quad (2.5)$$

and $c$ has to be an invariant element in $U(\mathfrak{g})$. It follows from (2.3) that $\varepsilon(C) = 1$.

From (2.4) follows that the elements of $f(Z_h^*)$ have the form

$$z = hx + o(h), \quad x \in \mathfrak{g}, \quad (2.6)$$

hence the subspace $L_1 = h^{-1}f(Z_h^*)$ forms a subrepresentation in $U_h(\mathfrak{g})$ under the left adjoint action of $U_h(\mathfrak{g})$ on itself, which is a deformation of the standard embedding of $\mathfrak{g}$ into $U(\mathfrak{g})$. It follows from (2.3) that $\varepsilon(L_1) = 0$.

The elements from $f(W^*)$ have the form $w = h^2b + o(h^2)$ and $\varepsilon(W^*) = 0$.

Denote $L = h^{-1}f(Z_h^* + W^*)$, so $\mathcal{T} = \mathbb{C}C_V \oplus hL$. Since $\mathcal{T}$ is a left coideal in $U_h(\mathfrak{g})$, for any $x \in L$ we have

$$\Delta(x) = x(1) \otimes x(2) = z \otimes C_V + v \otimes x',$$

where $z, v \in U_h(\mathfrak{g})$, $x' \in L$. Applying to the both sides $(1 \otimes \varepsilon)$ and multiplying we obtain $x = x(1)\varepsilon(x(2)) = z\varepsilon(C_V) + v\varepsilon(x') = z$. So, $z$ have to be equal to $x$. and we obtain

$$\Delta(x) = x(1) \otimes x(2) = x \otimes C_V + v \otimes x', \quad x, x' \in L. \quad (2.7)$$
From (2.7) we have for any \( y \in L \)
\[
x y = x^{(1)} y \gamma (x^{(2)}) x^{(3)} = x^{(1)} y \gamma (v^{(2)}) x^{(3)} + v^{(1)} y \gamma (v^{(2)}) x'.
\] (2.8)

Introduce the following maps:
\[
\sigma'_h : L \otimes L \to L \otimes L, \quad x \otimes y \mapsto v^{(1)} y \gamma (v^{(2)}) \otimes x',
\]
\[
[\cdot, \cdot]'_h : L \otimes L \to L, \quad x \otimes y \mapsto x^{(1)} y \gamma (x^{(2)}).
\] (2.9)

We may rewrite (2.8) in the form
\[
m(x \otimes y - \sigma'_h (x \otimes y)) - [x, y]'_h C V = 0.
\] (2.10)

Observe now that it follows from (2.5) that \( C V \) is an invertible element in \( U_h (g) \).
Put \( P = C V^{-1} \). Transfer the maps (2.9) to the space \( P \cdot L \), i.e. define
\[
\sigma_h (P x, P y) = (P \otimes P) \sigma'_h (x, y),
\]
\[
[P x, P y]'_h = P [x, y]'_h.
\] (2.11)

From (2.7) we obtain
\[
P (x^{(1)} \otimes P (2) x^{(3)}) = P (x \otimes P (2)) C V + P (1) v \otimes P (2) x',
\] (2.12)

Using this relation and taking into account that \( P \) commutes with all elements from \( U_h (g) \), we obtain as in (2.8)
\[
P x P y = P (x^{(1)} P y \gamma (x^{(2)}) \gamma (P (2)) P (3) x^{(3)}) = P (x^{(1)} P y \gamma (x^{(2)}) \gamma (P (2)) P (3) C V + P (1) v \gamma (v^{(2)}) \gamma (P (2)) P (3) x') = P [x, y]'_h + P^2 m \sigma'_h (x \otimes y) = [P x, P y]'_h + m \sigma_h (P x \otimes P y).
\]

This equality may be written as
\[
m(x \otimes y - \sigma_h (x \otimes y)) - [x, y]'_h = 0, \quad x, y \in C V^{-1} L.
\] (2.13)

Define \( L_V = C V^{-1} L \). Let \( T(L_V) = \bigoplus_{k=0}^{\infty} L_V^k \) be the tensor algebra over \( L_V \).
Notice, that \( T(L_V) \) is not supposed to be completed in \( h \)-adic topology. Let \( J \) be the ideal in \( T(L_V) \) generated by the relations
\[
(x \otimes y - \sigma_h (x \otimes y)) - [x, y]'_h = 0, \quad x, y \in L_V.
\] (2.14)

Due to (2.12) we have a homomorphism of algebras, \( T(L_V)/J \to U_h (g) \), extending the natural embedding \( \iota : L_V \to U_h (g) \). Introduce a new variable \( t \) and consider a homomorphism of algebras, \( T(L_V)[t] \to U_h (g)[t] \), which extends the embedding \( \iota t : L_V[t] \to U_h (g)[t] \). From (2.14) follows that it factors through the homomorphism of algebras
\[
\phi_{t,h} : T(L_V)[t]/J_t \to U_h (g)[t],
\] (2.15)
Remark 3.1. The completion of a quantum symmetric algebra. It is a free $C$-module equal to $U$-quantum group. Hence, the quantum group in some sense as a quadratic-linear algebra.

and $\Im(\phi)$ module. It follows that $\sigma$ of them is a quadratic one defined by the operator $g$ on $L_0$. Classical case. For this reason we call $L_V$ a quantum Lie (sub)algebra of $U_h(sl(n))$.

3 Double quantization on $sl(n)^*$ and quantum de Rham complex

In this section $g = sl(n)$. Let us apply the construction of the previous section to $V = \mathbb{C}^n[[h]]$, the deformed basic representation of $g$. In this case $End(V) = I \oplus Z_h$, where $Z_h$ is a deformed adjoint representation. So, $g_h = L_V = h^{-1}C^{-1}_V f(Z_h)$ is a deformation of the standard embedding of $g$ in $U(g)$. It is easy to see that in this case $\sigma_h$ is a deformation of the usual permutation: $\sigma_0(x \otimes y) = y \otimes x$, and $[\cdot, \cdot]_h$ is a deformation of the Lie bracket on $g$: $[x, y]_0 = [x, y]$, $x, y \in g \subset U(g)$.

It follows that the homomorphism of algebras $T(g_h)/J \to U_h(g)$ is a monomorphism, because it is an isomorphism for $h = 0$. Recall, that the ideal $J$ is defined by relations (2.13). Moreover, according to the PBW theorem the algebra $\Im(\phi_{t,h})$ at the point $h = 0$ is a free $\mathbb{C}[t]$-module and is equal to

$$(Sg)_t = T(g)/\{x \otimes y - y \otimes x - t[x, y]\}.$$ 

For $t = 0$ this algebra is the symmetric algebra $Sg$, the algebra of algebraic functions on $g^*$. For $t \neq 0$ this algebra is isomorphic to $U(g)$. Moreover, $U_h(g)$ is a flat $\mathbb{C}[[h]]$-module. It follows that $\phi_{t,h}$ in (2.14) is a monomorphism of algebras over $\mathbb{C}[[h]][t]$ and $\Im(\phi_{t,h})$ is a free $\mathbb{C}[[h]][t]$-module isomorphic to

$$(Sg)_{t,h} = T(g_h)[t]/\{x \otimes y - \sigma_h(x \otimes y) - t[x, y]_h\}.$$ 

Call the algebra

$$(Sg)_h = (Sg)_{0,h} = T(g_h)/\{x \otimes y - \sigma_h(x \otimes y)\}$$

a quantum symmetric algebra. It is a free $\mathbb{C}[[h]]$-module and a quadratic algebra equal to $Sg$ at $h = 0$.

Remark 3.1. The completion of $(Sg)_{1,h}$ in $h$-adic topology is isomorphic to the quantum group. Hence, the quantum group $U_h(g)$ for $g = sl(n)$ may by considered in some sense as a quadratic-linear algebra.

Two compatible Poisson brackets correspond to the deformation $(Sg)_{t,h}$. One of them is a quadratic one defined by the operator $\sigma_h$. Another one is the usual Lie bracket.
Now define a quantum exterior algebra, \((\Lambda g)_h\).

First, modify the operator \(\sigma_h\). Since the representation \(g_h^*\) is isomorphic to \(g_h\), there exists a \(U_h(g)\)-invariant bilinear form on \(g_h\), deformed Killing form. This form can be extended to all tensor degrees \(g^\otimes k\). Let \(\mathbb{C}[[h]]\)-submodule in \(g_h \otimes g_h\) orthogonal to \(V_h^2 = \text{Im}(\text{id} \otimes \text{id} - \sigma_h)\). Define an operator \(\bar{\sigma}_h\) on \(g_h \otimes g_h\) in such a way that it has the eigenvalues \(-1\) on \(V_h^2\) and \(1\) on \(W_h^2\). It is clear, that \(V_h^2\) and \(W_h^2\) are deformed skew symmetric and symmetric subspaces of \(g \otimes g\).

Now observe, that the third graded component in the quadratic algebra \((Sg)_h\) is the quotient of \(g_h^{\otimes 3}\) by the submodule \(V_h^2 \otimes g_h + g_h \otimes V_h^2\); hence this submodule and, therefore, the submodule \(V_h^2 \otimes g_h \cap g_h \otimes V_h^2\) are direct summands in \(g_h^{\otimes 3}\), i.e. they have complement submodules. As the complement submodules one can choose the submodules \(V_h^2 \otimes g_h \cap g_h \otimes W_h^2\) and \(V_h^2 \otimes g_h + g_h \otimes W_h^2\), respectively, since they are complement at the point \(h = 0\) and \(W_h^2\) is dual to \(V_h^2\) with respect to the Killing form extended to \(g_h \otimes g_h\). Hence, \(V_h^2 \otimes g_h + g_h \otimes W_h^2\) is a direct submodule. Moreover, the symmetric algebra \(Sg\) is Koszul. From a result of Drinfeld it follows that the quadratic algebra \((\Lambda g)_h = T((g_h)\cap \cdots \cap g_h)\cap \{W_h^2\}\) is a free \(\mathbb{C}[[h]]\)-module, i.e. is a flat \(U_h(g)\)-invariant deformation of the exterior algebra \(\Lambda g\).

Call \((\Lambda g)_h\) a quantum exterior algebra over \(g\).

Define a quantum algebra of differential forms over \(g^*\) as the tensor product \((\Omega g)_h = (Sg)_h \otimes (\Lambda g)_h\) in the tensor category of representations of the quantum group \(U_h(g)\). The multiplication of two elements \(a \otimes \alpha\) and \(b \otimes \beta\) looks like \(ab_{\alpha} \otimes \alpha_1 \beta\), where \(b_{\alpha} \otimes \alpha_1 = S(\alpha_1 \otimes b)\) and \(S = \sigma R\) is the permutation in that category.

As in the classical case, the algebras \((Sg)_h\) and \((\Lambda g)_h\) can be embedded in \(T(g_h)\) as a graded submodules in the following way. Call the submodule \(W_h^k = (W_h^2 \otimes g_h \otimes \cdots \otimes g_h) \cap (g_h \otimes W_h^2 \otimes g_h \otimes \cdots \otimes g_h) \cap \cdots \cap (g_h \otimes g_h \otimes \cdots \otimes W_h^2)\) of \(T^k(g_h)\) a \(k\)-th symmetric part of \(T(g_h)\). It is clear, that the natural map \(\pi_W : T(g_h) \rightarrow (Sg)_h\) restricted to \(W_h^k\) is a bijection onto the \(k\)-degree component \((S^k g)_h\) of \((Sg)_h\). Denote by \(\pi'_W : (S^k g)_h \rightarrow W_h^k\) the inverse bijection. Similarly we define \(V_h^k\), the \(k\)-th skew symmetric part of \(T(g_h)\), and the bijection \(\pi'_V : (\Lambda^k g)_h \rightarrow V_h^k\).

Now, define a differential \(d_h\) in \((\Omega g)_h\) as a homogeneous operator of degree \((-1,1)\). It acts on the element \(a \otimes \omega\) of degree \((k,m)\) in the following way. Let \(a \otimes \omega = (a_1 \otimes \cdots \otimes a_k) \otimes (\omega_1 \otimes \cdots \otimes \omega_m)\) be its realization as an element from \(W_h^k \otimes V_h^m\). Then the formula

\[
d_h(a \otimes \omega) = (a_1 \otimes \cdots \otimes a_{k-1} \otimes \pi'_V \pi_V (a_k \otimes \omega_1 \otimes \cdots \otimes \omega_m))\] (3.3)

presents the element \(d_h(a \otimes \omega)\) through its realization in \(W_h^{k-1} \otimes V_h^{m+1}\). One can prove that \(d_h^2 = 0\).

Call the algebra \((\Omega g)_h\) with operator \(d_h\) a quantum de Rham complex. It is easy to see that at the point \(h = 0\) this complex becomes the usual de Rham complex. The quantum de Rham complex is exact, since \(d_h^2 = 0\) and it is exact at \(h = 0\).

**Remark 3.2.** Up to now all our constructions were considered for the quantum group in sense of Drinfeld, \(U_h(g)\), defined over \(\mathbb{C}[[h]]\). But one can deduce all the constructions above for the quantum group in sense of Lusztig, \(U_q(g)\), defined...
over the algebra $\mathbb{C}[q, q^{-1}]$. We show, for example, how to obtain the quantum symmetric algebra over $\mathfrak{g}$. Let $E$ be a Grassmannian consisting of subspaces in $\mathfrak{g} \otimes \mathfrak{g}$ of dimension equal to $\text{dim}(\Lambda^2 \mathfrak{g})$, and $Z$ the closed algebraic subset of $E$ consisting of subspaces $J$ such that $\text{dim}(E \otimes J \cap J \otimes E) \geq \text{dim}(\Lambda^3 \mathfrak{g})$. Let $\mathcal{X}$ be the algebraic subset in $Z \times (\mathbb{C} \setminus 0)$ consisting of points $(J, q)$ such that $J$ is invariant under the action of $U_q(\mathfrak{g})$. The projection $\mathcal{X} \to (\mathbb{C} \setminus 0)$ is a proper map. One can check that over the point $q = 1$ there lies only one point of $\mathcal{X}$.

As follows from the existence of $(\mathcal{S} \mathfrak{g})_h$ (completed situation at $q = 1$), the dimension of $\mathcal{X}$ is equal to 1. Hence, the projection $p : \mathcal{X} \to \mathbb{C} \setminus 0$ is a covering. For $x \in \mathcal{X}$ let $J_x$ be the corresponding subspace in $\mathfrak{g} \otimes \mathfrak{g}$ and $(\mathcal{S} \mathfrak{g})_x = T(\mathfrak{g})/(J_x)$ the corresponding quadratic algebra. Due to the projection $p$ the family $(\mathcal{S} \mathfrak{g})_x, x \in \mathcal{X}$, is a module over $\mathbb{C}(q, q^{-1})$. Since $J_x$ is $U_{p(x)}(\mathfrak{g})$-invariant, $(\mathcal{S} \mathfrak{g})_x$ is a $U_{p(x)}(\mathfrak{g})$-algebra. At the “classical” point $x_0$, $p(x_0) = 1$, this module is flat. Hence, after possibly deleting from $\mathcal{X}$ some countable set of points, we obtain a flat family of algebras with the same Poincaré series as $\mathcal{S} \mathfrak{g}$. So, $(\mathcal{S} \mathfrak{g})_x$ is the quantum symmetric algebra over $U_q(\mathfrak{g})$.

4 Double quantization on semisimple orbits in $sl(n)^*$

In this section $G = SL(n)$, $\mathfrak{g} = sl(n)$.

Let $M$ be a semisimple orbit of $G$ in $\mathfrak{g}^*$ and $A$ the algebra of algebraic functions on $M$. It is known that $M$ is a closed algebraic submanifold in $\mathfrak{g}^*$, so $A$ can be presented as a quotient of $\mathcal{S} \mathfrak{g}$ by some ideal, $\mathcal{S} \mathfrak{g} \to A \to 0$. The Lie bracket on $\mathfrak{g}^*$ induces the Kirillov-Kostant-Souriau (KKS) bracket on $M$.

The problem is to construct a two parametric family of algebras, $A_{t,h}$, such that $A_{t,h}$ is a free $\mathbb{C}[[h]][t]$-module, $A_{0,0} = A$, it is invariant under the action of $U_h(\mathfrak{g})$, the algebra $A_t = A_{t,0}$ is a quantization of KKS bracket, and $A_{t,h}$ is a quantized $(\mathcal{S} \mathfrak{g})_{t,h}$ by some ideal, $(\mathcal{S} \mathfrak{g})_{t,h} \to A_{t,h} \to 0$.

As follows from [8], there exists such an algebra $A_{t,h}$ in case $M$ is a minimal orbit, i.e. $M$ is a hermitian symmetric space. That this $A_{t,h}$ can be presented as a quotient of $(\mathcal{S} \mathfrak{g})_{t,h}$ follows easily from the view of irreducible components of $A$.

We are going to show here, that the problem also has a positive solution for $M$ being a maximal orbit, i.e. can be defined as a set of zeros of invariant functions in $\mathcal{S} \mathfrak{g}$. Such orbits are the orbits of diagonal matrices with distinct elements on the diagonal.

The construction of $A_{t,h}$ is the following. There exists an isomorphism of $U_h(\mathfrak{g})$-modules $(\mathcal{S} \mathfrak{g})_h \to W_h$, where $W_h = \oplus_h W^k_h$, the direct sum of the $k$-th symmetric parts of $T(\mathfrak{g}_h)$ (see previous Section). Consider the composition $W_h[t] \to T(\mathfrak{g}_h)[t] \to (\mathcal{S} \mathfrak{g})_{t,h}$. It is an isomorphism, since it is an isomorphism at the point $h = 0$. It follows that $(\mathcal{S} \mathfrak{g})_{t,h}$ is isomorphic to $W_h[t]$ as a $U_h(\mathfrak{g})$-module.

Denote by $\mathcal{I}_{t,h}$ the submodule of $U_h(\mathfrak{g})$-invariant elements in $(\mathcal{S} \mathfrak{g})_{t,h}$. It is obvious that $\mathcal{I}_{t,h}$ is isomorphic to $\oplus_h \mathcal{I}^k_{t,h}[t]$, where $\mathcal{I}^k_{t,h}$ is the invariant submodule in $W^k_h$. Hence, $\mathcal{I}_{t,h}$ is a direct free $\mathbb{C}[[h]][t]$-submodule in $(\mathcal{S} \mathfrak{g})_{t,h}$. Moreover, $\mathcal{I}_{t,h}$
is a central subalgebra in \((Sg)_{t,h}\). Indeed, for \(t \neq 0\) the algebra \((Sg)_{t,h}\) can be invariantly embedded in \(U_h(g)\), but \(ad(U_h(g))\)-invariant elements in \(U_h(g)\) form the center of \(U_h(g)\). Yet \(I_{t,h}\) as an algebra is isomorphic to \(I_0\), the algebra of invariant elements in \(Sg\) which is a polynomial algebra \([8]\) and, therefore, admits only the trivial commutative deformation.

By the Kostant theorem \([8]\) \(U(g)\) is a free module over its center. It follows that at the point \(h = 0\) the module \((Sg)_{t,0}\) is a free module over the algebra \(I_{t,0}\). One can easily derive from this that \((Sg)_{t,h}\) is a free module over \(I_{t,h}\).

Now, let the orbit \(M\) be defined by invariant elements from \(I\). Consider a character defined by \(M\), the algebra homomorphism \(\lambda : I \to C\) which sends each element from \(I\) to its value on \(M\). Then, \(C\) may be considered as an \(I\)-module, and the function algebra \(A\) on \(M\) is equal to \(Sg/\text{Ker}(\lambda)Sg = Sg \otimes_I C\). Extend the character \(\lambda\) up to a character \(\lambda_{t,h} : I_{t,h} \to C[[h]][t]\) in the trivial way and consider \(C[[h]][t]\) as an \(I_{t,h}\)-module. The tensor product over \(I_{t,h}\)

\[ A_{t,h} = (Sg)_{t,h} \otimes C[[h]][t] \]

is a \(C[[h]][t]\)-algebra. It is a free \(C[[h]][t]\)-module, since \((Sg)_{t,h}\) is a free one over \(I_{t,h}\).

It is obvious, that \(A_{0,0} = A\), \(A_{t,0}\) gives a quantization of the KKS bracket on \(M\), and \(A_{t,h}\) is a quotient algebra of \((Sg)_{t,h}\).

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