Stationary black diholes

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Abstract

In this paper we present and analyze the simplest physically meaningful model for stationary black diholes – a binary configuration of counter-rotating Kerr-Newman black holes endowed with opposite electric charges – elaborated in a physical parametrization on the basis of one of the Ernst-Manko-Ruíz equatorially antisymmetric solutions of the Einstein-Maxwell equations. The model saturates the Gabach-Clement inequality for interacting black holes with struts, and in the absence of rotation it reduces to the Emparan-Teo electric dihole solution. The physical characteristics of each dihole constituent satisfy identically the well-known Smarr’s mass formula.

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I. INTRODUCTION

In the paper [1] Empanar and Teo considered an electrostatic solution of the Einstein-Maxwell equations for a non-extreme dihole – a configuration consisting of two non-extremal Reissner-Nordström black holes [2, 3] endowed with equal masses and opposite charges of the same magnitude. Extension of their results to the case of rotation was referred to by those authors as an interesting work to be done in the future. The problem of obtaining such an extension seems to have been attacked in a recent paper of Cabrera-Mungúia et al [4] who constructed a specific 4-parameter exact solution for two oppositely charged counter-rotating Kerr-Newman (KN) black holes [5]. Though technically the paper [4] is correct, the solution itself is, however, frankly unphysical because of the presence in it of non-vanishing magnetic charges induced by rotation of electric charges. As a consequence, the usual Smarr mass formula [6] does not hold for the black-hole constituents comprising that binary configuration and, moreover, the authors of [4] failed to find a concise expression for the quantity $\sigma$ which would be similar for instance to that of the physically parametrized Bretón-Manko (BM) solution [7, 8].

The present paper aims at working out a correct 4-parameter model for stationary diholes with antiparallel rotation of its constituents. To accomplish this task, we shall make use of a 5-parameter asymptotically flat specialization of the Ernst-Manko-Ruiz (EMR) equatorially antisymmetric solution [9, 10] possessing arbitrary parameters of electric and magnetic dipole moments, which will enable us to eliminate the individual magnetic charges of the constituents by choosing appropriately the values of the latter dipole parameters. This will secure the validity of Smarr’s formula for each black-hole constituent and, in turn, will enable us to find a simple expression for $\sigma$ in terms of the Komar quantities [11] which is a key point for elaborating the physical parametrization of the whole model. After having reached our main objective, we will prove that, similar to the BM model of equally charged counter-rotating black-hole constituents, the configuration obtained for KN black holes with opposite charges verifies (and actually saturates) the inequality for interacting black holes with struts recently derived by Gabach Clement [12].
II. THE 5-PARAMETER ASYMPTOTICALLY FLAT EMR SOLUTION IN $\sigma$-REPRESENTATION

A key result of the paper [4] is its expressions (14) for the axis data $e(z)$ and $f(z)$ allowing one to construct in a straightforward manner the corresponding entire metric with the aid of the general formulas of the extended $N$-soliton solution [13] (we also refer the interested reader to Appendix of Ref. [10] for a complete set of algebraic relations involved in the construction procedure of the $N = 2$ case). However, an explanation the authors of [4] give to the origin of those expressions – the solution (11) of a complicated system of algebraic equations for certain metric functions – is in fact misleading, and below we will give a veritable simple derivation of their data (14) with one additional arbitrary real parameter representing a magnetic dipole moment.

As a starting point of the derivation procedure we take the following axis data obtained in the paper [9] for an equatorially antisymmetric spacetime [14] with both electric and magnetic dipole moments:

$$e(z) = \frac{z^2 - b_1 z + b_2}{z^2 + b_1 z + b_2}, \quad f(z) = \frac{c_2}{z^2 + b_1 z + b_2},$$

where $b_1, b_2$ and $c_2$ are arbitrary complex constants. Mention that this data rewritten in an equivalent representation was used in the paper [9] for constructing the corresponding Ernst potentials [15] $E$ and $\Phi$ of a $6$-parameter EMR solution. If nevertheless one opts to work directly with the above (1), then the asymptotic flatness of the solution implies immediately that $b_1$ is a real constant related to the total mass $2M$ of the binary configuration as $b_1 = 2M$. Choosing then the constant $c_2$ in the form $c_2 = 2(q + ib)$, the real parameters $q$ and $b$ being associated, respectively, with the electric and magnetic dipole moments, and also formally setting $b_2 = c - i\delta$, we arrive at the 5-parameter axis data

$$e(z) = e_-, \quad f(z) = \frac{2(q + ib)}{e_+}, \quad e_+ = z^2 + 2Mz + c - i\delta,$$

in which the real constants $c$ and $\delta$ should yet be related to some physical or geometrical characteristics.

Recall now that the extended multi-soliton solutions involve the constants $\alpha_n$ which satisfy the algebraic equation [16]

$$e(z) + \bar{e}(z) + 2f(z)\bar{f}(z) = 0$$

in which the real constants $c$ and $\delta$ should yet be related to some physical or geometrical characteristics.
(the bar over a symbol means complex conjugation), and in the equatorially antisymmetric case these can be chosen in the form

\[ \alpha_1 = -\alpha_4 = \frac{1}{2}R + \sigma, \quad \alpha_2 = -\alpha_3 = \frac{1}{2}R - \sigma, \]  

(4)

where \( R \) is a real constant representing the coordinate separation of the sources, and the parameter \( \sigma \) can take non-negative real or pure imaginary values (see Fig. 1). Then instead of the constants \( c \) and \( \delta \) from the axis data (2) one is able to introduce the new parameters \( R \) and \( \sigma \) by equating coefficients at the same powers of \( z \) on the two sides of the equation

\[ \frac{e_-}{e_+} + \frac{\bar{e}_-}{\bar{e}_+} + \frac{4(q^2 + b^2)}{e_+ \bar{e}_+} = \frac{2 \prod_{n=1}^{4} (z - \alpha_n)}{e_+ \bar{e}_+}, \]  

(5)

thus yielding

\[ c = 2M^2 - \frac{1}{4}R^2 - \sigma^2, \quad \delta = \sqrt{(R^2 - 4M^2)(M^2 - \sigma^2) - 4(q^2 + b^2)}. \]  

(6)

Accounting for (6), the 5-parameter axis data (2) finally assumes the form

\[ e(z) = \frac{e_-}{e_+}, \quad f(z) = \frac{2(q + ib)}{e_+} + \bar{e}_-, \quad e_\pm = z^2 \mp 2Mz + 2M^2 - \frac{1}{4}R^2 - \sigma^2 - i\delta, \]  

(7)

and, by setting \( b = 0 \) in (3) and (7), one recovers the axis data (14) of [4] (and consequently the quantities \( \beta_{1,2} \) and \( f_{1,2} \) in (11) of [4] via the simple fraction decomposition of \( e(z) \) and \( f(z) \)). It is worth noting that this procedure of changing parameters in the axis data was already described in application to the case of identical counter-rotating uncharged black holes [17] and, moreover, has been recently used for obtaining a physical parametrization of the BM solution [8]. Furthermore, by virtue of the equatorial antisymmetry, the axis condition for the solution defined by the axis data (7) is satisfied automatically, and therefore there is no need to solve any additional algebraic equations for the metric functions.

As it is straightforward to elaborate by purely algebraic computing the explicit form of the Ernst potentials defined by the axis data (7), as well as the form of the corresponding metric functions \( f, \gamma \) and \( \omega \) entering the stationary axisymmetric line element

\[ ds^2 = f^{-1}\left[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\right] - f(d\tau - \omega d\varphi)^2, \]  

(8)

below we will restrict ourselves to only writing out the final expressions which reproduce and generalize the analogous formulas of the paper [4]. Then for \( E \) and \( \Phi \) we have

\[ E = \frac{A - B}{A + B}, \quad \Phi = \frac{C}{A + B}. \]  

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\[ A = R^2[M^2(R^2 - 4\sigma^2) - 4(q^2 + b^2)](R_+ - R_-)(r_+ - r_-) + 4\sigma^2[M^2(R^2 - 4\sigma^2) + 4(q^2 + b^2)](R_+ - r_+)(R_- - r_-) + 2R\sigma(R^2 - 4\sigma^2)[R\sigma(R_+ r_- + R_- r_+)] + i\delta(R_+ r_- - R_- r_+), \]
\[ B = 2MR\sigma(R^2 - 4\sigma^2)[R\sigma(R_+ + R_- + r_+ + r_-) - (2M^2 - i\delta)(R_+ - R_- - r_+ + r_-)], \]
\[ C = 4(q + ib)\sigma[(R + 2\sigma)(R\sigma - 2M^2 - i\delta)(r_+ - R_-) + (R - 2\sigma)(R\sigma + 2M^2 + i\delta) \times (r_- - R_+)], \]
where
\[ R_\pm = \sqrt{\rho^2 + (z + \frac{1}{2}R \pm \sigma)^2}, \quad r_\pm = \sqrt{\rho^2 + (z - \frac{1}{2}R \pm \sigma)^2}, \]
while the metric functions are given by the expressions
\[ f = \frac{AA - B\bar{B} + CC}{(A + B)(A + \bar{B})}, \quad e^{2\gamma} = \frac{AA - B\bar{B} + CC}{16\rho^4\sigma^4(R^2 - 4\sigma^2)^2 R_+ R_- r_+ r_-}, \]
\[ \omega = -\frac{\text{Im}[2G(A + \bar{B}) + C\bar{I}]}{AA - B\bar{B} + CC}, \]
\[ G = -zB + R\sigma\{2R[M^2(R^2 - 4\sigma^2) - 2(q^2 + b^2)](R_- r_+ + R_+ r_-) + 4\sigma[M^2(R^2 - 4\sigma^2) + 2(q^2 + b^2)](r_+ r_- - R_+ R_-) + M(R + 2\sigma)((R + 2\sigma)^2(R\sigma + 2M^2 - i\delta) - 8(q^2 + b^2))(R_- - r_+) + M(R - 2\sigma)((R - 2\sigma)^2(R\sigma - 2M^2 + i\delta) + 8(q^2 + b^2))(R_+ - r_-)\}, \]
\[ I = -zC + 4M(q + ib)[R^2(2M^2 - 2\sigma^2 + i\delta)(R_+ r_+ + R_- r_-) + 2\sigma^2(R^2 - 4M^2 - 4i\delta)(R_+ R_- + r_+ r_-)] - 2(q + ib)(R^2 - 4\sigma^2) \times [2M((R\sigma + 2M^2 + i\delta)R_+ r_- - (R\sigma - 2M^2 - i\delta)R_- r_+) + R\sigma \times [(R\sigma + 6M^2 + i\delta)(R_+ + r_-) + (R\sigma - 6M^2 - i\delta)(R_- + r_+) + 8MR\sigma]]. \] (11)

The \( t \) and \( \varphi \) components of the electromagnetic 4-potential are defined by the formulas
\[ A_t = -\text{Re}\left(\frac{C}{A + B}\right), \quad A_\varphi = \text{Im}\left(\frac{I}{A + B}\right), \]
and these complete the general mathematical description of the 5-parameter EMR solution in \( \sigma \)-parametrization.

At this point, several remarks on the formulas (9)-(12) might be appropriate. First, the above representation of the 5-parameter EMR solution is fully equivalent to the known description of that solution worked out in the papers \[9, 10\] (with the NUT parameter \( \nu \) set equal to zero). Second, it is highly important to underline that the arbitrary parameter \( \sigma \) of
the solution is not restricted exclusively to real values (contrary to what was assumed in [4]) but can also take pure imaginary values determining the hyperextreme part of the solution. The significance of this point will be fully understandable later on when we express \( \sigma \) in terms of the Komar quantities. Third, the \( \sigma \)-representation of the EMR solution should be only considered as an intermediate parametrization that could be suitable for elaborating the final physical representation in which \( \sigma \) must be replaced by a rotation parameter.

To gain a better insight into the structure of the EMR solution, let us consider its first four Beig-Simon multipole moments [18–20] which can be found with the aid of the Hoenselaers-Perjés procedure [21, 22]:

\[
M_0 = 2M, \quad M_1 = 0, \quad M_2 = \frac{1}{2}M(R^2 - 8M^2 + 4\sigma^2), \quad M_3 = 0, \\
J_0 = J_1 = 0, \quad J_2 = 2M\delta, \quad J_3 = 0, \\
Q_0 = 0, \quad Q_1 = 2q, \quad Q_2 = 0, \quad Q_3 = \frac{1}{2}q(R^2 - 8M^2 + 4\sigma^2) - 2b\delta, \\
B_0 = 0, \quad B_1 = 2b, \quad B_2 = 0, \quad B_3 = \frac{1}{2}b(R^2 - 8M^2 + 4\sigma^2) + 2q\delta
\]  

(13)

\( M_i, J_i, Q_i \) and \( B_i \) define, respectively, the mass, angular momentum, electric and magnetic multipole moments), whence it follows the asymptotic flatness of the solution \( (J_0 = 0) \), the total mass \( M_0 \) of the configuration being equal to \( 2M \), and total angular momentum \( J_1 \) being zero due to counterrotation. In the absence of net charges the parameters \( q \) and \( b \) define the electric and magnetic dipole moments, respectively, which means that the two sources in the EMR solution are endowed with opposite electric and magnetic charges.

It is clear from the above form of the multipole moments that the special \( b = 0 \) case of the EMR solution considered in [4] is characterized by zero total magnetic dipole moment, and this fact explains the intrinsically non-physical character of that particular solution. Indeed, the magnetic dipole moment \( 2b \) of the 5-parameter EMR solution is a result of the following two non-zero contributions – one coming from the rotating electric charges and the other originated by the opposite magnetic charges. The electric contribution is twice the magnetic dipole moment created by one rotating electric charge, so by demanding \( b = 0 \), Cabrera-Munguia et al introduced in [4] a specific non-vanishing magnetic dipole moment due to magnetic charges, antiparallel to that created by electric charges. Those authors apparently confused the case of counter-rotating opposite charges and the BM configuration in which the counter-rotating charges have the same signs and hence the total magnetic and electric dipole moments are both equal to zero intrinsically. Therefore, a physically
meaningful dihole solution arising from the 5-parameter EMR configuration must have zero individual magnetic charges and, at the same time, a non-zero magnetic dipole moment generated by counterrotation of opposite electric charges.

The individual magnetic charges in the 5-parameter EMR solution can be eliminated by means of the condition \[ A_t(\rho = 0, z = \alpha_1) - A_t(\rho = 0, z = \alpha_2) = 0, \] (14)
which can be easily solved for \( b \). Then from (9), (10) and (12) we get

\[ b^2 = \frac{4q^2[(R^2 - 4M^2)(M^2 - \sigma^2) - 4q^2]}{(R^2 - 4M^2)^2 + 16q^2}, \] (15)

and this condition, together with the formulas (9)-(12) with real \( \sigma \), provide one with a \( \sigma \)-representation of the physically meaningful 4-parameter model for a stationary black dihole whose constituents are counter-rotating. It can be verified by a direct calculation that each black-hole constituent of such a model verifies the well-known Smarr mass formula identically.

III. THE 4-PARAMETER DIHOLE SOLUTION IN PHYSICAL PARAMETRIZATION

As was already mentioned, the \( \sigma \)-representation of the solution is only an intermediate step on the way of obtaining the physical parametrization in terms of the Komar quantities. Once the \( \sigma \)-representation is known, our further actions are the following: we must first try to express the parameter \( q \) in terms of the individual Komar charge \( Q \) of any of the dihole constituents, thus rewriting the solution in the parameters \( M, R, Q \) and \( \sigma \), and then find the form of \( \sigma \) in terms of \( M, R, Q \) and \( J \), \( J \) being the individual Komar angular momentum, from Smarr’s mass formula, by considering the latter an algebraic equation for \( \sigma \).

The mass formula for black holes discovered by Smarr [6] relates the mass \( M \), angular momentum \( J \) and charge \( Q \) of a black hole to several quantities evaluated on the horizon: the surface gravity \( \kappa \), horizon’s area \( S \) and angular velocity \( \Omega^H \), and the electric potential \( \Phi^H \). The formula reads

\[ M = \frac{1}{4\pi} \kappa S + 2J\Omega^H + Q\Phi^H = \sigma + 2J\Omega^H + Q\Phi^H, \] (16)
the Komar quantities $M$, $J$ and $Q$ being defined by the integrals \[23\]

\[
M = -\frac{1}{8\pi} \int_H \omega \Omega_z d\varphi dz, \tag{17}
\]

\[
J = \frac{1}{8\pi} \int_H \omega \left[-1 - \frac{1}{2} \omega \Omega_z + \tilde{A}_\varphi A'_\varphi, z + (A_\varphi A'_\varphi), z\right] d\varphi dz, \tag{18}
\]

\[
Q = \frac{1}{4\pi} \int_H \omega A'_\varphi, z d\varphi dz, \tag{19}
\]

with $\Omega = \text{Im}(\mathcal{E})$, $A'_\varphi = \text{Im}(\Phi)$, $\tilde{A}_\varphi = A_\varphi + \omega A_t$ (note that the metric functions $\omega$ and $\gamma$, as well as the potential $\tilde{A}_\varphi$, take constant values on the horizon), while the form of the constants $\kappa$, $S$, $\Omega^H$ and $\Phi^H$ is given by the formulas \[23, 25\]

\[
\kappa = \sqrt{-\omega^{-2} e^{-2\gamma}}, \quad S = 4\pi \sigma \sqrt{-\omega^2 e^{2\gamma}}, \quad \Omega^H = \omega^{-1}, \quad \Phi^H = -A_t - \Omega^H A_\varphi. \tag{20}
\]

For our dihole solution, the calculation of the individual charge $Q$ of the upper black-hole constituent, whose horizon is represented by the null hypersurface $\rho = 0$, $\frac{1}{2} R - \sigma \leq z \leq \frac{1}{2} R + \sigma$, leads to a cubic equation for $q$ which has to be solved in order to pass from the latter $q$ to the Komar $Q$ in the formulas determining the solution. It is remarkable, however, that the need to solve a cubic equation can be circumvented by an appropriate change of the parameter $q$. Thus, by introducing a new parameter $q$ via the relation

\[
q^2 = q^2 + b^2, \tag{21}
\]

which has some analogy with a duality rotation of the electromagnetic Ernst potential, we find from (15) and (21) the form of $q$ and $b$ in terms of $q$:

\[
q = q(R^2 - 4M^2)/\tau, \quad b = 2q\delta'/\tau, \quad \delta' = \sqrt{(R^2 - 4M^2)(R^2 - \sigma^2) - 4q^2}, \quad \tau = \sqrt{(R^2 - 4M^2)(R^2 - 4\sigma^2) - 16q^2}, \tag{22}
\]

and this redefinition of the parameter $q$ permits us to obtain from (19) a simple expression for the Komar charge $Q$ in terms of $q$:

\[
Q = -2q(R + 2M)/\tau, \tag{23}
\]

whence we readily get the inverse dependence of $q$ on $Q$:

\[
q = -\frac{Q \sqrt{(R^2 - 4M^2)(R^2 - 4\sigma^2)}}{2 \sqrt{(R + 2M)^2 + 4Q^2}}. \tag{24}
\]
The above formula for $q$ permits us, by rewriting the dihole solution in terms of the parameters $M$, $R$, $Q$ and $\sigma$, to obtain the quantities $\Omega^H$ and $\Phi^H$ that we need for finding $\sigma$:

$$\Omega^H = \frac{\sqrt{(R - 2M)[(R + 2M)^2 + 4Q^2][(R + 2M)(M^2 - \sigma^2) - Q^2(R - 2M)]}}{(R + 2M)[2M(R + 2M)(M + \sigma) - Q^2(R - 4M - 2\sigma)]},$$

$$\Phi^H = \frac{Q(R - 2M)[(R + 2M)(M + \sigma) + 2Q^2]}{(R + 2M)[2M(R + 2M)(M + \sigma) - Q^2(R - 4M - 2\sigma)]}. \quad (25)$$

Finally, after the substitution of (25) into the Smarr formula (16) in which we can put $J = Ma$, $a$ being the angular momentum per unit mass of the upper black hole, we obtain by a simple calculation the desired expression for $\sigma$ in terms of the physical quantities $M$, $a$, $Q$ and $R$:

$$\sigma = \sqrt{M^2 - \left(\frac{M^2 a^2[(R + 2M)^2 + 4Q^2]}{M(R + 2M) + Q^2} + Q^2\right) \frac{R - 2M}{R + 2M}}. \quad (26)$$

This formula for $\sigma$ is the central result of our paper. Now the dihole solution can be rewritten in the physical parameters, its Ernst potentials $\mathcal{E}$ and $\Phi$ assuming the form

$$\mathcal{E} = \frac{A - B}{A + B}, \quad \Phi = \frac{C}{A + B},$$

$$A = R^2(M^2 - Q^2\nu)(R_+ + R_-)(r_+ - r_-) + 4\sigma^2(M^2 + Q^2\nu)(R_+ - r_+)(R_- - r_-) + 2R\sigma[R\sigma(R_+r_- + R_-r_+) + iMa\mu(R_+r_- - R_-r_+)],$$

$$B = 2MR\sigma[R\sigma(R_+ + R_- + r_+ + r_-) - (2M^2 - iMa\mu)(R_+ - R_- - r_+ + r_-)],$$

$$C = 2C_0R\sigma[(R + 2\sigma)(R\sigma - 2M^2 - iMa\mu)(r_+ - R_-) + (R - 2\sigma)(R\sigma + 2M^2 + iMa\mu)(r_- - R_+)], \quad (27)$$

where the dimensionless quantities $\mu$, $\nu$ and $C_0$ are defined as

$$\mu = \frac{R^2 - 4M^2}{M(R + 2M) + Q^2}, \quad \nu = \frac{R^2 - 4M^2}{(R + 2M)^2 + 4Q^2}, \quad C_0 = -\frac{Q(R^2 - 4M^2 + 2iMa\mu)}{(R + 2M)(R^2 - 4\sigma^2)}, \quad (28)$$

and the final form of the metric coefficients $f$, $\gamma$ and $\omega$ is the following:

$$f = \frac{A\bar{A} - B\bar{B} + C\bar{C}}{(A + B)(\bar{A} + \bar{B})}, \quad e^{2\gamma} = \frac{A\bar{A} - B\bar{B} + C\bar{C}}{16R^4\sigma^4R_+R_-r_+r_-}, \quad \omega = -\frac{\text{Im}[2G(\bar{A} + \bar{B}) + C\bar{I}]}{AA - BB + CC},$$

$$G = -zB + R\sigma\{(R(2\sigma - 2M^2 - iMa\mu)(R_+r_- + R_-r_+) + 2\sigma(2M^2 + Q^2\nu)(r_+r_- + R_+R_-)
+ M[(R + 2\sigma)(R\sigma - 2M^2 + iMa\mu) + 2(R - 2\sigma)Q^2\nu](R_+ - r_-)
+ M[(R - 2\sigma)(R\sigma + 2M^2 - iMa\mu) - 2(R + 2\sigma)Q^2\nu](R_- - r_+),$$

$$I = -zC + 2C_0M[R^2(2\sigma^2 + 2M^2 - iMa\mu)(R_+r_- + R_-r_+)
+ 2\sigma^2(R^2 - 4M^2 - 2iMa\mu)(R_+r_- + r_+r_-)] - C_0(R^2 - 4\sigma^2)$$
\begin{align}
\times & \{2M[R\sigma(R_r - R_- r_+)] + (2M^2 + iMa\mu)(R_r + R_- r_+) + R\sigma R \sigma \\
& \times (R_+ + R_- + r_+ + r_-) + (6M^2 + iMa\mu)(R_+ - R_- - r_+ + r_-) + 8MR\sigma \}. \quad (29)
\end{align}

It should be noted that \( \sigma \) in the above (27)-(29) is no longer an independent parameter, having conceded that role to the constant \( a \). From (26) it follows that \( \sigma \), depending on interrelations between the parameters \( M, a, Q \) and \( R \), can automatically take on (non-negative) real or pure imaginary values, thus describing not only the binary configurations of black holes but also of hyperextreme objects. That is why \( \sigma \)'s taking pure imaginary values (along with the real ones) in the EMR solution (9) is highly important for the mathematical equivalence of the parameter sets \( (M, Q, R, \sigma) \) and \( (M, Q, R, a) \), and consequently for the correctness of the entire reparametrization procedure.

IV. THE LIMITS AND PHYSICAL PROPERTIES OF DIHOLE SOLUTION

The main limits of the dihole solution can be well seen from the formula (26) for \( \sigma \). Thus, in the absence of rotation \( (a = 0) \) the solution reduces to the Emparan-Teo electrostatic non-extreme dihole spacetime [1] whose physical form was found in the paper [26]. In the pure vacuum limit \( (Q = 0) \) the solution represents a vacuum specialization of the BM equatorially antisymmetric binary configuration whose physical parametrization was elaborated in the paper [17] on the basis of Varzugin’s expression for the quantity \( \sigma \) [27]. When \( R \to \infty \) (no interaction between the dihole constituents), one gets from (26) \( \sigma = (M^2 - a^2 - Q^2)^{1/2} \), which is characteristic of a single KN black hole.

By construction, the upper KN constituent has mass \( M \), angular momentum \( Ma \) and charge \( Q \), whereas the analogous characteristics of the lower constituent are \( M, -Ma \) and \( -Q \), respectively. The strut separating the two constituents provides us with the information about the interaction force [28], the latter being defined by the expression
\begin{align}
\mathcal{F} = & \frac{1}{4}(e^{-\gamma_0} - 1) = \frac{M^2(R + 2M)^2 + Q^2 R^2}{(R + 2M)^2(R^2 - 4M^2)} \\
= & \frac{1}{R^2 - 4M^2} \left( M^2 + Q^2 - \frac{4MQ^2(R + M)}{(R + 2M)^2} \right), \quad (30)
\end{align}

where \( \gamma_0 \) is the value of the metric function \( \gamma \) on the strut, and one can see that \( \mathcal{F} \) cannot take zero value at any finite separation \( R \) of the constituents, so that the strut is irremovable generically. It is worth mentioning that the formula (30) differs from the analogous expres-
sion obtained in [4], as our $\mathcal{F}$ does not contain a non-vanishing unphysical contribution due to magnetic charges.

Turning now to the thermodynamical quantities $\kappa$, $S$, $\Omega^H$, and $\Phi^H$ entering the Smarr mass formula (16), it may be observed that these must be calculated only for the upper black-hole constituent because the analogous set for the lower constituent is just $\kappa$, $S$, $-\Omega^H$, and $-\Phi^H$. Then for the upper constituent we get

$$\kappa = \frac{R \sigma [(R + 2M)^2 + 4Q^2]}{(R + 2M)^2 [2M^2 + Q^2] - Q^2 (R - 2M)},$$

$$S = 4\pi \left(1 + \frac{2M}{R}\right) \left(2M(M + \sigma) - \frac{Q^2 (R - 2M)(R - 2\sigma)}{(R + 2M)^2 + 4Q^2}\right),$$

$$\Omega^H = \frac{Ma[2(M - \sigma)(MR + 2M^2 + Q^2) - Q^2 (R - 2M)]}{(4M^2a^2 + Q^4)(MR + 2M^2 + Q^2)},$$

$$\Phi^H = \frac{Q[Q^2(M - \sigma)(MR + 2M^2 + Q^2) + 2M^2a^2(R - 2M)]}{(4M^2a^2 + Q^4)(MR + 2M^2 + Q^2)},$$

where $S$ is given in two different forms suitable for recovering the known limiting cases. The substitution of (31)-(34) into (16) shows that the above expressions satisfy identically Smarr’s formula for black holes.

It has been recently shown [8] that the equally charged black-hole constituents of the BM configuration saturate the Gabach-Clement inequality for black holes with struts [12] which reads

$$\sqrt{1 + 4\mathcal{F}} \geq \frac{\sqrt{(8\pi J)^2 + (4\pi Q^2)^2}}{S}. \quad (35)$$

In this respect it would be interesting to clarify whether the oppositely charged constituents of our dihole model saturate the inequality (35) too. The saturation means that the extremal dihole constituents must satisfy (35) with the equality sign. The extremality condition $\sigma = 0$ yields from (26) the value of $a$ at which the black-hole degeneration occurs:

$$a^2 = \frac{(MR + 2M^2 + Q^2)[M^2(R + 2M) - Q^2 (R - 2M)]}{M^2(R - 2M)[(R + 2M)^2 + 4Q^2]}, \quad (36)$$

and by substituting this $a$ into (32) and (35) we get ($J = Ma$)

$$S = \frac{4\pi (R + 2M)^2 [2M^2(R + 2M) - Q^2(R - 4M)]}{R[(R + 2M)^2 + 4Q^2]},$$

$$(8\pi J)^2 + (4\pi Q^2)^2 = \frac{16\pi^2 (R + 2M)[2M^2(R + 2M) - Q^2(R - 4M)]^2}{(R - 2M)[(R + 2M)^2 + 4Q^2]}. \quad (37)$$
Taking into account that \( F \) does not depend explicitly on \( a \), it is easy to check that (30) and (37) verify the equality in (35) identically. Therefore, independently of whether the KN black holes in a binary system have equal or opposite charges, the interaction force between them is governed by the Gabach-Clement inequality. Mention here one more common feature shared by the BM and dihole configurations – the extreme limit is achieved in both of them at a larger absolute value of \( a \) (for some given \( M \) and \( Q \)) than in the case of a single KN black hole whose extremality condition is simply \( a^2 = M^2 - Q^2 \).

Mention that in the paper \[4\] the Ernst potentials defining the extreme limit are given with errors. Therefore, we find it useful to give below the expressions for these potentials and corresponding metric functions of the entire 5-parameter EMR solution in the extreme limit \( \sigma \rightarrow 0 \):

\[
\begin{align*}
\mathcal{E} &= \frac{A - B}{A + B}, & \Phi &= \frac{C}{A + B}, \\
f &= \frac{N}{D}, & e^{2\gamma} &= \frac{N}{\alpha^8(x^2 - y^2)^4}, & \omega &= -\frac{4\alpha^2 \delta y(x^2 - 1)(1 - y^2)W}{N}, \\
A &= M^2 \alpha^2(x^4 - 1) + \alpha^2(\alpha^2 - M^2)(x^2 - y^2)^2 + (q^2 + b^2)(1 - y^2) \\
&\quad + 2i\alpha^2 \delta(x^2 - 2x^2y^2 + y^2), \\
B &= 2M\alpha x[\alpha^2(x^2 - y^2) - (M^2 - i\delta)(1 - y^2)], \\
C &= -2(q + ib)y[\alpha^2(x^2 - y^2) - (M^2 + i\delta)(1 - y^2)], \\
I &= -\alpha xy C - 2(q + ib)(M + \alpha x)(1 - y^2)[(M + \alpha x)^2 + (M^2 - \alpha^2)y^2 + i\delta(1 + y^2)], \\
N &= [M^2 \alpha^2(x^2 - 1)^2 + \alpha^2(\alpha^2 - M^2)(x^2 - y^2)^2 - (q^2 + b^2)(1 - y^2)]^2 \\
&\quad - 16\alpha^4 \delta^2 x^2 y^2(x^2 - 1)(1 - y^2), \\
D &= \{M^2 \alpha^2(x^4 - 1) + \alpha^2(\alpha^2 - M^2)(x^2 - y^2)^2 + (q^2 + b^2)(1 - y^4) + 2M\alpha x \\
&\quad \times [\alpha^2(x^2 - y^2) - M^2(1 - y^2)]]^2 + 4\alpha^2 \delta^2[\alpha(x^2 - 2x^2y^2 + y^2) + Mx(1 - y^2)]^2, \\
W &= M\alpha^2[(\alpha^2 - M^2)(x^2 - y^2)(3x^2 + y^2) + M^2(3x^4 + 6x^2 - 1) + 8M\alpha x^3] \\
&\quad - (q^2 + b^2)[4\alpha xy^2 - M(1 - y^2)^2], \\
\delta &= \sqrt{M^2(\alpha^2 - M^2) - q^2 - b^2}, & \alpha &= \frac{1}{2}R, \quad (38)
\end{align*}
\]

where the prolate spheroidal coordinates \((x, y)\) are related to the cylindrical coordinates \((\rho, z)\) by the formulas

\[
\begin{align*}
x &= \frac{1}{2\alpha}(r_+ + r_-), & y &= \frac{1}{2\alpha}(r_+ - r_-), & r_\pm &= \sqrt{\rho^2 + (z \pm \alpha)^2},
\end{align*}
\]

\[(39)\]
and where we have also given the explicit extremal form of the function \( I \) defining the magnetic potential \( A_\phi \) via formula (12).

As it follows from (13), the magnetic field in our dihole solution differs considerably from that of the particular \( b = 0 \) specialization of the EMR solution considered in [4]: in the former case it has a dipole character, while in the latter case it behaves itself like a magnetic octupole \( (B_3 = 2q\delta) \). In Figs. 2 and 3 this difference is illustrated by the plots of magnetic lines of force for two characteristic particular cases.

V. CONCLUSIONS

In our paper we succeeded in elaborating a physically consistent 4-parameter model for stationary diholes which generalizes the known dihole electrostatic solution earlier obtained by Emparan and Teo. The model is comprised of two identical (up to the sign of charges) counter-rotating KN black holes supported from falling onto each other by a massless strut, and its finding and correct mathematical description has turned out more sophisticated than in the case of counter-rotating equally charged KN black holes represented by the BM solution because the knowledge of a more general 5-parameter EMR solution was needed for getting rid of the specific individual magnetic charges initially present in the dihole components. The solution’s physical representation was advantageous for a direct check that the binary configuration it describes really saturates the Gabach-Clement inequality for interacting black holes.

Since the aforementioned inequality also takes into account the possibility for the black holes to carry magnetic charges, we would like to mention that our dihole solution can be very easily generalized to the case when the two KN constituents, besides the opposite electric charges, would have arbitrary opposite magnetic charges too, thus representing a pair of dyons [29]. To introduce an arbitrary magnetic charge \( B \) into our dihole model, one only needs to make the following substitutions in the formulas (26)-(29): change \( Q \) to \( Q = Q - iB \), and \( Q^2 \) to \( |Q|^2 = Q^2 + B^2 \) in all the occurrences. For instance, our expression (26) for \( \sigma \) will then assume the form

\[
\sigma = \sqrt{M^2 - \left( \frac{M^2a^2[(R + 2M)^2 + 4|Q|^2]}{[M(R + 2M) + |Q|^2]^2} + |Q|^2 \right) \frac{R - 2M}{R + 2M}}. \tag{40}
\]

We underline that the parameter \( B \) thus introduced will be a genuine individual magnetic
charge of the upper black hole, and this can be readily verified by means of the formula
\[ B = \frac{1}{4\pi} \int_H \omega A_{t,z} d\varphi dz. \] (41)

It is easy to see that the dyonic dihole model, which is of course equivalent to the 5-parameter EMR solution, will also saturate the Gabach-Clement inequality because the electric and magnetic charges \( Q_i \) and \( B_i \) enter that inequality only in the combination \( Q_i^2 + B_i^2 \) [12]. Mention also that the introduction of the magnetic charge does not actually modify the Smarr mass formula [16], provided the substitutions described above are carried out properly in the term \( Q^2 \Phi_{H}^2 \).

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FIG. 1: Location of sources on the symmetry axis for two branches of the dihole solution: (a) a black dihole configuration composed of two KN black holes; (b) a hyperextreme dihole configuration composed of two superextreme KN constituents.
FIG. 2: Magnetic lines of force plotted for the dihole solution in the particular case $R = 8, M = 3/2, a = 1/8, Q = 1/2.$
FIG. 3: Magnetic lines of force plotted for the specific EMR solution considered in [4]; the parameter choice (in the original notation for the charge parameter) is $R = 8$, $M = 3/2$, $\sigma = 1/4$, $Q_0 = 1/2$. 