Surjectivity of the adelic Galois Representation associated to a Drinfeld module of prime rank

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Abstract

In this paper, let $\phi$ be the Drinfeld module over $\mathbb{F}_q(T)$ of prime rank $r$ defined by

$$\phi_T = T + r^{r-1} + T^{q-1}r^r.$$ 

We prove that under certain condition on $\mathbb{F}_q$, the adelic Galois representation

$$\rho_\phi : \text{Gal}(\mathbb{F}_q(T)_{\text{sep}}/\mathbb{F}_q(T)) \to \varprojlim_{a} \text{Aut}(\phi[a]) \cong \text{GL}_r(\mathbb{A})$$

is surjective.

1 Introduction

In [Ser72], Serre proved his famous “Open Image Theorem” for elliptic curves over number field without complex multiplication. Restricted to elliptic curves over $\mathbb{Q}$, the theorem says

**Theorem (Ser72).** If $E$ is an elliptic curve over $\mathbb{Q}$ without complex multiplication, then its associated adelic Galois representation

$$\rho_E : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \varprojlim_{m} \text{Aut}(E[m]) \cong \text{GL}_2(\hat{\mathbb{Z}})$$

has open image in $\text{GL}_2(\hat{\mathbb{Z}})$.

The following is then a natural question:

**Question.** Is it possible to have an elliptic curve $E$ over $\mathbb{Q}$ such that $\rho_E(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) = \text{GL}_2(\hat{\mathbb{Z}})$?

Serre [Ser72] showed that the answer to the above question is negative, the main reason being the existence of a non-trivial relation between the 2-torsion $E[2](\overline{\mathbb{Q}})$ and some $m$-torsion $E[m](\overline{\mathbb{Q}})$. Although Serre proved the adelic surjectivity problem is negative for elliptic curves over $\mathbb{Q}$, Greicius [Gre10] proved it is true for elliptic curves over a “good enough” number field. Several mathematicians studied the generalization of adelic surjectivity problem for abelian varieties of dimension $\geq 2$ over a number field, cf. [AD20, Hal11, Zyw15, and LSTX20]. Currently, it is known that there
is a genus 3 hyperelliptic curve over \( \mathbb{Q} \) whose Jacobian variety satisfies the adelic surjectivity. Not much is known for adelic surjectivity of higher dimensional abelian varieties.

Let \( A = \mathbb{F}_q[T], \) \( F = \mathbb{F}_q(T) \) and \( \hat{\tilde{A}} = \lim_{\frac{1}{a}} A/a. \) Pink and Rütsche [PR09a] proved the Drinfeld module analogue of Serre’s open image theorem:

**Theorem (PR09a).** Let \( \phi \) be a Drinfeld \( A \)-module over \( F \) of rank \( r \) without complex multiplication, then its associated adelic Galois representation

\[
\rho_\phi : \text{Gal}(F^{\text{sep}}/F) \to \lim_{\frac{1}{a}} \text{Aut}(\phi[a]) \cong \text{GL}_r(\hat{\tilde{A}})
\]

has open image in \( \text{GL}_r(\hat{\tilde{A}}) \)

Therefore, it is reasonable to study the adelic surjectivity problem for Drinfeld modules over \( F \) of rank \( r \). The answer for rank \( r = 1 \) is positive, which follows from Hayes’ work [Hay74] on the function field analogue of class field theory. More precisely, the Carlitz module \( C_T = T + \tau \) has surjective adelic Galois image. Moreover, the answer for rank \( r = 2 \) is also positive, assuming that \( q = p^e \geq 5 \) is an odd prime power. This was proved by Zywina [Zyw11] for the rank-2 Drinfeld module \( \phi_T = T + \tau - T^{q-1}\tau^2 \). Remark that for the case \( q = 2 \) and \( r = 2 \), the author [Che21] proved there is no rank-2 Drinfeld \( A \)-module over \( F \) with surjective adelic Galois representation due to a similar reason as Serre’s arguments for elliptic curves over \( \mathbb{Q} \). Therefore, some restrictions on \( q \) is necessary.

In [Che20], we have proved the adelic surjectivity of Galois representation associated to a Drinfeld module of rank 3 defined by \( \varphi_T = T + \tau^2 + T^{q-1}\tau^3 \). Thus for the case of Drinfeld modules of rank \( r \geq 4 \), one may expect the same result will hold for Drinfeld module \( \phi \) over \( F \) defined by \( \phi_T = T + \tau^{r-1} + (-1)^{r-1}T^{q-1}\tau^r \). Our main result is the following theorem that shows the expectation is true for prime rank Drinfeld modules:

**Main Theorem.** Let \( q = p^e \) be a prime power, \( A = \mathbb{F}_q[T], \) and \( F = \mathbb{F}_q(T) \). Assume \( r \geq 3 \) is a prime number and \( q \equiv 1 \mod r \), there is a constant \( c = c(r) \in \mathbb{N} \) depending only on \( r \) such that for \( p > c(r) \) the following statement is true:

Let \( \phi \) be a Drinfeld \( A \)-module over \( F \) defined by \( \phi_T = T + \tau^{r-1} + T^{q-1}\tau^r \). Then the adelic Galois representation

\[
\rho_\phi : \text{Gal}(\mathbb{F}_q(T)^{\text{sep}}/\mathbb{F}_q(T)) \to \lim_{\frac{1}{a}} \text{Aut}(\phi[a]) \cong \text{GL}_r(\hat{\tilde{A}})
\]

is surjective.

The general idea comes from the proof for \( r = 3 \). However, when we try to adapt the strategy of the proof for \( r = 3 \) to \( r \geq 3 \), several technical problems arise. To explain these problems we briefly recall the idea of the proof for \( r = 3 \). The problems happened when we try to prove the mod \( I \) Galois representations are irreducible and surjective for all prime ideals \( I \) of \( A \).

To prove irreducibility in the rank-3 case, we aim for a contradiction by assuming the mod \( I \) representation \( \tilde{\rho}_{\varphi,I} \) is reducible. Hence the characteristic polynomial \( \tilde{P}_{\varphi,p}(x) \in \mathbb{F}_p[x] \) of Frobenius elements \( \tilde{\rho}_{\varphi,I}(\text{Frob}_p) \) must contain a linear factor for all prime \( p \neq (T) \) or \( I \). Since the degree is 3, we can explicitly write down a linear factor of \( \tilde{P}_{\varphi,p}(x) \) when \( \deg_p(p) = 1 \). Furthermore, we can also determine the characteristic polynomials concretely. Hence a contradiction can be deduced by comparing coefficients of \( \tilde{P}_{\varphi,p}(x) \) and its factorization. In the case \( r \geq 4 \), the assumption of
reducibility does not imply the characteristic polynomial of a Frobenius element would contain a linear factor.

Following the strategy for the rank-3 case, one might approach the problem of surjectivity by contradiction. We assume the image \( \text{Im} \bar{\rho}_{\varphi,l}(G_F) \) is a proper subgroup of \( \text{GL}_r(\mathbb{F}_l) \). Thus \( \text{Im} \bar{\rho}_{\varphi,l}(G_F) \) must be contained in some maximal subgroup of \( \text{GL}_r(\mathbb{F}_l) \). By our knowledge of \( \text{Im} \bar{\rho}_{\varphi,l}(I_T) \) (see Lemma 3.2), we can see that \( |\text{Im} \bar{\rho}_{\varphi,l}(G_F)| \) is divisible by certain power of \( |A/l| \). Hence we can use this property to rule out possible maximal subgroups. In rank-3 case, we can rule out all possible maximal subgroups of \( \text{GL}_3(\mathbb{F}_l) \). As the rank increases, we have some maximal subgroups that cannot be ruled out by merely considering their sizes. The reason is that the growth rate of \( |\text{Im} \bar{\rho}_{\varphi,l}(I_T)| \) is much slower than the growth rate of \( p \)-power component of \( |\text{GL}_r(\mathbb{F}_l)| \) when \( r \) increases.

Another harder problem is the classification of maximal subgroups in \( \text{GL}_r(\mathbb{F}_l) \). From [BHRD13] Theorem 2.2.19, we can see that those maximal subgroups are divided into 9 classes in Aschbacher’s theorem. The first 8 classes (geometric classes) have general description, but the ninth class (special class) doesn’t have a known description for arbitrary \( r \) so far. (This is also the reason why the authors could only describe all the maximal subgroups for low dimensional finite classical groups.)

Fortunately, if we restrict to the case where \( r \) is a prime number, then we can combine Pink’s work (see [PR09a], section 3) on surjectivity of mod 1 representations

\[
\bar{\rho}_{\varphi,l}: G_F \longrightarrow \text{Aut}(\phi[l]) \cong \text{GL}_r(\mathbb{F}_l)
\]

with Aschbacher’s theorem ([BHRD13], Theorem 2.1.5) to prove “irreducibility of mod 1 Galois representations” toward “surjectivity of mod 1 Galois representations” assuming the characteristic of \( F \) is large enough. This procedure is described in section 3 and 4.

In section 5, we prove the irreducibility of the Galois representation. On the other hand, we prove the surjectivity of mod \((T)\) Galois representation directly using a result of Abhyankar.

The proof toward \( l \)-adic surjectivity is similar to the rank-3 case and is proved in section 6. In section 7, we prove the adelic surjectivity under a further assumption \( q \equiv 1 \pmod{r} \). The proof for adelic surjectivity is similar to the rank-3 case as well.

## 2 Preliminaries

### 2.1 Notation

- \( q = p^e \) is a prime power with \( p \geq 5 \)
- \( A = \mathbb{F}_q[T] \)
- \( F = \mathbb{F}_q(T) \)
- \( F^{\text{sep}} = \) separable closure of \( F \)
- \( F^{\text{alg}} = \) algebraic closure of \( F \)
- \( G_F = \text{Gal}(F^{\text{sep}}/F) \)
- \( l = (l) \) a prime ideal of \( A \), and define \( \deg_T(l) = \deg_T(l) \)
- \( A_p = \) completion of \( A \) at the nonzero prime ideal \( p \triangleleft A \)
• $\hat{A} = \lim_{\longrightarrow} A/a$

• $F_p = \text{fraction field of } A_p$

• $F_p = A/p$

2.2 Drinfeld module over a field

Let $K$ be a field, we call $K$ an $A$-field if $K$ is equipped with a homomorphism $\gamma : A \to K$. Let $K\{\tau\}$ be the ring of skew polynomials satisfying the commutation rule $e^b \cdot \tau = \tau \cdot c$.

A Drinfeld $A$-module over $K$ of rank $r \geq 1$ is a ring homomorphism

$$\phi : A \longrightarrow K\{\tau\}$$

$$a \mapsto \phi(a) = \gamma(a) + \sum_{i=1}^{r-\deg(a)} g_i(a)\tau^i.$$  

It is uniquely determined by $\phi_T = \gamma(T) + \sum_{i=1}^r g_i(T)\tau^i$, where $g_r(T) \neq 0$.

$\ker(\gamma)$ is called the $A$-characteristic of $K$, and we say $K$ has generic characteristic if $\ker(\gamma) = 0$.

Proposition 2.1. Let $\phi$ be a Drinfeld module over $K$ of rank $r$ with nonzero $A$-characteristic $p$.

For each $a \in A$, we may write $\phi$ as $\phi_a = c(a)\tau^{m(a)} + \cdots + C(a)\tau^{M(a)}$. There is a unique integer $0 < h \leq r$ such that $m(a) = hv_p(a)$ for all nonzero $a \in A$, where $v_p$ is the $p$-adic valuation of $F$.

This integer $h$ is called the height of $\phi$.

Proof. See [Gos96] Proposition 4.5.7. $$

An isogeny from a Drinfeld module $\phi$ to another Drinfeld module $\psi$ over $K$ is an element $u \in K\{\tau\}$ such that $u \cdot \phi_a = \psi_a \cdot u \ \forall \ a \in A$. Hence the endomorphism ring of $\phi$ over $K$ is defined as

$$\text{End}_K(\phi) = \{u \in K\{\tau\} \mid u \cdot \phi_T = \psi_T \cdot u\}.$$  

The Drinfeld module $\phi$ gives $K^{\text{alg}}$ an $A$-module structure, where $a \in A$ acts on $K^{\text{alg}}$ via $\phi_a$. We use the notation $\phi\phi(a)$ to emphasize the action of $A$ on $K^{\text{alg}}$.

The $a$-torsion $\phi[a] = \{\text{zeros of } \phi_a(x) = \gamma(a)x + \sum_{i=1}^{r-\deg(a)} g_i(a)x^i \} \subseteq K^{\text{alg}}$. The action $b \cdot \alpha = \phi_b(\alpha) \ \forall \ b \in A, \forall \alpha \in \phi[a]$ also gives $\phi[a]$ an $A$-module structure.

Proposition 2.2. Let $\phi$ be a rank $r$ Drinfeld module over $K$ and $a$ an ideal of $A$,

1. If $\phi$ has $A$-characteristic prime to $a$, then the $A/a$-module $\phi[a]$ is free of rank $r$

2. If $\phi$ has nonzero $A$-characteristic $p$, let $h$ be the height of $\phi$, then the $A/p$-module $\phi[p^e]$ is free of rank $r - h$ for all $e \in \mathbb{Z}_{\geq 1}$.

Proof. See [Gos96] Proposition 4.5.7. $$

Note 2.3. From now on, we consider $K = F$ and $\gamma : A \to F$ is the natural injection map.
Let \( \phi \) be a rank \( r \) Drinfeld module over \( F \) of generic characteristic, then \( \phi[a] \) is separable, so we have \( \phi[a] \subseteq F^{\text{sep}} \). This implies that \( \phi[a] \) has a \( G_F \)-module structure. Given a nonzero prime ideal \( \mathfrak{l} \) of \( A \), we can consider the \( G_F \)-module \( \phi[\mathfrak{l}] \). We obtain the so-called mod \( \mathfrak{l} \) Galois representation

\[
\bar{\rho}_{\phi, \mathfrak{l}} : G_F \longrightarrow \text{Aut}(\phi[\mathfrak{l}]) \cong GL_r(F). 
\]

Taking inverse limit with respect to \( \mathfrak{l} \), we have the \( \mathfrak{l} \)-adic Galois representation

\[
\rho_{\phi, \mathfrak{l}} : G_F \longrightarrow \varprojlim \text{Aut}(\phi[\mathfrak{l}]) \cong GL_r(A). 
\]

Combining all representations together, we get the adelic Galois representation

\[
\rho_\phi : G_F \longrightarrow \varprojlim \text{Aut}(\phi[a]) \cong GL_r(\hat{A}). 
\]

### 2.3 Carlitz module

The **Carlitz module** is the Drinfeld module \( C : A \rightarrow F\{t\} \) of rank 1 defined by

\[
C(t) = t + \tau.
\]

**Proposition 2.4.** (Hayes [Hay74]) For a nonzero ideal \( a \) of \( A \), the Galois representation

\[
\bar{\rho}_{C, a} : G_F \longrightarrow \text{Aut}(C[a]) \cong (A/a)^* 
\]

is surjective. This implies the adelic Galois representation of Carlitz module is surjective. Moreover, for prime ideals \( p \) of \( A \) such that \( p \nmid a \), we have \( \bar{\rho}_{C, a}(\text{Frob}_p) \equiv p \mod a \).

### 2.4 Reduction of Drinfeld modules

Let \( K \) be a local field with uniformizer \( \pi \), valuation ring \( R \), unique maximal ideal \( p := (\pi) \), normalized valuation \( v \) and residue field \( \mathbb{F}_p \). Let \( \phi : A \rightarrow K\{\tau\} \) be a Drinfeld module of rank \( r \). We say that \( \phi \) has **stable reduction** if there is a Drinfeld module \( \phi' : A \rightarrow R\{\tau\} \) such that

1. \( \phi' \) is isomorphic to \( \phi \) over \( K \); 
2. \( \phi' \mod p \) is still a Drinfeld module (i.e. \( \phi'_p \mod p \) has deg. \( \geq 1 \)).

\( \phi \) is said to have **stable reduction of rank** \( r_1 \) if \( \phi \) has stable reduction and \( \phi \mod p \) has rank \( r_1 \). \( \phi \) is said to have **good reduction** if \( \phi \) has stable reduction and \( \phi \mod p \) has rank \( r \).

**Remark 2.5.**

We sometimes denote \( \phi \mod p \) by \( \phi \otimes \mathbb{F}_p \).

The Drinfeld module analogue of Néron-Ogg-Shafarevich is the following:

**Proposition 2.6.** ([Tak82], Theorem 1) Let \( \phi : A \rightarrow K\{\tau\} \) be a Drinfeld module and \( \mathfrak{l} \) be a nonzero prime ideal different from the \( A \)-characteristic of \( \phi \otimes \mathbb{F}_p \). Then \( \phi \) has good reduction if and only if the \( \mathfrak{l} \)-adic Galois representation is unramified at \( p \). In other words, \( \rho_{\phi, \mathfrak{l}}(I_p) = 1 \) where \( I_p \) is the inertia subgroup of \( G_K \).
Let $u : \phi \to \psi$ be an isogeny between Drinfeld modules over $K$. We study the reduction type of the isogenous Drinfeld module. The isogeny $u$ induces an $G_K$-equivariant isomorphism between rational Tate modules

$$u : V_l(\phi) \to V_l(\psi),$$

where $V_l(\phi) := T_l(\phi) \otimes F_l$. Suppose $l$ is different from the $A$-characteristic of $\phi \otimes \mathbb{F}_p$, then the Drinfeld module analogue of Néron-Ogg-Shafarevich implies $\psi$ has good reduction at $p$. As a result, isogenous Drinfeld modules over a local field either both have good reduction or bad reduction.

On the other hand, we prove that isogenous Drinfeld modules also preserve stable bad reduction under a condition on the inertia action.

**Proposition 2.7.** Let $u : \phi \to \psi$ be an isogeny between Drinfeld modules over $K$. Suppose there is a prime ideal $l$ of $A$, and $l$ is different from the $A$-characteristic of $\phi \otimes \mathbb{F}_p$. Assume further that the inertia group $I_p$ acts on $V_l(\phi)$ via a group of unipotent matrices, then the isogenous Drinfeld module $\psi$ has stable reduction at $p$.

**Proof.** We may assume $\psi$ is defined over the valuation ring $R$ after replacing $\psi$ by an isomorphic copy. Moreover, the assumption that $I_p$ acts on $V_l(\phi)$ by unipotent matrices implies that the action of $I_p$ on $V_l(\psi)$ is also unipotent. Therefore, $I_p$ acts by unipotent matrices on $\psi[l]$, which means the ramification index of $K(\psi[l])/K$ is a power on $p$.

Suppose $\psi$ does not have stable reduction over $K$, then any extension $L/K$ over which $\psi$ has stable reduction has ramification index divisible by some prime not equal to $p$. Therefore, if $\psi$ does not have stable reduction over $K$, then $\psi$ does not have stable reduction over $K(\psi[l])$ either, so we may assume that $\psi[l]$ is rational over $K$.

On the other hand, consider the Newton Polygon $\text{NP}(\psi_1(x)/x)$ of the polynomial $\psi_1(x)/x$. If $\psi[l]$ is rational over $K$, then the roots of $\psi_1(x)$ have integer valuations, so the slope of the first line segment of $\text{NP}(\psi_1(x)/x)$ is an integer. Otherwise, we have the slope of the first line segment of $\text{NP}(\psi_1(x)/x)$ is a simplified fraction $\frac{a}{b}$ with denominator $d \neq 1$. Thus there are roots of $\psi_1(x)/x$ with valuation equal to $\frac{a}{b}$, which is a contradiction.

Now we write

$$\psi_1(x) = \sum_{i=0}^{r-\deg_{F}(l)} g_i(l)x^q^i, \text{ with } g_0(l) = l.$$ 

The first line segment of $\text{NP}(\psi_1(x)/x)$ has endpoints $(0,0)$ and $(q^m - 1, v(g_m(l)))$. The integrality of slope implies $q^m - 1 \mid v(g_m(l))$. Hence after taking a suitable isomorphic copy of $\psi$, we may assume $v(g_m(l)) = 0$. This implies one of the coefficient of $\psi_T(x)$ other than $T^r x$ must be a unit. Hence we deduce that $\psi_T$ has stable reduction over $K$, which is a contradiction.

\[\square\]

**Remark 2.8.** Unlike abelian varieties (Corollaire 3.8 in chapter IX of [SGA72]), stable bad reduction of Drinfeld module over local field does not imply the inertia group acts on Tate module via unipotent matrices. The following is a counterexample:

Let $p = (T)$ and $\phi_T = T + \tau + T\tau^2$ be the Drinfeld $A$-module defined over $F_p$. It’s clear that $\phi$ has stable bad reduction of rank 1. For any prime $l = (T - c) \neq p$, the Tate uniformization shows that the inertia group $I_p$ acts on $T_l(\phi)$ via matrices of the form

$$\begin{pmatrix} 1 & * \\ 0 & c \end{pmatrix}.$$
We know $\det \circ \rho_{\phi,l}(I_p) = \rho_{\psi,l}(I_p)$, where $\psi$ is the rank-1 Drinfeld module $\psi_T = T - T \tau$ by Proposition 7.1 in [vdH04]. Now we claim that $\rho_{\psi,l}(I_T)$ is nontrivial. We observe the Newton polygon of $\psi_1(x)/x$ with respect to the valuation $v_p$, the polygon is a single line with slope equal to $1/q - 1$. Hence the Galois extension $F_p(\psi[l])/F_p$ ramifies. This implies the inertia group $I_T$ acts nontrivially on $\psi[l]$, hence acts nontrivially on $T_1(\psi)$ as well. Therefore, $\det \circ \rho_{\phi,l}(I_p) = \rho_{\psi,l}(I_p)$ is nontrivial, so $c \neq 1$.

### 2.5 Determinant of $\rho_\phi$

Let $\phi$ be a Drinfeld module over $F$ defined by

$$\phi_T = T + g_1 \tau + g_2 \tau^2 + \cdots + g_r \tau^r.$$ 

Let $p \neq l$ be a prime of good reduction of $\phi$, the $l$-adic Galois representation $\rho_{\phi,l}$ is unramified at $p$ by Proposition [2.6]. Therefore, the matrix $\rho_{\phi,l}(\text{Frob}_p) \in \text{GL}_r(A_l)$ is well-defined up to conjugation, so we can consider the characteristic polynomial $P_{\phi,p}(x) = \det(xI - \rho_{\phi,l}(\text{Frob}_p))$ of the Frobenius element $\text{Frob}_p$.

The polynomial $P_{\phi,p}(x)$ has coefficients in $A$ which are independent of the choice of $l$. Moreover, $P_{\phi,p}(x)$ is equal to the characteristic polynomial of Frobenius endomorphism of $\phi \otimes \overline{\mathbb{F}}_p$ acting on $T_1(\phi \otimes \mathbb{F}_p)$. We may write $P_{\phi,p}(x)$ as follows:

$$P_{\phi,p}(x) = a_r + a_{r-1} x + a_{r-2} x^2 + \cdots + a_1 x^{r-1} + x^r \in A[x].$$

The constant term $a_r$ is equal to $(-1)^r \det \circ \rho_{\phi,a}(\text{Frob}_p)$.

**Proposition 2.9.** ([Yu95], Theorem 1) For $1 \leq i \leq r$, we have $\deg(a_i) \leq \frac{i \deg(p)}{r}$.

**Proposition 2.10.** ([HY00], p.268)

$$a_r = \epsilon(\phi) \cdot p.$$ 

Here $\epsilon(\phi) = (-1)^r (-1)^{\deg_T (r+1)} \text{Nr}_{\mathbb{F}_q}(\phi^r)_1^{-1}$, which belongs to $\mathbb{F}_q^*$ and is independent of $\phi$.

Let $\phi$ be the Drinfeld module over $F$ defined by $\phi_T = T + T^r - 1 + T^{q-1} \tau^r$, where $r$ is an odd prime. By [vdH04] Proposition 7.1, we have

$$\det \circ \bar{\rho}_{\phi,a} = \bar{\rho}_{\psi,a},$$

where $\psi$ is defined by $\psi_T = T + T^{q-1} \tau$. It’s clear that $\psi$ is isomorphic over $F$ to the Carlitz module $C_T = T + \tau$, so we have the following commutative diagram:

$$\begin{array}{ccc}
G_F & \xrightarrow{\bar{\rho}_{\phi,l}} & \text{GL}_r(F_l) \\
\| & & \downarrow \det \\
G_F & \xrightarrow{\bar{\rho}_{C,l}} & (A/l)^*
\end{array}$$

Hence we can deduce the following Corollary from Proposition [2.4]

**Corollary 2.11.** For prime ideals $p$ of $A$ such that $p \not| a$, we have $\det \circ \bar{\rho}_{\phi,a}(\text{Frob}_p) \equiv p \mod a$. 

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2.6 Drinfeld-Tate uniformization

Let $\phi : A \to A_p \{\tau\}$ be a Drinfeld module. A \textit{\phi-lattice} is a finitely generated free $A$-submodule of $\phi F_p^{\text{sep}}$ and stable under $G_{F_p}$-action. Here the discreteness is with respect to the topology of the local field $F_p^{\text{sep}}$.

A 	extit{Tate datum} over $A_p$ is a pair $(\phi, \Gamma)$ where $\phi$ is a Drinfeld module over $A_p$ and $\Gamma$ is a $\phi$-lattice. Two pairs $(\phi, \Gamma)$ and $(\phi', \Gamma')$ of Tate datum are isomorphic if there is an isomorphism from $\phi$ to $\phi'$ such that the induced $A$-module homomorphism $\phi F_p^{\text{sep}} \to \phi' F_p^{\text{sep}}$ gives an $A$-module isomorphism $\Gamma \to \Gamma'$.

**Proposition 2.12.** (Drinfeld) Let $r_1$, $r_2$ be two positive integers. There is a one-to-one correspondence between two sets:

1. The set of $F_p$-isomorphism classes of Drinfeld modules $\phi$ over $F_p$ of rank $r := r_1 + r_2$ with stable reduction of rank $r_1$

2. The set of $F_p$-isomorphism classes of Tate datum $(\psi, \Gamma)$ where $\psi$ is a Drinfeld module over $A_p$ of rank $r_1$ with good reduction, and $\Gamma$ is a $\psi$-lattice of rank $r_2$.

**Proof.** See chapter 4 in [Leh09].

**Remark 2.13.**

Fix $a \in A - F_q$. From the proof of Proposition 2.12 we have the following two properties:

(i) There is a $G_{F_p}$-equivariant short exact sequence of $A$-modules:

$$0 \to \psi[a] \to \phi[a] \xrightarrow{\psi} \Gamma/a \Gamma \to 0.$$

(ii) There is an element $u \in A_p \{\tau\}$ such that $u \psi = \phi u$, here $A_p \{\tau\}$ is the set of power series in $\tau$ with coefficients in $A_p$. This element $u$ induces an isomorphism of $A[G_{F_p}]$-modules from $\psi^{-1}(\Gamma)/\Gamma$ to $\phi[a]$ by mapping $z + \Gamma$ to $u(z)$.

Moreover, the function $u$ can be expressed in the following ways:

1. $u(x) = x + u_1 x^q + u_2 x^{q^2} + \cdots + u_i x^{q^i} + \cdots$, and $u_i$ belongs to the maximal ideal of $A_p$ for all $i$.

2. When $r_2 = 1$, $u(x) = x \cdot \prod_{0 \neq \alpha \in A} (1 - \frac{x}{\phi_{\alpha}(\gamma)})$ where $\gamma$ is a generator of $\Gamma$.

3 Image of $\hat{\rho}_{\phi, \mathbf{1}}$ and Aschbacher’s Theorem

From now on, we work under the following assumptions:

Let $r$ be a prime number, $A = F_q[T]$, and $F = F_q(T)$, where $q = p^r$ and $p > r!$. Let $\phi$ be a Drinfeld $A$-module over $F$ of rank $r$ with generic characteristic defined by $\phi_{T} = T + T^{q-1} + T^{q^2 - 1} r T$. Let $l$ be a place of $F$ where $\phi$ has good reduction at $l$. We denote $\hat{\rho}_{\phi, \mathbf{1}}(G_F)$ by $\Gamma_1$, so $\Gamma_1$ is a subgroup of $GL_r(F_l)$.

By Aschbacher’s Theorem (Theorem B.1 in Appendix B), $\Gamma_1$ lies in one of the Aschbacher classes. In this subsection, we’ll show that classes $C_2$, $C_3$, $C_4$, $C_7$, and $C_8$ can be ruled out.
Lemma 3.1. There is a basis of \( \phi[l] \) such that

\[
\tilde{\rho}_{\phi,1}(I_T) \subseteq \left\{ \begin{pmatrix} 1 & b_1 \\ \vdots & \vdots \\ \vdots & \vdots \\ b_{r-1} & 1 \end{pmatrix}, \ b_i \in \mathbb{F}_l \ \forall \ 1 \leq i \leq r-1 \right\}.
\]

Proof. As \( \phi_T = T + \tau r^{-1} - T \tau r^{-1} \) has stable reduction at \( (T) \) of rank \( r-1 \), we may use Tate uniformization to obtain a Tate datum \( (\psi, \Gamma) \). Here \( \psi \) has good reduction of rank \( r-1 \) and \( \Gamma \) has rank 1. The Drinfeld module \( \psi : A \to A/(T) \{ r \} \) has rank \( r-1 \) of good reduction, so the Galois representation \( \tilde{\rho}_{\psi,1} : G_{F(T)} \to \text{Aut}(\psi[l]) \) is unramified. Thus there is a basis \( \{ w_1, w_2, \ldots, w_{r-1} \} \) of \( \psi[l] \) such that \( \sigma(w_i) = w_i \ \forall \sigma \in I_T, \ \forall \ 1 \leq i \leq r-1 \).

Now since \( \Gamma \) is a free \( A \)-module of rank 1, we may fix a generator \( \gamma \) of \( \Gamma \). Choose \( z \in F_{sep}^{T} \) such that \( \psi(z) = \gamma \). The fact that \( \Gamma \) is stable under the Galois action implies that there is a character \( \chi_\Gamma : G_{F,T} \to \mathbb{F}_l^* \) such that \( \sigma(\gamma) = \chi_\Gamma(\sigma) \gamma, \ \forall \sigma \in I_T \). By Remark 2.13(ii), we have

\[
\psi(\sigma(z)) = \psi(\psi(z)) = \sigma(\gamma) = \chi_\Gamma(\sigma) \gamma = \chi_\Gamma(\sigma) \psi(z) = \psi(\chi_\Gamma(\sigma) z)
\]

Thus \( \sigma(z) - \chi_\Gamma(\sigma) z \in \psi[l] \), therefore there are some elements \( b_{\sigma,1}, b_{\sigma,2}, \ldots, b_{\sigma,r-1} \) in \( \mathbb{F}_l \) such that

\[
\sigma(z) = b_{\sigma,1} w_1 + b_{\sigma,2} w_2 + \cdots + b_{\sigma,r-1} w_{r-1} + \chi_\Gamma(\sigma) z.
\]

Therefore, the action of \( \sigma \in I_T \) on \( \psi^{-1}_l(\Gamma)/\Gamma \) with respect to the basis \( \{ w_1 + \Gamma, w_2 + \Gamma, \ldots, w_{r-1} + \Gamma, \ z + \Gamma \} \) is of the form

\[
\begin{pmatrix}
1 & b_{\sigma,1} \\
\vdots & \vdots \\
\vdots & \vdots \\
b_{\sigma,r-1} & 1
\end{pmatrix}.
\]

Because of our choice of \( \phi_T \), we have the determinant of mod \( I \) Galois representation \( \det \circ \tilde{\rho}_{\phi,1} \) is equal to the mod \( I \) Galois representation of the Carlitz module \( \tilde{\rho}_{\psi,1} \). Hence \( \chi_\Gamma(\sigma) = 1 \) for all \( \sigma \in I_T \) since the Carlitz module has good reduction at \( (T) \). Therefore, we have deduced

\[
\tilde{\rho}_{\phi,1}(I_T) \subseteq \left\{ \begin{pmatrix} 1 & b_1 \\ \vdots & \vdots \\ \vdots & \vdots \\ b_{r-1} & 1 \end{pmatrix}, \ b_i \in \mathbb{F}_l \ \forall \ 1 \leq i \leq r-1 \right\}
\]

with respect to the basis \( \{ w_1 + \Gamma, w_2 + \Gamma, \ldots, w_{r-1} + \Gamma, \ z + \Gamma \} \).

Lemma 3.2. The inclusion in Lemma 3.1 is an equality. In particular, \( \tilde{\rho}_{\phi,1}(I_T) \) has order equal to \( |\mathbb{F}_l|^{r-1} \).

Proof. So far we have from Lemma 3.1 that

\[
\tilde{\rho}_{\phi,1}(I_T) \subseteq \left\{ \begin{pmatrix} 1 & b_1 \\ \vdots & \vdots \\ \vdots & \vdots \\ b_{r-1} & 1 \end{pmatrix}, \ b_i \in \mathbb{F}_l \ \forall \ 1 \leq i \leq r-1 \right\}
\]
Let $F_{\text{un}}^{T}$ be the maximal unramified extension of $F_{T}$ in $F_{T}^{\text{sep}}$. Since $I_{T} = \text{Gal}(F_{T}^{\text{sep}}/F_{\text{un}}^{T})$ by definition, we have $\bar{\phi}_{T}(I_{T}) \cong \text{Gal}(F_{\text{un}}^{T}(\psi[1])/F_{\text{un}}^{T}(T))$. By Remark 2.13 and Lemma 3.1, we can derive that $F_{\text{un}}^{T}(\psi[1]) = F_{\text{un}}^{T}(w_{1}, w_{2}, \ldots, w_{r-1}, z)$. Furthermore, $w_{i}$ belongs to $F_{\text{un}}^{T}$ because $\bar{\phi}_{T} : G_{F_{T}} \to \text{Aut}(\psi[1])$ is unramified for all $1 \leq i \leq r - 1$. Therefore, $F_{\text{un}}^{T}(\psi[1]) = F_{\text{un}}^{T}(z)$ and its ramification index $e[F_{\text{un}}^{T}(z) : F_{\text{un}}^{T}(T)]$ at least the order of $v(z)$ in $Q/Z$ where $v$ is the normalized valuation of $F_{T}$.

From Remark 2.13, we have $\phi_{T}(x) = \text{Tr} \cdot \prod_{0 \neq \gamma \in \psi_{T}^{-1}(\Gamma)/\Gamma} \left( 1 - \frac{x}{u(\gamma)} \right)$. We compute the leading coefficients on both sides up to units of $A_{T}$:

$$\prod_{i=1}^{\deg_{T}(1)} q^{r(i-1)} \prod_{\gamma \in \psi_{T}^{-1}(\Gamma)/\Gamma} u(\gamma) = 0 - \sum_{a_{1}a_{2}a_{r-1}b \in F_{T}, \text{not all zero}} v(u(a_{1}w_{1} + a_{2}w_{2} + \cdots + a_{r-1}w_{r-1} + bz)).$$

Recall from the proof of Lemma 3.1 that we have $\gamma = \psi_{1}(z)$, where $\gamma$ is a generator of the rank 1 discrete $A$-module $\Gamma$. The discreteness of $\Gamma$ forces $v(\gamma) < 0$, which implies $v(z) < 0$ because every coefficient of $\psi_{1}(x)$ has nonnegative valuation. Moreover, the valuations $v(w_{i})$ are all nonnegative because they are roots of $\psi_{1}(x)$.

The proof of $v(u(w_{i})) = v(w_{i})$ is easy because $v(w_{i})$ are non-negative and non-constant coefficients of $u(x)$ lie in the maximal ideal of $A_{T}$. For the proof of $v(u(z)) = v(z)$, we use the chosen $\gamma$ to compare coefficients of two expressions of $u(x)$ in Remark 2.13. We have

$$v(u_{n}) \geq -(q^{n} - 1)v(\gamma).$$

We also know that $v(\gamma) = q^{(r-1)(\deg_{T}(1))}v(z)$. Hence

$$v(u_{n}z^{q^{n}}) = v(u_{n}) + q^{n}v(z) \geq -(q^{n} - 1)v(\gamma) + q^{(n-(r-1)(\deg_{T}(1)))v(\gamma)}.$$

For $n \geq 1$, $v(u_{n}z^{q^{n}})$ is always non-negative. Therefore, $v(u(z)) = v(z)$.

Thus we can compute the valuation of $u(a_{1}w_{1} + a_{2}w_{2} + \cdots + a_{r-1}w_{r-1} + bz)$ explicitly:

$$v(u(a_{1}w_{1} + a_{2}w_{2} + \cdots + a_{r-1}w_{r-1} + bz)) = \begin{cases} q^{(r-1)}v(z), & \text{if } b \neq 0, \deg_{T}(b) = i, \\ v(a_{1}w_{1} + a_{2}w_{2} + \cdots + a_{r-1}w_{r-1}), & \text{if } b = 0. \end{cases}$$
Hence we have
\[
(q - 1) \left( \sum_{i=1}^{\deg_T(t)} q^r(i-1) \right) = -(q^{(r-1)(\deg_T(t))})(q - 1) \left( \sum_{i=1}^{\deg_T(t)} q^r(i-1) \right) \nu(z)
\]
\[
-\nu \left( \prod_{\substack{a_1, a_2, \ldots, a_{r-1} \in \mathbb{F}_q \text{ not all zero} \atop \quad a_1 w_1 + a_2 w_2 + \cdots + a_{r-1} w_{r-1}} \right)
\]
In fact, \( \prod_{\substack{a_1, a_2, \ldots, a_{r-1} \in \mathbb{F}_q \text{ not all zero}} \quad (a_1 w_1 + a_2 w_2 + \cdots + a_{r-1} w_{r-1}) \) is equal to the constant term \( l \) of \( \psi(x)/x \). Finally, we are able to compute the valuation \( v(z) = -\frac{1}{q^{(r-1)(\deg_T(t))}} \), its order in \( \mathbb{Q}/\mathbb{Z} \) is equal to \( q^{(r-1)(\deg_T(t))} = |A/t|^r - 1 \). Therefore, \( |A/t|^r - 1 \leq c[F(T)^n(z) : F(T)] \) and so \( |\tilde{\rho}_{\phi,t}(I_T)| \geq |A/t|^r - 1 \). Combining with Lemma \( \ref{lem:valuation} \) we have \( |\tilde{\rho}_{\phi,t}(I_T)| = |A/t|^r - 1 \).

Now we can apply the Aschbacher’s theorem (Theorem \( \ref{thm:aschbacher} \) in the Appendix) to rule out certain Aschbacher classes.

- \( \Gamma_1 \) does not lie in Class \( C_2 \).

**Proof.** Suppose \( \Gamma_1 \) lies in \( C_2 \), then \( \Gamma_1 \) acting on \( \mathbb{F}_t^r \) must be of the type \( \text{GL}_1(\mathbb{F}_t) \wr S_r \), the wreath product of \( \text{GL}_1 \) and the symmetric group \( S_r \). Therefore, we have \( |\Gamma_1| \) divides \( |\mathbb{F}_t| \cdot r! \). We then get a contradiction from the fact that \( p \) doesn’t divide \( |\mathbb{F}_t| \cdot r! \).

- \( \Gamma_1 \) does not lie in Class \( C_3 \)

**Proof.** Suppose \( \Gamma_1 \) lies in \( C_3 \), then the action of \( \Gamma_1 \) on \( \mathbb{F}_t^r \) must be of the type \( \text{GL}_1(\mathbb{F}_r) \), here \( \mathbb{F}_r \) is a degree-\( r \) extension over \( \mathbb{F}_1 \). Thus we have \( |\Gamma_1| \) divides \( \text{GL}_1(\mathbb{F}_r) \), which contradicts to the fact that \( p \) divides \( |\Gamma_1| \).

- \( \Gamma_1 \) does not lie in Class \( C_4 \)

**Proof.** This is clear by the primality of \( r \). Since \( \mathbb{F}_t^r \) cannot have such tensor product decomposition \( \mathbb{F}_t^r = V_1 \otimes V_2 \), where \( V_1 \) (resp. \( V_2 \)) is a \( \mathbb{F}_t \)-subspace of \( \mathbb{F}_t^r \) of dimension \( n_1 \) (resp. \( n_2 \)) and \( 1 < n_1 < \sqrt{t} \).

- \( \Gamma_1 \) does not lie in Class \( C_7 \)

**Proof.** Suppose \( \Gamma_1 \) lies in \( C_7 \), then the action of \( \Gamma_1 \) on \( \mathbb{F}_t^r \) must be in a quotient of the standard wreath product \( \text{GL}_1(\mathbb{F}_t) \wr S_r \). Hence we still have \( |\Gamma_1| \) divides \( |\mathbb{F}_t^r| \cdot r! \), a contradiction.

- \( \Gamma_1 \) does not lie in Class \( C_8 \)
Proof. Suppose $\Gamma_1$ lies in $C_8$, then $\Gamma_1$ would preserve a non-degenerate classical form on $F_1^r$ up to scalar multiplication. By classical form we mean symplectic form, unitary form or quadratic form. As $r$ is odd, $F_1^r$ can only have unitary form or quadratic form structure. We refer to section 1.5 of [BHRD13] for the definitions and properties for classical forms on a vector space.

Case 1. $\Gamma_1$ preserves a non-degenerate unitary form on $F_1^r$ up to scalar multiplication.

In this case, we are dealing with unitary form $<.,.>$ on $F_1^r$. There is a basis $B$ such that $<.,.>$ corresponds to the identity matrix $I_r$ (see Proposition 1.5.29 in [BHRD13]). Let $M \in \Gamma_1$ be a matrix with respect to the basis $B$, the fact that $M$ preserves $<.,.>$ up to a scalar multiplication can be interpreted as the equality:

$$M \cdot M^\top = \lambda_M \cdot I,$$

Where $\text{id} \neq \sigma \in \text{Aut}(F_1)$ depends only on the unitary form $<.,.>$, $\sigma^2 = 1$, and $\lambda_M \in F_1^*$ depends on $M$.

Therefore, we can compare the characteristic polynomials of such $M$ and $M^{-1}$. As $\lambda_M \cdot M^{-1} = M^\sigma$, we have

$$\det(xI - M^\sigma) = \frac{-x^r}{\det(\lambda_M \cdot M)} \det(\frac{1}{x}I - \lambda_M^{-1} \cdot M) = \frac{-x^r}{\det(M)} \det(\frac{\lambda_M}{x}I - M)$$

Now we consider $M = \tilde{\rho}_{\phi,1}(\text{Frob}_p)$ where $p = (T - c) \neq (T)$ or 1. The characteristic polynomial of $\tilde{\rho}_{\phi,T}(\text{Frob}(T-c))$ is congruent to $P_{\phi,(T-c)}(x)$ modulo 1. By Proposition 2.10 we may write

$$P_{\phi,(T-c)}(x) = -(T - c) + a_{r-1}x + a_{r-2}x^2 + \cdots + a_1x^{r-1} + x^r \in A[x].$$

Proposition 2.9 then implies all the $a_i$’s are belong to $F_q$. Because $P_{\phi,(T-c)}(x)$ is also the characteristic polynomial of Frobenius endomorphism of $\phi \otimes F_p$ acting on $T_I(\phi \otimes F_p)$, we have

$$-(\phi \otimes F_p)_{T-c} + (\phi \otimes F_p)_{a_{r-1}}\tau + (\phi \otimes F_p)_{a_{r-2}}\tau^2 + \cdots + (\phi \otimes F_p)_{a_1}\tau^{r-1} + \tau^r = 0.$$ As $\phi_T = T + \tau^{r-1} + T^{r-1}\tau^r$, we have $(\phi \otimes F_p)_{T-c} = \tau^{r-1} + \tau^r$. Thus

$$a_{r-1} = a_{r-2} = \cdots = a_2 = 0 \text{ and } a_1 = 1.$$ Hence the characteristic polynomial of $\tilde{\rho}_{\phi,1}(\text{Frob}(T-c)) \in F_1[x]$ is

$$P_{\phi,(T-c)}(x) = x^r + x^{r-1} - \tilde{p},$$

where $\tilde{p}$ denotes the reduction of $p$ modulo 1.

Therefore, we have

$$x^r + x^{r-1} - \sigma(\tilde{p}) = \det(xI - M^\sigma) = \frac{-x^r}{\det(M)} \det(\frac{\lambda_M}{x}I - M) = x^r - \frac{\lambda_M^{-1}}{p}x - \frac{\lambda_M}{p},$$

This is a contradiction because the polynomials on both sides cannot be equal.
Case 2 $\Gamma_t$ preserves a non-degenerate quadratic form on $F_t^r$ up to scalar multiplication. In this case, we are dealing with quadratic form $\langle \cdot, \cdot \rangle$ on $F_t^r$. There is a basis $\mathcal{B}$ such that $\langle \cdot, \cdot \rangle$ corresponds to the the $r \times r$ matrix

$$A := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & d \end{pmatrix},$$

here $d$ is either 1 or some nonsquare element in $F_t^r$. Let $M \in \Gamma_t$ be a matrix with respect to the basis $\mathcal{B}$, then $M$ preserves $\langle \cdot, \cdot \rangle$ up to a scalar multiplication can be interpreted into the following equalities:

$$M \cdot A \cdot M^\top = \lambda_M \cdot A,$$

where $\lambda_M \in F_t^r$ depends on $M$. Thus we have $M^\top = \lambda_M \cdot A^{-1} M^{-1} A$, which implies $M$ and $\lambda_M \cdot M^{-1}$ have the same characteristic polynomial.

$$\det(x I - M) = \frac{-x^r}{\det(\lambda_M^1 \cdot M)} \det(\frac{1}{x} I - \lambda_M^1 \cdot M) = \frac{-x^r}{\det(M)} \det(\frac{\lambda_M}{x} I - M).$$

Consider $M = \rho_\phi, \iota(Frob_p)$ where $p = (T-c) \neq (T)$ or $l$, we can deduce

$$x^r + x^{r-1} - \overline{p} = \det(x I - M) = \frac{-x^r}{\det(M)} \det(\frac{\lambda_M}{x} I - M) = x^r - \frac{\lambda_M^{-1}}{\overline{p}} x - \frac{\lambda_M}{\overline{p}}.$$

Hence we have a contradiction.

In summary, we’ve proved the following Proposition:

**Proposition 3.3.** Let $r$ be a prime number, $A = F_q[T]$, and $F = F_q(T)$, where $q = p^r$ and $p > r^4$. Let $\phi$ be a Drinfeld $A$-module over $F$ of rank $r$ with generic characteristic defined by $\phi_T = T + \tau r^{-1} + T^{-1} - \tau r$. Let $l$ be a place of $F$ where $\phi$ has good reduction at $l$. Denote $\rho_\phi, \iota(G_F)$ by $\Gamma_t$. Then $\Gamma_t$ can only lie in Aschbacher classes $C_1$, $C_5$, $C_6$, or $S$.

4 Some algebraic group theory

Under the assumption in Proposition 3.3 we know that $\Gamma_t$ can only lie in Aschbacher classes $C_1$, $C_5$, $C_6$, or $S$. Now we assume further that $\Gamma_t$ is irreducible, then $\Gamma_t$ can not lie in class $C_1$. Hence $\Gamma_t$ lies in class $C_5$, $C_6$, or $S$, which implies $\Gamma_t$ contains a subgroup that acts on $F_t^r$ absolutely irreducibly. Hence $\Gamma_t$ acts on $F_t^r$ absolutely irreducibly.

For most part, the following arguments are the same as in [PR09a], p.888-p.889.

Consider the following exact sequence $0 \to I_t^d \to I_t \to I_t^1 \to 0$ of inertia group $I_t$. Here $I_t^d$ is the wild inertia group at $l$ and $I_t^1$ is the tame inertia group at $l$. Let $h_l$ be the height of $\phi \otimes F_l$, and let $F_n$ be an extension of $F_l$ in $F_t$ with $n := |F|^{h_l}$ elements. By [PR09a] Proposition 2.7, we have up to conjugation.
$$\tilde{\rho}_{\phi,i}(I_1^r) = \begin{pmatrix} \mathbb{F}_p^n & 0 \\ 0 & 1 \end{pmatrix} \subseteq \Gamma_1,$$
written in block matrices with size \(h_t, r-h_t\). Since \(\mathbb{F}_p^r \neq \{1\}\), the centralizer of \(\tilde{\rho}_{\phi,i}(I_1^r)\) in \(\text{GL}_{r,\mathbb{F}_p}\) is
$$\begin{pmatrix} T_l & 0 \\ 0 & \text{GL}_{r-h_t,\mathbb{F}_p} \end{pmatrix}.$$ 

Here \(T_l\) is the torus \(\text{Res}^G_{\mathbb{F}_p^*} \mathbb{G}_m,\mathbb{F}_p^r\) in the algebraic group \(\text{GL}_{h_t,\mathbb{F}_p}\). We embed \(T_l\) into \(\text{GL}_{r,\mathbb{F}_p}\) by setting
$$T_l = \begin{pmatrix} T_l & 0 \\ 0 & r-h_t \end{pmatrix} \subseteq \text{GL}_{r,\mathbb{F}_p}. $$

The \(\Gamma_1\)-conjugacy class of \(T_l\) in \(\text{GL}_{r,\mathbb{F}_p}\) is independent of the choice of place \(\bar{l}\) of \(\bar{F}\) above \(l\).

Let \(H_l^r\) be the connected algebraic subgroup of \(\text{GL}_{r,\mathbb{F}_p}\) generated by \(\gamma T_l \gamma^{-1}\) for all \(\gamma \in \Gamma_1\) (see [Hum73], Proposition 7.5). Let \(H_1^r\) be the algebraic subgroup of \(\text{GL}_{r,\mathbb{F}_p}\) generated by \(H_l^r\) and \(\Gamma_1\). Thus we have
$$H_l^r \subseteq H_1^r \subseteq \text{GL}_{r,\mathbb{F}_p}. $$

Since \(H_1^r\) contains \(\Gamma_1\), \(H_1^r\) acts on \(\mathbb{F}_p^r\) absolutely irreducibly. Now we can make the following claim

**Claim:** \(H_1^r\) acts on \(\mathbb{F}_p^r\) absolutely irreducibly. In other words, \(H_l^r\) acts on \(\mathbb{F}_p^r\) irreducibly.

**Proof of claim.** Let \(W\) be a non-trivial \(H_l^r\)-invariant subspace of \(\mathbb{F}_p^r\) of minimal dimension. As \(H_l^r\) is normalized by \(\Gamma_1\), we know that \(\gamma W\) is also invariant under \(H_l^r\) for all \(\gamma \in \Gamma_1\). Consider \(V = \sum_{\gamma \in \Gamma_1} \gamma W\), it is invariant under \(\Gamma_1\). Thus we have \(\mathbb{F}_p^r = \sum_{\gamma \in \Gamma_1} \gamma W\) because \(\Gamma_1\) acts on \(\mathbb{F}_p^r\) irreducibly. Since each \(\gamma W\) is irreducible under the action of \(H_l^r\), there is a natural number \(s_t\) and a decomposition
$$\mathbb{F}_p^r = W_1 \oplus W_2 \oplus \cdots \oplus W_{s_t}$$
into irreducible \(H_l^r\)-subspaces which are conjugate under \(H_l^r\).

Let \(t_t\) be the common dimension of \(W_i\). Then \(H_l^r\) acts on \(\mathbb{F}_p^r\) through matrices in \(\text{GL}_{s_t,\mathbb{F}_p}\). Thus \(H_l^r \subseteq \text{GL}_{s_t,\mathbb{F}_p}\). The algebraic subgroup of \(\text{GL}_{r,\mathbb{F}_p}\) mapping a summand \(W_i\) to some summand \(W_j\) is isomorphic to \(\text{GL}_{s_t,\mathbb{F}_p} \rtimes S_{s_t}\).

**Lemma 4.1.** \(H_{l,\mathbb{F}_p} \subseteq \text{GL}_{s_t,\mathbb{F}_p} \rtimes S_{s_t}\)

**Proof.** See Lemma 3.5 in [PR09a].

Combining the result \(\Gamma_1 \subseteq H_{l,\mathbb{F}_p} \subseteq \text{GL}_{s_t,\mathbb{F}_p} \rtimes S_{s_t}\) with the primality of \(r\), we can conclude either \(s_t = 1, t_t = r\) or \(s_t = r, t_t = 1\). If \(s_t = r, t_t = 1\), then we have
$$\Gamma_1 \subseteq \text{GL}_{s_t,\mathbb{F}_p} \rtimes S_r.$$ 

Since we assume \(p > r!\), there is no element in \(\text{GL}_{s_t,\mathbb{F}_p} \rtimes S_r\) whose order is divisible by \(p\). Thus we’ve deduced a contradiction because \(\Gamma_1\) has elements of order divisible by \(p\). Therefore we must have \(s_t = 1, t_t = r\). So \(H_{l,\mathbb{F}_p} \rtimes S_r\) acts on \(\mathbb{F}_p^r\) irreducibly. 

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So far we have the following information on $H_1^\circ$:

- $H_1^\circ$ is a connected algebraic group over $\mathbb{F}_t$.
- $H_1^\circ$ acts on $\mathbb{F}_t^r$ absolutely irreducibly.
- $H_{\text{et},\mathbb{F}_t}$ has a cocharacter of weight 1 with multiplicity 1 and weight 0 with multiplicity $r-1$ (In the proof of Lemma 4.1).

With these conditions in hand, we can apply Proposition A.3 in [Pin97] and prove that

$$H_1^\circ = \text{GL}_{r,\mathbb{F}_t}.$$

### 4.1 Image of residual representation

**Theorem 4.2.** Let $q = p^e$ be a prime power, $A = \mathbb{F}_q[T]$, and $F = \mathbb{F}_q(T)$. Assume $r \geq 3$ is a prime number, there is a constant $c = c(r) \in \mathbb{N}$ depending only on $r$ such that for $p > c(r)$ the following statement is true:

Let $\phi$ be a Drinfeld $A$-module over $F$ of rank $r$ defined by $\phi_T = T + \tau r^{-1} + T q^{-1} \tau r$. Let $I$ be a finite place of $F$ where $\phi$ has good reduction at $I$. Then the mod $I$ representation

$$\bar{\rho}_{\phi, I} : G_F \rightarrow \text{Aut}(\phi[I]) \cong \text{GL}_r(\mathbb{F}_t)$$

is either reducible or surjective.

**Proof.** Let $r$ be a prime number, $A = \mathbb{F}_q[T]$, and $F = \mathbb{F}_q(T)$ where $q = p^e$ where $p > r!$. Let $\phi$ be a Drinfeld $A$-module over $F$ of rank $r$ with generic characteristic defined by $\phi_T = T + \tau r^{-1} + T q^{-1} \tau r$. Let $I$ be a place of $F$ where $\phi$ has good reduction at $I$. Let $\Gamma_I = \bar{\rho}_{\phi, I}(G_F)$ be irreducible. From the argument above, we have $H_1^\circ = \text{GL}_{r,\mathbb{F}_t}$.

Now we can apply Lemma 3.12 in [PR09a] for all places $I$ of $F$ where $\phi$ has good reduction. Thus there exists a natural number $\hat{c}$ depends only on $r$ such that

$$[\text{GL}_r(\mathbb{F}_t) : \Gamma_I] \leq \hat{c}.$$

Therefore, we replace $p$ by a prime number $p > c := \max\{r!, \hat{c}!\}$ and run Lemma 3.12 in [PR09a] again, we get

$$[\text{GL}_r(\mathbb{F}_t) : \Gamma_I] \leq \hat{c} \text{ and } |\text{PGL}_r(\mathbb{F}_t)| > \hat{c}!.$$

By applying Proposition 2.3 in [PR09a], we have

$$\Gamma_I \supseteq \text{SL}_r(\mathbb{F}_t).$$

By Proposition 2.10 and Chebotarev density theorem, $\det \circ \rho_{\phi, I}$ is equal to the mod $I$ representation $\rho_{\bar{C}, I}$ of Carlitz module. Hence we have $\det(\Gamma_I) = \mathbb{F}_t^*$, which implies

$$\Gamma_I = \text{GL}_r(\mathbb{F}_t).$$
5 Irreducibility of the mod l representations

We assume \( l \neq (T) \) from here to subsection 5.2, the case when \( l = (T) \) will be dealt with in subsection 5.3.

For \( l = (T - c) \) a degree-1 prime ideal different from \( (T) \), the proof for irreducibility is straightforward.

Claim. \( \phi_{T-c}(x)/x = (T - c) + x^{q^{-1} - 1} + T^{q^{-1}}x^{q^{-1}} \) is irreducible over \( F \).

Proof of claim. First of all, \( \phi_{T-c}(x)/x \) is irreducible over \( F \) if and only if its reciprocal polynomial

\[
x^{q^{-1}}\phi_{T-c}(x^{-1})/x^{-1} = (T - c)x^{q^{-1}} + x^{q^{-r} - 1} + T^{q^{-1}}
\]

is irreducible over \( F \). As \( T - c \in F^* \), it is enough for us to prove the polynomial

\[
x^{q^{-1}} + \frac{1}{T - c}x^{q^{-r} - 1} + \frac{T^{q^{-1}}}{T - c}
\]

is irreducible over \( F \). As \( F = \mathbb{F}_q(T) = \mathbb{F}_q(T - c) \), we may consider the completion \( F' \) of \( F \) at the place \( \mathfrak{m} = (T - c) \). Then the polynomial \( x^{q^{-1}} + \frac{1}{T - c}x^{q^{-r} - 1} + \frac{T^{q^{-1}}}{T - c} \) is irreducible over \( F' \) by the Eisenstein criterion. Hence \( x^{q^{-1}} + \frac{1}{T - c}x^{q^{-r} - 1} + \frac{T^{q^{-1}}}{T - c} \) must be irreducible over \( F \), the proof is now complete.

For general prime ideal \( l \neq (T) \), we suppose \( \phi[l] \) viewed as a \( \mathbb{F}_l[G_F] \) is reducible, then it contains a \( G_F \)-invariant proper submodule \( X \). Let \( \dim_{\mathbb{F}_l} X = d \), we claim that \( d \neq 1 \) or \( r - 1 \). This follows the same strategy as in the rank-3 case (see section 3 in [Che20]). Firstly, there is a basis such that the action of \( G_F \) on \( \phi[l] \) is of one of the following forms:

(i) \( \begin{pmatrix} \chi & * \\ 0 & B \end{pmatrix} \) (if \( X \) has dimension 1),

(ii) \( \begin{pmatrix} B & * \\ 0 & \chi \end{pmatrix} \) (if \( X \) has codimension 1),

where \( B : G_F \rightarrow \text{GL}_{r-1}(\mathbb{F}_l) \) is a homomorphism and \( \chi : G_F \rightarrow \mathbb{F}_l^* \) is a character.

Now we consider the following exact sequence

\[
0 \rightarrow \phi[l]^{\circ} \rightarrow \phi[l] \rightarrow \phi[l]^{\circ I_l} \rightarrow 0
\]

of \( \mathbb{F}_l[I_l] \)-modules.

Proposition 5.1. ([PR09b], Proposition 2.7)

i. The inertia group \( I_l \) acts trivially on \( \phi[l]^{\circ I_l} \).

ii. The \( \mathbb{F}_l \)-vector space \( \phi[l]^{\circ} \) extends uniquely to a one dimensional \( \mathbb{F}_l^{(h)} \)-vector space structure such that the action of \( I_l \) on \( \phi[l]^{\circ} \) is given by the fundamental character \( \zeta_n \).

iii. The action of wild inertia group at \( I \) on \( \phi[l]^{\circ} \) is trivial.
Corollary 5.2. \( \phi[l]^o \) is an irreducible \( \mathbb{F}_l[I_l] \)-module.

Proof. This immediately follows from Proposition 5.1(ii).

By Corollary 5.2 we have either (i) \( X \cap \phi[l]^o = \{0\} \) or (ii) \( X \cap \phi[l]^o = \phi[l]^o \).

Claim. Either \( \det B \) is unramified at every prime ideal \( p \) of \( A \) or \( \chi \) is unramified at every prime ideal \( p \) of \( A \).

Proof. If \( p \) is a prime different from \( (T) \) and \( I \), then \( \tilde{\rho}_{\phi,1} \) is unramified at \( p \). Thus \( \det B \) and \( \chi \) are both unramified at \( p \). It suffices for us to prove the cases \( p = (T) \) and \( p = I \).

For \( p = (T) \), Lemma 5.1 implies \#\( \tilde{\rho}_{\phi,1}(I_T) \) is a \( q \)-power. This means the order of \( B(\sigma) \) and \( \chi(\sigma) \) are also \( q \)-powers. Therefore, the order of \( \det B(\sigma) \) and \( \chi(\sigma) \) are also \( q \)-powers for all \( \sigma \in I_T \). However, \( \det B(\sigma) \) and \( \chi(\sigma) \) belong to \( \mathbb{F}_l^* \), which is of order prime to \( q \). Hence \( \det B(\sigma) = \chi(\sigma) = 1 \) for all \( \sigma \in I_T \).

For \( p = I \), we have two cases.

Case (i): \( X \) has dimension 1. Then \( X \) is an irreducible \( \mathbb{F}_l[G_F] \)-module. This means either \( X = \phi[l]^o \) or \( X \cap \phi[l]^o = \{0\} \). Let \( \sigma \in I_T \). If \( X = \phi[l]^o \), then the matrix \( B(\sigma) \) describes how \( \sigma \) acts on the \( \mathbb{F}_l[I_l] \)-module \( \phi[l]/\phi[l]^o \approx \phi[l]^o \). Proposition 5.1 then implies \( \det B(\sigma) = 1 \) for all \( \sigma \in I_T \).

When \( X \cap \phi[l]^o = \{0\} \), we can view \( X \) as a \( \mathbb{F}_l[I_l] \)-submodule of \( \phi[l]/\phi[l]^o \approx \phi[l]^o \). Therefore, Proposition 5.1 implies \( \chi(\sigma) = 1 \) for all \( \sigma \in I_T \).

Case (ii): \( X \) has codimension 1. As \( \phi[l]^o \) is an irreducible \( \mathbb{F}_l[I_l] \)-module by Corollary 5.2 we get either \( X \supseteq \phi[l]^o \) or \( X \cap \phi[l]^o = \{0\} \). When \( X \supseteq \phi[l]^o \), we have the “modulo \( X \)” map \( \phi[l]^{\prime} \approx \phi[l]/\phi[l]^o \rightarrow \phi[l]/X \) as \( \mathbb{F}_l[I_l] \)-modules. Thus \( \chi(\sigma) = 1 \) for all \( \sigma \in I_T \) by Proposition 5.1. The argument for the case when \( X \cap \phi[l]^o = \{0\} \) is similar to what happened in case (i), so \( \det B(\sigma) = 1 \) for all \( \sigma \in I_T \).

We first assume that \( \det B \) is unramified at every prime \( p \). The homomorphism \( \det B : G_F \rightarrow \mathbb{F}_l^* \) factors through \( \text{Gal}(K/\mathbb{F}_q(T)) \), where \( K \) is a finite abelian extension of \( \mathbb{F}_q(T) \), unramified at every finite place and tamely ramified at infinity. Hayes [Hay74] studied the class field theory in function field analogue and showed that such finite abelian extension is just some constant extension of \( \mathbb{F}_q(T) \) (see section 5 and Theorem 7.1 in [Hay74]), hence we have \( K \subseteq \mathbb{F}_q(T) \). Now we can write \( \det B \) in the following way:

\[
\det B : G_F \rightarrow \text{Gal}(\bar{\mathbb{F}}_q(T)/\mathbb{F}_q(T)) \cong \text{Gal}(\mathbb{F}_l/\mathbb{F}_q) \rightarrow \mathbb{F}_l^*,
\]

where the first map is the restriction map.

This implies there is some element \( \xi \in \mathbb{F}_l^* \) such that \( \det B(\text{Frob}_p) = \xi^{\deg p} \) for every prime ideal \( p \) of \( A \) not equal to \( (T) \) or \( I \). Now we consider the Frobenius element for a degree-1 prime \( p = (T-c) \neq (T) \) or \( I \). The characteristic polynomial of \( \text{Frob}_p \) acting on \( \phi[l] \) is \( \tilde{P}_{\phi,p}(x) = x^r + x^{r-1} - \bar{p} \). Thus we have the following factorization

\[
x^r + x^{r-1} - \bar{p} = (x^{r-1} - \alpha_p x^{r-2} + \cdots + \xi)(x - \xi^{-1} \bar{p}) \in \mathbb{F}_l[x].
\]

Hence for distinct prime ideals \( p_1 \) and \( p_2 \) of degree 1, we can factorize the characteristic polynomial of \( \tilde{\rho}_{\phi,1}(\text{Frob}_{p_1}) \) and \( \tilde{\rho}_{\phi,1}(\text{Frob}_{p_2}) \), respectively. By computing their coefficients, we get
Proposition 5.3. \( X \) if \( 5.1 \) the case \( \varphi \)

\[ \begin{aligned}
\alpha_{p_1} + \xi^{-1}\bar{p}_1 &= \alpha_{p_2} + \xi^{-1}\bar{p}_2 = -1 \\
\alpha_{p_1}\xi^{-1}\bar{p}_1 + \xi &= \alpha_{p_2}\xi^{-1}\bar{p}_2 + \xi = 0
\end{aligned} \]

\[ \Rightarrow \begin{cases}
\alpha_{p_1} = -1 - \xi^{-1}\bar{p}_1 \\
\alpha_{p_2} = -1 - \xi^{-1}\bar{p}_2 \\
\alpha_{p_1}\bar{p}_1 = \alpha_{p_2}\bar{p}_2
\end{cases} \]

\[ \Rightarrow (-\xi - \bar{p}_1)\bar{p}_1 = (-\xi - \bar{p}_2)\bar{p}_2 \]
\[ \Rightarrow -\xi(\bar{p}_1 - \bar{p}_2) = (\bar{p}_1^2 - \bar{p}_2^2). \]

So \( \xi \equiv -(p_1 + p_2) \mod I \). As \( q > 5 \), there are at least three different prime ideals of degree 1 not equal to \( (T) \) or \( I \). Thus we can derive the above argument for any two such prime ideals, which implies that all these prime ideals are congruent to each other modulo \( I \). This gives us a contradiction. The arguments for “If \( \chi \) is unramified at every prime \( p \)” follows the same process as above.

In conclusion, we have shown that if there is an proper \( G_F \)-submodule \( X \) of \( \phi[I] \), then

\[ 2 \leq \dim F, X = d \leq r - 2. \]

Therefore, we may assume in the subsection 5.1 and 5.2 that

(a) \( 2 \leq \dim F, X = d \leq r - 2. \)

(b) \( \deg_T l \geq 2 \)

The following subsections will split into the cases (i) \( X \cap \phi[I]^\circ = \{0\} \) and (ii) \( X \cap \phi[I]^\circ = \phi[I]^\circ \) to derive contradictions. Then the irreducibility for mod \( I \) representations when \( I \neq (T) \) is proved.

5.1 The case \( X \cap \phi[I]^\circ = \{0\} \)

If \( X \cap \phi[I]^\circ = \{0\} \), then Proposition 5.1(i) shows the action of \( I_1 \) on \( X \) is trivial. Now we consider the action of \( I_T \) on \( X \). By Lemma 3.2 there is a basis \( \{w_1, w_2, \cdots, w_{r-1}, z\} \) of \( \phi[I] \) such that

\[ \bar{\rho}_{\phi,I}(I_T) = \begin{pmatrix}
1 & b_1 \\
\cdots & \ddots \\
\cdots & b_{r-1} & 1
\end{pmatrix}, \quad b_i \in F_l \forall 1 \leq i \leq r - 1. \]

Since \( X \) is a proper submodule of \( \phi[I] \), we have \( X \subseteq \text{span}\{w_1, w_2, \cdots, w_{r-1}\} \). Therefore, \( I_T \) acts trivially on \( X \). For places \( p \neq (T) \) or \( I \), we know that \( I_p \) acts trivially on \( \phi[I] \), hence \( I_p \) acts trivially on \( X \).

In conclusion, we have proved the following proposition:

Proposition 5.3. The action of \( G_F \) on \( X \) is unramified at every finite place.

Now we consider the ramification index at infinity of the extension \( F(\phi[I])/F \). Let \( F_\infty = F_q((\frac{1}{T})) \) be the local field of \( F \) at \( \infty \) with valuation \( v_\infty \), and \( |\cdot|_\infty \) be its corresponding absolute value. Let \( \Lambda \) be the period lattice of the polynomial \( \phi_T(x) = Tx + x^{q^{-1}} + T^{q^{-1}}x^{q^{-1}} \). From section 1.1 of [Gek19], we know \( \Lambda \) is a discrete free \( A \)-submodule of \( \hat{F}_\infty \), the completed algebraic closure of \( F_\infty \), of rank \( r \). We also have

\[ e_\Lambda(Tz) = \phi_T(e_\Lambda(z)) \quad \text{for all} \ z \in \hat{F}_\infty, \]

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where
\[ e_{\Lambda}(z) = z \prod_{0 \neq \lambda \in \Lambda} (1 - z/\lambda). \]

Let \( F_{\infty}(\Lambda) \) be the field extension generated by the period lattice \( \Lambda \). From Proposition 1.2 of \textit{Gek19}, we have
\[ F_{\infty}(\Lambda) = F_{\infty}(\text{tor}(\phi)), \]
where \( \text{tor}(\phi) = \bigcup_{0 \neq a \in \Lambda} \phi[a] \). Let \( \{\lambda_1, \lambda_2, \cdots, \lambda_r\} \) be a successive minimum basis of the lattice \( \Lambda \) (see \textit{Gek19}, 1.3 for the definition). By Proposition 1.4 (i) of \textit{Gek19}, the spectrum \((|\lambda_1|_{\infty}, |\lambda_2|_{\infty}, \cdots, |\lambda_r|_{\infty})\) is related to the Newton polygon of \( \phi_T(x) \) with respect to the valuation \( v_\infty \). The Newton polygon of \( \phi_T(x) \) is a line segment with slope \( \frac{2 - q}{q^r - 1} \). Hence, by Proposition 1.4(ii) in \textit{Gek19}, we have
\[ |\lambda_1|_{\infty} = |\lambda_2|_{\infty} = \cdots = |\lambda_r|_{\infty}. \]

Let \( \mu_i = e_{\Lambda}(\frac{\lambda_i}{T}) \), the set \( \{\mu_1, \mu_2, \cdots, \mu_r\} \) is a \( \mathbb{F}_q \)-basis of \( \phi[T] \). By formula 1.3.2 of \textit{Gek19}, we have
\[ |\mu_i|_{\infty} = |e_{\Lambda}(\frac{\lambda_i}{T})|_{\infty} = |\frac{\lambda_i}{T}|_{\infty}. \]

From the Newton polygon of \( \phi_T(x) \), we can compute \( v_\infty(\mu_i) = \frac{q - 2}{q^r - 1} \). Thus we can deduce
\[ v_\infty(\lambda_i) = v_\infty(\mu_i) - 1 = \frac{-q^r + q - 1}{q^r - 1}. \]

Now we can study the extension \( F_{\infty}(\Lambda)/F_{\infty} \). We use the notation set up in \textit{Gek19}, section 2.1. In our case, \( L = F_{\infty}, B_1 = \{\lambda_1, \lambda_2, \cdots, \lambda_r\}, \tau = t = 1, L_\tau = F_{\infty}(\Lambda) \). By applying diagram 2.8.4 of \textit{Gek19} to our situation, we have the followings:

1. \( L_\tau = L_0(V_\tau) = F_{\infty}(\Lambda) \)
2. \( L_{\tau-1} = L_0 = F_{\infty} \)
3. \( V_\tau \) is the \( \mathbb{F}_q \)-vector space generated by basis \( e_0(B_1) = \{\lambda_1, \cdots, \lambda_r\} \) (see \textit{Gek19}, 2.4)
4. \( V_\tau \) is pure with precise denominator of weight equal to \( q^r - 1 \) by our computation on \( v_\infty(\lambda_i) \), Definition 2.5 and Proposition 2.6 in \textit{Gek19}.
5. \( L_{\tau-1}' = \) a completely ramified separable extension of \( L_{\tau-1} \) of degree \( q^r - 1 \). (see \textit{Gek19}, 2.8)
6. \( M_\tau = L_{\tau-1}(V_\tau) \cap L_{\tau-1}' = F_{\infty}(\Lambda) \cap L_{\tau-1}', \) and \( [M_\tau : L_{\tau-1}] = e(L_\tau|L_{\tau-1}) = e(F_{\infty}(\Lambda)|F_{\infty}) \) (see \textit{Gek19}, 2.9.1)

Therefore, \( [M_\tau : L_{\tau-1}] \) must divide \( q^r - 1 \). This implies the ramification index \( e(F_{\infty}(\Lambda)|F_{\infty}) \) is some number prime to \( p \). Hence the field extension \( F_{\infty}(\text{tor}(\phi))/F_{\infty} \) is tamely ramified, which means the field extension \( F(\text{tor}(\phi))/F \) is tamely ramified at \( \infty \). Since \( F(\phi[l]) \) is an intermediate field of the extension \( F(\text{tor}(\phi))/F \), the extension \( F(\phi[l])/F \) is tamely ramified at \( \infty \). Combining with Proposition 5.3 we can make the following statement:

(*) The action of \( G_F \) on \( X \) is unramified at every finite place and tamely ramified at the place of infinity.

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Finally, we claim that the field extension $\overline{\mathbb{F}}_q F(X)/\overline{\mathbb{F}}_q F$ is nontrivial. This leads to a non-trivial Galois cover of $\mathbb{P}^1_{\overline{\mathbb{F}}_q}$ which is unramified away from infinity and tamely ramified at infinity, hence is a contradiction by Hurwitz genus formula.

Suppose the field extension $\overline{\mathbb{F}}_q F(X)/\overline{\mathbb{F}}_q F$ is trivial. Then for a element $0 \neq w \in X \subset \overline{\mathbb{F}}_q(T)$, the valuation $v_T(w)$ is an integer. We may write

$$w = \sum_{i=-m}^{n} c_i T^i,$$

where $m, n \in \mathbb{Z}$, and $c_i \in \overline{\mathbb{F}}_q$ with $c_{-m}, c_n \in \overline{\mathbb{F}}_q^*$. Hence we have $v_T(w)$ an integer. We may write

$$v_T(w) = l \phi,$$

where $l \phi$ is the valuation of $w$. Moreover, this term in $T^l \phi$ must lie in $\overline{\mathbb{F}}_q$. Hence the term of $v_T(w)$ is an integer. We may write

$$v_T(w) = l \phi \leq 0.$$ 

We show that $w$ must lie in $\overline{\mathbb{F}}_q[T]$. Suppose $m \geq 1$, we have the valuation $v_T(w) = -m$. Now we compute the valuation of $\phi_T(w) = T w + w q^{l-1} + T q^{-1} w q^{l}$. As $v_T(w q^{l-1}) = q^{l-1} \cdot (-m)$ and $v_T(T q^{l-1} w q^{l}) = q^{l} \cdot (-m) + (q - 1)$, we have

$$q^{l} \cdot (-m) + (q - 1) = v_T(\phi_T(w)) < v_T(w) = -m.$$ 

Therefore, we can see that

$$v_T(\phi_{T^j}(w)) < v_T(\phi_{T^j-1}(w))$$

for any integer $j \geq 1$. Hence the term of $\phi_{T^j}(w)$ with smallest valuation does not appear in $\phi_{T^j}(w)$ for any integer $0 \leq j < \deg_T(l)$. We can deduce from this that the term of $\phi_l(w)$ with smallest valuation cannot be eliminated, which contradicts to $w \in X \subset \phi[l]$. Thus $w$ must lie in $\overline{\mathbb{F}}_q[T]$.

On the other hand, we can replace $v_T$ by $v_{\infty}$ and run the same process as above to show that $w$ must lie in $\overline{\mathbb{F}}_q$. Indeed, we assume $v_{\infty}(w) = -m \leq -1$. We can deduce that

$$q^{l} \cdot (-m) - (q - 1) = v_{\infty}(\phi_T(w)) < v_{\infty}(w) = -m.$$ 

Hence we have $v_{\infty}(\phi_{T^j}(w)) < v_{\infty}(\phi_{T^j-1}(w))$ for any integer $j \geq 1$. The remains follow the same argument as above.

Now we have the element $0 \neq w \in X \subset \phi[l]$ lies in $\overline{\mathbb{F}}_q$. However, this implies the term of $\phi_l(w)$ with largest $T$-degree is the leading term

$$v = \left( \sum_{i=1}^{\deg_T(l)} q^{l(i-1)} \right) , w q^{-\deg_T(l)}.$$ 

Moreover, this term in $\phi_l(w)$ cannot be eliminated since the $T$-degree of the leading coefficient in $\phi_l(x)$ is strictly larger than the $T$-degree of any other coefficient. Thus we have a contradiction due to the fact that $w \in X \subset \phi[l]$. In conclusion, the field extension $\overline{\mathbb{F}}_q F(X)/\overline{\mathbb{F}}_q F$ is nontrivial.

### 5.2 The case $X \cap \phi[l]^0 = \phi[l]^0$

If $X \cap \phi[l]^0 = \phi[l]^0$, then we have $\phi[l]^0 \subseteq X \subseteq \phi[l]$. We consider the following isogeny over $F$:

$$u : \phi \rightarrow \psi := \phi/X.$$ 

Now we can study the action of $G_F$ on the proper $G_F$-submodule $u(\phi[l])$ of $\psi[l]$. We claim that the extension $F(u(\phi[l]))/F$ is unramified at every finite place and tamely ramified at the place of infinity.

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The isogeny induces an $G_F$-equivariant isomorphism between rational Tate modules

$$u : V_i(\phi) \rightarrow V_i(\psi),$$

where $V_i(\phi) := T_i(\phi) \otimes F$. For any prime ideal $p \neq (T)$ or $I$, since $\phi$ has good reduction at $p$ and $u$ is $G_F$-equivariant, we know $\psi$ has good reduction at $p$. Therefore, $G_F$ acts on $u(\phi[I])$ is unramified at $p \neq (T)$ or $I$.

For the prime $p = I$, the action of inertia group $I$ on $u(\phi[I])$ is determined by the action of $I$ on $\phi[I]/X$. Combining the condition $X \supseteq \phi[I]^q$ and Proposition 3.1(iii), we deduce that $I_1$ acts trivially on $\phi[I]/X$. Hence $I_1$ acts trivially on $u(\phi[I])$ as well.

For the place $p = \infty$, we proved the extension $F(\text{tor}(\phi))/F$ is tamely ramified at $\infty$ in the end of previous subsection. This implies the wild inertia group at $\infty$ acts trivially on $V(\phi)$. Therefore, we can conclude that the wild inertia group at $\infty$ of previous subsection. This implies the wild inertia group at $\infty$ acts trivially on $V(\phi)$. Hence $I_1$ acts trivially on $u(\phi[I])$ as well.

Finally, for the prime $p = (T)$, from Proposition 2.21 and Lemma 3.1 we know that $\psi$ has stable bad reduction at $p$.

Once $\psi$ has stable bad reduction at $p$, we know that $\psi$ has stable bad reduction at $p$ of rank $r - 1$. Moreover, both $u$ and $\psi$ can be defined over $A_p$. As $\dim_{\underline{\mathbb{F}}}X = d$, then the $\tau$-degree of $u$ is $d$. We may write

$$u = a_0 + a_1\tau + \cdots + a_d\tau^d$$

and

$$\psi_T = T + g_1\tau + \cdots + g_r\tau^r,$$

where $a_i$ and $g_j$ are elements in $A_p$ with $a_d, g_r \in A_p^\ast$. By comparing the leading coefficients of $u\phi_T = \psi_T u$, we get

$$T^{q^d-1} = g_r a_d^{q^d-1}.$$

Therefore, we can conclude $g_r = T^{q^d-1} \cdot m$ for some $m \in A_p^\ast$.

Now we can follow the same process as in Lemma 3.1 to deduce the following:

There is a basis $\{w_1 \cdots, w_{r-1}, z\}$ of $\psi[I]$ such that

$$\rho_{\psi, i}(I_p) \subseteq \begin{pmatrix}
1 & b_1 & \cdots & b_{r-1} \\
\vdots & & \ddots & \vdots \\
\vdots & & & 1
\end{pmatrix}, \quad b_i \in \mathbb{F}_q \quad 1 \leq i \leq r - 1,$$

with respect to the basis.

Furthermore, we can give an estimation on the size of $\rho_{\psi, i}(I_p)$ by using the same process as in Lemma 3.2. The computation only involves in the leading coefficient $g_r$ of $\psi_T$. Because $g_r = T^{q^d-1} \cdot m$ for some $m \in A_p^\ast$, the left hand side of equation $(1)$ in the proof of Lemma 3.2 becomes

$$T^{(q^d-1) \sum_{i=1}^{\deg_T(1)} q^r(i-1)}.$$

Hence we can deduce the estimation

$$|\rho_{\psi, i}(I_p)| \geq q^{(r-1)\deg_T 1 - d},$$

on
Claim. \( I_p \) acts trivially on \( u(\phi[l]) \subset \psi[l] \).

Proof of claim. We split \( \dim_{\mathbb{F}_1} X = d \) into two cases.

\( d \leq \deg_T(l) \): In this case, we can deduce from the estimation \( |\tilde{\rho}_{\psi,l}(I_p)| \geq q^{(r-1)\deg T l-d} \) that

\[
|\tilde{\rho}_{\psi,l}(I_p)| \geq q^{(r-2)\deg T l}.
\]

As \( \dim_{\mathbb{F}_1} u(\phi[l]) = r - d \) and \( 1 \leq d < r - 1 \), we know \( \dim_{\mathbb{F}_1} u(\phi[l]) \leq r - 2 \). Now we use the basis \( \{w_1, \ldots, w_{r-1}, z\} \) to present vectors in the \( \mathbb{F}_1 \)-space \( u(\phi[l]) \). Suppose there is an element \( w \in u(\phi[l]) \) with linear combination \( c_1w_1 + \cdots + c_{r-1}w_{r-1} + cz \) where \( c_i \in \mathbb{F}_1 \) and \( c \in \mathbb{F}_1^* \). Since we have

\[
\tilde{\rho}_{\psi,l}(I_p) \subseteq \left\{ \begin{pmatrix} 1 & b_1 & \cdots & b_i \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & \cdots & b_{r-1} & b_{r-1} \end{pmatrix} : b_i \in \mathbb{F}_1 \forall 1 \leq i \leq r-1 \right\}.
\]

For every \( \sigma = \begin{pmatrix} 1 & b_{1,\sigma} & \cdots & b_{r-1,\sigma} \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & \cdots & b_{r-1,\sigma} & 1 \end{pmatrix} \in \tilde{\rho}_{\psi,l}(I_p) \), we consider

\[ \sigma \cdot w - w = b_{1,\sigma}w_1 + \cdots + b_{r-1,\sigma}w_{r-1}. \]

We may start with an element \( \sigma_1 \in \tilde{\rho}_{\psi,l}(I_p) \) with \( b_{1,\sigma_1}, \ldots, b_{r-1,\sigma_1} \) not all zero. Hence there are at most \( |\mathbb{F}_1| = q^{\deg T l} \) many elements \( \sigma \in \tilde{\rho}_{\psi,l}(I_p) \) such that \( \sigma \cdot w - w \) is linearly dependent with \( \sigma_1 \cdot w - w \). From our estimation \( |\tilde{\rho}_{\psi,l}(I_p)| \geq q^{(r-2)\deg T l} \), we can find \( \sigma_2 \in \tilde{\rho}_{\psi,l}(I_p) \) such that \( \sigma_2 \cdot w - w \) is linearly independent with \( \sigma_1 \cdot w - w \). Now there are at most \( |\mathbb{F}_1|^2 = q^{2\deg T l} \) many elements \( \sigma \in \tilde{\rho}_{\psi,l}(I_p) \) such that \( \sigma \cdot w - w \) is linearly independent with \( \{\sigma_1, w-w, \sigma_2 \cdot w-w\} \).

From the estimation again, we can find \( \sigma_3 \in \tilde{\rho}_{\psi,l}(I_p) \) such that \( \{\sigma_1, w-w, \sigma_2 \cdot w-w, \sigma_3 \cdot w-w\} \) is a linearly independent set. By following this procedure, we can pick \( \sigma_1, \sigma_2, \ldots, \sigma_{r-2} \in \tilde{\rho}_{\psi,l}(I_p) \) such that

\[ \{\sigma_i \cdot w - w \mid 1 \leq i \leq r-2\} \]

is a linearly independent set. Therefore, \( \sigma_i \cdot w - w \) together with \( w \) produce \( r-1 \) many linearly independent vectors in \( u(\phi[l]) \), a contradiction. Thus all elements in \( u(\phi[l]) \) are linear combinations of the basis vectors \( w_1, w_2, \ldots, w_{r-1} \), which implies that \( I_p \) acts trivially on \( u(\phi[l]) \).

\( d > \deg_T(l) \): In this case, we are in the situation where

\[ 1 < \deg_T(l) < d < r - 2. \]

Therefore, the estimation \( |\tilde{\rho}_{\psi,l}(I_p)| \geq q^{(r-1)\deg T l-d} \) implies

\[ |\tilde{\rho}_{\psi,l}(I_p)| \geq q^{(r-1)\deg T l-d} > q^{(r-1-d)\deg T l}. \]
As \( \dim_{\mathbb{F}_q} u(\phi[\ell]) = r - d \), we use the basis \( \{ w_1, \ldots, w_{r-1}, z \} \) to present vectors in the \( \mathbb{F}_q \)-space \( u(\phi[\ell]) \) again. Again, we suppose there is an element \( w \in u(\phi[\ell]) \) with linear combination 
\[ c_1 w_1 + \cdots + c_{r-1} w_{r-1} + cz \]
where \( c_i \in \mathbb{F}_q \) and \( c \in \mathbb{F}_q^* \). For every \( \sigma \in \bar{G}_{\psi,1}(I_\mathfrak{p}) \), we consider
\[
\sigma \cdot w - w = b_{1,\sigma} \cdot w_1 + \cdots + b_{r-1,\sigma} \cdot w_{r-1}.
\]

By the procedure in the previous case and our estimation \( |\bar{\rho}_{\psi,1}(I_\mathfrak{p})| > q^{(r-1-d)\deg_F} \), we can find \( r - d \) many elements \( \sigma_1, \ldots, \sigma_{r-d} \in \bar{G}_{\psi,1}(I_\mathfrak{p}) \) such that the set
\[
\{ \sigma_i \cdot w - w \mid 1 \leq i \leq r - d \}
\]
is linearly independent. Thus \( \sigma_i \cdot w - w \) together with \( w \) produce \( r - d + 1 \) many linearly independent vectors in \( u(\phi[\ell]) \), which is a contradiction because \( \dim_{\mathbb{F}_q} u(\phi[\ell]) = r - d \). Therefore, all elements in \( u(\phi[\ell]) \) are linear combinations of the basis vectors \( w_1, w_2, \ldots, w_{r-1} \). Hence \( I_\mathfrak{p} \) acts trivially on \( u(\phi[\ell]) \).

From our study of the ramifications of \( G_F \)-action on \( u(\phi[\ell]) \), we conclude

**Lemma.** The action of \( G_F \) on \( u(\phi[\ell]) \) is unramified at every finite place and tamely ramified at the place of infinity.

Finally, we prove the field extension \( \overline{\mathbb{F}}_q F(u(\phi[\ell]))/\overline{F}_q F \) is nontrivial. This leads to a non-trivial Galois cover of \( \overline{\mathbb{F}}_q \) which is unramified away from infinity and tamely ramified at infinity, which is a contradiction by Hurwitz genus formula.

Suppose the field extension \( \overline{\mathbb{F}}_q F(u(\phi[\ell]))/\overline{F}_q F \) is trivial, then for any element \( \alpha \in \phi[\ell] \) the valuation \( v_\infty(u(\alpha)) \) is an integer. On the other hand, we take the minimum successive basis \( \{ \lambda_1, \ldots, \lambda_r \} \) from the previous subsection and set \( \alpha_i = e_{\lambda_i}(\frac{\lambda_i}{\ell}) \). The set \( \{ \alpha_1, \ldots, \alpha_r \} \) is a \( \mathbb{F}_q \)-basis of \( \phi[\ell] \).

Moreover, we have
\[
v_\infty(\alpha_i) = v_\infty(e_{\lambda_i}(\frac{\lambda_i}{\ell})) = v_\infty(\frac{\lambda_i}{\ell}) = \deg_F(\ell) + \frac{-q^r + q - 1}{q^r - 1}.
\]

Now we compute the valuation \( v_\infty(u(\alpha_i)) \). As \( u = a_0 + a_1 \tau + \cdots + a_d \tau^d \in F\{\tau\} \), we have
\[
u_\infty(\alpha_i) = a_0 \alpha_i + a_1 \alpha_i^q + \cdots + a_d \alpha_i^{q^d}.
\]

For each nonzero term \( a_j \alpha_i^{q^j} \) of \( u(\alpha_i) \) where \( 0 \leq j \leq d < r - 1 \), its valuation is
\[
v_\infty(a_j \alpha_i^{q^j}) = v_\infty(a_j) + q^j \cdot v_\infty(\alpha_i) = \text{some integer} + q^j \cdot \frac{-q^r + q - 1}{q^r - 1} = \text{some integer} + \frac{q^j \cdot (q - 2)}{q^r - 1}.
\]

Therefore, the valuation of each nonzero term of \( u(\alpha_i) \) has distinct fractional part. Thus \( v_\infty(u(\alpha_i)) \) is not an integer by the strong triangle inequality because the valuation of each summand \( v_\infty(a_j \alpha_i^{q^j}) \) is a distinct fraction. This shows the field extension \( \overline{\mathbb{F}}_q F(u(\phi[\ell]))/\overline{F}_q F \) is nontrivial.

In conclusion, we have proved the irreducibility of the mod \( I \) Galois representations for \( I \neq (T) \). Combining with Theorem 4.22, we have the following corollary:
Corollary 5.4. Let \( q = p^r \) be a prime power, \( A = \mathbb{F}_q[T] \), and \( F = \mathbb{F}_q(T) \). Assume \( r \geq 3 \) is a prime number, there is a constant \( c = c(r) \in \mathbb{N} \) depending only on \( r \) such that for \( p > c(r) \) the following statement is true:

Let \( \phi \) be a Drinfeld \( A \)-module over \( F \) of rank \( r \) with generic characteristic, which is defined by \( \phi_T = T + \tau r - 1 + T^{q-1} \tau^r \). Let \( \ell \neq (T) \) be a finite place of \( F \). Then the \( \ell \)-adic Galois representation \( \bar{\rho}_{\phi,1} : G_F \rightarrow \text{Aut}(\phi[\ell]) \cong \text{GL}_r(\mathbb{F}_\ell) \)

is surjective.

5.3 The case when \( \ell = (T) \)

For the mod \( \ell \) Galois representation \( \bar{\rho}_{\phi,T} : G_F \rightarrow \text{Aut}(\phi[\ell]) \cong \text{GL}_r(\mathbb{F}_\ell) \), we are actually computing the Galois group \( \text{Gal}(\phi[\ell]/F) \) of the field extension obtained by adjoining the roots of \( \phi_T(x) = Tx + x^{q-1} + T^{q-1}x^r \) to \( F \). This question has been studied by Abhyankar \cite{Abh94}. Here we use the same notation as in \cite{Abh94}, Theorem 3.2 (3.2.3).

Let \( k_0 = \mathbb{F}_q \), \( K = k_0(\sqrt[q]{T}) \), and

\[
V = V(Y) = Y^{q-1} + \frac{1}{T^{q-1}}Y^{q-1} - 1 + \frac{1}{T^{q-2}}.
\]

We have \( \mu = r - 1 \) and \( \text{GCD}(\nu, \tau) = 1 \), where \( \nu = 1 + q + \cdots + q^{r-1} \) and \( \tau = 1 + q + \cdots + q^{r-2} \).

Further, we have \( C_1 = \left( \frac{1}{T} \right)^{q-1} \) and \( C_r = \left( \frac{1}{T} \right)^{q-2} \). Thus \( \rho = q - 1 \) and \( \sigma = q - 2 \). Certainly we have \( \rho \neq \frac{\sigma(\nu - \tau)}{\nu} \). Thus we can apply \cite{Abh94}, Theorem 3.2 (3.2.3). Since \( \text{GCD}(\sigma, q - 1) = 1 \), the theorem implies

\[
\text{Gal}(\mathbb{F}_q(T)[\phi[\ell]]/\mathbb{F}_q(T)) = \text{GL}_r(\mathbb{F}_\ell).
\]

As a summary of section 5, we have proved the surjectivity of mod \( \ell \) Galois representation \( \bar{\rho}_{\phi,1} \) for any prime ideal \( \ell \) of \( A \) under the assumption on \( q \) in Theorem 5.2.

6 Surjectivity of \( \ell \)-adic Galois representations

Similar to the rank 3 case, we wish to apply \cite{PR09a}, Proposition 4.1 to prove surjectivity of \( \ell \)-adic representations. We separate \( \ell \) into two cases:

Case 1. \( \ell \neq (T) \)

Our proof of the equality \( \bar{\rho}_{\phi,1}(I_T) = \begin{pmatrix} 1 & b_1 & \cdots & \cdots & \cdots & b_{r-1} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \ddots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \ddots & \vdots \\ b_1 & \cdots & \cdots & \cdots & \cdots & 1 \end{pmatrix} \), \( b_i \in \mathbb{F}_\ell \ \forall \ 1 \leq i \leq r - 1 \)

only restrict \( \ell \) to be prime to \((T)\). Hence we can prove the above equality for mod \( \ell^2 \) representation. In other words, we have

\[
\bar{\rho}_{\phi,1^2}(I_T) = \begin{pmatrix} 1 & b_1 & \cdots & \cdots & \cdots & b_{r-1} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \ddots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \ddots & \vdots \\ b_1 & \cdots & \cdots & \cdots & \cdots & 1 \end{pmatrix} , \ b_i \in (A/\ell^2) \ \forall \ 1 \leq i \leq r - 1 \}
\]

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Therefore, the mod \( l^2 \) representation certainly contains a non-scalar matrix that becomes identity after modulo \( l \).

Case 2. \( l = (T) \)

We consider the following diagram and focus on the representations of decomposition subgroups.

\[
\begin{align*}
\tilde{\rho}_{\phi, l^2}(G_F) \quad &\text{modulo } l \to \tilde{\rho}_{\phi, l}(G_F) = \text{GL}_r(\mathbb{F}_l), \\
\tilde{\rho}_{\phi, l^2}(G_{F(T)}) \quad &\text{modulo } l \to \tilde{\rho}_{\phi, l}(G_{F(T)})
\end{align*}
\]

It’s clear that the modulo \( l \) homomorphism \( \tilde{\rho}_{\phi, l^2}(G_{F(T)}) \to \tilde{\rho}_{\phi, l}(G_{F(T)}) \) is surjective. Suppose it is injective, then we can conclude that the splitting field of \( \phi_{l^2}(x) \) and of \( \phi_l(x) \) are isomorphic over \( F(T) \). Now we can apply [PR09a], Proposition 4.1 to show that the splitting field of \( \phi_{l^2}(x) \) and of \( \phi_l(x) \).

\[
\phi_{T^2} = T^2 + T(T^{q-1}r^{-1} + 1)\tau^{r-1} + T^q(T^{q'}^{-1} + 1)\tau^r + \tau^{2r-2}
+ T^q(T^{q-1}r^{-1} - 1)\tau^{2r-1} + T(q-1)(q'+1)\tau^{2r}.
\]

There is some root of \( \phi_{l^2}(x)/x \) has valuation equal to \( -\frac{1}{q^{r-2}} \), but there is no element in the splitting field of \( \phi_l(x) \) with the same valuation. Thus the splitting fields of both polynomials are not isomorphic, which implies \( \tilde{\rho}_{\phi, l^2}(G_{F(T)}) \to \tilde{\rho}_{\phi, l}(G_{F(T)}) \) has nontrivial kernel.

Now we prove such a nontrivial element \( \tilde{\rho}_{\phi, l^2}(\sigma) \in \tilde{\rho}_{\phi, l^2}(G_{F_{T^2}}) \) can not be a scalar matrix. Suppose \( \tilde{\rho}_{\phi, l^2}(\sigma) \in \tilde{\rho}_{\phi, l^2}(G_{F_{T^2}}) \) is a scalar matrix, then there is some \( a \in \mathbb{F}_q^* \) such that \( \sigma \) maps every root \( \alpha \) of \( \phi_{T^2}(x)/x \) to \( \phi_{1+Ta}(\alpha) \). Furthermore, as \( \sigma \) lies in the decomposition subgroup \( G_{F_{T^2}} \), \( \sigma \) should preserve the valuation \( v_T \). We pick a root \( \alpha \) with valuation equal to \( -\frac{1}{q^{r-2}} \), then compare the valuation of \( \sigma \) with \( \sigma(\alpha) \).

\[
-\frac{1}{q^{r-2}} = v_T(\alpha) \neq v_T(\sigma(\alpha)) = v_T(\phi_{1+Ta}(\alpha)) = v_T([1+Ta]\alpha + a\alpha^{q'-1} - [aT^{q-1}]\alpha^{q'}) = -\frac{1}{q^{r-2}}.
\]

Therefore \( \sigma \) can not be a scalar matrix, and the mod \( (T^2) \) representation contains a non-scalar matrix that becomes identity after modulo \( l \).

Now we can apply [PR09a], Proposition 4.1 to show that the \( l \)-adic representation is surjective for every prime ideal \( l \) of \( A \).

### 7 Adelic surjectivity of Galois representations

In this section, we make a further assumption that \( q \equiv 1 \mod r \). The proof of adelic surjectivity is similar to the proof for rank 2 and 3 cases in [Zyw11] and [Che20].

**Lemma 7.1.** For each finite place \( l \) of \( F \), \( SL_r(A_l) \) is equal its commutator subgroup. The only normal subgroup of \( SL_r(A_l) \) with simple quotient is

\[
N := \{ B \in SL_r(A_l) | B \equiv \delta \cdot I_r \mod l \, | \, \delta \in \mathbb{F}_l \, \text{satisfies} \, \delta^r = 1 \}.
\]
Proof. Let $H$ be the commutator subgroup of $\text{SL}_r(A_1)$. It’s a closed normal subgroup of $\text{SL}_r(A_1)$ and $\text{GL}_r(A_1)$. We define $S^0 := \text{SL}_r(A_1)$ and for $i \geq 1$, we set $S^i := \{ s \in \text{SL}_r(A_1) \mid s \equiv 1 \text{ mod } t^i \}$. For $i \geq 0$, define $H^i = H \cap S^i$. For $i \geq 0$, we define $S^{[i]} := S^i / S^{i+1}$ and $H^{[i]} := H^i / H^{i+1}$. There is a natural injection form $H^{[i]}$ to $S^{[i]}$ and we claim that $H^{[i]} = S^{[i]}$ for all $i \geq 0$.

For $i = 0$, modulo $t$ induces an isomorphism $S^{[0]} \simeq \text{SL}_r(\mathbb{F}_1)$ and the image of $H^{[0]}$ under this isomorphism becomes the commutator subgroup of $\text{SL}_r(\mathbb{F}_1)$. It’s well known that $\text{SL}_r(\mathbb{F}_1)$ is quasi-simple, i.e. $\text{SL}_r(\mathbb{F}_1)$ equals to its commutator subgroup and the quotient of $\text{SL}_r(\mathbb{F}_1)$ by its center $Z(\text{SL}_r(\mathbb{F}_1))$ is a simple group. Therefore, we have $H^{[0]} = S^{[0]}$.

Now we fix $i \geq 1$, let $\mathfrak{s}_p(\mathbb{F}_1)$ be the additive subgroup in $M_r(\mathbb{F}_1)$ consisting of matrices with trace $0$. We have the following isomorphism:

$$S^{[i]} \ni \begin{bmatrix} 1 + t^iy \end{bmatrix} \mapsto \mathfrak{s}_p(\mathbb{F}_1),$$

where $l$ is the monic polynomial in $A$ that generates $l$. Consider $\text{GL}_r(A_1)$ acting on both sides via conjugation action, it factors through $\text{GL}_r(\mathbb{F}_1)$. By [PR09a] Proposition 2.1, $\mathfrak{s}_p(\mathbb{F}_1)$ is an irreducible $\text{GL}_r(\mathbb{F}_1)$-module (here we used the assumption $q = p^r$ is a prime power with $p \geq 5$). On the other hand, $H^{[i]}$ injects into $S^{[i]}$ and $H$ is normal in $\text{GL}_r(A_1)$ imply that $H^{[i]}$ is also stable under $\text{GL}_r(\mathbb{F}_1)$-action. Once we can show that $H^{[i]}$ is nontrivial, we have $H^{[i]} = S^{[i]}$ for all $i \geq 0$ and so $H = S^0$.

Consider the commutator map $S^0 \times S^i \to H^i$ that maps $(g, h)$ to $ghg^{-1}h^{-1}$. This induces a map $S^{[0]} \times S^{[i]} \to H^{[i]}$. Combining with the isomorphism $S^{[i]} \simeq \mathfrak{s}_p(\mathbb{F}_1)$, we obtain the following map:

$$\text{SL}_r(\mathbb{F}_1) \times \mathfrak{s}_p(\mathbb{F}_1) \to \mathfrak{s}_p(\mathbb{F}_1), \quad (s, X) \mapsto sXs^{-1} - X.$$ 

This map is not a zero map, so we have $H^{[i]} = S^{[i]}$ for all $i \geq 0$ and $H = S^0$.

Now let $N'$ be a normal subgroup of $\text{SL}_r(A_1)$ with simple quotient. We consider the subgroup $S^1 \triangleleft \text{SL}_r(A_1)$. Note that $S^1$ is a pro-$p$ group by the definition of $S^i$ and $S^0 / S^1 = \text{SL}_r(\mathbb{F}_1)$ is quasi-simple. Thus there is a composition series of $\text{SL}_r(A_1)$ and its composition factors are $\text{SL}_r(\mathbb{F}_1)/Z(\text{SL}_r(\mathbb{F}_1)), \mathbb{Z}/p\mathbb{Z}$ (comes from the composition factors of $S^1$) and $\mathbb{Z}/r\mathbb{Z}$ (comes from composition factors of $Z(\text{SL}_r(\mathbb{F}_1)$ if it is nontrivial). As $\text{SL}_r(A_1)$ equals to its commutator, it has no abelian quotient. Thus $\text{SL}_r(A_1) / N' \cong \text{SL}_r(\mathbb{F}_1) / Z(\text{SL}_r(\mathbb{F}_1))$. On the other hand, we also have $\text{SL}_r(A_1) / N \cong \text{SL}_r(\mathbb{F}_1) / Z(\text{SL}_r(\mathbb{F}_1))$. Therefore, we get $\text{SL}_r(A_1) / N \cong \text{SL}_r(A_1) / N'$ and so $N' = N$. Otherwise, we may have $N \subseteq NN' \triangleleft \text{SL}_r(A_1)$. Furthermore, we prove $NN' \neq \text{SL}_r(A_1)$. Suppose $NN' = \text{SL}_r(A_1)$, then we have

$$\text{PSL}_r(\mathbb{F}_1) \cong \text{SL}_r(A_1) / N' \cong NN' / N' \cong N / N \cap N'.$$

On the other hand, we look at the composition factors of $\text{SL}_r(A_1)$. We have $\text{SL}_r(A_1) / N \cong \text{SL}_r(\mathbb{F}_1) / Z(\text{SL}_r(\mathbb{F}_1)) \cong \text{PSL}_r(\mathbb{F}_1)$, and the composition factor $\text{PSL}_r(\mathbb{F}_1)$ appears only once. Therefore, the composition factors of $N$ are all abelian, either $\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Z}/r\mathbb{Z}$. It is impossible to have a quotient $N / N \cap N'$ of $N$ isomorphic to $\text{PSL}_r(\mathbb{F}_1)$. Thus $NN'$ is a proper subgroup sits between $\text{SL}_r(A_1)$ and $N$, which contradicts to the fact that $\text{SL}_r(A_1) / N \cong \text{SL}_r(\mathbb{F}_1) / Z(\text{SL}_r(\mathbb{F}_1))$ is simple. \hfill $\square$

**Lemma 7.2.** ([Zyw11], Lemma A.4) Let $B_1$ and $B_2$ be finite groups and suppose that $H$ is a subgroup of $B_1 \times B_2$ for which the two projections $p_1 : H \to B_1$ and $p_2 : H \to B_2$ are surjective. Let $N_1$ be the kernel of $p_2$ and $N_2$ be the kernel of $p_1$. We may view $N_1$ as a normal subgroup of $B_1$ and $N_2$ as a normal subgroup of $B_2$. Then the image of $H$ in $B_1 / N_1 \times B_2 / N_2$ is the graph of the isomorphism $B_1 / N_1 \cong B_2 / N_2$.  

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Lemma 7.3. ([Zyw11], Lemma A.6) Let $S_1, S_2, \ldots, S_k$ be finite groups with no non-trivial abelian quotients. Let $H$ be a subgroup of $S_1 \times \cdots \times S_k$ such that each projection $H \to S_i \times S_j$ ($1 \leq i < j \leq k$) is surjective. Then $H = S_1 \times \cdots \times S_k$.

Lemma 7.4. Let $I_1$ and $I_2$ be two prime ideals of $A$, and set $a = I_1I_2$. Let $\phi$ be the Drinfeld module as before, and let $H = \bar{\rho}_{\phi,a}(G_F) \subseteq \text{GL}_r(A/a)$. Then $H$ satisfies the following properties:

1. $\det(H) = (A/a)^*$;
2. the projections $p'_1 : H' \to \text{SL}_r(A/I_1)$ and $p'_2 : H' \to \text{SL}_r(A/I_2)$ are surjective, where $H' = H \cap \text{SL}_r(A/a)$;
3. the subring of $A/a$ generated by the set
\[ S = \{ \text{tr}(h)^r / \det(h) \mid h \in H \} \cup \{ \text{det}(h) / \text{tr}(h)^r \mid h \in H \text{ with } \text{tr}(h) \in (A/a)^* \} \]
is exactly $A/a$.

These three properties will imply $H = \text{GL}_r(A/a)$.

Proof. Condition 1 follows from the fact that $\det \circ \bar{\rho}_{\phi,a} = \bar{\rho}_{\phi,1}$, the mod $a$ Galois representation of the Carlitz module. Condition 2 follows from our result in section 5 and 6 that $\bar{\rho}_{\phi,i}$ is surjective for any prime ideal $I$. For condition 3, we may take $c \in F_q^*$ such that $p = T - c$ not equal to $I_1$ or $I_2$.

By looking at the characteristic polynomial of $\bar{\rho}_{\phi,a}(\text{Frob}_p)$, we have
\[ \frac{\text{det}(\bar{\rho}_{\phi,a}(\text{Frob}_p))}{\text{tr}(\bar{\rho}_{\phi,a}(\text{Frob}_p))^r} = \frac{T - c}{1^r} = T - c \mod a. \]

The subring generated by these $T - c$ is equal to $A/a$.

The following shows how these three properties imply $H = \text{GL}_r(A/a)$. Let $N'_1$ be the kernel of $p'_2$ and $N'_2$ be the kernel of $p'_1$. We may view $N'_1$ as a normal subgroup of $\text{SL}_r(F_{l_1})$. Lemma 7.2 then implies the image of $H'$ in $\text{SL}_r(F_{l_1})/N'_1 \times \text{SL}_r(F_{l_2})/N'_2$ is the graph of a group isomorphism $\text{SL}_r(F_{l_1})/N'_1 \cong \text{SL}_r(F_{l_2})/N'_2$. If one of $N'_1$ or $N'_2$ is the whole group, then the group isomorphism will force the other one to be the whole group. Therefore, if $N'_1 = \text{SL}_r(F_{l_1})$ or $N'_2 = \text{SL}_r(F_{l_2})$, we have $H' = \text{SL}_r(F_{l_1}) \times \text{SL}_r(F_{l_2})$. Combining with condition 1, we have $H = \text{GL}_r(A/a)$.

Now we suppose both $N'_i$ are proper normal subgroups of $\text{SL}_r(F_{l_1})$. By the proof of Lemma 7.2, we can see that $N'_i \subseteq Z(\text{SL}_r(F_{l_1}))$ for $i = 1, 2$. Since the order $|Z(\text{SL}_r(F_{l_1}))|$ is either 1 or 2, the order $|N'_i|$ is either 1 or 2. Comparing the order on both sides of the isomorphism $\text{SL}_r(F_{l_1})/N'_1 \cong \text{SL}_r(F_{l_2})/N'_2$, we can show that $|F_{l_1}| = |F_{l_2}|$, i.e. $F_{l_1}$ and $F_{l_2}$ are isomorphic fields. For $i = 1, 2$, define the projection $p_i : H \to \text{GL}_r(F_{l_i})$. Let $N_i$ be kernel of $p_2$ and $N_2$ be kernel of $p_2$. We may also view $N_1$ as normal subgroup of $\text{GL}_r(F_{l_1})$. Lemma 7.2 then implies the image of $H$ into $\text{GL}_r(F_{l_1})/N_1 \times \text{GL}_r(F_{l_2})/N_2$ is the graph of a group isomorphism $\text{GL}_r(F_{l_1})/N_1 \cong \text{GL}_r(F_{l_2})/N_2$.

As $N_i/N'_i \cong N_i\text{SL}_r(F_{l_i})/\text{SL}_r(F_{l_i}) \subseteq \text{GL}_r(F_{l_i})/\text{SL}_r(F_{l_i}) \cong F_{l_i}^*$, we see that $N_i/N'_i$ and $N_i$ are abelian. Thus $N_i$ is a solvable normal subgroup of $\text{GL}_r(F_{l_i})$. Furthermore, we see that $N_i \subseteq N_iZ(\text{GL}_r(F_{l_i}))/Z(\text{GL}_r(F_{l_i}))$ is a solvable normal subgroup of $\text{PGL}_r(F_{l_i})$. Hence we can deduce a normal series
\[ \{1\} \triangleleft N_iZ(\text{GL}_r(F_{l_i}))/Z(\text{GL}_r(F_{l_i})) \triangleleft \text{PGL}_r(F_{l_i}). \]

If $\text{PSL}_r(F_{l_i}) = \text{PGL}_r(F_{l_i})$, then we have $N_i \subseteq Z(\text{GL}_r(F_{l_i}))$ by the simplicity of $\text{PSL}_r(F_{l_i})$. If $\text{PSL}_r(F_{l_i}) \neq \text{PGL}_r(F_{l_i})$, then we consider the composition series
\[ \{1\} \triangleleft \text{PSL}_r(F_{l_i}) \triangleleft \text{PGL}_r(F_{l_i}). \]
Its factors are $\text{PSL}_r(\mathbb{F}_l)$ and $C_r$, the cyclic group of order $r$. Thus by the Jordan-Hölder theorem, $N_i \cong N_i Z(\text{GL}_r(\mathbb{F}_l))/Z(\text{GL}_r(\mathbb{F}_l)) \cong C_r$. Let $\delta$ be a generator of $N_i \subset \text{GL}_r(\mathbb{F}_l)$. We know that $\delta$ satisfies the polynomial $x^r - 1$. Since $q \equiv 1 \mod r$, $x^r - 1$ splits over $\mathbb{F}_l$. The minimal polynomial of $\delta$ over $\mathbb{F}_l$ is a product of distinct degree one polynomials. Now we prove that the minimal polynomial of $\delta$ has to be a degree one polynomial. Let $\zeta$ be a generator of $\mathbb{F}_q^*$ and $\omega = \zeta^{\frac{1}{r-1}}$. The roots of $x^r - 1$ in $\mathbb{F}_q$ are $\omega^i$ with $0 \leq i \leq r-1$. Suppose the minimal polynomial of $\delta \in N_i \subset \text{GL}_r(\mathbb{F}_l)$ is not a degree one polynomial, then $\delta$ can be written as a diagonal matrix with diagonal entries of the form $\omega^i$ but not all the same. Let’s assume there are two distinct diagonal entries $\omega_1, \omega_2 \in \{\omega^i, 0 \leq i \leq r-1\}$ in $\delta$. The proof for general situation will follow the same process. Now we may write $\delta$ as

$$\delta = \begin{pmatrix}
\omega_1 & & \\
& \ddots & \\
& & \omega_1 \\
\omega_2 & & \\
& \ddots & \\
& & \omega_2
\end{pmatrix}.$$ 

Now since $N_i$ is a normal subgroup of $\text{GL}_r(\mathbb{F}_l)$, we can conjugate $\delta$ by a suitable permutation matrix to get a matrix in $N_i$ where two distinct diagonal entries interchanged while the other diagonal entries fixed. Namely, we have

$$\begin{pmatrix}
\omega_1 & & \\
& \ddots & \\
& & \omega_1 \\
\omega_2 & & \\
& \ddots & \\
& & \omega_2
\end{pmatrix} \in N_i$$

As $N_i$ is generated by $\delta$, this matrix is equal to $\delta^j$ for some $1 \leq j \leq r-1$. By comparing the fixed entries in $\delta$ and $\delta^j$, we can deduce that $j$ must be equal to 1 because $\omega^i$ has order equal to $r$ for $1 \leq i \leq r-1$ (here we use $r$ is a prime number). This implies $\omega_1 = \omega_2$, a contradiction. Therefore, the minimal polynomial of $\delta$ is a degree one polynomial. Hence $\delta$ is a scalar matrix and we have $N_i \subseteq Z(\text{GL}_r(\mathbb{F}_l))$ in both cases.

By taking further quotient, the image of $H$ into $\text{PGL}_r(\mathbb{F}_l_1) \times \text{PGL}_r(\mathbb{F}_l_2)$ is the graph of a group isomorphism

$$\alpha : \text{PGL}_r(\mathbb{F}_l_1) \cong \text{PGL}_r(\mathbb{F}_l_2).$$

By [Die80] Theorem 2, $\alpha$ can be lifted to an isomorphism

$$\tilde{\alpha} : \text{GL}_r(\mathbb{F}_l_1) \cong \text{GL}_r(\mathbb{F}_l_2).$$

Let $\sigma : \mathbb{F}_{l_1} \cong \mathbb{F}_{l_2}$ be a field isomorphism and $\chi : \text{GL}_r(\mathbb{F}_{l_1}) \rightarrow \mathbb{F}_{l_2}^*$ be a homomorphism. Now we are able to create two group homomorphisms $\text{GL}_r(\mathbb{F}_{l_1}) \cong \text{GL}_r(\mathbb{F}_{l_2})$:
(i) \( A \mapsto \chi(A)gA^\sigma g^{-1} \),

(ii) \( A \mapsto \chi(A)g((A^T)^{-1})^\sigma g^{-1} \),

where \( A \in \text{GL}_r(\mathbb{F}_{l_1}) \), \( A^\sigma \) is the matrix that applies \( \sigma \) to each entry of \( A \), and \( g \in \text{GL}_r(\mathbb{F}_{l_2}) \). By \cite{Die80} Theorem 1, there are \( \sigma, \chi, \) and \( g \) such that \( \tilde{\alpha} \) is one of the homomorphisms above.

**Lemma.** \( \tilde{\alpha} \) must be of the first type.

**Proof.** Suppose \( \tilde{\alpha} \) is of second type, then we choose a degree 1 prime ideal \( p \) of \( A \) different from \( l_1 \) and \( l_2 \). We consider the image of \( \tilde{\alpha}(\text{Frob}_p) \in H \) in \( \text{PGL}_r(\mathbb{F}_{l_1}) \times \text{PGL}_r(\mathbb{F}_{l_2}) \) under

\[
H \quad \mapsto \quad \text{PGL}_r(\mathbb{F}_{l_1}) \times \text{PGL}_r(\mathbb{F}_{l_2})
\]

\[
\tilde{\alpha}(\text{Frob}_p) \quad \mapsto \quad (\tilde{\rho}_{\phi,1}(\text{Frob}_p) \cdot Z(\text{GL}_r(\mathbb{F}_{l_1})), \tilde{\rho}_{\phi,2}(\text{Frob}_p) \cdot Z(\text{GL}_r(\mathbb{F}_{l_2})))
\]

\[
(\tilde{\rho}_{\phi,1}(\text{Frob}_p) \cdot Z(\text{GL}_r(\mathbb{F}_{l_1})), \tilde{\rho}_{\phi,2}(\text{Frob}_p) \cdot Z(\text{GL}_r(\mathbb{F}_{l_2})))
\]

Therefore, \( \rho_{\phi,1}(\text{Frob}_p) \cdot Z(\text{GL}_r(\mathbb{F}_{l_1})) \) and \( g(((\tilde{\rho}_{\phi,1}(\text{Frob}_p)^\sigma)^T)^{-1}g^{-1} \cdot Z(\text{GL}_r(\mathbb{F}_{l_2})) \) are the same coset in \( \text{PGL}_r(\mathbb{F}_{l_1}) \). For each element in these two cosets, we can compute its trace. All elements in \( \rho_{\phi,1}(\text{Frob}_p) \cdot Z(\text{GL}_r(\mathbb{F}_{l_1})) \) has nonzero trace because the characteristic polynomial of \( \rho_{\phi,1}(\text{Frob}_p) \) is \( x^r+x^{r-1}p \) (mod \( l_2 \)). However, the trace of all elements in \( g(((\tilde{\rho}_{\phi,1}(\text{Frob}_p)^\sigma)^T)^{-1}g^{-1} \cdot Z(\text{GL}_r(\mathbb{F}_{l_2})) \) are equal to zero because the characteristic polynomial of \( (\tilde{\rho}_{\phi,1}(\text{Frob}_p))^{-1} \) is \( x^r - \frac{1}{p}x - \frac{1}{p} \) (mod \( l_1 \)). Hence we have a contradiction and \( \tilde{\alpha} \) must be of first type.

Writing \( \tilde{\alpha}(A) = \chi(A)gA^\sigma g^{-1} \) for all \( A \in \text{GL}_r(\mathbb{F}_{l_1}) \), we have

\[
\frac{\text{tr}(\tilde{\alpha}(A))}{\det(\tilde{\alpha}(A))} = (\frac{\text{tr}(A)^r}{\det(A)^r})^\sigma.
\]

Therefore, for each element \( (h_1, h_2) \in H \), we have

\[
\frac{\text{tr}(h_2)^r}{\det(h_2)^r} = (\frac{\text{tr}(h_1)^r}{\det(h_1)^r})^\sigma.
\]

Let \( W = \{(x_1, x_2) \mid \sigma(x_1) = x_2\} \) be the subring of \( A/\mathfrak{a} \cong \mathbb{F}_{l_1} \times \mathbb{F}_{l_2} \). We have \( S \subseteq W \). However, \( W \neq A/\mathfrak{a} \) by counting cardinality on both sides. Thus we get a contradiction from the assumption that \( N' \) is a proper normal subgroup of \( \text{SL}_r(\mathbb{F}_{l_1}) \) for \( i = 1 \) or 2. The proof is complete.

**Lemma 7.5.** Let \( l_1 \) and \( l_2 \) be two finite places of \( F \). Define

\[
\rho : G_F \to \text{GL}_r(A_{l_1}) \times \text{GL}_r(A_{l_2}), \quad \sigma \mapsto (\rho_{\phi,1}(\sigma), \rho_{\phi,2}(\sigma)).
\]

Then \( \rho(G_F) = \text{GL}_r(A_{l_1}) \times \text{GL}_r(A_{l_2}) \).
Proof. It’s enough to show that for any positive integers \( n_1 \) and \( n_2 \), we have
\[
\bar{\rho}_{\phi,a}(G_{F^{\mathrm{nr}}}) = \SL_r(A/\mathfrak{a})
\]
where \( \mathfrak{a} = \mathfrak{t}_1^{n_1} \mathfrak{t}_2^{n_2} \).

Suppose the equality doesn’t hold, then we can apply Lemma \( \ref{L2} \) with \( H = \bar{\rho}_{\phi,a}(G_{F^{\mathrm{nr}}}) \), \( B_1 = \SL_r(A/\mathfrak{t}_1^{n_1}) \) and \( B_2 = \SL_r(A/\mathfrak{t}_2^{n_2}) \). The image of \( H \) in \( \SL_r(A/\mathfrak{t}_1^{n_1})/N_1 \times \SL_r(A/\mathfrak{t}_2^{n_2})/N_2 \) is the graph of the isomorphism \( \SL_r(A/\mathfrak{t}_1^{n_1})/N_1 \xrightarrow{\sim} \SL_r(A/\mathfrak{t}_2^{n_2})/N_2 \). From the second paragraph in the proof of Lemma \( \ref{L4} \) we know that \( \SL_3(A/\mathfrak{t}_1^{n_1})/N_i \) are not trivial for \( i = 1, 2 \). By Lemma \( \ref{L1} \), \( N_i \) is a subgroup of
\[
\bar{N} := \{ B \in \SL_r(A/\mathfrak{t}_r) | B \equiv \delta \cdot I_r \mod I, \text{ where } \delta \in \mathbb{F}_r \text{ satisfies } \delta^r = 1 \}.
\]
Taking further quotient, the image of \( H \) in \( \SL_r(A/\mathfrak{t}_1)/Z(\SL_r(A/\mathfrak{t}_1)) \times \SL_r(A/\mathfrak{t}_2)/Z(\SL_r(A/\mathfrak{t}_2)) \) is the graph of an isomorphism \( \SL_r(A/\mathfrak{t}_1)/Z(\SL_r(A/\mathfrak{t}_1)) \xrightarrow{\sim} \SL_r(A/\mathfrak{t}_2)/Z(\SL_r(A/\mathfrak{t}_2)) \). However, Lemma \( \ref{L4} \) shows the reduction of \( H \) modulo \( I_1 I_2 \) is the whole group \( \SL_r(A/\mathfrak{t}_1) \times \SL_r(A/\mathfrak{t}_2) \). Thus the image of \( H \) into \( \SL_r(A/\mathfrak{t}_1)/Z(\SL_r(A/\mathfrak{t}_1)) \times \SL_r(A/\mathfrak{t}_2)/Z(\SL_r(A/\mathfrak{t}_2)) \) should be the whole group, this gives a contradiction. \( \square \)

7.1 Proof of the main theorem

Main Theorem. Let \( q = p^r \) be a prime power, \( A = \mathbb{F}_q[T] \), and \( F = \mathbb{F}_q(T) \). Assume \( r \geq 3 \) is a prime number and \( q \equiv 1 \mod r \), there is a constant \( c = c(r) \in \mathbb{N} \) depending only on \( r \) such that for \( \rho > c(r) \) the following statement is true:

Let \( \phi \) be a Drinfeld \( A \)-module over \( F \) of rank \( r \) with generic characteristic, which is defined by \( \phi_T = T + \tau^{r-1} + T^{q-1} \tau^r \). Then the adelic Galois representation
\[
\rho_\phi : \Gal(\mathbb{F}_q(T)^{\sep}/\mathbb{F}_q(T)) \longrightarrow \lim_{\rightarrow \mathfrak{a}} \Aut(\phi[\mathfrak{a}]) \cong \GL_r(\hat{A})
\]
is surjective.

Proof. Firstly, \( \det \rho_\phi = \rho_C \) is the adelic representation of Carlitz module. We know that the adelic representation of the Carlitz module is surjective, so \( \det \rho_\phi = \hat{A}^* \). It suffices for us to prove \( \rho_\phi(G_{F^{\mathrm{nr}}}) = \SL_r(\hat{A}) \). It’s equivalent to show that for every nonzero ideal \( \mathfrak{a} = \mathfrak{t}_1^{n_1} \mathfrak{t}_2^{n_2} \cdots \mathfrak{t}_k^{n_k} \) of \( A \), we have
\[
\bar{\rho}_{\phi,a}(G_{F^{\mathrm{nr}}}) = \SL_r(A/\mathfrak{a}) \cong \prod_i \SL_r(A/\mathfrak{t}_i^{n_i}).
\]
By Lemma \( \ref{L4} \), we know that each \( \SL_r(A/\mathfrak{t}_i^{n_i}) \) has no nontrivial abelian quotient. Thus we can apply Lemma \( \ref{L3} \) once we can prove that each projection \( \bar{\rho}_{\phi,a}(G_{F^{\mathrm{nr}}}) \rightarrow \SL_r(A/\mathfrak{t}_i^{n_i}) \times \SL_r(A/\mathfrak{t}_j^{n_j}) \) is surjective for \( 1 \leq i < j \leq k \). The surjectivity of each projection is proved by Lemma \( \ref{L1} \). Thus Lemma \( \ref{L3} \) implies \( \bar{\rho}_{\phi,a}(G_{F^{\mathrm{nr}}}) = \SL_r(A/\mathfrak{a}) \) for every nonzero ideal \( \mathfrak{a} \) of \( A \), the proof of theorem is complete. \( \square \)

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Appendices

In the Appendices, we restate Aschbacher’s theorem in [BHRD13] and focus on the classification of subgroups in general linear group over a finite field.

A Notations and Aschbacher classes

A.1 Notation

At here, $A, B, G$ are groups, and $a, b, n$ belong to $\mathbb{N}$.

- $\text{GL}_n(q) = \text{GL}_n(\mathbb{F}_q)$
- $Z(G)$ denote the center of $G$.
- $[G, G]$ or $G'$ denote the derived subgroup of $G$.
- For $n > 1$, $G^{(n)} = [G^{(n-1)}, G^{(n-1)}]$. $G^\infty = \bigcap_{i \geq 0} G^{(i)}$.
- $A \ast B$ is an extension of $A$ by $B$ with unspecified splitness.
- $A \wr B$ is the wreath product of $A$ by a permutation group $B$.
- $E_{p^n}$ or just $p^n$ is an elementary abelian group of order $p^n$.
- For an elementary abelian group $A$, $A^{m+n}$ is a group with elementary abelian normal subgroup $A^m$ such that the quotient is isomorphic to $A^n$.

A.2 Aschbacher classes

Let $H$ be a subgroup of $\text{GL}_n(q)$, where $n \geq 3$. In this subsection we give a summary of information when $H$ lies in an Aschbacher class. For a complete edition of Aschbacher classes and the definition of each class we refer to chapter 2 in [BHRD13].

- $H$ lies in class $C_1 \Rightarrow H$ stabilizes a proper non-zero subspace of $\mathbb{F}_q^n$.
- $H$ lies in class $C_2 \Rightarrow$ there is a direct sum decomposition $\mathcal{D}$ of $\mathbb{F}_q^n$ into $t$ subspaces, each of dimension $m = n/t$:
  \[ \mathcal{D} : \mathbb{F}_q^n = V_1 \oplus V_2 \oplus \cdots \oplus V_t, \text{ where } t \geq 2 \]
  The action of $H$ on $\mathbb{F}_q^n$ is of the type $\text{GL}_m(q) \wr S_t$.
- $H$ lies in class $C_3 \Rightarrow$ There is a prime divisor $s \geq 2$ of $n$ and $m = n/s$ such that $\mathbb{F}_q^n$ has a $\mathbb{F}_q^s$-vector space structure and $H$ acts $\mathbb{F}_q^s$-semilinear on $\mathbb{F}_q^n$. The action of $H$ on $\mathbb{F}_q^n$ is of the type $\text{GL}_m(q^s)$
- $H$ lies in class $C_4 \Rightarrow H$ preserves a tensor product decomposition $\mathbb{F}_q^n = V_1 \otimes V_2$, where $V_1$ (resp. $V_2$) is a $\mathbb{F}_q$-subspace of $\mathbb{F}_q^n$ of dimension $n_1$ (resp. $n_2$) and $1 < n_1 < \sqrt{n}$. The action of $H$ on $\mathbb{F}_q^n$ is of the type $\text{GL}_{n_1}(q) \otimes \text{GL}_{n_2}(q)$.
• $H$ lies in class $C_5$ $\Rightarrow$ $H$ acts on $\mathbb{F}_q^n$ absolutely irreducible and there is a subfield $\mathbb{F}_{q_0}$ of $\mathbb{F}_q$ such that a conjugate of $H$ in $\text{GL}_n(q)$ is a subgroup of $\langle Z(\text{GL}_n(q)), \text{GL}_n(q_0) \rangle$.

• $H$ lies in class $C_6$ $\Rightarrow$ $n = r^m$ with $r$ prime. There is an absolutely irreducible group $E$ such that $E \triangleleft H \subseteq N_{\text{GL}_n(q)}(E)$. Here $E$ is either an extraspecial $r$-group or a 2-group of symplectic type. The action of $H$ on $\mathbb{F}_q^n$ is of the type $r^{1+2m}.\text{Sp}_{2m}(r)$ when $n$ is odd. And $H$ is of the type $2^{2+2m}.\text{Sp}_{2m}(2)$ when $n$ is even.

• $H$ lies in class $C_7$ $\Rightarrow$ $H$ preserves a tensor induced decomposition $\mathbb{F}_q^n = V_1 \otimes V_2 \otimes \cdots \otimes V_t$ with $t \geq 2$, $\dim V_i = m$ and $n = m^t$. The action of $H$ on $\mathbb{F}_q^n$ is of the type $\text{GL}_m(q) \wr S_t$. Here the wreath product is a tensor wreath product, which is a quotient of standard wreath product.

• $H$ lies in class $C_8$ $\Rightarrow$ $H$ preserves a non-degenerate classical form on $\mathbb{F}_q^n$ up to scalar multiplication. By classical form we mean symplectic form, unitary form or quadratic form.

• $H$ lies in class $S$ $\Rightarrow$ $H$ doesn’t contain $\text{SL}_n(q)$ and $H^\infty$ acts on $\mathbb{F}_q^n$ absolutely irreducibly.

B Note on Aschbacher’s theorem

Theorem B.1. (Special case of Aschbacher’s Theorem)

Let $H$ be a subgroup of $\text{GL}_n(q)$, then $H$ lies in one of the Aschbacher classes $C_i$ or $S$.

In [BHRD13] Theorem 2.2.19, the authors state a detailed version of Aschbacher’s Theorem and describe the structure of $H \cap \text{SL}_n(q)$ when $H$ maximal in $\text{GL}_n(q)$ that lies in a class $C_i$ for $1 \leq i \leq 8$.

Conflict of interest statement

On behalf of the author, the author states that there is no conflict of interest.

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