On boundary primitive manifolds and a theorem of Casson-Gordon

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Abstract

In this paper it is shown that manifolds admitting minimal genus weakly reducible but irreducible Heegaard splittings contain an essential surface. This is an extension of a well known theorem of Casson-Gordon to manifolds with non-empty boundary. The situation for non-minimal genus Heegaard splittings is also investigated and it is shown that boundary stabilizations are stabilizations for manifolds which are boundary primitive.

Keywords

Heegaard splittings, weakly reducible, irreducible, primitive meridian

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1 Introduction

A well known result of Casson-Gordon (see [CG] Theorem 3.1) states that if $M$ is a closed orientable manifold and $(V_1, V_2)$ an irreducible but weakly reducible Heegaard splitting of $M$ then $M$ contains an essential surface of positive genus. This theorem is extremely useful and it was a natural thing to expect an extension of it to manifolds with boundary. Surprisingly, so far, the statement of [CG] Theorem 3.1 does not extend as is to manifolds with boundary and the emerging picture is rather complicated as will become clear from the following theorems. First the positive results:

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Theorem 3.1: Let $M^3$ be an orientable 3-manifold which has a weakly reducible Heegaard splitting of minimal genus, then $M$ contains an essential surface of positive genus.

This raises the question of what can be said about irreducible but weakly reducible Heegaard splittings which are not of minimal genus. We have:

Theorem 4.2: Let $M^3$ be an orientable 3-manifold with a single boundary component of genus $h$. Assume that $M$ has a weakly reducible but irreducible Heegaard splitting $(V_1, V_2)$ of genus $g$, then either $M$ contains an essential surface or $(V_1, V_2)$ is a boundary stabilization and $M$ has a strongly irreducible Heegaard splitting of genus $g - h$.

An extension of the Casson-Gordon theorem to non-minimal genus Heegaard splittings of manifolds with boundary can fail in only one way i.e., if there is a manifold with an irreducible but weakly reducible non-minimal Heegaard splitting, which contains only boundary parallel incompressible surfaces. Such a surface will separate a region homeomorphic to (boundary) $\times I$ and the Heegaard splitting will induce a standard Heegaard splitting on this region (otherwise the original Heegaard splitting will be reducible by [ST]).

A somewhat simple example for the failure of the Casson-Gordon theorem to manifolds with boundary is given in Example 6.1 of [Se], where a genus three weakly reducible and irreducible Heegaard splitting for the complement of the three component trivial open chain link is presented. Since the complement of the link is homeomorphic to a (pair of pants) $\times S^1$ it contains no closed essential surfaces. The Heegaard splitting in the example has all three boundary components contained in one compression body. However a minimal genus splitting for (pair of pants) $\times S^1$ is of genus two with one compression body containing two boundary components and the other one boundary component. The question is still open for manifolds with two or less boundary components.

We can take the opposite point of view: start with an irreducible Heegaard splitting and add a (boundary $\times I$) with a standard Heegaard splitting to it. This operation will be called a boundary stabilization. This does not change the manifold but will give a new amalgamated Heegaard splitting which if weakly reducible will give rise to an incompressible but boundary parallel surface and thus a candidate for a counter example to an extension of the Casson-Gordon theorem. In order to do this the notion of a $\gamma$-primitive
Heegaard splitting is defined in Section 4. This approach also runs into difficulty as we have:

**Theorem 4.5:** If an orientable 3-manifold $M$ has a $\gamma$-primitive Heegaard splitting then every boundary stabilization on the component containing the curve $\gamma$ is a stabilization.

Since $\gamma$-primitive Heegaard splittings are very common weakly reducible but irreducible non-minimal genus Heegaard splittings are hard to find and hence possible counter examples to the remaining cases are also hard to find.

**Remark:** In [LM] Theorem 1.3 we stated an extension of this result to manifolds with boundary but unfortunately the statement and the proof given there are not quite right (see the footnote in Section 2). I would like to thank T. Kobayashi for pointing this out to me. To the best of my knowledge no such extensions of [CG] Theorem 3.1 appeared before [LM], and the problems mentioned above do not affect the other results of that paper. In this paper the situation is corrected.

Extensions to [CG] Theorem 3.1 have been proved by E. Sedgwick [Sc] who proves a similar theorem to Theorem 3.1 and recently by T. Kobayashi in [Ko] Proposition 4.2 which states that, if $M$ has a weakly reducible Heegaard splitting then either the Heegaard splitting is reducible or $M$ contains incompressible surfaces of positive genus (which might be boundary parallel i.e., not essential). The theorems presented above are a strengthening of this result.

## 2 Preliminaries

In this paper it is assumed that all manifolds and surfaces will be orientable unless otherwise specified.

A compression body $V$ is a compact orientable and connected 3-manifold with a preferred boundary component $\partial_+ V$ and is obtained from a collar of $\partial_+ V$ by attaching 2-handles and 3-handles, so that the connected components of $\partial_- V = \partial V - \partial_+ V$ are all distinct from $S^2$. The extreme cases, where $V$ is a handlebody i.e., $\partial_- V = \emptyset$, or where $V = \partial_+ V \times I$, are allowed. Alternatively we can think of $V$ as obtained from $(\partial_- V) \times I$ by attaching 1-handles to $(\partial_- V) \times \{1\}$. An annulus in a compression body will be called a vertical (or
a spanning) annulus if it has its boundary components on different boundary components of the compression body.

Given a manifold $M^3$ a Heegaard splitting for $M$ is a decomposition $M = V_1 \cup V_2$ into two compression bodies $(V_1, V_2)$ so that $V_1 \cap V_2 = \partial V_1 = \partial V_2 = \Sigma$. The surface $\Sigma$ will be call the Heegaard splitting surface.

A Heegaard splitting $(V_1, V_2)$ for a manifold $M$ will be called reducible if there are essential disks $D_1 \subset V_1$ and $D_2 \subset V_2$ so that $\partial D_1 = \partial D_2 \subset \Sigma$.

A Heegaard splitting $(V_1, V_2)$ for a manifold $M$ will be called weakly reducible if there are disjoint essential disks $D_1 \subset V_1$ and $D_2 \subset V_2$. Otherwise it will be called strongly irreducible.

Let $M$ be a 3-manifold which is homeomorphic to a $(surface) \times I$. A Heegaard splitting $(V_1, V_2)$ of $M$ will be called standard if it is homeomorphic to one of the following types:

1. $V_1 \cong (surface) \times [0, \frac{1}{2}]$, $V_2 \cong (surface) \times [\frac{1}{2}, 1]$ and $\partial_+ V_1 = \partial_+ V_2 = (surface) \times [\frac{1}{2}]$

2. If $\{p\} \in (surface)$ is a point then for $0 < \epsilon < \frac{1}{2}$

   $V_1 \cong \left((surface) \times [0, \epsilon]\right) \cup (N(p) \times I) \cup \left((surface) \times [1 - \epsilon, 1]\right)$ and $V_2 = cl(M - V_1)$.

Note that $V_2$ is a regular neighborhood of a once punctured surface and hence is a handlebody and $V_1$ is a compression body with one boundary component $\partial_+$ of genus $2g$ and two boundary components $\partial_-$ of genus $g$, where $g = genus(surface)$. In [ST] it is proved that any irreducible Heegaard splitting of $(surface) \times I$ is homeomorphic to one of the above two types.

A closed surface $F \subset M$ will be called essential if it incompressible and non-boundary parallel.

Given a closed (possibly disconnected) surface $\Sigma \subset M$ and a system of pairwise disjoint non-parallel compressing disks $\Delta$ for $\Sigma$ define (as in [LM]) $\Sigma_0 = \sigma(\Sigma, \Delta)$ to be the surface obtained from $\Sigma$ by compressing along $\Delta$. Let $c(\Sigma) = \sum_i (1 - \chi(\Sigma_i))$, where the sum is taken over all components $\Sigma_i$ of $\Sigma$ which are not 2-spheres. The complexity of the system $\Delta$ is defined to be:

$$c(\Delta) = c(\Sigma) - c(\Sigma_0)$$

For a given Heegaard splitting surface $\Sigma$ for $M$ we will assume that a system of compressing disks $\Delta = \Delta_1 \cup \Delta_2$, where $\Delta_i \subset V_i$, satisfies:

(a) $\Delta_i \neq \emptyset$ for both $i = 1, 2$. i.e., $\Delta$ contains disks on both sides of $\Sigma$.

(b) $\Delta$ is maximal with respect to $c(\Delta)$ over all systems $\Delta$ satisfying (a).
**Definition 2.1.** Let $\Sigma^*$ be the surface $\Sigma_0$ less the 2-sphere components and the components which are contained in $V_1$ or $V_2$. Let $N_0$ denote the closure of a component of $M - \Sigma_0$ which is not a 3-ball and let $N$ denote the closure of a component of $M - \Sigma^*$ which contains $N_0$. By the symmetry between $V_1$ and $V_2$ we can assume that $N_0 \subset V_1 \cup \eta(\Delta_2)$. Now set $U_1 = (V_1 \cap N_0) - \eta(\Sigma \cup \Delta)$ and $U_2 = N - U_1$. By Lemma 1.2 (a) of [LM] the pair $(U_1, U_2)$ is a Heegaard splitting for $N$ and will be called the *induced Heegaard splitting on $N$.*

If $(V_1, V_2)$ is an irreducible Heegaard splitting of $M$ then $(U_1, U_2)$ is an irreducible Heegaard splitting of $N$ by Lemma 1.2 (c) of [LM].

Given two manifolds $M_1$ and $M_2$ with respective Heegaard splittings $(U_1^1, U_2^1)$ and $(U_1^2, U_2^2)$, assume further that there are homeomorphic boundary components $F_1 \subset \partial U_1^1$ and $F_2 \subset \partial U_2^2$. Let $M$ be a manifold obtained by gluing $F_1$ and $F_2$. We can obtain a Heegaard splitting $(V_1, V_2)$ for $M$ by a process called *amalgamation* (see [MS] and [Sc]). The process of amalgamation reconstructs the original Heegaard splitting $(V_1, V_2)$ of $M$ from the Heegaard splittings induced on the components $N_i$ of $M - \Sigma^*$ (see Fig.1).

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1 This definition of $\Sigma^*$ is different from that of [LM] in that it excludes the components of $\Sigma_0$ which are contained in $V_1$ or $V_2$. The problems in the proof of Theorem 1.3 of [LM] emanate from the fact that with the definition given there the component $N$ is not correctly defined. In particular, note that with this modified definition the mistake in the proof of Theorem 1.3 of [LM] disappears. However, as expected, this will not correct the given proof: The point is that, with the modified definition, the statement of Lemma 1.2 (b) becomes wrong (compare also [Ko]).
3 Essential surfaces

In this section we prove two of the main theorems:

**Theorem 3.1.** Let $M^3$ be an orientable 3-manifold which has a weakly reducible Heegaard splitting of minimal genus, then $M$ contains an essential surface of positive genus.

**Proof.** Let $(V_1, V_2)$ be an irreducible and weakly reducible Heegaard splitting of minimal genus for $M$ and $\Sigma, \Delta, \Sigma_0$ and $\Sigma^*$ be as above. Since $\Sigma$ is connected there must be at least one component $S$ of $\Sigma_0$ so that both of $S \cap V_1$ and $S \cap V_2$ are not empty. Since the Heegaard splitting is minimal it is irreducible and since the surface $S$ contains disks from both $V_1$ and $V_2$ it is not a 2-sphere and hence is in $\Sigma^*$. By the proof of Theorem 3.1 in [CG] since $\Delta$ is maximal then $S$ is incompressible.

It remains to show that $S$ is not boundary parallel when $\partial M \neq \emptyset$. If $S$ is boundary parallel then $M - S$ has two components and the closure of one $M_1$, is homeomorphic to $S \times I$. Let $M_0$ be the closure of the other component. Note that $M_0$ is homeomorphic to $M$ and that $(V_1, V_2)$ induces a Heegaard splitting on both of $M_0$ and $M_1$ as in Definition 2.1. Assume that the component of $\partial M$ homeomorphic to $S$ is contained in, say, $V_1$. Since $S \cap V_1 \neq \emptyset$ then $S \cap V_1$ must consist of a single disk as otherwise the induced Heegaard splitting on $M_1$ will be reducible: As all Heegaard splittings of $S \times I$ are standard and since $S \cap V_1 \neq \emptyset$ then the induced Heegaard splitting is of type II in the terminology of [ST] and is reducible if there is more than one such disk. Furthermore if the induced Heegaard splitting on $S \times I$ is reducible then by [LM] Lemma 1.2(c) (see also [Ko] Lemma 4.6) it follows that $(V_1, V_2)$ is reducible in contradiction.

This implies that the genus of the induced Heegaard splitting on $M_1$ is $2 \times g(S)$. The formula for computing the genus of the amalgamated Heegaard splitting from the genus of the Heegaard splitting of the components $M_0, M_1$ is $\bar{g}(M) = \bar{g}(M_1) + \bar{g}(M_0) - g(S)$ where $\bar{g}$ is the genus of the induced Heegaard splitting (see [Sc]). Hence $\bar{g}(M_0)$ must be strictly smaller than $\bar{g}(M)$ in contradiction to the fact that $M_0$ is homeomorphic to $M$ and that $(V_1, V_2)$ is a minimal genus Heegaard splitting of $M$. 

$\square$
Theorem 3.2. Let $M^3$ be an orientable 3-manifold with a single boundary component of genus $h$. Assume that $M$ has a weakly reducible but irreducible Heegaard splitting $(V_1, V_2)$ of genus $g$, then either $M$ contains an essential surface or $M$ has a strongly irreducible Heegaard splitting of genus $g - h$.

Proof. Let $(V_1, V_2)$ be an irreducible and weakly reducible Heegaard splitting of genus $g$ for $M$ and $\Sigma, \Delta, \Sigma_0, \Sigma^*$ and $S$ be as above. If $S$ is essential we are done. So we assume that $S$ is boundary parallel and hence $g(S) = h$. As in the proof of Theorem 3.1 let $M_0$ and $M_1$ be the closure of the components of $M - S$. The Heegaard splitting $(V_1, V_2)$ of $M$ induces a Heegaard splitting of genus $2h$ on $M_1 = S \times I$ and a Heegaard splitting $(U_1, U_2)$ of genus $g - h$ on $M_0$ as in Definition 2.1.

The Heegaard splitting $(U_1, U_2)$ is irreducible as otherwise it follows from [LM] Lemma 1.2(c) (see also [Ko] Lemma 4.6) that $(V_1, V_2)$ is reducible in contradiction. If $(U_1, U_2)$ is weakly reducible then by Theorem 3.1 $M_0$ has an incompressible surface $S_0$. If $S_0$ is essential in $M_0$ we are done since $M \cong M_0$. If $S_0$ is boundary parallel then since $M_0$ has a single boundary component which is homeomorphic to $S$ we have two incompressible surfaces $S$ and $S_0$ which are boundary parallel. Hence the closure of the component of $M - S_0$ is homeomorphic to $S_0 \times I \cong S_0 \times [0, \frac{1}{2}] \cup S \times [\frac{1}{2}, 1]$. The amalgamation of the genus $2h$ Heegaard splittings of $S_0 \times [0, \frac{1}{2}]$ and $S \times [\frac{1}{2}, 1]$ will induce a Heegaard splitting of genus $2h + 2h - h$ on $S_0 \times I$ which is reducible as all Heegaard splittings of a $(surface) \times I$ are standard by [ST]. But this implies as before that $(V_1, V_2)$ is reducible in contradiction. Hence $(U_1, U_2)$ is a strongly irreducible Heegaard splitting of genus $g - h$ of $M \cong M_0$ (see Fig.2).
Remark 3.3. The same argument as in the above proof could be used for manifolds with more than one boundary component. However the statement of the theorem in that case would be very complicated.

4 Boundary primitive manifolds

Given a manifold $M$ with boundary components $\partial M^1, \ldots, \partial M^k$ of corresponding genus $g^1, \ldots, g^k$ and a Heegaard splitting $(V_1, V_2)$ for $M$ of genus $g$ we can always obtain a new Heegaard splitting $(U^i_1, U^i_2)$ of genus $g + g^i, i = 1, \ldots, k$, by gluing a $\partial M^i \times I$ to the $\partial M^i$ boundary component and then amalgamating the standard Heegaard splitting of genus $2g^i$ of $\partial M^i \times I$ with the given Heegaard splitting $(V_1, V_2)$ of $M$ (as indicated in Fig.1).

Definition 4.1. The construction above will be called boundary stabilization on the $i$-th boundary component. If there is a single boundary component or no ambiguity we can just use boundary stabilization.

We can now restate a stronger form of Theorem 3.2 namely:

Theorem 4.2. Let $M^3$ be an orientable 3-manifold with a single boundary component of genus $h$. Assume that $M$ has a weakly reducible but irreducible Heegaard splitting $(V_1, V_2)$ of genus $g$, then either $M$ contains an essential surface or $(V_1, V_2)$ is a boundary stabilization and $M$ has a strongly irreducible Heegaard splitting of genus $g - h$.

Remark 4.3. The Heegaard splitting $(U^i_1, U^i_2)$ is clearly weakly reducible by the construction and the question arises of when is it irreducible? This question is of interest as it was shown in [LM1] that it is relatively easy to find manifolds with an arbitrarily large number of strongly irreducible Heegaard splittings. It is much more difficult to find manifolds with reducible but weakly reducible Heegaard splittings.

We say that an element $x$ in a free group $F_n$ is primitive if it belongs to some basis for $F_n$. A curve on a handlebody $H$ is primitive if it represents a
primitive element in the free group $\pi_1(H)$. An annulus $A$ on $H$ is primitive if its core curve is primitive. Note that a curve on a handlebody is primitive if and only if there is an essential disk in the handlebody intersecting the curve in a single point.

**Definition 4.4.** Let $M$ be a 3-manifold with incompressible boundary components $\partial M^1, \ldots, \partial M^k$. Let $\gamma \subset \partial M^i$ be an essential simple closed curve. A Heegaard splitting $(V_1, V_2)$ of $M$ will be called $(\gamma -$ primitive) if there is an annulus $A$ in $V_1$ or $V_2$, say $V_1$, with $\gamma$ as one boundary component of $A$ and the other a curve on the Heegaard surface $\Sigma$ which intersects an essential disk of $V_2$ in a single point.

**Theorem 4.5.** If an orientable 3-manifold $M$ has a $\gamma$-primitive Heegaard splitting then every boundary stabilization on the component containing the curve $\gamma$ is a stabilization.

**Proof.** Let $\partial M^i$ be the boundary component on which we are going to stabilize. Assume that $\text{genus}(\partial M^i) = g_i$, hence we amalgamate the given genus Heegaard splitting $(W_1, W_2)$ of $M$ with a genus $2g_i$ Heegaard splitting $(U_1, U_2)$ of $\partial M^i \times I$. The Heegaard splitting $(U_1, U_2)$ is standard of type II in the terminology of [ST].

Since the Heegaard splitting of $M$ is $\gamma$-primitive there is some curve $\gamma \subset \partial M^i$ which co bounds an annulus $A'$ so that the other boundary component of $A'$ meets an essential disk $D_2$ of $W_2$, say, in a single point. When $\partial M^i$ is identified with $\partial M^i \times I$ the curve $\gamma$ determines an annulus $A = \gamma \times I \subset \partial M^i \times I$. Assume that the handlebody component of the standard Heegaard $(U_1, U_2)$ splitting of $\partial M^i \times I$ is $U_2$. After an ambient isotopy of $\partial M^i \times I$ we can always assume that the vertical arc $\{p\} \times I$ is contained in $A$ hence $A \cap U_2$ is an essential disk $D_1$ in $U_2$. In the process of amalgamating the Heegaard splittings the handlebody $U_2$ gets glued to $W_1$ and the compression body $U_1$ gets glued to $W_2$. It is possible that $D_1$ and $D_2$ will get identified with disks which are not properly embedded. However this can be corrected by a small isotopy. Now the two disks $D_1 \subset W_1$ and $D_2 \subset W_2$ still intersect in a single point. In the amalgamated Heegaard splitting $(V_1, V_2)$ of $M$ we have that $D_1 \subset V_1$ and $D_2 \subset V_2$ and hence it is a stabilization (see Fig. 3).
Definition 4.6. Let $M$ be a 3-manifold with incompressible boundary components $\partial M^1, \ldots, \partial M^k$. The manifold $M$ will be called boundary primitive or $(\partial - primitive)$ if for each Heegaard splitting of minimal genus of $M$ and for each boundary component $\partial M^i$ there is some curve $\gamma \subset \partial M^i$ for which the Heegaard splitting is $\gamma - primitive$.

In particular if $M = S^3 - N(K)$ where $K \subset S^3$ is a knot and $\gamma = \mu$ is a meridian curve we will say that the Heegaard splitting is $\mu$-primitive and that $E(K)$ is $\mu$-primitive if all its Heegaard splittings of minimal genus are $\mu$-primitive.

Corollary 4.7. If an orientable 3-manifold $M$ is $\partial$-primitive then every boundary stabilization of a minimal genus Heegaard splitting is a stabilization.

Corollary 4.8. Let $K \subset S^3$ be a $\mu$-primitive knot. Assume that $g(E(K)) = g$ and that $E(K)$ has an irreducible but weakly reducible Heegaard splitting of genus $g + 1$ then $E(K)$ contains an essential surface.

Remark 4.9. Knot complements $E(K_i), i = 1, 2$, in $S^3$ which are tunnel number super additive, i.e., $t(K_1 \# K_2) = t(K_1) + t(K_2) + 1$ are examples of manifolds which are not $\mu$-primitive. These knots exist by independent results of Moriah, Rubinstein [MR] and Morimoto, Sakuma and Yokota [MSY].
Example 4.10. In [Ko] it is proved that all 2-bridge knots are $\mu$-primitive.

Example 4.11. Knots in $S^3$ which admit a $(g, 1)$-decomposition (see e.g. [MS1]) have a Heegaard splitting which is $\mu$-primitive.

Example 4.12. Any Heegaard splitting of a knot in a $2n$-plat projection which is induced by a top or bottom minimal tunnels is $\mu$-primitive (see [LM]).

Remark 4.13. All knots $K \subset S^3$ have $g(E(K)) + 1$ $\mu$-primitive Heegaard splittings obtained from a minimal genus Heegaard splitting by stabilizing once i.e., by drilling a small tunnel from $V_2$ and adding it to $V_1$ so that the new 1-handle of $V_2$ intersects a vertical annulus of $V_1$ in a single point.

Remark 4.14. Note that every Heegaard splitting which is a boundary stabilization is $\mu$-primitive. As the disk $D_1$ in $V_1$ will intersect a curve bounding a vertical annulus with a meridian in $V_2$ in a single point.

Remark 4.15. The previous definitions and examples raise the question of whether a given knot complement can have Heegaard splittings which are $\mu$-primitive and others which are not. The answer to this question is affirmative and examples of such knots are torus knots. Given a torus knot $K(p, q) \subset S^3$, where $g.c.d.(p, q) = 1$, then $E(K(p, q))$ has three genus two Heegaard splitting if and only if $p \neq \pm 1 \mod q$ and $q \neq \pm 1 \mod p$ (see [Mo], [BRZ]). One of the Heegaard splittings is obtained by considering a decomposition of $E(K(p, q))$ into two solid tori glued along a $p, q$-annulus and then drilling out a neighborhood of an essential arc from this annulus. Thus the two tori are glued along a disk and form a genus two handlebody the spine of which is composed of two loops $x$ and $y$. The complement of this handlebody is homeomorphic to $T^2 \times I \cup N(essential\ arc)$ which is a genus two compression body. The loops $x$ and $y$ generate the following presentation for the fundamental group of $E(K(p, q))$: $\pi_1(E(K(p, q)) \cong < x, y | x^p = y^q >$. Since $g.c.d.(p, q) = 1$ we can find positive integers $r < q, s < p$ so that $rp - sq = 1$. A curve $\mu$ representing the element $x^r y^s$ is a meridian of $E(K(p, q))$ (see for example Proposition 3.28 of [BZ]). Now choose $p$ and $q$ so that $\min\{r, s\} \geq 2$, The main theorem of [CMZ] shows that $\mu$ is not primitive in $F(x, y)$ and in
particular this Heegaard splitting is not $\mu$-primitive. However the other two Heegaard splittings are $\mu$-primitive as shown in [Mo] and [BRZ] also in [MS1].

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