ON PRODUCT AND GOLDEN STRUCTURES
AND
HARMONICITY

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Abstract: In this work, almost product and almost golden structures are studied. Conditions for those structures being integrable and parallel are investigated. Also harmonicity of a map between almost product or almost golden manifolds with pure or hyperbolic metric is discussed under certain conditions.

Key words: Golden structure, golden map, pure and hyperbolic metric, product structure, harmonic maps.

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1 1. Introduction

Let \( M^n \) be a smooth manifold of dimension \( n \) with a \((1,1)\)-tensor field \( \varphi \) of rank \( n \). Then (see [3, 4, 5, 8, 12, 13]) the pair \((M, \varphi)\) is called polynomial manifold provided \( U(\varphi) = 0 \) for some polynomial \( U(x) \) over the field of real numbers \( \mathbb{R} \). In particular, \( \varphi \) and \((M, \varphi)\) are respectively called

i) metallic structure and metallic manifold if \( U(x) = x^2 - \eta x - \delta \) for some positive integers \( \eta \) and \( \delta \), so that \( U(\varphi) = \varphi^2 - \eta \varphi - \delta = 0 \).

ii) almost complex structure and almost complex manifold if \( U(x) = x^2 + 1 \), so that \( U(\varphi) = \varphi^2 + I = 0 \).

iii) almost product structure and almost product manifold if \( U(x) = x^2 - 1 \). In this case we reserve the letter \( P \) for \( \varphi \). Thus \( U(P) = P^2 - I = 0 \).

iv) almost golden structure and almost golden manifold if \( U(x) = x^2 - x - 1 \). In this case we reserve the letter \( G \) for \( \varphi \). Thus \( U(G) = G^2 - G - I = 0 \).

Where \( I \) denotes the identity tensor field.

Note here that an almost golden manifold \((M, G)\) is in fact a metallic manifold with \( \eta = \delta = 1 \). Golden structure has been catching more attention of many geometers (see: for example [3, 4, 12, 13]) in the last few years as it is closely related to the golden ratio which plays an important role in various disciplines such as physics, topology, probability, field theory etc. (see [3, 4] and the references therein)
In this work, we have dealt with almost product and almost golden structures simultaneously as one can be obtained from the other, and provided the following results besides some side ones:

Let \( \varphi \) denote either almost product structure \( P \) or almost golden structure \( G \). To emphasize this we shall be writing \( \varphi (= P, G) \). On an almost product or an almost golden manifold \( (M, h, \varphi (= P, G)) \) with pure or hyperbolic metric \( h \), (Definition (2.1)).

1) By analogy with the result for the paracomplex case, we introduce a condition \( \varphi(*) \) (see page 12, just before Proposition (2.5)) which, together with the integrability condition of \( \varphi \), guarantees that \( \varphi \) is parallel, (Proposition (2.5)).

2) For the bilinear operator \( S_\varphi : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM) \) (see : right after Definition (2.4)) it is shown that vanishing of \( S_\varphi \) is equivalent to that of \( \varphi \) being parallel, (Proposition (2.6)), unlike the case in which the metric \( h \) is hyperbolic, vanishing of \( S_\varphi \) does not imply that \( \varphi \) is parallel. Instead, it provides a bigger class whose members are called quasi para-Hermitian manifolds, quasi golden-Hermitian manifolds (Definition (2.5)).

3) We introduced a subclass of \( (M, h, \varphi (= P, G)) \), namely, a class of semi decomposable product (or golden) Riemannian manifolds (Definition (2.4)) and that used later on for the harmonicity of certain map from, (Theorems (3.1) & (3.2)).

4) By analogy with the concept of anti-paraholomorphic map, a concept of antigolden map is introduced (Definition (3.2)) and that later it is used for its harmonicity, (Theorems (3.1) & (3.2)).

5) It is shown that being a golden (resp : paraholomorphic) map of an almost golden (resp : almost product) manifold with a pure metric is no way sufficient for its harmonicity where as it is sufficient when the metric is hyperbolic. However, on the same line, an alternative result is provided, (Theorem (3.1)).

6) Finally, (Theorems (3.1) & (3.2)), for a non-constant map \( F : (M, h, \varphi (= P, G)) \to (N, g, \varphi (= Q, K)) \), where \( h \) and \( g \) are hyperbolic, the harmonicity result given for \( \pm (P, Q) \)-paraholomorphic map \( F \), ([2, 7, 11]), is extended to the cases where

- \( F \) is \( \pm (P, Q) \)-paraholomorphic and \( h \) is hyperbolic, \( g \) is pure.
- \( F \) is \( \pm (G, K) \)-golden and \( h \) is hyperbolic, \( g \) is pure or hyperbolic.

7) Overall, we have managed so far to express the results involving almost golden structures in terms of almost product structures.

2 Definitions and some basic results

The structure \( \varphi (= P, G) \) on \( M^n \) has two distinct real eigenvalues, namely; \( k \) and \( \bar{k} \). Let denote the corresponding eigendistributions by \( E_{(k)} \), and \( E_{(\bar{k})} \).

Note that, (see [2, 3, 4, 5, 8, 12, 13]),

1) \( \varphi : TM \to TM \) is an isomorphism.
2) \( TM = E_{(k)} \oplus E_{(\bar{k})} \).
3) For an almost product manifold \((M, P)\) we have

- \(k = 1\) and \(\overline{k} = -1\).
- \(P^2(X) = X, \quad \forall \ X \in \Gamma(TM)\)

4) For an almost golden manifold \((M, G)\) we have

- \(k = \frac{1}{2} (1 + \sqrt{5})\) and \(\overline{k} = \frac{1}{2} (1 - \sqrt{5})\).

Throughout this work we shall be setting \(\sigma = \frac{1}{2} (1 + \sqrt{5})\) and \(\overline{\sigma} = \frac{1}{2} (1 - \sqrt{5})\). Observe that,

\[\sigma^2 = \sigma + 1, \quad \overline{\sigma}^2 = \overline{\sigma} + 1 \quad \text{and} \quad \sigma \overline{\sigma} = -1.\]

- \(G^2(X) = GX + X, \quad \forall \ X \in \Gamma(TM)\).

5)

- for every almost product structure \(P\), define a \(P\)-associated \((1, 1)\)-tensor field \(G_P = \mathcal{G}\) by
  \[G_p = \mathcal{G} = \frac{1}{2} (I + \sqrt{5}P)\]

- for every almost golden structure \(G\), define a \(G\)-associated \((1, 1)\)-tensor field \(P_G = \mathcal{R}\) by
  \[P_G = \mathcal{R} = \frac{1}{\sqrt{5}} (2G - I)\]

Note that,

- \(i)\) for every almost product structure \(P\) on \(M\), the corresponding \(G_P = \mathcal{G}\) is an almost golden structure on \(M\) and therefore it will be called \(P\)-associated almost golden structure.

- \(ii)\) for every almost golden structure \(G\) on \(M\), the corresponding \(P_G = \mathcal{R}\) is an almost product structure on \(M\) and therefore it will be called \(G\)-associated almost product structure.

- \(iii)\) we have

\[\mathcal{E}_G^\mathcal{G}(\sigma) = \mathcal{E}_P^P(1) \quad \text{and} \quad \mathcal{E}_G^\mathcal{G}(\overline{\sigma}) = \mathcal{E}_P^P(-1)\]

- \(iv)\) there is a one-to-one correspondence between the set of all almost product structures and the set of all almost golden structures on a manifold \(M\). We shall be calling \(\{P, G_P\}\) (or \(\{G, P_G\}\)) an associated pair or a twin pair. we also say that \(\{P, G_P\}\) (or \(\{G, P_G\}\)) are twins. It is easy to see that for a given pair of twin structures \(\{P, G_P\}\), the \(G_P\)-associated almost product structure is equal to \(P\), that is,

\[P_{(G_P)} = P\]
Similarly, for a twin pair \( \{ G, P_G \} \), the \( P_G \)-associated almost golden structure is equal to \( G \), that is,
\[
G(P_G) = G
\]

v) If \( P \) is an almost product structure on \( M \) then \( \hat{P} = -P \) is also an almost product structure on \( M \). Observe that \( P \) and \( \hat{P} \) have the same eigenvalues 1 and \(-1\). However for their corresponding eigendistributions we have
\[
\mathcal{E}_{(1)}^P = \mathcal{E}_{(-1)}^{\hat{P}} \quad \text{and} \quad \mathcal{E}_{(-1)}^P = \mathcal{E}_{(1)}^{\hat{P}}
\]
We shall be calling \( \hat{P} \), the conjugate almost product structure of \( P \) or the \( P \)-conjugate almost product structure.

vi) If \( G \) is an almost golden structure on \( M \) then \( \hat{G} = I - G \) is also an almost golden structure on \( M \). Observe that \( G \) and \( \hat{G} \) have the same eigenvalues \( \sigma \) and \( \bar{\sigma} \). However for their corresponding eigendistributions we have
\[
\mathcal{E}_{(\sigma)}^G = \mathcal{E}_{(\bar{\sigma})}^{\hat{G}} \quad \text{and} \quad \mathcal{E}_{(\bar{\sigma})}^G = \mathcal{E}_{(\sigma)}^{\hat{G}}
\]
We shall be calling \( \hat{G} \), the conjugate almost golden structure of \( G \) or the \( G \)-conjugate almost golden structure.

vii) If \( \{ P, G \} \) is a twin pair then \( \hat{G} = G_{\hat{P}} = \frac{1}{2} \left( I + \sqrt{5}P_G \right) \) is also a twin pair. Conversely, if \( \{ \hat{P}, \hat{G} \} \) is a twin pair then \( G = \frac{1}{2} \left( I + \sqrt{5}P \right) \), that is, \( \{ P, G \} \) is also a twin pair.

viii) If \( \{ \hat{P}, \hat{G} \} \) is a twin pair then
\[
\mathcal{E}_{(1)}^P = \mathcal{E}_{(-1)}^{\hat{P}} = \mathcal{E}_{(\sigma)}^G = \mathcal{E}_{(\bar{\sigma})}^{\hat{G}} \quad \text{and} \quad \mathcal{E}_{(-1)}^P = \mathcal{E}_{(1)}^{\hat{P}} = \mathcal{E}_{(\bar{\sigma})}^G = \mathcal{E}_{(\sigma)}^{\hat{G}}.
\]

An almost product manifold \((M, P)\) is called an almost paracomplex manifold if the eigendistributions \( \mathcal{E}_{(1)} \) and \( \mathcal{E}_{(-1)} \) are of the same rank, ([2, 5]). An almost golden manifold \((M, G)\) is called an almost para-golden manifold if the eigendistributions \( \mathcal{E}_{(\sigma)} \) and \( \mathcal{E}_{(\bar{\sigma})} \) are of the same rank. It clear from their definitions that an almost paracomplex manifold \((M, P)\) and an almost para-golden manifold \((M, G)\) are necessarily of even dimensions.

**Definition 1** (2.1/A): Let \( M \) be a smooth manifold together with a \((1,1)\) tensor field \( \varphi (= P, G) \) and a Riemannian metric \( h \) satisfying
\[
h(\varphi X, Y) = h(X, \varphi Y); \quad \forall X, Y \in \Gamma(TM).
\]
Then
i) \((M, h, P)\) is called almost product Riemannian manifold, [8].
ii) \((M, h, G)\) is called almost golden Riemannian manifold, ([4, 8])

We refer the condition \((*)\) as the compatibility of \( h \) and \( \varphi \). We also say "\( h \) is pure with respect to \( \varphi \)" if \( h \) and \( \varphi \) are compatible, and
call $h$ pure metric (with respect to $\varphi$). Note here that the eigendistributions $\mathcal{E}_k$ are $h$-orthogonal.

iii) An almost product Riemannian manifold $(M, h, P)$ and its metric $h$ are also called almost $B$-manifold and $B$-metric respectively if the eigendistributions $\mathcal{E}_{(1)}$ and $\mathcal{E}_{(-1)}$ are of the same rank, [12].

iv) An almost golden Riemannian manifold $(M, h, P)$ is also called almost para-golden Riemannian manifold if the eigendistributions $\mathcal{E}_{(\sigma)}$ and $\mathcal{E}_{(-\sigma)}$ are of the same rank.

**Definition 2** (2.1/B): Let $M$ be a smooth manifold together with a $(1,1)$ tensor field $\varphi (= P, G)$ and a nondegenerate metric $h$ satisfying

$h(\varphi X, Y) = h(X, \hat{\varphi} Y); \forall X, Y \in \Gamma(TM).$ ((**))

Then

i) $(M, h, P)$ is called almost para-Hermitian manifold, [2].

ii) $(M, h, G)$ is called almost golden-Hermitian manifold.

In this case, we refer the conditions (***) as the hyperbolic compatibility of $h$ and $\varphi$. We also say "$h$ is hyperbolic with respect to $\varphi$" if $h$ and $\varphi$ are hyperbolic compatible, and call $h$ hyperbolic metric (with respect to $\varphi$).

Note here that the hyperbolic case differs from the pure one. To be precise:

On a manifold $(M, h, \varphi)$ with a hyperbolic metric $h$ (with respect to $\varphi$) one has,

1) $h(\varphi X, Y) = h(PX, Y) = h(X, \hat{\varphi} Y) = -h(X, PY); \forall X, Y \in \Gamma(TM).$

Therefore we have

$h(PX, X) = 0, \forall X \in \Gamma(TM)$

unlike the pure case where, for example,

$h(PX, X) = h(X, X), \forall X \in \Gamma(\mathcal{E}_P^{(1)})$

2) $h(GX, Y) = h(X, \hat{G} Y); \forall X, Y \in \Gamma(D).$ Therefore we have

$2h(GX, X) = h(X, X) = 2h(\hat{G}X, X), \forall X \in \Gamma(TM).$

3) $h(X, Y) = 0; \forall X, Y \in \Gamma(\mathcal{E}_k^\varphi)$ or $\forall X, Y \in \Gamma(\mathcal{E}_k^\varphi).$

That is, hyperbolic metric $h$ is null on the eigendistributions $\mathcal{E}_k^\varphi$ and $\mathcal{E}_k^\varphi$ (and therefore the hyperbolic metric is necessarily semi-Riemannian where as the pure metric is taken to be Riemannian.)

Indeed, let $X, Y \in \Gamma(\mathcal{E}_k^\varphi)$ then $h(\varphi X, Y) = kh(X, Y)$ and $h(X, \varphi Y) = k h(X, Y).$ On the other hand, $h(\varphi X, Y) = h(X, \varphi Y)$ since


$h$ is hyperbolic. So $kh(X, Y) = \tilde{k}h(X, Y)$, which gives $(k - \tilde{k})h(X, Y) = 0$, so that $h(X, Y) = 0; \forall X, Y \in \Gamma \left( E^\varphi_k \right)$. By the same argument we get $\forall X, Y \in \Gamma \left( E^\varphi_\tilde{k} \right)$ $h(X, Y) = 0.$

**Lemma 3** (2.2/A): Let $(M, h, P)$ be an almost para-Hermitian manifold. Then

i) $h$ is of signature $(m, m)$ on $TM$ for some $m.$

ii) $\text{rank} \left( E^P_{(1)} \right) = \text{rank} \left( E^P_{(-1)} \right) = m.$

Having given an almost golden manifold $(M, h, G)$ with a hyperbolic metric $h$, since $h$ is also hyperbolic with respect to the product structure $P_G$, by considering the almost para-Hermitian manifold $(M, h, P_G)$ and using the above Lemma, we get:

**Lemma 4** (2.2/B): Let $(M, h, G)$ be an almost golden-Hermitian manifold. Then

i) $h$ is of signature $(m, m)$ on $TM$ for some $m.$

ii) $\text{rank} \left( E^G_{(1)} \right) = \text{rank} \left( E^G_{(-1)} \right) = \text{rank} \left( E^R_{(1)} \right) = m,$ where $R = P_G.$

**Proposition 5** (2.1): Let an almost product structure $P$ and an almost golden structure $G$ form a twin pair \{P, G\} on a smooth manifold $M.$ For a nondegenerate metric $h$ on $M$ the following statements are equivalent:

i) $h$ is pure \[\text{resp.} \] hyperbolic with respect to $P.$

ii) $h$ is pure \[\text{resp.} \] hyperbolic with respect to $\hat{P}.$

iii) $h$ is pure \[\text{resp.} \] hyperbolic with respect to $G.$

iv) $h$ is pure \[\text{resp.} \] hyperbolic with respect to $\hat{G}.$

**Proof.** We only be showing the equivalence of (i) and (iv) as the rest of the cases follow by the similar argument: ■

Assume (i), then $\forall X, Y \in \Gamma (TM)$

$h \left( X, \hat{G}Y \right) = h \left( X, \frac{1}{2} \left( I + \sqrt{5} \hat{P} \right) Y \right) = \frac{1}{2}h(X, Y) + \frac{\sqrt{5}}{2} h \left( X, \hat{PY} \right) = \frac{1}{2}h(X, Y) - \frac{\sqrt{5}}{2} h \left( X, PY \right)$

$= \frac{1}{2}h(X, Y) + \frac{\sqrt{5}}{2} h \left( \hat{PX}, Y \right) = h \left( \frac{1}{2} \left( I + \sqrt{5} \hat{P} \right) X, Y \right) = h \left( \hat{GX}, Y \right).$

Next assume (ii) then $\forall X, Y \in \Gamma (TM)$

$h \left( X, PY \right) = -h \left( X, \hat{PY} \right) = -h \left( X, \frac{1}{\sqrt{5}} \left( 2\hat{G} - I \right) Y \right)$

$= \left[ -\frac{1}{\sqrt{5}} h \left( X, Y \right) + \frac{2}{\sqrt{5}} h \left( \hat{GX}, Y \right) \right] = -h \left( \frac{1}{\sqrt{5}} \left( 2\hat{G} - I \right) X, Y \right)$

$= -h \left( \hat{PX}, Y \right) = h \left( PX, Y \right)$ ■

We immediately get, from Proposition (2.1), the following
Proposition 6 (2.2): Let an almost product structure $P$ and an almost
golden structure $G$ form a twin pair $\{P, G\}$ on a smooth manifold $M$.

(A): The following statements are equivalent:
i) $(M, h, P)$ is an almost product Riemannian manifold.
ii) $(M, h, \hat{P})$ is an almost product Riemannian manifold.
iii) $(M, h, G)$ is an almost golden-Riemannian manifold.
iv) $(M, h, \hat{G})$ is an almost golden-Riemannian manifold.

(B): The following statements are equivalent
i) $(M, h, P)$ is an almost para-Hermitian manifold.
ii) $(M, h, \hat{P})$ is an almost para-Hermitian manifold.
iii) $(M, h, G)$ is an almost golden-Hermitian manifold.
iv) $(M, h, \hat{G})$ is an almost golden-Hermitian manifold.

Definition 7 (2.2): An almost product manifold $(M, P)$ and an almost
golden manifold $(M, G)$ are said to be twins if $P$ and $G$ are twins (on the
same manifold $M$).

Remark 8 (2.1): It is obvious that $(M, P)$ and $(M, G)$ are twins if and
only if $(M, \hat{P})$ and $(M, \hat{G})$ are twins.

For an almost product (or golden) manifold $(M, \varphi)$, $\varphi$ is said to be
integrable if its Nijenhuis tensor field $N_{\varphi}$ vanishes, ([3, 9]). That is, $\forall X, Y \in \Gamma(TM)$

\[ N_{\varphi}(X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] = 0. \]

For an almost product (or golden) manifold $(M, \varphi)$ with integrable $\varphi$ we drop the adjective ”almost” and then simply call it product (or golden)
manifold.

Lemma 9 (2.3): [3], For a twin pair $\{P, G\}$ on a manifold $M$ with any
linear connection $\nabla$ one has

\[ 5N_P = 4N_G \quad \text{and} \quad \sqrt{5}\nabla P = 2\nabla G. \]

This lemma gives immediately:

Corollary 10 (2.1): Let $\{P, G\}$ be a twin pair on a manifold $M$, then we
have:

$P$ is integrable if and only if $\hat{P}$ is integrable if and only if $\hat{G}$ is integrable
if and only if $G$ is integrable.

Lemma 11 (2.4): ([9], Pg: 150 – 151) For an almost product manifold $(M, P)$,
i) There always exist a linear connection \( \nabla \) on \( M \) with \( \nabla P = 0 \).
(Note that despite that \( \nabla P = 0 \), the almost product structure \( P \) may not to be integrable unless \( \nabla \) is symmetric.)

ii) For any symmetric linear connection \( \tilde{\nabla} \) on \( M \)
\[
\mathcal{N}_P (X, Y) = (\tilde{\nabla}_X P) Y - (\tilde{\nabla}_Y P) X - P (\tilde{\nabla}_X P) Y + P (\tilde{\nabla}_Y P) X
\]
and therefore, If \( \nabla P = 0 \) then \( P \) is integrable.

iii) If \( P \) is integrable then there always exist a symmetric linear connection
\( \hat{\nabla} \) on \( M \) with \( \hat{\nabla} P = 0 \).

\[\blacksquare\]

From Corollary (2.1) and Lemma (2.4) one gets:

**Corollary 12 (2.2):** Let \( \{P, G\} \) be a twin pair of almost product and almost golden structures on a smooth manifold \((M, h)\) with a nondegenerate metric \( h \). Then, for the Levi-Civita connection \( \nabla \) on \((M, h)\) one has:

A) The following are equivalent:
   i) \( \nabla P = 0 \).
   ii) \( \nabla \hat{G} = 0 \).
   iii) \( \nabla G = 0 \).
   iv) \( \nabla \hat{P} = 0 \).

B) If \( \nabla P = 0 \) then \( P, \hat{P}, \hat{G} \) and \( G \) are all integrable.

**Remark 13 (2.2):** Note that

i) the above Corollary is true regardless of whether \( h \) is pure or hyperbolic or neither with respect to \( P \) (and therefore with respect to \( \hat{P}, \hat{G} \) and \( G \)).
   ii) Integrability of \( \varphi (= P, G) \) does not imply that \( \varphi \) is parallel (with respect to the metric (Levi-Civita) connection). \( \blacksquare \)

Let \((M, h, \varphi (= P, G))\) be an almost product (or golden) manifold with a metric \( h \) which is pure or hyperbolic with respect to \( \varphi \). Then due to the above lemma (2.4)/(iii), an integrable \( \varphi \) is always parallel with respect to some symmetric connection \( \hat{\nabla} \) anyway. However \( \hat{\nabla} h = 0 \) need not be true, that is, \( \hat{\nabla} \) need not be the Levi-Civita connection. The question here is that what extra condition should be imposed so that integrability of \( \varphi \), together with the imposed condition, guarantees that \( \varphi \) is parallel under the Levi-Civita connection? Answer to this question will differ depending on whether the metric \( h \) is pure or hyperbolic with respect to \( \varphi \).

From here on, unless otherwise stated, the connections involved will be the Levi-Civita ones and denoted by \( \nabla \).

**I: The hyperbolic case:** Even though this case is well known for \( \varphi = P \), (see \cite{2, 11}), we will give an outline to some extend.

Let \((M, h, \varphi (= P, G))\) be an almost para-Hermitian manifold with its Levi-Civita connection \( \nabla \). Set

\[
\Omega_P (X, Y) = \Omega (X, Y) = h (PX, Y) ; \quad \forall X, Y \in \Gamma (TM).
\]
\( \Omega_P \) is a \((P\text{-associated})\) 2-form on \(M\) and it is called "fundamental 2-form" or "para-Kaehler form". The exterior differential \( d\Omega \) is a 3-form on \(M\) given by, \((6)\),

\[
d\Omega(X, Y, Z) = \nabla_X (\Omega(Y, Z)) - \nabla_Y (\Omega(X, Z)) + \nabla_Z (\Omega(X, Y)) - \Omega([X, Y], Z) - \Omega([Y, Z], X) + \Omega([X, Z], Y)
\]

which can also be expressed as

\[
d\Omega(X, Y, Z) = (\nabla_X \Omega)(Y, Z) - (\nabla_Y \Omega)(X, Z) + (\nabla_Z \Omega)(X, Y). \quad (2.1)
\]

**Definition 14 (2.3):**

\( A : ([11]) \)

i) An almost para-Hermitian manifold \((M, h, P)\) is called **almost para-Kaehler** if its para-Kaehler form \(\Omega_P\) is closed, i.e. \(d\Omega_P = 0\).

ii) An almost para-Kaehler manifold \((M, h, P)\) with integrable \(P\) is called **para-Kaehler**.

\( B : \)

i) An almost golden-Hermitian manifold \((M, h, G)\) is called **almost golden-Kaehler** if the para-Kaehler form \(\Omega_R\) is closed, i.e. \(d\Omega_R = 0\), where \(R = PG\) is the \(G\)-associated product structure and \(\Omega_R\) is the \(R\)-associated 2-form.

ii) An almost golden-Kaehler manifold \((M, h, G)\) with integrable \(G\) is called **golden-Kaehler**.

**Proposition 15 (2.3):** Let \(\{P, G\}\) be twin structures on \((M, h)\) with a hyperbolic metric \(h\) with respect to \(P\) (and therefore with respect to \(G\)). Then the following statements are equivalent:

i) The manifold \((M, h, P)\) is almost para-Kaehler, that is, \(d\Omega_P = 0\).

ii) The manifold \((M, h, G)\) is an almost golden-Kaehler, that is, \(d\Omega_R = 0\).

iii) The manifold \((M, h, \tilde{P})\) is an almost para-Kaehler, that is, \(d\Omega_{\tilde{P}} = 0\).

iv) The manifold \((M, h, \tilde{G})\) is an almost golden-Kaehler, that is, \(d\Omega_{\tilde{G}} = 0\).

**Proof.** The result follows from the fact that \(R = PG = P\) since \(P\) and \(G\) are twins. ■

**Lemma 16 (2.5): ([5])** Let \((M, h, P)\) be an almost para-Hermitian manifold with its Levi-Civita connection \(\nabla\) and para-Kaehler form \(\Omega\). Then the following relation holds: \(\forall X, Y, Z \in \Gamma(TM)\)

\[
2h((\nabla_X P)Y, Z) + 3d\Omega(X, Y, Z) + 3d\Omega(X, PY, PZ) + h(N_P(Y, Z), PX) = 0.
\]

**Proposition 17 (2.4):** Let \((M, h, \varphi(=P, G))\) be an almost para-Hermitian or an almost golden-Hermitian manifold.
A: Then the following are equivalent:
i) $P$ is parallel with respect to the Levi-Civita connection $\nabla$, that is $\nabla P = 0$.
ii) $M$ is para-Kaehler (that is, $\mathcal{N}_P = 0$ and $d\Omega_P = 0$).

B: Then the following are equivalent:
i) $G$ is parallel with respect to the Levi-Civita connection $\nabla$, that is, $\nabla G = 0$.
ii) $M$ is golden-Kaehler (that is $\mathcal{N}_G = 0$ and $d\Omega_R = 0$), where $\mathcal{R} = PG$ is the $G$-associated product structure.

Proof: A ($[11]$):
(i) $\Rightarrow$ (ii): Note that
$$\nabla (\Omega (X, Y)) = (\nabla \Omega) (X, Y) + \Omega (\nabla X, Y) + \Omega (X, \nabla Y)$$
((2.2))

On the other hand, since $\nabla P = 0$ and $\nabla h = 0$, we have

$$\nabla (\Omega (X, Y)) = \nabla (h (PX, Y)) = (\nabla h) (PX, Y) + h (\nabla PX, Y) + h (PX, \nabla Y)$$
$$= h (P(\nabla X), Y) + h (PX, \nabla Y)$$
$$= \Omega (\nabla X, Y) + \Omega (X, \nabla Y)$$
((2.3))

But then the equalities (2.2) and (2.3) give us that $\nabla \Omega = 0$. So, from (2.1), we get $d\Omega_P = 0$. The equality $\mathcal{N}_P = 0$ follows from Lemma (2.5).

(ii) $\Rightarrow$ (i): This follows directly from Lemma (2.5).

B: Since $\{\mathcal{R}, G\}$ is a twin pair on $M$, the required equivalence follows from part (A).

Remark 18 (2.3): Let $J$ be a $(1, 1)$-tensor field with $J^2 = -I$ on a Riemannian manifold $(M, g)$, where $g$ is hyperbolic with respect to $J$ and $\tilde{J} = -J$ is the conjugate of $J$. Then $J$ and $(M, g, J)$ are called almost complex structure and almost Hermitian manifold respectively. In this case it is well known that Proposition (2.4)/A is also valid when $P$ is replaced by $J$. ■

II: The pure case:

Let $(M, h, \varphi (\equiv P, G))$ be an almost product (or an almost golden) Riemannian manifold (so that $h$ is pure) with its Levi-Civita connection $\nabla$. The so called "Tachibana operator"

$$\phi_\varphi : \mathfrak{S}_0^2 (M) \rightarrow \mathfrak{S}_3^0 (M)$$

from the set of all $(0, 2)$-tensor fields into the set of all $(0, 3)$-tensor fields over $M$ is given by (see [8, 12]): $\forall u \in \mathfrak{S}_0^2 (M)$ and $\forall X, Y, Z \in \Gamma (TM)$

$$\phi_\varphi u (X, Y, Z) = (\varphi X) (u (Y, Z)) - X (u (\varphi Y, Z))$$
$$+ u (\mathcal{L}_Y \varphi X, Z) + u (Y, \mathcal{L}_Z \varphi X)$$

where $\mathcal{L}_\varphi$ is the Lie derivative of $\varphi$. 

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In particular, for the pure metric $h$ with respect to $\varphi$, the above equality takes the form (see [8, 12]): \forall X, Y, Z \in \Gamma(TM)

$$(\phi_\varphi h)(X, Y, Z) = -h((\nabla_Y \varphi)Y, Z) + h((\nabla_Y \varphi)X, Z) + h((\nabla_Z \varphi)X, Y).$$

Now let us define another operator $\Psi_\varphi : \mathfrak{S}^3_2(M) \to \mathfrak{S}^3_2(M)$ by, \forall X, Y, Z \in \Gamma(TM)

$$(\Psi_\varphi u)(X, Y, Z) = (\phi_\varphi u)(X, Y, Z) + (\phi_\varphi u)(Z, Y, X)$$

Lemma 19 (2.6): ([8, 12]) Let $(M, h, \varphi (= P, G))$ be an almost product (or an almost golden) Riemannian manifold with its Levi-Civita connection $\nabla$. Then

i) \forall X, Y, Z \in \Gamma(TM)

$$(\Psi_\varphi h)(X, Y, Z) = 2h((\nabla_Y \varphi)X, Z)$$

ii) The following are equivalent:

a$^\circ$) $(\Psi_\varphi h) = 0$

b$^\circ$) $(\phi_\varphi h) = 0$

c$^\circ$) $\nabla \varphi = 0$

Now set a condition on $\Psi_\varphi$:

$$(\Psi_\varphi h)(X, Y, Z) = (\Psi_\varphi h)(Y, X, Z)+(\Psi_\varphi h)(\varphi Y, \varphi X, Z), \ \forall X, Y, Z \in \Gamma(TM)$$

Proposition 20 (2.5): Let $(M, h, \varphi (= P, G))$ be an almost product (or an almost golden) Riemannian manifold. Then

A: The following are equivalent:

i) $P$ is parallel with respect to the Levi-Civita connection $\nabla$, that is, $\nabla P = 0$.

ii) $P$ is integrable, that is, $\mathcal{N}_P = 0$ and the condition $P(*)$ holds.

B: The following are equivalent:

i) $G$ is parallel with respect to the Levi-Civita connection $\nabla$, that is, $\nabla G = 0$.

ii) $G$ is integrable, that is, $\mathcal{N}_G = 0$ and the condition $R(*)$ holds.

Here $R = P_G$ is the twin product structure of $G$, so that, $\{G, R\}$ form a twin pair.

Proof: A:

(i) $\Rightarrow$ (ii): Since $\nabla P = 0$ by the assumption, we have from Lemma (2.4)/(ii) that $\mathcal{N}_P = 0$, and from Lemma (2.6) the condition $P(*)$ follows.
(ii) ⇒ (i) : From Lemma (2.4) / (ii) we have: \( \forall X, Y, Z \in \Gamma(TM) \)

\[
\mathcal{N}_P (X, Y) = (\nabla_{P Y} P) Y - (\nabla_{P Y} P) X - P ((\nabla_X P) Y) + P ((\nabla_Y P) X) \\
= (\nabla_{P Y} P) Y - (\nabla_{P Y} P) X + (\nabla_X P) (PY) - (\nabla_Y P) (PX)
\]

So

\[
h (\mathcal{N}_P (X, Y), Z) = h ((\nabla_{P Y} P) Y, Z) - h ((\nabla_{P Y} P) X, Z) + h ((\nabla_X P) (PY), Z) - h ((\nabla_Y P) (PX), Z).
\]

That is,

\[
h (\mathcal{N}_P (X, Y), Z) - h ((\nabla_X P) (PY), Z) = h ((\nabla_{P Y} P) Y, Z) - h ((\nabla_{P Y} P) X, Z) - h ((\nabla_Y P) (PX), Z).
\]

Then using Lemma (2.6), we get: \( \forall X, Y, Z \in \Gamma(TM) \)

\[
2 \{ h (\mathcal{N}_P (X, Y), Z) - h ((\nabla_X P) (PY), Z) \} = (\Psi_p h) (X, PY, Z)
- (\Psi_p h) (X, PY, Z) - (\Psi_p h) (PX, Y, Z).
\]

Exchanging \( X, Y \), this equation reads:

\[
2 \{ h (\mathcal{N}_P (Y, X), Z) - h ((\nabla_Y P) (PX), Z) \} = (\Psi_p h) (X, PY, Z)
- (\Psi_p h) (X, PY, Z) - (\Psi_p h) (PY, X, Z).
\]

Then putting \( Y \) for \( PY \) in the last equation (doing this does not alter the equation since \( P \) is an isomorphism) we get: \( \forall X, Y, Z \in \Gamma(TM) \)

\[
2 \{ h (\mathcal{N}_P (PY, X), Z) - h ((\nabla_{PY} P) (PX), Z) \} = (\Psi_p h) (X, Y, Z)
- (\Psi_p h) (Y, PX, Z) - (\Psi_p h) (Y, X, Z).
\]

But then under the assumptions that \( \mathcal{N}_P = 0 \) and the condition \( P(*) \) holds, the last equation gives us that

\[
h ((\nabla_{PY} P) (PX), Z) = 0, \forall X, Y, Z \in \Gamma(TM)
\]

which means that \( \nabla P = 0 \).

\( B : \)

(i) ⇒ (ii) : By the assumption, \( \nabla G = 0 \) and therefore \( \nabla R = 0 \). So by part (A) above, we get \( \mathcal{N}_R = 0 \), and therefore \( \nabla G = 0 \) by Lemma (2.3). Also by part (A), we get that the condition \( R(*) \) holds.

(ii) ⇒ (i) : By the assumption, \( \nabla G = 0 \) and therefore \( \mathcal{N}_R = 0 \) and the condition \( \mathcal{R}(*) \) holds. So by part (A), we get \( \nabla R = 0 \), and therefore \( \nabla G = 0 \) by Corollary (2.2).

For an almost product (or an almost golden) manifold \( (M, h, \varphi) \) with a pure or hyperbolic metric \( h \) with respect to \( \varphi \), and with its Levi-Civita connection \( \nabla \), the divergence \( \text{div} \varphi \) of \( \varphi \) is given by, [6],

\[
\text{div} \varphi = \sum_{i=1}^{m} h_{ii} (\nabla_{e_i} \varphi) e_i.
\]

Here \( \{e_1, ..., e_m\} \) is a local orthonormal frame field for \( \Gamma(TM) \) and \( h_{ii} = h(e_i, e_i) \).
Definition 21 (2.4):

\((A)\): An almost product Riemannian manifold \((M, h, P)\) with its Levi-Civita connection \(\nabla\), is called

i) \(\text{locally product Riemannian manifold}\) if \(P\) is integrable, \([8]\).

ii) \(\text{almost decomposable product Riemannian manifold}\) if \(P(\ast)\) holds.

iii) \(\text{locally decomposable product Riemannian manifold}\) if both \(P\) is integrable and \(P(\ast)\) holds (that is, \(P\) is parallel), \([8]\).

In particular, if \((M, h, P)\) is a \(B\)-manifold (resp: almost \(B\)-manifold) holding the condition \(P(\ast)\) then it is also called \(\text{para-holomorphic } B\)-manifold, \([12]\), (resp: \(\text{almost para-holomorphic } B\)-manifold). Note here that by the virtue of Proposition (2.5), if \((M, h, P)\) is a para-holomorphic \(B\)-manifold then \(\nabla P = 0\), i.e. \(P\) is parallel.

iv) \(\text{Semi decomposable product Riemannian manifold}\) if \(\text{div } P = 0\)

\((B)\): An almost golden Riemannian manifold \((M, h, G)\) with its Levi-Civita connection \(\nabla\), is called

i) \(\text{locally golden Riemannian manifold}\) if \(G\) is integrable, \([8]\).

ii) \(\text{almost decomposable golden Riemannian manifold}\) if \(G(\ast)\) holds.

iii) \(\text{locally decomposable golden Riemannian manifold}\) if both \(G\) is integrable and \(G(\ast)\) holds (that is, \(G\) is parallel), \([8]\).

iv) \(\text{Semi decomposable golden Riemannian manifold}\) if \(\text{div } G = 0\)

Define a bilinear map, \(7\),

\[
S_\varphi : \Gamma (TM) \times \Gamma (TM) \to \Gamma (TM)
\]
on a manifold \((M, h, \varphi (= P, G))\) with the Levi-Civita connection \(\nabla\), by

\[
S_\varphi (X, Y) = (\nabla_X \varphi) Y + \varphi (\nabla_{\varphi X} \varphi) Y \quad \forall \ X, Y \in \Gamma (TM).
\]

Lemma 22 (2.7):

\((A)\): For \(\varphi = P\) and \(\forall \ X, Y \in \Gamma (TM)\) we have

i) \(S_P (X, Y) = (\nabla_X P)Y - (\nabla_{P X} P)(PY)\).

ii) \(P(S_P (X, Y)) = -S_P (X, PY) = S_P (PX, Y)\).

iii) \(S_P (X, Y) = 0, \ \forall \ X, Y \in \Gamma (\mathcal{E}_{(1)})\) and \(S_P (X, Y) = 0, \ \forall \ X, Y \in \Gamma (\mathcal{E}_{(-1)})\).

\((B)\): For \(\varphi = G\) and \(\forall \ X, Y \in \Gamma (TM)\) we have

i) \(S_G (X, Y) = (\nabla_X G) Y - (\nabla_{\varphi X} G)(GY) + (\nabla_{\varphi X} G)Y\).

ii) \(S_G (X, Y) = 0, \ \forall \ X, Y \in \Gamma (\mathcal{E}_{(\sigma)})\) and \(S_G (X, Y) = 0, \ \forall \ X, Y \in \Gamma (\mathcal{E}_{(\bar{\sigma})})\).
Proof. : ■

Using the facts that \( P((\nabla P)X) = - (\nabla P)(PX) \) and \( G((\nabla G)Y) = - (\nabla G)(GY) + (\nabla G)Y \) we get \( A/(i) \) and \( B/(i) \). Next, \( A/(ii) \) and \( A/(iii) \) are easy. For \( B/(ii) \) let \( X, Y \in \Gamma (E(\sigma)) \), then

\[
S_G (X, Y) = (\nabla_X G)Y - (\nabla_{\sigma X} G)(GY) + (\nabla_{\sigma X} G)Y = (\nabla_X G)Y - \sigma^2 (\nabla_X G)(Y) + \sigma (\nabla_X G)Y
\]

\[
= (1 - \sigma^2 + \sigma) (\nabla_X G)Y = 0, \quad \text{since} \quad \sigma^2 = 1 + \sigma.
\]

By the same argument we also get that

\[
S_G (X, Y) = 0, \quad \forall X, Y \in \Gamma (E(\sigma)),
\]

which completes the proof.

Lemma 23 (2.8): (c.f [7]) On an almost product Riemannian or an almost para Hermitian manifold \( (M, h, P) \), the following statements are equivalent:

- \( i \) \( S_P (X, Y) = 0 \) \( \forall X, Y \in \Gamma (TM) \).
- \( ii \) \( S_P (X, PX) = 0 \) \( \forall X \in \Gamma (TM) \).
- \( iii \) \( S_P (X, X) = 0 \) \( \forall X \in \Gamma (TM) \).

Proposition 24 (2.6): For an almost product Riemannian manifold \( M_P = (M, h, P) \) and an almost golden Riemannian manifold \( M_G = (M, h, G) \),

- \( A \): on \( M_P \)
  - \( i \) \( \nabla P = 0 \) if and only if \( S_P = 0 \) if and only if \( (\psi_P h) = 0 \) if and only if \( (\phi_P h) = 0 \).
  - \( ii \) \( \nabla G = 0 \) if and only if \( S_G = 0 \) if and only if \( (\psi_G h) = 0 \) if and only if \( (\phi_G h) = 0 \).

- \( B \): If \( M_P \) and \( M_G \) are twin manifolds then the following are equivalent:
  - \( i \) \( \nabla P = 0 \) on \( M_P \).
  - \( ii \) \( S_P = 0 \) on \( M_P \).
  - \( iii \) \( \nabla G = 0 \) on \( M_G \).
  - \( iv \) \( S_G = 0 \) on \( M_G \).
  - \( v \) \( \nabla \tilde{G} = 0 \) on \( M_{\tilde{G}} \).
  - \( vi \) \( S_{\tilde{P}} = 0 \) on \( M_{\tilde{P}} \).
  - \( vii \) \( \nabla \tilde{P} = 0 \) on \( M_{\tilde{P}} \).

Proof. : ■

\( A \):

- \( i \): If \( \nabla P = 0 \) then obviously \( S_P = 0 \).
  Conversely, assume that \( S_P = 0 \). Then for \( X \in \Gamma (E(1)) \) and \( Y \in \Gamma (E(-1)) \)

\[
S_P (Y, X) = (\nabla_X P)Y - (\nabla_{PX} P)PY = 2 (\nabla_X P)Y = 0,
\]

which gives

\[
(\nabla_X P)Y = 0; \quad \forall X \in \Gamma (E(1)) \quad \text{and} \quad \forall Y \in \Gamma (E(-1)). \quad \text{(2.4)}
\]
By a similar argument we get
\[
(\nabla_Y P) X = 0; \quad \forall \, X \in \Gamma \left( \mathcal{E}_1 \right) \quad \text{and} \quad \forall \, Y \in \Gamma \left( \mathcal{E}_{-1} \right). \quad \text{(2.5)}
\]

From (2.4) we get
\[
(\nabla_X P) Y = \nabla_X (PY) - P (\nabla_X Y) = -\nabla_X Y - P (\nabla_X Y) = 0.
\]

So,
\[
\nabla_X Y \in \Gamma \left( \mathcal{E}_{-1} \right) \quad \forall \, X \in \Gamma \left( \mathcal{E}_1 \right) \quad \text{and} \quad \forall \, Y \in \Gamma \left( \mathcal{E}_{-1} \right). \quad \text{(2.6)}
\]

On the other hand, \( \forall \, X, Z \in \Gamma \left( \mathcal{E}_1 \right) \quad \text{and} \quad \forall \, Y \in \Gamma \left( \mathcal{E}_{-1} \right) \)
\[
X (h (Y, Z)) = h (\nabla_X Y, Z) + h (Y, \nabla_X Z) = 0, \quad \text{since} \quad h (Y, Z) = 0.
\]

Using (2.6), this gives that \( h (Y, \nabla_X Z) = 0, \quad \forall \, Y \in \Gamma \left( \mathcal{E}_{-1} \right) \) and therefore \( \nabla_X Z \in \Gamma \left( \mathcal{E}_1 \right) \). But then,
\[
P (\nabla_X Z) = \nabla_X Z = \nabla_X (PZ)
\]
which gives
\[
(\nabla_X P) Z = 0; \quad \forall \, X, Z \in \Gamma \left( \mathcal{E}_1 \right) \quad \text{(2.7)}
\]

By a similar argument we also get
\[
(\nabla_X P) Z = 0; \quad \forall \, X, Z \in \Gamma \left( \mathcal{E}_{-1} \right) \quad \text{(2.8)}
\]

But then \( (2.4), (2.5), (2.7), \) and \( (2.8) \) give us that \( \nabla P = 0 \), i.e. \( P \) is parallel. The rest of the statements in \( (i) \) will follow from Lemma (2.6).

\( i) \) : This will follow by mimicking the arguments used in \( (i) \).

\( (B) \) : Now, observing that \( S_P = -S_{\hat{P}} \), \( \nabla P = -\nabla \hat{P} \) and \( \nabla G = -\sqrt{2} \nabla \hat{P} = -\nabla \hat{G} \), together with the part \( (A) \), proofs of the statements \( (i) \) to \( (vi) \) in part \( B \) will easily follow.

For an almost para-Hermitian manifold \( M_P = (M, h, P) \) and an almost golden Hermitian manifold \( M_G = (M, h, G) \), (note that, here the metric is hyperbolic with respect to the indicated structures rather than pure) we do not have Proposition \( (2.6/A) \) type of results. Instead, some conditions on the operator \( S_\varphi \), with \( \varphi (= P, G) \) induce some extra subclasses of those manifolds. To be precise:

**Definition 25** (2.5):

\( (A) \) : An almost para-Hermitian manifold \( M_P = (M, h, P) \) with its Levi-Civita connection \( \nabla \) is said to be, \( ([7]) \),

\( i) \) nearly para-Kaehler if \( (\nabla_X P) X = 0, \quad \forall \, X \in \Gamma (TM) \)

\( ii) \) quasi para-Kaehler if \( S_P = 0. \)
iii) semi para-Kaehler if \( \text{div} (P) = 0 \) (equivalently, \( \sum_{i=1}^{m} h_{ii} S_P (e_i, e_i) = 0 \)) where \( \{ e_1, \ldots, e_m; P e_1, \ldots, P e_m \} \) is a local orthonormal frame field for \( \Gamma (TM) \) and \( h_{ii} = h (e_i, e_i) \).

(B) An almost golden-Hermitian manifold \( (M, h, G) \) with its Levi-Civita connection \( \nabla \) is said to be
i) nearly golden-Kaehler \( (\nabla_X G) X = 0, \forall X \in \Gamma (TM) \)
ii) quasi golden-Kaehler if \( S_R = 0 \), where \( R = P_G, G \)-associated product structure.
iii) semi golden-Kaehler if \( \text{div} (G) = 0 \).

3 3. Harmonicity

Definition 26 (3.1): A distribution \( D \) over a (semi) Riemannian manifold \( (M, h) \) with its Levi-Civita connection \( \nabla \), is said to be
i) (c.f. [10]) Vidal if
\[
\nabla_X X \in D, \forall X \in \Gamma (D).
\]
ii) ([1]) critical if
\[
\sum_{i=1}^{n} h_{ii} \nabla v_i v_i \in D
\]

If the restriction \( h |_D \) of \( h \) to \( D \) is positive (or negative) definite then the critical distribution \( D \) is also called minimal. Here \( \{ v_1, \ldots, v_n \} \) is a local orthonormal frame field for \( D \) and \( h_{ii} = h (v_i, v_i) \).

Remark 27 (3.1):

1) For an almost product (or an almost golden) Riemannian manifold \( (M, h, \varphi (= P, G)) \)
   i) every Vidal distribution is critical.
   ii) if \( \varphi \) is parallel then the eigendistributions \( E_{(k)} \) and \( E_{(\overline{k})} \) of \( \varphi \) are both Vidal and therefore they are minimal. Here \( k = 1, \overline{k} = -1 \) for \( \varphi = P \) and \( k = \sigma, \overline{k} = \overline{\sigma} \) for \( \varphi = G \)
   iii) the eigendistributions \( E_{(1)} \) and \( E_{(-1)} \) of \( P \) (resp \( E_{(\sigma)} \) and \( E_{(\overline{\sigma})} \) of \( G \)) are both minimal if and only if \( \text{div} P = 0 \), that is, \( (M, h, P) \) is semi decomposable product Riemannian (resp \( \text{div} G = 0 \), that is, \( (M, h, G) \) semi decomposable golden Riemannian) manifold.

2) For an almost product (or an almost golden) manifold \( (M, h, \varphi (= P, G)) \) with a pure or hyperbolic metric \( h \), if \( \{ P, G \} \) is a twin pair then the following are equivalent:
   i) \( \text{div} P = 0 \)
   ii) \( \text{div} G = 0 \)
Lemma 28 (3.1): Let $F : (M, \varphi_M) \to (N, \varphi_N)$ be a smooth map with its differential map $F_* : TM \to TN$, where $\varphi (= P, G)$.

i) If $F_* \circ P_M = G_N \circ F_*$ or $F_* \circ G_M = P_N \circ F_*$ then $F$ is constant.

ii) If any one of the following

- $F_* \circ P_M = \hat{G}_N \circ F_*,$
- $F_* \circ \hat{P}_M = \hat{G}_N \circ F_*,$
- $F_* \circ \hat{P}_M = G_N \circ F_*,$
- $F_* \circ G_M = \hat{P}_N \circ F_*,$
- $F_* \circ \hat{G}_M = \hat{P}_N \circ F_*,$
- $F_* \circ \hat{G}_M = P_N \circ F_*$

holds then $F$ is constant.

Proof. The statement (i) is treated in ([13], Theorem 7&8). The argument used in [13] works for all the cases in (ii). ■

Definition 29 (3.2): A smooth map $F : (M, \varphi_M) \to (N, \varphi_N)$ with its differential map $dF = F_* : TM \to TN$ is said to be

i) ([2, 7]), $(P_M, P_N)$-paraholomorphic, [resp: $(P_M, P_N)$-anti-paraholomorphic] if

$F_* \circ P_M = P_N \circ F_*$, [resp: $F_* \circ P_M = \hat{P}_N \circ F_* = -P \circ F_*$]

ii) 

- ([13]), $(G_M, G_N)$-golden if

$F_* \circ G_M = G_N \circ F_*$

- $(G_M, G_N)$-antigolden if

$F_* \circ G_M = \hat{G}_N \circ F_* = I_N - G_N F_*$

where $\varphi = \hat{\varphi} = \hat{P}, \hat{G}$ is the conjugate of $\varphi$.

We shall be writing $\pm (P_M, P_N)$-paraholomorphic to mean either $(P_M, P_N)$-paraholomorphic or $(P_M, P_N)$-anti-paraholomorphic. Similarly, We shall be writing $\pm (G_M, G_N)$-golden to mean either $(G_M, G_N)$-golden or $(G_M, G_N)$-antigolden.

Note that since $\hat{\varphi} = \varphi$, we have:

If a map $F : (M, \varphi_M) \to (N, \varphi_N)$ is $(P_M, P_N)$-paraholomorphic then it is $(P_M, \hat{P}_N)$-anti-paraholomorphic as a map $F : (M, \hat{\varphi}_M) \to (N, \hat{P}_N)$. 18
and if \((G_M, G_N)\)-golden then it is \(\left(G_M, \hat{G}_N\right)\)-antigolden as a map \(F : (M, G_M) \to (N, \hat{G}_N)\). Conversely, if a map \(F : (M, \varphi_M) \to (N, \varphi_N)\) is \((P_M, P_N)\)-anti-paraholomorphic then it is \(\left(P_M, \hat{P}_N\right)\)-paraholomorphic as a map \(F : (M, P_M) \to (N, \hat{P}_N)\), and if \((G_M, G_N)\)-antigolden then it is \(\left(G_M, \hat{G}_N\right)\)-golden as a map \(F : (M, G_M) \to (N, \hat{G}_N)\).

**Proposition 30** (3.1): For twin pairs \(\{P_M, G_M\}\) and \(\{P_N, G_N\}\) let \(F : (M, \varphi_M (= P_M, G_M)) \to (N, \varphi_N (= P_N, G_N))\) be a smooth map. Then the following statements are equivalent:

i) \(F\) is \((P_M, P_N)\)-paraholomorphic [resp: \((P_M, P_N)\)-anti-paraholomorphic].

ii) \(F\) is \((G_M, G_N)\)-golden [resp: \((G_M, G_N)\)-antigolden].

**Proof.**

\(F\) is \((G_M, G_N)\)-antigolden

\[\Leftrightarrow \quad F_* \circ G_M = \hat{G}_N \circ F_*\]

\[\Leftrightarrow \quad F_* \circ (tI + rP_M) = \left(tI + r\hat{P}_N\right) \circ F_*\]

Since \(\{P_M, G_M\}\) is a twin pair and then so is \(\{\hat{P}_M, \hat{G}_M\}\), so that \(G_M = tI + rP_M\) and \(\hat{G}_M = tI + r\hat{P}_M\), where \(t = \frac{1}{5}, \quad r = \sqrt{\frac{2}{5}}\),

\[\Leftrightarrow \quad tF_* + r \left(F_* \circ P_M\right) = tF_* + r \left(\hat{P}_N \circ F_*\right)\]

\[\Leftrightarrow \quad F_* \circ P_M = \hat{P}_N \circ F_*\]

\[\Leftrightarrow \quad F\ is \ (P_M, P_N)\text{-anti-paraholomorphic}\]

The rest of the cases can be shown similarly. ■ ■

Let \(F : (M, h) \to (N, g)\) be a smooth map between (semi) Riemannian manifolds. The second fundamental form \(\nabla F_* : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TN)\) of \(F\) is given by \(\forall X, Y \in \Gamma(TM)\)

\[(\nabla F_*)(X, Y) = \nabla_{(F_*X)}(F_*Y) - F_* \left(\nabla^M_X Y\right)\]

where \(\nabla^M\) and \(\nabla^N\) are the Levi-Civita connections on \(M\) and \(N\) respectively. Note that the map \(\nabla F_*\) is bilinear and symmetric, \(\text{see [1, 6]}\)
For a given distribution $D$ over a (semi) Riemannian manifold $(M, h)$, the $D$-tension field $T_D(F)$ of $F : (M, h) \rightarrow (N, g)$ is given by (c.f. [1, 6, 7])

$$T_D(F) = \sum_{i,j=1}^{s} h^{ij} (\nabla F)(e_i, e_j) \in \Gamma (TN)$$

(3.1)

where $\{e_1, ..., e_s\}$ is a local frame field for $D$ and $(h^{ij}) = (h_{ij})^{-1}$, $h_{ij} = h(e_i, e_j)$. In particular, if $\{e_1, ..., e_s\}$ is a local $h$-orthonormal frame field for $D$ then the expression (3.1) takes the form

$$T_D(F) = \sum_{i=1}^{s} h^{ii} (\nabla F)(e_i, e_i) \in \Gamma (TN)$$

(3.2)

In the cases where $D = TM$, we simply write $T(F)$ for $T_{TM}(F)$ call it the "tension field of $F$".

**Definition 31 (3.3)** (c.f. [1, 6]) A smooth map $F : (M, h) \rightarrow (N, g)$ is said to be harmonic [resp: $D$-harmonic] if its tension field [resp: $D$-tension field] vanishes. In particular,

- for a map $F : (M, h, P) \rightarrow (N, g)$ from an almost product Riemannian manifold $M$, if $D = \mathcal{E}(1)$ [resp: $D = \mathcal{E}(-1)$] then $D$-harmonic $F$ is also called plus-eigen harmonic [resp: minus-eigen harmonic].

- for a map $F : (M, h, G) \rightarrow (N, g)$ from an almost golden Riemannian manifold $M$, if $D = \mathcal{E}(\sigma)$ [resp: $D = \mathcal{E}(\bar{\sigma})$] then $D$-harmonic $F$ is also called plus-eigen harmonic [resp: minus-eigen harmonic].

**Proposition 32 (3.2)**: For an almost product manifolds $(M, h, P)$, $(N, g, Q)$ with pure or hyperbolic metric $h$ with respect to $P$ and pure or hyperbolic metric $g$ with respect to $Q$, let $F : (M, h, P) \rightarrow (N, g, Q)$ be a $\pm (P, Q)$-paraholomorphic map. Then for every local sections $X, Y \in \Gamma (TM)$,

$$\left(\nabla F*(PX, PY) = \left(\nabla F*(X, Y) + \left(\nabla_{QX} \cdot Q\right)Y - \left(\nabla_{QY} \cdot Q\right)X\right)QX‘\right)$$

$$-F*\left(\left(\nabla_{PX} P\right)Y - \left(\nabla_{PY} P\right)PX\right)$$

(3.3)

In particular,

$$\left(\nabla F*(PX, PX) = \left(\nabla F*(X, X) + S_Q (QX‘, X‘) - F*[SP (PX, X)]\right)\right)$$

(3.4)
where \( X' = F_*X \), \( Y' = F_*Y \), and \( \lambda = 1 \) when \( F \) is \((P, Q)\)-holomorphic, \( \lambda = -1 \) when \( F \) is \((P, Q)\)-antiholomorphic.

**Proof.** Let \( F \) be a \((P, Q)\)-paraholomorphic map so that \( F_* \circ P = Q \circ F_* \) then

\[
(\nabla F_*) (X, PY) = \nabla^N X \cdot (PY)' - F_* \left( \frac{\partial}{\partial X} (PY) \right) + F_* \left( \frac{\partial}{\partial Y} (PY) \right) - F_* \left( \frac{\partial}{\partial Y} (PY) \right)
\]

But then this gives us

\[
(\nabla F_*) (X, PY) = Q [(\nabla F_*) (X, Y)] + \left( \nabla^N X \cdot Q \right) Y' \quad \text{and} \quad F_* \left( \frac{\partial}{\partial Y} (PY) \right)
\]

So we get

\[
(\nabla F_*) (X, PY) = Q [(\nabla F_*) (X, Y)] + \left( \nabla^N X \cdot Q \right) Y' - F_* \left( \frac{\partial}{\partial Y} (PY) \right)
\]

But then this gives us

\[
(\nabla F_*) (PX, PY) = Q [(\nabla F_*) (PX, Y)] + \left( \nabla^N (PX) \cdot Q \right) Y' \quad \text{and} \quad F_* \left( \frac{\partial}{\partial Y} (PY) \right)
\]

and

\[
(\nabla F_*) (Y, PX) = Q [(\nabla F_*) (Y, X)] + \left( \nabla^N Y \cdot Q \right) X' \quad \text{and} \quad F_* \left( \frac{\partial}{\partial X} (PX) \right)
\]

Using (3.6) in (3.5) and the symmetry of \( \nabla F_* \), we get

\[
(\nabla F_*) (PX, PY) = (\nabla F_*) (X, Y) + \left( \nabla^N (PX) \cdot Q \right) Y' + Q \left( \frac{\partial}{\partial Y} (PY) \right) X' - F_* \left( \frac{\partial}{\partial Y} (PY) \right) + P \left( \frac{\partial}{\partial Y} (PY) \right)
\]

Finally using the fact that \( P \circ (\nabla P) = - (\nabla P) \circ P \) in the equation (3.7) we get the desired result (3.3).

In particular, since

\[
S_Q (QX', X') = \left( \nabla^N Q_X \cdot Q \right) X' - \left( \nabla^N Q_X \cdot Q \right) (QX') = Q (S_Q (X', X'))
\]

\[
S_P (PX, X) = \left( \nabla^M P_X \cdot P \right) X - \left( \nabla^M P_X \cdot P \right) (PX) = P (S_P (X, X))
\]

we get the equation (3.4), that is,

\[
(\nabla F_*) (PX, PX) = (\nabla F_*) (X, X) + S_Q (QX', X') - F_* [S_P (PX, X)]
\]

\[
= (\nabla F_*) (X, X) + Q \{ S_Q (X', X') - F_* [S_P (X, X)] \}.
\]
For the case where \( F \) is \((P, Q)\)-anti-paraholomorphic, the same argument works so that the required result \( (3.4) \) follows.

For a pair of almost product [resp: almost golden] manifolds \((M, h, P), (N, g, Q)\) [resp: \((M, h, G), (N, g, K)\)] with pure or hyperbolic metric \( h \) with respect to \( P \) [resp: \( G \)] and pure or hyperbolic metric \( g \) with respect to \( Q \) [resp: \( K \)] we set the following conditions:

(I): \((M, h, P)\) is a para-Kaehler manifold and \((N, g, Q)\) is either a para-Kaehler manifold or a locally decomposable product Riemannian manifold.

(II): \((M, h, P)\) is a locally decomposable product Riemannian manifold and \((N, g, Q)\) is either a para-Kaehler manifold or a locally decomposable product Riemannian manifold.

(III): \((M, h, P)\) is a quasi para-Kaehler manifold and \((N, g, Q)\) is either quasi para-Kaehler manifold or a locally decomposable product Riemannian manifold.

(IV): \((M, h, P)\) is a locally decomposable product Riemannian manifold and \((N, g, Q)\) is either a quasi para-Kaehler manifold or a locally decomposable product Riemannian manifold.

Corollary 33 \((3.1)\): For a map \( F : (M, h, P) \to (N, g, Q) \)

i) let \( F \) be \( \pm (P, Q) \)-paraholomorphic between manifolds which are holding the condition (I) or (II) then for every local section \( X, Y \in \Gamma(TM) \),

\[
(\nabla F_\ast)(PX, PY) = (\nabla F_\ast)(X, Y)
\]

ii) let \( F \) be \( \pm (P, Q) \)-paraholomorphic between manifolds which are holding the condition (III) or (IV) then for every local section \( X \in \Gamma(TM) \),

\[
(\nabla F_\ast)(PX, PX) = (\nabla F_\ast)(X, X)
\] \((3.8)\)

Proof: \(
\]

i) Since \( \nabla^N Q = 0 \) and \( \nabla^M P = 0 \) in the case (I) or (II), the result follows from Proposition \((3.2)\).

ii) Since \( S^N_Q (QX', X') = 0 \) and \( S^P (PX, X) = 0 \) in the case (III) or (IV), the result follows from Proposition \((3.2)\) .

Theorem 34 \((3.1/A)\): Let \( F : (M, h, P) \to (N, g, Q) \) be a \( \pm (P, Q) \)-paraholomorphic map from a semi decomposable product Riemannian manifold \( M \) into either an almost product Riemannian manifold or an almost para-Hermitian manifold \( N \) with Vidal eigendistributions \( \mathcal{N}_{(1)}^N \) and \( \mathcal{N}_{(-1)}^N \) of \( Q \). Then the following statements are equivalent:

i) \( F \) is harmonic

ii) \( F \) is plus-eigen harmonic and minus-eigen harmonic

Proof: \(
\]

\((ii) \Rightarrow (i)\): This is obvious.
\[(i) \Rightarrow (ii):\] Let \( \{u_1, \ldots, u_s\} \) and \( \{v_1, \ldots, v_t\} \) be local orthonormal frame fields for \( M \mathcal{E}(1) \) and \( M \mathcal{E}(-1) \) respectively. Then observe that

\[ a^o \quad \text{Since} \div \mathcal{D} P = 0 \text{ and therefore } M \mathcal{E}(1) \text{ and } M \mathcal{E}(-1) \text{ are both minimal distributions,} \]

\[
u = \sum_{i=1}^{s} \nabla u_i u_i \in \Gamma \left( M \mathcal{E}(1) \right) \quad \text{and} \quad v = \sum_{i=1}^{t} \nabla v_i v_i \in \Gamma \left( M \mathcal{E}(-1) \right) \]

\[ b^o \quad \text{Since } F \text{ is } \pm (P, Q)\text{-paraholomorphic, } (a^o) \text{ gives us} \]

\[ F_* (u) \text{ and } u'_i = F_* (u_i) \in \Gamma \left( N \mathcal{E}(c) \right), \quad \forall \ i = 1, \ldots, s \]

\[ \text{and} \]

\[ F_* (v), \ v'_i = F_* (v_i) \in \Gamma \left( N \mathcal{E}(-c) \right), \quad \forall \ i = 1, \ldots, t \]

\[ c^o \quad \text{Since } N \text{ is Vidal, } (b^o) \text{ gives us} \]

\[ \nabla u'_i u'_i \in \Gamma \left( N \mathcal{E}(c) \right), \quad \forall \ i = 1, \ldots, s \quad \text{and} \quad \nabla v'_i v'_i \in \Gamma \left( N \mathcal{E}(-c) \right), \quad \forall \ i = 1, \ldots, t \]

\[ \text{and therefore} \]

\[ \sum_{i=1}^{s} \nabla u'_i u'_i \in \Gamma \left( N \mathcal{E}(c) \right) \quad \text{and} \quad \sum_{i=1}^{t} \nabla v'_i v'_i \in \Gamma \left( N \mathcal{E}(-c) \right) \]

\[ \text{So, from } (c^o), \text{ we have} \]

\[ T_{\mathcal{E}(1)} (F) = \sum_{i=1}^{s} (\nabla F_* (u_i, u_i) = \sum_{i=1}^{s} \left[ \nabla u'_i u'_i - F_* \left( \nabla u'_i u'_i \right) \right] \in \Gamma \left( N \mathcal{E}(c) \right) \]

\[ T_{\mathcal{E}(-1)} (F) = \sum_{i=1}^{t} (\nabla F_* (v_i, v_i) = \sum_{i=1}^{t} \left[ \nabla v'_i v'_i - F_* \left( \nabla v'_i v'_i \right) \right] \in \Gamma \left( N \mathcal{E}(-c) \right) \]

\[ \text{Where } c = 1, \text{ when } F \text{ is } (P, Q)\text{-paraholomorphic and } c = -1, \text{ when } F \]

\[ \text{is } (P, Q)\text{-anti-paraholomorphic. But then, since } \mathcal{E}(1) \cap \mathcal{E}(-1) = \{0\}, \text{ we have that} \]

\[ T_{\mathcal{E}(1)} (F) \text{ and } T_{\mathcal{E}(-1)} (F) \text{ are linearly independent. So, by the} \]

\[ \text{facts that } \{u_1, \ldots, u_s; \ v_1, \ldots, v_t\} \text{ is a local orthonormal frame fields for } TM \]

\[ \text{we have} \]

\[ T_{\mathcal{E}(1)} (F) = 0 = T_{\mathcal{E}(-1)} (F) \]
Corollary 35 (3.2/A) : Let \( F : (M, h, P) \rightarrow (N, g, Q) \) be a \( \pm (P, Q) \)-paraholomorphic map from a locally decomposable product Riemannian manifold \( M \) into either a locally decomposable product Riemannian manifold or nearly para-Kaehler (in particular, para-Kaehler) manifold \( N \). Then the following statements are equivalent:

i) \( F \) is harmonic.

ii) \( F \) is plus-eigen harmonic and minus-eigen harmonic.

Proof. : By Remark (3.1) one gets that

a°) for every locally decomposable product Riemannian manifold \( M \), the eigendistributions \( \mathcal{E}_{(1)}^M \) and \( \mathcal{E}_{(-1)}^M \) are both minimal.

b°) for every nearly para-Kaehler manifold \( N \), the eigendistributions \( \mathcal{E}_{(1)}^N \) and \( \mathcal{E}_{(-1)}^N \) are also both Vidal.

So the equivalence of (i) and (ii) follows from the observations (a°), (b°) and Theorem (2.1/A).

Theorem 36 (3.1/B) : Let \( F : (M, h, G) \rightarrow (N, g, K) \) be a \( \pm (G, K) \)-golden map from a semi decomposable golden Riemannian manifold \( M \) into either an almost golden Riemannian manifold or an almost golden-Hermitian manifold \( N \) with Vidal eigendistributions \( \mathcal{E}_{(\sigma)}^N \) and \( \mathcal{E}_{(\bar{\sigma})}^N \) of \( K \). Then the following statements are equivalent:

i) \( F \) is harmonic.

ii) \( F \) is plus-eigen harmonic and minus-eigen harmonic.

Proof. : Let \( P_G \) and \( Q_K \) denote the twin product structures of \( G \) and \( K \) respectively. Then by Lemma (2.1) and Proposition (3.1) the hypothesis of this theorem becomes equivalent to the hypothesis of Theorem (3.1/A), namely:

"Let \( F : (M, h, P) \rightarrow (N, g, Q_K) \) be a \( \pm (P, Q_K) \)-paraholomorphic map from a semi decomposable product Riemannian manifold \( M \) into either an almost product Riemannian manifold or an almost para-Hermitian manifold \( N \) with Vidal eigendistributions \( \mathcal{E}_{(\sigma)}^N \) and \( \mathcal{E}_{(\bar{\sigma})}^N \) of \( Q_K \)."

So the required conclusion of the theorem follows from Theorem (3.1/A).

Corollary 37 (3.2/B) : Let \( F : (M, h, G) \rightarrow (N, g, K) \) be a \( \pm (G, K) \)-golden map from a locally decomposable golden Riemannian manifold \( M \) into either a locally decomposable golden Riemannian manifold or nearly golden-Kaehler (in particular, golden-Kaehler) manifold \( N \). Then the following statements are equivalent:

i) \( F \) is harmonic.

ii) \( F \) is plus-eigen harmonic and minus-eigen harmonic.
Proposition 38 (3.3) : Let \( F : (M, h, P) \to (N, g, Q) \) be a \( \pm (P, Q) \)-paraholomorphic map from an almost para-Hermitian manifold \((M, h, P)\) into an almost para-Hermitian manifold or an almost product Riemannian manifold \((N, g, Q)\). Then the tension field \( \mathcal{T}(F) \) of \( F \) takes the form

\[
\mathcal{T}(F) = \sum_{i=1}^{m}\left\{ h(e_i, e_i)(\nabla F_*)(e_i, e_i) + h(Pre_i, Pre_i)(\nabla F_*)(Pre_i, Pre_i) \right\} - Q\left\{ \sum_{i=1}^{m} h_{ii}S_Q(e'_i, e'_i) - \lambda F_*(\text{div}(P)) \right\}
\]

where \( \{e_1, ..., e_m, Pre_1, ..., Pre_m\} \) is a local orthonormal frame field for \( TM \) and \( \lambda = 1 \) when \( F \) is \( (P, Q) \)-paraholomorphic, \( \lambda = -1 \) when \( F \) is \( (P, Q) \)-anti-paraholomorphic and \( h_{ii} = h(e_i, e_i) \), \( e'_i = F_*(e_i) \).

Proof. : For an orthonormal frame field \( \{e_1, ..., e_m, Pre_1, ..., Pre_m\} \) for \( TM \) we have, by definition,

\[
\mathcal{T}(F) = \sum_{i=1}^{m}\left\{ h(e_i, e_i)(\nabla F_*)(e_i, e_i) + h(Pre_i, Pre_i)(\nabla F_*)(Pre_i, Pre_i) \right\} = \sum_{i=1}^{m} h_{ii}\left\{ (\nabla F_*)(e_i, e_i) - (\nabla F_*)(Pre_i, Pre_i) \right\}
\]

On the other hand, from Proposition (3.2), we have

\[
(\nabla F_*)(Pre_i, Pre_i) = (\nabla F_*)(e_i, e_i) + Q\left\{ S_Q(e'_i, e'_i) - \lambda F_*(Sp(e_i, e_i)) \right\}
\]

Replacing this into (3.8) we get

\[
\mathcal{T}(F) = -Q\left\{ \sum_{i=1}^{m} h_{ii}\left[ S_Q(e'_i, e'_i) - \lambda F_*(Sp(e_i, e_i)) \right] \right\} - Q\left\{ \sum_{i=1}^{m} h_{ii}\left[ S_Q(e'_i, e'_i) - \lambda F_*(Sp(e_i, e_i)) \right] \right\} - Q\left\{ \sum_{i=1}^{m} h_{ii}S_Q(e'_i, e'_i) - \lambda F_*(\text{div}(P)) \right\}
\]

Theorem 39 (3.2/A) : Let \( F : (M, h, P) \to (N, g, Q) \) be a \( \pm (P, Q) \)-holomorphic map from an almost para-Hermitian manifold \((M, h, P)\) into an almost para-Hermitian manifold or an almost product Riemannian manifold \((N, g, Q)\). If either

i) \( (2, 7, 11) \), \((M, h, P)\) is a semi para-Kaehler manifold and \((N, g, Q)\) is a quasi para-Kaehler manifold,

or
(ii) \((M, h, P)\) is a semi para-Kaehler manifold and \((N, g, Q)\) is a locally decomposable product Riemannian manifold, then \(F\) is harmonic.

**Proof.**

For a local orthonormal frame field \(\{e_1, \ldots, e_m, Pe_1, \ldots, Pe_m\}\) for \(TM\) we have, by Proposition (3.3) that,

\[
\mathcal{T}(F) = -Q \left\{ \sum_{i=1}^{m} h_{ii} S_Q(e_i', e'_i) - \lambda F^* (\text{div}(P)) \right\}.
\]

But then, \(\lambda F^* (\text{div}(P)) = 0\) since \((M, h, P)\) is semi para-Kaehler and \(S_Q(e_i', e'_i) = 0\) since \((N, g, Q)\) is either quasi para-Kaehler or locally decomposable product Riemannian. So harmonicity of \(F\) follows.

**Theorem 40 (3.2/B):** For a \(\pm (G, K)\)-golden map \(F: (M, h, G) \rightarrow (N, g, K)\) from an almost golden-Hermitian manifold \((M, h, G)\) into an almost golden-Hermitian manifold or an almost golden Riemannian manifold \((N, g, K)\), if either

i) \((M, h, G)\) is a semi golden-Kaehler manifold and \((N, g, K)\) is a quasi golden-Kaehler manifold,

or

ii) \((M, h, G)\) is a semi golden-Kaehler manifold and \((N, g, K)\) is a locally decomposable golden Riemannian manifold,

then \(F\) is harmonic.

**Proof.** Let \(P_G\) and \(Q_K\) denote the twin product structures of \(G\) and \(K\) respectively. Then by Lemma (2.1) and Remark (3.1)/2 the hypothesis of this theorem becomes equivalent to the hypothesis of Theorem (3.2/A), namely:

\[
\text{"Let } F: (M, h, P_G) \rightarrow (N, g, Q_K) \text{ be a } \pm (P_G, Q_K)\text{-holomorphic map from a semi Kaehler manifold } (M, h, P_G) \text{ into either a quasi para-Kaehler manifold or a locally decomposable product Riemannian manifold."}
\]

Then the harmonicity of \(F\) follows from Theorem (3.2/A).

**Remark 41 (3.2):** For a non-constant map \(F: (M, h, \varphi (= P, G)) \rightarrow (N, g, \psi (= Q, K))\), when \(h\) is a pure metric (with respect to \(\varphi\)) the \(\pm\) paraholomorphicity of \(F\) (or \(F\) being a \(\pm\) golden) is not much of a help for the harmonicity of \(F\). The best results we seem to get are Theorems (3.1)/A and (3.1)/B. On the other hand, when \(h\) is hyperbolic, then \(\pm\) paraholomorphicity of \(F\) (or \(F\) being a \(\pm\) golden) gives its harmonicity under certain conditions as Theorems (3.2)/A and (3.2)/B show. On these lines we provide the following example:

**Example 42 (3.1):** On \(\mathbb{R}^2\) for \(X = (x_1, x_2), \ Y = (y_1, y_2) \in \Gamma(T\mathbb{R}^2)\), define

\[
h(X, Y) = \sum_{i=1}^{2} x_i y_i \quad \text{and} \quad P(X) = (x_1, -x_2), \quad G(X) = (\sigma x_1, \bar{\sigma} x_2).
\]
Then \((\mathbb{R}^2, h, P)\) becomes locally decomposable product Riemannian manifold and \((\mathbb{R}^2, h, G)\) becomes locally decomposable golden Riemannian manifold. Moreover, \((\mathbb{R}^2, h, P)\) and \((\mathbb{R}^2, h, G)\) are twin manifolds as \(\{P, G\}\) form a twin pair on \(\mathbb{R}^2\). Let \(f : (\mathbb{R}^2, h, \varphi(= P, G)) \rightarrow (\mathbb{R}^2, h, \varphi(= P, G))\) be defined by
\[
f(s, t) = (s, e^t).
\]

Observe that
- \(f\) is \((P, P)\)-paraholomorphic and also \((G, G)\)-golden and yet
- \(f\) is not harmonic since
\[
T(f) = \frac{\partial^2 f}{\partial s^2} + \frac{\partial^2 f}{\partial t^2} = (0, 0) + (0, e^t) = (0, e^t).
\]

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