A MINIMAX FRAMEWORK FOR QUANTIFYING RISK-FAIRNESS TRADE-OFF IN REGRESSION

BY EVGENII CHZHEN¹ AND NICOLAS SCHREUDER²

¹LMO, Université Paris-Saclay, CNRS, INRIA
²CREST, ENSAE, Institut Polytechnique de Paris

We propose a theoretical framework for the problem of learning a real-valued function which meets fairness requirements. Leveraging the theory of optimal transport, we introduce a notion of $\alpha$-relative (fairness) improvement of the regression function. With $\alpha = 0$ we recover an optimal prediction under Demographic Parity constraint and with $\alpha = 1$ we recover the regression function. For $\alpha \in (0, 1)$ the proposed framework allows to continuously interpolate between the two. Within this framework we precisely quantify the cost in risk induced by the introduction of the $\alpha$-relative improvement constraint. We put forward a statistical minimax setup and derive a general problem-dependent lower bound on the risk of any estimator satisfying $\alpha$-relative improvement constraint. We illustrate our framework on a model of linear regression with Gaussian design and systematic group-dependent bias. Finally, we perform a simulation study of the latter setup.

CONTENTS

1 Introduction ............................................................... 1
2 General setup ............................................................ 3
3 Minimax setup ............................................................ 9
4 Application to linear model with systematic bias ................. 11
5 Conclusion .................................................................. 16
A Reminder ...................................................................... 16
B Proofs for Section 2.1 .................................................. 18
C Proof of Theorem 3.3 .................................................... 21
D Proofs for Section 4 ....................................................... 21
References ..................................................................... 33

1. Introduction. Data driven algorithms are deployed in almost all areas of modern daily life and it becomes increasingly more important to adequately address the fundamental issue of historical biases present in the data (Barocas et al., 2019). The goal of algorithmic fairness is to bridge the gap between the statistical theory of decision making and the understanding of justice, equality, and diversity. The literature on fairness is broad and its volume increases day by day, we refer the reader to (Barocas et al., 2019, Mehrabi et al., 2019) for a general introduction on the subject and to (del Barrio et al., 2020, Oneto and Chiappa, 2020) for the review of the most recent theoretical advances.

Basically, the mathematical definitions of fairness are divided into two groups (Dwork et al., 2012): individual fairness and group fairness. The former notion reflects the principle that similar individuals must be treated similarly, which translates into the Lipschitz type constraints on possible prediction rule. The latter defines fairness on population level via (conditional)

Keywords and phrases: Algorithmic fairness, risk-fairness trade-off, regressions, Demographic Parity, least-squares, optimal transport, minimax analysis, statistical learning.
statistical independence of a prediction from a protected attribute (e.g., gender, ethnicity). A popular formalization of such notion is the Demographic Parity, initially introduced in the context of binary classification (Calders et al., 2009). Despite of some limitations (Hardt et al., 2016), demographic parity remains viable in a range of applied problems (Köeppen et al., 2014, Zink and Rose, 2019).

In this work we study the regression problem of learning a real-valued prediction function, which complies with an approximate notion of Demographic Parity while minimizing expected squared loss. Unlike its classification counterpart, the problem of fair regression has received far less attention in the literature. However, as argued by Agarwal et al. (2019), classifiers only provide binary decisions, while in practice final decisions are taken by humans based on prediction from the machine. In this case a continuous prediction is more informative than the binary one.

Related works. Until very recently, contributions on fair regression were almost exclusively focused on the practical incorporation of proxy fairness constraints in classical learning methods, such as random forest, ridge regression, kernel based methods to name a few (Berk et al., 2017, Calders et al., 2013, Fitzsimons et al., 2018, Komiyama and Shimao, 2017, Pérez-Suay et al., 2017, Raff et al., 2018). Several works empirically study the impact of (relaxed) fairness constraints on the risk (Bertsimas et al., 2012, Haas, 2019, Wick et al., 2019, Zafar et al., 2017, Zliobaite, 2015). Yet, the problem of precisely quantifying the effect of such constraints on the risk has not been tackled.

More recently, statistical and learning guarantees for fair regression were derived (Agarwal et al., 2019, Chiappa et al., 2020, Chzhen et al., 2020a,b, Fitzsimons et al., 2019, Le Gouic et al., 2020, Plečko and Meinshausen, 2019). The closest works to our contribution are that of Chiappa et al. (2020), Chzhen et al. (2020a), Le Gouic et al. (2020), who draw a connection between the problem of exactly fair regression of demographic parity and the multi-marginal optimal transport formulation (Agueh and Carlier, 2011, Gangbo and Święch, 1998). In particular, Chzhen et al. (2020a), Le Gouic et al. (2020) derive the form of optimal fair prediction, provide statistical guarantees on plug-in type estimators, and establish the exact value of the risk of the optimal fair prediction.

Organization and contributions. The main contributions of the present paper are the following.

- In Section 2, we propose a statistical framework for fair regression, which allows to continuously interpolate between unconstrained regression and regression under demographic parity constraint.
- In Section 2.1 we obtain a closed form expression for the optimal predictor under our statistical framework.
- In Section 3, we describe the minimax framework and derive a general problem-dependent minimax lower bound, quantifying the risk-fairness trade-off in Section 3.1.
- In Section 4, we apply our machinery to the problem of linear regression with systematic bias, deriving minimax upper and lower bounds, and support this part with empirical validation in Section 4.3.

The omitted proofs are postponed to appendix.

Notation. For any univariate probability measure \( \mu \) we denote by \( F_\mu \) (resp. \( F^{-1}_\mu \)) the cumulative distribution function (resp. the quantile function) of \( \mu \). For a probability measure \( \mu \) on \( \mathbb{R}^p \) and a measurable function \( g : \mathbb{R}^p \rightarrow \mathbb{R} \), we denote by \( g \# \mu \) the push-forward (image)
measure. For two random variables \( U, V \) we write \( U \overset{d}{=} V \) to denote their equality in distribution. We denote by \( \Delta^{K-1} \) the probability simplex in \( \mathbb{R}^K \). For any integer \( K \geq 1 \), we write \( [K] = \{1, \ldots, K\} \). For any \( a, b \in \mathbb{R} \) we denote by \( a \vee b \) (resp. \( a \wedge b \)) the maximum (resp. the minimum) between \( a, b \).

2. General setup. We study the regression problem when sensitive attribute is available. Let \( (X, S, Y) \sim \mathbb{P} \) on \( \mathbb{R}^p \times [K] \times \mathbb{R} \), where \( X \in \mathbb{R}^p \) is a feature vector, \( S \in [K] \) is some sensitive attribute, and \( Y \in \mathbb{R} \) is the target variable. Consider the following general regression model

\[
Y = f^*(X, S) + \xi ,
\]

where \( \xi \in \mathbb{R} \) is a centered random variable and \( f^* : \mathbb{R}^p \times [K] \rightarrow \mathbb{R} \) is the regression function. In this model the regression function \( f^*(X, S) = \mathbb{E}[Y|X, S] \) and the noise \( \xi = Y - \mathbb{E}[Y|X, S] \). We denote by \( w_s = \mathbb{P}(S = s) \) for all \( s \in [K] \), the marginal distribution of the sensitive attribute \( S \), by \( \mu_X \) the distributions of the feature vector \( X \), and by \( \mu_X|S \) the conditional distributions of \( X|S=s \). Throughout this work we assume that the random variable \( f^*(X, S) \) has a finite second moment. A prediction is any measurable function of the form \( f : \mathbb{R}^p \times [K] \rightarrow \mathbb{R} \) and its population risk is measured by the \( L_2 \) distance to the regression function \( f^* \), that is, the risk is defined as

\[
\mathcal{R}(f) := \| f - f^* \|_2^2 := \mathbb{E}(f(X, S) - f^*(X, S))^2 .
\]

(Risk Measure)

A prediction rule \( f : \mathbb{R}^p \times [K] \rightarrow \mathbb{R} \) is said to be (exactly) fair in the sense of Demographic Parity if for any \( s, s' \in [K] \) it holds that

\[
(f(X, S) | S = s) \overset{d}{=} (f(X, S) | S = s') ,
\]

(DP)

that is, the prediction is fair if it is statistically independent from the protected attribute. The notion of Demographic Parity as a way to impose exact fairness in the regression setup has been studied in several papers (see Barocas et al. (2019), Oneto and Chiappa (2020) and references therein).

Unless \( f^* \) is already fair, restricting ourselves to predictions satisfying DP incurs an unavoidable price in terms of the risk (Le Gouic et al., 2020). Depending on the application at hand, this price might or might not be reasonable. However, since the notion of Demographic Parity (exact fairness) is completely fairness driven, it does not allow to quantify the price of considering “fairer” predictions than the regression function \( f^* \). For this reason, several contributions relax this constraint, forcing a milder fairness requirement. A natural idea is to define a functional \( \mathcal{U} \) which quantifies the violation of the DP and to declare a prediction approximately fair if this functional does not exceed a user pre-specified threshold. In recent years a large variety of such relaxations has been proposed: correlation based (Baharlouei et al., 2019, Komiyama et al., 2018, Mary et al., 2019); Kolmogorov-Smirnov distance (Agarwal et al., 2019); Mutual information (Steinberg et al., 2020a,b); Total Variation distance (Oneto et al., 2019a,b); Equality of means and higher moment matching (Berk et al., 2017, Calders et al., 2013, Donini et al., 2018, Fitzsimons et al., 2019, Ollat et al., 2020, Raff et al., 2018); Maximum Mean Discrepancy (Madras et al., 2018, Quadrianto and Sharmanska, 2017); Wasserstein distance (Chiappa et al., 2020, Chzhen et al., 2020a, Gordaliza et al., 2019, Le Gouic et al., 2020). The following two dominate the literature:
1. Total Variation (TV) relaxation:
\[ U_{TV}(f) := \max_{s, s' \in [K]} \sup_{C \subset \mathbb{R}} |\mathbb{P}(f(X, S) \in C \mid S = s) - \mathbb{P}(f(X, S) \in C \mid S = s')| \leq \varepsilon . \]  
(2)

2. Kolmogorov-Smirnov (KS) relaxation:
\[ U_{KS}(f) := \max_{s, s' \in [K]} \sup_{t \in \mathbb{R}} |\mathbb{P}(f(X, S) \leq t \mid S = s) - \mathbb{P}(f(X, S) \leq t \mid S = s')| \leq \varepsilon . \]  
(3)

Importantly, for \( \varepsilon = 0 \) both relaxations reduce to the exact fairness constraint and for \( \varepsilon > 0 \) these formulations allow some slack. Motivated by recent advances in understanding of the connections between fairness and optimal transport, we will consider the relaxation of the exact fairness constraint using the Wasserstein-2 distance. We recall that the Wasserstein-2 distance between probability distributions \( \mu \) and \( \nu \) in \( \mathcal{P}_2(\mathbb{R}^d) \), the space of measures on \( \mathbb{R}^d \) with finite second moment, is defined as
\[ W_2^2(\mu, \nu) := \inf_{\gamma \in \Gamma(\mu, \nu)} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\|^2_2 d\gamma(x, y) \right\} , \]
where \( \Gamma(\mu, \nu) \) denotes the collection of measures on \( \mathbb{R}^d \times \mathbb{R}^d \) with marginals \( \mu \) and \( \nu \). See Santambrogio (2015), Villani (2003) for more details about Wasserstein distances and optimal transport. We define the unfairness of a prediction rule \( f : \mathbb{R}^p \times [K] \rightarrow \mathbb{R} \) by
\[ U(f) := \min_{\nu \in \mathcal{P}_2(\mathbb{R})} \sum_{s=1}^K w_s W_2^2(f(\cdot, s)\#\mu_X|s, \nu) . \]  
(4)

The functional \( U \) measures how far the group conditional distributions of a prediction are from their common barycenter. The bigger it is the more predictions differ from one group to another and vice-versa. Note that similarly to the case of the TV and KS distances the constraint \( U(f) = 0 \) recovers the definition of the \textbf{DP}. The main motivation to consider \( U \) instead of \( U_{TV} \) or \( U_{KS} \) relaxations is the following result, which provides an intrinsic connection between the risk \( R \) and the introduced unfairness measure \( U \).

**Theorem 2.1 (Chzhen et al. (2020a), Le Gouic et al. (2020)).** Assume that, for any \( s \in [K] \), \( f^*(\cdot, s)\#\mu_X|s \) is atomless. Then
\[ U(f^*) = \min \left\{ R(f) : f(X, S) \mid S = s \overset{d}{=} f(X, S) \mid S = s' \ \forall s, s' \in [K] \right\} . \]  
(5)

Moreover, the distribution of the minimizer of the above problem is given by the solution of
\[ \min_{\nu \in \mathcal{P}_2(\mathbb{R})} \sum_{s=1}^K w_s W_2^2(f(\cdot, s)\#\mu_X|s, \nu) . \]

An important consequence of Theorem 2.1 is that it puts the risk \( R \) and the unfairness \( U \) – two conflicting quantities, on the same scale. In particular, it allows to measure both fairness and risk using the same unit measurements, hence study the trade-off between the two. In order to build our framework, we remark that the problem on the right hand side of Eq. (5) can be equivalently written as
\[ \min \{ R(f) : U(f) \leq 0 \times U(f^*) \} . \]
We consider a natural relaxation of the above formulation and say that $f$ is an $\alpha$-relative improvement of the regression function $f^*$ if

$$U(f) \leq \alpha U(f^*) \ .$$

Consequently, for a fixed parameter of choice $\alpha \in [0, 1]$, we define the best $\alpha$-relative improvement of the regression function $f^*$ as

$$f^*_\alpha \in \arg \min \{ R(f) : U(f) \leq \alpha U(f^*) \} \ , \tag{6}$$

For instance, with $\alpha = 1/2$ the corresponding $1/2$-relative improvement $f^*_{1/2}$ halves the unfairness of $f^*$ and has the lowest risk among such predictions. Note that with $\alpha = 0$ we recover the formulation considered in Chzhen et al. (2020a), Le Gouic et al. (2020), that is, $f^*_0$ is the fair optimal prediction in the sense of Demographic Parity and Theorem 2.1 states that $R(f^*_0) = U(f^*)$. Because of the last relation Le Gouic et al. (2020) refer to $U(f^*)$ as to the price of fairness. Meanwhile, if $\alpha = 1$, then $f^*_1 = f^*$ and the problem is equivalent to the standard regression setup without any constraints. Thus, the $\alpha$-relative improvement $f^*_\alpha$ continuously interpolates between the standard regression problem and the problem of exactly fair regression.

Another appealing feature of this framework is the fact that the fairness is measured relatively to the fairness of the optimal prediction $f^*$, which makes it easier to fix the desired level of fairness $\alpha$.

2.1. Properties of $\alpha$-relative improvement. This section is devoted to the study of the $\alpha$-relative improvement $f^*_\alpha$. In particular, the next result establishes a closed form solution to Problem (6) under mild assumptions.

**Proposition 2.2.** Assume that for each $s \in [K]$ the measure $a_s := f^*(\cdot, s)\#\mu_{X|s}$ is atomless, then for all $\alpha \in [0, 1]$ and all $(x, s) \in \mathbb{R}^p \times [K]$ (up to a set of null measure) it holds that

$$f^*_\alpha(x, s) = \sqrt{\alpha} f^*(x, s) + \left(1 - \sqrt{\alpha}\right) \sum_{s' = 1}^{K} w_{s'} F^{-1}_{a_{s'}} \circ F_{a_s} \circ f^*(x, s)$$

$$= \sqrt{\alpha} f^*_1(x, s) + \left(1 - \sqrt{\alpha}\right) f^*_0(x, s) \ .$$

Recall that $f^* = f^*_1$, hence the $\alpha$-relative improvement $f^*_\alpha$ is the point-wise convex combination of exactly fair prediction $f^*_0$ and the regression function $f^*_1$. Besides, setting $\alpha = 0$ we recover the result of Chzhen et al. (2020a), Le Gouic et al. (2020) as a particular case of our framework. Let us also emphasize an attractive group-wise order preserving property of $f^*_\alpha$. Fix some $s \in [K], x, x' \in \mathbb{R}^p$, we remark that if $f^*(x, s) \geq f^*(x', s)$, then for any $\alpha \in [0, 1]$ it holds that $f^*_\alpha(x, s) \geq f^*_\alpha(x', s)$. For the special case of $\alpha = 0$, this observation has already been made in (Chzhen et al., 2020a). The proof of Proposition 2.2 relies on a general geometric Lemma 2.4 interesting on its own. First, let us introduce the following definition, which asks for existence of finitely supported barycenters in a metric space $(X, d)$.

**Definition 2.3.** We say that a metric space $(X, d)$ satisfies barycenter property if for any weights $w \in \Delta^{K-1}$ and tuple $a = (a_1, \ldots, a_K) \in X^K$ there exists a barycenter

$$C_a \in \arg \min_{C \in X} \sum_{s=1}^{K} w_s d^2(a_s, C) \ .$$

\footnote{When there is no ambiguity, we call $f^*_\alpha$ the $\alpha$-relative improvement instead of the best $\alpha$-relative improvement.}
FIG 1. Geometric lemma. Initial points $a_1, a_2, a_3$ are the vertices of the isosceles triangle. Weights: $w_1 = 0.1, w_2 = 0.4, w_3 = 0.5$

**Lemma 2.4 (General geometric lemma).** Let $(X, d)$ be a metric space satisfying the barycenter property. Let $a = (a_1, \ldots, a_K) \in X^K$, $w = (w_1, \ldots, w_K)^\top \in \Delta^{K-1}$ and let $C_a$ be a barycenter of $a$ with respect to weights $w$. For a fixed $\alpha \in [0, 1]$ assume that there exists $b = (b_1, \ldots, b_K) \in X^K$ which satisfies

\[
d(a_s, C_a) = d(a_s, b_s) + d(b_s, C_a), \quad s = 1, \ldots, K, \quad \text{(P_1)}
\]
\[
d(b_s, a_s) = (1-\sqrt{\alpha})d(a_s, C_a), \quad s = 1, \ldots, K. \quad \text{(P_2)}
\]

Then, $b$ is a solution of

\[
\inf_{b \in X^K} \left\{ \sum_{s=1}^{K} w_s d^2(b_s, a_s) : \sum_{s=1}^{K} w_s d^2(b_s, C_b) \leq \alpha \sum_{s=1}^{K} w_s d^2(a_s, C_a) \right\}.
\]

**Remark 2.5.** Property (P_1) essentially requires that each $b_i$ lies on the geodesic between $a_i$ and $C_a$ while Property (P_2) specifies the location of $b_i$ on this geodesic: $b_i$ should be $(1-\sqrt{\alpha})$ times closer to $a_i$ than $C_a$ to $a_i$. An illustration provided on Figure 1 describes these properties in Euclidean geometry. For general case, the straight lines should be replaced by geodesics.

The setting of Lemma 2.4 is quite general and only requires existence of barycenters (also known as the Fréchet means) for any weighted finite combination of points in accordance with Definition 2.3. For our purposes, Lemma 2.4 will be applied for $(X, d) = (P_2(\mathbb{R}), W_2)$, we refer to (Agueh and Carlier, 2011, Le Gouic and Loubes, 2017) who investigate and prove the existence of Wasserstein barycenters of random probabilities defined on a geodesic spaces.

**Proof of Lemma 2.4.** Fix some $a = (a_1, \ldots, a_K) \in X^K$, $w = (w_1, \ldots, w_K)^\top \in \Delta^{K-1}$ and let $C_a$ be a barycenter of $a$ with respect to weights $w$. Fix $\alpha \in [0, 1]$ and any $b = (b_1, \ldots, b_K) \in X^K$ which satisfies properties (P_1)--(P_2). Let $b_k = (b_{1k}, \ldots, b_{Kk}) \in X^K$ be a minimizing sequence of the problem (7) and for any $b' = (b'_{1}, \ldots, b'_{K}) \in X^K$ denote by $G(b') = \sum_{s=1}^{K} w_s d^2(b'_{s}, a_s)$ the objective function of the problem (7). Then, by the definition
of a minimizing sequence, the following two properties hold
\[
\lim_{k \to \infty} G(b_k) = \inf_{b \in \mathcal{K}} \left\{ G(b) : \sum_{s=1}^{K} w_s d^2(b_s, C_b) \leq \alpha \sum_{s=1}^{K} w_s d^2(a_s, C_a) \right\}, \quad (8)
\]
\[
\sum_{s=1}^{K} w_s d^2(b_s, C_b) \leq \alpha \sum_{s=1}^{K} w_s d^2(a_s, C_a), \quad \forall k \in \mathbb{N}. \quad (9)
\]
Furthermore, using properties \((P_1)-(P_2)\) we deduce that
\[
\sum_{s=1}^{K} w_s d^2(b_s, C_b) \leq \sum_{s=1}^{K} w_s d^2(b_s, C_a) \leq \alpha \sum_{s=1}^{K} w_s d^2(a_s, C_a),
\]
where \((a)\) follows from Lemma B.3 in appendix. Therefore, \(b = (b_1, \ldots, b_s) \in \mathcal{X}^K\) is feasible for the problem \((7)\).

By Lemma B.2 it holds for all \(k \in \mathbb{N}\) that
\[
\left\{ \sum_{s=1}^{K} w_s d^2(a_s, C_{b_k}) \right\}^{1/2} \leq \left\{ \sum_{s=1}^{K} w_s d^2(a_s, b_s^k) \right\}^{1/2} + \left\{ \sum_{s=1}^{K} w_s d^2(b_s^k, C_{b_k}) \right\}^{1/2}.
\]
We continue using the definition of \(C_a\) and Eq. \((9)\) to obtain for all \(k \in \mathbb{N}\)
\[
\left\{ \sum_{s=1}^{K} w_s d^2(a_s, C_a) \right\}^{1/2} \leq \left\{ \sum_{s=1}^{K} w_s d^2(a_s, C_{b_k}) \right\}^{1/2} \leq G^{1/2}(b_k) + \sqrt{\alpha} \left\{ \sum_{s=1}^{K} w_s d^2(a_s, C_a) \right\}^{1/2},
\]
which after rearranging implies that
\[
(1 - \sqrt{\alpha}) \left\{ \sum_{s=1}^{K} w_s d^2(a_s, C_a) \right\}^{1/2} \leq G^{1/2}(b_k), \quad \forall k \in \mathbb{N}.
\]
Finally, using property \((P_2)\) we derive that
\[
G(b) \leq G(b_k), \quad \forall k \in \mathbb{N}.
\]
Recall that we have already shown that \(b\) is feasible for the problem \((7)\), hence taking the limit \(w.r.t.\) to \(k\) concludes the proof of Lemma 2.4.

The complete proof of Proposition 2.2 is omitted in the main body. We only provide a short intuition.

**Sketch of the proof.** The idea of the proof is to apply Lemma 2.4 with \((\mathcal{X}, d) = (\mathcal{P}_2(\mathbb{R}), W_2)\) and with measures
\[
a_s := f^*(\cdot, s) \# \mu_{\mathcal{X}|s} \in \mathcal{P}_2(\mathbb{R}).
\]
Fig 2. Risk $R$ and unfairness $U$ of $f^*_\alpha$. Green curves (decreasing, convex) correspond to the risk, while orange curves (increasing, linear) correspond to the unfairness. Each pair of curves (solid, dashed, dashed dotted) corresponds to three regimes: high, moderate, and low unfairness of the regression function $f^*$ respectively.

Then, we need to construct measures $b = (b_1, \ldots, b_K)^T \in \mathcal{P}_2^K(\mathbb{R})$, which satisfy the properties $(P_1)$–$(P_2)$. To this end, let $\gamma_s$ be the (constant-speed) geodesic between $a_s$ and $C_a$ i.e., $\gamma_s(0) = a_s$, $\gamma_s(1) = C_a$. We define $b_s := \gamma_s(1-\sqrt{\alpha})$ for $s \in [K]$, similarly to the intuition provided by Figure 1. One can verify that that $b = (b_s)_{s \in [K]}$ satisfies $(P_1)$ and $(P_2)$. Then, by Lemma 2.4 we know that $b$ solves the minimization problem in Eq. (7). For the final part of the proof we propagate the optimality of $b$ in the space of distributions to the optimality of $f^*_\alpha$ in the space of predictions using the assumption on $a$ and an explicit construction of the geodesic $\gamma_s$.

The next key result of our framework establishes the fairness-risk trade-off provided by the parameter $\alpha \in [0, 1]$ on the population level. In particular, it establishes a simple user-friendly relation between the risk and unfairness of $\alpha$-relative improvement. Note that such a result is not available neither for $U_{TV}$ nor for $U_{KS}$, due to the fundamentally different geometry of the squared risk and the aforementioned distances.

**Lemma 2.6.** Assume that for each $s \in [K]$ the measure $f^*(\cdot, s)\#\mu_X|s$ is atomless, then for any $\alpha \in [0, 1]$ it holds that

$$R(f^*_\alpha) = (1-\sqrt{\alpha})^2 R(f^*_0) = (1-\sqrt{\alpha})^2 U(f^*) .$$

**Proof.** Proposition 2.2 gives the following explicit expression for the best $\alpha$-improvement of $f^*$:

$$f^*_\alpha(x, s) = \sqrt{\alpha} f^*(x, s) + (1-\sqrt{\alpha}) f^*_0(x, s) .$$

Plugging it in the risk gives

$$R(f^*_\alpha) = \|f^*_\alpha - f^*\|^2_2 = (1-\sqrt{\alpha})^2 \|f^*_0 - f^*\|^2_2 = (1-\sqrt{\alpha})^2 R(f^*_0) .$$

This proves the first equality. Given the definition of $f^*_0$, the second equality is exactly the result stated in Theorem 2.1.
Recall that thanks to Theorem 2.1 we have that $\mathcal{R}(f^*_\alpha) = \mathcal{U}(f^*)$. Hence, the $\alpha$-relative improvement $f^*_\alpha$ enjoys the following two properties

$$
\mathcal{R}(f^*_\alpha) = (1-\sqrt{\alpha})^2 \mathcal{R}(f^*_0) \quad \text{and} \quad \mathcal{U}(f^*_\alpha) = \alpha \mathcal{U}(f^*) .
$$

For instance, if $\alpha = 1/2$, that is, we want to half the unfairness of $f^*$, it incurs the risk which is equal to approximately 8.5% of the risk of exactly fair predictor $f^*_0$. We illustrate this general behaviour on Figure 2, where the risk and the unfairness of $f^*_\alpha$ are shown for different levels of $\mathcal{U}(f^*)$. A striking observation we can make from this plot is that the risk of $f^*_\alpha$ growth rapidly in the vicinity of zero, while behaves almost linearly in a large neighbourhood of one. That is, one can reduce the unfairness of $f^*$ by a constant factor without large increase in risk.

**Remark 2.7.** Let us remark that the results of this section apply also in the case when the risk is given by

$$
\mathcal{R}(f) = \sum_{s=1}^K w_s \mathbb{E}[(f(X,S) - f^*(X,S))^2 | S = s] ,
$$

with arbitrary probability vector $w = (w_1, \ldots, w_K)^\top$. It can be potentially useful for applications where the group-wise risks must be re-weighted. For instance, one can consider uniform weights $w = (1/K, \ldots, 1/K)^\top$ or weights which are proportional to $1/\mathbb{P}(S = s)$.

3. Minimax setup. While previous section was dealing with the general framework on the population level, the goal of this section is to put forward a minimax setup for the statistical problem of regression with the introduced fairness constraints.

Let $(X_1, S_1, Y_1), \ldots, (X_n, S_n, Y_n)$ be i.i.d. sample with joint distribution $P_{(f^*, \theta)}$, where the pair $(f^*, \theta) \in \mathcal{F} \times \Theta$ for some class $\mathcal{F}$ and $\Theta$. In this notation $f^*$ is the regression function and $\theta$ is a nuisance parameter. For example $\mathcal{F}$ can be a set of all affine or Lipschitz continuous functions and $\Theta$ defines additional assumptions on the model in Eq. (1) (see Section 4 for a concrete example). For a given fairness parameter $\alpha \in [0, 1]$ and a given confidence parameter $t > 0$, the goal of the statistician is to construct an estimator $\hat{f}$, which simultaneously satisfies the following two properties

1. Uniform fairness guarantee:

$$
\forall (f^*, \theta) \in \mathcal{F} \times \Theta \quad P_{(f^*, \theta)} \left( \mathcal{U}(\hat{f}) \leq \alpha \mathcal{U}(f^*) \right) \geq 1 - t .
$$

2. Uniform risk guarantee:

$$
\forall (f^*, \theta) \in \mathcal{F} \times \Theta \quad P_{(f^*, \theta)} \left( \mathcal{R}(\hat{f}) \leq r_{n,\alpha,f^*}(\mathcal{F}, \Theta, t) \right) \geq 1 - t .
$$

Eq. (11) states that the constructed estimator satisfies the fairness requirement with high probability uniformly over the class $\mathcal{F} \times \Theta$. Meanwhile, in Eq. (12) we seek for the rate $r_{n,\alpha,f^*}(\mathcal{F}, \Theta, t)$ being as small as possible to quantifying the statistical price for being $\alpha$-relatively fair. Note that $r_{n,\alpha,f^*}(\mathcal{F}, \Theta, t)$ depends explicitly on $f^*$. This is explained by the fact that the fairness of $\hat{f}$ is measured relatively to $f^*$, hence the price of this constraint also depends on the initial unfairness level of the regression function $f^*$.

The actual construction of the estimator $\hat{f}$ is problem dependent and the proof of Eqs. (11)–(12) requires a careful case-by-case study. In Section 4 we provide an example of the analysis for a simple statistical model of linear regression with systematic group-dependent bias.
3.1. Generic lower bound. While the upper bounds of Eqs. (11)–(12) require a problem dependent analysis, a general problem dependent lower bound can be derived. In this section we develop such lower bound. Let us first introduce some useful definitions.

**Assumption 3.1 (Unconstrained rate).** For a fixed confidence level \( t \in (0, 1) \) and a class \( \mathcal{F} \times \Theta \), there exists positive sequence \( \delta_n(\mathcal{F}, \Theta, t) \) such that

\[
\inf_{f} \sup_{(f^*, \theta) \in \mathcal{F} \times \Theta} \mathbb{P}_{(f, \theta)} \left( \mathcal{R}(\hat{f}) \geq \delta_n(\mathcal{F}, \Theta, t) \right) \geq t ,
\]

where the infimum is taken over all estimators.

Assumption 3.1 can be used with any sequence \( \delta_n(\mathcal{F}, \Theta, t) \), however, we implicitly assume that \( \delta_n(\mathcal{F}, \Theta, t) \) corresponds to the minimax optimal rate of estimation of \( f^* \) by any estimator (without constraints) in expected squared loss.

**Definition 3.2 (Valid estimators).** For some \( \alpha \in [0, 1] \), confidence level \( t' \in (0, 1) \) we say that an estimator \( \hat{f} \) is \((\alpha, t')\)-valid w.r.t. \((\mathcal{F}, \Theta)\) if

\[
\inf_{(f^*, \theta) \in \mathcal{F} \times \Theta} \mathbb{P}_{(f, \theta)} \left( \mathcal{U}(\hat{f}) \leq \alpha \mathcal{U}(f^*) \right) \geq 1 - t' .
\]

The set of all \((\alpha, t')\)-valid estimators w.r.t. \((\mathcal{F}, \Theta)\) is denoted by \( \hat{\mathcal{F}}_{(\alpha, t')} \).

Definition 3.2 introduces the estimators which satisfy the constraint of \( \alpha \)-relative improvement at least with constant probability uniformly over the \( \mathcal{F} \times \Theta \). Hence, these are the estimators which possess desirable fairness requirements.

Equipped with Assumption 3.1 and Definition 3.2 we are in position to state the main result of this section, which establishes the statistical risk-fairness trade-off. As we will see in Section 4, supported by appropriate upper bounds, this bound yields optimal rates of convergence up to a multiplicative factor.

**Theorem 3.3.** Let \( \delta_n(\mathcal{F}, \Theta, t) \) be the sequence that satisfies Assumption 3.1, then

\[
\inf_{f \in \hat{\mathcal{F}}_{(\alpha, t')}} \sup_{(f^*, \theta) \in \mathcal{F} \times \Theta} \mathbb{P}_{(f, \theta)} \left( \mathcal{R}^{1/2}(\hat{f}) \geq \delta_n^{1/2}(\mathcal{F}, \Theta, t) \lor (1-\sqrt{\alpha}) \mathcal{U}^{1/2}(f^*) \right) \geq t \land (1 - t') .
\]

Drawing an analogy with Lemma 2.6, the two terms of the derived bound have natural interpretation: the first term \( \delta_n(\mathcal{F}, \Theta, t) \) is the price of statistical estimation; the second term \((1-\sqrt{\alpha}) \mathcal{U}^{1/2}(f^*) \) is the price of fairness. Consequently, the rate \( r_{n, \alpha, f^*}(\mathcal{F}, \Theta, t) \) in Eq. (12) is lower bounded (up to a multiplicative constant factor) by \( \delta_n^{1/2}(\mathcal{F}, \Theta, t) \lor (1-\sqrt{\alpha}) \mathcal{U}^{1/2}(f^*) \). The confidence parameter on the r.h.s. of the bound is \( t \land (1 - t') \). The reasonable choice of \( t' \) is in the vicinity of zero, which corresponds to estimators satisfying the fairness constraint with high probability. Finally, observe that this bound in not conventional in the sense of classical statistics, where the bound should converge to zero with the growth of sample size. This behavior is not surprising, since the infimum is taken w.r.t. to \((\alpha, t')\)-valid estimators and not w.r.t. all possible estimators. One can draw an analogy of the obtained bound with recent results in robust statistics (Chen et al., 2016, 2018), where the minimax rate converges to the function of the proportion of outliers, different from zero.
4. Application to linear model with systematic bias.

Additional notation. We denote by \( \| \cdot \|_2 \) and by \( \| \cdot \|_n = (1/\sqrt{n}) \| \cdot \|_2 \) the Euclidean and the normalized Euclidean norm. The standard scalar product is denoted by \( \langle \cdot, \cdot \rangle \). We denote by \( 1_p \) the vector of all ones of size \( p \). For a Boolean valued function \( P \) we denote by \( \mathbb{I}\{P\} \) the indicator of \( P \). For a square matrix \( A \in \mathbb{R}^{n,n} \), we write \( A \succ 0 \) if for any \( x \in \mathbb{R}^n \setminus \{0\} \), we have \( x^T A x > 0 \).

The goal of this part is to demonstrate that the results of Section 3.1 provide minimax rate optimal bound. To this end we apply the developed theory to the following model of linear regression with systematic group-dependent bias

\[
Y = \langle X, \beta^* \rangle + b_s^* + \xi \, ,
\]  

(13)

where \( X \sim \mathcal{N}(0, \Sigma) \) is a feature vector independent from the sensitive attribute \( S \) with \( \Sigma \succ 0 \); \( \xi \sim \mathcal{N}(0, \sigma^2) \) is an additive independent noise; and the vector \( b^* = (b_1^*, \ldots, b_K^*) \) is the vector of systematic bias. We assume that the noise level \( \sigma \) is known to the statistician. Note that in this case the regression function \( f^* \) is given by the expression \( f^*(x, s) = \langle x, \beta^* \rangle + b_s^* \). We assume that the observations are

\[
Y_s = X_s \beta^* + b_s^* 1_{n_s} + \xi_s, \quad s = 1, \ldots, K ,
\]

(14)

with \( Y_s, \xi_s \in \mathbb{R}^{n_s}, X_s \in \mathbb{R}^{n_s \times p} \), and \( 1_{n_s} \) is the vector of all ones of size \( n_s \). The rows of \( X_s \) are i.i.d. realization of \( X \), the components of \( \xi_s \) are i.i.d. from \( \mathcal{N}(0, \sigma^2) \). Additionally, we set \( n = n_1 + \ldots + n_K \) and \( w_s = n_s/n \). The risk of a prediction rule \( f : \mathbb{R}^p \times [K] \to \mathbb{R} \) is defined as

\[
\mathcal{R}(f) = \sum_{s=1}^K w_s \mathbb{E} \left( \langle X, \beta^* \rangle + b_s^* - f(X, s) \right)^2 .
\]

Remark 4.1. We set \( w_s = n_s/n \) instead of \( w_s = \mathbb{P}(S = s) \) to simplify the presentation and proofs of the main results. Thanks to Remark 2.7, all of the statements of Sections 2-3 are applied for this choice. Finally, note that if \( \mathbb{P}(S = s) = w'_s \) and \( S_1, \ldots, S_n \) is an i.i.d. sample, then \( n_s = \sum_{i=1}^n \mathbb{I}\{S_i = s\} \) and \( \mathbb{E}[n_s/n] = w'_s \), that is our choice of weights essentially corresponds to the scenario of i.i.d. sampling of sensitive attribute.

Using the terminology of Section 3.1 the joint distribution of data sample \( P_{(f^*, \theta)} \) is uniquely defined by \( (\beta^*, b^*) \) and \( (\Sigma, \sigma) \). That is, \( (\beta^*, b^*) \) defines the regression function \( f^* \) and \( (\Sigma, \sigma) \) is the nuisance parameter \( \theta \). To simplify the notation we write \( P_{(\beta^*, b^*)} \) instead of \( P_{(\beta^*, b^*, \Sigma, \sigma)} \).

The following result is the application of Theorem 2.2 to the model in Eq. (13).

Proposition 4.2. For all \( \alpha \in [0, 1] \), the \( \alpha \)-relative improvement of \( f^* \) is given for all \( (x, s) \in \mathbb{R}^p \times [K] \) by

\[
f^*_\alpha(x, s) = \langle x, \beta^* \rangle + \sqrt{\alpha} b_s^* + (1 - \sqrt{\alpha}) \sum_{s=1}^K w_s b_s^* .
\]

In order to build an estimator \( \hat{f} \), which improves the fairness of \( f^* \), while providing minimal risk among such predictions, we first estimate parameters of model in Eq. (13) using least-squares estimators

\[
(\hat{\beta}, \hat{b}) \in \arg\min_{(\beta,b) \in \mathbb{R}^p \times \mathbb{R}^K} \sum_{s=1}^K w_s \| Y_s - X_s \beta - b_s 1_{n_s} \|^2_{n_s} .
\]

(15)
Based on the above quantities we define a family of linear estimators $\hat{f}_\tau$ parametrized by $\tau \in [0, 1]$ as

$$\hat{f}_\tau(x, s) = \langle x, \hat{\beta} \rangle + \sqrt{\tau} \hat{b}_s + (1-\sqrt{\tau}) \sum_{s'=1}^K w_{s'} \hat{b}_{s'}, \quad (x, s) \in \mathbb{R}^p \times [K].$$  \hspace{1cm} (16)

We would like to find a value of $\tau = \tau_n(\alpha)$ such that Eqs. (11)–(12) are satisfied. Note that the choice of $\tau = \alpha$ would not yield the desired fairness guarantee stated in Eq. (11). As it will be show later, $\tau$ should be smaller than $\alpha$, in order to account for finite sample effects and derive high confidence fairness guarantee. The next result shows that under the model in Eq. (13), the unfairness of $\hat{f}_\tau$ can be computed in a data-driven manner, which is crucial for the consequent choice of $\tau$.

**Lemma 4.3.** For any $\tau \in [0, 1]$, the unfairness of $\hat{f}_\tau$ is given by

$$\mathcal{U}(\hat{f}_\tau) = \tau \sum_{s=1}^K w_s \left( \hat{b}_s - \frac{\sum_{s'=1}^K w_{s'} \hat{b}_{s'}}{\sum_{s'=1}^K w_{s'}} \right)^2,$$

almost surely.

Apart from being computable in practice, Lemma 4.3 provides an intuitive result that $\mathcal{U}(\hat{f}_\tau)$ is the variance of the bias term $\hat{b}$.

### 4.1. Upper bound

Linear regression is one of the most well-studied problems of statistics (Audibert and Catoni, 2011, Catoni, 2004, Györfi et al., 2006, Hsu et al., 2012, Mourtada, 2019, Nemirovski, 2000, Tsybakov, 2003). In the context of fairness, linear regression is considered in (Berk et al., 2017, Calders et al., 2013, Donini et al., 2018), where the fairness constraint formulated via the approximate equality of group-wise means. In this section we establish a statistical guarantee on the risk and fairness of $\hat{f}_\tau$ for an appropriate data-driven choice of $\tau$. Our theoretical analysis in this part is inspired by that of Hsu et al. (2012), who derived high probability bounds on least squares estimator for linear regression with random design.

The following rate plays a crucial role in the analysis of this section

$$\delta_n(p, K, t) = 8 \left( \frac{p}{n} + \frac{K}{n} \right) + 16 \left( \sqrt{\frac{p}{n}} + \sqrt{\frac{K}{n}} \right) \sqrt{\frac{t}{n}} + \frac{32t}{n}.$$

Not taking into account the confidence parameter $t > 0$, $\delta_n(p, K, t) \asymp (p + K)/n$ up to a constant multiplicative factor, which as it is shown in Theorem 4.5 is the minimax optimal rate for the model in Eq. (13) without the fairness constraint.

**Theorem 4.4 (Fairness and risk upper bound).** Define

$$\hat{\tau} = \begin{cases} 
\alpha \left( 1 + \frac{\sigma_{n^{1/2}}(p, K, t)}{\mathcal{U}^{1/2}(\hat{f}_1) - \sigma_{n^{1/2}}^2(p, K, t)} \right)^{-2} & \text{if } \mathcal{U}^{1/2}(\hat{f}_1) > \sigma_{n^{1/2}}(p, K, t), \\
0, & \text{otherwise}
\end{cases}$$

Consider $p, K \in \mathbb{N}, t \geq 0$ and define $\gamma(p, K, t) = (4\sqrt{p} + 5\sqrt{t} + 6\sqrt{p})/(\sqrt{p} + \sqrt{t})$. Assume that $\sqrt{n} \geq 2(\sqrt{p} + \sqrt{t})/(\gamma(p, K, t) - \sqrt{\gamma(p, K, t) - 3})$. Then, for any $\alpha \in [0, 1]$, with probability at least $1 - 4 \exp(-t/2)$ it holds that

$$\mathcal{U}(\hat{f}_\tau) \leq o(\mathcal{U}(f^*)) \quad \text{and} \quad \mathcal{R}^{1/2}(\hat{f}_\tau) \leq 2\sigma(1+\sqrt{\alpha})\delta_n^{1/2}(p, K, t) + (1-\sqrt{\alpha})\mathcal{U}^{1/2}(f^*).$$
Theorem 4.4 simultaneously provides two results: first, it shows that the estimator \( \hat{f} \) is \((\alpha, 4e^{-t/2})\)-valid, that is, it satisfies the fairness constraint with high probability; second it provides the rate of convergence which consists of two parts. The first part of the rate \( \sigma_n^{1/2}(p, K, t) \) is the price of statistical estimation of \((\beta^*, b^*)\), while the second part \((1−\sqrt{\alpha})U^{1/2}(f^*) \) is the price for satisfaction of the fairness constraint. In order to achieve the fairness validity, we need to loosen the value of \( \alpha \) to reflect the base level of unfairness, that is, \( \hat{\tau} \) is adjusted by \( U(\hat{f}_1) \). Let us point out that the bound of Theorem 4.4 slightly differs from the conditions required by Eqs. (11)–(12). In particular, it provides joint guarantee on risk and fairness.

Let us remark that the previous result requires \( n \) to be sufficiently large, similarly to the conditions in (Audibert and Catoni, 2011, Hsu et al., 2012). One can obtain a more explicit, but more restrictive bound on \( n \) by finding sufficient conditions under which the assumption on \( n \) is satisfied. For instance, rough computations show that it is sufficient to assume that \( \sqrt{n} \geq 16\sqrt{K} \) and \( \sqrt{n} \geq 12.5(\sqrt{\bar{p}} + \sqrt{t}) \).

Besides, we emphasize that the choice of \( \hat{\tau} \) requires the knowledge of the noise level \( \sigma \), that is, this choice is not adaptive. However, our proof can effortlessly be extended to the case when only an upper bound \( \bar{\sigma} \) on the noise level \( \sigma \) is known. In this case \( \sigma \) should be replaced by \( \bar{\sigma} \) in the definition of \( \hat{\tau} \) and in the resulting rate. The question of adaptation to \( \sigma \) without any prior knowledge should be treated separately and is out of the scope of this work.

4.2. Lower bound. The goal of this section is to provide a lower bound, demonstrating that the result of Theorem 4.4 is minimax optimal up to a multiplicative constant factor. Recall that thanks to the general lower bound derived in Theorem 3.3 it is sufficient to prove a lower bound on the risk without constraining the set of possible estimators. Even though the problem of linear regression is well studied, to the best of our knowledge there is no known lower bound for the model in Eq. (13) which \( i) \) holds for the random design \( ii) \) is stated in probability \( iii) \) considers explicitly the confidence parameter \( t \). Next theorem establishes such lower bound.

**Theorem 4.5.** For all \( n, p, K \in \mathbb{N}, t \geq 0, \sigma > 0 \) it holds that
\[
\inf \sup_{f} \mathbb{P}_{(\beta^*, b^*)} \left( R(f) \geq \frac{\sigma^2}{3 \cdot 2^p n} \left( \sqrt{p + K} + \sqrt{32t} \right)^2 \right) \geq \frac{1}{12} e^{-t},
\]
where the infimum is taken w.r.t. all estimators.

The proof of Theorem 4.5 relies on standard information theoretic results. In particular, in order to prove optimal exponential concentration we follow similar strategy as that of Bellec et al. (2017), Kerkyacharian et al. (2014) who derived optimal exponential concentrations in the context of density aggregation and binary classification. Theorem 4.5 combined with generic lower bound derived in Theorem 3.3 yields the following corollary.

**Corollary 4.6.** Let \( \bar{\delta}_n(p, K, t) = \left( \sqrt{\left(\frac{p + K}{n} + \sqrt{32t/\alpha} \right)^2} / (3 \cdot 2^p) \right) \). For all \( n, p, K \in \mathbb{N}, t \geq 0, \sigma > 0, \alpha \in [0,1] \) it holds for all \( t \geq 0 \) and all \( t' \leq 1 - e^{-t/12} \) that
\[
\inf_{f \in \tilde{F}_{n, t'}} \sup_{(\beta^*, b^*)} \mathbb{P}_{(\beta^*, b^*)} \left( R^{1/2}(\hat{f}) \geq \sigma_n^{1/2}(p, K, t) \vee (1−\sqrt{\alpha})U^{1/2}(f^*) \right) \geq \frac{1}{12} e^{-t'}.
\]

Comparing the upper bound of Theorem 4.4 and the lower bound of Corollary 4.6 we conclude that the two obtained rates are the same up to a multiplicative constant factor. Hence confirming the tightness of the results derived in Section 3.1.
4.3. Simulation study. In this section we perform simulation study to empirically validate our theoretical analysis. Before continuing let us discuss the notion of signal-to-unfairness ratio. Setting $\beta^* = 0$ in the model (13), if the amplitudes of $b^*_i$ is much smaller than the noise level $\sigma^2$, then the observations $Y_s$ are mainly composed of noise.\footnote{For our empirical validation and illustrations we have relied on the following \texttt{python} packages: \texttt{scikit-learn} (Pedregosa et al., 2011), \texttt{numpy} (Van Der Walt et al., 2011), \texttt{matplotlib} (Hunter, 2007), \texttt{seaborn}.} While for the prediction problem it is not a problem, since our rates will scale with the noise level, it becomes important for the estimation of unfairness $U(f^*)$. Motivated by this discussion, we define the noise-to-unfairness ratio as

$$NUR^2 := \frac{\sigma^2}{U(f^*)}.$$  

The signal-to-unfairness ratio tells us how the level of unfairness compares to the noise level. The regime $NUR \gg 1$ means that the unfairness of the distributions is below the noise level, and it is statistically difficult to estimate it. In contrast, $NUR \ll 1$ implies that the unfairness dominates the noise. Instead of varying $U(f^*)$ and $\sigma$ we fix $\sigma$ and perform our study for different values of NUR.

We follow the following protocol. For some fixed $K, n_1, \ldots, n_K, p, \sigma, \text{NUR}$ we simulate the model in Eq. (14) with $\Sigma = I_p$. In all the experiments we set $\beta^* = (1, \ldots, 1)^T \in \mathbb{R}^p$. For $b^*$ we first define $v = (1, -1, 1, -1, \ldots)^T \in \mathbb{R}^K$ and set $b^* = v\sqrt{\sigma^2/NUR \cdot Var_S(\alpha)}$, where $Var_S(v)$ is the
Fig 4. Dashed green and brown lines correspond to the risk and unfairness of $\hat{f}^*$, respectively. Solid green and brown lines correspond to the average risk and unfairness of $\hat{f}_\tau(\alpha)$ and the shaded region shows three standard deviations over 50 repetitions. On the left $\tau(\alpha) = \bar{\tau}$ and on the risk $\tau(\alpha) = \alpha$.

variance of $v$ with weights $w_1, \ldots, w_K$. So that the unfairness of this model is exactly equal to $\sigma^2/NUR^2$. On each simulation round of the model, we compute the estimator in Eq. (16) with two choices of parameter $\tau$:

1. Proposed: $\tau(\alpha) = \bar{\tau}$ from Theorem 4.4;
2. Naive: $\tau(\alpha) = \alpha$.

Remark 4.7. While performing experiments we have noticed that setting $\bar{\tau}$ with $\delta_n(p, K, t)$ defined in Theorem 4.4 results in too pessimistic estimates in terms of unfairness, for this reason in all of our experiments we set $\delta_n(p, K, t) = (n/\alpha) + (K/\alpha)$, which is of the same order as that of Theorem 4.4.

Then, for each $\hat{f}_\tau(\alpha)$ we evaluate $R(\hat{f}_\tau(\alpha))$ and $U(\hat{f}_\tau(\alpha))$. This procedure is repeated 50 times, which results in 50 values of $R(\hat{f}_\tau(\alpha))$ and $U(\hat{f}_\tau(\alpha))$ for each $\alpha \in (0, 1)$. For these 50 values we compute mean and standard deviation. We considered $p = 10$, $K = 5$, $\sigma = 1$, and $NUR \in \{0.2, 0.5, 2\}$. Furthermore, for the choice of $n_1, \ldots, n_K$ we study the following two regimes

1. Balanced: $n_1 = \ldots = n_5 = 100$.
2. Unbalanced: $n_1 = 5, n_2 = 45, n_3 = 100, n_4 = 100, n_5 = 250$.

The reason we consider two regimes is to confirm the theoretical findings of Theorem 4.4, which indicate that the rate is governed by $n_1 + \ldots + n_K$ instead of the their individual values. Finally, for a given fairness parameter function $\alpha \mapsto \tau(\alpha)$ we report cumulative risk increase
over all $\alpha \in [0, 1]$ defined as

$$\Delta R(\tau) := \int_0^1 (R(\hat{f}_{\tau(\alpha)}) - R(f^*_\alpha)) \, d\alpha.$$  

This quantity describes the cumulative risk loss of the rule $\tau(\alpha)$ across all the levels of fairness $\alpha$ compared to the best $\alpha$-relative improvement $f^*_\alpha$.

On Figures 3–4 we draw the evolution of the risk and of the unfairness when $\alpha$ traverses the interval $[0, 1]$. We also report $\Delta R(\tau)$ defined above. Inspecting the plots we can see that that the main disadvantage of the naive choice of $\tau = \alpha$ is its poor fairness guarantee, that is, in almost half of the outcomes, the unfairness of $\hat{f}_\alpha$ exceeded the prescribed value. In contrast, the proposed choice of $\tau(\alpha) = \hat{\tau}$ consistently improves the unfairness of the regression function $f^*$, empirically validating our findings in Theorem 4.4. However, good fairness results come at the cost of consistently higher risk. One can also see that the effect of unbalanced distributions is negligible for the considered model (it only affects the variance of the result). This is explained by the definition of the risk, which weights the groups proportionally to their frequencies. Finally, observing the behavior of naive approach for NUR = 0.2 and NUR = 2 we note that in the latter case the unfairness of $\hat{f}_\alpha$ starts to deviate from the true value (with consistently positive bias). Meanwhile, since the proposed choice $\tau(\alpha) = \hat{\tau}$ is more conservative, the bias remains negative, that is, the unfairness of $f^*$ is still improved.

Table 1 presents the numeric results for $p = 10$, $K = 5$, NUR = 0.5. We remark the striking drop in the risk for $\alpha = 0.2$, indicating that a slight relaxation of the Demographic Parity constraint results in a significant improvement in terms of the risk. Of course, the justification of such a relaxation must be considered based on the application at hand.

### 5. Conclusion

In this work we proposed a theoretical framework for rigorous analysis of regression problems under fairness requirements. Our framework allows to interpolate between the regression of demographic parity and the unconstrained regression using univariate parameter between zero and one. Within this framework we precisely quantified the risk-fairness trade-off and derived general plug-n-play lower bound. To demonstrate the generality of our results we provided minimax analysis of the linear model with systematic group-dependent bias. Finally, we have performed empirical validation. In future it would be interesting to extend our analysis to other statistical model, providing estimators with high confidence fairness improvement.

**APPENDIX A: REMINDER**
A.1. The Wasserstein-2 distance. We recall basic results on the Wasserstein-2 distance on the real line.

The following lemma gives a closed form expression for the Wasserstein-2 distance between two univariate Gaussian distributions.

**Lemma A.1 (Fréchet (1957)).** For any $m_0, m_1 \in \mathbb{R}, \sigma_0, \sigma_1 \geq 0$ it holds that

$$W_2^2(\mathcal{N}(m_0, \sigma_0^2), \mathcal{N}(m_1, \sigma_1^2)) = (m_0 - m_1)^2 + (\sigma_0 - \sigma_1)^2.$$ 

The next lemma gives a closed form expression for the barycenter of $K$ univariate Gaussian distributions. It shows in particular that such barycenter is also a univariate Gaussian distribution.

**Lemma A.2 (Agueh and Carlier (2011)).** Let $w \in \mathbb{R}^K$ be a probability vector, then the solution of

$$\min_{\nu \in \mathcal{P}_2(\mathbb{R})} \sum_{s=1}^K w_s W_2^2(\mathcal{N}(m_s, \sigma_s^2), \nu),$$

is given by $\mathcal{N}(\bar{m}, \bar{\sigma}^2)$ with

$$\bar{m} = \sum_{s=1}^K w_s m_s \quad \text{and} \quad \bar{\sigma} = \sum_{s=1}^K w_s \sigma_s.$$

Finally we state a lemma giving an explicit form for the transport map to the barycenter of probability distributions supported on the real line and the corresponding constant speed geodesics. See (Agueh and Carlier, 2011, Section 6.1).

**Lemma A.3.** Let $a_1, \ldots, a_K$ be non-atomic probability measures on the real line that have finite second moments, and let $w_1, \ldots, w_K$ be positive reals that sum to 1. Denote by $\bar{a}$ a barycenter of those measures (w.r.t. to the Wasserstein-2 distance). For any $s \in [K]$, the transport map from $a_s$ to the barycenter $\bar{a}$ is given by

$$T_{a_s \to \bar{a}} = \left( \sum_{s'=1}^K w_{s'} F_{s'}^{-1} \circ F_s \right),$$

where $F_s$ is the cumulative distribution function of $a_s$ and $F_s^{-1}$ denotes the generalized inverse of $F_s$:

$$F_s^{-1}(t) = \inf \{ x : F_s(x) \geq t \}.$$ 

In particular, the constant speed geodesic $\gamma_s(\cdot)$ from $a_s$ to $\bar{a}$ is given by

$$\gamma_s(t) = ((1 - t) \text{Id} + t T_{a_s \to \bar{a}}) \# a_s, \quad t \in [0, 1].$$
A.2. Tail inequalities. The next result can be found in (Laurent and Massart, 2000, Lemma 1).

Lemma A.4. Let $\zeta_1, \ldots , \zeta_p$ be i.i.d. standard Gaussian random variables. Let $a = (a_1, \ldots , a_p)^\top$ be component-wise non-negative, then

$$
P \left( \sum_{j=1}^{p} a_j (\zeta_j^2 - 1) \geq 2 \|a\|_2 \sqrt{t} + 2 \|a\|_\infty t \right) \leq \exp(-t), \quad \forall t \geq 0.
$$

In particular, setting $\zeta = (\zeta_1, \ldots , \zeta_p)^\top$ and applying the previous result with $a_1 = \ldots = a_p = 1$ we get

$$
P \left( \|\zeta\|_2^2 \geq p + 2 \sqrt{pt} + 2t \right) \leq \exp(-t), \quad \forall t \geq 0.
$$

We need one result from random matrix theory to control the smallest and largest singular values of a Gaussian matrix, see (Vershynin, 2010, Corollary 5.35).

Lemma A.5. Let $A$ be an $N \times m$ matrix whose entries are independent standard normal random variables. Then,

$$
P \left( \sigma_{\min}(A) \leq \sqrt{N - \sqrt{m}} - t \right) \lor P \left( \sigma_{\max}(A) \geq \sqrt{N + \sqrt{m}} + t \right) \leq \exp(-t^2 / 2), \quad \forall t \geq 0.
$$

APPENDIX B: PROOFS FOR SECTION 2.1

B.1. Auxiliary results. The next result is taken from (Le Gouic et al., 2020, Theorem 3).

Lemma B.1. Let $f : \mathbb{R}^p \times [K] \to \mathbb{R}$ be any measurable function. If for all $s \in [K]$ random variable $f^*(X, S) \mid S = s$ is atomless, then

$$
\mathcal{R}(f) \geq \sum_{s=1}^{K} w_s W_2^2 \left( f(\cdot, s) \# \mu_X \mid s, f^*(\cdot, s) \# \mu_X \mid s \right).
$$

Lemma B.2 (Minkowski’s inequality). Let $(X, d)$ be a metric space. Fix an integer $K \geq 2$, a weight vector $w \in \Delta^{K-1}$ and define the mapping $d_w : \mathcal{X}^K \times \mathcal{X}^K \to \mathbb{R}$ as

$$
d_w(a, b) = \sqrt{\sum_{s=1}^{K} w_s d^2(a_s, b_s)}, \quad \text{for any } a, b \in \mathcal{X}^K.
$$

Then, $d_w$ is a pseudo-metric on the product space $\mathcal{X}^K$.

Proof. The mapping $d_w$ is clearly symmetric and non-negative. We only have to check the triangle inequality. Fix arbitrary $a, b, c \in \mathcal{X}^K$. Then, by triangular inequalities on the distance $d$ and Jensen’s inequality,

$$
\sum_{s=1}^{K} w_s d^2(a_s, b_s) \leq \sum_{s=1}^{K} w_s d(a_s, b_s) d(a_s, c_s) + \sum_{s=1}^{K} w_s d(a_s, b_s) d(c_s, b_s)
\leq \sqrt{\sum_{s=1}^{K} w_s d^2(a_s, b_s)} \sqrt{\sum_{s=1}^{K} w_s d^2(a_s, c_s)} + \sqrt{\sum_{s=1}^{K} w_s d^2(a_s, b_s)} \sqrt{\sum_{s=1}^{K} w_s d^2(c_s, b_s)}.
$$
That is,

\[ d_w(a, b) = \sqrt{\sum_{s=1}^{K} w_s d^2(a_s, b_s)} \leq \sqrt{\sum_{s=1}^{K} w_s d^2(a_s, c_s)} + \sqrt{\sum_{s=1}^{K} w_s d^2(c_s, b_s)} = d_w(a, c) + d_w(c, b). \]

\[ \square \]

**Lemma B.3.** Let \( a = (a_1, \ldots, a_K) \in X^K \), \( w = (w_1, \ldots, w_K)^\top \in \Delta^{K-1} \). Assume that \( b = (b_1, \cdots, b_K) \in X^K \) satisfies \((P_1)-(P_2)\), then

\[ \sqrt{\sum_{s=1}^{K} w_s d^2(b_s, C_b)} = \sqrt{\sum_{s=1}^{K} w_s d^2(b_s, C_a)}. \]

**Proof.** Let \( C_b \) be a barycenter of \((b_s)_{s \in [K]}\) with weights \((w_s)_{s \in [K]}\); then by Lemma B.2 it holds that

\[ \sqrt{\sum_{s=1}^{K} w_s d^2(a_s, C_b)} \leq \sqrt{\sum_{s=1}^{K} w_s d^2(a_s, b_s)} + \sqrt{\sum_{s=1}^{K} w_s d^2(b_s, C_b)}. \]

The following chain of inequalities holds thanks to Eq. (17) and properties \((P_1)-(P_2)\)

\[ \sqrt{\sum_{s=1}^{K} w_s d^2(b_s, C_b)} \geq \sqrt{\sum_{s=1}^{K} w_s d^2(a_s, C_b)} - \sqrt{\sum_{s=1}^{K} w_s d^2(a_s, b_s)} \]

\[ \geq \sqrt{\sum_{s=1}^{K} w_s d^2(a_s, C_a)} - \sqrt{\sum_{s=1}^{K} w_s d^2(a_s, b_s)} \]

\[ = \frac{1}{\sqrt{\alpha}} \sqrt{\sum_{s=1}^{K} w_s d^2(b_s, C_a)} - \frac{1-\sqrt{\alpha}}{\sqrt{\alpha}} \sqrt{\sum_{s=1}^{K} w_s d^2(b_s, C_a)} \]

\[ = \sqrt{\sum_{s=1}^{K} w_s d^2(b_s, C_a)}. \]

The converse inequality follows from the definition of \( C_b \), which concludes the proof. \( \square \)

**B.2. Proof of Proposition 2.2.** Let \( \alpha \in [0, 1] \). For any \( s \in [K] \), define

\[ a_s = f^*(\cdot, s) \# \mu_{X \mid s}, \] (18)

Let \( \gamma_s \) be the (constant-speed) geodesic between \( a_s \) and \( C_a \) i.e., \( \gamma_s(0) = a_s, \gamma_s(1) = C_a \) and \( \mathcal{W}_2(\gamma_s(t_1), \gamma_s(t_2)) = |t_2 - t_1| \mathcal{W}_2(a_s, C_a) \) for any \( t_1, t_2 \in [0, 1] \). Note that the uniqueness of the geodesic come from the particular structure of the Wasserstein-2 space on the real line, see e.g., (Kloeckner, 2010, Section 2.2). We define \( b_s := \gamma_s(1-\sqrt{\alpha}) \) for \( s \in [K] \). Let us show that \( b = (b_s)_{s \in [K]} \) satisfies the properties \((P_1)-(P_2)\) of the Geometric Lemma 2.4 when considering
We define \( \alpha = (a_s)_{s \in [K]} \) with the weights \((w_s)_{s \in [K]} \) and \( d \equiv W_2 \). By construction of \( b_s = \gamma_s(1 - \sqrt{\alpha}) \), we have
\[
W_2(b_s, C_α) = \sqrt{\alpha} W_2(a_s, C_α), \tag{19}
\]
\[
W_2(b_s, a_s) = (1 - \sqrt{\alpha}) W_2(a_s, C_α). \tag{20}
\]
This shows that \( b = (b_s)_{s \in [K]} \) satisfies \((P_1)\) and \((P_2)\). Therefore, using Lemma 2.4 we get
\[
\sum_{s=1}^{K} w_s W_2^2(b_s, a_s) = \inf_{b \in P_2^K(\mathbb{R})} \left\{ \sum_{s=1}^{K} w_s W_2^2(b_s, a_s) : \sum_{s=1}^{K} w_s W_2^2(b_s, C_b) \leq \alpha \sum_{s=1}^{K} w_s d^2(a_s, C_α) \right\}. \tag{21}
\]
Finally, thanks to the assumption that \( a_s = f^*(\cdot, s)\#\mu_{X|s} \) is atomless the constant speed geodesic \( \gamma_s \) between \( a_s \) and \( C_α \) can be written as
\[
\gamma_s(t) = \left( (1 - t) \text{Id} + t \left( \sum_{s' = 1}^{K} w_{s'} F_{a_{s'}}^{-1} \circ F_{a_s} \right) \right) # a_s = \left\{ \left( (1 - t) \text{Id} + t \left( \sum_{s' = 1}^{K} w_{s'} F_{a_{s'}}^{-1} \circ F_{a_s} \right) \right) \circ f^*(\cdot, s) \right\} # \mu_{X|s}, \quad t \in [0, 1].
\]
See Appendix A.1 for details about the first equality. Substituting \( t = 1 - \sqrt{\alpha} \) to \( \gamma_s \), the expression for \( b_s \) is
\[
b_s = \left\{ \left( \sqrt{\alpha} \text{Id} + (1 - \sqrt{\alpha}) \left( \sum_{s' = 1}^{K} w_{s'} F_{a_{s'}}^{-1} \circ F_{a_s} \right) \right) \circ f^*(\cdot, s) \right\} # \mu_{X|s}. \tag{22}
\]
We define \( f^*_α \) for all \((x, s) \in \mathbb{R}^p \times [K] \) as
\[
f^*_α(x, s) = \sqrt{\alpha} f^*(x, s) + (1 - \sqrt{\alpha}) \sum_{s' = 1}^{K} w_{s'} F_{a_{s'}}^{-1} (F_{a_s} (f^*(x, s))) \tag{23}\]
then after Eq. \( (22) \) it holds that \( b_s = f^*_α(\cdot, s)\#\mu_{X|s} \) and
\[
W_2^2(b_s, a_s) = \mathbb{E} \left[ (f^*(X, S) - f^*_α(X, S))^2 \mid S = s \right]. \tag{24}
\]
with \( \mathcal{U}(f^*_α) = \alpha \mathcal{U}(f^*) \). Moreover, Lemma B.1 implies that for any \( f \) such that \( \mathcal{U}(f) \leq \alpha \mathcal{U}(f^*) \) we have
\[
\mathbb{E}(f^*(X, S) - f(X, S))^2 \geq \sum_{s=1}^{K} w_s W_2^2(b_s, a_s) = \sum_{s=1}^{K} w_s \mathbb{E} \left[ (f^*(X, S) - f^*_α(X, S))^2 \mid S = s \right] = \mathcal{R}(f^*_α).
\]
Thus, \( f^*_α \) is the optimal fair prediction with \( \alpha \) relative improvement. The proof is concluded.
APPENDIX C: PROOF OF THEOREM 3.3

To ease the notation we write \( \delta_n \) instead of \( \delta_n(F, \Theta, t) \). We also define

\[
\Psi(\hat{f}, (f^*, \theta)) := P_{(f^*, \theta)} \left( \mathcal{R}^{1/2}(\hat{f}) \geq \delta_n^{1/2} \vee (1-\sqrt{\alpha}) \mathcal{U}^{1/2}(f^*) \right).
\]

We split the proof according to two complementary cases.

**Case 1:** there exists \( (f^*, \theta) \in F \times \Theta \) such that \( \delta_n \leq (1-\sqrt{\alpha})^2 \mathcal{U}(f^*) \). In this case, for such couple \( (f^*, \theta) \in F \times \Theta \) and for any estimator \( \hat{f} \in \hat{F}_{(a,t)} \) we have

\[
\Psi(\hat{f}, (f^*, \theta)) \geq P_{(f^*, \theta)} \left( \mathcal{R}^{1/2}(\hat{f}) \geq \delta_n^{1/2} \vee (1-\sqrt{\alpha}) \mathcal{U}^{1/2}(f^*) \right) \geq P_{(f^*, \theta)} \left( \mathcal{U}(\hat{f}) \leq \alpha \mathcal{U}(f^*) \right) \geq 1 - t'.
\]

Since in the considered case there exists a couple \( (f^*, \theta) \in \hat{F}_{(a,t')} \times \Theta \) such that \( \delta_n \leq (1-\sqrt{\alpha})^2 \mathcal{U}(f^*) \), by definition of \( \hat{F}_{(a,t')} \) we have

\[
\inf_{\hat{f} \in \hat{F}_{(a,t')}} \sup_{(f^*, \theta) \in F \times \Theta} \Psi(\hat{f}, (f^*, \theta)) \geq 1 - t'. \tag{25}
\]

**Case 2:** for any couple \( (f^*, \theta) \in F \times \Theta \) it holds that \( \delta_n > (1-\sqrt{\alpha})^2 \mathcal{U}(f^*) \). In this case, for any couple \( (f^*, \theta) \in F \times \Theta \) and for any estimator \( \hat{f} \in \hat{F}_{(a,t')} \),

\[
\Psi(\hat{f}, (f^*, \theta)) = P_{(f^*, \theta)} \left( \mathcal{R}(\hat{f}) \geq \delta_n \right).
\]

By definition of \( \delta_n \) it holds in this case that

\[
\inf_{\hat{f} \in \hat{F}_{(a,t')}} \sup_{(f^*, \theta) \in F \times \Theta} \Psi(\hat{f}, (f^*, \theta)) \geq \inf_{\hat{f}} \sup_{(f^*, \theta) \in F \times \Theta} \Psi(\hat{f}, (f^*, \theta)) = \inf_{\hat{f}} \sup_{(f^*, \theta) \in F \times \Theta} P_{(f^*, \theta)} \left( \mathcal{R}(\hat{f}) \geq \delta_n \right) \geq t. \tag{26}
\]

Putting two cases together, and in particular using Eqs. (25) and (26) we obtain

\[
\inf_{\hat{f} \in \hat{F}_{(a,t')}} \sup_{(f^*, \theta) \in F \times \Theta} \Psi(\hat{f}, (f^*, \theta)) \geq \begin{cases} 1 - t' & \text{if } \exists (f^*, \theta) \in F \times \Theta \text{ s.t. } \delta_n \leq (1-\sqrt{\alpha})^2 \mathcal{U}(f^*) \\ t & \text{otherwise} \end{cases}.
\]

We conclude the proof observing that the r.h.s. of the last inequality is lower bounded by \( t \land (1 - t') \).

APPENDIX D: PROOFS FOR SECTION 4

Additional notation. We denote by \( \mathbb{S}^{p-1} \) the unit sphere in \( \mathbb{R}^p \). For any matrix \( A \) we denote by \( \|A\|_{op} \) the operator norm of \( A \). We denote by \( \chi^2(p) \) the standard chi-square distribution with \( p \) degrees of freedom and by \( \mathcal{N}(\mu, \Sigma) \) the multivariate Gaussian with mean \( \mu \) and covariance \( \Sigma \). We denote by \( I_p \) the identity matrix of size \( p \times p \).
D.1. Proof of Lemma 4.3. Throughout the proof we implicitly condition on the observations. Let $\tau \in [0, 1]$. For each $s \in [K]$ we set $\hat{m}_s = \sqrt{\tau}b_s + (1 - \sqrt{\tau}) \sum_{s=1}^{K} w_s \hat{b}_s$. Note that for all $s \in [K]$, $(\hat{f}_s(X, S)|S = s) \sim \mathcal{N}(\hat{m}_s, \langle \hat{\beta}, \Sigma \hat{\beta} \rangle)$. Therefore, by the definition of the unfairness and Lemma A.2

$$
\mathcal{U}(\hat{f}_s) = \min_{\nu} \sum_{s=1}^{K} w_s W_2^2 \left( \mathcal{N}(\hat{m}_s, \langle \hat{\beta}, \Sigma \hat{\beta} \rangle), \nu \right) = \sum_{s=1}^{K} w_s W_2^2 \left( \mathcal{N}(\hat{m}_s, \langle \hat{\beta}, \Sigma \hat{\beta} \rangle), \mathcal{N}(\hat{m}, \langle \hat{\beta}, \Sigma \hat{\beta} \rangle) \right),
$$

where $\bar{m} = \sum_{s=1}^{K} w_s \hat{m}_s$. We conclude the proof by noticing that thanks to Lemma A.1 it holds that

$$
W_2^2 \left( \mathcal{N}(\hat{m}_s, \langle \hat{\beta}, \Sigma \hat{\beta} \rangle), \mathcal{N}(\hat{m}, \langle \hat{\beta}, \Sigma \hat{\beta} \rangle) \right) = (\hat{m}_s - \bar{m})^2 = \left\{ \sqrt{\tau}b_s + (1 - \sqrt{\tau}) \sum_{s=1}^{K} w_s \hat{b}_s - \sum_{s=1}^{K} w_s \hat{b}_s \right\}^2.
$$

The proof is concluded.

D.2. Auxiliary results for Theorem 4.4.

**Lemma D.1** (Fixed design analysis). Define the following matrix of size $(p + K) \times (p + K)$

$$
\hat{\Psi} = \left[ \frac{1}{\tau} \sum_{s=1}^{K} w_s X_s^\top X_s / n_s \right| \begin{bmatrix} O \\ \frac{O}{p} \end{bmatrix}, \frac{\tau}{w} \mathbf{W} \right],
$$

where $O = [w_1 \mathbf{X}_1, \ldots, w_K \mathbf{X}_K] \in \mathbb{R}^{pxK}$ and $\mathbf{W} = \text{diag}(w_1, \ldots, w_K)$. For all $t \geq 0$ it holds that

$$
\mathbb{P} \left( \|\hat{\Psi}^{1/2} \hat{\Delta}\|_2^2 \geq \sigma^2 \left\{ \left( \frac{p}{n} + \frac{K}{n} \right) + 2 \left( \sqrt{\frac{p}{n}} + \sqrt{\frac{K}{n}} \right) \sqrt{\frac{t}{n} + 4 \frac{t}{n}} \right\} \mid \mathbf{X}_{1:K} \right) \leq 2 \exp(-t),
$$

where $\hat{\Delta} = (\hat{\beta} - \beta^*, \hat{b} - b^*) \in \mathbb{R}^{p} \times \mathbb{R}^{K}$ and $\mathbf{X}_{1:K} = (\mathbf{X}_1, \ldots, \mathbf{X}_K)$.

**Proof.** By optimality of $(\hat{\beta}, \hat{b})$ and the linear model assumption in Eq. (14) it holds that

$$
\sum_{s=1}^{K} w_s \left\| Y_s - X_s \hat{\beta} - \hat{b}_s 1_{n_s} \right\|_{n_s}^2 \leq \sum_{s=1}^{K} w_s \left\| \xi_s / n_s \right\|_{n_s}^2.
$$

After simplification, the above yields

$$
\sum_{s=1}^{K} w_s \left\| X_s (\beta^* - \beta) + (b^*_s - \hat{b}_s) 1_{n_s} \right\|_{n_s}^2 \leq 2 \sum_{s=1}^{K} w_s \langle X_s (\beta^* - \beta) + (b^*_s - \hat{b}_s) 1_{n_s}, \xi_s / n_s \rangle
$$

$$
= 2 \langle \beta - \beta^*, \sum_{s=1}^{K} X_s^\top \xi_s / n \rangle + 2 \sum_{s=1}^{K} w_s (\hat{b}_s - b^*_s) \bar{\xi}_s,
$$

where $\bar{\xi}_s = \frac{1}{n} \sum_{s=1}^{K} \xi_s$. The proof is concluded.
where \( \tilde{\xi}_s = (1/n_s) \sum_{i=1}^{n_s} (\xi_i)_s \). Using Young’s inequality, we can write

\[
2 \left( \hat{\beta} - \beta^* \sum_{s=1}^{K} X_s' \xi_s/n \right) \leq \frac{1}{2} \sum_{s=1}^{K} w_s \| X_s (\beta^* - \hat{\beta}) \|^2_{n_s} + 2 \left( \frac{\| \beta - \beta^* \sum_{s=1}^{K} X_s' \xi_s/n \|_{n_s}^2}{\sqrt{\sum_{s=1}^{K} w_s \| X_s (\beta^* - \hat{\beta}) \|^2_{n_s}}} \right)^2
\]

\[
\leq \frac{1}{2} \sum_{s=1}^{K} w_s \| X_s (\beta^* - \hat{\beta}) \|^2_{n_s} + 2 \sup_{\Delta \in \mathbb{R}^p} \left( \frac{\| \Delta, \sum_{s=1}^{K} X_s' \xi_s/n \|_{n_s}^2}{\sqrt{\sum_{s=1}^{K} w_s \| X_s \Delta \|^2_{n_s}}} \right)^2.
\]

We also observe that again thanks to Young’s inequality

\[
2 \sum_{s=1}^{K} w_s (\hat{\beta}_s - b_s^*) \tilde{\xi}_s \leq \frac{1}{2} \sum_{s=1}^{K} w_s (\hat{\beta}_s - b_s^*)^2 + 2 \sum_{s=1}^{K} w_s \xi_s^2.
\]

Putting everything together, we have shown that

\[
\| \hat{\Psi}^{1/2} \hat{\Delta} \|^2_2 \leq 2 \sup_{\Delta \in \mathbb{R}^p} \left( \frac{\| \Delta, \sum_{s=1}^{K} X_s' \xi_s/n \|_{n_s}^2}{\sqrt{\sum_{s=1}^{K} w_s \| X_s \Delta \|^2_{n_s}}} \right)^2 + 2 \sum_{s=1}^{K} w_s \xi_s^2. \tag{27}
\]

Notice that since \( \xi_s \sim \mathcal{N}(0, \sigma^2 I_{n_s}) \), then conditionally on \( X_1, \ldots, X_K \),

\[
\sum_{s=1}^{K} X_s' \xi_s/n = \frac{\sigma}{n} \left( \sum_{s=1}^{K} X_s' X_s \right)^{1/2} \zeta,
\]

where \( \zeta \sim \mathcal{N}(0, I_p) \). Besides, since \( w_s = n_s/n \), it holds for all \( \Delta \in \mathbb{R}^p \) that

\[
\sum_{s=1}^{K} w_s \| X_s \Delta \|^2_{n_s} = \Delta' \left( \frac{1}{n} \sum_{s=1}^{K} X_s' X_s \right) \Delta = \left\| \left( \frac{1}{n} \sum_{s=1}^{K} X_s' X_s \right)^{1/2} \Delta \right\|^2_2.
\]

The above implies that conditionally on \( X_1, \ldots, X_K \),

\[
\sqrt{U} := \sup_{\Delta \in \mathbb{R}^p} \left( \frac{\| \Delta, \sum_{s=1}^{K} X_s' \xi_s/n \|_{n_s}}{\sqrt{\sum_{s=1}^{K} w_s \| X_s \Delta \|^2_{n_s}}} \right) \sim \frac{\sigma}{\sqrt{n}} \sup_{\Delta \in \mathbb{R}^p} \left\| \left( \frac{1}{n} \sum_{s=1}^{K} X_s' X_s \right)^{1/2} \Delta \right\|^2_2. \tag{28}
\]

Note that for any random variable \( \zeta \) taking values in \( \mathbb{R}^p \),

\[
\sup_{\Delta \in \mathbb{R}^p} \left\| \left( \sum_{s=1}^{K} X_s' X_s \right)^{1/2} \Delta, \zeta \right\|_2 \leq \| \zeta \|_2 \text{ almost surely.} \tag{29}
\]

Furthermore, recalling that \( \tilde{\xi}_s \sim \mathcal{N}(0, 1/n_s) \) we get

\[
V := \sum_{s=1}^{K} w_s \tilde{\xi}_s^2 \sim \frac{\sigma^2}{n} \chi^2(K). \tag{30}
\]
For any $u, v \in \mathbb{R}$ it holds that
\[
\mathbb{P}\left(\left\|\hat{\Psi}^{1/2} \hat{\Delta}\right\|_2^2 \geq 2(u + v) \mid X_{1:K}\right) \leq \mathbb{P}\left(2(U + V) \geq 2(u + v) \mid X_{1:K}\right)
\]
\[
\leq \mathbb{P}\left(\frac{\sigma^2}{n} \chi^2(p) \geq u \mid X_{1:K}\right) + \mathbb{P}\left(\frac{\sigma^2}{n} \chi^2(K) \geq v \mid X_{1:K}\right),
\]
where inequality (a) uses Eqs. (28) and (30) and the fact that $\mathbb{P}(U + V \geq u + v) \leq \mathbb{P}(U \geq u) + \mathbb{P}(V \geq v)$ for all random variables $U, V$ and all $u, v \in \mathbb{R}$. Finally, setting $u = u_n(\sigma, p, t), v = v_n(\sigma, p, t)$ with
\[
u_n(\sigma, p, t) = \frac{\sigma^2}{n} + 2\sigma^2 \sqrt{\frac{p}{n}} + \frac{t}{n} + 2\sigma^2 t, \quad v_n(\sigma, K, t) = \frac{\sigma^2 K}{n} + 2\sigma^2 \sqrt{\frac{K}{n}} + \frac{t}{n} + 2\sigma^2 t,
\]
we obtain the stated result after application of Lemma A.4 in appendix
\[
\mathbb{P}\left(\left\|\hat{\Psi}^{1/2} \hat{\Delta}\right\|_2^2 \geq 2(u_n(\sigma, p, t) + v_n(\sigma, p, t)) \mid X_{1:K}\right) \leq 2 \exp(-t).
\]

\textbf{Theorem D.2 (From fixed to random design).} Define,
\[
\delta_n(p, K, t) = 8\left(\frac{p}{n} + \frac{K}{n}\right) + 16\left(\sqrt{\frac{p}{n}} + \sqrt{\frac{K}{n}}\right) \sqrt{\frac{t}{n} + \frac{32t}{n}}.
\]
Consider $p, K \in \mathbb{N}, t \geq 0$ and define $\gamma(p, K, t) = (4\sqrt{p} + 5\sqrt{t} + 6\sqrt{p})/(\sqrt{p} + \sqrt{t})$. Assume that $\sqrt{n} \geq 2(\sqrt{p} + \sqrt{t})/(\gamma(p, K, t) - \sqrt{\gamma^2(p, K, t) - 3})$, then with probability at least $1 - 4 \exp(-t/2)$
\[
\left\|\Sigma^{1/2}(\beta^* - \hat{\beta})\right\|_2^2 + \sum_{s=1}^{K} w_s (b_s^* - \hat{b}_s)^2 \leq \sigma^2 \delta_n(p, K, t).
\]

\textbf{Proof.} Define the $(p + K) \times (p + K)$ matrix
\[
\Psi = \frac{1}{2} \begin{bmatrix} \Sigma & 0 \\ 0 & W \end{bmatrix},
\]
then under notation of Lemma D.1 we can write
\[
\left\|\hat{\Psi}^{1/2} \hat{\Delta}\right\|_2^2 = \hat{\Delta}^\top \Psi^{-1/2} \Psi^{-1/2} \hat{\Psi} \Psi^{-1/2} \hat{\Psi}^{-1/2} \hat{\Delta}
\]
\[
= \hat{\Delta}^\top \Psi^{-1/2} \hat{\Psi}^{-1/2} \left(\hat{\Psi} - \Psi\right) \Psi^{-1/2} \hat{\Psi} \Psi^{-1/2} \hat{\Delta} + \hat{\Delta}^\top \Psi \hat{\Delta}
\]
\[
\geq \left(1 + \lambda_{\text{min}} \left(\Psi^{-1/2} \left(\hat{\Psi} - \Psi\right) \Psi^{-1/2}\right)\right) \left\|\Psi^{1/2} \hat{\Delta}\right\|_2^2.
\]
If we set $\hat{\Sigma} = \sum_{s=1}^{K} w_s X_s^\top X_s/n_s$, then
\[
\Psi^{-1/2} \left(\hat{\Psi} - \Psi\right) \Psi^{-1/2} = \begin{bmatrix} \Sigma^{-1/2} \left(\hat{\Sigma} - \Sigma\right) \Sigma^{-1/2} & 2\Sigma^{-1/2} OW^{-1/2} \\ 2W^{-1/2} O \Sigma^{-1/2} & 0 \end{bmatrix}.
\]
Furthermore, by Courant-Fisher theorem it holds that
\[
\lambda_{\text{min}} \left( \Psi^{-1/2} \left( \hat{\Psi} - \Psi \right) \Psi^{-1/2} \right) \geq \lambda_{\text{min}} \left( \Sigma^{-1/2} \left( \hat{\Sigma} - \Sigma \right) \Sigma^{-1/2} \right) - 4\|\Sigma^{-1/2}OW^{-1/2}\|_{\text{op}} .
\] (33)

Using the definition of \( O \) we can write
\[
\Sigma^{-1/2}OW^{-1/2} = [w_1^{1/2} \Sigma^{-1/2} \tilde{X}_1, \ldots, w_K^{1/2} \Sigma^{-1/2} \tilde{X}_K] .
\]

Note that the random variable on right hand side of Eq. (33) is independent from \( \xi_1, \ldots, \xi_K \). Recall that since \( w_s = n_s/n \) and \( \tilde{X}_s \sim N(0, \Sigma/n) \), then for all \( s = 1, \ldots, K \) it holds that
\[
w_s^{1/2} \Sigma^{-1/2} \tilde{X}_s \sim N(0, I_p/n) ,
\]
and these vectors are independent. Hence, the matrix \( \Sigma^{-1/2}OW^{-1/2} \in \mathbb{R}^{p \times K} \) has i.i.d. Gaussian entries with variance \( 1/n \). Therefore, by Lemma A.5 we get
\[
P \left( \|\Sigma^{-1/2}OW^{-1/2}\|_{\text{op}} \geq \sqrt{\frac{p}{n}} + \sqrt{\frac{K}{n}} + \sqrt{\frac{t}{n}} \right) \leq \exp(-t/2) .
\] (34)

Furthermore, we observe that
\[
\Sigma^{-1/2} \Sigma_{-1/2} \overset{d}{=} \frac{1}{n} \sum_{i=1}^{n} \zeta_i \zeta_i^\top ,
\]
where \( \zeta_i \overset{\text{i.i.d.}}{\sim} N(0, I_p) \). It implies that
\[
\Sigma^{-1/2} \left( \hat{\Sigma} - \Sigma \right) \Sigma^{-1/2} \overset{d}{=} \frac{1}{n} \sum_{i=1}^{n} \zeta_i \zeta_i^\top - I_p = \frac{1}{n} \left( \mathbf{Z}^\top \mathbf{Z} - nI_p \right) ,
\]
where \( \mathbf{Z} \) is a matrix of size \( n \times p \) with \( i^\text{th} \)-row being equal to \( \zeta_i^\top \). Note that the spectral theorem and the relation between eigenvalues of \( \mathbf{Z}^\top \mathbf{Z} \) and the singular values of \( \mathbf{Z} \) imply that
\[
n \lambda_{\text{min}} \left( \Sigma^{-1/2} \left( \hat{\Sigma} - \Sigma \right) \Sigma^{-1/2} \right) \overset{d}{=} \lambda_{\text{min}} \left( \mathbf{Z}^\top \mathbf{Z} - nI_p \right) = \sigma_{\text{min}}^2(\mathbf{Z}) - n .
\]
where \( \sigma_{\text{min}}(\mathbf{Z}) \) is the maximal singular value of \( \mathbf{Z} \). Applying Lemma A.5 from appendix we get for all \( t \geq \sqrt{n} - \sqrt{p} \) that \( P \left( \frac{1}{n} \sigma_{\text{min}}^2(Z) - n \right) \leq \exp(-t^2/2) \) equals to
\[
P \left( \frac{1}{n} \sigma_{\text{min}}^2(\mathbf{Z}) - n \right) \leq \frac{p}{n} + 2\sqrt{\frac{p}{n}} \frac{t}{\sqrt{n}} + \frac{t^2}{n} - 2\sqrt{\frac{p}{n}} - 2 \frac{t}{\sqrt{n}} \right) \leq \exp(-t^2/2) .
\]
Changing variables \( t^2 \mapsto t \) we get
\[
P \left( \lambda_{\text{min}} \left( \Sigma^{-1/2} \left( \hat{\Sigma} - \Sigma \right) \Sigma^{-1/2} \right) \leq \frac{p}{n} + 2\sqrt{\frac{p}{n}} \frac{t}{\sqrt{n}} + \frac{t^2}{n} - 2\sqrt{\frac{p}{n}} - 2 \frac{t}{\sqrt{n}} \right) \leq \exp(-t/2) .
\] (35)
Combining Eqs. (33), (34), and (35) we deduce that
\[ P \left( \lambda_{\min} \left( \Psi^{-1/2} \left( \hat{\Psi} - \Psi \right) \Psi^{-1/2} \right) \leq \psi_n(p, K, t) \right) \leq 2 \exp(-t/2), \]
where \( \psi_n(p, K, t) = \frac{p}{n} - 6\sqrt{\frac{p}{n}} + 2\sqrt{\frac{p}{n}} \sqrt{\frac{t}{n}} - 4\sqrt{\frac{K}{n}} + \frac{t}{n} - 5\sqrt{\frac{t}{n}}. \) Applying Lemma D.3 we deduce that under the assumption on \( n \) that \( \psi_n(p, K, t) \geq -0.75. \) Thus,
\[ P \left( \lambda_{\min} \left( \Psi^{-1/2} \left( \hat{\Psi} - \Psi \right) \Psi^{-1/2} \right) \leq -0.75 \right) \leq 2 \exp(-t/2). \]
Combining the above fact with Eq. (32) and Lemma D.1 we conclude that with probability at least
\[ 1 - 2 \exp(-t) - 2 \exp(-t/2) \]
\[ \| \Psi^{1/2} \hat{\Delta} \|^2 \leq \sigma^2 \frac{4(p + K)}{\sqrt{p} + \sqrt{t}}. \]
The statement of the lemma follows from the fact that
\[ \| \Psi^{1/2} \hat{\Delta} \|^2 = \frac{1}{2} \left( \| \Sigma^{1/2} (\beta^* - \hat{\beta}) \|^2 + \sum_{k=1}^{K} w_k (b^*_k - \hat{b}_k)^2 \right). \]

**Lemma D.3.** Consider \( p, K \in \mathbb{N}, t \geq 0 \) and define
\[ \gamma(p, K, t) = \frac{4\sqrt{K} + 5\sqrt{t} + 6\sqrt{p}}{\sqrt{p} + \sqrt{t}}. \]
For all \( n, K, p \in \mathbb{N}, t \geq 0, \) the following two conditions are equivalent
1. \( n \geq \left( \frac{2(\sqrt{p} + \sqrt{t})}{\gamma(p, K, t) - \gamma^2(p, K, t) - 3} \right)^2; \)
2. \( \frac{p}{n} - 6\sqrt{\frac{p}{n}} + 2\sqrt{\frac{p}{n}} \sqrt{\frac{t}{n}} - 4\sqrt{\frac{K}{n}} + \frac{t}{n} - 5\sqrt{\frac{t}{n}} \geq -0.75. \)

**Proof.** To simplify the notation and to save space we write \( \gamma \) instead of \( \gamma(p, K, t). \) Let \( x = n^{-1/2}, \) we want to solve
\[ x^2 (\sqrt{p} + \sqrt{t})^2 - x(4\sqrt{p} + 4\sqrt{K} + 5\sqrt{t}) \geq -0.75 \]
Set \( y = x(\sqrt{p} + \sqrt{t}), \) then thanks to the definition of \( \gamma, \) the previous inequality amounts to
\[ y^2 - \gamma y + 0.75 \geq 0. \]
The roots of the polynomial above are
\[ x_-, x_+ = \frac{\gamma \pm \sqrt{\gamma^2 - 3}}{2}, \]
which are both positive. The polynomial is non-negative outside the interval \( (x_-, x_+) \subset \mathbb{R}_+. \) Hence, a sufficient condition is to have
\[ y \leq \frac{\gamma - \sqrt{\gamma^2 - 3}}{2}. \]
Substituting \( x = n^{-1/2} \) and the expression for \( \gamma \) we conclude.
Lemma D.4 (General unfairness control). Under notation of Lemma 4.3 it holds that, for any $\alpha \in [0,1]$, \[
\mathcal{U}(\hat{f}_\alpha) \leq \alpha \mathcal{U}(f^*) \left\{ 1 + \text{NUR} \sqrt{\frac{\sum_{s=1}^{K} w_s (\hat{b}_s - b^*_s)^2}{\sigma^2}} \right\}^2, \text{ almost surely.} \tag{36}
\]
Moreover, \[
\left| \mathcal{U}^{1/2}(\hat{f}_1) - \mathcal{U}^{1/2}(f^*) \right| \leq \left\{ \sum_{s=1}^{K} w_s (\hat{b}_s - b^*_s)^2 \right\}^{1/2}, \text{ almost surely.} \tag{37}
\]

Proof. Let $U$ and $V$ be discrete random variables such that $P(U = \hat{b}_s, V = b^*_s) = w_s \delta_{s,s'}$, for any $s, s' \in [K]$. Note that, in particular, $P(U = \hat{b}_s) = w_s$ and $P(V = b^*_s) = w_s$. Then, according to Lemma 4.3 and the definition of $\hat{f}_\alpha$ it holds that \[
\mathcal{U}(\hat{f}_\alpha) = \alpha \Var(U) \quad \text{and} \quad \mathcal{U}(\hat{f}_\alpha) = \alpha \mathcal{U}(f^*) = \alpha \Var(V).
\]
Therefore, with our notations we have \[
\mathcal{U}(\hat{f}_\alpha) - \alpha \mathcal{U}(f^*) = \alpha (\Var(U) - \Var(V)). \tag{38}
\]
Furthermore, for all $\varepsilon \in (0,1)$ we have that $\Var(U)$ equals to \[
\Var(U - V + V) = \Var(U - V) + 2 \mathbb{E}[ (U - V - \mathbb{E}[U] + \mathbb{E}[V])(V - \mathbb{E}[V]) ] + \Var(V)
\leq \Var(U - V) + 2 \sqrt{\Var(U - V) \Var(V)} + \Var(V)
\leq \sum_{s=1}^{K} w_s (\hat{b}_s - b^*_s)^2 + 2 \sqrt{\mathcal{U}(f^*)} \sqrt{\sum_{s=1}^{K} w_s (\hat{b}_s - b^*_s)^2 + \Var(V)} \tag{39}
\]
Finally, combining Eqs. (38) and (39) we deduce \[
\mathcal{U}(\hat{f}_\alpha) \leq \alpha \left( \sum_{s=1}^{K} w_s (\hat{b}_s - b^*_s)^2 + 2 \sqrt{\mathcal{U}(f^*)} \sqrt{\sum_{s=1}^{K} w_s (\hat{b}_s - b^*_s)^2 + \mathcal{U}(f^*)} \right) \tag{\frac{39'}{}
\]
The proof of Eq. (36) is concluded after factorizing the square of the r.h.s. of the above bound. To prove Eq. (37), we set $\alpha = 1$ in Eq. (36) to get \[
\mathcal{U}^{1/2}(\hat{f}_1) \leq \mathcal{U}^{1/2}(f^*) + \left\{ \sum_{s=1}^{K} w_s (\hat{b}_s - b^*_s)^2 \right\}^{1/2}.
\]
The converse bound derived in a similar fashion using $\Var(V) \leq \Var(U - V) + 2 \sqrt{\Var(U - V) \Var(U)}$.
Lemma D.5 (General risk control). Under notation of Lemma 4.3 it holds that
\[
\mathcal{R}(\hat{f}_\alpha) \leq \sum_{s=1}^{K} w_s \mathbb{E}(\langle X, \beta^* - \hat{\beta} \rangle + (b_s^* - \hat{b}_s))^2 \\
+ 2(1 - \sqrt{\alpha}) \sqrt{\sum_{s=1}^{K} w_s (b_s^* - \hat{b}_s)^2} \sqrt{\sum_{s'=1}^{K} w_{s'} (\hat{b}_{s'} - \hat{b}_s - \sum_{s''=1}^{K} w_{s''} \hat{b}_{s''})^2} \\
+ (1 - \sqrt{\alpha})^2 \sum_{s=1}^{K} w_s \left( \hat{b}_s - \sum_{s'=1}^{K} w_{s'} \hat{b}_{s'} \right)^2 .
\]

Proof. Recall the expression for \( \hat{f}_\alpha \)
\[
\hat{f}_\alpha(x, s) = \langle x, \hat{\beta} \rangle + \sqrt{\alpha} \hat{b}_s + (1 - \sqrt{\alpha}) \sum_{s=1}^{K} w_s \hat{b}_s.
\]
Using this expression, we can write for the risk of \( \hat{f}_\alpha \)
\[
\mathcal{R}(\hat{f}_\alpha) = \sum_{s=1}^{K} w_s \mathbb{E}(\langle X, \beta^* - \hat{\beta} \rangle + (b_s^* - \hat{b}_s) + (1 - \sqrt{\alpha}) \left( \hat{b}_s - \sum_{s'=1}^{K} w_{s'} \hat{b}_{s'} \right))^2 \\
\overset{(a)}{=} \sum_{s=1}^{K} w_s \mathbb{E}(\langle X, \beta^* - \hat{\beta} \rangle + (b_s^* - \hat{b}_s))^2 + 2(1 - \sqrt{\alpha}) \sum_{s=1}^{K} w_s (b_s^* - \hat{b}_s) \left( \hat{b}_s - \sum_{s'=1}^{K} w_{s'} \hat{b}_{s'} \right) \\
+ (1 - \sqrt{\alpha})^2 \sum_{s=1}^{K} w_s \left( \hat{b}_s - \sum_{s'=1}^{K} w_{s'} \hat{b}_{s'} \right)^2 \\
\overset{(b)}{\leq} \sum_{s=1}^{K} w_s \mathbb{E}(\langle X, \beta^* - \hat{\beta} \rangle + (b_s^* - \hat{b}_s))^2 + (1 - \sqrt{\alpha})^2 \sum_{s=1}^{K} w_s \left( \hat{b}_s - \sum_{s'=1}^{K} w_{s'} \hat{b}_{s'} \right)^2 \\
+ 2(1 - \sqrt{\alpha}) \sqrt{\sum_{s=1}^{K} w_s (b_s^* - \hat{b}_s)^2} \sqrt{\sum_{s=1}^{K} w_{s'} (\hat{b}_{s'} - \hat{b}_s - \sum_{s''=1}^{K} w_{s''} \hat{b}_{s''})^2}.
\]
where (a) follows from the fact that \( X \) is centered and (b) is due to the Cauchy-Schwarz inequality.

Theorem D.6 (Risk-unfairness bound for any \( \tau \)). Recall the definition of \( \delta_n(p, K, t) \)
\[
\delta_n(p, K, t) = 8 \left( \frac{p}{n} + \frac{K}{n} \right) + 16 \left( \sqrt{\frac{p}{n}} + \sqrt{\frac{K}{n}} \right) \sqrt{\frac{t}{n}} + \frac{32t}{n}.
\]
On the event
\[
\mathcal{A} = \left\{ \| \Sigma^{1/2}(\beta^* - \hat{\beta}) \|_2^2 + \sum_{s=1}^{K} w_s (b_s^* - \hat{b}_s)^2 \leq \sigma^2 \delta_n(p, K, t) \right\},
\]
it holds that
\[
\mathcal{R}(\hat{f}_r) \leq \left( \sigma \delta_{\hat{f}_r}(p, K, t) + (1 - \sqrt{\tau}) \mathcal{U}(\hat{f}_r) \right)^2.
\]
\[
\mathcal{U}(\hat{f}_r) \leq \tau \mathcal{U}(f^*) \left( 1 + \text{NUR} \delta_{\hat{f}_r}(p, K, t) \right)^2.
\]

**Proof.** We recall that Lemma 4.3 gives, for any \( \tau \in (0, 1) \),
\[
\mathcal{U}(\hat{f}_r) = \tau \sum_{s=1}^{K} w_s \left( \hat{b}_s - \sum_{s'=1}^{K} w_{s'} \hat{b}_{s'} \right)^2.
\]  
(40)

Let us start by proving the first part of the statement. Using Lemma D.5 to upper bound the risk \( \mathcal{R}(\hat{f}_r) \) and the definition of \( \mathcal{A} \) to control this upper bound, we obtain
\[
\mathcal{R}(\hat{f}_r) \leq \sigma^2 \delta_n(p, K, t) + 2 \sigma \sqrt{\delta_n(p, K, t)} \left( 1 - \sqrt{\tau} \right) \sqrt{\mathcal{U}(\hat{f}_1)} + (1 - \sqrt{\tau})^2 \mathcal{U}(\hat{f}_1)
\]
\[
= \left( \sigma \sqrt{\delta_n(p, K, t)} + (1 - \sqrt{\tau}) \sqrt{\mathcal{U}(\hat{f}_1)} \right)^2.
\]
The second part of the statement follow by applying Lemma D.4 and Theorem D.2 to get
\[
\mathcal{U}(\hat{f}_r) \leq \tau \mathcal{U}(f^*) \left( 1 + \text{NUR} \sqrt{\delta_n(p, K, t)} \right)^2.
\]  
(41)

**D.3. Proof of Theorem 4.4.** We set \( \tilde{f} := \hat{f}_1 \) and \( \delta_n = \delta_n(p, K, t) \). The proof relies on Eq. (37) of Lemma 4.3. Using notations of Theorem D.6 we also define the event
\[
\mathcal{A} = \left\{ \| \Sigma^{1/2} (\beta^* - \hat{\beta}) \|_2^2 + \sum_{s=1}^{K} w_s (b^*_s - \hat{b}_s)^2 \leq \sigma^2 \delta_n(p, K, t) \right\}
\]
(42)

which holds with probability at least \( 1 - 4 \exp(-t) \).

**Case 1.** Assume that \( \mathcal{U}^{1/2}(\tilde{f}) > \sigma \delta^{1/2}_n(p, K, t) \). Note that thanks to Theorem D.6, and the definition of \( \hat{\tau} \) we derive on the event \( \mathcal{A} \) that
\[
\mathcal{U}(\tilde{f}_r) \leq \tau \mathcal{U}(f^*) \left( 1 + \sqrt{\frac{\delta_n}{\mathcal{U}(f^*)}} \right)^2 = \sigma \mathcal{U}(f^*) \left( 1 + \sqrt{\frac{\delta_n}{\mathcal{U}(f^*)}} \right)^2 \left( 1 + \frac{\sigma \delta^{1/2}_n}{\mathcal{U}^{1/2}(\tilde{f}) - \sigma \delta^{1/2}_n} \right)^{-2}
\]
\[
\overset{(a)}{=} \sigma \mathcal{U}(f^*) \left( 1 + \frac{\sigma \delta^{1/2}_n}{\mathcal{U}^{1/2}(f^*)} \right)^{-2} \mathcal{U}(f^*)
\]

In the last equation, inequality (a) follows from Eq. (37) of Lemma 4.3 and thanks to the fact that on the event \( \mathcal{A} \) it holds that \( \mathcal{U}^{1/2}(\tilde{f}) \leq \mathcal{U}^{1/2}(f^*) + \left\{ \sum_{s=1}^{K} w_s (b^*_s - \hat{b}_s)^2 \right\}^{1/2} \leq \mathcal{U}^{1/2}(f^*) + \sigma \delta^{1/2}_n \). For the risk we have thanks to Theorems D.6 that
\[
\mathcal{R}(\tilde{f}_r) \leq \left( \frac{\sigma \sqrt{\delta_n} + (1 - \sqrt{\tau}) \sqrt{\mathcal{U}(f^*)}}{\sqrt{\mathcal{U}(f^*)}} \right)^2.
\]  
(43)
Furthermore, we note that
\[
\sqrt{\hat{\mathcal{U}}(\hat{f})} = \sqrt{\alpha} \frac{\sqrt{\mathcal{U}(\hat{f})}}{1 + \frac{\sigma \sqrt{\delta_n}}{\sqrt{\mathcal{U}(\hat{f})} - \sigma \sqrt{\delta_n}}} = \sqrt{\alpha} \left( \sqrt{\mathcal{U}(\hat{f})} - \sigma \sqrt{\delta_n} \right) \geq \sqrt{\alpha} \left( \sqrt{\mathcal{U}(f^*)} - 2\sigma \sqrt{\delta_n} \right),
\]
where inequality (b) again follows from Eq. (37) of Lemma 4.3 and thanks to the fact that on the event \( A \) it holds that \( \mathcal{U}^{1/2}(\hat{f}) \geq \mathcal{U}^{1/2}(f^*) - \left\{ \sum_{s=1}^{K} w_s(\hat{b}_s - b^*_s)^2 \right\}^{1/2} \geq \mathcal{U}^{1/2}(f^*) - \sigma \delta_n^{1/2} \).

Recall, that we have already shown that on the event \( A \) we have
\[
\mathcal{U}^{1/2}(\hat{f}) \leq \mathcal{U}^{1/2}(f^*) + \sigma \delta_n^{1/2}.
\]
Combining Eqs. (44) and (45) we obtain
\[
(1 - \sqrt{\pi}) \sqrt{\mathcal{U}(\hat{f})} \leq \sqrt{\mathcal{U}(f^*)} + \sigma \sqrt{\delta_n} - \sqrt{\alpha} \left( \sqrt{\mathcal{U}(f^*)} - 2\sigma \sqrt{\delta_n} \right) = (1 - \sqrt{\alpha}) \sqrt{\mathcal{U}(f^*)} + (1 + 2\sqrt{\alpha}) \sigma \sqrt{\delta_n}
\]
Thus since the function \((\sigma \delta_n^{1/2} + \cdot)^2\) is increasing on \([-\sigma \sqrt{\delta_n}, \infty)\) we get from Eq. (43) that
\[
\mathcal{R}(\hat{f}_z) \leq \left( 2(1 + \sqrt{\alpha}) \sigma \sqrt{\delta_n} + (1 - \sqrt{\alpha}) \sqrt{\mathcal{U}(f^*)} \right)^2,
\]
which concludes the proof of the first case.

**Case 2.** if \( \mathcal{U}^{1/2}(\hat{f}) \leq \sigma \delta_n^{1/2} \) \((p, K, t)\), then
\[
\hat{f}_0(x, s) = (x, \hat{\beta}) + \sum_{s=1}^{K} w_s \hat{b}_s.
\]
Furthermore, on the event \( A \) thanks to Theorem D.6 it holds that \( 0 = \mathcal{U}(\hat{f}_0) \leq \alpha \mathcal{U}(f^*) \) and
\[
\mathcal{R}(\hat{f}_0) \leq \left( \sigma \delta_n^{1/2} + \mathcal{U}^{1/2}(\hat{f}) \right)^2 = \left( \sigma \delta_n^{1/2} + \sqrt{\alpha} \mathcal{U}^{1/2}(\hat{f}) + (1 - \sqrt{\alpha}) \mathcal{U}^{1/2}(\hat{f}) \right)^2 \leq \left( (1 + \sqrt{\alpha}) \sigma \delta_n^{1/2} + (1 - \sqrt{\alpha}) \mathcal{U}^{1/2}(\hat{f}) \right)^2 \leq \left( (1 + \sqrt{\alpha}) \sigma \delta_n^{1/2} + (1 - \sqrt{\alpha}) \left( \mathcal{U}^{1/2}(f^*) + \sigma \delta_n^{1/2} \right) \right)^2 = \left( 2\sigma \delta_n^{1/2} + (1 - \sqrt{\alpha}) \mathcal{U}^{1/2}(f^*) \right)^2.
\]
The proof is concluded by application of Theorem D.2 to control the probability of event \( A \).

**D.4. Auxiliary results for Theorem 4.5.** Let us first present auxiliary results used for the proof of Theorem 4.5. The next lemma is known as Varshamov-Gilbert Lemma (Gilbert, 1952, Varshamov, 1957), its statement is taken from (Rigollet and Hütter, 2015, Lemma 4.12).

**Lemma D.7.** Let \( d \geq 1 \) be an integer. There exist binary vectors \( \omega_1, \ldots, \omega_M \in \{0, 1\}^d \) such that
1. $\rho(\omega_j, \omega_{j'}) \geq d/4$ for all $j \neq j'$,
2. $M = [e^{d/10}] \geq e^{d/32},$

where $\rho(\cdot, \cdot)$ is the Hamming’s distance on binary vectors.

The next lemmas can be found in (Bellec et al., 2017, Lemma 5.1), see also (Kerkyacharian et al., 2014, Lemma 3).

**Lemma D.8.** Let $(\Omega, A)$ be a measurable space and $M \geq 1$. Let $A_0, \ldots, A_M$ be disjoint measurable events. Assume that $Q_0, \ldots, Q_M$ are probability measures on $(\Omega, A)$ such that

$$\frac{1}{M} \sum_{j=1}^{M} KL(Q_j, Q_0) \leq \kappa < \infty.$$

Then,

$$\max_{j=0, \ldots, M} Q_j(A_j^c) \geq \frac{1}{12} \min(1, M \exp(-3\kappa)) .$$

Define the diagonal matrix $W = \text{diag}(w_1, \ldots, w_K)$.

**Lemma D.9.** Let $n \geq 1$ be an integer and $s > 0$ be a positive number. Let $M \geq 1$ and $(\beta_j, b_j) \in \mathbb{R}^p \times \mathbb{R}^K$, $j = 0, \ldots, M$, such that $\|\Sigma^{1/2}(\beta_j - \beta_k)\|^2_2 + \|W^{1/2}(b_j - b_k)\|^2_2 \geq 4s$ for $j \neq k$. Assume that

$$\frac{1}{M} \sum_{j=1}^{M} KL(P_{(\beta_j, b_j)}, P_{(\beta_0, b_0)}) \leq \kappa < \infty.$$

Then, for any estimator $\hat{f}$,

$$\max_{j=0, \ldots, M} P_{(\beta_j, b_j)} \left( R(\hat{f}) \geq s \right) \geq \frac{1}{12} \min(1, M \exp(-3\kappa)) .$$

**Proof.** Denote by $A_j$ the event $R_j(\hat{f}) < s$ for $j = 1, \ldots, M$. Note that the events $A_0, \ldots, A_M$ are pair-wise disjoint. Indeed, if they were not there would exist indices $j$ and $j'$, with $j \neq j'$, such that, on the non-empty event $A_j \cap A_{j'}$,

$$\|\Sigma^{1/2}(\beta_j - \beta_{j'})\|^2_2 + \|W^{1/2}(b_j - b_{j'})\|^2_2 \leq 2R_j(\hat{f}) + 2R_{j'}(\hat{f}) < 4s \quad (46)$$

contradicting our assumption on the $(\beta_j, b_j)$ and $(\beta_{j'}, b_{j'})$. We conclude applying Lemma D.8.

**D.5. Proof of Theorem 4.5.** Define the $(p + K) \times (p + K)$ matrix

$$\Psi = \begin{bmatrix} \Sigma & 0 \\ 0 & W \end{bmatrix} , \quad (47)$$

Apply Lemma D.7 to obtain $\omega_0, \ldots, \omega_M$ with $M + 1 \geq e^{(p+K)/32}$ and such that $\rho(\omega_j, \omega_k) \geq (p + K)/4$. Let $B_0 = (\beta_0, b_0), \ldots, B_M = (\beta_M, b_M)$ be such that

$$B_j = \varphi \sqrt{\frac{\sigma^2}{n} \left( 1 + \sqrt{1/(p + K)} \right)} \Psi^{-1/2} \omega_j , \quad (48)$$
with \( p + K \leq 32 \log(M) \) and \( \varphi > 0 \) to be determined later.

On the one hand we have
\[
\| \Sigma^{1/2}(\beta_j - \beta_k) \|_2^2 + \| W^{1/2}(b_j - b_k) \|_2^2 = \frac{\varphi^2 \sigma^2}{n} (1 + \sqrt{t/(p + K)})^2 \rho(\omega_j, \omega_j') \\
\geq \frac{\varphi^2 \sigma^2}{n} (1 + \sqrt{t/(p + K)})^2 (p + K)/4 \\
= \frac{\varphi^2 \sigma^2}{4n} (\sqrt{p + K} + \sqrt{t})^2.
\]

On the other hand, recall that \( \mathbb{P}_{X, S = s} = \mathcal{N}(\langle X, \beta \rangle + b_s, \sigma^2) \) and \( \mathbb{P}_X = \mathcal{N}(0, \Sigma) \), then, for a given \( (\beta, b) \in \mathbb{R}^p \times \mathbb{R}^K \) the joint distribution of observations is
\[
\mathbb{P}_{(\beta, b)} = \bigotimes_{s=1}^K \left( \mathcal{N}(\langle X, \beta \rangle + b_s, \sigma^2) \otimes \mathcal{N}(0, \Sigma) \right)^{\otimes n_s}.
\]

Given \( B = (\beta, b), B' = (\beta', b') \) in \( \mathbb{R}^p \times \mathbb{R}^K \) we can write
\[
\text{KL} \left( \mathbb{P}_{(\beta, b)}, \mathbb{P}_{(\beta', b')} \right) = \sum_{s=1}^K n_s \mathbb{E}_{X \sim \mathcal{N}(0, \Sigma)} \left[ \widetilde{\text{KL}} \left( \mathcal{N}(\langle X, \beta \rangle + b_s, \sigma^2), \mathcal{N}(\langle X, \beta' \rangle + b'_s, \sigma^2) \right) \right] \\
= \sum_{s=1}^K n_s \mathbb{E}_{X \sim \mathcal{N}(0, \Sigma)} \left( \frac{\| \Sigma^{1/2}(\beta - \beta') \|_2^2}{2\sigma^2} + \frac{(b_s - b'_s)^2}{2\sigma^2} \right) \\
= \sum_{s=1}^K n_s \left( \frac{\| \Sigma^{1/2}(\beta - \beta') \|_2^2}{2\sigma^2} + \frac{\| W^{1/2}(b - b') \|_2^2}{2\sigma^2} \right) \\
= \frac{n}{2\sigma^2} \| \Psi^{1/2}(B - B') \|_2^2 \\
\leq \frac{\varphi^2}{2} (\sqrt{p + K} + \sqrt{t})^2 \leq \varphi^2 (p + K) + \varphi^2 t \leq 32 \varphi^2 \log(M) + \varphi^2 t.
\]

Let \( \hat{f} \) be any estimator and define the risks
\[
\mathcal{R}_j(\hat{f}) = \sum_{s=1}^K w_s \mathbb{E} \left[ (\hat{f}(X, S) - \langle X, \beta_j \rangle - (b_j)_S)^2 \mid S = s \right], \quad j = 1, \ldots, M.
\]

Set \( u_n(p, K, t, \gamma, \sigma) = \frac{\gamma^2 \sigma^2}{16n} (\sqrt{p + K} + \sqrt{t})^2 \). Applying Lemma D.9 after reducing the supremum to a finite number of hypothesis, we get for all estimators \( \hat{f} \) that
\[
\sup_{(\beta^*, b^*) \in \mathbb{R}^p \times \mathbb{R}^K} \mathbb{P}_{(\beta^*, b^*)} \left( \mathcal{R}(\hat{f}) \geq u_n(p, K, t, \gamma, \sigma) \right) \geq \max_{j=0, \ldots, M} \mathbb{P}_{(\beta_j, b_j)} \left( \mathcal{R}_j(\hat{f}) \geq u_n(p, K, t, \gamma, \sigma) \right) \\
\geq \frac{1}{12} \min \left( 1, M \exp \left( -96 \gamma^2 \log(M) - 3\gamma^2 t \right) \right)
\]
Setting $\gamma = 1/\sqrt{96}$, we obtain

$$
\sup_{(\beta^*, b^*) \in \mathbb{R}^p \times \mathbb{R}^K} \mathbb{P}_{(\beta^*, b^*)} \left( R(\hat{f}) \geq \frac{\sigma^2}{1536n} \left( \sqrt{p + K} + \sqrt{t} \right)^2 \right) \geq \frac{1}{12} \exp \left( -\frac{t}{32} \right).
$$

The proof is concluded.

REFERENCES

A. Agarwal, M. Dudik, and Z. S. Wu. Fair regression: Quantitative definitions and reduction-based algorithms. In International Conference on Machine Learning, 2019.

M. Agueh and G. Carlier. Barycenters in the wasserstein space. SIAM Journal on Mathematical Analysis, 43 (2):904–924, 2011.

J.-Y. Audibert and O. Catoni. Robust linear least squares regression. The Annals of Statistics, 39(5):2766–2794, 2011.

S. Baharlouei, M. Nouiehed, A. Beirami, and M. Razaviyayn. Rényi fair inference. arXiv preprint arXiv:1906.12005, 2019.

S. Barocas, M. Hardt, and A. Narayanan. Fairness and Machine Learning. Fairmlbook.org, 2019. http://www.fairmlbook.org.

P. C. Bellec et al. Optimal exponential bounds for aggregation of density estimators. Bernoulli, 23(1):219–248, 2017.

R. Berk, H. Heidari, S. Jabbari, M. Joseph, M. Kearns, J. Morgenstern, S. Neel, and A. Roth. A convex framework for fair regression. In Fairness, Accountability, and Transparency in Machine Learning, 2017.

D. Bertsimas, V. F. Farias, and N. Trichakis. On the efficiency-fairness trade-off. Management Science, 58(12):2234–2250, 2012.

T. Calders, F. Kamiran, and M. Pechenizkiy. Building classifiers with independency constraints. In IEEE international conference on Data mining, 2009.

T. Calders, A. Karim, F. Kamiran, W. Ali, and X. Zhang. Controlling attribute effect in linear regression. In IEEE International Conference on Data Mining, 2013.

O. Catoni. Statistical learning theory and stochastic optimization. Ecole d’été de probabilités de Saint-Flour XXXI-2001. Springer, 2004. URL https://hal.archives-ouvertes.fr/hal-00104952. Collection : Lecture notes in mathematics n°1851.

M. Chen, C. Gao, Z. Ren, et al. A general decision theory for huber’s $\epsilon$-contamination model. Electronic Journal of Statistics, 10(2):3752–3774, 2016.

M. Chen, C. Gao, Z. Ren, et al. Robust covariance and scatter matrix estimation under huber’s contamination model. The Annals of Statistics, 46(5):1932–1960, 2018.

S. Chiappa, R. Jiang, T. Stepleton, A. Pacchiano, H. Jiang, and J. Aslanides. A general approach to fairness with optimal transport. In AAAI, 2020.

E. Chzhen, C. Denis, M. Hebiri, L. Oneto, and M. Pontil. Fair regression with wasserstein barycenters. arXiv preprint arXiv:2006.07286, 2020a.

E. Chzhen, C. Denis, M. Hebiri, L. Oneto, and M. Pontil. Fair regression via plug-in estimator and recalibration with statistical guarantees. arXiv preprint arXiv, 2020b.

E. del Barrio, P. Gordaliza, and J.-M. Loubes. Review of mathematical frameworks for fairness in machine learning. arXiv preprint arXiv:2005.13755, 2020.

M. Donini, L. Oneto, S. Ben-David, J. S. Shawe-Taylor, and M. Pontil. Empirical risk minimization under fairness constraints. In Neural Information Processing Systems, 2018.

C. Dwork, M. Hardt, T. Pitassi, O. Reingold, and R. Zemel. Fairness through awareness. In Proceedings of the 3rd innovations in theoretical computer science conference, pages 214–226, 2012.

J. Fitzsimons, A. A. Ali, M. Osborne, and S. Roberts. Equality constrained decision trees: For the algorithmic enforcement of group fairness. arXiv preprint arXiv:1810.05041, 2018.

J. Fitzsimons, A. Al Ali, M. Osborne, and S. Roberts. A general framework for fair regression. Entropy, 21(8):741, 2019.

M. Fréchet. Sur la distance de deux lois de probabilité. Comtes Rendus Hebdomadaires des Seances de l’Academie des Sciences, 244(6):689–692, 1957.

W. Gangbo and A. Święch. Optimal maps for the multidimensional monge-kantorovich problem. Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences, 51 (1):23–45, 1998.
E. Gilbert. A comparison of signalling alphabets. *The Bell system technical journal*, 31(3):504–522, 1952.

P. Gordaliza, E. Del Barrio, G. Fabrice, and J. M. Loubes. Obtaining fairness using optimal transport theory. In *International Conference on Machine Learning*, 2019.

L. Györfi, M. Kohler, A. Krzyzak, and H. Walk. *A distribution-free theory of nonparametric regression*. Springer Science & Business Media, 2006.

C. Haas. The price of fairness—a framework to explore trade-offs in algorithmic fairness. *arXiv preprint arXiv*, 2019.

M. Hardt, E. Price, and N. Srebro. Equality of opportunity in supervised learning. In *Neural Information Processing Systems*, 2016.

D. Hsu, S. M. Kakade, and T. Zhang. Random design analysis of ridge regression. In *Conference on learning theory*, pages 9–1, 2012.

J. D. Hunter. Matplotlib: A 2d graphics environment. *Computing in Science & Engineering*, 9(3):90–95, 2007.

G. Kerkyacharian, A. B. Tsybakov, V. Temlyakov, D. Picard, and V. Koltchinskii. Optimal exponential bounds on the accuracy of classification. *Constructive Approximation*, 39(3):421–444, 2014.

B. Kloeckner. A geometric study of wasserstein spaces: Euclidean spaces. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, 9(2):297–323, 2010.

J. Komiyama and H. Shimao. Two-stage algorithm for fairness-aware machine learning. *arXiv preprint arXiv:1710.04924*, 2017.

J. Komiyama, A. Takeda, J. Honda, and H. Shimao. Nonconvex optimization for regression with fairness constraints. In *International Conference on Machine Learning*, 2018.

M. Köeppen, K. Yoshida, and K. Ohnishi. Evolving fair linear regression for the representation of human-drawn regression lines. In *2014 International Conference on Intelligent Networking and Collaborative Systems*, pages 296–303, 2014.

B. Laurent and P. Massart. Adaptive estimation of a quadratic functional by model selection. *Annals of Statistics*, pages 1302–1338, 2000.

T. Le Gouic and J.-M. Loubes. Existence and consistency of wasserstein barycenters. *Probability Theory and Related Fields*, 168(3-4):901–917, 2017.

T. Le Gouic, J. Loubes, and P. Rigollet. Projection to fairness in statistical learning. *arXiv preprint arXiv:2005.11720*, 2020.

D. Madras, E. Creager, T. Pitassi, and R. Zemel. Learning adversarially fair and transferable representations. In *International Conference on Machine Learning*, pages 3384–3393, 2018.

J. Mary, C. Calauzènes, and N. El Karoui. Fairness-aware learning for continuous attributes and treatments. In *International Conference on Machine Learning*, pages 4382–4391, 2019.

N. Mehrabi, F. Morstatter, N. Saxena, K. Lerman, and A. Galstyan. A survey on bias and fairness in machine learning. *arXiv preprint arXiv:1908.09635*, 2019.

J. Mourtada. Exact minimax risk for linear least squares, and the lower tail of sample covariance matrices. *arXiv preprint arXiv:1912.10754*, 2019.

A. Nemirovski. Topics in non-parametric statistics. *Lecture Notes in Mathematics*, 1738:86–282, 2000.

M. Olfat, S. Sloan, P. Hespanhol, M. Porter, R. Vasudevan, and A. Aswani. Covariance-robust dynamic watermarking. *arXiv preprint arXiv:2003.13908*, 2020.

L. Oneto and S. Chiappa. Fairness in machine learning. In *Recent Trends in Learning From Data*, pages 155–196. Springer, 2020.

L. Oneto, M. Donini, A. Maurer, and M. Pontil. Learning fairness and transferable representations. *arXiv preprint arXiv:1906.10673*, 2019a.

L. Oneto, M. Donini, and M. Pontil. General fair empirical risk minimization. *arXiv preprint arXiv:1901.10080*, 2019b.

F. Pedregosa, G. Varoquaux, A. Gramfort, V. Michel, B. Thirion, O. Grisel, M. Blondel, P. Prettenhofer, R. Weiss, V. Dubourg, J. Vanderplas, A. Passos, D. Cournapeau, M. Brucher, M. Perrot, and E. Duchesnay. Scikit-learn: Machine learning in Python. *Journal of Machine Learning Research*, 12:2825–2830, 2011.

A. Pérez-Suay, V. Laparra, G. Mateo-García, J. Muñoz-Mari, L. Gómez-Chova, and G. Camps-Valls. Fair kernel learning. In *Joint European Conference on Machine Learning and Knowledge Discovery in Databases*, 2017.

D. Plečko and N. Meinshausen. Fair data adaptation with quantile preservation. *arXiv preprint arXiv:1911.06685*, 2019.

N. Quadrianto and V. Sharmanska. Recycling privileged learning and distribution matching for fairness. In *Advances in Neural Information Processing Systems*, pages 677–688, 2017.

E. Raff, J. Sylvester, and S. Mills. Fair forests: Regularized tree induction to minimize model bias. In
RISK-FAIRNESS TRADE-OFF IN REGRESSION

AAAI/ACM Conference on AI, Ethics, and Society, 2018.
P. Rigollet and J.-C. Hütter. High dimensional statistics. Lecture notes for course 18S997, 2015.
F. Santambrogio. Optimal transport for applied mathematicians. Springer, 2015.
D. Steinberg, A. Reid, and S. O’Callaghan. Fairness measures for regression via probabilistic classification. arXiv preprint arXiv:2001.06089, 2020a.
D. Steinberg, A. Reid, S. O’Callaghan, F. Lattimore, L. McCalman, and T. Caetano. Fast fair regression via efficient approximations of mutual information. arXiv preprint arXiv:2002.06200, 2020b.
A. B. Tsybakov. Optimal rates of aggregation. In Learning theory and kernel machines, pages 303–313. Springer, 2003.
S. Van Der Walt, S. C. Colbert, and G. Varoquaux. The numpy array: a structure for efficient numerical computation. Computing in Science & Engineering, 13(2):22, 2011.
R. Varshamov. Estimate of the number of signals in error correcting codes. Dokl. Akad. Nauk SSSR, 117:739—741, 1957.
R. Vershynin. Introduction to the non-asymptotic analysis of random matrices. arXiv preprint arXiv:1011.3027, 2010.
C. Villani. Topics in Optimal Transportation. American Mathematical Society, 2003.
M. Wick, S. Panda, and J.-B. Tristan. Unlocking fairness: a trade-off revisited. In Advances in Neural Information Processing Systems 32, pages 8783–8792. Curran Associates, Inc., 2019. URL http://papers.nips.cc/paper/9082-unlocking-fairness-a-trade-off-revisited.pdf.
M. B. Zafar, I. Valera, M. Gomez Rodriguez, and K. P. Gummadi. Fairness beyond disparate treatment & disparate impact: Learning classification without disparate mistreatment. In International Conference on World Wide Web, 2017.
A. Zink and S. Rose. Fair regression for health care spending. Biometrics, n/a(n/a), 2019.
I. Zliobaite. On the relation between accuracy and fairness in binary classification. arXiv preprint arXiv:1505.05723, 2015.