Annealed estimates on the Green function

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Abstract

We consider a random, uniformly elliptic coefficient field \( a(x) \) on the \( d \)-dimensional integer lattice \( \mathbb{Z}^d \). We are interested in the spatial decay of the quenched elliptic Green function \( G(a; x, y) \). Next to stationarity, we assume that the spatial correlation of the coefficient field decays sufficiently fast to the effect that a logarithmic Sobolev inequality holds for the ensemble \( \langle \cdot \rangle \). We prove that all stochastic moments of the first and second mixed derivatives of the Green function, that is, \( \langle |\nabla_x G(x, y)|^p \rangle \) and \( \langle |\nabla_x \nabla_y G(x, y)|^p \rangle \), have the same decay rates in \( |x - y| \gg 1 \) as for the constant coefficient Green function, respectively. This result relies on and substantially extends the one by Delmotte and Deuschel [8], which optimally controls second moments for the first derivatives and first moments of the second mixed derivatives of \( G \), that is, \( \langle |\nabla_x G(x, y)|^2 \rangle \) and \( \langle |\nabla_x \nabla_y G(x, y)| \rangle \). As an application, we are able to obtain optimal estimates on the random part of the homogenization error even for large ellipticity contrast.

Outline

The outline of this work is as follows: After introducing the discrete setting in Section 1, we present the statistical assumptions and the main result on the annealed moments of the Green function in Section 2. The following two sections contain applications of the main result: We present optimal estimates on the random part of the homogenization error in Section 3 and Section 4 contains an annealed Hölder-estimate in the spirit of De Giorgi. In Section 5 we explain our main assumption, a logarithmic Sobolev inequality, which in particular holds for all independent, identically distributed coefficient fields. Section 6 contains the main ingredients of the proof of the annealed Green function estimates — in particular we recall the result by Delmotte and Deuschel [8]. All proofs are postponed until Section 7.

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1 Discrete uniformly elliptic equations

In this paper we consider linear second-order difference equations with uniformly elliptic, bounded random coefficients of the form
\[ \nabla^*(a\nabla u)(x) = f(x) \quad \text{for all } x \in \mathbb{Z}^d. \] (1)

If there is no danger of confusion, we also write \( \nabla^*a\nabla u \) for \( \nabla^*(a\nabla u) \). In this equation we define the spatial derivatives as follows: Let \( E_d \) denote the set of edges of \( \mathbb{Z}^d \) consisting of all pairs \([x, x + e_i] \) of neighboring vertices with \( x \in \mathbb{Z}^d \), \( i = 1, \ldots, d \), where \( e_1, \ldots, e_d \) is the canonical basis of \( \mathbb{R}^d \). For functions on vertices \( \zeta : \mathbb{Z}^d \to \mathbb{R} \) and functions on edges \( \xi : E_d \to \mathbb{R} \) we set
\[ \nabla \zeta([x, x + e_i]) = \zeta(x + e_i) - \zeta(x), \]
\[ \nabla^* \xi(x) = \sum_{i=1}^d (\xi([x - e_i, x]) - \xi([x, x + e_i])). \]

The spatial derivatives \( \nabla \zeta \) and \( -\nabla^* \xi \) are the discrete gradient and divergence, respectively, on the lattice \( \mathbb{Z}^d \). As our notation suggests, the operators \( \nabla \) and \( \nabla^* \) are adjoint in the sense of
\[ \sum_{e \in E_d} \xi(e)\nabla \zeta(e) = \sum_{x \in \mathbb{Z}^d} \nabla^* \xi(x)\zeta(x). \]

In (1), the coefficient field \( a \) is a field on edges \( a : E_d \to \mathbb{R} \). Consequently \( \nabla^*a\nabla \) is well-defined as an operator on vertex fields \( \mathbb{Z}^d \to \mathbb{R} \). In this paper, we denote edges in \( E_d \) by the letters \( e \) and \( b \) and vertices in \( \mathbb{Z}^d \) by the letters \( x \), \( y \), and \( z \).

Throughout this work we consider coefficient fields \( a : E_d \to \mathbb{R} \) in the space \( \Omega \) of uniformly elliptic coefficient fields, i.e. we let
\[ \Omega := \{ a : E_d \to \mathbb{R} : \lambda \leq a(e) \leq 1 \text{ for all } e \in E_d \} = [\lambda, 1]^{E_d}. \] (2)

Here and below \( \lambda \in (0, 1) \) denotes the ellipticity ratio, which is fixed throughout the paper. This allows, for instance, to interpret \( \nabla^*a\nabla \) as either the operator of a “conductance model” (i.e. the solution of (1) is a potential on a network of resistors) or the generator of a random walk on \( \mathbb{Z}^d \) with jump rates across edges described by \( a \). Note that if we interpreted \( \nabla^*a\nabla \) as a discretization of a continuum operator \( -\nabla \cdot a\nabla \), the coefficient field \( a \in \Omega \) would be diagonal next to being symmetric and uniformly elliptic. In the discrete setting, diagonality is known to be a sufficient (but not necessary) condition for the maximum principle to hold for \( \nabla^*a\nabla u \). The maximum principle is a crucial ingredient for the estimates (23) and (24) on the quenched Green function, on which our results rely.

Our main object is the non-constant coefficient, elliptic, discrete Green function \( G(a; x, x') \) defined through \( \nabla^*a\nabla G(a; \cdot, x') = \delta(\cdot - x') \), where \( \delta \) stands for the discrete version of the Dirac distribution, i.e.
\[ \delta(x) = \begin{cases} 1 & \text{for } x = 0 \\ 0 & \text{otherwise} \end{cases}. \]
We usually drop the argument $a$ and just write $G(x, y)$. Often, it is more convenient to appeal to the distributional characterization:

$$\forall \zeta : \nabla \zeta(e) a(e) \nabla G(e, x') = \zeta(x').$$

(3)

Here and throughout the paper, derivatives are understood to all on the edge variable. We will always work in dimension $d \geq 2$. Dimension $d = 2$ needs a bit more care in terms of the definition of the Green function. Since we are only interested in gradient estimates, this is merely technical and will be ignored here. Sometimes, it is more convenient to think of $\nabla G$ as the limit of $\nabla G_T$ as $T \to \infty$ where $G_T$ is the Green’s function with a massive term in the sense that

$$T^{-1}G_T(\cdot, x') + \nabla^* a \nabla G_T(\cdot, x') = \delta(\cdot - x');$$

(4)

this is the case in the proof of Proposition 1. At other times, it is more convenient to think in terms of an approximation via periodization in the sense of

$$\nabla^* a(x) \nabla G_L(\cdot, x') = \sum_{z \in \mathbb{Z}^d} \delta(\cdot - x' - Lz) - L^{-d};$$

(5)

this is the case in the proof of Lemma 6.

2 Assumptions on the ensemble and main result

We are given a probability measure on the space $\Omega$ of uniformly elliptic, diagonal coefficient fields (endowed with the product topology), cf. (2) in the previous section. Following the convention in statistical mechanics, we call this probability measure an ensemble and denote the associated ensemble average (i.e. the expected value) by $\langle \cdot \rangle$. Functions $\zeta : \Omega \to \mathbb{R}$ will also be called random variables. Note that $\mathbb{Z}^d$ acts on $\mathbb{E}^d$ by translation and we denote by $b + x \in \mathbb{E}^d$ the edge $b \in \mathbb{E}^d$ shifted by $x \in \mathbb{Z}^d$. With this definition, we assume that $\langle \cdot \rangle$ is stationary in the sense that for any shift vector $z \in \mathbb{Z}^d$, the shifted coefficient field $a(\cdot + z) := (\mathbb{E}^d \ni e \mapsto a(e + z)) \in \Omega$ has the same distribution as $a$. We also note that the Green function is shift-invariant or stationary in the sense that

$$G(a(\cdot + z); x, y) = G(a; x + z, y + z).$$

Besides stationarity, the main assumption on the ensemble of coefficients and only probabilistic tool will be a variant of the logarithmic Sobolev inequality (LSI). It constitutes a quantification of ergodicity. In Section 5, we will comment on the LSI and the related spectral gap inequality — there we will also describe the relation between this LSI and the usual LSI.

**Definition 1.** [Logarithmic Sobolev inequality]. Let $\langle \cdot \rangle$ be a (not necessarily stationary) ensemble of coefficients $a$.

We say $\langle \cdot \rangle$ satisfies a logarithmic Sobolev inequality (LSI) with constant $\rho > 0$ if for all random variables $\zeta : \Omega \to \mathbb{R}$, we have that

$$\langle \zeta^2 \log \frac{\zeta^2}{\langle \zeta^2 \rangle} \rangle \leq \frac{1}{2\rho} \langle \sum_{e \in \mathbb{E}^d} \left( \text{osc}_{a(e)} \zeta \right)^2 \rangle,$$

(6)
where the oscillation is to be taken over all values of $a(e) \in [\lambda, 1]$, i.e. over all coefficient fields $\tilde{a} \in \Omega$ that coincides with a outside of $e \in \mathbb{E}^d$ (i.e. $\tilde{a}(b) = a(b)$ for all $b \neq e$). In formulas:

$$
\left( \text{osc } \zeta \right)_{a(e)}(a) = \sup \{\zeta(\tilde{a}) \mid \tilde{a} \in \Omega \text{ s.t. } \tilde{a}(b) = a(b) \forall b \neq e \} - \inf \{\zeta(\tilde{a}) \mid \tilde{a} \in \Omega \text{ s.t. } \tilde{a}(b) = a(b) \forall b \neq e \}.
$$

Note that the difference between the LSI (6) and the usual LSI, see (19), lies in the use of the oscillation instead of the partial derivative $\frac{\partial \zeta}{\partial a(e)}$. The merit of this form is that it is satisfied by any ensemble of independent, identically distributed coefficients $(a(e))_{e \in \mathbb{E}^d}$, cf. Lemma 1 below. Our main result is:

**Theorem 1.** Let $\langle \cdot \rangle$ be stationary and satisfy the LSI (6) with constant $\rho > 0$, see Definition 1. Then for all $1 \leq p < \infty$, $x \in \mathbb{Z}^d$ and $b, b' \in \mathbb{E}^d$, we have that

$$
\langle |\nabla G(b, b')|^{2p} \rangle^{\frac{1}{2p}} \leq C(d, \lambda, \rho, p)(|b - b'| + 1)^{-d}, \quad (7)
$$

$$
\langle |\nabla G(b, x)|^{2p} \rangle^{\frac{1}{2p}} \leq C(d, \lambda, \rho, p)(|b - x| + 1)^{1-d}. \quad (8)
$$

We furthermore let $|b|$ denote the Euclidean distance of the midpoint of the edge $b$ from the origin and $|b - b'|$ the distance between the midpoints of the two edges $b$ and $b'$. Recall that $b + x$ denotes the edge $b$ shifted by $x$. Here in the sequel, $C(d, \lambda, \rho, p)$ stands for a generic constant that only depends on dimension $d \geq 2$, on the ellipticity ratio $\lambda > 0$, on the LSI constant $\rho > 0$, and on the exponent of integrability $p < \infty$.

We defer the proof of Theorem 1 until Subsection 7.4. Clearly, the spatial decay rates in Theorem 1 are optimal, since those are the decay rates of the constant coefficient Green function, see for instance [17, Theorem 4.3.1]. Note that we may assume without loss of generality that $x = 0$ in (8) since stationarity of $\langle \cdot \rangle$ and $G$ implies

$$
\langle |\nabla G(a; b, x)|^{2p} \rangle^{\frac{1}{2p}} = \langle |\nabla G(a(-x); b, x)|^{2p} \rangle = \langle |\nabla G(a; b - x, 0)|^{2p} \rangle.
$$

An interesting aspect of Theorem 1 is the following: The *quenched* versions of (7) and (8) are false, i.e. the *uniform* in $a$ and *point-wise* in $x$ estimates $|\nabla G(a; e, b)| \leq C(d, \lambda)(|e - b| + 1)^{-d}$ and $|\nabla G(a; e, 0)| \leq C(d, \lambda)(|e| + 1)^{d-1}$ do *not* hold (while suitably *spatially* averaged versions of both estimates do hold uniformly in $a$); see our discussion in Section 4 below.

An easy consequence is the following generalized variance estimate on $G$ itself:

**Corollary 1.** Let $\langle \cdot \rangle$ be as in Theorem 1. Then we have that

$$
\langle |G(x, 0) - \langle G(x, 0) \rangle|^{2p} \rangle^{\frac{1}{2p}} \leq C(d, \lambda, \rho, p) \begin{cases} (|x| + 1)^{2(1-d)} & d > 2 \\ (|x| + 1)^{-2} \log(|x| + 2) & d = 2 \end{cases}
$$

for all $x \in \mathbb{Z}^d$ and $1 \leq p < \infty$.

The proof of Corollary 1 will be given in Subsection 7.5.

**Remark 1.** We note that the estimate in Corollary 1 is optimal in the scaling of the spatial decay. This can be seen by developing to leading order in a small ellipticity ratio $1 - \lambda \ll 1$. We expand upon this argument (for the special case of $p = 1$) in Subsection 7.6 after the proof of Corollary 1.
3 Homogenization error

In the same vein as Corollary 1, Theorem 1 allows to give optimal estimates on the random part of the homogenization error. These extend the results by Conlon and Naddaf [5, Theorem 1.2, Theorem 1.3] from small ellipticity ratio (i.e. $1 - \lambda \ll 1$) to arbitrary ellipticity ratio. For the “strong error” (see below for an explanation of this wording) [5, Theorem 1.2] in $d > 3$, this was already achieved by Gloria [11, Theorem 2]. For all other cases, our result appears to be new. Let us be more precise: For a coefficient field $a : \mathbb{R}^d \to \mathbb{R}$ and a right-hand side $f : \mathbb{Z}^d \to \mathbb{R}$ we consider the solution $u : \mathbb{Z}^d \to \mathbb{R}$ of

$$\nabla^* a \nabla u = f \quad \text{on } \mathbb{Z}^d. \tag{10}$$

In order for (10) to have a unique solution that decays (i.e. $\lim_{|x| \to \infty} u(x) = 0$), we assume for simplicity that $f$ is compactly supported (and furthermore is of zero spatial average in the case of $d = 2$). By the random part of the homogenization error, we understand the “fluctuations” $u(x) - \langle u(x) \rangle$. These are expected to be small (w. r. t. the size of $u(x)$ itself) if $f(x)$ varies only slowly w. r. t. to the lattice spacing. In our notation, the lattice spacing is unity, so a natural model for a right-hand side that has a large characteristic scale $L \gg 1$ is given by $f(x) = L^{-2} \hat{f}(\hat{x})$ for some bounded and compactly supported “mask” $\hat{f}(\hat{x}), \hat{x} \in \mathbb{R}^d$. The scaling $L^{-2}$ of the amplitude of $f$ is motivated as follows: In the rescaled variables $\hat{x}$, (10) now assumes the suggestive form of

$$(\nabla^*_\epsilon a(\hat{x}) \nabla_{\epsilon} u)(\hat{x}) = \hat{f}(\hat{x}) \quad \text{for all } \hat{x} \in \epsilon \mathbb{Z}^d, \tag{11}$$

where $\epsilon := L^{-1}$ is the ratio of the lattice spacing to the characteristic scale of the r.-h. s. and where $\nabla_{\epsilon}$ denote finite differences for the rescaled lattice $\epsilon \mathbb{Z}^d$ (i.e. $\nabla_{\epsilon} u([\hat{x}, \hat{x} + \epsilon e_i]) = \epsilon^{-1}(u(\hat{x} + \epsilon e_i) - u(\hat{x}))$).

The size of the fluctuations will be measured in two different ways.

- **Corollary 2**: Here, the fluctuations will be controlled in a **strong** way in the sense that we estimate the (discrete) $\ell^p(\mathbb{Z}^d)$-norm $(\sum_{x} |u - \langle u \rangle|^p)^{1/p}$ of the fluctuations. This will be done for arbitrary stochastic moments (the role played by $rp$). Corollary 2 is the generalization of [5, Theorem 1.2] as well as [11, Theorem 2]. For our model right-hand side, $f(x) = \epsilon^2 \hat{f}(\epsilon x)$ with bounded and compactly supported $\hat{f}$, the fluctuations are (up to a logarithmic correction for $d = 2$) of the order of $\epsilon$ in this measure, see (15).

- **Corollary 3**: Here, the fluctuations will be controlled in a **weak** way in the sense that we only estimate spatial averages $\sum_{x} (u - \langle u \rangle)q$ of the fluctuations, with deterministic averaging function $g(x)$. Again, this will be done for arbitrary stochastic moments (the role played by $r$). Corollary 3 is the generalization of [5, Theorem 1.3]. For our model right-hand side $f(x) = \epsilon^2 \hat{f}(\epsilon x)$ with bounded and compactly supported $\hat{f}$, and an averaging function of the form $g(x) = \hat{g}(\epsilon x)$ with bounded and compactly supported $\hat{g}$, the fluctuations are $O(\epsilon^{d/2})$ in this measure, see (16). (Here, there is no logarithmic correction even for $d = 2$.)
Corollary 2. Let $\langle \cdot \rangle$ be as in Theorem 1; for compactly supported right-hand side $f(x)$, consider the decaying solution $u(x)$ to (10). Let the spatial integrability exponents $2 \leq p < \infty$ and $1 < q < \infty$ be related through $\frac{1}{q} = \frac{1}{p} + \frac{1}{2}$. 

In case of $d > 2$, we have for all $r < \infty$:

$$
\left\langle \left( \sum_x |u - \langle u \rangle|^p \right)^{\frac{r}{p}} \right\rangle \leq C(d, \lambda, \rho, p, r) \left( \sum_x |f|^q \right)^{\frac{r}{q}}.
$$

(12)

In case of $d = 2$, we additionally require $p > 2$ (so that $q > 1$) and that $f$ is supported in $\{x : |x| \leq R\}$ for some $R \geq 1$. Then we have for all $r < \infty$:

$$
\left\langle \left( \sum_{|x| \leq R} |u - \langle u \rangle|^p \right)^{\frac{r}{p}} \right\rangle \leq C(\lambda, \rho, p, r) (\log^2 R) \left( \sum_x |f|^q \right)^{\frac{r}{q}}.
$$

(13)

Corollary 3. Let $\langle \cdot \rangle$ be as in Theorem 1; for compactly supported right-hand side $f(x)$, consider the decaying solution $u(x)$ to (10). Let the averaging function $g(x)$ be compactly supported. Let the two integrability exponents $1 < q, \tilde{q} < d$ be related by $\frac{1}{q} + \frac{1}{\tilde{q}} = \frac{1}{d} + \frac{1}{2}$. Then we have for all $r < \infty$:

$$
\left\langle \left( \sum_x (u - \langle u \rangle)g \right)^{\frac{r}{p}} \right\rangle \leq C(d, \lambda, \rho, \tilde{q}) \left( \sum_x |f|^q \right)^{\frac{r}{q}} \left( \sum_x |g|^\tilde{q} \right)^{\frac{r}{\tilde{q}}}.
$$

(14)

Corollaries 2 and 3 will be proved in Subsections 7.7. For the convenience of the reader, we express the results of both corollaries in terms of the rescaled variable $\hat{x} = \epsilon x$, the model right-hand side $f(x) = e^{\hat{x}} \hat{f}(cx)$ and the model averaging function $g(x) = e^{\hat{x}} \hat{g}(cx)$; we also rewrite the solution itself in terms of $u(x) = \hat{u}_\epsilon(cx)$. In this notation, (12) (multiplied by $\epsilon^{d/p}$) turns into

$$
\left\langle \left( \epsilon^d \sum_{\hat{x} \in \mathbb{Z}^d} |\hat{u}_\epsilon - \langle \hat{u}_\epsilon \rangle|^p \right)^{\frac{r}{p}} \right\rangle \leq C(d, \lambda, \rho, p, r) \epsilon \left( \epsilon^d \sum_{\hat{x} \in \mathbb{Z}^d} |\hat{f}|^q \right)^{\frac{1}{q}} \leq C(d, \lambda, \rho, r, \hat{f}) \epsilon.
$$

(15)

Note that this can be interpreted as the discrete version of

$$
\left\langle \left( \int_{\mathbb{R}^d} |\hat{u}_\epsilon - \langle \hat{u}_\epsilon \rangle|^p d\hat{x} \right)^{\frac{r}{p}} \right\rangle \leq C(d, \lambda, \rho, p, r) \epsilon \left( \int_{\mathbb{R}^d} |\hat{f}|^q d\hat{x} \right)^{\frac{1}{q}},
$$

which highlights the $O(\epsilon)$-nature of the “spatially strong” error.

Likewise, (14) turns into

$$
\left\langle \left( \epsilon^d \sum_{\hat{x} \in \mathbb{Z}^d} (\hat{u}_\epsilon - \langle \hat{u}_\epsilon \rangle) \hat{g} \right)^{\frac{r}{p}} \right\rangle \leq C(d, \lambda, \rho, r, \hat{f}, \hat{g}) \epsilon^{\frac{d}{2}}.
$$

(16)
As above, this can be seen as the discrete version of
\[
\left\langle \int_{\mathbb{R}^d} (\tilde{u}_x - \langle \tilde{u}_x \rangle \tilde{g})^r d\tilde{x} \right\rangle^{\frac{1}{r}} \leq C(d, \lambda, \rho, r) \epsilon^{\frac{2}{d}} \left( \int_{\mathbb{R}^d} |\tilde{f}|^q d\tilde{x} \right)^{\frac{1}{q}} \left( \int_{\mathbb{R}^d} |\tilde{g}|^q d\tilde{x} \right)^{\frac{1}{q}},
\]
uncovering the \(O(\epsilon^{d/2})\)-nature of the “spatially weak” error.

Let us make a couple of further more detailed remarks related to Corollaries 2 and 3. In case of Corollary 2 and \(d = 2\), we can use Hölder’s inequality to establish an estimate also for \(p = 2\). However, in that case we pay the price of an arbitrarily small power of \(R\) on the right-hand side of (13). We also note that the requirement that \(f\) has compact support and that \(u\) decays can be weakened: All we need is the Green function representation \(u(x) = \sum_y G(x, y) f(y)\). We conclude by pointing out that our argument does not require any smoothness assumptions on \(\tilde{f}(\tilde{x})\) and \(\tilde{g}(\tilde{x})\) beyond (uniform) boundedness to obtain (15) and (16).

The central limit theorem (CLT) scaling \(O(\epsilon^{d/2})\) of the weak error seems to suggests that \(u_\epsilon(x)\) behaves like a random field of amplitude \(O(1)\) and integrable correlations. In fact, this is misleading, as can be seen by distinguishing the scale \(\frac{1}{\epsilon}\) on which \(f\) varies from the scale \(1 \ll \frac{1}{\rho} \ll \frac{1}{\epsilon}\) on which we take the spatial average with help of the function \(g\). If Corollary 3 were true in the limiting case of \(q = d\) (which is not the case since the Hardy-Littlewood-Sobolev inequality in Step 3 in the proof of Corollary 3 requires \(q < d\)), we would obtain
\[
\left\langle \left| \epsilon^d \sum_{x \in \mathbb{Z}^d} (\tilde{u}_\epsilon \delta_x(x) - \langle \tilde{u}_\epsilon \delta_x(x) \rangle) \tilde{g}(\delta x) \right|^{r} \right\rangle^{\frac{1}{r}} \leq C(d, \lambda, \rho, r, \tilde{f}, \tilde{g}) \epsilon^{\frac{d}{2}} - 1.
\]
This refined estimate does suggest that \(\tilde{u}_\epsilon \delta_x(x)\) behaves like a random field of amplitude \(O(\epsilon)\) and correlations that decay like the Green’s function:
\[
\left| \langle (\tilde{u}_\epsilon \delta_x(x) - \langle \tilde{u}_\epsilon \delta_x(x) \rangle) (\tilde{u}_\epsilon \delta_y(y) - \langle \tilde{u}_\epsilon \delta_y(y) \rangle) \rangle \right| \leq C(d, \lambda, \rho, \tilde{f}) \epsilon^2 (|x - y| + 1)^{2 - d}
\]
for all \(x, y \in \mathbb{Z}^d\). This scaling is natural, since it would follow from the (higher-order, two-scale) expansion \(\tilde{u}_\epsilon(\tilde{x}) \approx u_{\text{hom}}(\tilde{x}) + \epsilon \sum_{k=1}^{d} \phi_k(\frac{\tilde{x}}{\epsilon}) \partial_k u_{\text{hom}}(\tilde{x})\) and the expected — but unproven — estimate on the covariance of this corrector:
\[
|\langle \phi_k(x) \phi_k(y) \rangle| \leq C(d, \lambda, \rho) (|x - y| + 1)^{2 - d}
\]
for all \(x, y \in \mathbb{Z}^d\). In the above, the function \(\phi_k\) is the corrector in direction \(e_k\) (which is an \(a\)-harmonic function of affine behavior on large scales) and \(u_{\text{hom}}\) is the solution to the elliptic equation with homogenized coefficients. We remark here that the above-mentioned expansion for \(u_\epsilon\) was recently quantified by Gloria, Neukamm and the second author [14] using Theorem 1. Indeed, there it is shown that the error in an \(H^1\)-norm in space and \(L^2\)-norm in probability for \(u_\epsilon - u_{\text{hom}} - \epsilon \sum_{k=1}^{d} \phi_k(\frac{\tilde{x}}{\epsilon}) \partial_k u_{\text{hom}}\) is still of order \(\epsilon\), cf. (15). In order to obtain this result, the authors also treat the so-called systematic error, which is the difference between \(\langle u_\epsilon \rangle\) and \(u_{\text{hom}}\).

A more traditional CLT-scaling has been established for the energy density. For \(g = f\), the weak measure of fluctuations turns into a measure of fluctuations of the energy:
\[
\sum_x (u - \langle u \rangle) g = \sum_x a(\nabla u)^2 - \left\langle \sum_x a(\nabla u)^2 \right\rangle.
\]
If we set $u = \phi_k$, then the (stationary) energy density defines the homogenized diffusion coefficient. In [12, Theorem 2.1], it is shown that in the case of independent, identically distributed (i. i. d.) coefficients, the energy density of the corrector has CLT scaling in the sense that spatial averages behave as if the energy density was independent from site to site; in [13, Proposition 7], that result has been generalized to ensembles that only satisfy a spectral gap condition. The scaling result has been substantially sharpened for i. i. d. ensembles: In this situation, the fluctuations of the energy of the corrector become more and more Gaussian as the box over which the spatial average is taken increases. The latter result has been obtained by three different techniques: Nolen [25] gives a quantitative estimate based on a differential characterization of Gaussian distributions (second-order Poincaré inequality) and relies on the corrector estimates from [12, Theorem 2.1]. Biskup, Salvi, and Wolff [3] obtain a more qualitative result using a Martingale decomposition of the spatially averaged energy density (their result assumes small ellipticity contrast $1 - \lambda \ll 1$, but presumably could be extended using the results of [13]). Rossignol [27] in turn uses an orthogonal decomposition of the space of coefficients (Walsh decomposition).

4 Relation to De Giorgi’s approach to elliptic regularity

While our result heavily relies on the celebrated regularity theory for scalar elliptic operators, connected with the names of De Giorgi, Nash, and Moser, it also gives a new perspective on these results. We will specify the input from regularity theory, namely Nash’s (upper) bounds on the parabolic Green function, in the next section. We now address what we see as a new perspective on these results, namely on De Giorgi’s result on Hölder continuity of $a$-harmonic functions.

An elementary consequence of the mean value property is the following Liouville principle: Harmonic functions that grow sub-linearly must be constant. This holds for the constant-coefficient Laplacian both on $\mathbb{R}^d$ and on $\mathbb{Z}^d$, but is no longer true for variable coefficients, even if they are uniformly elliptic. Indeed, a well-known example [1, Corollary 16.1.5] shows that for any $\alpha > 0$, there exists an explicit coefficient field $a^2 \leq a(z) \leq 1$ such that $u(z) = \Re(|z|^{\alpha-1}z)$ is $a$-harmonic in $z \ni \mathbb{C} \cong \mathbb{R}^2$. We believe that this example can be adapted to the lattice $\mathbb{Z}^2$ (provided the condition of diagonality is relaxed to the condition that the discrete maximum principle is valid, a setting to which our results presumably can be extended). A celebrated result of De Giorgi [6, Theorem 2] states that this is the worst-case scenario: For any dimension $d$ and any ellipticity ratio $\lambda$, there exists an exponent $a_0(d, \lambda) > 0$ with the following property: For any field of coefficients $\lambda \leq a(x) \leq 1$ and any $a$-harmonic function $u(x)$, a bound of the form $|u(x)| \leq C|x|^{a_0}$ for $|x| \geq 1$ implies that $u$ is constant. This result holds both in $\mathbb{R}^d$ and in $\mathbb{Z}^d$ [7, Proposition 6.2]. In this sense, while it is no longer true that “sub-linear implies constant”, it remains true that “very sub-linear implies constant”.

De Giorgi’s result is in fact more quantitative and can be rephrased as an inner regularity result in terms of Hölder continuity with Hölder exponent $a_0$. For any harmonic function $u(x)$ on $\{x : |x| \leq R\}$, the $a_0$-Hölder modulus of continuity
at zero is estimated by the supremum:

\[ \sup_{x:|x| \leq R} \frac{|u(x) - u(0)|}{|x|^\alpha} \leq C(d, \lambda) R^{-\alpha} \sup_{x:|x| \leq R} |u(x)|. \]

To contrast De Giorgi's result with our result below, let us rephrase it as follows:

\[ \forall \lambda \leq a(x) \leq 1, \forall R < \infty : \sup_u \frac{\sup_{|x| \leq R} |u(x) - u(0)|}{\sup_{|x| \leq R} |u(x)|} \leq C(d, \lambda), \quad (17) \]

where the outer supremum is taken over all \( u(x) \) that satisfy \( \nabla^* a \nabla u = 0 \) in \( \{ x : |x| \leq R \} \).

In this context, we will show in Subsection 7.8 that Theorem 1 has the following Corollary.

**Corollary 4.** For all \( 0 < \alpha < 1, p < \infty, \) and \( R < \infty, \) we have

\[ \left( \sup_u \frac{\sup_{|x| \leq R} |u(x)|}{\sup_{|x| \leq R} |u(x)|} \right)^p \leq C(d, \lambda, \alpha, p), \quad (18) \]

where the outer supremum is taken over all \( u(x) \) that satisfy \( \nabla^* a \nabla u = 0 \) in \( \{ x : |x| \leq R \} \).

Loosely speaking, Corollary 4 implies that for “most” coefficient fields, an \( a \)-harmonic function \( u(x) \) is H"older continuous with an exponent arbitrarily close to one. More precisely, the modulus of near-Lipschitz continuity of \( u(x) \) in some large ball is estimated by its supremum in the concentric ball of twice the radius with a “quenched” constant \( C(a) \) with all moments bounded independently of the radius. Indeed, with the same proof the numerator in Corollary 4 can be chosen as the full H"older-norm on \( \{ x : |x| \leq \frac{R}{2} \} \). Furthermore it is straightforward to extend the result to functions \( \nabla^* a \nabla u = f \) if we include the \( \ell^d \)-norm of \( f \) over \( \{ x : |x| \leq R \} \) in the denominator. The quantitative result of Corollary 4 has the Liouville principle as an easy corollary: For almost every \( a \), any sublinear \( a \)-harmonic function \( u \) must be constant. However, surprisingly for us, the (qualitative) Liouville principle holds without any assumption on the ensemble \( \langle \cdot \rangle \) besides stationarity! This is established in a very inspiring paper [2, Theorem 3]. The main ingredients for the short and elegant argument are

- The “annealed” estimate \( \langle \sum_x |x|^2 G(t, x, 0) \rangle \leq C t \) on the second moments of the parabolic Green function \( G(a; t, x, y) \) short \( G(t, x, y) \) (cf. [2, (SBD)]), see Subsection 6 below for the definition of \( G \), which in our uniformly elliptic context even holds in its stronger “quenched” version, that is, \( \sum_x |x|^2 G(t, x, 0) \leq C t \).

- The annealed estimate \( -\langle \sum_x G(t, x, 0) \log G(t, x, 0) \rangle \leq C \log t \) on the spatial entropy of the parabolic Green function \( G \) (cf. [2, p.12]), which in our context is an immediate consequence of the second moments estimate. This ingredient is shown to imply the following annealed continuity property of \( G \):

\[ \left( \sum_y G(1, 0, y) \sum_x \frac{G(t, 0, x) - G(t-1, y, x)}{G(t, 0, x) + G(t-1, y, x)} \right)^2 \leq \frac{C}{t} \]

for some sequence \( t \to \infty \).
5 Logarithmic Sobolev inequality

In the following, we give a more detailed description of our use of the logarithmic Sobolev inequality and prove that any i. i. d. ensemble satisfies Definition 1. LSI substitutes the spectral gap inequality (SG) in prior work on quantitative stochastic homogenization. SG has been introduced into the field by Naddaf and Spencer [20, Theorem 1] (in form of the Brascamp-Lieb inequality) and used most recently in [12, Lemma 2.3] in an indirect way and in [13] explicitly. The LSI follows like SG from the property that there is an integrable fall-off of correlations in the sense of a uniform mixing condition à la Dobrushin-Shlosman, see for instance [28, Theorem 1.8 c)] for a discrete setting. Both SG and LSI quantify ergodicity of the ensemble, see for instance the discussion in [13, Chapter 4]. Recall that the usual LSI in this setting (with continuum derivative) would read

\[
\left\langle \xi^2 \log \frac{\xi^2}{\langle \xi^2 \rangle} \right\rangle \leq \frac{1}{2\rho} \left\langle \sum_{e \in \mathbb{E}^d} \left( \frac{\partial \xi}{\partial a(e)} \right)^2 \right\rangle.
\]

(19)

In the LSI of Definition 1, we have simply changed the derivative by an oscillation in order to capture ensembles whose marginal distribution contains atoms, as we shall explain now.

Both SG and LSI are based on the notion of a vertical derivative (here, the oscillation) that defines a Dirichlet form and thus a reversible dynamics, namely Glauber dynamics, on the space of coefficient fields (the word “vertical” is used to distinguish this derivative from the “horizontal” derivative naturally arising in stochastic homogenization, but not used in this paper). In the earlier work on stochastic homogenization and motivated by field theories, see [21], the version of SG that is based on the continuum vertical derivative (as on the r.h.s. of (19)) has been used [20]. However, this assumption rules out the natural example of coefficients with a single-site distribution that only assumes a finite number of values (Bernoulli). Hence in order to treat arbitrary single-site distributions, we are forced to consider the version of LSI found in Definition 1. A SG inequality based on the oscillation was already considered in [12, Lemma 2.3].

The LSI has been of great use in the setting of stochastic processes and diffusion semi-groups, for the first time introduced in generality by Gross [15]. It implies SG as well as concentration of measure [18, Chapter 5] and is equivalent to the notion of hyper-contractivity, see [15, Theorem 1] or [16, Theorem 4.1]. Incidentally, hyper-contractivity was first observed in the Gaussian context by Nelson [23], see [24] for an improved result. It is thus the older notion and in fact motivated the (somewhat implicit) introduction of LSI by Federbush [10]. We refer to [16] for a recent exposition on LSI.

The result of this section is that any independent, identically distributed coefficient-field satisfies the LSI (6) of Definition 1.

Lemma 1. Consider an ensemble \( \langle \cdot \rangle \) of i. i. d. coefficients on each edge with arbitrary marginal distribution on \([\lambda, 1]\). Then (6) holds, i.e.

\[
\left\langle \xi^2 \log \frac{\xi^2}{\langle \xi^2 \rangle} \right\rangle \leq \frac{1}{2\rho} \left\langle \sum_{e \in \mathbb{E}^d} \left( \frac{\text{osc}_{a(e) \in [\lambda, 1]} \xi}{\xi} \right)^2 \right\rangle.
\]
for all functions $\zeta$ of the coefficient field $a$. The constant $\rho$ may be taken to be $\rho = \frac{1}{8}$.

Lemma 1 is an immediate consequence of the following two lemmas. The first one shows that any single-edge distribution on $[\lambda, 1]$ satisfies the LSI in Definition 1.

**Lemma 2.** Let $\langle \cdot \rangle$ be any distribution on $[\lambda, 1]$. Then we have that

$$\langle \zeta^2 \log \frac{\zeta^2}{\langle \zeta^2 \rangle} \rangle \leq \frac{1}{2\rho} \left( \text{osc } \zeta \right)^2$$

for all functions $\zeta : [\lambda, 1] \to \mathbb{R}$. In fact, the constant $\rho = \frac{1}{8}$ will do.

The next lemma shows that the LSI in Definition 1 satisfies the tensorization principle.

**Lemma 3.** Let $\langle \cdot \rangle$ be an ensemble consisting of independent distributions on the edges such that each single-edge distribution satisfies the LSI (20) with the same constant $\rho$. Then $\langle \cdot \rangle$ itself satisfies the LSI (6) with constant $\rho$.

The proofs of Lemmas 2 and 3 will be given in Subsection 7.9.

### 6 Main ingredients of the proof

Loosely speaking, our approach consists in upgrading the (optimal) annealed estimates of Delmotte and Deuschel [8, Theorem 1.1] in terms of the integrability exponent $p$.

**Proposition 1.** [Delmotte and Deuschel]. Let $\langle \cdot \rangle$ be stationary. Then we have for all $b, b' \in \mathbb{Z}^d$:

$$\langle |\nabla \nabla G(b, b')| \rangle \leq C(d, \lambda, \alpha)(|b - b'| + 1)^{-d},$$

$$\langle |\nabla G(b, x)| \rangle \leq C(d, \lambda)(|b - x| + 1)^{-d-a}.$$  \hspace{1cm} (21, 22)

More precisely, we refer to the estimates (1.4) and (1.5a) in [8, Theorem 1.1] on the discrete parabolic Green function $G(t, x, y) = G(a; t, x, y)$ (i.e. the solution of $\partial_t G(t, \cdot, y) + \nabla^* a \nabla G(t, \cdot, y) = 0$ with $G(t = 0, x, y) = \delta(x - y)$) that in our notation imply for any weight exponent $\alpha < \infty$:

$$\langle |\nabla \nabla G(t, b, b')| \rangle \leq C(d, \lambda, \alpha)(t + 1)^{-\frac{d}{2} - \frac{d}{2} - \frac{\alpha}{2}},$$

$$\langle |\nabla G(t, b, x)| \rangle \leq C(d, \lambda, \alpha)(t + 1)^{-\frac{d}{2} - \frac{d}{2} - \frac{\alpha}{2}}.$$  \hspace{1cm} (23, 24)

(In fact, [8] establishes (23) and (24) with exponentially decaying weights instead of just algebraically decaying ones.) Since the elliptic Green function can be inferred from the parabolic one via $G(x, y) = \int_0^\infty G(t, x, y)dt$, these estimates imply (21) and (22) (by fixing some $\alpha > d$ and performing the change of variables $\tilde{t} = |x|^{-2}(t + 1)$). Actually, [8] establishes (24) and thus (22) in the stronger form where the $L^1$-norm $\langle |\cdot| \rangle$ is replaced by the $L^2$-norm $\langle |\cdot|^2 \rangle^{1/2}$:

$$\langle |\nabla G(t, b, x)| \rangle^{1/2} \leq C(d, \lambda, \alpha)(t + 1)^{-\frac{d}{2} - \frac{d}{2} - \frac{\alpha}{2}}.$$  \hspace{1cm} (25)
Let us point out that the *spatially point-wise annealed* estimates (23) and (24) are consequences of the following *spatially averaged quenched* estimates

\[
\sum_{x \in \mathbb{Z}^d} \left( \frac{|x|^2}{t+1} + 1 \right) \frac{2}{\alpha} \left| G(t, x, 0) \right|^2 \leq C(d, \lambda, \alpha)(t+1)^{-\frac{d}{2}},
\]

(25)

\[
\sum_{b \in \mathbb{E}^d} \left( \frac{|b|^2}{t+1} + 1 \right) \frac{2}{\alpha} \left| \nabla G(t, b, 0) \right|^2 \leq C(d, \lambda, \alpha)(t+1)^{-\frac{d}{2}-1}.
\]

(26)

The first estimate (25) is the (upper, off-diagonal part of the) celebrated Nash estimate [22, Appendix]. The discrete case was treated in full generality in [4, Corollary 3.28]. The second estimate (26) is a consequence of the first one. For an elementary proof of both, we refer to [13, Lemmas 24 and 25], with the Nash inequality as only noteworthy ingredient. Let us point out how (26) is implies (23). Using the semi-group property in form of \( \nabla \nabla G(t, b, b') = \sum_y \nabla G(\frac{t}{2}, b, y) \nabla G(\frac{t}{2}, y, b') \) we obtain by the triangle inequality for the weight, Cauchy Schwarz in \( \sum_y \) and the symmetry of \( G(t, x, y) \) in \( x \) and \( y \):

\[
\left( \frac{|b-b'|^2}{t+1} + 1 \right) \frac{2}{\alpha} \left| \nabla \nabla G(t, b, b') \right| \leq \sum_{y \in \mathbb{Z}^d} \left( \frac{2|b-y|^2}{t+1} + 1 \right) \frac{2}{\alpha} \left| \nabla G(\frac{t}{2}, b, y) \right| \left( \frac{2|b-y|^2}{t+1} + 1 \right) \frac{2}{\alpha} \left| \nabla G(\frac{t}{2}, y, b') \right|
\]

\[
\leq \left( \sum_{y \in \mathbb{Z}^d} \left( \frac{2|b-y|^2}{t+1} + 1 \right) \frac{2}{\alpha} \left| \nabla G(\frac{t}{2}, b, y) \right| \right)^2 \quad \times \quad \sum_{y \in \mathbb{Z}^d} \left( \frac{2|b-y|^2}{t+1} + 1 \right) \frac{2}{\alpha} \left| \nabla G(\frac{t}{2}, y, b') \right| \right)^{\frac{1}{2}}.
\]

Note that the right-hand side of the last inequality does not allow for application of (26), since the sum is not in the variable in which the derivative is taken. However, we take the expectation, use the Cauchy-Schwarz inequality in \( \langle \cdot \rangle \) and stationarity and symmetry in form of \( \langle |\nabla G(\frac{t}{2}, b, y)|^2 \rangle = \langle |\nabla G(\frac{t}{2}, b - y, 0)|^2 \rangle \) to obtain

\[
\left( \frac{|b-b'|^2}{t+1} + 1 \right) \frac{2}{\alpha} \left\langle |\nabla \nabla G(t, b, b')| \right\rangle \leq \left( \sum_{y \in \mathbb{Z}^d} \left( \frac{2|b-y|^2}{t+1} + 1 \right) \frac{2}{\alpha} \left\langle |\nabla G(\frac{t}{2}, b, y)|^2 \right\rangle \right)^{\frac{1}{2}}
\]

\[
\times \sum_{y \in \mathbb{Z}^d} \left( \frac{2|b-y|^2}{t+1} + 1 \right) \frac{2}{\alpha} \left\langle |\nabla G(\frac{t}{2}, b', y)|^2 \right\rangle \right)^{\frac{1}{2}}
\]

\[
\leq \left( \sum_{y \in \mathbb{Z}^d} \left( \frac{2|b-y|^2}{t+1} + 1 \right) \frac{2}{\alpha} \left\langle |\nabla G(\frac{t}{2}, b - y, 0)|^2 \right\rangle \right)^{\frac{1}{2}}
\]

\[
\times \left( \sum_{y \in \mathbb{Z}^d} \left( \frac{2|b'-y|^2}{t+1} + 2 \right) \frac{2}{\alpha} \left| \nabla G(\frac{t}{2}, b' - y, 0) \right|^2 \right)^{\frac{1}{2}}.
\]
We now see that (26) implies (23). The estimate (24) is derived via the semigroup property in form of $\nabla G(t,b,x) = \sum_y \nabla G(\frac{t}{2},b,y)G(\frac{t}{2},y,x)$ from the combination of (25) and (26) by an analogous argument.

Note that the estimates of Proposition 1 make no assumptions on the ensemble besides stationarity. In order to pass from Proposition 1 to Theorem 1, we need the assumption on the ensemble from Definition 1. In fact, LSI enters only through the following lemma, which we shall prove in Subsection 7.1.

**Lemma 4.** Let $\langle \cdot \rangle$ satisfy LSI (6) with constant $\rho > 0$. Then for arbitrary $\delta > 0$ and $1 \leq p < \infty$ and for any $\zeta : \Omega \to \mathbb{R}$, we have that

$$
\langle |\zeta|^{2p} \rangle_{\frac{\delta}{p}} \leq C(d,\rho,p,\delta)\langle |\zeta|\rangle + \delta \left( \sum_{e} \left( \text{osc}_{a(e)} \zeta \right)^2 \right)^{\frac{p}{2}}. \tag{27}
$$

The preceding lemma may be seen as a reverse H"older inequality in probability: If one controls a bit (recall that $\delta > 0$ may be arbitrarily small) of the vertical derivative of a random variable $\zeta$, then its $L_{\langle \cdot \rangle}^1$-norm bounds its $L_{\langle \cdot \rangle}^{2p}$-norm. It can be seen as a softening of the concentration of measure phenomenon, which requires Lipschitz continuity of $\zeta$, cf. [18, Theorem 5.3].

In order to make use of Lemma 4, we need to estimate the vertical derivatives of $\nabla \nabla G$ and $\nabla G$. The following lemma is at the core of our result.

**Lemma 5.** There exists an integrability exponent $p_0 = p_0(d,\lambda) < \infty$ such that for all $p \geq p_0$, we have that

$$
\sup_{b,b' \in \mathbb{R}^d} \left\{ (|b-b'| + 1)^d \left( \sum_{e \in \mathbb{R}^d} \left( \text{osc}_{a(e)} \nabla \nabla G(b,b') \right)^2 \right)^{\frac{p}{2}} \right\} \lesssim C(d,\lambda,p) \sup_{b,b' \in \mathbb{R}^d} \left\{ (|b-b'| + 1)^d \langle |\nabla \nabla G(b,b')|^{2p} \rangle \right\}, \tag{28}
$$

and

$$
\sup_{b \in \mathbb{R}^d, x \in \mathbb{Z}^d} \left\{ (|b-x| + 1)^{d-1} \left( \sum_{e \in \mathbb{R}^d} \left( \text{osc}_{a(e)} \nabla G(b,x) \right)^2 \right)^{\frac{p}{2}} \right\} \lesssim C(d,\lambda,p) \sup_{b \in \mathbb{R}^d, x \in \mathbb{Z}^d} \left\{ (|b-x| + 1)^{d-1} \langle |\nabla G(b,x)|^{2p} \rangle \right\} + \sup_{b,b' \in \mathbb{R}^d} \left\{ (|b-b'| + 1)^d \langle |\nabla \nabla G(b,b')|^{2p} \rangle \right\}. \tag{29}
$$

For the proof, we refer to Subsection 7.3. Note that in contrast to Proposition 1, here the only assumption on the ensemble is LSI (6) — in particular, Lemmas 4 and 5 do not require stationarity and stationarity enters the proof of Theorem 1 only through Proposition 1. The formulation of Lemma 5 shows that with our method, we first have to estimate the mixed second derivatives $\langle |\nabla \nabla G(b,b')|^{2p} \rangle$ before we can tackle the first derivatives $\langle |\nabla G(b,0)|^{2p} \rangle$. It also reveals that it is necessary to estimate high moments $p \geq p_0$ in $\langle \cdot \rangle$ in order to estimate moderately low moments like the fourth moment $\langle |\nabla G(b,0)|^4 \rangle$ that is needed in the proof of Corollary 1.
The preceding lemma relies on the following suboptimal, but \textit{quenched} estimates on the (elliptic) Green function:

\textbf{Lemma 6.} [Gloria and Otto] \textit{There exists an exponent }\alpha_0 = \alpha_0(d, \lambda) > 0 \textit{such that for all }R > 0 \textit{and }b \in \mathbb{E}^d, \textit{we have that}

\begin{align}
R^{2\alpha_0} \sum_{e : R \leq |e - b| < 2R} |\nabla \nabla G(e, b)|^2 &\leq C(d, \lambda), \\
R^{2\alpha_0} \sum_{e : R \leq |e| < 2R} |\nabla G(e, 0)|^2 &\leq C(d, \lambda).
\end{align}

The estimate (31) was established in the stronger (dimensionally optimal) form of \(\sum_{R \leq |e| < 2R} |\nabla G(e, 0)|^2 \lesssim R^{2-d}\) in [12, Lemma 2.9]; in its weaker form of (31), it is straightforward for \(d > 2\). The proof of estimate (31) in [12] in case of \(d = 2\) is subtle and relied on an adaptation of [9]. In Subsection 7.2, we will give an elementary argument for the estimate (30) which we could not find in the literature. We remark that the proof presented here does not make use of the maximum principle (directly or indirectly) and therefore is also applicable to the case of systems, which we intend to use in future work.

\textbf{Remark 2.} We mention that with the same proof, one obtains a periodic version of Theorem 1 (with constants uniform in \(L\)) for the Green function defined in (5). In that case, one just replaces the Euclidean distance \(|x|\) on \(\mathbb{Z}^d\) by its periodic version \(\text{dist}(x, L\mathbb{Z}^d)\) on the torus \(\mathbb{R}/L\mathbb{Z}^d\). The periodic version of Proposition 1 follows as above from the quenched spatially averaged estimates of [13, Theorem 3(b)]. The same is true in the presence of a massive term, cf. (4).

\section{Proofs}

\subsection{Proof of Lemma 4}

\textbf{Step 1.} Result for \(p = 1\). We claim that for any \(\delta > 0\) and all \(\zeta(a)\):

\[\langle \zeta^2 \rangle^{\frac{1}{2}} \leq \left( \exp \left( \frac{2}{\rho \delta^2} \right) + \frac{\rho \delta^2}{2e} \right) \langle |\zeta| \rangle + \delta \left( \sum_{e} \langle \text{osc}_{a(e)} \zeta \rangle^2 \right)^{\frac{1}{2}},\]

where \(\rho\) denote the constant in the LSI, see Definition 1. By homogeneity, we may assume \(\langle \zeta^2 \rangle = 1\). For all real-valued \(\zeta\) we have that

\[\zeta^2 \leq \begin{cases} 
\exp \left( \frac{2}{\rho \delta^2} |\zeta| \right) & \text{if } |\zeta| \leq \exp \frac{2}{\rho \delta^2} \\
\frac{\rho \delta^2}{4} \zeta^2 \log \zeta^2 & \text{if } |\zeta| \geq \exp \frac{2}{\rho \delta^2} 
\end{cases}.
\]

Since \(x \log x\) is bounded from below by \(\frac{1}{e}\), we have that \(\frac{2}{e} |\zeta| + \zeta^2 \log \zeta^2 \geq 0\) for all \(\zeta\). It follows that

\[\zeta^2 \leq \left( \exp \left( \frac{2}{\rho \delta^2} \right) + \frac{\rho \delta^2}{2e} \right) |\zeta| + \frac{\rho \delta^2}{4} \zeta^2 \log \zeta^2.
\]

Hence taking the expectation \(\langle \cdot \rangle\) yields

\[\langle \zeta^2 \rangle \leq \left( \exp \left( \frac{2}{\rho \delta^2} \right) + \frac{\rho \delta^2}{2e} \right) \langle |\zeta| \rangle + \frac{\rho \delta^2}{4} \langle \zeta^2 \log \zeta^2 \rangle.
\]
Since \( \langle \zeta^2 \rangle = 1 \), Young’s inequality yields
\[
\langle |\zeta| \rangle \leq \frac{1}{2} \left( \exp \left( \frac{2}{\rho \delta^2} \right) + \frac{\rho \delta^2}{2e} \right) \langle |\zeta| \rangle^2 + \frac{1}{2} \left( \exp \left( \frac{2}{\rho \delta^2} \right) + \frac{\rho \delta^2}{2e} \right)^{-1} \langle \zeta \rangle
\]
Combining the last two estimates, we deduce
\[
\langle \zeta^2 \rangle \leq \frac{1}{2} \left( \exp \left( \frac{2}{\rho \delta^2} \right) + \frac{\rho \delta^2}{2e} \right)^2 \langle |\zeta| \rangle^2 + \frac{\rho \delta}{2} \langle \zeta^2 \rangle - \frac{\rho}{2} \langle \zeta \rangle \langle \zeta \rangle.
\]
Hence LSI yields
\[
\langle \zeta^2 \rangle \leq \frac{1}{2} \left( \exp \left( \frac{2}{\rho \delta^2} \right) + \frac{\rho \delta^2}{2e} \right)^2 \langle |\zeta| \rangle^2 + \frac{\rho \delta}{2} \langle \zeta^2 \rangle - \frac{\rho}{2} \langle \zeta \rangle \langle \zeta \rangle.
\]
and estimate (32) follows from taking the square root and applying the inequality \( \sqrt{\zeta + \xi} \leq \sqrt{\zeta} + \sqrt{\xi} \) for all numbers \( \zeta, \xi \geq 0 \).

**Step 2.** We finish the proof of (27), i.e. show that
\[
\langle \zeta^{2p} \rangle^{\frac{1}{2p}} \leq C(p, \rho, \delta) \langle |\zeta| \rangle + \delta \left( \left( \sum_e \left( \text{osc}_e \zeta \right)^p \right)^{\frac{1}{p}} \right)^p
\]
for general \( p \geq 1 \). To that end, we apply (32) to \( \zeta \) replaced by \( |\zeta|^p \):
\[
\langle |\zeta|^{2p} \rangle \leq C(p, \rho, \delta) \langle |\zeta|^p \rangle^2 + \delta \left( \sum_e \left( \text{osc}_e |\zeta|^p \right)^2 \right),
\]
where \( C(p, \rho, \delta) \) denotes a generic constant only depending on \( \rho, p, \) and \( \delta \). Since \( p < 2p \), an application of Hölder’s inequality in \( \langle \cdot \rangle \) and Young’s inequality on the first r.-h. s. term yields
\[
\langle |\zeta|^{2p} \rangle \leq C(p, \rho, \delta) \langle |\zeta|^p \rangle^2 + \delta \left( \sum_e \left( \text{osc}_e |\zeta|^p \right)^2 \right).
\]
(33)
Now we use that
\[
\text{osc}_{a(e)} |\zeta|^p \leq C(p) \left( |\zeta|^{p-1} \text{osc}_{a(e)} \zeta + \left( \text{osc}_{a(e)} \zeta \right)^p \right)
\]
which follows from the elementary inequality \( |\zeta^p - \xi^p| \leq C(p)(|\zeta^{p-1}| - |\xi^{p-1}|) + |\zeta - \xi|^p \) for all numbers \( \zeta, \xi > 0 \) and the triangle inequality in form of \( \text{osc}_{a(e)} |\zeta| \leq \text{osc}_{a(e)} \zeta \). Hence (33) yields
\[
\langle |\zeta|^{2p} \rangle \leq C(p, \rho, \delta) \langle |\zeta|^p \rangle^{2p} + 2C(p)\delta \left( \sum_e \left( \text{osc}_{a(e)} \zeta \right)^2 \right) + 2C(p)\delta \left( \sum_e \left( \text{osc}_{a(e)} \zeta \right)^{2p} \right). \quad (34)
\]
The last term on the right-hand side may be estimated by discreteness:

\[
\left\langle \sum_e \left( \text{osc}_{a(e)} \zeta \right)^2 \right\rangle \leq \left\langle \left( \sum_e \left( \text{osc}_{a(e)} \zeta \right)^2 \right)^p \right\rangle.
\] (35)

Furthermore, Hölder’s inequality followed by Young’s inequality yields

\[
\left\langle |\zeta|^{2p-2} \sum_e \left( \text{osc}_{a(e)} \zeta \right)^2 \right\rangle \leq \frac{1}{4C(p)\delta} \langle |\zeta|^{2p} \rangle + (4C(p)\delta)^p \langle \left( \sum_e \left( \text{osc}_{a(e)} \zeta \right)^2 \right)^p \rangle.
\] (36)

Hence collecting (34), (35) and (36) yields

\[
\langle |\zeta|^{2p} \rangle \leq C(\rho, p, \delta) \langle |\zeta| \rangle^{2p} + 2(2C(p)\delta + (4C(p)\delta)^p) \langle \left( \sum_e \left( \text{osc}_{a(e)} \zeta \right)^2 \right)^p \rangle,
\]

where we have absorbed the second term of (36) on the left-hand side. By redefining \(\delta\), we obtain (27).

### 7.2 Proof of Lemma 6

We just give the proof of (30); for (31), we refer to [12, Lemma 2.9]. Note that in the stronger form \(\sum_{e:R \leq |b-e| < 2R} |\nabla \nabla G(e, b)|^2 \leq C(d, \lambda)R^{2-d-2\alpha_0}\), Estimate (30) can also be seen as a consequence of the following classical ingredients (which however would not hold in the systems case):

- the optimal decay of \(G(x, y)\) itself, that is just needed in a spatially averaged sense of \(R^{-d} \sum_{y:R \leq |x-y| < 2R} |G(x, y) - \bar{G}| \leq C(d, \lambda)R^{2-d}\) (thanks to subtracting the average \(\bar{G}\) over the annulus \(\{y : R \leq |x-y| \leq 2R\}\), this estimate also holds in \(d = 2\)),
- De Giorgi’s Hölder continuity estimate, that then yields for some \(\alpha_0 = \alpha_0(d, \lambda) > 0\) that \(\sup_{x:R \leq |b-x| < 2R} |\nabla G(x, b)| \leq C(d, \lambda)R^{2-d-\alpha_0}\),
- Caccioppoli’s estimate, that then yields \(\sum_{e:R \leq |b-e| < 2R} |\nabla \nabla G(e, b)|^2 \leq C(d, \lambda)R^{2-d-2\alpha_0}\).

**Step 1.** In this step, we derive the a priori estimate

\[
\sum_e |\nabla \nabla G(e, b)|^2 \leq C(d, \lambda).
\] (37)

Indeed, recall the weak formulation (3) of the defining equation for \(G\), i.e.

\[
\forall \zeta(x) : \sum_e \nabla \zeta(e)a(e)\nabla G(e, x) = \zeta(x).
\]

Taking the derivative w. r. t. the variable \(x\) along some edge \(b\) yields

\[
\forall \zeta(x) : \sum_e \nabla \zeta(e)a(e)\nabla \nabla G(e, b) = \nabla \zeta(b).
\] (38)
The choice of \( \zeta(x) = \nabla G(x, b) \) (we address the question of admissibility of this test function below) yields

\[
\sum_e a(e)(\nabla \nabla G(e, b))^2 = \nabla \nabla G(b, b).
\]

Since \( a(b) \geq \lambda \), this implies (37) in the explicit form of

\[
\sum_e |\nabla \nabla G(e, b)|^2 \leq \lambda^{-2}.
\]  

We now turn to the question of admissibility of \( \zeta(x) = \nabla G(x, b) \) as a test function for (38), i.e. the question of decay as \( |x| \to \infty \) of this function and its gradient. This issue can be circumvented as in Step 3 below through approximation by the periodic problem. More precisely, we consider the periodic discrete elliptic Green function \( G_L(x, x') = G_L(a, x, x') \) of period \( L \). Up to additive constants, it is characterized by the weak equation

\[
\sum_{e \in E^d \cap [-\frac{L}{2}, \frac{L}{2})^d} \nabla \zeta(e)a(e) \nabla G_L(e, x') = \zeta(x') - L^{-d} \sum_{x \in E^d \cap [-\frac{L}{2}, \frac{L}{2})^d} \zeta(x)
\]

for all periodic \( \zeta(x) \). Here \( b \in E^d \cap [-\frac{L}{2}, \frac{L}{2})^d \) stands short for the set of all edges \( b \) whose midpoint is contained in the box \( [-\frac{L}{2}, \frac{L}{2})^d \). With the same argument as above, we obtain

\[
\sum_{e \in E^d \cap [-\frac{L}{2}, \frac{L}{2})^d} |\nabla \nabla G_L(e, b)|^2 \leq \lambda^{-2}.
\]  

Since \( G_L(x, x') \) converges point-wise to \( G(x, x') \), the latter implies (39) in the limit \( L \to \infty \) by Fatou’s lemma. Incidentally, \( \lim_{L \to \infty} \nabla G_L(e, x') \) may be taken as a definition of \( \nabla G(e, x') \) in the case of \( d = 2 \), where \( G \) itself is not unambiguously defined.

In the following steps, we use the fact that \( u(x) = \nabla G(x, b) \) is \( a \)-harmonic away from the endpoints of \( b \) to show that there exists a decay exponent \( \alpha_0(d, \lambda) > 0 \) such that for all \( R \geq C(d) \) we have

\[
\sum_{e : |b-e| \geq R} |\nabla u(e)|^2 \leq C(d, \lambda) R^{-2\alpha_0} \sum_{e : |b-e| \geq 1} |\nabla u(e)|^2.
\]  

Together with (37), this implies (30). In Step 2, we will formally treat the continuum whole-space case (where \( a(x) \) is a uniformly elliptic matrix). In Step 3, we will show how to make the continuum case rigorous by approximation through the continuum periodic case. More precisely, using (37), we will directly prove the estimate (30) in form of

\[
\sum_{e : |b-e| \geq R} |\nabla u(e)|^2 \leq C(d, \lambda) R^{-2\alpha_0}.
\]  

In Step 4, we indicate the changes necessary to treat the discrete case.

**Step 2.** Formal derivation of the continuum version of (41), that is

\[
\int_{\{x : |x| \geq R\}} |\nabla u|^2 dx \leq C(d, \lambda) R^{-2\alpha_0} \int_{\{x : |x| \geq 1\}} |\nabla u|^2 dx
\]
Indeed, let $\eta(x)$ be a cut-off function for $\{x : |x| \geq 2R\}$ in $\{x : |x| \geq R\}$. We test (44) with $\zeta = \eta^2(u - \bar{u})$, where $\bar{u}$ is the spatial average of $u$ on the annulus $\{x : R \leq |x| \leq 2R\}$. It is a priori not clear that this is an admissible test function for (44); we shall address this in the next step. We appeal to the identity

$$\nabla(\eta^2(u - \bar{u})) \cdot a \nabla u = \nabla(\eta(u - \bar{u})) \cdot a \nabla(\eta(u - \bar{u})) - (u - \bar{u})^2 \nabla \eta \cdot a \nabla \eta, \quad (45)$$

which in view of ellipticity in form of $\lambda |\xi|^2 \leq \xi \cdot a(x) \xi \leq |\xi|^2$ for all $\xi \in \mathbb{R}^d$ turns into the inequality

$$\nabla(\eta^2(u - \bar{u})) \cdot a \nabla u \geq \lambda |\nabla(\eta(u - \bar{u}))|^2 - (u - \bar{u})^2 |\nabla \eta|^2. \quad (46)$$

Hence from testing (44) we obtain

$$\lambda \int_{\mathbb{R}^d} |\nabla(\eta(u - \bar{u}))|^2 dx \leq \int_{\mathbb{R}^d} (u - \bar{u})^2 |\nabla \eta|^2 dx,$$

which by the choice of $\eta$ yields the Caccioppoli estimate

$$\int_{\{x : |x| \geq 2R\}} |\nabla u|^2 dx \leq C(d, \lambda) R^{-2} \int_{\{x : R \leq |x| \leq 2R\}} (u - \bar{u})^2 dx. \quad (47)$$

By Poincaré’s estimate on $\{x : R \leq |x| \leq 2R\}$ with mean value zero, this turns into

$$\int_{\{x : |x| \geq 2R\}} |\nabla u|^2 dx \leq C(d, \lambda) \int_{\{x : R \leq |x| \leq 2R\}} |\nabla u|^2 dx,$$

which can be reformulated as

$$\int_{\{x : |x| \geq R\}} |\nabla u|^2 dx \leq C(d, \lambda) \int_{\{x : R \leq |x| \leq 2R\}} |\nabla u|^2 dx. \quad (48)$$

A standard iteration argument now leads from (48) to (43): Introducing the notation $I_k := \int_{\{x : |x| \geq 2^k\}} |\nabla u|^2 dx$, estimate (48) reads

$$\forall k \in \{0, 1, \cdots\} \quad I_k \leq C(d, \lambda)(I_k - I_{k+1}),$$

which with help of $\theta = \theta(d, \lambda) := 1 - \frac{1}{\log 2} < 1$ can be reformulated

$$\forall k \in \{0, 1, \cdots\} \quad I_{k+1} \leq \theta I_k,$$

or with help of $\alpha_0 = \alpha_0(d, \lambda) := \frac{\log \theta}{2 \log 2}$ as

$$\forall k \in \{0, 1, \cdots\} \quad I_k \leq \theta^k I_0 = (2^k)^{-2\alpha_0} I_0.$$

In the original notation, this implies (43) in form of

$$\forall R \geq 1 \quad \int_{\{x : |x| \geq R\}} |\nabla u|^2 dx \leq \left(\frac{R}{2}\right)^{-2\alpha_0} \int_{\{x : |x| \geq 1\}} |\nabla u|^2 dx.$$
Step 3. In this step, as opposed to the previous step, we deal with the issue that we do not know a priori that \( \eta^2(u - \bar{u}) \) is an admissible test function for (44). This allows us to rigorously deduce the continuum version (42) for \( R \geq 1 \), and where \( u \) is now specified to be a partial derivative of the Green function, i.e. \( u(x) = \nabla_{i,x'} G(x,x')|_{x'=0} \) with \( i = 1, \cdots, d \). More precisely, we worry about the decay at \( |x| \to \infty \)— we do not worry about local smoothness since anyway, we will apply the argument to the discrete case in the next step. As in Step 1, we circumvent the problem of decay through approximation by the periodic problem. Indeed, we consider the periodic continuum elliptic Green function \( G_L(x,x') = G_L(x,x') \) of period \( L \). Up to additive constants, it is characterized by the weak equation

\[
\int_{[-\frac{L}{2}, \frac{L}{2})^d} \nabla \zeta(x) \cdot a(x) \nabla_x G_L(x,x') dx = \zeta(x') - L^{-d} \int_{[-\frac{L}{2}, \frac{L}{2})^d} \zeta(x) dx \tag{49}
\]

for all periodic \( \zeta(x) \). We note that \( u_L(x) = \nabla_{i,x'} G_L(x,x')|_{x'=0} \) thus is characterized by

\[
\int_{[-\frac{L}{2}, \frac{L}{2})^d} \nabla \zeta(x) \cdot a(x) \nabla_x u_L(x) dx = \nabla_i \zeta(0). \tag{50}
\]

Since \( \nabla \nabla G_L \) converges in the sense of distributions to \( \nabla \nabla G \) as \( L \to \infty \), it is enough to show (50) implies

\[
\int_{[-\frac{L}{2}, \frac{L}{2})^d \cap \{ x : |x| \geq R \}} |\nabla u_L|^2 dx \leq C(d,\lambda) R^{-2\alpha_0} \int_{[-\frac{L}{2}, \frac{L}{2})^d \cap \{ x : |x| \geq 1 \}} |\nabla u_L|^2 dx \tag{51}
\]

for \( 1 \leq R \leq C(d) L \). Indeed we can estimate the right-hand side of (51) using (40) and apply weak lower semi-continuity to take the limit as \( L \to \infty \) on the left-hand side to obtain (42). Now, disregarding smoothness issues, \( \eta^2(u_L - \bar{u}) \) is an admissible test function for (50). The argument for (51) is identical to the one in Step 2.

Step 4. Rigorous derivation of (41) for \( R \geq C(d) \). In this step, we indicate the modifications in Step 2 (or rather Step 3) that are necessary to treat the discrete case. The first modification results from the fact that Leibniz rule and thus the neat identity (45) does not hold anymore. However, we claim that the estimate (46) survives in form of

\[
\nabla (\eta^2(u - \bar{u}))(e)a(e) \nabla u(e) \geq \lambda (\nabla (\eta(u - \bar{u}))(e))^2 - (|u(e)| \eta(e) - \bar{u})^2(\nabla \eta(e))^2, \tag{52}
\]

where we denote by \( |u|([x, x + e_i]) = \frac{1}{2}(u(x) + u(x + e_i)) \) the local average of \( u \) along each edge \( e = [x, x + e_i] \). Indeed, since \( \lambda \leq a(e) \leq 1 \) is elliptic, this follows from the simple inequality on 4 numbers \( \eta = \eta(x), \tilde{\eta} = \eta(x + e_i), v = u(x) - \bar{u}, \) and \( \tilde{v} = u(x + e_i) - \bar{u} \):

\[
(\eta^2 v - \tilde{\eta}^2 \tilde{v})(v - \tilde{v}) - (\eta \eta - \tilde{\eta} \tilde{\eta})^2 \geq -(\eta - \tilde{\eta})^2 v \tilde{v} \geq -(\eta - \tilde{\eta})^2 \frac{1}{2} (v + \tilde{v})^2.
\]

Hence, if \( \eta(x) \) denotes the (slightly narrower) cut-off function for \( \{ x : |x| \geq 2R - 2 \} \) in \( \{ x : |x| \geq R + 2 \} \) (which is possible for \( R \geq 5 \)), from (52) we obtain
the following substitute of (47)

\[
\sum_{e: |e-b| \geq 2R} |\nabla u(e)|^2 \leq C(d, \lambda) R^{-2} \sum_{e:R+1 \leq |e-b| \leq 2R-1} ((u(e) - \bar{u})^2 \leq C(d, \lambda) R^{-2} \sum_{x:R \leq |x-b| \leq 2R} (u(x) - \bar{u})^2. \tag{53}
\]

The second modification comes from the fact that we need a discrete version of the Poincaré estimate with mean value zero on the annulus \( \mathbb{Z}^d \cap \{ R \leq |x| \leq 2R \} \), which obviously holds with a constant \( C(d) R^2 \) provided that \( R \geq C(d) \).

### 7.3 Proof of Lemma 5

**Step 1.** In this first step, we consider two coefficient fields \( \tilde{a}, a \in \Omega \) and their associated Green functions \( \tilde{G} = G(\tilde{a} ; \cdot , \cdot) \) and \( G = G(a ; \cdot , \cdot) \), respectively. We claim that if \( \tilde{a} \) and \( a \) differ only at some edge \( e \in \mathbb{E}^d \), then we have that:

\[
\tilde{G}(x, x') - G(x, x') = (a(e) - \tilde{a}(e)) \nabla \tilde{G}(x, e) \nabla G(e, x'), \tag{54}
\]

\[
\nabla \tilde{G}(b, x') - \nabla G(b, x') = (a(e) - \tilde{a}(e)) \nabla \tilde{G}(b, e) \nabla G(e, x'), \tag{55}
\]

\[
\nabla \tilde{G}(b, b') - \nabla G(b, b') = (a(e) - \tilde{a}(e)) \nabla \tilde{G}(b, e) \nabla G(e, b'). \tag{56}
\]

Indeed, the difference satisfies the equation

\[
\nabla^* \tilde{a} \nabla (\tilde{G} - G)(\cdot, x') = \nabla^* (a - \tilde{a}) \nabla G(\cdot, x')
\]

Since by assumption \( \tilde{a}(b) = a(b) \) for all edges \( b \neq e \), the Green function representation (3) immediately yields (54). Differentiating (54) then yields (55) and (56).

**Step 2.** In this step, we derive the following estimate on the oscillations:

\[
\osc_{a(e)} G(x, x') \leq 4 (1 + \frac{1}{R}) |\nabla G(x, e)||\nabla G(e, x')|, \tag{57}
\]

\[
\osc_{a(e)} \nabla G(b, x') \leq 4 (1 + \frac{1}{R}) |\nabla \nabla G(b, e)||\nabla G(e, x)|, \tag{58}
\]

\[
\osc_{a(e)} \nabla \nabla G(b, b') \leq 4 (1 + \frac{1}{R}) |\nabla \nabla G(b, e)||\nabla \nabla G(e, b')|. \tag{59}
\]

To do so, we first show that for any edge \( e \), the dependence of \( \nabla G(e, \cdot) \) on the value of \( a(e) \) of the conductivity is mild in the sense that

\[
|\nabla \tilde{G}(e, x') - \nabla G(e, x')| \leq \frac{1}{\lambda} |\nabla G(e, x')|, \tag{60}
\]

\[
|\nabla \nabla \tilde{G}(e, b') - \nabla \nabla G(e, b')| \leq \frac{1}{\lambda} |\nabla \nabla G(e, b')|, \tag{61}
\]

where \( \tilde{G} \) and \( G \) are given in Step 1. This indeed follows from letting \( b = e \) in (55) and (56) and recalling the a priori estimate \( |\nabla \nabla G(e, e)| \leq \lambda^{-1} \) from (39).

We turn to the proof of (58). It is clear that for any \( a \in \Omega \), there exist \( \tilde{a}_1, \tilde{a}_2 \in \Omega \) with \( \tilde{a}_1(b) = a(b) = \tilde{a}_2(b) \) for all \( b \neq e \) and associated Green functions \( \tilde{G}_1 \) and \( \tilde{G}_2 \) such that

\[
\osc_{a(e)} G(x, x') \leq 2 |\tilde{G}_1(x, x') - \tilde{G}_2(x, x')| \leq 2 |\tilde{G}_1(x, x') - G(x, x')| + 2 |G(x, x') - \tilde{G}_2(x, x')|.
\]
We insert (54) with $\tilde{a}_i$, $i = 1, 2$, into this estimate to obtain that

$$\text{osc}_{a(e)} G(x, x') \leq 2|\nabla \tilde{G}_1(x, e)||\nabla G(e, x')| + 2|\nabla \tilde{G}_2(x, e)||\nabla G(e, x')|$$

Consequently, symmetry $\nabla \tilde{G}_i(x, e) = \nabla \tilde{G}_i(e, x)$ and estimate (60) yield

$$\text{osc}_{a(e)} G(x, x') \leq 2\left(1 + \frac{1}{\lambda}\right)|\nabla G(x, e)||\nabla G(e, x')|.$$

This proves (57). The estimates (58) and (59) follow similarly using (61).

**Step 3.** In this step, we rephrase Lemma 6, more precisely (30), in a way more suitable for its application in Step 4. More specifically, we claim that there exists a weight exponent $\alpha(d, \lambda) > 0$ such that

$$\sup_{a \in \Omega} \sum_e \left(\frac{|e - b| + 1}{2} \nabla \nabla G(e, b)\right)^{2q} \leq C(d, \lambda, q),$$

for all $q \geq 1$. In fact, we claim that

$$\alpha := \frac{1}{2} \alpha_0$$

does the job. Because of $q \geq 1$, and thus $\ell^2(\mathbb{E}^d) \subset \ell^{2q}(\mathbb{E}^d)$, we have

$$\sum_e \left(\frac{|e - b| + 1}{2} \nabla \nabla G(e, b)\right)^{2q} \leq \left(\sum_e \left(\frac{|e - b| + 1}{2} \nabla \nabla G(e, b)\right)^{2}\right)^q.$$  

Using a dyadic decomposition, we see

$$\sum_e \left(\frac{|e - b| + 1}{2} \nabla \nabla G(e, b)\right)^{2}$$

$$= |\nabla \nabla G(b, b)|^2 + \sum_{n=0}^{\infty} \sum_{e : 2^{n-1} \leq |e - b| < 2^n} \left|\frac{|e - b| + 1}{2} \nabla \nabla G(e, b)\right|^2$$

$$\leq |\nabla \nabla G(b, b)|^2 + \sum_{n=0}^{\infty} 2^{2\alpha(n+1)} \sum_{e : 2^{n-1} \leq |e| < 2^n} |\nabla \nabla G(e, b)|^2.$$

We now may appeal to (30) to obtain

$$\sum_e \left(\frac{|e - b| + 1}{2} \nabla \nabla G(e, b)\right)^{2} \leq C(d, \lambda) \left(1 + \sum_{n=0}^{\infty} 2^{2\alpha(n+1)} 2^{-2\alpha n}\right) \leq C(d, \lambda).$$

(63)

(64)

**Step 4.** In this step, we establish the first statement of Lemma 5, namely (28). More precisely, we claim that for $p \geq \max\{\frac{d}{\alpha}, 1\}$ with $\alpha$ chosen in Step 3 and all $b, b' \in \mathbb{E}^d$:

$$\left(|b - b'| + 1\right)^{2pd} \left(\sum_e \left(\text{osc}_{a(e)} \nabla \nabla G(b, b')\right)^{2}\right)^p$$

$$\leq C(d, \lambda, p) \sup_{e, e'} \left\{\left(|e - e'| + 1\right)^{2pd} |\nabla \nabla G(e, e')|^{2p}\right\}.$$  

(65)
Indeed, we first square (59) and sum over $e$:

$$
\sum_{e} \left( \text{osc} \left( \nabla \nabla G(b, b') \right) \right)^2 \leq C(\lambda) \sum_{e} |\nabla \nabla G(b, e)|^2 |\nabla \nabla G(e, b')|^2.
$$

After taking the $p$-th power, we split the sum into its contributions over $\{e : |e - b| \leq |e - b'|\}$ and $\{e : |e - b| > |e - b'|\}$ to obtain

$$
\left( \sum_{e} \left( \text{osc} \left( \nabla \nabla G(b, b') \right) \right)^2 \right)^p
\leq C(\lambda, p) \left( \sum_{e : |e - b| \leq |e - b'|} |\nabla \nabla G(b, e)|^2 |\nabla \nabla G(e, b')|^2 \right)^p
\quad + \left( \sum_{e : |e - b| \geq |e - b'|} |\nabla \nabla G(b, e)|^2 |\nabla \nabla G(e, b')|^2 \right)^p.
$$

(66)

We first bound the first term. To this end, we smuggle in a weight $(|e - b| + 1)^{2\alpha}$ with $\alpha = \alpha(d, \lambda)$ from Step 3 and apply Hölder’s inequality with $p$ and its dual exponent $q$ (i.e. $\frac{1}{p} + \frac{1}{q} = 1$):

$$
\left( \sum_{e : |e - b| \leq |e - b'|} |\nabla \nabla G(b, e)|^2 |\nabla \nabla G(e, b')|^2 \right)^p
\leq \left( \sum_{e : |e - b| \leq |e - b'|} \left( (|e - b| + 1)^\alpha |\nabla \nabla G(b, e)| \right)^{2q} \right)^{p-1}
\quad \times \sum_{e : |e - b| \leq |e - b'|} \left( (|e - b| + 1)^{-\alpha |\nabla \nabla G(e, b)|} \right)^{2p}.
$$

The first term on the right-hand side is bounded by Step 3, that is (62). After taking the expectation, we smuggle in another weight $(|e - b'| + 1)^{2pd}$ and take the supremum over appropriate terms to obtain

$$
\left\langle \sum_{e : |e - b| \leq |e - b'|} \left( (|e - b| + 1)^{-\alpha |\nabla \nabla G(e, b')|} \right)^{2p} \right\rangle
\leq \left( \sum_{e : |e - b| \leq |e - b'|} (|e - b| + 1)^{-2p\alpha} (|e - b'| + 1)^{-2pd} \right)
\quad \times \sup_{e'} \left\{ (|e' - b'| + 1)^{2pd} |\nabla \nabla G(e', b')|^2 \right\}.
$$

Since $|e - b| \leq |e - b'|$ implies $|e - b'| \geq \frac{1}{2}|b - b'|$, we find for the first r.-h. s. factor that

$$
\sum_{e : |e - b| \leq |e - b'|} (|e - b| + 1)^{-2p\alpha} (|e - b'| + 1)^{-2pd}
\leq \left( \frac{1}{2} |b - b'| + 1 \right)^{-2pd} \sum_{e} (|e - b| + 1)^{-2p\alpha}.
$$
Since by assumption $2p\alpha \geq 2d > d$, we obtain for the last factor
\[ \sum_{e \in \mathbb{Z}^d} (|e - b| + 1)^{-2p\alpha} \leq C(d). \]

Combining these estimates yields the bound
\[ \left\langle \left( \sum_{e : |e - b| \leq |e - b'|} |\nabla \nabla G(b, e)|^2 |\nabla \nabla G(e, b')|^2 \right)^p \right\rangle \]
\[ \leq \left( C(d, \lambda, p)(|b - b'| + 1)^{-d} \sup_{e, e'} \left\{ (|e - e'| + 1)^d (|\nabla \nabla G(e, e')|^{2p})^{\frac{1}{p}} \right\} \right)^{2p}, \]
i.e. the expectation of the first term on the right-hand side of (66) is bounded as desired. The second term in (66) can be dealt with exactly as the first term by simply exchanging the roles of $b$ and $b'$.

**Step 5.** Like in Step 3, we rephrase Lemma 6, this time (31), in a way more suitable for its application in Step 6. We claim that for any integrability exponent $q \geq 1$ and any weight exponent $\beta > 0$ we have
\[ \sup_{a \in \Omega} \sum_{e} (|e| + 1)^{-\beta} \nabla G(e, 0)^{2q} = C(d, \lambda, q, \beta) \] (67)

We note that by (31) we have as soon as $\beta > 0$:
\[ \sum_{e} (|e| + 1)^{-\beta} \nabla G(e, 0)^{2q} \leq \left( \sum_{e} (|e| + 1)^{-\beta} \nabla G(e, 0)^{2} \right)^q \]
\[ \leq \left( \sum_{i=1}^{d} |\nabla G([0, e_i], 0)|^2 + \sum_{n=0}^{\infty} 2^{-q\beta n} \sum_{e : 2^n \leq |e| < 2^{n+1}} |\nabla G(e, 0)|^2 \right)^q \] (68)
\[ \leq C(d, \lambda, \beta). \]

**Step 6.** In this step we establish the second conclusion of Lemma 5, namely (29). More precisely, we show that for any integrability exponent $p < \infty$ at least as large as in Step 3 and for any weight exponent $\beta > 0$ sufficiently small such that
\[ 2p(\beta - d) + d < 0 \] (69)
we have
\[ (|b - x| + 1)^{d-1} \left\langle \left( \sum_{e} \left( \frac{\text{osc} G(b, x)}{a(e)} \right)^2 \right)^p \right\rangle^{\frac{1}{p}} \]
\[ \leq C(d, \lambda, p, \beta) \left( \sup_{e, e'} \left\{ (|e - e'| + 1)^d (|\nabla G(e, e')|^{2p})^{\frac{1}{p}} \right\} \right) \]
\[ + (|b - x| + 1)^{\beta - d - 1 + \frac{1}{p}} \sup_{e, e'} \left\{ (|e - e'| + 1)^d (|\nabla \nabla G(e, e')|^{2p})^{\frac{1}{p}} \right\}, \]
for all $x \in \mathbb{Z}^d$ and $b \in \mathbb{Z}^d$, where $C(d, \lambda, p, \beta)$ denotes a generic constant that only depends on $d, \lambda, p, \beta$. We note that by choosing $\beta$ small and $p$ large,
the exponent $\beta - 1 + \frac{d}{2p}$ can be made to be non-positive (in fact, as close to $-1$ as we want), which proves (29). In order to establish (70), we first square (58) and sum over $e \in E^d$ to obtain that
\[
\sum_e \left( \text{osc}_{a(e)} \nabla G(b, x) \right)^2 \leq C(\lambda) \sum_e |\nabla \nabla G(b, e)|^2|\nabla G(e, x)|^2.
\]
We now split the sum over $e$:
\[
\sum_e |\nabla \nabla G(b, e)|^2|\nabla G(e, x)|^2
\]
\[
\leq C(d, \lambda) \left( \sum_{e:|e-x| \geq \frac{1}{2}|b-x|} |\nabla \nabla G(b, e)|^2|\nabla G(e, x)|^2 + \sum_{e:|e-b| > \frac{1}{2}|b-x|} |\nabla G(e, x)|^2|\nabla \nabla G(b, e)|^2 \right). 
\]
Since $|e - x| < \frac{1}{2}|b - x|$ implies $|e - b| > \frac{1}{2}|b - x|$, it follows
\[
\sum_e |\nabla \nabla G(b, e)|^2|\nabla G(e, x)|^2
\]
\[
\leq C(d, \lambda) \left( \sum_{e:|e-x| \geq \frac{1}{2}|b-x|} |\nabla \nabla G(b, e)|^2|\nabla G(e, x)|^2 + \sum_{e:|e-b| > \frac{1}{2}|b-x|} |\nabla G(e, x)|^2|\nabla \nabla G(b, e)|^2 \right). 
\]
We start by treating the first term on the r.h.s. of (71) in an analogous way to Step 4. For that purpose, let $a$ be as in Step 3. We smuggle in the weight $(|e - b| + 1)^\alpha$ and apply Hölder’s inequality with $p$ and $q$ such that $\frac{1}{p} + \frac{1}{q} = 1$:
\[
\left( \sum_{e:|e-x| \geq \frac{1}{2}|b-x|} |\nabla \nabla G(b, e)|^2|\nabla G(e, x)|^2 \right)^p
\]
\[
\leq \left( \sum_e |(|e - b| + 1)^\alpha \nabla \nabla G(b, e)|^{2q}\right)^{p-1}
\times \sum_{e:|e-x| \geq \frac{1}{2}|b-x|} |(|e - b| + 1)^{-\alpha}|\nabla G(e, x)|^{2p}.
\]
The first term was bounded by a constant $C(d, \lambda, p)$ in Step 3. Now we take the expectation $\langle \rangle$ w.r.t. $a$ and then smuggle in a weight $(|e - x| + 1)^{2p(d-1)}$ to obtain as desired:
\[
\left( \langle \sum_{e:|e-x| \geq \frac{1}{2}|b-x|} |\nabla \nabla G(b, e)|^2|\nabla G(e, x)|^2 \rangle \right)^p
\]
\[
\leq \left( \sum_{e:|e-x| \geq \frac{1}{2}|b-x|} (|e - b| + 1)^{-2\alpha}(|e - x| + 1)^{-2p(d-1)}
\times \sup_{e'} \left\{ (|e' - x| + 1)^{2p(d-1)} |\nabla \nabla G(e', x)|^{2p} \right\} \right)^p
\]
\[
\leq C(d, \lambda, p) (|b - x| + 1)^{-2p(d-1)} \sup_{e'} \left\{ (|e' - x| + 1)^{2p(d-1)} |\nabla \nabla G(e', x)|^{2p} \right\} ,
\]
(72)
where we have used that \(2p\alpha > d\).

We now address the second term on the r.h.s. of (71) in a similar way, just exchanging the roles of \(\nabla G\) and \(\nabla^2 G\), of \(b\) and \(x\), and of \(\alpha\) and \(-\beta\), where the weight exponent \(\beta > 0\) needs to satisfy (69). By Hölder’s inequality we obtain:

\[
\left( \sum_{e:|e-b|\geq \frac{1}{2}|b-x|} |\nabla G(e, x)|^2 |\nabla^2 G(b, e)|^2 \right)^p \leq \left( \sum_{e} \left(|e - x| + 1\right)^{-\beta} |\nabla G(e, x)|^{2q} \right)^{p-1} \times \sum_{e:|e-b|\geq \frac{1}{2}|b-x|} \left(|e - x| + 1\right)^{\beta} |\nabla^2 G(b, e)|^{2p}.
\]

The first term is bounded by Step 5 in form of (67). Taking the expectation and smuggling in a weight \((|e-b| + |b-x|)\) yields

\[
\left\langle \sum_{e:|e-b|\geq \frac{1}{2}|b-x|} \left(|e - x| + 1\right)^{\beta} |\nabla^2 G(b, e)|^{2p} \right\rangle \leq \sum_{e:|e-b|\geq \frac{1}{2}|b-x|} \left(|e - x| + 1\right)^{2p\beta} (|e-b| + 1)^{-2pd} \times \sup_{e'} \left\{ (|e' - b| + 1)^{2p} \langle |\nabla G(b, e')|^2 \rangle \right\}.
\]

We note that by the triangle inequality in form of \(|e - x| \leq |e-b| + |b-x|\), in the range (69) the remaining sum is bounded as follows:

\[
\sum_{e:|e-b|\geq \frac{1}{2}|b-x|} \left(|e - x| + 1\right)^{2p\beta} (|e-b| + 1)^{-2pd} \leq C(p, \beta) \left(|b-x| + 1\right)^{2p\beta} \sum_{e:|e-b|\geq \frac{1}{2}|b-x|} (|e-b| + 1)^{-2pd} + \sum_{e:|e-b|\geq \frac{1}{2}|b-x|} (|e-b| + 1)^{2p(\beta - d)} \leq C(d, p, \beta) \left(|b-x| + 1\right)^{2p(\beta - d) + d}.
\]

Hence we have obtained

\[
\left\langle \left( \sum_{e:|e-b|\geq \frac{1}{2}|b-x|} |\nabla G(e, x)|^2 |\nabla^2 G(b, e)|^2 \right)^p \right\rangle \leq C(d, \lambda, p, \beta) \left(|b-x| + 1\right)^{2p(\beta - d) + d} \sup_{e, e'} \left\{ (|e - e'| + 1)^{2pd} \langle |\nabla G(e, e')|^2 \rangle \right\}.
\]

(73)

In view of (71), the combination of (72) and (73) as well as taking the \(2p\)-th root yields (70).
7.4 Proof of Theorem 1

We start with the proof of (7). For this purpose, we fix \( b, b' \in \mathbb{R}^d \) and \( p < \infty \); by Jensen’s inequality, we may assume that \( p \geq p_0 \) with \( p_0 \) from Lemma 5. Applying Lemma 4 to \( \zeta(a) = \nabla \nabla G(a; b, b') \) and inserting the estimate (28) of Lemma 5 yields (after redefining \( \delta \))

\[
(|b - b'| + 1)^d(|\nabla \nabla G(b, b')|^{2p})^{\frac{1}{2p}} \leq C(d, \lambda, \rho, p, \delta)(|b - b'| + 1)^d(|\nabla \nabla G(b, b')|) + \delta \sup_{e, e'} \left\{ (|e - e'| + 1)^d(|\nabla \nabla G(e, e')|^{2p})^{\frac{1}{2p}} \right\}.
\]

We now insert (21) and take the supremum over \( b \) and \( b' \):

\[
\sup_{b, b'} \left\{ (|b - b'| + 1)^d(|\nabla \nabla G(b, b')|^{2p})^{\frac{1}{2p}} \right\}
\]

\[
\leq C(d, \lambda, \rho, p, \delta) + \delta \sup_{e, e'} \left\{ (|e - e'| + 1)^d(|\nabla \nabla G(e, e')|^{2p})^{\frac{1}{2p}} \right\}.
\]

Choosing \( \delta = 1/2 \), we obtain (7). We deal with the objection that \( \sup_{e, b}( | |e - b| + 1)^d(\|\nabla \nabla G(e, b)\|^{2p}/2p) \) may be infinite by first working with the periodic Green function \( \mathcal{G}_L \) as in the proof of Lemma 6 and then letting \( L \to \infty \).

We now turn to the proof of (8). With help of the just established (7), we may upgrade the result of Lemma 5, cf. (29), to

\[
(|b - x| + 1)^{d-1}\left( \sum_{e} \left( \operatorname{osc}_{a(e)} \nabla G(b, x) \right)^{2p} \right)^{\frac{1}{2p}}
\]

\[
\leq C(d, \lambda, \rho, p) \left( \sup_{e, x} \left\{ (|e - x'| + 1)^{d-1}(|\nabla G(e, x')|^{2p})^{\frac{1}{2p}} \right\} + 1 \right). \quad (74)
\]

We apply Lemma 4 to \( \zeta = \nabla G(b, x) \) and insert (74) (after redefining \( \delta \)):

\[
(|b - x| + 1)^{d-1}(\|\nabla G(b, x)\|^{2p})^{\frac{1}{2p}}
\]

\[
\leq C(d, \lambda, \rho, p, \delta)(|b - x| + 1)^{d-1}(\|\nabla G(b, x)\|) + \delta \left( \sup_{e, x} \left\{ (|e - x'| + 1)^{d-1}(|\nabla G(e, x')|^{2p})^{\frac{1}{2p}} \right\} + 1 \right).
\]

We now insert (22) and take the supremum over \( b \) and \( x \):

\[
\sup_{b, x} \left\{ (|b - x| + 1)^{d-1}(\|\nabla G(b, x)\|^{2p})^{\frac{1}{2p}} \right\}
\]

\[
\leq C(d, \lambda, \rho, p, \delta) + \delta \sup_{e, x} \left\{ (|e - x'| + 1)^{d-1}(\|\nabla G(e, x')\|^{2p})^{\frac{1}{2p}} \right\}.
\]

As before, letting \( \delta = 1/2 \) yields (8).

7.5 Proof of Corollary 1

It is well known that an LSI implies a corresponding SG, see for instance [16, Theorem 4.9]. Indeed, using \( \zeta^2 = 1 + \epsilon f \) for some \( f(a) \) in (6) and expanding to
second order in $\epsilon \ll 1$ one obtains
\[
\langle (f - \langle f \rangle)^2 \rangle \leq \frac{1}{\rho} \sum_e \left( \text{osc}_e(f) \right)^2.
\]

As in Step 2 of the proof of Lemma 4, see also \[13, \text{Lemma 11}\], it follows that
\[
\|f - \langle f \rangle\|_{L^2} \leq C(\rho, p) \left( \sum_e |\nabla G(x, e)| |\nabla G(e, 0)| \right)^{1/p}.
\] (75)

We fix $x \in \mathbb{Z}^d$ and apply this inequality to $f(a) = G(a; x, 0)$ and use (57) from the proof of Lemma 5, i.e.
\[
\text{osc}_a G(x, 0) \leq C(\lambda) |\nabla G(x, e)||\nabla G(e, 0)|,
\]
to obtain
\[
\langle |G(x, 0) - \langle G(x, 0) \rangle|^{2p} \rangle^{1/p} \leq C(\lambda, \rho, p) \sum_e \left( |\nabla G(x, e)| |\nabla G(e, 0)| \right)^{1/p}.
\]

The triangle inequality in $\langle (\cdot)^p \rangle^{1/p}$ yields
\[
\langle |G(x, 0) - \langle G(x, 0) \rangle|^{2p} \rangle^{1/p} \leq C(\lambda, \rho, p) \sum_e \left( |\nabla G(x, e)| |\nabla G(e, 0)|^{2p} \right)^{1/p}.
\]

Using the Cauchy-Schwarz inequality in $\langle \cdot \rangle$ and appealing to stationarity, we obtain
\[
\langle |G(x, 0) - \langle G(x, 0) \rangle|^{2p} \rangle^{1/p} \leq C(\lambda, \rho, p) \sum_e \left( |\nabla G(x, e)|^{4p} |\nabla G(e, 0)|^{4p} \right)^{1/2}.
\]

where we recall that $e - x \in \mathbb{Z}^d$ is the edge $e$ shifted by $x$ and $\nabla$ always falls on the edge variable. Into this estimate, we insert the result of Theorem 1:
\[
\langle |G(x, 0) - \langle G(x, 0) \rangle|^{2p} \rangle^{1/p} \leq C(d, \lambda, \rho, p) \sum_e \left( |e - x| + 1 \right)^{2(1-d)}.
\] (76)

We now turn to the sum on the r.h.s. of (76): By symmetry, we have
\[
\sum_e \left( |e - x| + 1 \right)^{2(1-d)} \leq 2 \sum_{e: |e - x| \leq |e|} \left( |e - x| + 1 \right)^{2(1-d)}.
\] (77)

We note that in the case of $d > 2$ we have $2(1 - d) < -d$ so that
\[
\sum_e \left( |e| + 1 \right)^{2(1-d)} \leq C(d) < \infty.
\] (78)
Since $|e - x| \leq |e|$ implies $|e| \geq \frac{1}{2}|x|$ we thus have as desired for (77)
\[
\sum_{e:|e-x|\leq|e|} (|e - x| + 1)^{2(1-d)}(|e| + 1)^{2(1-d)}
\leq (\frac{1}{2}|x| + 1)^{2(1-d)} \sum_{e:|e-x|\leq|e|} (|e - x| + 1)^{2(1-d)}
\leq C(|x| + 1)^{2(1-d)}. \tag{79}
\]

We now turn to the case of $d = 2$. In this case, we split the sum on the r. h. s. of (77) according to
\[
\sum_{e:|e-x|\leq|e|} = \sum_{e:|e-x|\leq|e| \text{ and } |e|\geq2|x|} + \sum_{e:|e-x|\leq|e| \text{ and } |e|<2|x|}
\leq \sum_{e:|e-x|\geq\frac{1}{2}|e| \text{ and } |e|\geq2|x|} + \sum_{e:|e-x|\leq|e| \text{ and } |e|\geq\frac{1}{2}|e|}
\]
so that
\[
\sum_{e:|e-x|\leq|e|} (|e - x| + 1)^{-2}(|e| + 1)^{-2}
\leq \sum_{e:|e|\geq2|x|} (\frac{1}{2}|e| + 1)^{-4} + (\frac{1}{2}|x| + 1)^{-2} \sum_{e:|e-x|\leq|e|} (|e - x| + 1)^{-2}
\leq C(|x| + 1)^{-2} + C(|x| + 1)^{-2} \log(|x| + 2). \tag{80}
\]
Combining (79) and (80), we gather
\[
\sum_{e} (|e - x| + 1)^{2(1-d)}(|e| + 1)^{2(1-d)}
\leq C(d)(|x| + 1)^{2(1-d)} \begin{cases} 1 & \text{for } d > 2 \\ \log(|x| + 2) & \text{for } d = 2 \end{cases}, \tag{81}
\]
which we insert into (76) to obtain (9).

### 7.6 Optimality of Corollary 1 for $p = 1$

In this section we will show by formal calculations that Corollary 1 is optimal by considering the regime $1 - \lambda \ll 1$. Recall that the Green function satisfies $\nabla^* a \nabla G(\cdot, x') = \delta(\cdot - x')$. Now let $a(e) = 1 + \epsilon \tilde{a}(e)$ for $\epsilon \ll 1$, where $\tilde{a}$ is i. i. d. with values at each edge taken in $[-1, 1]$. Furthermore we assume $\langle \tilde{a}(e) \rangle = 0$ as well as $\langle \tilde{a}(e)^2 \rangle = 1$. Note that this implies $a \in [1 - \epsilon, 1 + \epsilon] \subset [1/2, 3/2]$ (w. l. o. g. $\epsilon < 1/2$), but (by linearity of the equation in $a$) all results remain true with this new upper bound on $a$. Let us expand the Green function corresponding to $a$ in powers of $\epsilon$:
\[
G(x, y) = G_0(x, y) + \epsilon G_1(x, y) + \ldots
\]
Substituting into the defining equation for $G$, we find that to zeroth order in $\epsilon$, we have
\[
\nabla^* \nabla G_0(\cdot, x') = \delta(\cdot - x'),
\]
i.e. $G_0$ is the constant-coefficient Green function. Then to first order, it follows
\[ \nabla^* \nabla G_1(\cdot, x') + \nabla^* \tilde{a} \nabla G_0(\cdot, x') = 0. \]

Hence we have that
\[ G_1(x, x') = -\sum_e \nabla G_0(x, e) \tilde{a}(e) \nabla G_0(e, x'). \]

Since $\langle \tilde{a}(e) \rangle = 0$, we deduce $\langle G_1 \rangle = 0$ and consequently
\[ \langle (G_1(x, 0))^2 \rangle = \langle (G_1(x, 0) - \langle G_1(x, 0) \rangle)^2 \rangle = \sum_{e, e'} \nabla G_0(x, e) \nabla G_0(x, e') (\tilde{a}(e) \tilde{a}(e')) \nabla G_0(e, 0) \nabla G_0(e', 0). \]

Since the coefficients $\tilde{a}(x)$ are i. i. d. with variance 1, it follows
\[ \langle (G_1(x, 0) - \langle G_1(x, 0) \rangle)^2 \rangle = \sum_e (\nabla G_0(x, e))^2 (\nabla G_0(e, 0))^2. \]

The behavior of the constant-coefficient Green function $G_0$ is well-known, cf. [17, Theorem 4.3.1], and yields that $\langle \nabla G_0(e, 0) \rangle^2$ scales like $(|e| + 1)^{1-d}$ with a similar expression for $\langle \nabla G_0(x, e) \rangle^2$. Hence we find that
\begin{align*}
\langle (G_1(x, 0) - \langle G_1(x, 0) \rangle)^2 \rangle &\leq C(d) \sum_e (|e - x| + 1)(|e| + 1)^{2(1-d)} \text{ and} \\
\langle (G_1(x, 0) - \langle G_1(x, 0) \rangle)^2 \rangle &\geq \frac{1}{C(d)} \sum_e (|e - x| + 1)(|e| + 1)^{2(1-d)}. \quad (82)
\end{align*}

Thus (81) and (82) yield the upper bound
\[ \langle (G_1(x, 0) - \langle G_1(x, 0) \rangle)^2 \rangle \leq \frac{1}{C(d)} |x| + 1)^{2(1-d)} \begin{cases} 
1 
& \text{for } d > 2 \\
\log(|x| + 2) 
& \text{for } d = 2
\end{cases}. \]

If $d > 2$, a lower bound can be obtained by considering only the summand $e = [0, e_i]$ in (82). If $d = 2$, we restrict the sum to all $e$ such that $|e| \leq |x|$ and use $|e - x| \leq 2|x|$ in that region to obtain
\[ \langle (G_1(x, 0) - \langle G_1(x, 0) \rangle)^2 \rangle \geq \frac{1}{C(d)} \sum_{e | |e| \leq |x|} (|e - x| + 1)^{-2}(|e| + 1)^{-2} \]
\[ \geq \frac{1}{C(d)} (2|x| + 1)^{-2} \sum_{e | |e| \leq |x|} (|e| + 1)^{-2} \]
\[ \geq \frac{1}{C(d)} (|x| + 1)^{-2} \log(|x| + 2). \]

Thus Corollary 1 is indeed optimal in scaling.

### 7.7 Proof of Corollaries 2 and 3

**Proof of Corollary 2**

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Step 1. Proof in dimension \( d > 2 \). First of all, the triangle inequality in \( \langle (\cdot)^r \rangle^{1/r} \) yields

\[
\left\langle \left( \sum_x |u(x) - \langle u(x) \rangle|^p \right)^r \right\rangle^{\frac{1}{rp}} \leq \left( \sum_x \left\langle |u(x) - \langle u(x) \rangle|^{rp} \right\rangle^{\frac{1}{rp}} \right)^{\frac{1}{p}}.
\]  

(83)

Since \( u \) is the decaying solution of (10) with compactly supported right-hand side \( f \), it can be represented via the Green function:

\[
u(x) = \sum_y G(x, y) f(y),\]

(84)

Consequently, an application of the triangle inequality in \( \langle (\cdot)^{rp} \rangle^{1/(rp)} \) yields

\[
\left\langle |u(x) - \langle u(x) \rangle|^{rp} \right\rangle^{\frac{1}{rp}} \leq \sum_y \left\langle |G(x, y) - \langle G(x, y) \rangle|^{rp} \right\rangle^{\frac{1}{rp}} |f(y)|,
\]

so that we may use Corollary 1 to the effect of

\[
\left\langle |u(x) - \langle u(x) \rangle|^{rp} \right\rangle^{\frac{1}{rp}} \leq C(d, \lambda, \rho, r, p) \sum_y (|x - y| + 1)^{1-d} |f(y)|.
\]

(85)

We now insert (85) in (83) to obtain

\[
\left\langle \left( \sum_x |u(x) - \langle u(x) \rangle|^p \right)^r \right\rangle^{\frac{1}{rp}} \leq C(d, \lambda, \rho, r, p) \left( \sum_x \left( \sum_y (|x - y| + 1)^{1-d} |f(y)| \right)^p \right)^{\frac{1}{p}}.
\]

(86)

Now let us recall the Hardy-Littlewood-Sobolev inequality in \( \mathbb{R}^d \), see [19, Section 4.3] for a proof:

\[
\left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |x - y|^{-\alpha} f(y) \, dy \right)^p \right)^{\frac{1}{p}} \leq C(d, \alpha, p) \left( \int_{\mathbb{R}^d} |f(y)|^q \, dy \right)^{\frac{1}{q}}
\]

for all weight exponents \( 0 < \alpha < d \) and for all integrability exponents \( 1 < p, q < \infty \) related by \( 1 + \frac{1}{p} = \frac{\alpha}{d} + \frac{1}{q} \). A discrete version can easily be obtained by applying the continuum version to piecewise constant functions. We use the discrete version for \( \alpha = d - 1 \), that is,

\[
\left( \sum_x \left( \sum_y (|x - y| + 1)^{1-d} |f(y)| \right)^p \right)^{\frac{1}{p}} \leq C(d, p) \left( \sum_y |f(y)|^q \right)^{\frac{1}{q}},
\]

(87)

in which case the relation turns as desired into \( \frac{1}{q} + \frac{1}{d} = \frac{1}{p} \). Our assumption \( p \geq 2 \) and \( d > 2 \) ensure that \( q \) is indeed admissible for Hardy-Littlewood-Sobolev in the sense of the strict inequality \( q > 1 \).
Step 2. Changes if \( d = 2 \). In this case, using that \( f(y) \) is supported in \( \{ y : |y| \leq R \} \), (86) assumes the form

\[
\left\langle \left( \sum_{x:|x| \leq R} |u(x) - \langle u(x) \rangle|^p \right)^{\frac{1}{p}} \right\rangle \leq C(d, \lambda, \rho, r, p) \left( \sum_{x:|x| \leq R} \left( \sum_{y:|y| \leq R} (|x-y| + 1)^{1-d} \left( \log \frac{1}{|x-y|} \right) |f(y)| \right)^p \right)^{\frac{1}{p}} \]

\[
\leq C(d, \lambda, \rho, r, p) \left( \sum_{x} \left( \sum_{y} (|x-y| + 1)^{1-d} |f(y)| \right)^p \right)^{\frac{1}{p}}.
\]

As in Step 1, it remains to apply the discrete Hardy-Littlewood-Sobolev inequality, where we note that our assumption \( p > 2 \) now ensures \( q > 1 \) even for \( d = 2 \).

Proof of Corollary 3

Step 1. In this step, we derive the estimate

\[
\left\langle \left| \sum_{x} (u(x) - \langle u(x) \rangle) g(x) \right|^r \right\rangle^{\frac{1}{r}} \leq C(\rho, r) \left( \sum_{e} \left( \sum_{x} \sum_{y} (\text{osc}_{a(e)} G(x,y)) |f(y)| |g(x)| \right)^2 \right)^{\frac{1}{r}}. \quad (88)
\]

Indeed, it follows from the representation (84) that

\[
\left\langle \left| \sum_{x} (u(x) - \langle u(x) \rangle) g(x) \right|^r \right\rangle^{\frac{1}{r}} = \left\langle \left| \sum_{x} \sum_{y} (G(x,y) - \langle G(x,y) \rangle) f(y) g(x) \right|^r \right\rangle^{\frac{1}{r}}.
\]

Hence the \( L^p \)-version of SG (75), with \( 2p \) replaced by \( r \) (w. l. o. g. we may assume \( r \geq 2 \)), yields

\[
\left\langle \left| \sum_{x} (u(x) - \langle u(x) \rangle) g(x) \right|^r \right\rangle^{\frac{1}{r}} \leq C(\rho, r) \left( \sum_{e} \left( \sum_{x} \sum_{y} G(x,y) f(y) g(x) \right)^2 \right)^{\frac{1}{r}}.
\]

Since the only dependence on the coefficients \( a \) is through \( G \), we may use sublinearity of the oscillation to obtain (88).
Step 2. In this step, we estimate the right-hand side of \((88)\) as follows:

\[
\left\langle \left( \sum_e \left( \sum_x \sum_y \left( \osc_{a(e)} G(x, y) \right) |f(y)||g(x)| \right)^2 \right)^{\frac{1}{2}} \right\rangle \\
\leq C(d, \lambda, \rho, r) \left( \sum_x \left( \sum_e (|e - x| + 1)^{1-d}|g(x)| \right)^2 \right) \right. \\
\left. \times \left( \sum_y (|e - y| + 1)^{1-d}|f(y)| \right)^{\frac{1}{2}} \right) .
\]

Indeed, expanding the square on the l.-h. s. of \((89)\) and inserting \((57)\) yields

\[
\left\langle \left( \sum_e \left( \sum_x \sum_y \left( \osc_{a(e)} G(x, y) \right) |f(y)||g(x)| \right)^2 \right)^{\frac{1}{2}} \right\rangle \\
\leq C(\lambda) \sum_{x, x', y, y'} |\nabla G(x, e)||\nabla G(x', e)||\nabla G(e, y)||\nabla G(e, y')| \\
\times |g(x)||g(x')||f(y)||f(y')|.
\]

Consequently we obtain by the triangle inequality w. r. t. \(|\cdot|/r^{2/\tau}\) that

\[
\left\langle \left( \sum_e \left( \sum_x \sum_y \left( \osc_{a(e)} G(x, y) \right) |f(y)||g(x)| \right)^2 \right)^{\frac{1}{2}} \right\rangle \\
\leq C(d, \lambda) \left( \sum_{e, x, x', y, y'} \langle |\nabla G(x, e)|^{\frac{2}{\tau}}|\nabla G(x', e)||^{\frac{2}{\tau}}|\nabla G(e, y)|^{\frac{2}{\tau}}|\nabla G(e, y')|^{\frac{2}{\tau}} \right)^{\frac{1}{2}} \\
\times |g(x)||g(x')||f(y)||f(y')| \right)^{\frac{1}{2}} .
\]

Hölder’s inequality with respect to the ensemble \(\langle \cdot | \rangle\) and Theorem 1 yield

\[
\langle |\nabla G(x, e)|^{\frac{2}{\tau}}|\nabla G(x', e)||^{\frac{2}{\tau}}|\nabla G(e, y)|^{\frac{2}{\tau}}|\nabla G(e, y')|^{\frac{2}{\tau}} \rangle^{\frac{1}{\tau}} \\
\leq C(d, \lambda, \rho, r)((|e - x| + 1)(|e - x'| + 1)(|e - y| + 1)(|e - y'| + 1))^{1-d}.
\]

Hence \((89)\) follows from partly undoing the expansion of the square:

\[
\left( \sum_{e, x, x', y, y'} (|e - x| + 1)^{1-d}(|e - x'| + 1)^{1-d}(|e - y| + 1)^{1-d} \right) \\
\times (|e - y'| + 1)^{1-d}|g(x)||g(x')||f(y)||f(y')|^{\frac{1}{2}} \\
= \left( \sum_e \left( \sum_x (|e - x| + 1)^{1-d}|g(x)| \right)^2 \right)^{\frac{1}{2}} \\
\times \left( \sum_y (|e - y| + 1)^{1-d}|f(y)| \right)^{\frac{1}{2}} .
\]

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We integrate (92):

\[ \left( \sum_e \left( \sum_x (|e - x| + 1)^{1-d}|g(x)| \right)^\frac{p}{q} \right)^{\frac{1}{p}} \times \left( \sum_e \left( \sum_y (|e - y| + 1)^{1-d}|f(y)| \right)^\frac{p}{q} \right)^{\frac{1}{q}}, \quad (90) \]

with \( \hat{p} \) and \( p \) such that \( \frac{1}{\hat{p}} + \frac{1}{p} = 1 \) to be chosen later. We recall the Hardy-Littlewood-Sobolev inequality (87), i.e.

\[ \left( \sum_e \left( \sum_x (|e - x| + 1)^{1-d}|f(x)| \right)^\frac{p}{q} \right)^{\frac{1}{p}} \leq C(d,q) \left( \sum_x |f(x)|^q \right)^{\frac{1}{q}}, \]

if we choose \( p \) such that \( \frac{1}{q} = \frac{1}{d} + \frac{1}{p} \). (Here we require \( 1 < q < d \) so that in particular \( q < \infty \).) The Hardy-Littlewood-Sobolev inequality likewise yields

\[ \left( \sum_e \left( \sum_x (|e - x| + 1)^{1-d}|g(x)| \right)^\frac{p}{q} \right)^{\frac{1}{p}} \leq C(d,q) \left( \sum_x |g(x)|^q \right)^{\frac{1}{q}}, \]

where \( \frac{1}{q} = \frac{1}{d} + \frac{1}{p} = \frac{1}{d} + \frac{1}{2} - \frac{1}{p} \) and we require \( 1 < \hat{q} < d \). Inserting these estimates into (90) and then into Steps 2 and 1 yields Corollary 3.

### 7.8 Proof of Corollary 4

#### Step 1. Let \( u \) satisfy \( \nabla^* a \nabla u = 0 \) in \( \{ x : |x| \leq R \} \). We claim that for any function \( \eta(x) \) supported in \( \{ x : |x| < R \} \), we obtain the representation

\[ (\eta u)(x) = \sum_{e=[y,y'] : |e| \leq R} (u(y) \nabla G(e,x)a(e) \nabla \eta(e) - G(y,x) \nabla \eta(a(e) \nabla u(e)), \quad (91) \]

where we sum over all edges in \( \mathbb{E}^d \) of the form \([y,y']\) such that their midpoint is of distance at most \( R \) from the origin. We start by noting that even on the discrete level, some aspects of Leibniz rule survive, such as

\[ \nabla \zeta(e) a(e) \nabla (\eta u)(e) - \nabla (\eta \zeta)(e) a(e) \nabla u(e) = u(y) \nabla \zeta(e) a(e) \nabla \eta(e) - \zeta(y) \nabla \eta(a(e) \nabla u(e) \quad (92) \]

for any function \( \zeta : \mathbb{Z}^d \to \mathbb{R} \) and \( e = [y,y'] \in \mathbb{E}^d \). Indeed, (92) reduces to the elementary identity

\[ (\tilde{\zeta} - \zeta)(\tilde{\eta} \tilde{u} - \eta u) - (\tilde{\eta} \tilde{\zeta} - \eta \zeta)(\tilde{u} - u) = u(\tilde{\zeta} - \zeta)(\tilde{\eta} - \eta) - \zeta(\tilde{\eta} - \eta)(\tilde{u} - u). \]

We integrate (92):

\[ \sum_e \nabla \zeta a \nabla (\eta u) - \sum_e \nabla (\eta \zeta)a \nabla u = \sum_e \left( u \nabla \zeta a \nabla \eta - \zeta \nabla \eta a \nabla u \right) \quad (93) \]
and use it for \( \zeta = G(\cdot, x) \). By definition of \( G \), the first term on the l.-h.s. of (92) yields \((\eta u)(x)\). Since \( \eta G(\cdot, x) \) is supported in \( \{ y : |y| \leq R \} \), the second term on the l.-h.s. of (92) vanishes. This completes the step.

**Step 2.** We now use the representation obtained in Step 1 to obtain bounds on the gradient of \( u \) and consequently on the \( \alpha \)-Hölder norm of \( u \). Specifically, we claim that

\[
\left( \frac{\sup_{x: |x| \leq \frac{R}{p}} |u(x) - u(0)|}{\sup_{x: |x| \leq R} |u(x)|} \right)^p \leq C(d, \lambda, p) R^{\alpha p} R^{-p} \left( R^{d(p-1)} \sum_{e: |e| \leq \frac{R}{p}} \sum_{b: \frac{R}{p} \leq |b| \leq \frac{R}{p}} |\nabla \nabla G(e, b)|^p \right. \\
+ \left. R^{d(p-1)-p} \sum_{e: |e| \leq \frac{R}{p}} \sum_{x: |x| \leq \frac{R}{p}} |\nabla G(e, x)|^p \right),
\]

(94)

if \( \alpha < 1 \) and \( p > d \) are related by \( \alpha p = p - d \). To this end, we choose a cut-off function \( \eta \) for \( \{ x : |x| \leq \frac{R}{p} + 1 \} \) in \( \{ x : |x| \leq \frac{R}{p} - 1 \} \) (w. l. o. g. \( R > 8 \)). We restrict to \( |x| \leq \frac{R}{4} \) and take the derivative of (91) along the edge \( b \) to obtain

\[
\nabla u(b) = \sum_{e: \in [y, y']} (u(y)\nabla \nabla G(e, b) a(e) \nabla \eta(e) - (\nabla \eta(a(e) \nabla u(e)) \nabla G(y, b)).
\]

This implies

\[
|\nabla u(b)| \leq \frac{C(d)}{R} \sum_{e: [y, y'] |y|, |y'| \leq \frac{R}{4}} \left( |u(y)||\nabla \nabla G(e, b)| + |\nabla G(y, b)||\nabla u(e)| \right).
\]

(95)

Applying Hölder’s inequality and summing the \( p \)-th power of (95), we obtain

\[
\sum_{b: |b| \leq \frac{R}{4}} |\nabla u(b)|^p \leq C(d, p) R^{-p} \left( \left( \sum_{y: |y| \leq \frac{R}{4}} |u(y)|^q \right)^{p-1} \sum_{b: |b| \leq \frac{R}{4}} \sum_{e: |e| \leq \frac{R}{p}} |\nabla \nabla G(e, b)|^p \right. \\
+ \left. \left( \sum_{e: |e| \leq \frac{R}{p}} |\nabla u(e)|^q \right)^{p-1} \sum_{b: |b| \leq \frac{R}{4}} \sum_{y: |y| \leq \frac{R}{4}} |\nabla G(y, b)|^p \right),
\]

(96)

where \( q \) is the dual Hölder exponent of \( p \). Now we apply the following (discrete) Sobolev inequality: If \( \alpha < 1 \) and \( p > d \) are related by

\[
\alpha = 1 - \frac{d}{p},
\]

(97)

then we have that

\[
\sup_{x: |x| \leq \frac{R}{4}} \frac{|u(x) - u(0)|}{|x|^\alpha} \leq C(d, p) \left( \sum_{b: |b| \leq \frac{R}{4}} |\nabla u(b)|^p \right)^{\frac{1}{p}}.
\]

(98)
poli’s estimate (53) implies
To estimate the second summand on the right-hand side, we note that Cacciop-
by
α < 1 for all α if
Substituting (99) and (100) into (96) yields the claim of this step.
Likewise the second double-sum is bounded by
Together with Jensen’s inequality (here we need q ≤ 2, that is p ≥ d from (97)), we obtain that
Substituting (99) and (100) into (96) yields the claim of this step.
Step 3. Using (98) and bounding the Green function, we conclude that
for all α < 1, p < ∞, and R < ∞, where the outer supremum is taken over all solutions u(x) to ∇*a∇u = 0 in \{x : |x| ≤ R\}. Indeed, Theorem 1 applied to the result (94) of Step 2 yields
if α and p are related by (97). In the domains of e and b, we have |e − b| + 1 ≥ |b| − |e| ≥ R/8. Therefore the first double-sum on the right-hand side is bounded by
C(d, p) R^{2d−pd},
Likewise the second double-sum is bounded by
C(d, p) R^{2d+p(1−d)}.

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If (97) holds, we thus conclude that

$$\left\langle \left( \sup_u \frac{\sup_{x:|x| \leq R} |u(x) - u(0)|}{|x|^\alpha} \right)^p \right\rangle \leq C(d, \lambda, \rho, p).$$

In the region \( \{x : \frac{R}{2} \leq |x| \leq R\} \), it obviously holds

$$\frac{|u(x) - u(0)|}{|x|^\alpha} \leq 2 \frac{8^\alpha}{R^\alpha} \sup_{x:|x|\leq R} |u(x)|.$$

Thus we have obtained (101) for \( p \) and \( \alpha \) such that (97) holds. Since in (97), \( \alpha \to 1 \) as \( p \to \infty \) and since we can always decrease \( p \) and \( \alpha \) in the conclusion (101) (in \( p \) this follows from Jensen’s inequality), the estimate (101) indeed holds for arbitrary \( p < \infty \) and \( \alpha < 1 \).

### 7.9 Proof of Lemmas 2 and 3

**Proof of Lemma 2**

Without loss of generality, we may assume \( \langle \zeta^2 \rangle = 1 \). The elementary inequality \( \zeta^2 \log \zeta^2 - \zeta^2 + 1 \leq (\zeta^2 - 1)^2 \) then yields

$$\langle \zeta^2 \log \zeta^2 \rangle = \langle \zeta^2 \log \zeta^2 - \zeta^2 + 1 \rangle \leq (\langle \zeta^2 - 1 \rangle)^2.$$

Since \( (\zeta^2 - 1)^2 = (|\zeta| - 1)^2 (|\zeta| + 1)^2 \), we find that

$$\langle \zeta^2 \log \zeta^2 \rangle \leq (\langle(|\zeta| + 1)^2 \rangle \sup_{a} (|\zeta| - 1)^2).$$

Since \( \langle \zeta^2 \rangle = 1 \), there exists \( a_* \in [\lambda, 1] \) such that \( |\zeta(a_*)| \leq 1 \). It follows that

$$|\zeta(a)| - 1 \leq |\zeta(a)| - |\zeta(a_*)| \leq |\zeta(a) - \zeta(a_*)| \leq \text{osc}_a \zeta(a).$$

Likewise there exists \( a^* \in [\lambda, 1] \) such that \( |\zeta(a^*)| \geq 1 \) and therefore

$$1 - |\zeta(a)| \leq |\zeta(a^*)| - |\zeta(a)| \leq |\zeta(a^*) - \zeta(a)| \leq \text{osc}_a \zeta(a).$$

Hence it follows that

$$\langle \zeta^2 \log \zeta^2 \rangle \leq (\langle(|\zeta| + 1)^2 \rangle \text{osc}_a \zeta)^2.$$ 

Finally we have that

$$\langle(|\zeta| + 1)^2 \rangle \leq \langle 2 \zeta^2 + 2 \rangle = 4,$$

and the combination of the previous two inequalities yields (20) with constant \( \rho = \frac{1}{8} \).

**Proof of Lemma 3**

The following is a simple adaptation of the usual tensorization proof, cf. [16, Theorem 4.4]. Take any enumeration \( (e_n)_{n \geq 1} \) of the edge set \( E^d \) and denote by \( \langle \cdot \rangle_n \) the \( e_n \)-marginal of the (product) ensemble \( \langle \cdot \rangle \). We assume that every marginal satisfies the LSI

$$\langle \zeta^2 \log \frac{\zeta^2}{\langle \zeta^2 \rangle_n} \rangle_n \leq \frac{2}{\rho} \left( \text{osc}_{a \in [\lambda, 1]} \zeta \right)^2.$$
for all $\zeta : [\lambda, 1] \to \mathbb{R}$. Replacing $\zeta^2$ by $f$ in the definition of the LSI, it suffices to prove

$$\langle f \log \frac{f}{\langle f \rangle} \rangle \leq \frac{2}{\rho} \sum_{n=1}^{\infty} \left( \frac{\text{osc}_{a(e_n)} \sqrt{f}}{f} \right)^2$$

for all positive random variables $f : \Omega \to (0, \infty)$. By a simple density argument, it suffices to consider local random variables, i.e. $f$ that depend on $a$ only through a finite number of sites so that the above sum is finite. We denote iteratively $f_0 := f$ and $f_n := \langle f_{n-1} \rangle_n$. Thus $f_n$ is the average of $f$ over the first $n$ edges. Then the left-hand side of (6) can be expressed as a telescope sum (a finite sum for local random variables):

$$\langle f \log f \rangle - \langle f \rangle \log \langle f \rangle = \sum_{n=1}^{\infty} \langle f_n \log f_{n-1} - f_n \log f_n \rangle$$
$$= \sum_{n=1}^{\infty} \langle f_{n-1} \log f_{n-1} - \langle f_{n-1} \rangle_n \log \langle f_{n-1} \rangle_n \rangle. \quad (102)$$

The assumption of single-edge LSI yields

$$\langle f_{n-1} \log f_{n-1} \rangle_n - \langle f_{n-1} \rangle_n \log \langle f_{n-1} \rangle_n \leq \frac{2}{\rho} \left( \text{osc}_{a(e_n)} \sqrt{f_{n-1}} \right)^2. \quad (103)$$

Notice that the definition of $f_{n-1}$ immediately yields $f_{n-1} = \langle f \rangle_{<n}$, where we have abbreviated the ensemble average over the first $n - 1$ edges as $\langle \cdot \rangle_{<n}$. We clearly have

$$\text{osc}_{a(e_n)} \sqrt{f_{n-1}} = \left( \sup_{a(e_n)} \langle f \rangle_{<n} \right)^{\frac{1}{2}} - \left( \inf_{a(e_n)} \langle f \rangle_{<n} \right)^{\frac{1}{2}} \leq \left( \sup_{a(e_n)} \langle f \rangle_{<n} \right)^{\frac{1}{2}} - \left( \inf_{a(e_n)} \langle f \rangle_{<n} \right)^{\frac{1}{2}}.$$

By monotonicity of the square root, it follows

$$\text{osc}_{a(e_n)} \sqrt{f_{n-1}} \leq \left( \langle \sup_{a(e_n)} \sqrt{f} \rangle_{<n} \right)^{\frac{1}{2}} - \left( \langle \inf_{a(e_n)} \sqrt{f} \rangle_{<n} \right)^{\frac{1}{2}}.$$

Consequently the triangle inequality w. r. t. $\langle (\cdot)^2 \rangle_{<n}$ on the right-hand side yields

$$\text{osc}_{a(e_n)} \sqrt{f_{n-1}} \leq \left( \langle \sup_{a(e_n)} \sqrt{f} - \inf_{a(e_n)} \sqrt{f} \rangle_{<n} \right)^{\frac{1}{2}},$$

which by definition of $\text{osc}_{a(e_n)}$ can be written as

$$\text{osc}_{a(e_n)} \sqrt{f_{n-1}} \leq \left( \langle \text{osc}_{a(e_n)} \sqrt{f} \rangle_{<n} \right)^{\frac{1}{2}}. \quad (104)$$

Finally we collect (102), (103), and (104) to obtain

$$\langle f \log f \rangle - \langle f \rangle \log \langle f \rangle \leq \frac{1}{2\rho} \sum_{n=1}^{\infty} \left( \langle \text{osc}_{a(e_n)} \sqrt{f} \rangle_{<n} \right)^2,$$

which is the LSI (6) for $f = \zeta^2$. 

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